The Bipartite Zero Forcing Set for a Full Sign Pattern Matrix

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Abstract: For an \( m \times n \) sign pattern \( P \), we define a signed bipartite graph \( B(U, V) \) with one set of vertices \( U = \{1, 2, \ldots, m\} \) based on rows of \( P \) and the other set of vertices \( V = \{1', 2', \ldots, n'\} \) based on columns of \( P \). The zero forcing number is an important graph parameter that has been used to study the minimum rank problem of a matrix. In this paper, we introduce a new variant of zero forcing set—bipartite zero forcing set and provide an algorithm for computing the bipartite zero forcing number. The bipartite zero forcing number provides an upper bound for the maximum nullity of a square full sign pattern \( P \). One advantage of the bipartite zero forcing is that it can be applied to study the minimum rank problem for a non-square full sign pattern.

Keywords: signed bipartite graph; bipartite zero forcing; directed constrained matching; minimum rank problem

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1. Introduction

A sign pattern matrix is a matrix whose entries are drawn from \{+, −, 0\}; a full sign pattern has entries in \{+, −\}. For a real matrix \( A \), \( \text{sgn}(A) \) is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of \( A \) by + (respectively, −, 0). If \( P \) is an \( m \times n \) sign pattern, the sign pattern class (or qualitative class) of \( P \), denoted \( \mathbb{P}(P) \), is the set of all \( A \in \mathbb{R}^{m \times n} \) such that \( \text{sgn}(A)=P \). The minimum rank of \( P \) is

\[
\text{mr}(P) = \min \{ \text{rank}(A) : A \in \mathbb{P}(P) \}
\]

and the maximum nullity of \( P \) is defined to be

\[
M(P) = \max \{ \text{null}(A) : A \in \mathbb{P}(P) \},
\]

Clearly, \( \text{rank}(A)+\text{null}(A)=\)the number of columns of \( A \).

The minimum rank problem, which asks us to determine the minimum rank among all real (symmetric or asymmetric) matrices whose zero-nonzero pattern of entries, is described by a given (undirected or directed) graph. Clearly, the solution of the minimum rank problem is equivalent to determining the maximum nullity problem. In order to study the minimum rank problem of a graph, some graph parameters have been used to bound the maximum nullity of a graph, and the relationship between the maximum nullity and the related graph parameters has received considerable attention (see [1–9]). The zero forcing number provides a method to bound the maximum nullity for a sign pattern. For a sign pattern, the signed zero forcing of a signed digraph was introduced in [10].
In order to present the zero forcing number of a graph, a color change rule was applied to the graph (see [1,7,10–12]). Zero forcing number is of interest in the study for an upper bound on the maximum nullity of a matrix with a given sign pattern. The signed zero forcing provided an upper bound for the maximum nullity of a given sign pattern (see [10]), while the signed zero forcing method is unsuitable for studying the minimum rank problem for a non-square sign pattern, such as

$$ P = \begin{bmatrix} - & + & - & - \\ - & - & + & - \\ - & - & - & + \\ - & - & - & + \end{bmatrix}. $$

In this paper, we give a new variant of zero forcing set—bipartite zero forcing set and provide an algorithm for computing the bipartite zero forcing number. The bipartite zero forcing number provides an upper bound for the maximum nullity of a matrix with a given full sign pattern. One advantage of the bipartite zero forcing set is that it can be applied to study the minimum rank problem for a non-square full sign pattern.

The outline of this paper is as follows: Section 2 presents an algorithm for a signed bipartite graph in order to construct a sub-signed bipartite graph with the maximum perfect matching $M'$ corresponding to every set of disjoint $M'$-interlacing cycles, which contain an even number of $M'$-interlacing $e$-cycles. In Section 3, we define the bipartite zero forcing set for a full sign pattern and obtain a bound for the maximum nullity of a real matrix $A$ with a given sign pattern $P$.

2. An Algorithm for a Signed Bipartite Graph

In this section, we introduce a signed constrained matching for the signed bipartite graph in order to study the minimum rank problem for a sign pattern.

A signed digraph is a digraph $\Gamma(P) = (V, E, sign)$, which consists of a digraph $(V, E)$ together with a sign function $sign: E \rightarrow \{1, -1\}$, denoted $sign(e), e \in E$, that is, the edges (arcs) of $\Gamma(P)$ have been signed positive or negative by sign. For a $n \times n$ sign pattern $P$, the graph of $P$ is a signed digraph $\Gamma(P)$. It is converted into a signed bipartite graph $G(U, V)$ with vertices $U = \{1, 2, \ldots, n\}$ based on rows of $P$ and $V = \{1', 2', \ldots, n'\}$ based on columns of $P$. If $(i, j)$-entry of $P$ is nonzero, then the sign of an edge $\{i, j\}$ of $G(U, V)$ is the same as the sign of the entry of $P$. The bipartite graph is important for the study of the regularity for graphs (see [13,14]).

In [2], the constrained matching in a bipartite graph has proven useful in the study of the zero forcing number of a loop directed graph, and the equivalence between a constrained matching in a bipartite graph and a zero forcing set in a directed graph was proved. We will give the definition of a signed constrained matching in a signed bipartite graph $G(U, V)$ in the following way:

- A set of $k$ signed edges $\{(i_1, j'_1), (i_2, j'_2), \ldots, (i_k, j'_k)\}$ with the same sign is said to be a signed $k$-matching between $U_1 = \{i_1, i_2, \ldots, i_k\}$ and $V_1 = \{j'_1, j'_2, \ldots, j'_k\}$ if vertices $i_1, i_2, \ldots, i_k$ are distinct as well as vertices $j'_1, j'_2, \ldots, j'_k$.
- A signed $k$-matching is called constrained if it is the only signed $k$-matching between $U$ and $V$.
- A signed constrained $k$-matching in $G(U, V)$ is said to be maximum if there is no (constrained) $s$-matching with $s > k$.

For a signed bipartite graph $G(U, V)$, if two edges with the same direction can meet at the same column, then they are called a c-pair. A cycle with an even (respectively, odd) number of c-pairs is called an e-cycle (respectively, o-cycle). If the entries of $P$ are resigned to make sure all diagonal entries of $P$ are non-zero, then these diagonal entries can correspond to a perfect matching $M$ in $G(U, V)$. A cycle in $G(U, V)$ containing each edge $(c, M(c))$ in $G(U, V)$, corresponding to any column $c$ it passes through, is called an interlacing relative to $M$, or an $M$-interlacing for short. The cycles relative to $M$ with no common vertices are called disjoint $M$-interlacing cycles (see [15]).
Lemma 1 ([15]). If \( \tau_1, \tau_2, \ldots, \tau_m \) are \( m \) cycles in \( M \)-interlacing cycles and \( \tau_i = (\tau_{i1} \tau_{i2} \cdots \tau_{il} \tau_{i1}) \), then

\[
\text{sign}(P) \prod_{i=1}^{n} P_{\dot{\tau}, \tau(i)} = \prod_{i=1}^{m} (-1)^{l_i-1} \prod_{i=1}^{l_i} P_{\dot{\tau}_j, \tau_{ij+1}} \prod_{k \notin \tau_j} P_{b_kk},
\]

where \( \text{sign}(P) \) denotes the sign of the permutation \( P \).

Lemma 2 ([15]). Let \( P \) be an \( n \times n \) sign pattern with each diagonal element nonzero, \( G(U,V) \) its associated signed bipartite graph and \( M \) the perfect matching in the bipartite graph \( G(U,V) \) of \( P \) corresponding to its diagonal.

1. For the determinant expansion of \( P \), the number of terms \( t(P) \) is equal to the number of disjoint \( M \)-interlacing cycles in \( G(U,V) \).

2. Let \( \varepsilon \) denote the sign of the product of the diagonal elements of \( P \). Then, the number of terms of \( (-\varepsilon) \) in the determinant expansion of \( P \) is equal to the cardinality of the set of all sets of disjoint \( M \)-interlacing cycles that contain an odd number of \( M \)-interlacing e-cycles.

We will change the condition in Lemma 2 to obtain the following theorem.

**Theorem 1.** Let \( P \) be an \( n \times n \) sign pattern with each diagonal element nonzero, \( G(U,V) \) its associated signed bipartite graph and \( A \) a real matrix in the sign pattern class of \( \mathbb{P}(P) \). Let \( M \) denote the perfect matching in \( B(U,V) \) of \( P \) corresponding to its diagonal. Then, the number of terms of \( \varepsilon \) in the determinant expansion of \( P \) is equal to the cardinality of the set of all sets of disjoint \( M \)-interlacing cycles that contain an even number of \( M \)-interlacing e-cycles.

**Proof.** According to Equation (1), every term \( x \) in the determinant expansion of \( P \) is \( \text{sign}(P) \prod_{i=1}^{n} P_{\dot{\tau}, \tau(i)} \) and the sign of \( x \) is determined by \( \text{sign}(P) \cdot \text{sign}(\text{disjoint } M \text{-interlacing cycles}) \), where the disjoint \( M \)-interlacing cycles are \( \tau_1, \tau_2, \ldots, \tau_l \) and \( \tau_i = (\tau_{i1} \tau_{i2} \cdots \tau_{il} \tau_{i1}) \) (\( i = 1, \ldots, l \)).

The cycle \( B(\tau_i) \) corresponding to \( \tau_i \) is defined to be the subgraph

\[
R(\tau_{i1}) - C(\tau_{i2}) - R(\tau_{i2}) - C(\tau_{i3}) - \cdots - R(\tau_{il}) - C(\tau_{il}) - R(\tau_{i1})
\]

of \( B(U,V) \).

By permutation, we suppose that all diagonal entries of \( P \) are negative. Then, \( \varepsilon = (-1)^n \) and we consider other terms with sign \( \varepsilon \). By Equation (1) of Lemma 1,

\[
x = \prod_{i=1}^{l}((-1)^{l_i-1} \prod_{i=1}^{l_i} P_{\dot{\tau}_j, \tau_{ij+1}} \prod_{k \notin \tau_j} P_{b_kk},
\]

where \( \prod_{i=1}^{l} P_{\dot{\tau}_j, \tau_{ij+1}} = (-1)^{\text{number of c-pairs in } B(\tau_i)} = : \text{sign}(B(\tau_i)). \) Because of all negative diagonal entries, the sign of \( x \) equals

\[
(-1)^{n-l} \prod_{i=1}^{l} \text{sign}(B(\tau_i)) = (-1)^n.
\]

In order for the term \( x \) to have sign \( (-1)^n \), \( \prod_{i=1}^{l} \text{sign}(B(\tau_i)) \) must be equal \( (-1)^l \). We will discuss the following two cases:
Case 1: Let \( l \) be odd, then

\[
\prod_{i=1}^{l} \text{sign}(B(\tau_i)) = (-1)^l
\]

\[\iff\]

\[
\prod_{i=1}^{l} \text{sign}(B(\tau_i)) = -1
\]

\[\iff\]

\[
(-1)^{N(\text{o-cycles among } \tau_1, \ldots, \tau_l)} = -1
\]

\[\iff\]

\[
N(\text{o-cycles among } \tau_1, \ldots, \tau_l) \text{ is odd}
\]

\[\iff\]

\[
N(\text{e-cycles among } \tau_1, \ldots, \tau_l) \text{ is even},
\]

where \( N(\text{o-cycles among } \tau_1, \ldots, \tau_l) \) is the number of o-cycles among \( \tau_1, \ldots, \tau_l \), a similar statement holds for e-cycles as well.

Case 2: Let \( l \) be even, then

\[
\prod_{i=1}^{l} \text{sign}(B(\tau_i)) = (-1)^l
\]

\[\iff\]

\[
\prod_{i=1}^{l} \text{sign}(B(\tau_i)) = 1
\]

\[\iff\]

\[
(-1)^{N(\text{o-cycles among } \tau_1, \ldots, \tau_l)} = 1
\]

\[\iff\]

\[
N(\text{o-cycles among } \tau_1, \ldots, \tau_l) \text{ is even}
\]

\[\iff\]

\[
N(\text{e-cycles among } \tau_1, \ldots, \tau_l) \text{ is even}.
\]

The proof of this theorem is similar to the proof of Theorem 2.9 in [15].

For an \( n \times n \) sign pattern \( P \), let \( \alpha, \beta \) be subsets of \( \{1, 2, \ldots, n\} \). We denote by \( P[\alpha|\beta] \) \((P(\alpha|\beta))\) the submatrix of \( P \) lying in (deleting) the rows indicated by \( \alpha \) and the columns indicated by \( \beta \). If \( \alpha = \beta \), the submatrix \( P[\alpha|\alpha] \) \((P(\alpha|\alpha))\) is abbreviated to \( P[\alpha] \) \((P(\alpha))\).

If \( P \) is a square full sign pattern associated with its signed bipartite graph \( G(U, V) \), with one set of vertices \( U = \{1, 2, \ldots, n\} \) based on rows of \( P \) and the other set of vertices \( V = \{1', 2', \ldots, n'\} \) based on columns of \( P \). For an \( k \times k \) square sub-sign pattern of \( P \), there is an associate sub-signed bipartite graph \( G(U', V') \) of \( G(U, V) \).

We will provide the following Algorithm 1 to construct a sub-signed bipartite graph with the maximum perfect matching \( M' \) corresponding to every set of disjoint \( M'\)-interlacing cycles, which contain an even number of \( M'\)-interlacing e-cycles.
Algorithm 1: The construction for a sub-signed bipartite graph with the maximum perfect matching

Let \( G(U_i, V_i) \) be a signed bipartite graph associated with its sign pattern \( P_i \) and \( M_i \) be the signed perfect matching in \( G(U_i, V_i) \) corresponding to \( P_i \)’s diagonal. This algorithm produces a sub-signed bipartite graph \( G(U', V') \) with the maximum perfect matching \( M' \) corresponding to every set of disjoint \( M'_i \)-interlacing cycles, which contain an even number of \( M'_i \)-interlacing \( e \)-cycles.

Set \( G(U_1, V_1) = G(U, V) \) with one set of vertices \( U = \{1, 2, \ldots, n\} \) based on rows of \( P \) and the other set of vertices \( V = \{1', 2', \ldots, n'\} \) based on columns of \( P \), \( G(U', V') = \emptyset \) and \( i = 1 \).

While \( M_i \neq M' \):
1. Search for every set of the disjoint \( M_i \)-interlacing cycles and count the number of each \( k_i \) of every set of the disjoint \( M_i \)-interlacing \( e \)-cycles.
2. If each \( k_i \) is even, then \( G(U', V') = G(U_i, V_i) \).
3. Else delete a vertex \( r \) from \( U_i \) and a vertex \( r' \) from \( V_i \) in a disjoint \( M_i \)-interlacing \( e \)-cycles and all edges adjacent to \( r \) and \( r' \), and obtain a sub-signed bipartite \( G'(U_i, V_i) \) of \( G(U_i, V_i) \).
4. \( i = i + 1 \).

Example 1. Let

\[
P = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}
\]

be a sign pattern of order 4 whose signed bipartite graph is \( G(U_1, V_1) \) with one set of vertices \( U_1 = \{1, 2, 3, 4\} \) based on rows of \( P \) and the other set of vertices \( V_1 = \{1', 2', 3', 4'\} \) based on columns of \( P \) (see Figure 1).

The perfect matching \( \{(1, 2'), (2, 3'), (3, 1'), (4, 4')\} \) is a signed 4-matching, but it is not constrained (see Figure 2). If we delete the vertex 1 from \( U \) and the vertex 1' from \( V \), then the perfect matching \( \{(2, 3'), (3, 2'), (4, 4')\} \) is a signed constrained 3-matching (see Figure 3).

The cycle \( 3 - 4' - 4 - 3' - 3 \) has one \( c \)-pair and thus is an \( o \)-cycle. On the other hand, the cycle \( 1 - 3' - 3 - 4' - 4 - 2' - 2 - 1' - 1 \) has two \( c \)-pairs and so is an \( e \)-cycle.

Let \( M_i \) be the perfect signed matching \( \{(1, 1'), (2, 2'), (3, 3'), (4, 4')\} \) corresponding to its diagonal. Search for every set of the disjoint \( M_i \)-interlacing cycles and the disjoint \( M_i \)-interlacing \( 2 - 2', 3 - 3', 1 - 4' - 4 - 1' - 1 \) or \( 1 - 1', 4' - 4, 2 - 3' - 3 - 2' - 2 \) has one \( e \)-cycle. We delete the vertices 2 and 2' from \( M_1 \)-interlacing and all edges adjacent 2 and 2' from \( B(U_1, V_1) \), and obtain a sub-signed bipartite graph \( B(U_2, V_2) \) of \( B(U_1, V_1) \) according to Algorithm 1. However, each set of disjoint \( M_2 \)-interlacing cycles does not contain an even number of \( M_2 \)-interlacing \( e \)-cycles in \( B(U_2, V_2) \), and we will continue (1) of Algorithm 1.

![Figure 1. The signed bipartite graph \( B(U_1, V_1) \).](image-url)
Figure 2. The perfect matching \(\{(1, 2'), (2, 3'), (3, 1'), (4, 4')\}\).

Figure 3. The signed constrained 3-matching \(\{(2, 3'), (3, 2'), (4, 4')\}\).

Remark 1. The dashed lines denote negative edges, and full lines denote positive edges in all of the above figures.

3. Bipartite Zero Forcing Set for a Full Sign Pattern

Zero forcing is a combinatorial game played on the vertices of a graph, during which the vertices are colored black or white. The coloring change rule is to change the color of a white vertex \(w\) to black; in this case, we say \(u\) forces \(w\) and write \(u \rightarrow w\). Then, a color-change rule is applied until no more changes are possible. This rule is different in a simple undirected graph or a directed graph or a signed digraph. However, the following definition is identical for graphs of any type (see [7]).

Definition 1. Let \(G = (V, E)\) be a graph.

- A subset \(Z \subseteq V\) defines an initial set of black vertices, called a coloring.
- Given a coloring of \(G\), the derived set is the set of black vertices obtained by applying the color change rule until no more changes are possible.
- A zero forcing set for \(G\) is a subset of vertices \(Z\) such that, if, initially, the vertices in \(Z\) are colored black and the remaining vertices are colored white, the derived coloring of \(G\) is all black. The zero forcing number \(Z(G)\) is the minimum number over all zero forcing sets.
- For a given zero forcing set, construct the derived set and list the forces in order in which they were performed to color the vertices of \(G\) in black. This list is called a chronological list of forces.

For example, a zero forcing set for a path is an endpoint and a zero forcing set for a cycle is any set of two adjacent vertices.

In order to give the definition of the bipartite zero forcing set, we will convert a signed bipartite graph \(G(U, V)\) into a directed bipartite graph \(B(U, V)\) in the following way:

- if sign\((i, j') = +\) (or sign\((i, j') = -\)) in \(G(U, V)\), vertex \(j'\) is an out-neighbor (or in-neighbor) of \(i\) in \(B(U, V)\); that is to say, \((i, j') \in E\) (or \((j', i) \in E\)) is a directed edge of \(B(U, V)\).
- the out-degree of \(i\), denoted by \(\deg^+(i)\), is the number of out-neighbors of \(i\) in \(B(U, V)\), and similarly for in-degree, denoted by \(\deg^-(i)\). The smallest out-degree among the vertex set \(U\) (or \(V\)) is denoted by \(\delta^+(U) = \min\{\deg^+(i) : i \in U\}\) (or \(\delta^+(V) = \min\{\deg^+(j') : j' \in V\}\)) and analogously for the smallest in-degree among the vertex set \(U\) (or \(V\)), denoted by \(\delta^-(U) = \min\{\deg^-(u) : u \in U\}\) (or \(\delta^-(V) = \min\{\deg^-(j') : j' \in V\}\)).
We will give the definition of directed constrained matching in a directed bipartite graph $B(U, V)$ in the following way:

- A set of $k$ directed edges $\{(i_1, j'_1), (i_2, j'_2), \ldots, (i_k, j'_k)\}$ with the same direction is said to be a directed $k$-matching between $U_1 = \{i_1, i_2, \ldots, i_k\}$ and $V_1 = \{j'_1, j'_2, \ldots, j'_k\}$ if vertices $i_1, i_2, \ldots, i_k$ are distinct as well as vertices $j'_1, j'_2, \ldots, j'_k$.
- A directed $k$-matching is called constrained if it is the only directed $k$-matching between $U$ and $V$.
- A directed constrained $k$-matching in $B(U, V)$ is a said to be maximum if there is no (constrained) $s$-matching with $s > k$.

**Observation 1.** Let $B(U, V)$ be a directed bipartite graph. If $j' \in V$ is a vertex with $\deg^-(j') = 0$ in $V$, then $j'$ is in every zero forcing set of $V$.

**Rule A** ([7]). (The color-change rule for a digraph $\Gamma$). If $\Gamma$ is a simple digraph with each vertex colored either white or black, $u$ is a black vertex, and exactly one out-neighbor $v$ of $u$ is white; then, change the color of $v$ to black.

Let $A$ be a matrix associated with a sign pattern $P$. The sign inversion function $l$ is defined by $l(+) = -$ and $l(-) = +$; the letters $s$ and $t$ will be used as indices taking values in $\{+, -, 0\}$ and $s \cdot t$ will be defined as $+$ if $s = t$ and as $-$ if $s \neq t$. The signed zero forcing rule was introduced in [10]: if $w$ is a white vertex, we will denote by $m(w)$ its marker if it is marked; otherwise, we will write $m(w) = *$.

**Rule B** ([10]). (Signed zero forcing rule for sign pattern $P$). Let $u$ be a vertex of $P$ such that either $u$ is black or $u$ is white and $P_{uw} \neq ?$. Let

$$W = \{w \mid w \text{ is white } \land w \rightarrow u\} \cup \{u\}.$$ Define

$$W_+ = \{w \in W \mid m(w) = P_{uw}\}, W_- = \{w \in W \mid m(w) \neq P_{uw}\}$$

and

$$W_* = \{w \in W \mid m(w) = *\}.$$

(a) If $W = \{w\}$, color $w$ black.
(b) If $W_+ = W$ or $W_- = W$, color all vertices $w$ black.
(c) If $W_\emptyset \neq \emptyset, W_{l(s)} = \emptyset$, and $W_* = \{w\}$, mark $w$ with $P_{uw} \cdot l(s)$.
(d) If no white vertices in the whole graph are marked and $u$ is white, then mark $u$ with $+$.

The signed zero forcing number of $P$, $Z_\pm(P)$, is the size of the minimum forcing set in signed zero forcing rule. It provided an upper bound for the maximum nullity $M^\pm(P)$ of a matrix with a given sign pattern, that is, $M^\pm(P) \leq Z_\pm(P)$. In order to do better at the signed zero forcing game, a new variant-branched signed zero forcing was defined, denoted by $Z_{bL}^\pm(P)$, and $Z_{bL}^\pm(P) < Z_\pm(P)$, while the signed zero forcing method is unsuitable for providing an upper bound for the maximum nullity of a non-square sign pattern. We define a new variant-bipartite zero forcing for a directed bipartite graph in order to study the minimum rank problems for a square full sign pattern and a non-square sign pattern.

Let $B(U, V)$ be a directed bipartite graph, where all vertices of $U$ are colored black and each vertex of $V$ is colored either white or black. A new zero forcing rule will be applied.

**Rule C.** Given an $n \times n$ full sign pattern $P$. Let $B(U, V)$ denote its directed bipartite graph with $U = \{1, 2, \ldots, n\}$ based on rows and $V = \{1', 2', \ldots, n'\}$ based on columns, and $M$ be the perfect matching corresponding to its nonzero diagonal.

(a) If there exists a vertex $j'$ with $\deg^-(j') = 0$ in $V$, then mark $j'$ as a black vertex and delete the vertex $j'$ (corresponding to the $j'$th column of $P$) and all edges adjacent to $j'$. Otherwise, we will return (b).
(b) If there exists a set of disjoint $M$-interlacing cycles with an odd number of $M$-interlacing $e$-cycle in $B(U, V)$, then delete some vertex $i$ (corresponding to the $i$’th row of $P$) from the $M$-interlacing $e$-cycle and all edges adjacent to $i$ from $B(U, V)$ according to Algorithm 1 (2), and we will return

(a). Let $B(U_1, V_1)$ denote a sub-directed bipartite graph with $U_1 = U - \{i\}$ based on rows of $P(\{i\}|\{j\})$ and $V_1 = V - \{j'\}$ based on columns of $P|\{i\}|\{j\}$.

(c) If there exists a maximum directed constrained matching \{$(i_1, j'_{1})$, $(i_2, j'_{2})$, ..., $(i_{n-1}, j'_{n-1})$\} in $B(U_1, V_1)$, then change the signs of all the edges $(i_1, j'_{1})$, $(i_2, j'_{2})$, ..., $(i_{n-1}, j'_{n-1})$ to zero. Otherwise, we will return (b). Let $P'(\{i\}|\{j\})$ be a sign pattern with these entries $p_{ij,k}(k = 1, 2, ..., n - 1)$ zero. Let $B(U_2, V_2)$ denote a directed sub-bipartite graph with $U_2 = U - \{i\}$ based on rows of $P'(\{i\}|\{j\})$, and $V_2 = V - \{j'\}$ based on columns of $P'(\{i\}|\{j\})$.

(d) If $P'(\{i\}|\{j\})$ is non-singular according to Theorem 1, then change these vertices $j'_{t}$ ($t = 1, 2, ..., n-1$) to black, $i_1 \rightarrow j'_{1}, i_2 \rightarrow j'_{2}, ..., i_{n-1} \rightarrow j'_{n-1}$ is a chronological list of forces. Otherwise, we will repeatedly apply (a) and (b).

Let $S$ be a deleted row set of $P$ and $T$ be a set of the black vertices from $V$ in Rule C. Let $P'$ be a sign pattern with entries $p_{ji,t}$ zero for $t = 1, 2, ..., k$ and $B(U', V')$ its associated directed bipartite graph with $U' = U - S$ and $V' = V - T$.

**Definition 2.** A bipartite zero forcing set for a sign pattern $P$ is a subset $T$ such that if, initially, the vertices in $V$ are colored black and the remaining vertices are colored white in $B(U, V)$, the derived coloring of $B(U, V)$ is all black. The bipartite zero forcing number of a square sign pattern $P$, denoted by $Z_b(P)$, is the size of the minimum forcing set in $V$.

In order to obtain the following theorem, we recall the zero forcing set for a graph (see [7]).

Let $A$ be an $n \times n$ real matrix and $x$ be a vector in the kernel of $A$:

- A black vertex is associated with a coordinate in $x$ that is required to be zero,
- A white vertex indicates a coordinate that can be either zero or non-zero,
- Changing a vertex from white to black, it is essentially noted that the corresponding coordinate is forced to be zero if $x$ is in the kernel of $A$ and all black vertices indicate coordinates assumed to be or previously forced to be 0.

The set of black vertices is called the zero forcing set.

**Theorem 2.** If \{$(i_1, j'_{1})$, $(i_2, j'_{2})$, ..., $(i_k, j'_{k})$\} is a directed constrained $k$-matching for $B(U, V)$ in Rule C, then the signs of all the edges $(i_1, j'_{1})$, $(i_2, j'_{2})$, ..., $(i_k, j'_{k})$ are changed to zero.

**Proof.** Vertex $i_1$ has only vertex $j'_{1}$ as a white out-neighbor. Thus, the vertex $i_1$ will color vertex $j'_{1}$ black by applying Rule A. Since $j'_{1}$ has been colored in black, vertex $i_2$ has only vertex $j'_{2}$ as white out-neighbor and $j'_{2}$ is colored in black, and so on for all vertices $j'_{3}, ..., j'_{k}$.

Now, let $A$ be a real matrix associated with a full sign pattern $P$ whose graph is a directed bipartite graph $B(U, V)$. The kernel of matrix $A$ will be denoted by $\ker(A)$ and let a real vector $x \in \ker(A)$. We have for any vertex $i_t$ of $U$:

$$\left( Ax \right)_{i_t} = a_{i_t,j}x_j + \sum_{j' \in T, j \neq j'} a_{i_t,j'}x_{j'} + \sum_{j'' \in V - T} a_{i_t,j''}x_{j''} = 0,$$

where $T$ is a bipartite zero forcing set for $B(U, V)$, $x_{j''} = 0$. Because $i_1 \rightarrow j'_{1}, ..., i_l \rightarrow j'_{l}, ..., i_k \rightarrow j'_{k}$ is a chronological list of forces $(j' \in \{j'_{1}, j'_{2}, ..., j'_{l}, ..., j'_{k}\}), x_{j''} = 0$. Then, (2) reduces to

$$\sum_{j'' \in V - T} a_{i_t,j''}x_{j''} = 0.$$
Theorem 3. Let $P$ be a square full sign pattern. Then, $M_B$ is the minimum zero forcing set for the first row of $P$ edges adjacent to 1 black vertex (see Figure 4) and delete the vertex 1.

Example 2. $P$ as signature equivalent or permutation equivalent to more of these operations is called equivalent to $P$. $P$ not change the minimum rank of $P$ does not have any zero entries on the main diagonal. The bipartite zero forcing number is of interest in the study of an upper bound on the maximum nullity of a matrix with a given sign pattern.

**Theorem 3.** Let $P$ be a square full sign pattern. Then, $M^R(P) \leq Z_b(P)$ according to Rule C.

**Proof.** Let $A$ be an $n \times n$ real matrix with a full sign pattern $P$ and $B(U, V)$ be a directed bipartite graph with vertices $U = \{1, 2, \ldots, n\}$ based on rows of $P$ and $V = \{1', 2', \ldots, n'\}$ based on columns of $P$. The sub-vector of real vector $x$ is formed on the entries indexed by a set $T$, denoted by $x|_T$, where $T$ is the minimum zero forcing set for $B(U, V)$.

We will argue that, for $x \in \ker(A)$, if $x|_T = 0$, then $x = 0$.

Indeed, by applying Rule C(a), Rule C(b), and Rule C(c), we have Equation (3) for every vertex $i \in U$ and obtain a linear system with $|V - T|$ equations in $|V - T|$ variables, and the sign pattern of the coefficient matrix $(a_{i,j,s})$ of (3) is $P'(S|T)$ and it is non-singular according to Theorem 1. Thus, all $x_{j,t} = 0$ and $x = 0$. \[\square\]

A permutation sign pattern is an $n \times n$ square sign pattern each of whose rows and columns contain exactly one + entry and $n-1$ zero entries. A signature pattern is a diagonal sign pattern that does not have any zero entries on the main diagonal.

Pre-or post-multiplication of a sign pattern $P$ by a signature pattern or permutation pattern does not change the minimum rank of $P$. A sign pattern $\bar{P}$ that is obtained from $P$ by performing one or more of these operations is called equivalent to $P$; if only one type of operation is used, $\bar{P}$ is referred to as signature equivalent or permutation equivalent to $P$.

**Example 2.** Let

$$P = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & - \end{bmatrix}$$

be the Hadamard sign pattern of order 4.

By pre-multiplying a negative diagonal sign pattern, we obtain

$$P_1 = \begin{bmatrix} - & - & - & - \\ - & + & + & - \\ - & - & + & + \\ - & + & + & - \end{bmatrix}.$$

Let $B(U, V)$ denote its directed bipartite graph. The vertex $1'$ with $\deg^-(1') = 0$ is marked as a black vertex (see Figure 4) and delete the vertex $1'$ (corresponding to the first column of $P_1$) and all edges adjacent to $1'$ by applying Rule C(a). We apply Rule C(b) to delete the vertex 1 (corresponding to the first row of $P_1$) from the disjoint interlacing $1 - 3' - 3 - 4' - 4 - 2' - 2 - 1'$.

$B(U_1, V_1)$ is a directed sub-bipartite graph with $U_1 = \{2, 3, 4\}$ based on rows of $P_1(\{1\}|\{1\})$ and $V_1 = \{2', 3', 4'\}$ based on columns of $P_1(\{1\}|\{1\})$ (see Figure 5).
There exists a directed constrained 3-matching \{(2, 3'), (3, 2'), (4, 4')\} in \(B(U_1, V_1)\), then change the signs of edges \((2, 3')\), \((3, 2')\) and \((4, 4')\) to zero according to Rule C(c) (see Figure 6). We obtain that

\[
P'_1(\{1\}|\{1\}) = \begin{bmatrix} + & 0 & + \\ 0 & + & + \\ + & + & 0 \end{bmatrix}
\]

is non-singular. Then, change the vertices 2', 3' and 4' to black according to Rule C(d). Therefore, the bipartite zero forcing set of \(P\) is \(\{1'\}\), \(Z_b(P) = 1\) and \(M(P) \leq 1\).

![Figure 4. The bipartite zero forcing set of \(P\).](image)

![Figure 5. The directed sub-bipartite graph \(B(U_1, V_1)\).](image)

![Figure 6. The blackened vertices 2', 3', 4'.](image)

**Example 3.** Let

\[
P = \begin{bmatrix} - & + & + \\ - & - & + \\ - & - & - + \\ - & - & - \\ - & - & - \end{bmatrix}
\]

be a full sign pattern of order 4.

By applying Rule B, color the vertex 1 black, mark vertex 2 with +, and consider three options for vertex 3: marked with +, marked with −, or black.
Case 1: \( m(3) = + \). Let \( u = 1 \), then we have \( W_+ = \{2, 3\} \) and \( W_* = \{4\} \). Therefore, we can mark 4 with \( P_{14} \cdot I(+) = + \cdot - = - \). Now, let \( u = 3 \) and we obtain \( W_- = \{2, 3, 4\} \). This means we can blacken all three white vertices.

Case 2: \( m(3) = - \). Let \( u = 2 \), then we have \( W_+ = \{2, 3\} \) and \( W_* = \{4\} \). Therefore, we can mark 4 with \( P_{24} \cdot I(-) = + \cdot + = + \). Now, let \( u = 3 \) (or 1, 4) and we can not obtain \( W_- = \{2, 3, 4\} \) (or \( W_* = \{2, 3, 4\} \))

Case 3: 3 is black. Let \( u = 1 \); then, we have \( W_+ = \{2\} \) and \( W_* = \{4\} \). Therefore, we can mark 4 with \( P_{14} \cdot I(+) = + \cdot - = - \). Now, let \( u = 3 \) and we obtain \( W_- = \{2, 4\} \). All white vertices can be blackened once again.

We have obtained \( Z^b_-(P) = 1 \) and \( Z^b(P) > 1 \).

We are going to show that \( Z_0(P) = 2 \) for \( P \) by applying Rule C.

Let \( B(U, V) \) denote its directed bipartite graph with \( U = \{1, 2, 3, 4\} \) based on rows of \( P \) and \( V = \{1', 2', 3', 4'\} \) based on columns of \( P \). Now, we apply Rule C(b) to delete the vertex 4 (corresponding to the fourth row of \( P \) ) from the disjoint interlacing \( 1 - 3' - 3 - 4' - 4 - 2' - 2 - 1' - 1 \). The vertex \( 1' \) with \( \deg^-(1') = 0 \) is marked as a black vertex and delete the vertex \( 1' \) (corresponding to the first column of \( P \)) and all edges adjacent to \( 1' \) by applying Rule C(a).

Let \( B(U_1, V_1) \) denote the directed sub-bipartite graph with \( U_1 = \{1, 2, 3\} \) based on rows of \( P(\{4\}\{1\}) \) and \( V_1 = \{2', 3', 4'\} \) based on columns of \( P(\{4\}\{1\}) \). There exists a directed constrained 3-matching \( \{1, 2', 3', 3', 4'\} \) in \( B(U_1, V_1) \), then change the signs of edges \( (1, 2'), (2, 3') \) and \( (3, 4') \) to zero according to Rule C(c). We obtain that

\[
P'((4)\{1\}) = \begin{bmatrix} 0 & + & + \\ - & 0 & + \\ - & - & 0 \end{bmatrix}
\]

is not non-singular.

By pre-multiplying a negative diagonal sign pattern, we obtain

\[
P((4)\{1\}) = \begin{bmatrix} + & - & - \\ + & - & - \\ + & + & - \end{bmatrix}.
\]

In \( B(U_1, V_1) \), we again apply Rule C(b) to delete the vertex 1 (corresponding to the first row of \( P \) ) from the disjoint interlacing \( 1 - 2' - 2 - 3' - 3 - 1' - 1 \). The vertex \( 4' \) with \( \deg^+(4') = 0 \) is marked as a black vertex and delete the vertex \( 4' \) (corresponding to the fourth column of \( P \)) and all edges adjacent to \( 4' \) by applying Rule C(a).

Let \( B(U_2, V_2) \) denote the directed sub-bipartite graph with \( U_2 = \{2, 3\} \) based on rows of \( P(\{1, 4\}\{1, 4\}) \) and \( V_2 = \{2', 3'\} \) based on columns of \( P(\{1, 4\}\{1, 4\}) \). There exists a directed constrained 2-matching \( \{2, 2', 3, 3'\} \) in \( B(U_2, V_2) \), then change the signs of edges \( (2, 2') \) and \( (3, 3') \) to zero according to Rule C(c). We obtain that

\[
P' = \begin{bmatrix} 0 & + \\ - & 0 \end{bmatrix}
\]

is non-singular. Then, change these vertices \( 2' \) and \( 3' \) to black according to Rule C(d). Therefore, the bipartite zero forcing set of \( P \) is \( \{1', 4'\} \), \( Z_0(P) = 2 \) and \( M(P) \leq 2 \).
In fact, the $4 \times 4$ integer Toeplitz matrix
\[
T = \begin{bmatrix}
-1 & 1 & 3 & 5 \\
-3 & -1 & 1 & 3 \\
-5 & -3 & -1 & 1 \\
-7 & -5 & -3 & -1
\end{bmatrix}
\]
whose sign pattern is $P$ has rank 2.

One advantage of the bipartite zero forcing is that it can be not only applied to a square sign pattern but also applied to a non-square sign pattern. A non-square minimum rank problem will be converted to a square minimum rank problem.

**Definition 3.** Suppose an $m \times n$ full sign pattern $P$ associated with directed bipartite graph is $B(U, V)$ with $U = \{1, 2, \ldots, m\}$ based on rows and $V = \{1', 2', \ldots, n'\}$ based on columns. $B(U, V)$ is balanced if $m = n$. If $m \neq n$, an $k \times k$ square sub-sign pattern of $P_0$ can be picked out from a non-square full sign pattern $P$ and hence induces a sub-bipartite graph $B(U_0, V_0)$ of $B(U, V)$. Such a sub-sign pattern and the sub-bipartite graph are both said to be balanced.

**Rule D** (Bipartite zero forcing rule for a non-square full sign pattern $P$) Given an $m \times n$ ($m \neq n$) full sign pattern $P$, let $B(U, V)$ denote its directed bipartite graph with $U = \{1, 2, \ldots, m\}$ based on rows and $V = \{1', 2', \ldots, n'\}$ based on columns. (a) Search for a maximum directed constrained $k$-matching for $B(U, V)$ by applying Algorithm 1. Let $R$ be a set consisting of all deleted vertices from $U$ and $L$ be a set consisting of all deleted vertices from $V$. Let $B(U_0, V_0)$ be a balanced directed bipartite graph with $U_0 = U - R$ and $V_0 = V - L$. Suppose $P_0$ is an $k \times k$ sign pattern associated with $B(U_0, V_0)$. (b) Compute the bipartite zero forcing number of $P_0$ by applying Rule C.

**Example 4.** Let
\[
P = \begin{bmatrix}
- & + & - & - & - \\
- & - & + & - & - \\
- & - & - & + & - \\
- & - & - & - & +
\end{bmatrix}
\]
be a $4 \times 5$ sign pattern.

Let $B(U, V)$ denote its directed bipartite graph with $U = \{1, 2, 3, 4\}$ based on rows of $P$ and $V = \{1', 2', 3', 4', 5'\}$ based on columns of $P$ (see Figure 7). Now, we apply Rule D(a) to obtain the directed constrained $\{(1', 2'), (2', 3'), (3, 4'), (4, 5')\}$ and delete $1'$ from $V$ and all edges adjacent to $1'$. We obtain a $4 \times 4$ sub-sign pattern $P_0$. Let $B(U_0, V_0)$ denote the balanced directed sub-bipartite graph with $U_0 = \{1, 2, 3, 4\}$ and $V_0 = \{2', 3', 4', 5'\}$ (see Figure 8). Then, change the signs of edges $(1, 2'), (2', 3'), (3, 4')$ and $(4, 5')$ to zero according to Rule C(c).

In this case, we obtain that
\[
P_0' = \begin{bmatrix}
0 & - & - & - \\
- & 0 & - & - \\
- & - & 0 & - \\
- & - & - & 0
\end{bmatrix}
\]
is not non-singular.
Figure 7. $B(U, V)$.

Figure 8. The balanced directed sub-bipartite graph.

For $B(U_0, V_0)$, we apply Rule C(b) to delete the vertex 1 (corresponding to the first row of $P$) from the disjoint interlacing $1 - 3' - 2 - 4' - 3 - 5' - 4 - 2' - 1$. The vertex $2'$ with deg $^-(2') = 0$ is marked as a black vertex and delete the vertex $2'$ (corresponding to the second column of $P$) and all edges adjacent to $2'$ by applying Rule C(a).

Let $B(U_1, V_1)$ denote the directed sub-bipartite graph with $U_1 = \{2, 3, 4\}$ based on rows of $P(\{1\} | \{1, 2\})$ and $V_1 = \{3', 4', 5'\}$ based on columns of $P(\{1\} | \{1, 2\})$. There exists a directed constrained 3-matching $\{(2, 3'), (3, 4'), (4, 5')\}$ in $B(U_1, V_1)$, then change the signs of edges $(2, 3'), (3, 4')$ and $(4, 5')$ to zero by applying Rule C(c). We obtain that

\[
P'(\{1\} | \{1, 2\}) = \begin{bmatrix} 0 & - & - \\ - & 0 & - \\ - & - & 0 \end{bmatrix}
\]

is non-singular. Then, change these vertices $3'$, $4'$ and $5'$ to black according to Rule C(d). Therefore, the bipartite zero forcing set of $P$ is $\{2'\}$, $Z_b(P) = 1$ and $M(P) \leq 1$.

4. Conclusions

In this paper, we have given a new variant—bipartite zero forcing of a directed bipartite graph $B(U, V)$ associated with its sign pattern $P$. The problem of computing bipartite zero forcing number $Z_b(P)$ is equivalent to that of searching for a maximum constrained matching in $B(U, V)$. Thus, we have provided an algorithm to search for a maximum constrained matching in $B(U, V)$ and have obtained an upper bound for the maximum nullity $M^R(P)$ of a real square matrix $A$ with a given full sign pattern $P$. One advantage of the bipartite zero forcing is that it can also be applied to study the minimum rank problem for a non-square full sign pattern. We have extended Rule C to a non-square sign pattern by converting the non-square minimum rank problem to the square minimum rank problem. The problem of controllability of the directed network may be considered by applying the bipartite zero forcing number. Finally, we put forward an open question how to solve an upper bound for the maximum nullity of a no full sign pattern $P$ by applying the bipartite zero forcing set.
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