UNIQUENESS OF QUASI-EINSTEIN METRICS ON $\mathbb{H}^n \times \mathbb{R}$

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Abstract. The aim of this note is to give an explicit description of quasi-Einstein metrics on $\mathbb{H}^n \times \mathbb{R}$. We shall construct two examples of quasi-Einstein metrics on this manifold and then we shall prove the uniqueness of these examples. Finally, we shall describe the closed relation between quasi-Einstein metrics and static metrics in the quoted space.

1. Introduction and statement of the results

In the last years very much attention has been given to Einstein metrics and its generalizations, for instance Ricci solitons and quasi-Einstein metrics. Ricci solitons model the formation of singularities in the Ricci flow and they correspond to self-similar solutions, i.e. they are stationary points of this flow in the space of metrics modulo diffeomorphisms and scalings; for more details in this subject we recommend the survey due to Cao \cite{7} and the references therein. On the other hand, one of the motivation to study quasi-Einstein metrics on a Riemannian manifold is its closed relation with warped product Einstein metrics, see e.g. \cite{5}, \cite{8}, \cite{12} and \cite{13}. In \cite{5} was proposed to find new examples of Einstein metrics on warped products. The authors wrote:

"Nevertheless warped products do give new examples of complete Einstein manifolds and the Einstein equations are quite interesting", [see chapter 9 page 265.]

Based on this problem we shall give explicit details of how to solve this question by using quasi-Einstein theory. Indeed, $m$-quasi-Einstein metrics, for simplicity quasi-Einstein metrics, are directly related with warped product Einstein metrics, see e.g. \cite{5}, \cite{8}, \cite{12} and \cite{13}. For instance, when $m$ is a positive integer, $m$-quasi-Einstein metrics correspond to exactly those $n$-dimensional manifolds which are the base of an $(n+m)$-dimensional Einstein warped product. One fundamental ingredient to understand the behavior of such a class of manifold is the $m$-Bakry-Emery Ricci tensor which appeared previously in \cite{15} and \cite{4}. It is given by

\begin{equation}
\text{Ric}_m^f = \text{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df,
\end{equation}

where $f$ is a smooth function on $M^n$ and $\nabla^2 f$ stands for the Hessian form.

This tensor was extended recently, independently, by Barros and Ribeiro Jr \cite{3} and Limoncu \cite{14}. More precisely, they extended (1.1) for an arbitrary vector field $X$ on $M^n$ as follows:

\begin{equation}
\text{Ric}_m^X = \text{Ric} + \frac{1}{2} \mathcal{L}_X g - \frac{1}{m} X^\flat \otimes X^\flat,
\end{equation}

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where $\mathcal{L}_X g$ and $X^\flat$ denote, respectively, the Lie derivative on $M^n$ and the canonical 1-form associated to $X$.

With this setting we say that $(M^n, g)$ is a quasi-Einstein metric, if there exist a vector field $X \in \mathfrak{X}(M)$ and constants $0 < m \leq \infty$ and $\lambda$ such that

$$Ric^g_X = \lambda g.$$  

(1.3)

On the other hand, when $m$ goes to infinity, equation (1.2) reduces to the one associated to a Ricci soliton. Whereas, when $m$ is a positive integer and $X$ is gradient, it corresponds to warped product Einstein metrics, for more details see for instance [8]. Following the terminology of Ricci solitons, a quasi-Einstein metric $g$ on a manifold $M^n$ will be called expanding, steady or shrinking, respectively, if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. Moreover, a quasi-Einstein metric will be called trivial if $X \equiv 0$. Otherwise, it will be nontrivial. We notice that the triviality implies that $M^n$ is an Einstein manifold.

Classically the study of such metrics are considered when $X$ is a gradient of a smooth function $f$ on $M^n$, which will be the case considered in this work. From now on, when quoting the quasi-Einstein metrics we will be referring to the gradient case. Therefore, a Riemannian manifold $(M^n, g)$, $n \geq 2$, will be called quasi-Einstein metric if there exist a smooth potential function $f$ on $M^n$ and a constant $\lambda$ satisfying the following fundamental equation

$$Ric^g_f = Ric + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g.$$  

(1.4)

In order to proceed we remember that on a compact manifold $M^n$ an $\infty-$quasi-Einstein metric (Ricci soliton) with $\lambda \leq 0$ is trivial, see [11]. The same result was proved previously in [13] for quasi-Einstein metric on compact manifold with $m$ finite. Besides, we known that compact shrinking Ricci solitons have positive scalar curvature, see e.g. [11]. An extension of this result for shrinking quasi-Einstein metrics with $1 \leq m < \infty$ was obtained in [8]. Recently, in [6] Brozos-Vázquez et al. proved that locally conformally flat quasi-Einstein metrics are globally conformally equivalent to a space form or locally isometric to a pp-wave or a warped product. In [12] was given some classification for quasi-Einstein metrics where the base has non empty boundary. Moreover, they proved a characterization for quasi-Einstein metrics when the base is locally conformally flat.

We point out that Bakry and Ledoux [4] proved an analogue of Myers’s theorem and also they presented a new analytic proof of Cheng’s theorem based on Sobolev inequalities. Bakry and Ledoux’s result implies that every shrinking quasi-Einstein metric must be compact. Moreover, Case proved nonexistence of steady quasi-Einstein metric with $\mu \leq 0$, where $\mu$ is a constant satisfying $\Delta_f f = -m\mu e^\frac{f}{m}$, save for the trivial ones, for more details see [9]. Combining Bakry-Ledoux’s result with Case’s theorem we conclude that every nontrivial noncompact quasi-Einstein metrics are expanding provided $\mu \leq 0$. For instance, it is well-known that $\mathbb{H}^n$ with its canonical metric admits a nontrivial expanding quasi-Einstein structure. Moreover, it is important to detach that a Euclidean space $\mathbb{R}^n$ and a Euclidean sphere $\mathbb{S}^n$ do not admit a nontrivial quasi-Einstein structure.

From now on, we shall fixe the standard metric on $\mathbb{H}^n \times \mathbb{R}$ which is given by

$$g = \frac{1}{x_n^2} \sum_{i=1}^n dx_i^2 + dt^2.$$  

(1.5)

Moreover, we shall prove that $(\mathbb{H}^n \times \mathbb{R}, g)$ admits only two quasi-Einstein structures. Our first example will be obtained with a Killing vector field. More precisely, we have the following example.
Example 1. We consider $\mathbb{H}^n \times \mathbb{R}$ with standard metric (1.5) and the potential function $f(x, t) = \pm \sqrt{(n-1)m}t$. It is easy to see that $\nabla f = \pm \sqrt{(n-1)m}\partial_t$, hence $\text{Hess} f = 0$. Therefore $(\mathbb{H}^n \times \mathbb{R}, g, \nabla f, -(n-1))$ is a quasi-Einstein metric.

Next we shall describe our second example, where its associated vector field is not Killing vector field.

Example 2. We consider $\mathbb{H}^n \times \mathbb{R}$ with standard metric (1.5) and the potential function $f(x, t) = -m \ln(\cosh(\mu t + a))$, where $a \in \mathbb{R}$ and $\mu = \sqrt{\frac{n-1}{m}}$, hence $\nabla f = -m\mu \tanh(\mu t + a)\partial_t$. Under these conditions $(\mathbb{H}^n \times \mathbb{R}, g, \nabla f, -(n-1))$ is a quasi-Einstein metric.

For sake of completeness it is important to detach that the basic object of study in general relativity is a Lorentzian manifold $(M^n, g)$ satisfying Einstein’s equation

$$\text{Ric} - \frac{1}{2} Rg = 9\pi T,$$

where $R$ and $T$ stand, respectively, for the scalar curvature and the stress-energy tensor of matter. The first solution of the Einstein equation (with $T = 0$) was obtained by Schwarzschild in 1916, for more information about this subject we recommend [10]. In this issue it is important to recall that a static space-time is a four-dimensional manifold which possesses a time-like vector field and a spacelike hypersurface which is orthogonal to the integral curves of this Killing field, see [16]. Static space-times are the special and important global solution to Einstein equation in general relativity. It is important also to detach that 1-quasi-Einstein metrics satisfying $\Delta e^{-f} + \lambda e^{-f} = 0$ are static metrics with cosmological constant $\lambda$. These static metrics have been studied extensively because their connection with scalar curvature, the positive mass theorem and general relativity, for more details see e.g. [1], [2] and [10].

On the other hand, we recall that for a Riemannian manifold $(M^n, g)$ the linearization $L_g$ of the scalar curvature operator is given by

$$L_g(h) = -\Delta_g(tr_g(h)) + \text{div}(\text{div}(h)) - g(h, \text{Ric}_g),$$

where $h$ is a 2-tensor. Moreover, the formal $L^2$-adjoint $L_{g}^*$ of $L_g$ is given by

$$(1.6) \quad L_{g}^*(u) = -(\Delta_g u)g + \text{Hess} u - u\text{Ric}_g,$$

where $u$ is a smooth function on $M^n$.

Under these conditions $u$ is a nontrivial element in the kernel of $L_{g}^*$ if and only if the warped product metric $\overline{g} = -u^2dt^2 + g$ is Einstein, for more details see Proposition 2.7 in [10].

Since $(M^n, \nabla f, \lambda)$ is a quasi-Einstein metric with $m < \infty$ we may consider $u = e^{-\frac{t}{m}}$ to rewrite the fundamental equation (1.4) as

$$(1.7) \quad \text{Ric} - \frac{m}{u}\text{Hess} u = \lambda g.$$

Therefore, combining (1.6) and (1.7) we may conclude from Examples [1] and [2] that $(\mathbb{H}^n \times \mathbb{R}, g)$ produces Einstein warped products.

Now, we are in position to announce our main result. It will show that the above examples are unique. More precisely, we have the following result.

Theorem 1. Let $(\mathbb{H}^n \times \mathbb{R}, g, \nabla f, \lambda)$ be a quasi-Einstein metric. Then this structure is given according either Example 1 or Example 2.

We point out that Theorem 1 shows that there exists a natural Einstein warped product on $\mathbb{H}^n \times \mathbb{R}$ for every $m$ with warped function given by $e^{-\frac{t}{m}}$, where $f$ is such as in Examples 1 and 2. Moreover, we recall that Riemannian metrics with $\text{Ker} L_{g}^*$ nontrivial are called static. Therefore, taking into account $m = 1$, it is easy to check that $\Delta e^{-f} + \lambda e^{-f} = 0$,
which permits to conclude from (1.6) that $\mathbb{H}^n \times \mathbb{R}$ endowed with such metric is a static metric.

2. Preliminaries and some basic results

Throughout this section we collect a couple of lemmas that will be useful in the proofs of our results. Our object of study is $\mathbb{H}^n \times \mathbb{R}$ given by

$$\{(x, t) \in \mathbb{R}^n \times \mathbb{R}; x_n > 0, \text{ where } x = (x_1, \ldots, x_n)\}$$

endowed with metric

$$g = \frac{1}{x_n^2} \sum_{i=1}^{n} dx_i^2 + dt^2.$$

It is easy to see that $\{E_i = x_n \partial_{x_i}, E_{n+1} = \partial_t\}$ with $i$ ranging from 1 to $n$, gives a global orthonormal frame. Moreover, the Lie brackets to $\mathbb{H}^n \times \mathbb{R}$ satisfies

$$[E_l, E_n] = -E_l, \text{ when } l = 1, \ldots, n - 1$$

and

$$[E_j, E_k] = 0,$$

otherwise.

Therefore, we may use the Koszul’s formula to obtain the following Riemannian connections

$$\nabla_{E_l} E_l = E_n$$

and

$$\nabla_{E_l} E_n = -E_l, \text{ with } l \text{ ranging from } 1 \text{ to } n - 1$$

and

$$\nabla_{E_j} E_k = 0 \text{ in the other cases.}$$

Now, we may use the Riemannian connection to deduce the following lemma.

**Lemma 1.** The Ricci tensor of $\mathbb{H}^n \times \mathbb{R}$ is given by

$$\text{Ric} = -(n - 1)g + (n - 1)dt^2.$$ 

**Proof.** Firstly, we recall that $\text{Ric}(X, Y) = \sum_{k=1}^{n+1} \langle R(E_k, X)Y, E_k \rangle$, where $\{E_1, \ldots, E_{n+1}\}$ is an orthonormal frame. From what it follows that for $l = 1, \ldots, n - 1$ we have

$$\text{Ric}(E_l, E_l) = \sum_{k \neq l} \langle \nabla_{E_k} \nabla_{E_l} E_l - \nabla_{E_l} \nabla_{E_k} E_l - \nabla_{[E_k, E_l]} E_l, E_k \rangle$$

$$= \langle \nabla_{E_n} E_n - \nabla_{[E_n, E_l]} E_l, E_n \rangle + \sum_{k \neq l,n} \langle \nabla_{E_k} E_n - \nabla_{[E_k, E_l]} E_l, E_k \rangle$$

$$= \langle -\nabla_{E_l} E_l, E_n \rangle + \sum_{k \neq l,n} \langle \nabla_{E_k} E_n, E_k \rangle$$

$$= -1 + \sum_{k \neq l, k < n} \langle -E_k, E_k \rangle$$

$$= -(n - 1).$$

In a similar way a straightforward computation gives $\text{Ric}(E_n, E_n) = -(n - 1), \text{ } \text{Ric}(E_{n+1}, E_{n+1}) = 0$ and $\text{Ric}(E_i, E_j) = 0$ if $i \neq j$, we left its checking for the reader. So, we finishes the proof of the lemma. □

Now, we consider that $\mathbb{H}^n \times \mathbb{R}$ admits a quasi-Einstein structure and we use Lemma 1 to deduce the following lemma.

**Lemma 2.** Let $(\mathbb{H}^n \times \mathbb{R}, g, \nabla f, \lambda)$ be a quasi-Einstein structure. Then the following statements hold:

1. $x_n^2 \frac{\partial^2 f}{\partial x_i^2} - x_n \frac{\partial f}{\partial x_n} = \frac{1}{m} x_n^2 \left( \frac{\partial f}{\partial x_i} \right)^2 + \lambda + (n - 1)$, if $i < n$;
(2) $x_n \frac{\partial f}{\partial x_n} + x_n^2 \frac{\partial^2 f}{\partial x_n^2} = \frac{1}{m} x_n^2 \left( \frac{\partial f}{\partial x_n} \right)^2 + \lambda + (n - 1)$;

(3) $\frac{\partial^2 f}{\partial x_i^2} = \frac{1}{m} \left( \frac{\partial f}{\partial x_i} \right)^2 + \lambda$;

(4) $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{1}{m} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}$, if $i < j < n$;

(5) $x_n^2 \frac{\partial^2 f}{\partial x_n^2} + x_n \frac{\partial f}{\partial x_n} = \frac{1}{m} x_n^2 \frac{\partial f}{\partial x_n} \frac{\partial f}{\partial x_n}$, if $i < n$;

(6) $\frac{\partial^2 f}{\partial x_i \partial t} = \frac{1}{m} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial t}$, if $i \leq n$.

Proof. In order to obtain the first item we may compute $\text{Ric}^m_i(E_i, E_i)$ with $i < n$ in equation (1.4) to obtain

$$\text{Ric}(E_i, E_i) + \text{Hess} f(E_i, E_i) - \frac{1}{m} df \otimes df(E_i, E_i) = \lambda.$$}

Since $E_i = x_n \partial_{x_n}$, we can apply Lemma 1 to obtain the first assertion.

Proceeding, we compute $\text{Ric}^m_i(E_n, E_n)$ and once more we use Lemma 1 to infer

$$-(n - 1) + (n - 1)(E_n, \partial_i)^2 + \text{Hess} f(E_n, E_n) - \frac{1}{m} df \otimes df(E_n, E_n) = \lambda.$$}

Thus, we use that $E_n = x_n \partial_{x_n}$ to deduce the second statement.

Analogously, computing $\text{Ric}^m_i(E_{n+1}, E_{n+1})$, $\text{Ric}^m_i(E_i, E_j)$ with $i < j < n$, $\text{Ric}^m_i(E_i, E_n)$ with $i < n$ and $\text{Ric}^m_i(E_i, E_{n+1})$ with $i \leq n$ straightforward computations give the desired statements, which complete the proof of the lemma.

As an application of Lemma 2 finally we have the following lemma.

Lemma 3. Let $(\mathbb{H}^n \times \mathbb{R}, g, \nabla f, \lambda)$ be a quasi-Einstein structure. Then either $\frac{\partial f}{\partial t}(x, t) = \pm \sqrt{-m\lambda}$ or $\frac{\partial f}{\partial t}(x, t) = -\sqrt{-m\lambda} \tanh(\mu t + a)$, where $a = a(x)$ and $\mu = \sqrt{-\frac{2}{m}}$.

Proof. First, we define $h(t) = \frac{\partial f}{\partial t}(x, t)$. Therefore, we use the third item of Lemma 2 to write

$$h' = \frac{h^2}{m} + \lambda.$$

If $h' \equiv 0$, then $h = \pm \sqrt{-m\lambda}$, which gives the first assertion.

On the other hand, supposing $h' \neq 0$, then (2.1) can be rewrite as

$$h' = \frac{1}{(m^{-1}g^2 + \lambda)^{-1}}$$

and thus it is a separable ODE. Therefore, its solutions are given by

$$h(t) = \sqrt{m\lambda} \tan \left( \sqrt{\frac{\lambda}{m}}(t + c) \right) \text{ if } \lambda > 0,$$

$$h(t) = -\frac{m}{t + c}, \text{ if } \lambda = 0$$

and

$$\frac{h(t) - \sqrt{-m\lambda}}{h(t) + \sqrt{-m\lambda}} = \exp \left( 2\sqrt{-\frac{\lambda}{m}}(t + c) \right), \text{ if } \lambda < 0,$$

where $c$ is a real constant. Now, we may delete the two first solution by using the differentiability of the metric. Therefore, it remains only the last solution. In this case we have two possibilities

$$h(t) - \sqrt{-m\lambda} = \exp \left( \frac{2}{\sqrt{-\frac{\lambda}{m}}(t + c)} \right)$$
and

\[(2.3)\quad \frac{h(t) - \sqrt{-m\lambda}}{h(t) + \sqrt{-m\lambda}} = -\exp \left(2\sqrt{\frac{-\lambda}{m}}(t + c)\right).\]

Taking into account \((2.2)\) we consider \(\mu = \sqrt{-\frac{\lambda}{m}}\) to arrive at

\[h(t) = \sqrt{-m\lambda} \coth(-\mu(t + c)),\]

which gives a contradiction with the differentiability of \(h\). Therefore, from \((2.3)\) we conclude that

\[h(t) = \sqrt{-m\lambda} \tanh(-\mu(t + c)) = -\sqrt{-m\lambda} \tanh(\mu t + a).\]

This is what we wanted to prove. \(\square\)

3. PROOF OF THEOREM \(1\)

**Proof.** First, since \(\mathbb{H}^n \times \mathbb{R}\) admits a quasi-Einstein structure, we obtain from Lemma 3 two possibilities given by

\[\frac{\partial f}{\partial t}(x, t) = \pm \sqrt{-m\lambda}\]

and

\[\frac{\partial f}{\partial t}(x, t) = -\sqrt{-m\lambda} \tanh(\mu t + a),\]

where \(a = a(x)\) and \(\mu = \sqrt{-\frac{\lambda}{m}}\).

In the first case we may use item (6) of Lemma 2 to arrive at

\[\frac{\partial f}{\partial x} \left(\frac{\partial f}{\partial t}\right) = \frac{1}{m} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial t} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial t},\]

Hence \(\frac{1}{m} \frac{\partial f}{\partial x_i} (\pm \sqrt{-m\lambda}) = 0\). From what it follows that \(\frac{\partial f}{\partial x_i} = 0\) for \(i \leq n\), where we used that \(\lambda < 0\). Moreover, we use item (1) of Proposition 2 to conclude \(\lambda = -(n - 1)\) and \(f(x, t) = \pm \sqrt{m(n - 1)t}\), which proves the first part.

Proceeding we consider \(\frac{\partial f}{\partial t}(x, t) = -\sqrt{-m\lambda} \tanh(\mu t + a)\). Once more, we use item (6) of Lemma 2 to obtain

\[\frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial t}\right) = \frac{1}{m} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial t},\]

which implies that

\[(3.1)\quad \text{sech}^2(\mu t + a) \frac{\partial a}{\partial x_i} = \frac{1}{m} \tanh(\mu t + a) \frac{\partial f}{\partial x_i},\]

for every \((x, t) \in \mathbb{H}^n \times \mathbb{R}\).

On the other hand, fixing \(x\) and choosing \(t\) such that \(\tanh(\mu t + a(x)) = 0\) we have from \((3.1)\) that \(\text{sech}^2(\mu t + a(x)) \frac{\partial a}{\partial x_i}(x) = 0\). Since \(\text{sech}^2(\mu t + a(x))\) cannot assume null value we conclude that \(\frac{\partial a}{\partial x_i}(x) = 0\). Thus, since \(x\) is arbitrary we obtain \(\frac{\partial a}{\partial x_i} \equiv 0\), which implies that \(a\) is constant. Therefore, we have \(\frac{1}{m} \tanh(\mu t + a) \frac{\partial f}{\partial x_i} = 0\) for every \((x, t) \in \mathbb{H}^n \times \mathbb{R}\). From what it follows that \(\frac{\partial f}{\partial x_i} = 0, \text{ for } i \leq n\). By using again the first item of Lemma 2 we arrive at \(\lambda = -(n - 1)\). So, we finish the proof of the theorem. \(\square\)

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REFERENCES

[1] Anderson, M.: Scalar curvature, metric degenerations and the static vacuum Einstein equations on 3-manifolds. I. Geom. Funct. Anal. 9 (1999) n.5, 855-967.

[2] Anderson, M. and Khuri, M.: The static extension problem in General relativity. arXiv:0909.4550v1 [math.DG], (2009).

[3] Barros, A. and Ribeiro Jr, E.: Integral formulae on quasi-Einstein manifolds and applications. Glasgow Math. J. 54 (2012), 213-223.

[4] Bakry, D. and Ledoux, M.: Sobolev inequalities and Myers diameter theorem for an abstract Markov generator. Duke Math. J. 85 (1996), no. 1, 253-270.

[5] Besse, A.: Einstein manifolds, Springer-Verlag, New York (2008).

[6] Brozos-Vázquez, M., García-Rio, E., Gavino-Fernández, S.: Locally conformally flat Lorentzian quasi-Einstein manifolds. J. Geom. Anal., to appear (doi: 10.1007/s12220-011-9283-z), arXiv:1202.1245v1 [math.DG], (2012).

[7] Cao, H.-D.: Recent progress on Ricci soliton. Adv. Lect. Math. (ALM), 11 (2009), 1-38.

[8] Case, J., Shu, Y. and Wei, G.: Rigidity of quasi-Einstein metrics. Differ. Geom. Appl., 29 (2011), 93-100.

[9] Case, J.: On the nonexistence of quasi-Einstein metrics. Pacific J. Math. 248 (2010), 227-284.

[10] Corvino, J.: Scalar curvature deformations and a gluing construction for the Einstein constraint equations. Comm. Math. Phys. 214 (2000) 137-189.

[11] Eminenti, M., La Nave, G. and Mantegazza, C.: Ricci solitons: the equation point of view. Manuscripta Math. 127 (2008) 345-367.

[12] He, C., Petersen, P. and Wylie, W.: On the classification of warped product Einstein metrics. Commun. in Analysis and Geometry, 20 (2012) 271-312.

[13] Kim, D. S. and Kim, Y. H.: Compact Einstein warped product spaces with nonpositive scalar curvature. Proc. Amer. Math. Soc. 131 (2003) 2573-2576.

[14] Limoncu, M.: Modifications of the Ricci tensor and applications. Arch. Math. 95 (2010) 191-199.

[15] Qian, Z.: Estimates for weighted volumes and applications, Quart. J. Math. Oxford Ser. (2) 48 (1997), no. 190, 235-242.

[16] Wald, R.: General Relativity. Chicago: U. Chicago Press, 1984.

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