BUNCHES OF CONES IN THE DIVISOR CLASS GROUP — A NEW COMBINATORIAL LANGUAGE FOR TORIC VARIETIES

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Abstract. As an alternative to the description of a toric variety by a fan in the lattice of one parameter subgroups, we present a new language in terms of what we call bunches — these are certain collections of cones in the vector space of rational divisor classes. The correspondence between these bunches and fans is based on classical Gale duality. The new combinatorial language allows a much more natural description of geometric phenomena around divisors of toric varieties than the usual method by fans does. For example, the numerically effective cone and the ample cone of a toric variety can be read off immediately from its bunch. Moreover, the language of bunches appears to be useful for classification problems.

Introduction

The most important feature of a toric variety $X$ is that it is completely described by a fan $\Delta$ in the lattice $N$ of one parameter subgroups of the big torus $T_X \subset X$. Applying a linear Gale transformation to the set of primitive generators $v_\varrho$ of the rays $\varrho \in \Delta^{(1)}$ of $\Delta$, gives a new vector configuration in a rational vector space $K_\mathbb{Q}$. This opens an alternative combinatorial approach to the toric variety $X$: The vector space $K_\mathbb{Q}$ is isomorphic to the rational divisor class group of $X$, and one can shift combinatorial information between the spaces $N_\mathbb{Q}$ and $K_\mathbb{Q}$.

In toric geometry, this principle has been used to study the projective case, compare [19]: Roughly speaking, if we consider all fans $\Sigma$ in $N$ having their rays among $\Delta^{(1)}$, then the (quasi-)projective $\Sigma$ correspond to so called Gelfand-Kapranov-Zelevinski cones in $K_\mathbb{Q}$. These cones subdivide the cone generated by the Gale transform of the vector configuration $\{v_\varrho; \varrho \in \Delta^{(1)}\}$, and the birational geometry of the associated toric varieties is reflected by the position of their Gelfand-Kapranov-Zelevinski cones.

If one leaves the (quasi-)projective setting, then there are generalizations of Gelfand-Kapranov-Zelevinski cones, compare e.g. [14]; but so far there seems to be no concept which is simple enough to serve for practical purposes in toric geometry. Our aim is to fill this gap and to propose a natural combinatorial language which also works in the non quasiprojective case. The combinatorial data are certain collections — which we call bunches — of overlapping cones in the vector space of rational divisor classes. As we shall see, this approach gives very natural descriptions of geometric phenomena connected with divisors.

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In order to give a first impression of the language of bunches, let us present it here for in a special case, namely for toric varieties with free class group. Consider a sequence \((w_1, \ldots, w_n)\) of not necessarily pairwise distinct points, called weight vectors, in a lattice \(K \cong \mathbb{Z}^k\). By a weight cone in \(K\) we then mean a (convex polyhedral) cone \(\tau \subset K\) generated by some of the \(w_i\).

**Definition.** A (free standard) bunch in \(K\) is a nonempty collection \(\Theta\) of weight cones in \(K\) with the following properties:

(i) a weight cone \(\sigma\) in \(K\) belongs to \(\Theta\) if and only if \(\emptyset \neq \sigma \cap \tau \neq \tau^\circ\) holds for all \(\tau \in \Theta\) with \(\tau \neq \sigma\).

(ii) for each \(i\), the set \(\{w_j; j \neq i\}\) generates \(K\) as a lattice, and there is a \(\tau \in \Theta\) such that \(\tau^\circ \subset \text{cone}(w_j; j \neq i)^\circ\) holds for the relative interiors.

How to construct a toric variety from such a bunch \(\Theta\)? The first step is to unpack the combinatorial information encoded in \(\Theta\). For this, let \(E := \mathbb{Z}^n\), and let \(Q: E \to K\) denote the linear surjection sending the canonical base vector \(e_i\) to \(w_i\). Denote by \(\gamma \subset E_Q\) the positive orthant. We define the covering collection of \(\Theta\) as

\[
\text{cov}(\Theta) := \{\gamma_0 \preceq \gamma; \gamma_0 \text{ minimal with } Q(\gamma_0) \supset \tau \text{ for some } \tau \in \Theta\}.
\]

The next step is to dualize the information contained in \(\text{cov}(\Theta)\). This is done by a procedure close to a linear Gale transformation, for the classical setup see e.g. [11] and [17]: Consider the exact sequence arising from \(Q: E \to K\), and the corresponding dualized exact sequence:

\[
0 \to M \to E \overset{Q}{\to} K \to 0,
\]

\[
0 \to L \to F \overset{P}{\to} N \to 0.
\]

Note that \(P: F \to N\) is not the dual homomorphism of \(Q: E \to K\). Let \(\delta := \gamma^\vee \subset F_Q\) be the dual cone of \(\gamma \subset E_Q\). The crucial observation then is that we obtain a fan \(\Delta(\Theta)\) in the lattice \(N\) having as its maximal cones the images

\[P(\gamma_0^\perp \cap \delta), \quad \gamma_0 \in \text{cov}(\Theta).\]

**Definition.** The toric variety associated to the bunch \(\Theta\) is \(X_\Theta := X_{\Delta(\Theta)}\).

The toric variety \(X_\Theta\) is nondegenerate in the sense that it has no torus factors. Moreover, \(X_\Theta\) is 2-complete, that means if \(X_\Theta \subset X\) is an open toric embedding such that the complement \(X \setminus X_\Theta\) is of codimension at least 2, then \(X = X_\Theta\).

**Example.** Consider the sequence \((1, 2, 3)\) of weight vectors in \(K := \mathbb{Z}\) and the bunch \(\Theta := \{Q_{\geq 0}\}\) in \(\mathbb{Z}\). Then we have \(E = \mathbb{Z}^3\), and the associated linear map \(Q: \mathbb{Z}^3 \to \mathbb{Z}\) sends \(e_i\) to \(i\). The covering collection \(\text{cov}(\Theta)\) consists of the following three faces of \(\gamma = Q_{\geq 0}^3\):

\[
\gamma_i := Q_{\geq 0} e_i, \quad i = 1, 2, 3.
\]

If we identify the dual space \(F = E^*\) with \(\mathbb{Z}^3\), then the cone \(\delta = \gamma^\vee\) is again \(Q_{\geq 0}^3\). Moreover, we may identify \(N\) with \(\mathbb{Z}^2\) and thus realize the map \(P: F \to N\) via the matrix

\[
\begin{bmatrix}
-2 & 1 & 0 \\
-3 & 0 & 1
\end{bmatrix}
\]
Each $\gamma_i^+ \cap \delta$ equals $\text{cone}(e_j, e_1)$, where $j \neq i$ and $l \neq i$. Hence the images $P(\gamma_i^+ \cap \delta)$, which are the maximal cones of the fan $\Delta(\Theta)$, are given in terms of the canonical base vectors $e_1, e_2 \in \mathbb{Z}^2$ as
\[
\text{cone}(e_1, e_2), \quad \text{cone}(-2e_1 - 3e_2, e_2), \quad \text{cone}(-2e_1 - 3e_2, e_1).
\]
Consequently, the toric variety $X_\Theta$ associated to the bunch $\Theta$ equals the weighted projective space $\mathbb{P}_{1,2,3}$.

Introducing a suitable notion of a morphism, we can extend the assignment $\Theta \mapsto X_\Theta$ from bunches to 2-complete nondegenerate toric varieties to a contravariant functor. In fact, we even obtain a weak antiequivalence, see Theorem 6.3:

**Theorem.** The functor $\Theta \mapsto X_\Theta$ induces a bijection on the level of isomorphism classes of bunches and nondegenerate 2-complete toric varieties. In particular, every complete toric variety arises from a bunch.

In order to read off geometric properties of $X_\Theta$ directly from the bunch $\Theta$, one has to translate the respective fan-theoretical formulations via the above Gale transformation into the language of bunches. This gives for example:

- $X_\Theta$ is $\mathbb{Q}$-factorial if and only if every cone $\tau \in \Theta$ is of full dimension, see Proposition 7.2.
- $X_\Theta$ is smooth if and only if for every $\gamma_0$ in $\text{cov}(\Theta)$, the image $Q(\text{lin}(\gamma_0) \cap E)$ equals $K$, see Proposition 7.3.
- The orbits $X_\Theta$ have fixed points in their closures if and only if all cones $Q(\gamma_0)$, where $\gamma_0 \in \text{cov}(\Theta)$, are simplicial, see Proposition 7.5.

As mentioned, the power of the language of bunches lies in the description of geometric phenomena around divisors, because $K_\Theta$ turns out to be the rational divisor class group of $X_\Theta$. For example, we obtain very simple descriptions for the classes of rational Cartier divisors, the cone $\text{C}^{\text{sa}}(X_\Theta)$ of semiample classes and the cone $\text{C}^{\text{a}}(X_\Theta)$ of ample classes, see Theorem 9.2:

**Theorem.** For the toric variety $X_\Theta$ arising from a bunch $\Theta$, there are canonical isomorphisms:
\[
\text{Pic}_\mathbb{Q}(X_\Theta) \cong \bigcap_{\tau \in \Theta} \text{lin}(\tau), \quad \text{C}^{\text{sa}}(X_\Theta) \cong \bigcap_{\tau \in \Theta} \tau, \quad \text{C}^{\text{a}}(X_\Theta) \cong \bigcap_{\tau \in \Theta} \tau^\circ.
\]

Note that the last isomorphism gives a quasiprojectivity criterion in the spirit of [26] and [27], see Corollary 9.3. Moreover, we can derive from the above Theorem a simple Fano criterion, see Corollary 9.4. Finally, we get back Reid’s Toric Cone Theorem, see [21], even with a new description of the Mori Cone, see Corollary 9.10.

**Corollary.** Suppose that $X_\Theta$ is complete and simplicial. Then the cone of numerically effective curve classes in $H_2(X, \mathbb{Q})$ is given by
\[
\text{NE}(X_\Theta) \cong \sum_{\tau \in \Theta} \tau^\vee.
\]
In particular, this cone is convex and polyhedral. Moreover, $X_\Theta$ is projective if and only if $\text{NE}(X_\Theta)$ is strictly convex.

Bunches can also be used for classification problems. For example, once the machinery is established, Kleinschmidt’s classification [15] becomes very simple and can even be slightly improved, see Proposition 10.1; below we represent a sequence of weight vectors as a set of vectors $w$, each of which carries a multiplicity $\mu(w)$:
Theorem. The smooth 2-complete toric varieties $X$ with $\text{Cl}(X) \cong \mathbb{Z}^2$ correspond to bunches $\Theta = \{\text{cone}(w_1, w_2)\}$ given by

- weight vectors $w_1 := (1, 0)$, and $w_i := (b_i, 1)$ with $0 = b_n < b_{n-1} < \cdots < b_2$,
- multiplicities $\mu_i := \mu(w_i)$ with $\mu_1 > 1$, $\mu_n > 0$ and $\mu_2 + \cdots + \mu_n > 1$.

Moreover, the toric variety $X$ defined by such a bunch $\Theta$ is always projective, and it is Fano if and only if we have

$$b_2(\mu_3 + \cdots + \mu_n) < \mu_1 + b_2\mu_3 + \cdots + b_{n-1}\mu_{n-1}.$$  

In general the functor from bunches to toric varieties is neither injective nor surjective on morphisms, see Examples 8.7 and 8.8. But if we restrict to $\mathbb{Q}$-factorial toric varieties, then the language of bunches provides also a tool for the study of toric morphisms, see Theorem 8.2:

Theorem. There is an equivalence from the category of simple bunches to the category of full $\mathbb{Q}$-factorial toric varieties.
By a cone in a lattice $N$ we always mean a polyhedral (not necessarily strictly) convex cone in the associated rational vector space $N_{\mathbb{Q}}$. Let $N$ be a lattice, and let $M := \text{Hom}(N, \mathbb{Z})$ denote the dual lattice of $N$. The orthogonal space and the dual cone of a cone $\sigma$ in $N$ are
\[ \sigma^\perp := \{ u \in M_{\mathbb{Q}} \mid u|_{\sigma} = 0 \}, \quad \sigma^\vee := \{ u \in M_{\mathbb{Q}} \mid u|_{\sigma} \geq 0 \}. \]

The relative interior of a cone $\sigma$ is denoted by $\sigma^\circ$. If $\sigma_0$ is a face of $\sigma$, then we write $\sigma_0 \preceq \sigma$. The dimension of $\sigma$ is the dimension of the linear space $\text{lin}(\sigma)$ generated by $\sigma$. The set of the $k$-dimensional faces of $\sigma$ is denoted by $\sigma(k)$, and the one-dimensional faces of a strictly convex cone are called rays.

The primitive generators of a strictly convex cone $\sigma$ in a lattice $N$ are the primitive lattice vectors of its rays. A strictly convex cone in $N$ is called simplicial if its primitive generators are linearly independent, and it is called regular if its primitive generators can be complemented to a lattice basis of $N$.

**Definition 1.1.**

(i) A **fan** in a lattice $N$ is a finite collection $\Delta$ of strictly convex cones in $N$ such that for each $\sigma \in \Delta$ also all $\sigma_0 \preceq \sigma$ belong to $\Delta$ and for any two $\sigma_1, \sigma_2 \in \Delta$ we have $\sigma_1 \cap \sigma_2 \preceq \sigma_i$.

(ii) A map of fans $\Delta_1$ in lattices $N_1$ is a lattice homomorphism $F : N_1 \to N_2$ such that for every $\sigma_1 \in \Delta_1$ there is a $\sigma_2 \in \Delta_2$ containing $F(\sigma_1)$.

Recall that the compatibility condition $\sigma_1 \cap \sigma_2 \preceq \sigma_i$ in the above definition is equivalent to the existence of a separating linear form for the cones $\sigma_1$ and $\sigma_2$, i.e., a linear form $u$ on $N$ such that
\[ u|_{\sigma_1} \geq 0, \quad u|_{\sigma_2} \leq 0, \quad u^\perp \cap \sigma_i = \sigma_1 \cap \sigma_2. \]

If we replace in Definition 1.1 “strictly convex” with “convex”, we obtain the category of quasifans. For a fan $\Delta$ in $N$, we denote by $|\Delta|$ its support, that is the union of all its cones. Moreover, $\Delta^{\text{max}}$ is the set of maximal cones of $\Delta$, and $\Delta(k)$ is the set of all $k$-dimensional cones of $\Delta$.

In the sequel, we shall often make use of a well known universal lifting construction, which makes the set of primitive generators of the rays of a given fan into a lattice basis, compare for example [6]:

**Construction 1.2.** Let $\Delta$ be a fan in a lattice $N$, and let $R := \Delta(1)$. Let $C : \mathbb{Z}^R \to N$ be the map sending the canonical base vector $e_\varrho$ to the primitive generator $v_\varrho \in \varrho$. For $\sigma \in \Delta^{\text{max}}$, set
\[ \tilde{\sigma} := \text{cone}(e_\varrho; \varrho \in \sigma(1)). \]
Then the cones $\tilde{\sigma}$, where $\sigma \in \Delta^{\text{max}}$, are the maximal cones of a fan $\overset{\longrightarrow}{\Delta}$ consisting of faces of the positive orthant in $Q^R$. Moreover, $C : \mathbb{Z}^R \to N$ is a map of the fans $\Delta$ and $\overset{\longrightarrow}{\Delta}$.

Now we turn to toric varieties. Throughout the entire paper, we work over an algebraically closed field $K$ of characteristic zero, and the word point refers to a closed point.

**Definition 1.3.**

(i) A **toric variety** is a normal variety $X$ containing an algebraic torus $T_X$ as an open subset such that the group structure of $T_X$ extends to a regular action on $X$.

(ii) A **toric morphism** is a regular map $X \to Y$ of toric varieties that restricts to a group homomorphism $T_X \to T_Y$. 
The correspondence between fans and toric varieties is obtained as follows: Let \( \Delta \) be a fan in a lattice \( N \), and let \( M := \text{Hom}(N, \mathbb{Z}) \) be the dual lattice of \( N \). For every cone \( \sigma \in \Delta \) one defines an affine toric variety:

\[
X_\sigma := \text{Spec}(K[\sigma^\vee \cap M]).
\]

For any two such \( X_{\sigma_1}, X_{\sigma_2} \), one has canonical open embeddings of \( X_{\sigma_1} \cap \sigma_2 \) into \( X_{\sigma_i} \). Patching together all \( X_\sigma \) along these open embeddings gives a toric variety \( X_\Delta \). The assignment \( \Delta \mapsto X_\Delta \) is functorial; it is even a (covariant) equivalence of categories.

In the sequel, we shall frequently restrict our investigations to toric varieties that behave reasonably. For that purpose, we consider the following geometric properties:

**Definition 1.4.**

(i) A toric variety \( X \) is called **nondegenerate** if it admits no toric decomposition \( X \cong X' \times \mathbb{K}^* \).

(ii) We call a toric variety \( X \) **2-complete** if it does not admit a toric open embedding \( X \subset X' \) with \( X' \setminus X \) nonempty of codimension at least two.

(iii) We call a toric variety \( X \) **full** if it is 2-complete and every \( T_X \)-orbit has a fixed point in its closure.

The notion of 2-completeness already occurs in [2]. It generalizes completeness in the sense that a toric variety is complete if and only if it is “1-complete”. For example, the affine space \( \mathbb{K}^n \) is 2-complete, whereas for a toric variety \( X \) of dimension at least two and a fixed point \( x \in X \), the variety \( X \setminus \{x\} \) is not 2-complete.

In terms of fans, the properties introduced in Definition 1.4 are characterized as follows:

**Remark 1.5.** Let \( X \) be the toric variety arising from a fan \( \Delta \) in a lattice \( N \).

(i) \( X \) is nondegenerate if and only if the support \( |\Delta| \) generates \( N_\mathbb{Q} \) as a vector space.

(ii) \( X \) is 2-complete if and only if the fan \( \Delta \) cannot be enlarged without adding new rays.

(iii) \( X \) is full if and only if \( \Delta \) is as in (ii) and every maximal cone of \( \Delta \) is of full dimension.

2. The category of bunches

In this section, we introduce the language of bunches. Intuitively, one should think of a bunch as a collection of pairwise overlapping lattice cones, which satisfies certain irredundancy and maximality properties.

The precise definition of the category of bunches is performed in three steps. The first one is to introduce the category of projected cones:

**Definition 2.1.**

(i) A **projected cone** is a pair \( (E \twoheadrightarrow K, \gamma) \), where \( Q: E \rightarrow K \) is an epimorphism of lattices and \( \gamma \subset E_\mathbb{Q} \) is a simplicial cone of full dimension.
(ii) A *morphism of projected cones* \((E_i \xrightarrow{Q_i} K_i, \gamma_i)\) is a homomorphism \(\Phi: E_1 \to E_2\) such that \(\Phi(\gamma_1) \subseteq \gamma_2\) holds and there is a commutative diagram

\[
\begin{array}{ccc}
E_1 & \xrightarrow{\Phi} & E_2 \\
Q_1 & \downarrow & \downarrow Q_2 \\
K_1 & \xrightarrow{\Phi} & K_2
\end{array}
\]

In the second step, we give the definition of bunches. Such a bunch will live in a *projected cone* \((E \xrightarrow{Q} K, \gamma)\). By a *projected face* in \(K\) we mean the image \(Q(\gamma_0)\) of a face \(\gamma_0 \preceq \gamma\).

**Definition 2.2.** A *bunch* in \((E \xrightarrow{Q} K, \gamma)\) is a nonempty collection \(\Theta\) of projected faces in \(K\) with the following property: A projected face \(\tau_0 \subset K\) belongs to \(\Theta\) if and only if

\[
\emptyset \neq \tau_0^\circ \cap \tau_0^\circ \neq \tau^\circ \quad \text{holds for all } \tau \in \Theta \text{ with } \tau \neq \tau_0.
\]

Let us reformulate this definition in a less formal way. We say that two cones \(\tau_1\) and \(\tau_2\) *overlap*, if \(\tau_1^\circ \cap \tau_2^\circ \neq \emptyset\) holds. Now, a nonempty collection \(\Theta\) of projected faces is a bunch if and only if it has the following properties:

- any two members of \(\Theta\) overlap,
- there is no pair \(\tau_1, \tau_2 \in \Theta\) with \(\tau_1 \subseteq \tau_2\),
- if a projected face \(\tau_0\) overlaps each \(\tau \in \Theta\), then \(\tau \subseteq \tau_0\) for a \(\tau_1 \in \Theta\).

**Example 2.3.** Let \(E := \mathbb{Z}^n\), and let \(K = \mathbb{Z}\). Moreover, fix a sequence \(w_1, \ldots, w_n\) of positive integers having greatest common divisor one. This gives an epimorphism

\[
Q: E \to K, \quad e_i \mapsto w_i,
\]

where \(e_i\) denotes the \(i\)-th canonical base vector. Setting \(\gamma := \text{cone}(e_1, \ldots, e_n)\), we obtain a projected cone \((E \xrightarrow{Q} K, \gamma)\), and \(\Theta := \{Q(\gamma)\}\) is a bunch.

Finally, as the third step, we have to fix the notion of a morphism of bunches. For this, we first have to “unpack” the combinatorial information contained in a bunch. This is done by constructing a further collection of cones:

**Definition 2.4.** Let \(\Theta\) be a bunch in a projected cone \((E \xrightarrow{Q} K, \gamma)\). The *covering collection* of \(\Theta\) is

\[
\text{cov}(\Theta) := \{\gamma_0 \preceq \gamma; \gamma_0 \text{ minimal with } Q(\gamma_0) \supset \tau \text{ for some } \tau \in \Theta\}.
\]

As Example 2.3 shows, \(\text{cov}(\Theta)\) will in general comprise much more cones than \(\Theta\) itself. We can reconstruct the bunch from its covering collection:

\[
\Theta = \{\tau; \tau \text{ minimal with } \tau = Q(\gamma_0) \text{ for some } \gamma_0 \in \text{cov}(\Theta)\}.
\]

In general, for an element \(\gamma_0 \in \text{cov}(\Theta)\), the image \(Q(\gamma_0)\) need not be an element of \(\Theta\). For later purposes, the following observation will be crucial:

**Lemma 2.5 (Overlapping Property).** Let \(\Theta\) be a bunch in \((E \xrightarrow{Q} K, \gamma)\). For any two \(\gamma_1, \gamma_2 \in \text{cov}(\Theta)\), we have \(Q(\gamma_1)^\circ \cap Q(\gamma_2)^\circ \neq \emptyset\).
Proof. Let \( \sigma_i \assign Q(\gamma_i) \). By the definition of \( \text{cov}(\Theta) \), there exist cones \( \tau_1, \tau_2 \in \Theta \) with \( \tau_i \subset \sigma_i \). Now assume that the relative interiors of the cones \( \sigma_i \) are disjoint. Then there is a proper face \( \sigma_0 \prec \sigma_1 \) such that \( \sigma_1 \cap \sigma_2 \) is contained in \( \sigma_0 \).

Clearly, \( \tau_1 \cap \tau_2 \) is contained in \( \sigma_0 \). Moreover, by the condition \( \ref{2.2.1} \) the intersection \( \tau^0 \cap \tau^2 \) is not empty. In particular, \( \tau^0 \) meets \( \sigma_0 \). Since \( \sigma_0 \) is a face of \( \sigma_1 \), we conclude \( \tau_1 \subset \sigma_0 \). Thus \( \gamma_0 \assign Q^{-1}(\sigma_0) \cap \gamma_1 \) is a proper face of \( \gamma_1 \) such that \( Q(\gamma_0) = \sigma_0 \) contains an element of \( \Theta \). This contradicts minimality of \( \gamma_1 \). \( \Box \)

We come back to the definition of a morphism of bunches. It is formulated in terms of the respective covering collections:

**Definition 2.6.** Let \( \Theta_i \) be bunches in projected cones \( (E_i \fib \Theta_i, \gamma_i) \). A morphism from \( \Theta_1 \) to \( \Theta_2 \) is a morphism \( \Phi \colon E_1 \to E_2 \) of the projected cones such that for every \( \alpha_2 \in \text{cov}(\Theta_2) \) there is an \( \alpha_1 \in \text{cov}(\Theta_1) \) with \( \Phi(\alpha_1) \subset \alpha_2 \).

This concludes the definition of the category of bunches. The notion of an isomorphism is characterized as follows:

**Proposition 2.7.** A morphism \( \Phi \) of bunches \( \Theta_1 \) and \( \Theta_2 \) is an isomorphism if and only if \( \Phi \) is an isomorphism of the ambient projected cones and the induced map \( \Phi^\sim \) defines a bijection \( \Theta_1 \to \Theta_2 \).

**Proof.** Let the bunch \( \Theta_i \) live in the projected cone \( (E_i \fib \Theta_i, \gamma_i) \). Suppose first that \( \Phi \colon E_1 \to E_2 \) is an isomorphism of the bunches. Then there is a morphism of bunches \( \Psi \colon E_2 \to E_1 \) from \( \Theta_2 \) to \( \Theta_1 \) such that \( \Phi \) and \( \Psi \) are inverse to each other as lattice homomorphisms. Note that \( \Phi \) and \( \Psi \) are as well inverse to each other as morphisms of projected cones.

In order to see that \( \Phi \colon K_1 \to K_2 \) defines a bijection \( \Theta_1 \to \Theta_2 \), it suffices to show that \( \Phi \) defines a bijection \( \text{cov}(\Theta_1) \to \text{cov}(\Theta_2) \). By bijectivity of \( \Phi \) and \( \Psi \), we only have to show that for every \( \alpha_1 \in \text{cov}(\Theta_1) \) the image \( \Phi(\alpha_1) \) belongs to \( \text{cov}(\Theta_2) \). This is done as follows:

Given \( \alpha_1 \in \text{cov}(\Theta_1) \), we apply Definition \( \ref{2.6} \) to \( \Psi \), and obtain an \( \alpha_2 \in \text{cov}(\Theta_2) \) with \( \Psi(\alpha_2) \subset \alpha_1 \). Applying \( \Phi \) gives \( \alpha_2 \subset \Phi(\alpha_1) \). Again by Definition \( \ref{2.6} \) we find an \( \alpha_1 \in \text{cov}(\Theta_1) \) with \( \Phi(\alpha_1) \subset \alpha_2 \). Thus we have \( \Phi(\alpha_1) \subset \Phi(\alpha_1) \). By the definition of \( \text{cov}(\Theta_1) \), we obtain \( \alpha_1 = \alpha_2 \). Consequently, \( \Phi(\alpha_1) = \alpha_2 \) belongs to \( \text{cov}(\Theta_2) \).

Now suppose that \( \Phi \) is an isomorphism of projected cones and that \( \Phi^\sim \) defines a bijection \( \Theta_1 \to \Theta_2 \). Let \( \Psi \colon E_2 \to E_1 \) denote the inverse of \( \Phi \) as a morphism of projected cones. The only thing we have to show is that \( \Psi \) is a morphism from \( \Theta_2 \) to \( \Theta_1 \). This is done below:

Let \( \alpha_1 \in \text{cov}(\Theta_1) \). Then \( \alpha_2 \assign \Phi(\alpha_1) \) is a face of \( \gamma_2 \). We check \( \alpha_2 \in \text{cov}(\Theta_2) \). By the definition of \( \text{cov}(\Theta_1) \), the face \( \alpha_1 \leq \gamma_1 \) is minimal with \( Q_1(\alpha_1) \supset \tau_1 \) for some \( \tau_1 \in \Theta_1 \). Thus \( \alpha_2 \leq \gamma_2 \) is minimal with \( Q_2(\alpha_2) \supset \Phi(\tau_1) \) for some \( \tau_1 \in \Theta_1 \). Since \( \Phi \) induces a bijection \( \Theta_1 \to \Theta_2 \), we obtain \( \alpha_2 \in \text{cov}(\Theta_2) \). \( \Box \)

The reminder of this section is devoted to the visualization of bunches. The idea is that one should be able to recover many basic properties of a bunch \( \Theta \) without knowing the ambient projected cone \( (E \fib \Theta, \gamma) \). This will work for the following important class of bunches:

**Definition 2.8.** By a free bunch we mean a bunch \( \Theta \) in a projected cone \( (E \fib \Theta, \gamma) \), where \( \gamma \) is a regular cone in \( E \).
The following construction shows that every free bunch arises from a certain collection of data in some lattice $K$:

**Construction 2.9.** Let $(w_1, \ldots, w_n)$ be a sequence in a lattice $K$ such that the $w_i$ generate $K$. We speak of the weight vectors $w_i$, and call any cone generated by some of the $w_i$ a weight cone. Let $\Theta$ be a collection of weight cones in $K$ satisfying Condition 2.2.1 i.e., a weight cone $\tau_0$ belongs to $\Theta$ if and only if

\[(2.9.1) \emptyset \neq \tau_0 \cap \tau^\circ \neq \tau^\circ \text{ holds for all } \tau \in \Theta \text{ with } \tau \neq \tau_0.\]

Then there is an associated projected cone $(E \overset{\omega}{\to} K, \gamma)$ with the lattice $E := \mathbb{Z}^n$, the cone $\gamma := \text{cone}(e_1, \ldots, e_n)$ spanned by the canonical base vectors, and the map $Q: E \to K$ sending $e_i$ to $w_i$. By construction, the collection $\Theta$ is a bunch in the projected cone $(E \overset{\omega}{\to} K, \gamma)$.

This construction allows us to visualize bunches. We regard a given sequence of weight vectors as a set $\{w_1, \ldots, w_n\}$, where each $w_i$ has a multiplicity $\mu_i$ counting the number of its repetitions. Then we may put these data as well as the cones of a given bunch in a picture. For example, in the setting of 2.8, the bunch $\Theta$ defined by the sequence $(1, 3, 5, 5)$ arises from the picture:

As one might expect, this picture describes the three-dimensional weighted projective space $\mathbb{P}_{1,3,5,5}$. Moreover, as we shall see in Proposition 10.1, the smooth complete toric varieties with Picard group $\mathbb{Z}^2$ arise from sequences in $\mathbb{Z}^2$ and a collection $\Theta = \{\tau\}$ according to the following picture:

In order to compare two bunches arising from Construction 2.9, there is no need to determine the covering collection. Namely, using Proposition 2.7, we obtain:

**Remark 2.10.** Two sets of data $(K; w_1, \ldots, w_n; \Theta)$ and $(K'; w'_1, \ldots, w'_n; \Theta')$ as in 2.9 have isomorphic associated free bunches if and only if there is a lattice isomorphism $\Phi: K \to K'$ such that

(i) $(w'_1, \ldots, w'_n)$ and $(\Phi(w_1), \ldots, \Phi(w_n))$ differ only by enumeration,

(ii) the collections $\Theta'$ and $\{\Phi(\tau); \tau \in \Theta\}$ coincide.

### 3. The Basic Duality Lemmas

In this section, we provide basic duality statements for translating from the language of bunches into the language of fans. First we need a concept of a dual of a given projected cone $(E \overset{\omega}{\to} K, \gamma)$. For this, note that $(E \overset{\omega}{\to} K, \gamma)$ determines two exact sequences of lattice homomorphisms

$$
0 \longrightarrow M \longrightarrow E \overset{Q}{\longrightarrow} K \longrightarrow 0,
$$

$$
0 \longrightarrow L \longrightarrow F \overset{P}{\longrightarrow} N \longrightarrow 0,
$$

where $M$ is the kernel of $Q: E \to K$, and the second sequence arises from the first one by applying $\text{Hom}(?, \mathbb{Z})$; note that $P$ is not the dual homomorphism of $Q$. Let
\( \delta := \gamma^\vee \) denote the dual cone. Then \( \delta \) is again strictly convex, simplicial and of full dimension.

**Definition 3.1.** We call \((F \stackrel{P}{\rightarrow} N, \delta)\) the dual projected cone of \((E \stackrel{Q}{\rightarrow} K, \gamma)\).

In the sequel, fix a projected cone \((E \stackrel{Q}{\rightarrow} K, \gamma)\), and denote the associated dual projected cone by \((F \stackrel{P}{\rightarrow} N, \delta)\). Recall that we have the face correspondence, see for example [1, Appendix A]:

**Remark 3.2.** The sets of faces of the cones \( \gamma \subset E_Q \) and \( \delta \subset F_Q \) are in order reversing correspondence via

\[
\text{faces}(\gamma) \rightarrow \text{faces}(\delta), \quad \gamma_0 \mapsto \gamma_0^\perp := \gamma_0^\perp \cap \delta.
\]

Our task is to understand the relations between the projected faces \(Q(\gamma_0)\), where \(\gamma_0 \preceq \gamma\), and the images \(P(\gamma_0^\perp)\) of the corresponding faces. The following observation is central:

**Lemma 3.3 (Invariant Separation Lemma).** Let \(\gamma_1, \gamma_2 \preceq \gamma\), let \(\delta_i := \gamma_i^\perp\), and let 
\(L := \ker(P)\). Then the following statements are equivalent:

(i) There is an \(L\)-invariant separating linear form for \(\delta_1 \) and \(\delta_2\).

(ii) For the relative interiors \(Q(\gamma_i)^\circ\) we have \(Q(\gamma_1)^\circ \cap Q(\gamma_2)^\circ \neq \emptyset\).

Here, by an \(L\)-invariant linear form we mean an element \(u \in E = \text{Hom}(F, \mathbb{Z})\) with \(L \subset u^\perp\). Moreover, recall from Section [4] that a separating linear form for the cones \(\delta_1 \) and \(\delta_2\) is an element \(u \in E\) with

\[
u|_{\delta_1} \geq 0, \quad u|_{\delta_2} \leq 0, \quad u^\perp \cap \delta_1 = u^\perp \cap \delta_2 = \delta_1 \cap \delta_2.
\]

**Proof of Lemma 3.3** As before, let \(M := \ker(Q)\). Then the \(L\)-invariant linear forms on \(F\) are precisely the elements of \(M\). Thus, since \(\delta_1 \cap \delta_2\) is a face of both \(\delta_i\), condition (i) is equivalent to

\[
M \cap (\delta_1^\perp \cap (\delta_1 \cap \delta_2)^\perp)^\circ \cap -(\delta_2^\perp \cap (\delta_1 \cap \delta_2)^\perp)^\circ \neq \emptyset.
\]

Note that \(\delta_i^\perp\) equals \(\text{lin}(\gamma_i) + \gamma\). Moreover, \((\delta_1 \cap \delta_2)^\perp\) equals \(\text{lin}(\gamma_1) + \text{lin}(\gamma_2)\), because the cone \(\delta\) is simplicial. Hence we obtain

\[
(\delta_1^\perp \cap (\delta_1 \cap \delta_2)^\perp)^\circ = (\text{lin}(\gamma_1) + \gamma) \cap (\text{lin}(\gamma_1) + \text{lin}(\gamma_2))^\circ = \text{lin}(\gamma_1) + \gamma_2^\perp.
\]

For the second equality we used simpliciality of \(\gamma\). By analogous arguments, the expression \(- (\delta_1^\perp \cap (\delta_1 \cap \delta_2)^\perp)^\circ\) simplifies to \(\text{lin}(\gamma_2) - \gamma_1^\perp\). Thus (3.3.1) holds if and only if

\[
M \cap (\text{lin}(\gamma_1) + \gamma_2^\perp) \cap (\text{lin}(\gamma_2) - \gamma_1^\perp) \neq \emptyset.
\]

We claim that the left hand side simplifies to \(M \cap (\gamma_2^\perp - \gamma_1^\perp)\). Indeed, any \(u \in M\) belonging to the left hand side of (3.3.2) has a unique representation in terms of the primitive generators of \(e_1, \ldots, e_n\) of \(\gamma\):

\[
u = \sum_{e_i \in \gamma_1 \setminus \gamma_2} a_i e_i + \sum_{e_j \in \gamma_1 \cap \gamma_2} b_j e_j + \sum_{e_k \in \gamma_2 \setminus \gamma_1} c_k e_k, \quad \text{where } a_i < 0 \text{ and } c_k > 0.
\]

Now, dividing the middle term into two sums, one with only positive coefficients and the other with only negative ones, gives \(u \in \gamma_2^\perp - \gamma_1^\perp\). The reverse inclusion is obvious.
Consequently, it is equivalent to $M \cap (\gamma_2^* - \gamma_1^*) \neq \emptyset$. This in turn is obviously equivalent to condition (ii). \qed

Let us mention here that simpliciality of the cones $\gamma$ and $\delta$ is essential for the Invariant Separation Lemma:

**Example 3.4.** Consider the “projected cone” $(E \xrightarrow{Q} K, \gamma)$, where the lattices are $E := \mathbb{Z}^2$ and $K := \mathbb{Z}$, the map $Q$ is the projection onto the third coordinate, and the cone $\gamma$ is given in terms of canonical base vectors by

$$\gamma = \text{cone}(e_1 + e_3, e_2 + 2e_3, e_1 - 2e_3, e_2 - e_3).$$

Denote the “dual projected cone” by $(F \xrightarrow{\delta} N, \gamma)$. Then $L := \ker(P)$ is the sublattice generated by the dual base vector $e_3^*$, and the cone $\delta$ is given by

$$\delta = \text{cone}(e_1^*, e_2^*, e_1^* + 2e_2^* - 2e_3^*, 2e_1^* + e_2^* + e_3^*).$$

The faces $\gamma_1 := \text{cone}(e_2 - e_3)$ and $\gamma_2 := \text{cone}(e_1 + e_3)$ do not satisfy Condition (ii).

Nevertheless, the corresponding faces

$$\gamma_1^* = \text{cone}(e_1^* + 2e_1^* + e_2^* + e_3^*), \quad \gamma_2^* = \text{cone}(e_2^* + e_1^* + 2e_2^* - e_3^*),$$

admit $L$-invariant separating linear forms. For example we can take the linear form $e_1 - e_2 \in E$.

Next we compare injectivity of $Q$ with surjectivity of $P$ along corresponding faces (of course, the roles of $Q, \delta_0$ etc. and $P, \gamma_0$, etc. can be interchanged in the statement):

**Lemma 3.5.** For a face $\gamma_0 \leq \gamma$ and $\delta_0 := \gamma_0^*$, the following statements are equivalent:

(i) $P$ maps $\text{lin}(\delta_0)$ onto $N_0$.

(ii) $Q$ is injective on $\text{lin}(\gamma_0)$.

Proof. Let $M := \ker(Q)$ and $L := \ker(P)$. Using the fact that $\text{lin}(\delta_0)$ and $\text{lin}(\gamma_0)$ are the orthogonal spaces of each other, we obtain the assertion by dualizing:

$$\text{lin}(\delta_0) + L_Q = F_Q \iff \text{lin}(\gamma_0) \cap M_Q = \{0\}. \quad \Box$$

If we take the lattice structure into consideration, then the situation becomes slightly more involved. The essential observation is:

**Lemma 3.6.** For a face $\gamma_0 \leq \gamma$ and $\delta_0 := \gamma_0^*$, the following statements are equivalent:

(i) $P$ maps $\text{lin}(\delta_0) \cap F$ onto $N$.

(ii) $Q$ maps $\text{lin}(\gamma_0) \cap E$ isomorphically onto a primitive sublattice of $K$.

Proof. Set $L := \ker(P)$. Assume that (i) holds. Then the snake lemma provides an exact sequence

$$0 \longrightarrow L \cap \text{lin}(\delta_0) \longrightarrow L \longrightarrow F/(\text{lin}(\delta_0) \cap F) \longrightarrow 0.$$ 

The dual lattice of $F/(\text{lin}(\delta_0) \cap F)$ is canonically isomorphic to $E \cap \text{lin}(\gamma_0)$. Hence, applying $\text{Hom}(?, \mathbb{Z})$ gives an exact sequence

$$0 \longrightarrow E \cap \text{lin}(\gamma_0) \xrightarrow{Q} K \longrightarrow \text{Hom}(L \cap \text{lin}(\delta_0), \mathbb{Z}) \longrightarrow 0.$$ 

This implies condition (ii). The reverse direction can be settled by similar arguments. \qed
Finally, we consider morphisms $\Phi: E_1 \to E_2$ of projected cones $(E_i \xrightarrow{Q_i} K_i, \gamma_i)$. These data define a commutative diagram of lattices with exact rows:

\[
\begin{array}{cccccccc}
0 & \to & M_1 & \to & E_1 & \xrightarrow{Q_1} & K_1 & \to & 0 \\
\downarrow & & \downarrow \Phi & & \downarrow \Phi & & \downarrow & \\
0 & \to & M_2 & \to & E_2 & \xrightarrow{Q_2} & K_2 & \to & 0
\end{array}
\]

Applying $\text{Hom}(?, \mathbb{Z})$ to this diagram, we obtain the following commutative diagram, again with exact rows:

\[
\begin{array}{cccccccc}
0 & \to & L_1 & \to & F_1 & \xrightarrow{P_1} & N_1 & \to & 0 \\
\downarrow & & \downarrow \Psi & & \downarrow \Psi & & \downarrow & \\
0 & \to & L_2 & \to & F_2 & \xrightarrow{P_2} & N_2 & \to & 0
\end{array}
\]

Remark 3.7. The dual map $\Psi: F_2 \to F_1$ is a morphism of the dual projected cones $(F_i \xrightarrow{P_i} N_i, \delta_i)$ of $(E_i \xrightarrow{Q_i} K_i, \gamma_i)$.

Now, consider faces $\alpha_i \preceq \gamma_i$, and let $\beta_i := \alpha_i^*$ denote the corresponding faces of the cones $\delta_i$.

Lemma 3.8. We have $\Phi(\alpha_1) \subset \alpha_2$ if and only if $\Psi(\beta_2) \subset \beta_1$ holds.

Proof. We only have to verify one implication. The other then is a simple consequence of $\alpha_i = \beta_i^*$. So, suppose $\Phi(\alpha_1) \subset \alpha_2$. Then we obtain

$\Psi(\beta_2) = \Psi(\alpha_1^\perp \cap \delta_2) \subset \Psi(\alpha_1^\perp) \cap \Psi(\delta_2) \subset \alpha_1^\perp \cap \delta_1 = \beta_1$. □

4. Bunches and fans

In this section, we compare bunches with fans. We shall show that the category of bunches is equivalent to the category of “maximal projectable fans”, see Theorem 4.6. The latter category is defined as follows:

Definition 4.1. (i) Let $(F \xrightarrow{P} N, \delta)$ be a projected cone, and let $L := \ker(P)$.

(a) A projectable fan in $(F \xrightarrow{P} N, \delta)$ is a fan $\Sigma$ consisting of faces of $\delta$ such that any two maximal cones of $\Sigma$ can be separated by an $L$-invariant linear form.

(b) We call a projectable fan $\Sigma$ in $(F \xrightarrow{P} N, \delta)$ maximal if any $\delta_0 \prec \delta$, which can be separated by $L$-invariant linear forms from the maximal cones of $\Sigma$, belongs to $\Sigma$.

(ii) A morphism of projectable fans $\Sigma_i$ (maximal or not) in projected cones $(F_i \xrightarrow{P_i} N_i, \delta_i)$ is a morphism $\Psi: F_1 \to F_2$ of projected cones which is in addition a map of the fans $\Sigma_i$.

Note that a projectable fan is the collection of all faces of the cones belonging to a “locally coherent costring” in the sense of [14, Def. 5.1], but the converse does not hold in general. We shall demonstrate later by means of an example the importance of the maximality condition (b), see 4.7.
We define now a functor $\mathfrak{F}$ from bunches to maximal projectable fans. Let $\Theta$ be a bunch in the projected cone $(E \xrightarrow{\sigma} K, \gamma)$. Consider the associated dual projected cone $(F \xrightarrow{\rho} N, \delta)$ and the following subfan of the fan of faces of $\delta$:

$$\Sigma := \{\sigma \leq \delta; \sigma \leq \gamma_0^* \text{ for some } \gamma_0 \in \text{cov}(\Theta)\}.$$ 

**Lemma 4.2.** $\Sigma$ is a maximal projectable fan in $(F \xrightarrow{\rho} N, \delta)$.

**Proof.** Let $L := \ker(F)$. By the Overlapping Property $2.5$ of the Covering Collection $\text{cov}(\Theta)$ and the Invariant Separation Lemma $3.3$, any two maximal cones of $\Sigma$ can be separated by $L$-invariant linear forms. So we only have to verify the maximality condition $(i)$ $(b)$ for $\Sigma$.

Suppose that the face $\sigma \leq \delta$ can be separated by $L$-invariant linear forms from the maximal cones of $\Sigma$ but does not belong to $\Sigma$. The projected face $\tau_0 := Q(\sigma^*)$ does not belong to $\Theta$, because otherwise any minimal face $\gamma_0 \leq \sigma^*$ projecting onto $\tau_0$ would belong to $\text{cov}(\Theta)$, which contradicts the choice of $\sigma$.

The Invariant Separation Lemma $3.3$ yields $\tau_0^* \cap \tau^\circ \neq \emptyset$ for every $\tau \in \Theta$. Since $\tau_0$ is not an element of $\Theta$, it has to contain some element of $\Theta$. But then some face of $\sigma^*$ belongs to the Covering Collection $\text{cov}(\Theta)$. Again this contradicts the choice of the face $\sigma \leq \delta$. $\square$

The assignment $\Theta \mapsto \Sigma$ extends canonically to morphisms. Namely, let $\Theta_i$ be bunches in projected cones $(E_i \xrightarrow{Q_i} K_i, \gamma_i)$. Let $\Sigma_i$ denote the associated maximal projectable fans in the respective dual projected cones $(F_i \xrightarrow{\rho_i} N_i, \delta_i)$.

Lemma $3.3$ tells us that for every morphism $\Phi: E_1 \rightarrow E_2$ of the bunches $\Theta_1$ and $\Theta_2$, the dual map $\Psi: F_2 \rightarrow F_1$ is a morphism of the maximal projectable fans $\Sigma_2$ and $\Sigma_1$. Thus, we obtain:

**Proposition 4.3.** The assignments $\Theta \mapsto \Sigma$ and $\Phi \mapsto \Psi$ define a contravariant functor $\mathfrak{F}$ from bunches to the category of maximal projectable fans. $\square$

Now we go the other way round. Consider a maximal projectable fan $\Sigma$ in a projected cone $(F \xrightarrow{\rho} N, \delta)$. Let $(E \xrightarrow{\sigma} K, \gamma)$ denote the associated dual projected cone. Define $\Theta$ to be the set of the minimal cones among all projected faces in $K$ arising from $\Sigma$:

$$\Theta := \{\tau_0; \tau_0 \text{ minimal with } \tau_0 = Q(\delta_0^*) \text{ for some } \delta_0 \in \Sigma^{\text{max}}\}.$$ 

**Lemma 4.4.** $\Theta$ is a bunch in $(E \xrightarrow{\sigma} K, \gamma)$.

**Proof.** We verify Property $2.2.1$ for a given $\tau_0 \in \Theta$. According to the Invariant Separation Lemma $3.3$, we have $\tau_0^* \cap \tau^\circ \neq \emptyset$ for any further $\tau \in \Theta$. Moreover, since $\Theta$ consists of minimal cones, $\tau^\circ$ is not contained in $\tau_0^*$ provided that $\tau \in \Theta$ is different from $\tau_0$.

Conversely, let the projected face $\tau_0$ satisfy $2.2.1$. Choose $\gamma_0 \leq \gamma$ with $Q(\gamma_0) = \tau_0$. The Invariant Separation Lemma $3.3$ tells us that $\delta_0 := \gamma_0^*$ belongs to $\Sigma$. Let $\delta_1 \in \Sigma$ be a maximal cone with $\delta_0 \leq \delta_1$, and consider the image $\tau_1 := Q(\delta_1^*)$. Then we have $\tau_1 \subset \tau_0$, because $\delta_1^* \leq \delta_0^*$ holds.

By the definition of the collection $\Theta$, there is a cone $\tau_2 \in \Theta$ with $\tau_2 \subset \tau_1$. In particular, we have $\tau_2 \subset \tau_0$. Applying once more the Invariant Separation Lemma $3.3$ gives even $\tau_2^* \subset \tau_0^*$, Thus Property $2.2.1$ yields $\tau_0 = \tau_2$. This shows $\tau_0 \in \Theta$. $\square$
According to Lemma 3.8, associating to a map $\Psi$ of maximal projectable fans its dual map $\Phi$ makes this construction functorial. Thus we have:

**Proposition 4.5.** The assignments $\Sigma \mapsto \Theta$ and $\Psi \mapsto \Phi$ define a contravariant functor $\mathcal{B}$ from the category of maximal projectable fans to the category of bunches.

Summing up, we arrive at the main result of this section, namely the following duality statement:

**Theorem 4.6.** The functors $\mathfrak{F}$ and $\mathcal{B}$ are inverse to each other. In particular, the categories of bunches and maximal projectable fans are dual to each other.

Let us emphasize here the role of the maximality condition 4.1 (i) (b) in this result. The following example shows that there is no hope for a similar statement on (non maximal) projectable fans:

**Example 4.7.** Consider the projected cone $(F \xrightarrow{P} N, \delta)$, where the lattices are $F := \mathbb{Z}^3$ and $N := \mathbb{Z}^2$, the cone $\delta$ is the positive orthant in $F_\mathbb{Q}$, and the projection map is given by

$$P : F \to N, \quad (v_1, v_2, v_3) \mapsto (v_1 - v_3, v_2 - v_3).$$

Then the fan $\Sigma$ in $F$ having $\delta_1 := \text{cone}(e_1, e_2)$ and $\delta_2 := \text{cone}(e_1, e_3)$ as its maximal cones is projectable. But $\Sigma$ is not maximal, because we may enlarge it to a (maximal) projectable fan $\Sigma'$ by adding the cone $\delta_3 := \text{cone}(e_2, e_3)$.

Let $(E \xrightarrow{\gamma} K, \gamma)$ be the dual projected cone of $(F \xrightarrow{P} N, \delta)$. Then we have $E \cong \mathbb{Z}^3$ and $K \cong \mathbb{Z}$. Moreover the projection $Q$ sends each dual base vector $e_i^*$ to $1 \in \mathbb{Z}$. In particular, we obtain

$$Q(\delta_i^*) = Q(\delta_i^\perp \cap \gamma) = \mathbb{Q}_{\geq 0}, \quad i = 1, 2, 3.$$

Thus, $\Sigma$ and $\Sigma'$ determine the same collection $\Theta = \{\mathbb{Q}_{\geq 0}\}$ of projected faces in $K$. In other words, there is no way to reconstruct $\Sigma$ via face duality from a collection of projected faces in $K$.

In the rest of this section, we associate to any maximal projectable fan its “quotient fan”. So, let $\Sigma$ be a maximal projectable fan in a weighted lattice $(F \xrightarrow{P} N, \delta)$. Then the images $P(\sigma)$, where $\sigma$ runs through the maximal cones of $\Sigma$ are the maximal cones of a quasifan $\Sigma'$ in $N$.

We reduce $\Sigma'$ to a fan as follows: Let $L' \subset N$ be the primitive sublattice generating the minimal cone of $\Sigma'$, let $N' := N/L'$, and let $P' : N \to N'$ denote the projection.

**Definition 4.8.** The quotient fan of $\Sigma$ is $\Delta := \{P'(\sigma') \mid \sigma' \in \Sigma'\}$.

Note that $R := P' \circ P : F \to N'$ is a map of the fans $\Sigma$ and $\Delta$. In fact, this is a special case of a more general construction, see [1, Theorem 2.3]. In our setting, it is easy to see that everything is compatible with morphisms. Thus we obtain:

**Proposition 4.9.** The assignment $\Sigma \mapsto \Delta$ defines a covariant functor $\mathcal{Q}$ from the category of maximal projectable fans to the category of fans.

The following simple example shows that dividing by $L'$ in the construction of quotient fan is indeed necessary:
Example 4.10. Consider the lattices $E := \mathbb{Z}^2$ and $K := \mathbb{Z}$, the map $Q: E \to K$, $(u_1, u_2) \mapsto u_1 + u_2$, and the positive orthant $\gamma \subset \mathbb{Q}^2$. Let $\Theta$ be the bunch consisting just of the trivial cone $\{0\}$. Then the quasifan $\Sigma'$ determined by $\Theta$ consists of the single cone $\sigma := Q$.

We note an observation on the composition $Q \circ F$. Consider a bunch $\Theta$ in $(E \xrightarrow{Q} K, \gamma)$ and its associated maximal projectable fan $\Sigma$ in $(F \xrightarrow{\nu} N, \delta)$. Let $\Delta$ be the quotient fan of $\Sigma$, and, as before, let $R: F \to N'$ be the projection.

Proposition 4.11. There is a canonical order reversing bijection $\{\gamma_0 \preceq \gamma; \tau \circ \subset Q(\gamma_0)\} \to \Delta, \gamma_0 \mapsto R(\gamma_0^\circ)$.

Proof. The inverse map is given by $\sigma \mapsto (R^{-1}(\sigma) \cap |\Sigma|)^\circ$. \hfill $\square$

5. Combinatorics of quotients

Here we present the first application of the language of bunches. We consider the action of a subtorus on a $Q$-factorial nondegenerate affine toric variety and give a combinatorial description of the maximal open subsets admitting a good quotient by this action. This complements results of [4] for torus actions on $X = \mathbb{C}^n$.

Let us first recall the basic concepts concerning good quotients. Let the reductive group $G$ act on a variety $X$ by means of a morphism $G \times X \to X$. A good quotient for this action is a $G$-invariant affine morphism $p: X \to Y$ such that the canonical map $O_Y \to p^*(O_X)^G$ is an isomorphism. If it exists, then the good quotient space is usually denoted by $X//G$.

In general, a $G$-variety $X$ need not admit a good quotient $X \to X//G$, but there frequently exist many invariant open subsets $U \subset X$ with good quotient $U \to U//G$. It is one of the central tasks of Geometric Invariant Theory to describe all these open subsets, see [3, Section 7.2]. In the course of this problem, one reasonably looks for maximal $U \subset X$ in the following sense, see [3, Section 7.2] and [4]:

Definition 5.1. An open subset $U \subset X$ is called $G$-maximal, if there is a good quotient $p: U \to U//G$ and there is no open $U' \subset X$ admitting a good quotient $p': U' \to U'///G$ such that $U$ is a proper $p'$-saturated subset of $U'$.

In the setting of subtorus actions, the maximal open subsets with good quotient can be characterized in terms of fans. This relies on the following observation due to Święcicka, see [22, Proposition 2.5]:

Proposition 5.2. Let $X$ be a toric variety, and let $T \subset T_X$ be a subtorus of the big torus. If $U \subset X$ is a $T$-maximal subset, then $U$ is invariant under $T_X$.

Now, let $X$ be the toric variety arising from a fan $\Delta$ in a lattice $N$, let $T \subset T_X$ be the subtorus corresponding to a primitive sublattice $L \subset N$. By the above proposition, the $T$-maximal subsets $U \subset X$ correspond to certain subfans of $\Delta$. The characterization of these fans is standard, see e.g. [12, Proposition 1.3]:

Proposition 5.3. Let $U \subset X$ be the open $T_X$-invariant subset defined by a subfan $\Sigma$ of $\Delta$.

(i) There is a good quotient $U \to U//T$ if and only if any two maximal cones of $\Sigma$ can be separated by an $L$-invariant linear form.
(ii) $U$ is $T$-maximal if and only if (i) holds and every $\sigma \in \Delta$ that can be separated by $L$-invariant linear forms from the maximal cones of $\Sigma$ belongs to $\Sigma$.

Though this is a complete combinatorial description of all $T$-maximal subsets, it has two drawbacks in practice: On the one hand, the ambient space of the combinatorial data might be of quite big dimension, and, on the other hand, for the explicit checking of the conditions there may be large numbers of cones to go through. The language of bunches makes the situation more clear.

Let $X$ be an affine nondegenerate $\mathbb{Q}$-factorial toric variety arising from a cone $\delta$ in a lattice $F$, and let $T \subset T_X$ be the subtorus corresponding to a sublattice $L \subset F$. Setting $N := F/L$, we obtain a projected cone $(F \xrightarrow{\delta} N, \delta)$. Moreover, we have the dual projected cone $(E \xrightarrow{\gamma} K, \gamma)$, where $K$ is canonically isomorphic to the lattice of characters of the small torus $T \subset T_X$.

In order to describe the $T$-maximal subsets of $X$, we use the functor $\mathfrak{G}$ associating to a bunch $\Theta$ in $(E \xrightarrow{\gamma} K, \gamma)$ a maximal projectable fan $\mathfrak{G}(\Theta)$ in $(F \xrightarrow{\delta} N, \delta)$. The resulting statement generalizes and complements the results of [4]:

**Theorem 5.4.** The assignment $\Theta \mapsto X_{\mathfrak{G}(\Theta)}$ defines a one-to-one correspondence between the bunches in $(E \xrightarrow{\gamma} K, \gamma)$ and the $T$-maximal open subsets of $X$.

**Proof.** By the definition of a maximal projectable fan and Proposition 5.3, the toric open subvariety $X_{\mathfrak{G}(\Theta)}$ is indeed $T$-maximal. Hence the assignment is well defined. Moreover, it is of course injective. Surjectivity follows from Proposition 5.2. □

**Remark 5.5.** In the setting of Theorem 5.4, the good quotient of the $T$-action on $X_{\mathfrak{G}(\Theta)}$ is the toric morphism $X_{\mathfrak{G}(\Theta)} \to X_{\Omega(\mathfrak{G}(\Theta))}$ arising from the projection $\mathfrak{G}(\Theta) \to \Omega(\mathfrak{G}(\Theta))$ onto the quotient fan.

Some of the good quotients are of special interest: A geometric quotient for an action of a reductive group $G$ on a variety $X$ is a good quotient that separates orbits. Geometric quotients are denoted by $X \to X/G$.

Again, for subtorus actions on toric varieties, there is a description in terms of fans. Let $X$ be the toric variety arising from a fan $\Delta$ in a lattice $N$, let $T \subset T_X$ be the subtorus corresponding to a primitive sublattice $L \subset N$. Existence of a geometric quotient is characterized as follows, see [13, Theorem 5.1]:

**Proposition 5.6.** The action of $T$ on $X$ admits a geometric quotient if and only if the projection $P: N \to N/L$ is injective on the support $\Sigma$.

Let us translate this into the language of bunches. As before, consider a projected cone $(F \xrightarrow{\rho} N, \delta)$ and its associated dual projected cone $(E \xrightarrow{\rho} K, \gamma)$.

**Definition 5.7.** A bunch $\Theta$ in $(E \xrightarrow{\rho} K, \gamma)$ is called geometric if $\dim(\tau) = \dim(K)$ holds for every $\tau \in \Theta$.

We consider the affine toric variety $X := X_\delta$ and the subtorus $T \subset T_X$ corresponding to the sublattice $L \subset N$. The above notion yields what we are looking for:

**Proposition 5.8.** Let $\Theta$ be a bunch in $(E \xrightarrow{\rho} K, \gamma)$. The open toric subvariety $X_{\mathfrak{G}(\Theta)} \subset X$ admits a geometric quotient by the action of $T$ if and only if $\Theta$ is geometric.
Proof: $X_{\tilde{\Theta}}$ admits a geometric quotient by $T$ if and only if $P: F \to N$ is injective on $|\Sigma|$. In our situation, the latter is equivalent to saying that $P: F \to N$ is injective on the maximal cones of $\Sigma$, that means on the cones $\gamma_0^* \gamma_0^*$ with $\gamma_0 \in \text{cov}(\Theta)$.

Thus Lemma 3.5 tells us that $X_{\tilde{\Theta}}$ admits a geometric quotient if and only if every cone $Q(\gamma_0), \gamma_0 \in \text{cov}(\Theta)$, is of full dimension in $K$. Since the elements of $\Theta$ occur among these cones and for any two cones of $\text{cov}(\Theta)$ their relative interiors intersect, we obtain the desired characterization. 

6. Standard bunches and toric varieties

We introduce the class of standard bunches. The main result of this section, Theorem 6.3, says that every nondegenerate 2-complete toric variety can be described by such a standard bunch, and, moreover, the isomorphism classes of free standard bunches correspond to the isomorphism classes of nondegenerate 2-complete toric varieties having free class group.

Definition 6.1. Let $\Theta$ be a bunch in a projected cone $(E \to QK, \gamma)$, and let $\gamma_1, \ldots, \gamma_n$ be the facets of $\gamma$. We say that $\Theta$ is a standard bunch if

(i) for all $i = 1, \ldots, n$ we have $K = Q(\text{lin}(\gamma_i) \cap E)$,

(ii) for every $i = 1, \ldots, n$ there is a $\tau \in \Theta$ with $\tau^\circ \subset Q(\gamma_i)^\circ$.

If $\Theta$ is a standard bunch in $(E \to QK, \gamma)$, and the cone $\gamma \subset EQ$ is regular, then we speak of the free standard bunch $\Theta$.

The constructions of Section 4 provide a functor from standard bunches to toric varieties:

Definition 6.2. Let $\Theta$ be a standard bunch in $(E \to QK, \gamma)$, and let $\Delta := \Omega(\tilde{\Theta})$ be the quotient fan of the maximal projectible fan corresponding to $\Theta$. The toric variety associated to $\Theta$ is $X_\Theta := X_\Delta$.

Recall from Section 1 that a toric variety $X$ is nondegenerate, if it does not admit a toric decomposition $X \cong X' \times K^*$. Moreover, $X$ is 2-complete, if any toric open embedding $X \subset X'$ with $\text{codim}(X' \setminus X) \geq 2$ is an isomorphism.

Theorem 6.3. The assignment $\Sigma: \Theta \mapsto X_\Theta$ defines a contravariant functor from the category of standard bunches to the category of nondegenerate 2-complete toric varieties. Moreover,

(i) Every nondegenerate 2-complete toric variety is isomorphic to a toric variety $X_\Theta$ with a standard bunch $\Theta$.

(ii) $\Sigma$ induces a bijection from the isomorphism classes of free standard bunches to the isomorphism classes of nondegenerate 2-complete toric varieties with free class group.

For the proof of Theorem 6.3 we have to do some preparation. We need a torsion free version of Cox’s construction 1.2 for nondegenerate fans, i.e. fans $\Delta$ in a lattice $N$ such that the support $|\Delta|$ generates the vector space $N_\mathbb{Q}$:

Definition 6.4. Let $(F \to P N, \delta)$ be a projected cone, $\Sigma$ a fan in $F$, and $\Delta$ a fan in $N$. We say that these data form a reduced Cox construction for $\Delta$ if

(i) $\Sigma^{(1)}$ equals $\delta^{(1)}$, and $P$ induces bijections $\Sigma^{(1)} \to \Delta^{(1)}$ and $\Sigma^{\max} \to \Delta^{\max}$,

(ii) $P$ maps the primitive generators of $\delta$ to primitive lattice vectors.
We show now that every nondegenerate fan admits reduced Cox constructions. In fact, these reduced Cox constructions will be even compatible with a certain type of maps of fans.

Let \( \Delta_i \) be nondegenerate fans in lattices \( N_i \). Moreover, let \( \Psi: N_1 \to N_2 \) be any isomorphism of lattices that is a map of the fans \( \Delta_1 \) and \( \Delta_2 \). Suppose that \( \Psi \) induces a bijection on the sets of rays \( \Delta_i(1) \).

**Lemma 6.5.**

(i) There exist a projected cone \( (F_1 \xrightarrow{p_1} N_1, \delta_1) \) and a fan \( \Sigma_1 \) in \( F_1 \) defining a reduced Cox construction for \( \Delta_1 \).

(ii) For every reduced Cox construction as in (i) there exist a reduced Cox construction for \( \Delta_2 \) given by \( (F_2 \xrightarrow{p_2} N_2, \delta_2) \) and \( \Sigma_2 \), a lattice isomorphism \( \Psi: F_1 \to F_2 \), and a commutative diagram of maps of fans

\[
\begin{array}{ccc}
F_1 & \xrightarrow{\Psi} & F_2 \\
\downarrow{p_1} & \downarrow{\Psi} & \downarrow{p_2} \\
N_1 & \xrightarrow{\Psi} & N_2
\end{array}
\]

where the map \( \Psi \) induces a bijection of the sets of rays \( \Sigma_1(1) \) and \( \Sigma_2(1) \).

Moreover, if \( \Psi \) maps \( \Delta_1 \) isomorphically onto a subfan of \( \Delta_2 \), then \( \Psi \) maps \( \Sigma_1 \) isomorphically onto a subfan of \( \Sigma_2 \).

**Proof.**

First we perform Cox’s original construction \([1,2]\) for the fans \( \Delta_i \). Denote by \( R_i \) the set of rays of \( \Delta_i \). For every maximal cone \( \sigma \in \Delta_i \) let

\[
\tilde{\sigma} := \text{cone}(e_\rho; \rho \in R_i, \rho \subset \sigma).
\]

Then these cones \( \tilde{\sigma} \) are the maximal cones of a fan \( \tilde{\Delta}_i \) in \( \mathbb{Z}^{R_i} \). Moreover, we have canonical projections sending the canonical base vectors to the primitive lattice vectors of the corresponding rays:

\[
C_i: \mathbb{Z}^{R_i} \to N_i, \quad e_\rho \mapsto v_\rho.
\]

Let us verify (i). Since \( C_1 \) needs not be surjective, it cannot serve as a projection of a projected cone. We have to perform a reduction step: Let \( L_1 := \ker(C_1) \), choose a section \( s_1: N_1 \to \mathbb{Q}^{R_1} \) of \( C_1 \), and set

\[
F_1 := s_1(N_1) \oplus L_1 \subset \mathbb{Q}^{R_1}.
\]

Then we can view \( \delta_1 \) and \( \Sigma_1 \) as well as objects in the lattice \( F_1 \). Note that \( \delta_1 \) needs no longer be regular, but remains simplicial. Together with the surjection \( p_1: F_1 \to N_1 \), the cone \( \delta_1 \) and the fan \( \Sigma_1 \) give the desired data.

We turn to (ii). Define a (invertible) linear map \( \Psi: \mathbb{Q}^{R_1} \to \mathbb{Q}^{R_2} \) of rational vector spaces by prescribing its values on the canonical base vectors as follows:

\[
\Psi(e_\rho) := e_{\Psi(\rho)} \quad \text{for every } \rho \in R_1.
\]

Similar to the proof of (i), we may reduce the Cox construction of \( \Delta_2 \) by refining the lattice \( \mathbb{Z}^{R_2} \) via \( F_2 := \Psi(F_1) \). Then the resulting \( (F_2 \xrightarrow{p_2} N_2, \delta_2) \) and \( \Sigma_2 \) and the lattice isomorphism \( \Psi: F_1 \to F_2 \) are as desired. \( \square \)

The above proof shows in particular that we cannot expect uniqueness of reduced Cox constructions for a given fan.
Lemma 6.6. Let $\Delta$ be a nondegenerate fan in a lattice $N$. Then the associated toric variety $X$ has free class group $\text{Cl}(X)$ if and only if $\Delta$ admits a reduced Cox construction $\Sigma$ in a projected cone $(F \xrightarrow{\rho} N, \delta)$ with a regular cone $\delta$.

Proof. Again, we consider Cox’s construction $[1, 2]$ for $\Delta$. Let $R$ denote the set of rays of $\Delta$, and let $C: \mathbb{Z}^R \rightarrow N$ be the map sending the canonical base vector $e_i$ to the primitive lattice vector $v_\phi \in \phi$.

If the toric variety $X$ has free class group then the lattice homomorphism $C$ is surjective, see [3]. Hence it defines the desired reduced Cox construction with $F = \mathbb{Z}^R$ and $\delta$ the positive orthant in $\mathbb{Q}^R$.

Conversely, let $(F \xrightarrow{\rho} N, \delta)$ and $\Sigma$ be a reduced Cox construction of $\Delta$ with $\delta$ regular. Then $C$ factors as $C = P \circ S$, where $S: \mathbb{Z}^R \rightarrow F$ maps the canonical base vectors to the primitive generators of the cone $\delta$. It follows that $C$ is surjective. Hence $\text{Cl}(X)$ is free.

Lemma 6.7. Let $(F_i \xrightarrow{\rho_i} N_i, \delta_i)$ and $\Sigma_i$ be reduced Cox constructions for $\Delta_i$ such that the $\delta_i$ are regular cones. Then every isomorphism $\Psi: N_1 \rightarrow N_2$ of the fans $\Delta_i$ admits a unique lifting $\Psi: F_1 \rightarrow F_2$ to an isomorphism of the fans $\Sigma_i$.

Proof. For $\phi \in \Delta_i^{(1)}$, let $e^i_\phi \in F_i$ denote the primitive generator of $\delta_i$ above the primitive vector of $\phi$. Then define $\Psi: F_1 \rightarrow F_2$ by setting $\Psi(e^1_\phi) := e^2_{\Psi(\phi)}$.

Let $\Theta$ be a bunch in $(E \xrightarrow{\rho} K, \gamma)$, and let $X$ denote the associated maximal projectable fan in the dual projected cone $(F \xrightarrow{\rho} N, \delta)$. Let $\Delta$ denote the quasifan in $N$ obtained by projecting the maximal cones of $\Sigma$.

Lemma 6.8. The following statements are equivalent:

(i) $\Delta$ is a fan having $\Sigma$ and $(F \xrightarrow{\rho} N, \delta)$ as a reduced Cox construction.

(ii) $\Theta$ is a standard bunch.

Proof. Recall that $\Sigma$ being projectable means that any two maximal cones of $\Sigma$ can be separated by a ker($P$)-invariant linear form. In particular, $P$ sets up a bijection $\Sigma^{\text{max}} \rightarrow \Delta^{\text{max}}$. Thus, the first statement holds if and only if we have:

(a) $P$ maps every primitive generator of $\delta$ to a primitive lattice vector in $N$.

(b) $\Delta$ is a fan, we have $\delta^{(1)} = \Sigma^{(1)}$, and $P$ induces a bijection $\Sigma^{(1)} \rightarrow \Delta^{(1)}$.

Hence the task is to show that the conditions 6.7(i) and (ii) hold if and only if (a) and (b) do so. Equivalence of (a) and 6.6(i) is a direct application of Lemma 3.6. Note that for this one has to interchange the roles of $P, \delta$ and $Q, \gamma$ in the lemma such that for the $\gamma_0$ of the lemma one can take a ray of $\Sigma$.

Now, suppose that the conditions (a) and (b) are valid. By equivalence of (a) and 6.6(i), we only have to check that 6.6(ii) holds.

Fix a facet $\gamma_i \preceq \gamma$. By assumption, there is a maximal cone $\hat{\sigma} \in \Sigma$ such that $\hat{\sigma} := \gamma_i^*$ is a ray of $\hat{\sigma}$ and $P(\hat{\sigma})$ is a ray of $P(\hat{\sigma}) \in \Delta$. Then $\gamma_0 := \sigma^*$ belongs to cov($\Theta$), we have $\gamma_0 \preceq \gamma_i$, and the Invariant Separation Lemma implies $Q(\gamma_0)^\circ \subset Q(\gamma_i)^\circ$. By the Overlapping Property 2.5 any $\tau \in \Theta$ with $\tau \subset Q(\gamma_0)$ is as in 6.6(ii).

Conversely, if 6.6(i) and (ii) are valid, then we have to show that $\Delta$ is a fan, $\delta^{(1)}$ equals $\Sigma^{(1)}$, and that $P$ induces a bijection $\Sigma^{(1)} \rightarrow \Delta^{(1)}$.

Consider any $\hat{\sigma} \in \delta^{(1)}$. Then $\hat{\sigma} = \gamma_i^*$ for some facet $\gamma_i \preceq \gamma$. By 6.6(ii), we have $\tau_i^0 \subset Q(\gamma_i)^0$ for some $\tau_i \in \Theta$. Thus we find a $\gamma_0 \preceq \gamma_i$ being minimal with the property that $Q(\gamma_0) \supset \tau_0$ holds for some $\tau_0 \in \Theta$. Then $\gamma_0 \in \text{cov}(\Theta)$, and
\(Q(\gamma_i) \cap Q(\gamma_j)\) is nonempty because it contains \(\tau^*_i \cap \tau^*_j\). Hence \(\delta := \gamma^*_i \) is a maximal cone of \(\Sigma\) with \(\delta \preceq \delta^*\), and the Invariant Separation Lemma yields \(P(\delta) \preceq P(\delta^*) \in \Delta\).

So, this consideration gives in particular \(\delta^{(1)} = \Sigma^{(1)}\). Moreover, since \(P(\delta)\) is strictly convex, it gives \(\delta \in \Delta\); in other words, the quasifan \(\Delta\) is a fan. Furthermore, since we already know that (a) holds, the image \(P(\delta)\) is in fact one-dimensional. Hence we obtain \(\Sigma^{(1)} \rightarrow \Delta^{(1)}\) is well defined.

Surjectivity of the map \(\Sigma^{(1)} \rightarrow \Delta^{(1)}\) follows from the fact that \(\Sigma^{\text{max}} \rightarrow \Delta^{\text{max}}\) is surjective. Injectivity follows from the observation that by \ref{thm:6.3} (ii) we always have \(Q(\gamma_i) \cap \Sigma(\gamma_j) \neq \emptyset\), and hence any two rays of \(\delta\) can be invariantly separated. □

**Proof of Theorem 6.3.** By Lemma \ref{lem:6.8}, the toric variety \(X_\Theta\) associated to a standard bunch \(\Theta\) is nondegenerate. It is also 2-complete: Otherwise there is an open toric embedding \(X_\Theta \subset X\) with nonempty complement of codimension at least two. Using Lemmas \ref{lem:6.5} and \ref{lem:6.8}, we can compare reduced Cox constructions of \(X_\Theta\) and \(X\), and we see that the projectable fan associated to \(\Theta\) is not maximal, i.e., does not satisfy \ref{thm:4.1} (i) (b). A contradiction.

So the functor \(\Sigma : \Theta \mapsto X_\Theta\) is well defined. The fact that it is surjective on isomorphism classes follows from existence of reduced Cox constructions, Proposition \ref{prop:5.3}, Theorem \ref{thm:4.6} and Lemma \ref{lem:6.8}. The correspondence of isomorphism classes of free bunches with isomorphism classes of nondegenerate 2-complete toric varieties with free class group is a direct application of Lemmas \ref{lem:6.6} and \ref{lem:6.7}. □

### 7. A very first dictionary

Fix a standard bunch \(\Theta\) in a projected cone \((E \xrightarrow{\phi} K, \gamma)\), and let \(X := X_\Theta\) denote the associated toric variety. In this section, we characterize basic geometric properties of \(X\) in terms of the bunch \(\Theta\).

Let \((F \xrightarrow{\phi} N, \delta)\) be the dual projected cone. Denote by \(\Sigma\) the maximal projectable fan associated to \(\Theta\), and let \(\Delta\) be the quotient fan of \(\Sigma\). Recall from Lemma \ref{lem:6.8} that these data form a reduced Cox construction of \(\Delta\). In particular, \(\Delta\) lives in \(N\), and we have

\[
X = X_\Delta, \quad \dim(X) = \text{rank}(E) - \text{rank}(K).
\]

We study now \(\mathbb{Q}\)-factoriality, smoothness, existence of fixed points and completeness. For this we need the following observation:

**Lemma 7.1.** Consider a face \(\gamma_0 \in \text{cov}(\Theta)\) and the corresponding maximal cone \(P(\gamma_0^*)\) of \(\Delta\). Then we have:

(i) \(Q(\gamma_0)\) is of full dimension if and only if \(P(\gamma_0^*)\) is simplicial.

(ii) \(Q(\gamma_0)\) is simplicial if and only if \(P(\gamma_0^*)\) is of full dimension.

**Proof.** We prove (i). Let \(Q(\gamma_0)\) be of full dimension. By Lemma \ref{lem:5.5}, the map \(P\) is injective along \(\gamma_0^*\). In particular, \(P(\gamma_0^*)\) is simplicial. Conversely, let \(P(\gamma_0^*)\) be simplicial. Since \(P\) induces a bijection from the rays of \(\gamma_0^*\) to the rays of \(P(\gamma_0^*)\), it is injective along \(\gamma_0^*\). Thus Lemma \ref{lem:5.5} yields that \(Q(\gamma_0)\) is of full dimension.

We turn to (ii). If \(P(\gamma_0^*)\) is of full dimension, we see as before that \(Q(\gamma_0)\) is simplicial. For the converse we show that \(Q\) is injective along \(\gamma_0\): For every ray \(\rho\) of \(Q(\gamma_0)\), choose a ray \(\tau\) of \(\gamma_0\) with \(Q(\tau) = \rho\). Then the cone \(\gamma_1 \geq \gamma_0 \) generated by these rays \(\tau\) is mapped bijectively onto \(Q(\gamma_0)\). By minimality of \(\gamma_0\) as an element of \(\text{cov}(\Theta)\), we conclude \(\gamma_1 = \gamma_0\). □
The first statement of this section is the following characterization of \( \mathbb{Q} \)-factoriality:

**Proposition 7.2.** The toric variety \( X \) is \( \mathbb{Q} \)-factorial if and only if \( \Theta \) consists of cones of full dimension in \( K \).

**Proof.** The toric variety \( X \) is \( \mathbb{Q} \)-factorial if and only if all cones of \( \Delta \) are simplicial. By Lemma 7.1 this is equivalent to saying that all cones \( Q(\gamma_0), \gamma_0 \in \text{cov}(\Theta) \), are of full dimension. The latter holds if and only if \( \Theta \) consists of cones of full dimension, because every \( Q(\gamma_0) \) contains some \( \tau \in \Theta \). \( \square \)

Characterizing smoothness of toric varieties in terms of bunches involves the lattice structure:

**Proposition 7.3.** The toric variety \( X_\Theta \) is smooth if and only if for every \( \gamma_0 \in \text{cov}(\Theta) \) we have

(i) \( \gamma_0^* \) is a regular cone in \( F \).

(ii) \( Q \) maps \( \text{lin}(\gamma_0) \cap E \) onto \( K \).

**Proof.** Suppose that (i) and (ii) hold. To verify smoothness of \( X \), we have to show that all cones \( P(\gamma_0^*), \gamma_0 \in \text{cov}(\Theta) \), are regular. By the properties 6.4 of a reduced Cox construction and Condition (i), the primitive generators of \( P(\gamma_0^*) \) span the sublattice \( P(\text{lin}(\gamma_0) \cap F) \) of \( N \). By Lemma 3.6 and Condition (ii), this sublattice is primitive. This proves regularity of the cone \( P(\gamma_0^*) \).

Conversely, let \( X \) be smooth. For \( \gamma_0 \in \text{cov}(\Theta) \), consider the sublattice \( F_0 \subset F \) spanned by the primitive generators of \( \gamma_0^* \). Since \( P(\gamma_0^*) \) is regular, the properties 6.4 of a reduced Cox construction yield that \( P(F_0) \) is a primitive sublattice of \( N \). Since \( P \) is injective along \( \gamma_0^* \), we see that also \( F_0 \) is primitive. This gives Condition (i). Condition (ii) then follows from Lemma 3.6. \( \square \)

Existence of global regular functions on a toric variety is characterized as follows:

**Proposition 7.4.** We have \( \mathcal{O}(X) = K \) if and only if \( Q \) contracts no ray of \( \gamma \) to a point and the image \( Q(\gamma) \) is strictly convex.

**Proof.** \( \mathcal{O}(X) = K \) holds if and only if the rays of \( \Delta \) generate \( N_\mathbb{Q} \) as a convex cone. This is valid if and only if \( \delta + \ker(P) \mathbb{Q} = \{0\} \). By dualizing, this condition is equivalent to \( \gamma \cap \ker(Q) \mathbb{Q} = \{0\} \). This in turn holds if and only if \( Q \) contracts no ray of \( \gamma \) and \( Q(\gamma) \) is strictly convex. \( \square \)

Recall that we speak of a full toric variety \( X \) if \( X \) is 2-complete and every \( T_X \)-orbit contains a fixed point in its closure.

**Proposition 7.5.** The toric variety \( X \) is full if and only if all cones \( Q(\gamma_0), \gamma_0 \in \text{cov}(\Theta) \), are simplicial.

**Proof.** Existence of fixed points in the \( T_X \)-orbit closures means that all maximal cones of \( \Delta \) are of full dimension. Thus Lemma 7.1 gives the assertion. \( \square \)

Finally, we also can characterize completeness of a toric variety in terms of its bunch:

**Proposition 7.6.** The toric variety \( X_\Theta \) is complete if and only if \( \Theta \) contains a simplicial cone and any face \( \gamma_0 \preceq \gamma \) satisfying \( \tau^0 \subset Q(\gamma_0)^0 \) for some \( \tau \in \Theta \) and \( \gamma_1 \preceq \gamma_0 \) for only one \( \gamma_1 \in \text{cov}(\Theta) \) belongs to \( \text{cov}(\Theta) \).
Proof. This is a direct translation of a well known characterization of completeness in terms of fans. Namely, \( X \) is complete if and only the fan \( \Delta \) has the following two properties: firstly, at least one of its (maximal) cones is of full dimension; secondly, any cone contained in only one maximal cone is itself maximal.

By Lemma \textbf{7.1}, the first property translates to the property that \( \Theta \) contains a simplicial cone. For the second, recall from Proposition \textbf{4.11}, that the cones \( \sigma_0 \in \Delta \) correspond to the faces \( \gamma_0 \preceq \gamma \) satisfying \( \tau^\circ \subset Q(\gamma_0)^\circ \) for some \( \tau \in \Theta \) via

\[
\sigma_0 \mapsto (P^{-1}(\sigma_0) \cap |\Sigma|)^*.
\]

Thereby the maximal cones \( \sigma_1 \) of \( \Delta \) correspond to the elements of \( \text{cov}(\Theta) \). Thus the statement that \( \sigma_0 \preceq \sigma_1 \) for only one maximal \( \sigma_1 \) implies \( \sigma_0 \in \Delta^{\text{max}} \) directly translates to the second characterizing condition of the assertion. \( \Box \)

One may ask if a full toric variety \( X \) with \( O(X) = \mathbb{K} \) is already complete. If \( \text{dim}(X) \leq 3 \) holds, then the answer is positive. For \( \text{dim}(X) \geq 4 \), there are counterexamples, compare [9, Remark 2]. However, we shall see in Section \textbf{10} that every smooth 2-complete toric variety with class group \( \mathbb{Z}_2 \) is complete.

8. Full \( \mathbb{Q} \)-factorial toric varieties

In this section we introduce the class of simple bunches, and we show that \( \Theta \mapsto X_\Theta \) defines an equivalence of categories between the simple bunches and the full \( \mathbb{Q} \)-factorial toric varieties.

Definition 8.1. By a simple bunch we mean a standard bunch \( \Theta \) in a projected cone \( (E \xrightarrow{\gamma} K, \gamma) \) such that \( Q \) maps \( E \cap \text{lin}(\gamma_0) \) isomorphically onto \( K \) for every \( \gamma_0 \in \text{cov}(\Theta) \).

Note that the cones of a simple bunch are of full dimension and simplicial, but they need not be regular. To state the main result of this section, recall from Definition \textbf{1.4} that a toric variety \( X \) is full, if it is 2-complete and every \( TX \)-orbit has a fixed point in its closure.

Theorem 8.2. The assignment \( \Theta \mapsto X_\Theta \) defines an equivalence of the category of simple bunches with the category of full \( \mathbb{Q} \)-factorial toric varieties.

The crucial observation for the proof is that full \( \mathbb{Q} \)-factorial toric varieties admit a universal reduced Cox construction in a certain sense. First we show existence:

Lemma 8.3. Let \( \Delta \) be a simplicial fan in a lattice \( N \) such that any maximal cone is of full dimension. Then there are \( (F \xrightarrow{\delta} N, \delta) \) and \( \Sigma \) defining a reduced Cox construction of \( \Delta \) such that

(i) \( P \) induces for every \( \hat{\sigma} \in \Sigma^{\text{max}} \) an isomorphism \( P_{\hat{\sigma}} : F \cap \text{lin}(\hat{\sigma}) \to N \).

(ii) \( F \) is the sum of the sublattices \( \text{lin}(\hat{\sigma}) \cap F \), where \( \hat{\sigma} \in \Sigma^{\text{max}} \).

Proof. As usual, let \( R \) be the set of rays of \( \Delta \), let \( P : \mathbb{Z}^R \to N \) be the homomorphism sending the canonical base vector \( e_\varrho \) to the primitive vector \( v_\varrho \in \varrho \), and let \( \delta \subset \mathbb{Q}^R \) be the cone generated by the \( e_\varrho \). For any cone \( \sigma \in \Delta \) set

\[
\hat{\sigma} := \text{cone}(e_\varrho; \varrho \in \sigma^{(1)}).
\]

Then these cones form a fan \( \Sigma \) in the lattice \( \mathbb{Z}^R \). In order to achieve the desired properties, we refine the lattice \( \mathbb{Z}^R \) as follows: Think of \( P \) for the moment as a map
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of the vector spaces \( Q^R \) and \( N_\Theta \) and consider the set

\[ P^{-1}(N) \cap |\Sigma| = \bigcup_{\sigma \in \Sigma} P^{-1}(N) \cap \hat{\sigma} \subset Q^R. \]

This set generates a lattice \( F \subset Q^R \), because by simpliciality of \( \Delta \) the map \( P \) is injective along the cones \( \hat{\sigma} \in \Sigma \), and hence each \( \hat{\sigma} \cap P^{-1}(N) \) is discrete. Moreover, the restriction \( P_{\hat{\sigma}}: F \cap \text{lin}(\hat{\sigma}) \to N \) of \( P \) is an isomorphism for every \( \hat{\sigma} \in \Sigma_{\text{max}} \).

Here we use that the maximal cones of \( \Delta \) are of full dimension.

We may view \( \delta \) and \( \Sigma \) as well as data in the lattice \( F \). Since \( P: F \to N \) is surjective, this gives in particular a projected cone \( (F \xrightarrow{P} N, \delta) \). Now it is straightforward to verify the defining properties of a reduced Cox construction for these data. By construction it satisfies (i) and (ii).

We shall call a reduced Cox construction with the properties of Lemma 8.3 a universal reduced Cox construction. Uniqueness of universal reduced Cox constructions is a consequence of the following lifting property:

**Lemma 8.4.** Let \( \Delta, \Sigma \) be fans in lattices \( N_i \), and let \( (F_i \xrightarrow{P_i} N_i, \delta_i) \) and \( \Sigma_i \) be universal reduced Cox constructions. Then every map \( \Psi: N_1 \to N_2 \) of the fans \( \Delta_i \) admits a unique lifting to a map \( \Psi: F_1 \to F_2 \) of the fans \( \Sigma_i \):

\[
\begin{array}{ccc}
F_1 & \xrightarrow{\Psi} & F_2 \\
\downarrow P_1 & & \downarrow P_2 \\
N_1 & \xrightarrow{\Psi} & N_2
\end{array}
\]

**Proof.** For every maximal cone \( \sigma_1 \in \Delta_1 \), fix a maximal cone \( \sigma_2 \in \Delta_2 \) with \( \Psi(\sigma_1) \subset \sigma_2 \). Let \( \hat{\sigma}_i \in \Sigma_i \) denote the cones lying over \( \sigma_i \). Then, by the Property 8.3 (i) of a universal reduced Cox construction, we obtain for every \( \sigma_1 \) a unique commutative diagram

\[
\begin{array}{ccc}
F_1 \cap \text{lin}(\hat{\sigma}_1) & \xrightarrow{\Psi_{\hat{\sigma}_1}} & F_2 \cap \text{lin}(\hat{\sigma}_2) \\
\downarrow s_{\hat{\sigma}_1} & & \downarrow s_{\hat{\sigma}_2} \\
N_1 & \xrightarrow{\Psi} & N_2
\end{array}
\]

where the \( s_{\hat{\sigma}_i} \) are the sections mapping the primitive generators of \( \sigma_i \) to those of \( \hat{\sigma}_i \). Any two \( \Psi_{\hat{\sigma}_1} \) and \( \Psi_{\hat{\sigma}_2} \) have the same values on the primitive generators of \( \delta \) that lie in \( \hat{\sigma}_1 \cap \hat{\sigma}_2 \). Thus the \( \Psi_{\hat{\sigma}_1} \) fit together to a linear map \( \Psi: F_1 \to (F_2)_{\text{Q}} \). Lemma 8.3 (ii) gives \( \Psi(F_1) \subset F_2 \). Hence we found the desired lifting. \( \square \)

**Proof of Theorem 8.2.** First observe that for a simple bunch \( \Theta \), the toric variety \( X_{\Theta} \) is indeed full and \( \mathbb{Q} \)-factorial: Since for every \( \gamma_0 \in \text{cov}(\Theta) \) the cone \( Q(\gamma_0) \) is simplicial and of full dimension, Lemma 7.1 ensures that also the maximal cones of the fan \( \Delta \) defining \( X \) are of full dimension and simplicial.

Next we show that, up to isomorphism, every full \( \mathbb{Q} \)-factorial toric variety \( X \) is of the form \( X_{\Theta} \) with a simple bunch. We may assume that \( X \) arises from a fan \( \Delta \) in a lattice \( N \). Lemma 8.3 provides a universal reduced Cox construction \( (F \xrightarrow{P} N, \delta) \) and \( \Sigma \) of \( \Delta \). By Lemma 8.3 the bunch \( \Theta \) corresponding to the maximal projectable fan \( \Sigma \) is as wanted.
Finally, we have to show bijectivity on the level of morphisms. Due to Lemma 8.3 the morphisms between full \( \mathbb{Q} \)-factorial simplicial toric varieties are in one-to-one correspondence with the maps of maximal projectable fans defined by their universal reduced Cox constructions. By Theorem 4.6, the latter morphisms correspond to the morphisms of the associated bunches. □

**Corollary 8.5.** \( \Theta \mapsto X_\Theta \) defines an equivalence of categories from free simple bunches to full smooth toric varieties.

**Proof.** By Propositions 7.3 and 7.5, every free simple bunch \( \Theta \) defines a full smooth toric variety \( X_\Theta \). Conversely, for a full smooth toric variety \( X \), the usual Cox construction \( 1.2 \) is a universal reduced Cox construction. Hence \( X \) arises from a free simple bunch. □

We conclude this section with some observations and examples. The first remark gives the geometric interpretation of the universal reduced Cox construction:

**Remark 8.6.** In terms of toric varieties, the universal reduced Cox construction \( p: \hat{X} \rightarrow X \) arises from the usual Cox construction \( c: \tilde{X} \rightarrow X \) by dividing \( \tilde{X} \) by the (finite) group generated by all isotropy groups of the action of \( \ker(Q) \) on \( \tilde{X} \).

The following example shows that for nonsimplicial toric varieties the functor \( \Theta \mapsto X_\Theta \) associating to a standard bunch its toric variety is not surjective on the level of morphisms:

**Example 8.7.** We present a toric morphism that cannot be lifted to the Cox constructions. In \( \mathbb{Z}^3 \) consider the vectors

\[
v_1 := (1, 0, 0), \quad v_2 := (1, 0, 1), \quad v_3 := (0, 1, 0), \quad v_4 := (-3, 2, 2), \quad v_5 := (0, 1, 1).
\]

Let \( \Delta_2 \) be the fan of faces of the cone generated by \( v_1, \ldots, v_4 \), and let \( \Delta_1 \) be the subdivision of \( \Delta_2 \) at \( v_5 \). Mapping the \( i \)-th canonical base vector to \( v_i \), we obtain projections

\[
P_1: \mathbb{Z}^5 \rightarrow \mathbb{Z}^3, \quad P_2: \mathbb{Z}^4 \rightarrow \mathbb{Z}^3.
\]

Let \( \Sigma_1 \) and \( \Sigma_2 \) be the fans above \( \Delta_1 \) and \( \Delta_2 \), respectively. Note that these data are in fact reduced Cox constructions.

We claim that the identity \( \varphi: \mathbb{Z}^3 \rightarrow \mathbb{Z}^3 \) does not admit a lifting. In fact, since \( v_5 \) equals \( 3/2v_1 + 1/2v_4 \), a possible lifting \( \Phi: \mathbb{Z}^5 \rightarrow \mathbb{Z}^4 \) must satisfy

\[
\Phi(0, 0, 0, 0, 1) \in (3/2, 0, 0, 1/2) + \mathbb{Q}(5, -2, -2, 1),
\]

where \( (5, -2, -2, 1) \) generates \( \ker(P_2) \). An explicit calculation shows that the right hand side does not contain integral points with nonnegative coefficients.

This excludes existence of a map \( \Phi: \mathbb{Z}^5 \rightarrow \mathbb{Z}^4 \) of the fans \( \Sigma_1 \) and \( \Sigma_2 \) lifting \( \varphi: \mathbb{Z}^3 \rightarrow \mathbb{Z}^3 \).

The next example shows that in the nonsimplicial case the functor \( \Theta \mapsto X_\Theta \) is not injective on the level of morphisms:

**Example 8.8.** We give a toric morphism that admits two different liftings. Let \( \Delta_2 \) be the fan of faces of the cone generated by the vectors

\[
(1, 0, 0), \quad (0, 1, 0), \quad (-1, 0, 1), \quad (0, -1, 1).
\]
Let $\Delta_1$ be the subdivision of $\Delta_2$ at $(0,0,1)$. Then there are two different liftings of the identity map $\mathbb{Z}^3 \to \mathbb{Z}^3$ to the respective Cox-constructions, namely the maps $\mathbb{Z}^5 \to \mathbb{Z}^4$ defined by the matrices
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}.
\]

9. Invariant divisors and divisor classes

In this section, we come to one of the most powerful parts of the language of bunches: We study geometric properties of divisors and divisor classes.

Let us fix the notation. As usual, $\Theta$ is a standard bunch in a projected cone $(E \to K, \gamma)$, and $X := X_\Theta$ is the associated toric variety with its big torus $T := TX$. Let $(F \to N, \delta)$ be the dual projected cone, $\Sigma$ the maximal projectable fan corresponding to $\Theta$, and $\Delta$ the quotient fan of $\Sigma$. Recall that these data define a reduced Cox construction of $\Delta$, and we have $X = X_\Delta$.

Our first task is to relate the lattice $K$ to the divisor class group $\text{Cl}(X)$. Let $v_1, \ldots, v_r$ be the primitive generators of the one-dimensional cones of $\Delta$, and let $\hat{v}_1, \ldots, \hat{v}_r$ be the primitive generators of the rays of $\delta$, numbered in such a way that we always have $P(\hat{v}_i) = v_i$.

Every ray $\varrho_i = Q_{\geq 0}v_i$ determines an invariant prime divisor $D_i$ in $X$. There is a canonical injection mapping $E$ into the lattice $W\text{Div}^T_T(X)$ of invariant Weil divisors on $X$:
\[
\mathcal{D}: E \to W\text{Div}^T_T(X), \quad \hat{w} \mapsto \mathcal{D}(\hat{w}) := \sum_{i=1}^r \hat{w}(\hat{v}_i)D_i.
\]
By construction, an element $u \in M$ is mapped to the principal divisor $\text{div}(\chi^u)$ of $X$. Hence, we obtain, compare [10]:

**Proposition 9.1.** There is a commutative diagram with exact rows and injective upwards arrows:
\[
\begin{array}{ccc}
0 & \longrightarrow & \text{PDiv}^T_T(X) \\
\mathcal{D} & \downarrow & \text{WDiv}^T_T(X) \\
0 & \longrightarrow & \text{Cl}(X)
\end{array}
\]
Moreover, tensoring this diagram with $\mathbb{Q}$ provides isomorphisms of rational vector spaces:
\[
\mathcal{D}: E_\mathbb{Q} \to W\text{Div}^T_T(X)_\mathbb{Q}, \quad \mathcal{T}_\mathbb{Q}: K_\mathbb{Q} \to \text{Cl}_\mathbb{Q}(X).
\]

In the main result of this section, we study the group $\text{Pic}_\mathbb{Q}(X) \subset \text{Cl}_\mathbb{Q}(X)$ of rational Cartier divisor classes. We obtain very simple descriptions of the cone $C^{\text{sa}}(X) \subset \text{Pic}_\mathbb{Q}(X)$ of semiample classes and the cone $C^{\text{a}}(X) \subset \text{Pic}_\mathbb{Q}(X)$ of ample classes:

**Theorem 9.2.** The map $\mathcal{T}_\mathbb{Q}: K_\mathbb{Q} \to \text{Cl}_\mathbb{Q}(X)$ defines canonical isomorphisms:
\[
\text{Pic}_\mathbb{Q}(X) \cong \bigcap_{\tau \in \Theta} \text{lin}(\tau), \quad C^{\text{sa}}(X) \cong \bigcap_{\tau \in \Theta} \tau, \quad C^{\text{a}}(X) \cong \bigcap_{\tau \in \Theta} \tau^o.
\]
Proof. Let $D \in \text{WDiv}^T_{\mathbb{Q}}(X)$, and set $\hat{w} := \Delta^{-1}(D) \in E_{\mathbb{Q}}$. Our task is to characterize the statements that $D$ is a $\mathbb{Q}$-Cartier, a semiample or an ample divisor in terms of the image $w := Q(\hat{w})$ in $K_{\mathbb{Q}}$.

For the description of $\text{Pic}(X)$, recall from [10, p. 66], that $D$ is $\mathbb{Q}$-Cartier if and only if it arises from a support function, i.e., there is a family $(u_\sigma)_{\sigma \in \Delta_{\max}}$ with $u_\sigma \in M_{\mathbb{Q}}$ such that $D = m^{-1}\text{div}(\chi^{m_\sigma})$ holds on each affine chart $X_\sigma \subset X$ with a positive integral multiple $m_\sigma$. If $D$ is $\mathbb{Q}$-Cartier, then the describing linear forms $u_\sigma$ are unique up to elements of $\sigma^\perp$.

Now, suppose that $D$ is $\mathbb{Q}$-Cartier, and let $(u_\sigma)$ be a describing support function. Then $u_\sigma(v_i) = \hat{w}(\tilde{v}_i)$ holds for every $v_i \in \sigma^{(1)}$. Define $\ell_\sigma := \hat{w} - P^*(u_\sigma)$ and denote the cone above $\sigma$ by $\hat{\sigma}$. Then $\ell_\sigma$ lies in $\hat{\sigma}^\perp = \text{lin}(\hat{\sigma}^*)$, and hence $Q(\hat{w}) = Q(\ell_\sigma)$ lies in $Q(\text{lin}(\hat{\sigma}^*))$. Since this applies for all $\sigma \in \Delta_{\max}$, we obtain

$$w = Q(\hat{w}) \in \bigcap_{\gamma_0 \in \text{cov}(\Theta)} \text{lin}(Q(\gamma_0)) = \bigcap_{\tau \in \Theta} \text{lin}(\tau).$$

Conversely, let $w = Q(\hat{w})$ belong to the last intersection. Since for $\sigma \in \Delta_{\max}$ the image $Q(\hat{\sigma}^*)$ contains an element of $\Theta$, we find for each $\sigma \in \Delta_{\max}$ an $\ell_\sigma \in \text{lin}(\hat{\sigma}^*)$ with $Q(\ell_\sigma) = w$. Then $u_\sigma := \hat{w} - \ell_\sigma$ maps to zero and can therefore be viewed as an element of $M_{\mathbb{Q}}$. Hence, $(u_\sigma)_{\sigma \in \Delta_{\max}}$ provides a support function describing $D$.

For the descriptions of $\text{C}^n(X)$ and $\text{C}^n_{\mathbb{Q}}(X)$, let $D$ be $\mathbb{Q}$-Cartier. Recall that $D$ is semiample (ample) if and only if it is described by a support function $(u_\sigma)$, which is convex (strictly convex) in the sense that $u_\sigma - u_{\sigma'}$ is nonnegative (positive) on $\sigma \setminus \sigma'$ for any two $\sigma, \sigma' \in \Delta_{\max}$.

Suppose that $D$ is semiample (ample) with convex (strictly convex) support function $(u_\sigma)$. In terms of $\ell_\sigma := \hat{w} - P^*(u_\sigma)$ this means that each $\ell_\sigma - \ell_{\sigma'}$ is nonnegative (positive) on $\hat{\sigma} \setminus \hat{\sigma}'$. Since $\ell_\sigma \in \hat{\sigma}^\perp$ holds, this is equivalent to nonnegativity (positivity) of $\ell_\sigma$ on every $\hat{\sigma} \setminus \hat{\sigma}'$.

Since all rays of the cone $\delta$ occur in the fan $\Sigma$, the latter is valid if and only if $\ell_{\sigma} \in \hat{\sigma}^*$ (resp. $\ell_{\sigma} \in (\hat{\sigma}^*)^\circ$) holds for all $\sigma$. This in turn implies that for every $\sigma \in \Delta_{\max}$ we have

$$(9.2.1) \quad w = Q(\hat{w}) = Q(\ell_\sigma) \in Q(\hat{\sigma}^*) \quad (\text{resp. } w \in Q((\hat{\sigma}^*)^\circ)).$$

Now, the $\hat{\sigma}^*$, where $\sigma \in \Delta_{\max}$, are precisely the cones of $\text{cov}(\Theta)$. Since any interior $Q(\hat{\sigma}^*)$ contains the interior of a cone of $\Theta$, we can conclude that $w$ lies in the respective intersections of the assertion.

Conversely, if $w$ belongs to one of the right hand side intersections, then we surely arrive at $[9.2.1]$. Thus, for every $\sigma \in \Delta_{\max}$, we find an $\ell_\sigma \in \hat{\sigma}^*$ (an $\ell_\sigma \in (\hat{\sigma}^*)^\circ$) mapping to $w$. Reversing the above arguments, we see that $u_\sigma := \hat{w} - \ell_\sigma$ is a convex (strictly convex) support function describing $D$. \hfill \Box

As an immediate consequence of Theorem 9.2 we obtain a quasiprojectivity criterion in the spirit of [8].

**Corollary 9.3.** The variety $X$ is quasiprojective if and only if the intersection of all relative interiors $\tau^\circ$, where $\tau \in \Theta$, is nonempty.

Combining Theorem 9.2 with Proposition 7.2 shows that a $\mathbb{Q}$-factorial quasiprojective toric variety has an ample cone of full dimension. Hence we get back a result of Oda and Park [19, Theorem 3.5]:
Corollary 9.4. Suppose that $X$ is quasiprojective and $\mathbb{Q}$-factorial. Then every rational Weil divisor $D$ of $X$ admits a representation $D = D_1 - D_2$ with ample rational Cartier divisors $D_1$ and $D_2$.

So far, we considered rational divisors and divisor classes. For toric varieties with free class group, we also obtain a simple picture for integral divisors. We need the following notation:

Every ray $\hat{\gamma}_i := \mathbb{Q}_{\geq 0} \hat{v}_i$ of $\delta$ has a unique “opposite” ray, namely the unique ray of $\gamma$ that is not contained in $\hat{\gamma}_i^\perp$. We denote the primitive generator of this opposite ray by $\hat{w}_i$, and its image in $K$ by $w_i := Q(\hat{w}_i)$.

Remark 9.5. The bunch $\Theta$ is free if and only if $\hat{w}_i(\hat{v}_i) = 1$ holds for all $1 \leq i \leq r$.

In the free case, we have the following integral version of the corresponding statements in Proposition 9.1 and Theorem 9.2.

Proposition 9.6. Assume that $\Theta$ is a free bunch. Then we have:

$$K \cong \text{Cl}(X), \quad \text{Pic}(X) \cong \bigcap_{\gamma_0 \in \text{cov}(\Theta)} Q(\text{lin}(\gamma_0) \cap E).$$

Proof. Using Remark 9.3 we infer from freeness of $\Theta$ that the map $D: E \to \text{WDiv}^T(X)$ is an isomorphism. Using the 5-Lemma in the diagram of Proposition 9.1, we obtain that also the map $\bar{D}: K \to \text{Cl}(X)$ is an isomorphism. This gives the first part of the assertion.

The rest of the proof is similar to that of Proposition 9.2. Suppose that $w \in K$ lies in all $Q(\text{lin}(\gamma_0) \cap E)$, where $\gamma_0 \in \text{cov}(\Theta)$. Let $\hat{w} \in E$ with $Q(\hat{w}) = w$. Then for every $\sigma \in \Delta^\text{max}$ we can choose $\ell_\sigma$ even in $\hat{\sigma}^\perp \cap E$ with $Q(\ell_\sigma) = w$. But then all $u_\sigma := \hat{w} - \ell_\sigma$ lie in $M$. Therefore $D(\hat{w})$ is a Cartier divisor and thus we have $\bar{D}(w) \in \text{Pic}(X)$.

Conversely, given $w \in K$ with $\bar{D}(w) \in \text{Pic}(X)$, choose $\hat{w} \in E$ with $Q(\hat{w}) = w$. Then $D(\hat{w})$ is a Cartier divisor, and hence it is described by a support function $(u_\sigma)_{\sigma \in \Delta^\text{max}}$ with $u_\sigma \in M$. But then each $\ell_\sigma := \hat{w} - P^*(u_\sigma)$ lies in $\hat{\sigma}^\perp \cap E$, which proves the assertion. \qed

We present some further applications. If $\Theta$ is free, then we can easily describe the canonical divisor class of $X$: By [10, Sec. 4.3], the negative of the sum over all invariant prime divisors is a canonical divisor on $X$; its class is given by

$$[K_X] = -\bar{D}(w_1 + \ldots + w_r).$$

Recall that a toric variety is said to be $\mathbb{Q}$-Gorenstein if some multiple of its anticanonical divisor is Cartier. In terms of bunches, we have the following characterization.

Corollary 9.7. Suppose that $\Theta$ is free. Then $X$ is $\mathbb{Q}$-Gorenstein if and only if we have

$$\sum_{i=1}^r w_i \in \bigcap_{\tau \in \Theta} \text{lin}(\tau). \quad \Box$$

Similarly we can decide whether a toric variety is a Fano variety, i.e. its anticanonical divisor is Cartier and ample:
Corollary 9.8.  
(i) Suppose that $\Theta$ is free. Then $X$ is Fano if and only if we have  
\[ \sum_{i=1}^{r} w_i \in \bigcap_{\gamma_0 \in \text{cov}(\Theta)} Q(\gamma_0 \cap E)^{\circ}. \]  
(ii) Suppose that $\Theta$ is free and $X$ is smooth. Then $X$ is Fano if and only if we have  
\[ \sum_{i=1}^{r} w_i \in \bigcap_{\tau \in \Theta} \tau^\circ. \] \hfill $\square$

We now apply our results to the case of simple bunches. Recall that these describe precisely the $\mathbb{Q}$-factorial full 2-complete toric varieties. The first observation is that in this case we immediately obtain the Picard group.

Proposition 9.9. Suppose that $\Theta$ is a simple bunch. Then $D: E \to \text{WDiv}^T(X)$ induces an isomorphism $K \cong \text{Pic}(X)$.

Proof. It suffices to show that the image of $\mathcal{D}$ is the group of invariant Cartier divisors $\text{CDiv}^T(X)$. As $\Theta$ is simple, Lemma 3.6 tells us that the dual projected cone $(F \xrightarrow{\pi} \mathbb{N}, \delta)$ and the maximal projectable fan $\Sigma$ associated to $\Theta$ are the universal reduced Cox construction of the fan $\Delta$ of $X$.

Now, for an element $\mathring{w} \in E$, consider the restrictions $\mathring{w}|_{\text{lin}(\sigma) \cap F}$, where $\sigma \in \Delta^{\text{max}}$. By the properties of a universal reduced Cox construction,  
\[ P: \text{lin}(\mathring{\sigma}) \cap F \to \mathbb{N} \]  
is an isomorphism for each $\sigma \in \Delta^{\text{max}}$. Hence, $\mathring{w}|_{\text{lin}(\sigma) \cap F}$ defines a $\mathbb{Z}$-valued linear function on $\mathbb{N}$. This shows that $\mathcal{D}(\mathring{w})$ is in fact a Cartier divisor.

On the other hand, if $D$ is a Cartier divisor, we have to show that $\mathring{w} := \mathcal{D}^{-1}(D)$ is $\mathbb{Z}$-valued on $F$. As above, we see that the restriction of $\mathring{w}$ to any sublattice $\text{lin}(\mathring{\sigma}) \cap F$ is $\mathbb{Z}$-valued. This implies the assertion, because Lemma 8.3 (ii) yields  
\[ F = \sum_{\sigma} \text{lin}(\mathring{\sigma}) \cap F. \] \hfill $\square$

For a $\mathbb{Q}$-factorial toric variety over $\mathbb{C}$, we can identify $H^2(X, \mathbb{Q})$ with $K_\mathbb{Q}$. Hence we may identify $H^2(X, \mathbb{Q})$ with $L_\mathbb{Q} = \text{Hom}(K_\mathbb{Q}, \mathbb{Q})$. Then the Mori Cone, i.e., the cone $\text{NE}(X) \subset L_\mathbb{Q}$ of numerically effective curve classes is dual to the cone of numerically effective divisor classes $\mathcal{N}(X) \subset K_\mathbb{Q}$.

Reid proved that the Mori Cone of a $\mathbb{Q}$-factorial complete toric variety is generated by the classes of the invariant curves, see [21, Cor. 1.7]. Using the fact that on a $\mathbb{Q}$-factorial complete toric variety semiample and numerically effective divisors coincide, Theorem 9.2 gives a new description of the Mori Cone:

Corollary 9.10. Suppose that $X_\Theta$ is complete and simplicial. Then the cone of numerically effective curve classes in $H_2(X, \mathbb{Q})$ is given by  
\[ \text{NE}(X_\Theta) \cong \sum_{\tau \in \Theta} \tau^\vee. \]  
In particular, this cone is convex and polyhedral. Moreover, $X_\Theta$ is projective if and only if $\text{NE}(X_\Theta)$ is strictly convex. \hfill $\square$
10. Applications and examples

In this section, we present some applications and examples. First, we perform Kleinschmidt’s classification in the setting of bunches. We use the visualization techniques introduced at the end of Section 6.

**Proposition 10.1.** The 2-complete smooth toric varieties $X$ with $\text{Cl}(X) \cong \mathbb{Z}^2$ and $\mathcal{O}(X) \cong \mathbb{K}$ correspond to free bunches $\Theta = \{\text{cone}(w_1, w_2)\}$ given by

- **weight vectors** $w_1 := (1, 0)$ and $w_i := (b_i, 1)$, with $0 = b_n < b_{n-1} < \cdots < b_2$,
- **multiplicities** $\mu_i := \mu(w_i)$ with $\mu_1 > 1$, $\mu_n > 0$ and $\mu_2 + \cdots + \mu_n > 1$.

Moreover, the toric variety $X$ defined by such a bunch $\Theta$ is always projective, and it is Fano if and only if we have

$$b_2(\mu_3 + \cdots + \mu_n) < \mu_1 + b_2\mu_3 + \cdots + b_{n-1}\mu_{n-1}.$$

**Proof.** We first show that every smooth 2-complete toric variety $X$ with $\text{Cl}(X) \cong \mathbb{Z}^2$ and $\mathcal{O}(X) \cong \mathbb{K}$ arises from a bunch as in the assertion. From Theorem 6.6(ii), we infer that the toric variety $X$ arises from a free bunch $\Theta$ in a projected cone $(\mathbb{Z}^m \rightarrow \mathbb{Z}^2, \gamma)$ with $\gamma = \text{cone}(e_1, \ldots, e_m)$. Let $\{w_1, \ldots, w_n\}$ be the set of weight vectors, and let $\mu_i$ be the multiplicity of $w_i$.

Smoothness of $X$ means that every image $Q(\gamma_0 \cap \mathbb{Z}^m)$, where $\gamma_0 \in \text{cov}(\Theta)$, generates $\mathbb{Z}^2$, see Proposition 7.3. Since we have $\mathcal{O}(X) = \mathbb{K}$, Proposition 7.4 tells us that the cone $\vartheta \subset K_{\mathbb{Q}}$ generated by the weight vectors is strictly convex. Being two-dimensional, $\vartheta$ is generated by two vectors, say $\vartheta = \text{cone}(w_1, w_n)$.

Now we make essential use of the fact that $\Theta$ lives in a two-dimensional space: Consider the intersection $\tau_0$ of all $\tau \in \Theta$. Then $\tau_0$ is a weight cone. Moreover, $\tau_0$ is of dimension two, because for any two cones of $\Theta$ their relative interiors intersect. Thus the defining property of a bunch implies $\Theta = \{\tau_0\}$.

As a strictly convex two-dimensional weight cone containing $\tau_0$, also $\vartheta$ occurs among the images $Q(\gamma_0)$, where $\gamma_0 \in \text{cov}(\Theta)$. Consequently, $\vartheta = \text{cone}(w_1, w_n)$ is a regular cone in $\mathbb{Z}^2$, and moreover, using Remark 2.11 we may assume that $w_1 = e_1$ and $w_n = e_2$ are the canonical base vectors of $\mathbb{Z}^2$.

We claim that the cone $\tau_0$ has at least one ray in common with $\vartheta$. Indeed, otherwise one of the cones generated by $\tau_0 \cup \mathbb{Q}_{\geq 0} e_1$ or $\tau_0 \cup \mathbb{Q}_{\geq 0} e_2$ would be non-regular. But both cones occur among $Q(\gamma_0)$, where $\gamma_0 \in \text{cov}(\Theta)$. A contradiction. Again by Remark 2.11 we may assume that $\tau_0$ contains $e_1$, and hence is generated by vectors $e_1$ and $b_2 e_1 + e_2$, where $b_2 \geq 0$.

Consider any $w_i \neq e_1$. Then $\tau_i := \text{cone}(w_i, e_1)$ overlaps $\tau_0$, and thus the properties of a bunch give $\tau_0 \subset \tau_i$. Consequently $\tau_i = Q(\gamma_0)$ for some $\gamma_0 \in \text{cov}(\Theta)$, and hence $w_i$ and $e_1$ generate $\mathbb{Z}^2$. But this means $w_i = b_i e_1 + e_2$ with $b_i \geq b_2$. So we arrive at the desired picture. Note that the conditions $\mu_1 > 1$ and $\mu_2 + \cdots + \mu_n > 1$ are due to Property 6.1(ii).

The fact that the toric variety $X$ associated to $\Theta$ is projective is clear by Proposition 7.6 and Corollary 9.8. Moreover, Corollary 9.8 tells us that $X$ is Fano if and
only if the weighted sum $\mu_1 w_1 + \ldots + \mu_n w_n$ lies in the relative interior of the cone $\tau$. But this holds if and only if the condition stated in the assertion is valid. □

Let us illustrate the quasiprojectivity criterion 9.3 by means of the following version of a classical example taken from Oda’s book [18, p. 84]:

**Example 10.2.** The simplest nonprojective complete simplicial fans $\Delta$ live in $\mathbb{Z}^3$ and are of the following type: Consider a prism $P \subset \mathbb{Q}^3$ over a 2-simplex such that 0 lies in the relative interior of $P$. Here we take $P$ with the vertices

$(-1, 0, 0), \ (0, -1, 0), \ (0, 0, -1), \ (0, 1, 1), \ (1, 0, 1), \ (1, 1, 0).$

Then subdivide the facets of $P$ according to the picture below, and let $\Delta$ be the fan generated by the cones over the simplices of this subdivision.

The bunch $\Theta$ corresponding to $\Delta$ is free and lives in a 3-dimensional lattice $K \cong \mathbb{Z}^3$. Combinatorially, it looks as indicated above. Explicitly it is given by the weight vectors $e_1, e_2, e_3, \ w_1 := e_1 + e_2, \ w_2 := e_1 + e_3, \ w_3 = e_2 + e_3$ in $\mathbb{Z}^3$, and the four cones

$$\text{cone}(e_3, w_1, w_2), \ \text{cone}(e_1, w_1, w_3)$$

$$\text{cone}(e_2, w_2, w_3), \ \text{cone}(w_1, w_2, w_3)$$

Using the Propositions 7.1, 7.6, and Corollary 9.3 it is immediately clear from the picture that $X := X_\Theta$ is $\mathbb{Q}$-factorial and nonprojective. Moreover, by Theorem 9.2 the cone $C^{sa}(X)$ of semample divisors is spanned by the class of the anticanonical divisor.

The last example concerns the problem whether or not the Betti numbers of a toric variety are determined by the combinatorial type of the defining fan. A first counterexample was given by McConnell, see [16]. Later Eikelberg [17] gave the following simpler one:

**Example 10.3.** Let $P \subset \mathbb{Q}^3$ be a prism over a 2-simplex such that 0 lies in the relative interior of $P$, and define $\Delta$ to be the fan generated the cones over the facets of $P$. As vertices of $P$ we take:

$$v_1 := (1, 0, 1), \ v_2 := (0, 1, 1), \ v_3 := (-1, -1, 1),$$

$$v_4 := (1, 0, -1), \ v_5 := (0, 1, -1), \ v_6 := (-1, -1, -1).$$

Eikelberg constructs a nonprojective fan $\Delta'$ from $\Delta$ by moving the ray through $v_2$ into the ray through $v_2' := (1, 2, 3)$. The picture is the following (the dotted diagonals indicate the edges of the convex hull over $v_1, v_2', v_3, \ldots, v_6$ that do not define cones of $\Delta'$):
What does this mean in terms of bunches? First note that the bunches $\Theta$ and $\Theta'$ corresponding to $\Delta$ and $\Delta'$ contain 2-dimensional cones, because $\Delta$ as well as $\Delta'$ have nonsimplicial cones. The loss of projectivity is reflected in terms of bunches as follows:

Here we draw the intersection of $\Theta$ and $\Theta'$ with a plane orthogonal to an inner vector of the (strictly) convex hulls $|\Theta|$ and $|\Theta'|$ of the unions of the respective bunch cones. So the fat lines correspond to 2-dimensional bunch cones, whereas the shaded simplices represent 3-dimensional bunch cones.

Now, both fans $\Delta$ and $\Delta'$ have the same combinatorial type. In terms of bunches, we see immediately that the associated toric varieties $X$ and $X'$ have different Betti numbers: The second Betti number $b_2(X)$ equals the dimension of $\text{Pic}_Q(X)$. Hence Theorem 9.2 gives us $b_2(X) = 1$. For $X'$, we obtain $b_2(X') = 0$ by the same reasoning.

We conclude with a remark concerning Ewald’s construction of “canonical extensions” of a given toric variety presented in [8, Section 3], compare also [5]. The bunch theoretical analogue is the following:

**Construction 10.4.** Consider a free bunch, represented in the sense of Construction 2.9 by a set weight vectors $\{w_1, \ldots, w_n\}$ in a lattice $K$, multiplicities $\mu_1, \ldots, \mu_n$ and a collection $\Theta$ of weight cones. Define a new free bunch by setting

$$K' := K, \quad w'_i := w_i, \quad \Theta' := \Theta$$

and replacing the multiplicities $\mu_i$ with bigger ones, say $\mu'_i$. For the toric varieties $X$ and $X'$ associated to these bunches, we have

$$\dim(X') - \dim(X) = \sum_{i=1}^{n} (\mu'_i - \mu_i), \quad \text{Cl}(X') \cong \text{Cl}(X), \quad \text{Pic}(X') \cong \text{Pic}(X).$$
Moreover, applying the respective characterizations of Section 7 and 9, one immediately verifies that $X'$ is non quasiprojective (complete, $\mathbb{Q}$-factorial, smooth) if $X$ was so.

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