Hermitian Curvature Flow on Compact Homogeneous Spaces

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Received: 29 March 2019 / Published online: 16 July 2019
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Abstract
We study a version of the Hermitian curvature flow on compact homogeneous complex manifolds. We prove that the solution has a finite extinction time $T > 0$ and we analyze its behavior when $t \to T$. We also determine the invariant static metrics and we study the convergence of the normalized flow to one of them.

Keywords Lie group actions · Homogeneous complex manifolds · Hermitian curvature flow

Mathematics Subject Classification 53C25 · 53C30

1 Introduction

Given a Hermitian manifold $(M, J, h)$, it is well known that there exists a family of metric connections leaving the complex structure $J$ parallel (see [6]). Among these, the Chern connection is particularly interesting and provides different Ricci tensors which can be used to define several meaningful parabolic metric flows preserving the Hermitian condition and generalizing the classical Ricci flow in the non-Kähler setting. In [7], Gill introduced an Hermitian flow on a compact complex manifold involving the first Chern–Ricci tensor, namely, the one whose associated 2-form represents the first Chern class of $M$ (see also [14] for further related results). In [13], Streets and Tian introduced a family of Hermitian curvature flows (HCFs) involving the second Ricci tensor $S$ together with an arbitrary symmetric Hermitian form $Q(T)$ which is quadratic in the torsion $T$ of the Chern connection:

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\[ h'_t = -S + Q(T). \]

For any admissible \( Q(T) \), the corresponding flow is strongly parabolic and the short-time existence of the solution is established. This family includes geometrically interesting flows as for instance the *pluriclosed flow* that was previously introduced in [12] and preserves the pluriclosed condition \( \partial \bar{\partial} \omega = 0 \). More recently Ustinovskiy focused on a particular choice of \( Q(T) \) obtaining another remarkable flow, which we will call HCFU for brevity, with several geometrically relevant features (see [15]).

In particular, Ustinovskiy proves that the HCFU on a compact Hermitian manifold preserves Griffiths non-negativity of the Chern curvature, generalizing the classical result that Kähler–Ricci flow preserves the positivity of the bisectional holomorphic curvature (see e.g., [10]). In [17], the author could prove stronger results showing that the HCFU preserves several natural curvature positivity conditions besides Griffiths positivity. In [16], Ustinovskiy focuses on complex homogeneous manifolds and proves that the finite-dimensional space of induced metrics (which are not necessarily invariant) is preserved by the HCFU.

Given a connected complex Lie group \( G \) acting transitively, effectively, and holomorphically on a complex manifold \( M \), a simple observation shows that \( M \) does not carry any \( G \)-invariant Hermitian metric unless the isotropy is finite. More recently in [9] Lafuente, Pujia, and Vezzoni considered the behavior of a general HCF in the space of left-invariant Hermitian metrics on a complex unimodular Lie group.

In this work, we focus on C-spaces, namely, compact simply connected complex manifolds \( M \) which are homogeneous under the action of a compact semisimple Lie group \( G \). By classical results \( M \) fibers over a flag manifold \( N \) with a complex torus as a typical fiber \( F \); in particular, we consider the case where \( N \) is a product of compact Hermitian symmetric spaces and \( F \) is non-trivial (so that \( M \) does not carry any Kähler metric). While the analysis of a generic HCF on these manifolds seems to be out of reach, some special flows may deserve attention. In view of the classification results obtained in [5] for C-spaces carrying invariant SKT metrics, the pluriclosed flow can be investigated only in very particular cases of such spaces (see also [3] for the analysis of the pluriclosed flow on compact locally homogeneous surfaces and [2] for the case of left-invariant metrics on Lie groups). On the other hand, the HCFU is geometrically meaningful and can be dealt with more easily. Actually, we are able to write down the flow equations in the space of \( G \)-invariant metrics in a surprisingly simple way. In particular, we prove the remarkable fact that the flow can be described by an induced flow on the base, which depends only on the initial conditions on the base itself, and an induced flow on the fiber. Moreover the maximal existence domain is bounded above and we can provide a precise description of the limit metric. Indeed we see that the kernel of the limit metric defines an integrable distribution whose leaves coincide with the orbits of the complexification of a suitable normal subgroup \( S \) of \( G \). When the leaves of this foliation are closed, we can prove the Gromov–Hausdorff convergence of the space to a lower-dimensional Riemannian homogeneous space. We are also able to establish the existence and the uniqueness up to homotheties of invariant static metrics for the HCFU, providing new examples of non-Kähler static metrics on compact Hermitian manifolds. These metrics turn out to be particularly...
meaningful when \( S = G \), as in this case the normalized flow with constant volume converges to one of them.

The work is organized as follows. In Sect. 2, we give some preliminary notions on C-spaces and the HCF_U. In Sect. 3, we compute the invariant tensors involved in the flow equations and in Sect. 4 we prove our main result, which is summarized in Theorem 4.4 and Proposition 4.5.

**Notation**

Throughout the following, we will denote Lie groups by capital letters and the corresponding Lie algebras by gothic letters. The Cartan Killing form of a Lie algebra will be denoted by \( \kappa \).

### 2 Preliminaries

We consider a compact simply connected complex manifold \((M, J)\) with \( \dim \mathbb{C} M = m \) and we assume that it is homogeneous under the action of a compact semisimple Lie group \( G \) of biholomorphisms, namely, \( M = G/L \) for some compact subgroup \( L \subseteq G \). By Tits fibration theorem, the manifold \( M \) fibers \(-\)equivariantly onto a flag manifold \( N := G/K \), say \( \pi : M \to G/K \), and the manifold \( N \) can be endowed with a \( G \)-invariant complex structure \( I \) so that the fibration \( \pi : (M, J) \to (N, I) \) is holomorphic. Since \( M \) is supposed to be simply connected, the typical fiber \( F := K/L \) is a complex torus of complex dimension \( k \) (see e.g., [1]). Such a homogeneous complex manifold, which will be called a simply connected C-space (see [18]), is not Kähler if the fiber \( F \) is not trivial.

In this work, we will assume that the base of the Tits fibration is a product of irreducible Hermitian symmetric spaces of complex dimension at least two. More precisely, if we write \( G \) as the (local) product of its simple factors, say \( G = G_1 \times \cdots \times G_s \), then \( K \) also splits accordingly as \( K = K_1 \times \cdots \times K_s \) with \( K_i \subseteq G_i \) and \((G_i, K_i)\) is a Hermitian symmetric pair with \( \dim \mathbb{C} G_i/K_i = n_i \). Note that \( m = k + \sum_{i=1}^{s} n_i \).

At the level of Lie algebras, we can write the Cartan decomposition \( g_i = \mathfrak{k}_i \oplus \mathfrak{n}_i \) for each \( i = 1, \ldots, s \). We recall that the center of \( \mathfrak{k}_i \) is one-dimensional and spanned by an element \( Z_i \) so that \( \mathfrak{k}_i = \mathfrak{s}_i \oplus \mathbb{R} Z_i \), \( \mathfrak{s}_i \) being the semisimple part of \( \mathfrak{k}_i \), and \( \text{ad}(Z_i) = I|_{\mathfrak{n}_i} \). We can now write the following decompositions

\[
\mathfrak{l} = \bigoplus_{i=1}^{s} \mathfrak{s}_i \oplus \mathfrak{b}, \quad \mathfrak{g} = \mathfrak{l} \oplus \mathfrak{f} \oplus \mathfrak{n},
\]

where \( \mathfrak{n} := \bigoplus_{i=1}^{s} \mathfrak{n}_i \) and \( \mathfrak{b}, \mathfrak{f} \) are abelian subspaces of \( \mathfrak{z}(\mathfrak{l}) = \bigoplus_{i=1}^{s} \mathbb{R} Z_i \) with \( \kappa(\mathfrak{b}, \mathfrak{f}) = 0 \). Note that \( \mathfrak{f} \) and \( \mathfrak{m} := \mathfrak{f} \oplus \mathfrak{n} \) identify with the tangent spaces to the fiber and to \( M \), respectively.

Since the fibration \( \pi : M \to N \) is holomorphic, the complex structure \( J \in \text{End}(\mathfrak{m}) \) can be written as \( J = I_F + I \), where \( I_F \) is a totally arbitrary complex structure on the fiber \( F \).
We now consider a $G$-invariant Hermitian metric $h$ on $M$, which can be seen as an ad(l)-invariant Hermitian inner product on $m$. As the $s_i$ are not trivial, we have that $l$ acts non-trivially on $n$ and trivially on $f$, therefore $h(f, n) = 0$. In particular, the restriction $h|_{f \times f}$ is an arbitrary Hermitian metric.

Moreover, the ad(l)-modules $n_i$ are mutually non-equivalent, hence $h(n_i, n_j) = 0$ if $i \neq j$.

If $n_i$ is $s_i$-irreducible, then Schur Lemma implies that

$$h|_{n_i \times n_i} := -h_i \kappa_i,$$

where $h_i \in \mathbb{R}^+$ and $\kappa_i$ denotes the Cartan Killing form on $g_i$. Note that this is always the case unless $g_i = \mathfrak{so}(n+2)$ and $\mathfrak{f}_i = \mathfrak{so}(2) \oplus \mathfrak{so}(n)$ ($n \geq 3$). Throughout the following we will assume that none of the Hermitian factors of the basis $N$ is a complex quadric.

Given a $G$-invariant Hermitian metric $h$ on $M$, we can consider the associated Chern connection $\nabla$, which is the unique metric connection ($\nabla h = 0$) that leaves $J$ parallel ($\nabla J = 0$) and whose torsion $T$ satisfies for $X, Y$ tangent vectors

$$T(JX, Y) = T(X, JY) = JT(X, Y).$$

The curvature tensor $R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ of $\nabla$ gives rise to different Ricci tensors and we are mainly interested in the second Chern–Ricci tensor $S$ which is given by

$$S(X, Y) = \sum_{a=1}^{2m} h(JR_{e_a, Ja} X, Y),$$

where $\{e_1, \ldots, e_{2m}\}$ denotes an $h$-orthonormal basis. In [13], Streets and Tian have introduced a family of Hermitian curvature flows on any complex manifold given by

$$h'_t = -S(h) + Q(T), \quad (2.1)$$

where $Q(T)$ is symmetric, $J$-invariant tensor which is an arbitrary quadratic expression involving the torsion $T$. In [13], the authors proved the short-time existence for all these flows for any initial Hermitian metric. In [15], Ustinovskiy considered a special Hermitian flow where the quadratic term $Q(T)$ is given in complex coordinates by

$$Q(T)_{ij} = -\frac{1}{2} h^{m\bar{m}} h^{p\bar{p}} T_{mpj} T_{\bar{n}\bar{s}i}. \quad (2.2)$$

For the corresponding Hermitian curvature flow (HCFU) Ustinovskiy could prove several important properties, in particular that it preserves the Griffiths positivity of the initial metric.

We now focus on this Hermitian flow on the class of homogeneous compact complex manifolds $(M, J)$ we have introduced in this section. In particular, we note that the flow evolves along invariant Hermitian metrics whenever the initial metric is so.
3 The Computation of the Tensors

In this section, we compute the Ricci tensor $\mathcal{R}$ and the quadratic expression $Q(T)$ in (2.2) for a $G$-invariant Hermitian metric $h$ on the complex manifold $M = G/L$. Throughout the following, we keep the notation introduced in the previous section.

We choose a maximal abelian subalgebra $a_i \subset \frak{k}_i$ of $\frak{g}_i$ ($i = 1, \ldots, s$). The complexification $\frak{a}^\mathbb{C} \subset \frak{g}^\mathbb{C}$ where $\frak{a} := \bigoplus_{i=1}^s a_i$, gives a Cartan subalgebra of $\frak{g}^\mathbb{C}$ and we will denote by $\mathcal{R}$ the associated root system of $\frak{g}^\mathbb{C}$. We denote by $R_i$ the subset of $R$ given by all the roots whose corresponding root vectors belong to $\frak{g}_i$ for $i = 1, \ldots, s$, so that $\mathcal{R} = R_1 \cup \ldots \cup R_s$. For each $i = 1, \ldots, s$ we have

$$\frak{t}_i^\mathbb{C} = a_i^\mathbb{C} \oplus \bigoplus_{\alpha \in R_{\frak{t}_i}} \frak{g}_\alpha, \quad \frak{n}_i^\mathbb{C} = \bigoplus_{\alpha \in R_{\frak{n}_i}} \frak{g}_\alpha$$

so that $\mathcal{R}_i = \mathcal{R}_{\frak{t}_i} \cup \mathcal{R}_{\frak{n}_i}$. Moreover, the invariant complex structure $I|_{\frak{n}_i}$ on $\frak{n}_i$ determines an invariant ordering of $\mathcal{R}_{\frak{n}_i} = \mathcal{R}_{\frak{n}_i}^+ \cup \mathcal{R}_{\frak{n}_i}^-$ with $I(v) = \pm \sqrt{-1} v$ for every $v \in \frak{a}_\alpha$ with $\alpha \in \mathcal{R}_{\frak{n}_i}$.

We will use a standard Chevalley basis $\{E_\alpha\}_{\alpha \in \mathcal{R}}$ of $\frak{g}^\mathbb{C}$ with

$$\frak{g}^\mathbb{C} = \frak{a}^\mathbb{C} \oplus \bigoplus_{\alpha \in \mathcal{R}} \mathbb{C}E_\alpha, \quad \overline{E_\alpha} = -E_{-\alpha}, \quad \kappa(E_\alpha, E_{-\alpha}) = 1, \quad [E_\alpha, E_{-\alpha}] = H_\alpha,$$

where $\kappa(H_\alpha, v) = \alpha(v)$ for every $v \in \frak{a}^\mathbb{C}$.

Let $h$ be an invariant Hermitian metric on $M$, i.e., an ad(l)-invariant symmetric Hermitian inner product on $\frak{m}$. We recall that $h|_{\frak{n}_i \times \frak{n}_i} := -h_{i\bar{i}}$ and by the ad($\frak{a}$)-invariance be have

$$h(E_\alpha, \overline{E_\beta}) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ h_{i\bar{i}} & \text{if } \alpha = \beta \in R_i. \end{cases}$$

In the complexified tangent space $\mathfrak{f}^\mathbb{C}$ of the fiber, we fix a complex basis $\mathcal{V} := \{V_1, \ldots, V_k\}$ of $\mathfrak{f}^{10}$. We also put $h_{ab} := h(V_a, \overline{V_b})$ and $H := (h_{ab})_{a,b=1,\ldots,k}$.

If we split $\frak{m}^\mathbb{C} = \frak{m}^{10} \oplus \frak{m}^{01}$ with respect to the extension of $J$ on $\frak{m}^\mathbb{C}$, the torsion $T$ can be seen as an element of $\Lambda^2(\frak{m}^\mathbb{C}) \otimes \frak{m}$ and it satisfies

$$T(\frak{m}^{10}, \frak{m}^{01}) = 0. \quad (3.1)$$

The Chern connection $\nabla$ is completely determined by the corresponding Nomizu’s operator $\Lambda \in \frak{m}^{*} \otimes \text{End}(\frak{m})$ (see e.g., [8]), which is defined as follows: for $X, Y \in \frak{m}$ we set $\Lambda(X)Y \in \frak{m}$ to be so that at $o := [L] \in M$

$$(\Lambda(X)Y)^*|_o = (\nabla_X^*Y^* - [X^*, Y^*])|_o,$$

where for every $Z \in \frak{m}$ we denote by $Z^*$ the vector field on $M$ induced by the one-parameter subgroup $\exp(itZ)$. Since $\nabla$ preserves $J$, the Nomizu’s operator satisfies
\[ \Lambda(v)(m^{10}) \subseteq m^{10} \text{ for every } v \in m^{c}. \] Now we recall that the torsion \( T \) can be expressed as follows: for \( X, Y \in m \) (see e.g., [8])

\[
T(X, Y) = \Lambda(X)Y - \Lambda(Y)X - [X, Y]_m. \tag{3.2}
\]

Therefore using (3.1) and (3.2), we see that

\[
\Lambda(A)\overline{B} = [A, \overline{B}]^{01} \quad \forall A, B \in m^{10}.
\tag{3.3}
\]

**Lemma 3.1** Given \( \nu \in \mathfrak{c} \), \( w \in \mathfrak{f}^{10} \), \( \alpha, \beta \in R_{n_i}^{+}, \gamma \in R_{n_j}^{+} \) with \( i, j = 1, \ldots, s, i \neq j \), we have

(a) \( \Lambda(E_\alpha)E_\beta = 0 \), \( \Lambda(E_\alpha)E_{-\gamma} = 0 \) and \( \Lambda(E_\alpha)\overline{w} = 0 \);

(b) \( \Lambda(E_\alpha)E_{-\beta} = 0 \) for \( \beta \neq \alpha \); \( \Lambda(E_\alpha)E_{-\alpha} = H^{01}_\alpha \);

(c) \( \Lambda(E_\alpha)w = \frac{1}{\nu_i}h(w, H_\alpha)E_\alpha \);

(d) \( \Lambda(v) = \text{ad}(v) \).

**Proof**

(a) If \( A \in n^{10} \) and \( w \in \mathfrak{f}^{10} \), we have by (3.3) and using the fact that \( [n_i, n_i] \subseteq \mathfrak{f}_i \):

\[
h(\Lambda(E_\alpha)E_\beta, \overline{A}) = -h(E_\beta, \Lambda(E_\alpha)\overline{A}) = -h(E_\beta, [E_\alpha, \overline{A}]^{01}) = 0,
\]

\[
h(\Lambda(E_\alpha)E_\beta, \overline{w}) = -h(E_\beta, \Lambda(E_\alpha)\overline{w}) = -h(E_\beta, [E_\alpha, \overline{w}]^{01}) = \alpha(\overline{w})h(E_\beta, E_\alpha^{01}) = 0,
\]

therefore \( \Lambda(E_\alpha)E_\beta = 0 \). Similarly we get \( \Lambda(E_\alpha)E_{-\gamma} = 0 \). Finally

\[
\Lambda(E_\alpha)\overline{w} = [E_\alpha, \overline{w}]^{01} = -\alpha(\overline{w})E_\alpha^{01} = 0.
\]

(b) We have \( \Lambda(E_\alpha)E_{-\beta} = [E_\alpha, E_{-\beta}]^{01} \). If \( \alpha \neq \beta \), then \( [E_\alpha, E_{-\beta}]^{01} = 0 \). If \( \alpha = \beta \), \( [E_\alpha, E_{-\alpha}]^{01} = H^{01}_\alpha \).

(c) Since we have, by (a) and (b), \( h(\Lambda(E_\alpha)w, E_{-\alpha}) = -h(w, \Lambda(E_\alpha)E_{-\alpha}) = -h(w, H_\alpha) \), and \( h(\Lambda(E_\alpha)w, E_{-\beta}) = h(\Lambda(E_\alpha)w, E_{-\gamma}) = h(\Lambda(E_\alpha)w, w') = 0 \) (here \( \beta \in R_{n_i}^{+}, \beta \neq \alpha, w' \in \mathfrak{f}^{10} \)), the assertion follows.

(d) First we have, if \( w, w' \in \mathfrak{f}^{10} \), \( \Lambda(w)\overline{w'} = [w, \overline{w'}]^{01} = 0 = \text{ad}(w)\overline{w'} \) and \( \Lambda(w)E_{-\alpha} = [w, E_{-\alpha}]^{01} = -\alpha(w)E_{-\alpha} = \text{ad}(w)E_{-\alpha} \). Furthermore,

\[
h(\Lambda(w)E_\alpha, \overline{w'}) = -h(E_\alpha, \Lambda(w)\overline{w'}) = 0 = \alpha(w)h(E_\alpha, \overline{w'}) = h(\text{ad}(w)E_\alpha, \overline{w'}),
\]

\[
h(\Lambda(w)E_\alpha, E_{-\beta}) = -h(E_\alpha, \Lambda(w)E_{-\beta}) = \beta(w)h(E_\alpha, E_{-\beta}) = \delta_{\alpha, \beta}\alpha(w)h(E_\alpha, E_{-\alpha}) = \alpha(w)h(E_\alpha, E_{-\beta}) = h(\text{ad}(w)E_\alpha, E_{-\beta}),
\]

therefore \( \Lambda(w)E_\alpha = \text{ad}(w)E_\alpha \). At this point it is easily seen that \( \Lambda(w)w' = 0 = \text{ad}(w)w' \), so \( \Lambda(w) = \text{ad}(w) \). Conjugation yields also \( \Lambda(\overline{w}) = \text{ad}(\overline{w}) \), hence assertion (d) follows. □
Using the previous lemma, we can compute the torsion tensor as follows.

**Lemma 3.2** Given $\alpha, \beta \in R^+_n$, $\gamma \in R^+_n$ with $i \neq j$, and $w, w' \in \mathfrak{f}^0$ we have

(a) $T(E_\alpha, E_\gamma) = T(E_\alpha, E_\beta) = 0$;
(b) $T(w, E_\alpha) = -\frac{1}{h_i} h(w, H_\alpha) E_\alpha$;
(c) $T(w, w') = 0$.

**Proof** (a) From Lemma 3.1(a) and (3.2), we see that

$$T(E_\alpha, E_\gamma) = -[E_\alpha, E_\gamma]_{m^c}, \quad T(E_\alpha, E_\beta) = -[E_\alpha, E_\beta]_{m^c}.$$ 

Now $\alpha + \gamma$ cannot be a root, therefore $[E_\alpha, E_\gamma] = 0$. On the other hand, since $G_i/K_i$ is symmetric, $[E_\alpha, E_\beta] \in \mathfrak{t}_i \cap \mathfrak{n}_i = 0$. This proves the assertion.

(b) By Lemma 3.1(c)–(d) and (3.2), we have

$$T(w, E_\alpha) = \alpha(w) E_\alpha - \frac{1}{h_i} h(w, H_\alpha) E_\alpha - \frac{1}{h_i} h(w, H_\alpha) E_\alpha = -\frac{1}{h_i} h(w, H_\alpha) E_\alpha.$$ 

(c) It follows immediately from Lemma 3.1(d).

In order to compute the curvature tensor $R$, we use the general formula given in [8, p. 192] (cf. also [11]): for $X, Y \in \mathfrak{m}$

$$R(X, Y) = [\Lambda(X), \Lambda(Y)] - \Lambda([X, Y]_m) - \text{ad}([X, Y]_i). \quad (3.4)$$

**Lemma 3.3** Given $\alpha, \beta \in R^+_n$, $\gamma \in R^+_n$, $i \neq j$, $v_1, v_2 \in \mathfrak{f}^c$, $w \in \mathfrak{f}^0$, we have

(a) $R(E_\alpha, \overline{E_\alpha}) E_\beta = \Lambda(E_\alpha) \Lambda(\overline{E_\alpha}) E_\beta + \beta(H_\alpha) E_\beta$,
(b) $R(E_\alpha, \overline{E_\alpha}) E_\gamma = 0$,
(c) $R(E_\alpha, \overline{E_\alpha}) w = \frac{1}{h_i} h(w, H_\alpha) \overline{H_\alpha^0}$,
(d) $R(v_1, v_2) = 0$.

**Proof** (a) Using (3.4) and Lemma 3.1, we have

$$R(E_\alpha, \overline{E_\alpha}) E_\beta = [\Lambda(E_\alpha), \Lambda(\overline{E_\alpha})] E_\beta - \Lambda([E_\alpha, \overline{E_\alpha}]_{m^c}) E_\beta - \text{ad}([E_\alpha, \overline{E_\alpha}]_{m^c}) E_\beta$$

$$= \Lambda(E_\alpha) \Lambda(\overline{E_\alpha}) E_\beta + \Lambda((H_\alpha)_{m^c}) E_\beta + \text{ad}((H_\alpha)_{m^c}) E_\beta$$

$$= \Lambda(E_\alpha) \Lambda(\overline{E_\alpha}) E_\beta + [H_\alpha, E_\beta]$$

$$= \Lambda(E_\alpha) \Lambda(\overline{E_\alpha}) E_\beta + \beta(H_\alpha) E_\beta.$$ 

(b) is proved similarly.

(c, d) We have, using (3.4), Lemma 3.1 and the fact that $\mathfrak{f}^c$ is abelian:

$$R(E_\alpha, \overline{E_\alpha}) w = -\Lambda(\overline{E_\alpha}) \Lambda(E_\alpha) w + \Lambda((H_\alpha)_{m^c}) w + \text{ad}((H_\alpha)_{m^c}) w$$

$$= -\frac{1}{h_i} h(w, H_\alpha) \Lambda(\overline{E_\alpha}) E_\alpha + [H_\alpha, w] = \frac{1}{h_i} h(w, H_\alpha) \overline{H_\alpha^0}.$$
and, for similar reasons,

\[ R(v_1, v_2) = [\Lambda(v_1), \Lambda(v_2)] = [\text{ad}(v_1), \text{ad}(v_2)] = \text{ad}([v_1, v_2]) = 0. \]

\[ \square \]

We can now compute the second Chern–Ricci tensor \( S \). If \( \beta \in R_{n_i}^+ \), we have by Lemmas 3.1, 3.3:

\[
S(E_\beta, E_\beta) = \sum_{j=1}^{s} \sum_{\alpha \in R_{n_i}^+} \frac{1}{h_j} h(R(E_\alpha, E_\alpha)E_\beta, E_\beta)
\]

\[
= \sum_{\alpha \in R_{n_i}^+} \frac{1}{h_i} h(R(E_\alpha, E_\alpha)E_\beta, E_\beta)
\]

\[
= - \sum_{\alpha \in R_{n_i}^+} \frac{1}{h_i} h(\Lambda(E_\alpha)E_\beta, \Lambda(E_\alpha)E_\beta) + \sum_{\alpha \in R_{n_i}^+} \beta(H_\alpha)
\]

\[
= - \frac{1}{h_i} h(\Lambda(E_\beta)E_{-\beta}, \Lambda(E_\beta)E_{-\beta}) + \sum_{\alpha \in R_{n_i}^+} \beta(H_\alpha)
\]

\[
= - \frac{1}{h_i} h(H_{01}^0, H_{01}^0) + \sum_{\alpha \in R_{n_i}^+} \beta(H_\alpha)
\]

\[
= - \frac{1}{h_i} h(H_{01}^0, H_{01}^0) + \frac{1}{2},
\]

where we have used that \( \sum_{\alpha \in R_{n_i}^+} H_\alpha = -\frac{\sqrt{-1}}{2} Z_i \) and that \( \beta(Z_i) = \sqrt{-1} \). Similarly, for \( a, b = 1, \ldots, k \),

\[
S(V_a, V_b) = \sum_{j=1}^{s} \sum_{\alpha \in R_{n_j}^+} \frac{1}{h_j} h(R(E_\alpha, E_\alpha)V_a, V_b)
\]

\[
= \sum_{j=1}^{s} \sum_{\alpha \in R_{n_j}^+} \frac{1}{h_j} h(V_a, H_\alpha) h(V_b, H_{01}^0)
\]

\[
= \sum_{j=1}^{s} \sum_{\alpha \in R_{n_j}^+} \frac{1}{h_j^2} h(V_a, H_\alpha) h(V_b, H_\alpha).
\]

We now compute the tensor \( Q(T) \). For \( \alpha \in R_{n_i}^+ \), \( i = 1, \ldots, s \), set

\[
e_\alpha := \frac{E_\alpha}{\sqrt{h_i}}.
\]
We fix a $h$-orthonormal basis $\{e_a\}_{a=1,...,k}$ of $\mathfrak{g}^{10}$. In the following, we shall use the Greek letters $\alpha, \beta, \ldots$ as indices varying among the positive roots, while the lowercase Latin letters $a, b, \ldots$ will denote indices varying in the set $\{1, \ldots, k\}$. Finally, Latin capital letters $A, B, \ldots$ will vary both among the positive roots and the elements of the set $\{1, \ldots, k\}$.

So we have, if $\beta \in \mathbb{R}^+_n$,

$$Q(T)(e_{\beta}, e_{\overline{\beta}}) = -\frac{1}{2} \sum_{A,B} T_{AB}^{\beta} \overline{T}_{AB}^{\beta} = -\frac{1}{2} \sum_{a,b} T_{ab}^{\beta} \overline{T}_{ab}^{\beta} - \frac{1}{2} \sum_{a,b} T_{ba}^{\beta} \overline{T}_{ba}^{\beta},$$

whence

$$Q(T)(e_{\beta}, e_{\overline{\beta}}) = -\frac{1}{h_i} \sum_b |h(e_b, H_\beta)|^2 = -\frac{1}{h_i} h(H_0^{01}, H_\beta^{01}).$$

The last equality in the previous formula holds noting that, if one writes $H_\beta^{01} = \sum_{a} \lambda_a e_a$ for suitable $\lambda_a \in \mathbb{C}$, then

$$\sum_{b=1}^k |h(e_b, H_\beta)|^2 = \sum_{b=1}^k |\lambda_b|^2 = h(H_0^{01}, H_\beta^{01}).$$

Moreover, using Lemma 3.2 we immediately see that $Q(T)(f, f) = 0$ and $Q(T)(f, n) = 0$.

We can now write an expression for the tensor

$$K(h) := -S(h) + Q(T),$$

which governs the flow (2.1). Namely,

$$\begin{align*}
K(h)(E_\alpha, E_{\overline{\alpha}}) &= -\frac{1}{2}, \quad \alpha \in \mathbb{R}^+_n; \\
K(h)(f, n) &= 0; \\
K(h)(V_a, V_b) &= -\sum_{j=1}^s \frac{1}{h_j^2} \sum_{\alpha \in R^+_n} h(V_a, H_\alpha) \overline{h}(V_b, H_\alpha).
\end{align*}$$

(3.5)

4 The Analysis of the Flow and Static Metrics

Starting from an invariant Hermitian metric $h_\phi$, the unique solution to the flow equation (2.1) consists of $G$-invariant Hermitian metrics on $M$. Therefore, using (3.5) we can write the flow equations as follows

$$\begin{align*}
h'_{ab} &= -\sum_{j=1}^s \frac{1}{h_j^2} \sum_{\alpha \in R^+_n} h(V_a, H_\alpha) \overline{h}(V_b, H_\alpha), \quad \text{for } a, b = 1, \ldots, k, \\
h'_i &= -\frac{1}{2}, \quad \text{for } i = 1, \ldots, s.
\end{align*}$$

(4.1)
In order to write the equations in (4.1) relative to the fiber in a nicer form, set $R_{n_j}^+ := \{\alpha_1^j, \ldots, \alpha_{n_j}^j\}$ for $j = 1, \ldots, s$. We note that for every $\alpha \in R_{n_j}^+$, we have $H_\alpha = -\frac{1}{2n_j} Z_j \text{ (mod } s_j\text{)},$ hence we can find coefficients $c_i^j$, $j = 1, \ldots, s$, $i = 1, \ldots, k$ so that

$$H^0_{\alpha_i} = \sum_{l=1}^{k} c_l^j \overline{V_l} \quad i = 1, \ldots, n_j, \quad j = 1, \ldots, s. \quad (4.2)$$

Thus

$$h_{ab}' = -\sum_{j=1}^{s} \frac{1}{n_j^2} \sum_{l,m=1}^{k} \left( n_j c_l^j c_m^j \right) h_{al} h_{\overline{m}b} = -\sum_{j=1}^{s} \frac{1}{h_j^2} \left( H \Gamma^j H \right)_{ab},$$

where we have set for $j = 1, \ldots, s$

$$(\Gamma^j)_{l\overline{m}} := n_j c_l^j c_m^j.$$ 

Therefore, we can write (4.1) as

$$\begin{cases} H' = -H \Gamma H \\ h_i' = \frac{1}{2}, \quad \text{for } i = 1, \ldots, s, \end{cases} \quad (4.3)$$

where

$$\Gamma(t) := \sum_{j=1}^{s} \frac{1}{h_j^2(t)} \Gamma^j.$$ 

We note that $\Gamma$ is positive semidefinite as each $\Gamma^j$ is so, $j = 1, \ldots, s$. The metric $h_o$ can be fully described by $s$ positive numbers $A_1, \ldots, A_s$ where

$$h_o|_{n_i \times n_i} = -A_i \kappa \quad (4.4)$$

and by a positive definite Hermitian $k \times k$ matrix $H_o$, which represents $h_o|_f \times f$ w.r.t. the basis $V$. From (4.3) we immediately see that

$$h_i(t) = -\frac{1}{2} t + A_i, \quad i = 1, \ldots, s. \quad (4.5)$$

If we set $A := \min_{i=1,\ldots,s} \{A_i\}$, we see that $h_i(t)$ are all positive when $t \in [0, 2A)$. The flow equation boils down to

$$\begin{cases} H' = -H \Gamma H \\ H(0) = H_0 \end{cases} \quad (4.6)$$
that can be explicitly integrated to

\[ H^{-1}(t) = H_0^{-1} + \int_0^t \Gamma(u) \, du. \]  

(4.7)

Note that the right-hand side of (4.7) is positive definite for all \( t \in [0, 2A] \), therefore the solution \( h(t) \) to the HCF exists on \([0, 2A]\). Moreover we notice that the maximal existence domain of \( h(t) \) is of the form \((-r, 2A)\), where \( r \in (0, +\infty)\).

In order to analyze the behavior of the metric along the fiber when \( t \) approaches the limit \( 2A \), we simply observe that (4.1) implies

\[ h(v, \bar{v})' = -\sum_{j=1}^s \frac{1}{h_j^2} \sum_{\alpha \in \mathcal{R}_+^{\alpha \neq j}} |h(v, H_\alpha)|^2 \leq 0, \quad \text{for any } v \in \mathfrak{T}^{10}; \]  

(4.8)

therefore, \( \lim_{t \to 2A} h(v, \bar{v}) \) exists and is non-negative. Thus when \( t \to 2A \), the metric along the fiber converges to a positive semidefinite Hermitian form \( \hat{h} \).

**Proposition 4.1** There is a compact normal subgroup \( S \) of \( G \) such that the orbits of \( S^c \) are the leaves of the distribution defined by the kernels of \( \hat{h} \).

**Proof** We consider the distribution \( Q \) on \( M \) which is defined for \( x \in M \) by \( Q_x := \{ v \in T_x M \mid \hat{h}_x(v, w) = 0, \forall w \in T_x M \} \). It is clear that \( Q \) is \( G \)-invariant and \( J \)-stable, so that it is enough to study it at \( o := [L] \in M = G/L \), where we can see \( q := Q_o \) as a \( J \)-stable subspace of \( m \). We write \( q^c = q^{10} \oplus q^{01} \). We rearrange the indices so that \( A = A_1 = \ldots = A_p < A_{j} \) for \( i = p + 1, \ldots, s \) and we define \( Z_p \) to be the complex subspace of \( f^{01} \) generated by \( Z_p^{01}, \ldots, Z_p^{01} \). The limit form \( \hat{h} \) is described by a pair \((\hat{h}^N, \hat{h}^F)\), where \( \hat{h}^N \) is a \( \text{Ad}(L) \)-invariant Hermitian form on \( n \) whose kernel is given by \( n_1 \oplus \ldots \oplus n_p \) and \( \hat{h}^F \) is a positive semidefinite Hermitian form on \( f \). \( \square \)

**Lemma 4.2** \( q^c = n_1^c \oplus \ldots \oplus n_p^c \oplus Z_p \oplus \overline{Z}_p. \)

**Proof** It is enough to prove that \( Z_p = (\ker \hat{h}^F)^{01} \). Throughout the following, we will identify \( f^{01} \) with \( \mathbb{C}^k \) by means of the basis \( \overline{v} \). Observe that (4.7) reads

\[ H^{-1}(t) = H_0^{-1} - \frac{2t}{A(t - 2A)} \Theta_p + \sum_{j=p+1}^s \left( \int_0^t \frac{1}{h_j^2(u)} \, du \right) \Gamma^j, \]

where \( \Theta_p := \sum_{j=1}^p \Gamma^j \). Since the image of \( \Gamma^j \) is spanned by \( Z_j^{01} \) for \( j = 1, \ldots, k \), we see that \( \Theta_p(Z_p) \subseteq Z_p \). Moreover, as each \( \Gamma^j \) is Hermitian positive semidefinite, we have that \( \ker \Theta_p = \bigcap_{j=1}^p \ker \Gamma^j = Z_p^p \), where the orthogonal space \( Z_p^p \) is taken with respect to the standard Hermitian structure on \( \mathbb{C}^k \). This implies that \( \ker \Theta_p \cap Z_p = \{0\} \) and therefore \( \Theta_p(Z_p) = Z_p \). We also observe that \( \Theta_p \) is diagonalizable and therefore its image \( Z_p \) is the sum of all eigenspaces with non-zero eigenvalues. If we set \( q := \overline{Springer} \)
\[ \dim \mathbb{C} Z_p, \text{ there exists } U \in U_k \text{ so that} \]
\[ \Delta := U \Theta_p U^T \]

is the diagonal matrix \( \text{diag}(\mu_1, \ldots, \mu_q, 0, \ldots, 0) \), \( \mu_i \neq 0 \) for \( i = 1, \ldots, q \). Then if \( H_u(t) := U H(t) U^T \) we have for \( t \in [0, 2A) \)

\[ H_u^{-1}(t) = \Lambda(t) - \frac{2t}{A(t - 2A)} \Delta, \tag{4.9} \]

where \( \Lambda(t) \) is positive definite for all \( t \in [0, 2A) \) and, when \( t \to 2A \), it converges to a positive definite matrix

\[ \hat{\Lambda} := \lim_{t \to 2A} \Lambda(t). \]

We have

\[ H_u(t) = \frac{\text{adj}(H_u^{-1}(t))}{\det H_u^{-1}(t)} \]

and using (4.9) we see that

\[ \lim_{t \to 2A} (2A - t)^q \det H_u^{-1}(t) = 4^q \mu_1 \cdot \ldots \cdot \mu_q \det \hat{\Lambda}_o > 0 \]

where \( \hat{\Lambda}_o \) is the minor of \( \hat{\Lambda} \) obtained intersecting the last \( k - q \) rows and the last \( k - q \) columns. Moreover

\[ \lim_{t \to 2A} (2A - t)^q \text{adj}(H_u^{-1}(t))_{a\overline{b}} = 0, \quad \forall a \in \{1, \ldots, q\}, \quad \forall b \in \{1, \ldots, k\}, \]

hence

\[ \lim_{t \to 2A} (H_u(t))_{\overline{a}b} = 0, \quad \forall a \in \{1, \ldots, q\}, \quad \forall b \in \{1, \ldots, k\}, \]

while for \( a, b = q + 1, \ldots, k \),

\[ \lim_{t \to 2A} (H_u(t))_{\overline{a}\overline{b}} = (\hat{\Lambda}_o^{-1})_{(a-q)(b-q)}. \]

Thus we have that

\[ \lim_{t \to 2A} H_u(t) = \begin{pmatrix} 0 & 0 \\ 0 & \hat{\Lambda}_o^{-1} \end{pmatrix}, \]

where \( \hat{\Lambda}_o^{-1} \) is \( (k - q) \times (k - q) \) and positive definite. The claim follows. \( \Box \)
We define $U := G_1 \cdot \ldots \cdot G_p$. We observe that the universal complexification $U^c$ acts on $M$ and that the $U^c$-orbits define a $G$-invariant foliation. At the point $o \in M$, we have that $T_o(U^c \cdot o) = n_1 \oplus \ldots \oplus n_p \oplus \text{Span}\{(Z_1)_f, \ldots, (Z_p)_f, I_F((Z_1)_f), \ldots, I_F((Z_p)_f)\} = q$ by Lemma 4.2. Therefore, the $U^c$-orbits coincide with the leaves of $Q$. \hfill \Box

Remark 4.3 Since the complex structure $I_F$ along the fiber is totally arbitrary, the $U^c$-orbits are not necessarily closed. Note that $G$ acts transitively on the set of $U^c$-orbits and therefore one such orbit is closed if and only if all are so.

4.1 Static Metrics

We say that an invariant Hermitian metric $h$ on $M$ is static for the HCF $U$ if there exists $\lambda \in \mathbb{R}$ such that

$$-\mathcal{K}(h) = \lambda h. \quad (4.10)$$

In terms of algebraic data on $\mathfrak{g}$, this equation becomes

$$\begin{cases}
\lambda H = H \Gamma H, \\
\lambda h_i = \frac{1}{2}, \quad \text{for } i = 1, \ldots, s.
\end{cases} \quad (4.11)$$

This immediately implies that (4.10) has no solution if $\lambda \leq 0$. On the other hand, if $\lambda > 0$, the second equation in (4.11) gives

$$h_i = \frac{1}{2\lambda} > 0, \quad \text{for } i = 1, \ldots, s. \quad (4.12)$$

Then the first equation in (4.11) reads

$$H = \frac{1}{4\lambda} \Theta^{-1}_s, \quad (4.13)$$

where we note that $\Theta_s$ is invertible as it is shown in the proof of Lemma 4.2. Using (4.12), (4.13) we can therefore define, for any $\lambda > 0$, a static metric satisfying (4.10) for the chosen $\lambda$.

Suppose now that in (4.4) the initial conditions satisfy $A = A_1 = \cdots = A_s$. Clearly in this case, we have $\lim_{t \to 2A} h_i(t) = 0$ for $i = 1, \ldots, s$ and

$$\int_0^t \Gamma(u)du = \frac{2t}{A(2A - t)} \Theta_s, \quad t \in [0, 2A).$$

On the other hand, we have

$$\lim_{t \to 2A} H(t) = \lim_{t \to 2A} \frac{\text{adj}(H^{-1}(t))}{\det H^{-1}(t)} = 0,$$
as one can easily verify that

\[
\begin{align*}
\lim_{t \to 2A} (2A - t)^k \det H^{-1}(t) &= 4^k \det \Theta_s > 0, \\
\lim_{t \to 2A} (2A - t)^{k-1} \text{adj}(H^{-1}(t)) &= 4^{k-1} \text{adj}(\Theta_s).
\end{align*}
\] (4.14)

Thus

\[
\lim_{t \to 2A} h(t) = 0.
\]

We now consider the Hermitian metric \( \tilde{h}(t) \) which is homothetic to \( h(t) \) and has unitary volume, namely, \( \tilde{h}(t) := c(t) h(t) \) with \( c(t) = (\text{vol}_{h(t)}(M))^{-1/m} \). Now we see that there exists a positive constant \( V \) so that

\[
\text{vol}_{h(t)}(M) = V \cdot \det H(t) \cdot \prod_{i=1}^{s} (2A - t)^{n_i} = V \cdot \frac{(2A - t)^m}{(2A - t)^k \det H^{-1}(t)},
\]

where we have used that \( m = k + \sum_{i=1}^{s} n_i \). We can then write

\[
c(t) = \frac{\xi(t)}{2A - t}, \quad \xi(t) := \left( V^{-1} (2A - t)^k \det H^{-1}(t) \right)^{1/m}
\]

and note that by (4.14),

\[
\lim_{t \to 2A} \xi(t) = \left( V^{-1} 4^k \det \Theta_s \right)^{1/m} := \xi > 0.
\]

We now have, for \( i = 1, \ldots, s \),

\[
\lim_{t \to 2A} c(t) \tilde{h}_i(t) = \lim_{t \to 2A} \frac{\xi(t)}{2A - t} \cdot \frac{2A - t}{2} = \frac{\xi}{2},
\]

while using (4.14)

\[
\lim_{t \to 2A} c(t) H(t) = \frac{\xi}{4} \cdot \frac{\text{adj}(\Theta_s)}{\det \Theta_s} = \frac{\xi}{4} \cdot \Theta_s^{-1}.
\]

Therefore, the solution \( \tilde{h}(t) \) to the normalized flow converges to the static metric satisfying (4.10) with \( \lambda = 1/\xi \).

We can then formulate our first result as follows

**Theorem 4.4** Let \( M = G/L \) be a simply connected C-space with Tits fibration over a Hermitian symmetric space \( N \) whose irreducible factors have complex dimension at least two and are not complex quadrics. Any \( G \)-invariant Hermitian metric \( h \) on \( M \) is determined by a pair \( (h^N, h^F) \), where \( h^N \) is an \( \text{Ad}(L) \)-invariant Hermitian metric on \( n \) and \( h^F \) is an arbitrary Hermitian metric on the fiber \( g \).

Given any initial invariant Hermitian metric \( (h^N_0, h^F_0) \) on \( M \), we have:
(i) the solution \( h(t) \) to the HCF\( U \) is given by the pair \( (h^N(t), h^F(t)) \), where \( h^N(t) \) depends only on \( h_0^N \);

(ii) the maximal existence domain of \( h^N(t) \) is an interval \((-\infty, T)\) with \( 0 < T < +\infty \) and the solution \( h^F(t) \) exists and is positive definite on an interval \((-r, T)\) with \( 0 < r \leq +\infty \). When \( t \to T \), the base \( N \) collapses to a product of some Hermitian symmetric spaces and the metric \( h(t) \) converges to a positive semidefinite Hermitian bilinear form \( \hat{h} \). The distribution given by the kernel of \( \hat{h} \) is integrable and its leaves coincide with the \( U_c \)-orbits for a suitable compact connected normal subgroup \( U \subseteq G \);

(iii) the manifold \( M \) admits a unique (up to homotheties) invariant Hermitian metric which is static for the HCF\( U \). If \( h^N(t) \to 0 \) when \( t \to T \), then \( h(t) \to 0 \) and the normalized flow with constant volume converges to a static metric.

The following proposition can be thought of as a complement of the main Theorem 4.4 when the \( U_c \)-orbits are closed. We keep the same notation as in Theorem 4.4 and we consider the foliation, again denoted by \( Q \), given by the \( U_c \)-orbits. If \( d_t \) denote the distance on \( M \) induced by the metric tensor \( h_t \) \((t \in [0, T))\), we describe the Gromov–Hausdorff limit of the metric spaces \((M, d_t)\) when \( t \to T \).

**Proposition 4.5** If the \( U_c \)-orbits are closed, the leaf space \( \overline{M} := M/Q \) has the structure of a smooth homogeneous manifold \( G/\hat{U} \) for some closed subgroup \( \hat{U} \subseteq G \). Moreover, the positive semidefinite tensor \( \hat{h} \) induces a \( G \)-invariant Riemannian metric \( \overline{h} \) on \( \overline{M} \) with induced distance \( \overline{d} \) and the metric spaces \((M, d_t)\) Gromov–Hausdorff converge to \((\overline{M}, \overline{d})\) when \( t \to T \).

**Proof** Note that \( G \) acts transitively on the leaves of the \( G \)-invariant foliation \( Q \). The closedness of the leaves implies that \( \hat{M} \) is Hausdorff and it can be expressed as a coset space \( G/\hat{U} \) for some closed subgroup \( \hat{U} \) that contains both \( U \) and \( L \). At the level of Lie algebras, we can write the decomposition

\[
g = \hat{u} \oplus \hat{m},
\]

where the subspace \( \hat{m} \) is defined as the \( \kappa \)-orthocomplement of \( \hat{u} \). As \( l \subseteq \hat{u} \) we have that \( \hat{m} \subseteq m \). Moreover \( u = \bigoplus_{j=1}^p \hat{g}_i \subseteq \hat{u} \) implies that \( \hat{m} \subseteq \bigoplus_{j=p+1}^s \hat{g}_j \). We now note that the \( G \)-equivariant projection \( \pi : M \to N \) maps the orbit \( U^c o \) onto the \( U \)-orbit \( \prod_{i=1}^p N_i \times \prod_{j=p+1}^s [K_j] \), where \( N_i := G_i/K_i \) for \( i = 1 \ldots s \), whence \( \hat{U} \subseteq \prod_{i=1}^p G_i \times \prod_{j=p+1}^s K_j \). Then we can write

\[
\hat{m} = \bigoplus_{j=p+1}^s n_j \oplus \hat{f}, \quad \hat{f} = \bigoplus_{j=p+1}^s \mathbb{R}Z_j.
\]

This implies in particular that \( \text{Ad}(\hat{U})|_{\hat{f}} = \text{Id} \). Therefore, the restriction \( \hat{h}|_{\hat{m} \times \hat{m}} \) gives a positive definite \( \text{Ad}(\hat{U}) \)-invariant inner product which descends to a \( G \)-invariant Riemannian metric \( \overline{h} \) on \( \overline{M} \).

We now prove the last statement, namely, that for \( x, y \in M \) we have

\[
\lim_{t \to T} d_t(x, y) = \overline{d}(p(x), p(y)),
\]

where \( \overline{d} \) is the distance induced by \( \overline{h} \) and \( p : \)
$M \rightarrow \overline{M}$ is the projection. We denote by $\gamma^t$ a minimizing geodesic for $h_t$ joining $x$ with $y$. As (4.5) and (4.8) imply that $h_t(v, v)$ is a non-increasing function of $t$ for every tangent vector $v$, we see that

$$d_t(x, y) = \int_0^1 h_t \left( \frac{d\gamma^t}{ds}, \frac{d\gamma^t}{ds} \right)^{1/2} ds \geq \int_0^1 \hat{h} \left( \frac{d\gamma^t}{ds}, \frac{d\gamma^t}{ds} \right)^{1/2} ds \geq \tilde{d}(p(x), p(y)).$$

This means that

$$\liminf_{t \to T} d_t(x, y) \geq \tilde{d}(p(x), p(y)).$$

On the other hand, let $\gamma$ be a minimizing geodesic for $\bar{h}$ connecting $p(x)$ with $p(y)$. Let $\tilde{\gamma}$ be a lift of $\gamma$ starting at $x$ with ending point $\tilde{y}$. We choose a path $\eta$ in $p^{-1}(p(y))$ connecting $y$ with $\tilde{y}$. Then

$$d_t(x, y) \leq d_t(x, \tilde{y}) + d_t(\tilde{y}, y) \leq \int_0^1 h_t(\tilde{\gamma}'(s), \tilde{\gamma}'(s))^{1/2} ds + \int_0^1 h_t(\eta'(s), \eta'(s))^{1/2} ds.$$

Now

$$\lim_{t \to T} \int_0^1 h_t(\eta'(s), \eta'(s))^{1/2} ds = 0,$$

while

$$\lim_{t \to T} \int_0^1 h_t(\tilde{\gamma}'(s), \tilde{\gamma}'(s))^{1/2} ds = \int_0^1 \hat{h}(\tilde{\gamma}'(s), \tilde{\gamma}'(s))^{1/2} ds = \int_0^1 \hat{h}(\gamma'(s), \gamma'(s))^{1/2} ds = \bar{d}(p(x), p(y)).$$

This implies that

$$\limsup_{t \to T} d_t(x, y) \leq \bar{d}(p(x), p(y))$$

and this concludes the proof. \hfill \Box

**Remark 4.6** Note that the homogeneous space $\overline{M}$ might not carry any $(G$-invariant) complex structure.
4.2 Example

A C-space $M$ will be called of Calabi–Eckmann type if $M = G/L$, where $G = G_1 \cdot G_2$, $L = L_1 \cdot L_2$, and $L_i$ is the semisimple part of $K_i$ for $i = 1, 2$ (see [4], [18]). The space $M$, which is $T^2$-bundle over a product of two Hermitian symmetric spaces, can be endowed with many invariant complex structures as described in Sect. 2. We consider now a more general C-space $M$ which is given by a product of C-spaces of Calabi–Eckmann type $M_1, M_3, \ldots, M_{2k-1}$ with $M_i = G_i \cdot G_{i+1}/L_i \cdot L_{i+1}$, endowed with the invariant complex structure $J$ given by the product of invariant complex structures on each factor $M_i$, $i = 1, 3, \ldots, 2k - 1$.

We now consider an initial datum $(h^N_0, h^F_0)$, where $h^N_0$ is determined by a sequence $(A_1, \ldots, A_{2k})$. We can rearrange the indices as in the proof of Proposition 4.1, i.e., $A = A_1 = \cdots = A_p < A_j$ for all $j = p + 1, \ldots, 2k$. Note that there is an involution $\sigma$ of the set $\{1, \ldots, 2k\}$ so that for each index $1 \leq l \leq 2k$ we have $JZ_l \in \text{Span}\{Z_l, Z_{\sigma(l)}\}$. Hence

$$T_0(U \cdot o) = n_1 \oplus \cdots \oplus n_p \oplus \text{Span}\{Z_1, \ldots, Z_p, Z_{\sigma(1)}, \ldots, Z_{\sigma(p)}\}.$$  

Note that in this case the $S^t$-orbits are closed. When $t \to 2A$, the manifold $M$ collapses to a product of Hermitian symmetric spaces and a lower-dimensional C-space $M'$ of Calabi–Eckmann type. Indeed, if we set $\mathcal{I}_1 := \{p + 1, \ldots, 2k\} \cap \sigma(\{p + 1, \ldots, 2k\})$ and $\mathcal{I}_2 := \{p + 1, \ldots, 2k\} \setminus \mathcal{I}_1$, then the manifold collapses to

$$\left(\prod_{i \in \mathcal{I}_2} G_i/K_i\right) \times M', \quad M' := \prod_{i \in \mathcal{I}_1} G_i/L_i.$$  

Acknowledgements The authors heartily thank Yury Ustinovskiy and Marco Radeschi for many valuable discussions and remarks as well as Luigi Vezzoni and Daniele Angella for their interest.

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