GLOBAL CONTROLLABILITY OF MULTIDIMENSIONAL RIGID BODY BY FEW TORQUES

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Abstract. We study global controllability of 'rotating' multidimensional rigid body (MRB) controlled by application of few torques. Study by methods of geometric control requires analysis of algebraic structure introduced by the quadratic term of Euler-Frahm equation. We discuss problems, which arise in the course of this analysis, and establish several global controllability criteria for damped and non-damped cases.

1. Introduction

In recent work [1, 2, 3] one studied controllability of Navier-Stokes (NS) equation, controlled by forcing applied to few modes on a 2D domain. Geometric control approach has been employed for establishing approximate controllability criteria for NS/Euler equation on 2D torus, sphere, hemisphere, rectangle and generic Riemannian surface with boundary.

In the present contribution we address controllability issues for a finite-dimensional "kin" of NS equation - Euler-Frahm equation for rotation of multidimensional rigid body (MRB) subject to few controlling torques and to possible damping. The equation evolves on so(n). We formulate global controllability criteria which are structurally stable with respect to the choice of 'directions' of controlled torques.

According to geometric approach to studying controllability, one starts with a system controlled by low-dimensional input and proceeds with a sequence of Lie extensions ([5, 6]) which add to the system new controlled vector fields. The latter are calculated via iterated Lie-Poisson brackets of the controlled vector fields and the drift (zero control vector field). The core of the method and the main difficulty is in finding proper Lie extensions and in tracing results of their implementation.

The Lie extension employed in [1, 2, 3] for studying controllability of NS equation, and similar one used equation below (see Subsection 4.1), involves double Lie bracket of drift vector field with a couple of constant controlled vector fields (they are identified with their values, or directions belonging to so(n)). At least one of the directions must be a steady state of MRB, i.e. an 'equilibrium points' of Euler-Frahm equation.

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The double Lie bracket results in constant controlled vector field (extending direction); the correspondence between couple of original controlled directions and the extending one defines bilinear operator \( \beta \) on \( \mathfrak{so}(n) \). More extending controlled directions are obtained by iterated application of \( \beta \). For proving global controllability of MRB we must verify saturating property - coincidence of the set of extending directions with \( \mathfrak{so}(n) \) after a number of iterations.

Tracing the iterations is by no means easy. For NS equation all cases, successfully analyzed in [1, 2, 3], are related to an explicit description of the basis of steady states and to specific representation of the operator \( \beta \) with respect to this basis. The results, so obtained, are heavily dependent on choice of original controlled directions and on geometry of the domain where the NS equation evolves.

Below we manage to establish several controllability criteria for damped and non damped MRB controlled by one, two or three torques. We pay special attention to deriving criteria which are structurally stable with respect to perturbation of (some of) the controlled directions.

2. Euler Equation for Generalized Rigid Body and Euler-Frahm equation for MRB

We follow [4] for definition of ‘generalized rigid body’. Let \( \mathcal{G} \) be a Lie group, \( \mathfrak{g} \) its Lie algebra and let left-invariant Riemannian metric on \( \mathcal{G} \) be defined by scalar product \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{g} \).

Introduce \( \mathcal{I} : \mathfrak{g} \rightarrow \mathfrak{g}^* \) - a symmetric operator, which corresponds to the Riemannian metrics by formula: \( \langle \xi, \eta \rangle = \mathcal{I} \xi | \eta \), where \( \cdot | \cdot \) is the natural pairing between \( \mathfrak{g} \) and \( \mathfrak{g}^* \). The operator \( \mathcal{I} \) is called inertia operator of generalized rigid body.

The trajectory of the motion of generalized rigid body is a curve \( g(t) \in \mathcal{G} \). Angular velocity, corresponding to this motion is: \( \Omega = L_{g^{-1}} \dot{g} \in \mathfrak{g} \), where \( L_g \) is left translation by \( g \). The image of angular velocity \( \Omega \) under \( \mathcal{I} \) is angular momentum \( M \in \mathfrak{g}^* \). Energy of the body equals \( \langle \Omega, \Omega \rangle = M | \Omega \).

Euler equation for the motion of generalized rigid body is \( \dot{\Omega} = \mathcal{B}(\Omega, \Omega) \), where bilinear operator \( \mathcal{B} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \) is defined by formula:

\[
\langle [a, b], c \rangle = \langle \mathcal{B}(c, a), b \rangle,
\]

\( [\cdot, \cdot] \) staying for Lie-Poisson bracket in \( \mathfrak{g} \).

MRB is particular case of generalized rigid body, where the Lie group \( \mathcal{G} = SO(n) \), and angular velocities \( \Omega \in \mathfrak{g} = \mathfrak{so}(n) \) are skew-symmetric matrices.

Identifying \( \mathfrak{so}(n) \) with \( \mathfrak{so}^*(n) \) by means of Killing form, we may think of momentum \( M \) as of skew-symmetric matrix. Then the inertia operator is a map \( \mathcal{I}_C : \Omega \rightarrow (\Omega C + C \Omega) = M \in \mathfrak{so}(n) \cong \mathfrak{so}^*(n) \), where \( C \) is some positive semidefinite matrix.
Operator $I_C$ is symmetric with respect to Killing form and is invertible (Sylvester theorem), whenever $C$ is positive definite.

We compute $B$ according to (1) ($\cdot$, $\cdot$ being matrix commutator):

$$B(\Omega^1, \Omega^2) = I^{-1}C[I_C\Omega^1, \Omega^2], \quad B(\Omega, \Omega) = I^{-1}[I_C\Omega, \Omega] = I^{-1}[C, \Omega^2].$$

Euler-Frahm equation for the motion of MRB is:

$$\dot{\Omega} = I^{-1}[I_C\Omega, \Omega] = I^{-1}[C, \Omega^2],$$

The motion, subject to damping, is described by the equation

$$\dot{\Omega} = I^{-1}[C, \Omega^2] - \nu \Omega, \quad \nu \geq 0.$$

3. Controllability of rotating MRB: problem setting and main results

Controlled rotation of MRB is described by equation

$$\dot{\Omega} = I^{-1}[C, \Omega^2] - \nu \Omega + \sum_{i=1}^{r} G^i u_i(t), \quad \nu \geq 0, \quad G^i \in so(n).$$

We are interested in global controllability of (3), meaning that for any $\tilde{\Omega}, \hat{\Omega} \in so(n)$ system (3) can be steered from $\tilde{\Omega}$ to $\hat{\Omega}$ in some time $T \geq 0$. We are interested in achieving global controllability by small number of controls; we prove that $r$ can be taken $\leq 3$ for all $n \geq 3$.

Equation (3) is particular case of control-affine system with quadratic(+linear) drift vector field and constant controlled vector fields.

The following genericity condition is assumed to hold furtheron: symmetric matrix $C$ is positive definite and has distinct eigenvalues.

Our first result claims global controllability of MRB by means of two controlled torques.

**Theorem 3.1.** There exists a pair of directions $G^1, G^2 \in so(n)$ (depending on $C$), such that the system (3) with $r = 2$ is globally controllable. $\square$

The proof of this Theorem, sketched below, is based on direct computation of Lie extensions in specially selected basis, related to $C$. More difficult is formulating criteria, which are structurally stable with respect to perturbation of controlled directions.

We start with non damped MRB, controlled by one torque. In this case - given recurrence of Euler-Frahm dynamics (2) - bracket generating property suffices for guaranteeing global controllability. This property means that evaluations (at each point) of iterated Lie brackets of drift and controlled vector fields span $so(n)$. Given high dimension of $so(n)$, verification of the bracket generating property for generic controlled direction is nontrivial task. We do this analyzing linearization of quadratic Euler operator. The result is

**Theorem 3.2.** For generic $G \in so(n)$ the system $\dot{\Omega} = I^{-1}_C[C, \Omega^2] + Gu(t)$, is globally controllable, also if control is bounded: $|u| \leq b$, $b > 0$. $\square$
We now pass to the damped case. Our method requires one of the controlled directions to be steady state for MRB. Recall that steady state or steady direction of MRB is equilibrium point of (2) - a matrix \( \hat{G} \) for which \([I_C \hat{G}, \hat{G}] = [C, \hat{G}^2] = 0\). Matrix \( \hat{G} \) is principal axis of MRB, if \( I_C \hat{G} = \mu \hat{G}, \mu \in \mathbb{R} \). These two sets coincide for \( n = 3 \), while for \( n \geq 4 \) the set of steady directions is much richer.

The results obtained for the damped case differ for odd and even \( n \).

**Theorem 3.3.** Let \( r = 2 \), \( n \) be odd in (3). For generic stationary direction \( G^1 \) and generic \( G^2 \in \text{so}(n) \) the system (3) is globally controllable. □

An additional symmetry in the case of even \( n \), obliges one to involve additional controlled direction for achieving global controllability.

**Theorem 3.4.** Let \( r = 3 \), \( n \) be even in (3). For generic stationary direction \( G^1 \in \text{so}(n) \) and generic pair \((G^2, G^3)\) of directions the system (3) is globally controllable. □

Generic element of a subset \( W \subseteq \text{so}(n) \) means an element of open dense subset of \( W \) in induced topology.

4. Sketch of the proof of Theorem 3.1

4.1. **Key Lie extension.** Lie extensions mean finding vector fields \( X \), which are compatible with control system, in the sense that closures of attainable sets of the control system are invariant for \( X \). If one is able to prove global controllability of the system extended by some compatible vector fields, then controllability of the original system can be concluded by standard argument.

Key Lie extension, we employ, is described by the following

**Proposition 4.1.** Let for control system
\[
\dot{x} = f(x) + \tilde{g}(x)u + \bar{g}(x)v,
\]

where \( f \) is the drift vector field, evolving on a manifold \( Q \), hold the relations
\[
\{\tilde{g}, \bar{g}\} = 0, \quad \{\tilde{g}, \{\tilde{g}, f\}\} = 0,
\]

\( \{\cdot, \cdot\} \) stays for Lie brackets of vector fields on \( Q \). Then the system \( \dot{x} = f(x) + \tilde{g}(x)u + \bar{g}(x)v + \{\tilde{g}, \{\tilde{g}, f\}\}(x)w \) is Lie extension of (4). □

**Remark 4.1.** Vector fields \( \pm\{\tilde{g}, \{\tilde{g}, f\}\} \) are extending controlled vector fields; they are also compatible with (4). □

We will repeatedly employ Proposition 4.1 for extending control system (3). At each step the first of the relations (5) will be trivially satisfied since all original and extending controlled vector fields will be constant. For drift vector field \( f(\Omega) = I \frac{1}{C}[C, \Omega^2] \) in (3), and constant controlled vector field \( \tilde{g} \equiv \tilde{G} \in \text{so}(n) \), the Lie bracket \( \{\tilde{g}, \{\tilde{g}, f\}\} \equiv I \frac{1}{C}[C, \tilde{G}^2] \) is constant vector field. The second relation (5) would hold if and only if \( \tilde{G} \) is steady state.
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state. When repeating the extension it is crucial to guarantee at each step disponibility of steady state controlled direction.

For two constant controlled vector fields \( \bar{g} \equiv \bar{G}, \tilde{g} \equiv \tilde{G}, \bar{G}, \tilde{G} \in so(n) \) the value of constant extending controlled vector field \( \{ \bar{g}, \{ \tilde{g}, f \} \} \) is

\[
\beta(\bar{G}, \tilde{G}) = \mathcal{I}^{-1}[C, \bar{G}\tilde{G} + \tilde{G}\bar{G}];
\]

(6) defines symmetric bilinear operator \( \beta \) on \( so(n) \).

4.2. Algebra of principal axes and controllability proof. Diagonalize matrix \( C \) presenting it as \( C = \text{Ad} SD = SDS^{-1} \) with \( S \) orthogonal and \( D = \text{diag}\{I_1, \ldots, I_n\} \), \( I_1 < I_2 < \cdots < I_n \).

Introduce matrices \( \Theta^{rs} = 1_{rs} - 1_{sr} \in so(n), (1 \leq r < s \leq n) \), with \( 1_{rs} \) being matrix with (the only nonvanishing) unit element at row \( r \) and column \( s \). Matrices \( \Omega^{rs} = \text{Ad} S\Theta^{rs} \) turn out to be 'eigenvectors' of the operators \( (\text{ad} C) \) and \( IC \). They form set of principal axes of the MRB.

'Multiplication table' for \( \beta \) with respect to the basis \( \Omega^{rs} \) is

\[
\beta(\Omega^{rs}, \Omega^{rs}) = 0, \quad \beta(\Omega^{rs}, \Omega^{r\ell}) = (I_\ell - I_s)(I_s + I_\ell)^{-1}\Omega^{s\ell},
\]

\[
\beta(\Omega^{rs}, \Omega^{kl}) = 0, \text{ whenever } r, s, k, \ell \text{ are distinct}.
\]

Take

\[
G^1 = \Omega^{12} - \text{principal axis, } G^2 = \Omega^{23} + \Omega^{34} + \cdots + \Omega^{n-1,n}.
\]

It suffices to prove that iterated applications of \( \beta \) to \( G^1, G^2 \) result in a basis of \( so(n) \), because then the extended system would possess full-dimensional input and therefore would be globally controllable. The original system \( (3) \) would be globally controllable as well.

According to the multiplication table \( G^3 = \beta(G^2, G^1) = \beta(\Omega^{12}, \Omega^{23}) \) coincides up to a multiplier with principal axis \( \Omega^{13} \). Calculating subsequently extending controlled 'directions' \( G^i = \beta(G^{i-1}, G^2), i > 2 \), we see that all \( G^i \) coincide up to a nonzero multiplier with \( \Omega^{1,i} \), i.e. are principal axes. Also \( \beta(\Omega^{1,i}, \Omega^{1,k}) \) coincides up to a multiplier with \( \Omega^{ik} \); this means that iterating applications of \( \beta \) to \( G^1, G^2 \) generate basis of \( so(n) \).

5. Concluding remarks

1. As one can see proof of Theorem 3.1 is "rigid construction", based on specific choice of controlled directions and on computation of iterated Lie extensions with respect to specific basis of principal axes of MRB. If one perturbs one of the original controlled directions the constructions fails, as far as first Lie extension does not result in new stationary direction of MRB, and the Proposition 4.1 can not be iterated.

This rigidity of controllability criteria with respect to the choice of controlled directions, manifested itself also in previous study of approximate controllability of NS system on particular 2D domains (\[1, 2\]). It does not seem natural, and is rather related to the proposed method.
Indeed one would expect structural stability of controllability criteria and this is achieved in the formulations of Theorems 3.2, 3.3, 3.4 which are structurally stable with respect to the choice of (some of) the controlled directions. The method for establishing these criteria differs from the previous one. It is based on study of linearization at a steady state of Euler operator for MRB. The proofs will appear elsewhere.

Besides its interest for studying controllability of MRB, the method can be extended onto infinite-dimensional case, and be applied to controlled NS/Euler equation for fluid dynamics on general 2D and 3D domains. The results will appear in further publications.

2. Publication [8] studies controllability of non damped MRB by using of a pair of controlled 'flywheels' - different type of "internal-force controls", with dynamics described by bilinear control system on Lie group.

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