Linear-fractional branching processes with countably many types

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Abstract

This paper is devoted to a special class of the Bienaymé-Galton-Watson processes with a countable type space whose probability generating functions are linear-fractional. For such processes using various tools (contour process, spinal representation, Perron-Frobenius theorem for countable matrices, renewal theory) we thoroughly investigate the case of $R$-positive recurrence with respect to the type space.

Keywords: multivariate linear-fractional distribution, contour process, spinal representation, Bienaymé-Galton-Watson process, Crump-Mode-Jagers process, Perron-Frobenius theorem, $R$-positive recurrence, renewal theory, phase-type distribution.

1. Introduction

Branching processes is a steadily growing body of mathematical research having applications in various areas, primarily in theoretical population biology [13], [16], [19]. A basic version of branching processes, called the Bienaymé-Galton-Watson (BGW) process, describes populations of particles which live one unit of time and at the moment of death give birth to a random number of new particles independently of each other. In the single type setting the consecutive population sizes $\{Z^{(n)}\}_{n \geq 0}$ form a Markov chain with the state space $\{0, 1, 2, \ldots\}$. An important analytical tool for studying branching processes is the probability generating functions. Given

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\[ \phi(s) = \mathbb{E}(s^{Z(1)}) \], the \( n \)-th generation’s size is characterized by the \( n \)-fold iteration of \( \phi(\cdot) \)

\[ \mathbb{E}(s^{Z(n)}) = \phi(\ldots(\phi(s))\ldots) \). 

Here and elsewhere in this paper we always assume that a branching process starts from a single particle.

The case of linear-fractional generating functions is of special interest as their iterations are again linear-fractional functions allowing for explicit calculations of various entities of importance (see [3], p. 7). Such explicit results, although being specific, illuminate the known asymptotic results concerning more general branching processes, and, on the other hand, may bring insight to less investigated aspects of the theory of branching processes.

In the multi-type setting particles still reproduce independently but now the number of offspring may depend on the mother’s type. A flexible family of population models is obtained by means of BGW-processes with countably many types ([4], [14], [16], [23]). These are infinitely dimensional Markov chains

\[ Z(n) = (Z_1^{(n)}, Z_2^{(n)}, \ldots), \quad n = 0, 1, 2, \ldots, \]

whose \( i \)-th component \( Z_i^{(n)} \) gives the number of particles of type \( i \) existing at time \( n \). In this paper we study the class of such branching processes with the generating functions for vectors \( Z^{(n)} \) all being linear-fractional. As shown in Section 2 a linear-fractional BGW-process with countably many types is fully specified by a triplet of parameters \((H, g, m)\), where \( H = (h_{ij})_{i,j=1}^{\infty} \) is a sub-stochastic matrix, \( g = (g_1, g_2, \ldots) \) is a proper probability distribution, and \( m \) is a positive constant.

A linear-fractional branching process with parameters \((H, g, m)\) has the following reproduction law. A particle of type \( i \) has no offspring with probability \( h_{i0} = 1 - \sum_{j \geq 1} h_{ij} \). Given that this particle has at least one offspring, the type its first daughter is \( j \) with probability \( h_{ij}/(1 - h_{i0}) \), and the number of subsequent daughters has a geometric distribution with mean \( m \). With the exception of the first daughter the types of all other offspring particles are chosen independently of mother’s type according to the common distribution \( g \).

Theorem 3 in Section 3 states that the probability generating function for the \( n \)-th generation of the branching process with parameters \((H, g, m)\) is linear-fractional. This theorem presents a general formula for the generating function of \( Z^{(n)} \). An important step in obtaining this formula uses a
spinal representation argument making the derivation more transparent as compared to the argument used in [12] for the finite-dimensional case.

Another way to treat a BGW-process with countably many types, which is mentioned in Section 4, is to see the sequence of first born daughters of a particle as a single individual evolving in time by changing its type. We emphasize these two different interpretations of the same branching system by referring to the reproducing entities either as particles living one unit of time and replaced by random numbers of offspring (non-overlapping generations) or individuals with random life lengths which are able to produce random numbers of offspring at different times along their life span (overlapping generations). In the linear-fractional case the evolution of individual’s type is governed by the sub-stochastic matrix $H$. Every time unit during its life span, except the moments of its birth and death, the individual gives birth to an independent geometric number of offspring with mean $m$ and the initial type for each offspring is chosen according to the distribution $g$.

Assume that the type of the initial particle has distribution $g$. Then the total population size $Z^{(n)} = Z^{(n)}1^t$ of the linear-fractional BGW-process has the mean $M_n := \mathbb{E}(Z^{(n)})$ which is computed as

$$M_n = \mathbf{gM}^n1^t,$$

where

$$\mathbf{M} = (m_{ij})_{i,j=1}^\infty, \quad m_{ij} = \mathbb{E}(Z_j^{(1)}|Z_j^{(0)} = e_i)$$

is the matrix of the offspring means. The sequence $\{Z^{(n)}\}_{n \geq 0}$ forms a single-type Crump-Mode-Jagers (CMJ) process, which we will call a linear-fractional CMJ-process. This CMJ-model is not restrictive about the individual’s life length $L \geq 1$ distribution (see Example 8 in Section 4) but individual’s reproduction point process must follow a very specific pattern: at times $1, \ldots, L - 1$ the individual produces independent and identically distributed geometric numbers of daughters. Making birth events in the linear-fractional CMJ-processes very rare (by choosing the row sums of $H$ to be close to zero) and rescaling time accordingly we arrive at the family of continuous time CMJ-processes with a Poisson point process reproduction. This family of branching processes was investigated in [17] emphasizing the role of the so-called contour processes of the corresponding planar genealogical trees.

In Section 5 we remind the concept of a contour process generated by a planar BGW-tree. In the multivariate linear-fractional framework a jumping version of the contour process (in the spirit of [17]) has a nice Markovian
structure of a constant speed descent with independent and identically distributed upward jumps until absorption at the level $-1$. We apply the contour process method to prove that given the reproduction parameters $(g, m)$ are independent on mother’s type, the vectors $Z^{(n)}$ have also linear-fractional distributions for all times $n$ as stated in Theorem 3.

Section 6 introduces a double classification of the branching processes based on the Perron-Frobenius theorem for countable matrices. Besides the usual classification into subcritical, critical, and supercritical branching processes in the case of infinitely many types one has to distinguish among $R$-transient, $R$-null recurrent, and $R$-positively recurrent cases depending on the corresponding property of the mean matrix $(2)$.

The main result of this paper, Theorem 10 in Section 7, among other statements contains a transparent criterion for $R$-positive recurrence of the linear-fractional branching processes. In the language of the CMJ-processes this criterion requires that the corresponding Malthusian parameter is well defined and the mean age at childbearing is finite. Theorem 10 is proven in Sections 8 and 9.

In Section 10 we present the basic asymptotic results for subcritical, critical, and supercritical linear-fractional BGW-processes with countably many types in the positively recurrent case. Finally, in Section 11 we summarize the implications of Section 10 results to the linear-fractional CMJ-processes.

All our results for the linear-fractional BGW-processes with parameters $(H, g, m)$ also apply to the case of finitely many, say $p$, types, after putting $Z^{(0)}_i = 0$, $g_i = 0$ for $i \geq p + 1$, and $h_{ij} = 0$ for $i \in [1, p]$, $j \geq p + 1$. The transient and null recurrent cases will be addressed in a separate paper.

2. Linear-fractional distributions

We are using the following notation for the vectors $x = (x_1, x_2, \ldots)$ of infinite dimension:

$$0 = (0, 0, \ldots), \ 1 = (1, 1, \ldots), \ e_i = (1_{i=1}, 1_{i=2}, \ldots),$$

the transpose of the vector $x$ will be denoted as $x^t$, and $I$ will stand for the unit matrix $(1_{i=j})_{i,j \geq 1}$. Recall that for a non-negative random vector $Z = (Z_1, Z_2, \ldots)$ with integer-valued components its probability generating function is defined by $E(s^Z) := E(s_1^{Z_1} s_2^{Z_2} \ldots)$. 

**Definition 1.** Let \((h_0, h_1, h_2, \ldots)\) be a probability distribution on \(\{0, 1, 2, \ldots\}\) and \((g_1, g_2, \ldots)\) be a probability distribution on \(\{1, 2, \ldots\}\). For a given positive constant \(m\) the function

\[
\phi(s) = h_0 + \frac{\sum_{i=1}^{\infty} h_is_i}{1 + m - m \sum_{i=1}^{\infty} g_is_i},
\]

(3)

generates a probability distribution to be called a linear-fractional distribution \(LF(h, g, m)\), where \(h = (h_1, h_2, \ldots)\), \(g = (g_1, g_2, \ldots)\).

Slightly modifying Theorem 1 from [12] (devoted to the finite-dimensional case) one can demonstrate that for a linear-fractional function

\[
\phi(s) = \frac{a_0 + a_1s_1 + a_2s_2 + \ldots}{b_0 + b_1s_1 + b_2s_2 + \ldots}
\]

to generate a proper probability distribution it is necessary that this function can be written in the form (3). It means that Definition 1 covers all possible linear-fractional probability generating functions.

Next we show that a multivariate linear-fractional distribution can be viewed as a multivariate geometric distribution modified at zero. Namely, we claim that a random vector \(Z\) with generating function (3) can be represented as

\[
Z = X + (Y_1 + \ldots + Y_N) \cdot 1_{\{X \neq 0\}}
\]

(4)
in terms of mutually independent random entities \((X, N, Y_1, Y_2, \ldots)\). Here vectors \(X\) and \(Y_j\) have multivariate Bernoulli distributions

\[
\mathbb{P}(X = e_i) = h_i, \quad i \geq 1, \quad \mathbb{P}(X = 0) = h_0,
\]

\[
\mathbb{P}(Y_j = e_i) = g_i, \quad i \geq 1, \quad j \geq 1,
\]

and \(N\) is a geometric random variable with distribution

\[
\mathbb{P}(N = k) = m^k(1 + m)^{-k-1}, \quad k = 0, 1, \ldots.
\]

Since

\[
\mathbb{E}(s^X \cdot 1_{\{X \neq 0\}}) = \sum_{i=1}^{\infty} h_is_i,
\]

\[
\mathbb{E}(s^{Y_j}) = \sum_{i=1}^{\infty} g_is_i,
\]

\[
\mathbb{E}(s^N) = \frac{1}{1 + m - ms},
\]
the linear-fractional formula (3) is equivalent to (4) due to
\[ E(s^Z) = h_0 + E(s^X \cdot 1_{X \neq 0})E((s^{Y_1})^N). \]

Representation (4) suggests a simple two-dice-one-coin experiment generating a random vector \( Z \) with a linear-fractional distribution LF\((h, g, m)\). Firstly, an \((h_0, h_1, \ldots)\)-die is thrown once producing an outcome \( i \in \{0, 1, \ldots\} \). If \( i = 0 \), the experiment is stopped and the vector \( Z \) is assigned value 0. Otherwise, we put \( X = e_i \) and continue the experiment. On the second stage we repeatedly throw an \( m \)-coin which shows heads with probability \( \frac{1}{1+m} \) and tails with probability \( \frac{m}{1+m} \). Let \( N \) be the number of tails before we see heads for the first time. Finally, a \((g_1, g_2, \ldots)\)-die is thrown \( N \) times to find the values of \( Y_j \). It remains to use (4) to compute the corresponding value of \( Z \).

**Definition 2.** Let \( H = (h_{ij})_{i,j=1}^{\infty} \) be a sub-stochastic matrix with rows \( h_i = (h_{i1}, h_{i2}, \ldots) \) having non-negative elements such that \( h_{i0} := 1 - h_{i1} - h_{i2} - \ldots \) take values in \([0, 1] \). Let \((g_1, g_2, \ldots)\) be a probability distribution on \( \{1, 2, \ldots\} \), and \( m \) be a positive constant. A branching process will be called linear-fractional with parameters \((H, g, m)\), if its particles of type \( i \) reproduce according to the LF\((h, g, m)\) distribution, \( i = 1, 2, \ldots \).

Notice the strong limitation on the reproduction law requiring parameters \((g, m)\) to be ignorant of mother’s type. This is needed for iterations of the corresponding generating functions
\[ \phi_i^{(n)}(s) = \mathbb{E}(s^{Z^{(n)}} | Z^{(0)} = e_i), \quad i = 1, 2, \ldots \]
to be also linear-fractional. It is easy to see that if the denominators in
\[ \phi_i(s) = h_{i0} + \frac{\sum_{j=1}^{\infty} h_{ij}s_j}{1 + m - m\sum_{j=1}^{\infty} g_js_j}, \]
were different for different \( i \), then the iterations of these generating functions \( \phi_i(\phi_1(s), \phi_2(s) \ldots) \) would lose the linear-fractional property.

3. Propagation of the linear-fractional property

The matrix (2) of the mean offspring numbers for a linear-fractional branching process with parameters \((H, g, m)\) is computed as
\[ M = H + mH1'g. \]
After multiplying this by $1^t$ we obtain

$$M1^t = (1 + m)H1^t,$$

which leads to a useful reverse expression of $H$ in terms of $M$

$$H = M - \frac{m}{1 + m}M1^tg.$$  \hspace{1cm} (6)

**Theorem 3.** Consider a linear-fractional branching process with parameters $(H, g, m)$ starting from a type $i$ particle. Its $n$-th generation size vector $Z(n)$ has a linear-fractional distribution $\text{LF}(h_i(n), g(n), m(n))$ whose parameters satisfy

$$m(n) = m \sum_{l=0}^{n-1} M_l,$$ \hspace{1cm} (7)

$$m(n)g(n) = mg(I + M + \cdots + M^{n-1}),$$ \hspace{1cm} (8)

$$H^{(n)} = M^n - \frac{m(n)}{1 + m(n)}M^n1^tg(n),$$ \hspace{1cm} (9)

where $M_l$ are given by (1) and $H^{(n)}$ is the matrix with the rows $(h_i^{(n)})_{i=1}^{\infty}$. In particular,

$$\mathbb{P}(Z(n) \neq 0) = (1 + m(n))^{-1}M^n1^t,$$ \hspace{1cm} (10)

where $\mathbb{P}(Z(n) \neq 0)$ is a column vector with elements $\mathbb{P}(Z(n) \neq 0|Z(0) = e_i)$.

**Proof.** Our proof of the fact that the $n$-th generation sizes have a linear-fractional joint distribution is based on the contour process method and is postponed until Section 5.

Turning to the proof of relations (7), (8), (9), and (10), observe that relation (9) is a straightforward counterpart of (6). The left hand side in (10) equals $H^{(n)}1^t$ so that (10) follows from (9). The only remaining relations to prove are (7) and (8). We do this using the spinal representation of the BGW-tree illustrated in Figure 1.

Both the spinal representation and the contour process methods (the latter discussed in Section 5) rely on a planar genealogical tree connecting the particles of the branching processes appeared up to the time of observation. For the current setting of linear-fractional branching processes with countably
many types it is important to use a particular plane version of the genealogical tree: given a group of siblings stemming from the same particle the \textit{leftmost branch} should connect the mother to its \textit{first daughter} (that one whose type may depend on mother’s type).

It is convenient to start with the proof of equality (7) even though it is an immediate consequence of (8) and (1). Suppose that at the observation time \( n \) population is not empty. The corresponding spine of the planar BGW-tree is the leftmost lineage of particles reaching the level \( n \). The number of all branches present at level \( n \) except the spinal one has the mean \( m^{(n)} \).

Equality (7) simply says that this mean is the sum of contributions from all the lineages stemming from the spine to the right of it, see Figure 1. In the linear-fractional case at each level \( l \in [0, n-1] \) there is a geometric with mean \( m \) number of branches growing off the spine to the right of it. Every one of such daughter branching processes, in accordance with (1), gives on average \( M_{n-l-1} \) descendants in generation \( n \). Summing the products \( mM_{n-l-1} \) over all spinal particles at levels \( l = 0, \ldots, n-1 \) we get (7).

Equality (8) is just a detailed version of (7) taking into account the numbers of particles of various types existing at time \( n \).

\[ \square \]

\textbf{Corollary 4.} \textit{Conditionally on non-extinction} \( Z^{(n)} \) \textit{has a multivariate shifted}
geometric distribution

\[ \mathbb{E}[s^{Z(n)} | Z^{(n)} \neq 0, Z^{(0)} = e_i] = \frac{\sum_{j=1}^{\infty} \tilde{h}^{(n)}_{ij} s_j}{1 + m^{(n)} - m^{(n)} \sum_{j=1}^{\infty} g^{(n)}_j s_j}, \]

where \( \tilde{h}^{(n)}_{ij} = h^{(n)}_{ij} / (1 - h^{(n)}_{i0}) \).

**Proof.** The asserted formula follows from

\[ \mathbb{E}[s^{Z(n)} 1_{\{Z(n) \neq 0\}} | Z^{(0)} = e_i] = \mathbb{E}[s^{Z(n)} | Z^{(0)} = e_i] - \mathbb{P}[Z^{(n)} = 0 | Z^{(0)} = e_i] \quad (11) \]

\[ = \frac{\sum_{j=1}^{\infty} h^{(n)}_{ij} s_j}{1 + m^{(n)} - m^{(n)} \sum_{j=1}^{\infty} g^{(n)}_j s_j} \]

after dividing the left and right sides by

\[ \mathbb{P}[Z^{(n)} \neq 0 | Z^{(0)} = e_i] = 1 - h^{(n)}_{i0}. \]

\[ \Box \]

**Example 5.** As a simple illustration take a single type BGW-process with a shifted geometric offspring distribution

\[ p_k = m^{k-1} (1 + m)^{-k}, \quad k = 1, 2, \ldots \quad (12) \]

Clearly, this is a supercritical process with the mean offspring number \( 1 + m \).

The following algorithm introduces a labeling system that will make this branching process look like a branching process with infinitely many types:

- the initial particle born at time zero gets label 1,
- for any particle of type \( i \) its first offspring is labelled \( i + 1 \) while the remaining offspring are assigned label 1.

The labeled process at time \( n \) will contain particles of types 1 through \( n + 1 \) so that the vector \( Z^{(n)} \) of sub-population sizes will have components \( Z^{(n)}_i \geq 0 \) for \( i = 1, \ldots, n \), \( Z^{(n)}_{n+1} = 1 \), and \( Z^{(n)}_i = 0 \) for all \( i \geq n + 2 \). This is a linear-fractional branching process with parameters \( (H, e_1, m) \), where \( H = (1_{\{j=i+1\}})_{i,j=1}^{\infty} \), see Definition 2.
In this case it is easy to compute the mean numbers of particles at any time

\[
M^n = \begin{pmatrix}
m(1 + m)^{n-1} & \ldots & m(1 + m) & m & 1 & 0 & 0 & 0 & \ldots \\
m(1 + m)^{n-1} & \ldots & m(1 + m) & m & 0 & 1 & 0 & 0 & \ldots \\
m(1 + m)^{n-1} & \ldots & m(1 + m) & m & 0 & 0 & 1 & 0 & \ldots \\
(1 + m)^{n-1} & \ldots & m(1 + m) & m & 0 & 0 & 0 & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}
\]

and according to (7) we get

\[
m^{(n)} = \sum_{j=0}^{n-1} \left( m + m^2 \sum_{k=0}^{j-1} (1 + m)^k \right) = m \sum_{j=0}^{n-1} (1 + m)^j = (1 + m)^n - 1,
\]

which is verified by the fact that the mean of the \(n\)-th generation size in the original branching process is \((1 + m)^n = 1 + m^{(n)}\). This and (8) yield

\[
g_i^{(n)} = \frac{m}{m^{(n)}} \left( 1 + \sum_{j=i}^{n-1} m(1 + m)^{j-i} \right) = \frac{m(1 + m)^{n-i}}{(1 + m)^n - 1}
\]

for \(i = 1, \ldots, n\) and \(g_i^{(n)} = 0\) for \(i \geq n + 1\). Finally, using (9) we obtain \(H^{(n)} = (1 \{j = i + n\})_{i,j=1}^{\infty}\).

Combining these results we conclude that the components of the vector \((Z_1^{(n)}, \ldots, Z_n^{(n)})\) are independent geometric random variables with means \(m(1 + m)^{n-1}, m(1 + m)^{n-2}, \ldots, m\).

4. Individuals evolving in the type space

A BGW-process is a population model of reproducing particles with non-overlapping generations. Here we give another interpretation of this population model in terms of individuals with overlapping generations. This view becomes very useful in the multi-type linear-fractional case.

We start by explaining what we view as the initial individual for a given realization of the underlying BGW-tree. The initial individual is represented by the starting particle, its first daughter if any, the first daughter’s first daughter if any, and so on. Such an individual has a random life length and it dies when a particle in the chain of the first-born descendants fails to produce offspring. In the multi-type setting the individual evolves in the type space as the particles forming the individual change their types in time.
Each moment of its life the initial individual is able to produce new individuals also evolving in the type space as sequences of the first-born particles. Turn for visual help to Figure 2.C giving the individual based picture of the same genealogical tree as in Figure 2.A. Each vertical arrowed branch in Figure 2.C represents an individual giving birth possibly multiple times during its life. In particular, the initial individual lives two units of time producing two daughters: one of them lives two units of time and the other only one. Notice that the first granddaughter of the initial individual produces two daughters at different ages.

The evolution of an individual can be modeled by a Markov chain whose state space \( \{0, 1, 2, \ldots\} \) is our type space \( \{1, 2, \ldots\} \) enhanced with a graveyard state \( \{0\} \). The corresponding transition probabilities are given by the sub-stochastic matrix \( H \) together with \( h_{0j} = 0 \) for \( j \geq 1 \) and \( h_{00} = 1 \). In terms this Markov chain the random lifespan \( L \) of a new-born individual is the time to visit the graveyard state starting form a state chosen at random among \( \{1, 2, \ldots\} \) according to distribution \( g \). Note that the resulting life length distribution belongs to the broad class of discrete phase-type distributions, see for example [1].

The above given description of the life length distribution implies that the tail probabilities for \( L \) are computed as
\[
d_n := \mathbb{P}(L > n) = gH^n1^1, \quad n \geq 0,
\]
and the distribution of the random type of an individual at age \( n \) is given by the vector \( gH^n/d_n \). The latter observation brings the following natural condition which excludes the phantom (never observed) types. Let \( S \) be the set of types ever observed in the linear-fractional branching process, that is \( j \in S \), if for some \( n \geq 1 \) the \( j \)-th element of \( gH^n \) is positive. Let \( p = |S| \) be the number of elements in \( S \). We can always relabel the non-phantom types in such a way that
\[
S = \{1, 2, \ldots, p\} \text{ if } 1 \leq p < \infty, \text{ or } S = \{1, 2, \ldots\} \text{ if } p = \infty.
\]

The generating function of the tail probabilities
\[
f(s) = \sum_{n=1}^{\infty} d_n s^n
\]
plays an important role in our analysis. Under condition (14) the function \( f(s) \) given by (15) is strictly monotonely growing from zero to infinity as \( s \) goes from zero to infinity.
Definition 6. Let
\[ R_f = \inf\{s > 0 : f(s) = \infty\}. \]

Define a positive finite constant \( R \) as \( R = R_f \) if \( f(R_f) < 1/m \), or else, if \( f(R_f) \geq 1/m \), as the unique positive solution of the equation
\[ mf(R) = 1. \] (16)

For such defined value of \( R \) set
\[ \beta = m \sum_{n=1}^{\infty} nd_n R^n. \] (17)

Lemma 7. Assume that condition (14) is valid. The following chain of implications is true
\[ \{ R < 1 \} \subset \{ R < R_f \} \subset \{ \beta < \infty \} \cap (16). \]

Proof. Since \( d_n \leq 1 \), we have \( f(s) < \infty \) for all \( s \in [0, 1) \) implying \( R_f \geq 1 \) and the first implication. The second implication is a straightforward corollary of Definition 6. □

To continue with important properties of the function (15) observe that
\[ f(s) = \frac{1 - \mathbb{E}(s)}{1-s} - 1 \]
implies that the life length has always a positive mean (finite or infinite)
\[ \lambda := \mathbb{E}(L) = 1 + f(1). \] (18)

At every age during its life (but not at the moment of death) each individual produces a geometric number of offspring with mean \( m \). Thus the mean total offspring number per individual is
\[ \mu = m(\lambda - 1) = mf(1). \] (19)

Observe also that given (16),
\[ mf(Rs) = \sum_{n=1}^{\infty} \hat{d}_n s^n, \quad \hat{d}_n = m R^n d_n, \] (20)
is the generating function for a probability distribution for a random variable (called the regeneration age of the immortal individual in [11]) with mean $\beta$. Notice that in the critical case $\hat{d}_n = \mathbb{P}(L > n)/\mathbb{E}(L) - 1$.

**Example 8.** Observe that for any sequence $1 = d_0 \geq d_1 \geq d_2 \geq \ldots$ with $d_n > 0$ for all $n \geq 1$ one can find a suitable pair $(H, g)$ such that (13) holds. Indeed, choosing

$$H = \begin{pmatrix} 0 & d_1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & d_1/d_2 & 0 & 0 & \ldots \\ 0 & 0 & 0 & d_2/d_3 & 0 & \ldots \\ 0 & 0 & 0 & 0 & d_3/d_4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad g = e_1,$$

we obtain $gH^1 \equiv d_n$. This observation allows us to illustrate the possibility of $\beta = \infty$ by letting $d_n \sim cn^{-kr-n}$ as $n \to \infty$ for some constants $r \geq 1$, $c > 0$, and $k \geq 0$. Clearly, in this case $R_f = r$ and

- if $0 \leq k \leq 1$, then (16) holds with $R < r$ and $\beta < \infty$,
- if $k > 1$ and $f(r) < 1/m$, then $R = r$ and (16) does not hold,
- if $k > 1$ and $f(r) = 1/m$, then (16) holds with $R = r$,
- if $k > 1$ and $f(r) > 1/m$, then (16) holds with $R < r$,
- if $1 < k \leq 2$, then $\beta = \infty$, while for $k > 2$ we always get $\beta < \infty$.

5. **Jumping contour processes in discrete time**

Recall that for any planar tree, one can define a contour profile of the tree by the depth-first search procedure. Figure 2 illustrates the basic definition of the contour process for a finite tree supplied with a path around the tree. The contour process is simply the seesaw line graph (panel B) representing the height of the location of an imaginary car driving with a constant speed along the path outlined on the panel A (the imaginary car stops when it reaches the level $-1$). If the realization of the genealogical tree is infinite, one still can work with the contour processes after cutting off the branches above the level $n$ corresponding to the observation time, as demonstrated in Figure 2. It is easy to reconstruct the tree from the contour process. For
Figure 2: A: a BGW-tree (thick line) stopped at level $n = 5$ and supplied with a contour (dashed) line. B: the corresponding unfolded contour process. C: the CMJ-view of the same tree when depicted in terms of individuals forming a branching process with overlapping generations. The vertices marked by arrows represent individuals which are dead at that time. The stopped tree gives no information about the fate of the three tip vertices. D: the modified contour process of a constant speed descent with iid upward jumps. To match the figure on panel B the jumping contour process starts at the level -1.

example, the number of particles alive at time $n$, if any, is 1 plus the number of excursions of the contour process starting at level $n$ downwards and coming back to the level $n$ escaping absorption at level -1. In particular, in Figure 2 there are two such excursions for $n = 5$ with the number of branches at this level being 3.

The contour process approach has proven to be very useful in the theory of branching processes (see for example [8] and references therein). In the single type linear-fractional case the contour process has a very simple structure of an alternating random walk. Alternating upwards and downwards stretches have independent lengths following two shifted geometric laws (one for upward and the other for downward moves).

In the multi-type linear-fractional setting, one can ensure a Markov property of the contour process using additional labeling so that the states of the contour process will be given by pairs $(l, i)$ with $l = -1, 0, 1, 2, \ldots$ and
The current state \((l, i)\) with \(i \geq 1\) tells three things about the contour process: it is visiting level \(l\), the last move was upwards, and the corresponding particle in the branching process is of type \(i\). The states \((l, 0)\) are attained by the contour process after the downward steps. This convention implies in particular, that at the absorption level \(l = -1\) the only possible label is \(i = 0\). Clearly, the labeled contour process is a Markov chain with transition probabilities formulae

\[
\begin{align*}
\mathbb{P}\{(l, i) \to (l + 1, j)\} &= h_{ij}, \quad \mathbb{P}\{(l, i) \to (l - 1, 0)\} = h_{i0}, \\
\mathbb{P}\{(l, 0) \to (l + 1, j)\} &= \frac{m}{1 + m} g_j, \quad \mathbb{P}\{(l, 0) \to (l - 1, 0)\} = \frac{1}{1 + m}, \\
\mathbb{P}\{(-1, 0) \to (-1, 0)\} &= 1,
\end{align*}
\]

valid for all \(i \geq 1, j \geq 1, l \geq 0\).

The following alternative way of introducing Markovian structure in the contour process of a linear-fractional multi-type branching process does not require additional labeling. What we call here the *jumping contour process* (cf [17]) has a trajectory of a constant speed descent with independent upward jumps each distributed as the individual life length \(L\). From any given current level \(l\) the jumping contour process moves one level down to \(l - 1\) and either settles there with probability \(\frac{1}{1+m}\) or, with probability \(\frac{m}{1+m}\), it instantaneously jumps \(L\) levels up coming to the level \(l - 1 + L\). Figure 2.D clearly illustrates the last construction.

**Proof of the first statement of Theorem 3.** Consider now a linear-fractional multi-type BGW-tree stopped at the level \(n\) so that the set of its labeled tips allows to compute the current generation sizes \(Z^{(n)}\). Using the contour process idea it is straightforward to see that the distribution of the vector \(Z^{(n)}\) must be linear-fractional.
We have to verify that if there is at least one branch reaching the level \( n \), then the number of tips and their types at the level \( n \) to the right of the leftmost tip has a multivariate geometric distribution. In terms of the jumping contour process the consecutive tips to the right of the leftmost tip are connected by a geometric number of independent excursions starting from the observation level \( n \) and returning from below back to the level \( n \). The parameter of the corresponding geometric distribution is the probability that the jumping contour process starting from level \( n \) will be absorbed at the level -1 without visiting the level \( n \) once again. This finishes the proof of the first statement of Theorem 3.

\[ \square \]

6. Classification of branching processes with countably many types

Branching processes are classified according to the asymptotic properties of the mean matrices \( \mathbf{M}^{(n)} = (m_{ij}^{(n)})_{i,j=1}^{\infty} \) with elements

\[
m_{ij}^{(n)} = \mathbb{E}(Z_{j}^{(n)} | Z^{(0)} = \mathbf{e}_i)
\]

as \( n \to \infty \). The assumed independence of particles implies a recursion \( \mathbf{M}^{(n)} = \mathbf{MM}^{(n-1)} \), where \( \mathbf{M} = \mathbf{M}^{(1)} \). It follows that \( \mathbf{M}^{(n)} = \mathbf{M}^n \). Given that all powers \( \mathbf{M}^n \) are element-wise finite (which is always true in the linear-fractional case) the asymptotic behavior of these powers is described by the Perron-Frobenius theory for countable matrices (see Chapter 6 in [22]).

Next we remind some crucial conclusions from this theory holding for an irreducible and aperiodic countable matrix \( \mathbf{M} \). Recall that a non-negative matrix \( \mathbf{M} \) is called irreducible, if for any pair of indices \( (i, j) \) there is a natural number \( n \) such that \( m_{ij}^{(n)} > 0 \). The period of an index \( i \) in an irreducible matrix \( \mathbf{M} \) is defined as the greatest common divisor of all natural numbers \( n \) such that \( m_{ij}^{(n)} > 0 \). In the irreducible case all such indices have the same period which is called the period of \( \mathbf{M} \). When this period equals one the matrix \( \mathbf{M} \) is called aperiodic.

Due to Theorem 6.1 from [22] all elements of the matrix power series

\[
\mathbf{M}(s) = \sum_{n=0}^{\infty} s^n \mathbf{M}^n
\]
have a common convergence radius $0 \leq R < \infty$, called the convergence parameter of the matrix $M$. Furthermore, one of the two alternatives holds:

\begin{align*}
R\text{-transient case: } & \sum_{n=0}^{\infty} m_{ii}^{(n)} R^n < \infty, \ i = 1, 2, \ldots, \quad (22) \\
R\text{-recurrent case: } & \sum_{n=0}^{\infty} m_{ii}^{(n)} R^n = \infty, \ i = 1, 2, \ldots. \quad (23)
\end{align*}

According to [22] (Theorem 6.2 and a remark afterwards) in the $R$-recurrent case there exist unique up to constant multipliers positive vectors $u$ and $v$ such that

\[ RMu^i = u^i, \ RvM = v. \]

A renormalization $Rv_j m_{ji}/v_i$ transforms the matrix $M$ into a stochastic matrix.

The $R$-recurrent case is further divided in two sub-cases: $R$-null, when $vu^i = \infty$, and $R$-positive with $vu^i < \infty$. In the $R$-null case (and clearly also in the $R$-transient case)

\[ R^n m_{ij}^{(n)} \to 0, \ i, j = 1, 2, \ldots \]

In the $R$-positive case (Theorem 6.5 from [22]) one can scale the eigenvectors so that $vu^i = 1$ and obtain

\[ R^n m_{ij}^{(n)} \to u_i v_j, \ i, j = 1, 2, \ldots. \quad (24) \]

These results suggest a double classification of the branching processes with countably many types having a mean matrix $M$. The usual classification depends on the value of the Perron-Frobenius eigenvalue $\rho = 1/R$. Given $\rho < 1$, $\rho = 1$, or $\rho > 1$ the branching process is called subcritical, critical, or supercritical. An additional classification is needed to distinguish among different asymptotic patterns due to recurrence property of the branching process with respect to the infinite type space.

**Definition 9.** A branching process with countably many types will be called subcritical (critical, supercritical) and transient \{recurrent, null-recurrent, positively recurrent\} in the type space, if its matrix of the mean offspring numbers $M$ has a convergence radius $R > 1$ ($R = 1$, $R < 1$) being $R$-transient \{R-recurrent, R-null recurrent, R-positively recurrent\}. 

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There are several published results for the BGW-processes with countably many types (see for example [2], [15]). One of them is Theorem 1 in [18] dealing with the $R$-positively recurrent supercritical ($R < 1$) case. It states that if
\[ \sum_{i=1}^{\infty} v_i \mathbb{E} ((Z^{(1)}u^i)^2|Z^{(0)} = e_i) < \infty, \]
then for any $w$ such that $w \leq cu$ for some positive constant $c$, the convergence $R^nZ_nw^t \to Yvw^t$ holds in mean square, where $Y \geq 0$ has a finite second moment. This statement is cited here just to illustrate the need for finding examples of branching processes, where conditions like $R$-positive recurrence could be verified and the values of $(R, u, v)$ be computed in terms of the basic model parameters. The linear-fractional framework allows for such amendments.

7. Main result

Consider a linear-fractional branching process with parameters $(H, g, m)$ whose matrix of the mean offspring numbers $M$ is given by (5). Clearly, if $M$ is irreducible, then starting at any type it is possible to observe any type in $S$ at some later time. Thus irreducibility of $M$ implies (14). The opposite is not true, if there exist so-called final types, that never produce offspring. To exclude the final types we will assume that
\[ \text{there are no zero rows in } H. \quad (25) \]

This is of course also a necessary condition for irreducibility of $M$.

Before stating our main result we remind that according to Lemma 7 parameter $\beta$ defined by (17) can be infinite only when $R \geq 1$, see Definition 6. Due to the same Lemma 7 condition $f(R) < 1/m$ is only possible again when $R \geq 1$.

**Theorem 10.** The matrix $M$ given by (5) is irreducible if and only if (14) and (25) hold. If $M$ is irreducible and aperiodic, the following five statements are valid:

(i) the convergence parameter $R$ of $M$ is computed from the generating function (15) as described by Definition 6,

(ii) if $f(R) < 1/m$, then $M$ is $R$-transient,
(iii) if \( f(R) = 1/m \), then \( M \) is \( R \)-recurrent,

(iv) if \( f(R) = 1/m \) and \( \beta = \infty \), then \( M \) is \( R \)-null recurrent, so that each element of \( R^n M^n \) converges to zero as \( n \to \infty \),

(v) if \( f(R) = 1/m \) and \( \beta < \infty \), then \( M \) is \( R \)-positively recurrent so that

\[
R^n M^n \to u^t v, \ n \to \infty,
\]

where element-wise positive and finite vectors \( u \) and \( v \) are given by

\[
\begin{align*}
\begin{array}{l}
u^t = (1 + m) \beta^{-1} \sum_{k=1}^{\infty} R^k H^k 1^t, \\
v = \frac{m}{1 + m} \sum_{k=0}^{\infty} R^k g H^k,
\end{array}
\end{align*}
\]

satisfying \( vu^t = v 1^t = 1 \) and \( gu^t = \frac{1 + m}{m \beta} \).

Example 11. Consider a triplet \( (H, g, m) \) such that \( g H = r g \) for some positive constant \( r \). Since \( H \) is sub-stochastic, we have necessarily \( r \leq 1 \):

\[
r = rg 1^t = g H 1^t \leq g 1^t = 1.
\]

It follows from (5) that \( g \) is a left eigenvector of the matrix \( M \) as well: \( g M = r (1 + m) g \). Using (16), (17), and (28) we obtain

\[
\rho = (1 + m) r, \ \beta = \frac{1 + m}{m}, \ v = g.
\]

Using (13) we obtain \( P(L > n) = r^n \). In this case the branching process is always \( R \)-positively recurrent.

Example 12. Assume now that \( H 1^t = r 1^t \), where necessarily \( r \leq 1 \). It follows from (5) that \( M 1^t = r (1 + m) 1^t \). In this case

\[
\rho = (1 + m) r, \ \beta = \frac{1 + m}{m}, \ u = 1.
\]

It follows from (13) that even in this case \( P(L > n) = r^n \). The case with \( r = 1 \), giving the infinite life length as in Example 5 from Section 3. is obtained whenever the matrix \( H \) is stochastic.
Remark 13. In the linear-fractional setting we have two alternative forms for the above given criterium for criticality recognizing three classes according to \( \rho < 1, \rho = 1, \) and \( \rho > 1. \) In terms of the mean number of offspring \( \mu \) produced by a single individual, see (19), the linear-fractional BGW-processes with countably many types can be called subcritical, critical, or supercritical if \( \mu < 1, \mu = 1 \) or \( \mu > 1 \) respectively. Observe that in the \( R \)-transient case with \( \rho = 1 \) we get \( \mu < 1, \) since \( f(1) = f(R) < 1/m. \)

On the other hand, super-criticality can be identified via a positive drift for the contour process. The drift \( \lambda - 1 - m^{-1} \) of the jumping contour process is computed as the difference between the mean jump size \( L \) and the length of a downward stretch \( 1 + m^{-1}. \) Clearly, the inequality \( \lambda - 1 - m^{-1} > 0 \) is equivalent to \( \mu > 1. \)

8. Renewal theory argument

This section contains two lemmata which will be used in the next section proving Theorem 10. The following well-known renewal theorem taken from Chapter XIII.4 in [5] will be used by us several times.

Lemma 14. Let \( A(s) = \sum_{n=0}^{\infty} a_n s^n \) be a probability generating function and \( B(s) = \sum_{n=0}^{\infty} b_n s^n \) is a generating function for a non-negative sequence so that \( A(1) = 1 \) while \( B(1) \in (0, \infty). \) Then the non-negative sequence defined by \( \sum_{n=0}^{\infty} t_n s^n = \frac{B(s)}{1 - A(s)} \) is such that \( t_n \to \frac{B(1)}{A(1)} \) as \( n \to \infty. \)

The next key lemma deals with the matrix power series (21).

Lemma 15. Let \( R \) be given by Definition 6. For any \( s \in [0, R) \) the matrix power series (21) satisfies

\[
M(s) = H(s) + \frac{m}{1 - mf(s)}(H(s) - I)1^t gH(s),
\]

where \( H(s) = \sum_{n=0}^{\infty} s^n H^n \) is element-wise finite. Moreover,

\[
gM(R)1^t < \infty, \quad \text{if } f(R) < 1/m,
\]

\[
gM(R)1^t = \infty, \quad \text{if } f(R) = 1/m,
\]

\[
gM(s)1^t = \infty, \quad \text{for } s > R.
\]
Proof. Due to (5) we have $M^{n+1} = (H + mHG)M^n$, where $G = 1^t g$.

Using induction we obtain

$$M^n = H^n + m \sum_{i=1}^{n} H^i GM^{n-i}. \quad (33)$$

This yields a renewal equation for the sequence $M_n = gM^n1^t$

$$M_n = d_n + m \sum_{i=1}^{n} d_i M_{n-i}$$

which translates in terms of generating functions into

$$\sum_{n=0}^{\infty} M_n s^n = 1 + f(s) + mf(s) \sum_{n=0}^{\infty} M_n s^n.$$

From here it follows that for $s \in [0, R)$ the generating function

$$gM(s)1^t = \frac{1 + f(s)}{1 - mf(s)} \quad (34)$$

takes finite values, and that relations (30), (31), and (32) hold.

Let $s \in [0, R)$. Relation (34) entails that $M(s)$ can not be element-wise infinite and therefore, the radius of convergence for $M(s)$ is larger or equal to the value $R$ given by Definition 6. Therefore, relation (33) implies

$$M(s) = H(s) + msH(s)GM(s)$$

$$= H(s) + msH(s)Gh(s) + \ldots + (msH(s)G)^n M(s)$$

with all terms being element-wise finite. Observe next that for all $n \geq 1$

$$\left(sH(s)G\right)^n = \left(\sum_{i=1}^{\infty} s^i H^i 1^t g\right)^n$$

$$= \left(\sum_{i=1}^{\infty} s^i H^i 1^t\right) \left(\sum_{i=1}^{\infty} s^i gH^i 1^t\right) \ldots \left(\sum_{i=1}^{\infty} s^i gH^i 1^t\right) g$$

$$= f^{n-1}(s) \sum_{i=1}^{\infty} s^i H^i 1^t g.$$
Thus due to $mf(s) < 1$, the term

$$(msH(s)G)^n M(s) = m^n f^{n-1}(s) \sum_{i=1}^{\infty} s^i H^i 1^t gM(s)$$

vanishes as $n \to \infty$ and the previous two relations yield (29)

$$M(s) = \sum_{n=0}^{\infty} (sH(s)A)^n H(s)$$

$$= H(s) + \sum_{n=1}^{\infty} m^n f^{n-1}(s) \sum_{i=1}^{\infty} s^i H^i 1^t gH(s)$$

$$= H(s) + \frac{m}{1 - mf(s)} (H(s) - I) 1^t gH(s).$$

9. Proof of Theorem 10

We show first that conditions (14) and (25) are sufficient for irreducibility. Namely, we will demonstrate that for any $j$ there is a $n = n_j$ such that $m_{ij}^{(n)} > 0$ for all $i$. To see this observe that (33) gives

$$M^n \geq mH^k 1^t gH^{n-k}$$

for any $k \in [1, n]$. Due to (25) for any $k$ the $i$-th component of the vector $H^k 1^t$ is positive. On the other hand, under (14) for the given $j$ there is an $l$ such that the $j$-th component of the vector $gH^l$ is positive.

Assume from now on that $M$ is irreducible and aperiodic. Statement (i) of Theorem 10 follows from Lemma 15 as $M(s)$ is element-wise finite for $s \in [0, R)$ and must be infinite for $s > R$ due to (32).

Statement (ii) is an immediate consequence of (30): given $f(R) < 1/m$, relation (23) is not possible, and we are in the transient case (22).

To prove (iii) we assume that $f(R) = 1/m$ and verify that $M(R)$ is element-wise infinite again using Lemma 15. Indeed, since $H(s) \geq I$, each element of the vector $gH(s)$ is larger or equal than 1, implying according to
that for any pair of indices \((i,j)\) and any \(s \in [0,R)\)

\[
e_i M(s) e_j^t \geq \frac{m}{1 - m f(s)} e_i (H(s) - I) 1^t = \frac{m}{1 - m f(s)} \sum_{n=1}^{\infty} s^n e_i H^n 1^t
\]

\[
= \frac{c_i(s)}{1 - m f(s)},
\]

where \(c_i(s)\) is strictly positive due to \((25)\). Letting here \(s \rightarrow R\) we see that every element of \(M(R)\) is infinite as soon as \(f(R) = 1/m\).

Suppose \((16)\) holds. Assertion (iv) is easily obtained using Lemma \(14\). It follows from \((34)\) that \(R^n g M^n 1^t \rightarrow 0\) given \(\beta = \infty\). Thus \((24)\) can not be true in this case implying null-recurrence.

It remains to prove (v). Let \(\beta < \infty\). We show next that \(H(R)\) is element-wise finite. To see this notice that \(g H(R) 1^t = 1/m\) according to \((16)\). On the other hand, under condition \((14)\) for any index \(i\) there is \(k = k_i\) and a positive \(c_i\) such that \(g H^k \geq c_i e_i\) implying

\[
c_i e_i H(R) \leq \sum_{n=0}^{\infty} R^n g H^{k+n} \leq R^{-k} \sum_{n=0}^{\infty} R^n g H^n
\]

and further

\[
e_i H(R) 1^t \leq c_i^{-1} R^{-k} m^{-1} < \infty.
\]

Consider vectors \(u\) and \(v\) which, thanks to the just proved finiteness of \(H(R)\), are well-defined by \((27)\) and \((28)\). The claimed equality \(v u^t = 1\) follows from

\[
m g H(R) (H(R) - I) 1^t = m \sum_{n=1}^{\infty} n R^n g H^n 1^t = \beta,
\]

which is a consequence of

\[
H(s) H(s) - H(s) = \sum_{i=1}^{\infty} H_i s^i \sum_{k=0}^{\infty} H_k s^k
\]

\[
= \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} H^n s^n = \sum_{n=1}^{\infty} \sum_{i=1}^{n} H^n s^n = \sum_{n=1}^{\infty} n H^n s^n.
\]

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Finally, to prove (26) define a sequence of matrices $B_n$ by
\[
\sum_{n=0}^{\infty} B_n s^n = \frac{m}{1-mf(Rs)}(H(Rs) - I)1^t gH(Rs).
\]
According to Lemma [14] we have an element-wise convergence $B_n \to u^tv$ as $n \to \infty$. It remains to see that due to (29)
\[
R^nM^n = R^nH^n + B_n \to u^tv,
\]
where each element of $R^nH^n$ converges to zero, since $H(R)$ is element-wise finite.

10. $R$-positively recurrent case

Propositions [16] [17], and [18] below deal with linear-fractional branching processes given that $M$ irreducible and aperiodic and that [16] holds with $\beta < \infty$. Due to Theorem [10] we are in the $R$-positively recurrent case, so that $M^n \sim \rho^n u^tv$, where $\rho = R^{-1}$. Notice that the left eigenvector $v$ describes the stable type distribution: $e_iM^n \sim u_i\rho^n v$, and the right eigenvector $u$ compares productivity of different types: $M^n1^t \sim \rho^n u^t$ (so that $u_i$ can be interpreted as the "reproductive value" of type $i$).

These propositions should be compared with the known versions of such statements in the finite-dimensional case given in [12] and [20]. We remind about formula (5) available for $M$, formula (17) for $\beta$, (27) for $u$, (28) for $v$, (18) for $\lambda$, and (19) for $\mu$.

**Proposition 16.** In the subcritical positively recurrent case when $\rho < 1$, or equivalently $\mu < 1$,
\[
\mathbb{P}[Z^{(n)} \neq 0] \sim \rho^n(1+m)^{-1}(1-\mu)u^t,
\]
and independently from the initial type $i$ we have a convergence in distribution
\[
[Z^{(n)} | Z^{(n)} \neq 0, Z^{(0)} = e_i] \overset{d}{\to} Y
\]
to a random vector $Y$ with a linear-fractional distribution $LF(\tilde{h}, \tilde{g}, \tilde{m})$, where
\[
\tilde{h} = (1+m)(1-\mu)^{-1}v - mg(I-M)^{-1},
\tilde{g} = \lambda^{-1}(1-\mu)g(I-M)^{-1},
\tilde{m} = m\lambda(1-\mu)^{-1},
\]
with $\tilde{h}1^t = 1$. In particular, $Y1^t$ has distribution [12].
Proof. From (7) and (34) we obtain

\[ m^{(n)} \rightarrow m \frac{1 + f(1)}{1 - mf(1)} = \tilde{m} \]

which together with (10) implies (35). The statement on the convergence of the conditional distribution of \( Z^{(n)} \) follows from Corollary 4:

\[ \mathbb{E}[s^{Z^{(n)}} | Z^{(n)} \neq 0, Z^{(0)} = e_i] \rightarrow \frac{\sum_{j=1}^{\infty} \tilde{h}_j s_j}{1 + m - m \sum_{j=1}^{\infty} \tilde{g}_j s_j}, \]

since \( m^{(n)}g^{(n)} \rightarrow mg(I - M)^{-1} = \tilde{m} \tilde{g} \) and

\[ H^{(n)} \sim \rho^n (u^t v - (1 - \mu)(1 + m)^{-1}u^t \tilde{m} \tilde{g}) = \rho^n (1 - \mu)(1 + m)^{-1}u^t \tilde{h}. \]

\[ \square \]

**Proposition 17.** In the critical positively recurrent case when \( \rho = 1 \) we have

\[ \mathbb{P}(Z^{(n)} \neq 0) \sim n^{-1}(1 + m)^{-1} \beta u^t, \]

and

\[ [n^{-1}Z^{(n)} | Z^{(n)} \neq 0, Z^{(0)} = e_i] \xrightarrow{d} Xv, \]

where \( X \) is exponentially distributed with mean \( (1 + m)\beta^{-1} \).

If furthermore a vector \( w \) is such that \( vw^t = 0 \), then

\[ [n^{-1/2}Z^{(n)}w^t | Z^{(n)} \neq 0, Z^{(0)} = e_i] \xrightarrow{d} Y \sqrt{(1 + m)(2\beta)^{-1}}w^t, \]

where \( w^2 = (w_1^2, w_2^2, \ldots) \) and \( Y \) has a Laplace (double exponential) distribution with parameter 1.

Proof. Lemma 14 and relations (7), (34) imply that in the critical case

\[ m^{(n)} \sim n(1 + m)\beta^{-1}. \]
Thus the stated asymptotics for the survival probability follows from (10). The second assertion readily follows from Corollary 4 as
\[ m^{(n)}g^{(n)} \sim n(1 + m) \beta^{-1} \mathbf{v} \]
and the multivariate shifted geometric distribution can be approximated by a one-dimensional exponential distribution along the vector \( \mathbf{v} \).

Convergence (36) is verified in terms of the moment generating functions
\[ \mathbb{E}\left(e^{(z/\sqrt{n})\mathbf{Z}^{(n)}\mathbf{w}^t}|\mathbf{Z}^{(n)} \neq \mathbf{0}, \mathbf{Z}^{(0)} = \mathbf{e}_i\right) = \frac{\mathbb{E}(s_n^{\mathbf{Z}^{(n)}}1_{\mathbf{Z}^{(n)} \neq \mathbf{0}}|\mathbf{Z}^{(0)} = \mathbf{e}_i)}{\mathbb{P}(\mathbf{Z}^{(n)} \neq \mathbf{0}|\mathbf{Z}^{(0)} = \mathbf{e}_i)}, \]
where \( s_n = (e^{(z/\sqrt{n})w_1}, e^{(z/\sqrt{n})w_2}, \ldots) \). Rewrite the numerator using (11) and observe that it is equal to (see Theorem 3)
\[ \sum_{j=1}^{\infty} h_{ij}^{(n)} e^{(z/\sqrt{n})w_j} \]
\[ \frac{\sum_{j=1}^{\infty} g_j^{(n)} e^{(z/\sqrt{n})w_j}}{1 + m^{(n)} - m^{(n)} \sum_{j=1}^{\infty} g_j^{(n)} e^{(z/\sqrt{n})w_j}}. \]

In view of
\[ \sum_{j=1}^{\infty} h_{ij}^{(n)} e^{(z/\sqrt{n})w_j} \sim 1 - h_{i0}^{(n)} = \mathbb{P}(\mathbf{Z}^{(n)} \neq \mathbf{0}|\mathbf{Z}^{(0)} = \mathbf{e}_i), \]
\[ m^{(n)} \sum_{j=1}^{\infty} g_j^{(n)} (1 - e^{(z/\sqrt{n})w_j}) \rightarrow -(z^2/2)(1 + m) \beta^{-1} \mathbf{w}^2 \mathbf{v}^t, \]
we conclude
\[ \mathbb{E}\left(e^{(z/\sqrt{n})\mathbf{Z}^{(n)}\mathbf{w}^t}|\mathbf{Z}^{(n)} \neq \mathbf{0}, \mathbf{Z}^{(0)} = \mathbf{e}_i\right) \rightarrow \frac{1}{1 - z^2(1 + m)(2\beta)^{-1} \mathbf{w}^2 \mathbf{v}^t}. \]

\[ \square \]

**Proposition 18.** In the supercritical positively recurrent case when \( \rho > 1 \)
\[ \mathbb{P}(\mathbf{Z}^{(n)} \neq \mathbf{0}) \rightarrow (\rho - 1)(1 + m)^{-1} \beta \mathbf{u}^t, \]
and
\[ [\rho^{-n}\mathbf{Z}^{(n)}|\mathbf{Z}^{(n)} \neq \mathbf{0}, \mathbf{Z}^{(0)} = \mathbf{e}_i] \xrightarrow{d} X \mathbf{v}, \]

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where $X$ is exponentially distributed with mean $(1 + m)(\rho - 1)^{-1}\beta^{-1}$.

If furthermore a vector $w$ is such that $vw^t = 0$, then

$$
\left[ \rho^{-n/2}Z^{(n)}w^t | Z^{(n)} \neq 0, Z^{(0)} = e_1 \right] \overset{d}{\to} Y \sqrt{(1 + m)(\rho - 1)^{-1}(2\beta)^{-1}}w^2v^t,
$$

where $w^2 = (w_1^2, w_2^2, \ldots)$ and $Y$ has a Laplace distribution with parameter 1.

**Proof.** Rewrite (7) as

$$
R^{n-1}m^{(n)} = m \sum_{l=0}^{n-1} R^l M_l R^{n-1-l}
$$

to obtain the following consequence of (34)

$$
\sum_{n=1}^{\infty} (Rs)^{n-1} m^{(n)} = \frac{m(1 + f(sR))}{(1 - mf(sR))(1 - Rs)}.
$$

Thus Lemma 14 entails

$$
m^{(n)} \sim \rho^n (1 + m)\beta^{-1}(\rho - 1)^{-1}.
$$

This together with (26) and (10) gives the stated formula for the survival probability. The remaining two assertions on weak convergence are proved in a similar way as in the critical case above.

□

**Remark 19.** To illuminate the above asymptotic results in the critical and supercritical cases leading to the Laplace distribution in the limit, we sketch a probabilistic argument for (36) in the two-type case when $Z^{(n)} = (X_n, N_n - X_n)$. Here conditionally on the total population size $N_n$ the subpopulation sizes are asymptotically binomially distributed

$$
X_n \approx \text{Bin}(N_n, v_1), \ N_n - X_n \approx \text{Bin}(N_n, v_2),
$$

and given $w_1v_1 + w_2v_2 = 0$, we are interested in the asymptotic distribution of the linear combination

$$
Y_n = w_1X_n + w_2(N_n - X_n) = (w_1 - w_2)(X_n - N_nv_1).
$$
With a large $N$, the distribution of $Y_n/\sqrt{n}$ is asymptotically normal with zero mean and variance $(N_n/n)(w_1 - w_2)^2v_1v_2 = N_nw^2v^t$. On the other hand, conditionally on non-extinction, $N_n/n$ is asymptotically exponentially distributed with parameter $(1 + m)^{-1}\beta$. The last two facts combined together result in a limit Laplace distribution for $Y_n/\sqrt{n}$ obtained as a normal distribution with a random variance having an exponential distribution.

11. The linear fractional CMJ-process

In this final section we focus on the process formed by the total population sizes $Z^{(n)}(t) = Z^{(n)}(t)$. As the particle types are neglected the process $\{Z^{(n)}\}_{n=0}^\infty$ is not a Markov chain anymore, however, it gives an interesting example of a discrete time CMJ-process [11] which will be called a linear-fractional CMJ-process. Its individual life length, if unbounded, can have an arbitrary distribution, see Example 8, and the numbers of births given at different ages (except at the moments of birth and death) are independent geometric random numbers with the same mean $m$.

For the linear-fractional CMJ-process to be fully described by the life length distribution [15] and parameter $m$ one has to assume that the initial particle of the underlying linear-fractional branching process with parameters $(H, g, m)$ has distribution $g$: $\mathbb{P}(Z^{(0)} = e_i) = g_i$, $i \geq 1$, so that the life length of the initial individual has the same distribution [15] as all new-born individuals appearing in the linear-fractional CMJ-process.

In the framework of CMJ-processes (see [9]) the established classification of branching processes is expressed in terms of the so-called Malthusian parameter $\alpha = -\ln R$. It is assumed that $R$, a positive solution of equation [16] is well defined, a restriction which in the current framework of multi-type BGW-processes excludes from consideration the transient case. The usual criterium of criticality $\{\alpha < 0, \alpha = 0, \alpha > 0\}$ for the CMJ-processes, when $\alpha$ is defined, is of course equivalent to the classification $\{\rho < 1, \rho > 1, \rho > 1\}$. Another key parameter $\beta$ is called the mean age at childbearing and plays an important role when the renewal theory methods are applied for the analysis of the CMJ-processes.

Next we give a summary of the results for the linear-fractional CMJ-process which follow from the more general results obtained earlier in this paper. In the case when the Maltusian parameter $\alpha$ exists and $\beta < \infty$, the
asymptotics of the survival probability

$$Q_n = P(Z^{(n)} > 0) = \sum_{i=1}^{\infty} g_i P[Z^{(n)} \neq 0 | Z^{(0)} = e_i]$$

according to Propositions \[16, 17, 18\] takes a particularly transparent form:

- $Q_n \sim e^{\alpha n} (1 - \mu)/(m\beta)$, if $\alpha < 0$,
- $Q_n \sim 1/(nm)$, if $\alpha = 0$,
- $Q_n \to (e^\alpha - 1)/m$, if $\alpha > 0$.

Moreover, in the subcritical case due to Proposition \[16\] there exist a discrete limit distribution

$$P(Z^{(n)} = k | Z^{(n)} > 0) \to m^{k-1}(1 + m)^{-k}, \quad k = 1, 2, \ldots.$$ 

In the critical case Propositions \[17\] implies that for any positive $x$

$$P(Z^{(n)} > nx | Z^{(n)} > 0) \to e^{-\beta x/(1+m)}.$$ 

Finally, in the supercritical case Propositions \[18\] yields a weak convergence result

$$P(Z^{(n)} > e^{\alpha n} x | Z^{(n)} > 0) \to e^{-ax}, \quad x > 0,$$

where $a = \beta(e^\alpha - 1)/(1 + m)$. These explicit results nicely illuminate much more general limit theorems obtained for the CMJ-processes, see for example \[9, 10, \text{and} 21\].

**Remark 20.** There are many triplets $(H, g, m)$ resulting in the same linear-fractional CMJ-process $\{Z^{(n)}\}$. For all of them the values of $R$ and $\beta$ will be the same as they are governed by the individual life distribution. The difference between such related branching processes is in the labeling rules for particles which can be seen only in the expressions for the Perron-Frobenius eigenvectors.

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References

[1] Asmussen, S. (2003) Applied probability and queues, Springer, New York.

[2] Athreya, K. and Kang, H. (1998). Some limit theorems for positive recurrent Markov Chains I and II. Adv. in Appl. Probab. 30, 693–722.

[3] Athreya, K. and Ney, P. (1972) Branching processes, John Wiley & Sons, London-New York-Sydney.

[4] Barbour, A. and Luczak, M. (2008) Laws of large numbers of epidemic models with countably many types. Ann. Appl. Probab. 18, 2208–2238.

[5] Feller, W. (1959). An introduction to probability theory and its applications, Vol I, 2nd ed. John Wiley & Sons, London-New York-Sydney.

[6] Feller, W. (1971). An introduction to probability theory and its applications, Vol II, 3rd ed. John Wiley & Sons, London-New York-Sydney.

[7] Geiger, J. (1999). Elementary new proofs of classical limit theorems for Galton-Watson processes. J. Appl. Probab. 36, 301–309.

[8] Geiger, J. and Kersting, G. (1997). Depth-first search of random trees, and Poisson point processes. In Classical and Modern Branching Processes (Minneapolis, MN, 1994). IMA Math. Appl. 84, pp. 111–126. Springer, New York.

[9] Jagers, P. (1975) Branching processes with biological applications, Wiley, New-York.

[10] Jagers, P. and Nerman, O. (1984) The growth and composition of branching populations. Adv. in Appl. Probab. 16, 221–259.

[11] Jagers, P. and Sagitov, S. (2008) General branching processes in discrete time as random trees. Bernoulli 14, 949–962.

[12] Joffe, A. and Letac, G. (2006) Multitype linear fractional branching processes. J. Appl. Probab. 43, 1091–1106.
[13] Haccou, P., Jagers, P., Vatutin, V.A. (2005) Branching Processes: Variation, Growth and Extinction of Populations, Cambridge University Press, Cambridge.

[14] Hoppe, FM. (1997) Coupling and the Non-degeneracy of the Limit in Some Plasmid Reproduction Models. Theor. Popul. Biol. 52, 27–31.

[15] Kesten, H. (1989) Supercritical branching processes with countably many types and the sizes of random cantor sets. In Probability, Statistics and Mathematics. Papers in Honor of Samuel Karlin, pp. 108–121, Academic Press, New York.

[16] Kimmel, M. and Axelrod, D. (2002) Branching Processes in Biology, Springer, New York.

[17] Lambert, A. (2010) The contour of splitting trees is a Levy process. Ann. Probab. 38, 348–395.

[18] Moy, S.-T. C. (1967) Extensions of a limit theorem of Everett Ulam and Harrison multi-type branching processes to a branching process with countably many types. Ann. Math. Statist. 38, 992-999.

[19] Pakes, A.G. (2003) Biological Applications of Branching Processes. In Handbook of Statistics 21, pp. 693–773, Elsevier Science, Amsterdam, Netherlands.

[20] Pollak, E. (1974) Survival probabilities and extinction times for some multitype branching processes. Adv. Appl. Prob. 6, 446–462.

[21] Sagitov, S. (1995) A key limit theorem for critical branching processes. Stoch. Proc. Appl. 56, 87–100.

[22] Seneta, E. (2006). Non-negative matrices and Markov chains, Springer Series in Statistics No. 21, Springer, New-York.

[23] Seneta, E. and Tavare, S. (1983) Some stochastic models for plasmid copy number. Theor. Popul. Biol. 23, 241–256.