New coherent states with Laguerre polynomials coefficients for the symmetric Pöschl–Teller oscillator

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Abstract

We construct new coherent states labeled by points \( z \) of the complex plane and depending on three parameters \( \gamma, \nu \) and \( \epsilon > 0 \) by replacing the coefficients \( z^n/\sqrt{n!} \) of the canonical coherent states by Laguerre polynomials with an order depending on \( \gamma \). These coherent states are superpositions of eigenstates of the Hamiltonian with a symmetric Pöschl–Teller potential indexed by \( \nu \), which solve an \( \epsilon \)-identity operator while the resolution of the identity of the states Hilbert space is achieved at the limit \( \epsilon \to 0^+ \). We obtain their wave functions in a closed form for a special case of parameters \( \gamma \) and \( \nu \). We also discuss their associated coherent states transform which leads to an integral representation of Hankel type for Laguerre functions.

Keywords: coherent states, Pöschl–Teller potential, Laguerre polynomials

1. Introduction

The recent works on quantum dots wells in nanophysics \[1, 2\] strongly motivate the construction of quantum states for infinite wells with localization properties comparable to those of Schrödinger states. Infinite wells are often modeled by a family of Pöschl–Teller (PT) potentials \[3\]. The symmetric PT potentials are a subclass of this family of potentials, which are widely used in molecular and solid state physics. Many other potentials can be obtained from PT potentials by appropriate limiting procedure and point canonical transformations.
There is an extensive literature in which many characteristic properties of the PT potential were examined at classical mechanics as well as at quantum mechanics levels.

Another interesting aspect of the PT potential lies in the fact that it has a quadratic spectrum leading to a rich revival structure for its coherent states, which makes possible the formation of Schrödinger cat and cat-like states. Different types of coherent states for quantum mechanical systems evolving in the PT potential have been discussed by many authors from various perspectives [4–14]. Here, the coherent states for the symmetric PT potential we are introducing are quite different from the existing ones in the above literature and are obtained by adopting a general Hilbertian probabilistic scheme [15] reminiscent to the classical construction of the Bargmann transform [16]. Our procedure can precisely be described as follows. In [17] the authors have introduced a family of coherent states for the isotonic oscillator [18] that is the harmonic oscillator with an inverse quadratic nonlinear term (like a centripetal barrier) through superpositions of the corresponding eigenstates where the role of coefficients \( z_n \) of the canonical coherent states was played by coefficients

\[
\left( \frac{\Gamma(n) + \gamma}{n! \Gamma(n)} \left( \frac{\gamma}{2} + n \right) \right)^{-\frac{1}{2}} e^{\frac{1}{2} \text{arg} z} L_n^{(\gamma-1)}(z \gamma), \quad n = 0, 1, 2, \ldots,
\]  
(1.1)

\( L_n^{(\alpha)}(\cdot) \) being the Laguerre polynomial [19]. Now, after noticing that coefficients (1.1) could also be used to define a new class of coherent states for the Hamiltonian with the symmetric PT potential \( V_{\nu}(x) \) (given below by (2.2)) via superpositions of its eigenstates, say \( |\phi_n^{\nu}\rangle \), which are expressed by Gegenbauer polynomials [19], we have proceeded to determine wave functions of these coherent states in a closed form. The latter leads to the kernel of the associated coherent states transform which maps the quantum states Hilbert space \( \mathcal{H} = L^2([0, \pi]) \) onto a subspace of a Hilbert space of square integrable functions with respect to a suitable measure on the complex plane (viewed as a phase space), ensuring a Bargmann-type representation. Since the kernel of such a transform is usually obtained by some generating function of orthogonal polynomials involved in expressions of the superposed eigenstates, we have modified the coefficients (1.1) by the factor \( e^{-\alpha \epsilon} \) in order to employ a generating relation for both Laguerre and Gegenbauer polynomials. In this way we here define the epsilon coherent states, which are denoted by \( \epsilon\text{-CS} \) for brevity.

Actually, these \( \epsilon\text{-CS} \) solve an \( \epsilon\)-identity operator which has the advantage of being a compact and trace class operator representing the thermodynamical quantum density for a Hamiltonian of harmonic type with the form \( \sum_{n=0}^{\infty} n |\phi_n^{\nu}\rangle \langle \phi_n^{\nu}| \) once we interpret the number \( \epsilon \) as the usual parameter \( \beta = 1/k_B T \) of statistical physics where \( k_B \) is the Boltzmann constant and \( T \) is the temperature. At the limit \( \epsilon \to 0^+ \), we can exploit a result due to Muckenhoop and Stein [20] to show that the \( \epsilon\)-identity operator goes to the identity operator of the states Hilbert space \( \mathcal{H} \). Furthermore, at this limit, the associated coherent states transform provides us with an integral representation of Hankel type for Laguerre functions, which constitutes a result of independent interest.

The paper is organized as follows. In section 2, we recall briefly some known facts on the Hamiltonian with the symmetric PT potential. Section 3 is devoted to the coherent states formalism we will be using. In section 4 we construct \( \epsilon\text{-coherent states} \) and we show that they solve an \( \epsilon\)-identity operator which reduces to the identity of the states Hilbert space at the limit \( \epsilon \to 0^+ \). In section 5, we give a closed form for these \( \epsilon\text{-coherent states} \). In section 6, we define a coherent states transform which leads to an integral representation for Laguerre functions. Section 7 is devoted to some concluding remarks.
2. The symmetric PT oscillator

The problem of finding the energy spectrum and the wavefunctions of a particle confined inside the infinite square well and submitted to a barrier of infinite potential at both frontiers has commonly been one of the most illustrative problems of quantum mechanics. This problem is generalized to a situation in which the confining potential is given by a family of continuously indexed potentials of the PT type [3]. Precisely, one considers the motion of a particle of mass $m$ living in the interval $[0, L]$ and evolving under the PT potential

$$V_{p,\delta}(x) := \frac{1}{4} E_0 \left( \frac{\eta(\eta - 1)}{\cos^2 \frac{x}{2L}} + \frac{\delta(\delta - 1)}{\sin^2 \frac{x}{2L}} \right), \quad 0 \leq x \leq L,$$

(2.1)

where $\eta, \delta > 0$, $E_0 := \hbar^2 \pi^2 (2m_\eta L^2)^{-1}$ is a coupling constant and $\hbar$ being the Planck's constant. The symmetric PT potential corresponds to the case $\eta = \delta$. For our purpose, we set $\eta = \delta = \nu + 1$ with $\nu > -1$. Thus, the potential in (2.1) reduces to

$$V_\nu(x) := \frac{\nu(\nu + 1)}{\sin^2 \frac{x}{L}}.$$

(2.2)

This potential corresponds the Hamiltonian operator or PT oscillator

$$\Delta_\nu := -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_\nu(x)$$

(2.3)

acting on the Hilbert space $\mathcal{H} := L^2(0, L, dx)$. Its eigenvalues are quadratic with the form

$$E_n = E_0(n + \nu + 1)^2, \quad n = 0, 1, 2, \ldots,$$

(2.4)

and the normalized eigenstates obeying the Dirichlet boundary conditions are given by

$$\phi_{\nu,\nu}^{\ell_n}(x) := \Gamma(\nu + 1) \frac{2^{\nu+1/2}}{\sqrt{\nu!}} \sin \left( \frac{\pi x}{L} \right)^{\nu+1} \cos \left( \frac{\pi x}{L} \right),$$

(2.5)

where $C_n^{\nu+1}(\cdot)$ are Gegenbauer polynomials [19]. For simplicity, we set $\hbar = 2m_\nu = 1$ and $L = \pi$ so that we will be dealing with eigenvalues $E_\nu := (n + \nu + 1)^2$, $n = 0, 1, 2, \ldots$, together with the eigenstates

$$\phi^\nu_n(x) := \phi_{\nu,\nu}^{\ell_n}(x) = \Gamma(\nu + 1) 2^{\nu+1/2} \frac{n!}{\pi \Gamma(n + 2\nu + 2)} (\sin x)^{\nu+1} C_n^{\nu+1}(\cos x),$$

(2.6)

which constitute a complete orthonormal basis in $\mathcal{H}$. A detailed mathematical discussion on the spectral theory of the Hamiltonian (2.3) can be found in [14, 21].

**Remark 2.1.** The case $\nu = 0$ corresponds to the infinite square well whose ground state energy is represented by the factor $E_0$. In this case, the Gegenbauer polynomial in (2.6) can be expressed in terms of the Tchebichef polynomial of the second kind ([19], p 97) as follows

$$C_n^1(\cos L^{-1} \pi x) = \frac{\sin \left( (n + 1)L^{-1} \pi x \right)}{\sin L^{-1} \pi x}.$$

(2.7)

Using (2.7) one can recover eigenstates of the Hamiltonian with the infinite square well potential as...
\[ \phi_{nL}^0(x) = \sqrt{\frac{2}{L}} \sin \left( \frac{(n+1)\pi}{L} x \right), \quad n = 0, 1, 2, \ldots \] (2.8)

### 3. Epsilon coherent states

Following the probabilistic Hilbertian scheme ([15], p 74), we will review a generalization of canonical coherent states by considering a kind of the identity resolution that we obtain at the zero limit with respect to a parameter \( \epsilon > 0 \). This formalism was introduced in [22, 23].

**Definition 3.1.** Let \( \mathcal{H} \) be a (complex, separable, infinite-dimensional) Hilbert space with an orthonormal basis \( \{ \psi_n \}_{n=0}^{\infty} \). Let \( \mathcal{D} \subseteq \mathbb{C} \) be an open subset of \( \mathbb{C} \) and let \( c_{n\epsilon} : \mathcal{D} \to \mathbb{C}; n = 0, 1, 2, \ldots \) be a sequence of complex functions. Define

\[
|z, \epsilon\rangle = \left( \mathcal{N}_\epsilon(z) \right)^{-1/2} \sum_{n=0}^{\infty} \frac{c_{n\epsilon}(z)}{\sigma_n(n)} |\psi_n\rangle,
\]

where \( \mathcal{N}_\epsilon(z) \) is a normalization factor and \( \sigma_n(n) \); \( n = 0, 1, 2, \ldots \), a sequence of positive numbers depending on \( \epsilon > 0 \). The vectors \( \{ |z, \epsilon\rangle, z \in \mathcal{D} \} \) are said to form a set of epsilon coherent states if

(i) for each fixed \( z \in \mathcal{D} \) and \( \epsilon > 0 \), the state in (3.1) is normalized, that is

\[ \langle z, \epsilon | z, \epsilon \rangle_{\mathcal{H}} = 1, \]

(ii) the following resolution of the identity operator on \( \mathcal{H} \)

\[
\lim_{\epsilon \to 0^+} \int_{\mathcal{D}} |z, \epsilon\rangle \langle z, \epsilon| \, d\mu_\epsilon(z) = \mathbf{1}_\mathcal{H}
\]

is satisfied with an appropriately chosen measure \( d\mu_\epsilon \).

In the above definition, the Dirac’s bra-ket notation \( |z, \epsilon\rangle \langle z, \epsilon| \) means the rank-one operator \( \phi \mapsto |z, \epsilon\rangle \langle z, \epsilon| \phi \), \( \phi \in \mathcal{H} \). Also, the limit in (3.2) is explained as follows. We define the integral of rank one operators as being the linear operator

\[
\mathcal{O}_\epsilon[\phi](\bullet) := \int_{\mathcal{D}} \langle z, \epsilon| \langle z, \epsilon| \phi \rangle \, d\mu_\epsilon(z).
\]

Then, the above limit is pointwise meaning \( \mathcal{O}_\epsilon[\phi](\bullet) \to \phi(\bullet) \) as \( \epsilon \to 0^+ \), almost everywhere with respect to \( (\bullet) \). Here, we should mention that the usual way is to understand the integral

\[
\int_{\mathcal{D}} |z, \epsilon\rangle \langle z, \epsilon| \, d\mu_\epsilon(z)
\]

in the weak sense, see for instance ([5], p 8). Namely, it is the sesquilinear form

\[
B_\epsilon(\phi, \psi) := \int_{\mathcal{D}} \langle \phi | z, \epsilon\rangle \langle z, \epsilon | \psi \rangle \, d\mu_\epsilon(z).
\]

Choosing this way, one has to check that the form (3.5) is bounded so that the Riesz lemma ensures the existence of a unique bounded operator \( \mathcal{O}_\epsilon \) satisfying \( B_\epsilon(\phi, \psi) = \langle \phi, \mathcal{O}_\epsilon \psi \rangle \). In our framework the resolution of the identity reads \( \lim_{\epsilon \to 0} B_\epsilon(\phi, \psi) = \langle \phi, \mathcal{O}_\psi \rangle \) meaning that \( \lim_{\epsilon \to 0} \mathcal{O}_\epsilon = \mathbf{1}_\mathcal{H} \) in the weak operator topology.
Remark 3.1. The formula (3.1) can be viewed as a generalization of the series expansion of the canonical (anti-holomorphic) coherent states

\[ |z\rangle := (e^{z^2})^{-1} \sum_{n=0}^{+\infty} \frac{z^n}{\sqrt{n!}} |\phi_n\rangle; \quad z \in \mathbb{C}, \]  

(3.6)

where \( \{ |\phi_n\rangle \} \) is an orthonormal basis in \( L^2(\mathbb{R}) \), which consists of eigenstates of the harmonic oscillator, given by \( \phi_n(y) = (\sqrt{\pi}2^n n!)^{-1/2}e^{-y^2/2}H_n(y) \) in terms of Hermite polynomials [19].

4. Epsilon CS with Laguerre coefficients for the symmetric PT potential

By adopting definition 3.1 we now construct a class of epsilon coherent states as follows.

Definition 4.1. Define a set of states labeled by points \( z, w \in \mathbb{C} \) and depending on three parameters \( \gamma, \nu, \epsilon > 0 \) by the following superposition

\[ |z; (\gamma, \nu); \epsilon\rangle := \left( \mathcal{N}_{\gamma,\nu}(z) \right)^{-1/2} \sum_{n=0}^{+\infty} \frac{1}{\sqrt{\sigma_{\gamma,\nu}(n)}} e^{-in\arg z} \mathcal{L}_n^{(\gamma-1)}(\sqrt{z\epsilon}) |\phi_n^\epsilon\rangle, \]  

(4.1)

where \( \mathcal{N}_{\gamma,\nu}(z) \) is a normalization factor, \( \sigma_{\gamma,\nu}(n) \) are a sequence of positive numbers given by

\[ \sigma_{\gamma,\nu}(n) = \frac{1}{n!} \frac{\Gamma^{2} (\nu+n)}{\Gamma^{2}(\nu)} \left( \frac{\nu}{2} + n \right)^{n} \epsilon^{n}, \quad n = 0, 1, 2, \ldots, \]  

(4.2)

and \( \{ |\phi_n^\epsilon\rangle \} \) is the orthonormal basis of \( \mathcal{H} = L^2([0, \pi], dx) \) as defined in (2.6).

In the next result (see appendix A) we give the overlap relation between two \( \epsilon \)-CS.

Proposition 4.1. For every \( z \) and \( w \) in \( \mathbb{C} \), the overlap relation between two \( \epsilon \)-CS in (4.1) is given through the scalar product

\[ \langle z; (\gamma, \nu); \epsilon | w; (\gamma, \nu); \epsilon \rangle_{\mathcal{H}} = \mathcal{N}_{\gamma,\nu}(z) \mathcal{N}_{\gamma,\nu}(w)^{-1/2} \int_{0}^{e^{-\gamma/2}} \frac{1}{\sqrt{t}} \exp \left( - \frac{(z\epsilon + w\epsilon)t}{1 - e^{i\arg z - \arg w}} \right) dt, \]  

(4.3)

where the normalization factor is given by

\[ \mathcal{N}_{\gamma,\nu}(z) = \Gamma (\gamma) e^{z^2/2} \int_{0}^{e^{-\gamma/2}} \frac{1}{\sqrt{t}} \exp \left( - \frac{2z\epsilon t}{1 - t} \right) I_{\nu-1} \left( \frac{2z\epsilon}{1 - t} \right) dt, \]  

(4.4)

In particular, when \( \epsilon \to 0^+ \), this factor becomes

\[ \mathcal{N}_{\gamma}(z) = \Gamma (\gamma) e^{z^2/2} \int_{0}^{e^{-\gamma/2}} \frac{z^2}{2} \exp \left( - \frac{z^2}{2} (2z\epsilon - 1) \right) I_{\nu} \left( \frac{z^2}{2} \right), \]  

(4.5)

where \( I_{\nu}(.) \) and \( K_{\nu}(.) \) are modified Bessel functions of first and second kind respectively. For \( \gamma = 2 \), equation (4.5) reduces further to...
\[ \mathcal{N}(z) = 2(z^2 - 2e^{-z/2}) \sinh(z/2). \] (4.6)

**Remark 4.1.** In the limit case \( \varepsilon \to 0^+ \), the coefficients in (4.2) becomes those in (1.1) which were used by the authors [17] in their construction of a set of coherent states for the isotonic oscillator \( H_A := -\frac{\partial^2}{\partial x^2} + x^2 + A/x^2 \), \( A \geq 0 \). The normalization factor in (4.5) is the same as in ([17], p 4) where it was referred to a proof in [24] based on the construction of the Green’s function for \( H_A \).

We now will proceed to determine a measure having the form \( \mathcal{N}_{r,\varepsilon}^{\gamma}(z) d\mu_{r,\varepsilon}(z) \) with respect to which the \( \varepsilon \)-CS satisfy a resolution of an \( \varepsilon \)-identity operator. Here, \( \mathcal{N}_{r,\varepsilon}^{\gamma}(z) \) is the normalization factor in (4.4) and the measure \( d\mu_{r,\varepsilon}(z) \) is not \( \varepsilon \)-dependent.

**Proposition 4.2.** Let \( \gamma > 0 \) and \( \nu > -1 \). Then, the \( \varepsilon \)-CS solve an \( \varepsilon \)-identity operator as follows

\[ \int_{C} |z; (\gamma, \nu); e\rangle\langle e; (\gamma, \nu); z| d\mu_{r,\varepsilon}(z) = e^{-\varepsilon H_r}, \] (4.7)

where \( H_r = \sum_{n=0}^{+\infty} \frac{1}{\sigma_{r,\varepsilon}(n)} \int_{0}^{+\infty} L_n^{(r-1)}(\rho^2) \frac{1}{\sqrt{\sigma_{r,\varepsilon}(n)(\rho^2)}} \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} \int_{0}^{2\pi} \frac{d\theta}{2\pi} e^{i(m-n)\theta} \langle \phi_n^\varepsilon | \phi_n^\varepsilon \rangle \langle \phi_n^\varepsilon | \phi_n^\varepsilon \rangle |\varphi\rangle \]

(4.10)

\[ [\varphi] = \sum_{n=0}^{+\infty} \left( \frac{n! \Gamma(\gamma)}{\Gamma(\gamma + n)(2n + \gamma)} \int_{0}^{+\infty} L_n^{(r-1)}(\rho^2) f_r(\rho^2) e^{-\varepsilon \rho^2} d\rho \right) \langle \phi_n^\varepsilon | \phi_n^\varepsilon \rangle |\varphi\rangle \]

(4.11)

\[ = \sum_{n=0}^{+\infty} \left( \frac{n! \Gamma(\gamma)}{\Gamma(\gamma + n)(2n + \gamma)} \int_{0}^{+\infty} L_n^{(r-1)}(\rho^2) f_r(\rho^2) e^{-\varepsilon \rho^2} d\rho \right) \langle \phi_n^\varepsilon | \phi_n^\varepsilon \rangle |\varphi\rangle. \] (4.12)
Now, we need to determine the function $f_r(r)$ such that

$$\int_0^{+\infty} \left( L_n^{(n-1)}(r) \right)^2 f_r(r) dr = \frac{(\gamma)^n}{n!} (2n + \gamma).$$  

(4.13)

where $(a)_n = \Gamma(a + n)/\Gamma(a)$ is the shifted factorial. To do that, we make appeal to the integral ([25], p 1133):

$$\int_0^{+\infty} y^s e^{-\gamma} L_k^{(a)}(y) L_m^{(b)}(y) dy = (-1)^{k+m} \Gamma(s + 1) \sum_{j=0}^{\min(k,m)} \binom{s - \alpha}{s - j} \binom{\beta}{m - j} \binom{s + j}{j}$$

(4.14)

with the condition $\Re(s) > -1$. In the case of parameters $y = r$, $\alpha = \beta = \gamma - 1 = s - 1$ and $k = m = n$, the integral (4.14) reads

$$\int_0^{+\infty} r e^{-\gamma} \left( L_n^{(n-1)}(r) \right)^2 dr = \Gamma(\gamma) \frac{\gamma^n}{n!} (2n + \gamma).$$

(4.15)

This suggests we should take $f_r(r) = r e^{-\gamma}$. Therefore, the measure in (4.8) has the form

$$d\mu_{r,e}(z) = \frac{1}{\Gamma(\gamma)} \frac{\gamma^n}{n!} (2n + \gamma).$$

(4.16)

With this measure equation (4.12) reduces to

$$\mathcal{O}_\epsilon[\varphi](\alpha) = \sum_{n=0}^{+\infty} x^\alpha \langle \varphi | \phi_n^\epsilon \rangle \langle \phi_n^\epsilon | \varphi \rangle.$$  

(4.17)

By defining a Hamiltonian operator of the harmonic type via the discrete spectral resolution

$$H_\epsilon := \sum_{n=0}^{+\infty} | \phi_n^\epsilon \rangle \langle \phi_n^\epsilon |,$$

then equation (4.17) also reads $\mathcal{O}_\epsilon[\varphi] = e^{-\epsilon H_\epsilon} [\varphi]$ for every $\varphi \in H$. This ends the proof.\[\square]\n
In the next result (see appendix B) we give the proof of the resolution of the identity operator.

**Proposition 4.3.** Let $\gamma > 0$. Then, the $\epsilon$-CS in (4.1) satisfies the following resolution of the identity

$$\lim_{\epsilon \to 0^+} \int \|z\rangle \langle \gamma, \nu, \epsilon | v \rangle \langle \gamma, \nu, \epsilon | z\| d\mu_{r,e}(z) = 1_{L^2([0,\infty])},$$

(4.18)

where $d\mu_{r,e}(z)$ is the measure given in (4.16).

**5. A closed form for the coherent states**

We now assume that the parameter $\gamma$ is of the form $\gamma = 2(\nu + 1)$ where $\nu$ is the parameter indexing the symmetric PT potential in (2.2).

**Proposition 5.1.** Let $\nu > -1$ and $\epsilon > 0$ be fixed parameters. Then, the wavefunctions of the $\epsilon$-CS in (4.1) can be written in a closed form as
\[ \langle x | z; \nu, \varepsilon \rangle = (\mathcal{N}_{2(\nu+1),\nu}(z))^{-1/2} \left( -\frac{ie^{\nu/2}}{\pi |z|} \right)^{\nu+1/2} \frac{1}{\sqrt{1 - 2e^{i(\theta+\varepsilon/2)} \sin x + e^{-(2i\theta+\varepsilon)}}} \]

\[ \times \exp \left\{ \frac{-i|z|e^{\nu/2}(\cos x - e^{-(\theta+\varepsilon/2)})}{1 - 2e^{i(\theta+\varepsilon/2)} \cos x + e^{-(2i\theta+\varepsilon)}} \right\} \times I_{\nu+\frac{1}{2}} \left( \frac{i|z|e^{\nu/2} \sin x}{1 - 2e^{i(\theta+\varepsilon/2)} \cos x + e^{-(2i\theta+\varepsilon)}} \right) \]

(5.1)

When \( \varepsilon \to 0^+ \) equation (5.1) reads

\[ \langle x | z; \nu \rangle = (-ie^{\nu})^{\nu+1/2} e^{-1/4} \left( I_{\nu+1/2} \left( \frac{z^2}{2} \right) K_{\nu+1/2} \left( \frac{z^2}{2} \right) \right)^{-1/2} \frac{1}{\sqrt{1 - 2e^{-\theta} \cos x + e^{-2i\theta}}} \]

\[ \times \exp \left\{ \frac{-i|z| \left( \cos x - e^{-\theta} \right)}{1 - 2e^{-\theta} \cos x + e^{-2i\theta}} \right\} I_{\nu+1/2} \left( \frac{i|z| \sin x}{1 - 2e^{-\theta} \cos x + e^{-2i\theta}} \right) \]

(5.2)

If moreover \( \nu = 0 \), which corresponds to the infinite square well then equation (5.2) reduces to

\[ \langle x | z \rangle = \frac{e^{i\theta}}{\sqrt{\pi}} \exp \left( -\frac{z^2}{4} \left( \frac{1 + 2e^{-\theta} \cos x - 3e^{-2i\theta}}{1 - 2e^{-\theta} \cos x + e^{-2i\theta}} \right) \right) \]

\[ \times \sinh^{-1/2} \left( \frac{z^2}{2} \right) \sin \left( \frac{|z| \sin x}{1 - 2e^{-\theta} \cos x + e^{-2i\theta}} \right) \]

(5.3)

for every \( x \in [0, \pi] \), where \( \theta = \text{arg} \ z \).

**Proof.** We start from the expression (4.1) with \( \gamma = 2(\nu + 1) \) as

\[ \langle x | z; \nu, \varepsilon \rangle = (\mathcal{N}_{2(\nu+1),\nu}(z))^{-1/2} \sum_{n=0}^{+\infty} \frac{e^{-in\text{arg} zL_{n}(2\nu+1)(z\xi)}}{\sqrt{\sigma_{2(\nu+1),\nu}(n)}} \langle x | \phi_{n}^{\nu} \rangle, \]

(5.4)

To get a closed form of the series

\[ S(x) = \sum_{n=0}^{+\infty} \frac{\sqrt{n!} \Gamma(2\nu+1)}{\Gamma(n+2(\nu+1))(n+\nu+1)} \frac{e^{-i(n+\varepsilon/2)}}{\sqrt{\sigma_{2(\nu+1),\nu}(n)}} \langle x | \phi_{n}^{\nu} \rangle \]

(5.5)

we replace \( \langle x | \phi_{n}^{\nu} \rangle \) by its expression in (2.6) and we use the Legendre’s duplication formula ([26], p 5):

\[ \Gamma(x) = \frac{1}{\sqrt{\pi}} \Gamma(x) \Gamma \left( x + \frac{1}{2} \right) \]

(5.6)

for \( \xi = \nu + 1 \). Then, (5.5) reads

\[ S(x) = \frac{\sqrt{\pi} \Gamma(2\nu+1)}{2^{\nu+1/2} \Gamma(\nu+1/2)} (\sin x)^{\nu+1} \sum_{n=0}^{+\infty} \frac{n!e^{-i(n+\varepsilon/2)}}{(2(\nu+1))_{n}} L_{n}(2\nu+1)(z\xi)C_{n+1}^{\nu+1}(\cos x). \]

(5.7)
We introduce the variable \( \tau \equiv \theta \epsilon - e (i 2) \) and we rewrite (5.7) as follows:

\[
\Gamma_{\nu} = + + \nu \nu - + + \nu \nu \sin (x) (1/2) \sin (x) (1/2)
\]

where

\[
G(x) = \sum_{n=0}^{+\infty} \frac{n! \tau^n}{(2\nu + 1)_n} I_n^{(2\nu+1)} (\nu \epsilon) C_n^{\nu+1} (\cos x). \tag{5.9}
\]

We now make use of the generating relation ([27], p 56):

\[
\sum_{n=0}^{+\infty} \frac{n! \nu^n}{(2\nu + 1)_n} I_n^{(2\nu-1)} (\nu \nu) C_n^{\eta} (u) = \frac{1}{(1 - 2\nu \nu + \nu^2)^{\eta}} \exp \left( \frac{-w u (\nu - \nu)}{1 - 2\nu \nu + \nu^2} \right) \times _2F_1 \left( -; \frac{w^2\nu^2 (\nu^2 - 1)}{4(1 - 2\nu \nu + \nu^2)^2}; \frac{1}{2}; \right) \tag{5.10}
\]

for parameters \( \eta = \nu + 1 \), \( \nu = \tau \), \( \nu = \epsilon \) and \( \nu = \cos x \). This gives

\[
G(x) = \left( 1 - 2e^{-i\theta + \epsilon/2}) \cos x + e^{-2i\theta + \epsilon}) \right)^{-1/2} \exp \left( \frac{-\nu \epsilon e^{-i\theta + \epsilon/2}) (\cos x - e^{-i\theta + \epsilon})}{1 - 2e^{-i\theta + \epsilon}) (\cos x + e^{-2i\theta + \epsilon})} \right) \times _2F_1 \left( -; \frac{1}{4} \left( 1 - 2e^{-i\theta + \epsilon}) (\cos x + e^{-2i\theta + \epsilon})^2 \right); \nu + 3/2 \right). \tag{5.11}
\]

Next, we apply the formula ([28], p 56):

\[
\left( \frac{1}{2} \right)^{\mu} \left( \mu + 1 \right) \frac{1}{4 \nu^2} = \Gamma (\mu + 1) I_{\nu} (y). \tag{5.12}
\]

for \( \mu = \nu + 1/2 \) and \( y = \left( 1 - 2e^{-i\theta + \epsilon}) (\cos x + e^{-2i\theta + \epsilon}) \right)^{-1} \nu \epsilon e^{-i\theta + \epsilon}) \sin x \). Therefore, equation (5.9) becomes

\[
G(x) = 2^{\nu + 1/2} \left( \nu + \frac{3}{2} \right) \left( 1 - 2e^{-i\theta + \epsilon}) (\cos x + e^{-2i\theta + \epsilon}) \right)^{-1/2} \times \exp \left( \frac{-\nu \epsilon e^{-i\theta + \epsilon}) (\cos x - e^{-i\theta + \epsilon})}{1 - 2e^{-i\theta + \epsilon}) (\cos x + e^{-2i\theta + \epsilon})} \right) \times \left( \nu e^{-i\theta + \epsilon}) (\sin x) (nu + 1/2) \right) \times \left( \frac{\nu \epsilon e^{-i\theta + \epsilon}) (\sin x)}{1 - 2e^{-i\theta + \epsilon}) (\cos x + e^{-2i\theta + \epsilon})} \right). \tag{5.13}
\]
By (5.8)–(5.13), we arrive at

\[
S(x) = \sqrt{F(2(\nu + 1))} \left( dz e^{-(i\theta + \varepsilon/2)} \right)^{-(\nu+1/2)} \sin x \left( 1 - 2e^{-(i\theta + \varepsilon/2)} \cos x + e^{-(2i\theta + \varepsilon)} \right)^{-1/2} \times \exp \left\{ \frac{-iz e^{-(i\theta + \varepsilon/2)}}{1 - 2e^{-(i\theta + \varepsilon/2)} \cos x + e^{-(2i\theta + \varepsilon)}} \right\} I_{\nu + \frac{1}{2}} \left( \frac{iz e^{-(i\theta + \varepsilon/2)}}{1 - 2e^{-(i\theta + \varepsilon/2)} \cos x + e^{-(2i\theta + \varepsilon)}} \right).
\]

(5.14)

which gives the expression (5.1). Now by letting \( \varepsilon \to 0^+ \) in equation (5.1) and by using the normalization factor given by (4.5) we arrive at (5.2). As mentioned above in remark 2.1, when \( \nu = 0 \), the symmetric PT potential becomes the infinite square well potential with eigenfunctions \( \phi_{0,\pi}^\nu(x) \) obtained by setting \( L = \pi \) in (2.8). So that the result (5.3) is deduced by setting \( \nu = 0 \) in (5.2) and applying formulae ([19], p 123):

\[
I_{1/2}(\xi) = \sqrt{\frac{2}{\pi \xi}} \sinh \xi, \quad K_{1/2}(\xi) = \sqrt{\frac{\pi}{2\xi}} e^{-\xi}
\]

(5.15)

to arising Bessel functions. This ends the proof. \( \square \)

6. An integral representation for Laguerre functions

Naturally, once we have obtained a closed form for the constructed coherent states we can look for the associated coherent states transform. In view of (4.1), this transform should map the space \( L^2([0, \pi]) \) spanned by eigenstates \( \{ \phi^\nu_n \} \) in (2.6) onto a subspace of the space of complex-valued square integrable functions with respect to the measure

\[
d\lambda^\nu(z) = \frac{1}{\Gamma(2\nu + 2)} d\mu(z),
\]

(6.1)

where \( d\mu(z) \) is the Lebesgue measure on \( \mathbb{C} \). Recalling that the resolution of the identity, which usually ensures the isometry property of such map, was obtained at the limit \( \varepsilon \to 0^+ \) in (4.18), then a convenient definition for such a transform could be as follows.

**Definition 6.1.** Let \( \nu > -1 \) be a fixed parameter. Define a coherent states transform by the map \( B^\nu : L^2([0, \pi], dx) \to L^2(\mathbb{C}, d\lambda^\nu(z)) \) as

\[
B^\nu [\varphi](z) = \lim_{\varepsilon \to 0^+} \sqrt{\frac{\nu(\nu+1)}{2\varepsilon}} \left( \varphi \right)^{\nu+1/2} d\mu(z),
\]

(6.2)

\[
= (-iz \left| z \right|)^{\nu+1/2} \int_0^\pi \frac{\sin x}{\sqrt{1 - 2e^{i\theta} \cos x + e^{2i\theta}}} \exp \left\{ \frac{-z \left| z \right| \left( \cos x - e^{i\theta} \right)}{1 - 2e^{i\theta} \cos x + e^{2i\theta}} \right\}
\]

\[
\times I_{\nu+\frac{1}{2}} \left( \frac{-iz \left| z \right| \sin x}{1 - 2e^{i\theta} \cos x + e^{2i\theta}} \right) \varphi(x) dx
\]

for every \( z = \left| z \right| e^{i\theta} \in \mathbb{C} \).

In particular, according to (6.2), the image of the basis vector \( \{ \phi^\nu_n \} \) under the transform \( B^\nu \) should exactly be the coefficient in (1.1) with \( \gamma = 2(\nu + 1) \). More precisely
This connection, suggests to us the following result on the Laguerre function

\[
\mathcal{L}_n^{(2\nu+1)}(x) := x^{\nu+1/2}e^{-x/2}L_n^{(2\nu+1)}(x), \quad x \geq 0,
\]

which can be checked by straightforward calculations (see appendix C for the proof).

**Lemma 6.1.** The following integral representation for the Laguerre function

\[
L_n^{(2\nu+1)}(x) = \frac{2^{2\nu+2}}{\sqrt{\pi}} \Gamma(\nu + 1)(n + \nu + 1) \int_0^{+\infty} J_{\nu+1/2}\left(\frac{x}{2}s\right)C_n^{\nu+1}\left(\frac{s^2 - 1}{s^2 + 1}\right)\frac{s^{\nu+3/2}}{(s^2 + 1)^{\nu+2}} ds
\]

(6.5)

holds true.

7. Concluding remarks

While dealing with the problem of a particle evolving under a symmetric PT potential indexed by a parameter \(\nu\), we have introduced new coherent states depending on a number \(\varepsilon > 0\) by replacing the coefficients \(z^n/\sqrt{n!}\) of the canonical coherent states by Laguerre polynomials of order \(\nu - 1\). These \(\varepsilon\)-coherent states solve an \(\varepsilon\)-identity which is a compact and trace class operator representing the thermodynamical quantum density for a Hamiltonian of harmonic type once we interpret the number \(\varepsilon\) as the usual parameter \(\beta = kT\) of statistical physics, \(k_B\) is the Boltzmann constant and \(T\) is the temperature. The \(\varepsilon\)-dependence has enabled us to employ a generating relation for both Laguerre and Gegenbauer polynomials leading to a closed form for the constructed coherent states. In the particular case \(\nu = 0\), we have obtained new coherent states for the Hamiltonian with the infinite square well. At the limit \(\varepsilon \to 0^+\), we have appealed to a result due to Muckenhoop and Stein in order to show that the \(\varepsilon\)-identity operator becomes the identity of the states Hilbert space \(L^2([0, \pi])\). Furthermore, at this limit we were able to define a Bargmann-type transform which maps isometrically \(L^2([0, \pi])\) onto a subspace of a Hilbert space of square integrable functions with respect to a suitable measure on the complex plane (phase space). Clearly, this subspace is spanned by the coefficients (given in terms of Laguerre polynomials) we have chosen in superposing eigenstates of the symmetric PT potential. But it would also be useful to get more precisions on the range of this integral transform. In fact, it is well known that in quantum mechanics, canonical coherent and affine coherent states, and in signal analysis Gabor and wavelet frames have offered an extremely flexible alternative method for describing functions in \(L^2(\mathbb{R})\) and in \(L^2(\mathbb{R}_+);\) however, few methods exist for the description of \(L^2\)-functions on an interval \([a, b]\), see [29]. Thus, we here are presenting a contribution in this direction. By an other side, this transform provides us with an integral representation of Hankel type for Laguerre functions. This constitutes a result which could be added to the literature of orthogonal polynomials and special functions.

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Appendix A.

Proof of proposition 4.1. Using the orthonormality relations of the basis elements (2.6), the scalar product in $H$ between two $\varepsilon$-CS can be written as

$$
\langle z; (\gamma, \nu); \varepsilon \mid w; (\gamma, \nu); \varepsilon \rangle_H = \left( N_{\gamma,\varepsilon}(z) \right)^{1/2} \left( N_{\gamma,\varepsilon}(w) \right)^{1/2} Q_{\gamma,\varepsilon}(z, w),
$$

where

$$
Q_{\gamma,\varepsilon}(z, w) = \sum_{n=0}^{+\infty} \frac{n! \Gamma(\gamma) e^{-nw}}{\Gamma(\gamma + n)} \exp\left[\frac{in \arg z}{2} L_n^{(\gamma - 1)}(zw) - in \arg w L_n^{(\gamma - 1)}(w\overline{w})\right]
$$

(A.2)

$$
= e^{\pi i/2} \Gamma(\gamma) \int_0^{e^\nu} t^{\gamma - 1} \left[ \sum_{n=0}^{+\infty} \frac{n!}{\Gamma(\gamma + n)} \left( te^{i(\arg z - \arg w)n} \right)^n L_n^{(\gamma - 1)}(z\overline{z}) L_n^{(\gamma - 1)}(w\overline{w}) \right] dt.
$$

(A.3)

Next, we make use of the Hille–Hardy formula ([30], p 135):

$$
\sum_{n=0}^{+\infty} \frac{n! u^n}{\Gamma(n + \alpha + 1)} L_n^{(\alpha)}(\xi) L_n^{(\alpha)}(\zeta) = (1 - u)^{-1} \exp\left( -\left(\xi + \zeta\right)u \right) \left(\xi \overline{\xi} u\right)^{-\alpha/2} L_0 \left( \frac{2\sqrt{\xi \overline{\xi} u}}{1 - u} \right).
$$

(A.4)

for parameters $\alpha = \gamma - 1, \xi = z\overline{z}, \zeta = w\overline{w}$ and $u = te^{i(\arg z - \arg w)}$. Therefore, (A.2) reads

$$
Q_{\gamma,\varepsilon}(z, w) = \frac{\Gamma(\gamma) e^{\pi i/2}}{\left|zw\right|^2 e^{i(\arg z - \arg w)}} \int_0^{e^\nu} t^{\gamma - 1} \left( 1 - te^{i(\arg z - \arg w)} \right)^{-1}
$$

$$
\times \exp\left( -\frac{\left|zw\right|^2 + \left|w\right|^2}{1 - te^{i(\arg z - \arg w)}} \right) I_{\gamma - 1} \left( \frac{2\left|zw\right|\left( te^{i(\arg z - \arg w)} \right)^{1/2}}{1 - te^{i(\arg z - \arg w)}} \right) dt.
$$

(A.5)

Finally, by (A.1)–(A.5) we arrive at the announced result (4.3). Next, by putting $z = w$ in (A.1) and using the condition \langle z; (\gamma, \nu); \varepsilon \mid z; (\gamma, \nu); \varepsilon \rangle = 1, we obtain the normalization factor in (4.4). This factor becomes

$$
N_{\gamma}(z) = \Gamma(\gamma) (z\overline{z})^{\gamma - 1} \int_0^{+i} \frac{1}{(1 - t)^{\gamma - 1}} \exp\left( -\frac{2zt}{1 - t} \right) I_{\gamma - 1} \left( \frac{2zt}{1 - t} \right) dt.
$$

(A.6)

at the limit $\varepsilon \to 0^+$. Next, by changing the variable as $\sqrt{t} = \tanh \rho$, equation (A.6) transforms to
\[ \mathcal{N}_f(z) = \Gamma(\gamma)(zz)^{1-\gamma} \int_0^{+\infty} \exp(-z\bar{z}(\cosh \rho - 1))I_{\gamma-1}(z\bar{z} \sinh \rho) d\rho. \]  

(A.7)

Next, recalling the relation between Bessel functions ([31], p 77):

\[ I_{\mu}(u) = e^{-i\mu\pi/2}J_{\mu}(iu), \]  

(A.8)

then (A.7) reads

\[ \mathcal{N}_f(z) = \Gamma(\gamma) e^{(1-\gamma)\bar{z}}(zz)^{1-\gamma} e^{iz\bar{z}} \int_0^{+\infty} \exp(-z\bar{z} \cosh \rho) J_{\gamma-1}(iz\bar{z} \sinh \rho) d\rho. \]  

(A.9)

We now apply the formula ([32], p 363):

\[ \int_0^{+\infty} \coth^{2\kappa} \left( \frac{\mu}{2} \right) e^{-b \cosh \gamma} J_{\kappa}(a \sinh \gamma) dy \]

\[ = \frac{\Gamma \left( \frac{1}{2} - \kappa + \mu \right)}{a\Gamma(2\mu + 1)} M_{\kappa,\mu} \left( \sqrt{a^2 + b^2} - b \right) W_{\kappa,\mu} \left( \sqrt{a^2 + b^2} + b \right). \]  

(A.10)

\[ \Re b > |\Re a| \text{ and } \Re(\mu - \kappa) > -\frac{1}{2}, \] where \( M_{\kappa,\mu}(\cdot) \) and \( W_{\kappa,\mu}(\cdot) \) are Whittaker functions. For parameters \( \kappa = 0, a = iz\bar{z}, b = z\bar{z}, \gamma = \rho \) and \( \mu = (\gamma - 1)/2 \), this gives

\[ \mathcal{N}_f(z) = -i \Gamma(\gamma) \Gamma \left( \frac{1}{2} \right) e^{(1-\gamma)\bar{z}}(zz)^{1-\gamma} e^{iz\bar{z}} M_{0,\frac{1}{2}}(-z\bar{z}) W_{0,\frac{1}{2}}(z\bar{z}). \]  

(A.11)

The Whittaker functions (A.11) can also be expressed in terms of Bessel functions of first and second kind respectively through the relations ([33], p 207):

\[ M_{0,\frac{1}{2}}(\zeta) = \frac{1}{\Gamma \left( \frac{1}{2} + \frac{1}{2} \right)} \sqrt{\pi} I_{\frac{1}{2}}(\zeta), \quad W_{0,\frac{1}{2}}(\xi) = \sqrt{\frac{\xi}{\pi}} K_{\frac{1}{2}} \left( \frac{\xi}{2} \right), \]  

(A.12)

in the case of parameters \( \zeta = -z\bar{z}, \xi = z\bar{z} \) and \( \lambda = \gamma - 1 \). By using the well known property \( I_{\mu}(-\zeta) = e^{i\mu\pi/2}I_{\mu}(\zeta) \), we arrive at the normalization factor in (4.5). Finally, by taking \( \gamma = 2 \), equation (A.11) reduces to

\[ \mathcal{N}(z) = e^{iz\bar{z}}(zz)^{-1/2} I_{\frac{1}{2}} \left( \frac{z\bar{z}}{2} \right) K_{\frac{1}{2}} \left( \frac{z\bar{z}}{2} \right) \]  

(A.13)

which leads to (4.6) by applying the relations in (5.15).

\[ \square \]

Appendix B.

Proof of proposition 4.3. We start from equation (4.17), for \( x \in [0, \pi] \), as follows

\[ \mathcal{O}_x \{ \varphi \}(x) = \sum_{n=0}^{+\infty} e^{-i\mu x} \left\{ \varphi \mid \varphi_n^k \right\}_H \left\{ x \mid \varphi_n^k \right\} \]  

(B.1)

\[ = \sum_{n=0}^{+\infty} e^{-i\mu x} \left( \int_0^{\pi} \varphi(y) \varphi_n^k(y) dy \right) \varphi_n^k(x). \]  

(B.2)
Replacing the \( \{ \phi_n^\nu \} \) by their expressions in (2.6), equation (B.3) becomes
\[
(s \pi)^{-\nu-1} \mathcal{O}_\nu [\varphi](x) = \int_0^\pi \left( \sum_{n=0}^{\infty} \omega_n C_{n+1}(\cos x) (\cos y) \right) h(y) \, dy. \tag{B.4}
\]
where
\[
\omega_n = \Gamma^2(\nu + 1) 2^{2\nu+1} n!(n + \nu + 1) \frac{\pi^\nu}{\pi^\nu}.
\tag{B.5}
\]
\( h(y) := (\sin y)^{-\nu-1} \varphi(y) \) and \( dm_{\nu+1}(y) = (\sin y)^{2(\nu+1)} \, dy \). In the above formal calculations (B.2) and (B.3), reversing the order of summation and integration can be justified in the more general setting of Jacobi polynomials (see for instance ([34], pp 3–4)). We recognize in (B.4) the Poisson kernel for Gegenbauer polynomial ([20], p 25):
\[
P_\nu(x, y) = \sum_{n=0}^{\infty} \omega_n C_n(x) C_n(y). \tag{B.6}
\]
This kernel function can also be written in a closed form by making use of the Bailey formula ([35], p 102). With these notations, equation (B.4) reads
\[
(s \pi)^{-\nu-1} \mathcal{O}_\nu [\varphi](x) = \int_0^\pi P_\nu(x, y) h(y) \, dy. \tag{B.7}
\]
Direct calculations show that \( h \in L^2([0, \pi]; dm_{\nu+1}(y)) \) and its norm is given by
\[
\|h\|_{L^2([0, \pi]; dm_{\nu+1}(y))} = \|\varphi\|_{L^2([0, \pi])}. \tag{B.8}
\]
Therefore, by applying a result due to Muchenhoopt and Stein ([20], theorem 2, (c), p = 2), the right-hand side of (B.7) can be denoted by \( h(x, \nu) \), and leads to the fact
\[
\lim_{\nu \to 0^+} (s \pi)^{-\nu-1} \mathcal{O}_\nu [\varphi](x) = \lim_{\nu \to 0^+} h(x, \nu) = (s \pi)^{-\nu-1} \varphi(x), \text{ a.e.}, \tag{B.9}
\]
which says that the limit \( \mathcal{O}_\nu [\varphi](x) \to \varphi(x) \), a.e. as \( \nu \to 0^+ \) is valid for every \( \varphi \in L^2([0, \pi]) \).
In other words,
\[
\lim_{\nu \to 0^+} \left( \int \langle \varphi; (y, \nu) \rangle; (y, \nu) \rangle; \, dm_{\nu+1}(z) = 1_{L^2([0, \pi])}. \tag{B.10}
\]
This completes the proof. \(\square\)

Appendix C.

Proof of lemma 6.1. We multiply the Laguerre function by \( t^\alpha \) and we use the generating function for the Laguerre polynomials ([30], p 84):
\[
(1 - t)^{-\alpha-1} \exp \left( \frac{-Nt}{1 - t} \right) = \sum_{n=0}^{\infty} t^n L_n^\alpha(t), \, |t| < 1, \, \alpha > -1, \tag{C.1}
\]
to obtain the sum
\[
\sum_{n=0}^{+\infty} t^n L_{n}^{(2\nu+1)}(s) = (1 - t)^{-2\nu-1} s^{\nu+1/2} \exp \left( -\frac{x}{2} \frac{1 + t}{1 - t} \right).
\] (C.2)

On the other hand, we consider the series
\[
S_t = \frac{2^{2\nu+2}}{\sqrt{\pi}} \Gamma(\nu + 1) \sum_{n=0}^{+\infty} (n + \nu + 1) t^n \int_{0}^{+\infty} J_{\nu+1/2} \left( \frac{x}{2} \right) C_n^{\nu+1} \left( \frac{s^2 - 1}{s^2 + 1} \right) \left( \frac{s^{\nu+3/2}}{(s^2 + 1)^{\nu+2}} \right) \, ds
\] (C.3)

which is absolutely convergent. Its sum can be obtained as follows. Reversing the order of summation and integration, gives
\[
S_t = \frac{2^{2\nu+2}}{\sqrt{\pi}} \Gamma(\nu + 1) \times \int_{0}^{+\infty} \left[ \sum_{n=0}^{+\infty} (n + \nu + 1) t^n C_n^{\nu+1} \left( \frac{s^2 - 1}{s^2 + 1} \right) \right] J_{\nu+1/2} \left( \frac{x}{2} \right) \frac{s^{\nu+3/2}}{(s^2 + 1)^{\nu+2}} \, ds.
\] (C.4)

By combining the two generating functions of Gegenbauer polynomials ([30], p 83) and ([36], p 449) respectively:
\[
\sum_{n=0}^{+\infty} t^n C_n^{\lambda}(y) = \left( 1 - 2yt + t^2 \right)^{-\lambda}
\] (C.5)

and
\[
\sum_{n=0}^{+\infty} (n + 2\lambda) t^n C_n^{\lambda}(y) = \frac{2\lambda(1 - ty)}{\left( 1 - 2yt + t^2 \right)^{\lambda+1}}
\] (C.6)

for \( \lambda = \nu + 1 \), we obtain the identity
\[
\sum_{n=0}^{+\infty} (n + \nu + 1) t^n C_n^{\nu+1} \left( \frac{s^2 - 1}{s^2 + 1} \right) = \frac{(\nu + 1)(1 - t^2)}{(1 - t)^2 s^2 + (1 + t)^2}.
\] (C.7)

Substituting the right-hand side of (C.7) into (C.4) gives
\[
S_t = \frac{2^{2\nu+2}}{\sqrt{\pi}} \Gamma(\nu + 2) \frac{(1 - t^2)}{(1 - t)^2 s^2 + (1 + t)^2} \int_{0}^{+\infty} J_{\nu+1/2} \left( \frac{x}{2} \right) s^{\nu+3/2} \, ds,
\] (C.8)

where \( a_t = (1 + t)(1 - t)^{-1} \). The integral in (C.8) can be evaluated with the aid of the formula ([31], p 434):
\[
\int_{0}^{+\infty} J_{\nu} \left( yu \right) \frac{y^{\beta+1}}{(y^2 + a^2)^{\nu+1}} \, dy = \frac{u^{\eta} a^{\beta-\eta}}{2^\nu \Gamma(\nu + 1)} K_{\nu+1/2}(2u),
\] (C.9)

with \(-1 < \Re(\beta) < 2\Re(\eta) + 3/2\). Indeed, for \( \beta = \nu + 1/2, \eta = \nu + 1, u = x/2, y = s \) and \( a = a_t \), in addition to the identity \( K_{\nu+1/2}(\zeta) = \sqrt{\pi} (2\zeta)^{-1/2} e^{-\zeta} \) see ([31], pp 79–80) we get, after some simplifications, that
Thus, for all $|t| < 1$, we have established the equality

$$S_t = (1 - t)^{-2i\nu - 2} x^{t+1/2} \exp\left(-\frac{x}{2} \left(\frac{1 + t}{1 - t}\right)\right).$$

(C.10)

and then, comparing coefficients of identical power of $t$, we obtain the result (6.5). □

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