ON MASS-MINIMIZING EXTENSIONS OF BARTNIK BOUNDARY DATA

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Abstract. We prove that the space of initial data sets which have fixed Bartnik boundary data and solve the constraint equations is a Banach manifold. Moreover, on this constraint manifold the critical points of the ADM mass are exactly the initial data sets which admit generalised Killing vector fields with asymptotic limit proportional to the ADM energy-momentum vector.

1. Introduction

The Bartnik quasi local mass is one of the most interesting and well-studied notions of quasi local mass in general relativity. For a bounded initial data set \((\Omega, g_0, K_0)\), which consists of a bounded 3-manifold \(\Omega\) with boundary \(\partial \Omega \neq \emptyset\), a Riemannian metric \(g_0\) and a symmetric 2-tensor \(K_0\) defined on \(\Omega\), its Bartnik quasi local mass is defined as (cf. [6])

\[
m_B(\Omega, g_0, K_0) = \inf \{ m_{\text{ADM}}(M, g, K) \}.
\]

Here the infimum is taken over all admissible extensions \((M, g, K)\) – asymptotically flat initial data sets such that the following data (called the Bartnik boundary data) of \(M\) equals those of \(\Omega\) along the boundary via some diffeomorphism \(\partial M \cong \partial \Omega\),

\[
g_{\partial M} = (g_0)_{\partial \Omega}, \quad H_{\partial M} = H_{\partial \Omega}, \quad \text{tr}_{\partial M} K = \text{tr}_{\partial \Omega} K_0, \quad \omega_{\partial M} = \omega_{\partial \Omega}.
\]

In the above, \(g_{\partial M}\) is the metric on the boundary induced by \(g\); \(H_{\partial M}\) is the mean curvature of the boundary \(\partial M \subset (M, g)\); \(\text{tr}_{\partial M} K\) is the trace (with respect to \(g_{\partial M}\)) of the tensor \(K|_{\partial M}\) on \(\partial M\) induced by \(K\); and \(\omega_{\partial M}\) is the connection 1-form on \(\partial M\) defined by \(\omega_{\partial M} = K(n)|_{\partial M}\), i.e. the normal-tangential components of the symmetric 2-tensor \(K\) on the boundary.

Various geometric conditions across the boundary \(\partial M\) have been studied in the literature, such as \((M, g, K)\) extends \((\Omega, g_0, K_0)\) smoothly or the mean curvature is non-increasing \((H_{\partial M} \leq H_{\partial \Omega})\). The Bartnik boundary condition \((1.2)\) ensures that the Hamiltonian constraint and momentum constraint are distributionally well-defined on the glued initial data set \((M \cup \partial M \Omega, g, K)\) so that dominant energy condition can be imposed. Moreover, it also arises naturally from a Hamiltonian analysis of the vacuum Einstein equations, which we will see in the discussion to follow.

A well-known conjecture on the Bartnik quasi local mass proposed by Bartnik is as follows:

**Conjecture 1.1.** If the infimum in \((1.1)\) is achieved, it must be realized by a stationary vacuum extension – an extension \((M, g, K)\) which can be embedded into a stationary vacuum spacetime as an initial data set.

Here a *stationary vacuum spacetime* is a spacetime equipped with a Lorentzian metric which is Ricci flat and admits a Killing vector field that is asymptotically time-like. We note that in the original conjecture the ambient vacuum spacetime admits a time-like Killing vector field. The conjecture was first studied in the time-symmetric case where \(K_0 \equiv 0\) so that the Bartnik boundary condition is reduced to

\[
(g_{\partial M}, H_{\partial M}) = (g_{\partial \Omega}, H_{\partial \Omega}).
\]

\(^1\)An extension \((M, g, K)\) is called admissible if it satisfies the dominant energy condition and certain decay conditions so that \(m_{\text{ADM}}\) is well-defined for the glued data \((M \cup_{\partial M} \Omega, g, K)\); in addition \((M \cup_{\partial M} \Omega, g, K)\) contains no apparent horizon (cf. [6]).
Corvino (cf. [12]) proved that if \((M, g)\) is a minimal ADM energy extension which extends \((Ω, g_0)\) smoothly, then it must be static in the sense that \(g\) admits a nontrivial static potential on \(M \setminus \partial M\). In addition, Miao (cf. [17]) proved when \(\partial M\) has positive Gauss curvature, a minimal mass extension for the Bartnik quasi local mass defined with non-increasing mean curvature boundary condition must satisfy the Bartnik boundary condition (1.3) as well as being static. For the general case where the spacetime is not time-symmetric, Corvino (cf. [13]) studies it using a modified constraint map and conformal argument. With further application of the modified constraint map, Huang-Lee (cf. [15]) prove that a minimizer can be embedded into a null dust spacetime which admits a global Killing vector field.

Besides the approaches mentioned above, Bartnik (cf. [8]) constructed a regularization \(\mathcal{H}\) of the Regge-Teitelboim Hamiltonian and analyzed the functional \(\mathcal{H}\) following an approach initiated by Brill-Deser-Fadeev (cf. [10]). By that he proved on a complete asymptotically flat manifold constrained critical points of the ADM mass must be stationary. Bartnik then suggested that a variational proof of the conjecture, based on extending his work to manifolds with boundary, would be more natural. By implementing the program suggested by Bartnik, Anderson-Jauregui (cf. [4]) proved the conjecture in the time-symmetric case and moreover, showed that the static potential function of the static metric must be positive and asymptotically decays to 1 at infinity, which has not been addressed by previous work. In this paper, we will generalize the method in [4, 8] to study the mass minimizing extensions for bounded initial data sets \((Ω, g_0, K_0)\) in general and extend the result on critical points of the ADM mass in [8] to initial data sets with boundary where the Bartnik boundary data is fixed, which will prove part of the conjecture.

Recall that the Einstein equation for a spacetime \((V^{(4)}, g^{(4)})\) is given by

\[
Ric_{g^{(4)}} - \frac{1}{2}R_{g^{(4)}} = 8\pi T,
\]

where \(Ric_{g^{(4)}}\) and \(R_{g^{(4)}}\) are the Ricci and scalar curvatures of \(g^{(4)}\) and \(T\) is the stress-energy tensor of matter. If an initial data set \((M, g, K)\) is embedded in such a spacetime, it must satisfy the constraint equations

\[
\begin{align*}
R - |K|^2 + (\text{tr} K)^2 &= u, \\
-\delta K - d\text{tr} K &= Z,
\end{align*}
\]

where the first equation above is called the Hamiltonian constraint and the second is called the momentum constraint. The constraint equations are obtained from the Gauss-Codazzi-Mainardi hypersurface equations on \(M \subset (V^{(4)}, g^{(4)})\) and \(u, Z\) are determined by the stress-energy tensor \(T\).

Consider the space \(\mathcal{C}(u, Z)\) of all asymptotically flat initial data sets \((M, g, K)\) which satisfy the constraint equations (1.5). For complete asymptotically flat manifolds \(M\), Bartnik proved (cf. [8]) the constraint space \(\mathcal{C}(u, Z)\) has Hilbert manifold structure. It is an interesting problem to extend this result to manifolds with boundary, where certain geometric boundary data is fixed. However, a crucial ingredient in Bartnik’s proof – surjectivity of the constraint map – becomes complicated and subtle when the manifold has nonempty boundary. In fact, McCormick (cf. [19]) worked with the boundary condition which requires the first derivatives of the metric to be fixed on \(\partial M\) and pointed out that the manifold structure theorem is almost certainly false in this case. In [4] Anderson-Jauregui proved the manifold structure theorem for the constraint space in the time-symmetric case where the boundary conditions are those in (1.3). In their work, the ellipticity of the boundary data (1.3) for static spacetimes (cf. [5]) plays an important role.

Inspired by the work [4], we apply the ellipticity of the Bartnik boundary data (cf. [3]) in this paper to prove that the space \(\mathcal{C}(u, Z)\) is a smooth Banach manifold when the Bartnik boundary data (1.2) is fixed. In §2, we construct a constraint map \(\Phi\) based on (1.5) and the boundary conditions (1.2). We will prove that the linearization \(D\Phi\) of this constraint map is surjective by

\[
\begin{align*}
Ric_{g^{(4)}} - \frac{1}{2}R_{g^{(4)}} &= 8\pi T, \\
R - |K|^2 + (\text{tr} K)^2 &= u, \\
-\delta K - d\text{tr} K &= Z,
\end{align*}
\]
showing it has closed range and trivial cokernel. The closed-range property is essentially due to the ellipticity of the Bartnik boundary data for stationary vacuum spacetimes. Then we will prove that the linearized constraint map has splitting kernel, using the idea developed in [22]. At last, based on the implicit function theorem for Banach spaces, we derive the manifold theorem for the constraint space $C_B(u, Z)$ – the space of initial data sets $(M, g, K)$ in $C(u, Z)$ which satisfy the Bartnik boundary condition (1.2).

**Theorem 1.2.** The space $C_B(u, Z)$ is an infinite-dimensional smooth Banach manifold.

In §3 we analyze the modified Regge-Teitelboim Hamiltonian $H$ constructed in [8] on $C_B(u, Z)$ and show the boundary terms appearing in the variational formula of $H$ vanish for infinitesimal deformations that fix the Bartnik boundary data. So we have a well-defined variational problem for the Hamiltonian $H$ on the constraint manifold $C_B(u, Z)$. Following the approach suggested by Bartnik, we study the critical points of the ADM mass on the constraint manifold. A rough version of the theorem is as follows, we refer to Theorem 3.2 for a precise statement.

**Theorem 1.3.** Critical points of the ADM mass on the constraint manifold $C_B(u, Z)$ which have positive ADM mass are exactly the initial data sets admitting generalised Killing fields that are asymptotically time-like.

Here we adopt the terminology *generalised Killing vector fields* from the work of Bartnik [8] – it refers to nontrivial kernel elements of the adjoint $D\Phi^*$ of the linearized constraint map (cf.§3 for the precise definition). Back to Conjecture 1.1, if an extension $(M, g, K)$ is a minimizer of the Bartnik mass of $(\Omega, g_0, K_0)$, then it must be a critical point the ADM mass on the constraint manifold that contains it. Assume in addition the infimum (1.1) is positive, then by the theorem above $(M, g, K)$ must admit a generalised Killing vector field which is asymptotically time-like. In the special case where the initial data set satisfies the vacuum constraint equations (1.5) with $u = Z = 0$ and has enough regularity, one can construct a vacuum spacetime starting from the initial data set $(M, g, K)$ by solving the Cauchy problem of the Einstein equations. Now in such a vacuum spacetime we have the following result from [18] (cf. also [14]):

**Theorem** 2 (Moncrief) Suppose $(M, g, K)$ is embedded as a Cauchy surface in a globally hyperbolic vacuum spacetime $(V^{(4)}, g^{(4)})$. Then a generalised Killing vector field $(X^0, X^i)$ of $(M, g, K)$ give rise to a standard Killing vector field $X^{(4)}$ in $(V^{(4)}, g^{(4)})$ such that the perpendicular and parallel components of $X^{(4)}$ are $X^0$ and $X^i$ on $M$.

Thus we can obtain the following corollary from Theorem 1.3:

**Corollary 1.4.** If $(M, g, K)$ is a smooth initial data set realizing the infimum in (1.1) with positive ADM mass and satisfying the vacuum constraint equations, then it must arise from a vacuum stationary spacetime.

It remains an open and interesting problem whether a minimizer of the Bartnik mass must belong to the vacuum constraint manifold $C_B(0, 0)$. We note that in a recent work by Huang-Lee [15] the constraints of a minimizer is well-studied and in particular they are shown to be vacuum outside a compact set of $M$.

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\(^2\)Moncrief worked with vacuum spacetimes with compact Cauchy hypersurfaces in order to discuss the linearization stability of the Einstein equations. But this particular result also holds in noncompact case.
2. The Constraint Manifold

Let $M$ be a smooth manifold diffeomorphic to $\mathbb{R}^3 \setminus B^3$, where $B^3$ denotes the open unit 3-ball. So $M$ has nonempty boundary $\partial M$ diffeomorphic to the unit sphere $S^2$. Via the diffeomorphism, $M$ can be equipped with a global coordinate chart $\{x^i\}, (i = 1, 2, 3)$, a radius function $\hat{r} \in [1, \infty)$ and a flat metric $\hat{g}$ which is the pull back of the flat metric on $\mathbb{R}^3 \setminus B^3$. Using this chart, we can define the weighted Hölder spaces of tensor fields on $M$ as follows.

**Definition 2.1.** The $C^m_\delta$-norm of a $C^m$ function $v$ on $M$ is given by

$$||v||_{C^m_\delta} = \Sigma_{k=0}^{m} \sup_i \hat{r}^{k+\delta} |\nabla^k v|$$

where $\nabla$ is the connection with respect to the pull back of the flat metric on $\mathbb{R}^3 \setminus B^3$. In addition, we use $tr$ with respect to the infinitesimal deformation $B$ space.

The space $C^m_\delta(M)$ (or $C^m_{\delta,\alpha}(M)$) is the space of all functions with bounded $C^m_\delta$-norm (or $C^m_{\delta,\alpha}$-norm). Various spaces of tensor fields on $M$ with respect to the weighted Hölder norm are defined as

$$Met^m_\delta(M) = \{\text{Riemannian metrics } g \text{ on } M : (g_{ij} - \hat{g}_{ij}) \in C^m_{\delta,\alpha}(M)\},$$

$$S^m_\delta(M) = \{\text{symmetric 2-tensors } K \text{ on } M : K_{ij} \in C^m_{\delta,\alpha}(M)\},$$

$$T^m_\delta(M) = \{\text{vector fields } Y \text{ on } M : Y^i \in C^m_{\delta,\alpha}(M)\},$$

$$(T^p)^m_\delta(M) = \{(p, q) - \text{tensors } \tau \text{ on } M : \tau^{ij}_{\alpha1\alpha2...\alpha p} \in C^m_{\delta,\alpha}(M)\},$$

$$(\wedge^p)^m_\delta(M) = \{p - \text{forms } \sigma \text{ on } M : \sigma_{i1i2...ip} \in C^m_{\delta,\alpha}(M)\}.$$

On $M$, an asymptotically flat initial data set consists of a Riemannian metric $g \in Met^m_\delta(M)$ and a symmetric 2-tensor $K \in S^{m-1,\alpha}_{\delta+1}(M)$. Based on the Bartnik conditions (1.2), we set up a space $\mathcal{B}$ of initial data on $M$ with fixed Bartnik boundary data:

$$\mathcal{B} = \{(g, K) \in [Met^m_{\delta} \times S^{m-1,\alpha}_{\delta+1}](M) : (g_{\partial M}, H_{\partial M}, tr_{\partial M} K, \omega_{\partial M}) = ((g_0)_{\partial M}, H_{\partial M}, tr_{\partial M} K_0, \omega_{\partial M}) \text{ on } \partial M\},$$

where $(\Omega, g_0, K_0)$ is a fixed bounded initial data set with boundary $\partial \Omega \cong S^2$. Throughout, we assume that $m \geq 2$ and $\frac{1}{2} < \delta \leq 1$. It is easy to show (by implicit function theorem) that for a fixed set of data $(g_0)_{\partial M}, H_{\partial M}, tr_{\partial M} K_0, \omega_{\partial M})$, $\mathcal{B}$ is a smooth closed Banach submanifold of $[Met^m_{\delta} \times S^{m-1,\alpha}_{\delta+1}](M)$. The tangent space at a point $(g, K) \in \mathcal{B}$ consists of infinitesimal deformations which fix the Bartnik boundary data, i.e.

$$T\mathcal{B}|_{(g,K)} = \{(h, p) \in [S^{m,\alpha}_{\delta} \times S^{m-1,\alpha}_{\delta+1}](M) : h^T = 0, H^T_h = 0, tr^T p = 0, p(n)^T + K(n)^T = 0 \text{ on } \partial M\}.$$

Here the superscript $^T$ on a tensor field denotes its components tangential to the boundary manifold $\partial M$. In addition, we use $tr^T$ to denote the trace of an induced tensor on the boundary manifold with respect to the induced metric $g^T$. The prime $'$ denotes the variation of a geometric tensor with respect to the infinitesimal deformation $h$ or $p$. For instance, $H^T_h = \frac{d}{dt}|_{t=0} H_{g+th}$ is the variation of the mean curvature at $g$.

Define the constraint map $\Phi$ on $\mathcal{B}$ as

$$\Phi : \mathcal{B} \to \mathcal{T}$$

$$\Phi(g, K) = (\Phi_0(g, K), \Phi_1(g, K), \Phi_2(g, K)).$$
where
\[ \Phi_0(g, K) = (R - |K|^2 + (\text{tr}K)^2)\sqrt{g}, \]
\[ \Phi_1(g, K) = -2(\delta K + d(\text{tr}K))\sqrt{g}, \]
with \( \sqrt{g} = \sqrt{\text{det}g}/\sqrt{\text{det}g} \). In (2.2) the target space of \( \Phi \) is \( \mathcal{T} = C_{\delta+2}^{m,2,\alpha}(M) \times (\wedge_1)^{m,2,\alpha}(M) \) since \( \Phi_0(g, K) \in C_{\delta+2}^{m,2,\alpha}(M) \) is a scalar field and \( \Phi_1(g, K) \in (\wedge_1)^{m,2,\alpha}(M) \) is a 1-form. By basic computation, the linearization of \( \Phi \) at a point \((g, K, \pi) \in \Phi^{-1}(u_0, Z_0)\) is
\[
D\Phi(g, K) : TB \to \mathcal{T}
\]
where
\[
(D\Phi_0)(g, K)(h, p) = R_h\sqrt{g} + (2K_{ik}K^k_i - 2(\text{tr}K)(\text{tr}h))\sqrt{g}
- 2((K, p) - (\text{tr}K)(\text{tr}p))\sqrt{g} + \frac{1}{2}(\text{tr}h)u_0,
\]
(2.4)
\[
(D\Phi_1)(g, K)(h, p) = -2(\delta p + d\text{tr}p)\sqrt{g} - 2(\delta h K - d(\text{tr}h))\sqrt{g} + \frac{1}{2}(\text{tr}h)Z_0.
\]
(2.5)
In the formulas above \( R_h \) is the Ricci tensor \( R_{ik}(\text{tr}h) \) and \( h_{ij} \) is the sectional curvature \( h_{ij} = \frac{1}{2}K_{ik}\nabla_j K_{kj} - K(\beta h)_i + \frac{1}{2}K_{jk}\nabla_i h_{kj} \) with \( \beta \) the Bianchi operator \( \beta h = \delta h + \frac{1}{2}d\text{tr}h \). Here and throughout the paper the Laplacian \( \Delta = -\text{tr}Hess \).

In this section we will prove for fixed \((u_0, Z_0) \in \mathcal{T}\) the level set \( \Phi^{-1}(u_0, Z_0) \) is a Banach manifold based on the implicit function theorem. Before starting the proof, we note that there is an equivalent way to express the constraint map. Let \( \pi \) be the conjugate momentum defined as
\[
\pi = (K - (\text{tr}gK)g)\frac{\sharp}{\sqrt{g}}.
\]
Here the superscript \( \frac{\sharp}{\sqrt{g}} \) means to raise the indices (or to take the dual) of a 2-tensor with respect to the metric \( g \). Let \( \mathcal{B} \) be the space of pairs \((g, \pi)\) parameterised by \((g, K)\) in \( \mathcal{B} \):
\[
\mathcal{B} = \{(g, \pi) \in [\text{Met}^{m,\alpha}_{\delta} \times (T_0^{2,m-1,\alpha}_{\delta+1})](M) : g = g_0, \pi = (K_0 - (\text{tr}g_0K_0)g_0)^{\sharp}/\sqrt{g_0}, \text{ for some } (g_0, K_0) \in \mathcal{B} \}.
\]
It is easy to observe that the space \( \mathcal{B} \) and \( \mathcal{B} \) are equivalent, so that \( \mathcal{B} \) is also a Banach manifold. The tangent space at \((g, \pi) \in \mathcal{B} \) is given by
\[
\mathcal{T}\mathcal{B}|(g, \pi) = \{(h, \sigma) \in [S^{m,\alpha}_{\delta} \times (T_0^{2,m-1,\alpha}_{\delta+1})](M) : \sigma \text{ is a symmetric (0,2)-tensor,}
\]
\[
h^T = 0, \quad H^T = 0, \quad \sigma^{11} + \frac{1}{2}\pi^{11}h_{11} = 0, \quad \sigma^{1A} + \pi^{11}h_{1A} = 0 (A = 2, 3) \text{ on } \partial M \}.
\]
(2.6)
Here and throughout the paper, we use the index 1 to denote normal direction to the boundary \( \partial M \subset (M, g) \) and indices 2, 3 to denote the tangential directions to the boundary \( \partial M \). Upper case Roman indices \( A \in \{2, 3\} \) and lower case Roman \( i \in \{1, 2, 3\} \). In addition we use index 0 to denote the time direction in the ambient spacetime which contains the initial data set \((M, g, K)\) and use Greek letters \( \alpha \in \{0, 1, 2, 3\} \) when needed.

The boundary conditions in (2.6) are equivalent to those listed in (2.1); we refer to appendix section 4.1 for the detailed calculation. The constraint map then can be defined equivalently as a map on \( \mathcal{B} \)
\[
\tilde{\Phi} : \mathcal{B} \to \mathcal{T}
\]
\[
\tilde{\Phi}(g, \pi) = (\tilde{\Phi}_0(g, \pi), \tilde{\Phi}_1(g, \pi)),
\]
with
\[
\tilde{\Phi}_0(g, \pi) = \frac{R(g)}{\sqrt{g}} - (|\pi|^2 - \frac{1}{2}(\text{tr}\pi)^2)/\sqrt{g}, \quad \tilde{\Phi}_1(g, \pi) = -2(\delta(\pi/\sqrt{g}))^\frac{\sharp}{\sqrt{g}}.
\]
Here the superscript $\flat$ means to lower the indices (or to take dual) of a tensor field with respect to the metric $g$. We refer to [8] for the explicit formula of the linearization $D\Phi$. Obviously, the maps $\Phi$ and $\tilde{\Phi}$ are related by the equivalence between $B$ and $\tilde{B}$, so their level sets are diffeomorphic. In the next section, we will switch between these two formulations as needed.

Now we turn to prove the constraint map $\Phi$ is a submersion, i.e. at a point $(g, K) \in \Phi^{-1}(u_0, Z_0)$ the linearized map $D\Phi_{(g, K)}$ as in (2.3) is surjective and its kernel splits in $TB$, so that we can apply the implicit function theorem on $\Phi$.

2.1. **Surjectivity.** We will prove surjectivity by showing the linearized constraint map has closed range and trivial cokernel. For simplicity we denote the linearized map $L$ the linearized map $\Phi$ and $\tilde{\Phi}$ are related by the equivalence between $B$ and $\tilde{B}$, so that the image $L(V)$ has finite codimension in $T$.

Fix $(g, K) \in \Phi^{-1}(u_0, Z_0)$. Define a space $W$ consisting of triples $(h, Y, v)$ of a symmetric 2-tensor $h$, a vector field $Y$ and a scalar field $v$ all of which are asymptotically zero on $M$ as follows

$$W = \{(h, Y, v) \in [S^m_{\delta} \times (TM)^{m,\alpha}_\delta \times C^{m,\alpha}_\delta](M) :$$

$$\delta h - 3dv = 0, \, h^T = 0, \, H'_h = 0, \, \text{tr}^T(\delta^* Y + (\delta Y) g) = 0, \, \delta^* Y(n)^T + K(n'_h)^T = 0$$

on $\partial M \}.

In the above and throughout the following, the divergence operator $\delta$ and its adjoint $\delta^*$ are both with respect to the metric $g$; and $\text{tr}^T$ is to take trace with respect to the induced metric $g^T$ on the boundary. Variation of the mean curvature $H'$ and unit normal $n'$ are both taken at the base point $g$. All the boundary conditions above are constructed based on the Bartnik boundary conditions (1.2), except that the first boundary condition is regarded as a gauge condition.

Let $V$ be the space obtained from projecting $W$ to the first two components, i.e.

$$V = \{(h, Y) \in [S^m_{\delta} \times (TM)^{m,\alpha}_\delta](M) : (h, Y, v) \in W \text{ for some } v \}.

Then it is easy to verify that the space $V$ generated by $V$:

$$V = \{(h, p) \in [S^m_{\delta} \times S^{m-1,\alpha}_{\delta+1}](M) : h = h_0, \, p = \delta^* Y_0 + (\delta Y_0) g \text{ for some } (h_0, Y_0) \in V \},

is a subspace of $TB_{(g, K)}$. Let $\Psi$ be the induced map on $V$ by the linearized constraint map $L$, i.e.

$$\Psi : V \to \mathcal{T}

(2.10)

$$\Psi(h, Y) = L(h, \delta^* Y + (\delta Y) g).

Based on the equations (2.4)-(2.5), $\Psi$ is of the form

$$\Psi(h, Y) = (\Delta (\text{tr} h) + \delta \delta h) \sqrt{g} + O_1(h, Y), \, -2[\delta \delta Y + d\delta Y] \sqrt{g} + O_1(h, Y)

(2.11)

where $O_1(h, Y)$ denote the terms that involve at most 1st order derivatives of $h$ and $Y$.

Observe the range of $\Psi$ satisfies $\text{Im} \Psi = L(V)$. Since $V$ is a subspace of $TB_{(g, K)}$, the closedness of the range of $L$ will hold if we show $\text{Im} \Psi$ has finite codimension in $\mathcal{T}$. As mentioned in the introduction, the constraint equations are actually part of the Einstein field equations on the spacetime $(V^4, g^{(4)})$ where $(M, g, K)$ is embedded as an initial data set. So its linearization $L$ is part of the linearized Einstein equations. Moreover, it is proved in [3] that the stationary Einstein equations (combined with proper gauge) and the Bartnik boundary conditions form an elliptic boundary value problem in the phase space consisting of triples $(g, X, N)$. Here $X$ and $N$ are the shift vector and lapse function on the hypersurface $(M, g) \subset (V^4, g^{(4)})$. So we can understand $(h, Y, v) \in W$ as the deformation of $(g, X, N)$ that preserves the Bartnik boundary data; and the map $\Psi$ can be taken as part of the linearized stationary Einstein field equations, which indicates that the map $\Psi$ is underdetermined elliptic.
To carry out this idea, we first construct a differential operator $P = (L, B)$, with $L$ being the interior operator

$$L : [S_{\delta}^{m,\alpha} \times (TM)_{\delta}^{m,\alpha} \times C_{\delta}^{m,\alpha}](M) \to [S_{\delta+2}^{m-2,\alpha} \times C_{\delta+2}^{m-2,\alpha} \times (\wedge 1)_{\delta+2}^{m-2,\alpha}](M)$$

(2.12)

$$L(h, Y, v) = (\mathcal{E}_0(h, v), \Delta \pi h + \delta \pi h, \delta \pi Y + d\pi Y)$$

and $B$ the boundary operator

$$B : [S_{\delta}^{m,\alpha} \times (TM)_{\delta}^{m,\alpha} \times C_{\delta}^{m,\alpha}](M) \to [(\wedge 1)^{m-1,\alpha} \times S_{\delta}^{m,\alpha} \times (C_{\delta}^{m,\alpha})^2 \times (\wedge 1)^{m-1,\alpha}](\partial M)$$

(2.13)

$$B(h, Y, v) = (\delta Y h - 3dv, h^T, H_h^T, \pi h^{T} \delta Y + (\delta Y)g, \delta Y (n)^T)$$

where the first term in $L(h, Y, v)$ is given by

$$\mathcal{E}_0(h, v) = Ein_h' + 2\pi h - (\delta \pi h)g - 4D^2v + 2(\Delta v)g.$$ (2.14)

Here the interior operator maps $(h, Y, v)$ to the principal part $^{(4)} Ein'_{(h, Y, v)}$ of the linearized stationary spacetime Einstein field equations combined with an extra term $[\delta \pi h - (\delta \pi h)g - 4D^2v + 2(\Delta v)g]$ which can be understood as a gauge term. We note it here that the combination in the gauge term is not unique and the one we choose here is for simplicity of the proof of ellipticity to follow. Observe the boundary operator maps $(h, Y, v)$ to the leading order part of the conditions listed in (2.8). Using the method developed in [5], one can prove that $P$ is an elliptic boundary value problem. We will give the detailed proof of ellipticity in the last part of this section.

Notice that the leading order terms in formula (2.11) of $\Psi$ differ from the 2nd and 3rd bulk terms in (2.12) only by non-vanishing rescalings $\sqrt{\gamma}$, $(-2)$ which preserve ellipticity. Moreover, adding lower order derivatives to a differential operator won’t affect its ellipticity either. So we can make the replacement with $\Psi$ in (2.12) and also modify the last boundary term in (2.13) to be the last one in (2.8). The resulting differential operator $P' = (L', B')$:

$$L' : [S_{\delta}^{m,\alpha} \times (TM)_{\delta}^{m,\alpha} \times C_{\delta}^{m,\alpha}](M) \to [S_{\delta+2}^{m-2,\alpha} \times C_{\delta+2}^{m-2,\alpha} \times (\wedge 1)_{\delta+2}^{m-2,\alpha}](M)$$

(2.15)

$$L'(h, Y, v) = (\mathcal{E}_0(h, v), \Psi(h, Y)),$$

$$B' : [S_{\delta}^{m,\alpha} \times (TM)_{\delta}^{m,\alpha} \times C_{\delta}^{m,\alpha}](M) \to [(\wedge 1)^{m-1,\alpha} \times S_{\delta}^{m,\alpha} \times (C_{\delta}^{m,\alpha})^2 \times (\wedge 1)^{m-1,\alpha}](\partial M)$$

$$B'(h, Y, v) = (\delta Y h - 3dv, h^T, H_h^T, \pi h^{T} \delta Y + (\delta Y)g, \delta Y (n)^T + K(n_h^T)^T).$$

is also elliptic, which further implies that the map $\mathcal{P}$ defined below is Fredholm:

$$\mathcal{P} : \mathcal{W} \to S_{\delta+2}^{m-2,\alpha}(M) \times \mathcal{T},$$

(2.16)

$$\mathcal{P}(h, Y, v) = (\mathcal{E}_0(h, v), \Psi(h, Y)).$$

Thus the range $\text{Im}\mathcal{P}$ is closed and has finite codimension in the target space $S_{\delta+2}^{m-2,\alpha}(M) \times \mathcal{T}$. Let $\Pi$ be the projection $\Pi : S_{\delta+2}^{m-2,\alpha}(M) \times \mathcal{T} \to \mathcal{T}$. Projecting the image of $\mathcal{P}$ to the second component, we obtain $\Pi(\text{Im}\mathcal{P}) = \text{Im}\Psi$. It must also be of finite codimension in $\mathcal{T}$. This completes the proof of the closed range of $\mathcal{L}$.

Since it is proved that the range of $\mathcal{L}$ has finite codimension, to prove the surjectivity of $\mathcal{L}$ it suffices to show its cokernel is trivial. We prove this by contradiction. Suppose $\text{Coker}\mathcal{L}$ is non-trivial. Then by the Hahn-Banach Theorem, there is a nontrivial element $\tilde{X}$ in the dual space $\mathcal{T}^*$ so that

$$\tilde{X}(\mathcal{L}(h, p)) = 0, \forall (h, p) \in TB.$$ (2.17)
Here \( \tilde{X} \) can be decomposed as \( \tilde{X} = (X^0, X) \) \((X = X^i i = 1, 2, 3)\) where \( X^0 \in [C^{m-2,\alpha}_{\delta+2}(M)]^* \) and \( X \in [\wedge_1 C^{m-2,\alpha}_{\delta+2}(M)]^* \) such that

\[
\begin{align*}
X^0((D\Phi_0)_{(g,K)}(h,p)) &= 0 \\
X((D\Phi_i)_{(g,K)}(h,p)) &= 0 
\end{align*}
\]

(2.18)

where \((D\Phi_0)_{(g,K)}(h,p), (D\Phi_i)_{(g,K)}(h,p)\) are given in (2.4)-(2.5). We first prove that \((X^0, X)\) must be a regular solution, i.e. it is locally \( C^{m,\alpha} \) in the interior \( \text{int} M \) of \( M \). Based on the construction of the space \( \mathcal{W} \) and the map \( \Psi \), we observe that (2.17) implies \( \tilde{X}(\Psi(h,Y)) = 0 \) \( \forall (h,Y,v) \in \mathcal{W} \). It then follows trivially from the construction of \( \mathcal{P} \) that \((0, \tilde{X})\) is a cokernel element of \( \mathcal{P} \), i.e. the pairing

\[
(0, \tilde{X})[\mathcal{P}(h,Y,v)] = 0, \forall (h,Y,v) \in \mathcal{W}.
\]

Thus \((0, \tilde{X})\) is a weak solution of the elliptic equation

\[
\mathcal{P}^*(0, \tilde{X}) = 0
\]

in the interior of \( M \). We follow the approach in [16] to prove regularity of \( \tilde{X} \). Take bounded domains \( V, U \subset M \) such that \( V \subset \tilde{V} \subset U \). Using the chart \( M \cong \mathbb{R}^3 \setminus B \), we can identify \( U \) as a bounded domain in \( \mathbb{R}^3 \) and \( X^\mu (\mu = 0, 1, 2, 3) \) as distributions in \( U \). Let \( \varphi \) be a smooth cutoff function which equals 1 in \( U \) and compactly supported in \( U \). So \( Y^\mu = \varphi X^\mu \) are compactly supported distributions in \( \mathbb{R}^3 \) and \((0,Y)\) is a weak solution to the elliptic equation (2.19) inside \( V \). Take the Fourier transform of \( Y^\mu \)

\[
F(Y^\mu)(y) = \frac{1}{2\pi^{3/2}} \int_{\mathbb{R}^3} e^{-ixy}Y^\mu(x)dx.
\]

Let \( Z^\mu \) be distributions in \( \mathbb{R}^3 \) such that their Fourier transform are given by

\[
F(Z^\mu)(y) = \left(\frac{1}{1 + |y|^2}\right)^k F(Y^\mu)(y)
\]

It then follows that

\[
(I + \Delta_0)^k Z^\mu = Y^\mu \quad (\mu = 0, 1, 2, 3) \quad \text{in} \ \mathbb{R}^3.
\]

(2.20)

Here \( \Delta_0 \) is the Laplacian with respect to the flat metric in \( \mathbb{R}^3 \). The order \( k \) is chosen so that \( 2k \geq m \). Since \( \tilde{Y} \in (C^{m-2,\alpha})^* \subset H^{-m} \), and the coefficients of the equations in (2.20) are smooth, it follows from Weyl’s lemma that \( Z^\mu \in L^2 \).

Now \( Z^\mu \) are \( L^2 \) functions solving the following elliptic system in \( V \)

\[
\mathcal{P}^*(0, (I + \Delta_0)^k Z^\mu) = 0.
\]

(2.21)

Note that the coefficients in (2.21) are continuous. Thus by interior regularity for elliptic equations (cf. [20]), we can obtain \( C^{m+2k,\alpha} \) control of \( Z^\mu \). Then it follows from equation (2.20) that \( Y^\mu \) has bounded \( C^{m,\alpha} \) norm in \( V \). By a partition of unity argument, it is easy to see that \( X^\mu \) is \( C^{m,\alpha} \) smooth in \( \text{int} M \).

Next we prove \( \tilde{X} = 0 \) in \( \text{int} M \). By basic computation with integration by parts, (2.18) shows that \( \tilde{X} \) is a solution of the following equations on \( M \) (cf. for example [18]):

\[
\begin{align*}
2X^0K + L_Xg &= 0 \\
D^2X^0 + L_XK + X^0[-Ricg + 2K \circ K - (trK)K + \frac{1}{4}u_0g] &= 0
\end{align*}
\]

(2.22)

Since \( \tilde{X} \) is \( C^{m,\alpha} \) in \( \text{int} M \) and by assumption \( m > 2 \), according to Proposition 2.1 in [9], there exist constants \( \Lambda_{\mu\nu} = \Lambda_{\mu\nu} \) \((\mu, \nu = 0, 1, 2, 3)\) such that

\[
X^i - \Lambda_{ij}x^j \in C^{m}_{\delta-1}(M), \quad X^0 - \Lambda_{0i}x^i \in C^{m}_{\delta-1}(M);
\]

(2.23)
or there exist constants $A^\mu$ such that
\begin{equation}
X^i - A^i \in C^m_\delta(M), \quad X^0 - A^0 \in C^m_\delta(M).
\end{equation}

On the other hand, $\hat{X}$ is also a bounded linear functional on $\mathcal{T}$, so we must have $\Lambda_{\mu\nu} = A^\mu = 0$ (cf. §4.5 for details). Then by the proposition in [9] again, we must have $\hat{X} = 0$ in int$M$. Therefore $\langle \hat{X}, (u, Z) \rangle = 0$ for any compactly supported $(u, Z) \in \mathcal{T}$ and hence the same for $(u, Z) \in \mathcal{T}$ which vanishes on $\partial M$.

Furthermore, it is easy to show that any $(u, Z) \in \mathcal{T}$ can be decomposed as $(u, Z) = (u_0, Z_0) + (u_1, Z_1)$ where $(u_0, Z_0)$ vanishes on the boundary and $(u_1, Z_1) \in \text{Im} L$ (cf.§4.2 for a detailed proof). Therefore, $\langle \hat{X}, (u, Z) \rangle = (\hat{X}, (u_0, Z_0)) + (\hat{X}, (u_1, Z_1)) = 0$ for all $(u, Z) \in \mathcal{T}$, i.e. $\hat{X} = 0$. This completes the proof of surjectivity.

2.2. Splitting Kernel. We apply the approach developed in [22] to prove the kernel of the linearized constraint map $\mathcal{L} = D\Phi_{(g,K)}$ at $(g, K) \in \Phi^{-1}(u_0, Z_0)$ splits in the domain space $T\mathcal{B}|_{(g, K)}$. We first introduce the following proposition.

**Proposition 2.2.** The tangent space $T\mathcal{B}|_{(g, K)}$ of deformations $(h, p)$ fixing the Bartnik boundary data admits a splitting
\begin{equation}
T\mathcal{B}|_{(g, K)} = S_1 \oplus S_2
\end{equation}
where the closed subspaces $S_1, S_2$ are such that range of the restricted map $\mathcal{L}|_{S_1} : S_1 \to \mathcal{T}$ has finite codimension. Moreover, the kernel of $\mathcal{L}$ splits in $S_1$, i.e. there is a closed subspace $S$ such that
\begin{equation}
S_1 = S \oplus [\mathcal{L}^{-1}(0) \cap S_1].
\end{equation}

Assuming the above proposition holds, we can then decompose the target space $\mathcal{T}$ as
\begin{equation}
\mathcal{T} = \mathcal{L}(S_1) \oplus K
\end{equation}
with dim$K < \infty$. Based on the decomposition (2.26), let $\mathcal{L}|_S$ denote the restricted map $\mathcal{L}|_S : S \to \mathcal{L}(S_1)$.

Then the map above is bounded linear and bijective. By the open map theorem, it admits a bounded inverse denoted by $\tilde{\mathcal{L}}$. Let $\tilde{\pi}$ denote the projection from $S_1$ onto $[\mathcal{L}^{-1}(0) \cap S_1]$, and $\pi_K$ denote the projection from $\mathcal{T}$ onto $K$. We then obtain the following description of the kernel Ker$\mathcal{L}$ in $T\mathcal{B}|_{(g, K)}$
\begin{equation}
\begin{aligned}
\text{Ker} \mathcal{L} &= \{(h, p) \in T\mathcal{B}|_{(g, K)} : \mathcal{L}(h, p) = 0\} \\
&= \{(h, p) = (h_1, p_1) + (h_2, p_2) : (h_1, p_1) \in S_1, (h_2, p_2) \in S_2, \text{ and } \mathcal{L}(h_1, p_1) = -\mathcal{L}(h_2, p_2)\} \\
&= \{(h, p) = (h_1, p_1) + (h_2, p_2) : (h_1, p_1) \in S_1, (h_2, p_2) \in \text{Ker}(\pi_K \circ \mathcal{L}) \cap S_2, \mathcal{L}(h_1, p_1) = -\mathcal{L}(h_2, p_2)\} \\
&= \{(h, p) = (h_1, p_1) + (h_2, p_2) : (h_2, p_2) \in \text{Ker}(\pi_K \circ \mathcal{L}) \cap S_2, (h_1, p_1) = \tilde{\mathcal{L}}(-\mathcal{L}(h_2, p_2)) + \tilde{\pi}(h_1, p_1)\}.
\end{aligned}
\end{equation}

The third equality above is based on that $\mathcal{L}(h_1, p_1) = -\mathcal{L}(h_2, p_2)$ implies $\mathcal{L}(h_2, p_2) \in \mathcal{L}(S_1)$ and hence $\pi_K \circ \mathcal{L}(h_2, p_2) = 0$ according to (2.27). In the last equality we use the inverse map $\tilde{\mathcal{L}}$ and projection $\tilde{\pi}$ to solve for $(h_1, p_1)$ from $\mathcal{L}(h_1, p_1) = -\mathcal{L}(h_2, p_2)$. Since the map $\pi_K \circ \mathcal{L} : S_2 \to K$ has a target space of finite dimension, its kernel must be of finite codimension and hence splits in $S_2$. So there is a bounded projection $P$ from $S_2$ onto Ker$(\pi_K \circ \mathcal{L}) \cap S_2$. Then we obtain a bounded projection from $T\mathcal{B}|_{(g, K)}$ onto Ker$\mathcal{L}$ given by
\begin{equation}
P : T\mathcal{B}|_{(g, K)} \to \text{Ker} \mathcal{L}
\end{equation}
\begin{equation}
P : (h, p) = (h_1, p_1) + (h_2, p_2) \mapsto \{P(h_2, p_2) + \tilde{\mathcal{L}}[-\mathcal{L}(P(h_2, p_2))] + \tilde{\pi}(h_1, p_1)\}.
\end{equation}
This completes the proof of splitting kernel.
Now we give the proof of Proposition 2.2. First notice that the space of 1-forms on the boundary manifold $\partial M$ can be decomposed as $\wedge_1(\partial M) = \text{Im}d^T \oplus \text{Ker}d^T$, where $d^T$ denotes the exterior derivative operator $d^T : C^{m,\alpha}(\partial M) \to (\wedge_1)^{m-1,\alpha}(\partial M)$ and $\delta^T$ denotes the divergence operator $\delta^T : (\wedge_1)^{m-1,\alpha}(\partial M) \to C^{m-2,\alpha}(\partial M)$ with respect to the induced metric $g^T$. So for the 1-form $(\delta h)^T$ on $\partial M$ induced by a general symmetric 2-tensor $h$ on $M$, there is $v_h \in C^{m,\alpha}(\partial M)$ and $\tau_h \in \text{Ker}\delta^T$ on $\partial M$ such that
\[(2.28) \quad (\delta h)^T = d^Tv_h + \tau_h,
\]
where the 1-forms $d^Tv_h$ and $\tau_h$ are uniquely determined by $h$. Construct a bounded linear map
\[E_1 : \text{Ker}\delta^T \to S_{\delta}^{m,\alpha}(M),\]
so that for any $\tau \in \text{Ker}\delta^T$, $h = E_1(\tau)$ is a symmetric 2-tensor on $M$ and the following conditions hold
\[(2.29) \quad [\delta h]^T = \tau, \quad h^T = 0, \quad H'_h = 0, \quad \mathbf{n}'_h = 0 \quad \text{on } \partial M.
\]
There are various ways to construct such a map. We refer to §4.3 for an appropriate candidate. Now given an element $(h, p) \in T\bar{B}_{(g, K)}$, we can decompose $h$ as
\[(2.30) \quad h = [h - E_1(\tau_h)] + E_1(\tau_h)
\]
where $\tau_h$ is as in (2.28). Notice that for the first part above, we have $[\delta(h - E_1(\tau_h))]^T = [\delta h]^T - \tau_h = d^Tv_h$ on $\partial M$. So $[\delta(h - E_1(\tau_h))]^T \in \text{Im}d^T$ and it is easy to construct a scalar field $v$ on $M$ such that $\delta(h - E_1(\tau)) = 3dv$ along the boundary which is the gauge condition in $W$.

Next construct a bounded linear (0-order in $h$) map $E_2 : S_{\delta}^{m,\alpha}(M) \to S_{\delta+1}^{m-1,\alpha}(M)$ such that for any $h \in S_{\delta}^{m,\alpha}(M)$, $\tilde{h} = E_2(h)$ is a symmetric 2-tensor belonging to $S_{\delta}^{m,\alpha}(M)$ and satisfying the following boundary conditions
\[(2.31) \quad \text{tr}^T\tilde{h} = 0, \quad \tilde{h}(\mathbf{n})^T = -K(\mathbf{n}')_h^T \quad \text{on } \partial M.
\]
Similar as the map $E_1$, we refer to §4.3 for a construction of $E_2$. Now given an element $(h, p) \in T\bar{B}_{(g, K)}$, one can first decompose $h$ as in equation (2.30) and then decompose $p$ as
\[(2.32) \quad p = E_2[h - E_1(\tau_h)] + \{p - E_2[h - E_1(\tau_h)]\}.
\]
Observe that conditions in (2.31) are chosen so that the second component above, $\{p - E_2[h - E_1(\tau_h)]\}$, belongs to the subspace
\[(2.33) \quad S_0 = \{p \in S_{\delta+1}^{m,\alpha}(M) : \text{tr}^Tp = 0, \quad p(\mathbf{n})^T = 0 \quad \text{on } \partial M\}.
\]
We have the following lemma for the space $S_0$.

**Lemma 2.3.** The space $S_0$ defined in (2.33) admits the following splitting:
\[S_0 = \text{Im}Q \oplus \text{Ker}Q^*,
\]
where $Q$ is the differential operator given by
\[Q : (TM)_0 \to S_0
\]
\[Q(Y) = \delta^*Y + (\delta Y)g,
\]
with $(TM)_0 = \{Y \in (TM)^{m,\alpha}_\delta : \text{tr}^T(\delta^*Y + (\delta Y)g) = 0, \delta^*Y(\mathbf{n})^T = 0 \quad \text{on } \partial M\}$.

**Proof.** The formal adjoint of $Q$ is $Q^* = \delta + d\text{tr}$, acting on the space of symmetric 2-tensors $p \in S_0$. For $Y \in (TM)_0$ and $p \in S_0$ the following equality holds
\[
\int_M \langle \delta^*Y + (\delta Y)g, p \rangle = \int_M \langle Y, \delta p + d\text{tr}p \rangle + \int_{\partial M} p(\mathbf{n}, Y) - Y(\mathbf{n})\text{tr}p = \int_M \langle Y, \delta p + d\text{tr}p \rangle,
\]
where the boundary integral vanishes because \( p(n, Y) - Y(n)tr_p = p(n, n)Y(n) - Y(n)tr_p = -Y(n)tr^T_p = 0 \). It follows that \( Q^*Q \) is a self-adjoint elliptic operator. In addition \( \text{Ker}Q^*Q = \text{Ker}Q \) since \((Q^*Q, Y) = (QY, QY)\). Thus for any \( p_0 \in S_0 \)

\[
\int_M \langle Q^*(p_0), Y \rangle = 0 \ \forall Y \in \text{Ker}Q^*Q,
\]

i.e. \( Q^*(p_0) \) is perpendicular to the kernel of \( Q^*Q \). By self-adjointness of \( Q^*Q \), \( \text{Ker}(Q^*Q) = \text{Coker}(Q^*Q) \). Thus \( Q^*(p_0) \in \text{Im}(Q^*Q) \), i.e. there exists a vector field \( Y_0 \in (TM)_0 \) such that \( Q^*p_0 = Q^*QY_0 \). So \( p_0 = QY_0 + w_0 \) with \( w_0 \in \text{Ker}Q^* \). Furthermore, it is easy to check this decomposition is unique because \( \text{Im}Q \cap \text{Ker}Q^* = \{0\} \).

Back to the decomposition (2.32), where the second component belongs to \( S_0 \). The lemma above implies \( p \) can be further decomposed as

\[
(2.34) \quad p = E_2[h - E_1(\tau_h)] + Q(Y_p) + w_p,
\]

with \( Y_p \in (TM)_0 \) and \( w_p \in \text{Ker}Q^* \) both of which are uniquely determined by \((h, p)\). Summing up the decompositions above, we conclude that every element \((h, p) \in TB|_{(g, K)}\) admits the following decomposition:

\[
(h, p) = (h - E_1(\tau_h), E_2[h - E_1(\tau_h)] + Q(Y_p)) + (E_1(\tau_h), w_p).
\]

It follows that

\[
(2.35) \quad TB|_{(g, K)} = S_1 + S_2,
\]

where

\[
S_1 = \{(h, p) \in TB|_{(g, K)} : (\delta h)^T \in \text{Im}d^T \text{ on } \partial M ; \ p = E_2(h) + Q(Y) \ \text{on } M \text{ for some } Y \in (TM)_0 \},
\]

\[
S_2 = \{(h, p) \in TB|_{(g, K)} : h \in \text{Im}E_1 ; \ p \in \text{Ker}Q^* \}.
\]

Then equation (2.25) will be true if the following lemma holds.

**Lemma 2.4.** Equation (2.35) is a splitting of \( TB|_{(g, K)} \), i.e. \( S_1, S_2 \) are closed subspaces and their intersection is trivial.

**Proof.** Observe \( S_1, S_2 \) are well-defined subspaces of \( TB|_{(g, K)} \). It suffices to show the following:

1. The intersection \( S_1 \cap S_2 = \{0\} \). Assume \((h_0, p_0) \in S_1 \cap S_2 \). So \((\delta h_0)^T = d^Tv_0 \) on \( \partial M \) for some scalar field \( v_0 \). On the other hand, there exist \( \tau_0 \in \text{Ker}\delta^T \) such that \( h_0 = E_1(\tau_0) \). It follows that \( \tau_0 = (\delta h_0)^T = d^Tv_0 \) on \( \partial M \). Then \( d^Tv_0 = 0 \), which implies that \( v_0 \) is a constant function and hence \( \tau_0 = 0 \). Thus \( h_0 = E_1(0) = 0 \) and it follows that \( p_0 \in \text{Im}Q \cap \text{Ker}Q^* \) which further implies that \( p_0 = 0 \).

2. The subspace \( S_1 \) is closed. Suppose there is a sequence \((h_i, p_i = E_2(h_i) + Q(Y_i)) \) in \( S_1 \) which converges to \((h_0, p_0) \in TB|_{(g, K)} \). For every \( i \), \((\delta h_i)^T \in \text{Im}d^T \) on the boundary. So \((\delta h_i)^T \) is a closed 1-form on \( \partial M \). It follows that \((\delta h_0)^T \) is also closed and hence exact i.e. \((\delta h_0)^T \in \text{Im}d^T \). Secondly, convergence of \( h_i \) and \( p_i \) implies the sequence \( Q(Y_i) = p_i - E_2(h_i) \) converges to \( p_0 - E_2(h_0) = Q(Y_0) \). So we can conclude the limit \((h_0, p_0) \in S_1 \).

3. The subspace \( S_2 \) is closed. Obviously \( \text{Ker}Q^* \) is closed. In addition the map \( E_1 \) must also has closed range, because for any 1-form \( \tau \in \text{Ker}\delta^T \) on \( \partial M \) we have \( \tau = [\delta(E_1(\tau))]^T \), i.e. the norm of \( \tau \) is controlled by the norm of its image \( E_1(\tau) \). This completes the proof.

\( \blacksquare \)
Next we prove the properties of $S_1$ stated in the second half of Proposition 2.2. Define the following subspace

$$\mathcal{W}' = \{(h, Y, v) \in [S_\delta^{m,\alpha} \times (TM)_\delta^{m,\alpha} \times C_\delta^{m,\alpha}](M) : 
\delta h - 3dv = 0, \; h^T = 0, \; H'_h = 0, \; tr^T(\delta'Y) + 2(\delta Y) = 0, \; \delta'Y(n)^T = 0 \text{ on } \partial M\}.$$ (2.36)

Notice that the only difference between $\mathcal{W}'$ and $\mathcal{W}$ in (2.8) is lower order terms of $h$ in the last boundary equation. As previously, let $\mathcal{V}'$ denote the space of pairs $(h, Y)$ such that $(h, Y, v) \in \mathcal{W}'$ for some function $v$. Then it is easy to observe that

$$\langle \delta h \rangle^T \in \text{Im}^T \; Y \in (TM)_0 \; \forall \; (h, Y) \in \mathcal{V'},$$ (2.37)

and the subspace $S_1$ in (2.35) can be equivalently written as

$$S_1 = \{(h, p) \in T\mathcal{B}|(g, K) : \; h = h_0, \; p = E_2(h_0) + Q(Y_0) \text{ for some } (h_0, Y_0) \in \mathcal{V}'\}.$$ (2.38)

Via this formula, we can construct a new operator

$$\hat{\Psi} : \mathcal{V}' \to \mathcal{T}$$

$$\hat{\Psi}(h, Y) = \mathcal{L}(h, E_2(h) + Q(Y))$$

$$= \mathcal{L}(h, \delta'Y + (\delta Y)g) + \mathcal{L}(0, E_2(h))$$

$$= \Psi(h, Y) + O_1(h),$$

where formula of $\Psi(h, Y)$ is the same as in equation (2.10)-(2.11), and $O_1(h)$ only involves zero and first order derivatives of $h$. Define an “Einstein-type” operator $\mathcal{E}$ on the space $\mathcal{W}'$ similar to $\mathcal{P}$ as in (2.16):

$$\mathcal{E} : \mathcal{W}' \to S_\delta^{m-2,\alpha}(M) \times \mathcal{T}$$

$$\mathcal{E}(h, Y, v) = (\mathcal{E}_0(h, v), \; \hat{\Psi}(h, Y)).$$ (2.40)

Notice that the leading order part of $\mathcal{E}$ is the same as that of $L$ in (2.12) and the domain space $\mathcal{W}'$ consists of exactly kernel elements of the operator $B$ in (2.13) by construction. It follows from the ellipticity of $P = (L, B)$ that $\mathcal{E}$ is Fredholm and hence its range has finite codimension. Let $\pi_2$ be the projection to the second component in (2.40). Obviously the image of $\pi_2 \circ \mathcal{E}$ is equal to the range $\mathcal{L}(S_1)$. Therefore, $\mathcal{L}(S_1)$ also has finite codimension in $\mathcal{T}$, as claimed in Proposition 2.2.

Lastly using the map $\mathcal{E}$ defined above we give the proof of equation (2.26).

**Lemma 2.5.** The subspace $\tilde{S}_1 := S_1 \cap \mathcal{L}^{-1}(0)$ splits in $S_1$, i.e.

$$S_1 = S \oplus \tilde{S}_1$$

for some closed subspace $S \subset S_1$.

**Proof.** The following proof is based on the equivalent expression (2.38) for the space $S_1$. The basic idea is to construct a splitting for $\mathcal{W}'$ and then derive a splitting for $\mathcal{W}'$ which would further yield the splitting for $S_1$. Let $W_1$ be the subspace of $\mathcal{W}'$ which consists of elements $(h, Y, v)$ such that $\tilde{\Psi}(h, Y) = 0$, i.e. $W_1 = \mathcal{E}^{-1}(\ast, 0)$. Similarly, define $W_2 = \mathcal{E}^{-1}(0, \ast)$ as the space consisting of $(h, Y, v)$ such that $\mathcal{E}_0(h, v) = 0$. Then $W_1$ and $W_2$ are closed subspaces of $\mathcal{W}'$. Moreover, $(W_1 + W_2)$ must be of finite codimension in $\mathcal{W}'$. In fact, we can construct a map

$$\mathcal{F} : \mathcal{W}'/(W_1 + W_2) \to [S_\delta^{m-2,\alpha}(M) \times \mathcal{T}]/\text{Im}\mathcal{E}$$

$$\mathcal{F}([v, h, Y]) = [\mathcal{E}_0(h, v), 0]$$

where $[v, h, Y]$ denotes an equivalence class in $\mathcal{W}'/(W_1 + W_2)$ and $[\mathcal{E}_0(h, v), 0]$ an equivalence class in $\mathcal{B}/\text{Im}\mathcal{E}$. It is easy to verify $\mathcal{F}$ is well-defined and injective. Since the range of the Fredholm
map $\mathcal{E}$ has finite codimension, $(W_1 + W_2)$ must also be of finite codimension in $W'$. Let $W_3$ be a complementary subspace, i.e.

$$W' = (W_1 + W_2) \oplus W_3.$$ 

Notice that $W_1 \cap W_2 = \mathcal{E}^{-1}(0,0)$ is of finite dimension and thus it splits in $W_2$, i.e. $W_2 = (W_1 \cap W_2) \oplus \hat{W}_2$ which further implies that

$$W' = W_1 \oplus \hat{W}_2 \oplus W_3. \tag{2.41}$$

Now consider the previously defined subspace $V'$. We will show that $V_1 = \hat{\Psi}^{-1}(0)$ splits in $V'$. Let $\pi$ be the projection $\pi : W' \to V'$, $\pi([h,Y,v]) = (h,Y)$. Obviously, $V_1 = \pi(W_1)$ and

$$V' = V_1 + \pi(\hat{W}_2) + \pi(W_3). \tag{2.42}$$

Let $V_2 = \pi(\hat{W}_2)$. It follows from the definition of $\hat{W}_2$ that $V_1 \cap V_2 = \{0\}$. Moreover, $V_2$ is also closed. In fact, given a Cauchy sequence $(h_i, Y_i)$ in $V_2$, there is a sequence $\{v_i\}$ such that $(v_i, h_i, Y_i) \in \hat{W}_2$. The sequence of their images $\mathcal{E}(v_i, h_i, Y_i) = (0, \hat{\Psi}(h_i, Y_i))$ must also converge since $\hat{\Psi}$ is a bounded operator. Observe $\mathcal{E}|_{\hat{W}_2} : \hat{W}_2 \to \{(0,*)\} \cap \text{Im} \mathcal{E}$ is a bijective and bounded linear operator. It follows that $(v_i, h_i, Y_i)$ must converge to $(\hat{v}_0, \hat{h}_0, \hat{Y}_0)$ in $\hat{W}_2$. Thus $(h_i, Y_i)$ converges to $(\hat{h}_0, \hat{Y}_0)$ in $V_2$. Thus equation (2.42) can be rewritten as $V' = (V_1 + V_2) + \pi(W_3)$. In this decomposition $V_1 + V_2$ must be of finite codimension, since $V_3$ has finite dimension. Thus there exist a closed subspace $V_3$ so that

$$V' = V_1 \oplus V_3. \tag{2.43}$$

Finally, using the splitting (2.43) we can finish the proof of the lemma. Define the map

$$T : V' \to S_1 \quad \text{where} \quad T([h,Y]) = (h, E_2(h) + Q(Y)).$$

Obviously $T$ is linear bounded and surjective with $\text{Ker} T = \{(0,Y) \in V' : Q(Y) = 0\}$. Since $\hat{\Psi}(h,Y) = \mathcal{L}(T(h,Y))$, we have $T(V_1) = \mathcal{L}^{-1}(0) \cap S_1 = \tilde{S}_1$. Thus

$$S_1 = \tilde{S}_1 + T(V_3).$$

According to (2.43) we see that $\tilde{S}_1 \cap T(V_3) = \{0\}$. So (2.26) will hold if $T(V_3)$ is closed. Suppose $(h_i, E(h_i) + Q(Y_i))$ is a Cauchy sequence in $T(V_3)$. Then $\hat{\Psi}(h_i, Y_i) = \mathcal{L}(h_i, E(h_i) + Q(Y_i))$ must converge in $\mathcal{L}(S_1)$ since $\mathcal{L}$ is bounded. Then $(h_i, Y_i)$ must converge to some $(\hat{h}_0, \hat{Y}_0)$ in $V_3$ because $\hat{\Psi}|_{V_3} : V_3 \to \text{Im} \hat{\Psi} = \mathcal{L}(S_1)$ is a bounded linear bijective map. Therefore $(h_i, E(h_i) + Q(Y_i))$ converges to $(\hat{h}_0, E(\hat{h}_0) + Q(\hat{Y}_0))$ in $T(V_3)$. This completes the proof.

Summarizing all the previous results, we conclude that the level set $\Phi^{-1}(u_0, Z_0)$ admits Banach manifold structure.

**Theorem 2.6.** Given fixed Bartnik data $(\gamma, l, k, \tau)$ on $\partial M$ and $(u, Z)$ in $\mathcal{T}$, the space $\mathcal{C}_B(u, Z)$ of initial data sets satisfying the constraint equations with fixed boundary data

$$\mathcal{C}_B(u, Z) = \{(g,K) \in [\text{Met}_\delta^{n,\alpha} \times S_\delta^{m-1,\alpha}](M) : \Phi(g, K) = (u, Z) \text{ on } M \}$$

is an infinite dimensional smooth Banach manifold.

**Proof.** It is proved above that the linearization $\mathcal{L} = D\Phi|_{(g,K)}$ at any $(g, K) \in \Phi^{-1}(u, Z)$ is surjective and has splitting kernel. The theorem is a natural consequence of the implicit function theorem in Banach spaces. ■
2.3. Ellipticity of the “Einstein-type” operator. In the last part of this section, we prove in detail that the operator \( P \) constructed as (2.12)-(2.13) in the proof of surjectivity is elliptic. First observe that in (2.12)-(2.13) the vector field \( Y \) is not coupled with \((h,v)\). So we can split \( P \) as an operator \( P_Y = (L_Y, B_Y) \) on \( Y \)

\[
L_Y : (TM)^{m,\alpha}_\delta(M) \to (\wedge_1)^{m-2,\alpha}_\delta(M)
\]

\[
L_Y(Y) =\delta \partial Y + dv
\]

\[
B_Y : (TM)^{m,\alpha}_\delta(M) \to [C^{m-1,\alpha} \times (\wedge_1)^{m-1,\alpha}](\partial M)
\]

\[
B_Y(Y) = (\text{tr}^T[\delta Y + (\delta Y)g], \delta Y(n)^T).
\]

and an operator \( P_h = (L_h, B_h) \) on \((h,v)\)

\[
L_h : [S^{m,\alpha}_\delta \times C^{m,\alpha}_\delta](M) \to [S^{m-2,\alpha}_\delta \times C^{m-2,\alpha}_\delta](M)
\]

\[
L_h(h, v) = (\xi_0(h, v), \Delta \text{tr}h + \delta \delta h),
\]

\[
B_h : [S^{m,\alpha}_\delta \times C^{m,\alpha}_\delta](M) \to [(\wedge_1)^{m-1,\alpha} \times S^{m,\alpha} \times C^{m,\alpha}](\partial M)
\]

\[
B_h(h, v) = (\delta h - 3dv, h^T, H'_h).
\]

It is easy to verify the ellipticity of (2.44) by applying the criterion given in [1]. Here we give the details. Since

\[
2(\delta \delta^* Y + dv) = -\partial_1 \partial_1 Y_j - \partial_1 \partial_2 Y_i - 2\partial_2 \partial_3 Y_i + O_1 = -\partial_1 \partial_1 Y_j - 3\partial_1 \partial_2 Y_i + 2\text{Ric}(Y) + O_1
\]

the interior principal symbol of \( P_Y \) is given by

\[
L_Y(\xi) = \frac{1}{2} \begin{bmatrix}
|\xi|^2 + 3\xi_1\xi_1 & 3\xi_1\xi_2 & 3\xi_1\xi_3 \\
3\xi_2\xi_1 & |\xi|^2 + 3\xi_2\xi_2 & 3\xi_2\xi_3 \\
3\xi_3\xi_1 & 3\xi_3\xi_2 & |\xi|^2 + 3\xi_3\xi_3
\end{bmatrix}
\]

Elementary calculation shows its determinant is \( l_Y = \frac{1}{4}|\xi|^6 \), and the adjoint matrix is given by

\[
L_Y^*(\xi) = \frac{1}{4}|\xi|^2 \begin{bmatrix}
|\xi|^2 + 3(\xi_2^2 + \xi_3^2) & -3\xi_1\xi_2 & -3\xi_1\xi_3 \\
-3\xi_2\xi_1 & |\xi|^2 + 3(\xi_1^2 + \xi_3^2) & -3\xi_2\xi_3 \\
-3\xi_3\xi_1 & -3\xi_3\xi_2 & |\xi|^2 + 3(\xi_1^2 + \xi_2^2)
\end{bmatrix}
\]

Since

\[
\text{tr}^T[\delta Y + (\delta Y)g] = \partial_2 Y_2 + \partial_3 Y_3 + 2(-\partial_1 Y_1 - \partial_2 Y_2 - \partial_3 Y_3) + O_0 = -2\partial_1 Y_1 - \partial_2 Y_2 - \partial_3 Y_3 + O_0
\]

\[
2\delta^* Y(n)^T = \partial_1 Y_A + \partial_3 Y_A + O_0 \quad A = 2, 3
\]

the boundary symbol is given by

\[
B_Y(\xi) = \begin{bmatrix}
-2\xi_1 & -\xi_2 & -\xi_3 \\
\xi_2 & \xi_1 & 0 \\
\xi_3 & 0 & \xi_1
\end{bmatrix}
\]

Thus we have

\[
B_Y(\xi)L_Y^*(\xi) = \frac{i}{4}|\xi|^2 \begin{bmatrix}
-2|\xi|^2\xi_1 - 3\xi_1(\xi_2^2 + \xi_3^2) & -|\xi|^2\xi_2 + 3\xi_1^2\xi_2 & -|\xi|^2\xi_3 + 3\xi_1^2\xi_3 \\
2\xi_2(2\xi_2^2 + 2\xi_3^2 - \xi_1^2) & 2\xi_1(-\xi_2^2 + 2\xi_1^2 + 2\xi_3^2) & -6\xi_1\xi_2\xi_3 \\
2\xi_3(2\xi_2^2 + 2\xi_3^2 - \xi_1^2) & -6\xi_1\xi_2\xi_3 & 2\xi_1(-\xi_2^2 + 2\xi_1^2 + 2\xi_3^2)
\end{bmatrix}
\]

Let \( \mu \) denote a 1-form normal to the boundary \( \partial M \), i.e. \( \mu^T = 0 \) on \( \partial M \) and \( \eta \) a nonzero 1-form tangential to the boundary, i.e. \( \eta(n) = 0 \) on \( \partial M \). Based on [1], the operator \( P_Y = (L_Y, B_Y) \) will be elliptic if there is no nonzero complex vector \( C \) such that \( C \cdot B_Y L_Y^*(z \mu + \eta) = 0 \mod(z - i|\eta|)^3 \), where \( z = i|\eta| \) is the root (of multiplicity 3) with positive imaginary part for \( l(z \mu + \eta) = 0 \). Denote
the matrix on the right side of the expression above as \( \hat{B}(\xi) = \frac{1}{|\xi|} B_{Y} L_{Y}^{*}(\xi) \). Then it suffices to verify that there is no nontrivial solution for \( C \cdot \hat{B}(z\mu + \eta) = 0 \mod(z - i|\eta|)^2 \).

It is easy to verify that \( \det \hat{B}(z\mu + \eta) = 0 \mod(z - i|\eta|) \) has no nontrivial solution, where \( \hat{B}'(z\mu + \eta) \) denotes the derivative of \( \hat{B}(z\mu + \eta) \) with respect to \( z \). This is equivalent to \( \det \hat{B}'(z\mu + \eta)|_{z = i|\eta|} = 0 \). Let \( \xi_1 = z \) and \( \xi_2 = \eta \), \( \xi_3 = \eta \) in \( \hat{B} \) with \( (\eta_1, \eta_2) \neq 0 \):

\[
\hat{B}(z\mu + \eta) = \begin{bmatrix}
-2z^2 - 5z|\eta|^2 & -(z^2 + |\eta|^2)\eta_2 + 3z^2\eta_2 & -(z^2 + |\eta|^2)\eta_3 + 3z^2\eta_3 \\
2\eta_2(2|\eta|^2 - z^2) & 2z(-\eta^2_2 + 2z^2 + 2\eta^2_2) & -6z\eta_2\eta_3 \\
2\eta_3(2|\eta|^2 - z^2) & -6z\eta_2\eta_3 & 2z(-\eta^2_3 + 2z^2 + 2\eta^2_3)
\end{bmatrix}.
\]

So its derivative is given by

\[
\hat{B}'(z\mu + \eta) = \begin{bmatrix}
-6z^2 - 5|\eta|^2 & -4z\eta_2 & 4z\eta_3 \\
-4z\eta_2 & -2\eta^2_2 + 12z^2 + 4\eta^2_3 & 4z\eta_3 \\
-4z\eta_3 & -6\eta_2\eta_3 & -2\eta^2_3 + 12z^2 + 4\eta^2_3
\end{bmatrix}.
\]

Plug in \( z = i|\eta| \), \( z^2 = -|\eta|^2 \) to obtain \( \det \hat{B}'(i|\eta|\mu + \eta) = 240|\eta|^4 \). Obviously it is never zero if \( \eta \neq 0 \). Thus \( (2.44) \) is an elliptic operator.

Next we prove ellipticity for the operator \( P_{h} \) in \( (2.45) \). Recall that this operator is constructed by combining the linearized Einstein tensor with gauge terms. Now it has been shown in [2] that the stationary Einstein field equations are elliptic with respect to certain boundary conditions. In particular it is shown that the operator

\[
L_{0} : [S^{m,\alpha}_{\delta} \times C^{m,\alpha}_{\delta}](M) \to [S^{m-2,\alpha}_{\delta+2} \times C^{m-2,\alpha}_{\delta+2}](M)
\]

\[
L_{0}(h, v) = (\text{Ric}_h + \delta^* \beta_g h, \Delta_4 v)
\]

\[
B_{0} : [S^{m,\alpha}_{\delta} \times C^{m,\alpha}_{\delta}](M) \to [(\Lambda)^{m-1,\alpha} \times S^{m,\alpha} \times C^{m,\alpha}](\partial M)
\]

\[
B_{0}(h, v) = (\beta_g h, h^T - 2vg^T, H_{h}^T - 2n(v)).
\]

is elliptic. Note that the operator above is obtained from a conformal transformation of a boundary value problem of Einstein field equations in the projection formalism of stationary spacetimes. So we first apply the same conformal transformation

\[
h = \bar{h} - 2\bar{v}g, \quad v = \bar{v}
\]

to \( (2.45) \) and obtain an equivalent operator

\[
L_{h} : [S^{m,\alpha}_{\delta} \times C^{m,\alpha}_{\delta}](M) \to [S^{m-2,\alpha}_{\delta+2} \times C^{m-2,\alpha}_{\delta+2}](M)
\]

\[
L_{h}(\bar{h}, \bar{v}) = (\text{Ein}'_{\bar{h}} + \delta^* \delta h - (\delta \bar{h})g + (\Delta \bar{v})g - D^2 \bar{v} + O_1, \Delta tr \bar{h} + \delta \bar{h} - 4\Delta \bar{v}),
\]

\[
B_{h} : [S^{m,\alpha}_{\delta} \times C^{m,\alpha}_{\delta}](M) \to [(\Lambda)^{m-1,\alpha} \times S^{m,\alpha} \times C^{m,\alpha}](\partial M)
\]

\[
B_{h}(\bar{h}, \bar{v}) = (\delta \bar{h} - d\bar{v}, \bar{h}^T - 2\bar{v}g^T, H_{\bar{h}}^T - 2n(\bar{v}) + \bar{v}H_{\bar{g}}).
\]

Take trace of the first term in \( L_{h}(\bar{h}, \bar{v}) \), multiply it by 2, add it to the second term of \( L_{h}(\bar{h}, \bar{v}) \). We obtain the following operator \( \bar{P} = (L_{h}^T, B_{h}^T) \) which behaves the same as the one above regarding to ellipticity:

\[
L_{h}^T : [S^{m,\alpha}_{\delta} \times C^{m,\alpha}_{\delta}](M) \to [S^{m-2,\alpha}_{\delta+2} \times C^{m-2,\alpha}_{\delta+2}](M)
\]

\[
L_{h}^T(\bar{h}, \bar{v}) = (\text{Ein}'_{\bar{h}} + \delta^* \delta h - (\delta \bar{h})g + (\Delta \bar{v})g - D^2 \bar{v}, 4\Delta \bar{v} - 8\delta \bar{h}).
\]

\[
B_{h}^T : [S^{m,\alpha}_{\delta} \times C^{m,\alpha}_{\delta}](M) \to [(\Lambda)^{m-1,\alpha} \times S^{m,\alpha} \times C^{m,\alpha}](\partial M)
\]

\[
B_{h}^T(\bar{h}, \bar{v}) = (\delta \bar{h} - d\bar{v}, \bar{h}^T - 2\bar{v}g^T, H_{\bar{h}}^T - 2n(\bar{v})).
\]
Here we throw away the terms involving only lower ($\leq 1$) derivatives of $(h,v)$. We use $\text{Ein}_h^I$ to denote leading part of $\text{Ein}_h^I$, i.e. $\text{Ein}_h^I = \frac{1}{2}D^*Dh - \delta^*\beta h - \frac{1}{2}(\Delta tr h + \delta dh)g$. The formal adjoint of $\bar{P}$ is given by $\bar{P}^* = (\bar{L}, \bar{B})$

$$\bar{L} : [S_{\delta}^{m,\alpha} \times C_{\delta}^{m,\alpha}](M) \rightarrow [S_{\delta+2}^{m-2,\alpha} \times C_{\delta+2}^{m-2,\alpha}](M)$$

$$\bar{L}(h,v) = \langle \text{Ein}_h^I + \delta^*\delta h - D^2tr h - 8D^2v, 4\Delta_g v - \delta dh + \Delta tr h \rangle,$$

$$\bar{B} : [S_{\delta}^{m,\alpha} \times C_{\delta}^{m,\alpha}](M) \rightarrow [(\land_1)^m \times C_{\delta}^{m,\alpha} \times C_{D}(\partial M)$$

$$\bar{B}(h,v) = (\delta h - dt h - 8dv, h^T + 2v^T, h_h + 2\mathbf{n}(v)).$$

It is straightforward to prove the adjointness via integration by parts. To do this, we proceed as follows. Consider the functional

$$I(g) = \int_M R_g + 2\int_{\partial M} H_g - 16\pi m_{ADM}(g).$$

The first variation of $I$ is given by (cf. for example [5])

$$I_g^I(h) = \int_M -\langle \text{Ein}_g, h \rangle + \int_{\partial M} \langle H_g g^T - A, h \rangle.$$

Take second variation of $I$

$$I_g''(h, \bar{h}) = \int_M -\langle \text{Ein}_g^I, h \rangle + \langle \text{Ein}_g, h \circ \bar{h} \rangle - \frac{1}{2}\text{tr}\bar{h}\langle \text{Ein}_g, h \rangle$$

$$\quad + \int_{\partial M} \langle H_g^T h + H_g h^T - A^I_h, h \rangle - \langle H_g g^T - A, h^T \circ \bar{h}^T \rangle + \frac{1}{2}\text{tr}\bar{h}\langle H_g g^T - A, h \rangle.$$

By symmetry of the second variation we obtain

$$\int_M -\langle \text{Ein}_g^I, h \rangle - \frac{1}{2}\text{tr}\bar{h}\langle \text{Ein}_g, h \rangle + \int_{\partial M} \langle H_g^T h - A^I_h, h \rangle - \frac{1}{2}\text{tr}\bar{h}\langle A, h \rangle$$

$$= \int_M -\langle \text{Ein}_g^I, \bar{h} \rangle - \frac{1}{2}\text{tr}\bar{h}\langle \text{Ein}_g, \bar{h} \rangle + \int_{\partial M} \langle H_g^T h - A^I_h, \bar{h} \rangle - \frac{1}{2}\text{tr}\bar{h}\langle A, \bar{h} \rangle.$$

If $B^I_h(h, \bar{v}) = 0$ then $\bar{h}^T = 2v^T$, $\text{tr}\bar{h} = 4v$, $H_h^I = 2\mathbf{n}(\bar{v})$ on $\partial M$. Similarly, if $\bar{B}(h, v) = 0$ then $h^T = -2v^T$, $\text{tr}h = -4v$, $H_h^I = -2\mathbf{n}(v)$ on $\partial M$. Plug these into the equation above and obtain that for all $(\bar{h}, \bar{v}), (h, v)$ such that $B^I_h(\bar{h}, \bar{v}) = 0, B(h, v) = 0$

$$\int_M \langle \text{Ein}_g^I, h \rangle = \int_M \langle \text{Ein}_g^I, \bar{h} \rangle + \int_{\partial M} [4\bar{v}\mathbf{n}(v) - v\mathbf{n}(\bar{v})].$$

Simple calculation of integration by parts on the remaining terms in $L^I_h$ and $\bar{L}$ yields

$$\int_M \langle \delta^*\delta h - (\delta dh)g + (\Delta v)g - D^2\bar{v}, h \rangle + \langle 4\Delta \bar{v} - 8\delta \bar{h}, v \rangle$$

$$= \int_M \langle \delta^*\delta h - D^2tr h - 8D^2v, h \rangle + \langle 4\Delta v - \delta dh + \Delta(tr h), \bar{v} \rangle + \int_{\partial M} B[(\bar{h}, \bar{v}), (h, v)]$$

where in the boundary integral $B$ is a bilinear form given by

$$B[(\bar{h}, \bar{v}), (h, v)] = h(\delta h, \mathbf{n}) - h(\delta h, \mathbf{n}) + \bar{h}(\mathbf{n}, d\text{tr}h) + (\text{tr}h)\delta h(\mathbf{n}) - (\text{tr}h)\mathbf{n}(\bar{v}) + \bar{v}\mathbf{n}(\text{tr}h)$$

$$- h(\mathbf{n}, d\bar{v}) - \bar{v}\delta h(\mathbf{n}) - 4\bar{v}\mathbf{n}(\bar{v}) + 4\mathbf{n}(v) + 8\mathbf{h}(\mathbf{n}, dv) + 8\delta \bar{h}(\mathbf{n}).$$

If $B^I_h(\bar{h}, \bar{v}) = 0$ and $B(h, v) = 0$, then $\delta h = d\bar{v}$ and $\delta h = d\text{tr}h + 8dv$ on $\partial M$. Plugging these equalities into the expression above we obtain $B[(\bar{h}, \bar{v}), (h, v)] = 4[v\mathbf{n}(v) - \bar{v}\mathbf{n}(\bar{v})]$. Combining this with (2.49)
we obtain that for all \((\bar{h}, \bar{v}), (h, v) \in [S^{m,\alpha}_\delta \times C^{m,\alpha}_\delta](M)\) such that \(B'_h(\bar{h}, \bar{v}) = 0\), \(\bar{B}(h, v) = 0\)

\[
\int_M \langle L_{\bar{h}}(\bar{h}, \bar{v}), (h, v) \rangle = \int_M \langle (\bar{h}, \bar{v}), L(h, v) \rangle,
\]

which justifies the adjointness between \(\bar{P}\) and \(\bar{P}^*\).

Now to prove that (2.45) is elliptic it suffices to prove that both the operator \(\bar{P}\) and its adjoint operator \(\bar{P}^*\) admit a uniform elliptic estimate (c.f. [1,21]). In the following we apply the idea in [5] to prove the elliptic estimate for \(\bar{P}^*\). The same proof works as well for \(\bar{P}\).

We observe if the boundary operator of (2.46) is replaced by the one in (2.48), ellipticity still holds. In fact since the principal symbol of \(L_\nu\) is simply a rescaling of the identity matrix, ellipticity of \((L_\nu, \bar{B})\) can be immediately verified by checking the symbol of \(\bar{B}(\xi)\) is a non-degenerate matrix when \(\xi = i\eta/\mu + \eta\) for \(\eta \neq 0\). Thus we have the following elliptic estimate

\[
||\langle (h, v) ||_{C^{m,\alpha}} \leq C(||L_\nu(h, v)||_{C^{m-2,\alpha}} + ||\bar{B}_i(h, v)||_{C^{m-k_i,\alpha}} + ||(h, v)||_{C^0}),
\]

where for each boundary term \(\bar{B}_i(h, v)\) in \(\bar{B}(h, v)\) the order \(k_i\) equals to highest order of derivatives involved in it. The interior operator \(\bar{L}\) and \(L_\nu\) differ by

\[
\bar{L}(h, v) - L_\nu(h, v) = ( - \frac{1}{2}(\Delta tr h + \delta\delta h)g - \frac{3}{2}D^2 tr h - 9D^2 v, 3\Delta v - \delta\delta h + \Delta(tr h)).
\]

So elliptic estimate for \(\bar{P}^*\) will hold if we can control \(\delta\delta h\), \(tr h\) and \(v\) by \(\bar{P}^*(h, v)\), i.e.

\[
||\delta\delta h||_{C^{m-2,\alpha}} \leq C(||\bar{L}(h, v)||_{C^{m-2,\alpha}} + ||\bar{B}_i(h, v)||_{C^{m-k_i,\alpha}} + ||(h, v)||_{C^0})
\]

and the same for \(tr h\) and \(v\). Taking the divergence of the first term of \(\bar{L}(h, v)\) we get:

\[
\delta(\bar{L}(h, v)) = \delta^* (\delta h - dtr h - 8dv)
\]

inside which we use the Bianchi identity \(\delta Ein = 0\). Note the expression above can be taken as \(\delta\delta^*\) – an elliptic operator – acting on the term \((\delta h - dtr h - 8dv)\) whose Dirichlet boundary data is included in \(\bar{B}(h, v)\). Thus \((\delta h - dtr h - 8dv)\) is controlled by \(\bar{P}^*(h, v)\) as well as \(Ein'_h\). So we get control of \((\delta\delta h - \Delta tr h - 8\Delta v)\). Compare this with the second component of \(\bar{L}(h, v)\), we see that \(\Delta v\) and \(\delta\delta h - \Delta tr h\) are controlled. In addition \(tr \bar{L}(h, v) = \frac{1}{2}\Delta(tr h) - \frac{3}{2}\delta\delta h + 8\Delta v\). So we obtain control (2.50) of \(\delta\delta h\) and a similar control for \(\Delta tr h\) and \(\Delta v\).

The Gauss equation at \(\partial M\) is given by \(|A|^2 - H^2 + R_g = R_g - 2Ric_g(n, n) = -2Ein_g(n, n)\). Its linearization is

\[
(|A|^2 - H^2 + R_g)n'_h = -2Ein'_h (n, n) - 4Ein_g(n'_h, n)
\]

where \(A'_h, H'_h\) and \(n'_h\) only involve 1st and 0th order behavior of \(h\), which can be ignored according to the interpolation inequality. So we obtain

\[
\Delta_{g,T} tr^{T} h^T + \delta^T \delta^T h^T = (R_g)_{h,T} + O_1 = -2Ein'_h (n, n) + O_1.
\]

The second boundary term in \(\bar{B}\) is \(\bar{B}_2(h, v) = h^T + 2vg^T\), so \(h^T = B_2 - 2vg^T\). Plug this into the equation above

\[
-2\Delta_{g,T}v = 2Ein'_h (n, n) - \Delta_{g,T} tr^{T} B_2 - \delta^T \delta^T B_2 + O_1.
\]

Recall that \(Ein'_h\) is already controlled by \(\bar{P}^*\). Thus \(||v||_{\partial M}||_{C^{m,\alpha}}\) is also controlled by the operator \(\bar{P}^*\) and so is the Dirichlet data \(h^T\). Since we already have control of \(\Delta v\) on \(M\), \(||v||_{C^{m,\alpha}}\) is controlled.

It remains to control \(tr h\). By the formula of variation of mean curvature we have

\[
n(tr h) = 2H'_h - \delta^T (h(n)^T) - (\delta h)(n) + O.
\]

In the equation above, \(\delta h(n)\) is not controlled, but we have control of the boundary data \((\delta h - dtr h - 8dv)\) with \(dv\) is already controlled. So we can rewrite the equation above as:

\[
2n(tr h) = 2H'_h - \delta^T (h(n)^T) - [(\delta h - d(tr h))(n)] + O.
\]
In addition, basic computation yields $-(\delta h - d tr h)^T = \nabla_n h(n)^T + \delta^T(h^T) + \nabla^T tr h + O$ inside which $\delta h - d tr h$ and $h^T$ are both controlled on $\partial M$. So one gets control of $\nabla_n h(n)^T + \nabla^T tr h$ on $\partial M$ and hence its tangential divergence

$$\delta^T[\nabla_n h(n)^T + \nabla^T tr h] = \nabla_n[\delta^T(h(n)^T)] + \Delta_{g^T tr h}$$

(2.52)

is also controlled. Combining (2.51) and (2.52), one obtains:

$$2nn(tr h) - \Delta_{g^T tr h} = 2n(H'_h) - [\delta^T(\nabla_n h(n)^T + \nabla^T tr h)] - n[(\delta h - d(tr h))(n)] + O_1.$$ 

Note $n(H'_h)$ above is controlled based on Riccati equation $n(H) + |A|^2 = -Ric(n, n)$, where $Ric'_h(n, n) = Ein'_h(n, n) + 1/2R'_h g + O$ and $R'_h$ is well-controlled because both $\Delta(tr h)$ and $\delta h$ are. So every term on the righthand side of the equation above is under control. Finally recall that $\Delta tr h$ is controlled by $P^*$ and it is elliptic when combined with the boundary term $2nn(tr h) - \Delta_{g^T tr h}$. Therefore, $tr h$ is also controlled by $P^*$. This completes the proof of elliptic estimate for $\hat{P}$, i.e.

$$||((h, v)||_{C^{m, \alpha}} \leq C(||L(h, v)||_{C^{m-2, \alpha}} + ||\tilde{B}(h, v)||_{C^{m-k, \alpha}} + ||(h, v)||_{C^0}).$$

It is easy to carry out the same process as above and derive the uniform elliptic estimate for $\hat{P}$.

We finish this section with the following remark.

**Remark 2.7.** Different from the work of Bartnik \[7,8\] where the functions and tensor fields belong to the weighted Sobolev spaces, we work with the weighted Hölder spaces in this paper. The main reason is that when taking trace of a function one loses an extra $\frac{1}{2}$ regularity $H^s(M) \rightarrow H^{s-1/2}(\partial M)$ which makes it complicated to discuss the ellipticity of the Bartnik boundary data.

### 3. Critical points of the ADM mass

In this section, we adopt the definitions of the ADM mass and the Regge-Teitelboim Hamiltonian from [8] and prove the corresponding result on the critical points for the ADM mass on the constraint manifold of initial data sets with fixed Bartnik boundary data.

We use the same notation as in [8]. A tensor field $\xi = (\xi^0, \xi^i)$ consisting of a scalar field $\xi^0$ and a vector field $\xi^i$ on $M$ is called a **spacetime vector field**. Let $T^{m, \alpha}_{\delta}(M)$ denote the asymptotically zero spacetime tangent bundle, i.e. $T^{m, \alpha}_{\delta}(M) = [C^{m, \alpha} \times T^{m, \alpha}_{\delta}](M)$. Fix a constant 4-vector $\xi_\infty = (\xi_\infty^0, \xi_\infty^i)$ ($i = 1, 2, 3$) defined on $\mathbb{R}^3 \setminus B^3$. Pull it back to $M$ and obtain a parallel spacetime vector field, still denoted as $\xi_\infty$, with respect to the metric $\hat{g}$. A smooth spacetime vector field $\hat{\xi}_\infty = (\xi_\infty^0, \xi_\infty^i)$ on $M$ is called a **constant translation near infinity** representing $\xi_\infty$, if there is a $R$ such that $\hat{\xi}_\infty = \xi_\infty$ on $E_{2R}$, and $\xi_\infty = 0$ on $M \setminus E_R$, where $E_R = \{p \in M : \hat{r}(p) > R\}$. Let $Z^{m, \alpha}_{\delta}(M)$ denote the space of **asymptotic translation vector fields**, i.e.

$$Z^{m, \alpha}_{\delta}(M) = \{\xi \in [C^{m, \alpha} \times T^{m, \alpha}_{\delta}](M) : \xi - \xi_\infty \in T^{m, \alpha}_{\delta}(M) \text{ for some constant translation near infinity } \hat{\xi}_\infty\}.$$ 

For convenience, we turn to the $(g, \pi, \Phi)$ formulation of the constraint map as described in §2. Denote $\hat{C}_B(u, z)$ as the level set

$$\hat{C}_B(u, z) = \{(g, \pi) \in \hat{B} : \Phi(\pi, g) = (u, z)\}$$

where $\hat{B}$ and $\hat{\Phi}$ are defined as in (2.6) and (2.7). Since the space $\hat{C}_B(u, z)$ is equivalent to $C_B(u, z)$, it is also a smooth Banach manifold.
The general ADM (total) energy-momentum vector $\mathbb{P}$ is defined in [8] by describing its pairing with a constant vector $\xi_\infty \in \mathbb{R}^{1,3}$

$$16\pi \xi_\infty^0 \mathbb{P}^0_0(g, \pi) = \int_M \tilde{\mathbb{P}}^0_0 R_0(g) + \tilde{\nabla}^i \tilde{\mathbb{P}}^0_0 (\tilde{\nabla}^j g_{ij} - \tilde{\nabla}^i tr_g g) d \text{vol}_g$$

$$16\pi \xi_\infty^i \mathbb{P}^i(g, \pi) = 2 \int_M (\tilde{\mathbb{P}}^i_0 \rho_0(\pi) + \pi^{ij} \tilde{\nabla}^i \tilde{\xi}_\infty^j) d \text{vol}_g$$

inside which $\tilde{\xi}_\infty$ is a representative translation vector at infinity for $\xi_\infty$ and

$$R_0(g) = \tilde{\nabla}^i g_{ij} - \Delta_0 tr_g g, \quad \rho_0(\pi) = \tilde{g} \tilde{\nabla}_k \pi^{jk}.$$

It is easy to generalize the result in [8] to obtain that $\mathbb{P}$ defines a smooth function on the Banach manifold $\tilde{\mathcal{C}}_B(u, Z)$ when $(u, Z) \in [C^k_\alpha \times (\Lambda_1^k)_\alpha](M)$ for some $\alpha \geq 4$ and $k \geq 0$. Moreover, in this case $\mathbb{P}$ agrees with the usual formal definition of ADM energy-momentum vector and it is independent of choice of the chart $M \cong \mathbb{R}^3 \setminus B$.

We adopt the Regge-Teitelboim Hamiltonian defined in [8] to our setting:

$$\mathcal{H} : \tilde{\mathcal{B}} \times \mathcal{Z}^{m,\alpha}_\delta(M) \rightarrow \mathbb{R}$$

$$\mathcal{H}(g, \pi; \xi) = \int_M \langle (\hat{\xi}_\infty - \xi), \tilde{\Phi}(g, \pi) \rangle + \int_M \tilde{\mathbb{P}}^0_0 (R_0(g) - \Phi_0(g, \pi)) + \int_M \tilde{\nabla}^i \tilde{\mathbb{P}}^0_0 (\tilde{\nabla}^j g_{ij} - \tilde{\nabla}^i tr_g g)$$

$$+ \int_M \tilde{\mathbb{P}}^i_0 \rho_0(\pi) + \tilde{\Phi}_i(g, \pi) + \int_M 2\pi^{ij} \tilde{\nabla}^i \tilde{\xi}_\infty^j,$$

inside which $\hat{\xi}_\infty$ is a constant translation at infinity such that $\xi - \hat{\xi}_\infty \in \mathcal{Z}^{m,\alpha}_\delta(M)$. Here and in the following we omit the volume form $d \text{vol}_g$. Based on [8], the functional $\mathcal{H}$ is smooth and bounded. In particular, on the constraint manifold $\tilde{\mathcal{C}}_B(u, Z)$ with $(u, Z) \in [C^k_\alpha \times (\Lambda_1^k)_\alpha](M)$ for some $\alpha \geq 4, k \geq 0$ the functional can be equivalently expressed as

$$\mathcal{H}(g, \pi; \xi) = 16\pi \xi_\infty^0 \mathbb{P}^0_0 - \int_M \xi^\alpha \tilde{\Phi}_\alpha(g, \pi),$$

where $\xi_\infty$ is the constant vector equal to the asymptotic limit of $\xi$. The following lemma describes the variation of $\mathcal{H}$.

**Lemma 3.1.** If $\xi \in \mathcal{Z}^{m,\alpha}_\delta(M)$ then for all $(g, \pi) \in \tilde{\mathcal{B}}$ and $(h, p) \in T\tilde{\mathcal{B}}_{(g, \pi)}$

$$D_{(g, \pi)} \mathcal{H}(g, \pi; \xi)(h, p) = -\int_M (h, p) \cdot D\Phi^*_\pi(g, \pi) \xi.$$

**Proof.** Using integration by parts, we can write the linearization of the first term in (3.1) as:

$$\int_M (\hat{\xi}_\infty - \xi), D\Phi^*_\pi(h, \sigma)) = \int_M D\Phi^*_\pi(\hat{\xi}_\infty - \xi), (h, \sigma)) + \int_{\partial M} \tilde{\mathcal{B}}[(\hat{\xi}_\infty - \xi), (h, \sigma)] + \lim_{r \to \infty} \int_{S_r} \tilde{\mathcal{B}},$$

where in the boundary integral $\tilde{\mathcal{B}}$ is a bilinear form given by (cf [8] equation (82)),

$$\tilde{\mathcal{B}}[(\mu, Y), (h, \sigma)] = n^i [\mu (\nabla^j h_{ij} - \nabla^j tr h) - h_{ij} \nabla^j \mu + tr h \nabla^j \mu] \sqrt{g}$$

$$+ 2n^i [Y_j \sigma^j_i + Y^j \pi^k_i h_{jk} - \frac{1}{2} Y_i \pi^i j k h_{jk}],$$

with $(\mu, Y) = \hat{\xi}_\infty - \xi = -\xi$ on the boundary $\partial M$. We also refer to [8] equations (6)-(9) for the formula of $D\Phi$ and its adjoint $D\Phi^*$. According to the boundary conditions in (2.6), any deformation
Given \((h, \sigma) \in T\tilde{\mathcal{B}}\) satisfies
\[
\begin{cases}
h_{AB} = 0, & \text{for } A, B = 2, 3 \\
n(\text{tr}^T h) + 2\delta^T (h(n)^T) - h_{11} H = 0 & \text{on } \partial M. \\
\sigma^{11} + \frac{1}{2}\pi^{11} h_{11} = 0 \\
\sigma_{1A} + \pi_{11} h_{1A} = 0 & A = 2, 3
\end{cases}
\] (3.6)

The second equation above implies that \(n(\text{tr}^T h) - 2(\nabla^T)^A h_{A1} - h_{11} H = 0\). Basic calculation gives
\[n^i \nabla^j h_{ij} = n(h_{00}) + (\nabla^T)^A h_{A1} + h_{11} H.\] Combining those two equalities we can derive that
\[n^i (\nabla^j h_{ij} - \text{tr}_M h) = -n(\text{tr}^T h) + (\nabla^T)^A h_{A1} + h_{11} H = -(\nabla^T)^A h_{A1}.\]

In addition, \[n^i [h_{ij} \nabla^j \mu + \text{tr}_M h \nabla_i \mu] = -h_{1A} \nabla^A \mu - h_{00} n(\mu) + h_{00} n(\mu) = -h_{1A} \nabla^A \mu\]
where we use the fact that \(h_{AB} = 0\) from (3.6). Summing up the two equations above we obtain that the first line in (3.5) can be written as
\[n^i [\mu (\nabla^j h_{ij} - \text{tr}_M h) - h_{ij} \nabla^j \mu + \text{tr}_M h \nabla_i \mu] = -\mu (\nabla^T)^A h_{A1} - h_{1A} \nabla^A u = -(\nabla^T)^A (\mu h_{A1})\]
which is a pure divergence term on the boundary \(\partial M\) and hence its integral is zero. As for the second line in (3.5), we have
\[n_i [Y_j \sigma^{ij} + Y_j \pi^{kj} h_{jk} - \frac{1}{2} Y_i \pi^{jk} h_{jk}] = Y_1 \sigma^{11} + Y_A \sigma^{A1} + Y_j \pi^{kj} h_{jk} - \frac{1}{2} Y_1 \pi^{jk} h_{jk}
= -\frac{1}{2} Y_1 \pi^{11} h_{11} - Y_A \pi^{11} h_{1A} + Y_A \pi^{11} h_{1A} + Y_1 \pi^{11} h_{11} - \frac{1}{2} Y_1 \pi^{11} h_{11} - Y_1 \pi^{11} h_{1A} = 0.
\]
In the second equality above, we use the last two equations in (3.6) to replace \(\sigma\) with \(\pi, h\) and the first equation in (3.6) to throw away terms involving \(h_{AB}\) \((A, B = 2, 3)\). Thus the boundary integral over \(\partial M\) in (3.4) must vanish. The integral at infinity is also zero because \(\xi - \tilde{\xi}_\infty\) and \((h, \sigma)\) decay fast enough to zero. Thus we obtain
\[
\int_M \langle (\tilde{\xi}_\infty - \xi), D\tilde{\Phi}^*_{(g, \pi)}(h, \sigma) \rangle = \int_M \langle D\tilde{\Phi}^*_{(g, \pi)}(\tilde{\xi}_\infty - \xi), (h, \sigma) \rangle
\]
The linearization of the remaining terms in (3.1) is given by
\[-\int_M \langle (h, \sigma), D\tilde{\Phi}^*_{(g, \pi)}(\tilde{\xi}_\infty) \rangle \] (cf. [8] Theorem 5.2). Combining this with the equation above we obtain (3.3).

When the energy-momentum vector \(P(g, \pi)\) is time-like, the ADM total mass of the initial data \((g, \pi)\) is defined as
\[
m_{\text{ADM}}(g, \pi) = \sqrt{-P^\alpha P_\alpha}.
\] (3.7)

With the lemma above we can now prove that any critical point of the ADM total mass on the constraint manifold \(\tilde{\mathcal{C}}_B(u, Z)\) must admit a generalised Killing vector field, which is the analog of Corollary 6.2 in [8]. Note a spacetime vector field \(\xi \in [T^m, \alpha \times [T^m, \alpha]](M)\) is called a generalised Killing vector field of the initial data set \((M, g, \pi)\) (or \((M, g, K)\)) if \(D\tilde{\Phi}^*_{(g, \pi)}(\xi) = 0\) (or \(D\tilde{\Phi}^*_{(g, K)}(\xi) = 0\)). In this case, the initial data set \((M, g, \pi)\) is called a generalised stationary initial data set.

**Theorem 3.2.** Suppose \((u, Z) \in [C^k \times (\Lambda^1)^k](M)\) \((k \geq 0, \delta \geq 4), (g_0, \pi_0) \in \tilde{\mathcal{C}}_B(u, Z)\) and \(P_0 = P(g_0, \pi_0)\) is a time-like vector. If \((Dm_{\text{ADM}})_{(g_0, \pi_0)}(h, \sigma) = 0\) for all \((h, \sigma) \in T\tilde{\mathcal{C}}_B(u, Z)\) \((g_0, \pi_0)\) then \((g_0, \pi_0)\) admits a generalised Killing vector field \(\xi\) which has a limit at infinity proportional to \(P_0\).
Conversely, if \((g_0, \pi_0)\) is a generalised stationary initial data set, then \(D m_{\text{ADM}}(g_0, \pi_0)(h, p) = 0\) for all \((h, p) \in T \mathcal{C}_B(u, Z)\). \((\xi_0)\)\infty

Proof. If \((g_0, \pi_0)\) is a critical point with time-like ADM energy-momentum vector \(\mathbb{P}_0\). Let \((\xi_0)\) be the constant vector \((\xi_0)\)\infty\ = -\((\mathbb{P}_0)\)\infty\/m_{\text{ADM}}(g_0, \pi_0) \in \mathbb{R}^4\) and define a functional on \(E\) on \(\mathcal{C}_B(u, Z)\) by

\[
E(g, \pi) = (\xi_0)\infty\mathbb{P}_0(g, \pi).
\]

Clearly, the derivative of the ADM mass at \((g_0, \pi_0)\) is

\[
D m_{\text{ADM}}(g_0, \pi_0) = -(m_{\text{ADM}})^{-1/2}(\mathbb{P}_0)\alpha D \mathbb{P}_0 = (\xi_0)\infty\mathbb{P}_0 = D E|_{(g_0, \pi_0)}.
\]

So \((g_0, \pi_0)\) is also a critical point of \(E\) on the constraint manifold. Let \((\xi_0)\)\infty\ be a constant translation near infinity representing \((\xi_0)\)\infty\. Choose \((\xi_0)\)\infty\ in \(Z_{m,\alpha}(M)\) such that \((\xi_0)\)\infty\ - \((\xi_0)\)\infty\ is a generalised stationary initial data set, then

\[
\mathcal{H}(g, \pi; \xi_0) = 16\pi (\xi_0)\alpha^\infty \mathbb{P}_0 - \int_M (\xi_0)\alpha^\infty \Phi_0(g, \pi).
\]

Observe on the constraint manifold \(\mathcal{C}_B(u, Z)\), \((g_0, \pi_0)\) is a critical point of the first term on the right side and the second term is constant. Thus we obtain

\[
D_{(g_0, \pi_0)} \mathcal{H}(g, \pi; \xi_0)(h, \sigma) = 0 \quad \forall (h, \sigma) \in T \mathcal{C}_B(u, Z).
\]

By a Lagrange-multiplier argument (cf. [8] Theorem 6.3), there is \((\tilde{X}^0, \tilde{X}) \in (\mathcal{T})^*\) such that

\[
\int_M \langle \langle \tilde{X}^0, \tilde{X} \rangle, D \tilde{\Phi}_{(g_0, \pi_0)}(h, \sigma) \rangle = D_{(g_0, \pi_0)} \mathcal{H}(g, \pi; \xi_0)(h, \sigma) \quad \forall (h, \sigma) \in T \mathcal{B}|_{(g_0, \pi_0)}.
\]

Apply Lemma 3.1 to the right side above,

\[
\int_M \langle \langle \tilde{X}^0, \tilde{X} \rangle, D \tilde{\Phi}_{(g_0, \pi_0)}(h, \sigma) \rangle = \int_M \langle D \tilde{\Phi}_{(g_0, \pi_0)}^*(\xi_0), (h, \sigma) \rangle \quad \forall (h, \sigma) \in T \mathcal{B}|_{(g_0, \pi_0)}.
\]

Let \((X^0, X) = ((\xi_0)^0 - \tilde{X}^0, (\xi_0)^I - \tilde{X})\), then the equality above implies that \((X^0, X)\) is a weak solution of \(D \tilde{\Phi}_{(g_0, K_0)}^*(\xi_0)\), i.e. for all compactly supported \((h, p) \in T \mathcal{B}|_{(g_0, K_0)}\)

\[
\int_M \langle \langle X^0, X \rangle, D \tilde{\Phi}_{(g_0, K_0)}^*(\xi_0) \rangle(h, p) = 0.
\]

Here we use the isomorphism between \((\mathcal{B}, \Phi)\) and \((\mathcal{B}, \tilde{\Phi})\) and \((g_0, K_0)\) denotes the correspondence in \(\mathcal{B}\) of \((g_0, \pi_0)\). We can prove in the same way as in §3.1 that \((X^0, X)\) is a regular solution, i.e. it is \(C^{m,\alpha}\) smooth in the interior of \(M\). In addition, we show in section §4.4 that \((X^0, X)\) is \(C^{m,\alpha}\) smooth up to the boundary. Moreover, \((\tilde{X}^0, \tilde{X}) = \xi_0 - (X^0, X)\) is also \(C^{m,\alpha}\) smooth in \(M\) and is a bounded linear functional on the space \([C^{m,\alpha}_{\delta+2} \times C^{m,\alpha}_{\delta+2}](\mathcal{M})]\). Consequently it must be asymptotically zero at the rate of \(\delta\) (cf. §4.5 for details). Therefore, \((X^0, X)\) is a generalised \(C^{m,\alpha}\) Killing vector field on \(M\) and \((X^0, X) - \xi_0 \in C^m\), i.e. its limit at infinity is proportional to the ADM energy-momentum vector.

Conversely, suppose \((g_0, \pi_0)\) admits a generalised Killing vector field \(\tilde{X} = (X^0, X) \in Z_{m,\alpha}(M)\) whose asymptotic limit \((\tilde{X})\)\infty\ is proportional to \(\mathbb{P}_0\). Based on Lemma 3.1, \(D \tilde{\Phi}_{(g_0, K_0)}(\tilde{X}) = 0\) implies that \(D_{(g_0, \pi_0)} \mathcal{H}(g, \pi; \tilde{X})(h, \sigma) = 0 \forall (h, \sigma) \in T \mathcal{C}_B(u, Z)|_{(g_0, \pi_0)}\). This further implies that \((\tilde{X})\infty\mathbb{P}_0 = 0\) on the constraint manifold. Since \((\tilde{X})\infty\ is proportional to \(\mathbb{P}_0\), we also have \(\mathbb{P}_0 D \mathbb{P}_0 = 0\) i.e. \(D m_{\text{ADM}}(g_0, \pi_0)(h, \sigma) = 0\) based on the discussion at the beginning of the proof.

We note that the condition \((u, Z) \in [C^k_\delta \times (\mathcal{L})^{k}_{\delta}](\mathcal{M})\) \((k \geq 0, \delta \geq 4)\) in the theorem can be replaced by the integrable condition that \(u, Z_i\) are \(L^1\) on \(M\).
4. Appendix

In this section we provide the details which are left open in some proof of this paper.

4.1. Transform from \( B \) to \( \tilde{B} \). Recall, in §2, we have defined the space \( B \) of pairs \((g,K)\) fixing the Bartnik boundary data. The reparametrization space \( \tilde{B} \) is equivalent to \( B \) via the map

\[
P : \tilde{B} \to B
\]

\[
P(g, \pi) = (g, (\pi^b - \frac{1}{2}(\text{tr}_g \pi)g)/\sqrt{g}).
\]

Let \((h, \sigma) \in T\tilde{B}|_{(g, \pi)}\) be an infinitesimal deformation at \((g, \pi)\). Then linearization of \( P \) is given by

\[
DP_{(g, \pi)}(h, \sigma) = (h, p(h, \sigma)),
\]

where

\[
p_{ij}(h, \sigma) = [\sigma_{ij} - \frac{1}{2}(\text{tr}\sigma)g_{ij} + \pi^b_i h_{kj} + \pi^b_j h_{ki} - \frac{1}{2}(\text{tr}\pi)h_{ij} - \frac{1}{2}(\pi^k \pi h_{kr})g_{ij} - \frac{1}{2}\text{tr}(\pi_{ij} - \frac{1}{2}(\text{tr}\pi g_{ij}))/\sqrt{g}.
\]

Since \((h, p) \in TB\), it must satisfy the boundary conditions listed in (2.1). It is obvious that the first two boundary conditions there yield the first two boundary conditions in (2.6). The third condition in (2.1) implies

\[
0 = \text{tr}^T p = \text{tr}^T \sigma - (\text{tr}\sigma) + 2\pi^{A1} h_{A1} - \pi^b h_{1k} - \frac{1}{2}\text{tr}(\text{tr}^T \pi - \text{tr}\pi) = -\sigma^{11} - \frac{1}{2}\pi^{11} h_{11},
\]

where we use the condition \( h^T = 0 \) on \( \partial M \). This gives the third boundary condition in (2.6). Finally along \( \partial M \) we have

\[
p(n)^T = [\sigma_{1A} + \frac{1}{2} h_{11} \pi_{1A} + \pi_{A} B h_{1A} + \pi_{11} h_{1A} - \frac{1}{2}(\text{tr}_g \pi)h_{1A}]/\sqrt{g}
\]

\[
K(n'_h)^T = (\pi^b(n'_h)^T - \frac{1}{2}\text{tr}_g \pi g(n'_h)^T)/\sqrt{g} = [-\pi_B A h_{11}^B - \frac{1}{2} h_{11} \pi_{1A} + \frac{1}{2}(\text{tr}_g \pi)h_{1A}]/\sqrt{g},
\]

where we apply \( h^T = 0 \) on \( \partial M \) and the variation formula of the unit normal \( n'_h = -h_{1A} - \frac{1}{2} h_{11} n \). Summing up the equations above, we derive that the last condition in (2.1) can be equivalently expressed in terms of \((h, \sigma)\) as \( 0 = p(n)^T + K(n'_h)^T = (\sigma_{1A} + \pi_{11} h_{1A})/\sqrt{g} \). This completes the proof of all the conditions listed in (2.6).

4.2. Decomposition at the boundary. Given any \((u, Z) \in T\) such that \((u, Z) = D\Phi_{(g,K)}(h, p)\) on \( \partial M \). Then it follows naturally that \((u, Z)\) can be decomposed as \((u, Z) = (u_1, Z_1) + (u_2, Z_2)\) with \((u_1, Z_1) = D\Phi_{(g,K)}(h, p)\) in \( \Im C \) and \((u_2, Z_2)\) vanishing on \( \partial M \). For simplicity we can first choose \((h, p)\) so that \( h \) vanishes to the first order on \( \partial M \) and \( p \) vanishes to the zero order on \( \partial M \), i.e.

\[
h_{ij} = 0, \quad n(h_{ij}) = 0, \quad p_{ij} = 0 \text{ on } \partial M.
\]

Obviously \((h, p) \in TB|_{(g,K)}\). Based on (2.4) the linearization \((D\Phi_0)_{(g,K)}(h, p)\) is given by:

\[
(D\Phi_0)_{(g,K)}(h, p) = (-\Delta(\text{tr}h) + \delta h)\sqrt{g} = -n(\text{tr}^T h)\sqrt{g}
\]

inside which we use the equality that \( \Delta(\text{tr}h) = \Delta^T(\text{tr}h) + n(\text{tr}h)H - mn(\text{tr}h) \) and \( \delta h = \nabla^i \nabla^j h_{ij} = mn(h_{11}) \). Set \( h \) satisfying (4.1) and such that

\[
-n(\text{tr}^T h)\sqrt{g} = u \text{ on } \partial M,
\]

then we have \((D\Phi_0)_{(g,K)}(h, p) = u\). Next plug \((h, p)\) into (2.5) and obtain

\[
(D\Phi_1)_{(g,K)}(h, p) = -2(\delta p + d\text{tr}p)\sqrt{g} = -2n(\text{tr}^T p)\sqrt{g} \cdot n - 2\nabla_n p(n)^T \sqrt{g}.
\]
Thus we can choose a vector field \( n \) satisfying (4.1) and such that \( n = (\text{tr}^TP)p = Z_1 \) and \( n(p_{1A}) = Z_A \), \( A = 2, 3 \) on \( \partial M \). It then follows that \( (D\Phi_t)(g, K)(h, p) = Z \).

4.3. Construction of the extension maps. We construct candidates for the maps \( E_1 \) and \( E_2 \) in the proof of splitting kernel for \( D\Phi \) in §2.2.

Fix \( \tau \in \text{Ker}\delta^T \) on the boundary \( \partial M \) of the Riemannian manifold \( (M, g) \). We can first fix a collar neighborhood \( U \) of the boundary \( \partial M \) inside which the flow of the distance function \( s \) to the boundary is well-defined. Without loss of generality, assume \( U = \{ s \in [0, 1] \} \). We also extend the unit normal vector \( n \) to \( \partial M \) to be the unit vector field perpendicular to the \( s \)-level set in \( U \). Define \( h \) so that \( h^T = 0 \) in \( U \) and \( h = 0 \) on \( \partial M \). It then follows that \( H_h' = 0 \) and \( n_h' = 0 \) on \( \partial M \). The 1-form \( h(n)^T \) is defined to be such that

\[
h(\partial_s)^T = 0 \quad \text{on} \, \partial M, \quad \nabla_{\partial_s}(h(\partial_s)^T) = \tau \quad \text{in} \, \Omega,
\]

where we think of \( \tau \) as being parallelly Lie-dragged by \( \partial_s \) in \( \Omega \). Then we have \( (\delta h)^T\langle h \rangle = -\nabla^1 h_{1A} - \nabla^B h_{BA} = \nabla_n(h(n)^T) = \tau_A \) on \( \partial M \). Next fix a smooth jump function \( f(s) \) in \( U \) such that \( f(s) = 1 \) for \( 0 \leq s \leq 1/4 \) and \( f(s) = 0 \) for \( s \geq 1/2 \). We can now define \( E_1(\tau) = f(s)h \) where \( h \) is as constructed above and \( fh \) is extended trivially from \( U \) to be a symmetric 2-tensor defined on \( M \). Then it is easy to check that \( E_1 \) is a linear bounded map.

Next fix \( h \in S^m_{00}(M) \) on the initial data set \( (M, g, K) \). Take the collar neighborhood \( U \) of \( \partial M \) inside of which \( \nabla^s \) is well-defined. Construct a symmetric 2-tensor in \( U \) such that \( h^T = \hat{h}(n, n) = 0 \) and \( h(n)^T = -K(n)^T \) in \( U \). Then extend \( \hat{h} \) to a global tensor, still denoted as \( \hat{h} \), on \( M \) which is equal to zero outside \( \partial M \) and equal to \( f(s)h \) in \( U \). Now define \( E_2(h) = \hat{h} \) as constructed. It follows that \( E_2 \) is a linear bounded map and only involves 0-order data of \( h \).

4.4. Boundary behavior of the generalised Killing vector field. In the proof of Theorem 3.2 we show that \( (X^0, X) \) is a weak solution of \( D\Phi^{(g_0, p_0)}(X^0, X) = 0 \) and thus is \( C^{m, \alpha} \) in the interior \( \text{int}M \) of \( M \) and satisfies

\[
\begin{align*}
2X^0K + LXg &= 0, \\
D^2X^0 + LXK + X^0(-Ric + 2K \circ K - (trK)K + \frac{1}{4}u_0g) &= 0.
\end{align*}
\]

We can apply the approach in [9] here to show that \( (X^0, X) \) is \( C^{m, \alpha} \) smooth up to the boundary \( \partial M \). From the equations above we can obtain

\[
\begin{align*}
\nabla_k \nabla_i X^0 &= -L_X K_{ij} + X^0(-Ric + 2K \circ K - (trK)K + \frac{1}{4}u_0g)_{ij}, \\
\nabla_i \nabla_j X_k &= R_{kjmi}X^m + D_k(X^0 K_{ij}) - D_i(X^0 K_{jk}) - D_j(X^0 K_{ki}),
\end{align*}
\]

where the second equation is obtained by taking covariant derivative of the first equation in (4.2) and applying the Binachi identity \( R_{kjim} + R_{ijkm} + R_{jikm} = 0 \) with \( R_{kimj}X^m = \nabla_k \nabla_i X_j - \nabla_l \nabla_l X_j \). Define the radius function on \( M \) as the pull back of \( r \) from \( \mathbb{R}^3 \setminus B \) and \( r = 1 \) on \( \partial M \). Let \( s = 1 - r \). So \( s \in (-\infty, 0] \), \( \partial_s = \frac{s}{s-1} \partial_i \) and we consider the limit \( \lim_{s \to 0}(X^0, X) \). Derivatives of \( (X^0, X) \) are given by

\[
\begin{align*}
\partial_s X^0 &= \frac{x^i}{s-1} \partial_i X^0, \\
\partial_s X_i &= \frac{x^j}{s-1} (\nabla_j X_i + \Gamma_{ji}^k X_k), \\
\partial_s \nabla_i X^0 &= \frac{x^j}{s-1} (\nabla_j \nabla_i X^0 + \Gamma_{ji}^k \nabla_k X^0), \\
\partial_s \nabla_i X_j &= \frac{x^k}{s-1} (\nabla_k \nabla_i X_j + \Gamma_{ki}^l \nabla_l X_j + \Gamma_{kj}^l \nabla_l X_i).
\end{align*}
\]

Here \( X_i = (g_0)_{ik}X^k \) and the covariant derivative \( \nabla \) and Christoffel symbol \( \Gamma_{ij}^k \) are with respect to the metric \( g_0 \). Notice that the terms \( \nabla_k \nabla_i X_j \) and \( \nabla_k \nabla_i X_j \) can be replaced by lower derivatives based on (4.3). So define \( f = (X^0, \nabla X^0, X, \nabla X) \) and let \( F = |f|^2 \). Then equations above imply
that $|\partial F| \leq CF$ for some constant $C > 0$. If $\partial_s F < 0$, then $F$ is not increasing along the $s$ curve. If $\partial_s F \geq 0$, then we have

$$\frac{\partial F}{\partial s} - CF \leq 0 \Rightarrow \partial_s (e^{-CS} F) \leq 0 \Rightarrow e^{-CS} F(s_1) \leq e^{-CS} F(s_2) \ \forall s_1 > s_2.$$ 

Thus $F(\varepsilon) \leq e^{C\varepsilon + C} F(-1) + C \ \forall \varepsilon > -1$. So in both cases $\lim_{s \to 0} F(s) \leq C$. Take an open neighborhood of the boundary $\partial M \subset U \subset M$. Then $X^0, \nabla X^0, X, \nabla X$ are uniformly bounded in $U \setminus \partial M$. Henceforth, $\nabla, \nabla X^0$ and $\nabla, \nabla X$ are also uniformly bounded in $U \setminus \partial M$ according to (4.3). This in return implies $\nabla X^0, \nabla X$ are uniformly continuous in $U \setminus \partial M$ and so is $(X^0, X)$. Based on (4.3) again, $\nabla^2 N, \nabla^2 Y$ are also uniformly continuous. Therefore, we can extend $(X^0, X)$ to be well-defined and second order differentiable in $U$. In fact by a bootstrap argument it can be extended to a $C^{m, \alpha}$ fields in $U$ and by continuity we also have $D\Phi^*(X^0, X) = 0$ up to the boundary.

4.5. **Asymptotic behavior of Killing vector field.** Since $(X^0, X)$ is a $C^{m, \alpha}$ solution of $D\Phi^*(X^0, X) = 0$, by [9] we know its asymptotic behavior must be as in (2.23) or (2.24). Thus the asymptotic behavior of $\tilde{X} = X - \xi$ is

$$\tilde{X}^i = -(\xi_0)_{\infty} + \Lambda_{ij} x^j + O_k(1), \quad \tilde{X}^0 = -(\xi_0)_{\infty} + \Lambda_{0i} x^i + O_k(1);$$

for some constant $\Lambda_{\mu \nu}$, or

$$\tilde{X}^i = -(\xi_0)^{i}_{\infty} + A^{i} + O_k(1), \quad \tilde{X}^0 = -(\xi_0)^{0}_{\infty} + A^0 + O_k(1).$$

for some constant $A^\mu$. In addition, $\tilde{X}^\alpha$ is a bounded functional on $C^{m-2, \alpha}_{\delta + 2}$, acting on a function $v \in C^{m-2, \alpha}_{\delta + 2}$ via the $L^2$-product $\int_M \tilde{X}^\alpha \cdot v \text{dvol}_g$.

If $\Lambda_{\mu \nu} \neq 0$ for some $\mu, \nu$, without loss of generality we can assume $\Lambda_{01} \neq 0$. Let $v = v(r)$ be the smooth positive function which equals to zero near the inner boundary and equals to $\frac{\xi_{01} x^1}{\nu - \delta} (\beta > \delta)$ for $r > R$. Then

$$\int_M \tilde{X}^0 \cdot v \text{dvol}_g = \int_{R}^{\infty} \int_{2}^{r+1} \frac{\Lambda_{01} x^1}{r^{\beta + 3}} \tilde{X}^0 r^2 \text{ds}^2 \text{d}r + O(1)$$

$$= \int_{R}^{\infty} \int_{2}^{r+1} \frac{\Lambda_{01} x^1}{r^{\beta + 1}} (- (\xi_0)_{\infty} + \Lambda_{0i} x^i + O(1)) ds^2 dr + O(1)$$

$$= \int_{R}^{\infty} \int_{2}^{r+1} \frac{\Lambda_{01} x^1}{r^{\beta + 1}} ds^2 dr - \int_{R}^{\infty} \int_{2}^{r+1} \frac{\Lambda_{0i} x^i}{r^{\beta + 1}} ds^2 dr + \int_{R}^{\infty} O(r^{1+\delta - \beta}) ds^2 dr + O(1)$$

Obviously if $1 < \beta \leq 2$, the above integral diverges which contradicts that $\tilde{X}^0$ is bounded.

If $A^\mu - (\xi_0)^{i}_{\infty} \neq 0$ for some $\mu$, again without loss of generality we assume $A^0 - (\xi_0)^{0}_{\infty} \neq 0$. Let $v = v(r)$ be a smooth positive function which equals to zero near the inner boundary and equals to $\frac{1}{r^{\beta + 2}} (\beta > \delta)$ for $r > R$. Then

$$\int_M \tilde{X}^0 \cdot v \text{dvol}_g = \int_{R}^{\infty} \int_{2}^{r+1} \frac{1}{r^{\beta + 2}} \tilde{X}^0 r^2 \text{ds}^2 \text{d}r + O(1)$$

$$= \int_{R}^{\infty} \int_{2}^{r} \frac{1}{r^{\beta}} (- (\xi_0)_{\infty} + A^0 + O(1)) ds^2 dr + O(1)$$

$$= (-(\xi_0)_{\infty} + A^0) \int_{R}^{\infty} \int_{2}^{r} \frac{1}{r^{\beta}} ds^2 dr + \int_{R}^{\infty} \int_{2}^{r+1} O(r^{1+\delta - \beta}) ds^2 dr + O(1)$$

If $\max\{\delta, 1 - \delta\} < \beta < 1$, the above integral diverges. So we must have $A^\mu = (\xi_0)^{i}_{\infty}$, i.e. $\tilde{X}$ decays to zero asymptotically.
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