FULLY NONLINEAR ELLIPTIC EQUATIONS WITH GRADIENT TERMS ON COMPACT ALMOST HERMITIAN MANIFOLDS

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ABSTRACT. We establish second order estimates for a general class of fully nonlinear elliptic equations with gradient terms on almost Hermitian manifolds including the deformed Hermitian-Yang-Mills equation and the equation in the proof of Gauduchon conjecture by Székelyhidi-Tosatti-Weinkove. As applications, we also consider the existence of Monge-Ampère equation and Hessian equations.

1. INTRODUCTION

Let \((M, \chi, J)\) be a compact almost Hermitian manifold of real dimension \(2n\), and \(\omega\) is a fixed real \((1,1)\)-form on \((M, J)\). For an arbitrary smooth function \(u\), we write
\[
\omega_u := \omega + \sqrt{-1} \partial \bar{\partial} u + Z(\partial u) = \omega + \frac{1}{2} (dJdu)^{(1,1)} + Z(\partial u),
\]
where \(Z(\partial u)\) denotes a smooth \((1,1)\)-form depending on \(\partial u\) linearly which will be specified later, and let \(\mu(u) = (\mu_1(u), \ldots, \mu_n(u))\) be the eigenvalues of \(\omega_u\) with respect to \(\chi\). For the sake of notational convenience, we sometimes denote \(\mu_i(u)\) by \(\mu_i\) when no confusion will arise. In the current paper, we consider the following fully nonlinear elliptic equations of the form
\[
F(\omega_u) = f(\mu_1, \ldots, \mu_n) = h,
\]
where \(h \in C^\infty(M)\) and \(f\) is a smooth symmetric function in \(\mathbb{R}^n\).

The equation (1.1) covers many important elliptic equations in (almost) complex geometry. A typical example of (1.1) is the following equation:
\[
\left( \eta + \frac{1}{n-1} (\Delta_\chi u \chi - \sqrt{-1} \partial \bar{\partial} u) + W(\partial u) \right)^n = e^h \chi^n.
\]
Here \(\eta\) is an almost Hermitian metric, \(\Delta_\chi\) denotes the canonical Laplacian operator of \(\chi\) and \(W = W(\partial u)\) is a Hermitian tensor that linearly depends on \(\partial u\). On a Hermitian manifold, the equation (1.2) was introduced by Popovici [33] and Tosatti-Weinkove [41] independently. Recently, Székelyhidi-Tosatti-Weinkove [36] confirmed the famous Gauduchon conjecture [17] by solving equation (1.2). When \(W \equiv 0\), the equation (1.2) is the notion of Monge-Ampère equation for \((n-1)\)-plurisubharmonic functions in pioneer works of Fu-Wang-Wu [15, 16].

The fully nonlinear elliptic equations with gradient terms on Hermitian manifolds have been researched extensively, we refer the reader to [14, 21, 22, 42, 44, 45] and references therein. On the framework of almost Hermitian manifolds, to our knowledge most of researches toward equation (1.1) are independent of \(\partial u\). Inspired

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by these works, we shall consider the equation (1.1) on compact almost Hermitian manifolds.

Let $\Gamma_n$ be the positive orthant in $\mathbb{R}^n$ and $\Gamma_1 = \{ \mu \in \mathbb{R}^n : \sum \mu_i > 0 \}$. In this paper, we always assume that $f$ is defined in a symmetric open and convex cone $\Gamma \subset \Gamma_1 \subseteq \mathbb{R}^n$ satisfying $\Gamma + \Gamma_n \subset \Gamma$, i.e. for any $\mu \in \Gamma$ and $\mu' \in \Gamma_n$, $\mu + \mu' \in \Gamma$. Furthermore, modifying the setup of Székelyhidi [35], suppose that

(i) $f_i = \frac{\partial f}{\partial \mu_i} > 0$ for all $i$ and $f$ is concave in $\Gamma$,
(ii) $\sup_{\partial \Gamma} f < \frac{\sup_{\Gamma} f}{N}$,
(iii) for any constant $\sup_{\partial \Gamma} f < \sigma < \sup_{\Gamma} f$, there exists a positive constant $N$, depending only on $\sigma$ and $\sigma'$, such that $\Gamma_{\sigma} + N1 \subset \Gamma_{\sigma'}$.

Here the sublevel set $\Gamma_{\sigma} = \{ \mu \in \Gamma : f(\mu) > \sigma \}$ is convex open for any $\sigma > \sup_{\partial \Gamma} f$ and $\sup_{\partial \Gamma} f = \sup_{\lambda' \in \partial \Gamma} \limsup_{\lambda \rightarrow \lambda'} f(\lambda)$, where $\lambda = (1, \ldots, 1) \in \mathbb{R}^n$.

Remark 1.1. The original setup in [35] assume the symmetric open and convex cone $\Gamma \subset \Gamma_1 \subseteq \mathbb{R}^n$ satisfying

(1.3) the vertex of $\Gamma$ is at the origin and $\Gamma_n \subset \Gamma$, $f$ is defined in $\Gamma$ and satisfies (i), (ii) and

(iii') for any $\sigma < \sup f$ and $\mu \in \Gamma$, we have $\lim_{t \rightarrow \infty} f(t\mu) > \sigma$.

Note that (iii') implies (iii) via [35, Lemma 9] if we further assume $\Gamma$ satisfies (1.3).

Motivated by Mirror Symmetry and Mathematical Physics, Jacob-Yau [30] studied the equation

$$\sum_i \text{arccot} \mu_i = \hat{\theta}, \quad \text{in } \Gamma_D = \{ \mu \in \mathbb{R}^n : 0 < \sum_i \text{arccot} \mu_i < \pi \}$$

for some real constant $\hat{\theta}$. We can verify that this equation satisfies (iii) (see §2) while not for (iii'), and $\Gamma_D$ satisfies the assumption $\Gamma + \Gamma_n \subset \Gamma$ rather than (1.3).

We have the following estimate:

Theorem 1.1. Let $(M, \chi, J)$ be a compact almost Hermitian manifold of real dimension $2n$. Suppose that $u$ (resp. $\underline{u}$) is a smooth solution (resp. $C$-subsolution) of (1.1). Then we have

$$||u||_{C^2(M, \chi)} \leq C(1 + \sup_M ||D\underline{u}||^2_{\chi}),$$

where $C$ is a constant depending on $u$, $h$, $Z$, $\omega$, $f$, $\Gamma$ and $(M, \chi, J)$.

As an application, to begin, we solve the equation (1.2). We have

Theorem 1.2. Let $(M, \chi, J)$ be a compact almost Hermitian manifold of real dimension $2n$ and $\eta$ be an almost Hermitian metric. There exists a unique pair $(u, c) \in C^\infty(M) \times \mathbb{R}$ such that

$$\left\{ \begin{array}{l}
\eta + \frac{1}{n-1} \left( (\Delta \chi) u - \sqrt{-1} \bar{\eta} \bar{\partial} u \right) + W(\partial u) = c^{h+c} \chi^n, \\
\eta + \frac{1}{n-1} \left( (\Delta \chi) u - \sqrt{-1} \bar{\eta} \bar{\partial} u \right) + W(\partial u) > 0, \\
\sup_M u = 0.
\end{array} \right.$$
For the complex Monge–Ampère equation, Yau \cite{43} solved it on a Kähler manifold and confirmed the famous Calabi’s conjecture (see \cite{4}). In the non-Kähler setting, we refer the reader to \cite{5, 9, 20, 23, 38, 39, 47}. The classical complex Hessian equations also have been studied extensively, see \cite{7, 12, 24, 25, 35, 46}. Similar to Theorem 1.2, we can solve the complex Monge-Ampère equation and complex Hessian equations with gradient terms.

**Theorem 1.3.** Let \((M,\chi,J)\) be a compact almost Hermitian manifold of real dimension \(2n\) and \(\omega\) be a smooth \(k\)-positive real \((1,1)\)-form. For any integer \(1 \leq k \leq n\), there exists a unique pair \((u,c)\) \(\in C^\infty(M) \times \mathbb{R}\) such that

\begin{equation}
\begin{cases}
\omega^k_u \wedge \chi^{n-k} = e^{h+c} \chi^n, \\
\omega^i_u \wedge \chi^{n-i} > 0, & i = 1, 2, \ldots, k, \\
\sup_M u = 0.
\end{cases}
\end{equation}

For the deformed Hermitian-Yang-Mills (dHYM) equation

\begin{equation}
\phi(\mu) = \sum_{i=1}^n \arccot \mu_i = h, \quad h \in C^\infty(M),
\end{equation}

we say (1.6) is hypercritical (resp. supercritical) if \(h \in (0, \frac{\pi}{2})\) (resp. \(h \in (0, \pi)\)). Jacob-Yau \cite{30} showed the existence of solution for dimension 2, and for general dimensions when \((M,\chi)\) has non-negative orthogonal bisectional curvature in the hypercritical phase setting. Pingali \cite{31, 32} obtained a solution when \(n = 3\). In general dimensions, the equation (1.6) was solved by Collins-Jacob-Yau \cite{10} under the existence of \(C\)-subsolutions. The equation (1.6) was also studied by Leung \cite{27, 28} to seek vector bundles over a symplectic manifold. Recently, Zhang and the authors \cite{26} provided a priori estimates on compact almost Hermitian manifolds for the hypercritical case. It was researched by Lin \cite{29} in the supercritical phase on compact Hermitian manifolds.

As a corollary, using Theorem 1.1, we are also able to derive a priori estimates for (1.6) in the supercritical case.

**Corollary 1.4.** Let \((M,\chi,J)\) be a compact almost Hermitian manifold of real dimension \(2n\). Suppose that \(u\) (resp. \(\bar{u}\)) is the solution (resp. \(C\)-subsolution) of equation (1.6) with \(h \in (0, \pi - \delta]\) (resp. \(h \in (0, \pi]\)). Then for each \(\alpha \in (0, 1)\), we have

\[\|u\|_{C^{2,\alpha}(M,\chi)} \leq C,\]

where \(C\) is a constant depending on \(\alpha, \bar{u}, h, \omega, \delta\) and \((M,\chi,J)\).

We now discuss the proof of Theorem 1.1. The zero order estimate can be proved by adapting the arguments of \cite{33, Proposition 11} and \cite{3, Proposition 3.1}, which are based on the method of Blocki \cite{2, 8}. For the second order estimate, following the idea of \cite{2, 4, 8, 33} and by some delicate calculations, the real Hessian \(\nabla^2 u\) can be controlled by the first gradient quadratically as follows:

\begin{equation}
\sup_M |\nabla^2 u|_\chi \leq C(1 + \sup_M |\partial u|^2_\chi).
\end{equation}
The paper is organized as follows. In §2, we will introduce some notations, and recall the definition and an important property of \( C \)-subsolution. We also verify that the dHYM equation satisfying the structural conditions. The zero order estimate will be established in §3.1. In §3.2, we shall prove the estimate (1.7). To see this, we apply the maximum principle to the quantity involving the largest eigenvalue \( \lambda_0 \) of real Hessian \( \nabla^2 u \) with respect to \( \chi \) of form

\[
Q = \log \lambda_1 + \varphi((\rho^2)_{\chi}) + \psi((\partial u^2)_{\chi}) + e^{-\lambda u}.
\]

In §3.3, we establish the second order estimate via the blowup argument and Liouville type theorem [32]. Theorem 20 when equation (1.1) satisfying the structural conditions (i), (ii) and (iii').

Given this, we are able to prove Theorems 1.2–1.3 in §3.4. In §4 we will prove Corollary 1.4 by using the maximum principle to establish the \( C^1 \) estimate for (1.6) which also implies the \( C^2 \) estimate.

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2. Preliminaries

2.1. Notations. Suppose that \((M, \chi, J)\) is an almost Hermitian manifold of real dimension \( 2n \). As pointed in [8, p.1564], we can define \((p, q)\)-forms and operators \( \partial, \bar{\partial} \) by using the almost complex structure \( J \). Let \( A^{1,1}(M) \) denote the set of smooth real \((1,1)\)-forms on \((M, J)\). For any \( u \in C^\infty(M) \), we see that \( \sqrt{-1}\partial\bar{\partial}u = \frac{1}{2}(dJdu)^{(1,1)} \) is a real \((1,1)\)-form in \( A^{1,1}(M) \). In the sequel, we set

\[
\omega_u = \omega + \sqrt{-1}\partial\bar{\partial}u + Z(\partial u),
\]

where \( Z(\partial u) \) is a real \((1,1)\)-form defined by \( Z_{ij} = Z^p_{ij}u_p + \bar{Z}^p_{ij}u_{\bar{p}} \).

For any point \( x_0 \in M \), let \( (e_1, \cdots, e_n) \) be a local unitary \((0,1)\)-frame with respect to \( \chi \) near \( x_0 \), and \( \{\theta^i\}_{i=1}^n \) be its dual coframe. Then in the local chart we have

\[
\chi = \sqrt{-1}\delta_{ij}\theta^i \wedge \theta^j.
\]

Suppose that

\[
\omega = \sqrt{-1}g_{ij}\theta^i \wedge \theta^j, \quad \omega_u = \sqrt{-1}\bar{g}_{ij}\theta^i \wedge \theta^j,
\]

as well as

\[
\bar{g}_{ij} = g_{ij} + \partial \bar{\partial}u(e_i, \bar{e}_j) + Z_{ij},
\]

\[
= g_{ij} + e_i\bar{e}_j(u) - [e_i, \bar{e}_j]^{(0,1)}(u) + u_pZ^p_{ij} + u_{\bar{p}}\bar{Z}^p_{ij},
\]

where \([e_i, \bar{e}_j]^{(0,1)}\) is the \((0, 1)\) part of the Lie bracket \([e_i, \bar{e}_j]\). Define

\[
G^{\sigma} = \frac{\partial F}{\partial g_{\sigma}} \quad \text{and} \quad G^{ij} = \frac{\partial^2 F}{\partial g_{ij} \partial g_{k\bar{l}}}.
\]

After making a unitary transformation, we may assume that \( \bar{g}_{ij}(x_0) = \delta_{ij}\bar{g}_{ij}(x_0) \). We denote \( \bar{g}_{ij}(x_0) \) by \( \mu_i \). It is useful to order \( \mu_i \) such that

\[
\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n.
\]

At \( x_0 \), we have the expressions of \( G^{\sigma} \) and \( G^{ij} \) (see e.g. [18, 34])

\[
G^{\sigma} = \delta_{ij}f_1, \quad G^{ij} = f_{ij} = f_{ij} + f_{k}[1 - (1 - \delta_{ik})\delta_{il}].
\]
where the quotient is interpreted as a limit if \( \mu_i = \mu_j \). Using \((2.4)\), we obtain (see e.g. \([13, 34]\))

\[
G_1^\Gamma \leq G_2^\Gamma \leq \cdots \leq G_m^\Gamma.
\]

On the other hand, the linearized operator of equation \((1.1)\) is

\[
L(v) = G_{ij} \left( e_i \bar{e}_j(v) - [e_i, \bar{e}_j] \right) (v) + e_p(v) Z_{ij}^p + \bar{e}_p(v) Z_{ij}^p.
\]

2.2. C-subsolution.

**Definition 2.1** \([35]\). We say that a function \( u \in C^2(M) \) is a C-subsolution of \((1.1)\) if at each point \( x \in M \), the set

\[
\{ \mu \in \Gamma : f(\mu) = h(x), \mu - u \in \Gamma_n \}
\]

is bounded.

By Definition 2.1, for each C-subsolution \( u \), there are constants \( \delta, R > 0 \) depending only on \( u, (M, \chi, J) \), \( f \) and \( \Gamma \) such that

\[
(\mu(u) - \delta \mathbf{1} + \Gamma_n) \cap \partial \Gamma^h(x) \subset B_R(0), \quad \forall \ x \in M,
\]

where \( B_R(0) \) denotes the Euclidean ball with radius \( R \) and center \( 0 \).

Similar to \([10, 33]\), we have the following proposition:

**Proposition 2.1.** Suppose that \( \sigma \in (\sup_{\partial \Gamma} f, \sup f) \) and \( \mu \in \mathbb{R}^n \) satisfying

\[
(\mu - \delta \mathbf{1} + \Gamma_n) \cap \partial \Gamma^\sigma \subset B_R(0)
\]

for some \( \delta, R > 0 \). Then there exists a constant \( \theta > 0 \) depending on \( \delta \) and the set in \((2.5)\) such that for each \( \mu' \in \partial \Gamma^\sigma \) and \( |\mu'| > R \), we have either

\[
\sum_i f_i(\mu')(\mu_i - \mu'_i) > \theta \sum_i f_i(\mu'),
\]

or \( f_k(\mu') > \theta \sum_i f_i(\mu') \) for each \( k = 1, 2, \ldots, n \).

**Proof.** The proof can be found in \([32, \text{Proposition 5}]\), we include it here for convenience to reader. Set

\[
A_\delta = \{ v \in \Gamma : f(v) \leq \sigma \ and \ v - (\mu - \delta \mathbf{1}) \in \Gamma_n \}.
\]

It follows from \((2.5)\) that \( A_\delta \) is compact. For each \( v \in A_\delta \), we define

\[
C_v = \{ w \in \mathbb{R}^n : v + tw \in (\mu - 2\delta \mathbf{1} + \Gamma_n) \cap \partial \Gamma^\sigma \ \text{for some} \ t > 0 \}.
\]

Note that \( f_i > 0 \) for all \( i \). We conclude that

\[
(\mu - \delta \mathbf{1} + \Gamma_n) \cap \partial \Gamma^\sigma \subset (\mu - 2\delta \mathbf{1} + \Gamma_n) \cap \partial \Gamma^\sigma,
\]

which implies that \( C_v \) is strictly larger than \( \Gamma_n \). Now we define the dual cone of \( C_v \) by

\[
C_v^* = \{ x \in \mathbb{R}^n : \langle x, y \rangle > 0 \ \text{for all} \ y \in C_v \}.
\]

We remark that \( C_v \supseteq \Gamma_n \) implies there exists a constant \( \epsilon > 0 \) such that if \( x = (x_1, \cdots, x_n) \in C_v^* \),

\[
x_k > \epsilon \ \text{for all} \ k.
\]

As \( A_\delta \) compact, we can find a uniform constant \( \epsilon \) such that \((2.6)\) holds for all \( v \in A_\delta \). Let \( \mu' \in \partial \Gamma^\sigma \), \( |\mu'| > R \) and \( T_{\mu'} \) be the tangent plane to \( \partial \Gamma^\sigma \) at \( \mu' \). Now we split the proof into two cases:
Case 1. Assume \( T_{\mu'} \cap A_\delta \neq \emptyset \) and let \( v \in T_{\mu'} \cap A_\delta \). Then the cone \( v + C_v \) lies above \( T_{\mu'} \), i.e. \( \langle x, n_{\mu'} \rangle > 0 \) for all \( x \in C_v \), where \( n_{\mu'} \) is the inward pointing unit normal vector of \( \partial \Gamma_\sigma \) at \( \mu' \). By the definition of \( C_v^* \), we obtain \( n_{\mu'} = Df(\mu')/|Df(\mu')| \in C_v^* \). It then follows (2.6) that for each \( k = 1, 2, \cdots, n \)
\[
f_k(\mu') > \epsilon |Df(\mu')|.
\]

Case 2. We now assume \( T_{\mu'} \cap A_\delta = \emptyset \), then \( \text{dist}(\mu, T_{\mu'}) > \delta \). Thus, \( (\mu - \mu') \cdot n_{\mu'} > \delta \), i.e.
\[
\sum_i f_i(\mu')(\mu - \mu') > \delta |Df(\mu')|.
\]

This completes the proof of proposition.

Using previous proposition, we have the following result originated from \(^{22}\). It will play an important role in the proof of Theorem \(^{32}\).

**Proposition 2.2.** Let \( \sigma \in [\inf_\mathcal{M} h, \sup_\mathcal{M} h] \) and \( A \) be a Hermitian matrix with eigenvalues \( \mu(A) \in \partial \Gamma_\sigma \).

(1) There exists a constant \( \tau \) depending on \( f, \Gamma \) and \( \sigma \) such that
\[
G(A) = \sum_i G_i^\sigma(A) > \tau.
\]

(2) For \( \delta, R > 0 \), there exists \( \theta > 0 \) depending only on \( f, \Gamma, h, \delta, R \) such that the following holds. If \( B \) is a Hermitian matrix satisfying
\[
(\mu(B) - 2\delta \mathbf{1} + \Gamma_n) \cap \partial \Gamma_\sigma \subset B_R(0),
\]
then we have either
\[
\sum_{p,q} G_i^\sigma(A)[B_{i,p} - A_{i,p}] > \theta \sum_p G_i^\sigma(A)
\]
or
\[
G_i^\sigma(A) > \theta \sum_p G_i^\sigma(A), \quad \forall i = 1, 2, \cdots, n.
\]

**Proof.** For (1), choosing \( \sigma' \) with \( \sup \mathcal{F} f > \sigma' > \sigma \). By assumption (iii) and concavity, there exists a large constant \( N' \) such that
\[
\sigma' < f(\mu(A) + N' \mathbf{1}) \leq f(\mu(A)) + N \sum_i f_i(\mu(A)).
\]
It follows \( G(A) \geq \frac{1}{\theta} (\sigma' - \sigma) \) which implies (1).

For (2), we divide into two possibilities:

- \( |\mu(A)| \geq R \). We note that the proof of \(^{32}\) Proposition 6] only needs assumption (i) and (ii). Then the conclusion follows.

- \( |\mu(A)| < R \). Using the argument of \(^{6}\) Proposition 2.1], we complete the proof.

\[\square\]
2.3. The dHYM equation. Let \( \Gamma = \{ \mu \in \mathbb{R}^n : 0 < \phi(\mu) < \pi \} \) and let \( \phi \) be the function defined in (1.6). We consider the dHYM equation

\[
f(\mu(u)) = \cot \phi(\mu(u)) = \cot h, \quad \mu(u) \in \Gamma.
\]

For any \( \sigma \in \mathbb{R} \), we have \( \Gamma^\sigma = \{ \mu \in \mathbb{R}^n : 0 < \phi(\mu) < \arccot \sigma \} \).

Now we prove the dHYM equation satisfying the structural condition (iii).

**Proposition 2.3.** Let \( f(\mu) = \cot \phi(\mu) \). For any \( \sigma, \sigma' \in \mathbb{R} \) with \( \sigma < \sigma' \), there exists a positive constant \( N \), depending only on \( \sigma \) and \( \sigma' \), such that

\[
(2.11) \quad \Gamma^\sigma + N I \subset \Gamma^\sigma'.
\]

**Proof.** We fix an arbitrary \( \mu \in \Gamma^\sigma \). By \([11, \text{Lemma 2.1}]\), there exists a constant \( N' \) such that \( \mu + N' I \in \Gamma \). It is straightforward that there exists a constant \( N'' \) such that \( f(N'' I) > \sigma' \). Then we have \( f(\mu + (N' + N'') I) > f(N'' I) > \sigma' \). This implies \( (2.11) \) by letting \( N = N' + N'' \).

\( \square \)

3. A priori estimates

3.1. Zero order estimate.

**Proposition 3.1.** Let \( u \) (resp. \( \underline{u} \)) be a smooth solution (resp. \( C^0 \)-subsolution) of (1.1) with \( \sup_M (u - \underline{u}) = 0 \). Then there exists a constant \( C \) depending on \( u, h, ||\omega||_{C^0}, f, \Gamma \) and \( (M, \chi, J) \) such that

\[
\|u\|_{L^\infty} \leq C.
\]

**Proof.** Without loss of generality, we may assume that \( \underline{u} = 0 \). Thanks to \([35, \text{(44)}]\), we have \( \text{tr}_\chi \omega_u > 0 \) and hence

\[
\Delta u = \Delta_\chi u + \chi^{ij} \tilde{Z}_{ij}(\partial u) = \text{tr}_\chi \omega_u - \text{tr}_\chi \omega \geq -C,
\]

where \( \Delta_\chi \) denotes the canonical Laplacian operator of \( \chi \). Following a similar argument of \([3, \text{Proposition 2.3}]\), then there exists a uniform constant \( C \) such that

\[
\int_M (-u) \chi^n \leq C.
\]

Now it suffices to establish the lower bound of the infimum \( I = \inf_M u \). We can adopt the arguments in \([3]\). We remark that the only difference here is the presence of the term \( Z(\partial u) \) in the definition of \( H(u) \). However, this term is linear in \( \partial u \), which can be controlled (by \( \varepsilon \)) on the contact set \( P \) in \([3]\).

\( \square \)

3.2. Second order estimate. In this subsection, we give the proof of Theorem 1.1. Our first goal is the following theorem:

**Theorem 3.2.** Under the same assumptions as in Proposition 3.1. Then there exists a constant \( C \) depending on \( u, h, ||\omega||_{C^2}, f, \Gamma \) and \( (M, \chi, J) \) such that

\[
(3.3) \quad \sup_M |\nabla^2 u|_\chi \leq C(\sup_M |\partial u|_\chi^2 + 1),
\]

where \( \nabla \) denotes the Levi-Civita connection with respect to \( \chi \).

Without loss of generality, we assume \( \underline{u} = 0 \) and \( \sup_M u = -1 \). Let \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\dim M \) be the eigenvalues of \( \nabla^2 u \) with respect to \( \chi \). For notational convenience, we write \( |\cdot| = |\cdot|_\chi \).
Let us define
\[(3.4)\quad K = \sup_M |\partial u|^2 + 1, \quad N = \sup_M |\nabla^2 u| + 1, \quad \rho = \nabla^2 u + N\chi.\]

On an open set \(\Omega = \{\lambda_1 > 0\} \subset M\), we consider
\[Q = \log \lambda_1 + \varphi(|\rho|^2) + \psi(|\partial u|^2) + e^{-\lambda u}\]
for a large constant \(A\) to be chosen later, where
\[\varphi(s) = -\frac{1}{4} \log(5N^2 - s), \quad \psi(s) = -\frac{1}{4} \log(2K - s).\]

By a directly calculation we see that
\[(3.5)\quad \varphi'' = 4(\varphi')^2, \quad \psi'' = 4(\psi')^2, \quad \frac{1}{20N^2} \leq \varphi' \leq \frac{1}{16N^2}, \quad \frac{1}{8K} \leq \psi' \leq \frac{1}{4K}.\]

We may assume \(\Omega \neq \emptyset\), otherwise we are done. Since \(Q(z) \to -\infty\) as \(z\) approaches to the boundary of \(\Omega\), we further assume \(Q\) achieves its maximum at a point \(x_0 \in \Omega\).

It is easy to show that (see [6])
\[\lambda_1(x_0) \leq CK.\]

Near \(x_0\), there exists a local unitary frame \(\{e_i\}_{i=1}^n\) with respect to \(\chi\) such that
\[(3.6)\quad \lambda_1(x_0) \leq CK.\]

Near \(x_0\), there exists a local unitary frame \(\{e_i\}_{i=1}^n\) with respect to \(\chi\) such that
\[\lambda_1(x_0) \leq CK.\]

We remark that \(\chi\) and \(J\) are compatible implies there exists a coordinate system \((U, \{x^n\}_{n=1}^2)\) in a neighborhood of \(x_0\) such that at \(x_0\),
\[\begin{align*}
\alpha) & \quad e_i = \frac{1}{\sqrt{2}} (\partial_{2i-1} - \sqrt{-1} \partial_{2i}) \text{ for } i = 1, 2, \ldots, n. \\
\beta) & \quad \partial_\alpha \chi_{\alpha\beta} = 0 \text{ for } \alpha, \beta = 1, 2, \ldots, 2n.
\end{align*}\]

Here \(\chi_{\alpha\beta} = \chi(\partial_\alpha, \partial_\beta)\) and \(\partial_\alpha = \frac{\partial}{\partial x^\alpha}\). Let us define \(u_{\alpha\beta} = (\nabla u)(\partial_\alpha, \partial_\beta)\) and \(\Phi^{\alpha}_{\beta} = \sum_{\gamma=1}^n \chi^{\alpha\gamma} u_{\gamma\beta}\), where \((\chi^{\alpha\gamma}) = (\chi_{\alpha\gamma})^{-1}\) denotes the inverse matrix of \((\chi_{\alpha\gamma})\).

Clearly, \(\lambda_\alpha\) are eigenvalues of \(\Phi\). Let \(V_1, V_2, \ldots, V_{2n}\) be the eigenvectors for \(\Phi\) at \(x_0\), corresponding to eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_{2n}\) respectively. Define \(V^\alpha_{\beta}\) by \(V_\alpha = V^\alpha_{\beta} \partial_\beta\) at \(x_0\), and extend \(V_\alpha\) to be vector fields near \(x_0\) by taking the components to be constants. Using a viscosity argument adapted in [8], we may assume that \(\lambda_1\) is smooth and \(\lambda_1 > \lambda_2\) at \(x_0\).

Applying the maximum principle at \(x_0\), we see that
\[(3.8)\quad \frac{(\lambda_1)_i}{\lambda_1} = -\varphi'(|\rho|^2)_i - \psi'(|\partial u|^2)_i + Ae^{-\lambda u}_i\]
for each $1 \leq i \leq n$, and

$$0 \geq L(Q) = \frac{L(\lambda_1)}{\lambda_1} - \frac{G^{\tilde{i}i}(\lambda_1)_{i}}{\lambda_1^2} + \frac{\varphi'(|\rho|^2)}{\lambda_1^2} + \frac{\varphi''G^{\tilde{i}i}(|\rho|^2)_{i}}{\lambda_1^2}$$

$$+ \psi' L(|\partial u|^2) + \frac{\psi''G^{\tilde{i}i}(||\partial u||^2)_{i}}{\lambda_1} - Ae^{-Au} L(u) + A^2 e^{-Au} G^{\tilde{i}i}|u_i|^2.$$  

In the sequel, we shall make the following conventions:

(i) all the calculations are done at $x_0$,

(ii) we will use the Einstein summation,

(iii) we usually use $C$ to denote a constant depending on $\|u\|_{C^0}$, $h$, $\omega$, $\Gamma$, $(M, \chi, J)$, and $C_A$ to denote a constant further depending on $A$,

(iv) we always assume without loss of generality, that $\lambda_1 \geq CK$ for some $C$, or $\lambda_1 \geq C_A K$ for some $C_A$,

(v) we use subscripts $i$ and $j$ to denote the partial derivatives $e_i$ and $\bar{e}_j$.

3.2.1. Lower bound for $L(Q)$.

**Proposition 3.3.** For $\varepsilon \in (0, \frac{1}{4}]$, at $x_0$, we have

$$0 \geq L(Q) \geq (2 - \varepsilon) \sum_{\alpha > 1} \frac{G^{\tilde{i}i}(|\nu_{\alpha}V_i|^2)}{\lambda_1(\lambda_1 - \lambda_\alpha)} - \frac{1}{\lambda_1} G^{\tilde{i}i}(\bar{g}_{i\bar{k}}) V_i(\bar{g}_{j\bar{k}})$$

$$+ \sum_{\alpha, \beta} \frac{G^{\tilde{i}i}(\nu_{\alpha} \beta)}{A_{\lambda_1}^2} \frac{G^{\tilde{i}i}(\lambda_1)_{i}}{\lambda_1^2} + \frac{\varphi''G^{\tilde{i}i}(|\rho|^2)_{i}}{\lambda_1^2}$$

$$+ \frac{3\psi'}{4} \sum_j G^{\tilde{i}i}(\nu_{i} e_j + \nu_{i} \bar{e} j) + \psi''G^{\tilde{i}i}(||\partial u||^2)_{i}$$

$$- Ae^{-Au} L(u) + A^2 e^{-Au} G^{\tilde{i}i}|u_i|^2 - \frac{C}{\varepsilon} \lambda_1 G.$$  

We remark that the fourth term is the bad term that we need to control. Since $F$ is both concave and elliptic, then the first, second and third term are nonnegative, which play an important role in our proof of Theorem 3.2. To prove Proposition 3.3 we shall estimate the lower bounds of $L(\lambda_1)$, $L(|\rho|^2)$ and $L(||\partial u||^2)$, respectively.

First, we give the lower bound of $L(\lambda_1)$.

**Lemma 3.4.** For each $\varepsilon \in (0, \frac{1}{4}]$, at $x_0$, we have

$$L(\lambda_1) \geq (2 - \varepsilon) \sum_{\alpha > 1} \frac{G^{\tilde{i}i}(\nu_{\alpha}V_i)}{\lambda_1(\lambda_1 - \lambda_\alpha)} - G^{\tilde{i}i}(\bar{g}_{i\bar{k}}) V_i(\bar{g}_{j\bar{k}}) - \varepsilon G^{\tilde{i}i}(\lambda_1)_{i} \frac{\lambda_1}{\lambda_1} - \frac{C}{\varepsilon} \lambda_1 G.$$  

**Proof.** The following formulas are well-known (see e.g. [8, 34, 35]):

$$\frac{\partial \lambda_1}{\partial \Phi^\alpha_{\beta}} = V_1^\alpha V_1^\beta,$$

$$\frac{\partial^2 \lambda_1}{\partial \Phi^\alpha_{\beta} \partial \Phi^\delta_{\bar{\gamma}}} = \sum_{\mu > 1} \frac{V_1^\alpha V_1^\beta V_1^\delta V_1^\gamma}{\lambda_1(\lambda_1 - \lambda_\mu)}.$$
Then we compute

\[
L(\lambda_1) = G^{\tilde{\nu}} \frac{\partial^2 \lambda_1}{\partial \Phi^\beta_\delta \partial \Phi^\beta_\delta} e_i(\Phi^i_\delta)e_i(\Phi^i_\delta) + G^{\tilde{\nu}} \frac{\partial \lambda_1}{\partial \Phi^\beta_\delta} (e_i\bar{e}_i - [e_i, \bar{e}_i])^{(0,1)}(\Phi^i_\delta)
\]

\[+ G^{\tilde{\nu}} \frac{\partial \lambda_1}{\partial \Phi^\beta_\delta} (e_p(\Phi^i_\delta)^p \bar{Z}^p_{\bar{\nu}} + \bar{e}_p(\Phi^i_\delta)^p \bar{Z}^p_{\bar{\nu}})
\]

\[
= G^{\tilde{\nu}} \frac{\partial^2 \lambda_1}{\partial \Phi^\beta_\delta \partial \Phi^\beta_\delta} e_i(u_{\gamma i})e_i(u_{\alpha \beta}) + G^{\tilde{\nu}} \frac{\partial \lambda_1}{\partial \Phi^\beta_\delta} (e_i\bar{e}_i - [e_i, \bar{e}_i])^{(0,1)}(u_{\alpha \beta})
\]

\[+ G^{\tilde{\nu}} \frac{\partial \lambda_1}{\partial \Phi^\beta_\delta} u_{\gamma \beta}e_i(\lambda^\gamma) + G^{\tilde{\nu}} \frac{\partial \lambda_1}{\partial \Phi^\beta_\delta} (e_p(\Phi^i_\delta)^p \bar{Z}^p_{\bar{\nu}} + \bar{e}_p(\Phi^i_\delta)^p \bar{Z}^p_{\bar{\nu}})
\]

\[
\geq 2 \sum_{\alpha > 1} G^{\tilde{\nu}} |e_i(u_{\alpha i}v)_{\alpha i}|^2 + G^{\tilde{\nu}} (e_i\bar{e}_i - [e_i, \bar{e}_i])^{(0,1)}(u_{\alpha i})v_{\alpha i}
\]

\[+ G^{\tilde{\nu}} (e_p(u_{\alpha i}v)_{\alpha i} \bar{Z}^p_{\bar{\nu}} + \bar{e}_p(u_{\alpha i}v)_{\alpha i} \bar{Z}^p_{\bar{\nu}}) - C\lambda_1 \mathcal{G}.
\]

**Claim 1.** At \(x_0\), we have

\[
G^{\tilde{\nu}} (e_p(u_{\alpha i}v)_{\alpha i} \bar{Z}^p_{\bar{\nu}} + \bar{e}_p(u_{\alpha i}v)_{\alpha i} \bar{Z}^p_{\bar{\nu}}) \geq G^{\tilde{\nu}} V_1 V_1 (u_p Z^p_{\bar{\nu}} + u_p Z^p_{\bar{\nu}}) - C\lambda_1 \mathcal{G}.
\]

**Proof.** By a direct calculation,

\[
G^{\tilde{\nu}} e_p(u_{\alpha i}v)_{\alpha i} \bar{Z}^p_{\bar{\nu}} = G^{\tilde{\nu}} e_p(V_1 V_1 u - (\nabla V_1 V_1) u) Z^p_{\bar{\nu}}
\]

\[= G^{\tilde{\nu}} e_p V_1 V_1 (u) \cdot Z^p_{\bar{\nu}} - O(\lambda_1) \mathcal{G}
\]

\[= G^{\tilde{\nu}} e_p V_1 V_1 (u) \cdot Z^p_{\bar{\nu}} - O(\lambda_1) \mathcal{G}
\]

\[= G^{\tilde{\nu}} V_1 V_1 (u_p Z^p_{\bar{\nu}}) - O(\lambda_1) \mathcal{G}.
\]

Here and hereafter \(O(\lambda_1)\) means the terms those can be controlled by \(C\lambda_1\). Similarly, we also obtain

\[
G^{\tilde{\nu}} \bar{e}_p (u_{\alpha i}v)_{\alpha i} \bar{Z}^p_{\bar{\nu}} = G^{\tilde{\nu}} V_1 V_1 (u_p Z^p_{\bar{\nu}}) - O(\lambda_1) \mathcal{G}.
\]

Then the claim follows. \(\square\)

**Claim 2.** At \(x_0\), we have

\(\text{(I)} = G^{\tilde{\nu}} (e_i\bar{e}_i - [e_i, \bar{e}_i])^{(0,1)}(u_{\alpha i}v_{\alpha i}) + G^{\tilde{\nu}} (e_p(u_{\alpha i}v)_{\alpha i} \bar{Z}^p_{\bar{\nu}} + \bar{e}_p(u_{\alpha i}v)_{\alpha i} \bar{Z}^p_{\bar{\nu}})
\]

\[
\geq - G^{\tilde{\nu}} e_i\bar{e}_i V_1 V_1 u - (\nabla V_1 V_1) u - C\lambda_1 \mathcal{G} - 2(\text{II}),
\]

where

\(\text{(II)} = G^{\tilde{\nu}} \{[V_1, \bar{e}_i] V_1 e_i(u) + [V_1, e_i] V_1 \bar{e}_i(u)\}.
\]

**Proof of Claim 2.** It is clear that

\[
G^{\tilde{\nu}} (e_i\bar{e}_i - [e_i, \bar{e}_i])^{(0,1)}(u_{\alpha i}v)_{\alpha i}
\]

\[
= G^{\tilde{\nu}} (e_i\bar{e}_i - [e_i, \bar{e}_i])^{(0,1)}(V_1 V_1 u - (\nabla V_1 V_1) u)
\]

\[
\geq G^{\tilde{\nu}} e_i\bar{e}_i V_1 V_1 u - G^{\tilde{\nu}} e_\bar{e}_i (\nabla V_1 V_1) u - G^{\tilde{\nu}} [e_i, \bar{e}_i]^{(0,1)} V_1 V_1 u - C\lambda_1 \mathcal{G}.
\]

Set \(W = \nabla V_1 V_1\). Then

\[
e_i\bar{e}_i W(u) = e_i W \bar{e}_i(u) + e_i e_i W(u)
\]

\[
= W \bar{e}_i(u) + [e_i, W] e_i(u) = e_i [\bar{e}_i, W] (u)
\]

\[
= W(\bar{e}_i(u)) + W(e_i, \bar{e}_i)^{(0,1)}(u) + [e_i, W] e_i(u) + e_i [\bar{e}_i, W](u) + O(\lambda_1).
\]
Applying \( W \) to the equation (1.1),

\[
G^{ii} W(\tilde{g}_{ii}) = W(h).
\]

It follows that

\[
|G^{ii} e_i \tilde{e}_i W(u)| = |G^{ii} e_i \tilde{e}_i (\nabla_{V_i} V_i) (u)| \leq C \lambda_1 G.
\]

Combining this with (3.12),

\[
G^{ii} (e_i \tilde{e}_i - [e_i, \tilde{e}_i]^{(0.1)} ) (u V_i V_i)
\]

Combining (3.11), (3.16) and Claim 2, we obtain Lemma 3.4.

By direct calculation, we see that

\[
G^{ii} \left\{ e_i \tilde{e}_i V_i V_i (u) - [e_i, \tilde{e}_i]^{(0.1)} V_i V_i (u) \right\}
\]

Combining (3.13),

\[
G^{ii} \left\{ e_i \tilde{e}_i V_i V_i (u) - [e_i, \tilde{e}_i]^{(0.1)} V_i V_i (u) \right\} 
\]

By direct calculation, we see that

\[
\begin{align*}
&= G^{ii} \left\{ e_i V_i \tilde{e}_i V_i (u) - e_i [V_i, \tilde{e}_i] V_i (u) - V_i [e_i, \tilde{e}_i] V_i (u) \right\} + O(\lambda_1) G \\
&= G^{ii} \left\{ e_i V_i \tilde{e}_i V_i (u) - V_i [e_i, \tilde{e}_i] V_i (u) - V_i [e_i, \tilde{e}_i] V_i (u) \right\} + O(\lambda_1) G \\
&= G^{ii} \left\{ V_i V_i (e_i \tilde{e}_i) - [e_i, \tilde{e}_i]^{(0.1)} (u) - V_i [e_i, \tilde{e}_i] (u) - V_i e_i [V_i, \tilde{e}_i] (u) \right\} + O(\lambda_1) G - 2(II)
\end{align*}
\]

Substituting this with Claim 1 into (3.13), we obtain

\[
(III) \geq G^{ii} V_i V_i (\tilde{g}_{ii}) + O(\lambda_1) G - 2(II).
\]

To deal with the first term, we apply \( V_i V_i \) to the equation (1.1) and obtain

\[
G^{ii} V_i V_i (\tilde{g}_{ii}) = -G^{ik,ij} V_i (\tilde{g}_{ik}) V_i (\tilde{g}_{ij}) + V_i V_i (h).
\]

Then Claim 2 follows from (3.14) and (3.15).

Using the similar argument of [6, Claim 2], for each \( \varepsilon \in (0, \frac{1}{3}] \), we deduce

\[
2(II) \leq \varepsilon \frac{G^{ii} |(\lambda_1)|^2}{\lambda_1} + \varepsilon \sum_{\alpha>1} \frac{G^{ii} |e_i (u V_i V_i)|^2}{\lambda_1 - \lambda_\alpha} + \frac{C}{\varepsilon} \lambda_1 G.
\]

Combining (3.11), (3.10) and Claim 2, we obtain Lemma 3.4.

Second, we estimate the lower bound of \( L(|\rho|^2) \).

**Lemma 3.5.** For each \( \varepsilon \in (0, \frac{1}{4}] \), at \( x_0 \), we have

\[
L(|\rho|^2) \geq (2 - \varepsilon) \sum_{\alpha, \beta} G^{ii} |e_i (u_{\alpha \beta})|^2 - \frac{C}{\varepsilon} N^2 F.
\]

**Proof.** We remark that the linear gradient terms in \( L \) can be absorbed by \( N^2 F \). Thus the proof is similar to [6].

Finally, we give the lower bound of \( L(|\partial u|^2) \).

**Lemma 3.6.** At \( x_0 \), we have

\[
L(|\partial u|^2) \geq \frac{3}{4} \sum_j G^{ii} |e_i e_j u|^2 + |e_i \tilde{e}_j u|^2 - CK G.
\]
Proof. By a direct calculation, we deduce

\[
L(|\partial u|^2) = G^{ii}(e_i \bar{e}_i |\partial u|^2) - |e_i, \bar{e}_i|^{(0,1)}|\partial u|^2 + e_p(|\partial u|^2)Z_{ii}^p + \bar{e}_p(|\partial u|^2)\overline{Z_{ii}^p})
\]

= \(I_1 + I_2 + I_3\),

where

\[
I_1 = G^{ii}(e_i \bar{e}_i e_j u - |e_i, \bar{e}_i|^{(0,1)}e_j u + e_p e_j(u)Z_{ii}^p + \bar{e}_p e_j(u)|\overline{Z_{ii}^p})e_j u,
\]

\[
I_2 = G^{ii}(e_i \bar{e}_i e_j u - |e_i, \bar{e}_i|^{(0,1)}\bar{e}_j u + e_p \bar{e}_j(u)Z_{ii}^p + \bar{e}_p \bar{e}_j(u)|\overline{Z_{ii}^p})e_j u,
\]

\[
I_3 = G^{ii}(|e_i e_j u|^2 + |e_i \bar{e}_j u|^2).
\]

Applying \(e_j\) to the equation (1.1),

\[
G^{ii}e_j(e_i \bar{e}_i u - |e_i, \bar{e}_i|^{(0,1)}u + e_p(u)Z_{ii}^p + \bar{e}_p(u)|\overline{Z_{ii}^p}) = h_j.
\]

Note that

\[
G^{ii}(e_i \bar{e}_i e_j u - |e_i, \bar{e}_i|^{(0,1)}e_j u + e_p e_j(u)Z_{ii}^p + \bar{e}_p e_j(u)|\overline{Z_{ii}^p})
\]

= \(G^{ii}(e_j e_i \bar{e}_i u + |e_i, \bar{e}_i|^{(0,1)}e_j u + e_i, \bar{e}_i|^{(0,1)}e_j u)
\]

+ \(G^{ii}(e_j e_p(u)Z_{ii}^p + e_j e_p(u)|\overline{Z_{ii}^p}) + O(\sqrt{K})G
\]

= \(G^{ii}(e_j e_i \bar{e}_i u + |e_i, \bar{e}_i|^{(0,1)}e_j u)
\]

+ \(G^{ii}(e_j e_p(u)Z_{ii}^p + e_p(u)|\overline{Z_{ii}^p}) + O(\sqrt{K})G
\]

= \(h_j + G^{ii}(e_j e_i, \bar{e}_i|^{(0,1)}u + G^{ii}(e_j e_i, \bar{e}_i|^{(0,1)}u + |e_i, \bar{e}_i|^{(0,1)}e_j u + O(\sqrt{K})G
\]

= \(h_j + G^{ii}(e_i \bar{e}_i, e_j|^{(0,1)}u + \bar{e}_i, e_j|^{(0,1)}u + |e_i, e_j, \bar{e}_i|^{(0,1)}u + |e_i, e_j, \bar{e}_i|^{(0,1)}u + O(\sqrt{K})G
\]

where \(O(\sqrt{K})\) means the terms those can be controlled by \(C\sqrt{K}\). Similarly,

\[
G^{ii}(e_i \bar{e}_i \bar{e}_j u - |e_i, \bar{e}_i|^{(0,1)}\bar{e}_j u + e_p \bar{e}_j(u)Z_{ii}^p + \bar{e}_p \bar{e}_j(u)|\overline{Z_{ii}^p})
\]

= \(h^\perp + G^{ii}(e_i \bar{e}_i, \bar{e}_j|^{(0,1)}u + \bar{e}_i, \bar{e}_j|^{(0,1)}u + |e_i, \bar{e}_j, \bar{e}_i|^{(0,1)}u + |e_i, \bar{e}_j, \bar{e}_i|^{(0,1)}u + O(\sqrt{K})G
\]

By the Cauchy-Schwarz inequality,

\[
I_1 + I_2 \geq 2\text{Re}(\sum_j h_j u_j) - C|\partial u|\sum_j G^{ii}(|e_i e_j u| + |e_i \bar{e}_j u|) - CK\mathcal{G}
\]

(3.18)

\[
\geq - C|\partial u| - \frac{1}{4} \sum_j G^{ii}(|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) - CK\mathcal{G}.
\]

Then we have

\[
L(|\partial u|^2) = I_1 + I_2 + I_3 \geq \frac{3}{4} \sum_j G^{ii}(|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) - CK\mathcal{G}.
\]

This proves the lemma. \(\Box\)

We will use the above computations to prove Proposition 3.3.
Proof of Proposition 3.3. Combining (3.9) and Lemmas 3.4–3.6, we obtain
\[ 0 \geq (2 - \varepsilon) \sum_{\alpha > 1} G^{i\bar{i}}(u_{\alpha\beta})^2 - \frac{1}{\lambda_1} G^{i\bar{i}}(\tilde{g}_{i\bar{i}})V_1(\tilde{g}_{i\bar{j}}) \]
\[ + (2 - \varepsilon) \sum_{\alpha,\beta} G^{i\bar{i}}(u_{\alpha\beta})^2 - (1 + \varepsilon) \frac{G^{i\bar{i}}(|\lambda_1|^2)}{\lambda_1^2} + \varphi'' G^{i\bar{i}}(|\rho|^2)_1^2 \]
\[ + \frac{3\psi''}{4} \sum_{j} G^{i\bar{i}}(|e_i e_j u|^2 + |e_i \tilde{e}_j u|^2) + \psi'' G^{i\bar{i}}(|\partial u|^2)_1^2 \]
\[ - A e^{-A u} L(u) + A^2 e^{-A u} G^{i\bar{i}}|u|_1^2 - \frac{C}{\varepsilon} (1 + \varphi' N^2 + \psi' K) G. \]
It suffices to deal with the third and last term. For the third term, using (3.5) and the fact \( N \leq C A \lambda_1 \),
\[ (2 - \varepsilon) \varphi' \sum_{\alpha,\beta} G^{i\bar{i}}(u_{\alpha\beta})^2 \geq \sum_{\alpha,\beta} \frac{G^{i\bar{i}}(u_{\alpha\beta})^2}{20 N^2} \geq \sum_{\alpha,\beta} \frac{G^{i\bar{i}}(u_{\alpha\beta})^2}{C_A \lambda_1^2}. \]
For the last term, using (3.3) again we infer that
\[ - \frac{C}{\varepsilon} (1 + \varphi' N^2 + \psi' K) G \geq - \frac{C}{\varepsilon} G. \]
Combining the above inequalities, we conclude Proposition 3.3. □

3.2.2. Proof of Theorem 3.2. First, we define the index set
\[ J = \left\{ 1 \leq j \leq n : \frac{\psi''}{2} \sum_{i} (|e_i e_j u|^2 + |e_i \tilde{e}_j u|^2) \geq A^5 n e^{-5n u} K \text{ at } x_0 \right\}. \]
If \( J = \emptyset \), then Theorem 3.2 follows. So we assume \( J \neq \emptyset \) and let \( j_0 \) be the maximal element of \( J \). If \( j_0 < n \), we denote
\[ (3.19) \quad S = \left\{ j_0 \leq i \leq n - 1 : G^{i\bar{i}} \leq A^{-2} e^{2A u} G^{i+1\bar{i}+1} \text{ at } x_0 \right\}. \]
According to the index sets \( J \) and \( S \), the proof of Theorem 3.2 can be divided into three cases:
Case 1. \( j_0 = n \).
Case 2. \( j_0 < n \) and \( S = \emptyset \).
Case 3. \( j_0 < n \) and \( S \neq \emptyset \).

For Case 1 and Case 2, the proof in 3 is still valid in our setting, we shall omit it here. Now we only need to establish Case 3.

Observe that \( S \neq \emptyset \). Let \( i_0 \) be the minimal element of \( S \) and define
\[ I = \{ i_0 + 1, \ldots, n \}. \]
Let us decompose the term

\[(1 + \varepsilon) \sum_{i} G^{\tilde{i}} |(\lambda_{1})i|^{2} \]

(3.20) \[= (1 + \varepsilon) \sum_{i \not\in I} G^{\tilde{i}} |(\lambda_{1})i|^{2} + 3\varepsilon \sum_{i \in I} G^{\tilde{i}} |(\lambda_{1})i|^{2} + (1 - 2\varepsilon) \sum_{i \in I} G^{\tilde{i}} |(\lambda_{1})i|^{2} \]

\[= B_{1} + B_{2} + B_{3} \]

into three terms based on \(I\).

**Lemma 3.7.** At \(x_{0}\), we have

\[B_{1} + B_{2} \leq \frac{\psi' l}{4} \sum_{j} G^{\tilde{i}} |(e_{1} e_{j} u|^{2} + |e_{1} \tilde{e}_{j} u|^{2}) + \phi'' G^{\tilde{i}} |(|\rho|^{2})_{i}|^{2} \]

\[+ \psi'' G^{\tilde{i}} |(|\partial u|^{2})_{i}|^{2} + 9\varepsilon A^{2} e^{-2\varepsilon A} G^{i} |u_{i}|^{2}. \]

**Proof.** See the proof of [6, Lemma 4.6]. \(\square\)

3.2.3. **Calculations of \(B_{3}\).** We now devote to prove the following proposition.

**Proposition 3.8.** Let \(\varepsilon = \frac{\varepsilon_{A} \nu}{y}\). Then at \(x_{0}\), we have

\[B_{3} \leq (2 - \varepsilon) \sum_{\alpha > 1} \frac{G^{\tilde{i}} |e_{1}(u_{\alpha} v_{1})|^{2}}{\lambda_{1}(\lambda_{1} - \lambda_{\alpha})} - \frac{1}{\lambda_{1}} G^{\tilde{j}k j} V_{1}(\tilde{g}_{ik}) V_{1}(\tilde{g}_{ji}) \]

(3.21) \[+ (2 - \varepsilon) \phi'' \sum_{\alpha, \beta} G^{\tilde{i}} |e_{1}(u_{\alpha} \beta)|^{2} + \frac{C_{4}}{\varepsilon}. \]

Let us define

\[W_{1} = \frac{1}{\sqrt{2}} (V_{1} - \sqrt{-1} JV_{1}) = \sum_{q} \nu_{q} e_{q}, \quad JV_{1} = \sum_{\alpha > 1} \mu_{\alpha} v_{\alpha}, \]

where we used \(V_{1}\) is orthogonal to \(JV_{1}\). At \(x_{0}\), \(V_{1}\) and \(e_{q}\) are \(\chi\)-unitary, which implies

\[\sum_{q=1}^{n} |\nu_{q}|^{2} = 1, \quad \sum_{\alpha > 1} \mu_{\alpha}^{2} = 1. \]

**Lemma 3.9.** At \(x_{0}\), we have

1. \(\omega_{u} \geq -C_{A} K \chi\),
2. \(|\nu_{i}| \leq \frac{C_{A} K}{\lambda_{i}} \) for any \(i \in I\).

**Proof.** Recalling the definitions of \(i_{0}\) and \(j_{0}\), we deduce \(i_{0} + 1 > i_{0} \geq j_{0}\) and hence \(I \cap J = \emptyset\). Therefore,

\[\frac{\psi' b}{4} \sum_{j} (|e_{1} e_{j} u|^{2} + |e_{1} \tilde{e}_{j} u|^{2}) \leq A^{5} n e^{-5 A n} K, \quad \text{for each } i \in I. \]

(3.23)

Furthermore, \(n \in I\) implies \(e_{n} e_{n} u \geq -C_{A} K\) and

\[\tilde{g}_{n} = g_{n} + e_{n} e_{n} u + [e_{n}, e_{n}]^{(0, 1)} u + Z_{n} \geq e_{n} e_{n} u - CK \geq -C_{A} K. \]

Using this together with (3.7), we conclude (1). The proof of (2) can be found in [6, Lemma 4.8]. \(\square\)
Now we give the proof of Proposition 3.8.

**Proof of Proposition 3.8.** By the definition of $W_1$ in (3.22), we see that $V_1 = \sqrt{2W_1} - \sqrt{-1}JV_1$. This implies

$$e_i(uV_1V_n) = -\sqrt{-1} \sum_{\alpha>1} \mu_\alpha e_i(uV_1V_n) + \sqrt{2} \sum_q \nu_q V_1 e_i \tilde{e}_q u + O(\lambda_1)$$

$$= -\sqrt{-1} \sum_{\alpha>1} \mu_\alpha e_i(uV_1V_n) + \sqrt{2} \sum_{q \neq l} \nu_q V_1 (\tilde{g}_q) + \sqrt{2} \sum_{q \neq l} \nu_q V_1 e_i \tilde{e}_q u + O(\lambda_1).$$

Using this together with Cauchy-Schwarz inequality and Lemma 3.9, we have

$$B_\lambda \leq (1 - \varepsilon) \sum_{i \in I} \frac{G^\alpha}{\lambda_i^2} \left| -\sqrt{-1} \sum_{\alpha>1} \mu_\alpha e_i(uV_1V_n) + \sqrt{2} \sum_{q \neq l} \nu_q V_1 (\tilde{g}_q) \right|^2$$

$$+ \frac{C_A}{\varepsilon \lambda_i^2} \sum_{i \in I} \sum_{q \in I} \frac{G^\alpha |V_1 e_i \tilde{e}_q u|^2}{\lambda_i^2} + \frac{CG}{\varepsilon}.$$

(3.24)

For the second term in RHS of (3.24), observing that $|V_1 e_i \tilde{e}_q u| \leq C \sum_{\alpha,\beta} |e_i(u_{\alpha,\beta})| + C\lambda_1$, we deduce

$$\frac{C_A}{\varepsilon \lambda_i^2} \sum_{i \in I} \sum_{q \in I} \frac{G^\alpha |V_1 e_i \tilde{e}_q u|^2}{\lambda_i^2} \leq \frac{C_A}{\varepsilon \lambda_i^2} \sum_{\alpha,\beta} \frac{G^\alpha |e_i(u_{\alpha,\beta})|^2}{\lambda_i^2} + \frac{C_A}{\varepsilon \lambda_i^2} G.$$

(3.25)

Under the assumption $\lambda_1 \geq \frac{C_A}{\varepsilon}$, we obtain

$$\frac{C_A}{\varepsilon \lambda_i^2} \sum_{i \in I} \sum_{q \in I} \frac{G^\alpha |V_1 e_i \tilde{e}_q u|^2}{\lambda_i^2} \leq \sum_{\alpha,\beta} \frac{G^\alpha |e_i(u_{\alpha,\beta})|^2}{C_A \lambda_i^2} + G.$$

(3.26)

Now we deal with the first term in RHS of (3.24). For a constant $\gamma > 0$ to be chosen later, we see that

$$\sum_{i \in I} \frac{G^\alpha}{\lambda_i^2} - \sqrt{-1} \sum_{\alpha>1} \mu_\alpha e_i(uV_1V_n) + \sqrt{2} \sum_{q \neq l} \nu_q V_1 (\tilde{g}_q)$$

$$\leq \left( 1 + \frac{1}{\gamma} \right) \sum_{i \in I} \frac{G^\alpha}{\lambda_i^2} \left| \sum_{\alpha>1} \mu_\alpha e_i(uV_1V_n) \right|^2 + \left( 1 + \gamma \right) \sum_{i \in I} \frac{2G^\alpha}{\lambda_i^2} \left| \sum_{q \neq l} \nu_q V_1 (\tilde{g}_q) \right|^2.$$

(3.27)

Using the Cauchy-Schwarz inequality again, for the first term,

$$\left( 1 + \frac{1}{\gamma} \right) \sum_{i \in I} \frac{G^\alpha}{\lambda_i^2} \left| \sum_{\alpha>1} \mu_\alpha e_i(uV_1V_n) \right|^2$$

$$\leq \left( 1 + \frac{1}{\gamma} \right) \sum_{i \in I} \frac{G^\alpha}{\lambda_i^2} \left( \sum_{\alpha>1} (\lambda_1 - \lambda_\alpha) \mu_\alpha \right) \left( \sum_{\alpha>1} \frac{|e_i(uV_1V_n)|^2}{\lambda_1 - \lambda_\alpha} \right)$$

$$= \left( 1 + \frac{1}{\gamma} \right) \sum_{i \in I} \frac{G^\alpha}{\lambda_i^2} \left( \lambda_1 - \sum_{\alpha>1} \lambda_\alpha \mu_\alpha \right) \left( \sum_{\alpha>1} \frac{|e_i(uV_1V_n)|^2}{\lambda_1 - \lambda_\alpha} \right).$$
and for the second term,
\[
(1 + \gamma) \sum_{i \in I} \frac{2G\vec{\pi}^i}{\lambda_1^2} \left| \sum_{q \notin I} \nu_q V_i(\tilde{g}_{\pi I}) \right|^2 \\
\leq (1 + \gamma) \sum_{i \in I} \frac{2G\vec{\pi}^i}{\lambda_1^2} \left( \sum_{q \notin I} \frac{(\tilde{g}_{\pi I} - \tilde{g}_{\pi I})|\nu_q|^2}{G\vec{\pi}^i - G\vec{\pi}} \right) \left( \sum_{q \notin I} \frac{(G\vec{\pi}^i - G\vec{\pi})|V_i(\tilde{g}_{\pi I})|^2}{\tilde{g}_{\pi I} - \tilde{g}_{\pi}} \right).
\]

Recalling the definition of the index set I, when q \notin I and i \in I,
\[G\vec{\pi}^i \leq G^{\text{no}_0} \leq A^{-2}e^{2Au}G^{\text{no}_0+1} \leq A^{-2}e^{2Au}G\vec{\pi}^i.
\]
Combining this with Lemma 3.3,\[
0 < \frac{(\tilde{g}_{\pi I} - \tilde{g}_{\pi I})|\nu_q|^2}{G\vec{\pi}^i - G\vec{\pi}} \leq \frac{\tilde{g}_{\pi I}|\nu_q|^2 - \tilde{g}_{\pi I}|\nu_q|^2}{(1 - A^{-2}e^{2Au})G\vec{\pi}} < \frac{\tilde{g}_{\pi I}|\nu_q|^2 + C_A K}{(1 - A^{-2}e^{2Au})G\vec{\pi}}.
\]
In addition, from (3.28) and the concavity of f, we get
\[
-\frac{1}{\lambda_1}G^{i,k,jl} V_i(\tilde{g}_{ik}) V_j(\tilde{g}_{jl}) \geq \frac{2}{\lambda_1} \sum_{i \in I} \sum_{q \notin I} \left( \frac{G\vec{\pi}^i - G\vec{\pi}}{g_{\pi I} - g_{\pi I}} \right) |V_i(\tilde{g}_{\pi I})|^2.
\]
It follows from (3.28) and (3.29) that
\[
(1 + \gamma) \sum_{i \in I} \frac{2G\vec{\pi}^i}{\lambda_1^2} \left| \sum_{q \notin I} \nu_q V_i(\tilde{g}_{\pi I}) \right|^2 \\
\leq \frac{(1 + \gamma)}{\lambda_1(1 - A^{-2}e^{2Au})} \left( \sum_{q \notin I} \tilde{g}_{\pi I}|\nu_q|^2 + C_A K \right) \cdot \left\{ -\frac{1}{\lambda_1}G^{i,k,jl} V_i(\tilde{g}_{ik}) V_j(\tilde{g}_{jl}) \right\}.
\]
Since \(\varepsilon = \frac{A\text{no}^i}{\tilde{g}}\), when A is large enough one have
\[
\frac{(1 - \varepsilon)(1 + \gamma)}{\lambda_1(1 - A^{-2}e^{2Au})} \leq \left( 1 - \frac{\varepsilon}{2} \right) \frac{1 + \gamma}{\lambda_1}.
\]
Together with (3.29), (3.27), (3.30) and (3.31), we conclude (3.32)
\[
(1 - \varepsilon) \sum_{i \in I} \frac{G\vec{\pi}^i}{\lambda_1^2} \left| -\sqrt{-1} \sum_{\alpha > 1} \mu_{\alpha} e_i(u_{V_i} V_{\alpha}) + \sqrt{2} \sum_{q \notin I} \nu_q V_i(\tilde{g}_{\pi I}) \right|^2 \\
\leq (1 - \varepsilon) \left( 1 + \frac{1}{\gamma} \right) \sum_{i \in I} \frac{G\vec{\pi}^i}{\lambda_1^2} \left( \lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_{\alpha}^2 \right) \left( \sum_{\alpha > 1} \frac{|e_i(u_{V_i} V_{\alpha})|^2}{\lambda_1 - \lambda_\alpha} \right) \\
+ \frac{(1 - \varepsilon)(1 + \gamma)}{\lambda_1(1 - A^{-2}e^{2Au})} \left( \sum_{q \notin I} \tilde{g}_{\pi I} |\nu_q|^2 + C_A K \right) \cdot \left\{ -\frac{1}{\lambda_1}G^{i,k,jl} V_i(\tilde{g}_{ik}) V_j(\tilde{g}_{jl}) \right\} \\
\leq \frac{1 - \varepsilon}{(2 - \varepsilon)\lambda_1} \left( 1 + \frac{1}{\gamma} \right) \left( \lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_{\alpha}^2 \right) \cdot \left\{ (2 - \varepsilon) \sum_{\alpha > 1} \frac{G\vec{\pi}^i |u_{V_i} V_{\alpha}|^2}{\lambda_1(\lambda_1 - \lambda_\alpha)} \right\} \\
+ \left( 1 - \frac{\varepsilon}{2} \right) \left( \frac{1 + \gamma}{\lambda_1} \right) \left( \sum_{q \notin I} \tilde{g}_{\pi I} |\nu_q|^2 + C_A K \right) \cdot \left\{ -\frac{1}{\lambda_1}G^{i,k,jl} V_i(\tilde{g}_{ik}) V_j(\tilde{g}_{jl}) \right\}.
\]
Now we prove the following lemma:
Lemma 3.10. At \( x_0 \), we have

\[
(1 - \varepsilon) \sum_{i \in I} G_i^2 \left( - \frac{1}{\lambda_i} \right) \geq \sqrt{1 - \sum_{\alpha > 1} \mu_\alpha e_i(u_{V_i}), + \sqrt{2} \sum_{q \notin I} g_q V_i(\hat{g}_q)}^2 \leq (2 - \varepsilon) \sum_{\alpha > 1} \mu_\alpha G_i^2 u_{V_i}^2 - \frac{1}{\lambda_i} G_i^2 V_i(\hat{g}_j)(\hat{g}_j). \tag{3.33}
\]

Proof. In light of (3.32), it suffices to prove

a) \( \frac{1}{(1 + \frac{1}{\lambda_1})} (\lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2) \leq 1. \)

b) \( (1 - \frac{\varepsilon}{2}) (\sum_{q \notin I} \hat{g}_q \nu_q^2 + C_A K) \leq 1. \)

We shall consider the following two cases:

Case A. \( \frac{1}{2} (\lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2) > (1 - \frac{\varepsilon}{2}) (\sum_{q \notin I} \hat{g}_q \nu_q^2 + C_A K). \)

It follows from (3.22) that

\[
\frac{1}{2} (\lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2) > (1 - \frac{\varepsilon}{2}) (\sum_{q \notin I} \hat{g}_q \nu_q^2 + C_A K) \geq 0.
\]

In this case we set \( \gamma = \frac{\lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2}{\lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2}. \) Note that \( \lambda_1 > \lambda_2 \) at \( x_0 \) and so \( \gamma \) is positive. This concludes a) and b).

Case B. \( \frac{1}{2} (\lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2) \leq (1 - \frac{\varepsilon}{2}) (\sum_{q \notin I} \hat{g}_q \nu_q^2 + C_A K). \)

For a), by Lemma 3.9 we deduce

\[
\sum_{q \notin I} \hat{g}_q \nu_q^2 + C_A K \leq \sum_{q} \hat{g}_q \nu_q^2 + C_A K = \hat{g}(W_1, W_1) + C_A K \tag{3.34}
\]

where we used (3.22) in the last inequality. Combining this with the assumption of Case B, we see that

\[
\sum_{q \notin I} \hat{g}_q \nu_q^2 + C_A K \leq \frac{C_A K}{\varepsilon}. \tag{3.35}
\]

Using Lemma 3.9 again and (3.34),

\[
\frac{1}{2} (\lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2) \geq \hat{g}(W_1, W_1) - C K = \sum_{q} \hat{g}_q \nu_q^2 - C K \geq -C_A K,
\]

which implies \( 0 < \lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 \leq 2\lambda_1 + C_A K \leq (2 + 2\varepsilon^2)\lambda_1 \) under the assumption \( \lambda_1 \geq \frac{C_A K}{\varepsilon^2}. \) Letting \( \gamma = \varepsilon^{-2} \), then

\[
\frac{1 - \varepsilon}{(2 - \varepsilon)\lambda_1} \left( 1 + \frac{1}{\gamma} \right) (\lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2) \leq \frac{2 - 2\varepsilon}{2 - \varepsilon} (1 + \varepsilon^2)^2.
\]

Since \( \varepsilon = \frac{A \kappa(x_0)}{9} \), for a large \( A \) we get \( \frac{2 - 2\varepsilon}{2 - \varepsilon} (1 + \varepsilon^2)^2 \leq 1. \) This proves a).
For b), using (3.35) and $\gamma = \varepsilon^{-2}$,
\[
\left(1 - \frac{\varepsilon}{2}\right)\left(\frac{1+\gamma}{\lambda_1}\sum_{q \in I} g_q |\nu_q|^2 + C_A K\right) \leq \frac{C_A}{\varepsilon^3 \lambda_1}.
\]
This proves b) provided by $\lambda_1 \geq \frac{C_A}{\varepsilon^3 \lambda_1}$.

\[\square\]

Consequently, the Proposition 3.8 follows from (3.24), (3.25) and (3.33).

\[\square\]

Now we are return to prove Case 3 of Theorem 3.2.

**Proof of Case 3.** Using Proposition 3.3 together with Lemma 3.7 and Proposition 3.8, we deduce
\[
0 \geq (A^2 e^{-Au} - 9\varepsilon A^2 e^{-2Au}) G_i |u_i|^2 - \frac{C}{\varepsilon} G + \frac{\psi' \varepsilon}{4} \sum_j G_j^i (|e_i e_j u|^2 + |e_i \tilde{e}_j u|^2) - Ae^{-Au} L(u).
\]

Since $\varepsilon = \frac{\varepsilon^{Au(u_0)}}{9}$,
\[
(3.36) \quad 0 \geq - \frac{C}{\varepsilon} G + \frac{\psi' \varepsilon}{4} \sum_j G_j^i (|e_i e_j u|^2 + |e_i \tilde{e}_j u|^2) - Ae^{-Au} L(u).
\]

Let $A = \frac{10C}{\theta}$, where $\theta$ is the constant given in Proposition 2.2. There are two possibilities:

- $-L(u) \geq \theta G$. In this setting, (3.36) yields that
  
  \[
  0 \geq \left( A \theta e^{-Au} - \frac{C}{\varepsilon} \right) G + \frac{\psi' \varepsilon}{4} \sum_j G_j^i (|e_i e_j u|^2 + |e_i \tilde{e}_j u|^2).
  \]
  Using the fact $A = \frac{10C}{\theta}$, we deduce
  \[
  A \theta e^{-Au} - \frac{C}{\varepsilon} = A \theta e^{-Au} - 9Ce^{-Au} = Ce^{-Au},
  \]
  which implies
  \[
  0 \geq Ce^{-Au} G + \frac{\psi' \varepsilon}{4} \sum_j G_j^i (|e_i e_j u|^2 + |e_i \tilde{e}_j u|^2) > 0.
  \]
  This is impossible.

- $G^{ii} \geq \theta G$. Using the Cauchy-Schwarz inequality,
  \[
  Ae^{-Au} L(u) = Ae^{-Au} \sum_i G_i^i (e_i \bar{e}_i u - |e_i, \bar{e}_i|^{(0,1)} u + e_p(u) Z_{ii}^p + \tilde{e}_p(u) \bar{Z}_{ii}^p)\]
  \[
  \leq Ae^{-Au} G \sum_i |e_i \bar{e}_i u| + CAe^{-Au} K G
  \leq \frac{\theta \psi' \varepsilon}{8} G \sum_i |e_i \bar{e}_i u|^2 + CA K G.
  \]
Plugging it into (3.36),
\[ \frac{\theta \psi'}{8} \sum_{i,j} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \leq C_A K \]
and hence
\[ \sum_{i,j} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \leq C_A K^2. \]
This yields \( \lambda_1 \leq C_A K \) and the proof is completely.

Now we give the proof of Theorem 1.1.

**Proof.** Combining Proposition 3.1 and Theorem 3.3, we obtain Theorem 1.1.

### 3.3. Higher order estimates.

**Proposition 3.11.** Let \((M, \chi, J)\) be a compact almost Hermitian manifold of real dimension \(2n\). Suppose \(f\) satisfies (i), (ii) and (iii') on a symmetric open and convex cone \(\Gamma \subseteq \mathbb{R}^n\) as in (1.3). Assume \(u\) is a \(C\)-subsolution and \(u\) is a smooth solution of (1.1). Then for each \(k = 0, 1, 2, \ldots\), we have
\[ \|u\|_{C^k(M, \chi)} \leq C_k, \]
where \(C_k\) is a constant depending on \(k, \omega, h, Z, \Gamma\) and \((M, \chi, J)\).

**Proof of Proposition 3.11.** With the estimate (3.3) at hand, a standard blow-up argument [6, Proposition 5.1] combining with Liouville theorem [35, Theorem 20] (see also [12, 36, 40, 41]), we conclude \( \sup_M |\partial u| \leq C \). Although the appearance of the term \(Z\) which depends on \(\partial u\) linearly, it does not matter under the rescaling procedure. The more details can be found in [6, §5].

We can then apply the Evans-Krylov-type estimate (see [37, Theorem 1.1] and [6, §5]). The higher estimates can be obtained by applying a standard bootstrapping argument, we shall omit the standard step here.

### 3.4. Proof of Theorems 1.2-1.3

We remark that equation (1.4) and equation (1.3) satisfying the structural conditions (i), (ii) and (iii'). Using Proposition 3.11 and a similar arguments in the proof of [9, Theorem 1.1] and [6, Theorems 1.2-1.3], we obtain Theorems 1.2-1.3.

### 4. Proofs of Corollary 1.4

In this section, we prove Corollary 1.4. First, we give the \(C^1\) estimates of the dHYM equation (2.10).

**Proposition 4.1.** Let \(u\) (resp. \(\underline{u}\)) be the solution (resp. \(C\)-subsolution) for (2.10) with \(\sup_M (u - \underline{u}) = 0\). Then we have
\[ \|u\|_{C^1} \leq C, \]
where \(C\) depending on \(\omega, h, \|\omega\|_{C^1}, \Gamma\) and \((M, \chi, J)\).
Proof. Let us define

\[ H(\eta) = \frac{1}{3} e^{D\eta}, \quad \eta = u - u. \]

Here \( D > 0 \) are certain constants to be picked up later. Consider the test function

\[ Q = e^{H(\eta)}|\partial u|^2. \]

Suppose \( Q \) achieves maximum at the \( x_0 \in M \). We may assume \( |\partial u|(x_0) \geq 1 \). Otherwise we are done. Then near \( x_0 \), we can choose a proper local frame \( \{e_i\}_{i=1}^n \) such that \( \chi_{ij} = \delta_{ij} \) and the matrix \( \{\tilde{g}_{ij}\} \) is diagonal at \( x_0 \). It follows from maximum principle that

\[ 0 \geq L(Q)(x_0) = L(\eta) + D(1 + H)G^\tilde{u}|\eta|^2 + \frac{L(|\partial u|^2)}{DH}|\partial u|^2 \]

\[ + \frac{2}{|\partial u|^2} \sum_{i,j} G^\tilde{u} \text{Re} \left\{ e_i(\eta)\bar{e}_i e_j(u)\bar{e}_j(u) + e_i(\eta)\bar{e}_i e_j(u)e_j(u) \right\}. \]

By a similar argument to Lemma 3.6, we get

**Lemma 4.1.** At \( x_0 \), we have, for every \( \varepsilon \in (0, \frac{1}{2}) \),

\[ L(|\partial u|^2) \geq (1 - \varepsilon) \sum_j G^\tilde{u} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) - \frac{C}{\varepsilon} |\partial u|^2 G. \]

Dividing by \( DH|\partial u|^2 \), we have

\[ \frac{L(|\partial u|^2)}{DH|\partial u|^2} \geq (1 - \varepsilon) \sum_{i,j} G^\tilde{u} \frac{|e_i e_j u|^2 + |e_i \bar{e}_j u|^2}{|\partial u|^2} - \frac{C G}{DH \varepsilon}. \]

For the last term of (4.1), Note that \( \varepsilon \in (0, \frac{1}{2}) \) implies \( 1 \leq (1 - \varepsilon)(1 + 2\varepsilon) \). Using the definition of Lie bracket again, we see

\[ 2 \sum_{i,j} G^\tilde{u} \text{Re} \left\{ e_i(\eta)\bar{e}_i e_j(u)\bar{e}_j(u) \right\} \]

\[ = 2 \sum_{i,j} G^\tilde{u} \text{Re} \left\{ \eta_i u_j \left\{ e_j, \bar{e}_i \right\}(u) - \left\{ e_j, \bar{e}_i \right\}^{1,0}(u) \right\} \]

\[ = 2 \sum_i G^\tilde{u} \text{Re} \left\{ \eta_i u_j \left\{ e_j, \bar{e}_i \right\} \right\} - 2 \sum_{i,j} G^\tilde{u} \text{Re} \left\{ \eta_i u_j \left\{ e_j, \bar{e}_i \right\} \right\} \]

\[ \geq 2 \sum_i G^\tilde{u} \text{Re} \left\{ \eta_i u_j \right\} - \varepsilon DH|\partial u|^2 \sum_i G^\tilde{u} |\eta_i|^2 - \frac{C}{DH \varepsilon} |\partial u|^2 G \]

and

\[ 2 \sum_{i,j} G^\tilde{u} \text{Re} \left\{ e_i(\eta)\bar{e}_i e_j(u)e_j(u) \right\} \]

\[ \geq - \frac{(1 - \varepsilon)}{DH} \sum_{i,j} G^\tilde{u} |\bar{e}_i e_j(u)|^2 - (1 + 2\varepsilon) DH|\partial u|^2 \sum_i G^\tilde{u} |\eta_i|^2. \]

\[ \footnote{From now on, the C below denotes the constants those may change from line to line, and it doesn’t depend on D that we yet to choose.} \]
It follows from (4.3) and (4.4) that
\[
\frac{2}{|\partial u|^2} \sum_{i,j} G^{\bar{i}} \text{Re} \left\{ e_i(\eta) \bar{e}_j(u) \bar{e}_j(u) + e_i(\eta) \bar{e}_i(u) e_j(u) \right\}
\geq \frac{2}{|\partial u|^2} \sum_i G^{\bar{i}} (\mu_i - g_{\bar{i}i}) \text{Re} \{ e_i(\eta) \bar{e}_i(u) \} - \frac{CG}{DH \varepsilon}
\]
\[- (1 + 3\varepsilon) DH \sum_i G^{\bar{i}} |\eta_i|^2 - (1 - \varepsilon) \sum_{i,j} G^{\bar{i}} |\eta_i| |\bar{e}_j(u)|^2 \frac{DH}{|\partial u|^2}.
\]
Combining (4.1), (4.2) and (4.5), and letting \( \varepsilon = \frac{1}{6H(x_0)} \),
\[
L(\eta) + \frac{2}{|\partial u|^2} \sum_i G^{\bar{i}} (\mu_i - g_{\bar{i}i}) \text{Re} \{ \eta_i u_i \} + \frac{D}{2} \sum_i G^{\bar{i}} |\eta_i|^2 \leq \frac{C}{DH|\partial u|} + \frac{CG}{D}.
\]
By the assumption \( |\partial u| \geq \max\{1, |\partial u|\} \), we obtain
\[
\frac{2}{|\partial u|^2} \sum_i G^{\bar{i}} (\mu_i - g_{\bar{i}i}) \text{Re} \{ \eta_i u_i \}
\geq - \frac{D}{4} \sum_i G^{\bar{i}} |\eta_i|^2 - \frac{C}{D|\partial u|^2} \sum_i (\mu_i - 1)^2 \frac{1 + \mu_i^2}{1 + \mu_i^2}
\geq - \frac{D}{4} \sum_i G^{\bar{i}} |\eta_i|^2 - \frac{C}{D|\partial u|^2}.
\]
Hence,
\[
L(\eta) + \frac{D}{4} \sum_i G^{\bar{i}} |\eta_i|^2 \leq \frac{C}{DH|\partial u|} + \frac{CG}{D} + \frac{C}{D|\partial u|^2}.
\]
There are two possibilities:

- If (2.8) holds. It follows from (4.6) that
  \[
  \theta + \theta G \leq \frac{C}{DH|\partial u|} + \frac{CG}{D} + \frac{C}{D|\partial u|^2}.
  \]
  Choose \( D \) large such that \( \theta > \frac{C}{D} \). Then we get
  \[
  \theta \leq \frac{C}{DH|\partial u|} + \frac{C}{D|\partial u|^2}.
  \]
  This implies \( |\partial u| \leq C \).

- If (2.9) is true. By (2.7), we have \( G^{11} \geq \theta G \geq \theta \tau \). Therefore,
  \[
  \sum_i G^{\bar{i}} |\eta_i|^2 \geq \theta \tau |\partial \eta|^2,
  \]
  and
  \[
  L(\eta) = G^{\bar{i}} ((g_{\bar{i}i} + u_{\bar{i}i}) - \mu_i) \geq -C - C \sum_i \frac{\mu_i}{1 + \mu_i^2} \geq -C.
  \]
  Plugging the above two inequalities into (4.6),
  \[
  \frac{D}{C} |\partial \eta|^2 \leq \frac{C}{DH|\partial u|} + C.
  \]
We may assume that $|\partial u| \geq 2|\partial u|$ and then $|\partial \eta| \geq \frac{1}{2}|\partial u|$. So

$$\frac{D}{C}|\partial u|^2 \leq \frac{C}{DH|\partial u|} + C.$$ 

As a consequence, $|\partial u| \leq C$.

Combining the Theorem 1.1, we establish the second order estimates. Therefore, the equation (1.6) is uniform elliptic. Based on Evans-Krylov theory, we obtain the higher order estimates. This completes the proof of Corollary 1.4. □

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