DYNAMICAL ASPECTS OF FOLIATIONS WITH AMPLE NORMAL BUNDLE

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Dedicated to Professor Takeo Ohsawa on the occasion of his 70th birthday

ABSTRACT. We prove the following result that was conjectured by Brunella: Let \( X \) be a compact complex manifold of dimension \( \geq 3 \). Let \( \mathcal{F} \) be a codimension one holomorphic foliation on \( X \) with ample normal bundle. Then every leaf of \( \mathcal{F} \) accumulates to the singular set of \( \mathcal{F} \).

1. INTRODUCTION

Let \( X \) be a compact complex manifold of dimension at least two, and let \( \mathcal{F} \) be a (singular) holomorphic foliation on \( X \). A natural question is whether or not every leaf of \( \mathcal{F} \) accumulates to the singular set \( \text{Sing}(\mathcal{F}) \). An answer to this question will depend on the properties of \( X \) and/or \( \mathcal{F} \).

It is known that the normal bundle \( N_{\mathcal{F}} \) of the foliation reflects some dynamical properties of \( \mathcal{F} \). In particular, as a consequence of Baum–Bott theory, \( \text{Sing}(\mathcal{F}) \) is nonempty if \( \mathcal{F} \) has ample normal bundle. One may therefore try to answer the above question under the assumption that \( N_{\mathcal{F}} \) is ample.

Our main result is as follows.

Main Theorem. Let \( X \) be a compact complex manifold of dimension \( \geq 3 \). Let \( \mathcal{F} \) be a codimension one holomorphic foliation on \( X \) with ample normal bundle \( N_{\mathcal{F}} \). Then every leaf of \( \mathcal{F} \) accumulates to \( \text{Sing}(\mathcal{F}) \).

This result was conjectured by Brunella in [5, Conjecture 1.1].

In the special case of \( X = \mathbb{CP}^n \), \( n \geq 3 \), the result goes back to Lins Neto [13]; note that in this situation, the normal bundle \( N_{\mathcal{F}} \) is automatically ample since \( \mathbb{CP}^n \) has positive holomorphic bisectional curvature. In [6], Brunella gave an affirmative answer to his conjecture when \( X \) is a complex torus or, more generally, a compact homogeneous manifold (cf. [9] for higher codimensional foliations). Also, under the assumption that \( \text{Pic}(X) = \mathbb{Z} \), Brunella and Perrone confirmed the conjecture in [7].

If \( \dim X = 2 \), the problem becomes more difficult, and even for the special case \( X = \mathbb{CP}^2 \), no answer is known.

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On the other hand, the Main Theorem might be seen as a further generalization of nonexistence theorems for compact Levi-flat real hypersurfaces that have attracted a great interest in the field of complex analysis over the last decades. Several results concerning the nonexistence of smooth real hypersurfaces invariant by a holomorphic foliation or, more generally, a Levi-flat real hypersurface, related to positivity of the normal bundle can be found in [13], [21], [17], [5], [19], [3] and [4]. In this setting, however, the assumption dim$X \geq 3$ is crucial: examples for dim$X = 2$ can be found in [5, Example 4.2]. More details will be given in §2.

We shall explain our main ideas by giving a sketch of the proof. We argue by contradiction. Assume that there exists a leaf $L$ whose closure $M := \overline{L}$ does not intersect Sing$(F)$. From Brunella’s convexity result (Theorem 4), we know that $X \setminus M$ is strongly pseudoconvex. Since dim$X \geq 3$, the codimension two components of Sing$(F)$ are contained in the maximal compact analytic set $A \subset X \setminus M$.

The first ingredient of the proof is the Baum–Bott theory (cf. [7, §2]), although we use it in a different way from the strategy of Brunella sketched in [5, §4]. In previous approaches [6, 7, 9], Baum–Bott’s formula was used to localize the square of the first Chern class $c_1^2(N_F)$ to Sing$(F)$. Instead, we use the vanishing of the Baum–Bott class over a neighborhood of $M$ to construct a holomorphic connection $\nabla_{\text{hol}}$ of $N_F$ over $X \setminus A$ (§3.1).

Using this connection $\nabla_{\text{hol}}$, we would like to localize the first Chern class $c_1(N_F)$, not its square, to $A$. If $\nabla_{\text{hol}}$ was integrable, the desired localization would follow from the residue formula of integrable connections as in [2,8]. Due to the lack of a residue formula ready-to-use, we accomplish this localization via that of the first Atiyah class $a_1(N_F)$, inspired by [1]. By rounding $\nabla_{\text{hol}}$ around $A$ with a Chern connection, we readily see that $a_1(N_F)$ is localized to a first Atiyah form supported in a small neighborhood $W$ of $A$, which can be chosen to have smooth strongly pseudoconvex boundary.

The main technical step in our proof is the comparison between $c_1(N_F)$ and $a_1(N_F)$. We take the harmonic projection of the localized first Atiyah form with respect to a complete Kähler metric on $W$, and show that the zero extension of this harmonic form gives a desired localization of $c_1(N_F)$. For this delicate analysis, we exploit $L^2$ Hodge theory on complete Kähler manifolds, inspired by [15,10] (§3.2).

Once we can localize $c_1(N_F)$ to $A$, a contradiction easily follows from the $\partial \bar{\partial}$-lemma and the maximum principle for strictly plurisubharmonic functions in the same way as in [2,8] (§3.3).

**Notations and conventions.** We will use the notation $|f| \lesssim |g|$ when there exists some positive constant $C$ such that $|f| \leq C|g|$. We write $|f| \sim |g|$ when both $|f| \lesssim |g|$ and $|g| \lesssim |f|$ hold. Smoothness means $C^\infty$-smoothness unless otherwise stated. We use the same notations for holomorphic line bundles and the sheaves of germs of their holomorphic sections.

2. Preliminaries

For the readers’ convenience, we explain some basic definitions and results on holomorphic foliations, the Atiyah class of holomorphic line bundles, and $L^2$ theory for the $\overline{\partial}$-operator on complete Kähler manifolds.
2.1. Holomorphic foliations. Let $X$ be a complex manifold of dimension $n \geq 2$. In this paper, we discuss foliations in the following sense:

**Definition 1.** We say that a collection of holomorphic 1-forms $\mathcal{F} = \{\omega_\mu\}$, where $\omega_\mu \in \Omega^1(U_\mu)$ and $\mathcal{U} = \{U_\mu\}$ is an open covering of $X$, define a **codimension one holomorphic foliation** on $X$ if they satisfy the following conditions: for any $\mu$ and $\nu$,

1. There exists $g_{\mu\nu} \in \mathcal{O}^*(U_\mu \cap U_\nu)$ such that $\omega_\mu = g_{\mu\nu} \omega_\nu$ on $U_\mu \cap U_\nu$;
2. The analytic set $\{p \in U_\mu \mid \omega_\mu(p) = 0\}$ has codimension $\geq 2$;
3. The integrability condition is fulfilled: $\omega_\mu \wedge d\omega_\mu = 0$ on $U_\mu$.

Here $\Omega^1$ denotes the sheaf of germs of holomorphic 1-forms on $X$. From the first condition, we see that $\{g_{\mu\nu}\}$ enjoys the cocycle condition and defines a holomorphic line bundle over $X$. We call it the **normal bundle** of $\mathcal{F}$ and denote it by $N_\mathcal{F}$. The dual bundle of $N_\mathcal{F}$ is called the **conormal bundle** and denoted by $N_\mathcal{F}^\ast$. Note that $\{\omega_\mu\}$ defines a global section of $N_\mathcal{F}^\ast$.

From the first and second condition, the zero sets of the $\omega_\mu$'s glue together and define an analytic set of codimension $\geq 2$ on $X$. We call it the **singular set** of $\mathcal{F}$ and denote it by $\text{Sing}(\mathcal{F})$. We also denote $X^\circ := X \setminus \text{Sing}(\mathcal{F})$ for a given foliation $\mathcal{F}$.

On $X^\circ$, the kernels of the $\omega_\mu$'s define a holomorphic subbundle of $T^1,0_{X^\circ}$ of corank one. We call it the **tangent bundle** of $\mathcal{F}$ and denote it by $T_\mathcal{F}$. The integrability condition implies that $T_\mathcal{F}$ is integrable in the sense of Frobenius, hence we can find an integral manifold through any point $p \in X^\circ$, that is, a pair of a connected complex manifold $\mathcal{L}$ of dimension $(n - 1)$ and an injective holomorphic immersion $\iota: \mathcal{L} \to X^\circ$ such that $p \in \iota(\mathcal{L})$ and $\iota_*: T^1,0_\mathcal{L} \to T^1,0_\mathcal{F}$. A maximal integral manifold is called a **leaf** of $\mathcal{F}$. It follows that $X^\circ$ is decomposed into the union of all the leaves of $\mathcal{F}$. Therefore, we may think of a foliation as a higher dimensional analogue of flows on $X$.

From this perspective, it would be natural to seek for a Poincaré–Bendixson type property for foliations. Sometimes a leaf $\mathcal{L}$ of $\mathcal{F}$ approaches the singular set, $\overline{\mathcal{L}} \cap \text{Sing}(\mathcal{F}) \neq \emptyset$, and sometimes we have a closed leaf $\mathcal{L}$ of $\mathcal{F}$, which could be thought of as a counterpart for a periodic orbit. We would like to know whether there is another possibility, namely, an exceptional minimal set defined as below:

**Definition 2.** Let $M$ be a nonempty closed subset of $X$.

1. We say $M$ is $\mathcal{F}$-**invariant** if $M \setminus \text{Sing}(\mathcal{F})$ is a union of some leaves of $\mathcal{F}$.
2. An $\mathcal{F}$-invariant subset $M$ is called a **minimal set** of $\mathcal{F}$ if it does not contain any proper $\mathcal{F}$-invariant subset.
3. A minimal set $M$ of $\mathcal{F}$ is said to be **non-trivial** if $M \cap \text{Sing}(\mathcal{F})$ is empty.
4. A non-trivial minimal set $M$ of $\mathcal{F}$ is said to be **exceptional** if $M$ is not a closed leaf of $\mathcal{F}$ nor a connected component of $X$.

It is a well-known fact that the closure of any leaf is $\mathcal{F}$-invariant and contains a minimal set by Zorn’s lemma. In an exceptional minimal set $M$, any leaf contained in $M$ is dense.

When $X$ is compact and $N_\mathcal{F}$ is ample, Baum–Bott theory (cf. [7 §2]) tells us that $\text{Sing}(\mathcal{F})$ cannot be empty. We can easily see that $\mathcal{F}$ does not have any compact leaf from a folklore argument explained below. Hence, nontrivial minimal sets of $\mathcal{F}$, if they exist, must be exceptional in this setting.
Proposition 3. Let $X$ be a complex manifold of dimension $\geq 2$ and $\mathcal{F}$ a codimension one holomorphic foliation on $X$. Assume that $N_\mathcal{F}$ admits a smooth Hermitian metric $h$ of positive curvature. Then $\mathcal{F}$ does not have any compact leaf.

Sketch of the proof. Suppose that there exists a compact leaf $L$ of $\mathcal{F}$. Using the description of a nonsingular holomorphic foliation by foliated charts, we obtain a system of local trivializations of $N_\mathcal{F}|_L$ whose transition functions are locally constant. Since $N_\mathcal{F}|_L$ admits a smooth Hermitian metric of positive curvature, $L$ is Kähler. Then we can construct a smooth flat Hermitian metric $h_0$ of $N_\mathcal{F}|_L$ (see, for instance, [22, Proposition 1]). This metric $h_0$ combined with the given $h$ yields a strictly plurisubharmonic function $\phi := -\log h/h_0$ on $L$. This contradicts the maximum principle. □

In §3, under the assumption that $X$ is compact, $N_\mathcal{F}$ is ample, and $\dim X \geq 3$, we will prove the nonexistence of a nontrivial minimal set, that is, an exceptional minimal set by extending this folklore argument substantially. One of the key tools is the following convexity result of Brunella.

Theorem 4 (Brunella [6, Proposition 3.1]). Let $X$ be a connected compact complex manifold of dimension $\geq 2$ and $\mathcal{F}$ a codimension one holomorphic foliation on $X$. Let $M \subset X$ be an $\mathcal{F}$-invariant subset which is disjoint from $\mathrm{Sing}(\mathcal{F})$. Assume that $N_\mathcal{F}$ admits a smooth Hermitian metric with positive curvature on a neighborhood of $M$. Then $X \setminus M$ is strongly pseudoconvex, namely it admits a smooth plurisubharmonic exhaustion function $\Psi \colon X \setminus M \to \mathbb{R}$ which is strictly plurisubharmonic on $W \setminus M$ for some neighborhood $W$ of $M$.

When an exceptional minimal set $M$ happens to be a smooth real hypersurface, it must be a Levi-flat hypersurface, that is, a smooth real hypersurface having a nonsingular smooth foliation by complex hypersurfaces. This foliation is called the Levi foliation of $M$. For Levi-flat hypersurfaces, a variant of Brunella’s convexity result holds:

Theorem 5 (cf. [5, Proposition 2.1], [19, Proposition 1.1], [3, Proposition 1]). Let $X$ be a connected compact complex manifold of dimension $\geq 2$ and $M$ a $C^3$-smooth closed Levi-flat hypersurface. Assume that the holomorphic normal bundle of $M$ admits a $C^2$-smooth Hermitian metric with positive curvature along the leaves of the Levi foliation. Then $X \setminus M$ is strongly pseudoconvex.

This convexity result played a fundamental role in the proof of the following theorem, which implies that an exceptional minimal set of a codimension one holomorphic foliation $\mathcal{F}$ cannot be a smooth compact real hypersurface when $\dim X \geq 3$ and $N_\mathcal{F}$ admits a smooth Hermitian metric of positive curvature.

Theorem 6 ([4, Theorem 1.1]). Let $X$ be a complex manifold of dimension $\geq 3$. Then there does not exist a smooth compact Levi-flat hypersurface $M$ in $X$ such that the holomorphic normal bundle of $M$ admits a smooth Hermitian metric with positive curvature along the leaves of the Levi foliation.

Theorem 6 generalizes nonexistence theorems for Levi-flat hypersurfaces previously obtained in [13], [21], [17], [5], [19], and [3].

When we show the nonexistence of compact leaves (Proposition 3), or Levi-flat hypersurfaces (Theorem 6), we may take advantage of the property that the normal bundle $N_\mathcal{F}$ is topologically trivial along these minimal sets. However, for a general
exceptional minimal set, we do not know whether such a property holds a priori (cf. [18, Theorem 1.1]). We shall overcome this difficulty by using the localization of the first Atiyah class of $N_F$ and exploiting the $L^2$ theory for $\bar{\partial}$ on complete Kähler manifolds, in particular, the $L^2$ Hodge theory, even further.

2.2. Atiyah class. Let $X$ be a complex manifold and $L$ a holomorphic line bundle over $X$, and let $\nabla$ be a connection on $L$.

**Definition 7.** $\nabla$ is of type $(1,0)$ (or a $(1,0)$-connection) if the connection forms with respect to a holomorphic frame are of type $(1,0)$.

**Example 8.** We will use two kinds of $(1,0)$-connections later.

1. A holomorphic connection is a special case of a $(1,0)$-connection. In this case, the connection forms with respect to a holomorphic frame are holomorphic 1-forms.

2. If $L$ moreover has the structure of a Hermitian holomorphic line bundle, then its Chern connection is a $(1,0)$-connection.

Note that a $(1,0)$-connection is equivalent to the following data: Let $\{g_{\mu\nu}\} \in H^1(\mathcal{U}, \mathcal{O}^*)$ be a cocycle defining the line bundle $L$ with respect to an open covering $\mathcal{U} = \{U_\mu\}$ of $X$. Then the connection forms $\{\eta_\mu\}$ satisfy the gauge transformation law

\[ \eta_\mu = \frac{dg_{\mu\nu}}{g_{\mu\nu}} + \eta_\nu \quad \text{in} \quad U_\mu \cap U_\nu. \]

If $\nabla$ is a $(1,0)$-connection, then its curvature form is given by $d\eta_\mu = \partial\eta_\mu + \bar{\partial}\eta_\mu$. This glues together to a well-defined 2-form thanks to (1). Since the $g_{\mu\nu}$ are holomorphic, also $\bar{\partial}\eta_\mu$ glues together to a global $(1,1)$-form on $X$. This form is called the first Atiyah form of $\nabla$ and denoted by $a_1(\nabla)$.

**Example 9.** We can easily describe the first Atiyah forms for Example 8:

1. If $\nabla$ is a holomorphic connection, then $a_1(\nabla) = 0$.

2. If $\nabla$ is the Chern connection of a Hermitian holomorphic line bundle, then $a_1(\nabla)$ is nothing but the first Chern form $c_1(\nabla)$.

Now suppose we have two $(1,0)$-connections $\nabla_1, \nabla_2$ on $L$, with connection forms $\{\eta^1_\mu\}, \{\eta^2_\mu\}$. Then (1) implies

\[ \eta^1_\mu - \eta^2_\mu = \eta^1_\nu - \eta^2_\nu \quad \text{in} \quad U_\mu \cap U_\nu, \]

hence $a_1(\nabla_1, \nabla_2) := \eta^1_\mu - \eta^2_\mu$ defines a global $(1,0)$-form satisfying

\[ \bar{\partial}a_1(\nabla_1, \nabla_2) = a_1(\nabla_1) - a_1(\nabla_2). \]

In particular, we see that the first Atiyah class of $L$, defined as the class represented by $a_1(\nabla)$ in $H^{1,1}(X)$ for an arbitrary $(1,0)$-connection on $L$, is well defined. We denote the first Atiyah class of $L$ by $a_1(L)$.

Here we discussed the first Atiyah class only. Atiyah classes of higher degrees can be defined for holomorphic vector bundles of arbitrary rank. We refer the reader to [11] for more details.
2.3. $L^2$ theory for $\overline{\partial}$ on complete Kähler manifolds. Let $X$ be a complex manifold of dimension $n$ endowed with a smooth Hermitian metric $\omega$. By $L^2_{p,q}(X, \omega)$ (resp. $L^2_k(X, \omega)$) we denote the space of $(p, q)$-forms on $X$ (resp. $k$-forms on $X$) that are integrable with respect to the global $L^2$-norm

$$\|u\|^2 = \int_X |u(x)|^2 \omega, \text{d}V_\omega,$$

where $|u(x)|\omega$ is the pointwise Hermitian norm and $\text{d}V_\omega$ is the volume form.

On $X$ we consider the differential operators $d, \partial, \overline{\partial}$, as well as the corresponding Laplace operators (depending on the metric $\omega$)

$$\Delta = dd^* + \partial^* \partial, \quad \Delta' = \partial d^* + \partial^* \partial, \quad \Delta'' = \overline{\partial} d^* + \partial^* \overline{\partial}.$$

All these operators extend naturally as linear, closed, densely defined operators on the previously defined $L^2$-spaces in the sense of distributions. We will then consider the spaces of harmonic forms

$$\mathcal{H}^k(X, \omega) = \{u \in L^2_k(X, \omega) \mid \Delta u = 0\},$$
$$\mathcal{H}^{p,q}(X, \omega) = \{u \in L^2_{p,q}(X, \omega) \mid \Delta' u = 0\}.$$

Of special importance will be the case of a complete metric, which is equivalent to the existence of a smooth exhaustion function $\psi: X \to \mathbb{R}$ satisfying $|d\psi|_\omega \leq 1$. In this case, the spaces $\mathcal{D}^{p,q}(X)$ of smooth compactly supported forms are dense in $L^2_{p,q}(X, \omega) \cap \text{Dom} \overline{\partial} \cap \text{Dom} \partial$ in the norm $\|u\| + \|\overline{\partial}u\| + \|\partial u\|$, and we have

$$\mathcal{H}^k(X, \omega) = \{u \in L^2_k(X, \omega) \mid du = d^* u = 0\},$$
$$\mathcal{H}^{p,q}(X, \omega) = \{u \in L^2_{p,q}(X, \omega) \mid \overline{\partial} u = \partial^* u = 0\}.$$

We will also need the following $L^2$ de Rham and Dolbeault cohomology groups

$$H^k_{L^2}(X, \omega) = \{u \in L^2_k(X, \omega) \cap \text{Ker} d / L^2_k(X, \omega) \cap \text{Im} d \},$$
$$H^{p,q}_{L^2}(X, \omega) = \{u \in L^2_{p,q}(X, \omega) \cap \text{Ker} \overline{\partial} / L^2_{p,q}(X, \omega) \cap \text{Im} \overline{\partial} \}.$$

Then, if $\omega$ is complete, we have a natural isomorphism

$$\mathcal{H}^{p,q}(X, \omega) \simeq H^{p,q}_{L^2}(X, \omega),$$

provided the latter cohomology group is finite dimensional, as well as the corresponding isomorphism for the $L^2$ de Rham cohomology.

If moreover the metric $\omega$ is Kähler, we have $\Delta = 2\Delta' = 2\Delta''$, which implies the usual Hodge decomposition

$$H^k_{L^2}(X, \omega) \simeq \bigoplus_{p+q=k} H^{p,q}_{L^2}(X, \omega)$$

if the $L^2$ de Rham and Dolbeault cohomology groups are finite dimensional.

In the following sections, we will also need weighted $L^2$-spaces of the type $L^2_{p,q}(X, \omega, \varphi)$, consisting of forms that are integrable with respect to the weighted $L^2$-norm

$$\|u\|_{\varphi}^2 = \int_X |u(x)|^2 \omega e^{-\varphi} \text{d}V_\omega,$$

where $\varphi: X \to \mathbb{R}$ is a (smooth) weight function. In this case, the adjoint $\overline{\partial}^* \varphi$ depends not only on the metric $\omega$, but also on the weight function $\varphi$.

We refer the reader to [12] for the details on the discussion above.
3. Proof of Brunella’s conjecture

Proof of the Main Theorem. The proof is by contradiction. Assume that there exists a leaf $L$ whose closure $M := \overline{L}$ does not intersect $S := \text{Sing}(F)$. We may assume that $X$ is connected by replacing it with the connected component containing $M$ if necessary. From Theorem 3, $X \setminus M$ is strongly pseudoconvex. Denote by $A$ the maximal compact analytic set of $X \setminus M$, and by $S^*$ the codimension two part of $S$. Note that $S^* \subset A$ since $\dim X \geq 3$.

3.1. First step. In this step, we shall construct a holomorphic connection of $N_F$ over $X \setminus A$.

Cover $X \setminus S^*$ by open sets $\{U_\mu\}$ and take the collection of holomorphic 1-forms $\{\omega_\mu\}$ defining $F$ on $X \setminus S^*$. By taking a refinement of $\{U_\mu\}$ if necessary, the integrability condition yields $\beta_\mu \in \Omega^1(U_\mu)$ such that $d\omega_\mu = \beta_\mu \wedge \omega_\mu$ thanks to Malgrange’s theorem [14, Théorème (0.1)]. We denote the cocycle defining the normal bundle $N_F$ by $\{g_{\mu\nu}\}$ as in Definition 1.

Claim 10. The cocycle $\{\gamma_{\mu\nu} := dg_{\mu\nu}/g_{\mu\nu} - \beta_\mu + \beta_\nu\}$ defines a cohomology class, called the Baum–Bott class, $BB_F \in H^1(X \setminus S^*, N^*_F)$.

Proof. On $U_\mu \cap U_\nu$, we have

$$\gamma_{\mu\nu} \wedge \omega_\nu = \frac{dg_{\mu\nu}}{g_{\mu\nu}} \wedge \omega_\nu - \beta_\mu \wedge \omega_\nu + \beta_\nu \wedge \omega_\nu = \frac{1}{g_{\mu\nu}} d(g_{\mu\nu} \omega_\nu) - d\omega_\nu - \beta_\mu \wedge \omega_\nu + d\omega_\nu = \frac{1}{g_{\mu\nu}} d\omega_\mu - \frac{1}{g_{\mu\nu}} \beta_\mu \wedge \omega_\mu = 0.$$ 

Hence, there exists $f_{\mu\nu} \in \mathcal{O}(U_\mu \cap U_\nu)$ such that $\gamma_{\mu\nu} = f_{\mu\nu} \omega_\nu$ thanks to Riemann’s extension theorem. Since $\{\omega_\nu\}$ defines a global section of $N^*_F$, we see that $\gamma_{\mu\nu}$ is a section of $N^*_F$ over $U_\mu \cap U_\nu$. The cocycle condition is clear. □

Take open connected neighborhoods $V$ and $W$ of $A$ so that $V \Subset W \Subset X \setminus M$ and $W$ has smooth strictly pseudoconvex boundary.

Claim 11. $BB_F|_{X \setminus \overline{W}} = 0$ in $H^1(X \setminus \overline{W}, N^*_F)$.

Proof. In this proof, we identify the sheaf cohomology groups with the Dolbeault cohomology groups. Let $g$ be a smooth $\overline{\partial}$-closed $(0,1)$-form on $X \setminus \overline{V}$ with values in $N^*_F$ that represents $BB_F|_{X \setminus \overline{V}}$. We would like to show that $g$ is $\overline{\partial}$-exact on $X \setminus \overline{W}$.

Take a compactly supported smooth function $\rho: W \to [0,1]$ such that $\rho = 1$ on $\overline{V}$. Then $\tilde{g} := (1 - \rho)g$ is a smooth $(0,1)$-form on $X$ with values in $N^*_F$, and $\overline{\partial}\tilde{g}$ is a $(0,2)$-form, compactly supported in $W$. Since $W$ is strongly pseudoconvex, $\dim X = n \geq 3$ and $N_F$ is positive, $H^{n,n-2}(W, N^*_F)$ vanishes as a consequence of the vanishing theorem by Grauert–Riemenschneider. But then Serre duality implies that $H^{n-2,n}(W, N^*_F)$ vanishes. Therefore we can solve the equation $\overline{\partial}u = \overline{\partial}\tilde{g}$ with $u$ compactly supported in $W$, and obtain a smooth $\overline{\partial}$-closed extension $\tilde{g} - u$ of $g|_{X \setminus \overline{V}}$ to $X$.

Since $N_F$ is assumed to be positive, $H^1(X, N^*_F)$ vanishes thanks to Kodaira’s vanishing theorem. It follows that $\tilde{g} - u$ is $\overline{\partial}$-exact on $X$, and hence, $g$ is $\overline{\partial}$-exact on $X \setminus \overline{W}$. □
By taking a refinement of \( \{ U_\mu \} \) if necessary, we find \( \gamma_\mu \in \Omega^1(U_\mu \setminus W) \) satisfying \( \gamma_{\mu\nu} = \gamma_\mu - \gamma_\nu \) in \( (U_\mu \cap U_\nu) \setminus W \). Then \( \hat{\beta}_\mu := \beta_\mu + \gamma_\mu \) satisfies the gauge transformation law \( \hat{\beta}_\mu = d\gamma_{\mu\nu}/g_{\mu\nu} + \hat{\beta}_\nu \), hence, \( \{ \hat{\beta}_\mu \} \) defines a holomorphic connection of \( N_F|_{X \setminus \partial W} \). By shrinking \( V \) and \( W \), we obtain a holomorphic connection \( \nabla_{\text{hol}} \) of \( N_F|_{X \setminus \partial W} \).

3.2. Second step. In this step, we shall show that the first Atiyah class \( a_1(N_F) \) of \( N_F \) is represented by a \( \bar{\partial} \)-closed \( (1,1) \)-form which is supported in \( W \) and smooth in \( X \setminus \partial W \).

We take an arbitrary smooth Hermitian metric \( h_0 \) of \( N_F \) and consider its Chern connection \( \nabla_0 \). Take a compactly supported smooth function \( \rho: W \to [0,1] \) such that \( \rho = 1 \) on \( \overline{V} \). Defining \( \nabla := \rho \nabla_0 + (1 - \rho) \nabla_{\text{hol}} \), we get a smooth \((1,0)\)-connection of \( N_F \) which is holomorphic in \( X \setminus \supp \rho \). We choose \( a_1(\nabla) \) as a representative of \( a_1(N_F) \). Since \( \nabla \) agrees with the holomorphic connection \( \nabla_{\text{hol}} \) over \( X \setminus \supp \rho \), \( a_1(\nabla) = 0 \) in \( X \setminus \supp \rho \). Hence, \( a_1(\nabla) \) can be seen as a \( \bar{\partial} \)-closed smooth \((1,1)\)-form compactly supported in \( W \).

We are going to find a \( \bar{\partial} \)-closed \((1,1)\)-form supported in \( \overline{W} \) which is \( \bar{\partial} \)-cohomologues to \( a_1(\nabla) \) by employing \( L^2 \) Hodge theory on \( W \). We equip \( W \) with a complete Kähler metric \( \omega \) of the form

\[
\omega = \omega_0 + i\bar{\partial}(-\log \delta),
\]

where \( \omega_0 \) is a Kähler metric on \( X \), which exists since \( X \) is projective, and \( -\delta: \overline{W} \to [-1,0] \) is a smooth plurisubharmonic defining function for \( W \) which is strictly plurisubharmonic on a neighborhood of \( \partial W \).

Lemma 12. \( H^{1,1}_{L^2}(W,\omega) \) and \( H^{1,1}(W,\omega) \) are finite dimensional and isomorphic if \( \dim X = n \geq 3 \).

Although this is a classical result (cf. [10,15,16,20]), we shall give a self-contained proof for the readers’ convenience.

Proof. The idea is to use the twisting trick of Berndtsson and Siu, making use of the advantage that the function \( -\log \delta \) satisfies the Donnelly–Fefferman condition outside a compact of \( W \). Precisely speaking, since \( \omega = \omega_0 + i\bar{\partial}(-\log \delta) = \omega_0 + \frac{i\delta + \delta}{2} + \frac{\delta}{2} \wedge \frac{\delta}{2} \), we may choose \( 0 < t < \frac{1}{4} \) and a compact \( K \subset W \) such that \( \varphi := -t\log \delta \) satisfies

\[
|\partial \varphi|^2_\omega \leq \frac{t}{4} \quad \text{in } W \setminus K.
\]

Let \( 0 \leq \lambda_1 \leq \cdots \leq \lambda_n \) be the eigenvalues of \( i\partial \bar{\partial} \varphi \) with respect to the metric \( \omega \). Then, since \( \omega = \omega_0 + \frac{1}{2}i\partial \bar{\partial} \varphi \), enlarging \( K \) if necessary, we may assume that

\[
\lambda_j = t + \varepsilon_j \quad \text{in } W \setminus K
\]

with \( |\varepsilon_j| \leq \frac{1}{2\tau_0} \) for all \( 1 \leq j \leq n \).

From the Bochner–Kodaira–Nakano inequality (see [11]) it follows that

\[
\|\bar{\partial}u\|_{2,\varphi}^2 + \|\bar{\partial} \omega u\|_{2,\varphi}^2 \geq \int_W ((\lambda_1 + \cdots + \lambda_{n-1}) - \lambda_n) |u|_\omega^2 e^\varphi dV_\omega
\]

for \( u \in \mathcal{D}^{1,1}(W) \). Using (3), we have the estimate

\[
\lambda_1 + \cdots + \lambda_{n-1} - \lambda_n \geq (n-2) - \frac{t}{2} \geq \frac{t}{2} \quad \text{in } W \setminus K
\]
since \( n \geq 3 \). Hence for every \( u \in \mathcal{D}^{1,1}(W \setminus K) \) we have the estimate
\[
(4) \quad \frac{t}{2} \|u\|_{-\varphi}^2 \leq \|\partial u\|_{-\varphi}^2 + \|\bar{\partial}_{-\varphi} u\|_{-\varphi}^2.
\]

Now we substitute \( u = ve^{-\varphi/2} \). It is not difficult to see that
\[
|\partial u|e^\varphi = |\partial v - \frac{1}{2} \partial \varphi \wedge v| \leq 2 \left( |\partial v| + \frac{1}{4} |\partial \varphi| v \right) \leq 2|\partial v| + \frac{t}{8}|v|^2.
\]

Since \( \partial_{-\varphi} = e^{-\varphi} \partial \), we likewise get
\[
|\partial_{-\varphi} u|e^\varphi = |\partial (e^{\varphi/2}v)|v = |\partial v + \frac{1}{2} (\partial \varphi) v| \leq 2 \left( |\partial v| + \frac{1}{4} |\partial \varphi| v \right) \leq 2|\partial v| + \frac{t}{8}|v|^2.
\]
Together with (4), these two inequalities imply
\[
(5) \quad \frac{t}{8} \|v\|^2 \leq \|\partial v\|^2 + \|\bar{\partial} v\|^2
\]
for all \( v \in \mathcal{D}^{1,1}(W \setminus K) \).

It is now standard to conclude that for any compact \( L \) containing \( K \) in its interior, there exists a positive constant \( C_L \) such that
\[
\|v\|^2 \leq C_L (\|\partial v\|^2 + \|\bar{\partial} v\|^2 + \int_L |v|^2 d\nu)
\]
for all \( v \in \mathcal{D}^{1,1}(W) \), hence for all \( v \in L^2_{1,1}(W, \omega) \) by completeness of the metric \( \omega \). But the above estimate implies that \( H^1_{L^2}(W, \omega) \) is finite dimensional and isomorphic to \( H^1_{L^2}(W, \omega) \).

Since \( a_1(\nabla) \) is a \( \partial \)-closed \((1,1)\)-form compactly supported in \( W \), Lemma [12] yields a smooth \((1,1)\)-form \( \gamma \in H^1_{L^2}(W, \omega) \) and \( \beta_1 \in L^2_{1,0}(W, \omega) \) such that \( a_1(\nabla) - \gamma = \bar{\partial} \beta_1 \) on \( W \).

Note that
\[
d\nu \sim \omega^n \sim \frac{1}{\beta^{n+1} \omega^n} \sim \delta^{-n-1} d\nu.
\]
Since \( \delta^n \omega \leq \omega_0 \) on \( W \), it holds that \( \delta^{-2} |\beta|_2^2 \geq |\beta|_2^2 \) for any \( 1 \)-form \( \beta \) on \( W \), and \( \delta^{-4} |\alpha|_2^2 \geq |\alpha|_2^2 \) for any \( 2 \)-form \( \alpha \) on \( W \). Hence,
\[
\int_W |\beta_1|_2^2 \delta^{-n-1} d\nu = \int_W |\beta_1|_2^2 \delta^{-n-1} d\nu \sim \int_W |\beta_1|_2^2 d\nu < \infty,
\]
\[
\int_W |\gamma|_2^2 \delta^{-n+3} d\nu \sim \int_W |\gamma|_2^2 d\nu < \infty,
\]
Therefore,
\[
\beta_1 \in L^2_{1,0}(W, \omega_0, (n-1) \log \delta) \subset L^2_{1,0}(W, \omega_0, \log \delta),
\]
\[
\gamma \in L^2_{1,1}(W, \omega_0, (n-3) \log \delta) \subset L^2_{1,1}(W, \omega_0)
\]
since \( \dim X = n \geq 3 \).

We denote the extensions by zero of \( \gamma \) and \( \beta_1 \) to \( X \) by \( \tilde{\gamma} \) and \( \tilde{\beta}_1 \) respectively. Then \( \tilde{\gamma} \) and \( \tilde{\beta}_1 \) belong to \( L^2_{\partial \phi}(X, \omega_0) \). Moreover, they satisfy the above \( \partial \)-equation not only on \( W \) but also on \( X \):
Claim 13. \( a_1(\nabla) - \tilde{\gamma} = \overline{\partial} \tilde{\beta}_1 \) on \( X \).

Proof. Setting \( a_1 = a_1(\nabla) - \tilde{\gamma} \), we want to show that \( \overline{\partial} \tilde{\beta}_1 = a_1 \) in the sense of distributions on \( X \). That is, we have to show that

\[
(6) \quad \int_W a_1 \wedge \overline{\partial}g = \int_W \alpha_1 \wedge g.
\]

for \( g \in C^\infty_{n-1,n-1}(X) \). Let \( \chi \in C^\infty(\mathbb{R}, \mathbb{R}) \) be a function such that \( \chi(t) = 0 \) for \( t \leq \frac{1}{2} \) and \( \chi(t) = 1 \) for \( t \geq 1 \). Set \( \chi_j = \chi(j\delta) \in D(W) \). Then \( \chi_j g \in D^{n-1,n-1}(W) \), and since \( \overline{\partial} \tilde{\beta}_1 = a_1 \) in \( W \), we therefore have

\[
\int_W a_1 \wedge \chi_j g = \int_W \beta_1 \wedge \overline{\partial} (\chi_j g) = \int_W \beta_1 \wedge (\overline{\partial} \chi_j \wedge g + \chi_j \overline{\partial}g).
\]

As \( \alpha_1 \) has \( L^2 \) coefficients on \( W \), the integral of \( a_1 \wedge \chi_j g \) converges to the integral of \( \alpha_1 \wedge g \) as \( j \) tends to infinity. Since \( \beta_1 \in L^1_{1,0}(X) \), the integral of \( \beta_1 \wedge \chi_j \overline{\partial}g \) converges to the integral of \( \beta_1 \wedge \overline{\partial}g \). The remaining term can be estimated as follows: using the Cauchy–Schwarz inequality we have

\[
(7) \quad \left| \int_W \beta_1 \wedge \overline{\partial} \chi_j \wedge g \right|^2 \leq \sup_W |g|^2 \int_{\{0 < \delta \leq \frac{1}{j}\}} |\beta_1|^2 \omega_0^{-1} \delta \omega_0 \cdot \int_W |\overline{\partial} \chi_j|^2 \omega_0^{-1} \delta \omega_0.
\]

Since \( \beta_1 \in L^2_{1,0}(W, \omega_0, \log \delta) \), the integral \( \int_{\{0 < \delta \leq \frac{1}{j}\}} |\beta_1|^2 \omega_0^{-1} \delta \omega_0 \) converges to 0 when \( j \) tends to infinity. We estimate the second integral as follows

\[
\int_W |\overline{\partial} \chi_j|^2 \omega_0^{-1} \delta \omega_0 \lesssim \int_{\{0 < \delta \leq \frac{1}{j}\}} j^2 \delta \omega_0 \lesssim j \text{Vol}_\omega(\{0 < \delta \leq \frac{1}{j}\}) \leq \text{cte}.
\]

Combining this estimate with (7), we have proved that \( \int_W \beta_1 \wedge \overline{\partial} \chi_j \wedge g \) converges to 0 when \( j \) tends to infinity. Equation (6) follows.

Since \( \omega \) is Kähler, the harmonic form \( \gamma \) is \( d \)-closed on \( W \) (see §2.3). The extension by zero, \( \tilde{\gamma} \), is also \( d \)-closed on \( X \) and gives the desired representative of \( a_1(N_F) \) over \( X \).

Claim 14. The \((1,1)\)-form \( \tilde{\gamma} \) is \( d \)-closed on \( X \).

Proof. Since \( a_1(\nabla) - \tilde{\gamma} = \overline{\partial} \tilde{\beta}_1 \) holds on \( X \), \( \tilde{\gamma} \) is \( \overline{\partial} \)-closed on \( X \). We show that \( \partial \tilde{\gamma} = 0 \) in the sense of distributions on \( X \). That is, we have to show that

\[
(8) \quad \int_W \tilde{\gamma} \wedge \partial g = 0
\]

for \( g \in C^\infty_{n-2,n-1}(X) \). Take \( \chi \) and \( \chi_j \) as in the proof of Claim 13. Then,

\[
\int_W \tilde{\gamma} \wedge \partial g = \lim_{j \to \infty} \int_W \chi_j \gamma \wedge \partial g = \lim_{j \to \infty} \int_W \gamma \wedge \partial (\chi_j g) - \lim_{j \to \infty} \int_W \gamma \wedge \partial \chi_j \wedge g.
\]

Since \( \partial \gamma = 0 \) in \( W \) this implies

\[
(9) \quad \int_W \tilde{\gamma} \wedge \partial g = - \lim_{j \to \infty} \int_W \gamma \wedge \partial \chi_j \wedge g.
\]
To compute (9), we need to decompose $\gamma$. Since $\partial \delta \wedge \overline{\partial} \delta$ does not vanish on $W \setminus K$ for some compact $K$, we can decompose the smooth $(1,1)$-form $\gamma$ as

$$\gamma = \gamma_1 + \gamma_2 \quad \text{over } W \setminus K,$$

where $\gamma_2$ is a smooth multiple of $\partial \delta \wedge \overline{\partial} \delta$ and $\gamma_1$ is orthogonal to $\partial \delta \wedge \overline{\partial} \delta$ with respect to the metric $\omega_0$.

We would like to show that $\gamma_1$ has better integrability than $\gamma$ with respect to the metric $\omega_0$. For this, let us use an orthogonal decomposition of the metric $\omega_0$ in its tangential and normal parts:

$$\omega_0 = \omega_t + \omega_n \quad \text{over } W \setminus K,$$

where $\omega_t$ corresponds to a smooth Hermitian metric of the subbundle $\text{Ker} \partial \delta \subset T^{1,0}_{W \setminus K}$, and $\omega_n$ is a smooth positive multiple of $i \partial \delta \wedge \overline{\partial} \delta$. Using this decomposition, we rescale $\omega_0$ to another smooth Hermitian metric $\omega'$ on $W \setminus K$ given by

$$\omega' := \frac{1}{\delta} \omega_t + \frac{1}{\delta^2} \omega_n = \frac{1}{\delta} \omega_t + \frac{1}{\delta^2} \omega_n.$$

Notice that $\gamma_1$ and $\gamma_2$ are still orthogonal with respect to $\omega'$. In particular, it holds that $|\gamma_1|_{\omega'}^2 \leq |\gamma|_{\omega_0}^2$. Also, we have $\omega \sim \omega'$ on $W \setminus K$, by enlarging $K$ if necessary, since

$$\omega \sim i \partial \overline{\partial}(- \log \delta) = i \frac{\partial \overline{\partial}(- \delta)}{\delta} + i \frac{\partial \delta \wedge \overline{\partial} \delta}{\delta^2} \sim \frac{1}{\delta} \omega_0 + \frac{1}{\delta^2} i \partial \delta \wedge \overline{\partial} \delta.$$

We thus have $\gamma_1 \in L^2_{1,1}(W \setminus K, \omega')$.

It follows from the definitions of $\gamma_1$ and $\omega'$ that $|\gamma_1|_{\omega'}^2 \geq \delta^3 |\gamma_1|_{\omega_0}^2$. Then,

$$\int_{W \setminus K} |\gamma_1|_{\omega_0}^2 \delta^{-1} dV_{\omega_0} \leq \int_{W \setminus K} |\gamma_1|_{\omega_0}^2 \delta^{-4} dV_{\omega_0} \lesssim \int_{W \setminus K} |\gamma_1|_{\omega_0}^2 dV_{\omega_0} < \infty$$

since $n \geq 3$ and $dV_{\omega'} \sim dV_{\omega} \sim \delta^{-(n+1)} dV_{\omega_0}$. Therefore, we have $\gamma_1 \in L^2_{1,1}(W \setminus K, \omega_0, \log \delta)$.

We continue with the computation of (9). Since $\partial \chi_j = j \chi'(j \delta) \partial \delta$, we get

$$\int_W \gamma \wedge \partial \chi_j \wedge g = \int_W \gamma_1 \wedge \partial \chi_j \wedge g$$

for $j$ enough large, and, as in the proof of Claim 13,

$$\left| \int_W \gamma_1 \wedge \partial \chi_j \wedge g \right| \leq \sup_W |g|_{\omega_0}^2 \int_{\{0 < \delta \leq \frac{1}{j}\}} |\gamma_1|_{\omega_0}^2 \delta^{-1} dV_{\omega_0} \cdot \int_W |\partial \chi_j|_{\omega_0}^2 \delta dV_{\omega_0} \lesssim \int_{\{0 < \delta \leq \frac{1}{j}\}} |\gamma_1|_{\omega_0}^2 \delta^{-1} dV_{\omega_0} \rightarrow 0.$$

\[ \square \]

Remark 15. When $\dim X = n \geq 4$, Claim 14 follows from the integrability $\gamma \in L^2_{1,1}(W, \omega_0, (n-3) \log \delta) \subset L^2_{1,1}(W, \omega_0, \log \delta)$ in the same way as in Claim 13. The delicate use of the orthogonal decomposition $\gamma = \gamma_1 + \gamma_2$ is required for $n = 3$. 

3.3. Third step. In this step, we construct a flat Hermitian metric of $N^F$ over $M$, and deduce a contradiction.

Since both $a_1(\nabla_0)$ and $a_1(\nabla)$ represent $a_1(N^F)$, there exists a smooth $(1,0)$-form $\beta_2$ on $X$ such that

$$\overline{\partial}\beta_2 = a_1(\nabla_0) - a_1(\nabla) = a_1(\nabla_0) - (\gamma + \overline{\partial}\beta_1).$$

Notice that $a_1(\nabla_0)$ is exactly the first Chern form of $\nabla_0$, a $d$-closed smooth $(1,0)$-form. Therefore,

$$a_1(\nabla_0) - \gamma = \overline{\partial}(\beta_1 + \beta_2).$$

is a $d$-closed, $\overline{\partial}$-exact $L^2(1,1)$-form which is smooth on $X \setminus \partial W$.

Since $X$ is projective, hence, Kähler, we may apply the $\partial\overline{\partial}$-lemma to obtain an $L^2$ function $\psi: X \to \mathbb{C}$ smooth on $X \setminus \partial W$ such that

$$a_1(\nabla_0) - \gamma = i\partial\overline{\partial}\psi.$$

Since $\gamma$ has its support in $W$, it holds on $X \setminus W$ that

$$\frac{i}{2\pi} \overline{\partial}\partial \log h_0 = i\partial\overline{\partial}\psi,$$

which implies

$$\frac{i}{2\pi} \overline{\partial}\partial \log h_0 = i\partial\overline{\partial} \text{Re } \psi$$

on $X \setminus \overline{W}$. Hence, $h_1 := h_0 e^{2\pi \text{Re } \psi}$ gives a smooth flat Hermitian metric of $N^F$ over $X \setminus \overline{W}$.

Since we assume that $N^F$ is ample, there exists a smooth Hermitian metric $h_2$ of $N^F$ with positive curvature. Then, $\phi := -\log h_2/h_1: X \setminus \overline{W} \to \mathbb{R}$ is a smooth strictly plurisubharmonic function. The existence of a compact $\mathcal{F}$-invariant subset $M \subset X \setminus \overline{W}$ violates the maximum principle: There is a point $p \in M$ where $\phi|_M$ takes its maximum since $M$ is compact. Denote by $\mathcal{L}_p$ the leaf passing through $p$. Then, $\phi|_{\mathcal{L}_p}$, a smooth plurisubharmonic function on $\mathcal{L}_p$, takes its maximum at $p$. Hence $\phi|_{\mathcal{L}_p}$ must be constant, but then cannot be a strictly plurisubharmonic function. This is a contradiction and completes the proof of the Main Theorem. □

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