MARKOV DYNAMICS ON THE DUAL OBJECT TO THE
INFINITE-DIMENSIONAL UNITARY GROUP

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References

1. Preface

These are notes for a mini-course of 3 lectures given at the St. Petersburg School
in Probability and Statistical Physics (June 2012). My aim was to explain, on the
example of a particular model, how ideas from the representation theory of big
groups can be applied in probabilistic problems. The material is based on the joint
paper [7] by Alexei Borodin and myself; a broader range of topics is surveyed in the
lecture notes by Alexei Borodin and Vadim Gorin [3].

The main result of [7] consisted in constructing a family of Feller Markov processes
living on the infinite-dimensional locally compact space $\hat{U}(\infty)$, a kind of dual object
to the infinite-dimensional unitary group $U(\infty)$. By definition, the group $U(\infty)$ is
the union of the chain of compact unitary groups $U(N)$, $N = 1, 2, \ldots$, embedded to
each other. Dually, the space $\hat{U}(\infty)$ appears as the “entrance boundary” of a chain

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of discrete sets $\widehat{U}(N)$ related to each other by certain stochastic matrices. This structure plays a key role in our construction of Markov dynamics on $\widehat{U}(\infty)$.

The problem solved in [7] is in some (nonconventional and not strictly defined) sense dual to the problem of constructing an infinite-dimensional analog of the fundamental Dyson’s model [14] of an $N$-particle non-colliding process coming from the Brownian motion on $U(N)$. The latter problem, initiated by Spohn [43], is investigated in recent works Katori–Tanemura [29], [30], [31], and Osada [41]. In our problem, the role of Dyson’s model is played by a family of continuous time Markov chains on $\widehat{U}(N)$. At first glance, it looks much more sophisticated than Dyson’s model but actually it turns out to be more friendly.

The method used in [7] was also applied to other models in Borodin–Gorin [2] and Borodin–Olshanski [10].

The prerequisites for reading the present notes are modest: an acquaintance with the basics of Markov processes is enough, and no real knowledge of representation theory is assumed.

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2. Dyson’s model

Let us start with recalling a classical fact. Consider the classical multidimensional Brownian motion in $\mathbb{R}^N$, $BM(\mathbb{R}^N)$, whose generator is half the Laplacian. Because the Laplacian admits a separation of variables in the polar coordinates, the radial part of $BM(\mathbb{R}^N)$ is still a Markov process. Namely, it is the Bessel process $BES^N$ on the halfline $\mathbb{R}^+ = \{ r \in \mathbb{R} : r \geq 0 \}$; the generator of $BES^N$ is the ordinary differential operator

$$\frac{1}{2} \left( \frac{d^2}{dr^2} + \frac{N - 1}{r} \frac{d}{dr} \right).$$

See, e.g., Itô-McKean [27].

A similar effect holds for a number of other multidimensional diffusion processes, in particular, for the Brownian motion on the unitary group, see, e.g., Dyson [14], McKean [33]. This diffusion process, which we denote by $BM(U(N))$, lives on the group $U(N)$ of $N \times N$ unitary matrices and is generated by a two-sided invariant second order differential operator on that group. The analog of the radial projection $\mathbb{R}^N \to \mathbb{R}^+$ is the map assigning to a generic unitary matrix $g \in U(N)$ the collection $(u_1, \ldots, u_N)$ of its eigenvalues, which we interpret as an unordered $N$-tuple of points on the unit circle $\mathbb{T} := \{ u \in \mathbb{C} : |u| = 1 \}$. Note that if $g \in U(N)$ is in general position, then the eigenvalues $u_i$ are pairwise distinct. The assignment $g \mapsto (u_1, \ldots, u_N)$ maps $U(N)$ onto $\mathbb{T}^N/S_N$, the quotient of the $N$-fold product space $\mathbb{T}^N$ with respect to the action of the symmetric group $S_N$ permuting the coordinates. Thus, $\mathbb{T}^N/S_N$ plays the role of the halfline.
It turns out that one can define the radial part of $BM(U(N))$, which is a diffusion process on $\mathbb{T}^N/S_N$; let us denote it by $X_N$.

To describe the generator of $X_N$, it is convenient to pass from the “multiplicative coordinates” $u_1, \ldots, u_N$ to the “additive coordinates” $x_1, \ldots, x_N$ by setting $u_k = \exp(\sqrt{-1} x_k)$, where $k = 1, \ldots, N$ and $x_k \in \mathbb{R}/2\pi\mathbb{Z}$. In these coordinates, the generator in question, denoted by $D_N$, can be written in the form

$$D_N = V_N^{-1} \circ \Delta_N \circ V_N + C_N,$$

(2.1)

where

$$V_N = V_N(x_1, \ldots, x_N) := \prod_{1 \leq i < j} |u_i - u_j| = \text{const} \prod_{1 \leq i < j \leq N} \sin \frac{x_i - x_j}{2},$$

$$\Delta_N := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2},$$

$$C_N = \frac{(N - 1)N(N + 1)}{12}.$$

In words, (2.1) means that to apply $D_N$ to a function $F$ we first multiply $F$ by $V_N$, then apply the Laplacian $\Delta_N$, then divide by $V_N$, and finally add $C_N F$. Since

$$\Delta_N V_N = -C_N V_N,$$

$D_N$ annihilates the constants.

Formula (2.1) is a kind of Doob’s $h$-transform (see Rogers–Williams [44]) applied to the “flat” Brownian motion generated by the Laplacian $\Delta_N$, where $h = V_N$. However, $V_N$ is not a harmonic function for $\Delta_N$ but only an eigenfunction; this explains the appearance of the compensating term $C_N$.

More explicitly, (2.1) can be rewritten as

$$D_N = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{N} \left( \sum_{\alpha: \alpha \neq i} \cot \frac{x_i - x_\alpha}{2} \right) \frac{\partial}{\partial x_i}.$$

(2.2)

Although the coefficients of the first order derivatives have singularities along the diagonals $x_i = x_j$, these singularities are cancelled when $D_N$ is applied to smooth \textit{symmetric} functions in $x_1, \ldots, x_N$. Note that natural “observables” on the quotient space $\mathbb{T}^N/S_N$ are just symmetric functions in the coordinates on $\mathbb{T}^N$.

In contrast to the Bessel process, the process $X_N$ generated by $D_N$ has a stationary distribution $\mu_N$; it is the radial part of the normalized Haar measure on $U(N)$. The density of $\mu_N$ with respect to Lebesgue measure $dx_1 \ldots dx_N$ is proportional to $V_N^2$. The measure $\mu_N$ first emerged in the context of Weyl’s character formula, see Weyl [49]. In Dyson’s interpretation, $\mu_N$ is the law of a system of $N$ interacting point particles on the unit circle $\mathbb{T}^N$, called the \textit{circular unitary ensemble} and usually denoted as $CUE_N$, see Dyson [14], Mehta [34].
A well-known result says that, in a suitable large-$N$ scaling limit regime, $CUE_N$ turns into an ensemble of infinitely many interacting particles on $\mathbb{R}$. The distribution of the particles is best described in terms of the correlation functions, which are determined by a simple translation invariant correlation kernel on $\mathbb{R}$, called the sine kernel. See, e.g., Mehta [34].

In view of this fact it is natural to ask what happens with the process $X_N$ in the same scaling limit regime: does there exist a Markov process $X_\infty$ which would be a large-$N$ limit (in some reasonable sense) of the diffusions $X_N$? Using the dynamical correlation functions one can check that if the initial distribution of the process $X_N$ is $\mu_N$, then its multi-time finite-dimensional distributions survive in a suitable scaling limit transition as $N \to \infty$. However, this is insufficient to conclude that $X_\infty$ does exist.

In the attempt to imagine the possible form of the generator of $X_\infty$, let us examine the limiting behavior of the operators $D_N$ (the informal argument below follows the discussion in the beginning of Spohn’s paper [45]).

The scaling limit in question consists in a change of variables,

$$x_i \leadsto y_i, \quad y_i := \frac{N}{2\pi} x_i, \quad 1 \leq i \leq N.$$ 

We assume that the initial coordinates $x_i$ range over the interval $(-\pi, \pi)$; then the new coordinates $y_i$ range over the interval $(-N/2, N/2)$ of length $N$, so that the mean density of particles equals 1. Writing

$$\cot \frac{x_i - x_\alpha}{2} = \cot \left( \frac{\pi}{N} (y_i - y_\alpha) \right) \sim \frac{N}{2\pi} \frac{2}{y_i - y_\alpha}$$

we get

$$\left( \frac{2\pi}{N} \right)^2 D_N \sim \sum_i \frac{\partial^2}{\partial y_i^2} + 2 \left( \sum_{\alpha: \alpha \neq i} \frac{1}{y_i - y_\alpha} \right) \frac{\partial}{\partial y_i}.$$ 

Next, the factor $\left( \frac{2\pi}{N} \right)^2$ can be eliminated by rescaling the time parameter, so that finally we are left with the formal differential operator

$$D_\infty := \sum_i \frac{\partial^2}{\partial y_i^2} + 2 \left( \sum_{\alpha: \alpha \neq i} \frac{1}{y_i - y_\alpha} \right) \frac{\partial}{\partial y_i},$$

where we may assume that the index $i$ ranges over $\mathbb{Z}$ and $\ldots < y_{-1} < y_0 < y_1 < \ldots$.

We see that the series in the brackets, in general, diverges, and even if we manage to regularize it, it is highly non-evident how to prove that a suitable regularization of $D_\infty$ serves as a (pre)generator of a Markov process.

This informal argument shows that a naive direct approach to constructing the generator of $X_\infty$ faces serious difficulties.
3. The one-particle dynamics: a bilateral birth-death process

I proceed to the model from our work \[7\]. It bears some resemblance with Dyson’s model and (in some informal sense to be clarified below) is dual to it.

Let us start with the simplest case when \(N\), the number of particles, equals 1. The one-particle Dyson model is very simple, it is the conventional Brownian motion on the unit circle \(\mathbb{T}\). I will explain what I mean by the corresponding “dual model”.

The space \(\mathbb{T}\) is a compact Abelian group, and its Pontryagin dual is the discrete Abelian group \(\mathbb{Z}\). So it is not surprising that the “dual” state space is \(\mathbb{Z}\). But what is a substitute of the Brownian motion? As \(\mathbb{Z}\) is discrete, it cannot be a diffusion process, it should be a jump process or, in other words, a Markov chain. We want a continuous time process, so that it is a continuous time Markov chain.

The generator of the Brownian motion on \(\mathbb{T}\) is the simplest second order differential operator, \(d^2/dx^2\), where \(x\) is the “additive coordinate” as above. A natural lattice analog of this operator should be a second order difference operator \(D\) on \(\mathbb{Z}\) transforming a test function \(F(l)\) to the function

\[
DF(l) = a^+(l)(F(l + 1) - F(l)) + a^-(l)(F(l - 1) - F(l)), \quad l \in \mathbb{Z},
\]

where the coefficients \(a^+(l)\) and \(a^-(l)\) represent the rates of the jumps \(l \to l + 1\) and \(l \to l - 1\), respectively (these are the only possible jumps).

At first glance, the most natural choice of the coefficients is to set \(a^+(l) = a^-(l) = \text{const} > 0\). This leads to the Markov chain which looks as the most natural discrete analog of the classical Brownian motion. However, this chain is not suitable for our purposes as it does not possess a stationary distribution. Certainly, the counting measure on \(\mathbb{Z}\) is invariant, but it is infinite, while we would like to have a finite measure, as in the case of \(\mathbb{T}\). For this reason we reject the constant coefficients.

The next possible variant would be to make the coefficients \(a^\pm(l)\) some linear functions in \(l\). This indeed allows one to get examples of processes with a stationary distribution (some birth-death processes). However, they cannot live on the whole lattice \(\mathbb{Z}\), because a linear function changes the sign, while the jump rate cannot take negative values. Since we want to deal with the whole lattice, we reject this variant as well.

Let us try now quadratic rates \(a^\pm(l)\). The leading terms in \(a^+(l)\) and \(a^-(l)\) must coincide to prevent a growing drift to \(+\infty\) or \(-\infty\) (such a drift is obviously incompatible with a stationary distribution). Then, without loss of generality, we may assume that \(a^\pm(l)\) equals \(l^2\) plus lower degree terms. Writing such a quadratic function as the product of two linear factors we set

\[
a^+(l) = (u - l)(u' - l), \quad a^-(l) = (v + l)(v' + l),
\]

where \((u, u', v, v')\) is a quadruple of parameters. Note that the change \(l \to -l\) amounts to switching \((u, u') \leftrightarrow (v, v')\).
Finally, we want \( a^\pm(l) \) to take strictly positive values for all \( l \in \mathbb{Z} \). Let us say that a couple \((z, z')\) of complex numbers is \textit{admissible} if \((u - l)(u' - l) > 0\) for all \( l \in \mathbb{Z} \). We will assume that both \((u, u')\) and \((v, v')\) are admissible.

It is not difficult to classify all admissible couples. Namely, \((u, u')\) is admissible if and only if

- either both \( u \) and \( u' \) are nonreal complex numbers and \( u' = \overline{u} \);
- or both \( u \) and \( u' \) are real and there exists \( m \in \mathbb{Z} \) such that \( m < u, u' < m + 1 \).

It turns out that quadratic rates give the desired result:

**Theorem 3.1.** Assume \((u, u')\) and \((v, v')\) are admissible couples of parameters.

(i) There exists a continuous time Markov chain \( X_1^{(u, u', v, v')} \) on \( \mathbb{Z} \), such that the only possible jumps are of the form \( l \to l \pm 1 \) and their rates \( a^\pm(l) \) are given by \((3.2)\).

(ii) The chain \( X_1^{(u, u', v, v')} \) possesses a unique, within a constant factor, symmetrizing measure. This measure is finite if and only if the parameters satisfy the additional constraint \( u + u' + v + v' > -1 \).

Note that \( u + u' + v + v' \) is a real number because \((u, u')\) and \((v, v')\) are admissible. Note also that a symmetrizing measure is automatically invariant. We denote the symmetrizing measure of our Markov chain by \( M_1^{(u, u', v, v')} \). Here is an explicit expression for it:

\[
M_1^{(u, u', v, v')}(l) = \text{const} \frac{1}{\Gamma(u + 1 - l)\Gamma(u' + 1 - l)\Gamma(v + 1 + l)\Gamma(v' + 1 + l)}. \tag{3.3}
\]

The normalization constant is found from a beautiful classical hypergeometric identity due to Dougall [12] (see also Erdelyi [15, §1.4]),

\[
\sum_{l \in \mathbb{Z}} \frac{1}{\Gamma(u + 1 - l)\Gamma(u' + 1 - l)\Gamma(v + 1 + l)\Gamma(v' + 1 + l)} = \frac{\Gamma(u + u' + v + v' + 1)}{\Gamma(u + v + 1)\Gamma(u + v' + 1)\Gamma(u' + v + 1)\Gamma(u' + v' + 1)}. \tag{3.4}
\]

If \( u + u' + v + v' > -1 \), then we may take as the constant factor in \((3.3)\) the quantity inverse to the right-hand side in \((3.4)\); with this normalization \( M_1^{(u, u', v, v')} \) becomes a probability measure.

The Markov chain \( X_1^{(u, u', v, v')} \) is an example of so-called bilateral birth and death processes, see Feller [19, Section 17], Pruitt [43], Yan [51].

The above arguments are intended to convince the reader that the definition of the chain \( X_1^{(u, u', v, v')} \) is quite natural. But in reality, this definition came from other considerations, related to our previous work on harmonic analysis on big groups: [4], [5], [6], [40].
4. The $N$-particle dynamics

In this section $(u, u', v, v')$ is a fixed quadruple of parameters such that $(u, u')$ and $(v, v')$ are admissible, and $N \geq 2$ is a fixed natural number.

We are dealing with the lattice $\mathbb{Z}^N$; its elements are denoted as $\ell = (l_1, \ldots, l_N)$. Denote by $D$ the 1-dimensional difference operator (3.1) with the coefficients given by (3.2), and let $D^{[i]}$ stand for a copy of $D$ acting on the $i$th coordinate of $\ell$, where $i = 1, \ldots, N$. The operator

$$D^{\text{free}}_N := \sum_{i=1}^{N} D^{[i]},$$  \hspace{1cm} (4.1)

generates a continuous time Markov chain on $\mathbb{Z}^N$, which is simply the product of $N$ independent copies of the chain $X^{(u, u', v, v')}_1$.

The next step is to apply to this chain the Doob $h$-transform (cf. (2.1)), taking as $h$ the function

$$V_N(\ell) := \prod_{1 \leq i < j \leq N} (l_i - l_j).$$

Note that $V_N$ is an eigenfunction of $D^{\text{free}}_N$:

$$D^{\text{free}}_N V_N = -C_N V_N,$$

where

$$C_N = C_N(u, u', v, v') := \frac{N(N-1)}{2} (u + u' + v + v') - \frac{N(N-1)(N-2)}{3}.$$

Consider the region $\Omega_N \subset \mathbb{Z}^N$ defined by

$$\Omega_N := \{ \ell \in \mathbb{Z}^N : l_1 > \cdots > l_N \}.  \hspace{1cm} (4.2)$$

**Theorem 4.1.** Assume $(u, u')$ and $(v, v')$ are admissible couples of parameters.

(i) There exists a continuous time Markov chain $\tilde{X}^{(u, u', v, v')}_N$ on $\Omega_N \subset \mathbb{Z}^N$, whose infinitesimal generator is given by the difference operator

$$D_N := V_N^{-1} \circ D^{\text{free}}_N \circ V_N + C_N.$$

(ii) The chain $\tilde{X}^{(u, u', v, v')}_N$ possesses a unique, within a constant factor, symmetrizing (and hence invariant) measure $\tilde{M}^{(u, u', v, v')}_N$:

$$\tilde{M}^{(u, u', v, v')}_N(\ell) = \text{const} \prod_{i=1}^{N} \left( \frac{1}{\Gamma(u - \ell_i + 1) \Gamma(u' - \ell_i + 1)} \times \frac{1}{\Gamma(v + \ell_i + 1) \Gamma(v' + \ell_i + 1)} \right) \cdot (V_N(\ell))^2, \hspace{1cm} \ell \in \Omega_N.  \hspace{1cm} (4.3)$$

This measure is finite if and only if the parameters satisfy the additional constraint $u + u' + v + v' > 2N - 3$. 


The possible jumps of the chain are of the form $\ell \to \ell \pm \varepsilon_i$, where $\varepsilon_i$ denotes the $i$th basis vector in $\mathbb{R}_N$, $i = 1, \ldots, N$. Note that if $\ell \in \Omega_N$ but $\ell + \varepsilon_i$ or $\ell - \varepsilon_i$ does not belong to $\Omega_N$, then the corresponding rate automatically vanishes, so that the chain does not exit from $\Omega_N$.

Explicitly, the rate of the jump $\ell \to \ell \pm \varepsilon_i$ equals
\[
\frac{V_N(\ell \pm \varepsilon_i)}{V_N(\ell)} \cdot \left\{ \left( u - l_i \right) \left( u' - l_i \right) \right\} \left\{ \left( v + l_i \right) \left( v' + l_i \right) \right\},
\]
where the upper/lower quantity in the braces corresponds to the plus/minus sign, respectively.

As will be shown in Section 7, $\Omega_N$ serves as the set of parameters for $\widehat{U}(N)$, the dual object to the unitary group $U(N)$. This is why we view the dynamics just introduced as “dual” to the Dyson model of Section 2.

5. The method of intertwiners

Here I describe a general formalism which will be used for constructing a model of infinite-dimensional Markov dynamics out of the Markov chains $X_N^{(u,u',v,v')}$. For more detail, see Borodin–Olshanski [7], [9].

An $m$-dimensional simplex $\Delta^m$ in a vector space has $m + 1$ vertices, and each point of $\Delta^m$ is uniquely represented as a convex linear combination of the vertices. It follows that $\Delta^m$ can be identified with the set of the probability measures on the set of the vertices.

Let us adopt this viewpoint and, more generally, given a finite or countably infinite abstract set $\mathcal{X}$, we define the simplex with the vertex set $\mathcal{X}$ as the set of probability measures on $\mathcal{X}$.

Even more generally, let $\mathcal{X}$ be a measurable space, that is, a set with a distinguished $\sigma$-algebra $\mathcal{B}(\mathcal{X})$ of subsets called measurable subsets (in another terminology, $\mathcal{X}$ is a Borel space). We assume that $\mathcal{B}(\mathcal{X})$ contains all singletons. Denote by $\mathcal{M}(\mathcal{X})$ the space of probability measures defined on $\mathcal{B}(\mathcal{X})$. We regard $\mathcal{M}(\mathcal{X})$ as a generalized simplex.

A Markov kernel between two measurable spaces $\mathcal{X}$ and $\mathcal{Y}$ is a function $K(x, B)$, where the first argument $x$ ranges over $\mathcal{X}$ and the second argument ranges over $\mathcal{B}(\mathcal{Y})$, and such that the following two conditions hold:

- $K(\cdot, B)$ is a measurable function on $\mathcal{X}$ for any fixed $B \in \mathcal{B}(\mathcal{Y})$;
- $K(x, \cdot)$ is a probability measure on $\mathcal{Y}$ for any fixed $x \in \mathcal{X}$.

If both $\mathcal{X}$ and $\mathcal{Y}$ are finite or countably infinite sets, then $K$ is simply a stochastic matrix of format $\mathcal{X} \times \mathcal{Y}$. About Markov kernels, see, e.g., Meyer [35].

We regard $K$ as a “link” between $\mathcal{X}$ and $\mathcal{Y}$ and write this symbolically as $K : \mathcal{X} \longrightarrow \mathcal{Y}$. Sometimes we use the word “link” as a synonym of “Markov kernel”. A link is not an ordinary map; this is why we represent it by a dash arrow. However,
it determines a true map \(\mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathcal{Y})\) taking a measure \(M \in \mathcal{M}(\mathcal{X})\) to the measure \(MK \in \mathcal{M}(\mathcal{Y})\) defined by
\[
(MK)(B) := \int_{x \in \mathcal{X}} M(dx)K(x, B), \quad B \in \mathcal{B}(\mathcal{Y}).
\]

We regard such a map \(\mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathcal{Y})\) as an “affine map” between (generalized) simplices. It is a conventional affine map if both \(\mathcal{X}\) and \(\mathcal{Y}\) are discrete.

Given two links, \(K: \mathcal{X} \to \mathcal{Y}\) and \(L: \mathcal{Y} \to \mathcal{Z}\), their composition \(KL: \mathcal{X} \to \mathcal{Z}\) is defined by
\[
(KL)(x, dz) = \int_{y \in \mathcal{Y}} K(x, dy)L(y, dz).
\]
This is a natural generalization of the matrix multiplication.

Thus, one may consider the category whose objects are measurable spaces and morphisms are links. We need the corresponding notion of projective limit. To avoid excessive formalism, I define this notion precisely in the degree of generality that we really need.

Assume we are given an infinite chain of finite or countably infinite sets together with links between them:
\[
\cdots \to \mathcal{X}_N \to \mathcal{X}_{N-1} \to \cdots \to \mathcal{X}_2 \to \mathcal{X}_1.
\]
Because the spaces are discrete, the links are simply stochastic matrices. The link between \(\mathcal{X}_N\) and \(\mathcal{X}_{N-1}\) will be denoted by \(\Lambda^N_{N-1}\). The “categorical” projective limit of (5.1) is explicitly constructed as follows.

Chain (5.1) gives rise to a chain of affine maps of simplices
\[
\cdots \to \mathcal{M}(\mathcal{X}_N) \to \mathcal{M}(\mathcal{X}_{N-1}) \to \cdots \to \mathcal{M}(\mathcal{X}_2) \to \mathcal{M}(\mathcal{X}_1).
\]

Let \(\lim_\leftarrow \mathcal{M}(\mathcal{X}_N)\) be the (conventional) projective limit of (5.2). By the very definition, elements of \(\lim_\leftarrow \mathcal{M}(\mathcal{X}_N)\) are sequences \(\{M_N \in \mathcal{M}(\mathcal{X}_N)\}_N\) such that \(M_N\Lambda^N_{N-1} = M_{N-1}\) for every \(N \geq 2\). Such a sequence is called a coherent family of measures; here “coherence” means that \(M_N\)’s are consistent with the links. The next theorem says that \(\lim_\leftarrow \mathcal{M}(\mathcal{X}_N)\) is a (possibly, generalized) simplex. More precisely, the claim is the following.

**Theorem 5.1.** There exists a measurable space \(\mathcal{X}_\infty\) and links \(\Lambda^\infty_N: \mathcal{X}_\infty \to \mathcal{X}_N\), where \(N = 1, 2, \ldots\), such that
\[
\Lambda^\infty_N\Lambda^N_{N-1} = \Lambda^\infty_{N-1}, \quad N \geq 2,
\]
and the correspondence \(M \mapsto \{M_N : N = 1, 2, \ldots\}\) defined by \(M_N := M\Lambda^\infty_N\) is a bijection between \(\mathcal{M}(\mathcal{X}_\infty)\) and \(\lim_\leftarrow \mathcal{M}(\mathcal{X}_N)\).

Such a space together with the links \(\Lambda^\infty_N\) is unique within a natural equivalence.

A proof based on Choquet’s theorem is given in Olshanski [40, §9], a more general result is contained in Winkler [50, Chapter 4].
Note that \( \mathcal{X}_\infty \) can be identified with the space of extreme points of the set \( \lim \mathcal{M}(\mathcal{X}_N) \) (which is obviously a convex set), and the nontrivial part of the theorem is that \( \lim \mathcal{M}(\mathcal{X}_N) \) coincides with \( \mathcal{M}(\mathcal{X}_\infty) \).

Note also that for infinite sets \( \mathcal{X}_N \) it may happen that \( \lim \mathcal{M}(\mathcal{X}_N) \) is empty (that is, there is no coherent families of probability measures) and then \( \mathcal{X}_\infty \) is empty, too. Here is a simple example: \( \mathcal{X}_N = \{N, N+1, \ldots\} \) and the link \( \mathcal{X}_N \rightarrow \mathcal{X}_{N-1} \) is induced by the inclusion \( \mathcal{X}_N \subset \mathcal{X}_{N-1} \). However, if all \( \mathcal{X}_N \) are finite sets, then \( \mathcal{X}_\infty \) is always nonempty.

Let us regard (5.1) as a kind of discrete time Markov chain with the transition probabilities determined by the links \( \Lambda^N_{N-1} \). (It does not matter that this chain looks a bit unusual, as the time parameter ranges from \(-\infty\) to 1 and the state space varies with time.) The space \( \lim \mathcal{M}(\mathcal{X}_N) = \mathcal{M}(\mathcal{X}_\infty) \) can be identified with the space of entrance laws (see Dynkin [13]) for this Markov chain; for this reason we call \( \mathcal{X}_\infty \) the boundary of (5.1), having in mind the entrance boundary.

If \( \{M_N\} \) is a coherent family of probability measures, then the corresponding measure \( M \in \mathcal{M}(\mathcal{X}_\infty) \) is called the boundary measure of the family.

By a Markov semigroup on a measurable space \( \mathcal{X} \) we mean a semigroup \( P(t) \) of Markov kernels \( \mathcal{X} \rightarrow \mathcal{X} \) depending on parameter \( t \geq 0 \). That is, the kernels are subject to the Chapman–Kolmogorov equation \( P(t_1)P(t_2) = P(t_1 + t_2) \) and \( P(0) \) is the trivial kernel corresponding to the identity map \( \mathcal{X} \rightarrow \mathcal{X} \), that is, \( P(0; x, \cdot) \) is the delta-measure at \( x \).

Under additional assumptions on a Markov semigroup \( P(t) \), one can prove that it serves as the transition function of a Markov process \( \mathcal{X} \); for instance, this is so if \( P(t) \) is Feller (see Section 10 below).

A stationary distribution for a Markov semigroup \( P(t) \) is a probability measure \( M \in \mathcal{M}(\mathcal{X}) \) such that \( MP(t) = M \) for all \( t \geq 0 \).

Assume \( P(t) \) and \( P'(t) \) are Markov semigroups with state spaces \( \mathcal{X} \) and \( \mathcal{X}' \), respectively, and \( \Lambda : \mathcal{X} \rightarrow \mathcal{X}' \) is a link. We say that \( \Lambda \) intertwines the processes if
\[
P(t)\Lambda = \Lambda P'(t), \quad t \geq 0.
\]

Now we are in a position to describe a general formalism that we call the method of intertwiners.

Let us return to the chain (5.1) of discrete spaces and the links \( \Lambda^N_{N-1} : \mathcal{X}_N \rightarrow \mathcal{X}_{N-1} \). Assume that for every \( N = 1, 2, \ldots \) we are given a Markov semigroup \( P_N(t) \) on \( \mathcal{X}_N \) (that is, simply a semigroup of stochastic matrices of format \( \mathcal{X}_N \times \mathcal{X}_N \)), and the links serve as intertwiners for these semigroups, so that
\[
P_N(t)\Lambda^N_{N-1} = \Lambda^N_{N-1}P_{N-1}(t) \quad (5.4)
\]
for every \( N \geq 2 \) and any \( t \geq 0 \). We call (5.4) the master relation. Finally, assume that the boundary \( \mathcal{X}_\infty \) of (5.1) is nonempty.
Theorem 5.2. (i) Under these hypotheses there exists a unique Markov semigroup 
\( P_\infty(t) \) on \( \mathcal{X}_\infty \) such that
\[
P_\infty(t)\Lambda_\infty^N = \Lambda_\infty^NP_N(t), \quad N = 1, 2, \ldots, \quad t \geq 0.
\]

(ii) Assume additionally that there exists a coherent family \( \{M_N\} \) of probability 
distributions such that \( M_N \) is a stationary distribution for \( P_N(t) \) for every \( N \). Then the 
corresponding boundary measure on \( \mathcal{X}_\infty \) is a stationary distribution for \( P_\infty(t) \).

These assertions are direct consequences of the definitions. We call \( P_\infty(t) \) the 
boundary Markov semigroup.

It may well happen that the semigroups \( P_N(t) \) are not given in an explicit form. 
Then, to check the master relation (5.4), one may try to reduce it to its infinitesimal 
version,
\[
D_N\Lambda_{N-1}^N = \Lambda_{N-1}^N D_{N-1},
\]
where \( D_N \) stands for the infinitesimal generator of \( P_N(t) \) and (5.5) should be 
understood as a relation for operators acting in suitable function spaces (see Section 10 below); when applied to a function \( F \), (5.5) should be read from right to left:
\[
D_N\Lambda_{N-1}^N F = \Lambda_{N-1}^N D_{N-1} F.
\]

6. Examples

Here I illustrate the formalism of the preceding section by two simple examples.

Consider the following chain of type (5.1) coming from the Pascal triangle: the 
spaces are finite sets,
\[
\mathcal{X}_N = \{0, 1, \ldots, N\} \subset \mathbb{Z},
\]
and the links are defined by
\[
\Lambda_{N-1}^N(n, m) = \begin{cases} 
\frac{N-n}{N}, & m = n, \\
\frac{n}{N}, & m = n-1, \\
0, & m \neq n, n-1.
\end{cases}
\]

The boundary \( \mathcal{X}_\infty \) of this chain can be identified with the closed unit interval 
[0, 1] and the links \( \Lambda_{N}^\infty \) are given by
\[
\Lambda_{N}^\infty(x, n) = \binom{N}{n} x^n (1-x)^{N-n}, \quad x \in [0, 1], \quad n = 0, 1, \ldots, N;
\]
that is, \( \Lambda_{N}^\infty(x, \cdot) \) is the binomial distribution with parameter \( x \). This fact is equiva-
 lent to de Finetti theorem or else to the solution of the Hausdorff moment problem. 
See, e.g., Gnedin–Pitman [23] and references therein.
Example 6.1. Fix two real parameters $a > 0$, $b > 0$. We are going to define, for every $N$, a continuous time Markov chain $X_N$ on $\mathbb{X}_N$. To do this we exhibit its generator $D_N$, which is a difference operator on $\mathbb{X}_N \subset \mathbb{Z}$. Its action on a test function $F$ is given by

$$(D_N F)(n) = (N - n)(n + a) [F(n + 1) - F(n)] + n(N + b - n) [F(n - 1) - F(n)].$$  \hspace{1cm} (6.1)$$

It is directly verified that the operators $D_N$ satisfy (5.5), from which one can deduce that the corresponding semigroups $P_N(t)$ (the transition functions of the chains $X_N$) satisfy the master relation (5.4).

Therefore, by virtue of Theorem 5.2, part (i), these semigroups give rise to a boundary Markov semigroup $P_\infty(t)$ on $[0, 1]$. One can prove that $P_\infty(t)$ is the transition function of a diffusion process $X_\infty$ on $[0, 1]$ with the infinitesimal generator

$$D_\infty = x(1 - x) \frac{d^2}{dx^2} + [a - (a + b)x] \frac{d}{dx}.$$  

For every $N$, the chain $X_N$ has a unique stationary distribution $M_N$,

$$M_N(n) = \frac{\Gamma(a + b)N!}{\Gamma(a)\Gamma(b)\Gamma(a + b + N)} \frac{\Gamma(a + n)\Gamma(b + N - n)}{n!(N - n)!}, \quad n = 0, 1, \ldots, N,$$

which is the well-known hypergeometric distribution (see, e.g., Feller [21, ch. II, §6]). The sequence $\{M_N\}$ is a coherent family, and the corresponding boundary measure $M_\infty$ is Euler’s beta distribution,

$$M_\infty(dx) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1 - x)^{b-1}dx, \quad x \in [0, 1].$$

It is a unique stationary distribution for $X_\infty$.

Example 6.2. Here the sets $\mathbb{X}_N$ and the links $\Lambda_{N-1}^N$ are as in the preceding example, but we choose different Markov chains: this time they are defined by the difference operators

$$(D'_N F)(n) = (N - n)c [F(n + 1) - F(n)] + n(1 - c) [F(n - 1) - F(n)],$$  \hspace{1cm} (6.2)$$

where $c \in [0, 1]$ is a fixed parameter. The corresponding Markov semigroups $P'_N(t)$ are again consistent with the links and so give rise to a boundary Markov semigroup $P'_\infty(t)$ on $[0, 1]$. But $P'_\infty(t)$ turns out to be degenerate in the sense that it corresponds to a deterministic process $X'_\infty$. Namely, the generator of $X'_\infty$ is a first order differential operator,

$$D'_\infty = (c - x) \frac{d}{dx},$$

so that $X'_\infty$ is not a genuine Markov process but a flow of endomorphisms of the interval $[0, 1]$ generated by a vector field. The dynamics is easily described: the
(deterministic) trajectory \(x(t)\) issued from a given point \(x(0)\) has the form
\[
x(t) = c - (c - x(0)) e^{-t}, \quad t \geq 0,
\]
so that the interval \([0, 1]\) is contracted to the point \(c\) exponentially fast.

Note that the \(N\)th chain \(X'_N\) has a unique stationary distribution \(M'_N\), which is the binomial distribution with parameter \(c\),
\[
M'_N(n) = \binom{N}{n} c^n (1 - c)^{N-n}, \quad n = 0, 1, \ldots, N.
\]
The distributions \(M'_N\) form a coherent system with the boundary measure \(\delta_c\), the delta-measure at \(c \in [0, 1]\). This agrees with the evident fact that \(\delta_c\) is a (unique) stationary distribution of the flow \(X'_\infty\).

As seen from the second example, it may happen that a boundary process constructed according to the general scheme of Section 5 degenerates to a deterministic process. So, if one wants to get a genuine Markov dynamics on the boundary, one needs additional arguments guaranteeing that such a degeneration does not occur.

### 7. Extremal characters of \(U(\infty)\) and the boundary \(\Omega_{\infty}\)

Here I introduce certain links \(\Lambda^N_{N-1}: \Omega_N \rightarrow \Omega_{N-1}\) between the subsets \(\Omega_N\) (see their definition in Section 4) and discuss the meaning of the boundary \(\Omega_{\infty}\) of the chain
\[
\cdots \rightarrow \Omega_N \rightarrow \Omega_{N-1} \rightarrow \cdots \rightarrow \Omega_2 \rightarrow \Omega_1.
\] (7.1)

In the end of Section 4, it was pointed out that \(\Omega_N\) parameterizes the dual object \(\hat{U}(N)\). I will explain this point in more detail.

By definition, the dual object \(\hat{G}\) to a compact group \(G\) is the set of equivalence classes of irreducible finite-dimensional representations of \(G\). As well known, the irreducible representations of the group \(G = U(N)\) are indexed by the vectors \(\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N\) with nonincreasing coordinates, \(\lambda_1 \geq \cdots \geq \lambda_N\); such vectors are called signatures of length \(N\) (see, e.g., Weyl [49], Zhelobenko [52]). There is a one-to-one correspondence \(\lambda \leftrightarrow \ell\) between signatures \(\lambda\) and elements \(\ell \in \Omega_N\) given by
\[
l_i = \lambda_i + N - i, \quad i = 1, \ldots, N.
\]

Thus, we may take \(\Omega_N\) as the set of parameters for the dual object to \(U(N)\).

Besides the parameterization of \(\hat{U}(N)\), the only extra fact about representations that we need is the Gelfand–Tsetlin branching rule which describes the decomposition of an irreducible representation of \(U(N)\) when restricted to \(U(N - 1) \subset U(N)\). Here \(U(N - 1)\) is considered as the subgroup of \(U(N)\) that fixes the last basis vector in \(\mathbb{C}^N\).
Let us introduce some notation. The irreducible representation of \( U(N) \) corresponding to a signature \( \lambda \) will be denoted by \( \pi^{\lambda,N} \). Two signatures \( \lambda = (\lambda_1, \ldots, \lambda_N) \) and \( \mu = (\mu_1, \ldots, \mu_{N-1}) \) are said to be \textit{interlaced} if
\[
\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N-1} \geq \mu_{N-1} \geq \lambda_N;
\]
then we write \( \mu \prec \lambda \).

The branching rule (Gelfand–Tsetlin [22], Zhelobenko [52]) says that
\[
\pi^{\lambda,N} \mid_{U(n-1)} = \bigoplus_{\mu: \mu \prec \lambda} \pi^{\mu,N-1}.
\]
Taking the dimensions of both sides gives the identity
\[
\dim \pi^{\lambda,N} \mid_{U(n-1)} = \sum_{\mu: \mu \prec \lambda} \dim \pi^{\mu,N-1}.
\]
We use it to define a link \( \Lambda_{N-1}^N : \Omega_N \to \Omega_{N-1} \), as follows. Let \( \ell \in \Omega_N, \ell' \in \Omega_{N-1} \), and let \( \lambda \leftrightarrow \ell \) and \( \mu \leftrightarrow \ell' \) be the corresponding signatures. We set
\[
\Lambda_{N-1}^N(\ell, \ell') := \begin{cases} 
\frac{\dim \pi^{\mu,N-1}}{\dim \pi^{\lambda,N}}, & \mu \prec \lambda, \\
0, & \text{otherwise.}
\end{cases}
\]
Because of the above identity, \( \Lambda_{N-1}^N \) is a stochastic matrix, so the definition is correct. Thus, we have constructed the chain (7.1).

Observe that the branching rule entails a direct combinatorial definition of the quantity \( \dim \pi^{\lambda,N} \): namely, it is equal to the total number of sequences
\[
\lambda^{(1)} \prec \lambda^{(2)} \prec \cdots \prec \lambda^{(N)} = \lambda,
\]
where \( \lambda^{(i)} \) is a signature of length \( i \). Such sequences are often written as triangular arrays, called \textit{Gelfand–Tsetlin schemes} or \textit{Gelfand–Tsetlin patterns}, see Gelfand–Tsetlin [22], Zhelobenko [52]. On the other hand, there is an explicit formula, which is a particular case of \textit{Weyl’s dimension formula} (Weyl [49], Zhelobenko [52]):
\[
\dim \pi^{\lambda,N} = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i} = \frac{\prod_{1 \leq i < j \leq N}(\ell_i - \ell_j)}{1! \ldots (N-1)!}.
\]
This makes the definition of the links formally independent of the representation theory of the unitary groups.

Let \( \Omega_\infty \) stand for the boundary of the chain (7.1). Observe that the chain of Section 6 can be embedded into (7.1). Namely, the element \( n \in \mathfrak{X}_N \) is identified with the signature of length \( N \) of the form \((1, \ldots, 1, 0, \ldots, 0)\), where the number of 1’s equals \( n \). This shows that the boundary \( \Omega_\infty \) contains the boundary of the chain of Section 6. In particular, it follows that \( \Omega_\infty \) is nonempty.

Let us define the group \( U(\infty) \) as the union of the groups \( U(N) \) embedded one into another as indicated above; \( U(\infty) \) belongs to the class of \textit{inductive limits of compact}
groups. We extend the definition of dual object to groups \( G = \lim_{\to} G_N \) from this class in the following way.

A function \( \chi : G \to \mathbb{C} \) is said to be an extremal character if it satisfies the following three conditions:

- First, \( \chi \) is normalized, that is, \( \chi(e) = 1 \).
- Second, \( \chi \) is central meaning that it is constant on each conjugacy class.
- Third, for any elements \( g_1, g_2 \in G \) one has

\[
\lim_{N \to \infty} \int_{h \in G_N} \chi(g_1 hg_2 h^{-1}) m_{G_N}(dh) = \chi(g_1) \chi(g_2),
\]

(7.2)

where \( m_{G_N} \) denotes the normalized Haar measure on the compact group \( G_N \). We define \( \hat{G} \) as the set of all such functions.

Here is an explanation why this new definition of \( \hat{G} \) extends the previous one. For a compact group \( G \), every irreducible representation is uniquely determined by its character \( \chi^\pi \): this is a function on \( G \) given by

\[
\chi^\pi(g) = \text{tr}(\pi(g)), \quad g \in G.
\]

The normalized function

\[
\tilde{\chi}^\pi(g) := \chi^\pi(g)/\chi^\pi(e) = \chi^\pi(g)/\dim \chi^\pi
\]

is called a normalized irreducible character. By the very definition, such functions may serve as parameters for the dual object \( \hat{G} \). On the other hand, it is well known that the normalized irreducible characters of a compact group \( G \) are precisely those functions \( \chi : G \to \mathbb{C} \) that satisfy the first and second conditions stated above and the simplified form of the third condition, the functional equation

\[
\int_{h \in G} \chi(g_1 hg_2 h^{-1}) m_G(dh) = \chi(g_1) \chi(g_2), \quad \forall g_1, g_2 \in G
\]

(7.3)

(here \( m_G \) is the normalized Haar measure on \( G \)).

There is another but equivalent definition of the extremal characters which shows that they are extreme points of a certain convex set of functions on \( G \), see Olshanski [38, §1], [37, §§23-24]. These references also explain how extremal characters are related to unitary representations.

**Theorem 7.1.** There exists a natural one-to-one bijective correspondence \( \Omega_\infty \leftrightarrow \hat{U}(\infty) \).

Thus, the boundary \( \Omega_\infty \) has a representation-theoretic meaning. Note, however, that this theorem is a purely abstract result that provides no information about the size of the boundary. Its explicit description is given below in Section 9.
Theorem 8.1. The semigroups just defined and the links $\Lambda_{N-1}^N : \Omega_N \to \Omega_{N-1}$ defined in Section 3 satisfy the master equation (5.4). That is,

$$P_{N-1}^N(t) \left[ \Lambda_{N-1}^N \right] = \Lambda_{N-1}^N P_{N-1}^N(t).$$

By virtue of Theorem 4.1, part (i), the semigroups $P_{N-1}^N(t)$ give rise to a boundary semigroup on $\Omega_\infty$; let us denote it by $P_{\infty}^N(z', w, w')(t)$.

Set $M_N^N(z, z', w, w') := M_N^N(u, u', v, v')$, where $M_N^N(u, u', v, v')$ is the invariant measure from Theorem 4.1, part (ii), defined by (4.3); here, as above, the quadruple $(u, u', v, v')$ is given by (8.1). Let us assume additionally that $z + z' + w + w' > -1$. Then the constant factor in (4.3) can be chosen so that $M_N^N(z, z', w, w')$ becomes a probability measure.

Theorem 8.2. Assume $z + z' + w + w' > -1$. The probability measures $M_N^N(z, z', w, w')$ just defined satisfy the relation

$$M_N^N(z, z', w, w') \Lambda_{N-1}^N = M_{N-1}^N(z, z', w, w'),$$

so that $\{M_N^N(z, z', w, w') : N \geq 1\}$ is a coherent family that determines a boundary measure $M(z, z', w, w')$ on $\Omega_\infty$.

This measure is a stationary distribution for the boundary semigroup $P_{\infty}^N(z, z', w, w')(t)$.

The first assertion of the theorem was proved in Olshanski [40]. The second assertion is a formal consequence of the first assertion and the fact that $M_N^N(z, z', w, w')$ is a stationary distribution for $P_{N}(t)$ for every $N$.

The measures $M_{\infty}^N(z, z', w, w')$ are called the boundary zw-measures. Note that $M_{\infty}^N(z, z', w, w')$ does not change under transposition $z \leftrightarrow z'$ or $w \leftrightarrow w'$.

Theorem 8.3 (Gorin [24]). The zw-measures corresponding to different, up to the above transpositions, quadruples of parameters are pairwise disjoint, that is, mutually singular.
9. The Edrei–Voiculescu theorem

Consider a two-sided infinite sequence \( \{ \varphi_n : n \in \mathbb{Z} \} \) of real numbers and assign to it the two-sided infinite Toeplitz matrix \( T \) with the entries \( T(i, j) := \varphi_{j-i} \), where \( i, j \in \mathbb{Z} \). The sequence \( \{ \varphi_n \} \) is called totally positive if all minors of \( T \) are nonnegative. In more detail, the minors of order 1 are the numbers \( \varphi_n \), so they must be nonnegative; next, the minors of order 2 are indexed by two arbitrary couples of integers, \( n_1 < n_2 \) and \( m_1 < m_2 \), which leads to the condition \( \varphi_{n_1} \varphi_{m_2} - \varphi_{n_2} \varphi_{m_1} \geq 0 \), and so on.

The problem of classification of the totally positive sequences was posed by Schoenberg and solved by Edrei [16]. The result is deep and the answer is beautiful. To avoid excessive and unnecessary complication we will impose the additional requirement that

\[
\sum_{n \in \mathbb{Z}} \varphi_n = 1. \tag{9.1}
\]

Then the result is conveniently stated in terms of the generating series

\[
\Phi(u) := \sum_{n \in \mathbb{Z}} \varphi_n u^n. \tag{9.2}
\]

Because of (9.1), the series converges on the unit circle \( T \subset \mathbb{C} \) and represents there a continuous function.

**Theorem 9.1** (Edrei [16]). The totally positive sequences with the normalization condition (9.1) are parameterized by sextuples \( \omega = (\alpha^+, \beta^+, \alpha^-, \beta^-, \delta^+, \delta^-) \), where \( \alpha^\pm \) and \( \beta^\pm \) are infinite sequences of nonincreasing nonnegative reals \( \{ \alpha^+_i : i = 1, 2, \ldots \} \) and \( \{ \beta^+_i : i = 1, 2, \ldots \} \), respectively, such that

\[
\sum_{i=1}^{\infty} (\alpha^+_i + \beta^+_i) \leq \delta^+, \quad \beta^+_1 + \beta^-_1 \leq 1.
\]

Given such a sextuple \( \omega \), the generating series of the corresponding sequence has the form

\[
\Phi(u; \omega) = e^{\gamma^+(u-1)+\gamma^-(u^{-1}-1)} \prod_{i=1}^{\infty} \frac{(1 + \beta^+_i (u-1)) (1 + \beta^-_i (u^{-1}-1))}{(1 - \alpha^+_i (u-1)) (1 - \alpha^-_i (u^{-1}-1))}, \tag{9.3}
\]

where

\[
\gamma^\pm := \delta^\pm - \sum_{i=1}^{\infty} (\alpha^\pm_i + \beta^\pm_i).
\]

In particular, the generating series converges in an annulus around \( T \) and extends to a meromorphic function in \( \mathbb{C} \). (About the theory of total positivity see Karlin’s fundamental monograph [28].)

Voiculescu discovered that the same functions (9.3) appear in the context of the representation theory of the group \( U(\infty) \). Namely, the following result holds. (Below
we use the fact that every matrix $U \in U(\infty)$ is conjugated to a diagonal matrix whose diagonal entries, the eigenvalues of $U$, lie on the unit circle $T$ and only finitely many of them are distinct from 1.)

**Theorem 9.2.** The extremal characters of $U(\infty)$ are precisely the functions of the form

$$\chi_\omega(U) = \prod_{j=1}^{\infty} \Phi(u_j; \omega), \quad U \in U(\infty),$$

where $\omega$ ranges over the same collection of parameters as in Theorem 9.1, and $u_1, u_2, \ldots$ are the eigenvalues of the matrix $U$. (Note that the infinite product here is actually finite, because $\Phi(1; \omega) = 1$ and $u_j = 1$ for $j$ large enough.)

Thus, the extremal characters of $U(\infty)$ and the totally positive sequences are in one-to-one correspondence, so that the classification problems for these two kinds of objects coincide.

**Remark 9.3.** Here are brief historical comments concerning Theorem 9.2. Voiculescu was the first person to study the extremal characters of $U(\infty)$ (see his paper [48]). He proved that all functions of the form (9.4) are extremal characters. He also explained why the extremal characters should be given by multiplicative expressions with respect to the eigenvalues. He did not prove that the list of Theorem 9.2 is exhaustive, but obtained some partial results in this direction. Then Vershik–Kerov [47] and Boyer [11] independently drew attention to the earlier work of Edrei [16], of which Voiculescu was unaware. Boyer explained how to deduce Theorem 9.2 from Edrei’s theorem. Vershik and Kerov sketched quite a different approach to Theorem 9.2, already tested on the example of the infinite symmetric group [46]. A detailed proof (in a broader context), based on the ideas of [17], appeared later in Okounkov–Olshanski [36]. Recently, one more proof was proposed in Borodin–Olshanski [8], and soon after that Petrov [42] found a simpler version of it together with a generalization.

**Remark 9.4.** It is worth noting that the multiplicativity property of extremal characters of $U(\infty)$ is related to specific properties of some infinite-dimensional groups and does not hold for finite-dimensional (noncommutative) groups. The nature of this phenomenon is analyzed in my expository paper [39] (see also [37]). One of the explanations given in [39] is related to a concentration property for the Haar measure of $U(N)$ (and other similar groups) as $N \to \infty$. Although the normalized irreducible characters of the groups $U(N)$ are not multiplicative, they become “approximately multiplicative” as $N$ gets large. This can be seen from [39], and recently, Gorin and Panova [25] found new character formulas which demonstrate this effect in a very clear manner.

Theorem 9.2 shows that the boundary $\Omega_{\infty}$, whose abstract definition was given in section 8, admits an explicit description. Namely, it can be identified with the
region in the product space
\[ \mathbb{R}^{4\infty+2} := \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R} \times \mathbb{R} \]
formed by the sextuples \( \omega = (\alpha^+, \beta^+, \alpha^-, \beta^-, \delta^+, \delta^-) \) from Theorem 9.1.

To complete the description of the boundary it remains to specify the links \( \Lambda^\infty_N : \Omega^\infty \rightarrow \Omega_N \):
\[ \Lambda^\infty_N(\omega, \ell) = \dim \pi^\lambda N \text{ det } [\varphi_{\lambda_i-i+j}]_{i,j=1}^N, \quad (9.5) \]
where \( \omega \in \Omega^\infty \subset \mathbb{R}^{4\infty+2} \), \( \ell \) ranges over \( \Omega_N \), \( \lambda \) is the signature corresponding to \( \ell \), and \( \{\varphi_n\} \) is the collection of the Laurent coefficients of the function \( \Phi_\omega(u) \) defined by (9.3). Note that the right-hand of (9.5) is nonnegative (as it should be), because the determinant \( \text{det } [\varphi_{\lambda_i-i+j}] \) is nonnegative due to the total positivity of \( \{\varphi_n\} \).

**Remark 9.5.** The determinants appearing in (9.5) do not exhaust all minors of the Toeplitz matrix \( T \) associated with the sequence \( \{\varphi_n\} \): indeed, these are minors with consecutive column numbers. However, the nonnegativity of these special minors already suffices to conclude that all minors of \( T \) are nonnegative, see Boyer [11]. This fine point is necessary for establishing the bijection between totally positive sequences and extremal characters.

**Remark 9.6.** Now one can explain how the boundary \([0, 1]\) of the Pascal triangle discussed in Section 7 is located in \( \Omega^\infty \). Namely, the interval \([0, 1]\) is identified with the set of those \( \omega \)'s for which \( \beta_1^+ = \delta^+ = x \in [0, 1] \) and all other coordinates of \( \omega \) equal 0.

### 10. The generator

I will start with a few definitions and facts concerning Feller Markov processes. For more detail, see Ethier–Kurtz [17], Liggett [32].

Assume \( \mathfrak{X} \) is a locally compact metrizable separable space and denote by \( C_0(\mathfrak{X}) \) the space of real-valued continuous functions vanishing at infinity. Let us equip \( C_0(\mathfrak{X}) \) with the supremum norm; then it becomes a separable Banach space. Note that the larger Banach space \( C(\mathfrak{X}) \) of bounded continuous functions is not separable unless \( \mathfrak{X} \) is compact, in which case \( C_0(\mathfrak{X}) = C(\mathfrak{X}) \); but we are interested in the case when \( \mathfrak{X} \) is not compact.

A semigroup \( P(t) \) of Markov kernels on \( \mathfrak{X} \) is said to be a Feller semigroup if it preserves the space \( C_0(\mathfrak{X}) \) and induces in it a strongly continuous operator semigroup. Then \( P(t) \) generates a Markov process \( X \) on \( \mathfrak{X} \) with sufficiently good sample trajectories; \( X \) is called a Feller process.

A Feller semigroup is uniquely determined by its generator \( A \), which is a closed dissipative operator on \( C_0(\mathfrak{X}) \). Its domain \( \text{Dom}(A) \) is formed by those elements \( F \in C_0(\mathfrak{X}) \) for which the limit
\[ AF := \lim_{t \to 0} \frac{P(t)F - F}{t} \]
exists. A subspace $\mathcal{F} \subset \text{Dom}(A)$ is called a core for $A$ if the closure of the restriction of $A$ to $\mathcal{F}$ coincides with $A$. In practice, it is usually problematic to explicitly describe $\text{Dom}(A)$, and then one is satisfied by indicating the action of $A$ on an appropriate core, because this suffices to specify $A$.

Now let us return to our boundary semigroup $P_{z,z',w,w'}^{(z,z',w,w')}(t)$. An important fact is that the boundary $\Omega_{\infty}$ is a locally compact space with respect to the topology induced by the product topology of the ambient infinite product space $\mathbb{R}^{4\infty+2}$.

**Theorem 10.1.** $P_{z,z',w,w'}^{(z,z',w,w')}(t)$ is a Feller semigroup, so it give rise to a Feller process $X_{z,z',w,w'}^{(z,z',w,w')}$ on $\Omega_{\infty}$.

This result raises the question about the semigroup generator as an operator in the Banach space $C_0(\Omega_{\infty})$. One can exhibit a core $\mathcal{F} \subset C_0(\Omega_{\infty})$ and prove that the action of the generator on $\mathcal{F}$ is implemented by a second order differential operator $D_{z,z',w,w'}^{(z,z',w,w')}$ with countably many variables.

At first glance, one would expect that $D_{z,z',w,w'}^{(z,z',w,w')}$ is somehow written in terms of the natural coordinate system $(\alpha^\pm_i; \beta^\pm_i; \delta^\pm)$ on $\Omega_{\infty}$, but it is not so. The natural coordinates are unsuitable, and we have to pass to other variables that are (in some sense) supersymmetric functions of the natural coordinates. These new variables are the Laurent coefficients of (9.3), which we denoted by $\varphi_n$, $n \in \mathbb{Z}$. As the core $\mathcal{F}$ we take a certain subspace in $\mathbb{R}^{\ldots, \varphi_{-1}, \varphi_0, \varphi_1, \ldots}$, the algebra of polynomials in variables $\varphi_n$. Then $D_{z,z',w,w'}^{(z,z',w,w')}$ is written in the form

$$D_{z,z',w,w'}^{(z,z',w,w')} = \sum_{n_1, n_2 \in \mathbb{Z}} \Gamma_{n_1, n_2}^{(2)} \frac{\partial^2}{\partial \varphi_{n_1} \varphi_{n_2}} + \sum_{m \in \mathbb{Z}} \Gamma_m^{(1)} \frac{\partial}{\partial \varphi_m},$$

(10.1)

where the coefficients $\Gamma_{n_1, n_2}^{(2)}$ are certain infinite quadratic expressions in variables $\varphi_n$ while the coefficients $\Gamma_m^{(1)}$ are certain finite linear combinations of these variables.

Note that only coefficients $\Gamma_m^{(1)}$ depend on the basic parameters $(z, z', w, w')$ while coefficients $\Gamma_{n_1, n_2}^{(2)}$ do not. This implies the following. Recall that the boundary process admits a stationary distribution, the zw-measure $M_{z,z',w,w'}^{(z,z',w,w')}$ (see Theorem 8.2). Consider the Hilbert space $H_{z,z',w,w'}^{(z,z',w,w')} := L^2(\Omega_{\infty}, M_{z,z',w,w'})$ and introduce the (pre)Dirichlet form corresponding to $D_{z,z',w,w'}^{(z,z',w,w')}$,

$$\mathcal{E}(F,G) := -(D_{z,z',w,w'}^{(z,z',w,w')})_F, \quad F, G \in \mathcal{F};$$

here the brackets in the right-hand side denote the inner product in $H_{z,z',w,w'}^{(z,z',w,w')}$. Then we get

$$\mathcal{E}(F,G) = \int_{\omega \in \Omega_{\infty}} \Gamma(F,G) M_{(z,z',w,w')}^{(z,z',w,w')}(d\omega), \quad \Gamma(F,G) := \sum_{n_1, n_2 \in \mathbb{Z}} \Gamma_{n_1, n_2}^{(2)} \frac{\partial F}{\partial \varphi_{n_1}} \frac{\partial G}{\partial \varphi_{n_2}},$$

(10.1)
where only $M(z,z',w,w')$ depends on the basic parameters while the form $\Gamma(F,G)$ does not. One may speculate that this form somehow expresses the “inner geometry” of the space $\Omega_\infty$.

The fact that $D_{\infty}(z,z',w,w')$ has second order implies that $X_{\infty}^{(z,z',w,w')}$ cannot degenerate to a deterministic process, as in Example 5.2. This conclusion can be also deduced from the fact that $M_{\infty}^{(z,z',w,w')}$ is not only a stationary distribution but also a symmetrizing measure.

11. Summary

The starting point of the story is a 4-parameter family $\{X_N^{(z,z',w,w')} : N = 1, 2, \ldots\}$ of continuous time Markov chains on the dual objects $\Omega_N = \hat{U}(N)$. For any fixed quadruple $(z, z', w, w')$ of parameters, the chains $X_N^{(z,z',w,w')}$ are consistent with some canonical “links” (stochastic matrices) relating the sets $\Omega_N$ to each other. This makes it possible to apply the abstract “method of intertwiners” and establish the existence of Markov semigroups $P_{\infty}^{(z,z',w,w')}(t)$ on an infinite-dimensional locally compact space $\Omega_\infty = \hat{U}(\infty)$. Every semigroup $P_{\infty}^{(z,z',w,w')}(t)$ possesses the Feller property and so determines a Feller Markov process $X_{\infty}^{(z,z',w,w')}$ on $\Omega_\infty$. This process has a unique stationary distribution $M_{\infty}^{(z,z',w,w')}$, which also serves as a symmetrizing measure. The action of the infinitesimal generator of the process on an appropriate core can be explicitly described, and it turns out that it is implemented by a second order differential operator with infinitely many variables.

12. Concluding remarks

The boundary zw-measures $M_{\infty}^{(z,z',w,w')}$ are of great interest for harmonic analysis (Olshanski [40]). However, they are defined indirectly, through an abstract existence theorem, which makes it difficult to work with them. As seen from Theorem 8.3, the zw-measures cannot be given by densities with respect to a reference measure on $\Omega_\infty$.

A way to describe the zw-measures is to interpret them as the laws of some determinantal processes whose correlation kernels can be explicitly computed (Borodin–Olshanski [4]), so every zw-measure is a determinantal measure. (About such measures, see Borodin [1] and references therein.)

The above results show that $M_{\infty}^{(z,z',w,w')}$ can also be characterized as the only invariant measure of the process $X_{\infty}^{(z,z',w,w')}$. In view of the results of [4], it seems plausible that the multi-time finite-dimensional distributions of the process $X_{\infty}^{(z,z',w,w')}$ started from corresponding the zw-measure also have the determinantal structure.
It would be very interesting to learn more about the properties of the processes \( X_{\infty}^{(z,z',w,w')} \). For instance, is it true that the sample trajectories of \( X_{\infty}^{(z,z',w,w')} \) are continuous?

The present notes do not cover all the results of the paper Borodin–Olshanski [7]. As shown in that paper (see also [4]), \( X_{\infty}^{(z,z',w,w')} \) can be interpreted as a time-dependent point process with infinitely many particles. Although the interaction between the particles is highly nonlocal, it turns out that \( X_{\infty}^{(z,z',w,w')} \) can be obtained as a projection of another Markov process, in which the interaction between the particles is local (see [7, §9]).

Finally, note that there exists a parallel theory in which the role of \( U(\infty) \) is played by the infinite symmetric group, see Borodin–Olshanski [9], [10] and references therein.

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