ALEKSANDROV-CLARK MEASURES AND SEMIGROUPS OF
ANALYTIC FUNCTIONS IN THE UNIT DISC

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Abstract. In this paper we prove a formula describing the infinitesimal generator of a continuous semigroup \((\varphi_t)\) of holomorphic self-maps of the unit disc with respect to a boundary regular fixed point. The result is based on Aleksandrov-Clark measures techniques. In particular we prove that the Aleksandrov-Clark measure of \((\varphi_t)\) at a boundary regular fixed points is differentiable (in the weak*-topology) with respect to \(t\).

1. Introduction

The aim of the present note is to study the incremental ratio of Aleksandrov-Clark measures (sometimes called spectral measures) of continuous semigroups of the unit disc at boundary regular fixed points, obtaining a measure-theoretic generalization of the well renowned Berkson-Porta formula at the Denjoy-Wolff point.

To state our results, we briefly recall the notion of Aleksandrov-Clark measures and semigroups as needed for our aims (for details on Aleksandrov-Clark measures we refer the reader to the recent surveys [8], [10], [11] and the references therein; while we refer to [1] and [12] for more about iteration theory and semigroups).

Let \(D := \{z \in \mathbb{C} : |z| < 1\}\) be the unit disc. Let \(f : D \to D\) be holomorphic. Fix \(\tau \in \partial D\) and consider the positive harmonic function \(\text{Re}((\tau + f(z))/(\tau - f(z)))\). Then there exists a non-negative and finite Borel measure \(\mu_{f,\tau}\) (called the Aleksandrov-Clark measure of \(f\) at \(\tau\)) on \(\partial D\) such that

\[
\text{Re}\left(\frac{\tau + f(z)}{\tau - f(z)}\right) = \int_{\partial D} P(\zeta, z) d\mu_{f,\tau}(\zeta) \quad \text{for all } z \in D,
\]

where \(P(\zeta, z) = \frac{1 - |z|^2}{|z - \zeta|^2}\) is the Poisson kernel.
We recall that a point $\zeta \in \partial \mathbb{D}$ is said to be a boundary contact point for $f : \mathbb{D} \to \mathbb{D}$ if $\lim_{\tau \to 1} f(\tau \zeta) = \tau \in \partial \mathbb{D}$. In such a case, as customary, we write $f^*(\zeta) := \tau$. It is a remarkable fact that the angular derivative at a boundary contact point $\zeta$ always exists (possibly infinity). Namely, the following non-tangential (or angular) limit exists in the Riemann sphere:

$$f'(\zeta) := \angle \lim_{z \to \zeta} \frac{f(z) - f^*(\zeta)}{z - \zeta}.$$ 

The modulus of $f'(\zeta)$ is known as the boundary dilatation coefficient of $f$ at $\tau$. A point $\tau \in \partial \mathbb{D}$ is a boundary regular fixed point, BRFP for short, for $f$ if $f^*(\tau) = \tau$ and $f'(\tau)$ is finite. If $\tau$ is a BRFP for $f$ then the classical Julia-Wolff-Carathéodory theorem (see, e.g., [1] Prop. 1.2.8, Thm. 1.2.7) asserts that the non-tangential limit $\angle \lim_{z \to \tau} f'(z) = f'(\tau)$ and $f'(\tau) \in (0, +\infty)$.

In 1929, R. Nevanlinna obtained a very deep relationship between angular derivatives and Aleksandrov-Clark measures. Namely, (see, e.g., [1] p. 61, [11] Thm. 3.1) he proved

**Theorem 1.1** (Nevanlinna). Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic self-map and $\zeta \in \partial \mathbb{D}$. Then $\zeta$ is a boundary contact point of $f$ with $f'(\zeta) \in \mathbb{C}$ if and only if for some $\tau \in \partial \mathbb{D}$ the Aleksandrov-Clark measure $\mu_{f,\tau}$ has an atom at $\zeta$ (that is, $\mu_{f,\tau}(\{\zeta\}) > 0$). In this case, it follows $f^*(\zeta) = \tau$ and $\mu_{f,\tau}(\{\zeta\}) = 1/|f'(\zeta)|$.

A (continuous) semigroup $(\varphi_t)$ of holomorphic self-maps is a continuous homomorphism from the additive semigroup of non-negative real numbers and the composition semigroup of all holomorphic self-maps of $\mathbb{D}$ endowed with the compact-open topology. It is well known after the basic work of Berkson and Porta [2] that in fact the dependence of the semigroup $(\varphi_t)$ on the parameter $t$ is real-analytic and there exists a holomorphic vector field $G : \mathbb{D} \to \mathbb{C}$, called the infinitesimal generator of the semigroup, such that

$$\frac{\partial \varphi_t}{\partial t}(z) = G(\varphi_t(z))$$

for all $z \in \mathbb{D}$.

A point $\tau \in \partial \mathbb{D}$ is a boundary regular fixed point, BRFP for short, for the semigroup $(\varphi_t)$ provided it is a BRFP for $\varphi_t$ for all $t \geq 0$ (and this is the case if and only if $\tau$ is a BRFP for $\varphi_t$ for some $t > 0$, see [3]). According to [4] Thm. 1, the point $\tau \in \partial \mathbb{D}$ is a BRFP for $(\varphi_t)$ if and only if $G(\tau) = 0$ as non-tangential limit and the non-tangential limit $\angle \lim_{z \to \tau} G'(z) = \lambda$ exists finitely. Moreover, if $\tau$ is a BRFP for $(\varphi_t)$ then $\varphi'_t(\tau) = e^\lambda$ and $\lambda \in \mathbb{R}$. It is well known that if $(\varphi_t)$ has no fixed point in $\mathbb{D}$ then there exists a unique boundary regular fixed point $\tau \in \partial \mathbb{D}$, called the Denjoy-Wolff point of the semigroup, such that $\varphi_t(z) \to \tau$ as $t \to \infty$ for all $z \in \mathbb{D}$.

In the rest of the paper we will denote by $dm$ the Lebesgue measure on $\partial \mathbb{D}$ normalized so that $m(\partial \mathbb{D}) = 1$ and by $\delta_\xi$ the Dirac atomic measure concentrated at $\xi \in \partial \mathbb{D}$.

Let $(\varphi_t)$ be a semigroup and $\tau \in \partial \mathbb{D}$. We will denote by $\mu_{t,\tau}$ the Aleksandrov-Clark measure of $\varphi_t$ at $\tau$. It can be checked (test with Poisson kernels and use the density of
their span in $C(\partial \mathbb{D})$) that $\{\mu_{t,\tau}\}$ is continuous (in the weak* topology) with respect to $t$. We will prove that it is actually differentiable at 0. Indeed, our first result is the following (note that $\mu_{0,\tau} = \delta_{\tau}$):

**Proposition 1.2.** Let $(\varphi_t)$ be a continuous semigroup of holomorphic self-maps of the unit disc $\mathbb{D}$. Let $\tau \in \partial \mathbb{D}$ be a boundary regular fixed point for $(\varphi_t)$ with boundary dilatation coefficients $(e^{\lambda t})$. Then there exists a positive measure $\mu$ on $\partial \mathbb{D}$ such that

$$\lim_{t \to 0} \frac{\mu_{t,\tau} - \delta_{\tau}}{t} \overset{w^*}{\longrightarrow} -\lambda \delta_{\tau} + \mu,$$

as $t \to 0$.

The measure $\mu$ in (1.3) is strictly related to the infinitesimal generator of the semigroup, see Proposition 4.1. From such a formula we will obtain our main result:

**Theorem 1.3.** Let $(\varphi_t)$ be a continuous semigroup of holomorphic self-maps of the unit disc $\mathbb{D}$ and let $G$ be its infinitesimal generator. Let $\tau \in \partial \mathbb{D}$ be a boundary regular fixed point for $(\varphi_t)$ with boundary dilatation coefficients $(e^{\lambda t})$. Then there exists a unique $p : \mathbb{D} \to \mathbb{C}$ holomorphic, with $\text{Re} p \geq 0$ and $\angle \lim_{z \to \tau}(z - \tau)p(z) = 0$ such that

$$G(z) = (\bar{\tau}z - 1)(z - \tau) \left[ p(z) - \frac{\lambda \tau + z}{2\tau - z} \right] \text{ for all } z \in \mathbb{D}. \tag{1.4}$$

Conversely, given $p : \mathbb{D} \to \mathbb{C}$ holomorphic with $\text{Re} p \geq 0$ and $\angle \lim_{z \to \tau}(z - \tau)p(z) = 0$, $\tau \in \partial \mathbb{D}$ and $\lambda \in \mathbb{R}$, the function $G$ defined as in (1.4) is the infinitesimal generator of a semigroup of holomorphic self-maps of the unit disc for which $\tau$ is a boundary regular fixed point with boundary dilatation coefficients $(e^{\lambda t})$.

In particular, $\tau$ is the Denjoy-Wolff point of $(\varphi_t)$ if and only if $\lambda \leq 0$ and, if this is the case,

$$\text{Re} \left( \frac{G(z)}{(\bar{\tau}z - 1)(z - \tau)} \right) \geq 0 \text{ for all } z \in \mathbb{D},$$

recovering in this way the celebrated Berkson-Porta representation formula when $\tau$ belongs to $\partial \mathbb{D}$ [2].

The plan of the paper is the following. In the second section we compute the singular part of Aleksandrov-Clark measures of $N$-to-1 mappings (in particular for $N = 1$, univalent maps). In the third section we will use such computation to prove Proposition 1.2 and Theorem 1.3. In the final section we discuss some consequences of our results.

2. **Singular parts of Aleksandrov-Clark measures for $N$-to-1 mappings**

P.J. Nieminen and E. Saksman [9, p. 3186] already remarked that for holomorphic $N$-to-1 self-maps of the unit disc the singular part of the corresponding Aleksandrov-Clark measures is discrete. In this section we enhance this result explicitly computing such a singular part.
Given a positive Borel measure \( \mu \) on \( \partial \mathbb{D} \) we will write \( \mu = \mu^s + \mu^a \) for its Lebesgue decomposition in the singular part \( \mu^s \) and the absolutely continuous part \( \mu^a \) with respect to the Lebesgue measure.

**Proposition 2.1.** Let \( f : \mathbb{D} \to \mathbb{D} \) be a \( N \)-to-1 \((N \geq 1)\) holomorphic map and let \( \tau \in \partial \mathbb{D} \). Then there exist \( 0 \leq m \leq N \) and \( \zeta_1, \ldots, \zeta_m \in \partial \mathbb{D} \) such that \( f^\ast(\zeta_j) = \tau \), the non-tangential limit \( f'(\zeta_j) \) of \( f' \) at \( \zeta_j \) exists finitely for \( j = 1, \ldots, m \) and

\[
\mu^s_{f,\tau} = \sum_{k=1}^{m} \frac{1}{|f'(\zeta_k)|} \delta_{\zeta_k}.
\]

Moreover, if \( x \in \partial \mathbb{D} \setminus \{\zeta_1, \ldots, \zeta_m\} \) is such that \( f^\ast(x) = \tau \) then \( \limsup_{z \to x} |f'(z)| = \infty \).

In order to prove Proposition 2.1 we need the following lemma:

**Lemma 2.2.** Let \( f : \mathbb{D} \to \mathbb{D} \) be holomorphic and let \( x, y \in \partial \mathbb{D} \) be such that \( f^\ast(x) = y \). If \( \{z_n\} \subset \mathbb{D} \) is a sequence converging tangentially to \( x \) such that \( \{f(z_n)\} \) converges non-tangentially to \( y \) then

\[
\lim_{n \to \infty} \frac{1 - |f(z_n)|}{1 - |z_n|} = +\infty.
\]

**Proof.** Since \( \{f(z_n)\} \) converges non-tangentially to \( y \) then there exists \( C > 0 \) such that for all \( n \in \mathbb{N} \) it holds

\[
\frac{1 - |f(z_n)|}{|y - f(z_n)|} \geq C.
\]

Therefore

\[
\lim_{n \to \infty} \frac{1 - |f(z_n)|}{1 - |z_n|} = \lim_{n \to \infty} \frac{1 - |f(z_n)|}{|y - f(z_n)|} \cdot \frac{|y - f(z_n)|}{|x - z_n|} \cdot \frac{|x - z_n|}{1 - |z_n|} \geq C \cdot \left[ \liminf_{n \to \infty} \frac{|y - f(z_n)|}{|x - z_n|} \right] \cdot \left[ \lim_{n \to \infty} \frac{|x - z_n|}{1 - |z_n|} \right] = +\infty,
\]

since \( \liminf_{n \to \infty} \frac{|y - f(z_n)|}{|x - z_n|} > 0 \) by Julia’s Lemma (see, e.g., [1]) and \( \lim_{n \to \infty} \frac{|x - z_n|}{1 - |z_n|} = +\infty \) by hypothesis. \( \square \)

**Proof of Proposition 2.1.** According to [9] Prop. 5.3 there exists a sequence \( \{z_n\} \subset \mathbb{D} \) which converges non-tangentially to \( \tau \) such that

\[
\|\mu^s_{f,\tau}\| = \lim_{n \to \infty} \sum_{w \in f^{-1}(z_n)} \frac{\log |w|}{\log |z_n|}.
\]

Since \( f \) is \( N \)-to-1, the number of preimages \( f^{-1}(z_n) \) for each \( n \in \mathbb{N} \) is (at most) \( N \). Let \( M := \limsup_{n \to \infty} \sharp \{f^{-1}(z_n)\} \) be the supremum limit of the number of preimages of \( z_n \). Notice that \( 0 \leq M \leq N \). If \( M = 0 \), namely, if \( f^{-1}(z_n) = \emptyset \) eventually, then \( \mu^s_{f,\tau} = 0 \). Assume that \( M > 0 \). Then there exists a subsequence \( \{z_{n_k}\} \) of \( \{z_n\} \) such that \( \sharp \{f^{-1}(z_{n_m})\} = M \). By (2.2) the mass \( \|\mu^s_{f,\tau}\| \) is expressed as a limit, thus we can replace...
\[
\{z_n\} \text{ with } \{z_{n_k}\} \text{ and assume directly that } \varepsilon f^{-1}(z_n) = M \text{ for all } n. \text{ Let us denote by } \\
\{w_{n,1}, \ldots, w_{n,M}\} \text{ the preimages of } z_n. \text{ Again by (2.2), up to extracting subsequences, we could assume that } \{w_{n,j}\} \text{ is converging to } \tau_j \in \overline{D} \text{ for } j = 1, \ldots, M.
\]

Fix \( j \in \{1, \ldots, M\} \). Clearly, since \( f \) is open, \( \tau_j \in \partial D \). Moreover, since \( \lim_{n \to \infty} \log |w_{n,j}| = 1 \) (and similarly for \( z_n \)), it follows that
\[
\lim_{n \to \infty} \frac{\log |w_{n,j}|}{\log |z_n|} = \lim_{n \to \infty} \frac{1 - |w_{n,j}|}{1 - |z_n|} = \lim_{n \to \infty} \frac{1 - |w_{n,j}|}{1 - |f(w_{n,j})|}.
\]

By Lemma 2.2 if \( \{w_{n,j}\} \) converges to \( \tau_j \) tangentially then \( \lim_{n \to \infty} \frac{1 - |w_{n,j}|}{1 - |f(w_{n,j})|} = 0 \). Suppose then that \( \{w_{n,j}\} \) converges to \( \tau_j \) non-tangentially. By the classical Julia-Wolff-Carathéodory theorem, either \( \lim_{n \to \infty} \frac{1 - |w_{n,j}|}{1 - |f(w_{n,j})|} = 0 \) or the non-tangential limit \( f'(\tau_j) \) of \( f \) at \( \tau_j \) exists finitely and \( \lim_{n \to \infty} \frac{1 - |w_{n,j}|}{1 - |f(w_{n,j})|} = |f'(\tau_j)|^{-1}. \)

Let \( 0 \leq m \leq M \) and \( \{\zeta_1, \ldots, \zeta_m\} \subseteq \{\sigma_1, \ldots, \sigma_M\} \) be the biggest possible subset such that the non-tangential limit \( f'(\zeta_j) \) exists finitely for every \( j \). Then we rewrite (2.2) as
\[
\|\mu^s_{f,\tau}\| = \sum_{j=1}^{m} \frac{1}{|f'(\zeta_j)|}.
\]

Moreover, by Theorem 1.1 we know that
\[
\sum_{j=1}^{m} \frac{1}{|f'(\zeta_j)|} \delta_{\zeta_j} \leq \mu^s_{f,\tau},
\]
Then (2.1) holds.

To conclude the proof, we need to show that if \( x \in \partial D \setminus \{\zeta_1, \ldots, \zeta_m\} \) is such that \( f^*(x) = \tau \) then \( |f'(x)| = \infty \). Indeed, if this were not true then by the Julia-Wolff-Carathéodory theorem the non-tangential derivative \( f'(x) \) would exist and \( |f'(x)|^{-1} \delta_{\zeta} \) would be part of \( \mu^s_{f,\tau} \) (see, e.g., [11 Thm. 3.1]), contradicting (2.1).

Corollary 2.3. Let \( f : D \to D \) be a univalent map and let \( \tau \in \partial D \). Then either \( \mu^s_{f,\tau} = 0 \) or there exists a unique point \( \zeta \in \partial D \) such that \( f^*(\zeta) = \tau \), the non-tangential limit \( f'(\zeta) \) of \( f' \) at \( \zeta \) exists finitely and
\[
\mu^s_{f,\tau} = \frac{1}{|f'(\zeta)|} \delta_{\zeta}.
\]
Moreover, if \( x \in \partial D \setminus \{\zeta\} \) is such that \( f^*(x) = \tau \) then \( \limsup_{z \to x} |f'(z)| = \infty \).

Remark 2.4. Corollary 2.3 implies in particular that for a univalent self-map \( f \) of the unit disc and any point \( \tau \in \partial D \) there exists at most one point \( x \in \partial D \) such that \( f^*(x) = \tau \) and the non-tangential limit of \( f' \) exists finitely at \( x \). This latter fact can also be proved directly, see [5 Lemma 8.2].
Corollary 2.5. Let \((\varphi_t)\) be a continuous semigroup of holomorphic self-maps of \(D\). Suppose that \(\tau \in \partial D\) is a BRFP for \((\varphi_t)\) with boundary dilatation coefficients \((e^{\lambda_t})\). Then
\[
\mu_{t,\tau}^t = e^{-\lambda t} \delta_{\tau}.
\]

Proof. For every \(t \geq 0\) the map \(\varphi_t\) is univalent (see, e.g., [1] or [12]). Therefore by Corollary 2.3 it follows that \(\mu_{s,t,\tau} = \frac{1}{|\varphi_t'(\tau)|} \delta_{\tau}\) and since \(\varphi_t'(\tau) = e^{\lambda t}\), we are done. \(\square\)

Remark 2.6. In the proof of Proposition 2.1 and as a byproduct of Theorem 1.1, we used that for an arbitrary holomorphic self-map of the unit disk \(f\), given \(\tau \in \partial D\) and \(\zeta_1, \ldots, \zeta_n\) different points in \(\partial D\) such that \(f^*(\zeta_j) = \tau\) with \(f'(\zeta_j) \in \mathbb{C}\) for all \(j = 1, \ldots, n\), then
\[
\sum_{j=1}^n \frac{1}{|f'(\zeta_j)|} \delta_{\zeta_j} \leq \mu_{f,\tau}^t.
\]
In particular, we have
\[
\sum_{j=1}^n \frac{1}{|f'(\zeta_j)|} \leq \|\mu_{f,\tau}\| = \int_{\partial D} d\mu_{f,\tau} = \text{Re} \frac{\tau + f(0)}{\tau - f(0)}.
\]
Moreover, equality holds in (2.5) if and only if \(\sum_{j=1}^n \frac{1}{|f'(\zeta_j)|} \delta_{\zeta_j} = \mu_{f,\tau}\) if and only if
\[
\text{Re} \frac{\tau + f(z)}{\tau - f(z)} = \int_{\partial D} P(\zeta, z) d\mu_{f,\tau}(\zeta) = \sum_{j=1}^n \frac{1}{|f'(\zeta_j)|} \int_{\partial D} P(\zeta, z) d\delta_{\zeta_j}(\zeta)
\]
\[
= \text{Re} \sum_{j=1}^n \frac{1}{|f'(\zeta_j)|} \frac{\zeta_j + z}{\zeta_j - z} \text{ for all } z \in D,
\]
namely, if and only if \(f\) is a finite Blaschke product of order \(n\). Inequality (2.5) was obtained in [5, Thm 8.1] by Cowen and Pommerenke with complete different techniques.

3. DIFFERENTIABILITY OF ALEKSANDROV-CLARK MEASURES AND THE REPRESENTATION FORMULA

First of all we prove Proposition 1.2.

Proof of Proposition 1.2. For the sake of simplicity, let us denote by \(\mu_t := \mu_{t,\tau}\) the Aleksandrov-Clark measure of \(\varphi_t\) at \(\tau\). Moreover, for \(t \geq 0\) we define
\[
\sigma_t := \frac{\mu_t - \delta_{\tau}}{t}.
\]
Let \(\sigma_t = \sigma_t^s + \sigma_t^a\) be the Lebesgue decomposition of \(\sigma_t\) with respect to the Lebesgue measure. By Corollary 2.5 it follows:
\[
\sigma_t^s = e^{-\lambda t} - \frac{1}{t} \delta_{\tau}.
\]
Taking the limit as $t \to 0$, we have

$$\sigma_t^w \xrightarrow{w^*} -\lambda \delta_\tau.$$ 

Now we examine the absolutely continuous part $\sigma_t^a$. Since $\sigma_t^a \geq 0$ we have

$$\|\sigma_t^a\| = \int_{\partial D} d\sigma_t^a = \int_{\partial D} d\sigma_t - \int_{\partial D} d\sigma_t^s \overset{3.1}{=} \int_{\partial D} d\sigma_t - \frac{e^{-\lambda t} - 1}{t}$$

$$= \Re \left[ \frac{2\varphi_t(0)}{t} \frac{1}{\tau - \varphi_t(0)} \right] - \frac{e^{-\lambda t} - 1}{t},$$

and then, taking the limit for $t \to 0$ and by (1.2), we obtain

$$\lim_{t \to 0} \|\sigma_t^a\| = 2\Re \mathcal{G}(0) + \lambda.$$ 

This implies that $\{\|\sigma_t\|\}$ is uniformly bounded for $t \ll 1$. Since the ball in the weak*-topology of measures on $\partial \mathbb{D}$ is compact and metrizable, the net $\{\sigma_t\}$ is sequentially compact. Now, by (1.1),

$$\int_{\partial D} P(\zeta, z) d\sigma_t(\zeta) = \frac{1}{t} \Re \left[ \frac{\tau + \varphi_t(z)}{\tau - \varphi_t(z)} - \frac{\tau + z}{\tau - z} \right]$$

$$= 2\Re \left[ \frac{\varphi_t(z) - z}{t} \cdot \frac{\tau}{(\tau - \varphi_t(z))(\tau - z)} \right],$$

and (1.2) yields

$$\lim_{t \to 0} \int_{\partial D} P(\zeta, z) d\sigma_t(\zeta) = 2\Re \left[ \frac{\mathcal{G}(z)\tau}{(\tau - z)^2} \right].$$

This implies that given two accumulation points $\sigma$ and $\sigma'$ of $\{\sigma_t\}$ we have $\int_{\partial \mathbb{D}} P(\zeta, z) d\sigma(\zeta) = \int_{\partial \mathbb{D}} P(\zeta, z) d\sigma'(\zeta)$. Hence $\sigma = \sigma'$ (see, e.g., [7, p. 10]). Therefore the net $\{\sigma_t\}$ is actually weak*-convergent for $t \to 0$.

Finally, denote by $\mu$ the limit of $\{\sigma_t^a\}$. Since $\sigma_t^a$ is a positive measure for all $t \geq 0$, so is $\mu$ and the proof is completed. \hfill \Box

Now we are in the good shape to prove our representation formula.

**Proof of Theorem 1.3** We retain the same notations as in the proof of Proposition 1.2.

Since

$$\Re \frac{\tau + \varphi_t(z)}{\tau - \varphi_t(z)} = \int_{\partial \mathbb{D}} \Re \frac{\zeta + z}{\zeta - z} d\mu_t(\zeta) \quad \text{for all } z \in \mathbb{D},$$

we have
and \( \frac{\tau + \varphi_t(z)}{\tau - \varphi_t(z)} \) and \( \int_{\partial \mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu_t(\zeta) \) are analytic functions, it follows

\[
\frac{\tau + \varphi_t(z)}{\tau - \varphi_t(z)} = \int_{\partial \mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu_t(\zeta) + i\text{Im} \left( \frac{\tau + \varphi_t(0)}{\tau - \varphi_t(0)} \right)
\]

for all \( z \in \mathbb{D} \).

Hence

\[
\frac{\tau + \varphi_t(z)}{\tau - \varphi_t(z)} - \frac{\tau + z}{\tau - z} = \int_{\partial \mathbb{D}} \frac{\zeta + z}{\zeta - z} d(\mu_t - \delta_t)(\zeta) + i\text{Im} \left( \frac{\tau + \varphi_t(0)}{\tau - \varphi_t(0)} - 1 \right)
\]

for all \( z \in \mathbb{D} \). After some computations and dividing by \( t \) we obtain

\[
\frac{\varphi_t(z) - z}{t} = \frac{(1 - \varphi_t(z))}{2} \left[ \int_{\partial \mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu_t(\zeta) + 2i\text{Im} \left( \frac{\varphi_t(0)}{t} - \frac{1}{\tau - \varphi_t(0)} \right) \right]
\]

for all \( z \in \mathbb{D} \). Now, by Proposition 1.2, passing to the limit as \( t \) goes to 0, we deduce

\[
G(z) = \frac{(1 - z^\tau)(\tau - z)}{2} \left[ -\frac{\lambda (\tau + z)}{\tau - z} + \int_{\partial \mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu_t(\zeta) + 2i\text{Im} \left( \frac{G(0)(1)}{\tau} \right) \right].
\]

Setting \( p(z) := \frac{1}{2} \int_{\partial \mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu_t(\zeta) + i\text{Im} \left( \frac{G(0)}{\tau} \right) \), we obtain (1.4).

Moreover, since \( \tau \) is a regular boundary fixed point, by [4] Theorem 1 it follows \( \angle \lim_{z \to \tau} \frac{G(z)}{z - \tau} = \lambda \). Then an easy computation shows that \( \angle \lim_{z \to \tau} (z - \tau)p(z) = 0 \).

In order to prove uniqueness, assume that \( q : \mathbb{D} \to \mathbb{C} \) is holomorphic and \( \gamma \in \mathbb{C} \) are such that \( \text{Re} q \geq 0 \), \( \angle \lim_{z \to \tau} (z - \tau)q(z) = 0 \), and

\[
G(z) = (\bar{\tau} - 1)(z - \tau) \left[ q(z) - \frac{\gamma (\tau + z)}{2(\tau - z)} \right]
\]

for all \( z \in \mathbb{D} \).

Then, by [4] Theorem 1,

\[
\lambda = \angle \lim_{z \to \tau} \frac{G(z)}{z - \tau} = \angle \lim_{z \to \tau} \bar{\tau}(z - \tau) \left[ q(z) - \frac{\gamma (\tau + z)}{2(\tau - z)} \right] = \gamma.
\]

From this, it follows immediately that \( p = q \).

Now we prove the converse: let \( p : \mathbb{D} \to \mathbb{C} \) holomorphic with \( \text{Re} p \geq 0 \), \( \tau \in \partial \mathbb{D} \) and \( \lambda \in \mathbb{R} \), and let \( G \) be defined by (1.4). We want to prove that \( G \) is an infinitesimal generator of some continuous semigroup of holomorphic self-maps of the unit disc.

Since \( \text{Re} p \geq 0 \), by [11] Theorem 1.4.19, the function

\[
H_1(z) := (\bar{\tau} - 1)(z - \tau)p(z)
\]

is the infinitesimal generator of a continuous semigroup of holomorphic functions with Denjoy-Wolff point \( \tau \). By [4] Theorem 1, \( \angle \lim_{z \to \tau} H_1(z) = 0 \) and

\[
\angle \lim_{z \to \tau} \frac{H_1(z)}{z - \tau} = \beta
\]
for some $\beta \leq 0$. Our hypothesis that $\angle \lim_{z \to \tau} (z - \tau)p(z) = 0$ implies that actually $\beta = 0$. Therefore the semigroup associated to $H_1$ has Denjoy-Wolff point $\tau$ with boundary dilatation coefficient 1 for all $t \geq 0$. In particular, if $\lambda = 0$ we are done.

Assume $\lambda \neq 0$. Then $H_2 : \mathbb{D} \to \mathbb{C}$ defined by $H_2(z) := \lambda(\tau z - 1)(z + \tau)$ is also the infinitesimal generator of a semigroup of linear fractional maps (in fact of hyperbolic automorphisms) with fixed points $\tau$ and $-\tau$ (see, [11 Corollary 1.4.16]). Since the set of infinitesimal generators is a real convex cone (see, e.g., [1, Corollary 1.4.15]), it follows that $G(z) = H_1(z) + H_2(z)$ is the infinitesimal generator of a semigroup of holomorphic self-maps of the unit disc. Moreover,

$$\angle \lim_{z \to \tau} G(z) = \angle \lim_{z \to \tau} H_1(z) + \angle \lim_{z \to \tau} H_2(z) = 0$$

and

$$\angle \lim_{z \to \tau} \frac{G(z)}{z - \tau} = \angle \lim_{z \to \tau} \frac{H_1(z)}{z - \tau} + \angle \lim_{z \to \tau} \frac{H_2(z)}{z - \tau} = \beta + \lambda = \lambda.$$

Therefore, by [4 Theorem 1], $\tau$ is a boundary regular fixed point of the semigroup with boundary dilatation coefficients $(e^{\lambda})$.

4. Final Remarks

1. The measure $\mu$ in formula (1.3) is strictly related to the infinitesimal generator $G$ of $(\varphi_t)$. In fact, from classical measure theory, if $\mu_r$ for $r \in (0, 1)$ is the measure defined by $\mu_r = \rho_r dm$ with density $\rho_r(\xi) := \int_{\partial D} P(\zeta, r \xi) d\mu(\zeta), \xi \in \partial D$, then

   (1) $\lim_{r \to 1} \mu_r(\xi) = \mu^\alpha(\xi)$ for $m$-almost every $\xi \in \partial D$ and

   (2) $\mu_r \xrightarrow{w^*} \mu$.

From (1.3) and (3.3) it follows that

$$\mu_r(\xi) = 2\Re \left[ \frac{G(r \xi)\tau}{(\tau - r \xi)^2} \right] dm(\xi) + \lambda \Re \left[ \frac{\tau + r \xi}{\tau - r \xi} \right] dm(\xi).$$

Thus, from (1) and (2) above we obtain

**Proposition 4.1.** Let $(\varphi_t)$ be a continuous semigroup of holomorphic self-maps of the unit disc $\mathbb{D}$ with infinitesimal generator $G$. Let $\tau \in \partial \mathbb{D}$ be a boundary regular fixed point for $(\varphi_t)$. Let $\mu$ be the positive measure defined in (1.3). Then

a) $\Re \left[ \frac{G(\xi)\tau}{(\tau - \xi)^2} \right] \in L^1(\partial \mathbb{D}, m)$ and $\mu^\alpha(\xi) = 2\Re \left[ \frac{G(\xi)\tau}{(\tau - \xi)^2} \right] dm(\xi)$.

b) $\int_{\partial \mathbb{D}} f(\xi) d\mu(\xi) = \lim_{r \to 1} \int_{\partial D} f(\xi) \left( 2\Re \left[ \frac{G(r \xi)\tau}{(\tau - r \xi)^2} \right] + \lambda \Re \left[ \frac{\tau + r \xi}{\tau - r \xi} \right] \right) dm(\xi)$ for all $f \in C(\partial \mathbb{D})$.

2. From the proof of Theorem [1.3] it follows that the condition $\angle \lim_{z \to \tau} (z - \tau)p(z) = 0$ is not necessary in order to show that (1.4) defines an infinitesimal generator, namely, what we really proved is:
Proposition 4.2. Let \( p : \mathbb{D} \to \mathbb{C} \) holomorphic with \( \text{Re} \ p \geq 0 \), \( \tau \in \partial \mathbb{D} \) and \( \lambda \in \mathbb{R} \). Then \( \angle \lim_{z \to \tau} (z - \tau) p(z) = \beta \) exists for some \( \beta \leq 0 \) and the function \( G \) defined as in (1.4) is the infinitesimal generator of a semigroup of holomorphic self-maps of the unit disc for which \( \tau \) is a boundary regular fixed point with boundary dilatation coefficients \( \left( e^{(\beta + \lambda)t} \right) \).

3. Theorem 1.3 shows that given a semigroup of holomorphic functions \( (\varphi_t) \) with a boundary regular fixed point \( \tau \), its infinitesimal generator \( G \) is the sum of the infinitesimal generator of a semigroup of parabolic holomorphic maps with Denjoy-Wolff point at \( \tau \) (namely, \( H_1(z) = (\bar{\tau}z - 1)(z - \tau)p(z) \)) plus, if \( \lambda \neq 0 \), the infinitesimal generator of a group of hyperbolic automorphisms of the unit disc (namely, \( H_2(z) = \frac{1}{2}(\bar{\tau}z^2 - \tau) \)) with a fixed point at \( \tau \). Notice that \( \tau \) is the Denjoy-Wolff point for \( (\varphi_t) \) if and only if it is the Denjoy-Wolff point for the group of hyperbolic automorphisms.

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