High-Temperature Properties of the $Z(3)$ Interface in (2+1)-D SU(3) Gauge Theory

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ABSTRACT

We study the high-temperature properties of the $Z(3)$ interface which forms between the various ordered phases of pure $SU(3)$ gauge theory above a critical temperature. On a (2+1)-D Euclidean lattice, we perform an accurate measurement of the interface tension, which shows good agreement with the prediction of perturbation theory. We also examine the behaviour of the Debye electric screening mass, and compare this with theoretical predictions.

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It has been known for some time that pure gauge theories have a non-confining high-temperature phase [1]. Therefore, any theory in a confining phase at zero temperature \((T = 0)\) must have a phase transition at some finite critical temperature, \(T_c\), separating the confining and non-confining phases. A consideration of QCD at finite temperatures reveals that the pure gauge \(SU(3)\) sector has the two expected phases. At low temperatures, there is the “disordered” phase, where the colour charge of QCD is confined and the vacuum is symmetric under the group \(Z(3)\), the centre of the \(SU(3)\) group. This is why no free quarks are seen in the relatively cool universe of today. At high temperatures, there is a non-confining, \(Z(3)\)-breaking “ordered” phase, corresponding to the free quark-gluon plasma believed to exist in the hot early universe. Owing to the \(Z(3)\) breaking, there exist three different, degenerate high-temperature phases, corresponding to the three members of \(Z(3)\), the cube roots of unity. Fermionic matter breaks the vacuum degeneracy, albeit on a small scale, so we consider only the pure gauge theory in what follows.

Two different types of interface are possible in the theory. First, there is one between the ordered and disordered phases; this type is only stable at the critical temperature, since only then do the temperature and pressure match on both of its sides. Second, an interface can form between two of the ordered phases with different \(Z(3)\) vacua, and this is usually called an “order-order interface”, or “\(Z(3)\) interface”. All thermodynamic quantities are the same on both sides of a \(Z(3)\) interface, but they dip slightly across the object itself. Clearly, this second type of interface can only exist above the critical temperature.

It has recently been emphasised [2][3][4] that the order parameter distinguishing the \(Z(3)\) phases is fundamentally a Euclidean object, with no counterpart in Minkowski space. Thus, whilst the \(Z(3)\) symmetry exists, and hence \(Z(3)\) domains and interfaces exist, in the Euclidean path integral, it is less certain that the interfaces can exist as physical objects in the universe. The claim has even been made [5] that only one true physical phase exists, even in Euclidean space, at high temperatures, and we shall address this when discussing our results later. In Minkowski space, the interfaces appear [3] to have unphysical thermodynamic properties when a certain number of fermion families is considered in addition to the gauge fields, resulting from the unusual fact that the ordered phases occur at high temperatures. Even setting this problem aside, the problem of interpretation of the interface remains, as the order parameter is non-local in Euclidean time, corresponding to an imaginary time-like gauge field in Minkowski space, and thus an imaginary chemical potential for the colour charge [6]. However, by virtue of their contribution to the partition function and to expectation values calculated using the Euclidean path integral, the interfaces must be included (like the instanton) in a non-perturbative analysis of the thermodynamics of QCD.

In the Euclidean formalism, the “time” dimension runs from 0 to \(\beta_T = 1/T\). The phases of a thermal gauge theory are characterised by the free energies of static configurations of quarks and antiquarks [7]. In particular, the self-energy of a single quark in the gluon medium, \(F_q\), is given by the expectation value of a single Polyakov line, \(L(x)\), a time-ordered Wilson loop which wraps around the time boundary for a fixed spatial location \(x\):

\[
L(x) = \frac{1}{16\pi^2 T} e^{i\beta_T} \int_0^\beta_T d\tau A_0(\tau, x) = e^{-\beta_T F_q}.
\]

If \(\langle L(x) \rangle = 0\) then the insertion of a single quark will require infinite energy, corresponding to a confining phase. In contrast, if \(\langle L(x) \rangle \neq 0\) then the insertion energy will be finite and the colour charge deconfined. Thus, the Polyakov line gives an effective test for confinement; unlike topologically trivial Wilson loops, it is aware of the phase structure of an \(SU(3)\) system and can map out the structure of any interface.

A general non-Abelian gauge transformation which leaves the Euclidean QCD action, \(S_E\), invariant will also leave \(L\) invariant if the transformation is periodic in time. However, \(S_E\) is actually invariant under a larger group than merely the periodic gauge transformations, being unaffected by transformations belonging to the centre of the gauge group (the set of elements which commute with all members of the group, \(Z(N)\) in the case of \(SU(N)\)). This global symmetry is an invariance under gauge transformations which are only periodic up to an element of the centre; it leaves local observables unchanged, and it remains after all gauge fixing. However, the topologically non-trivial Polyakov line, which wraps around the periodic boundary condition in the time direction, is not invariant under these transformations, but rather is rotated by an element, \(z\), of the centre: \(L(x) \rightarrow zL(x)\). Clearly, \(\langle L(x) \rangle\) acts as an order parameter for the centre symmetry, distinguishing between broken and unbroken phases, and between the different broken phases. This has been seen perturbatively [8][9] and by simulation [7].

In the deconfined phase, where the \(Z(N)\) symmetry has been spontaneously broken, the system can exist in any one of \(N\) degenerate vacua. It is possible to arrange the boundary conditions of the system so
that different parts of it exist in different vacua, i.e. different $Z(N)$ phases. The existence of these distinct domains forces the appearance of domain walls, or “$Z(N)$ interfaces”, where they meet [10][9]. Within these interfaces, the gauge fields interpolate between the different vacua, as does the expectation value of the Polyakov line.

The calculation of the interface tension is an instanton problem, in which $L(x)$ interpolates from one $Z(3)$ vacuum to another. An effective one-dimensional theory will describe the interface profile. The instanton is the solution of the classical equations of motion, with action $\sim 1/g^2$ (as seen by absorbing the dimensionless coupling through $gA^\mu \rightarrow A^\mu$), and so we might naively expect the interface tension to have the same behaviour. To check this, an effective action can be constructed from the classical piece, acting as a kinetic term, and a quantum piece, a potential term formed by integrating out fluctuations to one-loop order. The $Z(N)$ instanton is the stationary point of the total action. The instanton calculation has been performed [10][9][11], predicting that, in 2+1 Euclidean dimensions, the interface tension (the effective action per unit area of the interface) takes the form $\alpha = \alpha_0 T^{2.5}/g$, where $\alpha_0$ is a constant predicted to be 8.33 for $SU(3)$ in the continuum, in the limit as $T \rightarrow \infty$.

In the lattice formalism, one considers a hypercubic lattice whose sites are separated by lattice spacing $a$. Let us suppose there are two space dimensions, one ($x$) much longer than the other ($y$), and both much longer than the time dimension: $L_x \gg L_y \gg \beta L$. If we specify $N_t$ links in the time direction, this corresponds to a temperature $T = 1/\beta L = 1/N_t a$; similarly, $L_y = N_y a$ and $L_x = N_x a$. The conventional bare (dimensionless) coupling, $g$, is related to the lattice parameter, $\beta$, by $\beta = 2N/a^{d-1}g^2$ in $d$ dimensions. If $d = 3$ and $N = 3$ then $\beta = 6/a g^2$, and for $N_t = 2$ a lattice calculation [12] predicts that

$$\alpha_0 = 9.82. \quad (2)$$

To test these predictions, we have performed extensive simulations of the (2+1)-D $SU(3)$ theory, where the interface is one-dimensional (a string). A similar prediction for (2+1)-D $SU(2)$ gauge theory was borne out for a range of lattice sizes with $N_t \leq 5$ in a recent survey [11].

We impose periodic boundary conditions in all directions, and set the temperature, specified through the lattice parameter, to be $\beta \gg \beta_c$, where $\beta_c$ corresponds to the temperature of the deconfinement phase transition, previously estimated [13] as $\beta_c = 8.175(2)$. Left to its own devices, the entire system will settle into just one of the $Z(3)$ phases, this being the configuration of lowest energy. We need to force the system into producing a $Z(3)$ interface, and this is achieved by use of a “twist” [14].

The twist is introduced by modifying the action so that every plaquette in the $\tau - x$ plane at some particular value of $\tau$ and $x$ is to be premultiplied by an element, $z^\dagger$, of the centre group when it appears in the action. This means that if we order the system so that all the Polyakov lines to the left of the twist are in the $Z(3)$ phase given by $<L> = 1$, then the twist will cause the Polyakov lines to take the value $<L> = z$ to the right of the twist. No extra energy is associated with the twist itself; it merely acts as a change of variables, transforming one $Z(3)$ vacuum into another. However, the periodic boundary conditions force the appearance of a real physical $Z(3)$ interface somewhere else in the $x$ direction, where the Polyakov lines interpolate from $x$ back to 1. Thus, we have forced the creation of two $Z(3)$ domains on the lattice, with one $Z(3)$ interface between them. We place the twist at the boundary in our simulations, so that the low-$x$ end of the system is in one phase and the high-$x$ end in another.

For our Monte-Carlo simulations, we used a mix of Cabibbo-Marinari heat-bath steps [15], using the Kennedy-Pendleton algorithm [16] to update $SU(2)$ subgroups, and Brown-Woch over-relaxation steps [17], after using many initial heat-bath sweeps to equilibrate the system. Measurements of physical operators took place only every four sweeps, to reduce the correlations caused by adjacent configurations. All calculations were performed on a DEC 2100 A500MP machine, with CPU times tabulated later in table 1.

In fig. 1, an interface is shown on a $2 \times 24 \times 96$ lattice. The longitudinal direction goes across the page, and the transverse direction into it. The “height” in each case is determined by the value of the Polyakov line, and has nothing to do with the time dimension, which has been integrated over to form the Polyakov line. The twist has been placed at the boundary of the $x$ direction, as mentioned before, so the left-hand end of the lattice is in one $Z(3)$ phase with $<L> \approx e^{2\pi i/3}$, and the right-hand end in another with $<L> \approx 1$. The temperature is fairly high: $\beta = 50 \gg \beta_c$. Thus, the interface is fairly well-defined and stable, since its free energy is correspondingly high which suppresses fluctuations in phase. In this picture, the interface is roughly in the centre of the lattice. However, the interface is actually translation invariant; there is nothing
in the action to specify that it should be located at any particular value of $x$, and it can be seen to execute a random walk along the lattice, with a speed which increases slightly as the temperature decreases.

The free energy of a system is $F = -T \ln Z$, so $\frac{\partial}{\partial \beta} \langle F \rangle = \frac{1}{\beta} < S >$. This gives the behaviour of the free energy in terms of the average plaquette action. Simulations are performed for the same lattice with and without twisted plaquettes, i.e. with and without a $Z(3)$ interface, and the difference in free energy between the two gives the free energy of the interface itself, from which we can measure its surface tension: $F_{\text{interface}} = F_{\text{twisted}} - F_{\text{untwisted}} = \alpha A$, where $\alpha$ is the surface tension, and $A$ is the ‘area’ of the interface (in our case, the transverse size of the lattice). Therefore, substituting $A = L_y a$ into the previous equation,

$$\Delta S = \frac{\partial}{\partial \beta} \left( \frac{\alpha_0 T^{2.5} L_y a}{g T} \right) = \frac{L_y}{2 \sqrt{6} \beta N^{1.5}_{t}} \alpha_0.$$  

The prediction (2) that $\alpha_0 = 9.82$ for a system with $N_t = 2$ was made in the infinite-temperature limit. Hence, to test this prediction it is necessary to estimate $\alpha_0$ in this limit. The best way to do this is to carry out a survey of $\alpha_0$ for various values of $\beta$, and then to extrapolate the results to the infinite-$\beta$ limit. Several tens of thousands of heat-bath sweeps were used to ensure complete equilibration of the lattice before measurements began. “Main sweep” refers to a sweep after equilibration is complete.

| $\beta$  | Lattice Size | Heat-Bath + Main Sweeps (000’s) | CPU Time (hrs) |
|----------|--------------|---------------------------------|---------------|
| 16       | $2 \times 12 \times 30$ | 15 + 400                        | 80            |
| 28.125   | $2 \times 12 \times 36$ | 20 + 400                        | 120           |
| 50       | $2 \times 16 \times 48$ | 20 + 400                        | 260           |
| 112.5    | $2 \times 24 \times 72$ | 40 + 400                        | 800           |
| 200      | $2 \times 32 \times 96$ | 40 + 400                        | 1600          |
In table 2, we list the results for the average action for a single plaquette, \(< S^P >\), the total action divided by \(3N_xN_yN_z\). The difference between the twisted and untwisted values is then used in (3) to obtain the measurements of \(\alpha_0\) in the last column.

| \(\beta\) | \(< S^P >_{\text{twisted}}\) | \(< S^P >_{\text{untwisted}}\) | \(\alpha_0\) |
|--------|-----------------|-----------------|-----------|
| 16     | 0.17510(1)      | 0.17379(1)      | 13.1(1)   |
| 28.125 | 0.097575(6)     | 0.096832(6)     | 11.79(13) |
| 50     | 0.054310(3)     | 0.053916(3)     | 11.12(12) |
| 112.5  | 0.0239740(7)    | 0.0238074(7)    | 10.58(6)  |
| 200    | 0.0134549(5)    | 0.0133644(5)    | 10.21(8)  |

One would not expect the finite size of the lattices used to have a great effect on the untwisted results, as the whole lattice will be in one phase, but it is less clear how great the effect would be in the presence of the twist. The shape of the interface may not flatten out completely at opposite ends of the longitudinal dimension if the lattice is too short, skewing our measurement of the total interface free energy. The lattice sizes above were chosen to be large enough for the finite-size effects to be negligible, by studying results for \(\beta = 50\) on various sizes of lattice and scaling the spatial dimensions with the interface width, which is proportional to \(\sqrt{\beta}\) and which we choose as our measure of physical correlation lengths. The only exception to this was the measurement for \(\beta = 16\), for which the lattice could not safely be made any smaller. The results are given in table 3, each from 2+20 thousand sweeps. Similar runs without twist gave a result of \(< S^P >_{\text{untwisted}} \approx 0.05390(1)\), and this value was assumed for each lattice in the calculation of \(\alpha_0\). The results suggest, given the errors, that a lattice size of \(2 \times 16 \times 48\) is adequate for \(\beta = 50\).

| Lattice Size | \(< S^P >_{\text{twisted}}\) | \(\alpha_0\) |
|--------------|-----------------|-----------|
| \(2 \times 16 \times 32\)  | 0.05449(1)      | 11.1(3)   |
| \(2 \times 16 \times 40\)  | 0.05438(1)      | 11.3(3)   |
| \(2 \times 16 \times 48\)  | 0.05432(1)      | 11.9(3)   |
| \(2 \times 16 \times 64\)  | 0.05422(1)      | 12.0(3)   |
| \(2 \times 24 \times 48\)  | 0.05432(1)      | 11.9(3)   |
| \(2 \times 24 \times 60\)  | 0.05423(1)      | 11.6(3)   |
| \(2 \times 24 \times 72\)  | 0.05418(1)      | 11.9(3)   |

Now, as mentioned above, the prediction of (2) is only valid in the limit \(\beta \rightarrow \infty\), so the question arises of how to extrapolate our results to test this prediction. This is really a question of calculating infra-red divergences. In 3+1 dimensions, one would expect the one-loop correction to (2) to go like \(1/\beta\) in this limit, since the Debye mass, \(m \sim T\). However, in 2+1 dimensions, it is conceivable that a dependence on \(\ln \beta\) would need to be taken into account, as this appears in the infra-red correction for the Debye mass, discussed later. For this reason, it is difficult to predict the form of the \(\beta\)-dependence. In any case, it would be quite possible for the logarithmic dependence to masquerade as a power law over a restricted range of \(\beta\).

In the absence of a definite prediction for the form of the finite-\(\beta\) correction, we perform the extrapolation using a best fit to the form \(\alpha_0(\beta) = \alpha_0(1 + c/\beta^w)\). A computer-generated fit to the data, leaving the power, \(w\), as a free parameter, suggests a power of 0.85 with \(\chi^2 = 2.39\). This gives a \(\beta = \infty\) limit of \(\alpha_0 = 9.91(1)\), as extrapolated in fig. 2. Fits for various imposed powers give a lowest \(\chi^2\) of 2.34 for a power of 0.80, with
Fig. 2: Extrapolations of $\alpha_0$ are shown as $\beta \to \infty$ for various powers of $\beta$, with each extrapolation labelled accordingly: the five data points are plotted as triangles for an x-axis of $1/\beta^{0.95}$; as squares for the preferred power, $1/\beta^{0.85}$; and as diamonds for $1/\beta^{0.75}$.

$\alpha_0 = 9.83(1)$. However, $\chi^2 \leq 2.6$ for powers between 0.75 and 0.90. Therefore, in the absence of a definite prediction for the power of $1/\beta$, we allow for a reasonable error in the fit, and use $0.85(10)$, the range for which extrapolations are illustrated in fig. 2. Thus, our infinite-temperature extrapolation for the interface tension is

$$\alpha_0 = 9.91(1)(14),$$

in very good agreement with the instanton calculation result of (2). The first error is associated with the fit for $1/\beta^{0.85}$, and the second with the uncertainty in that value for the power.

It has been argued [5] that the different $Z(3)$ vacua distinguished by different values of the Polyakov line actually correspond to one and the same physical state. This claim comes from considering the rôle of infra-red divergences in the calculation of the surface tension, and, if true, would imply that the surface tension is actually zero (since no physical interface can exist). The agreement of our results for $\alpha_0$ with perturbation theory tends to disprove this argument. It is, of course, conceivable that our measurements for $\alpha_0$ would dramatically fall towards zero if we went to higher and higher $\beta$, meaning that our extrapolation is simply an artefact for the range of $\beta$ that we have considered. However, our highest value is $\beta \approx 25\beta_c$, a temperature well above the critical point, and it seems unlikely that any dramatically different behaviour will set in at a temperature higher than this. A sceptic could also argue that a lattice with $N_t = 2$ could show behaviour quite distinct from that in the continuum; but quite apart from the agreement with the lattice prediction of $\alpha_0$ for this value of $N_t$, a similar survey carried out for $SU(2)$ gauge theory for a range of $N_t \leq 5$ has found similar agreement with perturbation theory [11].

The electric (“Debye”) mass, or inverse Debye screening length, $m$, governs gauge-invariant correlation
functions of the time-like component of the gluon field \(A_0\) at large distances and high temperatures. The free energy of a quark-antiquark pair, over and above the sum of their separate free energies, vanishes as the quarks become infinitely far apart, and, from perturbation theory, we know that quarks are screened at a distance of the order of the inverse electric mass, so rather than being logarithmic in \(x\), the interaction potential takes the form

\[
F_{\gamma\gamma}(x) - 2F_q = V(|x|, T) \sim -Ce^{-2m|\mathbf{x}|} \quad \text{for } T > T_c.
\]

The factor of two arises because gauge invariance leads to an exchange of two gluons being the lowest-order contribution in perturbation theory. The electric mass is one of the fundamental parameters of a gauge theory at finite temperature. A self-consistent and gauge-invariant calculation of its value \[18\][19] gives

\[
m^2 = \frac{g^2 N T}{4\pi} \ln(T/g^2) + O\left[g^2T \ln(\ln(T/g^2))\right], \quad T/g^2 \gg 1,
\]

where the dimensional regularisation mass scale is set to \(m\). Clearly, this mass is a non-analytic function of coupling constant and temperature.

To measure the Debye mass, we use the correlation of the transverse average of Polyakov lines on the untwisted lattice. We estimate \(2m\) from the vacuum-subtracted correlations by assuming an exponential decay with the line separation, \(x\), using the function

\[
\mu(x) = \ln\left(\frac{<L^\dagger(x-1)L(0)> - <\mathcal{L}>^2}{<L^\dagger(x)L(0)> - <\mathcal{L}>^2}\right),
\]

where each \(<L(x)\>\) represents an average of \(L(x)\) over all configurations (sweeps), and \(<\mathcal{L}>\) is a further average over all \(x\). From the definition of the Debye mass, we expect \(\mu(x)\) to tend to a constant, \(2m\), as we increase the separation, \(x\), as long as \(x\) remains significantly less than \(L_x/2\). For each set of data at a particular \(\beta\), we extract a value of \(\mu(x)\) corresponding to the level at which the function flattens out before errors begin to increase wildly. These values are given in table 4 for the simulations listed in table 1.

| \(\beta\) | 16      | 28.125  | 50     | 112.5   | 200     |
|----------|---------|---------|--------|---------|---------|
| Twice Debye mass, \(\mu = 2m\) | 0.80(1) | 0.690(5)| 0.555(5)| 0.410(5)| 0.332(4)|

The values are plotted in fig. 3, together with the theoretical prediction of (4) (dashed line). The results show some qualitative agreement with the leading-order prediction of (4). However, although the shape of the rise of the mass with temperature looks correct, the results differ from the prediction by a factor of two or so. The reason for this becomes clearer when we consider the size of the ratio of coefficients of the free fit does at least confirm that most of the temperature dependence is contained within the leading \(\frac{3}{4\pi}\ln(\beta/12)\) term.

In conclusion, we have measured the interface tension on the lattice at high temperatures, and extrapolated our results to the infinite-temperature limit, where we have found good agreement with the
prediction of an instanton calculation. This reinforces the argument for the existence of $Z(3)$-breaking phases in the Euclidean functional integral, although their physical existence in the early universe remains in question, and suggests that the predictions of perturbation theory are indeed reliable in this regime. We have also examined the Debye profile of the interface, but have not been able to compare our results accurately with predictions at the temperatures concerned, though some measure of agreement seems apparent.

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