Quantum critical properties of a metallic spin density wave transition

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We report on numerically exact determinantal quantum Monte Carlo simulations of the onset of spin-density wave (SDW) order in itinerant electron systems captured by a sign-problem-free two-dimensional lattice model. Extensive measurements of the SDW correlations in the vicinity of the phase transition reveal that the critical dynamics of the bosonic order parameter are well described by a dynamical critical exponent $z = 2$, consistent with Hertz-Millis theory, but are found to follow a finite-temperature dependence that does not fit the predicted behavior of the same theory. The presence of critical SDW fluctuations is found to have a strong impact on the fermionic quasiparticles, giving rise to a dome-shaped superconducting phase near the quantum critical point. In the superconducting state we find a gap function that has an opposite sign between the two bands of the model and is nearly constant along the Fermi surface of each band. Above the superconducting $T_c$ our numerical simulations reveal a nearly temperature and frequency independent self energy causing a strong suppression of the low-energy quasiparticle spectral weight in the vicinity of the hot spots on the Fermi surface. This indicates a clear breakdown of Fermi liquid theory around these points.

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I. INTRODUCTION

Metallic spin-density wave (SDW) transitions are ubiquitous to strongly correlated materials such as the electron-doped cuprates [1], the Fe-based superconductors [2], heavy fermion systems [3], and organic superconductors [4]. In all these materials, unconventional superconductivity is found to emerge near the onset of SDW order, with the maximum of the superconducting $T_c$ occurring either at or near the underlying SDW quantum phase transition (QPT). In addition, the vicinity of the SDW transition is often characterized by strong deviations from Fermi liquid theory - both in thermodynamic and in single electron properties.

More broadly, understanding the properties of quantum critical points (QCPs) in itinerant fermion systems has attracted much interest over the past several decades [5-28]. Unlike thermal critical phenomena and QCPs in insulating systems, here the critical order parameter fluctuations are strongly interacting with low-energy fermionic quasiparticles near the Fermi surface. In the traditional approach to this problem due to Hertz [5], later refined by Millis [6], the fermions are integrated out from the outset, leading to an effective bosonic action for the order parameter fluctuations. The dynamics of the order parameter is found to be overdamped due to the coupling to the fermions, that act as a bath. The effective bosonic action is then treated using conventional renormalization group (RG) techniques. While physically appealing, this approach has the drawback that integrating out low-energy modes is dangerous, since it generates non-analytic terms in momentum and frequency that are difficult to treat within the RG scheme. An alternative popular approach has been to use a $1/N$ expansion [8], where $N$ is the number of fermion flavors. However, in the important case of two spatial dimensions, this approach turns out to be uncontrolled, as well [12] [15]. Alternative expansion parameters have been proposed [13] [18] [28], but a fully controlled analytical treatment of QCPs in itinerant electron systems has remained one of the grand challenges in strongly correlated electron physics.

In addition to the bosonic critical fluctuation dynamics, an important open question regards the behavior of the fermionic quasiparticles in the vicinity of the transition. The exchange of SDW critical fluctuations can mediate a superconducting instability; however, the same fluctuations also strongly scatter the quasiparticles, causing them to lose their coherence and leading to the formation of a non-Fermi liquid metal. It is not clear which of these effects dominates; i.e., is there a well-defined non-Fermi liquid regime that precedes superconductivity, or does pairing always preempt the formation of a non-Fermi liquid [29]?

In this work, we perform extensive numerically exact determinantal Monte Carlo (QMC) simulations [30-34] of a metallic system in the vicinity of an SDW transition. We use the approach of Refs. [17] [27], that introduced a two-dimensional multi-band lattice model that captures the generic structure of the “hot spots” - points on the Fermi surface where quasiparticles can scatter off critical SDW fluctuations resonantly. The universal properties of metallic SDW transitions are believed to depend only on the vicinity of the hot spots. At the same time, the model is amenable to QMC simulations without a sign problem [17]. Our goal here is both to understand the generic properties of the transition, and to provide a controlled benchmark to analytic theories. We present detailed information about the bosonic and fermionic correlations and the interplay with unconventional superconductivity in the vicinity of the QCP.

Previously, the finite-temperature phase diagram of the model has been characterized, and a dome-shaped superconducting phase peaked near the SDW transition was found [27] [35]. Here, we measure the SDW correlations in the vicinity of the transition, above the superconducting $T_c$. We find that,
over a broad range of parameters, the SDW susceptibility is well described by the following form:

$$\chi_0(q, \omega_n, r, T) = \frac{1}{a_g(q - Q)^2 + a_w|\omega_n| + a_r(r - r_c) + f(r, T)}.$$  (1)

Here $Q$ is the ordering wavevector [chosen to be $(\pi, \pi)$ in our model], $\omega_n = 2\pi n T$ is a Matsubara frequency, and $r$ is a parameter used for tuning through the SDW QCP, while $a_g$, $a_w$, $a_r$, and $r_c$ are nonuniversal constants. The function $f(r, T)$ extrapolates to 0 as $T \to 0$. Importantly, Eq. (1) captures the behavior of both the bosonic SDW correlations and the susceptibility of a fermion bilinear operator with the same symmetry, establishing the consistency of our analysis.

Interestingly, $\chi_0(q, \omega_n, r, T \to 0)$ has precisely the form predicted by Hertz and Millis; in particular, the bosonic critical dynamics are characterized by a dynamic critical exponent $z = 2$. The function $f(r, T)$ does not follow the predicted behavior, however. In a window of temperatures above $T_c$, we find a power-law dependence $f(r \approx r_c, T) \propto T^\alpha$ with $\alpha \simeq 2$, in contrast to the linear behavior predicted by Millis [6, 36].

The single-fermion properties above $T_c$ are found to depend strongly on the distance from the hot spots. Away from the hot spots, a behavior consistent with Fermi liquid theory is observed. At the hot spots, a substantial loss of spectral weight is seen upon approaching the QCP. In a temperature window above $T_c$, the fermionic self energy is only weakly frequency and temperature dependent, corresponding to a nearly constant lifetime of quasiparticles at the hot spots. This behavior indicates a strong breakdown of Fermi liquid theory at these points. It is not clear, however, whether it represents the asymptotic behavior at the putative underlying SDW QCP, since superconductivity intervenes before the QCP is reached.

Finally, in order to probe the interplay between magnetic quantum criticality and superconductivity, we measure the superconducting gap, $\Delta_k$, and the momentum-resolved superconducting susceptibility, $F_{k,k'}$, across the phase diagram. No strong feature is found in $\Delta_k$ at the hot spots. Rather, $\Delta_k$ and $F_{k,k'}$ vary smoothly on the Fermi surface. While the pairing interaction may be strongly peaked at wavevector $Q$, the resulting gap function does not reflect such a strong momentum dependence.

This paper is organized as follows. In Sec. II we describe the model and review its phase diagram. Sec. III presents a detailed analysis of the SDW susceptibility across the phase diagram. In Sec. IV we study the single-fermion properties, providing evidence for the breakdown of Fermi liquid theory in the vicinity of the hot spots. Sec. V analyzes the gap structure in the superconducting state. The cumulative results are put into perspective in Sec. VI. Supplementary data sets and some technical details are presented in the appendices.

II. THE MODEL AND THE PHASE DIAGRAM

Our lattice model is composed of two flavors of spin-$\frac{1}{2}$ fermions, $\psi_x$ and $\psi_y$, that are coupled to a real bosonic vector field $\phi$, which represents fluctuations of a commensurate SDW order parameter at wavevector $Q = (\pi, \pi)$. The two types of fermions exhibit quasi one-dimensional dispersions along momenta $k_x$ and $k_y$, respectively, which in the absence of interactions give rise to the Fermi surfaces illustrated in Fig. 1. It is precisely this two-flavor structure that fundamentally allows us to set up an action completely devoid of the fermion sign problem in QMC simulations [17]. Yet, the Fermi surfaces of this model capture the generic structure of the hot spots, which is generally believed to determine the universal physics near the QCP.

In our previous work on this model [27], we assume an $O(2)$ symmetric SDW order parameter, i.e. $\phi$ is restricted to the $XY$ plane. In contrast to the case of an $O(3)$ order parameter (studied in Refs. [17, 35]), the easy-plane order parameter implies the existence of a finite-temperature SDW phase transition of Berezinskii-Kosterlitz-Thouless (BKT) character, which we can track in our numerics. In addition, the reduced dimensionality of the order parameter brings a welcome computational benefit as it enables a reduction of the dimensions of all single-fermion matrices by half, greatly improving the efficiency of the numerical linear algebra.

Our lattice model is given by the action $S = S_F + S_\varphi = \int_{0}^{\beta} d\tau(L_F + L_\varphi)$ with

$$L_F = \sum_{i,j,s} \psi_{ijs}^\dagger \left[ (\partial_\tau - \mu) \delta_{ij} - t_{cis} \right] \psi_{jqs} + \lambda \sum_{i,s,s'} [\delta_{ss'} \phi_{i,s}^\dagger \phi_{i,s'}] + h.c.,$$

$$L_\varphi = \frac{1}{2} \sum_i \left( \frac{d \phi_i^2}{d\tau} \right)^2 + \frac{1}{2} \sum_{i,j} \left( e^{iQr_i} \phi_i^2 - e^{iQr_j} \phi_j^2 \right)^2 + \sum_{i} \frac{\tau}{2} \phi_i^2 + \frac{u}{4} (\phi_i^2)^4.$$  (2)

This action is defined on a square lattice with sites labeled by $i, j = 1, \ldots, N_x$, where $(i,j)$ are nearest neighbors. The two fermion flavors are indexed by $\alpha = x, y$, while $s, s' = \uparrow, \downarrow$.
FIG. 2. Finite-temperature phase diagram of model (2) for three different values of the Yukawa coupling $\lambda$ and bare bosonic velocity $c = 3$. Shown are the transition temperature $T_{\text{SDW}}$ to magnetic spin density wave (SDW) order as well as the estimated location of the zero-temperature phase transition point $r_c$ (indicated by the red star). The superconducting (SC) transition temperature $T_c$ is shown where applicable. In the same units the Fermi energy is $E_F = 2.5$. Dashed lines are a guide to the eye.

dex the spin polarizations and $s$ are the Pauli matrices. Imaginary time is denoted by $\tau$ and $\beta = 1/T$ is the inverse temperature. The fermionic dispersions are implemented by setting different hopping amplitudes along the horizontal and vertical lattice directions. For the $\psi_x$-fermions they are given by $t_{x,v} = 1$ and $t_{x,h} = 0.5$, respectively, and for the $\psi_y$-fermions by $t_{y,v} = 0.5$ and $t_{y,h} = 1$. Note that our model is fully $C_4$-symmetric with a $\pi/2$ rotation mapping the $\psi_x$-band to the $\psi_y$-band and vice versa. The tuning parameter $r$ allows to tune the system to the vicinity of an SDW instability. In an experimental system, $r$ could be proportional to a physical tuning parameter such as pressure or doping. We set the chemical potential to $\mu = -0.5$, such that the Fermi energy, measured relative to the band bottom, is $E_F = 2.5$. The quartic coupling is set to $u = 1$. In the following we mostly focus on the case of a bare bosonic velocity of $c = 3$ and a Yukawa coupling between fermions and bosons of $\lambda = 1.5$. Occasionally, we also consider other values of $c$ and $\lambda$.

Our numerical analysis of model (2) is based on extensive finite-temperature DQMC simulations. For the general setup and technical details on the implementation of our DQMC simulations we refer to our earlier paper [27] and its detailed supplemental online material. Here we want to single out a few conceptual aspects of our setup, which have allowed to push our simulations down to temperatures of $T = 1/40$ for system sizes up to $16 \times 16$ sites. First, using a replica-exchange scheme [37, 38] in combination with a global update procedure [27] thermal equilibration of our simulations is decidedly improved. Most of our simulations were performed in the presence of a weak fictitious perpendicular “magnetic” field (designed not to break time-reversal symmetry), which serves to greatly speed up convergence to the thermodynamic limit for metallic systems [27, 39, 40]. Since this technique breaks translational invariance of the fermionic Green’s function, we cannot make use of it to study $k$-resolved fermionic observables. For this reason we have carried out additional simulations without the magnetic flux, but with twisted boundary conditions, which allows to increase the momentum space resolution. We give details on these procedures in Appendix A.

To set the stage for our discussion in the following sections, we summarize the finite-temperature phase diagram of model (2) for $c = 3$ and three different values of the Yukawa coupling $\lambda = 1, 1.5$, and 2 in Fig. 2. Besides a paramagnetic regime for sufficiently high temperatures, the dominant feature of these phase diagrams is a quasi-long-range ordered SDW phase whose transition temperature $T_{\text{SDW}}$ is suppressed with increasing tuning parameter $r$. Extrapolating the SDW transition down to the zero temperature provides an estimate of the location of the quantum phase transition at $r = r_c$ (indicated by the red star). At low temperatures, the SDW transition may become weakly first order [27]. However, in the temperature range considered here, the transition is either continuous or (possibly) very weakly first order. While for $\lambda = 1$ this SDW phase is the only ordered phase down to temperatures of $T = 1/40$, there is an additional superconducting phase emerging in the vicinity of the QPT for the two larger values of the Yukawa coupling. For $\lambda = 1.5$ we barely observe the tip of this quasi-long-range ordered phase with a maximum critical temperature of $T_{c,\text{max}} \approx 1/40$, which is our numerical temperature limit. For $\lambda = 2$, we can clearly map out a superconducting dome with the maximum of the critical temperature $T_{c,\text{max}} \approx 1/20$.

At finite temperatures, both the SDW and the SC finite-temperature transitions are expected to be of BKT type. The SDW susceptibility $\chi = \int d\tau \sum_{Q} e^{iQ \cdot r} \langle \bar{\phi}_i(\tau) \phi_i(0) \rangle$ is found to follow a scaling law $\chi \propto L^{2-\eta}$ with a continuously changing exponent $\eta$. We identify the transition temperature $T_{\text{SDW}}$ with the point where the exponent takes the universal value $\eta = 1/4$. The SC transition temperature can be both determined via a similar $\eta$-fit or by the point where the superfluid density $\rho_s$ obtains the universal value of $2T_c/\pi$. Both estimates are found to agree. The error bars in Fig. 2 are mostly due to finite-size effects. For further details on the procedures employed to identify the different phase transitions see Ref. [27] and its accompanying supplementary online material.

A common feature to all three phase diagrams is a change of slope of the SDW phase boundary curve at low temperatures ($T \approx 0.07$). This apparent “bending” is more pronounced at larger Yukawa coupling $\lambda$. Such behavior is generally expected to occur at the onset of superconductivity [27, 41].
III. MAGNETIC CORRELATIONS

We start our discussion of the quantum critical behavior of model \( \mathcal{H} \) with an examination of magnetic fluctuations across its entire finite-temperature phase diagram. We probe the formation of magnetic correlations both through the susceptibility of the bosonic order parameter, \( \varphi \), and through a fermionic bilinear of the same symmetry, which we evaluate independently in the same numerical simulations of the action \( \mathcal{H} \). As we show below, both susceptibilities exhibit the same behavior, supporting the robustness of our conclusions to be presented.

A. Bosonic SDW susceptibility

We first consider the bosonic susceptibility calculated from the SDW order parameter \( \varphi \) in action \( \mathcal{H} \):

\[
\chi(q, i\omega_n, r, T) = \sum_i \int_0^\beta d\tau e^{i\omega_n \tau - i(\mathbf{r}_i \cdot \mathbf{q}) / \hbar} \langle \varphi(q, \mathbf{r}, \tau) \varphi(0) \rangle \tag{3}
\]

for a given momentum \( q \) and Matsubara frequency \( \omega_n = \frac{2\pi n}{\beta} \). The expectation values are estimated in a DQMC simulation run at finite temperature \( T \) and for a specific value of the tuning parameter \( r \), indicated here as explicit parameters. At low temperature, we use the following form to fit the data, inspired by Hertz theory [5]:

\[
\chi_0^{-1}(q, i\omega_n, r, T \to 0) = a_q(q - Q)^2 \omega_n + a_\omega |\omega_n| + a_r(r - r_c), \tag{4}
\]

where \( a_q, a_\omega \) and \( a_r \) are non-universal fitting parameters that describe the momentum dependence in the vicinity of the ordering wavevector \( Q = (\pi, \pi) \), Landau damping, and the dependence on the tuning parameter \( r \), respectively. The fitting parameter \( r_{c0} \) indicates the location of the divergence of \( \chi_0 \).

Due to the appearance of a superconducting phase at low temperatures, \( r_{c0} \) may differ from the actual location of the QPT at \( r = r_c \). However, within our numerical resolution, we find \( r_{c0} \approx r_c \), where \( r_c \) is obtained by extrapolating the finite-temperature transition line \( T_{SDW} \to 0 \), as shown in the phase diagrams of Fig. 2.

Running extensive DQMC simulations for system sizes \( L = 8, 10, \ldots, 14 \), we have evaluated \( \chi \) across the three principal phase diagrams of Fig. 2 for different values of the Yukawa coupling \( \lambda = 1, 1.5, 2 \) and bare bosonic velocity \( c = 3 \).

Restricting our analysis to the magnetically disordered side for each coupling and to temperature scales above the superconducting phase, we find that our calculated susceptibilities are in good agreement with the functional form of Eq. (1).

The consistency with Eq. (4) is illustrated in the panels of Fig. 5, which show data collapses for a range of small momenta \( q - Q \), small Matsubara frequencies, low temperatures \( T \leq 0.1 \) and tuning parameters \( r \geq r_{c0} \). Finite-size effects are rather small. Considering the variation of the Yukawa coupling \( \lambda \), we find that the fit to the functional form (4) is slightly worse for stronger coupling \( \lambda \), which is also indicated by the...
excluding superconducting dome

0.025 < \( T \) ≤ 0.1
−1.30 < \( r \) ≤ 0.40
0.00 < \( |\omega_n| \) ≤ 1.10
0.00 < \( |q - Q| \) ≤ 1.05

(a) \( \lambda = 1 \)

SDW susceptibility \( \chi \)

0.54(\( q - Q \))^2 + 0.29|\( \omega_n | + 0.36[| r + 1.31] \)

99,696 data points, \( \chi_{\text{dof}} = 3.3 \)

(b) \( \lambda = 1.5 \)

SDW susceptibility \( \chi \)

0.62(\( q - Q \))^2 + 0.57|\( \omega_n | + 0.40[| r - 0.60] \)

104,469 data points, \( \chi_{\text{dof}} = 4.4 \)

(c) \( \lambda = 2 \)

SDW susceptibility \( \chi \)

0.71(\( q - Q \))^2 + 0.94|\( \omega_n | + 0.43[| r - 3.03] \)

28,638 data points, \( \chi_{\text{dof}} = 6.5 \)

FIG. 5. Comparison between the inverse SDW susceptibility \( \chi^{-1} \) and the functional form \( \chi_0^{-1} = a_{\lambda}(q - Q)^2 + a_\omega|\omega_n| + a_r(r - r_{c0}) \), which has been fitted for small frequencies \( \omega_n \) and momenta \( q - Q \) at low temperatures \( T \) and tuning parameters \( r > r_{c0} \) in the magnetically disordered phase, for (a) \( \lambda = 1 \), (b) \( \lambda = 1.5 \), and (c) \( \lambda = 2 \). Data inside the superconducting phase has been excluded from the fit. For temperatures \( T \leq 2T_{\text{max}} \) we restrict the fit to finite frequencies \( |\omega_n| > \). The correspondence of \( \chi^{-1} \) with the fitted form is shown in the form of 2D histograms over all data points, which are normalized over the total area. In each fit we have minimized \( \chi_{\text{dof}}^2 = \frac{1}{N_{\text{dof}}} \sum \frac{\chi^{-1} - \chi_0^{-1}}{\varepsilon}^2 \), where \( N_{\text{dof}} \) is the number of degrees of freedom of the fit and \( \varepsilon \) is the statistical error of the data.

larger spread of the data points. This decreasing fit quality may be a consequence of the smaller temperature window available above the superconducting \( T_s \), as well as the associated regime of superconducting fluctuations at \( T \gtrsim T_s \), which increases with Yukawa coupling (see also Fig. 3).

With the data collapse of Fig. 5, asserting the general validity of the functional form (4), we now take a closer look at its individual dependence on tuning parameter, frequency and momentum. First, the dependence on the tuning parameter \( r \) is illustrated for the inverse susceptibility \( \chi^{-1}(q, i\omega_n = 0) \) in Fig. 6 (for \( \lambda = 1.5 \) and \( T = 0.1 \)). For tuning parameters \( r \gtrsim r_{c0} = 0.6 \) we find that the data for different system sizes follows a linear dependence. The moderate deviation from a perfect kink-like behavior at \( r_{c0} \) is likely a combination of finite-size and finite-temperature effects (see also the finite-size trend shown in the inset of Fig. 6). A very similar picture emerges for the two other coupling parameters \( \lambda = 1 \) and \( \lambda = 2 \), for which we show analogous plots in Fig. 19 of Appendix B.

Turning to the frequency dependence of \( \chi^{-1}(q, i\omega_n) \) next, we find that for a range of values \( r \gtrsim r_{c0} \) the frequency dependence is linear for small Matsubara frequencies \( \omega_n \) with an apparent cusp at \( \omega_n = 0 \), signaling overdamped dynamics of the order parameter field. This holds both for \( q = Q \) and for small finite momenta differences \( q - Q \). See Fig. 7 for an illustration at \( \lambda = 1.5 \) and Appendix B with Fig. 20 for \( \lambda = 1 \) and \( \lambda = 2 \). At finite Matsubara frequencies \( \omega_n \), finite-size effects are negligibly small, as evident in the data collapse of \( \chi^{-1} \) for different system sizes in the left panel in Fig. 7.

To establish the presence of a \( |\omega_n| \) term in \( \chi^{-1} \), we fit it at low frequencies to the form \( b_0 + b_1|\omega_n| + b_2\omega_n^2 \). The fits are shown in Fig. 8. The \( |\omega_n| \) contribution is clearly dominant in this frequency range. Inside the superconducting phase, the \( |\omega_n| \) term is suppressed (see Fig. 24 in Appendix B). This is presumably due to gaps coming out of the fermions.

Third, for the same range of \( r \) the momentum dependence of \( \chi^{-1}(q, i\omega_n) \) is consistent with a quadratic form in \( q - Q \), which holds both for \( \omega_n = 0 \) and small finite frequencies \( \omega_n \). See Fig. 8 for \( \lambda = 1.5 \) and appendix B with Fig. 21 for \( \lambda = 1 \) and \( \lambda = 2 \). Note that due to the discretization of the Brillouin zone finite-size effects are more pronounced here than for the frequency dependence.
fermionic SDW susceptibility

\[ \chi^{-1} \]

FIG. 7. Frequency dependence of the inverse bosonic SDW susceptibility \( \chi^{-1} \) for \( \lambda = 1.5 \) at \( T = 1/40 \) (a) shown at \( r \approx r_{c0} \) for various momenta \( q = Q + \bar{q} \) and (b) shown at various values \( r > r_{c0} \) for \( q = Q \). The black line is the best fit of a second degree polynomial \( \delta_0 + b_1|\omega_n| + b_2\omega_n^2 \) to the \( q = Q, L = 14 \) low-frequency data, yielding a basically straight line.

B. Fermion bilinear SDW susceptibility

An important independent confirmation that the form (4) is generic to the quantum critical regime is to affirm that it also holds for other SDW order parameters that have the same symmetry. We have examined the correlations of a fermion bilinear order parameter:

\[ S_{xx}(q, \omega_n, r, T) = \sum_i \int_0^\beta d\tau e^{i\omega_n \tau - iq \cdot r} \langle S^x_i(\tau) S^0_i(0) \rangle. \]

(5)

In the estimation of \( S_{xx} \) we make use of spin rotational symmetry around the \( z \) axis, \( \langle S^x_i(\tau) S^0_i(0) \rangle = \langle S^y_i(\tau) S^y_i(0) \rangle \).

Here \( S^x_i \) and \( S^y_i \) are inter-flavor fermion spin operators, which are given by

\[ \hat{S}_i^x = (S_i^x, S_i^y, S_i^z) = \sum_{s,s'} \hat{s}_{ss'} \psi_{i2s} \psi_{is's'} + \text{h.c.} \]

Indeed, we find that at small frequencies and momenta, the fermion bilinear SDW susceptibility \( S_{xx} \) follows the same functional form \( [4] \) as the bosonic SDW susceptibility \( \chi \) discussed above. The momenta and frequency dependences of the fermionic bilinear susceptibility at \( \lambda = 1.5 \) are shown in Fig. 9 with the respective dependences of the bosonic susceptibility appearing in Figs. 7 and 8. Additional data for the fermionic SDW susceptibility at \( \lambda = 1 \) and \( \lambda = 2 \) is given in Fig. 22 of Appendix B.

In summary, the dependence of both the bosonic and fermionic SDW susceptibilities on the tuning parameter, frequency, and momentum stand in good agreement with the form \( [4] \).

C. Temperature dependence

We now turn to the temperature dependence of the numerically computed bosonic and fermionic SDW susceptibilities \( \chi^{-1} \) and \( S_{xx}^{-1} \). Our numerical data for the temperature dependence of \( \chi^{-1} \) and \( S_{xx}^{-1} \) is shown in Fig. 10 for fixed Yukawa coupling \( \lambda = 1.5 \) and two different values of the tuning parameter on the paramagnetic side of the QCP, i.e. for \( r > r_{c0} \).

This data is complemented with similar results for \( \lambda = 1 \) and \( \lambda = 2 \) in Fig. 23 of Appendix B.
FIG. 10. Inverse bosonic SDW susceptibility \( \chi^{-1}(q = Q, i\omega_n = 0) \) and inverse fermionic SDW susceptibility \( S_{xx}^{-1}(q = Q, i\omega_n = 0) \) as a function of temperature for \( \lambda = 1.5 \) at (a) \( r = 0.7 \approx r_{c0} \) and at (b) \( r = 1.0 > r_{c0} \). Solid lines indicate fits of \( a_0 + a_2 T^2 \) to the \( L = 14 \) data at intermediate temperatures. Dashed lines are linear fits to the high-temperature data. In each figure the inset shows the same data as the main plot over a more extended temperature range.

Evidently, the data shows different scaling regimes with increasing temperature. At sufficiently high temperatures, \( T \gtrsim 0.35 \) the susceptibilities \( \chi^{-1}(q = Q, i\omega_n = 0) \) and \( S_{xx}^{-1}(q = Q, i\omega_n = 0) \) are approximately linearly dependent on temperature, as shown in the insets of Fig. 10. In an intermediate temperature regime, however, we observe a crossover to a different functional temperature dependence as shown in the main panels of Fig. 10. In this intermediate temperature window \( 0.05 \lesssim T \lesssim 0.35 \) our numerical data is found to reasonably fit functions of the quadratic form \( a_0 + a_2 T^2 \) with \( \alpha \approx 2 \pm 0.3 \). Unlike the leading dependences on the tuning parameter, frequency and momentum discussed in the previous section, this power-law dependence is not so robust. Note that the crossover temperature between the high-\( T \) linear and intermediate-\( T \) quadratic behaviors does not depend strongly on the tuning parameter \( r \). Notably, even for \( r \approx r_{c0} \) this intermediate regime does not disappear.

At still lower temperatures \( T \lesssim 0.05 \) our data might indicate a second crossover to yet different behavior. With the tuning parameter \( r \) tuned close to its critical value \( r_{c0} \) both \( \chi \) and \( S_{xx} \) are found to be non-monotonic for the smallest temperatures and largest system sizes accessed in this study. The apparent upturn, whose precise location is hard to determine due to finite-size effects (which are strongest for \( r \approx r_{c0} \) and the enhanced statistical uncertainty at low temperatures, is most likely a precursor effect of superconductivity \cite{27}, which for \( \lambda = 1.5 \) sets just at the lowest temperature we have accessed in this work, \( T_c \approx 1/40 \). For larger Yukawa coupling \( \lambda = 2 \), where \( T_c \) is higher, this non-monotonic behavior is indeed found to be more pronounced as shown in Fig. 23 of appendix \cite{29}.

Note that over the range of temperatures displayed in Fig. 3 the leading temperature dependence of \( \chi^{-1} \) is quadratic. This is reflected in the contour lines of \( \chi^{-1} \) in the \( r - T \) plane, which have a form \( T \sim \sqrt{\chi^{-1} - a(r - r_{c0})} \), approaching infinite slope at low temperatures.

Since the data does not allow us to identify a simple functional form for the temperature dependence of \( \chi^{-1} \), we have opted against taking into account any temperature dependence in the fits for the data collapse shown in Fig. 5. Instead we have constrained the included data to \( T < 0.1 \) where the overall temperature dependence is rather weak.

### IV. SINGLE-FERMION CORRELATIONS

We now turn to examine the fermionic spectral properties in the metallic state above the superconducting \( T_c \). As our DQMC simulations are performed in imaginary time, there is an inherent difficulty in probing real-time dynamics. To partially circumvent this issue, we use the relation \cite{43}

\[
G_k(\tau) = \int_{-\infty}^{\infty} d\omega e^{-\omega(\tau - \beta/2)} 2 \cosh(\beta\omega/2) A_k(\omega),
\]

which connects the readily available imaginary-time ordered Green’s function \( G_k(\tau) = \langle \psi_k(\tau) \psi_k^\dagger(0) \rangle \), where \( 0 \leq \tau \leq \beta \), with the spectral function \( A_k(\omega) \) of interest. Here and in the following, we focus on a single flavor of fermions \( \psi_\nu \), suppressing band and spin indices. Close inspection of \( G_k(\tau) \) reveals that the behavior of the Green’s function \( G_k(\tau) \) at long times, i.e for imaginary times close to \( \tau = \beta/2 \), provides information about the spectral function integrated over a frequency window of width \( T \).

In Fig. 11 we present the evolution of the Fermi surface across the phase diagram. For orientation, the Fermi surfaces of the noninteracting system are shown in panel (a). In panels (b-d) we show \( G_k(\tau = \beta/2) \) across a quadrant of the Brillouin zone for a low temperature \( T = 0.05 \approx 2T^{\text{max}} \). Near the magnetic QCP (panel c), there is a clear loss of spectral weight in the immediate vicinity of the hot spots, as compared to the magnetically disordered phase (panel d). Upon entering the magnetic phase (panel b), a gap opens around the hot spots. In this section we focus on the parameter set \( \lambda = 1.5 \) and \( c = 3 \). The DQMC simulations are carried out with different sets of twisted boundary conditions (see Appendix \( \lambda \) for details), providing a four-fold enhancement in \( k \)-space resolution.

![Graph](image-url)
A Fermi liquid is usually characterized by the quasiparticle weight $Z_{k_F}$ and the Fermi velocity $v_{k_F}$. We note that these quantities are only strictly defined at zero temperature. Given that the zero-temperature ground state of our model is probably always superconducting, our strategy is to consider finite-temperature proxies for $Z_{k_F}$ and $v_{k_F}$, and study their behavior over an intermediate temperature range $E_F > T > T_c$. Such proxies, $Z_{k_F}(T)$ and $v_{k_F}(T)$, can be extracted by considering the imaginary time dependence of $G_k(\tau)$ near $\tau = \beta/2$ and fitting it to the Fermi liquid form \cite{40}

$$G_k(\tau \sim \beta/2) = Z_k(T) \frac{e^{-\epsilon_k(\tau - \beta/2)}}{2 \cosh(\frac{\beta \epsilon_k}{2})},$$

(8)

where $\epsilon_k = v_{k_F}(T) \cdot (k - k_F)$.

In a complementary approach we consider the Matsubara frequency dependence of the Green’s function $G_k(\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau}G_k(\tau)$. In a Fermi liquid at low temperatures we have \cite{44}

$$G_k(\omega_n) \approx Z_k [i\omega_n - \epsilon_{k_F} \cdot (k - k_F)]^{-1}$$

(9)

up to higher order terms in temperature, frequency or the distance from the Fermi surface. It is then natural to define the finite-temperature quantities

$$Z_{k_F}(T) = \frac{\omega_1}{\text{Im} G_{k_F}^{-1}(\omega_1)}$$

(10)

and

$$v_{k_F}(T) = \omega_1 \frac{\partial}{\partial k} \frac{\text{Re} G_k(\omega_1)}{\text{Im} G_k(\omega_1)} \bigg|_{k = k_F},$$

(11)

where $\omega_1 = \pi T$ is the first Matsubara frequency at temperature $T$. In the zero-temperature limit, $Z_{k_F}(T \to 0) = Z_{k_F}'(T \to 0) = Z_{k_F}$, and similarly for $v_{k_F}$. We therefore use the finite-temperature observables \cite{10} and \cite{11} as alternative proxies for the quasiparticle spectral weight and Fermi velocity, respectively.

Figure \ref{fig12} shows the momentum dependence of $Z_{k_F}'$ for temperature $T = 1/20$. With $r$ tuned close to the location of the QCP at $r_c$, $Z_{k_F}'$ is suppressed in the vicinity of the hot spots, as shown for one quadrant of the Brillouin zone in Fig.\ref{fig12}(a) and along the Fermi surface in Fig.\ref{fig12}(c). This stands in sharp contrast to the featureless behavior of $Z_{k_F}$ in the magnetically disordered phase, as shown in Figs.\ref{fig12}(b,d). We find qualitative agreement between the two proxies $Z_{k_F}$ and $Z_{k_F}'$ throughout, as illustrated in panels (c) and (d) of Fig.\ref{fig12}. Here, we

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig11}
\caption{(a) Noninteracting Fermi surfaces. A pair of hot spots is connected by the magnetic ordering wavevector $Q$. The dashed curve corresponds to the Fermi surface of the $\psi_\pi$ band, shifted by $Q$, with a hot spot now at the intersection with the $\psi_\pi$ band (b-d) Color-coded Green’s function $G_k(\tau = \beta/2)$ evaluated for the $\psi_\pi$ fermions on a quadrant of the Brillouin zone, dotted in (a), for three values of the tuning parameter $r$. The dashed curve in panel (c) corresponds to the shifted noninteracting $\psi_\pi$ Fermi surface. The parameters used here are $L = 16$, $T = 0.05$, $\lambda = 1.5$, and $c = 3$. Results of simulations with different boundary conditions are combined for enhanced momentum resolution.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig12}
\caption{(a-b) The quasiparticle weight $Z_{k_F}'(T = 0.05)$ in a quadrant of the Brillouin zone. The dashed line in panel (a) corresponds to the noninteracting Fermi surface of the $\psi_\pi$ fermions, shifted by $Q$. (c-d) The quasiparticle weights $Z_{k_F}'(T = 0.05)$ and $Z_{k_F}'(T = 0.05)$ along the Fermi surface. The location of the hot spot is indicated by the red marker. Here we show data obtained for $L = 16$.}
\end{figure}
The temperature dependence of $Z_{k}^\tau(T)$ is shown in Fig. 13. For momenta away from the hot spots, we find $Z_{k}^\tau(T)$ to be nearly flat in temperature and to approach a constant as $T \to 0$. A different picture emerges at the hot spots, i.e. for $k = k_{hs}$. Here $Z_{k=k_{hs}}^\tau(T)$ remains flat only in the magnetically disordered phase $r > r_{c}$, whereas the quasiparticle weight $Z_{k=k_{hs}}^\tau(T)$ decreases substantially at the critical coupling $r_{c} \approx 0.7$ as the temperature is lowered towards the QCP, see Fig. 13(a). While our numerical data does not allow for a simple extrapolation towards $T = 0$, the results are not inconsistent with a vanishing of $Z_{k=k_{hs}}$, indicating a breakdown of Fermi-liquid behavior at this point.

Figure 14 shows the velocity $v_{k,p}(T = 0.05)$ along the Fermi surface. The qualitative behavior of the velocity does not differ substantially between $r = 0.7 \approx r_{c}$ [Fig. 14(a)] and $r = 1 > r_{c}$ [Fig. 14(b)]. The insets show the ratio $v_{k,p}(T = 0.05)/v_{k,p}^{\text{nonint}}$, where $v_{k}^{\text{nonint}}$ is the Fermi velocity of the noninteracting system. A small feature might be visible in the vicinity of the hot spot, but there is certainly no evidence of a substantial suppression of $v_{k,p}$ close to $r_{c}$.

Having found a substantial suppression of the quasiparticle weight tuned close to the QCP, we now directly examine the frequency dependence of the self energy, $\Sigma_{k}(\omega_{n})$, defined via $G_{k}(\omega_{n}) = (i\omega - \epsilon_{k} - \Sigma_{k}(\omega_{n}))^{-1}$. The imaginary part of the self energy is shown in Fig. 15. With $r = 0.7 \approx r_{c}$ tuned close to the QCP, see Fig. 15(a), the self energy close to the hot spots is found to be nearly frequency independent, consistent with a constant, yet small, scattering rate $\gamma = -\text{Im}\Sigma_{k}(\omega_{n} \to 0^{+}) \approx 0.13$ which is only weakly dependent on temperature. Away from the hot spot the self energy is linear with frequency, with a substantially smaller intercept. Moving away from the critical point [Fig. 15(b)], the self energy at all momenta decreases rapidly as the frequency is lowered.

![Graph showing the temperature dependence of the quasiparticle weight $Z_{k}^\tau(T)$ for different momenta $k_{x}$ along the Fermi surface (a) in the vicinity of the QCP at $r_{c}$ and (b) in the disordered side. Here we show data obtained for $L = 16$.](image)

**FIG. 13.** Temperature dependence of the quasiparticle weight $Z_{k}^\tau(T)$ for different momenta $k_{x}$ along the Fermi surface (a) in the vicinity of the QCP at $r_{c}$ and (b) in the disordered side. Here we show data obtained for $L = 16$.

**FIG. 14.** (a)-(b) The finite-temperature proxy to the velocity $v_{k}$ along the Fermi surface. (c)-(d) The velocity renormalization $v_{k}^{\tau}(T = 0.05)/v_{k}^{\text{nonint}}$. The location of the hot spot is indicated by the red marker. Here we show data obtained for $L = 16$ at $T = 0.05$.

**FIG. 15.** The imaginary part of the Matsubara self energy $\text{Im}\Sigma$ for different temperatures and two momenta along the Fermi surface, (a) in the vicinity of the QCP at $r_{c}$ and (b) on the disordered side. Here we show data obtained for $L = 14$. The data for $k = k_{hs}$ is indicated by full circles, the momentum away from the hotspot is indicated by empty squares.

V. SUPERCONDUCTING STATE

After concentrating our discussion on the manifestation of quantum critical behavior in the normal state, we now consider the effect of the QCP on the superconducting state that emerges in its vicinity.

We begin by considering the fermionic Green’s function for temperatures $T \ll T_{c}$. In this regime, the single-particle excitation energy $E_{k}$ can be extracted, as demonstrated in appendix D from the decay of the single-particle Green’s function at intermediate times $\tau_{0} < \tau < \beta/2$ (where $\tau_{0}$ is a microscopic scale). The so-extracted excitation energy $E_{k}$ is plotted...
in Fig. [16] which shows that, across the Brillouin zone, $E_k$ has a broad minimum in the vicinity of the noninteracting Fermi surface. From these momentum-resolved energy bands we extract the superconducting gap $\Delta_{k_x}$ as the minimum of $E_k$ with respect to $k_y$. As seen in Fig. [16](b), the superconducting gap $\Delta_{k_x}$ varies smoothly across momentum space, without any significant features at the hot spots. In this section we choose parameters $\lambda = 3$ and $c = 2$, as in Ref. [27]. The maximal $T_c$ for this value of $\lambda$ is high enough to allow us to explore properties of the superconducting state significantly below $T_c$.

At higher temperatures, close to $T_c$, additional information can be obtained by considering the momentum-resolved superconducting susceptibility

$$ P_{k_0; k'\alpha'} = \int_0^\beta d\tau \langle \Psi_{k_0}(\tau)\Psi_{k'\alpha'}^\dagger(0) \rangle, \quad (12) $$

where $\Psi_{k_0} = \frac{1}{\sqrt{2}} (\psi_{k_0+1} - \psi_{k_0-1})$ is the singlet superconducting pair amplitude on the band $\alpha = x, y$. Here we focus on the intraband, spin-singlet channel since it is the leading instability [17, 27]. Figure [17] shows the optimal pair amplitude $\Psi_{k_0}^{\text{opt}}$, corresponding to the maximal eigenvalue of the matrix $P_{k_0; k'\alpha'}$ at a temperature slightly above $T_c$. The pair amplitude of the band $\alpha = y$, shown in Fig. [17](a), is of the opposite sign to the amplitude on the band $\alpha = x$, shown in Fig. [17](b). In fact, the two amplitudes are related precisely by a $\pi/2$ rotation, highlighting the $d$-wave symmetry of the superconducting order parameter. The optimal pair amplitude is found to be maximal around the (noninteracting) Fermi surface. The variation of $\Psi_{k_0}^{\text{opt}}$ along the Fermi surface is weak, again showing no strong features at the hot spots.

VI. DISCUSSION

In this work, we have explored the properties of a metal on the verge of an SDW transition. We focused on the critical regime upon approaching the transition, characterized by a rapid growth of the SDW correlations, but still above the superconducting transition temperature. Our main conclusion is that, in this regime, the SDW correlations are remarkably well described by a form similar to that predicted by Hertz-Millis theory, Eq. (1) (although the temperature dependence of the SDW susceptibility deviates from the expected form). This holds both for the correlations of the bosonic SDW order parameter field, and for an SDW order parameter defined in terms of a fermion bilinear. In the same regime, we find evidence for strong scattering of quasiparticles near the hotspots, leading to a breakdown of Fermi liquid theory at these points on the Fermi surface. The scattering rate at the hotspots (extracted from the fermion self energy) is only weakly temperature and frequency dependent, down to $T \approx 2T_c$, where we suspect that superconducting fluctuations begin to play a role; it is out of this unusual metallic state that the superconducting phase emerges.

In addition, we have studied the structure of the superconducting gap near the SDW transition. Unlike the single-fermion Green’s function in the normal state, it does not have a sharp feature at the hot spots: rather, it is found to vary smoothly across the Fermi surface. Experimentally, a broad maximum of the superconducting gap near the hot spots was observed in a certain electron doped cuprate [45]. Eliashberg theory predicts a peak of the gap function at the hotspots at $T_c$, and it remains to be seen whether such behavior appears in our model at weaker coupling.

It is interesting to discuss our results in the context of the existing theories of metallic SDW transitions. First, the fact that Hertz-Millis theory successfully describes many features of our data is non-trivial, in view of the fact that it has no formal justification, even in the large $N$ limit [12, 14]. However, as we saw, an extension of the Hertz-Millis analysis to finite temperature predicts that at criticality, $\chi(T) \sim 1/T$, in apparent disagreement with our data. This may be due to the limited temperature window we can access without hitting the superconducting $T_c$, or to effects beyond the one-loop approximation.

An important conclusion of our study is that the SDW critical point is always masked by a superconducting phase [47]. As a result, it seems likely that the critical metallic regime is never parametrically broad, and one cannot sharply define scaling exponents within the metallic phase [48]. As mentioned above, the SDW correlations follow a Hertz-Millis form – and hence it is tempting to associate with them critical
exponents, i.e. a mean-field value \( \nu = 1/2 \) for the correlation length exponent, and a dynamical critical exponent \( z = 2 \). The fermionic quasiparticles at the hot spots, however, do not exhibit this scaling behavior. In particular, the expected scaling law \( \Sigma(\omega_n) \sim \sqrt{|\omega_n|} \) for the fermion self-energy at the hot spots is not seen within our accessible temperature range.

One can imagine trying to access the “bare” metallic quantum critical point by suppressing the superconducting transition. Presumably, this can be done by adding to the model a term that breaks time reversal and inversion symmetries (such a term would lift the degeneracy of fermionic states with opposite momenta, and hence remove the Cooper instability). Breaking time reversal symmetry, however, gives rise to a sign problem, so it is not clear whether the critical behavior can be accessed within the QMC technique.

Alternatively, one could try to understand the metallic regime above \( T_c \) in our model as a crossover regime of an underlying “nearby” metallic critical point, where some correlators already exhibit their asymptotic behavior (such as the SDW order parameter correlations), while others do not (e.g., the single-fermion Green’s function). Interestingly, a simple, non-self consistent one-loop calculation of the fermionic self energy in our model does show a broad range of temperature and frequency where the self energy at the hot spot is nearly constant, before eventually settling into the expected \( \sqrt{|\omega_n|} \) behavior (see Appendix C). This calls for a more detailed comparison between our numerically exact DQMC results and a detailed self-consistent one-loop analysis. Preliminary results show that this approximation is surprisingly successful in capturing at least some of the physics of our model [42].

To what extent such a crossover behavior, characterized by a nearly-constant fermionic lifetime at the hotspots, is ubiquitous across different models, as well as in real materials, remains to be seen. It is interesting to note, however, that a similar behavior has been observed in a study of a nematic transition in a metal [29]. It would be interesting to systematically look for such behavior in angle-resolved photoemission spectroscopy in the electron-doped cuprates, where anomalously large broadening of the quasiparticle peaks is seen near the hot spots [50].

Another important aspect of the metallic state in the critical regime, which we have not addressed in this work, is the electrical conductivity. The optical conductivity may be strongly affected by the presence of an SDW critical point, even without quenched disorder [16,51]. Extracting the conductivity from quantum Monte Carlo simulations requires an analytic continuation, and is therefore intrinsically more difficult (and involves more uncertainties) than calculating thermodynamic and imaginary-time quantities. Nevertheless, we have obtained preliminary results showing strong effects of the critical fluctuations on the low-frequency optical conductivity [52]. A full analysis of the conductivity is deferred to future work.

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Appendix A: DQMC simulations

In this Appendix we elaborate on a number of specific technical aspects of our numerical implementation of the determinantal quantum Monte Carlo (DQMC) approach. We refer readers looking for a more comprehensive discussion of the general DQMC setup to our previous paper [27] and in particular its supplementary online material.

We study the lattice model described by the action (2) at finite temperature in the grand canonical ensemble. After discretizing imaginary time and integrating out the fermionic degrees of freedom, the partition function reads

\[
Z = \int D\tilde{\varphi} e^{-\Delta \tau \sum \tau \phi(\tau)} \det G^{-1},
\]

where \( G \) is the equal-time Green’s function matrix for a fixed configuration of the bosonic order parameter field \( \varphi \). We use an imaginary time step of \( \Delta \tau = 0.1 \) in all calculations. The DQMC method samples configurations of \( \varphi \) according to their weight \( \exp (-\Delta \tau \sum \tau \phi(\tau)) \cdot \det G^{-1} \). For efficient Monte Carlo sampling, it is highly advantageous to consider models in which the determinant in (A1) is guaranteed to be positive, thereby avoiding the notorious fermion sign problem. For the two-band model that has been proposed in Ref. [17] this is ensured by an antiunitary symmetry of the action [53,55], written in first quantization as

\[
\mathcal{U} = is_y \tau_2 K \quad \text{with} \quad \mathcal{U}^2 = -1.
\]

Here \( K \) is the complex conjugation operator and \( s_y (\tau_2) \) are Pauli matrices acting in spin (flavor) space, respectively.

In this manuscript we have modified the model of Ref. [17] in two regards. First, as in our previous paper [27], we consider an easy-plane SDW order parameter, rather than an \( O(3) \) symmetric order parameter used in [17]. Second, we couple the system to a fictitious, spin and band dependent orbital “magnetic” field, whose flux through the system is given by \( \Phi_{\alpha, s}^{(x)} = \Phi_{\alpha, s}^{(x)} e^x + \Phi_{\alpha, s}^{(y)} e^y + \Phi_{\alpha, s}^{(z)} e^z \). Here, we place the two-dimensional lattice \((x, y)\) directions) on a torus. The \( x \) and \( y \)
components of the flux twist the boundary conditions along the $y$ and $x$ directions, respectively, while the $z$ component acts as an orbital magnetic field with a uniform flux per plaquette through the torus. As in the definition of the action $[\omega]$, $\alpha = x, y$ is a fermion flavor index and $s = \uparrow, \downarrow$ is a spin index. The flux $\Phi^z$ is restricted to be an integer multiple of the flux quantum $\Phi_0$.

In order to preserve the symmetry $[A2]$, fermions of different spin and flavor are coupled to this fictitious flux as

$$
\begin{align*}
\tilde{\Phi}_{\alpha,\uparrow} &= -\tilde{\Phi}_{\alpha,\downarrow} \\
\tilde{\Phi}_{x,s} &= -\tilde{\Phi}_{y,s},
\end{align*}
$$

with $\tilde{\Phi}_{x,\uparrow} = \tilde{\Phi}$. Note that the inter-band sign change is not strictly necessary to avoid the fermion sign problem. Importantly, in the thermodynamic limit $L \to \infty$ the fictitious field vanishes. In our previous paper [27] and for some of the results in the present one, we have chosen $\tilde{\Phi} = (0, 0, \Phi_0)$. Although such a flux is useful in reducing spurious finite-size effects at low temperatures $[39][40]$, it also breaks lattice translational symmetry, hampering the analysis of non-local, fermionic correlation functions such as the momentum-resolved Green’s function. In order to measure such quantities, for the present paper, we have run additional simulations without applying a perpendicular flux $\Phi^{\perp}$, but instead with in-plane fields, such that $\tilde{\Phi} = (n_x, n_y, 0) \frac{\Phi}{\pi}$, where $n_i = 0, 1, 2, 3$. This procedure is equivalent to having twisted boundary conditions, such that the allowed momenta for the $\psi_{x,\uparrow}$ fermions are

$$
k = \frac{2\pi}{4L}(4j_x - n_y, 4j_y + n_x),$$

where $j_x, j_y$ are integers, thereby enhancing the momentum-space resolution of fermionic observables fourfold.

**Appendix B: Magnetic correlations for $\lambda = 1$ and $\lambda = 2$, and inside the superconducting phase**

In this appendix we present additional data for the bosonic and fermionic SDW susceptibilities $\chi_q(i\omega_n, r, T)$ and $\Sigma^{xx}_q(i\omega_n, r, T)$.

We begin with Fig. 18 which shows $\chi^{-1}(q = Q, i\omega_n = 0)$ across the phase diagrams for $\lambda = 1.5$ and $\lambda = 2$. It illustrates how the bending of the SDW finite-temperature transition line is also visible in $\chi^{-1}$ for $r > r_{c0}$, similarly to the behavior for $\lambda = 1$ in Fig. 3 where, however, no superconducting phase has been observed within our temperature resolution.

Next, to complement the discussion in Sec. III where we have focused on the Yukawa coupling $\lambda = 1.5$ and a bosonic velocity of $c = 3$, we here show detailed data for values $\lambda = 1$ and $\lambda = 2$ at the same velocity $c$. In Fig. 19 we show $\chi^{-1}(q = Q, i\omega_n = 0)$ as a function of $r$ for constant $T = 0.1$, where the same linear dependence as for $\lambda = 1$ (Fig. 6) is apparent. As we show in Fig. 20 both for $r \approx r_{c0}$ and a range of $r > r_{c0}$ the leading frequency dependence of $\chi^{-1}$ is clearly linear, similarly to $\lambda = 1.5$ (Fig. 7), whereas the leading momentum dependence shown in Fig. 21 is quadratic in $q - Q$, which is again comparable to $\lambda = 1.5$ (Fig. 8). Note that for $\lambda = 2$ we show data at a higher temperature $T = 1/20$ rather than at $T = 1/40$, because otherwise the system would be in the superconducting phase, where the frequency dependence is significantly altered. For small frequencies and momenta the fermionic susceptibility $\Sigma^{-1}_{xx}$ in Fig. 22 behaves similarly to the bosonic $\chi^{-1}$ (see Fig. 9 for $\lambda = 1.5$).

The temperature dependence of both $\chi^{-1}$ and $\Sigma^{-1}_{xx}$ is demonstrated in Fig. 23. As in the case of $\lambda = 1.5$ (Fig. 10), we observe a linear regime at high temperatures, shown in the insets, and a crossover region, where we can fit a quadratic law. For $\lambda = 1$ this second region extends down to lower temperatures than for $\lambda = 1.5$, where $T_c$ is higher. At $\lambda = 2$ the data for $T < 0.05$ is from within the superconducting phase. Moreover, at $r = 3.1 \approx r_{c0}$ the system is partially inside the magnetic quasi-long range order phase (cf. the phase diagram in Fig. 2).

Finally, to illustrate the influence of superconductivity on the frequency dependence of the SDW susceptibility, we show data from deep within the superconducting phase in Fig. 24. Here we have chosen a data set with different values of the Yukawa coupling $\lambda = 3$ and the bosonic velocity $c = 2$ (as in Ref. [27]) since $T_{cmax}^{\max} \approx 0.08$ is about twice as high for

![Fig. 18. Companion figure to Fig. 3 for (a) $\lambda = 1.5$ and (b) $\lambda = 2$.](image-url)
these parameters as for \( \lambda = 2, c = 3 \). In contrast to the data at \( T > T_c \), shown in Figs. \ref{fig:7} and \ref{fig:20}, the low-frequency behavior is clearly no longer purely linear – indicative of a suppression of Landau damping in the superconducting phase.

Appendix C: Comparison with a one loop approximation for the fermion self energy

In this Appendix, we consider the fermionic self energy in a one-loop approximation. To this order, the self energy is given by

\[
\Sigma_{\mathbf{k},\alpha=y}(\omega_n) = \frac{\lambda^2}{\beta L^2} \sum_{\mathbf{q},\alpha} \chi(\Omega_m) G_{\mathbf{k}+\mathbf{q},\alpha=x}(\omega_n + \Omega_m),
\]

(C1)

where \( G^0 \) is the non-interacting Green’s function and \( \Omega_m = 2\pi m T \) is a bosonic Matsubara frequency. Deferring more systematic calculations for future work, here we do not attempt a full, self-consistent solution of the coupled Eliashberg equations \cite{3} for the SDW correlations and the fermionic Green’s function. Instead, we use the non-interacting Green’s function and \( \chi \) taken from a lattice, discretized imaginary time version of Eq. \ref{eq:4}

\[
\chi^{-1}_q(\Omega_m) = a_\tau (\tau - r_c) + 4a_q \left[ \sin^2 \left( \frac{q_x - Q_x}{2} \right) + \sin^2 \left( \frac{q_y - Q_y}{2} \right) \right] + 2a_\omega \frac{2}{\Delta \tau} \left[ \sin \left( \frac{\Delta \tau \Omega_m}{2} \right) \right],
\]

(C2)

where the parameters \( a_\tau, a_q, a_\omega \) are taken from a fit to the DQMC data, see Section IIIA. Strictly speaking, this procedure is not justified. However, within a self-consistent Eliashberg theory, \( \chi \) has the form \ref{eq:C2}, we expect our simplified approximation to capture the general behavior of the self-consistent theory.

In Fig. \ref{fig:25} we show the imaginary part of the self energy. The results bear some similarities to the DQMC data, shown in Fig. \ref{fig:15}. Whereas at moderate \( r - r_c \) or away from the hotspots the self energy is rapidly diminished as the frequency \( \omega_n \) is lowered, the behavior at the hotspots as \( r \) approaches \( r_c \) is different. There, as a function of temperature, a change of slope occurs in the frequency dependence of the self energy, where at intermediate temperatures \( T \approx 0.1 \) the self energy is nearly frequency independent. Only at lower temperatures, \( \Sigma_{\mathbf{k},\alpha}(\omega_n) \) starts resembling the expected \( \sqrt{\omega_n} \) form. In comparing with the DQMC results in Fig. \ref{fig:15} we note the similar magnitude of the self energy. However, the DQMC results show a far weaker temperature dependence at the hotspots for \( r \) close to \( r_c \).

Appendix D: Extracting the superconducting gap

In this Appendix we provide a detailed description of the procedure by which we extract the single-particle excitation energy \( E_k \) in the superconducting state, which we discuss in Sec. \ref{sec:v} of the main text. The single-particle Green’s function \( G_k(\tau) \) is found to exhibit a characteristic imaginary-time evolution as shown in Fig. \ref{fig:20}. At intermediate times, \( \tau_0 < \tau < \beta/2 \), where \( \tau_0 \approx 1 \) is some microscopic time scale, the single-particle Green’s function decays exponentially as

\[
G_k(\tau) \propto e^{-E_k^0 \tau},
\]

and similarly, for times \( \tau_0 < \beta - \tau < \beta/2 \),

\[
G_k(\tau) \propto e^{-E_h^0 (\beta - \tau)}. \quad (D2)
\]

At long times, \( \tau \approx \beta/2 \) the Green’s function is substantially suppressed and statistical errors dominate the signal. We extract the decay constants \( E_k^0 \) and \( E_h^0 \) from appropriate exponential fits and define the single-particle excitation energy as their minimum \( E_k = \min \{ E_k^0, E_h^0 \} \).

For a qualitative understanding of these results, we consider the behavior of the Green’s function in a Fermi liquid and in a Bardeen-Cooper-Schrieffer (BCS) superconductor. The fact that \( G_k(\tau) \) exhibits exponential behavior can be interpreted as arising from a peak in the spectral function \( A_k(\omega) \), occurring at a non-zero frequency, as can be seen from \ref{fig:7}. In a Fermi liquid, the spectral function at a given momentum has a single, sharp peak at the energy of the quasiparticle. It then follows that \( G_k(\tau) \) has the form of a single exponential, with hole-like quasiparticles obeying \ref{eq:D2} and particle-like quasiparticles obeying \ref{eq:D1}. Indeed, in our simulations in the normal state we find monotonic behavior of \( G_k(\tau) \) (not shown). It is illuminating to contrast this behavior with the BCS state, where a superposition of hole-like and particle-like excitations is allowed. In this case, the spectral function consists of two delta-function peaks at \( \omega = \mp E_k \), such that the Green’s function takes the form

\[
G_k(\tau) = \frac{1}{1 + e^{-\beta E_k}} \left( \frac{u_k^0 e^{-E_k \tau} + v_k^0 e^{-E_h (\beta - \tau)}}{u_k^0 + v_k^0} \right). \quad (D3)
\]

Here \( E_k = \sqrt{\Delta_k^2 + \epsilon_k^2} \) with the quasiparticle dispersion \( \epsilon_k \) and the gap \( \Delta_k \). \( u_k, v_k \) are particle and hole amplitudes, respectively. The resulting Green’s function is non-monotonic, showing a minimum at a finite imaginary time.

While our numerical data below \( T_c \), shares some similarities with the BCS form \ref{eq:D3}, it differs in two notable ways. First, clear exponential behavior is not seen at short times \( \tau \lesssim \tau_0 \approx 1 \). Second, the particle and hole excitation energies differ, i.e. \( E_k^0 \neq E_h^0 \), with the difference more pronounced away from the Fermi surface, as illustrated in Fig. \ref{fig:20}b.
FIG. 19. Companion figure of Fig. [ ] for Yukawa couplings (a) $\lambda = 1$ and (b) $\lambda = 2$. The black lines are linear fits of the $L = 14$ data for (a) $r > -1.2$ and (b) $r > 3.2$.

FIG. 20. Companion figure of Fig. [ ] for Yukawa couplings $\lambda = 1$ at $T = 1/40$ (left panel) and $\lambda = 2$ at $T = 1/20$ (right panel).

FIG. 21. Companion figure of Fig. [ ] for Yukawa couplings $\lambda = 1$ at $T = 1/40$ (left panel) and $\lambda = 2$ at $T = 1/20$ (right panel).
FIG. 22. Companion figure of Fig. 9 for Yukawa couplings $\lambda = 1$ at $T = 1/40$ (left panel) and $\lambda = 2$ at $T = 1/20$ (right panel).

FIG. 23. Companion figure of Fig. 10 for (top row) $\lambda = 1$ at (a) $r = -1.3 \approx r_{c0}$ and at (b) $r = -1.0 > r_{c0}$, and for (bottom row) $\lambda = 2$ at (c) $r = 3.1 \approx r_{c0}$ and at (d) $r = 3.4 > r_{c0}$.
FIG. 24. Frequency dependence of the SDW susceptibility $\chi^{-1}$ for $\lambda = 3$, $c = 2$ at $T = 1/30 < T_c$ and $r = 10.37 \approx r_{\text{opt}}$, close to where $T_c$ is highest for this set of parameters, shown for various momenta $\mathbf{q} = \mathbf{Q} + \tilde{\mathbf{q}}$. The black line is the best fit of a second degree polynomial $b_0 + b_1 |\omega_n| + b_2\omega_n^2$ to the $\mathbf{q} = \mathbf{Q}$, $L = 14$ low-frequency data.

FIG. 25. Imaginary part of the self energy $-\text{Im}(\Sigma)$, calculated within a one-loop approximation for several temperatures for two momenta on the Fermi surface and (a) close to the QCP at $\tau = r_c + 0.05$, (b) further away from the QCP. An SDW correlator of the form (C2) was used, with the parameters taken from the fit to the DQMC data for $\lambda = 1.5$, $c = 3$, with $\Delta \tau = 0.1$, and $L = 200$, as shown in Fig. 5(b). The data for $k = k_{\text{hs}}$ is indicated by full circles, the momentum away from the hotspot is indicated by empty squares.

FIG. 26. Imaginary time evolution of the single-particle Green’s function $G_k(\tau)$ for (a) several momenta along the noninteracting Fermi surface and (b) several momenta along a cut perpendicular to the Fermi surface. Here, $L = 12$, $T = 1/40$, $\lambda = 3$, and $c = 2$. Shaded regions indicate the statistical uncertainty. The solid lines are exponential fits.
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