Adaptive neural tracking control for a class of nonlinear systems with input delay and saturation

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ABSTRACT

For a class of non-strict-feedback nonlinear systems with input delay and saturation, the tracking control problem is addressed in this paper. An auxiliary system is constructed to handle the difficulty in control design caused by input delay. Moreover, hyperbolic tangent function is used to approximate the non-smooth saturation function to achieve controller design. The unknown nonlinear functions generated in backstepping control design are approximated by radial basis function neural networks. And then, with the help of backstepping approach, an adaptive neural control scheme is proposed. It is proved by Lyapunov stability theory that the tracking errors converge to a small neighbourhood of the origin and the other closed-loop signals are bounded. At last, a simulation example is able to verify the validity of this tracking control scheme.

ARTICLE HISTORY

Received 8 July 2020
Accepted 5 October 2020

KEYWORDS

Adaptive neural control; non-strict feedback; auxiliary system; backstepping; input delay and saturation

1. Introduction

In recent years, a lot of adaptive neural or fuzzy control schemes based on backstepping technology have emerged and are applied to control nonlinear systems with uncertainties (see for example Li & Tong, 2013; Tong & Li, 2014; Tong et al., 2020; H. Wang et al., 2013; M. Wang et al., 2011; Zhang et al., 2009). The adaptive Takagi-Sugeno type fuzzy control scheme is presented to gain fewer online parameters for nonlinear systems with pure-feedback form in Zhang et al. (2009). It is worth noting that M. Wang et al. (2011) uses dynamic surface control technique to avoid the difficulty of ‘the explosion of complexity’ in traditional backstepping control design. H. Wang et al. (2013) adopt the adaptive fuzzy control method to address the tracking control problem of stochastic nonlinear systems with pure-feedback form. For nonlinear systems with unmeasurable states, the adaptive fuzzy output-feedback control schemes are proposed in (Li & Tong, 2013; Tong & Li, 2014; Tong et al., 2020). However, these results on adaptive neural or fuzzy control require the systems must be strict-feedback or pure-feedback forms. Theoretically, such requirement limits the application of these control schemes to more general nonlinear systems, for example, the non-strict feedback systems.

In addition, the systems usually contain a variety of nonlinear factors in practical engineering applications. Moreover, input delay and input saturation are two very common cases. Specifically, the existence of input delay and saturation leads to poor stability and degraded performance of the systems. And the control design and stability analysis are not directly. Therefore, the research on control design and stability analysis for nonlinear systems with input delay and saturation has always been of theoretical and practical significance. Many scholars have conducted extensive research on the above subjects in recent years. Krstic (2009) gives an approach to compensate input delay for strict-feedforward nonlinear systems. The predictor-based technique is used to offset the input delay for nonlinear systems in (Bekiaris-Liberis & Krstic, 2012, 2016). Furthermore, Kamalapurkar et al. (2015) and Obuz et al. (2017) propose the proportional integral derivative control schemes that can solve the input delay problem of nonlinear systems. Using integral compensator technique, Li et al. (2020) solve the finite-time control problem for a class of nonlinear multiagent systems with input delay. On the other hand, to address the problem of input saturation, Wen et al. (2011) adopt a smooth function to approximate the saturated input signal. An auxiliary system method is presented in Gao et al. (2016). Although the meaningful results have been proposed to deal with input delay and input saturation, the control schemes in the above references are still limited to nonlinear systems with strict-feedback or...
pure-feedback forms. To the authors’ knowledge, there is little research on non-strict feedback nonlinear systems containing input delay and input saturation as yet.

Based on the above discussion, we consider the control design problem of a class of nonlinear systems with non-strict feedback form which contain input delay and saturation. Compared with previous studies, this paper has mainly the following contributions:

(1) The considered nonlinear systems have the non-strict feedback form, which takes the strict-feedback form and pure-feedback form as the special cases. So, the systems we considered are more general.

(2) A novel auxiliary system method is developed, which not only compensates the effects of input delay in theory but also successfully overcomes the difficulty of controller design caused by input delay.

(3) The tracking control problem of the nonlinear systems with input delay and saturation is addressed.

2. Preliminaries and problem formulation

Consider the nonlinear system with input delay and saturation described as follows:

\[
\begin{align*}
\dot{x}_i &= f_i(x) + g_i(x)x_{i+1}, & 1 \leq i \leq n - 1, \\
\dot{x}_n &= f_n(x) + u(\nu(t - \tau)), \\
y &= x_1,
\end{align*}
\]

where \(x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n\) is the state vector, \(u(\nu(t - \tau)) \in \mathbb{R}\) and \(y \in \mathbb{R}\) are the control input and output, respectively. \(f_i(\cdot)(1 \leq i \leq n), g_i(\cdot)(1 \leq i \leq n - 1)\) are the unknown smooth nonlinear functions. The saturated nonlinear input \(u(\cdot)\) can be described by

\[
u = \text{sat}(v) = \begin{cases} 
-u_{\max}, & v \geq u_{\max}, \\
n, & u_{\min} < v < u_{\max}, \\
u_{\min}, & v \leq u_{\min},
\end{cases}
\]

where \(v\) is actual control input, \(u_{\max} > 0\) and \(u_{\min} < 0\) are unknown constants.

An auxiliary system in the following is proposed to design and construct the controller for system (1).

\[
\dot{\eta} = -\rho \eta + u(\nu(t)) - u(\nu(t - \tau)),
\]

where \(\rho > 1\) is a design parameter.

To approximate the saturated input \(u\), a piecewise smooth function \(h(v)\) is introduced according to Wen et al. (2011):

\[
u = h(v) + d(v),
\]

where

\[
h(v) = \begin{cases} 
u_{\min} \cdot \tanh(v/\nu_{\min}), & v < 0, \\
u_{\max} \cdot \tanh(v/\nu_{\max}), & v \geq 0,
\end{cases}
\]

and \(d(v) = u - h(v)\) is bounded as

\[
|d(v)| = |u - h(v)| \leq D.
\]

Furthermore, it is easy to get \(D = \max[u_{\max}(1 - \tanh(1)), u_{\min}(\tanh(1) - 1)]\). By mean-value theorem, one has

\[
h(v) = h(v_0) + h_{v_0}(v - v_0),
\]

where \(\phi\) is a constant \((0 < \phi < 1)\), \(h_{v_0} = \partial h(v)/\partial v|_{v = v_0}\), \(v(\phi) = \phi v + (1 - \phi)v_0\).

Supposing \(v_0 = 0\), (6) becomes

\[
h(v) = h_{v_0} v.
\]

Substituting (7) into (4) gets

\[
u = h_{v_0} v + d(v).
\]

According to (2), define \(u_D = \max[u_{\max}, -u_{\min}]\) and it is obtained easily as

\[
u \leq u_D.
\]

To accomplish the control goal, the Assumptions and Lemmas shown as follows are essential.

**Assumption 2.1**: The input delay \(\tau \geq 0\) is a known constant.

**Assumption 2.2**: The tracking signal \(y_d(t)\) and its up to the \(n\)-order derivative, i.e. \(y_d^{(n)}(t)\), are known smooth functions and bounded.

**Assumption 2.3**: The functions \(g_i(x)\) for \(i = 1, \ldots, n - 1\) meet the restrictions as

\[
0 < b_m \leq g_i(x) \leq b_M < \infty, \quad \forall x \in \mathbb{R}^n,
\]

where both \(b_m\) and \(b_M\) are unknown constants.

**Assumption 2.4**: The function \(h_{v_0}\) in (6) meets the following conditions

\[
0 < h_m \leq h_{v_0} \leq 1,
\]

where \(h_m\) is an unknown constant.

Obviously, we can select \(b = \min(b_m, h_m)\) such that

\[
0 < b \leq b_m \leq g_i(x),
\]

\[
0 < b \leq h_m \leq h_{v_0}.
\]
Assumption 2.5: The unknown nonlinear functions \( f_i(x) \) for \( i = 1, \ldots, n \) are the smooth functions that are defined on a compact set \( \Omega_x \subset \mathbb{R}^n \).

The radial basis function neural network can be described in the following form:

\[
G_{nn}(\lambda) = W^T S(\lambda),
\]

where \( \lambda \in \Omega_\lambda \subset \mathbb{R}^q \) is a \( q \)-dimensional input vector, \( W = [w_1, \ldots, w_n] \) is a \( n \times n \) weight matrix, and \( S(\lambda) = [s_1(\lambda), \ldots, s_n(\lambda)] \) and its component \( s_i(\lambda) \) is selected as the Gaussian basis function defined below:

\[
s_i(\lambda) = \exp \left[ -\frac{(\lambda - \omega_i)^T(\lambda - \omega_i)}{\xi_i^2} \right],
\]

where \( \xi_i \) is the width of the Gaussian basis function, \( \omega_i = [\omega_{i1}, \ldots, \omega_{iq}]^T \) for \( i = 1, \ldots, n \) is the centre of the receptive field.

Lemma 2.1 (Sanner & Slotine, 1991): For any \( \varepsilon > 0 \) and a compact set \( \Omega_\lambda \subset \mathbb{R}^q \), there exists a radial basis function (RBF) neural network (13) with sufficiently large node number \( n \) to approximate the continuous function \( \tilde{f}(\lambda) \), which is defined on \( \Omega_\lambda \), such that:

\[
\tilde{f}(\lambda) = W^* S(\lambda) + \Phi(\lambda), \quad \forall \lambda \in \Omega_\lambda \subset \mathbb{R}^q,
\]

where \( W^* \) is the optimal weight vector shown as:

\[
W^* = \arg \min_{W \in \mathbb{R}^q} \left\{ \sup_{\lambda \in \Omega_\lambda} \left| \tilde{f}(\lambda) - W^T S(\lambda) \right| \right\},
\]

and \( \Phi(\lambda) \) is a estimation error satisfying \( |\Phi(\lambda)| \leq \varepsilon \).

Lemma 2.2 (Sun et al., 2016): For any positive integer \( m \leq n \), there exists the following inequality for two basis function vectors of a RBF NN:

\[
\|S(\tilde{\lambda}_n)\|^2 \leq \|S(\tilde{\lambda}_m)\|^2,
\]

where \( \tilde{\lambda}_n = [\lambda_1, \ldots, \lambda_n]^T \) and \( \tilde{\lambda}_m = [\lambda_1, \ldots, \lambda_m]^T \).

Lemma 2.3 (Polycarpou & Ioannou, 1993): For any \( \varepsilon > 0 \), given \( z \in \mathbb{R} \), one holds:

\[
0 \leq |z| - z \tanh \left( \frac{z}{\varepsilon} \right) \leq \varepsilon \gamma,
\]

where \( \gamma = 0.2785 \).

3. Main results

Define the Lyapunov function \( V(t) \) as:

\[
V(t) = \frac{1}{2} \tilde{x}_1^2 + \frac{b}{r_1} \hat{\theta}_i^2,
\]

where \( r_1 > 0 \) is a design parameter, \( \hat{\theta}_i \) is an adaptive parameter, \( \hat{\theta}_i \) is its estimation, and \( \tilde{\theta}_i \) is an estimation error satisfying this equation \( \tilde{\theta}_i = \hat{\theta}_i - \hat{\theta}_i \).

Differentiating \( V(t) \) and using (15)–(16) shows:

\[
\dot{V}_i = z_i \tilde{f}(Z_i) - \frac{b}{r_1} \hat{\theta}_i \dot{\hat{\theta}}_i - g_i(x)z_{i-1}z_i + g_i(x)z_{i-1}z_i,
\]

where \( z_0 = 0, \tilde{f}(Z_i) = f_i(x) - \dot{\hat{\theta}}_i - g_i(x)z_{i-1}, Z_i = [x^T, y_{d1}, y_{d2}[^T, \text{and for } i \geq 2, Z_i = [x^T, \tilde{y}_{d1}, \tilde{y}_{d2}, \tilde{y}_{d3}, \tilde{y}_{d4}, \ldots, \tilde{y}_{d N}[^T, \hat{\theta}_i = [\hat{\theta}_1, \ldots, \hat{\theta}_i]^T].

By virtue of RBF NN, the unknown function \( \tilde{f}(Z_i) \) is approximated. From Lemma 2.1, for a given \( \varepsilon_i > 0 \), the following holds:

\[
\tilde{f}(Z_i) = W_i^* S_i(Z_i) + \Phi_i(Z_i),
\]

where \( \Phi_i(Z_i) \) is an estimation error, which satisfies the condition:

\[
|\Phi_i(Z_i)| \leq \varepsilon_i.
\]

Combining Lemma 2.2, Lemma 2.3 and (19)–(20), one has:

\[
\begin{align*}
\tilde{z}_i \tilde{f}(Z_i) & \leq |z_i| ||W_i^*|| ||S_i(Z_i)|| + \varepsilon_i \\
& \leq |z_i| ||W_i^*|| ||S_i(Z_i)|| + \varepsilon_i \\
& \leq z_iQ_i(X_i)\dot{\theta}_i b \tanh \left( \frac{z_iQ_i(X_i)}{c_i} \right) + \kappa c_i \dot{\theta}_i b,
\end{align*}
\]
where $X_1 = [x_1, y_d, \dot{y}_d]^T$, and for $i \geq 2, X_i = [x_1, \ldots, x_i, \dot{y}_d]^T$.

Further, we construct the adaptive law shown below:

$$
\dot{\theta}_i = -k_i z_i Q_i(X_i) \tan \left( \frac{z_i Q_i(X_i)}{c_i} \right) - \sigma_i \hat{\theta}_i,
$$

where $k_i$ is a positive design parameter.

By combining (11) and (22), one can obtain

$$
g_i(x) \alpha_i z_i \leq -k_i b z_i^2 - b_i \dot{z}_i Q_i(X_i) \tan \left( \frac{z_i Q_i(X_i)}{c_i} \right). \tag{23}
$$

Further, we construct the adaptive law shown below:

$$
\dot{\theta}_i = r_i z_i Q_i(X_i) \tan \left( \frac{z_i Q_i(X_i)}{c_i} \right) - \sigma_i \hat{\theta}_i,
$$

where $\sigma_i > 0$ is a design parameter.

From (24) and applying Young’s inequality (Deng & Krstić, 1997) to deal with the term $\theta_i \dot{\theta}_i$, one gets

$$
\begin{aligned}
\dot{V}_i & \leq -k_i b z_i^2 - \frac{b}{2r_i} \sigma_i \dot{\theta}_i^2 - \frac{b}{2r_i} \sigma_i \dot{\theta}_i^2 \\
&\quad - b_i \dot{z}_i Q_i(X_i) \tan \left( \frac{z_i Q_i(X_i)}{c_i} \right).
\end{aligned} \tag{25}
$$

Substituting (21)–(25) into (18) yields

$$
\dot{V}_i \leq -k_i b z_i^2 - \frac{b}{2r_i} \sigma_i \dot{\theta}_i^2 + \frac{b}{2r_i} \sigma_i \dot{\theta}_i^2 + \kappa c_i \theta_i b \\
- g_{i-1}(x) z_{i-1} z_i + g_i(x) z_i z_{i+1} \\
\leq -k_{V_i} V_i + b_{V_i} + \Delta V_i, \tag{26}
$$

where

$$
k_{V_i} = \min \{2k_i b, \sigma_i \}, \\
b_{V_i} = \frac{b}{2r_i} \sigma_i \dot{\theta}_i^2 + \kappa c_i \theta_i b, \\
\Delta V_i = -g_{i-1}(x) z_{i-1} z_i + g_i(x) z_i z_{i+1}.
$$

Step $n-1$: Differentiating $z_{n-1}$ along (1) and (15) shows

$$
\dot{z}_{n-1} = f_{n-1}(x) - \dot{\alpha}_{n-2} + g_{n-1}(x) x_{n-1}, \tag{27}
$$

where

$$
\dot{\alpha}_{n-2} = \sum_{k=1}^{n-2} \frac{\partial \alpha_{n-2}}{\partial x_k} \dot{x}_k + \sum_{k=1}^{n-2} \frac{\partial \alpha_{n-2}}{\partial \dot{\theta}_k} \dot{\theta}_k + \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-2}}{\partial y_d^{(k-1)}} \dot{y}_d^{(k)}.
$$

Define the Lyapunov function $V_{n-1}(t)$ as shown below

$$
V_{n-1}(t) = \frac{1}{2} z_{n-1}^2 + \frac{b}{2r_{n-1}} \theta_{n-1}^2, \tag{28}
$$

where the design parameter $r_{n-1} > 0$.

From (15) and (27), differentiating $V_{n-1}(t)$ obtains

$$
\dot{V}_{n-1} = z_{n-1} \dot{f}_{n-1}(z_{n-1}) - g_{n-2}(x) z_{n-2} z_{n-1} \\
+ g_{n-1}(x) z_{n-1} z_n \\
- \frac{b}{r_{n-1}} \theta_{n-1} \dot{\theta}_{n-1} + g_{n-1}(x) z_{n-1} \alpha_{n-1}. \tag{29}
$$

where

$$
\dot{f}_{n-1}(z_{n-1}) = f_{n-1}(x) - g_{n-1}(x) \eta + g_{n-2}(x) z_{n-2} - \dot{\alpha}_{n-2},
$$

$$
Z_{n-1} = [x^{T}, \dot{y}_d^{T}, \dot{\theta}_{n-1}^{T}],
$$

$$
\tilde{y}_{d,n-1} = [y_d, y_d^{(1)}, \ldots, y_d^{(n-1)}]^T,
$$

$$
\tilde{\theta}_{n-1} = [\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_{n-1}]^T.
$$

Similar to (19)–(25), one has

$$
\dot{V}_{n-1} \leq -k_{n-1} b z_{n-1}^2 - \frac{b}{2r_{n-1}} \sigma_{n-1} \dot{\theta}_{n-1}^2 + \kappa c_{n-1} \theta_{n-1} b \\
+ \frac{b}{2r_{n-1}} \sigma_{n-1} \dot{\theta}_{n-1}^2 \\
- g_{n-2}(x) z_{n-2} z_{n-1} + g_{n-1}(x) z_{n-1} z_n \\
\leq -k_{V_{n-1}} V_{n-1} + b_{V_{n-1}} + \Delta V_{n-1}, \tag{30}
$$

where

$$
k_{V_{n-1}} = \min \{2k_{n-1} b, \sigma_{n-1} \},
$$

$$
b_{V_{n-1}} = \frac{b}{2r_{n-1}} \sigma_{n-1} \dot{\theta}_{n-1}^2 + \kappa c_{n-1} \theta_{n-1} b, \\
\Delta V_{n-1} = -g_{n-2}(x) z_{n-2} z_{n-1} + g_{n-1}(x) z_{n-1} z_n.
$$

Step $n$: By (1), (3) and (15), one has

$$
\dot{\alpha}_n = f_n(x) - \dot{\alpha}_{n-1} - \rho \eta + u(v(t)), \tag{31}
$$

where

$$
\dot{\alpha}_{n-1} = \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} \dot{x}_k + \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \dot{\theta}_k} \dot{\theta}_k + \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial y_d^{(k-1)}} \dot{y}_d^{(k)}.
$$

Define the Lyapunov function $V_n(t)$ as

$$
V_n(t) = \frac{1}{2} z_n^2 + \frac{b}{2r_n} \theta_n^2 + \frac{1}{2} \eta^2, \tag{32}
$$

where $r_n$ is a positive design parameter.

Differentiating $V_n(t)$ and substituting (31) and (8) yields

$$
\dot{V}_n = z_n f_n(x) + g_{n-1}(x) z_{n-1} - \dot{\alpha}_n - \rho \eta \\
- \frac{b}{r_n} \theta_n \dot{\theta}_n - g_{n-1}(x) z_{n-1} z_n \\
+ z_n h_v \nu + z_n d(\nu) - \rho \eta^2
$$
+ \eta u(v(t)) - \eta u(v(t - \tau)). \tag{33}

According to Lemma 2.3 and (5), one has
\[ z_n d(v) \leq |z_n| D \leq z_n \tanh \left( \frac{z_n}{c_n} \right) D + \kappa c_n^2 D, \tag{34} \]
where the design parameter \(c_n\) is positive.

**Remark 3.1:** Equation (34) shows the effect from approximation error \(d(v)\) on the closed-loop stability. Obviously, the effect from the first item on the right side of (34) can be compensated for by suitable control design. The second item on the right side is a small positive constant, which may bring slight effect on tracking precision.

Combining (9) with Young’s inequality (Deng & Krstić, 1997), one can be obtained easily
\[ \eta u(v(t)) - \eta u(v(t - \tau)) \leq \eta^2 + u_D^2. \tag{35} \]

By (33)–(35), one gets
\[ \dot{V}_n \leq z_n f_n(Z_n) - \frac{b}{r_n} \dot{\theta}_n \dot{\theta}_n + z_n h_v v - g_n(x) z_n - z_n \]
\[ + \kappa c_n^2 D - \rho \eta^2 + \eta^2 + u_D^2, \tag{36} \]
where
\[ \tilde{f}_n(Z_n) = f_n(x) - \rho \eta + \tanh \left( \frac{z_n}{c_n} \right) D \]
\[ = \tilde{z}_n + g_n(x) z_n - 1, \]
\[ Z_n = \left[ x, \tilde{y}_{\alpha, d}, \tilde{\theta}_n \right]^T, \]
\[ \tilde{y}_{\alpha, d} = [y_{\alpha, d}^{(1)} \ldots y_{\alpha, d}^{(n)}]^T, \]
\[ \tilde{\theta}_n = [\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_n]^T. \]

Using the similar approach such as (21), one has
\[ z_n \tilde{f}_n(Z_n) \leq z_n Q_n(Z_n) \dot{\theta}_n b \tanh \left( \frac{z_n Q_n(Z_n)}{c_n} \right) + \kappa c_n \dot{\theta}_n b, \tag{37} \]
where the design parameter \(c_n\) is positive, \(\dot{\theta}_n = \max \left\{ \frac{|W_t|}{b}, \frac{Q_n(Z_n)}{b} \right\}, Q_n(Z_n) = 1 + \| S_n(Z_n) \| \).

Construct the actual control input \(v\) and the adaptive law \(\dot{\theta}_n\) as
\[ v = -k_n z_n - \dot{\theta}_n Q_n(Z_n) \tanh \left( \frac{z_n Q_n(Z_n)}{c_n} \right), \tag{38} \]
\[ \dot{\theta}_n = r_n z_n Q_n(Z_n) \tanh \left( \frac{z_n Q_n(Z_n)}{c_n} \right) - \sigma_n \dot{\theta}_n, \tag{39} \]
where both \(k_n\) and \(\sigma_n\) are positive design parameters.

Combining (12) and (38), it can be easily obtained as
\[ z_n h_v v \leq -k_n b z_n^2 - b \dot{\theta}_n Q_n(Z_n) \tanh \left( \frac{z_n Q_n(Z_n)}{c_n} \right). \tag{40} \]

Similar to (25), one gets
\[ -\frac{b}{r_n} \dot{\theta}_n \dot{\theta}_n \leq \frac{b}{2r_n} \sigma_n^2 + \frac{b}{2r_n} \sigma_n^2 \]
\[ - b \dot{\theta}_n Q_n(Z_n) \tanh \left( \frac{z_n Q_n(Z_n)}{c_n} \right). \tag{41} \]

Substituting (37)–(41) into (36) obtains
\[ \dot{V}_n \leq z_n Q_n(Z_n) b (\theta_n - \dot{\theta}_n) \tanh \left( \frac{z_n Q_n(Z_n)}{c_n} \right) \]
\[ - k_n b z_n^2 - \frac{b}{2r_n} \sigma_n^2 + \frac{b}{2r_n} \sigma_n^2 \]
\[ + (1 - \rho) \eta^2 + \frac{b}{2r_n} \sigma_n^2 \]
\[ + \kappa c_n^2 D + u_D^2 - g_{n-1}(x) z_{n-1} z_n \]
\[ \leq -k_n b z_n^2 - \frac{b}{2r_n} \sigma_n^2 + \frac{b}{2r_n} \sigma_n^2 \]
\[ + \kappa c_n^2 D + u_D^2 - g_{n-1}(x) z_{n-1} z_n \]
\[ \leq -k_n b v_n + b v_n + \Delta v_n, \tag{42} \]
where
\[ k_{v_n} = \min (2k_n b, \sigma_n, 2(\rho - 1)), \]
\[ b v_n = \frac{b}{2r_n} \sigma_n^2 + \kappa c_n^2 b + \kappa c_n^2 D + u_D^2, \]
\[ \Delta v_n = -g_{n-1}(x) z_{n-1} z_n. \]

Therefore, combining with the above discussion, Theorem 3.1 is proposed as follows.

**Theorem 3.1:** The nonlinear system meeting Assumptions 2.1–2.5 is considered such as (1). Suppose that while meeting the bounded estimation error, the unknown function can be approximated by technique of the RBF neural network. With the effect of the control input (38) and adaptive law (24) for \(i = 1, \ldots, n\), all signals of the closed-loop system are SGUUB, and tracking errors between system output and tracking signal converge to a small neighbourhood of the origin.

**Proof:** Define the overall Lyapunov function \(V(t)\):
\[ V(t) = \sum_{i=1}^{n} \left( \frac{1}{2} z_i^2 + \frac{b}{2r_n} \sigma_i^2 \right) + \frac{1}{2} \eta^2. \tag{43} \]
Combining with the above derivation and differentiating \( V(t) \), one has

\[
V(t) \leq \sum_{i=1}^{n} \left( -k_{ibz_i^2} + \frac{b}{2ri} \sigma_i \beta_i^2 + \frac{b}{2ri} \sigma_i \beta_i^2 + \kappa \sigma_i \beta_i \right) + (1 - \rho) \eta^2 + \kappa \sigma_i \beta_i \] 
\leq \sum_{i=1}^{n} \left( k_{ibz_i^2} + \frac{b}{2ri} \sigma_i \beta_i^2 \right) + (1 - \rho) \eta^2 
+ \sum_{i=1}^{n} \left( \frac{b}{2ri} \sigma_i \beta_i^2 + \kappa \sigma_i \beta_i \right) + \kappa \sigma_i \beta_i \] 
\leq -k_{V} V + b_{V}, \quad (44)
\]

where

\[
k_{V} = \min \left[ 2k_{b}, \sigma_{i}, 2(\rho - 1) \right],
\]
\[
b_{V} = \sum_{i=1}^{n} \left( \frac{b}{2ri} \sigma_i \beta_i^2 + \kappa \sigma_i \beta_i \right) + \kappa \sigma_i \beta_i.
\]

From the above result, the closed-loop system is SGUUB.

Further, one can be derived by (44), which is shown below:

\[
0 \leq V(t) \leq \left( V(0) - \frac{b_{V}}{k_{V}} \right) e^{-k_{V}t} + \frac{b_{V}}{k_{V}}, \quad (45)
\]

As \( t \to \infty \), one gets

\[
\lim_{t \to \infty} z_i^2 \leq 2 \frac{b_{V}}{k_{V}}. \quad (46)
\]

It means that the tracking errors are bounded and eventually converge to the neighbourhood of the origin. 

\[ \Box \]

### 4. Simulation

The numerical example in the following form is considered to verify the validity of the adaptive neural controller designed above.

\[
\begin{align*}
\dot{x}_1 &= [0.25 \sin^2(x_1 + x_2) + 0.2]x_3 \\
&\quad + 0.1 \sin^2(x_1x_2x_3) + 0.1)x_2 \\
\dot{x}_2 &= \frac{0.1x_1x_2x_3}{1 + x_1^2 + x_2^2 + x_3^2} + [0.25 \cos^2(x_1 + x_2) + 0.2]x_3 \\
\dot{x}_3 &= (x_1 + x_2) \cos(x_3) + u[\nu(t - 0.2)] \\
y &= x_1
\end{align*}
\]

where \( f_1(x) = [0.25 \sin^2(x_1 + x_2) + 0.2]x_3 \), \( g_1(x) = 0.1 \sin^2(x_1x_2x_3) + 0.1 \), \( f_2(x) = 0.1x_1x_2x_3/(1 + x_1^2 + x_2^2 + x_3^2) \), \( f_3(x) = (x_1 + x_2) \cos(x_3) \), \( g_2(x) = 0.25 \cos^2(x_1 + x_2) + 0.2 \), \( f_4(x) = (x_1 + x_2) \cos(x_3) \).

Set the input delay \( \tau = 0.2 \), and the limits of input saturation function are selected as \( u_{\max} = 50 \), \( u_{\min} = -30 \). The signal being tracked is considered for \( y_{d}(t) = \sin(t) \).

From (3), we choose \( \rho = 60 \).

In RBF NN, the width of all Gaussian functions is chosen as \( \xi = 2 \). Further, neural network \( W_{1}^T S_1(X_1) \) contains 5 nodes with centres spaced evenly in the interval \([-2, 2]\), \( W_{2}^T S_2(X_2) \) contains \( 5^4 \) nodes with centres spaced evenly in the interval \([-2, 2] \times \cdots \times [-2, 2] \), and \( W_{3}^T S_3(Z_3) \) contains \( 5^6 \) nodes with centres spaced evenly in the interval \([-2, 2] \times \cdots \times [-2, 2] \) in this third-order system.

Using the theorem proposed in the above section, we can design the virtual control signals, actual control input

![Figure 1. System output \( y \) and its tracking signal \( y_d \).](image1)

![Figure 2. Tracking error between \( y \) and \( y_d \).](image2)
Figure 3. The system state variables $x_2$ and $x_3$.

Figure 4. The auxiliary system state variable $\eta$.

Figure 5. The adaptive parameters $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\theta}_3$.

Figure 6. The control input $u$.

and adaptive laws as

$$
\begin{align*}
\alpha_i &= -k_i z_i - \hat{\theta}_i Q_i(X_i) \tanh \left( \frac{z_i Q_i(X_i)}{c_i} \right), \quad (i = 1, 2) \\
\nu &= -k_3 z_3 - \hat{\theta}_3 Q_3(Z_3) \tanh \left( \frac{z_3 Q_3(Z_3)}{c_3} \right), \\
\dot{\hat{\theta}}_i &= r_i z_i Q_i(X_i) \tanh \left( \frac{z_i Q_i(X_i)}{c_i} \right) - \sigma_i \hat{\theta}_i, \quad (i = 1, 2, 3)
\end{align*}
$$

The values of control design parameters in this numerical example are given as: $k_1 = 10, k_2 = 10, k_3 = 30, c_i = 0.01, r_i = 1, \sigma_i = 0.1$ for $i = 1, 2, 3$. Set $x_1(0) = 0.1, x_2(0) = 0.1, x_3(0) = -0.1$ and the rest of the initial conditions are given as 0.

From the simulation results shown in Figures 1–6, it can be seen clearly that the controller designed in this paper can realize good control effect and tracking performance.

5. Conclusion

In this paper, a control scheme to address the tracking problem is designed in view of a kind of nonlinear systems with non-strict feedback form, which contains input delay and saturation. By virtue of an auxiliary system proposed and a smooth nonlinear function introduced, we can address the problem of input delay and saturation. Using the adaptive neural backstepping method, a tracking controller is designed. In the light of Lyapunov stability theory, it can be proved that all signals of the closed-loop system are SGUUB and the tracking errors converge to a small neighbourhood of the origin. And then numerical simulation results show the validity of our control scheme. In the future, some interesting control issues, such as unknown inputs or state time-delay (Zou et al., 2019, 2020) will be our main research directions.
Disclosure statement
No potential conflict of interest was reported by the author(s).

Funding
This work was supported by the National Natural Science Foundation of China (grant numbers 61873137 and 61673227).

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