Cluster Analysis of the Ising Model and Universal Finite-Size Scaling

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Abstract

The recent progress in the study of finite-size scaling (FSS) properties of the Ising model is briefly reviewed. We calculate the universal FSS functions for the Binder parameter $g$ and the magnetization distribution function $p(m)$ for the Ising model on $L_1 \times L_2$ two-dimensional lattices with tilted boundary conditions. We show that the FSS functions are universal for fixed sets of the aspect ratio $L_1/L_2$ and the tilt parameter. We also study the percolating properties of the Ising model, giving attention to the effects of the aspect ratio of finite systems. We elucidate the origin of the complex structure of $p(m)$ for the system with large aspect ratio by the multiple-percolating-cluster argument.

Key words: Ising model; Percolation; Finite-size scaling; Universality

1 Introduction

In the study of critical phenomena, universality and scaling are two important concepts [1]. Finite-size scaling (FSS) has been increasingly important [2,3], due to the progress in the theoretical understanding of finite-size effects, and the application of FSS in the analysis of simulative results. Privman and Fisher [4] first proposed the idea of universal FSS functions with nonuniversal metric factors in 1984, but only recently, the universality of FSS functions has received much attention [5–14].

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The first strong support for the idea of universal FSS functions was reported for critical phenomena in geometric percolation models [5]. Hu, Lin and Chen [5] applied a histogram Monte Carlo simulation method [15] to calculate the existence probability $E_p$ and the percolation probability $P$ of site and bond percolation on finite square (sq), plane triangular (pt), and honeycomb (hc) lattices, whose aspect ratios approximately have the relative proportions $1 : \sqrt{3}/2 : \sqrt{3}$ [16]. They found that the six percolation models have very nice universal FSS for $E_p$ and $P$ near the critical points of the percolation models with using nonuniversal metric factors. Using Monte Carlo simulation, Okabe and Kikuchi [7] found universal FSS for the Binder parameter $g$ [17] and magnetization distribution functions $p(m)$ of the Ising model on planar lattices; Wang and Hu discussed universal FSS for dynamical critical phenomena of the Ising model [8].

It is to be noted that universal FSS functions depend on boundary conditions and the shape of finite systems. The importance of the number of percolating clusters for anisotropic systems was pointed out by Hu and Lin [6,10]. The probability for the appearance of $n$ percolating clusters $W_n$ was investigated; the average number of percolating clusters increases linearly with aspect ratios of the lattices at the critical point. Pioneered by the work of Hu and Lin [6,10], the number of percolating clusters has captured current interest [18–20]. Moreover, the “nonuniversal scaling” of the low-temperature conductance peak heights for Corbino disks in the quantum Hall effect was discussed in terms of the number of the percolating clusters [21]. As for the effect of boundary conditions, the difference of the universal FSS functions between the systems with periodic boundary conditions and those with free boundary conditions was shown both in the percolation problem [5] and the Ising model [7].

Another interesting subject is the relation of the geometric percolation problem to the phase transition problem of the spin models. The understanding of critical phenomena of spin models in terms of geometric concepts has been enhanced by the cluster formalism introduced by Kasteleyn and Fortuin [22]. The problem of the thermal phase transition is mapped onto the geometric percolation problem in the cluster formalism [22–24]. The connection between the bond-correlated percolation model (BCPM) and the $q$-state Potts model was elucidated by Hu [24]; an equation for the spontaneous magnetization was formally derived by including the magnetic field in the subgraph expansion for the partition function of the $q$-state Potts model. The cluster formalism has been applied to the cluster update algorithms [25,26] in order to overcome the slow dynamics in the Monte Carlo simulation. It is important to study multiple percolating clusters for the spin models of anisotropic systems; especially, the aspect ratio dependence of the magnetization distribution function of the spin system is interesting to discuss in terms of the cluster formalism.
Fig. 1. $L_1 \times L_2$ square lattice with tilt parameter $c$. Here, $L_1 = 8$, $L_2 = 4$ and $c = 1/4$. The $i$-th site of the first row is identical with the mod($i + cL_1$, $L_1$)-th site in the last row. The left-most site and the right-most site on the same horizontal line are identical.

In the first part of this paper, we study the universal FSS functions of the Ising model with tilted boundary conditions. We find that the FSS functions are universal for fixed sets of aspect ratio $a$ and tilt parameter $c$. This part of work has been partially reported elsewhere [13]. We also discuss the relevance to the modular transformation [27]. In the second part we study the percolating property of the Ising model based on the connection between the BCPM and the Ising model. Special attention is paid to the number of percolating clusters and its dependence on the aspect ratio of the lattice. We discuss the origin of the complex structure of the magnetization distribution function $p(m)$ for the system with large aspect ratio. The details of this part have been given in a separate paper [14].

2 Ising model with tilted boundary conditions

We deal with the two-dimensional (2D) Ising model on $L_1 \times L_2$ square lattices with periodic boundary conditions in the horizontal $L_1$ direction and tilted boundary conditions in the vertical $L_2$ direction such that the $i$-th site in the first row is connected with the mod($i + cL_1$, $L_1$)-th site in the $L_2$ row of the lattice, where $1 \leq i \leq L_1$; see Fig. 1 for an example. We mainly calculate the universal FSS functions for the Binder parameter [17]

$$g = \frac{1}{2} (3 - \frac{\langle m^4 \rangle}{\langle m^2 \rangle^2})$$  

(1)

and the magnetization distribution function $p(m)$, using the Monte Carlo simulation method.

We have found very good FSS behavior for $g$ as a function of $(T - T_c)L^{1/\nu}$ and $p(m)L^{-\beta/\nu}$ as a function of $mL^{\beta/\nu}$ [13]. Here, $L = (L_1L_2)^{1/2}$, $T_c = 2.269 \cdots$ in units of the exchange coupling $J$, and the critical exponents are those of the 2D Ising exact values; $\nu = 1$ and $\beta = 1/8$. Moreover, FSS functions of $g$ and
Fig. 2. $g$ at $T = T_c$ as a function of $c$ for several values of $a = L_1/L_2$.

Fig. 3. $p(m)L^{-1/8}$ at $T = T_c$ as a function of $mL^{1/8}$ for $(a, c) = (5,0.1), (4,0), (5,0.4), \text{ and } (1,0)$.

$p(m)$ have been shown to depend strongly on the tilt parameter $c$. In order to see the tilt parameter dependence clearly, we plot the $c$ dependence of $g$ at $T = T_c$ for several values of the aspect ratio $a = L_1/L_2$ in Fig. 2.

We find a combination of $a$ and $c$ which gives universal FSS functions. As an example, we show the data for $p(m)$ at $T = T_c$ for $(a, c)$ of $(5,0.1), (4,0), (5,0.4), \text{ and } (1,0)$ together in Fig. 3. The figure shows that the pair $(5,0.1)$ and $(4,0)$ and the pair $(5,0.4)$ and $(1,0)$ share the universal FSS functions. There are many such combinations that share universal FSS functions.
Using the FSS argument in the momentum space, we have shown [13] that

\[ A = a/(c^2a^2 + 1) \]  

(2)

is an invariant, and can be regarded as the effective aspect ratio. It is easy to check that the pairs of \((a, c)\), which have the universal FSS functions shown in Fig. 3, have the same value of \(a/(c^2a^2 + 1)\). The invariant, Eq. (2), corresponds to the relative ratio of two primitive vectors \(k_1\) and \(k_2\) in the momentum space. More precisely, we may find other expressions than Eq. (2) for the invariant in case of some values of \(a\) and \(c\); the details will be given elsewhere.

It is interesting to discuss this problem in terms of the modular (conformal) transformation. According to Cardy [27], the shape of the 2D lattice may be represented by the imaginary number

\[ z = 1/a + i\, c. \]  

(3)

Then, Cardy asserted that the partition function becomes invariant under the transformations

\[ z \rightarrow z + i \]  

(4)

and

\[ z \rightarrow 1/z, \]  

(5)

in the limit that the system size becomes infinite. We have another invariant transformation

\[ z \rightarrow z^*, \]  

(6)

which corresponds to the fact that we can confine \(c\) to the interval of \(0 \leq c \leq 1/2\). Starting from the recurrence relation

\[ z_{n+1} = \frac{1}{z_n + i} + i, \]  

(7)

we can easily show that Eq. (2) becomes invariant. What we found in the Monte Carlo simulation is that the FSS functions are universal under the modular transformations based on the imaginary-number representation, Eq. (3). The detailed analysis of the modular transformation is now in progress.
We make a comment here that Ziff, Lorentz and Kleban [28] have recently studied the universal excess cluster numbers for percolation on lattices with tilted (twisted) boundary conditions. The related mathematical problems are also of interest to mathematical physicists.

3 Cluster analysis of the Ising model

Here, we study the FSS properties of the percolating clusters for the 2D Ising model based on the connection between the BCPM and the Ising model. The partition function of the Ising model, generally that of the $q$-state Potts model, can be expressed by the BCPM [22–24]. The essential point is that the problem of the Ising model is mapped to the percolation problem with the bond concentration of

$$p = 1 - e^{-2J/T},$$

where $J$ is the nearest-neighbor exchange coupling.

We perform the Monte Carlo simulation of the Ising model on the $L_1 \times L_2$ sq lattice with the periodic boundary conditions. We denote the aspect ratio as $a = L_1/L_2$ again. For the assignment of a bond-percolating cluster, we consider free and periodic boundary conditions in the vertical and horizontal directions, respectively; that is, a cluster which extends from the top row to the bottom row is a percolating cluster.

We calculate the probability for the appearance of $n$ percolating clusters $W_n$ for anisotropic lattices [6,10]. This quantity is related to the existence probability $E_p$ as

$$\sum_{n=1}^{\infty} W_n = 1 - W_0 = E_p.$$  

In Fig. 4, we show the FSS plot of $W_n$ as a function of $tL^{1/\nu}$, where $t = (T - T_c)/T_c$. The closed and open marks are the data for $144 \times 36$ and for $288 \times 72$, respectively. The aspect ratio $a$ is 4. We see interesting behavior for $W_1$; that is, from the high-temperature side $W_1$ increases as the temperature approaches the critical temperature, but decreases near $T_c$. Moreover, $W_1$ increases again for low temperatures, and finally approaches 1. This is the same behavior observed for the geometric percolation problem [6,10].

The average value $\langle n \rangle = \sum_n n W_n$ gives the direct measure of number of percolating clusters. To investigate the aspect ratio dependence, we give the tem-
Fig. 4. FSS plot of $W_n$ ($n = 0, 1, 2, 3$) for the sq lattice of $144 \times 36$ (closed marks) and $288 \times 72$ (open marks), where $t = (T - T_c)/T_c$.

Fig. 5. Temperature dependence of $\langle n \rangle$ for various aspect ratios $a = L_1/L_2$; $a = 8, 7, 6, 5, 4, 3, 2, 1$ from top to bottom.

Temperature dependence of $\langle n \rangle$ for various aspect ratios $a = L_1/L_2$ from 1 to 8 in Fig. 5. For high temperatures, $\langle n \rangle$ is small because there is no order. Then, $\langle n \rangle$ takes a maximum value near $T_c$, and decreases to 1 as $T$ lowers. There is only a single percolating cluster at low enough temperature. A systematic $a$ dependence is observed for the behavior of $\langle n \rangle$.

It is known that the magnetization distribution function $p(m)$ depends on the shape of the finite systems. In Fig. 3, we showed that the FSS functions for
\(a = 4\) and \(a = 1\) are quite different. For the system with the aspect ratio \(a = 4\), the total distribution function \(p(m)\) at \(T = T_c\) has a broad peak centered at \(m = 0\) in addition to two peaks of plus and minus \(m\); this is contrast to the case of \(a = 1\) where \(p(m)\) has only two distinct peaks of plus and minus \(m\). It is quite interesting to study this dependence on the aspect ratio by means of the cluster analysis. We may decompose \(p(m)\) by the number of percolating clusters as

\[
p(m) = \sum_{n=0}^{\infty} p_n(m).
\]

(10)

It should be noted that we have the relation,

\[
\int_{-1}^{1} p_n(m)dm = W_n, \quad (n = 0, \ldots, \infty),
\]

(11)

to connect \(p_n(m)\) and \(W_n\). By analyzing the FSS plot of \(p_n(m)\) at \(T = T_c\), we found that a broad peak centered at \(m = 0\) mainly comes from \(p_2(m)\) for \(a = 4\) [14]. There are two types of Ising clusters, that is, the clusters with up spins or the clusters with down spins. Therefore, if there are many percolating clusters, the combination of the percolating clusters with up spins and those with down spins gives the contribution to the broad peak around \(m = 0\) in \(p(m)\). Examples of snapshots of the Ising system with two percolating clusters are presented in Fig. 6. In case of (a), both percolating clusters are up; in contrast, in case of (b), one percolation cluster is up and another percolating cluster is down. It is known that the Binder parameter, Eq. (1), at the critical point has an aspect-ratio dependence [29]. The origin of such a dependence can be attributed to the structure of multiple percolating clusters.

In order to check the universal FSS, we have also made the calculation for the Ising model on the \(pt\) and \(hc\) lattices, whose aspect ratios are approximately \(4 \times \sqrt{3}/2\) and \(4 \times \sqrt{3}\), respectively. We have obtained very good universal FSS for the temperature dependence of \(W_n\) and other quantities. The details of the calculation have been reported in Ref. [14].

### 4 Summary and discussions

We have studied the universal FSS functions for the Ising model paying attention to the effects of the boundary conditions and the aspect ratio. For the system with the tilted boundary conditions we have found a combination of the aspect ratio \(a\) and the tilt parameter \(c\) which gives universal FSS func-
Fig. 6. Examples of snapshots of the Ising system with the aspect ratio $a = 4$. Up and down spins are represented by open and closed circles, and active bonds are represented by solid line. Percolating clusters are distinguished by shaded area. There are two percolating clusters with up spins in (a), whereas one percolating cluster is up and the other is down in (b).

We have discussed the relevance to the modular transformation in 2D systems [27].

We have also studied the FSS for the percolating properties of the Ising model based on the connection between the BCPM and the Ising model. Investigating the multiple percolating clusters for the system with large aspect ratio, we have clarified the origin of the complex structure of $p(m)$ near $T_c$. This is due to the combination of the clusters with up spins and those with down spins.

The extension of the present study to higher-dimensional systems is quite important. Conformal invariance plays a role in 2D systems [30], but is not so powerful for three-dimensional (3D) systems as for 2D ones. We have studied the FSS functions for anisotropic 3D Ising models of size $L_1 \times L_1 \times aL_1$ by Monte Carlo simulations [31]. We have observed the change of the FSS functions for $p(m)$ at $T_c$ as a function of $a$, which is the same situation as 2D systems.

Another interesting problem is the application of the present study to random spin systems. The study of the percolating properties of the diluted Ising model, especially the crossover from the percolation fixed point to the Ising fixed point, is now in progress.
Acknowledgments

We would like to thank T. Kawakatsu, N. Hatano and N. Kawashima for valuable discussions, and the Supercomputer Center of the ISSP, University of Tokyo, for providing the computing facilities. This work was supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Science, Sports and Culture, Japan and by the National Science Council of the Republic of China (Taiwan) under grant numbers NSC 88-2112-M-001-011.

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