Cooperation for interference management: A GDoF perspective

Soheyl Gherekhloo, Anas Chaaban, and Aydin Sezgin
Institute of Digital Communication Systems
Ruhr-Universität Bochum, Germany
Email: soheyl.gherekhloo@rub.de, anas.chaaban@kaust.edu.sa, aydin.sezgin@rub.de

Abstract

The impact of cooperation on interference management is investigated by studying an elemental wireless network, the so-called symmetric interference relay channel (IRC), from a generalized degrees of freedom (GDoF) perspective. This is motivated by the fact that the deployment of relays is considered as a remedy to overcome the bottleneck of current systems in terms of achievable rates. The focus of this work is on the regime in which the interference link is weaker than the source-relay link in the IRC. Our approach towards studying the GDoF goes through the capacity analysis of the linear deterministic IRC (LD-IRC). New upper bounds on the sum-capacity of the LD-IRC based on genie-aided approaches are established. These upper bounds together with some existing upper bounds are achieved by using four novel transmission schemes. Extending the upper bounds and the transmission schemes to the Gaussian case, the GDoF of the Gaussian IRC is characterized for the aforementioned regime. This completes the GDoF results available in the literature for the symmetric GDoF. It is shown that in the strong interference regime, in contrast to the IC, the GDoF is not a monotonically increasing function of the interference level.

I. INTRODUCTION

The demand for higher rates already exceeds today the supply of rates and thus resulting in a spectrum deficit. This is mainly due to an exclusive bandwidth allocation to each user. Hence, each user communicates over its own bandwidth without neither causing interference to other users nor be interfered by the other users. This is referred to as interference avoidance. In general, this approach is suboptimal and one possible solution is indeed to eliminate the constraint of exclusive bandwidth usage. Doing so results in the flexibility of spectrum usage, however it does not come for free. In more details, we obtain a system in which in addition to the additive Gaussian noise, users have to cope with the interference due to the concurrent transmissions and receptions. Thus, there is a need for sophisticated interference management, i.e., the handling of the communication traffic in a way such that the burden caused by the interference is limited. A key element to achieve this is to deploy additional nodes, referred to as relay nodes. The relay(s) then cooperate(s) with the transmitters and receivers and coordinate(s) in a distributed way the communication. The goal of this paper is to determine the fundamental limits of such a distributed coordination and cooperation.

To be more concrete, we add a causal, full-duplex relay node to a two user interference channel (IC) in which two transmitters want to communicate with their desired receiver. The obtained setup is known as an interference relay channel (IRC), which has been first introduced in [2]. Obviously, the achievable sum-rate of the IRC cannot be worse than that of the IC. However, in order to benefit from the relay, sophisticated relaying strategies, which mitigate the impact of interference, are required. In [3], an achievable sum-rate for the IRC based on compress-and-forward relaying strategy is studied. Moreover, special cases of IRC in which some channels are absent are studied in [4]–[6]. The IRC has been analyzed for the variant with a cognitive relay in [7]–[10]. While in [7], the capacity of the IRC with a cognitive relay has been derived for very strong interference, in [8] the capacity of this setup has been characterized within a constant gap for the case when no interference link is present. However, the capacity of IRC is still an open problem.

This work was presented in part at IEEE Allerton 2013 [1].
In order to obtain some progress on this research frontier, we study the so-called generalized degrees of freedom (GDoF) for the symmetric Gaussian IRC. This metric serves as an approximation of the capacity and has been used in several works. Essentially, it determines the number of available interference free streams in each channel use, where each stream has a capacity of a reference point-to-point channel (P2P). One of the most fundamental GDoF result is given in [11] in which the authors completed the GDoF characterization for the IC.

The GDoF of the symmetric Gaussian IRC has been characterized in [12] for the case when the source-relay channel is weaker than the interference channel. It has been shown that relaying can increase the GDoF performance of the IRC, although it was shown in [13] that relaying cannot increase the degrees of freedom (DoF). However, the GDoF of remaining regime, i.e., source-relay channel is stronger than the interference channel, was an open problem until now. In this work, we study the GDoF of the IRC for the complementary regime and settle this problem. Hence, this work completes the characterization of the GDoF for the symmetric Gaussian IRC.

To characterize the GDoF for the IRC, we consider first the deterministic model [14]. Essentially, the Gaussian IRC is modeled as a linear deterministic interference relay channel (LD-IRC) in which the relationship between inputs and outputs of the channel is deterministic. This makes analyzing the LD-IRC simpler than the Gaussian IRC. Generally, solving the capacity for the deterministic channel can lead to the GDoF for the original Gaussian channel. This has been shown for several setups such as IC [11] and X-channel [15]. In this work, first we characterize the capacity of the LD-IRC. Next, we extend the result to the Gaussian IRC and derive the GDoF of the Gaussian IRC.

To characterize the capacity of the LD-IRC, upper bounds and lower bounds for the capacity of the LD-IRC are needed. Besides two new upper bounds borrowed from [12], two cut-set bounds and four new upper bounds based on genie aided approach are used. All bounds are capacity-tight and required for the characterization of the capacity of the LD-IRC. On the other hand, four transmission schemes are introduced. The proposed schemes are based on rate splitting. The optimal rate splitting which achieves the upper bound is provided for each scheme. Hence, these schemes achieve the sum-capacity upper bounds in the whole regime in which the source-relay link is stronger than the interference link. The proposed schemes are combination of common and private signaling with different relaying strategies. While in previous work [12], only classical decode-and-forward [16] and compute-and-forward [17] were required to characterize the GDoF, in this work we need also cooperative interference neutralization [13] in addition to those relaying strategies. This is mainly due to the fact that in this work the source-relay link is stronger than the interference link and hence by providing some additional information to the relay which is not received at the destinations, the relay is able to neutralize the interference partially. Roughly speaking, using this relaying strategy, the causal relay is operating, to some extent, like a cognitive (Non-causal) relay.

As aforementioned, the upper bounds established in the LD-IRC are extended to the Gaussian model. They provide us upper bounds on the GDoF performance for the IRC. Next, we apply the proposed schemes for the LD-IRC and obtain lower bounds for the GDoF of Gaussian IRC.

The rest of the paper is organized as follows. In Section III we introduce the system model. The main results, i.e., the capacity of the LD-IRC and the GDoF of the Gaussian IRC, are presented in Section IV. The remainder of the paper is devoted to prove the results. The upper bounds on the capacity of the LD-IRC are presented in Section V. In Section VI the transmission schemes are presented for the LD-IRC. The upper bounds on the GDoF of the IRC are presented in Section VII. Finally, in Section VIII the extension of the transmission schemes to the Gaussian case is explained.

**Notation:** Throughout the paper, we use $\mathbb{F}_2$ to denote the binary field and $\oplus$ to denote the modulo 2 addition. We use normal lower-case, normal upper-case, boldface lower-case, and boldface upper-case letters to denote scalars, scalar random variables, vectors, and matrices, respectively, $X_{[a:b]}$ denotes the matrix formed by the $a$-th to $b$-th rows of a matrix $X$, and the vector $x_{[a:b]}$ is defined similarly. We write $X \sim \mathcal{N}(0,P)$ to indicate that the random variable $X$ is normal distributed with zero mean and variance $P$. Bern$(a)$ is a Bernoulli distribution with probability $a$. Furthermore, we define $x^n$ as the length-$n$ sequence $(x[1],\cdots,x[n])$. The vector $0_q$ denotes the zero-vector of length $q$, the matrix $I_q$ is the $q \times q$ identity matrix, the matrix $0_{l	imes m}$ represents the $l \times m$ zero matrix, and $x^T$ denotes the transposition of a vector $x$. Moreover, we define the functions $C(x)$
and \( C^+(x) \) as
\[
C(x) = \frac{1}{2} \log(1 + x), \quad C^+(x) = (C(x))^+, \tag{1}
\]
where \((x)^+ = \max\{0, x\}\). In this work, we suppose that all logarithms are binary unless the base of logarithm is given. We say that a set of random variables is i.i.d if its components are independent and identically distributed.

### II. System Model

We consider a network which consists of a two user interference channel with a causal and full-duplex relay as shown in Fig. [I]. The transmission is performed in \( n \) channel uses. While the transmitters are active in all channel uses, the relay is active in the last \( n - 1 \) channel uses due to the causality constraint.

Transmitter \( i \) (TX\( i \)), \( i \in \{1, 2\} \), has a message \( w_i \), which is a realization of a random variable \( W_i \) uniformly distributed over the set \( W_i \triangleq \{1, \ldots, [2^{nR_i}]\} \) for its respective receiver (RX\( i \)). The messages of the Tx’s are assumed to be independent from each other. Using an encoding function \( f_i \), the message \( w_i \) is encoded into a length \( n \) codeword \( x^n_i \in \mathbb{R}^n \), satisfying a power constraint
\[
\frac{1}{n} \sum_{k=1}^{n} E[x_i[k]^2] = P_i \leq P, \quad i \in \{1, 2\}. \tag{2}
\]

Then, the \( k \)th symbol \( x_i[k], k = 1, \ldots, n \), is transmitted in the \( k \)th channel use.

At the end of the \( k \)th channel use, the relay has \( y_r[k] \) which is given by
\[
y_r[k] = h_s x_1[k] + h_s x_2[k] + z_r[k], \tag{3}
\]
where \( h_s \) represents the real-positive channel gain value of the source-relay channel, and \( z_r[k] \) is a realization of an i.i.d. \( \mathcal{N}(0, 1) \) random variable which represents the additive white Gaussian noise (AWGN) at the relay. The relay re-encodes \( y_1[k], \ldots, y_r[k] \) using a function \( f_r \) into \( x_r[k+1] \), and sends the symbol \( x_r[k+1] \) in the \((k+1)\)th channel use. Since the transmit signal of the relay is generated from its received signal in previous channel uses, the relay is inactive in channel use \( k = 1 \), i.e., \( x_r[1] = 0 \). Furthermore, the relay signal satisfies the power constraint
\[
\frac{1}{n} \sum_{k=1}^{n} E[x_r[k]^2] = P_r \leq P. \tag{4}
\]

The destinations wait until end of the \( n \)th channel use, at which RX\( j \) has received the sequence \( y^n_j \), where \( y_j[k] \) is given by
\[
y_j[k] = h_d x_j[k] + h_c x_1[k] + h_r x_r[k] + z_j[k], \tag{5}
\]
where \( l \) is the index of the undesired Tx \((j \neq l)\) and \( h_d, h_c \) and \( h_r \) represent the real-positive channel gain value of the desired, interference and relay-destination channels, respectively, and the noise \( z_j \) is a realization of an \( \mathcal{N}(0, 1) \) random variable. The AWGN at the receivers and the relay are independent of each other. Moreover, it is assumed that all channel values are perfectly known at all nodes. By using a decoding function \( g_j \), RX\( j \) decodes \( \hat{w}_j \), i.e., \( \hat{w}_j = g_j(y^n_j) \). The messages sets, encoding functions, and decoding functions constitute a code for the channel which is denoted as a \( (n; 2^{nR_1}; 2^{nR_2}) \) code. Such a code induces an average error probability \( P^{(n)} \) defined as
\[
P^{(n)} = \frac{1}{2^{nR_2}} \sum_{w \in W_1 \times W_2} \text{Prob}(E), \tag{6}
\]
where \( R_2 = R_1 + R_2, \ w = (w_1, w_2) \), and \( E \) is the error event \( \hat{w}_i \neq w_i \) for some \( i \in \{1, 2\} \). Reliable communication is said to take place if this error probability can be made arbitrarily small by increasing \( n \). The achievability of a rate tuple \((R_1, R_2)\) is defined as the existence of a coding scheme which achieves reliable communication with these rates. In other words, a rate tuple \((R_1, R_2)\) is said to be achievable if there exists a sequence of \((n, 2^{nR_1}, 2^{nR_2})\) codes such that \( P^{(n)} \to 0 \) as \( n \to \infty \). The set of all achievable rate tuples is the capacity region of the IRC denoted by \( \mathcal{C} \). In this paper, we focus on the sum-capacity.
Fig. 1: System model of the symmetric Gaussian IRC.

defined as the maximum achievable sum-rate, i.e.,

\[
C_{\Sigma} = \max_{(R_1, R_2) \in \mathcal{C}} R_{\Sigma}. \tag{7}
\]

We consider the interference limited scenario, and hence, we assume that the received signal power from each node is larger than the noise variance, i.e.,

\[
\min\{h_d^2, h_c^2, h_s^2, h_r^2\} P > 1. \tag{8}
\]

For readability, we use the following parameters

\[
m_d = \frac{1}{2} \log(P h_d^2), \quad m_c = \frac{1}{2} \log(P h_c^2), \quad m_r = \frac{1}{2} \log(P h_r^2), \quad m_s = \frac{1}{2} \log(P h_s^2). \tag{9}
\]

Furthermore, since the focus of the paper is on the GDoF of the IRC, for a fixed \(h_d^2\) we define \(\alpha, \beta, \) and \(\gamma\) as

\[
\alpha = \frac{m_c}{m_d}, \quad \beta = \frac{m_r}{m_d}, \quad \gamma = \frac{m_s}{m_d}. \tag{10}
\]

Then, the GDoF of the Gaussian IRC, \(d(\alpha, \beta, \gamma)\) is defined as

\[
d(\alpha, \beta, \gamma) = \lim_{m_d \to \infty} \frac{C_{G, \Sigma}(\alpha, \beta, \gamma)}{m_d}, \tag{11}
\]

where \(C_{G, \Sigma}\) represents the sum-capacity of the Gaussian IRC. Our approach towards the GDoF analysis of Gaussian IRC starts with the linear-deterministic (LD) approximation of the wireless network introduced by Avestimehr et al. in [14]. Next, we introduce the linear deterministic IRC (LD-IRC).

A. Linear Deterministic Model

The Gaussian IRC shown in Fig. 1 can be approximated by the LD model as follows. An input symbol at Tx\(i\) is given by a binary vector \(x_i \in \mathbb{F}_2^q\) where \(q = \max\{n_d, n_c, n_r, n_s\}\) and the integers \(n_d, n_c, n_r,\) and \(n_s\) represent the channel strength and they are defined as follows

\[
n_d = \lfloor m_d \rfloor, \quad n_c = \lfloor m_c \rfloor, \quad n_r = \lfloor m_r \rfloor, \quad n_s = \lfloor m_s \rfloor. \tag{12}
\]

In the \(k\)th channel use, where \(k = 1, \ldots, n,\) the output signal vector \(y_r[k]\) at the relay is given by the following deterministic function of the inputs

\[
y_r[k] = S^{q-n_d} x_1[k] \oplus S^{q-n_s} x_2[k], \tag{13}
\]

\(^1\)For a fixed \(h_d^2,\) we can write \(m_d \to \infty\) is equivalent to \(P \to \infty.\)
where $S \in \mathbb{F}_2^{q \times q}$ is a down-shift matrix defined as

$$S = \begin{pmatrix} 0_q & 0 \\ I_{q-1} & 0_{q-1} \end{pmatrix}. \quad (14)$$

The relay generates a signal vector $x_r[k+1]$ at the end of the $k$th channel use from $y_r[1], \ldots, y_r[k]$ ($x_r[1] = 0_q$). The output signal vector $y_j[k]$ at Rx $j$ is given by the following deterministic functions of the inputs

$$y_j[k] = S^{q-n_d} x_j[k] \oplus S^{q-n_c} x_r[k], \quad (15)$$

where $j \neq l$. The input-output equations (13) and (15) approximate the input-output equations of the Gaussian IRC given in (3) and (5) in the high SNR regime. We denote the sum-capacity of the LD-IRC by $C_{\text{det,LD}}$.

In the next section, we present the main results of the paper, which are a complete characterization of the sum-capacity of the LD-IRC, and the approximate sum-capacity of the Gaussian counterpart. Therefore, in this paper we start by studying the LD-IRC, and in the process, we gather insights that are later used in the Gaussian case. The sum-capacity of the LD-IRC is summarized in the following subsection.

### III. Summary of the Main Results

The main results of the paper are the characterization of the sum-capacity of the LD-IRC, and the approximate sum-capacity of the Gaussian IRC given in terms of GDoF. The sum-capacity of the LD-IRC serves as a stepping stone towards the GDoF of the Gaussian counterpart. Therefore, in this paper we start by studying the LD-IRC, and in the process, we gather insights that are later used in the Gaussian case. The sum-capacity of the LD-IRC is summarized in the following subsection.

#### A. Sum-Capacity of the LD-IRC

The channel parameter space of the IRC can be split into two regimes, $n_s \leq n_c$ and $n_s > n_c$. The first regime corresponds to cases where the source-relay channels are weaker than the cross channels, i.e., $n_s \leq n_c$. The remaining regime was left open in (19). In this paper, we provide the complete solution of the problem by closing the whole regime where the source-relay channels are stronger than the cross channels, i.e., $n_s > n_c$. The sum-capacity in this regime is given in the following theorem.

**Theorem 1** (weaker source-relay channels LD-IRC (19)). The sum-capacity of the LD-IRC with $n_s \leq n_c$ is given by

$$C_{\text{det,LD}} = \min \left\{ \begin{array}{c} 2 \max\{n_d, n_r\} \\ 2 \max\{n_d, n_s\} \\ \max\{n_d, n_c, n_r\} + \max\{n_d, n_c\} - n_c \\ 2 \max\{n_d, n_c\} - n_c + n_s \\ 2 \max\{n_c, n_r, n_d - n_c\} \\ 2 \max\{n_c, n_d + n_s - n_c\} \end{array} \right\}. \quad (16)$$

The sum-capacity in this regime is given in the following theorem.

**Theorem 2** (stronger source-relay channels LD-IRC). The sum-capacity of the LD-IRC with $n_s > n_c$ is given by

$$C_{\text{det,LD}} = \min \left\{ \begin{array}{c} 2 \max\{n_d, n_r\} \\ \max\{n_d, n_c, n_r\} + \max\{n_d, n_c\} - (n_c - \max\{n_d, n_c\}^+) \\ n_r + 2 \max\{n_d, n_c\} - n_c \\ 2 \max\{n_s, n_r + n_s - n_c, n_d - n_c\} \\ \max\{n_d, n_c\} + \max\{n_d, n_s\} \\ 2 \max\{n_c, n_r + (n_d - n_c)^+\} \end{array} \right\} \quad \text{if } n_c \neq n_d, \text{ and by}$$

$$C_{\text{det,LD}} = \max\{n_d, \min\{n_r, n_s\}\} \quad (17)$$
The proof of this theorem for the special case $n_c = n_d$ is simple. In fact, the converse for this case follows from cut-set bounds, while its achievability follows by either ignoring the relay to achieve $n_d$, or using decode-forward at the relay to achieve $\min\{n_s, n_r\}$. The converse and achievability of the remaining case ($n_c \ne n_d$) are more involved, as they require genie-aided upper bounds, and transmission schemes which are based on different relay strategies. Details of the converse and achievability are presented in Sections IV and V, respectively.

B. GDoF Analysis of the Gaussian IRC

Using the capacity result to the LD-IRC and extending this for the Gaussian case, we obtain the GDoF characterization of the Gaussian IRC. Similar to above, the GDoF of the case where the source-relay channels are weaker than the cross channels was characterized in [12]. In the language of GDoF, this corresponds to $\gamma \le \alpha$. Basically, the GDoF in this case can be obtained from (16) by replacing $n_d, n_c, n_r$, and $n_s$ by $1, \alpha, \beta$, and $\gamma$, respectively. Using similar replacement in the statement of Theorem 2 gives the GDoF of the remaining case $\gamma > \alpha$. The following theorem presents this result.

**Theorem 3** (stronger source-relay channels Gaussian IRC). The GDoF of the IRC with $\gamma > \alpha$ is given by

$$d = \min \left\{ \begin{array}{l}
2 \max\{1, \beta\} \\
\max\{1, \alpha, \beta\} + \max\{1, \alpha\} - (\alpha - [\gamma - \max\{1, \alpha\}]^+) \\
\beta + 2 \max\{1, \alpha\} - \alpha \\
2 \max\{\gamma, \beta + \gamma - \alpha, 1 - \alpha\} \\
\max\{1, \alpha\} + \max\{1, \gamma\} \\
2 \max\{\alpha, \beta + (1 - \alpha)^+\} \\
\end{array} \right\} $$

if $\alpha \neq 1$, and by

$$d = \max\{1, \min\{\beta, \gamma\}\}$$

otherwise.

The proofs of the converse and achievability of this theorem are given in Sections VI and VII. The proofs are based on the insights obtained from the LD-IRC.

Before we continue, we recall the GDoF for the IC given in [11].

**Lemma 1** (ETW [11]). The GDoF for the IC is given by

$$d = \min \left\{ \begin{array}{l}
\max\{2 - 2\alpha, 2\alpha\} - 2 - \alpha, \quad \alpha \le 1 \\
\min\{\alpha, 2\}, \quad \alpha > 1.
\end{array} \right\} $$

C. Discussion

Before going into details of the proof of Theorems 2 and 3, we discuss the GDoF result and the obtained gain by using the relay. To this end, we study the GDoF for the IRC for different ranges of source-relay channel strength. First, consider the case where the source-relay channel is weak ($\gamma < 1$) and the relay-destination channel is weaker than the source-relay channel ($\beta < \gamma$). The GDoF for the IRC with $\gamma = 0.7$ and different relay-destination channel strength is illustrated in Fig. 2. As it is shown in this figure, the gain obtained by the relay is evident in the weak interference regime ($\alpha < 1$). Interestingly, despite the weak ingoing and outgoing links of the relay, the relay can increase the GDoF. While in Fig. 2(a) we cannot benefit from relay for $\alpha \le 1 - \gamma$, in Fig. 2(b) and 2(c) relay can increase the GDoF even for very small values of $\alpha$. It is worth noting that in the case shown in Fig. 2(c) the slop of GDoF is $-1$ with respect to $\alpha$ for $\alpha < \frac{2}{3}$, while in Fig. 2(a) the slop changes from $-2$ to 0 and then the GDoF of the IRC and that of the IC are parallel to each other. Now, consider the case shown in
Fig. 2: The GDoF of the IRC (solid-line) for different values of $\beta$ while $\gamma = 0.7$. The dashed line represents the GDoF of the IC. (a) $\beta = 0.1$, (b) $\beta = 0.4$, (c) $\beta = 0.7$.

Fig. 2(b) for values of $\alpha < \frac{2}{3}$. In this case, the slope of GDoF is $-1$ for $\alpha$’s in the beginning and the end of the interval $[0, \frac{2}{3}]$. However, in between, the GDoF behaviour is similar to that of shown in Fig. 2(a). This is due to the fact that in this case, we use a scheme which is a combination of the schemes used in other two extreme cases. To understand the GDoF behaviour shown in Fig. 2, we explain briefly the main idea of the transmission scheme. To benefit from the relay, we need to use superposition block Markov encoding [20], [21]. Using this encoding, some “future” signals are provided to the relay. These future signals can be used at the relay in the next channel uses. Hence, the relay can operate partially as a cognitive relay. For the case in which $\gamma < 1$, the future signal sent by Tx$i$ is received at relay and Rx$i$. However, by using backward decoding at the receiver side (details are provided in the following sections), Rx$i$ knows this future signal sent by Tx$i$ and thus, it removes the interference caused by that future signal. Doing this, some interference free dimension will be available at the Rx’s. This interference free dimension can be either used by the undesired Tx to send some common signal as in Han-Kobayashi scheme [22] or by the relay to provide some additional information to the receivers. While for the first choice (when common signals are provided), the interference channel has to be strong enough, for the second choice, the relay-destination link needs to be sufficiently strong. In the case shown in Fig. 2(a), the relay-destination channel is weak. Hence, relay cannot use this interference free dimension. However, for sufficiently strong interference, the interference free dimension will be accessible for common signaling. Thus, first for $1 - \gamma < \alpha$, we can benefit from the interference free dimension. On the other hand, in Fig. 2(c), the relay-destination channel is sufficiently strong. Hence, in this case, the interference free dimension will be used by the relay for providing some additional signal to the receivers. In this case, $\beta$ is so large that the relay can forward more information when the interference gets stronger as long as $\alpha < \frac{2}{3}$. Therefore, in whole regime $\alpha < \frac{2}{3}$, the GDoF performance of the IRC is higher than that of the IC. Now, consider the case shown in Fig. 2(b). In this case, the relay-destination channel is weaker than the previous case. Hence, the relay cannot benefit from increasing the interference channel in whole regime.
Fig. 3: The GDoF of the IRC (solid-line) for $\beta = 1.5$ and $\gamma = 0.7$. The dashed line represents the GDoF of the IC.

$\alpha < \frac{2}{3}$. Due to this, for very low values of $\alpha$ ($\alpha < 2(\beta + \gamma - 1)$), the GDoF behaviour is similar to the case shown in Fig. 2(c). On the other hand, when interference channel is strong enough ($\beta < \alpha$), we benefit from common signaling.

Generally, for each setup, there is a regime where the GDoF optimal scheme is to send only private signals. As long as the setup operates in this regime, the interference has to be completely ignored and the stronger the interference, the worse is the GDoF performance. This regime has been characterized for the IC in [11] and it is given by $\alpha \leq \frac{1}{2}$. As it is shown in Fig. 2 relaying shrinks this regime. In the extreme case, when relay-destination channel is sufficiently strong, this regime is completely vanished (cf. Fig. 2(c)).

Now, we consider the case that the source-relay channel is weaker than the direct channel ($\gamma < 1$) and the relay-destination channel is stronger than the direct channel ($1 < \beta$). The GDoF result for this case is shown in Fig. 3. In this case, the relay is so strong that it can forward some additional signals which are received at the receivers without any interference from other users as long as $\alpha < \beta$. This is shown in Fig. 3 where $\beta = 1.5$.

Finally, consider the case that the source-relay link $\gamma$ is strong. In Fig. 4 the GDoF for the IRC with $\gamma = 3$ is illustrated. In this case, the observation at the relay is so good that the relay performs as a cognitive relay. Therefore, the larger is the relay-destination channel, the more capable is the relay to neutralize the interference on air (see. Figures 4(a), 4(b)). In Fig. 4(c) the relay-destination link is large enough to neutralize the interference. However, at point $\alpha = \gamma/2$, the interference channel becomes so strong that the capacity of the source-relay channel is not sufficiently large for providing enough information to the relay. Hence, at this point the increase of the GDoF versus $\alpha$ stops. An interesting observation in Fig. 4(c) is that the GDoF can decrease versus the interference channel strength in strong interference regime. This is in contrast to the IC [11] and X-channel [15] with strong interference where the GDoF is a nondecreasing function of $\alpha$. The GDoF for the IRC for the case that the relay-destination channel is very strong is illustrated in Fig. 4(d). In this case, the bottleneck of the transmission will be the source-relay channel. This is shown in Fig. 4(d) where $\beta = 6$. In this case, the relay-destination channel is so strong that it is able to forward all its observation to the destinations without overlapping with other signals. In other words, we have a complete cooperation between relay and destinations since all received signal at the relay is available at the receivers. Due to this, the receiver is able to cancel the interference completely as long as the capacity of the source-relay channel is larger than the capacity of the interference channel ($\alpha < \gamma$).

Next, we discuss the LD-IRC. We start by presenting the upper bounds which provide the converse of Theorem 2.

IV. UPPER BOUNDS FOR THE LD-IRC

A standard bounding approach that can be applied for the IRC is the cut-set bound [23]. In addition to the cut-set bounds, further bounds that can be tighter than the cut-set bounds in some cases can be derived by using genie-aided methods. Those techniques are used in this paper for developing upper bounds that coincide with the statement of Theorem 2. These upper bounds are given in the following lemma.
Fig. 4: The GDoF of the IRC (solid-line) for $\gamma = 3$. The dashed line represents the GDoF of the IC. (a) $\beta = 0.2$, (b) $\beta = 1.5$, (c) $\beta = 2$, (d) $\beta = 6$.

Lemma 2. The sum-capacity of the LD-IRC is upper bounded by

$$C_{\text{det, } \Sigma} \leq \max\{n_d, \min\{n_r, n_s\}\} \quad \text{for } n_d = n_c$$  \hspace{1cm} (23)

$$C_{\text{det, } \Sigma} \leq \max\{n_d, n_c, n_e\} + \max\{n_d, n_c\}$$ \hspace{1cm} (24)

$$C_{\text{det, } \Sigma} \leq n_r + 2 \max\{n_d, n_c\} - n_c$$  \hspace{1cm} (25)

$$C_{\text{det, } \Sigma} \leq \max\{n_d, n_c\} + \max\{n_d, n_s\}$$ \hspace{1cm} (26)

$$C_{\text{det, } \Sigma} \leq 2 \max\{n_c, n_r, n_d - \max\{n_c, n_s\}\} + 2(n_s - n_c)^+$$ \hspace{1cm} (27)

$$C_{\text{det, } \Sigma} \leq 2 \max\{n_c, n_r + n_d - n_c\} \quad \text{for } n_r \leq n_c \leq n_d.$$ \hspace{1cm} (28)

Proof: Details of the proof of this lemma are given in Appendix A. Shortly, the first and second bounds are derived from the cut-set bounds. The remaining bounds are derived using genie-aided methods. The bounds (26) and (27) are tightened versions of the upper bounds given in [12, Theorems 3 and 4], tightened for the case where $n_c < n_s$. Finally, the bound (28) is inspired by a similar upper bound obtained for the IC [11, 24].

Note that similar to the X channel [15] and the $K$-user IC [25], the capacity of the LD-IRC has a discontinuity at $n_c = n_d$, i.e., if the cross channel is equal in strength to the direct channel.

In addition to these new bounds, some upper bounds are borrowed from [12, 19]. These bounds are given in the following lemma.

The bound given by $C_{\text{det-IC, } \Sigma} \leq 2 \max\{n_d - n_c, n_c\}$ for the LD-IC.
Lemma 3. (12) The sum capacity of the LD-IRC is upper bounded as follows

\[ C_{\text{det}, \Sigma} \leq 2 \max\{n_d, n_r\} \]  
(29)

\[ C_{\text{det}, \Sigma} \leq \max\{n_d, n_r, n_c\} + \max\{n_d, n_c\} - n_c + (n_s - \max\{n_d, n_c\})^+. \]  
(30)

The first of these bounds is in fact a cut-set bound, while the second is a genie-aided bound. The proof of these bounds can be found in [12].

Now, we need to show that the upper bounds (24)-(30) coincides with the capacity given in Theorem 2. First, it is clear that an upper bound is obtained by taking the minimum of all available upper bounds. At this point, in order to allow a direct comparison between the bounds, we need to get rid of the condition associated with the bound (28). First, we note that the first condition given by

\[ n_r \leq n_c \]

can be dropped since if this condition is not satisfied, i.e., if \( n_r > n_c \) with \( n_c \leq n_d \), then we have

\[
2 \max\{n_c, n_r + n_d - n_c\} = 2(n_r + n_d - n_c) \\
> n_r + 2n_d - n_c \\
= n_r + 2 \max\{n_d, n_c\} - n_c,
\]

which makes the bound (28) redundant given (25). Thus, we can write (28) as

\[ C_{\text{det}, \Sigma} \leq 2 \max\{n_c, n_r + n_d - n_c\} \text{ if } n_c \leq n_d. \]  
(31)

Furthermore, if \( n_c > n_d \), then by replacing \( n_r + n_d - n_c \) in this bound by \( n_r + (n_d - n_c)^+ \), we get the bound

\[ C_{\text{det}, \Sigma} \leq 2 \max\{n_c, n_r + (n_d - n_c)^+\}. \]  
(32)

This bound holds since

\[
2 \max\{n_c, n_r + (n_d - n_c)^+\} = 2n_r \\
> n_r + n_c \\
= n_r + 2 \max\{n_d, n_c\} - n_c,
\]

which shows that the bound (28) is redundant given (25) in this case. Thus, we can replace the bound (28) by (32).

Now, we can compare the upper bounds with the bounds in (17). The first term in (17) is (29). The second term in (17) is the minimum between (24) and (30). The third term is (25). The fourth term is (27) evaluated for \( n_c < n_s \). The fifth term is (26), and the last term is (32). Finally, the upper bound for the special case \( n_c = n_d \) (18) is given by (23).

This completes the proof of the converse of Theorem 2. Next, we propose a transmission scheme that achieves the sum-capacity of the LD-IRC.

V. SUM-CAPACITY ACHIEVING SCHEMES

The goal of this section is to introduce transmission schemes which achieve the sum-capacity in Theorem 2. To this end, different schemes are proposed to cover different operating regimes of the IRC. While only one scheme is enough to achieve the sum-capacity in the strong interference (SI) regime \( n_c > n_d \), three schemes are required to complete the characterization of the capacity in the weak interference (WI) regime \( n_c < n_d \). In addition, one simple scheme is required for the special case where the cross channels and the desired ones are equally strong \( n_d = n_c \). These schemes are described in the following paragraphs. But before we describe the schemes in detail, we describe the building blocks of these schemes.

A. Building blocks

The transmission schemes we propose are based on the
• private and common signaling approach of the Han-Kobayashi scheme [22], [11], [24]
in addition to three relaying strategies
  • compute-forward (CF) [17]
  • decode-forward (DF) [16]
  • cooperative interference neutralization (CN) [26].

Next, we introduce the three relaying schemes.

1) Compute-Forward (CF): In CF, Tx1 sends \( u_{i,cf}[k] \) in the \( k \)th channel use \((k = 1, \ldots, n-1)\). At the end of the \( k \)th channel use, the relay decodes \( u_{r,cf}[k+1] = u_{i,cf}[k] \oplus u_{2,cf}[k] \), and sends it in channel use \( k+1 \). Now, consider the decoding at the receiver side. We explain the decoding only for Rx1, since Rx2 does it similarly. Rx1 waits until the end of transmission block \( n \). Then it performs backward decoding starting with the \( n \)th channel use, where only the relay is active. Rx1 decodes \( u_{r,cf}[n] = u_{1,cf}[n-1] \oplus u_{2,cf}[n-1] \) in the \( n \)th channel use. Then, in channel use \((n-1)\), depending on the channel parameters, Rx1 has two possibilities:

• If interference is weak, i.e., \( n_c < n_d \), Rx1 decodes \( u_{1,cf}[n-1] \), then it adds it to \( u_{r,cf}[n] \) to extract \( u_{2,cf}[n-1] \). Next, it subtracts the contribution of \( u_{2,cf}[n-1] \) from the received signal, and decodes \( u_{r,cf}[n-1] \).

• If interference is strong, i.e., \( n_d < n_c \), Rx1 decodes \( u_{2,cf}[n-1] \) first, then it adds it to \( u_{r,cf}[n] \) to extract \( u_{1,cf}[n-1] \). Next, it subtracts the contribution of \( u_{1,cf}[n-1] \) from the received signal, and decodes \( u_{r,cf}[n-1] \).

Next, Rx1 proceeds backwards until reaching the first channel use. Since the relay is silent in the first channel use, Rx1 decodes

\[ u_{1,cf}[1] \] if \( n_c < n_d \),
\[ u_{2,cf}[1] \] if \( n_d < n_c \).

Note that in both cases, each receiver obtains the CF signals sent by both Tx’s, and thus, the CF signals can be interpreted as common relayed signals.

2) Decode-Forward (DF): In DF, Tx1 sends \( u_{i,df}[k] \) in the \( k \)th channel use \((k = 1, \ldots, n-1)\). The relay decodes both \( u_{1,df}[k] \) and \( u_{2,df}[k] \) as in a multiple access channel (MAC) in the \( k \)th channel use. Then, the relay forwards these two signals in channel use \( k+1 \). The sent signal by the relay is \( u_{r,df}[k+1] \). Similar to CF, the Rx’s wait until the end of the \( n \)th channel use and then they start with backward decoding. First, Rx1 processes the received signal in the \( n \)th channel use. As the Tx’s are silent in the \( n \)th channel use, Rx1 decodes only the relay signal, i.e. \( u_{r,df}[n] \). Thus, each Rx decodes \( u_{1,df}[n-1] \) and \( u_{2,df}[n-1] \) as in the MAC channel. Then, Rx1 starts processing the received signal in channel use \((n-1)\). It removes the interference caused by \( u_{1,df}[n-1] \) and \( u_{2,df}[n-1] \), since they are both already known at Rx1 (decoded in the \( n \)th channel use). Then, it decodes the relay signal \( u_{r,df}[n-1] \) and obtains \( u_{1,df}[n-2] \) and \( u_{2,df}[n-2] \). Rx1 proceeds backwards until the second channel use, where \( u_{r,df}[2] \) is decoded, and \( u_{1,df}[1] \) and \( u_{2,df}[1] \) are extracted. As a result, each receiver has \( u_{i,df}[k] \) for \( i = 1, 2 \) and \( k = 1, \ldots, n-1 \).

3) Cooperative interference neutralization (CN): In this scheme, the relay transmits its signal in such a way that the interference is completely neutralized at the Rx’s. To do this, Tx1 uses block-Markov encoding [21] by sending both \( u_{i,cn}[k] \) and \( u_{i,cn}[k-1] \) in the \( k \)th channel use \((k = 2, \ldots, n-1)\). In the first and the \( n \)th channel use, Tx1 sends \( u_{i,cn}[1] \) and \( u_{i,cn}[n-1] \), respectively. Note that while in CF and DF, the Tx’s are silent in \( n \)th channel use, in CN, the Tx’s are active in \( n \)th channel use. Similar to CF, the relay decodes the sum of the signals as follows. At the end of the first channel use, the relay decodes \( u_{1,con}[1] \oplus u_{2,con}[1] \). Note that in the second channel use, this sum is received again at the relay due to the block-Markov encoding. Therefore, the relay removes the interference caused by this sum, and then decodes \( u_{1,con}[2] \oplus u_{2,con}[2] \). Proceeding this way, the relay knows \( u_{1,con}[k] \oplus u_{2,con}[k] \) at the end of the \( k \)th channel use, \( k = 1, \ldots, n-1 \). This known sum can be used in the next channel use for interference neutralization as follows. The relay sends \( u_{r,con}[k] = u_{1,con}[k-1] \oplus u_{2,con}[k-1] \) in the \( k \)th channel use such that it overlaps with the transmitters’ signal \( u_{1,con}[k-1] \) and \( u_{2,con}[k-1] \) at Rx2 and Rx1, respectively. Similar to CF and DF, the receivers use backward decoding. Rx1 first processes the received signal in channel use \( n \), where it receives \( u_{r,con}[n] \oplus u_{2,con}[n-1] = u_{1,con}[n-1] \) as a superposition of the signals from Tx2 and the relay. This superposition of the relay signal and undesired signal neutralizes the interference and provides the receiver the desired signal as the aggregate.
In addition to this, Rx1 receives another copy of $u_{1,cn}[n-1]$ which is sent by Tx1. Depending on whether the IRC operates in the weak interference regime or the strong interference regime, Rx1 decodes $u_{1,cn}[n-1]$ either from Tx1 or from the superposition of the signals from relay and Tx2. Thus, Rx1 gets $u_{1,cn}[n-1]$. The contribution of the other received instance of $u_{1,cn}[n-1]$ can be removed from the received signal afterwards if needed. Then Rx1 proceeds to channel use $n-1$ where it removes the contribution of $u_{1,cn}[n-1]$, and then decodes $u_{1,cn}[n-2]$ similar to channel use $n$. Rx1 proceeds similarly until the first channel use, and hence gets $u_{1,cn}[k]$ for $k = 1, \ldots, n-1$.

Next, we introduce the achievable schemes for the LD-IRC which are combinations of private and common signaling with the three relaying strategies presented above.

### B. Scheme WI-1

The first scheme is developed for the WI regime $n_c < n_d$, and it performs optimally in several cases in this regime. We summarize the performance of the scheme in terms of achievable sum-rate in the following proposition.

**Proposition 1.** The achievable sum-rate with the scheme WI-1 for the IRC is given by

$$R_{\Sigma, WI-1} = \begin{cases} 
    \min \{n_s + n_d, n_r + n_s - n_c\} & \text{if } n_c < n_d \\
    \min \{2n_d, n_r + n_d - n_c\} & \text{if } n_c \leq n_d \leq n_r \\
    \min \{n_r + n_d, n_r + n_s - n_c\} & \text{if } n_c < n_d \leq n_r \\
    \min \{2n_d, n_s + n_d - n_c\} & \text{if } n_c \leq n_r \leq n_d \leq n_s
\end{cases}$$

(33)

This proves the achievability of Theorem 2 within the four regimes. Note that these sum-rate expressions coincide with the upper bounds given in Lemmas 2 and 2.

The rest of the subsection is devoted to the proof of the proposition [1]

In this transmission scheme, in addition to private signaling, CN, DF, and CF relaying strategies are used.

**Encoding at transmitters:** In the $k$th channel use, Tx1 transmits $x_1[k]$ given by

$$x_1[k] = \begin{bmatrix}
    0_{s} \\
    u_{1,cn}[k-1] + u_{1,df}[k] \\
    u_{1,cn}[k-1] \\
    u_{1,cf}[k] \\
    u_{1,cf}[k] \\
    u_{1,df}[k+1] \\
    0_{s}
\end{bmatrix}, \quad k = 1, \ldots, n,$$

(34)

where the subscript $p$ refers to the private signal vector, and $0_s$ is used to complete the length of $x_1[k]$ to $q$ which is equal to $\max\{n_r, n_s\}$ in these cases [65]. The vectors $u_{1,cn}[0], u_{1,cn}[n], u_{1,cf}[n], u_{1,cf}[n], u_{1,cf}[n], u_{1,df}[n], u_{1,df}[n], u_{1,df}[n]$ and $u_{1,df}[n]$ are zero vectors for $i \in \{1, 2\}$. Moreover, the length of the vectors $u_{1,df}, u_{1,df}, u_{1,cf}, u_{1,cf}, u_{1,cf}$ are $2R_{df}, 2R_{df}, R_{cf}, R_{cf},$ and $R_{df}$ respectively.

Note that the length of DF signal vector is twice as many information bits it contains. In particular, if $\hat{u}_{i,df}$ denotes the information bits of df1 from Tx$i$ with length $R_{df}$ then $u_{1,df}$ is constructed as follows $u_{1,df} = \begin{bmatrix} \hat{u}_{1,df}^T & 0_{R_{df}}^T \end{bmatrix}^T$. Similarly, $u_{2,df} = \begin{bmatrix} 0_{R_{df}}^T & \hat{u}_{2,df}^T \end{bmatrix}^T$. Therefore, in $u_{1,df} + u_{2,df}$, the information bits of the DF signal vectors from both Tx’s do not overlap.

It is worth mentioning that in $x_1[k]$, the CN signal vector with time index $[k]$ is desired at the relay and the CN signal vector with time index $[k-1]$ is neutralized at undesired Rx. Moreover, we define $u_{1,cn}$ as follows $u_{1,cn} = [u_{1,cn1}^T, u_{1,cdn2}^T]^T$, where $u_{1,cdn}$ is added to $u_{1,df}[k]$ before transmission. The length of the vectors $u_{1,cdn}$ and $u_{1,cdn}$ are $R_{cdn} = 2R_{df}$ and...
we need to keep in mind that $2R_{df} \leq R_{cn}$. 

Obviously, $x_1[k]$ can be generated as long as

$$2R_{cn} + R_{cf} + R_p + 2R_{df2} + \ell_1 \leq q$$  \hspace{1cm} (36)

Similarly, Tx'2 transmits $x_2[k]$.

**Decoding at the relay:** The relay receives in $k$th channel use the superposition of the signal vectors transmitted from both Tx’s as shown in Fig. 5(a). Notice that in this figure the time index of all signals is $[k]$ except $u_{r, cn}$ which has a time index of $[k-1]$. Supposing that the decoding in time slot $k-1$ is done successfully at the relay, the relay knows $u_{1, cn}[k-1] \oplus u_{2, cn}[k-1]$ in time slot $k$. Hence, it removes the interference caused by $u_{1, cn}[k-1] \oplus u_{2, cn}[k-1]$ before decoding process in time slot $k$. In time slot $k$, the relay wants to decode the sum of the CN signal vectors $u_{1, cn}[k] \oplus u_{2, cn}[k]$, the sum of the CF signal vectors $u_{1, cf}[k] \oplus u_{2, cf}[k]$, and the DF signal vectors $u_{1, df1}[k]$, $u_{1, df2}[k]$, $u_{2, df1}[k]$, and $u_{2, df2}[k]$. For successful decoding at the relay, the following constraints must be satisfied

$$\ell_1 + R_{cf} + R_p + 2(R_{cn} + R_{df2}) \leq n_q \text{ if } \max\{R_{cn}, R_{df2}\} > 0 \hspace{1cm} (37)$$

$$\ell_1 + R_{cf} \leq n_q \text{ if } \max\{R_{cn}, R_{df2}\} = 0. \hspace{1cm} (38)$$

Note that the case distinction in (37) and (38) is due to the fact that the relay does not need to decode the sum of private signal vectors if no CN and DF signal vectors are transmitted.

**Encoding at the relay:** As the previous (with time index $k-1$) CN, CF, and DF signal vectors are available at the relay in the $k$th channel use, the relay constructs the following signal vector at the time slot $k = 2, \ldots, n$

$$x_r[k] = \begin{bmatrix}
u_{r, cf}[k] \\ u_{r, df1}[k] \\ u_{r, df2}[k] \\ 0_{\ell_2} \\ u_{r, cn}[k] \\ 0_{\ell_3} \\ 0_r\end{bmatrix},$$

where we define $u_{r, R}[k] = u_{1, R}[k-1] \oplus u_{2, R}[k-1]$, with $R \in \{cf, df1, df2, cn\}$ for readability, and where $\ell_2$ is chosen so that $u_{r, df2}$ does not overlap with $u_{1, cn}$, and $\ell_3$ is chosen so that at the receiver side, the CN signal vector from the undesired transmitter is aligned with the CN signal vector sent by the relay. Moreover, $0_r$ is used to complete the length of $x_r[k]$ to $q$.

**Decoding at the receiver side:** We explain the decoding at Rx1 since the decoding at Rx2 is similar. Rx1 waits until end of the $n$th channel use. Then it starts with the backward decoding. Supposing that Rx1 decodes $y_1[n]$ successfully, it knows

- $u_{r, cf}[n] = u_{1, cf}[n-1] \oplus u_{2, cf}[n-1]$,
- $u_{r, df1}[n] \rightarrow \tilde{u}_{1, df1}[n-1], \tilde{u}_{2, df1}[n-1]$,
- $u_{r, df2}[n] \rightarrow \tilde{u}_{1, df2}[n-1], \tilde{u}_{2, df2}[n-1]$,
- $u_{1, cn}[n-1]$.

Next, it starts decoding $y_1[n-1]$. The received signal vector at Rx1 in time slot $2 \leq k \leq n-1$ is shown in Fig. 5(b).

Since Rx1 knows $\tilde{u}_{1, df1}[n-1]$ and $\tilde{u}_{2, df1}[n-1]$, it can remove the interference caused by $u_{1, df1}[n-1], u_{2, df1}[n-1]$ completely. Next, it decodes the interference free received bits from the relay. In order to avoid an overlap between the CF and DF signals from relay with the signal transmitted by the desired Tx, the top-most $\ell_1$ bits of the signal vectors from Tx’s
Note that the overlap between the two received versions of the bits successively as long as

\[ R_{cf} + 2(R_{df1} + R_{df2}) \leq (n_r - n_d + \ell_1)^+. \]  \hspace{1cm} (39)

Since this scheme is proposed for the WI regime \((n_c < n_d)\), Rx1 decodes next the top-most \(n_d - n_c\) desired CN bits \(i.e., u_{1,cn}[n-2]\) interference free. Moreover, the relay signal vector neutralizes the undesired CN signal vector, and replaces it by \(u_{1,cn}[n-2]\), i.e., \(u_{r,cn}[n-1] \oplus u_{2,cn}[n-2] = u_{1,cn}[n-2]\). Notice that this neutralization is possible if

\[ n_c - \ell_1 \leq n_r \quad \text{if} \quad 0 < R_{cn}. \]  \hspace{1cm} (40)

Note that the overlap between the two received versions of \(u_{1,cn}[n-2]\) (from TX1 and the relay) can be removed by decoding the bits successively as long as \(n_c \neq n_d\), which is satisfied in the WI regime where \(n_c < n_d\). After decoding the CN signal vector, Rx1 observes the top most \(n_d - n_c\) bits of the desired CF signal vector \(i.e., u_{1,cf}[n-1]\) interference free. Moreover, due to the backward decoding, the sum of the CF signal vectors \(i.e., u_{r,cf}[n] = u_{1,cf}[n-1] \oplus u_{2,cf}[n-1]\) is known at Rx1. Therefore, Rx1 can reconstruct the top-most \(n_d - n_c\) bits of the undesired CF bits and remove their interference. Similar to decoding the CN signal vector, decoding of the CF signal vector is also accomplished in a successive manner, which is possible as long as \(n_c \neq n_d\). After decoding the CF signal vector, Rx1 decodes the private signal vector. It can be done reliably as long as

\[ R_p \leq n_d - n_c. \]  \hspace{1cm} (41)

The other required rate constraint is that the desired CN, CF, and private signal vectors \(u_{1,cn}[n-2], u_{1,cf}[n-1], u_{1,p}[n-1]\) need to be observed at Rx1 and without any overlap with the CN signal vector corresponding to the next time slot (denoted by subscript \(cnF\) in Fig. 5). Therefore, we write

\[ \ell_1 + R_{cn} + R_{cf} + R_p + (R_{cn} + 2R_{df2} - (n_s - n_d)^+)^+ \leq n_d. \]  \hspace{1cm} (42)
The next goal is to maximize the sum-rate under the conditions in (35)-(42). Hence, we obtain the following optimization problem.

\[
\begin{align*}
\max_{\ell_1} & \quad R_{\Sigma, WI-1} \\
\text{s.t.} & \quad (35)-(42) \text{ are satisfied}
\end{align*}
\]

The solution of this optimization problem is given in Table I with an achievable sum-rate as given in (45). This proves the achievability of Theorem 2 for the four regimes corresponding to this scheme.

C. Scheme WI-2

Scheme WI-1 above does not achieve the sum-capacity of the LD-IRC for the whole WI regime. In what follows, we introduce another scheme for the WI regime which achieves the sum-capacity of the LD-IRC in parts of the WI regime that are not characterized by scheme WI-1. This scheme is called scheme WI-2. The performance of this scheme is summarized in the following proposition.

**Proposition 2.** The scheme WI-2 achieves the following sum-rate for the IRC

\[
R_{\Sigma, WI-2} = \min \left\{ 2n_d - n_c, 2 \max \{n_r + n_s - n_c, n_d - n_c \} \right\},
\]

in two regimes, the first of which is described by

\[
n_c \leq n_s \leq n_r \leq n_d,
\]

and the second by

\[
n_c \leq n_r \leq n_s \leq n_d - \frac{n_c}{2}
\]
This proves the achievability of Theorem 2 within these two regimes. Notice that this sum-rate expression coincides with the upper bounds given in Lemmas 2 and 3.

In what follows the proof of this proposition is presented. Notice that in both regimes where WI-2 is optimal (given in (47) and (48)), we have $q = n_d$. While both the WI-1 and WI-2 schemes use CN and CF and private signaling, the difference between the two schemes is that WI-1 uses DF while WI-2 uses common signaling instead. In what follows, we present this scheme in detail.

**Encoding at transmitters:** Tx1 constructs its transmit signal as follows

$$x_1[k] = \begin{bmatrix} u_{1,cm}[k] \\ u_{1,cm}[k-1] \\ u_{1,cf}[k] \\ 0_{\ell_1} \\ u_{1,p1}[k] \\ u_{1,cm}[k] \\ u_{1,p2}[k] \end{bmatrix}, \quad k = 1, \ldots, n, \quad (49)$$

where $u_{1,cm}$ represents the common signal vector with length $2R_{cm}$, and where $\ell_1$ will be specified later. Here, the private signals $u_{1,p1}$, $u_{1,p2}$, the CN signal $u_{1,cm}$, and the CF signal $u_{1,cf}$ have lengths $R_{p1}$, $R_{p2}$, $R_{cm}$, and $R_{cf}$, respectively. As described in Section V-A, the signals $u_{1,cm}[0]$, $u_{1,cm}[n]$, $u_{1,cf}[n]$, $u_{1,p1}[n]$, $u_{1,cm}[n]$, and $u_{1,p2}[n]$ are zero vectors. Tx2 generates $x_2[k]$ similarly. Clearly, this works only if

$$2R_{cm} + 2R_{cn} + R_{cf} + \ell_1 + R_{p1} + R_{p2} \leq q = n_d. \quad (50)$$

**Decoding at the relay:** The relay receives the sum of the top-most $n_s$ bits transmitted by Tx’s. In channel use $k$, the relay wants to decode the following sums of signals $u_{1,cf}[k] \oplus u_{2,cf}[k]$ and $u_{1,cm}[k] \oplus u_{2,cm}[k]$. This is possible if the relay observes the CF and CN signals, which requires following constraint

$$2R_{cm} + 2R_{cn} + R_{cf} + \ell_1 + R_{p1} \leq n_s. \quad (51)$$

Therefore, at the end of the $k$th channel use, the relay knows $u_{r,cf}[k+1] = u_{1,cf}[k] \oplus u_{2,cf}[k]$ and $u_{r,cm}[k+1] = u_{1,cm}[k] \oplus u_{2,cm}[k]$.

**Encoding at the relay:** In channel use $k$, the relay sends the following signal vector

$$x_r[k] = \begin{bmatrix} 0_{\ell_2} \\ u_{r,cf1}[k] \\ 0_{\ell_5} \\ u_{r,cf2}[k] \\ 0_{\ell_1} \end{bmatrix} \oplus \begin{bmatrix} 0_{\ell_5} \\ u_{r,cm1}[k] \\ 0_{r_2} \end{bmatrix},$$

where $0_{\ell_1}$ and $0_{\ell_2}$ are zero vectors which insure that the length of $x_r$ is $q$ (which equals $n_d$ in this regime), $u_{r,cf1}[k]$ consists of the top-most $\ell_4$ bits ($\ell_4$ is to be determined) of $u_{r,cf}[k]$ and $u_{r,cf2}[k]$ consists of remaining bits, and $u_{r,cm1}[k]$ consists of the top-most $\ell_6$ bits of $u_{r,cm}[k]$, where $\ell_6$ is the number of bits of $u_{2,cm}[k]$ that appear as interference at Rx1, i.e., $\ell_6 = \min\{R_{cm}, (n_c - 2R_{cm})^+\}$. Thus, $u_{r,cf1}[k] = (u_{r,cf}[k])_{[1:\ell_4]}$, $u_{r,cf2}[k] = (u_{r,cf}[k])_{[\ell_4+1:R_{cf}]}$, and $u_{r,cm1}[k] = (u_{r,cm}[k])_{[1:\ell_6]}$. The signals above ($u_{r,cf1}$, $u_{r,cf2}$, and $u_{r,cm1}$) will be received by receivers if

$$\ell_2 + R_{cf} + \ell_3 \leq n_r, \quad (52)$$

$$\ell_5 + \ell_6 \leq n_r. \quad (53)$$

The parameters $\ell_2$ to $\ell_6$ should be chosen in such a way that facilitates the decoding at the destinations, as we shall see next.
Decoding at the receiver side: Now, we describe decoding at Rx1. The decoding at Rx2 is done similarly. Similar to scheme WI-1, in this scheme, the destinations use backward decoding. Rx1 waits until the end of nth channel use. Assuming that decoding $y_1[n]$ is done successfully, Rx1 knows

- $u_{1, cn}[n - 1]$
- $u_{r, cf1}[n] = u_{1, cf1}[n - 1] \oplus u_{2, cf1}[n - 1]$
- $u_{r, cf2}[n] = u_{1, cf2}[n - 1] \oplus u_{2, cf2}[n - 1]$. 

Next, Rx1 start with processing $y_1[n - 1]$ which is shown in Fig. 6. First, Rx1 decodes the common signals as in an IC [24]. Thus, we treat the IRC at this stage as an IC, by treating the CN, CF, P1, and P2 signals as noise. Furthermore, to make sure that $u_{r, cf1}[n - 1]$ and $u_{r, cf1}[n - 2]$ do not interfere with the desired common signal, we require

$$n_r - \ell_2 \leq n_d - 2R_{cm} \quad \text{if } R_{cf} > 0. \quad (54)$$

Remark 1. As we discussed earlier, the CN signal sent by the relay needs to be received aligned with the CN signal sent by Tx2 to facilitate interference neutralization at Rx1. Moreover, since in weak interference regime $n_c < n_d$, $u_{2, cn}[n - 1]$ is received on lower level than $u_{1, cn}[n - 1]$. Now, due to (50), $u_{1, cn}[n - 1]$ cannot overlap the common signal $u_{1, cm}[n - 1]$ and hence, $u_{2, cn}[n - 1]$ and $u_{r, cn1}[n - 1]$ cannot do either.

Under this condition, we get an IC with $n_{d, IC} = 2R_{cm}$ and $n_{c, IC} = (n_c - n_d + 2R_{cm})^+$ which is an IC with weak interference. The decoding of both common signals at the receivers in this IC is done in a MAC fashion, achieving a rate of [24]

$$\min \left\{ \frac{n_{d, IC}}{2}, n_{c, IC} \right\}.$$ 

Therefore, under the aforementioned conditions (50), (54), the common rate $\min \{R_{cm}, (n_c - n_d + 2R_{cm})^+\}$ is achieved.

After removing the common signal vectors from the received signal, Rx1 observes a superposition of $x'_1$, $x'_2$, and $x'_r$ shown in Fig. 7 where we define

$$n'_d = n_d - 2R_{cm}$$
$$n'_c = n_c - 2R_{cm}$$
$$n'_r = n_r - \ell_2.$$ 

At this point, we need to make sure that $n'_d$, $n'_c$, $n'_r$, and $n'_c$ are all non-negative. The terms $n'_d$, $n'_s$, and $n'_c$ are clearly non-negative due to (50), (51), (52), and (53). To guarantee that $n'_c$ is non-negative, we require

$$2R_{cm} \leq n_c. \quad (55)$$
Fig. 7: The received signal at Rx1 in the kth channel use (2 ≤ k ≤ n − 1) after removing the common signal vectors based on scheme WI-2. Here, \( \mathbf{u}_{i, cn} \) denotes \( \mathbf{u}_{i, cn}[k] \), while \( \mathbf{u}_{i, cn} \) represents \( \mathbf{u}_{i, cn}[k - 1] \). The time index of all remaining signals is \( [k] \).

At this step, we are able to specify \( \ell_1, \ell_2, \ell_3, \ell_4, \) and \( \ell_5 \). We do this while describing the decoding at Rx1 step by step. First, we need to avoid interference between the relay signal \( \mathbf{x}_r \) and \( \mathbf{u}_{1, cn} \) and \( \mathbf{u}_{1, cf} \). Thus, we need \( n_r' \leq n_d' - R_{en} - R_{cf} \) leading to

\[
n_r - \ell_2 \leq n_d - 2R_{en} - R_{en} - R_{cf} \quad \text{if } 0 < R_{cf}.
\]

**Remark 2.** Notice that if \( R_{cf} = 0 \), \( \mathbf{x}_r' \) contains only CN signal. As we discussed earlier, CN relaying strategy can neutralize the interference at Rx1 as long as the relay CN signal is aligned with the CN signal from Tx2. Moreover, due to the condition of WI regime \( n_c < n_d \), the CN signal from Tx1 is received at Rx1 on higher level than that of Tx1. Thus, by using successive decoding (discussed below), the interference caused by the CN signals from Tx2 and relay are removed. Hence, the interference caused by the CN signal in \( \mathbf{x}_r' \) does not need to be considered for reliable decoding of \( \mathbf{u}_{1, cn} \) and \( \mathbf{u}_{1, cf} \).

Notice that (56) is stricter than (54). Thus, we set

\[
\ell_2 = (n_r - n_d + 2R_{en} + R_{en} + R_{cf})^+ \quad \text{if } 0 < R_{cf1}
\]

\[
\ell_2 = (n_r - n_d + 2R_{en} + R_{en} + R_{cf} + \ell_1 + R_{p1})^+ \quad \text{if } R_{cf1} = 0 \text{ and } 0 < R_{cf2}
\]

which satisfies both (54) and (56). Since \( n_c < n_d \), then Rx1 can decode the first bit of \( \mathbf{u}_{1, cn}[n - 2] \). If we align \( \mathbf{u}_{2, cn} \) and \( \mathbf{u}_{r, cn} \) at Rx1, then the interference of \( \mathbf{u}_{2, cn} \) and \( \mathbf{u}_{r, cn} \) is neutralized (as in Section V-A) since Rx1 receives \( \mathbf{u}_{r, cn}[n - 1] \oplus \mathbf{u}_{2, cn}[n - 2] = \mathbf{u}_{1, cn}[n - 2] \). This alignment is possible if the following two conditions are satisfied. Firstly, Rx1 has to receive \( \mathbf{u}_{r, cn} \) which requires condition (55). Secondly, the CN signal from the relay and Tx2 has to be aligned at Rx1. This is possible as long as \( n_r - \ell_5 = n_r' \) and hence

\[
\ell_5 = n_r - n_c + 2R_{en}.
\]

Note that \( \ell_5 \) is always positive since \( n_c \leq n_r \) in this regime. After decoding the first bit of \( \mathbf{u}_{1, cn}[n - 2] \), Rx1 removes the contribution of the first bit of \( \mathbf{u}_{r, cn}[n - 1] \oplus \mathbf{u}_{2, cn}[n - 2] \). Then, Rx1 decodes the second bit of \( \mathbf{u}_{1, cn}[n - 1] \) received from Tx1, and cancels the interference of the second bit of \( \mathbf{u}_{r, cn}[n - 1] \oplus \mathbf{u}_{2, cn}[n - 2] \). It continues this way until all bits of \( \mathbf{u}_{1, cn}[n - 1] \) are decoded, and all bits of \( \mathbf{u}_{r, cn}[n - 1] \oplus \mathbf{u}_{2, cn}[n - 2] \) are cancelled.
Therefore, the optimal achievable sum-rate of scheme 2 is obtained by solving the following optimization problem

$$\max \ R_{\Sigma, WI-2}$$

s.t. (50)-(62)

$$R_{cm}, R_{cn}, R_{cf}, R_{p1}, R_{p2}, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6 \geq 0.$$ 

The solution of this optimization problem is presented in Tables II and Tables III with

$$\ell_1 = n_d - 2R_{cm} - 2R_{cn} - R_{cf} - R_{p1} - R_{p2}.$$ 

| Regime | $n_r + n_s \leq n_d$ | $n_d < n_r + n_s \leq n_d + \frac{n_c}{2}$ | $n_d < \min\{n_r + n_s - \frac{n_c}{2}, \frac{3}{2}n_c\}$ | $\frac{3}{2}n_c \leq n_d < n_r + n_s - \frac{n_c}{2}$ |
|--------|----------------------|-----------------------------------|-----------------------------------|-----------------------------------|
| $R_{cm}$ | 0                    | 0                                 | $\frac{n_c}{2}$                     | 0                                 |
| $R_{cn}$ | 0                    | $n_s - \min\{n_c, n_d - n_c\}$ | 0                                 | $n_s - n_c - (n_s - \frac{3}{2}n_c)^+$ |
| $R_{cf}$ | 0                    | $n_r - n_d + n_s$               | 0                                 | $\frac{n_c}{2}$                     |
| $R_{p1}$ | 0                    | $\max\{n_c, n_d - n_c\} - n_c$ | $n_s - n_c$                     | $(n_s - \frac{3}{2}n_c)^+$          |
| $R_{p2}$ | $n_d - n_c$           | $n_d - n_s$                       | $n_d - n_s$                       | $n_d - n_s$                       |

**TABLE II:** Rate allocation parameters for the scheme WI-2 in the regime where $n_c \leq n_s \leq n_r \leq n_d$. 

Next, Rx1 decodes the first bit of $u_{1,cf}[n-1]$ interference free. Then, it uses this bit in combination with $u_{r,cf1}[n]$ and $u_{r,cf2}[n]$ (decoded in the $n$th channel use) to extract the first bit of the interference signal $u_{2,cf}[n-1]$ and subtract its contribution from the received signal. Then it proceeds to decode the second bit of $u_{1,cf}[n-1]$. Decoding proceeds this way until all bits of $u_{1,cf}[n-1]$ are decoded and all bits of $u_{2,cf}[n-1]$ are cancelled. Note that at this point there is no interference left from Tx2 if the signals $u_{2,p1}[n-1]$, $u_{2,cn}[n-1]$, and $u_{2,p2}$ are received below the noise floor at Rx1, i.e.,

$$n_c - 2R_{cm} - R_{cn} - R_{cf} - \ell_1 \leq 0.$$ 

Under this condition, Rx1 decodes $u_{r,cf1}$ which is received by Rx1 if (52) holds, and is interference free if

$$\ell_1 \geq \ell_4.$$ 

Then it decodes $u_{1,p1}$ which is also received interference free since

$$\begin{cases} 
\ell_3 = R_{p1} & \text{if } 0 < R_{cf1} \\
\ell_3 = 0 & \text{if } 0 = R_{cf1}.
\end{cases}$$ 

Notice that for the case in which $R_{cf1} = 0$, 57 guarantees that $u_{1,p1}$ is received interference free. Now, Rx1 wants to decode $u_{r,cf2}[n-1]$. To do this, it first removes the contribution of $u_{1,cn}[n-1]$ (denoted by $u_{1,cn,F}$ in Fig. 7) from the received signal, which is possible since Rx1 has decoded $u_{1,cn}[n-1]$ in channel use $n$. After removing $u_{1,cn}[n-1]$, Rx1 observes $u_{r,cf2}[n-1]$ interference free if

$$\ell_4 = (R_{cf} - R_{cn})^+.$$ 

Under this condition, Rx1 can decode $u_{r,cf2}[n-1]$. Finally, Rx1 decodes $u_{1,p2}$ interference free.

As a result, this scheme achieves

$$nR_{\Sigma, WI-2} = 2(n-1) \left( \min\{R_{cm}, (n_c - n_d + 2R_{cm})^+\} + R_{cn} + R_{cf} + R_{p1} + R_{p2}\right).$$ 

Dividing this expression by $n$ and letting $n \to \infty$, we obtain the sum-rate

$$R_{\Sigma, WI-2} = 2 \left( \min\{R_{cm}, n_c - n_d + 2R_{cm}\} + R_{cn} + R_{cf} + R_{p1} + R_{p2}\right).$$ 

Therefore, the optimal achievable sum-rate of scheme 2 is obtained by solving the following optimization problem

$$\max \ R_{\Sigma, WI-2}$$

s.t. (50)-(62)

$$R_{cm}, R_{cn}, R_{cf}, R_{p1}, R_{p2}, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6 \geq 0.$$ 

The solution of this optimization problem is presented in Tables II and Tables III with

$$\ell_1 = n_d - 2R_{cm} - 2R_{cn} - R_{cf} - R_{p1} - R_{p2}.$$
The given rate allocation satisfies constraints (50), (62) and achieves the sum-rate in (46). As a result, this proves the achievability of Theorem 2 for the two regimes \( n_c \leq n_r \leq n_s \leq n_d - \frac{n_r}{2} \).

### D. Scheme WI-3

Note that scheme WI-1 and WI-2 do not cover all possible regimes with weak interference \( n_c < n_d \) and with a source-relay channel stronger than the cross channel \( n_c < n_s \) (condition of Theorem 2). In particular, three possibilities remain given by (i) \( n_c \leq n_r \leq n_s \leq n_d < n_s + \frac{n_r}{2} \), (ii) \( n_r \leq n_c < n_d \leq n_s \), and (iii) \( n_r \leq n_c \leq n_s \leq n_d \). Next, we present the last scheme which covers these three cases and completes the proof of the achievability of Theorem 2 for the WI regime. This scheme is called WI-3. Its achievable sum-rate is presented in the following proposition.

**Proposition 3.** The achievable sum-rate with the scheme WI-3 for the IRC is given by

\[
R_{\Sigma, WI-3} = \begin{cases} 
\min\{n_s + n_d - n_c, 2n_d, n_r - n_c\}, & \text{if } n_c \leq n_r \leq n_s \leq n_d \\
\min\{2n_d - n_c, 2\max\{n_s, n_d - n_c\}, 2\max\{n_c, n_r + n_d - n_c\}\}, & \text{if } n_r \leq n_c \leq n_s \leq n_d \\
\min\{2n_d - n_c, 2\max\{n_r + n_s - n_c, n_d - n_c\}\}, & \text{if } n_c \leq n_r \leq n_s \leq n_d < n_s + \frac{n_r}{2}.
\end{cases}
\]  

(65)

This proves the achievability of Theorem 2 within the three regimes. Note that these sum-rate expressions coincide with the upper bounds given in Lemmas 2 and 3.

In what follows, we present scheme WI-3 in details. This transmission scheme is a combination of common and private signaling in addition to CN.

**Encoding at transmitters:** In \( k \)th channel use, Tx1 constructs the following signal vectors

\[
x_1[k] = \begin{bmatrix} u_{1, cm1}[k] \\
u_{1, cm2}[k] \\
u_{1, cn1}[k - 1] \\
u_{1, cn2}[k - 1] \\
u_{1, p1}[k] \\
u_{1, cm3}[k - 1] \\
u_{1, p2}[k] \\
u_{1, cn2}[k] \\
u_{1, cm3}[k] \\
u_{d_1} \\
u_{c_1} \end{bmatrix}, \quad k = 1, \ldots, n,
\]  

(66)
where $s$ is chosen so that the length of $x_1$ is $q$. The length of vectors $u_{1,a}$ is $R_a$, where $a \in \{cm2, cn1, cn2, cn3, p1, p2\}$. The length of the zero vector, i.e., $\ell_1^u$ and $\ell_1^d$, will be chosen later in such a way that facilitates reliable decoding. We define

$$\ell_1 = \ell_1^u + \ell_1^d.$$ 

The vector $u_{1,cm1}$ with rate $R_{cm1}$ has a length of $\ell_{cm1}$. We further set $u_{1,cm1}[n]$, $u_{1,cm2}[n]$, $u_{1,cm3}[n]$, $u_{1,cm1}[n]$, $u_{1,cm2}[n]$, $u_{1,cm3}[n]$, $u_{1,p1}[n]$, and $u_{1,p2}[n]$ to be zero vectors. Similarly, $Tx_2$ constructs $x_2[k]$. This construction requires

$$\ell_{cm1} + R_{cn2} + 2R_{cn1} + 2R_{cn2} + 2R_{cn3} + R_{p1} + R_{p2} + \ell_1 \leq q. \quad (67)$$

**Decoding at the relay:** The relay receives the top-most $n_s$ bits of the transmitted signal vectors. In channel use $k$, the relay decodes $u_{r,c}[k+1] = u_{1,c}[k] \oplus u_{2,c}[k]$, where $c \in \{cn1, cn2, cn3\}$. To enable this decoding, it is required to set the length of the transmitted signals in such a way that the relay is able to observe the desired signals $u_{r,c}[k+1]$. To write the constraint for reliability decoding of $u_{r,c}[k+1]$, we need to distinguish between several cases. This case distinction is necessary since the relay does not need to decode the sum of the private and common signal vectors when there is no CN signal vector in the lower level. The necessary constraint for decoding the CN signal vectors at the relay is given as follows

$$\begin{cases}
\ell_{cm1} + R_{cn2} + 2R_{cn1} + 2R_{cn2} + 2R_{cn3} + R_{p1} + R_{p2} + \ell_1 \leq n_s & \text{if } R_{cn2} \neq 0 \text{ or } R_{cn3} \neq 0 \\
\ell_{cm1} + R_{cn2} + 2R_{cn1} + R_{p1} + \ell_1^u \leq n_s & \text{if } R_{cn2} = R_{cn3} = 0 \text{ and } 0 < R_{cn1}.
\end{cases} \quad (68)$$

**Encoding at the relay:** After decoding the CN signals, the relay generates

$$x_r[k] = \begin{bmatrix}
0_{\ell_2} \\
u_{r,cm1}[k] \\
u_{r,cm2}[k] \\
0_{\ell_3} \\
u_{r,cm3}[k] \\
0_r
\end{bmatrix}$$

in channel use $k \in \{2, \ldots, n\}$, where $\ell_2$ and $\ell_3$ will be chosen later, and $r$ is chosen so that the length of $x_r$ is $q$. The signal $u_{r,cm1}$, $u_{r,cm2}$, and $u_{r,cm3}$ will be received by $Rx_1$ if

$$R_{cn1} + R_{cn2} + R_{cn3} + \ell_2 + \ell_3 \leq n_r. \quad (69)$$

**Decoding at the receiver side:** Here, we only discuss the decoding process at $Rx_1$ since decoding at $Rx_2$ is done similarly. Similar to previous schemes, receivers use backward decoding. Hence, $Rx_1$ waits until the end of $n$th channel use. Assuming that decoding $y_1[n]$ is done successfully, $Rx_1$ knows

- $u_{1,cm1}[n-1]$
- $u_{1,cm2}[n-1]$
- $u_{1,cm3}[n-1]$.

Next, it starts with processing $y_1[n-1]$. To do this, first, it decodes the first common signals of both transmitters $u_{1,cm1}[n-1]$ and $u_{2,cm1}[n-1]$ in a MAC fashion, while treating the remaining signals as noise. Decoding these signals requires that $x_r$ does not interfere with the desired common signal, thus

$$n_r - \ell_2 \leq n_d - \ell_{cm1} \quad \text{if } \ell_{cm1} > 0. \quad (70)$$

Similar to decoding the common signals in the WI-2 scheme (cf. Section V.C), this decoding is done as in a symmetric IC with channels $n_{d,IC} = \ell_{cm1}$ and $n_{c,IC} = n_c - n_d + \ell_{cm1}$. Decoding the two common signals can be done at a rate of
After removing the common signal vectors, Rx1 observes the superposition of signal vectors $u_{1,cn1}^k$ and $u_{2,cn1}^k$. Note that $u_{1,cn1}^k = u_{1,cn1}^k$, $u_{i,cn1} = u_{i,cn1}^k - 1$, and $u_{i,cn2} = u_{i,cn2}^k - 1$ for $i \in \{1, 2\}$. The time index of all other signal vectors is $k$.

Now, for the sake of simplicity, we distinguish between two cases and explain the decoding for each case separately.

- **Scheme WI3a:** In this case the signal vector $u_{i,cn3}$ does not appear. Hence, we have $R_{en3} = 0$ and we set $\ell_3 = 0$. The received signal vector at Rx1 after removing $u_{i,cn1}^k$ is illustrated for this case in Fig. 8. To guarantee that Rx1 receives the common signal vector $u_{1,cn2}[n - 1]$ the CN signal vectors $u_{1,cn1}[n - 2]$, $u_{1,cn2}[n - 2]$, and the private signal vectors $u_{1,p1}[n - 1]$, $u_{1,p2}[n - 1]$ without any overlap with each other, we write

$$\ell_{cn1} + R_{en1} + 2R_{en2} + R_{p1} + \ell_1 + R_{p2} \leq n_d. \tag{72}$$

First, Rx1 decodes $u_{1,cn2}[n - 1]$, $u_{1,cn1}[n - 2]$, $u_{1,cn2}[n - 2]$, and $u_{1,p1}[n - 1]$. These signals are received interference free as long as

$$R_{en1} + R_{en2} + R_{p1} \leq n'_d - n'_c, \tag{73}$$

$$R_{en1} + R_{en2} + R_{p1} \leq n'_d - n'_c. \tag{74}$$

Notice that condition (74) is tighter than (70). Now, Rx1 is ready to remove the contribution of the $u_{2,cn1}[n - 2]$ and $u_{2,cn2}[n - 2]$. To enable this, we require that the CN signal vectors of Tx2 and the relay are aligned at Rx1. This alignment
is possible if
\[ n'_c - R_{cn2} = n'_r \text{ if } R_{cn1} \neq 0 \text{ or } R_{cn2} \neq 0. \] (75)

From this condition, we obtain
\[ \ell_2 = \begin{cases} n_r - n'_c + R_{cn2} & \text{if } R_{cn1} \neq 0 \text{ or } R_{cn2} \neq 0 \\ n_r & \text{if } R_{cn1} = 0 \text{ and } R_{cn2} = 0. \end{cases} \] (76)

Under the condition in (75), interference neutralization takes place as shown in Section V-A and Rx1 receives \( u_{1, cn1}[n-2] \) and \( u_{1, cn2}[n-2] \) (or parts thereof) as an aggregate of the CN signals from Tx2 and the relay. Since Rx1 has already decoded \( u_{1, cn1}[n-2] \) and \( u_{1, cn2}[n-2] \), aggregate interference from these signals can be removed. This solves the problem of the CN interference.

Since Rx1 has decoded \( u_{1, cn1}[n-1] \) in channel use \( n \), it removes the contribution of this signal (\( u_{1, cn1,F} \) in Fig. 8) from its received signal. Next, Rx1 decodes \( u_{2, cm2}[n-1] \). This can be done reliably as long as
\[ n'_d - R_{cn2} - 2R_{cn1} - R_{cn2} - R_{p1} - \ell_1 \leq n'_c - R_{cm2} \text{ if } R_{cm2} > 0. \] (77)

Finally, Rx1 decodes \( u_{1, p2}[n-1] \). To guarantee that this signal vector is received interference free, following condition needs to be satisfied
\[ \begin{cases} n'_c - R_{cm2} - R_{cn1} - R_{cn2} = 0 & \text{if } 0 < R_{p1} \\ n'_c - R_{cm2} - R_{cn1} - R_{cn2} - \ell'_1 \leq 0 & \text{if } 0 = R_{p1} \end{cases} \] (78)

As a result this scheme achieves
\[ nR_{\Sigma, WI3a} = 2(n-1)\left( R_{cn1} + R_{cn2} + R_{cn1} + R_{cn2} + R_{p1} + R_{p2} \right). \] (79)

By dividing this expression by \( n \) and letting \( n \to \infty \), we obtain
\[ R_{\Sigma, WI3a} = 2\left( R_{cn1} + R_{cn2} + R_{cn1} + R_{cn2} + R_{p1} + R_{p2} \right). \] (80)

This sum-rate has to be maximized under the conditions in (68)-(78), which can be formulated as the following optimization problem
\[ \max \ R_{\Sigma, WI3a} \] (81)
\[ \text{s.t. (68)-(78) are satisfied} \]
\[ \ell_{cm1}, R_{cm2}, R_{cn1}, R_{cn2}, R_{p1}, R_{p2}, \ell'_1, \ell'_1 \geq 0. \]

- Scheme WI3b: Compared to previous case, here \( 0 < R_{cn3} \) while \( R_{cn1} = R_{p1} = 0 \). The received signal vector after removing the \( u_{1, cm1} \) and \( u_{2, cm1} \) is illustrated in Fig. 9. To guarantee that Rx1 receives the common signal vector \( u_{1, cm2} \), the CN signal vectors \( u_{1, cn2}, u_{1, cn3} \), and private signal vector \( u_{1, p2} \), we write
\[ \ell_{cm1} + R_{cm2} + R_{cn2} + \ell_1 + R_{cn3} + R_{p2} \leq n_d. \] (82)

First, Rx1 decodes \( u_{1, cm2}[n-1] \) and \( u_{1, cn2}[n-2] \). To guarantee that these signals are received interference free, we write
\[ R_{cm2} + R_{cn2} \leq n'_d - n'_c \] (83)
\[ R_{cm2} + R_{cn2} \leq n'_d - n'_r \] (84)

Next Rx1 decodes \( u_{2, cm2}[n-1] \). This can be done reliably as long as
Fig. 9: The received signal at Rx1 based on scheme WI3b in the kth channel use (2 ≤ k ≤ n − 1) after removing the common signal vectors \( u_{1, cn1}[k] \) and \( u_{2, cn1}[k] \). Note that \( u_{i, cn2} = u_{i, cn2}[k - 1] \), and \( u_{i, cn3} = u_{i, cn3}[k - 1] \) for \( i \in \{1, 2\} \). The time index of all other signal vectors is \([k]\).

Next, Rx1 removes the interference of \( u_{1, cn2}[n - 2] \) caused by Tx2 and the relay (i.e., \( u_{2, cn2}[n - 2] \oplus u_{r, cn2}[n - 1] \)). This can be done as long as the CN signal vectors \( u_{2, cn2}[n - 2] \) and \( u_{r, cn2}[n - 1] \) are aligned. Therefore, we write

\[
n'_{d} - R_{cn2} - R_{cn2} - \ell_{1} \leq n'_{c} - R_{cn2} \quad \text{if} \quad R_{cn2} \neq 0. \tag{85}
\]

Next, Rx1 decodes the top-most bit of \( u_{1, cn3}[n - 2] \) (sent by Tx1) without any overlap with \( u_{2, cn3}[n - 2] \oplus u_{r, cn3}[n - 1] = u_{1, cn3}[n - 2] \). This is possible since \( n_{c} < n_{d} \). After that Rx1 removes the interference caused by the top-most bit of \( u_{2, cn3}[n - 2] \oplus u_{r, cn3}[n - 1] \) and then it decodes the second bit of \( u_{1, cn3}[n - 2] \). It continues this decoding until the whole vector \( u_{1, cn3}[n - 2] \) is decoded. This can be done reliably as long as the interference caused by \( u_{2, cn3}[n - 2] \) can be neutralized by the relay CN signal \( u_{r, cn3}[n - 1] \). Hence, we write

\[
n'_{r} = n'_{c} - R_{cn2} \quad \text{if} \quad R_{cn2} \neq 0. \tag{86}
\]

From the conditions in (86) and (87), we can fix \( \ell_{2} \) and \( \ell_{3} \) as follows

\[
\begin{align*}
\ell_{2} &= n_{r} - n'_{c} + R_{cn2} \quad \text{and} \quad \ell_{3} = \ell_{1} \quad \text{if} \quad R_{cn2} \neq 0 \\
\ell_{2} &= n_{r} - R_{cn3} \quad \text{and} \quad \ell_{3} = 0 \quad \text{if} \quad R_{cn2} = 0 \quad \text{and} \quad R_{cn3} \neq 0 \\
\ell_{2} &= n_{r} \quad \text{and} \quad \ell_{3} = 0 \quad \text{otherwise.}
\end{align*} \tag{88}
\]

Finally, Rx1 decodes \( u_{1, p2}[n - 1] \). To guarantee that \( u_{1, p2}[n - 1] \) is received interference free, following condition is required

\[
R_{p2} \leq n'_{d} - n'_{c}. \tag{89}
\]

Now, Rx1 has completed the decoding its desired signals in channel use \( n - 1 \). It proceeds backwards till channel use 1. Therefore, the sum-rate is given as follows

\[
n R_{\Sigma, WI3b} = 2(n - 1) \left( R_{cm1} + R_{cm2} + R_{cn2} + R_{cn3} + R_{p2} \right). \tag{90}
\]

By dividing the expression by \( n \) and letting \( n \to \infty \), we obtain

\[
R_{\Sigma, WI3b} = 2 \left( R_{cm1} + R_{cm2} + R_{cn2} + R_{cn3} + R_{p2} \right). \tag{91}
\]

This sum-rate has to be maximized under the conditions in (68)-(71), (82)-(85), and (89) which can be formulated as...
TABLE IV: Rate allocation parameters for the scheme WI-3 in the regime where \( n_r \leq n_c < n_d \leq n_s \) and \( n_r + n_d - n_c > n_c \).

In these regimes, \( \ell^u_1 = R_{cm2} - R_{cn1} \), \( \ell^d_1 = 0 \) and \( \ell_1 = \ell^u_1 \).

TABLE V: Rate allocation parameters for the scheme WI-3 in the regime where \( n_r \leq n_c \leq n_d \leq n_s \) and \( n_r + n_d - n_c \leq n_c \).

In these regimes, \( \ell^u_1 = R_{cm2} - R_{cn1} \), \( \ell^d_1 = 0 \) and \( \ell_1 = \ell^u_1 \).

The following optimization problem

\[
\max \ R_{\Sigma, WI3b} \\
\text{s.t.} \quad (68)-(71), (82)-(85), \text{and (89) are satisfied} \\
\ell_{cm1}, R_{cm2}, R_{cn2}, R_{cn3}, R_{p1}, R_{p2}, \ell_1^u, \ell_1^d, \ell_2, \ell_3 \geq 0.
\]

The optimal parameters for the optimization problems (81) and (82) (with \( \ell^u_1 \) and \( \ell^d_1 \)) are given in Table IV-VII. Using these optimal values, the sum-rate in (65) is achieved. As a result, this scheme together with Schemes WI-1 and WI-2 proves the achievability of Theorem 2 for the WI regime.
TABLE VII: Rate allocation parameters for the scheme WI-3. In this case, \( R_{cn2} = 0 = R_{cn3} = 0. \)

E. Scheme SI

Finally, we present the scheme which is optimal is strong interference regime \( n_d < n_c \). The achievable sum-rate of this scheme is summarized in following proposition.

**Proposition 4.** The achievable sum-rate with the scheme SI for the IRC is given by

\[
R_{\Sigma} = \min \{ 2 \max \{ n_d, n_r \}, \max \{ n_r, n_c \} + (n_s - n_c), n_s + n_c, n_c + n_r \}, \tag{93}
\]

This proves the achievability of Theorem 2 within strong interference regime. Note that this sum-rate expression coincides with the upper bounds given in Lemmas 2 and 3.

In what follows we present scheme SI. In this scheme, we use CN, CF, and DF relaying strategies in addition to private and common signaling.

**Encoding at transmitters:** Suppose that Tx1 constructs in the \( k \)th channel use \( x_1[k] \) as follows

\[
x_1[k] = \begin{bmatrix}
    u_{1,cm1}[k] \\
    u_{1,cm2}[k] \\
    u_{1,cf1}[k] \\
    0_i \\
    u_{1,cf2}[k] \\
    u_{1,cm1}[k-1] \oplus u_{1,df1}[k] \\
    u_{1,cm2}[k-1] \\
    u_{1,df2}[k] \\
    u_{1,cm}[k] \\
    0_s
\end{bmatrix}, \quad k = 1, \ldots, n, \tag{94}
\]

the signal vectors \( u_{1,cm1}[0], u_{1,cm2}[0], u_{1,cm1}[n], u_{1,cm2}[n], u_{1,cf1}, u_{1,cf2}[n], u_{1,df1}[n], u_{1,df2}[n], \) and \( u_{1,cm}[n] \) are zero vectors. The signal vector \( u_{1,cm} \) is defined as \( \begin{bmatrix} u_{1,cm1}^T & u_{1,cm2}^T \end{bmatrix}^T \). Similar to previous schemes, subscripts \( cm, cn, cf, \) and \( df \) represent common, CN, CF, and DF signal vectors, respectively. Moreover, the length of the zero vector, i.e., \( \ell_1 \) is set later in such a way that facilitates reliable decoding. Note that the length of signal vector \( u_{1,a} \) is \( R_a \), where \( a \in \{ cm2, cf1, cf2, cm1, cn2 \} \) and the length of signal vectors \( u_{1,df1} \) and \( u_{1,df2} \) are \( 2R_{df1} \) and \( 2R_{df2} \), respectively. The signal vector \( u_{1,cm1} \) with rate \( R_{cm1} \) has a length of \( \ell_{cm1} \). The DF signal vectors are generated in a similar way as in Scheme WI-1. Therefore, \( u_{1,df1} = \begin{bmatrix} \tilde{u}_{1,df1}^T & 0_{R_{df1}}^T \end{bmatrix}^T \) and \( u_{2,df1} = \begin{bmatrix} 0_{R_{df1}}^T & \tilde{u}_{2,df1}^T \end{bmatrix}^T \), where \( \tilde{u}_{i,df1}, i \in \{ 1, 2 \} \) is a signal vector which contains \( R_{df1} \) information bits of Tx1. Note that the vectors \( u_{1,cm1} \) and \( u_{1,df1} \) have the same length, hence, we have \( R_{cm1} = 2R_{df1} \). This construction needs to satisfy

\[
\ell_{cm1} + R_{cm2} + R_{cf1} + R_{cf2} + 2R_{cm1} + 2R_{cn2} + 2R_{df2} + \ell_1 \leq q. \tag{95}
\]
**Decoding at the relay:** The relay receives the sum of the top-most \( n_s \) bits sent by Tx’s. Supposing that the decoding at the relay is done reliably in time slot \( k-1 \), the relay knows \( u_{1,cn}[k-1] \oplus u_{2,cn}[k-1] \) in the beginning of time slot \( k \). Hence, it remove the interference caused by this sum before decoding process in the \( k \)th channel use. In channel use \( k \), the relay decodes \( u_{r,c}[k+1] = u_{1,c}[k] \oplus u_{2,c}[k] \), where \( c = \{ cf1, ef2, df1, df2, cn1, cn2 \} \). This can be done reliably as long as the source-relay link is sufficiently strong. This can be formulated as follows

\[
\ell_{cm1} + R_{cm2} + R_{cf1} + \ell_1 + R_{cf2} + 2R_{cn1} + 2R_{cn2} + 2R_{df2} \leq n_s.
\]  

Therefore, at the end of \( k \)th channel use, the relay knows \( u_{r,c}[k+1] \), where \( c = \{ cf1, ef2, df1, df2, cn1, cn2 \} \).

**Encoding at the relay:** In the \( k \)th channel use (\( 2 \leq k \leq n \)), the relay constructs the following signal vector

\[
x_r[k] = \begin{bmatrix}
0_{\ell_2} \\
u_{r,df1}[k] \\
u_{r,df2}[k] \\
u_{r,cf1}[k] \\
u_{r,cf2}[k] \\
0_{\ell_3} \\
u_{r,cn1}[k] \\
u_{r,cn2}[k] \\
0_r
\end{bmatrix},
\]

where \( r \) is chosen so that the length of \( x_r[k] \) is \( q \). Note that \( u_{r,c}[k+1] = u_{1,c}[k-1] \oplus u_{2,c}[k-1] \), with \( c = \{ df1, df2, cf1, ef2, cn1, cn2 \} \). It is worth mentioning that \( u_{1,df1}[k] \oplus u_{2,df1}[k] = [\tilde{u}_{1,df1}[k] \quad \tilde{u}_{2,df1}[k]]^T \) and \( u_{1,df2}[k] \oplus u_{2,df2}[k] = [\tilde{u}_{1,df2}[k] \quad \tilde{u}_{2,df2}[k]]^T \).

**Decoding at the receiver side:** Here, we present the decoding only for Rx1 since decoding at Rx2 is similar. Rx1 waits until the end of \( n \)th channel use. Next, it starts with the backward decoding. Supposing that the decoding the received signal vector in the \( n \)th channel use is done successfully, Rx1 obtains

- \( u_{r,df1}[n] \rightarrow \tilde{u}_{1,df1}[n-1], \tilde{u}_{2,df1}[n-1] \)
- \( u_{r,df2}[n] \rightarrow \tilde{u}_{1,df2}[n-1], \tilde{u}_{2,df2}[n-1] \)
- \( u_{r,cf1}[n] \)
- \( u_{r,cf2}[n] \)
- \( u_{1,cn}[n-1] \).

Next, Rx1 starts decoding \( y_1[n-1] \). It decodes first \( u_{1,cn1}[n-1] \) and \( u_{2,cn1}[n-1] \) as in the MAC while ignoring the remaining signals. This can be done successfully as long as the relay signal does not cause any interference. Hence, we write

\[
n_r - \ell_2 \leq n_c - \ell_{cm1} \quad \text{if} \quad 0 < \ell_{cm1}.
\]  

For deciding these common signal vectors, we consider an IC with the channels \( n_{c,IC} = \ell_{cm1} \) and \( n_{d,IC} = n_d - n_c + \ell_{cm1} \). The common signal vectors \( u_{1,cm1} \) and \( u_{2,cm1} \) can be decoded reliably as long as their length does not exceed \( \min\{\frac{n_{c,IC}}{2}, n_{d,IC}\} \). Hence, we write

\[
R_{cm1} = \min\left\{\frac{\ell_{cm1}}{2}, (n_d - n_c + \ell_{cm1})^+\right\}.
\]  

After removing the common signal vectors \( u_{1,cm1} \) and \( u_{2,cm1} \) from the received signal, Rx1 observes a superposition of \( x_r' \), \( x_2' \), and \( x_r' \) shown in Fig. 10 where we define

\[
n_d' = (n_d - \ell_{cm1})^+ \\
n_c' = n_c - \ell_{cm1} \\
n_r' = n_r - \ell_2.
\]
Now, Rx1 are known at Rx constraints are required decoding process until the whole signal vector Fig. 10: The received signal vector at Rx1 in the Remark 3. Since Rx1 decodes the top-most bit of \( u_{1,cf1}[n-1] \) since they are both known at Rx1 due to the backward decoding. Then, it decodes \( u_{2,cm2}[n-1] \) and \( u_{2,cf1}[n-1] \). To do this following constraints are required

\[
R_{cm2} + R_{cf1} \leq (n'_c - n'_d)^{+} \tag{99}
\]

\[
R_{cm2} + R_{cf1} \leq n'_c - n'_d. \tag{100}
\]

Next, Rx1 constructs \( u_{1,cf1}[n-1] \) by adding \( u_{r,cf}[n] \) and \( u_{2,cf1}[n-1] \) which are both known at Rx1. Then it removes the interference caused by \( u_{1,cf1}[n-1] \) from the received signal.

Next, Rx1 decodes \( u_{r,df1}[n-1], u_{r,df2}[n-1], u_{r,cf1}[n-1], \) and \( u_{r,cf2}[n-1] \). This can be done successfully as long as

\[
2R_{df1} + 2R_{df2} + R_{cf1} + R_{cf2} \leq (n'_c - n'_d)^{+} \quad \text{if } 0 < R_{cm2} \tag{101}
\]

\[
2R_{df1} + 2R_{df2} + R_{cf1} + R_{cf2} \leq [n'_c - (n'_c - R_{cm2} - R_{cf1} - \ell_1)]^{+}. \tag{102}
\]

**Remark 3.** Note that for the case that \( R_{cm2} = 0 \), an overlap between the relay CF signals and \( u_{1,cf2}[n-1] \) is avoided using the condition in (102) and since \( n'_d < n'_c \).

Next, Rx1 decodes \( u_{1,cm2}[n-1] \). To do this the following constraint needs to be satisfied

\[
R_{cm2} \leq [n'_d - (n'_c - R_{cm2} - R_{cf1} - \ell_1)]^{+}. \tag{103}
\]

Since \( n'_d < n'_c \), Rx1 decodes the top-most bit of \( u_{2,cf2}[n-1] \) which is received without any interference from \( u_{1,cf2}[n-1] \). Then, it makes modulo 2 sum of this bit with the top-most bit of \( u_{r,cf}[n] \) (this is already known from decoding in \( n \)th channel use) to construct the top-most bit of \( u_{1,cf2}[n-1] \). Then it removes the interference caused by this bit. Rx1 repeats this decoding process until the whole signal vector \( u_{2,cf2}[n-1] \) is decoded. Therefore, Rx1 obtains \( u_{1,cf2}[n-1] \).

Next, Rx1 decodes the top-most bit of \( u_{2,cm1}[n-2] \oplus u_{r,cm1}[n-1] \). Again this bit is received on higher level than \( u_{1,cm1}[n-2] \) (which is sent from Tx1) since \( n'_d < n'_c \). Hence, Rx1 can decode the top-most bit of \( u_{1,cm1}[n-2] = u_{2,cm1}[n-2] \).
The optimal parameters are given in Table VIII and IX. Using these parameters, the sum-rate given in (93) is achieved. This shows the achievability of Theorem 2 for the strong interference regime ($n_d < n_c$).

### F. Scheme II

Until now, the achievability of Theorem 2 for the weak interference regime ($n_c < n_d$) and strong interference regime ($n_d < n_c$) has been shown. In what follows, we present a scheme which is optimal for the intermediate interference regime

| Regime | $n_c \leq n_r$ | $n_r \leq n_c$ | $n_d \leq n_r \leq n_c$ |
|--------|----------------|----------------|--------------------------|
| $\ell_{cm1}$ | 0 | 0 | 0 |
| $R_{cm2}$ | 0 | 0 | 0 |
| $R_{ef1}$ | 0 | 0 | 0 |
| $R_{df1}$ | $\min\{n_c - n_{c,n}n_1\}$ | $(n_c - n_{c,n} - 3n_c)^\circ$ | $\min\{n_c - n_{c,n} - n_c\}$ |
| $R_{df2}$ | 0 | $(n_c - 2n_{c,n})^\circ$ | $\min\{n_c - 2n_{c,n}\}$ |
| $R_{cn1}$ | $2R_{df1}$ | 0 | 0 |
| $R_{cn2}$ | $(2n_c - n_{c,n})^\circ$ | $(n_c - n_{c,n}) - R_{cn1}$ | 0 |
| $\ell_1$ | 0 | $(3n_c - n_{c,n})^\circ$ | 0 |
| $R_{\Sigma_1}$ | $n_r + n_c$ | $n_r + n_c - n_c$ | $n_r + n_c - 2n_r$ |
| $R_{\Sigma_2}$ | $n_r + n_c - n_{c,n} + n_c - 2n_r$ | $n_r + n_c$ | $n_r + n_c - 2n_r$ |
| $R_{\Sigma}$ | $n_r + n_c$ | $n_r + n_c - n_{c,n}$ | $n_r + n_c - 2n_r$ |

**TABLE VIII:** Rate allocation parameters for the scheme SI when $n_d \leq n_r$.  

$u_{r,cn1}[n-1]$ and remove the interference caused this bit which is also sent by Tx1. Then it decodes the remaining bits of $u_{r,cn1}[n-2] \oplus u_{r,cn1}[n-1]$ similarly. Doing this, the whole CN signal vector is decoded as long as the CN signal vector from Tx2 is received at Rx1 and this is aligned with that of the relay. Hence, this constraint needs to be satisfied

$$n_c' - R_{cm2} - R_{ef1} - \ell_1 - R_{df2} - R_{cn1} - R_{cn2} \geq 0$$  \hspace{1cm} (104)

$$R_{cn1} + R_{cn2} \leq n_c'$$  \hspace{1cm} (105)

To guarantee that the interference from $u_{2,cn}[n-1]$ is not received at Rx1, we have

$$n_c' - R_{cm2} - R_{ef1} - \ell_1 - R_{df2} - R_{cn1} - R_{cn2} - 2R_{df2} \leq 0.$$  \hspace{1cm} (106)

At this step, we can set the parameters $\ell_2$ and $\ell_3$ as follows

$$\ell_2 = [n_c - (n_c' - \ell_1 + R_{ef2} + 2R_{df1} + 2R_{df2})]^\circ$$  \hspace{1cm} (107)

$$\ell_3 = n_c' - 2R_{df1} - 2R_{df2} - R_{ef1} - R_{ef2} - R_{cn1} - R_{cn2}.$$  \hspace{1cm} (108)

By using this scheme, we achieve the sum-rate

$$nR_{\Sigma,SI} = 2(n-1)(R_{cm1} + R_{cm2} + R_{ef1} + R_{df1} + R_{df2} + R_{ef2} + R_{cn1} + R_{cn2}).$$  \hspace{1cm} (109)

Dividing the expression by $n$ and letting $n \to \infty$, we obtain the following sum-rate

$$R_{\Sigma,SI} = 2(R_{cm1} + R_{cm2} + R_{ef1} + R_{df1} + R_{df2} + R_{ef2} + R_{cn1} + R_{cn2}).$$  \hspace{1cm} (110)

This sum-rate has to be maximized under the constraints in (95)-(106). This is formulated as follows

$$\max \hspace{0.5cm} R_{\Sigma,SI}$$  \hspace{1cm} (111)

s.t. \hspace{0.5cm} (95)-(106) are satisfied

$$R_{cm1}, R_{cm2}, R_{ef1}, R_{df1}, R_{df2}, R_{ef2}, R_{cn1}, R_{cn2}, \ell_1, \ell_2, \ell_3 \geq 0$$

The optimal parameters are given in Table VIII and IX. Using these parameters, the sum-rate given in (93) is achieved. This shows the achievability of Theorem 2 for the weak interference regime ($n_c < n_d$).
In what follows, we present this scheme in details. Compared to the previous schemes, in which both transmitters send in all channel uses, in this scheme, only one Tx is active. One can use time division multiplexing access (TDMA) and assign the first $\frac{n}{2}$ channel uses to Tx1 and the second $\frac{n}{2}$ channel uses to Tx2, to achieve the same individual rate at both users. However, since in this work, we are interested in the achievable sum-rate and not in the individual rate, we explain the scheme for the case that Tx2 is inactive in all channel uses.

**Encoding at transmitters:** Tx1 generates in the $k$th channel use, the following signal vector

$$x_1[k] = \begin{bmatrix} u_{1,cm}[k] \\ u_{1,df}[k] \\ 0_s \end{bmatrix}, \quad k = 1, \ldots, n,$$

where $u_{1,cm}$ and $u_{1,df}$ represent the common and DF signal vectors, respectively. The length of the zero vector $s$ is chosen such that the length of $x_1[k]$ is $q$. The length of signal vectors $u_{1,cm}[k]$ and $u_{1,df}[k]$ are $R_{cm}$ and $R_{df}$, respectively, where

$$R_{cm} = n_d$$

$$R_{df} = \min\{(n_s - n_d)^+, (n_r - n_d)^+\}$$

Moreover, $u_{1,cm}[n]$ and $u_{1,df}[n]$ are both zero vectors. Notice that Tx2 is silent.

**Decoding at the relay:** In channel use $k = 1, \ldots, n - 1$, the relay receives the top-most $n_s$ bits of $x_1[k]$. Note that the length of $u_{1,df}[k]$ is chosen such that the relay is able to receive all bits in $u_{1,df}[k]$. Therefore, the relay knows $u_{1,df}[k]$ in the $(k + 1)$th channel use.

**Encoding at the relay:** In channel use $k = 2, \ldots, n$, the relay sends

$$x_r[k] = \begin{bmatrix} u_{r,df}[k] \\ 0_r \end{bmatrix},$$

where $u_{r,df}[k] = u_{1,df}[k - 1]$ and $r$ is chosen such that the length of $x_r[k]$ is $q$.  

| Regime | $n_d \leq \min\{\frac{n}{2}, n_d^{pr}\}$ | $n_s \leq \min\{n_r + n_c, 2n_d\}$ | $n_r + n_c \leq \min\{n_s, 2n_d\}$ |
|--------|---------------------------------|---------------------------------|---------------------------------|
| $\ell_{cm1}$ | $0$ | $2n_c - n_s$ | $n_c - n_r$ |
| $R_{cm2}$ | $n_d - \min\{n_r, n_s - n_c\}$ | $0$ | $0$ |
| $R_{ef1}$ | $0$ | $0$ | $0$ |
| $R_{ef2}$ | $0$ | $0$ | $0$ |
| $R_{df1}$ | $0$ | $0$ | $0$ |
| $R_{df2}$ | $0$ | $0$ | $0$ |
| $R_{cm1}$ | $2R_{df1}$ | $n_s$ | $n_r + n_c$ |
| $R_{cm2}$ | $\min\{n_r, n_s - n_c\}$ | $n_s - n_c$ | $n_r$ |
| $\ell_1$ | $n_c - n_d$ | $0$ | $0$ |
| $R_{\Sigma}$ | $2n_d$ | $0$ | $0$ |

**TABLE IX:** Rate allocation parameters for the scheme SI when $n_r < n_d$. 

Proposition 5. The achievable sum-rate with the scheme II for the IRC is given by

$$R_{\Sigma} = \max\{n_d, \min\{n_r, n_s\}\} \text{ if } n_c = n_d.$$ 

This proves the achievability of Theorem 2 when $n_c = n_d$. Note that this sum-rate expression coincides with the upper bounds given in Lemma 2.

In what follows, we present this scheme in details. Compared to the previous schemes, in which both transmitters send in all channel uses, in this scheme, only one Tx is active. One can use time division multiplexing access (TDMA) and assign the first $\frac{n}{2}$ channel uses to Tx1 and the second $\frac{n}{2}$ channel uses to Tx2, to achieve the same individual rate at both users. However, since in this work, we are interested in the achievable sum-rate and not in the individual rate, we explain the scheme for the case that Tx2 is inactive in all channel uses.

**Encoding at transmitters:** Tx1 generates in the $k$th channel use, the following signal vector

$$x_1[k] = \begin{bmatrix} u_{1,cm}[k] \\ u_{1,df}[k] \\ 0_s \end{bmatrix}, \quad k = 1, \ldots, n,$$

where $u_{1,cm}$ and $u_{1,df}$ represent the common and DF signal vectors, respectively. The length of the zero vector $s$ is chosen such that the length of $x_1[k]$ is $q$. The length of signal vectors $u_{1,cm}[k]$ and $u_{1,df}[k]$ are $R_{cm}$ and $R_{df}$, respectively, where

$$R_{cm} = n_d$$

$$R_{df} = \min\{(n_s - n_d)^+, (n_r - n_d)^+\}$$

Moreover, $u_{1,cm}[n]$ and $u_{1,df}[n]$ are both zero vectors. Notice that Tx2 is silent.

**Decoding at the relay:** In channel use $k = 1, \ldots, n - 1$, the relay receives the top-most $n_s$ bits of $x_1[k]$. Note that the length of $u_{1,df}[k]$ is chosen such that the relay is able to receive all bits in $u_{1,df}[k]$. Therefore, the relay knows $u_{1,df}[k]$ in the $(k + 1)$th channel use.

**Encoding at the relay:** In channel use $k = 2, \ldots, n$, the relay sends

$$x_r[k] = \begin{bmatrix} u_{r,df}[k] \\ 0_r \end{bmatrix},$$

where $u_{r,df}[k] = u_{1,df}[k - 1]$ and $r$ is chosen such that the length of $x_r[k]$ is $q$.  

| Regime | $n_d \leq \min\{\frac{n}{2}, n_d^{pr}\}$ | $n_s \leq \min\{n_r + n_c, 2n_d\}$ | $n_r + n_c \leq \min\{n_s, 2n_d\}$ |
|--------|---------------------------------|---------------------------------|---------------------------------|
| $\ell_{cm1}$ | $0$ | $2n_c - n_s$ | $n_c - n_r$ |
| $R_{cm2}$ | $n_d - \min\{n_r, n_s - n_c\}$ | $0$ | $0$ |
| $R_{ef1}$ | $0$ | $0$ | $0$ |
| $R_{ef2}$ | $0$ | $0$ | $0$ |
| $R_{df1}$ | $0$ | $0$ | $0$ |
| $R_{df2}$ | $0$ | $0$ | $0$ |
| $R_{cm1}$ | $2R_{df1}$ | $n_s$ | $n_r + n_c$ |
| $R_{cm2}$ | $\min\{n_r, n_s - n_c\}$ | $n_s - n_c$ | $n_r$ |
| $\ell_1$ | $n_c - n_d$ | $0$ | $0$ |
| $R_{\Sigma}$ | $2n_d$ | $0$ | $0$ |
**Lemma 4.** If of the LD-IRC in Section IV to establish the upper bounds for the GDoF of the Gaussian case. Before presenting the upper GDoF characterization for the Gaussian case. To do this, we use backward decoding. Rx uses backward decoding until the first channel use, Rx vector in the $n$th channel use. In the $(n-1)^{th}$ channel use, Rx receives
\[
y_1[n-1] = S^{r-d}x_1[n-1] + S^{r-n_r}x_r[n-1]
\]
(117)
\[
= \begin{bmatrix}
0_{q-\max\{n_d,n_r\}} \\
u_r,df[n-1] \\
0_{\ell_1} \\
u_{1,cm}[n-1]
\end{bmatrix},
\]
(118)
where $\ell_1 = (n_r-n_d-R_{df})^+$. Since $0 \leq \ell_1$, the signal vectors $u_{r,df}[n-1]$ and $u_{1,cm}[n-1]$ are received without an overlap at Rx1. Hence, Rx1 decodes both these signal vectors and obtains $u_{1,df}[n-2]$ and $u_{1,cm}[n-1]$. Doing this decoding backward until the first channel use, Rx1 receives $u_{1,cm}[k]$ and $u_{1,df}[k]$ for all $k = 1,\ldots,n-1$. Therefore, by using this scheme, we achieve
\[
R_s,11 = n_d + \min\{(n_s-n_d)^+,(n_r-n_d)^+\}
\]
(119)
\[
= \max\{n_d,\min\{n_r,n_s\}\}
\]
(120)
which shows the achievable sum-rate given in Proposition[5] This scheme completes the achievability of Theorem[2] together with schemes WI-1, WI-2, WI-3, and SI.

Until now, we characterized the sum-capacity for the LD-IRC when $n_c < n_s$. The rest of the paper is dedicated to the GDoF characterization for the Gaussian case.

**VI. UPPER BOUNDS FOR THE GAUSSIAN IRC**

In this section, we prove the converse of Theorem[3] To do this, we use the insights obtained from bounding the capacity of the LD-IRC in Section IV to establish the upper bounds for the GDoF of the Gaussian case. Before presenting the upper bounds on the capacity of Gaussian IRC, we present a lemma which will be required for establishing one of the upper bounds.

**Lemma 4.** If $\Gamma^n = \frac{h}{\sqrt{n}}X^n_i + U^n_i$ and $\Delta^n = h_cX^n_i + h_jX^n_j + Z^n_j$, where $i,j \in \{1,2\}$, $i \neq j$, and $U_i \sim \mathcal{N}(0,1)$ is i.i.d. over the time and independent from all other random variables, then $h(\Gamma^n) - h(\Delta^n|W_j)$ is upper bounded as follows
\[
h(\Gamma^n) - h(\Delta^n|W_j) \leq nC\left(2 + \frac{h_{ij}}{(h_c-h_{ij})^2}\right).
\]
(121)

**Proof:** The proof is given in Appendix[3]

Now, we present the upper bounds on the capacity of the Gaussian IRC in the following lemma.

**Lemma 5.** The GDoF of the Gaussian IRC is upper bounded by
\[
d \leq \max\{1,\min\{\beta,\gamma\}\} \quad \text{if } \alpha = 1
\]
(122)
\[
d \leq \max\{1,\alpha,\beta\} + \max\{1,\alpha\} \quad \text{if } \alpha = 2
\]
(123)
\[
d \leq \beta + 2\max\{1,\alpha\} - \alpha
\]
(124)
\[
d \leq \max\{1,\alpha\} + \max\{1,\gamma\}
\]
(125)
\[
d \leq 2\max\{\alpha,\beta,1-\max\{\alpha,\gamma\}\} + 2(\gamma-\alpha)^+
\]
(126)
\[
d \leq 2\max\{\alpha,\beta+1-\alpha\} \quad \text{if } \beta \leq \alpha \leq 1.
\]
(127)

**Proof:** While the first two upper bounds are cut-set bounds, the remaining bounds are established by using genie-aided
methods. The bounds given in (125) and (126) are inspired by similar bounds presented in [12] Theorems 3,4] which are tightened for the case where $\alpha < \gamma$. The complete proof of this lemma is given in Appendix C.

In addition to the upper bounds in Lemma 5, some upper bounds are borrowed from [12, 19]. In the following lemma, we present these bounds.

Lemma 6. (12) The GDoF of the Gaussian IRC is upper bounded by

$$d \leq 2 \max\{1, \beta\}$$

(128)

$$d \leq \max\{1, \beta, \alpha\} + \max\{1, \alpha\} - \alpha + (\gamma - \max\{1, \alpha\})^+.$$ (129)

While the first upper bound is a cut-set bound, the second upper bound is derived by using the genie-aided method. The proof of these bounds are given in [12].

Now, we need to show that the minimum of the upper bounds in (122)-(129) coincide with the GDoF expression in Theorem 3. This can be shown similar to the linear deterministic case by keeping in mind that the channel parameters $n_d, n_c, n_r,$ and $n_a$ in the LD-IRC are equivalent to $1, \alpha, \beta,$ and $\gamma$ in the Gaussian IRC, respectively.

VII. GDoF Achieving Schemes

In this section, we show the achievability of the GDoF in Theorem 3. This will be done by extending the achievability schemes presented for the LD-IRC to the Gaussian case in a similar manner as in [15, 27]. To do this, we decompose the Gaussian channel into $N$ sub-channels which can be accessed at the receiver side successively by using successive decoding. This decomposition reduces the rate allocation problem to a sub-channel allocation problem which can be solved as in the LD-IRC. In what follows, we present the idea of the channel decomposition for the point-to-point (P2P) channel. Then, we present the transmission scheme for the Gaussian IRC. At the end, we present the the strategies used in the Gaussian IRC over the sub-channels.

1) Point-to-point channel: Consider a received signal over a point-to-point channel in $n$ channel uses $y^n = x^n + z^n$, where $x$, $y$, and $z$ are the inputs with a power constraint $P (1 < P)$, output, and AWGN with unit variance, respectively. By decomposing the channel into $N$ sub-channels, the received signal $y$ can be rewritten as $y = \sum_{\ell=1}^{N} x_\ell + z$, where the power of $x_\ell$ is $\delta^\ell - \delta^{\ell-1}$ and its rate is $R_\ell$, where $\log \delta = \frac{1}{N} \log P$. Note that the power constraint is satisfied since $\sum_{\ell=1}^{N} \delta^\ell - \delta^{\ell-1} = P - 1 < P$. Notice that the signal in the $\ell$th sub-channel is received on a higher power level than in the $(\ell - 1)$th sub-channel. Therefore, by doing successive decoding at the receiver, the receiver decodes $x^n_\ell$ while $x^n_{\ell-1}, \ldots, x^n_1$ are treated as noise. Hence, the following rate is achievable

$$R_\ell \leq \frac{1}{2} \log \left( 1 + \frac{p_\ell}{1 + p_1 + p_2 + \ldots + p_{\ell-1}} \right)$$

$$= \frac{1}{2} \log \left( 1 + \frac{\delta^\ell - \delta^{\ell-1}}{\delta^{\ell-1}} \right)$$

$$= \frac{1}{2N} \log(P).$$ (130)

Using this for all sub-channels, we obtain the sum-rate $\frac{1}{2} \log(P)$ which is approximately equal to the capacity of the P2P channel in the high SNR regime.

2) Gaussian IRC: Now, we want to describe the transmission scheme for the Gaussian IRC. Suppose that Tx1 wants to send a message $W_1(b)$ to Rx1 in block $b$, where $b = 1, \ldots, B$. To do this, Tx1 uses a nested-lattice codebook (17], 28], and 29] $(\Lambda_f, \Lambda_c)$ with rate $R_\alpha$ and unit power, to generate the codewords $x^n_{1,\ell}(b)$, where $\ell = 1, \ldots, N$ and $\Lambda_f, \Lambda_c$ represent the fine, coarse lattices, respectively. The codeword $x^n_{1,\ell}(b)$ is given as follows

$$x^n_{1,\ell}(b) = \sqrt{P_\ell} \left[ (\lambda_1,\ell(b) - d_{1,\ell}) \mod \Lambda_c \right],$$ (131)
where $\lambda_{1,\ell}(b) \in \Lambda_f$ and $d_{1,\ell}$ is an $n$-dimensional random dither vector uniformly distributed over the fundamental Voronoi region $\mathcal{V}(\Lambda_c)$. Note that the dither vector is assumed to be known at all nodes. Tx1 sends in the $\ell$th sub-channel $x_{1,\ell}^n(b)$. Hence, $x_{1,\ell}^n(b) = \sum_{\ell=1}^N x_{1,\ell}(b)$. Similar to the P2P channel, the power of $x_{1,\ell}^n(b)$ is $p_\ell = \delta^\ell - \delta^{\ell-1}$ and its rate is $R_{1,\ell}$, where $\log(\delta) = \frac{1}{\lambda} \log(P)$. Note that the transmit power by Tx1 satisfies the power constraint $P$. Tx1 can decide whether it sends over the $\ell$th sub-channel or not. Hence, $R_{1,\ell} \in \{0, R_s\}$, where $R_s$ represents the maximum achievable rate by using a sub-channel. The same is done by Tx2. Note that both Tx's use the same coarse and fine lattices for generating the code-words.

Now, consider the relay side. The received signal at the relay in block $b$ is given by

$$y_r^n(b) = y_r^n(b) + \sum_{\ell=1}^{N-N_s} h_s[x_{1,\ell}^n(b) + x_{2,\ell}^n(b)] + z_r^n(b), \quad (132)$$

where $y_r^n$ is the part which is received at the relay higher than the noise level. This part is the sum of the transmitted signals by both Tx's in the top-most $N_s$ sub-channels. Hence, we can write

$$y_{r,\ell}^n(b) = \sum_{\ell' = 1}^{N_s} y_{r,\ell'}^n(b) = \sum_{\ell = N-N_s+1}^{N} h_s[x_{1,\ell}^n(b) + x_{2,\ell}^n(b)]. \quad (133)$$

To obtain $N_s$, consider the $(N - N_s + 1)$th sub-channel. The signal in this sub-channel is received at the relay on the lowest power level which is still higher than the noise level. Therefore, we write

$$1 < \delta^{(N - N_s)} h_s^2. \quad (134)$$

By solving this inequality, we obtain $N_s \leq \frac{\log(P h_s^2)}{\log(\delta)}$. Since $N_s$ is the maximum number of the sub-channels received at the relay higher than the noise level, we obtain $N_s = \left\lfloor \frac{\log(P h_s^2)}{\log(\delta)} \right\rfloor$.

In each block $b$, the relay decodes $y_{r}^n(b)$. To do this, it decodes first the received signal in the highest sub-channel, i.e., $y_{r,1}^n(b) = h_s(x_{1,1}^n + x_{2,1}^n)$. Hence, it decodes first the sum $\lambda_{1,N}(b) + \lambda_{2,N}(b) \mod \Lambda_c$ while it treats the interference caused by the lower sub-channels, i.e., $h_s(x_{1,1}^n + x_{2,1}^n)$ as noise. After decoding $\lambda_{1,N}(b) + \lambda_{2,N}(b) \mod \Lambda_c$ successfully, the relay constructs $x_{1,1}^n + x_{2,1}^n$ as shown in [30]. Then it removes the interference caused by $h_s(x_{1,1}^n + x_{2,1}^n)$. Next, it decodes $y_{r,1}^n(b)$ by treating all signals received in lower sub-channels as noise. Proceeding this decoding successively, relay completes decoding $y_{r}^n(b)$. Generally, the relay is able to decode the sum $\lambda_{1,\ell}(b) + \lambda_{2,\ell}(b) \mod \Lambda_c$ for all $\ell \in \{N - N_s + 1, \ldots, N\}$, as long as

$$R_{1,\ell}, R_{2,\ell} \leq R_s \leq \frac{1}{2} \log \left( \frac{1}{2} + \frac{h_s^2 p_\ell}{1 + 2h_s^2(p_{\ell-1} + p_{\ell-2} + \ldots + p_1)} \right) \quad (135)$$

$$= \frac{1}{2} \log \left( \frac{1}{2} + \frac{h_s^2 (\delta^\ell - \delta^{\ell-1})}{1 + 2h_s^2 (\delta^{\ell-1} - 1)} \right). \quad (136)$$

The condition in (136) is written for the worst case which is the case when both transmitters share all the sub-channels. Suppose that the $\ell$th sub-channel is used only by one of the transmitters or some sub-channels from the first to the $(\ell - 1)$th one are not used by both Tx's. Then, the rate constraint will be looser than that of in (136). Now, using the fact that $1 - 2h_s^2 < 1 \leq h_s^2 \delta^{\ell-1}$ for all $\ell \in \{N - N_s + 1, \ldots, N\}$, we tighten the condition in (136) as follows

$$R_{1,\ell}, R_{2,\ell} \leq R_s \leq \frac{1}{2} \log \left( \frac{1}{3} + \frac{h_s^2 (\delta^\ell - \delta^{\ell-1})}{3h_s^2 \delta^{\ell-1}} \right) \quad (137)$$

$$= \frac{1}{2} \log \left( \frac{\delta}{3} \right). \quad (138)$$

After decoding the received signal in block $b$, the relay generates $x_{r,\ell}^n(b+1), \ell = 1, \ldots, N$ which is sent over the $\ell$th sub-channel in the $(b+1)$th block. This signal is generated by using the nested-lattice codebook $(\Lambda_f, \Lambda_c)$ as follows

$$x_{r,\ell}^n(b+1) = \sqrt{P_{r,\ell}} [(\lambda_{r,\ell}(b+1) - d_{r,\ell}) \mod \Lambda_c], \quad (139)$$
where \( \lambda_{c,t}(b+1) \in \Lambda_f \) and \( d_{c,t} \) is an \( n \)-dimensional random dither vector uniformly distributed over the fundamental Voronoi region \( \mathcal{V}(\Lambda_f) \). The power and the rate of \( x_{c,t}^{n}(b+1) \) is \( p_{c,t} \leq P_r \) and \( R_{c,t} \), respectively. Similar to encoding at the transmitter side, relay can decide whether it uses the \( \ell \)th sub-channel or not. Therefore, \( R_{c,t} \in \{0, R_s \} \). The sent signal in the \((b+1)\)th block by the relay is \( x_{r}^{n}(b+1) = \sum_{\ell=1}^{N} x_{r,\ell}^{n}(b+1) \). The transmit power of the relay is given by \( P' = \sum_{\ell=1}^{N} p_{r,\ell} \leq P \).

Now, consider the decoding at the receiver side. Here, we explain the decoding at Rx1. The same is done by Rx2. Rx1 starts with decoding at the end of block \( B \). In the block \( B \), Rx1 receives \( y_{1}^{n}(B) = h_d x_{1,d}^{n}(B) + h_c x_{1,c}^{n}(B) + z_{1}^{n}(B) \). The signal which is received at Rx1 higher than noise power level is given by

\[
y_{1}^{n}(B) = \sum_{\ell=1}^{N_m} y_{1,\ell}^{n}(B) = \sum_{\ell=N-N_d+1}^{N} h_d x_{1,d,\ell}^{n}(B) + \sum_{\ell=N-N_c+1}^{N} h_c x_{1,c,\ell}^{n}(B) + \sum_{\ell=N-N_r+1}^{N} h_r x_{r,\ell}^{n}(B),
\]

where \( N_d, N_c, \) and \( N_r \) are the number of sub-channels which are received at Rx1 from Tx1, Tx2, and the relay higher than the noise level, respectively. Moreover, \( N_m \) is number of sub-channels which are observed at Rx1 higher than the noise level. Therefore, we obtain \( N_d = \lceil \log(P_{h_d}^2) / \log(\delta) \rceil \), \( N_c = \lceil \log(P_{h_c}^2) / \log(\delta) \rceil \), \( N_r = \lceil \log(P_{h_r}^2) / \log(\delta) \rceil \), and \( N_m = \max\{N_d, N_c, N_r\} \). This can be shown similar to obtaining \( N_s \).

**Remark 4.** To guarantee that the sub-channels used by both Tx’s are aligned at Rx1, we choose number of sub-channels \( N \) such that it exists an \( \ell \in \{1, \ldots, N\} \) where

\[
\begin{align*}
Ph_{c}^2 &= h_{d}^2 \delta^\ell \quad \text{if } h_{c}^2 < h_{d}^2 \\
Ph_{d}^2 &= h_{c}^2 \delta^\ell \quad \text{if } h_{d}^2 < h_{c}^2 
\end{align*}
\]

(141)

For aligning the sub-channels used by the relay and Tx’s at Rx1, the relay needs to reduce its transmit power to \( P' \) given as follows

\[
P' = \frac{\delta^{N_r}}{h_{r}^2} \leq P.
\]

(142)

Notice that reducing the transmit power of the relay from \( P \) to \( P' \) does not change the number of sub-channels received from relay at the Rx’s over the noise level, i.e., \( N_r = \lceil \log(P'h_{r}^2) / \log(\delta) \rceil \).

Decoding the received signal in block \( B \) is done at Rx1 in a successive manner. This is started with decoding \( y_{1,N_m}^{n}(B) \). After doing this, Rx1 removes the interference caused by \( y_{1,N_m-1}^{n}(B) \) and decodes \( y_{1,N_m-2}^{n}(B) \). This successive decoding is proceeded until end of decoding \( y_{1,1}^{n}(B) \). To write the rate constraint for successful decoding of \( y_{1,\ell}^{n}(B) \) for all \( \ell \in \{1, \ldots, N_m\} \), we consider the worst case which can occur. This is when for all \( \ell' \in \{1, \ldots, N_m\} \), \( y_{1,\ell'}^{n}(B) \) contains three signals which are from Tx1, Tx2, and the relay. Suppose that Rx1 wants to decode \( y_{1,\ell}^{n}(B) = h_d x_{1,d,\ell}^{n}(B) + h_c x_{1,c,\ell}^{n}(B) + h_r x_{r,\ell}^{n}(B) \), where \( \ell \in \{1, \ldots, N_m\} \) and \( \ell_1, \ell_2, \) and \( \ell_r \) are the index of the sub-channels used by Tx1, Tx2, and the relay which are received aligned at Rx1. Therefore, \( h_d p_{\ell_1} = h_c p_{\ell_2} = h_r p_{\ell_r} = p_{\ell'} \), where \( p_{\ell'} = \delta^{\ell'} - \delta^{\ell'-1} \). Rx1 is able to decode \( h_d \lambda_{1,\ell_1}(B) + h_c \lambda_{2,\ell_2}(B) + h_r \lambda_{r,\ell_r}(B) \) mod \( \Lambda_f \) as long as

\[
R_{\ell_1}, R_{\ell_2}, R_{r,\ell_r} \leq R_s \leq \frac{1}{2} \log \left( \frac{1}{3} + \frac{p_{\ell'}}{1 + 3(p_1 + p_2 + \ldots + p_{\ell'-1}) + 3} \right)
\]

(143)

\[
= \frac{1}{2} \log \left( \frac{1}{3} + \frac{\delta^{\ell'} - \delta^{\ell'-1}}{1 + 3(\delta^{\ell'-1} - 1) + 3} \right)
\]

(144)

\[
= \frac{1}{2} \log \left( \frac{1}{3} + \frac{\delta^{\ell'} - \delta^{\ell'-1}}{1 + 3(\delta^{\ell'-1})} \right)
\]

(145)
Since for all \( \ell' \in \{1, \ldots, N_m\}, 1 \leq \delta^\ell' - 1 \), we tighten the condition in (145) and obtain
\[
R_{t_1}, R_{t_2}, R_{r, t_2} \leq R_s \leq \frac{1}{2} \log \left( \frac{1}{4} + \frac{\delta^\ell' - \delta^{\ell' - 1}}{4\delta^{\ell' - 1}} \right) = \frac{1}{2} \log \left( \frac{\delta}{4} \right). \tag{146}
\]
By considering both conditions in (138) and (147), we conclude that the maximum achievable rate using one sub-channel is given by
\[
R_s = \frac{1}{2} \log \left( \frac{\delta}{4} \right). \tag{148}
\]
Note that \( \delta \) has to be larger than 4, which is equivalent to \( 4 < P_1 \). This is always satisfied when \( P \to \infty \).

After decoding \( y_1^n(B) \), Rx1 decodes the received signal in the block \((B - 1), \text{i.e., } y_1^n(B - 1)\). Due to the backward decoding, Rx1 knows parts of \( y_1^n(B - 1) \) a priori. Hence, it removes first the interference caused by these parts before decoding \( y_1^n(B - 1) \). Then, it starts decoding \( y_1^n(B - 1) \) similar to \( y_1^n(B) \). The backward decoding proceeds until the end of decoding \( y_1^n(1) \).

Now, we discuss different strategies which can be used over each sub-channel. Consider the \( \ell^{th} \) sub-channel at the receiver side in block \( b \), i.e., \( y_{1, \ell}^n(b) = h_{d, x_{1, \ell}^n} + h_{r, x_{2, \ell}^n} + h_{r, x_{1, \ell}^n}(b) \). After cancelling the interference caused by the a priori known signals (known due to the backward decoding or successive interference cancellation), this sub-channel can be used only for one of the following cases.

- **Common signaling**: Let suppose that Tx1 uses the \( \ell \)th sub-channel for sending the common signal. Since both receivers have to be able to decode this signal, this sub-channel has to be received at both receivers over the noise level. Therefore, we have
  \[
  N - \min\{N_d, N_c\} < \ell. \tag{149}
  \]

- **Private signaling**: Compared to common signal, only the desired Rx needs to be able to decode the private signal. Therefore, if Tx\( i \) \((i \in \{1, 2\})\) sends over \( \ell \)th sub-channel a private signal, then the following condition needs to be satisfied for reliable decoding of the private signal at Rx\( i \).
  \[
  N - N_d < \ell. \tag{150}
  \]

- **CF signaling**: In CF signaling, relay is also involved in the communication. This is done as follows. Suppose that Tx’s use \( \ell \)th sub-channel for transmitting the CF signal. This sub-channel must be in the top-most \( N_s \) sub-channels. Unless the relay does not observe this sub-channel over the noise level. Therefore, we write
  \[
  N - N_s < \ell. \tag{151}
  \]

Using nested lattice code, the relay decodes in block \( b \) the sum \( \lambda_{1, \ell}(b) + \lambda_{2, \ell}(b) \mod \Lambda_c \). Next, the relay encodes this sum into \( x_{r, \ell,}(b + 1) \) and sends it over \( \ell \)th sub-channel in the next block. At the receiver side, Rx1 receives in block \( b \), over sub-channel \( N_d - (N - \ell), N_c - (N - \ell), \) and \( N_r - (N - \ell) \) the CF signal from Tx1, Tx2, and the relay, respectively. As we mentioned, before Rx1 starts decoding with the last block to the first one. Suppose that the decoding of received signal in block \( B, B - 1, \ldots, b + 1 \) is done successfully. Therefore, Rx1 knows \( \lambda_{1, \ell}(b) + \lambda_{2, \ell}(b) \mod \Lambda_c \), since this sum is sent by the relay in the \((b + 1)\)th block. Using this sum, Rx1 can obtain the CF signals from both Tx’s if it decodes either \( \lambda_{1, \ell}(b) \) or \( \lambda_{2, \ell}(b) \). Depending on the channel strength, Rx1 decodes the CF signal which is received in the higher sub-channel and reconstructs the other one. For instance, suppose that the desired channel is stronger than the undesired channel \((N_c < N_d)\). Then, Rx1 obtains \( \lambda_{1, \ell}(b) \). Using \( \lambda_{1}(b) + \lambda_{2}(b) \mod \Lambda_c \) which is known at the Rx1, it obtains \( \lambda_{2, \ell}(b) \). Knowing \( \lambda_{2, \ell}(b) \), Rx1 reconstructs \( x_{2, \ell}(b) \) and removes the interference caused by \( x_{2, \ell}(b) \). Moreover,
Rx1 decodes the relay CF signal sent in channel use \( b \). This decoding can be done reliably as long as

\[
N - N_r < \ell_r
\]  
(152)

\[
N - \max\{N_d, N_c\} < \ell_c
\]  
(153)

\[
N_c \neq N_d
\]  
(154)

\[
\max\{N_d, N_c\} + \ell \neq N_r + \ell_r.
\]  
(155)

While conditions (152) and (153) guarantee that the CF signals sent by the relay and the Tx with stronger channel are received higher than the noise level at Rx1, conditions (154) and (155) avoid an overlap between the CF sub-channels of the Tx’s and the CF sub-channels of the relay and the stronger Tx.

- **DF signaling:** In this strategy, the relay needs to be able to decode both signals sent by Tx’s separately. Therefore, Tx1 and Tx2 have to use different sub-channels for transmitting their DF signals. Suppose that Tx1 and Tx2 use the sub-channels \( \ell_1 \) and \( \ell_2 \) to send \( x_{1,\ell_1}(b) \) and \( x_{2,\ell_2}(b) \) in block \( b \), respectively, where \( \ell_1 \neq \ell_2 \). Relay is able to observe both sub-channels over the noise level as long as

\[
N - N_s < \min\{\ell_1, \ell_2\}.
\]  
(156)

In next block \( (b + 1) \), the relay sends \( x_{1,\ell_1}(b) \) and \( x_{2,\ell_2}(b) \) in sub-channels \( \ell_{r1} \) and \( \ell_{r2} \) (\( \ell_{r1} \neq \ell_{r2} \)), respectively. At the receiver side, Rx’s use the backward decoding. Supposing that decoding received signal in blocks \( B, B - 1, \ldots, b + 1 \) is done successfully, Rx1 knows \( x_{1,\ell_1}(b) \) and \( x_{2,\ell_2}(b) \) since they are sent by relay in block \( b + 1 \). Therefore, Rx1 removes the interference caused by these signals before decoding the received signal in block \( b \). Next, Rx1 decodes \( x_{1,\ell_1}(b - 1) \) and \( x_{2,\ell_2}(b - 1) \). Note that these two signals are sent both by the relay. Decoding of these signals can be done successfully, as long as

\[
N - N_r < \min\{\ell_{r1}, \ell_{r2}\}.
\]  
(157)

- **CN signaling:** Using this strategy, Tx \( i \in \{1, 2\} \) sends \( x_{i,\ell}(b) \) and \( x_{i,\ell_F}(b) \) in block \( b \) over sub-channels \( \ell \) and \( \ell_F \), respectively. It is worth mentioning that

\[
x_{i,\ell_F}(b) = \sqrt{P_{F\ell}}x_{i,\ell}(b + 1).
\]  
(158)

In other words, in block \( b \) the \( \ell_F \)th sub-channel is used for sending the same signal as in the \( \ell \)th sub-channel in block \( (b + 1) \). Suppose that the decoding at the relay has been done successfully in block 1 to \( b - 1 \). Hence, relay knows \( x_{1,\ell}(b) + x_{1,\ell}(b) \) in beginning of block \( b \). Therefore, in block \( b \) relay removes the interference caused by \( x_{1,\ell}(b) + x_{1,\ell}(b) \) and then it decodes \( \lambda_{1,\ell_F}(b) + \lambda_{2,\ell_F}(b) \mod \Lambda_c \). This can be done successfully as long as

\[
N - N_s < \ell_F.
\]  
(159)

Knowing \( \lambda_{1,\ell_F}(b) + \lambda_{2,\ell_F}(b) \mod \Lambda_c \) in block \( b \), the relay constructs \( \lambda_{1,\ell}(b + 1) + \lambda_{2,\ell}(b + 1) \mod \Lambda_c \). Next, the relay sends in block \( b + 1 \) and sub-channel \( \ell_F \), the following sum

\[
x^*_{r,\ell_F}(b + 1) = -\sqrt{P_{r,\ell_F}}[\lambda_{1,\ell}(b + 1) + \lambda_{2,\ell}(b + 1) \mod \Lambda_c],
\]  
(160)

where \( \ell_F = \ell + N_c - N_r \). This is equivalent to \( P_{r,\ell_F} = P_r I_{r,\ell_F}^2 \). The decoding at the destination is done backward. Suppose that decoding the received signal at Rx1 in blocks \( B, B - 1, \ldots, b + 1 \) is done successfully. Hence, Rx1 knows \( \lambda_{1,\ell}(b + 1) \) before decoding block \( b \). Knowing \( \lambda_{1,\ell}(b + 1) \), Rx1 is able to reconstruct \( x_{1,\ell_F}(b) \). Therefore, Rx1 removes first the interference of \( x_{1,\ell_F}(b) \). Depending on the channel strength, the decoding order can be changed. First, suppose that \( N_d < N_c \). In this case, Rx1 decodes first the sum of \( h_c x_{2,\ell}(b) + h_r x_{r,\ell}(b) \). Note that \( \ell_r \) is chosen such that \( h_c x_{2,\ell}(b) \) and \( h_r x_{r,\ell}(b) \) are completely aligned over the same sub-channel. Therefore, over this sub-channel,
Rx1 observes $h_c x_{2,\ell}(b) + h_r x_{r,\ell}(b)$. To decode $\lambda_1,\ell(b)$, Rx1 divides this sum with $h_c \sqrt{P_\ell}$ and adds the dither vector $d_{2,\ell}$ then it calculates the quantization error with respect to $\Lambda_c$. Therefore, it obtains

$$[(\lambda_2,\ell(b) - d_{2,\ell}) \mod \Lambda_c - (\lambda_1,\ell(b) + \lambda_2,\ell(b)) \mod \Lambda_c + d_{2,\ell}] \mod \Lambda_c = -\lambda_1,\ell(b) \mod \Lambda_c. \quad (161)$$

In this way, Rx1 decodes $\lambda_1,\ell(b)$. Knowing $\lambda_1,\ell(b)$, Rx1 reconstructs $x_{1,\ell}(b)$ and removes its interference. This decoding can be done successfully, as long as

$$\ell + N_c - N_r = \ell_r \quad (162)$$

$$N - N_c \leq \ell \quad \text{if} \quad N_d < N_c. \quad (163)$$

While the condition in (162) guarantees that the relay CN signal is received over the same sub-channel as the undesired CN signal, the condition in (163) guarantees that these two aligned signals are received above the noise level.

Now, suppose that $N_c < N_d$. In this case, Rx1 decodes first $\lambda_1,\ell(b)$ (from $x_{1,\ell}(b)$) as in the P2P channel and then it removes the interference caused by $h_c x_{2,\ell}(b) + h_r x_{r,\ell}(b)$ which is observed in the sub-channel $N_c - (N - \ell)$. This interference cancellation is done by dividing the signal in sub-channel $N_c - (N - \ell)$ by $h_c \sqrt{P_\ell}$, adding $\lambda_1,\ell + d_{2,\ell}$ to it and finally calculating its quantization error with respect to $\Lambda_c$. This is given as follows.

$$[(\lambda_2,\ell(b) - d_{2,\ell}) \mod \Lambda_c - (\lambda_1,\ell(b) + \lambda_2,\ell(b)) \mod \Lambda_c + \lambda_1,\ell + d_{2,\ell}] \mod \Lambda_c = 0. \quad (164)$$

In this case ($N_c < N_d$), following conditions need to be satisfied.

$$\ell + N_c - N_r = \ell_r \quad (165)$$

$$N - N_d \leq \ell \quad \text{if} \quad N_c < N_d. \quad (166)$$

**Remark 5.** Note that when $N_c = N_d$, CN signaling cannot be used. This is due to the fact that the relay CN signal neutralizes both undesired and desired CN signals.

The schemes which are explained in the LD-IRC are combinations of private and common signaling, in addition to CF, DF, and CN relaying scheme. By using these strategies over the sub-channels in the same manner as it is shown for the LD-IRC, we achieve the upper bound for the GDoF. Notice that while in the LD-IRC, we optimize over the number of bits which should be use by each strategy, in the Gaussian case we optimize over number of sub-channels. Moreover, while in the LD-IRC using each bit level, we achieve one, in Gaussian IRC, by using one sub-channel, we achieve $R_s = \frac{1}{2} \log (\frac{N}{d})$. Notice that the parameters $n_d$, $n_c$, $n_r$, and $n_s$ in the LD-IRC are equivalent to $N_d$, $N_c$, $N_r$, and $N_s$, in the Gaussian case, respectively. In Appendix A we show how the achievable sum-rate for the LD-IRC can be extended to the achievable GDoF.

**APPENDIX A**

**PROOF OF THE UPPER BOUNDS FOR THE LD-IRC (LEMMA 2)**

**A. Proof of (23)**

The proof of the bound $C_{\text{det},\Sigma} \leq \max \{n_d, n_c, n_r, n_s\}$ follows from the cut-set bounds. Namely, consider the following cut-set bound $R_{S \rightarrow S^c} \leq \max_{P(x_1, x_2, x_r)} I(x_S; y_{S^c} | x_{S^c})$ where $S = \{\text{Tx1, Tx2, Relay}\}$ and $S^c = \{\text{Rx1, Rx2}\}$. This bound yields

$$R_{\Sigma} \leq \max_{P(x_1, x_2, x_r)} I(x_1, x_2, x_r; y_1, y_2). \quad (167)$$
The term $I(x_1, x_2, x_r; y_1, y_2)$ can be upper bounded as follows:

$$I(x_1, x_2, x_r; y_1, y_2) = H(y_1, y_2) - H(y_1, y_2|x_1, x_2, x_r)$$ (168)

$$(a) \leq H(y_1, y_2)$$ (169)

$$= H(y_1) + H(y_2|y_1),$$ (170)

where in step $(a)$, we used the fact that $y_1$ and $y_2$ are deterministic functions of $x_1$, $x_2$, and $x_r$. Note that $y_1$ and $y_2$ are equal for the case where $n_d = n_c$. Thus, $H(y_2|y_1) = 0$. It remains to maximize $H(y_1)$ with respect to the input distribution. Since $y_1$ is a binary vector of length $\max\{n_d, n_c, n_r\} = \max\{n_d, n_r\}$, $H(y_1)$ is maximized when the components of $y_1$ are i.i.d. Bern$(1/2)$, which corresponds to inputs $x_1$, $x_2$, and $x_r$ distributed also according to an i.i.d. Bern$(1/2)$ distribution. Therefore, $H(y_1) \leq \max\{n_d, n_r\}$ leading to

$$R_2 \leq \max\{n_d, n_r\}. \tag{171}$$

Using similar steps with the cut $S = \{T_1, T_2\}$ and $S^c = \{R_1, R_2, \text{Relay}\}$ leads to the bound $R_2 \leq \max\{n_d, n_s\}$. Namely, with this cut, we have

$$R_2 \leq \max_{p(x_1, x_2, x_r)} I(x_1, x_2; y_1, y_2, y_r|x_r). \tag{172}$$

Note that

$$I(x_1, x_2; y_1, y_2, y_r|x_r) = H(y_1, y_2, y_r|x_r) - H(y_1, y_2, y_r|x_r, x_1, x_2)$$ (173)

$$= H(y_1, y_2, y_r|x_r)$$ (174)

$$= H(y_1|x_r) + H(y_2|x_r, y_1) + H(y_r|x_r, y_1, y_2).$$ (175)

The first term $H(y_1|x_r)$ can be bounded as follows

$$H(y_1|x_r) = H(S^{n_d}x_1 + S^{n_c}x_2|x_r)$$ (176)

$$\leq H(S^{n_d}x_1 + S^{n_c}x_2)$$ (177)

$$\leq n_d,$$ (178)

where the first inequality follows since conditioning does not increase entropy, and the second follows since $n_d = n_c$ and since the entropy is maximized by i.i.d. Bern$(1/2)$ inputs. The second term $H(y_2|x_r, y_1)$ is zero since $y_1$ and $y_2$ are equal given $n_d = n_c$. Finally, the last term satisfies

$$H(y_r|x_r, y_1, y_2) = H(y_r|x_r, y_1)$$ (179)

$$= H(y_r|x_r, y_1 + S^{n_c}x_r)$$ (180)

$$\leq H(y_r|y_1 + S^{n_c}x_r)$$ (181)

$$= H(S^{n_c}x_1 + S^{n_c}x_2|S^{n_d}x_1 + S^{n_c}x_2)$$ (182)

$$\leq (n_s - n_d)^+,$$ (183)

where the first inequality follows since conditioning does not increase entropy, and the second follows since the number of unknown bits of $S^{n_c}x_1 + S^{n_c}x_2$ given $S^{n_d}x_1 + S^{n_c}x_2$ where $n_d = n_c$ is $n_s - n_d$ if $n_d \leq n_s$ and zero otherwise, and the entropy of these bits is maximized by the i.i.d. Bern$(1/2)$ distribution. Therefore, we can write

$$R_2 \leq \max\{n_d, n_s\}. \tag{184}$$
Combining (171) and (184), we get
\[ R_{\Sigma} \leq \max\{n_d, \min\{n_c, n_r\}\}, \tag{185} \]
which is the desired bound given in (23) in Lemma 2.

B. Proof of (24)

This bound is in fact derived from the cut-set bound given above in (167). As before, we have
\[ R_{\Sigma} \leq \max_{P(x_1, x_2, x_r)} H(y_1) + H(y_2|y_1), \tag{186} \]
where \( H(y_1) \) is maximized by i.i.d. Bern(1/2) distributed inputs, leading to \( H(y_1) \leq \max\{n_d, n_c, n_r\} \). The last term is non-zero, contrary to the \( n_d = n_c \) case. To bound it, we can use \( H(y_2|y_1) = H(y_2 \oplus y_1|y_1) \) and the property that conditioning does not increase entropy to write
\[ H(y_2|y_1) \leq H(y_2 \oplus y_1). \tag{187} \]
Notice that \( y_2 \oplus y_1 \) given by \( S^{q-n_d}x_2 \oplus S^{q-n_c}x_1 \oplus S^{q-n_r}x_2 \) has \( \max\{n_d, n_c\} \) non-zero components. Thus, the maximum value of \( H(y_2 \oplus y_1) \) is \( \max\{n_d, n_c\} \) and is achieved when \( x_1 \) and \( x_2 \) are i.i.d. Bern(1/2). Thus, \( H(y_2|y_1) \leq \max\{n_d, n_c\} \). Consequently, we get
\[ R_{\Sigma} \leq \max\{n_d, n_c, n_r\} + \max\{n_d, n_c\}, \tag{188} \]
which concludes the proof of the upper bound (24) in Lemma 2.

C. Proof of (25)

For establishing the upper bound (25) in Lemma 2 we use a genie-aided approach. In a general genie-aided approach, the side-information \( s_1 \) and \( s_2 \) is given to receivers 1 and 2, respectively. Then, using Fano’s inequality, we can write
\[ n(R_{\Sigma} - \epsilon_n) \leq I(W_1; y_1^n, s_1) + I(W_2; y_2^n, s_2), \tag{189} \]
where \( \epsilon_n \to 0 \) as \( n \to \infty \).

For this particular case, we use \( s_1 = S^{q-n_r}x_r^n \) and \( s_2 = (S^{q-n_r}x_r^n, y_1^n, W_1) \). By using the chain rule and the independence of the different messages, we can rewrite the bound as
\[ n(R_{\Sigma} - \epsilon_n) \leq I(W_1; S^{q-n_r}x_r^n) + I(W_1; y_1^n|S^{q-n_r}x_r^n) \]
\[ + I(W_2; S^{q-n_r}x_r^n|W_1) + I(W_2; y_1^n|S^{q-n_r}x_r^n, W_1) + I(W_2; y_2^n|y_1^n, S^{q-n_r}x_r^n, W_1), \]
\[ = I(W_1, W_2; S^{q-n_r}x_r^n) + I(W_1, W_2; y_1^n|S^{q-n_r}x_r^n) + I(W_2; y_2^n|y_1^n, S^{q-n_r}x_r^n, W_1). \tag{190} \]

Now we consider every term in (190) separately. The first term in (190) can be written as
\[ I(W_1, W_2; S^{q-n_r}x_r^n) = H(S^{q-n_r}x_r^n) - H(S^{q-n_r}x_r^n|W_1, W_2) \]
\[
\stackrel{(a)}{=} H(S^{q-n_r}x_r^n) \]
\[
\stackrel{(b)}{\leq} n \cdot n_r, \tag{191} \]
where (a) follows since \( H(S^{q-n_r}x_r^n|W_1, W_2) = 0 \), and (b) follows since \( H(S^{q-n_r}x_r^n) \) is the entropy of \( n \cdot n_r \) binary random variables, and thus it is maximized when these random variables are i.i.d. Bern(1/2) distributed.
The second term in (190) can be upper bounded as follows

\[ I(W_2; y_2^n | y_1^n, S^{q-n_r} x_1^n, W_1) = H(y_2^n | y_1^n, S^{q-n_r} x_1^n, W_1) - H(y_2^n | y_1^n, S^{q-n_r} x_1^n, W_1, W_2) \]

where (c) follows since \( H(A|B) = H(A \oplus B|B) \) and since \( y_1^n \) is deterministic given \( W_1 \) and \( W_2 \), (d) follows since conditioning does not increase entropy, and (e) follows since \( S^{q-n_d} x_1^n \oplus S^{q-n_r} x_2^n \) consists of \( n \cdot \max\{n_d, n_c\} \) binary random variables, and hence the maximizing distribution is the i.i.d. Bern(1/2) distribution.

Finally, the third term in (190) can be upper bounded by

\[ I(W_2; y_2^n | y_1^n, S^{q-n_r} x_1^n, W_1) = H(y_2^n | y_1^n, S^{q-n_r} x_1^n, W_1) - H(y_2^n | y_1^n, S^{q-n_r} x_1^n, W_1, W_2) \]

where (f) follows since \( H(A|B, C) = H(A \oplus f(C)|B \oplus f(C), C) \) for some function \( f(A) \), and since \( y_2^n \) is deterministic given \( W_1 \) and \( W_2 \), and (g) follows since conditioning does not increase entropy. Step (h) follows since \( S^{q-n_c} x_2^n \), the top-most \( n_c \) bits of \( x_2^n \) are known and can be removed from \( S^{q-n_d} x_2^n \). Thus, \( S^{q-n_d} x_2^n \) has only \( n \cdot (n_d - n_c)^+ \) random components (the lower-most ones), whose entropy is maximized by the i.i.d. Bern(1/2) distribution.

Now, by substituting the expressions in (192), (196), and (198) into (190), we obtain

\[ n(R_{\Sigma} - \epsilon_n) \leq n \cdot (n_r + 2 \max\{n_d, n_c\} - n_c). \] (199)

By dividing the expression by \( n \) and letting \( n \to \infty \), we get (25).

**D. Proof of (26)**

This is also a genie-aided upper bound. We set \( s_1 = y_1^n \) and \( s_2 = (y_2^n, W_1) \). Substituting in (189), we can write

\[ n(R_{\Sigma} - \epsilon_n) \leq I(W_1; y_1^n, y_2^n) + I(W_2; y_2^n, y_1^n, W_1) \]

\[ = I(W_1; y_1^n) + I(W_1; y_1^n | y_2^n) + I(W_2; y_2^n | W_1) + I(W_2; y_2^n | y_1^n) + I(W_2; y_2^n | y_1^n, W_1) \]

\[ = I(W_1, W_2; y_2^n) + I(W_1; y_1^n | y_2^n) + I(W_2; y_2^n | y_1^n), \] (201)

where the equalities follow from the independence of the messages and the chain rule. The first term in (201) can be bounded as follows

\[ I(W_1, W_2; y_2^n) = H(y_2^n) - H(y_2^n | W_1, W_2) \]

\[ \leq n \cdot n_s, \] (203)
where \((a)\) follows since \(y^n_r\) is a deterministic function of \(W_1\) and \(W_2\), and \((b)\) follows since the entropy of \(y^n_r\) is maximized by the i.i.d. \(\text{Bern}(\frac{1}{2})\) distribution. The second term in (201) satisfies

\[
I(W_1; y^n_1 | y^n_r, W_1) = H(y^n_1 | y^n_r, W_1) - H(y^n_1 | y^n_r, W_1)
\]

\[
\leq H(y^n_r)
\]

\[
= H(S^{q-n_s} x^n_1 \oplus S^{q-n_s} x^n_2 | y^n_r)
\]

\[
\leq H(S^{q-n_s} x^n_1 \oplus S^{q-n_s} x^n_2)
\]

\[
\leq n \cdot \max\{n_d, n_c\}
\]

where \((a)\) follows from the non-negativity of mutual information, \((b)\) follows since \(H(A|B) = H(A \oplus f(B)|B)\) for some function \(f(\cdot)\), \((c)\) follows since conditioning does not reduce entropy, and \((d)\) follows similar to (196). Finally, the last term in (201) can be bounded as follows

\[
I(W_2; y^n_2 | y^n_r, W_1) = H(y^n_2 | y^n_r, W_1) - H(y^n_2 | y^n_r, W_1, W_2)
\]

\[
\leq H(S^{q-n_d} x^n_2 | y^n_r, W_1)
\]

\[
\leq H(S^{q-n_s} x^n_2 | y^n_1, x^n_1, S^{q-n_s} x^n_2)
\]

\[
\leq H(S^{q-n_s} x^n_2 | S^{q-n_s} x^n_2)
\]

\[
\leq n \cdot (n_d - n_s)^+, \tag{210}
\]

where \((e)\) follows since \(y^n_1, y^n_2,\) and \(y^n_r\) are deterministic functions of \(W_1\) and \(W_2\), \((f)\) follows since knowing \(W_1\) and \(y^n_r, x^n_1\) and \(S^{q-n_s} x^n_2\) can be constructed, \((g)\) follows since conditioning does not reduce entropy, and \((h)\) follows similar to (198). Substituting (203), (206), and (210) in (201) leads to

\[
n(R_\Sigma - \epsilon_n) \leq n \cdot \max\{n_d, n_c\} + \max\{n_d, n_s\}. \tag{211}
\]

By dividing (211) by \(n\) and letting \(n \to \infty\), we obtain the upper bound in (26) in Lemma 2.

**E. Proof of (27)**

To establish the upper bound in (27), we use the upper bound given in [12, Theorem 4]. Writing this upper bound for the LD-IRC, we obtain

\[
C_{\text{det} , \Sigma} \leq 2 \max\{n_c, n_r, (n_d - n_c)\} + 2(n_s - n_c)^+. \tag{212}
\]

Now, we enhance the Rx’s observation by \((n_s - n_c)^+\) bits. Doing this, we replace \(n_d, n_c,\) and \(n_r\) by

\[
\bar{n}_d = n_d + (n_s - n_c)^+,
\]

\[
\bar{n}_c = n_c + (n_s - n_c)^+,
\]

\[
\bar{n}_r = n_r + (n_s - n_c)^+,
\]

respectively. We keep the source-relay channel intact, i.e., \(\bar{n}_s = n_s\). This operation is equivalent to reducing the noise power at the Rx’s in the Gaussian IRC. The sum-capacity of this enhanced channel is upper bounded by

\[
2 \max\{\bar{n}_c, \bar{n}_r, (\bar{n}_d - \bar{n}_c)\} + 2(\bar{n}_s - \bar{n}_c)^+ \tag{213}
\]
according to (212). Since the capacity of the enhanced channel is an upper bound for the capacity of the original channel, we get the bound
\[ C_{\text{det.}} \leq 2 \max\{\bar{n}_c, \bar{n}_r, (\bar{n}_d - \bar{n}_c)\} + 2(\bar{n}_s - \bar{n}_c)^+ \]  
(214)

Notice that $(\bar{n}_s - \bar{n}_c)^+ = 0$. Now, by substituting $\bar{n}_d$, $\bar{n}_c$, $\bar{n}_r$, and $\bar{n}_s$ into (214), we obtain
\[ C_{\text{det.}} \leq 2 \max\{n_c + (n_s - n_c)^+, n_r + (n_s - n_c)^+, (n_d - n_c)\} \].  
(215)

The expression in (215) can be rewritten as
\[ C_{\text{det.}} \leq 2 \max\{n_c, n_r, n_d - \max\{n_c, n_s\}\} + 2(n_s - n_c)^+ \],  
(216)

which completes the proof of (27) in Lemma 2.

**F. Proof of (28)**

This is also a genie-aided upper bound. Throughout this proof, we assume that $n_r \leq n_c \leq n_d$. Here, we set $s_1 = S^{q-(n_c-n_r)}x^n_1$ and $s_2 = S^{q-(n_c-n_r)}x^n_2$, and substitute them into (189) to obtain
\[
n(R_\Sigma - \epsilon_n) \leq I(W_1; y^n_1, s_1) + I(W_2; y^n_2, s_2) \]
\[
= I(W_1; s_1) + I(W_1; y^n_1|s_1) + I(W_2; s_2) + I(W_2; y^n_2|s_2) \]
\[
(a) H(s_1) + H(y^n_1|s_1) - H(S^{q-n_r}x^n_1 + S^{q-n_r}x^n_1|W_1) \]
\[
+ H(s_2) + H(y^n_2|s_2) - H(S^{q-n_r}x^n_1 + S^{q-n_r}x^n_1|W_2), \]

where in step $(a)$, we used the fact that knowing $W_i$, $x_i$ is deterministic. Since $n_c$ is larger than $n_r$, the top-most $n_c - n_r$ bits of interference signal is received without any overlap with the relay signal. Therefore, we can split $x^n_1$ in the term $H(S^{q-n_r}x^n_1 + S^{q-n_r}x^n_1|W_j)$ ($i, j \in \{1, 2\}$, $i \neq j$) into two parts: one part without overlap with $x^n_2$ and the other part with overlap with $x^n_2$. Doing this, we obtain
\[
n(R_\Sigma - \epsilon_n) \leq H(s_1) + H(y^n_1|s_1) - H(s_2, x^n_{[n_c-n_r+1:n_c]1:n_c} + x^n_{[1:n_c]}|W_1) \]
\[
+ H(s_2) + H(y^n_2|s_2) - H(s_1, x^n_{[n_c-n_r+1:n_c]} + x^n_{[1:n_c]}|W_2), \]

Here, we used the fact that $S^{q-n_r}x^n_r = x^n_{r, [1:n_c]}$. Next, we use chain rule and the fact that the messages are independent of each other to obtain
\[
n(R_\Sigma - \epsilon_n) \leq H(s_1) + H(y^n_1|s_1) - H(s_2) - H(x^n_{[n_c-n_r+1:n_c]} + x^n_{[1:n_c]}|W_1, s_2) \]
\[
+ H(s_2) + H(y^n_2|s_2) - H(s_1, x^n_{[n_c-n_r+1:n_c]} + x^n_{[1:n_c]}|W_2, s_1) \]
\[
\overset{(b)}{\leq} H(y^n_1|s_1) + H(y^n_2|s_2), \]
(217)

where in $(b)$, we used the non-negativity of entropy. Next, we replace $s_1$ and $s_2$ by their values, and use the given information bits to decrease the entropy as follows
\[
n(R_\Sigma - \epsilon_n) \leq H(x^n_1 + S^{q-n_r}x^n_2 + S^{q-n_r}x^n_r|S^{q-(n_c-n_r)}x_1) + H(x^n_2 + S^{q-n_r}x^n_1 + S^{q-n_r}x^n_r|S^{q-(n_c-n_r)}x_2), \]
(218)
where $\tilde{x}_i = S^{q-n_d}x_i \oplus (S^T)^{q-n_d+n_c-n_r}S^{q-(n_c-n_r)}x_i$. Therefore, $\tilde{x}_i$ can be written as

$$\tilde{x}_i = \begin{bmatrix} 0_{q-n_d+n_c-n_r} \\ X_{i,q-n_d+n_c-n_r+1} \\ \vdots \\ X_{i,q} \end{bmatrix}$$

(219)

where $X_{i,j}$ represents the $l$th element of the binary random vector $x_i$. Note that due to the assumption $n_d \geq n_c$, then $n_d - n_c + n_r > 0$ and the number of random components of $\tilde{x}_i$ is $n_d - n_c + n_r$. Now, similar to (190), we can upper bound (218) as

$$n(R_2^e - \epsilon_n) \leq n \cdot (2 \max \{n_d - n_c + n_c, n_r\})$$

(220)

Using the assumption $n_r \leq n_c$, and dividing the upper bound by $n$ and letting $n \rightarrow \infty$, we obtain

$$R_2^e \leq 2 \max \{n_d - n_c + n_r, n_c\}, \quad \text{if } n_c \leq n_c \leq n_d$$

(221)

which completes the proof of (28). With this, the proof of Lemma 4 is complete.

**APPENDIX B**

**PROOF OF LEMMA 4**

In this appendix, we prove Lemma 4 for the case that $i = 1$ and $j = 2$. The other case can be proved similarly. Hence, is what follows, we want to show that $h(\Gamma^n) - h(\Delta^n|W_2)$ is upper bounded by $nC \left( 2 + \frac{k^2}{(n_c-h_r)^2} \right)$, where $\Gamma^n = \frac{h_c}{\sqrt{P_1}} X_1^n + U_1^n$, $\Delta^n = h_c X_1^n + h_r X_r^n + Z_2^n$, and $U_1 \sim N(0, 1)$ is i.i.d. over the time and independent from other random variables. To do this, we write

$$h(\Gamma^n) - h(\Delta^n|W_2) = \begin{cases} h(\Gamma^n) - h(\Delta^n|W_2) - h(U_1^n) + h(Z_2^n|W_2) & \text{(a)} \\ = I(X_1^n; \Gamma^n) - I(X_1^n, X_1^n; \Delta^n|W_2) & \text{(222)} \\ = I(X_1^n; \Gamma^n) - I(X_1^n; \Delta^n|W_2) - I(X_r^n; \Delta^n|W_2, X_1^n) & \text{(223)} \\ \leq I(X_1^n; \Gamma^n) - I(X_1^n; \Delta^n|W_2) & \text{(224)} \\ \leq I(X_1^n; \Gamma^n) - I(\Gamma^n; \Delta^n|W_2) & \text{(225)} \\ = h(\Gamma^n) - h(U_1^n) - h(\Gamma^n) + h(\Gamma^n|\Delta^n, W_2) & \text{(226)} \\ = h(\Gamma^n|\Delta^n, W_2) - h(U_1^n) & \text{(227)} \\ \leq h(\Gamma^n|\Delta^n) - h(U_1^n) & \text{(228)} \\ \leq h(\Gamma^n|\Delta^n) - h(U_1^n) & \text{(229)} \\ \end{cases}$$

where, step (a) follows since the distribution of $U_1^n, Z_2^n \sim N(0, 1)$ and $Z_2^n$ is independent of $W_2$, step (b) follows from the fact that mutual information is non-negative, and in (c), we used the fact that $\Gamma^n, X_1^n$, and $\Delta^n$ form a Markov chain, i.e., $\Gamma^n \rightarrow X_1^n \rightarrow \Delta^n$. Therefore, $I(\Gamma^n; \Delta^n) \leq I(X_1^n; \Delta^n)$. Now, by using [31] Lemma 1 and the fact that $U_1$ is i.i.d. over the time, we write

$$h(\Gamma^n) - h(\Delta^n|W_2) \leq n \cdot \left( h(\Gamma^n) - h(U_1^n) \right)$$

(230)

where the subscript $G$ indicates that the inputs are i.i.d. and Gaussian distributed, i.e., $X_{i,G} \sim N(0, P_i)$, where $i \in \{1, 2, r\}$ and $\Gamma_G, \Delta_G$ are corresponding signal. Notice that $X_{r,G}$ and $X_{1,G}$ are correlated signals with a correlation coefficient $\rho_1 \in [-1, 1]$.

Note that $S^T$ is a shift-up matrix, and thus, multiplying $S^{q-(n_c-n_r)}x_i$ by $(S^T)^{q-n_d+n_c-n_r}$ aligns it with $S^{q-n_d}x_i$. 

\[3\]
Hence, we obtain

\[
\frac{n}{2} \log \left( 1 + \frac{P_i h_i^2}{P h_r^2} + \frac{P_i h_i^2 P_i h_i^2}{P h_r^2} + \frac{P_i h_i^2}{P h_r^2} \sqrt{P h_r^2 P h_r^2 \rho_i} \right)
\]  

\[
\frac{n}{2} \log \left( 1 + \frac{P_i h_i^2}{P h_r^2} + \frac{P_i h_i^2 P_i h_i^2}{P h_r^2} - \frac{P_i h_i^2}{P h_r^2} \sqrt{P h_r^2 P h_r^2 \rho_i} \right)
\]  

\[
\leq \frac{n}{2} \log \left( 1 + \frac{P_i h_i^2}{P h_r^2} \sqrt{P h_r^2 P h_r^2 \rho_i} \right)
\]  

\[
\leq \frac{n}{2} \log \left( 1 + \frac{P_i h_i^2}{P h_r^2} \right)
\]

where (d) follows since \( \rho_i \in [-1, 1] \). To upper bound the expression in (231), we upper bound the functions \( g_1(P_1, P_r) \) and \( g_2(P_1, P_r, \rho_1) \) separately, since log function is an increasing function. First, consider \( g_1(P_1, P_r) \). By computing sign of the derivative of \( g_1(P_1, P_r) \) with respect to \( P_1 \), we conclude that \( g_1(P_1, P_r) \) has a maximum at \( P_1 = \frac{(1 + P_i h_i^2)^2}{P h_r^2} \). Therefore, we obtain

\[
g_1(P_1, P_r) \leq g_1 \left( \frac{(1 + P_i h_i^2)^2}{P h_r^2}, P_r \right)
\]

\[
= \frac{1 + P_i h_i^2}{P h_r^2}
\]

\[
\leq 2,
\]

where step (e) is followed since \( P_r \leq P \) and \( 1 \leq P h_r^2 \). Now, consider \( g_2(P_1, P_r, \rho_1) \). Supposing that \( u^2 = \frac{1}{P_i} \) and \( v^2 = \frac{1}{P_i} \), we obtain

\[
g_2(P_1, P_r, \rho_1) = \frac{h_i^2 h_i^2 (1 - \rho_1^2)}{P h_r^2} \frac{1}{u^2 v^2 + u^2 h_i^2 + v^2 h_i^2 + 2h_i h_r u v \rho_1}
\]

\[
\leq \frac{h_i^2 h_i^2 (1 - \rho_1^2)}{P h_r^2} \frac{1}{u^2 v^2 + u^2 h_i^2 + v^2 h_i^2 + 2h_i h_r u v \rho_1}
\]

Maximizing the expression in (235) with respect to \( u \) and \( v \) is equivalent to minimizing its denominator with respect to \( u \) and \( v \). The denominator of (235) can be rewritten as

\[
u^2 h_r^2 + v^2 h_i^2 + 2h_i h_r u v \rho_1 = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} h_r^2 & \rho_1 h_i h_r \h_i^2 \rho_1 h_i h_r \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.
\]
Since $\rho_i^2 \in [0, 1]$, $A$ is a positive semi-definite matrix. Therefore, (236) is minimized by the lowest value of $u$ and $v$, i.e., $\frac{1}{\sqrt{p}}$. By substituting $u = v = \frac{1}{\sqrt{p}}$ into (235), we obtain

$$g_2(P_t, P_r, \rho_1) \leq \frac{h_r^2(1 - \rho_1)}{h_r^2 + h_c^2 + 2h_c h_r \rho_1}$$

(237)

where in (f), we used the fact that both $h_c$ and $h_r$ are positive. Now, by substituting (234) and (238) into (231), the proof of Lemma 4 is completed.

**APPENDIX C**

**Proof of the Upper Bounds for Gaussian IRC (Lemma 5)**

A. Proof of (122)

In what follows, we establish the upper bound $d \leq \max\{1, \min\{\beta, \gamma\}\}$ for the case that $\alpha = 1$. To do this, we establish two bounds namely $\max\{1, \beta\}$ and $\max\{1, \gamma\}$. The minimum of these two bounds gives us the bound in (122).

To establish the bound $d \leq \max\{1, \beta\}$, consider the following cut-set bound $R_S \rightarrow S^c \leq \max_{P(X_1, X_2, X_r)} I(X_S; Y_{S^c} | X_{S^c})$, where $S = \{T_1, T_2, \text{Relay}\}$ and $S = \{R_1, R_2\}$. Hence, we obtain

$$R_S \leq \max_{P(X_1, X_2, X_r)} I(X_1, X_2, X_r; Y_1, Y_2)$$

(239)

This term can be rewritten as

$$I(X_1, X_2, X_r; Y_1, Y_2) = h(Y_1, Y_2) - h(Y_1, Y_2 | X_1, X_2, X_r)$$

(a) $h(Y_1, Y_2) - h(Z_1, Z_2)$,

(240)

where in step (a), we used the fact that giving $X_1$, $X_2$, and $X_r$ all randomness of $Y_1$ and $Y_2$ is caused from the additive noises $Z_1$ and $Z_2$ which are independent from all other random variables. Now, by using the chain rule, we obtain

$$I(X_1, X_2, X_r; Y_1, Y_2) = h(Y_1) + h(Y_2 | Y_1) - h(Z_1) - h(Z_2)$$

(b) $h(Y_1) + h(Y_2 - Y_1 | Y_1) - h(Z_1) - h(Z_2)$

(242)

(c) $\leq h(Y_1) + h(Z_2 - Z_1) - h(Z_1) - h(Z_2)$

(d) $\leq h(Y_1) + h(Z_2 - Z_1) - h(Z_1) - h(Z_2)$,

(243)

(244)

(245)

where step (b) follows since $h(A - B | B) = h(A | B)$, in step (c), we used the fact that conditioning does not increase the entropy and $Y_2 - Y_1 = Z_2 - Z_1$ since $h_d = h_r^2$. Step (d) follows due to the fact that Gaussian distribution maximizes the differential entropy given the variance [23]. The subscript $G$ indicates that the inputs are i.i.d. and Gaussian distributed, i.e., $X_{iG} \sim \mathcal{N}(0, P_i)$, where $i \in \{1, 2\}$. Therefore, we obtain

$$I(X_1, X_2, X_r; Y_1, Y_2) \leq h(h_{d1} X_{1G} + h_c X_{2G} + h_r X_{rG} + Z_1) + h(Z_2) - h(Z_1) - h(Z_2)$$

$$= \frac{1}{2} \log \left( P_1 h_c^2 + P_r h_r^2 + P_1 h_d h_r + 2p_1 \sqrt{P_1 P_r h_d h_r} + 2p_2 \sqrt{P_1 P_r h_c h_r} + 1 \right) + \frac{1}{2} \log(2)$$

(246)

$$\leq C (2P h_d^2 + P h_r^2 + 4P h_d h_r) + C(1),$$

(247)

where the correlation coefficient between $X_i$, $X_r$ is $\rho_i \in [-1, 1]$ for $i = 1, 2$, in step (e), we used the fact that $\log(x)$ function is an increasing function with respect to $x$ and $h_d = h_{c}$. Due to (239), the sum-rate is upper bounded by the expression in

4Note that the condition $\alpha = 1$ corresponds to $h_d = h_{c}$ since $h_d$ and $h_{c}$ are both real and positive.
Now, by dividing the sum-rate by $\frac{1}{2} \log(P\bar{h}_d^2)$ and letting $P\bar{h}_d^2 \to \infty$, we obtain
\begin{equation}
d \leq \max\{1, \beta\}. \tag{248}\end{equation}

Now, we need to show that $d \leq \max\{1, \gamma\}$. To establish this upper bound, we use the cut $S = \{T_x 1, T_x 2\}$ and $S^c = \{R_x 1, R_x 2, Relay\}$. Hence, we can write
\begin{equation}
R_S \leq \max_{P(x_1, x_2, x_r)} I(X_1, X_2; Y_1, Y_2, Y_r|X_r). \tag{249}\end{equation}

The mutual information term can be rewritten as follows
\begin{align}
I(X_1, X_2; Y_1, Y_2, Y_r|X_r) &= h(Y_1, Y_2, Y_r|X_r) - h(Y_1, Y_2, Y_r|X_r, X_1, X_2) \\
&= h(Y_1, Y_2, Y_r|X_r) - h(Z_1, Z_2, Z_r) \\
&= h(Y_1|X_r) + h(Y_2|X_r, Y_1) + h(Y_r|X_r, Y_1, Y_2) - h(Z_1) - h(Z_2) - h(Z_r). \tag{252}\end{align}

First, by keeping in mind that $h_d = h_c$, we upper bound $h(Y_1|X_r) - h(Z_1)$ as follows
\begin{align}
h(Y_1|X_r) - h(Z_1) &\leq h(h_d(X_1 + X_2) + Z_1) - h(Z_1) \\
&= C(h_d^2(P_1 + P_2)) \tag{253} \\
&\leq C(2h_d^2P). \tag{254}\end{align}

Then, using the fact that for $\alpha = 1$, $Y_1 = h_d(X_1 + X_2) + h_cX_r + Z_1$ and $Y_2 = h_d(X_1 + X_2) + h_cX_r + Z_2$, we upper bound $h(Y_2|X_r, Y_1) - h(Z_2)$ and obtain
\begin{align}
h(Y_2|X_r, Y_1) - h(Z_2) &\leq h(h_d(X_1 + X_2) + Z_2|h_d(X_1 + X_2) + Z_1) - h(Z_2) \\
&\leq h(Z_2 - Z_1) - h(Z_2) \tag{257} \\
&= C(1). \tag{258}\end{align}

Step (f) follows since $h(A - B|B) = h(A|B)$ and conditioning does not increase the entropy. Finally, we upper bound $h(Y_r|X_r, Y_1, Y_2) - h(Z_r)$ as follows
\begin{align}
h(Y_r|X_r, Y_1, Y_2) - h(Z_r) &= h(Y_r|X_r, Y_1 - h_cX_r, Y_2) - h(Z_r) \\
&\leq h(Y_r|Y_1 - h_cX_r) - h(Z_r) \tag{259} \\
&\leq h(h(s(X_1 + X_2) + Z_r|h_d(X_1 + X_2) + Z_1) - h(Z_r) \tag{260} \\
&\leq h(h_s(X_{1G} + X_{2G}) + Z_r|h_d(X_{1G} + X_{2G}) + Z_1) - h(Z_r). \tag{261}\end{align}

where step (g) follows from the fact that Gaussian distribution maximizes the conditional differential entropy for a given covariance matrix [32 Lemma1]. Now, we define the variable $P_{12} = P_1 + P_2$ to obtain
\begin{align}
h(Y_r|X_r, Y_1, Y_2) - h(Z_r) &\leq \frac{1}{2} \log\left(\frac{P_{12}h_s^2 + 1}{h_s h_d P_{12}}\right) \tag{262} \\
&= C\left(\frac{P_{12}h_s^2}{P_{12}h_d^2 + 1}\right) \tag{263} \\
&\leq C\left(\frac{2P h_s^2}{2P h_d^2 + 1}\right). \tag{264}\end{align}
Step (h) follows since the function in (264) is increasing in \( P_{12} \) and \( \max P_{12} = 2P \).

By substituting (255), (258), and (265) into (252), we obtain

\[
I(X_1, X_2; Y_1, Y_2, Y_r | X_r) \leq C(2h_d^2 P) + C(1) + C \left( \frac{2Ph_s^2}{2Ph_d^2 + 1} \right). \tag{266}
\]

Due to (249), the sum-rate is upper bounded by (266). Similar to the previous case, we divide the upper bound for the sum-rate by \( \frac{1}{2} \log(Ph_d^2) \) and let \( Ph_d^2 \to \infty \) to obtain

\[
d \leq 1 + (\gamma - 1)^+ \tag{267}
\]

\[
= \max \{1, \gamma\} \tag{268}
\]

Now by combining (248) and (268), we complete the proof of (122).

**B. Proof of (123)**

To establish an upper bound \( d \leq \max \{1, \alpha, \beta\} + \max \{1, \alpha\} \), we use the cut-set bound in (239). Hence, write

\[
R_\Sigma \leq \max_{P(X_1, X_2, X_r)} I(X_1, X_2, X_r; Y_1) \tag{269}
\]

Next, by using the chain rule, we obtain

\[
R_\Sigma \leq \max_{P(X_1, X_2, X_r)} I(X_1, X_2, X_r; Y_1) + I(X_1, X_2, X_r; Y_2 | Y_1) \tag{270}
\]

First, we upper bound the mutual information \( I(X_1, X_2, X_r; Y_1) \) as follows

\[
I(X_1, X_2, X_r; Y_1) = h(Y_1) - h(Y_1 | X_1, X_2, X_r)
\]

\[
= h(h_d X_1 + h_c X_2 + h_r X_r + Z_1) - h(Z_1) \tag{271}
\]

\[
\leq C(P(h_d^2 + h_c^2 + h_r^2)) \tag{272}
\]

Now, we upper bound the expression \( I(X_1, X_2, X_r; Y_2 | Y_1) \)

\[
I(X_1, X_2, X_r; Y_2 | Y_1) \leq h(Y_2 | Y_1) - h(Y_2 | Y_1, X_1, X_2, X_r)
\]

\[
= h(Y_2 | Y_1) - h(Z_2) \tag{273}
\]

\[
= h(Y_2 - Y_1 | Y_1) - h(Z_2) \tag{274}
\]

\[
\leq h((h_d - h_c) X_2 + (h_c - h_d) X_1 + Z_2 - Z_1) - h(Z_2) \tag{275}
\]

\[
\leq C \left( 2P(h_d - h_c)^2 + 1 \right) \tag{276}
\]

Now, by substituting (273) and (276) into (270), we upper bound the sum-rate as follows

\[
R_\Sigma \leq C(P(h_d^2 + h_c^2 + h_r^2)) + C \left( 2P(h_d - h_c)^2 + 1 \right) \tag{277}
\]

By dividing the expression by \( \frac{1}{2} \log(Ph_d^2) \) and letting \( Ph_d^2 \to \infty \), we obtain

\[
d \leq \max \{1, \alpha, \beta\} + \max \{1, \alpha\} \tag{280}
\]

which completes the proof.
C. Proof of (124)

The upper bound in (124) is established by using a genie-aided approach. In genie-aided bounds, we provide the side information $s_1$ and $s_2$ to Rx1 and Rx2, respectively. Next, we can use Fano’s inequality, to upper bound $R_S$ as follows

$$n(R_S - \epsilon_n) \leq I(W_1; Y_1^n, s_1) + I(W_2; Y_2^n, s_2), \quad (281)$$

where $\epsilon_n \to 0$ when $n \to \infty$. In this case, we set $s_1 = S^n$ and $s_2 = (S^n, Y_1, W_1)$, where $S^n = h_r X^n_r + Z^n$ and $Z \sim \mathcal{N}(0, 1)$ is a Gaussian noise independent of all other random variables and i.i.d. over the time. Now, by using the chain rule and the fact that $W_1$ is independent from $W_2$, we obtain

$$n(R_S - \epsilon_n) \leq I(W_1; S^n) + I(W_1; Y_1^n|S^n) + I(W_2; S^n|W_1) + I(W_2; Y_2^n|S^n, W_1) + I(W_2; Y_2^n|S^n, Y_1^n, W_1) \quad (282)$$

$$\leq I(W_1, W_2; S^n) + I(W_1, W_2; Y_1^n|S^n) + I(W_2; Y_2^n|S^n, Y_1^n, W_1). \quad (283)$$

Next, we consider each term in (283) separately. The first term in (283) can be rewritten as follows

$$I(W_1, W_2; S^n) = h(S^n) - h(S^n|W_1, W_2) \quad (284)$$

$$\leq h(S^n) - h(S^n|W_1, W_2, X^n_r) \quad (285)$$

$$\leq h(S^n) - h(Z^n), \quad (286)$$

where (a) follows from the fact that conditioning does not increase the entropy. In step (b), we used the fact that knowing $X^n_r$, all randomness of $S^n$ is caused from $Z^n$. Now, by using [23, corollary to Theorem 8.6.2] and the fact that $Z^n$ is i.i.d. over the time, we obtain

$$I(W_1, W_2; S^n) \leq \sum_{t=1}^{n} h(S[t]) - h(Z[t]). \quad (287)$$

Due to the fact that differential entropy given a variance constraint is maximized by the Gaussian distribution [23], we upper bound (287) as follows

$$I(W_1, W_2; S^n) \leq n(h(h_r X_r + Z) - h(Z)) \quad (288)$$

$$= nC(P_r h_r^2) \quad (289)$$

$$\leq nC(P h_r^2), \quad (290)$$

where $X_r \sim \mathcal{N}(0, P_r)$. Now, we upper bound the second term in (283) as follows.

$$I(W_1, W_2; Y_1^n|S^n) = h(Y_1^n|S^n) - h(Y_1^n|S^n, W_1, W_2) \quad (291)$$

$$\leq h(Y_1^n - S^n) - h(Y_1^n - S^n|S^n, W_1, W_2) \quad (292)$$

$$\leq h(Y_1^n - S^n) - h(Z_1^n - Z^n|S^n, W_1, W_2, Z^n) \quad (293)$$

$$\leq h(Y_1^n - S^n) - h(Z_1^n). \quad (294)$$

Step (c) follows from the fact that $h(A|B) = h(A - B|B) \leq h(A - B)$. In step (d), we dropped all conditions in second term since $Z_1^n$ is independent of all other random variables. Similar to above, by using [23, corollary to Theorem 8.6.2], and the fact that additive noise is i.i.d. over time, and given the variance, Gaussian distribution maximizes the differential entropy,
we obtain

\[
I(W_1, W_2; Y^1_t | S^n) \leq n(h(Y_{1G} - S_G) - h(Z_1))
\]

and dividing the whole expression by the GDoF we obtain

\[
= n(h(h_d X_{1G} + h_e X_{2G} + Z_1 - Z) - h(Z_1))
\]

\[
= nC \left( P_1 h_d^2 + P_2 h_e^2 + 1 \right)
\]

\[
\leq nC \left( P(h_d^2 + h_e^2) + 1 \right),
\]

where \( X_{iG} \sim \mathcal{N}(0, P_i), i \in \{1, 2\} \). Finally, we consider the third term in (283). Similar to the previous case, we upper bound the third term in (283) as follows.

\[
I(W_2; Y^n_2 | S^n, Y^n_1, W_1) = h(Y^n_2 | S^n, Y^n_1, W_1) - h(Y^n_2 | S^n, W_1, W_2)
\]

\[
= h(h_d X_{2G}^n + h_e X_{2G}^n + Z_2^n | S^n, h_e X_{2G}^n + h_r X_{rG}^n + Z_1^n, W_1) - h(h_r X_{rG}^n + Z_2^n | S^n, Y^n_1, W_1, W_2)
\]

\[
= h(h_d X_{2G}^n + h_e X_{2G}^n + Z_2^n - Z^n | S^n, h_e X_{2G}^n + h_r X_{rG}^n + Z_1^n - Z^n, W_1) - h(Z_2^n - Z^n | S^n, Y^n_1, W_1, W_2)
\]

\[
\leq h(h_d X_{2G}^n + Z_2^n - Z^n | h_e X_{2G}^n + Z_1^n - Z^n) - h(Z_2^n - Z^n | S^n, Y^n_1, W_1, W_2, Z^n)
\]

\[
= h(h_d X_{2G}^n + Z_2^n - Z^n | h_e X_{2G}^n + Z_1^n - Z^n) - h(Z_2^n).
\]

In step (e), we dropped some conditions from first term and included an additional condition to the second term since conditioning does not increase the entropy. Now, we define \( \hat{Y}_{2G}[t] = h_d X_{2G}[t] + Z_2[t] - Z[t] \) and \( \hat{Y}_{1G}[t] = h_e X_{2G}[t] + Z_1[t] - Z[t] \), where \( X_{2G} \sim \mathcal{N}(0, P_2) \). By using Lemma 1 in [31] and the fact that \( Z_2 \) i.i.d. over the time, we obtain

\[
I(W_2; Y^n_2 | S^n, Y^n_1, W_1) \leq n(h(\hat{Y}_{2G} | \hat{Y}_{1G}) - h(Z_2))
\]

\[
= n / 2 \log \left( \frac{P_2 h_d^2 + 2}{P_2 h_d h_e + 1} \right)
\]

\[
\leq n / 2 \log \left( \frac{P_2 h_d^2 + 2(P_2 h_e^2 + 2) - (1 + P_2 h_e h_d)^2}{2 + P_2 h_e^2} \right)
\]

\[
= nC \left( 1 + \frac{2P_2 h_d(h_d - h_e)}{2 + P_2 h_e^2} \right)
\]

\[
\leq nC \left( 1 + \frac{2P h_d^2 + 2P \max\{h_e^2, h_d^2\}}{2 + P h_e^2} \right).
\]

Step (f) follows since the function in (308) is increasing in \( P_2 \) and \( P_2 h_d h_e \leq P_2 \max\{h_e^2, h_d^2\} \). Substituting (309) and (300) into (283) and dividing the whole expression by \( n \) and letting \( n \to \infty \), we obtain

\[
R_{\Sigma} \leq C(P h_d^2) + C(P(h_d^2 + h_e^2) + 1) + C \left( 1 + \frac{2P h_d^2 + 2P \max\{h_e^2, h_d^2\}}{2 + P h_e^2} \right).
\]

Similar to above, by dividing the expression by \( \frac{1}{2} \log(P h_d^2) \) and letting \( P h_d^2 \to \infty \), we obtain the following upper bound for the GDoF

\[
d \leq \beta + \max\{1, \alpha\} + \max\{1, \alpha\} - \alpha
\]

which completes the proof.
D. Proof of (125)

We use the genie-aided method to establish this upper bound. In this case, we set \( s_1 = Y^n_r \) and \( s_2 = (Y^n_r, W_1) \). Now, by using Fano’s inequality, we upper bound the sum-rate as follows

\[
n(R_S - \epsilon_n) \leq I(W_1; Y^n_r, W^n_r) + I(W_2; Y^n_r, Y^n_r, W_1).
\]

Then, by using the chain rule and the fact that the messages are independent from each other, we obtain

\[
n(R_S - \epsilon_n) \leq I(W_1; Y^n_r) + I(W_1; Y^n_r|Y^n_r) + I(W_2; Y^n_r|Y^n_r, W_1) + I(W_2; Y^n_r|Y^n_r, W_1)
= I(W_1; Y^n_r) + I(W_1; Y^n_r|Y^n_r) + I(W_2; Y^n_r|Y^n_r, W_1).
\]

Now, we consider each term in (314) separately. First, we write the first term as follows

\[
I(W_1, W_2; Y^n_r) = h(Y^n_r) - h(Y^n_r|W_1, W_2)
\]

\[
= h(Y^n_r) - H(Z^n_r),
\]

where \((a)\) follows since knowing \( W_1, W_2 \), all randomness of \( Y^n_r \) is from \( Z^n_r \). Moreover, we used the fact that \( Z^n_r \) is independent from all other variables. Now, by using \([23]\) corollary to Theorem 8.6.2 and the fact that \( Z^n_r \) is i.i.d. over the time, we obtain

\[
I(W_1, W_2; Y^n_r) \leq \sum_{i=1}^{n} h(Y_r[t]) - h(Z_r[t]).
\]

Using the fact that the differential entropy is maximized by the Gaussian distribution given the variance \([23]\), we upper bound \( I(W_1, W_2; Y^n_r) \) as follows

\[
I(W_1, W_2; Y^n_r) \leq n[h_s(X_{iG} + X_{2G}) + Z_r] - h(Z_r)]
= nC((P_1 + P_2)h_s^2)
\leq nC(2Ph_s^2),
\]

where \( X_{iG} \sim \mathcal{N}(0, P_i) \) for \( i \in \{1, 2\} \). Next, we consider the second term in (314).

\[
I(W_1; Y^n_r|Y^n_r) \overset{(b)}{=} I(W_1; Y^n_r|Y^n_r, X^n_r)
= h(Y^n_r|Y^n_r, X^n_r) - h(Y^n_r|Y^n_r, X^n_r, W_1)
\]

\[
\overset{(c)}{=} h(h_dX^n_r + h_cX^n_r + Z^n_1|Y^n_r, X^n_r) - h(h_cX^n_r + Z^n_1|Y^n_r, X^n_r, W_1)
\]

\[
\overset{(d)}{\leq} h(h_dX^n_r + h_cX^n_r + Z^n_1) - h(h_cX^n_r + Z^n_1|Y^n_r, X^n_r, W_1, X^n_2)
\]

\[
\overset{(e)}{=} h(h_dX^n_r + h_cX^n_r + Z^n_1) - h(Z^n_r).
\]

Step \((b)\) follows since encoding at the relay is known, hence knowing \( Y^n_r \), the signal \( X^n_r \) can be reconstructed, step \((c)\) follows since knowing \( X^n_r \) all randomness of \( Y^n_r \) is from \( X^n_r, X^n_r, \) and \( Z^n_1 \) and from \( W_1, X^n_r \) can be reconstructed. In step \((d)\), we used the fact that conditioning does not increase the entropy. In step \((e)\), we dropped the conditions since \( Z^n_1 \) is independent from \( Y^n_r, X^n_r, W_1, \) and \( X^n_2 \). Similar to above, by using \([23]\) corollary to Theorem 8.6.2 and the facts that \( Z^n_1 \) is i.i.d. over the time, and Gaussian distribution maximizes the differential entropy for a give variance, we obtain

\[
I(W_1; Y^n_r|Y^n_r) \leq n[h(h_dX_{1G} + h_cX_{2G} + Z_1) - h(Z_1)]
= nC(P_1h_d^2 + P_2h_c^2)
\leq nC(P(h_d^2 + h_c^2)).
\]
Finally, we bound the last term in (314) as follows:

\begin{align}
I(W_2; Y_2^n | Y_r^n, W_1) &\leq I(W_2; Y_2^n | Y_r^n, W_1, X_r^n) \\
&= h(Y_2^n | Y_r^n, W_1, X_r^n) - h(Y_2^n | Y_r^n, W_1, X_r^n, W_2) \\
&\leq h(h_d X_2^n + Z_2^n | h_s X_2^n + Z_2^n, W_1, X_r^n) - h(Z_2^n | Y_r^n, W_1, X_r^n, W_2) \\
&\leq h(h_d X_2^n + Z_2^n | h_s X_2^n + Z_2^n) - h(Z_2^n).
\end{align}

Step (f) follows since encoding at the relay is known hence \(X_r^n\) can be reconstructed from \(Y_r^n\). In (g), we used the fact that knowing \(W_1\) and \(W_2\), the randomness of \(X_r^n\) and \(X_2^n\) can be removed. Step (h) follows since conditioning does not increase the entropy and \(Z_2^n\) is independent of all other random variables. By using Lemma 1 in [31] and the fact that \(Z_2\) i.i.d. over the time, we obtain

\begin{align}
I(W_2; Y_2^n | Y_r^n, W_1) &\leq n[h(h_d X_2^n + Z_2 h_s X_2^n + Z_r^n) - h(Z_2^n)] \\
&= n \log \left( \frac{P h_d^2 + 1}{P h_d + h_s} \right) \\
&= n C \left( \frac{P h_d^2}{1 + P h_s^2} \right)
\end{align}

where \(X_2 \sim \mathcal{N}(0, P_2)\). Step (i) follows since the function in (335) is increasing in \(P_2\).

Now, by substituting (320), (328), and (336) in (314), we obtain

\begin{align}
n(R_\Sigma - \epsilon_n) &\leq n \left[ C \left( 2P h_s^2 \right) + C(P(h_d^2 + h_s^2)) + C \left( \frac{P h_d^2}{1 + P h_s^2} \right) \right].
\end{align}

Now, by dividing the whole expression by \(n\) and letting \(n \rightarrow \infty\), we obtain

\begin{align}
R_\Sigma &\leq C \left( 2P h_s^2 \right) + C(P(h_d^2 + h_s^2)) + C \left( \frac{P h_d^2}{1 + P h_s^2} \right).
\end{align}

In order to get an upper bound for the GDoF, we divide the sum-rate by \(\frac{1}{2} \log(P h_d^2)\) then we let \(P h_d^2 \rightarrow \infty\). Hence, we have

\begin{align}
d &\leq \gamma + \max\{1, \alpha\} + (1 - \gamma)^+ \\
&= \max\{1, \alpha\} + \max\{1, \gamma\}
\end{align}

This completes the proof of Lemma (125).

E. Proof of (126)

To establish this upper bound, we use the upper bound given in [12, Theorem 4]. This theorem bounds the capacity of the Gaussian IRC as follows

\begin{align}
R_\Sigma &\leq 2C \left( (|h_c| + |h_r|)^2 P + 4 \max \left\{ \frac{h_c^2 P}{1 + P h_c^2}, \frac{h_r^2 P}{h_r^2} \right\} \right) + 2C \left( \frac{h_r^2}{h_r^2} \right).
\end{align}

Now, suppose that the noise variance at the Rx’s is reduced to \(c^2 = \min \left\{ 1, \frac{h_r^2}{h_c^2} \right\} \). This enhances the channel. Therefore, an upper bound for the capacity of the new channel is an upper bound for the original channel. Reducing the noise variance to \(c^2\) is equivalent to increasing the channel strength \(h_d, h_c,\) and \(h_r\) while the noise variance is 1. Therefore, the upper bound
in (341) is upper bounded by
\[
R_\Sigma \leq 2C \left( (|\bar{h}_c| + |\bar{h}_r|)^2 P + 4 \max \left\{ \frac{\bar{h}_c^2 P}{1 + P h_{c}^2}, \bar{h}_r^2 P \right\} \right) + 2C \left( \frac{\bar{h}_c^2}{\bar{h}_c^2} \right).
\]  (342)
where \( \bar{h}_d, \bar{h}_c, \bar{h}_r, \) and \( \bar{h}_s \) represent the channel values for the enhanced IRC. They are defined as follows
\[
\bar{h}_d = h_d \max \left\{ 1, \frac{h_d}{h_c} \right\}, \quad \bar{h}_c = h_c \max \left\{ 1, \frac{h_c}{h_r} \right\}, \quad \bar{h}_r = h_r \max \left\{ 1, \frac{h_r}{h_s} \right\}, \quad \bar{h}_s = h_s.
\]  (343)
Now, we divide the expression in (342) by \( \frac{1}{2} \log_2(Ph_c^2) \) and then we let \( Ph_c^2 \to \infty \). Then, we obtain the following upper bound for the GDoF
\[
d \leq \lim_{m_d \to \infty} \frac{2 \max\{\bar{m}_d, \bar{m}_c, (\bar{m}_d - \bar{m}_c)^+\} + 2(\bar{m}_s - \bar{m}_c)^+}{m_d},
\]  (344)
where \( \bar{m}_d, \bar{m}_c, \bar{m}_r, \) and \( \bar{m}_s \) are defined as follows
\[
\bar{m}_d = m_d + (m_s - m_c)^+, \quad \bar{m}_c = m_c + (m_s - m_c)^+, \quad \bar{m}_r = m_r + (m_s - m_c)^+, \quad \bar{m}_s = m_s.
\]  (345)
By substituting (345) into (344), we obtain
\[
d \leq \lim_{m_d \to \infty} \frac{2 \max\{m_c + (m_s - m_c)^+, m_r + (m_s - m_c)^+, (m_d - m_c)^+\}}{m_d},
\]  (346)
Note that \( (\bar{m}_s - \bar{m}_c)^+ = 0 \). Now, we rewrite (346) as follows
\[
d \leq \lim_{m_d \to \infty} \frac{2 \max\{m_c, m_r, m_d - \max\{m_c, m_s\}\} + 2(m_s - m_c)^+}{m_d},
\]  (347)
Now, by using the definition of \( m_d, m_c, m_r, \) and \( m_s \), we obtain the following upper bound for the GDoF of the IRC
\[
d \leq 2 \max\{\alpha, \beta, 1 - \max\{\alpha, \gamma\}\} + 2(\gamma - \alpha)^+,
\]  (348)
which completes the proof of (126).

\textbf{F. Proof of (126)}

To establish this upper bound, we use the genie-aided method with \( S_1^n = S_1^n \) and \( S_2^n = S_2^n \), where \( S_1^n = \frac{h_c}{\sqrt{Ph_c^2}} X_1^n + U_1^n \) and \( S_2^n = \frac{h_r}{\sqrt{Ph_r^2}} X_2^n + U_2^n \). Here \( U_1 \) and \( U_2 \) are both \( \mathcal{N}(0, 1) \) distributed. Moreover, they are independent of all other random variables and i.i.d. over time. Now, we use Fano’s inequality to upper bound the sum-rate as follows
\[
n(R_\Sigma - \epsilon_n) \leq I(W_1; Y_1^n, S_1^n) + I(W_2; Y_2^n, S_2^n)
\]  (349)
\[\quad \overset{(a)}{=} I(W_1; S_1^n) + I(W_1; Y_1^n | S_1^n) + I(W_2; S_2^n) + I(W_2; Y_2^n | S_2^n),\]
where step \( (a) \) follows from the chain rule. Now, we proceed as follows
\[
n(R_\Sigma - \epsilon_n) \overset{(b)}{\leq} \bar{h}(S_1^n) - h(U_1^n) + h(Y_1^n | S_1^n) - h(h_c X_1^n + h_r X_r^n + Z_1 | S_1^n, W_1)
\]  (350)
\[\quad + h(S_2^n) - h(U_2^n) + h(Y_2^n | S_2^n) - h(h_c X_2^n + h_r X_r^n + Z_2 | S_2^n, W_2)
\]  (351)
\[\quad \overset{(c)}{=} h(S_1^n) - h(U_1^n) + h(Y_1^n | S_1^n) - h(h_c X_1^n + h_r X_r^n + Z_1 | W_1)
\]  (352)
\[\quad + h(S_2^n) - h(U_2^n) + h(Y_2^n | S_2^n) - h(h_c X_2^n + h_r X_r^n + Z_2 | W_2).
\]  (353)
In step \( (b) \) we used the fact knowing \( W_i \), where \( i \in \{1, 2\} \), all randomness of \( S_i^n \) is caused from \( U_i^n \). Step \( (c) \) follows since \( U_i^n \) is independent of all other random variables. Now, due to Lemma [4] the sum-rate is upper bounded as follows
\[
n(R_\Sigma - \epsilon_n) \leq 2nC \left( 2 + \frac{h_c^2}{(h_c - h_r)^2} \right) - h(U_1^n) + h(Y_1^n | S_1^n) - h(U_2^n) + h(Y_2^n | S_2^n).
\]  (354)
Next, we use [31] Lemma 1 and the fact that \( U^n_1 \) is i.i.d. over the time, to write
\[
N(R_S - \epsilon_n) \leq 2n \left[ 2C \left( 2 + \frac{h^2_c}{(h_c - h_r)^2} \right) - h(U_1) + h(Y_1G|S_1G) - h(U_2) + h(Y_2G|S_2G) \right], \tag{352}
\]
where the subscript \( G \) indicates that the inputs are i.i.d. and Gaussian distributed, i.e., \( X_{i,G} \sim \mathcal{N}(0, P_i) \), where \( i \in \{1, 2, r\} \) and \( S_{1,G}, S_{2,G}, Y_{1,G}, \) and \( Y_{2G} \) are corresponding signals. In what follows, we upper bound the expression \( h(Y_1G|S_1G) - h(U_1) \).

Similarly, we can bound \( h(Y_2G|S_2G) - h(U_2) \). To this end, we write
\[
h(Y_1G|S_1G) - h(U_1) = h(h_dX_1G + h_cX_2G + h_rX_rG + Z_1\frac{h_c}{\sqrt{P}}X_1G + U_1) - h(U_1)
\]
\[
= h(h_dX_1G + h_cX_2G + h_rX_rG + Z_1|h_dX_1G + \frac{h_d\sqrt{P}}{h_c}U_1) - h(U_1)
\]
\[
\leq h(h_cX_2G + h_rX_rG + Z_1 - \frac{h_d\sqrt{P}}{h_c}U_1) - h(U_1) \tag{355}
\]
\[
= C \left( P_rh^2_c + P_ch^2_r + \frac{h^2_dP_c}{h^2_c} + 2\rho_2h_ch_r\sqrt{P_cP_r} \right), \tag{356}
\]
where the parameter \( \rho_2 \in [-1, 1] \) is the correlation coefficient between \( X_2 \) and \( X_r \). Step (d) follows since \( h(A - B|B) = h(A|B) \) and conditioning does not increase the entropy. Since \( \rho_2 \in [-1, 1] \) and \( \log(x) \) is an increasing function in \( x \), the expression in (356) is upper bounded by
\[
h(Y_1G|S_1G) - h(U_1) \leq C \left( Ph^2_c + Ph^2_c + \frac{h^2_dP_c}{h^2_c}P + 2h_ch_rP \right) \tag{357}
\]
Similarly, we upper bound \( h(Y_2G|S_2G) - h(U_2) \). Doing this, the sum rate is upper bounded by
\[
n(R_S - \epsilon_n) \leq 2n \left[ 2C \left( 2 + \frac{h^2_c}{(h_c - h_r)^2} \right) + C \left( Ph^2_c + Ph^2_c + \frac{h^2_dP_c}{h^2_c}P + h_ch_rP \right) \right]. \tag{358}
\]
Now, by dividing the expression by \( n \) and letting \( n \to \infty \), we obtain
\[
R_S \leq 2 \left[ C \left( 2 + \frac{h^2_c}{(h_c - h_r)^2} \right) + C \left( Ph^2_c + Ph^2_c + \frac{h^2_dP_c}{h^2_c}P + h_ch_rP \right) \right]. \tag{359}
\]
To obtain an upper bound for the GDoF, we divide the sum-rate by \( \frac{1}{2} \log(Ph^2_d) \) and then we let \( Ph^2_d \to \infty \). Hence, we get
\[
d \leq 2 \max\{\alpha, \beta, (1 + \beta - \alpha)^+\}. \tag{360}
\]
Since in (127), we have the condition \( \beta \leq \alpha < 1 \), the upper bound is rewritten as
\[
d \leq 2 \max\{\alpha, \beta + 1 - \alpha\}, \tag{361}
\]
which completes the proof of (127).

**APPENDIX D**

**EXTENSION OF THE ACHIEVABLE SUM-RATE FROM LD-IRC TO THE ACHIEVABLE GDoF**

Suppose that by using a scheme in an LD-IRC, we achieve a following linear combination of \( n_d, n_c, n_r, \) and \( n_s \)
\[
R_S = k_dn_d + k_cn_c + k_rn_r + k_sn_s, \tag{362}
\]
where \( k_i \in \mathbb{Z} \) for \( i \in \{d, c, r, s\} \). By using the sub-channel allocation in the Gaussian IRC as the rate allocation for the LD-IRC, and keeping in mind that \( \frac{1}{2} \log \left( \frac{Q}{4} \right) \) is achieved by using each sub-channel, we achieve

\[
R_\Sigma = \frac{1}{2} \log \left( \frac{Q}{4} \right) \left( k_d N_d + k_c N_c + k_r N_r + k_s N_s \right)
\]

\[
= \frac{1}{2} \log(\delta) (k_d N_d + k_c N_c + k_r N_r + k_s N_s) - (k_d N_d + k_c N_c + k_r N_r + k_s N_s).
\]

Using the definition of \( N_d, N_c, N_r, \) and \( N_s \), we conclude that the following sum-rate is achievable as long as \( 364 \) is achievable

\[
R_\Sigma = \frac{1}{2} \log(\delta) \left( k_d \log(\frac{P h_d^2}{\log(\delta)}) + k_c \log(\frac{P h_c^2}{\log(\delta)}) + k_r \log(\frac{P h_r^2}{\log(\delta)}) + k_s \log(\frac{P h_s^2}{\log(\delta)}) - (|k_d| + |k_c| + |k_r| + |k_s|) \right)
\]

\[
- \left( k_d \log(\frac{P h_d^2}{\log(\delta)}) + k_c \log(\frac{P h_c^2}{\log(\delta)}) + k_r \log(\frac{P h_r^2}{\log(\delta)}) + k_s \log(\frac{P h_s^2}{\log(\delta)}) \right). \tag{365}
\]

Now, by dividing the sum-rate by \( \frac{1}{2} \log(\frac{P h_d^2}{\log(\delta)}) \) and using the definition in \( 16 \), we can write

\[
\frac{R_\Sigma}{\frac{1}{2} \log(\frac{P h_d^2}{\log(\delta)})} = (k_d + k_c \alpha + k_r \beta + k_s \gamma) \left( 1 - \frac{1}{\log(\delta)} \right) - (|k_d| + |k_c| + |k_r| + |k_s|) \frac{\log(\delta)}{\log(\frac{P h_d^2}{\log(\delta)})}. \tag{366}
\]

To obtain the achievable GDoF, we need to let \( P h_d^2 \rightarrow \infty \) in \( 366 \). For a fixed \( h_d^2 \), this is equivalent to \( P \rightarrow \infty \). Therefore, the term \( \frac{1}{\log(\delta)} = \frac{N}{\log(P)} \rightarrow 0 \) for a fixed \( N \). Therefore, we obtain

\[
\lim_{P h_d^2 \rightarrow \infty} \frac{R_\Sigma}{\frac{1}{2} \log(\frac{P h_d^2}{\log(\delta)})} = (k_d + k_c \alpha + k_r \beta + k_s \gamma) - (k_d + k_c + k_r + k_s) \frac{\log(\delta)}{\log(\frac{P h_d^2}{\log(\delta)})} \tag{367}
\]

\[
= (k_d + k_c \alpha + k_r \beta + k_s \gamma) - (k_d + k_c + k_r + k_s) \frac{\log(\frac{h_d^2}{\log(\delta)})}{1 + \frac{\log(h_d^2)}{\log(P)}}. \tag{368}
\]

For a fixed \( h_d^2 \), the term \( \frac{\log(h_d^2)}{\log(P)} \rightarrow 0 \). Notice that \( N \) is a constant which can be chosen arbitrarily. Therefore, by choosing \( N \) sufficiently large, the second term in \( 368 \) is negligible and we obtain the following achievable GDoF

\[
d = (k_d + k_c \alpha + k_r \beta + k_s \gamma). \tag{369}
\]
[11] R. H. Etkin, D. N. C. Tse, and H. Wang, “Gaussian interference channel capacity to within one bit,” IEEE Trans. on IT, vol. 54, no. 12, pp. 5534–5562, Dec. 2008.

[12] A. Chaaban and A. Sezgin, “On the generalized degrees of freedom of the Gaussian interference relay channel,” IEEE Trans. on IT, vol. 58, no. 7, pp. 4432–4461, July 2012.

[13] V. R. Cadambe and S. A. Jafar, “Degrees of freedom of wireless networks with relays, feedback, cooperation and full duplex operation,” IEEE Trans. on IT, vol. 55, no. 5, pp. 2334–2344, May 2009.

[14] A. S. Avestimehr, S. N. Diggavi, and D. N. C. Tse, “Wireless network information flow: A deterministic approach,” IEEE Trans. on Info. Theory, vol. 57, no. 4, pp. 1872 – 1905, Apr. 2011.

[15] C. Huang, S. A. Jafar, and V. R. Cadambe, “Interference Alignment and the Generalized Degrees of Freedom of the X Channel,” IEEE Transactions on Information Theory, vol. 58, no. 8, pp. 5130 – 5150, Aug. 2012.

[16] T. M. Cover, “Broadcast channels,” IEEE Trans. on Info. Theory, vol. IT-18, no. 1, pp. 2–14, Jan. 1972.

[17] B. Nazer and M. Gastpar, “Compute-and-forward: Harnessing interference through structured codes,” IEEE Trans. on IT, vol. 57, no. 10, pp. 6463 – 6486, Oct. 2011.

[18] A. Chaaban, A. Sezgin, and D. Tuninetti, “Cooperation strategies for the butterfly network: neutralization, feedback, and computation,” in Proc. of Asilomar on SSC, Nov. 2012.

[19] A. Chaaban, “Relaying for Interference Management,” Ph.D. dissertation, Ruhr-Universität Bochum, Bochum, Germany, July. 2013.

[20] T. M. Cover and A. El-Gamal, “Capacity theorems for the relay channel,” IEEE Trans. on IT, vol. 25, no. 5, pp. 572–584, Sep. 1979.

[21] F. M. J. Willems, “Information theoretical results for the discrete memoryless multiple access channel,” Ph.D. dissertation, Katholieke Univ. Leuven, Leuven, Belgium, Oct. 1982.

[22] T. S. Han and K. Kobayashi, “A new achievable rate region for the interference channel,” IEEE Trans. on Info. Theory, vol. IT-27, no. 1, pp. 49–60, Jan. 1981.

[23] T. Cover and J. Thomas, Elements of information theory (Second Edition). John Wiley and Sons, Inc., 2006.

[24] G. Bresler and D. Tse, “The two-user Gaussian interference channel: A deterministic view,” European Trans. in Telecommunications, vol. 19, pp. 333–354, Apr. 2008.

[25] V. R. Cadambe and S. A. Jafar, “Interference alignment and the degrees of freedom for the K user interference channel,” IEEE Trans. on Info. Theory, vol. 54, no. 8, pp. 3425–3441, Aug. 2008.

[26] A. Chaaban, A. Sezgin, and D. Tuninetti, “Achieving net feedback gain in the linear-deterministic butterfly network with a full-duplex relay,” Ahlswede Festschrift, Lecture Notes in Computer Science (LNCS) (arXiv:1207.6540), vol. 7777, pp. 167–208, 2013.

[27] S. A. Jafar and S. Vishwanath, “Generalized degrees of freedom of the symmetric Gaussian K user interference channel,” IEEE Trans. on Info. Theory, vol. 56, no. 7, pp. 3297–3303, Jul. 2010.

[28] M. P. Wilson, K. Narayanan, H. D. Pfister, and A. Sprintson, “Joint physical layer coding and network coding for bidirectional relaying,” IEEE Trans. on IT, vol. 56, no. 11, pp. 5641 – 5654, Nov. 2010.

[29] U. Erez and R. Zamir, “Achieving 1/2 log(1 + SNR) on the AWGN channel with lattice encoding and decoding,” IEEE Trans. on IT, vol. 50, no. 10, pp. 2293–2314, Oct. 2004.

[30] B. Nazer, “Successive compute-and-forward,” in Proc. of 22nd IZS on Communication, Zurich, Switzerland, March 2012.

[31] V. S. Annapureddy and V. V. Veeravalli, “Gaussian interference networks: Sum capacity in the low interference regime and new outer bounds on the capacity region,” IEEE Trans. on Info. Theory, vol. 55, no. 7, pp. 3032–3050, Jul. 2009.

[32] J. A. Thomas, “Feedback can at most double Gaussian multiple access channel capacity,” IEEE Trans. on Info. Theory, vol. 33, no. 5, pp. 711–716, Sep. 1987.