History operators in quantum mechanics

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Abstract

It is convenient to describe a quantum system at all times by means of a “history operator” $C$, encoding measurements and unitary time evolution between measurements. These operators naturally arise when computing the probability of measurement sequences, and generalize the “sum over position histories” of the Feynman path-integral.

As we argue in the present note, this description has some computational advantages over the usual state vector description, and may help to clarify some issues regarding nonlocality of quantum correlations and collapse.

A measurement on a system described by $C$ modifies the history operator, $C \rightarrow PC$, where $P$ is the projector corresponding to the measurement. We refer to this modification as ”history operator collapse”. Thus $C$ keeps track of the succession of measurements on a system, and contains all histories compatible with the results of these measurements. The collapse modifies the history content of $C$, and therefore modifies also the past (relative to the measurement), but never in a way to violate causality.

Probabilities of outcomes are obtained as $\text{Tr}(C^\dagger PC)/\text{Tr}(C^\dagger C)$. A similar formula yields probabilities for intermediate measurements, and reproduces the result of the two-vector formalism in the case of given initial and final states.

We apply the history operator formalism to a few examples: entangler circuit, Mach-Zehnder interferometer, teleportation circuit, three-box experiment.

Not surprisingly, the propagation of coordinate eigenstates $|q\rangle$ is described by a history operator $C$ containing the Feynman path-integral.
1 Introduction

Can measurements affect the past? The question is not new, and has been raised in various forms, see for ex. [1]. In the perspective we adopt here, the “history collapse” due to a measurement indeed affects the system also at times antecedent to the measurement, since it affects its history description involving all times back to an initial state. It could seem that such a retroaction is an artifact of the history approach. However consider for example entangled states of two qubits that are spacelike separated. If we insist on instantaneous collapse in every reference frame, a measurement on one qubit affects the composite system at different proper times of the second qubit, corresponding to the different reference frames where the collapse is observed. Therefore collapse must affect the whole history of the quantum system. This conclusion seems unavoidable if special relativity must hold together with simultaneity of collapse.

On the other hand, even nonrelativistic quantum mechanics has a spacetime description in terms of sum over histories, i.e. Feynman path-integrals [4]. Feynman’s approach has prompted various formalisms based on histories rather than on state vectors (for a very incomplete list of references see [5] - [15]). These formalisms must reproduce the standard probability rules of quantum mechanics, and in this

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1No experimental evidence for reference-dependent “speed of collapse” has ever been found, see for ex. [2] [3] and references therein.
sense do not add any fundamental novelty to the theory. However they can be of help in describing closed quantum systems (containing both the observer and the observed subsystem, as in cosmological models) and in interpreting some highly debated foundational issues.

We propose in this note to describe a quantum system by a “sum over histories” operator, in terms of which all the rules of standard quantum mechanics can be reproduced. This operator is just the evolution operator of the system, where projectors have been inserted to account for (projective) measurements.

This history operator is more versatile than the usual state vector: it contains in a transparent way all histories of the system, compatible with initial state and measurement results at different times. Note that in general these histories do not form a decoherent set. This is an important difference with the consistent histories approach of ref.s [5]-[8] and [10]-[14], where histories are required to decohere.

A further measurement produces a collapse of the history operator, by application of the projector corresponding to the measurement result. Its “history content” gets in general reduced, since the projector filters out histories not compatible with the result. The conceptual consequence of this description is that a measurement indeed affects the past, in the sense that it affects histories that go back to the initial state.

The plan of the paper is as follows. In Section 2 the familiar decomposition of the evolution operator is recalled, and chain operators are introduced, together with the probability of obtaining sequences of measurement results. In Section 3 the probability rules are formulated exclusively in terms of the history operator, and we discuss history amplitudes and interference. The collapse of the history operator is described in Section 4, and in Section 5 we recall the probability rules for measurement results at intermediate times. These rules reproduce the probability formula of the so called two-vector approach of [16]-[19] for a measurement between known initial and final states. In Section 6 the history operators of three simple quantum circuits are discussed. Section 7 deals with the three-box experiment of [23]. Section 8 contains some conclusions.

2 Evolution operator as a sum on histories

The time evolution operator $U(t_n, t_0)$ between the times $t_0$ and $t_n$ of a quantum system can be written in a “sum over histories” fashion as follows:

$$U(t_n, t_0) = U(t_n, t_{n-1}) U(t_{n-1}, t_{n-2}) \cdots U(t_1, t_0) = U(t_n, t_{n-1}) \sum_{\alpha_{n-1}} P^{(n-1)}_{\alpha_{n-1}} U(t_{n-1}, t_{n-2}) \cdots \sum_{\alpha_1} P^{(1)}_{\alpha_1} U(t_1, t_0)$$

(2.1)

with $t_0 < t_1 < \cdots < t_{n-1} < t_n$. The sum over projectors $P^{(i)}_{\alpha_i}$ are decompositions of the unity:

$$I = \sum_{\alpha_i} P^{(i)}_{\alpha_i}$$

(2.2)
inserted at times \( t_1, \ldots, t_{n-1} \). The \( P^{(i)}_{\alpha_i} \) are projectors on eigensubspaces of observables, satisfying

\[
P^{(i)}_{\alpha_i} P^{(i)}_{\beta_i} = \delta_{\alpha_i, \beta_i} P^{(i)}_{\alpha_i}
\]  
(2.3)

Thus the time evolution operator is given by a sum on all indices \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{n-1}) \)

\[
U(t_n, t_0) = \sum_{\alpha} C_{\alpha}
\]  
(2.4)

where the operators \( C_{\alpha} \) correspond to the single histories \( (\alpha_1, \alpha_2, \cdots, \alpha_{n-1}) \):

\[
C_{\alpha} = U(t_n, t_{n-1}) P^{(n-1)}_{\alpha_{n-1}} U(t_{n-1}, t_{n-2}) \cdots P^{(1)}_{\alpha_1} U(t_1, t_0)
\]  
(2.5)

and are usually called “chain operators”.

A well-known exercise is to compute the probability of obtaining a sequence of measurement results, at times \( t_1, t_2, \ldots, t_{n-1} \), on a system that starts in the initial state \( |\psi\rangle \) at time \( t_0 \). If the outcomes correspond to the projectors \( P^{(1)}_{\alpha_1}, P^{(2)}_{\alpha_2}, \cdots, P^{(n-1)}_{\alpha_{n-1}} \), and \( P_{\psi} = |\psi\rangle \langle \psi| \) is the projector on the initial state, the answer is given by:

\[
p(\psi, \alpha_1, \alpha_2, \cdots, \alpha_{n-1}) = Tr(C_{\alpha} P_{\psi} C_{\alpha}^\dagger)
\]  
(2.6)

and could be considered the “probability of the history” \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{n-1}) \) starting from state \( |\psi\rangle \). At first sight this seems reasonable, since we can easily prove that the sum of all these probabilities gives 1:

\[
\sum_{\alpha} Tr(C_{\alpha} P_{\psi} C_{\alpha}^\dagger) = 1
\]  
(2.7)

by using the completeness relations (2.2) and unitarity of the \( U(t_i, t_{i-1}) \) operators. We also find

\[
\sum_{\alpha} p(\psi, \alpha_1, \alpha_2, \cdots, \alpha_n) = p(\psi, \alpha_1, \alpha_2, \cdots, \alpha_{n-1})
\]  
(2.8)

However other standard sum rules for probabilities are not satisfied in general. For example relations of the type

\[
\sum_{\alpha_2} p(\psi, \alpha_1, \alpha_2, \alpha_3) = p(\psi, \alpha_1, \alpha_3)
\]  
(2.9)

hold only if the so-called \textit{decoherence condition} is satisfied:

\[
Tr(C_\beta P_{\psi} C_{\alpha}^\dagger) + c.c. = 0 \quad \text{when } \alpha \neq \beta
\]  
(2.10)

If all the histories we consider are such that the decoherence condition holds, they are said to form a \textit{consistent} set, and can be assigned probabilities satisfying all the standard sum rules.

In general, histories do not form a consistent set: interference effects between them can be important, as in the case of the double slit experiment. For this reason we do not limit ourselves here to consistent sets. The price to pay is to give up the possibility of assigning a probability to each history, but this is not the goal of the history operator formalism. Formula (2.6) for the probability of successive measurement outcomes holds in any case, and is all we need to compute the probabilities in terms of the history operator, as discussed in the next Sections.
3 History operator

The state vector after the $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{n-1})$ measurement outcomes is obtained by applying the chain operator \[2.5\] to the initial state $|\psi\rangle$:

$$|\psi_\alpha\rangle = C_\alpha |\psi\rangle$$ \hspace{1cm} (3.1)

up to a normalization factor$^2$.

Suppose now that we perform an additional measurement $P^{(n)}_\alpha$ on $|\psi_\alpha\rangle$. Using the standard formula we find:

$$p(\psi_\alpha, \alpha_n) = \frac{\langle \psi_\alpha | P^{(n)}_\alpha | \psi_\alpha \rangle}{\langle \psi_\alpha | \psi_\alpha \rangle}$$ \hspace{1cm} (3.2)

for the probability of obtaining the result $\alpha_n$. Substituting $|\psi_\alpha\rangle$ as given in (3.1) yields:

$$p(\psi_\alpha, \alpha_n) = p(\alpha_n | \psi, \alpha_1, \cdots, \alpha_{n-1}) = \frac{\langle \psi | C^\dagger_\alpha P^{(n)}_\alpha C_\alpha | \psi \rangle}{\langle \psi | C^\dagger_\alpha C_\alpha | \psi \rangle} = \frac{Tr(C^\dagger_\alpha P^{(n)}_\alpha C_\alpha P_\psi)}{Tr(C^\dagger_\alpha C_\alpha P_\psi)}$$ \hspace{1cm} (3.3)

This is the probability of obtaining the result $\alpha_n$ on a state that has evolved from $|\psi\rangle$ to $|\psi_\alpha\rangle$ through a sequence of measurements with results $\alpha_1, \cdots, \alpha_{n-1}$. Notice that the numerator is the joint probability $p(\psi, \alpha_1, \cdots, \alpha_{n-1}, \alpha_n)$ of obtaining $\alpha_1, \cdots, \alpha_{n-1}, \alpha_n$, and the denominator is the joint probability $p(\psi, \alpha_1, \cdots, \alpha_{n-1})$ of obtaining $\alpha_1, \cdots, \alpha_{n-1}$. The ratio correctly gives the conditional probability of obtaining $\alpha_n$, if $\alpha_1, \cdots, \alpha_{n-1}$ have been obtained.

Formulae (2.6) and (3.3) suggest to describe the system that has evolved from $|\psi\rangle$ to $|\psi_\alpha\rangle$ via the “history operator”

$$C_{\psi,\alpha} \equiv C_\alpha P_\psi$$ \hspace{1cm} (3.4)

encoding the sequence of measurements, the unitary evolution between them, and the initial state $|\psi\rangle$. All relevant information on the system can be extracted from $C_{\psi,\alpha}$: for example the joint probability in (2.6) can be rewritten more compactly as

$$p(\psi, \alpha_1, \alpha_2, \cdots, \alpha_{n-1}) = Tr(C^\dagger_{\psi,\alpha} C_{\psi,\alpha})$$ \hspace{1cm} (3.5)

and the conditional probability (3.3) takes the form

$$p(\alpha_n | \psi, \alpha_1, \cdots, \alpha_{n-1}) = \frac{Tr(C^\dagger_{\psi,\alpha} P^{(n)}_\alpha C_{\psi,\alpha})}{Tr(C^\dagger_{\psi,\alpha} C_{\psi,\alpha})}$$ \hspace{1cm} (3.6)

A particular history operator is the unitary evolution operator \[2.1\] times $P_\psi$, corresponding to absence of measurements, and containing all possible histories originating from the initial state $|\psi\rangle$. If measurements are performed, the history operator contains the corresponding projectors.

$^2$Due to the projectors in $C_\alpha$, the state $|\psi_\alpha\rangle$ is not normalized.
It is worthwhile to emphasize, even if it is tautological, that the decompositions of unity in (2.1), and hence the histories contained in the evolution operator, depend on the observables measured by the experimental apparatus at the various times $t_1, \ldots, t_{n-1}$. In this way the histories contained in history operators, even in absence of actual measurements, encode information on the measuring devices used to probe the system.

To summarize, the history operator of a system with initial state $|\psi\rangle$ at $t_0$, with measuring devices that can be activated at times $t_1, \cdots, t_{n-1}$, is given by

$$\sum_\alpha C_\alpha P_\psi$$  (3.7)

with the chain operators $C_\alpha$ given in (2.5). If actual measurements are performed at some of the times $t_i$, with results $\beta_i$, the history operator is obtained simply by replacing in (3.7) the identity decomposition at time $t_i$ with the single projector $P_{\beta_i}$.

The history content of (3.7) is defined to be the set of all histories $\psi, \alpha_1, \cdots, \alpha_{n-1}$ contained in the sum (3.7), where each history corresponds to a particular $C_\alpha P_\psi$.

It may be convenient to insert a decomposition of the unity also at time $t_n$, so that the history operator contains all histories $\psi, \alpha_1, \cdots, \alpha_n$ compatible with the measurement outcomes (if measurements occur) at times $t_1, \cdots, t_n$. Then the history operator becomes a sum of chain operators of the form

$$C_{\psi, \alpha} = P^{(n)}_{\alpha_n} U(t_n, t_{n-1}) P^{(n-1)}_{\alpha_{n-1}} U(t_{n-1}, t_{n-2}) \cdots P^{(1)}_{\alpha_1} U(t_1, t_0) P_\psi$$  (3.8)

The single chain operator vanishes if $\text{Tr}(C^\dagger_{\psi, \alpha} C_{\psi, \alpha}) = 0$, i.e. if the joint probability of successive measurement outcomes $p(\psi, \alpha_1, \cdots, \alpha_n)$ vanishes. In this case the particular history $(\psi, \alpha_1, \cdots, \alpha_n)$ is not present in the sum.

### 3.1 History amplitudes

If the initial and final states are respectively $|\psi\rangle$ and $|\phi\rangle$, the history operator becomes a sum on chain operators of the type:

$$C_{\psi, \alpha, \phi} = |\phi\rangle \langle \phi| U(t_n, t_{n-1}) P^{(n-1)}_{\alpha_{n-1}} U(t_{n-1}, t_{n-2}) \cdots P^{(1)}_{\alpha_1} U(t_1, t_0) |\psi\rangle \langle \psi|$$  (3.9)

The (complex) number $A(\psi, \alpha, \phi)$ is the amplitude for the history $(\psi, \alpha, \phi)$, and it is immediate to see that

$$|A(\psi, \alpha, \phi)|^2 = \text{Tr}(C^\dagger_{\psi, \alpha, \phi} C_{\psi, \alpha, \phi}) = p(\psi, \alpha, \phi)$$  (3.10)

The history operator is then

$$\left(\sum_\alpha A(\psi, \alpha, \phi)\right) |\phi\rangle \langle \psi|$$  (3.11)

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3This probability is also called the weight of the history $(\psi, \alpha_1, \cdots, \alpha_n)$.
In the sum of the amplitudes cancellations can occur when histories interfere de-
structively. Computing the amplitudes for each history enables to find which his-
tories are interfering, constructively or destructively.

4 Collapse

Suppose \( C_{\psi,\alpha} \) is the history operator of a system that has initial state \(|\psi\rangle\) and has been measured at times \(t_1, \cdots t_{n-1}\) with outcomes \(\alpha_1, \cdots, \alpha_{n-1}\). Immediately after a measurement at time \(t_n\) yielding \(\alpha_n\), the history operator becomes

\[
C_{\psi,\alpha'} = P^{(n)}_{\alpha_n} C_{\psi,\alpha}
\]

(4.1)

It describes the system that starts at \(|\psi\rangle\) and undergoes the \(\alpha' = (\alpha_1, \cdots, \alpha_n)\) measurements. The probability in (3.6) can be expressed in terms of the history operators before and after the collapse:

\[
p(\alpha_n|\psi, \alpha_1, \cdots \alpha_{n-1}) = \frac{Tr(C_{\psi,\alpha'}^t C_{\psi,\alpha'})}{Tr(C_{\psi,\alpha}^t C_{\psi,\alpha})}
\]

(4.2)

In general measurements project out some histories, but in some cases they may “open up” histories that are forbidden (due to interference) without the measurement. Explicit examples of “history reduction” and “history restoration” will be discussed in Section 6.

The collapse of the history operator describes the modification of the history content of the system when subjected to a measurement. Modifying history implies modifying the past, which may sound paradoxical. Note however that no causality violation is permitted. Whether Bob measures or not the system at time \(t_n\) cannot in any way communicate information to Alice (or to himself!) at time \(t_i < t_n\). Indeed suppose for example that Alice and Bob make measurements respectively at \(t_1\) and \(t_2\). The results will distribute themselves into the various values \(\alpha_1, \alpha_2\) according to the probabilities

\[
p(\psi, \alpha_1, \alpha_2) = Tr(C_{\psi,\alpha_1,\alpha_2}^t C_{\psi,\alpha_1,\alpha_2})
\]

(4.3)

The probability for Alice to get a particular \(\alpha_1\) is given by the sum

\[
\sum_{\alpha_2} p(\psi, \alpha_1, \alpha_2)
\]

(4.4)

On the other hand because of property (2.8) this sum is equal to \(p(\psi, \alpha_1)\), i.e. the probability that Alice measures the particular \(\alpha_1\) value without reference to future measurements by Bob. Thus the measuring act of Bob at \(t_2\) cannot influence the statistics of measurements by Alice at times antecedent to \(t_2\).
5 Probabilities at intermediate times

We have so far considered two kinds of probabilities:

- the joint probability of a sequence of measurements, eq. (3.5).
- the conditional probability of an outcome \(\alpha_n\) at \(t_n\), given the initial state \(|\psi\rangle\) and the outcomes \(\alpha_1, \ldots, \alpha_{n-1}\) at preceding times, eq. (3.6).

We can also compute the conditional probability of measurement outcomes \(\beta_i\) inside the time interval \([t_0, t_n]\), i.e. the probability that a measurement at \(t_i\) yields the outcome \(\beta_i\), given that the measurements at \(t_0, t_1, \ldots, t_n\) (\(t_i\) excluded) have as outcomes \(\psi, \alpha_1, \ldots, \alpha_n\) (\(\alpha_i\) excluded). This probability can be expressed via the history operator \(C_{\psi,\alpha} \leftrightarrow \beta\) corresponding\(^4\) to the measurement results \(\psi, \alpha_1, \ldots, \beta_i, \ldots, \alpha_n\):

\[
p(\beta_i|\psi, \alpha_1, \ldots, \phi_i, \ldots, \alpha_n) = \frac{p(\psi, \alpha_1, \ldots, \beta_i, \ldots, \alpha_n)}{\sum_{\gamma_i} p(\psi, \alpha_1, \ldots, \gamma_i, \ldots, \alpha_n)} = \frac{\text{Tr}(C_{\psi,\alpha}^\dagger C_{\psi,\beta} )}{\sum_{\gamma} \text{Tr}(C_{\psi,\alpha}^\dagger C_{\psi,\gamma} )}
\]

(5.1)

This formula generalizes the one for the conditional probability of outcomes at time \(t_n\) given in (4.2). Note however that

\[
\sum_{\gamma_i} p(\psi, \alpha_1, \ldots, \gamma_i, \ldots, \alpha_n) \neq p(\psi, \alpha_1, \ldots, \phi_i, \ldots, \alpha_n)
\]

(5.2)

cf. discussion after (2.8).

Consider the particular case when only the initial and final states of the system are given, respectively as \(|\psi\rangle\) at \(t_0\) and \(|\phi\rangle\) at \(t_2\). Then formula (5.1) yields

\[
p(\beta|\psi, \phi) = \frac{\text{Tr}(C_{\psi,\alpha}^\dagger C_{\psi,\beta} )}{\sum_{\gamma} \text{Tr}(C_{\psi,\alpha}^\dagger C_{\psi,\gamma} )}
\]

(5.3)

for the probability of obtaining the result \(\beta\) in a measurement at time \(t_1\). The history operator is in this case:

\[
C_{\psi,\beta,\phi} = P_{\phi} U(t_2, t_1) P_{\beta}^\dagger U(t_1, t_0) P_{\psi}
\]

(5.4)

so that

\[
\text{Tr}(C_{\psi,\beta,\phi}^\dagger C_{\psi,\beta,\phi}) = \text{Tr}(U(t_1, t_0) P_{\psi} U^\dagger(t_1, t_0) P_{\beta}^\dagger U^\dagger(t_2, t_1) P_{\phi} U(t_2, t_1) P_{\beta} )
\]

\[
= |\langle \psi(t_1) | P_{\beta}^\dagger | \phi(t_1) \rangle |^2
\]

(5.5)

using \(|\psi(t_1)\rangle = U(t_1, t_0)|\psi\rangle\) and \(|\phi(t_1)\rangle = U(t_1, t_2)|\phi\rangle\). Therefore (5.3) becomes

\[
p(\beta|\psi, \phi) = \frac{|\langle \psi(t_1) | P_{\beta}^\dagger | \phi(t_1) \rangle |^2}{\sum_{\gamma} |\langle \psi(t_1) | P_{\gamma}^\dagger | \phi(t_1) \rangle |^2}
\]

(5.6)

and reproduces the symmetric formula of the two-vector formalism of [16] - [19].

\(^4\)The notation \(\alpha \leftrightarrow \beta\) is short for the sequence \(\alpha_1, \ldots, \beta_i, \ldots, \alpha_n\).
6 Examples

In this Section we examine three examples of quantum systems evolving from a given initial state, and subjected to successive measurements. These examples are taken from simple quantum computation circuits\footnote{A review on quantum computation can be found for ex. in [20].} where unitary gates determine the evolution between measurements. Only two gates will be used: the Hadamard one-qubit gate $H$ defined by:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$  \hspace{1cm} (6.1)

and the two-qubit $CNOT$ gate:

$$CNOT|00\rangle = |00\rangle, \quad CNOT|01\rangle = |01\rangle, \quad CNOT|10\rangle = |11\rangle, \quad CNOT|11\rangle = |10\rangle$$  \hspace{1cm} (6.2)

Quantum computing circuits in the consistent history formalism have been discussed for example in in ref.s [6, 21].

6.1 The entangler circuit

This basic circuit uses a Hadamard gate and a $CNOT$ gate, and delivers entangled two-qubit states (the Bell states). Traditionally the upper qubit belongs to Alice, the lower one to Bob.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{entangler_circuit.png}
\caption{The entangler circuit, and some history diagrams: a) no measurements, or Bob measures 0 at $t_1$; b) Alice measures 0 at $t_1$; c) Alice measures 1 at $t_2$. Black triangles indicate measurements.}
\end{figure}
Consider for example the history operator describing the system with initial state $|\psi\rangle = |00\rangle$. The output of the circuit is the Bell state

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

(6.3)

Before any measurement, the history operator is:

$$C = \text{CNOT}(H \otimes I) |00\rangle\langle00| = \sum_{\alpha_1, \alpha_2} P_{\alpha_2} \text{CNOT} P_{\alpha_1} (H \otimes I) |00\rangle\langle00|$$

(6.4)

and is easily seen to contain the two histories $(\psi, \alpha_1, \alpha_2)$:

$$00 \rightarrow 00 \rightarrow 00$$

(6.5)

$$00 \rightarrow 10 \rightarrow 11$$

(6.6)

corresponding to Fig. 1 a).

Let us now introduce measurements. For example suppose a measurement at $t_2$ has given $\alpha_2 = 11$. The history operator becomes:

$$C' = \sum_{\alpha_1} |11\rangle\langle11| \text{CNOT} P_{\alpha_1} (H \otimes I) |00\rangle\langle00|$$

(6.7)

and contains only the second history (6.6). The same happens if the measurement at $t_2$ is performed only by Alice (then the projector at $t_2$ is $|1\rangle\langle1| \otimes I$). An interesting consequence of this history collapse is that it involves also the past relative to $t_2$. After Alice has obtained 1 at time $t_2$, the system is described by a history operator containing the single history $00 \rightarrow 10 \rightarrow 11$. Thus a measurement by Alice on her qubit does not “instantaneously” affect Bob’s measurement statistics, but affects the whole history of the system. Indeed, due to history collapse the state of Bob’s qubit is $|1\rangle$ even before the measurement (see Fig. 1c), and no “spooky action at a distance” is required.

If Bob measures his qubit at $t_1$ and obtains 0, the history diagram a) remains unchanged, since both histories “pass through” the projector $P_{\alpha_1} = I \otimes |0\rangle\langle0|$. On the other hand, if Alice measures 0 at $t_1$, only one history survives, see Fig. 1 b).

By means of formula (4.2) one finds the probability to measure 11 at time $t_2$:

$$\frac{\text{Tr}(C'^\dagger C')}{\text{Tr}(C'^\dagger C)} = \frac{1}{2}$$

(6.8)

and similar for the probability to find 00. Using the same formula, but with the projector at $t_2$ given by $|1\rangle\langle1| \otimes I$, yields the probability ($= \frac{1}{2}$) for Alice to measure 1 on her qubit at time $t_2$.

Note: the two histories of the unmeasured system are orthogonal, in the sense that $\text{Tr}(C'^\dagger C_{00,00,00} C_{00,01,11}) = 0$, and form therefore a consistent set.

The decoherence condition (2.10) takes the form $\text{Tr}(C_{\psi,\alpha}^\dagger C_{\psi,\beta}) + \text{c.c.} = 0$ in terms of history operators. When the scalar product between history operators $\text{Tr}(C_{\psi,\alpha}^\dagger C_{\psi,\beta})$ vanishes, the decoherence condition is satisfied “a fortiori”.

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6.2 Mach-Zehnder interferometer

The circuit in Fig. 2 mimics a particular setting of the Mach-Zehnder interferometer, which provides a convenient discretization of the double slit experiment. It is a one-qubit line with two Hadamard gates:

![Circuit Diagram]

**Fig. 2** Circuit analogue of the Mach-Zehnder interferometer and some history diagrams with initial state $|0\rangle$: a) no measurements, b) measurement giving 0 at time $t_1$, c) two measurements, giving 0 at time $t_1$ and 1 at time $t_2$.

The insertion of unity occurs at times $t_1$ and $t_2$. For an initial state $|\psi\rangle = |0\rangle$, the history operator of the unmeasured system is\(^7\)

$$C = |0\rangle\langle 0| = \sum_{\alpha_1, \alpha_2} P_{\alpha_2} H \; P_{\alpha_1} H \; |0\rangle\langle 0|$$  \hspace{1cm} (6.9)

and contains only the two histories:

$$0 \rightarrow 0 \rightarrow 0 \hspace{1cm} (6.10)$$

$$0 \rightarrow 1 \rightarrow 0 \hspace{1cm} (6.11)$$

\(^7\)Recall $H^2 = I$.  

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while the other two histories, i.e.

\[ 0 \rightarrow 0 \rightarrow 1 \]  
\[ 0 \rightarrow 1 \rightarrow 1 \]  

(6.12)  
(6.13)

give opposite contributions to the history operator, and therefore disappear from the sum. The diagram of the unmeasured history operator is given in Fig.2a: in this situation a measurement at \( t_2 \) can only have outcome 0. The histories with outcome 1 at \( t_2 \) have interfered and are not present in the history operator. The situation changes drastically if a measurement occurs at \( t_1 \). Supposing that its outcome is 0, the history operator becomes the one represented in Fig.2b, where two different histories open up after the second Hadamard gate, so that a further measurement at \( t_2 \) can give both results, 0 or 1. In Fig.2c a further measurement at \( t_2 \) with outcome 1 has been added, and the only history surviving is \( 0 \rightarrow 0 \rightarrow 1 \), a history that was absent without the measurement in \( t_1 \) because of destructive interference with \( 0 \rightarrow 1 \rightarrow 1 \). In the double slit experiment, the measure at \( t_1 \) corresponds to detect the photon at one of the two slits, thereby destroying the interference pattern on the screen (where the photon arrives at \( t_2 \)).

**Note:** the two histories of the “unmeasured” system are not orthogonal, since

\[ Tr(C_{000}^t C_{010}) = \frac{1}{2} \]  

(6.14)

and therefore are not a consistent set. Indeed interference occurs, constructive for the two histories (6.10), (6.11) and destructive for the two histories (6.12), (6.13).

Probabilities of measurement results at times \( t_1 \), \( t_2 \) or at both times can be computed easily with the general formulae (3.6) and (4.2) and give

\[ p(\alpha_1 = 0) = 1/2, \quad p(\alpha_1 = 1) = 1/2 \]
\[ p(\alpha_2 = 0) = 1, \quad p(\alpha_2 = 1) = 0 \]
\[ p(\alpha_1 = 0, \alpha_2 = 0) = 1/4, \quad p(\alpha_1 = 0, \alpha_2 = 1) = 1/4, \]
\[ p(\alpha_1 = 1, \alpha_2 = 0) = 1/4, \quad p(\alpha_1 = 1, \alpha_2 = 1) = 1/4 \]  

(6.15)

and we can verify that:

\[ p(\alpha_1 = 0, \alpha_2 = 0) + p(\alpha_1 = 1, \alpha_2 = 0) \neq p(\alpha_2 = 0) \]  

(6.16)

In this case the histories \((\psi, \alpha_1, \alpha_2)\) cannot be assigned individual probabilities.
6.3 Teleportation circuit

The teleportation circuit \[22\] is the three-qubit circuit given in Fig. 3, where the upper two qubits belong to Alice, and the lower one to Bob.

\[
|\psi\rangle \rightarrow |\beta_{00}\rangle \rightarrow (H \otimes I \otimes I) |\beta_{00}\rangle \rightarrow (X \otimes Z) |\beta_{00}\rangle
\]

\[
|\psi\rangle \rightarrow |\beta_{00}\rangle \rightarrow (H \otimes I \otimes I) |\beta_{00}\rangle \rightarrow (X \otimes Z) |\beta_{00}\rangle
\]

\[
|\psi\rangle \rightarrow |\beta_{00}\rangle \rightarrow (H \otimes I \otimes I) |\beta_{00}\rangle \rightarrow (X \otimes Z) |\beta_{00}\rangle
\]

\[
|\psi\rangle \rightarrow |\beta_{00}\rangle \rightarrow (H \otimes I \otimes I) |\beta_{00}\rangle \rightarrow (X \otimes Z) |\beta_{00}\rangle
\]

Fig. 3 Teleportation circuit: a) no measurements; b) Alice measures 00 at time \(t_3\); c) at time \(t_3\) Alice measures 00 and Bob measures 1.

The initial state is a three-qubit state, tensor product of the single qubit \(|\psi\rangle\) to be teleported and the 2-qubit entangled Bell state \(|\beta_{00}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)\). Before any measurement, the history operator is

\[
C = H_1 \text{CNOT}_{1,2} |\psi\rangle \otimes |\beta_{00}\rangle \langle \psi| \otimes \langle \beta_{00}| = \sum_{\alpha_1,\alpha_2,\alpha_3} P_{\alpha_3} H_1 P_{\alpha_2} \text{CNOT}_{1,2} P_{\alpha_1} |\psi\rangle \otimes |\beta_{00}\rangle \langle \psi| \otimes \langle \beta_{00}| \tag{6.17}
\]

where unity as sum of projectors has been inserted at times \(t_1, t_2, t_3\), and with obvious notations \(H_1 = H \otimes I \otimes I\) and \(\text{CNOT}_{1,2} = \text{CNOT} \otimes I\). For the moment we do not take into account the \(X\) and \(Z\) gates, activated by the results of Alice measurements at \(t_3\). The history operator has the representation given in Fig. 3a.
and contains 8 histories:

\[
000 \rightarrow 000 \rightarrow 000, \quad 000 \rightarrow 000 \rightarrow 100 \\
100 \rightarrow 110 \rightarrow 010, \quad 000 \rightarrow 110 \rightarrow 110 \\
011 \rightarrow 011 \rightarrow 011, \quad 011 \rightarrow 011 \rightarrow 111 \\
111 \rightarrow 101 \rightarrow 001, \quad 111 \rightarrow 101 \rightarrow 101
\]

(6.19)

Suppose now that Alice measures her two qubits. There are four possible outcomes, each with probability \(1/4\). This can be checked by use of formula (4.2). For example, if Alice obtains 00, the collapsed history operator is

\[
C' = (|00\rangle \langle 00| \otimes I) \sum_{\alpha_1, \alpha_2} H_1 P_{\alpha_2} \text{CNOT}_{1,2} P_{\alpha_1} (|\psi\rangle \otimes |\beta_{00}\rangle)(|\psi\rangle \otimes \langle \beta_{00}|)
\]

(6.20)

and contains only the two histories

\[
000 \rightarrow 000 \rightarrow 000, \\
111 \rightarrow 101 \rightarrow 001
\]

(6.21)

see Fig. 3b. If on this system Bob measures his qubit, and obtains 0 or 1, the history operator becomes

\[
C'' = (I_{2 \times 2} \otimes |0\rangle \langle 0|) C' \quad \text{or} \quad C'' = (I_{2 \times 2} \otimes |1\rangle \langle 1|) C'
\]

(6.22)

and contains only one of the two histories in (6.21), see Fig. 3c for the outcome 1. If \(|\psi\rangle\) is given by

\[
|\psi\rangle = \alpha |0\rangle + \beta |1\rangle
\]

(6.23)

formula (4.2) yields the probabilities \(|\alpha|^2\) and \(|\beta|^2\) for Bob to obtain 0 or 1 respectively, thus showing that the state \(|\psi\rangle\) has been correctly teleported. Similar arguments hold if Alice obtains 01 or 10 or 11. In these cases the gates \(X\) and \(Z\), represented by the Pauli matrices \(\sigma_x\) and \(\sigma_z\) on the (\(|0\rangle, |1\rangle\) basis, have to be added to the history operator.

Finally, if Bob measures his qubit without any preceding measurement by Alice, the surviving histories are the upper four (when Bob finds 0) or the lower four (when Bob finds 1) in Fig. 3a.

### 6.4 Path-integral operator

Consider a generic quantum system with initial state \(|q_0\rangle\) and final state \(|q_n\rangle\), controlled by a Hamiltonian \(H\). Subdividing time between \(t_0\) and \(t_n\) as usual \(t_0 < t_1 < t_2 \cdots < t_n\), and inserting unity as sum of projectors on coordinate eigenvectors \(|q_i\rangle\):

\[
I = \int dq \langle q_i| q_i \rangle
\]

(6.24)
yields the history operator of the system in the form:

\[
C_{q_0,q_n} = \int dq_1 \cdots dq_{n-1} |q_n\rangle \langle q_n| e^{-i\frac{\hbar}{\hbar}(t_n-t_{n-1})} |q_{n-1}\rangle \langle q_{n-1}| \cdots |q_1\rangle \langle q_1| e^{-i\frac{\hbar}{\hbar}(t_1-t_0)} |q_0\rangle \langle q_0|
\]

Taking the limit of infinitesimal time intervals we recover the familiar path-integral operator

\[
C_{q_0,q_n} = |q_n\rangle \langle q_0| \int \mathcal{D}q \ e^{-i\frac{\hbar}{\hbar}S(q_0,q_n)}
\]

where

\[
S(q_0, q_n) = \int_{q_0,t_0}^{q_n,t_n} L(q, \dot{q}) \ dt
\]

and \(L = \dot{p} - H\) is the Lagrangian. Formulae (3.6) and (4.2) reproduce the probability of propagation from \(|q_0\rangle\) to \(|q_n\rangle\). In fact the multiple insertion of identity as sum over projectors on coordinate eigenvectors is the central idea of Feynman’s path-integral, an idea that has inspired all history-based formulations of quantum mechanics.

7 The three-box experiment

It is sometimes called “the three-box paradox”, although no paradox is involved.\(^8\) It was introduced in ref. [23], and discussed in many subsequent works.

The ingredients are a (quantum) particle, three boxes \(A, B\) and \(C\), and measuring devices that probe the system at times \(t_1\) and \(t_2\). The particle is prepared in the initial state at \(t_0\):

\[
|\psi\rangle = \frac{1}{\sqrt{3}} (|A\rangle + |B\rangle + |C\rangle)
\]

where the mutually orthogonal states \(|A\rangle, |B\rangle\) and \(|C\rangle\) correspond to the particle being in box \(A, B,\) and \(C\).

Consider first the case of a measuring apparatus able to detect whether the particle is in \(A\) at \(t_1\), and in the state

\[
|\phi\rangle = \frac{1}{\sqrt{3}} (|A\rangle + |B\rangle - |C\rangle)
\]

at \(t_2\). The relevant projectors are therefore

\[
\begin{align*}
\text{at } t_0 : &\quad P_\psi = |\psi\rangle \langle \psi| \\
\text{at } t_1 : &\quad P_A = |A\rangle \langle A|, \quad P_A^\perp = I - |A\rangle \langle A| \\
\text{at } t_2 : &\quad P_\phi = |\phi\rangle \langle \phi|, \quad P_\phi^\perp = I - |\phi\rangle \langle \phi|
\end{align*}
\]

\(^8\)We thank R. B. Griffiths for calling our attention to this Gedanken experiment, and his treatment of it in ref. [6].
and the history operator of the system, before any measurement, is written as

\[(P_\phi + P_\tilde{\phi})(P_A + P_A\tilde{)}P_\psi\] (7.6)

By inspection it contains the histories:

\[\psi A\phi, \psi A\tilde{\phi}, \psi A\tilde{\phi}\] (7.7)

which form a consistent set. The particular history \(\psi A\tilde{\phi}\) is absent because

\[P_\phi P_A P_\psi = 0\] (7.8)

We can now compute the conditional probability of finding the particle in box A, given a final state \(|\phi\rangle\). Using formula (5.3) one finds:

\[p(A|\psi, \phi) = 1\] (7.9)

In other words, if a particle detector is turned on in box A, it will always detect the particle in A if the final state of the system is \(|\phi\rangle\).

The so-called paradox arises when one replaces A with B in the above reasoning. Since A and B play symmetrical roles, one finds that the conditional probability of finding the particle in box B, given a final state \(|\phi\rangle\), is

\[p(B|\psi, \phi) = 1\] (7.10)

which seems to contradict (7.9) since it means that the particle will always be found in B, if the same final state \(|\phi\rangle\) is post-selected.

In fact no contradiction arises, because the two situations are different. In the second case the measuring device at \(t_1\) is a particle detector in box B, differing from the particle detector in box A. One will always find the particle in A if “looking” in the box A, and always in B if “looking” in the box B, provided of course that the final state is postselected to be \(|\phi\rangle\). This may sound counter-intuitive using classical logic, but such “paradoxes” are the trademark of the quantum world.

We may ask what happens if at \(t_1\) the measuring apparatus is able to detect in which box the particle sits. We must then use the decomposition of unity

\[I = P_A + P_B + P_C\] (7.11)

at \(t_1\), instead of \(I = P_A + P_A\tilde{)}\). This gives rise to a different set of histories:

\[\psi A\phi, \psi A\tilde{\phi}, \psi B\phi, \psi B\tilde{\phi}, \psi C\phi, \psi C\tilde{\phi}\] (7.12)

a non-consistent set: for example the two histories \(\psi A\phi\) and \(\psi B\phi\) are not orthogonal. Nonetheless the history operator contains all of them, and in terms of it we can compute conditional probabilities, finding:

\[p(A|\psi, \phi) = \frac{1}{3}, \quad p(B|\psi, \phi) = \frac{1}{3}, \quad p(C|\psi, \phi) = \frac{1}{3}\] (7.13)

So if we look in each box, again with final state \(|\phi\rangle\) postselected, we find the particle in box A, B and C with equal probability.
8 Conclusions

We have proposed in this note to describe quantum systems by means of a history operator, that allows to compute probabilities in agreement with the usual state vector formulation. There is nothing fundamentally new in this formalism, but it does provide a more immediate and diagrammatic way to represent the “history content” of a quantum system. In particular, by shifting the point of view from state vectors to histories, it helps to alleviate a conflict between simultaneity of collapse and special relativity.

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