Wiener Indices of Maximal $k$-Degenerate Graphs

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Abstract
A graph is maximal $k$-degenerate if each induced subgraph has a vertex of degree at most $k$ and adding any new edge to the graph violates this condition. In this paper, we provide sharp lower and upper bounds on Wiener indices of maximal $k$-degenerate graphs of order $n \geq k \geq 1$. A graph is chordal if every induced cycle in the graph is a triangle and chordal maximal $k$-degenerate graphs of order $n \geq k$ are $k$-trees. For $k$-trees of order $n \geq 2k + 2$, we characterize all extremal graphs for the upper bound.

Keywords $k$-Tree · Maximal $k$-degenerate graph · Wiener index

1 Introduction

The Wiener index of a graph $G$, denoted by $W(G)$, is the summation of distances between all unordered vertex pairs of the graph. The concept was first introduced by Wiener in 1947 for applications in chemistry [17], and has been studied in terms of various names and equivalent concepts such as the total status [13], the total distance [10], the transmission [16], and the average distance (or, mean distance) [9].

A graph with a property $\mathcal{P}$ is called maximal if it is complete or if adding an edge between any two non-adjacent vertices results in a new graph that does not have the property $\mathcal{P}$. Finding bounds on Wiener indices of maximal planar graphs of a given order has attracted attention recently, see [7, 8, 12]. For a maximal planar graph of order $n \geq 3$, its Wiener index has a sharp lower bound $n^2 - 4n + 6$. An Apollonian...
network is a chordal maximal planar graph. Wiener indices of Apollonian networks of order \( n \geq 3 \) have a sharp upper bound \( \frac{1}{38} (n^3 + 3n^2) \), which also holds for maximal planar graphs of order \( 3 \leq n \leq 10 \), and was conjectured to be valid for all \( n \geq 3 \) in [7]. Recently, the conjecture was confirmed in [12]. With an extra condition on vertex connectivity, it was shown [8] that if \( G \) is a \( k \)-connected maximal planar graph of order \( n \), then the mean distance \( \mu(G) = \frac{W(G)}{\binom{n}{2}} \leq \frac{n}{3k} + O(\sqrt{n}) \) for \( k \in \{3, 4, 5\} \) and the coefficient of \( n \) is the best possible.

Let \( k \) be a positive integer. A graph is \( k \)-degenerate if its vertices can be successively deleted so that when deleted, they have degree at most \( k \). Note that Apollonian networks are maximal 3-degenerate graphs. In this paper, we provide sharp lower and upper bounds for Wiener indices of maximal \( k \)-degenerate graphs of order \( n \) and some extremal graphs for all \( n \geq k \geq 1 \). When the lower and upper bounds on Wiener indices are equal for maximal \( k \)-degenerate graphs of order \( n \), their diameters are at most 2, which implies that \( k \leq n \leq 2k + 1 \). The extremal graphs for the lower bound have a nice description for 2-trees of diameter at most 2. Maximal \( k \)-degenerate graphs with diameter at least 3 have order at least \( 2k + 2 \). For \( k \)-trees of order \( n \geq 2k + 2 \), we characterize all extremal graphs whose Wiener indices attain the upper bound. Our results generalize well-known sharp bounds on Wiener indices of some important classes of graphs such as trees and Apollonian networks.

## 2 Preliminaries

All graphs considered in the paper are simple graphs without loops or multiple edges. Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). Then the order of \( G \) is \( n = |V(G)| \) and the size of \( G \) is \( |E(G)| \). Let \( K_n \) and \( P_n \) denote the clique and the path of order \( n \) respectively. Let \( \overline{K}_n \) be the compliment of \( K_n \), that is, the graph on \( n \) isolated vertices. Let \( G + H \) be the graph obtained from \( G \) and \( H \) by adding all possible edges between vertices of \( G \) and vertices of \( H \). A complete bipartite graph \( K_{rs} \) is \( \overline{K}_r + \overline{K}_s \).

A graph is connected if there is a path between any two vertices of the graph. The distance between two vertices \( u, v \) of a graph \( G \) is the length of a shortest path joining \( u \) and \( v \) in \( G \), and denoted by \( d_G(u, v) \). The distance between two vertices from different components is infinite if \( G \) is disconnected. The eccentricity \( e_G(u) \) of a vertex \( u \) in \( G \) is the maximum distance between \( u \) and other vertices of \( G \). The set of all vertices with distance \( i \) from the vertex \( u \) in \( G \) is denoted by \( N_G(u, i) \) for \( 1 \leq i \leq e_G(u) \). In particular, the set of all vertices adjacent to vertex \( u \) in \( G \) is denoted by \( N_G(u) \), and its cardinality \( |N_G(u)| \) is called the degree of vertex \( u \). The diameter of \( G \), denoted by \( \text{diam}(G) \), is the maximum distance between any two vertices of \( G \). A subgraph \( H \) of \( G \) is said to be isometric in \( G \) if \( d_H(x, y) = d_G(x, y) \) for any two vertices \( x, y \) of \( H \). The status (or, transmission) of a vertex \( u \) in \( G \), denoted by \( \sigma_G(u) \), is the summation of the distances between \( u \) and all other vertices in \( G \).

**Lemma 1** [4, 10] Let \( G \) be a connected graph. Then
(i) \( W(G) \geq 2\binom{n}{2} - |E(G)| \), and the equality holds if and only if \( \text{diam}(G) \leq 2 \).
(ii) \( W(G) \leq W(G - v) + \sigma_G(v) \) for any vertex \( v \) of \( G \), and the equality holds if and only if \( G - v \) is isometric in \( G \).
(iii) \( W(G) = \sum_{i=1}^{\text{diam}(G)} i \cdot d_i \), where \( d_i \) is the number of unordered vertex pairs with distance \( i \) in \( G \).

We are interested in \( k \)-degenerate graphs and maximal \( k \)-degenerate graphs, introduced in [14]. A subclass of maximal \( k \)-degenerate graphs called \( k \)-trees [1] is particularly important. A \( k \)-tree is a generalization for the concept of a tree and can be defined recursively: a clique \( K_k \) of order \( k \geq 1 \) is a \( k \)-tree, and any \( k \)-tree of order \( n + 1 \) can be obtained from a \( k \)-tree of order \( n \geq k \) by adding a new vertex adjacent to all vertices of a clique of order \( k \), which is called the root of the newly added vertex, and we say that the newly added vertex is rooted at the specific clique. By definitions, the order of a maximal \( k \)-degenerate graph can be any positive integer, while the order of a \( k \)-tree is at least \( k \). A graph is a \( k \)-tree if and only if it is a chordal maximal \( k \)-degenerate graph of order \( n \geq k \) [2]. A graph is maximal 1-degenerate if and only if it is a tree [14]. It is known [15] that 2-trees form a special subclass of planar graphs extending the concept of maximal outerplanar graphs, and maximal outerplanar graphs are the only 2-trees that are outerplanar. Planar 3-trees are just Apollonian networks.

The \( k \)-th power of a path \( P_n \), denoted by \( P_n^k \), has the same vertex set as \( P_n \) and two distinct vertices \( u \) and \( v \) are adjacent in \( P_n^k \) if and only if their distance in \( P_n \) is at most \( k \). Note that the order \( n \) of \( P_n^k \) can be any positive integer. When \( n \geq k \), \( P_n^k \) is a special type of \( k \)-tree. For \( n \geq 2 \), \( P_n^k \) is an extremal graph for the upper bound on Wiener indices of maximal \( k \)-degenerate graphs of order \( n \).

A graph is called \( k \)-connected if the removal of any \( k - 1 \) vertices of the graph does not result a disconnected or trivial graph. It is well-known that for a \( k \)-connected graph \( G \) of order \( n \), \( \text{diam}(G) \leq \frac{n-2}{k} + 1 \). Since maximal \( k \)-degenerate graphs of order \( n \geq k + 1 \) are \( k \)-connected [14], this bound holds for them, and a characterization of the extremal graphs (among maximal \( k \)-degenerate graphs) appears in [2].

The following upper bound on vertex status of a \( k \)-connected graph of order \( n \) can be obtained by the fact that \( \sigma_G(x) = \sum_{i=1}^{\sigma_G(x)} i \cdot |N_G(x, i)| \) [4, 10]. An equivalent upper bound formula was first appeared in [11, Remark 2.6.1]. without reference papers available.

**Lemma 2** [6, 11] Let \( G \) be a \( k \)-connected graph of order \( n \geq k + 1 \) and \( k \geq 1 \). Then \( \sigma_G(x) \leq \left( \left\lfloor \frac{n-2}{k} \right\rfloor + 1 \right) (n - 1 - \frac{k}{2} \left\lfloor \frac{n-2}{k} \right\rfloor) \) for any vertex \( x \) of \( G \). Moreover, \( \sigma_G(x) \) attains the upper bound if and only if \( x \) satisfies both properties: (i) \( e_G(x) = \text{diam} (G) = \left\lfloor \frac{n-2}{k} \right\rfloor + 1 \), and (ii) \( |N_G(x, i)| = k \) for all \( 1 \leq i \leq \left\lfloor \frac{n-2}{k} \right\rfloor \).

If the graphs in consideration are maximal \( k \)-degenerate graphs, then the upper bound on vertex status in Lemma 2 can be achieved by any degree-\( k \) vertex of \( P_n^k \).
for all \( n \geq k + 1 \) and \( k \geq 1 \). Furthermore, the extremal graphs are exactly paths \( P_n \) when \( k = 1 \). If \( k \geq 2 \), then the extremal graphs can be different from \( P_n^k \) [2].

3 Sharp Bounds

Theorem 1 Let \( G \) be a \( k \)-degenerate graph of order \( n \geq k \geq 1 \). Then

\[
W(G) \geq n^2 - (k + 1)n + \binom{k + 1}{2}.
\]

The equality holds if and only if \( G \) is maximal \( k \)-degenerate with \( \text{diam}(G) \leq 2 \).

Proof By Lemma 1 (i), \( W(G) \geq 2 \binom{n}{2} - |E(G)| \) and the equality holds if and only if \( G \) has diameter at most 2. By Proposition 3 in [14], a \( k \)-degenerate graph \( G \) of order \( n \geq k \) has \( |E(G)| \leq kn - \binom{k + 1}{2} \). Moreover, a \( k \)-degenerate graph \( G \) of order \( n \geq k \) is maximal if and only if \( |E(G)| = kn - \binom{k + 1}{2} \), [2]. Therefore, \( W(G) \geq n(n - 1) - kn + \binom{k + 1}{2} = n^2 - (k + 1)n + \binom{k + 1}{2} \), and the equality holds exactly when \( G \) is maximal \( k \)-degenerate with \( \text{diam}(G) \leq 2 \).

This bound is sharp since for \( k \leq n \leq k + 1 \), the only maximal \( k \)-degenerate graph is \( K_n \). For \( n \geq k + 2 \), \( K_k + K_{n-k} \) achieves the bound.

Theorem 2 Let \( G \) be a maximal \( k \)-degenerate graph of order \( n \geq 2 \) and \( D = \lceil \frac{n-2}{k} \rceil \). Then

\[
W(G) \leq W(P_n^k) = \sum_{i=0}^{D} \text{C}_{n-ik} + \text{C}_{n-k} + \cdots + \text{C}_{n-Dk}.
\]

Proof We show that \( W(G) \leq W(P_n^k) \) using induction on order \( n \). When \( 2 \leq n \leq k + 2 \), \( P_n^k \) is the only such graph, so it is extremal. Let \( G \) be a maximal \( k \)-degenerate graph of order \( n \geq k + 3 \), and assume that the result holds for all maximal \( k \)-degenerate graphs of smaller orders. By [14], \( G \) has a vertex \( v \) of degree \( k \) and \( G - v \) is a maximal \( k \)-degenerate graph. Thus \( W(G - v) \leq W(P_n^{k-1}) \).

Label vertices of \( P_n^k \) along the path \( P_n \) as \( v_1, v_2, \ldots, v_n \) where \( n \geq k + 3 \). It is clear that \( P_n^k \) is \( k \)-connected and \( \sigma_{P_n^k}(v_n) \) achieves the bound in Lemma 2. By Lemma 1 (ii), \( W(G) \leq W(G - v) + \sigma_{P_n^k}(v) \leq W(P_n^k - v_n) + \sigma_{P_n^k}(v_n) = W(P_n^k) \).

Note \( W(P_n^k) = \text{C}_{n-k} \) when \( 2 \leq n \leq k + 1 \), so that the formula holds then. In \( P_n \), there are \( n - i \) pairs of vertices with distance \( i \). Now distances \( rk - k + 1 \) through \( rk \) in \( P_n \) become \( r \) in \( P_n^k \). Since \( \text{diam}(P_n^k) = D + 1 \), by Lemma 1 (iii),

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\[ W(P^k_n) = 1(n - 1) + \cdots + 1(n - k) \\
+ 2(n - k - 1) + \cdots + 2(n - 2k) \\
+ 3(n - 2k - 1) + \cdots + 3(n - 3k) \\
+ \cdots \\
+ D(n - (D - 1)k - 1) + \cdots + D(n - Dk) \\
+ (D + 1)(n - Dk - 1) + \cdots + (D + 1)1 \\
= (n - 1 + \cdots + 1) + (n - k - 1 + \cdots + 1) + (n - 2k - 1 + \cdots + 1) \\
+ \cdots + (n - (D - 1)k - 1 + \cdots + 1) + (n - Dk - 1 + \cdots + 1) \\
= \binom{n}{2} + \binom{n - k}{2} + \binom{n - 2k}{2} + \cdots + \binom{n - (D - 1)k}{2} + \binom{n - Dk}{2} \]

We now provide a closed form expression for \( W(P^k_n) \) for all \( n \geq 2 \).

**Corollary 1** Let \( n \geq 2 \) and \( n - 2 \equiv j \mod k \) for \( 0 \leq j \leq k - 1 \). Then

\[
W(P^k_n) = \frac{n^3}{6k} + \frac{(k - 1)n^2}{4k} + \frac{(k - 3)n}{12} + \frac{-2j^3 + 3j^2(k - 3) - j(k^2 - 9k + 12) - 2k^2 + 6k - 4}{12k}.
\]

**Proof** We have

\[
W(P^k_n) = \sum_{i=0}^{D} \binom{n - ik}{2} = \sum_{i=0}^{D} \frac{1}{2}(n - ik)(n - ik - 1) \\
= \sum_{i=0}^{D} \left[ \left( \frac{n^2}{2} - \frac{n}{2} \right) + \left( \frac{k}{2} - kn \right)i + \frac{k^2}{2}i^2 \right] \\
= \sum_{i=0}^{D} \left( \frac{n^2}{2} - \frac{n}{2} \right) + \sum_{i=0}^{D} \left( \frac{k}{2} - kn \right)i + \sum_{i=0}^{D} \frac{k^2}{2}i^2 \\
= (D + 1)\left( \frac{n^2}{2} - \frac{n}{2} \right) + D(D + 1)\left( \frac{k}{2} - kn \right) + \frac{D(D + 1)(2D + 1)k^2}{6} \\
= \frac{k^2}{6}D^3 + \left( \frac{k}{4} + \frac{k^2}{4} - \frac{kn}{2} \right)D^2 + \left( \frac{k}{4} + \frac{k^2}{12} - \frac{n}{2} - \frac{kn}{2} + \frac{n^2}{2} \right)D - \frac{n^2}{2} + \frac{n^2}{2}
\]

Since \( D = \lfloor \frac{n-2}{k} \rfloor \), \( n - 2 = Dk + j \) for \( 0 \leq j \leq k - 1 \). Substituting \( D = \frac{n-2-j}{k} \) into the above and simplifying, we obtain the formula.

If \( 1 \leq k \leq 5 \), this formula can be reduced to

\[
W(P^k_n) = \left\lfloor \frac{2n^3 + 3(k - 1)n^2 + k(k - 3)n}{12k} \right\rfloor.
\]

Formulas for small values of \( k \) and the beginnings of the resulting sequences are given in the following table. These sequences occur (shifted) in the On-Line Encyclopedia of Integer Sequences (OEIS). For \( 1 \leq k \leq 3 \), they have many different combinatorial interpretations, which are listed in OEIS.
\[ W(P_n^k) = \frac{n^3 - n}{6} \]

| \( k \) | Sequence | OEIS |
|-----|--------|------|
| 1   | 0, 1, 4, 10, 20, 35, 56, 84, 120, 165, ... | A000292 |
| 2   | 0, 1, 3, 7, 13, 22, 34, 50, 70, 95, ... | A002623 |
| 3   | 0, 1, 3, 6, 11, 18, 27, 39, 54, 72, ... | A014125 |
| 4   | 0, 1, 3, 6, 10, 16, 24, 34, 46, 61, ... | A122046 |
| 5   | 0, 1, 3, 6, 10, 15, 22, 31, 42, 55, ... | A122047 |

### 4 Extremal Graphs

Any graph of order \( n \) and diameter 1 is a clique and has Wiener index \( \frac{n^2}{2} \). Any maximal \( k \)-degenerate graph of diameter 1 is \( K_n \), \( 2 \leq n \leq k + 1 \), which is also \( P_n^k \).

Recall that a graph \( G \) of order \( n \) and diameter 2 has \( W(G) = n(n - 1) - |E(G)| \), and a maximal \( k \)-degenerate graph \( G \) of order \( n \geq k \) has \( |E(G)| = kn - \binom{k+1}{2} \). Then any maximal \( k \)-degenerate graph of order \( n \geq k \) and diameter 2 has

\[
W(G) = n(n - 1) - kn + \binom{k+1}{2} = \binom{n}{2} + \binom{n-k}{2}.
\]

Therefore, when \( k \leq n \leq 2k + 1 \), the lower bound given in Theorem 1 and the upper bound given in Theorem 2 are the same, and any maximal \( k \)-degenerate graph of order \( n \) has this value for its Wiener index.

Maximal 1-degenerate graphs are just trees and so all maximal 1-degenerate graphs of diameter 2 are just stars. For \( k \geq 2 \), the graphs \( K_k + K_{n-k} \) are maximal \( k \)-degenerate graphs of diameter 2, but there are others.

We are able to characterize 2-trees of diameter 2. But the situation becomes complicated as \( k \) gets larger.

**Proposition 1** The following statements are equivalent for a 2-tree \( G \):

1. \( G \) has diameter at most 2.
2. \( G \) does not contain \( P_6^2 \).
3. \( G \) is \( T + K_1 \) for any tree \( T \), or any graph formed by adding any number of vertices adjacent to pairs of vertices of \( K_3 \). See Fig. 1.

**Proof** (3 \( \Rightarrow \) 1) The graphs described all have diameter at most 2.

(1 \( \Rightarrow \) 2) (contrapositive) We see \( P_6^2 \) is a 2-tree with diameter 3. Adding a new degree 2 vertex \( v \) to a 2-tree cannot decrease its diameter, since \( v \)'s neighbors are adjacent. Thus a 2-tree containing \( P_6^2 \) has diameter at least 3.

(2 \( \Rightarrow \) 3) Assume \( G \) does not contain \( P_6^2 \). The 2-trees with orders 4 and 5 (\( K_4 - e, \ P_4 + K_1, \) and \( K_2 + \overline{K}_3 \)) don’t contain \( P_6^2 \) and can be described as \( T + K_1 \). Any 2-tree not containing \( P_4 + K_1 \) is \( K_1 + K_{1,r} \), because any additional vertices must be rooted.
at the edge $xy$ of $K_2 + \overline{K}_3$, see Fig. 1. Assume $G$ has order at least 6. Since it does not contain $P_6$, there are three possibilities.

Case 1. $G$ contains $P_5 + K_1$. Then any additional vertices must be rooted on edges incident with $K_1$ (the vertex $z$), or else it will contain $P_6$.

Case 2. $G$ contains the triangular grid $Tr_2$. Then the only edges that can be used as roots are those of the central clique $K_3$ (the triangle $abc$), or else it will contain $P_6$.

Case 3. $G$ roots all additional vertices on the edges between vertices of degree 3 and 4 in $P_4 + K_1$.

Graphs in Case 1 and Case 3 can be described as $T + K_1$, where $T$ is a tree.

Graphs in Case 2 are formed by adding vertices rooted at edges from a fixed clique $K_3$.

Maximal outerplanar graphs are exactly the 2-trees that are outerplanar [15]. A graph is outerplanar if and only if it does not contain a subdivision of $K_4$ or $K_{2,3}$ [5]. Thus we have the following corollary.

**Corollary 2** The maximal outerplanar graphs with diameter at most 2 are fans $P_{n-1} + K_1$ and the triangular grid $Tr_2$.

A characterization of all maximal 2-degenerate graphs with diameter 2, generalizing Proposition 1, has been proved in [3].
Since any maximal $k$-degenerate graph of order $n \geq k + 1$ is $k$-connected and $\text{diam}(G) \leq \left\lceil \frac{n-k-1}{k} \right\rceil + 1$ for a $k$-connected graph $G$ of order $n$, any maximal $k$-degenerate graph of diameter at least 3 has order $n \geq 2k + 2$.

**Theorem 3** Let $G$ be a $k$-tree of order $n \geq 2k + 2$ and $k \geq 1$. Then $W(G) = \sum_{i=0}^{\left\lceil \frac{n-2}{k} \right\rceil} \binom{n-i-1}{2}$ exactly when $G = P_n^k$.

**Proof** We use induction on order $n$. By definition, a $k$-tree can be constructed from a clique $K_k$, and the $i$-th vertex added is adjacent to at least $k - i + 1$ vertices of the starting clique. Thus the smallest order of a $k$-tree with diameter 3 is $n = 2k + 2$. To achieve this, there is a unique choice (up to isomorphism) for the neighborhood of each newly added vertex. Since $P_{2k+2}^k$ has diameter 3, this is the $k$-tree that is constructed. Thus the result holds for the base case of $n = 2k + 2$.

Let $G$ be a $k$-tree of order $n \geq 2k + 3$ that maximizes $W(G)$, and assume that the result holds for all $k$-trees of order $n - 1$. By the definition of a $k$-tree, $G$ has a vertex $v$ of degree $k$ such that $G - v$ is a $k$-tree. By Lemma 1(ii), $W(G) \leq W(G - v) + \sigma_G(v)$. We will show that $G$ simultaneously achieves the maximum possible values of $W(G - v)$ and $\sigma_G(v)$, which means that no extremal graph exists that does not do so.

Maximizing $W(G - v)$ requires that $G - v$ is the extremal graph $P_{n-1}^k$. Number the vertices of $G - v$ along the path from 1 to $n - 1$. Since $k$-trees of order at least $k + 1$ are $k$-connected, $\sigma_G(v)$ is maximized when $N_G(v) = \{1, 2, \ldots, k\}$ (or $N_G(v) = \{n-k, \ldots, n-1\}$) since it achieves the bound in Lemma 2. When $n \geq 2k + 3$, any other choice for $N_G(v)$ has $|N_G(v, 2)| > k$, so $\sigma_G(v)$ is not maximized. Thus $G = P_n^k$, and Theorem 2 provides the formula.

Note that for $k > 1$, there is a unique extremal graph for $k$-trees to achieve the upper bound in Theorem 2 when $k \leq n \leq k + 2$ or $n \geq 2k + 2$, but not when $k + 3 \leq n \leq 2k + 1$.

By Theorems 1, 2 and Corollary 1, we have the following sharp bounds on Wiener indices of maximal $k$-degenerate graphs for $1 \leq k \leq 3$.

**Corollary 3** Let $G$ be a maximal $k$-degenerate graph of order $n \geq k \geq 1$.

1. If $k = 1$, then $G$ is a tree and $n^2 - 2n + 1 \leq W(G) \leq \frac{n^3}{6} - \frac{n}{2}$. The extremal graphs for the bounds are exactly $K_1 + \overline{K}_{n-1}$ and $P_n$ respectively, see [10].

2. If $k = 2$, then $n^2 - 3n + 3 \leq W(G) \leq \frac{n^3}{12} + \frac{n^2}{8} - \frac{n}{12} - \frac{1}{6} + \frac{(-1)^n}{16}$.

For 2-trees, the extremal graphs for the lower bound are characterized in Proposition 1; the extremal graphs for the upper bound are $P_n^2$ and $K_2 + \overline{K}_3$ (of order 5), see Theorem 3.

For maximal outerplanar graph of order $n \geq 3$ (that is, outerplanar 2-trees), the extremal graphs for the lower bound are fans $P_{n-1} + K_1$ and the triangular grid graph $T_n$ if $n = 6$; and the extremal graphs for the upper bound are $P_n^2$.

3. If $k = 3$, then $n^2 - 4n + 6 \leq W(G) \leq \left\lfloor \frac{n^3}{18} + \frac{n^2}{6} \right\rfloor$.

For 3-trees, it is easily checked that the extremal graphs for the upper bound...
are $P_3^n$, $K_3 + \overline{K}_3$ of order 6 and four others of order 7 which are $K_3 + K_4$, $K_2 + T_5$, where $T_5$ is the tree of order 5 that is neither a path nor a star, $P_5 + K_2$, and the graph formed from $K_4$ by adding degree 3 vertices inside 3 regions. See Fig. 2.

For Apollonian networks (planar 3-trees), the upper bound was given in [7]. The extremal graphs for the upper bound are $P_3^n$ and the last two graphs of order 7 in Fig. 2.

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