Special points and Poincaré bi-extensions

by Daniel Bertrand, with an Appendix by Bas Edixhoven

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The context is the following:

i) in a joint project with D. Masser, A. Pillay and U. Zannier [7], we aim at extending to semi-abelian schemes the Masser-Zannier approach [11] to Conjecture 6.2 of R. Pink’s preprint [13]; this conjecture also goes under the name “Relative Manin-Mumford”. Inspired by Anand Pillay’s suggestion that the semi-constant extensions of [6] may bring trouble, I found a counter-example, which is described in Section 1 below.

ii) at a meeting in Pisa end of March, Bas Edixhoven found a more concrete way of presenting the counter-example, with the additional advantage that the order of the involved torsion points can be controlled in a precise way: this is the topic of the Appendix.

iii) finally, I realized that when rephrased in the context of mixed Shimura varieties, the construction, far from providing a counter-example, actually supports Pink’s general Conjecture 1.3 of [13]; a sketch of this viewpoint is given in Section 2.

1 A counter-example to relative Manin-Mumford ...

This counterexample is provided by a “Ribet section” on a semi-abelian scheme \( B/X \) of relative dimension 2 over a base curve \( X \). Roughly speaking, given an elliptic curve \( E_0/\mathbb{C} \) with complex multiplications and using an idea of L. Breen, K. Ribet constructed a non-torsion point \( \beta_0 \) with strange divisibility properties on any given non-isotrivial extension \( B_0 \) of \( E_0 \) by \( \mathbb{G}_m \), cf. [10]. In the relative situation \( B/X \), the very same construction yields:

**Theorem 1.** Let \( B/X \) be a non constant (hence non isotrivial) extension of \( E_0 \times X \) by \( \mathbb{G}_m \). There exists a section \( \beta : X \to B \) which does not factor through any proper closed subgroup scheme of \( B/X \), but whose image \( Y := \beta(X) \) meets the torsion points of the various fibers of \( B/X \) infinitely often (so, Zariski-densely, since \( X \) is a curve).

More precisely, let \( X \) be a smooth connected affine curve defined over (say) \( \mathbb{C} \), with function field \( K := \mathbb{C}(X) \). We may have to delete some points of \( X \), or consider finite covers of \( X \), but will still denote by \( X \) the resulting curve. We write \( x \) for the generic point
of $X$, i.e. $K = \mathbb{C}(x)$, and $\xi \in X(\mathbb{C})$ for its closed points. We start with a CM elliptic curve $E_0/\mathbb{C}$, denote by $\hat{E}_0 \simeq E_0$ its dual, and fix an antisymmetric isogeny

$$\varphi : \hat{E}_0 \to E_0.$$ 

This means that its transpose $\hat{\varphi} : \hat{E}_0 \to E_0$ is equal to $-\varphi$, i.e., in the identification $\hat{E}_0 \simeq E_0$, that $\varphi$ is a totally imaginary complex multiplication.

We consider the constant elliptic schemes $E = E_0 \times_{\mathbb{C}} X, \hat{E} = \hat{E}_0 \times_{\mathbb{C}} X$, and fix a non constant section $q : X \to \hat{E}$. In particular, $q$ is not a torsion section, and the semi-abelian scheme $B/X$ attached to $q$ is a non constant\footnote{In particular, $B/X$ is a semi-constant semi-abelian variety in the sense of \cite{6}. However, the counter-example to Lindemann-Weierstrass given there is of a different nature; see also Remark 1.(ii) below.} hence non isotrivial, extension of $E/X \times \mathbb{G}_m$; conversely, any such $B$ is of this type. Let $\pi : B \to E$ be the corresponding $X$-morphism. Extending the construction of [10], we will attach to these data $(q, \varphi)$ a section $\beta$ of $B/X$ such that the section $p := \pi \circ \beta$ of $E/X$ satisfies $p = 2 \varphi \circ q$. Furthermore, $\beta$ will have the following “lifting property” : for any $\xi \in X(\mathbb{C})$ such that $p(\xi)$ is a torsion point on the fiber $E_\xi \simeq E_0$ of $E \to X$, its lift $\beta(\xi)$ is automatically a torsion point on the fiber $B_\xi$ of $B \to X$.

Before describing this construction, let us show that such a section $\beta$ does satisfy the conditions announced in Theorem 1 :

- on the one hand, since $q \in \hat{E}(X)$ has infinite order, the only proper closed subgroup schemes of $B$ projecting onto $X$ are contained in finite unions of translates of $\mathbb{G}_m \times_{\mathbb{C}} X$ and of fibers of the projection; and the section $p = 2 \varphi \circ q \in E(X)$ too has infinite order. Therefore, $\beta$, which projects to $p$ on $E$, cannot factor through any proper closed subgroup scheme of $B/X$.

- on the other hand, since $q$ is not a constant section of $\hat{E}/X$, neither is $p \in E(X)$, and the set $X_{\text{tor}}^E = \{ \xi \in X(\mathbb{C}), p(\xi) \text{ is a torsion point on } E_\xi \simeq E_0 \}$ is infinite. But by the lifting property, this set coincide the set $X_{\text{tor}}^B = \{ \xi \in X(\mathbb{C}), \beta(\xi) \text{ is a torsion point on } B_\xi \}$. Therefore, the curve $Y = \beta(X) \subset B$ meets the set of torsion points of the various fibers of $B \to X$ Zariski-densely.

To perform the construction of $\beta$, we go to the generic fiber $B_x := B \otimes_S K := B_K$ of $B/S$, consider the non-constant point $q(x) := q_K \in \hat{E}_K = \hat{E}_0 \otimes_{\mathbb{C}} K$, and recall the construction of the Ribet point $\beta_K \in B_K(K)$ attached $q_K$ and to the antisymmetric isogeny $\varphi : \hat{E}_K \to E_K$, with transpose $\hat{\varphi} = -\varphi$. There are two ways to describe $\beta_K$ :

(i) The first one \cite{10} goes as follows : consider the pullback

$$\varphi^* B_K \in \text{Ext}(\hat{E}_K, \mathbb{G}_m) \text{ of } B_K \in \text{Ext}(E_K, \mathbb{G}_m)$$

under $\varphi$, and denote again by $\varphi : \varphi^* B_K \to B_K$ the natural extension of $\varphi$ to $\varphi^* B_K$. Since $B_K$ is parametrized by $q_K$, $\varphi^* B_K$ is parametrized by the point $\varphi(q_K)$ of the dual $E_K$ of $E_\xi$. Now, choose an arbitrary $K$-rational point $t_K$ in the fiber above $q_K$ of the extension
\(\varphi^*B_K\); in particular, its image \(t^1_K := \varphi(t_K) \in B_K(K)\) satisfies \(\pi_K(t^1_K) = \varphi(q_K)\), where 
\(\pi_K = \pi(x) : B_K \to E_K\). The point \(t_K\) defines a one-motive \(M_K : \mathbb{Z} \to \varphi^*B_K\), whose Cartier dual \(M_{K'} : \mathbb{Z} \to B_K(K)\) is given by a point \(t^2_K \in B_K(K)\) projecting to \(\hat{\varphi}(q_K) \in E(K)\).

Finally, set \(\beta_K = t^1_K - t^2_K\): this point of \(B_K(K)\) is independent of the choice of the auxiliary point \(t_K\) above \(q_K\) (this is clear on the symmetric definition of duals given in Hodge III), and its image under \(\pi_K\) is the point \(\varphi(q_K) - \hat{\varphi}(q_K) = 2\varphi(q_K) := p_K\) of \(E_K(K)\).

(ii) The second one \([4]\) is more geometric (and will actually not be used here): consider the Poincaré bundle \(\mathcal{P}_K\) on \(E_K \times E_K\), rigidified above \((0, 0)\). Since \(\varphi\) is antisymmetric, the square of its restriction to the graph \(\Phi_K \subset \hat{E}_K \times E_K\) of \(\varphi\) is trivial. Up to a 2-isogeny, we therefore get a unique non-zero \(K\)-regular section \(\sigma \in \mathcal{P}_K|_{\Phi_K}\). Now \(B_K\) (plus a zero section) identifies with the restriction of \(\mathcal{P}_K\) to \(\{q_K\} \times E_K \simeq E_K\), and its fiber over \(p_K' := \varphi(q_K)\) with the fiber of \(\mathcal{P}_K\) over \((q_K, p_K') \in \Phi_K(K)\). We then set \(\beta_K' = \sigma \varphi\left((q_K, p_K')\right)\), and view \(\beta_K'\) as a point of \(B_K(K)\) above \(p_K'\). Up to multiplication by 2, this is the same point as the \(\beta_K\) above.

Restricting \(X\) if necessary, we can extend this point \(\beta_K\) to a section \(\beta : X \to B\), which may be called the Ribet section attached to \(\varphi\) of the semi-abelian scheme \(B/X\) defined by the non-constant section \(q\) of \(\hat{E}/X\) we had started with. More precisely, we can extend the auxiliary point \(t_K\) of the first construction to a section \(t\) of \(\varphi^*B/X\), and repeat the whole process over \(X\) (minus some points), getting in particular a smooth one-motive \(M/X\), sections \(t^1, t^2\) over \(X\), etc. By definition, \(\pi \circ \beta\) is the section \(p = 2\varphi \circ q\) of \(E/X\) extending \(p_K\) over \(X\), and it remains to show that \(\beta\) satisfies the “lifting property”.

So, let \(\xi \in X(\mathbb{C})\) be a point such that \(p(\xi)\) is a torsion point on the fiber \(E_{\xi} (\simeq E_0)\) of \(E \to X\). We must show that \(\beta(\xi)\) is a torsion point on the fiber \(B_{\xi}\) of \(B \to X\). By the relation \(p(\xi) = 2\varphi(q(\xi))\), \(q(\xi)\) too is a torsion point on \(\hat{E}_{\xi}\) (in passing, this shows that \(B_{\xi}\) is an isotrivial extension). So, among the points which lie on the fiber \((\varphi^*B)_{\xi}\) of \(\varphi^*B/X\) above \(\xi\) (which is the pull-back

\[\varphi^*B_{\xi} \in \text{Ext}(\hat{E}_{\xi}, \mathbb{G}_m) \text{ of } B_{\xi} \in \text{Ext}(E_{\xi}, \mathbb{G}_m)\]),

and which project to \(q(\xi) \in (\hat{E}_{\xi})_{\text{tor}}\), we now have not only the value \(t(\xi)\) of the section \(t : X \to \varphi^*B\) at \(\xi\), but also plenty of torsion points of the complex semi-abelian variety \(\varphi^*B_{\xi}\). Choose one of them, and call it \(t_{\xi}\). Since \(t_{\xi}\) and \(t(\xi)\) differ by an element of \(\mathbb{G}_m\), the first construction, whether applied to \(t(\xi)\) or to \(t_{\xi}\), will yield the same point \(\beta_{\xi} \in B_{\xi}\), with \(\pi(\beta_{\xi}) = p(\xi)\). Using \(t(\xi)\), we see that \(\beta_{\xi} = \beta(\xi)\); using the torsion point \(t_{\xi}\) and the fact that \(\varphi(q(\xi))\) is a torsion point, we see that the weight filtrations of the corresponding complex one-motive \(\hat{M}_{\xi}\), hence of its dual \(\hat{M}_{\xi}\), split up completely up to isogeny. Consequently, the points \(t^1_{\xi}, t^2_{\xi}\) and \(\beta_{\xi}\) associated to \(t_{\xi}\) by the first construction are all torsion points, and \(\beta(\xi)\) is indeed a torsion point of \(B_{\xi}\).

**Remark 1** i) (from \(X\) to \(\hat{E}_0\)) : let \(\mathcal{B}\) be the “universal” extension of \(E_0\) by \(\mathbb{G}_m\), viewed as a group scheme over \(\text{Ext}^1(E_0, \mathbb{G}_m) \simeq \text{Pic}^0_{\mathbb{Q}/\mathbb{C}} = \hat{E}_0\). The extension \(B\) attached to the section \(q : X \to \hat{E}_0\) is the pull-back of \(\mathcal{B}\) under \(q\). Choosing \(X = \hat{E}_0\), and \(q = \text{the identity}\)
map, so that $K = \mathbb{C}(\hat{E}_0)$, we can therefore restrict to the case where $K = \mathbb{C}(\hat{E}_0)$ and $q_K$ is the generic point of $E_0(K)$. The Appendix - and most of §2 - concerns this generic case

$$X = \hat{E}_0, q = \text{id}, B = \mathcal{B}.$$

ii) (when $g > 1$): Ribet sections $\beta$ can be defined over any abelian scheme $A/X$, of relative dimension $g$, which admits an antisymmetric isogeny $\varphi : \hat{A} \to A$. If $g = 1$, this forces $A$ to be iso-constant, hence the $E_0/C$ above. But as soon as $g > 1$, there are examples of simple non constant $A/X$ with such a $\varphi = -\hat{\varphi}$. The section $\beta$ attached to $\varphi$ and to a section $q \in \hat{A}(X)$ will again satisfy the “lifting property”. However, in order to ensure that the set $X_{tor}^A = \{ \xi \in X(\mathbb{C}), \pi \circ \beta(\xi) \text{ is a torsion point on } A_\xi \}$ be infinite, one must in general insist that $\dim(X) \geq g$. So, the counterexample does not extend to extensions by $\mathbb{G}_m$ of higher dimensional abelian schemes over curves.

2 ... in support of Pink’s general conjecture.

In [13], R. Pink mentions the similarity of Conjecture 6.2 with Y. Andr´e’s result on special points on elliptic pencils [2], III, p. 9. Viewing the scheme $B/X$ above as a “semi-abelian pencil” over the fixed elliptic curve $E_0$, one could define its special points as the torsion points lying on a fiber which is an isotrivial extension. As mentioned in passing during the proof of the lifting property, the curve $Y = \beta(X)$ even contains infinitely many special points in this sense.

Going further in this direction, we will now construct a mixed Shimura variety $S(\varphi)$ into which the image $Y = \beta(X)$ of the Ribet section $\beta$ can be mapped in a natural way. Denoting this map by $i : Y \to S(\varphi)$, we have under the hypotheses of §1 (or more generally, of Footnote (2) below):

**Theorem 2.** The algebraic subvariety $Z = i(Y)$ of the mixed Shimura variety $S(\varphi)$ passes through a Zariski-dense set of special points of $S(\varphi)$, and is indeed a special subvariety of $S(\varphi)$.

This, of course, is in full concordance with the prediction of the general Conjecture 1.3 of [13] (more specifically, of the case $d = 0$ of Conjecture 1.1).

Here, I will merely give a set-theoretic description of the construction of $S(\varphi)$. We fix an integer $g \geq 1$ and a totally imaginary quadratic integer $\alpha = -\overline{\alpha}$, and denote by $S_0$ (a component of) the pure Shimura variety parametrizing abelian varieties $A$ endowed with a principal polarization $\psi : \hat{A} \to A$ (in particular, $\psi = \hat{\psi}$), with some level structure, and with an embedding $j : \mathbb{Z}[\alpha] \to \text{End}(A)$ such that $\psi \circ \hat{j(\alpha)} \circ \psi^{-1} = j(\overline{\alpha})$. Let

$$(\mathcal{A}, \psi, \varphi := \hat{j(\alpha)} \circ \psi)$$

be the corresponding universal abelian scheme over $S_0$ (in particular, $\varphi = -\hat{\varphi} : \hat{A} \to A$ is antisymmetric). Then, $S_1 := \mathcal{A} \times_{S_0} \hat{A}$ is a mixed Shimura variety parametrizing onemotives of the shape $M : \mathbb{Z} \to A \times \hat{A}$, i.e. couples of points $(p, q) \in A \times \hat{A}$, with $\{A\} \in S_0$, ...
and we can view
\[ S_1(\varphi) = \{(A, p, q) \in S_1, p = 2\varphi(q)\} \]
as a mixed Shimura subvariety of \( S_1 \). Finally, consider the Poincaré bi-extension
\[ \varpi = (\varpi_1, \varpi_2) : \mathcal{P}^* \to A \times_{S_0} \hat{A}. \]

This is a \( \mathbb{G}_m \)-torsor over \( S_1 = A \times_{S_0} \hat{A} \), which can again be viewed as a mixed Shimura variety, now parametrizing one-motives \( M : Z \to B \) of constant and toric ranks equal to 1, where on denoting by \( (A, p := \varpi_1(M), q := \varpi_2(M)) \) the point defined by \( \varpi(M) \) in \( S_1 \), \( B = B_q \) is the extension of \( A \) by \( \mathbb{G}_m \) attached to \( q \), while the image \( b \) of 1 \( \in \mathbb{Z} \) is a point on \( B \) projecting to \( p \in A \). Breaking the symmetry between \( A \) and \( \hat{A} \), we can alternatively consider \( \hat{A} \simeq \text{Ext}_1^1(S_0, \mathbb{G}_m) \) as a mixed Shimura variety, and view \( \varpi_2 : \mathcal{P}^* \to \hat{A} \) as the “universal” extension \( B \) of \( A \) by \( \mathbb{G}_m \), over its parameter space \( \hat{A} \). We at last define
\[ S(\varphi) := \varpi^{-1}(S_1(\varphi)) \]
as the mixed Shimura subvariety of \( \mathcal{P}^* \simeq B \) whose points parametrize one-motives \( M : Z \to B \) such that \( \varpi_1(M) = 2\varphi(\varpi_2(M)) \).

We now consider an abelian scheme \( A/X \) of the type parametrized by \( S_0 \), over some irreducible algebraic variety \( X/\mathbb{C} \). There then exists a unique morphism \( i_0 : X \to S_0 \) such that \( A/X \) is the pull-back of \( A \) under \( i_0 \). We fix a section \( q : X \to \hat{A} \), corresponding to a semi-abelian scheme \( \pi : B \to A \) over \( X \), and perform Ribet’s construction\(^2\), yielding sections \( p = 2\varphi \circ q : X \to A, \ \beta : X \to B \), with \( p = \pi \circ \beta \). By the universal property of \( S(\varphi) \), there exists a unique morphism
\[ i : X \to S(\varphi) \]
above \( i_0 \) such that the smooth \( X \)-one-motive \( M : Z \to B \) defined by \( \beta \) is the pull-back under \( i \) of the universal one-motive \( \mathcal{M} : Z \to B \). We again denote by \( i : B \to S(\varphi) \subset B \) the extension of \( i \) to \( B/X \). The image \( Z := i(Y) \subset S(\varphi) \) of \( Y = \beta(X) \subset B \) is the algebraic subvariety of \( S(\varphi) \) to be studied for Theorem 2.

We first check that \( Z \) contains a Zariski-dense set of special points of \( S(\varphi) \). By definition, these points represent complex one-motives such that the underlying abelian variety is CM, and whose weight filtrations are totally split up to isogeny. By the first hypothesis

\(^2\) When \( g = 1 \) as in the first Section, \( S_0 \) is reduced to a CM point \( \{E_0\} \). Taking into account Remark 1.(ii), we are now proving Theorem 2 for any \( g \geq 1 \). However, in the case \( g > 1 \), we must add the following hypotheses to its statement : the base \( X/\mathbb{C} \) is an irreducible variety of dimension \( \geq g \), and
(a) the image \( i_0(X) \) meets the set of CM points of \( S_0 \) Zariski-densely.
(b) \( q(X) \) meets the set of torsion points of the various CM fibers of \( \hat{A} \) Zariski-densely; For the sake of simplicity, we will assume, in what follows, that \( i_0(X) \) is a Shimura subvariety of \( S_0 \) (more or less \( \Leftrightarrow \) (a) under André-Oort), and that \( q : X \to \hat{A} \) dominates the “generic case” id : \( \hat{A} \to \hat{A} \) \( \Rightarrow \) (b)). Furthermore, the latter hypothesis implies the useful (but not necessary) property that :
(c) \( q \) factors through no proper closed subgroup scheme of \( \hat{A}/X \).
made in Footnote \(^{(2)}\), the projection \(i_0(X)\) of \(Z\) to \(S_0\) passes through a Zariski-dense set of CM points. In the fiber of any such point, the projection of \(Z\) to \(S_1(\varphi)\) passes through a Zariski-dense set of torsion points \((p = 2\varphi(q), q)\), because of the second hypothesis made in this foot-note. The lifting property established in §1 (more accurately, the sharper version mentioned in passing) now shows that \(Z\) does meet the set of special points of \(S(\varphi)\) Zariski-densely.

It remains to show that \(Z\) is a special subvariety of \(S(\varphi)\). We will check this Hodge-theoretically. Denote by \(G\), resp. \(P\), the generic Mumford-Tate group of the Shimura variety \(i_0(X)\), resp. of the mixed Shimura subvariety of \(S(\varphi)\) lying above \(i_0(X)\). Then, \(P\) is the semi-direct product of its unipotent radical \(W_{-1}(P)\) by \(G\), and \(W_{-1}(P)\) is the semi-direct product of \(W_{-2}(P) \subset \mathbb{G}_a\) by the vectorial group \(Gr_{-1}(P) \subset \mathbb{G}_a^{2g}\): notice that \(Gr_{-1}(P)\) is the unipotent radical of the generic Mumford-Tate group of \(S_1(\varphi)\), and the inclusion \(Gr_{-1}(P) \subset \mathbb{G}_a^{2g}\) follows from the linear dependence relation \(p = 2\varphi(q)\). Similarly, let \(P_z \subset P\) denote the Mumford-Tate group of a sufficiently general point \(z\) in \(Z\); then, \(P_z\) is the semi-direct product of its unipotent radical \(W_{-1}(P_z)\) by \(G\), and \(W_{-1}(P_z)\) is the semi-direct product of \(W_{-2}(P_z) \subset W_{-2}(P)\) by the vectorial group \(Gr_{-1}(P_z) \subset Gr_{-1}(P)\).

Now, by the second hypothesis of the footnote, \(Z, q(X)\) is Zariski-dense in \(\hat{A}\), and Proposition 1 of [4] shows that \(Gr_{-1}(P_z) = Gr_{-1}(P) = \mathbb{G}_a^{2g}\). On the other hand, Theorem 1 of [4] shows that \(W_{-2}(P) = \mathbb{G}_a\), while \(W_{-2}(P_z) = \{0\}\), and more precisely, that for any point \(s \in S(\varphi)(\mathbb{C})\) whose Mumford-Tate group \(P_s\) satisfies \(Gr_{-1}(P_s) = \mathbb{G}_a^{2g}\), we have

\[
W_{-2}(P_s) = \{0\} \iff \exists \tilde{s} \in [s], \tilde{s} \in Z,
\]

where \([s]\) denotes the Hecke orbit of \(s\). In other words, up to isogenies, the points of \(Z\) are characterized by the existence of an exceptional Hodge tensor in their Betti realization, which does not exist at the generic point of \(S(\varphi)\). So, \(Z\) is indeed a special subvariety of \(S(\varphi)\).

**Remark 2**: (i) This section shows that the special subvarieties of the mixed Shimura variety \(\mathcal{B}\) do not necessarily correspond to families of semi-abelian subvarieties, so that Theorem 5.7 and 6.3 of [13] must be modified. Similarly, the mixed Shimura varieties of Hodge type \(\{W_{-2}(P) = 0\}\) are not necessarily Kuga fiber varieties (compare [12], Example 1.10, and [4], Remark (v)).

(ii) The strange divisibility properties of Ribet points alluded to in the introduction of §1 are precisely reflected by the \(\ell\)-adic analogue of the vanishing of \(W_{-2}(P_z), z \in S(\varphi)(\overline{\mathbb{Q}})\); cf. [4], Theorem 1.(ii). Actually, K. Ribet gives in [14] an explicit description of the exceptional Galois invariant tensor occurring in the \(\ell\)-adic cohomology of these one-motives, which applies to all their realizations.

(iii) We refer to [3] for a complete description of the unipotent radical \(W_{-1}(P)\) of the Mumford-Tate group of one-motives of higher toric or constant ranks. When both ranks are equal to 1, the fact that \(W_{-2}(P)\) vanishes in the case of antisymmetrically self-dual one-motives is an exercise in group theory, cf. [5], Lemme 6.
3 Appendix, by Bas Edixhoven

We go back to the setting and the notations $E_0, X, q, \varphi = -\varphi, \pi : B \to E$ of §1, but to make later comparisons easier, we henceforth consider the extension

$$B = B_{2q}$$

of $E$ by $\mathbb{G}_m$ given by $2q \in \hat{E}(X)$, and denote by $\beta_R$ in $B_{2q}(X)$ the Ribet section which the construction of § 1 attaches to the data $\{2q, \varphi\}$; in particular, the projection $p := \pi \circ \beta_R = \varphi(2q) - \hat{\varphi}(2q) = 2\varphi(2q))$ of $\beta_R$ on $E(X)$ is now twice the section considered in §1.

Interpreting $B_{2q}/X$ as the generalized jacobian $\text{Pic}_{C/X}^0$ of a singular curve $C/X$ with normalization $\hat{E} = \text{Pic}_{E/X}^0 = \hat{E}_0 \times X/X$, we will here construct a concrete section $\beta_J$ (with “J” for “Jacobian”) of $B_{2q}/X$, which enjoys all the properties of the Ribet section $\beta_R$, and thereby provides a new proof of Theorem 1, but for which the “lifting property” takes the following sharper form:

**Theorem 3.** For any $\xi \in X$, the image of $\beta_J(\xi)$ under $\pi : B_\xi \to E_\xi \simeq E_0$ satisfies $\pi(\beta_J(\xi)) = \pi(\beta_R(\xi)) := p(\xi)$. And if $p(\xi)$ is a torsion point of $E_0$ of order $n$, with $n$ prime to $2\deg(\varphi)\deg(\varphi + \psi)\deg(\varphi - \psi)$, then $\beta_J(\xi)$ is a torsion point of $B_\xi$, of order dividing $n^2$.

Here, $\psi = \hat{\psi} : \hat{E} \to E$ denotes the standard principal polarization: its inverse sends a point $P$ to the class of the divisor $(P) - (0)$. Without loss of generality, we assume that we are in the generic case $X = \hat{E}_0, q = 1$, with $K := \mathbb{C}(X) = \mathbb{C}(\hat{E}_0)$, but we keep to the notation $X$ to indicate that some points of $X$ may have to be removed in the constructions which follow. We denote by $Q = \psi \circ q : X \to E$ the section of $E/X$ such that $q \in \hat{E}(X) = \text{Pic}^0(E/X)/\text{Pic}(X)$ is represented by the divisor $(Q) - (0)$ on $E$. By biduality, the section $Q$ in $E(X) = \text{Pic}^0(\hat{E}/X)/\text{Pic}(X)$ is represented by the divisor $(q) - (0)$ on $\hat{E}$.

Let $C/X$ be the singular curve over $X$ obtained by identifying the disjoint sections $q$ and $-q$ of $\hat{E}$ (we remove $\hat{E}_0[2]$ from $X$). As a set, it is the quotient of $\hat{E}$ by the equivalence relation generated by $q(\xi) \sim -q(\xi)$ with $\xi$ ranging over $X$. The topology on $C$ is the finest one for which the quotient map $\text{quot} : \hat{E} \to C$ is continuous: a subset $U$ of $C$ is open if and only if $\text{quot}^{-1}U$ is open in $\hat{E}$. The regular functions on an open set $U$ of $C$ are the regular functions $f$ on $\text{quot}^{-1}U$ such that $f(q(\xi)) = f(-q(\xi))$ whenever $\text{quot}(q(\xi))$ is in $U$. It is proved in Thm. 5.4 of [9] that this topological space with sheaf of $\mathbb{C}$-valued functions is indeed an algebraic variety over $\mathbb{C}$. In categorical terms, $\text{quot} : \hat{E} \to C$ is the equalizer of the pair of morphisms $(q, -q)$ from $X$ to $\hat{E}$.

The curve $C \to X$ is a family of singular curves, each with an ordinary double point; it is semi-stable of genus two (see §8 9.2/6]). Its normalization is $\text{quot} : \hat{E} \to C$. Its

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3 I thank Lenny Taelman for the suggestion to pay more attention to the symmetry between the points $q$ and $-q$ to be identified and the divisor $\beta_a$ that gives the section $\beta_J$.

4 In particular, $\beta_J$ does not factor through any proper closed subgroup scheme of $B$. For a discussion relating $\beta_J$ and $\beta_R$, see Remark 3.ii below.
generalized jacobian $B := \text{Pic}^0_{C/X}$ is described in \cite{S}, 8.1/4, 8.2/7, 9.2/1, 9.3/1. As $C \to X$ has a section (for example $\emptyset := \text{quot} \circ 0$ and $\tilde{q} := \text{quot} \circ q$), we have, for every $T \to X$, that $B(T) = \text{Pic}^0(T \times_X C/T)/\text{Pic}(T)$, where $\text{Pic}^0(T \times_X C/T)$ is the group of isomorphism classes of line bundles on $T \times_X C$ that have degree zero on the fibres of $T \times_X C \to T$. The group $\text{Pic}(T)$ is contained as direct summand in $\text{Pic}^0(T \times_X C/T)$ via pullback by the projection $T \times_X C \to X$ and a chosen section. In particular, a divisor $D$ on $C$ that is finite over $X$, disjoint from $\tilde{q}(X)$ and of degree zero after restriction to the fibers of $C \to X$ gives the invertible $\mathcal{O}_C$-module $\mathcal{O}_C(D)$ that has degree zero on the fibers and therefore gives an element denoted $[D]$ in $B(X)$.

For $\xi$ in $X$, the fiber $B_\xi$ is, as abelian group, the group $\text{Pic}^0(C_\xi)$. In terms of divisors this is the quotient of the group $\text{Div}^0(C_\xi)$ of degree zero divisors with support outside $\{\tilde{q}(\xi)\}$ by the subgroup of principal divisors $\text{div}(f)$ for nonzero rational functions $f$ in $\mathbb{C}(C_\xi)^\times$ that are regular at $\tilde{q}(\xi)$. As $C_\xi - \{\tilde{q}(\xi)\}$ is the same as $\hat{E}_0 - \{\tilde{q}(\xi), -\tilde{q}(\xi)\}$, $\text{Div}^0(C_\xi)$ is the group of degree zero divisors on $\hat{E}_0$ with support outside $\{\tilde{q}(\xi), -\tilde{q}(\xi)\}$. An element $f$ of $\mathbb{C}(C_\xi)^\times$ that is regular at $\tilde{q}(\xi)$ is an element of $\mathbb{C}(\hat{E}_0)^\times$ that is regular at $q(\xi)$ and $-q(\xi)$ and satisfies $f(q(\xi)) = f(-q(\xi))$. This gives us a useful description of $B_\xi$.

The normalization map quot: $\hat{E} \to C$ induces a morphism of group schemes over $X$

$$\pi: B = \text{Pic}^0_{C/X} \to \text{Pic}^0_{\hat{E}/X} = E/X,$$

and identifies $B$ with the extension of $E$ by $\mathbb{G}_m$ given by the section $2q \in \hat{E}(X)$. For $\xi$ in $X$, the class $[\delta]$ in $B_\xi$ of a divisor $\delta \in \text{Div}^0(C_\xi)$ lies in the kernel $\mathbb{C}^\times$ of $\pi_\xi$ if and only if there exists $f \in \mathbb{C}(\hat{E}_0)^\times$ such that $\delta = \text{div}(f)$ on $\hat{E}_0$, and it is then a torsion point in $\mathbb{C}^\times$ if and only if the quotient $f(q(\xi))/f(-q(\xi)) \in \mathbb{C}^\times$, which does not depend on the choice of $f$, is a root of unity.

We recall that for an elliptic curve $E$ over an algebraically closed field $k$, $\psi: \hat{E} \to E$ the standard polarization and $u$ in $\text{End}(E)$, the pullback map $u^*$ on $\text{Div}(E)$ induces $\hat{u}$ in $\text{End}(\hat{E})$, the dual of $u$, and then $\hat{u} := \psi \hat{u} \psi^{-1}$ in $\text{End}(\hat{E})$ is called the Rosati-dual of $u$; it is characterized by the property that in $\text{End}(E)$ we have $\hat{u}u = \text{deg}(u)$ and $u + \hat{u} \in \mathbb{Z}$. Also, the pushforward map $u_*$ on $\text{Div}(E)$ induces an element still denoted $u_*$ in $\text{End}(\hat{E})$ such that $\psi^{-1}u = u_\psi^{-1}$ and $u_*u^*$ is multiplication by $\text{deg}(u)$ in $\text{End}(\hat{E})$. Hence $u_*$ is the Rosati dual of $u^*$. We have $\hat{u} = u$, $\hat{u} = u$ and $\tilde{u}_* = u^*$ in $\text{End}(\hat{E})$. For $f$ a nonzero rational function on $E$ and $u \neq 0$ we have $u^*\text{div}(f) = \text{div}(f \circ u)$, and $u_*\text{div}(f) = \text{div}($\text{Norm}_u(f)$), where $\text{Norm}_u: k(E)^\times \to k(E)^\times$ is the norm map along $u$. Of course, all this applies to $E_0$ and $\hat{E}_0$ over $\mathbb{C}$, and to $E$ and $\hat{E}$ over the algebraic closure of the function field $K$ of $X$.

We will use Weil reciprocity: for $f$ and $g$ nonzero rational functions on a nonsingular irreducible projective curve $E$ over an algebraically closed field $k$, $\text{div}(f)$ and $\text{div}(g)$ have disjoint supports, one has $f(\text{div}(g)) = g(\text{div}(f))$, where for $D = \sum P D(P).P$ a divisor on $E$ one defines $f(D) = \prod P f(P)D(P)$, cf. \cite{L}, III, Prop. 7. In Remark 3.(i) after the proof, we will also use the Weil pairing on $E$. For $n$ a positive integer and $P$ and $Q$ in $\text{Pic}^0(E)[n]$ the element $e_n(P, Q)$ in $\mu_n(k)$ is defined as follows. Let $D_P$ and $D_Q$ in $\text{Div}^0(E)$ be disjoint divisors representing $P$ and $Q$. Let $f$ and $g$ be in $k(E)^\times$ such that
\(nD_p = \text{div}(f)\) and \(nD_q = \text{div}(g)\). Then \(e_n(P,Q) = f(D_q)/g(D_p)\). For \(n\) invertible in \(k\) this pairing \(e_n\) is a perfect alternating pairing.

To define \(\beta_J = \beta_J(a)\), we let \(a \in \text{End}(\hat{E}_0)\) be any endomorphism such that \(a^3 - a \neq 0\). We set \(\alpha := \hat{a}\) in \(\text{End}(E_0)\) and \(\varphi := \alpha \circ \psi: \hat{E}_0 \to E_0\). Just as in [10], we will not need that \(\hat{\varphi} = -\varphi\). However, if \(\hat{\varphi} = \varphi\) then \(\pi \beta_J\) will be zero in \(\hat{E}(X)\), so we will insist that \(\hat{\varphi} - \varphi \neq 0\). For \(s\in E(X)\) we let \((s)\) denote the relative divisor that it gives on \(E\), and, if \((s)\) is disjoint from the singular locus \(\bar{q}(X)\) of \(C\), also the relative divisor on \(C\) that it gives.

We set:

\[\beta_a := a^∗((q) - (-q)) - a_∗((q) - (-q)) \text{ in } \text{Div}^0(\hat{E}/X).\]  

The reader should note the antisymmetric use of both \(a^∗\) and \(a_∗\) in the definition of \(\beta_a\).

To get its support disjoint from \(\bar{q}(X)\) we remove from \(X\) the finite set \(\ker(a^2 - 1)\); note that, for \(\xi \in X\), \(\beta_a(\xi)\) and \((q(\xi)) = (-q(\xi))\) are not disjoint if and only if \(aq(\xi) = q(\xi)\) or \(aq(\xi) = -q(\xi)\). We can now also view \(\beta_a\) as element of \(\text{Div}^0(C/X)\), and we set:

\[\beta_J := [\beta_a] \text{ in } B(X).\]

The image of \(\pi \beta_J\) of \(\beta_J\) in \(E(X)\) is the class of the divisor \(\beta_a\) on \(\hat{E}\), hence we have, on denoting by \(\simeq\) linear equivalence on \(\text{Div}^0(\hat{E}/X)\):

\[\pi \beta_J = \tilde{a}_∗((q) - (-q)) - \alpha_∗((q) - (-q))\]
\[\simeq (\tilde{a}q) - (\tilde{a}(-q)) - ((aq) - (a(-q)))\]
\[\simeq (2(\tilde{a} - a)q) - (0) = 2(\alpha - \tilde{a})Q \text{ in } E(X).\]

In particular, since \(\tilde{a} - a\) is nonzero, then \(\pi \beta_J\) is not torsion in \(E(X)\), and in fact

\[\pi \beta_J = ((2\tilde{a}q) - (0)) - ((2aq) - (0)) = \varphi(2q) - \hat{\varphi}(2q) = p = \pi \beta_R;\]

as was to be checked for the first part of Theorem 3.

Now we start the proof of the second part of Theorem 3. Let \(n\) be a positive integer prime to \(2\deg(\varphi)\), \(2\deg(\varphi + \psi)\) \(\deg(\varphi - \psi) = 2\deg(a(a^2 - 1))\), and let \(\xi \in X\) be a point such that \(p(\xi) \in E_\xi = E_0\) is a torsion point of order \(n\). Actually, we will assume that \(q(\xi)\) is a point of order \(n\) in \(E_\xi = \hat{E}_0)\): in the antisymmetric case considered in Theorem 3, we have \(p(\xi) = 4\varphi(q(\xi))\), and the two conditions are equivalent by the primality assumption. To ease notations, we now drop the mention of \(\xi\), writing \(E, B, q, . . . \) instead of \(E_\xi, B_\xi, q(\xi), . . .\).

As \(nq = 0\) in \(\hat{E}_0\), we have \(n\pi \beta_J = 0\) in \(E_0\). This means that \(n\beta_a\) is a principal divisor on \(\hat{E}_0\). Let \(f \in \mathbb{C}(\hat{E}_0)^∗\) be such that \(\text{div}(f) = n(q) - n(-q)\) in \(\text{Div}(\hat{E}_0)\). Then we have, on \(\hat{E}_0\):

\[\text{div}(f \circ a) = a^∗\text{div}(f) = a^∗(n(q) - n(-q));\]
\[\text{div}(\text{Norm}_a(f)) = a_∗\text{div}(f) = a_∗(n(q) - n(-q)).\]
We define:
\[ g_a := (f \circ a)/\text{Norm}_a(f) \quad \text{in } \mathbb{C}(\hat{E}_0)^\times. \]
Then we have:
\[ n\beta_a = \text{div}(f \circ a) - \text{div}(\text{Norm}_a(f)) = \text{div}(g_a) \quad \text{on } \hat{E}_0. \]
This means that \( n[\beta_J] \) in \( B \) is the element \( g_a(q)/g_a(-q) \) of \( \mathbb{C}^\times \). As the divisor of \( f \) has support disjoint from that of \( g_a \) and of \( a^\ast\text{div}(f) \) and \( a_\ast\text{div}(f) \), Weil reciprocity gives us:
\[
\left( \frac{g_a(q)}{g_a(-q)} \right)^n = g_a(\text{div}(f)) = f(\text{div}(g_a)) = f(\text{div}(f \circ a) - \text{div}(\text{Norm}_a(f)))
= \frac{f(\text{div}(f \circ a))}{f(\text{div}(\text{Norm}_a(f)))} = \frac{(f \circ a)(\text{div}(f))}{f(a_\ast\text{div}(f))} = 1.
\]
This finishes the proof of Theorem 3.

**Remark 3**: i) We have shown that for \( \xi \) torsion in \( X \subset \hat{E}_0 \) and \( n \) its order, we have \( n^2\beta_J(\xi) = 0 \) in \( B_\xi \). We will show that for \( n > 1 \) odd and prime to \( \deg((a^2 - 1)(\bar{a} - a)) \) there exist \( \xi \) in \( X \) with \( p(\xi) \) of order \( n \) in \( E_0 \) such that the order of \( \beta_J(\xi) \) equals \( n^2 \).

Let \( n \) be such an integer. Recall the notation \( Q = \psi(q) \), and let \( \xi \) in \( \hat{E}_0 \) be of order \( n \) such that \( e_n(2(\alpha - \bar{\alpha})Q(\xi), 2Q(\xi)) \) is of order \( n \) in \( \mathbb{C}^\times \). Such \( \xi \) exist because \( \alpha - \bar{\alpha} \) is an automorphism of \( E_0[n] \) that is not scalar multiplication by an element of \( \mathbb{Z}/n\mathbb{Z} \). And such a \( \xi \) is in \( X \) because \( n \) is prime to \( 2\deg(a^2 - 1) \). By construction, \( p(\xi) = 2(\alpha - \bar{\alpha})Q(\xi) \) is represented by the divisor \( \beta_a(\xi) \), and \( 2Q(\xi) \) is represented by \( (q(\xi)) - (-q(\xi)) \). Again, let us drop the \( \xi \)'s from our notation. Then \( n\beta_a = \text{div}(g_a) \), and \( n((q) - (-q)) = \text{div}(f) \). We first compute:
\[
f(\beta_a) = f(a^\ast((q) - (-q))) - a_\ast((q) - (-q))) = \frac{(\text{Norm}_a(f)((q) - (-q)))}{(f \circ a)((q) - (-q))}
= g_a^{-1}((q) - (-q)) = \frac{g_a(-q)}{g_a(q)}.
\]
Hence:
\[
e_n(2(\alpha - \bar{\alpha})Q, 2Q) = \frac{g_a((q) - (-q))}{f(\beta_a)} = \frac{g_a(q)}{g_a(-q)f(\beta_a)} = \left( \frac{g_a(q)}{g_a(-q)} \right)^2.
\]
ii) The Ribet section \( \beta_R \) and the present section \( \beta_J \) of \( B/X \) differ by an element of \( \mathbb{G}_m(X) \), and it is natural to ask whether they are actually equal. In this respect, notice that by Theorem 3, \( \beta_J(X) \) passes through a Zariski-dense set of special points of the mixed Shimura variety \( S(\varphi) \) of \( \S2 \). If we assume Pink's general Conjecture 1.3 of \cite{13}, \( \beta_J(X) \) must then be a component of the Hecke orbit of the special subvariety \( Z \) defined by \( \beta_R(X) \). Hence, at least conjecturally, the section \( \beta_R - \beta_J \) of \( B/X \) is a root of unity.
References

[1] Y. André : Mumford-Tate groups of mixed Hodge structures and the theorem of the fixed part; Compo. math., 82, 1992, 1–24.

[2] Y. André : Shimura varieties, subvarieties and CM points; Taiwan, 2001.

[3] C. Bertolin : Le radical unipotent du groupe de Galois d’un 1-motif; Math. Annalen, 327, 2003, 585–607.

[4] D. Bertrand : Relative splittings of one-motives; Contemporary Maths 210, 1998, 3–17.

[5] D. Bertrand : Extensions panachées autoduales; arXiv: mathAG1011.4685, 2010.

[6] D. Bertrand, A. Pillay : A Lindemann-Weierstrass theorem for semi-abelian varieties over function fields; J. Amer. Math. Soc., 23, 2010, 491–533.

[7] D. Bertrand, D. Masser, A. Pillay, U. Zannier : (work in preparation).

[8] S. Bosch, W. Lütkebohmert, M. Raynaud : Néron models; Springer 1990.

[9] D. Ferrand : Conducteur, descente et pincement; Bull. Soc. Math. France 131, 2003, 553–585.

[10] O. Jacquinot, K. Ribet : Deficient points on extensions of abelian varieties by \( \mathbb{G}_m \); J. Number Th., 25, 1987, 133–151.

[11] D. Masser, U. Zannier : Torsion anomalous points and families of elliptic curves; CRAS Paris, 346, 2008, 491–494.

[12] J. Milne : Canonical models of (mixed) Shimura varieties and automorphic vector bundles; Persp. Maths 10, 1990, 283–414.

[13] R. Pink : A common generalization of the conjectures of André-Oort, Manin-Mumford, and Mordell-Lang; preprint (13 p.), April 2005.

[14] K. Ribet : Cohomological realization of a family of one-motives; J. Number Th., 25, 1987, 152–161.

[15] J-P. Serre : Groupes algébriques et corps de classes; Hermann, 1959.

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