THE SCALING LIMIT OF SENILE REINFORCED RANDOM WALK.

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Abstract
We prove that the scaling limit of nearest-neighbour senile reinforced random walk is Brownian Motion when the time $T$ spent on the first edge has finite mean. We show that under suitable conditions, when $T$ has heavy tails the scaling limit is the so-called fractional kinetics process, a random time-change of Brownian motion. The proof uses the standard tools of time-change and invariance principles for additive functionals of Markov chains.

1 Introduction

The senile reinforced random walk is a toy model for a much more mathematically difficult model known as edge-reinforced random walk (for which many basic questions remain open [e.g. see [15]]). It is characterized by a reinforcement function $f : \mathbb{N} \rightarrow [-1, \infty)$, and has the property that only the most recently traversed edge is reinforced. As soon as a new edge is traversed, reinforcement begins on that new edge and the reinforcement of the previous edge is forgotten. Such walks may get stuck on a single (random) edge if the reinforcement is strong enough, otherwise (except for one degenerate case) they are recurrent/transient precisely when the corresponding simple random walk is [9].

Formally, a nearest-neighbour senile reinforced random walk is a sequence $\{S_n\}_{n \geq 0}$ of $\mathbb{Z}^2$-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with corresponding filtration $\{\mathcal{F}_n = \sigma(S_0, \ldots, S_n)\}_{n \geq 0}$, defined by:

- $S_0 = o$, $\mathbb{P}$-almost surely, and $\mathbb{P}(S_1 = x) = (2d)^{-1}I_{|x| = 1}$.
- For $n \in \mathbb{N}$, $e_n = (S_{n-1}, S_n)$ is an $\mathcal{F}_n$-measurable undirected edge and

$$m_n = \max\{k \geq 1 : e_{n-l+1} = e_n \text{ for all } 1 \leq l \leq k\}$$

is an $\mathcal{F}_n$-measurable, $\mathbb{N}$-valued random variable.

1 RESEARCH SUPPORTED BY THE NETHERLANDS ORGANISATION FOR SCIENTIFIC RESEARCH (NWO)
• For \( n \in \mathbb{N} \) and \( x \in \mathbb{Z}^d \) such that \( |x| = 1 \),
\[
\mathbb{P}_f(S_{n+1} = S_n + x | \mathcal{F}_n) = \frac{1 + f(m_n)I_{((S_n, S_{n+1}) \equiv 1)}}{2d + f(m_n)}.
\] (1.2)

Note that the triple \((S_n, e_n, m_n)\) (equivalently \((S_n, S_{n-1}, m_n)\)) is a Markov chain. Hereafter we suppress the \( f \) dependence of the probability \( \mathbb{P}_f \) in the notation.

The diffusion constant is defined as \( \lim_{n \to \infty} n^{-1} \mathbb{E}[|S_n|^2] \) (=1 for simple random walk) whenever this limit exists. Let \( T \) denote the random number of consecutive traversals of the first edge traversed, and \( p = \mathbb{P}(T \text{ is odd}) \). Then when \( \mathbb{E}[T] < \infty \), the diffusion constant is given by ([9] and [11])
\[
v = \frac{d p}{(d - p) \mathbb{E}[T]},
\] (1.3)

which is not monotone in the reinforcement. Indeed one can prove that (1.3) holds for all \( f \) (in the case \( d = 1 \) and \( f(1) = -1 \) this must be interpreted as “1/0 = \infty”). The reinforcement regime of most interest is that of linear reinforcement \( f(n) = C n \) for some \( C \). In this case, by the second order mean-value theorem applied to \( \log(1 - x) \), \( x < 1 \) we have
\[
\mathbb{P}(T \geq n) = \prod_{j=1}^{n-1} \frac{1 + f(j)}{2d + f(j)} = \exp \left\{ \sum_{j=1}^{n-1} \log \left( 1 - \frac{2d - 1}{2d + C j} \right) \right\}
\]
\[
= \exp \left\{ - \sum_{j=1}^{n-1} \frac{2d - 1}{2d + C j} - \sum_{j=1}^{n-1} \frac{(2d - 1)^2}{2(2d + C j)^2(1 - u_j)^2} \right\}
\]
\[
= \exp \left\{ - \sum_{j=1}^{n-1} \frac{2d - 1}{2d + C j} - \sum_{j=1}^{\infty} \frac{(2d - 1)^2}{2(2d + C j)^2(1 - u_j)^2} + o(1) \right\}
\]
\[
= \exp \left\{ - \frac{2d - 1}{C} \log(2d + C(n - 1)) + \gamma + o(1) \right\} \sim \frac{\kappa}{n^\gamma},
\] (1.4)

where \( u_j \in (0, \frac{2d - 1}{2d + C}) \), and \( \gamma \) is a constant arising from the summable infinite series and the approximation of the finite sum by a log. An immediate consequence of (1.4) is that for \( f(n) = C n \), \( \mathbb{E}[T] \) is finite if and only if \( C < 2d - 1 \).

A different but related model, in which the current direction (rather than the current edge) is reinforced according to the function \( f \) was studied in [12,10]. For such a model, \( T \) is the number of consecutive steps in the same direction before turning. In [10], the authors show that in all dimensions the scaling limit is Brownian motion when \( \sigma^2 = \text{Var}(T) < \infty \) and \( \sigma^2 + 1 - 1/d > 0 \). In the language of this paper, the last condition corresponds to the removal of the special case \( d = 1 \) and \( f(1) = -1 \). Moreover when \( d = 1 \) and \( T \) has heavy tails (in the sense of (2.1) below) they show that the scaling limit is an \( \alpha \)-stable process when \( 1 < \alpha < 2 \) and a random time change of an \( \alpha \)-stable process when \( 0 < \alpha < 1 \). See [10] for more details.

Davis [4] showed that the scaling limit of once-reinforced random walk in one dimension is not Brownian motion (see [15] for further discussion).
In Section 2, we state and discuss the main result of this paper, which describes the scaling limit of $S_n$ when either $E[T] < \infty$ or $P(T \geq n) \sim n^{-\alpha}L(n)$ for some $\alpha > 0$ and $L$ slowly varying at infinity. When $P(T < \infty) < 1$ the walk has finite range since it traverses a random (geometric) number of edges before getting stuck on a random edge. To prove the main result, in Section 3 we observe the walk at the times that it has just traversed a new edge and describe this as an additive functional of a particular Markov chain. In Section 4 we prove the main result assuming the joint convergence of this time-changed walk and the associated time-change process. In Section 5 we prove the convergence of this joint process.

## 2 Main result

The assumptions that will be necessary to state the main theorem of this paper are as follows:

(A1) $P(T < \infty) = 1$, and either $d > 1$ or $P(T = 1) < 1$.

(A2a) Either $E[T] < \infty$, or for some $\alpha \in (0, 1]$ and $L$ slowly varying at infinity,

$$P(T \geq n) \sim L(n)n^{-\alpha}.$$  \hfill (2.1)

(A2b) If (2.1) holds but $E[T] = \infty$, then we also assume that

when $\alpha = 1$, $\exists \ell(n) \uparrow \infty$ such that $(\ell(n))^{-1}L(n\ell(n)) \to 0$, and $(\ell(n))^{-1}\sum_{j=1}^{\lfloor n\ell(n) \rfloor} j^{-1}L(j) \to 1$, \hfill (2.2)

when $\alpha < 1$, $P(T \geq n, T \text{ odd}) \sim L_o(n)n^{-\alpha_o}$, and $P(T \geq n, T \text{ even}) \sim L_e(n)n^{-\alpha_e}$,

where $\ell$, $L_o$ and $L_e$ are slowly varying at infinity and $L_o$ and $L_e$ are such that if $\alpha_o = \alpha_e$ then $L_o(n)/L_e(n) \to \beta \in [0, \infty]$ as $n \to \infty$.

Note that both (2.1) and $E[T] < \infty$ may hold when $\alpha = 1$ (e.g. take $L(n) = (\log n)^{-2}$).

By [6] Theorem XIII.6.2, when $\alpha < 1$ there exists $\ell(\cdot) > 0$ slowly varying such that

$$(\ell(n))^{-\alpha}L\left(n^{1/2}\ell(n)\right) \to \Gamma(1 - \alpha)^{-1}.$$  \hfill (2.3)

For $\alpha > 0$ let

$$g_\alpha(n) = \begin{cases} E[T]n & \text{if } E[T] < \infty \\ n^{2\alpha}/\ell(n) & \text{otherwise} \end{cases} \hfill (2.4)$$

By [3] Theorem 1.5.12, there exists an asymptotic inverse function $g_\alpha^{-1}(\cdot)$ (unique up to asymptotic equivalence) satisfying $g_\alpha(g_\alpha^{-1}(n)) \sim g_\alpha^{-1}(g_\alpha(n)) \sim n$, and by [3] Theorem 1.5.6 we may assume that $g_\alpha$ and $g_\alpha^{-1}$ are monotone nondecreasing.

A subordinator is a real-valued process starting at 0, with stationary, independent increments, such that almost every path is nondecreasing and right continuous. Let $B_d(t)$ be a standard $d$-dimensional Brownian motion. For $\alpha \geq 1$, let $V_\alpha(t) = t$ and for $\alpha \in (0, 1)$, let $V_\alpha$ be a standard $\alpha$-stable subordinator (independent of $B_d(t)$). This is a strictly increasing pure-jump Levy process whose law is specified by the Laplace transform of its one-dimensional distributions (see e.g. [2] Sections 1.1 and 1.2]

$$E[e^{-\lambda V_\alpha(t)}] := e^{-\tau^\alpha}.$$  \hfill (2.5)
Define the right-continuous inverse of $V_\alpha(t)$ and (when $\alpha < 1$ the fractional-kinetics process) $Z_\alpha(s)$ by
\[
V^{-1}_\alpha(s) := \inf\{t : V_\alpha(t) > s\}, \quad Z_\alpha(s) = B_\alpha(V^{-1}_\alpha(s)).
\] (2.6)

Since $V_\alpha$ is strictly increasing, both $V^{-1}_\alpha$ and $Z_\alpha$ are continuous (almost-surely). The main result of this paper is the following theorem, in which $D(E, \mathbb{R}^d)$ is the set of cadlag paths from $E$ to $\mathbb{R}^d$. Throughout this paper $\Rightarrow$ denotes weak convergence.

**Theorem 2.1.** Suppose that $f$ is such that (2.1) holds for some $\alpha > 0$, then as $n \to \infty$,
\[
\frac{S_{\lfloor nt \rfloor}}{\sqrt{d^{-p}g^{-1}_\alpha(n)}} \Rightarrow Z(t),
\] (2.7)

where the convergence is in $D([0,1], \mathbb{R}^d)$ equipped with the uniform topology.

### 2.1 Discussion

The limiting object in Theorem 2.1 is the scaling limit of a simple random walk jumping at random times $\tau_i$ with i.i.d. increments $T_i = \tau_i - \tau_{i-1}$ (e.g. see [13]) that are independent of the position and history of the walk. In [1] the same scaling limit is obtained for a class of (continuous time) trap models with $d \geq 2$, where a random jump rate or waiting time is chosen initially at each site and remains fixed thereafter. In that work, when $d = 1$, the mutual dependence of the time spent at a particular site on successive returns remains in the scaling limit, where the time change/clock process depends on the (local time of the) Brownian motion itself. The independence on returning to an edge is the feature which makes our model considerably easier to handle. For the senile reinforced random walk, the direction of the steps of the walk is dependent on the clock and we need to prove that the dependence is sufficiently weak so that it disappears in the limit.

While the slowly varying functions in $g_\alpha$ and $g^{-1}_\alpha$ are not given explicitly, in many cases of interest one can use [6] Theorem XIII.6.2 and [3] Section 1.5.7 to explicitly construct them. For example, let $L(n) = \kappa(\log n)^\beta$ for some $\beta \geq -1$. For $\alpha = 1$ we can take
\[
\ell(n) = \begin{cases} 
\kappa \log n, & \text{if } \beta = 0 \\
\kappa(\log n)^\beta, & \text{if } \beta = -1 \\
|\beta|^{-1}(\log n)^\beta + 1, & \text{otherwise,}
\end{cases} \quad \text{and} \quad g^{-1}_\alpha(n) = \begin{cases} 
(n(\kappa \log n)^{-1}, & \text{if } \beta = 0 \\
n(\kappa \log n)^{-1}, & \text{if } \beta = -1 \\
n|\beta|((\kappa \log n)^{-1})^{(\beta + 1)}, & \text{otherwise.}
\end{cases}
\] (2.8)

If $\alpha < 1$ we can take
\[
\ell(n) = \left(\kappa \Gamma(1-\alpha) \left(\frac{\log n}{\alpha}\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{\alpha}}, \quad \text{and} \quad g^{-1}_\alpha(n) = n^{\alpha} \left(\kappa(\alpha \log n)^{\beta}\right)^{-\alpha}. \] (2.9)

Assumption (A1) is simply to avoid the trivial cases where the walk gets stuck on a single edge (i.e. when $(1 + f(n))^{-1}$ is summable [9]) or is a self-avoiding walk in one dimension. For linear reinforcement $f(n) = Cn$, (1.4) shows that assumption (A2) holds with $\alpha = (2d - 1)/C$. It may be of interest to consider the scaling limit when $f(n)$ grows like $n\ell(n)$, where $\lim_{n \to \infty} \ell(n) = \infty$ but such that $(1 + f(n))^{-1}$ is not summable. An example is $f(n) = n \log n$, for which $P(T \geq n) \sim (C \log n)^{-1}$ satisfies (2.1) with $\alpha = 0$.

The condition (2.2) when $\alpha = 1$ is so that one can apply a weak law of large numbers. The condition holds for example when $L(n) = (\log n)^k$ for any $k \geq -1$. For the $\alpha < 1$ case, the
condition (2.2) holds (with \(\alpha_p = \alpha_e\) and \(L_0 = L_e\)) whenever there exists \(n_0\) such that for all \(n \geq n_0\), \(f(n) \geq f(n - 1) - (2d - 1)\) (so in particular when \(f\) is non-decreasing). To see this, observe that for all \(n \geq n_0\)

\[
\mathbb{P}(T \geq n, T \text{ even}) = \sum_{m=\lceil \frac{n}{2} \rceil}^{\infty} \mathbb{P}(T = 2m) = \sum_{m=\lceil \frac{n}{2} \rceil}^{\infty} \mathbb{P}(T = 2m + 1) \frac{2d + f(2m + 1)}{1 + f(2m)}
\]

(2.10)

Similarly, \(\mathbb{P}(T \geq n, T \text{ odd}) \geq \mathbb{P}(T \geq n + 1, T \text{ even})\) for all \(n \geq n_0\). If \(\alpha_p \neq \alpha_e\) in (2.2), then (2.1) implies that \(\alpha = \alpha_p \wedge \alpha_e\) and \(L\) is the slowly varying function corresponding to \(\alpha \in \{\alpha_p, \alpha_e\}\) in (2.2). If \(\alpha_p = \alpha_e\) then trivially \(L \sim L_p + L_e\) (\(\sim L_p\) if \(L_p(n)/L_e(n) \to \infty\)). One can construct examples of reinforcement functions giving rise to different asymptotics for the even and odd cases in (2.2), for example by taking \(f(2m) = m^2\) and \(f(2m + 1) = Cm\) for some well chosen constant \(C > 0\) depending on the dimension.

### 3 Invariance principle for the time-changed walk

In this section we prove an invariance principle for any senile reinforced random walk (satisfying (A1)) observed at stopping times \(\tau_n\) defined by

\[
\tau_0 = 0, \quad \tau_k = \inf\{n > (\tau_{k-1} \lor 1) : S_n \neq S_{n-2}\}. \quad (3.1)
\]

It is easy to see that \(\tau_n = 1 + \sum_{i=1}^{n} T_i\) for each \(n \geq 1\), where the \(T_i\), \(i \geq 1\) are independent and identically distributed random variables (with the same distribution as \(T\)), corresponding to the number of consecutive traversals of successive edges traversed by the walk.

**Proposition 3.1.** If (A1) is satisfied, then \((\frac{p}{1-p} n)^{-\frac{1}{2}} S_{\tau_{\lceil n \rceil}} \Rightarrow B_d(t)\) as \(n \to \infty\), where the convergence is in \(D([0, 1], \mathbb{R}^d)\) with the uniform topology.

The process \(S_{\tau_n}\) is a simpler one than \(S_n\) and one may use many different methods to prove Proposition 3.1 (see for example the martingale approach of [11]). We give a proof based on describing \(S_{\tau_n}\) as an additive functional of a Markov chain. This is not necessarily the simplest representation, but it is the most natural to the author.

Let \(\mathcal{X}\) denote the collection of pairs \((u, v)\) such that \(v\) is one of the unit vectors \(u_i \in \mathbb{Z}^d\), for \(i \in \{\pm 1, \pm 2, \cdots, \pm d\}\) (labelled so that \(u_{-i} = -u_i\)) and \(u\) is either \(0 \in \mathbb{Z}^d\) or one of the unit vectors \(u_i \neq -v\). The cardinality of \(\mathcal{X}\) is then \(|\mathcal{X}| = 2d + 2d(2d - 1) = (2d)^2\).

Given a senile reinforced random walk \(S_p\) with parameter \(p = \mathbb{P}(T \text{ odd}) \in (0, 1]\), we define an irreducible, aperiodic Markov chain \(X_n = (X_n^{[1]}, X_n^{[2]})\) with natural filtration \(\mathcal{F}_n = \sigma(X_1, \ldots, X_n)\), and finite state-space \(\mathcal{X}\), as follows.

For \(n \geq 1\), let \(X_n = (S_{\tau_n - 1} - S_{\tau_{n-1}}, S_{\tau_n} - S_{\tau_{n-1}})\), and \(Y_n = X_n^{[1]} + X_n^{[2]}\). It follows immediately that \(S_{\tau_n} = \sum_{m=1}^{n} Y_m\) and

\[
\mathbb{P}(X_1 = (0, u_i)) = \frac{1 - p}{2d}, \quad \mathbb{P}(X_1 = (u_i, u_j)) = \frac{p}{2d(2d - 1)} \quad \text{for each } i, j, \quad (j \neq -i). \quad (3.2)
\]
Now $T_n$ is independent of $X_1, \ldots, X_{n-1}$, and conditionally on $T_n$ being odd (resp. even), $S_{\tau_n} - S_{\tau_n - 1}$ (resp. $S_{\tau_n} - S_{\tau_n - 1}$) is uniformly distributed over the $2d - 1$ unit vectors in $\mathbb{R}^d$ other than $-X_{n-1}^{(2)}$ (resp. other than $X_{n-1}^{(2)}$). It is then an easy exercise to verify that $\{X_n\}_{n \geq 1}$ is a finite, irreducible and aperiodic Markov chain with initial distribution (3.2) and transition probabilities given by

$$\mathbb{P}(X_n = (u, v) | X_{n-1} = (u', v')) = \frac{1}{2d - 1} \times \begin{cases} p, & \text{if } u = 0 \text{ and } v \neq -v', \\ 1 - p, & \text{if } u = -v' \text{ and } v \neq v', \\ 0, & \text{otherwise.} \end{cases}$$

(3.3)

By symmetry, the first $2d$ entries of the unique stationary distribution $\bar{\pi} \in M_1(\mathscr{A})$ are all equal (say $\pi_a$) and the remaining $2d(2d - 1)$ entries are all equal (say $\pi_b$), and it is easy to check that

$$\pi_a = \frac{p}{2d}, \quad \pi_b = \frac{1 - p}{2d(2d - 1)}.$$

(3.4)

As an irreducible, aperiodic, finite-state Markov chain, $\{X_n\}_{n \geq 1}$ has exponentially fast, strong mixing, i.e. there exists a constant $c$ and $t < 1$ such that for every $k \geq 1$,

$$\alpha(k) := \sup_n \left\{ \mathbb{P}(F \cap G) - \mathbb{P}(F)\mathbb{P}(G) : F \in \sigma(X_j, j \leq n), G \in \sigma(X_j, j \geq n + k) \right\} \leq ct^k.$$

(3.5)

Since $Y_n$ is measurable with respect to $X_n$, the sequence $Y_n$ also has exponentially fast, strong mixing. To verify Proposition 3.1, we use the following multidimensional result that follows easily from [8] Corollary 1] using the Cramér-Wold device.

**Corollary 3.2.** Suppose that $W_n = (W_n^{(1)}, \ldots, W_n^{(d)})$, $n \geq 0$ is a sequence of $\mathbb{R}^d$-valued random variables such that $\mathbb{E}[W_n] = 0$, $\mathbb{E}[|W_n|^2] < \infty$ and $\mathbb{E}[n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} W_i^{(j)} W_j^{(j)}] \rightarrow \sigma^2 I_{j=i}$, as $n \rightarrow \infty$. Further suppose that $W_n$ is $\alpha$-strongly mixing and that there exists $\beta \in (2, \infty]$ such that

$$\sum_{k=1}^{\infty} \alpha(k)^{1-2/\beta} < \infty, \quad \text{and} \quad \limsup_{n \rightarrow \infty} \|W_n\|_\beta < \infty,$$

(3.6)

then $\mathcal{W}_n(t) := (\sigma^2 n)^{-\frac{1}{4}} \sum_{i=1}^{n} W_i \Longrightarrow B_d(t)$ as $n \rightarrow \infty$, where the convergence is in $D([0,1], \mathbb{R}^d)$ equipped with the uniform topology.

### 3.1 Proof of Proposition 3.1

Since $S_{\tau_n} = \sum_{m=1}^{n} Y_m$ where $|Y_m| \leq 2$, and the sequence $\{Y_n\}_{n \geq 0}$ has exponentially fast strong mixing, Proposition 3.1 will follow from Corollary 3.2 provided we show that

$$\mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} Y_i^{(j)} Y_j^{(j)} \right] \rightarrow \frac{p}{d - p} I_{j=l},$$

(3.7)

where the superscript $(j)$ denotes the $j$th component of the vector, e.g. $Y_m = (Y_m^{(1)}, \ldots, Y_m^{(d)})$. By symmetry, $\mathbb{E}[Y_i^{(j)} Y_j^{(j)}] = 0$ for all $i, j'$ and $j \neq l$, and it suffices to prove (3.7) with $j = l = 1$.

For $n \geq 2$, $\mathbb{E}[X_n^{[2]}(1) X_{n-1}^{(1)}] = \frac{2p - 1}{2d - 1} X_{n-1}^{[2]}(1)$, so by induction and the Markov property,

$$\mathbb{E}[X_n^{[2]}(1) | X_m] = \left( \frac{2p - 1}{2d - 1} \right)^{n-m} X_m^{[2]}(1), \quad \text{for every } n \geq m \geq 1.$$  

(3.8)
For \( n \geq 2 \), \( \mathbb{E}[Y_1^{(1)}|X_{n-1}] = \frac{2d(1-p)}{2d-1}X_{n-1}^{(2,1)} \), and the Markov property for \( X_n \) implies that

\[
\mathbb{E}[Y_1^{(1)}|X_m] = \frac{p - 2d(1-p)}{2d-1} \left( \frac{2p - 1}{2d - 1} \right) X_m^{(2,1)}, \quad \text{for } n > m \geq 1. \tag{3.9}
\]

For \( n > m \geq 1 \), and letting \( r = \frac{2p-1}{2d-1} \) we have

\[
\mathbb{E}[Y_1^{(1)}] = \mathbb{E}[Y_1^{(1)}|X_m] = \frac{p - 2d(1-p)}{2d-1} r^{n-1-m} \left( \mathbb{E}[X_m^{(1)}] X_m^{(2,1)} + \mathbb{E}[X_m^{(2,1)}]^2 \right) \tag{3.10}
\]

Lastly \( \mathbb{E}[Y_1^2] = (1-p) + \frac{4dp}{2d-1} \) and \( \mathbb{E}[Y_1^2] = p + \frac{4d(1-p)}{2d-1} \), for \( m \geq 2 \).

Combining these results, we get that

\[
\mathbb{E} \left[ \sum_{i=1}^{n} \sum_{m=1}^{n} Y_i^{(1)} Y_m^{(1)} \right] = 2 \sum_{i=2}^{n} \mathbb{E}[Y_1^{(1)}] \mathbb{E}[Y_1^{(1)}] + 2 \sum_{i=2}^{n} \mathbb{E}[Y_1^{(1)}] \mathbb{E}[Y_1^{(1)}] + \sum_{i=1}^{n} \mathbb{E}[Y_1^{(1)}]^2
\]

\[
= \frac{2p - 2d(1-p)}{2d - 1} \left[ \frac{2d - p}{2d - 1} \sum_{k=0}^{n-2} r^k + \frac{2d - 1}{2d - 1} \sum_{i=2}^{n} \frac{2d - 1 + p}{2d - 1} r^{i-2} \right] + \frac{1 - p}{d} + \frac{4p}{2d - 1} + (n - 1) \left( \frac{p}{d} + \frac{4(1-p)}{2d - 1} \right). \tag{3.11}
\]

Since \( r < 1 \), the second sum over \( i \) is bounded by a constant, uniformly in \( n \). Thus, this is equal to

\[
\frac{2p - 2d(1-p)}{d} \left( \frac{2d - p}{2d - 1} \right) \sum_{i=2}^{n} \frac{1 - r^{i-2}}{1 - r} + n \left( \frac{p}{d} + \frac{4(1-p)}{2d - 1} \right) + \mathcal{O}(1)
\]

\[
= n \left[ \frac{2p - 2d(1-p)}{d(1-r)} \left( \frac{2d - p}{2d - 1} \right) + \frac{4(1-p)}{2d - 1} \right] + \mathcal{O}(1) \tag{3.12}
\]

Dividing by \( n \) and taking the limit as \( n \to \infty \) verifies (3.7) and thus completes the proof of Proposition 3.1.

\[\Box\]

## 4 Proof of Theorem 2.1

Theorem 2.1 is a consequence of convergence of the joint distribution of the rescaled stopping time process and the random walk at those stopping times as in the following proposition.

**Proposition 4.1.** Suppose that assumptions (A1) and (A2) hold for some \( \alpha > 0 \), then as \( n \to \infty \),

\[
\left( \frac{S_{\tau_{nt}}}{\sqrt{\frac{p}{d-p} n}}, \frac{\tau_{nt}}{g_{\alpha}(n)} \right) \Rightarrow (B_d(t), V_d(t)), \tag{4.1}
\]

for some \( B_d(t), V_d(t) \).
where the convergence is in \((D([0,1],\mathbb{R}^d), \mathcal{U}) \times (D([0,1],\mathbb{R}), J_1)\), and where \(\mathcal{U}\) and \(J_1\) denote the uniform and Skorokhod \(J_1\) topologies respectively.

**Proof of Theorem 2.1 assuming Proposition 4.1** Since \(\lfloor g^{-1}_a(n) \rfloor\) is a sequence of positive integers such that \(\lfloor g^{-1}_a(n) \rfloor \to \infty\) and \(n/g_a(\lfloor g^{-1}_a(n) \rfloor) \to 1\) as \(n \to \infty\), it follows from (4.1) that as \(n \to \infty\),

\[
\left( \frac{S_{\lfloor g^{-1}_a(n) \rfloor}}, \frac{\tau_{\lfloor g^{-1}_a(n) \rfloor|t}}{n} \right) \rightarrow (B_d(t), V_a(t)), \tag{4.2}
\]

in \((D([0,1],\mathbb{R}^d), \mathcal{U}) \times (D([0,1],\mathbb{R}), J_1)\).

Let

\[
Y_n(t) = \frac{S_{\lfloor g^{-1}_a(n) \rfloor|t}}, \quad \text{and} \quad \mathcal{F}_n(t) = \frac{\tau_{\lfloor g^{-1}_a(n) \rfloor|t}}{n} \tag{4.3}
\]

and let \(\mathcal{F}^{-1}_n(t) := \inf\{s \geq 0 : \mathcal{F}_n(s) > t\} = \inf\{s \geq 0 : \tau_{\lfloor g^{-1}_a(n)|s} > nt\}\). It follows (e.g. see the proof of Theorem 1.3 in [1]) that \(Y_n(\mathcal{F}^{-1}_n(t)) \equiv B_d(V^{-1}_a(t))\) in \((D([0,1],\mathbb{R}^d), \mathcal{U})\). Thus,

\[
\frac{S_{\lfloor g^{-1}_a(n) \rfloor|\mathcal{F}^{-1}_n(t)}}{\sqrt{d-p g_a(n)}} \equiv B_d(V^{-1}_a(t)). \tag{4.4}
\]

By definition of \(\mathcal{F}^{-1}_n\), we have \(\tau_{\lfloor g^{-1}_a(n) \rfloor|\mathcal{F}^{-1}_n(t)} - 1 \leq nt \leq \tau_{\lfloor g^{-1}_a(n) \rfloor|\mathcal{F}^{-1}_n(t)}\) and hence \(|S_{\lfloor nt\rfloor} - S_{\lfloor g^{-1}_a(n)\mathcal{F}^{-1}_n(t)\rfloor}| \leq 3\). Together with (4.4) and the fact that \(g^{-1}_a(n)/\lfloor g^{-1}_a(n) \rfloor \to 1\), this proves Theorem 2.1 \(\square\)

## 5 Proof of Proposition 4.1

The proof of Proposition 4.1 is broken into two parts. The first part is the observation that the marginal processes converge, i.e. that the time-changed walk and the time-change converge to \(B_d(t)\) and \(V_a(t)\) respectively, while the second is to show that these two processes are asymptotically independent.

### 5.1 Convergence of the time-changed walk and the time-change.

**Lemma 5.1.** Suppose that assumptions (A1) and (A2) hold for some \(\alpha > 0\), then as \(n \to \infty\),

\[
\frac{S_{\tau_{\lfloor nt\rfloor}}}{{\sqrt{d-p n}}} \equiv B_d(t) \quad \text{in} \quad (D([0,1],\mathbb{R}^d), \mathcal{U}), \quad \text{and} \quad \frac{\tau_{\lfloor nt\rfloor}}{g_a(n)} \equiv V_a(t) \quad \text{in} \quad (D([0,1],\mathbb{R}), J_1). \tag{5.1}
\]

**Proof.** The first claim is the conclusion of Proposition 3.1 so we need only prove the second claim. Recall that \(\tau_n = 1 + \sum_{i=1}^{n} T_i\) where the \(T_i\) are i.i.d. with distribution \(T\). Since \(g_a(n) \to \infty\), it is enough to show convergence of \(\tau_{\lfloor nt\rfloor}/(\lfloor nt\rfloor - 1)/g_a(n)\).

For processes with independent and identically distributed increments, a standard result of Skorokhod essentially extends the convergence of the one-dimensional distributions to a functional central limit theorem. When \(\mathbb{E}[T]\) exists, convergence of the one-dimensional marginals \(\tau_{\lfloor nt\rfloor}/\lfloor nt\rfloor \to 0\) is immediate from the law of large numbers. The case \(\alpha < 1\) is well known, see for example [6] Section XIII.6 and [16] Section 4.5.3. The case where \(\alpha = 1\) but (2.1) is not summable is perhaps less well known. Here the result is immediate from the following lemma.
Lemma 5.2. Let $T_k \geq 0$ be independent and identically distributed random variables satisfying (2.7) and (2.2) with $\alpha = 1$. Then for each $t \geq 0$,
\[
\frac{\tau_{[nt]}}{n\ell(n)} \xrightarrow{p} t. \tag{5.2}
\]
Lemma 5.2 is a corollary of the following weak law of large numbers due to Gut [7].

Theorem 5.3 ([7], Theorem 1.3). Let $X_k$ be i.i.d. random variables and $S_n = \sum_{k=1}^{n} X_k$. Let $g_n = n^{1/\alpha} \ell(n)$ for $n \geq 1$, where $\alpha \in (0, 1)$ and $\ell(n)$ is slowly varying at infinity. Then
\[
\frac{S_n - n \mathbb{E} [XI_{|X| \leq g_n}]}{g_n} \xrightarrow{p} 0, \quad \text{as } n \to \infty, \tag{5.3}
\]
if and only if $n \mathbb{P}(|X| > g_n) \to 0$.

Proof of Lemma 5.2. Note that
\[
\mathbb{E} \left[ T I_{[T \leq nt]} \right] = \sum_{j=1}^{[nt]} \mathbb{P}(T \geq j) = \sum_{j=1}^{[nt]} \mathbb{P}(T \geq j) - \mathbb{P}(T \geq [nt]) \mathbb{P}(T \geq nt).
\]
Now by assumption (A2b),
\[
\frac{n}{\ell(n)} \mathbb{E} \left[ T I_{[T \leq nt]} \right] = \frac{\sum_{j=1}^{[nt]} \mathbb{P}(T \geq j)}{\ell(n)} - \frac{\mathbb{P}(T \geq [nt]) \mathbb{P}(T \geq nt)}{\ell(n)}
\]
\[
\sim \frac{\sum_{j=1}^{[nt]} j^{-1} L(j)}{\ell(n)} - \frac{[nt]^{-1} L([nt])^{-1} \tau_{[nt]}^{-1} \tau_{[nt]}^{-1}}{\ell(n)} \to 1.
\]
Theorem 5.3 then implies that $(n\ell(n))^{-1} \tau_n \xrightarrow{p} 1$, from which it follows immediately that
\[
(n\ell(n))^{-1} \tau_{[nt]} = (n\ell(n))^{-1} |nt| \ell([nt]) \ell([nt])^{-1} \tau_{[nt]} \xrightarrow{p} t. \tag{5.6}
\]
This completes the proof of Lemma 5.2 and hence Lemma 5.1.

5.2 Asymptotic Independence

Tightness of the joint process in Proposition 4.1 is an easy consequence of the tightness of the marginal processes (Lemma 5.1), so we need only prove convergence of the finite-dimensional distributions (f.d.d.s). For $\alpha \geq 1$ this is simple and is left as an exercise. To complete the proof of Proposition 4.1 it remains to prove convergence of the f.d.d.s when $\alpha < 1$ (hence $p < 1$).

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be convergence determining classes of bounded, $\mathbb{C}$-valued functions on $\mathbb{R}^d$ and $\mathbb{R}_+$ respectively, each closed under conjugation and containing a non-zero constant function, then \{g(x_1, x_2) := g_1(x_1)g_2(x_2) : g_i \in \mathcal{G}_i\} is a convergence determining class for $\mathbb{R}^d \times \mathbb{R}_+$. This follows as in [5] Proposition 3.4.6 where the closure under conjugation allows us to extend the proof to complex-valued functions. Therefore, to prove convergence of the finite-dimensional distributions in (4.1) it is enough to show that for every $0 \leq t_1 < \cdots < t_r \leq 1$, $k_1, \ldots, k_r \in \mathbb{R}^d$ and $\eta_1, \ldots, \eta_r \geq 0$,
\[
\mathbb{E} \left[ \exp \left( i \sum_{j=1}^{r} \frac{k_j \cdot S_{\tau_m} - \eta_j \tau_m}{\sqrt{d-p} m \alpha} \right) \right] \xrightarrow{p} \mathbb{E} \left[ \exp \left( i \sum_{j=1}^{r} k_j \cdot B_{\alpha}(t_j) \right) \right] \mathbb{E} \left[ \exp \left( -\sum_{j=1}^{r} \eta_j V_{\alpha}(t_j) \right) \right]. \tag{5.7}
\]
From (2.6) and the fact that $V_a$ has independent increments, the rightmost expectation can be written as $\exp\{-\sum_{i=1}^r (\eta_i^*) n(t_i - t_{i-1})\}$, where $\eta_i = \sum_{j=1}^r \eta_j$.

Let $\mathcal{S}_n = \{i \in \{1, \ldots, n\} : T_i \text{ is odd}\}$, $\mathcal{A} = (\mathcal{A}_{[nt]}^{(1)}, \ldots, \mathcal{A}_{[nt]}^{(r)})$ to denote an element of the range of the random variable $\mathcal{A}_{[nt]}$, where $A_i^{(i)} \subseteq \{[nt_{i-1}] + 1, \ldots, [nt_i]\}$ for each $i \in 1, \ldots, r$. Observe that $|\mathcal{A}_{[nt]}^{(i)}|$ has a binomial distribution with parameters $|[nt_i] - [nt_{i-1}]|$ and $p$. Then for $\epsilon \in (0, \frac{1}{2})$ and $B_n(\bar{r}) = \{A : \|A^{(i)} - ([nt_i] - [nt_{i-1}]) \| \leq n^{1-\epsilon}\}$ for each $i$, we have that $P(B_n(\bar{r})) \to 0$ as $n \to \infty$. Defining $Q_{\bar{r}}(\bar{r}) = \exp\{i \sum_{j=1}^r \frac{g_a(n)}{\sqrt{g_a(n)}}\}$, and conditioning on $\mathcal{A}_{[nt]}$, the left hand side of (5.7) is equal to

$$
e^{-\frac{1}{\eta_i} \sum_{j=1}^r \eta_j} \mathbb{E} \left[ Q_{\bar{r}}(\bar{r}) \exp\left\{ -\sum_{j=1}^r \frac{\tau^{[nt]}_{j}}{g_a(n)} \right\} \left| \mathcal{A}_{[nt]} = A \right. \right] P(\mathcal{A}_{[nt]} = A)$$

$$= \sum_{A \in B_n(\bar{r})} \mathbb{E} \left[ Q_{\bar{r}}(\bar{r}) \exp\left\{ -\sum_{j=1}^r \frac{\tau^{[nt]}_{j}}{g_a(n)} \right\} \left| \mathcal{A}_{[nt]} = A \right. \right] P(\mathcal{A}_{[nt]} = A) + o(1)$$

$$= \sum_{A \in B_n(\bar{r})} \mathbb{E} \left[ Q_{\bar{r}}(\bar{r}) \left| \mathcal{A}_{[nt]} = A \right. \right] \mathbb{E} \left[ \exp\left\{ -\sum_{j=1}^r \eta_j \frac{\tau^{[nt]}_{j}}{g_a(n)} \right\} \left| \mathcal{A}_{[nt]} = A \right. \right] P(\mathcal{A}_{[nt]} = A) + o(1),$$

(5.8)

where we have used the fact that $S_{\bar{r}}$ is conditionally independent of the collection $\{T_i\}_{i \geq 1}$ given $I_{[T, \text{even}]}$, $i = 1, \ldots, n$, to obtain the last equality.

Writing $\sum_{i=1}^{[nt]} T_i = \sum_{i=1}^{[nt]} \sum_{i=[nt_{i-1}]+1}^{[nt_i]} T_i$ and using the mutual independence of $T_i$, $i \geq 1$, the last line of (5.8) is equal to a term $o(1)$ plus

$$\sum_{A \in B_n(\bar{r})} \mathbb{E} \left[ Q_{\bar{r}}(\bar{r}) \left| \mathcal{A}_{[nt]} = A \right. \right] \mathbb{E} \left[ \exp\left\{ -\sum_{i=[nt_{i-1}]+1}^{[nt_i]} T_i \right\} \left| \mathcal{A}_{[nt]} = A \right. \right] P(\mathcal{A}_{[nt]} = A) \prod_{i=1}^r \mathbb{E} \left[ \exp\left\{ -\eta_i \frac{\sum_{j=[nt_{i-1}]+1}^{[nt_i]} T_j}{g_a(n)} \right\} \right],$$

(5.9)

Let $\{T_i^n\}_{n \in \mathbb{N}}$ be i.i.d. random variables satisfying $P(T_i^n = k) = P(T = k|T \text{ odd})$, and similarly define $T_i^n$ to be i.i.d. with $P(T_i^n = k) = P(T = k|T \text{ even})$. The $l^{th}$ term in the product in (5.9) is

$$\mathbb{E} \left[ \exp\left\{ -\eta_i^* \frac{\sum_{j=[nt_{i-1}]+1}^{[nt_i]} T_j}{g_a(n)} \right\} \right].$$

(5.10)

For $T^n$ we have $P(T^n \geq n) = p^{-1}P(T \geq n, T \text{ odd}) \sim p^{-1}n^{-\alpha}L_a(n)$ and there exists $\ell_o$ such that $(\ell_o(n))^{-\alpha}p^{-1}L_a(n) \rightarrow (\Gamma(1-\alpha))^{-1}$. Define $g_a^\infty(n) = n^{-\frac{1}{\alpha}}\ell_o(n)$. Similarly define $g_a^\sigma(n) = n^{-\frac{1}{\alpha}}\ell(o(n))$.

Observe that

$$\sum_{i=1}^{[nt]} T_i = \sum_{i=1}^{[nt]} T_i \frac{g_a(n_i)}{g_a(n)} g_a(n_i) + g_a^\sigma(n_i) g_a(n_i) + o \left( \frac{\sum_{j=1}^{[nt]} T_j - \sum_{j=1}^{[nt]} T_j \frac{g_a(n_i)}{g_a(n_i)}}{g_a(n_i)} \right),$$

(5.11)

where $n_i := [nt_i] - [nt_{i-1}]$ and $n_i^* := ||A^{(i)}| - n_i| \leq n^{1-\epsilon}$ since $A \in B_n(\bar{r})$. By definition of $g_a$ and standard results on regular variation we have that $g_a(n_i)/g_a(n) \to (p(t_i - t_{i-1}))^{\frac{1}{2}}$ and
Since \( \alpha = \alpha_o \wedge \alpha_e \leq \alpha_o \), the \( \Theta \) term on the right of (5.11) converges in probability to 0. Thus, as in the second claim of Lemma 5.1, we get that
\[
\sum_{i=1}^{\lfloor \alpha^u \rfloor} \frac{T_i^o}{g_a(n)} \Rightarrow V_a(1)(p(t_i - t_{i-1}))^{\frac{1}{2}} \lim_{n \to \infty} \frac{g_a^o(n_i)}{g_a(n_i)},
\]
where for \( \alpha < 1 \) the limit \( \rho_o := \lim_{n \to \infty} \frac{g_a^u(n_i)}{g_a(n_i)} \) exists in \([0, \infty]\) since \( \alpha \leq \alpha_o \) and in the case of equality, the limit \( L_o/L_e \) exists in \([0, \infty]\). Note that we were able to replace \( \alpha_o \) with \( \alpha \) in various places in (5.12) due to the presence of the factor \( g_a(n_i) \) which is zero when \( \alpha_o > \alpha \).

Therefore
\[
\mathbb{E} \left[ \exp \left\{ - \eta^1 \sum_{i=1}^{\lfloor \alpha^u \rfloor} \frac{T_i^o}{g_a(n)} \right\} \right] \to \mathbb{E} \left[ \exp \left\{ - \eta^1 V_a(1)(p(t_i - t_{i-1}))^{\frac{1}{2}} \rho_o \right\} \right], \text{ and similarly,}
\]
\[
\mathbb{E} \left[ \exp \left\{ - \eta^1 \sum_{i=1}^{\lfloor \alpha^u \rfloor} \frac{T_i^o}{g_a(n)} \right\} \right] \to \mathbb{E} \left[ \exp \left\{ - \eta^1 V_a(1)(1-p)(t_i - t_{i-1}))^{\frac{1}{2}} \rho_e \right\} \right].
\]

(5.13)

Since \( \mathbb{E}[e^{-\eta V_a(1)}] = \exp\{ -\eta^a \} \), it remains to show that
\[
(p(t_i - t_{i-1}))^{\frac{1}{2}} \rho_o^a + ((1-p)(t_i - t_{i-1}))^{\frac{1}{2}} \rho_e^a = t_i - t_{i-1}, \quad \text{i.e.} \quad p \rho_o^a + (1-p) \rho_e^a = 1.
\]

(5.14)

If \( \alpha_o < \alpha_e \) (or \( \alpha_o = \alpha_e \) and \( L_o/L_e \to \infty \)), then \( \alpha = \alpha_o \), and \( L \sim L_o \). It is then an easy exercise in manipulating slowly varying functions to show that \( \ell_o \sim p^{-1/a} \ell \) and therefore \( \rho_o \sim p^{-1/a} \) and \( \rho_e = 0 \), giving the desired result. Similarly if \( \alpha_o > \alpha_e \) (or \( \alpha_o = \alpha_e \) and \( L_o/L_e \to 0 \)) we get the desired result. When \( \alpha_o = \alpha_e < 1 \) and \( L_o/L_e \to 0 \) we have that \( L \sim L_o \sim (1+\beta)L_o \sim (1+\beta^{-1})L_o \). It follows that \( \ell_o \sim (1-p)(1+\beta)^{-1/a} \ell \) and \( \ell_e \sim (p(1+\beta^{-1}))^{-1/a} \ell \), and therefore \( \rho_o = ((1+\beta^{-1}))^{-1/a} \) and \( \rho_e = ((1-p)(1+\beta))^{-1/a} \). The result follows since \((1+\beta)^{-1} + (1+\beta^{-1})^{-1} = 1\).

\[\square\]

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