Quadratic $G$-BSDEs with convex generators and unbounded terminal conditions

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Abstract

In this paper, we first study one-dimensional quadratic backward stochastic differential equations driven by $G$-Brownian motions ($G$-BSDEs) with unbounded terminal values. With the help of a $\theta$-method of Briand and Hu \cite{BriandHu} and nonlinear stochastic analysis techniques, we propose an approximation procedure to prove existence and uniqueness result when the generator is convex (or concave) and terminal value is of exponential moments of arbitrary order. Finally, we also establish the well-posedness of multi-dimensional $G$-BSDEs with diagonally quadratic generators.

Key words: quadratic $G$-BSDEs, unbounded terminal value, convex generator
MSC-classification: 60H10, 60H30

1 Introduction

The present paper is devoted to the study of backward stochastic differential equations (BSDEs) on a $G$-expectation space, which was initiated by Peng \cite{Peng1, Peng2} motivated by mathematical finance problems with Knightian uncertainty. More precisely, we will investigate the case with quadratic convex generators and unbounded terminal conditions.

The nonlinear BSDE was firstly introduced by Pardoux and Peng \cite{PardouxPeng} on Wiener space $(\Omega, \mathcal{F}, P_0)$ with the natural filtration $(\mathcal{F}_t)_{t \in [0,T]}$. The solution of BSDE consists of a pair of progressively measurable processes $(Y, Z)$ such that

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad \forall t \in [0, T],$$

in which $W$ is a standard Brownian motion, and the generator $f$ is a progressively measurable function and the terminal condition $\xi$ is an $\mathcal{F}_T$-measurable random variable. Pardoux and Peng \cite{PardouxPeng} established the existence and uniqueness of solutions to BSDE via a contraction mapping approach when $f$ is uniformly Lipschitz continuous in both unknowns and $\xi$ is square integrable. Since then, great progress has been made in the field of BSDEs, as it has rich connections with partial differential equations, stochastic control and mathematical finance (cf. El Karoui et al. \cite{ElKaroui}). In particular, an extensive

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study has been given to BSDEs with generators having a quadratic growth in the 2nd unknown $z$ due to their several financial motivations, such as utility maximization problems and financial market equilibrium problems (cf. Hu et al. [19]).

For a one-dimensional quadratic BSDE, the monotone convergence method is a successful strategy to build a solution. Kobylanski [22] firstly established the existence and uniqueness theorem through the monotone convergence method and PDE-based approximation technique when the terminal condition is bounded. Subsequently, Briand and Hu [3] extended the existence result to the case of unbounded terminal conditions. Indeed, they developed a useful a priori estimate on $Y$, which allows to apply a monotone approximation technique when the terminal condition has exponential moments. However, the uniqueness of unbounded solutions to quadratic BSDE is not trivial. In [4], Briand and Hu formulated a $\theta$-method to obtain the uniqueness when the generator is convex (or concave) with respect to the 2nd unknown $z$. On the other hand, several efforts have been made towards proposing new methods for the research of quadratic BSDEs. When the terminal value is bounded, with the help of BMO martingale theory, Tevzadze [39] obtained the existence and uniqueness result using a Picard iteration, and Briand and Elie [2] gave a distinct approximation procedure to derive the solvability based on Malliavin calculus.

It is worth mentioning that the result of Tevzadze [39] still works for multi-dimensional quadratic BSDE with small enough terminal conditions. However, multi-dimensional quadratic BSDE (even with bounded terminal conditions) may not have a solution (see Frei and Dos Reis [12] for such a counterexample). Then, some structure conditions on the generator are introduced in order to guarantee that the system of quadratic BSDEs with bounded terminal values has a unique solution. For example, Hu and Tang [21] investigated BSDEs of diagonally quadratic generators (see also [6, 25]). Recently, by utilizing a $\theta$-method and iterative technique, Fan et al. [11] established the solvability of system of diagonally quadratic BSDEs with the terminal values of exponential moments of arbitrary order. For more research on this topic, we refer the reader to [1, 9, 10, 13, 26, 40] and the references therein.

In this paper, our probabilistic setup is the $G$-expectation space $(\Omega, L^1_G(\Omega), \hat{E}[\cdot], (\hat{E}[\cdot])_{t \in [0,T]})$, under which the canonical process $B$ is called $G$-Brownian motion. The $G$-expectation is a time-consistent sublinear expectation, and we could establish the corresponding stochastic calculus theory with respect to $G$-Brownian motion, such as $G$-Itô’s formula, $G$-stochastic differential equation and so on. Indeed, the $G$-expectation could be represented by an upper expectation over a weakly compact subset of mutually singular martingale measures (cf. Denis et al. [7]).

Due to the nonlinear structure, the quadratic process $\langle B \rangle$ is no longer a deterministic process, which results in the main difficulty compared to the linear case. For instance, there is a kind of non-increasing and continuous $G$-martingales $K$. Thus, a typical BSDE driven by $G$-Brownian motion ($G$-BSDE) is given by

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T g(s,Y_s,Z_s)d\langle B \rangle_s - \int_t^T Z_sdB_s - (K_T - K_t).$$

However, the Picard iteration involving the term $Z$ was found difficult to be applied to $G$-BSDE due to the presence of $K$. Then, Hu et al. [14] turned to a combined PDE and Galerkin approximation approach to obtain the well-posedness of Lipschitz $G$-BSDE when the terminal value has a finite moment of order $p > 1$. On the other hand, the application of monotone convergence theorem is restricted under $G$-expectation framework (see Lemma 2.3). So, Hu et al. [20] adapted the approximation approach of [14] to quadratic $G$-BSDE based on $G$-BMO martingale theory, and derived the existence and uniqueness result when the terminal value is bounded.

A notion quite related to $G$-BSDE is the second order BSDE (2BSDE) proposed by Soner et al. [33, 34]. By applying quasi-surely analysis and aggregations approach, Soner et al. [35] established the existence and uniqueness of solutions to 2BSDE with Lipschitz generators. Possamai and Zhou
Moreover, he introduced the conditional \( G \)-expectation setup \( \hat{E} \) that is more general than that of \( G \)-BSDE, whereas the solution of \( G \)-BSDE has more regularity, see \[12\] and the references therein for more research on this field.

This paper aims to fill the gap between boundedness and existence of exponential moments of the data for solution of quadratic \( G \)-BSDEs. We have to develop an alternative approximation approach, which is different from existing monotone approximation and Picard approximation. The key point is how to estimate the difference of two solutions, say \( Y \) and \( \hat{Y} \). Contrary to the case of bounded terminal values, the 2nd unknown \( Z \) may be unbounded in the BMO space and the conventional linearization technique fails to work in our context. Inspired by the arguments of [3] and [11], we will develop a \( \theta \)-method to our quadratic \( G \)-BSDE under the further assumption of either convexity or concavity on the generator, i.e., we estimate \( Y - \theta \hat{Y} \) for each \( \theta \in (0, 1) \), which allows to take advantage of the convexity of the generator.

In order to carry out the purpose, we firstly establish a priori estimate on exponential moments of the term \( Y \) of \( G \)-BSDE [24], as in [8] or [10]. Unlike the quadratic BSDE case, some delicate and technical computations are developed to deal with the new term \( K \) through nonlinear stochastic analysis theory, which generalizes the counterpart of [14] (see Lemma 3.6). Next, using the decreasing property of the new term \( Y \) inspired by [3, 31], we give a priori estimate on the term \( Z \), which involves exponential moments of the term \( Y \). Then, with the help of a \( \theta \)-method, we could develop an approximation procedure through a sequence of quadratic \( G \)-BSDEs with bounded terminal condition. Indeed, we prove existence and uniqueness of the global solution to quadratic \( G \)-BSDEs with bounded terminal value. Finally, we consider the solvability of systems of diagonally quadratic \( G \)-BSDEs with unbounded terminal values, and give some extension of [24]'s result to our quadratic case.

The rest of the paper is organized as follows. In section 2, we present some basic results on \( G \)-expectation. Section 3 is devoted to solution of quadratic \( G \)-BSDE with unbounded terminal conditions. In section 4, we discuss a multi-dimensional case.

2 The \( G \)-expectation setup

In this paper, we denote by \( \langle \cdot, \cdot \rangle \) and \( |\cdot| \) the scalar product and associated norm of a Euclidian space, respectively. Fix a constant \( T > 0 \). Let \( \Omega = C^0([0, T]) \) be the space of all \( \mathbb{R}^d \)-valued continuous functions \( \omega \) starting from the origin on \([0, T] \), and \( B_t(\omega) := \omega_t \) be the canonical process, equipped with the uniform norm, i.e., \( |\omega| := \sup_{t \in [0, T]} |\omega_t| \). We set \( \Omega_t := \{ \omega_{\cdot \land t} : \omega \in \Omega \} \) and denote by \( \mathcal{B}(\Omega) \) (resp. \( \mathcal{B}(\Omega_t) \)) the Borel \( \sigma \)-algebra of \( \Omega \) (resp. \( \Omega_t \)) for each \( t \in [0, T] \). We introduce the following space of cylinder functions as a counterpart of cylinder sets in the linear case:

\[
L_{ip}(\Omega_t) := \{ \varphi(B_{t_1}, \ldots, B_{t_k}) : k \in \mathbb{N}, t_1 < \cdots < t_k \in [0, t], \varphi \in C_{b,Lip}(\mathbb{R}^{k \times d}) \},
\]

and \( L_{ip}(\Omega) := L_{ip}(\Omega_T) \), where \( C_{b,Lip}(\mathbb{R}^{k \times d}) \) denotes the space of all bounded and Lipschitz functions on \( \mathbb{R}^{k \times d} \).

Given a monotonic and sublinear function \( G : \mathcal{S}(d) \to \mathbb{R} \), where \( \mathcal{S}(d) \) denotes the space of all \( d \times d \) symmetric matrices. Peng [28, 29] initiated the \( G \)-expectation \( \hat{\mathbb{E}}[\cdot] : L_{ip}(\Omega) \to \mathbb{R} \) satisfying that \( \hat{\mathbb{E}}[\varphi(B_t)] = u(t, 0) \), where \( u(t, x) \) is the viscosity solution to the following fully nonlinear PDE with initial condition \( u(0, x) = \varphi(x) \in C_{b,Lip}(\mathbb{R}^d) \):

\[
\frac{\partial}{\partial t} u(t, x) - G(\partial^2_{xx} u(t, x)) = 0, \quad \forall (t, x) \in (0, T) \times \mathbb{R}^d.
\]

Moreover, he introduced the conditional \( G \)-expectation \( \hat{\mathbb{E}}[\cdot] : L_{ip}(\Omega) \to L_{ip}(\Omega_t) \) for each \( t \in [0, T] \). Let \( L_{G}^p(\Omega) \) (resp. \( L_{G}^p(\Omega_t) \)) be the completion of \( L_{ip}(\Omega) \) (resp. \( L_{ip}(\Omega_t) \)) under the norm \( \hat{\mathbb{E}}[|\cdot|_p]^{1/p} \).
for each \( p \geq 1 \). The canonical process \( B_t = (B^i_t)_{i=1}^d \) is called a \( d \)-dimensional \( G \)-Brownian motion on the \( G \)-expectation space \( (\Omega, L^1_G(\Omega), \hat{\mathbb{E}}, [\cdot], (\hat{\mathbb{E}}[\cdot])_{t \in [0, T]} ) \). For each \( 1 \leq i, j \leq d \), denote by \( (B_t^i, B_t^j) \) the mutual variation process. Indeed, the \( G \)-expectation could be regarded as an upper expectation.

**Theorem 2.1** ([7, 16]) There exists a weakly compact set \( \mathcal{P} \) of probability measures on \( (\Omega, \mathcal{B}(\Omega)) \), such that

\[
\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E^P[X] \quad \text{for any } X \in L^1_G(\Omega).
\]

\( \mathcal{P} \) is called a set that represents \( \hat{\mathbb{E}} \).

**Remark 2.2** Denote by \( \mathcal{P}_{\text{max}} \) the collection of all probability measures \( P \) on \( (\Omega, \mathcal{B}(\Omega)) \) such that \( E^P[X] \leq \hat{\mathbb{E}}[X] \) for any \( X \in L^p(\Omega) \). Then, \( \mathcal{P}_{\text{max}} \) is a weakly compact set that represents \( \hat{\mathbb{E}} \). In what follows, we always assume that \( \mathcal{P} = \mathcal{P}_{\text{max}} \) to give the representation of conditional \( G \)-expectation. Note that the capacities induced by different weakly compact representation sets coincide with each other, see [17].

Then the \( G \)-expectation \( \hat{\mathbb{E}} \) could be generalized in the following way:

\[
\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E^P[X] \quad \text{for each } \mathcal{B}(\Omega)\text{-measurable function } X.
\]

For this \( \mathcal{P} \), we define capacity

\[
c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).
\]

A set \( A \in \mathcal{B}(\Omega) \) is polar if \( c(A) = 0 \). A property holds “quasi-surely” (q.s.) if it holds outside a polar set.

Due to the nonlinear structure of \( G \)-expectation space, we have the following nonlinear monotone convergence theorem, which is much more complicated.

**Lemma 2.3** ([7]) Let \( X_n, n \geq 1 \) be a sequence of \( \mathcal{B}(\Omega)\)-measurable random variables.

(i) Suppose \( X_n \geq 0 \). Then, \( \hat{\mathbb{E}}[\liminf_{n \to \infty} X_n] \leq \liminf_{n \to \infty} \hat{\mathbb{E}}[X_n] \).

(ii) Suppose \( X_n \in L^1_G(\Omega) \) are non-increasing. Then, \( \hat{\mathbb{E}}[X_n] \downarrow \hat{\mathbb{E}}[\lim_{n \to \infty} X_n] \).

Next, we introduce some useful spaces of stochastic processes, which will be used frequently in stochastic calculus theory with respect to \( G \)-Brownian motion. Set

\[
M^0_G(0, T) := \left\{ \eta = \sum_{i=0}^{n-1} \xi_i 1_{[t_i, t_{i+1})}(\cdot) : 0 = t_0 < t_1 < \cdots < t_n = T, \xi_i \in L_{lip}(\Omega_t) \right\};
\]

\[
S^0_G(0, T) := \{ h(t, B_{t_1}^{1 \land t}, \cdots, B_{t_n}^{n \land t}) : 0 < t_1 < t_2 < \cdots < t_n < T, h \in C_b,\text{Lip}(\mathbb{R}^{1+n \times d}) \};
\]

\[
M^p_G(0, T) := \text{the completion of } M^0_G(0, T) \text{ under the norm } \hat{\mathbb{E}}\left[ \int_0^T |\cdot|^p dt \right]^{\frac{1}{p}}, \forall p \geq 1;
\]

\[
H^p_G(0, T) := \text{the completion of } M^0_G(0, T) \text{ under the norm } \hat{\mathbb{E}}\left[ \left( \int_0^T |\cdot|^2 dt \right)^\frac{p}{2} \right]^{\frac{2}{p}}, \forall p \geq 1;
\]

\[
S^p_G(0, T) := \text{the completion of } S^0_G(0, T) \text{ under the norm } \hat{\mathbb{E}}\left[ \sup_{t \in [0, T]} |\eta(t)|^p \right]^{\frac{1}{p}}, \forall p \geq 1.
\]
Then, for two processes \( \eta \in M^2_G(0, T) \) and \( \xi \in H^1_G(0, T) \), the G-Itô integrals \( \int \eta_s d(B^i_s, B^j_s) \) and \( \int \xi_s dB^i_s \) could be constructed by a standard approximation method. Denote by \( M^d_G(0, T; \mathbb{R}) \) the \( \mathbb{R}^d \)-valued process such that each component belongs to \( M^d_G(0, T) \). Similarly, we can define \( H^p_G(0, T; \mathbb{R}) \), \( S^p_G(0, T; \mathbb{R}^d) \) and \( L^p_G(\Omega; \mathbb{R}^d) \). For the sake of convenience, set
\[
\int^T_0 \eta_s d(B^i_s) := \sum^d_{i,j=1} \int^T_0 \eta^i_j d(B^i_s, B^j_s) \quad \text{and} \quad \int^T_0 \xi_s dB^i_s := \sum^d_{i=1} \int^T_0 \xi^i_s dB^i_s
\]
for each \( \eta \in M^d_G(0, T; \mathbb{R}^d) \) and \( \xi \in H^1_G(0, T; \mathbb{R}) \).

For any \( p > 1 \), we denote by \( \mathcal{E}^p_G(\mathbb{R}^d) \) the collection of all stochastic processes \( Y \) such that \( e^Y \in S^p_G(0, T; \mathbb{R}^d) \), \( H^p_G(\mathbb{R}^d) \) the collection of all stochastic processes \( Z \in H^p_G(0, T; \mathbb{R}^d) \), and \( L^p_G(\mathbb{R}^d) \) the collection of all stochastic processes \( K \) such that \( K \) is a non-increasing \( G \)-martingale with \( K_0 = 0 \) and \( K_T \in L^p_G(\Omega; \mathbb{R}^d) \). We write \( Y \in \mathcal{E}^p_G(\mathbb{R}^d) \) if \( Y \in \mathcal{E}^p_G(\mathbb{R}^d) \) for any \( p \geq 1 \). Similarly, we define \( \mathcal{H}^p_G(\mathbb{R}^d) \) and \( \mathcal{L}^p_G(\mathbb{R}^d) \).

In the rest of this paper, we always assume that \( G \) is non-degenerate to ensure the solvability of \( G \)-BSDEs, i.e., there exist two constants \( 0 < \sigma^{-1} \leq \sigma < \infty \) such that
\[
\frac{1}{2\sigma^2} \text{tr}[A - B] \leq G(A) - G(B) \leq \frac{1}{2} \sigma^2 \text{tr}[A - B] \quad \text{for all } A \geq B.
\]
Then it follows from Corollary 3.5.8 of Peng [30] that
\[
\hat{\sigma}^{-2} I_d \leq |d(B^i_s, B^j_s)|^2_{t} \leq \hat{\sigma}^2 I_d. \quad (3)
\]
It is easy to verify that \( \int \xi_s dB_s \) is a \( P \)-martingale for each \( P \in \mathcal{P} \). Then we have the following BDG inequality.

**Lemma 2.4** Assume that \( \xi \in M^2_G(0, T; \mathbb{R}^d) \). Then, for each \( n \geq 1 \), there is a constant \( A(n) \) depending only on \( n \) and \( \sigma^2 \) so that have
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int^t_0 \xi_s dB_s \right|^n \right] \leq A(n) \mathbb{E} \left[ \left( \int^T_0 |\xi_s|^2 ds \right)^{\frac{n}{2}} \right]. \quad (4)
\]
We have Doob’s maximal inequality for \( G \)-martingale as follows. See [30, 34, 37] for details.

**Lemma 2.5** Suppose \( 1 < \alpha < \beta \). Then for each \( 1 < p < p^\bar{\beta} := \beta/\alpha \) with \( p \leq 2 \) and for all \( \eta \in L^\beta_G(\Omega T) \), there exists a constant \( C > 0 \) depending only on \( p, \alpha \) and \( \beta \) such that
\[
\hat{\mathbb{E}} \left[ \sup_{t \in [0, T]} \hat{\mathbb{E}}_t [|X|^\alpha] \right] \leq \frac{C p \bar{\beta}}{(\bar{\beta} - p)(p - 1)} \left( \hat{\mathbb{E}} \hat{\mathbb{E}}_t [|X|^\beta] \right) + \hat{\mathbb{E}} \hat{\mathbb{E}}_t [|X|^\beta].
\]

**Remark 2.6** Suppose \( e^X \in L^2_G(\Omega) \). Then, there exists a constant \( \hat{A}(G) \) depending only on \( \bar{\sigma} \) and \( \sigma \) such that (taking \( \alpha = 2, \beta = 4 \))
\[
\hat{\mathbb{E}} \left[ \sup_{t \in [0, T]} \hat{\mathbb{E}}_t [e^X] \right] = \hat{\mathbb{E}} \left[ \sup_{t \in [0, T]} \hat{\mathbb{E}}_t \left( e^\hat{X}_t \right)^2 \right] \leq \hat{A}(G) \hat{\mathbb{E}} [e^{\hat{X}}].
\]

### 3 One-dimensional quadratic \( G \)-BSDEs

In this section, we shall study the well-posedness of solutions to the following scalar-valued quadratic \( G \)-BSDEs with unbounded terminal values:
\[
Y_t = \xi + \int^T_t f(s, \omega, s, Y_s, Z_s) ds - \int^T_t Z_s dB_s - (K_T - K_t), \quad \forall t \in [0, T]. \quad (5)
\]
Throughout the paper, we always fix three positive constants $\lambda, \gamma, \kappa$, and two nonnegative stochastic processes $\alpha_t, \beta_t \in M^1_G(0, T)$. Consider the following assumptions on the terminal condition and generator.

**Remark 3.1** Assumptions $(H1)$ and $(H2)$ are also used in [20] to study quadratic $G$-BSDEs when the terminal condition and $f(s, 0, 0)$ are bounded. Assumption $(H3)$ allows us to use a $\theta$-method of [4] to establish the convergence of our approximating sequences of quadratic $G$-BSDEs. Assumption $(H4)$ relaxes the existing bounded condition on the terminal value and generator.

**Remark 3.2** Just for convenience of exposition, the type of quadratic $(H4)$ relaxes the existing bounded condition on the terminal value and generator.

(H1) For each $(t, \omega) \in [0, T] \times \Omega$ and $(y, z), (\bar{y}, \bar{z}) \in \mathbb{R} \times \mathbb{R}^d$,

$$|f(t, \omega, 0, 0)| \leq \alpha_t(\omega) \quad \text{and} \quad |f(t, \omega, y, z) - f(t, \omega, \bar{y}, \bar{z})| \leq \lambda |y - \bar{y}| + \gamma (1 + |z| + |\bar{z}|)|z - \bar{z}|.$$  

(H2) There exists a modulus of continuity $w : [0, \infty) \to [0, \infty)$ such that for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ and $(t, \omega), (\bar{t}, \bar{\omega}) \in [0, T] \times \Omega$,

$$|f(t, \omega, y, z) - f(\bar{t}, \bar{\omega}, y, z)| \leq w(|t - \bar{t}| + \|\omega - \bar{\omega}\|).$$

(H3) For each $(t, \omega, y) \in [0, T] \times \Omega \times \mathbb{R}$, $f(t, \omega, y, \cdot)$ is either convex or concave.

(H4) Both the terminal value $\xi \in L^1_G(\Omega)$ and $\int_0^T \alpha_t dt$ have exponential moments of arbitrary order, i.e.,

$$\mathbb{E} \left[ \exp \left( p|\xi| + p \int_0^T \alpha_t dt \right) \right] < \infty \quad \text{for any } p \geq 1.$$
Lemma 3.3 Let \( X \in S^1_G(0, T) \) be a stochastic process. Suppose that

\[
\hat{E} \left[ \exp \left( (1 + \varepsilon) \sup_{t \in [0, T]} |X_t| \right) \right] < \infty
\]

for some constant \( \varepsilon > 0 \). Then, \( \exp(X_t) \in S^1_G(0, T) \).

**Proof.** Denote by \( X^{(n)} = (X \wedge n) \vee (-n) \) for each \( n \geq 1 \). Then, it is easy to check that \( \exp(X_t^{(n)}) \in S^1_G(0, T) \). By Taylor’s expansion, we have that

\[
\hat{E} \left[ \sup_{t \in [0, T]} |\exp(X_t) - \exp(X_t^{(n)})| \right] \leq \hat{E} \left[ \sup_{t \in [0, T]} \exp(|X_t|)|X_t|1_{\{|X_t| \geq n\}} \right] \leq \frac{2}{n^2} \hat{E} \left[ \sup_{t \in [0, T]} \exp((1 + \varepsilon)|X_t|) \right],
\]

which ends the proof by sending \( n \to \infty \). □

Then, we have the following a priori estimates for quadratic \( G \)-BSDEs, which is crucial for our subsequent discussions.

Lemma 3.4 Assume that \( (Y, Z, K) \in \mathcal{S}^2_G(\mathbb{R}) \times \mathcal{H}^2_G(\mathbb{R}^d) \times \mathcal{L}^2_G(\mathbb{R}) \) satisfies the following equation

\[
Y_t = \xi + (\bar{K}_T - \bar{K}_t) + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t),
\]

where \( \bar{K} \in \mathcal{S}^2_G(0, T) \) is a non-increasing \( G \)-martingale. Suppose that there are two constants \( p \geq 1 \) and \( \varepsilon > 0 \) such that

\[
\hat{E} \left[ \exp \left( (2p + \varepsilon)\kappa \tilde{\sigma}^2 e^{\lambda T} \sup_{t \in [0, T]} |Y_t| + (2p + \varepsilon)\kappa \tilde{\sigma}^2 \int_0^T \beta_t e^{\lambda T} dt \right) \right] < \infty. \tag{7}
\]

Then, we have

(i) Let Assumption (H5) hold. Then, for each \( t \in [0, T] \),

\[
\exp \left\{ pk\tilde{\sigma}^2 e^{\lambda T} |Y_t| \right\} \leq \hat{E}_t \left[ \exp \left\{ pk\tilde{\sigma}^2 e^{\lambda T} |\xi| + pk\tilde{\sigma}^2 \int_t^T \beta_s e^{\lambda s} ds \right\} \right] .
\]

(ii) Let Assumption (H6) hold. Then, for each \( t \in [0, T] \),

\[
\exp \left\{ pk\tilde{\sigma}^2 e^{\lambda T} Y^+_t \right\} \leq \hat{E}_t \left[ \exp \left\{ pk\tilde{\sigma}^2 e^{\lambda T} \xi^+ + pk\tilde{\sigma}^2 \int_t^T \beta_s e^{\lambda s} ds \right\} \right] .
\]

**Proof.** The proof is based on the idea of \([10]\) and nonlinear stochastic analysis technique.

1. Proof of Assertion (i). From the representation theorem for \( G \)-expectation (Theorem 2.1), we see that \( (Y, Z) \) could be regarded as the solution to the following classical BSDE:

\[
Y_t = \xi + (\bar{K}_T - \bar{K}_t) - (K_T - K_t) + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad P \text{-a.s.}
\]

for each \( P \in \mathcal{P} \). Then, applying Itô-Tanaka’s formula (see \([33\text{ p. 234}]\)) to \(|Y_t|\), we have \( P \text{-a.s.} \)

\[
d|Y_t| = -\text{sgn}(Y_t)f(t, Y_t, Z_t)dt + \text{sgn}(Y_t)Z_t dB_t + \text{sgn}(Y_t)d(\bar{K}_t - \bar{K}_t) + dL_t,
\]
where \( \psi(x) = 1_{\{x > 0\}} - 1_{\{x < 0\}} \) and \( L \) is a continuous adapted and increasing process. It follows that \( P \)-a.s.

\[
d\psi(t, |Y_t|) \geq -\partial_x \psi(t, |Y_t|) \text{sgn}(Y_t)f(t, Y_t, Z_t)dt + \partial_x \psi(t, |Y_t|) \text{sgn}(Y_t)Z_t dB_t
\]

\[
+ \partial_x \psi(t, |Y_t|) \text{sgn}(Y_t)d(K_t - \tilde{K}_t) + \frac{1}{2} \partial^2_{xx} \psi(t, |Y_t|)Z_t^2dB_t + \partial_x \psi(t, |Y_t|)dt,
\]

where \( \psi(t, x) \) is given by

\[
\psi(t, x) = \exp \left\{ pk\bar{\sigma}^2 e^{\lambda t}x + pk\bar{\sigma}^2 \int_0^t \beta_s e^{\lambda s}ds \right\}, \quad (t, x) \in [0, T] \times [0, \infty).
\]

It follows from Assumption (H5) that

\[
-\partial_x \psi(t, |Y_t|) \text{sgn}(Y_t)f(t, Y_t, Z_t) \geq -\partial_x \psi(t, |Y_t|) \left( \beta_t + \lambda|Y_t| + \frac{\kappa}{2}|Z_t|^2 \right).
\]

Note that \( \bar{\sigma}^{-2}I_d \leq d(B)_t \leq \bar{\sigma}^2I_d \) according to Inequality (3). In spirit of the following fact

\[
\partial^2_{xx} \psi(t, |Y_t|) \geq 0,
\]

we have

\[
\partial^2_{xx} \psi(t, |Y_t|)Z_t^2dB_t \geq \bar{\sigma}^{-2}\partial^2_{xx} \psi(t, |Y_t|)|Z_t|^2dt.
\]

Since \( K \) and \( \tilde{K} \) are non-increasing and \( \partial_x \psi(t, |Y_t|) \geq pn\bar{\sigma}^2 \), we derive \( P \)-a.s.

\[
d\psi(t, |Y_t|) \geq \left[ \partial_t \psi(t, |Y_t|) - \partial_x \psi(t, |Y_t|) \left( \beta_t + \lambda|Y_t| \right) \right]dt + \partial_x \psi(t, |Y_t|) \text{sgn}(Y_t)Z_t dB_t
\]

\[
+ \frac{1}{2} \left[ -\kappa \partial_x \psi(t, |Y_t|) + \bar{\sigma}^{-2}\partial^2_{xx} \psi(t, |Y_t|) \right] |Z_t|^2dt + \partial_x \psi(t, |Y_t|) \left( 1_{\{Y_t > 0\}}dK_t + 1_{\{Y_t < 0\}}d\tilde{K}_t \right)
\]

\[
\geq \frac{1}{2}p(p - 1)\kappa^2 \bar{\sigma}^{-2}|Z_t|^2dt + \partial_x \psi(t, |Y_t|) \text{sgn}(Y_t)Z_t dB_t + \partial_x \psi(t, |Y_t|) \left( 1_{\{Y_t > 0\}}dK_t + 1_{\{Y_t < 0\}}d\tilde{K}_t \right),
\]

where we have used the fact that

\[
\partial_t \psi(t, x) - \partial_x \psi(t, x)(\lambda x + \beta_t) = 0 \quad \text{and} \quad -pn\bar{\sigma}^2\partial_x \psi(t, x) + \partial^2_{xx} \psi(t, x) \geq 0.
\]

In spirit of the condition (7), we could get that \( \partial_x \psi(t, |Y_t|) \in S^2_t(0, T) \) by Lemma 3.3. It follows that for any \( P \in \mathcal{P}, \int_0^T \partial_x \psi(s, |Y_s|) \text{sgn}(Y_s)Z_s dB_s \) is a \( P \)-martingale. Thus, recalling Equation (8), we deduce that \( P \)-a.s.

\[
\psi(t, |Y_t|) + E^P_t \left[ \int_t^T \partial_x \psi(s, |Y_s|)1_{\{Y_s > 0\}}dK_s + \int_t^T \partial_x \psi(s, |Y_s|)1_{\{Y_s < 0\}}d\tilde{K}_s \right] \leq E^P_t[\psi(T, |\xi|)],
\]

where \( E^P_t \) denotes the conditional expectation with respect to \( \mathcal{B}(\Omega_t) \). Noting \( \psi(T, |\xi|) \in L_G^1(\Omega) \) and using Lemma 4.6 and Lemma A.1 in the Appendix, we conclude that \( P \)-a.s.

\[
\psi(t, |Y_t|) \leq \mathbb{E}_t[\psi(T, |\xi|)],
\]

where \( \mathcal{P}(t, P) = \{ \tilde{P} \in \mathcal{P} | \tilde{P} = P \text{ on } \text{Lip}(\Omega_t) \} \). Since \( P \in \mathcal{P} \) is arbitrary, we get that

\[
\psi(t, |Y_t|) \leq \mathbb{E}_t[\psi(T, |\xi|)], \quad \text{q.s.,}
\]

which is the desired result.
2. Proof of Assertion (ii). Using another type of Itô-Tanaka’s formula (see \[33\] p. 222) to \( Y^+_t \), we have \( P \)-a.s.

\[
dY^+_t = -1_{\{Y_t > 0\}} f(t, Y_t, Z_t) dt + 1_{\{Y_t > 0\}} Z_t dB_t + 1_{\{Y_t > 0\}} d(K_t - \tilde{K}_t) + \frac{1}{2} d\tilde{L}_t,
\]

where \( \tilde{L} \) is also a continuous adapted and increasing process, and may differ from \( L \). Then, in view of \([33]\), we have that \( P \)-a.s.

\[
d\psi(t, Y^+_t) \geq \frac{1}{2} \rho(p - 1) \kappa^2 \sigma^2 |Z_t|^2 1_{\{Y_t > 0\}} dt + \partial_x \psi(t, Y^+_t) Z_t 1_{\{Y_t > 0\}} dB_t + \partial_y \psi(t, Y^+_t) 1_{\{Y_t > 0\}} dK_t.
\]

Proceeding identically as to prove Assertion (i), the proof is complete. \[\blacksquare\]

\textbf{Remark 3.5} From Itô-Tanaka’s formula ([33] p. 222), we have \( P \)-a.s.

\[
d|Y_t| = -\text{sgn}(Y_t) f(t, Y_t, Z_t) dt + \text{sgn}(Y_t) Z_t dB_t + \text{sgn}(Y_t) d(K_t - \tilde{K}_t) + d\tilde{L}_t,
\]

where \( \text{sgn}(x) = 1_{\{x > 0\}} - 1_{\{x \leq 0\}} \). In this case, it is difficult to prove \( \int_0^t \text{sgn}(Y_s)^+ dK_s + \text{sgn}(Y_s)^- d\tilde{K}_s \) is a \( G \)-martingale, see Lemma 3.6 below.

\textbf{Lemma 3.6} Let \((X^i, K^i) \) be in \( S^2_G(0, T) \times L^2_G(\mathbb{R}), \ i = 1, 2 \). If \( X^1 > 0 \), then \( P \)-a.s.

\[
\mathbb{E} \left[ \sup_{P \in \mathcal{P}(\tau, \mathcal{P})} \int_t^\tau X^2_i 1_{\{X^2_i > 0\}} dK^1_s + \int_t^\tau X^1_i 1_{\{X^2_i < 0\}} dK^2_s \right] = 0
\]

for each \( P \in \mathcal{P} \).

\textbf{Proof.} For each \( n \geq 1 \), we define

\[
\varphi_n(x) = \begin{cases} 
1, & \text{if } x \geq \frac{1}{n}; \\
\frac{nx}{\lambda}, & \text{if } x \in (-\frac{1}{n}, \frac{1}{n}); \\
-1, & \text{if } x \leq -\frac{1}{n}.
\end{cases}
\]

Then one could easily check that \( \varphi_n(X^2_i) X^1_i \in S^2_G(0, T) \) for each \( n \geq 1 \). Thus, from [14] Lemma 3.4], we see that \( \int_0^\tau \varphi^+_n(X^2_i) X^1_i dK^1_s + \int_0^\tau \varphi^-_n(X^2_i) X^1_i dK^2_s \) is a non-increasing \( G \)-martingale. Note that \( \varphi^+_n(x) \uparrow 1_{\{x > 0\}} \) and \( \varphi^-_n(x) \uparrow 1_{\{x < 0\}} \) as \( n \to \infty \). From Lemma [A1] in the Appendix, we derive that for each \( P \in \mathcal{P} \)

\[
\mathbb{E} \left[ \sup_{P \in \mathcal{P}(\tau, \mathcal{P})} \int_t^\tau X^2_i 1_{\{X^2_i > 0\}} dK^1_s + \int_t^\tau X^1_i 1_{\{X^2_i < 0\}} dK^2_s \right] = \lim_{n \to \infty} \mathbb{E} \left[ \int_t^\tau \varphi^+_n(X^2_i) X^1_i dK^1_s + \int_t^\tau \varphi^-_n(X^2_i) X^1_i dK^2_s \right] = 0, \ P\text{-a.s.}
\]

which ends the proof. \[\blacksquare\]

\textbf{Lemma 3.7} Assume that Assumption (H5) holds and \( \int_0^T \beta_t dt \) has exponential moments of arbitrary order. Let \((Y, Z, K) \in \mathcal{H}_G(\mathbb{R}) \times \mathcal{H}_G(\mathbb{R}^d) \times L^2_G(\mathbb{R}) \) be a solution to \( G \)-BSDE ([5]). Then, for any \( n \geq 1 \), there exists a constant \( \hat{A}(n) \) depending on \( \lambda, \bar{\sigma}, \bar{\sigma}, \kappa, T \) and \( n \), such that

\[
\mathbb{E} \left[ \left( \int_0^T |Z_i|^2 dt \right)^n + |K_T|^n \right] \leq \hat{A}(n) \mathbb{E} \left[ \exp \left( (4\kappa \bar{\sigma}^2 + 2\lambda)n \sup_{t \in [0, T]} |Y_t| + 2n \int_0^T \beta_t dt \right) \right].
\]
Proof. Recalling Inequality \ref{eq:inequality} and applying G-Itô's formula to \( e^{-2\kappa \bar{\sigma}^2 Y_t} \) yields that
\[
2\kappa^2 \bar{\sigma}^2 \int_0^T e^{-2\kappa \bar{\sigma}^2 Y_t} |Z_t|^2 dt \leq 2\kappa^2 \bar{\sigma}^4 \int_0^T e^{-2\kappa \bar{\sigma}^2 Y_t} Z_t Z_t^T d\langle B \rangle_t
\]
\[
\leq e^{-2\kappa \bar{\sigma}^2 \xi} - 2\kappa \bar{\sigma}^2 \int_0^T e^{-2\kappa \bar{\sigma}^2 Y_t} f(t, Y_t, Z_t) dt + 2\kappa \bar{\sigma}^2 \int_0^T e^{-2\kappa \bar{\sigma}^2 Y_t} (Z_t dB_t + dK_t)
\]
\[
\leq e^{-2\kappa \bar{\sigma}^2 \xi} + 2\kappa \bar{\sigma}^2 \int_0^T e^{-2\kappa \bar{\sigma}^2 Y_t} \left( \beta_t + \lambda |Y_t| + \frac{\kappa}{2} |Z_t|^2 \right) dt + 2\kappa \bar{\sigma}^2 \int_0^T e^{-2\kappa \bar{\sigma}^2 Y_t} Z_t dB_t,
\]
where we have used the fact that \( K \) is a non-increasing process in the last inequality. Thus, in view of the fact that \( e^{ix} \geq |x| \), we could derive that
\[
\kappa \int_0^T e^{-2\kappa \bar{\sigma}^2 Y_t} |Z_t|^2 dt \leq \kappa^{-1} \bar{\sigma}^2 e^{-2\kappa \bar{\sigma}^2 \xi} + 2 \int_0^T e^{-2\kappa \bar{\sigma}^2 Y_t} (\beta_t + \lambda |Y_t|) dt + 2 \int_0^T e^{-2\kappa \bar{\sigma}^2 Y_t} Z_t dB_t
\]
\[
\leq 2X + 2 \int_0^T e^{-2\kappa \bar{\sigma}^2 Y_t} Z_t dB_t,
\]
where \( X := (1 + \kappa^{-1} \bar{\sigma}^2 - T) \exp \left\{ (2\kappa \bar{\sigma}^2 + \lambda) \sup_{t \in [0, T]} |Y_t| + \int_0^T \beta_t dt \right\} \). It follows that
\[
\kappa^n \mathbb{E} \left[ \left( \int_0^T e^{-2\kappa \bar{\sigma}^2 Y_t} |Z_t|^2 dt \right)^n \right] \leq 2^{2n-1} \mathbb{E}[X^n] + 2^{2n-1} \mathbb{E} \left[ \int_0^T e^{-2\kappa \bar{\sigma}^2 Y_t} Z_t dB_t \right] \right]^n. \tag{9}
\]
Applying BDG inequality \ref{eq:bdg}, for each \( n \geq 1 \), we can find a constant \( A(n) \) depending only on \( n \) and \( \bar{\sigma}^2 \) so that
\[
\mathbb{E} \left[ \left( \int_0^T e^{-2\kappa \bar{\sigma}^2 Y_t} |Z_t|^2 dt \right)^n \right] \leq A(n) \mathbb{E} \left[ \exp \left\{ n \kappa \bar{\sigma}^2 \sup_{t \in [0, T]} |Y_t| \right\} \left( \int_0^T e^{-2\kappa \bar{\sigma}^2 Y_t} |Z_t|^2 dt \right)^{\frac{n}{2}} \right],
\]
which together with the inequality \( ab \leq \frac{1}{2} a^2 + \frac{1}{2} \varepsilon b^2 \) indicates that
\[
\frac{\kappa^n}{2^{2n-1} \mathbb{E} \left[ \int_0^T e^{-2\kappa \bar{\sigma}^2 Y_t} Z_t dB_t \right] \right]^n} \leq \frac{2^{4n-3} A^2(n)}{\kappa^n} \mathbb{E} \left[ \exp \left\{ 2n \kappa \bar{\sigma}^2 \sup_{t \in [0, T]} |Y_t| \right\} \right] + \frac{1}{2} \kappa^n \mathbb{E} \left[ \left( \int_0^T e^{-2\kappa \bar{\sigma}^2 Y_t} |Z_t|^2 dt \right)^n \right]. \tag{10}
\]
Putting \ref{eq:inequality} and \ref{eq:bdg} together, we could derive that
\[
\mathbb{E} \left[ \left( \int_0^T e^{-2\kappa \bar{\sigma}^2 Y_t} |Z_t|^2 dt \right)^n \right] \leq \bar{A}_1(n) \mathbb{E} \left[ \exp \left\{ (2\kappa \bar{\sigma}^2 + \lambda) \sup_{t \in [0, T]} |Y_t| + n \int_0^T \beta_t dt \right\} \right],
\]
where \( \bar{A}_1(n) \) is given by
\[
\bar{A}_1(n) = \frac{2^{2n}(1 + \kappa^{-1} \bar{\sigma}^2 - T) \kappa^n + 2^{4n-2} A^2(n)}{\kappa^{2n}}, \quad \forall n \geq 1.
\]
It follows from Hölder’s inequality that for any \( n \geq 1 \)
\[
\mathbb{E} \left[ \left( \int_0^T |Z_t|^2 dt \right)^n \right] \leq \mathbb{E} \left[ \exp \left\{ 2n \kappa \bar{\sigma}^2 \sup_{t \in [0, T]} |Y_t| \right\} \left( \int_0^T e^{-2\kappa \bar{\sigma}^2 Y_t} |Z_t|^2 dt \right)^n \right]
\]
\[
\leq \mathbb{E} \left[ \exp \left\{ 4n \kappa \bar{\sigma}^2 \sup_{t \in [0, T]} |Y_t| \right\} \right] \frac{1}{2} \mathbb{E} \left[ \left( \int_0^T e^{-2\kappa \bar{\sigma}^2 Y_t} |Z_t|^2 dt \right)^{2n} \right] \frac{1}{2} \tag{11}
\]
\[
\leq \sqrt{\bar{A}_1(2n)} \mathbb{E} \left[ \exp \left\{ (4\kappa \bar{\sigma}^2 + 2\lambda) \sup_{t \in [0, T]} |Y_t| + 2n \int_0^T \beta_t dt \right\} \right].
\]
On the other hand, from (5) and using Assumption (H5), we have that

$$-K_T \leq (2 + \lambda T) \sup_{t \in [0,T]} |Y_t| + \int_0^T \beta_t dt + \frac{\kappa}{2} \int_0^T |Z_t|^2 dt + \int_0^T Z_t dB_t,$$

Using BDG inequality (4) again, we get that for each $n \geq 1$

$$\mathbb{E}[K_T^n] \leq 4^{n-1} \mathbb{E}[(2 + \lambda T)^n \sup_{t \in [0,T]} |Y_t|^n + \left( \int_0^T \beta_t dt \right)^n] + 2^{n-2}\kappa n \mathbb{E}\left[ \left( \int_0^T |Z_t|^2 dt \right)^n \right] + 4^{n-1} A(n) \mathbb{E}\left[ \left( \int_0^T |Z_t|^2 dt \right)^\frac{n}{2} \right].$$

Consequently, by (11) and in view of the fact that $e^x \geq 1 + \frac{|x|^n}{n}$ for any $x \geq 0$, we could find a constant $A(n)$ depending only on $\lambda, \bar{\sigma}, \bar{\sigma}, \kappa, T$, and $n$, such that

$$\mathbb{E}\left[ \left( \int_0^T |Z_t|^2 dt \right)^n + |K_T|^n \right] \leq A(n) \mathbb{E}\left[ \exp\left\{ (4\kappa \bar{\sigma}^2 + 2\lambda)n \sup_{t \in [0,T]} |Y_t| + 2n \int_0^T \beta_t dt \right\} \right],$$

which ends the proof.

Next, using a $\theta$-method formulated by [4], we could get the comparison theorem for quadratic G-BSDEs with unbounded terminal values.

**Lemma 3.8** Let $(Y^l, Z^l, K^l)$ be a $\mathcal{E}_G(\mathbb{R}) \times \mathcal{H}_G(\mathbb{R}^d) \times \mathcal{L}_G(\mathbb{R})$-solution to G-BSDEs (5) with data $(\xi^l, f^l)$, $l = 1, 2$. Suppose $(\xi^1, f^1)$ (resp. $(\xi^2, f^2)$) verifies Assumptions (H1)-(H4). If $\xi^1 \leq \xi^2$ and $f^1(s, y, z) \leq f^2(s, y, z)$, then $Y^1_t \leq Y^2_t$ for any $t \in [0,T]$.

**Proof.** Without loss of generality, assume that $(\xi^1, f^1)$ verifies Assumptions (H1)-(H4), and the other case could be proved in a similar way.

First, we consider the case when $f^1$ is convex in $z$. For each $\theta \in (0, 1)$, we set

$$(\delta_\theta Y, \delta_\theta Z) := \left( \frac{Y^1 - \theta Y^2}{1 - \theta}, \frac{Z^1 - \theta Z^2}{1 - \theta} \right),$$

and then the triple $(\delta_\theta Y, \delta_\theta Z, \frac{1}{1-\theta}K^1)$ satisfies the following G-BSDE on the interval $[0,T]$:

$$\delta_\theta Y_t = \delta_\theta Y_T + \frac{\theta}{1 - \theta} (K^1_T - K^2_T) + \int_t^T \delta_\theta f(s, \delta_\theta Y_s, \delta_\theta Z_s) ds - \int_t^T \delta_\theta Z_s dB_s - \frac{1}{1 - \theta} (K^1_T - K^2_T),$$

with $\delta_\theta f(t, y, z) = \frac{1}{1-\theta} \left( f^1(t, (1-\theta)y + \theta Y^2_t, (1-\theta)z + \theta Z^2_t) - \theta f^2(t, Y^2_t, Z^2_t) \right)$. It is obvious that $\delta_\theta Y_T = \xi^1 + \frac{\theta (1 - \theta)\xi^2}{1-\theta} \leq \xi^1$. With the help of (5), we derive that

$$|f^1(t, \omega, y, z)| \leq \alpha_t(\omega) + \frac{\gamma}{2} + \lambda |y| + \frac{3\gamma}{2} |z|^2,$$

which together with convexity indicates that

$$\delta_\theta f(t, y, z) \leq \frac{1}{1 - \theta} \left( f^1(t, (1-\theta)y + \theta Y^2_t, (1-\theta)z + \theta Z^2_t) - \theta f^2(t, Y^2_t, Z^2_t) \right)$$

$$\leq \lambda |y| + \lambda |Y^2_t| + \frac{1}{1 - \theta} \left( f^1(t, Y^2_t, (1-\theta)z + \theta Z^2_t) - \theta f^2(t, Y^2_t, Z^2_t) \right)$$

$$\leq \lambda |y| + \lambda |Y^2_t| + f^1(t, Y^2_t, z) \leq \alpha_t + \frac{\gamma}{2} + 2\lambda |Y^2_t| + \lambda |y| + \frac{3\gamma}{2} |z|^2.$$
Using Assertion (ii) of Lemma 3.4 (taking \( p = 1, \beta_i = \alpha_t + \frac{\gamma}{2} + 2\lambda|Y_t^2| \) and \( \kappa = 3\gamma \)), we deduce that

\[
\exp \left\{ 3\gamma \tilde{\alpha}^2 e^{\lambda t} (\theta Y_t)^+ \right\} \leq \mathbb{E}_t \left[ \exp \left\{ 3\gamma \tilde{\alpha}^2 e^{\lambda T} \left( |\xi|^1 + \frac{\gamma T}{2} + \int_t^T (\alpha_s + 2\lambda|Y_s^2|) \, ds \right) \right\} \right],
\]

which implies that for every \( \theta \in (0, 1) \) and \( t \in [0, T] \),

\[
3\gamma \tilde{\alpha}^2 (Y_t^1 - Y_t^2)^+ \leq 3\gamma \tilde{\alpha}^2 (Y_t^1 - \theta Y_t^2)^+ + 3(1 - \theta) \gamma \tilde{\alpha}^2 (Y_t^2)^-
\]

\[
\leq (1 - \theta) \left( \mathbb{E}_t \left[ \exp \left\{ 3\gamma \tilde{\alpha}^2 e^{\lambda T} \left( |\xi|^1 + \frac{\gamma T}{2} + \int_t^T (\alpha_s + 2\lambda|Y_s^2|) \, ds \right) \right\} \right] \right) + 3\gamma \tilde{\alpha}^2 (Y_t^2)^-).
\]

Sending \( \theta \to 1 \) yields that \( Y_t^1 \leq Y_t^2 \) for any \( t \in [0, T] \).

Next, for the case that \( f \) is concave in \( z \), we need to use \( \theta \ell^1 - \ell^2 \) instead of \( \ell^1 - \theta \ell^2 \) in the definition of \( \delta \ell \) for \( \ell = Y, Z \). In this case, the triple \( (\delta Y, \delta Z, \frac{\theta}{1 - \theta} K^1) \) solves the following G-BSDE on \([0, T] \):

\[
\delta Y_t = \delta Y_T + \frac{1}{1 - \theta}(K_t^1 - K_T^1) + \int_t^T \delta \theta f(s, \delta Y_s, \delta Z_s) \, ds - \int_t^T \delta \theta Z_s dB_s - \frac{\theta}{1 - \theta} (K_T^1 - K_t^1)
\]

with

\[
\delta \theta f(t, y, z) = \frac{1}{1 - \theta} \left( \theta f^1(t, Y_t^1, Z_t^1) - f^2(t, -(1 - \theta)y + \theta Y_t^1, -(1 - \theta)z + \theta Z_t^1) \right).
\]

By Assumptions (H1), it is easy to check that \( \delta \theta Y_T \leq \frac{\theta \xi^1 - \xi^2}{1 - \theta} = -\xi^1 \) and

\[
\delta \theta f(t, y, z) \leq \lambda|y| + \lambda|Y_t^1| + \frac{1}{1 - \theta} (\theta f^1(t, Y_t^1, Z_t^1) - f^2(t, -(1 - \theta)y + \theta Y_t^1, -(1 - \theta)z + \theta Z_t^1))
\]

\[
\leq \lambda|y| + \lambda|Y_t^1| - f^1(t, Y_t^1, -(1 - \theta)z + \theta Z_t^1)) \leq \alpha_t + \frac{\gamma}{2} + 2\lambda|Y_t^1| + \lambda|y| + \frac{3\gamma}{2}|z|^2.
\]

Thus, in view of Assertion (ii) of Lemma 3.4 we have that for every \( \theta \in (0, 1) \) and \( t \in [0, T] \),

\[
3\gamma \tilde{\alpha}^2 (Y_t^1 - Y_t^2)^+ \leq 3\gamma \tilde{\alpha}^2 (\theta Y_t^1 - Y_t^2)^+ + 3(1 - \theta) \gamma \tilde{\alpha}^2 (Y_t^1)^+
\]

\[
\leq (1 - \theta) \left( \mathbb{E}_t \left[ \exp \left\{ 3\gamma \tilde{\alpha}^2 e^{\lambda T} \left( |\xi|^1 + \frac{\gamma T}{2} + \int_t^T (\alpha_s + 2\lambda|Y_s^2|) \, ds \right) \right\} \right] \right) + 3\gamma \tilde{\alpha}^2 (Y_t^1)^+.
\]

which completes the proof by sending \( \theta \to 1 \). ■

Now we are ready to state the main result of this section, which involves the existence and uniqueness of unbounded solutions to quadratic G-BSDE.

**Theorem 3.9** Assume that (H1)-(H4) are satisfied. Then, G-BSDE (11) admits a unique solution \( (Y, Z, K) \in \mathcal{E}_G(\mathbb{R}) \times \mathcal{H}_G(\mathbb{R}^d) \times \mathcal{L}_G(\mathbb{R}) \).

**Proof.** The uniqueness is immediate from Lemma 3.4. Indeed, let \( (Y^l, Z^l, K^l) \) be a \( \mathcal{E}_G(\mathbb{R}) \times \mathcal{H}_G(\mathbb{R}^d) \times \mathcal{L}_G(\mathbb{R}) \)-solution to G-BSDE (11), \( l = 1, 2 \). It follows from Lemma 3.4 that \( Y_1 = Y_2 \). Then, applying G-Itô’s formula to \( |Y^1 - Y^2|^2 \) yields that \( Z_1 = Z_2 \) and then \( K_1 = K_2 \). Thus, we only need to prove the existence. We will construct a solution through a sequence of quadratic G-BSDEs with bounded terminal value and generator. The proof will be divided into the following three steps.

**1. The uniform estimates.** Denote by \( f_0(t) = f(t, 0, 0) \) for narrative convenience. Then, for each positive integer \( m \geq 1 \), set

\[
f^{(m)} = (\ell \wedge m) \vee (-m) \quad \text{for} \quad \ell = \xi, f_0(t), \text{and} \quad f^{(m)}(t, y, z) = f(t, y, z) - f_0(t) + f_0^{(m)}(t).
\]
One can easily check that the terminal value $\xi^m$ and generator $f^{(m)}$ satisfies Assumption 2.14. Thus, it follows from Theorem 5.3 that, the $G$-BSDE with data $(\xi^{(m)}, f^{(m)})$ admits a unique solution $(Y^{(m)}, Z^{(m)}, K^{(m)}) \in \mathcal{E}_G(\mathbb{R}) \times \mathcal{H}_G(\mathbb{R}^d) \times \mathcal{L}_G(\mathbb{R})$. From Assumption (H1), we have that

$$|f^{(m)}(t, \omega, y, z)| \leq \alpha_t(\omega) + \frac{\gamma}{2} + \lambda |y| + \frac{3\gamma}{2}|z|^2. \tag{12}$$

In spirit of Lemma 3.4 and Remark 2.6, we get that for any $p \geq 1$

$$\sup_{m \geq 1} \hat{\mathbb{E}} \left[ \exp \left\{ 3p\gamma \hat{\sigma}^2 \sup_{0 \leq t \leq T} |Y_t^{(m)}| \right\} \right] \leq \hat{\mathbb{E}} \left[ \exp \left\{ 3p\gamma \hat{\sigma}^2 e^{\lambda T} \left( |\xi| + \frac{T}{2} + \int_0^T \alpha_t dt \right) \right\} \right]$$

$$\leq \bar{A}(G) \hat{\mathbb{E}} \left[ \exp \left\{ 6p\gamma \hat{\sigma}^2 e^{\lambda T} \left( |\xi| + \frac{T}{2} + \int_0^T \alpha_t dt \right) \right\} \right], \tag{13}$$

where the constant $\bar{A}(n)$ is independent of $m$. Therefore, in view of (13), we conclude that

$$\sup_{m \geq 1} \hat{\mathbb{E}} \left[ \left( \int_0^T |Z_t^{(m)}|^2 dt \right)^n + |K_T^{(m)}|^n \right] < \infty, \forall n \geq 1. \tag{14}$$

2. $\theta$-method. We first consider the case when the generator $f$ is convex in $z$. For each fixed $m, q \geq 1$ and $\theta \in (0, 1)$, we define

$$\delta_\theta Y^{(m, q)} := \frac{Y^{(m+q)} - \theta Y^{(m)}}{1 - \theta}, \quad \delta_\theta Z^{(m, q)} := \frac{Z^{(m+q)} - \theta Z^{(m)}}{1 - \theta}.$$

Then, the triple $(\delta_\theta Y^{(m, q)}, \delta_\theta Z^{(m, q)}, \frac{1}{1 - \theta} K^{(m)})$ solves the following $G$-BSDE:

$$\delta_\theta Y_t^{(m, q)} = \delta_\theta \xi^{(m, q)} + \frac{\theta}{1 - \theta} (K_T^{(m)} - K_t^{(m)}) - \int_t^T \delta_\theta Z_s^{(m, q)} dB_s - \frac{1}{1 - \theta} (K_T^{(m+q)} - K_t^{(m+q)})$$

$$+ \int_t^T \left( \delta_\theta f^{(m, q)}(s, \delta_\theta Y_s^{(m, q)}, \delta_\theta Z_s^{(m, q)}) + \delta_\theta f_0^{(m, q)}(s) \right) ds, \tag{15}$$

where the terminal value generator are given by

$$\delta_\theta \xi^{(m, q)} = \frac{\xi^{(m+q)} - \theta \xi^{(m)}}{1 - \theta}, \quad \delta_\theta f^{(m, q)}(t) = \frac{1}{1 - \theta} \left( f^{(m+q)}(t) - \theta f^{(m)}(t) \right) - f_0(t),$$

$$\delta_\theta f^{(m, q)}(t, y, z) = \frac{1}{1 - \theta} \left( f(t, (1 - \theta)y + \theta Y_t^{(m)}), (1 - \theta)z + \theta Z_t^{(m)} \right) - \theta f(t, Y_t^{(m)}, Z_t^{(m)}).$$

A direct computation yields that

$$\delta_\theta \xi^{(m, q)} = \xi^{(m)} + \frac{1}{1 - \theta} (\xi^{(m+q)} - \xi^{(m)}) \leq |\xi| + \frac{1}{1 - \theta} (|\xi| - m)^+, \quad \delta_\theta f_0^{(m, q)}(t) \leq f_0^{(m)}(t) - f_0(t) + \frac{1}{1 - \theta} (|f_0(t)| - m)^+ \leq 2 \frac{1}{1 - \theta} (|f_0(t)| - m)^+. \tag{13}$$
Applying Assertion (ii) of Lemma 3.4 to Equation (15), we derive that for any $p \geq 1$
\[
\exp \left\{ 3p \gamma \tilde{\sigma}^2 e^{\lambda t} \left( \delta Y_t^{(m,q)} \right)^+ \right\} \leq \tilde{E}_t \left[ \exp \left\{ 3p \gamma \tilde{\sigma}^2 e^{\lambda T} \left( \rho(\theta, m) + |\xi| + \frac{\gamma T}{2} + \int_t^T (\alpha_s + 2\lambda |Y_s^{(m)}|) \, ds \right) \right\} \right],
\]
(16)
where $\rho(\theta, m)$ is given by
\[
\rho(\theta, m) := \frac{1}{1-\theta}(|\xi| - m)^+ + \frac{2}{1-\theta} \int_0^T (|f_0(t)| - m)^+ \, dt.
\]
On the other hand, we define
\[
\delta Y_t^{(m,q)} := \frac{Y_t^{(m)} - \theta Y_t^{(m+q)}}{1-\theta}, \quad \delta Z_t^{(m,q)} := \frac{Z_t^{(m)} - \theta Z_t^{(m+q)}}{1-\theta}.
\]
Then, by a similar analysis, we conclude that
\[
\exp \left\{ 3p \gamma \tilde{\sigma}^2 e^{\lambda t} \left( \delta Y_t^{(m,q)} \right)^+ \right\} \leq \tilde{E}_t \left[ \exp \left\{ 3p \gamma \tilde{\sigma}^2 e^{\lambda T} \left( \rho(\theta, m) + |\xi| + \frac{\gamma T}{2} + \int_t^T (\alpha_s + 2\lambda |Y_s^{(m+p)}|) \, ds \right) \right\} \right],
\]
(17)
Note that
\[
\left( \delta Y_t^{(m,q)} \right)^- \leq \frac{\theta (Y_t^{(m)} - \theta Y_t^{(m+q)})^+ + (1-\theta) |Y_t^{(m+q)}|}{1-\theta} \leq \left( \delta Y_t^{(m,q)} \right)^+ + 2|Y_t^{(m+q)}|.
\]
Thus, it follows from (16) and (17) that
\[
\exp \left\{ 3p \gamma \tilde{\sigma}^2 e^{\lambda t} \left| \delta Y_t^{(m,q)} \right| \right\} \leq \exp \left\{ 3p \gamma \tilde{\sigma}^2 e^{\lambda t} \left( \left( \delta Y_t^{(m,q)} \right)^+ + \left( \delta Y_t^{(m,q)} \right)^+ + 2|Y_t^{(m+q)}| \right) \right\}
\]
\[
\leq \tilde{E}_t \left[ \exp \left\{ 3p \gamma \tilde{\sigma}^2 e^{\lambda T} \left( \rho(\theta, m) + |\xi| + \frac{\gamma T}{2} + |Y_t^{(m+q)}| + \int_t^T (\alpha_s + 2\lambda |Y_s^{(m)}| + 2\lambda |Y_s^{(m+q)}|) \, ds \right) \right\} \right]^2
\]
\[
\leq \tilde{E}_t \left[ \exp \left\{ 6p \gamma \tilde{\sigma}^2 e^{\lambda T} \left( \rho(\theta, m) + |\xi| + \frac{\gamma T}{2} + |Y_t^{(m+q)}| + \int_t^T (\alpha_s + 2\lambda |Y_s^{(m)}| + 2\lambda |Y_s^{(m+q)}|) \, ds \right) \right\} \right],
\]
where we have used Jensen’s inequality in the last inequality.

Consequently, from Remark 2.6 and Hölder’s inequality, we have
\[
\tilde{E} \left[ \exp \left\{ 3p \gamma \tilde{\sigma}^2 \sup_{t \in [0,T]} \left| \delta Y_t^{(m,q)} \right| \right\} \right] \leq A(G) \tilde{E} \left[ \exp \left\{ 12p \gamma \tilde{\sigma}^2 e^{\lambda T} \left( \rho(\theta, m) + |\xi| + \frac{\gamma T}{2} + (2\lambda T + 1) \sup_{t \in [0,T]} (|Y_t^{(m)}| + |Y_t^{(m+q)}|) + \int_0^T \alpha_t \, dt \right) \right\} \right]
\]
(18)
\[
\leq A(p) \tilde{E} \left[ \exp \left\{ \frac{24p \gamma \tilde{\sigma}^2 e^{\lambda T}}{1-\theta} \left( (|\xi| - m)^+ + \int_0^T 2(|f_0(t)| - m)^+ \, dt \right) \right\} \right]^{\frac{1}{2}},
\]
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where, in view of (13) and H"older's inequality again,

\[
\tilde{A}(p) := \tilde{A}(G) \sup_{m,q \geq 1} \mathbb{E} \left[ \exp \left\{ 24p \gamma \tilde{\sigma}^2 e^{\lambda T} \left( |\xi| + \frac{\gamma T}{2} + (2\lambda T + 1) \sup_{t \in [0,T]} (|Y_t^{(m)}| + |Y_t^{(m+q)}|) + \int_0^T \alpha_s dt \right) \right\} \right]^{\frac{1}{2}} < \infty.
\]

Next, we shall deal with the case that when the generator \( f \) is concave in \( z \). We shall use \( \theta \ell^{(m+q)} - \ell^{(m)} \) and \( \theta \ell^{(m+q)} - \ell^{(m)} \) instead of \( \ell^{(m+q)} - \theta \ell^{(m)} \) and \( \ell^{(m)} - \theta \ell^{(m+q)} \) in the definition of \( \delta \ell^{(m,q)} \) and \( \delta \ell^{(m,q)} \) for \( \ell = Y, Z \), respectively.

In this case, the triple \((\delta Y^{(m,q)}, \delta Z^{(m,q)}, \frac{\theta}{1-\theta} K^{(m)})\) solves the following G-BSDE:

\[
\delta Y_t^{(m,q)} = \delta \xi^{(m,q)} + \frac{1}{1-\theta}(K_T^{(m)} - K_t^{(m)}) + \int_t^T \left( \delta \theta f^{(m,q)}(s, \delta Y_s^{(m,q)}, \delta Z_s^{(m,q)}) + \delta \theta f_0^{(m,q)}(s) \right) ds
\]

\[- \int_t^T \delta \theta Z_s^{(m,q)} dB_s - \frac{\theta}{1-\theta}(K_T^{(m+q)} - K_t^{(m+q)})
\]

with

\[
\delta \xi^{(m,q)} = \frac{\theta \xi^{(m+q)} - \xi^{(m)}}{1-\theta}, \quad \delta \theta f_0^{(m,q)}(t) = \frac{1}{1-\theta} \left( \theta f_0^{(m+q)}(t) - f_0^{(m)}(t) \right) + f_0(t),
\]

\[
\delta \theta f^{(m,q)}(t, y, z) = \frac{1}{1-\theta} \left( \theta f(t, Y_t^{(m+q)}, Z_t^{(m+q)}) - f(t, -(1-\theta)y + \theta Y_t^{(m+q)}, -(1-\theta)z + \theta Z_t^{(m+q)}) \right).
\]

By Assumptions (H1) and (H3), it is easy to check that

\[
\delta \xi^{(m,q)} = -\xi^{(m+q)} + \frac{1}{1-\theta}(\xi^{(m+q)} - \xi^{(m)}) \leq |\xi| + \frac{1}{1-\theta}(|\xi| - m)^+,
\]

\[
\delta \theta f_0^{(m,q)}(t) \leq -f_0^{(m+q)}(t) + f_0(t) + \frac{1}{1-\theta}(|f_0(t)| - m)^+ \leq \frac{2}{1-\theta}(|f_0(t)| - m)^+ + \alpha_t + \frac{\gamma}{2} + 2\lambda|Y_t^{(m+q)}| + \lambda|y| + \frac{3\gamma}{2}|z|^2.
\]

Consequently, Inequalities (16) and (17) should be replaced with

\[
\exp \left\{ 3p \gamma \tilde{\sigma}^2 e^{\lambda T} \left( \delta \theta Y_t^{(m,q)} \right)^+ \right\} \leq \mathbb{E}_t \left[ \exp \left\{ 3p \gamma \tilde{\sigma}^2 e^{\lambda T} \left( \rho(\theta, m) + |\xi| + \frac{\gamma T}{2} + \int_t^T (\alpha_s + 2\lambda|Y_s^{(m+q)}|) ds \right) \right\} \right]
\]

\[
\exp \left\{ 3p \gamma \tilde{\sigma}^2 e^{\lambda T} \left( \delta \theta Y_t^{(m,q)} \right)^- \right\} \leq \mathbb{E}_t \left[ \exp \left\{ 3p \gamma \tilde{\sigma}^2 e^{\lambda T} \left( \rho(\theta, m) + |\xi| + \frac{\gamma T}{2} + \int_t^T (\alpha_s + 2\lambda|Y_s^{(m+q)}|) ds \right) \right\} \right].
\]

It follows from \((\delta Y^{(m,q)})^- \leq (\delta Y^{(m,q)})^+ + 2|Y^{(m)}|\) that

\[
\exp \left\{ 3p \gamma \tilde{\sigma}^2 e^{\lambda T} \left( \delta \theta Y_t^{(m,q)} \right)^+ \right\}
\]

\[
\leq \mathbb{E}_t \left[ \exp \left\{ 6p \gamma \tilde{\sigma}^2 e^{\lambda T} \left( \rho(\theta, m) + |\xi| + \frac{\gamma T}{2} + |Y_t^{(m)}| + \int_t^T (\alpha_s + 2\lambda|Y_s^{(m)|} + 2\lambda|Y_s^{(m+q)}|) ds \right) \right\} \right],
\]

and then Inequality (18) is still true when \( f \) is concave in \( z \).

3. The convergence. Since

\[
\exp \left\{ \frac{24p \gamma \tilde{\sigma}^2 e^{\lambda T}}{1-\theta} \left( (|\xi| - m)^+ + \int_0^T 2(|f_0(t)| - m)^+ dt \right) \right\} \in L_G^1(\Omega) \downarrow 1
\]
as \( m \to \infty \), from nonlinear monotone convergence theorem (Assertion (ii) of Lemma \ref{lem2.3}), we have that for each \( p \geq 1 \) and \( \theta \in (0, 1) \),

\[
\lim_{m \to \infty} \hat{E} \left[ \exp \left\{ \frac{24p \gamma \tilde{\sigma}^2 e^{\lambda t}}{1 - \theta} \left( \langle |\xi| - m \rangle^+ + \int_0^T 2(\langle |f_0(s)| - m \rangle^+ ds) \right) \right\} \right]^{1/2} = 1,
\]

which together with Inequality \((18)\) indicates that,

\[
\limsup_{m \to \infty} \sup_{q \geq 1} \hat{E} \left[ \sup_{t \in [0, T]} |\delta \gamma Y_t^{(m,q)}| \right] \leq \bar{A}(p), \forall \theta \in (0, 1).
\]

It follows that for each \( n \geq 1 \) and \( \theta \in (0, 1) \),

\[
\limsup_{m \to \infty} \sup_{q \geq 1} \hat{E} \left[ \sup_{t \in [0, T]} |\delta \gamma Y_t^{(m,q)}|^n \right] \leq \frac{\bar{A}(1)n!}{3n \gamma n \tilde{\sigma}^{2n}}.
\]

In view of the following fact

\[
Y_t^{(m+q)} - Y_t^{(m)} = (1 - \theta)(\delta \gamma Y_t^{(m,q)} - Y_t^{(m)}) \quad (\text{resp.} \quad (1 - \theta)(\delta \gamma Y_t^{(m,q)} + Y_t^{(m+q)}))
\]

when \( f \) is convex (resp. concave) in \( z \), we derive that for any \( n \geq 1 \) and \( \theta \in (0, 1) \),

\[
\limsup_{m \to \infty} \sup_{q \geq 1} \hat{E} \left[ \sup_{t \in [0, T]} \left| Y_t^{(m+q)} - Y_t^{(m)} \right|^n \right] \leq 2^{n-1}(1 - \theta)^n \left( \frac{\bar{A}(1)n!}{3n \gamma n \tilde{\sigma}^{2n}} + \sup_{m \geq 1} \hat{E} \left[ \sup_{t \in [0, T]} \left| Y_t^{(m)} \right|^n \right] \right).
\]

Sending \( \theta \to 1 \) and using \((19)\), we could find a continuous process \( Y \in \mathcal{C}(\mathbb{R}) \) such that

\[
\lim_{m \to \infty} \hat{E} \left[ \sup_{t \in [0, T]} \left| Y_t^{(m)} - Y_t \right|^n \right] = 0, \forall n \geq 1. \quad (19)
\]

Indeed, from \((13)\) and Assertion (i) of Lemma \ref{lem2.3} we have that for any \( p \geq 1 \)

\[
\hat{E} \left[ \exp \left\{ 3p \gamma \tilde{\sigma}^2 \sup_{0 \leq t \leq T} \left| Y_t \right| \right\} \right] \leq \bar{A}(G) \hat{E} \left[ \exp \left\{ 6p \gamma \tilde{\sigma}^2 e^{\lambda T} \left( |\xi| + \frac{\gamma T}{2} + \int_0^T \alpha_s ds \right) \right\} \right].
\]

Now, applying G-Itô’s formula to \( \left| Y_t^{(m+q)} - Y_t^{(m)} \right|^2 \) yields that

\[
\hat{E} \left[ \int_0^T \left| Z_t^{(m+q)} - Z_t^{(m)} \right|^2 dt \right] \leq \tilde{\sigma}^2 \hat{E} \left[ \int_0^T (Z_t^{(m+q)} - Z_t^{(m)})^2 d(B)_t \right]
\]

\[
\leq \tilde{\sigma}^2 \hat{E} \left[ \sup_{t \in [0, T]} \left| Y_t^{(m+q)} - Y_t^{(m)} \right|^2 + \sup_{t \in [0, T]} \left| Y_t^{(m+q)} - Y_t^{(m)} \right|^{1/2} \right] \]

\[
\leq \tilde{\sigma}^2 \hat{E} \left[ \sup_{t \in [0, T]} \left| Y_t^{(m+q)} - Y_t^{(m)} \right|^2 \right] + \tilde{\sigma}^2 \hat{E} \left[ \left| \Gamma^{(m,q)} \right|^2 \right] \hat{E} \left[ \sup_{t \in [0, T]} \left| Y_t^{(m+q)} - Y_t^{(m)} \right|^2 \right] \frac{1}{2},
\]

with

\[
\Gamma^{(m,q)} := \int_0^T f^{(m+q)}(t, Y_t^{(m+q)}, Z_t^{(m+q)}) - f^{(m)}(t, Y_t^{(m)}, Z_t^{(m)}) dt + |K_t^{(m+q)}| + |K_t^{(m)}|.
\]
In spirit of Inequalities (12), (13), and (14), we see that \( \sup_{m,q \geq 1} \hat{E}[|\Gamma(m,q)|^2] < \infty \). Thus, by (19) and (20), there is a process \( Z \in M^2_G(0,T) \) so that

\[
\lim_{m \to \infty} \hat{E}\left[ \int_0^T |Z_t^{(m)} - Z_t|^2 dt \right] = 0. \tag{21}
\]

In view of Assertion (i) of Lemma 2.3 and (14), we have that

\[
\hat{E}\left[ \left( \int_0^T |Z_t^2 dt \right)^n \right] < \infty,
\]

which together with Lemma 3.10 and (14), (21) implies that

\[
\lim_{m \to \infty} \hat{E}\left[ \left( \int_0^T |Z_t^{(m)} - Z_t|^2 dt \right)^n \right] = 0, \quad \forall n \geq 1. \tag{22}
\]

On the other hand, from Assumption (H1), we have

\[
|f^{(m)}(t, Y_t^{(m)}, Z_t^{(m)}) - f(t, Y_t, Z_t)| \leq \lambda |Y_t^{(m)} - Y_t| + \gamma (1 + |Z_t^{(m)}| + |Z_t|) |Z_t^{(m)} - Z_t| + (|f_0(t)| - m^+).
\]

Then, from Hölder’s inequality, we have for each \( n \geq 1, \)

\[
\hat{E}\left[ \left( \int_0^T |f^{(m)}(t, Y_t^{(m)}, Z_t^{(m)}) - f(t, Y_t, Z_t)| dt \right)^n \right] 
\leq 3^{n-1} \lambda^n T^n \hat{E}\left[ \sup_{t \in [0,T]} |Y_t^{(m)} - Y_t|^n \right] + 3^{n-1} \gamma^n \hat{E}\left[ \left( \int_0^T (|f_0(t)| - m^+) dt \right)^n \right] 
+ 3^{n-1} \gamma^n \hat{E}\left[ \left( \int_0^T (1 + |Z_t^{(m)}| + |Z_t|)^2 dt \right)^n \right] \frac{1}{2} \hat{E}\left[ \left( \int_0^T |Z_t^{(m)} - Z_t|^2 dt \right)^n \right] \frac{1}{2}.
\]

which converges to 0 as \( m \to \infty \) in view of Equations (19) and (22) and Assertion (ii) of Lemma 2.3.

We set

\[ K_t = Y_t - Y_0 + \int_0^t f(s, Y_s, Z_s) ds - \int_0^t Z_s dB_s. \]

Therefore, \( \hat{E}\left[ |K_t - K_t^{(m)}|^n \right] \to 0 \) for each \( n \geq 1 \). Thus, \( K \) is a non-increasing \( G \)-martingale, and then \((Y, Z, K) \in \mathcal{E}_G(\mathbb{R}) \times \mathcal{H}_G(\mathbb{R}) \times \mathcal{L}_G(\mathbb{R}) \) satisfies Equation (5). The proof is complete. \( \square \)

**Lemma 3.10** Let \( X_n \in L^1_G(\Omega) \) for \( n \geq 1 \) such that \( \sup_{n \geq 1} \hat{E}[|X_n|^{2p}] < \infty \) for some \( p \geq 1 \). If \( \hat{E}[|X_n|] \) converges to 0 as \( n \to \infty \), then \( \lim_{n \to \infty} \hat{E}[|X_n|^p] = 0. \)

**Proof.** For any \( \varepsilon > 0 \), we have

\[
\hat{E}[|X_n|^p] \leq \varepsilon^p + \hat{E}[|X_n|^p 1_{|X_n| > \varepsilon}] \leq \varepsilon^p + \varepsilon^{-\frac{p}{p-1}} \hat{E}[|X_n|^{2p}]^{\frac{1}{p}} \hat{E}[|X_n|^p]^{\frac{1}{p}}.
\]

Therefore, we have

\[
\limsup_{n \to \infty} \hat{E}[|X_n|^p] \leq \varepsilon^p.
\]

Sending \( \varepsilon \to 0 \), we complete the proof. \( \square \)
4 Multi-dimensional quadratic G-BSDEs

In this section, we consider multi-dimensional quadratic G-BSDEs on time interval \([0, T]\):

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t), \tag{23}
\]

where the generators

\[
f(t, \omega, y, z) = (f^1(t, \omega, y, z), \cdots, f^n(t, \omega, y, z))^T : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \to \mathbb{R}^n.
\]

For sake of convenience, denote by \(y^l\) and \(z^l\) the \(l\)-th component of \(y\) and the \(l\)-th row of \(z\) for each argument \((y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}\), respectively. Consider the following assumptions.

(B1) For each \(l = 1, \cdots, n\), \(f^l(t, \omega, y, z)\) depends only on the \(l\)-th row \(z^l\) of the argument \(z\) and is convex or concave in \(z^l\).

(B2) For each \((t, \omega) \in [0, T] \times \Omega\) and \((y, z), (\bar{y}, \bar{z}) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}\),

\[
|f(t, \omega, y, z) - f(t, \omega, \bar{y}, \bar{z})| \leq \lambda|y - \bar{y}| + \gamma(1 + |z| + |\bar{z}|)|z - \bar{z}|.
\]

(B3) There exists a modulus of continuity \(w : [0, \infty) \to [0, \infty)\) such that for each \((y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}\) and \((t, \omega), (\bar{t}, \bar{\omega}) \in [0, T] \times \Omega\),

\[
|f(t, \omega, y, z) - f(\bar{t}, \bar{\omega}, y, z)| \leq w(|t - \bar{t}| + \|\omega - \bar{\omega}\|).
\]

(B4) Both the terminal value \(\xi \in L^1_G(\Omega; \mathbb{R}^n)\) and \(\int_0^T \alpha_t dt\) have exponential moments of arbitrary order, i.e.,

\[
\hat{\mathbb{E}}\left[\exp\left\{p|\xi| + p \int_0^T \alpha_t dt\right\}\right] < \infty \quad \text{for any } p \geq 1.
\]

Lemma 4.1 Assume that all Assumptions (B1)-(B4) hold and \(U \in \mathcal{E}_G(\mathbb{R}^n)\). Then, the following multi-dimensional decoupled G-BSDE on \([0, T]\):

\[
Y^l_t = \xi^l + \int_t^T f^l(s, U_s, Z^l_s)ds - \int_t^T Z^l_s dB_s - (K^l_T - K^l_t), \quad \forall l = 1, \ldots, n,
\]

admits a unique solution \((Y, Z, K) \in \mathcal{E}_G(\mathbb{R}^n) \times \mathcal{H}_G(\mathbb{R}^{n \times d}) \times \mathcal{L}_G(\mathbb{R}^n)\).

Proof. From Assumptions (B1)-(B3), we have that for \(l = 1, \ldots, n\),

\[
|f^l(t, U_t, z^l)| \leq \alpha_t + \frac{\gamma}{2} + \lambda|U_t| + \frac{3\gamma}{2}|z^l|^2. \tag{24}
\]

From Assumption (B4) and Hölder’s inequality, we have

\[
\hat{\mathbb{E}}\left[\exp\left\{p|\xi| + p \int_0^T (\alpha_t + \lambda|U_t|) dt\right\}\right] < \infty \quad \text{for each } p \geq 1.
\]

Consequently, applying Theorem 3.3, we have the desired result.

Now, following the idea of [11], we study the well-posedness of solutions to multi-dimensional quadratic G-BSDE (23) of diagonally quadratic generators.
**Theorem 4.2** Assume that all Assumptions (B1)-(B4) are satisfied. Then, the multi-dimensional G-BSDE (23) admits a unique solution \((Y, Z, K) \in \mathcal{E}_G(\mathbb{R}^n) \times \mathcal{H}_G(\mathbb{R}^{n \times d}) \times \mathcal{L}_G(\mathbb{R}^n).

**Proof.** With the help of Lemma 4.2, the iterative method in the proof of [11, Theorem 2.8] still works here. Indeed, we firstly set \(Y^{(0)} = 0\), and define recursively the sequence of stochastic processes \((Y^{(m)})_{m=1}^\infty\) through solution of the following multi-dimensional G-BSDE on \([0, T]::

\[
Y^{(m):l}_t = \xi^l + \int_t^T f^l(s, Y^{(m-1)}_s, Z^{(m):l}_s)ds - \int_t^T Z^{(m):l}_s dB^l_s - (K^{(m):l}_T - K^{(m):l}_t), \quad \forall l = 1, \ldots, n. \quad (25)
\]

From Lemma 4.2 we get that \((Y^{(m)}, Z^{(m)}, K^{(m)}) \in \mathcal{E}_G(\mathbb{R}^n) \times \mathcal{H}_G(\mathbb{R}^{n \times d}) \times \mathcal{L}_G(\mathbb{R}^n)\). Next, we use Lemma 3.4 to establish a uniform estimate on \((Y^{(m)}, Z^{(m)}, K^{(m)})\), and then utilize a \(\theta\)-method to get the convergence of \(Y^{(m)}\) and the uniqueness without any further difficulty. For the reader’s convenience, we sketch the proof.

In view of (24) and Assertion (i) of Lemma 3.4 (taking \(\beta_t = \alpha_t + \frac{\gamma}{2} + \lambda |Y^{(m-1)}_t|, \lambda = 0\) and \(\kappa = 3\gamma\)), we have that for any \(p \geq 1\) and \(l = 1, \ldots, n,

\[
\exp\left\{3p\gamma|Y^{(m):l}_t|\right\} \leq \hat{E}_t\left[\exp\left\{3p\gamma|\xi| + \int_t^T \left(\alpha_s + \frac{\gamma}{2} + \lambda |Y^{(m-1)}_s|\right) ds\right]\right], \quad \forall m \geq 1.
\]

From Jensen’s inequality, we have

\[
\exp\left\{3p\gamma|Y^{(m)}_t|\right\} \leq \hat{E}_t\left[\exp\left\{3np\gamma|\xi| + \int_t^T \left(\alpha_s + \frac{\gamma}{2} + \lambda |Y^{(m-1)}_s|\right) ds\right]\right], \quad \forall m \geq 1.
\]

In spirit of Remark 2.6 we get that for any \(p \geq 1\), \(m \geq 1\) and \(t \in [0, T],

\[
\hat{E}\left[\exp\left\{3p\gamma^2 \sup_{0 \leq s \leq T} |Y^{(m)}_s|\right\}\right] \leq \hat{A}(G)\hat{E}\left[\exp\left\{6np\gamma^2 \left(|\xi| + \int_t^T \left(\alpha_s + \frac{\gamma}{2} + \lambda |Y^{(m-1)}_s|\right) ds\right)\right]\right]
\]

\[
\leq \sqrt{A(p)}\hat{E}\left[\exp\left\{12np\gamma^2 \lambda(T-t) \sup_{0 \leq s \leq T} |Y^{(m-1)}_s|\right\}\right]^{\frac{1}{2}}
\]

with

\[
A(p) = \hat{A}(G)^2\hat{E}\left[\exp\left\{12np\gamma^2 \left(|\xi| + \int_t^T \left(\alpha_s + \frac{\gamma}{2} \right) ds\right)\right\}\right] < \infty.
\]

Define

\[
\mu := \begin{cases}
4n\lambda T, & \text{if } 4n\lambda T \text{ is an integer;} \\
4n\lambda T + 1, & \text{otherwise.}
\end{cases}
\]

If \(\mu = 1\), it follows from (26) that for each \(p \geq 1\) and \(m \geq 1\)

\[
\hat{E}\left[\exp\left\{3p\gamma^2 \sup_{0 \leq s \leq T} |Y^{(m)}_s|\right\}\right] \leq (A(p))^{\frac{1}{2}}\hat{E}\left[\exp\left\{3p\gamma^2 \sup_{0 \leq s \leq T} |Y^{(m-1)}_s|\right\}\right]^{\frac{1}{2}},
\]

which implies that

\[
\hat{E}\left[\exp\left\{3p\gamma^2 \sup_{0 \leq s \leq T} |Y^{(m)}_s|\right\}\right] \leq (A(p))^{\frac{1}{2} + \frac{1}{2} + \frac{\mu}{2}}\hat{E}\left[\exp\left\{3p\gamma^2 \sup_{0 \leq s \leq T} |Y^{(0)}_s|\right\}\right]^{\frac{1}{2}}
\]

\[
\leq A(p) \leq \hat{A}(G)^2\hat{E}\left[\exp\left\{24np\gamma^2 |\xi|\right\}\right]\hat{E}\left[\exp\left\{24np\gamma^2 \int_0^T \left(\alpha_s + \frac{\gamma}{2}\right) ds\right\}\right].
\]
If $\mu = 2$, proceeding identically as in the above, we have for any $p \geq 1$,

$$
\mathbb{E}\left[ \exp \left\{ 3p_0^2 \sup_{0 \leq s \leq T} |Y^{(m)}_s| \right\} \right] 
\leq |\hat{A}(G)|^p \mathbb{E}\left[ \exp \left\{ 24n^2|Y^{(m)}_T| \right\} \right] 
\geq |\hat{A}(G)|^p \mathbb{E}\left[ \exp \left\{ 768n^2p_0^2 \int_0^T (\alpha_s + \frac{\gamma}{2}) ds \right\} \right].
$$

(28)

Then, consider the following $G$-BSDEs on time interval $[0, T - (4n\lambda)^{-1})$ for each $m \geq 1$:

$$
Y^{(m);l}_t = Y^{(m);l}_{T - (4n\lambda)^{-1}} + \int_t^{T - (4n\lambda)^{-1}} f(s, Y^{(m-1)}_s, Z^{(m);l}_s) ds - \int_t^{T - (4n\lambda)^{-1}} Z^{(m);l}_s dB_s - (K^{(m);l}_{T - (4n\lambda)^{-1}} - K^{(m);l}_t).
$$

Proceeding identically as to derive (24), we have

$$
\mathbb{E}\left[ \exp \left\{ 3p_0^2 \sup_{0 \leq s \leq T} |Y^{(m)}_s| \right\} \right] 
\leq |\hat{A}(G)|^p \mathbb{E}\left[ \exp \left\{ 192n^2p_0^2|Y^{(m-1)}_T| \right\} \right] 
\geq |\hat{A}(G)|^p \mathbb{E}\left[ \exp \left\{ 384n^2p_0^2 \int_0^T (\alpha_s + \frac{\gamma}{2}) ds \right\} \right].
$$

where we have used (28) in the last inequality. From the last two inequalities and Hölder’s inequality, we get

$$
\mathbb{E}\left[ \exp \left\{ 3p_0^2 \sup_{0 \leq s \leq T} |Y^{(m)}_s| \right\} \right] 
\leq |\hat{A}(G)|^p \mathbb{E}\left[ \exp \left\{ 384n^2p_0^2|Y^{(m)}_T| \right\} \right] 
\geq |\hat{A}(G)|^p \mathbb{E}\left[ \exp \left\{ 768n^2p_0^2 \int_0^T (\alpha_s + \frac{\gamma}{2}) ds \right\} \right].
$$

Iterating the above procedure $\mu$ times, we have

$$
\mathbb{E}\left[ \exp \left\{ 3p_0^2 \sup_{0 \leq s \leq T} |Y^{(m)}_s| \right\} \right] 
\leq |\hat{A}(G)|^{\mu+1} \mathbb{E}\left[ \exp \left\{ 24n(16n)^{\mu-1}p_0^2|Y^{(m)}_T| \right\} \right] 
\geq |\hat{A}(G)|^{\mu+1} \mathbb{E}\left[ \exp \left\{ 768n^{\mu-1}p_0^2 \int_0^T (\alpha_s + \frac{\gamma}{2}) ds \right\} \right].
$$

(29)

Furthermore, in view of (24), using Lemma 3.7, we see that for $p \geq 1$ and $l = 1, \ldots, n$,

$$
\mathbb{E}\left[ \left( \int_0^T |Z^{(m);l}_t|^2 dt \right)^p \right] + |K^{(m);l}_T|^p 
\leq \hat{A}(p) \mathbb{E}\left[ \exp \left\{ 12\gamma^2n \sup_{t \in [0,T]} |Y^{(m)}_t| + 2\lambda n T \sup_{t \in [0,T]} |Y^{(m-1)}_t| + 2n \int_0^T (\alpha_t + \frac{\gamma}{2}) dt \right\} \right],
$$

which is uniformly bounded with respect to $m$.

Finally, in view of Assertion (ii) of Lemma 3.7, the proof of Theorem 3.9 and the above derivation of (20), proceeding identically to that of [11] Theorem 2.8], we complete the proof. □

Appendix

In this appendix, we introduce the extended conditional $G$-expectation, which are needed in this paper. Consider the following two spaces of random variables

$$
L^1(\Omega) := \{ X \in \mathcal{B}(\Omega) : \mathbb{E}_G[|X|] < \infty \}, \text{ and } L^1_G(\Omega) := \{ X \in L^1(\Omega) : \exists X_n \in L^1_G(\Omega) \text{ such that } X_n \downarrow X \}.
$$
Then we could extend the conditional $G$-expectation to the space $L^1_G(\Omega_t)$:

$$\hat{E}_t[X] = \lim_{n \to \infty} \hat{E}_t[X_n],$$

which does not depend on the choice of approximating sequences, see [17] for more details.

**Lemma A.1** (Proposition 35 in [17]) Assume that $X \in L^1_G(\Omega)$. Then, for any $P \in \mathcal{P}$

$$\hat{E}_t[X] = \text{ess sup}_{\tilde{P} \in \mathcal{P}(t, P)} E^\tilde{P}_t[X] \ P\text{-a.s. for any } t \geq 0,$$

where $\mathcal{P}(t, P) = \{ \tilde{P} \in \mathcal{P} | \tilde{P} = P \text{ on } \text{Lip}(\Omega_t) \}$.

**References**

[1] Barrieu, P. and El Karoui, N. (2013) Monotone stability of quadratic semimartingales with applications to unbounded general quadratic BSDEs. Ann. Probab., 41(3B), 1831-1863.

[2] Briand, P. and Elie, R. (2013) A simple constructive approach to quadratic BSDEs with or without delay. Stochastic Process. Appl., 123(8), 2921-2939.

[3] Briand, P. and Hu, Y. (2006) BSDE with Quadratic Growth and Unbounded Terminal Value. Probability Theory and Related Fields, 136, 604-618.

[4] Briand, P. and Hu, Y. (2008) Quadratic BSDEs with convex generators and unbounded terminal conditions. Probability Theory and Related Fields, 141, 543-567.

[5] Cao, D. and Tang, S. (2020) Reflected Quadratic BSDEs Driven by $G$-Brownian Motions. Chinese Annals of Mathematics, Series B, 41, 873-928.

[6] Cheridito, P. and Nam, K. (2015) Multidimensional quadratic and subquadratic BSDEs with special structure. Stochastics, 87(5), 871-884.

[7] Denis, L., Hu, M. and Peng, S. (2011) Function spaces and capacity related to a sublinear expectation: application to $G$-Brownian motion paths. Potential Anal., 34, 139-161.

[8] El Karoui, N., Peng, S. and Quenez, M. C. (1997) Backward stochastic differential equations in finance. Math. Finance, 7(1), 1-71.

[9] Fan, S. and Hu, Y. (2021) Well-posedness of scalar BSDEs with sub-quadratic generators and related PDEs. Stochastic Process. Appl., 131, 21-50.

[10] Fan, S., Hu, Y. and Tang, S. (2020) On the uniqueness of solutions to quadratic BSDEs with non-convex generators and unbounded terminal conditions. C. R. Math. Acad. Sci. Paris, 358(2), 227-235.

[11] Fan, S., Hu, Y. and Tang, S. (2020) Multi-dimensional backward stochastic differential equations of diagonally quadratic generators: the general result, in arXiv:2007.04481.

[12] Frei, C. and Dos Reis, G. (2011) A financial market with interacting investors: does an equilibrium exist? Math. Financ. Econ., 4(3), 161-182.

[13] Harter, J. and Richou, A. (2019) A stability approach for solving multidimensional quadratic BSDEs. Electronic Journal of Probability, 24, No. 4, 51pp.
[14] Hu, M., Ji, S., Peng, S. and Song, Y. (2014) Backward stochastic differential equations driven by G-Brownian motion. Stochastic Process. Appl., 124, 759-784.

[15] Hu, M., Ji, S., Peng, S. and Song, Y. (2014) Comparison theorem, Feynman-Kac formula and Girsanov transformation for BSDEs driven by G-Brownian motion. Stochastic Process. Appl., 124, 1170-1195.

[16] Hu, M. and Peng, S. (2009) On representation theorem of G-expectations and paths of G-Brownian motion. Acta Math. Appl. Sin. Engl. Ser., 25(3), 539-546.

[17] Hu, M. and Peng, S. (2013) Extended conditional G-expectation and related stopping times, in arXiv:1309.3829.

[18] Hu, M., Wang, F. and Zheng, G. (2016) Quasi-continuous random variables and processes under the G-expectation framework. Stochastic Process. Appl., 126, 2367-2387.

[19] Hu, Y., Imkeller, P. and Müller, M. (2005) Utility maximization in incomplete markets. Ann. Appl. Probab., 15(3), 1691-1712.

[20] Hu, Y., Lin, Y. and Soumana Hima, A. (2018) Quadratic backward stochastic differential equations driven by G-Brownian motion: discrete solutions and approximation. Stochastic Process. Appl., 128(11), 3724-3750.

[21] Hu, Y. and Tang, S. (2016) Multi-dimensional backward stochastic differential equations of diagonally quadratic generators. Stochastic Process. Appl., 126(4), 1066-1086.

[22] Kobylanski, M. (2000) Backward stochastic differential equations and partial differential equations with quadratic growth. Ann. Probab., 28(2), 558-602.

[23] Lin, Y., Ren, Z., Touzi, N. and Yang, J. (2020) Second order backward SDE with random terminal time. Electronic Journal of Probability, 25, No. 99, 43pp.

[24] Liu, G. (2020) Multi-dimensional BSDEs with diagonal generators driven by G-Brownian motion. Stochastics, 92(5), 659-683.

[25] Luo, P. (2020) A type of globally solvable BSDEs with triangularly quadratic generators. Electronic Journal of Probability, 25, No. 112, 23pp.

[26] Morlais, M.-A. (2009) Quadratic BSDEs driven by a continuous martingale and applications to the utility maximization problem. Finance Stoch., 13(1), 121-150.

[27] Pardoux, E. and Peng, S. (1990) Adapted solution of a backward stochastic differential equation. Systems Control Lett., 14(1), 55-61.

[28] Peng, S. (2007) G-expectation, G-Brownian Motion and Related Stochastic Calculus of Itô type. Stochastic analysis and applications, 541-567, Abel Symp., 2, Springer, Berlin.

[29] Peng, S. (2008) Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation. Stochastic Process. Appl., 118(12), 2223-2253.

[30] Peng, S. (2019) Nonlinear expectations and stochastic calculus under uncertainty. Springer-Verlag Berlin Heidelberg.

[31] Possamaï, D. and Zhou, C. (2013) Second order backward stochastic differential equations with quadratic growth. Stochastic Process. Appl., 123, 3770-3799.
[32] Possamaï, D., Tan, X. and Zhou, C. (2018) Stochastic control for a class of nonlinear kernels and applications. The Annals of Probability, 46(1), 551-603.

[33] Revus, D. and Yor, M. (1999) Continuous Martingales and Brownian Motion. Springer-Verlag, New York, 3rd edition.

[34] Soner, H.M., Touzi, N. and Zhang, J. (2011) Martingale representation theorem for the $G$-expectation. Stochastic Process. Appl., 121, 265-287.

[35] Soner, H.M., Touzi, N. and Zhang, J. (2012) Wellposedness of Second Order Backward SDEs. Probability Theory and Related Fields, 153, 149-190.

[36] Soner, H.M., Touzi, N. and Zhang, J. (2013) Dual formulation of second order target problems. Ann. Appl. Probab., 23(1), 308-347.

[37] Song, Y. (2011) Some properties on $G$-evaluation and its applications to $G$-martingale decomposition. Science China Mathematics, 54, 287-300.

[38] Song, Y. (2019) Properties of $G$-martingales with finite variation and the application to $G$-Sobolev spaces. Stochastic Process. Appl., 129(6), 2066–2085.

[39] Tevzadze, R. (2008) Solvability of backward stochastic differential equations with quadratic growth. Stochastic Process. Appl., 118(3), 503-515.

[40] Xing, H. and Zitkovic, G. (2018) A class of globally solvable Markovian quadratic BSDE systems and applications. Ann. Probab., 46(1), 491-550.