Braneworld Cosmological Perturbations in Teleparallel Gravity

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Abstract. In this paper we find the fully gauge invariant cosmological perturbation equations in braneworld teleparallel gravity. In this theory, perturbations are the result of small fluctuations in the pentad field. We derive the gauge invariant 'potentials' for both geometric and matter variables. In teleparallel gravity, pentad perturbations can only contain scalar and vector modes. This is in contrast to the metric fluctuations in general relativity.

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1. Introduction and notation

The idea that some hidden extra dimensions exist in our spacetime, dates back to the pioneering works of Kaluza and Klein [1]. More recently, the so called ‘Braneworld’ models have attracted much attention in an attempt to explain various theoretical difficulties such as the Hierarchy problem [2, 3] and the origin of cosmological inflation [4, 5]. One of the most interesting of these models from a cosmological point of view, is the Randall-Sundrum single brane model (RSII). This model has a single positive tension brane embedded in an infinite AdS bulk. The bulk is assumed to be empty except for a cosmological constant and ordinary matter is confined to the brane. At low energies the gravity is localized on the brane via the curvature of the bulk and the standard 4D gravity is recovered on the brane, however at high energies the gravity ‘leaks’ into the bulk which leads to the modification of the standard general relativity [6]. Various aspects and properties of this model has been studied extensively in the literature; for example see [7, 8].

On the other hand, when studying any cosmological theory, a lot of information can be obtained by analyzing the behavior of cosmological perturbations and comparing it to the current observations of the Cosmic Microwave background and large scale structure data. This task has been done in the context of brane cosmology by various different authors [9]. As in the standard 4D gravity of general relativity, studying the braneworld cosmological perturbations is done by considering two manifolds: A background FRW and a perturbed 'physical' manifold. Perturbation of a quantity then defined by determining the difference between its values in the actual physical spacetime and the background reference manifold. The metric perturbations in GR can not be uniquely defined, but depend on the choice of 'gauge'. A gauge essentially can be regarded as an identification map which corresponds spacetime points in the two manifolds. Making a different choice of gauge, may result in the change of the values of the perturbation variables. It may also lead to un-physical i.e. not real perturbation modes.

The way around this problem is to work with purely gauge invariant equations. This means to find combinations of perturbation variables that remain invariant under a change of the identification map. This method, first introduced by Bardeen in [10], has the advantage of involving only real unambiguous physical quantities. For a comprehensive review of the gauge invariant theory of cosmological perturbations in general relativity and its applications in inflation theory and computing the spectrum of the cosmic microwave background radiation see references [11, 12, 13, 14].

General relativity, while highly successful in explaining cosmological observations, is not the only viable theory of gravity. More importantly some of its problems like the singularity problem and the issue of the unification of fundamental forces led people to try and modify or generalize it. Teleparallel theory of gravity first introduced by Einstein [15] in an attempt to unify gravity with electromagnetism. This theory in its general form is basically the gauge theory for the translation symmetry group [16]. The
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theory possesses a torsion field which can be regarded as the translational field strength corresponding to the coframe field \[ e^i_\mu \]. Generally the Lagrangian of this theory lacks Local Lorentz invariance. Restoring this symmetry mean imposing some restrictions on the form of the Lagrangian which results in a theory which is dynamically equivalent to general relativity and is usually called the teleparallel equivalent of general relativity (TEGR) in the literature \[ [18, 19] \]. The teleparallel gravity and its extensions \[ [20, 21] \] has generated renewed interest in recent years, in the hope that it may provide solutions to some of the cosmological problem like the origin and the nature of the dark energy \[ [22, 23] \]. It should be stated here that both teleparallel gravity and general relativity can be regarded as special cases of a more fundamental gauge theory of gravity called Poincare gauge theory which contains both torsion and curvature as translational and rotational field strengths respectively \[ [24] \]. It has been shown that a teleparallel setup naturally arises from the low-energy effective field theory induced on D-branes, described by a Yang-Mills theory on a flat noncommutative space which can be regarded as the low energy limit of string theory \[ [25] \].

In teleparallel gravity one considers a set of \( n \) linearly independent vectors \( e_i = e^\mu_i \partial_\mu \) which form a basis in the tangent space on every point of the manifold. The dual of this basis \( \vartheta^i = e^i_\mu dx^\mu \) are the coframes. The dynamical variable in the TEGR theory are \( e^i_\mu \) s, called the tetrads (or pentads in 5D) and they relate anholonomic tangent space indices to the coordinate ones. The spacetime metric is not an independent dynamical variable here and is related to the tetrad through the relations

\[
\begin{align*}
g_{\mu\nu} &= \eta_{ij} e^i_\mu e^j_\nu, \quad (1) \\
e_i \cdot e_j &= \eta_{ij} \quad (2)
\end{align*}
\]

The inverse of the tetrad is defined by the relation \( e^\mu_i e^j_\mu = \delta^j_i \).

Teleparallel geometry, \( T^4 \) is defined by the requirement of vanishing curvature. In the special case of TEGR the spin connection of the theory is also assumed to be zero. This assumption is usually called the absolute parallelism condition and by imposing it the connection of the theory will be the Weitzenb"ock connection defined as \[ [30] \]

\[
\Gamma^\rho_{\mu\nu} = e^\rho_i \partial_\nu e^i_\mu
\]

which unlike Livi-civita connection is not symmetric on its second and third indices. The curvature of this connection is identically zero and the torsion tensor is

\[
T^\rho_{\mu\nu} \equiv e^\rho_i (\partial_\mu e^i_\nu - \partial_\nu e^i_\mu).
\]

Contorsion tensor which denotes the difference between Livi-civita and Weitzenb"ock connections is

\[
K^\rho_{\mu\nu} = -\frac{1}{2}(T^\rho_{\mu\nu} - T^\rho_{\nu\mu} - T^\rho_{\mu\nu})
\]

and the superpotential tensor is defined as

\[
S_{\rho}^{\mu\nu} = \frac{1}{2}(K^{\mu\nu} + \delta^{\mu}_{\rho} T^{\alpha\nu}_{\alpha} - \delta^{\nu}_{\rho} T^{\alpha\mu}_{\alpha}).
\]
In correspondence with Ricci scalar, one can define torsion scalar
\[ T = S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu} \] (7)

Recently, braneworld models in the context of teleparallel gravity has been studied and its setup and fundamental equations has been derived [26, 27]. It is understood that the RS setup in TEGR is dynamically different from General Relativity. This fact has motivated the authors to study the dynamics of cosmological perturbations in this setup. In this paper, by perturbing the coframe components (or pentads in 5D), we present the fully gauge invariant cosmological perturbations equations in braneworld teleparallel gravity. Unlike the metric perturbation in general relativity which contains scalar, vector and tensor modes, pentads can only contain up to vector perturbations. The tensor part of the perturbation spectrum in teleparallel gravity originates directly from the contorsion tensor, which is the difference between Levi-Civita connection of general relativity and Weitzenböck connection of teleparallel gravity. As stated, to make the perturbation equations gauge invariant, one must find gauge invariant combinations of perturbation variables and rewrite the equations in terms of these parameters. These combinations in braneworld teleparallel gravity, in analogy to the Bardeen potentials of general relativity, has been derived for both geometric and matter perturbations in the following sections. 5D models based on teleparallel gravity are also studied in Refs. [28, 29].

Except when specifically stated, the notation we use throughout this paper is as follows: Lower case Latin letters \( a, b, ..., i, j, ... \) run over 1, 2, 3 and label spatial tangent space coordinates. The Greek indices \( \alpha, \beta, ..., \mu, \nu, .. \) run over 1, 2, 3 and refer to the spatial spacetime coordinates. The letter \( I \) refers to 5D tangent space indices. The rest of the upper case Latin Letters \( M, N, R, L \) refer to 5D coordinate space indices.

2. Geometry Perturbations

So far no trace of fifth dimension has been observed on the brane, therefore one can consider it separate from other dimensions. This will give the 5D spacetime, the layered structure [24]. Therefore the 5D tangent coordinate system \( \hat{e}_I = (\hat{e}_i, \hat{e}_5) \) can be written as
\[ \hat{e}_M = e^i_M \hat{e}_i + e^5_M \hat{e}_5 \] (8)
Here the first four vectors define the coordinate system on the layer and the fifth dimension can be chosen to be normal to the other four dimensions. The above orthogonality condition leads us to the point that the expansion of \( \hat{e}_5 \) cannot contain any of 4D \( \hat{e}_i \) vectors. Then we have
\[ \hat{e}_\mu = e^i_\mu \hat{e}_i + e^5_\mu \hat{e}_5 \quad \hat{e}_5 = e^5_5 \hat{e}_5 \] (9)
So the general form of pentad and inverse pentad respectively are
\[ e^I_M = \begin{pmatrix} e^i_\mu & 0 \\ e^5_\mu & e^5_5 \end{pmatrix} \]
\[
\begin{align*}
\bar{e}_0 & = n, & \bar{e}_5 & = 1, & e_\alpha^i = a\delta_\alpha^i \\
\bar{e}_0^5 & = n(1 + A), & e_0^\mu & = aC^\mu, & e_0^5 & = 0 \\
\bar{e}_5^0 & = n\alpha, & e_5^\mu & = aF^\mu, & e_5^0 & = 0, & e_5^5 & = 1 + \varphi, & e_\mu^i & = a(\delta_\mu^i + D_\mu^i)
\end{align*}
\]

And the inverse pentads are

\[
\begin{align*}
\bar{h}_0^0 & = \frac{1}{n}(1 - A), & \bar{h}_0^\mu & = -\frac{a}{n}B^\mu, & \bar{h}_0^5 & = -\frac{a}{n}C^5, & \bar{h}_5^0 & = -\alpha, & \bar{h}_5^0 & = 0 \\
\bar{h}_5^\mu & = 0, & \bar{h}_5^5 & = -F^i, & \bar{h}_5^5 & = 1 - \varphi, & \bar{h}_i^\mu & = \frac{1}{a}(\delta_\mu^i - D_\mu^i)
\end{align*}
\]

Where \( A \), \( \varphi \) and \( \alpha \) are scalar perturbations and \( C^\mu \), \( B^i \), \( F^\mu \), and \( D^i_\mu \) are vector perturbations and along with scalar ones depend on coordinates \((\tau, x^\alpha, y)\). \( \tau \) is the conformal time parameter and \( y \) is the coordinate of fifth dimension. Note that here we don’t have tensorial perturbations since pentad unlike metric is a vector and cannot be perturbed tensorially. The torsion, Contorsion, superpotential and torsion scalar of the above pentad is given in appendix A. The calculations has been done up to the first order of perturbations.

To have the gauge invariant geometry, the Lie derivative of the added perturbation fields should stay invariant under following coordinate transformations

\[
x^\mu \rightarrow \bar{x}^\mu = x^\mu + \xi^\mu, \quad \xi^\alpha = (T, L^\alpha, Y)
\]

In the language of teleparallel gravity, the Lie derivative of the pentad is

\[
\Delta \delta \bar{e}_M^I = \bar{e}_N^J \nabla_M \bar{\xi}_N
\]
This important result was first derived in [31] (see also [32]) and is usually called the 'Stewart Lemma’ in the literature.

Using the mentioned Stewart lemma, we can calculate the change in the pentad perturbation parameters under the gauge transformation as

\[ \Delta A = \dot{T} + \frac{n}{a} nT \]  
\[ \Delta C_\mu = \frac{n}{a} D_\mu T \]  
\[ \Delta B^i = \dot{L}^i + \frac{\dot{a}}{a} L^i \]  
\[ \Delta F_\mu = \frac{1}{a} D_\mu Y \]  
\[ \Delta D^\mu_\mu = D^\mu_\mu L^i \]  
\[ \Delta \Phi = Y' \]  
\[ \Delta \alpha = \frac{1}{n} \dot{Y} \]

Where a dot denotes differentiation with respect to proper time and a prime denotes differentiation with respect to fifth dimension. To consider only the scalar perturbations, we define the scalar part of the vector perturbations as

\[ B^i = D^i B \quad C_\mu = D_\mu C \quad F_\mu = D_\mu F \quad D^\mu_\mu = D \]

Then the transformations of scalar perturbations are

\[ A \to A + \dot{T} + \frac{n}{a} nT \]  
\[ B \to B + \dot{L} + \frac{\dot{a}}{a} L \]  
\[ C \to C + \frac{n}{a} nT \]  
\[ F \to F + \frac{1}{a} Y \]  
\[ D \to D + \dot{L} \]  
\[ \Phi \to \Phi + Y' \]  
\[ \alpha \to \alpha + \frac{1}{n} \dot{Y} \]

It turns out then that the theory is not gauge invariant. To make it invariant we repeat Bardeen’s method. We define new fields out of above old ones in a way that they stay invariant under (14).

\[ \Psi_1 = B - \dot{D} - \frac{\dot{a}}{a} D \]  
\[ \Psi_2 = A - \frac{\dot{a}}{n} \dot{C} - \frac{\dot{a}}{a} C \]  
\[ \Psi_3 = \Phi - a' F - a F' \]  
\[ \Psi_4 = \alpha - \frac{\dot{a}}{n} \dot{F} - \frac{\dot{a}}{n} F \]
These geometric objects, built from the pentad perturbation variables, clearly remain invariant under the gauge transformation (16). Comparing with the obtained potential in 5D RS setup in general relativity, one can easily find the above potentials are different from the ones derived in general relativity \[9\].

3. Matter Perturbation

We repeat the same calculations for the matter perturbations. In RS model of brane-world, matter is confined to the brane and bulk can be chosen to contain only a 5D cosmological constant. We consider the matter on the brane to be a perfect fluid \((-\rho, p, p, p)\).

We can generally perturb the energy-momentum as

\[
\delta T^0_0 = -\rho \delta \rho, \quad \delta T^0_\alpha = q_\alpha, \quad T^\beta_\alpha = p \rho \delta \rho^\beta + p \pi^\beta_\alpha \tag{35}
\]

Where \(\delta \rho\) and \(\delta p\) are scalar, \(q_\alpha\) vector and \(\pi^\beta_\alpha\) is purely tensor (\(\pi^\alpha_\alpha = 0\)).

Under a 4D coordinate transformation, the perturbed energy-momentum tensor will transform like

\[
\delta \rho \to \delta \rho + \dot{\rho} T + 2\dot{T} \tag{40}
\]
\[
q \to q + p(\dot{L} + \frac{\dot{a}}{a} L) - \rho D_\alpha T \tag{41}
\]
\[
\delta p \to \delta p + \dot{p} T - \frac{\dot{a}}{a} T + \frac{2}{3} D_\alpha L^\alpha \tag{42}
\]
\[
\pi_{\alpha\beta} \to \pi_{\alpha\beta} + 2D_\alpha (L_{\beta}) - \frac{2}{3} \gamma_{\alpha\beta} D_\delta L^\delta \tag{43}
\]

Considering again only the scalar parts, we have \(q_\alpha = D_\alpha q; \pi_{\alpha\beta} = \Delta_{\alpha\beta}\pi\) and \(L_\alpha = D_\alpha L\) then

\[
\delta \rho \to \delta \rho + \dot{\rho} T + 2\dot{T} \tag{40}
\]
\[
q \to q + p(\dot{L} + \frac{\dot{a}}{a} L) - \rho T \tag{41}
\]
\[
\delta p \to \delta p + \dot{p} T - \frac{\dot{a}}{a} T + \frac{2}{3} \Delta L \tag{42}
\]
\[
\pi \to \pi + 2L \tag{43}
\]

Where \((\dot{\rho} \text{ denotes } \frac{\dot{\rho}}{\rho} \text{ and } \dot{p} \text{ denotes } \frac{\dot{p}}{p})\). Similar to geometry potentials one can make matter potential in a way that they stay invariant under above transformations. The gauge invariant matter potentials then are

\[
\Psi_\rho = \delta \rho - 2A - \frac{a}{n} (\dot{\rho} - \frac{\dot{n}}{n}) C \tag{44}
\]
\[
\Psi_q = q - pB + \frac{a}{n} \rho C \tag{45}
\]
\[
\Psi_p = \delta p - \frac{a}{n} (\dot{p} - \frac{\dot{a}}{a}) C - \frac{2}{3} \Delta D \tag{46}
\]
\[
\Psi_\pi = \pi - 2D \tag{47}
\]
4. Gauge invariant 5D field equations

The gravitational action in TEGR is

\[ I = \frac{1}{16\pi G} \int d^5x |e| T \] (48)

Where \(|e|\) is the determinant of the pentad \(e_I^M\). The field equations then can be obtained by variation of (48) with respect to the pentad

\[ e^{-1}\partial_M(e_I^R S_R^{MN}) - e_N^T R^R_{\;\;ML} S_R^{LM} + \frac{1}{4} e_I^N T = 4\pi G e_I^R (-\Lambda_5 + \delta(y) S_R^N) \] (49)

Where \(\Lambda_5\) is the bulk cosmological constant. Here \(I\) refers to 5D tangent space and the rest refer to 5D coordinate space and \(S_R^N = (-\rho, p, p, p, 0)\) is the matter which is localized on the brane. Various components of the left-hand side of the perturbed 5D teleparallel field equation (49) is given in appendix B.

Brane is a 4D hypersurface which divides the 5D bulk in two regions. The two sides of this hypersurface is connected via the junction conditions. The junction conditions for the TEGR set-up of RS model has been derived in [26] as

\[ e_I^R [S_R^{MN}] n_M = 4\pi G S_I^N \] (50)

where \(n_M\) is the unit vector normal to the brane and can be chosen to be \((0, 0, 0, 0, 1)\).

writing (50) in components, we have

\((I = 0, N = 0)\) : \(-\frac{3a'}{2n}(1 - 2\Phi - A) - \frac{1}{n} D'_{\beta} = 4\pi G \frac{\rho}{n^3}(1 + \delta \rho - 3A) \) (51)

\((I = 0, N = \alpha)\) : \(\frac{a'}{2a} + \frac{n'}{2an} - \frac{a'}{a^2} C^\alpha + \left( \frac{2a'}{n} + \frac{an'}{2n^2} + \frac{a'}{an} - \frac{n'}{2n^2} \right) D^\alpha + \frac{1}{2n} B^\alpha - \frac{1}{2a} C^\alpha + \frac{\dot{\alpha}}{2an^2} F^\alpha + \frac{1}{2n} \dot{F}^\alpha - \frac{1}{2an} \dot{q}^\alpha = 4\pi G \left( -\left( \frac{P}{an} + \frac{\rho}{n^3} \right) B^\alpha + \frac{\rho}{an^2} C^\alpha + \frac{1}{na^2} q^\alpha \right) \) (52)

\((I = a, N = 0)\) : \(-\frac{1}{2an} \partial_a \alpha + \frac{5a}{2an^2} F_a + \frac{1}{2n^2} \dot{F}_a + \frac{n'}{2n^2} - \frac{5a'}{2n} \right) C_a + \frac{1}{2n} C'_a - \frac{n' a}{2n^3} (a - 1) B_a - \frac{a}{2n^2} B'_a = 4\pi G \left( \frac{P}{an^2} B_a - \left( \frac{p}{a^2 n} + \frac{\rho a}{n^3} \right) C_a - \frac{1}{an^2} q_a \right) \) (53)
(I = a, N = \alpha) : \left\{ \begin{array}{l}
- \frac{1}{a} \partial^{[\alpha} F_{\beta]} + \frac{1}{a} (D^{(\alpha} D_{\beta)} - \delta^{\alpha}_{\beta} D^{\beta}) + \frac{1}{a} \left( 2a' + n' \right) D_{\alpha} \\n- \frac{1}{a} A' \delta_{\alpha} + \left( \frac{4a'}{a^2} + \frac{2n'}{an} \right) \Phi \delta_{\alpha} - \frac{2a'}{a^2 n} \alpha \delta_{\alpha} - \left( \frac{2a'}{a^2} + \frac{n'}{an} \right) \delta_{\alpha} \\n= 4\pi G \frac{1}{a^3} \left( p(1 + \delta p) \delta_{\alpha} + p\pi_{\alpha} - p(D_{\alpha} + 2D_{\alpha}) \right)
\end{array} \right. (54)

(I = 5, N = 0) : \left\{ \begin{array}{l}
\frac{1}{n^2} \left( \partial_{\alpha} B_{\alpha} - \partial_{\alpha} D_{\alpha} \right) - \frac{3a}{an^2} (1 - 2A - \Phi) + \frac{3a'}{an} \alpha = 0
\end{array} \right. (55)

(I = 5, N = \alpha) : \left\{ \begin{array}{l}
- \frac{1}{a} \partial^{\alpha} A - \frac{2}{a^2} \partial^{[\alpha} D_{\beta]} - \frac{3a}{an^2} B_{\alpha} + \left( \frac{4a'}{a^2 n} - \frac{\dot{a}}{an} \right) C_{\alpha} \\n+ \frac{1}{na} \ddot{\delta}_{\alpha} + \left( \frac{4a'}{a^2} + \frac{n'}{an} \right) F_{\alpha} - \frac{2}{a^2} \partial^{\alpha} \Phi + \frac{2}{a} F'_{\alpha} = 0
\end{array} \right. (56)

Note that the left hand sides of the above equations should be evaluated at the position of the brane. Using the background field equations and the definition of geometric and matter potentials (31 – 34) and (44 – 47), we can rewrite various components of the field equation (49) in terms of gauge invariant variables. The result for the fully Gauge invariant field equations are as follows

(I = a, N = \alpha) : \left\{ \begin{array}{l}
\frac{1}{a} \left[ \partial_\alpha \partial^\alpha \left( \frac{1}{a^2} \Psi_1 + \frac{2a}{an^2} \Psi_2 \right) - \nabla^2 \left( \frac{1}{a^2} \Psi_1 + \frac{2a}{an^2} \Psi_2 \right) \delta^\alpha \right] \\n+ \frac{1}{2a^2 n^3} \Psi_1 \delta^\alpha - \frac{3a}{an^2} \Psi_1 \delta^\alpha - \frac{4a' n'}{a^2 n} \Psi_3 \delta^\alpha - \frac{2a'}{a^2 n} \Psi_2 \delta^\alpha \\n+ \dot{a} \left( \frac{1}{2 a^2 n} (\Psi_1 + \Psi_2) \delta^\alpha \right) + \frac{1}{2 a} \left( \Psi_3' + \Psi_2' \right) \delta^\alpha \\n+ \frac{a'}{a^2} \left( \Psi_1' + \Psi_2' \right) \delta^\alpha \right.
\end{array} \right. (57)

(I = a, N = 5) : \left\{ \begin{array}{l}
\frac{\dot{a}}{2a^2 n} \partial_{\alpha} \Psi_4 - \frac{3a}{2a^2} \partial_{\alpha} \Psi_2 + \frac{3 a'}{2 a^2} \partial_{\alpha} \Psi_1 + \left( \frac{3a n'}{4 n^3} + \frac{a n'}{4 n^3} - \frac{a n'}{2 n^2} \right) \partial_{\alpha} \Psi_1 \\n+ \left( - \frac{5 a'^2}{2 a^3} - \frac{2 a' n'}{2 a^2 n^2} + \frac{3 a^2}{2 a^2 n^2} \right) \partial_{\alpha} \Psi_4 = 0
\end{array} \right. (58)

(I = 5, N = \alpha) : \left\{ \begin{array}{l}
- \left( \frac{a'}{a^2} + \frac{n'}{a^2 n} \right) \partial^\alpha \Psi_3 + \left( \frac{a'}{2 a^2} + \frac{n'}{2 a^2 n} \right) \partial^\alpha \Psi_2 \\n- \frac{1}{a^2} \partial^\alpha \Psi_3 + \frac{1}{2 a^2} \partial^\alpha \Psi_2 + \frac{1}{4 a^2 n} \partial^\alpha \Psi_4 + \frac{\dot{a}}{a^2 n} \partial^\alpha \Psi_4 = 0
\end{array} \right. (59)
\[(I = 5, N = 5) : \frac{1}{a^2} \nabla^2 (\Psi_3 + \frac{1}{2} \Psi_2) + \left(\frac{\dot{n}}{an^3} - \frac{3\dot{a}}{2an^2}\right) \Psi_1 + \frac{3\dot{a}^2}{2a^2n^2} \Psi_2 = 0 \tag{60}\]

\[(I = a, N = 0) : \frac{a^2}{2n} \partial_a \Psi_1 + \left(\frac{a'a}{2} + \frac{n'a^2}{2n}\right) \partial_a \Psi_4 - \frac{a'}{2a^2n} \Psi_4 + \frac{3\dot{a}}{2a^2n} \partial_a \Psi_3 + \frac{a\dot{a}}{n} \partial_a \Psi_1 = 0 \tag{61}\]

\[(I = 5, N = 0) : - \frac{1}{4a^2n} \nabla^2 \Psi_4 - \frac{\dot{a}}{an^2} \Psi_3 + \frac{1}{2an^2} \nabla^2 \Psi_q + \left(\frac{3a'}{2n^2} + \frac{2n'}{2n^3}\right) \Psi'_4 = 0 \tag{62}\]

\[(I = 0, N = 0) : \left(\frac{5\dot{a}}{4a^2n^2} - \frac{\dot{a}}{2an^3}\right) \nabla^2 \Psi_2 - \frac{2\dot{a}}{an^3} \Psi_2 + 3\left(\frac{a'^2}{a^2n} + \frac{2a'n'}{an^2}\right) \Psi_3 + \left(\frac{\dot{a}'n'}{an^2} - \frac{3a'n'a}{2an^2}\right) \Psi_4 + \frac{3a'n'}{2na^2} \Psi_1 + \left(\frac{1 + 3a'}{na}\right) \Psi'_1 = 0 \tag{63}\]

\[(I = 0, N = \alpha) : \frac{1}{na} \partial^\alpha (\Psi'_3 + \left(\frac{n'}{a^2n} + \frac{a'}{a^3}\right) \partial^\alpha \Psi_3 - \frac{n'}{2an^2} \partial^\alpha \Psi_4 - \frac{1}{2a^2} \partial^\alpha \Psi'_2)
\quad - \left(\frac{n'}{2a^2n} + \frac{a'}{2a^3}\right) \partial^\alpha \Psi_2 = 0 \tag{64}\]

The first four of these equations are dynamical equations and the rest act as constraints. This set of equations can be solved for four potentials \(\Psi_1, \Psi_2, \Psi_3\) and \(\Psi_4\) in the bulk subject to boundary conditions provided by the junction conditions (eqs (51-56)). The resulting perturbation dynamics is in general different from general relativity. One can also bring the junction conditions into gauge invariant form. The results are as follows

\[(I = 0, N = 0) : \frac{3a'}{an} \Psi'_1 + \frac{6a'}{an} \Psi_3 + \frac{a'n}{an^2} \Psi_2 = 4\pi G \frac{1}{n^3} \Psi_\rho \tag{65}\]

\[(I = 0, N = \alpha) : - \frac{1}{2a^2} \partial^\alpha \Psi_4 + \frac{n'}{2na} \partial_\alpha \Psi_1 = 4\pi G \frac{1}{na^2} \Psi_q \tag{66}\]
\( (I = a, N = 0) : \frac{1}{2an} \partial_a \Psi_4 + \frac{3a^2n'}{2n^3} \partial_a \Psi_2 = \frac{1}{an^2} \partial_a \Psi_q \) \hspace{1cm} (67)

\( (I = a, N = \alpha) : \frac{1}{a} \partial_a \Psi_1 \delta_a^\alpha - \frac{2\dot{a}}{a^2} \Psi_4 \delta_a^\alpha \frac{1}{a} \partial_a \Psi_2 + \frac{4\alpha'}{a^2} \partial_a \Psi_3 \)

\[= 4\pi G \frac{1}{a^3} p \partial_a (\Psi_p + \Psi_\pi)\] \hspace{1cm} (68)

The other two junction conditions are constraints on geometric potentials and background pentad variables at the position of the brane and we use them to bring the other four equations into gauge invariant form. These equation act as boundary conditions for the bulk perturbation equations.

5. Conclusion and Discussion

Gauge properties and symmetries of the teleparallel theory of gravity is essentially different from those of general relativity. As a result, one should be careful when defining perturbations in a manifold with absolute parallelism. The difference in definitions of perturbations in teleparallel and GR, stems from the fact that when working with the Stewart lemma, the Lie derivative is different in two theories. Moreover pentad perturbations in teleparallel gravity can only contain up to vector modes. These items also implicate the cosmological perturbations in 4D teleparallel set-up. In 4 dimensions it has been shown that the results arisen by the perturbed FRW vierbein are different from the ones by perturbed FRW metric \cite{33}. This may bring out interests to study some cosmological issues to find out more about the conceptual role of torsion in such theories.

The five dimensional case is more complicated. Because of the different junction conditions in this model compared with GR, the results obtained on the brane cannot be retrieved by general relativity set-up. In the issue of inflation it has been shown that in RS model of TEGR, the inflation index grows faster \cite{27}. Considering these points, studying cosmological perturbations of this model could be interesting.In this paper, we presented the fully gauge invariant cosmological perturbation equations for scalar perturbations in teleparallel gravity by writing down the equations in terms of the gauge invariant geometric and matter potentials. These potentials are the teleparallel versions of Bardeen’s potentials in five dimensions. The resulting system of equations can be solved to uniquely determine the physical scalar perturbation modes. According to the result obtained in this paper, studying the cosmological issues in this setup will lead to different conceptual interpretations and also observational predictions.
Appendix A.

Non-zero components of the torsion

\[
\begin{align*}
T^\alpha_{\beta\gamma} &= \partial_\beta D^\alpha_\gamma - \partial_\gamma D^\alpha_\beta \\
T^0_{\alpha\beta} &= \frac{a}{n} \left( \partial_\alpha C_\beta - \partial_\beta C_\alpha \right) \\
T^0_{50} &= \frac{n}{n} + A' \\
T^5_{\alpha\beta} &= a(\partial_\alpha F_\beta - \partial_\beta F_\alpha) \\
T^5_{5\alpha} &= aF'_\alpha - \partial_\alpha \Phi \\
T^5_{\alpha\beta} &= -\frac{\dot{a}}{a} \delta^\alpha_\beta + \partial_\beta B^\alpha - \dot{D}^\alpha_\beta \\
T^0_{0\alpha} &= \left( \frac{\dot{a}}{n} - \frac{\dot{a}a}{n} \right) C_\alpha + \frac{a}{n} \dot{C}_\alpha - \partial_\alpha A \\
T^5_{5\alpha} &= \left( \frac{a'}{n} - \frac{a'a'}{n} \right) C_\alpha + \frac{a}{n} C'_\alpha \\
T^5_{0\alpha} &= n \partial_\alpha \alpha - a \dot{F}_\alpha \\
T^5_{05} &= \dot{\Phi} - n\alpha' \\
T^\alpha_{50} &= \left( \frac{a'}{a} - \frac{an'}{n} \right) B^\alpha + B'\alpha \\
T^\alpha_{\beta\gamma} &= -D^\alpha_{\beta\gamma} - \frac{a'}{a} \delta^\alpha_\beta \\
\end{align*}
\]

(A.1)

Contorsion

\[
\begin{align*}
K^{\beta\gamma}_\alpha &= -\frac{1}{a^2} \partial^{[\gamma} D^\beta_{\alpha]} + \frac{1}{a^2} \partial^{[\beta} D^\gamma_{\alpha]} + \frac{1}{a^2} \partial^{[\beta} D^\gamma_{\alpha]} - 2 \frac{\dot{a}}{an^2} \delta^{[\beta} B^{\gamma]} + 2 \frac{\dot{a}}{na^2} \delta^{[\beta} C^{\gamma]} \\
&+ 2 \frac{a'}{a} \delta^{[\beta} F^{\gamma]} \\
K^0_{\alpha} &= -\frac{1}{an} \partial^{[\gamma} C_{\alpha]} + \frac{1}{an^2} \left( \partial_{(\alpha} B^{\gamma)} + \frac{a'n}{a} \delta^\gamma_\alpha \right) - \frac{\dot{a}}{a} \delta^\gamma_\alpha (1 - 2A) - \partial_0 D^{(\gamma}_{\alpha)} \\
K^{5\gamma}_\alpha &= -\frac{1}{a} \partial^{[\gamma} F_{\alpha]} + \left( \frac{\dot{a}}{an} + \frac{a'}{a} - 2 \frac{a'}{a} \Phi \right) \delta^\gamma_\alpha + D^{(\gamma}_{\alpha} \\
K^{50}_\alpha &= \frac{1}{2} \left[ -\frac{1}{n} \partial_\alpha \alpha - \frac{\dot{a}}{n^2} F_\alpha + \frac{a}{n^2} \dot{F}_\alpha + \frac{an'}{n^2} C_\alpha + a \frac{C'_\alpha}{n} \\
&+ (\frac{a^3 n'}{n^3} - \frac{n'a^2}{n^3}) B_\alpha - \frac{a^2}{n^2} B'_\alpha \right] \\
K^{5\gamma}_0 &= -\frac{1}{a^2} \partial^{[\gamma} B^{\beta]} + \frac{1}{a^2} \partial_0 D^{[\beta\gamma]} - \frac{n}{a^2} \delta^{[\beta} C^{\gamma]} \\
K^0_{\gamma} &= -\frac{1}{a^2} \partial^{\gamma} A + \left( \frac{\dot{a}}{2an^2} - \frac{\dot{a}}{an} + \frac{3\dot{a}}{2a^2 n} \right) C^{\gamma} + \frac{n'}{an} F^{\gamma} - \frac{\dot{a}}{a} B^{\gamma} + \frac{1}{an} \dot{C}^{\gamma} \\
K^{5\gamma}_0 &= -\frac{1}{2} \left[ \left( \frac{n'}{n} + \frac{an'}{a^2} - \frac{2a'}{a} \right) B^{\gamma} - B'^{\gamma} + \left( \frac{2a'n}{a^2} - \frac{n'}{a} - \frac{a'n}{a} \right) C^{\gamma} + \frac{n}{a} C'^{\gamma} \\
&+ \frac{n}{a^2} \partial^{\gamma} \alpha - \frac{\dot{a}}{a^2} F^{\gamma} - \frac{1}{a} \dot{F}^{\gamma} \right]
\end{align*}
\]
\[
K^{50}_{5} = \frac{n'}{n}(1 - 2\Phi) + A'
\]
\[
K^{\beta\gamma}_{5} = \frac{1}{a^2}D^{[\beta\gamma]} + \frac{1}{a^2}\partial^{[\beta}F_{\gamma]}
\]
\[
K^{0\gamma}_{5} = \frac{1}{2}\left(\frac{1}{an}C^{\gamma} + \left(\frac{n'}{n^3} - \frac{an'}{n^3}\right)B^\gamma + \frac{1}{n^2}B^\gamma - \left(\frac{n'}{an^2} - \frac{a'}{na^2}\right)C^\gamma\right)
- \frac{1}{2}\frac{\dot{a}}{a^2n^2}F^\gamma + \frac{1}{2a^2n}\partial^\gamma\Phi - \frac{1}{2an^2}\dot{F}^\gamma
\]
\[
K^{5\gamma}_{5} = -\frac{1}{a^2}\partial^\gamma\Phi + \frac{a'}{a^2}F^\gamma + \frac{1}{a}F^\gamma
\]
\[
K^{50}_{5} = \frac{\dot{\Phi}}{n^2} - \frac{\alpha'}{n} - \frac{n'\alpha}{n^2}
\]

Superpotential
\[
S_{\alpha}^{\beta\gamma} = -\frac{1}{a^2}\partial^{\gamma}D^{\beta}_{\alpha} + \frac{1}{a^2}\partial^{[\beta}\partial_{\alpha]}D^\gamma + \frac{1}{2a^2}\partial^{[\beta\partial_{\alpha]}D^\gamma} - \left(\frac{3\dot{a}}{na^2} - \frac{\dot{a}}{an}\right)\delta^{[\beta}_{\alpha}C^\gamma
+ \frac{1}{a^2}\delta^{[\beta}_{\alpha}\partial_{\gamma}\Phi} + \frac{1}{a^2}\delta^{[\beta\partial_{\alpha]}D^\gamma} - \frac{1}{a^2}\delta^{[\beta}_{\alpha}\partial_{\partial_{\alpha]}D^\gamma}
- \frac{1}{na}\alpha^\alpha \delta^{[\beta}_{\alpha}\partial_{\gamma}C^\gamma + \frac{2\alpha}{a^{\gamma}}\delta^{[\beta}_{\alpha}B^\gamma - \left(\frac{2\alpha'}{a^2} + \frac{n'}{an}\right)\alpha^\alpha \partial_{\partial_{\alpha]}\partial_{\alpha]}D^\gamma - \frac{1}{a^2}\delta^{[\beta}_{\alpha}F^\gamma
\]
\[
S_{\alpha}^{0\gamma} = \frac{1}{2}\left[-\frac{1}{an}\partial^{\gamma}C_{\alpha} + \frac{1}{n^2}(\partial_{\alpha}B^\gamma - \delta_{\alpha}\partial_{\gamma}B^\gamma) - \frac{1}{n^2}(\partial_{\alpha}D^{\gamma}_{\alpha} - \delta_{\alpha}\partial_{\beta}D^\beta)
+ \left(\frac{2\alpha'}{an^2}(1 - 2A) + \frac{\dot{\Phi}}{n^2} - \frac{\alpha'}{a} + \frac{n'}{an} - \frac{\alpha'}{n}\right)\delta_{\alpha}^\gamma
\]
\[
S_{\alpha}^{5\gamma} = -\frac{1}{2a}\partial^{\gamma}D_{\alpha} + \frac{1}{2}(D^{\gamma}_{\alpha} + \partial_{\gamma}D_{\alpha} - \partial_{\gamma}\partial_{\beta}D^\beta - \frac{1}{2}\left(\frac{2\alpha'}{a} + \frac{n'}{an} + A + \frac{4\alpha'}{a} + 2n\right)\delta_{\alpha}^\gamma
\]
\[
S_{\alpha}^{50} = -\frac{1}{2n^2}\left[\partial_{\gamma}B^\gamma + \partial_{\eta}D^{\eta}_{\gamma} + 3\dot{\alpha}(1 - 2A\right]\right] + \frac{3\alpha'}{2an}
\]
\[
S_{\alpha}^{0\gamma} = \frac{1}{2}\left[\frac{\dot{a}}{2an^2} - \frac{5\dot{a}}{2a^2n}\right]C^\gamma - \frac{3\dot{a}'}{a^2}F^\gamma - \frac{2\dot{a}}{a^2}B^\gamma + \frac{2\dot{a}'}{a^2}\partial_{\gamma}D^{\eta}_{\eta}\]
\[
S_{\alpha}^{50} = -\frac{1}{2}\left(\frac{3\dot{a}'}{a} + \frac{1}{a}(-2\Phi + D^{\alpha}_{\alpha} - \frac{3\dot{a}}{an})
\]
\[
S_{\alpha}^{55} = \frac{1}{2a^2}(\partial^\gamma A + 2\partial^\gamma D^\alpha_{\alpha}) + \frac{3\dot{a}}{a^2}B^\gamma - \left(\frac{2\dot{a}}{a^2} - \frac{\dot{a}}{an}\right)C^\gamma - \frac{1}{2n}C^\gamma
- \left(\frac{2\alpha'}{a} + \frac{n'}{2an}\right)F^\gamma + \frac{1}{a^2}\partial^\gamma\Phi - \frac{1}{a}F^\gamma
\]
\[
S_{\alpha}^{50} = \frac{1}{2}K^{50}_{\alpha}
\]
\[
S_{\alpha}^{55} = \frac{1}{2}K^{55}_{\alpha}
\]
\[
S_{\alpha}^{50} = \frac{1}{2}K^{50}_{\alpha}
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S_{\alpha}^{55} = \frac{1}{2}K^{55}_{\alpha}
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S_{\alpha}^{50} = \frac{1}{2}K^{50}_{\alpha}
\]
\[
S_{\alpha}^{55} = \frac{1}{2}K^{55}_{\alpha}
\]
Torsion scalar

\[
T = \frac{6\dot{a}^2}{a^2 n^2} - \frac{6a'}{a} \left( \frac{a'}{a} + \frac{n'}{n} \right) - 4\dot{a} \left( \partial_\alpha B^\alpha - \partial_\beta D^\beta_a \right) - \frac{12\dot{a}^2}{a^2 n^2} A - \frac{6\dot{a}}{a} \left( \frac{2a'}{an} - \frac{n'}{n^2} \right) \alpha \\
- \frac{6\dot{a}}{an} \alpha' + \frac{6\dot{a}}{an^2} \dot{\Phi} - \left( \frac{4a'}{a} + \frac{2n'}{n} \right) D^\alpha_a + 12\frac{a'}{a} \left( \frac{a'}{a} + \frac{n'}{n} \right) \Phi - \frac{6a'}{a} A'
\]  

(A.4)

Appendix B.

In this appendix we present the left hand side of the perturbed teleparallel field equations in full.

\[
F^\alpha_a = \frac{1}{a} \left[ \partial_\beta \left( \frac{1}{a^2} \partial[\beta D^\alpha_a] - \frac{1}{a^2} \partial[\alpha D^\beta_a] + \frac{1}{2a^2} \partial[\beta D^\alpha_a] - \frac{3\dot{a}}{a^2 n} \partial[\beta C^\alpha_a] \right) \\
- \left( \frac{2a'}{a^2} + \frac{n'}{na} \right) \partial[\beta F^\alpha_a] - \frac{1}{a} \partial[\beta F^\alpha_a] + \frac{1}{a^2} \partial[\beta D^\gamma_a] A + \frac{1}{a^2} \partial[\beta \partial^\gamma_a] \Phi \\
+ \frac{1}{a^2} \partial_a \partial_\beta D^\gamma_a - \frac{1}{a} \partial_\beta A \Phi + \frac{1}{a} \partial_\beta B^\gamma_a \right] \\
+ \frac{1}{a^2 n^2} \partial_\beta \left[ - \frac{a}{2} \partial[\alpha C_a] + \frac{a^2}{2n} (\partial(\alpha B^\alpha) - \partial_\alpha \partial(\beta B^\beta)) + \frac{\dot{a}}{a} \frac{a}{n} D^\alpha_a \\
+ \frac{\dot{a}}{a} \frac{a}{n} \left( 1 - 2A \right) \partial_\alpha A_a + \frac{a^2}{2n} \left( \partial(\alpha B^\alpha) - \partial_\alpha \partial(\beta B^\beta) \right) \\
+ \frac{a^2}{2n} (\Phi - \frac{a^2}{2} A') \partial_\alpha A_a \right] \\
+ \frac{1}{a^2 n^2} \partial_\beta \left[ - \frac{a}{2} \partial[\alpha F_a] + \frac{a^2}{2} (\partial(\alpha B^\alpha) - \partial_\alpha \partial(\beta B^\beta)) + \frac{a^2}{2n} \left( \frac{a'}{a} + \frac{n'}{2n} \right) D^\alpha_a \\
- \left( \frac{a'}{a} + \frac{n'}{2n} \right) \partial_\alpha \partial_\beta A_a \right] \\
- \left( \frac{a'}{a^2} + \frac{n'}{2an} \right) \partial_\alpha \partial_\beta A_a \right] \\
- \left( \frac{a'}{a^2} + \frac{n'}{2an} \right) \partial_\alpha \partial_\beta A_a \right]
\]  

(B.1)
\begin{equation}
F_0^0 = \partial_\beta \left[ -\frac{1}{2an}(\dot{a}^2 - \frac{5\dot{a}}{2a^n})C^{\beta} - \frac{1}{2a^2n} \partial^{[\beta} D^{\gamma]} + \frac{3\dot{a}'}{2a^2n} F^{\beta} - \frac{3\dot{a}'}{2an} \partial_\delta (A + \Phi + |D|) \right. \\
+ \frac{1}{a^{3n}} \partial_\delta \left( \frac{a^3}{2} D^{\beta}_\delta \frac{3a^2 - 2}{a^2n} \frac{\dot{a}^2}{2} A + \frac{3\dot{a}'^2}{2a^2n} A + \frac{3\dot{a}'}{a^{2n} n^2} + \frac{2a'n'}{an^2} \Phi \\
- \frac{2\dot{a}}{a^n} (\partial_\beta B^{\beta} - \partial_\beta D^{\beta}_\beta) - \frac{3a'}{2an} \frac{\dot{a}'}{a^2n^2} + \frac{3\dot{a}'}{a^2n^2} |D| \right] \\
- \frac{3n'}{4an^2} - \frac{3n'\dot{a}}{2an^2} + \frac{3n'a'}{2an^2} \alpha - \frac{\dot{a}}{2an^2} (7n\alpha' + \dot{\Phi}) \tag{B.2}
\end{equation}

\begin{equation}
F_5^0 = \partial_\beta \left[ -\frac{1}{4an} C^{\beta} + \frac{1}{4} \left( \frac{\dot{a}'}{an^3} - \frac{n'}{an^3} \right) B^{\beta} - \frac{1}{4n^2} B^{\beta} + \frac{1}{4} \left( \frac{n'}{an^2} - \frac{\dot{a}'}{na^2} \right) C^{\beta} + \frac{\dot{a}}{4a^2n^2} F^{\beta} \\
- \frac{1}{4a^{2n}} \alpha + \frac{1}{4a^{2n}} \dot{F}^{\beta} \right] - \frac{3\dot{a}}{2an^2} \partial_\delta (A + \Phi + |D|) + \frac{\dot{a}'}{a^2n^2} (1 - 2A) \\
+ \frac{1}{a^{3n}} \partial_\delta \left[ \frac{a^3}{2n} (\partial_\beta B^{\beta} - \partial_\beta D^{\beta}_\beta) - \frac{3a^2}{2n} (1 - 2A + \frac{3\dot{a}'}{2n} \alpha + \frac{3\dot{a}^2}{2n} \Phi) \right] \\
- \frac{\dot{a}'}{2an} (\frac{2\dot{a}'}{a} + \frac{n'}{n}) \alpha + \frac{\dot{a}'}{2an^2} \dot{\Phi} - \frac{\dot{a}'}{2an} \alpha' - \frac{\dot{a}}{an^2} (D^{\alpha}_a + \Phi) \tag{B.3}
\end{equation}

\begin{equation}
F_a^0 = \partial_\beta \left[ \frac{a}{2} \partial^{[\beta} C_a + \frac{1}{2an} (\dot{a} B^{\beta} - \delta_a B^{\beta}) - \frac{a^2}{2n} \partial_\beta (A + \Phi + |D|) \\
- \frac{\dot{a}}{n} (1 - 2A) \delta_a + \left( \frac{\dot{a}'}{2n} + \frac{n'a^2}{2n} \right) \alpha \delta_a + \frac{a^2}{2n} \alpha' \delta_a \right] - \frac{\dot{a}}{n} \partial_\delta (A + \Phi + |D|) \\
+ \frac{1}{a^{3n}} \partial_\delta \left[ - \frac{a^3}{4} \partial_\alpha A + \frac{a^3}{4} F_a + \frac{a^3}{4n} (\frac{\dot{a}'}{an^3} + \frac{5\dot{a}'}{4n} A) C_a + \frac{a^3}{4} C_a + \frac{3a^2}{4n} F_a - \frac{a^2}{4n} B_a' \\
+ \frac{a^4}{4n^2} (a - 1) B_a \right] - \frac{2\dot{a}}{a^2n^2} \partial_\delta B^{\gamma} \right] + \left( - \frac{9\dot{a}^2}{2an^3} - \frac{3\dot{a}^2}{2an^3} + \frac{11\dot{a}'}{2an^3} + \frac{a'n'}{4n} + \frac{3a'n'}{2n^2} \right) C_a \\
+ \frac{3\dot{a}}{2a^{2n} n^2} \partial_\alpha F_a - \frac{3\dot{a}}{2an^2} F_a' + \frac{\dot{a}'}{4an^2} F_a - \frac{5\dot{a}'}{4an^3} C_a + \frac{a'n'}{4n^3} (a - 1) B_a \\
- \frac{\dot{a}}{4n^2} \partial_\gamma A - \frac{\dot{a}'}{4n^2} B_a' \tag{B.4}
\end{equation}

\begin{equation}
F_5^5 = \partial_\beta \left[ - \frac{2a'}{a^2} + \frac{n'}{2an} \right] F^{\beta} + \frac{1}{a} \partial^{\beta} \Phi - \frac{1}{a} F^{\beta} + \frac{1}{a^2} \partial^{[\beta} D^{\gamma]} + \frac{1}{2a^2} \partial^{\beta} A \\
+ \frac{3\dot{a}}{2a^{2n}} B^{\beta} - \frac{2\dot{a}}{a^{2n}} - \frac{\dot{a}}{2an} \right] C^{\beta} - \frac{1}{2an} C^{\beta} + \frac{\dot{a}}{2an^2} \partial_\delta (A + \Phi + |D|) 
\end{equation}
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\[ F^5_a = \left( \frac{a'}{a^2} + \frac{n'}{2an} \right) \partial_a (\Phi + A + |D|) + \partial_\beta \left[ \frac{1}{2a^2} \partial^{[\beta} F^a_{\alpha]} - \frac{1}{2a} (D^{(\beta}_{\alpha} - \delta^{\beta}_{\alpha} D^{\gamma}_{\gamma}) 
\right.
\]
\[ + \left( \frac{a'}{a^2} + \frac{n'}{2an} + \frac{1}{2an} A - \frac{2a'}{a^2} + \frac{n'}{an} F_a - \frac{3a^2}{4n} C_a + \frac{a^3}{4} C_a + \frac{a^4}{4n^2} (a - 1) B_a + \frac{a^4}{4n^2} (a - 1) B_a + \frac{\dot{a}'}{4an^2} B_a + \frac{\dot{a}'}{4an^2} C_a + \frac{\dot{a}'}{4an^2} F_a + \frac{\dot{a}'}{4an^2} F_a + \frac{\dot{a}'}{4an^2} F_a \right)
\]

F^\alpha_5 = \partial_\beta \left[ \frac{1}{2a^2} \partial^{[\beta} F^a_{\alpha]} + \frac{1}{2a^2} D^{[\beta\alpha]} \right] + \frac{1}{a^3 n} \partial_\beta \left[ \frac{an}{2} \partial^{\alpha} A + an \partial^{[\alpha} D^{\beta]} - \frac{3a^2}{2n} B^\alpha - an \partial^{\alpha} \Phi \right.
\]
\[ + (2aa - \frac{\dot{a}a^2}{2}) C^\alpha + \frac{a^2}{2} \dot{C}^\alpha + (2a'an + \frac{a^2 n'}{2}) F^\alpha + a^2 F' \right)
\]
\[ + \frac{1}{a^3 n} \partial_\beta \left[ \frac{a^2}{4} C^\alpha \right]
\]
\[ + \frac{a^3 n'}{4n^2} B^\alpha + \frac{a^3}{4n} B' - \frac{a^2 n'}{4n} C^\alpha + \frac{\dot{a}a}{4n} F^\alpha - \frac{a^2}{4n} \dot{F}^\alpha + \frac{a^2}{4} \partial^{\alpha} \Phi \]
\]
\[ - \frac{\dot{a}a}{a^2 n^2} B^\alpha + (\frac{a'}{a^3} - \frac{n'}{a^2 n}) \partial^{[\alpha} D^{\beta]} + \frac{\dot{a}a}{an^2} B^\alpha - \frac{\dot{a}n'}{4an^2} C^\alpha + \frac{\dot{a}n'}{4an^2} C^\alpha \right)
\]

F^\alpha_0 = \partial_\beta \left[ - \frac{1}{2na^2} (\partial^{[\alpha} B^{\beta]} - \partial_0 D^{[\alpha\beta]} \right] + \frac{1}{a^3 n} \partial_\beta \left[ \frac{\dot{a}a}{n^2} (1 - a) B^\alpha + (\frac{\dot{a}a^2}{4n^2} - \frac{5aa}{4n}) C^\alpha - 3a^2 F^\alpha + a \partial^{[\alpha} D^{\beta]} \right]
\]
\[ + \frac{1}{a^3 n} \partial_\beta \left[ \frac{\dot{a}a}{2n} - \frac{3aa}{n} B^\alpha + (\frac{a^2}{2} F^\alpha + \frac{a^2}{2} C^\alpha - an \partial^{\alpha} \Phi - \frac{an}{2} \partial^{\alpha} A \right]
\]

(B.5)

(B.6)

(B.7)
\[(2a'\dot{a} + \frac{a^2\ddot{a}}{2})F^\alpha + a^2nF'^\alpha - an\partial^{[\alpha}D^{\beta]}_{\beta]} \right] + \frac{3\ddot{a}}{2na^3}\partial^{[\alpha}D^{\beta]}_{\beta]} - \frac{a'}{an}B'^\alpha - \frac{n'}{2a^2n}\partial^{\alpha} + \frac{n'}{2an^2}\dot{F}^\alpha + \frac{n'\ddot{a}}{2n^2a^3}\dot{F}^\alpha - \frac{n'}{2an}C'^\alpha + \left( - \frac{n'^2}{2n^3} - \frac{5\ddot{a}}{2an^3} + \frac{5a'^2}{2an} + \frac{2an'\ddot{a}}{a^2n} - \frac{n'a'^2}{a^2n} + \frac{n'a'^2}{2an^2} + \frac{an'^2}{2n^3} \right)B^\alpha \]

(B.8)

Where \(|D|\) is the determinant of \(D^i_{\mu}\).

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