Inequalities for experimental tests of the
Kochen-Specker theorem

Koji Nagata
National Institute of Information and Communications Technology
4-2-1 Nukui kita, Koganei, Tokyo 184-8795, Japan

Abstract

We derive inequalities for $n$ spin-1/2 systems under the assumption that the hidden-variable theoretical joint probability distribution for any pair of commuting observables is equal to the quantum mechanical one. Fine showed that this assumption is connected to the no-hidden-variables theorem of Kochen and Specker (KS theorem). These inequalities give a way to experimentally test the KS theorem. The fidelity to the Bell states which is larger than 1/2 is sufficient for the experimental confirmation of the KS theorem. Hence, the Werner state is enough to test experimentally the KS theorem. Furthermore, it is possible to test the KS theorem experimentally using uncorrelated states. An $n$-partite uncorrelated state violates the $n$-partite inequality derived here by an amount that grows exponentially with $n$. 
1 Introduction

From the incompleteness argument of the EPR paper\cite{1}, hidden-variable interpretation of quantum mechanics (QM) has been an attractive topic of research\cite{2,3}. There are two main approaches to study this conceptual foundation of QM. One is the Bell-EPR theorem\cite{4}. This theorem says that the statistical prediction of QM violates the inequality following from the EPR-locality principle. The EPR-locality principle tells that a result of measurement pertaining to one system is independent of any measurement performed simultaneously at a distance on another system.

The other is the no-hidden-variables theorem of Kochen and Specker (KS theorem)\cite{5}. The original KS theorem says the non-existence of a real-valued function which is multiplicative and linear on commuting operators so that QM cannot be imbedded into classical theory. The proof of the KS theorem relies on intricate geometric argument. Fine connected\cite{6,7} the KS theorem to the assumption that the hidden-variable theoretical joint probability distribution for any pair of commuting observables. Greenberger, Horne, and Zeilinger discovered\cite{8} the so-called GHZ theorem for four-partite GHZ states and the KS theorem has taken very simple form since then (see also Refs. \cite{9,10,11,12}).

In 1990, Mermin considered the Bell-EPR theorem of multipartite systems and derived multipartite Bell’s inequality\cite{13}. It has shown that the $n$-partite GHZ state violates the Bell-Mermin inequality by an amount that grows exponentially with $n$. After this work, several multipartite Bell’s inequalities have been derived\cite{14,15}. They also exhibit that QM violates local hidden-variable theory by an amount that grows exponentially with the number of parties.

As for the KS theorem, most research is related to “all versus nothing” demolition of the existence of hidden variables\cite{16}. (Of course, Bell’s inequalities is available for a test of the KS theorem). Recently, it has begun to research the KS theorem using inequalities (see Refs. \cite{17}). To find such inequalities to test the KS theorem is particularly useful for experimental investigation\cite{18}. Since the KS theorem was purely related to the algebraic structure of quantum operators and was independent of states, it may be possible to find an inequality that is violated by quantum predictions of the result of measurement on uncorrelated states\cite{19}. That is to say, an entanglement of states might not be necessary in order to show a violation of such inequalities for the KS theorem unlike the ones for the Bell-EPR theorem. Here, we shall modify the Bell-Mermin inequality. Namely, the inequality derived in this paper is violated independently of entanglement effects. We may further ask, then, how the relation between the magnitude of the violation and the number of parties would be.

Motivated by these arguments, we shall derive two inequalities following from the assumption pointed out\cite{6,7} by Fine as a test for the KS theorem for $n$ spin-1/2 states. Fine’s assumption is that the hidden-variable theoretical joint probability distribution for any commuting pair of observables is equal to quantum mechanical one. That is, a violation of Fine’s assumption implies that there exists a pair of commuting observables such that the hidden-variable theoretical joint distribution does not agree with QM, or hidden variables cannot exist.

One of the inequalities says that the fidelity to the Bell states, which is larger than $1/2$, allows a proof of the KS theorem. This says the Werner state\cite{19} which admits local hidden-variable theory is enough to test experimentally the KS the-
orem. And we obtain modification of the Bell-Mermin inequality on combining Mermin’s geometric idea and a commutative operator group presented by Nagata et al. We show that when \( n \) exceeds 2, not only \( n \)-partite GHZ states but also \( n \)-partite uncorrelated states violate the modified inequality derived here. The amount of violations grows exponentially with \( n \), which is a factor of \( O(2^{n/2}) \) at the macroscopic level.

Our result provides a striking aspect of foundations of QM and impossibility of a classical reinterpretation of it. That is, QM exhibits an exponentially stronger refutation of the KS type of hidden-variable theory, as the number of parties constituting the state increases, irrespective of entanglement effects. In other words, we can say that the KS theorem is more serious in high-dimensional settings than in low-dimensional ones. Further, we can see the local hidden-variable theory violates the KS type of hidden-variable theory.

This paper is organized as follows. In Sec. 2 we fix several notations and prepare for arguments of this paper. In Sec. 3 we review the statistical KS theorem and mention that its inequality version is necessary for an experimental test. In Sec. 4 we present an inequality which follows from Fine’s assumption for two-partite states and derive a sufficient condition to allow a proof of the KS theorem, which states that the fidelity to the Bell states is larger than 1/2. Since the fidelity to the Bell states is 5/8, the two-spin 1/2 Werner state violates the inequality. In Sec. 5 we modify the Bell-Mermin inequality. We derive another inequality which follows from Fine’s assumption for \( n \)-partite states and show that not only \( n \)-partite GHZ states but also \( n \)-partite uncorrelated states violate the inequality by an amount that grows exponentially with \( n \).

Section 6 summarizes this paper.

### 2 Notation and preparations

Throughout this paper, we assume von Neumann’s projective measurements and we confine ourselves to the finite-dimensional and the discrete spectrum case. Let \( \mathbb{R} \) denote the reals where \( \pm \infty \not\in \mathbb{R} \). We assume every eigenvalue in this paper lies in \( \mathbb{R} \). Further, we assume that every Hermitian operator is associated with a unique observable because we do not need to distinguish between them in this paper.

We assume the validity of QM and we would like to investigate if the KS type of hidden-variable interpretation of QM is possible. Let \( \mathcal{O} \) be the space of Hermitian operators described in a finite-dimensional Hilbert space, and \( \mathcal{T} \) be the space of density operators described in the Hilbert space. Namely, \( \mathcal{T} = \{ \psi | \psi \in \mathcal{O} \wedge \psi \geq 0 \wedge \text{Tr}[\psi] = 1 \} \). Now we define the notation \( \theta \) which represents one result of quantum measurement. Suppose that the measurement of a Hermitian operator \( A \) for a system in the state \( \psi \) yields a value \( \theta(A) \in \mathbb{R} \). We assume that the following two propositions (BSF and QDJ) hold. Here, \( \chi_{\Delta}(x), x \in \mathbb{R} \) represents the characteristic function. \( \Delta \) is any subset of the reals \( \mathbb{R} \).

**Proposition: BSF (The Born statistical formula).**

\[
\text{Prob}(\Delta)^{\psi}_{\theta(A)} = \text{Tr}[\psi \chi_{\Delta}(A)]. \tag{2.1}
\]

The whole symbol \( (\Delta)^{\psi}_{\theta(A)} \) is used to denote the proposition that the value of
$\theta(A)$ lies in the set $\Delta$ in the quantum state $\psi$. And $Prob$ denotes the probability that the proposition holds.

**Proposition: QDJ** (The quantum-mechanical joint probability distribution for commuting observables).

$$Prob(\Delta, \Delta')_{\theta(A), \theta(B)} = Tr[\psi \chi(A) \chi(A')],$$  \hspace{1cm} (2.2)

where the notation on the LHS of (2.2) is a generalization of the symbol $(\Delta)_{\theta(A)}^\psi$ to express the proposition that measurement results of $A$ and $B$ will lie in the sets $\Delta$ and $\Delta'$, respectively.

Let us consider a classical probability space $(\Omega, \Sigma, \mu_\psi)$, where $\Omega$ is a nonempty sample space, $\Sigma$ is a $\sigma$-algebra of subsets of $\Omega$, and $\mu_\psi$ is a $\sigma$-additive normalized measure on $\Sigma$ such that $\mu_\psi(\Omega) = 1$. The subscript $\psi$ expresses that the probability measure is determined uniquely when the state $\psi$ is specified.

Let us introduce measurable functions (classical random variables) onto $\Omega$ ($f: \Omega \mapsto R$), which is written as $f_A(\omega)$ for an operator $A \in \mathcal{O}$. Here $\omega \in \Omega$ is a hidden variable. We introduce appropriate notation. $P(\omega) \simeq Q(\omega)$ means $P(\omega) = Q(\omega)$ holds almost everywhere with respect to $\mu_\psi$ in $\Omega$. One may assume the probability measure $\mu_\psi$ is chosen such that the following relation is valid:

$$tr[\psi A] = \int_{\omega \in \Omega} \mu_\psi(d\omega) f_A(\omega)$$ \hspace{1cm} (2.3)

for every Hermitian operator $A$ in $\mathcal{O}$. Please notice the assumption for the probability measure $\mu_\psi$ does not disturb the KS theorem. See the lemma (B.1) in Appendix B.

**Proposition: HV** (The deterministic hidden-variable interpretation of QM).

Measurable functions $f_A(\omega)$ exist for every Hermitian operator $A$ in $\mathcal{O}$.

**Proposition: D** (The probability distribution rule).

$$\mu_\psi(f_A^{-1}(\Delta)) = Prob(\Delta)_{\theta(A)}^\psi.$$

**Proposition: JD** (The joint probability distribution rule).

$$\mu_\psi(f_A^{-1}(\Delta) \cap f_B^{-1}(\Delta')) = Prob(\Delta, \Delta')_{\theta(A), \theta(B)}$$ \hspace{1cm} (2.5)

for every commuting pair $A, B$ in $\mathcal{O}$.

**Proposition: FUNC A.E.** (The functional rule holding almost everywhere).

$$f_{g(A)}(\omega) \simeq g(f_A(\omega))$$ \hspace{1cm} (2.6)

for every function $g: R \mapsto R$.

**Proposition: PROD A.E.** (The product rule holding almost everywhere).

If Hermitian operators $A$ and $B$ commute, then

$$f_{AB}(\omega) \simeq f_A(\omega) \cdot f_B(\omega).$$ \hspace{1cm} (2.7)

**Theorem.**

$$HV \land JD \Rightarrow HV \land D \land FUNC A.E.$$ \hspace{1cm} (2.8)
Proof. See (2.4) in Appendix 13.

Theorem [13]:

\[ \text{HV \& FUNC A.E. (2.6) \Rightarrow HV \& PROD A.E. (2.7).} \quad (2.9) \]

Proof. See (2.11) in Appendix 13.

3 The statistical Kochen-Specker theorem

In this section, we want to review the statistical KS theorem (see also Refs. [17]). In what follows, we assume HV and JD (2.5) hold. This implies that we can use D (2.4), FUNC A.E. (2.6), and PROD A.E. (2.7). We follow the statistical version of the KS theorem proposed by Peres [11] and refined by Mermin [12] for two spin-1/2 systems. One then can see that

\[
X(\omega) := f_{\sigma_1^z\sigma_2^z}(\omega)f_{\sigma_1^z\sigma_2^z}(\omega)f_{\sigma_1^z\sigma_2^z}(\omega)f_{\sigma_1^z\sigma_2^z}(\omega) = f_{-I}(\omega)
\]

\[
\Rightarrow \int_{\omega \in \Omega} \mu_\psi(d\omega)X(\omega) = Tr[\psi(-I)] = -1,
\]

where \( I \) represents the identity operator for the four-dimensional space. By the way we can factorize two of the terms as \( f_{\sigma_1^z\sigma_2^z} \approx f_{\sigma_1^z}\) and \( f_{\sigma_2^z}\). Further, we have \( f_{\sigma_1^z}\sigma_2^z \approx f_{\sigma_1^z}f_{\sigma_2^z} \) and \( f_{\sigma_2^z}\sigma_1^z \approx f_{\sigma_1^z}f_{\sigma_2^z} \). Hence we get

\[
X(\omega) \approx f_{\sigma_1^z\sigma_2^z}(\omega)f_{\sigma_1^z\sigma_2^z}(\omega)f_{\sigma_1^z\sigma_2^z}(\omega) = f_I(\omega)
\]

\[
\Rightarrow \int_{\omega \in \Omega} \mu_\psi(d\omega)X(\omega) = Tr[\psi(I)] = 1
\]

in contradiction to (3.1). Thereby, we see that HV does not hold if we accept JD (2.5).

We follow the statistical version of the KS theorem proposed in Refs. [9, 12] for three spin-1/2 systems. Then, one can see that

\[
Y(\omega) := f_{\sigma_1^z\sigma_2^z\sigma_3^z}(\omega)f_{\sigma_1^z\sigma_2^z\sigma_3^z}(\omega)f_{\sigma_1^z\sigma_2^z\sigma_3^z}(\omega)f_{\sigma_1^z\sigma_2^z\sigma_3^z}(\omega) = f_{-I}(\omega)
\]

\[
\Rightarrow \int_{\omega \in \Omega} \mu_\psi(d\omega)Y(\omega) = Tr[\psi(-I)] = -1,
\]

where \( I \) represents the identity operator for the eight-dimensional space. By the way, we can factorize each of the four terms as

\[
f_{\sigma_1^z\sigma_2^z}(\omega) \approx f_{\sigma_1^z}(\omega)f_{\sigma_2^z}(\omega)
\]

and so on to get

\[
Y(\omega) \approx (f_{\sigma_1^z}(\omega))^2(f_{\sigma_1^z}(\omega))^2(F_{\sigma_2^z}(\omega))^2(f_{\sigma_2^z}(\omega))^2(f_{\sigma_3^z}(\omega))^2
\]

\[
\approx f_I(\omega)f_I(\omega)f_I(\omega)f_I(\omega)f_I(\omega) \approx f_I(\omega)
\]

\[
\Rightarrow \int_{\omega \in \Omega} \mu_\psi(d\omega)Y(\omega) = Tr[\psi(I)] = 1
\]

(3.5)
in contradiction to \(\text{KS}\).

These two examples provide the statistical KS theorem, which says demolition of HV or of JD \(\text{(2.5)}\). We have the following result:

**Theorem (The statistical Kochen-Specker theorem).**

For every quantum state described in a Hilbert space \(\mathcal{H}_1 \otimes \mathcal{H}_2 \) or \(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3\), \((\text{Dim}(\mathcal{H}_j) = 2, (j = 1, 2, 3))\),

\[
\text{HV \& JD (2.5) } \Rightarrow \bot.
\]

That is, these two assumptions do not hold at the same time.

These examples are sufficient to show that, if we accept JD \(\text{(2.5)}\), HV cannot be possible in any state. However, they are not of suitable form to test experimentally the KS theorem. Because, in a real experiment, we cannot claim a sharp value as an expectation with arbitrary precision. Therefore, we need its inequality version is necessary for an experimental test of the KS theorem.

4 Inequality for two-partite systems

In this section, we shall derive the inequality version statistical KS theorem for two-partite systems. Then, we show that the two spin-1/2 Werner state violates the inequality. Since the Werner state satisfies all Bell’s inequalities, the inequality derived in this section does not belong to the category of Bell’s inequalities. (So does the inequality derived in the next section). The inequality is just the inequality concerned with the KS theorem. In the following, we assume that HV and JD \(\text{(2.5)}\) hold. Let \(x, y\) be real numbers with \(x, y \in \{-1, +1\}\), then we have

\[
1 + x + y - xy = \pm 2. \tag{4.1}
\]

**Theorem.** For every state \(\psi\) described in a Hilbert space \(\mathcal{H}_1 \otimes \mathcal{H}_2\), \((\text{Dim}(\mathcal{H}_j) = 2, (j = 1, 2))\),

\[
\text{HV \& JD (2.5)} (\wedge \sigma_1^x \sigma_2^x \sigma_1^y \sigma_2^y = \sigma_1^z \sigma_2^z) \\
\Rightarrow 1 + \text{Tr}[\psi \sigma_1^x \sigma_2^x] + \text{Tr}[\psi \sigma_1^y \sigma_2^y] - \text{Tr}[\psi \sigma_1^z \sigma_2^z] \leq 2. \tag{4.2}
\]

**Proof.** From PROD A.E. \(\text{(2.7)}\), we have

\[
(f_{\sigma_1^x \sigma_2^x}(\omega))^2 \simeq f_1(\omega) = +1 \iff f_{\sigma_1^y \sigma_2^y}(\omega) \simeq \pm 1, (k = x, y). \tag{4.3}
\]

Hence, the \(\text{(4.3)}\) says

\[
U(\omega) := 1 + f_{\sigma_1^x \sigma_2^x}(\omega) + f_{\sigma_1^y \sigma_2^y}(\omega) - f_{\sigma_1^x \sigma_2^y}(\omega)f_{\sigma_1^y \sigma_2^x}(\omega) \Rightarrow U(\omega) \simeq \pm 2 \tag{4.4}
\]

and

\[
\int_{\omega \in \Omega} \mu_\psi(d\omega) U(\omega) \leq 2. \tag{4.5}
\]

On using \(f_{\sigma_1^x \sigma_2^x} f_{\sigma_1^y \sigma_2^y} \simeq f_{\sigma_1^x \sigma_2^y} f_{\sigma_1^y \sigma_2^x} \simeq f_{\sigma_1^x \sigma_2^x}(\omega)\) we get

\[
\int_{\omega \in \Omega} \mu_\psi(d\omega) f_{\sigma_1^x \sigma_2^x}(\omega) f_{\sigma_1^y \sigma_2^y}(\omega) = \int_{\omega \in \Omega} \mu_\psi(d\omega) f_{\sigma_1^x \sigma_2^y}(\omega) f_{\sigma_1^y \sigma_2^x}(\omega)
\]

\[
= \int_{\omega \in \Omega} \mu_\psi(d\omega) f_{\sigma_1^x \sigma_2^x}(\omega), \tag{4.6}
\]
where we have used the quantum mechanical rule \( \sigma_x^1 \sigma_y^2 \sigma_y^1 \sigma_x^2 = \sigma_z^1 \sigma_x^2 \). Hence we conclude

\[
\int_{\omega \in \Omega} \mu_\psi (d\omega) U(\omega) \leq 2 \Leftrightarrow 1 + Tr[\psi \sigma_x^1 \sigma_x^2] + Tr[\psi \sigma_y^1 \sigma_y^2] - Tr[\psi \sigma_z^1 \sigma_z^2] \leq 2. \quad (4.7)
\]

QED.

Violation of the inequality (4.7) implies demolition of HV or of JD (2.5) in the state \( \psi \). Note the following quantum mechanical relation:

\[
1 + Tr[\psi \sigma_x^1 \sigma_x^2] + Tr[\psi \sigma_y^1 \sigma_y^2] - Tr[\psi \sigma_z^1 \sigma_z^2] \leq 2 \Leftrightarrow Tr[\psi |\pi \rangle \langle \pi |] \leq 1/2, \quad (4.8)
\]

where

\[
|\pi \rangle := \frac{|+1; -2 \rangle + |-1; +2 \rangle}{\sqrt{2}}. \quad (4.9)
\]

Therefore the statistical KS theorem holds if the fidelity to the Bell state \( |\pi \rangle \) is larger than \( 1/2 \). Note the fidelity to the Bell states of the two spin-1/2 Werner state \([19]\) is 5/8 (\( > 1/2 \)). The Werner state \( W \) is

\[
W = (1/2)|\pi \rangle \langle \pi | + (1/8)I, \quad (4.10)
\]

where \( I \) is the identity operator on the four-dimensional space. Hence, this quantum state which admits local hidden-variable theory allows a proof of the KS theorem.

5 Inequality for multipartite systems

In what follows, we shall modify the Bell-Mermin inequality \([13]\). We derive an \( n \)-party inequality which is satisfied if both HV and JD \([2, 3]\) hold. We show \( n \)-party uncorrelated states violate the inequality when \( n \geq 3 \), by an amount that grows exponentially with \( n \). Please note uncorrelated states satisfy all Bell’s inequalities \([19]\). Hence, the modified inequality does not belong to the category of Bell’s inequalities. In this section, we assume \( n \geq 2 \). Let us denote \( \{1, 2, \ldots, n\} \) by \( N_n \).

**Definition:** (Commutative group \( (\Lambda_n) \) of Hermitian operators).

\( O_p^n (p \in \{0, 1, \ldots, 2^n - 1\}, n \geq 2) \) are Hermitian operators defined by

\[
O_p^n := \prod_{j=1}^n (\sigma_z^1)^{b_j} (\sigma_z^2)^{b_0} = (\sigma_z^1)^{b_1} (\sigma_z^2)^{b_0} (\sigma_z^3)^{b_0} (\sigma_z^4)^{b_0} \cdots (\sigma_z^{n-1})^{b_0} (\sigma_z^n)^{b_0} \times (\sigma_z^1)^{b_0} (\sigma_z^2)^{b_0} (\sigma_z^3)^{b_0} \cdots (\sigma_z^{n-1})^{b_0} (\sigma_z^n)^{b_0}, \quad (5.1)
\]

where the superscript \( j \) of the Pauli operators denotes the party \( j \) and the \( n \)-bit sequence \( b_0 b_1 \cdots b_{n-1} \) is the binary representation of \( p \), and \( b_n \in \{0, 1\} \land b_0 = \sum_{j=1}^{n-1} b_j \) (mod2). Thus, the parity of \( b_1 b_2 \cdots b_n \) is even. (Here, \( \sigma_z^k \) means \( \sigma_z^1 \otimes_{j=2}^n I \) and so on. Omitting the identity operator, we abbreviate those as above.)

The operator \( O_0^n \) is the identity operator on the \( 2^n \)-dimensional space, and the other operators \( O_1^n, \ldots, O_{2^n-1}^n \) have two eigenvalues, \( \pm 1 \). In the following, there are the cases where we abbreviate \( O_0^n \) as \( I \).
**Example:** If \( p \in \{0, 1, \ldots, 2^n - 1\} \), then \( b_0 = 0 \) and \((\sigma_z^p)^{b_0} = \otimes_{k=1}^n I^k = O_0^n\) for all \( j \). That is, the binary representation of \( p \) takes, for example, the following form:

\[
p_1 := 01001 \cdots 01 (= b_0 b_1 \cdots b_{n-1}),
\]

where \( B_1 \) represents the sum of the number of 1. Then, \( b_n \in \{0, 1\} \land b_n \equiv B_1 \mod 2 \) holds. Suppose \( b_n = 1 \) holds, then \((\sigma_z^n)^{b_0} = \sigma_z^n\). Then, the corresponding Hermitian operator \( O_p^n \) is as follows:

\[
O_p^n = \sigma_1^1 I^2 \sigma_2^1 \cdots \sigma_{n-2}^1 \sigma_{n-1}^1 I^n \times \sigma_1^2 \sigma_2^2 \cdots \sigma_{n-1}^2 \sigma_n^2
\]

\[
= (\sigma_1^1 I^1)^2 \sigma_2^1 (\sigma_1^2 I^1) \cdots \sigma_{n-2}^1 (\sigma_{n-1}^1 I^{n-1}) (\sigma_n^1 I^n)
\]

\[
= \sigma_1^1 I^2 \sigma_2^1 \sigma_3^1 \cdots \sigma_{n-2}^1 \sigma_{n-1}^1 \sigma_n^1.
\]

where the number of \((\sigma_z I) = \sigma_z\) is even because of the definition of \( b_n \).

**Example:** If \( p \in \{2^n-1, 2^n-1+1, \ldots, 2^n - 1\} \), then \( b_0 = 1 \) and \((\sigma_z^p)^{b_0} = \sigma_z^2\) for all \( j \). That is, the binary representation of \( p \) takes, for example, the following form:

\[
p_2 := 11001 \cdots 01,
\]

where \( B_2 \) represents the sum of the number of 1. Then, \( b_n \in \{0, 1\} \land b_n \equiv B_2 \mod 2 \) holds. Suppose \( b_n = 0 \) holds, then \((\sigma_z^n)^{b_0} = O_0^n\). Then the corresponding Hermitian operator \( O_p^n \) is as follows:

\[
O_p^n = \sigma_1^1 I^2 \sigma_2^1 \cdots \sigma_{n-2}^1 \sigma_{n-1}^1 I^n \times \sigma_1^2 \sigma_2^2 \cdots \sigma_{n-1}^2 \sigma_n^2
\]

\[
= (\sigma_1^1 I^1)^2 \sigma_2^1 (\sigma_1^2 I^1) \cdots \sigma_{n-2}^1 (\sigma_{n-1}^1 I^{n-1}) (\sigma_n^1 I^n)
\]

\[
= (i\sigma_y^1)^2 \sigma_2^1 \sigma_3^1 \cdots \sigma_{n-2}^1 (i\sigma_y^{n-1}) (\sigma_y^n),
\]

where the number of \((\sigma_z \sigma_x) = i\sigma_y\) is even.

**Example:** The binary representation of \( 2^n - 1 \) takes the following form

\[
2^{n-1} = 10000 \cdots 00.
\]

Then the corresponding Hermitian operator \( O_{2^{n-1}}^n \) is as follows:

\[
O_{2^{n-1}}^n = I^1 I^2 I^4 \cdots I^{n-1} I^n \times \sigma_1^1 \sigma_2^1 \sigma_3^1 \cdots \sigma_{n-1}^1 \sigma_n^1
\]

\[
= \sigma_1^1 \sigma_2^2 \sigma_3^3 \cdots \sigma_{n-1}^1 \sigma_n^1.
\]

**Lemma.** If \( O_p^n, O_q^n \in \Lambda_n \), then

\[
O_p^n O_q^n = O_{p \oplus q}^n (\in \Lambda_n),
\]

where \( p \oplus q \) is the bitwise XOR (exclusive OR) of \( p \) and \( q \).

**Proof.** See [A.1](#A.1) in Appendix A.

From the lemma (5.5), the set of \( 2^n \) operators \( \{O_p^n\} \) forms a commutative group isomorphic to \((\mathbb{Z}_2)^n\). We have denoted this commutative group as \( \Lambda_n \).

Let us define another set of operators.
Definition.  
\( R^n_p(p \in \{0, 1, \ldots, 2^n - 1\}, n \geq 2) \) are operators defined by

\[
R^n_p := \prod_{j=1}^{n} (\sigma^j_x)^{e_j} (\sigma^j_z)^{e_{n-j}}, \tag{5.9}
\]

where the superscript \( j \) of the Pauli operators denotes the party \( j \) and the \( n \)-bit sequence \( e_0 e_1 \cdots e_{n-1} \) is the binary representation of \( p \), and \( e_{n} \in \{0, 1\} \land e_{n} \equiv \sum_{j=1}^{n-1} e_j + 1 \pmod{2} \). Thus, unlike \( O^n_p \), the parity of \( e_1 e_2 \cdots e_{n} \) is odd.

Example: If \( p \in \{0, 1, \ldots, 2^{n-1} - 1\} \), then \( e_0 = 0 \) and \( (\sigma^j_z)^{e_{n-j}} = \sigma^j_z \) for all \( j \). That is, the binary representation of \( p \) takes, for example, the following form:

\[
p_3 := 01001 \cdots 01 (= e_0 e_1 \cdots e_{n-1}), \tag{5.10}
\]

where \( B_3 \) represents the sum of the number of 1. Then, \( e_{n} \in \{0, 1\} \land e_{n} \equiv B_3 + 1 \pmod{2} \) holds. Suppose \( e_{n} = 1 \) holds, then \( (\sigma^j_z)^{e_{n-j}} = \sigma^j_z \). Then, the corresponding Hermitian operator \( R^n_{p_3} \) is as follows:

\[
R^n_{p_3} = \sigma^1_z I^2 \sigma^4_z \cdots I^{n-2} \sigma^{n-1}_z \sigma^n_z \times I^1 I^2 I^3 I^4 \cdots I^{n-1} I^n
= (\sigma^1_z I^1) I^2 \sigma^4_z \cdots I^{n-2} (\sigma^{n-1}_z I^{n-1}) (\sigma^n_z I^n)
= \sigma^1_z I^2 I^3 \sigma^4_z \cdots I^{n-2} \sigma^{n-1}_z \sigma^n_z, \tag{5.11}
\]

where the number of \( (\sigma_z I) = \sigma_z \) is odd because of the definition of \( e_{n} \).

Example: If \( p \in \{2^{n-1}, 2^{n-1} + 1, \ldots, 2^n - 1\} \), then \( e_0 = 1 \) and \( (\sigma^j_z)^{e_{n-j}} = \sigma^j_z \) for all \( j \). That is, the binary representation of \( p \) takes, for example, the following form:

\[
p_4 := 11001 \cdots 01, \tag{5.12}
\]

where \( B_4 \) represents the sum of the number of 1. Then, \( e_{n} \in \{0, 1\} \land e_{n} \equiv B_4 + 1 \pmod{2} \) holds. Suppose \( e_{n} = 0 \) holds, then \( (\sigma^j_z)^{e_{n-j}} = O^n_z \). Then the corresponding non-Hermitian operator \( R^n_{p_4} \) is as follows:

\[
R^n_{p_4} = \sigma^1_z I^2 I^3 \sigma^4_z \cdots I^{n-2} \sigma^{n-1}_z I^n \times \sigma^1_z \sigma^2_z \sigma^3_z \cdots \sigma^{n-1}_z \sigma^n_z
= (\sigma^1_z \sigma^1_z) \sigma^2_z \sigma^3_z \cdots \sigma^{n-2}_z \sigma^{n-1}_z \sigma^n_z
= (i \sigma^1_y \sigma^2_z \sigma^3_z (i \sigma^4_y) \cdots \sigma^{n-2}_z (i \sigma^{n-1}_y) \sigma^n_z), \tag{5.13}
\]

where the number of \( (\sigma_z \sigma_x) = i \sigma_y \) is odd. \( R^n_p / i \) and \( i R^n_p \) are Hermitian operators if \( p \in \{2^{n-1}, 2^{n-1} + 1, \ldots, 2^n - 1\} \).

Lemma:

\[
\frac{1}{2} \left( \prod_{j=1}^{n} (I^j + \sigma^j_z) + \prod_{j=1}^{n} (I^j - \sigma^j_z) \right) = \sum_{p=0}^{2^{n-1}-1} O^n_p,
\]

\[
\frac{1}{2} \left( \prod_{j=1}^{n} (I^j + \sigma^j_z) - \prod_{j=1}^{n} (I^j - \sigma^j_z) \right) = \sum_{p=0}^{2^{n-1}-1} R^n_p.
\]
\[
\frac{1}{2} \left( \prod_{j=1}^{n} (\sigma_j^i + i\sigma_j^y) + \prod_{j=1}^{n} (\sigma_j^i - i\sigma_j^y) \right) = \sum_{p=2^{n-1}}^{2^n-1} O_p^n,
\]
\[
\frac{1}{2} \left( \prod_{j=1}^{n} (\sigma_j^i + i\sigma_j^y) - \prod_{j=1}^{n} (\sigma_j^i - i\sigma_j^y) \right) = \sum_{p=2^{n-1}}^{2^n-1} R_p^n. \tag{5.14}
\]

**Proof.** See (A.7) and (A.14) in Appendix A.

**Lemma.**

\[
HV \land FUNC \text{ A.E.} \Rightarrow \quad \text{Re} \left( \prod_{j=1}^{n} (f_{\sigma_j^i}(\omega) + if_{\sigma_j^y}(\omega)) \right) \approx \sum_{p=2^{n-1}}^{2^n-1} f_{O_p^n}(\omega),
\]
\[
\text{Im} \left( \prod_{j=1}^{n} (f_{\sigma_j^i}(\omega) + if_{\sigma_j^y}(\omega)) \right) \approx \sum_{p=2^{n-1}}^{2^n-1} f_{R_p^n}(\omega). \tag{5.15}
\]

**Proof.** See (A.21) in Appendix A.

**Theorem.** For every state \( \psi \) described in a Hilbert space \( \otimes_{j=1}^{n} H_j \), \((\dim(H_j) = 2, (j \in \mathbb{N}_n, n \geq 2)),\)

\[
HV \land JD (\land (i\sigma_j^y)(i\sigma_k^y) \sigma_j^i \sigma_k^i = \sigma_j^i \sigma_k^i)) \Rightarrow \quad \sum_{p=0}^{2^{n-1}-1} \text{Tr}[\psi O_p^n] \leq \begin{cases} 
2^{n/2} & \text{if } n = \text{even} \\
2^{(n-1)/2} & \text{if } n = \text{odd}. 
\end{cases} \tag{5.16}
\]

**Proof.** From PROD A.E. (2.7), we have

\[
(f_{\sigma_j^k}(\omega))^2 \simeq f_{O_p^n}(\omega) \simeq +1 \Leftrightarrow f_{\sigma_j^k}(\omega) \simeq \pm 1, (j \in \mathbb{N}_n, k = x, y). \tag{5.17}
\]

Now, we define \( F^\psi \) by

\[
F^\psi := \int_{\omega \in \Omega} \mu_\psi(d\omega) G(\omega), \tag{5.18}
\]

where \( G(\omega) \) is defined by

\[
G(\omega) := \text{Re} \left( \prod_{j=1}^{n} (f_{\sigma_j^i}(\omega) + if_{\sigma_j^y}(\omega)) \right) \prod_{j=1}^{n} f_{\sigma_j^k}(\omega). \tag{5.19}
\]

From the geometric argument by Mermin in Ref. [13] and (5.17), we have

\[
G(\omega) \leq \begin{cases} 
2^{n/2} & \text{if } n = \text{even} \\
2^{(n-1)/2} & \text{if } n = \text{odd} \quad (\mu_\psi - \text{a.e.}) . 
\end{cases} \tag{5.20}
\]

In more detail, almost everywhere with respect to \( \mu_\psi \) in \( \Omega \), the maximum of \( G(\omega) \) is equal to the real part of a product of complex numbers each of which has magnitude of \( \sqrt{2} \) and a phase of \( \pm \pi/4 \) or \( \pm 3\pi/4 \) since absolute value of \( \prod_{j=1}^{n} f_{\sigma_j^k}(\omega) \) is unity almost everywhere with respect to \( \mu_\psi \). When \( n \) is even the product can lie along the real axis and can attain a maximum value of \( 2^{n/2} \),
when \( n \) is odd the product must lie along an axis at 45° to the real axis and its real part can only attain the maximum value \( 2^{(n-1)/2} \). Therefore, the value \( G(\omega) \) is bounded as \( 2^{(n-1)/2} \) almost everywhere in \( \Omega \), and hence \( F^\psi \) is bounded as

\[
F^\psi \leq \begin{cases} 
2^{n/2} & n = \text{even} \\
2^{(n-1)/2} & n = \text{odd}.
\end{cases}
\]

From (5.28), it is easy to see that

\[
\prod_{j=1}^{n} f_{\sigma_j}^p(\omega) \simeq f_{O_{2n-1}^n}(\omega).
\]

Therefore, from (5.19) and the lemma (5.16), we have

\[
G(\omega) \simeq \left( \sum_{p=2^{n-1}}^{2^{n-1}-1} f_{O_p^p}(\omega) \right) f_{O_{2n-1}^n}(\omega).
\]

Noting \( [O_p^p, O_q^n] = 0, \forall O_p^p, O_q^n \in \Lambda_n \) (See the lemma 5.8), PROD A.E. 2.7 tells the following relations,

\[
f_{O_p^p}(\omega)f_{O_q^n}(\omega) \simeq f_{O_{p+q}^{n+1}}(\omega), (\forall O_p^p, O_q^n \in \Lambda_n).
\]

It is easy to see that

\[
\{ O_p^pO_{2n-1}^p | p \in \{ 2^{n-1}, 2^{n-1} + 1, \ldots, 2^n - 1 \} \} = \{ O_p^p | p \in \{ 0, 1, \ldots, 2^{n-1} - 1 \} \}.
\]

Here, we have used the quantum mechanical rule \( (i\sigma_i^y)(i\sigma_j^y)\sigma_i^z = \sigma_i^z\sigma_j^z (i, j \in \mathbb{N}, i \neq j) \). (Eq. (5.23) is also obvious from the expression (5.1) and (5.7)). Therefore, we get

\[
G(\omega) \simeq \sum_{p=0}^{2^{n-1}-1} f_{O_p^p}(\omega).
\]

Thus from (5.18) we conclude

\[
F^\psi = \int_{\omega \in \Omega} \mu_\omega(d\omega) \left( \sum_{p=0}^{2^{n-1}-1} f_{O_p^p}(\omega) \right) = \sum_{p=0}^{2^{n-1}-1} Tr[\psi O_p^p].
\]

QED.

Now, it follows from the lemma (5.14) that

\[
\langle +1; +2; \ldots; +n | +1; +2; \ldots; +n \rangle + \langle -1; -2; \ldots; -n | -1; -2; \ldots; -n \rangle
\]

\[
= \frac{1}{2^n} \left( \prod_{j=1}^{n} (I^j + \sigma_j^z) + \prod_{j=1}^{n} (I^j - \sigma_j^z) \right) = \frac{1}{2^{n-1}} \left( \sum_{p=0}^{2^{n-1}-1} O_p^n \right)
\]

where \( O_p^n = \prod_{j=1}^{n} (\sigma_j^x)^{b_0} (\sigma_j^y)^{b_1} \) and \( \sigma_j^z \pm = \pm_j \pm_j \). Hence we have

\[
F^\psi = \sum_{p=0}^{2^{n-1}-1} Tr[\psi O_p^n] = Tr[\psi H_n]
\]
where, (see (5.28))
\[
H_n := 2^{n-1}(|+1;+2;\cdots;++n\rangle\langle+1;+2;\cdots;++n| \\
+ |−1;−2;\cdots;−n\rangle\langle−1;−2;\cdots;−n|).
\] (5.30)

Now, let \(\psi\) be \(|\Psi\rangle\langle\Psi|\) where
\[
|\Psi\rangle = \alpha|+1;+2;\cdots;++n\rangle + \beta|−1;−2;\cdots;−n\rangle, (|\alpha|^2 + |\beta|^2 = 1). \tag{5.31}
\]

This state \(|\Psi\rangle\) is an uncorrelated state if \(\alpha\) or \(\beta\) is zero and \(|\Psi\rangle\) is an \(n\)-partite GHZ state if \(\alpha = \beta = 1/\sqrt{2}\).

The quantum theoretical prediction says the expectation value \(\text{Tr}[|\Psi\rangle\langle\Psi|H_n]\) should take a value of \(2^{n-1}\) for the state \(|\Psi\rangle\) in spite of any value of \(\alpha\) and of \(\beta\), and we get
\[
F(\Psi) = 2^{n-1}. \tag{5.32}
\]

When \(n\) exceeds 2, this value \(F(\Psi)\) is larger than the bound (5.21), which exceeds (5.21) by the exponentially larger factor of \(2^{(n-2)/2}\) (for \(n\) even) or \(2^{(n-1)/2}\) (for \(n\) odd). This implies demolition of HV or of JD (2.5) in the state \(|\Psi\rangle\). Thus, we have derived the exponentially stronger violation of HV\(\land\)JD (2.5), irrespective of quantum entanglement effects.

6 Summary

In summary, we showed that the fidelity to the Bell states which is larger than 1/2 is sufficient to allow a proof of the KS theorem. Thus, the Werner state is enough to test experimentally the KS theorem. We also have derived an \(n\)-partite inequality following from HV\(\land\)JD (2.5). We have shown that an \(n\)-partite uncorrelated state violates the inequality by a factor of \(O(2^{n/2})\) at the macroscopic level. Hence, it turns out that QM exhibits an exponentially stronger violation of HV\(\land\)JD (2.5), as the number of parties constituting the state increases, irrespective of entanglement effects.

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A Appendix A

Lemma. If \(O_p^n, O_q^n \in \Lambda_n\), then
\[
O_p^n O_q^n = O_{p \oplus q}^n \in \Lambda_n, \tag{A.1}
\]
where \(p \oplus q\) is the bitwise XOR (exclusive OR) of \(p\) and \(q\).
Proof. Suppose that the binary representations of \( p \) and \( q \) are \( b_0b_1\cdots b_{n-1} \) and \( c_0c_1\cdots c_{n-1} \), respectively. Suppose that \( b_n \in \{0,1\} \) and \( b_n \equiv \sum_{j=1}^{n-1} b_j (\text{mod} 2) \), and \( c_n \in \{0,1\} \) and \( c_n \equiv \sum_{j=1}^{n-1} c_j (\text{mod} 2) \) hold. This means that

\[
c_j \in \{0,1\} \forall j \land \sum_{j=1}^{n} c_j \equiv 0 (\text{mod} 2). \tag{A.2}
\]

This yields \( (b_0 \in \{1,0\}) \)

\[
\prod_{j=1}^{n} \left[ (\sigma_x^j)^{b_0} \cdot \prod_{j=1}^{n} (\sigma_x^j)^{c_j} \right] = 0. \tag{A.3}
\]

Then from (5.1) we have

\[
O_n^p = \prod_{j=1}^{n} (\sigma_x^j)^{b_0} \cdot \prod_{j=1}^{n} (\sigma_x^j)^{c_j}, O_n^q = \prod_{j=1}^{n} (\sigma_x^j)^{c_j} (\sigma_x^j)^{c_0}
\]

\[
\Rightarrow O_n^p O_n^q = \prod_{j=1}^{n} (\sigma_x^j)^{b_0} (\sigma_x^j)^{c_j} (\sigma_x^j)^{c_0}
\]

\[
= \prod_{j=1}^{n} (\sigma_x^j)^{b_j} (\sigma_x^j)^{c_j} (\sigma_x^j)^{c_0}
\]

\[
= \prod_{j=1}^{n} (\sigma_x^j)^{d_j} (\sigma_x^j)^{d_0} = O_{n \oplus q}, \tag{A.4}
\]

where

\[
d_j \in \{0,1\} \land d_j \equiv b_j + c_j (\text{mod} 2). \tag{A.5}
\]

Here,

\[
d_n \equiv \sum_{j=1}^{n-1} b_j + \sum_{j=1}^{n-1} c_j (\text{mod} 2)
\]

\[
\equiv \sum_{j=1}^{n-1} (b_j + c_j) (\text{mod} 2)
\]

\[
\equiv \sum_{j=1}^{n-1} d_j (\text{mod} 2). \tag{A.6}
\]

Hence, \( d_n \) can be assumed such that \( d_n \in \{0,1\} \) and \( d_n \equiv \sum_{j=1}^{n-1} d_j (\text{mod} 2) \) hold.

QED.

Lemma.

\[
\frac{1}{2} \left( \prod_{j=1}^{n} (I^j + \sigma_z^j) + \prod_{j=1}^{n} (I^j - \sigma_z^j) \right) = \sum_{p=0}^{2^{n-1}-1} O^n_p,
\]

\[
\frac{1}{2} \left( \prod_{j=1}^{n} (I^j + \sigma_z^j) - \prod_{j=1}^{n} (I^j - \sigma_z^j) \right) = \sum_{p=0}^{2^{n-1}-1} R^n_p. \tag{A.7}
\]
Proof. If the following relations hold for all \( m, (2 \leq m \leq n) \),

\[
\frac{1}{2} \left( \prod_{j=1}^{m} (I^j + \sigma_j^z) + \prod_{j=1}^{m} (I^j - \sigma_j^z) \right) = \sum_{p=0}^{2^{m-1}-1} O_p^m, \tag{A.8}
\]

\[
\frac{1}{2} \left( \prod_{j=1}^{m} (I^j + \sigma_j^z) - \prod_{j=1}^{m} (I^j - \sigma_j^z) \right) = \sum_{p=0}^{2^{m-1}-1} R_p^m, \tag{A.9}
\]

then the theorem holds. Here, \( O_p^m \) means \( O_p^m \otimes_{j=m+1}^n I^j \) and so on. Omitting the identity operator, we abbreviate those as above. Remember, \( \sigma_1^k \) means \( \sigma_j^z \otimes_{j=2}^n I^j \) and so on.

In the case where \( m = 2 \), LHS of (A.8) is \((I^1I^2 + \sigma_2^z \sigma_3^z) \otimes_{j=3}^n I^j\) and RHS of (A.8) is also \((I^1I^2 + \sigma_2^z \sigma_3^z) \otimes_{j=3}^n I^j\). LHS of (A.9) is \((I^1 \sigma_2^z + \sigma_1^z I^2) \otimes_{j=3}^n I^j\) and RHS of (A.9) is also \((I^1 \sigma_2^z + \sigma_1^z I^2) \otimes_{j=3}^n I^j\). Therefore (A.8) and (A.9) hold when \( m = 2 \). In the following, if possible, we omit the identity operator.

Suppose that (A.8) and (A.9) hold for \( m = k - 1 \). Then we have

\[
\prod_{j=1}^{k-1} (I^j + \sigma_j^z) = \sum_{p=0}^{2^{k-2}-1} O_p^{k-1} + \sum_{p=0}^{2^{k-2}-1} R_p^{k-1},
\]

\[
\prod_{j=1}^{k-1} (I^j - \sigma_j^z) = \sum_{p=0}^{2^{k-2}-1} O_p^{k-1} - \sum_{p=0}^{2^{k-2}-1} R_p^{k-1}. \tag{A.10}
\]

On the other hand, we have

\[
(I^k + \sigma_z^k) \left( \sum_{p=0}^{2^{k-2}-1} O_p^{k-1} + \sum_{p=0}^{2^{k-2}-1} R_p^{k-1} \right)
\]

\[
= \left( \sum_{p=0}^{2^{k-2}-1} O_p^{k-1} I^k + \sum_{p=0}^{2^{k-2}-1} R_p^{k-1} \sigma_z^k \right) + \left( \sum_{p=0}^{2^{k-2}-1} O_p^{k-1} \sigma_z^k + \sum_{p=0}^{2^{k-2}-1} R_p^{k-1} I_z^k \right)
\]

\[
= \sum_{p=0}^{2^{k-1}-1} O_p^k + \sum_{p=0}^{2^{k-1}-1} R_p^k, \tag{A.11}
\]

and

\[
(I^k - \sigma_z^k) \left( \sum_{p=0}^{2^{(k-1)-1}-1} O_p^{k-1} - \sum_{p=0}^{2^{(k-1)-1}-1} R_p^{k-1} \right)
\]

\[
= \left( \sum_{p=0}^{2^{k-2}-1} O_p^{k-1} I^k + \sum_{p=0}^{2^{k-2}-1} R_p^{k-1} \sigma_z^k \right) - \left( \sum_{p=0}^{2^{k-2}-1} O_p^{k-1} \sigma_z^k + \sum_{p=0}^{2^{k-2}-1} R_p^{k-1} I_z^k \right)
\]

\[
= \sum_{p=0}^{2^{k-1}-1} O_p^k - \sum_{p=0}^{2^{k-1}-1} R_p^k. \tag{A.12}
\]

Therefore we have

\[
\prod_{j=1}^{k} (I^j + \sigma_j^z) = \sum_{p=0}^{2^{k-1}-1} O_p^k + \sum_{p=0}^{2^{k-1}-1} R_p^k,
\]

which completes the proof.

\[\text{\textcopyright\textregistered\texttrademark\textemdash\textregistered\textcopyright\textregistered\textregistered\textcopyright\textregistered}\]
\[
\prod_{j=1}^{k} (I^j - \sigma_z^j) = \sum_{p=0}^{2^{k-1} - 1} O_p^k - \sum_{p=0}^{2^{k-1} - 1} R_p^k. \tag{A.13}
\]

This implies that (A.8) and (A.9) hold for \( m = k \). QED.

**Lemma.**

\[
\frac{1}{2} \left( \prod_{j=1}^{n} (\sigma_x^j + i\sigma_y^j) + \prod_{j=1}^{n} (\sigma_x^j - i\sigma_y^j) \right) = \sum_{p=2^{n-1}}^{2^n-1} O_p^n, \tag{A.14}
\]

\[
\frac{1}{2} \left( \prod_{j=1}^{n} (\sigma_x^j + i\sigma_y^j) - \prod_{j=1}^{n} (\sigma_x^j - i\sigma_y^j) \right) = \sum_{p=2^{n-1}}^{2^n-1} R_p^n. \tag{A.15}
\]

**Proof.** If the following relations hold for all \( m, (2 \leq m \leq n) \),

\[
\frac{1}{2} \left( \prod_{j=1}^{m} (\sigma_x^j + i\sigma_y^j) + \prod_{j=1}^{m} (\sigma_x^j - i\sigma_y^j) \right) = \sum_{p=2^{m-1}}^{2^m - 1} O_p^m, \tag{A.15}
\]

\[
\frac{1}{2} \left( \prod_{j=1}^{m} (\sigma_x^j + i\sigma_y^j) - \prod_{j=1}^{m} (\sigma_x^j - i\sigma_y^j) \right) = \sum_{p=2^{m-1}}^{2^m - 1} R_p^m, \tag{A.16}
\]

then the theorem holds. Here, \( O_p^m \) means \( O_p^m \otimes \gamma_j^{m+1} I^j \) and so on.

In the case where \( m = 2 \): LHS of (A.15) is \( (\sigma_x^1 \sigma_x^2 + i\sigma_y^1 \sigma_y^2) \otimes I^3 \) and RHS of (A.15) is also \( (\sigma_x^1 \sigma_x^2 + i\sigma_y^1 \sigma_y^2) \otimes I^3 \). LHS of (A.16) is \( (\sigma_x^1 \sigma_x^2 + i\sigma_y^1 \sigma_y^2) \otimes I^3 \) and RHS of (A.16) is also \( (\sigma_x^1 \sigma_x^2 + i\sigma_y^1 \sigma_y^2) \otimes I^3 \). Therefore (A.15) and (A.16) hold when \( m = 2 \). In the following, if possible, we omit the identity operator.

Suppose that (A.15) and (A.16) hold for \( m = k - 1 \). Then we have

\[
\prod_{j=1}^{k-1} (\sigma_x^j + i\sigma_y^j) = \sum_{p=2^{k-2}}^{2^{k-1} - 1} O_p^{k-1} + \sum_{p=2^{k-2}}^{2^{k-1} - 1} R_p^{k-1}, \tag{A.17}
\]

On the other hand, we have

\[
(\sigma_x^k + i\sigma_y^k) \left( \sum_{p=2^{k-2}}^{2^{k-1} - 1} O_p^{k-1} + \sum_{p=2^{k-2}}^{2^{k-1} - 1} R_p^{k-1} \right)
\]

\[
= \left( \sum_{p=2^{k-2}}^{2^{k-1} - 1} O_p^{k-1} \sigma_x^k + \sum_{p=2^{k-2}}^{2^{k-1} - 1} R_p^{k-1} i\sigma_y^k \right) + \left( \sum_{p=2^{k-2}}^{2^{k-1} - 1} O_p^{k-1} i\sigma_y^k + \sum_{p=2^{k-2}}^{2^{k-1} - 1} R_p^{k-1} \sigma_x^k \right)
\]

\[
= \sum_{p=2^{k-2}}^{2^{k-1} - 1} O_p^k + \sum_{p=2^{k-2}}^{2^{k-1} - 1} R_p^k \tag{A.18}
\]

and

\[
(\sigma_x^k - i\sigma_y^k) \left( \sum_{p=2^{k-2}}^{2^{k-1} - 1} O_p^{k-1} - \sum_{p=2^{k-2}}^{2^{k-1} - 1} R_p^{k-1} \right)
\]
Therefore we have

\[
\prod_{j=1}^{k} (\sigma_x^j + i\sigma_y^j) = \sum_{p=2^{k-1}}^{2^{k-1} - 1} O_p^k + \sum_{p=2^{k-1}}^{2^{k-1}} R_p^k,
\]

\[
\prod_{j=1}^{k} (\sigma_x^j - i\sigma_y^j) = \sum_{p=2^{k-1}}^{2^{k-1} - 1} O_p^k - \sum_{p=2^{k-1}}^{2^{k-1}} R_p^k.
\] (A.20)

This implies that (A.15) and (A.19) hold for \(m = k\). QED.

Lemma.

HV \land \text{FUNC A.E.} \Rightarrow 
Re \left( \prod_{j=1}^{n} (f_{\sigma_x^j}(\omega) + i f_{\sigma_y^j}(\omega)) \right) \simeq \sum_{p=2^{n-1}}^{2^n - 1} f_{O_p}^n(\omega),

Im \left( \prod_{j=1}^{n} (f_{\sigma_x^j}(\omega) + i f_{\sigma_y^j}(\omega)) \right) \simeq \sum_{p=2^{n-1}}^{2^n - 1} f_{R_p}^{n/4}(\omega). \tag{A.21}

Proof. If the following relations hold for all \(m, (2 \leq m \leq n)\),

\[
\text{Re} \left( \prod_{j=1}^{m} (f_{\sigma_x^j}(\omega) + i f_{\sigma_y^j}(\omega)) \right) \simeq \sum_{p=2^{m-1}}^{2^m - 1} f_{O_p}^m(\omega), \tag{A.22}
\]

\[
\text{Im} \left( \prod_{j=1}^{m} (f_{\sigma_x^j}(\omega) + i f_{\sigma_y^j}(\omega)) \right) \simeq \sum_{p=2^{m-1}}^{2^m - 1} f_{R_p}^{m/4}(\omega), \tag{A.23}
\]

then the theorem holds. Here, \(f_{O_p}^m\) means \(f_{O_p}^m \otimes^{n}_{i=m+1} \omega_i\) and so on.

In the case where \(m = 2\): LHS of (A.22) is \(f_{(\sigma_1^1 \sigma_2^1) \otimes^{n}_{i=3} \omega_i} + f_{(i \sigma_1^1 \sigma_2^1) \otimes^{n}_{i=3} \omega_i}\) almost everywhere and RHS of (A.22) is \(f_{(\sigma_1^1 \sigma_2^1) \otimes^{n}_{i=3} \omega_i} + f_{(i \sigma_1^1 \sigma_2^1) \otimes^{n}_{i=3} \omega_i}\). LHS of (A.23) is \(f_{(\sigma_1^1 \sigma_2^1) \otimes^{n}_{i=3} \omega_i} + f_{(\sigma_1^1 \sigma_2^1) \otimes^{n}_{i=3} \omega_i}\) almost everywhere and RHS of (A.23) is \(f_{(\sigma_1^1 \sigma_2^1) \otimes^{n}_{i=3} \omega_i} + f_{(\sigma_1^1 \sigma_2^1) \otimes^{n}_{i=3} \omega_i}\). Therefore (A.22) and (A.23) hold when \(m = 2\).

Here, we have used PROD A.E. \(\text{HV} \land \text{FUNC A.E.} \Rightarrow \). In the following, if possible, we omit the identity operator.

Suppose that (A.22) and (A.23) hold for \(m = k - 1\). Then we have

\[
\prod_{j=1}^{k-1} (f_{\sigma_x^j} + i f_{\sigma_y^j}) \simeq \sum_{p=2^{k-2}}^{2^{k-1} - 1} f_{O_p}^{k-1} + \sum_{p=2^{k-2}}^{2^{k-1}} i f_{R_p}^{k-1/4},
\]

\[
\prod_{j=1}^{k-1} (f_{\sigma_x^j} - i f_{\sigma_y^j}) \simeq \sum_{p=2^{k-2}}^{2^{k-1} - 1} f_{O_p}^{k-1} - \sum_{p=2^{k-2}}^{2^{k-1}} i f_{R_p}^{k-1/4}. \tag{A.24}
\]
and so on. Hence, we have

\[(f_{\sigma^k} + if_{\sigma^k})(\sum_{p=2^{k-2}}^{2^k-1} f_{O^p_{k-1}} - \sum_{p=2^{k-2}}^{2^k-1} if_{R^p_{k-1}/i})\]

\[= \left(\sum_{p=2^{k-2}}^{2^k-1} f_{O^p_{k-1}} f_{\sigma^k} + \sum_{p=2^{k-2}}^{2^k-1} if f_{R^p_{k-1}/i} f_{\sigma^k} \right) + \left(\sum_{p=2^{k-2}}^{2^k-1} f_{O^p_{k-1}} if f_{\sigma^k} + \sum_{p=2^{k-2}}^{2^k-1} if f_{R^p_{k-1}/i} f_{\sigma^k} \right)\]

\[\simeq \left(\sum_{p=2^{k-2}}^{2^k-1} f_{O^p_{k-1}} - \sum_{p=2^{k-2}}^{2^k-1} if_{R^p_{k-1}/i} \right)\]

\[= \sum_{p=2^{k-2}}^{2^k-1} f_{O^p_{k-1}} + \sum_{p=2^{k-2}}^{2^k-1} if_{R^p_{k-1}/i} \]  \hspace{1cm} \text{(A.27)}

and

\[(f_{\sigma^k} - if_{\sigma^k})(\sum_{p=2^{k-2}}^{2^k-1} f_{O^p_{k-1}} - \sum_{p=2^{k-2}}^{2^k-1} if_{R^p_{k-1}/i})\]

\[= \left(\sum_{p=2^{k-2}}^{2^k-1} f_{O^p_{k-1}} f_{\sigma^k} + \sum_{p=2^{k-2}}^{2^k-1} if f_{R^p_{k-1}/i} f_{\sigma^k} \right) - \left(\sum_{p=2^{k-2}}^{2^k-1} f_{O^p_{k-1}} if f_{\sigma^k} + \sum_{p=2^{k-2}}^{2^k-1} if f_{R^p_{k-1}/i} f_{\sigma^k} \right)\]

\[\simeq \left(\sum_{p=2^{k-2}}^{2^k-1} f_{O^p_{k-1}} - \sum_{p=2^{k-2}}^{2^k-1} if_{R^p_{k-1}/i} \right)\]

\[= \sum_{p=2^{k-2}}^{2^k-1} f_{O^p_{k-1}} - \sum_{p=2^{k-2}}^{2^k-1} if_{R^p_{k-1}/i} \]  \hspace{1cm} \text{(A.28)}

Therefore we have

\[\prod_{j=1}^{k} (f_{\sigma^k_j} + if_{\sigma^k_j}) \simeq \sum_{p=2^{k-1}}^{2^k-1} f_{O^p_{k-1}} + \sum_{p=2^{k-1}}^{2^k-1} if_{R^p_{k-1}/i},\]

\[\prod_{j=1}^{k} (f_{\sigma^k_j} - if_{\sigma^k_j}) \simeq \sum_{p=2^{k-1}}^{2^k-1} f_{O^p_{k-1}} - \sum_{p=2^{k-1}}^{2^k-1} if_{R^p_{k-1}/i}.\]  \hspace{1cm} \text{(A.29)}

This implies that \text{(A.22)} and \text{(A.28)} hold for \(m = k\). QED.
Appendix B

Lemma.
Let $S_A$ stand for the spectrum of the Hermitian operator $A$. If
\[
Tr[\psi A] = \sum_{y \in S_A} \text{Prob}(\{y\})_{\theta(A)}^\psi y,
\]
\[
E_\psi(A) := \int_{\omega \in \Omega} \mu_\psi(d\omega)f_A(\omega),
\]
then
\[
HV \land D \Rightarrow Tr[\psi A] = E_\psi(A). \tag{B.1}
\]

Proof. Note
\[
\omega \in f_A^{-1}(\{y\}) \iff f_A(\omega) \in \{y\} \iff y = f_A(\omega),
\]
\[
\int_{\omega \in f_A^{-1}(\{y\})} \frac{\mu_\psi(d\omega)}{\mu_\psi(f_A^{-1}(\{y\}))} = 1,
\]
\[
y \neq y' \Rightarrow f_A^{-1}(\{y\}) \cap f_A^{-1}(\{y'\}) = \phi. \tag{B.2}
\]
Hence we have
\[
Tr[\psi A] = \sum_{y \in S_A} \text{Prob}(\{y\})_{\theta(A)}^\psi y = \sum_{y \in \mathbb{R}} \text{Prob}(\{y\})_{\theta(A)}^\psi y = \sum_{y \in \mathbb{R}} \mu_\psi(f_A^{-1}(\{y\})) y
\]
\[
= \sum_{y \in \mathbb{R}} \mu_\psi(f_A^{-1}(\{y\})) y \times \int_{\omega \in f_A^{-1}(\{y\})} \frac{\mu_\psi(d\omega)}{\mu_\psi(f_A^{-1}(\{y\}))} f_A(\omega)
\]
\[
= \sum_{y \in \mathbb{R}} \int_{\omega \in f_A^{-1}(\{y\})} \mu_\psi(f_A^{-1}(\{y\})) \times \frac{\mu_\psi(d\omega)}{\mu_\psi(f_A^{-1}(\{y\}))} f_A(\omega)
\]
\[
= \int_{\omega \in \Omega} \mu_\psi(d\omega)f_A(\omega) = E_\psi(A). \tag{B.3}
\]
QED.

Lemma.
\[
\chi_\Delta(g(x)) = \chi_{g^{-1}(\Delta)}(x), (x \in \mathbb{R})
\]
and
\[
\text{Prob}(\Delta)_{\theta(g(A))}^\psi = Tr[\psi \chi_\Delta(g(A))]
\]
\[
= Tr[\psi \chi_{g^{-1}(\Delta)}(A)] = \text{Prob}(g^{-1}(\Delta))_{\theta(A)}^\psi. \tag{B.4}
\]

Proof. Obvious.

Lemma.
\[
\text{QJD} \Rightarrow \text{BSF},
\]
\[
HV \land \text{JD} \Rightarrow HV \land D. \tag{B.5}
\]

Proof. Obvious.
Theorem\[6\].

\[ \text{HV} \land \text{JD} \Rightarrow \text{HV} \land \text{D} \land \text{FUNC A.E.}. \quad (B.6) \]

Proof. Suppose JD holds. Let \( y \) be any real number, and let \( S := \{ \omega | f_g(A) = y \} \) and \( T := \{ \omega | g(f_A(\omega)) = y \} \). We want \( \mu_P(S \cap T) = \mu_P(S \cap T) = 0 \). This is valid if we have \( \mu_P(S) = \mu_P(T) = \mu_P(S \cap T) \) since

\[
\begin{align*}
\mu_P(S \cap T) + \mu_P(S \cap T) &= \mu_P(S), \\
\mu_P(S \cap T) + \mu_P(S \cap T) &= \mu_P(T).
\end{align*}
\]

(B.7)

Note

\[
\omega \in f_g^{-1}(\{y\}) \Leftrightarrow f_g(A)(\omega) \in \{y\} \Leftrightarrow y = f_g(A)(\omega)
\]

(B.8)

and

\[
\omega \in f_A^{-1}(g^{-1}(\{y\})) \Leftrightarrow f_A(\omega) \in g^{-1}(\{y\}) \Leftrightarrow g(f_A(\omega)) \in \{y\} \Leftrightarrow y = g(f_A(\omega)).
\]

(B.9)

The lemma \[B.3\] says that JD yields D. Then, from the lemma \[B.4\], we have

\[
\mu_P(T) = \mu_P(\{\omega | \omega \in f_A^{-1}(g^{-1}(\{y\}))\}) = \text{Prob}(g^{-1}(\{y\}))_{\theta(A)}
\]

(B.10)

Using the spectral representation of \( A \), it follows that \( \chi(A) \chi(g(A)) = \chi(A) \) for any set \( \Delta \), where \( g(\Delta) = \{g(x) | x \in \Delta \} \). Because, \( \chi(A) = 1 \Leftrightarrow z \in \Delta \Rightarrow g(z) \in g(\Delta) \Leftrightarrow \chi(g(A)(z)) = 1 \) holds (\( z \in \mathbb{R} \)). Hence,

\[
\text{Prob}(\Delta, g(\Delta))_{\theta(A), \theta(g(A))} = \text{Tr}[\psi \chi(A) \chi(g(A))] = \text{Tr}[\psi \chi(A)] = \text{Prob}(\Delta)_{\theta(A)}.
\]

(B.11)

On the other hand, we have \( g(g^{-1}(\Delta)) = \Delta \) because \( g(g^{-1}(\Delta)) = \{g(x) | x \in g^{-1}(\Delta)\} = \{g(x)g(x) \in \Delta \} = \Delta \). Therefore, on substituting \( g^{-1}(\{y\}) \) into \( \Delta \), we have

\[
\text{Prob}(g^{-1}(\{y\}), \{y\})_{\theta(A), \theta(g(A))} = \text{Prob}(g^{-1}(\{y\}))_{\theta(A)} = \mu_P(T).
\]

(B.12)

But, from JD we have

\[
\text{Prob}(g^{-1}(\{y\}), \{y\})_{\theta(A), \theta(g(A))}
\]

(B.13)

QED.

Theorem\[6\].

\[ \text{HV} \land \text{FUNC A.E.} \Rightarrow \text{HV} \land \text{PROD A.E.}. \quad (B.14) \]

Proof. Suppose that \( A \) and \( B \) are two commuting Hermitian operators. This means that there exists a basis \( \{P_i\} \) by which we can expand \( A = \sum_i a_i P_i \), and
such that $B$ can also be expanded in the form $B = \sum_i b_i P_i$. Now construct a Hermitian operator $O := \sum_i o_i P_i$ with real numbers $o_i$. None of them is equal. Namely, $O$ is assumed to be nondegenerate by construction. Let us define functions $j$ and $k$ by $j(o_i) := a_i$ and $k(o_i) := b_i$, respectively. Then we can see that if $A$ and $B$ commute, there exists a nondegenerate Hermitian operator $O$ such that $A = j(O)$ and $B = k(O)$. Therefore, we can introduce a function $h$ such that $AB = h(O)$ where $h := j \cdot k$. So we have the following:

$$f_{AB}(\omega) = f_{h(O)}(\omega) \simeq h(f_O(\omega)) = j(f_O(\omega)) \cdot k(f_O(\omega))$$

$$\simeq f_{j(O)}(\omega) \cdot f_{k(O)}(\omega) = f_A(\omega) \cdot f_B(\omega),$$

where FUNC A.E. (2.6) has been used. QED.

Lemma [7]. If

$$\mu(\mathbf{S} \cap \mathbf{S}') = \mu(\overline{\mathbf{S}} \cap \mathbf{S}) = \mu(\overline{T} \cap \mathbf{T}') = \mu(\overline{T'} \cap \mathbf{T}) = 0,$$

then

$$\mu(\mathbf{S} \cap \mathbf{T}) = \mu(\mathbf{S}' \cap \mathbf{T}).$$

(B.16)

Proof. Note

$$\mu(\mathbf{S} \cap \mathbf{S}' \cap \mathbf{T}) + \mu(\mathbf{S} \cap \mathbf{S}' \cap \mathbf{T}) = \mu(\overline{\mathbf{S}} \cap \mathbf{S} \cap \mathbf{T}),$$

$$\mu(\overline{\mathbf{S}} \cap \mathbf{S} \cap \mathbf{T}) = \mu(\mathbf{S} \cap \mathbf{T} \cap \mathbf{T}).$$

(B.17)

If the following relation holds

$$\mu(\mathbf{S} \cap \mathbf{S}') = \mu(\overline{\mathbf{S}} \cap \mathbf{S}) = 0,$$

then

$$\mu(\mathbf{S} \cap \mathbf{S}' \cap \mathbf{T}) = \mu(\overline{\mathbf{S}} \cap \mathbf{S} \cap \mathbf{T}) = 0.$$ (B.18)

Therefore, from (B.17), we have

$$\mu(\mathbf{S} \cap \mathbf{S}' \cap \mathbf{T}) = \mu(\overline{\mathbf{S}} \cap \mathbf{S} \cap \mathbf{T}) = \mu(\mathbf{S} \cap \mathbf{T}).$$ (B.20)

Similar to the argument by changing $\mathbf{S}$ to $\mathbf{T}$, $\mathbf{S}'$ to $\mathbf{T}'$, and $\mathbf{T}$ to $\mathbf{S}'$, we get

$$\mu(\mathbf{T} \cap \mathbf{T}' \cap \mathbf{S}') = \mu(\overline{\mathbf{T}} \cap \mathbf{T} \cap \mathbf{S}') = \mu(\mathbf{T} \cap \mathbf{S}').$$

(B.21)

From the relations (B.20) and (B.21), we conclude

$$\mu(T \cap \mathbf{S}) = \mu(T' \cap \mathbf{S}).$$

(B.22)

QED.

Lemma.

HV $\land$ PROD A.E. $\Rightarrow f_{\chi\Delta} (\omega) \in \{0,1\}, (\mu - a.e.).$

(B.23)

Proof. Obvious.

Theorem [7].

$$HV \land D \land PROD A.E. \Rightarrow HV \land JD.$$ (B.24)

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Proof. Suppose \([A, B] = 0\) holds. It follows from QJD \((2.2)\), BSF \((2.1)\), and D \((2.4)\) that

\[
Prob(\Delta, \Delta')_{\theta(A), \theta(B)}^\psi
= Tr[\psi\chi_\Delta(A)\chi_{\Delta'}(B)] (see \((2.2)\))
= Tr[\psi\chi_{\{1\}}(\chi_\Delta(A)\chi_{\Delta'}(B))]
= Prob(\{1\})_{\theta(\chi_\Delta(A)\chi_{\Delta'}(B))}^\psi (see \((2.3)\))
= \mu_\psi(f_{\chi_\Delta(A)\chi_{\Delta'}(B)}^{-1}(\{1\})) (see \((2.5)\)).
\] (B.25)

PROD A.E. \((2.7)\) and the lemma \((B.23)\) say that

\[
\begin{align*}
\text{On the other hand, we have} & \quad \mu_\psi(f_{\chi_\Delta(A)}^{-1}(\{1\}) \cap f_A^{-1}(\Delta)) \\
& = \mu_\psi(\{\omega | f_{\chi_\Delta(A)}(\omega) = 1 \land f_A(\omega) \in \Delta\}) \\
& = \mu_\psi(\{\omega | f_{\chi_\Delta(A)}(\omega) \cdot f_A(\omega) \in \Delta\}) \\
& = \mu_\psi(\{\omega | f_{\chi_\Delta(A)}(\omega) = 1\}) (see \((2.7)\)) \\
& = \mu_\psi(f_{\chi_\Delta(A)}^{-1}(\{1\}) \cap f_{\chi_\Delta(A)}^{-1}(\{1\})).
\end{align*}
\] (B.26)

We also obtain

\[
\mu_\psi(f_{\chi_\Delta(A)}^{-1}(\{1\})) = Tr[\psi\chi_{\{1\}}(\chi_\Delta(A))] = Tr[\psi\chi_\Delta(A)] = \mu_\psi(f_A^{-1}(\Delta)).
\] (B.28)

Note, (see \((B.7)\))

\[
\begin{align*}
\mu_\psi(S \cap T) &= \mu_\psi(S) = \mu_\psi(T) \iff \mu_\psi(S \cap \overline{T}) = \mu_\psi(\overline{S} \cap T) = 0.
\end{align*}
\] (B.29)

Therefore, from Eq. \((B.27)\) and Eq. \((B.28)\), we have

\[
\mu_\psi(f_{\chi_\Delta(A)}^{-1}(\{1\}) \cap f_A^{-1}(\Delta)) = \mu_\psi(f_{\chi_\Delta(A)}^{-1}(\{1\}) \cap f_A^{-1}(\Delta)) = 0.
\] (B.30)

Similarly we can get

\[
\begin{align*}
\mu_\psi(f_{\chi_\Delta(B)}^{-1}(\{1\}) \cap f_B^{-1}(\Delta')) &= Tr[\psi\chi_{\Delta'}(B)] \\
\mu_\psi(f_{\chi_{\Delta'}(B)}^{-1}(\{1\})) &= \mu_\psi(f_B^{-1}(\Delta')) = Tr[\psi\chi_{\Delta'}(B)],
\end{align*}
\] (B.31)

and we have

\[
\mu_\psi(f_{\chi_{\Delta'}(B)}^{-1}(\{1\}) \cap f_B^{-1}(\Delta')) = \mu_\psi(f_{\chi_{\Delta'}(B)}^{-1}(\{1\}) \cap f_B^{-1}(\Delta')) = 0.
\] (B.32)
Hence, from the lemma \((B.16)\), we have

\[
\mu_\psi(f^{-1}_\Delta(A)(\{1\}) \cap f^{-1}_\Delta'(B)(\{1\})) = \mu_\psi(f^{-1}_A(A) \cap f^{-1}_B(A')).
\] (B.33)

Therefore, from \((B.26)\), we conclude

\[
\text{Prob}^\psi(\Delta, \Delta')_{\theta(A), \theta(B)} = \mu_\psi(f^{-1}_A(A) \cap f^{-1}_B(A')),
\] (B.34)

which is JD \((2.5)\). QED.

**Theorem** \((B.35)\).

\[
\text{HV } \wedge \text{D (2.4)} \wedge \text{FUNC A.E. (2.6)} \Rightarrow \text{HV } \wedge \text{JD (2.5)}.
\] (B.35)

**Proof.** Suppose \([A, B] = 0\) holds. It follows from BSF \((2.1)\), QJD \((2.2)\), D \((2.4)\), FUNC A.E. \((2.6)\), and PROD A.E. \((2.7)\) that

\[
\text{Prob}^\psi(\Delta, \Delta')_{\theta(A), \theta(B)}
\]

\[
= \text{Tr}[\psi \chi_\Delta(A) \chi_\Delta'(B)] (\text{see } (2.2))
\]

\[
= \text{Tr}[\psi \chi_{\{1\}}(\chi_\Delta(A) \chi_\Delta'(B))]
\]

\[
= \text{Prob}(\{1\})^\psi_{\theta(A), \theta(B)} (\text{see } (2.4))
\]

\[
= \mu_\psi(f^{-1}_A(A) \chi_\Delta'(B)(\{1\})) (\text{see } (2.4))
\]

\[
= \mu_\psi(\{\omega | \omega \in f^{-1}_A(A) \chi_\Delta'(B)(\{1\})\})
\]

\[
= \mu_\psi(\{\omega | f_A(A)(\omega) = 1\})
\]

\[
= \mu_\psi(\{\omega | f_A(A)(\omega) = 1\} \cap f_B(B)(\omega) = 1\}) (\text{see } (2.4))
\]

\[
= \mu_\psi(\{\omega | f_A(A)(\omega) = 1\} \cap f_B(B)(\omega) = 1\})
\]

\[
= \mu_\psi(\{\omega | f_A(A)(\omega) = 1\} \cap f_B(B)(\omega) = 1\})
\]

\[
= \mu_\psi(f^{-1}_A(A) \cap f^{-1}_B(A')).
\] (B.36)

QED.

Now we summarize the inclusion relation as follows:

\[
\text{HV } \wedge \text{JD (2.5)} \iff \text{HV } \wedge \text{D (2.4)} \wedge \text{FUNC A.E. (2.6)} \iff \text{HV } \wedge \text{D (2.4)} \wedge \text{PROD A.E. (2.7)}.
\] (B.37)

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[21] We know that every proposition is true if the presupposition is false (see (3.6)). Therefore, one might think that theorem (4.2) and theorem (5.16) are trivial. However, this is not the matter of our argument. We have used a quantum mechanical rule \( \sigma_x^1 \sigma_y^1 \sigma_z^1 \sigma_x^2 \sigma_y^2 \sigma_z^2 = -I \) in the proof of the theorem (3.6). But, we have not used the quantum mechanical rule in the proof of the theorem (1.2). Likewise, a quantum mechanical rule \( \sigma_x^1 \sigma_y^0 \sigma_z^0 \sigma_x^1 \sigma_y^1 \sigma_z^1 = -I \)

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\[ I, (i, j \in \mathbb{N}_n, i \neq j) \text{ is needless to prove the theorem (5.16), while we have used the quantum mechanical rule in the proof of the theorem (3.14). Obviously, } \sigma_i^x \sigma_j^x \sigma_k^y \sigma_i^y \sigma_j^y \sigma_k^y \sigma_i^y \sigma_j^x \sigma_k^x = -I, (i, j, k \in \mathbb{N}_n, i \neq j \neq k \neq i) \text{ is needless to prove the theorem (5.16). Of course, Gleason’s theorem is needless. Therefore, we can derive these inequalities (4.2) and (5.16) from more precise and weaker presupposition which ought not to be necessarily false.} \]