Fully Degenerate Self-Gravitating Fermionic Dark Matter: Implications to the Density Profile of the Cluster of Galaxies A1689, and the Mass Hierarchy of Black Holes

Tadashi Nakajima

National Astronomical Observatory of Japan
Osawa 2-21-1, Mitaka, 181-8588, Japan

and

Masahiro Morikawa

Department of Physics, Ochanomizu University
2-1-1 Otsuka, Bunkyo, Tokyo, 112-8610, Japan

ABSTRACT

Equilibrium configurations of weakly interacting fully degenerate fermionic dark matter are considered at various scales in the Universe. We treat the general situations for the gravity from Newtonian to general relativity and the degeneracy from nonrelativistic to relativistic. The formulation of the problem is exactly the same as the case treated by Oppenheimer and Volkoff in their paper on neutron stellar cores. A dimensionless equilibrium configuration is specified by a single parameter regardless of particle properties, the Fermi velocity at the center, and the scalings of mass and length are specified by the rest mass and statistical weight of the dark matter particle. We focus our attention to the flat-top nature of the mass column density profile of the cluster of galaxies, A1689, recently reported by Broadhurst et al. using gravitational lensing. We convert the column density profile to a volume density profile assuming spherical symmetry and derive a 3D encircled mass profile of A1689, which is compared with the model profiles of degenerate fermion structures. The flat-top profile is reproduced. The corresponding fermion mass ranges from 2 eV to 30 eV depending on the actual scale of the degenerate structure. If massive neutrinos are the dominant dark matter, the rest mass will be about 4.7 or 2.3 eV respectively for Majorana or Dirac neutrinos. The mass and size of the degenerate structure
are $10^{14}M_\odot$ and 100 kpc for Majorana neutrinos, and $5 \times 10^{14}M_\odot$ and 300 kpc for Dirac neutrinos. If we identify the fermions as heavier sterile neutrinos, they yield the characteristic mass hierarchy of black holes; giant black hole at the center of a galaxy and the intermediate mass black holes. Thus we propose the possibility that the mass hierarchy of fermions determines that of black holes in the Universe.

Subject headings: clusters:individual(A1689) — gravitational lensing — neutrinos — black hole physics

1. Introduction

Recent precision observations have revealed that the unknown dark matter dominates the matter contents of the Universe. We wish to study the possible dynamical structures of their existence, especially in a universal form. If the dark matter is in the form of ordinary thermal gas, the structure and the dimension would be strongly dependent on the initial conditions and the environment of the expanding universe. On the other hand if the dark matter is almost degenerate, we will naturally find a universal structure. Moreover, we would like to know the possibility that the dark matter forms black holes.

To make the problem setting better defined, we start our consideration from the typical structure made from ordinary baryonic matter, in which an electron and a nucleon form the basic ingredient through the electromagnetic force. In the total energy $E = \frac{1}{2}m_e v^2 - (e^2/r)$, the second term represents the attractive force making the system collapse and the first term represents the pressure against it through the Heisenberg’s uncertainty principle. Actually putting the expression of the de Broglie wave length $r = \hbar/(m_e v)$ in $E$, and extremizing it with respect to $r$, we obtain the ground state bounding energy of hydrogen $E = -m_e e^4/(2\hbar^2) = -2.19 \times 10^{-18}$ J and the Bohr radius $r = \hbar^2/(e^2 m_e) = 5.28 \times 10^{-9}$ cm. The corresponding energy density is $\rho = m_p/(\frac{4\pi}{3}r^3) = 2.71$ g·cm$^{-3}$. This is the basic ingredient of the structure formed by electromagnetism. The electron can collapse more. Setting $v \rightarrow c$ in de Broglie wave length yields the Compton wave length. The corresponding energy density becomes $\rho = m_p/(\frac{4\pi}{3}r_c^3) = 6.93 \times 10^6$ g·cm$^{-3}$. This is the basic ingredient of white dwarfs. Further collapse makes the electron and the proton into a neutron, through the inverse beta decay process. Putting $m_e \rightarrow m_p$, we have $r = \hbar/(m_p c)$, and therefore the corresponding energy density is $\rho = m_p^4 c^3/\hbar^3 = 4.29 \times 10^{16}$ g·cm$^{-3}$. This is the basic ingredient of neutron stars.

In general, the material formed by fully degenerate fermion of mass $m$ would have the
energy density \( \rho = m^4 c^3 / \hbar^3 \). A structure of this density with radius \( R \) has the mass \( M = R^3 \rho \). On the other hand the size \( R \) should be smaller than the limiting scale, Schwarzschild radius, \( 2GM \leq R \). This condition yields the maximum mass as

\[
M_{\text{fermi}} = G^{-3/4} m^{-2} = m_{\text{pl}}^3 / m^2
\]

where the Planck mass is defined as \( m_{\text{pl}} = (\hbar c / G)^{1/2} = 2.18 \times 10^{-5} \text{ g} \). Thus the quantum mechanics (\( \hbar \)), gravity (\( G \)), and the particle physics (\( m \)) as well as relativity (\( c \)) characterize the universal structure of a fully degenerate fermion star (FDFS).

Incidentally, degenerate bosons form much smaller structures since bosons have no exclusion principle like fermions. Therefore only the Heisenberg uncertainty principle (quantum pressure) can support the structure against the collapse due to gravity. The sole characteristic length scale is the Compton wave length \( \lambda_{\text{Compton}} = \hbar / (mc) \approx R \), which must be larger than the limiting scale, Schwarzschild radius, \( 2GM \leq R \). This condition yields the density \( \rho = m^2 m_{\text{pl}}^2 c^3 / \hbar^3 \) and

\[
M_{\text{bose}} = m_{\text{pl}}^2 / m
\]

which is known as the Kaup mass (Kaup 1968). This structure is a boson star. This structure is smaller than that formed by fermions by a factor \( m_{\text{pl}} / m \) and no further discussion will be given in this paper, apart from a brief comment regarding the equation of state in \( \S 2 \).

Now we proceed to consider the equilibrium structures made of fermionic dark matter. One may think that this problem is analogous to the white dwarf case and the Chandrasekhar mass is the limiting mass. However this is not the right answer. In the case of a white dwarf, the pressure is due to the degeneracy pressure of relativistic electrons, but the gravity is Newtonian because it is due to the rest mass of hadrons which are nonrelativistic. When we treat a degenerate star made purely of dark matter itself, the total rest mass is also due to dark matter particles themselves which can be relativistic. Therefore we need to consider the gravitational mass of the star in a general relativistic manner. This problem is therefore analogous to the case of a neutron star. In the case of the neutron star, there is a complication due to the fact that neutrons interact by nuclear forces. However in the case of weakly interacting fermionic dark matter, it can be considered as ideal gas and the equation of state is well defined. It turned out that this situation was treated by the classical paper by Oppenheimer & Volkoff (1939), since they assumed free neutrons and used the equation of state of ideal gas in their calculation. Basically fully degenerate fermions can form very compact dense structures including black holes (Bilić, Munyaneza & Viollier 1999; Bilić, Tupper & Viollier 2003).

On the other hand, from the observational side, we now have many candidates of dense
structures at various scales. The most massive example is a huge dark matter distribution of the cluster of galaxies, A1689, reported recently by Broadhurst et al. (2005a,b). They obtained the mass column density distribution of the cluster of galaxies, A1689 by gravitational lensing. The column density profile has a flat-top and we suspected that this flat-top nature might be due to the degeneracy pressure of fermions. We derive a volume density profile from the observed column density profile assuming spherical symmetry and compare the observed 3D encircled mass profile with our model profile of an FDFS.

The condense structure made from the fully degenerate fermion is not restricted to a center of a cluster of galaxies. If we consider more massive neutrinos, such as sterile neutrinos, the similar structures are realized in scaled-down form as in Eq.(1). If this structure is universal, we will find groups of black holes which have typical masses directly characterized by fermion masses.

The paper is organized as follows. The formalism by Oppenheimer & Volkoff (1939) is introduced and equilibrium solutions are discussed in §2. Readers who are not interested in the derivation of the solutions, are advised to skip to §2.3 where the properties of the solutions are discussed. The application of a nonrelativistic FDFS to the cluster of galaxies A1689, is discussed in §3. The possible relation between the mass hierarchies of black holes and sterile neutrinos are considered in §4.

2. Formalism

Here we review the derivation of the general relativistic equilibrium equations following Tolman (1934) and Oppenheimer & Volkoff (1939).

2.1. Relativistic treatment of equilibrium

The most general static line element exhibiting spherical symmetry may be expressed in the form

$$ds^2 = -e^\lambda dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + e^\nu dt^2,$$

$$\lambda = \lambda(r), \quad \nu = \nu(r). \quad (3)$$

If the matter supports no traverse stresses and has no mass motion, then its energy momentum tenor is given by
\[ T_1 = T_2 = T_3 = -p, \quad T_4 = \epsilon, \]

where \( p \) and \( \epsilon \) are respectively the pressure and the macroscopic energy density measured in proper coordinates. Einstein’s field equations without the cosmological constant reduce to

\[ 8\pi p = e^{-\lambda} \left( \nu' \frac{1}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2}, \quad (5) \]

\[ 8\pi \epsilon = e^{-\lambda} \left( \frac{\chi'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2}, \quad (6) \]

\[ \frac{dp}{dr} = -\frac{p + \epsilon}{2} \nu', \quad (7) \]

where primes denote differentiation with respect to \( r \). These three equations together with the equation of state of the material \( \epsilon = p(\epsilon) \) determine the mechanical equilibrium of the matter distribution as well as the dependence of the metric \( g_{\mu\nu} \)’s on \( r \).

The boundary of the matter distribution is the value of \( r = r_b \) for which \( p = 0 \), and such that for \( r < r_b, p > 0 \). For \( r < r_b \) the solution depends on the equation of state of the material connecting \( p \) and \( \epsilon \). For many equations of state a sharp boundary exists with a finite value of \( r_b \).

In the empty space, \( p = \epsilon = 0 \), surrounding the spherically symmetric distribution of matter, the Schwarzschild’s exterior solution is obtained:

\[ e^{-\lambda(r)} = 1 - \frac{2m}{r}, \quad e^{\nu(r)} = 1 - \frac{2m}{r}, \quad (8) \]

where \( m \) is the Newtonian mass of the matter as calculated by a distant observer.

Inside the boundary, Eqs. (5), (6), and (7) may be rewritten as follows. Using the equation of state \( \epsilon = \epsilon(p) \), Eq. (7) may be immediately integrated

\[ \nu(r) = \nu(r_b) - \int_{0}^{p(r)} \frac{2dp}{p + \epsilon(p)}, \quad e^{\nu(r)} = e^{\nu(r_b)} \exp \left[ - \int_{0}^{p(r)} \frac{2dp}{p + \epsilon(p)} \right]. \quad (9) \]

The constant \( e^{\nu(r_b)} \) is determined by making \( e^{\nu} \) continuous across the boundary.

\[ e^{\nu(r)} = (1 - \frac{2m}{r}) \exp \left[ - \int_{0}^{p(r)} \frac{2dp}{p + \epsilon(p)} \right]. \quad (10) \]
Thus $e^\nu$ is known as a function of $r$ if $p$ is known as a function of $r$. Further in Eq.(6) introduce a new variable

$$u(r) = \frac{1}{2} r (1 - e^{\lambda}) \quad \text{or} \quad e^\lambda = 1 - \frac{2u}{r}. \quad (11)$$

Then Eq. (5) becomes:

$$\frac{du}{dr} = 4\pi \epsilon(p) r^2. \quad (12)$$

In Eq. (5) we replace $e^{-\lambda}$ by its expression (11) and $\nu'$ by its expression (7). Then it becomes

$$\frac{dp}{dr} = -\frac{p + \epsilon(p)}{r(r - 2u)} [4\pi pr^3 + u]. \quad (13)$$

Eqs. (12) and (13) form a system of two first-order equations in $u$ and $p$. Starting with some initial values $u = u_0$ and $p = p_0$ at $r = 0$, the two equations are integrated simultaneously to the value $r = r_b$ where $p = 0$, i.e., until the boundary of the matter distribution is reached. The value of $u = u_b$ at $r = r_b$ determines the value of $e^{\lambda(r_b)}$ at the boundary to the exterior solution, making

$$u_b = \frac{r_b}{2} [1 - e^{-\lambda(r_b)}] = \frac{r_b}{2} \left[ 1 - (1 - \frac{2m}{r_b}) \right] = m. \quad (14)$$

Thus the mass of this spherical distribution of matter as measured by a distant observer is given by the value $u_b$ of $u$ at $r = r_b$.

The following restrictions must be made on the choice of $p_0$ and $u_0$, the initial values of $p$ and $u$ at $r = 0$:

(a) In accordance with its physical meaning as pressure, $p_0 \geq 0$.

(b) From Eq.(11) it is seen that for all finite values of $e^{-\lambda}$, $u_0 = 0$. Since $g_{11} = -e^\lambda$ must never be positive, $u_0 \leq 0$ for infinite values of $e^\lambda$ at the origin. However, it may be shown that of all the finite values of $p_0$ at the origin $p_0 = 0$ is the only one compatible with a negative value of $u_0$, and that for equations of state of the type occurring in this problem even this possibility is excluded, so that $u_0$ must vanish.

This can be seen from the following argument. Having chosen some particular value of $p_0$ one may usually represent the equation of state in that pressure range by $\epsilon = Cp^s$ with
some appropriate value of $s$. Using this equation of state and taking the approximate form of Eq (13) near the origin for the case $u_0 < 0$, and finite $p_0$, one obtains:

$$\frac{dp}{dr} = p + \epsilon(p) = p + C p_s^2. \quad (15)$$

Integration of this equation shows that for $s < 1$, $p_0 \geq 0$ can not be satisfied, and for $s \geq 1$ only the value $p_0 = 0$ is possible. This immediately excludes the possibility that degenerate bosons form an equilibrium structure, because $p$ is independent of $\epsilon$, if $p$ is solely due to thermal bosons (Landau & Lifshitz 1980). As we mentioned in §1, a boson star is supported by the quantum pressure of ground-state bosons, which is outside of the scope of the consideration of the equation of state (Kaup 1968). For the equations of state used for degenerate fermions, always $s < 1$ holds. It is also be noted that the above equation together with Eq. (10) show that $e^{\nu(r)} \to \infty$ as $r \to 0$.

(c) A special investigation for any particular equation of state must be made to see whether solutions exist in which $0 \leq u_0 \leq -\infty$ and $p \to \infty$ as $r \to 0$.

2.2. Equation of state for degenerate Fermi gas

If the matter consists of fermions of rest mass $\mu_0$ and statistical weight $g$, and their thermal energy and all forces between them are neglected, then a parametric form for the equation of state (Landau & Lifshitz 1980) is,

$$\epsilon = K(\sinh t - t), \quad (16)$$

$$p = \frac{1}{3} K \left( \sinh t - 8 \sinh \frac{t}{2} + 3t \right), \quad (17)$$

where

$$K = \frac{\pi g \mu_0^4 c^5}{8 h^3}, \quad (18)$$

and

$$t = 4 \arcsinh \frac{p_F}{\mu_0 c}, \quad (19)$$
where $p_F$ is the maximum momentum in the Fermi distribution and is related to the proper particle density $N/V$ by

$$ N/V = \frac{4\pi g}{3h^3} p_F^3 = \frac{4\pi g}{3} \left( \frac{\mu_0 c}{\hbar} \right)^3 \sinh^3 \frac{t}{4}. \tag{20} $$

If we define the Fermi velocity $v_F$ by

$$ \frac{p_F}{\mu_0 c} = \frac{v_F}{\sqrt{1 - (v_F/c)^2}}, \tag{21} $$

$$ t = 2 \log \left( \frac{1 + v_F/c}{1 - v_F/c} \right) \quad \text{or} \quad v_F/c = \tanh \frac{t}{4}. \tag{22} $$

$t$ and $v_F$ are independent of particle properties.

Substituting the above expressions for $p$ and $\epsilon$ into Eqs. (12) and (13) one obtains:

$$ \frac{du}{dr} = 4\pi r^2 K (\sinh t - t), \tag{23} $$

and

$$ \frac{dt}{dr} = -4/[r(r - 2u)] $$

$$ \times (\sinh t - 2 \sinh 1/2t)/(\cosh t - 4 \cosh 1/2t + 3) $$

$$ \times \left[ \frac{4\pi}{3} K r^3 (\sinh t - 8 \sinh 1/2t + 3) + u \right]. \tag{24} $$

These equations are to be integrated from the values $u = 0, t = t_0$ at $r = 0$ to $r = r_b$ where $t_b = 0$ (which makes $p = 0$), and $u = u_b$.

So far, the equations are written in relativistic units, i.e., such that $c = 1, G = 1$. This determines the unit of time and the unit of mass in terms of still arbitrary unit of length. The unit of length is now fixed by the requirement that $K = 1/(4\pi)$. From the dimensional analysis of Einstein’s field equations, this requirement fixes the unit of length to be

$$ a = \frac{1}{\pi} \left( \frac{2}{g} \right)^{1/2} \left( \frac{\hbar}{\mu_0 c} \right)^{3/2} \frac{c}{(\mu_0 G)^{1/2}}, \tag{25} $$
and the unit of mass to be

\[ b = \frac{c^2}{G} a = \frac{1}{\pi} \left( \frac{2}{g} \right)^{1/2} \left( \frac{h}{\mu_0 c} \right)^{3/2} \frac{c^3}{(\mu_0 G^3)^{1/2}}. \] (26)

Eqs. (12) and (24) written in a dimensionless form become:

\[ \frac{du}{dr} = r^2 (\sinh t - t), \] (27)

\[ \frac{dt}{dr} = -4/[r(r - 2u)] \]
\[ \times (\sinh t - 2 \sinh 1/2t)(\cosh t - 4 \cosh 1/2t + 3) \]
\[ \times \left[ \frac{1}{3} r^3 (\sinh t - 8 \sinh 1/2t + 3t) + u \right]. \] (28)

For a given \( t_0 \), Eqs. (27) and (28) can be integrated. Fixing \( t_0 \) is equivalent to fixing \( v_{F0} \). As long as dark matter particles are fully degenerate, the solution describes the equilibrium between the gravity and degeneracy pressure from Newtonian to general relativistic gravity and from nonrelativistic \( v_F \) to relativistic \( v_F \). Therefore the solution for a given \( t_0 \) (or \( v_{F0} \)) is independent of particle properties, while the units of length and mass (\( a \) and \( b \)) are fixed by the particle properties \( \mu_0 \) and \( g \).

### 2.3. Discussion on solutions

The Eqs.(27) and (28) are numerically integrated using the fourth-order Runge-Kutta method. Apart from the gravitational mass \( u \), there is another mass indicator, the dimensionless total rest mass \( y_b \), defined as

\[ y_b = \int_0^{r_b} \frac{32}{3} \sinh^3 \frac{t}{4} / \sqrt{(1 - 2u/r)r^2} \, dr, \] (29)

which is the integral of the number density with the proper volume inside the radius \( r_b \). \( t_0, v_{F0}/c, u_b, y_b, \) and \( r_b \) are given for \( t_0 = 0.1 \sim 14.0 \) in Table 1. At \( t_0 = 0.1 \), degenerate particles are nonrelativistic (\( v_{F0}/c = 0.025 \)), while at \( t_0 = 14 \), particles are extremely relativistic (\( v_{F0} = 0.998 \)). In Fig. 1, the relation between the gravitational mass \( u_b \) and outer radius \( r_b \) is plotted. For \( t_0 \leq 3 \), \( u_b \) is an increasing function of \( t_0 \), while \( r_b \) is a decreasing
function of $t_0$. The maximum of $u_b = 0.0766$ is reached for $t_0 = 3$ and $r = 0.663$. This is the maximum stable solution for which $u_{F0}/c = 0.635$, which is modestly relativistic. In Fig.2, the radial profiles of the energy density, $\epsilon$, and pressure, $p$, are plotted. The contribution of $p$ is relatively small compared to $\epsilon$. This solution is termed as quasi-Newtonian by Oppenheimer & Volkoff (1939). In the case of self-gravity of degenerate fermions themselves, a stable configuration never reaches an extremely relativistic situation unlike the case of white dwarfs. Therefore the formula of the Chandrasekhar mass applied to this case over-estimates the maximum stable mass by one order of magnitude. For $t_0 > 3.0$, there is no stable solution (Oppenheimer & Volkoff 1939). $u_b$ decreases takes a minimum value at $(u_b, r_b) = (0.0395, 0.364)$ for $t_0 = 8.0$. For a large $t_0$, $(u_b, r_b)$ spirals to $(\sim 0.041, \sim 0.29)$.

The gravitational mass defect, $\Delta = u_b - y_b$ is one measure of stability of a general relativistic equilibrium configuration. In Fig.3, both $y_b$ and $u_b$ are plotted against $t_0$ for $0.1 < t_0 < 14.0$. For a small $t_0$, $\Delta$ is negative and gradually decreases and takes a minimum value, $\Delta = -0.0029$ for the maximum mass stable solution ($t_0 = 3.0$) and then increases. The fractional mass defect, or the packing fraction, is $f = \Delta/y_b = -0.036$. $|f|$ is only 3.6% and may appear small. However, if we compare this value to a typical nuclear packing fraction, which is less than 0.1% (Fermi 1950), we will find it to be significant. $\Delta < 0$ is a necessary condition for stability, but not sufficient. $\Delta < 0$ still holds for $3 < t_0 < 5$, but these solutions correspond to unstable equilibria. When $\Delta < 0$, solutions are for stable equilibria for $d\Delta/dt_0 \leq 0$, while they are for unstable equilibria for $d\Delta/dt_0 > 0$.

The question of whether dark matter fermions can form a black hole would be answered in terms of the maximum gravitational mass, $M_{\text{max}}$ and the corresponding radius $R_{\text{max}}$, as functions of the particle mass $\mu_0$ and statistical weight $g$. The results are:

$$M_{\text{max}} = 1.73 \times 10^{51} \frac{1}{\sqrt{g}} \left(\frac{\mu_0}{\text{eV}}\right)^{-2} \quad \text{(g)} \quad (30)$$

$$R_{\text{max}} = 1.10 \times 10^{24} \frac{1}{\sqrt{g}} \left(\frac{\mu_0}{\text{eV}}\right)^{-2} \quad \text{(cm)} \quad (32)$$

$$M_{\text{rest}} = 8.70 \times 10^{17} \frac{1}{\sqrt{g}} \left(\frac{\mu_0}{\text{eV}}\right)^{-2} \quad \text{(M\odot)} \quad (31)$$

$$R_{\text{max}} = 3.54 \times 10^{2} \frac{1}{\sqrt{g}} \left(\frac{\mu_0}{\text{eV}}\right)^{-2} \quad \text{(kpc)} \quad (33)$$

It should be noted that Landau & Lifshitz (1980) define the maximum rest mass, $M_{\text{rest}} =$
$1.037M_{\text{max}}$ as the maximum mass, which is the total rest mass brought from infinity. In the case of a star made of free neutrons, $\mu_0 = 939$ MeV and $g = 2$. Then $M_{\text{max}} = 0.70M_\odot$, and $R_{\text{max}} = 8.8$ km. It should also be noted that $R_{\text{max}}$ is sensitive to the actual outer boundary condition in the numerical calculation, which in principle should be $p = 0$. In reality, integration must be stopped at a finite value of $p$ and $r_b$ is somewhat sensitive to this finite $p$. Physically speaking, it is unrealistic to assume the fully degenerate equation of state to a small $p$ and the outer boundary condition is not well defined. The total gravitational mass $u_b$ is not very sensitive to the choice of the actual outer boundary condition $p$ and is well defined. Although it is not explicitly stated in the original paper, Oppenheimer & Volkoff (1939) were probably aware of this limitation in the applicability of the equation of state, if we judge from the carefully chosen title “On Massive Neutron Cores”. We also admit that without this boundary condition, we cannot use the Schwarzschild’s exterior solution outside the boundary and the formulation becomes more complicated.

Before concluding this section, we comment on the nonrelativistic limit. For the non-relativistic limit, it is expected that between the mass, $M$, and the size, $R$, of an FDFS, the relation,

$$MR^3 = \text{const.}$$  \hspace{1cm} (34)

should hold (Landau & Lifshitz 1980).

For $t_0 < 0.5$, $u_b r_b^3$ is indeed nearly constant and

$$u_b r_b^3 = 0.135,$$  \hspace{1cm} (35)

within 4% precision. In physical units,

$$MR^3 = (u_b b)(r_b a)^3 = 1.4 \times 10^{124} \frac{1}{g^2} \left( \frac{\mu_0}{\text{eV}} \right)^{-8} \text{ (g \cdot cm^3)},$$  \hspace{1cm} (36)

$$= 2.3 \times 10^{26} \frac{1}{g^2} \left( \frac{\mu_0}{\text{eV}} \right)^{-8} \text{ (M}_\odot \cdot \text{kpc}^3).$$  \hspace{1cm} (37)

Nonrelativistic solutions are similar and in Fig.4, the profiles of the normalized mass density and 3D encircled mass are plotted. In the next section, we will find the behavior of the 3D encircled mass profile interesting and the logarithmic profile of the normalized 3D encircled mass is given in Fig.5.
3. Can degenerate fermions be the dominant dark matter in the cluster of galaxies, A1689?

Recently Broadhurst et al. (2005a,b) reported a mass column density profile of the cluster of galaxies, A1689, obtained from gravitational lensing. One of the important properties of the profile is that it has a flat top. We propose that this flat-top column density profile might be explained by the effects of degeneracy pressure of fermionic dark matter. Here we analyze this proposal.

First we briefly introduce the main results of Broadhurst et al. (2005a,b). In their analysis, $1'$ corresponds to $129 \, \text{kpc} \, h^{-1}$. In Broadhurst et al. (2005a), the central $250 \, \text{kpc} \, h^{-1}$ in radius of multi-color HST/ACS images were analyzed. The mass column density profile, $\Sigma(r)$, is not expressed as a single power law of radius. The mass column density profile flattens toward the center with a mean slope of $d\log \Sigma / d\log r \approx -0.55$ within $r < 250 \, \text{kpc} \, h^{-1}$. Inside the Einstein radius ($\theta_E \approx 50''$), they obtained the slope of $\approx -0.3$ from the ratio between $\theta_E$ and the radius of the radial critical curve, $\theta_r \approx 17''$. They fit their results with an inner region of an NFW profile (Navarro, Frenk & White 1996) with a relatively high concentration, $C_{\text{vir}} = 8.2$.

The mass column density, $\Sigma(r)$, is the integral of the volume density, $\rho(r)$, along the line of sight over the entire cluster scale of Mpc. In order to study the possibility of fermion degeneracy near the center of the cluster, we need information on the volume density, $\rho(r)$, instead of the column density, $\Sigma(r)$. Broadhurst et al. (2005b) present the weak-lensing analysis of the wide field data obtained by Subaru and obtained the column density profile at $r < 2 \, \text{Mpc} \, h^{-1}$. They fit the combined profile of HST/ACS and Subaru with an NFW profile with a very high concentration, $C_{\text{vir}} = 13.7$, significantly larger than theoretically expected value of $C_{\text{vir}} \approx 4$. They also fit the same observed column density profile with a power law profile with a core. They give this result in terms of the angular radius dependence of the convergence, $\kappa$, as

$$\kappa \propto (\theta + \theta_C)^{-n}. \tag{38}$$

$\theta_C = 1.65'$ and $n = 3.16$ give the best fit although $\theta_C$ and $n = 3.16$ are mutually dependent and a finite range of the combination $(\theta_C, n)$ gives equally good fits. In terms of $\chi^2$ and the degrees of freedom, this core power law profile fits the observation better than the best-fit NFW profile and we use this profile for further discussion. Although Broadhurst et al. (2005b) do not claim so explicitly, the two facts that the best-fit NFW profile shows a much higher concentration than the value predicted by the CDM cosmology and the phenomenological profile, Eq.(38), fits better than the best-fit NFW profile, indicate some
serious contradiction to the CDM cosmology.

We start our analysis from this core power-law profile, (38), for further discussion. We convert (38) to a column density profile, $\Sigma(r)$, in physical units of length and mass using the relations, $\kappa = \Sigma/\Sigma_{\text{crit}}$, $\Sigma_{\text{crit}} \approx 0.95 \, g \cdot cm^{-2}$, and the normalization of 2D encircled mass inside the Einstein radius, $r_E$, $\int_{0}^{r_E} \Sigma(r)2\pi rdr = \Sigma_{\text{crit}} \cdot \pi r_E^2$. The result is expressed as

$$\Sigma(r) = 25.2 \cdot (r/r_E + 2.2)^{-3.16}, \quad (g \cdot cm^{-2})$$

where $r_E = 97 \, kpc \, h^{-1}$ corresponds to $\theta_E = 45''$, the value used in Broadhurst et al. (2005b). The 2D encircled mass, $M_2(r) = \int \Sigma(r)2\pi rdr$, is analytically obtained and $M_2(r) = 1.3 \times 10^{14}h^{-2} M_\odot$ and $1.1 \times 10^{15}h^{-2} M_\odot$, respectively for $r = r_E$ and $r = \infty$. Therefore, a high concentration of the mass is expected on the scale of $r_E$. By assuming spherical symmetry, we wish to obtain the volume density $\rho(r)$ by solving

$$\Sigma(\sqrt{x^2 + z^2})dz,$$

but we were not able to obtain an analytic solution. Instead, we assumed another power law with a core for $\rho(r)$ and obtained the best fit parameters. The range of integration in $z$ is from $-2Mpc \, h^{-1}$ to $+2Mpc \, h^{-1}$. The result is

$$\rho(r) = 1.60 \times 10^{-23} (r/r_E + 1.28)^{-3.71} h, \quad (g \cdot cm^{-3})$$

Near the center, $\rho(r) = 6.4 \times 10^{-24}h$ and $7.5 \times 10^{-25}h$ (g cm$^{-3}$) respectively at $r = 0$ and $r_E$. As for the case with the column density profile, the core radius and power-law index are mutually dependent and a finite range of their combination gives equally good fits. The above volume mass densities may appear small, but the degeneracy depends on the number density and de Broglie wavelength, both of which are heavily dependent on the rest mass of the particle. Before proceeding to the analysis by an FDFS, we first confirm that eV-mass fermions can be degenerate at these low mass densities. Since 1 eV corresponds to $1.8 \times 10^{-33} \, g$, the number density, $N/V \approx 10^{11} \, cm^{-3}$, the mean inter-particle spacing, $(N/V)^{-1/3} \approx 2 \times 10^{-4} \, cm$. On the other hand, the de Broglie wavelength for a 1 eV particle with a velocity $v$ is, $\hbar \mu_0 v = \lambda_{\text{Compton}}(c/v) = 2 \times 10^{-5}(c/v) \, cm$. Therefore for nonrelativistic particles with $v < 0.1c$, the condition for degeneracy, $(N/V)^{-1/3} < \lambda_{\text{de Broglie}}$, is satisfied.

More quantitative discussion is possible for a 3D encircled mass profile given in Fig.6. For the purpose of an explicit comparison, here we fix $h = 0.7$. The 3D encircled mass profile also predicts the rotation curve profile as Fig.7, which can be compared with observations of
kinematics of the galaxies in the cluster in the future. The column density profile, Eq.(39), was derived as a phenomenological formula without assuming any background physics, and so was the volume density profile, Eq.(41). Here we wish to explain the presence of the core in the power-law profile, which causes the flat-top nature of Eq.(39), in terms of an FDFS. The comparison must be made either in the volume density profiles or in the 3D encircled mass profiles. Here we choose the latter, because the observed 3D encircled mass profile is much less sensitive to the actual choice of the combination of the core radius and the power-law index in the original Eq.(39). As we mentioned above, the finite range of the parameter combinations gives equally good fits to the observations. The basic reason for this is that our observables are integrated quantities rather than the local densities. First, we compare the slopes of the 3D encircled mass profiles of the observation (Fig.6) and our model (Fig.5). The observed slope is 2.93 at \( r < 10 \) kpc and 2.54 at \( 10 < r < 100 \) kpc. On the other hand, the slope of a nonrelativistic FDFS is 2.90 between \( r = 0.03r_b \) and \( 0.3r_b \) and becomes shallower at \( 0.3r_b < r < r_b \). Note that the slope of the 3D encircled mass profile for a constant volume density is 3.0 and both the observed and model inner slopes of 2.9 are consistent with the flat-top nature of the volume density profile. Now we proceed to the physical scaling of the model in terms of the mass and length. For this purpose, it is most convenient to use the relation Eq.(37) for the mass and size of a nonrelativistic FDFS. The Eq.(37) can be rewritten as

\[
\log M_{14} + 3 \log R_2 = f(g, \mu_0),
\]

where \( M_{14} = M/(10^{14} M_\odot) \), \( R_2 = R/(100 \text{kpc}) \) and \( f(g, \mu_0) = 6.36 - 2 \log g - 8 \log(\mu_0/\text{eV}) \).

In Fig.8, the possible combinations of \( M \) and \( R \) are plotted for different values of \( f(g, \mu_0) \). Intersections of the dotted lines and the solid line (3D encircled mass profile of A1689) correspond to solutions for representative values of \( f \). Actually \( f \) is continuous and the solutions are continuous. The probable range is \( f = -6 \sim 2 \). For this range of \( f \), the rest mass of a fermion ranges from 30 to 2 eV. The larger the rest mass, the smaller the degenerate structure. The possible range of \( g \) is limited. For a particle with spin 1/2, \( g \) is either 1 (Majorana particle) or 2 (Dirac particle).

The special case is the massive neutrinos with similar masses for which effective \( g = 3 \) (Majorana) or \( g = 6 \) (Dirac). Recent underground experiments have shown that the mass differences among three species of neutrinos are smaller than 0.05 eV (Shirai 2005). In order for massive neutrinos to have masses greater than 1 eV, they must have similar masses (degenerate in mass). The mean number density of relic neutrinos is cosmologically fixed and there is the well known relation for the contribution of the relic neutrinos to \( \Omega \),
\[ \Omega_\nu h^2 = \Sigma_i \mu_i/(93.5\text{eV}), \]  

(43)

where \(i = 1 \sim 3\) for Majorana neutrinos and \(i = 1 \sim 6\) for Dirac neutrinos. The maximum allowed neutrino mass can be estimated by setting \(\Omega_\nu = 0.3\) and \(h = 0.7\), to be 4.7 or 2.3 eV respectively for Majorana or Dirac neutrinos. The former gives \(f = 0\) or the degenerate mass of \(10^{14} M_\odot\) in 100 kpc, and the latter gives \(f = 2\) or that of \(5 \times 10^{14} M_\odot\) in 300 kpc. These are rather comfortable numbers for this cluster profile. If we question the CDM cosmology based on the possible inconsistency of the high concentration of the observed mass profile with the CDM predictions mentioned above, we might as well revive massive neutrinos (hot dark matter) as the candidate for the dominant dark matter.

4. Mass hierarchy of black holes from that of neutrino through FDFS?

In the previous section, we have studied a possibility that fully degenerate fermions forms a huge mass concentration at the center of a cluster of galaxies and we mainly examined the case for massive neutrinos. Is this possibility really true? In order to answer this question, we focus on the universality and the scalability of FDFS. We try to extend this idea of FDFS to other neutrinos and fermions with different masses. As we have already seen in Eq.(1,31), the rest mass of the fermion mostly determines the characteristic mass scale of the structure; more massive fermions form lighter structures.

The most extreme condensed structure is a black hole. We have already known that there exist many black holes of several species in the Universe (Gebhardt et al. 2002). They are the most familiar stellar mass black holes \((\approx M_\odot)\), giant black holes at the center of a galaxy \((\approx 10^7 M_\odot)\), and the intermediate mass black holes \((\approx 10^3 M_\odot)\). Although these massive black holes are actively studied based on the bottom-up scenarios that they are formed by the coalescence of the stellar sized black holes, we try to propose yet another scenario based on the context of FDFS. The most prominent property of those black holes are that they appear to have a hierarchy in mass range. Most of the black holes are classified in the above three types and those in other mass ranges is rare. Therefore it would be natural to suspect any definite mechanism to construct such hierarchy from the fundamental level. Our hypothesis is that such fundamental mechanism is the mass hierarchy in fermions or possibly neutrinos. If those black holes are formed by the overweight FDFSs, we can estimate the masses of those neutrinos as in Table 2.

The corresponding masses of fermions, except \(\nu_{e,\mu,\tau}\), through Eq.(1) are far heavier than eV and we may identify those fermions as more massive sterile neutrinos. Lighter neutrino
of mass $10^{-1} - 10^{-3}$ eV would yield FDFSs much extended dilute structures whose sizes well exceed the Horizon size. In the above, we have approximate values. However, the lower limit of the black hole mass within a class yields the precise value of the corresponding neutrino mass. The existence of such a lower limit would be the key ingredient of the FDFS model.

5. Conclusions and Discussions

We have examined the possible structures formed by the fully degenerate self-gravitating fermions (FDFS) at various scales, such as the mass concentration of a cluster of galaxies, the giant black holes at the center of a galaxy and the intermediate black holes in galaxies. As an order estimation, their characteristic masses directly reflect the constituent fermion masses through the simple relation, $M_{\text{fermi}} = G^{-3/4} m^{-2} = m^{3}_{\text{pl}}/m^{2}$. For the purpose of a quantitative analysis, exact masses and detailed mass density profiles of FDFSs were examined from nonrelativistic to relativistic situation, using the formalism of Oppenheimer and Volkoff.

These results were applied to the cluster of galaxies, A1689, whose mass distribution has been observationally obtained. We converted the observed column density profile to a volume density profile assuming spherical symmetry, and compared the observed and model 3D encircled mass profiles. We found that the flat-top nature of the observed profile is reproduced by the model and the particle mass range is between 2 eV and 30 eV depending on the actual scale of the degenerate structure.

For about black holes, our scenario will provide alternative mechanism of the black hole formation. Most of the present theories assume the coalescence of stellar mass black holes in the gravitational potential. Such processes seem to be quite complex compared to the FDFS scenario, and therefore it would be much difficult to explain, for example, the observed universal relation between the black hole mass at the center of a galaxy and the bulge mass: $M_{BH}/M_{\text{bul}} \approx 0.002$. This point will be discussed in detail in our future work.

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Fig. 1.— Relation between the dimensionless gravitational mass $u$ and the dimensionless radius $r$ of equilibrium configuration for various central Fermi velocity ($v_F$). For $u < 0.04$, there is only one $r$, while $0.04 < u < 0.0766$, there are multiple values of $r$. Among them, however, only the configuration with the largest $r$ is the stable equilibrium. The maximum dimensionless mass $u = 0.0766$ is the mass limit before collapsing to a black hole (square).
Fig. 2.— Dimensionless energy density ($\epsilon$) and pressure ($p$) as functions of dimensionless radius for $t_0 = 3.0$. This is the maximum mass stable configuration. $p$ is small compared to $\epsilon$. Degenerate particles are moderately relativistic and the gravity is quasi-Newtonian.
Fig. 3.— Dimensionless gravitational mass ($u_b$) and total rest mass ($y_b$) plotted as functions of $t_0$. The gravitational mass defect, $\Delta = u_b - y_b$, is negative for $t_0 < 3$ and takes the minimum value for the maximum mass stable configuration ($t_0 = 3$).
Fig. 4.— Normalized density and encircled mass profiles for a nonrelativistic FDFS. Non-relativistic solutions are similar.
Fig. 5.— Normalized 3D encircled mass profile of a nonrelativistic FDFS in logarithmic scale. The slope of the profile is 2.9 for $\log(\frac{r}{r_b}) = -1.5 \sim -0.5$. 

Normalized 3D Encircled Mass Profile

Slope=2.9 ($x=-1.5 \sim -0.5$)
Fig. 6.— 3D encircled mass profile of A1689 obtained from the best-fit volume density profile. $h=0.7$ is assumed.
Fig. 7.— Predicted rotation curve of A1689 estimated from the 3D encircled mass profile. h=0.7 is assumed.
Fig. 8.— Allowed combinations of the mass and size of an FDFS for different values of $f(g, \mu_0) = 6.36 - 2 \log g - 8 \log(\mu_0/\text{eV})$ (dotted lines). The solid line represents the 3D encircled mass profile of A1689. Intersections between the solid and dotted lines correspond to possible degenerate structures.
Table 1: Solutions for various $t_0$.

| $t_0$ | $v_{F0}/c$ | $u_b$  | $y_b$  | $r$  |
|-------|------------|--------|--------|------|
| 0.1   | 0.025      | 0.0012 | 0.0012 | 4.934|
| 0.2   | 0.050      | 0.0033 | 0.0033 | 3.484|
| 0.3   | 0.075      | 0.0060 | 0.0060 | 2.838|
| 0.4   | 0.100      | 0.0091 | 0.0091 | 2.450|
| 0.5   | 0.124      | 0.0126 | 0.0126 | 2.183|
| 1.0   | 0.245      | 0.0325 | 0.0328 | 1.495|
| 1.5   | 0.358      | 0.0516 | 0.0525 | 1.161|
| 2.0   | 0.462      | 0.0659 | 0.0677 | 0.943|
| 2.5   | 0.555      | 0.0740 | 0.0766 | 0.784|
| 3.0a  | 0.635      | 0.0766 | 0.0795 | 0.663|
| 4.0   | 0.762      | 0.0710 | 0.0730 | 0.493|
| 5.0   | 0.848      | 0.0598 | 0.0597 | 0.391|
| 6.0   | 0.905      | 0.0491 | 0.0470 | 0.334|
| 7.0   | 0.941      | 0.0420 | 0.0387 | 0.343|
| 8.0   | 0.964      | 0.0395 | 0.0360 | 0.364|
| 9.0   | 0.978      | 0.0416 | 0.0383 | 0.382|
| 10.0  | 0.987      | 0.0444 | 0.0414 | 0.367|
| 11.0  | 0.992      | 0.0471 | 0.0443 | 0.399|
| 12.0  | 0.995      | 0.0463 | 0.0435 | 0.362|
| 13.0  | 0.997      | 0.0442 | 0.0411 | 0.324|
| 14.0  | 0.998      | 0.0411 | 0.0376 | 0.285|

*aThis solution corresponds to the maximum mass stable configuration. The gravitational mass defect, $u_b - y_b$, takes the minimum.*
Table 2: Neutrino masses and possible hierarchy

|                  | $m_\nu$/eV $\leq$ | $M_{\text{fermi}}/M_\odot$ | $R_{\text{fermi}}$ | FDFS                  |
|------------------|-------------------|-----------------------------|-------------------|-----------------------|
| $\nu_{e,\mu,\tau}$ | 2.3               | $9.8 \times 10^{16}$       | 42 kpc            | the center of a cluster |
| $\nu_1$          | $0.18 \times 10^6$ | $2.5 \times 10^7$          | $460 R_\odot$     | giant BH at center of galaxy |
| $\nu_2$          | $18.2 \times 10^6$ | $2.7 \times 10^3$          | $0.050 R_\odot$   | intermediate BH in a galaxy |