SIGN-COHERENCE OF C-VECTORS AND MAXIMAL GREEN SEQUENCES FOR ACYCLIC SIGN-SKEW-SYMMETRIC MATRICES

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ABSTRACT. In this paper we construct an unfolding for $c$-vectors of acyclic sign-skew-symmetric matrices and we also prove that the sign-coherence property holds for acyclic sign-skew-symmetric matrices. Then we prove that every acyclic sign-skew-symmetric matrix admits a maximal green sequence.

1. INTRODUCTION AND PRELIMINARIES

The problem posed by A. Berenstein, S. Fomin and A. Zelevinsky in [2] on whether any acyclic sign-skew-symmetric integer matrix is totally sign-skew-symmetric or not, was a great motivation for many mathematicians to study such matrices. M. Huang and F. Li gave an affirmative answer to this problem and proved in [10] that acyclic sign-skew-symmetric matrices are totally mutable. The authors in [10] also proved that every acyclic sign-skew-symmetric matrix can be covered by an (infinite) skew-symmetric matrix which is represented by an (infinite) cluster quiver and this covering can perform arbitrary steps of orbit-mutations. This (infinite) quiver is called an unfolding of this acyclic sign-skew-symmetric matrix. The existence of such an unfolding quiver for every acyclic sign-skew-symmetric matrix allows us to tackle problems related to an acyclic sign-skew-symmetric cluster algebra by promoting these problems to an (infinite) skew-symmetric cluster algebra. In this note we try to find an unfolding for the $c$-vectors of an acyclic sign-skew-symmetric matrix and prove that it always exists (see paragraph 3 of Remark 2.7). In other words, we prove that every extended acyclic sign-skew-symmetric matrix $\tilde{B} = \begin{pmatrix} B \\ I_\infty \end{pmatrix}$ can be covered by an (infinite) extended skew-symmetric matrix $\tilde{B}^\oplus = \begin{pmatrix} B^\oplus \\ I_\infty \end{pmatrix}$. The construction of this covering keeps the principal part as it was constructed in [10] which makes the ability of performing arbitrary steps...
of orbit-mutations remain valid. Using the unfolding method, we prove that the sign-coherence property holds for acyclic sign-skew-symmetric matrices, which together with the fact that these matrices are totally mutable, means that maximal green sequences are well-defined for acyclic sign-skew-symmetric matrices. Finally we prove that every acyclic sign-skew-symmetric matrix admits a maximal green sequence (see Theorem 3.10).

A [skew-symmetric](#) matrix is an integer matrix \( B = (b_{ij}) \) of the size \( n \times n \), such that \( b_{ij} = -b_{ji} \) for all \( 1 \leq i, j \leq n \). A [skew-symmetrizable](#) matrix is an integer matrix \( B = (b_{ij}) \) of the size \( n \times n \), such that \( B = -(BD)^T \) for \( D \) is a diagonal matrix with positive integers. \( D \) is called the symmetrizing matrix. A [sign-skew-symmetric](#) matrix is an integer matrix \( B = (b_{ij}) \) of the size \( n \times n \), such that either \( b_{ij}, b_{ji} = 0 \) or \( b_{ij} \cdot b_{ji} < 0 \) for any \( 1 \leq i, j \leq n \). The mutation of a matrix \( B \) in direction \( k \) where \( 1 \leq k \leq n \) is the matrix \( \mu_k(B) = B' = (b'_{ij}) \) where:

\[
(1.1) \quad b'_{ij} = \begin{cases} 
- b_{ij}, & \text{if } i \text{ or } j = k \\
 b_{ij} + \frac{1}{2} (|b_{ik}| b_{kj} + |b_{ik}| b_{kj}) & \text{otherwise}
\end{cases}
\]

Equation (1.1) is called the [matrix mutation formula](#). The mutation is an involution i.e \( \mu_k \mu_k(B) = B \). A skew-symmetric matrix \( B = (b_{ij}) \in \text{Mat}_{n \times n}(\mathbb{Z}) \) can be represented by a directed diagram called a [quiver](#) with \( n \) vertices such that there are \( |b_{ij}| \) many arrows from \( j \) to \( i \) if \( b_{ij} \geq 0 \). \( Q_0 \) is the set of vertices in \( Q \) and \( Q_1 \) is the set of arrows in \( Q \). The mutation formula can be translated to the language of quivers such that for every \( k \in Q_0 \), the [quiver mutation](#) in direction \( k \) is obtained by the following steps

1. for each subquiver \( i \to k \to j \) add a new arrow \( i \to j \).
2. reverse all arrows with source or target \( k \).
3. remove the arrows in a maximal set of pairwise disjoint 2-cycles.

\( Q \) is finite if \( Q_0 \) and \( Q_1 \) are both finite. A vertex \( i \) falls in the neighbourhood of a vertex \( j \) if there is an arrow connecting \( i \) and \( j \).

We can easily check that the skew-symmetricity and the skew-symmetrizability are invariant under mutation, whereas the sign-skew-symmetricity is not necessarily invariant under mutation. A sign-skew-symmetric matrix which remains sign-skew-symmetric under any arbitrary finite sequence of mutation is called [totally sign-skew-symmetric matrix](#).

An \( n \times n \) sign-skew-symmetric matrix \( B \) can be associated with a (simple) quiver \( \Delta(B) \) with vertices \( 1, \cdots, n \) such that for each pair \( (i, j) \) with \( b_{ij} < 0 \), there is exactly one arrow from vertex \( i \) to vertex \( j \). Trivially, \( \Delta(B) \) has no loops and no 2-cycles. Recall that the sign-skew-symmetric matrix \( B \) is called [acyclic](#) if \( \Delta(B) \) is acyclic i.e, \( \Delta(B) \) does not admit any directed cycles [10].
Definition 1.1. Let $B$ be a totally sign-skew-symmetric matrix, we call $\tilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in \text{Mat}_{2n \times n}(\mathbb{Z})$ the extended matrix of $B$. And let $\tilde{B}_{\sigma_m} = \begin{pmatrix} B_{\sigma_m} \\ C_{\sigma_m} \end{pmatrix}$ be the matrix obtained from $\tilde{B}$ by a composition of mutations $\mu_{\sigma_m} = \mu_{k_m} \mu_{k_{m-1}} \cdots \mu_{k_0}$ such that $1 \leq k_j \leq n$ for $0 \leq j \leq m$. Then the lower part of $\tilde{B}_{\sigma_m}$ is called the $C$–matrix and its columns are called the $c$–vectors.

The mutation of a matrix $\tilde{B}$ in direction $k$ where $1 \leq k \leq n$ is the matrix $\mu_k(\tilde{B}) = \tilde{B}' = \begin{pmatrix} B' \\ C' \end{pmatrix}$ where $B'$ is given as in Equation (1.1) and $C' = (c'_{ij})$ such that:

(1.2) $c'_{ij} = \begin{cases} -c_{ij}, & \text{if } j = k \\ c_{ij} + \frac{1}{2}(|c_{ik}|b_{kj} + c_{ik}|b_{kj}|) & \text{otherwise} \end{cases}$

Remark 1.2. In this paper we refer to the mutation given in Equations (1.1) and (1.2) as ordinary mutation and the mutation given in Equation (1.3) as orbit-mutation.

By convention $\mu_{k_0}(\tilde{B}) = \tilde{B}$. ($\mu_{k_0}$ means no mutation has been applied yet and any c–vector in $\tilde{B}$ has its entries either all non-positive or all non-negative.) If the entries of any c–vector in the matrix $\tilde{B}_{\sigma_j}$ such that $0 \leq j < \infty$ are either all non-positive or all non-negative, then we say that the sign-coherence property for $C$–matrix holds for the matrix $\tilde{B}$.

The idea of the unfolding method of an acyclic sign-skew-symmetric matrix $B$ is to create an (infinite) quiver $Q$ which covers $B$ and can do orbit-mutations. We recall the way to create such quiver as it was mentioned in [10].

A locally-finite quiver is an infinite quiver which has finitely many arrows incident to each of its vertices. A locally-finite quiver $Q$ can be represented by an infinite and well-defined matrix $B^\# = (b^\#_{ij})$ called the adjacency matrix of $Q$ such that $b^\#_{ij} \geq 0$ if there are $|b^\#_{ij}|$ many arrows from $j$ to $i$ in $Q$.

Definition 1.3. Let $B^\#$ be the adjacency matrix of a locally-finite quiver $Q$ and let $g$ be a permutation acting on $Q_0$, then $g$ is said to be an automorphism of $B$ or an automorphism of $Q$ if $b^\#_{g_i,g_j} = b^\#_{ij}$ for every $i, j \in Q_0$.

Let $Q$ be a locally-finite quiver and $\Gamma$ be a subgroup of the symmetric group $S_{Q_0}$. If all the elements of $\Gamma$ are automorphisms of $Q$, then $\Gamma$ is said to be a group of automorphisms of this quiver. Let $Q$ be a locally-finite quiver equipped with a group of automorphisms $\Gamma$. We denote the orbits created under the action of $\Gamma$ by $i$ such that $i \in Q_0$. A $\Gamma$-loop at $\tilde{a}$ is an arrow $a \rightarrow h.a$ and a $\Gamma$-2 cycle at $\tilde{a}$ is a pair of arrows $a \rightarrow j \rightarrow h.a$ such that $a, j \in Q_0$, $j \notin \tilde{a}$ and $h \in \Gamma$.
automorphisms $\Gamma$ acting on it such that $Q$ does not admit a $\Gamma$-loop or a $\Gamma$-2 cycle at any of its orbits, the orbit-mutation in direction $\bar{k}$ is defined as follows

\[
\mu_{\bar{k}}(b_{\bar{i}\bar{j}}) = \begin{cases} 
-b_{\bar{i}\bar{j}} & \text{if } i \in \bar{k} \text{ or } j \in \bar{k} \\
b_{\bar{i}\bar{j}} + \sum_{t \in \bar{k}} |b_{\bar{i}\bar{t}}|b_{\bar{t}\bar{j}} + b_{\bar{t}\bar{i}}b_{\bar{t}\bar{j}}| & \text{otherwise}
\end{cases}
\]

Since $Q$ is locally-acyclic, the summation in Equation (1.3) is well-defined and mutations in directions which belong to the same orbit commute since the quiver does not admit a $\Gamma$–loop, hence we get the fact

\[
\mu_{\bar{k}}(b_{\bar{i}\bar{j}}) = \prod_{t \in \bar{k}_{\{i,j\}}} \mu_t(b_{\bar{i}\bar{j}})
\]

where $\bar{k}_{\{i,j\}}$ denotes the indices of $\bar{k}$ which are incident to $i$ or $j$ and $\prod$ denotes the composition of mutations in directions $t \in \bar{k}_{\{i,j\}}$.

**Definition 1.4.**  
(1) Let $Q$ be a locally-finite quiver represented by $B^\bar{k} = (b_{\bar{i}\bar{j}})$ with no $\Gamma$-loops or $\Gamma$-2 cycles and with the action of a group of automorphisms $\Gamma$ such that there are finitely many orbits $n < \infty$ under the action of this group. The matrix $B = (b_{ij}) \in \text{Mat}_{n \times n}(\mathbb{Z})$ obtained by $b_{ij} = \sum_{k \in i} b_{\bar{k}j}$ is called the **folding** of $Q$ and denoted by $B = B(Q)$.

(2) Conversely, let $B$ be a sing-skew-symmetric matrix such that there is a pair $(Q, \Gamma)$ where $Q$ is a (locally-finite) quiver and $\Gamma$ is a group of automorphisms and $B = B(Q)$, then $(Q, \Gamma)$ is called a **covering** of $B$.

(3) If $(Q, \Gamma)$ is a covering of a sign-skew-symmetric matrix $B$ and $Q$ can perform arbitrary steps of orbit-mutation (the quiver obtained by any finite sequence of orbit-mutation does not have a $\Gamma$-loop or $\Gamma$-2 cycles), then $(Q, \Gamma)$ is called an **unfolding** of $B$.

**Remark 1.5.** Throughout this paper, sometimes we drop the group of automorphisms $\Gamma$ when pointing to an unfolding of a sign-skew-symmetric matrix and write $Q$ is an unfolding of $B$.

In [10] the authors proved the following Lemma.

**Lemma 1.6.** Let $Q$ be a locally-finite quiver and $\Gamma$ a group of automorphisms acting on it with finitely many number of orbits $\{\bar{i}_1, \bar{i}_2, \ldots, \bar{i}_n\}$ such that $Q$ does not admit any $\Gamma$-loops or $\Gamma$-2 cycles, then the folding matrix $B$ of $Q$ is a sign-skew-symmetric matrix.
In what follows, we recall the construction that M. Huang and F. Li set up in [10] to find a covering for acyclic sign-skew-symmetric matrices which can take arbitrary steps of orbit-mutation.

Construction 1.7. Let $B = (b_{ij}) \in M_{n \times n}(\mathbb{Z})$ be an acyclic sign-skew-symmetric matrix. An infinite quiver $Q(B)$ will be constructed inductively.

- For each $i \in \{1, 2, ..., n\}$, we define a quiver $Q^i$ as follows: $Q^i$ has $\sum_{j=1}^{n} |b_{ji}| + 1$ vertices with one vertex labeled by $i$ and other $|b_{ji}|$ vertices labeled by $j$ $(i \neq j)$. If $b_{ji} > 0$ there is an arrow from each vertex labeled by $j$ to the unique vertex labeled by $i$. If $b_{ji} < 0$ there is an arrow from the unique vertex labeled by $i$ to each vertex labeled by $j$. No arrows between $i$ and $j$ if $b_{ij} = 0$.
- Suppose we start the constructing process at $i = 1$, we denote $Q^{(1)} = Q_1$. The unique vertex labeled by 1 in $Q^{(1)}$ is called the old vertex, while the other vertices are called new vertices.
- For a new vertex in $Q^{(1)}$ labeled by $i_1$, $Q^{i_1}$ and $Q^{(1)}$ share a common arrow denoted by $\alpha_{i_1}$. We glue $Q^{(1)}$ and $Q^{i_1}$ along this common arrow. By iterating the gluing procedure for all $i_1 \in I$ where $I$ is the set of the new vertices in $Q^{(1)}$, we get a new quiver $Q^{(2)}$ whose old vertices are the vertices of $Q^{(1)}$ and the other vertices are the new vertices. Clearly $Q^{(1)}$ is a subquiver of $Q^{(2)}$.
- Inductively, we obtain $Q^{(m+1)}$ from $Q^{(m)}$. Similarly, the old vertices are the vertices of $Q^{(m)}$ and the rest are new.
- Finally, we define the (infinite) quiver $Q(B) = \bigcup_{i=1}^{\infty} Q^{(i)}$, as $Q^{(m)}$ is always a subquiver of $Q^{(m+1)}$ for any $m$.

Remark 1.8. Clearly we have the following facts:

1. The underlying quiver $Q(B)$ is a acyclic, since it is a tree clearly.
2. The full subquiver of $Q(B)$ obtained by all the vertices incident to a vertex labeled by $i$ is $Q^i$.
3. Mostly, the quiver $Q(B)$ constructed as in Construction [1.7] is infinite but in some cases it might be finite. For example when $B$ is the adjacency skew-symmetric matrix of a finite tree $Q'$, then $Q(B) = Q'$ and thus $Q(B)$ is finite here.
4. Let $B^\$ be the (infinite) skew-symmetric matrix corresponding to the (infinite) quiver $Q(B)$. The entries of $B^\$ are either $-1, 0$ or 1.
5. Let $\Gamma$ be a subgroup of the symmetric group $S_{Q(B)}$ that sends a vertex of $Q$ constructed as above to another vertex with the same label. By (2) in Remark [1.8] the vertices which carry different labels are always connected to each other by the same way. That is if $b_{ij}^\ = a \in \{0, 1, -1\}$,
then $b_{g(i)g(j)}^g = a$ for every $g \in \Gamma$ and hence $\Gamma$ is the maximal subgroup of automorphisms which preserves the labels:
\[
\Gamma = \{ h \in \text{Aut}Q : \text{if } h.a_s = h.a_t \text{ for } a_s, a_t \in Q_0, \text{ then } a_s, a_t \text{ have the same label} \}
\]
By the action of $\Gamma$ all the vertices which have the same label lie in the same orbit.
M. Huang and F. Li in [10], proved the following very important two Theorems.

**Theorem 1.9.** [10, Theorem 2.17] Any acyclic sign-skew-symmetric matrix $B$ of the size $n$ is always totally sign-skew-symmetric.

**Theorem 1.10.** [10, Theorem 2.16] If $B$ is an acyclic sign-skew-symmetric matrix of the size $n \times n$, then $(Q(B), \Gamma)$ built from $B$ as in Construction 1.7 is an unfolding of $B$.

**Remark 1.11.** Through out the proof of Theorem 1.10 in [10], it was proved that the property of no $\Gamma$− loops and no $\Gamma$−2 cycles is preserved under orbit-mutation for the (infinite) quiver $Q(B)$ constructed as in Construction 1.7, i.e., for any finite sequence of orbit-mutations the quiver $\mu_k_s ... \mu_k_1(Q(B))$ does not admit any $\Gamma$− loops or $\Gamma$−2 cycles where $k_s \in Q_0(B)$ for every $1 \leq s \leq j$. This fact will be used later in this paper in places like the proof of Lemma 3.6.

2. **THE SIGN-COHERENCE OF $c$−VECTORS FOR AN ACYCLIC SIGN-SKEW-SYMMETRIC MATRIX**

In this section, we modify Construction 1.7 to find an unfolding of the $c$− vectors of an extended sign-skew-symmetric matrix $\tilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in \text{Mat}_{2n \times n}(\mathbb{Z})$.

Let $Q$ be a locally-finite quiver, the **locally-finite framed quiver** $\tilde{Q}$ is the quiver obtained from $Q$ by adding new vertices in a way that each vertex $a \in Q_0$ is connected to a new vertex $a'$ by a single arrow $a \rightarrow a'$ while $Q$ remains the same. The elements of the set $Q'_0 = \{a' \mid a \in Q_0 \}$ are called the **frozen vertices**. This quiver is represented by the extended infinite skew-symmetric matrix $\tilde{B}_3 = \begin{pmatrix} B_3 \\ I_\infty \end{pmatrix}$.

The bottom part of the matrix $\tilde{B}_3$ is called the **$C$-matrix** and $B_3$ is called the **principal part**.

We extend the action of the group $\Gamma$ to the frozen vertices in the quiver $\tilde{Q}$ such that for every $g \in \Gamma$, $g(a') = g(b')$ if and only if $g(a) = g(b)$, that is two frozen vertices lie in the same $\Gamma$-orbit if their mutable copies lie in the same orbit.

Let $\Gamma$ be a group of automorphisms acting on $B_3$ such that $Q$ does not admit any $\Gamma$− loops or $\Gamma$−2 cycles, clearly $\Gamma$ is also a group of automorphisms of $I_\infty$. Hence $\Gamma$ is said to be a **group of automorphisms** of an extended matrix $B_3$ if it is a group of automorphisms of its principal part $B_3$. 
We define the orbit-mutation on the $C$-matrix in direction $\bar{k}$ where $k \in Q_0$ as follows

\begin{equation}
\mu_{\bar{k}}(c^i_{ij}) = \begin{cases} 
-c^i_{ij} & \text{if } j \in \bar{k} \\
\frac{c^i_{ij} + \sum_{t \in k} |c^i_t|b^i_{tj} + c^i_{it}|b^i_{tj}|}{2} & \text{otherwise}
\end{cases}
\end{equation}

Again since $Q$ does not admit a $\Gamma$-loop, the orbit-mutation of the $C$-matrix can be defined as

\begin{equation}
\mu_{\bar{k}}(c^i_{ij}) = \prod_{t \in \bar{k} \setminus \{i,j\}} \mu_t(c^i_{ij})
\end{equation}

Where $t \in \bar{k} \setminus \{i,j\}$ denotes the indices of $\bar{k}$ which are incident to $i$ or $j$ and $\prod$ denotes the composition of mutations in directions $t \in \bar{k} \setminus \{i,j\}$.

By the definition of orbit-mutation for an extended infinite skew-symmetric matrix $\widetilde{B}^\infty = \begin{pmatrix} B^\infty \\ I_\infty \end{pmatrix}$ given in (1.3) and (2.1), it is easy to check that if $\Gamma$ is a group of automorphisms of $\widetilde{B}^\infty$, it will be a group of automorphisms of any extended infinite skew-symmetric matrix obtained from $\widetilde{B}^\infty$ by any finite sequence of orbit-mutations. Since the extended adjacency matrix $\widetilde{B}^\infty = \begin{pmatrix} B^\infty \\ I_\infty \end{pmatrix}$ of a locally-finite framed quiver $\widetilde{Q}$ is well-defined, we say that the sign-coherence property holds for a locally-finite framed quiver $\widetilde{Q}$ (\$\widetilde{Q}$ is sign-coherent) if after performing any finite sequence of ordinary mutations $\mu_{k_1}\mu_{k_2}\ldots \mu_{k_s}$ on $\widetilde{B}^\infty$ where $k_j \in Q_0$ for every $1 \leq j \leq s$, the entries of any $c$-vector in the matrix $\mu_{k_1}\mu_{k_2}\ldots \mu_{k_s}(\widetilde{B}^\infty)$ are either all non-negative or all non-positive. In other words a locally-finite framed quiver $\widetilde{Q}$ is sign-coherent if the arrows connecting any mutable vertex with the frozen vertices in the quiver $\mu_{k_1}\mu_{k_2}\ldots \mu_{k_s}(\widetilde{Q})$ are either all emerging from this mutable vertex or all reaching at this mutable vertex, where $k_j \in Q_0$ for every $1 \leq j \leq s$ and $s < \infty$.

**Remark 2.1.** The definitions of finite framed quivers and the sign-coherence property for finite framed quivers coincide with the definitions of these concepts for locally-finite quivers.

**Remark 2.2.** The quiver mutation for a framed quiver $\widetilde{Q}$ (finite or locally-finite) can be taken only in direction of a mutable vertex and is obtained as for the quiver $Q$ with one modification which is to remove all arrows that can be created during the mutation process between any two frozen vertices.

The sign-coherence property was proved for finite skew-symmetric matrices (finite quivers) in [5] and for finite skew-symmetrizable matrices in [9]. Here we prove that the sign-coherence property holds for locally-finite framed quivers.
Lemma 2.3. The sign-coherence property of $c-$vectors holds for locally-finite framed quivers.

Proof. Suppose that $\tilde{Q}$ is a locally-finite framed quiver, and $\mu_{k_m} \ldots \mu_{k_1}$ is a composition of mutations such that $k_j \in Q_0$ for every $1 \leq j \leq m$. We consider the whole quiver $\tilde{Q}$ as a quiver resulting from the gluing of two full subquivers of $\tilde{Q}$ as follows

- The first full subquiver of $\tilde{Q}$ is $\tilde{Q}|_{k_m \ldots k_1}$ which is obtained by the vertices $\{k_m, k_{m-1}, \ldots, k_1\}$ and their frozen copies and the mutable vertices in $N$ with their frozen copies such that $N$ is the set of mutable vertices that are in the neighbourhood of $\{k_m, k_{m-1}, \ldots, k_1\}$. Clearly $\tilde{Q}|_{k_m \ldots k_1}$ is a finite framed quiver.
- The other full subquiver of $\tilde{Q}$ is $\tilde{Q}|_S$ which is obtained by the vertices of $S$ and their frozen copies where $S$ is the set of mutable vertices in $\tilde{Q}$ which are not contained in $\tilde{Q}|_{k_m \ldots k_1}$. Clearly $\tilde{Q}|_S$ is a locally-finite framed quiver satisfying the sign-coherence property by construction.

The gluing procedure occurs between $\tilde{Q}|_{k_m \ldots k_1}$ and $\tilde{Q}|_S$ along the set of arrows $A$ in $\tilde{Q}$ connecting $N$ and $S$.

Since mutation at some mutable vertex $r$ reverses the direction of all arrows incident to the vertex $r$ and may affect the arrows between the vertices in the neighbourhood of $r$, the quiver $\tilde{Q}|_S$ and the set of arrows $A$ remain unchanged during the composition of mutations $\mu_{k_m} \ldots \mu_{k_1}$.

Hence $\mu_{k_m} \ldots \mu_{k_1}(\tilde{Q}|_S) = \tilde{Q}|_S$ and the quiver $\mu_{k_m} \ldots \mu_{k_1}(\tilde{Q})$ is the gluing of two quivers along the set of arrows $A$ described as above and which connects mutable vertices. These two quivers are:

- The first one is $\mu_{k_m} \ldots \mu_{k_1}(\tilde{Q}|_{k_m \ldots k_1})$ which is sign-coherent for $\tilde{Q}|_{k_m \ldots k_1}$ is a finite framed-quiver [5, 9].
- The other one is $\mu_{k_m} \ldots \mu_{k_1}(\tilde{Q}|_S) = \tilde{Q}|_S$ which is sign-coherent by construction.

Thus $\tilde{Q}$ is sign-coherent.

Let $\tilde{A} = \begin{pmatrix} A \\ C \end{pmatrix}$ be a matrix with $C$ as the $C-$matrix and its columns are the $c-$vectors. Suppose that this matrix is equipped with a group of automorphisms $\Gamma$ acting on its principal part $A$ such that the ordinary mutation and orbit-mutation are well-defined and $A$ can do arbitrary steps of ordinary mutation and orbit-mutation. If all the entries of any $c-$vector are either all non-positive or all non-negative after performing any finite sequence of orbit-mutation i.e, any $c-$vector in the matrix $\mu_{i_m} \ldots \mu_{i_0}(\tilde{A})$ has entries which are all non-negative or all non-positive where $i_j$ is used to index the matrix $A$ for $0 \leq j \leq m$ and $0 \leq m < \infty$, then we say
that the orbit-sign coherence property holds for this matrix. By convention
\( \mu_{0}(A) = \bar{A} \) (\( \mu_{0} \) means no orbit-mutation has been applied yet and any \( c \)-vector in \( A \) has its entries either all non-positive or all non-negative.)

**Corollary 2.4.** The orbit-sign coherence property holds for a locally-finite framed quiver \( \tilde{Q} \) equipped with a group of automorphisms \( \Gamma \) and can do arbitrary steps of orbit-mutation.

**Proof.** For a locally-finite framed quiver, any finite sequence of orbit mutation can be regarded as a longer but still finite sequence of ordinary mutation (see Equations (1.4) and (2.2)) and by Lemma 2.3 the entries of any \( c \)-vector of the quiver obtained by any finite sequence of ordinary mutation performed on a locally-finite framed quiver \( \tilde{Q} \) are either all non-negative or all non-positive. Hence the result follows. \( \square \)

We denote by \( sgn(i) \) the sign of the column indexed by \( i \) in the \( C \)-matrix and \( sgn(i) = + \) when the entries of the column \( i \) are non-negative while \( sgn(i) = - \) when the entries of the column \( i \) are non-positive.

**Lemma 2.5.** Let \( \tilde{Q} \) be a locally-finite framed quiver equipped with a group of automorphisms \( \Gamma \) such that \( \tilde{Q} \) does not admit a \( \Gamma \)-loop or \( \Gamma \)-2 cycle with \( \tilde{B} = \begin{pmatrix} B & C \end{pmatrix} \) as its adjacency matrix, and let \( \tilde{B} = \mu_{k}(\tilde{B}) = \begin{pmatrix} B_{k} & C_{k} \end{pmatrix} \) be the matrix obtained by orbit-mutation in direction \( k \) such that \( k \in Q_{0} \). If \( i_{1}, i_{2} \) fall in the same orbit, the columns indexed by \( i_{1}, i_{2} \) in the \( C \)-matrix \( C_{k} \) have the same sign.

**Proof.** Since \( i_{1} \) and \( i_{2} \) fall in the same orbit, there exists an automorphism \( g \in \Gamma \) such that \( i_{2} = g(i_{1}) \). Clearly \( \Gamma \) is a group of automorphisms of \( \tilde{B} = \mu_{k}(\tilde{B}) = \begin{pmatrix} B_{k} & C_{k} \end{pmatrix} \). Thus \( c_{i_{1}}^{k} = c_{g(i_{1})}^{k} = c_{g(i_{2})}^{k} \) for any index \( l \), thus \( sgn(i_{1}) = sgn(i_{2}) \). \( \square \)

When \( sgn(s) = + \) for every \( s \in \bar{i} \), then \( sgn(\bar{i}) = + \) and \( \bar{i} \) is said to be a green orbit. Respectively, when \( sgn(s) = - \) for every \( s \in \bar{i} \), then \( sgn(\bar{i}) = - \) and \( \bar{i} \) is said to be a red orbit.

Let \( \tilde{B} = B \) be the adjacency matrix of a locally-finite framed quiver \( \tilde{Q} \), in [10] the authors defined the folding matrix \( B = (b_{ij}) \) of a locally-finite quiver \( \tilde{Q} \) endowed with a group of automorphisms \( \Gamma \)

\[
b_{ij} = \sum_{k \in \bar{i}} b_{kj}^{k}
\]

Analogously we define the folding of the \( C \)-matrix

\[
c_{ij} = \sum_{k \in \bar{i}} c_{kj}^{k}
\]
Clearly, when we have a finite number \( n \) of orbits, the folding of the adjacency matrix \( \tilde{B}^\$ \) of a locally-finite framed quiver \( \tilde{Q} \), is the extended matrix \( \tilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in \text{Mat}_{2n \times n}(\mathbb{Z}) \).

Now we construct an unfolding for a given extended sign-skew-symmetric matrix.

**Construction 2.6.** Let \( \tilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in \text{Mat}_{2n \times n}(\mathbb{Z}) \) be an extended acyclic sign-skew-symmetric matrix. A (locally-finite) framed quiver \( \tilde{Q}(\tilde{B}) \) will be constructed inductively.

- For each mutable vertex \( i \in \{1, 2, \ldots, n\} \), we define a quiver \( \tilde{Q}^i \) as follows: \( \tilde{Q}^i \) has \( \sum_{j=1}^n |b_{ji}| + 2 \) vertices with one vertex labeled by \( i \) and other \( |b_{ji}| \) vertices labeled by \( j \) (\( i \neq j \)). If \( b_{ji} < 0 \) there is an arrow from each vertex labeled by \( j \) to the unique vertex labeled by \( i \). If \( b_{ji} > 0 \) there is an arrow from the unique vertex labeled by \( i \) to each vertex labeled by \( j \). No arrows between \( i \) and \( j \) if \( b_{ij} = 0 \). And finally with one vertex labeled by \( i' \) which is the frozen copy of \( i \) such that there is one arrow \( i \to i' \).

- We start by considering \( \tilde{Q}^{(1)} \) as the initial subquiver and we denote \( \tilde{Q}^{(1)} = \tilde{Q}^1 \). During the constructing process which has \( \tilde{Q}^{(1)} \) as its initial subquiver, the mutable vertices are either old or new while the frozen vertices are not considered old or new. For \( \tilde{Q}^{(1)} \) the vertex 1 is an old vertex and the other mutable vertices are new. For every new vertex \( i \), \( \tilde{Q}^i \) and \( \tilde{Q}^{(1)} \) share a common arrow \( \alpha_i \), we glue \( \tilde{Q}^i \) and \( \tilde{Q}^{(1)} \) along this common arrow to get a new subquiver \( \tilde{Q}^{(2)} \). The old vertices of \( \tilde{Q}^{(2)} \) are the mutable vertices of \( \tilde{Q}^{(1)} \) and the other mutable vertices are new.

- We continue inductively as in Construction 1.7 and build \( \tilde{Q}^{(m+1)} \) from \( \tilde{Q}^{(m)} \).

- We define \( \tilde{Q}(\tilde{B}) = \bigcup_{i=1}^{\infty} \tilde{Q}^{(i)} \).

**Remark 2.7.**

1. The matrix associated with the (locally-finite) quiver \( \tilde{Q}(\tilde{B}) \) obtained from Construction 2.6 is the (infinite) and well-defined matrix \( \tilde{B}^\$ = \begin{pmatrix} B^\$ \\ I_\infty \end{pmatrix} \) where the upper part of this matrix is the principal part such that \( b^\$_{ij} < 0 \) if there are \( |b^\$_{ij}| \) many arrows from the mutable vertex \( i \) to the mutable vertex \( j \) whereas \( b^\$_{ij} > 0 \) if there are \( |b^\$_{ij}| \) many arrows from the mutable vertex \( j \) to the mutable vertex \( i \) and \( b^\$_{ij} = 0 \) if there are no arrows between the mutable vertices \( i \) and \( j \). The lower part of this matrix is the \( C^- \) matrix such that \( c^\$_{ij} > 0 \) if there are \( |c^\$_{ij}| \) many arrows from the mutable vertex \( j \) to the frozen vertex \( i' \) whereas \( c^\$_{ij} < 0 \) if there are \( |c^\$_{ij}| \) many arrows from the frozen vertex \( i' \) to the mutable vertex \( j \) and \( c^\$_{ij} = 0 \).
if there are no arrows between the mutable vertex $j$ and the frozen vertex $i'$.

(2) Let $\Gamma$ be the maximum subgroup that preserves labels of the symmetric matrix $S_{\tilde{Q}(\tilde{B})}$ acting on the set of mutable and frozen vertices of $\tilde{Q}(\tilde{B})$. By Construction 2.6, $\Gamma$ is a group of automorphisms and the orbits obtained by its action are $\{1, \ldots, \tilde{n}, \tilde{i}', \ldots, \tilde{n}'\}$ such that the orbit $\tilde{i}$ contains all the mutable vertices labeled by $i$ and the orbit $\tilde{i}'$ contains all the frozen vertices labeled by $i'$ for every $1 \leq i \leq n$.

(3) Clearly, the folding of the adjacency matrix $\tilde{B}^\delta$ of the quiver $\tilde{Q}(\tilde{B})$ is $\tilde{B}$. The full subquiver $\tilde{Q}$ of $\tilde{Q}$ obtained by the mutable vertices is exactly as constructed in Construction 1.7 taking into consideration the way we follow in this paper to associate a quiver with a matrix thus it is an unfolding of $\tilde{B}$ by Theorem 1.10 that is $\tilde{Q}$ can take arbitrary steps of orbit mutations and since orbit-mutation is only taken in direction of an orbit whose elements are labels of mutable vertices, we conclude that $\tilde{Q}(\tilde{B})$ is an unfolding of $\tilde{B}$.

**Remark 2.8.** Let $\tilde{B}$ be the folding of $\tilde{B}^\delta$ associated with a quiver $\tilde{Q}$, to avoid ambiguity, when we mutate $\tilde{B}$ of in direction $\bar{i}$, the mutation will be denoted as $\mu_{i\bar{i}}$ since this mutation is an ordinary mutation here and not orbit-mutation.

By convection $\mu^{\sigma_0}(\tilde{Q}) = \tilde{Q}$ and $\mu^{\sigma_\delta}(\tilde{B}) = \tilde{B}$.

**Lemma 2.9.** Let $\tilde{B} = \begin{pmatrix} B & I_n \\ I_m & 0 \end{pmatrix}$ be an extended, finite and acyclic sign-skew-symmetric matrix and let $\tilde{Q}(\tilde{B})$ be the (locally-finite) framed quiver associated with the (infinite) and well-defined skew-symmetric matrix $\tilde{B}^\delta = \begin{pmatrix} B^\delta & I_m \\ I_n & 0 \end{pmatrix}$ obtained from Construction 2.6 as an unfolding of $\tilde{B}$. We denote by $\mu^{\sigma_m}(\tilde{B}^\delta)$ the composition of orbit mutations $\mu_{k_0} \cdots \mu_{k_m}(\tilde{B}^\delta)$ such that $k_j \in \mathbb{Q}_0$ for $0 \leq j \leq m$ and we denote by $\mu^{\sigma_m}(\tilde{B})$ the composition of ordinary mutation $\mu_{k_0} \cdots \mu_{k_m}(\tilde{B})$. Then $\mu^{\sigma_m}(\tilde{B}^\delta)$ is an unfolding of $\mu^{\sigma_m}(\tilde{B})$.

**Proof.** Since $\tilde{Q}(\tilde{B})$ can take arbitrary steps of orbit-mutation, $\mu^{\sigma_m}(\tilde{Q}(\tilde{B}))$ can also take arbitrary steps of orbit-mutation so we need only to prove that $\mu^{\sigma_m}(\tilde{Q}(\tilde{B}))$ is a covering of $\mu^{\sigma_m}(\tilde{B})$. We prove it by induction for the mutable part $Q(\tilde{B})$ represented by the matrix $B^\delta$ first, then for the frozen part represented by the $C$-matrix. Trivially, $\mu^{\sigma_0}(Q(\tilde{B}))$ is a covering of $\mu^{\sigma_\delta}(B)$. Suppose the result holds for every $v < m$.

$$\mu^{\sigma_m}(b^\delta_{ij}) = \begin{cases} -\mu^{\sigma_{m-1}}(b^\delta_{ij}) & \text{if } i \in \tilde{k}_m \text{ or } j \in \tilde{k}_m \\ \mu^{\sigma_m}(b^\delta_{ij}) + \sum_{t \in \tilde{k}_m} \frac{[\mu^{\sigma_{m-1}}(b^\delta_{it})]\mu^{\sigma_{m-1}}(b^\delta_{jt}) + \mu^{\sigma_{m-1}}(b^\delta_{jt})[\mu^{\sigma_{m-1}}(b^\delta_{it})]}{2} & \text{otherwise} \end{cases}$$

When $i \in \tilde{k}_m$ or $j \in \tilde{k}_m$, $\sum_{s \in \tilde{i}} \mu^{\sigma_m}(b^\delta_{sj}) = -\sum_{s \in \tilde{i}} \mu^{\sigma_{m-1}}(b^\delta_{sj}) = -\mu^{\sigma_{m-1}}(b^\delta_{ij}) = \mu^{\sigma_m}(b^\delta_{ij})$. 

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When $i \notin \bar{k}_m$ and $j \notin \bar{k}_m$,
\[
\sum_{s \in I} \mu^m_{s^{-1}}(b_{ij}^{\bar{s}}) = \sum_{s \in I} \left( \mu^m_{s^{-1}}(b_{ij}^{\bar{s}}) + \frac{[\mu^m_{s^{-1}}(b_{ij}^{\bar{s}})]|\mu^m_{s^{-1}}(b_{ij}^{\bar{s}})+\mu^m_{s^{-1}}(b_{ij}^{\bar{s}})|}{2} \right) + \\
\sum_{s \in I} \left( \frac{[\mu^m_{s^{-1}}(b_{ij}^{\bar{s}})]|\mu^m_{s^{-1}}(b_{ij}^{\bar{s}})+\mu^m_{s^{-1}}(b_{ij}^{\bar{s}})|}{2} \right) = \\
\mu^m_{s^{-1}}(b_{ij}) + \sum_{s \in I} \left( \frac{[\mu^m_{s^{-1}}(b_{ij}^{\bar{s}})]|\mu^m_{s^{-1}}(b_{ij}^{\bar{s}})+\mu^m_{s^{-1}}(b_{ij}^{\bar{s}})|}{2} \right)
\]

Since $\mu^m_{s^{-1}}(Q(B))$ does not have a $\Gamma$-$2$ cycles, when $s \in \bar{i}$ all the entries $\mu^m_{s^{-1}}(b_{ij}^{\bar{s}})$ have the same sign and when $t \in \bar{k}_m$, the entries $\mu^m_{s^{-1}}(b_{ij}^{\bar{s}})$ have the same sign, hence
\[
\sum_{s \in I} \mu^m_{s^{-1}}(b_{ij}^{\bar{s}}) = \mu^m_{s^{-1}}(b_{ij}) + \sum_{t \in \bar{k}_m} \left( \frac{[\mu^m_{s^{-1}}(b_{ij}^{\bar{s}})]|\mu^m_{s^{-1}}(b_{ij}^{\bar{s}})+\mu^m_{s^{-1}}(b_{ij}^{\bar{s}})|}{2} \right) = \\
\mu^m_{s^{-1}}(b_{ij}) + \mu^m_{s^{-1}}(b_{ij}) = \mu^m_{s^{-1}}(b_{ij}). \quad \text{And now}
\]

we will prove that a covering of the $C-$matrix is invariant under a composition of orbit mutation. Trivially, $\mu^{\sigma_0}(I_n)$ is a covering of $\mu^m(I_n)$.

\[
\mu^m(c_{ij}^{\bar{s}}) = \begin{cases} 
-\mu^m_{s^{-1}}(c_{ij}^{\bar{s}}) & \text{if } j \in \bar{k}_m \\
\mu^m_{s^{-1}}(c_{ij}^{\bar{s}}) + \sum_{t \in \bar{k}_m} \left( \frac{[\mu^m_{s^{-1}}(c_{ij}^{\bar{s}})]|\mu^m_{s^{-1}}(c_{ij}^{\bar{s}})+\mu^m_{s^{-1}}(c_{ij}^{\bar{s}})|}{2} \right) & \text{otherwise}
\end{cases}
\]

When $j \in \bar{k}_m$, $\sum_{s \in I} \mu^m_{s^{-1}}(c_{ij}^{\bar{s}}) = -\sum_{s \in I} \mu^m_{s^{-1}}(c_{ij}^{\bar{s}}) = -\mu^m_{s^{-1}}(c_{ij}) = \mu^m_{s^{-1}}(c_{ij})$.

When $j \notin \bar{k}_m$,
\[
\begin{align*}
\sum_{s \in I} \mu^m_{s^{-1}}(c_{ij}^{\bar{s}}) &= \sum_{s \in I} \left( \mu^m_{s^{-1}}(c_{ij}^{\bar{s}}) + \frac{[\mu^m_{s^{-1}}(c_{ij}^{\bar{s}})]|\mu^m_{s^{-1}}(c_{ij}^{\bar{s}})+\mu^m_{s^{-1}}(c_{ij}^{\bar{s}})|}{2} \right) + \\
\sum_{s \in I} \left( \frac{[\mu^m_{s^{-1}}(c_{ij}^{\bar{s}})]|\mu^m_{s^{-1}}(c_{ij}^{\bar{s}})+\mu^m_{s^{-1}}(c_{ij}^{\bar{s}})|}{2} \right) = \\
\mu^m_{s^{-1}}(c_{ij}) + \sum_{s \in I} \left( \frac{[\mu^m_{s^{-1}}(c_{ij}^{\bar{s}})]|\mu^m_{s^{-1}}(c_{ij}^{\bar{s}})+\mu^m_{s^{-1}}(c_{ij}^{\bar{s}})|}{2} \right)
\end{align*}
\]

By Corollary 2.4 when $s \in \bar{i}$ all the entries $\mu^m_{s^{-1}}(c_{ij}^{\bar{s}})$ have the same sign and hence
\[
\sum_{s \in I} \mu^m_{s^{-1}}(c_{ij}^{\bar{s}}) = \mu^m_{s^{-1}}(c_{ij}) + \sum_{s \in I} \left( \frac{[\mu^m_{s^{-1}}(c_{ij}^{\bar{s}})]|\mu^m_{s^{-1}}(c_{ij}^{\bar{s}})+\mu^m_{s^{-1}}(c_{ij}^{\bar{s}})|}{2} \right)
\]
\[
\mu^{\sigma^f m - 1}(c_{i j}) + \frac{|\mu^{\sigma^f m - 1}(c_{i j})| \sum_{t \in \mathbb{F}_m} (\mu^{\sigma^f m - 1}(b^{l}_{i t})) + \mu^{\sigma^f m - 1}(c_{j t})| \sum_{t \in \mathbb{F}_m} (\mu^{\sigma^f m - 1}(b^{l}_{t j}))|}{2}
\]

\[
= \mu^{\sigma^f m - 1}(c_{i j}) + \frac{|\mu^{\sigma^f m - 1}(c_{i j})| \mu^{\sigma^f m - 1}(c_{j t})| \mu^{\sigma^f m - 1}(b^{l}_{t j})|}{2} = \mu^{\sigma^f m}(c_{i j}).
\]

**Example 2.10.** The construction of an unfolding of the extended acyclic sign-skew symmetric matrix

\[
\tilde{B} = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 3 & 0 & -1 & 0 \\ 0 & 5 & 0 & -2 \\ 1 & 0 & 3 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]

built according to Construction 2.6 is shown in Figure 1 and Figure 2.

**Theorem 2.11.** The sign-coherence property of \(c\)-vectors holds for acyclic sign-skew-symmetric matrices.

**Proof.** Let \(\tilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in \text{Mat}_{2n \times n} \mathbb{Z}\) be an extended sign-skew-symmetric matrix and let \(\tilde{Q}(\tilde{B})\) be the locally-finite framed quiver obtained from Construction 2.6 as an unfolding of \(\tilde{B}\) with the adjacency matrix \(\tilde{B}^{\tilde{B}} = \begin{pmatrix} B^{\tilde{B}} \\ I_\infty \end{pmatrix}\). We denote by \(\mu^{\tilde{B}}(B^{\tilde{B}}) = \begin{pmatrix} B^{\tilde{B}}^{\tilde{\sigma}_k} \\ C^{\tilde{B}}^{\sigma_k} \end{pmatrix}\) the matrix obtained from \(\begin{pmatrix} B^{\tilde{B}} \\ I_\infty \end{pmatrix}\) after taking a finite sequence \((\mu_{i_0}, \ldots, \mu_{i_k})\)
of orbit-mutation. We denote by \( \mu^\sigma_k(\tilde{B}) = \begin{pmatrix} B^\sigma_k \\ C^\sigma_k \end{pmatrix} \) the matrix obtained from \( \begin{pmatrix} B \\ I_n \end{pmatrix} \) after taking a finite sequence \((\mu^f_{i_0}, \ldots, \mu^f_{i_k})\) of ordinary mutation such that \( i^f_j \) refers to the order of the row or column indexed by \( i_j \) in the folding matrix where \( i_j \in \mathbb{Q}_0 \) for \( 0 \leq j \leq k \). By convention \( \mu^\sigma_0(\tilde{B}^\dagger) = \tilde{B}^\dagger \) and \( \mu^\sigma_0(\tilde{B}) = \tilde{B} \). By Lemma 2.9, \( \mu^\sigma_j(\tilde{B}^\dagger) \) is an unfolding of \( \mu^\sigma_j(\tilde{B}) \) for \( 0 \leq j \leq k \). The orbit-sign coherence property holds for the locally-finite framed quiver \( \tilde{Q}(\tilde{B}) \) by Lemma 2.4. Thus the entries of any \( c \)-vector in the matrix \( \mu^\sigma_k(\tilde{B}^\dagger) \) are either all non-negative or all non-positive. By the definition of an unfolding, we find that the entries of any \( c \)-vector in \( B^\sigma_k \) are either all non-negative or non-positive for any \( 0 \leq k < \infty \) and thus the sign-coherence property holds for \( \tilde{B} \).

\[ \square \]

3. **Maximal green sequences for an acyclic sign-skew-symmetric matrix**

After proving that the sign-coherence property holds for an acyclic sign-skew-symmetric matrix, it makes sense to define maximal green sequences for such matrices.
Definition 3.1. Let \( \tilde{B} = \begin{pmatrix} B \\ I_n \end{pmatrix} \in M_{2n \times n}(\mathbb{Z}) \) be a totally sign-skew-symmetric matrix for which the sign coherence property holds and let \( \tilde{B}^{\sigma_\ast} = \begin{pmatrix} B^{\sigma_\ast} \\ C^{\sigma_\ast} \end{pmatrix} \) be the matrix obtained from \( \tilde{B} \) by a composition of mutations \( \mu_{\sigma } = \mu_{k_1} \mu_{k_2} \cdots \mu_{k_t} \), \( 1 \leq k_j \leq n \) for every \( 1 \leq j \leq s \),

- an index \( i \) in the matrix \( \tilde{B}^{\sigma_\ast} \) for \( 1 \leq i \leq n \) is called green (respectively, red) if the entries of the column indexed by \( i \) in the \( C \)-matrix of \( \tilde{B}^{\sigma_\ast} \) are non-negative (respectively, non-positive).
- A sequence of indices \( (k_1, k_2, \ldots, k_s) \), where \( 1 \leq k_j \leq n \) for all \( j \in \{1, 2, \ldots, s\} \), is called a green sequence if \( k_j \) is green in the matrix \( \mu_{k_1} \cdots \mu_{k_s}(\tilde{B}) \) for \( 1 \leq j \leq s \). Such sequence is called maximal if \( \mu_{k_1} \cdots \mu_{k_s}(\tilde{B}) \) does not have any green indices.

Definition 3.2. A source in a sign-skew-symmetric matrix \( B \) of the size \( n \times n \) is an index \( i \) where \( 1 \leq i \leq n \) and \( b_{ik} \leq 0 \) for all \( 1 \leq k \leq n \). In the associated simple quiver \( \Delta(B) \) of \( B \), the source \( i \) has all the arrows incident to it emerging from it.

Definition 3.3. An admissible numbering by sources of an acyclic sign-skew-symmetric matrix \( B \) of the size \( n \times n \) is an \( n \)-tuple \((i_1, i_2, \ldots, i_n)\) such that the indices of \( B \) are \( \{i_1, \ldots, i_n\} \) with \( i_1 \) a source in \( B \) and the vertex \( i_k \) is a source in \( \mu_{i_{k-1}} \cdots \mu_{i_1}(B) \) for any \( 2 \leq k \leq n \).

Lemma 3.4. Every acyclic sign-skew-symmetric matrix \( B \) admits an admissible numbering by source.

Proof. \( B \) is a finite acyclic matrix, thus it has a source \( i_1 \). When mutating at this source, \( \mu_{i_1}(b_{ij}) = -b_{ij} \) if \( l = i_1 \) or \( j = i_1 \) and \( \mu_{i_1}(b_{ij}) = b_{ij} \) otherwise. Let \( \{B - i_1\} \) denote the matrix obtained from \( B \) by deleting the \( i_1 \)-th row and column, hence \( \mu_{i_1}(\{B - i_1\}) = \{B - i_1\} \). Since \( B \) is acyclic, every submatrix is also acyclic. Then \( \mu_{i_1}(\{B - i_1\}) \) is also acyclic and thus it has a source \( i_2 \neq i_1 \). Again the submatrix \( \mu_{i_2}(\mu_{i_1}(\{B - i_1\} - i_2)) \) is the same submatrix \( \{B - i_1 - i_2\} \) obtained from \( B \) by deleting the rows and columns \( i_1, i_2 \), and it is acyclic with a new source \( i_3 \). In every step the submatrix formed by the indices which haven’t been mutated at is the same as the submatrix obtained by the same indices in the original \( B \) and the new source \( i_k \) in this submatrix is also a source in the whole matrix \( \mu_{i_{k-1}} \cdots \mu_{i_1}(B) \) for \( 1 \leq k \leq n - 1 \) since we are preforming mutations at sources so the entries of the other indices remain unchanged and moreover the sign of entry \( b_{i_k i} \) that connects the new source \( i_k \) with an old one \( i_d \) (which has already been mutated at) is non-positive in the matrix \( \mu_{i_{k-1}} \cdots \mu_{i_1}(B) \) where \( 1 \leq d \leq k - 1 \). By repeating this process \( n - 1 \) times, the index \( i_n \) will definitely be a source and we get an admissible numbering by source \((i_1, \ldots, i_n)\). \( \square \)
$Q$ is a locally-finite quiver with a group of automorphisms $\Gamma$ and a finite number of orbits $n$. If the index $i$ is a source in $Q$, then $b_{ij} \leq 0$ for every $j \in Q_0$. Suppose $l \in \tilde{i}$, i.e. there is an automorphism $g \in \Gamma$ such that $g(l) = i$ and suppose that $b_{lk} > 0$ for some $k \in Q_0$, then by the definition of automorphism $b_{ik} = b_{g(k)l} > 0$, contradiction. Thus any index in the orbit $\tilde{i}$ is also a source in $Q$. In this case we call $\tilde{i}$ an orbit-source in $Q$. If there is a sequence $(\tilde{i}_1, ..., \tilde{i}_n)$ of orbit-mutations such that the orbit $\tilde{i}_1$ is an orbit-source in $Q$ and the orbit $\tilde{i}_j$ is an orbit-source in $\mu_{\tilde{i}_{j-1}}(...\mu_{\tilde{i}_1}(Q))$ for $1 \leq j \leq n$ and the set $\{\tilde{i}_1, ..., \tilde{i}_n\}$ represents all the orbits under the action of $\Gamma$, then the sequence $(\tilde{i}_1, ..., \tilde{i}_n)$ is called orbit-admissible numbering by source in $Q$.

Getting back to the pair $(Q, \Gamma)$ as an unfolding of the principal part of an acyclic sign-skew-symmetric matrix $B$ constructed as in Construction 2.6 By the construction of $Q$, we notice that if a vertex labeled by $i$ is a source, then all the vertices labeled by $i$ are also sources.

**Corollary 3.5.** Let $(Q, \Gamma)$ be the unfolding of the principal part of an acyclic sign-skew-symmetric matrix $B$ built as in Construction 2.6 with a finite set of orbits $\{\bar{1}, ..., \bar{n}\}$ obtained by the action of $\Gamma$, and let $(\bar{i}_1, ..., \bar{i}_k)$ be a sequence of orbit-mutations. If a vertex labeled by $j$ is a source in $\mu_{\bar{i}_k}...\mu_{\bar{i}_1}(Q)$, then all the vertices labeled by $j$ are also sources in $\mu_{\bar{i}_k}...\mu_{\bar{i}_1}(Q)$ such that $\bar{i}_l \in \{1, ..., n\}$ for $1 \leq l \leq k$.

**Proof.** Clearly $\Gamma$ is a group of automorphism for $\mu_{\bar{i}_k}...\mu_{\bar{i}_1}(Q)$. The statement holds true by the definition of automorphisms and since the indices which have the same label lie in the same orbit. \hfill $\square$

**Lemma 3.6.** Let $(Q, \Gamma)$ be the unfolding of the principal part of an acyclic sign-skew-symmetric matrix $B \in \text{Mat}_{n \times n}(\mathbb{Z})$ built as in Construction 2.6 with the adjacency matrix $B^\delta$, then $Q$ admits an orbit-admissible numbering by source.

**Proof.** By Lemma 3.4, the matrix $B$ defines an admissible numbering by source $(i_1^f, ..., i_n^f)$. There are $n$ orbits obtained by the action of the group of automorphisms $\Gamma$ defined in Construction 2.6 each orbit has the vertices with the same label in $Q$ and since $\{i_1^f, ..., i_n^f\} = \{1, 2, 3, ..., n\}$, the set $(\bar{i}_1, ..., \bar{i}_n) = \{\bar{1}, \bar{2}, \bar{3}, ..., \bar{n}\}$. By Lemma 2.9 the adjacency matrix of the quiver $\mu_{\bar{i}_{j-1}}...\mu_{\bar{i}_1}(Q)$ is an unfolding of the matrix $\mu_{i_{j-1}}...\mu_{i_1}(B)$ for $1 \leq j \leq n$. $i_j^f$ is a source in $\mu_{i_{j-1}}...\mu_{i_1}(B)$, thus by the folding relation we get

$$\mu_{i_{j-1}}...\mu_{i_1}(b_{i_{j-1}i}) = \mu_{i_{j-1}}...\mu_{i_1}(b_{i_{j-1}i}) = \sum_{r \in i_{j-1}} \mu_{i_{j-1}}...\mu_{i_1}(b_{i_{j-1}r}) \leq 0$$

for every $l \in Q_0$. Since $\mu_{i_{j-1}}...\mu_{i_1}(Q)$ does not admit any $\Gamma - 2$ cycles, the entries $\mu_{i_{j-1}}...\mu_{i_1}(b_{i_{j-1}r})$ have the same sign for every $r \in i_{j-1}$ and every $l \in Q_0$. Thus each term in the summation above is non-positive and hence $r$ is a source in
For every $r \in \bar{i}_j$ and every $1 \leq j \leq n$. Therefore $(\bar{i}_1, ..., \bar{i}_n)$ is an orbit-admissible numbering by source. □

By Lemma 2.5, we can define orbit-green sequences and orbit-maximal green green sequences for a locally-finite framed quiver $\tilde{Q}$ with an adjacency matrix $\tilde{B}^\sigma$ equipped with a group of automorphisms $\Gamma$ such that $\tilde{Q}$ can do arbitrary steps of orbit-mutations.

**Definition 3.7.** Let $\tilde{Q}$ be a locally-finite framed quiver with an adjacency matrix $\tilde{B}^\sigma$ equipped with a group of automorphisms $\Gamma$ such that $\tilde{Q}$ can do arbitrary steps of orbit-mutations, and let $\bar{\sigma}_s := (\bar{k}_1, \bar{k}_2, ..., \bar{k}_s)$ be a sequence of orbit-mutations and let $C_{\bar{\sigma}_j}$ be the C-matrix of $\tilde{B}^\bar{\sigma}_j$ obtained from $\tilde{B}^\bar{\sigma}_j$ by the sequence of orbit-mutation $(\bar{k}_1, \bar{k}_2, ..., \bar{k}_j)$ for $1 \leq j \leq s$, then $(\bar{k}_1, \bar{k}_2, ..., \bar{k}_s)$ is said to be an orbit-green sequence if for every $1 \leq j \leq s$, $\bar{k}_j$ is a green orbit in $B^\bar{\sigma}_j-1$. The sequence $(\bar{k}_1, \bar{k}_2, ..., \bar{k}_s)$ is said to be orbit-maximal green sequence if $\tilde{B}^\bar{\sigma}_s$ doesn't have any green orbits.

We always suppose that we have finitely many orbits under the action of $\Gamma$ on the unfolding locally-finite quiver of a sign-skew-symmetric matrix.

**Lemma 3.8.** Let $\tilde{Q}$ with the adjacency matrix $\tilde{B}^\bar{\sigma} = \begin{pmatrix} B^\bar{\sigma} \\ I_\infty \end{pmatrix}$ be the unfolding of an acyclic sign-skew-symmetric matrix $\bar{B} = \begin{pmatrix} B \\ I_n \end{pmatrix}$ built as in Construction 2.6 then any orbit admissible numbering by source of $\tilde{B}^\bar{\sigma}$ is an orbit-maximal green sequence.

**Proof.** Suppose that $\{\bar{i}_1, ..., \bar{i}_n\}$ is an orbit-admissible numbering by source of $\tilde{B}^\bar{\sigma}$. By the definition of orbit-mutation, the mutation at a specific orbit-source reflects the arrows incident to the vertices of that orbit while keeping other arrows the same. Hence at each step we get a new red orbit while the colors of other orbits remain the same. □

The following Theorem shows the relation between maximal green sequences for acyclic sign-skew-symmetric matrices and orbit-maximal green sequences for their unfolding matrices.

**Theorem 3.9.** Let $\tilde{B}^\bar{\sigma} = \begin{pmatrix} B^\bar{\sigma} \\ I_\infty \end{pmatrix}$ be the unfolding of an acyclic sign-skew-symmetric matrix $\bar{B} = \begin{pmatrix} B \\ I_n \end{pmatrix}$ as constructed in Construction 2.6 then the sequence $(\bar{k}_1, \bar{k}_2, ..., \bar{k}_s)$ is an orbit-maximal green sequence for $\tilde{B}^\bar{\sigma}$ if and only if the corresponding sequence $(k^1_f, k^2_f, ..., k^s_f)$ is a maximal green sequence for its folding matrix $\bar{B}$. 
Proof. By Lemma 2.9 the matrix \( \mu_{\bar{k}_{j-1}} \ldots \mu_{\bar{k}_1}(\tilde{B}^\delta) \) is an unfolding of the matrix 
\( \mu_{k_{j-1}} \ldots \mu_{k_1}(\tilde{B}) \) for every \( 1 \leq j \leq s \). Thus

\[
\mu_{k_{j-1}} \ldots \mu_{k_1}(c_{r_{k_j} k_j}) = \sum_{t \in \bar{r}_{k_j}} \mu_{\bar{k}_{j-1}} \ldots \mu_{\bar{k}_1}(c_{t_{k_j} k_j})
\]

The sign-coherence property is satisfied for locally-finite framed quivers by Lemma 2.3. Consequently, the terms that compose the summation on the right hand side of equation (3.1) have the same sign. Thus the \( k_j^f \) is green (red) in the matrix \( \mu_{k_{j-1}} \ldots \mu_{k_1}(\tilde{B}) \) if and only if the index \( k_j \) is green (red) in the matrix \( \mu_{\bar{k}_{j-1}} \ldots \mu_{\bar{k}_1}(\tilde{B}^\delta) \) and equivalently by Lemma 2.5 the orbit \( \bar{k}_j \) is green (red) in the matrix \( \mu_{\bar{k}_{j-1}} \ldots \mu_{\bar{k}_1}(\tilde{B}^\delta) \), hence the result follows. \( \square \)

Now we can prove that every acyclic sign-skew-symmetric matrix admits a maximal green sequence.

**Theorem 3.10.** Every acyclic sign-skew-symmetric matrix \( \tilde{B} = \begin{pmatrix} B & I_n \end{pmatrix} \) admits a maximal green sequence.

Proof. Let \( \tilde{B}^\delta = \begin{pmatrix} B^\delta & I_{\infty} \end{pmatrix} \) be the unfolding of an acyclic sign-skew symmetric matrix 
\( \tilde{B} = \begin{pmatrix} B & I_n \end{pmatrix} \) as constructed in Construction 2.6. By Lemma 3.8 the locally-finite framed quiver \( \tilde{Q} \) with the adjacency matrix \( B^\delta \) admits an orbit-maximal green sequence \((\bar{i}_1,...,\bar{i}_n)\). Hence the sequence \((i_1',...,i_n')\) is a maximal green sequence of matrix \( \tilde{B} \) by Theorem 3.9. \( \square \)

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