Some inverse limits of Cuntz algebras

Katsunori Kawamura*

College of Science and Engineering, Ritsumeikan University,
1-1-1 Nogi Higashi, Kusatsu, Shiga 525-8577, Japan

Abstract

We construct a nontrivial inverse system of Cuntz algebras \( \{ \mathcal{O}_n : 2 \leq n < \infty \} \), whose inverse limit is \( \ast \)-isomorphic onto \( \mathcal{O}_\infty \). By using this result, it is shown that the \( K_0 \)-functor is discontinuous with respect to the inverse limit even if the limit is a \( C^* \)-algebra.

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1 Introduction

In this paper, we construct a nontrivial inverse system of Cuntz algebras \( \{ \mathcal{O}_n : 2 \leq n < \infty \} \), whose inverse limit is \( \mathcal{O}_\infty \):

\[
\lim_{\leftarrow} \mathcal{O}_n \cong \mathcal{O}_\infty.
\] (1.1)

In order to explain (1.1), we will recall inverse system of \( C^* \)-algebras and pro-\( C^* \)-algebra, and show the construction of the inverse system in this section.

1.1 Unital \( \ast \)-homomorphisms among Cuntz algebras

Our study is motivated by well-known facts of unital \( \ast \)-homomorphisms among Cuntz algebras. Hence we start with their explanation in this subsection. For unital \( C^* \)-algebras \( A \) and \( B \), let \( \text{Hom}(A, B) \) denote the set of all unital \( \ast \)-homomorphisms from \( A \) to \( B \) and let \( K_0(A) \) denote the \( K_0 \)-group of \( A \) [5].

*e-mail: kawamura@kurims.kyoto-u.ac.jp.
**Fact 1.1** For $2 \leq n \leq \infty$, let $\mathcal{O}_n$ denote the Cuntz algebra [11]. Then the following holds:

(i) For $2 \leq n \leq \infty$ and a unital $C^*$-algebra $A$, if $f \in \text{Hom}(A, \mathcal{O}_n)$, then the induced homomorphism $\hat{f}$ from $K_0(A)$ to $K_0(\mathcal{O}_n)$ is surjective.

(ii) For $2 \leq m, n < \infty$, $\text{Hom}(\mathcal{O}_m, \mathcal{O}_n) \neq \emptyset$ if and only if there exists a positive integer $k$ such that $m = (n-1)k + 1$.

(iii) For any $2 \leq m < \infty$, $\text{Hom}(\mathcal{O}_m, \mathcal{O}_\infty) = \emptyset$.

(iv) For any $2 \leq n \leq \infty$, $\text{Hom}(\mathcal{O}_\infty, \mathcal{O}_n) \neq \emptyset$.

**Proof.** (i) From Example 6.3.2 in [3], the class of the unit of $\mathcal{O}_n$ is the generator of $K_0(\mathcal{O}_n)$. Since $\hat{f}$ maps the class of the unit of $A$ to that of $\mathcal{O}_n$, the statement holds.

(ii) This holds from Lemma 2.1 in [28] (see also [14], p164, V.16.) In §1.3, we will give concrete *-homomorphisms among Cuntz algebras.

(iii) From [12], $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$ ($2 \leq n < \infty$) and $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$. From these and (i), the statement holds.

(iv) This will be shown by using concrete *-homomorphisms $\{f_{n,\infty}: n \geq 1\}$ in (1.10).

About homomorphisms among Cuntz algebras, more general results are known ([20], Lemma 7.1.) Since $\mathcal{O}_n$ is simple for each $n$, any unital *-homomorphism from $\mathcal{O}_n$ is injective, hence it is an embedding of $\mathcal{O}_n$. For examples of Fact 1.1(ii), the following embeddings among $\mathcal{O}_2, \ldots, \mathcal{O}_8$ (except endomorphisms) are illustrated as follows ([28], §2.1):

**Figure 1.2**

where an arrow “$A \rightarrow B$” means a unital embedding of $A$ into $B$. For $2 \leq n < m \leq 8$, there is no unital *-homomorphism from $\mathcal{O}_m$ to $\mathcal{O}_n$ if there is no oriented path from $\mathcal{O}_m$ to $\mathcal{O}_n$ in Figure 1.2.
Remark 1.3 In p184 of [11], there is a statement about embeddings among Cuntz algebras as follows:

\[ \mathcal{O}_2 \supset \mathcal{O}_3 \supset \mathcal{O}_4 \supset \cdots \supset \mathcal{O}_\infty. \]  

(1.2)

It is explained that (1.2) is given by using the induction for the construction of a certain unital embedding of \( \mathcal{O}_3 \) into \( \mathcal{O}_2 \). However, (1.2) never means unital embeddings because of Fact 1.1(ii) except “\( \mathcal{O}_2 \supset \mathcal{O}_3 \)” and “\( \supset \mathcal{O}_\infty \)”.

On the other hand, there exist well-known two orders on the set \( \mathbb{N} \) of all positive integers \( \{1, 2, 3, \ldots\} \). The first is the standard linear order \( \leq \), that is, \( 1 \leq 2 \leq 3 \leq \cdots \). The second is the order \( \preceq \) on \( \mathbb{N} \) ([6, §1.11) defined as

\[ m \preceq n \quad \text{if and only if} \quad m \text{ divides } n. \]  

(1.3)

This relation \( m \preceq n \) is usually written as \( m|n \) in number theory [21]. Both \( (\mathbb{N}, \leq) \) and \( (\mathbb{N}, \preceq) \) are directed sets, but \( (\mathbb{N}, \preceq) \) is not a totally ordered set, which is illustrated as the following directed graph (except relations \( n \preceq n \)):

Figure 1.4

\begin{center}
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (-1,-1) {2};
  \node (3) at (-2,-2) {3};
  \node (4) at (-3,-3) {4};
  \node (5) at (1,-1) {5};
  \node (6) at (2,-2) {6};
  \node (7) at (3,-3) {7};

  \draw[->] (1) to (2);
  \draw[->] (1) to (3);
  \draw[->] (1) to (4);
  \draw[->] (1) to (5);
  \draw[->] (1) to (6);
  \draw[->] (1) to (7);

\end{tikzpicture}
\end{center}

By comparison with Figure 1.2, it is clear that Figure 1.4 is just the graph with inverse direction of Figure 1.2 by rewriting their suffix numbers. This idea is rigorously verified by using Fact 1.1(ii) and we can restate Fact 1.1(ii), (iii) and (iv) by using the order \( \preceq \) as follows.

Corollary 1.5 Let \( \hat{\mathbb{N}} \equiv \mathbb{N} \cup \{\infty\} \) and extend \( \preceq \) on \( \hat{\mathbb{N}} \) as \( \infty \preceq \infty \) and \( n \preceq \infty \) for each \( n \in \mathbb{N} \). Then, for \( n, m \in \hat{\mathbb{N}} \),

\[ \text{Hom}(\mathcal{O}_{m+1}, \mathcal{O}_{n+1}) \neq \emptyset \quad \text{if and only if} \quad n \preceq m \]  

(1.4)

where \( n + \infty \) means \( \infty \) for convenience. Especially, (1.4) holds for each \( n, m \in \mathbb{N} \), and there exists no unital \(*\)-homomorphism from \( \mathcal{O}_m \) to \( \mathcal{O}_n \) when \( n > m \).

From Corollary 1.5 if there exist the following unital inclusions

\[ \mathcal{O}_{n_1+1} \supset \mathcal{O}_{n_2+1} \supset \mathcal{O}_{n_3+1} \supset \cdots \]  

(1.5)

for \( \{n_i \in \mathbb{N} : i \geq 1\} \), then we obtain order relations \( n_1 \preceq n_2 \preceq n_3 \preceq \cdots \).

It is well-known that the order \( \preceq \) is used in the theory of profinite groups [39]. A profinite group is defined as an inverse limit of finite groups. From this and Corollary 1.5 the following questions are inspired.

3
Question 1.6  

(i) Does there exist an inverse system of Cuntz algebras 
\( \{O_{n+1} : 1 \leq n < \infty \} \) over the directed set \((\mathbb{N}, \preceq)\) with respect to 
embeddings in Figure 1.2?

(ii) If such an inverse system in (i) is found, then what-like algebra is the 
inverse limit?

(iii) If the answer to (ii) is given, then what is the meaning of this result?

The purpose of this paper is to give answers to these questions.

1.2 Inverse limits of C*-algebras

In order to consider inverse limits of Cuntz algebras in Question 1.6, we 
recall previous works of inverse limits of C*-algebras in this subsection.

According to Phillips [37], Fragoulopoulou [18] and Joiţa [25], inverse 
limits (or projective limits [13, 36]) of C*-algebras were studied by differ-
ent names in the literature as follows: \( b^* \)-algebras (Allan [1], Apostol [2]), 
LMC*-algebras (Lassner [31], Schmudgen [43]), l.m.c.C*-algebras (Mallios 
[34]), locally C*-algebras (Inoue [23], Fragoulopoulou [17]), generalized op-
erator algebras (Weidner [46, 47]), \( F^* \)-algebras (Brooks [10]), \( \sigma \)-C*-algebras (Arveson [4]), or pro-C*-algebras (Voiculescu [44]).

Next, we recall definitions and basic facts, where notations are slightly 
changed from [37]. Let \((D, \preceq)\) be a directed set \([30]\), that is, \(D\) is a non-
empty set and \(\preceq\) is a binary relation on \(D\) which satisfies the conditions:
For all \(a, b, c \in D\), we have \(a \preceq a\); if \(a \preceq b\) and \(b \preceq c\), then \(a \preceq c\); if \(a, b \in D\), 
then there exists \(c \in D\) such that \(a \preceq c\) and \(b \preceq c\). Such \(\preceq\) is called a 
preorder \([6]\). We call \(\preceq\) an order in this paper for simplicity of description. 
For every concrete directed set \((D, \preceq)\) in this paper, the order \(\preceq\) satisfies 
the antisymmetric law, that is, if \(a \preceq b\) and \(b \preceq a\), then \(a = b\).

Definition 1.7 Let \((D, \preceq)\) be a directed set.

(i) A data \(\{ (A_d, \varphi_{d,e}) : d, e \in D \} \) is an inverse system (or projective 
system \([44]\)) of C*-algebras if \(A_d\) is a C*-algebra for each \(d \in D\) and 
\(\varphi_{d,e}\) is a *-homomorphism from \(A_e\) to \(A_d\) when \(d \preceq e\) such that 
\(\varphi_{d,e} \circ \varphi_{e,f} = \varphi_{d,f}\) when \(d \preceq e \preceq f\), and \(\varphi_{d,d} = \text{id}_{A_d}\).

(ii) The inverse limit \((A, \{ \pi_d \}_{d \in D})\) of an inverse system \(\{ (A_d, \varphi_{d,e}) : d, e \in 
D \} \) of C*-algebras is a topological *-algebra \(A\) and a *-homomorphism 
\(\pi_d\) from \(A\) to \(A_d\) such that the following conditions hold:

(a) \(\varphi_{d,e} \circ \pi_e = \pi_d\) when \(d \preceq e\),
(b) for any $\ast$-algebra $B$ and $\ast$-homomorphisms $\{\eta_d\}_{d \in D}$, $\eta_d : B \to A_d$, which satisfy $\varphi_{d,e} \circ \eta_e = \eta_d$ when $d \leq e$, there exists a unique $\ast$-homomorphism $\psi$ from $B$ to $A$ such that $\pi_d \circ \psi = \eta_d$ for each $d \in D$.

(c) the topology of $A$ is the weakest topology such that every $\pi_d$ is continuous.

In this case, $A$ is written as $\lim_{\leftarrow} D (A_d, \varphi_{d,e})$ or $\lim_{\leftarrow} (A_d, \varphi_{d,e})$, and $\pi_d$ is called the canonical homomorphism (or projection [39], canonical mapping [6]).

(iii) A pro-$C^*$-algebra $A$ is a complete Hausdorff topological $\ast$-algebra over $C$ whose topology is determined by its continuous $C^*$-seminorms in the sense that a net $\{a_\lambda\}$ converges to 0 if and only if $p(a_\lambda) \to 0$ for every continuous $C^*$-seminorm $p$ on $A$.

(iv) An inverse limit of $C^*$-algebras over a countable directed set is called a $\sigma$-$C^*$-algebra.

From “1.2. Proposition” in [39], a $C^*$-algebra $A$ is a pro-$C^*$-algebra if and only if it is the inverse limit of an inverse system of $C^*$-algebras. Any $C^*$-algebra is a pro-$C^*$-algebra. For any inverse system $\{(A_d, \varphi_{d,e}) : d, e \in D\}$, the inverse limit $\lim_{\leftarrow} A_d$ is given as the following subset of the product set $\prod_{d \in D} A_d$:

$$\{(x_d) \in \prod_{d \in D} A_d : \varphi_{d,e}(x_e) = x_d \text{ for all } d, e \in D \text{ such that } d \leq e\}.$$  (1.6)

In general, a pro-$C^*$-algebra is not a $C^*$-algebra (see “1.3. EXAMPLE” in [44]), but the inverse limit of a special inverse system of $C^*$-algebras is also a $C^*$-algebra as follows.

**Fact 1.8** Let $\{(A_d, \varphi_{d,e}) : d, e \in D\}$ be an inverse system of $C^*$-algebras. We write $\lim_{\leftarrow} (A_d, \varphi_{d,e})$ as $\lim_{\leftarrow} A_d$ for simplicity of description.

(i) If $\varphi_{d,e}$ is injective for each $d, e$, then $\lim_{\leftarrow} A_d$ is a $C^*$-algebra.

(ii) If $\{A_n : n \in N\}$ is a sequence of inclusions of $C^*$-algebras such that $A_n \supset A_{n+1}$ for each $n \in N$, then the inclusion map $i_n : A_{n+1} \hookrightarrow A_n$ induces an inverse system over $(N, \leq)$ such that $\lim_{\leftarrow} A_n$ is $\ast$-isomorphic onto $\bigcap_{n \geq 1} A_n$ as a $C^*$-algebra.

(iii) If $A_d$ is unital and simple, and $f_{d,e}$ is unital for each $d, e \in D$, then $\lim_{\leftarrow} A_d$ is a unital $C^*$-algebra.
(iv) If there exists the maximal element $\omega$ of $D$, then $\lim \leftarrow A_d \cong A_\omega$.

Proof. Since an injective $\ast$-homomorphism is an isometry, (i) holds from Definition [1.7](ii)(c). (ii) and (iii) hold from (i). (iv) holds from (1.6).

When an inverse limit of C*-algebras is a C*-algebra, it is not interesting as a pro-C*-algebra, but it does not mean the triviality of the inverse limit as a C*-algebra.

Next, we consider the inverse limit of Cuntz algebras in general setting. (About inductive (or direct) limits of Cuntz algebras, see [20, 33, 40].) Since any Cuntz algebra is unital and simple, the inverse limit of any inverse system of Cuntz algebras by unital $\ast$-homomorphisms is a unital C*-algebra from Fact (1.8)(ii). By Corollary 1.5 and Fact (1.8)(iv), the following holds.

**Fact 1.9** Let $(\hat{N}, \preceq)$ be as in Corollary [1.5] and let $\{(O_{n(d)+1}, \varphi_{d,e}) : d, e \in D\}$ be an inverse system of Cuntz algebras over a directed set $(D, \preceq)$ such that $\{n(d) \in \hat{N} : d \in D\}$. We assume that $\varphi_{d,e}$ is unital for each $d, e$.

(i) The map

$$F : D \to \hat{N}; \quad d \mapsto F(d) \equiv n(d)$$

(1.7)

is an ordered set homomorphism from $(D, \preceq)$ to $(\hat{N}, \preceq)$.

(ii) If there exists the maximal element $\omega$ of $D$, then $\lim \leftarrow O_{n(d)+1} \cong O_{n(\omega)+1}$.

Remark that $F$ in (1.7) is not injective in general. There are many endomorphisms of Cuntz algebras [7, 8, 29].

Let $(N, \succeq)$ be as in Corollary [1.5]. Define the order $\preceq_c$ on the set $\{O_{n+1} : n \in N\}$ of all Cuntz algebras as “$A \preceq_c B$ if and only if $\text{Hom}(A, B) \neq \emptyset.”$ Then $(\{O_{n+1} : n \in N\}, \preceq_c)$ is anti-isomorphic onto $(N, \succeq)$ as an ordered set with respect to the mapping $n \mapsto O_{n+1}.$

### 1.3 An inverse system of Cuntz algebras

In this subsection, we construct an example of inverse system of Cuntz algebras as an answer to Question [1.6](i). For the directed set $(N, \preceq)$ in (1.3), define the inverse system $\{(R_n, f_{n,m}) : n, m \in N\}$ of C*-algebras as follows: For $2 \leq n < \infty$, let $s_1^{(n)}, \ldots, s_n^{(n)}$ denote the Cuntz generators of $O_n$, that is, $(s_i^{(n)})^* s_j^{(n)} = \delta_{ij} I$ for $i, j = 1, \ldots, n$ and $s_1^{(n)} (s_1^{(n)})^* + \cdots + s_n^{(n)} (s_n^{(n)})^* = I$. For convenience, rewrite $O_{n+1}$ as

$$R_n \equiv O_{n+1} \quad (n \in N).$$

(1.8)
This notation is reasonable with respect to the $K$-theory of Cuntz algebras and Corollary 1.5. (About such a notation, see also [3], which is not related to the inclusions in Figure 1.2.) Remark that $R_n$ is generated by $s_1^{(n+1)}, \ldots, s_{n+1}^{(n+1)}$ by definition. When $n \preceq m$ and $n \neq m$, define the $*$-homomorphism $f_{n,m}$ from $R_m$ to $R_n$ by

$$
\begin{cases}
  f_{n,m}(s_{nl+i}^{(m+1)}) \equiv (s_{n+1}^{(n+1)})^l s_i^{(n+1)} & \text{for } l = 0, 1, \ldots, \frac{m}{n} - 1, \quad i = 1, \ldots, n \\
  f_{n,m}(s_{m+1}^{(m+1)}) \equiv (s_{n+1}^{(n+1)})^\frac{m}{n}
\end{cases}
$$

(1.9)

where $(s_{n+1}^{(n+1)})^0$ means the unit of $R_n$ for convenience, and define $f_{n,n}$ as the identity map $\text{id}_{R_n}$ on $R_n$. A graphical explanation of (1.9) will be given in §3.2. Then the following holds.

**Theorem 1.10** Let $R_n$ and $f_{n,m}$ be as in (1.8) and (1.9), respectively.

(i) The data \(\{(R_n, f_{n,m}) : n, m \in \mathbb{N}\}\) is an inverse system of $C^*$-algebras over the directed set \((\mathbb{N}, \preceq)\), that is, the relation $f_{n,m} \circ f_{m,l} = f_{n,l}$ holds when $n \preceq m \preceq l$.

(ii) Let \(\{s_1^{(\infty)}, s_2^{(\infty)}, \ldots\}\) denote the Cuntz generators of $\mathcal{O}_\infty$. For $n \in \mathbb{N}$, define the embedding $f_{n,\infty}$ of $\mathcal{O}_\infty$ into $R_n$ by

$$f_{n,\infty}(s_{ln+i}^{(\infty)}) \equiv (s_{n+1}^{(n+1)})^l s_i^{(n+1)} \quad (i = 1, \ldots, n, \quad l \geq 0).$$

(1.10)

Then \(\{f_{n,\infty} : n \in \mathbb{N}\}\) satisfies

$$f_{n,m} \circ f_{m,\infty} = f_{n,\infty} \quad \text{when } n \preceq m.$$  

(1.11)

(iii) Every $f_{n,m}$ in (1.9) and (1.10) is irreducible, where we state that $f \in \text{Hom}(A, B)$ is irreducible if $f(A)' \cap B = C\mathbb{I}$.

We illustrate relations of maps in Theorem 1.10 as the following commutative diagrams where we assume $n \preceq m$:  

7
**Remark 1.12** Every $f_{n,m}$ in (1.9) and (1.10) is unital and injective, but not surjective when $n \neq m$. Such $*$-homomorphisms were given in [26, 28], hence they are not new, but their relations of inverse system are new. The essential part of Theorem 1.10(i) is the construction of formulas in (1.9). The proof is given by simple algebraic calculation. We make a point that Corollary 1.5 itself does not show the existence of any inverse system of Cuntz algebras over the directed set $(\mathbb{N}, \preceq)$. Inversely, it is interesting question whether the existence of such an inverse system of Cuntz algebras with unital $*$-homomorphisms is shown only from Corollary 1.5 and general theory without use of concrete construction of $*$-homomorphisms.

For an example of $f_{n,m}$ in (1.9), the $*$-homomorphism $f_{1,2} : R_2(= \mathcal{O}_3) \to R_1(= \mathcal{O}_2)$ is given as follows:

$$f_{1,2}(\hat{s}_1) = s_1, \quad f_{1,2}(\hat{s}_2) = s_2 s_1, \quad f_{1,2}(\hat{s}_3) = s_2 s_2$$  \hspace{1cm} (1.12)

where $s_1, s_2$ and $\hat{s}_1, \hat{s}_2, \hat{s}_3$ denote Cuntz generators of $\mathcal{O}_2$ and $\mathcal{O}_3$, respectively. A similar $*$-homomorphism was given by Cuntz’s original paper ([11], p183). For the first time, the author knew (1.12) from Akira Asada who constructed $f_{1,2}$, and (1.9) is a generalization of (1.12).

### 1.4 Inverse limits of Cuntz algebras

In this subsection, we show the inverse limit of the inverse system $\{(R_n, f_{n,m}) : n, m \in \mathbb{N}\}$ in Theorem 1.10(i), which is the answer to Question 1.6(ii). The problem is solved in slightly generalized setting.

**Theorem 1.13** Let $\{(R_n, f_{n,m}) : n, m \in \mathbb{N}\}$ be as in Theorem 1.10. For a directed subset $\Lambda$ of $(\mathbb{N}, \preceq)$, let $\mathcal{O}(\Lambda)$ denote the inverse limit $\varprojlim_{\Lambda} R_n$ of
the subsystem \( \{(R_n, f_{n,m}) : n, m \in \Lambda\} \) of the inverse system \( \{(R_n, f_{n,m}) : n, m \in \mathbb{N}\} \):

\[
\hat{O}(\Lambda) \equiv \lim_{\Lambda} R_n. \tag{1.13}
\]

Then the following holds:

(i) Assume that \( \Lambda \) is an infinite totally ordered subset of \((\mathbb{N}, \preceq)\) such that \( \Lambda = \{n_1, n_2, \ldots\} \) and \( n_1 \preceq n_2 \preceq \cdots \). For \( \{f_{n,\infty} : n \in \mathbb{N}\} \) in (1.10), define the \( * \)-homomorphism \( \psi_\Lambda \) from \( O_\infty \) to \( \hat{O}(\Lambda) \) by

\[
\psi_\Lambda(x) \equiv (f_{n_1,\infty}(x), f_{n_2,\infty}(x), \ldots) \ (x \in O_\infty) \tag{1.14}
\]

where \( \hat{O}(\Lambda) \) is identified with the standard form (1.6). Then \( \psi_\Lambda \) is a \( * \)-isomorphism such that

\[
\pi_n \circ \psi_\Lambda = f_{n,\infty} \quad (n \in \Lambda) \tag{1.15}
\]

where \( \pi_n \) denotes the canonical homomorphism from \( \hat{O}(\Lambda) \) to \( R_n \).

(ii) For a directed subset \( \Lambda \) of \((\mathbb{N}, \preceq)\), the following holds:

\[
\hat{O}(\Lambda) \cong \begin{cases} O_{N+1} & (\exists N = \text{max } \Lambda), \\ O_\infty & \text{(otherwise)}. \end{cases} \tag{1.16}
\]

Especially, when \( \Lambda = \mathbb{N} \), we obtain

\[
\lim_{\mathbb{N}} R_n \cong O_\infty. \tag{1.17}
\]

The proof of Theorem 1.13 will be given in §2. The crucial part of the proof is the surjectivity of \( \psi_\Lambda \) in (1.14). From Theorem 1.13(i) for \( \Lambda = \mathbb{N} \), the data \( (O_\infty, \{f_{n,\infty}\}_{n \in \mathbb{N}}) \) is the inverse limit of \( \{(R_n, f_{n,m}) : n, m \in \mathbb{N}\} \) in the sense of Definition 1.7(ii).

As a counterview of Theorem 1.13(ii), we can say that \( O_\infty \) can be written as an inverse limit of \( O_n \)'s. From this and Definition 1.7(ii)(b), the following corollary immediately holds.

**Corollary 1.14** Let \( \{f_{n,m} : n, m \in \mathbb{N}\} \) and \( \{f_{n,\infty} : n \in \mathbb{N}\} \) be as in (1.9) and (1.10), respectively. If a \( * \)-algebra \( B \) and \( * \)-homomorphisms \( \{\eta_n\}_{n \in \mathbb{N}}, \eta_n : B \to O_{N+1} \), which satisfy \( f_{n,m} \circ \eta_m = \eta_n \) when \( n \preceq m \), there exists a unique \( * \)-homomorphism \( \psi \) from \( B \) to \( O_\infty \) such that \( f_{n,\infty} \circ \psi = \eta_n \) for each \( n \in \mathbb{N} \).
This is an essentially new universal property of \( O_\infty \). About other universal properties of \( O_\infty \), see Chapter 7 of [42].

**Remark 1.15** (i) Here we discuss the choice of notations. Both notations \( O_n \) and \( R_n \) are useful in different situations. The standard notations \( O_2, O_3, \ldots \) are used in the construction of \( C^* \)-bialgebra [27].

With respect to the standard notations of Cuntz algebras, the order \( \preceq \) in (1.3) can be rewritten as follows: A new order \( \ll \) on the set \( N_2 \equiv \{ 2, 3, 4, \ldots \} \) is defined as “\( n \ll m \) if \( n - 1 \preceq m - 1 \),” or “\( \frac{m-1}{n-1} \in N \).” Let \( g_{n,m} \equiv f_{n-1,m-1} \) for \( n,m \in N_2 \). Then \( \{(O_n, g_{n,m}) : n,m \in N_2\} \) is an inverse system over the directed set \( (N_2, \ll) \) which is isomorphic to the inverse system \( \{(R_n, f_{n,m}) : n,m \in N\} \) over \( (N, \preceq) \). By using \( (N_2, \ll) \), the formula (1.1) makes sense.

(ii) In “otherwise” in (1.16), we assume that \( \Lambda \) is neither a cofinal nor a totally ordered subset of \( (N, \preceq) \).

(iii) We consider inverse systems of \( C^* \)-subalgebras of Cuntz algebras in Theorem 1.10. Let \( \gamma^{(n+1)} \) and \( \eta^{(n+1)} \) denote the \( U(1) \)-gauge action and the standard torus \( (=T^{n+1}) \)-action on \( O_{n+1} \), respectively. Let \( A_n \) and \( C_n \) denote the fixed-point subalgebras of \( O_{n+1} \) with respect to \( \gamma^{(n+1)} \) and \( \eta^{(n+1)} \), respectively:

\[
A_n \equiv (O_{n+1})^{U(1)}, \quad C_n \equiv (O_{n+1})^{T^{n+1}}.
\]

Then we see that \( f_{n,m}(A_m) \not\subset A_n \) even if \( n \preceq m \), but \( f_{n,m}(C_m) \subset C_n \). In this way, \( \{ (C_n, f_{n,m}|_{C_m}) : n,m \in N \} \) is an inverse system of unital abelian \( C^* \)-algebras over \( (N, \preceq) \). Since \( f_{n,m}|_{C_m} \) is also injective and unital, \( \varprojlim (C_n, f_{n,m}|_{C_m}) \) is also a unital abelian \( C^* \)-algebra.

(iv) As analogy with previous works of inductive limits [32, 33, 40], the classification of inverse limits of Cuntz algebras is an interesting problem. In the case of inductive limits, classification theorems are given by using \( K \)-theory. On the other hand, it seems that \( K \)-theory is no use for the case of inverse limits, which will be shown in §1.5. In order to consider the problem, concrete examples are not sufficient yet. We will show other example of inverse system which limit is not a Cuntz algebra in §3.1.

(v) In the proof of “3.11 COROLLARY” in [12], \( O_\infty \) is written as an inductive limit of \( C^* \)-subalgebras of \( O_n \)’s as follows: For \( n \geq 1 \), let \( C^*(s_1^{(n+1)}, \ldots, s_n^{(n+1)}) \) denote the \( C^* \)-subalgebra of \( O_{n+1} \) generated by
\[ s_1^{(n+1)}, \ldots, s_n^{(n+1)} \text{ without } s_{n+1}^{(n+1)}. \text{ Remark } \mathcal{C}^*(s_1^{(n+1)}, \ldots, s_n^{(n+1)}) \not\cong \mathcal{O}_n \] because \( K_0(\mathcal{C}^*(s_1^{(n+1)}, \ldots, s_n^{(n+1)})) \cong \mathbb{Z}. \) Then it is identified with a \( \mathcal{C}^* \)-subalgebra of \( \mathcal{C}^*(s_1^{(n+2)}, \ldots, s_n^{(n+1)}) \) by \( \iota_n(s_i^{(n+1)}) \equiv s_i^{(n+2)} \) for \( i = 1, \ldots, n. \) Then the inductive system \( \{(\mathcal{C}^*(s_1^{(n+1)}, \ldots, s_n^{(n+1)}), \iota_n) : n \geq 1\} \) over the directed set \((\mathbb{N}, \leq)\) is obtained:

\[
\mathcal{C}^*(s_1^{(2)}) \subset \mathcal{C}^*(s_1^{(3)}, s_2^{(3)}) \subset \mathcal{C}^*(s_1^{(4)}, s_2^{(4)}, s_3^{(4)}) \subset \cdots, \tag{1.19}
\]

and its inductive limit is isomorphic onto \( \mathcal{O}_\infty : \)

\[
\varprojlim \mathcal{C}^*(s_1^{(n+1)}, \ldots, s_n^{(n+1)}) \cong \mathcal{O}_\infty. \tag{1.20}
\]

This is quite a contrast to our result.

### 1.5 Discontinuity of \( K_0 \)

We discuss the (dis-) continuity of \( K_0 \)-functor of \( \mathcal{C}^* \)-algebras as an answer to Question 1.6(iii). Related to the question of the continuity of \( K_0 \)-functor with respect to the inverse limit, we could not find similar results in the standard textbooks [5, 24, 41, 45]. In §3 of [38], the inverse limit for representable \( K \)-theory is considered for \( \sigma \)-\( \mathcal{C}^* \)-algebras as the Milnor \( \lim^{-1} \)-sequence. However, it is assumed that all maps in the inverse system are surjective (“3.2 THEOREM”, [38]). Hence it is no use for our example.

#### 1.5.1 Profinite groups

In order to discuss \( K \)-groups of inverse limits of Cuntz algebras, we recall basic examples of profinite group (especially, they are procyclic groups [39], §2.7) as follows. Let \((\mathbb{N}, \leq)\) be as in [1.3]. For \( n, m \in \mathbb{N} \), if \( n \leq m \), then the natural projection from \( \mathbb{Z}/m\mathbb{Z} \) onto \( \mathbb{Z}/n\mathbb{Z} \) induces an inverse system \( \{\mathbb{Z}/n\mathbb{Z} : n \in \mathbb{N}\} \) of finite cyclic groups (especially, they are rings) over \((\mathbb{N}, \leq)\). It is well-known that the inverse limit \( \hat{\mathbb{Z}} \) of \( \{\mathbb{Z}/n\mathbb{Z} : n \in \mathbb{N}\} \) is called the Prüfer ring [35]:

\[
\hat{\mathbb{Z}} \equiv \varprojlim \mathbb{Z}/n\mathbb{Z}. \tag{1.21}
\]

Remark \( \hat{\mathbb{Z}} \not\cong \mathbb{Z} \). For a fixed prime number \( p \), the subset \( \{p^n : n \geq 1\} \) of \( \mathbb{N} \) is a directed subset of \((\mathbb{N}, \leq)\) which is not cofinal. For the subsystem \( \{\mathbb{Z}/p^n\mathbb{Z} : n \geq 1\} \) of \( \{\mathbb{Z}/n\mathbb{Z} : n \in \mathbb{N}\} \), the pro-\( p \) group \( \mathbb{Z}_p \) is defined as follows [39]:

\[
\mathbb{Z}_p \equiv \varprojlim \mathbb{Z}/p^n\mathbb{Z}. \tag{1.22}
\]

Remark that \( \mathbb{Z}_p \) is identified with an uncountable proper subgroup of \( \hat{\mathbb{Z}} \) [39], Exercise 2.1.8].
1.5.2 \textit{K}-functor is not continuous with respect to the inverse limit

It is well-known that the \( K \)-functor of \( C^\ast \)-algebras is continuous with respect to the inductive limit as the following sense \([5]\):

\[ K_0(\lim_{\to} A_n) \cong \lim_{\to} K_0(A_n). \tag{1.23} \]

Since \( \lim_{\to} A_n \) is always a \( C^\ast \)-algebra for any inductive system, \((1.23)\) holds for any inductive system of \( C^\ast \)-algebra. On the other hand, the inverse limit of \( C^\ast \)-algebra is not always a \( C^\ast \)-algebra. Hence (standard) \( K \)-groups cannot be defined on a pro-\( C^\ast \)-algebra in general. (\( K \)-groups are generalized for \( \sigma \)-\( C^\ast \)-algebras \([38]\), see also \([13, 46, 47]\).)

From \((1.1)\), the \( K_0 \)-group of \( \lim_{\leftarrow} O_n \) is well-defined. Then the following holds:

\[ K_0(\lim_{\leftarrow} O_n) \cong K_0(O_\infty) \cong Z \not\cong \hat{Z} = \lim_{\leftarrow} \mathbb{Z}/n\mathbb{Z} \cong \lim_{\leftarrow} K_0(O_n). \tag{1.24} \]

This shows that \( K_0 \)-functor is discontinuous with respect to the inverse limit even if the limit is a \( C^\ast \)-algebra. Since \( \hat{Z} \) is the profinite completion of \( Z \) \([35]\), the profinite completion of \( K_0(\lim_{\leftarrow} O_n) \) coincides with \( \lim_{\leftarrow} K_0(O_n) \). However the profinite completion of \( K_0(\lim_{\leftarrow} O_{p^n+1}) \cong K_0(O_\infty) \) does not coincide with \( \lim_{\leftarrow} K_0(O_{p^n+1}) \cong \hat{Z}_p \) from Theorem \(1.13(ii)\) and \((1.22)\). Hence the profinite completion does not always recover the continuity of the \( K_0 \)-functor with respect to the inverse limit.

In §2, we will prove Theorem \(1.10\) and Theorem \(1.13\). In §3 we will show examples.

2 Proofs of main theorems

In this section, we prove main theorems in §1.

2.1 Proof of Theorem \(1.10\)

In this subsection, we prove Theorem \(1.10\). By simple calculation, both (i) and (ii) are directly verified from \((1.9)\) and \((1.10)\). In order to prove (iii), we recall a lemma.

\textbf{Lemma 2.1} \quad (i) \textit{Let} \( A \) \textit{and} \( B \) \textit{be unital} \( C^\ast \)-\textit{algebras and let} \( \rho \) \textit{be a unital} \( \ast \)-\textit{homomorphism from} \( A \) \textit{to} \( B \). \textit{If} \( B \) \textit{is simple and there exists an irreducible representation} \( \pi \) \textit{of} \( B \) \textit{such that} \( \pi \circ \rho \) \textit{is also irreducible, then} \( \rho \) \textit{is irreducible.}
Let \( \{s_i^{(n)}\} \) denote Cuntz generators of \( O_n \) for \( 2 \leq n \leq \infty \). Fix \( i \in \{1, \ldots, n\} \). Then there exists a unique state \( \omega \) on \( O_n \) such that \( \omega(s_i^{(n)}) = 1 \). Furthermore, \( \omega \) is pure.

**Proof.** (i) This holds from Proposition 3.1 in [28].

(ii) The uniqueness holds by Cuntz relations. Let \((H_\omega, \pi_\omega, \Omega_\omega)\) denote the GNS triple by \( \omega \). By definition, we see \( \pi_\omega(s_i^{(n)})\Omega_\omega = \Omega_\omega \). From [9, 15, 16], such a cyclic representation \( \pi_\omega \) exists and is irreducible. Hence \( \omega \) is pure.

Let \( \omega_n \) denote the state on \( R_n = O_{n+1} \) such that
\[
\omega_n(s_1^{(n+1)}) = 1. 
\] (2.1)

From Lemma 2.1(ii), the GNS representation \( \pi_n \) by \( \omega_n \) is irreducible. When \( n \preceq m \), we can verify \( \omega_n \circ f_{n,m} = \omega_m \) because \( f_{n,m}(s_1^{(n+1)}) = s_1^{(n+1)} \). Hence \( \pi_n \circ f_{n,m} \) is unitarily equivalent to \( \pi_m \), and \( \pi_n \circ f_{n,m} \) is also irreducible. From this and Lemma 2.1(i), Theorem 1.10(iii) is proved.

### 2.2 Proof of Theorem 1.13

In this subsection, we prove Theorem 1.13 except a certain equality of \( C^\ast \)-subalgebras, which will be proved in §2.3.

In order to reduce the problem, we show a lemma. A subset \( E \) of a directed set \((D, \leq)\) is **cofinal** if \( \{e \in D : e \geq d \} \cap E \neq \emptyset \) for any \( d \in D \). In this case, it is known that \( \varprojlim_E A_d \cong \varprojlim_D A_d \) for the subsystem \( \{(A_d, \varphi_{d,e}) : d, e \in E\} \) of the inverse system \( \{(A_d, \varphi_{d,e}) : d, e \in D\} \) over \((D, \leq)\). Then the following lemma holds.

**Lemma 2.2** For any countable directed set \((D, \leq)\), there exists a totally ordered cofinal subset \( D_0 \) of \( D \).

**Proof.** If the maximal element \( \omega \) of \( D \) exists, then let \( D_0 \equiv \{\omega\} \). If not, let \( D = \{x_1, x_2, \ldots\} \), where we do not assume \( x_1 \leq x_2 \leq \cdots \). Then we can inductively construct a subsequence \( y_1, y_2, \ldots \) of \( x_1, x_2, \ldots \) as follows: Let \( y_1 = x_1 \). For \( n \geq 2 \), there always exists \( z \in D \) such that \( y_{n-1} \leq z \) and \( x_n \leq z \). Choose such an element \( z \) and define \( y_n \equiv z \). Then \( D_0 \equiv \{y_1, y_2, \ldots\} \) is a totally ordered cofinal subset of \( D \).

For example, \( \{n! : n \in \mathbb{N}\} \) is a totally ordered cofinal subset of \((\mathbb{N}, \leq)\).
Proof of Theorem 1.13. (i) Here we will prove the statement except a certain equality. From Fact 1.8(iii), \( \mathcal{O}(\Lambda) \) is a C*-algebra. By the standard construction in (1.6), \( \mathcal{O}(\Lambda) \) is given as follows:

\[
\{(x_{n_1}, x_{n_2}, \ldots) \in \prod_{k \geq 1} R_{n_k} : f_{n_k, n_l}(x_{n_l}) = x_{n_k} \text{ for each } k \leq l\}. \tag{2.2}
\]

Define the set \( \{Q_n : 1 \leq n \leq \infty\} \) of C*-subalgebras of \( R_1 \) by

\[
Q_n \equiv f_{1,n}(R_n) \quad (1 \leq n < \infty), \quad Q_{\infty} \equiv f_{1,\infty}(O_{\infty}). \tag{2.3}
\]

Then we see that \( Q_1 = R_1 = O_2, \) \( Q_n \cong R_n = O_{n+1} \) when \( 1 \leq n < \infty, \) and \( Q_m \subset Q_n \) when \( n \leq m \) from Theorem 1.10(i). On the other hand, from (1.11), \( Q_{\infty} \subset Q_n \) for each \( n \geq 1. \) Since \( n_1 \preceq n_2 \preceq \cdots, \) we obtain the following unital inclusions

\[
O_2 = Q_1 \supset Q_{n_1} \supset Q_{n_2} \supset Q_{n_3} \supset \cdots \supset Q_{\infty}. \tag{2.4}
\]

Define \( \hat{\pi}_1 \equiv f_{1,n_1} \circ \pi_{n_1}. \) Then we see that

\[
\hat{\pi}_1(\mathcal{O}(\Lambda)) = \bigcap_{n \in \Lambda} Q_n. \tag{2.5}
\]

Since \( \hat{\pi}_1 \) is injective, \( \mathcal{O}(\Lambda) \) and \( \bigcap_{n \in \Lambda} Q_n \) are *-isomorphic, and the map \( f_{1,\infty} : O_{\infty} \to \bigcap_{n \in \Lambda} Q_n \) is well-defined. In consequence, we see that the following diagram is commutative:

\[
\begin{array}{ccc}
O_{\infty} & \xrightarrow{\psi_\Lambda} & \mathcal{O}(\Lambda) \\
\downarrow{\mathbb{C}} & \downarrow{\hat{\pi}_1} & \\
\bigcap_{n \in \Lambda} Q_n & \xleftarrow{f_{1,\infty}} & \hat{\pi}_1(\mathcal{O}(\Lambda))
\end{array}
\]

Figure 2.3

In order to prove the bijectivity of \( \psi_\Lambda, \) it is sufficient to show the bijectivity of \( f_{1,\infty}. \) Since \( f_{1,\infty} \) is an injective *-homomorphism, it is sufficient to show that

\[
Q_{\infty} = \bigcap_{n \in \Lambda} Q_n. \tag{2.6}
\]
We will prove (2.6) in § 2.3.

(ii) When there exists the maximal element of Λ, the statement holds from Fact 1.9(ii). Assume that Λ has no maximal element. In this case, it is sufficient to assume the condition in (i) for Λ by Lemma 2.2. Hence the statement holds from (i).

2.3 Proof of (2.6)

From the proof of Theorem 1.13(i), the problem is reduced to the relation (2.6) among C*-subalgebras \( \{Q_n : 1 \leq n \leq \infty\} \) of \( Q_1(= R_1 = O_2) \) in (2.3).

In this subsection, we prove (2.6).

2.3.1 Inclusions of free subsemigroups of \( Q_1 \)

In this subsubsection, we consider free subsemigroups of \( Q_1 \) and their relations. We rewrite the Cuntz generators of \( Q_1 \) as \( t_1, t_2 \) here. Let \( S(X) \) denote the subsemigroup of \( Q_1 \) generated by a subset \( X \) of \( Q_1 \). Define subsemigroups \( K_n, L_n \) of \( Q_1 \) as

\[
\begin{align*}
K_n &\equiv S(\{t_2^n\}) \quad (1 \leq n < \infty), \\
L_1 &\equiv S(\{t_1, t_2\}), \\
L_n &\equiv S(\{t_1, t_2, \ldots, t_2^{n-1}t_1, t_2^n\}) \quad (2 \leq n < \infty), \\
L_\infty &\equiv S(\{t_1, t_2^m t_1 : m \geq 1\}).
\end{align*}
\]

For each \( n \geq 1 \), \( K_n \) is abelian, and \( L_n \) is a (non-unital) free semigroup of rank \( n + 1 \) [22]. Both \( K_n \) and \( L_n \) are subsemigroups of \( Q_n \). If \( m \leq n \), then \( K_n \subset K_m \) and \( L_n \subset L_m \). For each \( n \geq 1 \), \( L_n \supseteq L_\infty \). Furthermore, we see that

\[
K_n = \{t_2^n, t_2^n t_2, t_2^{3n}, \ldots\}, \quad L_\infty = \{xt_1 : x \in L_1\} = L_1 t_1.
\]

Hence \( K_n \cap L_\infty = \emptyset \).

Since \( L_1 = L_1 t_1 \cup L_1 t_2 \) with respect to the ending of each word, the decomposition \( L_n = (L_n \cap L_1 t_1) \cup (L_n \cap L_1 t_2) \) holds. Let \( Y_n \equiv \{u, xu : x \in L_\infty, u \in K_n\} \). Then \( L_n \cap L_1 t_2 = Y_n \) and \( L_n \cap L_1 t_1 = L_\infty \). Hence the following decomposition into disjoint subsets holds:

\[
L_n = L_\infty \sqcup Y_n \quad (2 \leq n < \infty).
\]
2.3.2 Decomposition of algebras into linear subspaces

Let $K_n, L_n$ be as in (2.7). From definitions of $Q_n$ and $\{f_{n,m}, f_{n,\infty} : n, m \in \mathbb{N}\}$, we see that

$$Q_n = C^*(\mathcal{L}_n) \quad (1 \leq n \leq \infty)$$

(2.10)

where $C^*(X)$ denote the $C^*$-subalgebra of $Q_1$ generated by a subset $X$ of $Q_1$. As closed linear subspaces of $Q_1$, $Q_n$'s can be written as follows:

$$Q_n = \text{Lin}\langle \{xy^* : x, y \in \mathcal{L}_n\} \rangle \quad (1 \leq n < \infty),$$

$$Q_\infty = \text{Lin}\langle \{I, xy^* : x, y \in \mathcal{L}_\infty\} \rangle.$$

(2.11)

Remark that vectors in each generating set in (2.11) are not always linearly independent because of Cuntz relations.

Lemma 2.4 (i) For $n \geq 1$, $(t_2^n(t_2^*)^n) = I - \sum_{k=0}^{n-1}(t_2^k t_1^* (t_2^*)^k)$. 

(ii) Define closed linear subspaces $V_n$ and $V_n^*$ of $Q_n$ by

$$V_n \equiv \text{Lin}\langle \{ux, xu, xuy^* : x, y \in \mathcal{L}_\infty, u \in K_n\} \rangle, \quad V_n^* \equiv \{x^* : x \in V_n\}.$$ 

(2.12)

Then $V_n \subset V_m$ and $V_n^* \subset V_m^*$ when $m \preceq n$, $V_n \cap V_\infty = V_n^* \cap Q_\infty = \{0\}$.

(iii) For $u, v \in K_n$ and $x, y \in \mathcal{L}_\infty$, $xuv^* y^* \in Q_\infty \oplus V_n \oplus V_n^*$.

(iv) For each $n \geq 1$, the following decomposition of $Q_n$ into closed linear subspaces holds:

$$Q_n = Q_\infty \oplus V_n \oplus V_n^*.$$ 

(2.13)

(v) For $\Lambda$ in Theorem 1.13(i), $\bigcap_{n \in \Lambda} V_n = \bigcap_{n \in \Lambda} V_n^* = \{0\}$.

Proof. (i) By the Cuntz relations of $Q_1 = O_2$, the statement holds.

(ii) By definition, the statement holds.

(iii) From (ii), $Q_\infty \oplus V_n \oplus V_n^*$ makes sense as a subspace of $Q_n$. If $u = v$, then $xuv^* y^* \in Q_\infty$ from (i) and (2.11). Furthermore, from (i), $t_2^n(t_2^*)^{(l+k)(m+k)} = t_2^n - t_2^n(\sum_{j=0}^{l-1} t_2^j t_1^* t_2^j (t_2^*)^j)$ $\in V_n \oplus Q_\infty \oplus V_\infty$ for $l \geq 1$. Hence the statement also holds for the case $u \neq v$.

(iv) From (ii), $Q_n \supset Q_\infty \oplus V_n \oplus V_n^*$. Since $Q_n$ is the closure of the linear space spanned by the set

$$M_n \equiv \{x, x^*, x,y^* : x, y \in \mathcal{L}_n\} \quad (1 \leq n < \infty),$$

(2.14)
it is sufficient to show that $M_n$ is a subset of $Q_\infty \oplus V_n \oplus V_n^*$. From (2.9), $M_n$ is decomposed into the disjoint union as follows:

$$M_n = \{ x, x^*, xy^* : x, y \in L_\infty \cup Y_n \} = M_{n,1} \cup M_{n,2},$$

(2.15)

$$M_{n,1} \equiv \{ x, x^*, xy^* : x, y \in L_\infty \},$$

$$M_{n,2} \equiv \{ u, u^*, uv^*, xu^*, ux^* : x \in L_\infty, u, v \in Y_n \}.$$ (2.16)

Since $Q_\infty = \text{Lin}(M_{n,1} \cup \{ I \})$, it is sufficient to show that $M_{n,2}$ is a subset of $Q_\infty \oplus V_n \oplus V_n^*$. By the definition of $Y_n$ in (2.9),

$$M_{n,2} = M_{n,2,1} \cup M_{n,2,2},$$

$$M_{n,2,1} \equiv \{ u, u^*, xu, (xu)^* : x \in L_\infty, u \in K_n \},$$

(2.16)

$$M_{n,2,2} \equiv \{ xuy^*, xu^* y^*, xv^* y^* : x, y \in L_\infty, u, v \in K_n \}.$$ (2.17)

Then $M_{n,2,1} \subset V_n \oplus V_n^*$ by definition. From (iii), $M_{n,2,2} \subset Q_\infty \oplus V_n \oplus V_n^*$. Hence $M_{n,2} \subset Q_\infty \oplus V_n \oplus V_n^*$.

(v) By definition,

$$V_n = \text{Lin}(K_n) \oplus \text{Lin}(L_\infty \cdot K_n) \oplus \text{Lin}(L_\infty \cdot K_n \cdot L_\infty^*)$$

(2.18)

where $L_\infty^* \equiv \{ x^* : x \in L_\infty \}$. By assumption, $\Lambda$ is an infinite subset of $N$. Hence $\bigcap_{n \in \Lambda} K_n = \emptyset$. From this, (2.17) and (i),

$$\bigcap_{n \in \Lambda} V_n = \bigcap_{n \in \Lambda} \text{Lin}(K_n) \oplus \bigcap_{n \in \Lambda} \text{Lin}(L_\infty \cdot K_n) \oplus \bigcap_{n \in \Lambda} \text{Lin}(L_\infty \cdot K_n \cdot L_\infty^*) = \{ 0 \}.$$ (2.18)

In a similar way, we obtain $\bigcap_{n \in \Lambda} V_n^* = \{ 0 \}$. Hence the statement holds.

From Lemma (2.3) (iv) and (v),

$$\bigcap_{n \in \Lambda} Q_n = Q_\infty \oplus (\bigcap_{n \in \Lambda} V_n) \oplus (\bigcap_{n \in \Lambda} V_n^*) = Q_\infty.$$ (2.19)

Hence (2.6) is proved.

3 Examples

In order to explain theorems in §1 more, we show examples in this section.
3.1 Other inverse system

In this subsection, we show other example of inverse system of Cuntz algebras. Let $s_1^{(n)}, \ldots, s_n^{(n)}$ denote the Cuntz generators of $O_n$. Fix an integer $r \geq 2$. Let $r_n \equiv r^{2n-1}$ and rewrite $O_n$ as

$$A_{r,n} \equiv O_n \quad (n \geq 1).$$

Then $r_n^2 = r_{n+1}$ for $n \geq 1$. Define the $\ast$-homomorphism $q_n$ from $A_{r,n+1}$ to $A_{r,n}$ by

$$q_n(s^{(r_{n+1})}_n(i-1+j)) \equiv s^{(r_n)}_i s^{(r_n)}_j \quad (i, j = 1, \ldots, r_n).$$

Then $\{(A_{r,n}, q_n) : n \geq 1\}$ is an inverse system of Cuntz algebras over the directed set $(\mathbb{N}, \leq)$.

Remark $A_{r,n} = O_{(r_n-1)+1}$ for $n \geq 1$. Here we verify Fact 1.9(i) for the sequence $\{r_n - 1 : n \geq 1\}$. Define the map $F$ from $(\mathbb{N}, \leq)$ to $(\mathbb{N}, \preceq)$ by

$$F(n) \equiv r_n - 1 \quad (n \in \mathbb{N}).$$

Then $F(n) = r_n - 1 \preceq (r^{2n-1} - 1)(r^{2n-1} + 1) = r^{2n} - 1 = F(n + 1)$. Exactly, the map $F$ satisfies the statement in Fact 1.9(i).

Proposition 3.1 The inverse limit $\lim_{\leftarrow n} A_{r,n}$ of $\{(A_{r,n}, q_n) : n \in \mathbb{N}\}$ is $\ast$-isomorphic onto the uniformly hyperfinite algebra $UHF_r$ of the Glimm type $\{r^l : l \geq 1\}$:

$$\lim_{\leftarrow n} A_{r,n} \cong UHF_r.$$  

Proof. Let $\gamma$ denote the $U(1)$-gauge action on $O_r = O_1 = A_{r,1}$. For $l \in \mathbb{Z}$, define

$$A_{r,1}^{(l)} \equiv \{x \in A_{r,1} : \gamma_z(x) = z^l x \text{ for all } z \in U(1)\}. \quad (3.5)$$

Rewrite the Cuntz generators of $O_r = O_1$ as $t_1, \ldots, t_r$. By identifying $A_{r,n}$ with the $C^*$-subalgebra $(q_1 \circ q_2 \circ \cdots \circ q_{n-1})(A_{r,n})$ of $A_{r,1}$, $A_{r,n}$'s are rewritten as follows:

$$A_{r,n} = C^*(\{t_J : J \in \{1, \ldots, r\}^{2n-1}\}) \quad (n \geq 1) \quad (3.6)$$

where $t_J \equiv t_{j_1} \cdots t_{j_m}$ for $J = (j_1, \ldots, j_m)$. Then we obtain the following unital (rapidly decreasing) inclusions:

$$A_{r,1} \supset A_{r,2} \supset A_{r,3} \supset \cdots.$$  

Define

$$A_{r,n}^{(l)} = A_{r,n} \cap A_{r,1}^{(l)} \quad (l \in \mathbb{Z}, n \geq 1). \quad (3.7)$$

($l \in \mathbb{Z}, n \geq 1). \quad (3.8)$
Then $A_{r,n+1}^{(l)} \subset A_{r,n}^{(l)}$ for each $n,l$, and $A_{r,n}^{(l)}$ is the closure of $\text{Lin}(\{t_j t_K^r : J,K \in \bigcup_{a \geq 1} \{1, \ldots, r\}^{a \times 2^{n-1}}, |J| - |K| = l\})$. When $l \neq 0$, $\bigcap_{n \geq 1} A_{r,n}^{(l)} = \{0\}$ because $A_{r,n}^{(l)} = \{0\}$ if $2^{n-1} \nmid l$. Since $A_{r,n} = \bigoplus_{l \in \mathbb{Z}} A_{r,n}^{(l)}$ and $A_{r,n+1} = A_{r,n}$ for each $n$, $\lim_{\leftarrow n} A_{r,n} \cong \bigcap_{n \geq 1} A_{r,n} = A_{r,1}^{(0)} = UHF_r$. 

Since $K_0(UHF_r)$ is the group $\mathbb{Z}(\nu \infty) \subset \mathbb{Q}$ of all rational numbers whose denominators divide the generalized integer $r^{\infty}$ (§ 7.5, [5]), we see that

$$K_0(\lim_{\leftarrow n} A_{r,n}) \cong K_0(UHF_r) \cong \mathbb{Z}(\nu \infty),$$

$$\lim_{\leftarrow n} K_0(A_{r,n}) = \lim_{\leftarrow n} K_0(O_{r,n}) \cong \lim_{\leftarrow n} \mathbb{Z}/(r_n - 1)\mathbb{Z}. \tag{3.9}$$

The former is countable, but the latter is not. This case also shows that the $K_0$-functor is discontinuous with respect to the inverse limit.

### 3.2 Illustrations of embeddings

In this subsection, we illustrate embeddings in (1.9) by using decompositions of Hilbert spaces. When $O_n$ acts on a Hilbert space, by identifying a generator $s_i$ with the range of $s_i$, the following illustration is helpful in understanding $s_1, \ldots, s_n$:

**Figure 3.2**

![Diagram](image)

Recall that $R_1 = O_2$, $R_2 = O_3$, $R_4 = O_5$. Then $f_{1,2}, f_{1,4}, f_{2,4}$ in (1.9) are given as follows:

$f_{1,2} : R_2 \rightarrow R_1$:

$$f_{1,2}(s_1^{(3)}) = s_1^{(2)}, \quad f_{1,2}(s_2^{(3)}) = s_2^{(2)} s_1^{(2)}, \quad f_{1,2}(s_3^{(3)}) = (s_2^{(2)})^2. \tag{3.10}$$

$f_{1,4} : R_4 \rightarrow R_1$:

$$f_{1,4}(s_1^{(5)}) = s_1^{(2)}, \quad f_{1,4}(s_2^{(5)}) = s_2^{(2)} s_1^{(2)}, \quad f_{1,4}(s_3^{(5)}) = (s_2^{(2)})^2 s_1^{(2)}, \tag{3.11}$$
\[ f_{1,4}(s_4^{(5)}) = (s_2^{(2)})^3 s_1^{(2)}, \quad f_{1,4}(s_4^{(5)}) = (s_2^{(2)})^4. \quad (3.12) \]

\[ f_{2,4} : R_4 \to R_2; \]

\[ f_{2,4}(s_1^{(5)}) = s_1^{(3)}, \quad f_{2,4}(s_2^{(5)}) = s_2^{(3)}, \quad f_{2,4}(s_3^{(5)}) = s_3^{(3)} s_1^{(3)}, \quad (3.13) \]

\[ f_{2,4}(s_4^{(5)}) = s_3^{(3)} s_2^{(3)}, \quad f_{2,4}(s_5^{(5)}) = (s_3^{(3)})^2. \quad (3.14) \]

From these, we can directly verify the identity \( f_{1,2} \circ f_{2,4} = f_{1,4} \).

From Figure 3.2, inclusions \( \mathcal{O}_2 \supset \mathcal{O}_3 \supset \mathcal{O}_5 \) by embeddings \( f_{1,2} \) and \( f_{2,4} \) are illustrated as follows:

**Figure 3.3**

\[
\begin{array}{c}
\mathcal{O}_2 \\
\mathcal{O}_3 \\
\mathcal{O}_5
\end{array}
\]

By using a more rough analogy, Figure 3.3 shows that an embedding is represented as a refinement of a partition of a unit interval in the real line \( \mathbb{R} \). Then relations of inverse system in Theorem 1.10(i) mean that the set of such refinements is a directed set.

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