The Least Wrong Model Is Not in the Data

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Abstract

The true process that generated data cannot be determined when multiple explanations are possible. Prediction requires a model of the probability that a process, chosen randomly from the set of candidate explanations, generates some future observation. The best model includes all of the information contained in the minimal description of the data that is not contained in the data. It is closely related to the Halting Problem and is logarithmic in the size of the data. Prediction is difficult because the ideal model is not computable, and the best computable model is not “findable.” However, the error from any approximation can be bounded by the size of the description using the model.

Introduction

It is impossible to determine the true process that generated data when multiple explanations are possible. Each candidate process could generate different observations in the future, so predicting observations requires more than selecting a single best explanation. A model is needed that can characterize the distribution of potential observations generated by the candidate explanations. If one only considers computable processes, then the Church-Turing Hypothesis tells us that each such process can be described by a Turing machine which is specified by a binary string. If an incorrect Turing machine is used in place of the true process, then the data that the two machines generate will eventually disagree. A measurement of the disagreement or error is the minimal description required to convert a prediction into a future observation. That difference is bounded by the complexity required to convert the
binary string specifying the incorrect Turing machine into the string specifying the correct Turing machine. The Turing machine which is the minimal description of the initial observations has the lowest average error in that it requires the least complexity, on average, to convert from it to any other candidate process. However, it is not a model because it only describes the already observed data, not the distribution of potential observations. The ideal model gives the probability that a process, chosen at random from the set of candidate explanations, generates a potential observation. No computable function can determine this exactly, but it will be shown that the best computable model is the smallest prefix of a minimal description of the data that contains all of the information that is not contained in the observed data. The remaining information in the minimal description was present in the observed data and must be random.

Preliminary Definitions

Let the observed sequence of data be represented by $x$, with $\text{len}(x)$ indicating the number of bits in the representation. The Kolmogorov Complexity of $x$, indicated by $K(x)$ is the length of the shortest bitstring defining a Turing machine that, when given to a reference universal Turing machine, executes and halts leaving only $x$ on the output tape [LV08]. It is a measurement of the intrinsic complexity of $x$. If the data exhibits any regularity or non-independence between different portions of the string, then it is compressible and $K(x) < \text{len}(x)$. If the data is random, then the data is its own shortest description. The minimal program that generates $x$, or the first in lexical order if there are multiple such programs, is indicated by $x^*$, with $\text{len}(x^*) = K(x)$. The conditional Kolmogorov Complexity of a string, indicated by $K(a|b)$ is the length of the shortest program that generates string $a$, possibly using the information in $b$. The programs are all assumed to be “prefix-free” in that no program is a prefix of any other program, and each complexity is defined only to within an $O(1)$ factor to account for variation across different universal Turing machines. As a consequence of the Halting Problem, the Kolmogorov Complexity is not computable. The shortest program that halts after generating some string cannot be determined by any program because the set of programs that halt cannot be defined.

The set of all Turing machines that could have generated the data is indicated by $Y = \{y_i\}$. Each machine $y$ executes and halts after generating $x$ and possibly an additional, subsequent string $w$. The assumption is that only $x$ has been observed at model selection time, and that $w$ is not yet available. If multiple symbols are generated by a program, then some delimiter is used to indicate the end of each symbol.
Minimal Expected Error

If an incorrect program from $Y$ is used instead of the true program that generated the data, then the two programs will eventually disagree. There are many ways to define the error due to the incorrect selection. If we restrict ourselves to symmetrical error functions that satisfy the triangle inequality, then the error can also be interpreted as a distance function. The length of the shortest program that can generate either choice from the other is minimal among all non-degenerate distance functions [LV08, p.641]. Using this definition of error, if $y$ is the correct program and $z$ is the incorrect program, then it can be shown that the error is given by

$$\text{error}(y, z) = \max(\mathbb{K}(y|z), \mathbb{K}(z|y)).$$

In order to identify the program that minimizes the expected error, it is necessary to choose a specific probability distribution over the programs in $Y$. The universal probability distribution $p(\alpha) = 2^{-\mathbb{K}(\alpha)}$ introduced by Ray Solomonoff approximates any computable measure $\mu(\alpha)$ [LV08, p.348][Sol08]. Combining the universal probability with the error definition, the expected error is given by

$$\sum_{y \in Y} \frac{\max(\mathbb{K}(y|z), \mathbb{K}(z|y))}{2^\mathbb{K}(y)}.$$

Because each program $y \in Y$ generates $x$, the conditional complexity $\mathbb{K}(x|y) = 0$. The mutual information, or the information contained in both strings, is $\mathbb{K}(x) - \mathbb{K}(x|y)$. This version of mutual information is only symmetric to within a logarithmic factor, but a related version that conditions on the minimal description is symmetric: $\mathbb{K}(x) - \mathbb{K}(x|y^*) = \mathbb{K}(y) - k(y|x^*)$ [LV08, p.247]. Because $y^*$ includes all of the information in $y$, $\mathbb{K}(x|y^*)$ is also equal to zero, and we can write $\mathbb{K}(x) = \mathbb{K}(y) - k(y|x^*)$. Rearranging, we can write

$$\mathbb{K}(y) = \mathbb{K}(y|x^*) + \mathbb{K}(x).$$

This says that the conditional complexity required to generate each $y$ from $x^*$ is the additional complexity contained in $y$ beyond that contained in $x^*$. Because each $y$ generates $x$, the minimal description of $y$ must contain the minimal description of $x$. Using the mutual information, the expected error expands to

$$\sum_{y \in Y} \frac{\max(\mathbb{K}(y|x^*, z) + \mathbb{K}(x^*|z), \mathbb{K}(z|y))}{2^\mathbb{K}(y|x^*) + \mathbb{K}(x)}.$$

When $z$ includes the information in $x^*$, the $\mathbb{K}(x^*|z)$ term goes to zero for every item in the sum. Adding additional information to $z$ can decrease
some of the $K(y|x^*, z)$ terms, but it also increases the $K(z|y)$ terms when the added information is not contained in $y$. If the programs in $Y$ are independent, then adding more information beyond $x^*$ to $z$ can only reduce the error for an exponentially small fraction of the programs, each weighted by the universal prior, while increasing the error for the remaining programs. The expected error is therefore minimized by $x^*$, the minimal description of the data. With that choice, the expected error becomes

$$\sum_{y \in Y} \max\left(K(y|x^*), K(x^*|y)\right) 2^{K(y|x^*)+K(x)}.$$ 

**Ideal Model**

Although the minimal description of the data minimizes the expected error, it is not a model because it does not characterize the distribution of potential observations. The model needs to estimate the conditional probability $p(w|x)$ of observing a given string $w$ after $x$, which is given by the ratio

$$p(w|x) = \frac{p(xw)}{p(x)}.$$ 

It is possible to again use Solomonoff’s universal prior in place of the unknown probability distribution [LV08, p.350]. However, a perhaps more intuitive argument is to assume that programs are generated entirely randomly by, for example, flipping an unbiased coin [Cha90]. Each coin flip generates a bit that can be appended to a tape. Using the prefix-free assumption, any extra bits can be ignored. Even though there is no complexity prior or selection bias, simple programs are exponentially more likely to be drawn by this process. If $K(\alpha)$ is the minimal number of bits required to describe some string $\alpha$, then the fraction of random strings of length $n$ that start with that minimal description is

$$\frac{2^n - K(\alpha)}{2^n} = 2^{-K(\alpha)}.$$ 

The total fraction of strings that generate $\alpha$ may be larger than that because it includes non-minimal descriptions of $\alpha$. However, the non-minimal descriptions are more complex and, therefore, exponentially less frequent than the minimal description. Therefore, the probability of selecting a program at random that generates $\alpha$ is approximately

$$p(\alpha) = 2^{-K(\alpha)}.$$
Using this same argument, the conditional probability is given by
\[
p(\beta|\alpha) = \frac{p(\alpha \beta)}{p(\alpha)} = \frac{2^{-K(\alpha, \beta)}}{2^{-K(\alpha)}}.
\]

Using the symmetric mutual information, the term \(K(\alpha, \beta)\) expands to \(K(\alpha) + K(\beta|\alpha^*).\) The conditional probability of observing some string \(w\) after observing the data \(x\) is therefore given by
\[
p(w|x) = \frac{2^{-K(x, w)}}{2^{-K(x)}} = \frac{2^{-K(x) - K(w|x^*)}}{2^{-K(x)}} = 2^{-K(w|x^*)}.
\]

The ideal model predicts future observations using \(p(w|x) = 2^{-K(w|x^*)}\). However, it is not a computable function because the conditional complexity \(K(w|x^*)\) is not computable.

**Computable Model**

The ideal model determines the distribution of potential observations generated by candidate programs. That distribution is defined by the length of the shortest program that generates the predicted observation from the minimal description of the observed data: \(K(w|x^*).\) No computable function can always find the shortest program because the set of halting programs cannot be defined. However, if some program can be found that generates \(w\) from \(x^*\), then the length of that program is an upper bound on the length of the minimal program. Let \(h(w)\) indicate the best computable model. If the computable model determines the length of some program that generates \(w\) from \(x^*\), then it must be an upper approximation: \(h(w) >= K(w|x^*).\)

The model function \(h(w)\) returns an integer that is an upper approximation of \(K(w|x^*).\) The best choice of \(h(w)\) is difficult to characterize in this initial and general formulation. However, it is possible to define some additional, yet equivalent, functions for which the optimal solution is apparent. For any choice of \(h\), a function \(g\) can be defined that returns strings that are of length equal to the value returned by \(h:\ g(w) = r_w,\) and \(\text{len}(r_w) = h(w).\) Because there can be no more than \(2^{h(w)}\) programs of a given length \(h(w),\) it is always possible to uniquely assign the strings \(r_w\) returned by \(g(w)\) while preserving the correspondence \(\text{len}(r_w) = h(w).\) When the function \(g(w)\) is
unique, it is invertible. Let the inverse of $g$ be indicated by $f(r_w) = w$. The function $g$ is computable because the function $h$ is computable. This means that $g(w)$ can be expressed as a Turing machine: $[g \ w] \rightarrow r_w$. Because $f(r_w)$ is the inverse of $g$, it is also computable and can be expressed as a Turing machine: $[f \ r_w] \rightarrow w$. For any computable function, the output can be fully determined from the function and the input parameter, so the conditional complexity of the output given the input is zero: $K(r_w|g, w) = 0$ and $K(w|f, r_w) = 0$. Because the inverse of a computable function can be determined from the function by iterating through the input space until a given value is generated, the conditional complexity of the inverse function given the initial function is zero. The statement $K(r_w|f, w) = 0$ follows from the combination of $K(r_w|g, w) = 0$ with $K(g|f) = 0$. Here is a summary of the functions and relations.

\[
\begin{align*}
    h(w) & \geq K(w|x^*) \\
    g(w) & = r_w \\
    f(r_w) & = w \\
    f & = g^{-1} \\
    \text{len}(r_w) & = h(w) \\
    K(r_w|f, w) & = 0
\end{align*}
\]

An object is random when it is incompressible and therefore is its own shortest description: $K(\gamma) = \text{len}(\gamma)$. If an object is compressible, then it must exhibit some regularity or non-randomness. If that regularity can be specified, then the remaining information needed to describe the object is incompressible and the object can be considered conditionally random with respect to the description of the regularity. Just as the minimal information needed to fully describe a random object is contained in the object, so too is the remaining information needed to describe a conditionally random object fully contained in the object. If an object $\gamma$ is conditionally random with $\alpha$, then there is some minimal string $\beta$ of length $K(\gamma|\alpha)$ that can be determined from $\alpha$ and $\gamma$, and that can be combined with $\alpha$ to generate $\gamma$. Because $\beta$ is determined from $\alpha$ and $\gamma$, its conditional complexity is $K(\beta|\alpha, \gamma) = 0$. If $\gamma$ is not random with $\alpha$, then any string $\beta$ that is determined from the data, and for which $[\alpha\beta]$ generates $\gamma$ must be larger than the minimal $K(\gamma|\alpha)$. If $\alpha$ is the smallest string for which $\gamma$ is conditionally random, then $\gamma$ is maximally conditionally random with $\alpha$, and $\text{len}(\alpha) + \text{len}(\beta) = K(\gamma)$.

It will now be shown that the computable model $h(w)$ is a best approximation for the ideal model when $x$ is maximally random with $f$. For notational convenience, let $\gamma \in K(\alpha|\beta)$ indicate that $\gamma$ is a program of length $K(\alpha|\beta)$
that generates α from β. Any string \( x^* \) can be factored into two parts \([a b]\) where \( a \in \hat{K}(x^*|x) \) and \( b \in \hat{K}(x^*|a) \). Because the complete \( x^* \) can be generated from \( x \) and \( a \in \hat{K}(x^*|x) \), and because \( b \) is a component of \( x^* \), it is clear that \( K(b|a, x) = 0 \) and \( K(x|a) = b \). Therefore \( x \) is conditionally random with \( a \in \hat{K}(x^*|x) \). It is also clear that \( K(x^*|x) \) is the length of the smallest string for which \( x \) is conditionally random. By definition, it is the information that is needed to describe \( x^* \) that is not in \( x \), so if any information is removed, any subsequent choice of \( b \) determined from the data would have to exceed the minimal description. Therefore \( x \) is maximally conditionally random with \( a \in \hat{K}(x^*|x) \).

When \( x \) is maximally random with \( f \), if \( w \) is also maximally random with \( f \), then

\[
K(w|x^*) = K(f, r_w|f, r_x) = K(r_w|f) \leq \text{len}(r_w).
\]

With the exception of an exponentially small fraction of the strings, the \( r_w \) and \( r_x \) are independent of each other, and \( K(w|x^*) = \text{len}(r_w) \). The computable model’s approximation of the distribution is given by \( h(w) = \text{len}(r_w) \), and the ideal distribution is equal to \( K(w|x^*) \). Therefore the computable model exactly matches the ideal model for almost all strings that are maximally random with \( f \). Any increase in \( f \) to reduce \( r_w \) in the small fraction of cases where \( K(r_w|f) < K(r_w) \) would lead to an increase in the \( r_w \) for the rest of the maximally random strings in the set because the \( r_w \) must satisfy the Kraft Inequality \( \sum 2^{-\text{len}(r_w)} \leq 1 \). Any decrease in \( f \) would increase all \( r_w \) in the set because each \( w \) would no longer be random with \( f \) and so the \( r_w \) would have to increase by even more than the \( f \) decreased because \( r_w \) would have to grow larger than the conditional complexity. Therefore, the choice of \( f \) is optimal for the set of strings that are maximally random with \( f \).

In fact, the optimality extends to all \( w \) that are random with \( f \), even if they are not maximally random. Using the symmetric mutual information, the fact that \([f r_w]\) generates \( w \) and is incompressible means that if \( w \) is random, but not maximally random, with \( f \), then a portion of \( w \) is included in \( f \), and the rest is defined by \( r_w \):

\[
K(w|f, r_w) = 0
\]

\[
K(w) - K(w|f, r_w) = K(w) = K(f, r_w) - K(f, r_w|w)
\]

\[
K(f, r_w) - K(w) = \text{len}(f) + \text{len}(r_w) - \text{len}(w) = K(f, r_w|w)
\]

Therefore \( K(w|x^*) = K(f, r_w|f, r_x) = K(r_w|r_x) \leq \text{len}(r_w) \), as was the case for strings that were maximally random with \( f \).

When strings are random with \( f \), the approximation is optimal. When strings are not random with \( f \), the approximation defined by \( r_w \) will overestimate the conditional complexity \( K(w|x^*) \). Because most strings are incom-
pressible and independent of \( x \), the best estimate for \( K(w|x^*) \) in those cases is simply \( \text{len}(w) \). Even if strings are compressible, there is no computable function that can be expected to identify the regularity in general. The only regularity that can be exploited by \( f \) is information that has already been identified in \( x^* \). Therefore, the optimality of the approximation for the remaining strings depends on how closely \( r_w \) matches \( \text{len}(w) \). Because \( f(r_w) \) is computable, it corresponds to a set of prefix-free programs that begin with \( f \). The strings \( r_w \) form a prefix-free code tree. Any code paths that are not branching can be compacted to create an equivalent yet shorter codeword. If the function \( f(r_w) \) is also restricted so that each string \( r_w \) corresponds to a unique \( w \), then the set of codewords is incompressible. Given these assumptions, no \( r_w \) can be much larger than \( \text{len}(w) \) because \( \text{len}(r_w) = K(r_w) \), and \( K(r_w|f,w) = 0 \). The approximation of the computable model is therefore optimal for strings that are random with \( f \), and not much worse than any computable function for the remaining strings.

**Model Size and the Halting Problem**

The optimal computable model, which is the best approximation to the distribution of observations, is defined by a program prefix of size \( K(x^*|x) \). This is the information needed to construct the minimal description of the data that is not available from the data. The Kolmogorov Complexity is an undecidable function. It is not possible to run every Turing machine in order of size and select the smallest one that halts after generating the data because no program can determine which other programs will halt. However, if the size of the minimal description is known, a minimal description can be found by running all programs of that size in parallel and selecting the first one that halts after generating the data [LV08, p. 252]. Because the minimal description is no larger than the data, the size of the minimal description can be encoded using \( \log(\text{len}(x)) \) bits when \( \text{len}(x) \) is available. The size of the optimal model is therefore logarithmic with the data:

\[
K(x^*|x) <= \log(\text{len}(x)).
\]

**Kolmogorov’s Randomness Deficiency**

Kolmogorov proposed (but never published) an approach to “nonprobabilistic statistics” that is closely related to this work [VV04]. In his approach, models are defined as Turing machines that enumerate finite sets of objects. Given a model set \( S \), the index of the object within the set can be encoded using \( \log |S| \) bits, referred to as a data-to-model code, and this provides one way
to describe the object. The difference between $\log |S|$ and the conditional complexity is defined as the randomness deficiency: $\log(|S|) - K(x|S)$. When that quantity is small, the object is random or typical with respect to the set and there are no simple properties that distinguish the object from the majority of elements in the set. The best fit model at any complexity level is the model with the lowest randomness deficiency, and the least complex model achieving $K(S) + \log(|S|) = K(x)$ describes all of the regularity of the data. Although the randomness deficiency cannot be approximated and is therefore not a suitable criteria for model selection, it can be shown that a model with minimal $\log(|S|)$ or $\log(|S|) + \alpha$ among all candidate models of complexity less than $\alpha$ also achieves minimal randomness deficiency. Because the data-to-model code is computable, unlike the randomness deficiency, this can be used in an effective procedure for model selection.

In this work, the best model determines the distribution of potential strings that could be generated by the set of candidate programs. The best computable model is the best computable approximation of that distribution. It is interesting to compare the best model according to this criteria to the best model according to Kolmogorov’s Randomness Deficiency criteria. Although sets of objects are not explicitly enumerated by the models in this work, the $r_w$ component is analogous to the data-to-model code in Kolmogorov’s Randomness Deficiency approach. For a given inverse model function $f(r_w) = w$, a Turing machine could be constructed that enumerates the set of $w$ that can be generated from codewords of a given length. The distribution is optimally approximated when $x$ is maximally random with the inverse model function $f$. When $x$ is random with $f$, $\text{len}(r_x) = K(x|f)$. And when it is maximally random, $f$ is the smallest function for which that is true. Therefore $f$ also minimizes the randomness deficiency, and the best estimate of the distribution is also a best model according to Kolmogorov’s Randomness Deficiency criteria.

**Approximation, MDL, and Hillclimbing**

The best computable model may not be in the class of candidate models. Even if it is in the class, there is no way to know when it has been selected because it can be used to construct the minimal description of the data and no computable function can confirm that the minimal description has been selected. Although no strategy for selecting the best computable model can be devised, it is possible to bound the error from any model approximation. That bound can be used to select a model which is the closest approximation to the best computable model.

The best computable model is defined by the inverse model function
If a candidate model is described by $\tilde{\tilde{f}}(\tilde{\tilde{r}}_w) = w$, then the error is at least as large as $K(r_w|\tilde{r}_w)$ because that is the minimal complexity required to generate the correct $r_w$ from the approximate $\tilde{r}_w$. Because $r_w$ is fully determined from $f$ and $x$, the error is no less than $K(f|\tilde{f})$. The best approximation in the model class is the model with minimal $K(f|\tilde{f})$.

To simplify notation, let $A = fr_x$ and let $B = \tilde{f}\tilde{r}_x$. Using the symmetric mutual information, $K(A) - K(A|B*) = K(B) - K(B|A*)$. Because $B$ generates $x$, $K(x) - K(x|B*) = K(x) = K(B) - K(B|x*)$. The minimal description of $x$ is given by $A$, so $A = x*$, and $A* = A$. Therefore $K(B|A*) = K(B) - K(A)$. Combining, this gives

\[
K(A) - K(A|B*) = K(B) - K(B|A*)
\]

\[
K(A) - K(B) + K(B|A*) = K(A|B*)
\]

\[
K(A) - K(B) + (K(B) - K(A)) = K(A|B*)
\]

\[
K(A|B*) = 0
\]

\[
K(A|B) = K(A|B*) + K(B*|B)
\]

\[
K(A|B) = K(B*|B) \leq \log(\text{len}(B))
\]

The last inequality follows from the fact that the minimal description of a string can be generated from the length of the minimal description and the string. This expands to

\[
K(f, r_x|\tilde{\tilde{f}}, \tilde{\tilde{r}}_x) \leq \log(\text{len}(\tilde{\tilde{f}}) + \text{len}(\tilde{\tilde{r}}_x)).
\]

If $\tilde{\tilde{r}}_x$ is the non-compressive component of the candidate model, then it shares no mutual information with $f$, which only describes the compressive regularity of the data. Assuming that constraint, the expression expands to

\[
K(f|\tilde{f}) + K(r_x|\tilde{f}, \tilde{\tilde{r}}_x) \leq \log(\text{len}(\tilde{\tilde{f}}) + \text{len}(\tilde{\tilde{r}}_x))
\]

\[
K(f|\tilde{f}) \leq \log(\text{len}(\tilde{\tilde{f}}) + \text{len}(\tilde{\tilde{r}}_x))
\]

The error is bounded by the sum of the model description and the data description using the model. Minimizing the description length is therefore an effective strategy for selecting the best approximation. This is the strategy advocated by the minimum description length (MDL) principle. However, the error bound depends on the assumption that the model is the only component of the description that is compressive. Minimizing the description length without that constraint does not minimize the error.

Another caveat is that the model description may overestimate the model complexity to such an extent that it does not correctly identify the best
approximation. In practice, models can be very high dimensional. Starting from an initial, poorly performing model, gradient descent or some other hillclimbing strategy is typically applied to find a model that performs better on training data. However, hillclimbing can only be effective if there is mutual information between nearby points in the model space. In general terms, this means that the more amenable a class of models is to optimization, the more poorly its description length serves as a criteria for model selection. When description length is close to model complexity, the class of models is incompressible, and optimization is no better than exhaustive enumeration.

**Probability Models**

There is a direct equivalence between computable models and probability distributions. Every computable model defines a probability distribution, and every computable probability distribution is defined by a computable model. The inverse model functions $f(r_w) = w$ are constructed by prefix-free Turing machines, so the set of $r_w$ codewords is also prefix-free. The Kraft Inequality guarantees that the codewords can be interpreted as a probability distribution: $\sum 2^{-\text{len}(r_w)} \leq 1$. That distribution is the best computable approximation to the distribution of future observations, so it is reasonable to interpret $2^{-\text{len}(r_w)}$ as the probability of observing the string $w$. And every computable probability distribution can be expressed as a set of prefix-free codewords associated with some inverse model function.

Although the two kinds of models are fundamentally equivalent, classical probability models are generally defined over a restricted, finite set of symbols, whereas Turing machines can construct arbitrarily long strings. A string that does not exhibit any frequency regularity for the predefined set of symbols cannot be compressed by a restricted probability model. However, there may be regularity in the data that can be described by some program. For example, a sequence of coin flips that alternate perpetually (HTHTHT ...) exhibits predictable regularity [LV08, P.48]. But the frequency of each symbol is 1/2 and the sample entropy is 1, so the restricted probability model would require N bits to encode N coin flips. Restricting the class of models may exclude the optimal model. As with any restricted class of models, the error is minimized by choosing the model for which the sum of the model complexity and the remaining, non-compressive data complexity is minimal. In the case of probability models, that sum is given by $K(\text{model}) + \log(1/P(\text{data}))$. 
Entropy and Complexity

Important and subtle relationships between model complexity and entropy arise when considering probability models. Entropy measures the average information required to describe symbols from a distribution, independent of the objects associated with those symbols [CT06]. In contrast, Kolmogorov Complexity characterizes the information required to describe individual objects [LV08]. If data can be encoded more compactly using a given probability model, relative to the least complex uniform distribution, then

\[ K(q) + \log(1/q(x)) < \log(1/u(x)) = n \ast \log(m) \]

where \( q \) is the model distribution, and \( u \) is the uniform distribution that assigns equal probability to each of the \( m \) symbols. If \( p \) is the empirical distribution of the data (the frequency of each predefined symbol in the data), then this can be expressed as \( K(q) < n \ast \sum p(i) * \log(q(i)/u(i)) \). The complexity of the model must not exceed the reduction in the complexity due to the use of the model. Simplifying, this becomes

\[
\begin{align*}
K(q) &< n \ast \sum p(i) * \log(1/u(i)) - n \ast \sum p(i) * \log(1/q(i)) \\
&< n \ast \log(m) - n \ast \sum p(i) * \log(1/q(i)) \\
&< n \ast \log(m) - n \ast \sum p(i) * \log(1/p(i)) \\
&< n \ast (H(u) - H(p))
\end{align*}
\]

where \( H \) is the entropy of the distribution, and the Gibb’s Inequality has been used to substitute \( p \) for \( q \) in the third inequality [CT06]. The quantity \( n \ast (H(u) - H(p)) \) is an upper bound on the complexity of any viable probability model and is equal to the Kullback-Leibler divergence between the uniform and empirical distributions, scaled by the amount of data. When the entropy of the data is high, the complexity of the model must be low. The complexity of the model can increase only as the empirical distribution diverges from the uniform distribution.

As the model distribution approaches the empirical distribution, it is reasonable to view the empirical distribution also as an approximation of the model distribution. Although not strictly true, this suggests a roughly inverse relationship between the entropy and the complexity of the model. However, the intuition only applies to selected models, not to candidate models. Jaynes’ Maximum Entropy model selection criteria implies that entropy
and model complexity should have a more fundamental relationship \cite{Jay03}. That relationship is made clear by considering the two-part description of the Turing machine for the model. The total complexity of each symbol generated by the model is

\[ K(q) + \log(1/q(\alpha)) \geq K(q) + K(\alpha | q) \geq K(\alpha). \]

The model complexity must always satisfy

\[ K(q) \geq K(\alpha) - \log(1/q(\alpha)), \]

so the model complexity may need to increase if the codeword decreases. A distribution with maximum entropy has a uniform codeword for each symbol. As the entropy of the distribution decreases, the codewords for some of the symbols become small compared to others. If the symbols are of approximately equal complexity, the difference in the complexity must be shifted into the model. A more general interpretation is that if the entropy or uncertainty required to describe a system decreases, then either some compressible structure has been identified, or the information has been shifted into the model. As most data is not compressible, lower entropy descriptions are generally associated with higher complexity models.

**Conclusion**

This paper analyzes the problem of prediction from first principles. The ideal but uncomputable model, the best computable but “unfindable” model, and the best findable approximating model are each characterized. There is no way to identify the true, hidden process that generated data. However, the probability of any prediction can be defined as the probability of a randomly chosen candidate process generating it. That probability is equal to the uncomputable function \(2^{-K(w|x^*)}\), where \(K(w|x^*)\) is the shortest description of the prediction given the shortest description of the observed data. The best computable approximation to that uncomputable function is a model constructed from a string of length \(K(x^*|x)\) containing all of the regularity of the data that cannot be determined from the data. The best model is closely related to the Halting Problem. The undecidability of the Halting Problem is the reason that the information could not be determined from the data. The size of the best computable model is logarithmic with the data based on its relationship to the Halting Problem. Although the best model is computable, it is “unfindable” because it includes the uncomputable complexity of the data. Even if the correct model has been selected, there is no way to know that it has been selected. Although the best model cannot be found, the error
from using an approximation is bounded by the total description length of the data using the approximating model. The optimal computable model is also a best fit model by Kolmogorov’s Randomness Deficiency criteria, and the best approximate model is consistent with the MDL principle and the Maximum Entropy Principle in the context of probability models.

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