ROBUST OPTIMAL INVESTMENT AND REINSURANCE OF AN INSURER UNDER JUMP-DIFFUSION MODELS

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ABSTRACT. This paper studies a robust optimal investment and reinsurance problem under model uncertainty. The insurer’s risk process is modeled by a general jump process generated by a marked point process. By transferring a proportion of insurance risk to a reinsurance company and investing the surplus into the financial market with a bond and a share index, the insurance company aims to maximize the minimal expected terminal wealth with a penalty. By using the dynamic programming, we formulate the robust optimal investment and reinsurance problem into a two-person, zero-sum, stochastic differential game between the investor and the market. Closed-form solutions for the case of the quadratic penalty function are derived in our paper.

1. Introduction. The optimal investment-reinsurance problem has a long history in finance and actuarial science. To avoid different risk or to increase revenue, most insurance companies need to optimize their asset allocation by investing in money and share markets. Since the pioneering work of [5], the optimal investment problems for maximizing the expected utility of terminal wealth or minimizing the ruin

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probability have been widely considered in the finance and actuarial science literature, see \cite{20, 6, 7, 39, 35, 15} and so on. Besides the investment, reinsurance is a practical and effective tool for insurance companies to transfer their risk exposures to reinsurance companies. In the actuarial science literature, the optimal reinsurance problem usually comes with the problem of maximizing the expectation of the accumulated dividends, see for examples, \cite{2, 1, 17, 25, 26, 33, 30, 37, 38, 44, 16}. In this context, it is of practical importance to develop appropriate investment-reinsurance strategy to optimize their trades and to manage their risk exposures. In the past twenty years, the optimal investment problem and its fusion with the optimal reinsurance problem have been studied by many researchers, such as, \cite{31, 18, 19, 21, 42}, and so on.

Most of the literature mentioned above, however, overlooked the issue of model uncertainty in the optimal investment and reinsurance problem of an insurer. In practice, it is a common belief that there is no agreement on which model, or real world probability, should be used in the optimal investment and reinsurance problem. So model uncertainty exists widely in the finance and insurance optimal control problems. \cite{8} discussed an approach to the incorporation of parameter and model uncertainty in the process of making inferences about some quantity of interest. \cite{23} considered the dynamic portfolio and consumption problem of an investor who worries about model uncertainty and seeks robust decisions. \cite{11} introduced a quantitative framework for measuring model uncertainty in the context of derivatives pricing. \cite{4} studied the robust portfolio planning problem and analyzed the impact of uncertainty about diffusion and jump risks simultaneously.

Recently, there is an increasing number of research papers focused on the optimal investment-reinsurance problem under the framework of model uncertainty. \cite{36} considered robust optimal strategies for an insurer with reinsurance and investment under benchmark and mean-variance criteria but only for a diffusion model without jump. \cite{41} introduced a novel approach to optimal investment-reinsurance problems of an insurance company facing model uncertainty with jump via a game theoretic approach. Closed-form solutions for problems of the max-min expected exponential utility and max-min ruin probability are obtained. However, in their model, the uncertainty about diffusion risk and jump risk is modeled by only one parameter and the stock price is modeled by a geometric Brownian motion without jump. \cite{43} investigated the robust optimal portfolio and reinsurance problem under a Cramér-Lundberg risk model for an ambiguity-averse insurer. \cite{32} considered a robust optimal investment and reinsurance problem under model ambiguity and default risk for an insurer, who can trade in a saving account, a stock and a defaultable bond and aims to maximize the minimal expected utility. Although the above two papers considered the jump risk for the surplus model of the insurer, the stock price is modeled by a continuous stochastic process without jump and the parameters of uncertainty about diffusion risk and jump risk do not depend on the jump size. In this paper, we investigate the robust optimal investment-reinsurance problem for the jump-diffusion model with dependent jump risk. Both the surplus process of the insurer and the stock price are modeled by jump-diffusion processes and there are dependent jumps between the insurer’s surplus process and the stock price. Moreover, the parameters of modeling uncertainty in our paper depend on the jump size of the model.

This paper is organized as follows. We introduce the insurance risk process and the price dynamics of the investment assets in Section 2, and present the robust
investment–reinsurance problem in Section 3. In Section 4, we first introduce an auxiliary stochastic control problem and then find a relationship between the original robust stochastic control problem and the auxiliary stochastic control problem. We discuss the HJBI dynamic programming approach to solve the robust investment–reinsurance control problem in Section 5. In Section 6, we derive closed-form expressions for the quadratic penalty function and provide a numerical example.

2. The insurance and financial market model. In this section, we shall introduce the insurance and financial market model for our problem. We start with a stochastic basis \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) with a finite time horizon \(T < \infty\) and a filtration \(\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}\) satisfying the usual conditions of right continuity and completeness. On this stochastic basis, let \(W = \{W(t) | t \in [0, T]\}\) be a standard Brownian motion and let \(N\) denote an integer-valued random measure

\[
N(dt, dx) = (N(\omega, dt, dx)|\omega \in \Omega)
\]
on \(([0, T] \times \mathbb{R}_0, \mathcal{B}([0, T] \times \mathbb{R}_0))\) with compensator \(n(dt, dx)\), where \(\mathbb{R}_0 := \mathbb{R}\setminus\{0\}\).

In the following, we shall assume that the compensator of the random measure \(N(dt, dx)\) has the following form

\[
n(dt, dx) := \nu(dx)dt,
\]

where \(\nu\) is a \(\sigma\)-finite measure on \((\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0))\) satisfying \(\int_{\mathbb{R}_0} (1 + |x|^2)\nu(dx) < \infty\).

Now we introduce the risk process of an insurer. Suppose that the aggregated claim amount up to time \(t\) is given by

\[
C(t) = \int_0^t \int_{\mathbb{R}_0} x \mathbb{1}_{x \geq 0} N(ds, dx).
\]

We assume that \(C(t)\) and \(W(t)\) are stochastically independent under \(\mathbb{P}\).

Let \(p(t)\) denote the insurance premium accumulated up to time \(t\) and \(\delta > 0\) denote the insurer’s relative security loading. Following the convention of the actuarial risk theory literature, we set

\[
p(t) = (1 + \delta)\mathbb{E}[C(t)] = (1 + \delta) \int_0^t \int_{\mathbb{R}_0} x \mathbb{1}_{x \geq 0} \nu(dx)ds = (1 + \delta)\mu_1 t,
\]

where \(\mathbb{E}[\cdot]\) denotes the expectation under \(\mathbb{P}\) and \(\mu_1 := \int_{\mathbb{R}_0} x \mathbb{1}_{x \geq 0} \nu(dx)\). We suppose that the risk process of the insurance company is described by the following model:

\[
U(t) = x + p(t) - C(t), \quad U(0) = x. \tag{1}
\]

We next introduce the financial market in which the insurance company can invest. Suppose that \(r(t)\), \(\mu(t)\) and \(\sigma(t)\) are deterministic bounded functions, and \(\gamma(t, x)\) is also deterministic and satisfies

\[
\sup_{t \in [0, T]} \int_{\mathbb{R}_0} \gamma(t, x)^2 \nu(dx) < +\infty.
\]
The financial market consists of a risk-free asset \(S_0\) and a risky asset \(S_1\) whose price processes are described by the following stochastic differential equations

\[
dS_0(t) = r(t)S_0(t)dt, \quad S_0(0) = 1, \tag{2}
\]

\[
dS_1(t) = S_1(t-)[\mu(t)dt + \sigma(t)dW(t) + \int_{\mathbb{R}_0} \gamma(t, x)\tilde{N}(dt, dx)], \quad S_1(0) = s_1 > 0, \tag{3}
\]

where \(\tilde{N}(dt, dx) := N(dt, dx) - \nu(dx)dt\).
It is worth mentioning that a general marked point process is used to illustrate the influences of the insurance and finance events on the surplus of insurance company and the risky asset price. From equations (1) and (3), we see that only positive marks of the marked point process lead to insurance claims, while both the positive and negative marks affect the risky asset price. Thus, in our framework the jumps of the risky asset price happen not only at the times when insurance claims occur, but also at the times of negative marks of the marked point process. Indeed, if we take \( \gamma(t, x) \in (0, \infty) \) for \( x < 0 \), then the risky asset price experiences positive jumps upon the arrival of negative marks\(^1\). Such a modeling framework of dependent jumps is not unsupported. In fact, as reported by the National Association of Insurance Commissioners [28], among $727 billion of the industry’s aggregate investment in common stocks in 2016, insurance companies allocate 55% of total reported common stock holdings to affiliated common stocks. In this regard, it is not unreasonable to assume that negative jumps associated with insurance claims are contagious in the financial market and can induce synchronous downward jumps in the invested (affiliated) stock price.

Now suppose that the insurance company can purchase reinsurance/acquire new business as described in [3] and allocate its wealth in the financial market as described by (2) and (3). The strategy adopted by the insurance company is described by a two-dimensional stochastic process \( \pi := \{\pi(t)\}_{t \in [0, T]} \), where \( \pi(t) := (\pi_1(t), \pi_2(t)) \). Here \( \pi_1(t) \geq 0 \) is the retention level of reinsurance/new business acquired by the insurance company at time \( t \) and \( \pi_2(t) \) is the amount of money the insurance company invests in the stock \( S_1 \) at time \( t \). This means that for the claim \( \Delta C(t) := C(t) - C(t-) \) occurring at time \( t \), the insurance company only needs to pay the amount of \( \pi_1(t) \Delta C(t) \) and the reinsurance/new business is responsible to cover the rest amount \( (1 - \pi_1(t)) \Delta C(t) \). Here \( \pi_1(t) \in [0, 1] \) corresponds to a proportional reinsurance and \( \pi_1(t) > 1 \) corresponds to acquiring new business, (see [10] or [3] for more explanations). To purchase this reinsurance contract, the insurance company should pay the reinsurance premium for the period \([0, t]\):

\[
p_r(t) = (1 + \eta) \int_0^t \int_{\mathbb{R}_0} x 1_{x \geq 0} (1 - \pi_1(s)) \nu(dx) ds = (1 + \eta) \int_0^t (1 - \pi_1(s)) \mu_1 ds,
\]

where \( \eta > \delta \) is the safety loading for the reinsurance/new business. Therefore, the surplus process \( U^{\pi_1}(t) \) after purchasing reinsurance/new business is given by

\[
U^{\pi_1}(t) = x + p(t) - p_r(t) - \int_0^t \int_{\mathbb{R}_0} \pi_1(s) x 1_{x > 0} N(ds, dx)
\]

\[
= x + \int_0^t ((1 + \eta) \pi_1(s) - (\eta - \delta)) \mu_1 ds - \int_0^t \pi_1(s) x 1_{x > 0} N(ds, dx).
\]

In the following, the strategy \( \pi \) is said to be admissible if it is \( F \)-predictable and satisfies the following condition:

\[
\mathbb{E} \left[ \int_0^t (\pi_1^2(s) + \pi_2^2(s)) ds \right] < +\infty,
\]

for all \( t \in [0, T] \). We write \( \Pi \) for the set of admissible strategies of the insurance company. Let \( \{R^\pi(t)\}_{t \in [0, T]} \) denote the surplus process of the insurance company after adopting the strategy \( \pi \) and we write \( R(t) := R^\pi(t) \) for \( t \in [0, T] \) unless

\(^{1}\)In our numerical example (refer to Section 6), we choose \( \gamma(t, x) = e^{-x} - 1 \), which exactly gives rise to positive (resp. negative) jumps at negative (resp. positive) marks in the risky asset price.
where otherwise stated. Thus, the evolution of the surplus process is governed by:

\[ dR(t) = dU^{\pi_1}(t) + \frac{R(t) - \pi_2(t)}{S_0(t)} dS_0(t) + \frac{\pi_2(t)}{S_1(t)} dS_1(t) \]

\[ = [(1 + \eta)\pi_1(t) - (\eta - \delta)]_t \mu_1 dt - \int_{R_0} \pi_1(t)x_1 > 0 N(dt, dx) + r(t)(R(t) - \pi_2(t)) dt + \pi_2(t)\mu(t) dt + \pi_2(t)\sigma(t)dW(t) \]

\[ + \int_{R_0} \pi_2(t)\gamma(t, x)\tilde{N}(dt, dx) \]

\[ = [(\eta\pi_1(t) - (\eta - \delta))_t + r(t)R(t) + \pi_2(t)(\mu(t) - r(t))] dt + \pi_2(t)\sigma(t)dW(t) + \int_{R_0} [\pi_2(t)\gamma(t, x) - \pi_1(t)x_1 > 0] \tilde{N}(dt, dx). \quad (4) \]

3. **Robust optimal control under expected terminal wealth.** In this section, we present optimal investment and reinsurance of maximizing the minimal expected terminal wealth, which are equivalent with respect to the reference measure \( \mathcal{P} \).

We first specify the set of admissible controls by the market and then formulate the stochastic control problem into a two-player, zero-sum, stochastic differential game between the insurance company and the market. Here, we suppose that the market is a "fictitious" player of the game and selects a real-world probability measure so as to minimize the expected terminal wealth of the insurance company with the penalty. Whereas, the insurance company chooses its optimal investment and reinsurance strategies to maximize the expected terminal wealth.

Now we are going to construct the set of real-world measures. To begin with, define a process \( \theta(t) := (\theta_1(t), \theta_2(t, x)) \) such that

1. \( \theta(t) \) is \( \mathcal{F}_t \)-predictable process and \( \theta_2(t, x) > -1, \mathcal{P} \)-a.s.;
2. \( \int_0^T \{ \theta_1^2(t) + \int_{R_0} \theta_2^2(t, x)\nu(dx) \} dt < \infty, \mathcal{P} \)-a.s. .

We write \( \Theta \) for the set of all processes that satisfy the above two conditions. For each \( \theta \in \Theta \), we can define a new (real-world) probability measure \( \mathcal{Q}_\theta \) absolutely continuous with respect to \( \mathcal{P} \) on \( \mathcal{F}_T \) by putting

\[ \frac{dQ_\theta}{d\mathcal{P}}|_{\mathcal{F}_T} = M_\theta(T), \]

where

\[ dM_\theta(t) = M_\theta(t) \left[ \theta_1(t)dW(t) + \int_{R_0} \theta_2(t, x)\tilde{N}(dt, dx) \right], \quad M_\theta(0) = 1. \]

Thus we can generate a family \( \mathcal{M}(\Theta) \) of real-world probability measures \( \mathcal{Q}_\theta \) parameterized by \( \theta \in \Theta \) and the market can choose a real-world probability measure by selecting a process \( \theta \in \Theta \). Hence, \( \Theta \) is the set of admissible strategies of the market.

In what follows, we let \( y' \) denote the transpose of a vector or a matrix \( y \). To simplify notations, we define a vector-valued process \( \{ V(t) := (V_1^\theta(t), V_2^\theta(t))' | t \in [0, T] \} \) by

\[ dV(t) = (dV_1^\theta(t), dV_2^\theta(t))' = (dM_\theta(t), dR^\pi(t))', \]

which means that
Now we present the performance functional for the insurance company. The insurance company aims to choose an optimal investment-reinsurance strategy \( \pi(t) \in \Pi \) to maximize the following minimal expected terminal wealth with a penalty\(^2\) over \( \mathcal{M}(\theta) \):

\[
\inf_{\theta \in \Theta} \{ E_{Q_\theta} [V_2^\pi(T)] + \zeta(Q_\theta) \},
\]

where \( \zeta(\cdot) \) is the penalty for the market by selecting the real-world probability measure \( Q_\theta \). Following [29], we consider the penalty function \( \zeta(Q_\theta) \) of the following form:

\[
\zeta(Q_\theta) := E \left[ \int_t^T \int_{\mathbb{R}_0} \lambda(s, V_1^\pi(s), \theta(s), x) \nu(dx) ds + h(V_1^\pi(T)) \right],
\]

where \( \lambda \) and \( h \) are functions satisfying

\[
E \left[ \int_0^T \int_{\mathbb{R}_0} |\lambda(t, V_1^\pi(t), \theta(t), x)| \nu(dx) dt + |h(V_1^\pi(T))| \right] < \infty.
\]

By using a version of Bayes’ rule and the above form of the penalty function, we have

\[
\inf_{\theta \in \Theta} \{ E_{Q_\theta} [V_2^\pi(T)] + \zeta(Q_\theta) \} = \inf_{\theta \in \Theta} \mathbb{E} \left[ V_1^\theta(T) V_2^\pi(T) + \int_t^T \int_{\mathbb{R}_0} \lambda(s, V_1^\theta(s), \theta(s), x) \nu(dx) ds + h(V_1^\theta(T)) \right].
\]

Thus if we define the performance functional \( J^{\pi, \theta}(t, v_1, v_2) \) for the insurance company and the market by

\[
J^{\pi, \theta}(t, v_1, v_2) := \mathbb{E} \left[ V_1^\theta(T) V_2^\pi(T) + \int_t^T \int_{\mathbb{R}_0} \lambda(s, V_1^\theta(s), \theta(s), x) \nu(dx) ds \\
+ h(V_1^\theta(T))|V(t) = (v_1, v_2) \right],
\]

then we can present the problem of maximizing the minimal expected terminal wealth for the insurer as below:

\(^2\)In the literature, plenty of works study the stochastic control problems of maximizing the expected terminal wealth with or without the constraint or penalty, see [9, 12, 40, 27, 34], and so on. In our paper, not only the expected terminal wealth is considered, but also a penalty is added in the objective function. This form of objective function is related to some special convex risk measure, see [29] for more details. In fact, maximizing the expected terminal wealth is equivalent to maximizing the expected utility of terminal wealth if the linear utility function is adopted. Though seldom used in utility maximization problems, the linear utility function is closely related to the mean-variance problem in which the expected terminal wealth is maximized and the variance of the terminal wealth is constrained below a fixed level. Moreover, Maccheroni et al. [22] showed that the variational preference, which is of similar structure as our performance functional of the minimal expected terminal wealth subject to the penalty under the worst-case scenario, is equivalent to the mean-variance functional on the domain of monotonicity. This provides further support to the close relationship between our performance functional and that in mean-variance problems.
Problem 1. Solve
\[ \Phi(t, v_1, v_2) = \sup_{\pi \in \Pi} \inf_{\theta \in \Theta} J^{\pi, \theta}(t, v_1, v_2) \]
and find the optimal \( \pi^* \) and \( \theta^* \) such that
\[ \Phi(t, v_1, v_2) = J^{\pi^*, \theta^*}(t, v_1, v_2). \]

4. Auxiliary stochastic control problem. To solve Problem 1, we need to find the relation between Problem 1 and the following auxiliary stochastic control problem first.

Problem 2. Find \( \Psi(t, v_1) \) and \( \bar{\theta} \) such that
\[ \Psi(t, v_1) = \inf_{\theta \in \Theta} J^\theta(t, v_1) = J^{\bar{\theta}}(t, v_1), \]
where
\[ J^\theta(t, v_1) := E^{v_1} \left[ \int_t^T \int_{\mathbb{R}_0} \lambda(s, V^\theta(s), \theta(s), x) \nu(dx) ds + h(V^\theta(T)) \right] \]
and \( E^{v_1}[\cdot] \) is the conditional expectation given \( V_1^\theta(t) = v_1 \).

Let \( E^{v_1, v_2}[\cdot] \) denote the conditional expectation given \( V(t) = v = (v_1, v_2) \) and it is easy to verify that:
\[ J^{\pi, \theta}(t, v_1, v_2) = J^\theta(t, v_1) + E^{v_1, v_2}[V^\theta_1(T)V^\pi_2(T)]. \]

For any given Markovian control strategy \((\pi, \theta)\), the process \( V(t) \) is Markovian with the infinitesimal generator \( L^{\pi, \theta} \) defined by
\[ L^{\pi, \theta} \phi(t, v_1, v_2) := \frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial v_1^2} [\eta \pi_1 - \eta + \delta] \mu_1 + r(t)v_2 + \pi_2(\mu(t) - r(t))] \]
\[ + \frac{1}{2} \left[ \frac{\partial^2 \phi}{\partial v_1^2} V_1^2 + \frac{\partial^2 \phi}{\partial v_2^2} \pi_2^2 \sigma^2(t) + 2 \frac{\partial \phi}{\partial v_1 \partial v_2} \pi_2 \theta_1 \sigma(t)v_1 \right] \]
\[ + \int_{\mathbb{R}_0} \left[ \phi(t, v_1(1 + \theta_2(t), x)), v_2 + \pi_2 \gamma(t, x) - x1_{x>0} \right] \]
\[ - \phi(t, v_1, v_2) - \frac{\partial \phi}{\partial v_1} \theta_2(t, x)v_1 - \frac{\partial \phi}{\partial v_2} (\pi_2 \gamma(t, x) - x1_{x>0} \right) \nu(dx). \]

If we only consider the process \( \{V^\theta_1(t)|t \in [0, T]\} \), its generator \( L^\theta \) is given by
\[ L^\theta \psi(t, v_1) := \frac{\partial \psi}{\partial t} + \frac{1}{2} \frac{\partial^2 \psi}{\partial v_1^2} \theta_1^2 v_1^2 \]
\[ + \int_{\mathbb{R}_0} \left[ \psi(t, v_1(1 + \theta_2(t), x)) - \psi(t, v_1) - \frac{\partial \psi}{\partial v_1} \theta_2(t, x)v_1 \right] \nu(dx). \]

Now we present two simple results which will be used in further analysis.

Lemma 4.1. Let \( \psi \in C^{1,2}([0, T] \times \mathbb{R}_+), \Gamma(t, T) := \exp\{ \int_t^T r(s) ds \} \) and define
\[ \phi(t, v_1, v_2) = \psi(t, v_1) + v_1 \left( v_2 \Gamma(t, T) - \int_t^T (\eta - \delta) \mu_1 \Gamma(s, T) ds \right). \]

Write
\[ M(\theta) := \left( \begin{array}{c} \eta \mu_1 - \int_{\mathbb{R}_0} x1_{x>0} \theta_2(t, x) \nu(dx) \\ (\mu(t) - r(t)) + \theta_1(t) \sigma(t) + \int_{\mathbb{R}_0} \gamma(t, x) \theta_2(t, x) \nu(dx) \end{array} \right). \]
\[ L^{\pi,\theta}\phi(t,v_1,v_2) = L^\theta\psi(t,v_1) + v_1 \Gamma(t,T)(\pi_1,\pi_2)M(\theta). \]

**Proof.** The proof is just a direct calculation and we omit it. \(\square\)

**Lemma 4.2.** Let \(\psi\) and \(\phi\) be as in Lemma 4.1. Write

\[ \Lambda(\theta) := \int_{\mathbb{R}_0} \lambda(t,V_{\theta}(t),\theta(t),x)\nu(dx). \]

Suppose that for all \((\pi_1,\pi_2)\) and \((t,v_1,v_2)\), there exists a minimum point \(\hat{\theta} = \hat{\theta}(\pi)\) of the function

\[ \theta \to L^\theta\psi(t,v_1) + \Lambda(\theta) + v_1 \Gamma(t,T)(\pi_1,\pi_2)M(\theta) \]

and that \(\pi \to \hat{\theta}(\pi)\) is a \(C^1\)-function. Moreover, suppose the map

\[ \pi \to L^{\theta(\pi)}\psi(t,v_1) + L(\hat{\theta}(\pi)) + v_1 \Gamma(t,T)(\pi_1,\pi_2)M(\hat{\theta}(\pi)) \]

has a maximum point \(\hat{\pi} \in \mathbb{R}_+ \times \mathbb{R}\). Define

\[ \tilde{\theta} := \hat{\theta}(\hat{\pi}). \]

Then

\[ M(\tilde{\theta}) = 0 \]

and

\[ \sup_{\pi} \left( \inf_{\theta} \{ L^{\pi,\theta}\phi(t,v_1,v_2) + \Lambda(\theta) \} \right) = L^{\tilde{\theta}}\psi(t,v_1) + \Lambda(\tilde{\theta}) = \inf_{\theta:M(\theta)=0} \left( L^{\tilde{\theta}}\psi(t,v_1) + \Lambda(\tilde{\theta}) \right). \]

**Proof.** The first order condition for a minimum point \(\hat{\theta} = \hat{\theta}(\pi)\) of the map (9) is

\[ \nabla_{\theta} \left( L^{\theta}\psi(t,v_1) + \Lambda(\theta) + v_1 \Gamma(t,T)(\pi_1,\pi_2)M(\theta) \right)_{\theta = \hat{\theta}(\pi)} = 0, \]

where \(\nabla_{\theta} := \left( \frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \theta_2} \right)\) denotes the gradient operator. The first order condition for a maximum point \(\hat{\pi}\) of the map (10) is, by the chain rule,

\[ \nabla_{\theta} \left( L^{\theta}\psi(t,v_1) + \Lambda(\theta) + v_1 \Gamma(t,T)(\pi_1,\pi_2)M(\theta) \right)_{\theta = \hat{\theta}(\pi)} \left( \frac{d\hat{\theta}(\pi)}{d\pi} \right)_{\pi = \hat{\pi}} + v_1 \Gamma(t,T)M(\hat{\theta}(\hat{\pi})) = 0. \]

By (12) the first item is 0 and we conclude that \(M(\hat{\theta}(\hat{\pi})) = 0\). By the definition of \(\tilde{\theta}\), we have \(M(\tilde{\theta}) = 0\) as claimed. Therefore,

\[ \sup_{\pi} \left( \inf_{\theta} \left\{ L^{\theta}\psi(t,v_1) + \Lambda(\theta) + v_1 \Gamma(t,T)(\pi_1,\pi_2)M(\theta) \right\} \right) \]

\[ = \sup_{\pi} \left( L^{\theta(\pi)}\psi(t,v_1) + \Lambda(\hat{\theta}(\pi)) + v_1 \Gamma(t,T)(\pi_1,\pi_2)M(\hat{\theta}(\pi)) \right) \]

\[ = L^{\tilde{\theta}}\psi(t,v_1) + \Lambda(\tilde{\theta}) \geq \inf_{\theta:M(\theta)=0} \left( L^{\tilde{\theta}}\psi(t,v_1) + \Lambda(\tilde{\theta}) \right). \]
On the other hand, we always have
\[
\sup_{\pi} \left( \inf_{\theta} \left\{ \mathcal{L}^\theta \psi(t, v_1) + \Lambda(\theta) + v_1 \Gamma(t, T)(\pi_1, \pi_2)M(\theta) \right\} \right)
\leq \sup_{\pi} \left( \inf_{\theta: M(\theta) = 0} \left\{ \mathcal{L}^\theta \psi(t, v_1) + \Lambda(\theta) + v_1 \Gamma(t, T)(\pi_1, \pi_2)M(\theta) \right\} \right)
= \inf_{\theta: M(\theta) = 0} \left( \mathcal{L}^\theta \psi(t, v_1) + \Lambda(\theta) \right).
\]
Combining the above two equations, we obtain
\[
\sup_{\pi} \left( \inf_{\theta} \left\{ \mathcal{L}^\theta \psi(t, v_1) + \Lambda(\theta) + v_1 \Gamma(t, T)(\pi_1, \pi_2)M(\theta) \right\} \right) = \mathcal{L}^\theta \psi(t, v_1) + \Lambda(\theta) = \inf_{\theta: M(\theta) = 0} \left( \mathcal{L}^\theta \psi(t, v_1) + \Lambda(\theta) \right).
\]
Then by Lemma 4.1, the above equation is equivalent to (11).

5. HJBI equations for stochastic differential games. In this section, we first provide a verification theorem for the HJBI solution to Problem 1. Since the controlled state processes and the controls are Markovian, the dynamic programming approach of the stochastic optimal control applies (see, for example, [14]). So we have the following verification theorem for the HJBI solution to Problem 1. This verification theorem is a saddle-point result.

**Theorem 5.1.** Let \( \mathcal{O} := (0, T) \times \mathbb{R}_+ \times \mathbb{R} \) and \( \bar{\mathcal{O}} \) denote the closure of \( \mathcal{O} \). Suppose that there exists a function \( \phi(t, v_1, v_2) \in C^{1,2}(\mathcal{O}) \cap C(\bar{\mathcal{O}}) \) and a Markov control \((\hat{\pi}, \hat{\theta}) \in \Pi \times \Theta \) such that

1. \( \mathcal{L}^{\hat{\pi}, \hat{\theta}} \phi(t, v_1, v_2) + \Lambda(\hat{\theta}) \geq 0 \), for all \( \theta \in \Theta, (t, v_1, v_2) \in \mathcal{O}; \)
2. \( \mathcal{L}^{\pi, \theta} \phi(t, v_1, v_2) + \Lambda(\theta) \leq 0 \), for all \( \pi \in \Pi, (t, v_1, v_2) \in \mathcal{O}; \)
3. \( \mathcal{L}^{\hat{\pi}, \hat{\theta}} \phi(t, v_1, v_2) + \Lambda(\hat{\theta}) = 0 \), for all \( (t, v_1, v_2) \in \mathcal{O}; \)
4. \( \phi(T, v_1, v_2) = h(v_1) + v_1 v_2; \)
5. the family \( \{\phi(\tau, V(\tau))\}_{\tau \in \mathcal{T}} \) is uniformly integrable for all \( (\pi, \theta) \in \Pi \times \Theta \), where \( \mathcal{T} \) is the set of all \( \mathbb{F} \)-stopping times \( \tau \leq T \).

Then
\[
\phi(t, v_1, v_2) = \Phi(t, v_1, v_2) = \sup_{\pi} \inf_{\theta} J^{\pi, \theta}(t, v_1, v_2) = \inf_{\pi} \sup_{\theta} J^{\pi, \theta}(t, v_1, v_2)
= \inf_{\theta} J^{\pi, \theta}(t, v_1, v_2) = \sup_{\pi} J^{\pi, \theta}(t, v_2, v_2) = J^{\pi, \theta}(t, v_1, v_2)
\]
and \((\hat{\pi}, \hat{\theta})\) is an optimal Markov control.

**Proof.** The proof of this theorem is similar to the proof of Theorem 3.2 in [24] and so we do not repeat it here.

Next we shall present a relationship between Problem 1 and Problem 2.

**Theorem 5.2.** Suppose that the value function \( \Psi(t, v_1) \) for Problem 2 satisfies all the conditions for \( \psi(t, v_1) \) in Lemma 4.2 and the family \( \{\Phi(t, V_1(\tau))\}_{\tau \in \mathcal{T}} \) is uniformly integrable for all \( (\pi, \theta) \in \Pi \times \Theta \), where \( \mathcal{T} \) is the set of all \( \mathbb{F} \)-stopping times \( \tau \) such that \( \tau \leq T \). Then the value function for Problem 1 is given by
\[ \Phi(t, v_1, v_2) = \Psi(t, v_1) + v_1 \left( v_2 \Gamma(t, T) - \int_t^T (\eta - \delta) \mu_1 \Gamma(s, T) ds \right). \]

**Proof.** By the HJB equation for the stochastic control Problem 2, we know that
\[ \inf_{\theta \in \mathcal{M}(\theta)} \{ \mathcal{L}^\theta \Psi(t, v_1) + \Lambda(\theta) \} = \mathcal{L}^\theta \Psi(t, v_1) + \Lambda(\bar{\theta}) = 0 \quad (13) \]
and the terminal value is
\[ \Psi(T, v_1) = h(v_1). \quad (14) \]
Write
\[ \phi(t, v_1, v_2) = \Psi(t, v_1) + v_1 \left( v_2 \Gamma(t, T) - \int_t^T (\eta - \delta) \mu_1 \Gamma(s, T) ds \right). \quad (15) \]
Then by Lemma 4.1, we have
\[ \mathcal{L}^\pi \phi(t, v_1, v_2) = \mathcal{L}^\theta \Psi(t, v_1) + \Lambda(\theta) + v_1 \Gamma(t, T) (\pi_1, \pi_2) M(\theta), \]
where \(M(\theta)\) is defined by (7). Therefore, Conditions 1-3 of Theorem 5.1 get the form
1. \( \mathcal{L}^\theta \Psi(t, v_1) + \Lambda(\theta) + v_1 \Gamma(t, T) (\hat{\pi}_1, \hat{\pi}_2) M(\theta) \geq 0 \) for all \( \theta \in \Theta; \)
2. \( \mathcal{L}^\theta \Psi(t, v_1) + \Lambda(\hat{\theta}) + v_1 \Gamma(t, T) (\pi_1, \pi_2) M(\theta) \leq 0 \) for all \( \pi \in \Pi; \)
3. \( \mathcal{L}^\theta \Psi(t, v_1) + \Lambda(\hat{\theta}) + v_1 \Gamma(t, T) (\hat{\pi}_1, \hat{\pi}_2) M(\theta) = 0 \) for all \((t, v_1, v_2) \in \mathcal{S}.\)

Choose \( \hat{\pi} \) and \( \hat{\theta} = \hat{\theta}(\hat{\pi}) \) as in Lemma 4.2. Then we have
\[ \mathcal{L}^\theta \Psi(t, v_1) + \Lambda(\hat{\theta}) + v_1 \Gamma(t, T) (\hat{\pi}_1, \hat{\pi}_2) M(\theta) \]
\[ \geq \inf_{\theta} \mathcal{L}^\theta \Psi(t, v_1) + \Lambda(\theta) + v_1 \Gamma(t, T) (\hat{\pi}_1, \hat{\pi}_2) M(\theta) \]
\[ = \mathcal{L}^{\hat{\theta}(\hat{\pi})} \Psi(t, v_1) + \Lambda(\hat{\theta}(\hat{\pi})) + v_1 \Gamma(t, T) (\hat{\pi}_1, \hat{\pi}_2) M(\hat{\theta}(\hat{\pi})) \]
\[ = \inf_{\theta \in \mathcal{M}(\theta)} \mathcal{L}^\theta \Psi(t, v_1) + \Lambda(\theta) \]
which yields the first condition of Theorem 5.1. Moreover, since \( M(\hat{\theta}) = 0 \), by (13) we have
\[ \mathcal{L}^\theta \Psi(t, v_1) + \Lambda(\hat{\theta}) + v_1 \Gamma(t, T) (\pi_1, \pi_2) M(\hat{\theta}) = \mathcal{L}^\theta \Psi(t, v_1) + \Lambda(\hat{\theta}) = 0, \]
which proves the second and third condition of Theorem 5.1. Noting (15) and (14), we have
\[ \phi(T, v_1, v_2) = h(v_1) + v_1 v_2, \]
which implies Condition 4 of Theorem 5.1.

In what follows, we shall prove the uniform integrability of \( \{ \Phi(t, V(t)) \}_{t \in T}. \) Under the assumption of the uniform integrability of \( \{ \Psi(t, V^\theta(t)) \}_{t \in T}. \) we only need to prove the uniform integrability of \( \{ F(t, V^\theta(t), V^2(t)) \}_{t \in T}. \) where
\[ F(t, v_1, v_2) := v_1 (v_2 \Gamma(t, T) - \int_t^T (\eta - \delta) \mu_1 \Gamma(s, T) ds). \]
Observe that
\[ |F(t, V^\theta(t), V^2(t))| \leq K \left( |V^\theta(t)|^2 + |V^2(t)|^2 \right) + K, \]
where \( K \) is a positive constant and may differ from line to line in the sequel. Observing the stochastic differential equations in (5) and the boundness of \( r(t), \mu(t), \sigma(t), \) we can apply the Burkholder-Davis-Gundy inequality to derive...
Explicit solution for quadratic penalty function.

Thus combining Lemma 4.1, the compact form of HJBI equation (17) reduces to

\[ f(t) = \frac{1}{2(1 - \zeta)} \left( \theta_1(t)^2 + \int_{\mathbb{R}_0} \theta_2(t, x)^2 \nu(dx)dt \right) + K \]

where \( 1 - \zeta \) is a measure of an insurer’s relative risk aversion and \( \zeta \in (0, 1) \).

We first re-state the condition of Theorem 5.1 in the following compact form of the HJBI equation

\[
\sup_{\tau \in \mathcal{T}} \left( |\psi(\tau, \tau) + \Lambda(\theta)| \right) = 0,
\]

Motivated by Theorem 5.2, we try the following form of the value function

\[ \psi(t, v_1) = \psi(t, v_1) + v_1 \left( v_2 \Gamma(t, T) - \int_t^T (\eta - \delta) \mu_1 \Gamma(s, T) ds \right). \]

Thus combining Lemma 4.1, the compact form of HJBI equation (17) reduces to

\[
\sup_{\tau \in \mathcal{T}} \left( \inf_{\theta \in \Theta} \left[ \mathcal{L}^\theta(\psi(t, v_1) + \Lambda(\theta) + v_1 \Gamma(t, T)(\pi_1, \pi_2) M(\theta)) \right] \right) = 0,
\]

where \( M(\theta) \) is defined by (7). Motivated by the quadratic form of \( \Lambda(\theta) \), we try the following parametric form of \( \psi(t, v_1) \) to solve HJBI equation (18):

\[ \psi(t, v_1) = f(t)v_1^2, \]

where \( f(t) \) is some appropriate function satisfying the boundary condition \( f(T) = 0 \).

By this trial function, we have
The first-order condition for maximizing the function following equation:

\[ H(t, \pi, \theta) := \mathcal{L}^\theta \psi(t, v_1) + \Lambda(\theta) + v_1 \Gamma(t, T) (\pi_1, \pi_2) M(\theta) \]

\[ = v_1^2 \left[ f'(t) + g(t) \left( \theta_1^2(t) + \int_{\mathbb{R}_0} \theta_2^2(t, x) \nu(dx) \right) \right] + v_1 \Gamma(t, T) (\pi_1, \pi_2) M(\theta) \]

\[ = v_1^2 f'(t) + v_1^2 g(t) \theta_1^2(t) + v_1 \Gamma(t, T) \pi_2 \sigma(t) \theta_1(t) - \pi_1 \eta \mu_1 - \pi_2 (\mu(t) - r(t)) \]

\[ + \int_{\mathbb{R}_0} \left[ v_1^2 g(t) \theta_2^2(t, x) + v_1 \Gamma(t, T) (\pi_2 \gamma(t, x) - x_{1 > 0} \pi_1) \theta_2(t, x) \right] \nu(dx), \]

where \( g(t) := f(t) - \frac{1}{2} \theta_1. \)

The first-order condition for minimizing \( H(t, \pi, \theta) \) with respect to \( \theta_1 \) gives the following equation:

\[ \frac{\partial H}{\partial \theta_1} = 2v_1^2 g(t) \theta_1 + v_1 \Gamma(t, T) \pi_2 \sigma(t) = 0. \quad (19) \]

For the minimization of \( H \) with respect to \( \theta_2 \), we try to minimize the function

\[ v_1^2 g(t) \theta_2^2(t, x) + v_1 \Gamma(t, T) (\pi_2 \gamma(t, x) - x_{1 > 0} \pi_1) \theta_2(t, x) \]

pointwisely for each \((t, x)\) and obtain the following equation

\[ v_1^2 g(t) \theta_2(t, x) + v_1 \Gamma(t, T) (\pi_2 \gamma(t, x) - x_{1 > 0} \pi_1) = 0. \quad (20) \]

From (19) and (20), we know that the minimum point \( \hat{\theta} := (\hat{\theta}_1(t), \hat{\theta}_2(t, x)) \) for the function \( H(t, \pi, \theta) \) is given by

\[ \begin{cases} 
\hat{\theta}_1(t) = \frac{\Gamma(t, T) \pi_2 \gamma(t, x)}{2v_1^2 g(t)}, \\
\hat{\theta}_2(t, x) = \frac{\Gamma(t, T) (\pi_2 \gamma(t, x) - x_{1 > 0} \pi_1(t))}{2v_1^2 g(t)}. 
\end{cases} \quad (21) \]

The first-order condition for maximizing the function \( H(t, \pi, \hat{\theta} (\pi)) \) yields the following equation

\[ v_1 \Gamma(t, T) M(\hat{\theta}) = 0. \]

That is

\[ \begin{cases} 
\eta \mu_1 - \int_{\mathbb{R}_0} x_{1 > 0} \hat{\theta}_2(t, x) \nu(dx) = 0, \\
(\mu(t) - r(t)) + \hat{\theta}_1(t) \sigma(t) + \int_{\mathbb{R}_0} \gamma(t, x) \hat{\theta}_2(t, x) \nu(dx) = 0. 
\end{cases} \quad (22) \]

Combining (21) and (22), we obtain the maximum point \( \hat{\pi} \) of \( H(t, \pi, \theta) \) is given by

\[ \begin{cases} 
\hat{\pi}_1(t) = -\frac{2g(t) \gamma(t)}{|A(t)|} \left[ \eta \mu_1 (\sigma^2(t) + |\gamma(t, x)|_\nu^2) + (\mu(t) - r(t))(x_{1 > 0}, \gamma(t, x))_\nu \right], \\
\hat{\pi}_2(t) = -\frac{2g(t) \gamma(t)}{|A(t)|} \left[ x_{1 > 0}^2 (\mu(t) - r(t)) + \eta \mu_1 (x_{1 > 0}, \gamma(t, x))_\nu \right], 
\end{cases} \quad (23) \]

where we used the notation

\[ (u(x), v(x))_\nu := \int_{\mathbb{R}_0} u(x)v(x) \nu(dx), \quad |u(x)|_\nu^2 := (u(x), u(x))_\nu, \]

and

\[ |A(t)| := \left( \sigma^2(t) + |\gamma(t, x)|_\nu^2 \right) x_{1 > 0}^2 - (x_{1 > 0}, \gamma(t, x))_\nu^2. \]

Therefore, by (21), we have

\[ \begin{cases} 
\hat{\theta}_1(t) = \frac{\Gamma(t, T) \pi_2 \gamma(t)}{2v_1^2 g(t)} \hat{\pi}_1(t), \\
\hat{\theta}_2(t, x) = \frac{\Gamma(t, T) \hat{\pi}_2(t, x) - x_{1 > 0} \hat{\pi}_1(t)}{2v_1^2 g(t)} B_1(t), 
\end{cases} \quad (24) \]
Thus, Theorem 5.2 and the above analysis yield the following theorem by observing

\[ B_1(t) := |x_{1>0}|^2_{\nu}(\mu(t) - r(t)) + \eta \mu_1(x_{1>0}, \gamma(t, x)) \nu \]

and

\[ B_2(t, x) := [\gamma(t, x)|x_{1>0}|^2_{\nu} - x_{1>0}(x_{1>0}, \gamma(t, x)) \nu ] (\mu(t) - r(t)) \]

\[ + [\gamma(t, x)(x_{1>0}, \gamma(t, x)) \nu - x_{1>0}(\sigma^2(t) + |\gamma(t, x)|_{2\nu})] \eta_1. \]

Thus by Lemma 4.2 and Theorem 5.2, we know that \((\pi^*, \theta^*) := (\hat{\pi}, \hat{\theta})\) is the optimal strategy for our problem and so

\[ H(t, \pi^*, \theta^*) = \pi^2 \left[ f(t) + g(t) \left( \hat{\sigma}_1^2(t) + |\hat{\sigma}_2(t, x)|_{2\nu}^2 \right) \right] = 0. \]

Noting \(g(t) = f(t) - \frac{1}{2(1-\zeta)}\), and so \(f(t)\) must satisfy the following differential equation

\[
\begin{cases}
  f'(t) = -(f(t) - \frac{1}{2(1-\zeta)}) \left( \hat{\sigma}_1^2(t) + |\hat{\sigma}_2(t, x)|_{2\nu}^2 \right), \\
  f(T) = 0.
\end{cases}
\]

It is easy to obtain the solution of the above differential equation in the explicit form

\[
f(t) = \frac{1}{2(1-\zeta)} \left( 1 - \exp \left\{ - \int_t^T \left( \hat{\sigma}_1^2(s) + |\hat{\sigma}_2(s, x)|_{2\nu}^2 \right) ds \right\} \right). \tag{25}
\]

Thus, Theorem 5.2 and the above analysis yield the following theorem by observing that \(\hat{\theta}\) in (21) is a \(C^1\)-function on \(\pi\) and \(\psi(\tau, V_1^q(\tau)) = f(\tau)(V_1^q(\tau))^2\) is uniformly integrable.

**Theorem 6.1.** Suppose \(h(v_1) = 0\) and the penalty function \(\Lambda(\theta)\) is given by (16). Then the strategy \((\pi^*, \theta^*) := (\hat{\pi}, \hat{\theta})\) with \(\hat{\pi}\) and \(\hat{\theta}\) given by (23) and (24) respectively is the optimal strategy for Problem 1. Moreover the optimal value function is given by

\[
\Phi(t, v_1, v_2) = f(t) v_1^2 + v_1 \left( v_2 \Gamma(t, T) - \int_t^T \eta - \delta) \mu_1(s, T) ds \right),
\]

where \(f(t)\) is given by (25) and \(\Gamma(t, T) := \exp \{ \int_t^T r(s) ds \}. \]

To provide numerical analysis, we assume that the jump components of our models are described by a compound Poisson process, which is characterized by Poisson intensity \(\lambda\) and double-exponential PDF for individual jump sizes:

\[ q(x) = q \beta_1 e^{-\beta_1 x} 1_{x>0} + (1 - q) \beta_2 e^{\beta_2 x} 1_{x\leq 0}. \]

In this regards, both positive and negative jumps may be observed in the risky asset price, while the insurer’s aggregate claim is modeled by a compound Poisson process with Poisson intensity \(q\lambda\) and claim-size distribution

\[ q(x) = \beta_1 e^{-\beta_1 x}. \]

Hence, each random individual claim is a negative jump from the insurer’s perspective. Furthermore, we assume that all the coefficients are constant. More precisely, we specify the following set of parameter values as our benchmark:

\[
\begin{align*}
& r(t) = 0.02, \quad \mu(t) = 0.12, \quad \sigma(t) = 0.20, \quad \lambda = 1, \quad q = 0.3, \\
& \beta_1 = 1, \quad \beta_2 = 5, \quad \eta = 0.25, \quad \delta = 0.15, \quad t = 0, \quad T = 1,
\end{align*}
\]
and
\[ \gamma(t, x) \equiv e^{-x} - 1, \quad \mu_1 = 1/\beta_1 = 10. \]

The structure of the jump ratio function \( \gamma(t, x) \) indicates that the negative and positive jumps of the stock are described by \( \beta_1 e^{-\beta_1 x} \) and \( \beta_2 e^{\beta_2 x} \) with weights \( q \) and \( 1 - q \) in the double-exponential PDF \( q(x) \), respectively. At \( t = 0 \), we know the initial value is \( v_1 = 1 \), and we assume furthermore that the initial wealth is \( v_2 = 1 \). The parameter values for \( \beta_1 \), \( \beta_2 \) and \( q \) are chosen in such a way that the size of negative jumps is larger than that of positive ones but negative jumps happen less frequently than positive ones. This indicates that negative jumps, though less frequent, are more contagious and of higher magnitude than positive ones.

In Figs 1(a)-1(b), we show the effects of the Poisson intensity \( \lambda \) on the optimal reinsurance and investment strategies against different levels of risk aversion. Recall that the larger \( \zeta \) is, the less risk-averse the insurer is. Therefore, as \( \zeta \) increases, the insurer is opt to retain higher proportion of insurance claims and invest more in the risky asset (stock). This trend is clearly reflected in Figs. 1(a) and 1(b) as well as all the other figures in the remainder of the paper when the line colors change from red to black. On the other hand, a higher jump frequency results in a lower retention (i.e., higher ceding) level of insurance risk and a less amount to be invested in the stock. This observation is in accordance with our intuition. The increasing of \( \lambda \) makes insurance business and stock investment more risky, which discourages the insurer to undertake risk. Thus, the insurer becomes more conservative with smaller \( \hat{\pi}_1 \) and \( \hat{\pi}_2 \).

Figs. 2(a) and 2(b) depict the changes of the insurer’s optimal strategies \( \hat{\pi}_1 \) and \( \hat{\pi}_2 \) with respect to \( q \), i.e., the weight of negative jumps in the stock price model. As mentioned earlier, for the insurer’s collective risk model, the intensity of claim occurrence is \( q\lambda \). From the point of view of the insurance business, the appreciation of \( q \) plays essentially the same role as that of \( \lambda \). So, it is anticipated that the higher \( q \) is, the lower proportion of insurance claims is retained. However, the impact of \( q \) on the optimal investment strategy is different from that of \( \lambda \). In fact, as \( q \) increases, negative jumps occur more frequently while positive ones become relatively more infrequent. This does not necessarily lead to less or more stock investment. The
dominant factor is that the insurer has to shift capital from insurance business to other outlet. Under this circumstance, the stock holding is expected to rise.

In Figs. 3(a)-3(b) and Figs. 4(a)-4(b), we consider the effects of $\beta_1$ and $\beta_2$ on $(\hat{\pi}_1, \hat{\pi}_2)$, respectively. In can be seen from Figs. 3(a) and 3(b) that both $\hat{\pi}_1$ and $\hat{\pi}_2$ increase with $\beta_1$. Note that $1/\beta_1$ is the mean of the negative jump sizes. So, the negative jump sizes will be smaller on average when $\beta_1$ varies from 1 to 2. Following this fact, both insurance losses and stock price declines are expected to be less severe. This alleviates the negative effects of downward jumps, thereby encouraging the insurance company to undertake more risk in the insurance market and the stock market simultaneously. On the contrary, $\beta_2$, namely the parameter of the positive jumps in the stock price model, affects the optimal reinsurance strategy and the optimal investment strategy in opposite directions. This may not be easily understood at the first glance. However, if we look at the stock price model, it is revealed that other things being identical the variance of stock price
return decreases with $\beta_2$. As a risk-averse investor, the insurer would choose to increase the investment in the less volatile stock. Though the insurance risk model is independent of $\beta_2$, it is correlated with the stock price through the negative jumps. Compared with the less risky stock, the seemingly more fluctuating insurance claims would force the insurer to eventually shift capitals from the insurance business to the stock market.

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