Skein theory and the Murphy operators

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Abstract

The Murphy operators in the Hecke algebra $H_n$ of type $A$ are explicit commuting elements whose sum generates the centre. They can be represented by simple tangles in the Homfly skein theory version of $H_n$. In this paper I present a single tangle which represents their sum, and which is obviously central. As a consequence it is possible to identify a natural basis for the Homfly skein of the annulus, $C$.

Symmetric functions of the Murphy operators are also central in $H_n$. I define geometrically a homomorphism from $C$ to the centre of each algebra $H_n$, and find an element in $C$, independent of $n$, whose image is the $m$th power sum of the Murphy operators. Generating function techniques are used to describe images of other elements of $C$ in terms of the Murphy operators, and to demonstrate relations among other natural skein elements.

Keywords: skein theory; Murphy operators; power sums; symmetric functions; annulus; Hecke algebras.

Introduction.

The Hecke algebra $H_n$ of type $A$ is a deformed version of the group algebra $\mathbb{C}[S_n]$ of the symmetric group. It has a simple skein theory model in terms of $n$-tangles and the Homfly skein relations.

Certain sums of transpositions,

$$m(j) = \sum_{i=1}^{j-1} (i j) \in \mathbb{C}[S_n], \ j = 2, \ldots, n,$$

known as Jucys-Murphy elements, appeared in the work of Jucys [7] in 1974, and later in work of Murphy [14]. These elements all commute, and every
symmetric polynomial in them can be shown to lie in the *centre* of the algebra $C[S_n]$. For example $m(2) = (1\ 2)$, $m(3) = (1\ 3) + (2\ 3)$ and

$$m(2)m(3) = (1\ 3\ 2) + (1\ 2\ 3) = m(3)m(2).$$

Dipper and James [4] used a simple deformation of the transpositions to define analogous elements $M(j) \in H_n$, which they called the *Murphy operators*. These elements again all commute, and symmetric polynomials in them belong to the centre of $H_n$. Dipper and James showed that for generic values of the deformation parameter these account for the whole of the centre. Katriel, Abdessalam and Chakrabarti [8] noted that in fact any central element can be expressed as a polynomial in just the sum $M = \sum_{j=2}^n M(j)$ of the Murphy operators. This is a stronger result than for the non-deformed algebra $C[S_n]$, although the centre has the same dimension in each case, given by $\pi(n)$, the number of partitions of $n$.

In this paper I present a skein theory version of the Murphy operators and their sum, finishing with an elegant representation of their power sums and other symmetric polynomials. Starting in section 2 with a choice of $n$-string braids corresponding to the transpositions I exhibit a braid $T(j)$ representing each of the individual Murphy operators $M(j)$, following Ram [15], and then a very natural simple $n$-tangle $T(n)$ which represents their sum $M$, up to a linear combination with the identity element in $H_n$ in each case. From the tangle viewpoint it becomes immediately clear that $M$ is central.

The Homfly skein of the annulus, $C$, has been exploited for many years, for example in the work of Jun Murakami, and myself and Short, in constructing and analysing Homfly-based invariants by the use of satellites. These invariants cover the same ground as the quantum $sl(N)$ invariants for all choices of module and have been developed variously in algebraic and skein theoretic ways, for example by Wenzl [17], Aiston and myself [13] and Blanchet [3].

In section 3 I show how the tangle view of the Murphy operators gives a direct means for identifying the best linear basis for $C$ when studying these quantum invariants and establishing its multiplicative properties. I then introduce a rather overlooked homomorphism from $C$ to the centre of $H_n$. Using generating function methods I show how to realise other symmetric functions of the Murphy operators as images of explicit elements of $C$. The principal result is the identification of an element $P_m \in C$, independent of $n$, whose image determines the $m$th power sum of the Murphy operators in $H_n$.

Section 4 gives the final details of the results of section 3, based on a
further underused skein, $\mathcal{A}$, whose unexpected algebraic properties allow for some satisfyingly clean proofs.

1 The skein models.

The skein theory model of $H_n$ is based on the framed Homfly skein relations, in their simplest form

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=1cm]{skein1} \\
\includegraphics[width=1cm]{skein2}
\end{array} & = (s - s^{-1}) \\
\begin{array}{c}
\includegraphics[width=1cm]{skein3}
\end{array} \\
\begin{array}{c}
\includegraphics[width=1cm]{skein4}
\end{array} & = v^{-1}
\end{align*}
\]

The Homfly skein $\mathcal{S}(F)$ of a planar surface $F$, with some designated input and output boundary points, is defined as linear combinations of oriented tangles in $F$, modulo these two local relations, and Reidemeister moves II and III. The coefficient ring can be taken as $\Lambda = \mathbb{Z}[v^\pm 1, s^\pm 1]$ with powers of $s^k - s^{-k}$ in the denominators.

Every skein admits a mirror map, $- : \mathcal{S}(F) \to \mathcal{S}(F)$ induced by switching all crossings in a tangle, coupled with inverting $v$ and $s$ in $\Lambda$.

Write $R_n^n(v, s)$ for the skein $\mathcal{S}(F)$ of $n$-tangles, where $F$ is a rectangle with $n$ inputs at the bottom and $n$ outputs at the top. Composing $n$-tangles induces a product which makes $R_n^n(v, s)$ into an algebra. It has a linear basis of $n!$ elements, and is isomorphic to the Hecke algebra $H_n(z)$. This algebra has a presentation with generators $\{\sigma_i\}, i = 1, \ldots, n - 1$ satisfying the braid relations

\[
\begin{align*}
\sigma_i \sigma_j & = \sigma_j \sigma_i, \quad |i - j| > 1, \\
\sigma_i \sigma_{i+1} \sigma_i & = \sigma_{i+1} \sigma_i \sigma_{i+1},
\end{align*}
\]

and the quadratic relations $\sigma_i^2 = z \sigma_i + 1$.

In $R_n^n$ the generators are the elementary braids

\[
\sigma_i = \begin{array}{c}
\includegraphics[width=2cm]{elementary_braid}
\end{array}
\]
and the parameter $z$ is $s - s^{-1}$, giving the alternative form $(\sigma_i - s)(\sigma_i + s^{-1}) = 0$ for the quadratic relations.

In the special case $z = 0$ the Hecke algebra reduces to $C[S_n]$, with $\sigma_i$ becoming the transposition $(i \ i + 1)$. For general $i,j$ we can view the transposition $(i \ j)$ diagrammatically

$$(i \ j) = \begin{array}{c}
\text{diagram 1}
\end{array}.$$  

The corresponding positive permutation braid

$$\omega_{(i,j)} = \begin{array}{c}
\text{diagram 2}
\end{array}$$

with all crossings positive realises this transposition, as in [13]. Regarded via $R_n^n$ as an element of $H_n(z)$, it becomes the transposition $(i \ j)$ when we set $z = 0$.

**Definition.** The *Murphy operator* $M(j) \in R_n^n$, $j = 2, \ldots, n$ is defined as

$$M(j) = \sum_{i=1}^{j-1} \omega_{(i,j)}.$$  

These elements certainly project to the Jucys-Murphy elements $m(j)$ in $C[S_n]$. With a bit of algebraic work they can be shown to commute, and their sum

$$M = \sum_{j=2}^{n} M(j) = \sum_{i<j} \omega_{(i,j)}$$

can be shown to lie in the centre of $R_n^n$. These facts are immediately obvious from the skein representatives $T(j)$ and $T^{(n)}$, discussed shortly.

### 1.1 Variants of the Hecke algebras.

A simple adjustment of the skein relations, as in [13], allows for a skein model of $H_n$ whose parameters can be readily adapted to match any of the different appearances of the algebra.
Extend the coefficient ring to include an invertible parameter $x$, and define the skein $R_n^a(x, v, s)$ by linear combinations of oriented $n$-tangles modulo the relations

\[
x^{-1} \begin{tikzpicture} [thick, baseline = (X.base), scale=0.2] \draw (0,0) to (1,0); \draw (1,0) to (0,1); \draw (0,1) to (-1,0); \draw (-1,0) to (0,-1); \end{tikzpicture} - x \begin{tikzpicture} [thick, baseline = (X.base), scale=0.2] \draw (0,0) to (1,0); \draw (1,0) to (0,1); \draw (0,1) to (-1,0); \draw (-1,0) to (0,-1); \end{tikzpicture} = (s - s^{-1}) \begin{tikzpicture} [thick, baseline = (X.base), scale=0.2] \draw (0,0) to (1,0); \draw (1,0) to (0,1); \draw (0,1) to (-1,0); \draw (-1,0) to (0,-1); \end{tikzpicture}
\]

and \[
\begin{tikzpicture} [thick, baseline = (X.base), scale=0.2] \draw (0,0) to (1,0); \draw (1,0) to (0,1); \draw (0,1) to (-1,0); \draw (-1,0) to (0,-1); \end{tikzpicture} = xv^{-1} \begin{tikzpicture} [thick, baseline = (X.base), scale=0.2] \draw (0,0) to (1,0); \draw (1,0) to (0,1); \end{tikzpicture} .
\]

There is a natural algebra homomorphism $R_n^a(v, s) \rightarrow R_n^a(x, v, s)$ induced by replacing each tangle $T$ in the skein $R_n^a(v, s)$ by $x^{\text{wr}(T)}T$ in $R_n^a(x, v, s)$, where $\text{wr}(T)$ is the writhe of $T$.

We then get corresponding versions of the Murphy elements in our new skein; for example the new version of $T(j)$ is represented by $x^{2j-2}T(j)$ in $R_n^a(x, v, s)$.

The parameters in $R_n^a(x, v, s)$ can be adjusted to match the exact version of $H_n$ under study. Taking $x = s, q = s^2, v = 1$ we get the common algebraist’s version $H_n(q)$ of $\text{[4]}$, while the endomorphism rings arising from the fundamental representation of $sl(N)_q$ and its standard $R$-matrix correspond to the choice $x = s^{-1}/N, v = s^{-N}$ with $s = e^{h/2}$.

The choice $x = v$ gives the Homfly skein relation which is invariant under Reidemeister move I, and $x = q^f, s = q, v = a$ gives the version used by Kawagoe in $\text{[3]}$.

The elements $M(j)$ above do indeed become the Murphy operators used by Dipper and James under the isomorphism of $R_n^a(v, s)$ with the algebraic version $H_n(q)$ of the Hecke algebra where $x = s, q = s^2, v = 1$.

2 Geometric views of the Murphy operators.

The Murphy operators $M(j)$ and their sum $M$ have simple skein representatives as shown here, using $R_n^a(v, s)$ as the model for the Hecke algebra $H_n$.

**Theorem 2.1** (Ram). The Murphy operator $M(j)$ can be represented in $H_n$ by a single braid $T(j)$, up to linear combination with the identity.
Theorem 2.2 (Morton). The sum $M$ of the Murphy operators can be represented in $H_n$ by a single tangle $T^{(n)}$, again up to linear combination with the identity.

Proof of theorem 2.1: In [15] Ram notes that $M(j)$ can almost be represented by the single braid $T(j)$ pictured, where ‘almost’ means that $M(j)$ is a linear combination of $T(j)$ and the identity.

Set

$$T(j) = \begin{array}{c}
\includegraphics[width=2cm]{tangle1.png}
\end{array}$$

in $H_n$, for each $j \leq n$ including the case $T(1) = 1$. Skein theory shows quickly that $T(j) - 1 = zM(j)$, giving

$$M(j) = \frac{T(j) - 1}{s - s^{-1}}.$$  

The elements $T(j)$ will do equally well in place of $M(j)$, so long as $z \neq 0$, in other words away from $\mathbb{C}[S_n]$.

It is clear geometrically that the elements $T(j)$ all commute. It is not immediately clear that the sum of these elements is in the centre of $H_n$, although their product is the full twist, a well-known central element.

Proof of theorem 2.2 Set

$$T^{(n)} = \begin{array}{c}
\includegraphics[width=2cm]{tangle2.png}
\end{array}$$

in $R_n \cong H_n(z)$, with the coefficient ring extended to include $v^{\pm 1}$.  

\"
Apply the skein relation at one crossing to get

\[
\begin{array}{c}
\includegraphics[width=3cm]{skein_relation_1.png} \\
= \includegraphics[width=3cm]{skein_relation_2.png} + z \\
\end{array}
\]

Hence \( T^{(n)} = T^{(n-1)} + zv^{-1}T(n) \), using the standard inclusion of \( H_{n-1} \) in \( H_n \) to interpret \( T^{(n-1)} \) as an element of \( H_n \). Then

\[
T^{(n)} - T^{(0)} = zv^{-1}\sum_{j=1}^{n} T(j),
\]

by induction on \( n \). Here \( T^{(0)} \) is just a scalar multiple of the identity, represented by a single disjoint simple loop alongside the identity braid. Since a trivial loop in any Homfly skein contributes the scalar \( \delta = (v^{-1} - v)/z \) we can rewrite the equation as

\[
T^{(n)} - \frac{v^{-1} - v}{z} = zv^{-1}\sum_{j=1}^{n} T(j) = z^2v^{-1}M + nzv^{-1}.
\]

Then

\[
T^{(n)} = z^2v^{-1}M + \left(nzv^{-1} + \frac{v^{-1} - v}{z}\right) \times 1,
\]

and hence, again away from \( z = 0 \), we can write \( M \) as a linear combination of \( T^{(n)} \) and 1. \( \Box \)

From this representation it is quite obvious that \( T^{(n)} \), and thus \( M \), is in the centre of \( H_n \).

There is a known set of idempotent elements, \( E_\lambda \), one for each partition \( \lambda \) of \( n \), which were originally described algebraically by Gyoja \( \[5\] \). Skein pictures of these based on the Young diagram for \( \lambda \) were given by Aiston and myself \( \[13\] \). We showed there, in a skein based counterpart of the results of \( \[8\] \), that \( ME_\lambda = m_\lambda E_\lambda \) for an explicit scalar \( m_\lambda \) and that \( m_\lambda \neq m_\mu \) when \( \lambda \neq \mu \). It follows at once that \( T^{(n)}E_\lambda = t_\lambda E_\lambda \), where the scalars \( t_\lambda \) are different for each partition \( \lambda \).
3 The skein of the annulus.

The Homfly skein of the annulus, \( \mathcal{C} \), as discussed in \([12]\) and originally in \([16]\), is defined as linear combinations of diagrams in the annulus, modulo the Homfly skein relations. The element \( X \in \mathcal{C} \) will be indicated on a diagram as

\[
\begin{array}{c}
\bigcirc \\
\uparrow \\
X
\end{array}
\]

We shall make use of the results of section 2 to study \( \mathcal{C} \), and also to find simple skein representatives for the power sums of the Murphy operators in \( H_n \) in a way which is independent of \( n \).

The best known relation of \( \mathcal{C} \) with the Hecke algebra \( H_n \) is the closure map \( R_n : \mathcal{T} \rightarrow \mathcal{C} \), induced by taking a tangle \( T \) to its closure \( \hat{T} \) in the annulus, defined by

\[
\hat{T} = \begin{array}{c}
\bigcirc \\
\uparrow \\
T
\end{array}
\]

This is a linear map, whose image we will call \( \mathcal{C}_n \).

The skein \( \mathcal{C} \) has a product induced by placing one annulus outside another, under which \( \mathcal{C} \) becomes a commutative algebra;

\[
\begin{array}{c}
\bigcirc \\
\uparrow \\
X Y
\end{array} = \begin{array}{c}
\bigcirc \\
\uparrow \\
X \\
\downarrow \\
Y
\end{array}
\]

Write \( A_m \in \mathcal{C} \) for the closure of the \( m \)-braid

\[
\sigma_{m-1} \cdots \sigma_2 \sigma_1 = \begin{array}{c}
\uparrow \\
\downarrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\uparrow
\end{array}
\]

An explicit spanning set of \( \mathcal{C}_n \) consists of the monomials in \( \{A_j\} \) of total weight \( n \), where \( A_j \) has weight \( j \). Such monomials correspond bijectively with partitions of \( n \).
A very simple skein theory construction determines a natural linear map \( \varphi : \mathcal{C} \to \mathcal{C} \), induced by taking a diagram \( X \) in the annulus and linking it once with a simple loop to get
\[
\varphi(X) = \quad \quad \quad \quad .
\]

It is instructive to look at the eigenvectors of \( \varphi \). Clearly \( \varphi \) carries \( \mathcal{C}_n \) into itself. What is more, if we take an element \( S \in H_n \) with closure \( \hat{S} \in \mathcal{C}_n \) and compose it with the central element \( T^{(n)} \) then \( ST^{(n)} \) has closure \( \varphi(\hat{S}) \).

**Theorem 3.1** The eigenvalues of \( \varphi|\mathcal{C}_n \) are all distinct.

**Proof:** Take \( S = E_\lambda \) to see that the closure of \( T^{(n)}E_\lambda \) is \( t_\lambda \hat{E}_\lambda \) and also \( \varphi(\hat{E}_\lambda) \). The element \( Q_\lambda = \hat{E}_\lambda \in \mathcal{C}_n \) is then an eigenvector of \( \varphi \) with eigenvalue \( t_\lambda \). Now there are \( \pi(n) \) of these eigenvectors, (the number of partitions of \( n \)), and the eigenvalues are all distinct, by \([13]\). Since \( \mathcal{C}_n \) is spanned by \( \pi(n) \) elements we can deduce that the elements \( Q_\lambda \) form a basis for \( \mathcal{C}_n \), and that the eigenspaces of \( \varphi \) are all 1-dimensional.

It follows that any element of \( \mathcal{C}_n \) which is an eigenvector of \( \varphi \) must be a multiple of some \( Q_\lambda \).

The elements \( h_i \in \mathcal{C} \), discussed below, arise as \( Q_\lambda \) for the Young diagram \( \lambda \) consisting of a single row with \( i \) cells. Kawagoe \([4]\) constructed an element \( s_\lambda \) of \( \mathcal{C}_n \) for each partition \( \lambda \) of \( n \), as a Schur polynomial of the sequence \( \{h_i\} \), in Macdonald’s context of symmetric functions, \([4]\). He then used skein theory to show that each \( s_\lambda \) is also an eigenvector of the map \( \varphi \). His elements \( s_\lambda \) can now be identified almost at once with the elements \( Q_\lambda \), by theorem 3.1.

This gives an immediate proof of the multiplicative properties of the elements \( Q_\lambda \) in the skein of the annulus, since the elements \( s_\lambda \) automatically multiply according to the Littlewood-Richardson rules for Young diagrams \([4]\). Conversely it guarantees that Kawagoe’s elements \( s_\lambda \) are eigenvectors for a wide range of skein maps on \( \mathcal{C} \) as well as \( \varphi \), using the skein properties of \( E_\lambda \) from \([13]\).

Recently Lukac, \([11]\), has been able to simplify the proof that \( s_\lambda \) is an eigenvector of \( \varphi \), by using algebraic properties of the skein \( \mathcal{A} \) of the annulus with an input on one boundary and an output on the other. Discussion and
use of the skein $A$ in section 4 leads to simple descriptions for other symmetric functions of the Murphy operators, derived from a direct interplay between the skein of the annulus and the centre of the Hecke algebras.

3.1 Symmetric functions and the skein of the annulus.

In this section I recall some explicit results about elements in the Hecke algebras and their closure in $C$, and their interpretation in the context of symmetric functions, following the methods of Macdonald. This leads to a simple description in terms of generating functions of some interrelations in $C$, and also of the elements to be used later to represent the power sums of the Murphy operators.

The starting point is the description of the two simplest idempotents in $H_n$, corresponding to the single row and column Young diagrams. These are given algebraically in [6]; here I use the skein version as described in [12] in terms of the positive permutation braids $\omega_\pi$, $\pi \in S_n$. Define two quasi-idempotents by

$$a_n = \sum_{\pi \in S_n} s^{l(\pi)} \omega_\pi, \quad b_n = \sum_{\pi \in S_n} (-s)^{-l(\pi)} \omega_\pi,$$

where $l(\pi) = wr(\omega_\pi)$ is the writhe of the braid $\omega_\pi$.

The following result is straightforward.

**Lemma 3.2** We can write

$$a_n = a_{n-1} \gamma_n,$$

where $\gamma_n = 1 + s\sigma_{n-1} + s^2\sigma_{n-1}\sigma_{n-2} + \cdots + s^{n-1}\sigma_{n-1} \cdots \sigma_1$.

We have $\gamma_{n+1} = 1 + s\sigma_n \gamma_n$, and also an immediate skein relation

$$\gamma_{n+1} = \gamma_n + s^n,$$

in $R_{n+1}.$

The next lemma is proved in [13].
Lemma 3.3 For any braid $\beta \in B_n$ we have $a_n\beta = \varphi_s(\beta)a_n = \beta a_n$, where $\varphi_s(\beta) = s^{wr(\beta)}$.

Similar results, with $s$ replaced by $-s^{-1}$, hold for $b_n$.

The element $a_n$ then satisfies

$$a_n^2 = \varphi_s(a_n)a_n = \varphi_s(a_{n-1})\varphi_s(\gamma_n)a_n.$$

Since $\varphi_s(\gamma_n) = 1 + s^2 + \cdots + s^{2n-2} = s^{n-1}[n]$, we have an immediate corollary.

Corollary 3.4 We can write

$$s^{n-1}[n]h_n = h_{n-1}\gamma_n,$$

where $h_n = a_n/\varphi_s(a_n)$ is the true idempotent.

The element $h_n \in H_n$ is the idempotent which corresponds to the single row with $n$ cells. The single column idempotent is given from $h_n$ by using $-s^{-1}$ in place of $s$.

With a slight abuse of notation write $h_n \in C$ in place of $\hat{h}_n$ for the closure of this element in $C$, and $e_n \in C$ for the closure of the single column idempotent.

Remark. Aiston uses the notation $Q_{c_n}$ and $Q_{d_n}$ in [1], in place of $e_n$ and $h_n$ which are used here to suggest the terminology and techniques of symmetric functions from [11].

Write

$$H(t) = 1 + \sum_{n=1}^{\infty} h_n t^n$$

for the generating function of the elements $\{h_n\}$, regarded as a formal power series with coefficients in $C$, and similarly set $E(t) = 1 + \sum_{n=1}^{\infty} e_n t^n$. In [1] Aiston showed that

Theorem 3.5

$$E(-t)H(t) = 1,$$

as power series in $C$. 

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If we regard the elements $h_n$ formally as the $n$th complete symmetric functions in a large number $N$ of variables $x_1, \ldots, x_N$, setting

$$H(t) = \prod_{i=1}^{N} \frac{1}{1 - x_i t},$$

then $E(t) = \prod (1 + x_i t)$ and $e_n$ is the $n$th elementary symmetric function in the variables.

Following Macdonald, the Schur function $s_\lambda(x_1, \ldots, x_n)$ for each partition $\lambda$ can be expressed as a polynomial in the elements $h_n$ which does not depend on $N$ for large enough $N$, and is therefore determined formally by the series $H(t)$ and $\lambda$ as an element $s_\lambda$ of $\mathcal{C}$. This is the element $s_\lambda$ used by Kawagoe, which is identified with the idempotent closures $Q_\lambda$ by Lukac in [10].

The $\pi(n)$ elements $\{s_\lambda\}$ with $|\lambda| = n$ form a linear basis for $\mathcal{C}_n$, and can be expressed in terms of the monomials of weight $n$ in $\{h_r\}$ by the classical Jacobi-Trudy formulæ [11].

Indeed the skein $\mathcal{C}_n^+$, defined as $\cup \mathcal{C}_n$, is spanned by all monomials in $\{h_r\}$, and can be interpreted as the ring of symmetric polynomials in an unlimited number of variables $\{x_i\}$.

Monomials in the geometrically simpler closed braid elements $\{A_m\}$ also span $\mathcal{C}_n$, and it is thus interesting to relate these directly to $\{h_r\}$. An attractive formula connecting these two generating sets can be derived from corollary 3.4.

**Theorem 3.6** Write $A(t) = 1 + z \sum_{m=1}^{\infty} A_m t^m$, with $z = s - s^{-1}$.

Then

$$A(t) = \frac{H(st)}{H(s^{-1} t)} = H(st) E(-s^{-1} t).$$

The proof will be given in section 4.

**Lemma 3.7** The elements $\{h_n\}$ are invariant under the mirror map switching crossings and inverting $s$ and $v$.

**Proof**: In $H_n$ we have $\sigma_i a_n = \varphi_s(\sigma_i) a_n = s a_n$. Then $\overline{\sigma_i a_n} = \sigma_i^{-1} \overline{a_n} = s^{-1} \overline{a_n}$ giving $\sigma_i \overline{a_n} = s \overline{a_n}$. It follows that $\beta \overline{a_n} = \varphi_s(\beta) \overline{a_n}$ for any $\beta$ and so $a_n \overline{a_n} = \varphi_s(a_n) \overline{a_n}$.

On the other hand $a_n \overline{a_n} = a_n \varphi_s(\overline{a_n})$ by lemma 3.3, so that

$$h_n = \frac{a_n}{\varphi_s(a_n)} = \frac{\overline{a_n}}{\varphi_s(\overline{a_n})} = \overline{h_n}.$$

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Corollary 3.8 The inverse series to $A(t) = 1 + z \sum A_m t^m$ is $\overline{A}(t) = 1 - z \sum \overline{A}_m t^m$, where $\overline{A}_m$ is the negative closed braid corresponding to $A_m$.

Proof: Apply the mirror map in theorem 3.6 to get

$$\overline{A}(t) = \frac{H(st)}{H(s^{-1}t)} = H(s^{-1}t) / H(st) = A(t)^{-1}.$$ 

In our later description of the power sums of Murphy operators we use the element $P_m \in \mathcal{C}$ which can be interpreted formally as the $m$th power sum, $P_m = \sum x_i^m$, of the variables. This is determined unambiguously as a polynomial in $\{h_r\}$, and hence an element of $\mathcal{C}$, by Newton’s power series equation

$$\sum_{m=1}^{\infty} \frac{P_m}{m} t^m = \ln H(t).$$ 

Aiston showed in [2] that $[m]P_m$ is the sum of $m$ closed $m$-string braids given by switching the first $i$ crossings of the closed braid $A_m$ from positive to negative, for $i = 0, \ldots, m - 1$. Her proof depends on some quantum group translations to identify $s_\lambda$ and $Q_\lambda$, and to write $A_m$ in terms of $s_\lambda$. In another article I give a direct skein theory proof of her result, using theorem 3.6.

### 3.2 Symmetric functions of the Murphy operators.

The main result in this paper, besides theorem 2.2 and its consequences, is an expression for the $m$th power sum of the Murphy operators in $H_n$ in terms of the element $P_n \in \mathcal{C}$.

For this I use a previously unremarked relation between the Hecke algebras and the skein of the annulus. This relation takes the form of a very natural homomorphism $\psi_n$ from $\mathcal{C}$ to the centre of each algebra $H_n$.

The diagram

$$D = \begin{array}{c}
\bigcirc \\
\uparrow \\
1
\end{array}$$
determines a map $\psi_n : C \to H_n$, induced by placing $X \in C$ around the circle in $D$ and the identity of $H_n$ on the arc, to get

$$\psi_n(X) = \begin{array}{c}
\text{arc} \\
\text{circle}
\end{array} \in H_n.$$ 

It is clear that

$$\psi_n(XY) = \begin{array}{c}
\text{arc} \\
\text{circle}
\end{array} = \psi_n(X)\psi_n(Y),$$

so that $\psi_n$ is an algebra homomorphism. Moreover the elements $\psi_n(X)$ obviously all lie in the centre of $H_n$, and in fact it follows from theorem 3.9 below that the image of $\psi_n$ consists of all symmetric polynomials in the Murphy operators and so, by [4], makes up the whole of the centre in the generic case.

We already know from theorem 2.2 that the sum $\sum T(j)$ is essentially $T^n$, and we can write $T^n = \psi(X)$ with $X = h_1$, represented by a single string around the core of the annulus. We may then look for elements $X_2$ with $\psi(X_2) = \sum T(j)^2$ and more generally $\psi(X_m) = \sum T(j)^m$.

These can certainly be found for each fixed $n$, using for example a suitable polynomial in $h_1$ which depends on $n$. Much more surprising is the result of theorem 3.9, that there exists an element $X_m$ independent of $n$ with the property that

$$\psi_n(X_m) = \sum_{j=1}^{n} T(j)^m,$$

up to a multiple of the identity. Consequently any other symmetric function of the Murphy operators can be achieved as $\psi(X)$ where the choice of $X$ is essentially independent of $n$. 

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In fact, we can take $X_m = P_m$, to get the following explicit result, proved in section 4.

**Theorem 3.9** For any $n$ we have

$$\psi_n(P_m) - \psi_0(P_m) = (s^n - s^{-m})v^{-m} \sum_{j=1}^{n} T(j)^m.$$ 

This is particularly satisfactory in that $P_m$ itself arises as a power sum, although not of any identifiable objects in the ring $C$ itself.

Since any element $X \in C^+$ can be written as a polynomial in $\{P_m\}$ we can then express $\psi(X)$ as a symmetric function of the Murphy operators $T(j)$. It is useful to be able to write the elements $\psi(h_i)$ in terms of the Murphy operators; theorem 3.9 leads to a compact expression using formal power series with coefficients in the centre of the Hecke algebra.

I shall do this in terms of the *Murphy series* $HM(t)$ of the Hecke algebra, whose coefficients, lying in the centre of $H_n$, are defined to be the complete symmetric functions of the elements $T(j)$. Thus the Murphy series in $H_n$ can be written explicitly as

$$HM(t) = \prod_{j=1}^{n} (1 - T(j)t)^{-1}.$$ 

The series has a formal inverse, $EM(-t)$, where the coefficients of $EM(t) = \prod (1 + T(j)t)$ are the elementary symmetric functions of the elements $T(j)$, also in the centre of $H_n$.

**Theorem 3.10**

$$\psi_n(H(t)) = \psi_0(H(t)) \frac{HM(sv^{-1}t)}{HM(s^{-1}v^{-1}t)}.$$ 

**Proof:** It is enough to establish that the logarithms of the two sides are equal. Now

$$\ln(\psi_n(H(t))) = \psi_n(\ln(H(t))) = \sum_{m=1}^{\infty} \frac{\psi_n(P_m)}{m} t^m,$$

while

$$\ln(HM(t)) = \sum_{m=1}^{\infty} \frac{\sum T(j)^m}{m} t^m.$$
The coefficients of $t^m$ in $\ln(\psi_n(H(t)))$ and in

$$\ln\left(\frac{\psi_0(H(t))HM(sv^{-1}t)}{HM(s^{-1}v^{-1}t)}\right)$$

are equal, by theorem 3.9. $\Box$

4 The annulus with two boundary points.

To prove theorems 3.6 and 3.9 I introduce a third skein $S(F)$, whose underlying surface is the annulus with a single input on one boundary component and corresponding output on the other, as shown.

Write $A$ for this skein, whose elements are represented as linear combinations of oriented tangles in the annulus consisting of a single arc and a number of closed curves. As for the skein $C$ of the annulus there is a product on $A$ induced by placing one annulus outside the other. The identity element $e \in A$ is represented by the tangle

Write $a \in A$ for the element represented by
From this we can construct $a^m$ for each $m \in \mathbb{Z}$, giving for example

\[ a^2 = \quad , \quad a^{-1} = \quad . \]

There are two bilinear products, $l : \mathcal{C} \times \mathcal{A} \to \mathcal{A}$ and $r : \mathcal{A} \times \mathcal{C} \to \mathcal{A}$ induced by placing an element of $\mathcal{C}$ respectively under or over an element of $\mathcal{A}$. For example, $h_1 \in \mathcal{C}$, represented by a single counterclockwise loop, gives

\[ l(h_1, e) = \quad , \quad r(e, h_1) = \quad . \]

From the skein relation in $\mathcal{A}$ we have $l(h_1, e) - r(e, h_1) = (s - s^{-1})a$.

### 4.1 Algebraic properties of $\mathcal{A}$.

Kawagoe [9] and other authors have used a skein which is linearly isomorphic to $\mathcal{A}$, based on the annulus with input and output on the same component. The significant advantage of $\mathcal{A}$ lies in its algebraic properties.

Both $l$ and $r$ are algebra homomorphisms, since it is clear diagrammatically that $l(c_1, a_1)l(c_2, a_2) = l(c_1c_2, a_1a_2)$, while $r$ behaves similarly.

**Theorem 4.1** The algebra $\mathcal{A}$ is commutative.

**Proof**: Unlike the case of $\mathcal{C}$ this is not immediately clear. Using standard skein theory techniques we can represent any element of $\mathcal{A}$ as a linear combination of tangles consisting of a totally descending arc lying over a number of closed curves. Each such tangle represents $l(c_m, a^m) = l(c_m, e)a^m$ for some $m$ and some $c_m \in \mathcal{C}$. The general element of $\mathcal{A}$ can then be written as a Laurent polynomial

\[ \sum_{m \in \mathbb{Z}} l(c_m, e)a^m \]
in $a$, with coefficients in the commutative subalgebra $l(C,e) \subset A$. Since $a$ commutes with $l(C,e)$ it follows that any two elements of $A$ commute. $\square$

The subalgebras $l(C,e)$ and $r(e,C)$ are both isomorphic to $C$, but they are not equal. Their difference determines a sort of commutator map $[\cdot, e] : C \to A$, defined by $[c, e] = l(c,e) - r(e,c)$.

We have already noted that $[h_1, e] = (s - s^{-1})a$. We can deduce theorem 3.9 about the power sums of Murphy operators from the next theorem.

**Theorem 4.2** For $m \geq 1$ we have $[P_m, e] = (s^m - s^{-m})a^m$.

### 4.2 Skein interaction between $H_n$, $A$ and $C$.

We can make use of wiring diagrams, as described in [13], to induce linear maps between skeins. A **wiring diagram** $W$ is an inclusion of one surface $F$ into another $F'$, along with a fixed diagram of curves in $F'$ which avoid $F$, and connect any distinguished input and output points on the boundaries of both, respecting orientation. A wiring $W$ induces a linear map, denoted $S(W) : S(F) \to S(F')$, using $W$ to extend any diagram in $F$ to a diagram in $F'$.

To avoid over-elaborate notation I shall write $W$ in place of $S(W)$ where there is no risk of confusion.

The diagram

$$W_n = \begin{array}{c}
      \begin{array}{c}
        \text{\rotatebox{90}{\text{\hbox{\vspace{0.5em}}}}} \\
        \text{\rotatebox{90}{\text{\hbox{\vspace{0.5em}}}}} \\
        \text{\rotatebox{90}{\text{\hbox{\vspace{0.5em}}}}} \\
      \end{array}
    \end{array}
\end{array}$$

with $n$ strings running around the annulus then induces a linear map $W_n : R_{n+1} \to A$, by inserting an $(n+1)$-tangle in the empty box.

When an element $S \in H_n$ is included in $H_{n+1}$ in the standard way, using their skein versions, we can see from a diagram that $W_n(S) = W_{n-1}(S)a$. Then $W_n(1) = a^n$, where $a \in A$ is the element shown above.

**Theorem 4.3** The elements $W_n(h_{n+1}), W_n(h_n)$ and $l(h_n, e)$ in $A$ satisfy the linear relation

$$[n + 1]W_n(h_{n+1}) = s^{-1}[n]W_n(h_n) + l(h_n, e).$$
Proof : The relation

\[ \gamma_{n+1} = \gamma_n + s^n \]

gives \( W_n(h_n\gamma_{n+1}) = W_n(\gamma_nh_n) + s^n W_n(h_n\sigma_n\cdots\sigma_1) \), after taking \( \gamma_n \) from the second diagram around the strings of the wiring.

Now \( \gamma_nh_n = s^n[n]h_n \) and \( W_n(h_n\sigma_n\cdots\sigma_1) = l(h_n,e) \). Combined with the equation \( s^n[n+1]h_{n+1} = h_n\gamma_{n+1} \) from corollary 3.4 the result follows at once. \( \square \)

Now write \( Y_n = [n+1]W_n(h_{n+1}) \) and set \( Y(t) = \sum_{n=0}^{\infty} Y_n t^n \).

Corollary 4.4 As power series with coefficients in \( \mathcal{A} \) we have

\[ l(H(t),e) = (e - s^{-1}at)Y(t). \]

Proof : Since \( W_n(h_n) = W_{n-1}(h_n)a = aW_{n-1}(h_n) \) the relation can be written \( Y_n = s^{-1}aY_{n-1} + l(h_n,e) \). This gives \( Y(t) = s^{-1}atY(t) + l(H(t),e) \), and hence the result. \( \square \)

The mirror map, switching crossings and inverting \( s \) and \( v \), when applied to \( l(X,e) \) for any \( X \in C \) gives \( r(e,\overline{X}) \). Now \( \overline{H(t)} = H(t) \) and so \( \overline{Y(t)} = Y(t) \), giving

\[ r(e,H(t)) = (e - sat)Y(t). \]

Subtracting this from the equation in corollary 4.4 then gives \( [H(t),e] = (s - s^{-1})atY(t) \), a result used by Lukac in [10].

Proof of theorem 4.2: Take logarithms of the equations and then subtract. Then

\[ \ln(e - s^{-1}at) - \ln(e - sat) = \ln(l(H(t),e)) - \ln(r(e,H(t))) \]
\[ = l(\ln(H(t)),e) - r(e,\ln(H(t))) \]
\[ = \sum_{m=1}^{\infty} \frac{P_m t^m}{m},[e]. \]

Now \( \ln(e - s^{-1}at) = - \sum_{m=1}^{\infty} s^{-m}a^m t^m/m \), and it only remains to compare the coefficients of \( t^m \) to see that \( [P_m,e] = (s^m - s^{-m})a^m \). \( \square \)
Proof of theorem 3.9: Use the wiring

\[ V_n = \text{Diagram} \]

with \( n - 1 \) strings passing through the annulus to carry \( A \) into \( R_n^m \). Then for any \( X \) in \( C \) we have \( V_n(l(X, e)) = \psi_n(X) \) and \( V_n(r(e, X)) = \psi_{n-1}(X) \) with the standard inclusion. We can also see that \( V_n(a) = v^{-1}T(n) \), and extend this to \( V_n(a^m) = v^{-m}T(n)^m \). Then \( \psi_n(P_m) - \psi_{n-1}(P_m) = (s^m - s^{-m})v^{-m}T(n)^m \).

Induction on \( n \) completes the proof that

\[ \psi_n(P_m) - \psi_0(P_m) = (s^m - s^{-m})v^{-m} \sum_{j=1}^n T(j)^m. \]

\[ \square \]

To conclude this paper I give the proof of theorem 3.6, using one final wiring, this time to induce a map from \( A \) to \( C \).

Write

\[ C = \text{Diagram} \]

for the wiring diagram shown, which provides a form of closure for an element of \( A \), and determines a linear map \( C : A \rightarrow C \). It is clear that \( C(a^m) = A_{m+1} \) for every \( m \geq 0 \). We can also see readily that

\[ C(F, r(e, X)) = C(r(F, X)) = C(F)X \]

for every \( F \in A \) and \( X \in C \).

Proof of theorem 3.6: The relation \((e - \text{sat})Y(t) = r(e, H(t))\) gives

\[ Y(t) = \left( \sum_{m=0}^\infty s^m a^m t^m \right) r(e, H(t)). \]
Then

\[ C(Y(t)) = C \left( \sum_{m=0}^{\infty} s^m a^m t^m \right) H(t) \]

\[ = \left( \sum_{m=0}^{\infty} s^m A_{m+1} t^m \right) H(t), \]

giving

\[ zstC(Y(t)) = (A(st) - 1) H(t). \]

Now \( C(W_n(h_{n+1})) \) is the closure of the \((n+1)\)-tangle \( \sigma_1 \sigma_2 \cdots \sigma_n h_{n+1} = s^n h_{n+1} \). Then \( C(Y_n) = [n+1] s^n h_{n+1} \in C \). It follows that

\[ zstC(Y(t)) = \sum_{n=0}^{\infty} (s^{n+1} - s^{-(n+1)}) s^{n+1} h_{n+1} t^{n+1} \]

\[ = \sum_{n=0}^{\infty} (s^{2n+2} - 1) h_{n+1} t^{n+1} \]

\[ = H(s^2 t) - H(t). \]

This gives \( H(s^2 t) - H(t) = A(st) H(t) - H(t) \), and thus \( H(st) = A(t) H(s^{-1} t) \), to complete the proof. \( \square \)

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References

[1] A.K. Aiston, Skein theoretic idempotents of Hecke algebras and quantum group invariants. PhD dissertation, University of Liverpool, 1996.

[2] A.K. Aiston, Adams operators and knot decorations. Liverpool University preprint 1997, q-alg 9711015.

[3] C. Blanchet, Hecke algebras, modular categories and 3-manifolds quantum invariants. *Topology* 39 (2000), 193-223.

[4] R. Dipper and G.D. James, Blocks and idempotents of Hecke algebras of general linear groups. *Proc. London Math. Soc.* 54 (1987), 57-82.

[5] A. Gyoja, A q-analogue of Young symmetrisers. *Osaka J. Math.* 23 (1986), 841-852.
[6] V.F.R. Jones, Hecke algebra representations of braid groups and link polynomials. *Ann. Math.* 126 (1987), 335-388.

[7] A. Jucys, Factorization of Young’s projection operators for symmetric groups. *Litovsk. Fiz. Sb.* 11 (1971), 1-10.

[8] J. Katriel, B. Abdesselam and A. Chakrabarti, The fundamental invariant of the Hecke algebra $H_n(q)$ characterizes the representations of $H_n(q)$, $S_n$, $SU_q(N)$ and $SU(N)$. *J. Math. Phys.* 36 (1995), 5139-5158.

[9] K. Kawagoe, On the skeins in the annulus and applications to invariants of 3-manifolds. *J. Knot Theory Ramifications* 7 (1998), 187–203.

[10] S. Lukac, PhD dissertation, University of Liverpool, 2001.

[11] I.G. Macdonald, Symmetric functions and Hall polynomials. Clarendon Press, Oxford, 2nd edition (1995).

[12] H.R. Morton, Invariants of links and 3-manifolds from skein theory and from quantum groups. In ‘Topics in knot theory’, the Proceedings of the NATO Summer Institute in Erzurum 1992, NATO ASI Series C 399, ed. M.Bozhüyük. Kluwer 1993, 107-156.

[13] H.R. Morton and A.K. Aiston, Idempotents of Hecke algebras of type A. *J. Knot Theory Ramifications* 7 (1998), 463-487.

[14] G.E. Murphy, A new construction of Young’s seminormal representation of the symmetric groups. *J. Algebra* 69 (1981), 287-297.

[15] A. Ram, Seminormal representations of Weyl groups and Iwahori-Hecke algebras. *Proc. London Math. Soc.* 75 (1997), 99-133.

[16] V.G. Turaev, The Conway and Kauffman modules of a solid torus. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 167 (1988), *Issled. Topol.* 6, 79-89.

[17] H. Wenzl, Representations of braid groups and the quantum Yang-Baxter equation. *Pacific J. Math.* 145 (1990), 153-180.