The Analytical Solution of the Lag-Lead Compensator

Li Li, Zhengpeng Wu

Department of Automation, Tsinghua University, Beijing, China 100084

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Abstract

In this paper, we first give the analytical solution of the general lag-lead compensator design problem. Then, we show why a series of more than 5 phase-lead/phase-lag compensator cannot be solved analytically using the Galois Theory.

1 Introduction

The well known lag-lead compensator design problem is a typical frequency controller design problem; see also the related discussions in the textbooks listed in [1]. During the last four decades, different design methods were proposed [2]-[9]. The analytical design procedures for single continuous phase-lag and phase-lead compensator have been given in several literatures, e.g. [8]. An analytical solving procedure is constructed for three-parameter lag-lead compensators in [9]. But that method cannot be directly applied to four-parameter cases. A universal design chart based four-parameter lag-lead compensator design method was proposed in [6]. Though it makes great progress to avoid manual graphical manipulations in design, it is still a graph based approach and sometimes does not yield the accurate solution. To our best knowledge, the analytical solution of four-parameter or even more general lag-lead compensator remains unsolved till now.

In this paper, we will first give the analytical solution of the general lag-lead compensator. Then, we will show why a series of more than 5 phase-lead/phase-lag compensator usually cannot be analytically determined using the Galois Theory.

*Corresponding author: li-li@mail.tsinghua.edu.cn
2 The Analytical Solution for the General Lag-Lead Compensator

In general, a \( n \)th-order lag-lead compensator \( (n \geq 1) \) can be written as

\[
G_c(s) = \frac{s^n + b_1s^{n-1} + \ldots + b_n}{s^n + a_1s^{n-1} + \ldots + a_n}
\]  

(1)

where \( a_i, \ b_i \in \mathbb{R}^+ \cup \{0\} \), for \( i = 1, \ldots, n \), due to the requirement of casual stability.

Substitute \( s \) with \( j\omega \), we get

\[
\bar{G}_c(j\omega) = \frac{(j\omega)^n + b_1(j\omega)^{n-1} + \ldots + b_n}{(j\omega)^n + a_1(j\omega)^{n-1} + \ldots + a_n}
\]

(2)

Usually, the dedicated performance requirements are given as several pairs of gain and phase at certain frequencies. For the \( k \)th performance requirement, we have

\[
\bar{G}_c(j\omega_k) = \frac{(j\omega_k)^n + b_1(j\omega_k)^{n-1} + \ldots + b_n}{(j\omega_k)^n + a_1(j\omega_k)^{n-1} + \ldots + a_n} = g_k \cos(p_k) + g_k \sin(p_k)j
\]

(3)

where \( g_k \) and \( p_k \) are the corresponding gain and phase at frequency \( \omega_k \), for \( k \in \mathbb{N} \).

Eq. (3) can be rewritten as

\[
(j\omega)^n + b_1(j\omega)^{n-1} + \ldots + b_n = [(j\omega_k)^n + a_1(j\omega_k)^{n-1} + \ldots + a_n] [g_k \cos(p_k) + g_k \sin(p_k)j]
\]

(4)

1) If \( n \) is an even integer satisfying \( n = 2m \), \( m \in \mathbb{N} \). From Eq. (4), we can have

\[
(-1)^m \omega_k^{2m} + \sum_{q=1}^{m} (-1)^{m-q} \omega_k^{2m-2q}b_{2q} + j \sum_{q=1}^{m} (-1)^{m-q} \omega_k^{2m-2q+1}b_{2q-1}
\]

\[
\begin{aligned}
&= \bigg( (-1)^m \omega_k^{2m} + \sum_{q=1}^{m} (-1)^{m-q} \omega_k^{2m-2q}a_{2q} + j \sum_{q=1}^{m} (-1)^{m-q} \omega_k^{2m-2q+1}a_{2q-1} \bigg) [g_k \cos(p_k) + g_k \sin(p_k)j] \\
&= g_k \cos(p_k) \bigg( (-1)^m \omega_k^{2m} + \sum_{q=1}^{m} (-1)^{m-q} \omega_k^{2m-2q}a_{2q} \bigg) - g_k \sin(p_k) \sum_{q=1}^{m} (-1)^{m-q} \omega_k^{2m-2q+1}a_{2q-1} \\
&+ j \bigg[ g_k \sin(p_k) \bigg( (-1)^m \omega_k^{2m} + \sum_{q=1}^{m} (-1)^{m-q} \omega_k^{2m-2q}a_{2q} \bigg) + g_k \cos(p_k) \sum_{q=1}^{m} (-1)^{m-q} \omega_k^{2m-2q+1}a_{2q-1} \bigg]
\end{aligned}
\]

(5)

which finally leads to the following two linear equations of \( a_i, \ b_i \), for \( i = 1, \ldots, n \).
\[
\sum_{q=1}^{m} (-1)^{m-q} \omega_k^{2m-2q} b_{2q} - g_k \cos(p_k) \sum_{q=1}^{m} (-1)^{m-q} \omega_k^{2m-2q} a_{2q}
\]
\[+ g_k \sin(p_k) \sum_{q=1}^{m} (-1)^{m-q} \omega_k^{2m-2q+1} a_{2q-1} = -(-1)^m \omega_k^{2m} + g_k \cos(p_k)(-1)^m \omega_k^{2m-2q} \]
\[
\sum_{q=1}^{m} (-1)^{m-q} \omega_k^{2m-2q+1} b_{2q-1} - g_k \sin(p_k) \sum_{q=1}^{m} (-1)^{m-q} \omega_k^{2m-2q} a_{2q}
\]
\[+ g_k \cos(p_k) \sum_{q=1}^{m} (-1)^{m-q} \omega_k^{2m-2q+1} a_{2q-1} = g_k \sin(p_k)(-1)^m \omega_k^{2m} \quad (7)
\]

II) Similarly, if \( n \) is an odd integer satisfying \( n = 2m - 1 \), \( m \in \mathbb{N} \), the \( k \)th performance requirement will also lead to two linear equations of \( a_i \), \( b_i \), for \( i = 1, \ldots, n \).

In the rest of this paper, we will call \( r \) performance requirement pairs \((g_k, p_k, \omega_k), k = 1, \ldots, r\) are feasible, if they lead to a 2\( r \) consistent and linearly independent (irreducible) equation set defined as (6)-(7). As a result, we can reach the following conclusion.

**Theorem 1** Suppose we have \( r \) feasible performance requirement pairs \((g_k, p_k, \omega_k), k = 1, \ldots, r\). If \( r < n \), we may have infinite possible solutions of this compensator. If \( r > n \), we cannot find a feasible solution of this compensator. If \( r = n \), we can formulate a 2\( n \) consistent and linearly independent linear equation set for these 2\( n \) unknown parameters \( a_i, b_i \), for \( i = 1, \ldots, n \). Thus, we can get the analytical solution of this lag-lead compensator directly by solving this linear equations set (e.g. using Cramer’s rule).

It is easy to prove that the analytical solving methods of phase-lag/phase-lead and three-parameter lag-lead compensator design problem proposed in [8-9] are indeed special cases of the above method.

## 3 Further Discussions

There are two interesting questions concerning the lag-lead compensator design problems. The first question is

**Question 1:** Determine whether a set of performance requirement pairs \((g_k, p_k, \omega_k), k = 1, \ldots, n\) is feasible for a \( n \)th-order lag-lead compensator.

From the above discussion, we can see that a set of \( n \) performance requirement pairs is feasible unless they lead to 2\( n \) consistent and linearly independent. Moreover, it is often required the lag-lead compensator to be casual stable. Thus, we need to check the algebraic stability criterion for the following equation

\[ s^n + a_1 s^{n-1} + \ldots + a_n = 0 \quad (8) \]
after obtaining $a_i$, $i = 1, \ldots, n$.

The necessary and sufficient algebra stability criterion for Eq. (8) is hard to find. However, we can apply Routh-Hurwitz stability criterion which is necessary and frequently sufficient. Since readers are familiar with this issue, we will not discuss the details.

The second question is

**Question 2**: Determine whether we find a series of $n$ phase-lead/phase-lag compensator connected as

$$G_c(s) = \frac{s + d_1}{s + c_1} \cdot \frac{s + d_2}{s + c_2} \cdot \ldots \cdot \frac{s + d_n}{s + c_n}$$

which can satisfy a set of performance requirement pairs $(g_k, p_k, \omega_k)$, $k = 1, \ldots, n$. Here, $c_i, d_i \in \mathbb{R}$, for $i = 1, \ldots, n$.

From the above discussion, if this set of performance requirement pairs $(g_k, p_k, \omega_k)$, $k = 1, \ldots, n$, is feasible, we have

$$G_c(s) = \frac{(s + d_1)(s + d_2)(s + d_n)}{(s + c_1)(s + c_2)(s + c_n)} = \frac{s^n + b_1s^{n-1} + \ldots + b_n}{s^n + a_1s^{n-1} + \ldots + a_n}$$

(10)

where $a_i, b_i$, $i = 1, \ldots, n$ are calculated from the selected performance requirements using the above method.

Thus, **Question 2** is equal to finding the roots of $s^n + a_1s^{n-1} + \ldots + a_n = 0$ and $s^n + b_1s^{n-1} + \ldots + b_n = 0$.

Based on the well known Galois theory [10]-[12], we can always find the analytical solution of $c_i, d_i$, $i = 1, \ldots, n$ for $n \in \{1, 2, 3, 4\}$. But generally, we cannot find the analytical solution for $n \geq 5$.

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