The tetrahedral analog of Veneziano amplitude

Igor G. Korepanov

Abstract

In solv-int/9812016 it was shown that the Veneziano amplitude in string theory comes naturally from one of the simplest solutions of the functional pentagon equation (FPE). More generally, FPE is intimately connected with the duality condition for scattering processes. Here I find the amplitude that comes the same way from a solution of the functional tetrahedron equation, with the duality replaced by the local Yang–Baxter equation.

1 Introduction

It was shown in [1] that the famous Veneziano amplitude, from which all the string theory starts, comes naturally from one of the simplest solutions of the functional pentagon equation (FPE). More generally, FPE is intimately connected with the duality condition for scattering processes.

From the viewpoint of the theory of integrable models, FPE is a rather trivial equation whose solutions have transparent geometrical or group-theoretic meaning [1, section 5]. It looks natural to search for similar constructions with FPE replaced by the functional tetrahedron equation (FTE). As the relations between the pentagon and duality condition are like those between the tetrahedron and local Yang–Baxter equation (LYBE), the duality condition is likely to be replaced by LYBE.

In this paper, I find such FTE and LYBE solutions that are described by formulas very similar to those describing Veneziano amplitude in [1], including the fundamental property of Möbius invariance. They are what I mean by the tetrahedral analog of Veneziano amplitude.

2 A functional transformation for edge variables from refactorization equation

Consider the following “refactorization equation” for the product of three matrices:

\[
\begin{bmatrix}
a_1 & b_1 & 0 \\
c_1 & d_1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a_2 & 0 & b_2 \\
0 & 1 & 0 \\
c_2 & 0 & d_2
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & a_3 & b_3 \\
0 & c_3 & d_3
\end{bmatrix}
= ...
\]
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & a_3' & b_3' \\
0 & c_3' & d_3'
\end{pmatrix}
\begin{pmatrix}
a_2' & 0 & b_2' \\
0 & 1 & 0 \\
c_2' & 0 & d_2'
\end{pmatrix}
\begin{pmatrix}
a_1' & b_1' & 0 \\
c_1' & d_1' & 0 \\
0 & 0 & 1
\end{pmatrix},
\]  
(1)

\((a_1, \ldots, d_3')\) are numbers) for the case when all six submatrices \(\begin{pmatrix} a_i^{(i)} & b_i^{(i)} \\ c_i^{(i)} & d_i^{(i)} \end{pmatrix}\) have the form

\[
\begin{pmatrix}
a & b \\ c & d
\end{pmatrix} = \begin{pmatrix} \alpha & 1 - \alpha \\ 1 - \beta & \beta \end{pmatrix}.
\]  
(2)

In other words, each of the six matrices in (1) transforms the vector \(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\) into itself.

It is known from [2, 3, 4] that each side of (1) determines the other side to within some “gauge freedom”, and one can verify that the additional conditions (2) are exactly good for fixing that freedom.

The fate of an arbitrary vector \(\begin{pmatrix} p \\ q \\ r \end{pmatrix}\) under the action of both sides of (1) is more complicated. We present it in Figure 1, where we denote the matrices entering (1),

\[
\begin{array}{ccc}
X_1 & X_2 & X_3 \\
x & y & z \\
u & v & w \\
p & q & r
\end{array}
\quad \quad
\begin{array}{ccc}
Y_1 & Y_2 & Y_3 \\
x & y & z \\
f & g & h \\
p & q & r
\end{array}
\]

Figure 1:

in their order in that equation, by letters \(X_1, X_2, X_3, Y_3, Y_2, Y_1\). The meaning of the LHS of Figure 1 is that

\[
X_3 \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} p \\ v \\ w \end{pmatrix}, \quad X_2 \begin{pmatrix} p \\ v \\ w \end{pmatrix} = \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \quad X_1 \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},
\]

while the meaning of the RHS is that

\[
Y_1 \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} f \\ g \\ r \end{pmatrix}, \quad Y_2 \begin{pmatrix} f \\ g \\ r \end{pmatrix} = \begin{pmatrix} x \\ g \\ h \end{pmatrix}, \quad Y_3 \begin{pmatrix} x \\ g \\ h \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\]

One can see that if, vice versa, all the values \(x, y, z, \ldots\) in e.g. the LHS of Figure 1 are given, then matrices \(X_1, X_2, X_3\) of the form (2) are recovered unambiguously.
So, we can take some given values of nine numbers in the LHS, get the triple of matrices $X_1, X_2, X_3$ from them, then get $Y_1, Y_2, Y_3$ by (1), and then get the missing values $f, g, h$ in the RHS from $p, q, r$ using $Y_1, Y_2, Y_3$. We will formulate this the following way: for any fixed “outer” variables $x, y, z, p, q, r$, the transformation

$$R = R(x, y, z, p, q, r) : (u, v, w) \mapsto (f, g, h)$$

is given.

The transformations (3) satisfy the *functional tetrahedron equation* (FTE). To explain this, note that equation (1) can be naturally regarded as an equation in the direct sum of three one-dimensional complex linear spaces, each of the matrices acting nontrivially only in a direct sum of two of them. One can consider similar relations in a direct sum of *four* spaces (each of the matrices acting nontrivially again only in a direct sum of two spaces). Let us picture in Figure 2 the spaces as straight lines, put matrices at their intersections, and attach the results of matrix action upon some 4-vector to line segments like in Figure 1, and then consider the transition from the LHS of Figure 2 to its RHS as a composition of “elementary” transformations $R$ of type (3).

As was explained in the paper [4] (and the reader will verify it him-/herself easily), there exist two different compositions of four $R$s both transforming the LHS of Figure 1 in its RHS. The first of them starts with $R_{356}$, by which we mean “turning inside out” triangle 356, while the other—with $R_{123}$. We can write FTE in the same abstract form as in [4]:

$$R_{123} \circ R_{145} \circ R_{246} \circ R_{356} = R_{356} \circ R_{246} \circ R_{145} \circ R_{123},$$

but the sense of (4) is now different: $R$ is now a transformation of variables belonging to the *edges* rather than of matrices belonging to vertices.

To prove FTE (4) for edge variables, note that the variables belonging to *inner* edges (i.e., say, edges 12, 13, ..., 56 in the LHS of Figure 2) are unambiguously recovered if variables at *outer* edges and matrices at vertices are given. The FTE for matrices, according to [4], does hold, while the variables at outer edges are not changed by the transformations. Thus, the variables at inner edges do not depend on the way of transformations as well.

---

Figure 2:
3 Möbius invariance

The same way as we have traced the fate of vector \( \begin{pmatrix} p \\ q \\ r \end{pmatrix} \) under the action of LHS and RHS of (1) in Figure 1, we can trace the fate of two more vectors, namely

\[
\begin{pmatrix} p_n \\ q_n \\ r_n \end{pmatrix} = \kappa \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} p \\ q \\ r \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p_d \\ q_d \\ r_d \end{pmatrix} = \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \nu \begin{pmatrix} p \\ q \\ r \end{pmatrix},
\]

where \( \kappa, \lambda, \mu, \nu \) are some constants (and subscripts \( n \) and \( d \) stand for “numerator” and “denominator”, see formula (6) below). I do not draw here corresponding diagrams, differing from Figure 1 only in that \( n \) or \( d \) is added to all small letters.

Now let us do the following gauge transformations (in the sense of [2, 3, 4]) on matrices \( X_1, \ldots, Y_3 \):

\[
\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} \mapsto \begin{pmatrix} \tilde{a}_1 \\ \tilde{b}_1 \\ \tilde{c}_1 \\ \tilde{d}_1 \end{pmatrix} = \begin{pmatrix} x_d^{-1} & 0 \\ 0 & y_d^{-1} \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} \begin{pmatrix} u_d & 0 \\ 0 & v_d \end{pmatrix},
\]

\[
\begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} \mapsto \begin{pmatrix} \tilde{a}_2 \\ \tilde{b}_2 \\ \tilde{c}_2 \\ \tilde{d}_2 \end{pmatrix} = \begin{pmatrix} u_d^{-1} & 0 \\ 0 & z_d^{-1} \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} \begin{pmatrix} p_d & 0 \\ 0 & w_d \end{pmatrix},
\]

\[
\begin{pmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{pmatrix} \mapsto \begin{pmatrix} \tilde{a}_3 \\ \tilde{b}_3 \\ \tilde{c}_3 \\ \tilde{d}_3 \end{pmatrix} = \begin{pmatrix} v_d^{-1} & 0 \\ 0 & w_d^{-1} \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{pmatrix} \begin{pmatrix} q_d & 0 \\ 0 & r_d \end{pmatrix},
\]

\[
\begin{pmatrix} a'_1 \\ b'_1 \\ c'_1 \\ d'_1 \end{pmatrix} \mapsto \begin{pmatrix} \tilde{a}'_1 \\ \tilde{b}'_1 \\ \tilde{c}'_1 \\ \tilde{d}'_1 \end{pmatrix} = \begin{pmatrix} f_d^{-1} & 0 \\ 0 & g_d^{-1} \end{pmatrix} \begin{pmatrix} a'_1 \\ b'_1 \\ c'_1 \\ d'_1 \end{pmatrix} \begin{pmatrix} p_d & 0 \\ 0 & q_d \end{pmatrix},
\]

\[
\begin{pmatrix} a'_2 \\ b'_2 \\ c'_2 \\ d'_2 \end{pmatrix} \mapsto \begin{pmatrix} \tilde{a}'_2 \\ \tilde{b}'_2 \\ \tilde{c}'_2 \\ \tilde{d}'_2 \end{pmatrix} = \begin{pmatrix} x_d^{-1} & 0 \\ 0 & h_d^{-1} \end{pmatrix} \begin{pmatrix} a'_2 \\ b'_2 \\ c'_2 \\ d'_2 \end{pmatrix} \begin{pmatrix} f_d & 0 \\ 0 & r_d \end{pmatrix},
\]

\[
\begin{pmatrix} a'_3 \\ b'_3 \\ c'_3 \\ d'_3 \end{pmatrix} \mapsto \begin{pmatrix} \tilde{a}'_3 \\ \tilde{b}'_3 \\ \tilde{c}'_3 \\ \tilde{d}'_3 \end{pmatrix} = \begin{pmatrix} y_d^{-1} & 0 \\ 0 & z_d^{-1} \end{pmatrix} \begin{pmatrix} a'_3 \\ b'_3 \\ c'_3 \\ d'_3 \end{pmatrix} \begin{pmatrix} q_d & 0 \\ 0 & h_d \end{pmatrix}.
\]

Here, of course, \( x_d = \mu + \nu x \) etc., in analogy with \( p_d, q_d \) and \( r_d \) in (5).

Denote the so obtained matrices \( \bar{X}_1, \ldots, \bar{Y}_3 \). Now imagine a version of Figure 1 for these matrices with tildes. One can say that the transformation of vectors corresponding to the above gauge matrix transformation has brought all variables with subscript \( d \) into 1, and hence the matrices with tildes have again the form (2).

As for the variables with subscript \( n \), they turned into

\[
x \rightarrow \bar{x} = \frac{x_n}{x_d} = \frac{\kappa + \lambda x}{\mu + \nu x} \quad \text{etc.} \tag{6}
\]

We see from here that a linear-fractional (Möbius) transformation of variables \( x, y, \ldots \) commutes with the transformation \( R \). Clearly, the same conclusion could be made from the explicit formulas for \( R \) given in Section 4. The Möbius invariance is an argument in support of the idea that \( R \) really is an analog of “pentagonal” transformation from [1] connected with Veneziano amplitude.
4 Connection between volume elements and the explicit form of functional transformation

Let us now vary the edge variables in Figure 1, with matrices $X_1, \ldots, Y_3$ fixed. For instance, consider the variables at outer edges of the LHS of that Figure as functions of three inner variables $u, v, w$, and calculate the corresponding partial derivatives. The reader will easily check that

$$\frac{\partial x}{\partial u} = \frac{x - v}{u - v}, \quad \frac{\partial x}{\partial v} = \frac{x - u}{v - u}, \quad \frac{\partial y}{\partial u} = \frac{y - v}{u - v}$$

(7)

and so on.

Using formulas of the type (7), it is not hard to obtain the following relations for “volume elements”:

$$dx \wedge dy \wedge dz = \frac{x - y}{u} \frac{z - u}{w - u} du \wedge dv \wedge dw$$

(8)

from the LHS of Figure 1 and similarly

$$dx \wedge dy \wedge dz = \frac{x - h}{f - h} \frac{y - z}{g - h} df \wedge dg \wedge dh$$

(9)

from its RHS. The equalness of the RHSs of (8) and (9) can be called “the relation between $du \wedge dv \wedge dw$ and $df \wedge dg \wedge dh$ got via $dx \wedge dy \wedge dz$”.

Similarly, the equalness of the RHSs of relations

$$dy \wedge dz \wedge dp = \frac{y - u}{v - u} \frac{z - p}{w - u} du \wedge dv \wedge dw$$

(10)

and

$$dy \wedge dz \wedge dp = \frac{y - z}{g - f} \frac{p - g}{f - g} du \wedge dv \wedge dw$$

(11)

can be called “the relation between $du \wedge dv \wedge dw$ and $df \wedge dg \wedge dh$ got via $dy \wedge dz \wedge dp$”. There are four more pairs of relations of the type (8–11) with $dz \wedge dp \wedge dq$, $dp \wedge dq \wedge dr$, $dq \wedge dr \wedge dx$ and $dr \wedge dx \wedge dy$ respectively in their LHSs.

Certainly, one can exclude the differentials from those relations and obtain formulas giving explicitly the connection between edge variables, i.e. the transformation $R$, namely

$$\frac{x - y}{u - y} \frac{u - z}{p - z} = \frac{x - h}{f - h} \frac{f - g}{p - g},$$

(12)

$$\frac{y - x}{v - x} \frac{v - r}{q - r} = \frac{y - h}{g - h} \frac{g - f}{q - f},$$

(13)

$$\frac{z - p}{w - p} \frac{w - q}{r - q} = \frac{z - g}{h - g} \frac{h - f}{r - f},$$

(14)

$$\frac{x - v}{u - v} \frac{u - w}{p - w} = \frac{x - r}{f - r} \frac{f - q}{p - q},$$

(15)

$$\frac{y - u}{v - u} \frac{v - w}{q - w} = \frac{y - z}{g - z} \frac{g - p}{q - p},$$

(16)

$$\frac{z - u}{w - u} \frac{w - v}{r - v} = \frac{z - y}{h - y} \frac{h - x}{r - x}.$$  

(17)
5 Local Yang–Baxter equation

The local Yang–Baxter equation (LYBE) dealt with in this section differs from the conventional Yang–Baxter equation, first, in its continuous (instead of usual discrete) “set of colours” and, second (and this is what makes it “local”, or “twisted”), in that all six $R$-matrices (instead of which we will have, however, functions of 4 complex variables) entering it are different (in the usual Yang–Baxter equation, the LHS and RHS are made from the same 3 matrices, multiplied in different orders). Namely, our LYBE will have the following form (for real $x, y, \ldots$):

$$\int L(x, y, u, v) M(u, z, p, w) N(v, w, q, r) \, du \wedge dv \wedge dw = \int N'(y, z, g, h) M'(x, h, f, r) L'(f, g, p, q) \, df \wedge dg \wedge dh. \quad (18)$$

In the same way as the duality relation in [1], the equality (18) will hold if we require that the relation hold obtained from (18) by removing the integration signs, with the triples of variables $u, v, w$ and $f, g, h$ connected by some dependence. For such dependence, we will take the transformation $R$ from formula (3). Then, the following construction of functions $L, \ldots, N'$ can be proposed.

Take the relation

$$\frac{x - y}{u - v} \frac{z - u}{w - u} \, du \wedge dv \wedge dw = \frac{x - h}{f - h} \frac{y - z}{g - h} \, df \wedge dg \wedge dh \quad (19)$$

(see (8, 9)), and also the relations (12–17) raised in arbitrary degrees (the relations (12–17) are not independent, so one of those degrees can be set to zero). Then multiply separately the LHSs and RHSs of all so obtained relations (including (19)). The obtained LHS and RHS will be exactly the integrands in (18), and from them the multipliers $L, \ldots, N'$ depending on proper quadruples of variables are easily extracted.

I leave for further work the problem of possible choices of integration domains in (18) and integral regularization (if needed). The explicit form of functions $L, \ldots, N'$ will also be presented elsewhere. Let me just note that we can also regard all the variables $x, y, \ldots$ as complex. In such case, we should multiply the integrands (including differentials) by their complex conjugates and integrate over some domains of six real dimensions. This will be the tetrahedral analog of Virasoro–Shapiro amplitude.

6 Discussion

The LYBE of the form (18), as well as the duality equations from [1], are interesting because there exists a hope to construct from them interesting “exactly solvable” functional integrals, perhaps connected with 3-dimensional statistical physics. By the way, here I have presented the tetrahedral analog of one of two models in [1], and it seems fascinatingly interesting to construct the analog of the other model (and their generalizations). Very interesting will be also to clarify the relations between
pentagon and tetrahedron equations, where, despite the presence of the excellent work [5], many things are unclear.

Acknowledgements

I am grateful to Satoru Saito for valuable discussions during my stay at Tokyo Metropolitan University, in the course of which the idea started to revisit, from the modern integrability theory viewpoint, the algebraic structures from which string theory was born in its time. I am also grateful to Sergei Sergeev for many discussions on tetrahedron and pentagon equations. Finally, I am glad to thank Russian Foundation for Basic Research for its (mostly moral) support under grant no. 98-01-00895.

References

[1] I.G. Korepanov and S. Saito, Finite-dimensional analogs of string s ↔ t duality and pentagon equation, solv-int/9812016, accepted for publication in Theor. Math. Phys. (1999).

[2] I.G. Korepanov, A dynamical system connected with inhomogeneous 6-vertex model, Zapiski Nauch. Semin. POMI 215, 178–196 (1994), also available as hep-th/9402043.

[3] I.G. Korepanov, Integrable systems in discrete space–time and inhomogeneous models of two-dimensional statistical physics, Thesis for obtaining the “doktor nauk” scientific degree, 161 pp., S-Petersburg, 1995 (in Russian. English version is available as solv-int/9506003).

[4] R.M. Kashaev, I.G. Korepanov and S.M. Sergeev, Functional Tetrahedron Equation, Teor. Mat. Fiz. 117:3, 370–384 (1998), also solv-int/9801015.

[5] R.M. Kashaev and S.M. Sergeev, On pentagon, ten-term, and tetrahedron relations, Comm. Math. Phys. 195, 309–319 (1998), also q-alg/9607032.