Abstract

We tightly analyze the sample complexity of CCA, provide a learning algorithm that achieves optimal statistical performance in time linear in the required number of samples (up to log factors), as well as a streaming algorithm with similar guarantees.

Keywords: Canonical correlation analysis, sample complexity, shift-and-invert preconditioning, streaming CCA

1. Introduction

Let \( x \in \mathbb{R}^{d_x} \) and \( y \in \mathbb{R}^{d_y} \) be two random vectors with a joint probability distribution \( P(x, y) \). The objective of CCA (Hotelling, 1936) in the population setting is to find \( u \in \mathbb{R}^{d_x} \) and \( v \in \mathbb{R}^{d_y} \) such that projections of the random variables onto these directions are maximally correlated:

\[
\max_{u, v} \frac{\mathbb{E}[(u^\top x)(v^\top y)]}{\sqrt{\mathbb{E}[(u^\top x)^2] \sqrt{\mathbb{E}[(v^\top y)^2]}}.
\]

This objective can be written in the equivalent constrained form

\[
\max_{u, v} u^\top E_{xy} v \quad \text{s.t.} \quad u^\top E_{xx} u = v^\top E_{yy} v = 1
\]

where the cross- and auto-covariance matrices are defined as

\[
E_{xy} = \mathbb{E}[xy^\top], \quad E_{xx} = \mathbb{E}[xx^\top], \quad E_{yy} = \mathbb{E}[yy^\top].
\]

The global optimum of (2), denoted by \((u^*, v^*)\), can be computed in closed-form. Define

\[
T := E_{xx}^{-\frac{1}{2}} E_{xy} E_{yy}^{-\frac{1}{2}} \in \mathbb{R}^{d_x \times d_y},
\]

1. For simplicity (especially for the streaming setting), we assume that \( \mathbb{E}[x] = 0 \) and \( \mathbb{E}[y] = 0 \). Nonzero means can be easily handled in the ERM approach (see Remark 3).
and let \((a_1, b_1)\) be the (unit-length) top left and right singular vector pair associated with \(T\)'s largest singular value \(\rho_1 = \sigma_1(T)\). Then the optimal objective value, i.e., the canonical correlation between \(x\) and \(y\), is \(\rho_1 \leq 1\), achieved by \((u^*, v^*) = (\frac{1}{\sqrt{2}} a_1, \frac{1}{\sqrt{2}} b_1)\).

In practice, we do not have access to the population covariance matrices, but observe samples pairs \((x_1, y_1), \ldots, (x_N, y_N)\) drawn from \(P(x, y)\). In this paper, we are concerned with both the number of samples \(N(\epsilon)\) needed to approximately solve (2), and the time complexity for obtaining the approximate solution. Note that the CCA objective is not a stochastic convex program due to the ratio form (1), and standard stochastic approximation methods do not apply (Arora et al., 2012). Globally convergent stochastic optimization of CCA has long been a challenge, until the recent breakthrough by Ge et al. (2016); Wang et al. (2016) for solving the empirical objective.

Our contributions The contributions of our paper are summarized as follows.

- First, we provide the ERM sample complexity of CCA. We show that in order to achieve \(\epsilon\)-suboptimality in the alignment between the estimated canonical directions and the population solution (relative to the population covariances, see Section 2), we can solve the empirical objective exactly with \(N(\epsilon, \Delta, \gamma)\) samples where \(\Delta\) is the singular value gap of the whitened cross-covariance and \(1/\gamma\) is a upper bound of the condition number of the auto-covariance, for several general classes of distributions widely used in statistics and machine learning.

- Second, to alleviate the high computational complexity of exactly solving the empirical objective, we show that we can achieve the same learning accuracy by drawing the same level of samples and solving the empirical objective approximately with the stochastic optimization algorithm of Wang et al. (2016). This algorithm is based on the shift-and-invert power iterations. We provide tightened analysis of the algorithm’s time complexity, removing an extra \(\log \frac{1}{\epsilon}\) factor from the complexity given by Wang et al. (2016). Our analysis shows that asymptotically it suffices to process the sample set for \(O(\log \frac{1}{\epsilon})\) passes. While near-linear runtime in the required number of samples is known and achieved for convex learning problems using SGD, no such result was established for the nonconvex CCA objective previously.

- Third, we show that the streaming version of shift-and-invert power iterations achieves the same learning accuracy with the same level of sample complexity, given a good estimate of the canonical correlation. This approach requires only \(O(d)\) memory and thus further alleviates the memory cost of solving the empirical objective. This addresses the challenge of the existence of a stochastic algorithm for CCA proposed by Arora et al. (2012).

Notations We use \(\sigma_i(A)\) to denote the \(i\)-th largest singular value of a matrix \(A\), and use \(\sigma_{\text{max}}(A)\) and \(\sigma_{\text{min}}(A)\) to denote the largest and smallest singular values of \(A\) respectively. We use \(\|\cdot\|\) to denote the spectral norm of a matrix or the \(\ell_2\)-norm of a vector. For a positive definite matrix \(M\), the vector norm \(\|\cdot\|_M\) is defined as \(\|w\|_M = \sqrt{w^\top M w}\) for any \(w\). Denote \(d := d_x + d_y\). We use \(C\) and \(C'\) to denote universal constants that are independent of the problem parameters, and their specific values may vary among appearances.

2. Problem setup

Assumptions We assume the following properties of the input random variables.
1. **Bounded covariances**: The eigenvalues of population auto-covariance matrices are bounded:

\[
\max\left(\|\mathbf{E}_{xx}\|, \|\mathbf{E}_{yy}\|\right) \leq 1, \quad \gamma := \min\left(\sigma_{\min}(\mathbf{E}_{xx}), \sigma_{\min}(\mathbf{E}_{yy})\right) > 0.
\]

Hence \(\mathbf{E}_{xx}\) and \(\mathbf{E}_{yy}\) are invertible with condition numbers bounded by \(1/\gamma\).

2. **Concentration property**: For sufficiently large sample sizes \(N_0(\nu)\), the input variables satisfy the following inequality with high probability:

\[
\max\left(\left\|\mathbf{E}_{xx}^{-\frac{1}{2}} \mathbf{\Sigma}_{xx} \mathbf{E}_{xx}^{-\frac{1}{2}} \right\|, \left\|\mathbf{E}_{yy}^{-\frac{1}{2}} \mathbf{\Sigma}_{yy} \mathbf{E}_{yy}^{-\frac{1}{2}} \right\|, \left\|\mathbf{E}_{xx}^{-\frac{1}{2}} (\mathbf{\Sigma}_{xy} - \mathbf{E}_{xy}) \mathbf{E}_{yy}^{-\frac{1}{2}} \right\|\right) \leq \nu, \quad (5)
\]

where the empirical covariance matrices are defined as

\[
\mathbf{\Sigma}_{xy} = \frac{1}{N} \sum_{i=1}^{N} \mathbf{x}_i \mathbf{y}_i^\top, \quad \mathbf{\Sigma}_{xx} = \frac{1}{N} \sum_{i=1}^{N} \mathbf{x}_i \mathbf{x}_i^\top, \quad \mathbf{\Sigma}_{yy} = \frac{1}{N} \sum_{i=1}^{N} \mathbf{y}_i \mathbf{y}_i^\top. \quad (6)
\]

3. **Singular value gap**: For the purpose of learning the canonical directions \((\mathbf{u}^*, \mathbf{v}^*)\), we assume that there exists a positive singular value gap \(\Delta := \sigma_1(\mathbf{T}) - \sigma_2(\mathbf{T}) \in (0, 1)\), such that the top left- and right-singular vector pair of \(\mathbf{T}\) is uniquely defined.

**Measure of error** For an estimate \((\mathbf{u}, \mathbf{v})\) which need not be correctly normalized (i.e., they may not satisfy the constraints of (2)), we can define \((\overline{\mathbf{u}}, \overline{\mathbf{v}}) := \left(\frac{\mathbf{u}}{\|\mathbf{E}_{xx}^{\frac{1}{2}} \mathbf{u}\|}, \frac{\mathbf{v}}{\|\mathbf{E}_{yy}^{\frac{1}{2}} \mathbf{v}\|}\right)\) as the correctly normalized version. And we can measure the quality of these directions by the alignment (cosine of the angle) between \(\left(\frac{1}{\sqrt{2}} \left[\mathbf{E}_{xx}^{\frac{1}{2}} \overline{\mathbf{u}}, \mathbf{E}_{yy}^{\frac{1}{2}} \overline{\mathbf{v}}\right], \frac{1}{\sqrt{2}} \left[\mathbf{E}_{xx}^{\frac{1}{2}} \mathbf{u}^*, \mathbf{E}_{yy}^{\frac{1}{2}} \mathbf{v}^*\right]\right)\), or the sum of alignment between \(\left(\mathbf{E}_{xx}^{\frac{1}{2}} \overline{\mathbf{u}}, \mathbf{E}_{xx}^{\frac{1}{2}} \mathbf{u}^*\right)\) and alignment between \(\left(\mathbf{E}_{yy}^{\frac{1}{2}} \overline{\mathbf{v}}, \mathbf{E}_{yy}^{\frac{1}{2}} \mathbf{v}^*\right)\) (all vectors have unit length):

\[
\text{align} \left(\left((\mathbf{u}, \mathbf{v}); (\mathbf{u}^*, \mathbf{v}^*)\right) := \frac{1}{2} \left(\frac{\mathbf{u}^\top \mathbf{E}_{xx} \mathbf{u}}{\|\mathbf{E}_{xx}^{\frac{1}{2}} \mathbf{u}\|} + \frac{\mathbf{v}^\top \mathbf{E}_{yy} \mathbf{v}}{\|\mathbf{E}_{yy}^{\frac{1}{2}} \mathbf{v}\|}\right).\right.
\]

This measure of alignment is invariant to the lengths of \(\mathbf{u}\) and \(\mathbf{v}\), and achieves the maximum of 1 if \((\mathbf{u}, \mathbf{v})\) lie in the same direction as \((\mathbf{u}^*, \mathbf{v}^*)\). Intuitively, this measure respects the geometry imposed by the CCA constraints that the projections of each view have unit length. As we will show later, this measure is also closely related to the learning guarantee we can achieve with power iterations. Moreover, high alignment implies accurate estimate of the canonical correlation.

**Lemma 1** Let \(\eta \in (0, 1)\). If \(\text{align} \left(\left((\mathbf{u}, \mathbf{v}); (\mathbf{u}^*, \mathbf{v}^*)\right) \geq 1 - \frac{\eta}{2}, \right.\) then \(\frac{\mathbf{u}^\top \mathbf{E}_{xy} \mathbf{v}}{\sqrt{\mathbf{u}^\top \mathbf{E}_{xx} \mathbf{u}} \sqrt{\mathbf{v}^\top \mathbf{E}_{yy} \mathbf{v}}} \geq \rho_1(1 - \eta)\).

All proofs are deferred to the appendix.

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2. CCA is invariant to linear transformations of the inputs, so we could always rescale the data.
3. We refrain ourselves from specifying the failure probability as it only adds additional mild dependences to our results.
3. The sample complexity of ERM

One approach to address this problem is empirical risk minimization (ERM): We draw \( N \) samples \( \{(x_i, y_i)\}_{i=1}^{N} \) from \( P(x, y) \) and solve the empirical version of (2):

\[
\max_{u, v} \ u^T \Sigma_{xy} v \quad \text{s.t.} \quad u^T \Sigma_{xx} u = v^T \Sigma_{yy} v = 1.
\]

Similarly, define the empirical version of \( T \) as \( \hat{T} := \Sigma_{xx}^{-\frac{1}{2}} \Sigma_{xy} \Sigma_{yy}^{-\frac{1}{2}} \in \mathbb{R}^{d_x \times d_y} \).

In the following, we analyze three general distributions commonly used in the statistics and machine learning literature: Denote \( z = \begin{bmatrix} E_{xx} & E_{xy} \\ E_{xy}^T & E_{yy} \end{bmatrix}^{-\frac{1}{2}} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^d \).

- **(Sub-Gaussian)** Let \( z \) be isotropic and sub-Gaussian, that is, \( E[zz^T] = I \) and there exists constant \( C > 0 \) such that \( \mathbb{P}\left(\{q^Tz\} > t\right) \leq \exp(-Ct^2) \) for any unit vector \( q \).

- **(Regular polynomial-tail)** Let \( z \) be isotropic and regular polynomial-tail, that is, \( E[zz^T] = I \) and there exist constants \( r > 1, C > 0 \) such that \( \mathbb{P}\left(\|Vz\|^2 > t\right) \leq Ct^{-1+r} \) for any orthogonal projection \( V \) in \( \mathbb{R}^d \) and any \( t > C \text{rank}(V) \). Note that this class is general and only implies the existence of a \((4+\delta)\)-moment condition.

- **(Bounded)** Let \( x \) and \( y \) be bounded and in particular \( \sup(\|x\|^2, \|y\|^2) \leq 1 \) (which implies \( \max(\|E_{xx}\|, \|E_{yy}\|) \leq 1 \) as in Assumption 1).

We proceed to analyze the sample complexities, eventually obtained in Theorem 7 of Section 3.2.

3.1. Approximating the canonical correlation

We first discuss the error of approximating \( \rho_1 \) by \( \hat{\rho}_1 = \sigma_1(\hat{T}) \). Observe that, although the empirical covariance matrices are unbiased estimates of their population counterparts, we do not have \( E[\hat{T}] = T \) due to the nonlinear operations (matrix multiplication, inverse, and square root) involved in computing \( T \). Nonetheless, we can provide approximation guarantee based on concentrations.

We will separate the probabilistic property of data—Assumption 2—from the deterministic error analysis, and we show below that it is satisfied by the three classes considered here.

**Lemma 2** Let Assumption 1 hold for the random variables. Then Assumption 2 is satisfied with

\[
N_0(\nu) \geq \frac{C'}{\nu^2} \quad \text{for the sub-Gaussian class,}
\]

\[
N_0(\nu) \geq \frac{C'}{\nu^{2(1+r^{-1})}} \quad \text{for the regular polynomial-tail class,}
\]

\[
N_0(\nu) \geq \frac{1}{\nu^{2\gamma^2}} \quad \text{for the bounded class.}
\]

**Remark 3** When \( (x, y) \) have nonzero means, we use the unbiased estimate of covariance matrices

\[
\Sigma_{xy} = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})^T, \quad \Sigma_{xx} = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})(x_i - \bar{x})^T, \quad \Sigma_{yy} = \frac{1}{N-1} \sum_{i=1}^{N} (y_i - \bar{y})(y_i - \bar{y})^T
\]

instead of those in (6), where \( \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \) and \( \bar{y} = \frac{1}{N} \sum_{i=1}^{N} y_i \). We have similar concentration results, and all results in Sections 3 and 4 still apply.
In our decomposition of the error $T - \hat{T}$, we need to bound terms of the form $E_x^x \Sigma_x^x E_x^x - I$. Such bounds can be derived from our assumption on $\|E_x^x \Sigma_x^x E_x^x - I\|$ using the lemma below.

**Lemma 4 (Perturbation of matrix square root, main result of Mathias (1997))** Let $H \in \mathbb{R}^{d \times d}$ be positive definite, let $\delta H$ be Hermitian, and suppose that $\|H^{-\frac{1}{2}}(\delta H)H^{-\frac{1}{2}}\| = 1$. Then we have

$$\|H^{\frac{1}{2}}(H + \eta \cdot \delta H)^{\frac{1}{2}}H^{-\frac{1}{2}} - I\| \leq C_d \cdot \eta,$$

where $C_d = O(\log d)$ is independent of $\eta$.

We can now bound the perturbation $\|T - \hat{T}\|$ and the approximation error in canonical correlation.

**Lemma 5** Assume that we draw $N$ samples $\{(x_i, y_i)\}_{i=1}^N$ independently from the underlying joint distribution $P(x, y)$ for computing the sample covariance matrices in (6). We have

$$|\hat{\rho}_1 - \rho_1| \leq \|T - \hat{T}\| \leq 4C_d \cdot \nu$$

where $C_d$ is same constant in Lemma 4.

Let $\epsilon' \in (0, 1)$. Then for $N \geq N_0 \left(\frac{\epsilon'}{2C_d}\right)$, i.e,

$$N \geq C_d \log^2 d \quad \text{for the sub-Gaussian class},$$

$$N \geq C_d \log^2(1+\rho^{-1}) d \quad \text{for the regular polynomial-tail class},$$

$$N \geq C_d \log^2 2 \quad \text{for the bounded class},$$

we have with high probability that $|\hat{\rho}_1 - \rho_1| \leq \epsilon'$.

**Remark 6** Due to better concentration properties, the sample complexity for the sub-Gaussian and regular polynomial-tail classes are independent of the condition number $\frac{1}{\gamma}$ of the co-covariances.

### 3.2. Approximating the canonical directions

We now discuss the error in learning $(u^*, v^*)$ by ERM, when $T$ has a singular value gap $\Delta > 0$. Let the nonzero singular values of $T$ be $1 \geq \rho_1 \geq \rho_2 \geq \cdots \geq \rho_r$, where $r = \text{rank}(T) \leq \min(d_x, d_y)$, and the corresponding (unit-length) singular vector pairs be $(a_1, b_1), \ldots, (a_r, b_r)$. Define

$$C = \begin{bmatrix} 0 & T \\ T^T & 0 \end{bmatrix} \in \mathbb{R}^{d \times d}. \quad (8)$$

The eigenvalues of $C$ are $\rho_1 \geq \cdots \geq \rho_r > 0 = \cdots = 0 > -\rho_r \geq \cdots \geq -\rho_1$, with corresponding eigenvectors

$$\frac{1}{\sqrt{2}} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}, \ldots, \frac{1}{\sqrt{2}} \begin{bmatrix} a_r \\ b_r \end{bmatrix}, \ldots, \frac{1}{\sqrt{2}} \begin{bmatrix} a_r \\ -b_r \end{bmatrix}, \ldots, \frac{1}{\sqrt{2}} \begin{bmatrix} a_1 \\ -b_1 \end{bmatrix}.$$ 

Therefore, learning canonical directions $(u^*, v^*)$ reduces to learning the top eigenvector of $C$. 


We denote the empirical version of $C$ by $\hat{C}$, and the singular vector pairs of $\hat{T}$ by $\{(\hat{a}_i, \hat{b}_i)\}$. Due to the block structure of $C$ and $\hat{C}$, we have $\|C - \hat{C}\| = \|T - \hat{T}\|$. Let the ERM solution be $(\hat{u}, \hat{v}) = \left( \Sigma_{xx}^{-\frac{1}{2}} \hat{a}_1, \Sigma_{yy}^{-\frac{1}{2}} \hat{b}_1 \right)$, which satisfy $\|\Sigma_{xx}^{\frac{1}{2}} \hat{u}\| = \|\Sigma_{yy}^{\frac{1}{2}} \hat{v}\| = 1$. We now state the sample complexity for learning the canonical directions by ERM.

**Theorem 7** Let $\epsilon \in (0, 1)$. Then for $N \geq N_0 \left(\frac{\sqrt{7\Delta}}{16 \eta \epsilon^2} \right)$, i.e.,

$$N \geq C \frac{d \log^2 d}{\epsilon \Delta^2}$$

for the sub-Gaussian class,

$$N \geq C \frac{d \log^2 (1+\gamma^{-1}) d}{\epsilon (1+\gamma^{-1}) \Delta^2}$$

for the regular polynomial-tail class,

$$N \geq C \frac{\log^2 d}{\epsilon \Delta^2 \gamma^2}$$

for the bounded class,

we have with high probability that align ($(\hat{u}, \hat{v}); (u^*, v^*)$) $\geq 1 - \epsilon$.

**Proof sketch** The proof of Theorem 7 consists of two steps. We first bound the error between $\hat{C}$’s top eigenvector $\frac{1}{\sqrt{2}} \left[ \Sigma_{xx}^{\frac{1}{2}} \hat{u} \right]$ and $\hat{C}$’s top eigenvector $\frac{1}{\sqrt{2}} \left[ \Sigma_{xx}^{\frac{1}{2}} E_{xx} u^* \right]$ using a standard result on perturbation of eigenvectors, namely the Davis-Kahan sin $\theta$ theorem (Davis and Kahan, 1970). We then show that $\frac{1}{\sqrt{2}} \left[ \Sigma_{xx}^{\frac{1}{2}} \hat{u} \right]$ is very close to the “correctly normalized” $\frac{1}{\sqrt{2}} \left[ E_{xx} \hat{u} / \|E_{xx} \hat{u}\| \right]$, so the later still aligns well with the population solution.

**Comparison to prior analysis** For the sub-Gaussian class, the tightest analysis of the sample complexity upper bound we are aware of was by Gao et al. (2015). However, their proof relies on the assumption that $\rho_2 = o(\rho_1)$. In contrast, we do not require this assumption, and our bound is sharp in terms of the gap $\Delta = \rho_1 - \rho_2$. Up to the $\log^2 d$ factor, our ERM sample complexity for the same loss matches the minimax lower bound $\frac{d}{\epsilon \Delta^2}$ given by Gao et al. (2015) (see also Section 6).

**4. Stochastic optimization for ERM**

A disadvantage of the empirical risk minimization approach is that it can be time and memory consuming. To obtain the exact solution to (7), we need to explicitly form and store the covariance matrix and computing their singular value decompositions (SVDs); these steps have a time complexity of $O(N d^2 + d^3)$ and a memory complexity of $O(d^2)$.

In this section, we study the stochastic optimization of the empirical objective, and show that the computational complexity is low: We just need to process a large enough dataset (with the same level of samples as ERM requires) nearly constant times in order to achieve small error with respect to the population objective. The algorithm we use here is the shift-and-invert meta-algorithm proposed by Wang et al. (2016). However, in this section we provide refined analysis of the algorithm’s time complexity than that provided by Wang et al. (2016). We show that, using a better measure of progress and careful initializations for each least squares problem, the algorithm enjoys linear convergence (see Theorem 12), i.e., the time complexity for achieving $\eta$-suboptimality in the empirical objective depends on $\log \frac{1}{\eta}$, whereas the result of Wang et al. (2016) has a dependence of $\log^2 \frac{1}{\eta}$.
4.1. Shift-and-invert power iterations

Our algorithm runs the shift-and-invert power iterations on the following matrix

\[ \hat{M}_\lambda = \left( \lambda I - \hat{C} \right)^{-1} = \begin{bmatrix} \lambda I & -\hat{T}^T \\ -\hat{T} & \lambda I \end{bmatrix}^{-1} \in \mathbb{R}^{d \times d} \]  

(9)

where \( \lambda > \hat{\rho}_1 \). It is obvious that \( \hat{M}_\lambda \) is positive definite and its eigenvalues are \( \frac{1}{\lambda - \rho_1} \geq \cdots \geq \frac{1}{\lambda + \rho_2} \geq \cdots \geq \frac{1}{\lambda + \rho_1} \), with the same set of eigenvectors as \( \hat{C} \).

Assume that there exists a singular value gap for \( \hat{T} \) (this can be guaranteed by drawing sufficiently many samples so that the singular values of \( \hat{T} \) are within a fraction of the gap \( \Delta \) of \( T \)), denoted as \( \Delta = \hat{\rho}_1 - \hat{\rho}_2 \). The key observation is that, as opposed to running power iterations on \( \hat{C} \) (which is essentially done by Ge et al. 2016), \( \hat{M}_\lambda \) has a large eigenvalue gap when \( \lambda = \hat{\rho}_1 + c(\hat{\rho}_1 - \hat{\rho}_2) \) with \( c = O(1) \), and thus power iterations on \( \hat{M}_\lambda \) converge more quickly. In particular, we assume for now the availability of an estimated eigenvalue \( \lambda \) such that \( \lambda - \hat{\rho}_1 \in [l \Delta, u \Delta] \) where \( 0 < l < u < 1 \); locating such a \( \lambda \) is discussed later in Remark 14.

Define \( \hat{A}_\lambda := \begin{bmatrix} \lambda \Sigma_{xx} & -\Sigma_{xy} \\ -\Sigma_{xy} & \lambda \Sigma_{yy} \end{bmatrix} \), \( \hat{B} := \begin{bmatrix} \Sigma_{xx} \\ \Sigma_{yy} \end{bmatrix} \), and we have \( \hat{M}_\lambda = \hat{B}^T \hat{A}_\lambda^{-1} \hat{B} \).

Due to the block structure, \( \hat{B} \)'s eigenvalues of can be bounded: we have \( \sigma_{\max}(\hat{B}) \leq 1 \) and \( \sigma_{\min}(\hat{B}) \geq \gamma \). And by the relationship \( \hat{A}_\lambda = \hat{B}^T \hat{M}_\lambda^{-1} \hat{B} \), the eigenvalues of \( \hat{A}_\lambda \) can be bounded:

\[
\begin{align*}
\sigma_{\max}(\hat{A}_\lambda) &\leq \sigma_{\max}(\hat{M}_\lambda^{-1}) \cdot \sigma_{\max}(\hat{B}) \leq (\lambda + \hat{\rho}_1), \\
\sigma_{\min}(\hat{A}_\lambda) &\geq \sigma_{\min}(\hat{M}_\lambda^{-1}) \cdot \sigma_{\min}(\hat{B}) \geq (\lambda - \hat{\rho}_1)\gamma.
\end{align*}
\]

It is convenient to study the convergence in the concatenated variables

\[
w_t := \frac{1}{\sqrt{2}} \begin{bmatrix} u_t \\ v_t \end{bmatrix}, \quad r_t := \hat{B}^T w_t = \frac{1}{\sqrt{2}} \begin{bmatrix} \Sigma_{xx} u_t \\ \Sigma_{yy} v_t \end{bmatrix}.
\]

Denote \( \hat{w} := \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} \) and \( \hat{r} := \hat{B}^T \hat{w} = \frac{1}{\sqrt{2}} \begin{bmatrix} \Sigma_{xx}^\frac{1}{2} \hat{u} \\ \Sigma_{yy}^\frac{1}{2} \hat{v} \end{bmatrix} \) using the ERM solution, which satisfy \( \hat{w}^T \hat{B} \hat{w} = 1 \) and \( \hat{r}^T \hat{r} = 1 \) respectively.

4.2. Error analysis of one iteration

Our algorithm iteratively applies the approximate matrix-vector multiplications

\[ r_{t+1} \approx \hat{M}_\lambda r_t, \quad \iff \quad w_{t+1} \approx \hat{A}_\lambda^{-1} \hat{B} w_t, \quad t = 0, 1, \ldots 
\]

(10)

This equivalence allows us to directly work with \( (u_t, v_t) \) and avoids computing \( \Sigma_{xx}^\frac{1}{2} \) or \( \Sigma_{yy}^\frac{1}{2} \) explicitly. Note that we do not perform normalizations of the form \( w_t \leftarrow w_t / \| \hat{B}^T w_t \| \) at each iteration as done by Wang et al. (2016) (Phase-I of their SI meta-algorithm); the length of each iterate is irrelevant for the purpose of optimizing the alignment between vectors and we could always perform the
normalization in the end to satisfy the length constants. Exact power iterations is known to converge linearly when there exist an eigenvalue gap (Golub and van Loan, 1996).

The matrix-vector multiplication $\hat{A}_\lambda^{-1}\hat{B}w_t$ is equivalent to solving the least squares problem

$$
\min_w f_{t+1}(w) := \frac{1}{2}w^T\hat{A}_\lambda w - w^T\hat{B}w_t
$$

whose unique solution is $w_{t+1}^* = \hat{A}_\lambda^{-1}\hat{B}w_t$ with the optimal objective $f_{t+1}^* = -\frac{1}{2}w_t^T\hat{B}\hat{A}_\lambda^{-1}\hat{B}w_t$. Of course, solving the problem exactly is costly and we will apply stochastic gradient methods to it.

We will show that, when the least squares problems are solved accurately enough, the iterates are of the same quality as those of the exact solutions and enjoys linear convergence of power iterations.

We begin by introducing the measure of progress for the iterates. Denote the eigenvalues of $\hat{M}_\lambda$ by $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_d$, with corresponding eigenvectors $p_1, \ldots, p_d$ forming an orthonormal basis of $\mathbb{R}^d$. Recall that $p_1 = \hat{r}$, $p_i^T\hat{M}_\lambda p_i = \beta_i$ for $i = 1, \ldots, d$, and $p_i^T\hat{M}_\lambda p_j = 0$ for $i \neq j$.

We therefore can write each iterate as a linear combination of the eigenvectors:

$$
\frac{r_t}{\|r_t\|} = \sum_{i=1}^d \xi_{ti} p_i, \quad \text{where} \quad \xi_{ti} = \frac{r_t^T p_i}{\|r_t\|} \quad \text{for} \quad i = 1, \ldots, d, \quad \text{and} \quad \sum_{i=1}^d \xi_{ti}^2 = 1.
$$

The potential function we use to evaluate the progress of each iteration is

$$
G(r_t) = \frac{\left\| P_{\perp} \frac{r_t}{\|r_t\|} \right\|}{\left\| P_\parallel \frac{r_t}{\|r_t\|} \right\|} \hat{M}^{-1} = \frac{\sqrt{\sum_{i=2}^d \xi_{ti}^2/\beta_i}}{\sqrt{\xi_{t1}^2/\beta_1}},
$$

where $P_{\perp}$ and $P_\parallel$ denote the projections onto the subspaces that are perpendicular and parallel to $\hat{r}$ respectively. The same potential function was used by Garber et al. (2016) for analyzing the convergence of shift-and-invert for PCA. The potential function is invariant to the length of $r_t$, and is equivalent to the criterion $|\tan \theta_t| := \frac{\sqrt{\sum_{i=2}^d \xi_{ti}^2}}{\sqrt{\xi_{t1}^2}}$ where $\theta_t$ is the angle between $r_t$ and $\hat{r}$:

$$
|\sin \theta_t| = \sqrt{\sum_{i=2}^d \xi_{ti}^2} \leq \sqrt{\frac{\beta_1}{\beta_2}} |\tan \theta_t| \leq G(r_t) \leq \sqrt{\frac{\beta_1}{\beta_d}} |\tan \theta_t|.
$$

The lemma below shows that under the iterative scheme (10), $\{G(r_t)\}_{t=1,\ldots}$ converges linearly to 0.

**Lemma 8** Let $\eta \in (0, 1)$. Assume that for each approximate matrix-vector multiplication, we solve the least squares problem so accurately that the approximate solution $w_{t+1}$ satisfies

$$
\epsilon_t := \frac{f_{t+1}(w_{t+1}) - f_{t+1}^*}{w_t^T\hat{B}w_t} \leq \min \left( \frac{1}{\beta_1 \beta_2}, \frac{\xi_{t1}^2/\beta_1}{\sum_{i=2}^d \xi_{ti}^2/\beta_i} \right) \frac{(\beta_1 - \beta_2)^2}{32}.
$$

Let $T = \lceil \log_2 \left( \frac{G(r_0)}{\eta} \right) \rceil$. Then we have $|\sin \theta_t| \leq G(r_t) \leq \eta$ for all $t \geq T$.
4.3. Bounding the initial error for each least squares

On the other hand, we can minimize the initial suboptimality for the least squares problem \( f_{t+1} \) for reducing the time complexity of its solver. It is natural to use an initialization of the form \( \alpha w_t \), a scaled version of the previous iterate, which gives the following objective

\[
 f_{t+1}(\alpha w_t) = \frac{1}{2} w_t^\top A\lambda w_t - (w_t^\top B w_t)\alpha.
\]

This is a quadratic function of \( \alpha \), and minimizing \( f_{t+1}(\alpha w_t) \) over \( \alpha \) gives the optimal scaling \( \alpha_t^* = \frac{w_t^\top B w_t}{w_t^\top A\lambda w_t} \) (and this quantity is also invariant to the length of \( w_t \)). Observe that \( \alpha_t^* \) naturally measures the quality of \( w_t \): As \( w_t \) converges to \( \hat{w} \), \( \alpha_t^* \) converges to \( \beta_1 \). This initialization technique plays an important role in showing the linear convergence of our algorithm, and was used by Ge et al. (2016) for their standard power iterations (alternating least squares) scheme for CCA.

Lemma 9 (Warm start for least squares) Initializing \( \min_w f_{t+1}(w) \) with \( \alpha w_t \), it suffices to set the ratio between the initial and the final error to be 64 \cdot \max(1, G(r_t)). \)

This result indicates that in the converging stage \((G(r_t) \leq 1)\), we just need to set the ratio between the initial and the final error to the constant 64 (and set it to be the constant \( 64G(r_0) \) before that). This will ensure that the time complexity of least squares has no dependence on the final error \( \epsilon \).

4.4. Solving the least squares by SGD

The least squares objective (11) can be further written as the sum of \( N \) functions: \( f_{t+1}(w) = \frac{1}{N} \sum_{i=1}^{N} f_{t+1}^i(w) \) where

\[
 f_{t+1}^i(w) = \frac{1}{2} w^\top \left[ \frac{\lambda x_i^\top}{-y_i x_i^\top} - x_i y_i^\top \right] w - w^\top \left[ \frac{\Sigma_{xx}}{\Sigma_{yy}} \right] w_t.
\]

There has been much recent progress in developing linearly convergent stochastic algorithms for solving finite-sum problems. We use SVRG (Johnson and Zhang, 2013) here due to its memory efficiency. Although \( f_{t+1}(w) \) is convex, each component \( f_{t+1}^i \) may not be convex. We have the following time complexity of SVRG for this case (see, e.g., Garber and Hazan, 2015, Appendix B).

Lemma 10 (Time complexity of SVRG for (13)) With the initialization \( \alpha_t^* w_t \), SVRG outputs an \( w_{t+1} \) such that \( f_{t+1}(w_{t+1}) - f_{t+1}^* \leq \epsilon_t(w_t^\top B w_t) \) in time \( O \left( d(N + \kappa^2) \log(64 \max(G(r_t), 1)) \right) \),

where \( \kappa^2 = \max_i \frac{L_i^2}{\sigma^2} \) with \( L_i \) being the gradient Lipschitz constant of \( f_{t+1}^i \), and \( \sigma \) is the strongly-convex constant of \( f_{t+1} \). Furthermore, if we sample each component \( f_{t+1}^i \) non-uniformly with probability proportional to \( L_i^2 \) for the SVRG stochastic updates, we have instead \( \kappa^2 = \frac{\Lambda}{\alpha^2} \sum_{i=1}^{N} L_i^2 \).

4. Although not explicitly stated by Garber and Hazan (2015), the result for non-uniform sampling is straightforward by a careful investigation of their analysis, and the effect of improved dependence on \( L_i \)'s through non-uniform sampling agrees with related work (Xiao and Zhang, 2014). The purpose of the non-uniform sampling variant is to bound \( \kappa^2 \) with high probability for sub-Gaussian/regular polynomial-tail inputs.
The next lemma upper-bounds the “condition number” $\kappa^2$.

**Lemma 11** Solving $\min_w f_{t+1}(w)$ using SVRG with non-uniform sampling, we have $\kappa^2 = \mathcal{O}\left(\frac{d^2}{\Delta^2\gamma^2}\right)$ for the sub-Gaussian/regular polynomial-tail classes, and $\kappa^2 = \mathcal{O}\left(\frac{1}{\Delta^2\gamma^2}\right)$ for the bounded class.

### 4.5. Putting everything together

We first provide the time complexity for solving the empirical objective using the (offline) shift-and-invert CCA algorithm, regardless of the number of samples used.

**Theorem 12** Let $\eta \in (0, 1)$. For the ERM objective with $N$ samples, offline shift-and-invert outputs an $(\mathbf{u}_T, \mathbf{v}_T)$ satisfying

$$
\min \left\{ \frac{\mathbf{u}_T^T \Sigma_{xx} \mathbf{u}}{\|\mathbf{u}\|}, \frac{\mathbf{v}_T^T \Sigma_{yy} \mathbf{v}}{\|\mathbf{v}\|} \right\} \geq 1 - \eta
$$

in total time

$$
\mathcal{O}\left( d \left( N + \frac{d^2}{\Delta^2\gamma^2} \right) \log \frac{1}{\eta} \right)
$$

for the sub-Gaussian/regular polynomial-tail classes,

$$
\mathcal{O}\left( d \left( N + \frac{1}{\Delta^2\gamma^2} \right) \log \frac{1}{\eta} \right)
$$

for the bounded class.

We have already shown in Theorem 7 that the ERM solution aligns well with the population solution. By drawing slightly more samples and requiring our algorithm to find an approximate solution that aligns well with the ERM solution, we can guarantee high alignment for the approximate solution.

**Corollary 13** Let $\epsilon \in (0, 1)$. Draw $N = N_0 \left( \frac{\sqrt{\Delta}}{2\Delta d} \right)$ samples for the ERM objective. Then the total time for offline shift-and-invert to output $(\mathbf{u}_T, \mathbf{v}_T)$ with align $(\mathbf{u}_T, \mathbf{v}_T); (\mathbf{u}^*, \mathbf{v}^*) \geq 1 - \epsilon$ is

$$
\mathcal{O}\left( d \left( \frac{d \log^2 d}{\epsilon\Delta^2} + \frac{d^2}{\Delta^2\gamma^2} \right) \log \frac{1}{\epsilon} \right)
$$

for the sub-Gaussian class,

$$
\mathcal{O}\left( d \left( \frac{d \log^2 (1+r^{-1}) d}{\epsilon(1+r^{-1})\Delta^2} + \frac{d^2}{\Delta^2\gamma^2} \right) \log \frac{1}{\epsilon} \right)
$$

for the regular polynomial-tail class,

$$
\mathcal{O}\left( d \left( \frac{d \log^2 d}{\epsilon\Delta^2\gamma^2} + \frac{1}{\Delta^2\gamma^2} \right) \log \frac{1}{\epsilon} \right)
$$

for the bounded class.

The $\epsilon$-dependent term is near-linear in the ERM sample complexity $N(\epsilon, \Delta, \gamma)$ and is also the dominant term in the total runtime (when $\epsilon = o(\gamma^2)$ for the first two classes). For sub-Gaussian/regular polynomial-tail classes, we incur an undesirable $d^2$ dependence for the condition number $\kappa^2$, mainly due to weak concentration regarding the data norm (in fact we have stronger concentration for the streaming setting discussed next). One can alleviate the issue of large $\kappa^2$ using accelerated SVRG (Lin et al., 2015), or a sample splitting scheme.

---

5. Strictly speaking, this holds with high probability over the sample set.
6. That is, we draw $\log \frac{1}{\Delta}$ times more samples and solve each least squares on a different split. In such a way, the initialization for each least squares problem only depends on previous splits and not the current split, so that we have stronger concentration. The least squares condition number then depends on $d$ linearly.
Remark 14 We have assumed so far the availability of $\lambda = \hat{\rho}_1 + c(\hat{\rho}_1 - \hat{\rho}_2)$ with $c = \mathcal{O}(1)$ for shift-and-invert to work. There exists an efficient algorithm for locating such an $\lambda$, see the repeat-until loop of Algorithm 3 in Wang et al. (2016). This procedure computes $\mathcal{O}\left(\log \frac{1}{\lambda}\right)$ approximate matrix-vector multiplications, and its time complexity does not depend on $\epsilon$ as we only want to achieve good estimate of the top eigenvalue (and not the top eigenvector). So the cost of locating $\lambda$ is not dominant in the total time complexity.

5. Streaming shift-and-invert CCA

A disadvantage of the ERM approach is that we need to store all the samples in order to go through the dataset multiple times. We now study the shift-and-invert algorithms in the streaming setting in which we draw samples from the underlying distribution $P(x, y)$ and process them once. Clearly, the streaming approach requires only $\mathcal{O}(d)$ memory.

We assume the availability of a $\lambda = \rho_1 + c\Delta$, where $0 < c < 1$. Our algorithm is the same as in the ERM case, except that we now directly work with the population covariances through fresh samples instead of their empirical estimates. With slight abuse of notation, we use $(A_\lambda, B, M_\lambda)$ to denote the population version of $(\hat{A}_\lambda, \hat{B}, \hat{M}_\lambda)$:

$$A_\lambda := \begin{bmatrix} \lambda E_{xx} & -E_{xy} \\ -E_{yx}^\top & \lambda E_{yy} \end{bmatrix}, \quad B := \begin{bmatrix} E_{xx} \\ E_{yy} \end{bmatrix}, \quad M_\lambda = B^\top \hat{A}_\lambda^{-1} B^2,$$

use $\{(\beta_t, p_t)\}_{t=1}^d$ to denote the eigensystem of $M_\lambda$, and use $(u_t, v_t)$ as well as

$$w_t = \frac{1}{\sqrt{2}} \begin{bmatrix} u_t \\ v_t \end{bmatrix}, \quad r_t = B^\frac{1}{2} w_t = \frac{1}{\sqrt{2}} \begin{bmatrix} E_{xx}^\frac{1}{2} u_t \\ E_{yy}^\frac{1}{2} v_t \end{bmatrix}, \quad t = 0, \ldots,$$

to denote the iterates of our algorithm. Also, define $\xi_t, \theta_t$ and $G(r_t)$ similarly as in Section 4.

Our goal is to achieve high alignment between $\frac{u^*}{\|u^*\|} = \left[\frac{E_{xx}^\frac{1}{2} u_T}{\|E_{xx}^\frac{1}{2} u_T + v_T^\top E_{yy} v_T\|}\right]$ and $r^* = \frac{1}{\sqrt{2}} \begin{bmatrix} E_{xx}^\frac{1}{2} u^* \\ E_{yy}^\frac{1}{2} v^* \end{bmatrix}$, which implies high alignment between $\frac{1}{\sqrt{2}} \begin{bmatrix} E_{xx}^\frac{1}{2} u_T / \|E_{xx}^\frac{1}{2} u_T\| \\ E_{yy}^\frac{1}{2} v_T / \|E_{yy}^\frac{1}{2} v_T\| \end{bmatrix}$ and $r^*$. The following lemma makes this intuition precise.

Lemma 15 (Conversion from joint alignment to separate alignment) Let $\eta \in (0, 1)$. If the output $(u_T, v_T)$ of our online shift-and-invert algorithm satisfy that

$$\frac{1}{\sqrt{2}} \frac{\|E_{xx}^\frac{1}{2} u_T + v_T^\top E_{yy} v_T\|}{\|E_{xx}^\frac{1}{2} u_T + v_T^\top E_{yy} v_T\|} \geq 1 - \frac{\eta}{4},$$

we also have align $((u_T, v_T); (u^*, v^*)) = \frac{1}{2} \left(\frac{u^*^\top E_{xx} u_T}{\|u_T^\top E_{xx} u_T\|} + \frac{v^*^\top E_{yy} v_T}{\|v_T^\top E_{yy} v_T\|}\right) \geq 1 - \eta.$

We remark that Lemma 15 improves over a similar result by Wang et al. (2016, Theorem 5), which requires the joint alignment to be $\mathcal{O}(\eta^2)$-suboptimal for the separate alignment to be $\mathcal{O}(\eta)$-suboptimal.
Turning to the streaming algorithm, the least squares problem at iteration \( t + 1 \), \( t = 0, \ldots \) is now a stochastic program: \( \min_w f_{t+1}(w) = \frac{1}{2} w^\top A_t w - w^\top B w_t = E[\phi_{t+1}(w; x, y)] \) where

\[
\phi_{t+1}(w; x, y) := \frac{1}{2} w^\top \begin{bmatrix} \lambda xx^\top - xy^\top \\ -yx^\top \end{bmatrix} w - w^\top \begin{bmatrix} xx^\top \\ yy^\top \end{bmatrix} w_t,
\]

and the expectation is computed over \( P(x, y) \), with the optimal solution \( w^*_{t+1} = A_t^{-1} B w_t \). Due to the high sample complexity of accurately estimating \( \alpha^*_t = \frac{w_t^\top B w_t}{w_t^\top A_t w_t} \) in the streaming setting, we instead initialize each linear systems with the zero vector. With this initialization, we have

\[
f_{t+1}(0) - f^*_{t+1} = 0 - \left( -\frac{1}{2} w_t^\top B A_t^{-1} B w_t \right) = \frac{r_t M \| r_t \|^2}{2} \leq \frac{\beta_1 \| r_t \|^2}{2}, \tag{14}
\]

We then solve the linear system with the streaming SVRG algorithm proposed by Frostig et al. (2015), as detailed in Algorithm 1 (in Appendix E). This is the same approach taken by Garber et al. (2016) for streaming PCA, and our analysis follows the same structure.

To analyze the sample complexity of streaming SVRG, we need a few key quantities.

**Lemma 16 (Parameters of streaming SVRG)** For any \( w, w' \in \mathbb{R}^d \), we have

- (strong convexity) \( f_{t+1}(w) \geq f_{t+1}(w') + \langle \nabla f_{t+1}(w'), w - w' \rangle + \frac{\mu}{2} \| w - w' \|^2 \),
- (streaming smoothness) \( E \| \nabla \phi_{t+1}(w) - \nabla \phi_{t+1}(w^*_{t+1}) \|^2 \| \leq 2 S (f_{t+1}(w) - f^*_{t+1}) \),
- (streaming variance) \( E \left[ \frac{1}{2} \| \nabla \phi(w^*_{t+1}) \|^2 \right] \leq \sigma^2 \),

where \( \mu := \frac{\gamma^2}{\beta_1} \geq C \Delta \gamma \) for some \( C > 0 \), and

\[
S = \mathcal{O} \left( \frac{d \beta_1}{\gamma} \right), \quad \sigma^2 = \mathcal{O} \left( \frac{d \beta_1^3 \| r_t \|^2}{\gamma} \right) \quad \text{for the sub-Gaussian/regular polynomial-tail classes},
\]

\[
S = \mathcal{O} \left( \frac{\beta_1}{\gamma} \right), \quad \sigma^2 = \mathcal{O} \left( \frac{\beta_1^3 \| r_t \|^2}{\gamma^2} \right) \quad \text{for the bounded class}.
\]

**Remark 17** Because we always draw fresh samples in the streaming setting, the “condition number” \( \frac{\gamma}{\mu} \) for the sub-Gaussian/regular polynomial-tail classes depend on \( d \) only linearly (as opposed to quadratically in approximate ERM).

Based on these quantities, we can apply the structural result of Frostig et al. (2015) and give the sampling complexity for driving the final suboptimality to \( \eta_t \) times the initial suboptimality in (14).

**Lemma 18 (Sample complexity of streaming SVRG for least squares)** Let \( \eta_t \in (0, 1) \). Applying streaming-SVRG in Algorithm 1 to \( \min_w f_{t+1}(w) \) with initialization \( 0 \), we have

\[
E \left[ f_{t+1}(w^\top) - f^*_{t+1} \right] \leq \eta_t \left( \frac{\beta_1 \| r_t \|^2}{2} \right)
\]

for \( \tau \geq \Gamma = \mathcal{O} \left( \log \frac{1}{\eta_t} \right) \). The sample complexity of the first \( \Gamma \) iterations is \( \mathcal{O} \left( \frac{d^2}{\beta_1^2} + \frac{d^2}{\beta_1^2 \eta_t} \right) \) for the sub-Gaussian/regular polynomial-tail classes, and \( \mathcal{O} \left( \frac{d^2}{\beta_1^2 \eta_t} \right) \) for the bounded class.
Based on the linear convergence of shift-and-invert, we need only solve $O\left(\log \frac{1}{\eta}\right)$ linear systems, and we can bound $\frac{1}{\eta}$ by a geometrically increasing series where the last term is $O\left(\frac{1}{\epsilon}\right)$ (so the sum of this truncated series is $O\left(\frac{1}{\epsilon}\right)$). This results in the following total sample complexity.

**Theorem 19 (Total sample complexity for streaming shift-and-invert CCA)** Let $\epsilon \in (0, 1)$. Then after solving $T = O\left(\log \frac{1}{\epsilon}\right)$ linear systems to sufficient accuracy, our streaming shift-and-invert CCA algorithm outputs $(u_T, v_T)$ with align $((u_T, v_T); (u^*, v^*)) \geq 1 - \epsilon$. Our algorithm processes each sample in $O(d)$ time, and has a total sample complexity of

$$O\left(\frac{d}{\epsilon \Delta^2} + \frac{d}{\Delta^2 \gamma^2} \log^2 \frac{1}{\epsilon}\right)$$

for the sub-Gaussian/regular polynomial-tail classes,

$$O\left(\frac{1}{\epsilon \Delta^2 \gamma^2}\right)$$

for the bounded class.

Interestingly, the sample complexity of our streaming CCA algorithm (assuming the parameter $\lambda$) improves over that of ERM we showed in Theorem 7: it removes small $\log d$ factors for all classes, and most remarkably achieves polynomial improvement in $\epsilon$ for the regular polynomial-tail class. This is due to the fact that the sample complexity of streaming SVRG basically only uses the moments, and does not require concentration of the whole covariance in Lemma 2. As a result, it is not clear if our analysis of ERM is the tightest possible.

### 6. Lower bound

Consider the following Gaussian distribution named single canonical pair model (Chen et al., 2013):

$$\begin{bmatrix} x \\ y \end{bmatrix} \sim \mathcal{N}\left(0, \begin{bmatrix} I & \Delta \phi \psi^T \\ \Delta \psi \phi^T & I \end{bmatrix}\right),$$

where $\|\phi\| = \|\psi\| = 1$. It is straightforward to check that $T = \mathbb{E}_{xy} = \Delta \phi \psi^T$ for such a distribution. Observe that $T$ is of rank one and has a singular value gap $\Delta$, and the single pair of canonical directions are $(u^*, v^*) = (\phi, \psi)$. Denote this class of model by $\mathcal{F}(d_x, d_y, \Delta)$.

We have the following minimax lower bound for CCA under this model, which is an application of the result of Gao et al. (2015) for sparse CCA (by using rank $r = 1$ and hard sparsity, i.e., $q = 0$ and sparsity level $d$ in their Theorem 3.2).

**Lemma 20 (Lower bound for single canonical pair model)** Suppose the data is generated by the single canonical pair model. Let $(u, v)$ be some estimate of the canonical directions $(u^*, v^*)$ based on $N$ samples. Then, there is a universal constant $C$, so that for $N$ sufficiently large, we have:

$$\inf_{u,v} \sup_{u^*, v^* \in \mathcal{F}(d_x, d_y, \Delta)} \mathbb{E} \left[1 - \text{align}\left((u_T, v_T); (u^*, v^*)\right)\right] \geq C \frac{d}{\Delta^2 N}.$$

This lemma implies that, to estimate the canonical directions up to $\epsilon$-suboptimality in our measure of alignment, we expect to use at least $O\left(\frac{d}{\Delta^2 \epsilon}\right)$ samples. We therefore observe that, for Gaussian inputs, the sample complexity of the our streaming algorithm matches that of the minimax rate of CCA, up to small factors.
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References
Raman Arora, Andy Cotter, Karen Livescu, and Nati Srebro. Stochastic optimization for PCA and PLS. In 50th Annual Allerton Conference on Communication, Control, and Computing, pages 861–868, 2012.

Mengjie Chen, Chao Gao, Zhao Ren, and Harrison H. Zhou. Sparse CCA via precision adjusted iterative thresholding. arXiv:1311.6186 [math.ST], November 24 2013.

Chandler Davis and W. M. Kahan. The rotation of eigenvectors by a perturbation III. SIAM Journal of Numerical Analysis, 7(1):1–46, 1970.

Roy Frostig, Rong Ge, Sham M. Kakade, and Aaron Sidford. Competing with the empirical risk minimizer in a single pass. arXiv:1412.6606 [stat.ML], February 25 2015.

Chao Gao, Zongming Ma, and Harrison H. Zhou. Sparse CCA: Adaptive estimation and computational barriers. arXiv:1409.8565 [stat.ME], January 9 2015.

Dan Garber and Elad Hazan. Fast and simple PCA via convex optimization. arXiv:1509.05647 [math.OC], November25 2015.

Dan Garber, Elad Hazan, Chi Jin, Sham M. Kakade, Cameron Musco, Praneeth Netrapalli, and Aaron Sidford. Faster eigenvector computation via shift-and-invert preconditioning. In Proc. of the Int. Conf. Machine Learning, pages 2626–2634, 2016.

Rong Ge, Chi Jin, Sham M. Kakade, Praneeth Netrapalli, and Aaron Sidford. Efficient algorithms for large-scale generalized eigenvector computation and canonical correlation analysis. In Proc. of the Int. Conf. Machine Learning, 2016.

Gene H. Golub and Charles F. van Loan. Matrix Computations. Johns Hopkins University Press, 1996.

Harold Hotelling. Relations between two sets of variates. Biometrika, 28(3/4):321–377, 1936.

Daniel Hsu, Sham Kakade, and Tong Zhang. A tail inequality for quadratic forms of subgaussian random vectors. Electron. Commun. Probab., 17(52):1–6, 2012.

Rie Johnson and Tong Zhang. Accelerating stochastic gradient descent using predictive variance reduction. In Advances in Neural Information Processing Systems, pages 315–323, 2013.

Hongzhou Lin, Julien Mairal, and Zaid Harchaoui. A universal catalyst for first-order optimization. In Advances in Neural Information Processing Systems, pages 3366–3374, 2015.

Roy Mathias. A bound for the matrix square root with application to eigenvector perturbation. SIAM J. Matrix Anal. and Apps., 18(4):861–867, 1997.
Appendix A. Auxiliary Lemmas

**Lemma 21** The population canonical correlation is bounded by 1, i.e.,
\[ \rho_1 = \sigma_1 (T) \leq 1. \]

**Proof** By the Cauchy-Schwarz inequality of random variables, we have
\[ \rho_1 = \frac{\mathbb{E}[(\mathbf{u}^\top \mathbf{x})(\mathbf{v}^\top \mathbf{y})]}{\sqrt{\mathbb{E}[(\mathbf{u}^\top \mathbf{x})^2]} \cdot \sqrt{\mathbb{E}[(\mathbf{v}^\top \mathbf{y})^2]}} = \sqrt{\mathbf{u}^\top \mathbf{E}_{xx} \mathbf{u} \cdot \mathbf{v}^\top \mathbf{E}_{yy} \mathbf{v}} = 1. \]

**Lemma 22 (Distance between normalized vectors)** For two nonzero vectors \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^d \), we have
\[ \left\| \frac{\mathbf{a}}{\|\mathbf{a}\|} - \frac{\mathbf{b}}{\|\mathbf{b}\|} \right\| \leq 2 \frac{\|\mathbf{a} - \mathbf{b}\|}{\|\mathbf{a}\|}. \]

**Proof** By direct calculation, we have
\[
\left\| \frac{\mathbf{a}}{\|\mathbf{a}\|} - \frac{\mathbf{b}}{\|\mathbf{b}\|} \right\| \leq \left\| \frac{\mathbf{a}}{\|\mathbf{a}\|} - \frac{\mathbf{a}}{\|\mathbf{a}\|} \right\| + \left\| \frac{\mathbf{b}}{\|\mathbf{b}\|} - \frac{\mathbf{b}}{\|\mathbf{b}\|} \right\|
= \frac{\|\mathbf{a} - \mathbf{b}\|}{\|\mathbf{a}\|} + \|\mathbf{b}\| \cdot \frac{\|\mathbf{a} - \mathbf{b}\|}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|}
\leq \frac{\|\mathbf{a} - \mathbf{b}\|}{\|\mathbf{a}\|} + \frac{\|\mathbf{a} - \mathbf{b}\|}{\|\mathbf{a}\|}
= 2 \frac{\|\mathbf{a} - \mathbf{b}\|}{\|\mathbf{a}\|},
\]
where we have used the triangle inequality in the two inequalities.
Lemma 23 (Conversion from joint alignment to separate alignment) Let $\eta \in (0, \frac{1}{4})$. Consider the four nonzero vectors $a, x \in \mathbb{R}^d$ and $b, y \in \mathbb{R}^d$ such that $\|a\| = \|b\| = 1$. If
\[
\frac{1}{\sqrt{2}} \frac{a^T x + b^T y}{\|x\|^2 + \|y\|^2} \geq 1 - \eta,
\]we also have
\[
\frac{1}{2} \left( \frac{\|a^T x\|}{\|x\|} + \frac{\|b^T y\|}{\|y\|} \right) \geq 1 - 4\eta.
\]

Proof By the Cauchy-Schwarz inequality, we have
\[
a^T x + b^T y \sqrt{\|x\|^2 + \|y\|^2} = a^T x \cdot \frac{\|x\|}{\sqrt{\|x\|^2 + \|y\|^2}} + b^T y \cdot \frac{\|y\|}{\sqrt{\|x\|^2 + \|y\|^2}} \leq \sqrt{\left( \frac{a^T x}{\|x\|} \right)^2 + \left( \frac{b^T y}{\|y\|} \right)^2}.
\]
Thus according to (16), we obtain
\[
\left( \frac{a^T x}{\|x\|} \right)^2 + \left( \frac{b^T y}{\|y\|} \right)^2 \geq 2(1 - \eta)^2 \geq 2 - 4\eta.
\]
Since $\left( \frac{b^T y}{\|y\|} \right)^2 \leq 1$, this implies
\[
\frac{a^T x}{\|x\|} \geq \sqrt{1 - 4\eta} \geq 1 - 4\eta
\]
where the last step is due to the fact that $\sqrt{x} \geq x$ for $x \in (0, 1)$. Similarly we have $\frac{b^T y}{\|y\|} \geq 1 - 4\eta$. Then the theorem follows.

Lemma 24 (Moment inequalities of sub-Gaussian and regular polynomial-tail random vectors)
Let $z \in \mathbb{R}^d$ be isotropic and sub-Gaussian or regular polynomial-tail (see their definitions in Lemma 2). Then for some constant $C' > 0$, we have
\[
\mathbb{E} \|z\|^2 \leq d, \quad \mathbb{E} \|z\|^4 \leq C' d^2, \quad \mathbb{E} \left| q^T z \right|^4 \leq C'
\]
where $q$ is any unit vector.

Proof Sub-Gaussian case The first bound is by $\mathbb{E} \|z\|^2 = \mathbb{E} \text{tr} (zz^T) = \text{tr} (I) = d$. To prove the second one, note that according to Theorem 2.1 in Hsu et al. (2012), we have
\[
\mathbb{P} \left( \|z\|^2 > C_1 (d + t) \right) < e^{-t},
\]
for all \( t > 0 \). Therefore

\[
\mathbb{E} \| z \|^4 = \int_0^\infty \mathbb{P} \left( \| z \|^4 > s \right) ds
\]

\[
= \int_0^{C_1 d^2} \mathbb{P} \left( \| z \|^4 > s \right) ds + \int_{C_1 d^2}^\infty \mathbb{P} \left( \| z \|^4 > s \right) ds
\]

\[
\leq C_1^2 d^4 + \int_{C_1 d^2}^\infty \exp \left( -\frac{\sqrt{s}}{C_1} - d \right) ds
\]

\[
\leq C' d^2.
\]

Lastly,

\[
\mathbb{E} \left| q^\top z \right|^4 = \int_0^\infty \mathbb{P} \left( \left| q^\top z \right|^4 > s \right) ds
\]

\[
\leq \int_0^\infty e^{-C\sqrt{s}} ds
\]

\[
\leq C'.
\]

**Regular polynomial-tail case** The first bound is still by \( \mathbb{E} \| z \|^2 = \mathbb{E} \operatorname{tr} (zz^\top) = \operatorname{tr} (I) = d \). When \( r > 1 \), we have

\[
\mathbb{E} \| z \|^4 = \int_0^\infty \mathbb{P} \left( \| z \|^4 > s \right) ds
\]

\[
\leq \int_0^{C_2 d^2} \mathbb{P} \left( \| z \|^4 > s \right) ds + \int_{C_2 d^2}^\infty \mathbb{P} \left( \| z \|^4 > s \right) ds
\]

\[
\leq C_2^2 d^4 + \int_{C_2 d^2}^\infty Cs^{-\frac{r}{r+1}} ds
\]

\[
\leq C' d^2.
\]

To prove the last bound, take \( V = qq^\top \) in the definition of regular polynomial-tail random vectors, and then

\[
\mathbb{P} \left( \left| q^\top z \right|^2 > t \right) \leq Ct^{-1-r},
\]

for any \( t > C \). We have

\[
\mathbb{E} \left| q^\top z \right|^4 = \int_0^\infty \mathbb{P} \left( \left| q^\top z \right|^4 > s \right) ds
\]

\[
\leq \int_0^{C^2} \mathbb{P} \left( \left| q^\top z \right|^4 > s \right) ds + \int_{C^2}^\infty \mathbb{P} \left( \left| q^\top z \right|^4 > s \right) ds
\]

\[
\leq C^2 + \int_{C^2}^\infty Cs^{-\frac{r}{r+1}} ds
\]

\[
\leq C'.
\]
Appendix B. Proofs for Section 1

B.1. Proof of Lemma 1

Proof Using the fact that \( \frac{u^\top E_{xx} u^*}{\|E_{xx}^\top u\|} \) and \( \frac{v^\top E_{yy} v^*}{\|E_{yy}^\top v\|} \) are at most 1, the condition on alignment implies

\[
\frac{u^\top E_{xx} u^*}{\|E_{xx}^\top u\|} = a_i^\top \frac{1}{\|E_{xx}^\top u\|} \geq 1 - \frac{\eta}{4}, \quad \frac{v^\top E_{yy} v^*}{\|E_{yy}^\top v\|} = b_i^\top \frac{1}{\|E_{yy}^\top v\|} \geq 1 - \frac{\eta}{4}.
\]

Since \( \{a_i\}_{i=1}^r \) and \( \{b_i\}_{i=1}^r \) are orthonormal, we have

\[
\sum_{i=2}^r \left( a_i^\top \frac{1}{\|E_{xx}^\top u\|} \right)^2 \leq 1 - \left( 1 - \frac{\eta}{4} \right)^2 \leq \frac{\eta}{2},
\]

\[
\sum_{i=2}^r \left( b_i^\top \frac{1}{\|E_{yy}^\top v\|} \right)^2 \leq 1 - \left( 1 - \frac{\eta}{4} \right)^2 \leq \frac{\eta}{2}.
\]

Observe that

\[
\frac{u^\top E_{xy} v}{\sqrt{u^\top E_{xx} u v^\top E_{yy} v}} = \left( \frac{E_{xx}^\top u}{\|E_{xx}^\top u\|} \right)^\top T \left( \frac{E_{yy}^\top v}{\|E_{yy}^\top v\|} \right) = \sum_{i=1}^d \rho_i \left( a_i^\top \frac{1}{\|E_{xx}^\top u\|} \right) \left( b_i^\top \frac{1}{\|E_{yy}^\top v\|} \right)
\]

\[
\geq \rho_1 \left( 1 - \frac{\eta}{4} \right)^2 - \rho_1 \frac{\eta}{2} \geq \rho_1 (1 - \eta)
\]

where we have used the Cauchy-Schwarz inequality in the first inequality.

Appendix C. Proofs for Section 3

C.1. Proof of Lemma 2

Proof Sub-Gaussian/regular polynomial-tail cases Consider the random variable \( z \) defined in the lemma, and draw i.i.d. samples \( z_1, \ldots, z_n \) of \( z \). It is known that when the sample size \( n \) is large enough (as specified in the lemma), we have

\[
\left\| \frac{1}{N} \sum_{i=1}^N z_i z_i^\top - I \right\| \leq \frac{\nu}{2}
\]
with high probability for the sub-Gaussian class (Vershynin, 2012) and for the regular polynomial-tail class (Srivastava and Vershynin, 2013), given \( N > C' \frac{d}{\nu^2} \) and \( N \geq C' \frac{d}{\rho_1 \log \frac{1}{\delta}} \), respectively.

We then turn to bounding the error in each covariance matrix. We note that the covariance of \( f := \begin{bmatrix} \frac{1}{N} \sum_{i=1}^{N} E_{xx}^{-\frac{1}{2}} x_i x_i^\top E_{xx}^{-\frac{1}{2}} \frac{1}{N} \sum_{i=1}^{N} E_{yy}^{-\frac{1}{2}} y_i y_i^\top E_{yy}^{-\frac{1}{2}} \end{bmatrix} \) is \( \Sigma = \begin{bmatrix} I & T \\ T^\top & I \end{bmatrix} \) with \( \| \Sigma \| = 1 + \rho_1 \leq 2 \) (since the eigenvalues of \( \Sigma \) are of the form \( 1 \pm \sigma_i(T) \)). On the other hand, we have \( f = \Sigma^\frac{1}{2} z \) and \( f_i = \Sigma^\frac{1}{2} z_i = \begin{bmatrix} E_{xx}^{-\frac{1}{2}} x_i \\ E_{yy}^{-\frac{1}{2}} y_i \end{bmatrix} \), \( i = 1, \ldots, N \) are i.i.d. samples of \( f \). Therefore, it holds that

\[
\left\| \frac{1}{N} \sum_{i=1}^{N} f_i f_i^\top - \Sigma \right\| = \left\| \Sigma^\frac{1}{2} \left( \frac{1}{N} \sum_{i=1}^{N} z_i z_i^\top - I \right) \Sigma^\frac{1}{2} \right\| \leq \| \Sigma \| \cdot \left\| \frac{1}{N} \sum_{i=1}^{N} z_i z_i^\top - I \right\| \leq \nu.
\]

Since the norm of each block is bounded by the norm of the entire matrix, we conclude that the error in estimating each covariance matrix is bounded by \( \nu \), as required by Assumption 2.

**Remark 25** In view of Lemma 24 and the proof technique here, for the sub-Gaussian/regular polynomial-tail cases, the bound of \( \|z\|^2 \) leads to a bound for \( \|x\|^2 \) and \( \|y\|^2 \):

\[
\mathbb{E}(\|x\|^2 + \|y\|^2) \leq \|E_{xx}\| \cdot \mathbb{E}\left\| E_{xx}^{-\frac{1}{2}} x \right\|^2 + \|E_{yy}\| \cdot \mathbb{E}\left\| E_{yy}^{-\frac{1}{2}} y \right\|^2 \leq \mathbb{E}\|f\|^2 \leq 2\mathbb{E}\|z\|^2 \leq C d
\]

for some constant \( C > 0 \), where we have used Assumption 1 in the second inequality. And similarly, we have

\[
\mathbb{E}(\|x\|^2 + \|y\|^2)^2 \leq \mathbb{E}\|f\|^4 \leq 4\mathbb{E}\|z\|^4 \leq C' d^2
\]

for some constant \( C' > 0 \).

**Bounded case** Consider the joint covariance matrix

\[
\begin{bmatrix}
E_{xx} & E_{xy} \\
E_{xy} & E_{yy}
\end{bmatrix} \in \mathbb{R}^{d \times d}
\]

which has eigenvalue bounded by 2 due to the assumption that \( \|x\|^2 + \|y\|^2 \leq 2 \). Applying Vershynin (2012, Corollary 5.52), we obtain that

\[
\left\| \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{xy}^\top & \Sigma_{yy} \end{bmatrix} - \begin{bmatrix} E_{xx} & E_{xy} \\
E_{xy} & E_{yy} \end{bmatrix} \right\| \leq \nu'
\]

with probability at least \( 1 - d^{-t^2} \) when \( N \geq C(t/\nu')^2 \log d \) for some constant \( C > 0 \). Setting the failure probability \( \delta = d^{-t^2} \) gives \( t^2 = \frac{\log \frac{1}{\nu'}}{\log d} \), and thus we require \( N \geq C \frac{d}{\nu'^2} \log \frac{1}{\delta} \) for \( 1 - \delta \) success probability.
Due to the block structure of the joint covariance matrix, (17) implies
\[
\|\Sigma_{xy} - E_{xy}\| \leq \nu', \quad \|\Sigma_{xx} - E_{xx}\| \leq \nu', \quad \|\Sigma_{yy} - E_{yy}\| \leq \nu'
\]
hold simultaneously.
Now, to satisfy the first inequality of (5), observe that
\[
\left\| E_{xx}^{-\frac{1}{2}} \Sigma_{xx} E_{xx}^{-\frac{1}{2}} \right\| = \left\| E_{xx}^{-\frac{1}{2}} (\Sigma_{xx} - E_{xx}) E_{xx}^{-\frac{1}{2}} \right\|
\leq \left\| E_{xx}^{-\frac{1}{2}} \right\| \cdot \|\Sigma_{xx} - E_{xx}\| \cdot \left\| E_{xx}^{-\frac{1}{2}} \right\|
\leq \|\Sigma_{xx} - E_{xx}\| / \gamma
\]
where we have used the assumption that \( \sigma_{\min}(E_{xx}) \geq \gamma \) in the last inequality. Therefore, we obtain
\[
\left\| E_{xx}^{-\frac{1}{2}} \Sigma_{xx} E_{xx}^{-\frac{1}{2}} - I \right\| \leq \nu \text{ by setting } \nu' = \gamma \nu \text{ in (17), and this yields the } N_0(\nu) \text{ chosen in the lemma. The other two inequalities of Assumption 2 can be obtained analogously. } \]

C.2. Proof of Lemma 5

Proof In view of the Weyl’s inequality, we have
\[
|\hat{\rho}_1 - \rho_1| \leq \|\hat{T} - T\| = \left\| \Sigma_{xx}^{-\frac{1}{2}} \Sigma_{xy} \Sigma_{yy}^{-\frac{1}{2}} - E_{xx}^{-\frac{1}{2}} E_{xy} E_{yy}^{-\frac{1}{2}} \right\|.
\]
For the right hand side of (18), we have the following decomposition
\[
\Sigma_{xx}^{-\frac{1}{2}} \Sigma_{xy} \Sigma_{yy}^{-\frac{1}{2}} - E_{xx}^{-\frac{1}{2}} E_{xy} E_{yy}^{-\frac{1}{2}} = \left( \Sigma_{xx}^{-\frac{1}{2}} - E_{xx}^{-\frac{1}{2}} \right) \Sigma_{xy} \Sigma_{yy}^{-\frac{1}{2}} + E_{xx}^{-\frac{1}{2}} (\Sigma_{xy} - E_{xy}) \Sigma_{yy}^{-\frac{1}{2}} + E_{xx}^{-\frac{1}{2}} E_{xy} \left( \Sigma_{yy}^{-\frac{1}{2}} - E_{yy}^{-\frac{1}{2}} \right).
\]

By the equality
\[
A^{-\frac{1}{2}} - B^{-\frac{1}{2}} = B^{-\frac{1}{2}} \left( B^\frac{1}{2} - A^\frac{1}{2} \right) A^{-\frac{1}{2}},
\]
the first term of the RHS of (19) becomes
\[
\left( \Sigma_{xx}^{-\frac{1}{2}} - E_{xx}^{-\frac{1}{2}} \right) \Sigma_{xy} \Sigma_{yy}^{-\frac{1}{2}} = E_{xx}^{-\frac{1}{2}} \left( E_{xx}^\frac{1}{2} - \Sigma_{xx}^\frac{1}{2} \right) \Sigma_{xx}^{-\frac{1}{2}} \Sigma_{xy} \Sigma_{yy}^{-\frac{1}{2}}.
\]

When \( \left\| E_{xx}^{-\frac{1}{2}} \Sigma_{xx} E_{xx}^{-\frac{1}{2}} - I \right\| \leq \nu \), according to Lemma 4, we have (by making the identification that \( H = E_{xx} \) and \( \delta H = \Sigma_{xx} - E_{xx} \))
\[
\left\| E_{xx}^{-\frac{1}{2}} \left( E_{xx}^\frac{1}{2} - \Sigma_{xx}^\frac{1}{2} \right) \right\| \leq C_d \cdot \nu.
\]
Combining with the fact that 

\[ \left\| \Sigma_{xx}^{-\frac{1}{2}} \Sigma_{xy} \Sigma_{yy}^{-\frac{1}{2}} \right\| \leq 1, \]

we have

\[ \left\| \left( \Sigma_{xx}^{-\frac{1}{2}} - E_{xx}^{-\frac{1}{2}} \right) \Sigma_{xy} \Sigma_{yy}^{-\frac{1}{2}} \right\| \leq C_d \cdot \nu. \]

A similar bound can be obtained for the third term of (19). Observe that when \( \left\| E_{yy}^{-\frac{1}{2}} \Sigma_{yy} E_{yy}^{-\frac{1}{2}} - I \right\| \leq \nu < 1 \), we have all eigenvalues of \( E_{yy}^{-\frac{1}{2}} \Sigma_{yy} E_{yy}^{-\frac{1}{2}} \) bounded away from 0, and \( \Sigma_{yy}^{-\frac{1}{2}} \Sigma_{xy} \Sigma_{yy}^{-\frac{1}{2}} \) have all eigenvalues in \( \left[ \frac{1}{1+\nu}, \frac{1}{1-\nu} \right] \), implying that \( \left\| \Sigma_{yy}^{-\frac{1}{2}} \Sigma_{xy} \Sigma_{yy}^{-\frac{1}{2}} - I \right\| \leq \frac{\nu}{1-\nu} \). According to Lemma 4, we have (by making the identification that \( H = \Sigma_{yy} \) and \( \delta H = \Sigma_{yy} - E_{yy} \))

\[ \left\| \left( \Sigma_{yy}^{-\frac{1}{2}} - \Sigma_{xy}^{-\frac{1}{2}} \right) \Sigma_{yy}^{-\frac{1}{2}} \right\| \leq \frac{C_d \cdot \nu}{1-\nu}. \]  

Therefore, we can bound the third term of (19) as

\[ \left\| E_{xx}^{-\frac{1}{2}} \Sigma_{xy} \left( \Sigma_{yy}^{-\frac{1}{2}} - E_{yy}^{-\frac{1}{2}} \right) \right\| = \left\| E_{xx}^{-\frac{1}{2}} E_{xy} E_{yy}^{-\frac{1}{2}} \left( E_{yy}^{-\frac{1}{2}} - \Sigma_{yy}^{-\frac{1}{2}} \right) \Sigma_{yy}^{-\frac{1}{2}} \right\| \]

\[ \leq \left\| E_{xx}^{-\frac{1}{2}} E_{xy} E_{yy}^{-\frac{1}{2}} \right\| \cdot \left\| \left( E_{yy}^{-\frac{1}{2}} - \Sigma_{yy}^{-\frac{1}{2}} \right) \Sigma_{yy}^{-\frac{1}{2}} \right\| \]

\[ \leq \frac{C_d \cdot \nu}{1-\nu} \]

where we have used the fact that \( \left\| E_{xx}^{-\frac{1}{2}} E_{xy} E_{yy}^{-\frac{1}{2}} \right\| \leq 1 \) by Lemma 21. Assuming that \( \nu \leq \frac{1}{2} \), we obtain \( \frac{\nu}{1-\nu} \leq 2\nu \).

For the second term of (19), we have by assumption that

\[ \left\| E_{xx}^{-\frac{1}{2}} \left( \Sigma_{xy} - E_{xy} \right) \Sigma_{yy}^{-\frac{1}{2}} \right\| \leq \nu. \]

Applying the triangle inequality, we obtain from (19) that

\[ \left\| \Sigma_{xx}^{-\frac{1}{2}} \Sigma_{xy} \Sigma_{yy}^{-\frac{1}{2}} - E_{xx}^{-\frac{1}{2}} E_{xy} E_{yy}^{-\frac{1}{2}} \right\| \leq 4C_d \cdot \nu. \]  

(21)

To sum up, it suffices to set \( \nu = \frac{\epsilon'}{4C_d} \) (which also implies \( \nu \leq \frac{1}{2} \) as assumed earlier) to ensure \( \left\| \hat{T} - T \right\| \leq \epsilon' \), and this yields the desired sample complexity.

\[ \Box \]

**C.3. Proof of Theorem 7**

**Proof** Apply Lemma 5 with \( \epsilon' = \frac{\sqrt{T\Delta}}{4} \). Since \( \epsilon' < \frac{\Delta}{4} \), there exists a positive eigenvalue gap of \( \Delta \) for \( \hat{C} \) due to the Weyl’s inequality, and therefore its top eigenvector is unique. Then according to
the Davis-Kahan sin $\theta$ theorem (Davis and Kahan, 1970), with the number of samples given in the theorem, the top eigenvectors of $C$ and $\hat{C}$ are well aligned:

$$\sin^2 \theta \leq \frac{\|C - \hat{C}\|^2}{\Delta^2} \leq \frac{\epsilon}{16} \quad (22)$$

where $\theta$ is the angle between the top eigenvector of $C$ and that of $\hat{C}$. This is equivalent to

$$\left\| E_{xx}^{\frac{1}{2}} u^* - \Sigma_{xx}^{\frac{1}{2}} \hat{u} \right\|^2 + \left\| E_{yy}^{\frac{1}{2}} v^* - \Sigma_{yy}^{\frac{1}{2}} \hat{v} \right\|^2 \leq \frac{\epsilon}{8} \quad (23)$$

and so

$$\max \left( \left\| E_{xx}^{\frac{1}{2}} u^* - \Sigma_{xx}^{\frac{1}{2}} \hat{u} \right\|^2, \left\| E_{yy}^{\frac{1}{2}} v^* - \Sigma_{yy}^{\frac{1}{2}} \hat{v} \right\|^2 \right) \leq \frac{\epsilon}{8}.$$

In the rest of the proof, we fix the issue of incorrect normalization of $(\hat{u}, \hat{v})$. Recall we have shown in the proof of Lemma 5 that (see e.g., (20))

$$\left\| I - E_{xx}^{\frac{1}{2}} \Sigma_{xx}^{-\frac{1}{2}} \right\| \leq \epsilon' \leq \frac{\sqrt{\epsilon}}{4}.$$

Consequently, we have

$$\left\| E_{xx}^{\frac{1}{2}} u^* - E_{xx}^{\frac{1}{2}} \hat{u} \right\|^2 = \left\| E_{xx}^{\frac{1}{2}} u^* - \left( E_{xx}^{\frac{1}{2}} \Sigma_{xx}^{-\frac{1}{2}} \right) \left( \Sigma_{xx}^{\frac{1}{2}} \hat{u} \right) \right\|^2 \leq \left( \left\| E_{xx}^{\frac{1}{2}} u^* - \Sigma_{xx}^{\frac{1}{2}} \hat{u} \right\|^2 + \| I - E_{xx}^{\frac{1}{2}} \Sigma_{xx}^{-\frac{1}{2}} \Sigma_{xx}^{\frac{1}{2}} \hat{u} \|^2 \right) \leq 2 \left\| E_{xx}^{\frac{1}{2}} u^* - \Sigma_{xx}^{\frac{1}{2}} \hat{u} \right\|^2 + 2 \left\| I - E_{xx}^{\frac{1}{2}} \Sigma_{xx}^{-\frac{1}{2}} \right\|^2 \leq \frac{\epsilon}{4} + \frac{\epsilon}{8} \leq \frac{\epsilon}{2}$$

where we have used the facts that $(x + y)^2 \leq 2x^2 + 2y^2$ and $\left\| \Sigma_{xx}^{\frac{1}{2}} \hat{u} \right\| = 1$ in the second inequality.

According to Lemma 22, we then have

$$\left\| \frac{\hat{u}^T E_{xx}^{\frac{1}{2}} u^*}{\left\| E_{xx}^{\frac{1}{2}} \hat{u} \right\|} - \frac{\hat{u}^T E_{xx}^{\frac{1}{2}} \hat{u}}{\left\| E_{xx}^{\frac{1}{2}} \hat{u} \right\|} \right\|^2 \leq \frac{4 \left\| E_{xx}^{\frac{1}{2}} u^* - E_{xx}^{\frac{1}{2}} \hat{u} \right\|^2}{\left\| E_{xx}^{\frac{1}{2}} u^* \right\|^2} = 4 \left\| E_{xx}^{\frac{1}{2}} u^* - E_{xx}^{\frac{1}{2}} \hat{u} \right\|^2 \leq 2\epsilon$$

and thus the alignment between these two vectors is

$$\frac{\hat{u}^T E_{xx}^{\frac{1}{2}} u^*}{\left\| E_{xx}^{\frac{1}{2}} \hat{u} \right\|} = 1 - \frac{1}{2} \left\| \frac{E_{xx}^{\frac{1}{2}} u^*}{\left\| E_{xx}^{\frac{1}{2}} \hat{u} \right\|} - \frac{E_{xx}^{\frac{1}{2}} \hat{u}}{\left\| E_{xx}^{\frac{1}{2}} \hat{u} \right\|} \right\|^2 \geq 1 - \epsilon.$$
A similar bound is obtained for $\hat{v}$:

$$\frac{\hat{v}^T E_{yy} \hat{v}^*}{\|E_{yy}^\frac{1}{2}\|} \geq 1 - \epsilon.$$  

Averaging the above two inequalities yields the desired result.

**Appendix D. Proofs for Section 4**

**D.1. Proof of Lemma 8**

**Proof** If we obtain an approximate solution $w_{t+1}$ to (11), such that $f_{t+1}(w_{t+1}) - f_{t+1}(w^*_t) = \epsilon_t (w_t^T \hat{B}w_t)$, it holds that

$$\epsilon_t \|\hat{B}^\frac{1}{2}w_t\|^2 = \frac{1}{2} (w_{t+1} - w^*_t)^T \hat{A} \lambda (w_{t+1} - w^*_t)$$

$$= \frac{1}{2} \left( \hat{B}^\frac{1}{2}w_{t+1} - \hat{B}^\frac{1}{2}w^*_t \right)^T \hat{B}^{-\frac{1}{2}} \hat{A} \lambda \hat{B}^{-\frac{1}{2}} \left( \hat{B}^\frac{1}{2}w_{t+1} - \hat{B}^\frac{1}{2}w^*_t \right)$$

$$= \frac{1}{2} (r_{t+1} - r^*_t)^T \hat{M}^{-1} (r_{t+1} - r^*_t) = \frac{1}{2} \|r_{t+1} - r^*_t\|^2 \hat{M}^{-1},$$

or equivalently

$$\|r_{t+1} - r^*_t\| \hat{M}^{-1} = \sqrt{2\epsilon_t} \cdot \|r_t\|.$$  

Note that our choice of $\epsilon_t$ is also invariant to the length of $r_t$ (or whether normalization is performed).

For the exact solution to the linear system, we have

$$r^*_t = \hat{M}_\lambda r_t = \|r_t\| \sum_{i=1}^d \beta_i \xi_{ti} p_i.$$  

As a result, we can bound the numerator and denominator of $G(r_{t+1})$ respectively:

$$\left\|P_{\perp} \frac{r_{t+1}}{\|r_{t+1}\|} \right\| \hat{M}_\lambda^{-1} \leq \frac{1}{\|r_{t+1}\|} \left( \left\|P_{\perp} r^*_t \right\| \hat{M}^{-1} + \left\|P_{\perp} (r_{t+1} - r^*_t) \right\| \hat{M}^{-1} \right)$$

$$\leq \frac{1}{\|r_{t+1}\|} \left( \left\|P_{\perp} r^*_t \right\| \hat{M}^{-1} + \|r_{t+1} - r^*_t\| \hat{M}^{-1} \right)$$

$$= \frac{\|r_t\|}{\|r_{t+1}\|} \left( \left( \sum_{i=2}^d \beta_i \xi_{ti}^2 + \sqrt{2\epsilon_t} \right) \right),$$

23
and

\[
\| P \| \frac{r_{t+1}}{\parallel r_{t+1} \parallel} \| \leq \frac{1}{\| r_{t+1} \|} \left( \| P \| \frac{r_{t+1}^*}{\parallel r_{t+1} \parallel - 1} - \| P \| (r_{t+1} - r_{t+1}^*) \| \right)
\]

\[
= \frac{1}{\| r_{t+1} \|} \left( \| P \| \frac{r_{t+1}^*}{\parallel r_{t+1} \parallel - 1} - \| r_{t+1} - r_{t+1}^* \| \right)
\]

\[
= \| r_t \| \left( \sqrt{\beta_1 \xi_{t+1}^2} - \sqrt{2 \epsilon_t} \right).
\]

Consequently, we have

\[
G(r_{t+1}) \leq \sqrt{\sum_{i=2}^{d} \beta_i \xi_{ti}^2 + \sqrt{2 \epsilon_t}} \leq \frac{\beta_2 \sqrt{\sum_{i=2}^{d} \xi_{ti}^2 / \beta_i + \sqrt{2 \epsilon_t}}}{\sqrt{\xi_{t+1}^2 / \beta_1} - \sqrt{2 \epsilon_t}}
\]

\[
= \frac{\beta_2 + \sqrt{2 \epsilon_t}}{\beta_1 - \frac{\sqrt{2 \epsilon_t}}{\sqrt{\xi_{t+1}^2 / \beta_1}}}
\]

As long as \( \sqrt{2 \epsilon_t} \leq \min \left( \sqrt{\sum_{i=2}^{d} \xi_{ti}^2 / \beta_i}, \sqrt{\xi_{t+1}^2 / \beta_1} \right) \cdot \frac{\beta_1 - \beta_2}{4}, \) i.e.,

\[
\epsilon_t \leq \min \left( \sum_{i=2}^{d} \frac{\xi_{ti}^2}{\beta_i}, \frac{\xi_{t+1}^2}{\beta_1} \right) \cdot \frac{(\beta_1 - \beta_2)^2}{32},
\]

we are guaranteed that

\[
G(r_{t+1}) \leq G(r_t) \cdot \frac{\beta_1 + 3 \beta_2}{3 \beta_1 + \beta_2}.
\]

Substituting in \( \beta_i = \frac{1}{\lambda - \tilde{\rho}_i} \) with \( \lambda - \tilde{\rho}_1 \leq \tilde{\Delta} \), we obtain that

\[
\frac{\beta_1 + 3 \beta_2}{3 \beta_1 + \beta_2} \leq \frac{5}{7} < 1.
\]

This means that if (12) holds for each least squares problem, the sequence \( \{ G(r_t) \}_{t=0,...} \) decreases (at least) at a constant geometric rate of \( \frac{5}{7} \). Therefore, the number of inexact matrix-vector multiplications \( T \) needed to achieve \( |\sin \theta_T| \leq \eta \) is \( \log_\frac{5}{7} \left( \frac{G(r_0)}{\eta} \right) \). \( \blacksquare \)
D.2. Proof of Lemma 9
Proof With the given initialization, we have
\[ f_{t+1}(\alpha_t \mathbf{w}_t) - f^*_t - f_{t+1}^* \leq f_{t+1}^* (\beta_1 \mathbf{w}_t) - f^*_{t+1} \]
\[ = \frac{\beta_1^2 \mathbf{r}_t^\top \hat{\mathbf{M}}_\lambda^{-1} \mathbf{r}_t}{2} - \beta_1 \mathbf{r}_t^\top \mathbf{r}_t + \frac{\mathbf{r}_t^\top \hat{\mathbf{M}}_\lambda \mathbf{r}_t}{2} \]
\[ = \frac{\|\mathbf{r}_t\|^2}{2} \sum_{i=1}^{d} \xi_{ti}^2 \left( \frac{\beta_1^2}{\beta_i} - 2\beta_1 + \beta_i \right) \]
\[ = \frac{\|\mathbf{r}_t\|^2}{2} \sum_{i=1}^{d} \xi_{ti}^2 (\beta_1 - \beta_i)^2 \]
\[ \leq \frac{(\mathbf{w}_t^\top \hat{\mathbf{B}} \mathbf{w}_t)}{2} \cdot \beta_1^2 \sum_{i=2}^{d} \frac{\xi_{ti}^2}{\beta_i}. \]

Therefore, in view of (12), it suffices to set the ratio between the initial and the final error of \( f_{t+1} \) to
\[ \max \{1, G(\mathbf{r}_t)\} \cdot \frac{16 \beta_1^2}{(\beta_1 - \beta_2)^2}. \]

In the initial phase, \( G(\mathbf{r}_t) \) is large, we can set the ratio to be \( G(\mathbf{r}_0) \cdot \frac{16 \beta_1^2}{(\beta_1 - \beta_2)^2} \), until it is reduced to 1 after \( O(\log G(\mathbf{r}_0)) \) iterations. Afterwards, we can set the ratio to be the constant of \( \frac{16 \beta_1^2}{(\beta_1 - \beta_2)^2} \), until we reach the desired accuracy. Observe that
\[ \frac{\beta_1^2}{(\beta_1 - \beta_2)^2} = \left( \frac{1}{\lambda - \tilde{\rho}_1} \right)^2 \left( \frac{1}{\lambda - \tilde{\rho}_2} \right)^2 \leq (u + 1)^2 \leq 4. \]

\[ \square \]

D.3. Proof of Lemma 11
Proof The gradient Lipschitz constant \( L_i \) is bounded by the largest eigenvalue (in absolute value) of its Hessian
\[ Q^i_\lambda = \begin{bmatrix} \lambda x_i x_i^\top & -x_i y_i^\top \\ -y_i x_i^\top & \lambda y_i y_i^\top \end{bmatrix}, \]
and the largest eigenvalue is defined as
\[ \max_{g_x \in \mathbb{R}^{d_x}, g_y \in \mathbb{R}^{d_y}} \beta := \left| \left[ g_x^\top, g_y^\top \right] Q^i_\lambda \left[ \begin{array}{c} g_x \\ g_y \end{array} \right] \right| \quad \text{s.t.} \quad \|g_x\|^2 + \|g_y\|^2 = 1. \]
We have
\[
\beta = \left| \lambda (g_x^\top x_i)^2 + \lambda (g_y^\top y_i)^2 - 2(g_x^\top x_i)(g_y^\top y_i) \right|
\leq \lambda (g_x^\top x_i)^2 + \lambda (g_y^\top y_i)^2 + 2 \left| g_x^\top x_i \right| \left| g_y^\top y_i \right|
\leq \lambda (g_x^\top x_i)^2 + \lambda (g_y^\top y_i)^2 + (g_x^\top x_i)^2 + (g_y^\top y_i)^2
= (\lambda + 1) \left( (g_x^\top x_i)^2 + (g_y^\top y_i)^2 \right)
\leq (\lambda + 1) \left( \|g_x\|^2 \|x_i\|^2 + \|g_y\|^2 \|y_i\|^2 \right)
\leq (\lambda + 1) \cdot \left( \|x_i\|^2 + \|y_i\|^2 \right)
\]
where we have used the Cauchy-Schwarz inequality in the third inequality.

Note that, for bounded inputs, we have \(\|x_i\|^2 + \|y_i\|^2 \leq 2\) and so \(L_i^2 \leq 4(\lambda + 1)^2\) for all \(i = 1, \ldots, N\). For sub-Gaussian/regular polynomial-tail inputs, we have
\[
\frac{1}{N} \sum_{i=1}^{N} L_i^2 \leq (\lambda + 1)^2 \cdot \frac{1}{N} \sum_{i=1}^{N} \left( \|x_i\|^2 + \|y_i\|^2 \right) = O((\lambda + 1)^2 d^2)
\]
with high probability in view of Remark 25.

On the other hand, we have shown that \(\sigma = \sigma_{\min}(A_{\lambda}) \geq (\lambda - \hat{\rho}_1)\gamma\). Recalling \(\lambda = \hat{\rho}_1 + c\hat{\Delta}\) with \(c \in (0, 1)\), we have \(\lambda \leq 2\) and \(\sigma \geq c\hat{\Delta}\gamma\). Combining this with the data norm bound above yields the desired result.

\[\square\]

\section*{D.4. Proof of Theorem 12}

\textbf{Proof} Since \(\frac{v_T^\top \Sigma_{ux} u}{\|\Sigma_{ux} u\|} \leq 1\) and \(\frac{v_T^\top \Sigma_{uy} \hat{v}}{\|\Sigma_{uy} v\|} \leq 1\), it suffices to require
\[
\frac{u_T^\top \Sigma_{ux} \hat{u}}{\|\Sigma_{ux} u\|} + \frac{v_T^\top \Sigma_{uy} \hat{v}}{\|\Sigma_{uy} v\|} \geq 2 - \eta.
\]
According to Lemma 23 (making the identification that \(a = \Sigma_{ux}^{\frac{1}{2}} \hat{u}\), \(x = \Sigma_{ux}^{\frac{1}{2}} u_T\), \(b = \Sigma_{uy}^{\frac{1}{2}} \hat{v}\), and \(y = \Sigma_{uy}^{\frac{1}{2}} v_T\)), it then suffices to have
\[
\cos \theta_T = \frac{1}{\sqrt{2}} \frac{\hat{u}^\top \Sigma_{ux} u_T + \hat{v}^\top \Sigma_{uy} v_T}{\sqrt{u_T^\top \Sigma_{ux} u_T + v_T^\top \Sigma_{uy} v_T}} \geq 1 - \frac{\eta}{8}.
\]
Since \(\cos \theta_T = \sqrt{1 - \sin^2 \theta_T} \geq 1 - \sin^2 \theta_T\), we just need \(|\sin \theta_T| \leq \frac{\sqrt{\eta}}{\sqrt{8}}\), and we ensure it by requiring \(G(r_T) \leq \frac{\sqrt{\eta}}{\sqrt{8}}\).
Applying results from the previous sections, we need to solve $O\left(\log \frac{1}{\eta}\right)$ linear systems, and the time complexity for solving each is $O\left(N + \kappa^2\right)$ for SVRG. Considering the term depending on $\eta$, we therefore obtain the total time complexity as stated in the theorem.

D.5. Proof of Corollary 13

**Proof** Denote $\tilde{r} := \frac{1}{\sqrt{2}} \begin{bmatrix} \Sigma_{xx}^{\frac{1}{2}} u_T / \left\| \Sigma_{xx}^{\frac{1}{2}} u_T \right\| \\ \Sigma_{yy}^{\frac{1}{2}} v_T / \left\| \Sigma_{yy}^{\frac{1}{2}} v_T \right\| \end{bmatrix}$, with $\|r\| = 1$. Assume without loss of generality that $\left\| \Sigma_{xx}^{\frac{1}{2}} u_T \right\| = \left\| \Sigma_{yy}^{\frac{1}{2}} v_T \right\| = 1$; this does not affect our measure of alignment, and can be ensured by a final (separate) normalization step with cost $O(Nd)$ (Wang et al., 2016).

Apply Lemma 5 with $\epsilon' = \sqrt{\epsilon \Delta}$. With the specified sample complexity, we have that with high probability

$$\left\| r - \hat{r} \right\| \leq \sqrt{\epsilon \Delta} \leq \Delta \frac{\epsilon}{8}. \quad (25)$$

In view of the Weyl's inequality, (25) implies that $\hat{\Delta} \geq \frac{3\Delta}{4}$.

Let $r^* = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} u^* \\ \Sigma_{yy}^{\frac{1}{2}} v^* \end{bmatrix}$ be the top eigenvector of $C$. And recall $\hat{r} := \frac{1}{\sqrt{2}} \begin{bmatrix} \Sigma_{xx}^{\frac{1}{2}} \hat{u} \\ \Sigma_{yy}^{\frac{1}{2}} \hat{v} \end{bmatrix}$ is the top eigenvector of $\hat{C}$. According to the Davis-Kahan sin $\theta$ theorem (Davis and Kahan, 1970), with the number of samples given in the theorem, the top eigenvectors of $C$ and $\hat{C}$ are well aligned:

$$\sin^2 \theta \leq \frac{\| C - \hat{C} \|^2}{\Delta^2} \leq \frac{\epsilon}{64}$$

where $\theta$ is the angle between $r^*$ and $\hat{r}$. This implies that

$$\hat{r}^T r^* = \cos \theta = \sqrt{1 - \sin^2 \theta} \geq 1 - \sin^2 \theta \geq 1 - \frac{\epsilon}{64}.$$

We now show that the theorem follows if we manage to solve the ERM objective so accurately that

$$\hat{r}^T \hat{r} = \frac{1}{2} \left( \frac{\hat{u}^T \Sigma_{xx} u_T}{\sqrt{u_T^T \Sigma_{xx} u_T}} + \frac{\hat{v}^T \Sigma_{yy} v_T}{\sqrt{v_T^T \Sigma_{yy} v_T}} \right) \geq 1 - \frac{\epsilon^2}{8192}. \quad (26)$$

To see this, first observe that (26) implies

$$\| \hat{r} - \hat{r} \| \leq \sqrt{2 - 2(\hat{r}^T \hat{r})} \leq \frac{\epsilon}{64},$$

and as a result

$$\hat{r}^T r^* \geq \hat{r}^T r^* - \| (\hat{r} - \hat{r})^T r^* \| \geq \hat{r}^T r^* - \| \hat{r} - \hat{r} \| \geq 1 - \frac{\epsilon}{32}.$$
Algorithm 1 Streaming SVRG for \( \min_w f(w) \).

**Input:** Initialization \( w^0 = 0, s = \frac{1}{352} \).

**for** \( \tau = 1, \ldots, \Gamma \) **do**

\[ \bar{z} \leftarrow w^{\tau - 1} \]

\[ m_\tau \leftarrow \left\lceil \frac{44S}{\mu} \right\rceil, \quad k_\tau \leftarrow \max \left( \left\lceil \frac{44S}{\mu} \right\rceil, \left\lceil \frac{20s^2}{\rho_1 \| r \|_1} \right\rceil \right) \]

Draw \( k_\tau \) samples \( (x_1, y_1), \ldots, (x_{k_\tau}, y_{k_\tau}) \) and estimate the batch gradient

\[ g \leftarrow \frac{1}{k_\tau} \sum_{i=1}^{k_\tau} \nabla \phi(\bar{z}; x_i, y_i) \]

Sample \( \tilde{m}_\tau \) uniformly at random from \( \{1, \ldots, m_\tau\} \)

**for** \( i = 1, \ldots, \tilde{m}_\tau \) **do**

Draw sample \( (x_i, y_i) \)

\[ z \leftarrow z - \frac{\eta_S}{\tilde{m}_\tau} (\nabla \phi(z; x_i, y_i) - \nabla \phi(\bar{z}; x_i, y_i) + g) \]

**end for**

\[ w^\tau \leftarrow z \]

**end for**

**Output:** Return \( w^\Gamma \) as the approximate solution.

Consequently, we have

\[ \frac{1}{2} \left( \left\| E_{xx} u^* - \Sigma_{xx} u_T \right\|^2 + \left\| E_{yy} v^* - \Sigma_{yy} v_T \right\|^2 \right) = \left\| \bar{r} - r^* \right\|^2 = 2 \left( 1 - r^\top r^* \right) \leq \frac{\epsilon}{16} \]

and so \( \max \left( \left\| E_{xx} u^* - \Sigma_{xx} u_T \right\|^2, \left\| E_{yy} v^* - \Sigma_{yy} v_T \right\|^2 \right) \leq \frac{\epsilon}{8} \). We are now in the same situation as (23); we can fix the incorrect normalization of \( \bar{r} \) analogously and then our lemma follows.

It remains to show the time complexity to achieve (26). According to Lemma 23, it suffices to have

\[ \cos \theta_T = \frac{1}{\sqrt{2}} \frac{\bar{u}^\top \Sigma_{xx} u_T + \bar{v}^\top \Sigma_{yy} v_T}{\sqrt{\bar{u}^\top \Sigma_{xx} u_T + \bar{v}^\top \Sigma_{yy} v_T}} \geq 1 - \frac{\epsilon^2}{2^{15}}. \]  \hspace{1cm} (27)

In turn, it suffices to have \( |\sin \theta_T| \leq \frac{\epsilon}{256} \) and we ensure it by requiring \( G(r_T) \leq \frac{\epsilon^2}{256} \).

Applying Theorem 12, the total time complexity of our algorithm, considering the terms depending on \( \epsilon \), is of the order

\[ \mathcal{O} \left( d \left( N + \frac{1}{\Delta^2 \gamma^2} \right) \log \frac{1}{\epsilon} \right). \]  \hspace{1cm} (28)

**Appendix E. Proofs for Section 5**
E.1. Proof of Lemma 15

**Proof** The desired result is a direct consequence of Lemma 23, by making the identification that

\[
a = \mathbf{E}_{\mathcal{F}_s} u^*, \quad x = \mathbf{E}_{\mathcal{F}_s} u_T, \quad b = \mathbf{E}_{\mathcal{G}_y} v^*, \quad y = \mathbf{E}_{\mathcal{G}_y} v_T.
\]

\[\square\]

E.2. Proof of Lemma 16

We divide the desired results into the following three lemmas.

**Lemma 26 (Strong convexity)** For any \( w, w' \in \mathbb{R}^d \), we have

\[
f_{t+1}(w) \geq f_{t+1}(w') + \langle \nabla f_{t+1}(w'), w - w' \rangle + \frac{\mu}{2} \|w - w'\|^2
\]

where \( \mu := \frac{\lambda}{\rho_1} \geq C \Delta \gamma \) for some \( C > 0 \).

**Proof** Just observe that the Hessian of \( f_{t+1}(w) \) is \( A_{\lambda} = B^2 M^{-1}_\lambda B^2 \), whose eigenvalues are bounded from below: \( \sigma_{\text{min}}(A_{\lambda})\geq(\lambda - \rho_1) \cdot \sigma_{\text{min}}(B) = \gamma / \beta_1 \). The lemma follows from the assumption that \( \lambda = \rho_1 + c \Delta \) for \( c \in (0, 1) \).

**Lemma 27 (Streaming smoothness)** For any \( w \in \mathbb{R}^d \), we have

\[
\mathbb{E} \left[ \| \nabla \phi_{t+1}(w) - \nabla \phi_{t+1}(w^*_t) \|^2 \right] \leq 2S (f_{t+1}(w) - f^*_t)
\]

where \( S = O \left( \frac{d \beta_1}{\gamma} \right) = O \left( \frac{d}{\Delta \gamma} \right) \) for the sub-Gaussian/regular polynomial-tail classes, and \( S = O \left( \frac{\beta_1}{\gamma} \right) = O \left( \frac{1}{\Delta \gamma} \right) \) for the bounded class.

**Proof** Observe that

\[
\nabla \phi_{t+1}(w) = \begin{bmatrix} \lambda x x^T & -x y^T \\ -y x^T & \lambda y y^T \end{bmatrix} w - \begin{bmatrix} x x^T \\ y y^T \end{bmatrix} w_t.
\]

As shown in Lemma 11, this gradient function is Lipschitz continuous:

\[
\| \nabla \phi_{t+1}(w) - \nabla \phi_{t+1}(w^*_t) \| \leq (\lambda + 1) \cdot \sup (\|x\|, \|y\|) \cdot \|w - w^*_t\|.
\]

Note that \( \lambda \leq \rho_1 + u \Delta \) where \( \rho_1 \leq 1, \Delta \leq 1, \) and \( u < 1 \) by assumption, and thus \( \lambda \leq 2 \). As a result, we obtain

\[
\mathbb{E} \left[ \| \nabla \phi_{t+1}(w) - \nabla \phi_{t+1}(w^*_t) \|^2 \right] \leq 9 \mathbb{E} \left[ \|x\|^2 + \|y\|^2 \right] \cdot \|w - w^*_t\|^2.
\]

For the distributions of \( P(x, y) \) considered here, \( \mathbb{E} \|x\|^2 \) and \( \mathbb{E} \|y\|^2 \) are both \( O(d) \) for the sub-Gaussian/regular polynomial-tail inputs (see Remark 25), and bounded by 1 for the bounded inputs.
On the other hand, according to Lemma 26, we have

\[ f(w) - f(w_{t+1}^*) \geq C \Delta \gamma \|w - w_{t+1}^*\|^2 \]

for some \( C > 0 \).

Combining the above two inequalities gives the desired result. \( \square \)

**Lemma 28 (Streaming variance)** We have

\[ \mathbb{E} \left[ \frac{1}{2} \| \nabla \phi(w_{t+1}^*) \|^2 \right] \leq \sigma^2, \]

where \( \sigma^2 = O \left( \frac{d \beta_1^3 \| r_t \|^2}{\gamma^2} \right) \) for the sub-Gaussian/regular polynomial-tail classes, and \( \sigma^2 = O \left( \frac{\beta_1^3 \| r_t \|^2}{\gamma^2} \right) \) for the bounded class.

**Proof** Observe that \( w_{t+1}^* = A_{\lambda}^{-1}Bw_t \) and

\[ \nabla \phi(w_{t+1}^*) = \left[ \begin{array}{cc} \lambda xx^T - xy^T & -y\lambda y^T \\ -y\lambda x^T & \lambda yy^T \end{array} \right] A_{\lambda}^{-1}B - \left[ \begin{array}{cc} xx^T & yy^T \end{array} \right] B_{\lambda}^{-\frac{1}{2}}. \]

Define the shorthands \( D = B_{\lambda}^{-\frac{1}{2}} \left[ \begin{array}{cc} \lambda xx^T - xy^T & -y\lambda y^T \end{array} \right] B_{\lambda}^{-\frac{1}{2}} \) and \( E = B_{\lambda}^{-\frac{1}{2}} \left[ \begin{array}{cc} xx^T & yy^T \end{array} \right] B_{\lambda}^{-\frac{1}{2}}. \)

Then we have

\[ \mathbb{E} \left[ \frac{1}{2} \| \nabla \phi(w_{t+1}^*) \|^2 \right] = \mathbb{E} \left[ \frac{1}{2} \left\| \nabla \phi(w_{t+1}^*) \right\|^2 \right] \]

\[ = \mathbb{E} \left[ \frac{1}{2} \left\| B_{\lambda}^{-\frac{1}{2}} \nabla \phi(w_{t+1}^*) \right\|^2 \right] \]

\[ = \frac{1}{2} \mathbb{E} \left[ \left( B_{\lambda}^{-\frac{1}{2}}w_t \right)^T (M_{\lambda}D - E) \cdot M_{\lambda} \cdot (DM_{\lambda} - E) \left( B_{\lambda}^{-\frac{1}{2}}w_t \right) \right] \]

\[ = \frac{1}{2} \mathbb{E} \left[ r_t^T (M_{\lambda}D - E) \cdot M_{\lambda} \cdot (DM_{\lambda} - E)r_t \right]. \quad (29) \]

**Bounded case** For the bounded case where \( \sup \left( \|x\|^2, \|y\|^2 \right) \leq 1 \), the derivation is relatively simple. We can bound \( \|D\| \leq \frac{3}{\gamma} \) and \( \|E\| \leq \frac{1}{\gamma} \), and thus

\[ \mathbb{E} \left[ \frac{1}{2} \| \nabla \phi(w_{t+1}^*) \|^2 \right] \leq \frac{M_{\lambda}r_t^2}{2} \mathbb{E} \|DM_{\lambda} - E\|^2 \]

\[ \leq \beta_1 \|r_t\|^2 \left( \mathbb{E} \|DM_{\lambda}\|^2 + \mathbb{E} \|E\|^2 \right) \]

\[ = O \left( \frac{\beta_1^3 \|r_t\|^2}{\gamma^2} \right) \]

where we have used the triangle inequality and the fact that \( (x + y)^2 \leq 2x^2 + 2y^2 \) in the second inequality.
Sub-Gaussian/regular polynomial tail cases  We now omit the subscript \( \lambda \) in \( M_\lambda \) and \( t \) from
iterates for convenience. Using the fact that \( |x + y|^2 \leq 2 |x|^2 + 2 |y|^2 \) with \( x = M_\lambda^T \mathbf{D} \mathbf{r} \) and
\( y = M_\lambda^T \mathbf{E} \mathbf{r} \), we continue from (29) and obtain
\[
\mathbb{E} \left[ \frac{1}{2} \| \nabla \phi (w^*_{t+1}) \|^2 \left( \nabla^2 f (w^*_{t+1}) \right)^{-1} \right] \leq \mathbb{E} \left[ \mathbf{r}^T \mathbf{D} \mathbf{M} \mathbf{D} \mathbf{r} \right] + \mathbb{E} \left[ \mathbf{r}^T \mathbf{E} \mathbf{M} \mathbf{E} \mathbf{r} \right].
\]
Introduce the notation \( \mathbf{u} = \mathbb{E}_{xx}^{-\frac{1}{2}} \mathbf{x} \), \( \mathbf{v} = \mathbb{E}_{yy}^{-\frac{1}{2}} \mathbf{y} \), and partition \( \mathbf{r} \) and \( \mathbf{M} \) according to \((\mathbf{x}, \mathbf{y})\):
\[
\mathbf{r} = \begin{bmatrix} \mathbf{r}_x \\ \mathbf{r}_y \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \mathbf{M}_{xx} & \mathbf{M}_{xy} \\ \mathbf{M}_{yx} & \mathbf{M}_{yy} \end{bmatrix}.
\]
In view of Lemma 24, we can assume \( \max \left( \mathbb{E} \| \mathbf{u} \|^4, \mathbb{E} \| \mathbf{v} \|^4 \right) \leq C d^2 \). From now on, we use \( C \) for
a generic constant whose specific value may change between appearances.
We have
\[
\mathbb{E} \mathbf{M} \mathbf{E} = \begin{bmatrix} \mathbf{u}^\top \mathbf{M}_{xx} \mathbf{u} & \mathbf{u}^\top \mathbf{M}_{xy} \mathbf{v} \\ \mathbf{v}^\top \mathbf{M}_{yx} \mathbf{u} & \mathbf{v}^\top \mathbf{M}_{yy} \mathbf{v} \end{bmatrix}.
\]
and thus
\[
\mathbf{r}^\top \mathbb{E} \mathbf{M} \mathbf{E} \mathbf{r} = \mathbf{r}_x^\top \mathbf{u}^\top \mathbf{M}_{xx} \mathbf{u} \mathbf{r}_x + \mathbf{r}_y^\top \mathbf{v}^\top \mathbf{M}_{yy} \mathbf{v} \mathbf{r}_y + 2 \mathbf{r}_x^\top \mathbf{u}^\top \mathbf{M}_{xy} \mathbf{v} \mathbf{r}_y.
\]
We have for the first term that
\[
\begin{align*}
\mathbb{E} \left[ \mathbf{r}_x^\top \mathbf{u}^\top \mathbf{M}_{xx} \mathbf{u} \mathbf{r}_x \right] &= \mathbb{E} \left[ \mathbf{r}_x^\top \mathbf{u} \right]^2 \mathbf{M}_{xx} \mathbf{u} \\
&\leq \sqrt{\mathbb{E} \| \mathbf{r}_x^\top \mathbf{u} \|^4} \sqrt{\mathbb{E} \| \mathbf{u} \|^2 \mathbf{M}_{xx} \mathbf{u}^2} \\
&\leq C \| \mathbf{r}_x \| \sqrt{\mathbb{E} \| \mathbf{u} \|^2 \mathbf{M}_{xx} \mathbf{u}^2} \\
&\leq C \| \mathbf{M} \| \| \mathbf{r}_x \| \| \mathbf{r}_y \| d.
\end{align*}
\]
Similar arguments also lead to
\[
\begin{align*}
\mathbb{E} \left[ \mathbf{r}_y^\top \mathbf{v}^\top \mathbf{M}_{yy} \mathbf{v} \mathbf{r}_y \right] &\leq C \| \mathbf{M} \| \| \mathbf{r}_y \|^2 d.
\end{align*}
\]
For the third term, we have
\[
\begin{align*}
\mathbb{E} \left[ \mathbf{r}_x^\top \mathbf{u}^\top \mathbf{M}_{xy} \mathbf{v} \mathbf{r}_y \right] &\leq \sqrt{\mathbb{E} \| \mathbf{r}_x^\top \mathbf{u} \|^2 \| \mathbf{r}_y^\top \mathbf{v} \|^2 \sqrt{\mathbb{E} \| \mathbf{u} \|^2 \mathbf{M}_{xy} \mathbf{v}^2}} \\
&\leq \| \mathbf{M} \| \left( \mathbb{E} \| \mathbf{r}_x^\top \mathbf{u} \|^4 \right)^{\frac{1}{4}} \left( \mathbb{E} \| \mathbf{r}_y^\top \mathbf{v} \|^4 \right)^{\frac{1}{4}} \left( \mathbb{E} \| \mathbf{u} \|^4 \right)^{\frac{1}{4}} \left( \mathbb{E} \| \mathbf{v} \|^4 \right)^{\frac{1}{4}} \\
&\leq C \| \mathbf{M} \| \| \mathbf{r}_x \| \| \mathbf{r}_y \| d.
\end{align*}
\]
Therefore,
\[ \mathbb{E} \left[ r^\top \text{EMEr} \right] \leq C \| M \| \| r \|^2 d. \]

Now we need to bound \( \mathbb{E} \left[ r^\top \text{MDMDMr} \right] \). Using the fact that \( \| x + y \|^2 \leq 2 \| x \|^2 + 2 \| y \|^2 \) with \( x = M_2 D_1 Mr \) and \( y = M_2 D_2 Mr \), this can be bounded by two terms:
\[ \mathbb{E} \left[ r^\top \text{MDMDMr} \right] \leq 2 \mathbb{E} \left[ r^\top \text{MDMD}_1 \text{MD}_1 Mr \right] + 2 \mathbb{E} \left[ r^\top \text{MD}_2 \text{MD}_2 Mr \right] \]
where
\[
D_1 = \lambda \begin{bmatrix} uu^\top & 0 \\ 0 & vv^\top \end{bmatrix}, \quad D_2 = -\begin{bmatrix} 0 & uv^\top \\ vu^\top & 0 \end{bmatrix}.
\]
The bound for \( \mathbb{E} \left[ r^\top \text{MDMD}_1 \text{MD}_1 Mr \right] \) can be derived using the same argument that bounds \( \mathbb{E} \left[ r^\top \text{EMEr} \right] \) (now \( Mr \) plays the role of \( r \) in bounding \( \mathbb{E} \left[ r^\top \text{EMEr} \right] \)), and thus we have
\[ \mathbb{E} \left[ r^\top \text{MDMD}_1 \text{MD}_1 Mr \right] \leq C \lambda^2 \| M \| \| Mr \|^2 d \leq C \| M \|^3 \| r \|^2 \lambda^2 d. \]

Finally, we bound \( \mathbb{E} \left[ r^\top \text{MD}_2 \text{MD}_2 Mr \right] \). Note that
\[
-\text{MD}_2 \text{MD}_2 Mr = \begin{bmatrix} \text{MD}_2 \text{MD}_2 Mr \end{bmatrix}.
\]
Let
\[
Mr = \begin{bmatrix} m_x \\ m_y \end{bmatrix},
\]
and then
\[
-\text{MD}_2 \text{MD}_2 Mr = m_x^\top uv^\top M_{yy} vu^\top m_x + m_y^\top vu^\top M_{xx} uv^\top m_y + 2m_x^\top uv^\top M_{yx} uv^\top m_y.
\]

 Similarly, to what we have done above,
\[
\mathbb{E} \left| m_x^\top uv^\top M_{yy} vu^\top m_x \right| \leq \sqrt{\mathbb{E} \left| m_x^\top u \right|^4} \sqrt{\mathbb{E} \left| v^\top M_{yy} v \right|^2} \\
\leq C \| M \| \| m_x \|^2 d \\
\leq C \| M \|^3 \| r \|^2 d.
\]

The same bound also holds for \( \mathbb{E} \left| m_y^\top vu^\top M_{xx} uv^\top m_y \right| \) with the same argument. For the term \( \mathbb{E} \left| m_x^\top uv^\top M_{yx} uv^\top m_y \right| \), we have
\[
\mathbb{E} \left| m_x^\top uv^\top M_{yx} uv^\top m_y \right| \leq \| M \| \left( \mathbb{E} \left| m_x u \right|^4 \right)^{\frac{1}{4}} \left( \mathbb{E} \left| m_y v \right|^4 \right)^{\frac{1}{4}} \left( \mathbb{E} \| u \|^4 \right)^{\frac{1}{4}} \left( \mathbb{E} \| v \|^4 \right)^{\frac{1}{4}} \\
\leq C \| M \| \| m_x \| \| m_y \| d \\
\leq C \| M \|^3 \| r \|^2 d.
\]
Combining all the terms, and noting that \( \lambda \leq 2 \), we have shown that

\[
\mathbb{E} \left[ r^\top M D M D M r \right] \leq C \|M\|^3 \|r\|^2 d.
\]

And the final bound is

\[
\mathbb{E} \left[ r^\top (M D - E) M (D M - E) r \right] \leq C \left[ \|M\|^3 + \|M\| \right] \|r\|^2 d = \mathcal{O} \left( \beta_1^3 \|r\|^2 d \right).
\]

**E.3. Proof of Lemma 18**

**Proof** For notational simplicity, we omit the subscript \( t + 1 \) below.

According to Frostig et al. (2015, Theorem 4.1), we have that for iteration \( \tau \) of Algorithm 1

\[
\mathbb{E} \left[ f(w^\tau) - f^* \right] \leq \frac{1}{1 - 4s} \left[ \left( \frac{S}{\mu m_r s} + 4s \right) \mathbb{E} \left[ f(w^{\tau - 1}) - f^* \right] + \frac{1 + 2s}{k_\tau} \left( \sqrt{\frac{S}{\mu} \mathbb{E} \left[ f(w^{\tau - 1}) - f^* \right] + \sigma} \right)^2 \right].
\]

Using the inequality \((x + y)^2 \leq 2(x^2 + y^2)\), it holds that

\[
\left( \sqrt{\frac{S}{\mu} \mathbb{E} \left[ f(w^{\tau - 1}) - f^* \right] + \sigma} \right)^2 \leq \frac{2S}{\mu} \mathbb{E} \left[ f(w^{\tau - 1}) - f^* \right] + 2\sigma^2.
\]

Now, set for this iteration \( s = \frac{c_2}{8} \), \( m_r = \lceil \frac{S}{\mu c_3} \rceil \), and \( k_\tau = \max \left( \lceil \frac{S}{\mu c_2} \rceil, \left\lceil \frac{\sigma^2}{\beta_1 \|r\|^2 c_3} \right\rceil \right) \), for some \( c_2, c_3 \in (0, 1) \). We continue from (30) and have

\[
\mathbb{E} \left[ f(w^\tau) - f^* \right] \leq \frac{1}{1 - 4s} \left[ \left( \frac{S}{\mu m_r s} + 4s + \frac{2 + 4s S}{k_\tau} \frac{S}{\mu} \right) \mathbb{E} \left[ f(w^{\tau - 1}) - f^* \right] + \frac{2 + 4s}{k_\tau} \sigma^2 \right]
\]

\[
\leq \frac{1}{1 - c_2/2} \left[ \left( 8c_2 + c_2^2/2 + \frac{2 + 4c_2}{2} \right) \mathbb{E} \left[ f(w^{\tau - 1}) - f^* \right] + \frac{4 + c_2}{2k_\tau} \sigma^2 \right]
\]

\[
\leq 22c_2 \cdot \mathbb{E} \left[ f(w^{\tau - 1}) - f^* \right] + 10c_3 \cdot \frac{\beta_1 \|r\|^2}{2}.
\]

We can now calculate the number of samples used in this iteration, which is

\[
k_\tau + m_r = \mathcal{O} \left( \frac{d \beta_1^2}{c_3} + \frac{d \beta_1^2}{\gamma^2 c_2^2} \right) = \mathcal{O} \left( \frac{d}{\Delta^2 c_3} + \frac{d}{\Delta^2 \gamma^2 c_2^2} \right)
\]

for sub-Gaussian/regular polynomial-tail inputs, and

\[
k_\tau + m_r = \mathcal{O} \left( \frac{\beta_1^2}{\gamma^2 c_3} + \frac{\beta_1^2}{\gamma^2 c_2^2} \right) = \mathcal{O} \left( \frac{1}{\Delta^2 \gamma^2 c_3} + \frac{1}{\Delta^2 \gamma^2 c_2^2} \right)
\]
Let us fix $c_2 = \frac{1}{d^2}$ for $\tau = 1, \ldots, \Gamma$. In view of our initialization strategy (14), setting $c_3 = \frac{1}{2d^2}$ for $\tau = 1$ gives $\mathbb{E} \left[ f(w^1) - f^* \right] \leq \frac{\beta_1 \| r_{t1} \|^2}{2}$. Afterwards, we halve $c_3$ at each outer loop $\tau = 2, \ldots,$ and this makes sure that $\mathbb{E} \left[ f(w^\tau) - f^* \right] \leq \frac{\beta_1 \| r_{t1} \|^2}{2^\tau}$. To achieve the desired accuracy, we need $\Gamma = \log \frac{1}{\eta_t}$ outer iterations. Summing (31) and (31) over $\tau = 1, \ldots, \Gamma$, and noting $\sum_{\tau=1}^\Gamma 2^{\tau-1} = O \left( \frac{1}{\eta_t} \right)$, the total sample complexity is

$$O \left( \frac{d}{\Delta^2} \cdot 20 \sum_{\tau=1}^\Gamma 2^{\tau-1} + \frac{44^2d}{\Delta^2 \gamma^2} \cdot \log \frac{1}{\eta_t} \right) = O \left( \frac{d}{\Delta^2 \eta_t} + \frac{d}{\Delta^2 \gamma^2 \log \frac{1}{\eta_t}} \right)$$

for sub-Gaussian/regular polynomial-tail inputs, and

$$O \left( \frac{1}{\Delta^2 \gamma^2} \left( 20 \sum_{\tau=1}^\Gamma 2^{\tau-1} + 44^2 \cdot \log \frac{1}{\eta_t} \right) \right) = O \left( \frac{1}{\Delta^2 \gamma^2 \eta_t} \right)$$

for bounded inputs.

**E.4. Proof of Theorem 19**

**Proof** Recall that our streaming CCA algorithm performs shift-and-invert power iterations on the population matrices directly. Following the same argument in the ERM case in Corollary 13, as long as each least squares objective is solved to sufficient accuracy, i.e.,

$$\frac{f_{t+1}(w_{t+1}) - f_t^*}{w^T_t B w_t} \leq \min \left( \sum_{i=2}^d \xi_{ti}^2 / \beta_i, \frac{\xi_{t1}^2}{\beta_1} \right) \cdot \frac{(\beta_1 - \beta_2)^2}{32},$$

(33)

the algorithm converges linearly, and therefore we only need to solve $T = O \left( \log \frac{1}{\epsilon} \right)$ linear systems. But due to the zero initialization we use in the online setting, the ratio between initial error and final error for each $f_{t+1}$ is different from the offline setting. When $G(r_t) > 1$, we are in the regime where $\sum_{i=2}^d \xi_{ti}^2 / \beta_i \geq \xi_{t1}^2 / \beta_1$, and we can ensure the sufficient accuracy in (33) by setting the ratio between the initial and the final error to be

$$\eta_t = \frac{(\beta_1 - \beta_2)^2}{16 \beta_1^2} \frac{\xi_{t1}^2}{\max_{i} \xi_{ti}^2}$$

in Lemma 18. Since $\frac{\beta_1^2}{(\beta_1 - \beta_2)^2} \leq 4$, this implies that

$$\frac{1}{\eta_t} \leq \frac{64}{\cos^2 \theta_t} = 64 \left( 1 + \tan^2 \theta_t \right) \leq 64 \left( 1 + \frac{\beta_2}{\beta_1} G^2(r_t) \right) \leq 64 \left( 1 + G^2(r_t) \right) \leq 64 \left( 1 + G^2(r_0) \right).$$

Note that the sample complexity of this phase does not depend on the final accuracy in alignment. When $G(r_t) \leq 1$, indicating that we are in the converging regime where $\sum_{i=2}^d \xi_{ti}^2 / \beta_i \leq \xi_{t1}^2 / \beta_1$, we can ensure the sufficient accuracy in (33) by setting

$$\eta_t = \frac{(\beta_1 - \beta_2)^2}{16 \beta_1^2} \frac{\sum_{i=2}^d \xi_{ti}^2}{\max_{i} \xi_{ti}^2}$$
in Lemma 18. This implies that

\[
\frac{1}{\eta_t} \leq \frac{64}{\sin^2 \theta_t}.
\]

Our goal is to have \( \sin^2 \theta_T \leq \frac{\epsilon}{4} \), as this implies \( \cos \theta_T = \sqrt{1 - \sin^2 \theta_T} \geq 1 - \sin^2 \theta_T \geq 1 - \frac{\epsilon}{4} \), and by Lemma 15 this further implies align \( ((u_T, v_T); (u^*, v^*)) \) \( \geq 1 - \epsilon \) as desired. Since \( \sin^2 \theta_t \leq G^2(r_t) \), and we have shown that \( G^2(r_t) \) decreases at a geometric rate, we can bound \( \frac{1}{\sin^2 \theta_t} \) by a geometrically increasing series where the last term is \( \frac{4}{\epsilon} \), and the sum of the truncated series up to time \( T \) is of the same order of the last term, i.e., \( \sum_{t=1}^{T-1} \frac{1}{\eta_t} = \mathcal{O} \left( \frac{1}{\epsilon} \right) \).

And the theorem follows from Lemma 18, by summing the sample complexity of least squares problems over the outer shift-and-invert iterations.

We remark that to achieve the result with probability \( 1 - \delta \), we require each least squares problem to be solved to the desired accuracy with failure probability \( \delta / \log(1/\epsilon) \) (using the Markov inequality) and finally apply the union bound. This would only cause additional \( \log(1/\epsilon) \) factors in the total sample complexity. \( \blacksquare \)