Unknown input observer design for linear time-invariant systems—A unifying framework

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Abstract
This article presents a new observer design approach for linear time invariant multivariable systems subject to unknown inputs. The design is based on a transformation to the so-called special coordinate basis (SCB). This form reveals important system properties like invertability or the finite and infinite zero structure. Depending on the system's strong observability properties, the SCB allows for a straightforward unknown input observer design utilizing linear or nonlinear observers design techniques. The chosen observer design technique does not only depend on the system properties, but also on the desired convergence behavior of the observer. Hence, the proposed design procedure can be seen as a unifying framework for unknown input observer design.

KEYWORDS
higher order sliding mode, infinite zero structure, special coordinate basis, strong detectability, unknown input observer

1 | INTRODUCTION

State estimation in the presence of unknown inputs has a long history in research1-11 and applications.12,13 So-called unknown input observers can provide state estimates even if not all system inputs are known. During the last decades, such observers proved useful in a large variety of applications ranging from robust control and uncertainty compensation over robust residual generation in fault detection problems to networked and decentralized control scenarios.15-17

The majority of the available works deals with the design of linear unknown input observers for linear time-invariant systems, see, for example, References 3, 4, 6, 15, and 17. The necessary and sufficient existence condition for such an observer is the system's strong detectability as introduced in Reference 3, Definition 1.3, see also Definition 3. Such systems do not possess unstable invariant zeros and have to fulfill the so-called rank condition. The latter condition allows to express the unknown input via the output signal and its first derivative.4

Strong detectability is also necessary and sufficient for the existence of first order sliding mode observers, see Reference 18, Chapter 6. The advantage of such observers compared to linear ones is that they can also provide estimates of the unknown input via an equivalent control approach. A comparison between linear and sliding mode observers for fault...
reconstruction is presented in Reference 19. Such first order sliding mode techniques were also successfully applied to, for example, fault detection and fault tolerant control problems in aviation.13,20

If the system is not strong detectable but merely strongly detectable,3 that is, the aforementioned rank condition is not fulfilled, it is still possible to estimate the states by taking higher order derivatives of the output signal into account using, for example, higher-order sliding mode approaches.10,21-23 Strong observability, which is a stronger condition than strong detectability, even allows to obtain the estimates in finite or fixed time.

Popular techniques for obtaining the required derivatives are based on higher order sliding mode techniques and in particular on Levant’s arbitrary order robust exact differentiator (RED)24 and its extensions to fixed-time convergent differentiators, see, for example, References 25-27. Recently, Levant’s observer was generalized by Moreno to a family of finite- or fixed-time bi-homogeneous differentiators28 with an appealingly simple structure.

Concerning higher-order sliding mode based unknown input observers, the early works proposed a hierarchical super-twisting algorithm.8,21 The underlying idea is Molinari’s algorithm2 for determining the weakly unobservable subspace. It is based on successive differentiation of the output vector and the decoupling from the influence of the unknown inputs. The approach was extended in order to employ higher order sliding mode differentiators.10,14,23 This improves the accuracy of the obtained estimates w.r.t. discretization and bounded measurement noise.14 All mentioned higher order sliding mode approaches require a Luenberger observer cascaded with the sliding mode reconstruction scheme in order to fulfill the requirements for the higher order sliding mode differentiators.23 Hence, such estimation schemes require at least twice the number of states of the considered system. For strongly observable single-input single-output (SISO) systems, a direct higher order sliding mode observer design which avoids this disadvantage and requires less tuning parameters is presented in Reference 29. The design is based on a generalization of Ackermann’s formula. Recently, a generalization to strongly observable multi-input multi-output (MIMO) systems was proposed in Reference 30. The design is based on a new observer normal form for multivariable systems. This normal form allows a direct application of the RED without a cascaded Luenberger observer and hence simplifies the design procedure and the tuning compared to previous approaches.

A generalization of the above normal form is the so-called special coordinate basis (SCB), which explicitly reveals the finite and infinite zero structure of the considered system and was introduced by Sannuti and Saberi.31 Since then, the SCB is utilized to solve many analysis and design problems for multivariable linear time-invariant systems, like, for example, determining invariant subspaces from the geometric control perspective,32 squaring down,33 model order reduction,34 loop transfer recovery,35 $H_2$ and $H_\infty$ optimal control,36,37 and many more. A numerically reliable algorithm for obtaining the SCB transformations is proposed in Reference 38. Xiong and Saif39-41 were among the first to utilize the SCB for unknown input observer design. In Reference 39, they propose a linear functional unknown input observer. A robust linear fault isolation observer for strongly detectable systems is presented in Reference 40. In Reference 41, a first order sliding mode observer design technique for fault diagnosis based on the equivalent control principle is proposed for strongly detectable systems.

It turns out that due to the relation of the SCB with the system’s zero structure, this form is particularly suited for the unknown input observer design.

This article presents a novel observer design approach for linear time invariant MIMO systems subject to unknown inputs. The design is based on a transformation to the SCB.31,32 In this form, and depending on the system’s observability properties, the design of a linear unknown input observer and the design of first order or higher order sliding mode observers can be performed in a similar fashion. For linear unknown input observers, it is shown that a full order observer has no benefit over a specific reduced order observer. Moreover, it turns out that the observer rank condition is a condition on the system’s infinite zero structure that only allows infinite zeros of degree zero or one.

If the system is merely strongly detectable but not strongly observable, it is only possible to asymptotically estimate the system states by utilizing higher order sliding mode techniques. It follows directly from the construction of the proposed observer that the number of differentiation operations is minimal. The design is straightforward and does not require a “stabilizing” Luenberger observer as in previous works.10,21,25 Hence, its observer order corresponds to the system order.

For strongly observable systems, it is well known that it is possible to reconstruct the states in finite time.8 The present work proposes a finite- or fixed-time unknown input observer design for strongly observable systems. To that end, depending on whether the observer rank condition is fulfilled or not, a continuous or discontinuous (bi)-homogenous observer based on Moreno’s differentiator is proposed, respectively. To sum up, the proposed design procedure can be regarded as a unifying unknown input observer design framework for linear time invariant systems. It incorporates the design of linear unknown input observers, sliding mode observers, and bi-homogeneous observers allowing for asymptotic, finite- or fixed-time convergence of the estimation error.
The contributions of this article can be summarized as follows:

- In contrast to References 10, 21, and 23, the proposed class of observers does not require a cascaded observer structure and hence yields an observer of the same order as the system.
- Compared to Reference 30, the present article considers the most general case for at least asymptotic state estimation in the presence of unknown inputs, while in Reference 30 only the strongly observable case is considered.
- Depending on the system properties and the desired error convergence properties, the proposed approach allows to design linear or nonlinear asymptotic-, finite- or fixed-time unknown input observers in a similar and systematic way. It thus constitutes a unified framework for the design of such observers.
- The approach uses the minimum number of required differentiation operations.
- The considered transformations rely on well known and extensively studied algorithms. 31,32,38

This article is structured as follows: Section 2 introduces the problem statement and points out the underlying assumptions. Section 3 recalls preliminaries such as important properties of linear time invariant systems. Moreover, it summarizes Moreno’s arbitrary order fixed-time differentiator and discusses Levant’s RED as a special case. The transformation to the SCB is introduced in Section 4. In order to utilize the SCB for observer design, a specific block triangular form of the subsystem related to the infinite zero structure is proposed. The existence of such a transformation is guaranteed by Theorem 1. The main contribution, that is, the unifying observer design framework, is presented in Section 5. There, the general structure of the observers is discussed and the particular designs are presented in detail. Section 6 exemplarily shows a sliding mode observer design for a specific strongly detectable system with direct feed-through. The numerical results underline the applicability and the appealing simplicity of the proposed design procedure. The conclusion together with possible future research directions is given in Section 7.

Notation: Matrices are printed in boldface capital letters, whereas column vectors are boldface lower case letters. The elements of a matrix $M$ are denoted by $m_{ij}$. The matrix $I_n$ is the $n \times n$ identity matrix and $J_n$ denotes a Jordan block of dimension $n \times n$ according to

$$J_n = \begin{bmatrix} 0 & I_{n-1} \\ 0 & 0 \end{bmatrix}. \quad (1)$$

Moreover, $M = \text{diag}(M_1, \ldots, M_j)$ denotes a (block) diagonal matrix with entries $M_1, \ldots, M_j$. In dynamical systems, differentiation of a vector $x$ with respect to time $t$ is expressed as $\dot{x}$. Time dependency of state (usually $x$), input (usually $u$) and output (usually $y$) is omitted. The signed power is represented by $|x|^a = \text{sign}(x) |x|^a$ and $|x|^0 = \text{sign}(x)$. For differential equations with a discontinuous right-hand-side, the solutions are understood in the sense of Filippov.42

## 2 PROBLEM STATEMENT

This article considers the linear time invariant MIMO system $\Sigma$ denoted by the quadruple $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ and given by

\begin{align*}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0, \\
\mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{2a}
\end{align*}

with the state $\mathbf{x} \in \mathbb{R}^n$, the unknown input $\mathbf{u} \in \mathbb{R}^m$, and the output $\mathbf{y} \in \mathbb{R}^p$. Without loss of generality, it is assumed that rank $\mathbf{D} = m_0$ and that the matrices $[\mathbf{B}^T \mathbf{D}^T]$ and $[\mathbf{C} \mathbf{D}]$ have full rank. Moreover, only unknown inputs are considered for simplicity. This is no restriction, because known inputs can always be easily integrated in any estimation scheme.21

The unknown input $\mathbf{u} = [u_1 \ u_2 \ \cdots \ u_m]^T$ is assumed to be bounded component-wise according to

$$u_i \in [u_{i,\min}, u_{i,\max}], \quad i = 1, \ldots, m. \quad (3)$$

The concept of an unknown input observer is introduced in
**Definition 1.** An unknown input observer is a (dynamical) system providing an estimate \( \hat{x} \) for the system state \( x \) without knowledge of the input \( u \). Moreover

(i) It is an asymptotic unknown input observer if the estimation error \( e = x - \hat{x} \) vanishes asymptotically for any initial condition, that is, \( \lim_{t \to \infty} e(t) = 0 \).
(ii) It is a finite-time unknown input observer, if \( e(t) = 0 \) for all \( t \geq T_f(e_0) \) with some finite time \( T_f > 0 \) depending on the initial error \( e_0 \).
(iii) It is a fixed-time unknown input observer if \( T_f \) is independent of \( e_0 \).

The goal is to derive a generic design procedure which, depending on the system properties, allows to design asymptotic, finite-time or fixed-time unknown input observers whose order is at most the order of the system.

## 3 | PRELIMINARIES

In the following, important concepts required for the unknown input observer design are briefly recalled. Most of the results presented in Sections 3.1 and 3.2 can be found in classical textbooks on multivariable control systems. Section 3.3 recalls a family of differentiators recently proposed in Reference 28.

### 3.1 | Zeros

Conditions for strong detectability and observability are often stated in terms of invariant zeros of system \( \Sigma \). These zeros are characterized by the so-called Rosenbrock matrix, see, for example, Reference 43, Chapter 7.

**Definition 2** (Invariant zeros). The invariant zeros of \( \Sigma \) are the values \( \lambda \in \mathbb{C} \) such that the Rosenbrock matrix

\[
P(s) = \begin{bmatrix} sI_n - A & -B \\ C & D \end{bmatrix}
\]  

exhibits a rank loss, that is,

\[
\text{rank } P(\lambda) < \text{normrank } P,
\]

where the normal rank of \( P(s) \) is defined as

\[
\text{normrank } P = \max \{ \text{rank } P(s) \mid s \in \mathbb{C} \}.
\]

It holds, that normrank \( P(s) = n + \text{normrank } G(s) \), with the transfer matrix

\[
G(s) = C(sI_n - A)^{-1}B + D,
\]

see Reference 43, Lemma 8.9. Note that this definition does not reveal the multiplicity or order of the zeros. In Reference 32, Definition 3.6.3, the orders are introduced via the Kronecker canonical form of \( P(s) \). There, also the definition of the infinite zero structure is stated. This definition is omitted here and the infinite zero structure will be introduced with the aid of the SCB in Section 4. If all invariant zeros of \( \Sigma \) are contained in \( \mathbb{C}^- \), the system is said to have the minimum phase property.

### 3.2 | Strong detectability and observability

This section recalls the basic concepts of strong detectability and strong observability and discusses several important aspects of systems with these properties.
Definition 3 (Strong detectability\(^3\)). System (2) is called

(i) Strongly observable, if \(y(t) = 0\) for all \(t \geq 0\) implies \(x(t) = 0\) for all \(t \geq 0\), all \(u(t)\) and all \(x(0) = x_0\).

(ii) Strongly detectable, if \(y(t) = 0\) for all \(t \geq 0\) implies \(x(t) \to 0\) for \(t \to \infty\), all \(u(t)\) and all \(x(0) = x_0\).

(iii) Strong\(^*\) detectable, if \(y(t) \to 0\) for \(t \to \infty\) implies \(x(t) \to 0\) for \(t \to \infty\), all \(u(t)\) and all \(x(0) = x_0\).

It is well known that strong\(^*\) detectability is the minimum requirement for the existence of a linear unknown input observer,\(^3\) whereas strong detectability is the minimum requirement for (at least asymptotically) reconstructing the state with any estimation scheme.\(^{21}\) If the system is strongly observable, it is possible to exactly reconstruct the state within finite or fixed time by employing higher-order sliding mode techniques.\(^{10,21,23}\) The following relations are stated in Reference 3.

Proposition 1 (Strong observability and detectability conditions). System \(\Sigma\) is

(i) Strongly observable, if and only if \(\text{rank } P(\lambda) = n + m\) for all \(\lambda \in \mathbb{C}\).

(ii) Strongly detectable, if and only if \(\text{rank } P(\lambda) = n + m\) for all \(\lambda \in \mathbb{C}\) with \(\text{Re}(\lambda) \geq 0\).

(iii) Strong\(^*\) detectable, if and only if it is strongly detectable and additionally

\[
\text{rank } \begin{bmatrix} CB & D \\ D & 0 \end{bmatrix} = \text{rank } D + \text{rank } B = \text{rank } D + m. \tag{8}
\]

Remark 1. For the characterization of strong observability and detectability, Hautus\(^3\) introduces a slightly different definition of zeros than in Definition 2. Both definitions coincide under the assumption that \(\text{normrank } P(s) = n + m\). In this case, condition (i) essentially requires that the system has no invariant zeros. Condition (ii) then states that the system is strongly detectable if and only if it is minimum phase. Moreover, \(\text{normrank } P(s) = n + m\) if and only if \(\text{normrank } G(s) = m\). A system for which the latter relation holds is also called left-invertible. A necessary condition for left-invertibility of \(G(s)\) is that \(p \geq m\), that is, that there are at least as many linearly independent measurements as unknown inputs. Consequently, this condition is also necessary for strong detectability.

Condition (8) is the so-called rank-condition and is a basic requirement for the design of a linear unknown input observer without using derivatives of the output signal. One can see from the above conditions that strong observability implies strong detectability. Moreover, strong\(^*\) detectability implies strong detectability. The converse is not true as shown with an example in Reference 3. An alternative characterization of strong observability and detectability can be given in terms of invariant subspaces:

Definition 4 (Weakly unobservable subspace\(^43\)). A point \(x_0 \in \mathbb{R}^n\) is called weakly unobservable if there exists an input \(u(t)\) such that the corresponding output satisfies \(y(t) = 0\) for \(t \geq 0\) and \(x(0) = x_0\). The set of all weakly unobservable points is denoted by \(\mathcal{Y}^\ast(\Sigma)\) and is called the weakly unobservable subspace of \(\Sigma\).

Definition 5 (Controllable weakly unobservable subspace\(^32,43\)). A point \(x_0 \in \mathbb{R}^n\) is called controllable weakly unobservable, if there exists an input signal \(u(t)\) and a \(T > 0\), such that \(y(t) = 0\) for all \(t \in [0, T]\) and \(x(T) = 0\). The set of all controllable weakly unobservable points is denoted by \(\mathcal{R}^\ast(\Sigma)\) and is called the controllable weakly unobservable subspace of \(\Sigma\).

It follows directly from Definitions 4 and 5 that \(\mathcal{R}^\ast(\Sigma) \subseteq \mathcal{Y}^\ast(\Sigma)\). A thorough introduction to the geometric subspace approach for linear systems is given in the book of Trentelman et al.\(^43\) There, the following two results are presented

Lemma 1 (Geometric conditions for strong observability and strong detectability\(^43\)(chapter 7)).

(i) \(\Sigma\) is strongly observable if and only if \(\mathcal{Y}^\ast(\Sigma) = 0\).

(ii) \(\Sigma\) is strongly detectable if and only if \(\mathcal{R}^\ast(\Sigma) = 0\) and \(\Sigma\) is minimum phase.

3.3 Moreno’s arbitrary order fixed-time differentiator

Recently, Moreno\(^28\) proposed a family of finite-/fixed-time convergent differentiators or observers. Fixed-time convergence is often desirable, if no bounds on the initial conditions are known. In this case, the finite-time convergent
observers may take arbitrarily long to converge because the convergence time grows with the initial condition; fixed-time convergence guarantees an upper bound for the convergence time. Finite-time convergent observers and in particular Levant’s RED are included as special cases within the family of arbitrary order finite-/fixed-time differentiators. Important results from Reference 28 are recalled in the following.

The system under consideration is given by the integrator chain

\[
\begin{align*}
\dot{x}_1 &= x_2, & y &= x_1, \\
\dot{x}_2 &= x_3, \\
&\vdots \\
\dot{x}_n &= u,
\end{align*}
\]

with the state \( x = [x_1 \ x_2 \ \cdots \ x_n]^T \) and \( n \) as the system order. The input signal \( u \) is assumed to be bounded with \( |u(t)| \leq \Delta \) for all \( t \geq 0 \) and some constant \( \Delta \geq 0 \). The observer based on Reference 28 can then be stated as

\[
\begin{align*}
\dot{\hat{x}}_1 &= \hat{x}_2 + \kappa_1 \Phi_1^n(e_1), \\
\dot{\hat{x}}_2 &= \hat{x}_3 + \kappa_2 \Phi_2^n(e_1), \\
&\vdots \\
\dot{\hat{x}}_n &= \kappa_n \Phi_n^n(e_1),
\end{align*}
\]

with positive parameters \( \kappa_i \) and nonlinear output injection terms \( \Phi_i^n : \mathbb{R} \mapsto \mathbb{R}, \ i = 1, \ldots, n \). The output injection terms are given by

\[
\Phi_i^n(z) = (\varphi_1^n \circ \cdots \circ \varphi_2^n \circ \varphi_1^n)(z)
\]

with the monotonic growing functions

\[
\varphi_i^n(z) = \mu \left[ z \right]^{r_{i+1}} + (1 - \mu) \left[ z \right]^{r_{\infty}}
\]

and a constant parameter* \( 0 < \mu < 1 \). The powers are completely determined by two parameters \(-1 \leq d_0 \leq d_\infty \leq \frac{1}{n-1}\) according to the recursive definitions

\[
\begin{align*}
r_{i+1} &= r_{0,i+1} - d_0 = 1 - (n - i)d_0, \\
r_{\infty,i+1} &= r_{\infty,i+1} - d_\infty = 1 - (n - i)d_\infty,
\end{align*}
\]

for \( i = 1, \ldots, n + 1 \). An insightful discussion of the design idea can be found in the introductory section of Reference 28. The dynamics of the estimation error \( e = x - \hat{x} \) can be stated according to

\[
\begin{align*}
\dot{e}_1 &= e_2 - \kappa_1 \Phi_1^n(e_1), \\
\dot{e}_2 &= e_3 - \kappa_2 \Phi_2^n(e_1), \\
&\vdots \\
\dot{e}_n &= u - \kappa_n \Phi_n^n(e_1).
\end{align*}
\]
It follows from Reference 28, Theorem 1 that there exist appropriate gains \( \kappa_i > 0 \) for \( i = 1, \ldots, n \), such that (14) is asymptotically stable and converges to zero in fixed time if

(i) \(-1 < d_0 < 0 < d_\infty < \frac{1}{n-1} \) and \( \Delta = 0 \), or
(ii) \(-1 = d_0 < 0 < d_\infty < \frac{1}{n-1} \) and \( \Delta \geq 0 \).

The error dynamics (14) is bi-homogeneous as defined in Reference 45, in the sense that near to the origin it is approximated by a homogeneous system of degree \( d_0 \) and far from the origin its approximation corresponds to a system with homogeneity degree \( d_\infty \), see Reference 28. A sequence of stabilizing gains \( \kappa_i, i = 1, \ldots, n \) can be selected according to Reference 28, Proposition 4. Note, that for \( d_0 = -1 \), the error dynamics (14) has a discontinuous right-hand side. This allows for robustness with respect to unknown inputs \( u \) with \( \Delta > 0 \). For \( d_0 = d_\infty = d \), the error dynamics become homogeneous and in particular, \( d = -1 \) yields the error dynamics of Levant’s RED.46 In this case, the functions \( \Phi_i^n \) are given by

\[
\Phi_i^n(z) = |z|^{n_i} \quad \text{for } i = 1, \ldots, n
\]

and the error dynamics (14) read as

\[
\begin{align*}
\dot{e}_1 &= e_2 - \kappa_1 |e_1|^{n_1} , \\
\dot{e}_2 &= e_3 - \kappa_2 |e_1|^{n_2} , \\
&\vdots \\
\dot{e}_n &= u - \kappa_n |e_1|^0 .
\end{align*}
\]

For \( |u| \leq \Delta \) and suitable gains \( \kappa_i > 0 \), the solution of the error dynamics (16) converges to zero in finite-time.24,46 Well established parameter settings for the differentiator gains up to order \( n = 6 \) are proposed in Reference 47, Section 6.7. A necessary condition for any suitable parameter set is \( \kappa_n > \Delta \) and hence \( \kappa_n = 1.1 \Delta \) is proposed in Reference 46. In this case, the discontinuous error injection in (16d) is able to dominate the unknown input \( u \).

### 4 | THE SPECIAL COORDINATE BASIS

The SCB was introduced by Sannuti and Saberi31 in order to investigate structural properties of LTI systems. After transformation to the SCB, the state space is decomposed into four parts \( \mathbb{R}^n = \mathcal{X}_a \oplus \mathcal{X}_b \oplus \mathcal{X}_c \oplus \mathcal{X}_d \) with corresponding state vectors \( x_a, x_b, x_c, \) and \( x_d \). The related subsystems are denoted by (a), (b), (c), and (d) or \( \Sigma_a, \Sigma_b, \Sigma_c, \) and \( \Sigma_d \), respectively. The SCB reveals explicitly the invariant zeros, which govern the dynamics of subsystem (a). It also shows the system’s invertability structure. The system is right invertable if and only if subsystem (b) is nonexistent and it is left invertable if and only if subsystem (c) is non-existent. The properties of the SCB are extensively treated in Reference 32, Chapter 5 and briefly summarized in Section 4.1.

For system (2), the transformation to the SCB is summarized in

**Proposition 2** (SCB32(Theorem 5.4.1)). There exist nonsingular state, input and output transformations \( x = T \bar{x}, u = T \bar{u}, \) and \( y = T_o \bar{y} \) for system (2), such that the transformed system is given by

\[
\begin{bmatrix}
x_a \\
x_b \\
x_c \\
x_d
\end{bmatrix} = \begin{bmatrix}
A_a & H_{ab} C_b & 0 & H_{ad} C_d \\
0 & A_b & 0 & H_{bd} C_d \\
B_c F_{ca} & H_{cb} C_b & A_c & H_{cd} C_d \\
B_d F_{da} & B_d F_{db} & B_d F_{dc} & A_d
\end{bmatrix} \begin{bmatrix}
x_a \\
x_b \\
x_c \\
x_d
\end{bmatrix} + \begin{bmatrix}
B_{oa} \\
B_{ob} \\
B_{oc} \\
B_{od}
\end{bmatrix} \begin{bmatrix}
C_{oa} & C_{ob} & C_{oc} & C_{od}
\end{bmatrix} \begin{bmatrix}
x_a \\
x_b \\
x_c \\
x_d
\end{bmatrix} + \begin{bmatrix}
B_{oa} \\
B_{ob} \\
B_{oc} \\
B_{od}
\end{bmatrix} \begin{bmatrix}
u_0 \\
u_d \\
u_c \\
u_e
\end{bmatrix},
\]

(17a)
where with positive integers \( q \) and positive integers \( l \) such that \( \sum_{i=1}^{p} l_i = n_b \). Moreover, \( H_{bb} \) is a constant \( n_b \times p_b \) matrix. The matrices \( B_d \) and \( C_d \) are given by

\[
B_d = \text{diag} \left( B_{q_1}, B_{q_2}, \ldots, B_{q_{m_d}} \right) \quad \text{and} \quad C_d = \text{diag} \left( C_{q_1}, C_{q_2}, \ldots, C_{q_{m_d}} \right)
\]

with positive integers \( q_1 \geq q_2 \geq \cdots \geq q_{m_d}, \sum_{i=1}^{m_d} q_i = n_d \),

\[
B_{q_i} = \begin{bmatrix} 0_{(q_i-1) 	imes 1} \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad C_{q_i} = \begin{bmatrix} 1 \\ 0_{1 \times (q_i-1)} \end{bmatrix}
\]

Moreover,

\[
A_d = A_d^* + B_d F_{dd} + H_{dd} C_d,
\]

where \( F_{dd} \) and \( H_{dd} \) are constant matrices of appropriate dimension and

\[
A_d^* = \text{diag} \left( J_{q_1}, J_{q_2}, \ldots, J_{q_{m_d}} \right).
\]

From Proposition 2, it follows that

\[
T_s^{-1} A T_s = \overline{A} + B_0 C_0, \quad T_s^{-1} B T_s = \overline{B},
\]

\[
T_0^{-1} C T_s = \overline{C}, \quad T_0^{-1} D T_s = \overline{D}.
\]

A numerically stable algorithm for obtaining the involved transformations is proposed in Reference 38 and implemented in the Linear Systems Toolkit.\(^{48}\) For the unknown input observer design, the system is not yet in a suitable form. Before the details of the SCB are discussed, a special choice of the transformation related to subsystem (d) is proposed. The transformation is introduced in

**Theorem 1.** For the transformation to an SCB according to Proposition 2, the nonsingular state transformation \( x = T_s \tilde{x} \) can be chosen such that the matrix \( F_{dd} \) has the following particular structure:

\[
F_{dd} = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \beta_{1,2,1} & \cdots & \beta_{1,2,q_1,1} & 0 & \cdots & 0 & \cdots & 0 \\
0 & \beta_{1,3,1} & \cdots & \beta_{1,3,q_1,1} & 0 & \beta_{2,3,1} & \cdots & \beta_{2,3,q_2,1} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \beta_{1,m_1,1} & \cdots & \beta_{1,m_1,q_1,1} & \cdots & 0 & \beta_{m_1-1,m_1,1} & \cdots & \beta_{m_1-1,m_1,q_{m_1-1},1} & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
0^T \\
\beta_2 \\
\vdots \\
\beta_{m_d}
\end{bmatrix}
\]
In order to proof this result, an auxiliary step is required. Note, that with this special choice, the system is still in an SCB. In particular, if the system is already in an SCB according to Proposition 2, it is sufficient to apply a change of coordinates to subsystem $\Sigma_d$. This is summarized in

**Lemma 2.** Assume that system (2) is already in the SCB presented in Proposition 2. Then, there exists a regular state transformation matrix $x = T_x \bar{x}$ with $T_x = \text{diag} (I_{n_x}, I_{n_b}, I_{n_c}, T_d)$, such that $F_{dd}$ has the structure presented in Theorem 1. Moreover, it holds that $C_d T_d = C_d$ and $T_d^{-1} B_d = B_d$.

The proof of Lemma 2 is based on the proof of Reference 30, Theorem 3.1 and presented in the Appendix. The proof of Theorem 1 then follows straightforwardly from Proposition 2 and Lemma 2. Note that Theorem 1 results from the application of Reference 30, Theorem 3.1 to subsystem $\Sigma_d$ in the SCB. The resulting system is then in a particular SCB form which enables the observer design procedures presented in Section 5.

**Remark 2.** In both SCBs presented in Proposition 2 and Theorem 1, subsystem $\Sigma_d$ is formed by $m_d$ chains of integrators with length $q_i, i = 1, \ldots, m_d$ up to an output feedback with feedback matrix $H_{dd}$, respectively. The difference is, that for the special form of $F_{dd}$ according to (25) in Theorem 1, the resulting matrix $A_d$ is a block lower triangular matrix up to an output feedback. This will be essential for the construction of the unknown input observers.

The SCB reveals important structural system properties, which are discussed in the following. Further details can be found in References 31 and 32.

### 4.1 Properties of the SCB

As already mentioned, the state space of (17) is decomposed into $\mathbb{R}^{n} = \mathcal{X}_a \oplus \mathcal{X}_b \oplus \mathcal{X}_c \oplus \mathcal{X}_d$ corresponding to the states $x_a, x_b, x_c$, and $x_d$.

Subsystem (a) is related to the invariant zeros, that is, the eigenvalues of $A_a$. Subsystems (b) and (c) are related to the invertability properties of the system and subsystem (d) reveals the infinite zero structure.

Subsystem (b) consists of $p_b$ decoupled chains of integrators up to an output feedback. More specifically, one can split its state space according to

$$x_b = \begin{bmatrix} x_{b,1}^T & x_{b,2}^T & \cdots & x_{b,p_b}^T \end{bmatrix}^T$$

and each subsystem corresponding to $x_{b,i}$ for $i = 1, \ldots, p_b$ takes the specific form

$$\dot{x}_{b,i,1} = x_{b,i,2} + h_{b,i,1}^T y_b + h_{db,i,1}^T y_d,$$

$$\dot{x}_{b,i,2} = x_{b,i,3} + h_{b,i,2}^T y_b + h_{db,i,2}^T y_d,$$

$$\vdots$$

$$\dot{x}_{b,i,i} = h_{b,i,i}^T y_b + h_{db,i,i}^T y_d,$$

$$y_{b,i} = x_{b,i,1}.$$

with appropriate row vectors $h_{b,i,1}, \ldots, h_{b,i,i}$ and $h_{db,i,1}, \ldots, h_{db,i,i}$.

The infinite zero structure represented by subsystem (d) is of great importance for state estimation in the presence of unknown inputs. The state $x_d$ can be decomposed according to

$$x_d = \begin{bmatrix} x_{d,1}^T & x_{d,2}^T & \cdots & x_{d,m_d}^T \end{bmatrix}^T.$$

In particular, each subsystem $x_{d,i}$ for $i = 1, \ldots, m_d$ takes the specific form

$$\dot{x}_{d,i,1} = x_{d,i,2} + h_{dd,i,1}^T y_d,$$

$$\dot{x}_{d,i,2} = x_{d,i,3} + h_{dd,i,2}^T y_d,$$

$$\vdots$$

$$\dot{x}_{d,i,i} = h_{dd,i,i}^T y_d.$$
\[
\dot{x}_{d,iq_i} = f_{d,iq_i}^T x + f_{d,iq_i}^T x_b + f_{d,iq_i}^T x_c + f_{d,iq_i}^T x_d + u_{d,i},
\]
\[
y_{d,i} = x_{d,i1}.
\]

with appropriate row vectors \( h_{d,i1}^T, \ldots, h_{d,iq_i}^T, f_{d,iq_i}^T, f_{d,iq_i}^T, f_{d,iq_i}^T, \) and \( f_{d,iq_i}^T \). Each particular subsystem is an integrator chain of length \( q_i \) up to an output injection. In other words, \( q_i \) corresponds to the number of integrations between the input \( u_{d,i} \) and the output \( y_{d,i} \). The list \( S_\infty^\star(\Sigma) = \{q_1, q_2, \ldots, q_m\} \) represents the so-called infinite zero structure of the system in the sense that \( \Sigma \) has \( m_d \) infinite zeros of order \( q_1, q_2, \ldots, q_m \). It should be remarked, that \( S_\infty^\star(\Sigma) \) corresponds to the list \( I_4 \) of Morse’s structural invariant indices as defined in References 32 and 49. The following lemma summarizes important properties of the SCB that will be required for the proposed observer design.

**Lemma 3** (SCB properties\(^{31,32}\)(Section 5.4)).

1. \( \text{norm} \text{rank } G(s) = m_0 + m_d \).
2. The invariant zeros of \( \Sigma \) are the eigenvalues of \( A_a \).
3. The pair \( (A_b, C_b) \) is observable.
4. The triple \( (A_d, B_d, C_d) \) is strongly observable.
5. System \( \Sigma \) is left invertible if and only if \( n_c = 0 \). In this case, \( x_c \) and \( u_c \) are nonexistent.
6. \( X_a \oplus X_c = \mathcal{V}^* \Sigma \).
7. \( X_c = \mathcal{R}^* \Sigma \).

The following implications follow directly from the properties of the SCB.

**Lemma 4** (Strong detectability in SCB). System (2) is

(i) Strongly observable, if and only if for its SCB (17) it holds that \( n_a = n_c = 0 \).

(ii) Strongly detectable, if and only if for its SCB (17) it holds that \( n_c = 0 \) and \( A_a \) is a Hurwitz matrix.

(iii) Strong* detectable, if and only if it is strongly detectable and for its SCB (17) it holds that \( q_i = 1 \) for all \( i = 1, \ldots, m_d \).

**Proof.** Item (i) follows directly from SCB property 3. Item (ii) is due to properties 3 and 3 and it remains to prove item (iii). It is assumed that system (17) is strongly detectable, and hence \( n_c = 0 \). The rank condition (8) is equivalent to the condition

\[
\text{rank } \begin{bmatrix} KCB & D \end{bmatrix} = \text{rank } \begin{bmatrix} B \\ D \end{bmatrix},
\]

where \( K \) is a \((p-m_0) \times p\) full row rank matrix satisfying \( KD = 0 \), see Reference 6, Proposition 4. For system (17) in the SCB, one can choose

\[
K = \begin{bmatrix} 0_{m_i \times m_0} & I_{m_d} & 0 \\ 0 & 0 & I_{p_d} \end{bmatrix}
\]

and hence condition (30) results in

\[
\text{rank } \begin{bmatrix} C_d B_{0d} & C_d B_{d} \\ C_d B_{0b} & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \end{bmatrix} = m_0 + m_d.
\]

This condition is equivalent to \( \text{rank } C_d B_d = m_d \). Due to the structure of \( C_d \) and \( B_d \), the latter condition can be fulfilled, if and only if \( q_i = 1 \) for all \( i = 1, \ldots, m_d \), that is, if the length of each integrator chain in subsystem (d) is equal to 1. ■
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Because strong detectability is the minimum requirement for the existence of any estimation scheme in the presence of unknown inputs, \(^{21}\) this property is assumed in the following. Hence, \(n_c = 0\), which yields the new system representation:

\[
\begin{align*}
\dot{x}_a &= \begin{bmatrix} A_a & H_{ab}C_b & H_{ad}C_d \\ 0 & A_b & H_{bd}C_d \\ B_dF_{da} & B_dF_{db} & A_d \end{bmatrix} \begin{bmatrix} x_a \\ x_b \\ x_d \end{bmatrix} + \begin{bmatrix} B_{oa} \\ B_{ob} \\ B_{od} \end{bmatrix} u_0 \\
y_0 &= \begin{bmatrix} C_{oa} & C_{ob} & C_{od} \end{bmatrix} \begin{bmatrix} x_a \\ x_b \\ x_d \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u_0 \\
y_d &= 0
\end{align*}
\]

The overall system order is \(n = n_a + n_b + n_d\). By using the output relations (33b) in (33a), system (33) reduces to the three subsystems \(\Sigma_a\), \(\Sigma_b\), and \(\Sigma_d\). The first subsystem

\[
\Sigma_a : \quad \dot{x}_a = A_a x_a + H_{ab}y_b + H_{ad}y_d + B_{oa}y_0
\]

has no unknown input and no output. It is, however, influenced by the outputs of the other subsystems and the output \(y_0\) given by

\[
y_0 = C_{oa}x_a + C_{ob}x_b + C_{od}x_d + I_{m_0}u_0.
\]

Due to the assumption of strong detectability, the matrix \(A_a\) is a Hurwitz matrix as shown in Lemma 4, item (ii). Subsystem \(\Sigma_b\) is given by

\[
\Sigma_b : \begin{aligned}
\dot{x}_b &= A_b x_b + H_{bd}y_d + B_{ob}y_0, \\
y_b &= C_b x_b,
\end{aligned}
\quad \text{or equivalently} \quad \Sigma_b : \begin{aligned}
\dot{x}_b &= A_b^* x_b + H_{bb}y_b + H_{bd}y_d + B_{ob}y_0, \\
y_b &= C_b x_b.
\end{aligned}
\]

Again, this subsystem is not directly influenced by the unknown input, but by the outputs \(y_d\) and \(y_0\). The last subsystem

\[
\Sigma_d : \begin{aligned}
\dot{x}_d &= A_d x_d + B_d(F_{da}x_a + F_{db}x_b + u_d) + B_{od}y_0, \\
y_d &= C_d x_d,
\end{aligned}
\]

or equivalently

\[
\Sigma_d : \begin{aligned}
\dot{x}_d &= (A_d^* + B_dF_{dd})x_d + B_d(F_{da}x_a + F_{db}x_b + u_d) + H_{dd}y_d + B_{od}y_0, \\
y_d &= C_d x_d,
\end{aligned}
\]

is influenced by the unknown input \(u_d\) and the states \(x_a\) and \(x_b\) of the subsystems \(\Sigma_a\) and \(\Sigma_b\), respectively. With these representations, it is possible to design separate observers for subsystems \(\Sigma_a\), \(\Sigma_b\), and \(\Sigma_d\).

For subsystem \(\Sigma_a\), a trivial observer of the form

\[
\hat{\Sigma}_a : \quad \dot{\hat{x}}_a = A_a \hat{x}_a + H_{ab}y_b + H_{ad}y_d + B_{oa}y_0
\]

is chosen and it directly follows that the dynamics of the estimation error \(e_a = x_a - \hat{x}_a\) given by

\[
e_a = A_a e_a
\]

are asymptotically stable. For subsystem \(\Sigma_b\), the following observer is proposed:

\[
\hat{\Sigma}_b : \quad \dot{\hat{x}}_b = A_b^* \hat{x}_b + H_{bb}y_b + H_{bd}y_d + B_{ob}y_0 + e^*_b(y_b - C_b \hat{x}_b),
\]
where $E_b : \mathbb{R}^{p_b} \mapsto \mathbb{R}^{n_b}$ is a continuous output injection function. The error dynamics for $e_b = x_b - \hat{x}_b$ are governed by

$$\dot{e}_b = A^* b e_b - E_b(C_b e_b),$$

where $(A^*_b, C_b)$ is an observable pair. The proposed observer for subsystem $\Sigma_d$ is given by

$$\hat{\Sigma}_d : \dot{x}_d = (A^*_d + B_d F_{dd})\hat{x}_d + B_d(F_{da}\hat{x}_a + F_{db}\hat{x}_b) + B_{0d}y_0 + H_{dd}y_d + E_d(y_d - \hat{x}_d)$$

with the (discontinuous) output injection $E_d : \mathbb{R}^{m_d} \mapsto \mathbb{R}^{n_d}$.

The corresponding estimation error dynamic for $e_d = x_d - \hat{x}_d$ are then governed by

$$\dot{e}_d = (A^*_d + B_d F_{dd})e_d + B_d(F_{da}e_a + F_{db}e_b + u_d) - E_d(C_d e_d).$$

The proposed generic observer structure in the original (not SCB) coordinates is depicted in Figure 1. The design of the output injection gains $E_b$ and $E_d$ depends on the system and on the desired convergence properties. The available design possibilities are summarized in Table 1 and discussed in detail in the following sections.

### 5.1 Observer design for $\Sigma_b$

#### 5.1.1 Asymptotic linear observer

In order to design a linear observer (LO), a Luenberger-type observer of the form

$$\hat{\Sigma}_b : \dot{x}_b = A^*_b \hat{x}_b + H_{bb}y_b + H_{bd}y_d + B_{0b}y_0 + L_b(y_b - C_b \hat{x}_b)$$

is used.
with the output injection matrix $L_b \in \mathbb{R}^{n_b \times p_b}$ is proposed. The corresponding estimation error dynamics is governed by

$$\dot{e}_b = (A_b^* - L_bC_b)e_b.$$  

(46)

It follows from the SCB that $(A_b^*, C_b)$ is observable, and hence $L_b$ can be chosen such that $(A_b^* - L_bC_b)$ is a Hurwitz matrix with arbitrary eigenvalues. Consequently, the resulting error dynamics are asymptotically stable and the convergence speed can be assigned arbitrarily. A proper design of $L_b$ is required throughout the rest of this article.

### 5.1.2 Finite- or fixed-time observer

In some cases, for example, if the overall system is strongly observable, a finite or fixed time convergent observer for $\Sigma_b$ may be desirable. To that end, and, because subsystem $\Sigma_b$ is not influenced by the unknown input, the following continuous bi-homogeneous observer (CBHO) is proposed:

$$\hat{\dot{x}}_b : \hat{x}_b = A_b^* \hat{x}_b + H_{bd}y_b + H_{bd}y_d + B_{bd}y_0 + \mathcal{E}_b(e_{b,y}).$$  

(47a)

where

$$e_{b,y} = \begin{bmatrix} e_{b,r_1} & e_{b,r_2} & \cdots & e_{b,r_{p_b}} \end{bmatrix}^T$$  

(47b)

with $r_i = \sum_{j=1}^{i-1} l_j + 1$ and $i = 1, 2, \ldots, p_b$. The nonlinear output injection $\mathcal{E}_b : \mathbb{R}^{p_b} \mapsto \mathbb{R}^{n_b}$ is given by

$$\mathcal{E}_b(e_{b,y}) = \left[ v_{1,i} \Phi_1^l(e_{b,r_1}) \cdots v_{1,i-1} \Phi_{l-1}^l(e_{b,r_1}) v_{1,i} \Phi_1^l(e_{b,r_2}) \cdots v_{p_{b,i}} \Phi_{l-1}^l(e_{b,r_{p_{b,i}}}) \right]^T.$$  

(47c)

with positive parameters $v_{ij}, i = 1, \ldots, p_b, 1 \leq j \leq l_i$ and the nonlinear functions designed according to (11) and (12) in Section 3.3. This allows to state the following

**Theorem 2.** For any constants $0 < \mu < 1$ and $-1 < d_0 < 0 < d_{\infty} < \min_{i=1}^{l-1} \frac{1}{l-1}$, there exist appropriate gains $v_{ij} > 0, i = 1, \ldots, p_b, j = 1, \ldots, l_i$, such that (47) with the output injection (47c) is a continuous fixed-time observer for (47), that is, $\Sigma_b$.

**Remark 3.** By choosing $-1 < d_0 = d_{\infty} < 0$ the error dynamics correspond to those of the homogeneous observer with finite time convergence proposed in Reference 50. For $d_0 = d_{\infty} = -1$, Levant’s RED 24 is obtained.

**Proof.** Following Section 3.3, the error dynamics of $e_b = x_b - \hat{x}_b$ are given by

$$\Sigma_{e,b,1} : \begin{cases} \dot{e}_{b,1} = e_{b,2} - v_{1,1} \Phi_1^l(e_{b,r_1}), \\ \vdots \\ \dot{e}_{b,l_i-1} = e_{b,l_i} - v_{1,l_i-1} \Phi_{l_i-1}^l(e_{b,r_1}), \\ \dot{e}_{b,l_i} = -v_{1,l_i} \Phi_{l_i}^l(e_{b,r_1}), \\ \vdots \\ \dot{e}_{b,r_2} = e_{b,r_2+1} - v_{2,1} \Phi_1^l(e_{b,r_2}), \\ \vdots \\ \dot{e}_{b,r_2+l_2-1} = -v_{2,l_2} \Phi_{l_2}^l(e_{b,r_2}), \\ \vdots \\ \dot{e}_{b,r_{p_{b,i}}} = e_{b,r_{p_{b,i}}+1} - v_{p_{b,i},1} \Phi_1^{l_{p_{b,i}}}(e_{b,r_{p_{b,i}}}), \\ \vdots \\ \dot{e}_{b,n_{p_{b,i}}} = e_{b,n_{p_{b,i}}}, \\ \dot{e}_{b,n_{p_{b,i}}+1} = -v_{p_{b,i},1} \Phi_{l_{p_{b,i}-1}}^{l_{p_{b,i}}}(e_{b,r_{p_{b,i}}}), \\ \vdots \\ \dot{e}_{b,n_{p_{b,i}}+l_{p_{b,i}}-1} = -v_{p_{b,i},l_{p_{b,i}}} \Phi_{l_{p_{b,i}}}(e_{b,r_{p_{b,i}}}). \end{cases}$$  

(48)
The systems $\Sigma_{e,b,i}$ for $i = 1, \ldots, p_b$ are decoupled and each coincides with the error dynamics of Moreno’s arbitrary order fixed-time differentiator, that is, (14) with $u = 0$. Hence, the result follows from Reference 28, Theorem 1.

5.2 Observer design for $\Sigma_d$

5.2.1 Linear observer

If the system is strong detectable, it follows from Lemma 4 that $q_i = 1$ for $i = 1, \ldots, m_d$. Together with (29), one immediately concludes that $B_d = C_d = I_{m_d}$ and $y_d = x_d$. Hence, instead of the full order observer (43) one may utilize a reduced order observer, that is,

$$\hat{\Sigma}_d : \dot{x}_d = y_d$$

(49)

There is no benefit in designing a full-order unknown input observer in comparison with the reduced order observer (49). To see this, consider a classical linear unknown input observer design for the strongly observable subsystem (d), that is, the triple $(A_d, B_d, C_d)$. There, the rank condition $\text{rank } C_d B_d = \text{rank } B_d = m_d$ is trivially fulfilled. Following the design procedure proposed in Reference 12 (see also the algorithm in Reference 12, Table 3.1), one can verify that

$$z_d = Fz_d, \quad z_d(0) = z_{d,0} \in \mathbb{R}^{n_d},$$

(50a)

$$\dot{x}_d = z_d + y_d,$$

(50b)

with an arbitrary Hurwitz matrix $F$ is the resulting full order unknown input observer for subsystem (d). For $z_{d,0} = 0$, the observer reduces to (49), which shows that there is no benefit in designing a full-order unknown input observer. This is due to the direct feed-through of the output $y_d$ to the estimate $\dot{x}_d$ which, contrary to the case without unknown inputs, doesn’t allow to mitigate effects from, for example, measurement noise acting on this output by using a dynamic observer. If one is interested in an estimate of the unknown input, it is possible to employ a first order sliding mode algorithm (1-SMO). This is a special case of the observer presented in the following section.

5.2.2 Case 2: Sliding mode observer

If $q_i > 1$ for some $i$, then derivatives of the output $y_{d,i}$ are required in order to estimate the states of $\Sigma_d$. This can be achieved by employing sliding mode techniques, which is a very common technique for the unknown input observer design. The sliding mode differentiator is used here to derive the idea and the stability proof for the observer design. It is then generalized to the arbitrary order bi-homogeneous differentiator in Section 5.2.3.

First, a component-wise bound on the unknown input $u_d$ is required. Note that the bounds for $u$ in the original coordinates (3) are not necessarily symmetric. Let the lower and upper bounds for $u$ be given by $u_{\text{min}}$ and $u_{\text{max}}$, respectively. In order to obtain the bounds in the SCB, it is possible to compute an offset and the remaining symmetric part, that is,

$$u_{\text{off}} = \frac{1}{2} (u_{\text{max}} + u_{\text{min}}) \quad \text{and} \quad u_s = \frac{1}{2} (u_{\text{max}} - u_{\text{min}}),$$

(51)

where one can verify that $u_{\text{min}} = u_{\text{off}} - u_s$ and $u_{\text{max}} = u_{\text{off}} + u_s$. The input transformation $\bar{u} = T_i^{-1}u = G_i u$ can be partitioned according to

$$\begin{bmatrix} u_0 \\ u_d \end{bmatrix} = \begin{bmatrix} G_0 \\ G_d \end{bmatrix} u,$$

(52)
with $G_d$ as an $m_d \times m$ matrix. This allows to derive a tight upper bound on the unknown input $u_d$ in the SCB according to

$$|u_d| = \left| \sum_{j=1}^{m} g_{d,i,j}(u_j - u_{off,j}) + \sum_{j=1}^{m} g_{d,i,j}u_{off,j} \right| \quad (53a)$$

where $G_d = [g_{d,i,j}]$.

The proposed observer for $\Xi_d$ is given by

$$\dot{\hat{x}}_d = \left( A_d^* + B_d F_{dd} \right) \hat{x}_d + B_d (F_{da} \hat{x}_a + F_{db} \hat{x}_b) + B_{0d} y_0 + H_d y_d + \epsilon_d(e_{d,y}),$$

where

$$e_{d,y} = y_d - \hat{y}_d = \begin{bmatrix} e_{d,w_1} & e_{d,w_2} & \cdots & e_{d,w_{m_d}} \end{bmatrix}^T \quad (54b)$$

is the output error with $w_i = \sum_{j=1}^{q_i} 1 + 1$ and $i = 1, 2, \ldots, m_d$. Moreover, $\epsilon_d : \mathbb{R}^{m_d} \mapsto \mathbb{R}^{m_d}$ is the nonlinear output injection with

$$\epsilon_d(e_{d,y}) = \left[ \kappa_1 \Phi_{q_1}^0(e_{d,w_1}) \cdots \kappa_{q_1} \Phi_{q_1}^{q_1-1}(e_{d,w_1}) \cdots \kappa_{m_d} \Phi_{m_d}^0(e_{d,w_{m_d}}) \cdots \kappa_{q_d} \Phi_{q_d}^{q_d}(e_{d,w_{m_d}}) \right]^T \quad (54c)$$

with positive parameters $\kappa_{i,j}$, $i = 1, \ldots, m_d$, $j = 0, \ldots, q_i - 1$. The nonlinear functions $\Phi_{j}^0$ with $i = 1, \ldots, m_d$ and $j = 1, \ldots, q_i$ are designed following the higher order sliding mode approach given by (15).

**Theorem 3.** Assume that system (17) with $F_{dd}$ as in (25) is strongly detectable and that the output injection matrix $L_b$ is chosen such that $(A_d^* - L_b C_b)$ is a Hurwitz matrix. Then, there exist sufficiently large gains $\kappa_{i,j} > 0$ in (54c), such that (54) is a finite-time unknown input observer for (37), that is, for every initial condition $e_{d,0} = x_{d,0} - \hat{x}_{d,0}$, there exists a finite time $T_f$ such that $e_{d}(t) = x_{d}(t) - \hat{x}_{d}(t) = 0$ for all $t \geq T_f$.

**Proof.** Let the rows of the matrices $F_{da}$ and $F_{db}$ be denoted by

$$F_{da} = \begin{bmatrix} f_{da,1}^T \\ f_{da,2}^T \\ \vdots \\ f_{da,m_d}^T \end{bmatrix} \quad \text{and} \quad F_{db} = \begin{bmatrix} f_{db,1}^T \\ f_{db,2}^T \\ \vdots \\ f_{db,m_d}^T \end{bmatrix},$$

respectively. It follows from Theorem 1 and the structure of $A_d$, that the dynamics of the estimation error

$$e_d = x_d - \hat{x}_d = \begin{bmatrix} e_{d,1} & e_{d,2} & \cdots & e_{d,m_d} \end{bmatrix}^T \quad (56)$$

are governed by

$$\Sigma_{e,d,1} : \begin{cases} e_{d,1} = e_{d,2} - \kappa_{1,1} [e_{d,w_1}]^{\frac{q_1 - 1}{q_1}}, \\
\vdots \\
 e_{d,q_1} = e_{d,q_1} - \kappa_{q_1,1} [e_{d,w_1}]^{\frac{1}{q_1}}, \\
 e_{d,q_1} = -\kappa_{q_1,1} [e_{d,1}]^0 + u_{d,1} + f_{da,1}^T e_a + f_{db,1}^T e_b, \end{cases}$$
The dynamics of each subsystem $\Sigma_{e,d,i}$, $i = 1, \ldots, m_d$ coincides with that of the RED (16). Note that the errors $e_a$ and $e_b$ decay exponentially. Hence, for every $\epsilon > 0$ there exists a finite time $T_\epsilon$, such that $|\hat{f}_{db,i}e_a(t) + f_{db,i}e_b(t)| < \epsilon$ for all $i = 1, \ldots, m_d$ and $t \geq T_\epsilon$.

Now, consider the first subsystem $\Sigma_{e,d,1}$. Define a new unknown input $\tilde{u}_{d,1}$ according to

$$\tilde{u}_{d,1} = u_{d,1} + f_{da,1}^T e_a + f_{db,1}^T e_b. \quad (58)$$

For any (arbitrarily small) $\epsilon > 0$, there exists a finite time $T_{\epsilon,1}$ such that this input is bounded by $|\tilde{u}_{d,1}| \leq \Delta_{d,1} + \epsilon$. Hence, there exist sufficiently large gains $\kappa_{1,1}, \ldots, \kappa_{1,q_1}$ such that the states in $\Sigma_{e,d,1}$ are exactly zero after a finite transient time.

The unknown input acting on $\Sigma_{e,d,2}$ is given by

$$\tilde{u}_{d,2} = \beta_2^T e_d + u_{d,2} + f_{da,2}^T e_a + f_{db,2}^T e_b. \quad (59)$$

It follows from the structure of $F_{dd}$ given in (25), that the first part $\beta_2^T e_d$ only depends on the states in $\Sigma_{e,d,1}$ and hence this term vanishes after a finite transient time. Consequently, there exists a finite time $T_{\epsilon,2}$ such that the input is bounded by $|\tilde{u}_{d,2}| \leq \Delta_{d,2} + \epsilon$ and the states of this subsystem converge to zero in finite time for sufficiently large gains $\kappa_{2,1}, \ldots, \kappa_{2,q_2}$. The rest of the proof follows analogously by induction.

**Remark 4.** If the observer is used in a closed control loop it has to be considered that the estimation error convergence time of $\hat{\Sigma}_a$ is significantly influenced by the convergence time constants of $\hat{\Sigma}_a$ and $\hat{\Sigma}_b$. The convergence of $\hat{\Sigma}_a$ is governed by the system’s invariant zeros and hence cannot be influenced. This may lead to undesired estimation error transients.

**Remark 5.** If $q_i = 1$ for some $\Sigma_{e,d,i}$, the corresponding error dynamics reduces to a first order sliding mode dynamics. Hence, if the system is strong detectable, that is, $q_i = 1$ for all $i = 1, \ldots, m_d$, this design procedure results in a first order sliding mode observer (1-SMO).

The proposed unknown input observer design can be seen as a generalization of Reference 30, which considers only the strongly observable case. It has some advantages compared to already existing higher order sliding mode observers, which are discussed in the following. Compared to Reference 30, its construction builds upon the SCB. This form is well studied in the literature and there are numerically reliable algorithms to obtain the transformation. Moreover, the direct feed-through case is explicitly included within the proposed framework. In contrast to References 10, 14, and 21-23, it does not require the design of an additional “stabilizing” Luenberger observer, see Reference 30. This reduces the design complexity and simplifies the tuning procedure because both, the observer order and the number of tuning parameters, are less than or equal to the system order. Moreover, the infinite zeros structure $S_{\infty}^*(\Sigma) = \{q_1, q_2, \ldots, q_d\}$ or equivalently the list $I_q$ of Morse’s structural invariant indices, see Section 4, represents the number of required signal derivatives. This is the minimum number of derivatives required for the reconstruction of the states. In the proposed design, the derivatives of the output error signal $e_{dy}$ are obtained component-wise with possibly distinct differentiator orders. In contrast to Reference 22, which requires the least number of vector-valued derivatives, our observer architecture typically requires less derivatives if the lengths of the integrator chains in subsystem (d) are different. In contrast to works like References 10 or 21, the design is based on the SCB representation. This allows to use well established and numerically
reliable algorithms to transform the system into the desired form. Moreover, the proposed design could be easily extended in the sense that if some $q_i = 1$, that is, if the corresponding state can be directly measured, then it is possible to use this measurement in the spirit of the reduced order unknown input observer design in Section 5.2.1. In practice, this reduces chattering effects. If the unknown input is not bounded but there exist bounds on some higher-order derivatives of the input, it is possible to formulate an auxiliary system following the ideas in Reference 22. The observer design can then be performed for this auxiliary system.

A generalization of the higher-order sliding mode approach is presented in the following.

5.2.3 Discontinuous bi-homogeneous observer

As an alternative to the sliding mode observer, the following discontinuous bi-homogeneous observer (DBHO) is proposed for $\Sigma_d$ in (38):

$$\hat{\Sigma}_d : \dot{x}_d = (A_d^* + B_d F_{dd}^*) \hat{x}_d + B_d F_{dl} \hat{x}_l + B_{dd} y_0 + H_{dd} y_d + \mathcal{E}_d(e_{d,y}),$$

where

$$e_{d,y} = y_d - \hat{y}_d = \left[ e_{d,w_1}, e_{d,w_2}, \ldots, e_{d,w_{m_d}} \right]^T$$

as the output error with $w_i = \sum_{j=1}^{i-1} q_j + 1$ and $i = 1, 2, \ldots, m_d$. Moreover, $\mathcal{E}_d : \mathbb{R}^{m_d} \mapsto \mathbb{R}^{n_d}$ is the nonlinear output injection with

$$\mathcal{E}_d(e_{d,y}) = \left[ \kappa_{1,1} \Phi_{q_1}^0(e_{d,1}) \cdots \kappa_{1,q_1-1} \Phi_{q_1}^{q_1-1}(e_{d,1}) \kappa_{1,q_1} \Phi_{q_1}^0(e_{d,1}) \cdots \kappa_{m_d,q_{m_d}} \Phi_{q_{m_d}}^0(e_{d,r_{m_d}}) \cdots \kappa_{m_d,q_{m_d}} \Phi_{q_{m_d}}^{q_{m_d}}(e_{d,r_{m_d}}) \right]^T$$

with positive parameters $\kappa_{ij}, i = 1, \ldots, m_d, j = 0, \ldots, q_i - 1$ and the nonlinear functions $\Phi_{q_i}^j$ with $i = 1, \ldots, m_d$ and $j = 1, \ldots, q_i$ (11) and (12) in Section 3.3.

Based on this observer design, it is possible to achieve fixed-time convergence of the overall estimation error according to

**Theorem 4.** Suppose that $\hat{\Sigma}_b$ in (47) is designed according to Theorem 2 and that the unknown input $u_d$ is uniformly bounded. Then, for $d_0 = -1$ and any constants $0 < d_\infty < \min_{i=1}^{m_d} \frac{1}{q_i-1}$ and $0 < \mu < 1$, there exist appropriate gains $\kappa_{ij} > 0$, $i = 1, \ldots, m_d$, $1 \leq j \leq q_i$, such that (60) with the output injection (60c) is a fixed-time observer for (38), that is, for $\Sigma_d$.

The proof is analogous to the proof of Theorem 3. Hence, the combination of Theorems 2 and 4 allows to design a fixed-time unknown input observer for strongly observable systems by using (47) and (60).

5.3 Summary of the design procedure

This section summarizes the design procedure and discusses important design aspects. In general, it is possible to combine various observer design approaches from the previous sections, which yields a variety of observers. The choice for one specific observer is a design question. The proposed design techniques are summarized in Table 1 and Figure 2.

For system $\Sigma_a$, a trivial observer $\hat{\Sigma}_a$ as in (39) has to be designed. The output injection of the observer $\hat{\Sigma}_b$ in (41) could be either linear (45) or nonlinear (47) but continuous. For the latter, either finite- or fixed-time convergence can be achieved. For the observer design for $\Sigma_d$, two cases are distinguished. If the rank condition (8) is fulfilled, all states of $\Sigma_d$ can be measured directly, see (49). If one is interested in disturbance reconstruction, it may be reasonable to employ a first order sliding mode observer (54). If the rank condition is not fulfilled, differentiation is required and hence a robust differentiation algorithm with discontinuous output injection for the observer $\hat{\Sigma}_d$ in (43) is employed. This could
FIGURE 2 Observer design for the specific subsystems depending on the observability properties and the convergence requirements

either be a higher-order sliding mode observer (54) or (a more general) discontinuous bi-homogeneous observer (60). The higher-order sliding mode observer provides finite time convergence whereas the bi-homogeneous observer allows to achieve finite- or fixed time convergence of the estimation error of \( \Sigma_d \).

If system (2) is merely strongly detectable, only asymptotic convergence of the overall estimation error can be achieved in general. If the system is strongly observable, it is possible to design either fixed-time or finite-time observers. If the rank condition (8) holds additionally, one may design a fixed-time observer with continuous output injection by employing the simple linear estimate (49) for \( \Sigma_d \).

6 | EXAMPLE

As a numerical example, system (2) is considered with the following coefficient matrices:

\[
A = \begin{bmatrix}
-11 & 0 & -1 & 11 & 2 & 0 & 15 & -12 \\
3 & -3 & 3 & 2 & -4 & -1 & -10 & -6 \\
2 & -7 & 4 & -7 & 8 & -9 & 10 \\
1 & 0 & 0 & -1 & 1 & -1 & -3 & 0 \\
1 & 3 & 2 & 2 & 4 & 8 & 5 \\
2 & 6 & 5 & 7 & 6 & -9 & 7 & -10 \\
1 & 0 & 0 & -2 & 0 & 1 & 0 & 1 \\
0 & -3 & 2 & -5 & -4 & 2 & -10 & -4
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 & -2 \\
-3 & 0 & 1 \\
1 & 1 & 1 \\
-1 & 0 & 0 \\
1 & 0 & -1 \\
-1 & 1 & -1 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad \text{(61)}
\]

\[
C = \begin{bmatrix}
-1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -2 & 1 & 0 & -2 & 1 & -2 & 0
\end{bmatrix}, \quad D = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}. \quad \text{(62)}
\]

The following observer design together with the simulation is provided as MATLAB and SIMULINK code as a supplementary material. For this unstable system, it can be verified that the rank condition (8) is not fulfilled and the system is not strong* detectable. Hence, no linear unknown input observer exists. A transformation to the SCB (17) using the linear systems toolkit and the transformation in Theorem 1 reveals that \( n_a = 1, \ n_b = 2, \ n_c = 0, \) and \( n_d = 5. \) For \( \Sigma_d, \) there are two chains of integrators with \( q_1 = 3 \) and \( q_2 = 2, \) respectively. The system is strongly detectable, because it has a stable invariant zero at \( \lambda_1 = -10 \) and \( n_c = 0. \)
The transformation to the proposed SCB (17) with $\mathbf{F}_{dd}$ as in (25) results in the transformation matrix

$$
\mathbf{T}_s = 
\begin{bmatrix}
-0.7071 & 0.3297 & -0.0330 & 5.6804 & -1.0328 & 0 & 3.7947 & -1.2649 \\
0 & -0.2857 & -0.1429 & 0.7746 & -0.7746 & 0 & -2.2136 & 0.6325 \\
0 & 0.4286 & -0.2857 & -11.8771 & 4.6476 & -1.2910 & -0.3162 & 0.9487 \\
0 & 0 & 0 & 0.7746 & 0 & 0 & -0.6325 & 0 \\
0 & 0 & 0 & -1.2910 & 0.7746 & 0 & 1.5811 & -0.6325 \\
0 & 0 & 0 & 11.8771 & -4.6476 & 1.2910 & 0.3162 & -0.9487 \\
0 & 0 & 0 & 0.5164 & 0 & 0 & 0.6325 & 0 \\
0 & 0 & 0 & -2.8402 & 0.5164 & 0 & -1.8974 & 0.6325 \\
\end{bmatrix}.
$$

(63)

The transformed system in the SCB is given by

$$
\begin{align*}
\mathbf{B}_0 &= \begin{bmatrix} -0.6216 & 2 & 3 & 0 & 1.1619 & 5.8095 & 1.5811 & 3.7947 \end{bmatrix}^T, \\
\mathbf{C}_0 &= \begin{bmatrix} 0.7071 & -0.3297 & 0.0330 & 5.9386 & -3.6148 & 1.2910 & -2.2136 & 0.3162 \end{bmatrix}, \\
\mathbf{A}_d &= -10, \quad \mathbf{H}_{ab} = 0, \quad \mathbf{H}_{ad} = [-2.6844 \ 2.4376], \quad \mathbf{H}_{bb} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \mathbf{H}_{bd} = \begin{bmatrix} -1.0328 & 1.8974 \\ 0.2582 & 5.0596 \end{bmatrix}, \\
\mathbf{F}_{da} &= \begin{bmatrix} 1.6432 \\ 0 \end{bmatrix}, \quad \mathbf{F}_{db} = \begin{bmatrix} -1.7620 & 0.3533 \\ 2.7105 & -0.2259 \end{bmatrix}, \quad \mathbf{H}_{dd} = \begin{bmatrix} -6.2000 & -0.2449 \\ -15.6000 & -3.1843 \\ -21.5000 & -4.8990 \\ -1.4697 & -2.8000 \\ -11.3493 & 0.6000 \end{bmatrix}, \quad \mathbf{F}_{dd} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.8165 & 0 \end{bmatrix}.
\end{align*}
$$

(64a)  (64b)  (64c)  (64d)

and the matrices $\mathbf{A}_b^\ast$, $\mathbf{C}_b$, $\mathbf{A}_d^\ast$, $\mathbf{B}_d$, and $\mathbf{C}_d$ as in Proposition 2. The input and output transformation matrices are given by

$$
\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.2910 & -0.3162 \\ 0 & 0 & 0.6325 \end{bmatrix} \quad \text{and} \quad \mathbf{T}_0^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.7746 & 0 & 0 \\ 0 & -0.6325 & 1.5811 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

(65)

For the simulation, the three components of the unknown input are chosen as $u_1(t) = \sin(2t)$, $u_2(t) = 2.5 \sin(4t) + 0.5$, and $u_3(t) = \sigma(t-1) - \sigma(t-4)$, where $\sigma(t)$ is the unit step function. The components of the unknown input are thus bounded according to $u_1 \in [-1 \ 1]$, $u_2 \in [-1 \ 3]$, and $u_3 \in [0 \ 1]$. Together with (65) and (53a), this results in the input bounds $|u_{d,1}| \leq 2.7111 = \Delta_{d,1}$ and $|u_{d,2}| \leq 1.5811 = \Delta_{d,2}$ in the SCB.

Following Section 5, a sliding mode based unknown input observer is designed for the strongly detectable system. For subsystem $\Sigma_a$, a trivial observer according to (39) is employed. For subsystem $\Sigma_b$, a Luenberger observer following (41) is designed, that is, $\dot{\mathbf{e}}_b(\mathbf{y}_b - \mathbf{C}_b \hat{\mathbf{x}}_b) = \mathbf{L}_b(\mathbf{y}_b - \mathbf{C}_b \hat{\mathbf{x}}_b)$ such that the eigenvalues of $(\mathbf{A}_b^\ast - \mathbf{L}_b \mathbf{C}_b)$ are given by the set $\{ -8 \ -6 \}$. This results in $\mathbf{L}_b = [13 \ 47]^T$. For subsystem $\Sigma_d$, the sliding mode observer proposed in (54) is employed. The nonlinear output injection $\mathbf{\vartheta}_d : \mathbb{R}^2 \mapsto \mathbb{R}^3$ is given by

$$
\mathbf{\vartheta}_d(\mathbf{e}_{d,y}) = \begin{bmatrix} \kappa_{1,1}[\mathbf{e}_{d,1}]^\frac{1}{2} & \kappa_{1,2}[\mathbf{e}_{d,1}]^\frac{1}{2} & \kappa_{1,3}[\mathbf{e}_{d,1}]^0 & \kappa_{2,1}[\mathbf{e}_{d,1}]^\frac{1}{2} & \kappa_{2,2}[\mathbf{e}_{d,1}]^0 \end{bmatrix}^T.
$$

(66)
**Figure 3** Estimation errors in SCB coordinates

**Figure 4** True (solid) and estimated system states (dashed red) for the original system (61)
The gains for the sliding mode observer are chosen according to

\[ \kappa_{1,1} = 2\Delta_{d,1}, \quad \kappa_{1,2} = 2.12\Delta_{d,1}, \quad \kappa_{1,3} = 1.1\Delta_{d,1}, \]
\[ \kappa_{2,1} = 1.5\Delta_{d,2}, \quad \kappa_{2,2} = 1.1\Delta_{d,2}. \]

see also Reference 47, Chapter 6.

The initial condition of system (2) is chosen as \( x_0 = T_s \left[ -0.5 \quad 0.2 \quad 0.1 \quad -0.15 \quad 0.25 \quad -0.1 \quad -0.1 \quad 0.3 \right]^T \) and the observer is initialized with zero. The components of the estimation error in SCB are depicted in Figure 3. Here, the finite-time convergence properties of the errors in subsystem (d) can be verified. Figure 4 shows a comparison of the states and their corresponding estimates in the original coordinates. It can be seen that the estimates (asymptotically) converge to the true states.

7 | DISCUSSION AND OUTLOOK

This article presents a unifying design framework for linear and nonlinear unknown input observers for linear time-invariant systems. It is shown that after the transformation to the SCB, the design for asymptotic and finite- or fixed-time can be carried out in a similar fashion. Depending on the system properties and the desired estimation error dynamics, the design procedure allows a straightforward design of asymptotic, finite- or fixed-time unknown input observers. If derivatives are required, the number of differentiation operations is kept at a minimum. The design can be straightforwardly extended to descriptor systems \(^{32,51}\) or unbounded unknown inputs. \(^{22}\)

In future work, the proposed observer design will be extensively evaluated in simulation studies and real-world experiments. The proposed method could also be employed to reconstruct the unknown input by following the ideas in Reference 30 and it can hence be utilized in a robust control framework. For this, the performance in a control loop needs thorough investigation. Moreover, the influence of measurement noise and model uncertainty will be investigated.

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CONFLICT OF INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

DATA AVAILABILITY STATEMENT

The data related to the example in Section 6 are available in the supplementary material of this article.

ENDNOTES

*This is a special choice discussed in the introduction of Reference 28.
†MATLAB code available via http://www.mae.cuhk.edu.hk/bmchen/
‡MATLAB code available via http://www.reichhartinger.at/index.php?id=38

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**APPENDIX. PROOF OF LEMMA 2**

Due to the special structure of $T$ it is sufficient to proof the existence of a transformation $x_d = T_d^{-1} \bar{x}_d$ such that subsystem (d) takes the desired form as in Proposition 2 with $F_{dd}$ as in (25). The existence of such a transformation for strongly observable systems follows from Reference 30, Theorem 3.1. It is noted that subsystem (d), that is, the triple $(A_d, B_d, C_d)$, is strongly observable, because it possesses no invariant zeros. The constructive proof of Reference 30, Theorem 3.1, that is, the transformation algorithm presented in Reference 30, Section 4.1 provides a solution to proof the relations $C_d T_d = C_d$ and $T_d^T B_d = B_d$.

In fact, the transformation algorithm presented in Reference 30, Section 4.1 can be drastically simplified since subsystem (d) already exhibits a special structure. The decomposition of the dynamic matrix in Step 1, Equation (24) of the algorithm is straightforward as argued in the following. The dynamic matrix of subsystem (d) obtained from any transformation to the SCB is given by

$$ \bar{A}_d = A_d^* + B_d \bar{F}_{dd} + \bar{H}_{dd} C_d $$

(A1)

with $\bar{F}_{dd}$ and $\bar{H}_{dd}$ as matrices of appropriate dimensions. Using the notation of Reference 30, this matrix is decomposed according to

$$ \bar{A}_d = \hat{A}_d - \Pi C_d. $$

(A2)
where $\tilde{A}_d = A_d^* + B_d F_{dd}$ and $\Pi = -H_{dd}$. Furthermore, an additional output transformation is not necessary, that is, $\bar{C} = C_d$ and $\Gamma = I_{m_d}$. The orders of the subsystems (denoted as $\mu_i$ in Reference 30) are already given in sorted order by the lengths $q_1 \geq q_2 \geq \cdots \geq q_{m_d}$ of the integrator chains. Then, the transformation algorithm yields the output matrix $C_d = \text{diag}(C_{q_1}, C_{q_2}, \ldots, C_{q_{m_d}})$ for the transformed system which is ensured by Niederwieser et al. 30 (Lemma 4.1.f) and, thus, $C_d T_d = C_d$ holds. In Equation (B23) in the proof of Reference 30, Lemma 4.1.e, it is shown that the input matrix of the transformed system is given by

$$T_d^{-1}B_d = \mathcal{O}_R B_d.$$  \hspace{1cm} (A3)

It can be easily shown that the reduced observability matrix

$$\mathcal{O}_R = I_{n_d}$$  \hspace{1cm} (A4)

reduces to the identity matrix in this special case and, thus, $T_d^{-1}B_d = B_d$ is satisfied which completes the proof.