ON SINGULAR SOERGEL BIMODULES

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Abstract. We establish a theory of singular Soergel bimodules which is a generalization of (a part of) Williamson’s theory. We use a formulation of Soergel bimodules developed by the author.

1. Introduction

Attached to a Coxeter system \((W,S)\), we have the Hecke algebra \(H\). By a Hecke category, we mean a categorification of \(H\). Such a category is now playing important roles in representation theory. For example, a tilting character formula for algebraic representations of a reductive group over a positive characteristic field is described in terms of such a categorification [RW18, AMRW19].

There are two types of categorifications of \(H\). The first one is geometric. Assume that \((W,S)\) is the Weyl group of a Kac-Moody group \(G\) over \(\mathbb{C}\) with the Borel subgroup \(B\). Over characteristic zero field, \(B\)-equivariant semisimple complexes on the flag variety gives a Hecke category. However, this category does not work well with a positive characteristic field. The category of parity sheaves, introduced in [JMW14], works well with positive characteristic field and it gives a good geometric Hecke category.

The other type of the categorifications is combinatorial one. Soergel defined such combinatorial version of the categorifications which is now called the category of Soergel bimodules [Soe07]. His theory works well over characteristic zero field, but does not work over positive characteristic field. (To be precisely, his theory needs a representation of \(W\) and the real assumption for his theory is that the representation is reflection faithful. Over a characteristic zero field, we always has a reflection faithful representation. However, for example, when \((W,S)\) is an affine Weyl group, which is interesting case from the viewpoint of modular representation theory, the natural representation arising from a reductive group is not reflection faithful over positive characteristic fields.) The author introduced a generalization of Soergel bimodules and proved that this works well even with positive characteristic field. In particular, it gives a Hecke category [Abe19].

We also mention two other combinatorial categories. Both are defined earlier than [Abe19]. Fiebig introduced a certain full-subcategory of the sheaves on moment graph [Fie08] and proved that this is equivalent to the category of Soergel bimodules over characteristic zero field. He used this category for giving an upper bound for primes \(p\) with which Lusztig conjecture does not hold [Fie12, Fie11]. The other category was introduced by Elias-Williamson [EW16] and it is used for finding a counterexample of Lusztig conjecture [Wil17] and also used for a formulation of a conjecture by Riche-Williamson [RW18].

Even there are several Hecke categories, these are equivalent to each other. In characteristic zero case, the equivalence of semisimple complexes on the flag variety and the category of Soergel bimodules is proved by Soergel and it is used to prove the Koszul duality of the category \(\mathcal{O}\) of complex reductive Lie algebras [Soe90, BGS96]. In a positive characteristic case, under mild assumptions, it is proved that the three combinatorial categories are equivalent to each other in [Abe19] and the equivalence between parity sheaves and the category of Elias-Williamson is proved in [RW18].

The aim of this paper is to give a singular version of these stories. Over a characteristic zero field, Williamson [Wil11] established the theory of singular Soergel bimodules based on Soergel
bimodules. Here, a singular Hecke category categorifies the module $\text{triv}_{S_0} \otimes_{H_{S_0}} H \otimes_{H_{S_1}} \text{triv}_{S_1}$ with $S_0, S_1$ are subsets of $S$ such that the group generated by $S_0, S_1$ are both finite, $H_{S_0}$ is the Hecke algebra attached to the parabolic subgroup for $S_1$ and $\text{triv}_{S_1}$ is the trivial $H_{S_1}$-module.

Geometrically, this corresponds to the category of semi-simple complexes over a generalized flag variety with an action of some Levi subgroup. Since the theory is based on Soergel’s one, it only works with characteristic zero field. So we have the following natural questions.

1. What is a combinatorial singular Hecke category in positive characteristic case?
2. Is a combinatorial singular Hecke category equivalent to the category of parity sheaves on a generalized flag variety?

The aim of this paper is to answer these questions. As we have mentioned in the above, Williamson constructed a category which categorifies $\text{triv}_{S_0} \otimes_{H_{S_0}} H \otimes_{H_{S_1}} \text{triv}_{S_1}$. In this paper, we only consider the case of $S_1 = \emptyset$.

Now we are going to more details. Fix a field $\mathbb{K}$ and a let $(V, \{\alpha_s\}_{s \in S}, \{\alpha^\vee_s\}_{s \in S})$ be a realization over $\mathbb{K}$ [EW16 Definition 3.1]. We can attach $\alpha_t \in V$ (only up to $\mathbb{K}^\times$) for any reflection $t \in W$. Let $R$ be the symmetric algebra of $V$. Let $S_0 \subset S$ be a subset and denote the group generated by $S_0$ by $W_{S_0}$. We assume that

- $\#W_{S_0} < \infty$.
- As a graded $R^{W_{S_0}}$-module, we have $R \simeq \bigoplus_{w \in W_{S_0}} R^{W_{S_0}}(-2\ell(w))$ where $\ell(w)$ is the length of $w$ and $(-2\ell(w))$ is a grading shift.
- For any distinct reflection $t_1, t_2 \in W_{S_0}$, $\{\alpha_{t_1}, \alpha_{t_2}\}$ is linearly independent.

Then we define the category of singular Soergel bimodules $S_0\text{Sbimod}$. We skip the definition in the introduction. See section 2 for the definition. If $S_0 = \emptyset$, then this is the category introduced in [Abe19] and it is showed that the category $\emptyset\text{Sbimod}$ is a categorification of the Hecke algebra. From the definition of the category, we have the following two extra structures:

- A right action of $S\text{bimod} = \emptyset\text{Sbimod}$.
- A grading shift $M \mapsto M(1)$.

Let $[S_0\text{Sbimod}]$ be the split Grothendieck group of $S_0\text{Sbimod}$. By the above two structures, this is a right $H$-module.

**Theorem 1.1** (Theorem 2.1). (1) There is a bijection between indecomposable objects in $S_0\text{Sbimod}$ up to grading shift and $W_{S_0} \backslash W$.

(2) $[S_0\text{Sbimod}] \simeq \text{triv}_{S_0} \otimes_{H_{S_0}} H$.

Next we state a theorem with geometric setting. Let $B \subset G$ be a Kac-Moody group and the Borel subgroup of $G$. Assume that the Weyl group of $G$ is $(W, S)$. Then $S$ corresponds to the set of simple roots. The subset $S_0 \subset S$ defines a standard parabolic subgroup $P_{S_0}$ and the generalized flag variety $S_0X = P_{S_0}\backslash G$. Let $\text{Parity}_B(S_0X)$ be the category of $B$-equivariant parity sheaves on $S_0X$. Let $X$ be the flag variety. Then by the convolution product we have a right action of $\text{Parity}_B(X)$ on $\text{Parity}_B(S_0X)$. Recall that $S_0\text{Sbimod}$ has a right action of $\text{Sbimod}$.

**Theorem 1.2** (Theorem 3.1). We have $S_0\text{Sbimod} \simeq \text{Parity}_B(S_0X)$ which is compatible with the right actions.

When $S_0 = \emptyset$, this can be obtained by combing the results in [RW18, Abe19]. The proof of the above theorem is different from this proof and is closer to the original proof in characteristic zero case.

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2. **Singular Soergel bimodules**

2.1. **Notation.** In this paper, we use the following notation. Let $(W, S)$ be a Coxeter system, $\mathbb{K}$ a noetherian integral domain. The unit element of $W$ is denoted by $e$ and the length function is denoted by $\ell$. We fix a realization $(V, \{(\alpha_s, \alpha^\vee_s)\}_{s \in S})$ over $\mathbb{K}$ [EW16 Definition 3.1].
We assume that $\alpha_s \neq 0$ and $\alpha^\vee_s : V \to K$ is surjective. For each $t = wsw^{-1}$ with $s \in S$ and $w \in W$, we put $\alpha_t = w(\alpha_s)$. This depends on a choice of $(w,s)$ but $K^\vee \alpha_t$ does not depend on $(w,s)$ [Abe-9]. We fix such $(w,s)$ for each $t$ to define $\alpha_t$. Let $R$ be the symmetric algebra of $V$ over $K$ and $Q$ the fractional field of $R$. For a subset $S_0 \subset S$, let $W_{S_0}$ be the group generated by $S_0$, $R_{W_{S_0}}$ the subalgebra of $R$ consisting of $W_{S_0}$-invariants. Each $w \in W_{S_0}$ has the minimal length representative. We denote this representative by $w_{\cdots}$.

We denote the Bruhat order on $W$ by $\leq$. We also define the order $\leq$ on $W_{S_0} \setminus W$ by $x \leq y$ if and only if $x_{\cdots} \leq y_{\cdots}$. We define a topology on $W_{S_0} \setminus W$ as follows: a subset $I \subset W_{S_0} \setminus W$ is open if $y \in I$, $x \in W$, $x \geq y$ implies $x \in I$.

The algebra $R$ is a graded algebra with deg($V$) = 2, here by graded we always mean $\mathbb{Z}$-graded. For a graded $R$-module $M = \bigoplus_{i \in \mathbb{Z}} M^i$, we define a graded $R$-module $M(k)$ by $M(k) = \bigoplus_{i \in \mathbb{Z}} M^i(k)^i$, $M(k)^i = M^{i+k}$. A graded $R$-module $M$ is called graded free if it is isomorphic to $\bigoplus_{i=1}^r R(n_i)$ for some $n_1, \ldots, n_r \in \mathbb{Z}$. Note that in this paper graded free means graded free of finite rank. If $M \cong \bigoplus_{i=1}^r R(n_i)$ is graded free, the graded rank $\text{grk}(M)\mathbb{Z}[v,v^{-1}]$ of $M$ is defined by $\text{grk}(M) = \sum_{i=1}^r v^{n_i}$ where $v$ is an indeterminate.

2.2. A category. Throughout this section, we fix a subset $S_0 \subset S$ such that $W_{S_0}$ is finite and $V$ is faithful as a $W_{S_0}$-representation. Let $Q_{W_{S_0}}$ be the set of $W_{S_0}$-invariants in $Q$.

Lemma 2.1. (1) The algebra $Q_{W_{S_0}}$ is equal to the fractional field of $R_{W_{S_0}}$.

(2) Let $S_1 \subset S_0$. Then the multiplication map $R_{W_{S_1}} \otimes_{R_{W_{S_0}}} Q_{W_{S_0}} \to Q_{W_{S_1}}$ is an isomorphism.

Proof. (1) Let $Q_1$ be the fractional field of $R_{W_{S_0}}$ and $Q_2$ the $W_{S_0}$-invariants of $Q$. Then we have $Q_1 \subset Q_2$. Let $f \in Q_2$ and denote $f = f_1/f_2$ where $f_1, f_2 \in R$. Then we have $f = (\prod_{w \in W_{S_0}} w(f_1))/((\prod_{w \in W_{S_0}, w \neq 1} w(f_1)))$. Since $f$ is $W_{S_0}$-invariant, the denominator is also $W_{S_0}$-invariant. Hence $f \in Q_1$.

(2) Since the map is induced by $R_{W_{S_1}} \hookrightarrow Q_{W_{S_1}}$ with a localization to $Q_{W_{S_0}}$, this is injective. Let $f_1/f_2 \in Q_{W_{S_1}}$ where $f_1, f_2 \in R_{W_{S_1}}$. Then the element $f_1/f_2$ is the image of $(\prod_{w \in W_{S_0}, w \neq 1} w(f_2)) f_1 \otimes (1/(\prod_{w \in W_{S_0}, w \neq 1} w(f_2)))$.

We define the category $S_0\mathcal{C}$ as follows. An object $M$ of $S_0\mathcal{C}$ is a graded $(R_{W_{S_0}}, R)$-bimodule with a decomposition $M \otimes_R Q = \bigoplus_{x \in W_{S_0} \setminus W} M^x_Q$ such that $m = mx^{-1}(f)$ for any $x \in W_{S_0} \setminus W$, $m \in M^x_Q$ and $f \in R_{W_{S_0}}$. A morphism $\varphi : M \to N$ in $S_0\mathcal{C}$ is an $(R_{W_{S_0}}, R)$-bimodule homomorphism $M \to N$ of degree zero such that $\varphi(M^x_Q) \subset N^x_Q$ for any $x \in W_{S_0} \setminus W$. For $n \in \mathbb{Z}$ and $m \in S_0\mathcal{C}$, a graded $(R_{W_{S_0}}, R)$-bimodule $M(n)$ is again an object of $S_0\mathcal{C}$. We put $\text{Hom}_{S_0\mathcal{C}}(M, N) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{S_0\mathcal{C}}(M, N(n))$.

Note that the action of $R$ in both sides coincide with each other. Hence, if $M \to M \otimes_R Q$ is injective, $M$ is a graded $R_{W_{S_0}} \otimes_R R_{W}$-module. We put $\text{supp}_W(M) = \{w \in W_{S_0} \setminus W \mid M^w_Q \neq 0\}$. Set $\mathcal{C} = \gamma\mathcal{C}$.

Remark 2.2. Assume $M \in S_0\mathcal{C}$. Then the action of $0 \neq f \in R_{W_{S_0}}$ on each direct summand $M^x_Q$, hence on $M \otimes_R Q$ is invertible. Therefore $M \otimes_R Q$ is also a left $Q_{W_{S_0}}$-module and we have $M \otimes_R Q \cong Q_{W_{S_0}} \otimes_{R_{W_{S_0}}} M \otimes_R Q$.

One can define an object $R_{\mathcal{C}}$ of $S_0\mathcal{C}$ for $w \in W_{S_0} \setminus W$ as follows. As a right $R$-module, $R_w = R$ and the left action of $f \in R_{W_{S_0}}$ is given by $fr = r^{-1}w^{-1}(f)$ for $r \in R_w$ where the right hand side means the multiplication of $r$ and $w^{-1}(f)$. We put $(R_w)^x_Q = 0$ if $x \neq w$ and $(R_w)^w_Q = Q$.

Let $S_1 \subset S_0$. For $M \in S_1\mathcal{C}$, let $\pi_{S_0, S_1, x}(M)$ be the restriction of $M$ to $(R_{W_{S_0}}, R)$ with

$$\pi_{S_0, S_1, x}(M)^x_Q = \bigoplus_{y \in W_{S_0} \setminus W, \tau = x} M^y_Q$$

where $\tau$ is the image of $y$ in $W_{S_1} \setminus W$. This defines a functor $\pi_{S_0, S_1, x} : S_1\mathcal{C} \to S_0\mathcal{C}$. 
We also define a functor $\pi_{S_0,S_1}^*: S_0C \to S_1C$ as follows. For $M \in S_0C$, we put $\pi_{S_0,S_1}^*(M) = R^{W_{S_1}} \otimes_{R^{W_{S_0}}} M$. Since $M \otimes_R Q$ is a left $Q^{W_{S_0}}$-module, we have

$$\pi_{S_0,S_1}^*(M) \otimes_R Q \simeq R^{W_{S_1}} \otimes_{R^{W_{S_0}}} M \otimes_R Q \simeq R^{W_{S_1}} \otimes_{R^{W_{S_0}}} Q^{W_{S_0}} \otimes Q^{W_{S_0}} M \otimes_R Q \simeq Q^{W_{S_1}} \otimes Q^{W_{S_0}} M \otimes_R Q \simeq \bigoplus_{x \in W_{S_0}} Q^{W_{S_1}} \otimes Q^{W_{S_0}} M_x^{Q_x}.$$

To give a structure of an object in $S_1C$, we use the following lemma.

**Lemma 2.3.** Let $x \in W_{S_0} \setminus W$. The map $f \otimes m \mapsto (my^{-1}(f))_y$ gives an isomorphism $Q^{W_{S_1}} \otimes Q^{W_{S_0}} M_x^{Q_x} \simeq \bigoplus_y M_x^{Q_y}$ where $y$ runs through $y \in W_{S_1} \setminus W$ such that the image of $y$ in $W_{S_0} \setminus W$ is $x$. Moreover, if the image of $m \in Q^{W_{S_1}} \otimes Q^{W_{S_0}} M_x^{Q_x}$ is in only one-component corresponding to $y$, then it satisfies $fm = my^{-1}(f)$ for any $f \in Q^{W_{S_1}}$.

**Proof.** Since $M_x^{Q_x}$ is a direct sum of $(R_x)_Q$, we may assume $M = R_x$. In particular, $M_x^{Q_x} \simeq Q$ as a right $Q$-module and we have $fm = mx^{-1}(f)$ for $f \in Q^{W_{S_0}}$ and $m \in M_x^{Q_x}$. We identify $M_x^{Q_x}$ with $Q$ and denote it by $Q_x$ to emphasis the left $Q^{W_{S_0}}$-module structure.

Fix a representative of $x$ in $W$ and denote it by the same letter $x$. Then the following diagram is commutative:

$$\begin{array}{ccc}
Q^{W_1} \otimes Q^{W_{S_0}} Q_x & \longrightarrow & \bigoplus_y Q_x \\
\downarrow \text{id} \otimes x & \text{ } & \text{ } \\
Q^{W_1} \otimes Q^{W_{S_0}} Q_x & \longrightarrow & \bigoplus_{y_0 \in W_{S_1} \setminus W_{S_0}} Q_x \ni (x(m_{y_0}x))
\end{array}$$

Therefore we may assume $x$ is the unit element $e$.

The representation $V$ of $W_{S_0}$ is faithful by the assumption. Hence $W_{S_0} \to \text{Aut}(Q)$ is injective. Therefore the extension $Q^{W_{S_0}} / Q$ is Galois with the Galois group $W_{S_0}$. Hence the desired isomorphism $Q^{W_{S_1}} \otimes Q^{W_{S_0}} Q \simeq \bigoplus_y Q_y$ follows.

The last part is obvious. □

Let $(\pi_{S_0,S_1}^*, M)_Q$ be the $y$-part in the lemma where $x \in W_{S_0} \setminus W$ is the image of $y$. Then this defines an object of $S_1C$. Let $N \in S_0C$ and $\varphi: M \to N$ be a morphism in $S_0C$. Then $(\text{id} \otimes \varphi)(Q^{W_{S_1}} \otimes Q^{W_{S_0}} M_x^{Q_x}) \subset Q^{W_{S_1}} \otimes Q^{W_{S_0}} N_y^{Q_y}$ for each $x \in W_{S_0} \setminus W$. By the lemma below, $\text{id} \otimes \varphi$ is a morphism $\pi_{S_0,S_1}^*(M) \to \pi_{S_0,S_1}^*(N)$. Hence we get a functor $\pi_{S_0,S_1}^*: S_0C \to S_1C$.

**Lemma 2.4.** Let $x \in W_{S_0} \setminus W$ and assume that for each $y \in W_{S_1} \setminus W$ with the image $x$, $(Q^{W_{S_1}}, Q)$-bimodules $M_x^1$, $M_x^2$ are given. We also assume that $fm = my^{-1}(f)$ for any $f \in Q^{W_{S_1}}$, $m \in M_x^i$ where $i = 1, 2$. Then any $(Q^{W_{S_1}}, Q)$-bimodule homomorphism $\bigoplus_y M_x^1 \to \bigoplus_y M_x^2$ sends $M_x^1$ to $M_x^2$.

**Proof.** As in the proof of the previous lemma, we may assume that $x = e$. Both $\bigoplus_y M_x^1$ and $\bigoplus_y M_x^2$ are $Q^{W_{S_1}} \otimes Q^{W_{S_0}} Q$-bimodules and the homomorphism is a $Q^{W_{S_1}} \otimes Q^{W_{S_0}} Q$-module homomorphism. Let $z \in W_{S_1} \setminus W_{S_0}$. We have $Q^{W_{S_1}} \otimes Q^{W_{S_0}} Q \simeq \bigoplus_y e \in W_{S_1} \setminus W_{S_0} Q$ and define $e_z \in Q^{W_{S_1}} \otimes Q^{W_{S_0}} Q$ such that the image of $e_z$ in $\bigoplus_y Q$ is given by $(\delta_{xy})_y$ where $\delta_{xy}$ is Kronecker’s delta. Then $\varphi(M_x^1) = \varphi(e_z(\bigoplus_y M_x^1)) \subset e_z(\bigoplus_y M_x^2) = M_x^2$. □

**Lemma 2.5.** The pair $(\pi_{S_0,S_1}^*, \pi_{S_0,S_1})$ is an adjoint pair.
Proof. We have $\text{Hom}_{(R^{W_{S_1}})}(R^{W_{S_1}} \otimes R^{W_{S_0}} M, N) \simeq \text{Hom}_{(R^{W_{S_0}})}(M, N)$. We prove that this isomorphism preserves the morphisms in $S_1 \mathcal{C}$ and $S_2 \mathcal{C}$. Assume that $\psi: \pi_{S_0, S_1}^*(M) \to N$ and $\psi: M \to \pi_{S_0, S_1, \ast}(N)$ corresponds to each other by this isomorphism. The correspondence is given by $\psi(m) = \varphi(1 \otimes m)$ and $\varphi(f \otimes m) = f \psi(m)$.

Assume that $\psi$ is a morphism in $S_1 \mathcal{C}$. By the definition, for $m \in M_{Q}^y$ with $x \in W_{S_0} \setminus W$, we have $1 \otimes m \in \bigoplus_{y \in W_{S_1} \setminus W} \varphi(y) \in W_{S_0} \setminus W$ is the image of $y$. Hence $\psi(m) = \varphi(1 \otimes m) \in \bigoplus_{y \in W_{S_1} \setminus W} \varphi(y) = \varphi(S_{0, S_1, \ast}(N))Q_y^y$.

On the other hand, assume that $\psi$ is a morphism in $S_0 \mathcal{C}$. Recall that we have $\pi_{S_0, S_1}^* M \otimes_R Q = \bigoplus_{x \in W_{S_0} \setminus W} Q_{W_1} \otimes_{Q_{W_{S_0}}} M_{Q}^y$ and this decomposition induces $Q_{W_1} \otimes_{Q_{W_{S_0}}} M_{Q}^x \simeq \bigoplus_{y \in W_{S_1} \setminus W} (\pi_{S_0, S_1}^* M_{Q}^y)$. Therefore

$$\varphi \left( \bigoplus_{y \in W_{S_1} \setminus W} (\pi_{S_0, S_1}^* M_{Q}^y) \right) = Q_{W_1} \psi(M_{Q}^y) \subset (\pi_{S_0, S_1, \ast} N)_{Q}^y = \bigoplus_{y \in W_{S_1} \setminus W} N_{Q}^y.$$

By Lemma 2.24, $\varphi$ is a morphism in $S_1 \mathcal{C}$. □

Proposition 2.6. Let $S_2\subset S_1\subset S_0$. Then $\pi_{S_0, S_1, \ast} \circ \pi_{S_1, S_2, \ast} \simeq \pi_{S_0, S_2, \ast}$ and $\pi_{S_1, S_2} \circ \pi_{S_0, S_1} \simeq \pi_{S_0, S_2}$. Proof. The first part is obvious and the second follows from the first and the previous proposition. □

Let $I \subset W_{S_0} \setminus W$ be a subset. For $M \in S_0 \mathcal{C}$, we define $M^I$ be the image of $M \to M \otimes_R Q = \bigoplus_{x \in W} M_{Q}^y \to \bigoplus_{x \in I} M_{Q}^y$ and $M_I$ the inverse image of $\bigoplus_{x \in I} M_{Q}^y \subset \bigoplus_{x \in W} M_{Q}^y$ in $M$. It is easy to see that $M^I \otimes_R Q \simeq M \otimes_R Q \simeq \bigoplus_{x \in I} M_{Q}^y$. Therefore $M_I, M^I \in S_0 \mathcal{C}$. We write $M_w, M^w$ for $M(w), M^w$, respectively. The proof of the following proposition is straightforward.

Proposition 2.7. Let $S_1 \subset S_0$, $\pi: W_{S_1} \setminus W \to W_{S_0} \setminus W$ be the natural projection and $I \subset W_{S_0} \setminus W$. Then we have $\pi_{S_0, S_1, \ast}(M)_I = \pi_{S_0, S_1, \ast}(M_{\pi^{-1}(I)})$, $\pi_{S_0, S_1}^*(M)_I = \pi_{S_0, S_1}^*(M_{\pi^{-1}(I)})$, $\pi_{S_0, S_1, \ast}(M)_I^I = \pi_{S_0, S_1, \ast}(M_{\pi^{-1}(I)})$ and $\pi_{S_0, S_1}^*(M)_I^I = \pi_{S_0, S_1}^*(M_{\pi^{-1}(I)})$.

Let $N \in S_0 \mathcal{C}$ and $M \in \mathcal{C}$. Then we define $N \otimes M \in S_0 \mathcal{C}$ as follows. As an $(R^{W_{S_0}}, R)$-bimodule, $N \otimes M = N \otimes_R M$ and we put

$$(N \otimes M)_w^Q = \bigoplus_{x \in W} N_{Q_{W_{S_0}}}^Q \otimes M_{Q_{w^{-1}}},$$

for $w \in W_{S_0} \setminus W$.

2.3. Soergel bimodules. We introduced the category of Soergel bimodules $S\text{bimod} \subset \mathcal{C}$ in [Abe19]. We recall the definition. For each $s \in S$, set $B_s = R \otimes_R R(1)$. This is a graded $R$-bimodule and this has the unique structure of an object in $\mathcal{C}$ such that $\text{supp}_W(B_s) = \{e, s\}$. We also have $R_e \subset \mathcal{C}$. Then $S\text{bimod}$ is the smallest full-subcategory of $\mathcal{C}$ which contains $\{R_e\} \cup\{B_s\mid s \in S\}$ and closed under $\otimes, (1), (-1), \otimes$ and the direct summand.

In this subsection, we assume the following.

- If $t_1, t_2$ are distinct reflections in $W_{S_0}$ then $\alpha_{t_1}$ and $\alpha_{t_2}$ are linearly independent in a $K/m$-vector space $V/mV$ for any maximal ideal $m \subset K$ (GKM condition).
- As an $R^{W_{S_0}}$-module, we have $R \simeq \bigoplus_{w \in W_{S_0}} R^{W_{S_0}}(-2\ell(w))$.
- [Abe19] Assumption 3.2 holds.

We also assume that $K$ is complete local.

Remark 2.8. If $V$ comes from a root system of a Kac-Moody algebra, then the third assumption are satisfied [Abe20]. If $V|_{W_{S_0}}$ comes from a root datum whose torsion primes are invertible in $K$, the second assumption is satisfied [Dem73].
Remark 2.9. Assume the second condition holds. Since there is no degree zero homomorphism from $R^{W_0}$ to $R^{W_0}(-2f(w))$ except $w \neq 1$, the factor of the right hand side corresponding to $w = 1$ is $R^{W_0} \subset R$. In particular, $R^{W_0} \hookrightarrow R$ splits.

By [Abe19], for each $w \in W$ there exists an indecomposable object $B(w) \in \text{Sbimod}$ such that $\text{supp}_B(B(w)) \subseteq \{x \in W \mid x \leq w\}$ and $B(w)^w \simeq R_w(\ell(w))$. In this subsection, we prove the following. (The author thinks this is well-known, but cannot find the proof in the literature.)

**Proposition 2.10.** Let $w_{S_0}$ be the longest element in $W_{S_0}$. Put $Z_{S_0} = \{(z_w) \in \bigoplus_{w \in W_{S_0}} z_{w_{S_0}} = z_w \ (\text{mod } \alpha_i) \ (t \in T_{S_0})\}$ where $T_{S_0}$ is the set of reflections in $W_{S_0}$. Then we have $B(w_{S_0}) \simeq R \otimes_{R^{W_{S_0}}} R(\ell(w_{S_0})) \simeq Z_{S_0}(\ell(w_{S_0}))$. Here $Z_{S_0}$ is regarded as an $R$-bimodule via $f(z_w)g = (w^{-1}(f)z_w)g$ for $(z_w) \in Z$ and $f, g \in R$.

Replacing $(W, S)$ with $(W_{S_0}, S_0)$, we assume that $S = S_0$. We write $w_0, Z$ for $w_{S_0}, Z_{S_0}$. We have $Z \otimes R Q \simeq \bigoplus_{w \in W} Q$ and from this $Z$ is an object of $C$. We have a map $R \otimes R^w R \to Z$ defined by $f \otimes g \mapsto (w^{-1}(f)g)_{w \in W}$.

**Lemma 2.11.** The above map is injective. Moreover the map is bijective after tensoring $Q$.

**Proof.** By Lemma 2.9, the map is bijective after tensoring $Q$. By the assumption, $R \otimes R^w R$ is free $R$-module, hence torsion-free. Therefore the injectivity of $R \otimes R^w R \to Z$ follows. □

Hence $R \otimes R^w R$ has a structure of an object of $C$. The following lemma follows from the argument in [Sca92, 2.2].

**Lemma 2.12.** For each $w \in W$, there exists $F_w \in R \otimes R_{w_{S_0}} R$ whose image $(z_x) \in Z$ satisfies

1. $z_w$ is of degree $2\ell(w_{S_0})$ non-zero element.
2. If $z_x \neq 0$, then $x \leq w$.

**Lemma 2.13.** Let $I \subset W$ be a closed subset and $w \in I$ a maximal element. Set $I' = I \setminus \{w\}$. We have an embeddings $(R \otimes R^w R)_I/(R \otimes R^w R)_{I'} \hookrightarrow (Z)_I/(Z)_{I'} \hookrightarrow R$ where last map is defined by $(z_x)_{x \in W} \mapsto z_w$. Then the first map is an isomorphism and the image of the second map is $(\prod_{w' > w} \alpha_i)R$ where $t$ runs through reflections in $W$ such that $w t > w$.

**Proof.** Let $M$ be the image of $(R \otimes R^w R)_I/(R \otimes R^w R)_{I'} \hookrightarrow R$. Let $z = (z_x) \in Z_I$. Then since $w$ is maximal in $I$, for any reflection $t \in W$ such that $w t > w$, we have $w t \notin I$. Hence $z_{w t} = 0$. Therefore $z_w \equiv 0 \ (\text{mod } \alpha_i)$. Hence $z_w \in (\prod_{w' > w} \alpha_i)R$. In particular we have $M \subset (\prod_{w' > w} \alpha_i)R$. Take $F \in R \otimes R^w R$ such that the image $z' = \langle z_w \rangle \in Z$ of $F$ satisfies the following: $z' \neq 0$ implies $x \leq w$ and $z'_w \neq 0$ is of degree $2\ell(w_{S_0})$. Hence $z'_w R \subset M$. If $k$ is a field, then since the graded rank of $z'_w R$ and $(\prod_{w' > w} \alpha_i)R$ are equal, we conclude $M = (\prod_{w' > w} \alpha_i)R$. In general, the embedding $M \hookrightarrow (\prod_{w' > w} \alpha_i)R$ is surjective after tensoring $k$ for any maximal ideal $m \subset k$. Hence by Nakayama’s lemma, $M \hookrightarrow (\prod_{w' > w} \alpha_i)R$ is surjective after localizing at $m$ for any maximal ideal $m$ and hence it is surjective. □

**Lemma 2.14.** The map $R \otimes R^w R \to Z$ is an isomorphism.

**Proof.** We prove that for any closed subset $I \subset W$ we have $(R \otimes R^w R)_I \sim (Z)_I$ by induction on $#I$. Let $w \in I$ be a maximal element and set $I' = I \setminus \{w\}$. Then we have the following commutative diagram with exact rows.

\[
\begin{array}{ccc}
0 & \to & (R \otimes R^w R)_{I'} \\
\downarrow & & \downarrow \\
0 & \to & (Z)_{I'}
\end{array}
\quad
\begin{array}{ccc}
0 & \to & (R \otimes R^w R)_{I} \\
\downarrow & & \downarrow \\
0 & \to & (Z)_{I}
\end{array}
\quad
\begin{array}{ccc}
0 & \to & (R \otimes R^w R)_{I}/(R \otimes R^w R)_{I'} \\
\downarrow & & \downarrow \\
0 & \to & (Z)_{I}/(Z)_{I'}
\end{array}
\to 0.
\]

The left vertical map is an isomorphism by inductive hypothesis and the right vertical map is an isomorphism by the above lemma. Hence the middle vertical map is also an isomorphism. □
Next we prove $R \otimes_R W R \in \text{Sbimod}$. Define the map $p_w: R \otimes_R W R \to R$ by $p_w(f \otimes g) = w^{-1}(f)g$. This is a composition of the isomorphism $R \otimes_R W R \simeq Z$ and the projection $Z \to R$ to $w$-part. Fix a reduced expression $w_0 = s_1 \cdots s_{l(w_0)}$.

To prove $R \otimes_R W R(\ell(w_0)) \in \text{Sbimod}$, it is sufficient to prove that this is a direct summand of $B_{s_1} \otimes \cdots \otimes B_{s_{l(w_0)}}$. We construct a morphism $\phi: B_{s_1} \otimes \cdots \otimes B_{s_{l(w_0)}} \to R \otimes_R W R(\ell(w_0))$ which will be correspond to an embedding to the direct summand. The construction will be done inductively, so for each $l$, we construct

$$\phi_l: B_{s_1} \otimes \cdots \otimes B_{s_l} \to R \otimes_R W R(2\ell(w_0) - l)$$

such that $p_{s_1,\ldots,s_l}(\phi_l((1 \otimes \cdots \otimes (1 \otimes 1))) = \prod_{i=1}^{\ell(w_0)} s_{i+1} \cdots s_{i-1}(\alpha_{s_i})$.

First we construct $\phi_0: R \to R \otimes_R W R(\ell(w_0))$. Define an element $z = (z_w) \in Z$ by $z_e = \prod_{i=1}^{\ell(w_0)} s_1 \cdots s_{i-1}(\alpha_{s_i})$ and $z_w = 0$ for $w \neq e$. Then $R \ni f \mapsto fz \in Z(\ell(w_0))$ is a morphism. We define $\phi_0$ as the composition of this morphism with an isomorphism $Z(\ell(w_0)) \simeq R \otimes_R W R(\ell(w_0))$.

We assume that we have already defined $\phi_{l-1}$. Set $M = B_{s_1} \otimes \cdots \otimes B_{s_{l-1}}$, $w = s_1 \cdots s_l$ and $s = s_l$. We define $\phi_l: M \otimes B_s \to R \otimes_R W R(\ell(w_0) - l)$ by

$$M \otimes B_s \xrightarrow{\varphi_l \otimes \text{id}} R \otimes_R W R \otimes_R B_s(\ell(w_0) - l + 1) = R \otimes_R W R \otimes_R R(2\ell(w_0) - l + 2)$$

$$\xrightarrow{f \otimes g \otimes h \mapsto f \otimes \delta_h(g)} R \otimes_R W R(\ell(w_0) - l),$$

here $\delta_h: R \to R$ is defined by $\delta_h(f) = (f - s(f))/\alpha_s$. For simplicity put $u_l = (1 \otimes \cdots \otimes (1 \otimes 1)) \in B_{s_1} \otimes \cdots \otimes B_{s_l}$ and we calculate $\varphi_l(u_l)$. For any $f, g \in R \otimes_R W R$, we have

$$p_w(f \otimes g) = w^{-1}(f)g - s(g) = p_w(f \otimes g) - s(p_{ws}(f \otimes g)).$$

Hence we have

$$\varphi_l(u_l) = p_w(\varphi_{l-1}(u_{l-1})) - s(p_{ws}(\varphi_{l-1}(u_{l-1}))).$$

Since $\text{supp}_W(M) \subset \{ x \in W | x \leq ws_l \}$, we have $w \notin \text{supp}_W(M)$. Therefore the $w$-part of the image of $\varphi_{l-1}$, regarding as an element in $Z$, is zero. Hence $p_w(\varphi_{l-1}(u_{l-1})) = 0$. On the other hand, by inductive hypothesis, we know what $p_{ws}(\varphi_{l-1}(u_{l-1}))$ is. Therefore we get

$$\varphi_l(u_l) = -s_i(\prod_{i=1}^{\ell(w_0)} s_{i+1} \cdots s_{i-1}(\alpha_{s_i}))$$

$$= -s_i(\alpha_{s_i}) \prod_{i=1}^{\ell(w_0)} s_{i+1} \cdots s_{i-1}(\alpha_{s_i}) = \prod_{i=l+1}^{\ell(w_0)} s_{i+1} \cdots s_{i-1}(\alpha_{s_i}).$$

Hence the morphism $\varphi_l$ has the desired property.

Set $N = B_{s_1} \otimes \cdots \otimes B_{s_l}$ and $\varphi = \varphi(\ell(w_0))$. This is a morphism $N \to R \otimes_R W R(\ell(w_0))$ such that $p_{w_0}(\varphi(u_0)) = 1$. Then we have an $R$-bimodule homomorphism $R \otimes R(\ell(w_0)) \to N$ defined by $f \otimes g \mapsto fu_0(g)$. For each $f \in RW$, the left action of $f$ and the right action of $f$ on $N$ is equal. Hence this gives $\psi: R \otimes RW R(\ell(w_0)) \to N$. By Lemma 2.4, this is a morphism in $\mathcal{C}$. Consider $\psi(1 \otimes 1) = \varphi(\alpha_{u_0}) \in R \otimes RW R(\ell(w_0))$. This has the degree $-\ell(w_0)$. On the other hand, the degree $-\ell(w_0)$-part of $R \otimes RW R(\ell(w_0))$ is $\{ c(1 \otimes 1) | c \in K \}$. Hence we have $\varphi(\alpha_{u_0}) = c(1 \otimes 1)$ for some $c \in K$. We have $c = p_{w_0}(c(1 \otimes 1)) = p_{w_0}(\varphi(u_0))$ and this is 1 by the construction of $\varphi$. Hence $c = 1$ and we have $\varphi(\psi(1 \otimes 1)) = 1 \otimes 1$. Since $R \otimes RW R(\ell(w_0))$ is generated by $1 \otimes 1$ as an $R$-bimodule, we get $\varphi \circ \psi = \text{id}$. Hence $R \otimes RW R(\ell(w_0)) = \text{End}_R(\otimes RW R(\ell(w_0)))$ is a direct summand of $N$ and therefore is an object of $\text{Sbimod}$.

**Proof of Proposition 2.4.** We prove $R \otimes_R W R(\ell(w_0)) \simeq B(w_0)$. The object $B(w_0)$ is characterized as the unique indecomposable summand of $B_{s_1} \otimes \cdots \otimes B_{s_{l(w_0)}}$ such that $B(w_0)w_0 \simeq R(\ell(w_0))$. We have $(R \otimes_R W R(\ell(w_0)))w_0 \simeq (Z(\ell(w_0)))w_0 \simeq R(\ell(w_0))$. Therefore it is sufficient to prove that $R \otimes R W R$ is indecomposable. By Lemma 2.4, $\text{End}_\mathcal{C}(R \otimes_R W R) = \text{End}_R(\otimes_R W R)$. 

Proposition 2.17. Let \( M \in S_0 \mathcal{C} \). Then \( \pi_{S_0,0}^*, \pi_{S_0,0}^* (M) \simeq \bigoplus_{w \in W_{S_0}} M (-2(\ell(w))) \).

Proof. We define the category \( S_0 \mathcal{C} \) of singular Soergel bimodules as follows.

Definition 2.16. An object \( M \in S_0 \mathcal{C} \) is in \( S_0 \mathcal{C} \) if and only if there is \( B \in \mathcal{C} \) such that \( M \) is a direct summand of \( \pi_{S_0,0}^* (B) \).

Proposition 2.18. Let \( S_1 \subset S_0 \). Then the functors \( \pi_{S_0,S_1}^* \) and \( \pi_{S_0,S_1}^* \) preserve the singular Soergel bimodules. Namely we have \( \pi_{S_0,S_1}^* (S_1 \mathcal{C}) \subset S_0 \mathcal{C} \) and \( \pi_{S_0,S_1}^* (S_0 \mathcal{C}) \subset S_1 \mathcal{C} \).

Proof. The first part is obvious. We prove the second part. Let \( M \in \mathcal{C} \) and we prove that \( \pi_{S_0,S_1}^* (\pi_{S_0,0}^* (M)) \in S_1 \mathcal{C} \). First we prove this when \( S_1 = \emptyset \). Then we have \( \pi_{S_0,0}^* (\pi_{S_0,0}^* (M)) \simeq R \otimes_{R^w S_0} M = (R \otimes_{R^w S_0} R) \otimes_R M \). By Proposition 2.10 \( R \otimes_{R^w S_0} R \in \mathcal{C} \) and by the construction of \( \pi_{S_0,0}^* \), \( (R \otimes_{R^w S_0} R) \otimes_R M \) is isomorphic to the tensor product in the category \( \mathcal{C} \). Hence \( (R \otimes_{R^w S_0} R) \otimes_R M \in \mathcal{C} \) since \( \mathcal{C} \) is closed under the tensor product. For general \( S_1 \), for \( N \in S_0 \mathcal{C} \), \( \pi_{S_0,S_1}^* (N) \) is a direct summand of \( \pi_{S_1,0}^* (\pi_{S_0,0}^* (N)) \simeq \pi_{S_1,0}^* (\pi_{S_0,0}^* (N)) \) by Lemma 2.15 and since \( \pi_{S_0,0}^* (N) \in \mathcal{C} \), we have \( \pi_{S_1,0}^* (\pi_{S_0,0}^* (N)) \in S_1 \mathcal{C} \). Hence \( \pi_{S_0,S_1}^* (N) \in S_1 \mathcal{C} \).

The following lemma shows a certain projectivity of Soergel bimodules.

Proposition 2.19. Let \( I' \subset I \subset W_{S_0} \backslash W \) be closed subsets. Then for \( M, N \in S_0 \mathcal{C} \), the natural map \( \text{Hom}(M, N_I) \to \text{Hom}(M, N_{I'/I}) \) is surjective.

Proof. We may assume \( N = \pi_{S_0,0}^* (N_0) \) for \( N_0 \in \mathcal{C} \). Let \( \tilde{I} \) be the inverse image of \( I \) by \( W \to W_{S_0} \backslash W \). Then these are also closed and \( N_I = \pi_{S_0,0}^* (\pi_{S_0,0}^* (N_0)) \). By Lemma 2.15 we have the commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(M, N_I) & \longrightarrow & \text{Hom}(M, N_{I'/I}) \\
\downarrow & & \downarrow \\
\text{Hom}(\pi_{S_0,0}^* (M), (N_0)_I) & \longrightarrow & \text{Hom}(\pi_{S_0,0}^* (M), (N_0)_{I'/I}).
\end{array}
\]

Therefore we may assume \( S_0 = \emptyset \).

By induction on \( \#(I \backslash I') \), we may assume \( \#(I \backslash I') = 1 \). We may assume \( M = B_{s_1} \otimes \cdots \otimes B_{s_t} \) for some \( (s_1, \ldots, s_t) \in S^t \). Let \( w \in I \backslash I' \). Fix a reduced expression \( w = t_1 \cdots t_r \) of \( w \). The object \( N_I/N_{I'} \) is isomorphic to a direct sum of objects of a form \( R_w(n) \) with \( n \in \mathbb{Z} \) and since any \( B_{s_1} \otimes \cdots \otimes B_{s_t} \to R_w(n) \) factors through \( B_{t_1} \otimes \cdots \otimes B_{t_r} (n - \ell(w)) \to B_{t_1} \otimes \cdots \otimes B_{t_r} (n - \ell(w)) = R_w(n) \) for some \( k \in \mathbb{Z} \). Hence we may assume \( (s_1, \ldots, s_t) = (t_1, \ldots, t_r) \). This is \( \text{[Ab}19 \] Corollary 3.18 \).
First assume that $I_1, I_2$ are closed. There is a finite closed subset $J \subset W$ which contains $\text{supp}_W(M)$. By replacing $I_i$ with $I_i \cap J$, we may assume that $I_i$ are finite. There exists a sequence of closed subsets $I_1 = J_1 \subset J_2 \subset \cdots \subset J_r = I_2$ such that $\#(J_{i+1} \setminus J_i) = 1$. Since $M_{I_{r+1}}/M_{I_i}$ is free as a right $R$-module [Abel9 Corollary 3.18], we also have that $M_{I_2}/M_{I_1}$ is free.

**Theorem 2.20.**

(1) For each $w \in W_{S_0} \setminus W$, there exists an indecomposable object $\mathfrak{s}_n B(w) \in s_0 S \text{bimod}$ such that $\text{supp}_W(s_0 B(w)) \subset \{ x \in W_{S_0} \setminus W \mid x \leq w \}$ and $s_0 B(w)^w \simeq R_w(\ell(w,-)).$ Moreover $s_0 B(w)$ is unique up to isomorphism.

(2) For any indecomposable object $B \in s_0 S \text{bimod}$ there exists unique $(w, n) \in (W_{S_0} \setminus W) \times \mathbb{Z}$ such that $B \simeq s_0 B(w)^n$.

(3) For any $w \in W_{S_0} \setminus W$, $\pi_{S_0,0}(B(w)) = s_0 B(w) \oplus \bigoplus_{y<w, n \in \mathbb{Z}} s_0 B(y)^m y^n$ for some $m_{y,n} \in \mathbb{Z}_{\geq 0}$.

(4) Let $S_1 \subset S_0$, $w \in W_{S_0} \setminus W$ and $w_{1^+} \in W_{S_0} \setminus W$ the image of the maximal length representative of $w$. Then $\pi_{S_0,0}^* (s_0 (B(w))) \simeq s_1 (B(w_{1^+})) (\ell(w_{1^+}) - \ell(w_0))$.

**Proof.** Let $\pi : W \rightarrow W_{S_0} \setminus W$ be the natural projection. Then since $\pi^{-1}(w) \cap \text{supp}_W(B(w)) = \{w\}$, $\pi_{S_0,0}(B(w)) = s_0 B(w)^w \simeq B(w)_{\pi^{-1}(w)} = B(w)^w - \simeq R_w(\ell(w,-))$. Therefore there exists a unique indecomposable direct summand $s_0 B(w) \in s_0 S \text{bimod}$ of $\pi_{S_0,0}(B(w))$ such that $s_0 B(w)^w \simeq R_w(\ell(w,-))$. This object satisfies the conditions of (1).

For the uniqueness in (1) and (2), (3), it is sufficient to prove that any object $M \in s_0 S \text{bimod}$ is a direct sum of $s_0 B(w)$ where $w \in W_{S_0} \setminus W$ and $n \in \mathbb{Z}$. Let $w \in \text{supp}_W(M)$ be a maximal element and set $I = \{ x \in W_{S_0} \setminus W \mid x \neq w \}$, $I' = I \setminus \{w\}$. Then $I$ and $I'$ are both closed and we have $M_I = M$, $M_I/M_{I'} = M^w$. In particular, $M^w$ is graded free by Proposition 2.19. Hence there exists $n \in \mathbb{Z}$ such that $s_0 B(w)^w \simeq R_n(n + \ell(w,-))$ is a direct summand of $M^w$. Let $i : s_0 B(w)^w \rightarrow M^w$ and $p : M^w \rightarrow s_0 B(w)^w$ be the embedding and the projection from the direct summand, respectively. By Proposition 2.18 there exists $i : s_0 B(w)^w \rightarrow M$ and $p : M \rightarrow s_0 B(w)^w$ which induce $i$ and $p$, respectively. Then $p \circ i$ induces identity on $s_0 B(w)^w$. In particular, it is not nilpotent, hence an isomorphism since $s_0 B(w)^w$ is indecomposable. Therefore $s_0 B(w)^w$ is a direct summand of $M$.

We prove (4). Put $n = \ell(w_{1^+}) - \ell(w_0)$. Let $\pi_0 : W_{S_1} \setminus W \rightarrow W_{S_0} \setminus W$ be the natural projection. Then we have $\text{supp}_W(\pi_{S_0,0}^* s_0 B(w))) = \pi_0^{-1}(\text{supp}_W(s_0 B(w))) \subset \{ x \in W_{S_1} \setminus W \mid x \leq w_{1^+} \}$. We also have $\pi_{S_0,0}^* s_0 B(w)^w \simeq s_0 B(w)^w \simeq R_n(n + \ell(w,-)) = s_1 (B(w_{1^+})) \oplus \ell(w,-) = \ell((w_{1^+}),-) = n$. Hence $s_1 (B(w_{1^+}))(n)$ is a direct summand of $\pi_{S_0,0}^* s_0 B(w)$. Take $M \in s_0 S \text{bimod}$ such that $\pi_{S_0,0}^* s_0 B(w) \simeq s_0 B(w_{1^+})(n) \oplus M$. We prove $M = 0$. Let $w \in W$ be the maximal length representative of $w$. We have $\pi_{S_0,0}^* s_0 B(w)^w = \pi_{S_1,0}^* s_1 B((w_{1^+}))(n) \oplus \pi_{S_1,0}^* s_1 B(M)$ and, by the above argument, $\pi_{S_1,0}^* s_1 B((w_{1^+}))(n)$ has a direct summand $B(\omega_+)(-\ell(w_0))$. Assume that we can prove $\pi_{S_0,0}^* s_0 B(w)^w \simeq B(\omega_+)(-\ell(w_0))$ and $\pi_{S_1,0}^* s_0 B(\omega_+)(n) = 0$. By the definition, it implies $M = 0$. Therefore it is sufficient to prove that $\pi_{S_0,0}^* s_0 B(w)^w \simeq B(\omega_+)(-\ell(w_0))$, namely we may assume $S = 0$.

We have $\pi_{S_0,0}^* s_0 B(w)^w \simeq s_0 B(w)_{\mathfrak{a}(x)}$ and this is one-dimensional if $\mathfrak{a}(x) = w$. On the other hand, we also have that $B(\omega_+)^w_{\mathfrak{a}(x)}$ is one-dimensional if $\mathfrak{a}(x) = w$. Hence $M^w_{\mathfrak{a}(x)} = 0$ if $\mathfrak{a}(x) = w$. Therefore we have $w \notin \text{supp}_W(\pi_{S_0,0}(M))$. The object $\pi_{S_0,0}(s_0 B(w)^w)$ is a direct sum of $s_0 B(w)$. Hence $\pi_{S_0,0}(M)$ is a direct sum of $s_0 B(w)$. Since $w \notin \text{supp}_W(\pi_{S_0,0}(M))$, $\pi_{S_0,0}(M) = 0$ and this implies $M = 0$. □

2.5. **Duality.** For $M \in S_0 C$, we define $D(M) = s_0 D(M) = \text{Hom}_R^*(M, R)$. Here $\text{Hom}_R^*$ is the space of right $R$-homomorphisms. This is a graded $(R^{W_{S_0}}, R)$-bimodule via $(f \cdot g)(m) = \varphi (f(m) g)$ for $f \in R^{W_{S_0}}, g \in R, \varphi \in D(M)$ and $m \in M$. We have $D(M) \otimes_R Q \simeq \text{Hom}_Q(M \otimes_R Q, Q) \simeq \bigoplus_{w \in W_{S_0} \setminus W} \text{Hom}_Q(M^w_{\mathfrak{a}(w)} , Q)$. By putting $D(M^w_{\mathfrak{a}(w)}) = \text{Hom}_Q(M^w_{\mathfrak{a}(w)} , Q)$, we have $D(M) \in s_0 C$. Since any $M \in s_0 S \text{bimod}$ is free as a right $R$-module, we have $D^2(M) \simeq M$. 


Proposition 2.21. Let $M \in s_0 S\text{bimod}$ and $B \in S\text{bimod}$, $D(M \otimes B) \simeq D(M) \otimes D(B)$.

Proof. There is a natural map $D(M) \otimes D(B) \to D(M \otimes B)$ defined by $\varphi \otimes \psi \mapsto ((m \otimes b) \mapsto \varphi(m)\psi(b))$ and since $M$ is a free right $R$-module, this is an isomorphism. It is easy to see that this morphism is a morphism in $s_0 C$.

Proposition 2.22. Let $S_1 \subset S_0$. Then $\pi_{S_0, S_1, r}(S_1 D(M)) \simeq s_0 D(\pi_{S_0, S_1, r}(M))$ for $M \in S_1 C$.

In particular, $s_0 D$ preserves $s_0 S\text{bimod}$.

Proof. This is obvious.

Lemma 2.23. Let $M \in S_0 C$ and $I$ a subset of $W_{S_0} \setminus W$. Then $D(M I) \simeq D(M) I$. If moreover $M$ and $D(M) I$ are free right $R$-modules, then $D(M I) \simeq D(M) I$.

Proof. The first one is easy. To prove the second one, apply the first one to $D(M)$. Then $D(D(M) I) \simeq D(D(M) I) \simeq M I$ because $M$ is a free right $R$-module. If $D(M) I$ is free, then $D(D(M) I) \simeq D(M) I$.

Proposition 2.24. Let $I_1 \subset I_2 \subset W_{S_0} \setminus W$ be both open or both closed. Then for $M \in S_0 S\text{bimod}$, $M_{I_2} / M_{I_1}$ is graded free as a right $R$-module.

Proof. We have proved when $I_1, I_2$ are closed in Proposition [2.19] We assume that $I_1, I_2$ are open and set $J_i = W \setminus I_i$. Then $N = D(M) I_i \simeq D(N I_i) \simeq D(N/N I_i)$. Therefore we have $D(N) I_2 / D(N) I_1 \simeq D(Ker(N/N I_2 \to N/N I_1)) \simeq D(N I_1 / N I_2)$. The right hand side is free since $N I_1 / N I_2$ is free as we have proved.

Remark 2.25. Let $I \subset W_{S_0} \setminus W$ be a closed or an open subset. Then by putting $I_2 = W_{S_0} \setminus W$ and $I_1 = (W_{S_0} \setminus W) \setminus I$, we have $M_{I_2} / M_{I_1} \simeq M I$ is a free right $R$-module.

2.6. Hecke algebras and Hecke modules. Let $[s_0 S\text{bimod}]$ be the split Grothendieck group of $s_0 S\text{bimod}$. Then $[M] [B] = [M \otimes B]$ gives a structure of a right $[S\text{bimod}]$-module on $[s_0 S\text{bimod}]$. By $v[M] = [M(1)]$, $[s_0 S\text{bimod}]$ is a $Z[v, v^{-1}]$-module where $v$ is an indeterminate.

Let $H$ be the Hecke algebra associated to $(W, S)$. Here we use the following definition for $H$: The $Z[v, v^{-1}]$-algebra $H$ is generated by $\{H_s \mid w \in W\}$ with relations: $(H_s - v^{-1}) (H_s + v) = 0$ for $s \in S$ and $H_{w_1 w_2} = H_{w_1} H_{w_2}$ for $w_1 w_2 = \ell(w_1 + \ell(w_2)$.

It is known that $\{H_w \mid w \in W\}$ is a $Z[v, v^{-1}]$-basis of $H$. It is proved in [Abe19] that the map $\text{ch} : [S\text{bimod}] \to H$ defined by $\text{ch}([B]) = \sum_{x \in W} v^{-\ell(x)} \text{grk}(B^x) H_x$ is a $Z[v, v^{-1}]$-algebra isomorphism. Therefore $[s_0 S\text{bimod}]$ is a right $H$-module.

It is straightforward to prove the following.

Lemma 2.26. Let $S_1 \subset S_0$.

1. We have $\pi_{S_0, S_1, r}(M \otimes B) \simeq \pi_{S_0, S_1, r}(M) \otimes B$ for $M \in S_1 C$ and $B \in C$.

2. We have $\pi_{S_0, S_1}^\circ (M \otimes B) \simeq \pi_{S_0, S_1}^\circ (M) \otimes B$ for $M \in s_0 C$ and $B \in C$.

Therefore the functor $\pi_{S_0, S_1, r}$ (resp. $\pi_{S_0, S_1}^\circ$) induces an $[S\text{bimod}] \simeq H$-module homomorphism $[s_0 S\text{bimod}] \to [s_0 S\text{bimod}]$ (resp. $[s_0 S\text{bimod}] \to [s_0 S\text{bimod}]$).

For a subset $S_0 \subset S$, let $H_{S_0}$ be the $Z[v, v^{-1}]$-subalgebra of $H$ generated by $\{H_s \mid s \in S_0\}$. This is isomorphic to the Hecke algebra attached to $(W_{S_0}, S_0)$. The trivial character $\text{triv} : H \to Z[v, v^{-1}]$ is defined by $\text{triv}(H_w) = v^{-\ell(w)}$. The restriction of $\text{triv}$ on $H_{S_0}$ is denoted by $\text{triv}_{S_0}$. Then we have a right $H$-module $\text{triv}_{S_0} \otimes_{H_{S_0}} H$. We define $\text{ch}_{S_0} : [s_0 S\text{bimod}] \to \text{triv}_{S_0} \otimes_{H_{S_0}} H$ by

$$\text{ch}_{S_0}([M]) = \sum_{w \in W_{S_0} \setminus W} 1 \otimes v^{\ell(w)} \text{grk}(M_{\geq w} / M_{> w}) H_w$$

where $M_{\geq w} = M_{\{x \in W_{S_0} \setminus W \mid x \geq w\}}$ and $M_{> w} = M_{\{x \in W_{S_0} \setminus W \mid x > w\}}$. By the lemma below, $\text{ch}$ is the same as $\text{ch}$ defined above.

Lemma 2.27. For $M \in S\text{bimod}$ and $w \in W$, we have $\text{grk}(M^w) = v^{2\ell(w)} \text{grk}(M_{\geq w} / M_{\geq w})$. 
Proof. We have \( \text{grk}(D(M)w) = v^{-2\ell(w)} \text{grk}(D(M)_{\neq w}/D(M)_{\neq w}) \) by [Abe19] Corollary 3.18, Proposition 3.19. Note that, as in Lemma 2.23, \( D(M)w \simeq D(M^w) \) and \( D(M)_{\neq w}/D(M)_{\neq w} \simeq D(M^w_{\neq w})/D(M^w_{\neq w}) = D(M/M^w_{\neq w})/D(M/M^w_{\neq w}) \simeq D(M_{\geq w}/M_{\geq w}) \). Therefore \( \text{grk}(D(M^w)) = v^{-2\ell(w)} \text{grk}(D(M_{\geq w}/M_{\geq w})) \). Hence \( \text{grk}(M^w) = v^{-2\ell(w)} \text{grk}(M_{\geq w}/M_{\geq w}) \).

\[ \square \]

Lemma 2.28. Let \( J_1 \subset J_2, J'_1 \subset J'_2 \) be open subsets of \( W_{S_0}\setminus W \). If \( J_2 \setminus J_1 = J'_2 \setminus J'_1 \), then \( M_{J_2}/M_{J_1} \) and \( M_{J'_2}/M_{J'_1} \) are naturally isomorphic to each other.

Proof. Assume that \( J'_1, J'_2 \subset W_{S_0}\setminus W \) are open subsets such that \( J'_1 \subset J_1, J'_2 \subset J_2 \) and \( J_2 \setminus J_1 = J'_2 \setminus J'_1 \). We prove that the natural homomorphism \( M_{J'_1}/M_{J_1} \to M_{J'_2}/M_{J_2} \) is an isomorphism. We may assume that \( M = \pi_{S_0,0,*}(N) \) for some \( N \in \text{Sbimod} \). Let \( \pi: W \to W_{S_0}\setminus W \) be the natural projection. Then we have \( M_J = N_{\pi^{-1}(J)} \) for \( J = J_1, J_2, J'_1, J'_2 \). Therefore, by replacing \( M, J_1, J_2, J'_1, J'_2 \) with \( N, \pi^{-1}(J_1), \pi^{-1}(J_2), \pi^{-1}(J'_1), \pi^{-1}(J'_2) \), respectively, we may assume \( S_0 = 0 \). We prove the lemma by induction on \( \#(J_2 \setminus J_1) \). If \( \#(J_2 \setminus J_1) = 1 \), then this is [Abe19] Corollary 3.18. Assume that \( \#(J_2 \setminus J_1) > 1 \) and take an open subset \( J_3 \) such that \( J_1 \subset J_3 \subset J_2 \). Set \( J'_3 = J_3 \cap J'_2 \). Then we have \( J_2 \setminus J'_3 = J_2 \setminus J_3, J'_2 \setminus J'_3 = J'_3 \setminus J_1 \) and \( J'_3 \subset J_3 \). By the commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \rightarrow & M_{J'_2}/M_{J'_1} & \rightarrow & M_{J'_2}/M_{J'_3} & \rightarrow & 0 \\
0 \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & \rightarrow & M_{J_2}/M_{J_1} & \rightarrow & M_{J_2}/M_{J_3} & \rightarrow & 0,
\end{array}
\]

we get the lemma.

\[ \square \]

Proposition 2.29. Let \( S_1 \subset S_0 \). We define \( p: \text{triv}_{S_1} \otimes_{H_{S_1}} H \to \text{triv}_{S_0} \otimes_{H_{S_0}} H \) by \( p(1 \otimes h) = 1 \otimes h \). Then we have \( s_0 \text{ch}([\pi_{S_0, S_1,*}(M)]) = p(s_1 \text{ch}([M])) \) for \( M \in S_1\text{Sbimod} \).

Proof. Fix \( w \in W_{S_0}\setminus W \). Denote the image of \( w \) in \( W_{S_1}\setminus W \) by \( \overline{w} \). Let \( \pi: W_{S_1}\setminus W \to W_{S_0}\setminus W \) be the natural projection. Then we have \( (\pi_{S_0, S_1,*}(M))_{\geq w} = M_{\pi^{-1}(\{x \in W_{S_0}\setminus W \mid x \geq w\})} \) and \( (\pi_{S_0, S_1,*}(M))_{> w} = M_{\pi^{-1}(\{x \in W_{S_0}\setminus W \mid x > w\})} \). Then \( \pi^{-1}(\{x \in W_{S_0}\setminus W \mid x \geq w\}) = \{x \in W_{S_1}\setminus W \mid x \geq \overline{w}\} \) and \( \pi^{-1}(\{x \in W_{S_0}\setminus W \mid x > w\}) = \{x \in W_{S_1}\setminus W \mid x > \overline{w}\} \). Take a sequence of open subsets \( \{x \in W_{S_1}\setminus W \mid x \geq \overline{w}\} \setminus W_{S_1}\setminus W_{w-} = J_0 \subset J_1 \subset \cdots \subset J_r = \{x \in W_{S_1}\setminus W \mid x \geq \overline{w}\} \) such that \( \#(J_r \setminus J_{r-1}) = 1 \). Pick \( x_i \in J_i \setminus J_{i-1} \). Then by Lemma 2.28 we have \( M_{J_i}/M_{J_{i-1}} \simeq M_{\geq x_i}/M_{> x_i} \). Therefore

\[
\text{grk}((\pi_{S_0, S_1,*}(M))_{\geq w}/(\pi_{S_0, S_1,*}(M))_{> w}) = \text{grk}(M_{\{x \in W_{S_1}\setminus W \mid x \geq \overline{w}\}}/M_{\{x \in W_{S_1}\setminus W \mid x \geq \overline{w}\}}) = \sum_{i=1}^{r} \text{grk}(M_{\geq x_i}/M_{> x_i}) = \sum_{x \in W_{S_1}\setminus W_{S_0}} \text{grk}(M_{\geq xw_-}/M_{> xw_-}).
\]

Hence

\[
s_0 \text{ch}([\pi_{S_0, S_1,*}(M)]) = \sum_{w \in W_{S_0}\setminus W} 1 \otimes v^{\ell(w_-)} \text{grk}((\pi_{S_0, S_1,*}(M))_{\geq w}/(\pi_{S_0, S_1,*}(M))_{> w})H_{w-} = \sum_{w \in W_{S_0}\setminus W} \sum_{x \in W_{S_1}\setminus W_{S_0}} 1 \otimes v^{\ell(w_-)} \text{grk}(M_{\geq xw_-}/M_{> xw_-})H_{w-}.
\]

For \( w \in W_{S_0}\setminus W \) and \( x \in W_{S_1}\setminus W_{S_0} \), we have \( (xw_-)_- = x_- w_- \). Moreover, since \( \ell(x_- w_-) = \ell(x_-) + \ell(w_-) \), we have \( 1 \otimes v^{\ell(x_- w_-)}H_{(xw_-)_-} = 1 \otimes v^{\ell(x_-)}v^{\ell(w_-)}H_{x_-}H_{w_-} = \text{triv}(H_{x_-}) \otimes v^{\ell(x_-)}v^{\ell(w_-)}H_{w_-} \).
We denote the above map by $H \to H^w$. The split Grothendieck group $Z_{N^1}$ Noriyuki Abe

When $\exists (S, x)$, we put $W_S = Z_{N^1}$. Hence $\pi_0$ is a $Z_{N^1}$-module isomorphism. Define $p$ asi in Proposition 2.25. Obviously $p$ is surjective and an $H$-module homomorphism. Therefore, with the previous lemma and the fact that $\pi_0, \pi_1$ are surjective, $\{\pi_{S_0, H^s}(N) | N \in \text{Sbimod}\}$ generates $[\text{Sbimod}]$ as a $Z[v, v^{-1}]$-module. Therefore, to prove $\pi_0 \circ [\text{Sbimod}] = [\text{Sbimod}]$, we may assume $M = \pi_{S_0, H^s}(N)$. Note that $\pi_0$ is an algebra homomorphism. Hence we have

$$\pi_0 \circ \text{Sbimod} = \pi_0 \circ \text{Sbimod} = \text{Sbimod}.$$ 

We get the theorem. 

By Lemma 2.13, we have $\text{ch}(R \otimes_{S_0} R) = \sum_{v \in W_{S_0}} v^{-l(w)} H_w \in H$. The map $\text{triv}_{S_0} \otimes_{H_{S_0}} H \to H$ defined by $1 \otimes h \mapsto \text{ch}(R \otimes_{S_0} R) h$ is well-defined.

Lemma 2.31. We denote the above map by $i$. Then we have $\text{ch}(\pi_{S_0}^{\ast}(M)) = i(\pi_0 \circ \text{Sbimod}).$

Proof. By Lemma 2.25 and Theorem 2.30, the map $[\text{Sbimod}] \to [S_0, \text{Sbimod}]$ induced by $\pi_{S_0, \ast}$ is surjective. Hence we may assume $M = \pi_{S_0, H^s}(N)$ for some $N \in \text{Sbimod}$. By Theorem 2.29, $\pi_0 \circ \text{Sbimod} = \text{Sbimod}$. Hence $\text{ch}(\pi_{S_0}^{\ast}(M)) = \text{ch}(R \otimes_{S_0} R) \text{ch}(N) = i(1 \otimes \text{ch}(N)) = i(\pi_0 \circ \text{Sbimod}).$ 

We define some notation.

- Let $h \mapsto \overline{h}$ be the $Z$-algebra involution on $H$ defined by $\overline{f(v)} = f(v^{-1})$, where $f(v) \in Z[v, v^{-1}]$. We put $f(v) = f(v^{-1})$.
- The map $a \otimes h \mapsto a \otimes \overline{h}$ is well-defined on $\text{triv}_{S_0} \otimes_{H_{S_0}} H$. We denote this map also by $m \mapsto \overline{m}$.
- Let $\omega : H \to H$ be a $Z$-algebra anti-involution defined by $\omega(\pi_{S_0, H^s}(N) \otimes R) = \pi_{S_0, H^s}(N) \otimes R$. For $m, m' \in \pi_{S_0, H^s}(N)$, take $a_x, b_x \in Z[v, v^{-1}]$ for each $x \in W_{S_0} \setminus W$ such that $m = \sum_x a_x \otimes H_x$ and $m' = \sum_x b_x \otimes H_x$. Then we define $\langle m, m' \rangle H_{S_0} = \sum a_x b_x$.

It is straightforward to see that $\langle mh, m' \rangle H_{S_0} = \langle m, m' \omega(h) \rangle H_{S_0}$ for $m, m' \in \pi_{S_0, H^s}(N)$, $h \in H$. When $S_0 = \emptyset$, we also have $\langle mh, m' \rangle H_{S_0} = \langle m, m' \omega(h) \rangle H_{S_0}$.

Theorem 2.32. For $M, N \in \text{Sbimod}$, the $R$-module $\text{Hom}_{S_0}^{\ast}(M, N)$ is graded free and the graded rank is given by

$$\text{grk} \text{Hom}_{S_0}^{\ast}(M, N) = \langle S_0 \text{ch}(M), S_0 \text{ch}(N) \rangle H_{S_0}.$$
Proof. If $S_0 = \emptyset$, then this is [Abe19, Theorem 4.6].

We prove the general case. By Lemma 2.29 and Theorem 2.30 the map $[\text{Sbimod}] \to [S_0^\bullet \text{Sbimod}]$ induced by $\pi_{S_0,0,*}$ is surjective. Therefore we may assume $M = \pi_{S_0,0,*}(M_0)$ and $N = \pi_{S_0,0,*}(N_0)$ for some $M_0, N_0 \in \text{Sbimod}$. Then

$$\text{Hom}^{*}_{S_0^\bullet \text{Sbimod}}(M, N) \simeq \text{Hom}^{*}\text{bimod}(\pi_{S_0,0,*}(M_0), N_0)$$

is graded free and the graded rank is $\langle \text{ch}(\pi_{S_0,0,*}(M_0)), \text{ch}(N_0) \rangle_{H,0}$. Take $a_x, b_x \in \mathbb{Z}[v, v^{-1}]$ such that $\text{ch}(M_0) = \sum_x a_x H_x$ and $\text{ch}(N_0) = \sum_x b_x H_x$. For each $w \in W_{S_0} \setminus W$, set $a'_w = \sum_{y \in W_{S_0}} v^{-\ell(y)} a_{yw^{-1}}$ and $b'_w = \sum_{y \in W_{S_0}} v^{-\ell(y)} b_{yw^{-1}}$. Then we have $\pi_{S_0,0,*}(M_0) = \sum_{w \in W_{S_0}} a'_w H_{w^{-1}}$. Similarly, we also have $\pi_{S_0,0,*}(M_0) = \sum_{w \in W_{S_0}} b'_w \otimes H_{w^{-1}}$. Hence we get the proposition.

We get the theorem. $\square$

Proposition 2.33. For $M \in S_0^\bullet \text{Sbimod}$, we have $\pi_{S_0,0,*}(D(M)) \simeq \text{ch}(M)$.

Proof. Since $[\text{Sbimod}] \to [S_0^\bullet \text{Sbimod}]$ defined by $[M] \mapsto [\pi_{S_0,0,*}(M)]$ is surjective, we may assume $M = \pi_{S_0,0,*}(M_0)$ for some $M_0 \in \text{Sbimod}$. Let $p: \mathcal{H} \to \text{triv}_{S_0^\bullet \text{Sbimod}} \otimes_{\mathcal{H}^{S_0^\bullet \text{Sbimod}}} \mathcal{H}$ be the map defined by $h \mapsto 1 \otimes h$. Then $\pi_{S_0,0,*}(D(M)) = p(\text{ch}(M_0))$ and $\pi_{S_0,0,*}(D(M)) = p(\text{ch}(M_0))$. Therefore we may assume $S_0 = \emptyset$.

The $\mathbb{Z}[v, v^{-1}]$-algebra $[\text{Sbimod}] \simeq \mathcal{H}$ is generated by $[B_s]$ with $s \in S$. Hence we may assume $M = B_s$. In this case, $D(B_s) \simeq B_s$ and $\text{ch}(B_s) = \mathcal{H}_s = H_s$. Hence we get the proposition. $\square$

3. Parity sheaves and singular Soergel bimodules

3.1. General Notation. For an algebraic variety $X$ with an action of an algebraic group $B$, let $D_B^b(X)$ be the bounded $B$-equivariant derived category of constructible $\mathbb{K}$-coefficient sheaves. Let $f: X \to Y$ be a morphism of algebraic varieties with $B$-actions and assume that $f$ commutes with the $B$-actions. Then we have functors $f^!, f^* : D_B^b(Y) \to D_B^b(X)$ and...
Theorem 3.1. Let \( G \) be a Kac-Moody group over \( \mathbb{C} \) attached to a generalized Cartan matrix. We also have a Borel subgroup \( B \subset G \), the unipotent radical \( U \subset B \) and a Cartan subgroup \( T \subset B \) such that \( B = TU \).

Let \( \Phi \) be the set of roots, \( \Pi \) the set of simple roots and \( W \) the Weyl group. For each \( \alpha \in \Phi \), we have the reflection \( s_\alpha \in W \). The subset \( S = \{ s_\alpha \mid \alpha \in \Pi \} \) gives a structure of a Coxeter system to \( W \). Let \( X^*(T) \) be the character group of \( T \) and set \( V = X^*(T) \otimes \mathbb{Z} \mathbb{K} \). For each \( s = s_\alpha \in S \) with \( \alpha \in \Pi \), we put \( \alpha_s = \alpha \) and \( \alpha_s^\vee = \alpha^\vee \). Then with \( (V, \{ (\alpha_s, \alpha_s^\vee) \}_{\alpha \in S}) \), we have the category of Soergel bimodules \( \mathcal{S} \text{-bimod} \). We say that a subset \( I \subset \Pi \) of finite type if the subgroup of \( W \) generated by \( S_I = \{ s_\alpha \mid \alpha \in I \} \) is finite. We fix such \( I \) and put \( \mathcal{S} \text{-bimod} = S_I \mathcal{S} \text{-bimod}, \ W_I = W_{S_I} \).

With \( I \), we have a parabolic subgroup \( P_I \subset G \). Let \( J \subset I \) be the parabolic flag variety attached to \( I \). We also put \( u = u_I \). For each \( w \in W_I \setminus W \), we have the Schubert variety \( I \times_{\leq w} \subset I \times \) and the Schubert cell \( I \times_{\leq w} \subset I \times_{< w} \). We denote the inclusion maps \( I \times_{\leq w} \to I \times, I \times_{< w} \to I \times \) and \( I \times_{< w} \to I \times \) by \( j_{\leq w}, j_{< w}, j_w \), respectively. If \( J \) is a subset of \( I \), we have the projection \( \pi_{I,J} : J \times \to I \times \). Let \( \text{Parity}_B(I) \subset D^b_B(I) \) the category of \( B \)-equivariant parity sheaves on \( I \times \) with respect to the stratification by Schubert cells \( [\text{JMWM}] \). For each \( w \in W_I \setminus W \), there exists an indecomposable parity sheaf \( \mathcal{F}(w) \) such that \( \text{supp}(\mathcal{F}(w)) \subset I \times_{\leq w} \) and \( \mathcal{F}(w)|_{I \times_{< w}} \cong \mathbb{K}[\ell(w)] \). The functors \( \pi_{I,J} \) and \( \pi_{I,J}^* \) preserve the parity sheaves \( [\text{JMWM}] \).

Throughout this section, we assume the following.

- The torsion primes of \( L_I \) are invertible in \( \mathbb{K} \). (See \([\text{JMWM}] 2.6\).)
- Let \( \alpha, \beta \) be the distinct positive root of \( L_I \). Then \( \{ \alpha, \beta \} \) is linearly independent in \( V/M(V) \) for any maximal ideal \( m \subset \mathbb{K} \).

Let \( \mathcal{F} \in D^b_B(I) \) and \( \mathcal{G} \in D^b_B(X) \). We define the convolution product \( \mathcal{F} \ast \mathcal{G} \in D^b_B(I) \) as follows. Let \( p : G \to I \times \) be the natural projection and \( m : I \times \to I \times \), \( q : I \times \to I \times \) be the action map of \( G \) on \( X \) and the natural projection, respectively. Then there exists unique \( \mathcal{F} \boxtimes p^* \mathcal{G} \in D^b_B(I \times \) such that \( q^*(\mathcal{F} \boxtimes p^* \mathcal{G}) \cong \mathcal{F} \boxtimes p^* \mathcal{G} \). Now we put \( \mathcal{F} \ast \mathcal{G} = m_* (\mathcal{F} \boxtimes p^* \mathcal{G}) \).

If \( \mathcal{F} \in \text{Parity}_B(I) \) and \( \mathcal{G} \in \text{Parity}_B(X) \) then \( \mathcal{F} \ast \mathcal{G} \in \text{Parity}_B(I) \) \([\text{JMWM}] \). Theorem 4.8].

In this section we prove the following. We denote \( \pi_{I,J} \ast \pi_{J,J'} \ast \pi_{J',J} \ast \pi_{J,J'} \ast \pi_{J,J} \ast \pi_{I,J} \) (resp. \( \pi_{I,J}^* \ast \pi_{J,J'} \ast \pi_{J',J} \ast \pi_{J,J'} \ast \pi_{J,J} \ast \pi_{I,J}^* \)) where \( S_I = \{ s_\alpha \mid \alpha \in I \} \). Note that this notation is the same as the push-forward (resp. pull-back) with respect to \( \pi_{I,J} : J \times \to I \times \). The author thinks that we are not confused by this.

**Theorem 3.1.** There exists an equivalence of categories \( \mathcal{I} \mathbb{H} : \text{Parity}_B(I) \to \mathcal{S} \text{-bimod} \). The functor satisfies the following.

1. For \( \mathcal{F} \in \text{Parity}_B(I) \) and \( \mathcal{G} \in \text{Parity}_B(X) \), we have \( \mathcal{I} \mathbb{H}(\mathcal{F} \ast \mathcal{G}) \cong \mathcal{I} \mathbb{H}(\mathcal{F}) \otimes \mathcal{I} \mathbb{H}(\mathcal{G}) \), here we put \( \mathbb{H} = \mathbb{H} \).
2. For \( J \subset I \), we have \( \mathcal{I} \mathbb{H} \circ \pi_{I,J} \ast \pi_{J,J'} \ast \pi_{J',J} \ast \pi_{J,J'} \ast \pi_{J,J} \ast \pi_{I,J} \) and \( \mathcal{I} \mathbb{H} \circ \pi_{I,J} \ast \pi_{J,J'} \ast \pi_{J',J} \ast \pi_{J,J'} \ast \pi_{J,J} \ast \pi_{I,J} \).
3. We have \( \mathcal{I} \mathbb{H} \circ \mathcal{I} \mathbb{H} \cong \mathcal{I} \mathbb{H} \).

The functor \( \mathcal{I} \mathbb{H} \) is given by taking the global sections. We will give the definition in the next subsection.

3.3. The functor \( \mathcal{I} \mathbb{H} \). Let \( \mathcal{F} \in D^b_B(I) \) and we put \( \mathcal{I} \mathbb{H}(\mathcal{F}) = H^*_B(I \times \), \( \mathcal{F} = \bigoplus_{n \in \mathbb{Z}} H^*_B(I \times, \mathcal{F}) \). This is an \( H^*_B(I) \)-module. Recall that \( R = S(V) = S(X^*(T) \otimes \mathbb{Z} \mathbb{K}) \) and \( R^W \) the subalgebra of
We have a natural homomorphism $R^{W_{T}} \otimes_{R} R \simeq H^{*}_{P_{T} \times B}(pt) \rightarrow H^{*}_{P_{T} \times B}(G) \simeq H^{*}_{B}(\mathcal{I}X)$. Hence $\mathcal{I}(\mathcal{F})$ is an $(R^{W_{T}}, R)$-bimodule.

Recall that $Q$ is a fractional field of $R$. Note that we have $H^{*}_{B} = H^{*}_{F_{T}}$. By the localization theorem,

$$\mathcal{I}(\mathcal{F}) \otimes_{R} Q = H^{*}_{T}(\mathcal{I}X^{T}, \mathcal{F}|_{\mathcal{I}X^{T}}) \otimes_{R} Q.$$  

The $T$-fixed points of $\mathcal{I}X^{T}$ is parametrized by $W_{T}|W$. For $w \in W_{T}|W$, we denote the corresponding $T$-fixed point by the same letter $w$. Then we have $\mathcal{F}|_{\mathcal{I}X^{T}} = \bigoplus_{w \in W_{T}|W} \mathcal{F}_{w}$. Therefore we get

$$\mathcal{I}(\mathcal{F}) \otimes_{R} Q = \bigoplus_{w \in W} H^{*}_{T}(\{w\}, \mathcal{F}_{w}) \otimes_{R} Q.$$  

For $f \in R^{W_{T}}$ and $m \in H^{*}_{T}(\{w\}, \mathcal{F}_{w})$, we have $fm = mw^{-1}(f)$. Therefore by putting $\mathcal{I}(\mathcal{F})w = H^{*}_{T}(\{w\}, \mathcal{F}_{w}) \otimes_{R} Q$, we have $\mathcal{I}(\mathcal{F}) \in \mathcal{I}C$.

For $\mathcal{F} \in D^{b}_{B}(I X)$, the complex $\mathcal{R}(\mathcal{I}X, \mathcal{F})$ can be regarded as a $H^{*}_{B}(\mathcal{I}X)$-module. Hence this is a $(R^{W_{T}}, R)$-bimodule. The proof of the following proposition is taken from [BY13 Proposition 3.2.1].

**Proposition 3.2.** Let $\mathcal{F} \in D^{b}_{B}(I X)$ and $\mathcal{G} \in D^{b}_{B}(X)$. Then, as $(R^{W_{T}}, R)$-bimodules, we have $\mathcal{R}(\mathcal{I}X, \mathcal{F}) \otimes_{R} \mathcal{R}(\mathcal{I}X, \mathcal{G}) \simeq \mathcal{R}(\mathcal{I}X, \mathcal{F} \otimes \mathcal{G})$.

\textbf{Proof.} Set $\mathcal{X} = U \backslash G$. Then $T$ acts on $\mathcal{X}$ from the left. Consider the action of $T \times T$ on $\mathcal{X}$ defined by $(t_{1}, t_{2})(x, y) = (xt_{1}^{-1}, t_{2}x)$. The action of $\text{diag}(T) \subset T \times T$ is free and the quotient space is $\mathcal{X}^{T} \times \mathcal{X}$. On this space, we have an action of $(T \times T)/\text{diag}(T)$.

Let $p: \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ be the natural projection. Then there exists a unique object $\mathcal{F} \boxtimes p^{*}\mathcal{G} \in D^{b}_{(T \times T)/\text{diag}(T) \times B}(I X \times \mathcal{X})$ such that $q^{*}(\mathcal{F} \boxtimes \mathcal{G}) \simeq \mathcal{F} \boxtimes p^{*}(\mathcal{G})$ where $q^{*}: I X \times \mathcal{X} \rightarrow I X^{T}$ is the natural projection. By [BY13 Corollary B.4.2], we have $\mathcal{R}(\mathcal{I}X^{T} \times \mathcal{X}, \mathcal{F} \boxtimes p^{*}(\mathcal{G}))$ here $B$ acts on $\mathcal{X}$ from the right. The left hand side is isomorphic to $\mathcal{R}(\mathcal{I}X^{T} \times \mathcal{X}, \mathcal{F} \boxtimes p^{*}(\mathcal{G}) \otimes_{R} H^{*}_{T \times T/\text{diag}(T)}(pt)) \simeq \mathcal{R}(\mathcal{I}X^{T} \times \mathcal{X}, \mathcal{F} \boxtimes p^{*}(\mathcal{G})) \otimes_{R} \mathcal{R}(\mathcal{I}X^{T} \times \mathcal{X}, \mathcal{F} \boxtimes p^{*}(\mathcal{G}))$ where $\mathcal{F} \boxtimes p^{*}(\mathcal{G}) \simeq \mathcal{F} \boxtimes p^{*}(\mathcal{G})$ is a $(\mathcal{I}X^{T} \times \mathcal{X}, \mathcal{F} \boxtimes p^{*}(\mathcal{G}))$-module.

For $s \in S$, let $j_{s}: X_{s} \hookrightarrow X$ be the inclusion map. Since $X_{s} \simeq \mathbb{P}^{1}$, it is easy to see that $j_{s*}\mathbb{K}[1]$ is an indecomposable parity sheaf. Therefore it is isomorphic to $\mathcal{E}(s)$. Hence we have $\mathbb{H}(\mathcal{E}(s)) \simeq H^{*}_{B}(X_{s}, \mathbb{K}[1]) \simeq B_{s}$. Since this is free as a left $R$-module, with the above proposition, we get $\mathcal{R}(\mathcal{I}(\mathcal{F} \otimes \mathcal{E}(s))) \simeq \mathcal{I}(\mathcal{F}) \otimes B_{s}$ as an $(R^{W_{T}}, R)$-bimodules for any $\mathcal{F} \in D^{b}_{B}(I X)$.

**Lemma 3.3.** We have $\mathcal{I}(\mathcal{F}) \otimes B_{s} = \mathcal{I}(\mathcal{F}) \otimes B_{s}$ as objects in $\mathcal{I}C$.

\textbf{Proof.} Set $\mathcal{X} = U \backslash G$. Let $\mathcal{F} \in D^{b}_{B}(I X)$ and $\mathcal{G} \in D^{b}_{B}(X)$. For $x \in W_{T}|W$ and $y \in W_{T}|W$, we fix representatives in $G$ and denote the representatives by the same letter $x, y$. Let $p: G \rightarrow X$, $q: \mathcal{X} \rightarrow X$ be the natural projections and $Y$ the inverse image of $P_{1}xy \in \mathcal{X}$ by the action map $\mathcal{X} \times G \rightarrow \mathcal{X}$. Then we have

$$\mathcal{R}(\mathcal{I}X^{T}(\mathcal{F} \otimes \mathcal{G})P_{1}xy) \simeq \mathcal{R}(\mathcal{I}X^{T}(\mathcal{F} \otimes \mathcal{G})|_{Y})$$
Let \( z_0 \) be the image of \((P_I x, y)\) in \( _I X \times G \). Then \( z_0 \in Y \) and it is fixed by \( T \). Hence we have a natural map 
\[
RT_T(Y, (\mathcal{F} ^{\mathsf{B}} \otimes p^* \mathcal{G})|_Y) \to RT_T(\{z_0\}, (\mathcal{F} ^{\mathsf{B}} \otimes p^* \mathcal{G})|_{z_0}).
\]

Denote the inverse image of \( z_0 \) under \( _I X \times G \to _I X \times G \) by \( Z_2 \) and the image of \( Z_2 \) under \( _I X \times \tilde{X} \) by \( Z_1 \). Then \( U \cong Z_2 \) by \( u \mapsto [(P_I xu^{-1}, uy)] \) and \( Z_1 \) is the image of \( _I X_x \times \{Uy\} \subset X \times \tilde{X} \) in \( _I X \times \tilde{X} \). Therefore \( Z_2 \to \{z_0\} \) and \( Z_2 \to Z_1 \) are fibrations whose fibers are isomorphic to pro-affine spaces. Hence 
\[
RT_T(\{z_0\}, (\mathcal{F} ^{\mathsf{B}} \otimes p^* \mathcal{G})|_{z_0}) \cong RT_T(Z_1, (\mathcal{F} ^{\mathsf{T}} \otimes q^* \mathcal{G})|_{Z_1})
\]

Let \( z_1 \) be the image of \((P_I x, uy)\) in \( _I X \times \tilde{X} \) in \( _I X \times \tilde{X} \). This is a \((T \times T)/\text{diag}(T)\)-fixed point and containing \( Z_1 \). Hence we have a natural morphism 
\[
RT_T(Z_1, (\mathcal{F} ^{\mathsf{T}} \otimes q^* \mathcal{G})|_{Z_1}) \to RT_T(\{z_1\}, (\mathcal{F} ^{\mathsf{T}} \otimes q^* \mathcal{G})|_{z_1})
\]

As in the proof of the above proposition, we have 
\[
RT_T(\{z_1\}, (\mathcal{F} ^{\mathsf{T}} \otimes q^* \mathcal{G})|_{z_1}) \cong RT_T((T \times T)/\text{diag}(T)) \times T(\{z_1\}, (\mathcal{F} ^{\mathsf{T}} \otimes q^* \mathcal{G})|_{z_1}) \otimes H^*_T(\text{diag}(pt)) \mathbb{K}.
\]

Let \( Z \) be the inverse image of \( z_1 \) by \( _I X \times \tilde{X} \to _I X \times \tilde{X} \). Then \( Z \cong \{P_I x\} \times Z_0 \) where \( Z_0 = \{Uy \mid t \in T\} \) and \( Z \to \{z_1\} \) is a \( T \)-torsor. We have 
\[
RT_T((T \times T)/\text{diag}(T)) \times T(\{z_1\}, (\mathcal{F} ^{\mathsf{T}} \otimes q^* \mathcal{G})|_{z_1}) \otimes H^*_T(\text{diag}(pt)) \mathbb{K}
\]
\[
\cong RT_T(X, (\mathcal{F} ^{\mathsf{T}} \otimes q^* \mathcal{G})) \otimes_R RT_T(Z_0, q^* \mathcal{G}) \otimes_R RT_T(\{B_y\}, \mathcal{G}_B y)
\]

since \( T \) acts on \( Z_0 \) freely. Hence we get a map 
\[
RT_T((\mathcal{F} \otimes \mathcal{G})|_{P_I x y}) \to RT_T(x, \mathcal{F}_x) \otimes_R RT_T(y, \mathcal{G}_y).
\]

By the construction, the following diagram is commutative:

\[
\begin{array}{ccc}
RT_B(X, \mathcal{F} \otimes \mathcal{G}) & \sim & RT_B(I X, \mathcal{F}) \otimes_R RT_B(X, \mathcal{G}) \\
\downarrow & & \downarrow \\
RT_T(\{P_I x y\}, (\mathcal{F} \otimes \mathcal{G})|_{P_I x y}) & \to & RT_T(\{x\}, \mathcal{F}_x) \otimes_R RT_T(\{y\}, \mathcal{G}_y).
\end{array}
\]

Now let \( \mathcal{G} = \mathcal{E}(s) \). Then both \( H^*(\mathcal{G}) \) and \( H^*_T(\{y\}, \mathcal{G}_y) \) are free as a left \( R \)-module. Therefore, by taking the cohomology and tensoring \( Q \), we get the following commutative diagram:

\[
\begin{array}{ccc}
j_H(\mathcal{F} \otimes \mathcal{G})_Q & \sim & i_H(\mathcal{F})_Q \otimes_R H(\mathcal{G})_Q \\
\downarrow & & \downarrow \\
j_H(\mathcal{F} \otimes \mathcal{G})^x_y & \to & j_H(\mathcal{F})_Q^x_y \otimes_R H(\mathcal{G})_Q^y.
\end{array}
\]

Therefore, for each \( w \in W \), we have 
\[
\begin{array}{ccc}
j_H(\mathcal{F} \otimes \mathcal{G})_Q & \sim & i_H(\mathcal{F})_Q \otimes_R H(\mathcal{G})_Q \\
\downarrow & & \downarrow \\
j_H(\mathcal{F} \otimes \mathcal{G})_Q^w & \to & \bigoplus_{x = w} i_H(\mathcal{F})_Q^x \otimes_Q H(\mathcal{G})_Q^y \hspace{1cm} (i_H(\mathcal{F}) \otimes H(\mathcal{G}))_Q^w.
\end{array}
\]

Hence the isomorphism \( j_H(\mathcal{F} \otimes \mathcal{E}(s)) \cong j_H(\mathcal{F}) \otimes_R B_s \) is an isomorphism in \( j^* \mathcal{C} \).
\[\square\]
Proposition 3.4. Let $J \subset I$ be a subset. Then we have $j^\mathbb{H} \circ \pi_{I,J,*} \simeq \pi_{I,J,*} \circ j^\mathbb{H}$.

Proof. Let $\mathcal{F} \in D^b_B(jX)$. Since $H^p_B(jX, \pi_{I,J,*}) \simeq H^p_B(jX, \mathcal{F})$, we have $j^\mathbb{H}(\pi_{I,J,*}(\mathcal{F})) \simeq \pi_{I,J,*}(j^\mathbb{H}(\mathcal{F})) \text{ as } (R^{W_I}_1, R)$-bimodules. For each $w \in W_I \setminus W$, by the localization theorem, we have

$$H^p_T(\{w\}, \pi_{I,J,*}(\mathcal{F})) \otimes_R Q \simeq H^p_T(\pi_{I,J,*}(\{w\}), \mathcal{F}|_{\pi_{I,J,*}(\{w\})}) \otimes_R Q \simeq H^p_T((\pi_{I,J,*}(\{w\}))^T, \mathcal{F}|_{(\pi_{I,J,*}(\{w\}))^T}) \otimes_R Q.$$ 

Since $(\pi_{I,J,*}(\{w\}))^T = \{x \in W_I \setminus W | \varpi = w\}$ where $\varpi$ is the image of $x$ in $W_I \setminus W$, we have

$$H^p_T((\pi_{I,J,*}(\{w\}))^T, \mathcal{F}|_{(\pi_{I,J,*}(\{w\}))^T}) \otimes_R Q \simeq \bigoplus_{x \in W_I \setminus W, \varpi = w} H^p_T(\{x\}, \mathcal{F}|_{(x)}) \otimes_R Q = (\pi_{I,J,*} j^\mathbb{H}(\mathcal{F}))^w.$$ 

Therefore $j^\mathbb{H}(\pi_{I,J,*}(\mathcal{F})) \simeq \pi_{I,J,*}(j^\mathbb{H}(\mathcal{F}))$ in $j^\mathbb{C}$.

Corollary 3.5. We have $j^\mathbb{H} (\text{Parity}_B(jX)) \subset j^\text{Sbimod}$.

Proof. First we assume that $I = \emptyset$. Any object in $\text{Parity}_B(X)$ is a direct summand of a direct sum of objects of a form $E(s_1) \cdots E(s_l)[n]$ where $s_1, \ldots, s_l \in S$ and $n \in \mathbb{Z}$. Hence the corollary follows from Lemma 3.3.3.

In general, let $\mathcal{F} \in \text{Parity}_B(jX)$. We may assume that $\mathcal{F} \simeq j^E(w)$ for some $w \in W_I \setminus W$. The object $\pi_{I,\emptyset,*}(E(w)[n])$ is a parity sheaf. By the support estimating, $\pi_{I,\emptyset,*}(E(w)[n])$ contains $j^E(w)[n]$ as a direct summand for some $l \in \mathbb{Z}$. Hence $j^\mathbb{H}(j^E(w))$ is a direct summand of $\pi_{I,\emptyset,*}(\mathbb{H}(E(w)[n][l])) \simeq \pi_{I,\emptyset,*}(j^\mathbb{H}(E(w)[n][l]))$. Since $j^\mathbb{H}(E(w)[n][l]) \in \text{Sbimod}$, we have $j^\mathbb{H}(j^E(w)) \in j^\text{Sbimod}$. 

Corollary 3.6. For $\mathcal{F} \in \text{Parity}_B(jX)$ and $\mathcal{G} \in \text{Parity}_B(X)$, we have $j^\mathbb{H}(\mathcal{F} \ast \mathcal{G}) \simeq j^\mathbb{H}(\mathcal{F}) \otimes j^\mathbb{H}(\mathcal{G})$.

Proof. Since $\mathbb{H}(\mathcal{G}) \in \text{Sbimod}$ is free as a left $R$-module, the same proof of Lemma 3.3.3 can apply.

Lemma 3.7. For $\mathcal{F} \in \text{Parity}_B(jX)$, we have $j^\mathbb{H}(D(\mathcal{F})) \simeq D(j^\mathbb{H}(\mathcal{F}))$.

Proof. This follows the equivariant Poincaré duality and the freeness of $j^\mathbb{H}(\mathcal{F})$ as a right $R$-module.

3.4. Proof of Theorem 3.1. Let $Z$ be a closed subset of $jX$ which is a union of Schubert cells and $U = jX \setminus Z$. By putting $\mathcal{F}$ to be the constant sheaf in [JMW14, Corollary 2.9], for $j: U \hookrightarrow jX$ and $i: Z \hookrightarrow jX$, the sequence

$$0 \to j^\mathbb{H}(i_*i^!\mathcal{G}) \to j^\mathbb{H}(\mathcal{G}) \to j^\mathbb{H}(j_*j^!\mathcal{G}) \to 0$$

is exact for a $!$-parity sheaf $\mathcal{G} \in D^b_B(jX)$.

Remark 3.8. [JMW14] Corollary 2.9] only states that $j^\mathbb{H}(\mathcal{G}) \to j^\mathbb{H}(j_*j^!\mathcal{G})$ is surjective. The long exact sequence attached to the triangle $i_*i^!\mathcal{G} \to \mathcal{G} \to j_*j^!\mathcal{G} \overset{+1}{\to}$ implies that the kernel is isomorphic to $j^\mathbb{H}(i_*i^!\mathcal{G})$.

Since $\mathcal{G}$ is a $!$-parity sheaf, $j^!\mathcal{G}$ and $i^!\mathcal{G}$ are $!$-parity. Hence by [JMW14, Proposition 2.6], we have a (non-canonical) isomorphism $j^\mathbb{H}(i_*i^!\mathcal{G}) \simeq \bigoplus_{w \in W_I \setminus W, X_w \subset Z} H^p_B(j^!\mathcal{G}).$ Since $j^!\mathcal{G}$ is isomorphic to a direct sum of shifts of the constant sheaf, $H^p_B(j^!\mathcal{G})$ is graded free as a $H^*_B(\text{pt}) \simeq R$-module. Hence $j^\mathbb{H}(i_*i^!\mathcal{G})$ is graded free. By the same argument, $j^\mathbb{H}(j_*j^!\mathcal{G})$ is also graded free.

We apply the above argument to $\mathcal{G} = D(\mathcal{F})$ where $\mathcal{F} \in D^b_B(jX)$ is a $*$-parity sheaf. Since $j^\mathbb{H}(D(\mathcal{F}))$ is free, we have $D(j^\mathbb{H}(D(\mathcal{F}))) \simeq j^\mathbb{H}(D^2(\mathcal{F})) \simeq j^\mathbb{H}(\mathcal{F}).$ Hence $j^\mathbb{H}(\mathcal{F})$ is graded free.
Similarly we have $D(I\mathbb{H}(j, i^*D(F))) \simeq I\mathbb{H}(i_iF)$ and $D(I\mathbb{H}(j, j^*D(F))) \simeq I\mathbb{H}(j,j^*F)$ and both are graded free. Therefore we get an exact sequence

$$(3.1) \quad 0 \to I\mathbb{H}(j, j^*F) \to I\mathbb{H}(F) \to I\mathbb{H}(i_iF) \to 0$$

and each term is a graded free $R$-module.

**Lemma 3.9.** Let $M_1, M, M_2 \in \mathcal{C}$ with a sequence $0 \to M_1 \to M \to M_2 \to 0$ in $\mathcal{C}$ which is exact as $(R^\mathcal{W}, R)$-bimodules. Let $A \subset W_i \backslash W$ and assume that $\text{supp}_W(M_1) \subset A$, $\text{supp}_W(M_2) \subset (W_i \backslash W) \setminus A$ and $M_2$ is torsion-free as a right $R$-module. Then we have $M_1 \simeq M_A$ and $M_2 \simeq M(W_i \backslash W) \setminus A$.

**Proof.** We regard $M_1$ as a submodule of $M$. Then $M_1 \subset M_A$ is obvious. Let $m \in M_A$. Then the image of $m$ in $M_2 \otimes_R Q$ is zero. By the assumption, $m = 0$ in $M_2$. Hence $m \in M_1$.

**Lemma 3.10.** Keep the above notation and let $A \subset W_i \backslash W$ be the subset such that $Z = \bigcup_{w \in A} I X_w$. Set $A^c = (W_i \backslash W) \setminus A$.

1. If $F \in D_B^b(I)X$ is $s$-parity, $I\mathbb{H}(j, j^*F) \simeq I\mathbb{H}(F)_{A^c}$ and $I\mathbb{H}(i_iF) \simeq I\mathbb{H}(F)^A$.

2. If $F \in D_B^b(I)X$ is $!$-parity, $I\mathbb{H}(j, j^*F) \simeq I\mathbb{H}(F)^A$ and $I\mathbb{H}(i_iF) \simeq I\mathbb{H}(F)_{A^c}$.

**Proof.** We have $I\mathbb{H}(F)^w = H^*_I(\{w\}, (j, j^*F)^w) \otimes_R Q$ and it is zero if $w \notin A^c$. Therefore $\text{supp}_W(I\mathbb{H}(j, j^*F)) \subset A^c$. Similarly we also have $\text{supp}_W(I\mathbb{H}(i_iF)) \subset A^c$. Therefore, by the exact sequence (3.1) and since $I\mathbb{H}(i_iF)$ is free (hence torsion-free) $R$-module, we get (1) by Lemma 3.11.

We prove (2). If $w \notin A$, then $(i_i^*F)^w = 0$. Hence $I\mathbb{H}(i_iF)^w = 0$. Therefore we have $\text{supp}_W(I\mathbb{H}(i_iF)) \subset A^c$. Let $w \in A$. We have $\dim_Q I\mathbb{H}(j, j^*F)^w_Q = \dim_Q D(I\mathbb{H}(j, j^*F))_{D_f}^w_Q = \dim_Q I\mathbb{H}(j, j^*(D(F)))_{D_f}^w_Q = 0$. Hence $\text{supp}_W(I\mathbb{H}(j, j^*F)) \subset A^c$. We get (2) with the previous lemma.

**Lemma 3.11.** Let $Z' = \bigcup_{w \in A'} I X_w \subset Z$ be a closed subset and set $U' = X \setminus Z'$. Put $Y = Z \cap U'$ and denote the inclusions $Z \hookrightarrow X$, $U' \hookrightarrow X$, $Y \hookrightarrow X$ by $i'$, $j'$, $a$, respectively. Let $F \in D_B^b(I)X$.

1. If $F$ is $s$-parity, then $0 \to I\mathbb{H}(j, j^*F) \to I\mathbb{H}(j', j'^*)F \to I\mathbb{H}(a, a^*F) \to 0$ is exact.

2. If $F$ is $!$-parity, then $0 \to I\mathbb{H}(i^*F) \to I\mathbb{H}(i^*_aF) \to I\mathbb{H}(a, a^*F) \to 0$ is exact.

**Proof.** Note that $j'(j'^*)^F$ is $s$-parity if $F$ is $s$-parity. By (3.1), we have an exact sequence $0 \to I\mathbb{H}(j, j^*F) \to I\mathbb{H}(j', j'^*)F \to I\mathbb{H}(j,j^*F) \to I\mathbb{H}(i_iF) \to I\mathbb{H}(i_iF) \to I\mathbb{H}(i_iF) \to I\mathbb{H}(a, a^*F) = 0$. We have $j(j')^F \simeq j(j')^F$ and $i_i^*F \simeq a^*F$. (2) follows from a similar argument.

In particular, $I\mathbb{H}(j, j^*F) \simeq I\mathbb{H}(F)_{\geq w} / I\mathbb{H}(F)_{> w}$.

**Lemma 3.12.** Let $F \in \text{Parity}_B(I)X$ and $w \in W_i \backslash W$.

1. Take $n_{w, k} \in \mathbb{Z}_{\geq 0}$ such that $j^*_w F \simeq \bigoplus_k \mathbb{K}[k]^{\oplus n_{w, k}}$. Then the coefficient of $1 \otimes H_{w, -}$ in $I\text{ch}(I\mathbb{H}(F))$ is $v^{\ell(w, -)} \sum_k n_{w, k} v^k$.

2. Take $m_{w, k} \in \mathbb{Z}_{\geq 0}$ such that $\bar{j}_w F \simeq \bigoplus_k \mathbb{K}[k]^{\oplus n_{w, k}}$. Then the coefficient of $1 \otimes H_{w, -}$ in $I\text{ch}(I\mathbb{H}(F))$ is $v^{\ell(w, -)} \sum_k m_{w, k} v^{-k}$.

**Proof.** We prove (2) first. We have

$$D(I\mathbb{H}(j, j^*F)) \simeq I\mathbb{H}(j, j^*F) \simeq \bigoplus_k H^*_B(I X_w, \mathbb{K}[k]^{n_{w, k}}) \simeq \bigoplus_k R(k)^{n_{w, k}}.$$

By the previous lemma, we have $I\mathbb{H}(j, j^*F) \simeq I\mathbb{H}(F)_{\geq w} / I\mathbb{H}(F)_{> w}$. Hence the coefficient of $1 \otimes H_{w, -}$ in $I\text{ch}(I\mathbb{H}(D(F)))$ is $v^{\ell(w, -)} \sum_k m_{w, k} v^{-k}$. Ad we have $I\text{ch}(I\mathbb{H}(D(F))) = I\text{ch}(I\mathbb{H}(F))$, we get (1).
We prove (1). We have \( j_w^* D(\mathcal{F}) \simeq \bigoplus_k D(\mathbb{K}[x,w]) [-k]^{n_{w,k}} \). Since \( j_X \simeq \mathbb{K}^{(w,-)} \), we have 
\( D(\mathbb{K}[x,w]) \simeq \mathbb{K}[x,w][2\ell(w,-)] \). Hence \( j_w^* D(\mathcal{F}) \simeq \bigoplus_k \mathbb{K}[x,w][2\ell(w,-) - k]^{n_{w,k}} \). By (2), the coefficient of \( H_w \) in \( j(\mathcal{H}(D(\mathcal{F}))) = j(\mathcal{H}(\mathcal{F})) \) is \( \ell(w,-) \sum n_{w,\ell(w,-) - k} u^{-k} = u^{-\ell(w,-)} \sum n_{w,k} u^{k} \).

**Lemma 3.13.** For any \( \mathcal{F}, \mathcal{G} \in \text{Parity}_B(jX) \), \( \text{grk} \text{Hom} \text{Parity}_B(jX)(\mathcal{F}, \mathcal{G}) \) is graded free and we have 
\[ \text{grk} \text{Hom} \text{Parity}_B(jX)(\mathcal{F}, \mathcal{G}) = \text{grk} \text{Hom} \text{Parity}_B(jX)(\mathcal{H}(\mathcal{F}), \mathcal{H}(\mathcal{G})). \]

**Proof.** Take \( n_{w,k} \in \mathbb{Z}_{\geq 0} \) (resp. \( m_{w,k} \in \mathbb{Z}_{\geq 0} \)) such that \( j_w^* \mathcal{F} \simeq \bigoplus_k \mathbb{K}[k]^{n_{w,k}} \) (resp. \( j_w^* \mathcal{G} \simeq \bigoplus_k \mathbb{K}[k]^{m_{w,k}} \)). By [JMW14, Proposition 2.6], we have 
\[ \text{Hom} \text{Parity}_B(jX)(\mathcal{F}, \mathcal{G}) \simeq \bigoplus_w \text{Hom} D_B(jX_w)(j^*_w \mathcal{F}, j^*_w \mathcal{G}) \]
\[ \simeq \bigoplus_{w,k,l} \text{Hom} D_B(jX_w)(\mathbb{K}[k]^{n_{w,k}}, \mathbb{K}[l]^{m_{w,l}}) \]
\[ = \bigoplus_{w,k,l} R(-k + l) \bigoplus n_{w,k} m_{w,l}. \]
This is graded free and, by Theorem 2.32 and Lemma 3.12, the graded rank is equal to 
\[ \text{grk} \text{Hom} \text{Parity}_B(jX)(\mathcal{H}(\mathcal{F}), \mathcal{H}(\mathcal{G})). \]

Now we are ready to prove that \( j_\mathcal{H} \) is fully-faithful. By the following lemma, we may assume \( \mathbb{K} \) is a field.

**Lemma 3.14.** Let \( \mathbb{K}' \) be a \( \mathbb{K} \)-algebra.

(1) Let \( \mathcal{F}, \mathcal{G} \in \text{Parity}_B(jX) \). Then \( \mathbb{K}' \otimes _\mathbb{K} \text{Hom}(\mathcal{F}, \mathcal{G}) \simeq \text{Hom}(\mathbb{K}' \otimes _\mathbb{K} \mathcal{F}, \mathbb{K}' \otimes _\mathbb{K} \mathcal{G}). \)

(2) Let \( M, N \in j\text{Sbimod} \). Then \( \mathbb{K}' \otimes _\mathbb{K} \text{Hom}(M, N) \simeq \text{Hom}(\mathbb{K}' \otimes _\mathbb{K} M, \mathbb{K}' \otimes _\mathbb{K} N). \)

**Proof.** (1) follows from an argument of [JMW14 Proposition 2.6]. For (2), we may assume \( N = \pi_{i,0}^* (N_0) \). Then \( \text{Hom}(M, N) \simeq \text{Hom}(\pi_{i,0}^* M, N_0) \) and \( \text{Hom}(\mathbb{K}' \otimes _\mathbb{K} M, \mathbb{K}' \otimes _\mathbb{K} N) \simeq \text{Hom}(\mathbb{K}' \otimes _\mathbb{K} \pi_{i,0}^* M, \mathbb{K}' \otimes _\mathbb{K} N_0) \). Hence we may assume \( I = \emptyset \). Moreover we may assume \( M = B_{s_1} \otimes \cdots \otimes B_{s_l} \) and \( N = B_{t_1} \otimes \cdots \otimes B_{t_s} \) for some \( s_1, \ldots, s_l, t_1, \ldots, t_s \in S \). Then \( \text{Hom}(M, N) \) and \( \text{Hom}(\mathbb{K}' \otimes _\mathbb{K} M, \mathbb{K}' \otimes _\mathbb{K} N) \) has a basis called double leaves [ABC19 Theorem 5.5]. From the construction of double leaves basis, they correspond to each other by the base change to \( \mathbb{K}' \). Hence we have (2).

**Lemma 3.15.** The functor \( j_\mathcal{H} \colon \text{Parity}_B(jX) \to \text{jSbimod} \) is fully-faithful.

**Proof.** We assume \( \mathbb{K} \) is a field. By Lemma 3.13, it is sufficient to prove that the natural map \( \text{Hom} \text{Parity}_B(jX)(\mathcal{F}, \mathcal{G}) \to \text{Hom} \text{jSbimod}(j_\mathcal{H}(\mathcal{F}), j_\mathcal{H}(\mathcal{G})) \) is injective for \( \mathcal{F}, \mathcal{G} \in \text{Parity}_B(jX) \). Let \( Y = \bigcup_{w \in A} jX_w \subset jX \) be a closed subset such that \( \# A < \infty \) and \( f \colon Y \to jX \) the inclusion map. We prove \( \text{Hom}(\mathcal{F}, f_* f^! \mathcal{G}) \to \text{Hom}(j_\mathcal{H}(\mathcal{F}), j_\mathcal{H}(f_* f^! \mathcal{G})) \) is an isomorphism by induction on \( \# A \).

Let \( w \in A \) be a maximal element and set \( Z = Y \setminus jX_w, U = jX \setminus Z \). Let \( i \colon Z \to jX \) and \( j \colon U \to jX \) be the inclusion maps. Set \( A' = A \setminus \{w\} \). Then we have the following commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Hom}(\mathcal{F}, i_* i^! \mathcal{G}) \\
\downarrow & & \downarrow \\
\text{Hom}(\mathcal{F}, f_* f^! \mathcal{G}) & \longrightarrow & \text{Hom}(j_\mathcal{H}(\mathcal{F}), j_\mathcal{H}(f_* f^! \mathcal{G})) \\
\downarrow & & \downarrow \\
\text{Hom}(\mathcal{F}, j_{w*} j^!_w \mathcal{G}) & \longrightarrow & \text{Hom}(j_\mathcal{H}(\mathcal{F}), j_\mathcal{H}(j_{w*} j^!_w \mathcal{G})).
\end{array}
\]
Here we use $i_{s!}f_* f^! \simeq i_{s!} i^!$ and $j_* j^! f_* f^! \simeq j_! j_* j_!$ By Lemma 3.11 the right column is exact. The first row is injective by inductive hypothesis. It is sufficient to prove that the last row is injective.

Note that we have an equality

$$\text{Hom}_{D^b(I X_w)}(H^*_B(I X_w, j_*^! j^* w^! F), H^*_B(I X_w, j_*^! j^* w^! G)) = \text{Hom}_{C}(j^!(I \mathbb{H}(j_*^! j^* w^! F)), j^!(I \mathbb{H}(j_*^! j^* w^! G))).$$

Indeed, we have $H^*_B(I X_w, j_*^! j^* w^! F) = I \mathbb{H}(j_*^! j^* w^! F)$, $H^*_B(I X_w, j_*^! j^* w^! G) = I \mathbb{H}(j_*^! j^* w^! G)$ and $H^*_B(I X_w) \simeq R$. Hence the right hand side is contained in the left hand side. We have supp$_W(I \mathbb{H}(j_*^! j^* w^! F)) \subset \{w\}$ as in the proof of Lemma 3.10. Since $H^*_B(I X_w, j_*^! j^* w^! F) = I \mathbb{H}(j_*^! j^* w^! F)$ is free $R$-module, we have $I \mathbb{H}(j_*^! j^* w^! F) \to I \mathbb{H}(j_*^! j^* w^! F)_{I \mathbb{H}(j_*^! j^* w^! G)}$ and the same is true for $j_*^! j^* w^! G$. Therefore any $R$-module homomorphism $I \mathbb{H}(j_*^! j^* w^! F) \to I \mathbb{H}(j_*^! j^* w^! G)$ is a morphism in $\mathcal{C}$, hence we have the above equality.

The last row of (3.2) is decomposed into

$$\text{Hom}_{D^b(I X)}(F, j_*^! j^* w^! F) \simeq \text{Hom}_{D^b(I X_w)}(j_*^! j^* w^! F, j_*^! j^* w^! G)$$

$$\to \text{Hom}_{D^b(I X_w)}(H^*_B(I X_w, j_*^! j^* w^! F), H^*_B(I X_w, j_*^! j^* w^! G))$$

$$= \text{Hom}_{C}(j^!(I \mathbb{H}(j_*^! j^* w^! F)), I \mathbb{H}(j_*^! j^* w^! G))$$

$$\to \text{Hom}_{C}(I \mathbb{H}(F), I \mathbb{H}(j_*^! j^* w^! G)),$$

here the last map is induced by $F \to j_*^! j^* w^! F$. The second morphism is an isomorphism since $j_*^! j^* w^! F$ and $j_*^! j^* w^! G$ are constant. Therefore it is sufficient to prove that the last map is injective.

We have the commutative diagram

$$\begin{array}{ccc}
\text{Hom}(j^!(I \mathbb{H}(F)), I \mathbb{H}(j_*^! j^* w^! G)) & \longrightarrow & \text{Hom}(j^!(I \mathbb{H}(j_*^! j^* w^! F)), I \mathbb{H}(j_*^! j^* w^! G)) \\
\uparrow & & \uparrow \\
\text{Hom}(I \mathbb{H}(j_*^! j^* w^! F), I \mathbb{H}(j_*^! j^* w^! G)) & \longrightarrow & \text{Hom}(I \mathbb{H}(j_*^! j^* w^! F), I \mathbb{H}(j_*^! j^* w^! G)).
\end{array}$$

The right row is injective by Lemma 3.11. Therefore it is sufficient to prove the lower column is injective.

Note that $H^*_B(I X_w) \simeq H^*_B(pt) \simeq R$. Since $j_*^! j^* w^! F$ is constant and $I X_w \simeq \mathcal{A}^{w}(\mathcal{X}_w)$, the natural map $H^*_B(I X_w, j_*^! j^* w^! F) \otimes H^*_B(I X_w) \to H^*_B(I X_w, j_*^! j^* w^! F)$ is an isomorphism and $H^*_B(I X_w)$ is free of rank one as a $H^*_B(I X_w) \simeq R$-module generated by some $u \in H^2(I X_w)$. Let $a \in R \simeq H^2(I X_w)$ be the image of $u$ under $H^*_B(I X_w) \to H^*_B(I X_w)$. Then we have $j^!(I \mathbb{H}(j_*^! j^* w^! F)) = H^*_B(I X_w, j_*^! j^* w^! F) \simeq H^*_B(I X_w, j_*^! j^* w^! F) a = I \mathbb{H}(j_*^! j^* w^! F) a$. The $R$-module $I \mathbb{H}(j_*^! j^* w^! F)$ is a free, hence it is torsion-free. Hence the map $\text{Hom}(j^!(I \mathbb{H}(j_*^! j^* w^! F)), I \mathbb{H}(j_*^! j^* w^! G)) \to \text{Hom}(j^!(I \mathbb{H}(j_*^! j^* w^! F)), I \mathbb{H}(j_*^! j^* w^! G))$ which is given by the restriction to $I \mathbb{H}(j_*^! j^* w^! F)$, is injective.

\begin{proof}
We prove that $I \mathbb{H}$ is essentially surjective. Let $M \in I \mathcal{B}$. Then $M$ is a direct summand of a direct sum of objects of a form $\pi_{I, j, s}(B_{s_1} \otimes \cdots \otimes B_{s_l})$ for some $s_1, \ldots, s_l \in S$ and $n \in \mathbb{Z}$. By Lemma 3.3 and 3.4, there exists $F \in \mathcal{P}(\pi_{I, j} R)$ such that $M$ is a direct summand of $I \mathbb{H}(F)$. Since $I \mathbb{H}$ is fully-faithful, there exists $F'$ such that $M \simeq I \mathbb{H}(F')$.

By taking the left adjoint functors of $I \mathbb{H} \circ \pi_{I, j, s} \simeq \pi_{I, j, s} \circ I \mathbb{H}$, we get $\pi_{I, j} \circ I \mathbb{H} \simeq j_* \circ I \mathbb{H}$.
\end{proof}

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