Phase transition in the spanning-hyperforest model on complete hypergraphs

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June 24, 2009

Abstract

By using our novel Grassmann formulation we study the phase transition of the spanning-hyperforest model of the $k$-uniform complete hypergraph for any $k \geq 2$. The case $k = 2$ reduces to the spanning-forest model on the complete graph. Different $k$ are studied at once by using a microcanonical ensemble in which the number of hyperforests is fixed. The low-temperature phase is characterized by the appearance of a giant hyperforest. The phase transition occurs when the number of hyperforests is a fraction $(k - 1)/k$ of the total number of vertices. The behaviour at criticality is also studied by means of the coalescence of two saddle points. As the Grassmann formulation exhibits a global supersymmetry we show that the phase transition is second order and is associated to supersymmetry breaking and we explore the pure thermodynamical phase at low temperature by introducing an explicit breaking field.
1 Introduction

The phase transition in a model of spanning forests is particularly interesting because only the geometric properties of connection of different parts are involved and this extremely reduced structure is probably at the root of many critical phenomena, within, and even outside, natural sciences.

A possible way to attack this problem on a generic graph by the tools of statistical mechanics goes back to the formulation as a Potts model \[1, 2, 3\] in the limit of vanishing number of states. The Potts model on any finite graph \( G = (V, E) \), with vertex set \( V \) and edge set \( E \), is characterized by the coupling \( v_e \), for each edge \( e \in E \), which is related to the inverse temperature \( \beta \) and exchange coupling \( J_e \) by the relation \( v_e = e^{\beta J_e} - 1 \). By definition \( q \) is a positive integer and the set of couplings \( \{v_e\}_{e \in E} \) is of real numbers in the interval \([-1, \infty] \).

The Fortuin-Kasteleyn representation \[4, 5\] expresses the partition function \( Z_G(q; v) \) of the Potts model as a sum on all subgraphs \( H \subseteq G \) of monomials in both \( q \) and \( v_e \)'s

\[
Z_G(q; v) := \sum_{H \subseteq G} q^{K(H)} \prod_{e \in E(H)} v_e \tag{1.1}
\]

where \( K(H) \) is the number of connected components of the subgraph \( H \). Therefore the model is easily extended to more general values of its parameters. In this form it takes the name of random cluster model \[6\]. More generally, it is convenient to introduce the redundant description in terms of two global parameters \( \lambda \) and \( \rho \)

\[
Z_G(\lambda, \rho; w) := \sum_{H \subseteq G} \lambda^{K(H)-K(G)} \rho^{L(H)} \prod_{e \in E(H)} w_e \tag{1.2}
\]

where \( L(H) \) is the cyclomatic number of the subgraph \( H \). The redundancy is easily shown by using the Euler relation

\[
V - K = E - L \tag{1.3}
\]

and the relations

\[
q = \lambda \rho \tag{1.4}
\]

\[
v = w \rho. \tag{1.5}
\]

Indeed, this form is suitable for taking two different limits when \( q \to 0 \). In the former limit \( \lambda \to 0 \) at \( \rho \) fixed only maximally-connected subgraphs will survive and will be weighted by a factor \( \rho^{L(H)} \). In the latter one \( \rho \to 0 \) at \( \lambda \) and \( w \) fixed only spanning forests will survive,
weighted by a factor $\lambda^K(H)$ as $\lambda^{-K(G)}$ is an overall constant. Remark that, when $G$ is planar, maximally-connected subgraphs are in one-to-one correspondence with spanning forests by graph duality, and that when both $\lambda \to 0$ and $\rho \to 0$ only spanning trees will survive in each connected component of $G$.

As shown in [7, 8], the model in the limit $\rho \to 0$, that is the spanning forest model, admits a representation in terms of fermionic fields, which means that the partition function can be written as a multiple Berezin integral over anti-commuting variables which belong to a Grassmann algebra. Moreover, this representation [8, 9] is powerful enough to describe the model with many body interactions which gives rise to hyperforests defined on a hypergraph [10], which is a natural generalization of the concept of graph where the edges can connect more than two vertices at once.

In two dimensions, the critical behaviour of the ferromagnetic Potts/random-cluster model is quite well understood, thanks to a combination of exact solutions [11], Coulomb gas methods [12], and conformal field theory [13]. Information can also be deduced from the study of the model on random planar lattices [14, 15, 16]. Also in the $q \to 0$ limit detailed results are available both for the tree model, in particular in connection with the abelian sandpile model [17], as for spanning forest on a regular lattice [7, 18] and directly in the continuum [19]. Also the model on random planar lattices has been considered [20].

But in more than two dimensions the only quantitative information we have about the spanning-forest model come from numerical investigations [21]. Monte Carlo simulations performed at increasing dimensionality ($d = 3, 4, 5$) show a second-order phase transition.

Much less results are available for the case of hyperforests. Also in two dimensions or in the limit of hypertrees. Even the problem of determining whether there exists a spanning hypertree in a given $k$-uniform hypergraph, is hard, technically NP-complete, for $k \geq 4$, whereas for $k = 3$, there exists a polynomial-time algorithm based on Lovasz’ theory of polymatroid matching [22]. See [23] for a randomized polynomial-time algorithm in the case $k = 3$ whose main ingredients is a Pfaffian formula for a polynomial that enumerates spanning hypertrees with some signs [24], which is quite similar to our Grassmann representation [25].

In [26] a phase transition is detected in the random $k$-uniform hypergraph when a number of hyperedges $|E| = n/k(k-1)$ of the total number of vertices $n = |V|$ is chosen uniformly at random. In the case of random graphs, that is for $k = 2$, Erdős and Rényi showed in their classical paper [27] that at the transition an abrupt change
occurs in the structure of the graph, for low density of edges it consists of many small components, while, in the high-density regime a giant component occupies a finite fraction of the vertices. Remark that their ensemble of subgraphs is the one occurring in the microcanonical formulation, at fixed number of edges, of the Potts model at number of states $q = 1$. The connected-component structure of the random $k$-uniform hypergraph has been analyzed in [26] where it has been shown that if $|E| < n/k(k-1)$ the largest component occupies order $\log n$ of vertices, for $|E| = n/k(k-1)$ it has order $n^{2/3}$ vertices and for $|E| > n/k(k-1)$ there is a unique component with order $n$ vertices. More detailed information on the behaviour near the phase transition when $|E| \to n/k(k-1)$ have been recovered in [28, 29] for the case of the random graph, but see also [30, 31], and in [32] for the general case of hypergraphs.

By using the new Grassmann representation, we present here a study of the phase transition for the hyperforest model on the $k$-uniform complete hypergraph, for general $k$, where the case $k = 2$ corresponds to spanning forests on the complete graph. The random-cluster model [33] on the complete graph has already been developed but it cannot be extended to the $q \to 0$ case, exactly like the mean-field solution for the Potts model [34, 35]. The fermionic representation, instead, describes the Potts model directly at $q = 0$ as it provides an exact representation of the partition function of the spanning-hyperforest model.

As usual with models on the complete graph, the statistical weight reduces to a function of only one extensive observable, which here is quadratic in the Grassmann variables. Under such a condition the partition function can be expressed as the integration over a single complex variable in a closed contour around the origin [9]. Indeed, counting the spanning forests over a complete (hyper-)graph is indeed a typical problem of **analytical combinatorics**. And, exactly like in the case of ordinary graph, when the number of connected components in the spanning forests is macroscopic, that is a finite fraction of the number of vertices, there are two different regimes, which can be well understood by means of two different saddle points of a closed contour integration over a single complex variable as presented in [36] (but see also the probabilistic analysis in [37]). And even the behaviour at the critical point can be studied as the coalescence of these two saddle points.

In this paper we shall first review for reader convenience the Grassmann formulation of the spanning-forest model in Sec. 2 and in Sec. 3 how it is possible to recover, in the case of the $k$-uniform complete hypergraphs, a representation of the partition function suitable for
the asymptotic analysis for large number of vertices $n$. In this same Section we shall also present a full discussion of the saddle points in the micro-canonical ensemble, that is at fixed number of connected components, and of the associated different phases. We shall see that the universality class of the transition is independent from $k$. And we will exhibit the relation with the canonical ensemble. In Sec. 4, we will provide an interpretation of the transition as the appearance of a giant component by introducing a suitable observable which is sensible to the size of the different hypertrees in the hyperforest.

More interestingly, our Grassmann formulation exhibits a global continuous supersymmetry, non-linearly realized. We shall show that the phase transition is associated to the spontaneous breaking of this supersymmetry. By the introduction of an explicit breaking we shall be able to investigate the expectation values in the broken pure thermodynamical states. We shall therefore be able to see in Sec. 5 that the phase transition is of second order. This seems at variance with the supersymmetric formulation of polymers given by Parisi and Sourlas [38] where it appeared to be of zeroth order.

2 The spanning-forest model

Given the complete hypergraph $K_n = (V, E)$ with vertex set $V = \{n\}$, and complete in the $k$-hyperedges for all $2 \leq k \leq n$, so that the hyperedge set $E$ is the collection of all $A \subseteq V$ with cardinality at least 2, let’s introduce on each vertex $i \in V$ a pair of anti–commuting variables $\psi_i, \bar{\psi}_i$:

\[
\{\psi_i, \psi_j\} = \{\bar{\psi}_i, \bar{\psi}_j\} = \{\bar{\psi}_i, \psi_j\} = 0 \quad \forall i, j \in V(G) \tag{2.1}
\]

which generate the Grassmann algebra $A[\psi_1, \ldots, \psi_n, \bar{\psi}_1, \ldots, \bar{\psi}_n]$ of dimension $2^{2n}$.

Then, for each hyperedge $A \subseteq E$, we define the monomial

\[
\tau_A := \prod_{i \in A} \bar{\psi}_i \psi_i, \tag{2.2}
\]

and, for each indeterminate $t$, the Grassmann element

\[
f_A^{(t)} := t(1 - |A|)\tau_A + \sum_{i \in A} \tau_{A \setminus i} - \sum_{i,j \in A, i \neq j} \bar{\psi}_i \psi_j \tau_{A \setminus \{i, j\}}. \tag{2.3}
\]

In [8] it has been shown that the generating function of unrooted spanning forest on a generic hypergraph admits the following repre-
sentations: given a set of edge weights \( w = \{w_A\}_{A \in E} \) we have

\[
Z_G(w, t) := \sum_{F \in \mathcal{F}} t^{K(F)} \prod_{A \in F} w_A
\]

\[
= \int \mathcal{D}_n(\bar{\psi}, \psi) \exp \left\{ t \sum_{i \in V} \bar{\psi}_i \psi_i + \sum_{A \in E} w_A f_A^{(t)} \right\}
\]

\[
= \int \mathcal{D}_n(\bar{\psi}, \psi) \exp (-\mathcal{H})
\]

where the indeterminate \( t \) plays the role of the parameter \( \lambda \) we had in the random cluster model formulation, \( \mathcal{F} \) is the set of hyperforests, \( K(F) \) the number of connected components in the hyperforest \( F \), that is the number of hypertrees,

\[
\int \mathcal{D}_n(\bar{\psi}, \psi) := \int \prod_{i \in V} d\bar{\psi}_i d\psi_i
\]

is the Berezin integration and we denoted by \(-\mathcal{H}\), as usual in statistical mechanics, the exponential weight.

The fermionic model we introduced above presents a non-linearly realized osp(1|2) supersymmetry [7, 8]. Firstly, we have the elements of the sp(2) subalgebra, with

\[
\delta \psi_i = -\alpha \psi_i + \gamma \bar{\psi}_i
\]

\[
\delta \bar{\psi}_i = +\alpha \bar{\psi}_i + \beta \psi_i
\]

where \( \alpha, \beta, \gamma \) are bosonic (Grassmann-even) global parameters. Secondly, we have the transformations parametrized by fermionic (Grassmann-odd) global parameters \( \epsilon, \bar{\epsilon} \):

\[
\delta \psi_i = t^{-1/2} \epsilon (1 - t \bar{\psi}_i \psi_i)
\]

\[
\delta \bar{\psi}_i = t^{-1/2} \bar{\epsilon} (1 - t \bar{\psi}_i \psi_i)
\]

In terms of the differential operators \( \partial_i = \partial / \partial \psi_i \) and \( \bar{\partial}_i = \partial / \partial \bar{\psi}_i \), the transformations (2.9) can be represented by the generators

\[
X_0 = \sum_{i \in V} (\bar{\psi}_i \bar{\partial}_i - \psi_i \partial_i)
\]

\[
X_+ = \sum_{i \in V} \bar{\psi}_i \partial_i
\]

\[
X_- = \sum_{i \in V} \psi_i \bar{\partial}_i
\]
corresponding to the parameters $\alpha, \beta, \gamma$, respectively, while the transformations (2.10) and (2.11) can be represented by the generators

$$ Q_+ = t^{-1/2} \sum_{i \in V} (1 - t \bar{\psi}_i \psi_i) \partial_i $$

$$ Q_- = t^{-1/2} \sum_{i \in V} (1 - t \bar{\psi}_i \psi_i) \bar{\partial}_i $$

(2.15)

(2.16)

corresponding to the parameters $\epsilon, \bar{\epsilon}$, respectively. These transformations satisfy the commutation/anticommutation relations

$$ [X_0, X_\pm] = \pm 2X_\pm \quad [X_+, X_-] = X_0 $$

$$ \{Q_+, Q_\pm\} = \pm 2X_\pm \quad \{Q_+, Q_-\} = X_0 $$

(2.17)

(2.18)

$$ [X_0, Q_\pm] = \pm Q_\pm \quad [X_\pm, Q_\pm] = 0 \quad [X_\pm, Q_\mp] = -Q_\mp $$

(2.19)

Note in particular that $X_\pm = Q_\pm^2$ and $X_0 = Q_+Q_- + Q_-Q_+$.

3 Uniform complete hypergraphs

The $k$-uniform complete hypergraph is the hypergraph whose vertices are connected in groups of $k$ in all possible ways or, alternatively, whose edge set $E$ is the set of all $k$-sets over the vertex set $V$. In our general formulas for the complete hypergraph we must set the weights $w_A = 0$ for all the hyperedges $A$ with cardinality different from $|A| = k$. In the following we will set all the nonzero weights to one, so that we shall restrict ourselves to a simple, one-parameter, counting problem. In this case the expansion of the partition function in series of $t$

$$ Z(t) = \sum_p Z_p t^p $$

(3.1)

provides $Z_p$ the number of hyperforests, with all hyperedges with cardinality $k$, composed by $p$ hypertrees. Please remark that, in the $k$-uniform complete hypergraph, the number $p$ of hyperforests must be such that

$$ s = \frac{n - p}{k - 1} $$

(3.2)

must be an integer. Indeed it is the total number of hyperedges in the hyperforest.
By definition
\[ Z_p = \frac{1}{p!} \langle U^p \rangle_{t=0} \] (3.3)

\[ = \frac{1}{p!} \int \mathcal{D}_n(\bar{\psi}, \psi) U^p \exp \left\{ \sum_{A:|A|=k} f^{(0)}_A \right\} \] (3.4)

where
\[ U = \sum_{i \in V} \bar{\psi}_i \psi_i + (1 - k) \sum_{A:|A|=k} \tau_A \] (3.5)

and \( \langle \cdot \rangle_{t=0} \) is the un-normalized expectation value in the ensemble of hypertrees.

The interested reader can find a full comparison of our approach with respect to the standard tools of combinatorics in our previous paper [9]. We introduce a mean-field variable
\[ \bar{\psi}\psi := \sum_{i \in V} \bar{\psi}_i \psi_i = (\bar{\psi}, \psi) \] (3.6)

and we observe that
\[ U = \bar{\psi}\psi + (1 - k) \frac{(\bar{\psi}\psi)^k}{k!} \] (3.7)
\[ \sum_{A:|A|=k} f^{(0)}_A = n \frac{(\bar{\psi}\psi)^{k-1}}{(k-1)!} - (\bar{\psi}, J\psi) \frac{(\bar{\psi}\psi)^{k-2}}{(k-2)!}, \] (3.8)

where J is the matrix with 1 on all entries, so that \( (\bar{\psi}, J\psi) = \sum_{i,j} \bar{\psi}_i \psi_j \).

The following lemma then applies:

**Lemma 1 ([9])** Let \( n \) be the number of vertices, \( g \) and \( h \) generic functions on the Grassmann algebra, then
\[ \int \mathcal{D}_n(\bar{\psi}, \psi) \ (\bar{\psi}\psi)^r e^{h(\bar{\psi}\psi) + (\bar{\psi}, J\psi) g(\bar{\psi})} \]
\[ = \int \mathcal{D}_n(\bar{\psi}, \psi) \ (\bar{\psi}\psi)^r e^{h(\bar{\psi}\psi)} \left[ 1 + \bar{\psi}\psi g(\bar{\psi}) \right] . \] (3.9)

By this lemma, \( Z(t) \) can be written in terms of the sole mean-field variable \( \bar{\psi}\psi \)
\[ Z(t) = \int \mathcal{D}_n(\bar{\psi}, \psi) \exp \left\{ tU + n \frac{(\bar{\psi}\psi)^{k-1}}{(k-1)!} \right\} \left[ 1 - \frac{(\bar{\psi}\psi)^{k-1}}{(k-2)!} \right] . \] (3.10)

In order to perform an estimate for the asymptotic value of the integral for large \( n \) we recall that for an analytic function \( f \)
\[ \int \mathcal{D}_n(\bar{\psi}, \psi) f(\bar{\psi}\psi) \equiv n! \int \frac{d\xi}{2\pi i} \frac{f(\xi)}{\xi^{n+1}} \] (3.11)
where the integration contour in the complex plane is around the origin. We have the following complex integral representation form for the partition function $Z(t)$

$$Z(t) = n! \oint \frac{d\xi}{2\pi i} \frac{1}{\xi^{n+1}} \exp \left\{ t \left[ \xi + (1-k)\frac{\xi^k}{k!} \right] \right\} \exp \left\{ n\frac{\xi^{k-1}}{(k-1)!} \right\} \left[ 1 - \frac{\xi^{k-1}}{(k-2)!} \right]. \quad (3.12)$$

Let us first work at fixed number of hypertrees, in a micro–canonical ensemble in the physics terminology.

Expanding (3.12) in powers of $t$ we obtain the number $Z_p$ of spanning hyperforests on the complete $k$-uniform hypergraph which is the number of states in the micro-canonical ensemble

$$Z_p = n! \frac{p}{p!} \oint \frac{d\xi}{2\pi i} \frac{1}{\xi^{n+1}} \left[ \xi + (1-k)\frac{\xi^k}{k!} \right]^p \exp \left\{ n\frac{\xi^{k-1}}{(k-1)!} \right\} \left[ 1 - \frac{\xi^{k-1}}{(k-2)!} \right]. \quad (3.13)$$

Since we are interested in obtaining $Z_p$ in the thermodynamical limit $n \to \infty$ also for large values of $p$, we define $p = \alpha n$ with fixed $\alpha$ as $n, p \to \infty$. Changing the variable of integration to $\eta = (k-1)\frac{\xi^{k-1}}{k!}$, we obtain the following integral expression:

$$Z_{\alpha n} = \frac{n!}{\Gamma(\alpha n + 1)} \left[ \frac{k-1}{k!} \right]^{\alpha n} I(\alpha) \quad (3.14)$$

where

$$I(\alpha) := \oint \frac{d\eta}{2\pi i} A(\eta) e^{\alpha B(\eta)} \quad (3.15)$$

with

$$A(\eta) = \frac{1 - k\eta}{\eta} \quad (3.16)$$

$$B(\eta) = \frac{k}{k-1} \eta + \alpha \log(1 - \eta) + \frac{\alpha - 1}{k-1} \log \eta. \quad (3.17)$$

Please note that the factor $k-1$ coming from the change of variable in the integral is exactly compensated by the fact that a full turn around the origin in the $\eta$ plane is equivalent to $k-1$ turns of the $\xi$ variable.

Precise estimates of integrals of this kind for $n \to \infty$ can be obtained by the saddle point method (see [36] for a very complete discussion of this method).
3.1 The saddle point method

A saddle point of a function $B(\eta)$ is a point $\eta_0$ where $B'(\eta_0) = 0$, it is said to be a simple saddle point if furthermore $B''(\eta_0) \neq 0$. In this case it is easy to see that the equilevel lines divide a neighborhood of $\eta_0$ in four regions where Re $B(\eta)$ is alternately higher and lower than the saddle point value Re $B(\eta_0)$. We will refer to the two lower regions as the valleys.

Analogously, a multiple saddle point has multiplicity $p$ if all derivatives up to $B^{(p)}(\eta_0)$ are equal to zero while $B^{(p+1)}(\eta_0) \neq 0$. In this case there are $p + 1$ higher and lower regions.

When evaluating Cauchy contour integrals of the form (3.15), saddle points of $B(\eta)$ play a central role in the asymptotic estimate for large $n$. The method essentially consists of two basic ingredients: an accurate choice of the contour and Laplace’s method for the evaluation of integrals depending on a large parameter.

The contour has to be chosen to pass through a point which is a global maximum of the integrand along the contour and that a neighborhood of which (the central region) dominates the rest of the contour (the tails) as $n$ grows. Since an analytic function cannot have an isolated maximum, this implies that the contour should pass through a saddle point.

The existence of a contour surrounding the origin and that crosses a saddle point along its direction of steepest descent requires that two of its valleys are topologically connected and the region connecting them surrounds the origin.

Once we have a contour, we proceed neglecting the tails and approximating the functions $A(\eta)$ and $B(\eta)$ with their Taylor series about the chosen saddle point $\eta^*$. Then, after having absorbed the factor $n$ into a rescaled variable $x = (\eta - \eta^*)/n^{1/(p+1)}$ (where $p$ is the multiplicity of the saddle point), we can easily obtain an asymptotic expansion of the integral in inverse powers of $n$.

We collect here the first few terms of the asymptotic expansion for the case of a simple saddle

$$ I \sim e^{nB(\eta^*)}/\sqrt{2\pi nB''(\eta^*)} \left[ A(\eta^*) + \frac{1}{n} C(\eta^*) + \frac{1}{n^2} D(\eta^*) + O\left(\frac{1}{n^3}\right) \right] \quad (3.18) $$

where the terms in the square brackets with half-integer inverse-power of $n$ vanish, and of a double saddle

$$ I \sim e^{nB(\eta^*)}/n^{3/2} B^{(3)}(\eta^*)^{3/2} \left[ \gamma_0 A(\eta^*) + \frac{1}{n^{3/2}} \tilde{C}(\eta^*) + \frac{1}{n} \tilde{D}(\eta^*) + O\left(\frac{1}{n^{3/2}}\right) \right] \quad (3.19) $$
where the terms in the square brackets with powers \( n^{-\left(l+\frac{2}{3}\right)} \), with integer \( l \), vanish. In these formulae \( C, \tilde{C}, D, \) and \( \tilde{D} \) are rational functions of \( A(\eta^*) \), \( B(\eta^*) \) and their derivatives, whose expression is reported in the Appendix A together with the value of the constant \( \gamma_0 \).

For our integral (3.15) in the large \( n \) limit the relevant saddle-point equation \( B'(\eta) = 0 \) has two solutions, \( \eta_a \) and \( \eta_b \):

\[
\eta_a = \frac{1}{k} \quad \eta_b = 1 - \alpha. \tag{3.20}
\]

If \( \alpha \neq \alpha_c \equiv (k - 1)/k \) the two solutions are distinct and correspond to simple saddle points. To understand which one is relevant to our discussion we need to study the landscape of the function \( B(\eta) \) beyond the neighborhood of the saddles.

In our specific case, as illustrated in figures from 1 to 3 when \( \alpha < \alpha_c \) among the two saddles only \( \eta_a \) is accessible, while, if \( \alpha > \alpha_c \), only \( \eta_b \) is so. When \( \alpha = \alpha_c \) the two saddle points coalesce into a double saddle point, thus with three valleys, having steepest-descent directions \( e^{\frac{2\pi ik}{3}} \), with \( k = 0, 1, 2 \). Of these valleys, the ones with the appropriate global topology are those with indices \( k = 1 \) and 2.

As a first result of this discussion, in order to study the asymptotic behaviour of \( Z_\alpha n \) will need to distinguish two different phases, and a critical point, upon the value of \( \alpha \) being below, above or equal to \( \alpha_c \).

We will name the phases with a smaller and a larger number of hypertrees, respectively, the low temperature and high temperature phase, the reason being that, as we shall see, in the low temperature phase there is a spontaneous symmetry breaking and the appearance of a non-zero residual magnetization.

### 3.1.1 Low temperature phase

In the case \( \alpha < \alpha_c \) the relevant saddle point is \( \eta_a = 1/k \). See Fig. 1. Since \( A(\eta_a) = 0 \), we are in the case in which the leading order of (3.18) vanishes and the next order has to be considered. The expansion of \( A(\eta) \) and \( B(\eta) \) in a neighborhood of the saddle \( \eta_a \) is as follows:

\[
A(\eta_a + u) \simeq -k^2 u + k^3 u^2 + O(u^3) \tag{3.21}
\]

\[
B(\eta_a + u) \simeq \frac{1 + (1 - \alpha) \log k}{k \alpha_c} + \alpha \log \alpha_c + k \frac{\alpha_c - \alpha}{\alpha_c^2} \frac{u^2}{2} - \left[ k^2 \frac{1 - \alpha}{\alpha_c} + \frac{\alpha}{\alpha_c^3} \right] \frac{u^3}{3} + O(u^4). \tag{3.22}
\]
Figure 1: Contour levels for $\text{Re} B(\eta)$ when $\alpha < \alpha_c$. More precisely, the figure shows the case $k = 2$ and $\alpha = \frac{1}{2}\alpha_c$. The two bold contour lines describe the level lines of $\text{Re} B(\eta)$ for the values at the two saddle points (located at the bullets). Darker tones denote higher values of $\text{Re} B(\eta)$. The crosses and the dotted lines describe the cut discontinuities due to logarithms in $B(\eta)$. The dashed path surrounding the origin going through one of the saddle points is an example of valid integration contour, and the solid straight portion of the path describes an interval in which the perturbative approach is valid.
Figure 2: Contour levels for \( \text{Re} B(\eta) \) when \( \alpha > \alpha_c \). More precisely, the figure shows the case \( k = 2 \) and \( \alpha = \frac{3}{2} \alpha_c \). Description of notations is as in figure 1.

Figure 3: Contour levels for \( \text{Re} B(\eta) \) when \( \alpha = \alpha_c \). More precisely, the figure shows the case \( k = 2 \). Description of notations is as in figure 1.
Using formula (3.18) we obtain for (3.13) the following asymptotic expression:

\[
Z_{an} \simeq \frac{n!}{\Gamma(an + 1)} \frac{\alpha \sqrt{k-1} e^{-\frac{\alpha}{n}} \left(\frac{k-1}{1}\right)^{n-1}}{2\pi n^3} \left[1 - \frac{k\alpha}{k-1}\right]^{-5/2} (3.23)
\]

\[
\simeq \frac{n^{n-2}}{(an)^{n-1}} \frac{\sqrt{k-1} e^{\left(\frac{\alpha}{n} - \frac{1}{k}\right) n} \left(\frac{k-1}{1}\right)^{n-1} \left[1 - \frac{k\alpha}{k-1}\right]^{-5/2}}{2\pi n^3} (3.24)
\]

where in the second line we used the Stirling formula to approximate the large factorial \(n!\). In a previous work [9] we already gave an asymptotic formula for the number of forests with a given number \(p\) of connected components. That formula has been obtained keeping \(p\) fixed while doing the limit \(n \to \infty\), in the notation of this paper this means taking \(\alpha\) infinitesimal. By setting \(\alpha n \to p\) in (3.23) and using

\[
\frac{\alpha}{\Gamma(an + 1)} = \frac{\alpha}{an \Gamma(an)} = \frac{n^{-1}}{(p-1)!} (3.25)
\]

and then taking the limit \(\alpha \to 0\), we can re-obtain the result in [9] by using again the Stirling formula to approximate the large factorial \(n!\):

\[
Z_p \simeq \frac{n^{n-2}}{e^{\frac{1}{k-1}}} \sqrt{\left(\frac{k-1}{1}\right)^{n-1} \left[\frac{1}{(k-2)!} \frac{1}{(p-1)!}\right]^{n-1}} \left(\frac{k-1}{k}\right)^{p-1}. (3.26)
\]

### 3.1.2 High temperature phase

When \(\alpha_c < \alpha < 1\) the relevant saddle point changes into \(\eta_b = 1 - \alpha\) (see Fig. 2) where the functions \(A(\eta_b + u)\) and \(B(\eta_b + u)\) can be approximated at \(O(u^4)\) with

\[
A(\eta_b + u) \simeq k \frac{\alpha - \alpha_c}{1 - \alpha} - \frac{u}{(\alpha - 1)^2} - \frac{u^2}{(\alpha - 1)^3} - \frac{u^3}{(\alpha - 1)^4} (3.27)
\]

\[
B(\eta_b + u) \simeq \frac{1 - \alpha}{\alpha_c} \left[1 - \frac{1}{k} \log(1 - \alpha)\right] + \alpha \log \alpha
\]

\[
+ \frac{1}{\alpha_c} \frac{\alpha - \alpha_c}{1 - \alpha} \frac{u^2}{2} - \frac{1}{\alpha^2} + \frac{1}{\alpha_c} \frac{1}{1 - \alpha} \frac{u^3}{3} (3.28)
\]

The situation is quite analogous to the previous one, with the exception that \(A(\eta_b) \neq 0\), and using formula (3.18) we obtain for (3.13) the
following asymptotic expression:

\[ Z_{\alpha n} \simeq n! \alpha^n (k-1) \frac{e^k}{(k-1) (k-2)!} \left[ \frac{\alpha k}{k-1} \right]^{1/2} \sqrt{\frac{\alpha}{2\pi n (1-\alpha)}} \]

\[ \simeq n^{(1-\alpha)n} \frac{e^{n^{1/1-\alpha}}}{\sqrt{2\pi n^{1-\alpha}}} \left[ \frac{\alpha k}{k-1} \right]^{1/2} (\alpha k + 1)^{1/2}. \]  

(3.29)

(3.30)

Remark that the saddle point method cannot be applied when \( \alpha = 1 \), but if we replace

\[ \frac{e^{n^{1/1-\alpha}}}{\sqrt{2\pi n^{1-\alpha}}} \simeq \frac{n^{n^{1/1-\alpha}}}{\Gamma \left( n^{1-\alpha} + 1 \right)} \]  

(3.31)

for \( \alpha \simeq 1 \) we get

\[ Z_n \simeq 1. \]  

(3.32)

as we should.

### 3.1.3 The critical phase

When \( \alpha \) is exactly \( \alpha_c = (k-1)/k \), the saddle points \( \eta_b \) and \( \eta_a \) coalesce into a double saddle point in \( \eta_a \) in which the second derivative vanishes along with the first one. The expansion of \( A(\eta) \) and \( B(\eta) \) are as in (3.21)-(3.22) with \( \alpha = \alpha_c \):

\[ A(\eta_c + u) \simeq -k^2 u + k^3 u^2 + O(u^3) \]  

(3.33)

\[ B(\eta_c + u) \simeq \frac{1}{k-1} + \frac{1}{k(k-1)} \log(k-1) + \frac{k-2}{k-1} \log k \]

\[ - \frac{k^3}{(k-1)^2} \frac{u^3}{3} + \frac{k^4(k-2)}{(k-1)^3} \frac{u^4}{4} + O(u^5) \]  

(3.34)

Using (3.19) we obtain the following result

\[ Z_{\alpha_c n} = \frac{n!}{\Gamma \left( \frac{k-1}{k} n + 1 \right)} \frac{e^{n^{1/1-k}}} {((k-2)!)^{n^{1-1-k}}} \frac{3^{1/6} \Gamma(2/3)(k-1)^{4/3}}{2\pi n^{2/3}} \]  

(3.35)

\[ \simeq n^{n^{1/1-k}} \frac{e^{n^{1/1-k}}}{((k-2)!)^{n^{1-1-k}}} \frac{3^{1/6} \Gamma(2/3)(k-1)^{4/3}}{2\pi n^{2/3}}. \]  

(3.36)

This formula, for \( k = 2 \), can be, in principle, compared with the result presented in [36, Proposition VIII.11], but unfortunately there is a discrepancy in the numerical pre-factor.
3.2 The canonical ensemble

According to the definition (3.14) for the number of forests with \( p = \alpha n \) trees \( Z_{\alpha n} \), by evaluating the integral \( I \) defined in (3.15) by the saddle-point method, when the saddle point \( \eta^*(\alpha) \) is simple and thus away from the critical point \( \alpha_c \), we get the following asymptotic expansion for large number of vertices \( n \):

\[
Z_{\alpha n} \simeq \frac{n!}{\Gamma(\alpha n + 1)} \left[ \frac{k - 1}{k!} \right]^{\frac{n}{k-1}} \frac{e^{nB(\eta^*)}}{\sqrt{2\pi nB''(\eta^*)}} \left[ A(\eta^*) + \frac{1}{n} C(\eta^*) \right].
\]

(3.37)

We define the entropy density \( s(\alpha) \) as

\[
s(\alpha) = \frac{1}{n} \log \frac{Z_{\alpha n}}{n!}
\]

(3.38)

so that we can recover the partition function \( Z(t) \) by a Legendre transformation

\[
Z(t) = \sum_p Z_p t^p
\]

(3.39)

\[
\simeq \int_0^1 d\alpha Z_{\alpha n} t^{n\alpha}
\]

(3.40)

\[
= n! \int_0^1 d\alpha \exp \{n[s(\alpha) + \alpha \log t]\}
\]

(3.41)

that can be evaluated for large \( n \) once more by the saddle-point method. Calling \( \bar{\alpha}(t) \) the mean number of trees at given \( t \), we have:

\[
s'(\bar{\alpha}(t)) + \log t = 0.
\]

(3.42)

From (3.37) we see that \( s(\alpha) \) still has an \( \alpha \)-dependent leading order in \( n \)

\[
s(\alpha) \simeq -\alpha \log n - \alpha \log \alpha + \frac{\alpha - 1}{k - 1} \log \left[ \frac{k!}{k - 1} \right] + B(\eta^*(\alpha))
\]

(3.43)

that would shift the solution down to 0. By the rescaling

\[
t = n \tilde{t}
\]

(3.44)

which is usual in the complete graph, in order to obtain a correct thermodynamic scaling, we can reabsorb this factor. The saddle-point equation now reads

\[
s'(\bar{\alpha}) + \log n + \log \tilde{t} = 0
\]

(3.45)
whose solution is

\[
\bar{\alpha} = \begin{cases} 
\frac{k-1}{k} \frac{1}{(k-2)!} \tilde{t} & \text{for } \tilde{t} < \tilde{t}_c \\
1 - \frac{k-1}{k} \tilde{t} & \text{for } \tilde{t} > \tilde{t}_c 
\end{cases}
\]

(3.46)

where \( \tilde{t}_c = [(k-2)!]^{-1/(k-1)} \). And by inversion

\[
\tilde{t} = \begin{cases} 
\frac{k}{k-1} \frac{1}{(k-2)!} \bar{\alpha} & \text{for } \bar{\alpha} < \bar{\alpha}_c \\
\left(\frac{k-1}{k-2}\right)^{1/2} \bar{\alpha} & \text{for } \bar{\alpha} > \bar{\alpha}_c.
\end{cases}
\]

(3.47)

In the ordinary graph case, this means

\[
\bar{\alpha} = \begin{cases} 
\frac{1}{2} \tilde{t} & \tilde{t} < 1 \\
1 - \frac{1}{2} \tilde{t} & \tilde{t} > 1
\end{cases}
\]

\[
\tilde{t} = \begin{cases} 
2\bar{\alpha} & \bar{\alpha} < \frac{1}{2} \\
\frac{1}{2(1-\bar{\alpha})} & \bar{\alpha} > \frac{1}{2}.
\end{cases}
\]

(3.48)

4 Size of the hypertrees

We have shown in the previous sections that the system admits two different phases. We want now to characterize these two regimes. Our field-theoretical approach provides us a full algebra of observables, as polynomials in the Grassmann fields, which we could study systematically. However, it is interesting to note that some of these observables have a rephrasing in terms of combinatorial properties of the forests (cfr. [8]). Furthermore, we are induced by the results of [21] to investigate the possibility of a transition of percolative nature, with the emergence of a giant component in the typical forest for a given ensemble.

A possibility of this sort is captured by the mean square size of the trees in the forest, as the following argument shows at least at a heuristic level. If we have all trees with size of order 1 in the large \( n \) limit, (say, with average \( a \) and variance \( \sigma \) both of order 1), then the sum of the squares of the sizes of the trees in a forest scales as \( (a^2 + \sigma^2/a)n \). If, conversely, in the large \( n \) limit one tree occupies a finite fraction \( p \) of the whole graph, the same sum as above would scale as \( p^2n^2 + O(n) \).

Furthermore, it turns out that the combinatorial observable above has a very simple formulation in the field theory, corresponding to the natural susceptibility for the fermionic fields, as we will see in a moment.

Let’s start our analysis with the un-normalized expectation

\[
t \langle \bar{\psi}_i \psi_i \rangle = t \int D_n(\bar{\psi}, \psi) \bar{\psi}_i \psi_i \exp(-\mathcal{H})
\]

(4.1)

\[
= Z(t)
\]

(4.2)
because the insertion of the operator \( \bar{\psi}_i\psi_i \) simply marks the vertex \( i \) as a root of a hypertree, and in a spanning forest every vertex can be chosen as the root of a hypertree. If we now sum over the index \( i \) we gain a factor \(|T|\) for each hypertree. Therefore we have:

\[
t \langle \bar{\psi}_i\psi_i \rangle = t \sum_{i \in V} \langle \bar{\psi}_i\psi_i \rangle = t \sum_{F \in \mathcal{F}} t^{K(F)} \sum_{T \in F} |T| \prod_{A \in T} w_A = n \mathcal{Z}(t)
\]  

(4.3)

as in each spanning hyperforest the total size of the hypertrees is the number of vertices in the graph, that is \( n \). By expanding in the parameter \( t \) and by taking the \( p \)-th coefficient we get the relation

\[
\frac{1}{\mathcal{Z}_p} \frac{\langle (\bar{\psi}_i\psi_i)^p \rangle_{t=0}}{(p - 1)!} = n.
\]

(4.4)

For the un-normalized two-point function

\[
\langle \bar{\psi}_i\psi_j \rangle = \int D_n(\bar{\psi}, \psi) \bar{\psi}_i\psi_j \exp(-\mathcal{H})
\]

(4.5)

we know (see [8]) that

\[
t \langle \bar{\psi}_i\psi_j \rangle = \sum_{i,j \text{ connected}} F \in \mathcal{F} t^{K(F)} \sum_{T \in F} \prod_{A \in T} w_A.
\]

(4.6)

As \( i \) and \( j \) are connected if they belong to the same hypertree, if we sum on both indices \( i \) and \( j \) we gain a factor \(|T|^2\) for each hypertree

\[
t \langle (\bar{\psi}, J\psi) \rangle = t \sum_{i,j \in V} \langle \bar{\psi}_i\psi_j \rangle = t \sum_{F \in \mathcal{F}} t^{K(F)} \sum_{T \in F} |T|^2 \prod_{A \in T} w_A.
\]

(4.7)

The effect of this observable is to introduce an extra weight for hypertrees in the spanning forests which is the square of its size.

The average of the square-size of hypertrees in the microcanonical ensemble of hyperforests with fixed number \( p \) of hypertrees is easily obtained from the previous relation by expanding in the parameter \( t \) and by taking the \( p \)-th coefficient, so that

\[
\langle |T|^2 \rangle_p := \frac{1}{\mathcal{Z}_p} \frac{\langle (\bar{\psi}, J\psi)^{p-1} \rangle_{t=0}}{(p - 1)!}.
\]

(4.8)

The very same method of the preceding Section can be used to
evaluate this quantity. Still in a mean-field description, we have:

\[
\langle \langle \bar{\psi}, J\psi \rangle U^{p-1} \rangle_{t=0} =
\]

\[
= \int D_n(\bar{\psi}, \psi) \langle \bar{\psi}, J\psi \rangle U^{p-1} \exp \left\{ n \frac{(\bar{\psi}\psi)^{k-1}}{(k-1)!} \right\} \left[ 1 - (\bar{\psi}, J\psi) \frac{(\bar{\psi}\psi)^{k-2}}{(k-2)!} \right]
\]

\[
= \int D_n(\bar{\psi}, \psi) \bar{\psi}\psi U^{p-1} \exp \left\{ n \frac{(\bar{\psi}\psi)^{k-1}}{(k-1)!} \right\}
\]

\[
= n! \int \frac{d\xi}{2\pi i} \frac{1}{\xi^{n+1}} \xi \left[ \xi + (1-k) \frac{\xi^{k-1}}{k!} \right] \exp \left\{ n \frac{\xi^{k-1}}{(k-1)!} \right\}
\]

\[
= n! \left[ \frac{k-1}{k!} \right]^{\frac{1}{1-\alpha}} \int \frac{dn}{2\pi i} \tilde{A}(\eta) \ e^{nB(\eta)},
\]

where now

\[
\tilde{A}(\eta) = \frac{1}{\eta (1-\eta)},
\]

and \( B(\eta) \) is the same as before. To evaluate this integral we again use the saddle point method. Please note that since the function \( B(\eta) \) is unchanged so are the saddle points.

Using the general expansion for \( p = \alpha n \) \( (3.18) \) we have

\[
\langle |T|^2 \rangle_{\alpha n} \sim \frac{1}{Z_{\alpha n}} \frac{\langle \langle \bar{\psi}, J\psi \rangle U^{\alpha n-1} \rangle_{t=0}}{\Gamma(\alpha n)} = an \frac{\tilde{A}(\eta^*) + \frac{1}{n} \tilde{C}(\eta^*) + O(\frac{1}{n})}{\tilde{A}(\eta^*) + \frac{1}{n} \tilde{C}(\eta^*) + O(\frac{1}{n})}.
\]

Now in the low temperature phase we have \( A(\eta_a) = 0 \) so in order to get the leading term we need \( C(\eta^*) \) and as

\[
\tilde{A}(\eta_a) = \frac{k^2}{k-1} \quad \text{and} \quad C(\eta_a) = \frac{\alpha(k-1)}{(\alpha - \alpha_c)^2},
\]

\( (4.15) \) at leading order gives

\[
\langle |T|^2 \rangle_{\alpha n} \sim an^2 \frac{\tilde{A}(\eta_a)}{C(\eta_a)} = n^2 \left( \frac{\alpha_c - \alpha}{\alpha_c} \right)^2 \]

so that, as soon as \( \alpha < \alpha_c \), a giant hypertree appears in the typical forest, which occupies on average a fraction \( 1 - \alpha/\alpha_c \) of the whole graph. In the high temperature instead we have

\[
\tilde{A}(\eta_h) = \frac{1}{\alpha(1-\alpha)} \quad \text{and} \quad A(\eta_h) = \frac{k \alpha - \alpha_c}{1 - \alpha},
\]

\( (4.18) \) giving (always at leading order)

\[
\langle |T|^2 \rangle_{\alpha n} \sim an \frac{\tilde{A}(\eta_h)}{A(\eta_h)} = \frac{n}{k} \frac{1}{\alpha - \alpha_c}.
\]

(4.19)
So that

\[
\lim_{n \to \infty} \frac{1}{n^2} \langle |T|^2 \rangle_{\alpha n} = \begin{cases} 
\frac{(\alpha - \alpha_c)^2}{\alpha_c} & \text{for } \alpha \leq \alpha_c \\
0 & \text{for } \alpha \geq \alpha_c
\end{cases}
\] (4.20)

is an order parameter, but it is represented as the expectation value of a non-local operator. We shall see in the next Section how to construct a local order parameter.

5 The symmetry breaking

In this section we will describe the phase transition in terms of the breaking of the global osp(1|2) supersymmetry. According to the general strategy (see for example [39]) let’s add an exponential weight with an external source \( h \) coupled to the variation of the fields (2.10) and (2.11):

\[
h \sum_{i \in V} (1 - t \bar{\psi}_i \psi_i) = h (n - t \bar{\psi}\psi),
\] (5.1)

The partition function becomes now:

\[
Z(t,h) = \int D_n(\bar{\psi},\psi) e^{-\mathcal{H}[\psi,\bar{\psi}] - h(n-t\bar{\psi}\psi)}.
\] (5.2)

We have chosen to add the exponential weight with a minus sign because in this way when \( t \) is sent to zero with the product \( ht \) kept fixed, we get, aside from a vanishing trivial factor, the generating function of rooted hyperforests.

More generally, for finite \( t \) and \( h \), we have that \( Z(t,h) \) can be expressed as a sum over spanning hyperforests with a modified weight

\[
Z(t,h) = \sum_{F \in \mathcal{F}} \prod_{T \in F} t e^{-h|T|} (1 + h |T|)
\] (5.3)

which is always positive, at any \( n \), only for \( h \geq 0 \).

On the \( k \)-uniform complete hypergraph the partition function (5.2) is expressed

\[
Z(t,h) = \int D_n(\bar{\psi},\psi) \exp \left\{ t \mathcal{U} + ht \bar{\psi}\psi \right\}
\exp \left\{ -nh + n \frac{(\bar{\psi}\psi)^{k-1} \cdot (k-2)! / (k-1)!} {1 - (\bar{\psi}\psi)^{k-1} \cdot (k-2)! / (k-1)!} \right\}
\] (5.4)

To work in the micro-canonical ensemble we again expand in powers of \( t \)

\[
Z(t,h) = \sum_{p=0}^{n} Z_p(h) t^p,
\] (5.5)

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where each term of the above expansion gives the partition function at fixed number of components:

\[
Z_p(h) = \frac{1}{p!} \int D_n(\tilde{\psi}, \psi) \left( U + h\tilde{\psi}\psi \right)^p \\
\exp \left\{ -nh + n \frac{(\tilde{\psi}\psi)^{k-1}}{(k-1)!} \right\} \left[ 1 - \frac{(\tilde{\psi}\psi)^{k-1}}{(k-2)!} \right] (5.6)
\]

\[
= e^{-nh} \frac{1}{p!} \langle (U + h\tilde{\psi}\psi)^p \rangle_{t=0}. (5.7)
\]

Following the very same steps of the previous section, we can write this expression in terms of a complex integral:

\[
Z_p(h) = \frac{n!}{\Gamma(\alpha n + 1)} \left[ \frac{k-1}{k!} \right]^{\frac{1}{k-1-n}} I(\alpha, h) (5.8)
\]

with

\[
I(\alpha, h) := \oint \frac{d\eta}{2\pi i} A(\eta) e^{nB(\eta, h)} (5.9)
\]

where \(A(\eta)\) is the same as in (3.16) while

\[
B(\eta, h) := -h + \frac{k}{k-1} \eta + \alpha \log(1 + h - \eta) + \frac{\alpha - 1}{k-1} \log \eta. (5.10)
\]

\(I(\alpha, h)\) can be again evaluated with the same technique as above. Let’s call \(\eta^*(h)\) the position of the relevant saddle point, which is the accessible solution of the saddle point equation

\[
\frac{\partial}{\partial \eta} B(\eta, h) \bigg|_{\eta=\eta^*(h)} = 0. (5.11)
\]

If \(h > 0\) the two solutions are real valued and distinct for every value of \(\alpha\) and the accessible saddle is simple and turns out to be always the one closer to the origin.

In the following we are going to consider all the functions \(A, B, C\) and \(D\) as evaluated on \(\eta^*(h)\) and therefore as functions of the single parameter \(h\).

\[
A(h) \equiv A(\eta^*(h)) \quad B(h) \equiv B(\eta^*(h), h) (5.12)
\]

\[
C(h) \equiv C(\eta^*(h), h) \quad D(h) \equiv D(\eta^*(h), h) (5.13)
\]

The asymptotic behaviour of (5.8) is given by the general formula (3.18):

\[
Z_{\alpha n}(h) \propto \frac{e^{nB(h)}}{\sqrt{2\pi nB'(h)}} \left[ A(h) + \frac{C(h)}{n} + O\left( \frac{1}{n^2} \right) \right]. (5.14)
\]

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The density of entropy is obtained by taking the logarithm of the partition function $Z_{\alpha n}(h)$

$$s(\alpha, h) := \frac{1}{n} \log \frac{Z_{\alpha n}(h)}{n!}.$$  \hspace{1em} (5.15)

The magnetization is then the first derivative of the entropy

$$m(\alpha, h) = -\frac{\partial s}{\partial h} = 1 - \alpha \frac{\langle \bar{\psi} \psi (U + h \bar{\psi} \psi)^{\alpha n-1} \rangle_{t=0}}{\langle (U + h \bar{\psi} \psi)^{\alpha n} \rangle_{t=0}}$$

$$= 1 - \alpha n \frac{\langle \bar{\psi} \psi (U + h \bar{\psi} \psi)^{\alpha n-1} \rangle_{t=0}}{\langle (U + h \bar{\psi} \psi)^{\alpha n} \rangle_{t=0}}$$  \hspace{1em} (5.16)

which is written as the expectation of a local operator, and if we set $h = 0$ in this formula we get

$$m(0) = 1 - \frac{1}{Z_{\alpha n}(0)} \frac{\langle \bar{\psi} \psi U^{\alpha n-1} \rangle_{t=0}}{\Gamma(\alpha n)} = 0$$  \hspace{1em} (5.18)

because of (4.4). In order to evaluate first the limit of large number of vertices we use the asymptotic expression for $Z_{\alpha n}(h)$ to get

$$m(\alpha, h) = -\frac{\partial B(h)}{\partial h} + \frac{1}{2n} \frac{1}{B''(h)} \frac{\partial B''(h)}{\partial h}$$

\[ - \frac{1}{n} \frac{\partial A(h)}{\partial h} + \frac{1}{n} \frac{\partial C(h)}{\partial h} + O\left(\frac{1}{n^2}\right) \]  \hspace{1em} (5.19)

The vanishing of $A(0)$ in the low temperature phase ($\alpha < \alpha_c$) has the consequence that the two limits $n \to \infty$ and $h \to 0$ do not commute, indeed:

$$\lim_{n \to \infty} \lim_{h \to 0} m(\alpha, h) = -\frac{\partial B(h)}{\partial h} \bigg|_{h=0} - \frac{1}{C(0)} \frac{\partial A(h)}{\partial h} \bigg|_{h=0} = 0$$  \hspace{1em} (5.20)

$$\lim_{h \to 0} \lim_{n \to \infty} m(\alpha, h) = -\frac{\partial B(h)}{\partial h} \bigg|_{h=0} = \frac{\alpha_c - \alpha}{\alpha_c} \geq 0.$$  \hspace{1em} (5.21)

Remark that the magnetization $m$ vanishes at the critical point linearly and not with critical exponent $1/2$ as it is common in mean-field theory, the reason being that here the order parameter is not linear but quadratic in the fundamental fields.

In the high temperature phase $A(0) \neq 0$ and the two limits above commute.

$$\lim_{n \to \infty} \lim_{h \to 0} m(\alpha, h) = \lim_{h \to 0} \lim_{n \to \infty} m(\alpha, h) = -\frac{\partial B(h)}{\partial h} \bigg|_{h=0} = 0.$$  \hspace{1em} (5.22)
In the study of phase transitions the thermodynamical limit \( n \to \infty \) has to be taken first. Indeed, the ergodicity is broken in the thermodynamical limit first and then a residual spontaneous magnetization appears even when the external field vanishes. Remark that both the free energy and the magnetization vary continuously passing from one phase to the other.

The longitudinal susceptibility \( \chi_L \)

\[
\chi_L(\alpha, h) = \frac{\partial^2 s(\alpha, h)}{\partial h^2} = \alpha (\alpha n - 1) \frac{((\bar{\psi}\psi)(U + h\bar{\psi}\psi)_{\alpha n-2})_{t=0}}{((U + h\bar{\psi}\psi)_{\alpha n})_{t=0}} - n [1 - m(\alpha, h)]^2 \tag{5.23}
\]

can be obtained from the magnetization:

\[
\chi_L(\alpha, h) = -\frac{\partial m(\alpha, h)}{\partial h} = \frac{\partial^2 B(h)}{\partial h^2} + \frac{1}{2n} \frac{1}{B''(h)} \left( \frac{\partial B''(h)}{\partial h} \right)^2 - \frac{1}{2n} \frac{1}{B''(h)} \frac{\partial^2 B''(h)}{\partial h^2} + \frac{1}{n} \frac{\partial^2 A(h)}{\partial h^2} + \frac{1}{n} \frac{\partial^2 C(h)}{\partial h^2} + O\left(\frac{1}{n^2}\right) \tag{5.24}
\]

and taking the two limits in the appropriate order we get

\[
\lim_{h \to 0} \lim_{n \to \infty} \chi_L(\alpha, h) = \frac{\partial^2 B(h)}{\partial h^2} \bigg|_{h=0} = \begin{cases} 
\frac{-\alpha(1-\alpha)}{\alpha_c} \left( \frac{\alpha-\alpha_c}{\alpha_c} \right)^{-1} & \alpha < \alpha_c \\
\frac{-1-\alpha_c}{\alpha_c} \left( \frac{\alpha-\alpha_c}{\alpha_c} \right)^{-1} & \alpha > \alpha_c 
\end{cases} \tag{5.26}
\]

which shows that the susceptibility is discontinuous at the transition, with a singularity \( \chi(\alpha) \sim |\alpha - \alpha_c|^{-1} \), so that the transition is second order. Remark that the longitudinal susceptibility appears to be negative. This means that in our model of spanning hyperforest there are events negatively correlated. It is well known that in the model of spanning trees on a finite connected graph the indicator functions for the events in which an edge belongs to the tree are negatively correlated. This is proven by Feder and Mihail [40] in the wider context of balanced matroids (and uniform weights). See also [41] for a purely combinatorial proof of the stronger Raileigh condition, in the weighted case. The random cluster model for \( q > 1 \) is known to be positive associated. When \( q < 1 \) negative association is conjectured.
to hold. For an excellent description of the situation about negative
association see [42].

Still following the analogy with magnetic systems, let us introduce
the transverse susceptibility

\[ \chi_T(\alpha, h) := \frac{2}{Z_{\alpha n}} \left\langle \frac{1}{n \Gamma(\alpha n)} \langle \bar{\psi} J \psi \rangle_{U_{\alpha n}^t=0} \right\rangle \]  

(5.27)

which, by comparison with (4.15), provides, at \( h = 0 \)

\[ \chi_T(\alpha, 0) = \frac{2}{n} \left\langle |T|^2 \right\rangle_{\text{an}}. \]  

(5.28)

In Appendix B we prove the identity

\[ m(\alpha, h) = h \chi_T(\alpha, h). \]  

(5.29)

This relation is the bridge between the average square-size of hyper-
trees and the local order parameter.

At finite \( n \), when the symmetry-breaking field \( h \) is set to zero, we
get

\[ m(\alpha, 0) = 0 \]  

(5.30)

in agreement with formula (4.4) and

\[ \chi_T(\alpha, 0) = \lim_{h \to 0} \frac{2 m(\alpha, h)}{h} = 2 \left. \frac{\partial m(\alpha, h)}{\partial h} \right|_{h=0} = -2 \chi_L(\alpha, 0) \]  

(5.31)

which should be compared with the analogous formula for the \( O(N) \)-
model where it is

\[ \chi_T(\alpha, 0) = (N - 1) \chi_L(\alpha, 0) \]  

(5.32)

and, in our case, as the symmetry is \text{osp}(1|2), N should be set to \(-1\)
as we have one bosonic direction and two fermionic ones which give a
negative contribution.

The leading \( n \) contribution is

\[ \frac{1}{2} \chi_T(\alpha, 0) = - \chi_L(\alpha, 0) = \begin{cases} \frac{1}{2} \alpha_c - \alpha & \text{for } \alpha \leq \alpha_c \\ \frac{1}{2} \alpha - \alpha_c & \text{for } \alpha \geq \alpha_c \end{cases} \]  

(5.33)

But, for \( \alpha \leq \alpha_c \), if we first compute the large \( n \) limit and afterwards
send \( h \to 0 \), we know that we get a non-zero magnetization and therefore
the transverse susceptibility diverges as

\[ \chi_T(\alpha, 0) \sim 2 \frac{m(\alpha, 0)}{h} = \frac{2}{h} \frac{\alpha_c - \alpha}{\alpha_c}. \]  

(5.34)
which corresponds to the idea that there are massless excitations, Goldstone modes associated to the symmetry breaking. Remark that, at finite $h$, the transverse susceptibility does not increase with $n$, which shows that the average square-size of hypertrees stays finite.

The longitudinal susceptibility instead

$$\chi_L(\alpha,0) \sim -\frac{1}{m(\alpha,0)} \frac{\alpha (1 - \alpha)}{\alpha_c^2} \ .$$  \hspace{1cm} (5.35)

diverges only at $\alpha = \alpha_c$, when the magnetization vanishes.

6 A symmetric average

At the breaking of an ordinary symmetry the equilibrium states can be written as a convex superposition of pure, clustering, states, which cab be obtained, one from the other, by applying the broken symmetry transformations. The pure state we have defined in this Section uses a breaking field in the only direction we have at disposal where the Grassmann components are null. A more general breaking field would involve a direction in the superspace to which we are unable to give a combinatorial meaning. However, if we take the average in the invariant Berezin integral of these fields we give rise to a different, non-pure but symmetric, low-temperature state.

In this Section we shall set $t = 1$.

The most general breaking field, with total strength $h$, but arbitrary direction in the super-space, would give a weight

$$h \sum_{i=1}^{n} \left[ \lambda (1 - \bar{\psi}_i \psi_i) + \bar{\epsilon} \psi_i + \psi_i \epsilon \right]$$  \hspace{1cm} (6.1)

where $(\lambda; \bar{\epsilon}, \epsilon)$ is a unit vector in the $1|2$ supersphere, i.e. $\epsilon$ and $\bar{\epsilon}$ are Grassmann coordinates and $\lambda$ is a formal variable satisfying the constraint

$$\lambda^2 + 2 \bar{\epsilon} \epsilon = 1.$$ 

Let us introduce the normalized generalized measure

$$d\Omega := d\lambda d\epsilon d\bar{\epsilon} \delta \left( \lambda^2 + 2 \bar{\epsilon} \epsilon - 1 \right) \ .$$  \hspace{1cm} (6.2)

A symmetric equilibrium measure can be constructed by considering
the factor

\[ F[h; \bar{\psi}, \psi] := \quad (6.3) \]

\[ = \int d\Omega \exp \left\{ -h \sum_{i=1}^{n} \left[ \lambda (1 - \bar{\psi}_i \psi_i) + \bar{\epsilon}_i \psi_i + \bar{\psi}_i \epsilon_i \right] \right\} \quad (6.4) \]

\[ = \int d\epsilon d\bar{\epsilon} \exp \left\{ \bar{\epsilon} \epsilon - h \sum_{i=1}^{n} \left[ (1 - \bar{\epsilon} \epsilon)(1 - \bar{\psi}_i \psi_i) + \bar{\epsilon}_i \psi_i + \bar{\psi}_i \epsilon_i \right] \right\} \quad (6.5) \]

\[ = \left[ 1 - h^2 (\bar{\psi}, J\psi) + h (n - \bar{\psi} \psi) \right] \exp \left[ -h (n - \bar{\psi} \psi) \right] . \quad (6.6) \]

where only the last expression is specific to our model, but the previous are the appropriate expressions for the model of unrooted spanning hyperforests on an arbitrary weighted hypergraph. This function is symmetric, for every strength \( h \), as it can be easily checked that

\[ Q_{\pm} F = 0 . \quad (6.7) \]

If we send \( h \to 0 \) this factor is simply 1, but if we first take the \( n \to \infty \) limit and then \( h \to 0 \) the expectation value of non-symmetric observables can be different.

The partition function is not changed because of the identity \[(6.7)\]. Indeed

\[ \langle F \rangle = \langle \exp \left[ -h (n - \bar{\psi} \psi) \right] \rangle = \langle 1 \rangle \quad (6.8) \]

for un-normalized expectation values, because of the relation between the transverse susceptibility and the magnetization, equation \[(6.7)\], which is

\[ 0 = \langle \left[ -h^2 (\bar{\psi}, J\psi) + h (n - \bar{\psi} \psi) \right] \exp \left[ -h (n - \bar{\psi} \psi) \right] \rangle \quad (6.9) \]

\[ = h \langle \left[ -h (\bar{\psi}, J\psi) + (n - \bar{\psi} \psi) \right] \rangle \quad (6.10) \]

for every \( h \), and therefore also for the derivatives with respect to \( h \). But consider for example the magnetization. The insertion of the given factor \( F \) in the un-normalized expectation provides the relation

\[ \langle (n - \bar{\psi} \psi) \rangle^{\text{un}}_{h} := \langle (n - \bar{\psi} \psi) F \rangle = \langle (n - \bar{\psi} \psi) \rangle_{h} + \quad (6.11) \]

\[ + \langle \left[ -h^2 (\bar{\psi}, J\psi) + h (n - \bar{\psi} \psi) \right] \left( -\frac{\partial}{\partial h} \right) \exp \left[ -h (n - \bar{\psi} \psi) \right] \rangle \quad (6.12) \]

\[ = 2 \langle (n - \bar{\psi} \psi) \rangle_{h} - 2 h \langle (\bar{\psi}, J\psi) \rangle_{h} = 0 \quad (6.13) \]
Similarly
\[
\langle (\bar{\psi}, J\psi)^{\text{sym}} \rangle_h = \langle (\bar{\psi}, J\psi) F \rangle
\]
\[
= \langle (\bar{\psi}, J\psi) \rangle_h - h \frac{\partial}{\partial h} \langle (\bar{\psi}, J\psi) \rangle_h
\]
\[
= \langle (\bar{\psi}, J\psi) \rangle_h - h \frac{\partial}{\partial h} \frac{\langle (n - \bar{\psi}\psi) \rangle_h}{h}
\]
\[
= \langle (\bar{\psi}, J\psi) \rangle_h + \left( \frac{1}{h} - \frac{\partial}{\partial h} \right) \langle (n - \bar{\psi}\psi) \rangle_h
\]
\[
= 2 \langle (\bar{\psi}, J\psi) \rangle_h + \langle (n - \bar{\psi}\psi)^2 \rangle_h
\]
\[
= \sum_{i,j} \langle \bar{\psi}_i \psi_j + \bar{\psi}_j \psi_i + (1 - \bar{\psi}_i \psi_i)(1 - \bar{\psi}_j \psi_j) \rangle_h
\]
\[
= \sum_{i,j} \langle 1 - f_{(i,j)} \rangle_h
\]
is the total, not-connected, susceptibility, that is the sum of the longitudinal and transverse not-connected ones. And also
\[
\langle (n - \bar{\psi}\psi)^{2\text{sym}} \rangle_h = -2 \langle (\bar{\psi}, J\psi) \rangle_h - \langle (n - \bar{\psi}\psi)^2 \rangle_h
\]
so that
\[
2 \langle (\bar{\psi}, J\psi)^{\text{sym}} \rangle_h + \langle (n - \bar{\psi}\psi)^{2\text{sym}} \rangle_h = 2 \langle (\bar{\psi}, J\psi) \rangle_h + \langle (n - \bar{\psi}\psi)^2 \rangle_h
\]
as it must occur for a symmetric observable.

7 Conclusions

We have found that in the \( k \)-uniform complete hypergraph with \( n \) vertices, in the limit of large \( n \), the structure of the hyperforests with \( p \) hypertrees has an abrupt change when \( p = \alpha_c n \) with \( \alpha_c = (k - 1)/k \). This change of behaviour is related to the appearance of a giant hypertree which covers a finite fraction of all the vertices. As the number of hyperedges in the hyperforests with \( p \) hypertrees is \( (n - p)/(k - 1) \), this means that this change occurs when the number of hyperedges becomes \( 1/k(k - 1) \), which is exactly the critical number of hyperedges in the phase transition of random hypergraphs at fixed number of hyperedges [26].

If \( Z(t) \) is the generating partition function of hyperforests, where the coefficient of \( t^p \) is the total number of those with \( p \) hypertrees, in the limit of large \( n \) there is a corresponding singularity at \( t_c = n \left[ (k - 2)! \right]^{-1/(k-1)} \).

In our Grassmann formulation this singularity can be described as a second-order phase transition associated to the breaking of a
global osp(1|2) supersymmetry which is non-linearly realised. The equilibrium state occurring in the broken phase can be studied by the introduction of an explicit breaking of the supersymmetry.

A  Saddle-point constants

Let us use the notation

\[ X(n) := \frac{\partial^n}{\partial \eta^n} X(\eta) \bigg|_{\eta=\eta^*}. \]

For the simple saddle point we report the combinations \( C \) and \( D \) in terms of functions \( A \) and \( B \)

\[
C = \frac{1}{24B^3_{(2)}} \left[ 12A_{(1)}B_{(2)}B_{(3)} - 12A_{(2)}B^2_{(2)} + A \left( 3B_{(2)}B_{(4)} - 5B^2_{(3)} \right) \right]
\]

\[
D = \frac{1}{1152B^3_{(2)}} \left[ 385 A B^4_{(3)} + 144 B^4_{(2)}A_{(4)} - 210 B_{(2)}B^2_{(3)} \left( 4A_{(1)}B_{(3)} + 3AB_{(4)} \right) + 21 B^2_{(2)} \left( 40A_{(2)}B^2_{(3)} + 40A_{(1)}B_{(4)}B_{(2)} + 5AB^2_{(3)} + 8AB_{(3)}B_{(4)} \right) - 24 B^3_{(2)} \left( 20A_{(3)}B_{(3)} + 15A_{(2)}B_{(4)} + 6A_{(1)}B_{(5)} + AB_{(6)} \right) \right].
\] (A.1)

For the double saddle points the necessary combinations are instead

\[
\tilde{C} = \frac{AB_{(4)} \gamma_4}{B^4_{(3)} 4!} - \frac{A_{(1)}}{B^4_{(3)} 3!} \gamma_1
\]

\[
\tilde{D} = -\frac{A_{(3)} \gamma_3}{B^3_{(3)} 3!} + \frac{A_{(2)}B_{(4)} \gamma_6}{B^2_{(3)} 2!} + \frac{A_{(1)}B_{(5)} \gamma_6}{B^2_{(3)} 5!} - \frac{A_{(1)}B^2_{(4)} \gamma_9}{B^3_{(3)} 2! 4!} - \frac{AB_{(4)} B_{(5)} \gamma_9}{4! 5!} + \frac{AB^3_{(4)} \gamma_{12}}{B^4_{(3)} 3! (4!)^3} + \frac{AB_{(6)} \gamma_6}{B^2_{(3)} 6!}.
\] (A.2)

The constants \( \gamma_k \) are given by

\[
\gamma_k := -\frac{1}{\pi} \sin \left( 2\pi \frac{1+k}{3} \right) \int_0^\infty du u^k e^{-\frac{u^3}{\pi}} \]

\[
= -\frac{(3!)^{\frac{1+k}{3}}}{3\pi} \sin \left[ 2\pi \left( \frac{1+k}{3} \right) \right] \Gamma \left( \frac{1+k}{3} \right). \] (A.5)
B Ward identities

As a result of the underlying symmetry, there are relations among the correlation functions, called Ward identities [39]. In this Appendix we give a more direct derivation of one of them which simply uses integration by parts.

By definition
\[ \mathcal{U}(\xi) = \xi + (1 - k) \frac{\xi^k}{k!} \quad (B.1) \]
so that
\[ \frac{\partial \mathcal{U}}{\partial \xi} = 1 - \frac{\xi^{k-1}}{(k-2)!} \quad (B.2) \]
and therefore the un-normalized expectation value of \( \bar{\psi} \psi \) in presence of the symmetry breaking is
\[ t \langle \bar{\psi} \psi \rangle_h = \]
\[ = n! \int \frac{d\xi}{2\pi i} \frac{1}{\xi^{n+1}} \exp \left\{ t \mathcal{U} + n \frac{\xi^{k-1}}{(k-1)!} - h t (n - \xi) \right\} t \xi \frac{\partial \mathcal{U}}{\partial \xi} \]
\[ = n! \int \frac{d\xi}{2\pi i} \frac{1}{\xi^{n+1}} \exp \left\{ n \frac{\xi^{k-1}}{(k-1)!} - h t (n - \xi) \right\} \xi \frac{\partial}{\partial \xi} \exp \{ t \mathcal{U} \}. \quad (B.3) \]

Perform now an integration by parts
\[ t \langle \bar{\psi} \psi \rangle_h = \]
\[ = n! \int \frac{d\xi}{2\pi i} \frac{1}{\xi^{n+1}} \left[ n \left( 1 - \frac{\xi^{k-1}}{(k-2)!} \right) - h t \xi \right] \]
\[ \exp \left\{ t \mathcal{U} + n \frac{\xi^{k-1}}{(k-1)!} - h t (n - \xi) \right\} \]
\[ = n \mathcal{Z}(t, h) - h t \langle \bar{\psi}, J \psi \rangle_h. \quad (B.4) \]

So that
\[ n \mathcal{Z}(t, h) - t \langle \bar{\psi} \psi \rangle = h t \langle \bar{\psi}, J \psi \rangle \]
which expanded in series of \( t \) implies that
\[ \frac{n}{p} \langle (\mathcal{U} + h \bar{\psi} \psi)^p \rangle_{t=0} - \langle \bar{\psi} \psi (\mathcal{U} + h \bar{\psi} \psi)^{p-1} \rangle_{t=0} =
\[ h \langle \bar{\psi}, J \psi \rangle (\mathcal{U} + h \bar{\psi} \psi)^{p-1} \rangle_{t=0} \quad (B.5) \]
or for \( p = \alpha n \)
\[ 1 - \alpha \frac{\langle \bar{\psi} \psi (\mathcal{U} + h \bar{\psi} \psi)^{\alpha n-1} \rangle_{t=0}}{\langle (\mathcal{U} + h \bar{\psi} \psi)^{\alpha n} \rangle_{t=0}} =
\[ \alpha h \frac{\langle \bar{\psi}, J \psi \rangle (\mathcal{U} + h \bar{\psi} \psi)^{\alpha n-1} \rangle_{t=0}}{\langle (\mathcal{U} + h \bar{\psi} \psi)^{\alpha n} \rangle_{t=0}} \quad (B.6) \]
which means that, in the microcanonical ensemble, for every $h$, we have

$$m(\alpha, h) = \frac{h}{2} \chi_T(\alpha, h). \quad (B.10)$$

References

[1] R. B. Potts, Some generalized order-disorder transformations, *Proceedings of the Cambridge Philosophical Society* **48**, 106 (1952).

[2] F. Y. Wu, The Potts model, *Review of Modern Physics* **54**, 235 (1982) and **55**, 315 (1983).

[3] F. Y. Wu, Potts model of ferromagnetism, *Journal of Applied Physics* **55**, 2421 (1984).

[4] P. W. Kasteleyn and C. M. Fortuin, Phase transitions in lattice systems with random local properties, *Journal of the Physical Society of Japan* **26** (Suppl.), 11 (1969).

[5] C. M. Fortuin and P. W. Kasteleyn, On the random-cluster model. I. Introduction and relation to other models, *Physica* **57**, 536 (1972).

[6] G. Grimmett, *The random cluster model*, Springer, New York, 2006.

[7] S. Caracciolo, J. L. Jacobsen, H. Saleur, A. D. Sokal, and A. Sportiello, Fermionic field theory for trees and forests, *Physical Review Letters* **93**, 080601 (2004) [arxiv:cond-mat/0403271].

[8] S. Caracciolo, A. D. Sokal, and A. Sportiello, Grassmann integral representation for spanning hyperforests, *Journal of Physics A: Mathematical and Theoretical* **40**, 13799 (2007) [arXiv:0706.1509].

[9] A. Bedini, S. Caracciolo, and A. Sportiello, Hyperforests on the complete hypergraph by Grassmann integral representation, *Journal of Physics A: Mathematical and Theoretical* **41**, 205003 (2008) [arXiv:0802.1506].

[10] G. Grimmett, Potts models and random-cluster processes with many-body interactions, *Journal of Statistical Physics* **75**, 67 (1994).

[11] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Dover Publications, 2007.

[12] B. Nienhuis, Coulomb gas formulation of 2D phase transitions, in *Phase Transitions and Critical Phenomena*, vol. 11, edited by C. Domb and J. Lebowitz, Academic Press, 1987.
[13] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal field theory*. Springer, New York, 1997.

[14] V. A. Kazakov, Exactly solvable Potts models, bond- and tree-like percolation on dynamical (random) planar lattice, *Nuclear Physics B (Proc. Suppl.)* 4, 93 (1988).

[15] P. Zinn-Justin, The dilute Potts model on random surfaces, *Journal of Statistical Physics* 98, 245 (2001) [arXiv:cond-mat/9903385].

[16] B. Eynard and G. Bonnet, The Potts-$q$ random matrix model: loop equations, critical exponents, and rational case, *Physics Letters B* 463, 273 (1999) [arXiv:hep-th/9906130].

[17] S. N. Majumdar and D. Dhar, Equivalence between the Abelian sandpile model and the $q \to 0$ limit of the Potts model, *Physica A* 185, 129 (1992).

[18] S. Caracciolo, C. De Grandi and A. Sportiello, Renormalization flow for unrooted forests on a triangular lattice, *Nuclear Physics B* 787, 260 (2007) [arXiv:0705.3891].

[19] J. L. Jacobsen and H. Saleur, The arboreal gas and the supersphere sigma model, *Nuclear Physics B* 716, 439 (2005) [arXiv:cond-mat/0502052].

[20] S. Caracciolo and A. Sportiello, Spanning forests on random planar lattices, *Journal of Statistical Physics*, (2009) doi:10.1007/s10955-009-9733-1 [arXiv:0903.4432].

[21] Y. Deng, T. M. Garoni, and A. D. Sokal, Ferromagnetic phase transition for the spanning-forest model ($q \to 0$ limit of the Potts model) in three or more dimensions, *Physical Review Letters* 98, 030602 (2007) [arXiv:cond-mat/0610193].

[22] L. Lovász, M. D. Plummer, *Matching Theory*, Annals of Discrete Mathematics 29, North-Holland, 1986.

[23] S. Caracciolo, G. Masbaum, A. D. Sokal and A. Sportiello, A randomized polynomial-time algorithm for the Spanning Hypertree Problem on 3-uniform hypergraphs, [arXiv:0812.3593].

[24] G. Masbaum and A. Vaintrob, A New Matrix-Tree Theorem, *International Mathematics Research Notices* 27, 1397 (2002).

[25] A. Abdesselam, Grassmann–Berezin calculus and theorems of the matrix-tree type, *Advances in Applied Mathematics* 33, 51 (2004) [arxiv:math.CO/0306396].

[26] J. Schmidt-Pruzan and E. Shamir, Component structure in the evolution of random hypergraphs, *Combinatorica* 5, 81 (1985).
[27] P. Erdős and A. Rényi, On the evolution of random graphs, *Publications of the Mathematical Institute of the Hungarian Academy of Sciences* 5, 16 (1960).

[28] B. Bollobás, The evolution of random graphs. *Transactions of the American Mathematical Society* 286, 257 (1984).

[29] B. Bollobás, *Random Graphs*, Academic Press, London, 1985.

[30] S. Janson, D. E. Knuth, T. Łuczak and B. Pittel, The birth of the giant component, *Random Structures and Algorithms* 4, 233 (1993).

[31] S. Janson, T. Łuczak and A. Ruciński, *Random Graphs*, Wiley, New York, 2000.

[32] M. Karoński and T. Łuczak, The phase transition in a random hypergraph, *Journal of Computational and Applied Mathematics* 142, 125 (2002).

[33] B. Bollobás, G. Grimmett, and S. Janson, The random-cluster model on the complete graph, *Probability Theory and Related Fields* 104, 283 (2005).

[34] L. Mittag, and M. J. Stephen, Mean-field theory of the many component Potts model, *Journal of Physics A: Math. Nucl. Gen.* 7, L109 (1974).

[35] A. Baracca, M. Bellesi, R. Livi, R. Rechtman, and S. Ruffo, On the mean field solution of the Potts model, *Physics Letters* 99 A, 160 (1983).

[36] P. Flajolet and R. Sedgewick, *Analytical Combinatorics*, Cambridge University Press, 2009.

[37] V. F. Kolchin, *Random graphs*, Encyclopedia of Mathematics and its Applications. vol 53, Cambridge University Press, 1999.

[38] G. Parisi and N. Sourlas, Self avoiding walk and supersymmetry, *Journal de Physique Lettres* 41, L403 (1980).

[39] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 3rd ed., Clarendon Press, Oxford, 1996.

[40] T. Feder and M. Mihail, Balanced Matroids, *Proceedings 24th Annual STOC*, 26 (1992).

[41] J. Cibulka, J. Hladky, M. A. La Croix and D. G. Wagner, A combinatorial proof of Rayleigh monotonicity for graphs, [arXiv:0804.0027].

[42] R. Pemantle, Towards a theory of negative dependence, *Journal of Mathematical Physics* 41, 1371 (2000) [arXiv:math/0404095].

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