Characterisations of purity in a locally finitely presented additive category: A short functorial proof

Samuel Dean

April 25, 2023

Abstract

In this expository article, we will give an efficient functorial proof of the equivalence of various characterisations of purity in a finitely accessible additive category $C$. The complications of the proofs for specific choices of $C$ are contained in the description of fp-injective and injective objects in $(fpC, Ab)$, the category of additive functors $fpC \to Ab$. For example, the equivalence of many characterisations of purity in a module category $A$-Mod is a simple corollary of what we will prove here, since we know which objects are fp-injective, and which objects are injective, in $(A$-mod, $Ab)$.

1 Extending functors over direct limits

Acknowledgements

I thank Sergio Estrada and Pedro Guil Asensio for inviting me to visit the University of Murcia, where I wrote this, and for helpful discussions. Nothing herein deserves to be called original, but it is meant to be helpful.

All categories and functors mentioned in this paper are additive. We assume some background on locally finitely presented categories, which can be gotten from [4]. We write $Ab$ for the category of abelian groups and, for a small category $A$, we write $(A, Ab)$ for the category of functors $A \to Ab$, and $Flat(A, Ab)$ for the category of flat functors $A \to Ab$. I will write direct limit to mean the same as directed colimit.

Definition 1. Let $C$ be a category with direct limits. An object $A \in C$ is finitely presented if the representable functor $C(A, -) : C \to Ab$ preserves direct limits. We write $fpC$ for the full subcategory of finitely presented objects in $C$.

Definition 2. Let $C$ be a category with direct limits. We say that $C$ is locally finitely presented if $fpC$ is skeletally small and every object is a direct limit of finitely presented objects.

Theorem 1 ([4]). For any locally finitely presented category $C$, the functor $C \to ((fpC)^{op}, Ab) : C \mapsto C(-, C)_{|(fpC)^{op}}$ is fully faithful and restricts to an equivalence $C \simeq Flat((fpC)^{op}, Ab)$.

We will use tensor products of functors. For this, there are references such as [7] and [9], but we offer the following definition.
Definition 3. Let $A$ be a small category. For functors $G : A^{op} \to \text{Ab}$ and $F : A \to \text{Ab}$, the tensor product $G \otimes_A F$ is an abelian group given by the coend formula (see [5] for coends)

$$G \otimes_A F = \int^{A \in A} (GA) \otimes_Z (FA).$$

Lemma 1. Let $A$ be a small category. For any object $A \in A$ and any functor $F : A \to \text{Ab}$, there is an isomorphism

$$A(-, A) \otimes_A F \cong FA$$

which is natural in $F$ and $A$.

Proof. See [7, Proposition 1.1] or take this as an exercise in the calculus of coends. □

Definition 4. Let $C$ be a locally finitely presented category. For any functor $F : \text{fp}C \to \text{Ab}$, define $\overline{F} : C \to \text{Ab}$ by

$$\overline{F}C = C(-, C)|_{(\text{fp}C)^{op}} \otimes_{\text{fp}C} F$$

for any $C \in C$.

Theorem 2. Let $C$ be a locally finitely presented category. For any functor $F : \text{fp}C \to \text{Ab}$, $\overline{F}$ preserves direct limits and there is an isomorphism $\overline{F}|_{\text{fp}C} \cong F$ which is natural in $F$. If $E : C \to \text{Ab}$ preserves direct limits and $E|_{\text{fp}C} \cong F$ then $E \cong \overline{F}$.

Proof. Variations of this statement appear in many places, but we will give a proof, for the sake of self-containment, which similar to that at [5] 3.16. See [2] for a very simple argument when $F$ is finitely presented.

The property that $\overline{F}$ preserves direct limits and restricts to $F$ on $\text{fp}C$ follows directly from the definition of $\overline{F}$ and Lemma 1.

For such a functor $E : C \to \text{Ab}$, let $\alpha : F \to E|_{\text{fp}C}$ be an isomorphism and, for each $C \in C$, assemble the morphisms

$$C(A, C) \otimes_Z FA \to EC : f \otimes x \mapsto ((Ef)\alpha_A)x \quad (A \in \text{fp}C)$$

each of which is natural in $C$, into a morphism

$$\overline{F}C = C(-, C)|_{(\text{fp}C)^{op}} \otimes F \to EC$$

which is natural in $C$. This morphism is an isomorphism when $C \in \text{fp}C$. Since both $\overline{F}$ and $E$ preserve direct limits, it follows that this morphism is an isomorphism for any $C \in C$. □

2 Purity in a locally finitely presented category

Definition 5. Let $C$ be a locally finitely presented category. A sequence

$$0 \to A \to B \to C \to 0$$

in $C$ is pure-exact if and only if the induced sequence

$$0 \to C(-, A)|_{(\text{fp}C)^{op}} \to C(-, B)|_{(\text{fp}C)^{op}} \to C(-, C)|_{(\text{fp}C)^{op}} \to 0$$

is exact.

Definition 6. For a functor $F : A \to \text{Ab}$, we define its dual to be the functor $F^* : A^{op} \to \text{Ab}$ defined by $F^*A = \text{Hom}_Z(FA, \mathbb{Q}/\mathbb{Z})$. 

2
If the reader is working in a slightly different context, with a $k$-linear locally finitely presented category, and prefers to replace $\mathbb{Z}$ by $k$ and $\mathbb{Q}/\mathbb{Z}$ by some injective cogenerator in $k$-Mod, then they may do so. The following theorem will still hold.

**Definition 7.** A functor $F : \mathcal{A} \to \text{Ab}$ is said to be **fp-injective** if $\text{Ext}^1(\neg, F)|_{(\text{fp}(\mathcal{A}, \text{Ab}))^{\text{op}}} = 0$. We write $\text{Fpinj}(\mathcal{A}, \text{Ab})$ for the category of all fp-injective functors $\mathcal{A} \to \text{Ab}$ and $\text{Inj}(\mathcal{A}, \text{Ab})$ for the category of all injective functors $\mathcal{A} \to \text{Ab}$.

**Remark 1.** Let $\mathcal{C}$ be a locally finitely presented category. It is equivalent to $\text{Flat}((\text{fp}\mathcal{C})^{\text{op}}, \text{Ab})$, which is closed under extensions in $((\text{fp}\mathcal{C})^{\text{op}}, \text{Ab})$. By [3, Lemma 10.20], this implies that any exact structure on $((\text{fp}\mathcal{C})^{\text{op}}, \text{Ab})$ restricts to an exact structure on $\text{Flat}((\text{fp}\mathcal{C})^{\text{op}}, \text{Ab})$. The abelian exact structure $((\text{fp}\mathcal{C})^{\text{op}}, \text{Ab})$ restricts to the exact structure on $\text{Flat}((\text{fp}\mathcal{C})^{\text{op}}, \text{Ab})$ which corresponds to the class of pure-exact sequences on $\mathcal{C}$. Therefore, the pure-exact sequences on $\mathcal{C}$ form an exact structure on $\mathcal{C}$. In particular, for any pure-exact sequence

$$0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$$

in $\mathcal{C}$, the pullback of $g$ exists along any morphism to $C$, as does the pushout of $f$ along any morphism from $A$.

**Theorem 3.** Let $\mathcal{C}$ be a locally finitely presented category. For a sequence of maps

$$0 \to A \to B \to C \to 0$$

in $\mathcal{C}$, the following are equivalent.

1. It is pure-exact.
2. It is a direct limit of split exact sequences.
3. For any $F \in \text{fp}(\text{fp}\mathcal{C}, \text{Ab})$, the induced sequence

$$0 \to \overrightarrow{F}A \to \overrightarrow{F}B \to \overrightarrow{F}C \to 0$$

is exact in $\text{Ab}$.
4. For any $F \in (\text{fp}\mathcal{C}, \text{Ab})$, the induced sequence

$$0 \to \overrightarrow{F}A \to \overrightarrow{F}B \to \overrightarrow{F}C \to 0$$

is exact in $\text{Ab}$.
5. For any $F \in \text{Fpinj}(\text{fp}\mathcal{C}, \text{Ab})$, the induced sequence

$$0 \to \overrightarrow{F}A \to \overrightarrow{F}B \to \overrightarrow{F}C \to 0$$

is exact in $\text{Ab}$.
6. For any $F \in \text{Inj}(\text{fp}\mathcal{C}, \text{Ab})$, the induced sequence

$$0 \to \overrightarrow{F}A \to \overrightarrow{F}B \to \overrightarrow{F}C \to 0$$

is exact in $\text{Ab}$.
7. The induced sequence

$$0 \to C(-, C)|_{(\text{fp}\mathcal{C})^{\text{op}}} \to C(-, B)|_{(\text{fp}\mathcal{C})^{\text{op}}} \to C(-, A)|_{(\text{fp}\mathcal{C})^{\text{op}}} \to 0$$

is split exact in $(\text{fp}\mathcal{C}, \text{Ab})$. 3
Proof. 1 implies 2: This argument is well-known and standard, but we give it for the sake of self-containment. Express \( C \) as a direct limit of finitely presented objects, \( C = \lim_{\lambda \in \Lambda} C_\lambda \). Since pure-exact sequences form an exact structure on \( C \), the pullback of any pure epimorphism along any other morphism exists and is a pure epimorphism. Take the pullback of our sequence along the morphisms

\[
C_\lambda \to C.
\]

We obtain a directed system of pure-exact sequences

\[
0 \to A \to B_\lambda \to C_\lambda \to 0,
\]

each of which must be split since \( C_\lambda \) is finitely presented. The direct limit of this sequence is our original sequence.

2 implies 3: Obvious: Every functor preserves split exact sequences, and the direct limit of any split exact sequence is exact.

3 implies 4: For any object \( D \in C \), the functor \((\text{fp}C, \text{Ab}) \to \text{Ab} : F \mapsto \overrightarrow{F}D\) clearly preserves direct limits because it is a tensor product. By expressing \( F \) as a direct limit of finitely presented functors, \( F = \lim_{\lambda \in \Lambda} F_\lambda \), we obtain the sequence

\[
0 \to \overrightarrow{F}A \to \overrightarrow{F}B \to \overrightarrow{F}C \to 0
\]
as a direct limit of pure-exact sequence

\[
0 \to \overrightarrow{F}_\lambda A \to \overrightarrow{F}_\lambda B \to \overrightarrow{F}_\lambda C \to 0,
\]

which is exact since since direct limits are exact.

4 implies 5: Obvious.

5 implies 6: Obvious.

6 implies 7: To show that our sequence is split, we need only show that, for any \( F \in \text{Inj}(\text{fp}C, \text{Ab}) \), the sequence

\[
0 \to (F, C(-, C))_{\text{fp}C}^* \to (F, C(-, B))_{\text{fp}C}^* \to (F, C(-, A))_{\text{fp}C}^* \to 0
\]
is exact. The reason for this is that, since it is the dual of a flat functor, \((C(-, A))_{\text{fp}C}^*\) is injective (there is a standard argument for this – see e.g. [1, 19.14] for something similar), and therefore we may substitute \( F = C(-, A)^* \) to obtain a splitting.

Indeed, if \( F \in \text{Inj}(\text{fp}C, \text{Ab}) \) then, by the hom-tensor duality, this sequence is isomorphic to

\[
0 \to (C(-, C))_{\text{fp}C}^* \otimes_{\text{fp}C} F^* \to (C(-, B))_{\text{fp}C}^* \otimes_{\text{fp}C} F^* \to (C(-, A))_{\text{fp}C}^* \otimes_{\text{fp}C} F^* \to 0,
\]

which is equal to

\[
0 \to \overrightarrow{F}B^* \to \overrightarrow{F}A^* \to \overrightarrow{F}A^* \to 0
\]
which is exact by hypothesis.

7 implies 1: Easy since \( Q/Z \) is an injective cogenerator. \(\square\)

Corollary 1 (Well-known). For a pre-additive category \( A \) and a sequence

\[
0 \to L \to M \to N \to 0
\]
in \( A\text{-Mod} = (A, \text{Ab}) \), the following are equivalent.

1. It is pure-exact.
2. It is a direct limit of split exact sequences.
3. For any pp-pair $\varphi/\psi$ in the language of left $A$-modules, the sequence
\[ 0 \to \varphi L/\psi L \to \varphi M/\psi M \to \varphi N/\psi N \to 0 \]
is exact.

4. For any $Y \in \text{Mod-}A$, the induced sequence
\[ 0 \to Y \otimes_A L \to Y \otimes_A M \to Y \otimes_A N \to 0 \]
is exact. (This condition need only be checked when $Y$ is finitely presented.)

5. For any pure-injective $Y \in \text{Mod-}A$, the induced sequence
\[ 0 \to Y \otimes_A L \to Y \otimes_A M \to Y \otimes_A N \to 0 \]
is exact.

6. The induced sequence
\[ 0 \to N^* \to M^* \to L^* \to 0 \]
is split exact in $\text{Mod-}A$.

Proof. A functor $F \in (\text{A-mod}, \text{Ab})$ is:

- finitely presented if and only if it comes from a pp-pair [10] Section 10.2.5].
- fp-injective if and only if it is of the form $Y \otimes_A -$ for some $Y \in \text{Mod-}A$ by [10] Theorem 12.1.6].
- injective if and only if it is of the form $Y \otimes_A -$ for some pure-injective $Y \in \text{Mod-}A$ by [10] Theorem 12.1.6] (uses the fact that 1 is equivalent to 4).

For each $X \in A$-Mod, there is an isomorphism $(-,X)|^*_\text{A-mod} \cong X^* \otimes_A -$ which is natural in $X$ [6, 3.2.11]. Therefore,
\[ 0 \to (-,N)|^*_\text{A-mod} \to (-,M)|^*_\text{A-mod} \to (-,L)|^*_\text{A-mod} \to \]
is split exact if and only if
\[ 0 \to N^* \otimes_A -|_\text{A-mod} \to M^* \otimes_A -|_\text{A-mod} \to L^* \otimes_A -|_\text{A-mod} \to 0 \]
is split exact. Since $\text{Mod-}A \to (\text{A-mod}, \text{Ab}) : Y \mapsto Y \otimes -|_{\text{A-mod}}$ is fully faithful, this is equivalent to 6. \qed

References

[1] Frank W. Anderson and Kent R. Fuller, *Rings and Categories of Modules*, 2 ed., Graduate Texts in Mathematics, vol. 13, Springer-Verlag, 1974.

[2] Maurice Auslander, *Large modules over artin algebras*, Algebra, Topology, and Category Theory (Alex Heller and Myles Tierney, eds.), Academic Press, 1976, pp. 1–17.

[3] Theo Bühler, *Exact categories*, Expositiones Mathematicae 28 (2010), no. 1, 1–69.

[4] William Crawley-Boevey, *Locally finitely presented additive categories*, Communications in Algebra 22 (1994), no. 5, 1641–1674.

[5] Samuel Dean, *Duality and contravariant functors in the representation theory of artin algebras*, Journal of Algebra and Its Applications 18 (2019), no. 06, 1950111.
[6] Edgar E. Enochs and Overtoun M. G. Jenda, Relative Homological Algebra, De Gruyter, Berlin, New York, 2000.

[7] Janet L Fisher, The tensor product of functors; satellites; and derived functors, Journal of Algebra 8 (1968), no. 3, 277–294.

[8] Saunders Mac Lane, Categories For the Working Mathematician, 2 ed., Springer, 1971.

[9] J. Fisher Palmquist and David C. Newell, Bifunctors and adjoint pairs, Transactions of the American Mathematical Society 155 (1971), no. 2, 293–303.

[10] Mike Prest, Purity, Spectra and Localisation, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2009.