Hitting probabilities for fractional Brownian motion with deterministic drift

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Abstract
Let $B^H$ be a $d$-dimensional fractional Brownian motion with Hurst index $H \in (0, 1)$, $f : [0, 1] \to \mathbb{R}^d$ a Borel function, and $E \subset [0, 1]$, $F \subset \mathbb{R}^d$ are given Borel sets. The focus of this paper is on hitting probabilities of the fractional Brownian motion $B^H$ with the deterministic drift $f$. It aims to highlight the role of the regularity properties of the drift $f$ as well as that of the dimension of $E$ in determining the upper and lower bounds of $P\{(B^H + f)(E) \cap F \neq \emptyset\}$ for $F$ a subset of $\mathbb{R}^d$ and also for $F$ a singleton.

Keywords: Fractional Brownian motion, Hitting probabilities, Capacity, Hausdorff measure

Mathematics Subject Classification: 62134, 60J45, 60G17, 28A78

1 Introduction

The hitting probability describes the probability that a given process will ever reach some state or set of states $F$. To find upper and lower bounds for the hitting probabilities in terms of the Hausdorff measure and the capacity of the set $F$, is a fundamental problem in probabilistic potential theory. For $d$-dimensional Brownian motion $B$ the probability that a path, will ever visit a given set $F \subset \mathbb{R}^d$, is classically estimated using the Newtonian capacity of $F$. Kakutani [15] was the first to establish this result linking capacities and hitting probabilities for Brownian motion. Precisely, he showed that, for $(d \geq 3)$, a compact set $F$ is hit with positive probability by $B$ if and only if $F$ has positive Newtonian capacity. Since then, considerable efforts have been carried out to establish a series of extensions to other processes. This has given rise to a large and rapidly growing body of scientific literature on the subject. To cite a few examples, we refer to Xiao [30] for developments on hitting probabilities of stationary Gaussian random fields and fractional Brownian motion; to Pruitt and Taylor [26] and Khoshnevisan [16] for hitting probabilities results for general stable processes and Lévy processes; to Khoshnevisan and Shi [18] for hitting probabilities of the Brownian sheet; to Dalang and Nualart [7] for hitting probabilities for the solution of a system of nonlinear hyperbolic stochastic partial differential equations; to Dalang, Khoshnevisan and Nualart [8] and [9], for hitting probabilities for the solution of a non-linear stochastic heat equation with additive

1Supported by National Center for Scientific and Technological Research (CNRST)
and multiplicative noise respectively; to Xiao [31] Bierné, Lacaux and Xiao [4] for hitting probabilities of Gaussian random fields. Finally, we refer to Khoshnevisan [17] for more information on the latter as well as on potential theory of random fields.

It should be noted that the above characterization is not common to all the processes and this is generally due to the dependence structures thereof leading to an upper and lower bounds on hitting probabilities in terms of capacity and Hausdorff measure. In this context, Chen and Xiao [6] improved the results established by Xiao (Theorem 7.6 [31]) and by Bierné, Lacaux and Xiao (Theorem 2.1 [4]) on hitting probabilities of the $\mathbb{R}^d$-valued Gaussian random field $X$ satisfying conditions $(C_1)$ and $(C_2)$, see Xiao [31] for precise definition, through the following

$$c^{-1}C_{\rho_H,d}(E \times F) \leq \mathbb{P}\{X(E) \cap F \neq \emptyset\} \leq c\mathcal{H}_{\rho_H}^d(E \times F),$$  \hspace{1cm} (1.1)

where $E \subseteq [\varepsilon_0,1]^N$, $\varepsilon_0 \in (0,1)$ and $F \subseteq \mathbb{R}^d$ are Borel sets and $c$ is a finite constant which depends on $[\varepsilon_0,1]^N$, $F$ and $H$ only. We emphasize that, in addition to fractional Brownian motion, various processes are part of those satisfying conditions $(C_1)$ and $(C_2)$ namely, fractional Brownian sheets (Ayache and Xiao [11]), solutions to stochastic heat equation driven by space-time white noise (Dalang, Khoshnevisan and Nualart [8] and [9], Dalang and Nualart [7]), Mueller and Tribe [22] and many more. See Xiao [31] for more examples and further information on conditions $(C_1)$ and $(C_2)$. $C_{\rho_H,d}$ and $\mathcal{H}_{\rho_H}^d$ denotes the Bessel-Riesz type capacity and the Hausdorff measure with respect to the parabolic metric $\rho_H$ of order $d$. Both of these terms are defined next and will be referred as parabolic capacity and parabolic Hausdorff measure respectively.

The corresponding problem for $d$-dimensional Brownian motion $B$ with drift $f$, $d \geq 2$, has been considered by Peres and Souissi [23]. Precisely they showed that for $f : \mathbb{R}^+ \rightarrow \mathbb{R}^d$ $(1/2)$-Hölder continuous function there exists positives constants $\alpha_1, \alpha_2$ such that for all $x \in \mathbb{R}^d$ and all closed set $F \subseteq \mathbb{R}^d$

$$\alpha_1\text{Cap}_M (F) \leq \mathbb{P}_x\{(B + f)(0, \infty) \cap F \neq \emptyset\} \leq \alpha_2\text{Cap}_M (F),$$

where $\text{Cap}_M (\cdot)$ denotes the Martin capacity. At the heart of their method is the strong Markov property which can’t be used for fractional Brownian motion. Naturally, this begs the question : can we provide similar estimate to (1.1) for $d$-dimensional fractional Brownian motion $B^H$ of Hurst index $H$ with drift $f$?

Our first objective in this work is to give an answer to this question. In fact, we established the desired estimates by adjusting the standard proof, which relies on the covering argument for the upper bound and the second moment argument for the lower bound, to take into account the presence of the $H$-Hölder continuous drift $f$. This allowed us to obtain, this time, according to the usual Hausdorff measure and Bessel-Riesz capacity the upper and lower bounds on hitting probabilities of the following type

$$c_1C_{d-\beta_1/H}(F) \leq \mathbb{P}\{(B^H + f)(E) \cap F \neq \emptyset\} \leq c_2\mathcal{H}_{d-\beta_2/H}^d(F).$$

Worthy of special mention is the fact that $\beta_1$ and $\beta_2$ are two different constants closely related to the Hausdorff and Minkowski dimensions of $E$ respectively. In the event that the two dimensions coincide, often this is a consequence of the existence of a sufficiently regular measure see condition (S) below, we obtain the above estimates with the same constant $\beta = \text{dim}(E)$ the Hausdorff dimension of $E$. With these bounds in hand we draw the conclusion that $\mathbb{P}\{(B^H + f)(E) \cap F \neq \emptyset\} > 0$ (resp. $\mathbb{P}\{(B^H + f)(E) \cap F \neq \emptyset\} = 0$) for any compact set $F$ such that $\text{dim}(F) > d - \beta/H$ (resp. $\text{dim}(F) < d - \beta/H$). This leads us to consider the question: is there an $\alpha$-Hölder continuous function $f$, $\alpha < H$, for which $\mathbb{P}\{(B^H + f)(E) \cap F \neq \emptyset\} > 0$?

The idea is then to take $\alpha < H$ such that $\text{dim}(F) > d - \beta/\alpha$ and $B^\alpha$ another fractional Brownian motion with Hurst index $\alpha$ possibly defined on different probability space and thereafter to consider
\( B^H \) as a drift of \( B^\alpha \). This induces us to bring them together on the same space while preserving their distributions. The best way to do this is to work on the product space and to consider processes on this latter. Unfortunately, we are unable to have both estimates for the same drift. These results are proved in Section 2.

A problem related to estimating hitting probabilities for a random process \( X \) with drift is determining which Borel functions are polar for \( X \). Now we recall the definition of polar function for \( X \). A Borel function \( f : \mathbb{R}^+ \to \mathbb{R}^d \) is called polar for \( X \) if for any \( x \in \mathbb{R}^d \),

\[
P \{ X_t + x = f(t) \text{ for some } t > 0 \} = 0,
\]

which means that the process \( X - f \) does not hit points. The first study of polar functions for Brownian motion in 2 dimensions appears in Graversen [11]. Precisely, he showed that for all \( 0 < \gamma < 1/2 \), there exists a \( \gamma \)-Hölder continuous function \( f : \mathbb{R}^+ \to \mathbb{R}^2 \) for which \( B + f \) hits points. In [19], Le Gall proved that for any \( 1/2 \)-Hölder continuous function \( f \), the process \( B + f \) do not hits points and asked, for \( d \geq 3 \), whether for each \( \gamma < 1/d \) there exist \( \gamma \)-Hölder continuous functions for which \( B + f \) hits points. This problem has also been studied for the stable process by Mountford in [21]. Recently Antunović, Peres and Vermesi [2] prove first that, for \( d \geq 2 \) and for each \( \gamma < 1/d \) there exist \( \gamma \)-Hölder continuous functions for which the range of \( B + f \) covers an open set almost surely, thereby ensuring that \( B + f \) hits points. Moreover, for \( d \geq 3 \), there exists a \( 1/d \)-Hölder continuous function, accurately the \( d \)-dimensional Hilbert curve, such that \( B + f \) hits points. Considering this problem for fractional Brownian motion is our second focus. We begin by establishing, for a general measurable drift, an upper and lower bounds on hitting probabilities as follows

\[
C_1^{-1} C_{p_H,d}(Gr_E(f)) \leq P \{ \exists t \in E : (B^H + f)(t) = x \} \leq C_1 H^d_{p_H}(Gr_E(f)), \tag{1.2}
\]

where \( Gr_E(f) = \{(t, f(t)) : t \in E\} \) is the graph of \( f \) over the set \( E \). The above estimates are aimed first and foremost to seek conditions on the drift \( f \) for which \( B^H + f \) does or does not hit points. The first conclusion that we can draw is that functions with a positive parabolic capacity \( C_{p_H,d}(Gr_E(f)) \) hit points, on the other hand those who have parabolic Hausdorff measure \( H^d_{p_H}(Gr_E(f)) = 0 \) does not hit points. As a first step, we prove that for any \( \alpha < \dim(E)/d \land H \) there exists a \( \alpha \)-Hölder continuous function which is non-polar for \( B^H \) obtained as a realization of an independent fractional Brownian motion with Hurst parameter \( \alpha \).

The relationship between lack of regularity and fractal properties for special classes of functions has been highlighted a long time ago. Frequently graphs of continuous but sufficiently irregular functions are fractal sets what connects the lack of regularity of such functions to the Hausdorff dimension of their graphs. We consider the Weierstrass function as a prototype example of such functions. Our second step is to show that the one dimensional fractional Brownian motion with drift given by the Weierstrass function hits points with positive probability. The two keys ingredients in the proof are a recent result of Shen [28], which is an improvement on the result of Barański, Bárány and Romanowska [3] on a long-standing conjecture concerning the Hausdorff dimension of the graph of the Weierstrass function, giving the exact value of the latter and a comparison result for the Hausdorff parabolic dimensions with different parameters established by the authors in [10]. Among the properties of Weierstrass nowhere differentiable function most often used are \( \alpha \)-Hölder continuity and reverse \( \alpha \)-Hölder continuity for some \( 0 < \alpha < 1 \). Przytycki and Urbański in [27] proved that if \( f : [0,1] \to \mathbb{R} \) is both \( \alpha \)-Hölder and reverse \( \alpha \)-Hölder for some \( 0 < \alpha < 1 \), it satisfies \( \dim(Gr_{[0,1]}(f)) > 1 \). Replacing Weierstrass function by such function \( f \) we obtain the same result. We thought and hoped that this result continues to be true for in higher dimensions, i.e. \( d \geq 2 \), but it does not. Precisely, we consider a \( d \)-dimensional vector-valued function \( f \) where each component is the Weierstrass function for which \( \dim_{p_H}(Gr(f)) < d \) leading us to conclude that \( B^H + f \) does not hit points. The above mentioned results constitute the content of Section 3.
2 Hitting sets

In this section, we consider the problem on hitting probabilities of fractional Brownian motion with deterministic drift. Let $H \in (0,1)$ and $B^H_0 = \{B^H_0(t), t \geq 0\}$ be a real-valued fractional Brownian motion of Hurst index $H$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. a real valued Gaussian process with stationary increments and covariance function given by

$$E(B^H_0(s)B^H_0(t)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

Let $B^H_1, ..., B^H_d$ be $d$ independent copies of $B^H_0$, then the stochastic process $B^H = \{B^H(t), t \geq 0\}$ given by

$$B^H(t) = (B^H_1(t), ..., B^H_d(t)),$$

is called a $d$-dimensional fractional Brownian motion of Hurst index $H \in (0,1)$.

We consider the following parabolic metric $\rho_H$ on $\mathbb{R}_+ \times \mathbb{R}^d$ defined by

$$\rho_H((s,x),(t,y)) = \max\{|t-s|^H, |x-y|\} \quad \forall (s,x),(t,y) \in \mathbb{R}_+ \times \mathbb{R}^d,$$

where $||.||$ denotes the euclidean metric on $\mathbb{R}^d$. For $\beta > 0$ and $E \subset \mathbb{R}_+ \times \mathbb{R}^d$, the $\beta$-dimensional Hausdorff measure of $E$ with respect to the metric $\rho_H$ is defined by

$$\mathcal{H}^\beta_{\rho_H}(E) = \lim_{\delta \to 0} \left\{ \sum_{n=1}^{\infty} (2r_n)^\beta : E \subset \bigcup_{n=1}^{\infty} B_{\rho_H}(r_n), r_n \leq \delta \right\},$$

(2.2)

where $B_{\rho_H}(r)$ denotes an open ball of radius $r$ in the metric space $(\mathbb{R}_+ \times \mathbb{R}^d, \rho_H)$. The Bessel-Riesz type capacity of order $\alpha$ on the metric space $(\mathbb{R}_+ \times \mathbb{R}^d, \rho_H)$ is defined by

$$\mathcal{C}_{\rho_H,\alpha}(E) = \left[ \inf_{\mu \in \mathcal{P}(E)} \int_{\mathbb{R}_+ \times \mathbb{R}^d} \int_{\mathbb{R}_+ \times \mathbb{R}^d} \varphi_\alpha(\rho_H(u,v))\mu(du)\mu(dv) \right]^{-1},$$

(2.3)

where $\mathcal{P}(E)$ is the family of probability measures carried by $E$ and the function $\varphi_\alpha : (0, \infty) \to (0, \infty)$ is defined by

$$\varphi_\alpha(r) = \begin{cases} r^{-\alpha} & \text{if } \alpha > 0 \\ \log \left( \frac{e}{\alpha^{1/\alpha}} \right) & \text{if } \alpha = 0 \\ 1 & \text{if } \alpha < 0 \end{cases}.$$  

(2.4)

Remark 2.1. Let $\dim_{\rho_H}$ be the Hausdorff dimension associated to the measure $\mathcal{H}^\beta_{\rho_H}$ which is defined as

$$\dim_{\rho_H}(E) = \inf\{\beta > 0 : \mathcal{H}^\beta_{\rho_H}(E) = 0\} \quad \forall E \subset \mathbb{R}_+ \times \mathbb{R}^d,$$

we can verify that $\dim_{\Psi,H}(. \equiv H \times \dim_{\rho_H}(.)$, where $\dim_{\Psi,H}$ is the $H$-parabolic Hausdorff dimension which was used by Peres and Sousi in [24] in order to study the Hausdorff dimension of the graph and the image of $B^H + f$.

The usual $\beta$-dimensional Hausdorff measure, $\beta > 0$, and Bessel-Riesz capacity of order $\alpha$ in Euclidean metric $||.||$ are denoted by $\mathcal{H}^\beta$ and $\mathcal{C}_\alpha$ respectively. $\mathcal{H}^\beta$ is assumed equal to $1$ whenever $\beta \leq 0$. Let $I = [\varepsilon_0,1]$, where $\varepsilon_0 \in (0,1)$ is a fixed constant. First we have the following result.

Theorem 2.2. Let $\{B^H(t), t \in [0,1]\}$ be a $d$-dimensional fractional Brownian motion and $f : [0,1] \to \mathbb{R}^d$ a Hölder continuous function with order $H$ and constant $K$. If $F \subset \mathbb{R}^d$ is a compact subset of $\mathbb{R}^d$ and $E$ is a Borel subset of $I$, then

$$c_1^{-1} \mathcal{C}_{\rho_H,d}(E \times F) \leq \mathcal{P}\{(B^H + f)(E) \cap F \neq \emptyset\} \leq c_1 \mathcal{H}^d_{\rho_H}(E \times F),$$

(2.5)

where $c_1 \geq 1$ is finite constant which depends on $I$, $F$, $H$ and $K$ only.
An important property of the fractional Brownian motion that will serve us well into the proof is
the strong local nondeterminism which follows from Lemma 7.1 of [25]. Precisely, there exists a constant
$0 < C < \infty$ such that for all integers $n \geq 1$ and all $t_1, \ldots, t_n, t \in [0,1]$, we have
\[
Var \left( B_H^H(t)/B_H^H(t_1), \ldots, B_H^H(t_n) \right) \geq C \min_{0 \leq j \leq n} |t - t_j|^{2H},
\]
where $Var \left( B_H^H(t)/B_H^H(t_1), \ldots, B_H^H(t_n) \right)$ denotes the conditional variance of $B_H^H(t)$ given $B_H^H(t_1), \ldots, B_H^H(t_n)$ and $t_0 = 0$.

To prove the above theorem, we will make use of the following two lemmas proved by Biemeré, Lacaux
and Xiao in [4] for a general class of Gaussian processes to which fractional Brownian motion belongs. In
fact they will be used to obtain the upper and lower bounds of (2.5) respectively.

**Lemma 2.3** (Lemma 3.1, [4]). Let $\{B_H^H(t) : t \in [0, 1]\}$ be a fractional Brownian motion. For any constant
$M > 0$, there exist a positive constants $c_2$ and $\delta_0$ such that for all $r \in (0, \delta_0)$, $t \in I$ and all $x \in [-M, M]^d$
we have
\[
P \left\{ \inf_{s \in I, |s - t|^H \leq r} \|B_H^H(s) - x\| \leq r \right\} \leq c_2 r^d.
\]

**Lemma 2.4** (Lemma 3.2, [4]). Let $B_H^H$ be a fractional Brownian motion. Then there exists a positive
and finite constants $c_3$ and $c_4$ such that for all $\epsilon \in (0, 1)$, $s, t \in I$ and $x, y \in \mathbb{R}^d$ we have
\[
\int_{\mathbb{R}^{2d}} e^{-i(\xi, x) + (\eta, y)} \exp \left( -\frac{1}{2}(\xi, \eta) \left( \epsilon I_{2d} + Cov(B_H^H(s), B_H^H(t)) \right) (\xi, \eta)^T \right) d\xi d\eta
\leq \frac{(2\pi)^d}{\text{det}(\Gamma_{\epsilon}(s, t))^{d/2}} \exp \left( -\frac{c_3}{2} \frac{\|x - y\|^2}{\text{det}(\Gamma_{\epsilon}(s, t))} \right)
\leq \frac{c_4}{(\rho_H((s, x), (t, y)))^d},
\]
where $\Gamma_{\epsilon}(s, t) := I_{2\epsilon} + Cov(B_H^H(s), B_H^H(t))$, $I_{2\epsilon}$ and $I_2$ are the identities matrices of order $2d$ and $2$
respectively, $Cov(B_H^H(s), B_H^H(t))$ and $Cov(B_H^H(s), B_H^H(t))$ denote the covariance matrix of the random
vectors $(B_H^H(s), B_H^H(t))$ and $(B_H^H(s), B_H^H(t))$ respectively, and $(\xi, \eta)^T$ is the transpose of the row vector
$(\xi, \eta)$.

**Proof of Theorem 2.2** First of all, we note that the proof of the upper bound in (2.5) is similar to that
of Theorem 2.1 in [6] which relies on the use of a simple covering argument. Since $F$ is compact set there
exists a constant $M_0 > 0$ such that $F \subset [-M_0, M_0]^d$. Let $M_1 = \sup_{s \in I} \|f(s)\|$ and $M_2 = M_0 + M_1$. Then
for all $(t, y) \in I \times F$ we have $y + f(t) \in [-M_2, M_2]^d$. Applying Lemma 2.3 with the constant $M_2$ leads to
the existence of a positive constants $c_5$ and $\delta_1$ such that for all $r \in (0, \delta_1)$ and $(t, y) \in I \times F$ we have
\[
P \left\{ \inf_{s \in I, |s - t|^H \leq r} \|B_H^H(s) - (y - f(t))\| \leq r \right\} \leq c_5 r^d,
\]
where $c_5$ and $\delta_1$ depend only on $I, H$ and $M_2$. Now let us choose an arbitrary constant $\gamma > \mathcal{H}_d^d(E \times F)$.
Then there is a covering of $E \times F$ by balls $\{B_{\rho_H^d}((t_i, y_i), r_i), i \geq 1\}$ in $\mathbb{R}_+ \times \mathbb{R}^d$ such that $r_i < \delta_1/1 + K$
for all $i \geq 1$, where $K$ is the Hölder constant of $f$, and
\[
E \times F \subseteq \bigcup_{i=1}^{\infty} B_{\rho_H^d}((t_i, y_i), r_i) \quad \text{with} \quad \sum_{i=1}^{\infty} (2r_i)^d \leq \gamma.
\]
It follows that

\[ \{ (B^H + f)(E) \cap F \neq \emptyset \} = \{ \exists (t, y) \in E \times F : (B^H + f)(t) = y \} \]
\[ \subseteq \bigcup_{i=1}^{\infty} \{ (B^H + f) \left( \left( t_i - r_i^{1/H}, t_i + r_i^{1/H} \right) \right) \cap B(y_i, r_i) \neq \emptyset \}. \]  

(2.11)

As a first step, it is easy to see that

\[ \left\{ (B^H + f) \left( \left( t_i - r_i^{1/H}, t_i + r_i^{1/H} \right) \right) \cap B(y_i, r_i) \neq \emptyset \right\} = \left\{ \inf_{|s-t_i|^H < r_i} \| (B^H + f)(s) - y_i \| < r_i \right\}. \]

On the other hand since \( f \) is \( H \)-Hölder continuous then for all \( s \in \left( t_i - r_i^{1/H}, t_i + r_i^{1/H} \right) \) we have

\[ \{ \| (B^H + f)(s) - y_i \| < r_i \} \subset \{ \| B^H(s) - (y_i - f(t_i)) \| < (1 + K)r_i \}. \]

This enables us to obtain

\[ \left\{ (B^H + f) \left( \left( t_i - r_i^{1/H}, t_i + r_i^{1/H} \right) \right) \cap B(y_i, r_i) \neq \emptyset \right\} \subset \left\{ \inf_{|s-t_i|^H < c_6r_i} \| B^H(s) - (y_i - f(t_i)) \| < c_6r_i \right\}. \]

where \( c_6 := K + 1 > 1 \). Combining (2.9), (2.10) and (2.11) we derive that

\[ \mathbb{P}\{ (B^H + f)(E) \cap F \neq \emptyset \} \leq c_7g, \]

where \( c_7 \) depends only on \( I, H \) and \( K \). Let \( \gamma \downarrow \mathcal{H}_{E \times F}^d \), the upper bound in (2.3) follows.

The lower bound in (2.5) can be proved by using a second moment argument. We assume that \( \mathcal{C}_{\rho_H,d}(E \times F) > 0 \) otherwise the lower bound is obvious. We can see easily from (2.3) that there is a probability measure \( \mu \) on \( E \times F \) such that

\[ \mathcal{E}_{\rho_H,d}(\mu) := \int_{\mathbb{R}_+ \times \mathbb{R}^d} \int_{\mathbb{R}_+ \times \mathbb{R}^d} \frac{\mu(du)\mu(dv)}{\rho_H(u,v)} \leq \frac{2}{\mathcal{C}_{\rho_H,d}(E \times F)}. \]  

(12.2)

We consider the family of random measures \( \{ \mu_n, n \geq 1 \} \) on \( E \times F \) defined by

\[ \int_{E \times F} g(s, x) \mu_n(ds, dx) = \int_{E \times F} \left( 2\pi n \right)^{d/2} \exp \left( -\frac{n\|B^H(s) + f(s) - x\|^2}{2} \right) g(s, x) \mu(ds, dx) \]
\[ = \int_{E \times F} \int_{\mathbb{R}^d} \exp \left( -\frac{\|\xi\|^2}{2n} + i \langle \xi, B^H(s) + f(s) - x \rangle \right) g(s, x) d\xi \mu(ds, dx), \]

thanks to the characteristic function of a Gaussian vector. Here \( g \) is an arbitrary measurable function on \( \mathbb{R}_+ \times \mathbb{R}^d \). Our aim is to show that \( \{ \mu_n, n \geq 1 \} \) has a subsequence which converges weakly to a finite measure \( \nu \) supported on the set \( \{ (s, x) \in E \times F : B^H(s) + f(s) = x \} \). To carry out this goal, we will start by establishing the following inequalities

\[ \mathbb{E}(\|\mu_n\|) \geq c_8, \quad \mathbb{E}(\|\mu_n\|^2) \leq c_9 \mathcal{E}_{\rho_H,d}(\mu), \]  

(12.4)

which constitute together with the Paley-Zygmund inequality the cornerstone of the proof. Here \( \|\mu_n\| \) denotes the total mass of \( \mu_n \). We emphasize that the positive constants \( c_8 \) and \( c_9 \) are independent of \( \mu \).
and $n$. By (2.13), Fubini’s theorem and the use of the characteristic function of a Gaussian vector we have

$$
\mathbb{E}(\|\mu_n\|) = \int_{E \times F} \int_{\mathbb{R}^d} e^{-i\langle \xi, x-f(s) \rangle} \exp \left( - \frac{\|\xi\|^2}{2n} \right) \mathbb{E} \left( e^{i\langle \xi, B^H(s) \rangle} \right) \, d\xi \, \mu(ds, dx)
= \int_{E \times F} \int_{\mathbb{R}^d} e^{-i\langle \xi, x-f(s) \rangle} \exp \left( - \frac{1}{2} \left( \frac{1}{n} + s^{2H} \right) \|\xi\|^2 \right) \, d\xi \, \mu(ds, dx)
= \int_{E \times F} \left( \frac{2\pi}{n-1 + s^{2H}} \right)^{d/2} \exp \left( - \frac{\|x - f(s)\|^2}{2(n-1 + s^{2H})} \right) \, \mu(ds, dx)
\geq \int_{E \times F} \left( \frac{2\pi}{1 + s^{2H}} \right)^{d/2} \exp \left( - \frac{\|x - f(s)\|^2}{2s^{2H}} \right) \, \mu(ds, dx)
\geq c_8 > 0.
$$

Since $F$ and $f$ are bounded and $\mu$ is a probability measure we conclude that $c_7$ is independent of $\mu$ and $n$. This gives the first inequality in (2.14).

We will now turn our attention to the second inequality in (2.14). By (2.13) and Fubini’s theorem again we obtain

$$
\mathbb{E}(\|\mu_n\|^2) = \int_{E \times F} \int_{E \times F} \mu(ds, dx) \mu(dt, dy) \int_{\mathbb{R}^{2d}} e^{-i\langle \xi, x-f(s) \rangle + \langle \eta, y-f(t) \rangle} \exp \left( - \frac{1}{2} \langle \xi, \eta \rangle \right) \mathbb{E}(B^H(s), B^H(t)) \mathbb{E}(B^H(s), B^H(t)) \, d\xi \, d\eta
\leq \int_{E \times F} \int_{E \times F} \frac{(2\pi)^d}{\left[ \det \left( \Gamma_{1/n}(s, t) \right) \right]^{d/2}} \exp \left( - \frac{c_3}{2} \frac{\|x - y + f(s) - f(t)\|^2}{\det \left( \Gamma_{1/n}(s, t) \right)} \right) \, \mu(ds, dx) \mu(dt, dy),
$$

where the last inequality follows from Lemma 2.3. We denote by $I_n((s, x), (t, y))$ the last integrand. Since $\|x - y + f(t) - f(s)\| \geq \|x - y\| - \|f(t) - f(s)\|$, we have that

$$
I_n((s, x), (t, y)) \leq \frac{(2\pi)^d}{\left[ \det \left( \Gamma_{1/n}(s, t) \right) \right]^{d/2}} \exp \left( - \frac{c_3}{2} \frac{\|x - y\|^2}{\det \left( \Gamma_{1/n}(s, t) \right)} \right) \exp \left( - \frac{c_3}{2} \frac{\|f(s) - f(t)\|^2}{\det \left( \Gamma_{1/n}(s, t) \right)} \right).
$$

Using the strong local nondeterminism property (2.6) of $B^H_0$, there exists a constant $c_{10}$ such that

$$
\det \Gamma_{1/n}(s, t) \geq \det \text{Cov}(B^H_0(s), B^H_0(t)) = \text{Var}(B^H_0(s)) \text{Var}(B^H_0(t)) \geq c_{10} |t - s|^{2H}.
$$

Since $f$ is $H$-Hölder continuous we have

$$
\frac{\|f(t) - f(s)\|}{\sqrt{\det \left( \Gamma_{1/n}(s, t) \right)}} \leq \frac{K}{\sqrt{c_{10}}} \text{ for all } s \neq t \in E.
$$

It follows that

$$
I_n((s, x), (t, y)) \leq \frac{(2\pi)^d}{\left[ \det \left( \Gamma_{1/n}(s, t) \right) \right]^{d/2}} \exp \left( - \frac{c_3}{2} \frac{\|x - y\|^2}{\det \left( \Gamma_{1/n}(s, t) \right)} \right) \exp \left( - \frac{c_3 K}{\sqrt{c_{10}}} \frac{\|x - y\|}{\sqrt{\det \left( \Gamma_{1/n}(s, t) \right)}} \right),
$$

7
which implies that
\[ I_n((s, x), (t, y)) \leq \frac{c_{11}(2\pi)^d}{[\det(\Gamma_{1/n}(s, t))]^{d/2}} \exp \left( -\frac{c_3}{4} \frac{\|x - y\|^2}{\det(\Gamma_{1/n}(s, t))} \right) \]
where \( c_{11} \) is a positive constant such that \( \sup_{x \geq 0} \exp \left( -\frac{c_3}{4} \frac{x^2}{\det(\Gamma_{1/n}(s, t))} \right) \leq c_{11} \).

It is now straightforward to deduce that there exists a positive constant \( c_9 \) such that
\[ I_n((s, x), (t, y)) \leq \frac{c_9}{\rho_H((s, x), (t, y))^d} = \max \left\{ \|s - t\|^d, \|x - y\|^d \right\} \]
(2.17)
Indeed, if \( \det\Gamma_{1/n}(s, t) \geq \|x - y\|^2 \) we have
\[ I_n((s, x), (t, y)) \leq \frac{(2\pi)^d}{(\det\Gamma_{1/n}(s, t))^{d/2}} \leq \frac{c_9}{\|x - y\|^d}, \] (2.18)
where we use (2.16) for the last the inequality. Otherwise, if \( \det\Gamma_{1/n}(s, t) < \|x - y\|^2 \) the elementary inequality \( \sup_{x > 0} x^{d/2} e^{-c_3 x/4} \leq c_{12} \) enables us to obtain
\[ I_n((s, x), (t, y)) \leq \frac{c_9}{\|x - y\|^d}. \] (2.19)
Combining (2.18) and (2.19) leads to (2.17). Hence the second inequalities in (2.14) follows immediately.

Plugging the moment estimates of (2.14) into the Paley–Zygmund inequality (c.f. Kahane [14], p.8), allows us to confirm that \( \{\mu_n, n \geq 1\} \) has a subsequence that converges weakly to a finite measure \( \tilde{\mu} \) supported on the set \( \{ (s, x) \in E \times F : B^H(s) + f(s) = x \} \), positive with positive probability and also satisfying the moment estimates of (2.14). Consequently,
\[ \mathbb{P} \{ (B^H + f)(E) \cap F \neq \emptyset \} \geq \mathbb{P} \{ ||\tilde{\mu}|| > 0 \} \geq \frac{[\mathbb{E}(||\tilde{\mu}||)^2]}{\mathbb{E}[||\tilde{\mu}||]^2} \geq c_{13}c_{\rho_H,d}(E \times F), \]
where \( c_{13} = \frac{c_8}{c_9} \). All that remains to be done is take \( c_1 = c_7 \vee 1/c_{13} \). Thus the lower and upper bounds in (2.5) will follow immediately which completes the proof.

Recall that in the precise case where \( E \) is an interval, Corollary 2.2. in [6] ensures that there exists a finite constant \( c \geq 1 \) depending only on \( E, F \) and \( H \) such that
\[ c^{-1}c_{d-1/H}(F) \leq \mathbb{P} \{ B^H(E) \cap F \neq \emptyset \} \leq c_{H^{d-1/H}}(E \times F), \]
for any Borel set \( F \subseteq \mathbb{R}^d \). Our next goal is to establish such estimates for \( (B^H + f) \) and any Borel set \( E \) by means of its Hausdorff measure. However, to achieve our stated goal, we need to make use of the Minkowski dimension as well. We introduce now the Minkowski dimension of \( E \subseteq [0, 1] \). Let \( N(E, r) \) be the smallest number of open intervals of length \( r \) required to cover \( E \). The lower and upper Minkowski dimensions of \( E \) are respectively defined as
\[ \dim_M(E) := \liminf_{r \to 0^+} \frac{\log N(E, r)}{\log(1/r)}, \]
\[ \overline{\dim}_M(E) := \limsup_{r \to 0^+} \frac{\log N(E, r)}{\log(1/r)}. \]
Equivalently, the upper Minkowski dimension of \( E \) can be written as
\[ \overline{\dim}_M(E) = \inf \{ \gamma : \exists C < \infty \text{ such that } N(E, r) \leq C r^{-\gamma} \text{ for all } r > 0 \}. \] (2.20)
Proposition 2.5. Let $B^H$, $f$ and $F$ as in Theorem 2.2. Let $E$ be a subset of $I$ such that $\dim(E) > 0$. Then for any $0 < \beta_1 < \dim(E) \leq \dim_M(E) < \beta_2 < H d$, we have

$$c_1 c_{d-\beta_1/H}(F) \leq \mathbb{P}\{(B^H + f)(E) \cap F \neq \emptyset\} \leq c_2 H^{d-\beta_2/H}(F),$$

where $c_1$ and $c_2$ are two positive constants which depend on $E$, $F$, $H$, $K$, $\beta_1$ and $\beta_2$.

We need the following lemma to establish the lower bound in (2.21).

Lemma 2.6. Let $H \in (0, 1)$, $d \geq 1$ and $0 < \beta \leq H d$. Let $\nu$ a Borel probability measure on $[0, 1]$ such that, for all $a \in [0, 1]$ and $\delta > 0$

$$\nu([a, a + \delta]) \leq c_3 \delta^\beta,$$

where $c_3$ is a positive constant which depends on $\beta$ only. Then, for all $r > 0$, we have

$$\sup_{t \in [0, 1]} \int_{[0, 1]} \nu(ds) \max\{|r^d|s - t|Hd\} \leq c_4 \varphi_{d-\beta/H}(r),$$

where $\varphi_{d-\beta/H}(\cdot)$ is the function defined in (2.4) and $c_4$ is a positive constant depending only on $\beta$, $H$ and $d$.

Proof. First, it is worthwhile pointing out that for $r \geq 1$ we have

$$\sup_{t \in [0, 1]} \int_{[0, 1]} \nu(ds) \max\{|r^d|s - t|Hd\} \leq r^d \leq \varphi_{d-\beta/H}(r).$$

(2.24)

Now we assume that $0 < \beta < H d$. For $r \in (0, 1)$, we divide the integral in (2.23) into two parts $I_1, I_2$ as follows

$$I_1 = \int_{|s - t| < r^{1/H}} \frac{\nu(ds)}{r^d} \text{ and } I_2 = \int_{|s - t| \geq r^{1/H}} \frac{\nu(ds)}{|s - t|Hd}.$$ 

By using (2.22) we obtain

$$I_1 \leq c_3 2^\beta \varphi_{d-\beta/H}(r).$$

(2.25)

Let us set $k(r) := \min\{k \geq 0 : 2^{-k} \leq r^{1/H}\}$. Then it is easy to see that

$$[r^{1/H}, 1) \subset \bigcup_{k=1}^{k(r)} [2^{-k}, 2^{-k+1}).$$

(2.26)

A second use of (2.22) gives

$$I_2 \leq \sum_{k=1}^{k(r)} 2^{kH} \nu(\{s \in [0, 1] : 2^{-k} \leq |s - t| < 2^{-k+1}\}) \leq 2 c_3 \sum_{k=1}^{k(r)} 2^{k(Hd-\beta)} \leq 2 c_3 \frac{2^{(Hd-\beta)}}{2^{(Hd-\beta)} - 1} r^{-(d-\beta/H)} = 2 c_3 \frac{2^{(Hd-\beta)}}{2^{(Hd-\beta)} - 1} \varphi_{d-\beta/H}(r).$$

(2.27)

Finally, putting it all together enables us to deduce (2.23). For $\beta = H d$ the same techniques as above can give that

$$I_1 \leq 2^\beta c_3 \quad \text{and} \quad I_2 \leq 2 c_3 k(r).$$

It follows from the definition of $k(r)$ that $r^{1/H} < 2^{-k(r)+1}$. Hence, we have

$$I_1 \leq 2^\beta c_3 \varphi_0(r) \quad \text{and} \quad I_2 \leq 2 c_3 (1 + 1/H \log(2)) \varphi_0(r),$$

which ends the proof. 

\qed
Proof of Proposition 2.5. Using Theorem 2.2, it suffices to prove that there exists a positive constant \( c_6 \) such that

\[
C_{d-\beta_1/H}(F) \leq c_5 C_{\rho_H,d}(E \times F) \quad \text{and} \quad \mathcal{H}_H^d(E \times F) \leq c_6 \mathcal{H}^{d-\beta_2/H}(F). \tag{2.28}
\]

Indeed for \( \beta_1 \in (0, \dim(E)) \), by Frostman’s theorem there is a Borel probability measure \( \nu \) supported on \( E \) such that

\[
\nu([a, a + \delta]) \leq c_7 \delta^{\beta_1}, \tag{2.29}
\]

for all \( a \in E \) and \( \delta > 0 \), where \( c_7 \) is a positive constant which depends on \( \beta_1 \) only. Let us suppose that \( C_{d-\beta_1/H}(F) > 0 \), otherwise there is nothing to prove. It follows that for all \( \gamma \in (0, C_{d-\beta_1/H}(F)) \) there is a probability measure \( m \) supported on \( F \) such that

\[
\mathcal{E}_{d-\beta_1/H}(m) := \int_F \int_F \frac{m(dx)m(dy)}{\|x - y\|^{d-\beta_1/H}} \leq \gamma^{-1}. \tag{2.30}
\]

Since \( \nu \otimes m \) is a probability measure on \( E \times F \), then applying Fubini’s theorem and (2.23) of Lemma 2.6 we obtain

\[
\mathcal{E}_{\rho_H,d}(\nu \otimes m) = \int_{E \times F} \int_{E \times F} \frac{\nu \otimes m(du)\nu \otimes m(dv)}{(\rho_H(u, v))^d} \leq \gamma_4 \int_F \int_F \frac{m(dx)m(dy)}{\|x - y\|^{d-\beta_1/H}} \leq \gamma_4 \gamma^{-1}. \tag{2.31}
\]

Consequently we have \( C_{\rho_H,d}(E \times F) \geq \gamma_4^{-1} \gamma \). Then we let \( \gamma \uparrow C_{d-\beta_1/H}(F) \) to conclude that the first inequality in (2.28) holds true.

Now let us prove the second inequality in (2.28). Let \( l > \mathcal{H}^{d-\beta_2/H}(F) \) be arbitrary with \( d - \beta_2/H > 0 \). Then there is a covering of \( F \) by open balls \( B(r_n) \) of radius \( r_n \) such that

\[
F \subset \bigcup_{n=1}^{\infty} B(r_n) \quad \text{and} \quad \sum_{n=1}^{\infty} (2r_n)^{d-\beta_2/H} \leq l. \tag{2.32}
\]

For all \( n \geq 1 \), let \( E_{n,j}, j = 1, \ldots, N(E, 2r_n^{1/H}) \) be a family of open intervals of length \( 2r_n^{1/H} \) covering \( E \). It follows that the family \( E_{n,j} \times B(r_n), j = 1, \ldots, N(E, 2r_n^{1/H}) \), \( n \geq 1 \) gives a covering of \( E \times F \) by open balls of radius \( r_n \) for the parabolic metric \( \rho_H \).

It follows from (2.20) that for all \( \delta > 0 \) the number of open intervals of length \( \delta \) needed to cover \( E \) satisfies

\[
N(E, \delta) \leq c_8 \delta^{-\beta_2}, \tag{2.33}
\]

where \( c_8 \) is a positive and finite constant which depend on \( E \) only. Together with the estimates (2.32) and (2.33) that have been established above, we have

\[
\sum_{n=1}^{\infty} \sum_{j=1}^{N(E, 2r_n^{1/H})} (2r_n)^d \leq c_8 2^{-\beta_2(1-1/H)} \sum_{n=1}^{\infty} (2r_n)^{d-\beta_2/H} \leq c_8 2^{-\beta_2(1-1/H)} l. \tag{2.34}
\]

Then let \( l \downarrow \mathcal{H}^{d-\beta_2/H}(F) \), the second inequality in (2.28) follows with \( c_6 = c_8 2^{-\beta_2(1-1/H)} \).

It is well known that Hausdorff and Minkowski dimensions agree for many sets \( E \). Often this is linked on the one hand to the geometric properties of the set, on the other hand it is a consequence of the existence of a sufficiently regular measure. Among the best known are Ahlfors-David regular sets defined as follows:
(S): Let \( E \subset I \) and \( \beta \in ]0,1] \). We say that \( E \) is \( \beta \)-regular if there exists a finite positive Borel measure \( \nu \) supported on \( E \) and positive constant \( c_9 \) and such that
\[
c_9^{-1} \delta^\beta \leq \nu([a - \delta, a + \delta]) \leq c_9 \delta^\beta \quad \text{for all } a \in E, \quad 0 < \delta \leq 1. \tag{2.35}
\]

Remark 2.7. 1. If \( E \) is the whole interval \( I \) then \( \beta \) in the condition (S) should be equal to 1. This leads to the conclusion that the measure \( \nu \) can be chosen as the normalized Lebesgue measure on \( I \). In this case the above proposition is simply Corollary 2.2 in [6].

2. The Cantor set \( C(\lambda) \), \( 0 < \lambda < 1/2 \), subset of \( I \) with \( \nu \) is the \( \beta \)-dimensional Hausdorff measure restricted to \( C(\lambda) \) where \( \beta = \dim C(\lambda) = \log(2)/\log(1/\lambda) \). For more details see Theorem 4.14 p.67 in Mattila [20]. In general, self similar subsets of \( \mathbb{R} \) satisfying the open set condition are standard examples of regular sets, see [13].

According to Theorem 5.7 p.80 in [20], for a set \( E \) satisfying the condition (S) we have
\[
\beta = \dim(E) = \dim_M(E) = \dim_M(E).
\]

In such case Proposition 2.5 becomes

**Proposition 2.8.** Let \( B^H \), \( f \) and \( F \) as in Theorem 2.2. Let \( E \) be a subset of \( I \) satisfying the condition (S). Then there is a positive and finite constant \( c_{10} \) which depends on \( E, F, H, K \) and \( \beta \), such that
\[
c_{10}^{-1} C_{d-\beta/H}(F) \leq \mathbb{P}\{(B^H + f)(E) \cap F \neq \emptyset\} \leq c_{10} H^{d-\beta/H}(F). \tag{2.36}
\]

Proof. Three cases are to be discussed here: (i) \( \beta < Hd \), (ii) \( \beta = Hd \) and (iii) \( \beta > Hd \). Let us point out first that for the lower bound, the interesting cases are (i) and (ii) while for the upper bound it is the case (i) which requires proof. Indeed we have from [2.4] that \( C_\alpha(\cdot) = 1 \) for \( \alpha < 0 \) and \( H^\alpha(\cdot) \) is assumed to be equal to 1 whenever \( \alpha \leq 0 \). In this regard, a close reading of the proof of Proposition 2.5 is required. Thus, we can clearly see that it is based on two key estimates, namely (2.29) and (2.33) for the lower and upper bound respectively. In what follows we will establish such estimates under the condition (S). The estimation (2.29) with \( \beta \) is now a part of the condition (S). In order to establish (2.33), we will show that for all \( \delta > 0 \)
\[
N(E, \delta) \leq C\delta^{-\beta}, \tag{2.37}
\]
where \( C \) is a positive and finite constant which depend on \( I \) only. Indeed, let \( 0 < \delta \leq 1 \) and \( P(E, \delta) \) be the greatest number of disjoint intervals \( I_j \) centred in \( x_j \in E \) with length \( \delta \) required to cover \( E \). Condition (S) ensures that
\[
c_{9}^{-1} P(E, \delta)(\delta/2)^\beta \leq \sum_{j=1}^{P(E, \delta)} \nu(I_j) = \nu(E) \leq 1.
\]

Using the fact that
\[
N(E, 2\delta) \leq P(E, \delta),
\]
we obtain the desired estimation (2.33). The rest of the proof follows closely the lines of that of Proposition 2.5 especially given that Lemma 2.6 takes into account the case (ii). \( \square \)

Following the same pattern as above we get the following proposition, which can be considered also as a corollary of Theorem 2.1 in [6], for the subset \( E \) of \( I \) satisfying the condition (S).

**Proposition 2.9.** Let \( B^H \) and \( F \) as in Theorem 2.2. Let \( E \) be a subset of \( I \) satisfying the condition (S). Then there is a positive and finite constant \( c_{11} \) which depends on \( E, F, H, K \) and \( \beta \), such that
\[
c_{11}^{-1} C_{d-\beta/H}(F) \leq \mathbb{P}\{B^H(E) \cap F \neq \emptyset\} \leq c_{11} H^{d-\beta/H}(F). \tag{2.38}
\]
We would like to point out that, when the drift $f$ is $H$-Hölder continuous and $E$ satisfies the condition (S), Propositions 2.19 and 2.21 assert that the hitting probabilities of $(B^H + f)$ behave like the ones of $B^H$ in the following sense:

- if $\dim(F) < d - \beta / H$, then $\mathbb{P}((B^H + f)(E) \cap F \neq \emptyset) = \mathbb{P}(B^H(E) \cap F \neq \emptyset) = 0$,
- if $\dim(F) > d - \beta / H$, then $\mathbb{P}((B^H + f)(E) \cap F \neq \emptyset) > 0$ and $\mathbb{P}(B^H(E) \cap F \neq \emptyset) > 0$.

This brings us to the following question: when $\dim(F) < d - \beta / H$, is it possible to get a function, with smaller Hölder order than $H$, for which $\mathbb{P}((B^H + f)(E) \cap F \neq \emptyset) > 0$? In order to address this question, we need to consider $\alpha < H$ such that $d - \beta / \alpha < \dim(F)$ and $(\Omega', \mathcal{F}', \mathbb{P}')$ another probability space on which we define a fractional Brownian motion $B^\alpha$ with Hurst parameter $\alpha$. We will work with the mixed process $Z^{H,\alpha}$ defined on the probability space $(\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', \mathbb{P} \otimes \mathbb{P}')$ by

$$Z^{H,\alpha}(t, \omega) = B^H(t, \omega) + B^\alpha(t, \omega')$$

for all $t \geq 0$ and $(\omega, \omega') \in \Omega \times \Omega'$. \hspace{1cm} (2.39)

It is easy to see that $Z^{H,\alpha} = (Z_1^{H,\alpha}, \ldots, Z_d^{H,\alpha})$ where $Z_i^{H,\alpha}$ are independent copies of a real valued Gaussian process $Z_0^{H,\alpha}$ on $(\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', \mathbb{P} \otimes \mathbb{P}')$ with stationary increment and the covariance function given by

$$\mathcal{E}(Z_0^{H,\alpha}(s), Z_0^{H,\alpha}(t)) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}) + \frac{1}{2}(s^{2\alpha} + t^{2\alpha} - |t-s|^{2\alpha}),$$

where $\mathcal{E}$ denote the expectation under the probability $\mathbb{P} \otimes \mathbb{P}'$. The following lemma is about the strong local nondetermination property of the process $Z$. Let $I \subset (0,1]$, be a closed interval, then we have

**Lemma 2.10.** The real-valued process $\{Z_t^{H,\alpha} : t \geq 0\}$ satisfy the following:

1. For all $s, t \in [0,1]$,

$$|s - t|^{2\alpha} \leq \mathcal{E}(Z_0^{H,\alpha}(t) - Z_0^{H,\alpha}(s))^2 \leq 2|s - t|^{2\alpha}. \hspace{1cm} (2.40)$$

2. There exists a positive constant $C$ depending on $\alpha$, $H$ and $I$ only, such that

$$\text{Var} \left( Z_0^{H,\alpha}(u) | Z_0^{H,\alpha}(t_1), \ldots, Z_0^{H,\alpha}(t_n) \right) \geq C \left[ \min_{0 \leq k \leq n} |u - t_k|^{2\alpha} + \min_{0 \leq k \leq n} |u - t_k|^{2H} \right], \hspace{1cm} (2.41)$$

for all integers $n \geq 1$, all $u, t_1, \ldots, t_n \in I$ and $t_0 = 0$.

3. There exists a positive constant $C$ depending on $\alpha$, $H$ and $I$ only. Such that for any $t \in I$ and any $0 < r \leq t$,

$$\text{Var} \left( Z_0^{H,\alpha}(s) | Z_0^{H,\alpha}(t) : |s - t| \geq r \right) \geq Cr^{2\alpha}. \hspace{1cm} (2.42)$$

The proof of the lemma is that of Proposition 4.2. in [10]. Now we are able to provide an answer to the above question.

**Theorem 2.11.** Let $E$ be a compact set satisfying the condition (S) and $F \subset \mathbb{R}^d$ be a compact set such that $\dim(F) \leq d - \beta / H$. Then for all $\alpha < \frac{\beta}{d - \dim(F)}$ and for all $\varepsilon > 0$ small enough such that $C_{d-\beta/(\alpha+\varepsilon)}(F) > 0$, there exists a $\alpha$-Hölder continuous function $f : [0,1] \rightarrow \mathbb{R}^d$, satisfying

$$\mathbb{P}((B^H + f)(E) \cap F \neq \emptyset) \geq C_1 C_{d-\beta/(\alpha+\varepsilon)}(F), \hspace{1cm} (2.43)$$

where $C_1$ is a positive constant which depends on $\alpha, \beta, H, d$ and $\varepsilon$ only.
On the other hand we have a positive probability which is stronger than (3.1).

Our goal is to shed some light on this question. In fact we will need some additional information about (2.28) to obtain

\[ c \] for a simply measurable Borel function \( f \) for some fixed positive constant \( c \), such that

\[ \dim(B^H + f(\omega'))(E) \leq \dim(E) \],

then according to Theorem 1.2. in [24] we have

\[ c \geq 1 \] is a finite constant which depends on \( I, F \) and \( H \) only. Since \( E \) satisfies the condition (S) we use (2.28) to obtain

\[ c_3 c_{d-\beta/\alpha'}(F) \leq \tilde{P}(Z(E) \cap F) \leq c_4 \mathcal{H}^{d-\beta/\alpha'}(F), \]

where \( c_3 \) and \( c_4 \) are positive constants which depends on \( d, \beta, \) and \( \alpha' \) only. We can choose \( \varepsilon \) small enough such that \( c_{d-\beta/\alpha'}(F) > 0 \). Therefore by Fubini’s theorem we get

\[ \mathbb{E}'(\tilde{P}((B^H + B^{\alpha'}(\omega'))(E) \cap F) - c_5 c_{d-\beta/\alpha'}(F)) > 0, \]

for some fixed positive constant \( c_5 \in (0, c_3) \). The above inequality lead to

\[ \mathbb{P}'(\tilde{P}((B^H + B^{\alpha'}(\omega'))(E) \cap F) - c_5 c_{d-\beta/\alpha'}(F) > 0) > 0. \]

We therefore choose the function \( f \) among the paths of \( B^{\alpha'} \) satisfying the above. \( \square \)

Remark 2.12. The same reasoning should also apply to check for all \( \alpha < \frac{\beta}{d - \dim(F)} \) and for all \( \varepsilon > 0 \) small enough such that \( \mathcal{H}^{d-\beta/(\alpha+\varepsilon)}(F) < +\infty \), there exists a \( \alpha \)-Hölder continuous function \( f : [0, 1] \to \mathbb{R}^d \), satisfying

\[ \mathbb{P}((B^H + f)(E) \cap F) \leq c_6 \mathcal{H}^{d-\beta/(\alpha+\varepsilon)}(F), \] (2.44)

where \( c_6 \) is a positive constant which depends on \( \alpha, \beta, H, d \) and \( \varepsilon \) only.

3 Hitting points

Let’s start with the following fact:

Since \( f \) is \( H \)-Hölder continuous Proposition 2.7. in [10] ensures that \( \dim_{\Psi,H}(Gr_E(f)) = \dim(E) \) and then according to Theorem 1.2. in [24] we have

\[ \dim(B^H + f)(E) = \frac{\dim(E)}{H} \wedge d. \]

Hence, if \( H d < \dim(E) \) Theorem 3.2. in [10] implies that \( \mathbb{E}(\lambda_d(B^H + f)(E)) > 0, \) and a simple application of Fubini’s theorem leads to

\[ \lambda_d\{x \in \mathbb{R}^d : \mathbb{P}\{\exists t \in E : B^H(t) + f(t) = x\} > 0\} > 0. \] (3.1)

On the other hand we have \( d < \dim(E)/H = \dim_{\rho_H}(E \times \{x\}) \) for all \( x \in \mathbb{R}^d \), from which follows that \( c_{\rho_H,d}(E \times \{x\}) > 0 \). Using Theorem 2.2 we conclude that \( B^H + f \) restricted on \( E \) hits all points with positive probability which is stronger than (3.1).

When the function \( f \) loses the Hölder property, thus one wonders what about the less smooth functions? Our goal is to shed some light on this question. In fact we will need some additional information about the set \( Gr_E(f) \) in order to study the hitting probabilities points for the process \( B^H + f \). First we provide, for a simply measurable Borel function \( f \), the lower and upper bounds of hitting probabilities of points in terms of the parabolic capacity of \( Gr_E(f) \) of order \( d \) and the \( d \)-dimensional parabolic Hausdorff measure of \( Gr_E(f) \) respectively. It can be also seen as an extension of Theorem 2.2 to a measurable drift \( f \) and \( F = \{x\} \).
Proposition 3.1. Let \( \{B^H(t) : t \in [0,1]\} \) be a \( d \)-dimensional fractional Brownian motion with Hurst index \( H \in (0,1) \). Let \( f : [0,1] \to \mathbb{R}^d \) be a bounded Borel measurable function and let \( E \subset (0,1) \) be a Borel set. Then for all \( x \in \mathbb{R}^d \) there is a finite constant \( C_1 \geq 1 \) such that

\[
C_1^{-1}C_{\rhoH,d}(Gr_E(f)) \leq \mathbb{P}\{ \exists t \in E : (B^H + f)(t) = x \} \leq C_1 \mathcal{H}_{\rhoH,d}^d(Gr_E(f)). \tag{3.2}
\]

Proof. We will closely follow the same steps as in the proof of Theorem 2.2. We start with the upper bound using again the covering argument. Choose an arbitrary constant \( \gamma > \mathcal{H}_{\rhoH,d}(Gr_E(f)) \). Then there is a covering of \( Gr_E(f) \) by balls \( \{B_{\rhoH}((ti, yi), i, t) \), \( i \geq 1 \} \in \mathbb{R}^d \) such that

\[
Gr_E(f) \subseteq \bigcup_{i=1}^{\infty} B_{\rhoH}((ti, yi), 1) \text{ and } \sum_{i=1}^{\infty} (2r_i)^d \leq \gamma. \tag{3.3}
\]

It is easy to see that

\[
\{ \exists s \in E : (B^H + f)(s) = x \} \subseteq \bigcup_{i=1}^{\infty} \left\{ \exists (s, f(s)) \in \left( t_i - r_i^{1/H}, t_i + r_i^{1/H} \right) \times B(y_i, r_i) \text{ s.t. } (B^H + f)(s) = x \right\}.
\]

Since for every fixed \( i \geq 1 \) we have

\[
\left\{ \exists (s, f(s)) \in \left( t_i - r_i^{1/H}, t_i + r_i^{1/H} \right) \times B(y_i, r_i) \text{ s.t. } (B^H + f)(s) = x \right\} \subseteq \left\{ \inf_{\|s - ti\| < r_i} \|B^H(s) - x - y_i\| \leq r_i \right\}, \tag{3.4}
\]

then we get from Lemma 2.3 that

\[
\mathbb{P}\left\{ \exists (s, f(s)) \in \left( t_i - r_i^{1/H}, t_i + r_i^{1/H} \right) \times B(y_i, r_i) \text{ s.t. } (B^H + f)(s) = x \right\} \leq \mathbb{P}\left\{ \inf_{\|s - ti\| < r_i} \|B^H(s) - x - y_i\| \leq r_i \right\} \leq C_2 r_i^d, \tag{3.5}
\]

where \( C_2 \) depends on \( H, E \) and \( f \) only. Combining (3.3), (3.4), (3.5) and (3.6) we derive that

\[
\mathbb{P}\{ \exists s \in E : (B^H + f)(s) = x \} \leq C_3 \gamma,
\]

where \( C_3 \) depends only on \( I, H \) and \( f \). Let \( \gamma \downarrow \mathcal{H}_{\rhoH,d}^d(Gr_E(f)) \), the upper bound in (3.2) follows.

The lower bound in (3.2) holds also from the second moment argument. We assume that \( C_{\rhoH,d}(Gr_E(f)) > 0 \), then let \( \sigma \) be a measure supported on \( Gr_E(f) \) such that

\[
\mathcal{E}_{\rhoH,d}(\sigma) = \int_{Gr_E(f)} \int_{Gr_E(f)} \frac{d\sigma(s, f(s))d\sigma(t, f(t))}{\rhoH((s, f(s)), (t, f(t)))^d} \leq \frac{2}{C_{\rhoH,d}(Gr_E(f))}. \tag{3.7}
\]

Let \( \nu \) be the measure on \( E \) satisfying \( \nu := \sigma \circ P_1^{-1} \) where \( P_1 \) is the projection mapping on \( E \), i.e. \( P_1(s, f(s)) = s \). For \( n \geq 1 \) we consider a family of random measures \( \nu_n \) on \( E \) defined by

\[
\int_E g(s)\nu_n(ds) = \int_E (2\pi n)^{d/2} \exp \left( -\frac{n\|B^H(s) + f(s) - x\|^2}{2} \right) g(s)\nu(ds), \tag{3.8}
\]

where \( g \) is an arbitrary measurable function on \( \mathbb{R}_+ \). Following the same steps as in (2.14) we obtain that there exists two positives constants \( C_4 \) and \( C_5 \) such that
In other words, that the restriction of 

\[ \mathbb{E}(\|\nu_n\|) \geq \int_{E} \left( \frac{2\pi}{1 + s^{2H}} \right)^{d/2} \exp \left( -\frac{\|x - f(s)\|^{2}}{2s^{2H}} \right) \nu(ds) \geq c_4 > 0, \tag{3.9} \]

and

\[ \mathbb{E} \left( \|\nu_n\|^2 \right) = \int_{E} \int_{E} \nu(ds)\nu(dt) \int_{\mathbb{R}^{2d}} e^{-i\langle \xi, x - f(s) \rangle + \langle \eta, x - f(t) \rangle} \times \exp \left( -\frac{1}{2} (\xi, \eta) (n^{-1} I_{2d} + \text{Cov}(B^H(s), B^H(t))) (\xi, \eta)^T \right) d\xi d\eta \]

\[ \leq c_5 \int_{GrE(f)} \int_{GrE(f)} \frac{d\sigma(s, f(s))d\sigma(t, f(t))}{\max\{|t - s|^{H}, \|f(t) - f(s)\|\}^{d}} = c_5 \mathcal{E}_{\rho_H,d}(\sigma) < \infty. \tag{3.10} \]

where the last inequality is a direct consequence of Lemma 2.3. Using once again the Paley-Zygmund inequality, we conclude that \((\nu_n)_{n \geq 1}\) admits a subsequence converging weakly to a finite measure \(\tilde{\mu}\) supported on the set \(\{(s, x) \in E \times F : B^H(s) + f(s) = x\}\), positive with positive probability and also satisfying the moment estimates of 2.14. Hence we have

\[ \mathbb{P} (\exists s \in E : (B^H + f)(s) = x) \geq \mathbb{P} (\|\tilde{\mu}\| > 0) \geq \frac{\mathbb{E}(\|\tilde{\mu}\|^2)}{\mathbb{E}(\|\tilde{\mu}\|^2)} \geq \frac{c_4}{c_5 \mathcal{E}_{\rho_H,d}(\sigma)}. \tag{3.11} \]

Combining this with (3.7) yields the lower bound in (3.2). The proof is completed. \(\square\)

**Corollary 3.2.** Let \(\{B^H(t) : t \in [0, 1]\}\) be a \(d\)-dimensional fractional Brownian motion of Hurst index \(H \in (0, 1)\) and \(E \subset [0, 1]\) be a Borel set. Then for any \(\alpha < \text{dim}(E)/d \wedge H\) there exists a \(\alpha\)-Hölder continuous function \(f : [0, 1] \to \mathbb{R}^d\) such that, for all \(x \in \mathbb{R}^d\), we have

\[ \mathbb{P} \{\exists t \in E : (B^H + f)(t) = x\} > 0. \tag{3.12} \]

In other words, that the restriction of \(f\) to \(E\) is non-polar for \(B^H\).

**Proof.** Let \(\varepsilon > 0\) such that \(\alpha'' = \alpha + \varepsilon < \text{dim}(E)/d \wedge H\) and \(B^{\alpha''}\) be the fractional Brownian motion defined above. It is known from Theorem 2.9 and Corollary 2.11 in [10] that, for \(\alpha'' < \text{dim}(E)/d \wedge H\), we have

\[ \text{dim}_{\rho_H} (Gr_E(B^{\alpha''})) = \text{dim}(E)/\alpha'' \wedge (\text{dim}(E)/H + d(1 - \alpha''/H)) < d, \mathbb{P}'a.s. \]

Then for any fixed \(x \in \mathbb{R}^d\), Proposition 3.1 tells us that for \(\mathbb{P}'\) almost all \(\omega' \in \Omega'\) there is a positive constant \(C = C(\omega')\) such that

\[ \mathbb{P} \{\exists s \in E : (B^H + B^{\alpha}(\omega'))(s) = x\} \geq C \mathcal{E}_{\rho_H,d} \left( Gr_E(B^{\alpha''}(\omega')) \right) > 0. \]

Hence, if we choose \(f\) to be one of the trajectories of \(B^{\alpha''}\), which is \(\alpha\)-Hölder continuous, we obtain

\[ \text{dim}_{\rho_H} (Gr_E(f)) > d. \]

Therefore for any fixed \(x \in \mathbb{R}^d\), \(\{(B^H + f(s), s \in E\}\) hits \(x\) with positive probability. \(\square\)

**Remark 3.3.** 1. 2. We mention that the covering argument used to prove the upper bound in (3.2) can also serve to show that for any Borel set \(F \subset \mathbb{R}^d\), there exists a positive finite constant \(C\) such that

\[ \mathbb{P} \{(B^H + f)(E) \cap F \neq \emptyset\} \leq C \mathcal{H}^d_{\rho_H}(Gr_E(f) \times F). \tag{3.13} \]
Here $\mathcal{H}_E^\alpha (\cdot )$ is the $\alpha$-dimensional Hausdorff measure on the metric space $(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d, \tilde{\rho}_H)$, where the $\tilde{\rho}_H$ is defined by
\[
\tilde{\rho}_H((s, x, u), (t, y, v)) := \max\{|t - s|^H, \|x - y\|, \|u - v\|\}.
\]
But we have difficulty in proving the lower band in terms of $C_{\tilde{\rho}_H,d}(\text{Gr}_F(f) \times F)$ even when $F$ has some smooth structure.

Accordingly, in view of the foregoing, can we expect the same result for others functions with fewer restrictions? This impels us to consider the class of reverse $\alpha$-Hölder continuous functions whose definition is as follows

**Definition 3.4.** We say that a continuous function $f : [0, 1] \to \mathbb{R}^d$ is reverse $\alpha$-Hölder continuous, for $0 < \alpha < 1$, if there exists a constant $C > 0$ such that for any interval $J \subset [0, 1]$, we have
\[
\sup_{s, t \in J} \|f(s) - f(t)\| \geq C|J|^\alpha,
\]
where $|J|$ is the diameter of the interval $J$.

Recall that this notion is closely linked to the geometric properties of the graph of the function $f$. For $d = 1$, a famous example of a function satisfying the reverse $\alpha$-Hölder continuity condition is the Weierstrass function given by
\[
W_{\tau, \theta}(t) = \sum_{n=0}^{\infty} \tau^n \cos(2\pi \theta^n t), \quad t \in [0, 1],
\]
for $\tau < 1 < \theta$ and $\tau \theta > 1$ and $\alpha = -\log(\tau)/\log(\theta)$. See [5] or [12] for the proof. Recently Shen [28], improving result of Barański, Bárány and Romanowska [3], proved that for any integer $\theta \geq 2$ and any $\tau \in (\theta^{-1}, 1)$, the Hausdorff dimension of the graph of the Weierstrass function $W_{\tau, \theta}$ is equal to $2 + \log(\tau)/\log(\theta)$.

It is worth pointing out that, for $H = 1$, the metric $\rho_H$ on $\mathbb{R}_+ \times \mathbb{R}$ defined by
\[
\rho_H((s, x), (t, y)) = \max\{|t - s|^H, \|x - y\|\} \quad \forall (s, x), (t, y) \in \mathbb{R}_+ \times \mathbb{R}
\]
is nothing but the metric derived from the Maximum norm on $\mathbb{R}^2$. Now using a comparison result for the Hausdorff parabolic dimensions with different parameters, see Proposition 2.5 in Erraoui and Hakiki [10] (which remains valid also for $H$ and $1$), one can check that
\[
1 < 2 + \frac{\log(\tau)}{\log(\theta)} = \dim(\text{Gr}_{[0,1]}(W_{\tau, \theta})) \leq \dim_{\rho_H}(\text{Gr}_{[0,1]}(W_{\tau, \theta})).
\]
Thus $\dim_{\rho_H}(\text{Gr}_{[0,1]}(W_{\tau, \theta})) > 1$ and therefore $C_{\rho_H,1}(\text{Gr}_{[0,1]}(W_{\tau, \theta})) > 0$. This is expressed in the following

**Proposition 3.5.** For any integer $\theta \geq 2$ and any $\tau \in (\theta^{-1}, 1)$, the Weierstrass function $\{W_{\tau, \theta}(t), t \in [0, 1]\}$ is non-polar for real valued fractional Brownian motion $\{B^H(t), t \in [0, 1]\}$ with Hurst index $H \in (0, 1)$.

**Remark 3.6.** It is worth mentioning that the Hölder continuity of order $\alpha = -\log(\tau)/\log(\theta)$ is also met by the Weierstrass function $\{W_{\tau, \theta}(t), t \in [0, 1]\}$, cf. Lemma 5.1.8 in [5].

It is also interesting to note that, in the same context, Theorem 4 in [27] affirms that, for any $\alpha$-Hölder and reverse $\alpha$-Hölder continuous function $f$ with $\alpha \in (0, 1)$, we have $\dim_{[0,1]}(\text{Gr}(f)) > 1$ which was the key element in the proof of the above proposition. Therefore, we will come to the same conclusion as the one for the Weierstrass function stated as follows
Proposition 3.7. Suppose that $0 < \alpha < 1$ and $f : [0,1] \to \mathbb{R}$ is $\alpha$-Hölder and reverse $\alpha$-Hölder continuous function. Then $f$ is non-polar for real valued fractional Brownian motion $\{B^H(t), t \in [0,1]\}$ with Hurst index $H \in (0,1)$.

This raises the question whether the result remains valid in higher dimensions. However Proposition 3.7 cannot be extended to $d$-dimensional case as is shown by the following

Proposition 3.8. Let $\{B^H(t) : t \in [0,1]\}$ be a $d$-dimensional fractional Brownian motion of Hurst index $H \in (0,1)$ such that $Hd > 1$. Then there exists a $\alpha$-Hölder and reverse $\alpha$-Hölder continuous function $f : [0,1] \to \mathbb{R}^d$ such that the process $\{(B^H + f)(t) : t \in [0,1]\}$ does not hit $x$ for all $x \in \mathbb{R}^d$, i.e.

$$
P \{ \exists t \in [0,1] : (B^H + f)(t) = x \} = 0.
$$

Proof. Let $d \geq 2$ and define the function $f$ from $[0,1]$ to $\mathbb{R}^d$ by $f(t) = (W_{\tau,\theta}(t), \ldots, W_{\tau,\theta}(t))$ where $W_{\tau,\theta}$ is the Weierstrass function given in (3.15). It is easy to see that $f$ satisfies the $\alpha$-Hölder and reverse $\alpha$-Hölder conditions with $\alpha = -\frac{\log(\tau)}{\log(\theta)}$. Moreover we have $\dim_{\rho_H}(Gr_{[0,1]}(f)) = \dim_{\rho_H}(Gr_{[0,1]}(W_{\tau,\theta}))$. It follows from Proposition 2.5 in Erraoui and Hakiki [10] that

$$
\dim_{\rho_H}(Gr_{[0,1]}(f)) \leq \dim(Gr_{[0,1]}(f)) - 1 + \frac{1}{H} = 1 + \frac{\log(\tau)}{\log(\theta)} + \frac{1}{H}.
$$

Then for any $\theta > 1$ and $\tau \in (\theta^{-1}, \theta^{d-(1/H)-1} \wedge 1)$ we have $\dim_{\rho_H}(Gr_{[0,1]}(f)) < d$. The later entails that $\mathcal{H}^d_{\rho_H}(Gr_{[0,1]}(f)) = 0$. Proposition (3.1) will allow us to achieve the desired outcome. \qed

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