THE LAGRANGIAN COBORDISM GROUP OF $T^2$

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ABSTRACT. We compute the Lagrangian cobordism group of the standard symplectic 2-torus and show that it is isomorphic to the Grothendieck group of its derived Fukaya category.

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1. INTRODUCTION

The Fukaya category $\mathcal{F}(M)$ of a symplectic manifold $(M, \omega)$ is an $A_\infty$-category whose objects are the Lagrangian submanifolds of $M$, and whose morphism spaces are Floer cochain groups. Its derived category $D\mathcal{F}(M)$ is triangulated, and thus it possesses a mechanism for generating objects by taking cones of morphisms. Recent work of Biran–Cornea [BC13a, BC13b] provides a way of understanding cone decompositions of objects in $D\mathcal{F}(M)$ geometrically via Lagrangian cobordisms. According to their definition, a Lagrangian cobordism $V : (L_1, \ldots, L_r) \leadsto (L'_1, \ldots, L'_s)$ between tuples of Lagrangians in $M$ is a Lagrangian submanifold $V$ of $\mathbb{R}^2 \times M$ with cylindrical ends corresponding to the $L_i$ and $L'_i$. Figure 1 displays the projection of such a cobordism to $\mathbb{R}^2$. The main result of [BC13a, BC13b] is that a Lagrangian cobordism of the form

$$V : L \leadsto (L_1, \ldots, L_s)$$

leads to an iterated cone decomposition of $L$ in $D\mathcal{F}(M)$ whose “building blocks” are the $L_i$.  

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Figure 1. Projection of a Lagrangian cobordism $V : L \leadsto (L_1, L_2, L_3)$.

Part of the information about the triangulated structure of $\mathcal{D}F(M)$ is captured by its Grothendieck group $K_0(\mathcal{D}F(M))$. It is generated by the objects of $\mathcal{F}(M)$, that is, the Lagrangians in $M$, with relations coming from exact triangles in $\mathcal{D}F(M)$. On the cobordism side, one can naturally define a Lagrangian cobordism group $\Omega_{\text{Lag}}(M)$. It is generated by the Lagrangians in $M$, or a suitable subset thereof, and has relations coming from Lagrangian cobordisms. As an immediate consequence of Biran–Cornea’s results, there exists a surjective group homomorphism

$$\Theta : \Omega_{\text{Lag}}(M) \to K_0(\mathcal{D}F(M)),$$

induced by $L \mapsto L$. It is natural to ask if $\Theta$ is an isomorphism, and if not, what its kernel is. This question formalises the question to what extent the triangulated structure of $\mathcal{D}F(M)$ can be explained geometrically by the existence of Lagrangian cobordisms. In general, nothing is known about the kernel of $\Theta$.

Main result. The case we consider is when $(M, \omega)$ is the standard symplectic 2-torus $(\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2, \omega_{\text{std}} = dx \wedge dy)$. The version of the Lagrangian cobordism group $\Omega_{\text{Lag}}(\mathbb{T}^2)$ we study is generated by non-contractible simple closed curves in $\mathbb{T}^2$. We will compute this group in Section 6, with the following result.

**Proposition 1.1.** There exists a canonical short exact sequence

$$0 \to \mathbb{R}/\mathbb{Z} \xrightarrow{\zeta} \Omega_{\text{Lag}}(\mathbb{T}^2) \xrightarrow{\eta} H_1(\mathbb{T}^2; \mathbb{Z}) \to 0.$$

The map $\zeta$ takes $x \in \mathbb{R}/\mathbb{Z}$ to the class represented by the boundary of a cylinder of area $x$. In other words, $\zeta(x)$ is represented by the difference $L - L'$, with $L$ any non-contractible curve, and $L'$ any isotopic curve such that the area swept out during any isotopy from $L'$ to $L$ is $x$ (this area is determined by the two curves up to an integer). The map $\eta$ is the obvious one, given by $[L]_{\Omega} \mapsto [L]_{H_1}$.

Our main result is the following statement.

**Theorem 1.2.** The natural group homomorphism

$$\Theta : \Omega_{\text{Lag}}(\mathbb{T}^2) \to K_0(\mathcal{D}F(\mathbb{T}^2))$$

is an isomorphism.

Theorem 1.2 has implications of three different kinds. First, as already indicated above, it tells us that the set of relations in $K_0(\mathcal{F}(\mathbb{T}^2))$ is generated by ones coming from Lagrangian cobordisms. Second, together with Proposition 1.1 it can be regarded as a computation of $K_0(\mathcal{D}F(\Sigma))$, which to the author’s knowledge has not been carried out before (but cf. [Abo08] for a computation of $K_0(\mathcal{D}F(\Sigma))$ for higher genus surfaces). Third, it gives information about Lagrangian cobordisms
in $\mathbb{R}^2 \times T^2$. There are two known constructions of such cobordisms based on Lagrangian suspension and on Lagrangian surgery, but we do not know if the resulting cobordisms are the only ones that exist. Theorem 1.2 does not rule out that there are more, but it (or rather its proof) shows that the known cobordisms generate the set of relations in $\Omega^\text{Lag}_L(T^2)$.

Proving Theorem 1.2 using HMS. We will first prove an extended version of Theorem 1.2 for which we consider a Fukaya category $\mathcal{F}^\sharp(T^2)$ defined over a Novikov field $\Lambda$, whose objects are Lagrangians in $T^2$ that are decorated with certain local systems of $\Lambda$-vector spaces. The definition of the cobordism group is also modified accordingly: The Lagrangians as well as the cobordisms carry local systems, and there are additional relations coming from short exact sequences of local systems. We denote the resulting group by $\Omega^\sharp_L(T^2)$. As in the case without local systems, there is a natural surjective group homomorphism $\Theta^\sharp : \Omega^\sharp_L(T^2) \to K_0(D\mathcal{F}^\sharp(T^2))$.

Theorem 1.3. The natural group homomorphism

$$\Theta^\sharp : \Omega^\sharp_L(T^2) \to K_0(D\mathcal{F}^\sharp(T^2))$$

is an isomorphism.

As the main ingredient in the proof of Theorem 1.3, apart from Biran–Cornea’s theory, we use that $T^2$ is one of the symplectic manifolds for which the homological mirror symmetry conjecture has been proven. The statement of relevance to us is the result of Abouzaid–Smith [AS10], who, on their way to HMS for $T^4$, construct a triangulated equivalence $D^b(X) \simeq D^e\mathcal{F}(T^2)$ between the derived category of coherent sheaves of an elliptic curve $X$ defined over the Novikov field $\Lambda$ and the split-closed derived Fukaya category of $T^2$. (A more refined version of HMS for $T^2$ was recently proven by Lekili–Perutz [LPT12], but this is not needed for our purposes.)

One can adapt Abouzaid–Smith’s result to our setting, such as to obtain a triangulated equivalence

$$D^b(X) \simeq D\mathcal{F}^\sharp(T^2).$$

(In particular, this shows that taking the split-closure of $D\mathcal{F}(T^2)$ or using appropriate local systems is equivalent here.) What needs to be checked is that Abouzaid–Smith’s functor, which is given explicitly only on a small collection of split-generators, takes every sheaf to a Lagrangian with a local system (as opposed to some summand of some non-trivial complex). We do so by matching up sheaves and Lagrangians with local systems in an inductive manner, using Atiyah’s classification of vector bundles on elliptic curves [Ati57] respectively surgery and cobordisms to understand the structures of $D^bX$ and $D\mathcal{F}^\sharp(T^2)$. The result of this procedure is stated in Proposition 7.3.

The resulting isomorphism between Grothendieck groups

$$K_0(D^b(X)) \cong K_0(D\mathcal{F}^\sharp(T^2))$$

We remark that from certain perspectives $\mathcal{F}^\sharp(T^2)$ might actually be a more natural version of the Fukaya category to look at than $\mathcal{F}(T^2)$.
allows us to understand relations in $K_0(D,\mathcal{F}^\sharp(T^2))$ via the well-understood group $K_0(D^b(X))$, and to check that the “obvious” inverse to the map $\Theta^\sharp : \Omega^\sharp_{\text{Lag}}(T^2) \to K_0(D,\mathcal{F}^\sharp(T^2))$ is well-defined. This proves Theorem 1.3 from which Theorem 1.2 follows.

Organisation of the paper. In Section 2 we recall the definition of Lagrangian cobordisms and define the groups $\Omega_{\text{Lag}}(M)$ and $\Omega^\sharp_{\text{Lag}}(M)$. Section 3 serves to define $\mathcal{F}^\sharp(T^2)$ and to provide some general background on Fukaya categories. In Section 4 we explain how Lagrangian cobordisms lead to iterated cone decompositions in the derived Fukaya category, focusing on the (small) modifications necessary to make Biran–Cornea’s proofs work in our setting. In Section 5, we describe in detail the Lagrangian cobordisms resulting from surgering curves in surfaces and discuss some specifics in the case of $T^2$. Section 6 contains the computation of $\Omega_{\text{Lag}}(T^2)$, that is, the proof of Proposition 1.1. In Section 7, we examine Abouzaid–Smith’s mirror functor (or rather, a version of it that’s adapted to our setting); we provide a description of its action on objects that’s explicit enough to enable the computations in $K_0(D,\mathcal{F}^\sharp(T^2))$ required for the proofs of the main theorems in Section 8. Appendix A collects some facts on triangulated categories, cone decompositions and Grothendieck groups; Appendix B explains how to get exact triangles from short exact sequences of local systems; and finally, Appendix C assembles a couple of statements from Atiyah’s classification of vector bundles on elliptic curves [Ati57], which are used in Section 7.

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2. The Lagrangian cobordism group

2.1. Lagrangian cobordisms. We first recall some definitions from [BC13a]. For a symplectic manifold $(M, \omega)$, we denote by $(\tilde{M}, \tilde{\omega})$ the symplectic manifold obtained by equipping $\tilde{M} = \mathbb{R}^2 \times M$ with the split symplectic form $\tilde{\omega} = \omega_{\text{std}} \oplus \omega$, where $\omega_{\text{std}} = dx \wedge dy$ is the standard symplectic form on $\mathbb{R}^2$. We denote by $\pi : \tilde{M} \to \mathbb{R}^2$ the projection to the first factor, and given any subset $S \subset \mathbb{R}^2$, we write $V|_S = V \cap \pi^{-1}(S)$.

We say that two ordered collections $(L_i)_{i=1}^r$ and $(L'_j)_{j=1}^s$ of Lagrangian submanifolds of $M$ are Lagrangian cobordant if there exists a compact cobordism $(V; \bigsqcup_i L_i, \bigsqcup_j L'_j)$ together with a Lagrangian embedding $V \to [0,1] \times \mathbb{R} \times M \subset \mathbb{R}^2 \times M$ with cylindrical ends, in the sense that there is some $\varepsilon > 0$ such that

$$V|_{[0,\varepsilon] \times \mathbb{R}} = \bigsqcup_{i=1}^r [0, \varepsilon) \times \{i\} \times L_i \quad \text{and} \quad V|_{[1-\varepsilon,1] \times \mathbb{R}} = \bigsqcup_{j=1}^s (1-\varepsilon, 1] \times \{j\} \times L'_j.$$

The Lagrangian submanifold $V$ of $\tilde{M}$ is called a Lagrangian cobordism with positive ends $(L'_j)_{j=1}^s$ and negative ends $(L_i)_{i=1}^r$. The terminology is that $V$ goes from $(L'_j)_{j=1}^s$ to $(L_i)_{i=1}^r$ and we denote this relationship by $V : (L'_1, \ldots, L'_s) \leadsto (L_1, \ldots, L_r)$. 
Example 1. Lagrangian suspension: Let \( \Phi : M \to M \) be a Hamiltonian diffeomorphism, and let \( \phi : [0, 1] \times M \to M \) be a Hamiltonian isotopy with \( \phi(0, \cdot) = \text{id} \), \( \phi(1, \cdot) = \Phi \) and which is generated by a Hamiltonian \( H : [0, 1] \times M \to \mathbb{R} \) such that \( H(t, \cdot) \equiv 0 \) for \( t \) close to 0 and 1 (this condition can be achieved by suitably reparametrising any given Hamiltonian isotopy). Then for any Lagrangian submanifold \( L \subset M \), the map
\[
[0, 1] \times L \to \mathbb{R}^2 \times M, \quad (t, x) \mapsto (t, -H(t, \phi_t(x)), \phi_t(x)),
\]
defines a Lagrangian cobordism \( V : \phi_1(L) \leadsto L \).

Example 2. Lagrangian surgery: Let \( L_0, L_1 \subset M \) be two transversely intersecting Lagrangian submanifolds. One can resolve the intersection points by cutting out small neighbourhoods and gluing in Lagrangian handles diffeomorphic to \( [-1, 1] \times S^{n-1} \) (where \( n = \frac{1}{2} \text{dim} M \)). This produces a new Lagrangian submanifold which we denote by \( L_0 \#_L L_1 \). Biran–Cornea [BC13a] show that there exists a Lagrangian cobordism
\[
V : L_0 \#_L L_1 \leadsto (L_0, L_1).
\]
This construction will be of prime importance later on; we will describe it in more detail in Section 5.

2.2. Extra data. Whenever Lagrangians come equipped with extra data, such as orientations, \( \text{Spin} \) or \( \text{Pin} \) structures, gradings, or local systems, it makes sense to consider cobordisms over which these data extend. We say that two collections \( (L_i)_{i=1}^r \) and \( (L'_j)_{j=1}^s \) of Lagrangians decorated with such extra data are Lagrangian cobordant if there exists a Lagrangian cobordism \( V : (L'_j)_{j=1}^s \leadsto (L_i)_{i=1}^r \) between the underlying Lagrangians together with choices of the same types of extra data for \( V \) which restrict to the given data on the ends (provided there exists a suitable notion of restricting to boundary components).

2.3. The Lagrangian cobordism group. Let \( \mathcal{L} \) be the set of (suitably qualified) Lagrangian submanifolds of \( M \), and let \( \langle \mathcal{L} \rangle \) be the free Abelian group generated by \( \mathcal{L} \). Denote by \( R \subset \langle \mathcal{L} \rangle \) the subgroup generated by all expressions
\[
L_1 + \cdots + L_r - L'_1 - \cdots - L'_s \in \langle \mathcal{L} \rangle
\]
such that there is a (suitably qualified) Lagrangian cobordism \( V : (L'_1, \ldots, L'_s) \leadsto (L_1, \ldots, L_r) \). The Lagrangian cobordism group corresponding to \( \mathcal{L} \) and \( R \) is then defined as
\[
\Omega_{\text{Lag}}(M) = \langle \mathcal{L} \rangle / R,
\]
where we suppress the dependence of \( \mathcal{L} \) and \( R \) in the notation. An analogous definition applies when Lagrangians carry extra data, in which case one imposes relations coming from cobordisms with the same types of extra data.

As indicated, it usually makes sense to constrain the Lagrangians and cobordisms one admits in the definition of \( \Omega_{\text{Lag}}(M) \), because Lagrangian cobordism without any additional condition is a too flexible notion. One possibility is to require all Lagrangians and cobordisms to be (uniformly) monotone, as is done in [BC13a, BC13b]. In the case \( M = T^2 \) this paper deals with, we will require that the Lagrangians are non-contractible curves and that the cobordisms have vanishing Maslov class.
Remark 2.1. Constraining the cobordisms one allows in the definition of $\Omega_{\text{Lag}}(M)$ obviously has an effect on what equality in $\Omega_{\text{Lag}}(M)$ means. For example, the identity $[L] = [L']$ in the monotone version of $\Omega_{\text{Lag}}(M)$ does not necessarily mean that there exists a monotone cobordism $L' \leadsto L$. The identity might instead come from a monotone cobordism $(L', K) \leadsto (L, K)$, from which one can create a cobordism $L' \leadsto L$ by connecting the ends corresponding to $K$, but in general at the cost of monotonicity.

2.4. Additional relations from local systems. Next we define a variant of the Lagrangian cobordism group that one can define whenever the Lagrangians one studies carry local systems. Let $L^\#$ be the set of all pairs $(L, E)$ where $L$ is a Lagrangian submanifold in $M$ and where $E$ is a local system on $L$ (both with suitable qualifications depending on the context), and let $\langle L^\# \rangle$ be the free Abelian group generated by $L^\#$.

To define the subgroup $R^\# \subset \langle L^\# \rangle$ of relations we impose, consider all elements of $\langle L^\# \rangle$ of the form

$$(L_1, E_1) + \ldots (L_r, E_r) - (L'_1, E'_1) - \ldots - (L'_s, E'_s)$$

such that there is a Lagrangian cobordism $V : (L'_1, \ldots, L'_s) \leadsto (L_1, \ldots, L_r)$ together with a local system $E$ on $V$ which restricts to the $E'_i$ or $E_j$ on the respective ends. Moreover, consider all elements of the form

$$(L, E) - (L, E') - (L, E'')$$

such that there exists a short exact sequence $0 \to E' \to E \to E'' \to 0$ of local systems on $L$. Now define $R^\#$ to be the subgroup of $\langle L^\# \rangle$ generated by all elements of types [1] and [2] and set

$$\Omega^\#_{\text{Lag}}(M) = \langle L^\# \rangle / R^\#.$$ 

Similar to before, we usually suppress $L^\#$ and $R^\#$ from the notation, keeping in mind that the group really depends on them.

3. The Fukaya category of $T^2$

In this section we describe the constructions of the Fukaya categories $\mathcal{F}(T^2)$ and $\mathcal{F}^\#(T^2)$ appearing in Theorems 1.2 and 1.3, following essentially the general recipe in [Sei08]. The ground field over which these $A_{\infty}$-categories are defined is the Novikov field

$$\Lambda = \left\{ \sum_{i=0}^{\infty} c_i q^{a_i} \mid c_i \in \mathbb{C}, \ a_i \in \mathbb{R}, \ a_i < a_{i+1}, \ \lim_{i \to \infty} a_i = \infty \right\}.$$

We will first describe the objects of $\mathcal{F}^\#(T^2)$ and $\mathcal{F}(T^2)$, and then sketch the general construction of morphism spaces, $A_{\infty}$-compositions and derived categories.

3.1. Objects. The objects of $\mathcal{F}^\#(T^2)$ are tuples $(L, \alpha, P, E)$ as follows:

- $L \subset T^2$ is a non-contractible simple closed curve,
- $\alpha : L \to \mathbb{R}$ is a grading of $L$,
- $P$ is a $Pin$ structure on $L$,
- $E$ is a local system of $\Lambda$-vector spaces on $L$. 


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These structures are subject to certain conditions which we will describe below. The pair $(\alpha, P)$ is called a \textit{brane structure} on $L$, and the triple $(L, \alpha, P)$ is called a \textit{Lagrangian brane}. We usually suppress the brane structure and/or the local system from the notation.

The objects of $\mathcal{F}(T^2)$ are Lagrangian branes $L = (L, \alpha, P)$ with $L$, $\alpha$ and $P$ as above. We regard these as objects of $\mathcal{F}^\natural(T^2)$ by equipping them with trivial rank one local systems, such as to make $\mathcal{F}(T^2)$ a full $A_\infty$-subcategory of $\mathcal{F}^\natural(T^2)$ (in view of that, we won’t define the morphism spaces of $\mathcal{F}(T^2)$ separately).

3.1.1. \textit{Gradings}. The Lagrangian Grassmannian $\text{Gr}(\mathbb{R}^{2n})$ of $(\mathbb{R}^{2n}, \omega_{\text{std}})$ can be naturally identified with $\mathbb{U}(n)/\mathbb{O}(n)$, and this identification induces a map $\det^2 : \text{Gr}(\mathbb{R}^{2n}) \to S^1$. A \textit{grading} of a Lagrangian subspace $\Lambda \in \text{Gr}(\mathbb{R}^{2n})$ is a number $\alpha \in \mathbb{R}$ such that $e^{2\pi i \alpha} = \det^2(\Lambda)$. (See [Sei08] for the general definition.)

Since the tangent bundle of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is $T^2 \times \mathbb{R}^2$, the Gauß map associated to a Lagrangian $L \subset T^2$ can be viewed as a map $\Gamma_L : L \to \text{Gr}(\mathbb{R}^2)$. A \textit{grading} of $L$ is a continuous function $\alpha : L \to \mathbb{R}$ such that $\alpha(x)$ is a grading of $T_x L \in \text{Gr}(\mathbb{R}^2)$ in the previous sense; that is, $\alpha$ is a lift of the map $\det^2 \circ \Gamma_L : L \to S^1$ with respect to the covering $\mathbb{R} \to S^1$, $\alpha \mapsto e^{2\pi i \alpha}$. Note that every non-contractible $L \subset T^2$ possesses a grading because its $\Gamma_L$ is null-homotopic. Objects of $\mathcal{F}^\natural(T^2)$ are allowed to carry all possible gradings.

A grading $\alpha$ on a curve $L \subset T^2$ induces an orientation of $L$ because we can view $e^{\pi i \alpha} \in S^1$ as a point in $\text{Gr}^\text{or}(\mathbb{R}^2)$, the Grassmannian of oriented lines in $\mathbb{R}^2$; changing the grading by $\pm 1$ reverses the induced orientation. In the following it will sometimes be convenient to have an orientation present, and we will always equip graded curves with this induced orientation. Conversely, given an oriented curve $L$, we can assign a \textit{standard grading} to it as follows. Suppose that $L$ has slope $(p, q) \in H_1(T^2; \mathbb{Z})$. Then there is a unique number $\alpha_0 \in [-1, 1)$ such that

$$e^{\pi i \alpha_0} = \frac{p + iq}{\sqrt{p^2 + q^2}}.$$  

Viewed as a constant function, $\alpha_0$ is a grading for any of the linear Lagrangians to which $L$ is isotopic. We define the standard grading of $L$ to be the function $\alpha : L \to \mathbb{R}$ induced by $\alpha_0$ via any isotopy connecting $L$ to a linear Lagrangian.

3.1.2. \textit{Pin structures}. A \textit{Pin} structure on a rank $n$ vector bundle $F$ is a principal $\text{Pin}_n$-bundle $P$ together with a choice of two-sheeted covering $P \to P_0(F)$ which is equivariant with respect to the action of $\text{Pin}_n$ on both sides. Here $P_0(F)$ is the bundle of orthonormal frames of $F$ with respect to some auxiliary metric, which is

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Oriented curves on $T^2$ and their standard grading.}
\end{figure}
acted upon by $\text{Pin}_n$ via the homomorphism $\text{Pin}_n \to O_n$. A $\text{Pin}$ structure on a manifold is a $\text{Pin}$ structure on its tangent bundle. $\text{Pin}$ structures generalize the more familiar notion of $\text{Spin}$ structures to non-oriented (even non-orientable) bundles. If bundle $F$ is orientable, a $\text{Pin}$ structure is the same as an equivalence class of $\text{Spin}$ structures, where two $\text{Spin}$ structures are equivalent if they are obtained from one another by “reversing the orientation”. (See [Sei08, Section(11i)] for some and [LM89] for more background on $\text{Pin}_n$ and $\text{Pin}$ structures.)

The orthonormal frame bundle of $L$ is simply $L \sqcup L$, and we can therefore think of a $\text{Pin}$ structure on $L$ as a double cover of $L$. Hence $L$ admits precisely two different $\text{Pin}$ structures, corresponding to the trivial resp. the non-trivial 2-sheeted cover of $L$. The latter is referred to as the bounding $\text{Pin}$ structure, because it is obtained by restricting the unique $\text{Pin}$ structure on the disc to the boundary circle.

The Lagrangians appearing in the definition of $F^\sharp(T^2)$ will be equipped with the bounding $\text{Pin}$ structure (allowing also the other $\text{Pin}$ structure would be possible, but would lead to some redundancy).

3.1.3. Local systems. Let $X$ be a topological space. A local system $E$ on $X$ assigns a vector space $E_x$ to every point $x \in X$ and an isomorphism $\pi_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ to every path $\gamma : [0,1] \to X$ which depends only on the homotopy class of the path relative to the end points, in a way which is compatible with concatenation of paths. We call $E_x$ the fibre of $E$ at $x$, and $\pi_\gamma$ the parallel transport in $E$ along $\gamma$. A local system $E$ yields a representation $\eta_{x_0} : \pi_1(X,x_0) \to \text{GL}(E_{x_0})$ for every $x_0 \in X$. Assuming that $X$ is path-connected, $E$ is determined up to isomorphism by one single such representation $\eta_{x_0}$; in fact, $E$ is isomorphic to a local system whose fibres are all equal to $E_{x_0}$ and with parallel transport maps constructed from $\eta_{x_0}$.

If the base space is an oriented circle $L$, a local system $E$ on $L$ is hence determined by specifying a single vector space $F_E$ and a monodromy isomorphism $M_E \in \text{GL}(F)$ which encodes the parallel transport along the preferred generator of $\pi_1(L)$. The local systems $E$ we allow in the definition of $\mathcal{F}^\sharp(T^2)$ are as follows: They have fibre $F_E = \Lambda^\oplus n$ for some $n \geq 1$, and all eigenvalues of the monodromy $M_E \in \text{GL}_n(\Lambda)$ have norm 1 with respect to the non-Archimedean norm on $\Lambda$ defined by

$$\sum_{i=0}^{\infty} c_i q^{a_i} = e^{-a_0}.$$ 

3.2. Morphisms. The content of the rest of this section is not $T^2$-specific, so we write $M$ instead of $T^2$ from now on. We assume that the objects of $\mathcal{F}^\sharp(M)$ have been defined to be suitable Lagrangian branes equipped with local systems.

Let $L_i \equiv (L_i, \alpha_i, P_i, E_i)$, $i = 0, 1$, be objects of $\mathcal{F}^\sharp(M)$ such that $L_0$ and $L_1$ intersect transversely. The corresponding space of Floer cochains is the graded $\Lambda$-vector space with $i$-th graded component

$$CF^\alpha(L_0, L_1) = \bigoplus_{y \in L_0 \cap L_1, i(y) = i} \text{Hom}(E_{0,y}, E_{1,y}),$$
where $\text{Hom}(E_{0,y}, E_{1,y})$ is the space of homomorphisms between the fibres of the local systems at $y \in L_0 \cap L_1$, and $i(y)$ is the index of $y$, defined by

$$i(y) \equiv i(y; L_0, L_1) = \lfloor \alpha_1(y) - \alpha_0(y) \rfloor + 1,$$

where $\lfloor \cdot \rfloor$ is the next lowest integer.

Let now $y_0, y_1 \in L_0 \cap L_1$ and let $Z = \mathbb{R} \times [0, 1]$ equipped with the usual complex structure $j_Z$. Denote by $\mathcal{M}_Z(y_0, y_1)$ the space of all maps $u : Z \to M$ satisfying

$$Du + J \circ Du \circ j_Z = 0$$

for a generic $\omega$-compatible almost complex structure $J$ on $M$, and with boundary and asymptotic conditions given by

$$u(s, 0) \in L_0, \quad u(s, 1) \in L_1, \quad \lim_{s \to -\infty} u(s, \cdot) = y_0, \quad \lim_{s \to +\infty} u(s, \cdot) = y_1.$$

$\mathcal{M}_Z(y_0, y_1)$ carries a natural $\mathbb{R}$-action given by translation in the $\mathbb{R}$-variable. We denote by $\mathcal{M}^{1+1}(y_0, y_1) = \mathcal{M}_Z(y_0, y_1)/\mathbb{R}$ the moduli space obtained by quotienting out this action.

The Floer differential $\partial : CF(L_0, L_1) \to CF(L_0, L_1)[1]$ is then defined on generators $\phi_1 \in \text{Hom}(E_{0,y_1}, E_{1,y_1})$ of $CF(L_0, L_1)$ by

$$\partial \phi_1 = (-1)^{i(y_1)} \bigoplus_{y_0 \in L_0 \cap L_1} \sum u \text{sgn}(u) \pi_1^u \circ \phi \circ \pi_0^u q^{\omega(u)},$$

where the second sum runs over the zero-dimensional part $\mathcal{M}^{1+1}(y_0, y_1)^0$ of the moduli space (which is a discrete set). Here $\pi_i^u$, for $i = 0, 1$, denotes parallel transport in the local system $E_i$ along the boundary component $u(\mathbb{R} \times \{i\}) \subset L_i$ of the strip $u(Z)$, and $q^{\omega(u)}$ is an element of $\Lambda$ that encodes the symplectic area of the strip $u$. Finally, $\text{sgn}(u) \in \{\pm 1\}$ is a sign whose determination we will describe in Section 3.4.

The definition of $CF(L_0, L_1)$ for non-transversely intersecting $L_0, L_1$ requires the use of Hamiltonian perturbations: One fixes a Floer datum $(H, J)$ for every such pair, consisting of a Hamiltonian function $H$ such that $\phi_1^H(L_0) \cap L_1$ and an almost complex structure $J$, and then considers an analogue of equation (3) with an additional perturbation term on the right-hand side.

In either case, the space of morphisms in $\mathcal{F}(M)$ from $L_0$ to $L_1$ is defined as a graded vector space over $\Lambda$ by

$$\text{hom}(L_0, L_1) = CF(L_0, L_1),$$

and the first order $A_\infty$-structure map $\mu^1 : \text{hom}(L_0, L_1) \to \text{hom}(L_0, L_1)[1]$ is the Floer differential $\partial$.

### 3.3. $A_\infty$-compositions

We give a brief description of the higher $A_\infty$-compositions

$$\mu^d : \text{hom}(L_{d-1}, L_d) \otimes \cdots \otimes \text{hom}(L_0, L_1) \to \text{hom}(L_0, L_d)[2 - d], \quad d \geq 2,$$

again limiting ourselves to the case of mutually transverse $L_i$. For the proof that these really define an $A_\infty$-structure, we refer to [ScIo8].

Let $L_0, \ldots, L_d$ be objects of $\mathcal{F}(M)$, and let $y_0 \in L_0 \cap L_d$ and $y_i \in L_{i-1} \cap L_i$, for $i = 1, \ldots, d$. Moreover, let $S$ be a disc with one incoming and $d$ outgoing boundary punctures. The moduli space of $\mathcal{M}^{d+1}(y_0, \ldots, y_d)$ of pseudo-holomorphic
polygons associated to this collection Lagrangians and points is the set of all maps \( u \in C^\infty(S, M) \) solving the equation

\[
Du + J \circ Du \circ j_S = 0,
\]

where again \( J \) is an \( \omega \)-compatible almost complex structure, with boundary conditions given by the \( L_i \) and asymptotic conditions at the punctures given by the \( y_i \) as indicated in Figure 3. The precise general definitions require again choices of Floer data and additional choices of perturbation data which lead to the appearance of an inhomogeneity on the righthand side of the equation; see [Sei08, Section 8(f)] for details.

Given now \( \phi_1, \ldots, \phi_d \) with \( \phi_i \in \text{Hom}(E_{i-1}, y_i, E_i, y_i) \), the corresponding output of \( \mu^d \) is

\[
\mu^d(\phi_d, \ldots, \phi_1) = (-1)^\dagger \sum_{y_0 \in L_0 \cap L_d} \text{sgn}(u) \pi^u_d \circ \phi_d \circ \pi^u_{d-1} \circ \cdots \circ \phi_1 \circ \pi^u_0 q^{\omega(u)},
\]

which is an element of \( \text{Hom}(E_{0,y_0}, E_{d,y_0}) \subset CF(L_0, L_d) \). Here the second sum runs over all elements \( u \in \mathcal{M}^{d+1}(y_0, \ldots, y_d) \), the zero-dimensional component of the moduli space. \( \pi^u_i \) is the parallel transport in \( E_i \) along the boundary component of \( u(S) \) that lies in \( L_i \), and \( q^{\omega(u)} \in \Lambda \) encodes the symplectic area of \( u \). The determination of the sign \( \text{sgn}(u) \in \{\pm 1\} \) will be explained below; \( (-1)^\dagger \) is an additional sign, with \( \dagger = \sum_{k=1}^d k i(y_k) \).

**Remark 3.1.** We remark that for \( M = T^2 \), the set \( \mathcal{M}^{d+1}(y_0, \ldots, y_d) \) is infinite in general (cf. [PZ01] for the case \( d = 2 \)). Nevertheless, the sums above converge in \( \text{Hom}(E_{0,y_0}, E_{d,y_0}) \) with respect to the topology induced by the non-Archimedean norm on \( \Lambda \). This follows from two facts: First, the monodromies of the \( E_i \) have norm 1; second, the areas \( \omega(u) \) of the polygons entering the count tend to \( \infty \), as one can see by thinking of their lifts to the universal cover of \( T^2 \).

### 3.4. Signs

Let \( L_0, L_1 \) be two Lagrangian branes intersecting transversely at some point \( y \in L_0 \cap L_1 \). One can associate to \( y \), considered as a morphism from \( L_0 \) to \( L_1 \), a 1-dimensional real vector space \( o(y) \) called the orientation space of \( y \). We refer to [Sei08] for the precise definition (also cf. Section 4.3 below). At this point we only remark that \( o(y) \) depends on a choice of homotopy class of paths from \( T_y L_0 \) to \( T_y L_1 \) in the Lagrangian Grassmannian; this homotopy class in turn is canonically determined by the brane structures.
Given Lagrangian branes $L_0, \ldots, L_d$ and intersection points $y_i$ as in Section 3.3, there exists a preferred isomorphism

$$\Lambda^\text{top}(T_u \mathcal{M}^{d+1}(y_0, \ldots, y_d)) \cong o(y_0) \otimes o(y_1)^\vee \otimes \cdots \otimes o(y_d)^\vee$$

for every regular $u \in \mathcal{M}^{d+1}(y_0, \ldots, y_d)$, see [Sei08] Section (12b). In particular, whenever $u$ is isolated (i.e., an element of the zero-dimensional component of the moduli space), this isomorphism yields a preferred element

$$c_u \in o(y_0) \otimes o(y_1)^\vee \otimes \cdots \otimes o(y_d)^\vee,$$

because $\Lambda^\text{top}(T_u \mathcal{M}^{d+1}(y_0, \ldots, y_d)) = \mathbb{R}$ for isolated $u$.

The signs $\text{sgn}(u)$ are then defined as follows: We choose, arbitrarily and once and for all, an orientation $o_u$ of $o(y_i)$ for every $y_i$, which induces an orientation for every $o(y_0) \otimes o(y_1)^\vee \otimes \cdots \otimes o(y_d)^\vee$. Then we set $\text{sgn}(u) = \pm 1$ according to whether $c_u$ is positive or negative with respect to this orientation.

**Remark 3.2.** An equivalent way of dealing with signs would be to adopt Seidel’s basis-free approach from [Sei08] and to define Floer complexes as $CF(L_0, L_1) = \bigoplus_y |o(y)|_\Lambda \otimes \text{Hom}(E_{0,y}, E_{1,y})$, where $|o(y)|_\Lambda$ is the $\Lambda$-normalization of $o(y)$; that is, $|o(y)|_\Lambda$ is the 1-dimensional $\Lambda$-vector space obtained by taking the vector space generated by the two orientations of $o(y)$, and imposing the relation that the two generators add up to zero. The $A_\infty$-compositions would then defined by combining the preferred elements $|c_u|_\Lambda \in |o(y_0)|_\Lambda \otimes |o(y_1)|_\Lambda \otimes \cdots \otimes |o(y_d)|_\Lambda$ induced by the $c_u$ with the parallel transport maps $\pi^u_\Lambda$. The approach to signs we have chosen is slightly more geometric (though a bit less elegant), since the generators of Floer complexes are actual homomorphisms between fibres of local systems. To translate between the two approaches, one identifies the two versions of the Floer complexes by mapping a generator $\phi \in \text{Hom}(E_{0,y}, E_{1,y})$ of the first version to $|o_y| \otimes \phi \in |o(y)|_\Lambda \otimes \text{Hom}(E_{0,y}, E_{1,y})$, which is a generator of the second version; here $o_y$ is the chosen orientation for $o(y)$. This intertwines the respective versions of the $A_\infty$-compositions.

### 3.5. The derived Fukaya category

The derived Fukaya category $D\mathcal{F}^\sharp(M)$ is constructed from $\mathcal{F}^\sharp(M)$ by a purely algebraic procedure. One first completes it to a triangulated $A_\infty$-category $\mathcal{F}^\sharp(M)$, and then sets $D\mathcal{F}^\sharp(M) = H^0(\mathcal{F}^\sharp(M))$. A further completion that formally introduces images for all idempotent morphisms yields the split-closed derived Fukaya category $D^c\mathcal{F}^\sharp(M)$. These categories are triangulated categories in the classical sense, meaning that they possess a shift functor and a class of exact triangles (see Appendix A for a brief description of what that means). The shift functor is realized geometrically by changing the brane structure in a certain way. In the case $M = T^2$, its effect is simply a shift of the grading by 1,

$$(L, \alpha, P, E)[1] = (L, \alpha - 1, P, E).$$

(See [Sei08] (11k); the $Pin_\text{str}$ structure remains unchanged since we are dealing with curves, which have trivial tangent bundle.)
4. Cobordisms and cone decompositions

Let \((M, \omega)\) be a symplectic manifold and suppose that \(\mathcal{F}^\dagger(M)\) is defined as a graded \(A_\infty\)-category over the Novikov field \(\Lambda\), with objects Lagrangian branes carrying local systems.

**Theorem 4.1.** Let \(L, L_1, \ldots, L_r \in \mathcal{F}^\dagger(M)\) and suppose that there exists a Lagrangian cobordism \(V : L \sim (L_1, \ldots, L_k)\). Then \(L\) admits an iterated cone decomposition in \(D\mathcal{F}^\dagger(M)\) with linearization \((L_1, \ldots, L_k)\).

Here we suppress brane structures and local systems from the notation, i.e., we write \(L\) instead of \((L, \alpha, P, E)\). By a Lagrangian cobordism between such objects we mean a Lagrangian cobordism between the underlying Lagrangians which is equipped with the same type of extra data in a compatible way. The definitions of “iterated cone decomposition” and “linearization” are given in Appendix A.

Theorem 4.1 (which will follow from Proposition 4.8 below) is an adaptation of an analogous statement in [BC13b] to the present setting, and this section serves to explain the (small) modifications of Biran-Cornea’s arguments required for its proof.

The differences to the setting in [BC13b] are as follows. First, the version of the Fukaya category considered there is linear over \(\mathbb{Z}/2\mathbb{Z}\) and ungraded, and has as objects plain Lagrangians \(L\), without brane structures or local systems. Our main point will be to explain the inclusion of gradings and signs in the proofs given in [BC13b], including the local systems is standard and doesn’t pose any additional difficulties. Second, Biran-Cornea require that \((M, \omega)\), as well as all Lagrangians \(L\) and cobordisms \(V\) involved are uniformly monotone, and that the maps \(\pi_1(L) \to \pi_1(M)\) and \(\pi_1(V) \to \pi_1(\tilde{M})\) induced by the inclusions vanish for all of them. These conditions are needed to prove compactness of moduli spaces of pseudo-holomorphic curves. More precisely, the conditions on fundamental groups are used to obtain bounds on the areas of such curves; we do not have to impose these conditions, because we use the Novikov field \(\Lambda\) to encode series of curves with areas tending to infinity. As for the monotonicity, we assume for the time being that our setting is as in [BC13b], so that there are no additional complications of analytic nature. (But cf. Sections 5.3-5.5 for a discussion of related questions in the case \(M = T^2\), which does not precisely fit into this setting.) Third, there are some minor differences because we work with Floer cohomology, in contrast to Biran-Cornea, who work with homology.

**Remark 4.2.** The analogue of Theorem 4.1 in [BC13b] is one (the most important) ingredient of the construction of a certain functor \(\text{Cob}(M) \to T^S D\mathcal{F}(M)\) from the Lagrangian cobordism category of \(M\) to the category of stable cone decompositions of \(D\mathcal{F}(M)\), whose existence is the main result of [BC13b]. For our purposes, only the existence of cone decompositions is needed, not the functorial way in which they are associated to cobordisms.

4.1. The categories \(\mathcal{B}_{\gamma, h}\). As in [BC13b], we consider \(A_\infty\)-categories \(\mathcal{B}_{\gamma, h}\) associated to pairs \((\gamma, h)\) as follows: First, \(\gamma \subset \mathbb{R}^2\) is a properly embedded curve diffeomorphic to \(\mathbb{R}\) with “horizontal ends” whose \(y\)-coordinates are in \(\frac{1}{2} \mathbb{Z}\); second,
$h : \mathbb{R}^2 \to \mathbb{R}$ is a profile function, whose associated extended profile function we denote by $h' : \mathbb{R}^2 \to \mathbb{R}$.

These terms are defined in Sections 3.2 and 4.1 of [BC13b]. What is really only important for us, is that $h$ and $h'$ are assumed to be such that $\Phi^h_1(\gamma)$ looks as in Figure 4, where $\Phi^h_1$ is the time one map of the Hamiltonian flow of $h'$, and $\gamma$ is a typical specimen of the type of curves considered. That is, there should be an odd number of intersection points $o_1, \ldots, o_\ell \in \gamma \cap \Phi^h_1(\gamma)$, and $\Phi^h_1(\gamma)$ should have horizontal ends such that its negative end lies below the negative end of $\gamma$.

Remark 4.3. Note that in [BC13b] the requirement for $h'$ is that $(\Phi^h_1)^{-1}(\gamma)$ looks as in Figure 4. The difference is due to our use of cohomology.

The objects of $\mathcal{B}_{\gamma,h}$ are pairs $(\tilde{L}, \tilde{E})$, where $\tilde{L}$ is a Lagrangian brane in $\tilde{M} = \mathbb{R}^2 \times M$ whose underlying Lagrangian is of the form $\tilde{L} = \gamma \times L$ for some $L \in \text{Ob} \mathcal{F}(M)$, and $\tilde{E}$ is a local system on $\tilde{L}$. We usually abbreviate $\tilde{L} \equiv (\tilde{L}, \tilde{E})$.

The space of morphisms in $\mathcal{B}_{\gamma,h}$ between two such $\tilde{L}_0$ and $\tilde{L}_1$ is

$$\text{hom}_{\mathcal{B}_{\gamma,h}}(\tilde{L}_0, \tilde{L}_1) = CF(\tilde{L}_0, \tilde{L}_1),$$

where the Floer complex on the right-hand side is defined with respect to a Floer datum $(\tilde{H}, \tilde{J})$ for $(\tilde{L}_0, \tilde{L}_1)$ which is chosen compatibly with the Floer datum $(H, J)$ for $(L_0, L_1)$ used to define the Floer complex $CF(L_0, L_1) = \text{hom}_{\mathcal{F}(M)}(L_0, L_1)$.

Concretely, the first part of the Floer datum is the time-dependent Hamiltonian function $\tilde{H} = h' \oplus H$. With this choice, the intersection points of $\Phi^\tilde{H}_1(\tilde{L}_0)$ and $\tilde{L}_1$ are of the form $(o_j, y)$, with $o_j \in \Phi^\tilde{H}_1(\gamma) \cap \gamma$ and $y \in \Phi^\tilde{H}_1(L_0) \cap L_1$. Hence $CF(L, L')$ is generated by elements of the form

$$\phi^{o_j} \in \text{Hom}((\tilde{E}_0, (o_j, y)), (\tilde{E}_1, (o_j, y))),$$

$j = 1, \ldots, \ell$, and there is an obvious splitting of vector spaces

$$CF(\tilde{L}_0, \tilde{L}_1) = \bigoplus_{j=1}^\ell CF(\tilde{L}_0, \tilde{L}_1)^{o_j},$$

where $CF(L_0, L_1)^{o_j}$ is the summand consisting of homomorphisms between the fibres of the local systems over $(o_j, y)$. The second part of the Floer datum is the time-dependent almost complex structure $\tilde{J}(t) = i_{h'}(t) \oplus J(t)$, with $i_{h'}(t) = (\phi^{o_j})_t$, where $i$ is the standard complex structure on $\mathbb{R}^2$.

With these choices of Floer data and further choices of perturbation data, one constructs the $A_{\infty}$-compositions $\mu^\mathcal{B}_{\gamma,h}^{\mathcal{F}}$ by combining the description in [BC13b].

**Figure 4.** The curves $\gamma$ and $\Phi^h_1(\gamma)$ used to define $\mathcal{B}_{\gamma,h}$. 
4.2. The functors \( c_j : B_{\gamma,h} \to \mathcal{F}^j(M) \). For every odd \( j \) with \( 1 \leq j \leq \ell \), there exists a natural \( A_\infty \)-functor

\[
c_j \equiv c_{\gamma,h,j} : B_{\gamma,h} \to \mathcal{F}^j(M),
\]

which will turn out to be a quasi-isomorphism. We start by describing the action of these functors on objects; their full definition will be given in Section 4.5.

Let \((\bar{L}, \bar{E})\) be an object of \( B_{\gamma,h} \), and denote by \( \tilde{\alpha} : \bar{L} \to \mathbb{R} \) the grading and by \( \tilde{P} \) the \( Pin \) structure which together form the brane structure on \( \bar{L} \). Recall that \( \bar{L} = \gamma \times L \) for some Lagrangian \( L \subset M \). The image of \((\bar{L}, \bar{E})\) under \( c_j \), for all \( j \), is

\[
c_j(\bar{L}, \bar{E}) = (L, E),
\]

where \( L \) and \( E \) obtained from \( \bar{L} \) and \( \bar{E} \) as follows. Consider the inclusion \( L \hookrightarrow \bar{L} \) of \( L \) as a fibre of \( \bar{L} \to \gamma \) over the negative horizontal end of \( \gamma \). Then the grading, the \( Pin \) structure, and the local system on \( L \) are the pullbacks of the corresponding data on \( \bar{L} \) via this inclusion. With the definitions of gradings, \( Pin \) structures and local systems given in Section 3.1, it should be fairly obvious what pulling back means. As for pulling back a \( Pin \) structure \( \bar{S} \) from \( \bar{L} \), we include \( P_O(TL) \hookrightarrow P_O(TL) \) by completing frames in \( TL \) to frames in \( T\bar{L} \) using the right-pointing tangent vector to \( \gamma \); then we restrict \( \bar{P} \) accordingly, thinking of it as a double cover of \( P_O(TL) \).

4.3. Curves in \( M \) vs. curves in \( \tilde{M} \). To relate the \( A_\infty \)-compositions in \( B_{\gamma,h} \) in terms with those of \( \mathcal{F}^j(M) \), we need to add to the discussion in [BC13b] a verification that \( \tilde{J} \)-holomorphic curves \( \tilde{u} \) in \( \tilde{M} \) living in a fibre of \( \tilde{M} \to \mathbb{R}^2 \) carry the same sign as the corresponding \( J \)-holomorphic curves \( u \) in \( M \).

We need a bit of preparation for that. Take objects \( \bar{L}_0 \) and \( \bar{L}_1 \) of \( B_{\gamma,h} \) and let \( L_0 = c_j(\bar{L}_0) \) and \( L_1 = c_j(\bar{L}_1) \) be the corresponding objects of \( \mathcal{F}^j(M) \). Let

\[
\tilde{y} = (o_j, y) \in \Phi^h_0(\bar{L}_0) \cap L_1,
\]

where \( o_j \in \Phi^h_0(\gamma) \cap \gamma \) for some \( j = 1, \ldots, \ell \), and \( y \in \Phi^h_1(L_0) \cap L_1 \). Here \((\tilde{H}, \tilde{J})\) and \((H, J)\) are Floer data corresponding to \((\bar{L}_0, \bar{L}_1)\) and \((L_0, L_1)\) which are related as described in Section 4.1.

We now associate an index and an orientation space to the intersection points \( y \) and \( \tilde{y} \) as described in [Sei08, Section 11]. To explain this for \( y \), consider the Lagrangian subspaces

\[
\Lambda_0 = T_y\Phi^h_1(L_0) \quad \text{and} \quad \Lambda_1 = T_yL_1
\]

of \( T_yM \); these spaces carry natural brane structures induced by those on \( L_0 \) and \( L_1 \). Choose a generic path

\[
\lambda \in \Omega^-(Gr(T_yM); \Lambda_0, \Lambda_1),
\]

such that the Lagrangian subbundle of \([0, 1] \times T_yM \) obtained from \( \lambda \) admits a grading which restricts to the given gradings of \( \Lambda_0 \) and \( \Lambda_1 \) on the ends; here \( \Omega^-(Gr(T_yM); \Lambda_0, \Lambda_1) \) is the space of all paths from \( \Lambda_0 \) to \( \Lambda_1 \) in \( Gr(T_yM) \) such...
that the crossing form \( q_\lambda(s) \) is negative definite at \( s = 1 \) (see [Sci08, Section (11f)] for the definition). The index and orientation space of \( y \) are then defined as

\[
i(y) = \sum_{s < 1} \text{sign}(q_\lambda(s)) \quad \text{and} \quad o(y) = \bigotimes_{s < 1} (\lambda(s) \cap \lambda(1))^{\text{sign}(q_\lambda(s))}.
\]

The genericity assumption on \( \lambda \) implies that the intersection \( \lambda(s) \cap \lambda(1) \), \( s < 1 \), is \( \{0\} \) except at finitely many points, where it is a one-dimensional real vector space. Thus the definitions make sense, and \( o(y) \) is a one-dimensional real vector space.

The definitions of \( i(\tilde{y}) \) and \( o(\tilde{y}) \) are analogous.

**Lemma 4.4.** Given \( y \) and \( \tilde{y} = (o_j, y) \) as above, the indices satisfy \( i(\tilde{y}) = i(y) \) if \( j \) is odd, and \( i(\tilde{y}) = i(y) + 1 \) if \( j \) is even. Moreover, there are canonical isomorphisms \( o(\tilde{y}) \cong o(y) \) if \( j \) is odd, and \( o(\tilde{y}) \cong T_{o_j} \gamma \otimes o(y) \) if \( j \) is even.

**Proof.** Set \( \ell_0 = T_{o_j} (\Phi^h(\gamma)) \) and \( \ell_1 = T_{o_j} \gamma \). Choose two generic paths \( \lambda \in \Omega^- (\text{Gr}(\mathbb{R}^2); \ell_0, \ell_1) \) and \( \lambda' \in \Omega^- (\text{Gr}(T_y M); \Lambda_0, \Lambda_1) \) which satisfy the required compatibility with the gradings. For \( \lambda \) this means that it looks as shown in Fig.

![Figure 5. The paths \( \lambda \in \Omega^- (\text{Gr}(\mathbb{R}^2); \ell_0, \ell_1) \).](image)

ure depending on the parity of \( j \) (the negative definiteness condition on the crossing form means \( \lambda \) must approach \( \ell_1 \) from above at \( s = 1 \)). Then the path \( \tilde{\lambda} = (\lambda, \lambda') : [0, 1] \to \text{Gr}(T_y M) \) lies in \( \Omega^- (\text{Gr}(T_y M); \Lambda_0, \Lambda_1) \) and is also compatible with the gradings, as a consequence of how the gradings on the \( L_i \) and the \( \tilde{L}_i \) are related. The claimed statements now follow immediately from the definitions of the indices and of the orientation spaces.

Recall that the moduli spaces \( \mathcal{M}^{d+1}(y_0, \ldots, y_d) \) appearing in the definition of the \( A_\infty \)-compositions of \( \mathcal{F}_U(M) \) are spaces of maps \( u : S \to M \) satisfying a Cauchy-Riemann type equation, where \( S \) is a \((d + 1)\)-pointed disc. One commonly views \( \mathcal{M}^{d+1}(y_0, \ldots, y_d) \) as the zero set of a section of \( \mathcal{E} \to \mathcal{B} \), where \( \mathcal{B} \) is a Banach manifold of maps \( u : S \to M \) which are locally of class \( W^{1,p} \) and satisfy appropriate boundary and asymptotic conditions, and where \( \mathcal{E} \to \mathcal{B} \) is the vector bundle whose fibre at \( u \) is \( L^p(S; \Lambda^{0,1} T^* S \otimes E) \), with \( E = u^* TM \). The linearization of the defining section at \( u \in \mathcal{M}^{d+1}(y_0, \ldots, y_d) \) is a non-degenerate Cauchy-Riemann operator (in the sense of [Sci08, Section (8h)])

\[
D_u : W^{1,p}(S; E, F) \to L^p(S; \Lambda^{0,1} T^* S \otimes E),
\]

where \( F \subset E \mid_{\partial S} \) is the Lagrangian subbundle over \( \partial S \) induced by the Lagrangians \( L_i \) to which \( u \) maps the components of \( \partial S \). Analogous statements apply to the moduli spaces \( \mathcal{M}^{d+1}(\tilde{y}_0, \ldots, \tilde{y}_d) \) used to define \( \mathcal{B}_{\gamma, h} \).
Suppose now that choices of Floer data have been made as in Section 4.2, and choices of perturbation data as in [BC13b, Section 4.2]. Let \( o_j \in \Phi_1^b(\gamma) \cap \gamma \) for some odd \( j \). Then for every curve \( u \in \mathcal{M}^{d+1}(y_0, \ldots, y_d) \), the curve \( \tilde{u} = (o_j, u) \) lies in \( \mathcal{M}^{d+1}(\tilde{y}_0, \ldots, \tilde{y}_d) \) (after applying a small perturbation which only affects the \( \mathbb{C} \)-component of the curve – in [BC13b] this is called a naturality transformation). This is a consequence of the choice of perturbation datum used in the equation defining \( \mathcal{M}^{d+1}(\tilde{y}_0, \ldots, \tilde{y}_d) \), which around the odd points splits into a planar part and a vertical part that is identical to the perturbation datum appearing in the equation defining \( \mathcal{M}^{d+1}(y_0, \ldots, y_d) \). On the level of linearized operators, we have a splitting

\[
D_{\tilde{u}} = D_{u,C} \oplus D_u
\]

with respect to the canonical identifications

\[
W^{1,p}(S; \tilde{E}, \tilde{F}) \cong W^{1,p}(S; C, T_o, \gamma) \oplus W^{1,p}(S; E, F) \\
L^p(S; \Lambda^{0,1} T^* S \otimes \tilde{E}) \cong L^p(\Lambda^{0,1} T^* S \otimes C) \oplus L^p(\Lambda^{0,1} T^* S \otimes E)
\]

of the relevant spaces of sections (here \( E = u^* TM, \tilde{E} = \tilde{u}^* \tilde{T}M \), and \( \tilde{F}, F \) are the Lagrangian subbundles over \( \partial S \) corresponding to the boundary conditions for \( \tilde{u} \) and \( u \)). Again as a consequence of the form the Floer and perturbation data around the \( o_j \) with \( j \) odd, the planar operator \( D_{\tilde{u}, C} \) has index \( D_{u,C} = 0 \), \( \ker D_{\tilde{u}, C} = \{0\} \) and \( \ker D_{u,C} = \{0\} \). Therefore, there are canonical isomorphisms

\[
\ker D_{\tilde{u}} \cong \ker D_u,
\]

and moreover the map \( u \mapsto \tilde{u} \) identifies \( \mathcal{M}^{d+1}(y_0, \ldots, y_d) \cong \mathcal{M}^{d+1}(\tilde{y}_0, \ldots, \tilde{y}_d) \).

Remark 4.5. In fact, if the number of outgoing points is \( d = 1 \), all these statements are also true in case \( j \) is even, because the index of the planar operator \( D_{\tilde{u}, C} \) is zero in this case. In contrast, this index is negative for even \( j \) if \( d \geq 2 \) (cf. the relation between \( i(y) \) and \( i(\tilde{y}) \) in Lemma 4.4 and the index formula for operators in [Sei08, Prop. 11.13]).

4.4. **Compatible choices of signs.** We orient all moduli spaces \( \mathcal{M}^{d+1}(y_0, \ldots, y_d) \), \( d \geq 1 \), by choosing, as in Section 3.4, an orientation for every \( o(y) \), and requiring that the canonical identification

\[
\Lambda^{\text{top}}(T_u \mathcal{M}^{d+1}(y_0, \ldots, y_d)) \cong o(y_0) \otimes o(y_1) \otimes \cdots \otimes o(y_d)
\]

be orientation-preserving. In particular, this yields a sign \( \text{sgn}(u) \in \{\pm 1\} \) for every regular curve \( u \in \mathcal{M}^{d+1}(y_0, \ldots, y_d) \) of index zero.

To express the \( A_\infty \)-compositions of \( \mathcal{B}_{\gamma,h} \) (partially) in terms of those of \( \mathcal{F}^t(M) \), we have to orient the moduli spaces \( \mathcal{M}^{d+1}(\tilde{y}_0, \ldots, \tilde{y}_d) \) in a compatible way. The requirement is as follows: Given some \( \tilde{y} = (o_j, y) \) with odd \( j \) and an orientation of \( o(y) \) as above, we orient \( o(\tilde{y}) \) such that the canonical identification

\[
o(\tilde{y}) \cong o(y)
\]

from Lemma 4.4 matches up the orientations. (For the \( \tilde{y} = (o_j, y) \) with even \( j \) it will not matter how the \( o(\tilde{y}) \) are oriented.) The orientations of the \( \mathcal{M}^{d+1}(\tilde{y}_0, \ldots, \tilde{y}_d) \)
and in particular the signs \( \text{sgn}(\tilde{u}) \in \{ \pm 1 \} \) of isolated curves \( \tilde{u} \) are then determined as described above.

**Lemma 4.6.** Given \( y_0, \ldots, y_d \) and corresponding \( \tilde{y}_0, \ldots, \tilde{y}_d \) with \( \tilde{y}_i = (o_j, y_i) \) for some odd \( j \), the canonical identification \( \mathcal{M}^{d+1}(y_0, \ldots, y_d) \cong \mathcal{M}^{d+1}(\tilde{y}_0, \ldots, \tilde{y}_d) \) is orientation-preserving. In particular, \( \text{sgn}(u) = \text{sgn}(\tilde{u}) \) for all isolated curves \( u \) and \( \tilde{u} = (o_j, u) \).

**Proof.** The statement follows from the commutativity of the diagram

\[
\begin{array}{ccc}
\Lambda^{\text{top}}(T_u \mathcal{M}^{d+1}(y_0, \ldots, y_d)) & \overset{\cong}{\longrightarrow} & \Lambda^{\text{top}}(T_{\tilde{u}} \mathcal{M}^{d+1}(\tilde{y}_0, \ldots, \tilde{y}_d)) \\
\cong & \sim & \cong \\
\Lambda^{\text{top}}(\ker D_u) & \overset{\cong}{\longrightarrow} & \Lambda^{\text{top}}(\ker D_{\tilde{u}}) \\
\cong & \sim & \cong \\
o(y_0) \otimes o(y_1)^{\nu_1} \otimes \cdots \otimes o(y_d)^{\nu_d} & \overset{\cong}{\longrightarrow} & o(\tilde{y}_0) \otimes o(\tilde{y}_1)^{\nu_1} \otimes \cdots \otimes o(\tilde{y}_d)^{\nu_d}
\end{array}
\]

in which the vertical isomorphisms are the canonical ones, and where the first row is induced by \( \mathcal{M}^{d+1}(y_0, \ldots, y_d) \cong \mathcal{M}^{d+1}(\tilde{y}_0, \ldots, \tilde{y}_d) \), the second by the splitting \( D_{\tilde{u}} = D_{\tilde{u},C} \oplus D_u \) and the fact that \( \ker D_{\tilde{u},C} = 0 \), and the third by the canonical identifications from Lemma 4.4. \( \square \)

**4.5. Definition of the functors** \( c_j : \mathcal{H}_{\gamma,h} \to \mathcal{F}^{\sharp}(M) \). We now complete the definition of the (presumable) \( A_{\infty} \)-functor

\[
c_j \equiv c_{\gamma,h,j} : \mathcal{H}_{\gamma,h} \to \mathcal{F}^{\sharp}(M)
\]

for odd \( j \) with \( 1 \leq j \leq \ell \). On objects, \( c_j \) takes \((\tilde{L}, \tilde{E})\) to \((L, E)\), as already mentioned in Section 4.2. As for morphisms, consider objects \( \tilde{L}_i \equiv (\tilde{L}_0, \tilde{E}_i) \) for \( i = 0, 1 \) and the corresponding \( L_i \equiv (L_0, E_i) \). Note that for each of the summands \( CF(\tilde{L}_0, \tilde{L}_1)^{\nu_i} \subset CF(\tilde{L}_0, \tilde{L}_1) \) appearing in the splitting \( 4 \), we have a canonical isomorphism of vector spaces

\[
CF(\tilde{L}_0, \tilde{L}_1)^{\nu_i} \cong CF(L_0, L_1).
\]

This is compatible with gradings, because the indices of corresponding intersection points \( y \) and \( \tilde{y} = (o_j, y) \) satisfy \( i(\tilde{y}) = i(y) \) whenever \( j \) is odd (cf. Lemma 4.4). We define the first order component of \( c_j \) on morphisms to be the composition

\[
c_j^1 : CF(\tilde{L}_0, \tilde{L}_1) \to CF(\tilde{L}_0, \tilde{L}_1)^{\nu_i} \cong CF(L_0, L_1),
\]

where the first map is the projection onto morphisms of type \( o_j \). The higher order components \( c_j^d \) for \( d \geq 2 \) are defined to be identically zero.

**Proposition 4.7** (Cf. Proposition 4.2.3 in [BC13]). The \( c_j : \mathcal{H}_{\gamma,h} \to \mathcal{F}^{\sharp}(M) \) are \( A_{\infty} \)-quasi-isomorphisms. Moreover, \( c_i \) and \( c_j \) are homotopic for any two odd \( i, j \) with \( 1 \leq i, j \leq \ell \).

**Proof.** The proof is an adaptation of the proof of Proposition 4.2.3 in [BC13] that takes into account gradings and signs. We have already noted that the maps \( c_j^1 \) are compatible with gradings. Using the arguments in [BC13] together with Lemma 4.6 one sees that the differentials of \( CF(\tilde{L}_0, \tilde{L}_1) \) and \( CF(L_0, L_1) \) are related as
described in [BC13b] Remark 4.2.2, which implies that the complexes are quasi-isomorphic. Then one checks that the \( \{ c_j \} \) really define \( A_\infty \)-functors \( c_j : \mathcal{F}(M) \to \mathcal{B}_{\gamma,h} \).

4.6. Exact triangles from cobordisms. The construction of the exact triangles associated to a Lagrangian cobordism now follows the scheme in [BC13b] Sections 4.3, 4.4. Since even to outline this would require quite a bit of additional notation, we content ourselves with saying that all relevant statement carry over with minor modifications. The bridge between \( \mathcal{F}(M) \) and the world of cobordisms is now provided by the \( A_\infty \)-quasi-isomorphisms

\[
\mathcal{B}_{\gamma,h} \xrightarrow{c_j} \mathcal{F}(M)
\]

from Proposition 4.7.

To state the upshot of all this, let \( L \equiv (L,E) \) and \( L_1 \equiv (L_1, E_1), \ldots, L_k \equiv (L_k, E_k) \) be Lagrangian branes in \( M \) carrying local systems, and let

\[
V : L \rightsquigarrow (L_1, \ldots, L_k)
\]

be a Lagrangian cobordism equipped with a brane structure and a local system which restrict to the given ones on the ends.

**Proposition 4.8.** There exist \( \mathcal{F}(M) \)-modules \( \mathcal{M}^1_V, \ldots, \mathcal{M}^k_V \) such that \( \mathcal{M}^1_V = \text{Yon}(L_1) \) and such that there are exact triangles

\[
\mathcal{M}^{i-1}_V \to \text{Yon}(L_j) \to \mathcal{M}^i_V \to \mathcal{M}^{i-1}_V[1]
\]

in \( \text{mod}(\mathcal{F}(M)) \) for \( j = 2, \ldots, k \). Moreover, there is an \( A_\infty \)-quasi-isomorphism \( \text{Yon}(L) \to \mathcal{M}^k_V \).

Here \( \text{Yon} : \mathcal{F}(M) \to \text{mod}(\mathcal{F}(M)) \) denotes the Yoneda embedding of \( \mathcal{F}(M) \) into the \( A_\infty \)-category \( \text{mod}(\mathcal{F}(M)) \) of \( A_\infty \)-modules over itself. The Proposition summarizes what would be the analogues of Corollary 4.3.3 and Proposition 4.4.1 of [BC13b] in our setting (and the \( \mathcal{M}^V \) are the analogues of the \( \mathcal{M}_{V,\gamma,h} \), there).

Note that the directions of the arrows in the exact triangles are reversed compared to [BC13b] because we use cohomological conventions.

Theorem 4.1 is an immediate consequence of Proposition 4.8.

4.7. Cobordism and Grothendieck groups. Denote by \( \mathcal{F}(M) \) the full subcategory of \( \mathcal{F}(M) \) consisting of Lagrangian branes with trivial rank one local systems. Let \( \Omega^\text{Lag}(M) \) and \( \Omega^\text{Lag}_0(M) \) be the Lagrangian cobordism group defined as in Sections 2.3 and 2.4 with respect to \( \mathcal{L} = \text{Ob} \mathcal{F}(M), \mathcal{L}^\text{r} = \text{Ob} \mathcal{F}(M) \) and relation subgroup \( R, R^\text{r} \) induced by appropriate Lagrangians cobordisms between these generators. Denote by \( K_0(D\mathcal{F}(M)) \) and \( K_0(D\mathcal{F}(M)) \) be the Grothendieck groups of the derived Fukaya categories.

**Proposition 4.9.** There exist canonical surjective group homomorphisms

\[
\Theta : \Omega^\text{Lag}(M) \to K_0(D\mathcal{F}(M)) \quad \text{and} \quad \Theta : \Omega^\text{Lag}_0(M) \to K_0(D\mathcal{F}(M))
\]

induced by \( L \mapsto L \) resp. \( (L,E) \mapsto (L,E) \).
The Lagrangian cobordism group of $T^2$

The statement for $\Omega_{\text{Lag}}(M) \to K_0(D\mathcal{F}(M))$ follows immediately from Theorem 4.1 (together with Lemma A.3), while that for $\Omega_{\#}\text{Lag}(M) \to K_0(D\mathcal{F}^\#(M))$ needs in addition Proposition B.1, which implies that the relations in $\Omega_{\#}\text{Lag}(M)$ coming from short exact sequences of local systems are respected.

5. Lagrangian surgery and cobordisms in $\tilde{T}^2$

Surgering transversely intersecting Lagrangians $L_0, L_1$ means cutting out small neighbourhoods in $L_0$ and $L_1$ of each intersection point and gluing in Lagrangian handles, such as to produce a new Lagrangian submanifold $L_0\# L_1$ (see [Pol91]).

Biran–Cornea [BC13a] describe how to construct a Lagrangian cobordism $R^n \# iR^n \to (R^n, iR^n)$.

In general, one can produce cobordisms $L_0 \# L_1 \to (L_0, L_1)$ by gluing in this local model via Darboux charts. We describe this construction for $n = 1$ in Sections 5.1 and 5.2.

Throughout this section, we identify $\mathbb{R}^2 \cong \mathbb{C}$ in the usual way, so that Lagrangian cobordisms now live in $\tilde{M} = \mathbb{C} \times M$.

5.1. Model cobordisms in dimension 1. To surger $\mathbb{R}$ and $i\mathbb{R} \subset \mathbb{C}$, one cuts out two neighbourhoods of $0 \in \mathbb{C}$ and connects the ends on $\mathbb{R}$ thus created with the ends on $i\mathbb{R}$ by two curve segments. More formally, one chooses a smooth curve $\gamma : \mathbb{R} \to \mathbb{C}$, $\gamma = a + ib$ such that

- $\gamma(t) = t$ for $t \in (-\infty, -\varepsilon]$,
- $\gamma(t) = it$ for $t \in [\varepsilon, \infty)$,
- $a'(t), b'(t) > 0$ for $t \in (-\varepsilon, \varepsilon)$

for some $\varepsilon > 0$. Then one defines $\mathbb{R}\#_\gamma i\mathbb{R} = \{\pm \gamma(t) \mid t \in \mathbb{R}\}$. See Figure 6.

To construct the corresponding Lagrangian cobordism $V_\gamma : \mathbb{R}\#_\gamma i\mathbb{R} \to (\mathbb{R}, i\mathbb{R})$, consider the embedding $\psi_\gamma : \mathbb{R} \times S^1 \to \mathbb{C}^2$, $(t, (x_1, x_2)) \mapsto (\gamma(t) \cdot x_1, \gamma(t) \cdot x_2)$, where $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$. As a first step, we set

$$V_\gamma' = \text{im}(\psi_\gamma) \cap \pi^{-1}\left(\{x + iy \in \mathbb{C} \mid x \leq y, -2\varepsilon \leq x, y \leq 2\varepsilon\}\right)$$
where \( \pi : \mathbb{C}^2 \to \mathbb{C} \) is the projection to the first factor; in other words, \( V'_\gamma \) is the image of \( \{(t, (x_1, x_2)) \in \mathbb{R} \times S^1 \mid 0 \leq x_1, -2\varepsilon \leq tx_1 \leq 2\varepsilon \} \) under \( \psi_\gamma \). Note that \( V'_\gamma \) is a manifold with boundary 
\[ \partial V'_\gamma = \{-2\varepsilon \} \times \mathbb{R} \cup \{2\varepsilon i\} \times i\mathbb{R} \cup \{0\} \times \mathbb{R} \# i\mathbb{R}. \]
To complete the construction of \( V_\gamma \), we extend the part of \( \partial V'_\gamma \) lying over \( 0 \in \mathbb{C} \) to a cylindrical end as explained in the proof of Lemma 6.1.1 of [BC13a] (the other two ends are already cylindrical).

Figure 7 shows a schematic picture of \( V_\gamma \). The \( t \)-axis is horizontal, and the vertical dashed lines correspond to segments of \( S^1 \). The two curved lines correspond to the negative ends \( \mathbb{R} \) and \( i\mathbb{R} \) of the cobordism, whereas the horizontal ones correspond to the two components of the positive end \( \mathbb{R} \# i\mathbb{R} \). The striped areas represent the cylindrical ends we attached in the end.

We will refer to the following subsets of \( V_\gamma \) as its fat and its thin parts:
\[
V'_f \gamma = \psi_\gamma \left( \{(t, (x_1, x_2)) \in \mathbb{R} \times S^1 \mid 0 \leq x_1, -\varepsilon \leq t \leq \varepsilon \} \right), \\
V'_t \gamma = \psi_\gamma \left( \{(t, (x_1, x_2)) \in \mathbb{R} \times S^1 \mid x_1 = 0, x_2 = \pm 1 \} \right).
\]
In Figure 7, \( V'_f \gamma \) is represented by the shaded square, and \( V'_t \gamma \) is represented by the union of the two horizontal lines. Note that \( V'_f \gamma \) is precisely the part of \( V_\gamma \) lying over the fat part of \( \pi(V_\gamma) \) (the shaded region in Figure 8), whereas \( V'_t \gamma \) projects via \( \pi \) to \( 0 \in \mathbb{C} \). Moreover, note that \( V_\gamma \) deformation retracts onto \( V'_f \gamma \cup V'_t \gamma \).

5.2. Surgery in surfaces and cobordisms. Let now \( L_0 \) and \( L_1 \) be two (possibly disconnected) Lagrangians in a surface \( \Sigma \) intersecting transversely in \( m \) points.
THE LAGRANGIAN COBORDISM GROUP OF $T^2$

Figure 9. Surgering compatibly and non-compatibly with given orientations.

$q_0, \ldots, q_{m-1}$. Fix small neighbourhoods $U_j \subset \Sigma$ of the $q_j$ and Darboux charts $\phi_j : U_j \to \mathbb{C}$ such that $\phi_j(L_0 \cap U_j) \subset \mathbb{R}$ and $\phi_j(L_1 \cap U_j) \subset i\mathbb{R}$. Moreover, choose curves $\gamma_j : \mathbb{R} \to \mathbb{C}$ as in the previous section. The surgered manifold $L_0 \# L_1$ corresponding to these data is obtained by gluing in the local surgery models corresponding to the $\gamma_j$ using the charts $\phi_j$.

To construct the corresponding cobordism $V : L_0 \# L_1 \sim (L_0, L_1)$, we first define subsets of $\mathbb{C}$ as follows:

$$I_0 = \{x \in \mathbb{R} \mid -2\varepsilon \leq x \leq 0\},$$

$$I_1 = \{iy \in i\mathbb{R} \mid 0 \leq y \leq 2\varepsilon\},$$

$$I_2 = \{x - ix \in \mathbb{C} \mid 0 \leq x \leq 2\varepsilon\}.$$

Moreover, we set $\tilde{L}_i = L_i \setminus \bigcup_{j=0}^{m-1}(L_i \cap \phi_j^{-1}(B_{2\varepsilon}(0)))$ for $i = 0, 1$, and $\tilde{V}_{\gamma_j} = V_{\gamma_j} \cap (\mathbb{C} \times B_{2\varepsilon}(0))$ for $j = 0, \ldots, m - 1$, where the $V_{\gamma_j}$ are the model cobordisms from the previous section. Then we set

$$V = (I_0 \times \tilde{L}_0) \cup (I_1 \times \tilde{L}_1) \cup (I_2 \times (\tilde{L}_0 \cup \tilde{L}_1)) \cup \bigcup_{j=0}^{m-1}(\text{id} \times \phi_j^{-1})(\tilde{V}_{\gamma_j}).$$

Remark 5.1. For this to work, it was necessary to choose all charts $\phi_j$ such that $L_0 \cap U_j$ was mapped to $\mathbb{R}$ and $L_1 \cap U_j$ to $i\mathbb{R}$. This reflects that one needs to select one of the $L_i$ whose corresponding cylindrical end lies over $\mathbb{R}$, and the other for which it lies over $i\mathbb{R}$. Of course, one can swap $\mathbb{R}$ and $i\mathbb{R}$ by a $\frac{\pi}{2}$-rotation, so that this requirement does not impose any restriction at first. However, if one wants to glue in all local surgery models in a way compatible with given orientations of the $L_i$ (which implies that the resulting cobordism is orientable), then the local topological type of the surgery at one intersection point determines that at all others.

As in the local case, we define fat and thin parts of $V$ by

$$V^f = \bigcup_{j=0}^{m-1}(\text{id} \times \phi_j^{-1})(\tilde{V}_{\gamma_j}^f),$$

$$V^t = \bigcup_{j=0}^{m-1}(\text{id} \times \phi_j^{-1})(\tilde{V}_{\gamma_j}^t) \cup \{0\} \times (\tilde{L}_0 \cup \tilde{L}_1),$$
where \( \tilde{V}_f^i = V_f^i \cap (C \times B_{2e}(0)) \) and \( \hat{V}_j^i = V_j^i \cap (C \times B_{2e}(0)) \). By combinedly deformation retracting the glued-in \( \tilde{V}_j^i \) onto their fat and thin parts, and the remaining (cylindrical) parts of \( V \) onto \( \{0\} \times (\hat{L}_0 \cup \hat{L}_1) \), one deformation retracts \( V \) onto \( V_f^j \cup V_t^i \).

One sees readily from this description that \( V \) is homotopy equivalent to a 1-dimensional CW-complex with \( m \) cells of dimension 0 and \( 2m \) cells of dimension 1, where \( m = \# L_0 \cap L_1 \). Denoting by \( b \) the number of boundary components of \( V \) (which is equal to the total number of connected components of \( L_0, L_1 \) and \( L_0 \# L_1 \)), its Euler characteristic and its genus are hence given by

\[
\chi(V) = -m \quad \text{and} \quad g(V) = 1 + \frac{1}{2}(m - b),
\]

provided that \( V \) is orientable.

5.3. Cobordisms in \( \overline{T}^2 \) and their Maslov class. We identify the tangent bundle of \( \overline{T}^2 = C \times T^2 \) with \( \overline{T}^2 \times \mathbb{R}^4 \) in the obvious way, so that the Gauß map of a Lagrangian in \( \overline{T}^2 \) takes values in the Lagrangian Grassmannian \( \text{Gr}(\mathbb{R}^4) \).

In the following, we call a Lagrangian \( L \subset T^2 \) straight if it lifts to a straight line in the universal cover. The cobordisms of relevance for us come from surgering oriented straight Lagrangians in \( T^2 \) in a way that is compatible with the orientations in the sense of Remark 5.1. (This ensures not only orientability of the resulting cobordisms, but also that \( L_0 \# L_1 \) has no contractible components.)

**Lemma 5.2.** Let \( L_0, L_1 \) be oriented straight Lagrangians in \( T^2 \) of different slopes, let \( L_0 \# L_1 \) be the result of an orientation-compatible surgery, and let \( V : L_0 \# L_1 \leadsto (L_0, L_1) \) be the resulting cobordism. Then the Gauß map \( \Gamma_V : V \to \text{Gr}(\mathbb{R}^4) \) is null-homotopic. In particular, \( V \) has vanishing Maslov class.

**Proof.** We will show that \( \Gamma_V \) is homotopic to a map which factors through a map defined on a contractible domain; this statement immediately implies the assertion. Note that we need only show this for \( \Gamma_V|_{V_f \cup V_t} \), since \( V \) deformation retracts onto \( V_f^j \cup V_t^i \). Moreover, we can assume that the local surgeries at all intersection points \( q_i \) are defined with respect to the same local model, and thus that \( V \) is obtained by gluing in the same \( V_\gamma \) at every \( q_i \); namely, isotopies of the curves \( \gamma_j \) (with respect to which the surgery is defined) to one given curve \( \gamma \) induce a Lagrangian isotopy of the resulting cobordisms.

Under these assumptions, all maps \( \Gamma_V \circ (\text{id} \times \phi_j^{-1}) : V_f^j \to \text{Gr}(\mathbb{R}^4) \) are equal to one and the same map \( \hat{\Gamma} : V_f^j \to \text{Gr}(\mathbb{R}^4) \). Since \( \Gamma_V \) is constant on every connected component of \( (V_f^j \cup V_t^i) \setminus V_f^j \), it factors through \( \hat{\Gamma} \), whose domain \( V_f^j \) is contractible. \( \square \)

5.4. Non-monotonicity of cobordisms in \( \overline{T}^2 \). We next explain that, for cobordisms \( V : L_0 \# L_1 \leadsto (L_0, L_1) \) in \( \overline{T}^2 \) as above, there always exist classes \([u] \in \pi_2(\overline{T}^2, V)\) that have positive symplectic area. Together with the vanishing of the Maslov class, this implies that these cobordisms are not monotone.

Choose an intersection point \( q_0 \in L_0 \cap L_1 \), and label the remaining intersection points by \( q_1, \ldots, q_{m-1} \) in the order in which they occur when following \( L_0 \) in positive direction, starting from \( q_0 \) (for simplicity, we assume that \( L_0 \) is connected). Figure
shows an example. Viewing the indices of the $q_j$ as elements of $\mathbb{Z}/m\mathbb{Z}$, we define a function $\kappa : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ by the requirement that the first intersection point encountered when starting from $q_j$ and then travelling along $L_1$ in positive direction is $q_{\kappa(j)}$.

Recall from Section 5.2 that $V$ deformation retracts onto $V^f \cup V^t \subset V$, and that $V^f$ is the disjoint union of the subsets $V^f_j := (\text{id} \times \phi_j^{-1})(\check{V}^t_j) \subset V, j = 0, \ldots, m - 1$. By collapsing each $V^f_j \subset V^f \cup V^t$ to a point $F_j$, one obtains a graph $G_V$ whose edges correspond to the connected components of $V \setminus V^f$. Since each of these connected components projects either to $L_0$ or to $L_1$, the edges of $G_V$ are naturally oriented and labelled by $L_0$ or $L_1$. Note that each vertex $F_j$ of $G_V$ is connected to $F_{j-1}$, $F_{j+1}$ by $L_0$-edges, and to $F_{\kappa^{-1}(j)}$ and $F_{\kappa(j)}$ by $L_1$-edges (of course, these vertices need not be mutually distinct.) Figure 11 displays $G_V$ for the cobordism coming from surgery of the Lagrangians in Figure 10.

Define now a path $\eta$ in $G_V$ as follows: $\eta$ starts at some $F_j$, then travels to $F_{j+1}$ along an $L_0$-edge, then to $F_{\kappa(j+1)}$ along an $L_1$-edge, then to $F_{\kappa(j)}$ backwards along an $L_0$-edge, and finally back to $F_j$ backwards along an $L_0$-edge.

This description yields a well-defined class $[\eta] \in \pi_1(V)$, which is represented by paths $\eta_1$ and $\eta_2$ in $V^f \cup V^t \subset V$ with the following behaviour under the projections of $V \subset \mathbb{C} \times T^2$ to $\mathbb{C}$ and $T^2$: $\eta_1$ projects to the coordinate axes in $\mathbb{C}$, while $\eta_2$
projects to the quadrilateral in $T^2$ with vertices $q_j, q_{j+1}, q_{\kappa(j)+1}$ and $q_{\kappa(j)}$ and edges on $L_0$ and $L_1$ (in Figure 10, the quadrilateral obtained for $j = 0$ is shaded). Figure 12 shows what $\eta_1$ and $\eta_2$ look like on a fat piece of $V$.

Consider now the class $[u] \in \pi_2(\tilde{T}^2, V)$ corresponding to $[\eta]$ via the isomorphism $\pi_2(\tilde{T}^2, V) \to \pi_1(V)$ (cf. the fact that $\pi_2(\tilde{T}^2) = 0$). The descriptions of the projections of $\eta_1$ and $\eta_2$ imply that the first component of $\tilde{\omega} = \omega_{\text{std}} \oplus \omega_{T^2}$ vanishes on $[u]$, while the second component gives a positive contribution.

5.5. Implications for the analysis. The non-monotonicity of the cobordisms relevant to us imply that the compactness arguments of [BC13b] for the spaces of $J$-holomorphic polygons appearing in the definition of Fukaya-type categories of cobordisms (such as the $B_{\gamma,h}$ from Section 4) do not apply a priori. Specifically, the potential bubbling off of $J$-holomorphic discs of Maslov index zero might obstruct the compactness of the moduli spaces.

Note that the index of a $J$-holomorphic disc $u$ (that is, the Fredholm index of its linearized Cauchy-Riemann operator) of Maslov index $\mu(u) = 0$ with boundary on a Lagrangian in a symplectic 4-manifold is

$$2\chi(D^2) + \mu(u) = 2,$$

and therefore the expected dimension of the corresponding moduli space is $2 - 3 = -1$. Such discs exist therefore only for almost complex structures $J$ belonging to a codimension one stratum $\mathcal{J}_0 \subset \mathcal{J}$ of the space of compatible almost complex structures, and thus bubbling can be excluded as long as one works with a fixed generic $J$. However, when considering one-parameter of families $\{J_t\}$ of almost complex structures and associated moduli spaces of polygons, which is necessary to prove invariance properties, one may encounter Maslov zero disc bubbles even when $\{J_t\}$ is generic.

One solution to this problem is to work only with almost complex structures in a fixed chamber of $\mathcal{J}$ (i.e., component of $\mathcal{J} \setminus \mathcal{J}_0$). The resulting objects are then only invariant with respect to changing $J$ within this chamber. For our specific purposes, that’s not a problem, because we are ultimately not interested in invariants of Lagrangian cobordisms in $\tilde{T}^2$; rather, we only use them to deduce statements about the Fukaya category of $T^2$ (which is essentially of a topological nature, cf. [Sei08 Section 13]).

5.6. Grading of cobordisms in $\tilde{T}^2$. A grading of a Lagrangian $L \subset \tilde{T}^2$ is a function $\alpha : L \to \mathbb{R}$ lifting the composition $\det^2 \circ \Gamma_L : L \to \text{Gr}(\mathbb{R}^4) \to S^1$ (cf. Section 3.1.1), where $\Gamma_L : L \to \text{Gr}(\mathbb{R}^4)$ denotes the Gauß map. Note that every Lagrangian cobordism $V : L_0 \# L_1 \leadsto (L_0, L_1)$ coming from surgery of two linear
Lagrangians in $T^2$ admits a grading $\alpha_V : V \to \mathbb{R}$ because its $\Gamma_V$ is null-homotopic by Lemma 5.2. The next lemma tells how the restrictions of $\alpha_V$ to the ends are related.

Lemma 5.3. The restrictions of the grading $\alpha_V : V \to \mathbb{R}$ to the ends satisfy $\alpha_V|_{L_0} < \alpha_V|_{L_0 \# L_1} < \alpha_V|_{L_1}$.

Here we assume the positive end $L_0 \# L_1$ has been “linearized” by a Hamiltonian isotopy, so that $\alpha_V|_{L_0 \# L_1}$ is constant. The lemma can be proven easily by examining the local models from Section 5.1.

6. Computation of $\Omega_{\text{Lag}}(T^2)$

The Lagrangian cobordism group $\Omega_{\text{Lag}}(T^2)$ we study is defined as in Section 2.3 with $\mathcal{L}(T^2) = \text{Ob} \mathcal{F}(T^2)$. That is, $\Omega_{\text{Lag}}(T^2)$ has as generators non-contractible curves equipped with brane structures, and relations coming from cobordisms of vanishing Maslov class equipped with compatible brane structures.

6.1. Notation for curves. Let $m, n \in \mathbb{Z}$ such that $\gcd(m, n) = 1$, and let $x \in \mathbb{R}$. If $(m, n) \neq (\pm 1, 0)$, we define

$$L_{(m,n),x} \subset T^2$$

to be the oriented straight curve of slope $(m, n) \in \mathbb{Z}^2 \cong H_1(T^2; \mathbb{Z})$ passing through the point $(x, 0) \in T^2$. We define $L_{(\pm 1,0),x}$ to be the straight horizontal curve through $(0, x) \in T^2$ oriented such that it represents $(\pm 1,0) \in H_1(T^2; \mathbb{Z})$. When $x = 0$, we abbreviate $L_{(m,n)} \equiv L_{(m,n),0}$.

6.2. Cylinders. A cylinder in $T^2$ is a smooth map $u : C = [0,1] \times S^1 \to T^2$. Given such a cylinder $u$, we equip its boundary curves $L^i = u(i) \times S^1 \subset T^2$, $i = 0, 1$, with the boundary orientation induced from the standard orientation of $[0,1] \times S^1$, and with the standard brane structure. Note that the area $\int_C u^* \omega \in \mathbb{R}$ of $u$ is determined up to an integer by its boundary, as can be seen by considering the
cover of $T^2$ corresponding to the subgroup of $\pi_1(T^2)$ generated by $u$. This justifies the wording in the following lemma.

**Lemma 6.1.** There exists a canonical injective group homomorphism $\zeta : \mathbb{R}/\mathbb{Z} \to \Omega_{\text{Lag}}(T^2)$ taking $x \in \mathbb{R}/\mathbb{Z}$ to the class represented by the boundary of a cylinder of area $x$.

**Proof.** We first show that the map $\mathbb{R}/\mathbb{Z} \to \Omega_{\text{Lag}}(T^2)$ as described is well-defined and a group homomorphism, and then that it’s injective. Note that to prove well-definedness, we can replace curves by the straight representatives of their Hamiltonian isotopy classes, as Hamiltonian isotopies change neither areas nor cobordism classes.

Our first claim is that the cobordism class of the boundary of a cylinder of area $x \in \mathbb{R}/\mathbb{Z}$ with boundary circles representing $(0, \pm 1) \in H_1(T^2; \mathbb{Z})$ is well-defined, i.e., that in $\Omega_{\text{Lag}}(T^2)$ we have

$$[L_{(0,1),x+y}] - [L_{(0,1),y}] = [L_{(0,1),x}] - [L_{(0,1)}]$$

(5)

for all $x, y \in \mathbb{R}/\mathbb{Z}$. Before we justify that, note that subtracting $[L_{(0,-1)}]$ from both sides of (5) and rearranging yields

$$([L_{(0,1),x+y}] - [L_{(0,1)}]) = ([L_{(0,1),x}] - [L_{(0,1)}]) + ([L_{(0,1),y}] - [L_{(0,1)}]),$$

(6)

which shows that our map (if well-defined) is a group homomorphism.

To obtain (5), observe that surgering $L_{(1,0)}$ and $L_{(0,1),x+y}$ has the same result as surgering $L_{(1,0),-x}$ and $L_{(0,1),y}$ up to Hamiltonian isotopy (namely $L_{(1,1),\frac{1}{2}+x+y}$ in both cases, which is a right shift by $x + y$ of what one would get when surgering $L_{(1,0)}$ and $L_{(0,1)}$). This observation tells us that $[L_{(1,0)}] + [L_{(0,1),x+y}] = [L_{(1,0),-x}] + [L_{(0,1),y}]$, or equivalently that

$$[L_{(0,1),x+y}] - [L_{(0,1),y}] = [L_{(1,0),-x}] - [L_{(1,0)}]$$

(7)

for all $x, y \in \mathbb{R}/\mathbb{Z}$. The identity (5) follows from that as the right-hand side of (7) is independent of $y$.

To complete the proof of well-definedness, we have to show that for any two general curves $L^0$ and $L^1$ bounding a cylinder of area $x$ (after reversing the orientation on $L^0$), we have $[L^1] - [L^0] = [L_{(0,1),x}] - [L_{(0,1)}]$. One way of seeing this is by thinking about how to build the $L^1$ using iterated surgery of copies of $L_{(1,0)}$ and translated copies of $L_{(0,1)}$. For example, if $L^0$ can be constructed by iteratively surgering $m$ copies of $L_{(1,0)}$ and $n$ copies of $L_{(0,1)}$, then $L^1$ can be constructed by iteratively surgering $m$ copies of $L_{(1,0)}$ and $n$ copies of $L_{(0,1)}, \frac{\pi}{2}$. From this we obtain

$$[L^1] - [L^0] = n([L_{(0,1)}, \frac{\pi}{2}] - [L_{(0,1)}]) = [L_{(0,1),x}] - [L_{(0,1)}],$$

where the second equality follows from (6). (Alternatively, one can directly deduce from (5) and (7) that the boundaries of all “vertical” and all “horizontal” cylinders of area $x$ represent the same class, and then get the general case by applying suitable maps $A \in \text{SL}(2, \mathbb{Z})$.)

To finally see that $\zeta$ is injective, we have to rule out the existence of a Lagrangian cobordism between $L_{(0,1)}$ and $L_{(0,1),x}$ for $x \neq 0$, or, equivalently, between any two isotopic but not Hamiltonian isotopic curves $L$ and $L'$. The existence of such a
cobordism $L \sim L'$ would imply that $HF(N, L) \cong HF(N, L')$ for any other curve $N$, as a consequence of spelling out what Proposition 4.8 says. But for $N = L$, we have $HF(L, L) \cong \mathbb{Z}^2$ and $HF(L, L') = 0$. \hfill \Box

6.3. **Proof of Proposition 1.1** Recall that Proposition 1.1 states that the sequence

$$0 \rightarrow \mathbb{R}/\mathbb{Z} \xrightarrow{\zeta} \Omega_{\text{Lag}}(T^2) \xrightarrow{\eta} H_1(T^2; \mathbb{Z}) \rightarrow 0$$

is exact, where $\zeta$ is as described in the previous subsection, and with $\eta : \Omega_{\text{Lag}}(T^2) \rightarrow H_1(T^2; \mathbb{Z})$ the canonical map given by $[L]_0 \mapsto [L]_{H_1}$.

Injectivity of $\zeta$ was shown in Lemma 6.1. Moreover, it is clear that the canonical map $\eta : \Omega_{\text{Lag}}(T^2) \rightarrow H_1(T^2; \mathbb{Z})$ is surjective and that $\text{im}\ \zeta \subseteq \ker\ \eta$. To prove $\ker\ \eta \subseteq \text{im}\ \zeta$, consider the map

$$\mathbb{R}/\mathbb{Z} \oplus H_1(T^2; \mathbb{Z}) \rightarrow \Omega_{\text{Lag}}(T^2), \quad (x, (m, n)) \mapsto \zeta(x) + m[L_{(1,0)}] + n[L_{(0,1)}],$$

where we identify $H_1(T^2; \mathbb{Z}) \cong \mathbb{Z}^2$ in the obvious way.

Observe that $\Omega_{\text{Lag}}(T^2)$ is generated by $[L_{(1,0)}]$ and the elements of the family $\{L_{(1,1)}, x \in \mathbb{R}/\mathbb{Z}\}$, because every Lagrangian can be obtained by iteratively surgering these. An alternative set of generators of $\Omega_{\text{Lag}}(T^2)$ is given by $[L_{(1,0)}], [L_{(0,1)}]$ and the family $\zeta(x) = [L_{(1,1)}, x] - [L_{(0,1)}]$, $x \in \mathbb{R}/\mathbb{Z}$, which shows that the map (8) is surjective. Together with the fact that the composition $H_1(T^2; \mathbb{Z}) \hookrightarrow \mathbb{R}/\mathbb{Z} \oplus H_1(T^2; \mathbb{Z}) \rightarrow \Omega_{\text{Lag}}(T^2)$ is a section of $\eta$, this implies that $\ker\ \eta \subseteq \text{im}\ \zeta$. \hfill \Box

7. Homological mirror symmetry for $T^2$

7.1. **Abouzaid-Smith’s mirror functor.** Abouzaid and Smith [AS10] prove that the split-closed derived Fukaya category $D^\sigma_{\text{F}}(T^2)$ is equivalent to the derived category $D^b(X)$ of the Tate curve $X$, which is an elliptic curve over the Novikov field $\Lambda$ given by a specific Weierstrass equation (see [AS10]). To state the precise result, let $P_0 \in X$ be a base point, denote by $\mathcal{O}(nP_0)$ the line bundle corresponding to the divisor $nP_0$, and by $\mathcal{O}_{P_0}$ the skyscraper sheaf with one-dimensional stalk supported at $P_0$.

**Theorem 7.1.** There exists an equivalence of triangulated categories

$$\Phi : D^b(X) \simeq D^\sigma_{\text{F}}(T^2)$$

which takes $\mathcal{O}(nP_0)$ to $L_{(1,-n)}$ for every $n \in \mathbb{Z}$, and $\mathcal{O}_{P_0}$ to $L_{(0,-1)}$.\frac{1}{2}.

The proof is based on parts of Polishchuk-Zaslow’s computations in [PZ01], who work over an elliptic curve defined over $\mathbb{C}$. To make the connection to the current setting, one studies $X$ by means of its analytification

$$X^\text{an} = \Lambda^*/q^\mathbb{Z},$$

the quotient of $\Lambda^* = \Lambda \setminus \{0\}$ by the discrete subgroup generated by $q \in \Lambda$. $X^\text{an}$ can be given the structure of a rigid-analytic space over $\Lambda$ (we refer the reader to [FvdP04] for general background on rigid-analytic geometry, and to [FvdP04, Shi94] for specific information on the Tate curve). The complex-analytic $\theta$-functions appearing in [PZ01] can be interpreted as formal power series and hence functions on $\Lambda^*$, which give rise to sections of rigid-analytic vector bundles over $X^\text{an}$, analogously to the complex case. A rigid-analytic GAGA principle says that the categories
of coherent algebraic sheaves on $X$ and of coherent analytic sheaves on $X^{an}$ are equivalent, which allows to translate back to algebraic geometry. We will drop the notational distinction between $X$ and $X^{an}$ in the following.

To outline Abouzaid-Smith’s proof of Theorem 7.1, consider on the algebraic side the full subcategory $\Gamma \mathcal{A} \subset D^b(X)$ consisting of the line bundles $O(nP_0)$, $n \in \mathbb{Z}$ (where $D^b(X)$ is a dg-enhancement of $D^b(X)$, see [AS10]); on the symplectic side, consider the full $A_\infty$-subcategory $\Gamma \mathcal{A} \subset F^\#(T^2)$ with objects the Lagrangians $L_{(1,n)}$, $n \in \mathbb{Z}$. Both collections of objects split-generate on their respective sides. Polishchuk-Zaslow’s computations [PZ01] imply that the cohomological categories $H^0(\Gamma \mathcal{A}^\vee)$ and $H^0(\Gamma \mathcal{A})$ are equivalent by a functor taking $O(nP_0)$ to $L(1,-n)$. Then a deformation theoretic result by Polishchuk [Pol03] says that $H^0(\Gamma \mathcal{A}^\vee)$ can be equipped with an essentially unique non-formal $A_\infty$-structure. As the $A_\infty$-structures on $\Gamma \mathcal{A}^\vee$ and $\Gamma \mathcal{A}$ are both non-formal, Abouzaid-Smith conclude the existence of an $A_\infty$-quasi-equivalence between $\Gamma \mathcal{A}^\vee$ and $\Gamma \mathcal{A}$, which extends to an $A_\infty$-quasi-equivalence between the respective split-closures. The triangulated equivalence of Theorem 7.1 is then obtained by taking cohomology.

Remark 7.2. Abouzaid-Smith [AS10] really prove Theorem 7.1 for $\mathcal{F}(T^2)$, the version of the Fukaya category without local systems. However, the proof in [AS10] goes through for both $\mathcal{F}(T^2)$ and $\mathcal{F}^\#(T^2)$, because the $L_{(1,n)}$ split-generate both versions. One can prove this, for example, with the machinery from Section 4 by iteratively surgering the $L_{(1,n)}$ and equipping the resulting cobordisms with suitable local systems. Cf. Remark 6.7 in [AS10], which indicates that the effects of allowing appropriate non-trivial local systems and of taking the split-closure of $D\mathcal{F}(T^2)$ are equivalent.

7.2. Recovery of the mirror functor. The equivalence $\Phi : D^b(X) \to D^\pi \mathcal{F}^\#(T^2)$ of Theorem 7.1 is established using a deformation theoretic argument, and a priori it is not clear how precisely $\Phi$ acts on arbitrary objects. It is not even obvious that the mirror object of every indecomposable sheaf is a Lagrangian with a local system, as opposed to some “abstract” object introduced when passing to the split-closure. In contrast, the equivalence constructed by Polishchuk-Zaslow [PZ01] is given by explicit formulae, but it is not clear how compatible it is with the triangulated structure.

The aim here is to partially recover the effect of $\Phi$ on objects. We denote by $\mathcal{V}(r,d)$ the set of isomorphism classes of indecomposable vector bundles of rank $r$ and degree $d$ on $X$, and we identify $H_1(T^2; \mathbb{Z}) \cong \mathbb{Z}^2$ in the standard way. As usual, we include objects of Coh $X$ in $D^b(X)$ by viewing them as complexes concentrated in degree 0.

Proposition 7.3. (i) Let $\mathcal{S}$ be an indecomposable skyscraper sheaf on $X$ with stalk of rank $h$. Then $\Phi(\mathcal{S})$ is isomorphic to a Lagrangian of slope $(0,-1) \in H_1(T^2; \mathbb{Z})$ equipped with a local system of rank $h$.

(ii) Let $E \in \mathcal{V}(r,d)$ and set $h = \gcd(r,d)$. Then $\Phi(E)$ is isomorphic to a Lagrangian of slope $\frac{1}{h}(r,-d) \in H_1(T^2; \mathbb{Z})$, equipped with an indecomposable local system of rank $h$. 

Every object of $D^b(X)$ is isomorphic to a direct sum of shifted copies of vector bundles and skyscraper sheaves (see, e.g., Corollary 3.15 in [Huy06]). Since moreover the shift functor of $D^\pi \mathcal{F}^1(T^2)$ just shifts the grading of Lagrangian branes, Theorem 7.1 and Proposition 7.3 together imply the following statement.

**Corollary 7.4.** Every object of $D^\pi \mathcal{F}^2(T^2)$ is, up to isomorphism, a direct sum of objects of $\mathcal{F}^1(T^2)$, i.e., of Lagrangian branes with local systems.

**Corollary 7.5.** The inclusion $D\mathcal{F}^1(T^2) \hookrightarrow D^\pi \mathcal{F}^2(T^2)$ is an equivalence.

The proof of Proposition 7.3 will occupy the rest of this section.

7.3. **Proof of Proposition 7.3** (i) for $h = 1$. Every point $Q \in X$ can be written as $Q = [q^x M]$ with $x \in \mathbb{R}/\mathbb{Z}$ and $M \in S^1 \Lambda$ that are uniquely determined. From now on we fix the base point of $X$ to be $P_0 = [-q^{1/2}]$.

This choice determines an equivalence $\Phi : D^b(X) \rightarrow D^\pi \mathcal{F}^1(T^2)$ as described in Theorem 7.1 i.e., such that $\Phi$ takes $O(nP_0)$ to $L_{(1,-n)}$.

We denote by $O = [q^0]$ the neutral element for the natural group structure on $X$ induced by multiplication on $\Lambda^*$. Since $X$ is an elliptic curve, it also carries an “elliptic curve group structure” with of $O$ as neutral element, see [Har77, Section IV.4]. In fact, both group structures must coincide (cf. [FvdP04, p.127]), and we denote the operation in this group by $\oplus$. It is related to the operation in the divisor group by

$$Q \oplus Q' = Q'' \iff Q + Q' \sim Q'' + O,$$

where $\sim$ denotes linear equivalence of divisors (again, see [Har77, Section IV.4]).

To determine the mirror images of the 1-dimensional skyscraper sheaves, we will use that (almost) every such skyscraper sheaf can be obtained as a direct summand of a cone on a morphism $O(-2P_0) \rightarrow O_X$. To see this, note that

$$\text{Hom}_{D^b(X)}(O(-2P_0), O_X) \cong H^0(X; O(2P_0)),$$

and hence the cone of every morphism $O(-2P_0) \rightarrow O_X$ is of the form $O_D$ for a divisor $D$ belonging to the linear system $|2P_0|$. Observe that, by (9), $|2P_0|$ is the set of all divisors $D = Q + Q' = P_0 \oplus P_0$, hence the set of divisors of the form

$$D = [q^x M] + [q^{-x} M^{-1}]$$

with $x \in \mathbb{R}/\mathbb{Z}$ and $M \in S^1 \Lambda$. Whenever the points $[q^x M]$ and $[q^{-x} M^{-1}]$ are distinct, which is the case unless $x \in \{0, \frac{1}{2}\}$ and $M \in \{ \pm q^0 \}$, the sheaf $O_D$ corresponding to such $D$ is a direct sum of the corresponding 1-dimensional skyscraper sheaves, that is,

$$O_D = O_{[q^x M]} \oplus O_{[q^{-x} M^{-1}]}.$$

(And $O_D = O_{2[q^x M]}$ in the four cases in which $[q^x M] = [q^{-x} M^{-1}]$.)

\[2\text{Recall that by slight abuse of notation we write } X \text{ for the analytification } X^{\text{an}} = \Lambda^*/q^\mathbb{Z}.\]
The space $\text{Hom}_{D^b(X)}(\mathcal{O}(-2P_0), \mathcal{O}_X) \cong H^0(X; \mathcal{O}(2P_0))$ has a preferred basis given by the theta functions $\theta^0, \theta^1 : \Lambda^* \to \Lambda$ defined by

$$\theta^0(w) = \sum_{n \in \mathbb{Z}} w^{2n} q^{n^2},$$
$$\theta^1(w) = \sum_{n \in \mathbb{Z}} w^{2n+1} q^{(n+\frac{1}{2})^2}.$$  

To understand this, observe that these functions satisfy the functional equation $\theta^i(qw) = q^{-1}w^{-2}$, and therefore can be considered as sections of the line bundle $\Lambda^* \times \Lambda/((w, \xi) \sim (qw, q^{-1}w^{-2}\xi))$ over $X$, which is $\mathcal{O}(2P_0)$. (In fact, the reason for choosing the base point to be $P_0 = [-q^{1/2}]$ was to ensure that $H^0(X; \mathcal{O}(2P_0))$ is spanned by these “standard” theta functions.) We refer to [PZ01, Section 2.3] or [GH94] for the description of vector bundles on elliptic curves (over $\mathbb{C}$) via “multipliers”, to [FvdP04, Section 4.7] for information on vector bundles over rigid analytic spaces, and to [Mum07] for background on theta functions.

Now we turn to the symplectic side and study the cones on morphisms $L_{(1,2)} \to L_{(1,0)}$ in $D^b(\mathcal{T}(T^2))$ (recall that $L_{(1,2)}$ and $L_{(1,0)}$ are the mirror images of $\mathcal{O}(-2P_0)$ and $\mathcal{O}_X$). The space of morphisms between these Lagrangians is spanned by the two intersection points $c_1^0, c_1^1 \in L_{(1,2)} \cap L_{(1,0)}$, see Figure 14.

**Lemma 7.6.** For every non-zero morphism $c_1 = \sigma^0c_1^0 + \sigma^1c_1^1 : L_{(1,2)} \to L_{(1,0)}$, Cone($c_1$) is isomorphic either to a direct sum

$$(L_{(0,-1),x}, E^1_M) \oplus (L_{(0,-1),-x}, E^1_{M-1})$$

with $x \in \mathbb{R}/\mathbb{Z}, M \in S^1\Lambda$ and where $E^1_M$ is the rank 1 local system with monodromy $M$, or to one of the four objects $(L_{(0,-1),x}, E^2_M)$ with $x \in \{0, \frac{1}{2}\}, M \in \{\pm 1\}$ and where $E^2_M$ is the unique non-trivial extension of $E^1_M$ by itself.

**Proof.** Set $Y_0 := L_{(1,2)}$ and $Y_1 := L_{(1,0)}$. We first rephrase a bit what needs to be proven: Our task is to show that for every $c_1 = \sigma^0c_1^0 + \sigma^1c_1^1 \in \text{hom}^0(Y_0, Y_1)$, there exists an object $Y_2$ as described in the statement of the lemma, together with morphisms $c_2 \in \text{hom}^0(Y_1, Y_2)$ and $c_3 \in \text{hom}^1(Y_2, Y_0)$ such that the triangle

$$Y_0 \xrightarrow{\sigma^0 c_1^0} Y_1 \xrightarrow{\sigma^1 c_1^1} Y_2 \xrightarrow{c_3} Y_0[1]$$

is exact in $H(\mathcal{T}(T^2))$.

We will use Lemma 3.7 in [Sei08] to prove this. According to this lemma, it suffices to show that

$$\mu^2(c_3, c_2) = 0,$$
$$\mu^2(c_1, c_3) = 0,$$
$$\mu^3(c_1, c_3, c_2) = e_{Y_1},$$

where $e_{Y_1}$ is a chain level representative of the identity in $\text{Hom}^0(Y_1, Y_1)$, and to show the acyclicity of the complex $\text{hom}(X, Y_2)[1] \oplus \text{hom}(X, Y_0)[1] \oplus \text{hom}(X, Y_1, \partial)$ for every test object $X$ (see [Sei08] for the description of the differential $\partial$).

That $\Phi$ restricts essentially to the Polishchuck-Zaslow functor [PZ01] on the full subcategory of $D^b(X)$ consisting of the $\mathcal{O}(nP_0)$ tells us that $L_{(1,2)}$, $L_{(1,0)}$ are the
mirror images of $\mathcal{O}(-2P_0)$ and $\mathcal{O}_X$, and that the corresponding isomorphism between morphism spaces is determined by

$$\text{Hom}_{D^b(X)}(\mathcal{O}(-2P_0), \mathcal{O}_X) \ni \theta_i \leftrightarrow c_i \in \text{Hom}_{D^b(T^2)}(L_{(1,2)}, L_{(1,0)}),$$

for $i = 0, 1$.

Let now $c_1 = \sigma_0 c_0^0 + \sigma_1 c_1^0 : L_{(1,2)} \to L_{(1,0)}$ be a non-zero morphism and consider the corresponding morphism $\mathcal{O}(-2P_0) \to \mathcal{O}_X$, i.e., the section $s = \sigma_0 \theta_0 + \sigma_1 \theta_1$ of $\mathcal{O}(2P_0)$.

**Claim.** Suppose that $s$ vanishes at two distinct points $[-q^x M], [-q^{-x} M^{-1}] \in X$, as opposed to having a double zero. Then $\text{Cone}(c_1)$ is isomorphic to

$$Y_2 = (L_{(0,-1)}, x, E_1^0 \oplus (L_{(0,-1)}, x, E_1^0).$$

**Proof.** To prove the claim, we will verify parts of the criterion for exact triangles [Sei08, Lemma 3.7], namely that the middle equation

$$\mu^2(c_1, c_3) = 0$$

in (11) holds whenever $c_3 \propto c_3^0 + c_3^1$. Then we use an additional argument to see that this is already sufficient. Here the $c_3^0$ are the intersection points generating the space $\text{hom}^1(Y_2, Y_0)$, as depicted in Figure 14 (more precisely, they are the identity homomorphisms between the fibres of the respective local systems over these intersection points, which in our model for local systems are all equal to $\Lambda$).

The signs with which the polygons encountered in the verification of (11) contribute will be determined according to the recipe described in [Sei11, Section 7] or [LP11, Section 2] (replacing $\text{Spin}$ by $\text{Pin}$). In particular, we think of the bounding $\text{Pin}$ structures on our Lagrangians as double covers which are trivialized except over one point where the two sheets are interchanged; these points are indicated by $\otimes$ in Figure 14.

Observe first that the vanishing of

$$\mu^2(c_1, c_3) = \mu^2(c_1, c_3^0) + \mu^2(c_1, c_3^1)$$

is equivalent to the individual vanishing of $\mu^2(c_1, c_3^0)$ and $\mu^2(c_1, c_3^1)$, because the first term is a multiple of $c_3^0$, while the second term is a multiple of $c_3^1$; we will only show $\mu^2(c_1, c_3^0) = 0$.

The triangles contributing to $\mu^2(c_1, c_3^0)$ are precisely the images of the family of triangles $\Delta_n, n \in \mathbb{Z}$, in the universal cover $\mathbb{R}^2 \to T^2$ described as follows: $\Delta_n$ has
one vertex at \((x, 2x)\), one at \((x, n)\), and one at \((\frac{1}{2}n, n)\), see Figure 15. The first two points project to \(c_0^0\) and \(c_0^3\), respectively, while the third one projects to \(c_1^0\) or to \(c_1^1\), according to whether \(n\) is even or odd. The area of \(\Delta_n\) is \[A_n = \left(\frac{1}{2}(n - 2x)\right)^2.\]

According to the recipe in [Sei11, Section 7], the sign with which \(\pi(\Delta_n)\) contributes to \(\mu^2(c_1, c_3^0)\) is \((-1)^{s_n+1}\) in the case at hand, where \(s_n\) is the number of lifted \(\otimes\) symbols encountered when travelling around the edges of \(\Lambda_n\). One sees easily that \(s_n\) has the same parity as \(n\), and therefore the sought-for sign is \((-1)^{n+1}\).

The remaining ingredient for the computation of \(\mu^2(c_1, c_3^0)\) is the parallel transport around the edges of \(\pi(\Delta_n)\). Since the local systems on \(Y_0\) and \(Y_1\) are trivial, this is equal to the parallel transport in the local system \(E_1^M\) along the vertical segment of (oriented) length \(2x - n\) (the orientation of \(L(0, -1)\) points downwards); we denote this parallel transport map by \(M^{2x-n}\).

Assembling all these ingredients, we obtain

\[
\mu^2(c_1, c_3^0) = \left(\sigma^0 \sum_{n \in \mathbb{Z}} -M^{2x-2n}q^{(n-x)^2} + \sigma^1 \sum_{n \in \mathbb{Z}} M^{2x-(2n+1)}q^{(n+1/2-x)^2}\right) c_2^0.
\]

Some basic arithmetic shows that the vanishing of this is equivalent to

\[
\sigma^0 \sum_{n \in \mathbb{Z}} (-q^{-x}M^{-1})^{2n}q^{n^2} + \sigma^1 \sum_{n \in \mathbb{Z}} (-q^{-x}M^{-1})^{2n+1}q^{(n+\frac{1}{2})^2} = 0,
\]

which is nothing but \((\sigma^0\theta^0 + \sigma^1\theta^1)(-q^{-x}M^{-1}) = 0\). Similarly, \(\mu(c_1, c_3^1) = 0\) is equivalent to \((\sigma^0\theta^0 + \sigma^1\theta^1)(-q^xM) = 0\). That these equalities hold is the assumption made earlier, which finishes the proof of \(\mu^2(c_1, c_3) = 0\).

We now argue that this is already sufficient to establish our claim. Observe that there is a cobordism

\[V : L(1,0) \leadsto (L(0,1), x, L(0, -1), -x, L(1,2)),\]

obtained from the iterated surgery of \(L(1,2)\) with \(L(0, -1), x\) and \(L(0, -1), -x\), which results in \(L(1,0)\). This cobordism \(V\) can be equipped with a local system restricting to
$E^1_M$ and $E^1_{M-1}$ on the two negative ends corresponding to $L_{(0,-1),x}$ and $L_{(0,-1),-x}$, and to trivial local systems on the other ends. Moreover, it can be equipped with a brane structure restricting to the standard brane structures of these Lagrangians as described in Section 6 (to verify this for the grading, use Lemma 5.3).

The existence of such a cobordism $V$ implies that every object $Y_2$ as described in the claim is the cone of some morphism $\tilde{c}_1 = \tilde{\sigma}^0 c^0_1 + \tilde{\sigma}^1 c^1_1 \in \hom^0(Y_0, Y_1)$ for certain $\tilde{\sigma}_0, \tilde{\sigma}_1 \in \Lambda$. To see this, one can “bend” the negative end of $V$ corresponding to $L_{(1,2)}$ such as to become positive, and then use a slight generalization of Theorem 4.1 for cobordisms with multiple ends on both sides.

It remains to check that $\tilde{c}_1 = c_1$ up to a non-zero factor. To see this, it suffices to verify that $\tilde{s} = \tilde{\sigma}^0 \theta^0 + \tilde{\sigma}^1 \theta^1 \in H^0(X; \mathcal{O}(2P_0))$ vanishes at $q^* M$ and $q^* M^{-1}$, because this implies $\tilde{\sigma}^0 = \sigma^0$ and $\tilde{\sigma}^1 = \sigma^1$ up to a common non-zero factor. In fact, it suffices to show $\tilde{s}$ vanishes at one of these points, because the other zero is then determined.

We use the criterion for exact triangles [Sei08 Lemma 3.7] to show this. It tells us that there is some (non-zero) morphism $c_3 = \eta_1 c^0_3 + \eta_1 c^1_3 \in \hom^1(Y_1, Y_2)$ such that $\mu^2(c_1, c_3) = 0$. This is equivalent to $\eta^i \mu^2(c_1, c^1_3) = 0$ for $i = 0, 1$, as follows from the discussion above. Assuming $\eta^1 \neq 0$, the corresponding equation $\eta^1 \mu^2(c_1, c^1_3) = 0$ is equivalent to $\mu_2^2(c_1, c^1_3) = 0$, which is equivalent to $(\tilde{\sigma}^0 \theta^0 + \tilde{\sigma}^1 \theta^1)(q^* M) = 0$ by the computations further up in the proof.

This finishes the proof of the claim.

To complete the proof of the lemma, we still have to show that the cones of the four remaining morphisms, for which the corresponding sections $s \in H^0(X; \mathcal{O}(2P_0))$ have a double zero, are as claimed. By a similar cobordism argument as in the proof of the claim, one can infer that for every $x \in \{0, \frac{1}{2}\}$ and every $M \in \{\pm 1\}$, there exist a morphism $c_1 \in \hom^0(Y_0, Y_1)$ and an automorphism $\sigma$ of $(L_{(0,-1),x}, E^1_M)$ such that $\text{Cone}(c_1) \cong \text{Cone}(\sigma)$. These morphisms $c_1$ must be the four remaining ones, because we have already found the cones of all others to be different from what we get here. Moreover, their cones must be indecomposable, since the same is true on the algebraic side. But the only indecomposable cone on an automorphism of $(L_{(0,-1),x}, E^1_M)$ is $(L_{(0,-1),x}, E^1_M)$.

**Remark 7.7.** One could also prove Lemma 7.6 more directly by verifying all of the criterion of [Sei08 Lemma 3.7]. The remaining equations in (11) are easy to check for $c_2$ a suitable multiple of $c^0_2 + c^1_2$, and the acyclicity of the complex could be shown by a filtration argument similar to the one appearing in the proof of Theorem 1 of [LP11].

**Proof of Proposition 7.3(i) for $h = 1$.** Recall that $\Phi$ identifies the morphism spaces $\text{Hom}(\mathcal{O}(-2P_0), \mathcal{O}_X) \cong \text{Hom}(L_{(1,2)}, L_{(1,0)})$ via $\theta^i \mapsto c^i_1$, $i = 0, 1$. Moreover, since $\Phi$ is an equivalence of triangulated categories, it takes cones to cones. Using Lemma 7.6 and the discussion preceding it, we can therefore infer that

$$\Phi(\mathcal{O}_{[-q^* M]} \oplus \mathcal{O}_{[-q^* M^{-1}]} \cong (L_{(0,-1),x}, E^1_M) \oplus (L_{(0,-1),-x}, E^1_{M-1}$$

whenever $x \notin \{0, \frac{1}{2}\}$ and $M \notin \{\pm 1\}$. Since there are no other ways of writing the object on the right-hand side as direct sums, we conclude that the mirror images of $\mathcal{O}_{[-q^* M]}$ and $\mathcal{O}_{[-q^* M^{-1}]}$ are $(L_{(0,-1),x}, E^1_M)$ and $(L_{(0,-1),-x}, E^1_{M-1})$. The
corresponding statements for the four skyscrapers \( \mathcal{O}_{[q^r,M]} \) with \( x \in \{0, \frac{1}{2}\} \) and \( M \in \{\pm 1\} \) follow easily, using that we also know \( \Phi(\mathcal{O}(nP_0)) = L_{(1,-n)} \).

\[ \square \]

7.4. **Proof of Proposition 7.3** Proposition 7.3 (i) for \( h = 1 \) has already been proven. We break the rest of the proof up into several steps. The common strategy is to exhibit for a given sheaf \( \mathcal{F} \) a cone decomposition in \( D^b(X) \) with linearization consisting of objects for which we already know Proposition 7.3 holds, and with the property that \( \mathcal{F} \) is determined by the cone decomposition. We then construct, using iterated surgery, an object of \( D^b,\mathcal{F}^2(T^2) \) which has a cone decomposition mirror to the one of \( \mathcal{F} \) and therefore must be mirror to \( \mathcal{F} \).

**Step 1.** Proposition 7.3 (ii) holds for all line bundles \( \mathcal{E} \in \mathcal{V}(1,d) \), \( d \in \mathbb{Z} \).

**Proof.** We can write \( \mathcal{E} = \mathcal{O}((d + 1)P_0 - Q) \) for some \( Q \in X \), and there is an exact triangle

\[ \mathcal{O}_Q[-1] \to \mathcal{E} \to \mathcal{O}((d + 1)P_0) \to \mathcal{O}_Q \]

in \( D^b(X) \). Since \( \hom_{D^b(X)}(\mathcal{O}((d + 1)P_0), \mathcal{O}_Q) \) is one-dimensional, \( \mathcal{E} \) is in fact the only indecomposable object fitting into an exact triangle with the other two middle fixed. To find its mirror object, it suffices therefore to exhibit an exact triangle in \( D^\pi,\mathcal{F}^2(T^2) \) in which two rightmost objects are mirror to \( \mathcal{O}((d + 1)P_0) \) and \( \mathcal{O}_Q \).

We know already that these mirror objects are \( L_{(1,-(d+1))} \) and \( (L_{(0,-1)},x, E_M^1) \) for certain \( x \in \mathbb{R}/\mathbb{Z} \) and \( M \in S^1 \Lambda \). Now there exists some \( x' \in \mathbb{R}/\mathbb{Z} \) such that surgering \( L_{(0,-1),x} \) and \( L_{(1,-d),x'} \) produces \( L_{(1,-(d+1))} \), up to Hamiltonian isotopy. After equipping the corresponding cobordism \( V : L_{(1,-(d+1))} \sim (L_{(0,-1),x}, L_{(1,-d),x'}) \) with an appropriate brane structure and a local system, Theorem 4.1 yields an exact triangle

\[ (L_{(0,-1),x}, E_M^1)[−1] \to (L_{(1,-d),x'}, E_{M−1}^1) \to (L_{(1,-(d+1))}) \to (L_{(0,-1),x}, E_M^1) \]

We conclude that \( \Phi(\mathcal{E}) = (L_{(1,-d),x'}, E_{M−1}^1) \) up to isomorphism, as required.

\[ \square \]

**Step 2.** Proposition 7.3 holds for every \( \mathcal{E} \in \mathcal{V}(r,d) \) whenever \( \gcd(r,d) = 1 \).

**Proof.** Theorem C.1 implies that there exists an integer \( n \) such that \( \mathcal{E} \) fits into an exact triangle

\[ \mathcal{L}[-1] \to (\mathcal{L}')^\otimes r−1 \to \mathcal{E} \to \mathcal{L} \]

with \( \mathcal{L} = (\det \mathcal{E})((r - 1)n) \) and \( \mathcal{L}' = \mathcal{O}_X(-n) \), where \(-k\) denotes tensoring by the \( k\)th power of the hyperplane bundle. Hence \( \mathcal{E} \) admits an iterated cone decomposition with linearization \( (\mathcal{L}, \mathcal{L}', \ldots, \mathcal{L}') \), where \( \mathcal{L}' \) appears \( r - 1 \) times (cf. Lemma A.2).

We claim that \( \mathcal{E} \) is actually the only indecomposable object of \( D^b(X) \) admitting a cone decomposition with this linearization. Suppose that \( \tilde{\mathcal{E}} \) is another indecomposable object with the same property. It follows that \( [\tilde{\mathcal{E}}] = [\mathcal{E}] \in K_0(X) \) by Lemma A.3 and hence \( \det \tilde{\mathcal{E}} = \det \mathcal{E} \) and \( \rk \tilde{\mathcal{E}} = \rk \mathcal{E} \), because \( (\det, \rk) : K_0(X) \to \pic(X) \oplus \mathbb{Z} \) is an isomorphism (see [Har 17]). Since \( \mathcal{E} \) is indecomposable, we conclude that \( \tilde{\mathcal{E}} \in \mathcal{V}(r,d) \). By Theorem C.3, the condition \( \gcd(r,d) = 1 \) implies that \( \det : \mathcal{V}(r,d) \to \mathcal{V}(1,d) \subset \pic(X) \) is bijective, and hence \( \mathcal{E} = \tilde{\mathcal{E}} \).

To find \( \Phi(\mathcal{E}) \), it suffices therefore to construct an object of \( D^\pi,\mathcal{F}^2(T^2) \) admitting a cone decomposition with linearization mirror to \( (\mathcal{L}, \mathcal{L}', \ldots, \mathcal{L}') \). The line bundles

\[ ^3 \text{Here and in the following, we should often add "up to isomorphism" to be really precise.} \]
there have degrees \( \deg \mathcal{L} = d + 3n(r - 1) \) and \( \deg \mathcal{L}' = -3n \), since the hyperplane bundle has degree 3. We conclude, using the previous step, that their mirror objects are of the form
\[
\Phi(\mathcal{L}) = (L, E) \quad \text{and} \quad \Phi(\mathcal{L}') = (L', E'),
\]
with Lagrangians \( L \) and \( L' \) of slopes \((1, -d - 3n(r - 1))\) and \((1, 3n)\) that are equipped with rank one local systems \( E \) and \( E' \). Starting with \( L \) and iteratively surgering \( r - 1 \) times with \( L' \) leads to a sequence of Lagrangians \( N_0 = L, N_1, \ldots, N_{r - 1} \) of slopes
\[
[N_j] = (1 + j, -d - 3n(r - 1 - j)) \in H_1(T^2; \mathbb{Z}), \quad j = 0, \ldots, r - 1.
\]
(Note that \( N_j \) might have multiple components, and a priori these might have slopes equal to that of \( L' \), which would be problematic. But the only way this can happen is that \( N_j \) has \( 1 + j \) components of slope \((1, -d - 3n(r - 1 - j))/(1 + j)\); to exclude this being equal to \((1, 3n) = [L']\), we assume that \( n \) has been chosen such that \(-d < 3nr\). This is no restriction, since any sufficiently big \( n \) leads to an exact triangle as above.)

To this sequence of surgeries corresponds a sequence of cobordisms
\[
V_j : N_j \rightsquigarrow (L, L', \ldots, L'), \quad j = 0, \ldots, r - 1.
\]
These can be equipped with brane structures and rank one local systems that extend those on the negative ends and determine a brane structure and a rank one local system \( E_j \) on the positive end. Theorem 4.1 applied to \( V_{r - 1} \) says that there is an iterated cone decomposition of \((N_{r - 1}, E_{r - 1})\) with linearization
\[
((L, E), (L', E'), \ldots, (L', E')).
\]
This is mirror to the one for \( \mathcal{E} \), and we infer \( \Phi(\mathcal{E}) \cong (N_{r - 1}, E_{r - 1}) \). That proves the claimed statement, since \( N_{r - 1} \) has slope \((r, -d)\).

**Step 3.** Proposition 7.3 holds for all \( \mathcal{E} \in \mathcal{V}(hr, hd) \) with \( \gcd(r, d) = 1 \) and \( h \geq 1 \), and for all indecomposable skyscraper sheaves with stalks of rank \( h \geq 1 \).

**Proof.** By Lemma C.5, every indecomposable vector bundle \( \mathcal{E} \) on \( X \) is of the form
\[
E_{\mathcal{E}_i}(hr, hd) \otimes \mathcal{L}_0 \quad \text{for some } (r, d) \text{ with } \gcd(r, d) = 1, h \in \mathbb{N}, \quad \text{and certain line bundles } \mathcal{L}_0, \mathcal{L}_1 \text{ of degrees zero and one (see Appendix C for the notation). Moreover, for every } h \geq 1 \text{ and every } Q \in X, \text{ there is a unique indecomposable skyscraper sheaf } \mathcal{O}_{hQ} \text{ with stalk of rank } h \text{ supported at } Q, \text{ and every indecomposable skyscraper sheaf is of this form.}
\]

Denote by \( \mathcal{Y}_h \) either the vector bundle \( E_{\mathcal{E}_i}(hr, hd) \otimes \mathcal{L}_0 \) for fixed \( r, d, \mathcal{L}_0 \) and \( \mathcal{L}_1 \), or the skyscraper sheaf \( \mathcal{O}_{hQ} \) for fixed \( Q \). We will prove that for any such choice, the mirror objects of the \( \mathcal{Y}_h \), \( h \geq 1 \), are of the form \((L, E^h)\), with \( L \) a Lagrangian of slope \((r, d)\), or \((0, -1)\), and \( E^h \) an indecomposable local system of rank \( h \) over \( L \) such that there exists a short exact sequence
\[
0 \rightarrow E^1 \rightarrow E^{h+1} \rightarrow E^h \rightarrow 0
\]
of local systems on \( L \) (which in this case is equivalent to saying that the unique eigenvalue of the monodromy is the same for all of them). The proof will be by induction on \( h \). The claim for \( h = 1 \) is what was proven in the previous steps.
In both of the above cases and for every \( h \in \mathbb{N} \), there is an exact triangle
\[
\mathcal{Y}_h[-1] \to \mathcal{Y}_1 \to \mathcal{Y}_{h+1} \to \mathcal{Y}_h
\]
in \( D^b(X) \). For \( \mathcal{Y}_h = E_{L_1}(hr, hd) \otimes \mathcal{L}_0 \), this comes from the short exact sequence obtained by tensoring the short exact sequence in Theorem (C.2.1) with \( E_{L_1}(hr, hd) \otimes \mathcal{L}_0 \) (also cf. (C.4). For \( \mathcal{Y}_h = \mathcal{O}_{hQ} \), it comes from the short exact sequence \( 0 \to \mathcal{O}_Q \to \mathcal{O}_{(h+1)Q} \to \mathcal{O}_{hQ} \to 0 \). Moreover, we have
\[
\dim \text{Ext}^1(\mathcal{Y}_h, \mathcal{Y}_1) = 1
\]
in both cases (to see this in the first case, use Serre duality and (Ati57, Lemma 22).) Hence \( \mathcal{Y}_{h+1} \) is the only indecomposable object of \( D^b(X) \) that can arise as a cone on a morphism \( \mathcal{Y}_h[-1] \to \mathcal{Y}_1 \).

By inductive assumption, the mirror images of \( \mathcal{Y}_1 \) and \( \mathcal{Y}_h \) share the same Lagrangian brane that’s equipped with indecomposable local systems \( E^1, E^h \) of ranks 1 and \( h \), and whose monodromies have the same eigenvalue (because there exists a morphism of local systems \( E^1 \to E^h \)). There’s a unique indecomposable local system \( E^{h+1} \) of rank \( h + 1 \) fitting into a short exact sequence \( 0 \to E^1 \to E^{h+1} \to E^h \to 0 \) of local systems on \( L \) (namely, the unique indecomposable local system whose monodromy has this eigenvalue). Hence there exists an exact triangle
\[
(L, E^h)[-1] \to (L, E^1) \to (L, E^{h+1}) \to (L, E^h)
\]
in \( D\mathcal{F}^+(T^2) \) by Proposition (B.1). We conclude that \( \Phi(\mathcal{Y}_{h+1}) = (L, E^{h+1}) \) by the same argument as in the previous steps. This is as required. \( \square \)

This ends the proof of Proposition 7.3. \( \square \)

8. Proofs of the main theorems

We will first prove Theorem 1.3 and then deduce Theorem 1.2.

8.1. Proof of Theorem 1.3. The cobordism group \( \Omega^L_{\text{Lag}}(T^2) \) appearing in Theorem 1.3 is defined as in Section 2.4 with \( \mathcal{L}^+(T^2) = \text{Ob}\mathcal{F}^+(T^2) \), i.e., it has as generators Lagrangian branes with local systems and relations coming from cobordisms with vanishing Maslov class carrying compatible local systems and gradings, as well as additional relations induced by short exact sequences of local systems.

Let \( X \) be the Tate curve mirror to \( T^2 \), and denote by
\[
K_0(X) = K_0(D^b(X))
\]
the Grothendieck group of its derived category of coherent sheaves. The mirror functor \( \Phi : D^b(X) \to D^\pi\mathcal{F}^+(T^2) \) of Theorem 7.1 induces an isomorphism of Grothendieck groups as it is an equivalence of triangulated categories. Since moreover the inclusion \( D\mathcal{F}^+(T^2) \hookrightarrow D^\pi\mathcal{F}^+(T^2) \) is an equivalence by Corollary 7.5 and hence also induces an isomorphism of Grothendieck groups, we obtain an isomorphism
\[
K_0(D\mathcal{F}^+(T^2)) \xrightarrow{\cong} K_0(X).
\]

Denote now by \( \text{Coh} X \subset D^b X \) the full subcategory consisting of direct sums of shifted sheaves; as mentioned before, \( \text{Coh} X \) is equivalent to \( D^b X \). Consider now the map
\[
F_\Phi : \langle D^b(X) \rangle \to \langle \mathcal{L}^+(T^2) \rangle
\]
induced by \( \Phi \), in the sense that it first replaces an arbitrary object of \( D^bX \) by an isomorphic object of \( \text{Coh} X \), and then applies the mirror functor to get a sum of Lagrangian branes with local systems. We will prove that \( F_\# \) descends to a well-defined group homomorphism \( K_0(X) \to \Omega^2_{\text{Lag}}(T^2) \), thus completing the diagram

\[
\begin{array}{ccc}
\Omega^2_{\text{Lag}}(T^2) & \xrightarrow{\Theta^\#} & K_0(D,\mathcal{F}^+)(T^2) \\
\downarrow & & \downarrow \cong \\
K_0(X) & & \\
\end{array}
\]

(12)

where \( \Theta^\# : \Omega^2_{\text{Lag}}(T^2) \to K_0(D,\mathcal{F}^+)(T^2) \) is the canonical surjective homomorphism from Proposition [4.9]. Once well-definedness is proven, it is clear by construction that the composition of the two lower homomorphisms in (12) provides a left-inverse for \( \Theta^\# \), showing in particular that \( \Theta^\# \) is injective. Given that we already have surjectivity, this will conclude the proof of Theorem [1.3].

8.1.1. Well-definedness. Denote by \( R \) the set of \( K_0 \)-relations among objects of \( \langle \text{Coh} X \rangle \), i.e., the kernel of \( \langle \text{Coh} X \rangle \to K_0(X) \). It follows from Proposition [A.4] that the inclusion \( \text{Coh} X \to D^bX \) induces an isomorphism \( \langle \text{Coh} X \rangle/R \cong K_0(X) \).

To prove that \( F_\# \) induces a map \( K_0(X) \to \Omega^2_{\text{Lag}}(T^2) \), we must therefore show that \( F_\# \) takes \( R \) to \( R^\# \), the kernel of \( \langle \mathcal{F}^+(T^2) \rangle \to \Omega^2_{\text{Lag}}(T^2) \). The essential part is to prove that this is true for the subset \( R_0 = R \cap \langle \text{Ind Coh} X \rangle \), the set of \( K_0 \)-relations among indecomposable objects of \( \text{Coh} X \).

**Proposition 8.1.** \( R_0 \) is generated by relations coming from short exact sequences of the following types:

- \( 0 \to \mathcal{F} \to \mathcal{G} \to 0 \) with indecomposable \( \mathcal{F}, \mathcal{G} \in \text{Coh} X \);
- \( 0 \to \mathcal{O}(D - Q) \to \mathcal{O}(D) \to \mathcal{O}_Q \to 0 \) with \( D \) a divisor and \( Q \in X \);
- \( 0 \to (\mathcal{O}(D - 1)(-n) \to \mathcal{E} \to (\det \mathcal{E})(r-1)n \to 0 \) with \( \mathcal{E} \in \mathcal{V}(r,d) \) such that \( \gcd(r,d) = 1 \), and \( n \in \mathbb{Z} \) such that \( \mathcal{E}(n) \) is generated by global sections;
- \( 0 \to \mathcal{Y}_1 \to \mathcal{Y}_{h+1} \to \mathcal{Y}_h \to 0 \) for \( h \geq 1 \).

In the last line, \( \mathcal{Y}_h \) denotes either a vector bundle of the form \( E_{\mathcal{L}_0}(rh,dh) \otimes \mathcal{L}_1 \) for some fixed \( r,d \) with \( \gcd(r,d) = 1 \) and fixed line bundles \( \mathcal{L}_0, \mathcal{L}_1 \) of degree 0 resp. 1, or an indecomposable skyscraper sheaf \( \mathcal{O}_{hq} \) for some \( q \in X \). (The SESs of the second type are the obvious ones; those of the third type are as in Theorem [C.1]; and those of the fourth type are as mentioned in Step 3 of the proof of Proposition [7.3].)

**Proof.** Denote by \( S \subset \langle \text{Ind Coh} X \rangle \) the subgroup generated by the relations induced by the short exact sequences of the types stated. Moreover, denote by \( \lbrack \mathcal{F} \rbrack \) the class of \( \mathcal{F} \in \text{Ind Coh} X \) in \( \langle \text{Ind Coh} X \rangle/S \). We will show that

\[
\lbrack \mathcal{F} \rbrack = \lbrack \det \mathcal{F} \rbrack + (\mathrm{rk} \mathcal{F} - 1)\lbrack \mathcal{O}_X \rbrack.
\]

(13)

From this it follows immediately that the canonical map \( \langle \text{Ind Coh} X \rangle/S \to K_0(X) \), which is easily seen to be surjective using Proposition [A.4], is also injective: Namely, since \( \det, \mathrm{rk} : K_0(X) \to \text{Pic}(X) \oplus \mathbb{Z} \) is an isomorphism, the equality \( \lbrack \mathcal{F} \rbrack = \lbrack \mathcal{G} \rbrack \) in \( K_0(X) \) is equivalent to \( \det \mathcal{F} = \det \mathcal{G} \) and \( \mathrm{rk} \mathcal{F} = \mathrm{rk} \mathcal{G} \), from which \( \lbrack \mathcal{F} \rbrack = \lbrack \mathcal{G} \rbrack \) follows by (13). This implies that the described relations generate all of \( R_0 \).
We first claim that every sheaf can be written as a sum or difference of line bundles in \((\text{Ind Coh} \ X)/S\). Now an indecomposable sheaf on \(X\) is either a vector bundle of the form \(E_{\mathcal{L}_0}(r_h, dh) \otimes \mathcal{L}_1\) for certain \(r, d\) with \(\gcd(r, d) = 1\) and \(h \geq 1\), or a skyscraper sheaf \(\mathcal{O}_{hq}\) for some \(q \in X\) and \(h \geq 1\). Using the relations of the last type, we can inductively reduce to considering the case \(h = 1\), i.e., the case of vector bundles \(\mathcal{E} \in \mathcal{V}(r, d)\) with \(\gcd(r, d) = 1\) or of skyscraper sheaves \(\mathcal{O}_{L}\). Now the relations of second and third types show that these satisfy
\[
[\mathcal{E}] = (r - 1)[\mathcal{O}_X(-n)] + [\det \mathcal{E}(r - 1)n], \quad [\mathcal{O}_L] = [\mathcal{O}(D)] - [\mathcal{O}(D - Q)],
\]
and since the classes on the right-hand sides are those of lines bundles, our claim is shown.

We now prove that
\[
[\mathcal{L}] + [\mathcal{L}'] = [\mathcal{L} \otimes \mathcal{L}'] + [\mathcal{O}_X], \quad [\mathcal{L}] - [\mathcal{L}'] = [\mathcal{L} \otimes \mathcal{L}'^{-1}]
\]
(14)
for all lines bundles \(\mathcal{L}\) and \(\mathcal{L}'\). Observe that every line bundle is of the form \(\mathcal{O}(D)\) for a divisor \(D = \sum_{i=1}^n P_i - \sum_{j=1}^m Q_j\). Using the relations of the second type in the statement of the lemma, one obtains inductively that
\[
[\mathcal{O}(D)] = [\mathcal{O}_X] + \sum_{i=1}^n [\mathcal{O}_{P_i}] - \sum_{j=1}^m [\mathcal{O}_{Q_j}].
\]
The identities in (14) follow from this because \(\mathcal{O}(D) \otimes \mathcal{O}(D') = \mathcal{O}(D + D')\) and \(\mathcal{O}(D) \otimes \mathcal{O}(D')^{-1} = \mathcal{O}(D - D')\).

Equation (13) now follows easily for all \(\mathcal{F} \in \text{Ind Coh} \ X\). As we have seen, we can write \([\mathcal{F}] = \sum_{i=1}^k s_i[\mathcal{L}_i]\) with lines bundles \(\mathcal{L}_i\) and \(s_i \in \{\pm 1\}\), and then \(\det \mathcal{F} = \bigotimes_{i=1}^k \mathcal{L}_i^{s_i}\) and \(\text{rk} \mathcal{F} = \sum_{i=1}^k s_i\). Applying (14) inductively yields (13).

Generators of \(R_0\) of the first type listed in Proposition 8.1 identify isomorphic indecomposable sheaves; their mirror images are Hamiltonian isotopic curves with isomorphic local systems, and we know that there exists a cobordism between these. Cobordisms corresponding to the generators of \(R_0\) of the other types were constructed in the proof of Proposition 7.3. This implies that \(F_\gamma\) takes \(R_0\) to \(R^\delta\).

The remaining relations in \(R\) come from taking direct sums and shifting; that is, we can write \(R = R_0 + R_1\) where \(R_1\) is generated by elements of the form \(\mathcal{F} \oplus \mathcal{G} - (\mathcal{F} + \mathcal{G})\) and \(\mathcal{F}[1] + \mathcal{F}\). That \(F_\gamma\) takes these generators and hence all of \(R_1\) to \(R^\delta\) is clear by definition for those of type 'direct sum', and follows for those of type 'shift' from the fact that \((L[1], L)\) is null-cobordant for every Lagrangian brane \(L\): A null-cobordism is given by \(V = \gamma \times L\) equipped with a suitable grading, where \(\gamma \subset \mathbb{R}^2\) is a curve with two negative ends. 

\[\Box\]

8.2. Proof of Theorem 1.2 Consider the diagram
\[
\begin{array}{ccc}
\Omega_{\text{Lag}}(T^2) & \xrightarrow{\Theta} & \Omega_{\text{Lag}}^2(T^2) \\
\downarrow \Theta & & \downarrow \Theta^t \\
K_0(D\mathcal{F}(T^2)) & \xrightarrow{} & K_0(D\mathcal{F}^2(T^2))
\end{array}
\]
in which the upper arrow is the canonical map induced by \(\mathcal{L}(T^2) \hookrightarrow \mathcal{L}^2(T^2)\), and where the lower arrow is the map induced by the inclusion \(D\mathcal{F}(T^2) \hookrightarrow D\mathcal{F}^2(T^2)\).
The square commutes because both compositions take the class of any given Lagrangian $L$ in $\Omega_{\text{Lag}}(T^2)$ to the class of $L$ in $K_0(D^b(\Omega^4(T^2)))$. Note that the upper horizontal map is injective, as it is a section of the group homomorphism $\Omega^2_{\text{Lag}}(T^2) \to \Omega_{\text{Lag}}(T^2)$ induced by $(L,E) \mapsto \text{rk}(E)L$ (that this is well-defined is immediate from the definition of the relations in both groups, see Section 2). Since $\Theta^d$ is already known to be an isomorphism, this implies that $\Theta$ is an isomorphism and hence concludes the proof of Theorem 1.2.

□

APPENDIX A. ITERATED CONE DECOMPOSITIONS AND $K_0$

A.1. Triangulated categories. A triangulated category $\mathcal{D}$ is an additive category equipped with an additive autoequivalence $S : \mathcal{D} \to \mathcal{D}$ called the shift functor, and a set of exact triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} S(X)$. These data are required to satisfy a list of axioms, for which we refer to [Wei94]. The most relevant for us is that every morphism $f : X \to Y$ in $\mathcal{D}$ can be completed to such an exact triangle. The object $Z$ is then determined up to isomorphism, and we call it a cone on the morphism $f$. Moreover, we write $X[n] = S^nX$ and $f[n] = S^n(f)$ for the effect of iterates of the shift functor $S$ on objects and morphisms. This is in reminiscence of the homotopy category of complexes $K(\mathcal{A})$ over an Abelian category $\mathcal{A}$, which is the prototypical example of a triangulated category.

A.2. Generation and iterated cone decompositions. Given a full subcategory of a triangulated category $\mathcal{D}$ with objects a collection $\{X_i \mid i \in I\}$, one can consider the full subcategory consisting of all objects that are cones on morphisms between the $X_i$. Iterating this construction, i.e., including in each step all cones on morphisms between the previously constructed objects, one ends up with the subcategory of $\mathcal{D}$ generated by the $X_i$.

In the other direction, one can ask if and how an object $X$ of $\mathcal{D}$ can be constructed as an iterated cone on morphisms between other objects. The following notion is useful to formalize this.

Definition A.1. Let $\mathcal{D}$ be a triangulated category and let $X \in \mathcal{D}$. An iterated cone decomposition of $X$ is a sequence of exact triangles $C_{i-1}[−1] \to X_i \to C_i \to C_{i−1}, \quad i = 1, \ldots, k,$ with objects $C_0, C_1, \ldots, C_k \in \mathcal{D}$ that satisfy $C_0 = 0$ and $C_k = X$. The tuple $(X_1, \ldots, X_k)$ is called the linearization of the cone decomposition.

Remark A.1. This definition is adapted to the cohomological conventions we use and therefore differs from the one in [BC13b Section 2.6], where homological conventions are used.

Iterated cone decompositions can themselves be iterated and are well-behaved with respect to that, as the following lemma shows (cf. the composition in Biran–Cornea’s category $T^S\mathcal{D}$ of iterated cone decompositions [BC13b Section 2.6]).

Lemma A.2. Suppose that $X \in \mathcal{D}$ admits an iterated cone decomposition with linearization $(X_1, \ldots, X_k)$, and that one of the $X_h$, $1 \leq h \leq k$, admits an iterated
cone decomposition with linearization \((X_h^1, \ldots, X_h^k)\). Then \(X\) admits an iterated cone decomposition with linearization

\[
(X_1, \ldots, X_{h-1}, X_h^1, \ldots, X_h^k, X_{h+1}, \ldots, X_k).
\]

Proof. By assumption, there exist exact triangles

\[
\Delta_i = (C_{i-1}[-1] \to X_i \to C_i \to C_{i-1}), \quad i = 1, \ldots, k,
\]

with \(C_0 = 0\) and \(C_k = X\), as well as exact triangles

\[
\Delta'_i = (Z_{i-1}[-1] \to X_h^i \to Z_i \to Z_{i-1}), \quad i = 1, \ldots, \ell,
\]

with \(Z_0 = 0\) and \(Z_\ell = X_h\). For every \(i = 1, \ldots, \ell\), we define \(f_i : C_{h-1}[-1] \to Z_i\) to be the composition \(C_{h-1}[-1] \to X_h = Z_\ell \to Z_i\), where the first arrow is the left arrow in \(\Delta_h\), and the second one is the composition of the \(Z_j \to Z_{j-1}\) in \(\Delta'_j\) for \(j = i + 1, \ldots, \ell\). Then we define \(C_{h,i}\) to be a cone on \(f_i\), i.e., by the requirement that it fit into an exact triangle \(C_{h-1}[-1] \to Z_i \to C_{h,i} \to C_{h-1}\). Note that we can take \(C_{h,0} = C_{h-1}\) and \(C_{h,\ell} = C_h\), in particular. By [Wei94, Exercise 10.2.6] there exists, for every \(i = 1, \ldots, \ell\), a diagram

\[
\begin{array}{cccc}
C_{h-1}[-1] & \longrightarrow & Z_i & \longrightarrow & C_{h,i} \longrightarrow & C_{h-1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_{h-1}[-1] & \longrightarrow & Z_{i-1} & \longrightarrow & C_{h,i-1} \longrightarrow & C_{h-1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & X_h^i[1] & \longrightarrow & X_h^{i}[1] \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_{h-1} & \longrightarrow & Z_i[1] & \longrightarrow & C_{h,i}[1] \longrightarrow & C_{h-1}[1] \\
\end{array}
\]

in which all rows and columns are exact triangles. By rotating and shifting down the third columns we thus obtain exact triangles

\[
\Delta^i_h = (C_{h,i-1}[-1] \to X_h^i \to C_{h,i} \to C_{h,i-1}), \quad i = 1, \ldots, \ell.
\]

The list of triangles \(\Delta_1, \ldots, \Delta_{h-1}, \Delta_h^1, \ldots, \Delta_h^i, \Delta_{h+1}, \ldots, \Delta_k\) then constitutes an iterated cone decomposition of \(X\) with the required linearization. \(\square\)

A.3. Grothendieck groups. Let \(\mathcal{D}\) be a triangulated category. The Grothendieck group \(K_0(\mathcal{D})\) is defined as the quotient \(K_0(\mathcal{D}) = (\text{Ob } \mathcal{D})/R\) of the free Abelian group generated by the objects of \(\mathcal{D}\) by the subgroup \(R\) generated by all expression \(X - Y + Z \in (\text{Ob } \mathcal{D})\) such that there exists an exact triangle \(X \to Y \to Z \to X[1]\). The following lemma is straightforward.

Lemma A.3. Suppose that \(X \in \mathcal{D}\) admits an iterated cone decomposition with linearization \((X_1, \ldots, X_k)\). Then \([X] = [X_1] + \cdots + [X_k]\) in \(K_0(\mathcal{D})\).

One can also define the Grothendieck group \(K_0(\mathcal{A})\) of an Abelian category \(\mathcal{A}\), by starting with the free Abelian group on objects and imposing a relation \([A] + [C] = [B]\) for every short exact sequence \(0 \to A \to B \to C \to 0\). Recall the canonical inclusion \(\mathcal{A} \hookrightarrow \mathcal{D}^b(\mathcal{A})\), which on objects is given by viewing \(X \in \text{Ob } \mathcal{A}\) as a complex concentrated in degree zero. The following statement is well-known.
Proposition A.4. The canonical inclusion \( \mathcal{A} \hookrightarrow D^b(\mathcal{A}) \) induces an isomorphism \( K_0(\mathcal{A}) \cong K_0(D^b(\mathcal{A})) \).

Appendix B. Exact triangles from SESs of local systems

Consider a symplectic manifold \((M, \omega)\) for which we can define the Fukaya category \( \mathcal{F}^\sharp(M) \) with gradings and signs as outlined in Section 3 such that the objects are Lagrangian branes with local systems. There appears to be no reference in the literature for the following statement.

Proposition B.1. Let \( L \) be a Lagrangian brane and let \[ 0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0 \]
be a short exact sequence of local systems on \( L \). Then there exists an exact triangle \((L, E'')[-1] \rightarrow (L, E') \rightarrow (L, E) \rightarrow (L, E'')\) in \( D,\mathcal{F}^\sharp(M) \).

We view local systems as assignments of vector spaces and parallel transport maps as described in Section 3.1.3. By a short exact sequence of local systems \[ 0 \rightarrow E'_x \rightarrow E_x \rightarrow E''_x \rightarrow 0 \]
for every \( x \in L \) such that the \( i_x \) and \( p_x \) define morphisms of local systems, i.e., such that they commute with parallel transport maps. For the following proof, we choose a splitting
\[ 0 \rightarrow E'_x \xrightarrow{i_x} E_x \xrightarrow{p_x} E''_x \rightarrow 0 \]
for every \( x \in L \), that is, maps \( q_x : E_x \rightarrow E'_x \) and \( j_x : E''_x \rightarrow E_x \) such that \( q_x \circ i_x = \text{id}_{E'_x} \), \( p_x \circ j_x = \text{id}_{E''_x} \) and \( i_x \circ q_x + j_x \circ p_x = \text{id}_{E_x} \). Note that the \( j_x \) and \( q_x \) do generally not define morphisms of local systems, unless the short exact sequence splits globally.

In the proof of Proposition B.1 it will be convenient to model the relevant morphism spaces in \( \mathcal{F}^\sharp(M) \) as spaces of Morse cochains with coefficients in local systems. This is possible under certain conditions on \( M \) and \( L \), the fundamental example being that \( M \) is a cotangent bundle and \( L \) is exact (from which the case of interest in this paper follows immediately by a covering argument).

We will adapt a construction used in [Abo11] (which goes back to [FO97]) and consider an \( A_\infty \)-category \( \mathcal{M}(L) \) defined as follows: The objects of \( \mathcal{M}(L) \) are all those objects of \( \mathcal{F}^\sharp(M) \) whose underlying Lagrangian is \( L \). For the morphism spaces, we fix a Morse function \( f : L \rightarrow \mathbb{R} \) and define
\[ \text{hom}_\mathcal{M}(L)(E_0, E_1) = \bigoplus_{x \in \text{Crit} f \atop |x| = i} \text{Hom}(E_{0,x}, E_{1,x}) , \]
where \(| \cdot |\) denotes the Morse index; here we denote objects of \( \mathcal{M}(L) \) simply by their local system. The \( A_\infty \)-operations \( \mu^\lambda_\mathcal{M}(L) \) are defined by considering rigid perturbed gradient flow trees with vertices at critical points, and summing up parallel transport maps in the relevant local systems along edges of these trees. See [Abo11] for the description of the relevant moduli spaces.
By adapting the arguments in [Abo11], one can show (under certain conditions, as indicated above) that there is an \( A_{\infty} \)-quasi-isomorphism

\[
\mathcal{F}(L) \rightarrow \mathcal{M}(L),
\]

where \( \mathcal{F}(L) \) denotes the full \( A_{\infty} \)-subcategory of \( \mathcal{F}(M) \) consisting of objects whose underlying Lagrangian is \( L \). It will therefore suffice to prove that there exists an exact triangle of the claimed form in \( H^0(Tw.\mathcal{M}(L)) \), where \( Tw.\mathcal{M}(L) \) denotes the category of twisted complexes over \( \mathcal{M}(L) \) which we use to model the triangulated closure of that category (see [Sei08 Section (3i)]).

**Proof of Proposition B.1.** We assume that the Morse function \( f : L \rightarrow \mathbb{R} \) defining the morphism spaces in \( \mathcal{M}(L) \) has a single local minimum \( x_0 \in L \). As for notation, we write \( \pi_\gamma, \pi_x, \pi_\gamma^0 \) for the parallel transport in \( E', E, E'' \) along a path \( \gamma \), and we denote by \( \tau \) the path obtained by reversing \( \gamma \).

Let \( c_2 \in \text{hom}^0(E', E) = \text{Hom}(E'_{x_0}, E_{x_0}) \) and \( c_3 \in \text{hom}^0(\mathcal{M}(L)) \) be the morphisms in \( \mathcal{M}(L) \) determined by the short exact sequence, that is, \( c_2 = i_{x_0} \) and \( c_3 = p_{x_0} \). Then define \( c_1 \in \text{hom}^0(E''_{x_0}, E') = \bigoplus_y \text{Hom}(E''_{y_0}, E'_{y}) \) by

\[
c_1 = \bigoplus_y \sum_{\gamma} \pi_\gamma^0 \circ q_{x_0} \circ \tau \circ j_y,
\]

where the first sum runs over all critical points \( y \) of \( f \) with Morse index 1, and the second over all gradient flow lines of \( f \) from \( x_0 \) to \( y \), and where the sign \( \pm \) is associated to \( \gamma \) as indicated above. In fact, we view \( c_1 \) as living in \( \text{hom}^0(E'', E') \) and \( c_3 \) as living in \( \text{hom}^0(E, E''[-1]) \).

We claim that these morphisms fit into an exact triangle

\[
E''[-1] \xrightarrow{c_1} E' \xrightarrow{c_2} E \xrightarrow{c_3} E''
\]

in \( H^0(Tw.\mathcal{M}(L)) \). In this model, the cone \( C = \text{Cone}(c_1) \) is the twisted complex

\[
C = \left( E'' \oplus E', \delta = \begin{pmatrix} 0 & 0 \\ -c_1 & 0 \end{pmatrix} \right),
\]

see [Sei08 Section 3(p)] (we suppress the shift of \( c_1 \)). It comes together with morphisms \( p_C = (e_{E''}, 0) \in \text{hom}_{Tw}(C, E') \) and \( i_C = (0, e_{E'})^T \in \text{hom}_{Tw}(E', C) \), where \( e_{E''} = \text{id}_{E''_{x_0}} \) and \( e_{E'} = \text{id}_{E'_{x_0}} \) denote the chain level identity morphisms of \( E'', E' \) in \( \mathcal{M}(L) \).

To prove our claim, we will make use of Lemma 3.27 in [Sei08], which gives a criterion for exactness of triangles in \( H^0(Tw.\mathcal{M}(L)) \). According to that, we have to find a cocycle \( b \in \text{hom}_{Tw}(E, C) \) such that \( [b] \) is an isomorphism and \( [\mu^2_{Tw}(p_C, b)] = [c_3], [\mu^2_{Tw}(b, c_2)] = [i_C] \) in \( H^0(Tw.\mathcal{M}(L)) \). (Again, we suppress some shifts.)

We claim that \( b = (b'', b') \) with \( b'' = p_{x_0} \in \text{hom}^0(E, E'') \), \( b' = q_{x_0} \in \text{hom}^0(E, E') \) satisfies these requirements. The necessary computations are straightforward. We start by verifying that \( b \) is a cocycle, i.e., that \( \mu^1_{Tw}(b) = 0 \). Unravelling the definition of \( \mu^1_{Tw} \), cf. [Sei08 Section (3i)], we obtain

\[
\mu^1_{Tw}(b) = \left( \begin{pmatrix} \mu^1(b'') \\ \mu^1(b') - \mu^2(c_1, b'') \end{pmatrix} \right),
\]

where the \( \mu^i \)'s are those of \( \mathcal{M}(L) \). Now \( \mu^1(b'') = \mu^1(p_{x_0}) \) vanishes because the \( p_x \) form a morphism of local systems. To compute \( \mu^2(c_1, b'') \), note that for every
critical point \( y \) there’s a unique perturbed \( Y \)-shaped gradient tree with outgoing edges converging to \( x_0 \) and \( y \) that contributes to the count, and that the incoming edge of this tree also converges to \( y \) (recall that \( x_0 \) is the unique local minimum). In view of this and recalling the definition of \( c_1 \), we obtain

\[
\mu^2(c_1, b'') = \bigoplus_y \left( \sum \pm \pi'_y \circ q_{x_0} \circ \pi_\tau \circ j_y \circ p_y \right)
\]

\[
= \bigoplus_y \left( \sum \pm \pi'_y \circ q_{x_0} \circ \pi_\tau \circ (id_{E_y} - i_y \circ q_y) \right)
\]

\[
= \bigoplus_y \left( \sum \pm \pi'_y \circ q_{x_0} \circ \pi_\tau - \sum \pm q_y \right)
\]

\[
= \bigoplus_y \sum \pm \pi'_y \circ q_{x_0} \circ \pi_\tau.
\]

Here we use that the \( p_x \) give a morphism of local systems, i.e., commute with parallel transport maps (which makes the additional parallel transport maps disappear that would appear in the first line). Moreover, we use that \( \pi'_y \circ q_{x_0} \circ \pi_\tau \circ i_y = \pi'_y \circ q_{x_0} \circ i_{x_0} \circ \pi_\tau = id_{E_y} \), and that \( \sum \gamma \pm q_y = 0 \) as \( x_0 \) is a cocyle in the usual Morse complex. The result of the computation is equal to \( \mu^1(b') \), which shows that also the second component of \( \mu^1_{Tw}(b) \) vanishes.

It remains to check that \([\mu^2_{Tw}(p_C, b)] = [c_3], [\mu^2_{Tw}(b, c_2)] = [i_C]\) in \( H^0(Tw.\mathcal{M}(L)) \).

After unravelling again definitions, the first identity follows immediately (on the chain level). As for the second, we obtain

\[
\mu^2_{Tw}(b, c_2) = \left( \mu^2(b', c_2) - \mu^3(c_1, b'', c_2) \right).
\]

The first component is \( \mu^2(b'', c_2) = p_{x_0} i_{x_0} = 0 \), as required. As for the second component, we have \( \mu^2(b', c_2) = q_{x_0} i_{x_0} = id_{E_{x_0}} = i_C \), and hence we are done if we can show that \( \mu^3(c_1, b'', c_2) = 0 \). Recall that \( b'' = p_{x_0} \) and \( c_2 = i_{x_0} \), and that \( p_{x_0} i_{x_0} = 0 \); this together with the fact that the \( i_x \) commute with parallel transport maps suffices to conclude that the term vanishes. Hence the second required identity also holds on the chain level. \( \square \)

**Appendix C. Vector bundles on elliptic curves**

For the convenience of the reader, we collect here a couple of facts from Atiyah’s classification [Ati57] of vector bundles over an elliptic curve \( X \) which are used in Sections 7 and 8. We denote by \( \mathcal{V}(r, d) \) the set of isomorphism classes of vector bundles on \( X \) of rank \( r \) and degree \( d \).

**Theorem C.1** (Cf. Theorem 3 in [Ati57]). There exists an integer \( N(r, d) \) such that for every \( n \geq N(r, d) \), every \( E \in \mathcal{V}(r, d) \) fits into a short exact sequence

\[
0 \to \mathcal{O}_X^{\oplus (r-1)}(-n) \to E \to \det E(-(r-1)n) \to 0.
\]

(Here \( -(k) \) denotes tensoring by the \( k \)-th power of the hyperplane bundle.)

**Theorem C.2** (Cf. Theorem 5 in [Ati57]). (i) There is a unique \( F_r \in \mathcal{V}(r, 0) \) such that \( H^0(X; F_r) \neq 0 \), and there is a short exact sequence \( 0 \to \mathcal{O}_X \to F_r \to F_{r-1} \to 0 \) for every \( r > 1 \). (ii) Every \( E \in \mathcal{V}(r, 0) \) is of the form \( F_r \otimes L \) for a unique \( L \in \mathcal{V}(1, 0) \).
Theorem C.3 (Cf. Theorems 6 and 7 in [Ati57]). The choice of a line bundle \( \mathcal{L} \in \mathcal{V}(1,1) \) determines natural 1-1 correspondences \( \alpha_{r,d} : \mathcal{V}(h,0) \to \mathcal{V}(r,d) \), where \( h = \gcd(r,d) \). These are such that \( \det \alpha_{r,d}(\mathcal{E}) = \det \mathcal{E} \otimes \mathcal{L}^\otimes d \), which implies that \( \det : \mathcal{V}(r,d) \to \mathcal{V}(1,d) \) is an \( h \)-1 map.

Given \( \mathcal{L} \in \mathcal{V}(1,1) \) and the corresponding map \( \alpha_{r,d} : \mathcal{V}(h,0) \to \mathcal{V}(r,d) \) from Theorem C.3, we write \( E_{\mathcal{L}}(r,d) := \alpha_{r,d}(\mathcal{F}_h) \).

Lemma C.4 (Cf. Lemma 24 in [Ati57]). Suppose that \( \gcd(r,d) = 1 \). Then \( E_{\mathcal{L}}(r,d) \otimes \mathcal{F}_h \cong E_{\mathcal{L}}(hr,hd)\).

Lemma C.5 (Cf. Lemma 26 in [Ati57]). Let \( \mathcal{L}_1 \in \mathcal{V}(1,1) \). Then for every \( \mathcal{E} \in \mathcal{V}(r,d) \) there exists some \( \mathcal{L}_0 \in \mathcal{V}(1,0) \) such that \( \mathcal{E} = E_{\mathcal{L}_1}(r,d) \otimes \mathcal{L}_0 \).

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