Wave function renormalization constants and one-particle form factors in $D_l^{(1)}$ Toda field theories.

Vadim A. Brazhnikov *

Department of Physics, The University of Miami
James L. Knight Physics building
1320 Campo Sano Drive
Coral Gables, Fl 33146, USA

Abstract

We apply the method of angular quantization to calculation of the wave function renormalization constants in $D_l^{(1)}$ affine Toda quantum field theories. A general formula for the wave function renormalization constants in ADE Toda field theories is proposed. We also calculate all one-particle form factors and some of the two-particle form factors of an exponential field.

August, 1998

* e-mail address: vadim@physics.miami.edu
1. Introduction

Two dimensional affine Toda quantum field theories (ATQFT) have attracted a great deal of attention in recent years. These theories can be associated to any simple Lie algebra $\mathcal{G}$ of rank $l \geq 2$. Let $\{\vec{\alpha}_k\}_{k=1}^l$ be a set of positive simple roots of the Lie algebra $\mathcal{G}$, $C_{pq}$ is its Cartan matrix

$$C_{pq} = \frac{2 \vec{\alpha}_p \cdot \vec{\alpha}_q}{\vec{\alpha}_p \cdot \vec{\alpha}_p}.$$ (1.1)

and $\vec{\alpha}_0$ is the negative of the highest root

$$-\vec{\alpha}_0 = \sum_{k=1}^l n_k \vec{\alpha}_k.$$ (1.2)

The affine $\mathcal{G}$ Toda model describes the dynamics of an $l$-component real scalar field

$$\vec{\phi}(x) = \left(\phi^{(1)}(x), \ldots, \phi^{(l)}(x)\right).$$ (1.3)

governed by the Euclidean action

$$A = \int d^2 x \left\{ \frac{1}{8\pi} (\partial_{\nu} \vec{\phi})^2 + \mu \sum_{k=0}^l n_k e^{b \vec{\alpha}_k \vec{\phi}} \right\}.$$ (1.4)

where the integers $n_k$, $k = 1, \ldots, l$ are defined by (1.2), $n_0 = 1$ and $b$ is a real coupling constant. The affine Toda field theories are completely integrable at the classical level. The quantum integrability of the ATQFT is best understood for the simply laced algebras, though the exact $S$-matrices were proposed for all simple Lie algebras.

An important feature of the ATQFTs based on simply laced algebras is that the classical mass ratios are unaffected by renormalization. Another important property of the of these models is that the $S$-matrices are invariant under the duality transformation $b \to b^{-1}$. A possible way to study off-shell properties of ATQFTs is provided by a form factor approach originated in the works.

Recently a new powerful method has been designed for studying integrable two dimensional models. The key point of this method is to use angular quantization of the theories. In this approach the representation of the algebra of local fields is associated with a half-infinite line. The space of representation is usually called the angular quantization space. The roots of this approach go back to Baxter’s corner matrix method. The angular quantization space is a field-theoretical analog of the space where the lattice corner transfer matrix acts, and inherits the remarkable
features of the latter. Using this method, the form factors of exponential operators and
the wave functions renormalization constants for $A_l^{(1)}$ \text{ATQFT} were found in [26–27]. The
renormalization constants coincide with the ones calculated in [28]. The method was also
successfully applied [29] to the calculation of form factors in $A_2^{(2)}$ \text{ATQFT} [30]. It was
proposed in [31]–[29] that the angular quantization space for a massive integrable model
can be treated as a “scaling” limit of a representation of some “deformed” algebra. In this
limit the currents of the deformed algebra become Zamolodchikov-Faddeev (ZF) operators
corresponding to a diagonal scattering theory. In [32]–[29] [33] a striking similarity of the
free field representations of ZF operators and the Baxter $T−Q$ equation was observed.

In this paper we will deal with $D_1^{(1)}$ \text{ATQFT}. Using a free field representation of the
angular quantization space we obtain all one-particle and some of the two-particle form
factors of an exponential operator $e^{\vec{a} \vec{\phi}}$. This provides enough information to calculate
the wave function renormalization constants $Z_k$, $k = 1, \ldots, l$, which define residues of the
two-point functions of fields $\phi^{(k)}$ that diagonalize the mass matrix of (1.4)

$$
\langle \text{vac} | \phi^{(k)}(p) \phi^{(k)}(-p) | \text{vac} \rangle \to \frac{4\pi i Z_k}{p^2 - m_k^2 + i\epsilon}, \quad p^2 \to m_k^2 .
$$

(1.5)

where $m_k$ is the mass of the $k$-th particle (2.8) of the theory. The expression for $Z_k$ is
given by

$$
Z_k = \exp \left\{ -4 \int_0^\infty \frac{d\nu}{\nu} \sinh(\frac{\pi b\nu}{\hbar Q}) \sinh(\frac{\pi \nu}{\hbar Q b}) \left( \coth(\pi \nu) \left( C^{-1}(\nu) \right)_{kk} - e^{-\frac{\pi \nu}{\hbar}} \right) \right\} ,
$$

(1.6)

where $Q = b + b^{-1}$ and $h = 2(l - 1)$ is the dual Coxeter number. The matrix $C$ is a
“deformed” Cartan matrix

$$
C_{pq}(\nu) = 4 \sinh^2(\frac{\pi \nu}{2\hbar}) \delta_{pq} + C_{pq} .
$$

(1.7)

The renormalization constants for $A_l^{(1)}$ Toda model [28] can also be written in the form
(1.6) where the matrix $C$ is the “deformed” $A_l$ Cartan matrix. We expect that the formula
(1.6) is also valid for $E_l^{(1)}$ \text{ATQFT}.

The exact result for $Z_k$ (1.6) satisfy necessary field-theoretical requirements. In
particular, $0 < Z_k \leq 1$, and it matches the one loop perturbative check.
A plot of the functions $Z_1(b)$ for the first few $D_l^{(1)}$ Toda theories is given in Figure 1. The Figure 2 shows the functions $Z_k(b)$ for $D_6^{(1)}$ Toda theory. Note, that while $Z_1(b)$ is very close to 1, the deviation of $Z_5(b) = Z_6(b)$ from 1 is considerable.

Here is the layout of the paper. In Section 2 we introduce basic notations and facts about the $D_l^{(1)}$ Toda model. In Section 3 the method of reconstruction of form factors (the angular quantization method) is briefly described. In Section 4 we construct the free field representation for form factors and test it in Section 5 on the example of $D_4^{(1)}$ ATQFT. Section 6 contains the main results of the paper - the one-particle form factors and the wave function renormalization constants. Finally, we conclude with general remarks in Section 7.

### 2. S-matrix and spectrum of $D_l^{(1)}$ Toda field theory

The affine $D_l^{(1)}$ Toda model describes the dynamics of an $l$-component real scalar field $\vec{\varphi}(x)$ (1.3) governed by the Euclidean action (1.4). The vectors $\vec{\alpha}_k$, $k = 1, \ldots, l$ are the simple positive roots of the Lie algebra $D_l$, $\vec{\alpha}_k^2 = 2$. The integers $n_k = 1$ if $k = 0, 1, l, l-1$ and $n_k = 2$ otherwise. In terms of an orthonormal basis in $\mathbf{R}^l$

$$\vec{\varepsilon}_a \cdot \vec{\varepsilon}_b = \delta_{ab}, \quad a, b = 1, \ldots, l,$$  \hspace{1cm} (2.1)

the simple positive roots can be expressed as

$$\vec{\alpha}_k = \vec{\varepsilon}_k - \vec{\varepsilon}_{k+1}, \quad k = 1, \ldots, l-1,$$

$$\vec{\alpha}_l = \vec{\varepsilon}_{l-1} + \vec{\varepsilon}_l.$$  \hspace{1cm} (2.2)
The lagrangian (1.4) possesses explicit symmetry under the action of a group \( G = Z_2 \times Z_2 \) for even \( l \) or \( G = Z_4 \) if \( l \) is odd. It is convenient to introduce new fields \( \phi^{(k)}(x) \), \( k = 1, \ldots, l \) corresponding to the irreducible representations of the group \( G \):

\[
\varphi^{(k)} = 2h^{1/2} \sum_{p=1}^{l-2} \sin \left( \frac{2\pi p(k-1)}{h} \right) \phi^{(p)}, \quad k \neq 1, l, \quad (2.3)
\]

for even \( l \) we put

\[
\varphi^{(1)} = \frac{\phi^{(l)} + \phi^{(l-1)}}{\sqrt{2}}, \quad \varphi^{(l)} = (-1)^{l/2} \frac{\phi^{(l)} - \phi^{(l-1)}}{\sqrt{2}}, \quad (2.4)
\]

for odd \( l \) we put

\[
\varphi^{(1)} = \frac{\phi^{(l)} + \phi^{(l-1)}}{\sqrt{2}}, \quad \varphi^{(l)} = (-1)^{l+1} \frac{\phi^{(l)} - \phi^{(l-1)}}{i \sqrt{2}}. \quad (2.5)
\]

In terms of these new fields the mass matrix for the lagrangian (1.4) becomes diagonal. The action of the group \( G \) for even \( l \) is generated by the group elements \( g_1 \) and \( g_2 \), \( g_1^2 = g_2^2 = 1 \)

\[
g_1 : \quad \phi^{(k)} \rightarrow (-1)^k \phi^{(k)}, \quad k = 1, \ldots, l - 2, \quad \phi^{(l-1)} \rightarrow -(-1)^{l/2} \phi^{(l-1)}, \quad \phi^{(l)} \rightarrow (-1)^{l/2} \phi^{(l)}. \quad (2.6)
\]

\[
g_2 : \quad \phi^{(k)} \rightarrow \phi^{(k)}, \quad k = 1, \ldots, l - 2, \quad \phi^{(l-1)} \rightarrow -\phi^{(l-1)}, \quad \phi^{(l)} \rightarrow -\phi^{(l)}. \quad (2.6)
\]

For odd \( l \) the action of the group \( G \) is generated by the element \( g \), \( g^4 = 1 \)

\[
g : \quad \phi^{(k)} \rightarrow (-1)^k \phi^{(k)}, \quad k = 1, \ldots, l - 2, \quad \phi^{(l-1)} \rightarrow -i^l \phi^{(l-1)}, \quad \phi^{(l)} \rightarrow i^l \phi^{(l)}. \quad (2.7)
\]

The spectrum of the \( D_l^{(1)} \) ATQFT consists of \( l \) particles \( \{B_k\}_{k=1}^l \). These particles are in one-to-one correspondence with fundamental representations \( \{\pi_k\}_{k=1}^l \) of the Lie algebra \( D_l \) and their masses are given by

\[
m_k = 2m \sin \left( \frac{k\pi}{h} \right), \quad k = 1, \ldots, l - 2, \quad (2.8)
\]

\[
m_{l-1} = m_l = m.
\]
where \( h = 2(l - 1) \) is the dual Coxeter number. A linear basis in the physical Hilbert space \( \pi_A \) of the theory is provided by a set of asymptotic states

\[
| B_{k_1}(\theta_1) \ldots B_{k_n}(\theta_n) \rangle ,
\]  

(2.9)

where rapidities are ordered as \( \theta_1 > \ldots > \theta_n \).

The two-particle \( S \)-matrix, describing \( B_a B_b \rightarrow B_a B_b \) scattering was proposed in \cite{7}. In a compact form it can be written \cite{34}, \cite{33} as

\[
S_{ab}(\theta) = \exp \left\{ -4 \int_{-\infty}^{\infty} \frac{d\nu}{\nu} e^{i\nu \theta} \sinh(\frac{\pi b \nu}{hQ}) \sinh(\frac{\pi \nu}{hQb})(C^{-1}(\nu))_{ab} \right\} ,
\]  

(2.10)

where the matrix \( C \) was defined in (1.7) To analyze the analytical structure of the two-particle \( S \)-matrix it is useful to present its matrix elements as a product of meromorphic functions,

\[
\begin{align*}
S_{ab}(\theta) &= \prod_{p=1}^{\min(a,b)} \frac{F(\theta - i\frac{\pi}{h}(a + b + 1 - 2p))}{F(\theta + i\frac{\pi}{h}(a + b + 1 - 2p))} , \\ S_{l_a}(\theta) &= \prod_{p=0}^{a-1} \frac{F(\theta - i\frac{\pi}{h}(l - a + 2p))}{F(\theta + i\frac{\pi}{h}(l - a + 2p))} , \\ S_{l_2}(\theta) &= \prod_{p=0}^{[\frac{r_2}{2}]} \frac{F(\theta - i\frac{\pi}{h}(4p + 1))}{F(\theta + i\frac{\pi}{h}(4p + 1))} , \\ S_{l_2-1}(\theta) &= \prod_{p=1}^{[\frac{r_2-1}{2}]} \frac{F(\theta - i\frac{\pi}{h}(4p - 1))}{F(\theta + i\frac{\pi}{h}(4p - 1))} , \\ S_{l_1-1}(\theta) &= S_{l_2}(\theta), \\ S_{l_1}(\theta) &= S_{l_2}(\theta).
\end{align*}
\]  

(2.11)

Here the symbol \([a]\) stands for integer part of \( a \) and the rapidity variable \( \theta \) is defined as

\[
(p_n + p_k)^2 = m_n^2 + m_k^2 + 2m_nm_k \cosh(\theta) .
\]  

(2.12)

\footnote{Our convention for the normalization of the asymptotic states is

\[
\langle \text{vac} | \text{vac} \rangle = 1 , \quad \langle B_p(\theta) | B_k(\theta') \rangle = 2\pi \delta_{pk}\delta(\theta - \theta') .
\]}

\]
The functions \( F(\theta) \) and \( F(\theta) \) are given by

\[
F(\theta) = \frac{\tanh(\frac{\theta}{2} + \frac{i\pi}{2h} - \frac{i\pi}{2h}Q_b)}{\tanh(\frac{\theta}{2} - \frac{i\pi}{2h})} \frac{\tanh(\frac{\theta}{2} + \frac{i\pi}{2h} - \frac{i\pi}{2h}Q)}{\tanh(\frac{\theta}{2} - \frac{i\pi}{2h})},
\]

\[
F(\theta) = \frac{\sinh(\frac{\theta}{2} + \frac{i\pi}{2h} - \frac{i\pi}{2h}Q_b)}{\sinh(\frac{\theta}{2} - \frac{i\pi}{2h})} \frac{\sinh(\frac{\theta}{2} + \frac{i\pi}{2h} - \frac{i\pi}{2h}Q)}{\sinh(\frac{\theta}{2} - \frac{i\pi}{2h})}.
\]  

(2.13)

where \( Q = b + b^{-1} \).

In the physical strip \( 0 < \Im m \theta < \pi \) the amplitudes \((2.11)\) possess simple poles at \( \theta = i\theta_{ab}^c \) corresponding to the bound states of particles \( B_a \) and \( B_b \). Specifically,

(i) the amplitude \( S_{ll}(\theta) \) has simple poles at

\[
\theta = \frac{2i\pi(2k+1)}{h}, \quad k = 0, \ldots, \left[\frac{l-1}{2}\right] - 1
\]

(2.14)

which represent particles \( B_{l-2-2k} \);

(ii) the amplitude \( S_{ll-1}(\theta) \) has simple poles at

\[
\theta = \frac{4i\pi k}{h}, \quad k = 1, \ldots, \left[\frac{l}{2}\right] - 1
\]

(2.15)

which represent particles \( B_{l-3-2k} \);

(iii) the amplitudes \( S_{la}(\theta) \) and \( S_{l-1a}(\theta) \) have a simple pole at

\[
\theta = \frac{i\pi}{2} + \frac{i\pi a}{h}
\]

(2.16)

which represents particle \( B_{l-1} \) or \( B_l \) correspondingly;

(iv) the amplitudes \( S_{ac}(\theta), \ a \leq c \) have simple poles at

\[
\theta = \frac{i\pi(c + a)}{h}, \quad \theta = i\pi - \frac{i\pi(c - a)}{h}
\]  

(2.17)

representing particles \( B_{c+a} \) and \( B_{c-a} \).

There are also simple poles at \( \theta = i\pi - i\theta_{ab}^c \). These poles correspond to the particles in the cross channel. The analytical structure of the \( S \)-matrix respects the discrete \( G \)-symmetry \((2.6), (2.7)\) of the model.
3. Heuristic framework for form factors

Form factors are on-shell amplitudes of a local field $O$

$$F_O(\theta_1, ... \theta_n) = \langle \text{vac} | \pi_A(O) | B(\theta_1)...B(\theta_n) \rangle ,$$  \hspace{1cm} (3.1)

where the matrix of the field $O$ in the basis of asymptotic states is denoted by $\pi_A(O)$. The form factors satisfy a set of requirements [33] which constitute a complicated Riemann-Hilbert problem. In our calculation of form factors we will follow ideas of [27][29][33]. The main tool we will exploit is a special representation $\pi_Z$ of the formal Zamolodchikov-Faddeev algebra associated with the $S$-matrix (2.11). Defining properties of $\pi_Z$ were discussed in [19][29] and were motivated by form factor axioms [33]. In particular, if we denote by $B_k$ ZF operators acting in $\pi_Z$, they satisfy exchange relations

$$B_k(\theta_1)B_p(\theta_2) = S_{kp}(\theta_1 - \theta_2) B_p(\theta_2)B_k(\theta_1) , \hspace{1cm} \Im m (\theta_1 - \theta_2) = 0 .$$ \hspace{1cm} (3.2)

It is also assumed that there exists an operator $K$ acting in the space $\pi_Z$ in the following manner

$$B_k(\theta + \alpha) = e^{-\alpha K} B_k(\theta) e^{\alpha K} .$$ \hspace{1cm} (3.3)

Unitarity and crossing symmetry of the $S$-matrix allow us to equip $\pi_Z$ with a conjugation operation

$$B_k^+(\theta) = B_k(\theta + i\pi) , \hspace{1cm} K^+ = -K .$$ \hspace{1cm} (3.4)

There exists an embedding of the linear space of asymptotic states $\pi_A$ in the tensor product of $\pi_Z$ and its dual $\bar{\pi}_Z$,

$$\pi_A \hookrightarrow \bar{\pi}_Z \otimes \pi_Z .$$ \hspace{1cm} (3.5)

In other words, we can identify an arbitrary vector $|X\rangle \in \pi_A$ with some endomorphism (linear operator) $X$ of the space $\pi_Z$. To describe the embedding, we identify an arbitrary vector $|B_{k_1}(\theta_1)...B_{k_n}(\theta_n)\rangle \in \pi_A$ with an element of $\text{End}[\pi_Z]$ as

$$|B_{k_1}(\theta_1)...B_{k_n}(\theta_n)\rangle \equiv B_{k_1}(\theta_1)...B_{k_n}(\theta_n) e^{i\pi K} .$$ \hspace{1cm} (3.6)

The asymptotic states generate a basis in $\pi_A$, therefore (3.4) unambiguously specifies the embedding of the linear space. As well as $\pi_A$, the space $\bar{\pi}_Z \otimes \pi_Z$ possesses a canonical Hilbert space structure with the scalar product given by

$$\text{Tr}_{\pi_Z} \left[ Y^+ X \right] / \text{Tr}_{\pi_Z} \left[ e^{2\pi i K} \right] .$$
The conjecture that allows effective calculation of form factors is that the embedding \( \pi_A \hookrightarrow \bar{\pi}_Z \otimes \pi_Z \), preserves the structure of the Hilbert spaces,

\[
\langle Y | X \rangle = \text{Tr}_{\pi_Z} \left[ Y^+ X \right] / \text{Tr}_{\pi_Z} \left[ e^{2\pi i K} \right], \quad \text{if } | X \rangle \equiv X, \quad | Y \rangle \equiv Y. \quad (3.7)
\]

Let us define \( \pi_Z(\mathcal{O}) \in \text{End}[\pi_Z] \) associated with the state \( \pi_A(\mathcal{O})|\text{vac}\rangle \) in the following way

\[
\pi_A(\mathcal{O})|\text{vac}\rangle \equiv \pi_Z(\mathcal{O}) e^{\pi i K}. \quad (3.8)
\]

We also require that \( \pi_Z(\mathcal{O}) \in \text{End}[\pi_Z] \) associated with a local Hermitian field must commute with \( B(\theta) \),

\[
[\pi_Z(\mathcal{O}), B_k(\theta)] = 0. \quad (3.9)
\]

Using (3.7) the form-factors can be written as traces over the space \( \pi_Z \),

\[
F_{\mathcal{O}}(\theta_1, \ldots \theta_n) = \text{Tr}_{\pi_Z} \left[ e^{2\pi i K} \pi_Z(\mathcal{O}) B_{k_1}(\theta_1) \cdots B_{k_n}(\theta_n) \right] / \text{Tr}_{\pi_Z} \left[ e^{2\pi i K} \right]. \quad (3.10)
\]

4. Free field representation for form factors

To construct the representation \( \pi_Z \) for \( D_{l}^{(1)} \) ATQFT consider a set of oscillators

\[
\lambda_{\nu}^{(a)}, \quad a \in J = \{1, \ldots, l, \bar{l}, \ldots, \bar{1}\}. \quad (4.1)
\]

We specify an order in the set \( J \) as

\[
\bar{1} \succ \cdots \succ \bar{l} \succ \bar{l} - 1 \succ \cdots \succ 1. \quad (4.2)
\]

These oscillators satisfy commutation relations

\[
[\lambda_{\nu}^{(a)}, \lambda_{\nu'}^{(a)}] = \frac{4 \cosh(\pi \nu (\frac{1}{2} - \frac{1}{l}))}{\nu \cosh(\frac{\pi \nu}{2})} \sinh(\frac{\pi b \nu}{hQ}) \sinh(\frac{\pi \nu}{hQb}) \delta_{\nu + \nu', 0}, \quad (4.3a)
\]

\[
[\lambda_{\nu}^{(a)}, \lambda_{\nu'}^{(b)}] = -4\nu^{-1} \left\{ \frac{e^{\pi \nu} \sinh(\frac{\pi b \nu}{hQ})}{\cosh(\frac{\pi \nu}{2})} + \frac{e^{\pi \nu(2l-2a-1)}}{\cosh(\frac{\pi \nu}{2})} \right\} \sinh(\frac{\pi b \nu}{hQ}) \sinh(\frac{\pi \nu}{hQb}) \delta_{\nu + \nu', 0}, \quad (4.3b)
\]

\[
[\lambda_{\nu}^{(a)}, \lambda_{\nu'}^{(b)}] = -4 \epsilon(a, b) \frac{e^{(a, b) \pi \nu}}{\nu \cosh(\frac{\pi \nu}{2})} \sinh(\frac{\pi b \nu}{hQ}) \sinh(\frac{\pi \nu}{hQb}) \delta_{\nu + \nu', 0}, \quad b \neq \bar{a}, a
\]
The function $\epsilon(a,b)$ is a step function: $\epsilon(a,b) = -1$ if $a \prec b$ or $(a,b) = (l,\bar{l})$, and $\epsilon(a,b) = 1$ if $a \succ b$ or $(a,b) = (\bar{l},l)$. The “reflected” oscillators $\lambda^{(\bar{a})}_{\nu}$ are not independent and can be expressed in terms of $\lambda^{(a)}_{\nu}$,

$$\lambda^{(\bar{a})}_{\nu} = -\lambda^{(a)}_{\nu}e^{\frac{2\pi(l-a)\nu}{h}} - 2\sinh\left(\frac{\pi\nu}{h}\right)\sum_{k=1}^{a-1}\lambda^{(k)}_{\nu}e^{\frac{\pi(2l-2k-1)\nu}{h}}. \quad (4.4)$$

It is convenient to extend the Heisenberg algebra (4.3) by canonical conjugate pairs of operators $P^{(a)}$, $Q^{(a)}$ commuting with the oscillators $\lambda^{(a)}_{\nu}$,

$$[P^{(a)},Q^{(b)}] = -i(\delta_{a,b} - \delta_{a-1,b}), \quad b < n, \quad (4.5)$$

$$[P^{(a)},Q^{(n)}] = -i(\delta_{a,n} + \delta_{a-1,n}).$$

The extended algebra admits a representation in the direct sum of Fock spaces,

$$\pi_Z = \oplus_{\vec{p}} F_{\vec{p}}, \quad \text{where} \quad \vec{P} F_{\vec{p}} = \vec{p} F_{\vec{p}}. \quad (4.6)$$

and each of the spaces $F_{\vec{p}}$ is a span

$$F_{\vec{p}}: \oplus \lambda^{(a_1)}_{-\nu_1}\ldots\lambda^{(a_n)}_{-\nu_n}|\vec{p}\rangle, \quad \nu_k > 0. \quad (4.7)$$

The highest vector $|\vec{p}\rangle$ (not to be confused with the physical vacuum $|\text{vac}\rangle$ of the model) obeys the equations $\lambda^{(a)}_{\nu}|\vec{p}\rangle = 0, \nu > 0$. Let $\rho$ be a half sum of the positive roots of the Lie algebra $D_l$,

$$\rho \vec{\alpha}_k = 1, \quad k = 1, \ldots, l, \quad (4.8)$$

and $\vec{h}_a, a \in J$ are the weights of of the first fundamental representation $\pi_1$

$$\vec{h}_a = \vec{\varepsilon}_a, \quad a = 1, \ldots, l,$$

$$\vec{h}_{\bar{a}} = -\vec{\varepsilon}_{\bar{a}}. \quad (4.9)$$

Now we can define vertex operators $[27,30]$,

$$\Lambda_a(\theta) = \exp\left(\frac{2i\pi}{h}(\vec{p} - \frac{\vec{P}}{Q}) \cdot \vec{h}_a\right) : \exp\left(-i \int_{-\infty}^{+\infty} d\nu \lambda^{(a)}_{\nu}e^{i\nu(\theta - \frac{\vec{P}}{Q})}\right) : \quad (4.10)$$

which are in one-to-one correspondence with the weights of the representation $\pi_1$ and satisfy exchange relations

$$\Lambda_a(\theta_1) \Lambda_b(\theta_2) = S_{11}(\theta_1 - \theta_2) \Lambda_b(\theta_2) \Lambda_a(\theta_1), \quad 3m(\theta_1 - \theta_2) = 0. \quad (4.11)$$
The operators $\Lambda_\alpha(\theta)$ will be used to build ZF operators $\{B_k(\theta)\}_{k=1}^{l-2}$ associated with the vectorial representations $\{\pi_k\}_{k=1}^{l-2}$. The operators $B_{l-1}(\theta), B_l(\theta)$ associated with spinorial representations $\pi_{l-1}, \pi_l$ cannot be expressed solely in terms of $\Lambda_\alpha(\theta)$’s and we need to introduce additional vertex operators $A_k(\theta)$ and $Y_k(\theta)$ [33] corresponding to the simple positive roots $\vec{\alpha}_k$ and the highest weight vectors $\vec{\omega}_k$ of the fundamental representations $\pi_k$ of the Lie algebra $D_l$. Consider new oscillators

$$a^{(p)}_\nu = (\lambda^{(p)}_\nu - \lambda^{(l)}_\nu)e^{-\frac{\pi p \nu}{h}}, \quad p = 1, \ldots, l - 1,$$

$$a^{(l)}_\nu = (\lambda^{(l-1)}_\nu - \lambda^{(l)}_\nu)e^{-\frac{\pi (l-1) \nu}{h}}. \quad (4.12)$$

and

$$y^{(p)}_\nu = \sum_{k=1}^{l} \lambda^{(k)}_\nu e^{\frac{\pi (p-2k+1) \nu}{h}}, \quad p = 1, \ldots, l - 2,$$

$$y^{(l-1)}_\nu = (2 \cosh(\frac{\pi \nu}{h}))^{-1} \left( \lambda^{(1)}_\nu e^{\frac{\pi (l-3) \nu}{h}} + \cdots + \lambda^{(l-1)}_\nu e^{\frac{\pi (l-2) \nu}{h}} - \lambda^{(l)}_\nu e^{\frac{\pi (l-1) \nu}{h}} \right), \quad (4.13)$$

$$y^{(l)}_\nu = (2 \cosh(\frac{\pi \nu}{h}))^{-1} \left( \lambda^{(1)}_\nu e^{\frac{\pi (l-1) \nu}{h}} + \cdots + \lambda^{(l-1)}_\nu e^{\frac{\pi (l-2) \nu}{h}} + \lambda^{(l)}_\nu e^{\frac{\pi (l-1) \nu}{h}} \right).$$

The vertex operators $A_k(\theta)$ and $Y_k(\theta)$ are defined analogously to (4.10)

$$A_k(\theta) = \exp \left(\frac{2i\pi}{h} (\vec{\rho} - \frac{\vec{P}}{Q}) \cdot \vec{\alpha}_k \right) : \exp \left( -i \int_{-\infty}^{+\infty} d\nu a^{(k)}_\nu e^{i\nu(\theta - \frac{i\pi}{2})} \right) :,$$

$$Y_k(\theta) = \exp \left(\frac{2i\pi}{h} (\vec{\rho} - \frac{\vec{P}}{Q}) \cdot \vec{\omega}_k \right) : \exp \left( -i \int_{-\infty}^{+\infty} d\nu y^{(k)}_\nu e^{i\nu(\theta - \frac{i\pi}{2})} \right) :. \quad (4.14)$$

where the highest weight vectors $\vec{\omega}_p$ are

$$\vec{\omega}_p = \sum_{k=1}^{l} \vec{\varepsilon}_k, \quad p = 1, \ldots, l - 2,$$

$$\vec{\omega}_{l-1} = \frac{1}{2} (\vec{\varepsilon}_1 + \cdots + \vec{\varepsilon}_{l-1} - \vec{\varepsilon}_l),$$

$$\vec{\omega}_l = \frac{1}{2} (\vec{\varepsilon}_1 + \cdots + \vec{\varepsilon}_{l-1} + \vec{\varepsilon}_l). \quad (4.15)$$

Using results from Appendix A one can show that for $\Im m (\theta_1 - \theta_2) = 0$

$$A_b(\theta_1) A_c(\theta_2) = A_c(\theta_2) A_b(\theta_1),$$

$$A_b(\theta_1) Y_c(\theta_2) = Y_c(\theta_2) A_b(\theta_1), \quad (4.16)$$

$$Y_b(\theta_1) Y_c(\theta_2) = S_{bc}(\theta_1 - \theta_2) Y_c(\theta_2) Y_b(\theta_1).$$
Now we proceed to the construction of ZF operators $B_k(\theta)$. It was mentioned in the Introduction that the free field representations of ZF operators are similar to the Baxter $T-Q$ equations \[23\]. This observation is very useful in finding explicit forms of $B_k(\theta)$’s. Unfortunately, it seems that in the case of $D^{(1)}_l$ ATQFT the reconstruction of ZF operators from the Baxter equations is not straightforward. The general form of $B_k(\theta)$ remains obscure and the best we can do is to use an operator product expansion to generate ZF operators step by step.

First, following \[33\], define $B_1(\theta)$,

$$B_1(\theta) = Q \sqrt{\frac{\hbar \kappa_1}{2\pi}} \sum_{a \in J} \Lambda_a(\theta). \quad (4.17)$$

where $\kappa_1$ is some constant. Obviously, due to (4.11),

$$B_1(\theta_1) B_1(\theta_2) = S_{11}(\theta_1 - \theta_2) B_1(\theta_2) B_1(\theta_1). \quad (4.18)$$

We expect that operator products $B_p(\theta_1) B_1(\theta_2)$ develop simple poles in accordance with (2.17)

$$B_p(\theta_1) B_1(\theta_2) \to \frac{B_{p+1}(\theta_2 + \frac{i\pi}{\hbar})}{\theta_1 - \theta_2 - \frac{i\pi(p+1)}{\hbar}}, \quad \theta_2 \to \theta_1 - \frac{i\pi(p+1)}{\hbar}. \quad (4.19)$$

After some calculations one can find the next few operators. For example,

$$B_2(\theta) = Q \sqrt{\frac{\hbar \kappa_2}{2\pi}} \left\{ \sum_{\{a_1, a_2\} \in I} \gamma_{a_1 a_2} \Lambda_{a_1} \Lambda_{a_2} + \gamma \Lambda_{l-2} \Lambda'_{l-2} - \gamma \Lambda_{l-1} \Lambda'_{l-1} \right\}. \quad (4.20)$$

The $\kappa_p$’s are constants which at this point are irrelevant. The summation in (4.20) extends over the standard set $I$. We will call a set $\{a_1, \ldots, a_n\}$, $a_k \in J$ and a monom $\Lambda_{a_1} \cdots \Lambda_{a_k}$ standard if for any $k$ either $a_k \succ a_{k+1}$ or $(a_k, a_{k+1}) = (\bar{l}, l)$ or $(l, \bar{l})$. All monoms in (4.20) depend on $\theta$ in same way, for example

$$\Lambda_{a_1} \Lambda'_{a_2} \equiv \Lambda_{a_1}(\theta + \frac{i\pi}{\hbar}) \Lambda'_{a_2}(\theta - \frac{i\pi}{\hbar}). \quad (4.21)$$

\[2\] Another possible rout is to use a recurrent procedure of \[37\] to generate currents of deformed $W(D^{(1)}_l)$ algebra. Then, as it was proposed in \[29\], one might hope to recover ZF operators by taking a “scaling” limit of the currents.
The numerical coefficients $\gamma_{a_1 a_2}$ and $c$ are found to be

$$\gamma_{a_1 a_2} = \frac{(l - 1 - a - \frac{k}{h}) (l - 1 - a - \frac{1}{Q})}{(l - 1 - a) (l - 2 - a)}, \quad a \neq l - 1, l - 2,$$

$$\gamma_{l-2l-2} = \gamma_{l-1l-1} = 1 - \frac{1}{Q^2}, \quad \gamma_l = \gamma_{ll},$$ (4.22)

and $\gamma_{ab} = 1$ for the rest of the cases. The definition of $\gamma_{l-2l-2}$ and $\gamma_{l-1l-1}$ has an ambiguity. As a matter of fact some of the monoms can coincide due to (4.4). For example,

$$\Lambda_{l-p-1}(\theta + \frac{i\pi p}{h}) \Lambda_{l-p-1}(\theta - \frac{i\pi p}{h}) = \Lambda_{l-p}(\theta + \frac{i\pi p}{h}) \Lambda_{l-p}(\theta - \frac{i\pi p}{h}).$$ (4.23)

Therefore, only the sum $\gamma_{l-2l-2} + \gamma_{l-1l-1}$ enters the definition of $B_2(\theta)$. We choose $\gamma_{l-2l-2} = \gamma_{l-1l-1}$ for convenience. The identities (4.23) and similar identities allow also to cancel second and higher order poles in the operator product expansions of $B_k$’s. The explicit form of the operator $B_3(\theta)$ and the discussion of a general form of ZF operators are given in Appendix B.

The operators $B_{l-1}(\theta)$ and $B_l(\theta)$ cannot be obtained by the bootstrap procedure described above because the corresponding particles never appear as bound states of the particles $B_k$, $k = 1, \ldots, l - 2$. The weights of the spinorial representations $\pi_{l-1}$, $\pi_l$ of the Lie algebra $D_l$ are non-degenerate. Therefore, we expect, as in the case of $B_1(\theta)$, that the form of $B_{l-1}(\theta)$ and $B_l(\theta)$ is essentially the same as of the corresponding $T - Q$ equation [38][32]. For $D^{(1)}_4$ ATQFT ($h = 6$) we obtain

$$B_3(\theta) = Q \sqrt{\frac{h \kappa_3}{2\pi}}.$$

$$\left\{ \begin{array}{l}
Y_3(\theta) + :Y_3(\theta)A_3^{-1}(\theta - \frac{i\pi}{h}):\ + \\
:Y_3(\theta)A_3^{-1}(\theta - \frac{i\pi}{h})A_2^{-1}(\theta - \frac{2i\pi}{h}):\ + \\
:Y_3(\theta)A_3^{-1}(\theta - \frac{i\pi}{h})A_2^{-1}(\theta - \frac{2i\pi}{h})A_4^{-1}(\theta - \frac{3i\pi}{h}):\ + \\
:Y_3(\theta)A_3^{-1}(\theta - \frac{i\pi}{h})A_2^{-1}(\theta - \frac{2i\pi}{h})A_1^{-1}(\theta - \frac{3i\pi}{h})A_4^{-1}(\theta - \frac{3i\pi}{h}):\ +
\end{array} \right.$$

(4.24)
The operator $B_4(\theta)$ can be obtained from (4.24) by interchange of indices $3 \leftrightarrow 4$. To build $B_{l-1}(\theta)$ and $B_l(\theta)$ for general $D_l^{(1)}$ ATQFT one should use either the recurrent procedure of [38] or a general formula from [32].

In the following sections we are going to deal with one- and two-particle form factors of an exponential operator

$$\mathcal{O} = e^{\vec{a}\vec{\varphi}}. \quad (4.26)$$

To this end we need to specify the endomorphism $\pi_Z(e^{\vec{a}\vec{\varphi}}) \in \text{End}[\pi_Z]$ (3.8). It was observed in [27][29] that that for exponential operators the proper endomorphism is a projector on the Fock space $\mathcal{F}_{\vec{a}}$ with a given value of “zero modes”

$$\hat{\mathcal{P}} \mathcal{F}_{\vec{a}} = \vec{a} \mathcal{F}_{\vec{a}}. \quad (4.27)$$

We will see that this assertion is also true for $D_l^{(1)}$ ATQFT. With this choice of $\pi_Z(e^{\vec{a}\vec{\varphi}})$ we have

$$\langle \text{vac} | e^{\vec{a}\vec{\varphi}} | B(\theta_1) \ldots B(\theta_n) \rangle = \langle e^{\vec{a}\vec{\varphi}} \rangle \text{Tr}_{\mathcal{F}_{\vec{a}}} [e^{2\pi iK} B(\theta_1) \ldots B(\theta_n)]/\text{Tr}_{\mathcal{F}_{\vec{a}}} [e^{2\pi iK}],$$

where

$$\langle e^{\vec{a}\vec{\varphi}} \rangle = \text{Tr}_{\mathcal{F}_{\vec{a}}} [e^{2\pi iK}]/\text{Tr}_{\pi_Z} [e^{2\pi iK}] \quad (4.29)$$

is the one-point function of the exponential operator which was calculated in [39].

Before we go to the general case we will present results for the first nontrivial case - $D_4^{(1)}$ ATQFT. This will also provide a test for the free field representation constructed in this chapter.

5. $D_4^{(1)}$ Toda model

For any simple Lie algebra one can define characters $\chi_{\omega_p}(\vec{\lambda})$ of the $p$-fundamental representation $\pi_p$ by the formula

$$\chi_{\omega_p}(\vec{\lambda}) = \text{Tr}_{\pi_p} \left[ e^{2\pi i (\vec{\varphi} - \vec{\lambda}) \vec{H}} \right]. \quad (5.1)$$
Here \( \vec{H} = (H_1, \ldots, H_l) \) is a basis in the Cartan subalgebra of \( D_l \), normalized with respect to the Killing form \( \langle \cdot, \cdot \rangle \), \( \langle H_a, H_b \rangle = \delta_{ab} \).

Using the free field representation given in the last section we immediately derive the one-particle form factors,

\[
\langle \text{vac} | e^{\vec{a} \vec{\varphi}} | B_p(\theta) \rangle = \langle e^{\vec{a} \vec{\varphi}} \rangle Q \sqrt{\frac{3Z_p}{\pi}} \mathcal{X}_p(\vec{a}) .
\] (5.2)

The functions \( \mathcal{X}_p \) can be expressed in terms of the the characters of \( D_4 \)

\[
\mathcal{X}_p(\vec{\lambda}) = \chi_{\omega_p}(\vec{\lambda}), \quad p = 1, 3, 4
\] (5.3)

In the next section we will show that the constants \( Z_p \) are the wave functions renormalization constants and calculate them for general \( D_l^{(1)} \) ATQFT.

The calculation of two-particle form factors is also straightforward. In particular, we find form factors involving the particle \( B_1 \)

\[
\langle \text{vac} | e^{\vec{a} \vec{\varphi}} | B_1(\theta_1)B_3(\theta_2) \rangle = \langle e^{\vec{a} \vec{\varphi}} \rangle \frac{3Q^2}{\pi} \sqrt{Z_1Z_3} R_{13}(\theta) \left( \mathcal{X}_4 \mathcal{K}_4(\theta) + \mathcal{X}_1 \mathcal{X}_3 \right) ,
\]

\[
\langle \text{vac} | e^{\vec{a} \vec{\varphi}} | B_1(\theta_1)B_1(\theta_2) \rangle = \langle e^{\vec{a} \vec{\varphi}} \rangle \frac{3Q^2}{\pi} Z_1 R_{11}(\theta) \left( \mathcal{X}_2 \mathcal{K}_2(\theta) + \mathcal{X}_1^2 \right) ,
\]

\[
\langle \text{vac} | e^{\vec{a} \vec{\varphi}} | B_1(\theta_1)B_2(\theta_2) \rangle = \langle e^{\vec{a} \vec{\varphi}} \rangle \frac{3Q^2}{\pi} \sqrt{Z_1Z_2} R_{12}(\theta) .
\]

\[
\left( \tilde{\mathcal{X}}_3 \mathcal{K}_3(\theta) + \mathcal{X}_1 \mathcal{X}_2 - (1 - \eta) \mathcal{X}_1 \mathcal{K}_1(i\pi - \theta) \right) .
\] (5.4)

The functions \( R_{1p}(\theta), p = 1, 2, 3 \) are the “minimal” form factors \([14]\). They admit an integral representation

\[
R_{1p}(\theta) = \exp \left\{ - 2 \int_{-\infty}^{\infty} \frac{d\nu}{\nu} e^{i\nu(\theta - i\pi)} \frac{\sinh(\frac{\pi b \nu}{hQ}) \sinh(\frac{\pi \nu}{hQ}) \cosh(\frac{\pi \nu(\frac{1}{2} - \frac{1}{h})}{h})}{\sinh(\pi \nu) \cosh(\frac{\pi \nu}{2})} \right\} .
\] (5.5)

The representation (5.3) is valid in the strip

\[
-\frac{\pi(p-1)}{h} < \Im \theta < 2\pi + \frac{\pi(p-1)}{h} ,
\] (5.6)

while outside the strip it must be understood in a sense of analytical continuation. In (5.3), (5.6) \( h = 6 \). The constant \( \eta \), the “character” \( \tilde{\mathcal{X}}_3 \) and the functions \( \mathcal{K}_p(\theta) \) are given
by
\[ \tilde{X}_3 = \chi_{\omega_3 + \omega_4} + (1 + \eta) \chi_{\omega_1}, \]
\[ \eta = \frac{4}{\sqrt{3}} \sinh\left(\frac{i\pi}{6Qb}\right) \sinh\left(\frac{i\pi b}{6Q}\right), \]
\[ \mathcal{K}_p(\theta) = -\frac{2i \sinh\left(\frac{i\pi}{6Qb}\right) \sinh\left(\frac{i\pi b}{6Q}\right) \sinh\left(\frac{i\pi p}{6}\right)}{\sinh\left(\frac{\theta}{2} + \frac{i\pi p}{12}\right) \sinh\left(\frac{\theta}{2} - \frac{i\pi p}{12}\right)}. \] (5.7)

The $D_4^{(1)}$ Toda model possesses a symmetry under the action of the permutation group $G = S_3$. The group $G$ permutes particles $B_1, B_3, B_4$ or, equivalently, it acts as a permutation on the set of weights $\{\omega_1, \omega_3, \omega_4\}$. We can use this symmetry to obtain the rest of the form factors except for $\langle \text{vac} | e^{\vec{a} \vec{\varphi}} | B_2(\theta_1)B_2(\theta_2) \rangle$. The expression for it is somewhat complicated and we do not present it here.

One can show that for $j = 0, \ldots, 4$
\[ \chi_{\omega_1}(\lambda \vec{a}_j) = 8 \sinh\left(\frac{i\pi \lambda}{6}\right) \sinh\left(\frac{i\pi (1 - \lambda)}{6}\right) \cosh\left(\frac{i\pi (2j - 1)}{3}\right), \]
\[ \chi_{\omega_2}(\lambda \vec{a}_j) = 8\sqrt{3} \sinh\left(\frac{i\pi \lambda}{6}\right) \sinh\left(\frac{i\pi (1 - \lambda)}{6}\right) \cosh\left(\frac{i\pi (2j - 1)}{3}\right) + 16 \sinh^2\left(\frac{i\pi \lambda}{6}\right) \sinh^2\left(\frac{i\pi (1 - \lambda)}{6}\right) - 1, \] (5.8)

and
\[ \chi_{\omega_3}(\lambda \vec{a}_0) = -\chi_{\omega_3}(\lambda \vec{a}_1) = -\chi_{\omega_3}(\lambda \vec{a}_3) = \chi_{\omega_3}(\lambda \vec{a}_4) = \]
\[ \chi_{\omega_4}(\lambda \vec{a}_0) = -\chi_{\omega_4}(\lambda \vec{a}_1) = \chi_{\omega_4}(\lambda \vec{a}_3) = -\chi_{\omega_4}(\lambda \vec{a}_4) = 4\sqrt{3} \sinh\left(\frac{i\pi \lambda}{6}\right) \sinh\left(\frac{i\pi (1 - \lambda)}{6}\right), \]
\[ \chi_{\omega_3}(\lambda \vec{a}_2) = \chi_{\omega_4}(\lambda \vec{a}_2) = 0, \] (5.9)

This allows to check that all one- and two-particle form factors satisfy the quantum equations of motion
\[ \partial_\mu \partial^\mu (\vec{\alpha}_j \vec{\varphi}) = \mathcal{M}^2 \sum_{k=1}^l C_{jk} n_k (e^{b \vec{\alpha}_k \vec{\varphi}} - e^{b \vec{\alpha}_0 \vec{\varphi}}). \] (5.10)

with
\[ \mathcal{M}^2 = \frac{\pi m^2 \sin\left(\frac{\pi}{h}\right)}{2\hbar OQ \sin\left(\frac{\pi}{hQ}\right) \sin\left(\frac{\pi}{hQ^0}\right)}, \] (5.11)

and $O = \langle e^{b \vec{\alpha}_j \vec{\varphi}} \rangle$, $j = 0, \ldots, l$, $h = 6$. We intentionally keep the general notation for the dual Coxeter number $h$ in (5.5), (5.11) because, as we will see in the next section, the formulae are valid for general $h$. 

15
Note also that the two-particle form factors exhibit the cluster property \[40\]
\[
\langle \text{vac} | e^{\vec{a}\vec{\phi}} | B_k(\theta_1)B_p(\theta_2) \rangle \to \frac{\langle \text{vac} | e^{\vec{a}\vec{\phi}} | B_k \rangle \langle \text{vac} | e^{\vec{a}\vec{\phi}} | B_p \rangle}{\langle e^{\vec{a}\vec{\phi}} \rangle},
\]
as \( |\theta_1 - \theta_2| \to \infty \).

6. The wave function renormalization constants

To find the wave function renormalization constants we need an explicit form of one- and two-particle form factors. Although a general form of ZF operators is not known, it is still possible to carry out calculations building \( B_k(\theta) \) one by one and analyzing the general case using formulae from Appendices B and C. The results can be summarized as follows.

6.1. The one-particle form factors

The one-particle form factors for \( D_1^{(1)} \) ATQFT are given by
\[
\langle \text{vac} | e^{\vec{a}\vec{\phi}} | B_p(\theta) \rangle = \langle e^{\vec{a}\vec{\phi}} \rangle Q \sqrt{\frac{h Z_p}{2\pi}} \mathcal{X}_p \left( \frac{\vec{a}}{Q} \right). \tag{6.1}
\]
The functions \( \mathcal{X}_p \) can be expressed in terms of the the characters of \( D_1 \),
\[
\mathcal{X}_p(\vec{\lambda}) = \sum_{s=0}^{[\frac{l}{2}]} \zeta_{p \omega p \omega} \chi_{\omega p \omega} (\vec{\lambda}) , \quad p = 1, \ldots, l - 2 ,
\tag{6.2}
\]
\[
\mathcal{X}_p(\vec{\lambda}) = \chi_{\omega p} (\vec{\lambda}) , \quad p = l - 1, l.
\]
Unfortunately, the constants \( \zeta_{pk} \) cannot be calculated in a closed form. Instead, we found the following representation for them,
\[
\zeta_{pk} = 1 + \xi_{pp-2} + \xi_{pp-4} + \cdots + \xi_{pk} , \tag{6.3}
\]
where \( \xi_{pk} \) is obtained by recursion,
\[
\xi_{pk} = \left[ \frac{h}{Q} \right]_x \left[ \frac{1}{Q} \right]_x \sum_{s=1}^{\frac{p-k}{2}} \zeta_{p-k+2s} \left[ k \right]_x \left[ k + s \right]_x \left[ s \right]_x \left[ p - k \right]_x \left[ p + k \right]_x. \tag{6.4}
\]
Here a notation \([ \cdot ]_x \) was introduced,
\[
[a]_x = x^a - x^{-a} , \quad x = \frac{i\pi}{h}. \tag{6.5}
\]
In the above formulae we set \( \xi_{pp} \equiv 1 \) and one can also find that \( \xi_{p0} \equiv 0 \). In particular it means that \( \zeta_{p} \equiv 0 \).

Taking a limit of small \( \vec{a} \) in (5.1) we find the one-particle form factors of the field \( \vec{\varphi} \) itself,

\[
\langle \text{vac} | \vec{\alpha} j \vec{\varphi} | B_p(\theta) \rangle = -\sqrt{\frac{2\pi Z_p \hbar}{\kappa}} \mathcal{R}_p(j) .
\] (6.6)

The functions \( \mathcal{R}_p(j) \) are found to be

\[
\mathcal{R}_p(j) = -i \sum_{k=1}^{p} \xi_{pk} \frac{[k]_x [2k(2j - 1)]_x}{[k(2j - 1)]_x} , \quad p = 1, \ldots, l - 2 ,
\] (6.7)

and \( \mathcal{R}_{l-1}(j), \mathcal{R}_{l}(j) \) take a form

\[
\mathcal{R}_{l-1}(0) = -\mathcal{R}_{l-1}(1) = -i^{-l} \mathcal{R}_{l-1}(l - 1) = i^{-l} \mathcal{R}_{l-1}(l) = \sqrt{\frac{h}{2}} ,
\]
\[
\mathcal{R}_{l}(0) = -\mathcal{R}_{l}(1) = i^{-l} \mathcal{R}_{l}(l - 1) = -i^{-l} \mathcal{R}_{l}(l) = \sqrt{\frac{h}{2}} ,
\]
\[
\mathcal{R}_{l-1}(j) = \mathcal{R}_{l}(j) \equiv 0 , \quad j = 2, \ldots, l - 2 .
\] (6.8)

In the course of derivation of (6.7),(6.8) we used the generating function,

\[
G(t, j, \lambda) = (1 - e^{2t}) \frac{[t + j - \lambda]_x [t - j + \lambda]_x [t + j - 1 + \lambda]_x [t - j + 1 - \lambda]_x}{[t + j]_x [t - j]_x [t + j - 1]_x [t - j + 1]_x} .
\] (6.9)

The first \( l - 2 \) terms of the expansion of \( G(t, j, \lambda) \),

\[
G(t, j, \lambda) = \sum_{s=0}^{\infty} e^{st} \chi_{\omega_s}(\lambda \vec{\alpha}_j) .
\] (6.10)

give the values of \( \chi_{\omega_s}(\lambda \vec{\alpha}_j) \) for \( s = 0, \ldots, l - 2 \). The values of \( \chi_{\omega_{l-1}}(\lambda \vec{\alpha}_j) \) and \( \chi_{\omega_l}(\lambda \vec{\alpha}_j) \) can be found explicitly,

\[
\chi_{\omega_{l-1}}(\lambda \vec{\alpha}_0) = -\chi_{\omega_{l-1}}(\lambda \vec{\alpha}_1) = -i^{-l} \chi_{\omega_{l-1}}(\lambda \vec{\alpha}_{l-1}) = i^{-l} \chi_{\omega_{l-1}}(\lambda \vec{\alpha}_l) =
\]
\[
\chi_{\omega_l}(\lambda \vec{\alpha}_0) = -\chi_{\omega_l}(\lambda \vec{\alpha}_1) = i^{-l} \chi_{\omega_l}(\lambda \vec{\alpha}_{l-1}) = -i^{-l} \chi_{\omega_l}(\lambda \vec{\alpha}_l) =
\]
\[
\frac{i\sqrt{2\hbar}}{\sinh(\frac{\pi \lambda}{\hbar})} \frac{\sinh(\frac{\pi(1-\lambda)}{\hbar})}{\sinh(\frac{\pi}{\hbar})} ,
\]
\[
\chi_{\omega_{l-1}}(\lambda \vec{\alpha}_j) = \chi_{\omega_l}(\lambda \vec{\alpha}_j) \equiv 0 , \quad j = 2, \ldots, l - 2 .
\] (6.11)

One can show, using (6.9) and (6.11), that the one-particle form factors satisfy the quantum equations of motion (5.10) for general \( h \).
The fields $\phi^{(a)}$ are normalized in accordance with the short distance behavior,

$$
\langle \text{vac} | \phi^{(a)}(x) \phi^{(b)}(y) | \text{vac} \rangle = -2\delta^{ab} \log(M|x - y|) + O(1), \quad |x - y| \to 0. \quad (6.12)
$$

This normalization and (2.4) - (2.5), (6.6) allow to identify $Z_p$ in (6.1) as the wave function renormalization constants and to find one-particle form factors of the field $\vec{\phi}$

$$
\langle \text{vac} | \phi^{(k)} | B_p \rangle = \sqrt{2\pi Z_p \xi_{pk}}, \quad p = 1, \ldots, l - 2,
$$

$$
\langle \text{vac} | \phi^{(k)} | B_p \rangle = \sqrt{2\pi Z_p \delta_{pk}}, \quad p = l - 1, l. \quad (6.13)
$$

6.2. Two-particle form factors

For the purpose of calculating $Z_p$ it is enough to calculate the simplest two-particle form factors

$$
\langle \text{vac} | e^{i\vec{\phi}} | B_1(\theta_1)B_1(\theta_2) \rangle = \langle e^{i\vec{\phi}} \rangle \frac{hQ^2}{2\pi} \tilde{Z}_{11} \mathcal{R}_{11}(\theta) \left( \chi_2 \kappa_2(\theta) + \chi_1^2 \right),
$$

$$
\langle \text{vac} | e^{i\vec{\phi}} | B_1(\theta_1)B_2(\theta_2) \rangle = \langle e^{i\vec{\phi}} \rangle \frac{hQ^2}{2\pi} \sqrt{Z_2Z_1} \mathcal{R}_{21}(\theta).
$$

$$
\langle \text{vac} | e^{i\vec{\phi}} | B_1(\theta_1)B_{l-2}(\theta_2) \rangle = \langle e^{i\vec{\phi}} \rangle \frac{hQ^2}{2\pi} \sqrt{Z_1Z_{l-2}} \mathcal{R}_{11}^{2}(\theta) \left( \chi_{l-1} \kappa_{l-1}(\theta) + \chi_{l-1} \chi_{l-2} \right),
$$

$$
\langle \text{vac} | e^{i\vec{\phi}} | B_1(\theta_1)B_{l-1}(\theta_2) \rangle = \langle e^{i\vec{\phi}} \rangle \frac{hQ^2}{2\pi} \sqrt{Z_1Z_l} \mathcal{R}_{11}(\theta) \left( \chi_{l-1} \kappa_{l-1}(\theta) + \chi_{l-1} \chi_{l} \right).
$$

(6.14)

and form factors involving both of the particles $B_{l-1}$ and $B_l$

$$
\langle \text{vac} | e^{i\vec{\phi}} | B_{l-1}(\theta_1)B_{l-1}(\theta_2) \rangle =
$$

$$
\langle e^{i\vec{\phi}} \rangle \frac{hQ^2}{2\pi} \tilde{Z}_{l-1} \mathcal{R}_{l-1l}(\theta) \left( \sum_{p=1}^{l-1} \left( \prod_{k=1}^{p-1} \eta_{l-1-2k} \chi_{l-2p} \kappa_{4p-2}(\theta) + \chi_{l-1}^2 \right) \right),
$$

$$
\langle \text{vac} | e^{i\vec{\phi}} | B_{l}(\theta_1)B_{l}(\theta_2) \rangle =
$$

$$
\langle e^{i\vec{\phi}} \rangle \frac{hQ^2}{2\pi} \tilde{Z}_{l} \mathcal{R}_{l}(\theta) \left( \sum_{p=1}^{l} \left( \prod_{k=1}^{p-1} \eta_{l-1-2k} \chi_{l-2p} \kappa_{4p-2}(\theta) + \chi_{l}^2 \right) \right),
$$

$$
\langle \text{vac} | e^{i\vec{\phi}} | B_{l-1}(\theta_1)B_{l}(\theta_2) \rangle =
$$

$$
\langle e^{i\vec{\phi}} \rangle \frac{hQ^2}{2\pi} \sqrt{Z_1Z_l} \mathcal{R}_{l-11}(\theta) \left( \sum_{p=1}^{l} \left( \prod_{k=1}^{p-1} \eta_{l-1-2k} \chi_{l-1-2p} \kappa_{4p}(\theta) + \chi_{l-1} \chi_{l} \right) \right).
$$

(6.15)
where the functions $\mathcal{K}_p(\theta)$, the constants $\eta_p$ and the “character” $\tilde{X}_{l-1}(\tilde{\lambda})$ are similar to the ones introduced in the case of the $D_4^{(1)}$ Toda model:

$$
\tilde{X}_{l-1}(\tilde{\lambda}) = \chi_{\omega_l + \omega_{l-1}}(\tilde{\lambda}) + \sum_{s=1}^{[\frac{l-1}{2}]} \zeta_{p-2s} \chi_{\omega_{p-2s}}(\tilde{\lambda}) ,
$$

$$
\eta_p = \frac{[l-1-p-\frac{1}{Q^0}]_x [l-1-p-b]_x}{[l-1-p]_x [l-p]_x} ,
$$

$$
\mathcal{K}_p(\theta) = \frac{[p]_x}{4[1]_x} \frac{[\frac{1}{Q^0}]_x [\frac{b}{Q}]_x}{\sinh(\frac{\theta}{2} + \frac{i\pi p}{2h}) \sinh(\frac{\theta}{2} - \frac{i\pi p}{2h})} .
$$

The “minimal” form factors $R_{1p}(\theta)$, $p = 1, \ldots, l-1$, $R_{l-1}(\theta) = R_{1l}(\theta)$ are given by (5.5) with general $h$. For $R_{l-1l}(\theta)$ and $R_{l-1l-1}(\theta) = R_{ll}(\theta)$ we have

$$
R_{l-1l}(\theta) = \exp \left\{ - \int_{-\infty}^{\infty} \frac{d\nu}{\nu} e^{i\nu(\theta - i\pi)} \frac{\sinh(\frac{\pi\nu}{hQ}) \sinh(\frac{\nu(l-2)}{hQ})}{\sinh(\pi\nu) \cosh(\frac{\pi\nu}{2}) \sinh(\frac{2\pi\nu}{h})} \right\} ,
$$

$$
R_{ll}(\theta) = \exp \left\{ - \int_{-\infty}^{\infty} \frac{d\nu}{\nu} e^{i\nu(\theta - i\pi)} \frac{\sinh(\frac{\pi\nu}{hQ}) \sinh(\frac{\nu l}{hQ})}{\sinh(\pi\nu) \cosh(\frac{\nu}{2}) \sinh(\frac{2\nu}{h})} \right\} .
$$

The dependence on the vector $\vec{a}$ enters the form factors through the “characters” $X_p(\frac{\vec{a}}{Q})$.

6.3. The calculation of the renormalization constants $Z_p$

The form factors (6.14), (6.15) satisfy crossing symmetry and the Watson equations [35] and exhibit the required analytical structure. In particular, they have simple poles at the points $\theta = i\theta_{ab}^c$ (2.16), (2.17) which correspond to bound states of particles. The singularity of the form factors

$$
\langle \text{vac} | e^{i\vec{a}\vec{\varphi}} | B_a(\theta_1)B_b(\theta_2) \rangle \rightarrow \frac{i\Gamma_{ab}^c}{\theta_1 - \theta_2 - i \theta_{1a}^b} \langle \text{vac} | e^{i\vec{a}\vec{\varphi}} | B_c \rangle , \quad \theta \rightarrow i\theta_{ab}^c ,
$$

defines [35] the residue of the two-particle $S$-matrix

$$
S_{ab}(\theta) \rightarrow \frac{i \left( \Gamma_{ab}^c \right)^2}{\theta - i \theta_{1a}^b} , \quad \theta \rightarrow i\theta_{ab}^c .
$$
Comparison of $\Gamma_{ab}$ obtained from the $S$-matrix and the form factor singularities leads to the equations

\[
\frac{Z_1 Z_p}{Z_{p+1}} = \frac{4i \pi}{h Q^2} \left[ \begin{array}{c} 2 \\ \frac{b}{Q} \end{array} \right] \left[ \begin{array}{c} 2 \\ \frac{Qb}{Q} \end{array} \right] \mathcal{F}(i \pi - \frac{i \pi (2p+1)}{h}) \frac{\mathcal{R}_{2 p} \left( \frac{i \pi (p+1)}{h} \right)}{\mathcal{R}_{2 p} \left( \frac{i \pi (p-1)}{h} \right)},
\]

\[
\frac{Z_1 Z_p}{Z_{p-1}} = \frac{4i \pi}{h Q^2} \left[ \begin{array}{c} 2 \\ \frac{b}{Q} \end{array} \right] \left[ \begin{array}{c} 2 \\ \frac{Qb}{Q} \end{array} \right] \mathcal{F}(i \pi - \frac{i \pi (2p-1)}{h}) \frac{\mathcal{R}_{2 p} \left( \frac{i \pi (p-1)}{h} \right)}{\mathcal{R}_{2 p} \left( \frac{i \pi (p+1)}{h} \right)},
\]

\[
Z_l^2 = \frac{2i \pi}{h Q^2} \left[ \begin{array}{c} 1 \\ \frac{b}{Q} \end{array} \right] \left[ \begin{array}{c} 1 \\ \frac{Qb}{Q} \end{array} \right] \mathcal{R}_{1 l} \left( \frac{i \pi (4p-2)}{h} \right) \prod_{k=0}^{p-2} F \left( \frac{i \pi (4p-4k-3)}{h} \right) \prod_{k=p}^{l-2} F \left( \frac{i \pi (4k-4p+3)}{h} \right) \prod_{k=1}^{l-1} \eta_{l-1-2k} \prod_{k=0}^{l} F \left( \frac{i \pi (4p+4k-1)}{h} \right).
\]

which unambiguously determine the constants $Z_p$,

\[
Z_p = \exp \left\{ -4 \int_0^\infty \frac{d \nu}{\nu} \sinh(\frac{\pi b \nu}{h Q}) \sinh(\frac{\pi \nu}{h Q b}) \left( \frac{\cosh(\pi \nu) \sinh(\frac{\pi p \nu}{h})}{\sinh(\pi \nu) \sinh(\frac{\pi \nu}{2})} - e^{-\frac{\pi \nu}{h}} \right) \right\},
\]

for $p = 1, \ldots, l-2$, and $Z_{l-1} = Z_l$

\[
Z_l = \exp \left\{ -2 \int_0^\infty \frac{d \nu}{\nu} \sinh(\frac{\pi b \nu}{h Q}) \sinh(\frac{\pi \nu}{h Q b}) \left( \frac{\cosh(\pi \nu) \sinh(\frac{\pi l \nu}{h})}{\sinh(\pi \nu) \sinh(\frac{\pi l \nu}{2})} - e^{-\frac{\pi l \nu}{h}} \right) \right\}.
\]

The above formulae coincide with (1.6) if we recall the explicit form of $C^{-1}$ [33]. The exact result for $Z_p$ (6.21),(6.22) also matches the one loop perturbative check.

\textbf{7. Conclusion}

In this paper we have calculated the wave function renormalization constants. The formula (1.9), together with the formulae for the one-particle form factors (5.1) may be considered as the main results of the paper. Although the absence of a general expression for ZF generators $B_k(\theta)$ acting in $\pi_Z$ makes the analysis of multi-particle form factors difficult, it is still possible to obtain two-particle form factors involving the particle $B_1$ and the particles $B_{l-1}, B_l$ in an asymptotic state.
It would be interesting (if possible) to find the explicit form of ZF operators or a generating function for them. Another interesting problem is to find one-particle form factors for other Toda field theories and to understand their group-theoretical meaning. Note, that in the limit $b \to 0$ the one-particle form factors of $D_l^{(1)}$, as well as of $A_l^{(1)}$, ATQFT become the characters of finite dimensional representations of the Yangian $Y(D_l)$ or $Y(A_l)$ correspondingly [11]. And, finally, we consider as a challenging problem to generalize the result (1.6) to Toda theories associated with non-simply laced Lie algebras.

Acknowledgments

I am grateful to S. Lukyanov for interesting discussions. I also would like to thank R. Nepomechie and L. Mezincescu for helpful discussions and the Department of Physics and Astronomy, Rutgers University for hospitality. This work was supported in part by the National Science Foundation under Grants PHY-9507829 and PHY-9870101.

8. Appendix A

Here we collect commutation relations and pairings of the operators $\Lambda_k(\theta)$, $A_k(\theta)$ and $Y_k(\theta)$. In all formulae below we denote $\theta = \theta_1 - \theta_2$.

\[
\langle \vec{p} | \Lambda_a(\theta_1) \Lambda_a(\theta_2) | \vec{p} \rangle = g(\theta) e^{\frac{4\pi}{h} (\vec{p} - \vec{\theta}) \vec{h}_a},
\]

\[
\langle \vec{p} | \Lambda_a(\theta_1) \Lambda_\bar{a}(\theta_2) | \vec{p} \rangle = g(\theta) f(\theta + \frac{i\pi}{h}) f(\theta + \frac{i\pi(2l - 2a - 1)}{h}),
\]

\[
\langle \vec{p} | \Lambda_\bar{a}(\theta_1) \Lambda_a(\theta_2) | \vec{p} \rangle = g(\theta) f(\theta - \frac{i\pi}{h}) f(\theta - \frac{i\pi(2l - 2a - 1)}{h}),
\]

and for $b \neq a, \bar{a}$ we obtain

\[
\langle \vec{p} | \Lambda_a(\theta_1) \Lambda_b(\theta_2) | \vec{p} \rangle = g(\theta) f(\theta - \frac{i\pi\epsilon(a,b)}{h}) e^{\frac{2i\pi}{h} (\vec{p} - \vec{\theta}) (\vec{h}_a + \vec{h}_b)},
\]

where the function $g(\theta)$ is defined for $\Im \theta \geq 0$ by

\[
g(\theta) = \exp \left\{ -4 \int_0^{\infty} \frac{d\nu}{\nu} e^{i\nu \theta} \sinh(\frac{\pi b \nu}{hQ}) \sinh(\frac{\pi \nu}{hQ}) \frac{\cosh(\frac{\pi \nu}{2})}{\cosh(\frac{\pi \nu}{2})} \right\},
\]

\[
f(\theta) = \frac{(\frac{\theta}{2} + \frac{i\pi}{2h}) (\frac{\theta}{2} - \frac{i\pi b}{hQ}) (\frac{\theta}{2} - \frac{i\pi}{2h})}{(\frac{\theta}{2} + \frac{i\pi}{2h}) (\frac{\theta}{2} - \frac{i\pi}{2h})}.
\]

For $\Im \theta < 0$ relations (8.1) - (8.3) must be understood in the sense of analytical continuation.
The commutation relations
\[
[a^{(p)}_\nu, a^{(q)}_{\nu'}] = 4\nu^{-1} C_{pq} \sinh\left(\frac{\pi b\nu}{hQ}\right) \sinh\left(\frac{\pi \nu}{hQb}\right) \delta_{\nu+\nu',0},
\]
\[
[y^{(p)}_\nu, y^{(q)}_{\nu'}] = 4\nu^{-1} (C^{-1})_{pq} \sinh\left(\frac{\pi b\nu}{hQ}\right) \sinh\left(\frac{\pi \nu}{hQb}\right) \delta_{\nu+\nu',0},
\]
\[
[a^{(p)}_\nu, y^{(q)}_{\nu'}] = 4\nu^{-1} \delta_{pq} \sinh\left(\frac{\pi b\nu}{hQ}\right) \sinh\left(\frac{\pi \nu}{hQb}\right) \delta_{\nu+\nu',0},
\]
allows us to calculate pairings between $Y_a(\theta)$ and $A_b(\theta)$
\[
\begin{align*}
\langle \vec{p} | Y_a(\theta_1)Y_b(\theta_2) | \vec{p} \rangle &= R_{ab}(\theta) e^{\frac{2i\pi}{h} (\vec{p}-\vec{\rho})(\vec{\omega}_a+\vec{\omega}_b)}, \\
\langle \vec{p} | Y_a(\theta_1)A_b(\theta_2) | \vec{p} \rangle &= f^{-1}(\theta) e^{\frac{2i\pi}{h} (\vec{p}-\vec{\rho})(\vec{\omega}_a+\vec{\omega}_b)}, \\
\langle \vec{p} | A_a(\theta_1)A_b(\theta_2) | \vec{p} \rangle &= f^{-1}(\theta + i\frac{\pi}{h}) f^{-1}(\theta - i\frac{\pi}{h}) e^{\frac{4i\pi}{h} (\vec{p}-\vec{\rho})(\vec{\omega}_a+\vec{\omega}_b)}, \quad \text{if } C_{ab} = -1,
\end{align*}
\]
and the rest of the pairings are equal to 1. The function $R_{ab}(\theta)$, $a \leq b$ is given by
\[
R_{ab}(\theta) = \exp \left\{ -4 \int_0^\infty \frac{d\nu}{\nu} e^{i\nu\theta} \sinh\left(\frac{\pi b\nu}{hQ}\right) \sinh\left(\frac{\pi \nu}{hQb}\right) (C^{-1}(\nu))_{ab} \right\}.
\]
This integral representation is valid for $\Im m \theta \geq -\frac{\pi (b-a)}{h}$ and must be understood in a sense of analytical continuation for $\Im m \theta \leq -\frac{\pi (b-a)}{h}$.

9. Appendix B

Here we give the explicit form of the operator $B_3(\theta)$ for $p = 3$ and discuss the case of general $p$.

\[
B_3(\theta) = Q \sqrt{\frac{h\kappa_3}{2\pi}} \left\{ \sum_{\{a_1,a_2,a_3\} \in I} \gamma (\Lambda_{a_1}, \Lambda_{a_2}, \Lambda_{a_3}) \Lambda_{a_1} \Lambda_{a_2} \Lambda_{a_3} + \right.
\]
\[
\gamma \sum_{a=1}^{l-3} \left( \Lambda_{l-2}(\Lambda_{l-2} \Lambda_a)' + \Lambda_{\bar{a}} \Lambda_{l-2} \Lambda_{l-2}' \right) - \gamma \sum_{a=1}^{l-2} \left( \Lambda_{l-1}(\Lambda_{l-1} \Lambda_a)' + \Lambda_{\bar{a}} \Lambda_{l-1} \Lambda_{l-1}' \right) +
\]
\[
\gamma \sum_{a=l-2}^{l} \left( \Lambda_{l-3}(\Lambda_a \Lambda_{l-3})' + \Lambda_{\bar{a}} \Lambda_{l-3} \Lambda_{l-3}' \right) - \gamma \sum_{a=l-1}^{l} \left( \Lambda_{l-2}(\Lambda_a \Lambda_{l-2})' + \Lambda_{\bar{a}} \Lambda_{l-2} \Lambda_{l-2}' \right)
\]
The numerical coefficients $\gamma(\Lambda_{a_1}, \ldots, \Lambda_{a_n})$ have a factorized form

$$\gamma(\Lambda_{a_1}, \ldots, \Lambda_{a_p}) = \prod_{m<k} \gamma(\Lambda_{a_m}, \Lambda_{a_k}), \quad (9.2)$$

where

$$\gamma(\Lambda_{\pi}(\theta + \frac{2i\pi p}{h}), \Lambda_{a}(\theta)) = c_{a+p-1}, \quad a + p \neq l, l - 1,$$

$$\gamma(\Lambda_{l-2}(\theta + \frac{2i\pi p}{h}), \Lambda_{l-2}(\theta)) = \gamma(\Lambda_{l-1}(\theta + \frac{2i\pi}{h}), \Lambda_{l-1}(\theta)) = 1 - \frac{1}{Q^2}, \quad (9.3)$$

$$\gamma(\Lambda_{l}(\theta + \frac{2i\pi p}{h}), \Lambda_{l}(\theta)) = \gamma(\Lambda_{l}(\theta + \frac{2i\pi p}{h}), \Lambda_{l}(\theta)) = c_{l+p-2},$$

$$\gamma(\Lambda_{l}(\theta + \frac{2i\pi p}{h}), \Lambda_{l}(\theta)) = \gamma(\Lambda_{l}(\theta + \frac{2i\pi p}{h}), \Lambda_{l}(\theta)) = c_{l+p-1},$$

$$\gamma(\Lambda_{a}, \Lambda_{b}) = 1, \quad \text{for the rest of the cases}.$$

The constants $c_p$ and $\gamma$ are given by

$$c_p = 1 + \frac{1}{(l-1-p)(l-2-p) Q^2}, \quad (9.4)$$

$$\gamma = \frac{2i\pi}{hQ^2}.$$

For general $p$ an operator $B_p(\theta)$ can be written in a form

$$B_p(\theta) = Q \sqrt{\frac{h\kappa_p}{2\pi}} \left\{ \sum_{\{a_1, \ldots, a_p\} \in I} \gamma(\Lambda_{a_1}, \ldots, \Lambda_{a_p}) \Lambda_{a_1} \cdots \Lambda_{a_p} + \text{“unpleasant“ terms} \right\}. \quad (9.5)$$

The derivative terms are not the only possible “unpleasant” terms that can appear in $B_p(\theta)$. Considering $B_4(\theta)$ one can find, for example, that besides derivative terms it contains also the nonstandard monom $\Lambda_{l-3}\Lambda_{l-2}\Lambda_{l-3}\Lambda_{l-2}$. All “unpleasant” terms disappear in the limit $b \to 0$ and $B_p(\theta)$ acquire a form of the Baxter $T - Q$ equation.

10. Appendix C

Here we give some details of the trace calculations.

The problem is to calculate traces over a Fock space

$$\langle\langle \mathcal{O}(\lambda) \rangle\rangle = \frac{\text{Tr}_{\mathcal{F}_p}[e^{2\pi i K} \mathcal{O}(\lambda)]}{\text{Tr}_{\mathcal{F}_p}[e^{2\pi i K}]}, \quad (10.1)$$

23
where $O$ is an arbitrary operator. This can be achieved by adopting a method of [42]. Let the oscillators $\lambda^{(a)}_\nu$ satisfy

\[ [\lambda^{(a)}_\nu, \lambda^{(b)}_{\nu'}] = \nu^{-1} g^{ab}(\nu) \delta_{\nu+\nu',0} . \] (10.2)

We introduce a complementary set of oscillators $\gamma^{(a)}_\nu$ with commutation relations

\[ [\gamma^{(a)}_\nu, \gamma^{(b)}_{\nu'}] = \nu^{-1} g^{ab}(-\nu) \delta_{\nu+\nu',0} , \]

\[ [\gamma^{(a)}_\nu, \gamma^{(b)}_{\nu'}] = 0 . \] (10.3)

Then the traces can be rewritten as vacuum averages

\[ \text{Tr}_{F\tilde{r}}[e^{2\pi iK} O(\lambda)] = \langle v | \exp \left( \int_0^\infty d\nu g^{ab}(\nu) \gamma^{(a)}_\nu \lambda^{(b)}_\nu \right) O(\lambda) \exp \left( \int_0^\infty d\nu g^{ab}(-\nu) \gamma^{(a)}_{-\nu} \lambda^{(b)}_{-\nu} \right) | v \rangle . \] (10.4)

where

\[ g^{ab}(\nu) g^{bc}(\nu) = \delta^c_a , \]

\[ \gamma^{(a)}_\nu | v \rangle = \lambda^{(a)}_\nu | v \rangle = 0 , \quad \nu > 0 . \] (10.5)

After some manipulations one can arrive at

\[ \text{Tr}_{F\tilde{r}}[e^{2\pi iK} O(\lambda^{(a)}_\nu, \lambda^{(b)}_\nu)] = \langle v | O(\lambda^{(a)}_{-\nu} + \frac{e^{-2\pi i\nu}}{1 - e^{-2\pi i\nu}} \gamma^{(a)}_{-\nu}, \gamma^{(a)}_\nu + \frac{1}{1 - e^{-2\pi i\nu}} \lambda^{(a)}_\nu) | v \rangle \text{Tr}_{Fa}[e^{2\pi iK}] . \] (10.6)

It is implied in (10.6) that $\nu > 0$. Now one can find, for example

\[ \langle \langle \Lambda_a(\theta_1) \Lambda_a(\theta_2) \rangle \rangle = \mathcal{N} R_{11}(\theta) e^{\frac{4\pi i}{\hbar}(\vec{p} - \vec{q})\vec{h}} , \]

\[ \langle \langle \Lambda_a(\theta_1) \Lambda_{\tilde{a}}(\theta_2) \rangle \rangle = \mathcal{N} R_{11}(\theta) F(\theta + \frac{i\pi}{\hbar}) F(\theta + \frac{i\pi(2l - 2a - 1)}{\hbar}) , \] (10.7)

\[ \langle \langle \Lambda_{\tilde{a}}(\theta_1) \Lambda_a(\theta_2) \rangle \rangle = \mathcal{N} R_{11}(\theta) F(\theta - \frac{i\pi}{\hbar}) F(\theta - \frac{i\pi(2l - 2a - 1)}{\hbar}) , \]

and for $b \neq a, \tilde{a}$ we obtain

\[ \langle \langle \Lambda_a(\theta_1) \Lambda_b(\theta_2) \rangle \rangle = \mathcal{N} R_{11}(\theta) F(\theta - \frac{i\pi e(a,b)}{\hbar}) e^{\frac{2\pi a}{\hbar} (\vec{p} - \vec{q})\vec{h} + \vec{h} + \vec{h}} . \] (10.8)

The constant $\mathcal{N}$ can be absorbed in the definition of $Z_p$ and its exact value is not important.

24
References

[1] Mikhailov A.V.: The reduction problem and the inverse scattering method. Physica D3, 73 (1981)
[2] Mikhailov A.V., Olshanetskii M.A., Perelomov A.M.: Two-dimensional generalized Toda lattice. Commun.Math.Phys. 179, 401 (1981)
[3] Vergeles S., Gryanik V.: Two dimensional quantum field theories having exact solutions. Yad.Fiz. 23, 1324 (1976)
[4] Zamolodchikov A.B.: Quantum Sine-Gordon model. The total S matrix. ITEP-12-1977, 1977
[5] Arinshtein A.E., Fateev V.A., Zamolodchikov A.B.: Quantum S matrix of the (1+1)-dimensional Toda chain. Phys.Lett. B87, 389 (1979)
[6] Fateev V.A., Zamolodchikov A.B.: Conformal field theory and purely elastic S matrices. Int.J.Mod.Phys.A5, 1025 (1990)
[7] Braden H.W., Corrigan E., Dorey P.E., Sasaki R.: Affine Toda field theory and exact S matrices. Nucl.Phys. B338, 689 (1990)
[8] Christie P., Mussardo G.: Elastic S matrices in (1+1)-dimensions and Toda field theories. Int.J.Mod.Phys.A5, 4581 (1990)
[9] Christie P., Mussardo G.: Integrable systems away from criticality: the Toda field theory and S matrix of the tricritical Ising model. Nucl.Phys. B330, 465 (1990)
[10] Berg B., Karowski M., Weisz P.: Construction of Green functions from an exact S matrix. Phys.Rev. D19, 2477 (1979)
[11] Zamolodchikov A.B.: Two point correlation function in scaling Lee-Yang model. Nucl.Phys. B348, 619 (1991)
[12] Fring A., Mussardo G., Simonetti P.: Form factors of the elementary field in the Bullough-Dodd model. Phys.Lett. B307, 83 (1993)
[13] Fring A., Mussardo G., Simonetti P.: Form factors for integrable lagrangian field theories, the Sinh-Gordon model. Nucl.Phys. B393, 413 (1993)
[14] Oota T.: Functional equations of form factors for diagonal scattering theories. Nucl.Phys. B466, 361 (1996)
[15] Pillin M.: Polynomial recursion equations in form factors of $A - D - E$ Toda field theories. Lett.Math.Phys.43, 211 (1998)
[16] Watson K.M.: Phys.Rev. 95, 228 (1954)
[17] Karowski M, Weisz P.: Exact form factors in $(1+1)$-dimensional field theoretic models with soliton behavior. Nucl.Phys. B139, 455 (1978)
[18] Smirnov F.: The quantum Gelfand-Levitan-Marchenko equations and form factors in the Sine-Gordon model. J.Phys. A17, L873 (1984)
[19] Lukyanov S.: Free field representation for massive integrable models. Commun. Math. Phys. 167,183 (1995)
[20] Lukyanov S.: Correlators of the Jost functions in the Sine-Gordon model. Phys.Lett. B325, 409 (1994)
[21] Lukyanov S., Pugai Ya.: Bosonization of ZF algebras: direction toward deformed Virasoro algebra. J.Exp.Theor.Phys. 82, 1021 (1996)
[22] Zamolodchikov A.B.: Unpublished
[23] Baxter R.J.: Exactly solved models in statistical mechanics. London: Academic Press 1982
[24] Jimbo J., Miwa T.: Algebraic analysis of Solvable Lattice Models. Kyoto Univ., RIMS-981 (1994)
[25] Davies B., Foda O., Jimbo J., Miwa T., Nakayashiki A.: Diagonalization of the XXZ Hamiltonian by vertex operators. Commun. Math. Phys. 151, 89 (1993)
[26] Lukyanov S.: Form factors of exponential fields in the Sine Gordon model. Mod. Phys. Lett. A12, 2543 (1997)
[27] Lukyanov S.: Form factors of exponential fields in the affine $A_{N-1}^{(1)}$ Toda model. Phys.Lett. B408, 192 (1997)
[28] Destri C., de Vega H.J.: New exact results in affine Toda field theories: free energy and wave function renormalization. Nucl.Phys. B358, 251 (1991)
[29] Brazhnikov V., Lukyanov S.: Angular quantization and form factors in massive integrable models. Nucl.Phys. B512, 616 (1998)
[30] Acerbi C.: Form factors of exponential operators and exact wave function renormalization constant in the Bullough-Dodd model. Nucl.Phys. B497, 589 (1997)
[31] Lukyanov S.: A note on the deformed Virasoro algebra. Phys.Lett. B367, 121 (1996)
[32] Frenkel E., Reshetikhin N.: Quantum Affine Algebras and Deformations of the Virasoro and $W$-algebras. q-alg/9505025
[33] Frenkel E., Reshetikhin N.: Deformations of $W$-algebras associated to simple Lie algebras. q-alg/9708006
[34] Oota T.: Q deformed Coxeter element in nonsimply laced affine Toda field theories. Nucl.Phys. B504, 738 (1997)
[35] Smirnov F.A.: Form factors in completely integrable models of quantum field theory. Singapore: World Scientific (1992) 208 p. (Advanced series in mathematical physics, 14).
[36] Feigin B., Frenkel E.: Quantum $W$ algebras and elliptic algebras. Commun.Math.Phys. 178, 653 (1996)
[37] Bouwknegt P., Pilch K.: On deformed $W$-algebras and quantum affine algebras. q-alg/9801112
[38] Kuniba A., Suzuki J.: Analytic Bethe ansatz for fundamental representations of Yangians. Commun.Math.Phys. 173, 225 (1995)
[39] Fateev V.A.: To be published
[40] Delfino G., Simonetti P., Cardy J.L.: Asymptotic factorization of form factors in two dimensional quantum field theory. Phys. Lett. B387, 327 (1996)

[41] Fateev V.A.: Unpublished.

[42] Clavelli L., Shapiro J.A.: Pomeron factorization in general dual model. Nucl. Phys. B57, 490 (1973)