The dynamical origin of opinion polarization in the real world is an interesting topic physical scientists may help to understand. To properly model the dynamics, the theory must be fully compatible with findings by social psychologists on microscopic opinion change. Here we introduce a generic model of opinion formation with homogeneous agents based on the well-known social judgment theory in social psychology by extending a similar model proposed by Jager and Amblard. The agents’ opinions will eventually cluster around extreme and/or moderate opinions forming three phases in a two-dimensional parameter space that describes the microscopic opinion response of the agents. The dynamics of this model can be qualitatively understood by mean-field analysis. More importantly, first-order phase transition in opinion distribution is observed by evolving the system under a slow change in the system parameters, showing that punctuated equilibria in public opinion can occur even in a fully connected social network.

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I. INTRODUCTION

Opinion formation and evolution are interesting and important subject of research in social psychology. Many experiments and theories have been conducted and proposed [1, 2], including the elaboration likelihood model, the heuristic-systematic model and the cognitive dissonance theory. In particular, Sherif et al. proposed the well-known social judgment theory (SJT) [1–4] in the 1960’s to explain the microscopic behavior of how individuals evaluate and change their opinions based on interaction with others.

The basic idea of SJT is that attitude change of an individual is a judgmental process. According to SJT, describing the stand of an individual as a point in a continuum of possible opinions is not adequate because the individual’s degree of tolerance is also important in determining his/her response to external stimuli and persuasion [3, 4]. In particular, a presented opinion is acceptable (unacceptable) to a person if it is perceived to be sufficiently close to (far from) his/her own stand point. This presented opinion is said to be in his/her latitude of acceptance (rejection). A presented opinion is neither acceptable nor objectable if it is perceived to be neither close to or far from the individual’s own stand point. This opinion is said to be in his/her latitude of noncommitment. Clearly, these three latitudes differ from person to person and they depend on factors such as individual’s ego involvement and the person’s familiarity of the subject of discussion [1, 2]. When the presented opinion is in one’s latitude of acceptance (rejection) or perhaps also in the nearby latitude of noncommitment, assimilation (contrast) occurs in the sense that the presented opinion is perceived to be closer to (farther from) one’s stand point than it truly is. Moreover, this positively-evaluated (negatively-evaluated) opinion may cause the person to move his stand point towards (away from) it. The greater the difference between the individual’s and the presented opinions, the more the resultant attitude change in general. The phenomenon of moving away from the presented opinion through contrast is called the boomerang effect [1, 4]. The opinion change due to boomerang effect, however, is generally smaller than the opinion change induced by assimilation. Thus, not every psychological experiment unambiguously shows its existence [2], making it perhaps the most controversial part of the SJT. In fact, some social psychologists do not consider the boomerang effect to be one of the core thesis of SJT and some even cast doubt on its existence [1]. Here we adopt the view that the boomerang effect is one of the central themes of SJT whose effect, in general, is rather weak in comparison to the opinion change due assimilation. Finally, whenever the presented opinion is in the person’s latitude of noncommitment which is not close to his/her latitudes of acceptance or rejection, then there is little chance for him/her to change his/her mind. Consequently, the most effective method to successfully persuade an individual is to present the opinion near the boundary of his/her latitudes of acceptance and noncommitment [2]. And just like most theories in social science, the above findings should be interpreted in statistical sense rather than as definitive rules governing every single persuasion and discussion [1, 4]. Thoroughly studied and advanced by social psychologists, SJT is one of the most important theories in the field and is strongly supported by many psychological experiments especially concerning the latitudes of acceptance and noncommitment [2, 6].

Recently, physical scientists entered this field by studying the more macroscopic aspects of the problem such as opinion formation and evolution in a social network using simple models and computer simulations [7]. The variety
of models proposed include the use of discrete or continuous opinions, discrete or continuous time, homogeneous or heterogeneous agents, fully connected or more realistic social networks [7,27]. Of particular importance is the continuous opinion agent-based model in a fully connected network introduced by Defuant et al. (D-W Model) with the feature that players only have latitudes of acceptance and noncommitment so that only the effect of assimilation is considered [9,11]. The appeal of this model is that it can be simulated efficiently by computers and its dynamics can be qualitatively understood. This model is also consistent with the social psychologists’ finding that opinions can be reasonably well represented and measured as a continuum [12,28]. However, the absence of contrast and boomerang effect imply that D-W Model cannot be used to simulate opinion polarization in the real world in which opinions of the supporters of very different viewpoints become much more extreme.

Various modifications of the D-W Model have been proposed [15–21,23–27,29–31]. To account for opinion polarization, some modified this model by introducing inflexible or contrarian players [20,23,24,26,29–31], stochastic boomerang effect in the region of assimilation [19,25] and vector-valued opinions [25]. These models are not fully compatible with the SJT as the agents’ response in the latitude of rejection due to contrast are not properly treated. This is not ideal because in order to understand the macroscopic origin of opinion formation and polarization, one should combine the strengths of social psychology and physical science communities by introducing D-W-based models of opinion evolution whose rules are consistent with SJT. In fact, this approach is beginning to gain acceptance among social psychologists [32]. Actually, the only SJT-based models we are aware of are the ones proposed by Jager and Ambland (J-A Model) [15] and its recent extension by Crawford et al. [27] as well as the model of Huet et al. [16]. Jager and Ambland studied their model only by Monte Carlo simulation with very limited sample and agent sizes [15]. The work of Crawford et al. was more extensive, which included a simple analysis on eventual opinion distribution of the agents [27]. Note that both the models of Jager and Ambland [15] and Crawford et al. [27] involved agents with opinions on one issue only. In contrast, the model of Huet et al. [16] studied the response of agents based on their opinions on two issues by Monte Carlo simulation up to 5000 agents.

While these works [15,16,27] point to the right direction, we argue in Sec. [V] that the microscopic rules adopted in their models are questionable. Here we first proposed a minimalist SJT-based model of opinion formation by extending the J-A Model in Ref. [15]. This minimalist model is free of the questionable assumption implicitly used in Refs. [15,16,27]. Then in Sec. [VI] we report that our minimalist model is simple enough to be studied both semi-analytically and numerically, and at the same time refined enough to show opinion polarization even in the case of homogeneous agents.

By studying the agents’ dynamics in Sec. [VI], we can understand the process of opinion clustering. In particular, using a simple mean-field analysis, we find that the most important parameters to determine the formation of extreme opinion clusters as well as the coexistence of both extreme and moderate opinion clusters are the values of two parameters $d_1$ and $d_2$ to be defined in Sec. [II], which determine the widths of the regions for assimilation and boomerang effect to occur. Our analysis also shows that other factors such as network topology, agent’s heterogeneity, and the detailed response dynamics due to assimilation and boomerang effect chiefly affect the opinion formation timescales. More importantly, we find in Sec. [VI] that first-order phase transition in opinion clustering can occur occasionally when the widths of the assimilation and boomerang effect regions change very slowly. This shows that punctuated equilibrium in opinion distribution — the observation that opinion distribution change often comes in a short burst between a long period of stasis, a notion first pointed out by Gould and Eldredge [33] in evolution biology — can occur even in a fully connected network, repudiating one of the criticisms [34] to the punctuated equilibrium theory in social science [35]. Finally, we give a brief outlook in Sec. [VII].

II. THE MODEL

Just like the D-W Model [9,11] and the J-A Model [15], we consider a fixed connected network of $N$ agents each with a randomly and uniformly assigned initial opinion $x_i$ in a bounded interval, say, $[0,1]$. We call the opinions 0 and 1 extreme while those in between moderate. At each time step, we randomly pick two neighboring agents, say, $a$ and $b$, in the network and simulate their opinion changes after they meet and discuss by the following rules:

- **The assimilation rule:** If $|x_a - x_b| < d_1$, then $x_a$ and $x_b$ are simultaneously updated as

  $x_a \leftarrow x_a + \mu(x_b - x_a), \quad (1a)$

  $x_b \leftarrow x_b + \mu(x_a - x_b), \quad (1b)$

  where $\mu \in (0,0.5]$ is the convergence parameter.

- **The boomerang effect rule:** If $|x_a - x_b| \geq d_2$, then $x_a$ and $x_b$ are simultaneously updated as

  $x_a \leftarrow \mathcal{N}(x_a - \lambda(x_b - x_a)), \quad (2a)$

  $x_b \leftarrow \mathcal{N}(x_b - \lambda(x_a - x_b)), \quad (2b)$

  where $\lambda > 0$ is the divergence parameter, and

  $\mathcal{N}(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } x < 0, \\ 1 & \text{if } x > 1 \end{cases} \quad (3)$

  is the normalization function which maps extreme opinions back to the range $[0,1]$. 


• The neutral rule: The values of $x_a$ and $x_b$ do not change otherwise.

This serial opinion updating is repeated until the system is equilibrated.

Clearly, our model is well-defined if $d_2 \geq d_1$ and is compatible with the SJT with $d_1$ and $d_2$ reflecting the widths of the assimilation and boomerang effect regions, respectively. Furthermore, the rules are symmetric about $x = 1/2$. More importantly, our model is highly flexible. Adapting it to model heterogeneous agents (in which each has different values of $d_1$ and $d_2$), different opinion change dynamics (by modifying Eqs. (1)–(2) — something that we are going to do in Sec. IV below), and network topology are easy.

Our model is very different from that of Huet et al. since theirs is based on the repeated interaction of randomly picked pairs of agents whose responses are based on their opinion differences on two issues. Also, the most important difference between our model in the above form and the J-A Model model as well as its extension by Crawford et al. is that we use different convergence and divergence parameters $\mu$ and $\lambda$; while they set both to the same value. Their choice is not very natural since $d_1 \leq d_2$ would then imply the magnitude of opinion change due to boomerang effect must be greater than or equal to that due to assimilation, whose validity is not without doubt. Note that both groups used Monte Carlo simulations to study their models. In fact, Jager and Ambraud did not perform any analytical or semi-analytical study and Crawford et al. only carried out a basic mean-field analysis which focused mainly on the asymptotic behavior rather than the detailed opinion dynamics of the agents. In contrast, our detailed mean-field analysis in Sec. IV below shows that the dynamics of this type of models are so general that asymptotic behavior is very robust against any change in the assimilation and boomerang effect rules as well as the network topology provided that the average connectivity of the network is not too low.

III. SIMULATION RESULTS

We first present our findings for agents in a fully connected network with $\mu = 0.20$ and $\lambda = 0.05$. These values are chosen to reflect the reality that agents generally have to interact several times before becoming extremists or sharing almost identical opinions. Besides, this choice makes sure that the magnitude of opinion change due to boomerang effect need not be greater than due to assimilation, which is consistent with findings of psychological experiments. Since the network is fully connected, the dynamics of opinion distribution can be written as a master equation in the mean-field approximation. The master equation approach is computationally more efficient and generally more suitable to study the steady state opinion distribution than Monte Carlo method for a fully connected network. The results reported below are found by both Monte Carlo simulations and numerically solving the master equation. Both methods give similar results.

By numerically solving the master equation, Fig. also shows that the equilibrated opinion distribution for different values of $d_1$ and $d_2$ can be divided into three regions. In region A ($1 - d_2 \leq d_1$), the system evolves to clusters of moderate opinions similar to that of the D-W Model. In region B ($d_1 \leq 1 - d_2 \leq 1/2$), the system equilibrates to two clusters of extreme opinions plus one or more clusters of moderate opinions similar to the J-A Model. And in region C ($1 - d_2 \leq 1/2$), the system evolves to two clusters of extreme opinions only. Again, this is similar to the results of the J-A Model. These findings are consistent with our Monte Carlo simulations except for the small region C’ in which $d_1, d_2 \leq 1/2$. We shall discuss this difference when we talk about the agents’ dynamics below.

Fig. also shows that the fraction of agents in an opinion cluster upon equilibration vary greatly for different values of $d_1$ and $d_2$. In fact, Ben-Naim et al. found that equilibrated opinion clusters of vastly different sizes can be present in the D-W Model. They called an opinion cluster with $\gg 10^{-3} (\ll 10^{-5})$ fraction of agents a major (minor) cluster. Here, we clarify what an opinion cluster means in this paper. In our subsequent theoretical analysis, it refers to a connected subgraph of the network such that each agent in this subgraph has the same opinion. In addition, the ratio of agents in this subgraph to N is non-zero in the large N limit. Whereas in our Monte Carlo program, an opinion cluster is a connected subgraph of the network with at least $2 \times 10^{-3}$ fraction of the agents such that opinion difference between any two agents in the subgraph is less than $d_1$ after equilibration. On the other hand, in our master equation program, consecutive discretized opinion bins each with fraction of opinion greater than a threshold of $2 \times 10^{-3}$ upon equilibration is considered to be a cluster. In other words, unless otherwise stated, we do not consider minor opinion clusters in our simulations.

Finally, we remark that we have tried several parameters pairs $(\mu, \lambda)$ in our simulations and they all exhibit similar dynamics. In fact, our mean-field analysis in Sec. IV shows why this is the case.

IV. UNDERSTANDING OUR SIMULATION RESULTS

Consider the following mean-field analysis. Let $y$ be the opinion of one of the agents chosen to interact at time $t = 0$. Since the initial opinions are randomly and uniformly assigned, the net rate for $y$ to increase after the interaction equals

$$f(y) \equiv \Pr(y \text{ increases}) - \Pr(y \text{ decreases}) = \min(1 - y, d_1) + \max(y - d_2, 0) - \min(y, d_1) - \max(1 - d_2 - y, 0)$$

(4)
We only need to analyze the situation for opinions $x < \mu$ and (b) number of moderate (major) opinion clusters in our model in a complete network found by numerically solving the master equation with the opinions divided into 1002 bins for $\mu = 0.20$ and $\lambda = 0.05$.

for $0 < y < 1$. In addition, $f(0) = d_1 > 0$, $f(1) = -d_1 < 0$. There are three cases to consider.

Case (1): $d_2 < 1/2$, namely, most of the region C. We only need to analyze the situation for opinions $x \in [1/2, 1]$ as our model is symmetric about $x = 1/2$. Eq. (1) implies $f(x) > 0$ for $1/2 < x < 1$ and $f(1/2) = 0$. Thus, initially opinion tends to move towards $x = 1$; and $x = 1/2$ is an unstable equilibrium point. Whereas those with initial opinion $x = 1$ may change to an opinion in the range $R = [1 - \mu d_1, 1]$ after its first interaction due to the assimilation rule. Hence, opinions pile up around $R$ in the large $N$ limit shortly after $t = 0$. Note that $f(y)$ is close to a linear function, increasing from $f(1/2) = 0$ to $f(1 - d_1) = 1 - d_1 - d_2$. So the number of agents with opinions around $1/2$ almost stays constant shortly after $t = 0$. Provided that $1 - \mu d_1 - 1/2 \geq d_1$, the assimilation rule has no effect between these piled up opinions in $R$ and those near $x = 1/2$. In this case, the net rate for opinion $x \in (1/2, 1 - \mu d_1)$ to increase at time $t \geq 1$ is greater than $f(x)$. More importantly, this positive feedback mechanism quickly kicks opinions out of $(1/2, 1 - \mu d_1)$. Finally, the assimilation rule among opinions in $R$ and the boomerang effect rule between opinions in $R$ and $[0, \mu d_1]$ assure that only the extreme opinions $x = 0$ and 1 will be present in the long run. (See Videos 1 and 2 in the Supplemental Material [37] as well as Videos 1 and 2 in the Supplemental Material [37]) show that this is indeed the observed dynamics in region C.

The situation is more complex when $d_1 \geq 1/[2(1 + \mu)]$ due to the competing dynamics of the assimilation and boomerang effect rules between opinions in $R$ and opinions near $x = 1/2$. Depending on the details of the dynamics, the master equation method finds that the assimilation rule may win resulting in a single moderate peak around $x = 1/2$; whereas the Monte Carlo method shows that this moderate peak may then be repelled to one of the extreme ends by a handful of remaining extreme opinion agents at the other end via the boomerang effect rule. (See Videos 1 and 2 as well as the discussions in the Supplemental Material [37] on why the results of the two methods differ.)

Note that one point is certain — extreme and moderate opinion clusters cannot coexist for $d_2 < 1/2$.

Case (2): $d_1 < 1 - d_2 < 1/2$, that is, most of the region B. Here Eq. (1) in the region becomes $f(x) > 0$ for $d_2 < x < 1$, and $f(x) = 0$ for $1/2 \leq x \leq d_2$. So we have a pile up of opinions in the interval $R$ and a migration of opinions from $R' = (d_2, 1 - \mu d_1)$ to $R$ in the large $N$ limit shortly after $t = 0$. The same positive feedback mechanism acting on region C then leads to the formation of the two extremist clusters provided that $d_1 \lesssim 1/[2(1 + \mu)]$. Note that the opinion interval $(1 - d_2, d_2)$ is in unstable equilibrium initially because local opinion clustering by the assimilation rule can grow. Besides, the depletion of opinions in $R'$ due to migration will in effect pull the opinions slightly less than $d_2$ to a lower value. These are precisely the effects governing the dynamics of the D-W Model. Thus, we end up with two extreme clusters plus several moderate ones as shown in Fig. 1. Moreover, the distance between two successive moderate (major) opinion clusters are separated by $\approx 2d_1$ in case of a fully connected network [37] and so that there are about $(2d_2 - 1)/2d_1$ of them. (See Fig. 3 as well as Videos 1 and 2 in the Supplemental Material [37].)

There are two exceptions to this rule. Just like case (1), if $d_1 \gtrsim 1/[2(1 + \mu)]$, it is possible for opinions in $R$ to merge with opinions near $x = 1/2$ forming a single moderate cluster due to the assimilation rule. (See Fig. 3 as well as Videos 1 and 2 in the Supplemental Material [37].) Unlike region C, both master equation and Monte Carlo approaches give the same conclusion here.) Another situation is when $d_2 \approx 1/2$ so that the region $R'' = (1 - d_2, d_2)$, where $f(x) = 0$, is very small. Depending on the details of the dynamics, the proportion of agents in $R''$ may not be high enough to keep them in place before the region $R'$ is depleted. If this happens, the system will evolve to two extreme opinion peaks at $x = 0$ and 1; and this is what we find in Fig. 1 as well as Videos 1 and 2 in the Supplemental Material [37].

Case (3): $1 - d_2 < d_1$, namely, region A and part of region B. Here, $f(x) < 0$ for $\max(1 - d_1, 1/2) \leq x < 1$, and $f(x) = 0$ for $1/2 \leq x \leq \max(1 - d_1, 1/2)$. Hence, there is an initial migration of opinions from $(1 - d_1, 1]$ to opinions around $x \lesssim 1 - d_1$. Similar analysis in case (2) shows that at least one moderate opinion cluster will form. (See Fig. 4 as well as Videos 1 and 2 in the Supplemental Material [37].) Nevertheless, there is a subtlety. If $1 - d_2$ is close to $d_1$ and $\lambda$ is sufficiently large, it is still possible for a small portion of agents to become extremists before they have time to join a moderate opinion cluster. The boundary between regions A and B, however, depends on the detailed dynamics of the system. Nevertheless, it is not possible to have extreme opinion peaks only in $R$. Fig. 1. [Color online] (a) Fraction of extreme opinion agents and (b) number of moderate (major) opinion clusters in our model in a complete network found by numerically solving the master equation with the opinions divided into 1002 bins for $\mu = 0.20$ and $\lambda = 0.05$. The situation is more complex when $d_1 \geq 1/[2(1 + \mu)]$ due to the competing dynamics of the assimilation and boomerang effect rules between opinions in $R$ and opinions near $x = 1/2$. Depending on the details of the dynamics, the master equation method finds that the assimilation rule may win resulting in a single moderate peak around $x = 1/2$; whereas the Monte Carlo method shows that this moderate peak may then be repelled to one of the extreme ends by a handful of remaining extreme opinion agents at the other end via the boomerang effect rule. (See Videos 1 and 2 as well as the discussions in the Supplemental Material [37] on why the results of the two methods differ.)

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\begin{enumerate}

\item Agent’s opinions can be described by a real number in [0, 1].
\item All agents have the same \(d_1\) and \(d_2\).
\item The assimilation (boomerang effect) rule makes the opinions of the two agents closer (farther) whereas the opinions are unchanged by the neutral rule.
\item The criteria for applying the assimilation and boomerang effect rules are based only on the distance between two opinions \(|x_a - x_b|\).
\item The three rules governing the microscopic opinion change are symmetric about \(x = 1/2\). The last two conditions ensures that there is no prior bias toward one of the extreme opinions.
\end{enumerate}

In other words, the appeal of the above analysis is that conclusions can be drawn that are insensitive to factors such as network topology and the precise form of the agent dynamics as long as the average network connectivity is not too low and the agent dynamics is consistent with the SJT. Indeed, Fig. 5 shows a similar phase diagram of our model in the Barabási-Albert (B-
A) scale-free network [38] even when the assimilation and boomerang effect rules for $x_a$ and $x_b$ in Eqs. (1) and (2) are changed to
\[
x_a \leftarrow x_a + \mu'(x_b - x_a)(1 - |x_b - x_a|), \tag{5a}
\]
\[
x_b \leftarrow x_b + \mu'(x_a - x_b)(1 - |x_b - x_a|), \tag{5b}
\]
and
\[
x_a \leftarrow \mathcal{N}\left(x_a - \frac{\lambda'(x_b - x_a)(|x_b - x_a| - d_2)}{1 - d_2}\right), \tag{6a}
\]
\[
x_b \leftarrow \mathcal{N}\left(x_b - \frac{\lambda'(x_a - x_b)(|x_b - x_a| - d_2)}{1 - d_2}\right), \tag{6b}
\]
respectively, where $\mu'$ and $\lambda'$ are fixed positive parameters. Clearly, these modified assimilation and boomerang effect rules are very different from those used in the D-W Model and the J-A Model. More importantly, unlike our original rules in Eqs. (1) and (2), the modified assimilation and boomerang effect rules are chosen such that agent's response is continuous across different latitudes, thereby demonstrating that the phase diagram is not sensitive to discontinuity in response across different latitudes. Note further that the number of moderate clusters in this case can be more than $\approx (2d_2 - 1)/2d_1$, which is consistent with the behavior of the D-W Model model in the B-A network [13, 14]. (See Videos 3a–h in the Supplemental Material [39].)

The major shortcoming in our mean-field analysis is that we cannot predict the height of each opinion cluster and the most likely location of each of them. Actually, by choosing $d_1$ and $d_2$ near the boundaries between regions A, B and C, some of the equilibrated opinion clusters may contain less than 1% of the population.

V. OPINION DYNAMICS OF SLOWLY DRIVING $d_1$ AND $d_2$

We go on to study the situation that the thresholds $d_1$ and $d_2$ in Eqs. (1) and (2) change to reflect the change in the level of opinion tolerance in the society. While one's opinion may change by interacting with another agent once, it probably takes a much longer time for $d_1$ and $d_2$ to change since it reflects a fundamental change in the way the agents evaluate and respond to the opinions of others. Here we consider the idealized situation that $d_1$ and $d_2$ change gradually in a timescale much longer than the opinion equilibration time of the system in a way analogous to the study of quasi-static equilibrium processes in thermodynamics.

From the above analysis, we only need to consider the evolution of equilibrated opinions, which consists of extreme and/or moderate clusters, upon a small change in $d_1$ and $d_2$. Note that upon equilibration, the opinion difference between two adjacent agents belonging to two different opinion clusters must either be (i) outside both the regions of assimilation and boomerang effect or (ii) in the boomerang effect region and the two agents hold extreme opinions of $x = 0$ and 1. Consequently, the opinion distribution will not change as one perturbs $d_1$ and $d_2$ unless $d_1$ increases above or $d_2$ decreases below the opinion difference between two adjacent clusters. Some opinion clusters will merge in either cases. Thus, the opinion distribution stays constant most of the time and then suddenly change by opinion merging in a first-order phase transition. More importantly, a single moderate opinion cluster or two extreme opinion clusters are the only two stable fixed points of the system due to slowly random drifting of $d_1$ and $d_2$.

Actually, opinion sudden changes in social issues after a long period of stasis, known as punctuated equilibria in social theory, is commonly observed [35]. Our analysis here shows that they may occur even in a fully connected social network, therefore repudiating the criticism by Tilcsik and Marquis [34]. In fact, punctuated equilibrium in our model originates from the separation of timescales between agents' interactions and the change in $d_1$ and $d_2$ similar to the case of first-order phase transition in a quasi-statically evolving thermal system.

VI. OUTLOOK

In summary, we have proposed an agent-based and SJT-compatible model by extending the works of Jager and Amblard [15] and Crawford et al. [27]. Our model can serve as a blueprint to study opinion formation dynamics. In our model, formation of extreme and/or moderate opinion clusters as well as punctuated equilibria are observed even in the case of homogeneous agents in a complete network. Besides, we identify the most important conditions for forming extreme and moderate opinion clusters by mean-field analysis. Further works should be done, including the addition of noise to the agent’s response and a more detailed model of how $d_1$ and $d_2$ change, to make our model more realistic. Our next goal is to model opinion cluster splitting.

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SUPPLEMENTAL MATERIAL FOR “SOCIAL JUDGMENT THEORY BASED MODEL ON OPINION FORMATION, POLARIZATION AND EVOLUTION”

Videos 1 and 2 depict the dynamics of our model in a fully connected network in various regions of the parameter space \((d_1, d_2)\) found by numerically solving the master equation and in typical runs of Monte Carlo simulation using \(N = 1000\) agents, respectively. These two approaches give similar results except in region \(C'\) (as shown in Videos 1b and 2b) where \(d_1, d_2\) are slightly less than \(1/2\). In this exceptional case, the master equation approach gives a single moderate peak at \(x = 1/2\) in the steady state; while our Monte Carlo simulation shows that this moderate peak can be meta-stable. More precisely, after a long time, the moderate peak at \(x = 1/2\) sometimes move towards one of the extreme ends giving eventually a major extreme peak plus a very small minor extreme peak at the other end. In fact, by finite-size scaling analysis, our Monte Carlo simulation suggests that all the steady states in region \(C'\) are made up of one major and one minor extreme peaks in the large \(N\) limit.

This discrepancy may be caused by the followings. For our numerical solution to the master equation, numerical truncation error and a long decay time of the meta-stable state may lead to a wrong conclusion. More importantly, as an approach that deals with the evolution of opinion distribution, the master equation approach fails to capture the dynamics of certain opinion distributions in our model. For example, consider the opinion distribution in which agents are of opinion \(x = 1/2\) almost surely and at the same time with a measure zero number of agents with opinion \(x = 0\). (One may think of the system configuration in which there is only one agent with \(x = 0\) and all the remaining \(N-1\) agents has \(x = 1/2\). Then we take the limit \(N \to +\infty\).) For \(d_2 < 1/2\) and \(d_1 \lesssim d_2\), this configuration will evolve to the steady state with one major opinion peak at \(x = 1\) and the opinions of those agents with \(x = 0\) are unchanged. It takes a long time for the system to evolve to this state though. Surely, this is what we will find by running the Monte Carlo simulation for a sufficiently long time. However, since the opinion distribution in this example is equal to that of the Dirac delta function at \(x = 1/2\), the master equation approach will wrongly predicts that the system is already in steady state. However, we are not sure if this is the cause of the discrepancy because we do not know if a uniformly distributed initial opinion will almost surely evolve to a meta-stable state like the one in the above example.

In addition, our Monte Carlo simulation is not without trouble in region \(C'\) because the convergence time is too long to be computationally feasible to perform finite-size scaling when \(d_1, d_2\) are very close to \(1/2\). So we cannot rule out the existence of a single stable moderate peak at \(x = 1/2\) in the large \(N\) limit when \(d_1\) and \(d_2\) are very close to \(1/2\).

Finally, Video 3 shows the dynamics of our model in the B-A network with \(N = 1000\) using different assimilation and boomerang effect rules. It demonstrates that it has very similar dynamics as in the case of the complete network although the number of moderate peaks may differ.
Video 1. Due to file size limit, all videos are not uploaded. Please contact the first author at hfchan@hku.hk if you want a copy. Videos showing the dynamics of our model in a fully connected network in different regions of the parameter space $(d_1, d_2)$ found by numerically solving the master equation with the opinions divided into 1002 bins. In particular, Video a shows the typical dynamics in region C using parameters $(d_1, d_2) = (0.30, 0.40)$; Video b shows the dynamics in region C’ using parameters $(0.47, 0.49)$; Video c shows the dynamics in region B using parameters $(0.10, 0.85)$ which results in three moderate opinion clusters plus two extreme opinion clusters; Video d shows the dynamics in region B using parameters $(0.15, 0.80)$ which results in two moderate opinion clusters plus two extreme opinion clusters; Video e shows the dynamics in region A with $d_1 + d_2 < 1$ using parameters $(0.42, 0.55)$ which results in a single moderate opinion cluster; Video f shows the dynamics in region C with $d_2 > 1/2$ using parameters $(0.37, 0.52)$; Video g shows the dynamics in region A using parameters $(0.25, 0.90)$; Video h shows the dynamics in region B with $d_1 + d_2 > 1$ using parameters $(0.13, 0.90)$.

Video 2. The same as Videos a–h except that these are typical runs of Monte Carlo simulations for a network of 1000 agents.

Video 3. The same as Videos a–h but for the case of the B-A network using different assimilation and contrast rules as mentioned in the main text.