A NEW BLOWUP CRITERION FOR STRONG SOLUTIONS TO A VISCOUS LIQUID-GAS TWO-PHASE FLOW MODEL WITH VACUUM IN THREE DIMENSIONS

YINGSHAN CHEN AND MEI ZHANG*

Department of Mathematics
South China University of Technology
Guangzhou 510641, China

(Communicated by Huijiang Zhao)

Abstract. In this paper, we establish a new blowup criterion for the strong solutions in a smooth bounded domain $\Omega \subset \mathbb{R}^3$. In [13], Wen, Yao, and Zhu prove that if the strong solutions blow up at finite time $T^*$, the mass in $L^\infty(\Omega)$ norm must concentrate at $T^*$. Here we extend Wen, Yao, and Zhu’s work in the sense of the concentration of mass in $BMO(\Omega)$ norm at $T^*$. The method can be applied to study the blow-up criterion in terms of the concentration of density in $BMO(\Omega)$ norm for the strong solutions to compressible Navier-Stokes equations in smooth bounded domains. Therefore, as a byproduct, we can also improve the corresponding result about Navier-Stokes equations in [11]. Moreover, the appearance of vacuum is allowed in the paper.

1. Introduction. The viscous liquid-gas two-phase flow model [8, 10], modeling the motion of the mixture of liquid and gas, has been widely studied since Evje-Karlsen’s work [4]. Mathematically, the model can be written as follows:

$$
\begin{cases}
    m_t + \text{div}(mu) = 0, \\
    n_t + \text{div}(nu) = 0, \\
    (mu)_t + \text{div}(mu \otimes u) + \nabla P(m,n) = \mu \Delta u + (\mu + \lambda)\nabla \text{div} u,
\end{cases}
$$

in $\Omega \times (0, \infty)$, where $\Omega \subseteq \mathbb{R}^3$ is a smooth bounded domain. The system (1) is supplemented with the initial-boundary conditions:

$$
\begin{align*}
    (m, n, u)|_{t=0} &= (m_0, n_0, u_0), & \text{in } \Omega, \\
    u(x, t) &= 0, & \text{on } \partial \Omega \times [0, \infty).
\end{align*}
$$

Here $m = \alpha_l \rho_l$ and $n = \alpha_g \rho_g$ denote the liquid mass and gas mass, respectively; $\mu$, $\lambda$ are viscosity constants, satisfying

$$
\mu > 0, \quad 2\mu + 3\lambda \geq 0,
$$

which implies $\mu + \lambda \geq \frac{1}{3}\mu > 0$. The unknown variables $\alpha_l, \alpha_g \in [0, 1]$ denote the liquid and gas volume fractions, respectively, which naturally satisfy the fundamental relation: $\alpha_l + \alpha_g = 1$. The other unknown variables $\rho_l$ and $\rho_g$ denote liquid and gas

2010 Mathematics Subject Classification. 76T10, 76N10, 35L65.

Key words and phrases. Liquid-gas two-phase flow model, strong solution, BMO criterion, vacuum.

* Corresponding author: Mei Zhang.
density, respectively. As stated in [4], $\rho_l$ and $\rho_g$ satisfy the equations of state, i.e.,

$$
\rho_l = \rho_{l,0} + \frac{P - P_{l,0}}{a_l^2}, \quad \rho_g = \frac{P}{a_g^2},
$$

where $a_l$, $a_g$ are sonic speeds, respectively, in the liquid and gas, and $P_{l,0}$ and $\rho_{l,0}$ are the reference pressure and density given as constants. $u$ denotes velocity of the liquid and gas. $P$ is the common pressure for both phases, which satisfies

$$
P(m, n) = C^0 \left(-b(m, n) + \sqrt{b(m, n)^2 + c(n)}\right),
$$

with $C^0 = \frac{1}{2}a_l^2$, $k_0 = \rho_{l,0} - \frac{P_{l,0}}{a_l^2} > 0$, $a_0 = \left(\frac{a_g}{a_l}\right)^2$ and

$$
b(m, n) = k_0 - m - \left(\frac{a_g}{a_l}\right)^2 n = k_0 - m - a_0 n,
$$

$$
c(n) = 4k_0 \left(\frac{a_g}{a_l}\right)^2 n = 4k_0 a_0 n.
$$

Let us briefly review some previous works about the viscous liquid-gas two-phase flow model (1). More precisely, when both of the two fluids are compressible (liquid is considered slightly compressible), Evje and Karlsen [4] derived the first work on the global existence of weak solutions of Cauchy problem of (1) in one dimension. In [4], both of $m$ and $n$ are positive initially. In higher dimensions, Yao, Zhang and Zhu [17] obtained the existence of the global weak solution to the two-dimensional case of (1) when the initial energy is small, and both of $m$ and $n$ are positive initially. Furthermore, they [18] established a blow-up criterion in terms of the upper bound of the liquid mass for the strong solutions to the model in a smooth bounded domain of $\mathbb{R}^2$. In [18], the authors only deal with the case: there is no initial vacuum, i.e., $m_0 > 0$, $n_0 > 0$. The global well-posedness of classical solutions with small initial energy in $\mathbb{R}^3$ can be referred to Cui-Wen-Yin’s work [2]. In the framework of Besov space, please refer to [7].

When vacuum is allowed, i.e., $m_0 \geq 0$ and $n_0 \geq 0$, things become more difficult. For example, we can not estimate $\frac{1}{m}$ or $\frac{1}{n}$ when we do the estimates. By using the iteration arguments, Wen, Yao, and Zhu obtained the local well-posedness of strong solutions of system (1) with vacuum in a smooth bounded domain of $\mathbb{R}^3$. Moreover, Wen, Yao, Zhu derived a blowup criterion for the strong solutions in terms of the concentration of $m$ in $L^\infty(\Omega)$ at the finite maximal time $T^*$ for existence. For the global well-posedness of the strong solutions with small initial energy in three dimensions, please refer to [6]. When the liquid is incompressible and the gas is polytropic, i.e., $P(m, n) = C\rho_l^n \left(\frac{n}{\rho_l - m}\right)^\gamma$, please refer to [5, 15, 3, 16] and references therein.

However, it is still unknown that the global smooth solutions with large initial data in three dimensions exist or not. This motivates us to study some sharp blowup criteria which can give some insight into the mechanics of possible blowing up.

Before stating our main result, we would like to mention the definition of strong solutions here.

**Definition 1.1 (Strong solutions).** Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^3$ and $q \in (3, 6)$. $(m, n, u)$ is called a strong solution of (1)-(3) over $\Omega \times [0, T]$, if $(m, n, u)$
satisfies (1)-(3) almost everywhere in $\Omega \times [0, T]$, with the following regularity:

$$0 \leq s_0 m \leq n \leq \pi_0 m, \quad (m, n) \in C([0, T]; W^{1,q}(\Omega)),$$

$$m_t, n_t \in L^\infty(0, T; L^q(\Omega)), \quad P \in L^\infty(0, T; W^{1,q}(\Omega)),$$

$$u \in C([0, T]; H^1_0(\Omega) \cap H^2(\Omega)) \cap L^2(0, T; W^{2,q}(\Omega)),$$

$$\mu u_t \in L^\infty(0, T; L^2(\Omega)), \quad u_t \in L^2(0, T; H^1_0(\Omega)).$$

(6)

Actually, given some suitable initial data, the local existence and uniqueness of the strong solutions of (1)-(3) has been obtained by Wen, Yao, Zhu in their work[13]. More precisely,

**Proposition 1.** ([13]) Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^3$ and $q \in (3, 6]$. Assume that the initial data $m_0, n_0, u_0$ satisfy $m_0, n_0 \in W^{1,q}(\Omega), \quad u_0 \in H^1_0(\Omega) \cap H^2(\Omega)$, $0 \leq s_0 m_0 \leq n_0 \leq \pi_0 m_0$ in $\Omega$, where $s_0$ and $\pi_0$ are positive constants. The following compatible condition is also valid:

$$-\mu \Delta u_0 - (\mu + \lambda) \nabla \text{div} u_0 + \nabla P(m_0, n_0) = \sqrt{m_0} g, \quad \text{for some } g \in L^2(\Omega).$$

(7)

Then, there exist a $T_0 > 0$ and a unique strong solution $(m, n, u)$ to the problem (1)-(3) over $\Omega \times [0, T_0]$.

Furthermore, under the assumption

$$\lambda < \frac{29}{3} \mu,$$

(8)

Wen, et al in [13, 14] established the blow-up criterion for the strong solutions:

**Proposition 2.** ([13, 14]) Under the assumptions of Proposition 1, if $T^* < \infty$ is the maximal existence time for the strong solution $(m, n, u)(x, t)$ to the problem (1)-(3) stated in Theorem 1.2, then

$$\limsup_{t \to T^*} \|m(\cdot, t)\|_{L^\infty(\Omega)} = \infty,$$

(9)

provided that (8) holds.

In this paper, we establish a new blow-up criterion for the strong solutions. Our main result is stated as follows:

**Theorem 1.2.** Under the assumptions of Proposition 1, if $T^* < \infty$ is the maximal existence time of the strong solution, then

$$\limsup_{t \to T^*} \|m(\cdot, t)\|_{BMO(\Omega)} + \limsup_{t \to T^*} \|n(\cdot, t)\|_{BMO(\Omega)} = \infty,$$

(10)

provided $\frac{29\mu}{3} > \lambda$. Here $BMO(\Omega)$ denotes the John-Nirenberg’s space of bounded mean oscillation whose norm is given by (18) and (19).

**Remark 1.** Under the conditions of Proposition 1, we can prove Lemma 3.1 which implies that (9) is equivalent to

$$\limsup_{t \to T^*} \|n(\cdot, t)\|_{L^\infty(\Omega)} = \infty.$$

Observe that for a given function $f$, it is easy to verify

$$\|f(\cdot, t)\|_{BMO(\Omega)} \leq \tilde{C} \|f(\cdot, t)\|_{L^\infty(\Omega)}$$

for some known positive constant $\tilde{C}$ independent of $t$ and $m$. Therefore, it is obvious that our main result (10) is a relaxation of Wen et al’s result (9).
Lemma 2.2. Unlike [14], the domain here is bounded. The estimates of the effective viscous flux and the vorticity in [14] are not valid any more, since we do not know anything about the boundary conditions of them. Here we use some ideas of Sun et al [11] to handle some challenges due to the bounded domain.

Remark 3. Our methods can also be applied to compressible Navier-Stokes equations in bounded domains. Therefore, as a byproduct, we extend the blowup criterion in terms of the concentration of the density in $L^\infty(\Omega)$ in Sun et al’s work [11] for the strong solutions of compressible Navier-Stokes equations in bounded domains.

2. Preliminaries. In this section, like in [13], we give some useful lemmas which will be used afterwards, where $N = 3$.

Lemma 2.1. Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded domain with piecewise smooth boundaries. Then the following inequality is valid for every function $u \in W_0^{1,p}(\Omega)$ or $u \in W^{1,p}(\Omega)$, $\int_\Omega u dx = 0$:

$$\|u\|_{L^{r'}(\Omega)} \leq C_1 \|\nabla u\|_{L^p(\Omega)} \|u\|_{L^{r'}(\Omega)}^{1-\alpha},$$

where $\alpha = (1/r' - 1/p')(1/r' - 1/\alpha N)^{-1}$; moreover, if $p < N$, then $p' \in [r'/pN/(N-p)]$ for $r' \leq pN/(N-p)$, and $p' \in [pN/(N-p), r']$ for $r' > pN/(N-p)$. If $p \geq N$, then $p' \in [r', \infty]$ is arbitrary; moreover, if $p > N$, then inequality (11) is also valid for $p' = \infty$. The positive constant $C_1$ in inequality (11) depends on $N$, $p$, $r'$, $\alpha$ and the domain $\Omega$ but independent of the function $u$.

Lemma 2.2. Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded domain with piecewise smooth boundaries. Then the following inequality is valid for every function $u \in W^{1,p}(\Omega)$:

$$\|u\|_{L^{r'}(\Omega)} \leq C_2(\|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^p(\Omega)} \|u\|_{L^{r'}(\Omega)}^{1-\alpha}),$$

where $N$, $r'$, $p'$ and $\alpha$ are the same as those in Lemma 2.1. The positive constant $C_2$ in inequality (12) depends on $N$, $p$, $r'$, $\alpha$ and the domain $\Omega$ but independent of the function $u$.

The above two lemmas can be found in [9, 12] and the references therein.

Next, we give some $L^p$ ($p \in (1, \infty)$) regularity estimates for the solution of the following boundary problem:

$$\begin{cases}
LU := \mu\Delta U + (\mu + \lambda)\nabla \text{div} U = F, & \text{in } \Omega, \\
U(x) = 0, & \text{on } \partial\Omega.
\end{cases}$$

(13)

Here $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $L$ is the Lamé operator, $U = (U_1, U_2, \ldots, U_N)$, $F = (F_1, F_2, \ldots, F_N)$. From (4), we know that (13) is a strong elliptic system. If $F \in W^{-1,2}(\Omega)$, then there exists an unique weak solution $U \in H_0^1(\Omega)$. In the subsequent context, we will use $L^{-1}F$ to denote the unique solution $U$ of the system (13) with $F$ belonging to some suitable space such as $W^{-1,p}(\Omega)$. The following two lemmas can be found in [11] and references therein:

Lemma 2.3. Let $p \in (1, \infty)$, and $U$ be a solution of (13). Then there exists a constant $C$ depending only on $\mu$, $\lambda$, $p$, $N$ and $\Omega$ such that

1. if $F \in L^p(\Omega)$, then

$$\|U\|_{W^{2,p}(\Omega)} \leq C\|F\|_{L^p(\Omega)};$$

(14)
(2) if $F \in W^{-1,p}(\Omega)$ (i.e., $F = \text{div} f$ with $f = (f_{ij})_{N \times N}$, $f_{ij} \in L^p(\Omega)$), then
\[ \|U\|_{W^{1,p}(\Omega)} \leq C \|f\|_{L^p(\Omega)}; \tag{15} \]

(3) if $F = \text{div} f$ with $f_{ij} = \partial_k h^k_{ij}$ and $h^k_{ij} \in W^{1,p}_0(\Omega)$ for $i, j, k = 1, 2, \ldots, N$, then
\[ \|U\|_{W^{1,p}(\Omega)} \leq C \|h\|_{L^p(\Omega)}. \tag{16} \]

**Lemma 2.4.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$ and $f \in W^{1,p}(\Omega)$ with $p \in (N, \infty)$. Then there exists a constant $C$ depending on $p$, $N$ and the Lipschitz property of the domain $\Omega$ such that
\[ \|f\|_{L^\infty(\Omega)} \leq C \left( 1 + \|f\|_{\text{BMO}(\Omega)} \ln(e + \|\nabla f\|_{L^p(\Omega)}) \right). \tag{17} \]
Here $\text{BMO}(\Omega)$ denotes the John-Nirenberg’s space of bounded mean oscillation whose norm is defined by
\[ \|f\|_{\text{BMO}(\Omega)} = \|f\|_{L^2(\Omega)} + [f]_{\text{BMO}(\Omega)}, \tag{18} \]
with the semi-norm
\[ [f]_{\text{BMO}(\Omega)} = \sup_{x \in \Omega, r \in (0, d)} \frac{1}{\Omega_r(x)} \int_{\Omega_r(x)} |f(y) - f_{\Omega_r(x)}|dy, \tag{19} \]
where $\Omega_r(x) = B_r(x) \cap \Omega$, $B_r(x)$ is the ball with center $x$ and radius $r$ and $d$ is the diameter of $\Omega$. For a measurable subset $E$ of $\mathbb{R}^N$, $|E|$ denotes its Lebesgue measure and
\[ f_{\Omega_r(x)} = \frac{1}{\Omega_r(x)} \int_{\Omega_r(x)} f(y)dy. \]

The last lemma in the section can be found in [1]:

**Lemma 2.5.** If $F = \text{div} f$ with $f = (f_{ij})_{N \times N}$, $f_{ij} \in \text{BMO}(\Omega)$, then $\nabla U \in \text{BMO}(\Omega)$ and there exists a constant $C$ depending only on $\mu$, $\lambda$ and $\Omega$ such that
\[ \|\nabla U\|_{\text{BMO}(\Omega)} \leq C \|f\|_{\text{BMO}(\Omega)}. \tag{20} \]

### 3. Proof of Theorem 1.2
Some of ideas used by Sun et al and Wen et al in [11, 13, 14] play important roles in the proof of Theorem 1.2. Let $(m, n, u)$ be a strong solution to the problem (1)-(5) in $Q_T$. We assume that the opposite holds, i.e.
\[ \lim_{T \to T^*} \sup_{T \to T^*} \|m\|_{L^\infty(0,T;\text{BMO}(\Omega))} + \lim_{T \to T^*} \sup_{T \to T^*} \|n\|_{L^\infty(0,T;\text{BMO}(\Omega))} \leq M < \infty. \tag{21} \]
It is easy to verify from (21) that
\[ \lim_{T \to T^*} \sup_{T \to T^*} \|m\|_{L^\infty(0,T;L^{q_1}(\Omega))} + \lim_{T \to T^*} \sup_{T \to T^*} \|n\|_{L^\infty(0,T;L^{q_1}(\Omega))} \leq M_1 < \infty, \tag{22} \]
for any $q_1 < \infty$ and some positive constant $M_1$.

In this section, we denote by $C$ a generic positive constant which may depend on $\mu$, $\lambda$, $\Omega$, $m_0$, $n_0$, $u_0$, $M$, $T^*$, and the parameters in the expression of $P$ in (5). For simplicity, we omit the domain of the spatial integrability and spatial norm when it does not cause any confusion.

Similar to Lemma 5.1 and Lemma 5.2 in [13], we get the following lemmas. In lemma 3.2, in order to relax the restriction $\frac{2n}{3} > \lambda$ in [13], we use the technique in [14].

**Lemma 3.1.** Under the conditions of Proposition 1, we have for all $0 \leq T < T^*$
\[ \mathbb{E}_0 m \leq n \leq \mathbb{E}_0 m, \text{ in } Q_T. \]
Lemma 3.2. Under the conditions of Proposition 1 and 22, if \( \frac{29\mu}{T} > \lambda \), there exists some \( r > 3 \) such that

\[
\sup_{0 \leq t \leq T} \int_{\Omega} m|u|^r \, dx \leq C, \quad 0 \leq T < T^*.
\]

For later use, as in [11], we denote \( w = u - h \), where \( h \) is the unique solution to

\[
\begin{aligned}
Lh &= \nabla \cdot \, \nabla P, \quad \text{in } \Omega \times (0, T], \\
h|_{\partial \Omega} &= 0.
\end{aligned}
\]  

(23)

From Lemma 2.3, we get for any \( p \in (1, \infty) \)

\[
\begin{aligned}
&\|h\|_{W^{1,p}} \leq C \|P\|_{L^p}, \\
&\|h\|_{W^{2,p}} \leq C \|
abla P\|_{L^p}.
\end{aligned}
\]  

(24)

(1), (3) and (23) imply

\[
\begin{aligned}
mw_t - Lw &= mF, \quad \text{in } \Omega \times (0, T], \\
w(x, 0) &= u_0 - L^{-1}\nabla P(m_0, n_0), \quad \text{in } \Omega, \\
w|_{\partial \Omega} &= 0,
\end{aligned}
\]  

(25)

where

\[
F = -u \cdot \nabla u - L^{-1}\nabla P_t \\
= -u \cdot \nabla u + L^{-1}\nabla \text{div}(Pu) + L^{-1}\nabla [mP_m + nP_n - P]\text{div}u.
\]

Lemma 3.3. Under the conditions of Proposition 1 and (22), for \( 0 \leq T < T^* \), we have

\[
\begin{aligned}
&\|P\|_{L^\infty(0, T; L^{q_1})} + \|P_m\|_{L^\infty(Q_T)} + \|P_n\|_{L^\infty(Q_T)} \leq C, \\
&\|P\|_{L^\infty(0, T; BMO)} \leq C, \\
&\|\nabla u\|_{L^2(0, T; L^{p_1})} + \|\nabla u\|_{L^\infty(0, T; L^2)} \leq C,
\end{aligned}
\]  

(26)

for any \( q_1 < \infty \) and \( p_1 < 6 \).

Proof. Similar to Lemma 4.3 in [13], we can easily obtain \( \|P_m\|_{L^\infty(Q_T)} + \|P_n\|_{L^\infty(Q_T)} \leq C \) where we have used Lemma 3.1. Next we aim to show \( \|P\|_{L^\infty(0, T; L^{q_1})} \leq C \). A direct calculus yields that

\[
|P(m, n)| \leq 2C^0 \sqrt{(k_0 - m - a_0 n)^2 + 4k_0 a_0 n^2} \\
\leq 2C^0 (k_0 + m + a_0 n).
\]

This together with (22) completes the proof of (26).
In the following we prove (26).2.

\[ [P]_{BMO(\Omega)} = \sup_{x \in \Omega, r \in (0, d)} \int_{\Omega_r(x)} |P(y) - P_{\Omega_r(x)}|dy \]

\[ \leq \sup_{x \in \Omega, r \in (0, d)} \int_{\Omega_r(x)} \int_{\Omega_r(x)} |P(y) - P(z)|dzdy \]

\[ = \sup_{x \in \Omega, r \in (0, d)} \int_{\Omega_r(x)} \int_{\Omega_r(x)} \left| P_n(z_1) (m(y) - m(z)) + P_n(z_2) (n(y) - n(z)) \right|dzdy \]

\[ \leq C \sup_{x \in \Omega, r \in (0, d)} \left[ \int_{\Omega_r(x)} \int_{\Omega_r(x)} |m(y) - m(z)|dzdy + \int_{\Omega_r(x)} \int_{\Omega_r(x)} |n(y) - n(z)|dzdy \right] \]

\[ \leq C \sup_{x \in \Omega, r \in (0, d)} \left( \int_{\Omega_r(x)} |n(y) - n(z)|dz + C \sup_{x \in \Omega, r \in (0, d)} \int_{\Omega_r(x)} |m(z) - m_{\Omega_r(x)}|dz \right) \]

\[ \leq C[m_{BMO} + C|n_{BMO}|, \sup_{x \in \Omega, r \in (0, d),} |n_{\Omega_r(x)}|] \]

where we have used the boundness of \( P_m \) and \( P_n \). This together with (21) and (26) completes the proof of (26).2.

What left is to show (26)3. Multiplying (25)1 by \( w_t \), integrating over \( \Omega \), and using integration by parts and Cauchy inequality, we have

\[ \int_{\Omega} m|w_t|^2 + \frac{1}{2} \frac{d}{dt} \int_{\Omega} [\mu|\nabla w|^2 + (\mu + \lambda)|\text{div} w|^2] \]

\[ \leq \frac{1}{2} \int_{\Omega} m|w_t|^2 + \frac{1}{2} \int_{\Omega} m|F|^2, \]

which implies

\[ \int_{\Omega} m|w_t|^2 + \frac{d}{dt} \int_{\Omega} [\mu|\nabla w|^2 + (\mu + \lambda)|\text{div} w|^2] \leq \int_{\Omega} m|F|^2. \]

(27)

Observe that

\[ \int_{\Omega} m|F|^2 \]

\[ \leq C \int_{\Omega} m|u|^2|\nabla u|^2 + C \int_{\Omega} m \left[ |L^{-1}\text{div}(Pu)|^2 + |L^{-1}\text{div}((mP_m + nP_n - P)\text{div}u)|^2 \right] \]

\[ = I_1 + I_2. \]

(28)

Next we estimate \( I_1 \) and \( I_2 \) respectively. Assume \( p_2 \in (\frac{2p}{p - 2}, 6) \), \( \frac{3p_2}{3p_2 - 2} < p_3 < 2 \) and let \( \alpha = (\frac{1}{2} - \frac{1}{p_2})/(\frac{3p_2}{3p_2 - 2} - \frac{1}{2}) \), then

\[ I_1 \leq C \int_{\Omega} m|u|^2|\nabla w|^2 + C \int_{\Omega} m|u|^2|\nabla h|^2 \]

\[ \leq C \left( m^\frac{1}{2} u_{L^2}^2 \left[ m \right]_{L^{p_2(p_2 - 2)}}^2 \left( \|
abla w\|_{L^{p_2}}^2 + \|
abla h\|_{L^{p_2}}^2 \right) \right. \]

\[ \leq C \left( \|
abla w\|_{L^2}^2 + \|
abla^2 w\|_{L^2}^2 \right) \left( \|
abla w\|_{L^2}^{2(1 - \alpha)} + \|P\|_{L^2}^2 \right) \]

\[ \leq C \left( \epsilon \|
abla^2 w\|_{L^2}^2 + (1 + \epsilon^{\alpha/(\alpha - 1)}) \|
abla w\|_{L^2}^2 + 1 \right), \]

(29)
Combining (27) and (33), we get inequality, Lemma 2.3, (22) and (26), we have

\[
\|\nabla^2 w\|_{L^p}^2 \leq C \left( \|mw_i\|_{L^p}^2 + \|MF\|_{L^p}^2 \right)
\]

\[
\leq C \|m\|_{L^{\frac{2q^2}{q^2-1}}} \left( \int_{\Omega} |w_t|^2 + \int_{\Omega} |F|^2 \right)
\]

\[
\leq C \left( \int_{\Omega} |w_t|^2 + \int_{\Omega} |F|^2 \right).
\]

(30)

Substituting (30) into (29), we have

\[
I_1 \leq C \left( \epsilon \int_{\Omega} |w_t|^2 + \epsilon \int_{\Omega} |F|^2 + (1 + \epsilon^{a/(a-1)}) \|\nabla w\|_{L^2}^2 + 1 \right).
\]

(31)

On the other hand, we can choose \( q_1 > \frac{3}{2} \) and use Hölder inequality, Sobolev inequality, Lemma 2.3, (22) and (26) to obtain

\[
I_2 \leq C \|m\|_{L^{q_1}} \left[ \|L^{-1} \nabla \text{div}(Pu)\|^2_{L_{\Omega}^{\frac{2q_1}{q_1-1}}} + \|L^{-1} \nabla [(mp_m + np_n - P)\text{div}u]\|^2_{L_{\Omega}^{\frac{2q_1}{q_1-1}}} \right]
\]

\[
\leq C \|Pu\|^2_{L_{\Omega}^{\frac{2q_1}{q_1-1}}} + C \|\nabla L^{-1} \nabla [(mp_m + np_n - P)\text{div}u]\|^2_{L_{\Omega}^{\frac{2q_1}{q_1-1}}}
\]

\[
\leq C \|P\|^2_{L_{\Omega}^{\frac{2q_1}{q_1-1}}} + \|u\|^2_{L_{\Omega}^{q_1}} + C \|(mp_m + np_n - P)\text{div}u\|^2_{L_{\Omega}^{q_1}}
\]

\[
\leq C \|\nabla u\|^2_{L^2} + C \|(mp_m + np_n - P\|^2_{L_{\Omega}^{q_1}} |\text{div}u|_{L^2}^2
\]

\[
\leq C (\|\nabla w\|_{L^2}^2 + ||\text{div}w||_{L^2}^2 + ||\nabla h||_{L^2}^2) + (||\text{div}h||_{L^2}^2)
\]

\[
\leq C (\|\nabla w\|_{L^2}^2 + ||\text{div}w||_{L^2}^2 + 1).
\]

(32)

Substituting (31)-(32) into (28), and taking \( \epsilon = \frac{1}{10} \), we obtain

\[
\int_{\Omega} |F|^2 \leq \frac{1}{3} \int_{\Omega} |w_t|^2 + C \int_{\Omega} |\nabla w|^2 + C \int_{\Omega} (\mu + \lambda)|\text{div}w|^2 + C.
\]

(33)

Combining (27) and (33), we get

\[
\frac{2}{3} \int_{\Omega} |w_t|^2 + \frac{d}{dt} \int_{\Omega} |\mu|\nabla w|^2 + (\mu + \lambda)|\text{div}w|^2
\]

\[
\leq C \int_{\Omega} |\mu|\nabla w|^2 + C \int_{\Omega} (\mu + \lambda)|\text{div}w|^2 + C.
\]

Integrating over \((0,t)\), we have

\[
\frac{2}{3} \int_{0}^{t} \int_{\Omega} |w_t|^2 + \int_{\Omega} |\mu|\nabla w|^2 + (\mu + \lambda)|\text{div}w|^2
\]

\[
\leq C \int_{0}^{t} \int_{\Omega} |\mu|\nabla w|^2 + C \int_{0}^{t} \int_{\Omega} (\mu + \lambda)|\text{div}w|^2 + C(t + 1).
\]

(34)

(34) together with Gronwall inequality yields

\[
\int_{0}^{t} \int_{\Omega} |w_t|^2 + \int_{\Omega} |\mu|\nabla w|^2 + (\mu + \lambda)|\text{div}w|^2 \leq C, \text{ for } t \in [0,T^\star).
\]

(35)
Based on (33) and (35), we can use (22), (24), (25), Hölder inequality and Sobolev inequality to further obtain

\[
\|\nabla w\|_{L^{p_1}}^2 \leq \|w\|^2_{W^{2, \frac{3p_1}{3+p_1}}} \leq C\|mw_t + mF\|_{L^{\frac{3p_1}{3+p_1}}}^2
\]

\[
\leq C\|m\|_{L^{\frac{3p_1}{3}}} \left( \int_{\Omega} |w_t|^2 + \int_{\Omega} |F|^2 \right)
\]

\[
\leq C \left( \int_{\Omega} m|w_t|^2 + 1 \right),
\]

provided that \( p_1 < 6 \). (35) and (36) together with (23), (24) and (26) complete the proof of Lemma 3.3.

\[\square\]

**Lemma 3.4.** Under the conditions of Proposition 1 and (22), for \( 0 \leq T < T^* \), we have

\[
\int_{\Omega} m|\dot{u}|^2 + \int_0^T \int_{\Omega} \mu|\nabla \dot{u}|^2 + (\mu + \lambda)|\text{div}\dot{u}|^2 \leq C,
\]

where \( \dot{u} = u_t + u \cdot \nabla u \).

**Proof.** (1)\(_3\) can be rewritten as

\[
m\dot{u} + \nabla P - Lu = 0.
\]

Differentiating (37) with respect to \( t \), and using (1)\(_1\), we conclude

\[
m\dot{u}_t + m\dot{u} \cdot \nabla \dot{u} + \nabla \dot{P} + \text{div} (\nabla P \otimes u) = L\dot{u} - L(u \cdot \nabla u) + \text{div}(Lu \otimes u).
\]

(38)

Multiplying (38) by \( \dot{u} \), integrating the resulting equation over \( \Omega \), and using integration by parts, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} m|\dot{u}|^2 + \int_{\Omega} (\mu|\nabla \dot{u}|^2 + (\mu + \lambda)|\text{div}\dot{u}|^2)
\]

\[
= \int_{\Omega} (P_t \text{div}\dot{u} + (u \cdot \nabla \dot{u}) \cdot \nabla P) + \mu \int_{\Omega} \nabla (u \cdot \nabla u) : \nabla \dot{u} + (\mu + \lambda) \int_{\Omega} \text{div}(u \cdot \nabla u) \text{div}\dot{u}
\]

\[
- \int_{\Omega} (u \cdot \nabla \dot{u}) \cdot (\mu \Delta u + (\mu + \lambda) \nabla \text{div}u)
\]

\[
= \int_{\Omega} (P_t \text{div}\dot{u} + (u \cdot \nabla \dot{u}) \cdot \nabla P) + \mu \int_{\Omega} [\nabla (u \cdot \nabla u) : \nabla \dot{u} - (u \cdot \nabla \dot{u}) \cdot \Delta u]
\]

\[
+ (\mu + \lambda) \int_{\Omega} [\text{div}(u \cdot \nabla u) \text{div}\dot{u} - (u \cdot \nabla \dot{u}) \cdot \nabla \text{div}u]
\]

\[= N_1 + N_2 + N_3.
\]
Now we estimate $N_1$, $N_2$ and $N_3$ as follows:

$$
N_1 = \int_{\Omega} \left[ (P_{m} m + P_{n} n) \nabla u \cdot \nabla P \right] \ dx
$$

$$
= \int_{\Omega} \left[ (-mP_m - nP_n) \nabla u \cdot \nabla P \right] \ dx
$$

$$
= \int_{\Omega} \left[ (-mP_m - nP_n) \nabla u \cdot \nabla P \right] \ dx
$$

$$
\leq C \|P - mP_m - nP_n\|_{L^2} \|\nabla u\|_{L^{2q_1}} \|\nabla P\|_{L^2} + C \|P\|_{L^{2q_1}} \|\nabla u\|_{L^{2q_1}} \|\nabla \tilde{u}\|_{L^2}
$$

(39)

where we have assumed $q_1 > 2$ and where we have used integration by parts, (1), (1)$_2$, (22), (26) and Hölder inequality.

$$
N_2 = \mu \int_{\Omega} \left[ \nabla (u \cdot \nabla u) \cdot \nabla \tilde{u} + \nabla (u \cdot \nabla \tilde{u}) \cdot \nabla u \right]
$$

$$
= \mu \int_{\Omega} \left[ \nabla (u \cdot \nabla u) \cdot \nabla \tilde{u} + (\nabla u \cdot \nabla \tilde{u}) \cdot \nabla u \right]
$$

$$
= \mu \int_{\Omega} \left[ \nabla (u \cdot \nabla u) \cdot \nabla \tilde{u} + (\nabla u \cdot \nabla \tilde{u}) \cdot \nabla u - \nabla u \nabla \tilde{u} \cdot \nabla u \right]
$$

$$
\leq C \|\nabla \tilde{u}\|_{L^2} \|\nabla u\|_{L^4}^2,
$$

(40)

where we have used integration by parts and Hölder inequality.

$$
N_3 = (\mu + \lambda) \int_{\Omega} \left[ \nabla (u \cdot \nabla u) \cdot \nabla \tilde{u} + (u \cdot \nabla \tilde{u}) \cdot \nabla u \right]
$$

$$
= (\mu + \lambda) \int_{\Omega} \left[ \nabla (u \cdot \nabla u) \cdot \nabla \tilde{u} + \nabla \tilde{u} \cdot (u \cdot \nabla \tilde{u}) \right]
$$

$$
= (\mu + \lambda) \int_{\Omega} \left[ \nabla (u \cdot \nabla u) \cdot \nabla \tilde{u} + \nabla \tilde{u} \cdot (u \cdot \nabla \tilde{u}) - (u \cdot \nabla \tilde{u}) \cdot (u \cdot \nabla \tilde{u}) \right]
$$

$$
\leq C \|\nabla \tilde{u}\|_{L^2} \|\nabla u\|_{L^4}^2.
$$

(41)

Substituting (39)-(41) into (39), and using Cauchy inequality we have

$$
\frac{d}{dt} \int_{\Omega} \|\tilde{u}\|^2 + \int_{\Omega} \left( \mu |\nabla u|^2 + (\mu + \lambda) |\nabla \tilde{u}|^2 \right) \leq C \|\nabla u\|_{L^{2q_1}}^2 + C \|\nabla u\|_{L^4}^4.
$$

(42)

In the following, we estimate the term $\|\nabla u\|_{L^4}^2$. From equation (1)$_3$ and (23), we know that $w$ satisfies

$$
\left\{ \begin{array}{l}
Lw = m\dot{u}, \quad \text{in } \Omega, \\
\dot{w}(x) = 0, \quad \text{on } \partial\Omega.
\end{array} \right.
$$

(43)
By (43), Lemma 2.3 and Hölder inequality, we get
\[ \|w\|_{L^2(0, T; \mathbb{R})}^2 \leq C\|m \dot{u}\|_{L^4}^2 \leq C\|m\|_{L^6}\|\sqrt{m}\dot{u}\|_{L^2}^2, \]
which together with Sobolev inequality, (24)1 and (22) yields
\[ \|\nabla u\|_{L^4}^4 \leq C\|\nabla u\|_{L^4}^2\left(\|\nabla w\|_{L^4}^2 + \|\nabla h\|_{L^4}^2\right) \leq C\|\nabla u\|_{L^4}^2\left(\|w\|_{L^2}^2 + \|P\|_{L^2}^2\right) \]
\[ \leq C\|\nabla u\|_{L^4}^2\left(\|\sqrt{m}\dot{u}\|_{L^2}^2 + 1\right) \]
Substituting (44) into (42), we get
\[ \frac{d}{dt} \int_{\Omega} m|\dot{u}|^2 + \int_{\Omega} (\mu|\nabla \dot{u}|^2 + (\mu + \lambda)|\text{div}\dot{u}|^2) \leq C\|\nabla u\|_{L^4}^2 + C\|\nabla u\|_{L^4}^2 \frac{2m}{L^{4q}} \]
\[ + C\|\nabla u\|_{L^4}^2\|\sqrt{m}\dot{u}\|_{L^2}^2. \]
Integrating the resulting inequality over (0, t), and using (26)3, we get
\[ \int_{\Omega} m|\dot{u}|^2 + \int_0^t \int_{\Omega} (\mu|\nabla \dot{u}|^2 + (\mu + \lambda)|\text{div}\dot{u}|^2) \leq C + C\int_0^t \|\nabla u\|_{L^4}^2, \int_{\Omega} m|\dot{u}|^2. \]
This together with Gronwall inequality and (26)3 complete the proof of Lemma 3.4.

Due to (22), (43), Lemma 2.3, Lemma 3.4 and Sobolev inequality, we immediately give the following result.

**Corollary 1.** Under the conditions of Proposition 1 and (22), for \(0 \leq T < T^*\), we have
\[ \|w\|_{L^2(0, T; W^{2, q})} \leq C, \]
provided that \(q < 6\).

In the following, we give the estimates of the derivatives of \(m\) and \(n\).

**Lemma 3.5.** Under the conditions of Proposition 1 and (22), for \(0 \leq T < T^*\), we have
\[ \sup_{t \in [0, T]} \|\nabla m, \nabla n\|_{L^6} \leq C. \]

**Proof.** Differentiating the equation (1)1 with respect to \(x_i\), then multiplying both sides of the resulting equation by \(q|\partial_i m|^q - 2\partial_i m\), we get
\[ \partial_i|\partial_i m|^q + \text{div}(\partial_i m)^q(\partial_i u) + (q - 1)|\partial_i m|^q \text{div} u + qm|\partial_i m|^q \text{div} u + q|\partial_i m|^q - 2\partial_i m\partial_i u \cdot \nabla m = 0. \]
Integrating (45) over \(\Omega\), we obtain
\[ \frac{d}{dt} \int_{\Omega} |\nabla m|^q \leq C \int_{\Omega} |\nabla u||\nabla m|^q + q \int_{\Omega} m|\text{div} u||\nabla m|^{q-1} \]
\[ \leq C\|\nabla u\|_{L^\infty}\|\nabla m\|_{L^q}^q + C\|\nabla^2 u m\|_{L^6}\|\nabla m\|_{L^6}^{q-1}, \]
which implies that
\[ \frac{d}{dt} \|\nabla m\|_{L^6} \leq C\|\nabla u\|_{L^\infty}\|\nabla m\|_{L^6} + C\|\nabla^2 u m\|_{L^6}. \]

Similarly,
\[ \frac{d}{dt} \|\nabla n\|_{L^6} \leq C\|\nabla u\|_{L^\infty}\|\nabla n\|_{L^6} + C\|\nabla^2 u m\|_{L^6}. \]
By (46)-(47), we have
\[
\frac{d}{dt} (\|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}) \\
\leq C (\|\nabla u\|_{L^\infty} (\|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}) + C (\|m\|_{L^\infty} + \|n\|_{L^\infty}) (\|\nabla^2 w\|_{L^q} + \|\nabla^2 h\|_{L^q})) \\
\leq C (\|\nabla w\|_{L^\infty} + \|\nabla h\|_{L^\infty}) (\|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}) \\
+ C (\|m\|_{L^\infty} + \|n\|_{L^\infty}) (\|m \dot{u}\|_{L^q} + \|\nabla P\|_{L^q}) \\
\leq C (\|\nabla w\|_{L^\infty} + \|\nabla h\|_{L^\infty}) (\|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}) \\
+ C (\|\nabla u\|_{L^\infty} + \|\nabla h\|_{L^\infty}) (\|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}) \\
+ C (\|m\|_{L^\infty} + \|n\|_{L^\infty}) \|\nabla \dot{u}\|_{L^2},
(48)
\]
where we have used (26)\textsubscript{1}, (43), Lemma 2.3, Sobolev inequality and H"older inequality.

Using (23), (26)\textsubscript{1}, (26)\textsubscript{2} and Lemmas 2.3-2.5, we obtain
\[
\|\nabla h\|_{L^\infty} \leq C \left( 1 + \|\nabla h\|_{BMO(\Omega)} \ln(e + \|\nabla^2 h\|_{L^q}) \right) \\
\leq C \left( 1 + \|\nabla h\|_{BMO(\Omega)} \ln(e + \|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}) \right) \\
\leq C \left( 1 + \ln(e + \|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}) \right).
(49)
\]
Moreover, by Lemma 2.5 and (21)
\[
\|m\|_{L^\infty} \leq C \left[ 1 + \|m\|_{BMO(\Omega)} \ln(e + \|\nabla m\|_{L^q}) \right] \leq C \left[ 1 + \ln(e + \|\nabla m\|_{L^q}) \right],
(50)
\]
\[
\|n\|_{L^\infty} \leq C \left[ 1 + \|n\|_{BMO(\Omega)} \ln(e + \|\nabla n\|_{L^q}) \right] \leq C \left[ 1 + \ln(e + \|\nabla n\|_{L^q}) \right].
(51)
\]
Substituting (49)-(51) into (48), we get
\[
\frac{d}{dt} (e + \|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}) \\
\leq C \left[ \|w\|_{W^{2,q}} + 1 + \ln(e + \|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}) \right] \cdot (e + \|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}) \\
+ C \|\nabla \dot{u}\|_{L^2} \left[ 1 + \ln(e + \|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}) \right] \\
\leq C \left[ \|w\|_{W^{2,q}} + 1 + \|\nabla \dot{u}\|_{L^2} + (1 + \|\nabla \dot{u}\|_{L^2}) \ln(e + \|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}) \right] \\
\cdot (e + \|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}).
(52)
\]
Denote \(G(t) = \ln(e + \|\nabla m(t)\|_{L^q} + \|\nabla n(t)\|_{L^q})\), we have from (52)
\[
\frac{d}{dt} G(t) \leq C (1 + \|w\|_{W^{2,q}} + \|\nabla \dot{u}\|_{L^2}) + (1 + \|\nabla \dot{u}\|_{L^2}) G(t).
(53)
\]
Using Gronwall inequality, Corollary 1, Lemma 3.4 and (53), we complete the proof of Lemma 3.5.

Due to Lemma 3.5 and Sobolev inequality, we immediately give the following result.
Corollary 2. Under the conditions of Proposition 1 and (21), for \(0 \leq T < T^*\), we have

\[
\lim \sup_{T \to T^*} \|m\|_{L^\infty(0,T;L^\infty)} \leq C < \infty.
\]

By Proposition 2 and Corollary 2, \(T^*\) is not the maximum time of existence of strong solution, which is the desire contradiction. Thus, the proof of Theorem 1.2 is completed.

Acknowledgments. Chen was supported by the Fundamental Research Funds for the Central Universities \#D2154460 and \#D2154560. Zhang was supported by the natural science foundation of Tianyuan \#11126072 and the national research foundation for the doctoral program of higher education of China 20110172120031.

REFERENCES

[1] P. Acquistapace, On BMO regularity for linear elliptic systems, Ann. Mat. Pura Appl., 161 (1992), 231–269.
[2] H. B. Cui, H. Y. Wen and H. Y. Yin, Global classical solutions of viscous liquid-gas two-phase flow model, Math. Meth. Appl. Sci., 36 (2013), 567–583.
[3] S. Evje, T. Fløtten and H. A. Friis, Global weak solutions for a viscous liquid-gas model with transition to single-phase gas flow and vacuum, Nonlinear Anal., TMA, 70 (2009), 3864–3886.
[4] S. Evje and K. H. Karlsen, Global existence of weak solutions for a viscous two-phase model, J. Differential Equations, 245 (2008), 2660–2703.
[5] S. Evje and K. H. Karlsen, Global weak solutions for a viscous liquid-gas model with singular pressure law, Commun. Pure Appl. Anal., 8 (2009), 1867–1894.
[6] Z. H. Guo, J. Yang and L. Yao, Global strong solution for a three-dimensional viscous liquid-gas two-phase flow model with vacuum, Journal of Mathematical Physics, 52 (2011), 093102, 14pp.
[7] C. C. Hao and H. L. Li, Well-posedness for a multidimensional viscous liquid-gas two-phase flow model, SIAM J. Math. Anal., 44 (2012), 1304–1332.
[8] M. Ishii, Thermo-Fluid Dynamic Theory of Two-Phase Flow, Eyrolles, Paris, 1975.
[9] O. A. Ladyzenskaja, V. A. Solonikov and N. N. Ural’ceva, Linear and Quasilinear Equation of Parabolic Type, Amer. Math. Soc., Providence RI, 1968.
[10] A. Prosperetti and G. Tryggvason (Editors), Computational Methods for Multiphase Flow, Cambridge University Press, Cambridge, 2009.
[11] Y. Z. Sun, C. Wang and Z. F. Zhang, A Beale-Kato-Majda blow-up criterion for the 3-D compressible Navier-Stokes equations, J. Math. Pures Appl., 95 (2011), 36–47.
[12] V. A. Vaigant and A. V. Kazhikhov, On existence of global solutions to the two-dimensional Navier-Stokes equations for a compressible fluid, Siberian Math. J., 36 (1995), 1108–1141.
[13] H. Y. Wen, L. Yao and C. J. Zhu, A blow-up criterion of strong solution to a 3D viscous liquid-gas two-phase flow model with vacuum, J.Math. Pures Appl., 97 (2012), 204–229.
[14] H. Y. Wen and C. J. Zhu, Blow-up criterions of strong solutions to 3D compressible Navier-Stokes equations with vacuum, Advances in Mathematics, 248 (2013), 534–572.
[15] L. Yao and C. J. Zhu, Free boundary value problem for a viscous two-phase model with mass-dependent viscosity, J. Differential Equations, 247 (2009), 2705–2739.
[16] L. Yao and C. J. Zhu, Existence and uniqueness of global weak solution to a two-phase flow model with vacuum, Math. Ann., 349 (2011), 903–928.
[17] L. Yao, T. Zhang and C. J. Zhu, Existence and asymptotic behavior of global weak solutions to a 2D viscous liquid-gas two-phase flow model, SIAM J. Math. Anal., 42 (2010), 1874–1897.
[18] L. Yao, T. Zhang and C. J. Zhu, A blow-up criterion for a 2D viscous liquid-gas two-phase flow model, J. Differential Equations, 250 (2011), 3362–3378.

Received November 2015; revised December 2015.
E-mail address: mayshchen@scut.edu.cn
E-mail address: scmzh@scut.edu.cn