Multiple-correction and Faster Approximation

Xiaodong Cao, Hongmin Xu and Xu You

Abstract

In this paper, we formulate a new multiple-correction method. The goal is to accelerate the rate of convergence. In particular, we construct some sequences to approximate the Euler-Mascheroni and Landau constants, which are faster than the classical approximations in literature.

1 Introduction

Euler constant was first introduced by Leonhard Euler (1707-1783) in 1734 as the limit of the sequence

$$\gamma(n) := \frac{1}{n} - \ln n.$$  

(1.1)

It is also known as the Euler-Mascheroni constant. There are many famous unsolved problems about the nature of this constant. For example, it is a long-standing open problem if it is a rational number. See e.g. the survey papers or books of Brent and Zimmermann [3], Dence and Dence [15], Havil [22] and Lagarias [23]. A good part of its mystery comes from the fact that the known algorithms converging to $\gamma$ are not very fast, at least, when they are compared to similar algorithms for $\pi$ and $e$.

The sequence $(\gamma(n))_{n \in \mathbb{N}}$ converges very slowly toward $\gamma$, like $(2n)^{-1}$. To evaluate it more accurately, we need to accelerate the convergence. This can be done using the Euler-Maclaurin summation formula, Stieltjes approach, exponential integral methods, Bessel function method, etc. See e.g. Gourdon and Sebah [19].

Up to now, many authors are preoccupied to improve its rate of convergence. See e.g. Chen and Mortici [11], DeTemple [16], Gavrea and Ivan [18], Lu [25, 26], Mortici [27], Mortici and...
Chen [31] and references therein. We list some main results as follows: as \( n \to \infty \),

\[
\sum_{m=1}^{n} \frac{1}{m} - \ln \left( n + \frac{1}{2} \right) = \gamma + O(n^{-2}), \quad (\text{DeTemple [16]}, 1993),
\]

\[
\sum_{m=1}^{n} \frac{1}{m} - \ln \left( \frac{3n^2 + 2n + 97}{240} \right) = \gamma + O(n^{-3}), \quad (\text{Mortici [27]}, 2010),
\]

\[
\sum_{m=1}^{n} \frac{1}{m} - \ln \rho(n) = \gamma + O(n^{-5}), \quad (\text{Chen and Mortici [11]}, 2012), \quad (1.4)
\]

where \( \rho(n) = 1 + \frac{1}{2n} + \frac{1}{24n^2} - \frac{1}{48n^3} + \frac{23}{25920n^4} \).

Recently, Mortici and Chen [31] provided a very interesting sequence

\[
\nu(n) = \sum_{m=1}^{n} \frac{1}{m} - \frac{1}{2} \ln \left( n^2 + n + \frac{1}{3} \right)
\]

\[
- \left( -\frac{1}{180} \left( \frac{8}{2835} \right)^2 + \frac{8}{2835} \left( \frac{5}{1412} \right)^3 + \frac{5}{1412} \left( \frac{592}{2835} \right)^4 + \frac{592}{2835} \right),
\]

and proved

\[
\lim_{n \to \infty} n^{12} (\nu(n) - \gamma) = -\frac{796801}{43783740}.
\]

Hence, the rate of convergence of the sequence \( (\nu(n))_{n \in \mathbb{N}} \) is \( n^{-12} \).

Let \( R_1(n) = \frac{a_1}{n} \) and for \( k \geq 2 \)

\[
R_k(n) := \frac{a_1}{n} + \frac{a_{2n}}{n+\frac{1}{3}} + \cdots + \frac{a_{2k}}{n+\frac{1}{3}^k},
\]

where \( (a_1, a_2, a_4, a_6, a_8, a_{10}, a_{12}) = (1, 1, \frac{3}{5}, 79, 120, 5241, 9133446) \), \( a_{2k+1} = -a_{2k} \) for \( 1 \leq k \leq 6 \), and

\[
r_k(n) := \sum_{m=1}^{n} \frac{1}{m} - \ln n - R_k(n).
\]

Lu [25] introduced a continued fraction method to investigate this problem, and showed

\[
\frac{1}{120(n+1)^4} < r_3(n) - \gamma < \frac{1}{120(n-1)^4}.
\]

In fact, Lu [25] determined the constants \( a_1 \) to \( a_4 \). Xu and You [39] continued Lu’s work to find \( a_5, \cdots, a_{13} \) with the help of Mathematica software, and obtained

\[
\lim_{n \to \infty} n^{k+1} (r_k(n) - \gamma) = C'_k,
\]
where \((C'_1, \ldots, C'_{13}) = (-\frac{1}{12}, -\frac{1}{36}, \frac{1}{120}, \frac{1}{720}, -\frac{79}{510720}, \frac{241}{22018240}, -\frac{58081}{91974960}, -\frac{262445}{892586949408}, -\frac{71521421431}{450711521003463200}, -\frac{20169451}{3821257440}, -\frac{406806753641401}{450711521003463200}). Moreover, they improved (1.8) to

\[
\begin{align*}
C'_{10} \left(\frac{1}{n+1}\right)^{11} &< \gamma - r_{10}(n) < \frac{1}{n^{11}}, \\
C'_{11} \left(\frac{1}{n+1}\right)^{12} &< r_{11}(n) - \gamma < \frac{1}{n^{12}}.
\end{align*}
\]

However, it seems difficult for us to find more constants \(a_k\). One of the main reasons is due to the recursive algorithm. The other reason is that the parameter \(a_j\) appears many times in the coefficients of polynomials \(P_l(x)\) and \(Q_m(x)\), and this causes that expanding function \(P_l(x)\) needs a huge of computations. To overcome this difficulty, the purpose of this paper is to formulate a new multiple-correction method to accelerate the convergence. In addition, we will use this method to study the sharp bounds for the constants of Landau.

The Landau’s constants are defined for all integers \(n \geq 0\) by

\[
G(n) = \sum_{k=0}^{n} \frac{1}{16^k} \left(\frac{2k}{k}\right)^2.
\]

The constants \(G(n)\) are important in complex analysis. In 1913, Landau \[24\] proved that if \(f(z) = \sum_{k=0}^{\infty} a_k z^k\) is an analytic function in the unit disc which satisfies \(|f(z)| < 1\) for \(|z| < 1\), then \(|\sum_{k=0}^{n} a_k| \leq G(n)\), and that this bound is optimal. Landau \[24\] showed that

\[
G(n) \sim \frac{1}{\pi} \ln n, (n \to \infty).
\]

In 1930, Watson \[37\] obtained the following more precise asymptotic formula

\[
G(n) \sim \frac{1}{\pi} \ln(n+1) + c_0 - \frac{1}{4\pi(n+1)} + O\left(\frac{1}{n^2}\right), (n \to \infty),
\]

where

\[
c_0 = \frac{1}{\pi}(\gamma + 4 \ln 2) = 1.0662758532089143543\ldots
\]

The work of Watson opened up a novel insight into the asymptotic behavior of the Landau sequences \((G(n))_{n \geq 0}\). Inspired by formula (1.14), many authors investigated the upper and lower bounds of \(G(n)\). Some of the main results are listed as follows:

\[
\begin{align*}
\frac{1}{\pi} \ln(n+1) + 1 &\leq G(n) < \frac{1}{\pi} \ln(n+1) + c_0 \quad (n \geq 0), \quad (\text{Brutman \[4\], 1982}), \\
\frac{1}{\pi} \ln \left(n + \frac{3}{4}\right) + c_0 &< G(n) \leq \frac{1}{\pi} \ln \left(n + \frac{3}{4}\right) + 1.0976 \quad (n \geq 0), \quad (\text{Falaleev \[17\], 1991}), \\
\frac{1}{\pi} \ln \left(n + \frac{3}{4}\right) + c_0 &< G(n) < \frac{1}{\pi} \ln \left(n + \frac{3}{4}\right) + \frac{11}{192n} + c_0 \quad (n \geq 1), \quad (\text{Mortici \[30\], 2011}).
\end{align*}
\]
Very recently, Chen [9] presented the following better approximation to \( G(n) \): as \( n \to \infty \),

\[
G(n) = c_0 + \frac{1}{\pi} \ln \left( n + \frac{3}{4} + \frac{11}{192(n + \frac{3}{4})} \right) - \frac{2009}{184320(n + \frac{3}{4})^3} + \frac{2599153}{371589(n + \frac{3}{4})^5} + O\left(\frac{1}{(n + \frac{3}{4})^8}\right),
\]

and the better upper bound:

\[
G(n) < c_0 + \frac{1}{\pi} \ln \left( n + \frac{3}{4} + \frac{11}{192(n + \frac{3}{4})} \right), (n \geq 0).
\]

Another direction for developing the approximation to \( G(n) \) was initiated by Cvijović and Klinowski [12], who established estimates for \( G(n) \) in terms of the Psi(or Digamma) function \( \psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} \):

\[
\frac{1}{\pi} \psi \left( n + \frac{5}{4} \right) + c_0 < G(n) < \frac{1}{\pi} \psi \left( n + \frac{5}{4} \right) + 1.0725, \quad (n \geq 0),
\]

\[
\frac{1}{\pi} \psi \left( n + \frac{3}{2} \right) + 0.9883 < G(n) < \frac{1}{\pi} \psi \left( n + \frac{3}{2} \right) + c_0, \quad (n \geq 0).
\]

Since then, many authors have made significant contributions to sharper inequalities and asymptotic expansions for \( G(n) \). See e.g. Alzer [11], Chen [8], Cvijović and Srivastava [14], Granath [21], Mortici [30], Nemes [32 33], Popa [34], Popa and Secelean [35], Zhao [41], Gavrea and M. Ivan [18], Chen and Choi [5 7], etc. To the best knowledge of authors, the latest upper bound is due to Chen [9], who proved

\[
G(n) < c_0 + \frac{1}{\pi} \psi \left( n + \frac{5}{4} + \frac{1}{64(n + \frac{3}{4})} \right), (n \geq 0).
\]

Here, the authors would like to thank Alzer, Chen, Choi, DeTemple, Granath, Lu, Mortici, etc., it is their important works that makes the present work becomes possible.

**Notation.** Throughout the paper, the notation \( P_k(x) \) (or \( Q_k(x) \)) as usual denotes a polynomial of degree \( k \) in terms of \( x \). The notation \( \Psi(k; x) \) means a polynomial of degree \( k \) in terms of \( x \) with all of its non-zero coefficients being positive, which may be different at each occurrence. Notation \( \Phi(k; x) \) denotes a polynomial of degree \( k \) in terms of \( x \) with the leading coefficient being equal to one, which may be different at different subsection.

## 2 Some Lemmas

The following lemma gives a method for measuring the rate of convergence, for the proof of which, see Mortici [21 28].
Lemma 1. If the sequence \((x_n)_{n \in \mathbb{N}}\) is convergent to zero and there exists the limit
\[
\lim_{n \to +\infty} n^s(x_n - x_{n+1}) = l \in [-\infty, +\infty]
\]
with \(s > 1\), then
\[
\lim_{n \to +\infty} n^{s-1}x_n = \frac{l}{s-1}.
\]

In the study of Landau constants, we need to apply a so-called Brouncker’s continued fraction formula.

Lemma 2. For all integer \(n \geq 0\), we have
\[
q(n) := \left(\frac{\Gamma(n + \frac{1}{2})}{\Gamma(n + 1)}\right)^2 = \frac{4}{1 + 4n + \frac{1^2}{2+8n+\frac{3^2}{2+8n+\ldots}}}.
\]

In 1654 Lord William Brouncker found this remarkable fraction formula, when Brouncker and Wallis collaborated on the problem of squaring the circle. Formula (2.3) was not published by Brouncker himself, but first appeared in \[36\]. For a general \(n\), Formula (2.3) follows from Entry 25 in Chapter 12 in Ramanujan’s notebook \[2\], which gives a more general continued fraction formula for quotients of gamma functions, and which have several proofs published by different authors.

Writing continued fractions in this way of (2.3) takes a lot of space. So instead we use the following shorthand notation
\[
q(n) = \frac{4}{1 + 4n + \frac{1^2}{2+8n+\frac{3^2}{2+8n+\ldots}}},
\]
and its \(k\)-th approximation \(q_k(n)\) is defined by
\[
q_1(n) = \frac{4}{1 + 4n}, \quad q_k(n) = \frac{4}{1 + 4n + \frac{1^2}{2+8n+\frac{3^2}{2+8n+\ldots}} \frac{(2k-3)^2}{2+8n}}, \quad (k \geq 2).
\]

In the proof of our inequalities for the constants of Euler-Mascheroni and Landau, we also use the following simple inequality.

Lemma 3. Let \(f''(x)\) be a continuous function. If \(f''(x) > 0\), then
\[
\int_a^{a+1} f(x)dx > f(a + 1/2).
\]
Proof. By letting \( x_0 = a + 1/2 \) and Taylor’s formula, we have
\[
\int_a^{a+1} f(x) \, dx = \int_a^{a+1} \left( f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(\theta_x)(x - x_0)^2 \right) \, dx > \int_a^{a+1} \left( f(x_0) + f'(x_0)(x - x_0) \right) \, dx = f(a + 1/2).
\]
This completes the proof of Lemma 3. Also see Lemma 2 in Xu and You [39].

3 Two Examples for Euler-Mascheroni Constant

In this section, to illustrate quickly and clearly the main ideas of this paper, we consider the simplest case of Euler-Mascheroni constant by using the correction-process again.

**Example 1.** We choose an initial-correction function \( \theta_0(n) \) given by
\[
\theta_0(n) = \frac{13 + 30n}{6(1 + 6n + 10n^2)}.
\]
and define
\[
\nu_0(n) = \sum_{m=1}^{n} \frac{1}{m} - \ln n - \theta_0(n).
\]
Applying Lemma 1, one can check (See Theorem 1 in [39], or (1.9))
\[
\lim_{n \to \infty} n^6 (\nu_0(n) - \nu_0(n + 1)) = \frac{1}{40},
\]
\[
\lim_{n \to \infty} n^5 (\nu_0(n) - \gamma) = \frac{1}{200}.
\]
By using the similar idea of Kummer’s acceleration method and inserting the correction function \(-\frac{1}{200}n^5\) in (3.2) again, one can use Lemma 1 again to show
\[
\lim_{n \to \infty} n^6 \left( \nu_0(n) - \frac{1}{200} \frac{1}{n^5} - \gamma \right) = -\frac{773}{126000}.
\]
Furthermore, we try to obtain an algorithm with a faster convergent rate by using \( \Phi(5; n) \) instead of \( n^5 \). To do that, let
\[
\theta(n) = \frac{\frac{1}{200}}{\Phi(5; n)} = \frac{\frac{1}{200}}{n^5 + a_4n^4 + a_3n^3 + a_2n^2 + a_1n + a_0},
\]
\[
\nu(n) = \sum_{m=1}^{n} \frac{1}{m} - \ln n - \theta_0(n) - \theta(n).
\]
First, we use the method of undetermined coefficients to find \(a_j(0 \leq j \leq 4)\). By using the Mathematica software, we expand the difference \(\nu(n) - \nu(n + 1)\) into a power series in terms of \(n^{-1}\):

\[
\nu(n) - \nu(n + 1) = \frac{-773 + 630a_4}{21000} \frac{1}{n^7} + \frac{4033 + 1050a_3 - 3150a_4 - 1050a_4^2}{30000} \frac{1}{n^8} + \frac{-37657 + 4500a_2 - 15750a_3 + 31500a_4 - 9000a_3a_4 + 15750a_4^2 + 4500a_4^3}{112500} \frac{1}{n^9} + \frac{1}{500000} \frac{1}{n^{10}} + \frac{\varphi_{10}}{4125000} \frac{1}{n^{11}} + \frac{\varphi_{11}}{7500000} \frac{1}{n^{12}} + O\left(\frac{1}{n^{13}}\right),
\]

where

\[
\begin{align*}
\varphi_{10} &= 350191 + 2250a_1 - 90000a_2 + 210000a_3 - 22500a_4^2 - 315000a_4 - 45000a_2a_4 + 180000a_3a_4 - 210000a_4^2 + 67500a_3a_4^2 - 90000a_4^3 - 22500a_4^4, \\
\varphi_{11} &= -5465923 + 2062500a_0 - 928125a_1 + 2475000a_2 - 4331250a_3 - 4125000a_2a_3 + 928125a_4^2 + 618750a_3a_4^2 + 31500a_2a_4^2 - 2784375a_3a_4^2 + 2475000a_4^3 - 825000a_3a_4^4 + 928125a_4^5, \\
\varphi_{12} &= 175990871 - 20625000a_0 + 61875000a_1 - 123750000a_2 - 4125000a_2^2 + 173250000a_3 - 8250000a_1a_3 + 41250000a_2a_3 - 61875000a_4^2 + 12375000a_2a_4 + 173250000a_3a_4 - 8250000a_1a_4 + 41250000a_2a_4 - 12375000a_2a_4 + 475000000a_3a_4 + 24750000a_2a_4 - 61875000a_3a_4^2 - 173250000a_4^2 + 12375000a_1a_4 + 123750000a_2a_4 + 8250000a_3a_4^2 - 173250000a_4^2 - 12375000a_4^3 - 165000000a_2a_4^3 + 8250000a_3a_4^2 - 61875000a_4^4 + 20625000a_3a_4^4 - 20625000a_4^5 - 4125000a_4^6.
\end{align*}
\]

According to Lemma 1, we have five parameters \(a_4, a_3, a_2, a_1\) and \(a_0\) which produce the fastest convergence of the sequence from (3.3):

\[
\begin{align*}
-773 + 630a_4 &= 0, \\
4033 + 1050a_3 - 3150a_4 - 1050a_4^2 &= 0 \\
-37657 + 4500a_2 - 15750a_3 + 31500a_4 - 9000a_3a_4 + 15750a_4^2 + 4500a_4^3 &= 0 \\
\varphi_{10} &= 0, \\
\varphi_{11} &= 0,
\end{align*}
\]

namely if

\[
\theta(n) = \frac{1}{209} \frac{1}{n^5} + \frac{773}{630} \frac{1}{n^4} + \frac{21361}{15876} \frac{1}{n^3} + \frac{1348075}{200376} \frac{1}{n^2} - \frac{91207415}{252047376} \frac{1}{n} - \frac{178345771079}{1746688315680}.
\]
Thus, we get

\begin{equation}
\nu(n) - \nu(n + 1) = -\frac{10992878936527}{160060165655040} n^{12} + O\left(\frac{1}{n^{13}}\right).
\end{equation}

We can apply another approach to find \( a_4, a_3, a_2, a_1 \) and \( a_0 \) step by step, which is achieved by using \( n^5 + a_4 n^4, n^5 + a_4 n^4 + a_3 n^3, n^5 + a_4 n^4 + a_3 n^3 + a_2 n^2, n^5 + a_4 n^4 + a_3 n^3 + a_2 n^2 + a_1 n, n^5 + a_4 n^4 + a_3 n^3 + a_2 n^2 + a_1 n + a_0 \) instead of \( \Phi(5; n) \) in turn. For the reader’s convenience, here we give an example to explain how Mathematica software generates \( \nu(n) - \nu(n + 1) \) into power series in the terms of \( \frac{1}{n} \). For example, find \( a_3 \). We manipulate Mathematica program

\begin{verbatim}
Normal[Series[(-1/n + Log[1 + 1/n]) - (13 + 30n/6(1 + 6n + 10n^2)) + 1/200
+n^5/(773/630)n^4 + a_3 n^3)/n -> n + 1]/.n -> 1/x,
{x, 0, 10}]/.x -> 1/n
to generate
\end{verbatim}

\begin{equation}
\nu(n) - \nu(n + 1) = -\frac{21361}{45300} + \frac{7a_3}{200} + \frac{9173092}{31255875} - \frac{3751a_3}{15750} + O\left(\frac{1}{n^{10}}\right).
\end{equation}

By solving the equation \(-\frac{21361}{45300} + \frac{7a_3}{200} = 0\), we also find \( a_3 = \frac{21361}{18850} \). In what follows, we always use this approach.

By Lemma 1, we obtain finally

\begin{equation}
\lim_{n \to \infty} n^{11} (\nu(n) - \gamma) = -\frac{10992878936527}{176066182205440}.
\end{equation}

We observe that the above twice-correction improves the rate of convergence from \( n^{-6} \) to \( n^{-11} \). \( \square \)

**Remark 1.** The main idea of twice-correction is that from \( n^5 + a_4 n^4, n^5 + a_4 n^4 + a_3 n^3 + a_2 n^2, n^5 + a_4 n^4 + a_3 n^3 + a_2 n^2 + a_1 n \) to \( n^5 + a_4 n^4 + a_3 n^3 + a_2 n^2 + a_1 n + a_0 \), their approximations in turn become better and better.

**Remark 2.** It should be noted that once we find the exact values of the parameters \( a_4 \) to \( a_0 \), it is not very difficult for us to check the formula (3.10) with the help of Mathematica software.

**Example 2.** We would like to give another example. Now we take the initial-correct function \( \eta_0(n) = \frac{6n - 1}{72n^2} \) (see Theorem 1.1 in Lu [24] or (1.9), which is found by the continued fraction method), and define

\begin{equation}
\mu_0(n) = \sum_{m=1}^{n} \frac{1}{m} - \ln n - \eta_0(n).
\end{equation}

One may check by using Lemma 1

\begin{equation}
\lim_{n \to \infty} n^{4} (\mu_0(n) - \gamma) = \frac{1}{120}.
\end{equation}
Similarly, we insert a correction function $-\eta(n)$ in (3.13) again, which has the form of

$$(3.15) \quad \eta(n) = \frac{1}{120} n^4 + b_3 n^3 + b_2 n^2 + b_1 n + b_0.$$ 

Applying the same method as Example 1, we can find $(b_3, b_2, b_1, b_0) = (0, \frac{10}{27}, 0, -\frac{241}{882})$. The details is omitted here. Now we define

$$(3.16) \quad \mu(n) = \sum_{m=1}^{n} \frac{1}{m} - \ln n - \eta_0(n) - \eta(n),$$

where

$$(3.17) \quad \eta_0(n) = \frac{6n - 1}{12n^2} \quad \text{and} \quad \eta(n) = \frac{1}{n^4 + \frac{10}{27}n^2 - \frac{241}{882}}.$$ 

By using Mathematica software and Lemma 1, we can attain

$$(3.18) \quad \mu(n) - \mu(n + 1) = -\frac{13775}{3056130} \frac{1}{n^{11}} + O \left( \frac{1}{n^{12}} \right),$$

$$(3.19) \quad \lim_{n \to \infty} n^{10} (\mu(n) - \gamma) = -\frac{13775}{3056130}.$$ 

Remark 3. We observe that the above twice-correction improves the rate of convergence from $n^{-4}$ to $n^{-10}$, which is the desired result. However, it is interesting to note that both $b_3$ and $b_1$ equal zero. The reason of why inserting the sub-correction term $b_3 n^3$ (or $b_1 n$) does not improves the rate of convergence (i.e. compare $n^4 + b_3 n^3$ with $n^4$, or $n^4 + b_3 n^3 + b_2 n^2 + b_1 n$ with $n^4 + b_3 n^3 + b_2 n^2$) may be that the function $n^3$ (or $n$) changes too rapidly when $n$ tends to infinity. Fortunately, these losses are made up by the sub-correction terms $b_2 n^2$ and $b_0$.

More precisely, we will improve (3.19), and prove the following double-sides inequalities.

**Theorem 1.** Let $\mu(n)$ be defined by (3.16). Then for all positive integer $n$, we have

$$(3.20) \quad \frac{13775}{3056130} \frac{1}{(n + \frac{3}{4})^{10}} < \gamma - \mu(n) < \frac{13775}{3056130} \frac{1}{(n - \frac{1}{4})^{10}}.$$ 

Remark 4. In fact, Theorem 1 implies that $\mu(n)$ is a strictly increasing function of $n$.

**Proof.** It follows from (3.16)

$$(3.21) \quad \mu(n) - \mu(n + 1) = -\frac{1}{n + 1} + \ln \left( 1 + \frac{1}{n} \right) + \eta_0(n + 1) + \eta(n + 1) - \eta_0(n) - \eta(n).$$ 

We write $D = \frac{13775}{27783}$, and define for $x \geq 1$

$$(3.22) \quad -\omega(x) = -\frac{1}{x + 1} + \ln \left( 1 + \frac{1}{x} \right) + \eta_0(x + 1) + \eta(x + 1) - \eta_0(x) - \eta(x).$$
By Mathematica software, it is not difficult to check

\[ -\omega'(x) - \frac{D}{(x + \frac{1}{4})^{12}} \]

\[ \Psi_1(20; x)(x - 1) + 4032098201877889488940287625 \]

\[ \frac{277830x^3(1 + x)^3(1 + 4x)^{12}(-241 + 420x^2 + 882x^4)^2(1061 + 4368x + 5712x^2 + 3528x^3 + 882x^4)^2}{277830} \times x^3(1 + x)^3(1 + 4x)^{12}(-241 + 420x^2 + 882x^4)^2(1061 + 4368x + 5712x^2 + 3528x^3 + 882x^4)^2 < 0, \]

and

\[ -\omega'(x) - \frac{D}{(x + \frac{3}{4})^{12}} \]

\[ \Psi_2(20; x)(x - 1) + 51726219719747325679290363431 \]

\[ \frac{277830x^3(1 + x)^3(3 + 4x)^{12}(-241 + 420x^2 + 882x^4)^2(1061 + 4368x + 5712x^2 + 3528x^3 + 882x^4)^2}{277830} \times x^3(1 + x)^3(3 + 4x)^{12}(-241 + 420x^2 + 882x^4)^2(1061 + 4368x + 5712x^2 + 3528x^3 + 882x^4)^2 > 0. \]

Note that \( \omega(\infty) = 0 \). From (3.23) and Lemma 3, one has

\[ \omega(n) = \int_n^\infty -\omega'(x)dx < D \int_n^\infty \frac{dx}{(x + \frac{1}{4})^{12}} \]

\[ \frac{D}{11} \frac{1}{(n + \frac{1}{4})^{11}} < D \int_{n-\frac{1}{4}}^{n+\frac{3}{4}} \frac{dx}{x^{11}}. \]

Note that \( \mu(\infty) = \gamma \). Combining (3.21), (3.22) and (3.25), we have

\[ \gamma - \mu(n) = \sum_{m=n}^{\infty} (\mu(m + 1) - \mu(m)) < D \frac{1}{11} \sum_{m=n}^{\infty} \int_{m-\frac{1}{4}}^{m+\frac{3}{4}} \frac{dx}{x^{11}} \]

\[ \frac{D}{11} \int_{n-\frac{1}{4}}^{n+\frac{3}{4}} \frac{dx}{x^{11}} = \frac{D}{110} \frac{1}{(n - \frac{1}{4})^{10}}. \]

This finishes the proof of right-hand inequality in (3.20). Similarly, it follows from (3.24)

\[ \omega(n) = \int_n^\infty -\omega'(x)dx > D \int_n^\infty \frac{dx}{(x + \frac{3}{4})^{12}} \]

\[ \frac{D}{11} \frac{1}{(n + \frac{3}{4})^{11}} > D \int_{n+\frac{3}{4}}^{n+\frac{7}{4}} \frac{dx}{x^{11}}. \]

Finally, by (3.21), (3.22) and (3.27), one has

\[ \gamma - \mu(n) = \sum_{m=n}^{\infty} (\mu(m + 1) - \mu(m)) > D \frac{1}{11} \sum_{m=n}^{\infty} \int_{m+\frac{3}{4}}^{m+\frac{7}{4}} \frac{dx}{x^{11}} \]

\[ \frac{D}{11} \int_{n+\frac{3}{4}}^{n+\frac{7}{4}} \frac{dx}{x^{11}} = \frac{D}{110} \frac{1}{(n + \frac{3}{4})^{10}}. \]

This completes the proof of Theorem 1. 

\[ \square \]
4 The multiple-correction method

Based on the work of Section 2, we will formulate a new multiple-correction method to study faster approximation problem for the constants of Euler-Mascheroni and Landau.

Let \((v(n))_{n \geq 1}\) be a sequence to be approximated. Throughout the paper, we always assume that the following three conditions hold.

Condition (i). The initial-correction function \(\eta_0(n)\) satisfies
\[
\lim_{n \to \infty} (v(n) - n^{l_0} (v(n) - v(n + 1) - \eta_0(n) + \eta_0(n + 1)) = C_0 \neq 0,
\]
with some a positive integer \(l_0 \geq 2\).

Condition (ii). The \(k\)-th correction function \(\eta_k(n)\) has the form of \(-\frac{C_{k-1}}{\Phi_k(l_{k-1}; n)}\), where
\[
\lim_{n \to \infty} n^{l_k-1} \left( v(n) - v(n + 1) - \sum_{j=0}^{k-1} (\eta_j(n) - \eta_j(n + 1)) \right) = C_{k-1} \neq 0.
\]

Condition (iii). The difference \((v(1/x) - v(1/x + 1) - \eta_0(1/x) + \eta_0(1/x + 1))\) is an analytic function in a neighborhood of point \(x = 0\).

4.1 Euler-Mascheroni Constant

(Step 1) The initial-correction. We choose \(\eta_0(n) = 0\), and let
\[
\nu_0(n) = \sum_{m=1}^{n} \frac{1}{m} - \ln n - \eta_0(n) = \sum_{m=1}^{n} \frac{1}{m} - \ln n.
\]
By lemma 1, it is not difficult to prove that
\[
\lim_{n \to \infty} n^2 (\nu_0(n) - \nu_0(n + 1)) = \frac{1}{2},
\]
\[
\lim_{n \to \infty} n (\nu_0(n) - \gamma) = \frac{1}{2} =: C_0.
\]

(Step 2) The first-correction. We let
\[
\eta_1(n) = \frac{C_0}{\Phi_1(1; n)} = \frac{\frac{1}{2}}{n + b_{(1,0)}},
\]
and define
\[
\nu_1(n) = \sum_{m=1}^{n} \frac{1}{m} - \ln n - \eta_0(n) - \eta_1(n).
\]
By the same method as (3.11), we find 
\[ b_{(1,0)} = \frac{1}{6}. \] Applying Lemma 1 again, one has

\begin{align*}
\text{(4.6)} & \quad \lim_{n \to \infty} n^4 (\nu_1(n) - \nu_1(n + 1)) = -\frac{1}{24}, \\
\text{(4.7)} & \quad \lim_{n \to \infty} n^3 (\nu_1(n) - \gamma) = -\frac{1}{72} := C_1.
\end{align*}

**(Step 3) The second-correction.** Similarly, we set the second-correction function in the form of \( \eta_2(n) = \Phi_2(3;n) \), and define

\[ \nu_2(n) = \sum_{m=1}^{n} \frac{1}{m} - \ln n - \eta_0(n) - \eta_1(n) - \eta_2(n). \]

By using similar approach of (3.11), we can find

\[ \eta_2(n) = \frac{-1}{72} + \frac{126901}{70610 n^7} + \frac{302657774122}{78525910575 n^6} + \frac{5227873714517850}{13964478610281140426652539 n^5} \\
- \frac{11362407540057138240300}{2713180197918474605475 n^4} + \frac{298861139889036647352440010}{1335669056713380727335512329403306 n^3} \\
+ \frac{4586975399311716291806}{243734857761374337083369364175455 n^2} + \frac{1787275723433572920281319954725767947341}{103260709839138516487402648265732653000 n}.
\]

By Lemma 1, one can obtain

\begin{align*}
\text{(4.10)} & \quad \lim_{n \to \infty} n^8 (\nu_2(n) - \nu_2(n + 1)) = -\frac{7061}{5400}, \\
\text{(4.11)} & \quad \lim_{n \to \infty} n^7 (\nu_2(n) - \gamma) = -\frac{7061}{3780000} := C_2.
\end{align*}

**(Step 4) The third-correction.** We set \( \eta_3(n) = \frac{C_2}{\Phi_3(7; n)} \), and define

\[ \nu_3(n) = \sum_{m=1}^{n} \frac{1}{m} - \ln n - \eta_0(n) - \eta_1(n) - \eta_2(n) - \eta_3(n). \]

By using *Mathematica* software, we can find

\[ \Phi_3(7; n) = \frac{70610}{126901 n^7} + \frac{78525910575}{302657774122 n^6} + \frac{13964478610281140426652539}{203050524517143511 n^5} \\
+ \frac{298861139889036647352440010}{5227873714517850 n^4} + \frac{1335669056713380727335512329403306}{4586975399311716291806 n^3} \\
+ \frac{243734857761374337083369364175455}{2713180197918474605475 n^2} + \frac{1787275723433572920281319954725767947341}{103260709839138516487402648265732653000 n}.
\]

Now by Lemma 1 again, we obtain

**Theorem 2.** Let \( \nu_3(n) \) be defined by (4.12). Then we have

\[ \lim_{n \to \infty} n^{16} (\nu_3(n) - \nu_3(n + 1)) = 15C_3, \]

and

\[ \lim_{n \to \infty} n^{15} (\nu_3(n) - \gamma) = -\frac{6044981017774921659252823535814990412377703}{10246043937930176798462527774167322493925000} := C_3. \]
Remark 5. It could be imagined that if we apply the correction-process many times, then, we can obtain $k$th-correction sequence

$$
\nu_k(n) = \sum_{m=1}^{n} \frac{1}{m} - \ln n - \eta_0(n) - \sum_{j=1}^{k} \frac{C_{j-1}}{\Phi_j(l_{j-1}; n)}
$$

with the rate of convergence $\frac{1}{n^k}$, here $l_k \geq 2l_{k-1} + 1$, i.e.

$$
\lim_{n \to \infty} n^k (\nu_k(n) - \gamma) = C_k \neq 0,
$$

$$
\gamma = \sum_{m=1}^{n} \frac{1}{m} - \ln n - \eta_0(n) - \sum_{j=1}^{k} \frac{C_{j-1}}{\Phi_j(l_{j-1}; n)} + O \left( \frac{1}{n^k} \right).
$$

Remark 6. For comparison, the result $\nu_1(n)$ in Theorem 2 is the same as $r_2(n)$ in (1.9), and $\lim_{n \to \infty} \frac{\nu_1(n) - \gamma}{r_2(n) - \gamma} = \frac{C_0}{C_0^*} = 0.950367 \cdots < 1$. Theoretically at least, for a large $n$ the above formula may reduce or eliminate numerically computations compared with Euler-Maclaurin summation formula. For example, if we take $n = 2^{15} = 32768$ in Theorem 2, then $-1.09418 \cdot 10^{-69} < \nu_3(n) - \gamma < 0$.

Remark 7. We can take different initial-correction function to find some other simple faster approximations. For example, we choose the initial-correction function $\eta_0(n) = -\ln n + \frac{1}{2} \ln \left( n^2 + n + \frac{1}{3} \right)$, see, Chen and Li [10]. By Lemma 1, it is not very difficult for us to check that

$$
\nu_0(n) = \sum_{m=1}^{n} \frac{1}{m} - \ln n - \eta_0(n) = \sum_{m=1}^{n} \frac{1}{m} - \frac{1}{2} \ln \left( n^2 + n + \frac{1}{3} \right),
$$

$$
\lim_{n \to \infty} n^4 (\nu_0(n) - \gamma) = \frac{-1}{180} =: C_0.
$$

Let

$$
\eta_1(n) = \frac{C_0}{\Phi_1(4; n)} \quad \text{and} \quad \eta_2(n) = \frac{C_1}{\Phi_2(10; n)},
$$

where $C_1 = \frac{457528}{1237474667}$.

$$
\eta_1(n) = (n + \frac{1}{2})^4 + \frac{85}{126} \left( n + \frac{1}{2} \right)^2 - \frac{18287}{63504},
$$

$$
\eta_2(n) = (n + \frac{1}{2})^{10} + \frac{28038237821}{5995446912} \left( n + \frac{1}{2} \right)^8 + \frac{11612938185328451401}{3594538367461033544} \left( n + \frac{1}{2} \right)^6
$$

$$
+ \frac{163544744039006129564874642269}{8307589279805355451415033136} \left( n + \frac{1}{2} \right)^4
$$

$$
- \frac{2762081970439756978947606523226093661017}{1502137300658621011789214048600457216} \left( n + \frac{1}{2} \right)^2
$$

$$
+ \frac{277100747560698973680958970352585491511233080640189551}{13508976604761479541649384438829437061829754880}.
$$
Now define

\[ \nu_2(n) = \sum_{m=1}^{n} \frac{1}{m} - \ln n - \eta_0(n) - \eta_1(n) - \eta_2(n). \]  

(4.24)

By using Lemma 1, one may check

\[ \lim_{n \to \infty} n^{22} (\nu_2(n) - \gamma) = C_2, \]

(4.25)

where

\[ C_2 = \frac{186484155415412379058939871158437373230857062719102654146029}{1865356576717684121096754889225439763685362998641415946240}. \]

(4.26)

Some other interesting correction functions can be found in Gourdon and Sebah [20].

4.2 Landau Constants

(Step 1) The initial-correction. Let \( c_0 \) be defined by (1.15). Motivated by inequalities (1.17) and (1.18), we choose \( \eta_0(n) = \frac{1}{\pi} \ln(n + \frac{3}{4}) + c_0 \), and define

\[ u_0(n) = G(n) - \eta_0(n) = G(n) - \frac{1}{\pi} \ln(n + \frac{3}{4}) - c_0. \]

(4.27)

Now we consider the difference \( u_0(n) - u_0(n + 1) \). It follows immediately from (4.27)

\[ u_0(n) - u_0(n + 1) = G(n) - G(n + 1) - \frac{1}{\pi} \ln(n + \frac{3}{4}) + \frac{1}{\pi} \ln(n + \frac{7}{4}). \]

(4.28)

First, from the duplication formula (Legendre, 1809)

\[ 2^{2z-1} \Gamma(z) \Gamma(z + \frac{1}{2}) = \sqrt{\pi} \Gamma(2z), \]

(4.29)

one can prove

\[ G(n) - G(n - 1) = \left( \frac{(\Gamma(2n + 1))^2}{16^2(\Gamma(n + 1))^4} \right) = \left( \frac{(2n)!}{4^n(n!)^2} \right)^2 = \frac{1}{\pi} q(n), \]

(4.30)

where \( q(n) \) is defined by (2.3). Also see p.739 in Granath [21] or p.306 in Chen [9]. By (4.28) and (4.30), one has

\[ u_0(n) - u_0(n + 1) = -\frac{1}{\pi} q(n + 1) - \frac{1}{\pi} \ln(n + \frac{3}{4}) + \frac{1}{\pi} \ln(n + \frac{7}{4}). \]

(4.31)

From Lemma 2 and (2.6), on one hand, it can be observed that for all positive integer \( j \), one has

\[ q_2(n) < q_4(n) < \cdots < q_{2j}(n) < q(n) < q_{2j+1}(n) < \cdots < q_3(n) < q_1(n). \]

(4.32)
On the other hand, we can check by using Mathematica software

\[ q_9(n) - q_8(n) = O\left(\frac{1}{n^{17}}\right). \] (4.33)

Combining (4.32) and (4.33) gives

\[ q(n + 1) = q_8(n + 1) + O\left(\frac{1}{n^{16}}\right). \] (4.34)

By using the Mathematica software, we expand \( q_8(n + 1) \) into a power series in terms of \( n^{-1} \).

Noting formula (4.34), we obtain

\[ q(n + 1) = q_8(n + 1) + O\left(\frac{1}{n^{16}}\right) \] (4.35)

\[ = \frac{1}{n} - \frac{5}{4 n^2} + \frac{49}{32 n^3} - \frac{235}{128 n^4} + \frac{4411}{4096 n^5} - \frac{20275}{16384 n^6} + \frac{183077}{65536 n^7} \]
\[ - \frac{815195}{1677721 n^8} + \frac{28754131}{1048576 n^9} - \frac{125799895}{2097152 n^{10}} + \frac{1091975567}{65536 n^{11}} \]
\[ - \frac{262144 n^8}{8388608 n^9} - \frac{2554432}{33554432 n^{10}} + \frac{268435456}{1073741824 n^{11}} \]
\[ + \frac{4702048685}{17179869184 n^{12}} + \frac{1073741824 n^8}{68719476736 n^{13}} + O\left(\frac{1}{n^{14}}\right). \]

The above expression is also used in the first and second-correction below. In addition, it is not difficult to obtain

\[ -\ln(n + \frac{3}{4}) + \ln(n + \frac{7}{4}) = \frac{1}{n} - \frac{5}{4 n^2} + \frac{79}{48 n^3} + O\left(\frac{1}{n^4}\right). \] (4.36)

Inserting (4.35) and (4.36) into (4.31) yields

\[ u_0(n) - u_0(n + 1) = \frac{11}{96\pi n^3} + O\left(\frac{1}{n^4}\right). \] (4.37)

Note that the inequalities (1.18) implies \( u_0(\infty) = 0. \) Applying Lemma 1, we obtain

\[ \lim_{n \to \infty} n^2 u_0(n) = \frac{11}{192\pi} = C_0. \] (4.38)

(Step 2) The first-correction. We let

\[ \eta_1(n) = \frac{C_0}{\Phi_1(2;n)} = \frac{C_0}{n^2 + a_1 n + a_0}, \] (4.39)

and define

\[ u_1(n) = G(n) - \eta_0(n) - \eta_1(n). \] (4.40)
Hence

\[ u_1(n) - u_1(n+1) = (u_0(n) - \eta_1(n)) - (u_0(n+1) - \eta_1(n+1)) \]
\[ = (u_0(n) - u_0(n+1)) - (\eta_1(n) - \eta_1(n+1)). \]  

Note that the first term of (4.41) can be treated by the same method in (step 1). Here we only need to replace (4.36) by the following more accurate power series expansion

\[ \ln(n + \frac{3}{4}) + \ln(n + \frac{7}{4}) = \frac{1}{n} - \frac{5}{4n^2} + \frac{1}{48n^3} - \frac{145}{64n^4} + \frac{4411}{1280n^5} - \frac{14615}{3072n^6} \]
\[ + \frac{205339}{28672n^7} - \frac{179945}{16384n^8} + \frac{10083481}{589824n^9} - \frac{7060405}{262144n^{10}} \]
\[ + \frac{11534336}{494287399n^{11}} - \frac{12582912}{865047235n^{12}} + \frac{24221854021}{24221854021n^{13}} \]
\[ - \frac{84777286235}{469762048n^{14}} + \frac{1186886790259}{4026531840n^{15}} + O\left(\frac{1}{n^{16}}\right). \]

By applying Mathematica software again, we have

\[ \frac{1}{\Phi_1(2;n)} - \frac{1}{\Phi_1(2; n+1)} \]
\[ = \frac{2}{n^3} - \frac{3 - 3a_1}{n^4} + \frac{4 + 6a_1 + 4a_1^2 - 4a_0}{n^5} \]
\[ - \frac{5 - 10a_1 - 10a_1^2 - 5a_1^3 + 10a_0 + 10a_1a_0}{n^6} \]
\[ + \frac{6 + 15a_1 + 20a_1^2 + 15a_1^3 + 6a_1^4 - 20a_0 - 30a_1a_0 - 18a_1^2a_0 + 6a_0^2}{n^7} + O\left(\frac{1}{n^8}\right). \]

Now combining (4.41), (4.34), (4.35), (4.42) and (4.43), and performing some simplifications, we can obtain

\[ \pi (u_1(n) - u_1(n+1)) = \frac{235}{128} - \frac{134 + 11a_1}{64n} \]
\[ + \frac{4411}{2048} - \frac{11543 - 1320a_1 - 880a_1^2 + 880a_0}{3840n^5} \]
\[ - \frac{20275}{8192} + \frac{5(-2747 + 352a_1 + 352a_1^2 + 176a_1^3 - 352a_0 - 352a_1a_0)}{3072n^6} \]
\[ + \frac{183077}{35350} + \frac{\sigma}{n^7} + O\left(\frac{1}{n^8}\right), \]

where

\[ \sigma = 586449 - 73920a_1 - 98560a_1^2 - 73920a_1^3 - 29568a_1^4 + 98560a_0 + 147840a_1a_0 + 88704a_1^2a_0 - 29568a_0^2. \]
The fastest sequence \((u_1(n))_{n \geq 1}\) is obtained when the first two coefficients of this power series vanish. In this case

\begin{equation}
(4.45) \quad a_1 = \frac{3}{2}, \quad a_0 = \frac{5501}{7040}
\end{equation}

thus

\begin{equation}
(4.46) \quad u_1(n) - u_1(n + 1) = \frac{89684299}{3027763200\pi n^7} + O \left( \frac{1}{n^8} \right).
\end{equation}

Finally, by using Lemma 1, one has

\begin{equation}
(4.47) \quad \lim_{n \to \infty} n^6 u_1(n) = \frac{89684299}{18166579200\pi} = C_1.
\end{equation}

(Step 3) The second-correction. We let

\begin{equation}
(4.48) \quad \eta_2(n) = \frac{C_1}{\Phi_2(6; n)} = \frac{C_1}{n^6 + b_5 n + b_4 n^4 + b_3 n^3 + b_2 n^2 + b_1 n + b_0},
\end{equation}

and define

\begin{equation}
(4.49) \quad u_2(n) = G(n) - \eta_0(n) - \eta_1(n) - \eta_2(n).
\end{equation}

Thus

\begin{equation}
(4.50) \quad u_2(n) - u_2(n + 1) = (u_0(n) - u_0(n + 1)) - (\eta_1(n) + \eta_2(n) - \eta_1(n + 1) - \eta_2(n + 1)).
\end{equation}

We use (4.35) and (4.32) to expand \(u_0(n) - u_0(n + 1)\) into a power series as in terms of \(n^{-1}\). In addition, as mentioned already in Section 3, one can use a similar Mathematica program in Example 1 to find \(b_5, b_4, b_3, b_2, b_1\) and \(b_0\) in turn. Here we omit the details. We write

\begin{equation}
(4.51) \quad \Phi_1(2; n) = (n + \frac{3}{4})^2 + \frac{1541}{7040},
\end{equation}

\begin{equation}
(4.52) \quad \Phi_2(6; n) = (n + \frac{3}{4})^6 + \frac{1092000370209}{631377464960}(n + \frac{3}{4})^4 - \frac{111862508515629162375}{181198865117870921728}(n + \frac{3}{4})^2 + \frac{1824588073050833974528912179250963}{540823069619183303269309779804160}
\end{equation}

Then, by using Lemma 1 again, it is not very difficult for us to check the following assertion.

**Theorem 3.** Let \(c_0, C_2\) be defined by (1.15) and (4.50) respectively, and

\begin{equation}
(4.53) \quad u_2(n) := G(n) - \left( \frac{1}{\pi} \ln(n + \frac{3}{4}) + c_0 + \frac{11}{\Phi_1(2; n)} + \frac{89684299}{\Phi_2(6; n)} \right),
\end{equation}

where

\begin{equation}
(4.54) \quad \Phi_1(2; n) = (n + \frac{3}{4})^2 + \frac{1541}{7040},
\end{equation}

\begin{equation}
(4.55) \quad \Phi_2(6; n) = (n + \frac{3}{4})^6 + \frac{1092000370209}{631377464960}(n + \frac{3}{4})^4 - \frac{111862508515629162375}{181198865117870921728}(n + \frac{3}{4})^2 + \frac{1824588073050833974528912179250963}{540823069619183303269309779804160}
\end{equation}
Then we have

\begin{align*}
\lim_{n \to \infty} n^{15} (u_2(n) - u_2(n + 1)) &= 14C_2, \\
\lim_{n \to \infty} n^{14} u_2(n) &= C_2.
\end{align*}

Remark 8. It should be stressed that that a “good” initial-correction is very important for us to accelerate the convergence. In addition, one may study analogous question by choosing different initial-correction.

The following Theorem tells us how to improve (1.17) and (1.20).

**Theorem 4.** Let \( c_0 \) be defined by (1.15). Then for all integer \( n \geq 0 \), we have

\[
\frac{C_1}{(n + \frac{3}{2})^6} < G(n) - \frac{1}{\pi} \ln(n + \frac{3}{4}) - c_0 - \frac{11}{192\pi} \Phi_1(2; n) - \frac{11}{192\pi} \Phi_1(2; n + 1) < \frac{C_1}{(n + \frac{3}{2})^6},
\]

where \( C_1 = \frac{89684299}{18166579200\pi} \).

Remark 9. In fact, Theorem 4 implies that \( u_1(n) \) is a strictly decreasing function of \( n \).

**Proof.** Although the method used in this section is very similar to that in proof of Theorem 1, we would like to give a full proof for the sake of completeness. First, we can see that the inequalities (4.56) are true for \( n = 0 \). Hence, in the following we only need to prove that these inequalities are also true for \( n \geq 1 \). To this end, let

\[
u_1(n) = G(n) - \frac{1}{\pi} \ln(n + \frac{3}{4}) - c_0 - \frac{11}{192\pi} \Phi_1(2; n),
\]

it follows easily from (4.30)

\[
u_1(n) - \nu_1(n + 1) = -\frac{1}{\pi} q(n + 1) - \frac{1}{\pi} \ln(n + \frac{3}{4}) - \frac{11}{192\pi} \Phi_1(2; n) - \frac{11}{192\pi} \Phi_1(2; n + 1).
\]

Let

\[
f(x) = -\frac{1}{\pi} q_0(x + 1) - \frac{1}{\pi} \ln(x + \frac{3}{4}) - \frac{11}{192\pi} \Phi_1(2; x) + \frac{1}{\pi} \ln(x + \frac{7}{4}) + \frac{11}{192\pi} \Phi_1(2; x + 1),
\]

\[
g(x) = -\frac{1}{\pi} q_0(x + 1) - \frac{1}{\pi} \ln(x + \frac{3}{4}) - \frac{11}{192\pi} \Phi_1(2; x) + \frac{1}{\pi} \ln(x + \frac{7}{4}) + \frac{11}{192\pi} \Phi_1(2; x + 1).
\]

From (4.32) and (4.60), one has

\[
g(n) < \nu_1(n) - \nu_1(n + 1) < f(n).
\]
Firstly, we give the lower bound for $g(n)$, and the upper bound for $f(n)$, respectively. We set $D_1 = \frac{89684299432537600}{432537600}$.

By using the Mathematica software, we easily obtain

$$ -f'(x) - \frac{D_1}{\pi(x + 1)^8} = -\frac{1}{\pi} \Psi_1(21; n) \Psi_2(30; n) < 0. \tag{4.62} $$

Noting $f(\infty) = 0$ and utilizing (4.62) and Lemma 3, one has

$$ f(n) = -\int_n^\infty f'(x)dx < \int_n^\infty \frac{D_1}{\pi(x + 1)^8}dx = \frac{D_1}{7\pi(n + 1)^7}. \tag{4.63} $$

Similarly, we can check

$$ -g'(x) - \frac{D_1}{\pi(x + \frac{3}{2})^8} = \frac{1}{\pi} \Psi_3(19; n) \Psi_4(28; n) > 0. \tag{4.64} $$

Applying $g(\infty) = 0$ and (4.64), we obtain

$$ g(n) = -\int_n^\infty g'(x)dx > \int_n^\infty \frac{D_1}{\pi(x + \frac{3}{2})^8}dx = \frac{D_1}{7\pi(n + \frac{3}{2})^7}. \tag{4.65} $$

On the other hand, from $u_1(\infty) = 0$ and (4.63), we have

$$ u_1(n) = \sum_{m=n}^\infty (u_1(m) - u_1(m + 1)) < \sum_{m=n}^\infty \frac{D_1}{7\pi} \int_{m+\frac{1}{2}}^{m+\frac{3}{2}} \frac{1}{x^7}dx = \frac{D_1}{7\pi} \int_{n+\frac{1}{2}}^{n+\frac{3}{2}} \frac{1}{x^7}dx = \frac{D_1}{42\pi(n + \frac{1}{2})^6}. \tag{4.66} $$

Similarly, it follows from (4.65)

$$ u_1(n) = \sum_{m=n}^\infty (u_1(m) - u_1(m + 1)) > \sum_{m=n}^\infty \frac{D_1}{7\pi} \int_{m+\frac{1}{2}}^{m+\frac{5}{2}} \frac{1}{x^7}dx = \frac{D_1}{7\pi} \int_{n+\frac{1}{2}}^{n+\frac{5}{2}} \frac{1}{x^7}dx = \frac{D_1}{42\pi(n + \frac{1}{2})^6}. \tag{4.67} $$

This completes the proof of Theorem 4. \qed
References

[1] H. Alzer, Inequalities for the constants of Landau and Lebesgue, J. Comput. Appl. Math. 139(2002)215–230.

[2] B.C. Berndt, Ramanujan’s Notebooks, Part II, Springer-Verlag, 1999.

[3] R. P. Brent and P. Zimmermann, Modern computer arithmetic. Cambridge Monographs on Applied and Computational Mathematics, 18. Cambridge University Press, Cambridge, 2011. xvi+221 pp.

[4] L. Brutman, A sharp estimate of the Landau constants, J. Approx. Theory 34 (1982) 217–220.

[5] C.-P. Chen and J. Choi, Asymptotic expansions for the constants of Landau and Lebesgue, Advances in mathematics, 254(2014)622–641.

[6] C.-P. Chen, J. Choi, Inequalities and asymptotic expansions for the constants of Landau and Lebesgue, RGMIA Res. Rep. Collect. 17 (2014). Article 9, 11. pp. (Available at: http://rgmia.org/papers/v17/v17a09.pdf).

[7] C.-P. Chen, J. Choi, Inequalities and asymptotic expansions for the constants of Landau and Lebesgue, Preprint.

[8] C.-P. Chen, Approximation formulas for Landau’s constants, J. Math. Anal. Appl. 387 (2012) 916–919.

[9] C.-P. Chen, Sharp bounds for the Landau constants, Ramanujan J. 31 (2013) 301–313.

[10] C.-P. Chen, L. Li, Two accelerated approximations to the Euler-Mascheroni constant, Sci. Magana, 6 (2010) 161–164.

[11] C.P. Chen, C. Mortici, New sequence converging towards the Euler-Mascheroni constant, Comput. Math. Appl. 64 (2012) 391–398.

[12] D. Cvijović, J. Klinowski, Inequalities for the Landau constants, Math. Slovaca, 50 (2000) 159–164.

[13] D. Cvijović, H.M. Srivastava, Asymptotics of the Landau constants and their relationship with hypergeometric functions, Taiwanese J. Math. 13 (2009) 855–870.

[14] Mark W. Coffey and Jonathan Sondow, Rebuttal of Kowalenkos Paper As Concerns the Irrationality of Euler’s Constant γ, Acta Appl Math (2012) 121:1–3.

[15] T.P. Dence, J.B. Dence, A survey of Eulers constant, Math. Mag. 82 (2009) 255–265.

[16] D.W. DeTemple, A quicker convergence to Euler’s constant, Amer. Math. Monthly 100 (5) (1993) 468–470.
[17] L.P. Falaleev, Inequalities for the Landau constants, Sib. Math. J. 32 (1991) 896–897.

[18] I. Gavrea and M. Ivan, Optimal rate of convergence for sequences of a prescribed form. J. Math. Anal. Appl. 402 (2013), no. 1, 35–43.

[19] X. Gourdon and P. Sebah, Euler’s constant γ,
http://numbers.computation.free.fr/Constants/constants.html?5000

[20] X. Gourdon and P. Sebah, Collection of formulae for the Euler constant. http://numbers.computation.free.fr/Constants/Gamma/gammaFormulas.pdf

[21] H. Granath, On inequalities and asymptotic expansions for the Landau constants, J. Math. Anal. Appl. 386 (2012) 738–743.

[22] J. Havil, Gamma: Exploring Eulers Constant, Princeton University Press, Princeton, NJ, 2003.

[23] Jeffrey C. Lagarias, Euler’s constant: Euler’s work and modern developments. Bull. Amer. Math. Soc. (N.S.) 50 (2013), no. 4, 527–628.

[24] E. Landau, Abschätzung der Koeffizientensumme einer Potenzreihe, Arch. Math. Phys. 21 (42-50) (1913) 250–255.

[25] Dawei Lu, A new quicker sequence convergent to Euler’s constant, J. Number Theory, 136(2014), 320–329.

[26] Dawei Lu, Some quicker classes of sequences convergent to Euler’s constant. Appl. Math. Comput. 232 (2014), 172–177.

[27] C. Mortici, On new sequences converging towards the Euler-Mascheroni constant, Comput. Math. Appl. 59 (8) (2010) 2610–2614.

[28] C. Mortici, Product approximations via asymptotic integration, Amer. Math. Monthly, 117 (5) (2010) 434–441.

[29] C. Mortici, New approximations of the gamma function in terms of the digamma function, Applied Mathematics Letters, 23 (2010) 97–100.

[30] C. Mortici, Sharp bounds of the Landau constants, Math. Comp. 80 (2011) 1011–1018.

[31] C. Mortici and C.-P. Chen, On the harmonic number expansion by Ramanujan, J. Inequal. Appl. 2013, 2013:222, 10 pp.

[32] G. Nemes, A. Nemes, A note on the Landau constants, Appl. Math. Comput. 217 (2011) 8543–8546.

[33] G. Nemes, Proofs of two conjectures on the Landau constants, J. Math. Anal. Appl. 388 (2012) 838–844.
[34] E.C. Popa, Note of the constants of Landau, Gen. Math. 18 (2010) 113–117.
[35] E.C. Popa and N.-A. Secelean, On some inequality for the Landau constants, Taiwanese J. Math. 15 (2011) 1457–1462.
[36] J. Wallis, Arithmetica Infinitorum, Oxford, England, 1656; Facsimile of relevant pages available in: J.A. Stedall, Catching Proteus: The collaborations of Wallis and Brouncker. I. Squaring the circle, Notes and Records Roy. Soc. London 54 (3) (2000) 293–316.
[37] G.N. Watson, The constants of Landau and Lebesgue, Quart. J. Math. Oxford Ser. 1 (1930) 310–318.
[38] R. Wong, Asymptotic Approximations of Integrals, Classics in Applied Mathematics, 34. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001. xviii+543 pp. ISBN: 0-89871-497-4
[39] Hongmin Xu and Xu You, Continued fraction inequalities for Euler-Mascheroni constant, J. Inequal. Appl. to appear. (Available at: http://arxiv.org/abs/1407.3865v1).
[40] Shijun Yang, On an open problem of Chen and Mortici concerning the Euler-Mascheroni constant, J. Math. Anal. Appl. 396 (2012) 689–693.
[41] D. Zhao, Some sharp estimates of the constants of Landau and Lebesgue, J. Math. Anal. Appl. 349 (2009) 68–73.

Xiaodong Cao
Department of Mathematics and Physics,
Beijing Institute of Petro-Chemical Technology,
Beijing, 102617, P. R. China
e-mail: caoxiaodong@bipt.edu.cn

Hongmin Xu
Department of Mathematics and Physics,
Beijing Institute of Petro-Chemical Technology,
Beijing, 102617, P. R. China
e-mail: xuhongmin@bipt.edu.cn

Xu You
Department of Mathematics and Physics,
Beijing Institute of Petro-Chemical Technology,
Beijing, 102617, P. R. China
e-mail: youxu@bipt.edu.cn