The Topological Nature Of Defects

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Abstract

The subject of topological defects has become a very attractive field of study given its apparent relevance to as diverse systems as the early universe and condensed matter. As usually envisaged the topology of the manifold $M$ of the minima of the relevant to our physical system effective potential $V$, provides reliable information about the capacity of that system to accommodate topological defects. Here we will examine the premises for that statement to be true.

1 Introduction

An ordered medium corresponds to some real smooth manifold $W$ where a function $\varphi : W \rightarrow M$ is defined which assigns to every point of the medium an order parameter. For $x$ a point of $W$, the order parameter corresponds to the value of $\varphi(x)$. In our discussion, in fact, we will refer to $\varphi$ itself by the name order parameter even though this title might be confusing since one could think in terms of a map from the ordered medium to a space of functions, allocating a function (the order parameter) at every point of the manifold. In our discussion, we will call $\varphi$ by what it actually does: it designates a value, the order parameter, at every point of $W$.

Excluding the trivial case where the order parameter is constant throughout the medium (which is called, thus, uniform), we will focus our attention to non uniform media where the function, through connected space, varies continuously apart perhaps (depending on the specific configuration) at isolated regions.$^9$

The ordered medium has no defects if the order parameter is everywhere continuous. The defect will be associated with a space region containing the discontinuity of the order parameter. For example, assume that the order parameter $\varphi : W = \mathbb{R} \rightarrow M = \mathbb{R}$ is given by

$$\varphi(x) = \begin{cases} f(x) & x > x_0 \\ c & x = x_0 \\ f(x) & x < x_0 \end{cases}$$

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for some constant $c$ and $f : \mathbb{R} \to \mathbb{R}$. For example, $f(x)$ could be 
\[
\frac{(c_1 \Theta(x - x_0) + c_2 \Theta(x_0 - x))}{(1 + \Theta(x - x_0) \Theta(x_0 - x))}
\]
with $\Theta(x) = 1$ for $x \geq 0$ and zero otherwise. Take $f(x)$ to be a continuous function for all $x \neq x_0$. Thus, we can calculate the 
\[
\lim_{x \to x_0^+} f(x)
\]
which we will call as $f^+(x_0)$ and $f^-(x_0)$ accordingly. If $f^+(x_0) \neq f^-(x_0)$ then, regardless of the actual value of $c = \varphi(x_0)$, $\varphi$ will be discontinuous at $x = x_0$. Otherwise, for $f^+(x_0) = f^-(x_0)$ then, if there is a discontinuity of $\varphi$ at $x = x_0$, it will be due to the value of $c$ which should satisfy the relation: 
\[
c \neq \left[ f^+(x_0) = f^-(x_0) \right].
\]
Both cases assimilate a topological defect (at $x = x_0$) since $\varphi$, for different reasons for each situation, is discontinuous at $x_0$. For the record, we will associate the first type of discontinuity with a stable defect and the second with an unstable one.

Correlating discontinuities in $\varphi$ with topological defects, is a quite controversial way forward. The main reason is that one does not need discontinuities to establish whether or not topological defects exist within a certain field configuration. However, here we will advocate in favor of the argument maintaining that we can always appropriately associate a topological defect with a discontinuity, the latter being the primordial source of defects.

Consider a function $\Psi : W \to M$ to be continuous and to incorporate defects. We should be able to recover a discontinuity on the defects. This can be done by substituting $\Psi : W \to M$ with a $\varphi : W \to M$ where $M \subset M$, and $\varphi = \Psi$ away from the defects. At the latter, $\varphi$ is discontinuous.

The converse is not necessarily true. That is, suppose we know $\Psi : W \to M$ to be a continuous function but we have no information about the existence of topological defects in its configuration. If we can find an $M \subset M$ and define a $\varphi : W \to M$ so that $\varphi$ to be discontinuous on $W$, then we cannot automatically draw an analogy between the points of discontinuity in $\varphi$ with possible embodied topological defects in $\Psi$.

If we know of the existence of topological defects, we can either directly or indirectly recover discontinuities at the defects. However, discontinuities by themselves do not necessarily indicate the presence of topological defects. For the latter to be securely identified one needs more information than simply the reassurance of the feasibility of singularities for some appropriate choice of field $\varphi$. There are energetic conditions one should verify whether or not they are satisfied.

Most physical systems are described by a continuous function. The ordered medium corresponds to some physical system which is being described by a continuous function $\Psi$. For classical physical systems, at least, one assumes a tendency for the system to acquire the lowest possible energy state. Since our discussion will be mainly on a classical level, we will adopt the viewpoint that this tendency for the lowest energy state is incorporated in the behavior of $\Psi$. The lowest energy state we will call the “ideal” or “perfect” state which $\Psi$ aims for.
If some $\Psi$ does not correspond to the lowest energy we will say that it must incorporate defects, being faithful perhaps to the true meaning of the word, since such a configuration will have to be “imperfect”. The energy of some $\Psi$ will be measured in terms of the Hamiltonian energy density $\mathcal{H} = K + V$ with $K$ and $V$ the kinetic and potential energy density contributions respectively. We are going to postulate here that the occurrence of any type of topological defect will be a consequence of the departure from the ground state of $\mathcal{H}$. If this is some inevitable result of certain conditions then the arising defect will be called stable. Otherwise, it will be identified as an unstable defect.

When $\Psi$ corresponds to the lowest energy state, it should be a continuous map from $W$ to $M$, where $M$ the manifold of the minima of the potential energy $V$. In that case, $\Psi$ does not incorporate defects. When $\Psi$ does not correspond to the lowest energy, topological defects must be there. We are going to postulate that the topological defects within $\Psi$ exist as discontinuities in $\varphi$. In fact, a defect corresponds to an appropriate change in $\varphi$, applied only to those regions where it is discontinuous, having as a main aim to restore the continuity of that function throughout the original medium. Thus, the discontinuities will indicate where the introduction of a defect is needed. Associating, however, those configurations with a particular order parameter behavior is another matter which we will look into later.

It is convenient to call $M$ the manifold of the minima of the potential energy density $V$ and assume that $\varphi$, should be a function from the ordered medium $W$ to $M$. Ideally, $\varphi : W \rightarrow M$ ought to be a continuous function on $W$ and equal $\Psi$, describing the lowest energy state. However, the minimal energy state cannot always be achieved. We will suppose that this will be due to the existence of discontinuities in $\varphi$ and therefore assimilated topological defects in $\Psi^1$. Therefore, the ground state cannot be reached because $\varphi$ cannot be a continuous function to $M$.

Our contemplation to consider order parameters with discontinuities might seem quite far fetched to someone having in mind the situation in physical systems where the order parameter is always continuous. We shall see, though, that thinking about discontinuities is quite interesting and leads to intriguing results. First, we are going to investigate whether there is a pattern in the occurrence of discontinuities for some order parameter $\varphi$, physical space $W$ and minimal energy manifold $M$.

2 Considering discontinuities

In our discussion here we will focus on stable defects for these are the configurations that can be uniquely associated with certain topological characteristics of specific manifolds. Unstable defects, on the other hand, are heavily dependent on the definition of $\varphi$ and they are not necessarily derived from the topological

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1 In case $\Psi : W \rightarrow M$ incorporates defects then $M$ cannot be the manifold of the minima of $V$. However, it should be possible to define a function $\varphi : W \rightarrow M$ with $M$ the manifold of the minima of $V$, which will be discontinuous at the defects.
properties of the manifolds \( W \) or \( M \). We shall postpone a discussion about them for later.

Consider the manifold of the minima to be \( M = \{ -1, 1 \} \), or any discrete set, and \( W \) to be any simply connected manifold of dimension \( D \geq 1 \). A function \( \varphi : W \to M \) will necessarily be discontinuous on \( W \) when any two points \( a \) and \( b \) on \( W \) have \( \varphi(a) \neq \varphi(b) \). The stable defect will happen if we try to associate two points on \( W \) that can drop to one another on \( W \) with two points of \( M \) that cannot drop to one another on \( M \). The arising stable defects are identified as domain walls.

Take \( M = S^1 \) and \( W \) either \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \). It is clear that any closed contour on \( W \) can shrink continuously to a point. This is in contrast to what happens on \( M \) where no closed contour\(^2\) can contract to a point while remaining a closed contour. The stable defect will happen if we try to associate a loop and a point on \( W \) that can turn into one another on \( W \) with a loop and a point on \( M \) that cannot share that property on \( M \). Those kind of stable defects are called cosmic strings.

Consider \( M = S^2 \) and \( W \) to be \( \mathbb{R}^w \) with \( w > 2 \). As before, one links a 2-sphere and a point in \( W \) that can transform into one another on \( W \), with a 2-sphere and a point on \( M \) that cannot do that. The associated stable topological defects are called magnetic monopoles.

In general, consider two configurations on \( W \) that can continuously drop to one another. Those configurations are homotopic on \( W \). Define a function \( \varphi \) on them so that their corresponding configurations on \( M \) will not share their homotopy property. Inevitably, \( \varphi \) will incorporate at least one stable defect. Going a step ahead, the postulation we are advocating for here is that the stable defects are a consequence of the forced association of configurations that belong to different homotopy classes of a certain homotopy group. That association happens because of the difference in the topological properties of \( W \) and \( M \) and of the way \( \varphi \) is defined on \( W \).

### 3 Homotopy Groups

Our postulation here is that stable topological defects happen when one tries to link two different values of a quantized quantity. In particular, we argue that the discontinuity arises because we have to link two maps that belong to different homotopy classes of \( \pi_k(M) \) for some \( k \). The quantized quantity therefore corresponds to the Hopf invariant or winding number that turns out only to depend on the homotopy class a map to \( M \) belongs to.

We can start by recalling some basic notions of the theory of homotopy groups of spheres\(^[1] \). There are often lots of topologically different ways of wrapping an \( k \)-dimensional sphere around a \( m \)-dimensional sphere. The group of all homotopy classes of ways of wrapping an \( k \)-sphere around an \( m \)-sphere is the \( \pi_k(S^m) \). Only the maps which belong to the same homotopy class can

\(^2\)Of the form \( g : I \equiv [0, 1] \subset \mathbb{R} \to M \) such that \( g(0) = g(1) \) and \( g(t_1) \neq g(t_2) \) for \( t_1 \neq t_2 \) for each \( t_1, t_2 \epsilon (0, 1) \)
be deformed into one another continuously. The different homotopy classes are classified by an integer, the “winding number” which indicates how many times the $k$-sphere has been wrapped around the $m$-sphere.

The task of calculating the $\pi_k(S^m)$ for all $k$ and $m$ is not at all simple. Regardless of the actual form of $\pi_k(S^m)$, the reassurance that it is non trivial can be enough to conclude that topological defects are possible. Since we need to associate two maps that belong to different homotopy classes of $\pi_k(M)$ in order to create a topological defect, it is clear that $\pi_k(M)$ must accommodate different homotopy classes and thus be non trivial. There are some results we need to keep in mind. The homotopy group is trivial, thus different homotopy classes and thus be non trivial. There are some results we need to keep in mind. The homotopy group is trivial, thus $\pi_k(S^m) = \{0\}$, when $k < m$. Further, $\pi_k(S^k) = \mathbb{Z}$ for all $k \geq 1$ and $\pi_k(S^1) = \{0\}$ for all $k > 1$.

For example, consider maps from the unit circle in the complex plane to itself. Two functions $g_0(\theta) = 1$ and $g_1(\theta) = e^{i2\pi \cos^2 \frac{\theta}{2}}$ with $\theta$ the polar angle, are homotopic since one can define a homotopy $G$ as $G(t, \theta) = e^{i2\pi \cos^2 \frac{\theta}{2}}$ so that $g_0$ becomes $g_1$ and vice versa. That is $G(0, \theta) = g_0$ and $G(1, \theta) = g_1$ since $t \in I \equiv [0, 1]$. In fact, any continuous function from the (complex unitary) circle to itself can be continuously deformed to exactly one of the functions $f(z) = z^n$ with $n$ the winding number, an integer, and $\|z\| = 1$. We know the homotopy class of a map from a circle to itself if we know its winding number.

Let $f : S^k \to \mathbb{R}^{k+1}$ be the inclusion of the $k$-sphere. Call

$$F : I \otimes S^k \to \mathbb{R}^{k+1}$$

a homotopy that drops $f$ to a point of $\mathbb{R}^{k+1}$. Due to convexity of $\mathbb{R}^{k+1}$, we can write $F$ as

$$F(t, z) = F_t(z) = (1 - t)f(z) + tc$$

with $c$ a point on $\mathbb{R}^{k+1}$ and $z$ a point on the $k$-sphere.

Suppose $h : S^k \to S^m$ a map from a $k$-sphere to an $m$-sphere. That map should belong to a non trivial class of $\pi_k(S^m) \neq \{0\}$ where $k \geq m$. Let

$$H : I \otimes S^k \to S^m$$

be a homotopy written as $H(q, z) = H_q(z)$ such that $H(0, z) = h(z)$ and $H(1, z) = h_f(z)$ with $h_f$ another function of the homotopy class $h$ belongs to and $z$ a point on the $k$-sphere.

**QUESTION:** Is there a way to make $q$ a function of $t$ so as to allow

$$G : \text{Im} F_t \to \text{Im} H_q$$

(4)

to be a continuous well defined onto function for every $t$?

The reason for requiring $G$ to be surjective is that we would not like to give $G$ the freedom of choosing its range. This can restrict the way $G$ is defined over its domain. For each $0 \leq t < 1$, $\text{Im} F_t$ corresponds to a $k$-sphere in $\mathbb{R}^{k+1}$ and we can define $G$ as

$$G \equiv H \circ F : I \otimes I \otimes S^k \to S^m$$

(5)

For the considered $t$ interval, $q$ can be any smooth function of $t$ and the construction in $\otimes$ is well defined. However, when $t = 1$, $F_1$ changes to dimension
zero and $H \circ F_1$ is always a point on the $S^m$. This means that $G$ cannot be onto $\text{Im}H_q(1)$ with the latter being the area spanned by $H(q(1), z)$ for every $z$ on a $k$-sphere. If we assume $q(1) = 1$, we can say that there is no $H(1, z) = h_f(z)$ that will permit $G$ to be well defined on $F_1$.

On the other hand, if $H$ is a homotopy between functions of the trivial homotopy class of $\pi_k(S^m)$, we can require $H(1, z) = c_0$, a constant, and $h_i(z) = H(0, z)$ some null homotopic function. We can define $G$ as in (4) and rest assured that so long as $q(t)$ satisfies the equation $q(1) = 1$, we can always find an $h_i(z)$ so that $G$ is well defined.

Stable topological defects will occur when $G$ cannot be well defined for a particular combination of $F$ and $H$ homotopies.

4 The occurrence of stable topological defects

Consider $\varphi$ the order parameter defined on $W$. The $\text{Im}F_1$ with $F$ as defined in (1) must be homeomorphic to a subset of $W$ for each $t \in I$. We can suppose $W$ to be a convex subset of $\mathbb{R}^w$ with $w \geq k + 1$. Assume we know $\varphi_0 = \varphi(F_0(z))$ and $\varphi_1 = \varphi(F_1(z))$ with $F_1(z)$ a point on $W$ and $\text{Im}F_0$ homeomorphic to an $S^k$. Let $\varphi \circ F_0 : S^k \to M$ belong to some homotopy class $[\alpha]$ of $\pi_k(M) \neq \{0\}$. The $\varphi \circ F_1 : S^k \to M$ will necessarily belong to the trivial class of the $k$-homotopy group of $M$. The latter is considered some $m$-sphere. We must have $1 \leq m \leq k \leq w - 1$.

Call $H : I \otimes S^k \to M = S^m$ a homotopy between $\varphi : S^k \subset W \to M$ and some $\varphi_f : S^k \subset W \to M$. The stable topological defect will occur when we cannot find a continuous function $H(q(t), F(t, z))$ so that $H(q(0), F(0, z)) = \varphi_0 = \varphi(F(0, z))$ and $H(q(1), F(1, z)) = \varphi_1 = \varphi(F(1, z))$ with $q(t)$ a continuous function of $t \in I$ with $q(1) = 1$ and $q(0) = 0$. In the absence of defects, $H(q(t), F(t, z))$ may be considered to be the order parameter.

To be sure, the discontinuity in $G$, when stable topological defects exist, will be evident as $t$ changes. That is, if we do not keep track of the changes in $F$ and $G$ as $t$ changes then we will not be able to detect the singularity. Consider, for example, $\varphi : W = S^1 \to M = S^1$ as

$$\varphi(\theta) = e^{i\theta \exp(-\sin^2(\frac{\theta}{2}))}$$

with $\theta$ the polar angle on $W$ and $\theta \in [0, 2\pi]$. Call $\vartheta = \theta \exp(-\sin^2(\frac{\theta}{2}))$. One could introduce another variable, say $t$, which would alter the definition of $\varphi$ on $W$ as

$$e^{t(\vartheta_0 \vartheta(t - t_1) \vartheta(t - t_2) \vartheta(t - t_3) \cdots)}$$

(6)

with increasing $t \in \mathbb{R}^+$ (e.g $t_2 = t_1 + \Delta t_2$, with $\Delta t_2$ a certain step and $t_1 > 0$). The function $\Theta$ corresponds to the Theta function which, here, is defined as $\Theta(x \geq 0) = 1$ and $\Theta(x < 0) = 0$. We assume that $\vartheta_0 = 1$ and $\vartheta$ is always finite. We see that the function in (6), as $t \to \infty$, tends to behave in a similar fashion as the $\Theta(\vartheta - 2\pi)$ with $\theta \in [0, 2\pi]$. At every individual $t$, the function on $W$ would be given by (6) and it would be continuous on $W$. However, if we were to
draw the changes the order parameter sustained as \( t \) changed we would discover discontinuities. The discontinuities would be visible on the space \( \mathbb{R}^+ \otimes S^1 \) and not the \( S^1 \) alone.

5 Conclusions

The usual premise for the feasibility of stable topological defects is the \( \pi_k(M) \neq \{0\} \). If \( \pi_k(W) \neq \{0\} \) then it is not necessary that \( \pi_k(M) \neq \{0\} \) can provide reliable information on the occurrence of topological defects. Consider, for example, \( M = S^m = W = S^w \). We must have \( k \leq w \) if we want to find an inclusion map of a \( k \)-sphere to \( W \). Further, \( k \geq m \geq 1 \) since \( \pi_k(M) \neq \{0\} \). Therefore, \( k = w \).

In that situation \( \pi_k(S^w) = \pi_k(S^k) = \mathbb{Z} \). The homotopy \( F(t, z) \) may relate maps of a non trivial homotopy class of \( \pi_k(W) \). If that happens then \( \varphi \circ F(0, z) \) and \( \varphi \circ F(1, z) \) have no reason to belong to different homotopy classes of \( \pi_k(M) \). In fact, stable topological defects cannot happen in that situation.

Consider \( \Phi_0 = \varphi \circ F(0, z) \) a function to a non trivial homotopy class of \( \pi_k(M) \). The \( \varphi \circ F(1, z) \) will necessarily belong to a trivial class, because \( F(1, z) \) is a single point on \( W \). That is, we can write \( \varphi \circ F(1, z) = \tilde{\varphi} \circ F(0, z) \) with \( \tilde{\varphi} \) a constant function. We should not be able to find a homotopy between \( \Phi_0 \) and \( \Phi_1 \). That is the cause of the stable defect. In other words, the stable defect arises not only because \( \Phi_0 \) belongs to a non trivial class but also because \( F(1, z) \) becomes such a configuration on \( W \) that \( \varphi \), when defined on \( F(1, z) \) will have to create a trivial map \( \Phi_1 \).

Usually, one assumes that \( M \) is the boundary of \( W = D^w \), a disk or otherwise a convex subset of \( \mathbb{R}^w \). Thus, \( W = D^w \). Therefore, one fixes \( M \) to some \( S^{w-1} \) and therefore the only way that \( \pi_k(S^{w-1}) \) is non trivial and the \( k \)-sphere could contract to point in \( W \), is if \( k = w - 1 \). To be sure, \( \pi_k(S^m) \) can be non trivial for various values of \( k \). The ordered medium as much as the dimension \( m \), will dictate the values of \( k \) that can be relevant. The general framework we presented here aims at providing the means to creating various situations where defects should arise.

This can be achieved by considering, for instance, different forms for the \( F \) homotopy in (1). We can also examine what may happen for different \( W \).s. For example, assume that \( W \) is a 2-sphere and \( m = 1 \). Therefore, \( 1 \leq k \leq 1 \Rightarrow k = 1 \). We see that there are two degenerate functions \( F \) as in (1) that can realize the homotopy to a point on \( W \). There are, thus, two points of discontinuity and two stable topological defects arising as a result. The study of such situations will be reported elsewhere.

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