Efficient Peer Effects Estimators with Random Group Effects

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Abstract

We study linear peer effects models where peers interact in groups, individual’s outcomes are linear in the group mean outcome and characteristics, and group effects are random. Our specification is motivated by the moment conditions imposed in Graham (2008). We show that these moment conditions can be cast in terms of a linear random group effects model and lead to a class of GMM estimators that are generally identified as long as there is sufficient variation in group size. We also show that our class of GMM estimators contains a Quasi Maximum Likelihood estimator (QMLE) for the random group effects model, as well as the Wald estimator of Graham (2008) and the within estimator of Lee (2007) as special cases. Our identification results extend insights in Graham (2008) that show how assumptions about random group effects as well as variation in group size can be used to overcome the reflection problem in identifying peer effects. Our QMLE and GMM estimators can easily be augmented with additional covariates and are valid in situations with a large but finite number of different group sizes. Because our estimators are general moment based procedures, using instruments other than binary group indicators in estimation is straightforward. Monte-Carlo simulations show that the bias of the QMLE estimator decreases with the number of groups and the variation in group size, and increases with group size. We also prove the consistency and asymptotic normality of the estimator under reasonable assumptions.

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1 Introduction

Peer effects and unobserved group heterogeneity are both of great interest to empirical researchers. The idea that individuals are affected by their peers motivates policies that try to manipulate peer compositions for better outcomes. Unobserved group level heterogeneity, usually measured by between-class variance, is a popular measure of teacher efficiency in education studies. However, it is difficult to distinguish these two effects in practice if they coexist. Firstly, identifying peer effects is notoriously challenging due to the reflection problem (Manski, 1993; Angrist, 2014), arising from the simultaneous equation nature of peer effects models. Moreover, unobserved group heterogeneity generates spurious peer effects, which further complicates identification. In this paper we consider a set of specific moment restrictions that were proposed by Graham (2008) in the context of peer effects identification. These restrictions essentially amount to a random group effects specification combined with assumptions of independently distributed unobserved individual specific characteristics. We show that estimators exploiting these moment restrictions identify endogenous peer effects under some conditions.

Incorporating random group effects in peer effects models is of empirical significance. Firstly, more and more peer effect studies use data where people are randomly assigned into groups (Sacerdote 2001; Guryan et al. 2009; Carrell et al. 2009, 2013, etc.). Assuming group effects to be independent of observed individual and group characteristics is plausible in this setting. Ignoring group effects or assuming fixed groups effects (Lee 2007) leads to consistent but less efficient estimators. Second, random group effects themselves have important empirical implications. For example, researchers in education policy often treat random class effects as unobserved teacher effects (e.g., Nye et al. 2004; Rivkin et al. 2005; Chetty et al. 2011). Combining these teacher effects with endogenous peer effects is empirically important for the proper identification of teacher effects. For example, directly measuring group effects with between group variances can be misleading because peer effects also affect the between group variance. The moment based methods we consider in this paper account for the combined group level and peer effects and allow to isolate them consistently.

We use the moment restrictions proposed by Graham (2008) as the starting point for our analysis. Random effects assumptions for the group level effects combined with assumptions of cross-sectional independence for idiosyncratic errors are the basis for a set of moment conditions we impose. We give an interpretation of the conditional variance estimator (CVE) of Graham (2008) in terms of a GMM estimator based on moment conditions for the within-group variance and between-group variance. We also show that the moment conditions underlying Graham (2008) are the score function of a quasi maximum likelihood estimator (QMLE) for a random group effects model. The QMLE can be shown to be the best GMM estimator in the class of estimators using the moment conditions utilized by Graham (2008). One limitation of the conditional variance estimator proposed by Graham is the fact that it amounts to a difference in difference identification strategy for the variances that requires groups fall into two size categories. As shown in Graham (2008) the resulting procedure takes on the form of a Wald estimator for a set of binary instruments. This
setting is restrictive in applications where groups may not be easily separated into two categories or where a more general set of instruments needs to be considered. The estimators that we propose are general GMM based procedures that can easily be augmented with additional covariates as well as offer flexibility in terms of the instruments and the number of moment conditions that are being used. We illustrate these points by explicitly considering moment based estimators that exploit exogenous variation in group size, as opposed to a binary group size indicator, as instruments. This leads us to study a general linear random group effects model estimated using QML.

Manski (1993) highlights the difficulty of disentangling endogenous peer effects from exogenous effects. This problem is known as the reflection problem, in recognition of the simultaneous determination of individual outcomes with outcomes of their peers. It is well known that identification of linear peer effects models using linear instrumental variables strategies is often not possible. Two important contributions to overcome the reflection problem are the conditional variance method by [Graham (2008)] and the conditional maximum likelihood (CMLE) method developed by [Lee (2007)]. We analyze both methods in the context of a set of moment conditions for the within and between sample variances of outcomes. [Graham (2008)] achieves identification by combining both moments while [Lee (2007)] only uses the within variance. We show that the identification result in [Lee (2007)] hinges on a model specification in terms of 'leave-out' means rather than full group mean peer effects. We also show that the conditional variance method can be adapted to the 'leave-out' specification under certain conditions. Our analysis shows that for the 'leave-out' specification variation in group size is sufficient, although not necessary in all scenarios, for identification. We also cover cases where identification is possible without variation in group size but instead is based on heteroskedastic idiosyncratic errors alone. We extend Graham’s Wald estimator to the specification with 'leave-out' means and show under what conditions it can be used to identify the endogenous peer effects parameter.

The QMLE we develop in this paper uses moment conditions for the within and between variances of outcomes individually, rather than combining them into a single moment condition as is the case for the CV estimator. This leads to a more flexible procedure that is able to identify the endogenous peer effects parameter in a broader class of settings. Our setup also facilitates the inclusion of additional covariates in a unified joint estimation framework which is important for statistical inference. In contrast to the CMLE of [Lee (2007)] our procedure is based on both the within and between variance. This leads to efficiency gains under correct specification but comes at the cost of potential misspecification bias if the random group effect assumption is incorrect. The trade-offs are similar to related results for fixed and random effects in the panel literature.

Our work is also related to the literature in spatial econometrics started by the work of Cliff and Ord (1973), Cliff and Ord (1981), and Anselin (1988).

Recently, there is a growing literature using spatial methods to model social network effects, e.g., Lee (2007), Bramoulle et al. (2009), and Kuersteiner and Prucha (2020). The strength of social links can be characterized by proximity

\footnote{Anselin (2010) offers a brief review of the development of spatial econometrics literature over the past thirty years.}
in the social network space. We extend Kelejian et al. (2006) and Lee (2007) by considering a random group effects specification. Spatial models were traditionally estimated with maximum likelihood (ML), e.g., Ord (1975). Kelejian and Prucha (1998, 1999) develop generalized method of moments (GMM) estimator based on linear and quadratic moments. While this paper utilizes a quasi-maximum likelihood estimation method, the score function depends on linear quadratic forms of the error terms. Properties of quadratic moment conditions were introduced by Kelejian and Prucha (1998, 1999) in the cross section case, and Kapoor et al. (2007) and Kuersteiner and Prucha (2020) in a panel setting. Moreover, Kelejian and Prucha (2001) and Kelejian and Prucha (2010) develop a central limit theorem for linear quadratic forms, which is the basis for the asymptotic analysis in this paper.

The linear-in-means peer effect model in Manski (1993) is a special case of a spatial model with group-wise equal dependence, see Kelejian and Prucha (2002) and Kelejian et al. (2006). Kelejian and Prucha (2002) were the first to study the group-wise equal dependence spatial model. They show that if there is one group in a single cross section and the model has equal spatial weights, two-stage least squares (2SLS), GMM and QMLE methods all yield inconsistent estimators, although consistent estimation with 2SLS and GMM is possible for panel data. However, Kelejian et al. (2006) point out that if group fixed effects are incorporated and the panel is balanced, the estimators are inconsistent. The results in Kelejian et al. (2006) show the importance of variation in group size in identification of spatial models with blocks of equal weights. The QMLE developed in this paper and the conditional maximum likelihood estimator in Lee (2007) both rely on group size variation for identification although we show that identification exploiting heteroskedastic errors is also possible. Extensions include Lee et al. (2010) who allow for specific social structure within each group and Liu and Lee (2010) and Liu et al. (2014) who allow for non-row normalized weight matrices. The linear spatial model has also been applied to the empirical evaluation of peer effects by Liu (2010) and Boucher et al. (2014). Bramoullé et al. (2009) study a broader range of social interaction models and give conditions for identification.

The paper is organized as follows. In Section 2 we consider identification of endogenous peer effects in a simple setting without covariates for the CV, CML and QML estimators. We consider specifications for the full mean and 'leave-out' mean of the peer effect. Section 3 presents the full model that allows for covariates and general variation in group size. Section 4 summarizes the technical conditions we impose and presents theoretical results for the QMLE. Section 5 contains a small Monte Carlo experiment. Proofs are collected in an appendix.

2 Peer Effects with Random Group Effects

We start the discussion by presenting the simple peer effects model of Graham (2008). The model decomposes variation in outcomes of a cross-section of individuals into idiosyncratic noise, group level random effects and correlation that is due to group level interaction. The moment conditions that underlie the conditional variance (CV) estimator of Graham (2008) can be interpreted in terms
of a random effects specification that leads to efficient GMM, quasi maximum likelihood, and under additional distributional assumptions, maximum likelihood estimators. We interpret the estimator of [Graham (2008)] as a special case of a class of moment based estimators that include the QMLE as well as the conditional likelihood estimator of [Lee (2007)].

Let \( y_{ir} \) be an observed outcome of individual \( i \) in group \( r \) which has \( m_r \) members, let \( v_r \) be an unobserved group level effect and let \( \epsilon_{ir} \) be unobserved individual specific characteristics. We observe data for \( R \) groups as well as an indicator \( D_r \) for small groups which is \( D_r = 1 \) whenever \( m_r \leq \bar{m} \) for some constant \( \bar{m} \) and \( D_r = 0 \) otherwise. More generally, we allow \( D_r \) to be a categorical variable. An example is when there are three class sizes such that \( D_r \in \{ ‘small’, ‘medium’, ‘large’ \} \). However, \( D_r \) could be a characteristic that is not necessarily related to class size. An example is when classrooms are in schools in urban or rural districts and \( D_r \) is used to denote these characteristics. The model does not have covariates. One interpretation, which lends itself towards the inclusion of covariates, is that measured outcomes \( y_{ir} \) are residuals from a regression model of actual outcomes on a set of individual specific covariates. We make this interpretation precise in subsequent sections and for now work under the assumption of no included covariates.

The peer effects model of [Graham (2008)] can be written as

\[
y_{ir} = v_r + \epsilon_{ir} + (\gamma - 1) \bar{\epsilon}_r, \tag{1}
\]

where \( \bar{\epsilon}_r = m_r^{-1} \sum_{i=1}^{m_r} \epsilon_{ir} \) is the group average of unobserved characteristics. The parameter \( \gamma \) captures the peer effect, see [Manski (1993)]. Letting \( Y_r = (y_{1r}, \ldots, y_{m_r})' \), \( \epsilon_r = (\epsilon_{1r}, \ldots, \epsilon_{m_r})' \), \( \iota_{m_r} = (1, \ldots, 1)' \) and using the notation \( I^*_r = I_{m_r} - \iota_{m_r}\iota_{m_r}'/m_r \) and \( J^*_r = \iota_{m_r}\iota_{m_r}'/m_r \), the model can be represented in matrix notation as

\[
Y_r = v_r \iota_{m_r} + (I_{m_r} + (\gamma - 1) J^*_r) \epsilon_r.
\]

By pre-multiplying the above equation by \( (I_{m_r} + (\gamma - 1) J^*_r)^{-1} = I_{m_r} - (1 - 1/\gamma) J^*_r \), we can rewrite the model in what we call the structural form as

\[
Y_r = \left( 1 - \frac{1}{\gamma} \right) \bar{y}_r \iota_{m_r} + \frac{1}{\gamma} v_r \iota_{m_r} + \epsilon_r,
\]

where \( \bar{y}_r \) denotes the mean of \( y_{ir} \) in group \( r \). We later use the notation \( \lambda = 1 - 1/\gamma \) as well as \( \alpha_r = v_r/\gamma \), which permits the following scalar representation of the structural model:

\[
y_{ir} = \lambda \bar{y}_r + \alpha_r + \epsilon_{ir}.
\]

The structural form emphasizes the decomposition of \( y_{ir} \) into a social interaction term \( \lambda \bar{y}_r \), a group level effect \( \alpha_r \) and an idiosyncratic error term \( \epsilon_{ir} \). A literature on linear instrumental variables methods gives conditions under which \( \lambda \) can be identified in models that have additional exogenous covariates \( Z_r \). However, in the context of Model (1) the results in [Manski (1993)], [Kelejian et al. (2006)] or [Bramoulle et al. (2009)] show that \( \lambda \) cannot be identified by instrumenting \( \bar{y}_r \iota_{m_r} = J^*_r Y_r \)
with $J_{m_r}^2$, $Z_r$ when $J_{m_r}^*, Z_r$ is included as a covariate, observing that $J_{m_r}^2 = J_{m_r}^*$. While conventional instrumental variables strategies do not deliver the desired identification results alternative strategies are available, see Graham (2008) or Kuersteiner and Prucha (2020). To isolate or identify the social interaction effect Graham (2008) imposes restrictions on the unobservables. These assumptions can sometimes be motivated by specific empirical designs. For example, in the application Graham (2008) considers, kindergarten students and teachers are randomly assigned to classrooms. This random assignment mechanism justifies interpreting $\alpha_r$ as the classroom or teacher effect. It also justifies assuming that $\alpha_r$ and $\epsilon_{ir}$ are mutually independent random variables, see Graham (2008), Assumption 1.1.

Further restrictions on the conditional second moments of $\alpha_r$ and $\epsilon_{ir}$ and/or group sizes are required for identification. To describe these restrictions define the composite error term

$$U_r = \alpha_r t_m + \epsilon_r,$$

where $U_r$ is an $m_r \times 1$ vector with elements $u_{ir} = \alpha_r + \epsilon_{ir}$. Let $\bar{u}_r$ and $\bar{\epsilon}_r$ be the mean of $u_{ir}$ and $\epsilon_{ir}$ in group $r$. Let $\bar{U}_r = U_r - \bar{u}_r t_m$, be the vector of within-group deviation from the mean of $U_r$ and let $\bar{Y}_r$ and $\bar{\epsilon}_r$ be defined in a similar manner, where in particular $\bar{y}_r = \bar{u}_r/(1 - \lambda)$ with $\bar{u}_r = \alpha_r + \bar{\epsilon}_r$, and $\bar{Y}_r = \bar{U}_r = \bar{\epsilon}_r$. Graham (2008) implicitly assumes that the variance co-variance matrix of $\epsilon_r$ is of the form $E[\epsilon_r \epsilon_r' | D_r = d] = \sigma_{\epsilon,d}^2 I_{m_r} + \sigma_{\epsilon,\epsilon_d}^2 (t_{m_r} t_{m_r} - I_{m_r})$ with $\sigma_{\epsilon,d}^2 = E[\epsilon_r^2 | D_r = d]$ and $\sigma_{\epsilon,\epsilon_d}^2 = E[\epsilon_{ir} \epsilon_{jr} | D_r = d]$ for $i \neq j$.

For the following discussion we maintain assumptions in line with Assumptions 1.1-1.3 of Graham (2008). Assumption 1.1 of Graham (2008) implies that $\sigma_{\epsilon,\epsilon_d}^2 = 0$, while Assumption 1.3 imposes that $E[\bar{Y}_r \bar{Y}_r' | D_r = 1] \neq E[\bar{Y}_r \bar{Y}_r' | D_r = 0]$. Observing that $E[\bar{Y}_r \bar{Y}_r' | D_r = d] = \sigma_{\epsilon,d}^2 (m_r - 1)$ the latter condition is seen to be satisfied if either $m_r$ varies or if the idiosyncratic errors $\epsilon_{ir}$ are heteroskedastic across small and large groups such that $\sigma_{\epsilon,0}^2 \neq \sigma_{\epsilon,1}^2$ in a way that is not proportional to the group size. Assumption 1.1 of Graham (2008) maintains further that the random group effects $\alpha_r$ are independent of the idiosyncratic disturbances, and Assumption 1.2 maintains that $\alpha_r$ have the same marginal distribution up to a non-random shift, and thus $Var[\alpha_r | D_r = d] = \sigma_{\alpha}^2$ for $d = 0, 1$. The parameter $\gamma^2$ is identified under Assumptions 1.1-1.3, see Proposition 1.1 in Graham (2008). If $D_r$ is categorical with $J + 1$ categories and $J \geq 0$ such that $D_r \in \{0, 1, ..., J\}$ these definitions readily extend to the case where $d$ takes values in $\{0, 1, ..., J\}$. Also note that the random variable $E[\epsilon_{ir}^2 | D_r] = \sigma_{\epsilon,d}^2 D_r$ has the representation $\sigma_{\epsilon,D_r}^2 = \sigma_{\epsilon,0}^2 \{D_r = 0\} + \ldots + \sigma_{\epsilon,J}^2 \{D_r = J\}$ where $\sigma_{\epsilon,d}^2$ are fixed parameters to be estimated. This formulation contains the case considered by Graham (2008) where $J = 1$ as a special case.

Graham’s conditional variance (CV) estimator exploits the moment structure imposed on the unobservables, and employs moment conditions corresponding to the within and between variance. Adopting the terminology of Graham (2008), let $G_r^W = \frac{1}{m_r} \frac{1}{m_r - 1} \bar{Y}_r \bar{Y}_r$ and $G_r^B = \bar{y}_r^2$ denote the within sum of squares and between squares respectively. Let the parameter vector be $\vartheta = (\lambda, \sigma_{\epsilon,0}^2, ..., \sigma_{\epsilon,J}^2, \sigma_{\alpha}^2)^T$ and, for clarity, let the true parameter vector be denoted by $\vartheta_0 = \ldots$
Consider the following two empirical moment functions,

\[
\tilde{\chi}_r^w(\vartheta) = \hat{Y}_r'\hat{Y}_r - (m_r - 1)\sigma_{\tilde{\chi}_r,D_r}, \tag{3}
\]

\[
\tilde{\chi}_r^b(\vartheta) = (1 - \lambda)^2\hat{y}_r^2 - \sigma_{\tilde{\chi}_r}^2 - \frac{\sigma_{\tilde{\chi}_r,D_r}^2}{m_r} \tag{4}
\]

where \(\sigma_{\tilde{\chi}_r,D_r} = \sigma_{\tilde{\chi}_{01}}^2 \{D_r = 0\} + ... + \sigma_{\tilde{\chi}_{rs}}^2 \{D_r = J\}\). Note that by design \(E[\tilde{\chi}_r^w(\vartheta_0)|m_r, D_r] = 0\) and \(E[\tilde{\chi}_r^b(\vartheta_0)|m_r, D_r] = 0\) at the true parameter vector \(\vartheta_0\). Graham uses what amounts to a difference in difference strategy in conditional variances. To illustrate his method, focus on the case where \(J = 1\). To eliminate the parameters \(\sigma_{\tilde{\chi}_{01}}^2\) and \(\sigma_{\tilde{\chi}_{11}}^2\) he combines \(\tilde{\chi}_r^w(\vartheta)\) and \(\tilde{\chi}_r^b(\vartheta)\) into one moment function \(\nu_r(\lambda, \sigma_{\tilde{\chi}}^2)\) defined as

\[
\nu_r(\lambda, \sigma_{\tilde{\chi}}^2) = \frac{1}{(1 - \lambda)^2}\left[(\tilde{\chi}_r^w(\vartheta) - \frac{\tilde{\chi}_r^w(\vartheta)}{m_r - 1}) = \hat{y}_r^2 - \frac{\sigma_{\tilde{\chi}}^2}{(1 - \lambda)^2} - \frac{1}{m_r(m_r - 1)}\right], \tag{5}
\]

with \(E[\nu_r(\lambda, \sigma_{\tilde{\chi}}^2)|m_r, D_r = 1] = 0\) at the true parameter values. The restriction that group effect variances are homoskedastic can be exploited by taking differences \(E[\nu_r(\lambda, \sigma_{\tilde{\chi}}^2)|m_r, D_r = 1] - E[\nu_r(\lambda, \sigma_{\tilde{\chi}}^2)|m_r, D_r = 0]\) to eliminate \(\sigma_{\tilde{\chi}}^2\). This implies the following population equation for \((1 - \lambda)^2\):

\[
\frac{1}{(1 - \lambda)^2} = \frac{E[\hat{y}_r^2|m_r, D_r = 1] - E[\hat{y}_r^2|m_r, D_r = 0]}{E\left[\frac{Y_r'Y_r}{m_r(m_r - 1)}|m_r, D_r = 1\right] - E\left[\frac{Y_r'Y_r}{m_r(m_r - 1)}|m_r, D_r = 0\right]}. \tag{6}
\]

The Wald estimate for \(\lambda\) can then be calculated from the sample analog of the right-hand side above but is not unique. This means that while \(\gamma^2 = 1/(1 - \lambda)^2\) is identified, the sign of \(\gamma = 1/(1 - \lambda)\) is not identified, unless \(\lambda\) is constrained to take values in \((-1, 1)\). Two points are worth noting. First, identification is possible even when \(\sigma_{\tilde{\chi}_{11}}^2 > \sigma_{\tilde{\chi}_{00}}^2/m_s\) as long as there is variation in \(m_r\). Second, since by construction \(m_r < m_s\) for \(D_r = 1\) and \(D_r = 0\) it is possible that \(\sigma_{\tilde{\chi}_{11}}^2/m_r < \sigma_{\tilde{\chi}_{00}}^2/m_s\) for some pairs of \(r\) and \(s\). The implication is that the difference \(E\left[\frac{Y_r'Y_r}{m_r(m_r - 1)}|m_r, D_r = 1\right] - E\left[\frac{Y_r'Y_r}{m_r(m_r - 1)}|m_r, D_r = 0\right]\), when estimated over the whole sample, may be close to zero even if the condition \(E\left[\frac{Y_r'Y_r}{m_r(m_r - 1)}|m_r, D_r = 1\right] \neq E\left[\frac{Y_r'Y_r}{m_r(m_r - 1)}|m_r, D_r = 0\right]\) holds for individual pairs. These problems do not arise for example, when there are only two class sizes, one for small and one for large classes.

Let \(\tilde{\chi}_r(\vartheta) = \left(\tilde{\chi}_r^w(\vartheta), \tilde{\chi}_r^b(\vartheta)\right)^t\) and consider the moment conditions \(E[\tilde{\chi}(\vartheta)|m_r, m_s] = 0\) for \(\tilde{\chi}(\vartheta) = \left(\tilde{\chi}_r(\vartheta)^t, \tilde{\chi}_s(\vartheta)^t\right)^t\) and \(m_r \neq m_s\). Inspection of (3) and (4) shows that these moment conditions have a unique solution for \((1 - \lambda)^2, \sigma_{\tilde{\chi}_{00}}^2, \sigma_{\tilde{\chi}_{11}}^2\) and \(\sigma_{\tilde{\chi}}^2\) even when \(\sigma_{\tilde{\chi}_{00}}^2 = \sigma_{\tilde{\chi}_{11}}^2\), as long as \(m_r \neq m_s\). Now consider the case where \(m_r = m_s\), but \(D_r\) is a group indicator of some feature other than group size and \(D_r \neq D_s\) for some \(r\) and \(s\). An example is an indicator for whether a school is in a poor or in an affluent neighborhood. The moment conditions \(E[\tilde{\chi}(\vartheta)|m_r, m_s] = 0\) again have a unique solution for \((1 - \lambda)^2, \sigma_{\tilde{\chi}_{00}}^2, \sigma_{\tilde{\chi}_{11}}^2\) and \(\sigma_{\tilde{\chi}}^2\) as long as \(\sigma_{\tilde{\chi}_{00}}^2 \neq \sigma_{\tilde{\chi}_{11}}^2\). It can be shown that the score of a Gaussian likelihood estimator is a function of \(\tilde{\chi}(\vartheta)\) and therefore that the Gaussian maximum
likelihood estimator shares the same identification properties.

Figure 1 illustrates how $\lambda$ is identified through the relationship between within-group variance and between-group variance, the key idea underlying Graham’s CV method. It depicts the correlation between the variance of $\bar{\epsilon}_r$ and excess variance where excess variance is defined by Graham (2008) as "residual between-group variance...that remains after accounting for the contribution of individual heterogeneity", i.e. $\text{var}(\bar{y}_r) - \text{var}(\bar{\epsilon}_r)$. Note that $\text{var}(\bar{\epsilon}_r) = \sigma^2_{\epsilon,D_r}/m_r$ can be obtained from the within-group variance. The excess variance is

$$\text{var}(\bar{y}_r) - \text{var}(\bar{\epsilon}_r) = \frac{\sigma^2_{\alpha}}{(1-\lambda)^2} + \left[ \frac{1}{(1-\lambda)^2} - 1 \right] \frac{\sigma^2_{\epsilon,D_r}}{m_r}.$$  

While $\lambda$ cannot be separated from $\sigma^2_{\alpha}/(1 - \lambda)^2$, the term involving $\sigma^2_{\alpha}$ is eliminated with the difference in difference specification of the CV estimator. We thus focus on the term involving $\sigma^2_{\epsilon,D_r}$ which is the remaining component of the procedure. We can identify $|1 - \lambda|$, or $\lambda$ as long as the parameter space is restricted to $(-1, 1)$, from $\frac{1}{(1-\lambda)^2} - 1$, the slope of the curve of the excess variance against $\text{var}(\bar{\epsilon}_r)$ in Figure 1. When $\lambda = 0$, the excess variance is $\sigma^2_{\alpha}$ and is constant across $m_r$. This creates a horizontal line for $\lambda = 0$ in the graph. A more upward slopped curve indicates a larger $\lambda$.

It is well known that identification for the case of peer effects captured by full group means is difficult, see Manski (1993), Bramoullé et al. (2009) or Angrist (2014). In the case of the conditional variance restrictions or likelihood approaches considered here, this manifests itself in the fact that $(1 - \lambda)^2$ but not $\lambda$ is identified without additional constraints on the parameter space. We now slightly modify the setup to a situation where the model is expressed in terms of “leave-out” means rather than full group means. Such specifications are often preferred in the literature, see for example Angrist (2014) for a discussion. Identification of $\lambda$ based on group size variation for this specification appears for example in Lee (2007) who considers within estimators for a group level fixed effects specification. Here we consider a random effects specification and relate it to a modified version of Graham’s CV method as well as Lee’s within estimator. We give an identification result for the simple scenario with two and three group sizes and defer a general treatment to later sections. We also show that the identification result of Graham (2008) continues to hold in a modified form and in fact can be strengthened under certain conditions.

The peer effects model is stated directly in terms of the structural equation

$$y_{ir} = \lambda \bar{y}_{(-i)r} + \alpha_r + \epsilon_{ir}, \quad (7)$$

where $\bar{y}_{(-i)r} = \frac{1}{m_r - 1} \sum_{j \neq i}^m y_{jr}$. This specification is preferred since it parametrizes the endogenous peer effect independently of group size, a feature that would not hold if we replaced $\bar{\epsilon}_r$ with $\bar{\epsilon}_{(-i)r}$ in (1). Similarly, it can be shown that the reduced form corresponding to (7) has group size dependent parameters. Define the operator $W_{m_r} = \frac{1}{m_r - 1} (I_{m_r} - I_{m_r})$ such that the model can be written in matrix notation as

$$Y_r = \lambda W_{m_r} Y_r + \alpha_r t_{m_r} + \epsilon_r. \quad (8)$$
We continue to maintain the assumptions regarding the group level effect $\alpha_r$ and the idiosyncratic disturbances $\epsilon_r$ discussed above, which are in line with Assumptions 1.1-1.3 of Graham (2008). Using the earlier notation $\hat{U}_r = \alpha_r m_r + \epsilon_r$ it can be shown that $\hat{y}_r = \frac{\bar{y}_r}{1-\lambda} = \frac{\alpha_r + \epsilon_r}{1-\lambda}$ and $\bar{Y}_r = \frac{m_r-1}{m_r-1+\lambda} \hat{U}_r = \frac{m_r-1}{m_r-1+\lambda} \epsilon_r$.

In the Gaussian case the between and within variance are sufficient statistics. Correspondingly two conditional moment conditions, one for the within-group variance, the other for the between group variance, arise for the model in (7) under Assumptions 1.1-1.3 in Graham (2008) but without the assumption of Gaussian distributions. The expected value of the within-group squares of group $r$ is

$$\text{var}_r^w = E \left[ \frac{\hat{y}_r^2}{m_r-1} | m_r, D_r \right] = E \left[ \frac{(m_r - 1)\hat{U}_r^2 \hat{Y}_r}{(m_r - 1 + \lambda)^2} | m_r, D_r \right] = \frac{(m_r - 1)^2}{(m_r - 1 + \lambda)^2} \sigma^2_{\epsilon, D_r}. \quad (9)$$

The expected value of the between groups squares of group $r$ is

$$\text{var}_r^b = E \left[ \bar{y}_r^2 | m_r, D_r \right] = E \left[ \bar{u}_r^2 | m_r, D_r \right] = \frac{1}{(1-\lambda)^2} \left( \sigma^2_{\alpha} + \sigma^2_{\epsilon, D_r} \right). \quad (10)$$

The Wald estimator of Graham (2008) does not have the same simple interpretation for the specification in (7) as it does when full sample means are used. Nevertheless, it may still be possible to identify the parameters of interest. A possible scenario is one where the indicator variable $D_r$ satisfies an overlap condition in the sense that there are groups $r$ and $s$ of the same size $m_r = m_s = m$ for which $D_r = 1$ and $D_s = 0$ with positive probability. This requirement is not satisfied if $D_r$ is a deterministic function of group size as in the leading case considered here and in Graham (2008). However, if an overlap type condition holds, it can be shown that

$$\frac{(m-1+\lambda)^2}{(1-\lambda)^2} = \frac{E \left[ \bar{y}_r^2 | m_r, D_r = 1 \right] - E \left[ \bar{y}_r^2 | m_s, D_s = 0 \right]}{E \left[ \bar{y}_r^2 | m_m(m-1)^2 | m_r, D_r = 1 \right] - E \left[ \bar{y}_r^2 | m_m(m-1)^2 | m_s, D_s = 0 \right]}, \quad (11)$$

leading to an expression that in principle can be solved for $\lambda$. In Lemma 2.2 below we outline the exact conditions under which this is possible.

The procedures we propose in this paper do not rely on the availability of a binary instrument but rather can be implemented for a general set of valid instruments and are valid for cases where $J \geq 0$ as long as $J$ is fixed and finite. Here we focus on the case where random variation in group sizes serves as an instrument. This may arise if assignment to groups is random in a way that generates random variation in group size. Using group size $m_r$ as instruments directly leads to a vector of two conditional moment restrictions $E[\chi_r(\vartheta) | m_r] = 0$, with $\chi_r(\vartheta) = (\chi_r^w(\vartheta), \chi_r^b(\vartheta))$, and where

$$\chi_r^w(\vartheta) = \frac{(m_r - 1 + \lambda)^2 \hat{Y}_r \hat{Y}_r}{(m_r - 1)^2} - (m_r - 1) \sigma^2_{\epsilon, D_r}, \quad (12)$$
\[ \chi_r^b(\vartheta) = (1 - \lambda)^2 y_r^2 - \sigma^2 - \frac{\sigma_{\epsilon,D_r}^2}{m_r}. \]  

(13)

Identification of the parameter \( \vartheta \) is possible as long as there is sufficient variation in group size. This is summarized in the following lemma.

**Lemma 2.1.** Let \( \Lambda \) be a compact subset of \((-1,1)\) and assume that \( \lambda_0 \in \Lambda \). Assume that \( D_r \in \{0,...,J\} \) where \( J \) is fixed and finite with \( J \geq 0 \). Further assume that \( \alpha_r \) and \( \epsilon_{ir} \) are independently distributed across all \( i \) and \( r \), with \( E[\epsilon_{ir}|m_r,D_r] = 0 \) and \( E[\epsilon_{ir}^2|m_r,D_r] = \sigma_{\epsilon_{0,D_r},r}^2 \), where \( 0 < \bar{\omega} < \sigma_{\epsilon_{0,D_r},r}^2 \leq \bar{\alpha} < \infty \), and where \( \sigma_{\epsilon_{0,D_r},r}^2 \) is a function only of \( D_r \). Further assume that \( E[\alpha_r|m_r,D_r] = 0 \) and \( E[\alpha_r^2|m_r,D_r] = \sigma_{\alpha_0}^2 \), where \( 0 \leq \sigma_{\alpha_0}^2 \leq \bar{\alpha} < \infty \) and where \( \sigma_{\alpha_0}^2 \) is a constant. Then, the parameter \( \vartheta \) is identified under the following two scenarios:

(i) If the errors \( \epsilon_{ir} \) are homoskedastic such that \( \sigma_{\epsilon_{0,D_r},r}^2 = \sigma_{\epsilon_0}^2 \) where \( \sigma_{\epsilon_0}^2 \) is a constant, this restriction is imposed on the parameter space such that \( \vartheta = (\lambda, \sigma_{\epsilon}^2, \sigma_{\alpha}^2)^\prime \) and there are at least two distinct group sizes, i.e. for some indices \( r \) and \( s \) it follows that \( m_r \neq m_s \), then the moment conditions \( E(\chi_r(\vartheta)|m_r,D_r) = 0 \) and \( E(\chi_s(\vartheta)|m_s,D_s) = 0 \), with \( \chi_r(\vartheta) \) and \( \chi_s(\vartheta) \) defined in (12) and (13) uniquely identify \( \vartheta \).

(ii) If the parameter \( \vartheta = (\lambda, \sigma_{\epsilon_{0,1},r}^2,...,\sigma_{\epsilon_{r,J},r}^2,\sigma_{\alpha}^2)^\prime \) is unconstrained, there are \( J+1 \) categories and at least 2 distinct group sizes, w.l.o.g. \( 1 < m_{q1} < m_{q2} \) for some indices \( q_1, q_2 \) such that \( D_{q1} = D_{q2} \). Further, let \( q_j \) for \( j = 1,...,J+2 \) be indices such that \( D_{q_j} \) for \( j = 2,...,J+2 \) all take different values. Then the moment conditions \( E[\chi_q(\vartheta)|m_{q_j},D_{q_j}] = 0 \), for \( j=1,...,J+2 \) with \( \chi_q(\vartheta) \) defined in (12) and (13) uniquely identify \( \vartheta \).

The proof of Lemma 2.1 in Appendix D shows that the identification result continues to hold when \( \sigma_{\epsilon_{0,D_r},r}^2 = \sigma_{\epsilon_0,D_s}^2 \) for all \( r \neq s \) as long as there is variation in group size. The result also holds irrespective of whether the constraint of homoskedastic errors is imposed on the model or not. Thus, variation in group size alone can provide variation that is sufficient for identification. Furthermore, in the homoskedastic case where only a common variance parameter \( \sigma_{\epsilon_r}^2 \) is specified, two distinct group sizes are sufficient for identification. The results in Section 4 focus on this case and show that under the assumption of homoskedastic innovations \( \epsilon_{ir} \), but with sufficient variation in group size the parameter \( \vartheta \) is identified. Case (ii) requires group size variation in at least one category, for example \( D_{q1} = D_{q2} = 0 \), while the remaining categories \( d = 1,...,J \) may have the same group sizes. If the number of distinct group sizes exceeds the number of categories \( J + 1 \) then it automatically must be the case that for two group sizes, there exist groups that belong to the same category.

While variation in group size is a natural feature of the application considered in Graham (2008), identification based on the moment condition \( E[\chi_r(\vartheta)|m_r] = 0 \) is also possible without group size variation as long as there is some other form of group heterogeneity. This is shown in the next lemma.

**Lemma 2.2.** Impose the same assumptions as in Lemma 2.1 except that \( 1 < m = m_r \) for some integer \( m \) and all \( r \) and \( J \geq 1 \). Also, assume that for some \( r \) and \( s \), \( D_r \neq D_s \) and \( \sigma_{\epsilon_{0,D_r},r}^2 \neq \sigma_{\epsilon_{0,D_s},s}^2 \).
Then, $\lambda$ is identified by (11). In addition, $E[\chi_r(\vartheta)|m_r, D_r] = 0$ and $E[\chi_s(\vartheta)|m_s, D_s] = 0$ with $\chi_r(\vartheta)$ and $\chi_s(\vartheta)$ defined in (12) and (13) uniquely identify $\vartheta$.

As is shown in the proof of Lemma 2.2 in Appendix D, endogenous peer effects in classes of the same size are identified by the estimator in (11) in this scenario as long as there is heteroskedasticity between groups in different categories.

In proving Lemma 2.1 we actually show that when $\epsilon_{ir}$ are homoskedastic, $\lambda$ can be identified from the moment function for the within-group variance $E(\chi_r^w(\vartheta)|m_r, D_r) = 0$ as long as there are two distinct group sizes. This is the idea of the conditional maximum likelihood estimator (CMLE) in Lee (2007), the score function of which can be written as $\varphi(m_r)\chi_r^w(\vartheta)$, where $\varphi(m_r)$ is a function of $m_r$. Figure 2 illustrates how moment conditions based on within-group variance identify $\lambda$ when we have at least two group sizes. The graph depicts the correlation between the logarithm of the within-group variance $ln(var_r^w)$ and group size $m_r$ for different values of $\lambda$. Note that the slope of the curve of $ln(var_r^w|m_r = m)$ is $\lambda/[(m - 1)(m - 1 + \lambda)]$, which is strictly increasing in $\lambda$. A larger $\lambda$ indicates a larger slope of $ln(var_r^w|m_r = m)$ at any $m$, or between any two points (sizes) on the curve. The slope between any two points thus uniquely identifies $\lambda$. To further illustrate this argument consider any two groups $r, s$ with size $m_r$ and $m_s$. The difference in their logarithm of the within-variance ratio is

$$ln \left( \frac{var_r^w}{var_s^w} \right) = 2ln \left[ \frac{(m_r - 1)/(m_r - 1 + \lambda_0)}{(m_s - 1)/(m_s - 1 + \lambda_0)} \right] = 2ln \left[ 1 + \frac{(m_r - m_s)\lambda_0}{(m_r - 1 + \lambda_0)(m_s - 1)} \right].$$

If we treat $ln \left( \frac{var_r^w}{var_s^w} \right)$ as a function of $\lambda$, for some $\hat{\lambda}$ to satisfy the equation $ln \left( \frac{var_r^w}{var_s^w} \right)(\hat{\lambda}) = ln \left( \frac{var_r^w}{var_s^w} \right)(\lambda_0)$, we need

$$\frac{(m_r - m_s)\hat{\lambda}}{(m_r - 1 + \hat{\lambda})(m_s - 1)} = \frac{(m_r - m_s)\lambda_0}{(m_r - 1 + \lambda_0)(m_s - 1)},$$

which gives

$$(\hat{\lambda} - \lambda_0)(m_r - m_s)(m_s - 1) = 0.$$ 

As long as $m_r \neq m_s$ the equation above can only hold if $\hat{\lambda} = \lambda_0$. As a result, we can identify $\lambda$ from the within-variance as long as there are two different sizes.

A second aspect of the moment conditions imposed on the unobservables besides identification is efficient inference. As a leading case, suppose that $\alpha$ and $\epsilon$ follow a Gaussian distribution, then the optimal moment function corresponding to $\chi_r(\vartheta_0)$ is given by $\chi_r^*(\vartheta) = \varphi^*(m_r, D_r)\chi_r(\vartheta)$ where, focusing on the case with $J = 1$ for exposition, \footnote{See our Online Appendix for details. The derivation uses Lemma C.1 and the special properties of matrices $\Omega(\vartheta)$, $I - \lambda W$ and $W$ described in Appendix C.1. In the Online Appendix we also give an explicit expression for the VC matrix of $\chi_r(\vartheta)$.}
\[ \varphi^* (m_r, D_r) = E \left[ \frac{\partial}{\partial \theta} \chi_r(\theta_0) | m_r, D_r \right] \left\{ E \left[ \chi_r(\theta_0) \chi_r(\theta_0)' | m_r, D_r \right] \right\}^{-1} \]

\[
= \begin{pmatrix}
\frac{1}{(m_r-1+\lambda)s_{0,0,1}^2} & -\left(1-\lambda\right)s_{0,0,0}^2 m_r & -\frac{m_r}{2(\sigma_{0,0,1}^2 + m_r\sigma_{0,0}^2)} \\
-\left(1-\lambda\right)s_{0,0,0}^2 m_r & \frac{1}{2^2(\sigma_{0,0,0}^2 + m_r\sigma_{0,0}^2)^2} & \frac{-2(m_r\sigma_{0,0}^2 + m_r\sigma_{0,0}^2)}{D_r m_r} \\
-\frac{m_r}{2(\sigma_{0,0,1}^2 + m_r\sigma_{0,0}^2)} & \frac{2(m_r\sigma_{0,0}^2 + m_r\sigma_{0,0}^2)}{D_r m_r} & \frac{-2(m_r\sigma_{0,0}^2 + m_r\sigma_{0,0}^2)}{2^2(\sigma_{0,0,0}^2 + m_r\sigma_{0,0}^2)^2}
\end{pmatrix}
\]

(14)

Clearly \( E[\chi_r^*(\theta_0)] = 0 \) by iterated expectations. In the following we show furthermore that \( \partial\ln L_r(\vartheta) / \partial \vartheta = -\chi_r^*(\vartheta) \) where \( \ln L_r(\vartheta) \) denotes the log-likelihood function corresponding to group \( r \). From these observations we see that the matrices \( \varphi^* (m_r, \delta_0) \) can be viewed to provide the optimal weighting for the basic moment functions \( \chi_r(\delta) \); compare also the corresponding discussion for the general model for more details.

To develop the quasi maximum likelihood estimator note that the moment condition

\[ E(U_r | m_1, ..., m_R, D_1, ..., D_R) = 0 \]

holds and the true variance-covariance (VC) matrix of \( U_r \) conditional on \( (m_1, ..., m_R, D_1, ..., D_R) \) is

\[ \Omega_r(\vartheta_0) = \sigma_{\epsilon 0, D_r}^2 I_{m_r} + \sigma_{\epsilon 0, m_r}^2 I_{m_r}' + (\sigma_{\epsilon 0, D_r}^2 + m_r\sigma_{\epsilon 0, m_r}^2)J_{m_r}^* \]

(15)

Let \( Y = (Y_1', ..., Y_R')' \) and \( U = (U_1', ..., U_R')' \). Given the independence of \( \alpha_r \) and \( \epsilon_r \) across groups, the variance-covariance matrix for the whole sample is

\[ \Omega(\vartheta_0) = var(U | m_1, ..., m_R, D_1, ..., D_R) = diag_{r=1}^R \{ \Omega_r(\vartheta_0) \} = diag_{r=1}^R \{ \sigma_{\epsilon 0, D_r}^2 I_{m_r} + (\sigma_{\epsilon 0, D_r}^2 + m_r\sigma_{\epsilon 0, m_r}^2)J_{m_r}^* \}. \]

(16)

Under regularity conditions we introduce formally in Section 4 \( \Omega(\vartheta) \) is always invertible with inverse

\[ \Omega(\vartheta)^{-1} = diag_{r=1}^R \{ \frac{1}{\sigma_{\epsilon, D_r}^2} I_{m_r}^* + \frac{1}{\sigma_{\epsilon, D_r}^2 + m_r\sigma_{\epsilon, m_r}^2} J_{m_r}^* \}. \]

(17)

Further define \( W = diag_{r=1}^R \{ W_m \} \). Under Gaussianity the log likelihood function conditional on \( (m_1, ..., m_R) \) then can be written as

\[ \ln L_N(\vartheta) = -\frac{N}{2} \ln(2\pi) + \ln |I - \lambda W| - \frac{1}{2} \ln |\Omega(\vartheta)| \]

\[ -\frac{1}{2} (Y - \lambda W Y)' \Omega(\vartheta)^{-1} (Y - \lambda W Y) \]

(18)

\[ \text{See Appendix C.1 for the properties of } pI_m + sJ_m^* \text{ type of matrices}. \]
By exploiting the block diagonal structure of the model the log likelihood can also be expressed as

\[ \ln L_N(\vartheta) = -\frac{N}{2} \ln(2\pi) + \sum_{r=1}^{R} \ln L_r(\vartheta) \]

where

\[ \ln L_r(\vartheta) = \ln |I - \lambda W_{m_r}| - \frac{1}{2} \ln |\Omega_{m_r}(\vartheta)| - \frac{1}{2} (Y_r - \lambda W_{m_r} Y_r)' \Omega_{m_r}(\vartheta)^{-1} (Y_r - \lambda W_{m_r} Y_r). \]

At the true parameter, the score function of the log likelihood for group \( r \) is exactly the negative of \( \chi^*(\vartheta) \), that is

\[ \partial \ln L_r(\vartheta_0) / \partial \vartheta = -\chi^*_r(\vartheta_0). \]

This result establishes the asymptotic efficiency of the GMM estimator based on \( E[\chi_r(\vartheta)|m_r, D_r] = 0 \) under the assumption of Gaussian distributions for the unobservables. When the unobservables are not Gaussian then the GMM estimator has the interpretation of a quasi maximum likelihood estimator (QMLE). Similarly in Lee (2007) the score function of the conditional maximum likelihood estimator (CMLE) for group \( r \) is the optimal moment function corresponding to \( E(\chi^w_r(\vartheta)|m_r) = 0 \) under the assumption of homoskedastic and normally distributed errors \( \epsilon_{ir} \). While the CMLE of Lee (2007) is not efficient under the assumptions we postulate in this paper, it shares robustness properties of within group panel estimators in cases where the group effects are possibly correlated with covariates in the model. Under those circumstances, random effects quasi maximum likelihood estimators are generally not expected to be consistent.

Our discussion so far highlights group size variation and variance as the source of identification. We show that variation in group size is a source of identification in the QMLE, CVE and CMLE. In all, the CMLE utilizes how within-group variance changes with \( \lambda \) and size, while the CVE exploits the relationship between the within-group variance and between-group variance in relation to \( \lambda \) and size, and our QMLE uses both pieces of information. All three estimators remain valid without covariates, and may achieve identification as long as there are at least two different group sizes in the limit. This complements other results in the literature. For example, Proposition 4 in Bramoullé et al. (2009) states that in the setting of Lee (2007), \( \lambda \) is identified by instrumenting \((I-W)WY\) with \((I-W)W^2Z\), \((I-W)W^3Z\), etc. Their result is due to the fact that they only exploit restrictions for the conditional mean of \( \epsilon \). In Graham (2008) as well as in this paper additional constraints on the distribution of \( \epsilon \) are imposed and shown to be useful in the identification of peer effects. Under these conditions including \( Z \) offers additional sources of variation, but identification is possible with or without it.

Adding covariates is critically important in empirical applications. Consider adding the covariate matrix \( Z \). This leads to two additional moment conditions \( E[\tilde{Z}_r'\tilde{U}_r|m_r] = 0 \) and \( E[\tilde{z}_r'\tilde{u}_r|m_r] = 0 \), where \( \tilde{z}_r = u_{m_r} Z_r/m_r \) is the group mean of \( Z_r \) and \( \tilde{Z}_r = Z_r - u_{m_r} \tilde{z}_r \) is the deviation from group mean. Moreover, \( \tilde{Y}_r \) and \( \tilde{y}_r \) in (12) and (13) now need to be replaced by \( \tilde{Y}_r - \frac{m_r-1}{m_r-1+\lambda} \tilde{Z}_r \beta \) and \( \tilde{y}_r - \frac{\tilde{z}_r \beta}{1+\lambda} \), respectively. The score function of the QMLE discussed in this section then is the same as
the moment conditions of the best GMM corresponding to these two moment functions in addition to the moments $E[\chi_r(\hat{\theta})|m_r] = 0$. In the same way, in the presence of covariates, Lee’s CMLE estimator is based on $E(\hat{Z}_r^t\hat{U}_r) = 0$ in addition to $E[\chi_r^w(\theta)|m_r] = 0$ and the relative efficiency considerations discussed in this section continue to apply to the situation with covariates.

3 General Model

In this section we generalize the model to allow for individual characteristics, average individual characteristics and group level covariates. We assume that we have access to observations on $R$ groups. We consider asymptotics where the number of groups $R$ tends to infinity and where groups are formed with a finite number of group sizes. This setup assumes that asymptotically we observe infinitely many groups for at least two group sizes. Let $r = 1, ..., R$ denote the group index, and let $m_r$ denote the size of group $r$. The total sample size is then given by $N = \sum_{r=1}^R m_r$. Suppose further that interactions occur within each group, but not across groups, and that peer effects work through the mean outcome and mean characteristics of peers in the same group. The linear-in-means peer effects model that includes endogenous as well as exogenous peer effects then is given by

$$ y_{ir} = \beta_1 + \lambda \bar{y}_{(-i)r} + x_{1,ir}\beta_2 + \bar{x}_{2,(-i)r}\beta_3 + x_{3,r}\beta_4 + \alpha_r + \epsilon_{ir}, \quad (19) $$

where $y_{ir}$ is the outcome variable of individual $i$ in group $r$, $\bar{y}_{(-i)r} = \frac{1}{m_r-1} \sum_{j \neq i}^m y_{jr}$ is the average outcome of $i$’s peers, $x_{1,ir}$ and $x_{2,ir}$ are both row vectors of predetermined characteristics of individual $i$ in group $r$, $\bar{x}_{2,(-i)r} = \frac{1}{m_r-1} \sum_{j \neq i}^m x_{2,jr}$ is a vector of average characteristics of $i$’s peers, $x_{3,r}$ is a vector of observed group characteristics. The variables in $x_{1,ir}$ and $x_{2,ir}$ can be non-overlapping, partially overlapping or totally overlapping. The error term consists of two components, the group effect $\alpha_r$ and the disturbance term $\epsilon_{ir}$. We treat $x_{1,ir}$, $x_{2,ir}$, $x_{3,r}$ and $m_r$ as non-stochastic, while noting that at the expense of more complex notation we could also think of the analysis as being conditional on those variables. In this model, peer effects work through the mean peer outcome $\bar{y}_{(-i)r}$ and mean peer characteristics $\bar{x}_{2,(-i)r}$. The two terms are also known as the leave-out-mean of $y$ and $x_2$, as they are means of the group leaving out oneself. In Manski’s terminology, $\lambda$ in (19) reflects endogenous peer effects, $\beta_3$ reflects exogenous peer effects.

Let $z_{ir} = (1, x_{1,ir}, \bar{x}_{2,(-i)r}, x_{3,r})$ be the row vector of all exogenous variables, let $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)'$ be the corresponding coefficients vector, and let $k_Z$ denote the number of columns in $z_{ir}$. A compact form of model (19) is

$$ y_{ir} = \lambda \bar{y}_{(-i)r} + z_{ir}\beta + \alpha_r + \epsilon_{ir}. \quad (20) $$

The model can be further written as a Cliff-Ord type spatial model. To see this let $I_m$ denote the $m-$dimensional identity matrix, let $\iota_m$ denote the $m-$dimensional column vector of ones, and define the weight matrix $W_{m_r}$ for group $r$ as $W_{m_r} = \frac{1}{m_r-1}(\iota_m\iota_m' - I_{m_r})$. The off-diagonal elements of this matrix are all equal to $\frac{1}{m_r-1}$ and diagonal elements are 0. Let $Y_r = (y_{1r}, ..., y_{m_r})'$, $Z_r = (z_{1r}', ..., z_{m_r}')'$, $\epsilon_r = (\epsilon_{1r}, ..., \epsilon_{m_r})'$, then the model for group $r$ can be expressed in matrix
form as
\[ Y_r = \lambda W_m Y_r + Z_r \beta + U_r, \]  
(21)

where \( U_r = \alpha_r \epsilon_{m_r} + \epsilon_r \). Let \( Y = [Y'_1, Y'_2, \ldots, Y'_R]' \), \( Z = [Z'_1, Z'_2, \ldots, Z'_R]' \), \( U = [U'_1, U'_2, \ldots, U'_R]' \), and \( W = diag_{r=1}^R \{W_{m_r}\} \). Then the model for the whole sample is given by
\[ Y = \lambda W Y + Z \beta + U. \]  
(22)

In the spatial literature \( W \) is referred to as a spatial weight matrix and \( W Y \) as a spatial lag. The model in (22) is analyzed under the random effects assumptions introduced in Section 2 that \( \alpha_r \) and \( \epsilon_{ir} \) are i.i.d. \( (0, \sigma^2_{\alpha}) \) and \( (0, \sigma^2_{\epsilon}) \) respectively.

In principle we could allow for heteroskedasticity at the group level as long as there are only a finite number of different parameters. For example, we could allow for \( \sigma^2_{\epsilon} \) to be different for small and large groups, or more generally for all groups of a certain size \( m_r \). On the other hand we do not cover the case where \( \sigma^2_{\epsilon} \) differs for each individual group \( r \), as this would lead to an infinite dimensional parameter space. For ease of exposition we focus on the homoskedastic case in what follows.

The parameters of interest are \( \lambda, \sigma^2_{\epsilon}, \sigma^2_{\alpha} \) and \( \beta \). Their respective true values are \( \lambda_0, \sigma^2_{\epsilon_0}, \sigma^2_{\alpha_0} \) and \( \beta_0 \). In analyzing the model it will be convenient to concentrate the log-likelihood function with respect to \( \beta \) for given values of \( \theta = (\lambda, \sigma^2_{\epsilon}, \sigma^2_{\alpha})' \). Let \( \Theta \) denote the parameter space for \( \theta \) , and let \( \delta = (\theta', \beta')' \) denote the vector of all parameters.

Under homoskedasticity the expression for the variance covariance matrix \( \Omega_0 \) of \( U \) given in (16) in Section 2 simplifies to
\[ \Omega_0 = \Omega(\theta_0) = diag_{r=1}^R \{\sigma^2_{\epsilon_0} I_{m_r} + \sigma^2_{\alpha_0} \epsilon_{m_r} \epsilon_{m_r}'\} = diag_{r=1}^R \{\sigma^2_{\epsilon_0} I_{m_r} + (\sigma^2_{\epsilon_0} + m_r \sigma^2_{\alpha_0}) J_{m_r}\}. \]

The quasi-maximum likelihood estimator (QMLE) for the peer effects model in (20) is now introduced. Solving \( Y \) from (22) yields the reduced from:
\[ Y = (I - \lambda W)^{-1} Z \beta + (I - \lambda W)^{-1} U. \]  
(23)

If \( \alpha_r \) and \( \epsilon_{ir} \) follow normal distributions,
\[ Y \sim N((I - \lambda W)^{-1} Z \beta, (I - \lambda W)^{-1} \Omega(\theta)(I - \lambda W')^{-1}). \]  
(24)

The corresponding log likelihood function is
\[ lnL_N(\theta, \beta) = -\frac{N}{2} ln(2\pi) + \frac{1}{2} ln| (I - \lambda W)^2 \Omega(\theta)^{-1}| \]
\[ -\frac{1}{2}(Y - \lambda WY - Z \beta)' \Omega(\theta)^{-1} (Y - \lambda WY - Z \beta). \]  
(25)
and the corresponding QMLE is given by
\[
\hat{\delta}_N = (\hat{\theta}_N', \hat{\beta}_N')' = \arg \max_{\theta, \beta} lnL_N(\theta, \beta).
\] (26)

It is convenient to concentrate out \(\beta\) and to obtain the QMLE for \(\theta\) first. The first order condition for \(\beta\) is
\[
\frac{\partial \ln L_N(\theta, \beta)}{\partial \beta} = (Y - \lambda W Y - Z \beta)' \Omega(\theta)^{-1} Z = 0,
\] (27)
which leads to
\[
\hat{\beta}_N(\theta) = (Z' \Omega(\theta)^{-1} Z)^{-1} Z' \Omega(\theta)^{-1} (I - \lambda W) Y.
\] (28)
Plugging \(\hat{\beta}_N(\theta)\) back into (25) yields the following concentrated log likelihood function,
\[
Q_N(\theta) = \frac{1}{N} \ln L_N(\theta, \hat{\beta}_N(\theta)) = -\frac{\ln(2\pi)}{2} + \frac{1}{2N} \ln|I - \lambda W|^2 \Omega(\theta)^{-1}| - \frac{1}{2N} Y'(I - \lambda W)' M_Z(\theta)(I - \lambda W) Y,
\] (29)
where
\[
M_Z(\theta) = \Omega(\theta)^{-1} - \Omega(\theta)^{-1} Z(Z' \Omega(\theta)^{-1} Z)^{-1} Z' \Omega(\theta)^{-1}.
\] (30)
Then the QMLE for \(\theta, \hat{\theta}_N = (\hat{\lambda}_N, \hat{\sigma}_{\epsilon, N}^2, \hat{\sigma}_{\alpha, N}^2)'\) is given by
\[
\hat{\theta}_N = \arg \max_{\theta} Q_N(\theta).
\] (31)
Plugging \(\hat{\theta}_N\) back into (28), the QMLE for \(\beta\) is
\[
\hat{\beta}_N = \hat{\beta}_N(\hat{\theta}_N) = (Z' \Omega(\hat{\theta}_N)^{-1} Z)^{-1} Z' \Omega(\hat{\theta}_N)^{-1} (I - \hat{\lambda}_N W) Y.
\] (32)

A formal result regarding the identification of the model parameters is given in the next section.

We next provide some intuition for that result, by extending our earlier discussion of identification for the canonical model without covariates to our model (22) with covariates. Let \(|.||\) be the Euclidean norm on \(\mathbb{R}^k\). Using the relationships \(\bar{U}_r = (m_r - 1 + \lambda_0) \bar{Y}_r - \bar{Z}_r \beta_0\) and \(\bar{u}_r = (1 - \lambda_0) \bar{y}_r - \bar{z}_r \beta_0\), the moment functions related to the full model can be written as follows
\[
\chi_r^w(\delta) = \left\| \frac{(m_r - 1 + \lambda)}{m_r - 1} \bar{Y}_r - \bar{Z}_r \beta \right\|^2 - (m_r - 1) \sigma^2_{\epsilon}.
\] (33)
\[
\chi_r^b(\delta) = [(1 - \lambda) \bar{y}_r - \bar{z}_r \beta]^2 - \sigma^2_{\alpha} - \sigma^2_{\epsilon}.
\] (34)
\[
\chi_r^{zw}(\delta) = \bar{Z}_r' \left( \frac{(m_r - 1 + \lambda)}{m_r - 1} \bar{Y}_r - \bar{Z}_r \beta \right),
\] (35)
\[
\chi_r^{zb}(\delta) = \bar{Z}_r' ((1 - \lambda) \bar{y}_r - \bar{z}_r \beta),
\] (36)
where \(\chi_r^w(\delta)\) and \(\chi_r^b(\delta)\) summarize the restrictions on the unobservables, and are natural extensions of the moment conditions considered before in (12) and (13) for the model without covariates. The
additional moment restrictions $\chi_r^{zw}(\delta)$ and $\chi_r^{zb}(\delta)$ relate to the exogeneity of $Z_r$ relative to $\epsilon_r$ and $\alpha_r$. As for the model without excoriates there is a representation of the score of the log-likelihood in terms of the fundamental moment conditions. To describe the relationship between moments and the score define the matrix

$$
\varphi_{m_r} = \begin{pmatrix}
\frac{1}{(m_r-1+\lambda_0)\sigma_{00}^+} & -\frac{m_r}{(1-\lambda_0)(\sigma_{00}^-+m_r\sigma_{00}^+)} & \frac{1}{(m_r-1+\lambda_0)\sigma_{00}^-} & -\frac{m_r}{(1-\lambda_0)(\sigma_{00}^-+m_r\sigma_{00}^-)}
-\frac{1}{2\sigma_{00}^-} & -\frac{m_r}{2(\sigma_{00}^-+m_r\sigma_{00}^+)^2} & 0 & 0
0 & 0 & -\frac{1}{\sigma_{00}^-}I_{kZ} & -\frac{m_r}{\sigma_{00}^-+m_r\sigma_{00}^-}I_{kZ}
0 & 0 & 0 & 0
\end{pmatrix}.
$$

Furthermore observe that the log-likelihood function can be written as $lnL_N(\delta) = -\frac{N}{2} ln(2\pi) + \sum_{r=1}^{R} \ln L_r(\delta)$ where

$$
lnL_r(\delta) = \frac{1}{2} \ln|I_{m_r} - \lambda W_{m_r})^2\Omega_r(\theta)^{-1}|
- \frac{1}{2}(Y_r - \lambda W_{m_r}Y_r - Z_r\beta)^\prime\Omega_r(\theta)^{-1}(Y_r - \lambda W_{m_r}Y - Z_r\beta).
$$

is the log-likelihood function for group $r$. Then it can be shown that

$$
\frac{\partial lnL_r(\delta_0)}{\partial \delta} = -\chi_r^+(\delta_0) = -\varphi_{m_r} \chi_r(\delta_0),
$$

with $\chi_r(\delta) = (\chi_r^{zw}(\delta),\chi_r^{zb}(\delta),\chi_r^{zw}(\delta),\chi_r^{zb}(\delta))'$. As is well known, the score of the log-likelihood function, $S(\delta) = -\sum_{r=1}^{R} \frac{\partial lnL_r(\delta)}{\partial \delta}$ can be interpreted as a moment function corresponding to the moments $E[S(\delta_0)] = -\sum_{r=1}^{R} E\left[\frac{\partial lnL_r(\delta_0)}{\partial \delta}\right] = 0$. Furthermore, under a Gaussian assumption the score is an optimal moment function. From this we see that the matrices $\varphi_{m_r}$ can be viewed to provide the optimal weighting for the basic moment functions $\chi_r(\delta)$. Under Gaussian assumptions the optimal GMM estimator coincides with the maximum likelihood estimator and is asymptotically efficient under the stated assumptions.

4 Theoretical Results

We next state our maintained assumptions for the general model. Let $\mathcal{I}_m \subset \{1,\ldots,R\}$ be the index set of all groups with size equal to $m$. Thus if $r \in \mathcal{I}_m$, then $m_r = m$. Let $R_m$ be the cardinality of

---

4See our Online Appendix for details. The derivation uses Lemma C.1 and the special properties of matrices $\Omega(\theta)$, $I - \lambda W$ and $W$ described in Appendix C.1. In the Online Appendix we also give an explicit expression for the VC matrix of $\chi_r(\delta)$.

5Observe that

$$
E\left[\frac{\partial S(\delta_0)}{\partial \delta'}\right] Var\left(S(\delta_0)\right)^{-1} S(\delta_0) = \left(-\sum_{r=1}^{R} E\left[\frac{\partial^2 lnL_r(\delta_0)}{\partial \delta \partial \delta'}\right]\right) \left(\sum_{r=1}^{R} E\left[\frac{\partial lnL_r(\delta_0)}{\partial \delta} \frac{\partial lnL_r(\delta_0)}{\partial \delta'}\right]\right)^{-1} S(\delta_0)
$$

in light of the information matrix equality.
\( \mathcal{I}_m \), or the number of groups with size equal to \( m \), and let \( \omega_m = R_m / R \) denote the share of groups with size \( m \).

**Assumption 1.** (a) The sample size \( N \) goes to infinity; (b) The group size is bounded in the sense that there exists some positive constant \( \bar{a} \) such that for \( r = 1, 2, ..., R \), \( 2 \leq m_r \leq \bar{a} < \infty \); (c) The limit \( \omega_m^* = \lim_{N \to \infty} \omega_m \) exists, and \( 0 \leq \omega_m^* < 1 \) for \( 2 \leq m \leq \bar{a} \).

Observe that \( N = \sum_{r=1}^{R} m_r = \sum_{m=2}^{\bar{a}} m R_m \). The restriction that the minimal group size is two rules out singleton groups. A member of such groups has no peers. Since the group size is bounded, the number of groups \( R \) goes to infinity as \( N \) goes to infinity. Since \( \sum_{m=2}^{\bar{a}} R_m = R \), we have \( \sum_{m=2}^{\bar{a}} \omega_m = 1 \) and thus \( \sum_{m=2}^{\bar{a}} \omega_m^* = 1 \). The condition \( \omega_m^* < 1 \) in Assumption 1(c) ensures that, in the limit, the group size is not the same for all groups. As discussed earlier, identification requires variation in the group size, and so Assumption 1(c) is vital. Assumption 1(c) also implies that, in the limit, the group size is not the same for all groups. The parameter of the endogenous peer effects \( \lambda_0 \in \Lambda \), where \( \Lambda \) is a compact subset of \((-1, 1)\).

**Assumption 2.** The parameter of the endogenous peer effects \( \lambda_0 \in \Lambda \), where \( \Lambda \) is a compact subset of \((-1, 1)\).

**Assumption 3.** The disturbance terms \( \epsilon_{ir} \) are independently and identically distributed across all \( i \) and \( r \), with \( E[\epsilon_{ir}] = 0 \) and \( E[\epsilon_{ir}^2] = \sigma_{\epsilon}^2 \), where \( 0 < \sigma_\epsilon \leq \sigma_{\epsilon}^2 \leq \bar{\sigma}_{\epsilon} < \infty \). There exists some \( \eta_\epsilon > 0 \) such that \( E[|\epsilon_{ir}|^{1+\eta_\epsilon}] < \infty \).

**Assumption 4.** For \( r = 1, ..., R \), the group effects \( \alpha_r \) are independently and identically distributed, with \( E[\alpha_r] = 0 \) and \( E[\alpha_r^2] = \sigma_{\alpha}^2 \), where \( 0 \leq \sigma_{\alpha} \leq \sigma_{\alpha}^2 \leq \bar{\sigma}_\alpha < \infty \). There exists some \( \eta_\alpha > 0 \) such that \( E[|\alpha_i|^{1+\eta_\alpha}] < \infty \). Also, \( \{\alpha_r : r = 1, ..., R\} \) are independent of \( \{\epsilon_{ir} : i = 1, ..., m_r; r = 1, ..., R\} \).

From Assumptions 2, 3, and 4, \( \Theta \) is a compact subset of the Euclidean space \( \mathbb{R}^3 \). Observe that

\[
I_{m_r} - \lambda W_{m_r} = (1 + \frac{\lambda}{m_r - 1}) I_{m_r} - \lambda J_{m_r}^*,
\]

where \( I_{m_r}^* = I_{m_r} - \epsilon_{m_r} \epsilon_{m_r}' / m_r \) and \( J_{m_r}^* = \epsilon_{m_r} \epsilon_{m_r}' / m_r \) are symmetric, idempotent, orthogonal, and sum to the identity matrix. Furthermore from the results in Appendix C.1, we have \( |I_{m_r} - \lambda W_{m_r}| = [1 + \lambda / (m_r - 1)]^{m_r} (1 - \lambda)^{\frac{m_r}{2}} \). Thus the matrix \( I_{m_r} - \lambda W_{m_r} \) is nonsingular if \( 1 + \lambda / (m_r - 1) \neq 0 \) and \( 1 - \lambda \neq 0 \). Assumption 2 ensures the non-singularity of \( I_{m_r} - \lambda W_{m_r} \), and hence the non-singularity of matrices of the form \( pI_{m_r} + sJ_{m_r}^* \), which will be used repeatedly in this paper. In particular, their multiplication is commutative. The products of such matrices are also of the form \( pI_{m_r} + sJ_{m_r}^* \), and \( |pI_{m_r} + sJ_{m_r}^*| = p^{m_r - 1} s, (pI_{m_r} + sJ_{m_r}^*)^{-1} = \frac{1}{p} I_{m_r} + \frac{1}{s} J_{m_r}^* \).
of \( I - \lambda W = \operatorname{diag}_{r=1}^{R} \{ I_{m_r} - \lambda W_{m_r} \} \), since for \( m_r \geq 2 \) and \( \lambda < 1 \) we have \( 1 + \lambda/(m_r - 1) > 0 \) and \( 1 - \lambda > 0 \).

Let \( \tilde{z}_r = \frac{1}{m_r} \zeta_r' Z_r \) be the row vector of column means of \( Z_r \), and let \( \tilde{Z}_r = Z_r - t_{m_r} \tilde{z}_r \) be the deviations from the column means. Then \( Z_r' I_{m_r} Z_r = \tilde{Z}_r' \tilde{Z}_r, Z_r' J_{m_r} Z_r = m_r \zeta_r' \tilde{z}_r \).

**Assumption 5.** (a) The \( N \times k_Z \) matrix \( Z \) is non-stochastic, with \( \text{rank}(Z) = k_Z > 0 \) for \( N \) sufficiently large. The elements of \( Z \) are uniformly bounded in absolute value.

(b) For \( 2 \leq m \leq \bar{a} \),
\[
\lim_{N \to \infty} N^{-1} \sum_{r \in \mathcal{I}_m} \tilde{Z}_r' \tilde{Z}_r = \tilde{z}_m,
\]
\[
\lim_{N \to \infty} N^{-1} \sum_{r \in \mathcal{I}_m} m \zeta_r' \tilde{z}_r = \tilde{z}_m,
\]
\[
\lim_{N \to \infty} N^{-1} \sum_{r \in \mathcal{I}_m} \zeta_r = \tilde{z}(m).
\]

(c) For all \( m \) with \( \omega_m > 0 \) and \( N \) sufficiently large, the smallest eigenvalues of \( N^{-1} \sum_{r \in \mathcal{I}_m} \tilde{Z}_r' \tilde{Z}_r \) and \( N^{-1} \sum_{r \in \mathcal{I}_m} m \zeta_r' \tilde{z}_r \) are bounded away from zero, uniformly in \( N \), by some finite constant \( \xi_{Z} > 0 \).

Since the elements of \( Z \) are uniformly bounded in absolute value, \( \tilde{z}_m \) and \( \tilde{z}_m \) are finite \( k_Z \times k_Z \) matrices, and \( \tilde{z}(m) \) is a \( 1 \times k_Z \) vector of finite elements. Suppose we have some \( N \times N \) matrix \( A_N(\theta) = \operatorname{diag}_{r=1}^{R} \{ p(m_r, \theta) J_{m_r}^* + s(m_r, \theta) J_{m_r}^* \} \), where \( p(m_r, \theta) \) and \( s(m_r, \theta) \) are positive, uniformly bounded and continuous on \( \Theta \), e.g., \( \Omega(\theta)^{-1} \) as described in Equation (C.3) in Appendix C.1. Then under Assumption 5(b), the limiting matrix of \( N^{-1} Z' A_N(\theta) Z \) always exists, is continuous in \( \theta \) and takes the form
\[
\lim_{N \to \infty} \frac{1}{N} Z' A_N(\theta) Z = \sum_{m=2}^{\bar{a}} [p(m, \theta) \tilde{z}_m + s(m, \theta) \tilde{z}_m].
\]
Furthermore, \( N^{-1} Z' A_N(\theta) Z \) converges to its limiting matrix uniformly on \( \Theta \). With \( p(m_r, \theta) > 0 \) and \( s(m_r, \theta) > 0 \), Assumption 5(c) ensures that \( N^{-1} Z' A_N(\theta) Z \) and its limiting matrix are invertible, with the elements of the inverse matrix uniformly bounded in absolute value. See Lemma C.5 for details and a proof.

Below we give results on the consistency and asymptotic normality of the QMLE \( \hat{\delta}_N = (\hat{\theta}_N, \hat{\beta}_N)' \) defined in (26).

**Theorem 4.1.** Suppose Assumptions 1-5 hold. Then

(a) The parameter \( \delta_0 \) is identified.

(b) The QMLE \( \hat{\delta}_N \) is consistent, i.e., \( \hat{\delta}_N \xrightarrow{p} \delta_0 \) as \( N \to \infty \).

As we shall see in the proof, variation in group size plays a significant role in identification. A detailed proof of the theorem is in Appendices E.1 and E.2. Here is a sketch of the proof to provide some intuition. The limiting expected value of the concentrated log likelihood function \( Q_N(\theta) \) is
\[
\hat{Q}^*(\theta) = C^* + \frac{1}{2m^*} \sum_{m=2}^{\bar{a}} \omega_m^* g(m, \theta) + \lim_{N \to \infty} \hat{Q}_N^{(2)}(\theta),
\]
Theorem 4.2. strongly. Of course, the condition holds, e.g., for the Gaussian distribution.

and

\[ \bar{G} = \frac{\sigma^2_0}{\sigma_x^2} (\frac{m - 1 + \lambda}{m - 1 + \lambda_0})^2 I_m^* + \frac{(\sigma^2_0 + m\sigma^2_0)}{(\sigma_x^2 + m\sigma^2_0)} (\frac{1 - \lambda}{1 - \lambda_0})^2 J_m^*, \]

and

\[ \tilde{Q}^2_N (\theta) = -1 \frac{1}{2N} \bar{G} \tilde{M}_Z(\theta) \bar{M}_Z(\theta)^T \] with

\[ \tilde{M}_Z(\theta) = I - \Omega(\theta)^{-1/2} Z' \Omega(\theta)^{-1} Z' \Omega(\theta)^{-1/2} \]

and

\[ \bar{G} \tilde{M}_Z(\theta) \bar{M}_Z(\theta) = \Omega(\theta)^{-1/2} (I - \lambda W)(I - \lambda_0 W)^{-1} Z \beta_0. \]

Given that \(-\tilde{Q}^2_N (\theta)\) is the quadratic form of an idempotent and thus positive semi-definite matrix, it is easy to see that \(\theta_0\) is a global maximizer of \(Q^2(\theta) = \lim_{N \to \infty} \tilde{Q}^2_N (\theta)\), observing that \(Q^2(\theta_0) = 0\). However, this does not ensure that \(\theta_0\) is a unique global maximizer. Identification thus comes from \(\sum_{i=2}^{N} \omega^* i g(m, \theta)\). Note that for any symmetric positive definite matrix \(A\), \(ln|A| - tr(A) \leq -m\) with equality if and only if \(A\) is an identity matrix.

For any \(m, g(m, \theta)\) is maximized if and only if \(G(m, \theta) = I_m\). For all \(m, G(m, \theta_0) = I_m\). For any \(\theta \neq \theta_0\), \(G(m', \theta) = I_{m'}\) and \(G(m'', \theta) = I_{m''}\) cannot hold simultaneously if \(m' \neq m''\). As a result, \(\theta_0\) is the unique global maximizer of \(\tilde{Q}^2(\theta)\) when there are at least one pair of \(m' \neq m''\) such that \(\omega_{m'}^* > 0\) and \(\omega_{m''}^* > 0\).

To study the asymptotic distribution of the estimator, first note that under Assumptions 3 and 4, the third and fourth moments of \(\epsilon_{ir}\) and \(\alpha_r\) exist. Let \(E(\epsilon_{ir})^3 = \mu^{(3)}_{\epsilon_0}, \ E\epsilon^4_{ir} = \mu^{(4)}_{\epsilon_0}, \ E(\alpha_r)^3 = \mu^{(3)}_{\alpha_0}\) and \(E\alpha^4_r = \mu^{(4)}_{\alpha_0}\). Also, define \(\Gamma_0\) and \(\Upsilon_0\) as

\[ \Gamma_0 = \lim_{N \to \infty} N^{-1} E \left[ \frac{-\partial^2 \ln L_N(\delta_0)}{\partial \delta \partial \delta'} \right], \]

\[ \Upsilon_0 = \lim_{N \to \infty} N^{-1} E \left[ \frac{\partial^2 \ln L_N(\delta_0)}{\partial \delta \partial \delta'} \frac{\partial \ln L_N(\delta_0)}{\partial \delta} \right]. \]

As shown in Appendix E.3, the two limiting matrices exist. Specific expressions are given in Appendix F. When \(\epsilon_{ir}\) and \(\alpha_r\) both follow normal distributions, \(\Upsilon_0 = \Gamma_0\). The next lemma shows that \(\Gamma_0\) is p.d. under the maintained assumptions. The lemma also provides a sufficient condition on the moments of \(\epsilon\) under which \(\Upsilon_0\) is p.d..

**Lemma 4.1.** Under Assumptions 1-5, \(\Gamma_0\) is positive definite. Under the additional assumption that \(\mu^{(4)}_{\epsilon_0} - \sigma^4_{\epsilon_0} > (\mu^{(3)}_{\epsilon_0})^2/\sigma^2_{\epsilon_0}\), \(\Upsilon_0\) is also positive definite.

The proof of the lemma is in Appendix F. Note that from Holder’s inequality we have \(\mu^{(4)}_{\epsilon_0} - \sigma^4_{\epsilon_0} \geq (\mu^{(3)}_{\epsilon_0})^2/\sigma^2_{\epsilon_0}\). The sufficient condition is mild in that it only postulates that the inequality holds strongly. Of course, the condition holds, e.g., for the Gaussian distribution.

With both \(\Upsilon_0\) and \(\Gamma_0\) ensured to be positive definite, we have the following theorem.

**Theorem 4.2.** Under Assumptions 1-5, and assuming that \(\mu^{(4)}_{\epsilon_0} - \sigma^4_{\epsilon_0} > (\mu^{(3)}_{\epsilon_0})^2/\sigma^2_{\epsilon_0}\), we have

\[ \sqrt{N} (\hat{\delta}_N - \delta_0) \overset{d}{\to} N(0, \Gamma_0^{-1} \Upsilon_0 \Gamma_0^{-1}) \]

as \(N \to \infty\).
The proof of the theorem is given in Appendix E.3. We next discuss consistent estimators for the matrices $\Gamma_0$ and $\Upsilon_0$ composing the asymptotic variance covariance matrix. An inspection shows that $\Gamma_0 = \Gamma(\delta_0, s_0)$ and $\Upsilon_0 = \Upsilon(\delta_0, \mu_{r0}^{(3)}, \mu_{r0}^{(4)}, \mu_{a0}, s_0)$, with

$$s_0 = [\bar{x}_2, \ldots, \bar{x}_a, \bar{x}_2, \ldots, \bar{x}_a, \bar{z}(2), \ldots, \bar{z}(a), \bar{\omega}_2, \ldots, \bar{\omega}_a, m^*],$$

and where the functions $\Gamma(.)$ and $\Upsilon(.)$ are continuous. Since the functions $\Gamma(.)$ and $\Upsilon(.)$ are continuous, consistent estimators for $\Gamma_0$ and $\Upsilon_0$ can be readily obtained by replacing the arguments of those functions by consistent estimators thereof. Let $\hat{s}_N$ be the sample analogue of $s_0$, then clearly $\hat{s}_N \xrightarrow{p} s_0$ in light of Assumptions 1 and 5. Recall further that by Theorem 4.1 the QMLE estimator $\hat{\delta}_N$ is consistent for $\delta_0$, and suppose we have consistent estimators for $\mu_{r0}^{(3)}, \mu_{r0}^{(4)}, \mu_{a0}$, and $\mu_{a0}^{(4)}$, denoted as $\hat{\mu}_{r0}^{(3)}, \hat{\mu}_{r0}^{(4)}, \hat{\mu}_{a0}$, and $\hat{\mu}_{a0}^{(4)}$. Now define $\hat{\Gamma}_N$ and $\hat{\Upsilon}_N$ as

$$\hat{\Gamma}_N = \Gamma(\hat{\delta}_N, \hat{s}_N),$$
$$\hat{\Upsilon}_N = \Upsilon(\hat{\delta}_N, \hat{\mu}_{r0}^{(3)}, \hat{\mu}_{r0}^{(4)}, \hat{\mu}_{a0}, \hat{\mu}_{a0}^{(4)}, \hat{s}_N),$$

then it follows from Slutsky’s theorem that $\hat{\Gamma}_N$ and $\hat{\Upsilon}_N$ are consistent estimators for $\Gamma_0$ and $\Upsilon_0$. A consistent estimator for the variance covariance matrix of the limiting distribution is given by $\hat{\Gamma}_N^{-1} \hat{\Upsilon}_N \hat{\Gamma}_N^{-1}$.

The above discussion assumed the availability of consistent estimators for the third and fourth moment of the error components. In the following we now define consistent estimators for $\mu_{r0}^{(3)}, \mu_{r0}^{(4)}, \mu_{a0}$, and $\mu_{a0}^{(4)}$. To motivate the estimators consider the composite error term for individual $i$ in group $r$, $u_{ir} = \alpha_r + \epsilon_{ir}$, and let $\bar{u}_r = \frac{1}{m_r} \sum_{i=1}^{m_r} u_{ir}$, and $\bar{u}_{ir} = u_{ir} - \bar{u}_r$. Then $\bar{u}_r = \alpha_r + \bar{\epsilon}_r$ and $\bar{u}_{ir} = \epsilon_{ir} - \bar{\epsilon}_r$, where $\bar{\epsilon}_r$ is the group mean of $\epsilon_{ir}$. It is readily verified that under Assumptions 3 and 4, we have for $m_r \geq 3$\footnote{The formula for $E \bar{u}_{ir}^4$ is different from the one in Galvao et al. 2013 because of a small mistake in their calculation.}

$$E \left[ \bar{u}_{ir}^3 \right] = (1 - \frac{3}{m_r} + \frac{2}{m_r^2}) \mu_{r0}^{(3)},$$
$$E \left[ \bar{u}_{ir}^3 \right] = \mu_{a0}^{(3)} + \frac{\mu_{a0}}{m_r^2},$$

$$E \left[ \bar{u}_{ir}^4 \right] = \frac{m_r^3 - 4m_r^2 + 6m_r - 3}{m_r^3} \mu_{r0}^{(4)} + \frac{3(m_r - 1)(2m_r - 3)}{m_r^3} \sigma_{\epsilon_0}^2,$$
$$E \left[ \bar{u}_{ir}^4 \right] = \mu_{a0}^{(4)} + \frac{1}{m_r} \mu_{a0}^{(4)} + \frac{3}{m_r^2} \sigma_{\epsilon_0}^2 + \frac{6}{m_r} \sigma_{\epsilon_0}^2 \sigma_{\epsilon_0}^2.$$

Next define for group $r$ with $m_r \geq 3$,

$$\hat{f}_{r}^{(3)} = \frac{1}{m_r} \sum_{i=1}^{m_r} \hat{u}_{ir}^3 / (1 - \frac{3}{m_r} + \frac{2}{m_r^2}),$$

$$\hat{f}_{r}^{(4)} = \frac{1}{m_r} \sum_{i=1}^{m_r} \hat{u}_{ir}^4 / (1 - \frac{3}{m_r} + \frac{2}{m_r^2}),$$
the first case,

\[ f_{α,r}^{(3)} = \bar{u}_r^3 - f_{ε,r}^{(3)}/m_r^2, \]

\[ f_{ε,r}^{(3)} = \frac{m_r^3}{m_r^3 - 4m_r^2 + 6m_r - 3} \left[ \left( \sum_{i=1}^{m_r} \bar{u}_{ir}^4 \right) - \frac{3(m_r - 1)(2m_r - 3)}{m_r^3} \sigma_{α0}^4 \right], \]

\[ f_{ε,r}^{(4)} = \bar{u}_r^4 - f_{ε,r}^{(4)}/m_r^2 - \frac{3(m_r - 1)}{m_r^2} \sigma_{α0}^4 - 6 \frac{m_r}{m_r} \sigma_{α0}^2 \sigma_{ε0}^2. \]

Then \( E \left[ f_{ε,r}^{(3)} \right] = \mu_{ε0}, E \left[ f_{α,r}^{(3)} \right] = \mu_{α0}, E \left[ f_{ε,r}^{(4)} \right] = \mu_{ε0}, E \left[ f_{α,r}^{(4)} \right] = \mu_{α0} \). Furthermore, by Lemma C.4(a), as \( R \) goes to infinity, the respective averages of \( f_{ε,r}^{(3)}, f_{α,r}^{(3)}, f_{ε,r}^{(4)}, f_{α,r}^{(4)} \) across \( R \) groups converge to their respective mean values in probability. That is, \( \frac{1}{R} \sum_{r=1}^{R} f_{ε,r}^{(3)} \to_p \mu_{ε0} \), etc.

To construct feasible counterparts of these estimates, consider the estimated disturbances \( \hat{u}_{ir} = y_{ir} - \hat{λ}\bar{y}_{(-i)r} - z_{ir}\hat{β} \), where \( \hat{λ} \) and \( \hat{β} \) denote the QML estimators, and let \( \hat{u}_r = \frac{1}{m_r} \sum_{i=1}^{m_r} \hat{u}_{ir} \) and \( \hat{z}_{ir} = \hat{u}_{ir} - \hat{u}_r \). Feasible counterparts, say, \( \tilde{f}_{ε,r}^{(3)}, \tilde{f}_{α,r}^{(3)}, \tilde{f}_{ε,r}^{(4)}, \tilde{f}_{α,r}^{(4)} \) of \( f_{ε,r}^{(3)}, f_{α,r}^{(3)}, f_{ε,r}^{(4)}, f_{α,r}^{(4)} \) can now be defined by replacing \( \bar{u}_r \) and \( \bar{u}_{ir} \) with \( \hat{u}_r \) and \( \hat{u}_{ir} \), and \( σ_{ε0}^2 \) and \( σ_{α0}^2 \) with their QML estimators. Now consider the following estimators for the third and fourth moments of the error components:

\[ \hat{μ}_{ε}^{(3)} = \frac{1}{R} \sum_{r=1}^{R} \tilde{f}_{ε,r}^{(3)}, \hat{μ}_{ε}^{(4)} = \frac{1}{R} \sum_{r=1}^{R} \tilde{f}_{ε,r}^{(4)}, \hat{μ}_{α}^{(3)} = \frac{1}{R} \sum_{r=1}^{R} \tilde{f}_{α,r}^{(3)}, \hat{μ}_{α}^{(4)} = \frac{1}{R} \sum_{r=1}^{R} \tilde{f}_{α,r}^{(4)}. \]

The next theorem establishes their consistency.

**Theorem 4.3.** Suppose Assumptions 1-5 hold, then \( \hat{μ}_{ε}^{(3)} \to_p μ_{ε0}, \hat{μ}_{ε}^{(4)} \to_p μ_{ε0}, \hat{μ}_{α}^{(3)} \to_p μ_{α0}, \hat{μ}_{α}^{(4)} \to_p μ_{α0}. \)

Proof of the theorem is in Appendix E.

## 5 Monte Carlo Results

We conduct Monte-Carlo experiments to assess the finite sample properties of the quasi-maximum likelihood (QML) estimator \( \hat{δ}_N \). The data generating mechanism is determined by the main model in (20). For simplicity, \( x_{1,ir}, x_{2,ir} \) and \( x_{3,ir} \) each only includes one regressor. We treat them as scalars in this section. We set the true value of the parameters to \( λ_0 = 0.5, \sigma_{ε0}^2 = 0.5^2, \sigma_{α0}^2 = 1, \beta_{10} = 1, \beta_{20} = 1, \beta_{30} = 1 \) and \( β_{40} = 1 \). The model for the data generating process is thus

\[ y_{ir} = 0.5\bar{y}_{(-i)r} + 1 + x_{1,ir} + x_{3,ir} + α_r + ε_{ir}. \] (40)

We consider both the case when \( x_1 \) and \( x_2 \) are identical, and when \( x_1 \) and \( x_2 \) are distinct. In the first case, \( x_{1,ir} = x_{2,ir} \) are i.i.d. \( N(0,1) \). In the second case, \( x_{1,ir} \) and \( x_{2,ir} \) are independently drawn and each is i.i.d. \( N(0,1) \). With \( x_{2,ir} \), we then calculate \( \bar{x}_{2,(-i)r} = 1/m_r \sum_{j \neq i} x_{2j,r} \). Observed

\footnote{As stated, the above observations hold for \( m_r \geq 3 \), which will also be maintained for subsequent discussion for ease of presentation. Theorems 1 and 2 only maintain that \( m_r \geq 2 \). To accommodate the weaker assumption we note that under our Assumption 1 as \( R \) goes to infinity also the number of groups with \( m_r \geq 3 \) has to go to infinity. Consequently, under the relaxed condition \( m_r \geq 2 \), all of the subsequent discussion remains valid if we only average over groups with \( m_r \geq 3 \) rather than over all groups.}
group characteristic \( x_{3,r} \) is i.i.d. \( N(0, 1) \) and is drawn independently of \( x_{1,ir} \) and \( x_{2,ir} \). The idiosyncratic term \( \epsilon_{ir} \) is i.i.d \( N(0, 1) \). The group effect \( \alpha_r \) is i.i.d \( N(0, 0.5^2) \). Both \( \epsilon_{ir} \) and \( \alpha_r \) are drawn independently of \( x_{1,ir}, x_{2,ir}, x_{3,r} \), and of each other. The dependent variable \( y_{ir} \) is calculated using equation (23).

The number of groups \( R \) is selected from the set \{50, 100, 200, 400, 800\}. Group size \( m_r \) is drawn from a normal distribution, and then rounded to the nearest integer. We consider cases of both large and small variation in group size \( m_r \). In the case of large size variation, the coefficient of variation of \( m_r \) is 30% and \( m_r \) is i.i.d \( N(10, 3^2) \) or i.i.d \( N(20, 6^2) \). In the case of small size variation, coefficient of variation of \( m_r \) is 15% and \( m_r \) is i.i.d \( N(10, 1.5^2) \) or i.i.d \( N(20, 3^2) \). Groups with size smaller than 2 are dropped out from the sample.

We compare our QML estimator with the conditional maximum likelihood (CML) estimator by Lee (2007). The two estimators use the same data generated by the process described above. When group effects are in fact independent of the observed characteristics, the CML estimator is still consistent but less efficient than our QML estimator. The comparison thus helps to evaluate the efficiency gain of our estimator over the CML estimator in finite samples. The CML estimator is based on the within-group variation hence \( \sigma^2_{\alpha}, \beta_1 \) and \( \beta_4 \) are not identified.

We generate 300 repetitions for each of the experiments. Tables 1, 2, 3, and 4 summarize the results of the Monte Carlo experiments. Each row displays the mean values and standard errors (in the parentheses) of estimates across 300 repetitions in an experiment. Table 1 uses large size variation \( (m_r \) is \( N(10, 3^2) \) or \( N(20, 6^2) \)), and \( x_{1,ir} \) and \( x_{2,ir} \) are independently drawn. Table 2 also uses large size variation but sets \( x_{1,ir} = x_{2,ir} \). Table 3 uses small size variation \( (m_r \) is \( N(10, 1.5^2) \) or \( N(20, 3^2) \)), and \( x_{1,ir} \) and \( x_{2,ir} \) are independently drawn. Table 4 uses small size variation, and assumes \( x_{1,ir} = x_{2,ir} \). QML estimates are in Columns 2-8, and CML estimates are in Columns 9-12 as a comparison. True values of the parameters are on the top of the panel. Our discussion about the simulation results will focus on estimates of \( \lambda \). This is our key parameter of interest. As we can see, the performance of the estimates of \( \lambda \) is representative of the overall performance of the QML and the CML estimators.

As expected, the QML estimator converges to the true parameter values when the number of groups increases. Within each table, both bias and standard errors of the QML estimates decrease as \( R \) increases. In all four tables, QML estimates have significantly smaller bias and standard errors than CML estimates. On average of all experiments, CML estimates have 10 times larger bias and 7 times larger standard errors than the QML estimates. While the CML estimator is generally biased upwards and can therefore overestimate peer effects, the QML estimator is in general biased downwards and thus more conservative.

All else equal, the QML estimator has less bias and smaller standard errors when \( x_{1,ir} \) and \( x_{2,ir} \) are distinct than when they are identical. Estimates in Tables 1 and 3 perform better than those in Tables 2 and 4 respectively. Technically, including \( x_{2,ir} \) as a regressor hinders identification because

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\[ ^{10} \text{These consist only around 0.2\% of all groups. For example, when there are 800 groups, less than 2 groups are dropped on average.} \]
peer characteristics $\bar{x}_{2,(-i)r}$ and peer outcomes $\bar{y}_{(-i)r}$ are highly collinear in this case. This reflects the difficulty of distinguishing endogenous peer effects $\lambda$ from exogenous peer effects $\beta_3$. Still, with enough variation in group size and sufficient number of groups, the QML estimator performs well even when $x_{1,ir} = x_{2,ir}$.

The distribution of group size plays a significant role in the performance of the QML estimator. Bias and standard errors of the QML estimator are positively related to the average group size and negatively related to the standard deviation of group size. This is evident from a comparison between Table 1 and 3 or between Table 2 and 4. For example, comparing the estimates when $m_r$ is $N(10, 3)$ in Table 1 and when $m_r$ is $N(10, 1.5^2)$ in Table 3, the former have smaller standard errors and less bias. Holding everything else constant, the QML estimator performs better when $m_r$ is $N(10, 3)$ than when $m_r$ is $N(20, 3)$. This reveals that when the standard deviation of group size is fixed, identification improves as the average group size declines.

In all, the bias and standard errors of the QML estimator decrease as $R$ increases, the mean group size decreases, the standard deviation of group size increases, and when $x_{1,ir}$ and $x_{2,ir}$ are different. The QML estimator is less biased and more efficient than the CML estimator, regardless of the value of $R$, the distribution of group size, and the relation between $x_{1,ir}$ and $x_{2,ir}$. The QML estimator generally works well in small samples, unless we have the worst case scenario: small variation in group size, $x_{1,ir} = x_{2,ir}$, and a small ($R \leq 200$) number of groups. The top panels in Table 4 show that the estimates are significantly smaller than the true values in such a case.

6 Conclusion

In this paper, we show that moment conditions underlying the conditional variance method of Graham (2008) lead to a general class of linear peer effects models with random group effects. Random group effects in the context of peer effects models are appropriate in settings where people are randomly assigned to groups. We show that Graham’s estimator can be interpreted as a special case of a more general class of GMM estimators. We show that the quasi maximum likelihood estimator (QMLE) related to a linear Gaussian specification, as well as the fixed effects estimator of Lee (2007) are contained in the class of GMM estimators. Under Gaussian error assumptions the QMLE is the most efficient estimator in this class. We study conditions of identification, extending results in Graham (2008) and Lee (2007) for a simple model without covariates and a general model with covariates estimated by QML. We also establish that our QMLE is asymptotically normal and we construct consistent standard error formulas. Monte Carlo results show that our QML estimator has good small sample properties.
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Appendix

A Figures

Figure 1: Excess Variance v.s. Variance of $\bar{\epsilon}_r$

Note: [1] Variance of the group mean of $\epsilon$ when group size is $m$ on the x-axis.
Excess variance, defined as between group variance – variance of the group mean of $\epsilon$, on the y-axis.
[2] As $m$ increases, $\sigma_{\epsilon}^2/m$ gets closer to the origin.

Figure 2: Logarithm of Within-Group Variance v.s. Group Size

Note: [1] The logarithm of within-group variance is $\ln(\sigma_{\epsilon}^2 - 2\ln(1.9)) - 2\ln(\sigma_{m}^2/m) - 2\ln(1+\lambda) - 2\ln(1+\lambda - 1) - 2\ln(1+\lambda - 1 - 1) - 2\ln(1+\lambda - 1 - 1 - 1)$. The intercept at $m=2$ is therefore $\ln(\sigma_{\epsilon}^2 - 2\ln(1.9))$.
[2] As group size $m$ goes to infinity, the logarithm of within-group variance goes to $\ln(\sigma_{\epsilon}^2)$ for all $\lambda$. 

### B Monte Carlo Simulation Results

Table 1: Monte Carlo Results: Large Variation in Group Size, $x_1 \neq x_2$

| $m_r$ | QMLE for random group effects model | CMLE by Lee(2007) |
|-------|------------------------------------|-------------------|
|       | $\lambda$ | $\sigma_\epsilon$ | $\sigma_\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\lambda$ | $\sigma_\epsilon$ | $\beta_2$ | $\beta_3$ |
|       | True value | 0.500 | 1.000 | 0.500 | 1.000 | 0.500 | 1.000 | 0.500 | 1.000 |
|       | $\sigma_\alpha$ | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
|       | $\beta_1$ | 0.500 | 1.000 | 0.500 | 1.000 | 0.500 | 1.000 |
|       | $\beta_2$ | 0.500 | 1.000 | 0.500 | 1.000 | 0.500 | 1.000 |
|       | $\beta_3$ | 0.500 | 1.000 | 0.500 | 1.000 | 0.500 | 1.000 |
|       | $\beta_4$ | 0.500 | 1.000 | 0.500 | 1.000 | 0.500 | 1.000 |

#### Number of Groups $R = 50$

| $N(10,3^2)$ | 0.479 | 0.996 | 0.481 | 1.048 | 0.998 | 0.990 | 1.043 | 0.616 | 1.025 | 1.012 | 0.994 |
|             | (0.124) | (0.040) | (0.188) | (0.282) | (0.046) | (0.267) | (0.266) | (0.828) | (0.193) | (0.096) | (0.453) |
| $N(20,6^2)$ | 0.471 | 0.997 | 0.499 | 1.064 | 0.998 | 1.027 | 1.052 | 0.621 | 1.011 | 1.005 | 1.013 |
|             | (0.181) | (0.025) | (0.229) | (0.378) | (0.032) | (0.355) | (0.373) | (0.939) | (0.106) | (0.056) | (0.573) |

#### Number of Groups $R = 100$

| $N(10,3^2)$ | 0.495 | 0.997 | 0.484 | 1.014 | 1.002 | 1.020 | 1.010 | 0.588 | 1.015 | 1.012 | 1.023 |
|             | (0.080) | (0.024) | (0.126) | (0.174) | (0.035) | (0.196) | (0.175) | (0.530) | (0.120) | (0.064) | (0.314) |
| $N(20,6^2)$ | 0.481 | 0.999 | 0.508 | 1.034 | 0.998 | 1.030 | 1.038 | 0.602 | 1.011 | 1.004 | 1.039 |
|             | (0.105) | (0.017) | (0.130) | (0.211) | (0.023) | (0.242) | (0.216) | (0.760) | (0.086) | (0.045) | (0.390) |

#### Number of Groups $R = 200$

| $N(10,3^2)$ | 0.501 | 1.000 | 0.494 | 0.999 | 1.001 | 1.003 | 0.999 | 0.571 | 1.016 | 1.008 | 1.012 |
|             | (0.050) | (0.017) | (0.079) | (0.104) | (0.024) | (0.123) | (0.111) | (0.355) | (0.081) | (0.044) | (0.197) |
| $N(20,6^2)$ | 0.486 | 0.999 | 0.508 | 1.029 | 0.998 | 1.008 | 1.028 | 0.468 | 0.996 | 0.997 | 0.982 |
|             | (0.076) | (0.012) | (0.100) | (0.160) | (0.017) | (0.187) | (0.159) | (0.550) | (0.058) | (0.033) | (0.324) |

#### Number of Groups $R = 400$

| $N(10,3^2)$ | 0.496 | 0.999 | 0.500 | 1.008 | 0.999 | 1.001 | 1.008 | 0.514 | 1.003 | 1.001 | 1.003 |
|             | (0.038) | (0.013) | (0.056) | (0.083) | (0.016) | (0.090) | (0.082) | (0.255) | (0.057) | (0.034) | (0.146) |
| $N(20,6^2)$ | 0.500 | 1.000 | 0.497 | 1.001 | 1.000 | 0.995 | 0.999 | 0.515 | 1.002 | 1.001 | 0.980 |
|             | (0.042) | (0.008) | (0.054) | (0.091) | (0.012) | (0.121) | (0.089) | (0.334) | (0.038) | (0.021) | (0.200) |

#### Number of Groups $R = 800$

| $N(10,3^2)$ | 0.501 | 1.000 | 0.497 | 1.002 | 0.999 | 1.000 | 1.000 | 0.498 | 0.999 | 0.999 | 0.999 |
|             | (0.027) | (0.009) | (0.042) | (0.057) | (0.012) | (0.064) | (0.055) | (0.168) | (0.039) | (0.021) | (0.108) |
| $N(20,6^2)$ | 0.499 | 0.999 | 0.499 | 1.003 | 1.001 | 1.006 | 1.004 | 0.523 | 1.001 | 1.003 | 1.005 |
|             | (0.031) | (0.006) | (0.039) | (0.067) | (0.009) | (0.083) | (0.064) | (0.263) | (0.029) | (0.015) | (0.146) |

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1. Means and standard errors (in the parentheses) of estimates across 300 repetitions.
2. Columns 2-8 show the quasi-maximum likelihood estimator (QMLE) in our paper. As a comparison, Columns 9-12 show the conditional maximum likelihood estimator (CMLE) for the fixed group effects model by Lee(2007). The CMLE is based on the within-group variation hence $\sigma_\alpha, \beta_1, \beta_4$ are not estimated.
3. Simulation is based on model (19): $y_{ir} = \beta_1 + \lambda y_{(i-1)r} + x_{1,ir} \beta_2 + \bar{x}_{2,(i-1)r} \beta_3 + x_{3,r} \beta_4 + \alpha_r + \epsilon_{ir}$, with the true parameter values on the top panel of the table.
4. Group size $m_r$ is drawn from $N(10,3^2)$ or $N(20,6^2)$ and rounded to the nearest integer.
5. Sample is generated by: $x_{1,ir} \sim N(0,1)$, $x_{2,ir} \sim N(0,1)$, $x_{3,r} \sim N(0,1)$, $\alpha_r \sim N(0,0.5^2)$, and $\epsilon_{ir} \sim N(0,1)$. All variables are independent of each other across $i$ and $r$. 

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Table 2: Monte Carlo Results: Large Variation in Group Size, $x_1 = x_2$

| $m_r$ | QMLE for random group effects model | CMLE by Lee(2007) |
|------|-----------------------------------|------------------|
|      | $\lambda$ | $\sigma_\epsilon$ | $\sigma_\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\lambda$ | $\sigma_\epsilon$ | $\beta_2$ | $\beta_3$ |
| True value | 0.500 | 1.000 | 0.500 | 1.000 | 1.000 | 1.000 | 1.000 |
| Number of Groups $R = 50$ |
| N(10, 3$^2$) | 0.442 | 0.991 | 0.506 | 1.113 | 1.015 | 1.172 | 1.112 | 0.500 | 1.000 | 1.000 | 1.000 |
| (0.228) | (0.040) | (0.341) | (0.459) | (0.088) | (0.814) | (0.466) | (1.028) | (0.232) | (0.153) | (1.417) |
| N(20, 6$^2$) | 0.447 | 0.996 | 0.508 | 1.113 | 1.007 | 1.190 | 1.095 | 0.777 | 1.028 | 0.997 | 0.716 |
| (0.293) | (0.029) | (0.373) | (0.597) | (0.056) | (1.151) | (0.578) | (1.240) | (0.136) | (0.091) | (1.931) |
| Number of Groups $R = 100$ |
| N(10, 3$^2$) | 0.487 | 0.996 | 0.479 | 1.029 | 1.007 | 1.042 | 1.028 | 0.723 | 1.045 | 1.000 | 0.794 |
| (0.163) | (0.030) | (0.255) | (0.329) | (0.062) | (0.573) | (0.335) | (0.711) | (0.161) | (0.100) | (0.966) |
| N(20, 6$^2$) | 0.451 | 0.997 | 0.533 | 1.094 | 1.005 | 1.166 | 1.101 | 0.665 | 1.018 | 1.003 | 0.927 |
| (0.235) | (0.020) | (0.289) | (0.468) | (0.041) | (0.867) | (0.485) | (0.940) | (0.103) | (0.068) | (1.378) |
| Number of Groups $R = 200$ |
| N(10, 3$^2$) | 0.493 | 0.999 | 0.500 | 1.013 | 1.004 | 1.032 | 1.013 | 0.610 | 1.025 | 1.006 | 0.963 |
| (0.095) | (0.019) | (0.142) | (0.191) | (0.040) | (0.359) | (0.190) | (0.432) | (0.098) | (0.065) | (0.544) |
| N(20, 6$^2$) | 0.468 | 0.998 | 0.524 | 1.065 | 1.003 | 1.100 | 1.062 | 0.470 | 0.997 | 0.998 | 1.005 |
| (0.144) | (0.014) | (0.173) | (0.293) | (0.028) | (0.546) | (0.286) | (0.618) | (0.065) | (0.049) | (0.895) |
| Number of Groups $R = 400$ |
| N(10, 3$^2$) | 0.493 | 0.999 | 0.501 | 1.013 | 1.001 | 1.018 | 1.013 | 0.535 | 1.007 | 1.000 | 0.972 |
| (0.077) | (0.014) | (0.110) | (0.157) | (0.030) | (0.271) | (0.155) | (0.299) | (0.066) | (0.051) | (0.404) |
| N(20, 6$^2$) | 0.485 | 0.999 | 0.514 | 1.031 | 1.002 | 1.052 | 1.028 | 0.545 | 1.005 | 0.999 | 0.940 |
| (0.086) | (0.009) | (0.104) | (0.175) | (0.017) | (0.316) | (0.174) | (0.436) | (0.047) | (0.032) | (0.694) |
| Number of Groups $R = 800$ |
| N(10, 3$^2$) | 0.498 | 0.999 | 0.500 | 1.007 | 1.000 | 1.010 | 1.005 | 0.509 | 1.002 | 0.997 | 0.981 |
| (0.050) | (0.010) | (0.074) | (0.102) | (0.020) | (0.187) | (0.101) | (0.209) | (0.047) | (0.033) | (0.289) |
| N(20, 6$^2$) | 0.498 | 0.999 | 0.500 | 1.004 | 1.002 | 1.005 | 1.005 | 0.533 | 1.002 | 1.002 | 0.983 |
| (0.053) | (0.006) | (0.066) | (0.108) | (0.012) | (0.220) | (0.109) | (0.316) | (0.034) | (0.023) | (0.454) |

1. Means and standard errors (in the parentheses) of estimates across 300 repetitions.
2. Columns 2-8 show the quasi-maximum likelihood estimator (QMLE) in our paper. As a comparison, Columns 9-12 show the conditional maximum likelihood estimator (CMLE) for the fixed group effects model by Lee(2007). The CMLE is based on the within-group variation hence $\sigma_\alpha, \beta_1, \beta_4$ are not estimated.
3. Simulation is based on model (19): $y_{ir} = \beta_1 + \lambda \bar{y}_{(-i)r} + x_{1,ir} \beta_2 + \bar{x}_{2,(-i)r} \beta_3 + x_{3,r} \beta_4 + \alpha_r + \epsilon_{ir}$, with the true parameter values on the top panel of the table.
4. Group size $m_r$ is drawn from $N(10, 3^2)$ or $N(20, 6^2)$ and rounded to the nearest integer.
5. Sample is generated by: $x_{1,ir} = x_{2,ir} \sim N(0, 1), x_{3,r} \sim N(0, 1), \alpha_r \sim N(0, 0.5^2)$, and $\epsilon_{ir} \sim N(0, 1)$. All variables are independent of each other across $i$ and $r$ except that $x_{1,ir} = x_{2,ir}$.
| $m_r$          | QMLE for random group effects model | CMLE by Lee(2007) |
|---------------|----------------------------------|------------------|
|               | $\lambda$ $\sigma_t$ $\sigma_\alpha$ $\beta_1$ $\beta_2$ $\beta_3$ $\beta_4$ | $\lambda$ $\sigma_t$ $\beta_2$ $\beta_3$ |
| **True value** | $0.500$ $1.000$ $0.500$ $1.000$ $1.000$ $1.000$ $1.000$ | $0.500$ $1.000$ $1.000$ $1.000$ |
| Number of Groups $R = 50$ | | |
| $N(10, 1.5^2)$ | $0.456$ $0.994$ $0.520$ $1.097$ $0.994$ $1.023$ $1.093$ | $0.935$ $1.149$ $1.043$ $1.062$ |
| (0.231) (0.041) (0.325) (0.443) (0.053) (0.307) (0.489) | (2.474) (0.940) (0.254) (0.489) |
| $N(20, 3^2)$ | $0.457$ $0.997$ $0.515$ $1.084$ $0.997$ $1.050$ $1.087$ | $0.997$ $1.068$ $1.024$ $1.084$ |
| (0.205) (0.025) (0.251) (0.410) (0.034) (0.403) (0.413) | (2.761) (0.322) (0.144) (0.697) |
| Number of Groups $R = 100$ | | |
| $N(10, 1.5^2)$ | $0.487$ $0.997$ $0.499$ $1.026$ $0.997$ $1.003$ $1.018$ | $0.812$ $1.078$ $1.032$ $1.018$ |
| (0.084) (0.026) (0.125) (0.180) (0.033) (0.198) (0.176) | (1.226) (0.275) (0.130) (0.344) |
| $N(20, 3^2)$ | $0.483$ $0.999$ $0.506$ $1.033$ $0.998$ $1.032$ $1.037$ | $0.591$ $1.016$ $1.004$ $1.017$ |
| (0.108) (0.016) (0.136) (0.220) (0.022) (0.413) (0.227) | (1.683) (0.179) (0.089) (0.448) |
| Number of Groups $R = 200$ | | |
| $N(10, 1.5^2)$ | $0.493$ $1.001$ $0.500$ $1.015$ $0.999$ $1.007$ $1.015$ | $0.564$ $1.022$ $1.006$ $1.018$ |
| (0.055) (0.018) (0.084) (0.119) (0.024) (0.140) (0.116) | (0.747) (0.164) (0.083) (0.236) |
| $N(20, 3^2)$ | $0.494$ $0.999$ $0.498$ $1.018$ $0.997$ $1.005$ $1.012$ | $0.600$ $1.012$ $1.002$ $0.991$ |
| (0.067) (0.014) (0.083) (0.144) (0.016) (0.181) (0.142) | (1.179) (0.123) (0.064) (0.300) |
| Number of Groups $R = 400$ | | |
| $N(10, 1.5^2)$ | $0.502$ $1.000$ $0.491$ $0.996$ $1.000$ $1.001$ $0.996$ | $0.561$ $1.014$ $1.006$ $0.992$ |
| (0.038) (0.012) (0.058) (0.081) (0.016) (0.099) (0.086) | (0.517) (0.111) (0.055) (0.161) |
| $N(20, 3^2)$ | $0.494$ $1.000$ $0.505$ $1.110$ $0.999$ $1.014$ $1.013$ | $0.501$ $1.002$ $1.000$ $1.009$ |
| (0.048) (0.008) (0.062) (0.098) (0.012) (0.141) (0.097) | (0.831) (0.088) (0.044) (0.227) |
| Number of Groups $R = 800$ | | |
| $N(10, 1.5^2)$ | $0.497$ $1.000$ $0.503$ $1.007$ $1.000$ $1.006$ $1.003$ | $0.526$ $1.008$ $1.003$ $1.007$ |
| (0.026) (0.009) (0.040) (0.055) (0.011) (0.070) (0.055) | (0.370) (0.082) (0.040) (0.117) |
| $N(20, 3^2)$ | $0.500$ $0.999$ $0.499$ $1.001$ $1.000$ $1.000$ $1.000$ | $0.556$ $1.005$ $1.003$ $0.994$ |
| (0.030) (0.006) (0.038) (0.063) (0.008) (0.082) (0.062) | (0.508) (0.054) (0.027) (0.155) |

1. Means and standard errors (in the parentheses) of estimates across 300 repetitions.
2. Columns 2-8 show the quasi-maximum likelihood estimator (QMLE) in our paper. As a comparison, Columns 9-12 show the conditional maximum likelihood estimator (CMLE) for the fixed group effects model by Lee(2007). The CMLE is based on the within-group variation hence $\sigma_\alpha, \beta_1, \beta_4$ are not estimated.
3. Simulation is based on model (19): $y_{ir} = \beta_1 + \lambda \bar{y}_{(i)r} + x_{1,ir} \beta_2 + \bar{x}_{2,(i)r} \beta_3 + x_{3,r} \beta_4 + \alpha_r + \epsilon_{ir}$, with the true parameter values on the top panel of the table.
4. Group size $m_r$ is drawn from $N(10, 1.5^2)$ or $N(20, 3^2)$ and rounded to the nearest integer.
5. Sample is generated by: $x_{1,ir} \sim N(0, 1)$, $x_{2,ir} \sim N(0, 1)$, $x_{3,r} \sim N(0, 1)$, $\alpha_r \sim N(0, 0.5^2)$, and $\epsilon_{ir} \sim N(0, 1)$. All variables are independent of each other across $i$ and $r$. 

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Table 4: Monte Carlo Results: Small Variation in Group Size, $x_1 = x_2$

| $m_r$ | QMLE for random group effects model | CMLE by Lee(2007) |
|-------|-----------------------------------|------------------|
|       | $\lambda$ | $\sigma_x$ | $\sigma_\alpha$ | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\lambda$ | $\sigma_x$ | $\beta_2$ | $\beta_3$ |
|       |       |       |       |       |       |       |       |       |       |       |       |
|       |       |       |       |       |       |       |       |       |       |       |       |
|       | True value |       |       |       |       |       |       |       |       |       |       |       |
|       | 0.500 | 1.000 | 0.500 | 1.000 | 1.000 | 1.000 | 1.000 | 0.500 | 1.000 | 1.000 | 1.000 |       |
| Number of Groups $R = 50$ |       |       |       |       |       |       |       |       |       |       |       |       |
| $N(10, 1.5^2)$ | 0.355 | 0.983 | 0.594 | 1.307 | 1.042 | 1.509 | 1.296 | 1.148 | 1.212 | 1.016 | 0.616 |       |
|       | (0.440) | (0.059) | (0.599) | (0.910) | (0.143) | (1.589) | (0.886) | (2.743) | (0.833) | (4.629) |       |
| $N(20, 3^2)$ | 0.297 | 0.989 | 0.677 | 1.397 | 1.031 | 1.785 | 1.413 | 1.163 | 1.100 | 1.017 | 0.731 |       |
|       | (0.576) | (0.038) | (0.655) | (1.133) | (0.097) | (2.239) | (1.170) | (3.597) | (0.329) | (3.111) |       |
| Number of Groups $R = 100$ |       |       |       |       |       |       |       |       |       |       |       |       |
| $N(10, 1.5^2)$ | 0.362 | 0.984 | 0.627 | 1.275 | 1.039 | 1.472 | 1.268 | 0.861 | 1.100 | 1.039 | 1.052 |       |
|       | (0.334) | (0.042) | (0.464) | (0.678) | (0.105) | (1.174) | (0.670) | (1.603) | (0.024) | (2.141) |       |
| $N(20, 3^2)$ | 0.382 | 0.994 | 0.609 | 1.234 | 1.017 | 1.440 | 1.237 | 0.621 | 1.022 | 1.004 | 0.987 |       |
|       | (0.394) | (0.025) | (0.458) | (0.779) | (0.066) | (1.500) | (0.801) | (2.047) | (0.146) | (3.118) |       |
| Number of Groups $R = 200$ |       |       |       |       |       |       |       |       |       |       |       |       |
| $N(10, 1.5^2)$ | 0.455 | 0.997 | 0.533 | 1.090 | 1.013 | 1.163 | 1.092 | 0.591 | 1.032 | 1.004 | 0.960 |       |
|       | (0.185) | (0.027) | (0.270) | (0.372) | (0.063) | (0.698) | (0.373) | (0.971) | (0.128) | (1.343) |       |
| $N(20, 3^2)$ | 0.467 | 0.998 | 0.526 | 1.071 | 1.002 | 1.125 | 1.066 | 0.626 | 1.017 | 1.003 | 0.983 |       |
|       | (0.184) | (0.016) | (0.225) | (0.371) | (0.034) | (0.739) | (0.367) | (1.525) | (0.098) | (2.231) |       |
| Number of Groups $R = 400$ |       |       |       |       |       |       |       |       |       |       |       |       |
| $N(10, 1.5^2)$ | 0.480 | 0.997 | 0.515 | 1.041 | 1.007 | 1.076 | 1.040 | 0.594 | 1.023 | 1.001 | 0.932 |       |
|       | (0.118) | (0.018) | (0.167) | (0.240) | (0.041) | (0.441) | (0.238) | (0.641) | (0.096) | (0.975) |       |
| $N(20, 3^2)$ | 0.486 | 0.999 | 0.514 | 1.029 | 1.001 | 1.046 | 1.030 | 0.551 | 1.008 | 0.996 | 0.880 |       |
|       | (0.105) | (0.010) | (0.122) | (0.212) | (0.020) | (0.418) | (0.207) | (0.998) | (0.073) | (1.593) |       |
| Number of Groups $R = 800$ |       |       |       |       |       |       |       |       |       |       |       |       |
| $N(10, 1.5^2)$ | 0.496 | 1.000 | 0.502 | 1.011 | 1.001 | 1.010 | 1.007 | 0.568 | 1.017 | 0.997 | 0.910 |       |
|       | (0.082) | (0.012) | (0.117) | (0.165) | (0.028) | (0.299) | (0.165) | (0.480) | (0.067) | (0.693) |       |
| $N(20, 3^2)$ | 0.501 | 1.000 | 0.497 | 0.999 | 1.000 | 0.995 | 0.998 | 0.612 | 1.012 | 0.998 | 0.856 |       |
|       | (0.047) | (0.006) | (0.058) | (0.097) | (0.011) | (0.198) | (0.096) | (0.655) | (0.050) | (1.146) |       |

1. Means and standard errors (in the parentheses) of estimates across 300 repetitions.
2. Columns 2-8 show the quasi-maximum likelihood estimator (QMLE) in our paper. As a comparison, Columns 9-12 show the conditional maximum likelihood estimator (CMLE) for the fixed group effects model by Lee(2007). The CMLE is based on the within-group variation hence $\sigma_\alpha, \beta_1, \beta_4$ are not estimated.
3. Simulation is based on model \[ y_{ir} = \beta_1 + \lambda y_{(i-1)r} + x_{1,ir} \beta_2 + x_{2,(i-1)r} \beta_3 + x_{3,r} \beta_4 + \alpha_r + \epsilon_{ir}, \] with the true parameter values on the top panel of the table.
4. Group size $m_r$ is drawn from $N(10, 1.5^2)$ or $N(20, 3^2)$ and rounded to the nearest integer.
5. Sample is generated by: $x_{1,ir} ~ N(0, 1), x_{3,r} ~ N(0, 1), \alpha_r ~ N(0, 0.5^2),$ and $\epsilon_{ir} ~ N(0, 1)$. All variables are independent of each other across $i$ and $r$ except that $x_{1,ir} = x_{2,ir}$. 

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## C Preliminaries

In proving consistency and asymptotic normality of the QMLE estimator we encounter linear quadratic forms of the form

$$ S_N(\theta) = U' A_N(\theta) U + U' a_N(\theta) $$

where $A_N(\theta)$ is an $N \times N$ non-stochastic matrix, $a_N(\theta)$ is an $N$–dimensional non-stochastic column vector, and where $A_N(\theta)$ and $a_N(\theta)$ exhibit some special structures. In the following we describe that structure in more detail, and collect some basic lemmata used in proving the consistency and asymptotic normality of the QMLE.

We adopt the following notation: Partition an $N \times N$ matrix $A_N$ into $R \times R$ submatrices, with the $(r,r')$-th submatrix being an $m_r \times m_{r'}$ matrix, $r, r' = 1, \ldots, R$. We then denote the $(r,r')$-th submatrix of $A_N$ as $A_{(r,r'),N}$, and the $(i,j)$-th element of $A_N$ as $a_{ij,N}$, $1 \leq i \leq N$, $1 \leq j \leq N$. Partition an $N \times 1$ vector $a_N$ into $R$ subvectors, with the $r$-th subvector being an $m_r \times 1$ vector. We then denote the $r$-th subvector of $a_N$ as $a_{(r),N}$ and the $i$-th element of $a_N$ as $a_{i,N}$. In line with Kelejian and Prucha (2001), we call the column and row sums of an $N \times N$ matrix $A_N(\theta)$ uniformly bounded in absolute value if there exists some finite constant $C$ (which does not depend on $N$ or $\theta$) such that

$$ \sup_{\theta \in \Theta} \sum_{i=1}^{N} |a(\theta)_{ij,N}| \leq C, \quad \sup_{\theta \in \Theta} \sum_{j=1}^{N} |a(\theta)_{ij,N}| \leq C. $$

A corresponding definition applies to rectangular matrices. Of course, if the row sums of $A_N(\theta)$ are uniformly bounded in absolute value, and the elements of $a_N(\theta)$ are uniformly bounded in absolute value, then the elements of $A_N(\theta) a_N(\theta)$ are uniformly bounded in absolute value. Note that if the row and column sums of $A(\theta)$ and $B(\theta)$ are uniformly bounded in absolute value, then $A(\theta) + B(\theta)$ and $A(\theta) B(\theta)$ (if dimension permits addition or multiplication) also have row and column sums uniformly bounded in absolute value.\footnote{This is readily seen by argumentation in line with Kelejian and Prucha (1999).}

### C.1 Basic Properties of Matrices Forming the Log-Likelihood Function

Recall that $\theta = (\theta_1, \theta_2, \theta_3)' \in \Theta$ with $\theta_1 = \lambda$, $\theta_2 = \sigma_v^2$, $\theta_3 = \sigma_a^2$, and that in light of Assumptions 2, 3 and 4 the parameter space $\Theta$ is compact. Furthermore, there exists an open bounded set $\Theta_o$ such that $\Theta \subset \Theta_o \subset (-1,1) \times R^2$. An inspection of the expression of the log-likelihood function shows that it depends on the following set of matrices: $I - \lambda W$, $(I - \lambda W)^{-1}$, $\Omega(\theta)$, $\Omega(\theta)^{-1}$, $W$. All those matrices are symmetric block diagonal matrices of the form, say,

$$ A_N(\theta) = diag_{r=1}^{R} \{ p(m_r, \theta) I_{m_r}^* + s(m_r, \theta) J_{m_r}^* \}. $$

\footnote{This is readily seen by argumentation in line with Kelejian and Prucha (1999).}
In particular,

\[ I - \lambda W = \text{diag}_{r=1}^{R} \{ \phi_S(m_r, \theta) I_{m_r}^* + \psi_S(m_r, \theta) J_{m_r}^* \} \quad (C.3) \]

\[ (I - \lambda_0 W)^{-1} = \text{diag}_{r=1}^{R} \{ \phi_S^{-1}(m_r, \theta_0) I_{m_r}^* + \psi_S^{-1}(m_r, \theta_0) J_{m_r}^* \}, \]

\[ \Omega_0 = \text{diag}_{r=1}^{R} \{ \phi_{\Omega}(m_r, \theta_0) I_{m_r}^* + \psi_{\Omega}(m_r, \theta_0) J_{m_r}^* \}, \]

\[ \Omega(\theta)^{-1} = \text{diag}_{r=1}^{R} \{ \phi_{\Omega}^{-1}(m_r, \theta) I_{m_r}^* + \psi_{\Omega}^{-1}(m_r, \theta) J_{m_r}^* \}, \]

\[ W = \text{diag}_{r=1}^{R} \{ \phi_W(m_r, \theta) I_{m_r}^* + \psi_W(m_r, \theta) J_{m_r}^* \}, \]

with

\[ \phi_S(m_r, \theta) = \frac{m_r - 1 + \alpha}{m_r}, \quad \psi_S(m_r, \theta) = 1 - \lambda, \]

\[ \phi_{\Omega}(m_r, \theta) = \sigma_{\epsilon}, \quad \psi_{\Omega}(m_r, \theta) = \sigma_{\epsilon} + m_r \sigma_{\alpha}. \quad (C.4) \]

The functions \( p(m_r, \theta) \) and \( s(m_r, \theta) \) are continuously differentiable on \( \Theta \). Thus, by Bolzano-Weierstrass’ extreme value theorem there exists a positive constant \( C \), which does not depend on \( \theta \), such that

\[ 0 \leq |p(m_r, \theta)|, |s(m_r, \theta)|, |\partial p(m_r, \theta)/\partial \theta_i|, |\partial s(m_r, \theta)/\partial \theta_i| \leq C < \infty, \quad (C.5) \]

for all \( \theta \in \Theta \). \footnote{Of course, since the \( m_r \) only takes on finitely many values, the constants can also be taken such that they do not depend on \( m_r \). We note, although not stated explicitly, all subsequent uniformity results also hold uniformly for \( m_r \in \{m : 2 \leq m \leq \bar{a} \} \).} This implies that \( p(m_r, \theta) \) and \( s(m_r, \theta) \) are both uniformly continuous on \( \Theta \). Observing that \( \phi_{\Omega}(m_r, \theta) \) and \( \psi_{\Omega}(m_r, \theta) \) are positive on \( \Theta \) it follows further that there exists a positive constant \( c \), which does not depend on \( \theta \), such that

\[ 0 < c \leq \phi_{\Omega}(m_r, \theta), \psi_{\Omega}(m_r, \theta) \leq C < \infty. \quad (C.6) \]

Since \( I_{m_r}^* \) and \( J_{m_r}^* \) are orthogonal, the multiplication of block diagonal matrices, where the blocks are of the form \( p(m_r, \theta) I_{m_r}^* + s(m_r, \theta) J_{m_r}^* \), yields a matrix with the same structure. Furthermore the multiplication of those matrices is commutative. More specifically, let

\[ A_N(\theta) = \text{diag}_{r=1}^{R} \{ p(m_r, \theta) I_{m_r}^* + s(m_r, \theta) J_{m_r}^* \} \]

and \( \tilde{A}_N(\theta) = \text{diag}_{r=1}^{R} \{ \tilde{p}(m_r, \theta) I_{m_r}^* + \tilde{s}(m_r, \theta) J_{m_r}^* \}, \) then

\[ A_N(\theta)\tilde{A}_N(\theta) = \text{diag}_{r=1}^{R} \{ p(m_r, \theta)\tilde{p}(m_r, \theta) I_{m_r}^* + s(m_r, \theta)\tilde{s}(m_r, \theta) J_{m_r}^* \} . \]

Also,

\[ |A_N(\theta)| = \prod_{m=2}^{\tilde{a}} |p(m, \theta) I_{m}^* + s(m, \theta) J_{m}^*|^{R_m} = \prod_{m=2}^{\tilde{a}} \left[ p(m, \theta)^{m-1}s(m, \theta) \right]^{R_m}, \quad (C.7) \]
as is readily checked observing that \( pI_m^* + sJ_m^* = p\{I_m + [(s-p)/(pm)]t_{m't_m}\} \) and applying Proposition 31 in Dhrymes (1978) to compute the determinant of the matrix in curly brackets. Furthermore,

\[
tr[A_N(\theta)] = \sum_{m=2}^\alpha R_m[(m - 1)p(m, \theta) + s(m, \theta)].
\]

and

\[
\frac{1}{N} Z' A_N(\theta) Z = \frac{1}{N} \sum_{r=1}^R \left[ p(m_r, \theta)Z_r I_m^* Z_r + s(m_r, \theta)Z_r J_m^* Z_r \right] \quad \text{(C.8)}
\]

We note that the row and column sums of any matrix \( A_N(\theta) \) of the form (C.2) are uniformly bounded in absolute value, if \( p(m_r, \theta) \) and \( s(m_r, \theta) \) are positive and bounded away from zero, then also the elements of \( [N^{-1}Z'\Omega(\theta)Z]^{-1} \) are uniformly bounded; see Lemma C.5. Consequently the elements of \( [N^{-1}Z'\Omega(\theta)Z]^{-1} \) are uniformly bounded, and the row and column sums of \( Z'(Z'\Omega(\theta)Z)^{-1}Z' = N^{-1}Z[N^{-1}Z'^*\Omega(\theta)Z]^{-1}Z', \quad \Omega(\theta)^{-1}Z[N^{-1}Z'^*\Omega(\theta)Z]^{-1}Z\Omega(\theta)^{-1} \) and \( M_Z(\theta) \) are uniformly bounded in absolute value. As a result, \( M_Z(\theta) \) and \( \partial M_Z(\theta)/\partial \theta_i = -M_Z(\theta)[\partial \Omega(\theta)/\partial \theta_i]M_Z(\theta) \) have row and column sums uniformly bounded in absolute value.

In all, if a matrix \( A_N(\theta) \) is the product of \( I - \lambda W, \quad (I - \lambda W)^{-1}, \quad \Omega(\theta)^{-1}, \quad W, \quad \partial \Omega(\theta)/\partial \theta_i, \quad \) and \( M_Z(\theta), \) then both \( A_N(\theta) \) and \( \partial A_N(\theta)/\partial \theta_i \) have row and column sums uniformly bounded in absolute value, and the elements of \( A_N(\theta)Z/\theta_0 \) are uniformly bounded in absolute value over \( \theta \in \Theta \) and \( N \).

### C.2 Limit Theorems for Linear Quadratic Forms in \( U \)

The following result follows trivially from Lemma A.1 in Kelejian and Prucha (2010), and is only given for the convenience of the reader.

**Lemma C.1.** [Mean and Covariance] Let \( A \) and \( B \) be \( N \times N \) nonstochastic symmetric matrices, which are partitioned into \( R^2 \) submatrices and let \( a \) and \( b \) be \( N \times 1 \) vectors, which are conformably partitioned into \( R \) subvectors. Let \( a_{ij} \) and \( b_{ij} \) denote the \((i, j)\)-th element of \( A \) and \( B \), let \( A_{(r,r')} \) and \( B_{(r,r')} \) denote the \((r, r')\)-th block of dimension \( m_r \times m_{r'} \), let \( a_i \) and \( b_i \) denote the \( i \)-th element of \( a \) and \( b \), and \( a_{(r)} \) and \( b_{(r)} \) denote the \( r \)-th subvectors of dimension \( m_r \times 1 \). Then, under Assumptions
Proof. Let $F = \text{diag} \left[ \iota_{m_r} \right]_r^R$ where $\iota_{m_r}$ denotes the $m_r \times 1$ vector of ones, and let $H = [\sigma_0 F, \sigma_0 I_{N \times N}]$. Consider the $(N + R) \times 1$ dimensional vector

$$\xi = (\alpha_1/\sigma_0, \ldots, \alpha_R/\sigma_0, \epsilon_1/\sigma_0, \ldots, \epsilon_R/\sigma_0)',$$

Then $U = H\xi$, and

$$U'AU + U'a = \xi'(H'AH)\xi + \xi'(H'a). \quad (C.9)$$

Note that by Assumptions 3 and 4, the elements of $\xi$ are independently distributed with $E(\xi) = 0_{(N+R)\times 1}$, $\text{Var}(\xi) = I_{N+R}$. Denote the $i$-th entry of $\xi$ as $\xi_i$, $1 \leq i \leq N + R$. Denote the third and fourth moments of $\xi_i$ as $\mu_{\xi_i}^{(3)}$ and $\mu_{\xi_i}^{(4)}$ respectively. Under Assumptions 3 and 4, when $1 \leq i \leq R$, $\mu_{\xi_i}^{(3)} = \mu_{\xi_i}^{(3)}/\sigma_0^3$ and $\mu_{\xi_i}^{(4)} = \mu_{\xi_i}^{(4)}/\sigma_0^4$. When $R + 1 \leq i \leq R + N$, $\mu_{\xi_i}^{(3)} = \mu_{\xi_i}^{(3)}/\sigma_0^3$ and $\mu_{\xi_i}^{(4)} = \mu_{\xi_i}^{(4)}/\sigma_0^4$. Furthermore, there exists some $\eta_{\xi} > 0$ such that $E(|\xi_i|^{4+\eta_{\xi}}) < \infty$.

Using the transformation of linear quadratic forms in (C.9) and applying Lemma A.1 in Kelejian and Prucha (2010) yields,

$$E \left[ U'AU + U'a \right] = E[\xi'(H'AH)\xi + \xi'(H'a)] = tr(H'AH) = tr(A\Omega_0),$$

observing that

$$HH' = \sigma_0^2 F F' + \sigma_0^2 I = \Omega_0.$$
Furthermore the variance of the linear quadratic forms in $U$ is given by

$$
\text{cov}(U'AU + U'a, U'BU + U'b)
$$

$=\text{cov} (\xi'(H'AH)\xi + \xi'(H'a), \xi'(H'BH)\xi + \xi'(H'b))$

$=2tr (H'AHH'BH) + a'HH'b$

$$
=2tr (H'AHH'BH) + a'HH'b
+ \sum_{i=1}^{N+R} (H'AH)_{ii}(H'BH)_{ii}(\mu_{\xi_i}^{(4)} - 3) + \sum_{i=1}^{N+R} [(H'AH)_{ii}(H'b)_{i} + (H'BH)_{ii}(H'a)_{i}]\mu_{\xi_i}^{(3)}
$$

$=2tr (A\Omega_0B\Omega_0) + a'\Omega_0b
+ \sum_{i=1}^{N} a_{ii}b_{ii}(\mu_{\xi_i}^{(4)} - 3\sigma_{\xi_i}^4) + \sum_{r=1}^{R} [tr(A_{(rr)}J_m) [tr(B_{(rr)}J_m)] (\mu_{\alpha_0}^{(4)} - 3\sigma_{\alpha_0}^4)

+ \sum_{i=1}^{N} (a_{ii}b_{ii} + b_{ii}a_{ii}) \mu_{\xi_i}^{(3)} + \sum_{r=1}^{R} (b_{m, A_{(rr)}J_m}b_{(r)} + b_{m, B_{(rr)}J_m}a_{(r)}) \mu_{\xi_i}^{(3)}.$

\[ \square \]

**Lemma C.2.** [Central Limit Theorem] Suppose Assumptions 1, 3, 4 hold. For $l = 1, \ldots, L$ let $A_N^{(l)}$ be non-stochastic $N \times N$ matrices where the row and column sums of the absolute elements are uniformly bounded in $N$, and let $a_N^{(l)}$ be $N \times 1$ non-stochastic vectors where the absolute elements are uniformly bounded in $N$. Let $S_N = [S_N^{(1)}, S_N^{(2)}, \ldots, S_N^{(L)}]'$ be an $L \times 1$ vector of linear quadratic forms of $U$, with

$$
S_N^{(l)} = U'A_N^{(l)}U + U'a_N^{(l)}, \quad l = 1, \ldots, L.
$$

Let $\Sigma_{S,N}$ denote variance covariance matrix of $S_N$, where explicit expressions for the elements of $\Sigma_{S,N}$ are readily obtained from Lemma C.1 and assume that $\rho_{\min}(\Sigma_{S,N}) \geq c$ for some constant $c > 0$. Let $\Sigma_{S,N} = \sum_{S,N}^{1/2} \Sigma_{S,N}^{1/2}$, then

$$
\Sigma_{S,N}^{-1/2} (S_N - E[S_N]) \overset{D}{\to} N(0, I_L)
$$

as $N \to \infty$.

(Note that under Assumption 5 the conditions postulated for $a_N^{(l)}$ hold if $a_N^{(l)} = B_N^{(l)}Z\beta_0$, and the $B_N^{(l)}$ are non-stochastic $N \times N$ matrices where the row and column sums of the absolute elements are uniformly bounded in $N$.)

**Proof.** Let $H$ and $\xi$ be defined as in the proof of Lemma C.1, so that $U = H\xi$. Upon substitution of this expression for $U$ we have

$$
S_N^{(l)} = \xi'\tilde{A}_N^{(l)}\xi + \xi'\tilde{a}_N^{(l)}
$$

where $\tilde{A}_N^{(l)+R} = (1/2)H'(A_N^{(l)} + A_N^{(l)'})H$, $\tilde{a}_N^{(l)+R} = H'a_N^{(l)}$. Clearly, in light of Assumptions 3, 4, $\xi$ satisfies Assumptions A.1. and A.3 in Kelejian and Prucha (2010). Furthermore, given the maintained assumptions on $A_N^{(l)}$ and $a_N^{(l)}$, and since the row an column sums of $H$ are uniformly
bounded in absolute value, it follows that the row an column sums of \( A_{N+R}^{(l)} \) and the elements of \( \bar{a}_{N+R}^{(l)} \) are uniformly bounded in absolute value. This verifies that those matrices and vectors satisfy the conditions of Assumption A.2 in Kelejian and Prucha (2010). The lemma now follows from Theorem A.1 in Kelejian and Prucha (2010).

As above, let \( \Theta_o \) be an open bounded set with \( \Theta \subset \Theta_o \subset (-1, 1) \times R^2_+ \).

**Lemma C.3.** [Uniform Convergence] Let \( \Theta_o \) be an open set containing \( \Theta \). Let \( A_N(\theta) \) and \( B_N(\theta) \) be \( N \times N \) matrices and let \( S_N(\theta) \) be a linear-quadratic form of \( U \):

\[
S_N(\theta) = U' A_N(\theta) U + U' B_N(\theta) Z \beta_0
\]

were \( A_N(\theta) \) and \( B_N(\theta) \) are differentiable \( N \times N \) matrices defined for \( \theta \in \Theta_o \). Suppose Assumptions 1-5 hold, and suppose the row and column sums of \( A_N(\theta) \), \( B_N(\theta) \), \( \partial A_N(\theta)/\partial \theta_i \), and \( \partial B_N(\theta)/\partial \theta_i \), \( i = 1, 2, 3 \), are bounded in absolute value uniformly in \( N \) and \( \theta \). Then \( N^{-1} S_N(\theta) - N^{-1} E[S_N(\theta)] \) converges uniformly to zero i.p., i.e.,

\[
p \lim_{N \to \infty} \sup_{\theta \in \Theta} |N^{-1} S_N(\theta) - N^{-1} E[S_N(\theta)]| = 0.
\]

**Remark C.1.** Given the uniform convergence in probability of \( N^{-1} S_N(\theta) \) to its mean and the equicontinuity of \( N^{-1} S_N(\theta) \), we have \( p \lim_{N \to \infty} |N^{-1} S_N(\hat{\theta}_N) - N^{-1} E[S_N(\theta_0)]| \) as \( N \) goes to infinity if \( \hat{\theta}_N \to \theta_0 \).

**Proof.** To prove the lemma we verify that \( N^{-1} S_N(\theta) \) and \( N^{-1} \tilde{S}_N(\theta) = N^{-1} E[S_N(\theta)] \) satisfy the conditions postulated by Corollary 2.2 of Newey (1991); cp., also Theorem 3.1(a) and the discussion after eq. (2.7) in Pötscher and Prucha (1994).

The parameter space \( \Theta \) is compact by assumption. We next verify that \( N^{-1} \tilde{S}_N(\theta) \) is uniformly equicontinuous. By Lemma C.1, \( N^{-1} \tilde{S}_N(\theta) = N^{-1} tr(\Omega_0 A(\theta)) \). Let \( \theta, \theta' \in \Theta \), then by the mean value theorem

\[
tr(\Omega_0 A_N(\theta)) = tr(\Omega_0 A_N(\theta')) + \left[ tr(\Omega_0 \frac{\partial A_N(\theta^*)}{\partial \theta_1}), tr(\Omega_0 \frac{\partial A_N(\theta^*)}{\partial \theta_2}), tr(\Omega_0 \frac{\partial A_N(\theta^*)}{\partial \theta_3}) \right](\theta - \theta').
\]

where \( \theta^* \) is a “vector of between values”. Note that the row and column sums of \( A_N(\theta) \), \( \nabla_\theta A_N(\theta) = \partial A_N(\theta)/\partial \theta_i \), \( \Omega_0 \), and consequently the row and column sums of \( \Omega_0 A_N(\theta) \) and \([\Omega_0 \nabla_\theta A_N(\theta)]\), are uniformly (in \( \theta \) and \( N \)) bounded in absolute value. Consequently there exists a constant \( C_A \) which does not depend on \( \theta, \theta' \), or \( N \) such that

\[
|N^{-1} tr(\Omega_0 A_N(\theta)) - N^{-1} tr(\Omega_0 A_N(\theta'))| \leq C_A \|\theta - \theta'\|,
\]

which establishes that \( N^{-1} \tilde{S}_N(\theta) = N^{-1} tr(\Omega_0 A_N(\theta)) \) is uniformly equicontinuous on \( \Theta \).

We next prove point-wise convergence i.p. of \( N^{-1} S_N(\theta) - N^{-1} E[S_N(\theta)] \) to zero. In light of Chebychev’s inequality it suffices to show that the variance of \( N^{-1} S_N(\theta) \) converges to 0 for any
\[ \theta \in \Theta. \] Let \( A_N = [A_N(\theta) + A_N'(\theta)]/2 \) and \( \bar{\sigma}_N = B_N(\theta)Z\beta_0 \), then Lemma C.1 the variance of \( S_N(\theta) \) is

\[
\text{var}(N^{-1/2}S_N(\theta)) = N^{-1} \text{tr} \left( A_N(\Theta_0)A_N\Omega_0 \right) + N^{-1} \bar{\sigma}_N^\prime \Omega_0 \bar{\sigma}_N
\]

\[
+ N^{-1} \sum_{i=1}^{N} (\bar{\sigma}_{i,N})^2 (\mu_4^{(1)} - 3\sigma_0^4) + N^{-1} \sum_{r=1}^{R} \left[ \text{tr}(A_{(rr)}N^J m_r) \right]^2 (\mu_4^{(1)} - 3\sigma_0^4)
\]

\[
+ 2N^{-1} \sum_{i=1}^{N} \bar{\sigma}_{i,N} \mu_3^{(\prime)} + 2N^{-1} \sum_{r=1}^{R} \left[ \mu_3^{(\prime)}A_{(rr)}N^J m_r \mu_3^{(N)} \right] \Omega_0^4 .
\]

Under our assumptions the row and column sums of \( \Omega_0, A_N \), and thus of \( A_N(\Theta_0)A_N\Omega_0 \) are uniformly bounded. Furthermore, it is readily seen that the elements of \( \bar{\sigma}_N \) are uniformly bounded in absolute value. Consequently \( N^{-1} \text{tr} \left( A_N(\Theta_0)A_N\Omega_0 \right) \), \( N^{-1} \bar{\sigma}_N^\prime \Omega_0 \bar{\sigma}_N \), and all sums in the above expression are seen to be bounded by a finite constant uniformly in \( N \). In turn this implies that \( \text{var}(N^{-1}S_N(\theta)) \to 0 \).

Finally we prove that \( N^{-1}S_N(\theta) \) satisfies the following Lipschitz condition:

\[
\left| N^{-1}S_N(\theta) - N^{-1}S_N(\theta') \right| \leq C_N \| \theta - \theta' \|
\]

for all \( \theta, \theta' \in \Theta \) and some nonnegative random variable \( C_N \) that does not depend on \( \theta, \theta' \) and where \( C_N = O_p(1) \). It prove again convenient to rewrite as \( S_N(\theta) \) as \( S_N(\theta) = \xi^\prime A_N(\theta)\xi + \xi^\prime \bar{a}_N(\theta) \) with \( A_N(\theta) = H^\prime A_N(\theta)H \) and \( \bar{a}_N(\theta) = H^\prime B_N(\theta)Z\beta_0 \), where \( H \) and \( \xi \) are defined as in the proof of Lemma C.1. Under the maintained assumptions the \( \bar{A}_N(\theta) \) and \( \bar{a}_N(\theta) \) are differentiable for \( \theta \in \Theta_a \), and the row and column sums of \( \bar{A}_N(\theta) \), \( \partial \bar{A}_N(\theta)/\partial \theta_i \) and the elements of \( \bar{a}_N(\theta), \partial \bar{a}_N(\theta)/\partial \theta_i \) are uniformly bounded in absolute value in \( \theta \) and \( N \), with \( i = 1, 2, 3 \). Consequently, for some finite constant, say \( K \), we have \( |\bar{a}_{i,N}(\theta)| \leq K/2 \) and, using the mean value theorem,

\[
\sum_{j=1}^{N+R} |\bar{a}_{i,j,N}(\theta) - \bar{a}_{i,j,N}(\theta')| \leq \sum_{j=1}^{N+R} \| \partial \bar{a}_{i,j,N}(\theta)/\partial \theta \| \| \theta - \theta' \| \leq K \| \theta - \theta' \| ,
\]

with \( \theta_* \) a “between value”. Observing further that \( |\xi_i\xi_j| \leq (\xi_i^2 + \xi_j^2)/2 \) we have for any \( \theta, \theta' \in \Theta \)

\[
\left| N^{-1}S_N(\theta) - N^{-1}S_N(\theta') \right| = \left| N^{-1}\xi^\prime \left[ \bar{A}_N(\theta) - \bar{A}_N(\theta') \right] \xi + N^{-1}\xi^\prime \left[ \bar{a}_N(\theta) - \bar{a}_N(\theta') \right] \right|
\]

\[
\leq \frac{2}{N + R} \sum_{i=1}^{N+R} \sum_{j=1}^{N+R} |\bar{a}_{i,j,N}(\theta) - \bar{a}_{i,j,N}(\theta')| (\xi_i^2 + \xi_j^2)/2 + \frac{2}{N + R} \sum_{i=1}^{N+R} ||\bar{a}_{i,N}(\theta) + |\bar{a}_{i,N}(\theta')|| \xi_i|
\]

\[
\leq \frac{1}{N + R} \sum_{i=1}^{N+R} \xi_i^2 \sum_{j=1}^{N+R} |\bar{a}_{i,j,N}(\theta) - \bar{a}_{i,j,N}(\theta')| + \frac{1}{N + R} \sum_{j=1}^{N+R} \xi_j^2 \sum_{i=1}^{N+R} |\bar{a}_{i,j,N}(\theta) - \bar{a}_{i,j,N}(\theta')|
\]

\[
+ \frac{2K}{N + R} \sum_{i=1}^{N+R} |\xi_i|
\]
and consequently $|N^{-1}S_N(\theta) - N^{-1}S_N(\theta')| \leq C_N \|\theta - \theta'\|$ where

$$C_N = \frac{2K}{N + R} \sum_{i=1}^{N+R} (\xi_i^2 + |\xi_i|).$$

Since $EC_N \leq 4KE(\xi_i^2 + |\xi_i|) \leq 8K$, since $E\xi_i^2 = 1$, it follows that $C_N$ is $O_p(1)$. Having verified all conditions of Corollary 2.2 of [Newey 1991], this concludes the proof of the lemma.

The following Lemma is helpful in proving Theorem 4.3 later. Motivated by the proof of Theorem 4.3, we define the following variables: $\tilde{\phi} = \frac{(1-\lambda)\beta_0}{1-\lambda_0} - \beta$, $\tilde{\varphi} = \frac{1-\lambda}{1-\lambda_0}$, $\tilde{z}_r = \frac{1}{m_r} \sum_{i=1}^{m_r} z_{ir}$, $\tilde{\phi}_r = \frac{m_r-1+\lambda_0}{m_r-1+\lambda_0} \beta_0 - \hat{\beta}$, $\tilde{\varphi}_r = 1 + \frac{\lambda-\lambda_0}{m_r-1+\lambda_0}$, $\tilde{\tilde{z}}_{ir} = z_{ir} - \tilde{z}_r$.

**Lemma C.4.** Suppose Assumptions 1-5 hold, and let $\psi(.)$ be a finite positive scalar function of $m_r$, $0 < c_\psi \leq \psi(m_r) \leq C_\psi < \infty$ for $r = 1, \ldots, R$, then:

(a) For $1 \leq p \leq 4$, the expected values of $\frac{1}{R} \sum_{r=1}^{R} \sum_{i=1}^{m_r} \psi(m_r) \bar{u}_{ir}^{p}$ and $\frac{1}{R} \sum_{r=1}^{R} \sum_{i=1}^{m_r} \psi(m_r) \bar{u}_{ir}^{p}$ are finite, and their deviations from the expected values converge in probability to zero as $R \to \infty$.

(b) For a finite constant $s \geq 1$, $\frac{1}{R} \sum_{r=1}^{R} \sum_{i=1}^{m_r} \psi(m_r) |\tilde{z}_{ir} \tilde{\phi}_r|^s \to 0$, and $\frac{1}{R} \sum_{r=1}^{R} |\tilde{z}_{ir} \tilde{\phi}_r|^s \to 0$.

(c) As $R$ goes to infinity, $\frac{1}{R} \sum_{r=1}^{R} \psi(m_r) \sum_{i=1}^{m_r} |(\tilde{\varphi}_r \bar{u}_{ir})^4 - \bar{u}_r^4| \to 0$ and $\frac{1}{R} \sum_{r=1}^{R} \psi(m_r) \sum_{i=1}^{m_r} |(\tilde{\varphi}_r \bar{u}_{ir})^4 - \bar{u}_r^4| \to 0$.

(d) For finite constants $s \geq 1$, and $1 \leq p \leq 4$, $\frac{1}{R} \sum_{r=1}^{R} \psi(m_r) \sum_{i=1}^{m_r} |\tilde{z}_{ir} \tilde{\phi}_r|^s |(\tilde{\varphi}_r \bar{u}_{ir})^p| \to 0$ and $\frac{1}{R} \sum_{r=1}^{R} |\tilde{z}_{ir} \tilde{\phi}_r|^s |(\tilde{\varphi}_r \bar{u}_{ir})^p| \to 0$.

**Proof.** (a) With both $m_r$ and $\psi(m_r)$ being finite and $1 \leq p \leq 4$, Assumptions 3 and 4 imply that $|\sum_{i=1}^{m_r} \psi(m_r) \bar{u}_{ir}^{p+1}| < \infty$ for some $\eta_\mu > 0$, and that the sums $\sum_{i=1}^{m_r} \psi(m_r) \bar{u}_{ir}^{p}$ are independently distributed across $r$. Consequently, $\frac{1}{R} \sum_{r=1}^{R} \sum_{i=1}^{m_r} \psi(m_r) \bar{u}_{ir}^{p}$ has finite expected value. By Corollary 3.9 in [White 1984], the difference between $\frac{1}{R} \sum_{r=1}^{R} \sum_{i=1}^{m_r} \psi(m_r) \bar{u}_{ir}^{p}$ and its expected value goes to 0 in probability. Similarly, $\frac{1}{R} \sum_{r=1}^{R} \sum_{i=1}^{m_r} \psi(m_r) \bar{u}_{ir}^{p}$ has finite expected value and its difference from the expected value goes to 0 in probability.

(b) That $\frac{1}{R} \sum_{r=1}^{R} |\tilde{z}_{ir} \tilde{\phi}_r|^s \to 0$ is straightforward, as $\tilde{\phi} \to_p 0$ and the elements of $\tilde{z}_r$ are uniformly bounded in absolute value. Let $|\tilde{\phi}_r|, |\tilde{\phi}_r|, |\beta_0 - \hat{\beta}|$ be the $\ell_1$ norm of $\tilde{\phi}_r$, $\beta_0$ and $\beta_0 - \hat{\beta}$ respectively. Observe that $m_r - 1 + \lambda_0 \geq 1$, and thus

$$|\tilde{\phi}_r| = \frac{m_r - 1 + \lambda_0}{m_r - 1 + \lambda_0} |\beta_0 - \hat{\beta}|$$

$$= \frac{(\lambda - \lambda_0)\beta_0}{m_r - 1 + \lambda_0} + |(\beta_0 - \hat{\beta})|$$

$$\leq |\lambda - \lambda_0| |\beta_0| + |\beta_0 - \hat{\beta}|,$$

and

$$\psi(m_r)|\tilde{z}_{ir} \tilde{\phi}_r|^s \leq C_Z \psi C_Z (|\lambda - \lambda_0| |\beta_0| + |\beta_0 - \hat{\beta}|)^s,$$

where $C_Z$ is upper bound of the elements in $|\tilde{z}_{ir}|$. Consequently, we have $\frac{1}{R} \sum_{r=1}^{R} \sum_{i=1}^{m_r} \psi(m_r)|\tilde{z}_{ir} \tilde{\phi}_r|^s \to_p 0.$
(c) Since \(m_r - 1 + \lambda_0 > 1\) and \(|\hat{\lambda} - \lambda_0| < 2\),
\[
|\hat{\varphi}_r - 1| = \frac{|\hat{\lambda} - \lambda_0|}{|m_r - 1 + \lambda_0|} \leq |\hat{\lambda} - \lambda_0|,
\]
\[
|\hat{\varphi}_r| \leq 1 + |\hat{\lambda} - \lambda_0| \leq 3,
\]
\[
|\hat{\varphi}_r^4 - 1| = |\hat{\varphi} - 1||\hat{\varphi} + 1|(|\hat{\varphi}^2 + 1)| \leq 40|\hat{\lambda} - \lambda_0|.
\]
Consequently,
\[
\frac{1}{R} \sum_{r=1}^{R} \psi(m_r) \sum_{i=1}^{m_r} (\hat{\varphi}_r^4 - 1) \hat{u}_ir \leq 40C\psi|\hat{\lambda} - \lambda_0|\left(\frac{1}{R} \sum_{r=1}^{R} \sum_{i=1}^{m_r} \hat{u}_ir\right).
\]
By part (a) of the lemma, \(\frac{1}{R} \sum_{r=1}^{R} \sum_{i=1}^{m_r} \hat{u}_ir\) is bounded in probability. The lemma then follows.
(d) Using equation (C.11), we have
\[
\frac{1}{R} \sum_{r=1}^{R} \psi(m_r) \sum_{i=1}^{m_r} (\hat{\varphi}_r^4 - 1) \hat{u}_ir \leq C\psi C_Z|\hat{\lambda} - \lambda_0||\beta_0|1 + |\beta_0 - \hat{\beta}_0|1\frac{1}{R} \sum_{r=1}^{R} \sum_{i=1}^{m_r} |\hat{u}_ir|^p.
\]
Since \(1 \leq p \leq 4\), by the same argument as in part (a), \(\frac{1}{R} \sum_{r=1}^{R} \sum_{i=1}^{m_r} |\hat{u}_ir|^p\) is bounded in probability. Consequently \(\frac{1}{R} \sum_{r=1}^{R} \psi(m_r) \sum_{i=1}^{m_r} (\hat{\varphi}_r^4 - 1) \hat{u}_ir \to 0\). The other claim follows similarly.

**Lemma C.5.** Suppose Assumption \(\text{5}\) holds. Let
\[
A_N(\theta) = diag_{r=1}^{R} \{p(m_r, \theta)I_{m_r}^* + s(m_r, \theta)I_{m_r}^*\},
\]
where, for \(2 \leq m_r \leq \bar{a}\), the scalar functions \(p(m_r, \theta)\) and \(s(m_r, \theta)\) are positive and continuous on the compact parameter space \(\Theta\). Let \(S_N(\theta) = N^{-1}Z' A_N(\theta) Z\), then there exist positive constants \(c\) and \(C\) that do not depend on \(\theta\) and \(N\) such that
\[
0 < c \leq \lambda_{\min}[S_N(\theta)] \leq \lambda_{\max}[S_N(\theta)] \leq C < \infty. \tag{C.12}
\]
Furthermore
\[
\sup_{\theta \in \Theta} |S_N(\theta) - S(\theta)| \to 0 \text{ as } N \to \infty, \tag{C.13}
\]
where \(S(\theta) = \sum_{m=2}^{\bar{a}} [p(m, \theta)\bar{z}_m + s(m, \theta)\bar{z}_m]\). The elements of \(S(\theta)\) are continuous on \(\Theta\), and
\[
0 < c \leq \lambda_{\min}[S(\theta)] \leq \lambda_{\max}[S(\theta)] \leq C < \infty. \tag{C.14}
\]

**Remark C.2.** It follows from the uniform convergence of \(S_N(\theta)\) and the continuity of \(S(\theta)\) that if \(\hat{\theta}_N \to P \theta_0\), then \(|S_N(\hat{\theta}_N) - S(\theta_0)| \to P 0\) as \(N \to \infty\).
Proof. Observe that by the Bolzano-Weierstrass’ extreme value theorem there exist positive constants $c$ and $C$, which do not depend on $\theta$ such that

$$0 < c \leq p(m_r, \theta), s(m_r, \theta) \leq C < \infty.$$ 

Since $m_r$ only takes on finitely many values the constants $c$ and $C$ can be chosen such that the above inequalities also hold for all $m$. By Assumption 5 we have $0 < \xi_Z \leq \lambda_{\text{min}}[N^{-1}\sum_{r \in \mathcal{I}_m} \bar{Z}'_r \bar{Z}_r]$ and $0 < \xi_Z \leq \lambda_{\text{min}}[N^{-1}\sum_{r \in \mathcal{I}_m} \bar{m}'_r \bar{m}_r]$ for some $m$. Since the elements of $Z$ are bounded in absolute value it follows further that there exists a finite constant $\xi_Z$ such that $\lambda_{\text{max}}[N^{-1}\sum_{r \in \mathcal{I}_m} \bar{Z}'_r \bar{Z}_r] \leq \xi_Z < \infty$ and $\lambda_{\text{max}}[N^{-1}\sum_{r \in \mathcal{I}_m} \bar{m}'_r \bar{m}_r] \leq \xi_Z < \infty$; see, e.g., Horn and Johnson (1985), Lemma 5.6.10 and the discussion of the equivalence of matrix norms on pp. 202. Next observe that

$$S_N(\theta) = N^{-1}\sum_{r=1}^R [p(m_r, \theta)Z'_r Z_r + s(m_r, \theta)Z'_r J^*_{m_r} Z_r]$$

$$= \sum_{m=2}^a [p(m, \theta)N^{-1}\sum_{r \in \mathcal{I}_m} \bar{Z}'_r \bar{Z}_r + s(m, \theta)N^{-1}\sum_{r \in \mathcal{I}_m} \bar{m}'_r \bar{m}_r].$$

Consequently

$$\lambda_{\text{min}}[S_N(\theta)] = \inf_{\phi \in \mathbb{R}^kZ} \phi' S_N(\theta) \phi$$

$$\geq \sum_{m=2}^a p(m, \theta) \inf_{\phi \in \mathbb{R}^kZ} \phi'[N^{-1}\sum_{r \in \mathcal{I}_m} \bar{Z}'_r \bar{Z}_r] \phi + s(m, \theta) \inf_{\phi \in \mathbb{R}^kZ} \phi'[N^{-1}\sum_{r \in \mathcal{I}_m} \bar{m}'_r \bar{m}_r] \phi$$

$$\geq \sum_{m=2}^a c\xi_Z > 0$$

and

$$\lambda_{\text{max}}[S_N(\theta)] \leq \sup_{\phi \in \mathbb{R}^kZ} \phi' S_N(\theta) \phi$$

$$\leq \sum_{m=2}^a p(m, \theta) \sup_{\phi \in \mathbb{R}^kZ} \phi'[N^{-1}\sum_{r \in \mathcal{I}_m} \bar{Z}'_r \bar{Z}_r] \phi + s(m, \theta) \sup_{\phi \in \mathbb{R}^kZ} \phi'[N^{-1}\sum_{r \in \mathcal{I}_m} \bar{m}'_r \bar{m}_r] \phi$$

$$\leq \sum_{m=2}^a c\xi_Z < \infty.$$
This proves the first part of the lemma. Next observe that

\[
\sup_{\theta \in \Theta} |S_N(\theta) - S(\theta)| \\
\leq \sup_{\theta \in \Theta} \left[ \sum_{m=2}^{\bar{a}} p(m, \theta) \left| N^{-1} \sum_{r \in I_m^m} \hat{Z}_r \bar{Z}_r - \bar{x}_m \right| + s(m, \theta) \left| N^{-1} \sum_{r \in I_m} m \bar{z}_r \bar{z}_r - \bar{x}_m \right| \right] \\
\leq C \sum_{m=2}^{\bar{a}} \left| N^{-1} \sum_{r \in I_m^m} \hat{Z}_r \bar{Z}_r - \bar{x}_m \right| + \left| N^{-1} \sum_{r \in I_m} m \bar{z}_r \bar{z}_r - \bar{x}_m \right| \to 0
\]

by Assumption 5. Clearly \(S(\theta)\) is continuous given the assumptions maintained w.r.t. \(p(m, \theta)\) and \(s(m, \theta)\). Furthermore, since the eigenvalues of a matrix are continuous functions of the elements of the matrix we have for some \(m\)

\[
0 < \xi_Z \leq \lim_{N \to \infty} \lambda_{\min}\left[N^{-1} \sum_{r \in I_m^m} \hat{Z}_r \bar{Z}_r \right] = \lambda_{\min}\left[\lim_{N \to \infty} N^{-1} \sum_{r \in I_m^m} \hat{Z}_r \bar{Z}_r \right] = \lambda_{\min}(\bar{x}_m),
\]

\[
0 < \xi_Z \leq \lim_{N \to \infty} \lambda_{\min}\left[N^{-1} \sum_{r \in I_m} m \bar{z}_r \bar{z}_r \right] = \lambda_{\min}\left[\lim_{N \to \infty} N^{-1} \sum_{r \in I_m} m \bar{z}_r \bar{z}_r \right] = \lambda_{\min}(\bar{x}_m),
\]

and we have for all \(m\)

\[
\lambda_{\max}(\bar{x}_m) = \lambda_{\max}\left[\lim_{N \to \infty} N^{-1} \sum_{r \in I_m^m} \hat{Z}_r \bar{Z}_r \right] = \lim_{N \to \infty} \lambda_{\max}\left[N^{-1} \sum_{r \in I_m^m} \hat{Z}_r \bar{Z}_r \right] \leq \xi_Z < \infty,
\]

\[
\lambda_{\max}(\bar{x}_m) = \lambda_{\max}\left[\lim_{N \to \infty} N^{-1} \sum_{r \in I_m} m \bar{z}_r \bar{z}_r \right] = \lim_{N \to \infty} \lambda_{\max}\left[N^{-1} \sum_{r \in I_m} m \bar{z}_r \bar{z}_r \right] \leq \xi_Z < \infty.
\]

The remainder of the proof of (C.14) is analogous to the proof of (C.12).
D Proofs of Lemmas 2.1, 2.2

D.1 Proof of Lemma 2.1

We first consider case (i) and assume that the errors are homoskedastic and that this condition is imposed on the parameter vector. By assumption there are two groups $r$ and $s$ such that $E [\epsilon_r^2 | m_r, D_r] = E [\epsilon_s^2 | m_s, D_s] = \sigma_0^2$ and $m_r \neq m_s$. Now consider

$$E [\chi_r^w(\vartheta) | m_r, D_r] = \frac{(m_r - 1 + \lambda)^2 E [\tilde{Y}_r \tilde{Y}_r]}{(m_r - 1)^2} - \sigma_{\epsilon, D_r}^2 (m_r - 1)$$

$$= (m_r - 1) \frac{\left(\frac{(m_r - 1 + \lambda)^2 \sigma_{e0, D_r}^2}{(m_r - 1 + \lambda_0)^2} - \sigma_{\epsilon, D_r}^2 \right)}{\sigma_\epsilon, D_r} = 0$$

which leads to the equation

$$(m_r - 1 + \lambda)^2 \sigma_{e0, D_r}^2 = \sigma_{\epsilon, D_r}^2 (m_r - 1 + \lambda_0)^2.$$

Now use the same moment condition for group $s$ such that it follows that

$$\frac{(m_r - 1 + \lambda)^2}{(m_s - 1 + \lambda)^2} = \frac{(m_r - 1 + \lambda_0)^2}{(m_s - 1 + \lambda_0)^2}. \quad (D.1)$$

Clearly the equation in (D.1) holds for $\lambda = \lambda_0$. The RHS is constant in $\lambda$. The LHS is a monotonic function in $\lambda$. To see this, compute the derivative $\partial h(\lambda)/\partial \lambda$ of

$$h(\lambda) = \frac{(m_r - 1 + \lambda)^2}{(m_s - 1 + \lambda)^2}$$

given by

$$\frac{\partial h(\lambda)}{\partial \lambda} = \frac{2(m_r - 1 + \lambda)}{(m_s - 1 + \lambda)^2} - \frac{2(m_r - 1 + \lambda)^2 (m_s - 1 + \lambda)}{(m_s - 1 + \lambda)^4}$$

$$= \frac{2(m_r - 1 + \lambda)(m_s - 1 + \lambda)(m_s - m_r)}{(m_s - 1 + \lambda)^4}.$$ 

Since $\lambda > -1$, $m_s > 1$ and $m_r > 1$ then $\text{sign}(\partial h(\lambda)/\partial \lambda) = \text{sign}(m_s - m_r)$. This implies that (D.1) can only have one solution. Now consider $E [\chi^b_r(\vartheta) | m_r, D_r] = 0$. We already established that $E [\chi^w_r(\vartheta) | m_r, D_r] = 0$ and $E [\chi^w_s(\vartheta) | m_s, D_s] = 0$ imply that $\lambda = \lambda_0$. Then,

$$E [\chi^b_r(\vartheta) | m_r, D_r] = \frac{(1 - \lambda_0^2)}{(1 - \lambda_0)^2} \left(\frac{\sigma_{\alpha,0}^2 + \sigma_{\alpha, D_r}^2}{m_r} - \left(\frac{\sigma_\alpha^2 + \sigma_{\epsilon, D_r}^2}{m_r}\right)\right) = 0 \quad (D.2)$$
defines which is different from zero as long as irrespective of whether the errors are homoskedastic or not. 

\[
E \left[ \chi_r^b(\vartheta) \right] - E \left[ \chi_s^b(\vartheta) \right] = \sigma^2_{\epsilon,D_r} \left( \frac{1}{m_r} - \frac{1}{m_s} \right) - \sigma^2_{\epsilon,D_s} \left( \frac{1}{m_r} - \frac{1}{m_s} \right) = 0 \tag{D.3}
\]

or \( \sigma^2_{\epsilon,D_r} = \sigma^2_{\epsilon,D_s} \). Substituting back into \( E \left[ \chi_r^b(\vartheta) \right] = 0 \) leads to \( \sigma^2_{\alpha,0} = \sigma^2_{\alpha} \). If there are additional groups with sizes different from \( m_r \) and \( m_s \) then moment conditions related to these groups constitute overidentifying restrictions.

Now consider case (ii) where we assume that no restriction of homoskedasticity is imposed on the parameter space. By assumption there exist at least two groups \( r = q_1 \) and \( s = q_2 \) with sizes \( m_r \neq m_s \) that are in the same category, i.e. where \( D_r = D_s \). Without loss of generality assume that the category is labeled as 0 such that \( D_r = D_s = 0 \). Since for these two groups \( \sigma^2_{\epsilon,D_r} = \sigma^2_{\epsilon,D_s} = \sigma^2_{\epsilon,0} \) by construction and \( m_r \neq m_s \) it follows from (D.1), (D.2) and (D.3) that \( \lambda, \sigma^2_{\alpha} \) and \( \sigma^2_{\alpha,0} \) are identified. Since the parameters \( \lambda, \sigma^2_{\alpha} \) and \( \sigma^2_{\alpha,0} \) are identified from moment conditions involving only groups \( r \) and \( s \), it follows that \( E \left[ \chi_r^b(\vartheta) \right] \mid m_q, D_q \) identifies \( \sigma^2_{\epsilon,D_q} \) for \( D_q \in \{1, \ldots, J\} \) irrespective of whether the errors are homoskedastic or not.

### D.2 Proof of Lemma 2.2

Consider

\[
E \left[ \bar{y}_r^2 \right]_{m_r, D_r} - E \left[ \bar{y}_s^2 \right]_{m_s, D_s} = \frac{\sigma^2_{\alpha} + \sigma^2_{\alpha,D_r}}{(1 - \lambda)^2} - \frac{\sigma^2_{\alpha} + \sigma^2_{\alpha,D_s}}{(1 - \lambda)^2} = \frac{\sigma^2_{\alpha,D_r} - \sigma^2_{\alpha,D_s}}{m(1 - \lambda)^2},
\]

such that, using \( m = m_r = m_s \), it follows that

\[
E \left[ \frac{\bar{Y}_r^2}{m_r(m_r - 1)^3} \right]_{m_r, D_r} - E \left[ \frac{\bar{Y}_s^2}{m_s(m_s - 1)^3} \right]_{m_s, D_s} = \frac{1}{m(m - 1 - \lambda)^2} \left( \sigma^2_{\alpha,D_r} - \sigma^2_{\alpha,D_s} \right)
\]

which is different from zero as long as \( \sigma^2_{\alpha,D_r} - \sigma^2_{\alpha,D_s} \neq 0 \). Then,

\[
\frac{(m - 1 + \lambda)^2}{(1 - \lambda)^2} = \frac{E \left[ \frac{\bar{y}_r^2}{m_r(m_r - 1)^3} \right]_{m_r, D_r} - E \left[ \frac{\bar{y}_s^2}{m_s(m_s - 1)^3} \right]_{m_s, D_s} = 0}{(m - 1 + \lambda)^2} = \frac{(m - 1 + \lambda)^2}{(1 - \lambda)^2}.
\]

Define \( c = \frac{(m - 1 + \lambda)^2}{(1 - \lambda)^2} \). Then the equation is equivalent to

\[
(m - 1 + \lambda)^2 = c (1 - \lambda)^2
\]

or equivalently,

\[
(m - 1)^2 - c + 2 ((m - 1) + c) \lambda + (1 - c) \lambda^2 = 0
\]
which is a polynomial in $\lambda$. Clearly, $\lambda = \lambda_0$ is a solution. Consider the derivative

$$
\frac{\partial}{\partial \lambda} \left( \frac{(m-1+\lambda)^2}{(1-\lambda)^2} \right) = \frac{2(m-1+\lambda)}{(1-\lambda)^2} + \frac{2(m-1+\lambda)^2}{(1-\lambda)^3}
$$

$$
= \frac{2((m-1+\lambda)(1-\lambda) + (m-1+\lambda)^2)}{(1-\lambda)^3}
$$

$$
= 2(m-1+\lambda) \frac{(1-\lambda) + (m-1+\lambda)}{(1-\lambda)^3}
$$

$$
= \frac{2(m-1+\lambda)m}{(1-\lambda)^3} > 0.
$$

Since $m \geq 2$ and $\lambda \in (-1, 1)$ it follows that $(m-1+\lambda)m > 0$ and $1 - \lambda > 0$ such that the derivative is positive for all values of $\lambda$ on the parameter space. This implies that $\lambda = \lambda_0$ is the only solution to the moment condition. Once $\lambda$ is identified, one can use $E[\chi_r^m(\vartheta)|m_r, D_r] = 0$ to identify $\sigma_{\epsilon,1}^2$. Substituting $\lambda$ and $\sigma_{\epsilon,1}^2$ back into $E[\chi_r^l(\vartheta)|m_r, D_r] = 0$ leads to $\sigma_{\alpha,0}^2 = \sigma_{\alpha}^2$ as in Lemma 2.1. Finally, note that (11) is a function of $E[\chi_r(\vartheta)|m_r, D_r]$ and $E[\chi_s(\vartheta)|m_s, D_s]$ which implies that these moment functions are sufficient to identify the parameter $\vartheta$. 

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E Proofs of Theorem 4.1, 4.2, and 4.3

In this section, we will first prove Theorem 4.1: identification and the consistency of the quasi-maximum likelihood estimator in Section E.1 and E.2. Then we will prove Theorem 4.2: the asymptotic distribution of the QMLE in Section E.3 and Theorem 4.3: consistency of the third and fourth moments in Section E.4.

E.1 Proof of Theorem 4.1(a)

For the un-concentrated log likelihood function in (25), let
\[ \bar{R}(\theta, \beta) = \lim_{N \to \infty} E \left[ \frac{1}{N} \ln L(\theta, \beta) \right]. \]

Let \( \bar{\beta}(\theta) \) be the maximizer of \( \bar{R}(\theta, \beta) \) with respect to \( \beta \),
\[ \bar{R}(\theta, \bar{\beta}(\theta)) = \max_{\beta} \bar{R}(\theta, \beta), \]
and let
\[ \bar{Q}^{**}(\theta) = \bar{R}(\theta, \bar{\beta}(\theta)). \]

For the concentrated log likelihood function in (29), let \( \bar{Q}^*(\theta) = \lim_{N \to \infty} E [Q_N(\theta)] \). To prove that \( (\theta_0, \beta_0) \) is identified, it suffices to show that Condition 1(a) and Condition 2 below hold.

Condition 1. (a) The non-stochastic functions \( \bar{Q}^*(\theta) \) and \( \bar{Q}^{**}(\theta) \) exist, and \( \bar{Q}^*(\theta) = \bar{Q}^{**}(\theta) \) are continuous and finite; 
(b) As \( N \) goes to infinity, \( \sup_{\theta \in \Theta} |E Q_N(\theta) - \bar{Q}^*(\theta)| \to 0 \).

Condition 2. The parameter space \( \Theta \) is compact, the true value \( \theta_0 \) is the unique maximizer of \( \bar{Q}^*(\theta) \) (and hence \( \bar{Q}^{**}(\theta) \)) on \( \Theta \) and \( \bar{\beta}(\theta_0) = \beta_0 \).

Condition 1(b) is necessary for proof of consistency later. We combine conditions 1(a) and 1(b) as they can be proved together. With the two conditions above, \( \delta_0 = (\theta_0', \beta_0') \) is identified in the sense that it is the unique maximizer of \( \bar{R}(\theta, \beta) \).

■ Verification of Condition 1. To prove that \( \bar{Q}^*(\theta) = \lim_{N \to \infty} E Q_N(\theta) \) exists, it is readily seen that
\[
E [Q_N(\theta)] = -\frac{\ln(2\pi)}{2} + \frac{1}{2} \ln |I - \lambda W^2\Omega(\theta)^{-1}| - \frac{1}{2N} tr \{(I - \lambda W)' M_Z(\theta)(I - \lambda W)(E [YY'])\}
\]
\[
= -\frac{\ln(2\pi)}{2} + \frac{1}{2} \ln |I - \lambda W^2\Omega(\theta)^{-1}|
- \frac{1}{2N} tr \{(I - \lambda W)' M_Z(\theta)(I - \lambda W)(I - \lambda_0 W)^{-1} (\Omega_0 + Z\beta_0\beta_0' Z') (I - \lambda_0 W)^{-1}\}
\]
\[
= \bar{Q}_N^{(1)}(\theta) + \bar{Q}_N^{(2)}(\theta) + \bar{Q}_N^{(3)}(\theta),
\]

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where

\[ Q_N^{(1)}(\theta) = -\frac{\ln(2\pi)}{2} + \frac{1}{2N} \ln|I - \lambda W|^2 \Omega(\theta)^{-1} - \frac{1}{2N} \operatorname{tr} [(I - \lambda_0 W)^{-2}(I - \lambda W)^2 \Omega(\theta)^{-1} \Omega_0], \tag{E.2} \]

\[ Q_N^{(2)}(\theta) = -\frac{1}{2N} \operatorname{tr} \left[ \beta_0' Z'(I - \lambda_0 W)^{-1}(I - \lambda W) M_Z(\theta)(I - \lambda W)(I - \lambda_0 W)^{-1} Z \beta_0 \right], \]

\[ \tilde{Q}_N^{(3)}(\theta) = \frac{1}{2N} \operatorname{tr} \left[ (I - \lambda_0 W)^{-2}(I - \lambda W)^2 \Omega(\theta)^{-1} Z (Z' \Omega(\theta)^{-1} Z)^{-1} Z' \Omega(\theta)^{-1} \Omega_0 \right], \]

recalling that \( M_Z(\theta) = \Omega(\theta)^{-1} - \Omega(\theta)^{-1} Z (Z' \Omega(\theta)^{-1} Z)^{-1} Z' \Omega(\theta)^{-1}. \)

We show the limits of \( \tilde{Q}_N^{(1)}(\theta), \tilde{Q}_N^{(2)}(\theta) \) and \( \tilde{Q}_N^{(3)}(\theta) \) exist in reverse order. Observe that

\[ \tilde{Q}_N^{(3)}(\theta) = \frac{1}{2N} \operatorname{tr} \left\{ \left[ \frac{1}{N} Z' \Omega(\theta)^{-1} \Omega_0 (I - \lambda_0 W)^{-2}(I - \lambda W)^2 \Omega(\theta)^{-1} Z \right] \left[ \left( \frac{1}{N} Z' \Omega(\theta)^{-1} Z \right)^{-1} \right] \right\}. \]

Both matrices in square brackets are of the form considered in \( \Box \) with \( p(m_r, \theta) \) and \( s(m_r, \theta) \) satisfying the assumptions of Lemma \( \Box \). Thus their elements, and in turn the trace, are bounded in absolute value by respective constants that do not depend on \( \theta \) and \( N \). Consequently \( \sup_{\theta \in \Theta} \tilde{Q}_N^{(3)}(\theta) \leq \text{const}/N \to 0 \) as \( N \to \infty \) and \( \lim_{N \to \infty} \tilde{Q}_N^{(3)}(\theta) = 0 \).

Second, observe that

\[
2\tilde{Q}_N^{(2)}(\theta) = \beta_0' \left[ \frac{1}{N} Z'(I - \lambda_0 W)^{-2}(I - \lambda W)^2 \Omega(\theta)^{-1} Z \right] \beta_0 \\
- \beta_0' \left[ \left[ \frac{1}{N} Z'(I - \lambda_0 W)^{-1}(I - \lambda W) \Omega(\theta)^{-1} Z \right] \left[ \frac{1}{N} Z' \Omega(\theta)^{-1} Z \right]^{-1} \right] \\
\times \left[ \frac{1}{N} Z' \Omega(\theta)^{-1}(I - \lambda W)(I - \lambda_0 W)^{-1} Z \right] \beta_0. \tag{E.3} \]

In light of \( \Box \) and \( \Box \) and using Lemma \( \Box \), we see that \( \sup_{\theta \in \Theta}|\tilde{Q}_N^{(2)}(\theta) - \tilde{Q}^{(2)*}(\theta)| \to_p 0, \) where

\[ \tilde{Q}^{(2)*}(\theta) = \frac{1}{2} \beta_0' \left\{ \mathcal{Y}_1(\theta) - \mathcal{Y}_2(\theta) \mathcal{Y}_3^{-1}(\theta) \mathcal{Y}_2(\theta) \right\} \beta_0, \]

\[ \mathcal{Y}_1(\theta) = \sum_{m=2}^{\tilde{a}} \left[ \frac{\phi_W^2(m, \theta)}{\phi_W^2(m, \theta_0) \phi_0(m, \theta)} \tilde{\zeta}_m + \frac{\psi_W^2(m, \theta)}{\psi_W(m, \theta_0) \psi_0(m, \theta)} \tilde{\eta}_m \right], \tag{E.4} \]

\[ \mathcal{Y}_2(\theta) = \sum_{m=2}^{\tilde{a}} \left[ \frac{\phi_W(m, \theta)}{\phi_W(m, \theta_0) \phi_0(m, \theta)} \tilde{\zeta}_m + \frac{\psi_W(m, \theta)}{\psi_W(m, \theta_0) \psi_0(m, \theta)} \tilde{\eta}_m \right], \tag{E.5} \]

\[ \mathcal{Y}_3(\theta) = \sum_{m=2}^{\tilde{a}} \left[ \frac{1}{\phi_0(m, \theta)} \tilde{\zeta}_m + \frac{1}{\psi_0(m, \theta)} \tilde{\eta}_m \right], \tag{E.6} \]

and where \( \tilde{Q}^{(2)*}(\theta) \) is finite and continuous on \( \Theta \).
Third, observe that
\[
\tilde{Q}_N^{(1)}(\theta) = -\frac{\ln(2\pi)}{2} - \frac{1}{2N} \ln |(I - \lambda_0W)^{-2}\Omega_0| \\
+ \frac{1}{2N} \ln |(I - \lambda_0W)^{-2}(I - \lambda W)^2\Omega(\theta)^{-1}\Omega_0| - \frac{1}{2N} \text{tr} \left[ (I - \lambda_0W)^{-2}(I - \lambda W)^2\Omega(\theta)^{-1}\Omega_0 \right]
\]
\[
= C_N + \frac{1}{2} \sum_{m=2}^{\tilde{a}} R_m \ln |G(m, \theta)| - \frac{1}{2} \sum_{m=2}^{\tilde{a}} \frac{R_m}{N} \text{tr}[G(m, \theta)]
\]
with
\[
G(m, \theta) = (I_m - \lambda W_m)^2\Omega_m(\theta)^{-1}(I_m - \lambda_0 W_m)^{-2}\Omega_{m0}
\]
\[
= \frac{\phi_W^2(m, \theta)\phi_\Omega(m, \theta_0)}{\phi_W^2(m, \theta_0)\phi_\Omega(m, \theta)} I_m^* + \frac{\psi_W^2(m, \theta)\psi_\Omega(m, \theta_0)}{\psi_W^2(m, \theta_0)\psi_\Omega(m, \theta)} J_m^*
\]
\[
= \frac{\sigma^2_\theta (m - 1 + \lambda)}{\sigma^2_\theta (m - 1 + \lambda_0)} I_m^* + \frac{(\sigma^2_\theta + m\sigma^2_{\alpha_0}) (1 - \lambda)}{(\sigma^2_\theta + m\sigma^2_{\alpha_0}) (1 - \lambda_0)} J_m^*,
\]
(E.7)
and
\[
C_N = -\frac{\ln(2\pi)}{2} - \frac{1}{2} \sum_{m=2}^{\tilde{a}} \frac{R_m}{N} \ln |(I - \lambda_0W_m)^{-2}\Omega_{m0}|. \text{ Under Assumption } R_m/N \to \omega^*_m/m^*.
\]
Let \( C^* = \lim_{N \to \infty} C_N \) and
\[
\tilde{Q}^{(1)*}(\theta) = C^* + \frac{1}{2m^*} \sum_{m=2}^{\tilde{a}} \omega^*_m g(m, \theta)
\]
(E.8)
with
\[
g(m, \theta) = \ln |G(m, \theta)| - \text{tr}[G(m, \theta)],
\]
then clearly \( \sup_{\theta \in \Theta} |\tilde{Q}_N^{(1)}(\theta) - \tilde{Q}^{(1)*}(\theta)| \to 0 \) with \( \tilde{Q}^{(1)*}(\theta) \) finite and continuous on \( \Theta \). In all, \( \sup_{\theta \in \Theta} |Q_N(\theta) - Q^*(\theta)| \to 0 \), where
\[
Q^*(\theta) = \tilde{Q}^{(1)*}(\theta) + \tilde{Q}^{(2)*}(\theta) = C^* + \frac{1}{2m^*} \sum_{m=2}^{\tilde{a}} \omega^*_m g(m, \theta) + \tilde{Q}^{(2)*}(\theta),
\]
and where \( Q^*(\theta) \) is continuous and finite.
For the un-concentrated likelihood function,

\[
\begin{align*}
\tilde{R}(\theta, \beta) &= \lim_{N \to \infty} \frac{1}{N} E \left[ \ln L_n(\theta, \beta) \right] \\
&= \lim_{N \to \infty} \frac{1}{N} \left\{ -\frac{N}{2} \ln(2\pi) + \frac{1}{2} \ln(\det(I - \lambda W)^2\Omega(\theta)^{-1}) \right\} \\
&\quad - \frac{1}{2} \text{tr} \left[ (I - \lambda_0 W)^{-2}(I - \lambda W)^2\Omega(\theta)^{-1}\eta_0 \right] \\
&\quad - \frac{1}{2} \left[ (I - \lambda W)(I - \lambda_0 W)^{-1}(Z\beta_0 - Z\beta)'\Omega(\theta)^{-1}\left( (I - \lambda W)(I - \lambda_0 W)^{-1}(Z\beta_0 - Z\beta) \right) \right] \\
&= \lim_{N \to \infty} \bar{Q}^{(1)}_N(\theta) - \lim_{N \to \infty} \frac{1}{2N} \left[ \beta_0' Z'(I - \lambda W)^2(I - \lambda_0 W)^2\Omega(\theta)^{-1}Z\beta_0 \right] \\
&\quad + \lim_{N \to \infty} \frac{1}{N} \beta_0' Z'(I - \lambda W)(I - \lambda_0 W)\Omega(\theta)^{-1}Z\beta - \lim_{N \to \infty} \frac{1}{2N} \beta' Z'\Omega(\theta)^{-1}Z\beta \\
&= \bar{Q}^{*}(\theta) - \frac{1}{2} \beta_0 \bar{Y}_1(\theta)\beta_0 + \beta_0 \bar{Y}_2(\theta)\beta - \frac{1}{2} \beta' \bar{Y}_3(\theta)\beta,
\end{align*}
\]

where \( \bar{Y}_1(\theta), \bar{Y}_2(\theta), \bar{Y}_3(\theta) \) are defined in Equations (E.4), (E.5), and (E.6). Taking the derivative of \( \tilde{R}(\theta, \beta) \) with respect to \( \beta \),

\[
\frac{\partial \tilde{R}(\theta, \beta)}{\partial \beta} = \bar{Y}_2(\theta) - \bar{Y}_3(\theta) = 0.
\]

Since \( \bar{Y}_3(\theta) \) is non-singular by Assumption 5

\[
\tilde{\beta}(\theta) = \bar{Y}_3(\theta)^{-1}\bar{Y}_2(\theta)\beta_0.
\]  

(E.9)

Let \( \bar{Q}^{**}(\theta) = \tilde{R}(\theta, \tilde{\beta}(\theta)) \) and plug \( \tilde{\beta}(\theta) \) above back to \( \tilde{R}(\theta, \beta) \),

\[
\bar{Q}^{**}(\theta) = \bar{Q}^{(1)*}(\theta) - \frac{1}{2} \beta_0' [\bar{Y}_1(\theta) - \bar{Y}_3(\theta)^{-1}\bar{Y}_2(\theta)\bar{Y}_3(\theta)^{-1}]\beta_0 \\
= \bar{Q}^{(1)*}(\theta) + \bar{Q}^{(2)*}(\theta) = \bar{Q}^{*}(\theta).
\]

Note that the second order derivative

\[
\frac{\partial^2 \tilde{R}(\theta, \beta)}{\partial \beta_0 \partial \beta'} = -\bar{Y}_3(\theta) = - \lim_{N \to \infty} Z'\Omega(\theta)^{-1}Z
\]

is negative definite by Assumption 5, thus \( \tilde{\beta}(\theta) \) is the unique maximizer of \( \tilde{R}(\theta, \beta) \) over \( \beta \). In all, we have \( \bar{Q}^{*}(\theta) \) and \( \bar{Q}^{**}(\theta) \) both exist and \( \bar{Q}^{*}(\theta) = \bar{Q}^{**}(\theta) \).

\[\text{ Verification of Condition 2} \]

Since \( \bar{Y}_3(\theta_0) = \bar{Y}_2(\theta_0), \tilde{\beta}(\theta_0) = \beta_0 \) is readily seen. Next we show that \( \theta_0 \) is the unique global maximizer of \( \bar{Q}^{*}(\theta) \) on \( \Theta \). We first show that \( \theta_0 \) is a global maximizer of \( \bar{Q}^{(2)*}(\theta) \). To see this observe that we can rewrite \( \bar{Q}^{(2)}_N(\theta) \) as \( \bar{Q}^{(2)}_N(\theta) = -\frac{1}{2N} \tilde{\eta}_Z(\theta)' \tilde{M}_Z(\theta) \tilde{\eta}_Z(\theta) \), where \( \tilde{M}_Z(\theta) = I - \Omega^{-1/2}Z'(\Omega(\theta)^{-1}Z\Omega^{-1/2} \Omega(\theta)^{-1/2}(I - \lambda W)(I - \lambda_0 W)^{-1}Z\beta_0 \). Thus \( \bar{Q}^{(2)}_N(\theta) \leq 0 \) and consequently also \( \bar{Q}^{(2)*}(\theta) \leq 0 \). Next observe, as is readily checked, that \( \bar{Q}^{(2)*}(\theta_0) = 0 \). Therefore \( \bar{Q}^{(2)*}(\theta) \leq \bar{Q}^{(2)*}(\theta_0) \) for all \( \theta \in \Theta \).

To show that \( \theta_0 \) is the unique global maximizer of \( \bar{Q}^{*}(\theta) \) it thus suffices to show that \( \theta_0 \) is the
unique maximizer of $\sum_{m=2}^{\tilde{m}} \omega_m^* g(m, \theta)$. Observe that

$$\sum_{m=2}^{\tilde{m}} \omega_m^* g(m, \theta) = \sum_{m=2}^{\tilde{m}} \omega_m^* \{ \ln \{ G(m, \theta) \} - \ln \{ G(\tilde{m}, \theta) \} \} \leq -m^*,$$

(E.10)

where $m^* = \sum_{m=2}^{\tilde{m}} \omega_m^* m$.\(^{13}\) Equality holds if and only if $G(m, \theta) = I_m$ for all $m$ with $\omega_m^* > 0$.

Under Assumption 1, \( G(m, \theta) = I_m \) for all $m$ with $\omega_m^* > 0$ if and only if $\theta = \theta_0$, which establishes that $\theta_0$ is the unique maximizer of $\sum_{m=2}^{\tilde{m}} \omega_m^* g(m, \theta)$. To see this, observe that in light of (E.7) the equality $G(m, \theta) = I_m$ only holds if

$$\left( \frac{m - 1 + \lambda}{m - 1 + \lambda_0} \right)^2 \frac{\sigma_{\theta_0}^2}{\sigma_\epsilon^2} = 1,$$

(E.11)

$$\left( \frac{\sigma_{\theta_0}^2 + m \sigma_{\alpha_0}^2}{\sigma_\epsilon^2 + m \sigma_0^2} \right) \left( \frac{1 - \lambda}{1 - \lambda_0} \right)^2 = 1.$$

(E.12)

Consequently $\theta$ is a global maximizer of $\sum_{m=2}^{\tilde{m}} \omega_m^* g(m, \theta)$ if and only if it satisfies both (E.11) and (E.12) for all $m$ with $\omega_m^* > 0$.

Under Assumption 1, we have at least one pair $m' \neq m''$ with $\omega_{m'} > 0$ and $\omega_{m''} > 0$. For equality (E.11) to hold for both $m'$ and $m''$, we need

$$\frac{m' - 1 + \lambda}{m' - 1 + \lambda_0} = \frac{m'' - 1 + \lambda}{m'' - 1 + \lambda_0} = \frac{\sigma_{\theta_0}^2}{\sigma_\epsilon^2},$$

(E.13)

which only holds when $\lambda = \lambda_0$. Plugging $\lambda = \lambda_0$ into (E.12) we obtain $\sigma_{\theta_0}^2 - \sigma_\epsilon^2 = m'(\sigma_{\alpha}^2 - \sigma_{\alpha_0}^2)$ and $\sigma_{\theta_0}^2 - \sigma_\epsilon^2 = m''(\sigma_{\alpha}^2 - \sigma_{\alpha_0}^2)$, which hold for both $m'$ and $m''$ only if $\sigma_{\alpha}^2 = \sigma_{\alpha_0}^2$ and $\sigma_\epsilon^2 = \sigma_\epsilon^2$. Therefore, $\theta_0$ is the unique global maximizer of $\sum_{m=2}^{\tilde{m}} \omega_m^* g(m, \theta)$ and thus of $\tilde{Q}^*(\theta)$.

### E.2 Proof of Theorem 4.1(b)

To prove the consistency of the QMLE estimator $\hat{\theta}_N$ we utilize Lemma 3.1 of Pötscher and Prucha (1991). Previously we have shown that $\theta_0$ is the unique maximizer of $\tilde{Q}^*(\theta)$ on $\Theta$, where $\tilde{Q}^*(\theta)$ is finite and continuous. The compactness of $\Theta$ follows from Assumptions 2, 3 and 4. To prove consistency of $\hat{\theta}$, it then suffices to have Condition 3. Since $\tilde{\beta}(\theta_0) = \beta_0$, once we have shown that $\hat{\theta}_N \to_p \theta_0$, consistency of $\tilde{\beta}_N(\hat{\theta}_N)$ follows from Condition 4.

**Condition 3.** As $N$ goes to infinity, $\sup_{\theta \in \Theta} | Q_N(\theta) - \tilde{Q}^*(\theta) | \to_p 0$.

**Condition 4.** As $N$ goes to infinity, $\sup_{\theta \in \Theta} | \tilde{\beta}_N(\theta) - \tilde{\beta}(\theta) | \to_p 0$.

**Verification of Condition 3.** Verification of Condition 1 has shown that $\sup_{\theta \in \Theta} | E Q_N(\theta) - \tilde{Q}^*(\theta) | \to 0$ as $N \to \infty$. It remains to show that as $N$ goes to infinity, $\sup_{\theta \in \Theta} | Q_N(\theta) - E [ Q_N(\theta) ] | \to_p 0$. Upon substitution of $Y = (I - \lambda_0 W)^{-1}(Z \beta_0 + U)$ into (29) we have

$$Q_N(\theta) - E [ Q_N(\theta) ] = \frac{1}{N} \{ U' A Q_N(\theta) U + 2 U' A Q_N(\theta) Z \beta_0 - tr [ A Q_N(\theta) \Omega_0 ] \},$$

\(^{13}\)See Footnote 7 for details.
where

\[ A_{QN}(\theta) = -\frac{1}{2} (I - \lambda_0W)^{-1}(I - \lambda W)'M_Z(\theta)(I - \lambda W)(I - \lambda_0W)^{-1}. \]

The row and column sums in absolute value of \((I - \lambda_0W)^{-1} (I - \lambda W), M_Z(\theta)\) and their first derivatives are all uniformly bounded in absolute value. It now follows from Lemma C.3 that \(Q_N(\theta) - E[Q_N(\theta)] \rightarrow_p 0\) uniformly in \(\theta\).

\textbf{Verification of Condition 4.1}

In light of (32) we have

\[
\hat{\beta}_N(\theta) = (Z'\Omega(\theta)^{-1}Z)^{-1}Z'\Omega(\theta)^{-1}(I - \lambda W)Y
\]

\[
= [N^{-1}Z'\Omega(\theta)^{-1}Z]^{-1}[N^{-1}Z'\Omega(\theta)^{-1}(I - \lambda W)(I - \lambda_0W)^{-1}Z]\beta_0
\]

\[
+ [N^{-1}Z'\Omega(\theta)^{-1}Z]^{-1}[N^{-1}Z'\Omega(\theta)^{-1}(I - \lambda W)(I - \lambda_0W)^{-1}U].
\]

By Lemma C.3

\[
\sup_\theta [N^{-1}Z'\Omega(\theta)^{-1}(I - \lambda W)(I - \lambda_0W)^{-1}U] \rightarrow_p 0.
\]

Also \([N^{-1}Z'\Omega(\theta_N)^{-1}Z]^{-1}\) is uniformly bounded in absolute value. By Lemma C.5, \(\sup_{\theta \in \Theta} \{[N^{-1}Z'\Omega(\theta)^{-1}Z]^{-1} - \gamma_3(\theta)^{-1}\} \rightarrow 0\) and \(\sup_{\theta \in \Theta} [N^{-1}Z'\Omega(\theta)^{-1}(I - \lambda W)(I - \lambda_0W)^{-1}Z] - \gamma_2(\theta) \rightarrow 0\). In all, we have \(\sup_{\theta \in \Theta} |\hat{\beta}_N(\theta) - \beta(\theta)| \rightarrow_p 0\).

\textbf{E.3 Proof of Theorem 4.2}

To derive the limiting distribution of the QMLE \(\hat{\delta}_N = (\hat{\theta}_N, \hat{\beta}_N)'\) it proves more convenient to work with the unconcentrated log-likelihood function defined in (25). Applying the mean value theorem, the first order condition for the QMLE can be written as

\[
0 = \frac{1}{N^{1/2}} \frac{\partial \ln L_N(\tilde{\delta}_N)}{\partial \hat{\delta}} = \frac{1}{N^{1/2}} \frac{\partial \ln L_N(\tilde{\delta}_0)}{\partial \hat{\delta}} + \frac{1}{N} \frac{\partial \ln L_N(\tilde{\delta}_N)}{\partial \delta \partial \hat{\delta}} N^{1/2}(\tilde{\delta}_N - \tilde{\delta}_0),
\]

where \(\tilde{\delta}_N\) denotes a “between” value vector. Given that \(\hat{\delta}_N\) was shown to be consistent, it follows that also the “between” value \(\tilde{\delta}_N\) is consistent for \(\tilde{\delta}_0\). It is not difficult to see that

\[
\frac{\partial \ln L_N(\delta)}{\partial \delta} = \begin{bmatrix}
-\text{tr}[(I - \lambda W)^{-1}W] + U'(\delta)'\Omega^{-1}WY \\
-\frac{1}{2} \text{tr}[\Omega^{-1}] + \frac{1}{2} U'(\delta)'\Omega^{-2}U(\delta) \\
-\frac{1}{2} \text{tr}[\Omega^{-1} \text{diag}_{r=1}^R(m_rJ^*_{mr})] + \frac{1}{2} U'(\delta)'\Omega^{-1}[\text{diag}_{r=1}^R(m_rJ^*_{mr})]\Omega^{-1}U(\delta) \\
Z'\Omega^{-1}U(\delta)
\end{bmatrix}
\]

with \(U(\delta) = Y - \lambda WY - Z\beta\), and thus

\[
\frac{\partial \ln L_N(\tilde{\delta}_0)}{\partial \delta} = \begin{bmatrix}
-\text{tr}[(I - \lambda_0W)^{-1}W] + U'\Omega_0^{-1}W(I - \lambda_0W)^{-1}(Z\beta_0 + U) \\
-\frac{1}{2} \text{tr}[\Omega_0^{-1}] + \frac{1}{2} U'\Omega_0^{-2}U \\
-\frac{1}{2} \text{tr}[\Omega_0^{-1} \text{diag}_{r=1}^R(J^*_{mr})] + \frac{1}{2} U'\Omega_0^{-1}[\text{diag}_{r=1}^R(J^*_{mr})]\Omega_0^{-1}U \\
Z'\Omega_0^{-1}U
\end{bmatrix}.
\]

(E.14)
Furthermore, it is not difficult to see that with \( \theta_2 = \sigma^2_\epsilon \), \( \theta_3 = \sigma^2_\alpha \), the elements of the Hessian matrix are

\[
\begin{align*}
\frac{\partial^2 \ln L_N(\delta)}{\partial \lambda^2} &= -\text{tr}[(I - \lambda W)^{-2}W^2] - Y'W'\Omega(\theta)^{-1}WY, \\
\frac{\partial^2 \ln L_N(\delta)}{\partial \lambda \partial \theta_i} &= -U(\delta)'\Omega(\theta)^{-1} \frac{\partial \Omega(\theta)}{\partial \theta_i} \Omega(\theta)^{-1}WY, \\
\frac{\partial^2 \ln L_N(\delta)}{\partial \theta_i \partial \theta_j} &= \frac{1}{2} \text{tr}[\Omega(\theta)^{-2} \frac{\partial \Omega(\theta)}{\partial \theta_i} \Omega(\theta)^{-1} \frac{\partial \Omega(\theta)}{\partial \theta_j} \Omega(\theta)^{-1}U(\delta)], \\
\frac{\partial^2 \ln L_N(\delta)}{\partial \theta_i \partial \beta} &= -Z'\Omega(\theta)^{-1} \frac{\partial \Omega(\theta)}{\partial \theta_i} \Omega(\theta)^{-1}U(\delta), \\
\frac{\partial^2 \ln L_N(\delta)}{\partial \lambda \partial \beta} &= -Z'\Omega(\theta)^{-1}WY, \\
\frac{\partial^2 \ln L_N(\delta)}{\partial \beta \partial \beta'} &= -Z'\Omega(\theta)^{-1}Z,
\end{align*}
\]

with \( i, j = 2, 3 \) and

\[
\frac{\partial \Omega(\theta)}{\partial \theta_2} = I_N, \quad \frac{\partial \Omega(\theta)}{\partial \theta_3} = \text{diag}_{r=1}^R \{ J_{m_r}, m_r \}.
\]

Since \( Y = (I - \lambda_0 W)^{-1}(Z\beta_0 + U) \) and \( U(\delta) = (I - \lambda W)(I - \lambda_0 W)^{-1}(Z\beta_0 + U) - Z\beta \), each element of \( N^{-1} \frac{\partial^2 \ln L_N(\delta)}{\partial \delta \partial \delta'} \) is a linear quadratic form of \( U \) or \( Z \) in the form of \( \frac{1}{N} U'A(\theta)U \), \( \frac{1}{N} Z'A(\theta)U \), \( \frac{1}{N} Z'A(\theta)Z \), and their products with \( \beta \): \( \frac{1}{N} \beta' Z' A(\theta) U \), \( \frac{1}{N} Z' A(\theta)Z \beta \), \( \frac{1}{N} \beta' Z' A(\theta)Z \beta \), etc., where \( A(\theta) = \text{diag}_{r=1}^R \{ p(m_r, \theta)I_{m_r} + q(m_r, \theta)J_{m_r} \} \) satisfies the conditions in Lemma C.3 and Lemma C.5. By these two lemmas, if \( \hat{\theta}_N \to_p \theta_0 \), all three types of linear quadratic forms converge to the limit of their expected value at \( \theta_0 \) in probability. That is as \( N \) goes to infinity,

\[
\begin{align*}
\left| \frac{1}{N} U'A(\hat{\theta}_N)U - \lim_{N \to \infty} \frac{1}{N} \text{tr}[A(\theta_0)\Omega_0] \right| &\to_p 0, \\
\left| \frac{1}{N} Z'A(\theta_0)U \right| &\to_p 0, \\
\left| \frac{1}{N} Z'A(\hat{\theta}_N)Z - \lim_{N \to \infty} \frac{1}{N} Z'A(\theta_0)Z \right| &\to_p 0.
\end{align*}
\]

Also, we have \( \hat{\beta}_N \to_p \beta_0 \). Thus Slutsky’s theorem, also the products of the linear quadratic forms with \( \hat{\beta}_N \) converge in probability to the products of the expected values with \( \beta_0 \). Therefore, as \( \delta_N \to_p \delta_0 \),

\[
\frac{1}{N} \frac{\partial \ln L_N(\delta_N)}{\partial \delta} \overset{p}{\to} \lim_{N \to \infty} \frac{1}{N} E \frac{\partial^2 \ln L_N(\delta_0)}{\partial \delta_0 \partial \delta'} = -\Gamma_0,
\]

where the specific structure of \( \Gamma_0 \) is given in Appendix F.

We next show that \( N^{1/2}(\partial \ln L_N(\delta_0) / \partial \delta_0) \delta_0 \overset{d}{\to} N(0, \Sigma_0) \). Each element of the score function in
\[ (E.14) \text{can be written as a linear quadratic form of } U \text{ in the form of } U'A_N(\theta_0)U + U'B_N(\theta_0)Z\beta_0 + C(\delta_0), \text{ which has zero mean and where the row and column sums of } A_N(\theta_0) \text{ and } B_N(\theta_0) \text{ are uniformly bounded in absolute value. Using Theorem } \boxed{C.2}, N^{-1/2}(\partial \ln L_N(\delta)/\partial \delta)_{\delta_0} \xrightarrow{d} N(0, \Upsilon_0) \text{ with } \Upsilon_0 = \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \left[ \frac{\partial \ln L_N(\delta)}{\partial \delta} \frac{\partial \ln L_N(\delta)}{\partial \delta'} \right], \text{ where } \Upsilon_0 = \Gamma_0 + \Xi_0 \text{ with expressions of } \Gamma_0 \text{ and } \Xi_0 \text{ given in Appendix F. In all, } \sqrt{N}(\hat{\delta}_N - \delta_0) \xrightarrow{d} N(0, \Gamma_0^{-1} \Upsilon_0 \Gamma_0^{-1}) \text{ as } N \text{ goes to infinity.}

E.4 Proof of Theorem 4.3

Let \( \hat{U}_r = (\hat{u}_{1r}, ..., \hat{u}_{mr})' \), then

\[
\hat{U}_r = (I - \hat{\lambda}W)Y_r - Z_r\hat{\beta} = (I - \hat{\lambda}W)(I - \lambda_0W)^{-1}(Z_r\beta_0 + U_r) - Z_r\hat{\beta}
\]

\[
= [(I - \hat{\lambda}W)(I - \lambda_0W)^{-1}Z_r\beta_0 - Z_r\hat{\beta}] + (I - \hat{\lambda}W)(I - \lambda_0W)^{-1}U_r.
\]

Note that

\[
(I - \hat{\lambda}W)(I - \lambda_0W)^{-1} = \frac{m_r - 1 + \hat{\lambda}}{m_r - 1 + \lambda_0} I_{mr} + \frac{1 - \hat{\lambda}}{1 - \lambda_0} J^*_{mr},
\]

where \( J^*_{mr} = \iota_{mr}/m_r \) and \( I_{mr} = I_{mr} - J^*_{mr} \) are two orthogonal idempotent matrices that generate vectors of group means and vectors of deviations from the group means. Thus

\[
\hat{u}_r = \iota_{mr}U_r/m_r = \hat{z}_r(\frac{1 - \hat{\lambda})\beta_0}{1 - \lambda_0} - \hat{\beta}) + \frac{1 - \hat{\lambda}}{1 - \lambda_0} \hat{u}_r = \hat{z}_r\hat{\phi} + \hat{\varphi}\hat{u}_r,
\]

where \( \hat{\phi} = \frac{(1 - \hat{\lambda})\beta_0}{1 - \lambda_0} - \hat{\beta}, \hat{\varphi} = \frac{1 - \hat{\lambda}}{1 - \lambda_0}, \hat{z}_r = \frac{1}{m_r} \sum_{i=1}^{m_r} \hat{z}_{ir} \). Let \( \hat{U}_r = (\hat{u}_{1r}, ..., \hat{u}_{mr})' \), then

\[
\hat{U}_r = I^*_{mr} \hat{U}_r = \hat{Z}_r(\frac{m_r - 1 + \hat{\lambda}}{m_r - 1 + \lambda_0} \beta_0 - \hat{\beta}) + \frac{m_r - 1 + \hat{\lambda}}{m_r - 1 + \lambda_0} \hat{U}_r,
\]

and

\[
\hat{u}_{ir} = \hat{z}_{ir}\hat{\phi}_r + \hat{\varphi}_r\hat{u}_{ir},
\]

where \( \hat{\phi}_r = \frac{m_r - 1 + \hat{\lambda}}{m_r - 1 + \lambda_0} \beta_0 - \hat{\beta}, \hat{\varphi}_r = 1 + \frac{\lambda - \lambda_0}{m_r - 1 + \lambda_0} \), \( \hat{z}_{ir} = z_{ir} - \hat{z}_r \).

We first show that \( \hat{\mu}_e^{(4)} \to_{p} \mu_{e0}^{(4)} \). Recall that \( \frac{1}{R} \sum_{r=1}^{R} f_{e,r}^{(4)} \to_{p} \mu_{e0}^{(4)} \), and that \( \hat{\mu}_e^{(4)} = \frac{1}{R} \sum_{r=1}^{R} \hat{f}_{e,r}^{(4)} \) with

\[
\hat{f}_{e,r}^{(4)} = \frac{m_r^2}{m_r^2 - 4m_r^2 + 6m_r - 3} \left[ \frac{1}{m_r} \sum_{i=1}^{m_r} \hat{a}_{ir}^4 \right] \frac{3(m_r - 1)(2m_r - 3)}{m_r^3} \sigma_e^4.
\]
Thus to prove the claim it suffices to show that \( \frac{1}{R} \sum_{r=1}^{R} (\hat{f}_{\epsilon,r}^{(4)} - f_{\epsilon,r}^{(4)}) \to_p 0 \). Now
\[
\frac{1}{R} \sum_{r=1}^{R} (\hat{f}_{\epsilon,r}^{(4)} - f_{\epsilon,r}^{(4)}) = \frac{1}{R} \sum_{r=1}^{R} \frac{m_r^3}{m_r^3 - 4m_r^2 + 6m_r - 3} \left[ \frac{1}{m_r} \sum_{i=1}^{m_r} (\hat{u}_{ir}^4 - \bar{u}_{ir}^4) \right] - (\hat{\sigma}^4_0 - \sigma^4_0) \frac{1}{R} \sum_{r=1}^{R} \frac{3(m_r - 1)(2m_r - 3)}{m_r^3 - 4m_r^2 + 6m_r - 3}
= \frac{1}{R} \sum_{r=1}^{R} \frac{m_r^2}{m_r^3 - 4m_r^2 + 6m_r - 3} \sum_{i=1}^{m_r} (\hat{u}_{ir}^4 - \bar{u}_{ir}^4) + o_p(1),
\]
since the sum in the second line is bounded and \( \hat{\sigma}^4_0 \to_p \sigma^4_0 \). Next observe that
\[
\hat{u}_{ir}^4 - \bar{u}_{ir}^4 = (\bar{z}_{ir}\bar{\phi}_r + \bar{\varphi}_r\bar{u}_{ir})^4 - \bar{u}_{ir}^4
= (\bar{z}_{ir}\bar{\phi}_r)^4 + 4(\bar{z}_{ir}\bar{\phi}_r)^3(\bar{\varphi}_r\bar{u}_{ir}) + 6(\bar{z}_{ir}\bar{\phi}_r)^2(\bar{\varphi}_r\bar{u}_{ir})^2 + 4(\bar{z}_{ir}\bar{\phi}_r)(\bar{\varphi}_r\bar{u}_{ir})^3 + (\bar{\varphi}_r\bar{u}_{ir})^4 - \bar{u}_{ir}^4.
\]
Upon substitution of this expression into equation (E.22), the claim now follows immediately from Lemma C.4(b)-(d). Similarly, \( \hat{\mu}_c^{(3)} \), \( \hat{\mu}_d^{(3)} \), and \( \hat{\mu}_s^{(4)} \) are consistent estimators for \( \mu_{c0}^{(3)} \), \( \mu_{d0}^{(3)} \), and \( \mu_{s0}^{(4)} \) respectively.

### F Variance-Covariance Matrix and Proof of Lemma 4.1

#### F.1 Variance-Covariance Matrix

Recall that \( \Gamma_0 = \text{lim}_{N \to \infty} -\frac{1}{N} E \left[ \frac{\partial^2 \ln L_N(\delta_0)}{\partial \delta_0 \partial \delta_0} \right] \) and \( \Upsilon_0 = \text{lim}_{N \to \infty} -\frac{1}{N} E \left[ \frac{\partial \ln L_N(\phi_0)}{\partial \phi_0} \frac{\partial \ln L_N(\delta_0)}{\partial \delta_0} \right] \). Those matrices are of dimension \((3 + k_Z) \times (3 + k_Z)\) and symmetric, and underlie the expression for the limiting variance covariance matrix of the QMLE estimator for \( \delta_0 \). In the following we give explicit expressions for \( \Gamma_0 \) and \( \Upsilon_0 \). Detailed derivations are provided in the Online Appendix. We have
\[
\Upsilon_0 = \sum_{m=2}^{\hat{a}} \varphi_m \bar{\Psi}(m) \varphi'_m, \quad (F.1)
\]
and
\[
\Gamma_0 = \sum_{m=2}^{\hat{a}} \varphi_m \bar{\Psi}_G(m) \varphi'_m,
\]
where
\[
\varphi_m = \begin{pmatrix}
\frac{1}{(m-1+\lambda_0)\sigma^2_0} & -\frac{m}{2\sigma^2_0} & \frac{m}{2\sigma^2_0} & \frac{m}{2\sigma^2_0} & \frac{m}{2\sigma^2_0} & \frac{m}{2\sigma^2_0} & \frac{m}{2\sigma^2_0} \\
\frac{1}{2\sigma^4_0} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{2}{(\sigma^2_0+\sigma^2_0)^2} & -\frac{2}{(\sigma^2_0+\sigma^2_0)^2} & -\frac{2}{(\sigma^2_0+\sigma^2_0)^2} & -\frac{2}{(\sigma^2_0+\sigma^2_0)^2} & -\frac{2}{(\sigma^2_0+\sigma^2_0)^2} & -\frac{2}{(\sigma^2_0+\sigma^2_0)^2} \\
0 & 0 & -\frac{1}{\sigma^4_0} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad (F.2)
\]
which is given in (37) and repeated here for the convenience of the reader,
\[
\bar{\Psi}_G(m) = \text{diag}\{2(m-1)\sigma^4_{\epsilon0} + \omega^*_m, 2(\sigma^2_{\epsilon0} + \sigma^2_{\epsilon0}/m)\omega^*_m, \sigma^2_{\epsilon0}, (\sigma^2_{\epsilon0} + \sigma^2_{\epsilon0}/m)\bar{\xi}_m, (\sigma^2_{\epsilon0} + \sigma^2_{\epsilon0}/m)\bar{\Xi}_m\}, \tag{F.3}
\]
\[
\bar{\Psi}(m) = \begin{bmatrix} \bar{\Psi}_{11}(m) & \bar{\Psi}_{12}(m) \\ \bar{\Psi}_{21}(m) & \bar{\Psi}_{22}(m) \end{bmatrix}, \tag{F.4}
\]
with
\[
\bar{\Psi}_{11}(m) = \frac{\omega^*_m}{m^*} \begin{bmatrix} 2(m-1)\sigma^4_{\epsilon0} & 0 \\ 0 & 2(\sigma^2_{\epsilon0} + \sigma^2_{\epsilon0}/m) + (\mu^{(4)}_{\epsilon0} - 3\sigma^4_{\epsilon0}) \frac{\omega^*_m}{m^*} \end{bmatrix} + (\mu^{(4)}_{\epsilon0} - 3\sigma^4_{\epsilon0}) \frac{\omega^*_m}{m^*} \begin{bmatrix} (m-1)^2 \frac{m}{m^*} & (m-1)^2 \frac{m}{m^*} \\ (m^*)^2 & 1 \\ \frac{m}{m^*} \end{bmatrix},
\]
\[
\bar{\Psi}_{21}(m) = \begin{bmatrix} m-1 \mu^{(3)} z^{(m)}_{\epsilon0} & 0 \\ m \mu^{(3)} z^{(m)}_{\epsilon0} & \mu^{(3)} \end{bmatrix} = \Psi'_{12}(m),
\]
\[
\bar{\Psi}_{22}(m) = \begin{bmatrix} \sigma^2_{\epsilon0} \bar{\xi}_m & 0 \\ 0 & (\sigma^2_{\epsilon0} + \sigma^2_{\epsilon0}/m) \bar{\Xi}_m \end{bmatrix}.
\]

Note that \(\bar{\Psi}_G(m)\) can be obtained by setting \(\mu^{(4)}_{\epsilon0} - 3\sigma^4_{\epsilon0} = \mu^{(4)}_{\epsilon0} - 3\sigma^4_{\epsilon0} = \mu^{(3)}_{\epsilon0} = \mu^{(3)}_{\epsilon0} = 0\) in \(\bar{\Psi}(m)\).

When \(\epsilon\) and \(\alpha\) are both Gaussian, \(\Upsilon_0 = \Gamma_0\), consistent with what is expected from the information matrix equality.

**F.2 Proof of the Positive Definiteness of \(\Upsilon_0\) and \(\Gamma_0\)**

Let \(\varphi_{\epsilon m}, \bar{\Psi}_G(m)\) and \(\bar{\Psi}(m)\) be as defined in [F.2], [F.3] and [F.4] respectively. To prove Lemma [4.1] we introduce the two lemmas below.

**Lemma F.1.** Suppose the Assumptions 1-5 hold and \(\omega^*_m > 0\), then \(\bar{\Psi}_G(m)\) is positive definite.

**Lemma F.2.** Suppose the Assumptions 1-5 hold and assume further that \(\mu^{(4)}_{\epsilon} - \sigma^4_{\epsilon} > (\mu^{(3)}_{\epsilon})^2/\sigma^2_{\epsilon}\) and \(\omega^*_m > 0\), then \(\bar{\Psi}(m)\) is positive definite.

Lemma [F.1] follows easily from (F.3), observing that \(\omega^*_m > 0, \sigma^2_{\epsilon0} > 0\) and \(\bar{\xi}_m\) and \(\bar{\Xi}_m\) are p.d. if \(\omega^*_m > 0\) under Assumption [5]. Utilizing Equation (F.4) and that \(\mu^{(4)}_{\epsilon} - \sigma^4_{\epsilon} > (\mu^{(3)}_{\epsilon})^2/\sigma^2_{\epsilon}\), it can be shown that when \(\omega^*_m > 0\) for any nonzero \(3 + k_Z\) dimensional vector \(\ell, \ell \bar{\Psi}(m)\ell > 0\) unless \(\ell = 0\) and hence \(\bar{\Psi}(m)\) is p.d. A detailed proof of Lemma [F.2] is in the Online Appendix.

We have shown in the previous subsection that
\[
\Upsilon_0 = \sum_{m=2}^{\tilde{a}} \varphi_{\epsilon m} \bar{\Psi}(m) \varphi'_{\epsilon m},
\]
\[
\Gamma_0 = \sum_{m=2}^{\tilde{a}} \varphi_{\epsilon m} \bar{\Psi}_G(m) \varphi'_{\epsilon m}.
\]
We can now utilize the above lemmas to prove that under the maintained assumptions \(\Gamma_0\) is positive definite, and that the matrix \(\Upsilon_0\) is positive definite for \(\mu^{(4)}_{\epsilon} - \sigma^4_{\epsilon} > (\mu^{(3)}_{\epsilon})^2/\sigma^2_{\epsilon}\). We present a proof
for the positive definiteness of $\Upsilon_0$. The proof for the positive definiteness of $\Gamma_0$ is analogous. By Assumption 1 in the main text, there exist $m' \neq m''$ such that $\omega^*_m > 0$ and $\omega^*_{m''} > 0$, let

$$
\lambda_s = \min \{ \lambda_{\min}(\Psi(m')), \lambda_{\min}(\Psi(m'')) \}
$$

be the minimum of the smallest eigenvalues of $\Psi(m')$ and $\Psi(m'')$, and observe that by Lemma F.2 we have $\lambda_s > 0$. Next observe that $[\varphi_{m'}, \varphi_{m''}]$ has full row rank. Consequently $\alpha'[\varphi_{m'}, \varphi_{m''}] [\varphi_{m'}, \varphi_{m''}] \alpha$ is positive definite and thus for any nonzero $3 + k_Z$ dimensional column vector $\alpha$,

$$
\alpha' \Upsilon_0 \alpha \geq \alpha'[\varphi_{m'} \bar{\Psi}(m') \varphi_{m'} + \varphi_{m''} \bar{\Psi}(m'') \varphi_{m''}] \alpha

\geq \lambda_s \alpha'[\varphi_{m'}, \varphi_{m''}] [\varphi_{m'}, \varphi_{m''}] \alpha > 0,
$$

which proves that $\Upsilon_0$ is positive definite.

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$$
\begin{pmatrix}
\frac{1}{\sigma_0} \left( \frac{1}{m' - 1 + \lambda_0} - \frac{1}{m'' - 1 + \lambda_0} \right) & -m' \lambda_s \\
0 & -\frac{1}{2 \sigma_0} \left( \frac{1}{m' - 1 + \lambda_0} - \frac{1}{m'' - 1 + \lambda_0} \right) - m'' \lambda_s \\
0 & 0
\end{pmatrix}
$$

first column is nonzero. It is then easy to see from the first three columns that $(A_{m'}, A_{m''})$ has full row rank.