About the second neighborhood problem in tournaments missing disjoint stars

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Abstract

Let $D$ be a digraph without digons. Seymour’s second neighborhood conjecture states that $D$ has a vertex $v$ such that $d^+(v) \leq d^{++}(v)$. Under some conditions, we prove this conjecture for digraphs missing $n$ disjoint stars. Weaker conditions are required when $n = 2$ or $3$. In some cases we exhibit two such vertices.

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1. Introduction

In this paper, a digraph $D$ is a pair of two disjoint finite sets $(V, E)$ such that $E \subseteq V \times V$. $E$ is the arc set and $V$ is the vertex set and they are denoted by $E(D)$ and $V(D)$ respectively. An oriented graph is a digraph without loop and digon (directed cycles of length two). If $K \subseteq V(D)$ then the induced restriction of $D$ to $K$ is denoted by $D[K]$. As usual, $N_D^+(v)$ (resp. $N_D^-(v)$) denotes the (first) out-neighborhood (resp. in-neighborhood) of a vertex $v \in V$. $N_D^{++}(v)$ (resp. $N_D^{--}(v)$) denotes the second out-neighborhood (resp. in-neighborhood) of a vertex $v \in V$.
Conjecture 1. In 1990, Seymour conjectured the following: \(N_D^-(v))\) denotes the second out-neighborhood (in-neighborhood) of \(v\), which is the set of vertices that are at distance 2 from \(v\) (resp. to \(v\)). We also denote \(d_D^+(v) = |N_D^+(v)|\), \(d_D^+(v) = |N_D^+(v)|\), \(d_D^+(v) = |N_D^+(v)|\) and \(d_D^+(v) = |N_D^+(v)|\). We omit the subscript if the digraph is clear from the context. For short, we write \(x \rightarrow y\) if the arc \((x, y) \in E\). A vertex \(v \in V(D)\) is called whole if it is adjacent to every vertex in \(V(D) - \{v\}\). A sink \(v\) is a vertex with \(d^+(v) = 0\), while a source \(v\) is a vertex with \(d^-(v) = 0\). For \(x, y \in V(D)\), we say \(xy\) is a missing edge of \(D\) if neither \((x, y)\) nor \((y, x)\) are in \(E(D)\). The missing graph \(G\) of \(D\) is the graph whose edges are the missing edges of \(D\) and whose vertices are the non whole vertices of \(D\). In this case, we say that \(D\) is missing \(G\). So, a tournament does not have any missing edge. A star of center \(x\) is a graph whose edge set has the form \(\{a_ix; i = 1, ..., k\}\). In this paper, \(n\) stars are said to be disjoint if any two of them do not share a common vertex.

A vertex \(v\) of \(D\) is said to have the second neighborhood property (SNP) if \(d_D^+(v) \leq d_D^+(v)\). In 1990, Seymour conjectured the following:

**Conjecture 1. (Seymour’s Second Neighborhood Conjecture (SNC))**[1] Every oriented graph has a vertex with the SNP.

In 1996, Fisher [3] solved the SNC for tournaments by using a certain probability distribution on the vertices. Another proof of Dean’s conjecture was established in 2000 by Havet and Thomassé [7]. Their short proof uses a tool called median orders. Furthermore, they have proved that if a tournament has no sink vertex then there are at least two vertices with the SNP. In 2007 Fidler and Yuster [2] proved, using median orders and dependency digraphs, that SNC holds for digraphs missing a matching, a star or a complete graph. Ghazal proved more general statements in [4, 6] and proved that the SNC holds for some other classes of digraphs [5].

2. Definitions and Preliminary Results

Let \(L = v_1v_2...v_n\) be an ordering of the vertices of a digraph \(D\). An arc \(e = (v_i, v_j)\) is forward with respect to \(L\) if \(i < j\). Otherwise \(e\) is a backward arc. The weight of \(L\) is \(\omega(L) = \{|(v_i, v_j) \in E(D); i < j\}|\). \(L\) is called a median order of \(D\) if \(\omega(L) = \max\{\omega(L'); L'\text{ is an ordering of the vertices of } D\}; that is \(L\) maximizes the number of forward arcs. In fact, the median order \(L\) satisfies the feedback property: For all \(1 \leq i \leq j \leq n:\)

\[
d^+_{D[i,j]}(v_i) \geq d^+_{D[i,j]}(v_i)
\]

and

\[
d^+_{D[i,j]}(v_j) \geq d_{D[i,j]}^+(v_j)
\]

where \([i, j] := \{v_i, v_{i+1}, ..., v_j\}\) (See [7]).

It is also known that if we reverse the orientation of a backward arc \(e = (v_i, v_j)\) of \(D\) with respect to \(L\), then \(L\) is again a weighted median order of the new digraph \(D' = D - (v_i, v_j) + (v_j, v_i)\)
(See [5]).

Let \( L = v_1v_2 \cdots v_n \) be a median order. Among the vertices not in \( N^+(v_n) \) two types are distinguished: A vertex \( v_j \) is good if there is \( i \leq j \) such that \( v_n \rightarrow v_i \rightarrow v_j \), otherwise \( v_j \) is a bad vertex. The set of good vertices of \( L \) is denoted by \( G_L^+ \) [7] (or \( G_L \) if there is no confusion). Clearly, \( G_L \subseteq N^+(v_n) \). The last vertex \( v_n \) is called a good vertex of \( D \).

We say that a missing edge \( x_1y_1 \) loses to a missing edge \( x_2y_2 \) if: \( x_1 \rightarrow x_2, y_2 \notin N^+(x_1) \cup N^+(x_1), y_1 \rightarrow y_2 \) and \( x_2 \notin N^+(y_1) \cup N^+(y_1) \). The dependency digraph \( \Delta \) of \( D \) is defined as follows: Its vertex set consists of all the missing edges and \( (ab, cd) \in E(\Delta) \) if \( ab \) loses to \( cd \) [2, 5]. Note that \( \Delta \) may contain digons.

\textbf{Definition 1.} [4] In a digraph \( D \), a missing edge \( ab \) is called a good missing edge if:

\begin{enumerate}[(i)]
  \item \((\forall v \in V \setminus \{a, b\})([v \rightarrow a] \Rightarrow [b \in N^+(v) \cup N^+(v)]))\) or
  \item \((\forall v \in V \setminus \{a, b\})([v \rightarrow b] \Rightarrow [a \in N^+(v) \cup N^+(v)]))\).
\end{enumerate}

If \( ab \) satisfies (i) we say that \( (a, b) \) is a convenient orientation of \( ab \).

If \( ab \) satisfies (ii) we say that \( (b, a) \) is a convenient orientation of \( ab \).

We will need the following observation:

\textbf{Lemma 2.1.} ([2], [5]) Let \( D \) be an oriented graph and let \( \Delta \) denote its dependency digraph. A missing edge \( ab \) is good if and only if its in-degree in \( \Delta \) is zero.

Let \( D \) be a digraph and let \( \Delta \) denote its dependency digraph. Let \( C \) be a connected component of \( \Delta \). Set \( K(C) = \{ u \in V(D) \mid \text{there is a vertex } v \text{ of } D \text{ such that } uv \text{ is a missing edge and belongs to } C \} \). The interval graph of \( D \), denoted by \( I_D \) is defined as follows. Its vertex set consists of the connected components of \( \Delta \) and two vertices \( C_1 \) and \( C_2 \) are adjacent if \( K(C_1) \cap K(C_2) \neq \emptyset \).

So \( I_D \) is the intersection graph of the family \( \{K(C) \mid C \text{ is a connected component of } \Delta \} \). Let \( \xi \) be a connected component of \( I_D \). We set \( K(\xi) = \cup_{C \in \xi} K(C) \). Clearly, if \( uv \) is a missing edge in \( D \) then there is a unique connected component \( \xi \) of \( I_D \) such that \( u \) and \( v \) belong to \( K(\xi) \).

For \( f \in V(D) \), we set \( J(f) = \{ f \} \) if \( f \) is a whole vertex, otherwise \( J(f) = K(\xi) \), where \( \xi \) is the unique connected component of \( I_D \) such that \( f \in K(\xi) \). Clearly, if \( x \in J(f) \) then \( J(f) = J(x) \) and if \( x \notin J(f) \) then \( x \) is adjacent to every vertex in \( J(f) \).

Let \( L = x_1 \cdots x_n \) be a median order of a digraph \( D \). For \( i < j \), the sets \([i, j] := [x_i, x_j] := \{x_i, x_{i+1}, \ldots, x_j\}\) and \([i, j] = [i, j]\setminus\{x_i, x_j\}\) are called intervals of \( L \). We recall that \( K = \cup V(D) \) is an interval of \( D \) if for every \( u, v \in K \) we have: \( N^+(u) \setminus K = N^+(v) \setminus K \) and \( N^-(u) \setminus K = N^-(v) \setminus K \).

The following shows a relation between the intervals of \( D \) and the intervals of \( L \).

\textbf{Proposition 2.1.} [6] Let \( I = \{I_1, \ldots, I_r\} \) be a set of pairwise disjoint intervals of \( D \). Then for every median order \( L \) of \( D \), there is a weighted median order \( L' \) of \( D \) such that: \( L \) and \( L' \) have the same feed vertex and every interval in \( I \) is an interval of \( L' \).

We say that \( D \) is good digraph if the sets \( K(\xi) \)'s are intervals of \( D \). By the previous proposition, every good digraph has a median order \( L \) such that the \( K(\xi) \)'s form intervals of \( L \). Such an
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enumeration is called a good median order of the good digraph $D$ [6].

**Theorem 2.1.** [6] Let $D$ be a good oriented graph and let $L$ be a good median order of $D$, with feed vertex $f$. Then for every $x \in J(f)$, we have $|N^+(x) \setminus J(f)| \leq |G_L \setminus J(f)|$. So if $x$ has the SNP in $D[J(f)]$, then it has the SNP in $D$.

**Corollary 2.1.** ([7]) Let $L$ be a median order of a tournament with feed vertex $f$. Then $|N^+(f)| \leq |G_L|$.

Let $L$ be a good median order of a good oriented graph $D$ and let $f$ denote its feed vertex. By theorem 2.1, for every $x \in J(f)$, $|N^+(x) \setminus J(f)| \leq |G_L \setminus J(f)|$. Let $b_1, \ldots, b_r$ denote the bad vertices of $L$ not in $J(f)$ and $v_1, \ldots, v_s$ denote the non bad vertices of $L$ not in $J(f)$, both enumerated in increasing order with respect to their index in $L$.

If $|N^+(x) \setminus J(f)| < |G_L \setminus J(f)|$, we set Sed$(L) = L$. If $|N^+(x) \setminus J(f)| = |G_L \setminus J(f)|$, we set sed$(L) = b_1 \cdots b_r J(f)v_1 \cdots v_s$. This new order is called the sedimentation of $L$.

**Lemma 2.2.** [6] Let $L$ be a good median order of a good oriented graph $D$. Then Sed$(L)$ is a good median order of $D$.

In the rest of this section, $D$ is an oriented graph missing a matching and $\Delta$ denotes its dependency digraph. We begin by the following lemma:

**Lemma 2.3.** [2] The maximum out-degree of $\Delta$ is one and the maximum in-degree of $\Delta$ is one. Thus $\Delta$ is composed of vertex disjoint directed paths and directed cycles.

**Proof.** Assume that $a_1b_1$ loses to $a_2b_2$ and $a_1b_1$ loses to $a_2'b_2$, with $a_1 \to a_2$ and $a_1 \to a_2'$. The edge $a_2'b_2$ is not a missing edge of $D$. If $a_2' \to b_2$ then $b_1 \to a_2' \to b_2$, a contradiction. If $b_2 \to a_2'$ then $b_1 \to b_2 \to a_2'$, a contradiction. Thus, the maximum out-degree of $\Delta$ is one. Similarly, the maximum in-degree is one.

In the following, $C = a_1b_1, \ldots, a_kb_k$ denotes a directed cycle of $\Delta$, namely $a_i \to a_{i+1}$, $b_{i+1} \notin N^{++}(a_i) \cup N^+(a_i)$, $b_i \to b_{i+1}$ and $a_{i+1} \notin N^{++}(b_i) \cup N^+(b_i)$, for all $i < k$.

**Lemma 2.4.** ([2]) If $k$ is odd then $a_k \to a_1$, $b_1 \notin N^{++}(a_k) \cup N^+(a_k)$, $b_k \to b_1$ and $a_1 \notin N^{++}(b_k) \cup N^+(b_k)$. If $k$ is even then $a_k \to b_1$, $a_1 \notin N^{++}(a_k) \cup N^+(a_k)$, $b_k \to a_1$ and $b_1 \notin N^{++}(b_k) \cup N^+(b_k)$.

**Lemma 2.5.** [2] $K(C)$ is an interval of $D$.

**Proof.** Let $f \notin K(C)$. Then $f$ is adjacent to every vertex in $K(C)$. If $a_1 \to f$ then $b_2 \to f$, since otherwise $b_2 \in N^{++}(a_1) \cup N^+(a_1)$ which is a contradiction. So $N^+(a_1) \setminus K(C) \subseteq N^+(b_2) \setminus K(C)$. Applying this to every losing relation of $C$ yields: $N^+(a_1) \setminus K(C) \subseteq N^+(b_2) \setminus K(C) \subseteq N^+(a_3) \setminus K(C) \subseteq \cdots \subseteq N^+(b_k) \setminus K(C) \subseteq N^+(b_1) \setminus K(C) \subseteq N^+(a_2) \setminus K(C) \subseteq \cdots \subseteq N^+(a_k) \setminus K(C) \subseteq N^+(a_1) \setminus K(C)$ if $k$ is even. So these inclusion are equalities. An analogous argument proves the same result for odd cycles.  

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3. Main Results

3.1. Removing n stars

We recall that a vertex \( x \) in a tournament \( T \) is a king if \( \{ x \} \cup N^+(x) \cup N^{++}(x) = V(T) \). It is well known that every tournament has a king. However, for every natural number \( n \not\in \{2, 4\} \), there is a tournament \( T_n \) on \( n \) vertices, such that every vertex is a king for this tournament.

A digraph is called non trivial if it has at least one arc.

**Proposition 3.1.** Let \( D \) be a digraph missing disjoint stars. If the connected components of its dependency digraph are non-trivial strongly connected, then \( D \) is a good digraph.

**Proof.** Let \( \xi \) be a connected component of \( \Delta \). First, suppose that \( K(\xi) = K(C) \) for some directed cycle \( C = a_1b_1, a_2b_2, \ldots, a_nb_n \in \Delta \), namely \( a_i \to a_{i+1} \) and \( b_{i+1} \not\in N^+(a_i) \cup N^{++}(a_i) \). If the set of the missing edges \( \{a_ib_i; i = 1, \ldots, n\} \) forms a matching, then by lemma 2.5, \( K(C) \) is an interval of \( D \).

So we will suppose that a center \( x \) of a missing star appears twice in the list \( a_1, b_1, a_2, b_2, \ldots, a_n, b_n \) and assume without loss of generality that \( x = a_1 \). Suppose that \( n \) is even. Set \( K_1 = \{a_1, b_2, \ldots, a_{n-1}, b_n\} \) and \( K_2 = K(C) \setminus K_1 \).

Suppose that \( a_n \to b_1 \) and \( a_1 \not\in N^+(a_n) \cup N^{++}(a_n) \). Then by following the proof of lemma 2.5 we get the desired result.

Suppose \( a_n \to a_1 \) and \( b_1 \not\in N^+(a_n) \cup N^{++}(a_n) \). Then by following the proof of lemma 2.5 we get that \( K_1 \) and \( K_2 \) are intervals of \( D \). Assume, for contradiction that \( K_1 \cap K_2 = \emptyset \) and let \( i > 1 \) be the smallest index for which \( x \) is incident to \( a_ib_i \). Clearly \( i > 2 \). However, \( b_3 \not\in K_1 \) and \( x = a_1 \to a_2 \to a_3 \) implies that \( i > 3 \). Suppose that \( x = a_i \). Note that \( i \) must be odd by definition of \( K_1 \). Since \( b_2 \to a_1 = x = a_i \) and \( a_3 \not\in N^+(x) \cup N^{++}(x) \) then \( a_3 \to x \). Similarly \( b_4, a_5, \ldots, b_{i-1} \) are in-neighbors of \( x \). However, \( b_{i-1} \) is an out-neighbor of \( a_i = x \), a contradiction. Suppose that \( x = b_i \). Similarly, \( a_3, b_4, \ldots, a_{i-1} \) are in-neighbors of \( x \). However, \( a_{i-1} \) is an out-neighbor of \( b_i = x \), a contradiction. Thus \( K_1 \cap K_2 \neq \emptyset \). Whence, \( K = K_1 \cup K_2 \) is an interval of \( D \). Similar argument is used to prove it when \( n \) is odd.

This result can be easily extended to the case when \( K(\xi) = K(C) \) and \( C \) is a non trivial strongly connected component of \( \Delta \), because between any two missing edges \( uv \) and \( zt \) there is directed path from \( uv \) to \( zt \) and a directed path from \( zt \) to \( uv \). These two directed paths creat many directed cycles that are used to prove the desired result.

This also is extended to the case when \( K(\xi) = \bigcup_{C \in \xi} K(C) \): Let \( u \) and \( u' \) be two vertices of \( K(\xi) \). There are two non trivial strongly connected components of \( \Delta \) such that \( u \in K(C) \) and \( u' \in K(C') \). Since \( \xi \) is a connected component of \( \mathcal{I}_D \), there is a path \( C = C_0C_1 \ldots C_n = C' \). For all \( i > 0 \), there is \( u_i \in K(C_{i-1}) \cap K(C_i) \), by definition of edges in \( \mathcal{I}_D \). Therefore, \( N^+(u) \backslash K(\xi) = N^+(u_1) \backslash K(\xi) = \ldots = N^+(u_n) \backslash K(\xi) = N^+(u') \backslash K(\xi) \).
and $N^−(u)\setminus K(ξ) = N^−(u_1)\setminus K(ξ) = ... = N^−(u_i)\setminus K(ξ) = ... = N^−(u_n)\setminus K(ξ) = N^−(u')\setminus K(ξ)$.

**Theorem 3.1.** Let $D$ be a digraph obtained from a tournament by deleting the edges of disjoint stars. Suppose that, in the induced tournament by the centers of the missing stars, every vertex is a king. If $δ_Δ > 0$ then $D$ satisfies SNC.

**Proof.** Orient every missing edge of $D$ towards the center of its star. Let $L$ be a median order of the obtained tournament $T$ and let $f$ be its feed vertex. Then $f$ has the SNP in $T$. We prove that $f$ has the SNP in $D$ as well.

First, suppose that $f$ is a whole vertex. Then $N^+(f) = N^+_T(f)$. Let $v \in N^+_T(f)$. Then there $\exists u \in V(T) = V(D)$ such that $f \rightarrow u \rightarrow v \rightarrow f$ in $T$. Since $f$ is whole, then $(f, u)$ and $(v, f) \in D$. If $(u, v) \in D$ then $v \in N^+(f)$. Otherwise, $uv$ is a missing edge and hence, $\exists ab$ that loses to $uv$, say $b \rightarrow v$ and $u \notin N^+(b) \cup N^+(b)$. But $fb$ is not a missing edge, since $f$ is whole. Then $(f, b) \in D$, since otherwise, $b \rightarrow f \rightarrow u$ in $D$ which is a contradiction. Therefore, $f \rightarrow b \rightarrow v$ in $D$. Whence, $v \in N^+(f)$. So $N^+_T(f) \subseteq N^+(f)$. Therefore, $d^+(f) = d^+_T(f) \leq d^+_T(f) \leq d^+(f)$.

Now suppose that $f$ is the center of a missing star. Then $N^+(f) = N^+_T(f)$. Let $v \in N^+_T(f)$. Then there $\exists u \in V(T) = V(D)$ such that $f \rightarrow u \rightarrow v \rightarrow f$ in $T$. Then $(f, u) \in D$ while $(f, v) \notin D$. If $(u, v) \in D$ then $v \in N^+(f)$. Otherwise, $uv$ is a missing edge and $v$ is the center of a missing star. Then $v \in N^+(f) \cup N^+(f)$, because $f$ is a king for the centers of the missing stars. Note that $v \notin N^+(f)$. So $N^+_T(f) \subseteq N^+(f)$. Therefore, $f$ has the SNP in $D$.

Finally, suppose that $f$ is not whole and not the center of a missing star. Then $\exists x$ a center of a missing star such that $fx$ is a missing edge. We distinguish between two cases.

In the first case, we suppose that $fx$ does not lose to any missing edge. We reorient $fx$ as $(x, f)$. Since $(f, x) \in T$ is a backward arc with respect to $L$, the again $L$ is a median order of the new tournament $T'$ obtained by reversing the orientation of $fx$. Moreover, $N^+(f) = N^+_T(f)$ and $f$ has the SNP in $T'$. Let $v \in N^+_T(f)$. Then there $\exists u \in V(T) = V(D)$ such that $f \rightarrow u \rightarrow v \rightarrow f$ in $T'$. Then $(f, u) \in D$ while $(f, v) \notin D$. If $(u, v) \in D$ then $v \in N^+(f)$. Otherwise $uv$ is a missing edge and $v$ is the center of a missing star. Since $Δ$ has no source, there is a missing edge that loses to $uv$. Suppose that this edge is of the form $ax$. Then we must have $x \rightarrow v$ and $u \notin N^+(x) \cup N^+(x)$, by definition of losing relation and due to the fact that $v \in N^+(x) \cup N^+(x)$ ($x$ is a king for the centers of the missing stars). If $v \notin N^+(f)$, then $fx$ loses $uv$ which is a contradiction to the supposition of this case. Hence, $v \notin N^+(f)$. Now, suppose that the missing edge that loses to $uv$ is of the form $by$ with $x \notin \{b, y\}$. Suppose without loss of generality that $y$ is the center of a missing star containing $by$. Then $y \rightarrow v$ and $u \notin N^+(y) \cup N^+(y)$, by definition of losing relation and due to the fact that $v \in N^+(y) \cup N^+(y)$ ($y$ is a king for the centers of the missing stars). But $(f, u) \in D$ and $fy$ is not a missing edge, then $(f, y) \in D$. Thus $f \rightarrow y \rightarrow v$. Whence, $v \in N^+(f) \cup N^+(f)$. So $N^+_T(f) \subseteq N^+(f)$. Therefore, $f$ has the SNP in $D$ as well.
In the second case, we suppose that \( fx \) loses to some missing edge \( by \). We may assume without loss of generality that \( y \) is the center of a missing star containing \( by \). Then we must have \( x \to y \) and \( b \notin N^+(x) \cup N^+(x) \). Clearly, \( N^+(f) \cup \{y\} = N^+_T(f) \). We prove that \( N^+_T(f) \subseteq N^+(f) \cup \{y\} \). Let \( v \in N^+_T(f) \setminus y \). Then there \( \exists u \in V(T) = V(D) \) such that \( f \to u \to v \to f \) in \( T \). Suppose that \( u = x \). Since \( bv \) is not a missing edge, \( x = u \to v \) and \( b \notin N^+(x) \cup N^+(x) \) then we must have \((b,v) \in D \). Whence, \( f \to b \to v \in D \). Therefore \( v \in N^+(f) \). Now suppose that \( u \neq x \). Then \((f,u) \in D \). If \((u,v) \in D \) then \( v \in N^+(f) \). Otherwise, \( uv \) is a missing edge. Hence there is a missing edge \( pq \) that loses to \( uw \), namely, \( q \to v \) and \( u \notin N^+(q) \cup N^+(q) \). If \( q = x \), then we have \( f \to x \to v \to f \) in \( T \), which is the same as the case when \( u = x \). So we may suppose that \( q \neq x \). Note that \( q \) must be the center of a missing star. So \( f, x \notin \{p, q\} \). Thus \( fq \) is not a missing edge, \( u \notin N^+(q) \cup N^+(q) \) and \((f,u) \in D \). Then we must have \((f,q) \in D \), since otherwise we get \( q \to f \to u \in D \) which is a contradiction. Thus \( f \to q \to v \in D \). Whence \( v \in N^+(f) \). So \( N^+_T(f) \subseteq N^+(f) \cup \{y\} \). Therefore \( d^+(f) + 1 = d^+_T(f) \leq d^+_T(f) \leq d^+(f) + 1 \). Whence \( f \) has the SNP in \( D \).

3.2. Removing a star

A more general statement to the following theorem is proved in [4]. Here we introduce another prove that uses the sedimentation technique of a median order.

**Theorem 3.2.** [2] Let \( D \) be an oriented graph missing a star. Then \( D \) satisfies SNC.

**Proof.** Orient all the missing edges of \( D \) towards the center \( x \) of the missing star. The obtained digraph is a tournament \( T \). Let \( L \) be a median order of \( T \) that maximizes \( \alpha \), the index of \( x \) in \( L \), and let \( f \) denote its feed vertex. Reorient the missing edges incident to \( f \) towards \( f \) (if any). \( L \) is also a median order of the new tournament \( T' \). Note that \( N^+(f) = N^+_T(f) \) and we have \( d^+_T(f) \leq |G^T_L| \). If \( x \in G^T_L \) and \( d^+_T(f) = |G^T_L| \) then \( sed(L) \) is a median order of \( T' \) in which the index of \( x \) is greater than \( \alpha \), and also greater than the index of \( f \). So we can give the missing edge incident to \( f \) (if it exists then it is \( xf \)) its initial orientation (as in \( T \)) such that \( sed(L) \) is a median order of \( T' \), a contradiction to the fact that \( L \) maximizes \( \alpha \). So \( x \notin G^T_L \) or \( d^+(T'(f) < |G^T_L| \). If \( f = x \) then, clearly, \( d^+(f) = d^+_T(f) \leq |G^T_L| \leq d^+_T(f) = d^+(f) \). Now suppose that \( f \neq x \). We have that \( x \) is the only possible gained second out-neighbor vertex for \( f \). If \( x \notin G^T_L \) then \( G^T_L \subseteq N^+(f) \), whence the result follows. If \( d^+_T(f) < |G^T_L| \) then \( d^+(f) = d^+_T(f) \leq |G^T_L| - 1 \leq d^+(f) \). So \( f \) has the SNP in \( D \).

3.3. Removing 2 disjoint stars

In this section, let \( D \) be a digraph obtained from a tournament by deleting the edges of 2 disjoint stars and let \( \Delta \) denote its dependency digraph. Let \( S_x \) and \( S_y \) be the two missing disjoint stars with centers \( x \) and \( y \) respectively, \( A = V(S_x) \setminus x \), \( B = V(S_y) \setminus y \), \( K = V(S_x) \cup V(S_y) \) (the set of non whole vertices) and assume without loss of generality that \( x \to y \). In [4] it is proved that if the dependency digraph of any digraph consists of isolated vertices only then it satisfies SNC. Here we consider the case when the \( \Delta \) has no isolated vertices.

**Theorem 3.3.** Let \( D \) be an oriented graph missing 2 disjoint stars. If \( \Delta \) has no isolated vertex, then \( D \) satisfies SNC.
Proof. Assume without loss of generality that \( x \to y \). We note that the condition \( \Delta \) has no isolated vertex, implies that for every \( a \in A \) and \( y \in B \) we have \( y \to a \) and \( b \to x \). We shall orient all the missing edges of \( D \). First, we give every good edge a convenient orientation. For the other missing edges, let the orientation be towards the center of the 2 missing stars \( S_x \) or \( S_y \). The obtained digraph is a tournament \( T \). Let \( L \) be a median order of \( T \) such that the index \( k \) of \( x \) is maximum and let \( f \) denote its feed vertex. We know that \( f \) has the SNP in \( T \). We have only 5 cases:

Suppose that \( f \) is a whole vertex. In this case \( N^+(f) = N^-T(f) \). Suppose \( f \to u \to v \) in \( T \). Clearly \( (f, u) \in D \). If \( (u, v) \in D \) or is a convenient orientation then \( v \in N^+(f) \cup N^{++}(f) \). Otherwise there is a missing edge \( zt \) that loses to \( uv \) with \( t \to v \) and \( u \notin N^+(f) \cup N^{++}(f) \). But \( f \to u \), then \( f \to t \), whence \( f \to t \to v \) in \( D \). Therefore, \( N^{++}(f) = N^{++}_T(f) \) and \( f \) has the SNP in \( D \) as well.

Suppose \( f = x \). Orient all the edges of \( S_x \) towards the center \( x \). \( L \) is a median order of the modified completion \( T' \) of \( D \). We have \( N^+(f) = N^T(f) \). Suppose \( f \to u \to v \) in \( T' \). If \( (u, v) \in D \) or is a convenient orientation then \( v \in N^+(f) \cup N^{++}(f) \). Otherwise \( (u, v) = (b, y) \) for some \( b \in B \), but \( f = x \to y \). Thus, \( N^{++}(f) = N^{++}_T(f) \) and \( f \) has the SNP in \( T' \) and \( D \).

Suppose \( f = b \in B \). Orient the missing edge \( by \) towards \( b \). Again, \( L \) is a median order of the modified tournament \( T' \) and \( N^+(f) = N^{++}_T(f) \). Suppose \( f \to u \to v \) in \( T' \). If \( (u, v) \in D \) or is a convenient orientation then \( v \in N^+(f) \cup N^{++}(f) \). Otherwise \( (u, v) = (b', y) \) for some \( b' \in B \) or \( (u, v) = (a, x) \) for some \( a \in A \), however \( x, y \in N^{++}(f) \cup N^+(f) \) because \( f = b \to x \to y \) in \( D \). Thus, \( N^{++}(f) = N^{++}_T(f) \) and \( f \) has the SNP in \( T' \) and \( D \).

Suppose \( f = y \). Orient the missing edges towards \( y \) and let \( T' \) denote the new tournament. We note that \( B \subseteq N^{++}(y) \cap N^{++}_T(y) \) due to the condition \( \delta \Delta > 0 \). Also, \( x \) is the only possible new second neighbor of \( y \) in \( T' \). If \( B \cup \{x\} \subseteq G_L \) or \( d^+_T(y) < d^+_T(y) \), then \( d^+_T(y) = d^+_T(y) - 1 \leq d^+_T(y) \). Otherwise, \( B \cup \{x\} \subseteq G_L \) and \( d^+_T(y) = |G_L| \). In this case we consider the median order \( Sed(L) \) of \( T' \). Now the feed vertex of \( sed(L) \) is different from \( y \), the index of \( x \) had increased, and the index of \( y \) became less than the index of any vertex of \( B \) which makes \( Sed(L) \) a median order of \( T \) also, in which the index of \( x \) is greater than \( k \), a contradiction.

Suppose \( f = a \in A \). Orient the missing edge \( ax \) as \( (x, a) \) and let \( T' \) denote the new tournament. Note that \( y \) is the only possible new second neighbor of \( a \) in \( T' \) and not in \( D \). Also \( x \in N^{++}_T(a) \cap N^{++}(a) \). If \( d^+_T(a) < d^+_T(a) \), then \( d^+(a) = d^+_T(a) \leq d^+_T(a) - 1 \leq d^+(a) \), hence \( a \) has the SNP in \( D \). Otherwise, \( d^+_T(a) = |G_L| \neq d^+_T(a) = d^+_T(a) \) and in particular \( x \in G_L \). In this case we consider \( sed(L) \) which is a median order of \( T' \). Note that the feed vertex of \( sed(L) \) is different from \( a \) and the index of \( a \) is less than the index of \( x \) in the new order \( Sed(L) \). Hence \( Sed(L) \) is a median of \( T \) as well, in which the index of \( x \) is greater than \( k \), a contradiction.

So in all cases \( f \) has the SNP in \( D \). Therefore \( D \) satisfies SNC.

\[ \square \]

Theorem 3.4. Let \( D \) be a digraph obtained from a tournament by deleting the edges of 2 disjoint
stars. If \( \Delta \) has neither a source nor a sink and \( D \) has no sink, then \( D \) has at least two vertices with the SNP.

**Proof.**

**claim 1:** Suppose \( K = V(D) \). If \( \Delta \) has no isolated vertex, then \( D \) has at least two vertices with the SNP.

**Proof of claim 1:** The condition \( \Delta \) has no isolated vertex implies that for every \( a \in A \) and \( b \in B \) we have \( y \to a \) and \( b \to x \). Clearly, \( N^+(x) = \{ y \} \), \( N^+(y) = A \), \( d^+(x) \leq 1 \leq |A| \leq d^+(x) \), thus \( x \) has the SNP. Let \( H \) be the tournament \( D - \{ x, y \} \). Then \( H \) has a vertex \( v \) with the SNP in \( H \). If \( v \in A \), then \( d^+(v) = d^+_{H^+}(v) \leq d^+_{H^+}(v) = d^+(v) \). If \( v \in B \), then \( d^+(v) = d^+_{H^+}(v) + 1 \leq d^+_{H^+}(v) + 1 = d^+(v) \). Whence, \( v \) also has the SNP in \( D \).

**Claim 2:** \( D \) is a good digraph.

**Proof of claim 2:** Let \( I_D \) be the interval graph of \( D \). Let \( C_1 \) and \( C_2 \) be two distinct connected components of \( \Delta \). Then the centers \( x \) and \( y \) appear in each of these two connected components, whence \( K(C_1) \cap K(C_2) \neq \phi \). Therefore, \( I_D \) is a connected graph, having only one connected component \( \xi \). Then, \( K = K(\xi) \).

So if \( \Delta \) is composed of non trivial strongly connected components, the result holds by lemma 3.1. Due to the condition \( \Delta \) has neither a source nor a sink, \( \Delta \) has a non trivial strongly connected component, hence \( N^+(x) \setminus K = N^+(y) \setminus K \). Now let \( v \in K \) and assume without loss of generality that \( xv \) is a missing edge. Due to the condition \( \Delta \) has neither a source nor a sink, we have that \( xv \) belongs to a non trivial strongly connected component of \( \Delta \), and in this case \( v \in R \) and \( N^+(v) \setminus K = N^+(x) \setminus K = N^+(y) \setminus K \). or \( xv \) belongs to a directed path \( P = xa_1, yb_1, \cdots , xa_p, yb_p \) joining 2 non trivial strongly connected components \( C_1 \) and \( C_2 \) with \( xa_1 \in C_1 \) and \( yb_p \in C_2 \). There is \( i > 1 \) such that \( v = a_i \). \( L = xa_{i-1}, yb_{i-1}, xa_i, yb_i \) is a path in \( \Delta \). By the definition of losing cycles we have \( N^+(x) \setminus K \subseteq N^+(b_{i-1}) \setminus K \subseteq N^+(a_i) \setminus K \subseteq N^+(y) \setminus K = N^+(x) \setminus K \). Hence \( N^+(x) \setminus K = N^+(v) \setminus K \) for all \( v \in K \). Since every vertex outside \( K \) is adjacent to every vertex in \( K \) we also have \( N^+(x) \setminus K = N^+(v) \setminus K \) for all \( v \in K \). This proves the second claim.

Since \( D \) is a good digraph, then it has a good median order \( L = x_1x_2\cdots x_n \). If \( J(x_n) = K \), then the result follows by claim 1 and theorem 2.1. Otherwise, \( x_n \) is whole, that is \( J(x_n) = \{ x_n \} \). By theorem 2.1, \( x_n \) has the SNP in \( D \). So we need to find another vertex with the SNP in \( D \). Consider the good median order \( L' = x_1x_2\cdots x_{n-1} \) of the good digraph \( D' = D\{ x_1, \ldots , x_{n-1} \} \). Suppose first that \( L' \) is stable. There is \( q \) for which \( Sed^q(L') = y_1\cdots y_{n-1} \) and \( N^+(y_{n-1}) \setminus J(y_{n-1}) \leq G_{Sed^q(L')} \setminus J(y_{n-1}) \) due to (\( * \)). Note that \( y_1\cdots y_{n-1} x_n \) is also a good median order of \( D \). By theorem 2.1 and claim 1, there is \( y \in J(y_{n-1}) \) that has the SNP in \( D' \), more precisely \( |N^+(y)| < |N^+(y')| \) due to (\( * \)). Since \( y \in J(y_{n-1}) \) and \( y_{n-1} \to x_n \) then \( y \to x_n \). So \( |N^+(y)| = |N^+_D(y)| + 1 \leq |N^+(y')| \).

Now suppose that \( L' \) is periodic. Since \( D \) has no sink then \( x_n \) has an out-neighbor \( x_j \). Choose \( j \) to be the greatest (so that it is the last vertex of its corresponding interval). Note that for every \( q \), \( x_n \) is an out-neighbor of the feed vertex of \( Sed^q(L') \). So \( x_j \) is not the feed vertex of any \( Sed^q(L') \). Since \( L' \) is periodic, \( x_j \) must be a bad vertex of \( Sed^q(L') \) for some integer \( q \), otherwise the index
of $x_j$ would always increase during the sedimentation process. Let $q$ be such an integer and set $Sed^q(L') = y_1 \ldots y_{n-1}$. By theorem 2.1 and claim 1, there is $y \in J(y_{n-1})$ that has the SNP in $D'$, more precisely $|N^+_D(y) \setminus J(y_{n-1})| < |G_{Sed^q(L')} \setminus J(y_{n-1})|$ due to $(\ast)$. Since $y \in J(y_{n-1})$ and $y_{n-1} \rightarrow x_n$ then $y \rightarrow x_n$. Note that $y \rightarrow x_n \rightarrow x_j, G_{Sed^q(L')} \cup \{x_j\} \setminus J(y_{n-1}) \subseteq N^+(y) \setminus J(y_{n-1})$ and $|N^+_D(y) \setminus J(y_{n-1})| = |G_{Sed^q(L')} \setminus J(y_{n-1})|$.

Therefore, $|N^+(y)| = |N^+_D(y)| + 1 = |N^+_D(y) \setminus J(y_{n-1})| + 1 + |N^+_D(y) \cap J(y_{n-1})| = |G_{Sed^q(L')} \setminus J(y_{n-1})| + 1 + |N^+_D(y) \cap J(y_{n-1})| = |G_{Sed^q(L')} \cup \{x_j\} \setminus J(y_{n-1})| + |N^+_D(y) \cap J(y_{n-1})| \leq |N^+_D(y) \setminus J(y_{n-1})| + |N^+_D(y) \cap J(y_{n-1})| \leq |N^+(y)|$.

\[\square\]

3.4. Removing 3 disjoint stars

In this section, $D$ is an oriented graph missing three disjoint stars $S_x, S_y$ and $S_z$ with centers $x$, $y$ and $z$ respectively. Set $A = V(S_x) - x$, $B = V(S_y) - x$, $C = V(S_z) - z$ and $K = A \cup B \cup C \cup \{x, y, z\}$. Let $\Delta$ denote the dependency digraph of $D$. The triangle induced by the vertices $x$, $y$ and $z$ is either a transitive triangle or a directed triangle.

First we will deal with the case when this triangle is directed, and assume without loss of generality that $x \rightarrow y \rightarrow z \rightarrow x$. This is a particular case of the case when the missing graph is a disjoint union of stars such that, in the induced tournament by the centers of the missing stars, every vertex is a king.

**Theorem 3.5.** Let $D$ be an oriented graph missing 3 disjoint stars whose centers form a directed triangle. If $\Delta$ has no isolated vertices, then $D$ satisfies SNC.

**Proof.**

**Claim:** The only possible arcs in $\Delta$ have the forms $xa \rightarrow yb$ or $yb \rightarrow zc$ or $zc \rightarrow xa$, where $a \in A, b \in B$ and $c \in C$.

**Proof of the claim:** $xa$ can not lose to $zc$ because $z \rightarrow x$ and $z \in N^+(x)$. Similarly $yb$ can not lose to $xa$ and $zc$ can not lose to $yb$.

Orient the good missing edges in a convenient way and orient the other edges toward the centers. The obtained digraph $T$ is a tournament. Let $L$ be a median order of $T$ such that the sum of the indices of $x, y$ and $z$ is maximum. Let $f$ denote the feed vertex of $L$. Due to symmetry, we may assume that $f$ is a whole vertex or $f = x$ or $f = a \in A$.

Suppose $f$ is a whole vertex. Clearly, $N^+(f) = N^+_D(f)$. Suppose $f \rightarrow u \rightarrow v$ in $T$. If $(u, v) \in E(D)$ or $uv$ is a good missing edge then $v \in N^+(f) \cup N^+(f)$. Otherwise, there is missing edge $rs$ that loses to $uv$ with $r \rightarrow v$ and $u \notin N^+(r) \cup N^+(r)$. But $f \rightarrow u$, then $f \rightarrow r$, whence $f \rightarrow r \rightarrow v$ and $v \in N^+(f) \cup N^+(f)$. Thus, $N^+_f(f) = N^+_f(f)$ and $f$ has the SNP in $D$.

Suppose $f = x$. Reorient all the missing edges incident to $x$ toward $x$. In the new tournament $T'$ we have $N^+(x) = N^+_f(x)$ and $x$ has the SNP in $T'$. Since $y \in N^+(x)$ and $z \in N^+(x)$ we
have that \( N^{++}(x) = N^{++}_{T_T}(x) \). Thus \( x \) has the SNP in \( D \).

Suppose that \( f = a \in A \). Reorient \( ax \) toward \( a \). Suppose \( a \to u \to v \) in the new tournament \( T' \) with \( v \neq y \). If \( (u, v) \in E(D) \) or \( uv \) is a good missing edge then \( v \in N^+(a) \cup N^{++}(a) \). Otherwise, there is \( b \in B \) and \( c \in C \) such that \( (u, v) = (c, z) \) and \( by \) loses to \( cz \), then \( f \to c \) implies that \( a \to y \), but \( y \to z \), whence \( z \in N^+(a) \cup N^+(a) \). So the only possible new second out-neighbor of \( a \) is \( y \), hence if \( y \notin N^{++}_{T_T}(a) \) then \( a \) has the SNP in \( D \). Suppose \( y \in N^{++}_{T_T}(a) \). If \( d^{T_T}(a) < d^{T_T+}(a) \) then \( d^+(a) = d^{T_T+}(a) = d^{T_T}(a) \), hence \( a \) has the SNP in \( D \). Otherwise, \( d^{T_T}(a) = |G_L| \) and \( G_L = N^{++}_{T_T}(a) \). So \( x, y \) and \( z \) are not bad vertices, hence the index of each increases in the median order \( Sed(L) \) of \( T' \). But the index of \( a \) is less than the index of \( x \), then we can give \( ax \) its initial orientation as in \( T \) nd the same order \( Sed(L) \) is a median order of \( T \). However, the sum of indices of \( x, y \) and \( z \) has increased. A contradiction. Thus \( f \) has the SNP in \( D \) and \( D \) satisfies \( SNC \).

**Theorem 3.6.** Let \( D \) be an oriented graph missing 3 disjoint stars whose centers form a directed triangle. If \( \Delta \) has neither a source nor a sink and \( D \) has no sink, then \( D \) has at least two vertices with the SNP.

**Proof.** **Claim 1:** For every \( a \in A \), \( b \in B \) and \( c \in C \) we have:
\( b \to x \to c \to y \to a \to z \to b \).
**Proof of Claim 1:** This is due to the claim in the previous proof and the condition that \( \Delta \) has neither a source nor a sink.

**Claim 2:** If \( K = V(D) \) then \( D \) has at least 3 vertices with the SNP.

**Proof of Claim 2:** Let \( H = D - \{x, y, z\} \). \( H \) is a tournament with no sink (dominated vertex). Then \( H \) has 2 vertices \( u \) and \( v \) with SNP in \( H \). Without loss of generality we may assume that \( u \in A \). But \( y \to u \to z \), the adding the vertices \( x, y \) and \( z \) makes \( u \) gains only one vertex to its first out-neighborhood and \( x \) to its second out-neighborhood. Thus, also \( u \) has the SNP in \( D \). Similarly, \( v \) has the SNP in \( D \). Suppose, without loss of generality, that \( |A| \geq |C| \). We have \( C \cup \{y\} = N^+(x) \) and \( A \cup \{z\} = N^{++}(x) \). Hence, \( d^+(x) = |C| + 1 \leq |A| + 1 \leq d^{++}(x) \), whence, \( x \) has the SNP in \( D \).

**Claim 3:** \( D \) is a good oriented graph.

**Proof of Claim 3:** Let \( I_D \) be the interval graph of \( D \). Let \( C_1 \) and \( C_2 \) be two distinct connected components of \( \Delta \). The three centers of the missing disjoint stars must appear in each of the these two connected components, whence \( K(C_1) \cap K(C_2) \neq \phi \). Therefore, \( I_D \) is a connected graph, having only one connected component \( \xi \). Then, \( K = K(\xi) \).  
So if \( \Delta \) is composed of non trivial strongly connected components, the result holds by proposition 3.1.

Due to the condition that \( \Delta \) has neither a source nor a sink, \( \Delta \) has a non trivial strongly connected component \( C \).

Since \( x, y \) and \( z \) must appear in \( C \), we have \( N^+(x) \setminus K = N^+(y) \setminus K = N^+(z) \setminus K \). Now let \( v \in K \). If \( v \) appears in a non trivial strongly connected component of \( \Delta \) then \( N^+(v) \setminus K = N^+(x) \setminus K = N^+(y) \setminus K = N^+(z) \setminus K \).
Otherwise, due to the condition that $\Delta$ has neither a source nor a sink, $v$ appears in a directed path $P$ of $\Delta$ joining two non trivial strongly connected components $C_1$ and $C_2$ of $\Delta$. By the definition of losing relations we can prove easily that for all $a \in K(C_1)$, $b \in K(P)$ and $c \in K(C_2)$ we have $N^{+}(a) \setminus K(\xi) \subseteq N^{+}(b) \setminus K(\xi) \subseteq N^{+}(c) \setminus K(\xi)$. In particular, for $a = x = c$ and $b = v$. So $N^{+}(v) \setminus K = N^{+}(x) \setminus K$. Similarly, $N^{-}(v) \setminus K = N^{-}(x) \setminus K$. This proves claim 3.

To conclude we apply the same argument of the proof of theorem 3.4.

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