DISCONTINUITY OF THE LYAPUNOV EXPONENT

ZHENG GAN AND HELGE KRÜGER

Abstract. We study discontinuity of the Lyapunov exponent. We construct a limit-periodic Schrödinger operator, of which the Lyapunov exponent has a positive measure set of discontinuities. We also show that the limit-periodic potentials, whose Lyapunov exponent is discontinuous, are dense in the space of limit-periodic potentials.

1. Introduction

In this paper, we construct examples of ergodic Schrödinger operators \( H_\omega \), whose Lyapunov exponent \( L(E) \) is discontinuous. In addition to being a result of interest of its own, we were motivated by the following observation. Denote by

\[
Z = \{ E : L(E) = 0 \}
\]

the set where the Lyapunov exponent vanishes. It was shown by Deift and Simon in [5] that the measure of this set satisfies

\[
|Z| \leq 4.
\]

Introduce the essential closure of \( Z \) by

\[
Z^{\text{ess}} = \{ E : \forall \varepsilon > 0 : |Z \cap (E - \varepsilon, E + \varepsilon)| > 0 \}.
\]

Denote by \( \sigma_{\text{ac}}(H_\omega) \) the absolutely continuous spectrum of \( H_\omega : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \). By Kotani theory, we have that for almost every \( \omega \)

\[
\sigma_{\text{ac}}(H_\omega) = Z^{\text{ess}}.
\]

This result can for example be found in [3], [10], or Theorem 5.17 in [11]. Being naive, one might assume that from these two statements that

\[
|\sigma_{\text{ac}}(H_\omega)| \leq 4
\]

holds for almost every \( \omega \). However, we do not know if this is true and trying to show this was a big motivation to write this paper. To see that (1.5) cannot be a simple consequence of (1.2) and (1.4) let \( Z \) be the complement of a Cantor set of measure > 4 in \([-4, 4]\). Then (1.2) holds, but \( Z^{\text{ess}} = [-4, 4] \) is a set of measure 8.

Let us now discuss our main result and its relevance to this conjecture. We begin by introducing some notation. Let \( (\Omega, \mu) \) be a probability space, \( T : \Omega \to \Omega \) an invertible ergodic transformation, and \( f : \Omega \to \mathbb{R} \) a bounded measurable function. For \( \omega \in \Omega \) we introduce the potentials \( V_\omega(n) = f(T^n \omega) \). \( H_\omega = \Delta + V_\omega \) denotes the
Schrödinger operator with potential $V_\omega$. Here $\Delta u(n) = u(n+1) + u(n-1)$ denotes the discrete Laplacian. We introduce the transfer matrices
\begin{equation}
A_N(E, V_\omega) = \prod_{n=1}^{N} \begin{pmatrix} E - V_\omega(N-n) & -1 \\ 1 & 0 \end{pmatrix}.
\end{equation}

The Lyapunov exponent is then defined by
\begin{equation}
L(E) = \lim_{N \to \infty} \frac{1}{N} \int_{\Omega} \log \| A_N(E, V_\omega) \| d\mu(\omega).
\end{equation}

Our main result is

**Theorem 1.1.** There exists $(\Omega, T, f)$ such that $L(E)$ vanishes for generic $E \in [-4, 4]$. Furthermore for every $\omega \in \Omega$, we have that $[-4, 4] \subseteq \sigma(H_\omega)$.

Our proof does not give any control on the measure of the set $Z$, where the Lyapunov exponent vanishes. However, it is sufficient to show that

**Corollary 1.2.** We have that $L(E)$ is discontinuous on a set of $E$ of positive Lebesgue measure.

**Proof.** Denote by $A$ the set of $E_0 \in [-4, 4]$ such that $L(E)$ is continuous at $E_0$. By Theorem 1.1, we have that $A \subseteq Z$. Hence by (1.2), we have that $|A| \leq 4$. In particular, the Lyapunov exponent is discontinuous on $B = [-4, 4] \setminus A$ with measure $|B| \geq 4$. \hfill \Box

Our proof of Theorem 1.1 is done by constructing a limit-periodic potential explicitly with these properties. We will discuss the details in the next section, where we also show that for a dense set of limit-periodic potentials the Lyapunov exponent has at least one discontinuity. This is achieved by adapting the argument of Johnson \[7\] to the discrete setting.

We furthermore wish to point out that Thouvenot has constructed in \[12\] discontinuity points of the Lyapunov exponent for matrix-valued cocycles.

Let us now discuss the relevance to whether (1.5) holds. One can show that if $A$ is a closed set, then $|A^{\text{ess}}| = |A|$. Now, the existence of large sets of discontinuity of the Lyapunov exponent stops us from concluding that $Z$ is closed and thus (1.5).

Artur Avila has informed us of an alternative proof of Theorem 1.1, which is based on the work of Bochi and Viana \[2\]. The main idea is that if $T$ has plenty of periodic points, then one can ensure that $|\sigma(\Delta + f(T^n\omega))| > 4$ for almost every $\omega$. Then one uses to show the results of \[2\] to conclude that by perturbing $f$ slightly the Lyapunov exponent $L(E)$ vanishes on a dense set in $\sigma(\Delta + f(T^n\omega))$, and still $|\sigma(\Delta + f(T^n\omega))| > 4$. Then discontinuity of the Lyapunov exponent follows as in the proof of Corollary 1.2.

2. LIMIT-PERIODIC POTENTIAL AND FURTHER RESULTS

We will begin this section by discussing some basics about limit-periodic potentials. For further informations, we refer to the appendix of the paper \[4\] of Avron and Simon, and to the survey by Gan \[6\].
Given a bounded sequence $V : \mathbb{Z} \to \mathbb{R}$, we denote by $\Omega_V$ the hull of its translates. That is
\[ \Omega_V = \{ V_m, \ m \in \mathbb{Z} \}^{\ell^\infty(\mathbb{Z})}, \]
where $V_m(n) = V(n - m)$. If $\Omega_V$ is compact in the $\ell^\infty(\mathbb{Z})$ topology, then $V$ is called almost-periodic. The shift map on $\ell^\infty(\mathbb{Z})$ becomes a translation on the group $\Omega_V$ and it is uniquely ergodic with respect to the Haar measure of $\Omega_V$.

$V$ is called limit-periodic, if there exists a sequence of periodic potentials $V^k$ such that
\[ V = \lim_{k \to \infty} V^k \]
in the $\ell^\infty(\mathbb{Z})$ topology. It should be remarked that limit-periodic $V$ are almost-periodic. In fact, then $\Omega_V$ has the extra structure of being a Cantor group, see [6] for details.

The proof of Theorem 1.1 will proceed by explicitly constructing such sequences of $V^k$. We will furthermore denote by $L(E, V^k)$ the Lyapunov exponent of such a sequence of periodic potentials.

Let us recall some properties of a $p_k$ periodic potentials $V^k$.

(i) The spectrum $\sigma(\Delta + V^k)$ is a finite union of intervals. That is
\[ \sigma(\Delta + V^k) = \bigcup_{j=1}^g [\alpha_j, \beta_j], \]
where $g \geq 2$ and $\alpha_j < \beta_j$.

(ii) The Lyapunov exponent $L(E, V^k)$ is continuous and vanishes on $L(E, V^k)$. It should furthermore be pointed out that, one always has $|\sigma(\Delta + V^k)| \leq 4$.

Let us now comment further on Theorem 1.1.

Remark 2.1. The potential constructed in Theorem 1.1 is limit-periodic. In particular, we provide an example of a limit-periodic potential whose spectrum contains an interval.

This is not the first known example with this property, since Pöschel provided some in [9]. However, our construction has the advantage of being relatively elementary in comparison to Pöschel’s KAM-type proof.

In order to state our other result, we denote by $\mathcal{L}$ the space of all limit-periodic potentials. We have

Theorem 2.2. There is a dense set of $V$ in $\mathcal{L}$ for which the Lyapunov exponent $L(E)$ is discontinuous.

Here the dense set cannot be improved to a generic set, since Damanik and Gan in [4] show that there is a generic set of $V$ in $\mathcal{L}$ such that the Lyapunov exponent is continuous.

Furthermore, we should point out that the proof of the previous theorem largely parallels the construction of Johnson in [7] of similar examples in the continuum setting. However, the observation of denseness is new.

Define the individual Lyapunov exponent by
\[ \overline{L}(E, \omega) = \limsup_{N \to \infty} \frac{1}{N} \log \| A_N(E, V_\omega) \|. \]
An important aspect of our construction will be to replace the Lyapunov exponent $L(E)$ defined in (1.7) by $\overline{L}(E,\omega)$. This is possible by the following result.

**Proposition 2.3.** Assume that $T : \Omega \to \Omega$ is uniquely ergodic. For each $\omega \in \Omega$, there exists a set $\mathcal{E}_\omega$ of zero Lebesgue measure such that for $E \in \mathbb{R} \setminus \mathcal{E}_\omega$

(2.5) \[
L(E) = \overline{L}(E,\omega).
\]

**Proof.** This is a consequence of Theorem 2.1. in [8]. \qed

Thus, we obtain that for fixed $\omega$

(2.6) \[
L(E) = \lim_{N \to \infty} \frac{1}{N} \log \|A_N(E,V_\omega)\|
\]

for almost every $E$. We will furthermore need Thouless’ formula

(2.7) \[
L(E) = \int \log |t - E| dN(t),
\]

where $N(t)$ is the integrated density of states. See for example Theorem 5.15 in Teschl’s book [11].

3. Proof of Theorem [11]

In order to simplify the notation, we introduce

**Definition 3.1.** A collection of open intervals $\Sigma$ is called $\varepsilon$-dense in $[-4,4]$ if for $E \in [-4,4]$, there exists $I \in \Sigma$ such that $I \subseteq [E - \varepsilon, E + \varepsilon]$.

We will construct a sequence of periodic potentials $V^k$ with the following properties:

(i) $V^k$ is $p_k$ periodic and $\|V^k - V^{k-1}\|_{\infty} \leq \frac{1}{2^{k-1}}$.

(ii) $V^k(n) = V^{k-1}(n)$ for $0 \leq n \leq p_{k-1}$.

(iii) For $1 \leq l \leq k$ and $E \in \sigma(\Delta + V^l)$

(3.1) \[
\frac{1}{p_k} \log \|A_{p_k}(E,V^k)\| \leq \sum_{s=l+1}^{k+1} \frac{1}{2^s}.
\]

(iv) There is a set $\Sigma_k$ consisting of open intervals $I \subseteq \sigma(\Delta + V^k)$ which is $2^{-k}$-dense in $[-4,4]$.

(v) For each $I_{k-1} \in \Sigma_{k-1}$ there exists $I_k \in \Sigma_k$ such that $I_k \subseteq I_{k-1}$.

We begin by showing that the existence of such $V^k$ implies Theorem [11]. From (i), one obtains that the limit

(3.2) \[
V(n) = \lim_{k \to \infty} V^k(n)
\]

exists uniformly and is bounded.

**Lemma 3.2.** Let $l \geq 1$, then for $E \in \sigma(\Delta + V^l)$

(3.3) \[
\liminf_{N \to \infty} \frac{1}{N} \log \|A_N(E,V)\| \leq \frac{1}{2^l}.
\]
Proof. By (ii), we have
\[ \frac{1}{p_k} \log \| A_{p_k} (E, V) \| = \frac{1}{p_k} \log \| A_{p_k} (E, V^k) \|. \]
By (iii), we obtain that for \( E \in \sigma(\Delta + V^l) \) and \( k \geq l \)
\[ \frac{1}{p_k} \log \| A_{p_k} (E, V) \| \leq \sum_{s=l+1}^{k+1} \frac{1}{2^s} \leq \frac{1}{2^l}, \]
which implies the claim. \( \square \)

Combining this lemma with (2.6), we obtain that for almost every \( E \in \sigma(\Delta + V^l) \)
\[ (3.4) \quad L(E) \leq \frac{1}{2^l}. \]
We also have

**Proposition 3.3.** For \( E_0 \in [-4, 4] \) we have
\[ (3.5) \quad \liminf_{E \to E_0} L(E) = 0. \]

**Proof.** Pick any \( E_0 \in [-4, 4] \) and \( k \geq 1 \). By (iv), we can choose \( I_k \in \Sigma_k \) such that \( I_k \subseteq [E_0 - \frac{1}{2^k}, E_0 + \frac{1}{2^k}] \). By (v), we can choose a sequence of intervals
\[ I_k \supseteq I_{k+1} \supseteq \ldots \]
Let \( \hat{E} \in \bigcap_{l \geq k} \bar{I}_l \). Since \( \hat{E} \in \bar{I}_k \), we have \( |E_0 - \hat{E}| < \frac{1}{2^k} \). By (3.4), we can choose \( E_l \in I_l \) with \( |E_l - \hat{E}| \leq \frac{1}{2^l} \) and \( L(E_l) \leq \frac{1}{2^l} \). Hence
\[ 0 \leq \liminf_{E \to \hat{E}} L(E) \leq \lim_{l \to \infty} L(E_l) = 0, \]
showing (3.5). Since \( k \geq 1 \) was arbitrary, the claim follows. \( \square \)

We now come to

**Proof of Theorem 1.1.** Let
\[ L_N(E) = \frac{1}{2^N} \int_{\Omega} \log \| A_{2^N}(E, V_\omega) \| d\mu(\omega). \]
\( L_N(E) \) is continuous in \( E \) and decreasing in \( N \). In particular, this implies
\[ L(E) = \inf_{N \geq 1} L_N(E). \]
Introduce
\[ U_l = \bigcup_{N \geq 1} \left\{ E : L_N(E) < \frac{1}{2^l} \right\}. \]
Since \( L_N(E) \) is continuous, \( U_l \) is open. Let us now show that \( U_l \) is also dense. Let \( E_0 \in [-4, 4] \) and \( \varepsilon > 0 \). By the previous proposition, there exists \( E \in [E_0 - \varepsilon, E_0 + \varepsilon] \) such that \( L(E) < \frac{1}{2^l} \). Furthermore, there must be an \( N_0 \geq 1 \) such that \( |L(E) - L_{N_0}(E)| < \frac{1}{2^l} \) and thus that \( L(E) < \frac{1}{2} \). This implies \( E \in U_l \) and thus that \( U_l \) is dense.
Introduce
\[ U = \bigcap_{l \geq 1} U_l. \]
Since each of the \( U_l \) is open and dense, we have by the Baire category theorem that also \( U \) is dense. The claim follows. \( \square \)
In the following subsections, we will explain how to construct \( V^k \) given \( V^{k-1} \). In the next subsection, we will construct a sequence of potentials \( \hat{V}_m \) such that properties (i), (ii), (iv), and (v) hold. Then, we will show in Subsection 3.2 that (iii) holds if \( m \) is chosen large enough.

3.1. **A sequence of potentials such that (i), (ii), (iv) and (v) hold.** We will need the following lemma, describing the generalized eigenfunctions of a periodic operator. A proof can be given using that \( A_p(E,V) \) has an eigenvalue of modulus 1 for \( E \in \sigma(\Delta + V) \) and we omit it for brevity.

**Lemma 3.4.** Let \( V \) be a periodic potential and \( E \in \sigma(\Delta + V) \). There is a bounded solution \( \psi \) of \( (\Delta + V) \psi = E \psi \) such that for every \( m \in \mathbb{Z} \)

\[
\sum_{k=1}^{p} |\psi(m + k)|^2 = 1.
\]

Let \( I^1, \ldots, I^L \) be an enumeration of the intervals in \( \Sigma_{k-1} \). For each \( I^l \), we choose a subinterval \( \tilde{I}^l \) of length \( < \frac{1}{2k+1} \). Denote by \( \tilde{\Sigma}_{k-1} \) the collection of intervals \( \tilde{I}^l \). Introduce \( \delta \) by

\[
\delta = \min_{I \in \tilde{\Sigma}} |I|.
\]

Clearly, \( 0 < \delta < \frac{1}{2k+1} \).

Choose \( m_0 \) so large that \( \delta \sqrt{m_0} > 4 \) and let \( \hat{p}_{k-1} = m_0 \hat{p}_{k-1} \). We will treat \( V^{k-1} \) as a \( \hat{p}_{k-1} \)-periodic potential. Write

\[
\Lambda_{j,m} = [m \hat{p}_{k-1} + (j + 4) \hat{p}_{k-1}, m \hat{p}_{k-1} + (j + 5) \hat{p}_{k-1} - 1].
\]

Introduce the potentials \( \hat{V}_m \) by

\[
\hat{V}_m(n) = \begin{cases} 
V^{k-1}(n), & 0 \leq n \leq m \hat{p}_{k-1} - 1; \\
V^{k-1}(n) + \frac{1}{2k+1}, & n \in \Lambda_{j,m} - 4 \leq j \leq 3.
\end{cases}
\]

Note that \( \hat{V}_m \) will be \( \hat{p}_m = m \hat{p}_{k-1} + 8 \hat{p}_{k-1} \) periodic. We will later let \( V^k = \hat{V}_m \) for some large \( m \).

We see that the claimed properties (i) and (ii) are straightforward. It remains to prove (iv) and (v).

**Lemma 3.5.** For each \( I \in \tilde{\Sigma}_{k-1} \) and \(-4 \leq j \leq 3\), there exists an open interval \( J \) such that

\[
J \subseteq \tilde{I} + \frac{j}{2k+1} \quad \text{and} \quad J \subseteq \sigma(\Delta + \hat{V}_m).
\]

**Proof.** Let \( I = (E_-, E_+) \) and \( \hat{E} = \frac{E_+ - E_-}{2} \). By Lemma 3.1 there exists a function \( \psi \) such that

\[
(\Delta + V^{k-1}) \psi = \hat{E} \psi, \quad \text{and} \quad \sum_{n=1}^{p_{k-1}} |\psi(n)|^2 = 1.
\]
Let \( \varphi \) be the restriction of \( \psi \) to \( \Lambda_{j,m} \). A computation shows \( \| \varphi \| = \sqrt{m_0} \) and
\[
(\Delta + \hat{V}_m - \hat{E} - \frac{j}{2^{k+1}}) \varphi(n) = \begin{cases}
\psi(mp_{k-1} + (j + 4)p_{k-1}), & n = mp_{k-1} + (j + 4)p_{k-1} - 1; \\
-\psi(mp_{k-1} + (j + 4)p_{k-1} - 1), & n = mp_{k-1} + (j + 4)p_{k-1}; \\
-\psi(mp_{k-1} + (j + 5)p_{k-1}), & n = mp_{k-1} + (j + 5)p_{k-1} - 1; \\
\psi(mp_{k-1} + (j + 5)p_{k-1} - 1), & n = mp_{k-1} + (j + 5)p_{k-1}; \\
0, & \text{otherwise.}
\end{cases}
\]
So we have
\[
\frac{\| (\Delta + \hat{V}_m - \hat{E} - \frac{j}{2^{k+1}}) \varphi \|}{\| \varphi \|} \leq \frac{2}{\sqrt{m_0}} < \frac{\delta}{2}.
\]
The above inequality implies that
\[
\text{dist}(\hat{E} + \frac{j}{2^{k+1}}, \sigma(\Delta + \hat{V}_m)) < \frac{\delta}{2}.
\]
Hence, we see that \( \sigma(\Delta + \hat{V}_m) \cap (I + \frac{j}{2^{k+1}}) \) is non empty since \( |I + \frac{j}{2^{k+1}}| \geq \delta \). Since \( \sigma(\Delta + \hat{V}_m) \) consists of bands, we may choose an open interval \( J \) such that (3.10) holds.

We denote by \( \Sigma_k \) the collection of open intervals \( J \) obtained from the previous lemma for all possible choices of \( I \) and \( j \). It is clear that property (v) holds.

**Proof of Property (iv).** Given any \( E \in [-4,4] \), we may find an \( I \in \tilde{\Sigma}_{k-1} \) such that
\[
I \subseteq [E - \frac{1}{2^{k-1}}, E + \frac{1}{2^{k-1}}].
\]
Let \( I = (E_-, E_+) \) and \( \hat{E} = \frac{E_+ + E_-}{2} \). Since \( \hat{E} \in I \) and
\[
[E - \frac{1}{2^{k-1}}, E + \frac{1}{2^{k-1}}] = \bigcup_{j=-4}^{3} \left[ E + \frac{j}{2^{k+1}}, E + \frac{j+1}{2^{k+1}} \right],
\]
we can find \(-4 \leq j \leq 3 \) such that
\[
\hat{E} \in \left[ E + \frac{j}{2^{k+1}}, E + \frac{j+1}{2^{k+1}} \right]
\]
By the construction of \( \Sigma_k \) in Lemma 3.5, there exists \( J \in \Sigma_k \) such that \( J \subseteq I + \frac{j}{2^{k+1}} \). By \( |I| \leq \frac{1}{2^{k-1}} \), we obtain
\[
J \subseteq I + \frac{j}{2^{k+1}} \subseteq \left[ \hat{E} + \frac{j-1}{2^{k+1}}, \hat{E} + \frac{j+1}{2^{k+1}} \right] \subseteq \left[ E - \frac{1}{2^k}, E + \frac{1}{2^k} \right],
\]
which finishes the proof.

**3.2. Ensuring (iii).**

**Lemma 3.6.** Let \( V \) be a \( p \) periodic potential. For any \( \mu > 0 \), there exists \( M \geq 1 \) such that for \( E \in \sigma(\Delta + V) \) and \( N \geq M \), we have
\[
\frac{1}{N} \log \| A_N(E,V) \| \leq \mu.
\]
Proof. Introduce the continuous functions

\[ f_i(E) = \frac{1}{2i} \log \| A_{2^i}(E,V) \|. \]

One can easily check that \( f_{i+1}(E) \leq f_i(E) \) and that for \( E \in \sigma(\Delta + V) \), we have \( \lim_{i \to \infty} f_i(E) = L(E,V) = 0 \). Hence, we obtain by Dini’s theorem that there exists \( M_1 \geq 1 \) such that for \( i \geq M_1 \) and \( E \in \sigma(\Delta + V) \)

\[ \frac{1}{2i} \log \| A_{2^i}(E,V) \| \leq \frac{\mu}{2}. \]

Let \( N = q2^{M_1} + r \), where \( q \geq 1 \) and \( 0 \leq r \leq 2^{M_1} - 1 \). For \( E \in \sigma(\Delta + V) \), we get

\[
\frac{1}{N} \log \| A_N(E,V) \| \leq \frac{1}{q2^{M_1} + r} (\log \| A_{q2^{M_1}}(E,V) \| + \log \| A_r(E,V) \|)
\leq \frac{\mu}{2} + \frac{\log \| A_r(E,V) \|}{q2^{M_1} + r}.
\]

Choose \( M_2 \geq 1 \) so large that for \( E \in \sigma(\Delta + V) \) and \( 0 \leq r \leq 2^{M_1} - 1 \)

\[
\frac{\log \| A_r(E,V) \|}{M_22^{M_1} + r} \leq \frac{\mu}{2}.
\]

The claim follows by taking \( q \geq M_2 \) or equivalently \( N \geq M = M_2 \cdot 2^{M_1} \). \( \square \)

By this lemma, we can ensure (iii) for \( l = k \) as long as \( m \) large enough. Let \( 1 \leq l \leq k - 1 \). By assumption,

\[
\frac{1}{p_{k-1}} \log \| A_{p_{k-1}}(E,V^{k-1}) \| \leq \sum_{s=l+1}^{k} \frac{1}{2^s}.
\]

By submultiplicativity, we get

\[
\frac{1}{p_{k-1}} \log \| A_{\hat{p}_{k-1}}(E,V^{k-1}) \| \leq \sum_{s=l+1}^{k} \frac{1}{2^s}.
\]

We recall that \( \hat{V}_m \) is a \( \hat{p}_m = (m + 8)\hat{p}_{k-1} \) periodic potential, such that \( \hat{V}_m(n) = V^{k-1}(n) \) for \( 0 \leq m\hat{p}_{k-1} - 1 \). Thus, we have

\[
A_{\hat{p}_m}(E,\hat{V}_m) = (A_{p_{k-1}}(E,V^{k-1}))^m \cdot \hat{A},
\]

where \( \hat{A} \) is some fixed matrix, whose norm is independent of \( m \). Hence, we obtain

\[
\frac{1}{p_m} \log \| A_{\hat{p}_m}(E,\hat{V}_m) \| \leq \sum_{s=l+1}^{k} \frac{1}{2^s} + \frac{1}{p_m} \log \| \hat{A} \|.
\]

The claim now follows by choosing \( m \) large enough.

3.3. Construction of the initial potential. Last, we construct an initial potential \( V^0 \). Define for \( L \geq 1 \), the potential \( \hat{V}_L \) by

\[
\hat{V}_L(n) = \begin{cases} 2, & 0 \leq n \leq L - 1; \\ -2, & L \leq n \leq 2L - 1. \end{cases}
\]
and continue it to be 2L periodic. For \( k \in [0, \pi] \) the function \( \varphi(n) = e^{ikn} \) solves \( \Delta \varphi = 2 \cos(k) \varphi \). Set \( E = 2 \cos(k) \) and

\[
\psi(n) = \begin{cases} \varphi(n), & 0 \leq n \leq L - 1; \\ 0, & \text{otherwise}. \end{cases}
\]

Clearly, \( \|\psi\| = \sqrt{L} \), and

\[
\|(\Delta + V^0 - E - 2)\psi\| \leq 2.
\]

This implies that \( \text{dist}(E + 2, \sigma(\Delta + V^0)) \leq \frac{2}{\sqrt{L}} \). When \( L > 4 \), we have

\[
\text{dist}(E + 2, \sigma(\Delta + V^0)) < 1.
\]

So for any \( E_0 \in [0, 4] \) we can find an interval \( I \subset \sigma(\Delta + V^0) \) such that \( I \subseteq [E_0 - 1, E_0 + 1] \).

Similarly, we conclude that for any \( E_0 \in [-4, 0] \) there is an interval \( I \subset \sigma(\Delta + V^0) \) such that \( I \subseteq [E_0 - 1, E_0 + 1] \). Thus, for \( V^0 \) we can find a collection \( \Sigma_0 \) of intervals \( I \subset \sigma(\Delta + V^0) \) which is 1-dense in \( [-4, 4] \). (iv) follows.

By Lemma 4.6 (iii) follows since we can treat \( V_0 \) as a \( p_0 \) periodic where \( p_0 = 2Lm \), and \( m \in \mathbb{Z}^+ \) is large. There is nothing to check for (i), (ii), and (v).

4. Proof of Theorem 2.2

The first step in the proof of Theorem 2.2 is to prove the following proposition, which we postpone to a later subsection.

**Proposition 4.1.** Let \( V_0 \) be a \( p_0 \) periodic potential such that

\[
E_0 = \inf(\sigma(\Delta + V_0)) > 0.
\]

Introduce \( \gamma = \frac{1}{2}L(0, V^0) \). There exists a limit-periodic potential \( V \) such that \( \|V - V_0\| \leq 5E_0, \inf(\sigma(\Delta + V)) = 0, \) and

\[
\limsup_{E \to 0} L(E) \geq \gamma > \liminf_{E \to 0} L(E) = 0.
\]

In particular, the Lyapunov exponent \( L(E, V) \) is discontinuous at 0.

We are now ready for

**Proof of Theorem 2.2** First note that the periodic potentials are dense in the space \( \mathcal{L} \) of all limit periodic potentials.

Let \( V_0 \) be any periodic potential and \( \varepsilon > 0 \). Introduce the potential

\[
\tilde{V}_0 = V_0 - \inf(\Delta + V_0) + \frac{\varepsilon}{5}.
\]

One can check that \( \tilde{V}_0 \) satisfies the assumptions of the previous proposition, and we thus obtain a potential \( \tilde{V} \) such that the Lyapunov exponent of \( \tilde{V} \) is discontinuous and \( \|\tilde{V}_0 - \tilde{V}\| \leq \varepsilon \).

We now see that \( V = \tilde{V} + \inf(\Delta + V_0) - \frac{\varepsilon}{5} \) satisfies that its Lyapunov exponent is discontinuous and \( \|V - V_0\| \leq \varepsilon \). Hence, the potentials with discontinuous Lyapunov exponent are dense. \( \square \)
4.1. **Proof of Proposition 4.1.** In the following subsection, we will construct a sequence of potentials $V^k$ with the following properties.

(i) $V^k$ is $p_k$ periodic and satisfies

\[
\|V^k - V^{k-1}\| \leq \frac{2}{5} E_{k-1},
\]

and $V^k(n) = V^{k-1}(n)$ for $0 \leq n \leq p_{k-1}$.

(ii) The bottom of the spectrum $E_k = \inf \sigma(\Delta + V^k)$ satisfies

\[
\left(\frac{3}{5}\right)^k E_0 \leq E_k \leq \left(\frac{4}{5}\right)^k E_0.
\]

(iii) The Lyapunov exponent at energy 0 satisfies

\[
L(0, V^k) \geq \left(2 - \sum_{s=1}^{k} \frac{1}{2^s}\right) \gamma.
\]

(iv) For $1 \leq l \leq k$ and every $E \in \sigma(\Delta + V^l)$

\[
\frac{1}{p_k} \log \|A_{p_k}(E, V^k)\| \leq \sum_{s=l+1}^{k+1} \frac{1}{2^s}.
\]

From properties (i) and (ii), we see that

\[
\|V^k - V^{k-1}\| \leq \left(\frac{4}{5}\right)^k E_0.
\]

Hence the limit

\[
V = \lim_{k \to \infty} V^k,
\]

exists in $\ell^\infty(\mathbb{Z})$ and $\|V - V^0\| \leq 5 E_0$.

**Lemma 4.2.** We have that

\[
L(0) \geq \gamma.
\]

**Proof.** By (ii), we have that $\inf(\sigma(\Delta + V^k)) > 0$. Hence by Thouless’ formula (2.7)

\[
L(E, V^k) = \int \log |t - E| dN^k(t),
\]

we have that $E \mapsto L(E, V^k)$ is decreasing in $E \leq 0$.

This and property (iii) imply for $E < 0$ and $k \geq 1$ that

\[
L(E, V^k) \geq \gamma.
\]

Since $N^k \to N$ weakly and $\log |t - E|$ is a bounded and continuous function for $E < 0$, we also obtain

\[
L(E) \geq \gamma
\]

for $E < 0$. Thouless formula even implies that

\[
L(0) = \lim_{E \uparrow 0} L(E).
\]

This implies the claim. \qed
Similarly as for (3.4) in the previous section, we have that properties (i) and (iv) imply that

\[
L(E) \leq \frac{1}{2^k}
\]

for almost every \( E \in \sigma(\Delta + V^k) \). Hence, we have that

\[
\limsup_{E \to 0} L(E) > \liminf_{E \to 0} L(E) = 0.
\]

This finishes the proof of Proposition 4.1.

4.2. Construction of the sequence of potentials. In order to construct the sequence of potentials \( V^k \), we will prove the following lemmas. These imply the existence of \( V^k \) given \( V^{k-1} \) by applying them to \( V = V^{k-1} \).

**Lemma 4.3.** Let \( V \) be a \( p \) periodic potential and \( E_0 = \inf \sigma(\Delta + V) \). Let \( \tilde{p} = m_0 p \) for sufficiently large \( m_0 \). Define for \( 1 \leq l \leq h \) the intervals \( I_l \) by

\[
I_l = [m \tilde{p} + (l-1)\tilde{p}, m \tilde{p} + l\tilde{p} - 1].
\]

Define the \( \hat{p} = (m+h)\tilde{p} \) periodic potential \( \hat{V}_{m,h} \) by

\[
\hat{V}_{m,h}(n) = \begin{cases} 
V(n) & n \in I_l, 1 \leq l \leq h; \\
V(n) - \frac{2E_0}{5} & [0, m \tilde{p} - 1]; 
\end{cases}
\]

Then we have

\[
3 \frac{E_0}{5} \leq \inf \sigma(\Delta + \hat{V}_{m,h}) \leq 4 \frac{E_0}{5}.
\]

**Proof.** As in the proof of Lemma 3.5, let \( E_- = E_0 \) and \( \delta = \frac{E_0}{4} \). Pick \( \hat{E} \) in the first band of \( \sigma(\Delta + V) \) such that \( \hat{E} - E_0 < \frac{E_0}{10} \). Also, \( m_0 \) will be picked sufficiently large so that \( \delta \sqrt{m_0} > 4 \). Then, we will get

\[
\text{dist}(\hat{E} - \frac{2E_0}{5}, \sigma(\Delta + \hat{V}_{m,h})) \leq \frac{\delta}{2} = \frac{E_0}{10}.
\]

The lemma follows. \( \square \)

We can view the \( \hat{p} \) period of \( \hat{V}_{m,h} \) as a concatenation of two parts:

- \( m \) pieces of the \( \hat{p} \) period of \( V \),
- \( h \) pieces of the \( \hat{p} \) period of \( \hat{V} \), where \( \hat{V} = V - \frac{2E_0}{5} \).

Denote \( \hat{A} = A_{\hat{p}}(E, \hat{V}) \), then we have

\[
A_{\hat{p}}(E, \hat{V}_{m,h}) = (\hat{A})^h \cdot (A_{\hat{p}}(E, V))^m.
\]

Let \( E \notin \inf(\sigma(\Delta + V)) \). Then we can chose two normalized vectors \( v, u \in C^2 \), such that

\[
A_{\hat{p}}(E, V)v = e^{\hat{p}L(E,V)}v, \quad A_{\hat{p}}(E, V)u = e^{-\hat{p}L(E,V)}u.
\]

Denote by \( v^\perp = av + bu \) a vector orthonormal to \( v \). We will first show

**Lemma 4.4.** There exists \( h \in \{1, 2\} \) such that

\[
\langle v, \hat{A}^h v \rangle + a\langle v^\perp, \hat{A}^h v \rangle \neq 0.
\]
Proof. From the Cayley–Hamilton theorem, we have that \( \tilde{A}^2 - \text{tr}(\tilde{A})\tilde{A} + I = 0 \).
Taking \( \langle v, v \rangle \) and \( \langle v^\perp, v \rangle \), we obtain
\[
\langle v, \tilde{A}^2 v \rangle - \text{tr}(\tilde{A})\langle v, \tilde{A}v \rangle + 1 = 0,
\]
\[
\langle v^\perp, \tilde{A}^2 v \rangle - \text{tr}(\tilde{A})\langle v^\perp, \tilde{A}v \rangle = 0.
\]
Multiplying the second equation by \( a \) and adding the two together, we obtain
\[
\left( \langle v, \tilde{A}^2 v \rangle + a\langle v^\perp, \tilde{A}^2 v \rangle \right) - \text{tr}(\tilde{A})\left( \langle v, \tilde{A}v \rangle + a\langle v^\perp, \tilde{A}v \rangle \right) = -1.
\]
Hence, the claim follows. \( \square \)

We now come to

Lemma 4.5. Let \( E \notin \sigma(\Delta + V) \). Then there exists \( h \in \{1, 2\} \) such that
\[
(4.19) \quad L(E, \hat{V}_{m,h}) \to L(E, V)
\]
as \( m \to \infty \).

Proof. Let \( h \) be as in the previous lemma. The lower bound can be obtained by a similar argument as in Subsection 3.2.

By (7.10) in [11] and a computation, we have
\[
L(E, \hat{V}_{m,h}) = \frac{1}{\bar{p}} \log \left( \frac{\text{tr}(A_{\hat{p}}(E, \hat{V}_{m,h}))}{2} \right) + \sqrt{\left( \frac{\text{tr}(A_{\hat{p}}(E, \hat{V}_{m,h}))}{2} \right)^2 - 1} \geq \frac{1}{\bar{p}} \log \left| \text{tr}(A_{\hat{h}}(E, V))^m \right| - \frac{\log(2)}{\bar{p}}.
\]

Let us now evaluate \( \text{tr}(A_{\hat{h}}(E, V))^m \). We have
\[
\text{tr}(A_{\hat{h}}(E, V))^m = \langle v, A_{\hat{h}}(A_{\hat{p}}(E, V))^m v \rangle + \langle v^\perp, A_{\hat{h}}(A_{\hat{p}}(E, V))^m v^\perp \rangle
\]
\[
= e^{m\hat{p}L(E,V)}\langle v, \hat{A}^h v \rangle + ae^{m\hat{p}L(E,V)}\langle v^\perp, \hat{A}^h v \rangle
\]
\[
+ be^{-m\hat{p}L(E,V)}\langle v^\perp, \hat{A}^h u \rangle
\]
\[
= e^{m\hat{p}L(E,V)} \left( \langle v, \hat{A}^h v \rangle + a\langle v^\perp, \hat{A}^h v \rangle \right) + o(1).
\]

Combining this with the previous formula, we obtain that
\[
L(E, \hat{V}_{m,h}) \geq L(E, V) + o(1).
\]

Hence, we see that the claim holds, if we choose \( m \) large enough. \( \square \)

In order to show the existence of \( V^k \) such that (i) to (iv) hold. Use Lemma 4.3 to find a sequence of potentials \( \hat{V}_{m,h} \) such that properties (i) and (ii) hold. The previous lemma implies the existence of \( h \in \{1, 2\} \) such that property (iii) for \( m \geq 1 \) large enough. Finally, by arguments similar to the ones in Subsection 3.2, we can show that property (iv) holds for \( m \geq 1 \) large enough. This finishes the proof of the existence of the sequence \( V^k \) and so also of Proposition 4.4.

Acknowledgments

We thank Artur Avila and David Damanik for useful discussions and Svetlana Jitomirskaya for informing us of [12].
References

[1] J. Avron, B. Simon, *Almost periodic Schrödinger operators. I. Limit periodic potentials*, Commun. Math. Phys. **82** (1981), 101–120.

[2] J. Bochi, M. Viana, *The Lyapunov exponents of generic volume-preserving and symplectic maps*, Ann. of Math. (2) 161 (2005), no. 3, 1423–1485.

[3] D. Damanik, *Lyapunov exponents and spectral analysis of ergodic Schrödinger operators: a survey of Kotani theory and its applications*, Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon’s 60th birthday, 539–563, Proc. Sympos. Pure Math., **76**, Part 2, Amer. Math. Soc., Providence, RI, 2007.

[4] D. Damanik, Z. Gan, *Spectral properties of limit-periodic Schrödinger operators*, to appear in Discrete Contin. Dyn. Syst. Ser. S

[5] P. Deift, B. Simon, *Almost periodic Schrödinger operators, III. The absolutely continuous spectrum in one dimension*, Commun. Math. Phys. **90** (1983), 389-411.

[6] Z. Gan, *An exposition of the connection between limit-periodic potentials and profinite groups*, to appear in Math. Model. Nat. Phenom.

[7] J. Johnson, *Lyapunov numbers for the almost periodic Schrödinger equation*, Illinois J. Math. **28**:3 (1984), 397–419.

[8] H. Krüger, *Probabilistic averages of Jacobi operators*, Commun. Math. Phys. **295**:3 (2010), 853–875.

[9] J. Pöschel, *Examples of Discrete Schrödinger Operators with Pure Point Spectrum*, Commun. Math. Phys. 88, 447-463 (1983)

[10] B. Simon, *Kotani theory for one dimensional stochastic Jacobi matrices*, Commun. Math. Phys. **89** (1983), 227-234.

[11] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Math. Surv. and Mon. **72**, Amer. Math. Soc., Rhode Island, 2000.

[12] J.-P. Thouvenot, *An example of discontinuity in the computation of the lyapunov exponents*, Tr. Mat. Inst. Steklova **216** (1997), 370–372

Department of Mathematics, Rice University, Houston, TX 77005, USA

E-mail address: zheng.gan@rice.edu

URL: [http://math.rice.edu/~zg2/](http://math.rice.edu/~zg2/)

Department of Mathematics, Rice University, Houston, TX 77005, USA

E-mail address: helge.krueger@rice.edu

URL: [http://math.rice.edu/~hk7/](http://math.rice.edu/~hk7/)