NON-DIAGONAL INVARIANT EINSTEIN METRICS ON REAL
FLAG MANIFOLDS

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Abstract. We find the Einstein invariant metrics on the real flag manifolds
$SO(4)/SO(2)\times O(1)\times O(1))$ and $(SO(l)\times SO(l))/SO(l-1)\times O(1))$, $l \geq 4$ and
classify them by isometry. These are the only real flag manifolds of classical type
where the isotropy representation decomposes into three irreducible submodules
and two of them are equivalent.

1. Introduction

A Riemannian metric $g$ on a differentiable manifold $M$ is called an Einstein metric if its
associated Ricci tensor satisfies the condition $\text{Ric} = cg$ for some constant $c$. In general, finding Einstein metric is
equivalent to solve a system of non-linear partial differential equation and its is a very difficult task.
When $M$ is a compact homogeneous space, the problem of finding invariant Einstein metrics reduces
to solve a system of polynomial equations. This problem has been widely studied for the case where $M$ is a
complex generalized flag manifold (e.g. [2], [3] and [11]), in this case, the isotropy representation
is multiplicity free and $M$ admits only invariant metrics of diagonal type. In [1], Alekseevsky, Dotti
and Ferraris obtained a unique (up to homothety) non-diagonal $S^3 \times S^3$–invariant Einstein metric
on $S^3 \times S^3/S^1$. In this paper, we classify all invariant Einstein metrics for a family of generalized
real flag manifolds (which admit non-diagonal metrics) associated to a split real form $\mathfrak{g}$ of a complex
simple Lie algebra.

A real flag manifold of $\mathfrak{g}$ is the homogeneous space $F_\Theta = G/P_\Theta$ where $G$ is a connected non-compact
Lie group with Lie algebra $\mathfrak{g}$ and $P_\Theta$ is a parabolic subgroup of $G$. Several authors have studied some
geometric and topological aspects of these spaces as their fundamental group, orientability, almost
complex structures, their isotropy representation, invariant metrics, etc. (see for instance, [5], [6], [9],
[10], [15]).

In this paper we find all the invariant Einstein metrics for the real flag manifolds of classical type
with following property: the isotropy representation decomposes into 3 irreducible sub-modules and
only two them are equivalent. This correspond precisely to the homogeneous spaces $SO(4)/SO(2)\times
O(1)\times O(1))$ and $(SO(l)\times SO(l))/SO(l-1)\times O(1))$, $l \geq 4$.

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The contents are organized as follows: Section 2 contains a review about compact homogeneous
spaces and the Ricci curvature tensor. In Section 3 we give a description of a flag manifold from a
Lie theoretical point of view, we describe the invariant metrics on $SO(4)/SO(2)\times O(1)\times O(1))$
and $(SO(l)\times SO(l))/SO(l-1)\times O(1))$, $l \geq 4$ and determine sufficient and necessary conditions
Conversely, given an operator 
reductive decomposition. Then, every 
Ad \cdot \text{by a unique } (\text{Proposition 3.1}). For 
(SO(l) \times SO(l))/S(O(l - 1) \times O(1)), l \geq 4 we find two diagonal and four non-diagonal invariant 
Einstein metrics (Proposition 3.2). Section 4 contains a discussion about which of the metrics obtained 
in Section 3 are isometric. We can summarize all the results of this paper in the following Main 
Theorem:

**Theorem.** Let \( M = SO(4)/S(O(2) \times O(1) \times O(1)) \) or \( (SO(l) \times SO(l))/S(O(l - 1) \times O(1)) \) with 
l \geq 4. Then, there exist exactly two (up to isometry) invariant Einstein metrics on \( M \) and one of them 
is non-diagonal.

2. The Ricci tensor

We consider a homogeneous space \( M = G/H \), where \( G \) is a compact, connected Lie group and \( H \) 
is a closed subgroup of \( G \). A Riemannian metric \((\cdot, \cdot)\) on \( G/H \) is said to be \( G\text{-invariant} \) if, for every 
a \in G, the map 

\[
(1) \quad \phi_a : (G/H, (\cdot, \cdot)) \rightarrow (G/H, (\cdot, \cdot)); \quad \phi_a(bH) = abH
\]
is an isometry. Let \( \mathfrak{g} \) and \( \mathfrak{h} \) be the Lie algebras of \( G \) and \( H \) respectively, by compactness of \( G \), there 
exists a reductive decomposition \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \), i.e., \( Ad(h)\mathfrak{m} \subseteq \mathfrak{m} \). The subspace \( \mathfrak{m} \) is identified with the 
tangent space of \( G/H \) at the identity coset \( eH \) via the isomorphism 

\[
(2) \quad X \mapsto X^*(eH) = \left. \frac{d}{dt} \exp(tX)H \right|_{t=0}.
\]

This map also gives rise to an equivalence between the isotropy representation of \( G/H \) and the ad-
joint representation of \( H \) on \( \mathfrak{m} \). Thus, every \( G\text{-invariant} \) metric on \( G/H \) can be identified with an 
\( Ad(H)\text{-invariant} \) inner product \( g \) on \( \mathfrak{m} \), i.e., 

\[
(3) \quad g(Ad(h)X, Ad(h)Y) = g(X, Y) \text{ for all } h \in H, \ X, Y \in \mathfrak{m}.
\]

Let \((\cdot, \cdot)\) be a fixed \( Ad(G)\text{-invariant} \) inner product \((\cdot, \cdot)\) on \( \mathfrak{g} \) such that \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \) is an \((\cdot, \cdot)\text{-orthogonal} \) reductive decomposition. Then, every \( Ad(H)\text{-invariant} \) inner product \( g \) on \( \mathfrak{m} \) is completely determined 
by a unique \((\cdot, \cdot)\text{-self-adjoint}, positive operator } A : \mathfrak{m} \rightarrow \mathfrak{m} \text{ that commutes with } Ad(h), \text{ for all } h \in H. \text{The operator } A \text{ is defined implicitly by the formula}

\[
(4) \quad g(X, Y) = (AX, Y) \text{ for all } X, Y \in \mathfrak{m}.
\]

Conversely, given an operator \( A \) satisfying the conditions above, we have that the formula (4) defines 
an \( Ad(H)\text{-invariant} \) inner product on \( \mathfrak{m} \). We call \( A \) the metric operator corresponding to \( g \). From 
now, we identify a \( G\text{-invariant} \) Riemannian metric on \( G/H \) with its corresponding \( Ad(H)\text{-invariant} \) 
product on \( M \) and its corresponding metric operator \( A \).

By compactness of \( H \) (as a closed subgroup of the compact group \( G \)), the adjoint representation of \( H \) 
on \( \mathfrak{m} \) induces a \((\cdot, \cdot)\text{-orthogonal splitting}

\[
(5) \quad \mathfrak{m} = \bigoplus_{i=1}^{s} \mathfrak{m}_i
\]
of \( \mathfrak{m} \) into \( H\text{-invariant} \), irreducible subspaces \( \mathfrak{m}_i, \ i = 1, \ldots, s \). When all the submodules \( \mathfrak{m}_i \) have 
multiplicity one, every \( G\text{-invariant} \) metric \( A \) is equal to a positive scalar multiple of the identity 
map when it is restricted to each \( \mathfrak{m}_i \). When \( \mathfrak{m}_i \) and \( \mathfrak{m}_j \) are equivalent for some \( i, j \), we have metric
operators $A$ mapping vectors of $m_i$ to vectors with non-zero projection on $m_j$. This motivates the following definition.

**Definition 2.1.** Let $G/H$ be a homogeneous spaces, where $G$ is a compact, connected Lie group and $H$ is a closed subgroup. We say that $G/H$ admits a non-diagonal invariant metric if the adjoint representation $Ad : H \rightarrow GL(m)$ decomposes into irreducible submodules and at least two of such different submodules are equivalent.

For a $G$-invariant metric $g$ on $G/H$, define $U : m \times m \rightarrow m$ by the formula

$$2g(U(X,Y),W) = g([W,X]_m,Y) + g([W,Y]_m,X)$$

for all $W \in m$. We may apply Corollary 7.38 of [3] and obtain an explicit formula for the Ricci tensor:

$$Ric(X,Y) = -\frac{1}{2} \sum_i g([X,X_i]_m,[Y,X_i]_m) - \frac{1}{2}\langle X,Y \rangle$$

$$+ \frac{1}{4} \sum_{i,j} g([X_i,X_j]_m,X)g([X_i,X_j]_m,Y) - g(U(X,Y),Z)$$

where $\langle \cdot, \cdot \rangle$ is the Killing form of $G$, $\{X_i\}$ is an $g$-orthonormal basis of $m$, $Z = \sum_i U(X_i,X_i)$ and $X, Y \in m$. We say that $g$ is an Einstein metric if there exists a real number $c$ such that

$$Ric = cg.$$  

In the next section, we shall study the equation (8) for some split real flag manifolds admitting non-diagonal invariant metrics. The key is to use the description of the invariant metrics for real flag manifolds given in [6] and the fact that, for a non-diagonal invariant metric $g$, one can have $g$-orthogonal vectors $X, Y \in m$ with $Ric(X,Y)$ not necessarily zero. In this situation, $Ric(X,Y)$ gives us an expression in terms of the parameters of the metric $g$ which can be factored. If $g$ satisfies equation (8), then $Ric(X,Y) = 0$ and we obtain strong necessary conditions for $g$.

### 3. Einstein metrics on real flag manifolds

Let $\mathfrak{g}$ be a non-compact, simple, real Lie algebra which is a split real form of a complex Lie algebra, $G$ a connected Lie group with Lie algebra $\mathfrak{g}$ and $P_\Theta$ a parabolic subgroup of $G$. A generalized flag manifold of $\mathfrak{g}$ is the homogeneous space $F_\Theta = G/P_\Theta$. If $K$ is a maximal compact subgroup of $G$, then $K$ acts transitively on $F_\Theta$ with isotropy $K_\Theta = K \cap P_\Theta$, so $F_\Theta$ can be identified with the quotient $K/K_\Theta$. We consider a reductive decomposition $\mathfrak{k} = \mathfrak{k}_\Theta \oplus \mathfrak{m}_\Theta$, where $\mathfrak{k}$ and $\mathfrak{k}_\Theta$ are the Lie algebras of $K$ and $K_\Theta$ respectively.

We can also describe generalized flag manifolds of $\mathfrak{g}$ by considering a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ and a maximal abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{s}$ as follows: let $\Pi$ denote the set of roots of $\mathfrak{g}$ corresponding to $\mathfrak{a}$ and

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Pi^+} \mathfrak{g}_\alpha$$

the associated root space decomposition. Fix a set $\Pi^+$ of positive roots and consider $\Sigma$ the corresponding set of simple roots. Each $\Theta \subseteq \Sigma$ determines a parabolic subalgebra.
Proposition 3.1. Let $A$ be an invariant metric on the flag $\mathbb{F}_{\lambda_1-\lambda_2}$ of $A_3$ written as in (3). Then $A$ is an Einstein metric if and only if its entries satisfy one of the following conditions:

1. $b = 0, \mu_1 = \mu_2 = \frac{3}{2} \mu_0$
2. $b > 0, b = \mu_1 = \frac{\mu_0}{3} = \frac{\mu_0}{2}$
3. $b > 0, b = \mu_2 = \frac{\mu_1}{3} = \frac{\mu_0}{2}$
4. $b < 0, -b = \mu_1 = \frac{\mu_0}{3} = \frac{\mu_0}{2}$
5. $b < 0, -b = \mu_2 = \frac{\mu_1}{3} = \frac{\mu_0}{2}$

Proof. We separate two cases:

• $A$ diagonal ($b = 0$):
The vectors

\[ X_0 = \frac{w_{43}}{\sqrt{\mu_0}}, \quad X_1 = \frac{w_{31}}{\sqrt{\mu_1}}, \quad X_2 = \frac{w_{32}}{\sqrt{\mu_2}}, \quad X_3 = \frac{w_{42}}{\sqrt{\mu_2}}, \quad X_4 = \frac{w_{41}}{\sqrt{\mu_2}} \]

form an \( A \)-orthonormal basis of \( m_{(\lambda_1-\lambda_2)} \). The non-zero bracket relations between these vectors are given by

\[ [X_0, X_1]_{m_{(\lambda_1-\lambda_2)}} = \sqrt{\frac{\mu_2}{\mu_0\mu_1}} X_4, \quad [X_0, X_2]_{m_{(\lambda_1-\lambda_2)}} = \sqrt{\frac{\mu_2}{\mu_0\mu_1}} X_3, \]

\[ [X_0, X_3]_{m_{(\lambda_1-\lambda_2)}} = -\sqrt{\frac{\mu_1}{\mu_0\mu_2}} X_2, \quad [X_0, X_4]_{m_{(\lambda_1-\lambda_2)}} = -\sqrt{\frac{\mu_1}{\mu_0\mu_2}} X_1, \]

\[ [X_1, X_4]_{m_{(\lambda_1-\lambda_2)}} = \sqrt{\frac{\mu_0}{\mu_1\mu_2}} X_0, \quad [X_2, X_3]_{m_{(\lambda_1-\lambda_2)}} = \sqrt{\frac{\mu_0}{\mu_1\mu_2}} X_0. \]

Since \([X_i, X_j]\) is \( A \)-orthogonal to \( X_i \) and \( X_j \) for all \( i, j \in \{0, 1, 2, 3, 4\} \), then, equation (6) implies

\[ Z = \sum_i U(X_i, X_i) = 0. \]

A direct application of the formula (7) gives us the nonzero components of the Ricci tensor:

\[ r_0 := Ric(X_0, X_0) = \frac{2}{\mu_0} + \frac{\mu_0}{\mu_1\mu_2} - \frac{\mu_2}{\mu_0\mu_1} - \frac{\mu_1}{\mu_0\mu_2} \]

\[ r_1 := Ric(X_1, X_1) = Ric(X_2, X_2) = \frac{2}{\mu_1} + \frac{\mu_1}{2\mu_0\mu_2} - \frac{\mu_2}{2\mu_0\mu_1} - \frac{\mu_0}{2\mu_1\mu_2} \]

\[ r_2 := Ric(X_3, X_3) = Ric(X_4, X_4) = \frac{2}{\mu_2} + \frac{\mu_2}{2\mu_0\mu_1} - \frac{\mu_1}{2\mu_0\mu_2} - \frac{\mu_0}{2\mu_1\mu_2}. \]

Thus, Einstein condition for the diagonal metric \( A \) reduces to \( r_0 = r_1 = r_2 \), which gives us

\[ \mu_1 = \mu_2 = \frac{3}{4}\mu_0. \]

- \( A \) non-diagonal \( (b \neq 0) \):

The eigenvalues of \( A \) are

\[ \xi_0 = \mu_0, \quad \xi_1 = \frac{\mu_1 + \mu_2 - \sqrt{4b^2 + (\mu_1 - \mu_2)^2}}{2}, \quad \xi_2 = \frac{\mu_1 + \mu_2 + \sqrt{4b^2 + (\mu_1 - \mu_2)^2}}{2} \]

and satisfy the relations

\[ \xi_1 + \xi_2 = \mu_1 + \mu_2, \quad \xi_1\xi_2 = \mu_1\mu_2 - b^2, \]

\[ b^2 = (\xi_2 - \mu_2)(\xi_2 - \mu_1) = (\mu_1 - \xi_1)(\xi_2 - \mu_1) \]

\[ = (\mu_2 - \xi_1)(\mu_1 - \xi_1) = (\mu_2 - \xi_1)(\xi_2 - \mu_2). \]

Since \( A \) is positive definite then \( \xi_1, \xi_2 > 0 \). Let us set
Then the vectors

\[ X_0 = \frac{w_{43}}{\sqrt{\xi_0}}, \quad X_1 = \frac{(\xi_1 - \mu_2)w_{31} + bw_{12}}{\sqrt{c_1}}, \quad X_2 = \frac{(\xi_2 - \mu_1)w_{32} + bw_{11}}{\sqrt{c_1}}, \]

\( X_3 = \frac{(\xi_2 - \mu_2)w_{31} + bw_{12}}{\sqrt{c_2}}, \quad X_4 = \frac{(\xi_1 - \mu_1)w_{32} + bw_{11}}{\sqrt{c_2}} \)

form an \( A \)-orthonormal basis for \( m_{(\lambda_1-\lambda_2)} \). The non-zero bracket relations are given by

\[ [X_0, X_1]_{m_{(\lambda_1-\lambda_2)}} = \frac{b}{\xi_1 - \xi_2} \left( \frac{2}{\sqrt{\xi_0}} X_2 + \sqrt{\frac{\xi_2}{\xi_0 \xi_1}} \left( \frac{\mu_2 - \mu_1}{|b|} \right) X_4 \right), \]

\[ [X_0, X_2]_{m_{(\lambda_1-\lambda_2)}} = \frac{b}{\xi_2 - \xi_1} \left( \frac{2}{\sqrt{\xi_0}} X_1 + \sqrt{\frac{\xi_2}{\xi_0 \xi_1}} \left( \frac{\mu_2 - \mu_1}{|b|} \right) X_3 \right), \]

\[ [X_0, X_3]_{m_{(\lambda_1-\lambda_2)}} = \frac{b}{\xi_2 - \xi_1} \left( \sqrt{\frac{\xi_1}{\xi_0 \xi_2}} \left( \frac{\mu_1 - \mu_2}{|b|} \right) X_2 + \frac{2}{\sqrt{\xi_0}} X_4 \right), \]

\[ [X_0, X_4]_{m_{(\lambda_1-\lambda_2)}} = \frac{b}{\xi_1 - \xi_2} \left( \sqrt{\frac{\xi_1}{\xi_0 \xi_2}} \left( \frac{\mu_1 - \mu_2}{|b|} \right) X_1 + \frac{2}{\sqrt{\xi_0}} X_3 \right), \]

\[ [X_1, X_2]_{m_{(\lambda_1-\lambda_2)}} = \frac{2b\sqrt{\xi_0}}{\xi_1(\xi_1 - \xi_2)} X_0, \]

\[ [X_1, X_4]_{m_{(\lambda_1-\lambda_2)}} = \frac{b(\mu_2 - \mu_1)}{|b|(\xi_1 - \xi_2)} \sqrt{\frac{\xi_0}{\xi_1 \xi_2}} X_0, \]

\[ [X_2, X_3]_{m_{(\lambda_1-\lambda_2)}} = \frac{b(\mu_2 - \mu_1)}{|b|(\xi_2 - \xi_1)} \sqrt{\frac{\xi_0}{\xi_1 \xi_2}} X_0, \]

\[ [X_3, X_4]_{m_{(\lambda_1-\lambda_2)}} = \frac{2b\sqrt{\xi_0}}{\xi_2(\xi_2 - \xi_1)} X_0. \]

Observe that \([X_i, X_j]\) is \( A \)-orthogonal to \( X_i \) and \( X_j \) for all \( i, j \in \{0, 1, 2, 3, 4\} \), thus, \( Z = 0 \). If \( A \) is an Einstein metric, then \( Ric(X_1, X_3) = 0 \) (because \( Ric = cg \)). By (7) we have

\[ Ric(X_1, X_3) = -\frac{1}{2} g \left( [X_1, X_2]_{m_{(\lambda_1-\lambda_2)}}, [X_3, X_2]_{m_{(\lambda_1-\lambda_2)}} \right) - \frac{1}{2} g \left( [X_1, X_4]_{m_{(\lambda_1-\lambda_2)}}, [X_3, X_4]_{m_{(\lambda_1-\lambda_2)}} \right) \]
\[ -\frac{1}{2} g \left( [X_1, X_0]_{m(\lambda_1 - \lambda_2)}, [X_3, X_0]_{m(\lambda_1 - \lambda_2)} \right) - \frac{1}{2} \langle X_1, X_3 \rangle \]

\[ + \frac{1}{2} g \left( [X_0, X_2]_{m(\lambda_1 - \lambda_2)}, X_1 \right) g \left( [X_0, X_2]_{m(\lambda_1 - \lambda_2)}, X_3 \right) \]

\[ + \frac{1}{2} g \left( [X_0, X_4]_{m(\lambda_1 - \lambda_2)}, X_1 \right) g \left( [X_0, X_4]_{m(\lambda_1 - \lambda_2)}, X_3 \right) \]

\[ = -\frac{1}{2} \left( \frac{2b\sqrt{\xi_0}}{\xi_1(\xi_1 - \xi_2)} \right) \left( \frac{b(\mu_1 - \mu_2)}{|b|(\xi_2 - \xi_1)} \sqrt{\xi_0} \right) - \frac{1}{2} \left( \frac{2b\sqrt{\xi_0}}{\xi_2(\xi_2 - \xi_1)} \right) \left( \frac{b(\mu_2 - \mu_1)}{|b|(\xi_1 - \xi_2)} \sqrt{\xi_0} \right) \]

\[ - \frac{1}{2} \left( \frac{2b}{\sqrt{\xi_0}(\xi_2 - \xi_1)} \right) \left( \frac{b(\mu_2 - \mu_1)}{|b|(\xi_2 - \xi_1)} \sqrt{\xi_0} \right) + \frac{1}{2} \left( \frac{2b}{\sqrt{\xi_0}(\xi_1 - \xi_2)} \right) \left( \frac{b(\mu_2 - \mu_1)}{|b|(\xi_1 - \xi_2)} \sqrt{\xi_0} \right) \]

\[ = \frac{b^2(\mu_2 - \mu_1)}{|b|(\xi_1 - \xi_2)^2} \left( \frac{-\xi_0}{\xi_1\sqrt{\xi_1\xi_2}} + \frac{\xi_0}{\xi_2\sqrt{\xi_1\xi_2}} - 2\sqrt{\xi_1} + \frac{2\sqrt{\xi_2}}{\xi_0\sqrt{\xi_1}} \right) \]

\[ = \frac{b^2(\mu_2 - \mu_1)}{|b|(\xi_1 - \xi_2)^2} \left( \frac{-\xi_0^2(\xi_2 - \xi_1) + 2\xi_1\xi_2(\xi_2 - \xi_1)}{\xi_0\xi_1\xi_2\sqrt{\xi_1\xi_2}} \right) \]

\[ = \frac{|b|(\mu_2 - \mu_1)(2\xi_1\xi_2 - \xi_0^2)}{\xi_0(\xi_1\xi_2)^2(\xi_2 - \xi_1)}, \]

so, a necessary condition for \( A \) to be an Einstein metric is that \( \mu_1 = \mu_2 \) or \( 2\xi_1\xi_2 - \xi_0^2 = 0 \).

**Case 1.** \( \mu := \mu_1 = \mu_2 \).

We can use formula (12) to obtain

\[ r_0 = Ric(X_0, X_0) = \frac{\xi_0(\xi_1\xi_2 + 2b^2)}{(\xi_1\xi_2)^2}, \]

\[ r_1 = Ric(X_1, X_1) = Ric(X_2, X_2) = \frac{4\xi_1 - \xi_0}{2\xi_1^2}, \]

\[ r_2 = Ric(X_3, X_3) = Ric(X_4, X_4) = \frac{4\xi_2 - \xi_0}{2\xi_2^2}. \]
If $r_0 = r_1 = r_2$, then $\xi_1 = \xi_2$, which is not possible since $b \neq 0$. Therefore, we have no solutions in this case.

**Case 2.** $2\xi_1\xi_2 - \xi_0^2 = 0$.

The non-zero components of the Ricci tensor are $r_0 = Ric(X_0, X_0)$, $r_1 = Ric(X_1, X_1) = Ric(X_2, X_2)$ and $r_2 = Ric(X_3, X_3) = Ric(X_4, X_4)$. By (7) we have

$$r_1 - r_2 = \frac{\xi_1\xi_2(\mu_2 - \mu_1)^2}{\xi_0(\xi_2 - \xi_1)^2} \left( \frac{1}{\xi_2} - \frac{1}{\xi_1} \right) + \frac{2b^2\xi_0}{(\xi_2 - \xi_1)^2} \left( \frac{1}{\xi_2} - \frac{1}{\xi_1} \right) - 2 \left( \frac{1}{\xi_2} - \frac{1}{\xi_1} \right)$$

$$= \left( \frac{1}{\xi_2} - \frac{1}{\xi_1} \right) \left( \frac{\xi_1\xi_2(\mu_2 - \mu_1)^2(\xi_1 + \xi_2) + 2b^2\xi_0^2(\xi_1 + \xi_2) - 2\xi_1\xi_2\xi_0(\xi_2 - \xi_1)^2}{\xi_0(\xi_2 - \xi_1)^2(\xi_1 + \xi_2)} \right)$$

$$= \left( \frac{1}{\xi_2} - \frac{1}{\xi_1} \right) \left( \frac{\xi_1\xi_2(\xi_1 + \xi_2)(\xi_2 - \xi_1)^2 - 2\xi_1\xi_2\xi_0(\xi_2 - \xi_1)^2}{\xi_0(\xi_2 - \xi_1)^2(\xi_1 + \xi_2)} \right)$$

$$= \left( \frac{1}{\xi_2} - \frac{1}{\xi_1} \right) \left( \frac{\xi_1\xi_2(\xi_1 + \xi_2 - 2\xi_0)}{\xi_0(\xi_1 + \xi_2)} \right)$$

$$= \frac{(\xi_1 - \xi_2)(\xi_1 + \xi_2 - 2\sqrt{2\xi_1\xi_2})}{\sqrt{2}(\xi_1\xi_2)^{\frac{3}{2}}}$$

and

$$r_0 - \frac{r_1 + r_2}{2} = \frac{b^2}{(\xi_2 - \xi_1)^2} \left( \frac{3\xi_0}{\xi_1^2} + \frac{3\xi_0}{\xi_2^2} - \frac{8}{\xi_0} \right) + \frac{(\mu_2 - \mu_1)^2}{2(\xi_2 - \xi_1)^2} \left( \frac{3\xi_0}{\xi_1\xi_2} - \frac{2\xi_2}{\xi_0\xi_1} - \frac{2\xi_1}{\xi_0\xi_2} \right)$$

$$- \left( \frac{2}{\xi_0} - \frac{1}{\xi_1} - \frac{1}{\xi_2} \right)$$

$$= \frac{10\sqrt{2}b^2 + 3\sqrt{2}\xi_1\xi_2 - \sqrt{2}(\xi_1 - \xi_2)^2 - 2(\xi_1 + \xi_2)\sqrt{\xi_1\xi_2}}{\sqrt{2}(\xi_1\xi_2)^{\frac{3}{2}}}.$$

Since the systems of equations

$$\begin{cases} r_0 - r_1 = 0 \\ r_1 - r_2 = 0 \end{cases} \quad \begin{cases} r_1 - r_2 = 0 \\ r_0 - \frac{r_1 + r_2}{2} = 0 \end{cases}$$

are equivalent, we have that $A$ is an Einstein metric if and only if $\xi_1, \xi_2$ satisfy the equations
\[
\begin{aligned}
10\sqrt{2}b^2 + 3\sqrt{2}\xi_1\xi_2 - \sqrt{2}(\xi_1 - \xi_2)^2 - 2(\xi_1 + \xi_2)\sqrt{\xi_1\xi_2} &= 0 \\
\xi_1 + \xi_2 - 2\sqrt{2}\xi_1\xi_2 &= 0.
\end{aligned}
\]

By solving this system for \(\xi_1 < \xi_2\), we obtain the solutions:

\[
\begin{aligned}
\xi_1 &= (-2 + \sqrt{2})b, \quad b < 0, \\
\xi_2 &= (-2 - \sqrt{2})b, \\
\xi_1 &= (2 - \sqrt{2})b, \quad b > 0, \\
\xi_2 &= (2 + \sqrt{2})b.
\end{aligned}
\]

or, equivalently,

\[
\begin{aligned}
\mu_1 &= -3b, \quad b < 0, \\
\mu_2 &= -b, \\
\mu_1 &= -b, \quad b < 0, \\
\mu_2 &= -3b,
\end{aligned}
\]

\[
\begin{aligned}
\mu_1 &= b, \quad b > 0, \\
\mu_2 &= 3b, \\
\mu_1 &= 3b, \quad b > 0, \\
\mu_2 &= b.
\end{aligned}
\]

In all these cases, \(\mu_0 = \xi_0 = \sqrt{2\xi_1\xi_2} = \sqrt{4b^2} = 2|b|\). \(\square\)

**Remark** (Equivalent flag manifolds). Let us consider \(g = A_3\) and \(\Theta = \{\lambda_2 - \lambda_3\} \text{ or } \{\lambda_3 - \lambda_4\}\). For \(i = 2, 3\), the map

\(\varphi_i : \mathbb{F}_{\{\lambda_i-\lambda_{i+1}\}} \rightarrow \mathbb{F}_{\{\lambda_1-\lambda_2\}} ; \quad \varphi_{i}(kK_{\lambda_i-\lambda_{i+1}}) = e_ie_i^TK_{\lambda_1-\lambda_2}\),

where

\[
e_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
e_2 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},
\]

is a diffeomorphism. It is easy to show that every invariant metric on \(\mathbb{F}_{\{\lambda_i-\lambda_{i+1}\}}\) has the form \(\varphi_i^*g\), where \(g\) is an invariant metric on \(\mathbb{F}_{\{\lambda_1-\lambda_2\}}\). By [7] Lemma 7.2 we have that the Ricci tensor associated to \(\varphi_i^*g\) is equal to the pull-back by \(\varphi_i\) of the Ricci tensor associated to \(g\), thus, \(\varphi_i^*g\) is an Einstein metric if and only if is so \(g\). Therefore, Einstein metrics on \(\mathbb{F}_{\{\lambda_i-\lambda_{i+1}\}}\) are obtained by taking the pullback by \(\varphi_i\) of the Einstein metrics on \(\mathbb{F}_{\{\lambda_1-\lambda_2\}}\). \(\square\)

### 3.2. Invariant Einstein metrics on \(SO(l) \times SO(l)/S(O(l-1) \times O(1))\)

Let us consider \(g = D_l\) and \(\Theta = \{\lambda_1 - \lambda_2, \ldots, \lambda_{l-2} - \lambda_{l-1}\}\). In this case \(K\) is diffeomorphic to \(SO(l) \times SO(l)\) and \(K_\Theta\) is diffeomorphic to \(S(O(l-1) \times O(1))\). The Lie algebra \(\mathfrak{k}\) is the set

\[
\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\
B & A \end{pmatrix} \in \mathfrak{gl}(2l, \mathbb{R}) : A + A^T = B + B^T = 0 \right\}.
\]

We fix the \(\text{Ad}(K)\)-invariant inner product \((\cdot, \cdot)\) on \(\mathfrak{k}\) given by

\[
(\begin{pmatrix} A & B \\
B & A \end{pmatrix}, \begin{pmatrix} C & D \\
D & C \end{pmatrix}) = -\frac{\text{Tr}(AC) + \text{Tr}(BD)}{2}
\]

and the \((\cdot, \cdot)\)-orthogonal basis

\[
w_{ij} = E_{ij} - E_{ji} + E_{li,l+j} - E_{l+j,l+i},
\]

\[
u_{ij} = E_{l+i,j} - E_{l+j,i} + E_{i,l+j} - E_{j,l+i}, \quad 1 \leq j < i \leq l.
\]
According to [9], the isotropy representation of $K_\Theta$ on $m_\Theta$ decomposes into the irreducible submodule $W_0 = \text{span}\{u_{ij} : 1 \leq t < s \leq l-1\}$, which is not equivalent to any other submodule, and the equivalent irreducible sumodules $W_1 = \text{span}\{w_{ij} : 1 \leq j \leq l-1\}$ and $W_2 = \text{span}\{u_{ij} : 1 \leq j \leq l-1\}$. As a particular case of [6, Proposition 3.14] we have that every invariant metric $A$ is given by

$$A|_{W_0} = \mu_0 I_{W_0}, \quad A_{w_{ij}} = \mu_1 w_{ij} + bu_{ij}, \quad A_{u_{ij}} = bw_{ij} + \mu_2 u_{ij}$$

for some $\mu_0, \mu_1, \mu_2 > 0$ and $b \in \mathbb{R}$.

**Proposition 3.2.** Let $A$ be an invariant metric on the flag $F_{\{\lambda_1 - \lambda_2, \ldots, \lambda_{l-2} - \lambda_{l-1}\}}$ written as in (18). Then $A$ is an Einstein metric if and only if $A$ satisfies one of the following conditions:

- **(F1)** $b = 0$, $\mu_1 = \left(1 - \frac{\sqrt{2^l - 4} + 1}{2(l-1)}\right) \mu_0$ and $\mu_2 = \left(1 + \frac{\sqrt{2^l - 4} + 1}{2(l-1)}\right) \mu_0$

- **(F2)** $b = 0$, $\mu_1 = \left(1 + \frac{\sqrt{2^l - 4} + 1}{2(l-1)}\right) \mu_0$ and $\mu_2 = \left(1 - \frac{\sqrt{2^l - 4} + 1}{2(l-1)}\right) \mu_0$

- **(F3)** $b > 0$, $b = \mu_1 = \frac{\mu_2}{\sqrt{2^l - 4}}$

- **(F4)** $b > 0$, $b = \mu_2 = \frac{\mu_1}{\sqrt{2^l - 4}}$

- **(F5)** $b < 0$, $-b = \mu_1 = \frac{\mu_2}{\sqrt{2^l - 4}}$

- **(F6)** $b < 0$, $-b = \mu_2 = \frac{\mu_1}{\sqrt{2^l - 4}}$

**Proof.** The proof is analogous to the proof of Proposition 3.1. As before, we consider two cases:

- $A$ diagonal ($b = 0$):

In this case, we have the $A$–orthonormal basis of $m_\Theta$ given by the vectors

$$Y_{ij} = \frac{u_{ij}}{\sqrt{\mu_0}}, \quad 1 \leq j < i \leq l-1, \quad X_{lj} = \frac{w_{lj}}{\sqrt{\mu_1}}, \quad Y_{lj} = \frac{u_{lj}}{\sqrt{\mu_1}}, \quad 1 \leq j \leq l-1,$$

which satisfy the following bracket relations

$$[Y_{ij}, X_{kl}]_{m_\Theta} = -\sqrt{\frac{\mu_2}{\mu_0 \mu_1}} Y_{ij}, \quad [Y_{ij}, X_{lj}]_{m_\Theta} = \sqrt{\frac{\mu_2}{\mu_0 \mu_1}} Y_{ij},$$

$$[Y_{ij}, Y_{lj}]_{m_\Theta} = -\sqrt{\frac{\mu_1}{\mu_0 \mu_2}} X_{ij}, \quad [Y_{ij}, Y_{lj}]_{m_\Theta} = \sqrt{\frac{\mu_1}{\mu_0 \mu_2}} X_{ij},$$

$$[X_{lt}, Y_{ls}]_{m_\Theta} = \sqrt{\frac{\mu_0}{\mu_1 \mu_2}} Y_{st}, \quad \text{for } s \neq t,$$

where, $Y_{st} = -Y_{ts}$ if $s < t$. Observe that $[X, Y]$ is $A$–orthogonal to $X$ and $Y$, for all $X, Y$ in the basis (19), thus, $Z = \sum U(X, X) = 0$ (here $X$ extends over the basis (19)). By formula (7), we obtain that

$$r_0 := \text{Ric}(Y_{ij}, Y_{ij}) = \frac{2(l-2)}{\mu_0} + \frac{\mu_0}{\mu_1 \mu_2} - \frac{\mu_1}{\mu_0 \mu_2} - \frac{\mu_2}{\mu_0 \mu_1}, \quad 1 \leq j < i \leq l-1,$$

$$r_1 := \text{Ric}(X_{lj}, X_{lj}) = (l-2) \left(\frac{2}{\mu_1} + \frac{\mu_1}{2 \mu_0 \mu_2} - \frac{\mu_2}{2 \mu_0 \mu_1} - \frac{\mu_0}{2 \mu_1 \mu_2}\right), \quad 1 \leq j \leq l-1,$$

$$r_2 := \text{Ric}(Y_{lj}, Y_{lj}) = (l-2) \left(\frac{2}{\mu_2} + \frac{\mu_2}{2 \mu_0 \mu_1} - \frac{\mu_1}{2 \mu_0 \mu_2} - \frac{\mu_0}{2 \mu_1 \mu_2}\right), \quad 1 \leq j \leq l-1.$$
Thus, if \(A\)

We may use formula (7) to obtain

where \(c\)

As in Proposition 3.1, the eigenvalues of \(A\) so equation (14) also holds in this case. An non-diagonal \((b \neq 0)\):

\[
\begin{align*}
\mu_1 &= \left(1 - \frac{\sqrt{l^2 - 5l + 4}}{2(l - 1)} \right) \mu_0 \\
\mu_2 &= \left(1 + \frac{\sqrt{l^2 - 5l + 4}}{2(l - 1)} \right) \mu_0
\end{align*}
\]

\[
\begin{align*}
\mu_1 &= \left(1 + \frac{\sqrt{l^2 - 5l + 4}}{2(l - 1)} \right) \mu_0 \\
\mu_2 &= \left(1 - \frac{\sqrt{l^2 - 5l + 4}}{2(l - 1)} \right) \mu_0
\end{align*}
\]

- A non-diagonal \((b \neq 0)\):

As in Proposition 3.1 the eigenvalues of \(A\) are given by

\[
\xi_0 = \mu_0, \quad \xi_1 = \frac{1}{2} \left( \mu_1 + \mu_2 - \sqrt{4b^2 + (\mu_2 - \mu_1)^2} \right) \quad \text{and} \quad \xi_2 = \frac{1}{2} \left( \mu_1 + \mu_2 + \sqrt{4b^2 + (\mu_2 - \mu_1)^2} \right),
\]

so equation (14) also holds in this case. An \(A\)-orthonormal basis of \(m_\Theta\) is given by

\[
Y_{ij} = \frac{u_{ij}}{\sqrt{\xi_0}}, \quad 1 \leq j < i \leq l - 1, \quad X_{lj} = \frac{(\xi_1 - \mu_2)w_{lj} + bu_{lj}}{\sqrt{c_1}}, \quad Y_{lj} = \frac{(\xi_2 - \mu_2)w_{lj} + bu_{lj}}{\sqrt{c_2}}, \quad 1 \leq j \leq l - 1,
\]

where \(c_1 = \xi_1(\xi_1 - \mu_2)(\xi_1 - \xi_2)\) and \(c_2 = \xi_2(\xi_2 - \mu_2)(\xi_2 - \xi_1)\). These vectors satisfy the relations

\[
[X_{lt}, X_{ls}]_{m_\Theta} = \frac{2b\sqrt{\xi_0}}{\xi_1(\xi_1 - \xi_2)} Y_{st}, \quad [X_{lt}, Y_{ls}]_{m_\Theta} = \frac{b(\mu_2 - \mu_1)}{[b](\xi_1 - \xi_2)} \sqrt{\xi_1 \xi_2} Y_{st}, \quad [Y_{lt}, Y_{ls}]_{m_\Theta} = \frac{2b\sqrt{\xi_0}}{\xi_2(\xi_2 - \xi_1)} Y_{st}, \quad s \neq t
\]

\[
[X_{lj}, X_{lj}]_{m_\Theta} = \frac{b}{\xi_1 - \xi_2} \left( \frac{\sqrt{\xi_0}}{\xi_1 - \xi_2} \left( \frac{\mu_2 - \mu_1}{b} \right) Y_{lj} \right),
\]

\[
[X_{lj}, Y_{lj}]_{m_\Theta} = \frac{b}{\xi_1 - \xi_2} \left( \frac{\sqrt{\xi_0}}{\xi_1 - \xi_2} \left( \frac{\mu_2 - \mu_1}{b} \right) Y_{lj} \right),
\]

\[
[Y_{lj}, X_{lj}]_{m_\Theta} = \frac{b}{\xi_2 - \xi_1} \left( \frac{\sqrt{\xi_0}}{\xi_2 - \xi_1} \left( \frac{\mu_2 - \mu_1}{b} \right) X_{lj} \right),
\]

\[
[Y_{lj}, Y_{lj}]_{m_\Theta} = \frac{b}{\xi_2 - \xi_1} \left( \frac{\sqrt{\xi_0}}{\xi_2 - \xi_1} \left( \frac{\mu_2 - \mu_1}{b} \right) Y_{lj} \right), \quad 1 \leq j < i \leq l - 1.
\]

We may use formula (7) to obtain

\[
Ric(X_{lj}, Y_{lj}) = \frac{(l - 2)|b|(\mu_1 - \mu_2)(\xi_1^2 - 2\xi_1 \xi_2)}{(\xi_2 - \xi_1)\xi_0(\xi_1 \xi_2)^{3/2}}, \text{for all } j \in \{1, \ldots, l - 1\}.
\]

Thus, if \(A\) is an Einstein metric then \(\mu_1 = \mu_2\) or \(\xi_0 = \sqrt{2\xi_1 \xi_2}\).
If $\mu_1 = \mu_2$, we have

$$ r_0 := \text{Ric}(Y_{ij}, Y_{ij}) = \frac{2(l - 3)}{\xi_0} + \frac{\xi_0}{2} \left( \frac{1}{\xi_1^2} + \frac{1}{\xi_2^2} \right), \quad 1 \leq j < i \leq l - 1,$$

(22)

$$ r_1 := \text{Ric}(X_{ij}, X_{ij}) = (l - 2) \left( \frac{2}{\xi_1} - \frac{\xi_0}{2\xi_1^2} \right), \quad 1 \leq j \leq l - 1,$$

$$ r_2 := \text{Ric}(Y_{ij}, Y_{ij}) = (l - 2) \left( \frac{2}{\xi_2} - \frac{\xi_0}{2\xi_2^2} \right), \quad 1 \leq j \leq l - 1.$$

The equations $r_0 - r_1 = 0$, $r_2 - r_1 = 0$ have no solutions, hence, there are no Einstein metrics $A$ satisfying $b \neq 0$ and $\mu_1 = \mu_2$.

If $\xi_0 = \sqrt{2\xi_1\xi_2}$ then

$$ r_0 := \text{Ric}(Y_{ij}, Y_{ij}) = \frac{2(l - 2)}{\xi_0} + \frac{8b^2 - (\xi_2 - \xi_1)^2}{\xi_0\xi_1\xi_2},$$

(23)

$$ r_1 := \text{Ric}(X_{ij}, X_{ij}) = (l - 2) \left( \frac{2}{\xi_1} - \frac{2b^2\xi_0}{\xi_1^2(\xi_2 - \xi_1)^2} + \frac{(\mu_2 - \mu_1)^2}{2(\xi_2 - \xi_1)^2} \left( \frac{\xi_1^2 - \xi_2^2 - \xi_0^2}{\xi_0\xi_1\xi_2} \right) \right),$$

$$ r_2 := \text{Ric}(Y_{ij}, Y_{ij}) = (l - 2) \left( \frac{2}{\xi_2} - \frac{2b^2\xi_0}{\xi_2^2(\xi_2 - \xi_1)^2} + \frac{(\mu_2 - \mu_1)^2}{2(\xi_2 - \xi_1)^2} \left( \frac{\xi_2^2 - \xi_1^2 - \xi_0^2}{\xi_0\xi_1\xi_2} \right) \right).$$

Thus

$$ r_1 - r_2 = (l - 2) \left( 2 \left( \frac{1}{\xi_1} - \frac{1}{\xi_2} \right) - \frac{2b^2\xi_0}{(\xi_2 - \xi_1)^2} \left( \frac{1}{\xi_1^2} - \frac{1}{\xi_2^2} \right) + \frac{(\mu_2 - \mu_1)^2}{\xi_0(\xi_2 - \xi_1)^2} \left( \frac{\xi_1^2 - \xi_2^2}{\xi_0\xi_1\xi_2} \right) \right)$$

$$ = (l - 2) \left( 2 \left( \frac{\xi_2 - \xi_1}{\xi_1\xi_2} \right) - \frac{2b^2\xi_0}{(\xi_2 - \xi_1)^2} \left( \frac{\xi_1^2 - \xi_2^2}{\xi_1^2\xi_2^2} \right) + \frac{(\mu_2 - \mu_1)^2}{(\xi_2 - \xi_1)^2} \left( \frac{\xi_1^2 - \xi_2^2}{\xi_0\xi_1\xi_2} \right) \right)$$

$$ = (l - 2) \left( 2 \left( \frac{\xi_2 - \xi_1}{\xi_1\xi_2} \right) - \frac{4b^2}{(\xi_2 - \xi_1)^2} \left( \frac{\xi_2^2 - \xi_1^2}{\xi_0\xi_1\xi_2} \right) + \frac{(\mu_2 - \mu_1)^2}{(\xi_2 - \xi_1)^2} \left( \frac{\xi_1^2 - \xi_2^2}{\xi_0\xi_1\xi_2} \right) \right)$$

$$ = (l - 2) \left( 2 \left( \frac{\xi_2 - \xi_1}{\xi_1\xi_2} \right) - \frac{\xi_2^2 - \xi_1^2}{\xi_0\xi_1\xi_2} \left( 4b^2 + (\mu_2 - \mu_1)^2 \right) \right)$$

$$ = (l - 2) \left( 2 \left( \frac{\xi_2 - \xi_1}{\xi_1\xi_2} \right) - \frac{\xi_2^2 - \xi_1^2}{\xi_0\xi_1\xi_2} \right)$$

$$ = \frac{(l - 2)(\xi_2 - \xi_1)(2\xi_0 - (\xi_1 + \xi_2))}{\xi_0\xi_1\xi_2}$$

$$ = \frac{(l - 2)(\xi_2 - \xi_1)(2\sqrt{2\xi_1\xi_2} - (\xi_1 + \xi_2))}{\sqrt{2(\xi_1\xi_2)^2}}.$$
Observe that $\xi_2 - \xi_1 \neq 0$ (since $b \neq 0$), therefore, an Einstein metric $A$ satisfying $b \neq 0$ and $\xi_0 = \sqrt{2\xi_1 \xi_2}$ also satisfies $2\sqrt{2\xi_1 \xi_2} = \xi_1 + \xi_2$. In this situation we have

$$r_0 - \frac{r_1 + r_2}{2} = \frac{2(l-2)}{\xi_0} + \frac{8b^2 - (\xi_2 - \xi_1)^2}{\xi_0 \xi_1 \xi_2}$$

$$= \left(\frac{l-2}{2}\right) \left(2 \left(\frac{1}{\xi_1} + \frac{1}{\xi_2}\right) - \frac{2b^2}{(\xi_2 - \xi_1)^2} \left\{\frac{1}{\xi_1^2} + \frac{1}{\xi_2^2}\right\} - \frac{(\mu_2 - \mu_1)^2}{(\xi_2 - \xi_1)^2} \left\{\frac{\xi_0}{\xi_1 \xi_2}\right\}\right)$$

$$= 2(l-2) \left(\frac{1}{\xi_0} - \frac{1}{2\xi_1} - \frac{1}{2\xi_2}\right) + \frac{8b^2 - (\xi_2 - \xi_1)^2}{\xi_0 \xi_1 \xi_2} + \frac{2b^2(\xi_2 - \xi_1)^2}{(\xi_2 - \xi_1)^2} \left(\frac{\xi_0}{\xi_0 \xi_1 \xi_2}\right)$$

$$+ \frac{(\mu_2 - \mu_1)^2(l-2)}{(\xi_2 - \xi_1)^2} \left(\frac{\xi_0}{2\xi_1 \xi_2}\right)$$

$$= \frac{(l-2)(2\xi_1 \xi_2 - \xi_0(\xi_1 + \xi_2))}{\xi_0 \xi_1 \xi_2} + \frac{8b^2 - (\xi_2 - \xi_1)^2}{\xi_0 \xi_1 \xi_2}$$

$$+ \left(\frac{l-2}{(\xi_2 - \xi_1)^2}\right) \left(\frac{2b^2(\xi_1^2 + \xi_2^2) + (\mu_2 - \mu_1)^2 \xi_1 \xi_2}{\xi_0 \xi_1 \xi_2}\right)$$

$$= \frac{(l-2)(-2\xi_1 \xi_2)}{\xi_0 \xi_1 \xi_2} + \frac{8b^2 - (\xi_2 - \xi_1)^2}{\xi_0 \xi_1 \xi_2} + \left(\frac{l-2}{(\xi_2 - \xi_1)^2}\right) \left(\frac{2b^2(\xi_2 - \xi_1)^2 + (\xi_2 - \xi_1)^2 \xi_1 \xi_2}{\xi_0 \xi_1 \xi_2}\right)$$

$$= \frac{(l-2)(-2\xi_1 \xi_2)}{\xi_0 \xi_1 \xi_2} + \frac{8b^2 - (\xi_2 - \xi_1)^2}{\xi_0 \xi_1 \xi_2} + \frac{2b^2(l-2) + \xi_1 \xi_2(l-2)}{\xi_0 \xi_1 \xi_2}$$

$$= \frac{2b^2(l+2) + (4-l)\xi_1 \xi_2 - \xi_1^2 - \xi_2^2}{\sqrt{2(\xi_1 \xi_2)^\frac{3}{2}}}$$

$$= \frac{2b^2(l+2) - (l+2)\xi_1 \xi_2}{\sqrt{2(\xi_1 \xi_2)^\frac{3}{2}}}$$

(since $\xi_1 + \xi_2 = 2\sqrt{2\xi_1 \xi_2} \Rightarrow \xi_1^2 + \xi_2^2 = 6\xi_1 \xi_2$).

Then,

$$r_0 - \frac{r_1 + r_2}{2} = 0 \iff \xi_1 \xi_2 = 2b^2.$$
by solving $\xi_1 + \xi_2 = 2\sqrt{2}\xi_1\xi_2$ and $\xi_1\xi_2 = 2b^2$ for $\xi_1 < \xi_2$, we obtain

\[
\begin{cases}
\xi_1 = (-2 + \sqrt{2})b, & b < 0, \\
\xi_2 = (-2 - \sqrt{2})b, & b > 0,
\end{cases}
\]

and $\xi_0 = 2|b|$. Hence, an Einstein metric satisfying $\xi_0 = \sqrt{2}\xi_1\xi_2$ must satisfy one of the conditions (F2), (F3), (F4), (F5) or (F6). Conversely, it is easy to verify that any invariant metric satisfying one of the conditions (F2), (F3), (F4), (F5), (F6) is in fact an Einstein metric. □

**Remark (Equivalent flag manifolds).** Let us consider $\mathfrak{g} = D_l$ and $\Theta = \{\lambda_2 - \lambda_3, ..., \lambda_{l-1} - \lambda_l\}$ or $\{\lambda_2 - \lambda_3, ..., \lambda_{l-2} - \lambda_{l-1}, \lambda_{l-1} + \lambda_l\}$. Consider the automorphisms $\rho$ and $\eta$ of $\mathfrak{so}(l, l)$ given by

\[
\begin{align*}
\rho(w_{ij}) &= w_{l-j+1,l-i+1}, & \rho(u_{ij}) &= u_{l-j+1,l-i+1}, & 1 \leq j < i \leq l, \\
\eta(w_{ij}) &= w_{ij}, & \eta(u_{ij}) &= u_{ij}, & 1 \leq j < i \leq l - 1, \\
\end{align*}
\]

We have that

\[
\rho(m_{\{\lambda_1 - \lambda_2 - \lambda_{l-2} - \lambda_{l-1}\}}) = m_{\{\lambda_2 - \lambda_3, ..., \lambda_{l-2} - \lambda_{l-1}\}} \quad \text{and} \quad \eta(m_{\{\lambda_2 - \lambda_3, ..., \lambda_{l-2} - \lambda_{l-1}\}}) = m_{\{\lambda_2 - \lambda_3, ..., \lambda_{l-2} - \lambda_{l-1} + \lambda_l\}},
\]

also, $\rho^{-1}$, $\eta$ and $\eta^{-1}$ take an invariant inner product to an invariant inner product. Consequently, the components of the Ricci tensor of invariant metrics for $\{\lambda_2 - \lambda_3, ..., \lambda_{l-1} - \lambda_l\}$ or $\{\lambda_2 - \lambda_3, ..., \lambda_{l-2} - \lambda_{l-1}, \lambda_{l-1} + \lambda_l\}$ are the same as in the case of $\{\lambda_1 - \lambda_2, ..., \lambda_{l-2} - \lambda_{l-1}\}$. Therefore, Einstein invariant metrics on $\mathbb{F}_{\{\lambda_2 - \lambda_3, ..., \lambda_{l-2} - \lambda_{l-1}\}}$ and $\mathbb{F}_{\{\lambda_2 - \lambda_3, ..., \lambda_{l-2} - \lambda_{l-1}, \lambda_{l-1} + \lambda_l\}}$ have the form $(\rho^{-1})^* g$ and $(\rho^{-1} \circ \eta^{-1})^* g$, respectively, where $g$ is an Einstein invariant metric on $\mathbb{F}_{\{\lambda_1 - \lambda_2, ..., \lambda_{l-2} - \lambda_{l-1}\}}$. □

### 4. ISOMETRIC EINSTEIN METRICS

Now, we decide which of the Einstein metrics found in section 3 are isometric. As observed in [8], the Einstein constant corresponding to an Einstein Riemannian metric $g$ of volume 1 on a compact manifold $M$ is equal to $S/\dim(M)$, where $S$ is the scalar curvature of $g$. Since any isometry preserves the scalar curvature, then two Einstein metrics of volume 1 with different Einstein constant cannot be isometric.

Given an invariant metric $A$ on $SO(4)/S(O(2) \times O(1) \times O(1))$ written as in [9], we have that its volume is given by $\mu_0(\mu_1\mu_2 - b^2)^2$. If $A(b)$ and $A(\mu_0)$ denote the invariant metrics satisfying (E2) and (E1) respectively, their corresponding volumes are given by $8b^5$ and $(\frac{32\mu_0^2}{\mu_1^2})$, and their corresponding Einstein constants are $\frac{1}{b}$ and $\frac{16}{9\mu_0}$. When $b = 2\sqrt{2}$ and $\mu_0 = \left(\frac{4}{3}\right)^{\frac{3}{2}}$, we have volume 1, but $2\sqrt{2} \neq \left(\frac{4}{3}\right)^{\frac{3}{2}}$, so (E1) and (E2) cannot be isometric. Now, if we consider the diffeomorphisms

\[
\psi_i : \mathbb{F}_{\{\lambda_1 - \lambda_2\}} \rightarrow \mathbb{F}_{\{\lambda_1 - \lambda_2\}}, \quad \psi_i(kK_{\{\lambda_1 - \lambda_2\}}) = s_i kS_i^T kK_{\{\lambda_1 - \lambda_2\}}, \quad i = 3, 4, 5;
\]

where

\[
s_3 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad s_4 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad \text{and} \quad s_5 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix};
\]
Then it is easy to verify that the invariant metric $ψ_i^* A(b)$ satisfies (Ei). This shows that (E2), (E3), (E4) and (E5) are isometric. We can use a similar argument to show that metrics (F1), (F2) are isometric as well as metrics (F3), (F4), (F5), (F6) and that (F1) is not isometric to (F3).

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