DISTRIBUTED STOCHASTIC APPROXIMATION
WITH LOCAL PROJECTIONS

SUHAIL MOHMAD SHAH AND VIVEK S. BORKAR†

Department of Electrical Engineering,
Indian Institute of Technology Bombay,
Powai, Mumbai 400076, India.
(suhailshah@ee.iitb.ac.com, borkar.vs@gmail.com).

Abstract We propose a distributed version of a stochastic approximation scheme constrained
to remain in the intersection of a finite family of convex sets. The projection to the intersection
of these sets is also computed in a distributed manner and a ‘nonlinear gossip’ mechanism is
employed to blend the projection iterations with the stochastic approximation using multiple
time scales.

Key words distributed algorithms; stochastic approximation; projection; differential inclusions;
multiple time scales

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1 Introduction

In a landmark paper, Tsitsiklis et al [20] laid down a framework for distributed computation, notably for distributed optimization algorithms. They developed it further in [3]. There was a lot of subsequent activity and variations, an extensive account of which can be found in [18]. The key idea of [20] was to combine separate iterations by different processors/agents with an averaging mechanism that couples them and leads to a ‘consensus’ among the separate processors, often on the desired objective (e.g., convergence to a common local minimum). The averaging mechanism itself has attracted much attention on its own as gossip algorithm for distributed averaging [19] and various models for dynamic coordination [16]. One way to view these algorithms is as a two time scale dynamics with averaging on the fast or ‘natural’ time scale dictated by the iterate count $n = 1, 2, \cdots$ itself, with the rest (e.g., a stochastic gradient scheme) being a regular perturbation thereof on a slower time scale, dictated by the chosen stepsize schedule. Then the convergence results can be viewed as the fast averaging process leading to the confinement of the slow dynamics to the former’s invariant subspace, which is the one dimensional space spanned by constant vectors. This is just a fancy way of interpreting consensus, but lends itself to some natural generalizations which were taken up in [14]. Here the averaging was replaced by some nonlinear operation with the conclusion that it forced asymptotic confinement of the slow iterates given by a stochastic approximation scheme (SA for short) to its invariant set.

Our aim here is to leverage this viewpoint to propose a distributed algorithm for constrained computation wherein we want asymptotic confinement to the intersection of a finite family of compact convex sets. A prime example of such an exercise is constrained optimization, though the scheme we analyze covers the much broader class of projected stochastic approximation algorithms. In particular, the nonlinear operation gets identified with a distributed scheme for projection onto the intersection of a finite family of convex sets.

To contrast this with the classical projected stochastic approximation [13], note that in the latter, a projection is performed at each step. In practice this may entail another iterative scheme to compute the projection as a subroutine, so that one waits for its near-convergence at each step. In our scheme, this iteration is embedded in the stochastic approximation iteration as a fast time scale component so that it can be carried out concurrently.

Our scheme is inspired by the results of [14]. The results of [14], however, use strong regularity conditions such as Fréchet differentiability of the nonlinear map (among others) which are unavailable in the present case, making the proofs much harder. An additional complication is that the ‘fast’ dynamics in one of the algorithms is itself a two time scale dynamics with stochastic approximation-like time-dependent iteration. This creates several additional difficulties. Thus the proofs of ibid. cannot be applied here directly. This is even more so for the test of stability in section 4, which requires a significantly different proof.

Relevant literature: The literature on distributed algorithms is vast, mostly building upon the seminal work of [20]. The most relevant works for our purposes are [21]-[25], where distributed optimization algorithms with local constraints are considered. However, all of these are concerned primarily with convex optimization (and without noise). In [22], the convergence analysis is done for projected convex optimization for the special case when the network is completely connected. The work in [24], [25] extends the algorithm of [22] and its analysis to a more general setting including the presence of noisy links. [14] considers distributed optimization for non-convex functions but again without distributed projection.
**Contributions:** The main contribution of this paper is that algorithms are provided for projected distributed SA where the projection component is distributed (so that only local constraints are required at the nodes). This is helpful when projection onto the entire constraint set is not possible (constraints are known only locally) or computationally prohibitive (a large number of constraints). Moreover, in our algorithms the projection is tackled on a faster time scale which is a new feature that has not been explored previously and is of independent interest. In both the algorithms proposed here, the projection component gives the exact projection of the point provided and not just a feasible point. This fact seems to be crucial in extending previous works, which mainly consider convex optimization with noiseless gradient measurements, to a fully distributed algorithm for the much more general case of a Robbins-Monro type stochastic approximation scheme. The feasibility and convergence properties of previous works depend critically on the specifics of convex optimization such as the convexity of the function being optimized. The main compromise here is that our convergence results are established under a stability assumption which would not be required for a compact constraint set if exact projection was performed at every step.

The remainder of this section sets up the notation and describes our algorithm. The next section summarizes the algorithm and some key results from projected dynamical systems. Section 3 details the main convergence proof assuming boundedness of iterates. The latter is separately proved in section 4. Section 5 provides some numerical results.

### 1.1 Notation

The projection operator onto a constraint set $\mathcal{X}$ is denoted by $P_{\mathcal{X}}(\cdot)$, i.e.

$$P_{\mathcal{X}}(y) = \arg\min_{x \in \mathcal{X}} \|y - x\|$$

is the Euclidean projection onto $\mathcal{X}$. We will be considering the case where

$$\mathcal{X} = \bigcap_{i=1}^{N} \mathcal{X}_i.$$

The projection operator onto an individual constraint set $\mathcal{X}_i$ is denoted by $P^i(\cdot)$.

Since we are dealing with distributed computation, we use a stacked vector notation. In particular $x_k = [(x^1_k)^T \cdots (x^N_k)^T]^T$ where $x^i_k$ is the value stored at the $i$'th node, so that the superscript indicates the node while the subscript the iteration count. Similarly any function under consideration denoted by $h(\cdot) : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ represents $h(x_k) = [h^1(x^1_k)^T \cdots h^n(x^N_k)^T]^T$ where each $h^i : \mathbb{R}^n \to \mathbb{R}^n$ is a component function. We let a bold faced $\mathbf{P}(\cdot)$ denote the following operator on the space $\mathbb{R}^{Nn}$:

$$\mathbf{P}(x_k) = [P^1(x^1_k)^T, \cdots, P^N(x^N_k)^T]^T$$

with $P^i$ as described above. Let $\mathbf{1}$ denote the constant vector of all 1's of appropriate dimension. We let $\otimes$ denote the Kronecker product between two matrices and $\langle x_k \rangle$ denote the average of $x_k$, so that

$$\langle x_k \rangle = \frac{1}{N}(\mathbf{1}^T \otimes I_n)x_k = \frac{x^1_k + \cdots + x^N_k}{N}$$

with a bold faced $\langle \mathbf{x}_k \rangle = [(\langle x_k \rangle)^T, \cdots, (x_k)^T]^T$. A differential inclusion is denoted as

$$\dot{x} \in F(x)$$
where $F(\cdot)$ is a set valued map. Let $F^\delta(\cdot)$ denote the following set:

$$F^\delta(x) = \{z \in \mathbb{R}^n : \exists x' \text{ such that } \|x - x'\| < \delta, \ d(z, F(x')) < \delta\}$$

where $d(z, A) = \inf_{y \in A} \|z - y\|$ is the distance of a point $z$ from a set $A$. The notation $f(x) = o(g(x))$ is used to denote the fact

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} \to 0,$$

while $f(x) = O(g(x))$ represents

$$\limsup_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| \leq M$$

for some constant $M < \infty$.

## 2 Background

### 2.1 Set-up

Suppose we have a network of $N$ agents indexed by $1, ..., N$. We associate with each agent $i$, a function $h_i : \mathbb{R}^n \to \mathbb{R}^n$ and a constraint set $X_i$. The global constraint, given as the intersection of all $X_i$’s, is denoted by $X$:

$$\mathcal{X} = \bigcap_{i=1}^{n} X_i.$$

Also, let $H : \mathbb{R}^n \to \mathbb{R}^n$ denote

$$H(\cdot) := \frac{1}{N} \sum_{i=1}^{N} h_i(\cdot). \quad (1)$$

In many applications, one has $h_i(\cdot) = f(\cdot) \forall i$. Let the communication network be modeled by a static undirected graph $G = \{\mathcal{V}, \mathcal{E}\}$ where $\mathcal{V} = \{1, ..., N\}$ is the node set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of links $(i, j)$ indicating that agent $j$ can send information to agent $i$. All of the arguments presented here can be extended to a time-varying graph under suitable assumptions as in [17]. Here we deal only with a static network for ease of notation.

We associate with the network a non-negative weight matrix $Q = [q_{ij}]_{i,j \in \mathcal{V}}$ such that

$$q_{ij} > 0 \iff (i, j) \in \mathcal{E}.$$ 

In addition, the following assumptions are made on the matrix $Q$ and the constraint set:

**Assumption 1:**

i) **[Convex constraints]** For all $i$, $X_i$ are convex and compact. Also, the set $\mathcal{X}$ has a non-empty interior.

ii) **[Double Stochasticity]** $1^T Q = 1^T$ and $Q 1 = 1$.

iii) **[Irreducibility and aperiodicity]** We assume that the underlying graph is irreducible, i.e., there is a directed path from any node to any other node, and aperiodic, i.e., the g.c.d. of lengths...
of all paths from a node to itself is one. It is known that the choice of node in this definition is immaterial. This property can be guaranteed, e.g., by making \( q_{ii} > 0 \) for some \( i \).

This implies that the spectral norm \( \gamma \) of \( Q - \frac{11^T}{N} \) satisfies \( \gamma < 1 \). This guarantees in particular that

\[
\|(Q^k - Q^*)u\| \leq \kappa \beta^{-k} \|u\|
\]

for some \( \kappa > 0, \beta > 1 \), with \( Q^* \) denoting the matrix \( \frac{11^T}{N} \).

### 2.2 Distributed Projection Algorithms

We now give details of two algorithms for computing a distributed projection. “Distributed Projection” here means that the projection onto a particular constraint set \( X_i \) is performed by just one processor/agent and they communicate information with each other in order to compute projection onto the intersection \( X \).

**A. Gradient Descent** : The first approach involves viewing the projection problem as the minimization of the error norm subject to the appropriate constraints, so that the projection of a point \( x_0 \) can be thought of as the solution to the following optimization problem :

\[
\min_{z \in \mathbb{R}^n} \|z - x_0\|^2 \quad \text{s.t.} \quad z \in X = \bigcap_{i=1}^N X_i, \quad i = 1, \ldots, N
\]

In the distributed setting we associate each constraint \( X_i \) with an agent \( i \). To solve the projection problem in a distributed fashion we first re-cast it as :

\[
\min_{z \in \mathbb{R}^n} \|z - x_0\|^2 \quad \text{s.t.} \quad z \in X \leftrightarrow \min_{\{z^i \in \mathbb{R}^n, i = 1, \ldots, N\}} \frac{1}{N} \sum_{i=1}^N \|z^i - x_0\|^2 \quad \text{s.t.} \quad z^i \in X_i \forall i
\]

It is obvious that both the problems have the same unique minimizer. The problem on the right can be solved by using a distributed gradient descent of the form of equations 2a-2b, [21] which for our case becomes :

\[
z_{k+1}^i = P^i\left\{\sum_{j=1}^N q_{ij} z_j^k - b_k \left[\sum_{j=1}^N q_{ij} z_j^k - x_0\right]\right\}
\]

where \( b_k \) satisfies \( \sum_k b_k = \infty, \quad \sum_k b_k^2 < \infty \). Note that the term inside the square brackets is proportional to the gradient of the function in (3) evaluated at \( \sum_{j=1}^N q_{ij} z_j^k \).

**Lemma 1.** For any \( x_0 \in \mathbb{R}^n \) the iteration [4] converges to the projection of \( x_0 \) upon \( X \), i.e.,

\[
z_k^i \rightarrow P_X(x_0) \quad \forall i.
\]

**Proof.** The function being optimized is strongly convex so that the optimal point is unique. We can directly invoke Prop. 1 [21] which guarantees convergence to the unique minimum in the set \( X \) and this unique minimum is the projection point \( P_X(x_0) \). (Note that Assumption 1 here is necessary for the equations 2a-2b of [21] to converge to the projection point. Specifically Assumptions 1-5 of [21] are satisfied for our case.) \( \square \)
B. Distributed Boyle-Dykstra-Han: The algorithm originally proposed in [17] is as follows:

Algorithm 1 Distributed Projection Scheme

Input: \( y_0 \in \mathbb{R}^n \)

1: Set \( z_i^0 = y_0 \) and \( x_i^0 = 0 \) for all \( i \).

2: for \( k=1,2,\ldots \) do

3: At each node \( i \in \mathcal{I} \) do

3.1: \( x_i^k = \sum_{j=1}^{N} q_{ij} \{ x_{k-1}^j + P^j(z_k^j) \} - P^i(z_k^i) \).

4: \( z_{k+1}^i = z_k^i + b_k x_k^i \).

5: end for

The step size \( b_k \) is assumed to satisfy:

\[
\sum_k b_k = \infty, \quad \sum_k b_k^2 < \infty
\]

and for any \( \epsilon > 0 \), there exists an \( \alpha \in (1, 1 + \epsilon) \) and some \( k_0 \) such that

\[
\alpha b_{k+1} \geq b_k, \quad \forall k > k_0,
\]

(5)

Let us write the above in vector notation as

\[
\begin{align*}
x_k &= (Q \otimes I_n) \{ x_{k-1} + P(z_k) \} - P(z_k) \\
z_{k+1} &= z_k + b_k x_k.
\end{align*}
\]

(6)

(7)

The next theorem states that the above algorithm gives the exact projection of the initial point \( y_0 \).

**Theorem 2.** Suppose \( z_0^i = y_0 \forall i \). Then \( z_k \to z^* = [z_1^*, \ldots, z_n^*] \), such that

\[
P^i(z_i^*) = P(y_0).
\]

The detailed proof can be found in [17] where an ODE approximation is used along with the associated Lyapunov function \( z \mapsto \| z - z^* \|^2 \).

**Remark 3.** The above algorithm is inspired from a parallel version of Boyle-Dykstra-Han originally proposed in [12]. The main difference from [12] is that the weights in [17] are derived from the related graph of the communication network. The compromise is that a decaying time step is required to ensure convergence which may affect the convergence rate. We invite the reader to go through [17] to get some more intuition regarding the algorithm and what the exact roles of \( x_k \) and \( z_k \) are in the context of Boyle-Dykstra-Han algorithm.
2.3 The algorithm

The first algorithm for distributed projected SA is stated below along with the assumptions on the various terms involved.

Algorithm 2 Distributed SA with Gradient Descent (DSA-GD)

1: Initialize $y_i^0$ and set $z_i^k = 0$ for all $i$.

2: for $k=1,2,\ldots$ do:

3: At each node $i \in \mathcal{I}$ do:

a: [Fast Time Scale] Distributed Projection Step:

\begin{align*}
& a1: z_i^{k+1} = P^i\left(\sum_{j=1}^N q_{ij}z_j^k - b_k(\sum_{j=1}^N q_{ij}z_j^k - y_i^k)\right)
\end{align*}

b: Derive a noisy sample "$h_i^i(y_i^k) + M_{k+1}^i$" of $h_i^i(y_i^k)$ from a sampling oracle.

c: [Slow Time Scale] Distributed Stochastic Approximation Step:

\begin{align*}
& c1: y_i^{k+1} = \sum_{j=1}^N q_{ij}y_j^k + a_k(z_i^k - y_i^k) + a_k(h_i^i(y_i^k) + M_{k+1}^i)
\end{align*}

4: end for

We make the following key assumptions:

**Assumption 2:**

i) For each $i$, the function $h_i^i : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz.

ii) For each $i$, \{\{M_i^k\}\} is a martingale difference sequence with respect to the the filtration $\mathcal{F}_k^i := \sigma(y_i^\ell, M_i^\ell, \ell \leq k)$, i.e., it is a sequence of zero mean random variables satisfying:

\begin{align*}
\mathbb{E}\left[M_{k+1}^i | \mathcal{F}_k^i\right] = 0.
\end{align*}

where $\mathbb{E}[\cdot | \cdot]$ denotes the conditional expectation. In addition we also assume a conditional variance bound

\begin{align*}
\mathbb{E}\left[\|M_{k+1}^i\|^2 | M_i^\ell, y_i^\ell, \ell \leq k\right] \leq K(1 + \|y_i^k\|^2) \quad \forall k, i \text{ a.s.} 
\end{align*}

for some constant $K > 0$.

iii) Stepsizes \{a_k\} and \{b_k\} are positive scalars which satisfy :

\begin{align*}
& a_k = o(b_k), \quad \sum_{k} a_k = \sum_{k} b_k = \infty, \quad \sum_{k} (a_k^2 + b_k^2) < \infty,
\end{align*}

iv) The iterates $y_k$ are a.s. bounded, i.e

\begin{align*}
& \sup_{k} \|y_k\| < \infty \quad \text{a.s.}
\end{align*}

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As in classical analysis of stochastic approximation algorithms by the o.d.e. method, we prove convergence assuming the stability condition Assumption 2(iv). In Section 4 we give sufficient conditions for the latter to hold.

We now give a variant of the above algorithm using the distributed Boyle-Dykstra-Han algorithm (Algorithm 1):

**Algorithm 3** Distributed SA with Boyle-Dykstra-Han (DSA-BDH)

1: Initialize $y^i_0$ and set $z^i_k = 0, x^i_k = 0$ for all $i$

2: for $k=1,2,...$ do:

3: At each node $i \in \mathcal{I}$:

   a: **[Fast Time Scale]** Distributed Projection Step:

   a1: $x^i_k = \sum_{j=1}^{N} q_{ij} (x^{j}_{k-1} + P^j (z^j_k + y^j_k)) - P^i (z^i_k + y^i_k)$

   a2: $z^{i+1}_k = z^i_k + b_k x^i_k$

   b: Derive a noisy sample "$h^i(y^i_k) + M^i_{k+1}$" of $h^i(y^i_k)$ from a sampling oracle.

   c: **[Slow Time Scale]** Distributed Stochastic Approximation Step:

   c1: $\bar{y}^i_k = P^i (y^i_k + z^i_k)$

   c2: $y^i_{k+1} = \sum_{j=1}^{N} q_{ij} y^j_k + a_k (\bar{y}^j_k - y^j_k) + a_k (h^i(y^i_k) + M^i_{k+1})$

4: end for

**Remark 4.** The assumption of compact $\mathcal{X}_i$ is not necessary for Algorithm 2. As long as the iterates are assumed to be bounded (both $y_k$ and $z_k$), ‘compact sets’ can be replaced by ‘closed sets’.

**Remark 5.** For the various terms involved in Algorithm 3, Assumption 2 continues to apply. The only addition is that Assumption 2(iii) now includes the condition (5).

**Remark 6.** Both algorithms 2 and 3 have the same objective, i.e. distributed projected SA. For the rest of the paper we refer to Algorithm 2 as DSA-GD and Algorithm 3 as DSA-BDH.

### 2.4 Projected dynamical systems

We analyze the algorithms using the ODE (for ‘Ordinary Differential Equations’) approach for analyzing stochastic approximation, extended to differential inclusions \cite{2}. Define the normal cone $N_{\mathcal{X}}(x)$ to be the set of outward normals at any point $x \in \partial \mathcal{X}$ where $\partial \mathcal{X}$ is the boundary of the set $\mathcal{X}$, i.e.,

\[ N_{\mathcal{X}}(x) = \{ \gamma \in \mathbb{R}^n : \langle \gamma, x - y \rangle \geq 0 \ \forall \ y \in \mathcal{X} \} \tag{10} \]

with $N_{\mathcal{X}}(x) = \{0\}$ for any point $x$ in the interior of $\mathcal{X}$. The relevant differential inclusion for our problem is

\[ \dot{x} \in H(x(t)) - N_{\mathcal{X}}(x(t)), \tag{11} \]
\( x(t) \in \mathcal{X} \quad \forall t \in [0, T] \)

where \( H(\cdot) \) is as in [11]. This inclusion is identical to the well known "Projected Dynamical System" considered in [11]:

\[
\dot{x} = \Pi(x, h(x))
\]

where \( \Pi(x, h(x)) \) is defined to be the following limit for any \( x \in \mathcal{X} \):

\[
\Pi(x, h(x)) = \lim_{\delta \to 0} \frac{P_X(x + \delta h(x)) - x}{\delta}
\]

The proof of the fact that the operator \( \Pi(\cdot, \cdot) \) is identical to the RHS of (11) is provided in ([10], Lemma 4.6). The following theorem is borrowed from ([8], Corollary 2) and ([9], Theorem 3.1).

**Theorem 7.** For a convex \( \mathcal{X} \), ([11]) is well posed, i.e. a unique solution exists.

We recall the following notion from [2] where more general differential inclusions are considered:

**Definition 8.** ([2]) Suppose \( F \) is a closed set valued map such that \( F(x) \) is a non-empty compact convex set for each \( x \). Then a perturbed solution \( y \) to the differential inclusion

\[
\dot{x} \in F(x)
\]

is an absolutely continuous function which satisfies:

i) \( \exists \) a locally integrable function \( t \to U(t) \) such that for any \( T > 0 \),

\[
\lim_{t \to \infty} \sup_{0 \leq v \leq T} \left| \int_{t}^{t+v} U(s)ds \right| = 0,
\]

ii) \( \exists \) a function \( \delta : [0, \infty) \to [0, \infty) \) with \( \delta(t) \to 0 \) as \( t \to \infty \) such that

\[
\dot{y} - U(t) \in F^{\delta(t)}(y).
\]

There is no guarantee that the perturbed solution remains close to a solution of ([12]), however, the following assumption helps in establishing some form of convergence. Let \( \Lambda \) denote the equilibrium set of ([11]), assumed to be non-empty. Then:

\[
\Lambda \subset \{ x : H(x) \in N_X(x) \}.
\]

**Assumption 3 :** There exists a Lyapunov function for the set \( \Lambda \), i.e., a continuously differentiable function \( V : \mathbb{R}^n \to \mathbb{R} \) such that any solution \( x(t) \) to ([11]) satisfies

\[
V(x(t)) \leq V(x(0)) \forall t > 0
\]

and the inequality is strict whenever \( x(0) \notin \Lambda \).

If this assumption does not hold, the asymptotic behavior is a bit more complex. Specifically, under reasonable assumptions, an SA scheme converges a.s. to an invariant internally chain
transitive set of the limiting o.d.e. This behavior extends to differential inclusions as well (see [2] or Chapter 5, [6]) and is what we would expect for our algorithm if we remove Assumption 3.

For instance if \( H = -\nabla g \) for some continuously differentiable \( g \), then the above set represents the KKT points and the function \( g \) itself will serve as a Lyapunov function for the set \( \Lambda \).

The following result is from ([2], Prop. 3.27):

**Proposition 9.** Let \( y \) be a bounded perturbed solution to (11) and there exist a Lyapunov function for a set \( \Lambda \) with \( V(\Lambda) \) having an empty interior. Then

\[
\bigcap_{t \geq 0} y([t, \infty)) \subset \Lambda
\]

**Remark 10.** To prove the main result, we shall show that the suitably interpolated iterates generated by the algorithm form a perturbed solution to the differential inclusion (11), so that using Theorem 7 and Proposition 9 (along with Assumption 3), the algorithm is shown to converge to its equilibrium set.

To conclude this section, we provide some intuition behind the proposed algorithms. The SA part in DSA-GD is

\[
y_{k+1}^i = \sum_{j=1}^{N} q_{ij} y_k^j + a_k(z_k^i - y_k^i) + a_k(h(y_k^i) + M_{k+1}^i)
\]

If the term \( z_k^i \) asymptotically tracks the projection \( P_X(y_k^i) \) (see Lemma 13), then this can be written as (modulo some asymptotically vanishing error):

\[
y_{k+1}^i \approx \sum_{j=1}^{N} q_{ij} y_k^j + a_k(h(y_k^i) + \underbrace{P_X(y_k^i) - y_k^i}_{\text{projection error term}} + M_{k+1}^i)
\]

The projection error term strives to keep the iterates inside the constraint set (compare this to the inclusion (11) where the constraining term belongs to the normal cone). So although the above iteration may appear to behave like an unconstrained one, it achieves the same asymptotic behavior as what one would get by projecting onto the entire constraint set \( X \) at each step. An analogous intuition applies to DSA-BDH.

### 3 Convergence proof

We first deal with DSA-GD, the proof details for DSA-BDH are nearly the same and we give a brief outline in the second part of this section. Assumptions 1, 2 and 3 are assumed to hold throughout this section.

**A. Convergence of DSA-GD:** We rewrite some of the main steps in DSA-GD with a stacked vector notation:

\[
z_{k+1} = P \{(Q \otimes I_N)z_k - b_k((Q \otimes I_N)z_k - y_k)\}, \quad (13)
\]

\[
y_{k+1} = (Q \otimes I_N)y_k + a_k(z_k - y_k) + a_k(h(y_k) + M_{k+1}). \quad (14)
\]

Our main convergence result is:
Theorem 11. Under Assumptions 1-3, we have almost surely
\[ y_k \to \{ 1 \otimes y : y \in \Lambda \} . \]

The above theorem states that the variables \{y_k^i, i = 1, \ldots, N\} achieve consensus as expected and moreover, their limit points lie in the equilibrium set \( \Lambda \) of (11). For distributed optimization, this set corresponds to the KKT points of the related minimization problem.

Outline of the analysis: To analyze the above algorithm, we proceed in three steps:

a) Consensus: We first show that the iterates \( y_k^i \) achieve consensus (Lemma 12). This will help us analyze the algorithm by studying it in the average sense at each node, i.e., with
\[ y_k^i \approx \langle y_k \rangle := \frac{1}{N} \sum_{i=1}^{N} y_k^i. \]

b) Feasibility: Next, the condition \( a_k = o(b_k) \) is exploited to do a two time scale analysis. Specifically, in the distributed projection scheme operating on a fast time scale, the variable \( y_k \), evolving on a slow time scale, is quasi-static, i.e., treated as a constant. In turn the (slow) stochastic approximation step sees the faster iterations (13) as quasi-equilibrated. Therefore it asymptotically behaves as its projected version where the projection is taken to be upon the entire set at each node rather than only its own particular constraint set.

c) Convergence: Finally, using the above analysis, the slow iterates are shown to track the desired projected dynamical system.

We first show that the above algorithm achieves consensus, i.e., as \( k \to \infty \), \( \| y_k^i - \langle y \rangle \| \to 0 \).

Lemma 12. \( \lim_k \max_{i,j=1,\ldots,N} \| y_k^i - y_k^j \| = 0 \) a.s. Also,
\[ \| y_k^i - \langle y \rangle \| \to 0 \ \ \forall \ i. \]

Proof. Set
\[ Y_k = z_k - y_k + h(y_k) + M_{k+1}, \]
so that we can write (14) as
\[ y_{k+1} = (Q \otimes I_n) y_k + a_k Y_k. \]
We have, for a fixed \( k \leq n \) and any large \( n \),
\[ y_{n+k} = (Q \otimes I_n) y_{n+k-1} + a_{n+k-1} Y_{n+k-1} \]
\[ = (Q^2 \otimes I_n) y_{n+k-2} + a_{n+k-2} (Q \otimes I_n) Y_{n+k-2} \]
\[ + a_{n+k-1} Y_{n+k-1}, \]
where we have used the fact that \( (Q \otimes I_n)(Q \otimes I_n) = (Q^2 \otimes I_n) \). Iterating the above equation further, we have
\[ y_{n+k} = (Q^k \otimes I_n) y_n + a_n (Q^{k-1} \otimes I_n) Y_n + \cdots, \]
\[ \cdots + a_{n+k-2} (Q \otimes I_n) Y_{n+k-2} + a_{n+k-1} Y_{n+k-1} \]
i.e.,
\[ y_{n+k} = (Q^k \otimes I_n) y_n + \Gamma(Y_{n+k-1}, \ldots, Y_n) \] (15)
where \( \Gamma(\cdot) \) is some linear combination of its arguments. In view of Assumption 2(iv) (\( y_k \) is bounded),
\[ \| Y_k \| \leq M \quad \text{w.p.} \ 1 \]
for some random $M < \infty$. So we have,

$$
\|\Gamma(Y_{n+k-1}, \ldots, Y_n)\| = \| \sum_{i=n}^{n+k-1} a_i (Q^{n+k-1-i} \otimes I_n) Y_i \|
\leq M \left( \sum_{i=n}^{n+k-1} a_i \right) \quad (\because \|Q^{n+k-1-i} \otimes I_n\| = 1)
= O(\sum_{i=n}^{n+k-1} a_i)
$$

Subtracting $(Q^* \otimes I_n)y_n$ from both sides in (15), we get

$$
y_{n+k} - (Q^* \otimes I_n)y_n = [(Q^k - Q^*) \otimes I_n] y_n + \{ \Gamma(Y_{n+k-1}, \ldots, Y_n) \} \quad (16)
$$

Using equation (2) and taking norms in (16), we have :

$$
\|y_{n+k} - (Q^* \otimes I_n)y_n\| = O(\beta^{-k}) + O(\sum_{i=n}^{n+k-1} a_i).
$$

Letting $n \to \infty$ followed by $k \to \infty$, it follows that any limit point $y_*$ of the sequence $\{y_k\}$ satisfies

$$
y_* = (Q^* \otimes I_n)y_*.
$$

That is, $y_i = \frac{1}{N} \sum_{j=1}^{N} y_j$ for any $i$, so that consensus is achieved and the consensus value is the average of all the node estimates.

We next argue that the algorithm can be regarded as a two time scale iteration so that while analyzing the behavior of $z_k$ (fast variable), $y_k$ (slow variable) can be regarded as a constant (cf. [5] or [6], Chapter 6). We have

$$
z_{k+1} = \mathbf{P}\{ (Q \otimes I_n) z_k - b_k ((Q \otimes I_n) z_k - y_k) \}
= \mathbf{P}\{ Q \otimes I_n) z_k + b_k (\mu(z_k, y_k)) \}
$$

for a suitably defined $\mu$. The slow time scale iteration here is:

$$
y_{k+1} = (Q \otimes I_n) y_k + a_k (z_k - y_k + h(y_k) + M_{k+1})
= (Q \otimes I_n) y_k + a_k (\nu(z_k, y_k) + M_{k+1})
$$

for a suitably defined $\nu$. Since $a_k = o(b_k)$ in (18)-(19), the above pair of equations form a two time scale iteration (5 or 6, Chapter 6). So while analyzing (18), we can assume that $y_k \approx$ a constant, say $\langle y \rangle := [y, \ldots, y]$ (we take the same value at all the nodes because of consensus proved in Lemma 12), so that (18) becomes :

$$
z_{k+1} = \mathbf{P}\{ (Q \otimes I_n) z_k - b_k ((Q \otimes I_n) z_k - \langle y \rangle) \}.
$$

This can be viewed as iteration 4 for the problem (3) with $x_0 = \langle y \rangle$. So Lemma 1 implies $z_k^* \to P_X(\langle y \rangle)$.

**Lemma 13.** If $z^*(\langle y \rangle)$ is the limit point of (20) for any $\langle y \rangle$, then $z^*(\langle y \rangle) = P_X(\langle y \rangle)$. Also for all $i$,

$$
\|z_k^* - P_X(\langle y_k \rangle)\| \to 0.
$$
Proof. This first statement follows directly from the above discussion. The second is a direct consequence of Lemma 1, Chapter 6, [6].

We now prove the main result whose proof uses the techniques of [2], [4].

**Proof of Theorem 11:**

Proof. The consensus part was already proved in Lemma 12. Multiplying both sides of (14) by \( \frac{1}{N} (1^T \otimes I_n) \), and using the double stochasticity of \( Q \), we have

\[
\langle y_{k+1} \rangle = (1 - a_k)\langle y_k \rangle + a_k\langle z_k \rangle + a_k(\langle h(y_k) \rangle + \langle M_{k+1} \rangle)
\]

We used in the above the fact

\[
\frac{1}{N} (1^T \otimes I_n)(Q \otimes I_n)y_k = \frac{1}{N} (1^T \otimes I_n)y_k = \langle y_k \rangle = \frac{1}{N}(y_k^1 + \ldots + y_k^N).
\]

Adding and subtracting \( a_k P_X(\langle y_k \rangle) \) on the right hand side,

\[
\langle y_{k+1} \rangle = (1 - a_k)\langle y_k \rangle + a_k P_X(\langle y_k \rangle) + a_k(\langle z_k \rangle - P_X(\langle y_k \rangle)) + \langle h(y_k) \rangle + \langle M_{k+1} \rangle)
\]

(21)

Note that,

\[
\langle y_k \rangle - P_X(\langle y_k \rangle) \in \mathcal{N}_X(P_X(\langle y_k \rangle))
\]

Set \( Z_k = \langle z_k \rangle - P_X(\langle y_k \rangle) \) and \( p_k = \langle h(y_k) \rangle - H(P_X(\langle y_k \rangle)) \). So (21) becomes:

\[
\langle y_{k+1} \rangle = \langle y_k \rangle + a_k \left( H(P_X(\langle y_k \rangle)) - \mathcal{N}_X(P_X(\langle y_k \rangle)) + p_k + Z_k + \langle M_{k+1} \rangle \right)
\]

(22)

Let \( t_0 = 0 \) and \( t_k = \sum_{i=0}^{k} a_i \) for any \( k \geq 1 \), so that \( t_k - t_{k-1} = a_{k-1} \). Define the interpolated trajectory \( \Theta : [0, \infty) \rightarrow \mathbb{R}^n \) as:

\[
\Theta(t) = \langle y_k \rangle + (t - t_k) \frac{\langle y_{k+1} \rangle - \langle y_k \rangle}{t_{k+1} - t_k}, \quad t \in [t_k, t_{k+1}], \quad k \geq 1
\]

By differentiating the above we have

\[
\frac{d\Theta(t)}{dt} = \frac{\langle y_{k+1} \rangle - \langle y_k \rangle}{a_k} \quad \forall t \in [t_k, t_{k+1}]
\]

where we use the right, resp., left derivative at the end points. Now define the following set valued map

\[
F(x) = \{ H(x) - W : W \in \mathcal{N}(x), \|W\| \leq K \}
\]

(23)

where \( 0 < K < \infty \) is a suitable constant.

We get from (22) using (23),

\[
\frac{d\Theta(t)}{dt} \in F(P_X(\langle y_k \rangle)) + p_k + Z_k + \langle M_{k+1} \rangle, \quad t \in [t_k, t_{k+1}].
\]

(24)
To finish the proof, $\Theta(\cdot)$ is first shown to be a perturbed solution of (11). Let $\eta(t) := \|p_k\| + \|\Theta(t) - P_X(\langle y_k \rangle)\|$, $t \in [t_k, t_{k+1})$, $k \geq 0$. Also define
\[ U(t) = U_k := Z_k + (M_{k+1}), \]
for $t \in [t_k, t_{k+1}), k \geq 0$. Then we have
\[ \frac{d\Theta(t)}{dt} - U(t) \in F^{\eta(t)}(\Theta(t)). \]
We have used the following fact in the above: for any set valued map $F(\cdot)$ we have
\[ \forall (x, \dot{x}) \in \mathbb{R}^n \times \mathbb{R}^n, \quad p + F(x) \subset F(\|x - \dot{x}\|). \]
If we show that $\sum_k a_k U_k < \infty$ and $\eta(t) \to 0$, then by (25) $\Theta(\cdot)$ can be interpreted as a perturbed solution of the differential inclusion
\[ \hat{\Psi}(t) \in F(\Psi(t)). \]
Convergence to the set $\Lambda$ then follows by Assumption 3 and the proof is complete.

I: We know from Lemma 13 that the mapping $y \to P_X(\langle y \rangle)$ maps $y$ to the $y$-dependent limit point of (20). Then we have:
\[ \|\langle z_k \rangle - P_X(\langle y_k \rangle)\| = \|\langle z_k \rangle - \langle P_X(\langle y_k \rangle)\rangle\| \quad [\text{because } P_X(\langle y_k \rangle) = \langle P_X(\langle y_k \rangle)\rangle]\]
\[ = \|\frac{1}{N} \sum_{i=1}^{N} (z^i_k - P_X(\langle y_k \rangle))\| \]
\[ \leq \frac{1}{N} \sum_{i=1}^{N} \|z^i_k - P_X(\langle y_k \rangle)\| \quad [\text{Jensen's Inequality}] \]
\[ \to 0 \quad [\text{from Lemma 13}]. \]

For $T > 0$, let $m(n) := \min\{k \geq n : \sum_{j=n}^{n+k} a_j \geq T\}, n \geq 0$. Then
\[ \sup_{\ell \leq m(n)} \sum_{k=n}^{\ell} a_k \|\langle z_k \rangle - P_X(\langle y_k \rangle)\| \to 0 \text{ as } n \to \infty. \]

II: This term is the error induced by the noise. Note that the process $\sum_{m=0}^{k-1} a_m M^i_{m+1}, k \geq 1$, is a zero mean square integrable martingale w.r.t. the increasing $\sigma$-fields $\mathcal{F}^i_k := \sigma(M^i_m, y^i_m, m \leq k), k \geq 1$, with $\sum_{m=0}^{\infty} a_m^2 \mathbb{E}[\|M^i_{m+1}\|^2|\mathcal{F}^i_m] < \infty$ by (8) and (9), along with the square-summability
of \{a_m\}. It follows from the martingale convergence theorem (Appendix C, [6]), that this martingale converges a.s. Therefore

\[
\sup_{t \leq m(n)} \left\| \sum_{k=n}^{\ell} a_k M_{k+1} \right\| \to 0 \text{ a.s. } \forall i
\]

The claim follows for \( \langle M_{k+1} \rangle \).

To prove that \( \eta(t) \to 0 \), consider for \( t \in [t_k, t_{k+1}), k \geq 0 \).

\[
\eta(t) = \left\| y_k \right\| + \left\| \Theta(t) - P_X(\langle y_k \rangle) \right\| = \left\| h(y_k) \right\| - H(P_X(\langle y_k \rangle)) + \left\| \langle y_k \rangle - P_X(\langle y_k \rangle) \right\|
\]

\[
+ \left\| \langle y_{k+1} \rangle - \langle y_k \rangle \right\| \left( \frac{t - t_k}{t_{k+1} - t_k} \right)
\]

\[
\leq \frac{1}{N} \sum_{i=1}^{N} \left\| h^i(y_k^i) - h^i(P_X(\langle y_k \rangle)) \right\| + \left\| \langle y_k \rangle - P_X(\langle y_k \rangle) \right\|
\]

\[
+ \left\| \langle y_{k+1} \rangle - \langle y_k \rangle \right\| \frac{\left\| t - t_k \right\|}{t_{k+1} - t_k}
\]

\[
\leq \frac{C}{N} \sum_{i=1}^{N} \left\| y_k^i - P_X(\langle y_k \rangle) \right\| + \left\| \langle y_k \rangle - P_X(\langle y_k \rangle) \right\| + O(a_k)
\]

where \( C > 0 \) is a common Lipschitz constant for the \( h^i \)'s. Note that if we prove that \( \left\| P_X(\langle y_k \rangle) - \langle y_k \rangle \right\| \to 0 \), all the terms in the above inequality go to zero as \( k \uparrow \infty \). So to finish the proof, we prove this fact :

**Claim :** \( \lim_{k \to \infty} \inf_{x \in \mathcal{X}} \| y_k^i - x \| = 0 \) \( \forall i \), as \( k \uparrow \infty \)

**Proof.** Let us first consider the following fixed point iteration :

\[
\tilde{y}_{k+1} = (1 - a_k)\tilde{y}_k + a_k P_X(\tilde{y}_k).
\]

By the arguments of [6], Chapter 2, this has the same asymptotic behavior as the o.d.e.

\[
\dot{\tilde{y}} = P_X(\tilde{y}) - \tilde{y}.
\]

Consider the Lyapunov function \( V(\tilde{y}) = \frac{1}{2} \| \tilde{y} \|^2 \). Then

\[
\frac{d}{dt} V(\tilde{y}(t)) = \tilde{y}(t)^T (P_X(\tilde{y}(t)) - \tilde{y}(t))
\]

\[
= \tilde{y}(t)^T P_X(\tilde{y}(t)) - \| \tilde{y}(t) \|^2.
\]

For any \( v \in \mathbb{R}^n \), the non-expansive property of \( P_X \) leads to

\[
\left\| P_X(v) \right\| \leq \| v \|.
\]

By the Cauchy-Schwartz inequality,

\[
\frac{d}{dt} V(\tilde{y}(t)) \leq 0.
\]
By Lasalle’s invariance principle we have any trajectory $\bar{y}()$ converge to the largest invariant set where $\frac{d}{dt}V(\bar{y}(t)) = 0$, which is precisely the set $\mathcal{X}$. The claim for $\{\bar{y}_k\}$ now follows by a standard argument as in Lemma 1 and Theorem 2, pp. 12-16. [6].

Next, define maps $f_k$ by

$$f_k(y) = (1 - a_k)y + a_kP_X(y).$$

Then (21) becomes

$$\langle y_{k+1} \rangle = f_k(\langle y_k \rangle) + a_k(h(\langle y_k \rangle) + M_{k+1} + \epsilon_k).$$  \hspace{1cm} (26)

with $\epsilon_k = \langle z_k \rangle - P_X(\langle y_k \rangle)$. From the preceding discussion, we have the following: for any fixed $k \geq 0$,

$$\lim_{m \to \infty} F_{k,m}(\cdot) := f_{m+k} \circ \cdots \circ f_k(\cdot) \to P_X(\cdot).$$  \hspace{1cm} (27)

The family $\{F_{k,m}\}$ of functions is non-expansive, therefore equi-continuous, and bounded (because $\{y_k\}$ is assumed to be bounded), hence relatively sequentially compact in $C(\mathbb{R}^n)$ by the Arzela-Ascoli theorem. Hence the above convergence is uniform on compacts, uniformly in $k$. Consider any convergent subsequence of $\{y_k\}$ with limit (say) $y^*$ and by abuse of notation, index it by $\{k\}$ again. From (26), for any $k, m$ we have,

$$\langle y_{m+k+1} \rangle = f_{m+k}(\langle y_{m+k} \rangle) + a_{m+k}(h(\langle y_{m+k} \rangle) + M_{m+k+1} + \epsilon_{m+k}).$$  \hspace{1cm} (28)

We also have,

$$\|f_{m+k}(\langle y_{m+k} \rangle) - f_{m+k} \circ f_{m+k-1}(\langle y_{m+k-1} \rangle)\| \leq \|\langle y_{m+k} \rangle - f_{m+k-1}(\langle y_{m+k-1} \rangle)\|$$

$(\because f_{m+k}$ is non-expansive.)

$$= \|\langle y_{m+k} \rangle - \langle y_{m+k-1} \rangle + a_{m+k-1}(\langle y_{m+k-1} \rangle - P_X(\langle y_{m+k-1} \rangle))\|$$

$$= O(a_{m+k-1})$$

$$= o(1).$$

By iterating, we get :

$$\|f_{m+k}(\langle y_{m+k} \rangle) - f_{m+k} \circ \cdots \circ f_k(\langle y_k \rangle)\| = o(1).$$

Combining this with (28), we have

$$\|\langle y_{m+k+1} \rangle - f_{m+k} \circ \cdots \circ f_k(\langle y_k \rangle)\| = o(1).$$

Let $\epsilon > 0$ and pick $m$ large enough so that

$$\|\langle y_{m+k+1} \rangle - f_{m+k} \circ \cdots \circ f_k(\langle y_k \rangle)\| < \frac{\epsilon}{3}$$

and from (27),

$$\|f_{m+k} \circ \cdots \circ f_k(y) - P_X(y)\| < \frac{\epsilon}{3}$$

for all $y \in$ a closed ball containing $\{\langle y_k \rangle\}$ along the chosen subsequence. Along the same subsequence, choose $k$ large enough so that

$$\|\langle y_k \rangle - \langle y^* \rangle\| < \frac{\epsilon}{3}.$$
Combining and using non-expansivity of projection, we have

\[
\|y_{m+k+1} - P_X(\langle y^* \rangle)\| \leq \|y_{m+k+1} - f_{m+k} \circ \cdots \circ f_1(\langle y_k \rangle)\| + \|f_{m+k} \circ \cdots \circ f_1(\langle y_k \rangle) - P_X(\langle y_k \rangle)\| \\
+ \|P_X(\langle y_k \rangle) - P_X(\langle y^* \rangle)\| \\
\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\]

The claim follows.

\[\square\]

\subsection*{B. Convergence of DSA-BDH}

The convergence analysis is similar to the previous case and Theorem 11 also holds for DSA-BDH. With a stacked vector notation, the main steps of the algorithm are:

\begin{align}
    x_k &= (Q \otimes I_n)\{x_{k-1} + P(z_k + y_k)\} - P(z_k + y_k), \quad (29) \\
    z_{k+1} &= z_k + b_k x_k, \quad (30) \\
    y_{k+1} &= (Q \otimes I_n)y_k + a_k(P(y_k + z_k) - y_k) + a_k(h(y_k) + M_{k+1}). \quad (31)
\end{align}

The consensus part for \(y_k\) is proved along the same lines as Lemma 12. All that is required to show is that \(\|P(y_k + z_k) - P_X(y_k)\| \to 0\), then the proof of Theorem 11 goes through with only slight modification. (Here \(P(y_k + z_k)\) performs the same job as \(z_k\) of DSA-GD, i.e., asymptotically track the projection of \(y_k\).) We proceed to show this now with the help of the following lemma proved in the appendix.

\textbf{Lemma 14.} \(\|x_k\|\) is bounded and

\[
\|x_k - ((Q \otimes I_n)P(z_k + y_k) - P(z_k + y_k))\| \to 0.
\]

Using Lemma 14, let us write (30) as

\[
z_{k+1} = z_k + b_k ((Q \otimes I_n)P(z_k + y_k) - P(z_k + y_k) + o(1)) \quad (32)
\]

Since \(a_k = o(b_k)\) in (31), (32), they constitute a two time scale iteration ([15] or [6], Chapter 6). So while analyzing (30), we can assume that \(y_k\) is \(\approx\) a constant, say \(y := [y, ..., y]\). Add \(y\) to both sides of (30) to obtain

\[
z_{k+1} + y = z_k + y + b_k x_k \\
\text{i.e.,} \quad r_{k+1} = r_k + b_k x_k \quad (33)
\]

with \(r_k = z_k + y\). Thus (29), (33) can be written as

\[
x_k = (Q \otimes I_n)\{x_{k-1} + P(r_k)\} - P(r_k), \quad (34)
\]

\[
r_{k+1} = r_k + b_k x_k \quad (35)
\]

This is exactly the distributed Boyle-Dykstra-Han projection algorithm in the variable \(r_k\) (see [16-17]).

\textbf{Lemma 15.} If \(r^*\) is any limit point of (33) (or equivalently, (30)) with \(r^* = z^*(y) + y\) for some \(z^*\) and a fixed \(y\), then

\[
P(r_k) \to P(r^*) = P_X(y).
\]
Proof. Since \( r_k = z_k + y \) and \( z_0 = 0 \), we have \( r_0 = y \). We first show that \( r_k \) remains invariant under averaging. From \([29]\),

\[
(Q^* \otimes I_n)x_{k+1} = (Q^* \otimes I_n)x_k = \cdots = (Q^* \otimes I_n)x_0 = 0.
\]

Multiply both sides of \([35]\) by \( Q^* \otimes I_n \) to get :

\[
(Q^* \otimes I_n)r_{k+1} = (Q^* \otimes I_n)r_k.
\]

By iterating we get

\[
(Q^* \otimes I_n)x_k = (Q^* \otimes I_n)x_0 = 0.
\]

That is, \( \frac{1}{N} \sum_i r_k^i \) remains a constant equal to \( y \) and since \( r^* \) is a limit point of \( \{r_n\} \), \( \frac{1}{N} \sum_i r_n^i = y \).

Furthermore, we have the existence of a point \( c \in \mathbb{R}^n \) such that (see Lemma 4.4, \([17]\))

\[
P(r^*) = \left[ P^1(r_1^*), \cdots, P^N(r_N^*) \right] = [c, \cdots, c]
\]

\[
\frac{1}{N} \sum_i r_n^i = \frac{1}{N} \sum_k r_k^i = y,
\]

Since \( P^i(r_n^i) = c \) \( \forall i \) we have \( c \in \mathcal{X} = \cap_i \mathcal{X}_i \). This in turn implies for all \( i \), \( P^i(\mathcal{X}) = c \) because \( \mathcal{X} \subset \mathcal{X}_i \). Hence for all \( i \), \( r_n^i \) lie in the normal cone at the point \( c \). Hence so does \( y \). This means that \( c = P^i(\mathcal{X})(y) \) which proves the claim.

For the original algorithm, this translates into:

**Corollary 16.** \( \|P(r_k) - P_{\mathcal{X}}(y_k)\| \to 0 \)

As in Lemma \([13]\) this is a direct consequence of Lemma 1, Chapter 6, \([3]\). The rest of the analysis is exactly the same as for DSA-GD and is therefore omitted.

## 4 Stability

In this section we give a sufficient condition for the proposed algorithms to satisfy Assumption 2(iv) (i.e., have a.s. bounded iterates).

### A. Stability of DSA-GD :

Boundedness of \( z_k \) is obvious because it is projected onto a compact set at every step. To establish stability of \( y_k \) we adapt a stability test from \([6]\), Chapter 3, which was originally proposed in \([7]\).

Let \( h_i^e(y^i) := \frac{h^i(cy^i)}{e} \). Consider the following scaling limit for each \( i \), assumed to exist:

\[
h_i^\infty(y^i) := \lim_{e \to \infty} \frac{h^i(cy^i)}{e}.
\]

Note that the \( h_i^\infty \)'s have a common Lipschitz constant and hence are equicontinuous, implying that the above convergence is uniform on compact sets. Suppose for each \( i \) the following limiting ODE has the origin as the unique globally asymptotically stable equilibrium :

\[
y^i(t) = h_i^\infty(y^i(t)) - y^i(t).
\]
Then if we let $H_c(y) := \frac{1}{N} \sum_{i=1}^{N} h^i_{\Delta}(y^i) = \frac{1}{N} \sum_{i=1}^{N} h^i_{\Delta}(c y^i)$, $c \geq 1$, $y \in \mathbb{R}^{nN}$, it will satisfy

$$H_c(y) \to H_\infty(y) := \frac{1}{N} \sum_{i=1}^{N} h^i_{\infty}(y^i) \text{ as } c \to \infty$$

uniformly on compacts. Consider the following limiting ODE in $y(t) = [y^1(t), \ldots, y^N(t)]$, which will have the origin ($0 \in \mathbb{R}^{nN}$) as the unique globally asymptotically stable equilibrium:

$$\dot{y}(t) = h_\infty(y(t)) - y(t)$$

$$\implies (1^T \otimes I_n) \dot{y}(t) = (1^T \otimes I_n)\{h_\infty(y(t)) - y(t)\}$$

$$\implies \frac{1}{N} \sum_{i=1}^{N} \dot{y}^i(t) = \frac{1}{N} \sum_{i=1}^{N} \{h^i_{\infty}(y^i(t)) - y^i(t)\}.$$ 

If there is consensus so that $y^i(t) = y^j(t) = \langle y(t) \rangle$, we can write the above as

$$\langle \dot{y}(t) \rangle = H_\infty(\langle y(t) \rangle) - \langle y(t) \rangle. \quad (36)$$

Again, (36) has the origin ($0 \in \mathbb{R}^n$) as the unique globally asymptotically stable equilibrium. Let $T_0 = 0$ and for $n \geq 0$, $T_{n+1} = \min\{t_m : t_m > T_n + T\}$ where $T > 0$ and $t_m = \sum_{k=0}^{m} a_k$. Without loss of generality, let $\sup_k a_k \leq 1$. Then we have $T_{n+1} \in [T_n + T, T_n + T + 1] \ \forall n$. Also, we write $T_n = t_{m(n)}$ for a suitable $m(n)$. Let $y(t) = [y^1(t), \ldots, y^N(t)]$ define a continuous, piecewise linear trajectory linearly interpolated between $y(t_k) := y_k$ and $y(t_{k+1}) := y_{k+1}$ on $[t_k, t_{k+1}]$.

We construct another piecewise linear trajectory $\hat{y}(t)$ derived from $y(t)$ by setting $\hat{y}(t) = \frac{y(t)}{r_n}$ for $t \in [T_n, T_{n+1})$, where $r_n = \max_{i=1, \ldots, N}(\|y^i(T_n)\|, 1)$. This implies in particular that

$$\hat{y}^i(T_n) \leq 1 \ \forall n, i. \quad (37)$$

Let $y^\infty(t)$ denote a generic solution to the equation (36) and $\hat{y}^\infty_n(t)$ denote its solution which starts at the point $\langle \hat{y}(T_n) \rangle$. For later use, we let $\hat{y}(T_{n+1}) = y(T_{n+1})/r_n$. The following lemma is from [3]:

**Lemma 17.** (i) $\exists T > 0$ such that for all initial conditions $y$ on the unit sphere, $\|y^\infty(t)\| < \frac{1}{8}$ for all $t > T$.

(ii) The sequence $\xi_k = \sum_{p=0}^{k-1} a_p \hat{M}_{p+1}, k \geq 1$ with $\hat{M}_\ell = \frac{M_\ell}{r_n}$ for $m(n) \leq \ell < m(n + 1)$ is square integrable and a.s. convergent.

(iii) The sequence $\xi_k = \sum_{p=0}^{k-1} a_p A_p \hat{M}_{p+1}, k \geq 1$ with $\{A_p\}_{p \geq 0}$ being a sequence of doubly stochastic matrices is a.s. convergent.

**Proof.** The proof of (i) is the same as in Chapter 3, Lemma 1, [3], pp. 22-23, whereas (ii) is proved in Chapter 3, Lemma 5, [6], p. 25. The latter lemma establishes and uses the fact

$$\sum_{p=0}^{\infty} a^2_p \mathbb{E}[\|\hat{M}_{p+1}\|^2 | \mathcal{F}_p] < \infty.$$ 

To prove (iii), note that multiplication by a linear operator doesn’t affect the martingale property. Also,

$$\sum_{p=0}^{\infty} a^2_p \mathbb{E}[\|A_p \hat{M}_{p+1}\|^2 | \mathcal{F}_p] \leq \sum_{p=0}^{\infty} a^2_p \mathbb{E}[\|\hat{M}_{p+1}\|^2 | \mathcal{F}_p] < \infty \text{ a.s.}$$

Convergence follows from the martingale convergence theorem (Appendix C, [6]).
The following theorem proves that the iterates $y_k$ remain bounded. Since the proof is an adaptation of the arguments of [17] or [18], Chapter 2, we give only a sketch that highlights the significant points of departure.

**Theorem 18.** $\sup_k \|y_k\| < \infty$.

**Proof.** (Sketch) Suppose that $\|y'(t)\| \to \infty$ for some $i$ along a subsequence. We obtain a contradiction using the following argument:

**Claim:** $\lim_{n \to \infty} \sup_{t \in [T_n, T_{n+1}]} \|\dot{y}(t) - \bar{y}_{n}^{\infty}(t)\| = 0 \text{ a.s.}$

**Proof.** For $m(n) < k < m(n+1)$, we have, on dividing both sides of (14) by $r_n$,

$$
\dot{y}(t_{k+1}) = (Q \otimes I_n)\dot{y}(t_k) + a_k(h_{r_n}(\dot{y}(t_k)) - \bar{y}_k(t) + \bar{M}_{k+1} + \epsilon_k),
$$

where $\epsilon_k = \frac{z_{k}}{r_n}$. Since $z_k$ is bounded, $\epsilon_k \to 0$ \forall $n \to \infty$ (since $r_n \to \infty$ by assumption). Iterating the above equation we get,

$$
\dot{y}(t_{m(n)+k}) = (Q^k \otimes I_n)\dot{y}(t_{m(n)}) + \sum_{i=0}^{k-1} a_{m(n)+i}(Q^{k-i-1} \otimes I_n)\{h_{r_n}(\dot{y}(t_{m(n)+i})) - \bar{y}(t_{m(n)+i}) + \bar{M}_{m(n)+i+1} + \epsilon_{m(n)+i}\}.
$$

Taking norms in the above we have:

$$
\|\dot{y}(t_{m(n)+k})\| \leq \|\dot{y}(t_{m(n)})\| + \sum_{i=0}^{k-1} a_{m(n)+i}\{\|h_{r_n}(\dot{y}(t_{m(n)+i})) - \bar{y}(t_{m(n)+i}) + \epsilon_{m(n)+i}\| + \|h_{r_n}(0)\| + L\|\dot{y}(t_k)\| + \|\dot{y}(t_k)\|
$$

Let $L$ be a common Lipschitz constant for the functions $h_{r_n}(\cdot)$ (which in fact is the same as that for $h$). Then we have the following bound on $h_{r_n}(\cdot)$:

$$
\|h_{r_n}(\dot{y}(t_k))\| \leq \|h_{r_n}(0)\| + L\|\dot{y}(t_k)\|
$$

Using the above in (39), we have:

$$
\|\dot{y}(t_{m(n)+k})\| \leq \|\dot{y}(t_{m(n)})\| + \sum_{i=0}^{k-1} a_{m(n)+i}\{(L + 1)\|\dot{y}(t_{m(n)+i})\| + \epsilon_{m(n)+i}\| + \|h_{r_n}(0)\| + L\|\dot{y}(t_k)\| + \|\dot{y}(t_k)\|
$$

We first prove that $\sup_k \sum_{i=0}^{k-1} a_{m(n)+i}(Q^{k-i-1} \otimes I_n)\bar{M}_{m(n)+i+1} < \infty$. Define the sequence $\{\xi_p\}$ as:

$$
\xi_p = \epsilon_{m(n)} + \sum_{i=0}^{p-m(n)-1} a_{m(n)+i}(Q^{p-m(n)-i-1} \otimes I_n)\bar{M}_{m(n)+i+1} \text{ if } m(n) < p \leq m(n+1)
$$

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with \( \hat{\xi}_0 = 0 \) and

\[
\hat{\xi}_{m(n)} = \sum_{j=0}^{n-1} \left( \sum_{i=0}^{m(j+1)-m(j)-1} a_{m(j)+i} (Q^{m(j+1)-m(j)-i-1} \otimes I_n) \hat{M}_{m(j)+i+1} \right)
\]

The sequence \( \{\hat{\xi}_p\} \) is convergent by Lemma [17](iii). Then

\[
\| \sum_{i=0}^{k-1} a_{m(n)+i} (Q^{k-i} \otimes I_n) \hat{M}_{m(n)+i+1} \| = \| \hat{\xi}_{m(n)+k} - \hat{\xi}_{m(n)} \| := B_n,
\]

where \( B_n \to 0 \) a.s. Also \( \sum_{0 \leq p \leq m(n+1)-m(n)} a_{m(n)+p} \leq T + 1 \), so that

\[
\| \dot{y}(t_{m(n)+k}) \| \leq \| \dot{y}(t_{m(n)}) \| + C + (L + 1) \sum_{i=0}^{k-1} a_{m(n)+i} \| \dot{y}(t_{m(n)+p}) \|,
\]

where \( C \geq (T + 1) \| h(0) \| + B_n + \sum_{i=0}^{m(n+1)-m(n)} a_{n+i} \| \epsilon_{k+i} \| \) is a random constant. The last term is finite a.s. because the \( \epsilon_k \) term goes to zero as stated earlier. Since \( \| \dot{y}(t_{m(n)}) \| \leq 1 \) by (37), we have

\[
\| \dot{y}(t_{m(n)+k}) \| \leq (C + 1) + (L + 1) \sum_{i=0}^{k-1} a_{m(n)+i} \| \dot{y}(t_{m(n)+p}) \|,
\]

By discrete Gronwall inequality, we have

\[
\sup_{0 \leq k \leq m(n+1)-m(n)} \| \dot{y}(t_{m(n)+k}) \| \leq [C + 1] \exp\{(L + 1)(T + 1)\} = K^*.
\]

(40)

Since the bound is independent of \( n \), we have that

\[
\| \dot{y}(t_{m(n)+1}) \| < \infty.
\]

(41)

Now consider (38) again:

\[
\dot{y}(t_{k+1}) = (Q \otimes I_n) \dot{y}(t_k) + a_k (h_{r_n}(\dot{y}(t_k)) - \dot{y}(t_k) + \dot{M}_{k+1} + \epsilon_k).
\]

Multiply it on both sides by \( \frac{1}{N} (I^T \otimes I_n) \) to get

\[
\langle \dot{y}(t_{k+1}) \rangle = \langle \dot{y}(t_k) \rangle + a_k \left\{ \frac{1}{N} \sum_{i=1}^{N} \hat{h}_{r_n}^{i} (\dot{y}^i(t_k)) - \langle \dot{y}(t_k) \rangle + \langle \dot{M}_{k+1} \rangle + \langle \epsilon_k \rangle \right\}
\]

\[
= \langle \dot{y}(t_k) \rangle + a_k \left\{ H_{\infty}(\langle \dot{y}(t_k) \rangle) - \langle \dot{y}(t_k) \rangle + \delta_{k}^{1} + \delta_{k}^{2} + \langle \dot{M}_{k+1} \rangle + \langle \epsilon_k \rangle \right\}.
\]

(42)

where

a) \( \delta_{k}^{1} = \frac{1}{N} \sum_{i=1}^{N} \hat{h}_{r_n}^{i} (\dot{y}^i(t_k)) - \frac{1}{N} \sum_{i=1}^{N} h_{r_n}^{i} (\langle \dot{y}^i(t_k) \rangle) \). Since \( \{\langle \dot{y}_k \rangle\} \) is bounded by (41) we can adapt the arguments of Lemma [12](ii) of Section 3 to show that we achieve consensus and hence \( \| \delta_{k}^{1} \| \to 0 \): For any \( m(n) < k < m(n + 1) \) we have from (38):

\[
\dot{y}(t_{m(n)+k}) = (Q \otimes I_n) \dot{y}(t_{m(n)+k-1}) + a_{m(n)+k-1} \dot{Y}_{m(n)+k-1},
\]

where

\[
\dot{Y}_{m(n)+k-1} = h_{r_n} (\dot{y}(t_{m(n)+k-1})) - \dot{y}(t_{m(n)+k-1}) + \dot{M}_{m(n)+k} + \epsilon_{m(n)+k-1}.
\]
Iterating this equation we get,
\[ \hat{y}(t_{m(n)+k}) = (Q^K \otimes I_n)\hat{y}(t_{m(n)}) + \{\Gamma(\hat{Y}_{m(n)+k-1}, \ldots, \hat{Y}_{m(n)})\}. \]

where \( \Gamma(\cdot) \) is defined as in Lemma 12 and is of order \( \mathcal{O}(\sum_{i=m(n)}^{m(n)+k-1} a_i) \) because \( \hat{y}(\cdot) \) (and hence \( \hat{Y} \)) is bounded. Then as in Lemma 12 we have,
\[ \|\hat{y}(t_{m(n)+k}) - (Q^k \otimes I_n)\hat{y}(t_{m(n)})\| = \mathcal{O}(\beta^{-k}) + \mathcal{O}(\sum_{i=m(n)}^{m(n)+k-1} a_i), \]

where the \( \mathcal{O}(\beta^{-k}) \) term is uniform w.r.t. \( n \). Taking the limit \( n \to \infty \) (which means \( m(n) \to \infty \)) followed by \( k \to \infty \), we get that \( \|\hat{y}(\cdot) - \langle \hat{y}(\cdot)\rangle\| \to 0 \) and hence \( \|\delta^k\| \to 0 \).

b) \( \delta^k_2 = N \sum_{i=1}^{N} h_i^t (\langle \hat{y}(t_k)\rangle) - \frac{1}{N} \sum_{i=1}^{N} h_i^t (\langle \hat{y}(t_k)\rangle) \). Since \( r_n \to \infty \), we have by assumption \( H_{r_n} \to H_{\infty} \) uniformly on compact sets, so that by \( 41 \), \( \|\delta^k_2\| \to 0 \).

Now we can use standard arguments, e.g., of Lemma 1 and Theorem 2, Chapter 2, of \( 6 \), pp. 12-15, which show that \( 42 \) has the same asymptotic behavior as the ODE \( 36 \) a.s., which proves the claim.

The above claim gives the contradiction we require. Suppose without loss of generality that \( \|y(T_n)\| > 1 \) along the above subsequence, which we denote by \( \{n\} \) again by abuse of notation. Then \( r_n \to \infty \) and we have a sequence \( T_{n_1}, T_{n_2}, \ldots \) such that \( \|y(T_{n_k})\| \uparrow \infty \), i.e., \( r_{n_k} \uparrow \infty \). We have \( \|y(T_{n})\| = \|y^\infty_n(T_{n})\| \leq 1 \) and by Lemma 17 (i) we may take \( \|y^\infty_n(T_{n+1})\| < \frac{1}{2} \) since \( T_{n+1} > T + T_n \). So by the above claim there exists an \( N' \) such that for all \( n > N' \), we get \( \|\langle y(T_{n+1})\rangle\| < \frac{1}{4} \). Then for all sufficiently large \( n \),
\[ \frac{\|\langle y(T_{n+1})\rangle\|}{\|\langle y(T_n)\rangle\|} = \frac{\|\langle y(T_{n+1})\rangle\|}{\|\langle y(T_n)\rangle\|} < \frac{1}{4}. \]

We conclude that if \( \|y(T_n)\| > 1, \|y(T_k)\| \) for \( k \geq n \) falls back to the unit ball at an exponential rate. Thus if \( \|y(T_{n})\| > 1, \|y(T_{n-1})\| \) is either even greater than \( \|y(T_n)\| \) or inside the unit ball. Thus there must exist an instance prior to \( n \) when \( y(\cdot) \) jumps from inside the unit ball to a radius of \( 0.9 r_n \). Then we have a sequence of jumps of \( y(T_n) \), corresponding to the sequence \( r_{n_k} \to \infty \), from inside the unit ball to points increasingly far away from the origin. But, by a discrete Gronwall argument analogous to the one used in the above claim, it follows that there is a bound on the amount by which \( y(\cdot) \) can increase over an interval of length \( T + 1 \) when it is inside the unit ball at the beginning of the interval. This leads to a contradiction, implying \( \hat{C} = \sup_n \|y(T_n)\| < \infty \). This implies by \( 40 \) that \( \sup_n \|y_n\| \leq \hat{C} \). \( K^* < \infty \).

B. Stability of DSA-BDH : The proof that \( y_k \) remains stable for DSA-BDH is identical to Theorem 18. The stability of \( z_k \) and \( x_k \) can be handled as in \( 17 \) by minor modifications of the arguments therein. The proof uses a routine ODE approximation technique and we skip it here as it is quite lengthy.

5 Numerical Experiment

In this section, we present some numerical results to validate the proposed algorithms. We demonstrate the results on a stochastic optimization problem known as the stochastic utility
problem. We consider this problem for a max function with linear arguments as the objective and the unit simplex as the constraint set (Section 4.2, [27]):

\[
\min_{y \in \mathcal{X}} \left\{ \mathbb{E}[F(y, \xi)] = \mathbb{E} \left[ \phi \left( \sum_{i=1}^{n} \left( \frac{i}{n} + \xi(i) \right) y(i) \right) \right] \right\}
\]

\[\mathcal{X} = \{ y \in \mathbb{R}^n : y(i) \geq 0 \forall i, \sum_{i=1}^{n} y(i) = 1 \}, \ \xi(i) \in \mathcal{N}(0,1)\]

where \( y = [y(1), \ldots, y(n)] \) and \( \phi(t) = \max\{v_1 + s_1 t, \ldots, v_m + s_m t\} \) with \( v_k \) and \( s_k \) being constants. Also, \( \xi(i) \) are independent zero mean Gaussian random variables with unit standard deviation. The constants \( v_k \) and \( s_k \) are generated from a uniform distribution with \( m = 10 \). We test the DSA-GD algorithm with this setup for \( N = 10, 20 \) and 30, where \( N \) denotes the number of constraints (equal to \( n + 1 \)) and hence the number of nodes we require to do the projection. If we put the constraints in the intersection form \( \bigcap_i \mathcal{X}_i \), then \( \mathcal{X}_i \) will be:

\[\mathcal{X}_i = \begin{cases} \{ y \in \mathbb{R}^n : y(i) \geq 0 \}, & \text{for } 1 \leq i \leq N - 1, \\ \{ y \in \mathbb{R}^n : \sum_{i=1}^{N} y(i) \geq 1 \}, & \text{for } i = N. \end{cases}\]

Node \( i \) is assigned the constraint set \( \mathcal{X}_i \) in order to do a distributed projection. The matrix \( Q \) is generated using Metropolis weights\(^2\) and for \( N = 10 \) is explicitly given as:

\[
Q = \begin{bmatrix}
0.25 & 0.25 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0.25 \\
0.25 & 0.4167 & 0.333 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.333 & 0.333 & 0.333 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.333 & 0.333 & 0.333 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.333 & 0.333 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.333 & 0.4167 & 0.25 & 0 \\
0.25 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0.25 & 0.25 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0.4167 & 0.333 \\
0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.333 & 0.4167 
\end{bmatrix}
\]

The above matrix is consistent with Assumption 1. The time steps employed are \( b_k = \frac{1}{m + 1} \) and \( a_k = \frac{1}{m + 1} \). Also, \( h'(y^t, \xi) = \partial F(y^t, \xi) \), where \( \partial F \) denotes the sub-gradient. Being a sub-gradient descent, the problem does not satisfy the regularity hypotheses imposed in our analysis above, nevertheless the proposed schemes work well as we show below.

We plot our results in Figures 1, 2 and 3:

- Figure 1 shows the plot of the optimality error vs. the number of iterations. The optimality error is the difference \( \|y_k^1 - y_1^*\| \), where \( y_1^* \) is the output after running the algorithm long enough (\( k > 10^4 \)) and the error tolerance \( \frac{\|y_{k+1}^1 - y_k^1\|}{\|y_k^1\|} \) is sufficiently small. As expected, the number of iterations required increases with the dimension of the problem.

- Figure 2 shows the feasibility error, \( \|y_k^1 - P_{\mathcal{X}}(y_k^1)\| \), against the iteration count.

- Figure 3 shows the disagreement estimate, \( \|y_i^k - y_j^k\| \), between the various agents for \( i, j = 1, 2, 3, 4 \).

\(^2\)The code to generate it is borrowed from [28]
Figure 1: Optimality Error vs. Iteration Count

Figure 2: Feasibility Error vs. Iteration Count
Appendix

The proof of Lemma 14 is along the same lines as in [17] and we provide it here for sake of completeness. Recall the condition (see eq. (5)) on $b_k$: For any $\epsilon > 0$, there exists an $\alpha \in (1, 1 + \epsilon)$ and some $k_0$ such that

$$\alpha b_{k+1} \geq b_k, \ \forall k > k_0. \quad (43)$$

Proof. For any $n$ and $k$ in (29), we have

$$x_{n+k} = (Q \otimes I_n)x_n - P(r_{n+k}) + \sum_{i=1}^{k-1} (Q^i \otimes I_n)\{P(r_{n+k-i+1}) - P(r_{n+k-i})\} + (Q^k \otimes I_n)P(r_{n+1}).$$

Setting $r_k = z_k + y_k$ and iterating the above equation we get,

$$x_{n+k} = (Q^k \otimes I_n)x_n - P(r_{n+k}) + \sum_{i=1}^{k-1} (Q^i \otimes I_n)\{P(r_{n+k-i+1}) - P(r_{n+k-i})\} + (Q^k \otimes I_n)P(r_{n+1})$$

Using the fact that $(Q^* \otimes I_n)x_n = 0$ and adding the telescopic sum inside the curly brackets, we have:

$$x_{n+k} = ((Q^k - Q^*) \otimes I_n)x_n - P(r_{n+k})$$

$$+ \sum_{i=1}^{k-1} (Q^i \otimes I_n)\{P(r_{n+k-i+1}) - P(r_{n+k-i})\} + (Q^k \otimes I_n)P(r_{n+1})$$

$$- \sum_{i=1}^{k-1} (Q^* \otimes I_n)\{P(r_{n+k-i+1}) - P(r_{n+k-i})\} + (Q^* \otimes I_n)P(r_{n+k}) - (Q^* \otimes I_n)P(r_{n+1})$$

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\[
\Rightarrow x_{n+k} = ((Q^k - Q^*) \otimes I_n)(x_n + P(r_{n+1})) + (Q^* \otimes I_n)P(r_{n+k}) - P(r_{n+k}) \\
+ \sum_{i=1}^{k-1}((Q^i - Q^*) \otimes I_n)\{P(r_{n+k-i+1}) - P(r_{n+k-i})\}. \quad (44)
\]

Taking \( n = 0 \) in (44) and using \( \|P(r)\| \leq C < \infty \) and (2) to bound the norm of \((Q^i - Q^*) \otimes I_n\), we get
\[
\|x_k\| \leq \beta^{-k}\|x_0 + P(r_1)\| + 2C + 2\sum_{i=1}^{k-1}\kappa\beta^i. \quad (45)
\]

Since the RHS in the above is uniformly bounded, we have \( x_k \) uniformly bounded. Now consider (44) again:
\[
x_{n+k} - (Q^* \otimes I_n)P(r_{n+k}) + P(r_{n+k}) = ((Q^k - Q^*) \otimes I_n)(x_n + P(r_{n+1})) \\
\quad \quad + \sum_{i=1}^{k-1}((Q^i - Q^*) \otimes I_n)\{P(r_{n+k-i+1}) - P(r_{n+k-i})\}. \quad (46)
\]

(I): This can be bounded by using (2) and (45):
\[
\|(Q^k - Q^*) \otimes I_n)(x_n + P(r_{n+1}))\| \leq \kappa\beta^{-k}\|x_n + P(r_{n+1})\| = O(\beta^{-k})
\]

(II): To bound this term, we first consider (30) and add \( y_{k+1} \) on both sides of
\[
z_{k+1} + y_{k+1} = z_k + y_k + b_kx_k + a_k\{P(r_k) - y_k + h(y_k) + M_{k+1}\} + (Q \otimes I_n)y_k - y_k \\
\Rightarrow r_{k+1} = r_k + b_k\{x_k + \frac{a_k}{b_k}\{P(r_k) - y_k + h(y_k) + M_{k+1}\} + (Q \otimes I_n)y_k - y_k
\]
Since \( y_k \) is bounded (cf. Assumption 2(iv)) and \( x_k \) is bounded from (45),
\[
\|r_{k+1} - r_k\| \leq b_kM_{x,y} + \epsilon_k
\]
for some random constant \( M_{x,y} < \infty \) a.s, with \( \epsilon_k = \|(Q \otimes I_n)y_k - y_k\| \). We have, by assumption 43 on the step size \( b_k \), that there exists an \( \alpha \in (1, \beta) \) and \( k_0 \) such that
\[
\alpha^{n+k-i}b_{n+k-i} \leq \alpha^{n+k}b_{n+k} \quad \forall \ 1 \leq i \leq k - 1 \quad (47)
\]
Letting \( \beta = \frac{\beta}{\alpha} > 1, \)
\[
\|\sum_{i=1}^{k-1}((Q^i - Q^*) \otimes I_n)\{P(r_{n+k-i+1}) - P(r_{n+k-i})\}\| \leq \sum_{i=1}^{k-1}\kappa\beta^{-i}\|P(r_{n+k-i+1}) - P(r_{n+k-i})\| \\
\leq \sum_{i=1}^{k-1}\kappa\beta^{-i}\|r_{n+k-i+1} - r_{n+k-i}\| \\
\leq \sum_{i=1}^{k-1}\kappa\beta^{-i}(M_{x,y}b_{n+k-i} + \epsilon_{n+k-i}) \\
\leq \sum_{i=1}^{k-1}\kappa\beta^{-i}M_{x,y}b_{n+k} + \sum_{i=1}^{k-1}\kappa\beta^{-i}\epsilon_{n+k-i} \\
\leq \frac{\kappa}{\beta - 1}M_{x,y}b_{n+k} + \frac{\kappa}{\beta - 1}\epsilon_{n,k}
\]
where $\tilde{\epsilon}_{n,k} = \sup_{0 \leq i \leq k-1} \epsilon_{n+k-i} \to 0$ as $n \to \infty$. The lemma follows by substituting the above bounds on (I) and (II) in (46) and taking the limit $n \to \infty$ followed by $k \to \infty$ to obtain
\[
\|x_{n+k} - \{(Q^* \otimes I_n)P(r_{n+k}) - P(r_{n+k})\}\| \to 0
\]

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