A Generalized Itô’s Formula in Two-Dimensions and Stochastic Lebesgue-Stieltjes Integrals

Chunrong Feng¹,², Huaizhong Zhao¹

¹ Department of Mathematical Sciences, Loughborough University, LE11 3TU, UK. C.Feng@lboro.ac.uk, H.Zhao@lboro.ac.uk
² School of Mathematics and System Sciences, Shandong University, Jinan, Shandong Province, 250100, China

Summary. In this paper, a generalized Itô formula for time dependent functions of two-dimensional continuous semi-martingales is proved. The formula uses the local time of each coordinate process of the semi-martingale, left space and time first derivatives and second derivative \( \nabla_1 \nabla_2 f \) only which are assumed to be of locally bounded variation in certain variables, and stochastic Lebesgue-Stieltjes integrals of two parameters. The two-parameter integral is defined as a natural generalization of the Itô integral and Lebesgue-Stieltjes integral through a type of Itô isometry formula.

Keywords: local time, continuous semi-martingale, generalized Itô’s formula, stochastic Lebesgue-Stieltjes integral.

AMS 2000 subject classifications: 60H05, 60J55

1 Introduction

The classical Itô’s formula for twice differentiable functions has been extended to less smooth functions by many mathematicians. Progresses have been made mainly in one-dimension beginning with Tanaka’s pioneering work [28] for \( |X_t| \) to which the local time was beautifully linked. Further extensions were made to time independent convex functions in [20] and [30]; to the case of absolutely continuous function with the first derivative being locally bounded in [3]; to \( W_{1,2}^{1,2} \) functions of a Brownian motion in [11] for one dimension and [12] for multi-dimensions. It was proved in [11] that \( f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}[f(B), B]_t \), where \([f(B), B]_t\) is the covariation of the processes \( f(B) \) and \( B \) and is equal to \( \int_0^t f(B_s)d[B_s] \) as a difference of backward and forward integrals. See [27] for the case of continuous semi-martingale. The multi-dimensional case was considered by [12], [27] and [21]. An integral \( \int_{-\infty}^{\infty} f'(x) d_L(x) \) was introduced in [3] through the existence
of the expression \( f(X(t)) - f(X(0)) - \int_0^t \frac{\partial}{\partial s} f(X(s))dX(s) \), where \( L_t(x) \) is the local time of the semi-martingale \( X_t \). This work was extended further to define \( \int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial x} f(s, X(s))d\nu_{s,x} L_s(x) \) for a time dependent function \( f(s, x) \) using forward and backward integrals for Brownian motion in \([5]\) and to semi-martingales other than Brownian motion in \([6]\). This integral was also defined in \([20]\) as a stochastic integral with excursion fields, and in \([13]\) through Itô’s formula without assuming the reversibility of the semi-martingale which was required in \([5]\). Other generalizations include \([10]\) where it was also proved that if \( X \) is a semi-martingale, then \( f(X(t)) \) is a semi-martingale if and only if \( f \in W^{1,2}_{loc} \) and its weak derivative is of bounded variation using backward and forward integrals \([18]\).

The above mentioned extensions are useful in many problems. However, to use probabilistic methods to study problems arising in partial differential equations with singularities and mathematics of finance, we often need a generalization of Itô’s formula for time dependent \( f(t, x) \). In a special case that is if there exists a Radon measure \( \nu \) and locally bounded Borel function \( H \) such that \( d\nu(\nabla f(t, x)) = H(t, x)\nu(dx) \), a generalized Itô’s formula was obtained by \([2]\). In a recent work \([7]\), a new generalized Itô’s formula for one-dimensional continuous semi-martingales was proved. It is given in terms of a Lebesgue-Stieltjes integral of the local time \( L_t(x) \) with respect to the two-dimensional variation of \( \nabla^- f(t, x) \) as follows

\[
\begin{align*}
  f(t, X(t)) - f(0, X(0)) &= \int_0^t \frac{\partial^-}{\partial s} f(s, X(s))ds + \int_0^t \nabla^- f(s, X(s))dX_s \\
  &+ \frac{1}{2} \int_0^t \Delta f_h(s, X(s))d\langle X \rangle_s + \int_{-\infty}^\infty L_s(x)d_x \nabla^- f_v(t, x) \\
  &- \int_{-\infty}^{+\infty} \int_0^t L_s(x)d_{s,x} \nabla^- f_v(x, s). \text{ a.s.} (1.1)
\end{align*}
\]

Here \( f(t, x) = f_h(t, x) + f_v(t, x) \) is left continuous with \( f_h(t, x) \) being \( C^1 \) in \( x \) and \( \nabla f_h(t, x) \) being absolutely continuous whose left derivative \( \Delta^- f_h(t, x) \) is left continuous and locally bounded, and \( \nabla^- f_v(t, x) \) being of locally bounded variation in \((t, x)\) and of locally bounded variation in \( x \) at \( t = 0 \). Note the last two integrals are pathwise well defined due to the well-known fact that the local time \( L_t(x) \) is jointly continuous in \( t \) and càdlàg in \( x \) and has a compact support in space \( x \) for each \( t \) \((24, 14)\). In a special case, when there exists a curve \( x = \gamma(t) \) of locally bounded variation and the function \( f \) is continuous but the first order derivative \( \nabla f \) has jumps across the curve and second order derivative \( \Delta f \) has left limit when \( x \to \gamma(t)^- \), i.e. \( \Delta^- f \) exists and locally bounded and left continuous off the curve(s) \( x = \gamma(t) \), and there may be jumps of \( \nabla f \) along \( x = \gamma(t) \) (\( \Delta f \) is still undefined), define \( \Delta^- f \) on the curve \( x = \gamma(t) \) as the left limit of \( \Delta f \). Then the following formula was derived from \((1.1)\) using the integration by parts formula \((7)\):
Two-dimensional generalized Itô Formula

\[ f(t, X(t)) = f(0, X(0)) + \int_0^t \frac{\partial}{\partial s} f(s, X(s))ds + \int_0^t \nabla^- f(s, X(s))dX_s + \frac{1}{2} \int_0^t \Delta^- f(s, X(s))d<X>_s + \int_0^t (\nabla^- f(s, \gamma(s)) - \nabla^- f(s, \gamma(s)))dL_s(\gamma(s)). \text{ a.s.} \quad (1.2) \]

Here \(dL_s(a)\) refers to the Lebesgue-Stieltjes integral with respect to \(s \mapsto \int_{\cdot} L_s(\cdot) d\).

Formula (1.2) was also observed in [22] independently. These two new formulae have been proved useful in analysing asymptotics of solutions of partial differential equations in the presence of caustics ([8]) and studying the smooth fitting problem in American put option ([24]). Formula (1.1) is in a very general form. It includes the classical Itô formula, Tanaka’s formula, Meyer’s formula for convex functions, the formula given by Azéma, Jeulin, Knight and Yor [25] and formula (1.2).

The purpose of this paper is to extend formula (1.1) to two dimensions. This is a nontrivial extension as the local time in two-dimensions does not exist. But we observe for a smooth function \(f\), formally by the occupation times formula

\[ \frac{1}{2} \int_0^t \Delta_1 f(s, X_1(s), X_2(s))d<X>_s^1 = \int_{-\infty}^{+\infty} \int_0^t \Delta_1 f(s, a, X_2(s))dsL_1(s, a)da \]

\[ = \int_{-\infty}^{+\infty} \Delta_1 f(t, a, X_2(t))L_1(t, a)da - \int_{-\infty}^{+\infty} \int_0^t L_1(s, a)ds_n \nabla_1 f(s, a, X_2(s)), \quad (1.3) \]

if the integral \(\int_{-\infty}^{+\infty} \int_0^t L_1(s, a)ds_n \nabla_1 f(s, a, X_2(s))\) is properly defined. Here \(\nabla_1 f(s, a, X_2(s))\) is a semi-martingale for any fixed \(a\), following the one-dimensional generalized Itô’s formula (1.1). For this, we study this kind of the integral \(\int_{-\infty}^{+\infty} \int_0^t g(s, a)ds_n h(s, a)\) in section 2. Here \(h(s, x)\) is a continuous martingale with cross variation \(<h(\cdot, a), h(\cdot, b)>_s\) of locally bounded variation in \((a, b)\), and \(E \left[ \int_0^t \int_{\mathbb{R}^2} |g(s, a)g(s, b)||d_{a, b, s} < h(\cdot, a), h(\cdot, b) >_s \right] < \infty.\)

The integral is different from the Lebesgue-Stieltjes integral and Itô’s stochastic integral. But it is a natural extension to the two-parameter stochastic case and therefore called a stochastic Lebesgue-Stieltjes integral. According to our knowledge, our integral is new. It’s different from integration with Brownian sheet defined by Walsh ([29]) and integration w.r.t. Poisson random measure (see [14]). A generalized Itô’s formula in two dimensions is proved in section 3. Applications e.g. in the study of the asymptotics of the solutions of heat equations with caustics in two dimensions, are not included in this paper.
These results will be published in some future work. Furthermore, it has been observed by us in [9] that the local time $L_t(x)$ can be considered naturally as a rough path in $x$ of finite 2-variation and $\int_0^1 \int_{-\infty}^\infty \nabla^- f(s,x)ds_xL_s(x)$ is defined pathwisely by using and extending Lyons’ idea of rough path integration ([17]).

2 The definition of stochastic Lebesgue-Stieltjes integrals and the integration by parts formula

For a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, denote by $\mathcal{M}_2$ the Hilbert space of all processes $X = (X_t)_{0 \leq t \leq T}$ such that $(X_t)_{0 \leq t \leq T}$ is a $(\mathcal{F}_t)_{0 \leq t \leq T}$ right continuous square integrable martingale with inner product $(X,Y) = E(X_T Y_T)$. A three-variable function $f(s, x, y)$ is called left continuous iff it is left continuous in all three variables together i.e. for any sequence $(s_1, x_1, y_1) \leq (s_2, x_2, y_2) \leq \cdots \leq (s_k, x_k, y_k) \to (s, x, y)$, as $k \to \infty$, we have $f(s_k, x_k, y_k) \to f(s, x, y)$ as $k \to \infty$. Here $(s_1, x_1, y_1) \leq (s_2, x_2, y_2)$ means $s_1 \leq s_2, x_1 \leq x_2$ and $y_1 \leq y_2$. Define

$$V_1 := \{ h : [0, t] \times (-\infty, \infty) \times \Omega \to R \text{ s.t. } (s, x, \omega) \mapsto h(s, x, \omega) \text{ is } B([0, t] \times R) \times \mathcal{F} \text{-measurable, and } h(s, x) \text{ is }$$

$$\mathcal{F}_s \text{-adapted for any } x \in \mathbb{R} \},$$

$$V_2 := \{ h : h \in V_1 \text{ is a continuous (in s) } \mathcal{M}_2 \text{-martingale for each x, and the crossvariation } < h(\cdot, x), h(\cdot, y)>_s \text{ is left continuous and of locally bounded variation in } (s, x, y) \}.$$ 

In the following, we will always denote $< h(\cdot, x), h(\cdot, y)>_s$ by $< h(x), h(y)>_s$.

We now recall some classical results for the sake of completeness of the paper (see [1] and [19]). A three-variable function $f(s, x, y)$ is called monotonically increasing if whenever $(s_2, x_2, y_2) \geq (s_1, x_1, y_1)$, then

$$f(s_2, x_2, y_2) - f(s_1, x_1, y_1) + f(s_2, x_1, y_1) - f(s_1, x_2, y_2) \geq 0.$$ 

For a left-continuous and monotonically increasing function $f(s, x, y)$, one can define a Lebesgue-Stieltjes measure by setting

$$\nu([s_1, s_2] \times [x_1, x_2] \times [y_1, y_2]) = f(s_2, x_2, y_2) - f(s_2, x_1, y_1) - f(s_1, x_2, y_2) + f(s_1, x_2, y_1) - f(s_1, x_1, y_2) + f(s_1, x_1, y_1).$$

For $h \in V_2$, define
Two-dimensional generalized Itô Formula

\[ < h(x), h(y) >_{t_1}^{t_2} := < h(x), h(y) >_{t_2} - < h(x), h(y) >_{t_1}, \ t_2 \geq t_1. \]

Note as \( < h(x), h(y) >_{s} \) is left continuous and of locally bounded variation in \((s, x, y)\), so it can be decomposed to the difference of two increasing and left continuous functions \( f_1(s, x, y) \) and \( f_2(s, x, y) \) (see McShane [19] or Proposition 2.2 in Elworthy, Truman and Zhao [7] which also holds for multi-parameter functions). Note each of \( f_1 \) and \( f_2 \) generates a measure, so for any measurable function \( g(s, x, y) \), we can define

\[
\begin{align*}
\int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} d_{x,y,s} < h(x), h(y) >_s &= \int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} g(s, x, y) d_{x,y,s} < h(x), h(y) >_s \\
&= \int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} g(s, x, y) d_{x,y,s} f_1(s, x, y) \\
&\quad - \int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} g(s, x, y) d_{x,y,s} f_2(s, x, y).
\end{align*}
\]

In particular, a signed product measure in the space \([0, T] \times R^2\) can be defined as follows: for any \([t_1, t_2] \times [x_1, x_2] \times [y_1, y_2] \subset [0, T] \times R^2\)

\[
\int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} d_{x,y,s} < h(x), h(y) >_s \\
= \int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} d_{x,y,s} f_1(s, x, y) - \int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} d_{x,y,s} f_2(s, x, y) \\
= < h(x_2), h(y_2) >_{t_1}^{t_2} - < h(x_2), h(y_1) >_{t_1}^{t_2} \\
- < h(x_1), h(y_2) >_{t_1}^{t_2} + < h(x_1), h(y_1) >_{t_1}^{t_2} \\
= < h(x_2) - h(x_1), h(y_2) - h(y_1) >_{t_1}^{t_2}. \tag{2.1}
\]

Define

\[ |d_{x,y,s} < h(x), h(y) >_s | = d_{x,y,s} f_1(s, x, y) + d_{x,y,s} f_2(s, x, y). \tag{2.2} \]

Moreover, for \( h \in \mathcal{V}_2 \), define:

\[ \mathcal{V}_3(h) := \{ g : g \in \mathcal{V}_1 \ has \ a \ compact \ support \ in \ x, \ and \ \ E \left[ \int_0^t \int_{R^2} |g(s, x)g(s, y)||d_{x,y,s} < h(x), h(y) >_s | \right] < \infty \}. \]

Consider now a simple function

\[ g(s, x, \omega) = \sum_{i=1}^{n} \sum_{j=1}^{m} e(t_j, x_i) 1_{(t_j, t_{j+1})} (s) 1_{(x_i, x_{i+1})} (x) \tag{2.3} \]

where \( t_1 < t_2 < \cdots < t_{m+1}, x_1 < x_2 < \cdots < x_{m+1}, e(t_j, x_i) \) are \( \mathcal{F}_{t_j} \)-measurable. For \( h \in \mathcal{V}_2 \), define an integral as:
\[ I_t(g) := \int_0^t \int_{-\infty}^\infty g(s,x) d_{s,x} h(s,x) \]
\[ = \sum_{i=1}^n \sum_{j=1}^m e(t_j \wedge t, x_i) \left[ h(t_{j+1} \wedge t, x_{i+1}) - h(t_j \wedge t, x_{i+1}) \right. \]
\[ - \left. h(t_{j+1} \wedge t, x_i) + h(t_j \wedge t, x_i) \right]. \quad (2.4) \]

This integral is called the stochastic Lebesgue-Stieltjes integral of the simple function \( g \). It’s easy to see for simple functions \( g_1, g_2 \in V_3(h) \),
\[ I_t(\alpha g_1 + \beta g_2) = \alpha I_t(g_1) + \beta I_t(g_2), \quad (2.5) \]
for any \( \alpha, \beta \in \mathbb{R} \). The following lemma plays a key role in extending the integral of simple functions to functions in \( V_3(h) \). It is equivalent to the Itô’s isometry formula in the case of the stochastic integral.

**Lemma 2.1** If \( h \in V_2 \), \( g \in V_3(h) \) is simple, then \( I_t(g) \) is a continuous martingale with respect to \( (\mathcal{F}_t)_{0 \leq t \leq T} \) and
\[ E \left( \int_0^t \int_{-\infty}^\infty g(s,x) d_{s,x} h(s,x) \right)^2 \]
\[ = E \int_0^t \int_{\mathbb{R}^2} g(s,x)g(s,y) d_{x,y,s} < h(x), h(y) >_s. \quad (2.6) \]

**Proof:** From the definition of \( \int_0^t \int_{-\infty}^\infty g(s,x) d_{s,x} h(s,x) \), it is easy to see that \( I_t \) is a continuous martingale with respect to \( (\mathcal{F}_t)_{0 \leq t \leq T} \). As \( h(s,x,\omega) \) is a continuous martingale in \( \mathcal{M}_2 \), using a standard conditional expectation argument to remove the cross product parts, we get:
\[ E \left( \int_0^t \int_{-\infty}^\infty g(s,x) d_{s,x} h(s,x) \right)^2 \]
\[ = E \sum_{j=1}^m \left( \int_{-\infty}^\infty \sum_{i=1}^n e(t_j \wedge t, x_i) \left[ h(t_{j+1} \wedge t, x_{i+1}) - h(t_j \wedge t, x_{i+1}) \right. \right. \]
\[ - \left. \left. h(t_{j+1} \wedge t, x_i) + h(t_j \wedge t, x_i) \right] \right)^2 \]
\[ = E \sum_{j=1}^m \left( \int_{-\infty}^\infty \sum_{i=1}^n \sum_{k=1}^n e(t_j \wedge t, x_i) e(t_j \wedge t, x_k) \cdot \right. \]
\[ \left[ h(t_{j+1} \wedge t, x_{i+1}) - h(t_j \wedge t, x_{i+1}) - h(t_{j+1} \wedge t, x_i) + h(t_j \wedge t, x_i) \right] \cdot \]
\[ \left[ h(t_{j+1} \wedge t, x_{k+1}) - h(t_j \wedge t, x_{k+1}) - h(t_{j+1} \wedge t, x_k) + h(t_j \wedge t, x_k) \right]. \]
\[ E \sum_{j=1}^{m} \left\{ \sum_{i=1}^{n} \sum_{k=1}^{n} e(t_j \wedge t, x_i) e(t_j \wedge t, x_k) \left[ (h(t_{j+1} \wedge t, x_{i+1}) - h(t_j \wedge t, x_{i+1})) (h(t_{j+1} \wedge t, x_{k+1}) - h(t_j \wedge t, x_{k+1})) \\ - (h(t_{j+1} \wedge t, x_{i+1}) - h(t_j \wedge t, x_{i+1})) (h(t_{j+1} \wedge t, x_k) - h(t_j \wedge t, x_k)) \\ - (h(t_{j+1} \wedge t, x_i) - h(t_j \wedge t, x_i)) (h(t_{j+1} \wedge t, x_{k+1}) - h(t_j \wedge t, x_{k+1})) \\ + (h(t_{j+1} \wedge t, x_i) - h(t_j \wedge t, x_i)) (h(t_{j+1} \wedge t, x_k) - h(t_j \wedge t, x_k)) \right] \right\} \]

\[ = E \int_{0}^{t} \sum_{i=1}^{n} \sum_{k=1}^{n} e(s, x_i) e(s, x_k) \left[ d_s < h(x_{i+1}), h(x_{k+1}) > s \\ - d_s < h(x_{i+1}), h(x_k) > s - d_s < h(x_i), h(x_{k+1}) > s \\ + d_s < h(x_i), h(x_k) > s \right] \]

\[ = E \int_{0}^{t} \sum_{i=1}^{n} \sum_{k=1}^{n} e(s, x_i) e(s, x_k) \left[ d_s < h(x_{i+1}) - h(x_i), h(x_{k+1}) - h(x_k) > s \right] \]

\[ = E \left[ \int_{0}^{t} \int_{\mathbb{R}^2} g(s, x) g(s, y) d_{x,y,s} < h(x), h(y) > s \right]. \]

So we prove the desired result. \( \diamond \)

The idea is to use (2.6) to extend the definition of the integrals of simple functions to integrals of functions in \( \mathcal{V}_3(h) \), for any \( h \in \mathcal{V}_2 \). We achieve this goal in several steps:

**Lemma 2.2** Let \( h \in \mathcal{V}_2 \), \( f \in \mathcal{V}_3(h) \) be bounded uniformly in \( \omega \), \( f(\cdot, \cdot, \omega) \) be continuous for each \( \omega \) on its compact support. Then there exist a sequence of bounded simple functions \( \varphi_{m,n} \in \mathcal{V}_3(h) \) such that

\[ E \int_{0}^{t} \int_{\mathbb{R}^2} |(f - \varphi_{m,n})(s, x)(f - \varphi_{m',n'})(s, y)| \, d_{x,y,s} < h(x), h(y) > s | \]

\[ \rightarrow 0, \]

as \( m, n, m', n' \rightarrow \infty \).

**Proof:** Let \([0,T] \times [a, b]\) be a rectangle covering the compact support of \( f \) and \( 0 = t_1 < t_2 < \cdots < t_{m+1} = T \), and \( a = x_1 < x_2 < \cdots < x_{n+1} = b \) be a partition of \([0,T] \times [a, b]\). Assume when \( n, m \rightarrow \infty \), \( \max_{1 \leq j \leq m} (t_{j+1} - t_j) \rightarrow 0 \), \( \max_{1 \leq i \leq n} (x_{i+1} - x_i) \rightarrow 0 \). Define

\[ \varphi_{m,n}(t, x) := \sum_{j=1}^{m} \sum_{i=1}^{n} f(t_j, x_i) 1_{[(t_j, t_{j+1}))(x_i, x_{i+1})}(x). \quad (2.7) \]
Then \( \varphi_{m,n}(t,x) \) are simple and \( \varphi_{m,n}(t,x) \to f(t,x) \) a.s. as \( m,n \to \infty \). The result follows Lebesgue’s dominated convergence theorem.

\[ E \int_0^t \int_{\mathbb{R}^2} |(g - f_n)(s,x)(k - f_n')(s,y)| \, dx, dy \lesssim h(x), h(y) \rightarrow 0, \]
as \( n, n' \to \infty \).

**Proof:** Define

\[ f_n(s,x) = n^2 \int_{x-n}^{x-n} \int_{y-n}^{y-n} k(\tau, y) d\tau dy. \]

Then \( f_n(s,x) \) is continuous in \( s, x \), and when \( n \to \infty \), \( f_n(s,x) \to k(s,x) \) a.s.. The desired convergence follows by Lebesgue’s dominated convergence theorem.

**Lemma 2.4** Let \( h \in \mathcal{V}_2 \) and \( g \in \mathcal{V}_3(h) \). Then there exist functions \( k_n \in \mathcal{V}_3(h) \), bounded uniformly in \( \omega \) for each \( n \), and

\[ E \int_0^t \int_{\mathbb{R}^2} |(g - k_n)(s,x)(g - k_n')(s,y)| \, dx, dy \lesssim h(x), h(y) \rightarrow 0, \]
as \( n, n' \to \infty \).

**Proof:** Define

\[
k_n(t,x,\omega) := \begin{cases} 
-n & \text{if } g(t,x,\omega) < -n \\
g(t,x,\omega) & \text{if } -n \leq g(t,x,\omega) \leq n \\
n & \text{if } g(t,x,\omega) > n.
\end{cases} \tag{2.8}
\]

Then as \( n \to \infty \), \( k_n(t,x,\omega) \to g(t,x,\omega) \) for each \( (t,x,\omega) \). Note \( |k_n(t,x,\omega)| \leq |g(t,x,\omega)| \) and \( g \in \mathcal{V}_3(h) \). So applying Lebesgue’s dominated convergence theorem, we obtain the desired result.

From Lemmas 2.3, 2.4, 2.2, for each \( h \in \mathcal{V}_2 \), \( g \in \mathcal{V}_3(h) \), we can construct a sequence of simple functions \( \{\varphi_{m,n}\} \) in \( \mathcal{V}_3(h) \) such that,

\[ E \int_0^t \int_{\mathbb{R}^2} |(g - \varphi_{m,n})(s,x)(g - \varphi_{m',n'})(s,y)| \, dx, dy \lesssim h(x), h(y) \rightarrow 0, \]
as $m, n, m', n' \to \infty$. For $\varphi_{m,n}$ and $\varphi_{m',n'}$, we can define stochastic Lebesgue-Stieltjes integrals $I_t(\varphi_{m,n})$ and $I_t(\varphi_{m',n'})$. From Lemma 2.1 and (2.5), it is easy to see that

$$E[I_T(\varphi_{m,n}) - I_T(\varphi_{m',n'})]^2$$

$$= E[I_T(\varphi_{m,n} - \varphi_{m',n'})]^2$$

$$= E \int_0^T \int_{\mathbb{R}^2} (\varphi_{m,n} - \varphi_{m',n'})(s,x)(\varphi_{m,n} - \varphi_{m',n'})(s,y)d_{x,y,s} < h(x), h(y) >_s$$

$$= E \int_0^T \int_{\mathbb{R}^2} [(\varphi_{m,n} - g) - (\varphi_{m',n'} - g)](s,x) \cdot$$

$$\cdot [(\varphi_{m,n} - g) - (\varphi_{m',n'} - g)](s,y)d_{x,y,s} < h(x), h(y) >_s$$

$$= E \int_0^T \int_{\mathbb{R}^2} (\varphi_{m,n} - g)(s,x)(\varphi_{m,n} - g)(s,y)d_{x,y,s} < h(x), h(y) >_s$$

$$-E \int_0^T \int_{\mathbb{R}^2} (\varphi_{m',n'} - g)(s,x)(\varphi_{m',n'} - g)(s,y)d_{x,y,s} < h(x), h(y) >_s$$

$$-E \int_0^T \int_{\mathbb{R}^2} (\varphi_{m,n} - g)(s,x)(\varphi_{m',n'} - g)(s,y)d_{x,y,s} < h(x), h(y) >_s$$

$$+E \int_0^T \int_{\mathbb{R}^2} (\varphi_{m',n'} - g)(s,x)(\varphi_{m',n'} - g)(s,y)d_{x,y,s} < h(x), h(y) >_s$$

$$\leq E \int_0^T \int_{\mathbb{R}^2} |(\varphi_{m,n} - g)(s,x)(\varphi_{m,n} - g)(s,y)|d_{x,y,s} < h(x), h(y) >_s|$$

$$+E \int_0^T \int_{\mathbb{R}^2} |(\varphi_{m,n} - g)(s,x)(\varphi_{m',n'} - g)(s,y)|d_{x,y,s} < h(x), h(y) >_s|$$

$$+E \int_0^T \int_{\mathbb{R}^2} |(\varphi_{m',n'} - g)(s,x)(\varphi_{m,n} - g)(s,y)|d_{x,y,s} < h(x), h(y) >_s|$$

$$+E \int_0^T \int_{\mathbb{R}^2} |(\varphi_{m',n'} - g)(s,x)(\varphi_{m',n'} - g)(s,y)|d_{x,y,s} < h(x), h(y) >_s|$$

$$\to 0,$$

as $m, n, m', n' \to \infty$. Therefore $\{I_t(\varphi_{m,n})\}_{m,n=1}^\infty$ is a Cauchy sequence in $\mathcal{M}_2$ whose norm is denoted by $\|\cdot\|$. So there exists a process $I(g) = \{I_t(g), 0 \leq t \leq T\}$ in $\mathcal{M}_2$, defined modulo indistinguishability, such that

$$\|I(\varphi_{m,n}) - I(g)\| \to 0, \text{ as } m, n \to \infty.$$

By the same argument as for the stochastic integral, one can easily prove that $I(g)$ is well-defined (independent of the choice of the simple functions), and (2.6) is true for $I(g)$. We now can have the following definition.

**Definition 2.1** Let $h \in \mathcal{V}_2$, $g \in \mathcal{V}_2(h).$ Then the integral of $g$ with respect to $h$ can be defined as:
\[
\int_0^t \int_{-\infty}^\infty g(s, x) d_{s,x} h(s, x) = \lim_{m,n \to \infty} \int_0^t \int_{-\infty}^\infty \phi_{m,n}(s, x) d_{s,x} h(s, x), \quad \text{(limit in } \mathcal{M}_2)\]

is a continuous martingale with respect to \((\mathcal{F}_t)_{0 \leq t \leq T}\) and for each \(t \leq T\), (2.6) is satisfied. Here \(\{\phi_{m,n}\}\) is a sequence of simple functions in \(V_3(h)\), s.t.

\[
\mathbb{E} \int_0^t \int_{\mathbb{R}^2} |(g - \phi_{m,n})(s, x)(g - \phi_{m',n'})(s, y)| \, d_{x,y,s} < h(x), h(y) > s \to 0,
\]

as \(m, n, m', n' \to \infty\). Note \(\phi_{m,n}\) may be constructed by combining the three approximation procedures in Lemmas 2.4, 2.3, 2.2.

The following integration by parts formula will be useful in the proof of our main theorem. Although the conditions are strong and may be unnecessary, the proposition is enough for our purpose. We don’t strive to weaken the conditions here.

**Proposition 2.1** If \(h \in V_2\), \(g \in V_3(h)\), and \(g(t, x)\) is \(C^2\) in \(x\), \(\Delta g(t, x)\) is bounded uniformly in \(t\), then a.s.

\[
- \int_{-\infty}^{+\infty} \int_0^t \nabla g(s, x) d_s h(s, x) dx = \int_0^t \int_{-\infty}^{+\infty} g(s, x) d_{s,x} h(s, x). \quad (2.9)
\]

**Proof.** If \(g\) is a simple function as given in (2.3), one can always add some points in the partition to make \(e(t_j \wedge t, x_1) = 0\) and \(e(t_j \wedge t, x_{n+1}) = 0\) for all \(j = 1, 2, \ldots, m\) as \(g\) has a compact support in \(x\). So for \(h \in V_2\),

\[
\int_0^t \int_{-\infty}^{+\infty} g(s, x) d_{s,x} h(s, x)
\]

\[
= \sum_{i=1}^n \sum_{j=1}^m e(t_j \wedge t, x_i) \left[ h(t_{j+1} \wedge t, x_{i+1}) - h(t_j \wedge t, x_{i+1}) - h(t_{j+1} \wedge t, x_i) + h(t_j \wedge t, x_i) \right]
\]

\[
= - \sum_{i=0}^{n-1} \sum_{j=1}^m e(t_j \wedge t, x_{i+1}) \left[ h(t_{j+1} \wedge t, x_{i+1}) - h(t_j \wedge t, x_{i+1}) \right]
\]

\[
+ \sum_{i=1}^n \sum_{j=1}^m e(t_j \wedge t, x_i) \left[ h(t_{j+1} \wedge t, x_{i+1}) - h(t_j \wedge t, x_{i+1}) \right]
\]

\[
= - \sum_{i=1}^n \sum_{j=1}^m \left[ e(t_j \wedge t, x_{i+1}) - e(t_j \wedge t, x_i) \right] \left[ h(t_{j+1} \wedge t, x_{i+1}) - h(t_j \wedge t, x_{i+1}) \right].
\]
If \( g(t, x) \) is \( C^2 \) in \( x \) and \( \Delta g(t, x_2) \) is bounded uniformly in \( t \), let

\[
\varphi_{m,n}(t, x) := \sum_{j=1}^{m} \sum_{i=1}^{n} g(t_j, x_i) 1_{[t_j, t_{j+1})}(t) 1_{[x_i, x_{i+1})}(x),
\]

so

\[
\varphi_{m,n}(t, x) \to g(t, x) \text{ a.s. as } m, n \to \infty.
\]

Then by the intermediate value theorem, there exist \( \xi_i \in [x_i, x_{i+1}] \) (\( i = 1, 2, \ldots, n \)) such that,

\[
\int_{-\infty}^{+\infty} \int_{0}^{t} g(s, x) d_s h(s, x) = -\lim_{\delta_t, \delta_x \to 0} \sum_{i=1}^{n} \sum_{j=1}^{m} \left[ g(t_j \wedge t, x_{i+1}) - g(t_j \wedge t, x_i) \right] \left[ h(t_{j+1} \wedge t, x_{i+1}) - h(t_j \wedge t, x_i) \right] \quad \text{(limit in } \mathcal{M}^2)\]

\[
= -\lim_{\delta_t, \delta_x \to 0} \sum_{i=1}^{n} \int_{0}^{t} \nabla g(s, x_i) d_s h(s, x_i+1)(x_{i+1} - x_i) \quad \text{(limit in } \mathcal{M}^2)\]

\[
= -\lim_{\delta_t, \delta_x \to 0} \sum_{i=1}^{n} \int_{0}^{t} \nabla g(s, x_{i+1}) d_s h(s, x_{i+1})(x_{i+1} - x_i)
- \lim_{\delta_t, \delta_x \to 0} \sum_{i=1}^{n} \int_{0}^{t} \left( \nabla g(s, x_i) - \nabla g(s, x_{i+1}) \right) d_s h(s, x_{i+1})(x_{i+1} - x_i)
= -\int_{-\infty}^{+\infty} \int_{0}^{t} \nabla g(s, x) d_s h(s, x) dx.
\]

Here \( \delta_t = \max_{1 \leq j \leq m} |t_{j+1} - t_j|, \delta_x = \max_{1 \leq s \leq m} |x_{i+1} - x_i| \). To prove the last equality, first notice that

\[
\lim_{\delta_x \to 0} \sum_{i=1}^{n} \int_{0}^{t} \nabla g(s, x_{i+1}) d_s h(s, x_{i+1})(x_{i+1} - x_i)
= \int_{-\infty}^{+\infty} \int_{0}^{t} \nabla g(s, x) d_s h(s, x) dx.
\]

Second, by the intermediate value theorem again, the second term can be estimated as:
is continuous, and nondecreasing in $t$ for each $X$ with $\eta$

Let $X$ be a two-dimensional continuous local martingale and 

$$E \left[ \sum_{i=1}^{n} \int_{0}^{t} (\nabla g(s, \xi_i) - \nabla g(s, x_{i+1}))ds \right]^2$$

$$= E \sum_{i=1}^{n} \sum_{k=1}^{n} \left[ \int_{0}^{t} (\nabla g(s, \xi_i) - \nabla g(s, x_{i+1}))ds \right] \cdot \left[ \int_{0}^{t} (\nabla g(s, \xi_k) - \nabla g(s, x_{k+1}))ds \right]$$

$$= n \sum_{i=1}^{n} E \int_{0}^{t} (\nabla g(s, \xi_i) - \nabla g(s, x_{i+1}))(\nabla g(s, \xi_k) - \nabla g(s, x_{k+1}))ds$$

$$\leq n \sum_{i=1}^{n} \sum_{k=1}^{n} E \sup_{\xi \in [x_i, x_{i+1}]} |\nabla g(s, \xi_i) - \nabla g(s, x_{i+1})| \cdot \sup_{\xi \in [x_k, x_{k+1}]} |\nabla g(s, \xi_k) - \nabla g(s, x_{k+1})| \cdot |h(x_{i+1}) < h(x_{k+1}) >_t (x_{i+1} - x_i)(x_{k+1} - x_k)$$

$$\leq E \left[ \sup_{s} \sup_{\xi \in [x_i, x_{i+1}]} |\nabla g(s, \eta_i)(\xi_i - x_{i+1})| \cdot \sup_{s} \sup_{\xi \in [x_k, x_{k+1}]} |\nabla g(s, \eta_k)(\xi_k - x_{k+1})| \cdot \sup_{s} \sup_{\xi \in [x_k, x_{k+1}]} |\nabla g(s, \eta_k)(\xi_k - x_{k+1})| \cdot \left( \sum_{i=1}^{n} \sum_{k=1}^{n} (x_{i+1} - x_i)(x_{k+1} - x_k) \right) \right] \to 0, \text{ as } \delta_x \to 0,$$

where $\eta_i \in [\xi_i, x_{i+1}], \eta_k \in [\xi_k, x_{k+1}]$. The desired result is proved. \hfill \Box

### 3 The generalized Itô’s formula in two-dimensional space

Let $X(s) = (X_1(s), X_2(s))$ be a two-dimensional continuous semi-martingale with $X_i(s) = X_i(0) + M_i(s) + V_i(s) (i = 1, 2)$ on a probability space $(\Omega, F, P)$. Here $M_i(s)$ is a continuous local martingale and $V_i(s)$ is an adapted continuous process of locally bounded variation (in $s$). Let $L_i(t, a)$ be the local time of $X_i(t)$ ($i=1,2$)

$$L_i(t, a) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_{0}^{t} 1_{[a, a+\epsilon]}(X_i(s))d \langle M_i \rangle_s, \text{ a.s. } i = 1, 2 \quad (3.1)$$

for each $t$ and $a \in R$. Then it is well known for each fixed $a \in R$, $L_i(t, a, \omega)$ is continuous, and nondecreasing in $t$ and right continuous with left limit.
Two-dimensional generalized Itô Formula 13

(càdlàg) with respect to \( a \) \( [15, 25] \). Therefore we can define a Lebesgue-Stieltjes integral \( \int_0^\infty \phi(s) dL_i(s, a, \omega) \) for each \( a \) for any Borel-measurable function \( \phi \). In particular

\[
\int_0^\infty 1_{R \setminus \{a\}}(X_i(s)) dL_i(s, a, \omega) = 0, \quad a.s. \quad i = 1, 2.
\] (3.2)

Furthermore if \( \phi \) is differentiable, then we have the following integration by parts formula

\[
\int_0^t \phi(s) dL_i(s, a, \omega) = \phi(t)L_i(t, a, \omega) - \int_0^t \phi'(s)L_i(s, a, \omega) ds, \quad a.s. \quad i = 1, 2.
\] (3.3)

Moreover, if \( g(s, x_i, \omega) \) is measurable and bounded, by the occupation times formula (e.g. see \([15, 25]\)),

\[
\int_0^t g(s, X_i(s)) d<M_i>_s = 2 \int_{-\infty}^\infty \int_0^t g(s, a) dL_i(s, a, \omega) da. \quad a.s. \quad i = 1, 2
\]

If \( g(s, x_i) \) is differentiable in \( s \), then using the integration by parts formula, we have

\[
\int_0^t g(s, X_i(s)) d<M_i>_s = 2 \int_{-\infty}^\infty \int_0^t g(s, a) dL_i(s, a, \omega) da \\
= 2 \int_{-\infty}^\infty \int_0^t g(t, a)L_i(t, a, \omega) da \\
- 2 \int_{-\infty}^\infty \int_0^t \frac{\partial}{\partial s} g(s, a)L_i(s, a, \omega) ds da, \quad a.s.,
\] (3.4)

for \( i = 1, 2 \). On the other hand, by Tanaka formula

\[
L_i(t, a) = (X_1(t) - a)^+ - (X_1(0) - a)^+ - \hat{M}_1(t, a) - \hat{V}_1(t, a),
\]

where \( \hat{Z}_1(t, a) = \int_0^t 1_{\{X_1(s) > a\}} dZ_1(s), \ Z_1 = M_1, V_1, X_1 \). By a standard localizing argument, we may assume without loss of generality that there is a constant \( N \) for which

\[
\sup_{0 \leq s \leq t} |X_1(s)| \leq N, \quad <M_1>_t \leq N, \quad Var_t V_1 \leq N,
\]

where \( Var_t V_1 \) is the total variation of \( V_1 \) on \([0, t] \). From the property of local time (see Chapter 3 in \([15]\)), for any \( \gamma \geq 1 \),
\[ E[\hat{M}_1(t,a) - \hat{M}_1(t,b)]^{2\gamma} \]
\[ = E\left| \int_0^t 1_{a < X_s \leq b} d \langle M_1 \rangle_s \right|^\gamma \]
\[ \leq C(b - a)^\gamma, \quad a < b \]

where the constant \( C \) depends on \( \gamma \) and on the bound \( N \). From Kolmogorov’s tightness criterion (see [16]), we know that the sequence \( Y_n(a) := \frac{1}{n} \hat{M}_1(t,a), n = 1, 2, \cdots \), is tight. Moreover for any \( a_1, a_2, \cdots, a_k \),

\[ P\left( \sup_{a_i} \left| \frac{1}{n} \hat{M}_1(t,a_i) \right| \leq 1 \right) \]
\[ = P\left( \left| \frac{1}{n} \hat{M}_1(t,a_1) \right| \leq 1, \left| \frac{1}{n} \hat{M}_1(t,a_2) \right| \leq 1, \cdots, \left| \frac{1}{n} \hat{M}_1(t,a_k) \right| \leq 1 \right) \]
\[ \geq 1 - \sum_{i=1}^k P\left( \left| \frac{1}{n} \hat{M}_1(t,a_i) \right| > 1 \right) \]
\[ \geq 1 - \frac{1}{n^2} \sum_{i=1}^k E[\hat{M}_1^2(t,a_i)] \]
\[ \geq 1 - \frac{k}{n^2} C(N - a), \]

so by the weak convergence theorem of random fields (see Theorem 1.4.5 in [16]), we have

\[ \lim_{n \to \infty} P\left( \sup_{a} \left| \frac{1}{n} \hat{M}_1(t,a) \right| \leq n \right) = 1. \]

Furthermore it is easy to see that

\[ \frac{1}{n} \hat{V}_1(t,a) \leq \frac{1}{n} \text{Var}_t V_1(t,a) \to 0, \quad \text{when } n \to \infty, \]

so it follows that,

\[ \lim_{n \to \infty} P\left( \sup_{a} \left| L_1(t,a) \right| \leq n \right) = 1. \]

Therefore in our localization argument, we can also assume \( L_1(t,a) \) and \( L_2(t,a) \) are bounded uniformly in \( a \).

In the following we assume some conditions on \( f : R^+ \times R \times R \to R \):

**Condition (i)** \( f(\cdot, \cdot, \cdot) : R^+ \times R \times R \to R \) is left continuous and locally bounded and jointly continuous from the right in \( t \) and left in \( (x_1, x_2) \) at each point \((0, x_1, x_2)\);

**Condition (ii)** the left derivative \( \frac{\partial}{\partial a} f(t,x_1, x_2) \) exists at all points of \((0, \infty) \times R^2\), and \( \nabla_{x_1} f(t,x_1, x_2), \nabla_{x_2} f(t,x_1, x_2) \) exist at all points \([0, \infty) \times R^2\) and are jointly left continuous and locally bounded;
Condition (iii) $\nabla^-_i f(t, x_1, x_2)$ is of locally bounded variation in $x_i$, $i = 1, 2$;

Condition (iv) $\frac{\partial}{\partial t} f(t, x_1, x_2)$ $(i = 1, 2)$ and $\nabla^-_1 \nabla^-_2 f(t, x_1, x_2)$ exist at all points of $(0, \infty) \times R^2$ and $[0, \infty) \times R^2$ respectively, and are left continuous and locally bounded;

Condition (v) $\nabla^-_1 \nabla^-_2 f(t, x_1, x_2)$ is of locally bounded variation in $(t, x_1)$ and $(t, x_2)$ and $\nabla^-_1 \nabla^-_2 f(0, x_1, x_2)$ is of locally bounded variation in $x_1$ and $x_2$ respectively.

From the assumption of $\nabla^-_i f$, we can use the one-dimensional generalized Itô formula (Theorem 1.1 in [7])

$$\nabla^-_i f(t, a, X_2(t)) - \nabla^-_i f(0, a, X_2(0)) = \int_0^t \frac{\partial}{\partial s} \nabla^-_i f(s, a, X_2(s)) ds + \int_0^t \nabla^-_2 f(s, a, X_2(s)) dX_2(s)$$

$$+ \int_{-\infty}^{\infty} L_2(t, x_2) dx_2 \nabla^-_i \nabla^-_2 f(t, a, x_2)$$

$$- \int_{-\infty}^{t} \int_t^t L_2(s, x_2) ds, dx_2 \nabla^-_1 \nabla^-_2 f(s, a, x_2). \quad \text{a.s.} \tag{3.5}$$

Therefore $\nabla^-_1 f(t, a, X_2(t))$ is a continuous semi-martingale, and can be decomposed as $\nabla^-_1 f(t, a, X_2(t)) = \nabla^-_1 f(0, a, X_2(0)) + h(t, a) + v(t, a)$, where $h$ is a continuous local martingale and $v$ is a continuous process of locally bounded variation (in $t$). In fact $h(t, a) = \int_0^t \nabla^-_1 \nabla^-_2 f(s, a, X_2(s)) dM_2(s)$. Define

$$F_s(a, b) := < h(a), h(b) >_s = < \nabla^-_1 f(a), \nabla^-_1 f(b) >_s$$

$$F_{s+k}^{s+1}(a, b) := < h(a), h(b) >_{s+k} = < \nabla^-_1 f(a), \nabla^-_1 f(b) >_{s+k}$$

$$= \int_{s+k}^{s+k+1} \nabla^-_1 \nabla^-_2 f(r, a, X_2(r)) \nabla^-_1 \nabla^-_2 f(r, b, X_2(r)) dr \quad \text{and} \quad < M_2 >_r.$$

We need to prove $h(s, a) \in \mathcal{V}_2$. To see this, as $\nabla^-_1 \nabla^-_2 f(t, x_1, x_2)$ is of locally bounded variation in $x_1$, so for any compact set $G$, $\nabla^-_1 \nabla^-_2 f(t, x_1, x_2)$ is of bounded variation in $x_1$ for $x_1 \in G$. Also on this set, let $\mathcal{P}$ be the partition on $R^2 \times [0, t]$, $\mathcal{P}_i$ be a partition on $R$ $(i = 1, 2)$, $\mathcal{P}_3$ be a partition on $[0, t]$ such that $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3$. Then we have:

$$\text{Var}_{s, a, b}(F_s(a, b))$$

$$= \sup_{\mathcal{P}} \sum_k \sum_i \sum_j \left| F_{s+k}^{s+1}(a_{i+1}, b_{j+1}) - F_{s+k}^{s+1}(a_{i+1}, b_j) - F_{s+k}^{s+1}(a_i, b_{j+1}) + F_{s+k}^{s+1}(a_i, b_j) \right|$$
Theorem 3.1

Therefore under the localizing assumption, \(\int_{-\infty}^{\infty} \int_{-\infty}^{t} L_1(s, a) d_{s,a} \nabla^2_1 f(s, a, X_2(s))\) and \(\int_{-\infty}^{\infty} \int_{-\infty}^{t} L_2(s, a) d_{s,a} \nabla^2_2 f(s, X_1(s), a)\) can be defined by Definition 2.1. A localizing argument implies they are semi-martingales.

We will prove the following generalized Itô’s formula in two-dimensional space.

**Theorem 3.1** Under conditions (i)-(v), for any continuous two-dimensional semi-martingale \(X(t) = (X_1(t), X_2(t))\), we have

\[
f(t, X_1(t), X_2(t)) = f(0, X_1(0), X_2(0)) + \int_{0}^{t} \frac{\partial f}{\partial s}(s, X_1(s), X_2(s)) ds \\
+ \sum_{i=1}^{2} \int_{0}^{t} \nabla_1^i f(s, X_1(s), X_2(s)) dX_i(s) \\
+ \int_{-\infty}^{\infty} L_1(t, a) d_{a} \nabla^2_1 f(t, a, X_2(t)) - \int_{-\infty}^{+\infty} \int_{-\infty}^{t} L_1(s, a) d_{s,a} \nabla^2_1 f(s, a, X_2(s)) \\
+ \int_{-\infty}^{\infty} L_2(t, a) d_{a} \nabla^2_2 f(t, X_1(t), a) - \int_{-\infty}^{+\infty} \int_{-\infty}^{t} L_2(s, a) d_{s,a} \nabla^2_2 f(s, X_1(s), a) \\
+ \int_{0}^{t} \nabla_1^2 f(s, X_1(s), X_2(s)) d<M_1, M_2>_s \quad a.s. \quad (3.6)
\]
Proof: By a standard localization argument, we can assume \( X_1(t), X_2(t) \) and their quadratic variations \( <X_1>_t, <X_2>_t \) and \( <X_1, X_2>_t \) and the local times \( L_1, L_2 \) are bounded processes so that \( f, \frac{\partial}{\partial t} f, \nabla_1 f, \nabla_2 f, \nabla_1 \nabla_2 f, \text{Var}_x \nabla_1 f, \text{Var}_x \nabla_2 f, \text{Var}_x \nabla_1 \nabla_2 f \) are bounded. Note the left derivatives of \( f \) agree with the generalized derivatives, so condition (ii) and (iv) imply that \( f \) is absolutely continuous with respect to \( t \) and \( x_2 \) respectively and \( \nabla_2 f \) is absolutely continuous with respect to \( t \) and \( x_1 \) respectively.

We divide the proof into several steps:

(A) Define

\[
\rho(x) = \begin{cases} 
    ce^{(c-1)|x|}, & \text{if } x \in (0, 2), \\
    0, & \text{otherwise.}
\end{cases}
\]

Here \( c \) is chosen such that \( \int_0^2 \rho(x)dx = 1 \). Take \( \rho_n(x) = n\rho(nx) \) as mollifiers. Define

\[
f_n(s, x_1, x_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho_n(s-\tau)\rho_n(x_1-y)\rho_n(x_2-z)f(\tau, y, z)d\tau dy dz, \ n \geq 1,
\]

where we set \( f(\tau, y, z) = f(-\tau, y, z) \) if \( \tau < 0 \). Then \( f_n(s, x_1, x_2) \) are smooth and

\[
f_n(s, x_1, x_2) = \int_0^2 \int_0^2 \rho(t)\rho(y)\rho(z)f(s-t/n, x_1-y/n, x_2-z/n)dtdydz, \ n \geq 1. \tag{3.8}
\]

Because of the absolute continuity mentioned above, we can differentiate under the integral \( \Box \) to see \( \frac{\partial}{\partial s} f_n, \nabla_1 f_n, \nabla_2 f_n, \nabla_1 \nabla_2 f_n, \text{Var}_x \nabla_1 f_n, \text{Var}_x \nabla_2 f_n, \text{Var}_x \nabla_1 \nabla_2 f_n \) and \( \text{Var}_x \nabla_1 \nabla_2 f_n \) are bounded. Furthermore using Lebesgue’s dominated convergence theorem, one can prove that as \( n \to \infty \),

\[
f_n(s, x_1, x_2) \to f(s, x_1, x_2), \ s \geq 0 \tag{3.9}
\]

\[
\frac{\partial}{\partial s} f_n(s, x_1, x_2) \to \frac{\partial}{\partial s} f(s, x_1, x_2), \ s > 0 \tag{3.10}
\]

\[
\nabla_1 f_n(s, x_1, x_2) \to \nabla_1 f(s, x_1, x_2), \ s \geq 0 \tag{3.11}
\]

\[
\nabla_2 f_n(s, x_1, x_2) \to \nabla_2 f(s, x_1, x_2), \ s \geq 0 \tag{3.12}
\]

\[
\nabla_1 \nabla_2 f_n(s, x_1, x_2) \to \nabla_1 \nabla_2 f(s, x_1, x_2), \ s \geq 0 \tag{3.13}
\]

and each \( (x_1, x_2) \in \mathbb{R}^2 \).

(B) It turns out for any \( g(t, x_1) \) being continuous in \( t \) and \( C^1 \) in \( x_1 \) and having a compact support, using the integration by parts formula and Lebesgue’s dominated convergence theorem, we see that
\[
\lim_{n \to +\infty} \int_{-\infty}^{+\infty} g(t, x_1) \Delta f_n(t, x_1, X_2(t)) \, dx_1 \\
= - \lim_{n \to +\infty} \int_{-\infty}^{+\infty} \nabla g(t, x_1) \nabla f_n(t, x_1, X_2(t)) \, dx_1 \\
= - \int_{-\infty}^{+\infty} \nabla g(t, x_1) \nabla f(t, x_1, X_2(t)) \, dx_1. \tag{3.14}
\]

Note \( \nabla f(t, x_1, x_2) \) is of locally bounded variation in \( x_1 \) and \( g(t, x_1) \) has a compact support in \( x_1 \), so
\[
- \int_{-\infty}^{+\infty} \nabla g(t, x_1) \nabla f(t, x_1, X_2(t)) \, dx_1 \\
= \int_{-\infty}^{+\infty} g(t, x_1) \, dx_1 \nabla f(t, x_1, X_2(t)). \tag{3.15}
\]

Thus
\[
\lim_{n \to +\infty} \int_{-\infty}^{+\infty} g(t, x_1) \Delta f_n(t, x_1, X_2(t)) \, dx_1 \\
= \int_{-\infty}^{+\infty} g(t, x_1) \, dx_1 \nabla f(t, x_1, X_2(t)). \tag{3.16}
\]

(C) If \( g(s, x_1) \) is \( C^2 \) in \( x_1 \), \( \Delta g(s, x_1) \) is bounded uniformly in \( s \), \( \frac{\partial}{\partial s} \nabla g(s, x_1) \) is continuous in \( s \) and has a compact support in \( x_1 \), and
\[
E \left[ \int_0^t \int_{\mathbb{R}^2} |g(s, x)g(s, y)||dx_2, dx_3 < h(x), h(y) > s \right] < \infty,
\]
where \( h \in \mathcal{V}_2 \), then applying Itô’s formula, Lebesgue’s dominated convergence theorem and the integration by parts formula,
\[
\lim_{n \to +\infty} \left( \int_0^t \int_{-\infty}^{+\infty} g(s, x_1) \frac{\partial}{\partial s} \Delta f_n(s, x_1, X_2(s)) \, dx_1 \, ds \\
+ \int_0^t \int_{-\infty}^{+\infty} g(s, x_1) \nabla_2 \Delta f_n(s, x_1, X_2(s)) \, dx_1 \, dX_2(s) \\
+ \frac{1}{2} \int_0^t \int_{-\infty}^{+\infty} g(s, x_1) \Delta_2 \Delta f_n(s, x_1, X_2(s)) \, dx_1 \, d<M_2>_s \right) \\
= - \lim_{n \to +\infty} \left( \int_0^t \int_{-\infty}^{+\infty} \nabla g(s, x_1) \frac{\partial}{\partial s} \nabla f_n(s, x_1, X_2(s)) \, dx_1 \, ds \\
+ \int_0^t \int_{-\infty}^{+\infty} \nabla g(s, x_1) \nabla_2 f_n(s, x_1, X_2(s)) \, dx_1 \, dX_2(s) \\
+ \frac{1}{2} \int_0^t \int_{-\infty}^{+\infty} \nabla g(s, x_1) \Delta_2 \nabla f_n(s, x_1, X_2(s)) \, dx_1 \, d<M_2>_s \right) \\
= - \lim_{n \to +\infty} \int_{-\infty}^{t} \nabla g(s, x_1) \, ds \nabla f_n(s, x_1, X_2(s)) \, dx_1
\]
= - \lim_{n \to +\infty} \left( \int_{-\infty}^{\infty} \nabla g(s, x_1) \nabla \mathcal{L} f_n(s, x_1, X_2(s)) \right)^t dx_1 \\
- \int_{0}^{t} \int_{-\infty}^{+\infty} \frac{\partial}{\partial s} \nabla g(s, x_1) \nabla \mathcal{L} f_n(s, x_1, X_2(s)) dx_1 ds \\
= - \int_{-\infty}^{\infty} \nabla g(s, x_1) \nabla \mathcal{L} f(s, x_1, X_2(s)) \right)^t dx_1 \\
+ \int_{0}^{t} \int_{-\infty}^{+\infty} \frac{\partial}{\partial s} \nabla g(s, x_1) \nabla \mathcal{L} f(s, x_1, X_2(s)) dx_1 ds \\
= - \int_{-\infty}^{t} \int_{0}^{+\infty} \nabla g(s, x_1) ds \nabla f(s, x_1, X_2(s)) dx_1.

It turns out by applying Proposition 2.1 that

\[
\lim_{n \to +\infty} \left( \int_{0}^{t} \int_{-\infty}^{+\infty} g(s, x_1) \frac{\partial}{\partial s} \Delta \mathcal{L} f_n(s, x_1, X_2(s)) dx_1 ds \\
+ \int_{0}^{t} \int_{-\infty}^{+\infty} g(s, x_1) \nabla \Delta f_n(s, x_1, X_2(s)) dx_1 dX_2(s) \\
+ \frac{1}{2} \int_{0}^{t} \int_{-\infty}^{+\infty} g(s, x_1) \Delta^2 f_n(s, x_1, X_2(s)) dx_1 dM_2(s) \right) \\
= \int_{0}^{t} \int_{-\infty}^{+\infty} g(s, x_1) \nabla \mathcal{L} f(s, x_1, X_2(s)). \tag{3.17}
\]

(D) But any càdlàg function with a compact support can be approximated by smooth functions with a compact support uniformly by the following standard smoothing procedure

\[g_m(t, x_1) = \int_{-\infty}^{\infty} \rho_m(y - x_1) g(t, y) dy = \int_{0}^{2} \rho(z) g(t, x_1 + \frac{z}{m}) dz.\]

Then we can prove that \((3.16)\) also holds for any càdlàg function \(g(t, x_1)\) with a compact support in \(x_1\). Moreover, if \(g \in \mathcal{V}_3\), \((3.17)\) also holds.

To see \((3.17)\), note that there is a compact set \(G \subset \mathbb{R}^1\) such that

\[
\max_{x_1 \in G} |g_m(t, x_1) - g(t, x_1)| \to 0 \text{ as } m \to +\infty, \\
g_m(t, x_1) = g(t, x_1) = 0 \text{ for } x_1 \notin G.
\]

Note

\[
= \int_{-\infty}^{+\infty} g(t, x_1) \Delta \mathcal{L} f_n(t, x_1, X_2(t)) dx_1 \\
= \int_{0}^{t} \int_{-\infty}^{+\infty} g_m(t, x_1) \Delta \mathcal{L} f_n(t, x_1, X_2(t)) dx_1 \\
+ \int_{-\infty}^{+\infty} (g(t, x_1) - g_m(t, x_1)) \Delta \mathcal{L} f_n(t, x_1, X_2(t)) dx_1. \tag{3.18}
\]
It is easy to see from (3.16) and Lebesgue’s dominated convergence theorem, that
\[
\lim_{m \to \infty} \lim_{n \to \infty} \int_{-\infty}^{\infty} g_m(t, x_1) \Delta_1 f_n(t, x_1, X_2(t)) dx_1 = \lim_{m \to \infty} \int_{-\infty}^{\infty} g_m(t, x_1) dx_1 \nabla_1^{-} f(t, x_1, X_2(t))
\]
\[
= \int_{-\infty}^{\infty} g(t, x_1) dx_1 \nabla_1^{-} f(t, x_1, X_2(t)). \tag{3.19}
\]

Moreover,
\[
| \int_{-\infty}^{+\infty} \left( g(t, x_1) - g_m(t, x_1) \right) \Delta_1 f_n(t, x_1, X_2(t)) dx_1 |
\]
\[
\leq \left( \max_{x_1 \in G} | g(t, x_1) - g_m(t, x_1) | \right) \text{Var}_{x_1 \in G} \nabla_1 f_n(t, x_1, X_2(t)). \tag{3.20}
\]

But,
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \left( \max_{x_1 \in G} | g(t, x_1) - g_m(t, x_1) | \right) \text{Var}_{x_1 \in G} \nabla_1 f_n(t, x_1, X_2(t)) = 0.
\]

So inequality (3.20) leads to
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \int_{-\infty}^{+\infty} \left( g(t, x_1) - g_m(t, x_1) \right) \Delta_1 f_n(t, x_1, X_2(t)) dx_1 = 0. \tag{3.21}
\]

Now we use (3.19), (3.19) and (3.21)
\[
\limsup_{n \to \infty} \int_{-\infty}^{+\infty} g(t, x_1) \Delta_1 f_n(t, x_1, X_2(t)) dx_1
\]
\[
= \lim_{m \to \infty} \limsup_{n \to \infty} \int_{-\infty}^{+\infty} g_m(t, x_1) \Delta_1 f_n(t, x_1, X_2(t)) dx_1
\]
\[
+ \lim_{m \to \infty} \limsup_{n \to \infty} \int_{-\infty}^{+\infty} \left( g(t, x_1) - g_m(t, x_1) \right) \Delta_1 f_n(t, x_1, X_2(t)) dx_1
\]
\[
= \int_{-\infty}^{\infty} g(t, x_1) dx_1 \nabla_1^{-} f(t, x_1, X_2(t)).
\]

Similarly we also have
\[
\liminf_{n \to \infty} \int_{-\infty}^{+\infty} g(t, x_1) \Delta_1 f_n(t, x_1, X_2(t)) dx_1
\]
\[
= \int_{-\infty}^{\infty} g(t, x_1) dx_1 \nabla_1^{-} f(t, x_1, X_2(t)). \tag{3.22}
\]
So (3.16) holds for a càdlàg function \( g \) with a compact support in \( x_1 \).

Now we prove that (3.17) also holds for a càdlàg function \( g \in V_3 \). Obviously,

\[
\begin{align*}
\int_0^t \int_{-\infty}^{+\infty} g(s, x_1) \frac{\partial}{\partial s} \Delta_1 f_n(s, x_1, X_2(s)) dx_1 ds \\
+ \int_0^t \int_{-\infty}^{+\infty} g(s, x_1) \nabla_2 \Delta_1 f_n(s, x_1, X_2(s)) dx_1 dX_2(s) \\
+ \frac{1}{2} \int_0^t \int_{-\infty}^{+\infty} g(s, x_1) \Delta_2 \Delta_1 f_n(s, x_1, X_2(s)) dx_1 d\langle M_2 \rangle_s \\
= \int_0^t \int_{-\infty}^{+\infty} g(s, x_1) d_{s,x_1} \nabla_1 f_n(s, x_1, X_2(s)).
\end{align*}
\]

Define

\[
g_m(s, x_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho_m(y-x_1) \rho_m(\tau-s) g(\tau, y) d\tau dy.
\]

Then there is a compact \( G \subset R^1 \) such that

\[
\max_{0 \leq s \leq t, x_1 \in G} |g_m(s, x_1) - g(s, x_1)| \to 0 \quad \text{as} \quad m \to +\infty,
\]

\[
g_m(s, x_1) = g(s, x_1) = 0 \quad \text{for} \quad x_1 \notin G.
\]

Then it is trivial to see

\[
\begin{align*}
\int_0^t \int_{-\infty}^{+\infty} g(s, x_1) d_{s,x_1} \nabla_1 f_n(s, x_1, X_2(s)) \\
= \int_0^t \int_{-\infty}^{+\infty} g_m(s, x_1) d_{s,x_1} \nabla_1 f_n(s, x_1, X_2(s)) \\
+ \int_0^t \int_{-\infty}^{+\infty} (g(s, x_1) - g_m(s, x_1)) d_{s,x_1} \nabla_1 f_n(s, x_1, X_2(s)).
\end{align*}
\]

But from (3.17), we can see that

\[
\begin{align*}
\lim_{m \to \infty} \lim_{n \to \infty} \int_0^t \int_{-\infty}^{+\infty} g_m(s, x_1) d_{s,x_1} \nabla_1 f_n(s, x_1, X_2(s)) \\
= \lim_{m \to \infty} \int_0^t \int_{-\infty}^{+\infty} g_m(s, x_1) d_{s,x_1} \nabla_1 f_n(s, x_1, X_2(s)) \\
= \int_0^t \int_{-\infty}^{+\infty} g(s, x_1) d_{s,x_1} \nabla_1 f_n(s, x_1, X_2(s)). \quad (\text{limit in } M_2) \quad (3.23)
\end{align*}
\]

The last limit holds because of the following:
Now we use the multi-dimensional Itô’s formula to the function $f_n(s, x_1, X_2(s))$ to prove (3.24).

In fact, we can use an argument similar to the one in the proof of (3.21) and (3.22) to prove (3.24).

(E) Now we use the multi-dimensional Itô’s formula to the function $f_n(s, X_1(s), X_2(s))$, then a.s.

As $n \to \infty$, it is easy to see from Lebesgue’s dominated convergence theorem and (3.21), (3.22), (3.23), (3.24) that, $(i = 1, 2)$
\[ f_n(t, X_1(t), X_2(t)) - f_n(0, z_1, z_2) \rightarrow f(t, X_1(t), X_2(t)) - f(0, z_1, z_2), \text{ a.s.} \]

\[
\int_0^t \frac{\partial}{\partial s} f_n(s, X_1(s), X_2(s)) ds \rightarrow \int_0^t \frac{\partial}{\partial s} f(s, X_1(s), X_2(s)) ds, \text{ a.s.}
\]

\[
\int_0^t \nabla_i f_n(s, X_1(s), X_2(s)) dV_i(s) \rightarrow \int_0^t \nabla_i f(s, X_1(s), X_2(s)) dV_i(s), \text{ a.s.}
\]

\[
\int_0^t \nabla_1 \nabla_2 f_n(s, X_1(s), X_2(s)) d \langle M_1, M_2 \rangle_s
\]

\[
\rightarrow \int_0^t \nabla_1 \nabla_2 f(s, X_1(s), X_2(s)) d <M_1, M_2>_s. \text{ a.s.}
\]

and

\[
E \int_0^t (\nabla_i f_n(s, X_1(s), X_2(s)))^2 d <M_i>_s
\]

\[
\rightarrow E \int_0^t (\nabla_i f(s, X_1(s), X_2(s)))^2 d <M_i>_s.
\]

Therefore in \( M_2 \),

\[
\int_0^t \nabla_i f_n(s, X_1(s), X_2(s)) dM_i(s) \rightarrow \int_0^t \nabla_i f(s, X_1(s), X_2(s)) dM_i(s), (i = 1, 2).
\]

To see the convergence of \( \frac{1}{2} \int_0^t \Delta_1 f_n(s, X_1(s), X_2(s)) d <M_1>_s \), we recall the well-known result that the local time \( L_1(s, a) \) is jointly continuous in \( s \) and càdlàg with respect to \( a \) and has a compact support in space \( a \) for each \( s \) (\[25\], \[15\]). As \( L_1(s, a) \) is an increasing function of \( s \) for each \( a \), so if \( G \subset \mathbb{R}^1 \) is the support of \( L_1(s, a) \), then \( L_1(s, a) = 0 \) for all \( a \not\in G \) and \( s \leq t \). Now we use the occupation times formula, the integration by parts formula and \[3.10\], \[3.17\] for the case when \( g \) is càdlàg with compact support,

\[
\frac{1}{2} \int_0^t \Delta_1 f_n(s, X_1(s), X_2(s)) d <M_1>_s
\]

\[
= \int_{-\infty}^{+\infty} \int_0^t \Delta_1 f_n(s, a, X_2(s)) ds L_1(s, a) da
\]

\[
= \int_{-\infty}^{+\infty} \Delta_1 f_n(t, a, X_2(t)) L_1(t, a) da
\]

\[
- \int_{-\infty}^{+\infty} \left[ \int_0^t \frac{d}{ds} \Delta_1 f_n(s, a, X_2(s)) L_1(s, a) ds \right] 
\]

\[
+ \int_0^t \nabla_2 \Delta_1 f_n(s, X_1(s), a) L_1(s, a) dX_2(s)
\]
derivatives \( f \) continuous, then the same method to get a similar result. So we proved the desired formula.

The above smoothing procedure can be used to prove that if \( f : R^+ \times R^2 \to R \) is left continuous and locally bounded, \( C^1 \) in \( x_1 \) and \( x_2 \), and the left derivatives \( \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} \) exist at all points of \((0,\infty) \times R^2\) and \([0,\infty) \times R^2\) respectively and are locally bounded and left continuous, then

\[
\begin{align*}
&f(t, X(t)) - f(0, X(0)) \\
&\quad = \int_0^t \frac{\partial f}{\partial s}(s, X_1(s), X_2(s))ds + \sum_{i=1}^2 \int_0^t \nabla_i f(s, X_1(s), X_2(s))dX_i(s) \\
&\quad + \frac{1}{2} \sum_{i,j=1}^2 \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_1(s), X_2(s))dX_i(s, X_j)\].
\end{align*}
\]

This can be seen from the convergence in the proof of Theorem 3.1 and the fact that \( \frac{\partial^2 f}{\partial x_i \partial x_j}(s, x_1, x_2) \to \frac{\partial^2 f}{\partial x_i \partial x_j}(s, x_1, x_2) \) under the stronger condition on \( \frac{\partial^2 f}{\partial x_i \partial x_j}, f \).

The next theorem is an easy consequence of the methods of the proofs of Theorem 3.1 and 3.2.

**Theorem 3.2** Let \( f : R^+ \times R^2 \to R \) satisfy conditions (i),(ii) and \( f(t, x_1, x_2) = f_h(t, x_1, x_2) + f_v(t, x_1, x_2) \). Assume \( f_h \) is \( C^1 \) in \( x_1, x_2 \) and the left derivatives \( \frac{\partial f_h}{\partial x_i}, \frac{\partial f_h}{\partial x_j} \) exist and are left continuous and locally bounded; \( f_v \) satisfies conditions (iii)-(v). Then

\[
\begin{align*}
&f(t, X_1(t), X_2(t)) - f(0, X_1(0), X_2(0)) \\
&\quad = \int_0^t \frac{\partial f}{\partial s}(s, X_1(s), X_2(s))ds + \sum_{i=1}^2 \int_0^t \nabla_i f(s, X_1(s), X_2(s))dX_i(s) \\
&\quad + \frac{1}{2} \sum_{i=1}^2 \int_0^t \Delta_i f_h(s, X_1(s), X_2(s))dX_i(s) \\
&\quad + \int_{-\infty}^\infty L_1(t, a)ds \nabla_1 f_v(t, a, X_2(t)) - \int_{-\infty}^\infty \int_0^t L_1(s, a)ds \nabla_1 f_v(s, a, X_2(s)).
\end{align*}
\]
Corollary 3.1 Assume \( f : \mathbb{R}^2 \to \mathbb{R} \) is left continuous and locally bounded and there exists a continuous curve \( x_2 = b(x_1) \) such that

(i) the left derivative \( \frac{\partial}{\partial x_1} b(x_1) \) exists and is locally bounded and \( b(X_1(t)) \) is a semi-martingale;

(ii) \( f(x_1, x_2) \) is twice differentiable with continuous second order derivatives \( \frac{\partial^2}{\partial x_1 \partial x_2} f \) \((i, j = 1, 2)\) in regions \( x_2 \leq b(x_1) \) and \( x_2 \geq b(x_1) \) respectively.

Then for any two-dimensional continuous semi-martingale \((X_1(t), X_2(t))\)

\[
\begin{align*}
\frac{d}{dt} f(X_1(t), X_2(t)) &= f(X_1(0), X_2(0)) \\
&= \sum_{i=1}^{2} \int_{0}^{t} \nabla_1 f(X_1(s), X_2(s))dX_1(s) + \frac{1}{2} \sum_{i=1}^{2} \int_{0}^{t} \Delta^- f(X_1(s), X_2(s))d <X_i>_s \\
&+ \int_{0}^{t} \left[ \nabla_2^- f(X_1(s), b(X_1(s))+) - \nabla_2^- f(X_1(s), b(X_1(s))-) \right] dL^*_2(s, 0) \\
&+ \int_{0}^{t} \nabla_1^- \nabla_2^- f(X_1(s), X_2(s))d <M_1, M_2>_s. \quad a.s.
\end{align*}
\]

Proof: Formula \((3.28)\) can be read from \((3.27)\) by considering

\[
f_h(x_1, x_2) = f(x_1, x_2) + \int_{0}^{x_1} (\nabla_2 f(y, b(y)+) - \nabla_2 f(y, b(y)-))(x_2 - b(y))^+ dy,
\]

\[
f_v(x_1, x_2) = \int_{0}^{x_1} (\nabla_2 f(y, b(y)+) - \nabla_2 f(y, b(y)-))(x_2 - b(y))^+ dy,
\]

and the integration by parts formula. To verify conditions of Theorem \((3.2)\) on \( f_v \), first note

\[
\begin{align*}
\nabla_1^- f_v(x_1, x_2) &= (\nabla_2 f(x_1, b(x_1)+) - \nabla_2 f(x_1, b(x_1)-))(x_2 - b(x_1))^+, \\
\nabla_2^- f_v(x_1, x_2) &= \int_{0}^{x_1} (\nabla_2 f(y, b(y)+) - \nabla_2 f(y, b(y)-))1_{\{x_2 > b(y)\}} dy, \\
\nabla_1^- \nabla_2^- f_v(x_1, x_2) &= (\nabla_2 f(x_1, b(x_1)+) - \nabla_2 f(x_1, b(x_1)-))1_{\{x_2 > b(x_1)\}}.
\end{align*}
\]
It’s trivial to prove that \( \nabla_x^1 \nabla_x^2 f_r(x_1, x_2 + b(x_1)) \) is of locally bounded variation in \( x_2 \). To see \( \nabla_x^1 f_r(x_1, x_2 + b(x_1)) \) is of locally bounded variation for \( x_2 \), for any partition \(-N = x_2^0 < x_2^1 < \cdots < x_2^N = N\),

\[
\sum_{i=0}^{n-1} |\nabla_x^2 f_r(x_1, x_2^{i+1} + b(x_1)) - \nabla_x^2 f_r(x_1, x_2^i + b(x_1))| \leq \sum_{i=0}^{n-1} \int_{x_2^i}^{x_2^{i+1}} |\nabla_x^2 f(y, b(y)) + \nabla_x^2 f(y, b(y))| 1_{\{x_2^i + b(x_1) \leq b(y) \leq x_2^{i+1} + b(x_1)\}} dy
\]

\[
\leq \int_{x_2^0}^{x_2^N} |\nabla_x^2 f(y, b(y)) + \nabla_x^2 f(y, b(y))| 1_{\{-N + b(x_1) \leq b(y) \leq N + b(x_1)\}} dy < \infty.
\]

In order to prove that \( \nabla_x^1 \nabla_x^2 f_r(x_1, x_2 + b(x_1)) \) is of locally bounded variation in \( x_1 \), we only need to prove that \( \nabla_x^2 f(x_1, b(x_1)+) - \nabla_x^2 f(x_1, b(x_1)–) \) is of locally bounded variation in \( x_1 \). This is true, because for \( x_2^* > 0 \)

\[
\frac{D}{Dx_1} \nabla_x^2 f(x_1, x_2^* + b(x_1)) = \nabla_1 \nabla_x^2 f(x_1, x_2^* + b(x_1)) + \nabla_2 \nabla_x^2 f(x_1, x_2^* + b(x_1)) \frac{d}{dx_1} b(x_1).
\]

So as \( x_2^* \to 0+ \),

\[
\frac{D}{Dx_1} \nabla_x^2 f(x_1, x_2^* + b(x_1)) \to \nabla_1 \nabla_x^2 f(x_1, b(x_1)+) + \nabla_2 \nabla_x^2 f(x_1, b(x_1)+) \frac{d}{dx_1} b(x_1)
\]

\[
= \frac{d}{dx_1} \nabla_x^2 f(x_1, b(x_1)+).
\]

It follows that \( \nabla_x^2 f(x_1, b(x_1)+) \) is of locally bounded variation. Similarly, one can prove that \( \nabla_x^2 f(x_1, b(x_1)–) \) is also of locally bounded variation.

\[\Box\]

**Acknowledgement**

We would like to acknowledge partial financial supports to this project by the EPSRC research grants GR/R69518 and GR/R93582. CF would like to thank the Loughborough University development fund for its financial support. It is our great pleasure to thank N. Eisenbaum, D. Elworthy, Y. Liu, Z. Ma, S. Peng, G. Peskir, A. Truman, J. A. Yan, M. Yor and W. Zheng for helpful discussions. We would like to thank G. Peskir and N. Eisenbaum for
invitation to the mini-workshop of local time-space calculus with applications in Oberwolfach May 2004 where the results of this paper were announced; to S. Peng for invitation to speak at the 9-th Chinese mathematics summer school (Weihai) 2004; to F. Gong to the workshop on stochastic analysis in Chinese Academy of Sciences 2004 and M. Chen to the workshop on stochastic processes and related topics in Beijing 2004. We would like to thank the referee for careful reading of the manuscript and pointing out an error in the early version of the paper and other useful suggestions.

References

1. R. B. Ash and C. A. Doléans-Dade, Probability and Measure Theory, Second Edition, Academic Press (2000).
2. J. Azéma, T. Jeulin, F. Knight and M. Yor, Quelques calculs de compensateurs impliquant l’injectivité de certains processus croissants, Séminaire de Probabilités XXXII (1998), LNM1686, 316-327.
3. N. Bouleau and M. Yor, Sur la variation quadratique des temps locaux de certains semimartingales, C.R.Acad. Sci. Paris, Ser.I Math 292 (1981), 491-494.
4. K. L. Chung, R. J. Williams, Introduction to Stochastic Integration, Birkhauser 1990.
5. N. Eisenbaum, Integration with respect to local time, Potential analysis 13 (2000), 303-328.
6. N. Eisenbaum, Local time-space calculus for revisible semi-martingales, submitted to Séminaire de Probabilités XXXIX, Lecture Notes in Mathematics, Springer-Verlag (to appear).
7. K. D. Elworthy, A. Truman and H. Z. Zhao, Generalized Itô Formulae and space-time Lebesgue-Stieltjes integrals of local times, Séminaire de Probabilités, vol 40, Lecture Notes in Mathematics, Springer-Verlag (to appear).
8. K. D. Elworthy, A. Truman and H. Z. Zhao, Asymptotics of Heat Equations with Caustics in One-Dimension, Preprint (2003).
9. C. R. Feng and H. Z. Zhao, Two-parameter p,q-variation Path and Integration of Local Times, Preprint (2005).
10. F. Flandoli, F. Russo and J. Wolf, Some stochastic differential equations with distributional drift, Osaka J. Math. 40 (2003), no. 2, 493-542.
11. H. Föllmer, P. Protter and A. N. Shiryaev, Quadratic covariation and an extension of Itô’s Formula, Bernoulli 1 (1995), 149-169.
12. H. Föllmer and P. Protter, On Itô Formula for multidimensional Brownian motion, Probability Theory and Related Fields 116 (2000), 1-20.
13. R. Ghomrasni and G. Peskir, Local time-space calculus and Extensions of Itô’s Formula, Preprint (2003).
14. N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, 2nd Edition, North-Holland Publ. Co., Amsterdam Oxford New York; Kodansha Ltd., Tokyo, 1981.
15. I. Karatzas and S. E. Shreve, Brownian Motion and Stochastic Calculus, Second Edition, Springer-Verlag, New York, 1998.
16. H. Kunita, *Stochastic Flows and Stochastic Differential Equations*, Cambridge University Press, Cambridge, 1990.
17. T. Lyons and Z. Qian, *System Control and Rough Paths*, Clarendon Press Oxford, 2002.
18. T. J. Lyons and W. A. Zheng, *A crossing estimate for the canonical process on a Dirichlet space and tightness result*, Colloque Paul Levy sur les Processus Stochastiques (Palaiseau, 1987), Astérisque 157-158 (1988), 249-271.
19. McShane, *Integration*, Princeton University Press, Princeton, 1944.
20. P. A. Meyer, *Un cours sur les intégrales stochastiques*, Sém. Probab 10, Lecture Notes in Math. No. 511, Springer-Verlag (1976), 245-400.
21. S. Moret and D. Nualart, *Generalization of Itô’s formula for smooth nondegenerate martingales*, Stochastic Process. Appl., 91 (2001), 115-149.
22. G. Peskir, *A change-of-variable formula with local time on curves*, J.Theoret Probab. (to appear).
23. G. Peskir, *A change-of-variable formula with local time on surfaces*, Séminaire de Probabilités, vol 40, Lecture Notes in Mathematics, Springer-Verlag (to appear).
24. G. Peskir, *On the American option problem*, Math.Finance (to appear).
25. D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion, Second Edition*, Springer-Verlag, Berlin, Heidelberg, 1994.
26. L. C. G. Rogers and J. B. Walsh, *Local time and stochastic area integrals*, Annals of Probab. 19(2) (1991), 457-482.
27. F. Russo and P. Vallois, *Itô’s Formula for $C^1$-functions of semimartingales*, Probability Theory and Related Fields, 104 (1996), 27-41.
28. H. Tanaka, *Note on continuous additive functionals of the 1-dimensional Brownian path*, Z.Wahrscheinlichkeitstheorie and Verw Gebiete 1 (1963), 251-257.
29. J. B. Walsh, *An Introduction to Stochastic Partial Differential Equations*, École d’Été de Saint-Flour XIV-1984, Lecture Notes in Math. Vol 1180, (1986), 265-439.
30. A. T. Wang, *Generalized Itô’s formula and additive functionals of Brownian motion*, Z.Wahrscheinlichkeitstheorie and Verw Gebiete, 41(1977), 153-159.