Exponents of diophantine approximation in dimension 2 for numbers of Sturmian type

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Abstract

We generalize the construction of Roy’s Fibonacci type numbers to the case of a Sturmian recurrence and we determine the classical exponents of approximation $\omega_2(\xi)$, $\hat{\omega}_2(\xi)$, $\lambda_2(\xi)$, $\hat{\lambda}_2(\xi)$ associated with these real numbers. This also extends similar results established by Bugeaud and Laurent in the case of Sturmian continued fractions. More generally we provide an almost complete description of the combined graph of parametric successive minima functions defined by Schmidt and Summerer in dimension two for such Sturmian type numbers. As a side result we obtain new information on the joint spectra of the above exponents as well as a new family of numbers for which it is possible to construct the sequence of the best rational approximations.

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1 Introduction

If $\xi \in \mathbb{R}$ is not an algebraic number of degree $\leq 2$ we may study the following two problems which are standard in diophantine approximation:

**Problem $E_{\omega,X}$:** We search for non-zero integer points $x = (x_0, x_1, x_2) \in \mathbb{Z}^3$ solutions of the system

\[
\begin{cases}
|x_0 + x_1 \xi + x_2 \xi^2| \leq X^{-\omega} \\
\max \left( |x_1|, |x_2| \right) \leq X.
\end{cases}
\]

**Problem $E'_{\lambda,X}$:** We search for non-zero integer points $x = (x_0, x_1, x_2) \in \mathbb{Z}^3$ solutions of the system

\[
\begin{cases}
|x_0| \leq X \\
\max \left( |x_0 \xi - x_1|, |x_0 \xi^2 - x_2| \right) \leq X^{-\lambda}.
\end{cases}
\]

We denote by $\omega_2(\xi)$ (resp. $\hat{\omega}_2(\xi)$) the supremum of real numbers $\omega$ for which the problem $E_{\omega,X}$ admits a non-zero integer solution for arbitrarily large values of $X$ (resp. for each sufficiently large value of $X$).

Similarly we denote by $\lambda_2(\xi)$ (resp. $\hat{\lambda}_2(\xi)$) the supremum of real numbers $\lambda$ for which the problem $E'_{\lambda,X}$ admits a non-zero integer solution for arbitrarily large values of $X$ (resp. for each sufficiently large value of $X$).

We call spectrum of an exponent the set of values taken by this exponent at real numbers which are not algebraic of degree $\leq 2$. We also call joint spectrum of a family of diophantine exponents $(\nu_1, \nu_2, \ldots)$ the set of values taken by $(\nu_1(\xi), \nu_2(\xi), \ldots)$ at real numbers which are not algebraic of degree $\leq 2$. We denote this set by spec$(\nu_1, \nu_2, \ldots)$. 
The previous four exponents have been studied a lot during the last decade. The reader may refer to [4] which contains a well supplied summary of the subject and to [5] for a generalization to the approximation of a vector \((1, \xi_1, \xi_2) \in \mathbb{R}^3\) (in this paper we study the special case \((1, \xi_1, \xi_2) = (1, \xi, \xi^2)\)). Let us recall briefly some important properties they satisfy. We shall first start with Jarník’s identity [11] Theorem 1) which binds \(\hat{\lambda}_2\) and \(\hat{\omega}_2\) together:

**Proposition 1.1** (Jarník’s identity). For each real number \(\xi\) which is not algebraic of degree \(\leq 2\), we have

\[
\frac{1}{\hat{\lambda}_2(\xi)} = 1 - \frac{1}{\hat{\omega}_2(\xi)},
\]

with the convention that the right hand side of this equality is 1 if \(\hat{\omega}_2(\xi) = +\infty\).

In this paper \(\gamma\) denotes the golden ratio \(\frac{1 + \sqrt{5}}{2}\). It is known that these exponents satisfy

\[
\frac{1}{2} \leq \frac{1}{\hat{\lambda}_2(\xi)} \leq \frac{1}{\gamma} = 0.618... \quad \text{and} \quad 2 \leq \hat{\omega}_2(\xi) \leq \gamma^2 = 2.618...
\]

for any \(\xi\) which is not algebraic of degree \(\leq 2\). The lower bounds are obtained by the Dirichlet box principle (or equivalently, Minkowski’s Theorem), the upper bounds follow respectively from Theorem 1a of [8] and from [17]. Roy proved [12] the existence of real numbers \(\xi\) called extremal numbers for which \(\hat{\lambda}_2(\xi) = \frac{1}{\gamma}\) and \(\hat{\omega}_2(\xi) = \gamma^2\). Then, generalizing his construction he proved [14] that the spectrum of \(\hat{\lambda}_2\) (resp. \(\hat{\omega}_2\)) forms a dense subset of the interval \([\frac{1}{\gamma}, \frac{1}{2}]\) (resp. \([2, \gamma^2]\)). The numbers given by his generalized construction are said to be of the Fibonacci type. It is still an open problem to describe precisely the spectra of the two exponents \(\hat{\lambda}_2, \hat{\omega}_2\).

On the contrary, the spectra of \(\lambda_2\) and \(\omega_2\) are well known:

\[
\text{spec}(\lambda_2) = \left[\frac{1}{2}, +\infty\right] \quad \text{and} \quad \text{spec}(\omega_2) = [2, +\infty].
\]

The first equality of (1.3) results directly from the works of Beresnevich, Dickinson, Vaughan and Velani [2, 23], who computed the Hausdorff dimension of the set of real numbers \(\xi\) for which \(\lambda_2(\xi) = \lambda\). The second equality of (1.3) was proved by Bernik in [3], computing for all \(\omega \geq 2\) the Hausdorff dimension of the set of real numbers \(\xi\) for which \(\omega_2(\xi) = \omega\). Finally, note that for almost all \(\xi\) with respect to the Lebesgue’s measure, one has

\[
\omega_2(\xi) = \hat{\omega}_2(\xi) = \frac{1}{\hat{\lambda}_2(\xi)} = \frac{1}{\lambda_2(\xi)} = 2.
\]

(Cf [20] Ch. VI, Corollaries 1C, 1E) or [5] Theorem 2.3)). This equality is also achieved for each \(\xi\) algebraic number of degree \(\geq 3\) by virtue of Schmidt’s subspace theorem [20]. Cf also [4] Theorem 2.10.

To construct extremal numbers, Roy [12] associates to distinct positive integers \(a, b\) the real number \(\xi_{a,b}\) whose partial quotients are successively 0, and the letters of the Fibonacci word \(w = abaaba...\) constructed on \(\{a, b\}\) (i.e. \(\xi_{a,b} = [0; a, b, a, a, b, ...]\)). The property \(\hat{\lambda}_2(\xi_{a,b}) = 1/\gamma\) is related to the fact that the Fibonacci word has many palindromic prefixes, and each of them gives a good approximation of the vector \((1, \xi_{a,b}, \xi_{a,b}^2)\). Bugeaud and Laurent generalized this approach and calculated (among others) the four exponents \(\lambda_2(\xi), \hat{\lambda}_2(\xi), \omega_2(\xi)\) and \(\hat{\omega}_2(\xi)\) for a real number \(\xi\) whose partial quotients are given by a Sturmian characteristic word (see [5]; also note that Schleischitz deals with the problem of the cubic approximation of such numbers in [18, 19]). Again, the fact that Sturmian characteristic words have many palindromic prefixes that one knows how to describe precisely plays an essential role. Note that for all extremal numbers and all numbers \(\xi\) constructed by Bugeaud and Laurent, we have \(\lambda_2(\xi) = 1\). On the other hand, for Fibonacci type numbers Roy generalized the construction of extremal numbers in a different way using Fibonacci sequences in \(\text{GL}_2(\mathbb{R})\). One crucial point is that he did not consider directly the continued fraction representations (and the \(2 \times 2\) associated matrices) which allowed him to leave out the condition \(\lambda_2(\xi) = 1\) and show the density of the spectra of the exponents \(\hat{\lambda}_2\) and \(\hat{\omega}_2\).
in the intervals defined by $[1;2]$.

In this paper we construct numbers of Sturmian type, which generalize both Roy’s Fibonacci type numbers and the numbers constructed by Bugeaud and Laurent. For such a number $\xi$ we compute the four exponents $\widetilde{\lambda}_2(\xi)$, $\widetilde{\omega}_2(\xi)$, $\omega_2(\xi)$ and $\lambda_2(\xi)$ and (at least partially) the associated 3-system coming from Schmidt and Sumner’s parametric geometry of numbers (see the end of this introduction for more details, including the definition of Sturmian type numbers). As far as we know all real numbers $\xi$ satisfying $\widetilde{\omega}_2(\xi) > 2$ and for which one knows how to compute the associated diophantine exponents and the best rational approximations sequence of $(1, \xi, \xi^2)$ are of Sturmian type.\(^1\)

As an application of the construction and study of Sturmian type numbers, we prove density results on the spectra. To state these results we need the following definition.

**Definition 1.2.** Let $\underline{s} = (s_k)_{k\geq 0}$ be a sequence of positive integers. We associate to $\underline{s}$ the quantity

$$\sigma(\underline{s}) = \frac{1}{\limsup_{k\to+\infty} |s_{k+1}; s_k, \ldots, s_1|},$$

where $[a_0; a_1, a_2, \ldots]$ denotes the continued fraction whose partial quotients are $(a_0, a_1, \ldots)$. We denote by $A$ the set of all bounded sequences of positive integers, and as in [5], $S$ denotes the set of $\sigma(\underline{s})$ for $\underline{s} \in A$.

Cassaigne studied the elements of $S$ in [6]. This set is connected to the spectrum of Fischer’s exponent $\beta_0(\xi)$ [10]. To any $\underline{s} \in A$ is associated a characteristic Sturmian word of angle $[0; s_1, s_2, \ldots]$ and Sturmian type numbers $\xi$. In the following result $\xi$ is of Sturmian type and associated to $\underline{s}$.

**Theorem 1.1.** For each $\underline{s} \in A$, there exists a set $\Delta_{\underline{s}}$ which contains 0 and is dense in the interval $[0, \frac{\sigma}{1 + \sigma}]$ (with $\sigma = \sigma(\underline{s})$) and such that for each $\delta \in \Delta_{\underline{s}}$, there is a real number $\xi$ satisfying

$$\omega_2(\xi) = \frac{2 - \delta}{\sigma} + 1 - \delta,$$

$$\widetilde{\omega}_2(\xi) = 1 + (1 - \delta)(1 + \sigma),$$

$$\widetilde{\lambda}_2(\xi) = \frac{1 - \delta}{1 + (1 - \delta)(1 + \sigma)},$$

$$1 - \delta \leq \lambda_2(\xi) \leq \max \left(1 - \delta, \frac{1}{1 - \delta + \sigma}\right).$$

Moreover if we have $\delta \in [0, h(\sigma)]$ with

$$h(\sigma) = \frac{\sigma}{2} + 1 - \sqrt{\left(\frac{\sigma}{2}\right)^2 + 1} \leq \frac{\sigma}{1 + \sigma},$$

then

$$\lambda_2(\xi) = 1 - \delta.$$

Note that the case $\delta = 0$ of Theorem 1.1 corresponds to the values of the four exponents for the number $\xi_{\phi}$ constructed by Bugeaud and Laurent (see Theorem 3.1 of [5]); for a Fibonacci type number we have $s_k = 1$ for all $k \geq 1$ and the associated $\sigma$ is $\sigma = 1/\gamma$.

We may deduce from Theorem 1.1 the following corollary about the spectrum of $\omega_2$:

**Corollary 1.3.** We have

$$\{\omega_2(\xi) \mid \xi \text{ of Sturmian type}\} = [1 + \sqrt{5}, 2 + \sqrt{5}] \cup [2 + 2\sqrt{2}, 3 + 2\sqrt{3}] \cup [3 + \sqrt{13} + \infty],$$

where $\overline{A}$ denotes the topological closure of the subset $A \subset \mathbb{R}$.

\(^1\)Note that we also have a construction which gives for every $\omega \geq 3$ a number $\xi$ for which $\omega_2(\xi) = \omega$, but for such $\xi$ we always have $\widetilde{\omega}_2(\xi) = 2$ (see the proof of Theorem 5.1 and Theorem 5.5 of [4]).
One may compare this result with the special case where \( s_k = 1 \) for all \( k \):\[
\{\omega_2(\xi) \mid \xi \text{ of Fibonacci type} \} = [1 + \sqrt{5}, 2 + \sqrt{5}].
\]
These new constructions do not seem to give additional information on the individual spectra of each exponent. However, they bring new information on their joint spectrum. In this respect, a direct corollary of Theorem \([11]\) is the following.

**Corollary 1.4.** For each \( \sigma \in \mathcal{S} \), there is \( c = c(\sigma) \) satisfying \( 0 \leq c < 1 \), such that the joint spectrum of \( (\lambda_2, \hat{x}_2, \omega_2) \) is dense in the curve
\[
\left\{ \left( x, 1 - \frac{1}{1 + (1 + \sigma)x}, \frac{1 + (1 + \sigma)x}{\sigma}, 1 + (1 + \sigma)x \right) \mid x \in [c, 1] \right\}.
\]

Using their powerful theory of parametric geometry of numbers \([21], [22]\) Schmidt and Summerer proved that the study of the previous diophantine approximation exponents may be reduced to the study of what they call a \((3,0)\)-system (and that Roy calls a 3-system in [15], cf Definition \( 7.8 \) below). In this paper, we give an almost complete description of the 3-system associated to a number \( \xi \) of Sturmian type (see Figure \( 2 \) in Section \( 7 \)), and from it we compute the four exponents of Theorem \([11]\).

To conclude this introduction, let us summarize our strategy, parallel to that of Roy [14]. Let \( (s_k)_{k \geq 0} \) be a sequence of positive integers (except for \( s_0 \)) with \( s_0 = -1, s_1 = 1 \). For \( k \geq 0 \), we set \( t_k = s_0 + s_1 + \cdots + s_k \), and we associate to \( (s_k)_k \) a function \( \psi \) (called a Sturmian function, see Definition \( 2.3 \)) which is defined on \([0; +\infty[\) by \( \psi(t_k) = t_{k-1} - 1 \) for \( k \geq 1 \) and \( \psi(i) = i - 1 \) if \( i \) is not among the \( t_k \). We denote by \( \mathcal{F}_{\text{stur}} \) the set of all Sturmian functions for which the corresponding sequence \( (s_k)_k \) is bounded. Such a function \( \psi \) corresponds to a characteristic Sturmian word through the recurrence relation of its palindromic prefixes (see Section 3.1 of [9]).

In order to define a proper \( \psi \)-Sturmian number, we need the two following definitions. A \( \psi \)-Sturmian sequence in \( \mathcal{M} = \text{GL}_2(\mathbb{R}) \cap \text{Mat}_{2 \times 2}(\mathbb{Z}) \) is a sequence \( (w_i)_{i \geq 0} \) of matrices such that
\[
w_{i+1} = w_i^{-1}w_{i-1} \quad (i \geq 1).
\]
Such a sequence is **admissible** if there exists a matrix \( N \in \mathcal{M} \) such that \( w_1N, w_0N \) and \( w_1w_0N \) are symmetric matrices (see Definition \( 4.1 \)). We set \( N_k = N \) if \( k \) is even, \( N_k = \overline{N} \) if \( k \) is odd.

We say that a \( \psi \)-Sturmian sequence \( (w_i)_{i \geq 0} \) has **multiplicative growth** if
\[
\|w_k^l w_{k-1}\| \asymp \|w_k\| \times \|w_{k-1}\|
\]
for \( k \geq 1, 1 \leq l \leq s_{k+1} + 1 \) (see Definition \( 5.1 \)). Sequences with multiplicative growth are studied in Section \( 5 \). In the following we identify \( \mathbb{R}^2 \) (resp. \( \mathbb{Z}^2 \)) with the space of \( 2 \times 2 \) symmetric matrices with real (resp. integer) coefficients under the map \((x_0, x_1, x_2) \rightarrow (\begin{array}{cc} x_0 & x_1 \\ x_1 & x_2 \end{array})\). Accordingly, it makes sense to define the determinant \( \det(x) \) of a point \( x \in \mathbb{R}^3 \). Similarly, given symmetric matrices \( x, y \) we write \( x \wedge y \) to denote the inner product of the corresponding points.

From now on, we consider an admissible \( \psi \)-Sturmian sequence in \( \mathcal{M} \) with multiplicative growth such that the content of \( w_i \) is bounded and \( (\|w_i\|_i) \) tends to infinity. For instance, the \( \psi \)-Sturmian sequence defined by Roy’s matrices [14]
\[
w_0 = \begin{pmatrix} 1 & b \\ a & a(b + 1) \end{pmatrix}, \quad w_1 = \begin{pmatrix} 1 & c \\ a & a(c + 1) \end{pmatrix}
\]
satisfies these hypotheses (see Section 8.1).

Then we define two sequences of symmetric matrices \( (y_i)_i \) and \( (z_i)_i \) by the formulas
\[
y_{tk+l} = w_k^{l+1}w_{k-1}N_k \quad \text{and} \quad z_{tk+l} = \frac{1}{\det(w_k)}y_{tk+l} \wedge y_{tk+l} \quad (0 \leq k, 0 \leq l < s_{k+1})
\]
(see Definitions \( 3.3 \) and \( 4.2 \)). Combinatorial properties of these sequences are studied in Sections \( 3 \) and \( 4 \). By multiplicative growth (and because \( (\|w_i\|_i) \) tends to infinity) we show in
Proposition 5.6 that there exists $\delta \geq 0$ such that $|\det(w_i)| \asymp \|w_i\|^\delta$. If $\delta < \frac{\sigma}{1+\sigma}$, with $\sigma = 1/\limsup_{k \to \infty} |s_{k+1}^k : s_k, \ldots, s_1|$, then Proposition 6.1 ensures that the associated sequence $(y_i)$ converges in $\mathbb{R}^2$ to a point $y = (1, \xi, \xi^2)$. The number $\xi$ therefore produced is called a proper $\psi$-Sturmian number (and a Sturmian type number is a proper $\psi$-Sturmian number for some $\psi \in \mathcal{F}_\text{stur}$; see Definition 6.2). Then we show that the sequences $(y_i)$ and $(z_i)$ are “good” solutions of Problems $E_{\lambda \cdot X}$ and $E_{\omega \cdot X}$ respectively. In Section 6 (see Proposition 6.5) we prove the existence of $\xi$ and give precise estimates of $\|y_i\|, \|z_i\|, \|\langle z_i, y \rangle\|$ and $\|y_i \wedge y\|$. This allows us to build in Section 7 (see Propositions 7.14, 7.16 and Figure 2) a partial 3-system associated to $\xi$. Finally, we deduce from this 3-system the Diophantine exponents associated to $\xi$ (see Theorem 7.3). For $\lambda_2(\xi)$ another hypothesis—which is discussed in Section 7.3—is required in order to ignore the gray areas of the partial 3-system of Figure 2.

2 Notations

In this section we set some notations (in particular the sequence $(s_k)_{k \geq 0}$ and the function $\psi$ that we will constantly use throughout this paper) and we introduce the notion of Sturmian functions.

**Notation.** Let $I$ be a set (the set of index, typically $I$ of the form $\mathbb{N}^r$), $(a_i)_{i \in I}$ and $(b_i)_{i \in I}$ two sequences of non-negative real numbers indexed by $I$. Let $J$ be a subset of $I$ (the set of conditions).

We denote by “$a_i \ll b_i$ for $i \in J$” or “$b_i \gg a_i$ for $i \in J$” if there is a constant $c > 0$ such that for each $i \in J$ we have $a_i \leq cb_i$. We write “$a_i \asymp b_i$ for $i \in J$” if $a_i \ll b_i$ for $i \in J$ and $b_i \ll a_i$ for $i \in J$.

In the special case $I = \mathbb{N}$, without more precision we will always implicitly take $J$ of the form $\{0 + \infty\}$ for $j_0$ large enough, and we will simply write $a_i \ll b_i, b_i \gg a_i$ and $a_i \asymp b_i$.

**Definition 2.1.** For each non-zero $x \in \mathbb{R}^3$, we denote by $|x|$ its image in $\mathbb{P}^2(\mathbb{R})$. Commonly, we define the projective distance between two non-zero vectors $x$ and $y$ of $\mathbb{R}^3$ by

\[ \text{dist}(x, y) = \frac{|x \wedge y|}{\|x\| \cdot \|y\|}. \]

The projective distance satisfies in particular the triangle inequality:

\[ \text{dist}(x, z) \leq \text{dist}(x, y) + \text{dist}(y, z) \quad \text{for all } x, y, z \in \mathbb{R}^3 \setminus \{0\}. \]

We also recall the following inequalities, valid for $x, y, z \in \mathbb{R}^3$ (see [13, Lemma 2.2]):

**Lemma 2.2.** We have

\[ \|x \wedge y\| \leq 2 \|x\| \cdot \|y\|, \quad \text{(2.1)} \]

\[ \|y\| \cdot \|x \wedge y\| \leq \|x\| \cdot \|y\| + 2 \|x\| \cdot \|y\| \cdot \|z\|. \quad \text{(2.2)} \]

**Definition 2.3.** Let $(s_k)_{k \geq 0}$ be a sequence of positive integers (except for $s_0$) with $s_0 = -1, s_1 = 1$. For $k \geq 0$, we set $t_k = s_0 + s_1 + \cdots + s_k$, and we associate to $(s_k)_k$ a function $\psi$ defined on $[0; +\infty]$ by

\[ \begin{cases} \\ \psi(t_k) = t_{k-1} - 1 & \text{for } k \geq 1 \\ \psi(i) = i - 1 & \text{if } i \text{ is not among the } t_k. \end{cases} \]

The function $\psi$ is called Sturmian.

We denote by $\mathcal{F}_\text{stur}$ the set of all Sturmian functions for which the corresponding sequence $(s_k)_k$ is bounded.

**Remark.** Such a sequence is entirely characterised by its associated Sturmian function (the sequence $(t_k)_{k \geq 1}$ being exactly the integers $n$ such that $\psi(n) \leq n - 2$).

The Sturmian functions $\psi \in \mathcal{F}_\text{stur}$ belong to a bigger class of functions called asymptotically reduced functions and studied by Fischler in [9] and [10] (cf [10, Definition 2.1]). Generalizing the construction of Roy and the use of his bracket (cf [12] and [13]) Fischler shows that for each word $w$ with abundant palindromic prefixes, we may associate a function $\psi$ asymptotically reduced (if $w$ is a Sturmian word, the associated function $\psi$ is Sturmian) and that we may build a real number $\xi$ whose certain Diophantine exponents are intimately connected to $\psi$. 

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Notation. For the rest of the paper, we set a sequence \((s_k)_{k \geq 0}\) of positive integers (except for \(k = 0\), with \(s_0 = -1, s_1 = 1\), which is not necessarily bounded. We set \(t_k = s_0 + \cdots + s_k\) and we denote by \(\psi\) the associated Sturmian function. We also set \(\varphi = [0; s_2, s_3, \ldots]\).

3 \(\psi\)-Sturmian sequences in GL\(_2\)(\(\mathbb{C}\))

The goal of this section is to generalize Roy’s definition of Fibonacci sequence in GL\(_2\)(\(\mathbb{C}\)) (see [14] §3) and to identify in Proposition 3.6 some useful properties of the sequence \((y_i)\), (see Definition 3.5). This sequence will give “good” solutions of Problem \(E'_{2, N}\). Several properties have combinatorial analogues for characteristic Sturmian words (see [5] §5).

Definition 3.1. A \(\psi\)-Sturmian sequence in a monoid is a sequence \((w_i)_{i \geq 0}\) such that

\[w_{i+1} = w_i^{s_{i+1}} w_{i-1} \quad (i \geq 1).\]

Clearly, such a sequence is entirely determined by its first two elements \(w_0\) and \(w_1\). We say that a \(\psi\)-Sturmian sequence is a Sturmian sequence of angle (or slope) \(\varphi\) (where \(\varphi = [0; s_2, s_3, \ldots]\)).

Example 3.2. In the special case \(\varphi = [0; 1, 1, 1, \ldots] = 1/\gamma\) (then we have \((t_k)_{k \geq 0} = (k - 1)_{k \geq 0}\) and \(\psi(n) = n - 2\) for all \(n\)), a Sturmian sequence of angle \(1/\gamma\) in a monoid is a Fibonacci sequence according to Roy (cf at the beginning of §3 of [14]).

The following Proposition is a direct consequence of [14] Proposition 3.1.

Proposition 3.3 (Roy, 2007). There exists a non-empty Zariski open subset \(U\) of GL\(_2\)(\(\mathbb{C}\))\(^2\) such that for all \((w_0, w_1) \in U\), there is \(N \in \text{GL}_2(\mathbb{C})\) such that \(w_1 N, w_0 N\) and \(w_1 w_0 N\) are symmetric. When \(w_0\) and \(w_1\) have integer coefficients, we may take \(N\) with integer coefficients.

Proposition 3.4.

(a) Let \(w_0, w_1, N \in \text{GL}_2(\mathbb{C})\) be such that \(w_1 N, w_0 N\) and \(w_1 w_0 N\) are symmetric. Then \(w_1 w_0 N = w_0 w_1 N\).

(b) Let \((w_i)_{i \geq 0}\) be a \(\psi\)-Sturmian sequence of \(\text{GL}_2(\mathbb{C})\) and let \(N\) be \(\text{GL}_2(\mathbb{C})\). We set \(N_k = N\) if \(k\) is even, \(N_k = -N\) if \(k\) is odd. Suppose that \(w_1 w_0 N = w_0 w_1 N\). Then for all \(k \geq 1\) we have

\[w_{k-1} w_k N_{k+1} = w_k w_{k-1} N_k.\] (3.1)

Proof. For (a) we have the equality \(w_1 w_0 N = (w_1 N)^N (w_0 N)\). By taking the transpose of these expressions and using the fact that \(w_1 N, w_0 N\) and \(w_1 w_0 N\) are symmetric, we find

\[w_1 w_0 N = (w_0 N)^N (w_1 N) = w_0 w_1 N.\]

For (b) (one may compare this Proposition with Lemma 5.1 of [5]): one may adapt the arguments of the Proposition 1 of [11] to our context.

The case \(k = 1\) is true by hypothesis. Suppose that for \(k \geq 1\) one has \(w_{k-1} w_k N_{k+1} = w_k w_{k-1} N_k\). Then, one has

\[w_k w_{k+1} N_{k+2} = w_k \left( w_k^{s_k+1} w_{k-1} N_{k+2} \right) = w_k^{s_k+1} (w_k w_{k-1} N_k) = w_k^{s_k+1} (w_{k-1} w_k N_{k+1}) = w_{k+1} w_k N_{k+1},\]

and we can conclude by induction.

Definition 3.5. Let \((w_i)_{i \geq 0}\) be a \(\psi\)-Sturmian sequence in \(\text{GL}_2(\mathbb{C})\) and let \(N \in \text{GL}_2(\mathbb{C})\) be such that \(w_1 w_0 N = w_0 w_1 N\). We set \(N_k = N\) if \(k\) is even, \(N_k = -N\) if \(k\) is odd. We define a sequence
\((y_j)_{j \geq -2}\) in the following way: we set \(y_{-2} = w_0 N, y_{-1} = w_1 N\), and for \(1 \leq k, 0 \leq l < s_{k+1}\), we set
\[
y_{tk+l} = w_{k}^{l+1}w_{k-1}N_k.
\] (3.2)

(note that (3.2) remains valid for \(l = s_{k+1}\) by Eq. (3.1)).

In particular, we have
\[
y_{\psi(t_k)} = w_{k-1}N_k \quad (k \geq 1).
\] (3.3)

**Proposition 3.6.** We have the following properties:

(a) \(y_{tk+l} = w_{k}^{l+1}y_{\psi(tk)}\) for \(k \geq 1\) and \(0 \leq l \leq s_{k+1}\)

(b) \(y_{j+1} = w_ky_j\), for \(k \geq 0\) and \(t_k \leq j < t_{k+1}\)

(c) \(y_j = w_ky_{\psi(j)}\), for \(k \geq 1\), \(t_k \leq j < t_{k+1}\)

(d) For all \(j \geq 0\), we have
\[
y_{j+1} = y_j y_{\psi(j)}^{-1} y_j
\] (3.4)

In particular, if \(w_1 N, w_0 N\) and \(w_1 w_0 N\) are symmetric, then for each \(j \geq -2\), \(y_j\) is symmetric.

**Proof** The property (a) (resp. (b)) results from the definition of \(y_m\) and from (3.3).

The property (c) is implied by (b) if \(j > t_k\) (since the we have \(\psi(j) = j - 1\)) and by (a) if \(j = t_k\).

Finally, (3.4) is implied by (b) and (c). Furthermore, the property that \(w_0 N, w_1 N\) and \(w_1 w_0 N\) are symmetric is equivalent by definition to \(y_{-2}, y_{-1}\) and \(y_0\) are symmetric: we can conclude by induction using (3.4).

\[\square\]

**Remark.** Equation (3.4) may be rewritten as
\[
\det(y_{\psi(j)}) y_{j+1} = [y_j, y_j, y_{\psi(j)}],
\] (3.5)

where \([\cdot, \cdot, \cdot]\) is Roy’s bracket introduced in [13]. This bracket is used in a crucial way in [10] by Fischler (see [10, §2] and the first remark below Theorem 4.1 that may be linked to (3.5)). Equation (3.4) reflects common combinatorial properties of Sturmian words. One may for instance compare it with Eq. (4) of [10].

### 4 Admissible \(\psi\)-Sturmian sequences

In this section we establish several arithmetical properties of admissible \(\psi\)-Sturmian sequences in the monoid \(\mathcal{M}\) (see Definition 1.1), including the generalized recurrence found by Bugeaud and Laurent (cf Lemma 7.1 of [5] and its proof). We also generalize in Definition 4.2 the construction of the sequence \((z_i)\) of [14] Proposition 4.1. This sequence will give “good” solutions of Problem \(E_{\omega, \lambda}\) and plays a crucial role to build a 3-system associated to a proper \(\psi\)-Sturmian number and it allows us to calculate standard diophantine exponents \(\hat{\omega}_2, \lambda_2, \omega_2\) and \(\omega_2\). The main result of this section is Proposition 4.3.

The following definition is directly inspired by Definition 3.2 of [14].

**Definition 4.1.** Let \(\mathcal{M} = \text{Mat}_{2 \times 2}(\mathbb{Z}) \cap \text{GL}_2(\mathbb{C})\) denote the monoid of \(2 \times 2\) integer matrices with non-zero determinant. We say that a \(\psi\)-Sturmian sequence in \(\mathcal{M}\) is admissible if there is a matrix \(N \in \mathcal{M}\) such that \(w_1 N, w_0 N\) and \(w_1 w_0 N\) are symmetric. In this case, matrices of the sequence \((y_i)_{i \geq -2}\) associated by the Definition 3.5 are symmetric.

The examples of admissible sequences used by Roy in [14] lead to the construction of admissible \(\psi\)-Sturmian sequences. Details are gathered in Section 8.1. Following Roy’s approach (cf §4 of
Proposition 4.3.\footnote{\textsuperscript{14}} we identify \( \mathbb{R}^3 \) (resp. \( \mathbb{Z}^3 \)) with the space of \( 2 \times 2 \) symmetric matrices with real (resp. integer) coefficients under the map 

\[
x = (x_0, x_1, x_2) \rightarrow \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix}.
\]

If we write \( x = (x_0, x_1, x_2) \in \mathbb{R}^3 \), then set \( \det(x) = x_0x_2 - x_1^2 \) and \( \text{Tr}(x) = x_0 + x_2 \). Likewise, given three symmetric matrices \( x, y, z \), let \( x \wedge y \), \( \langle x, y \rangle \) and \( \det(x, y, z) \) denote respectively the inner product, the scalar product and the determinant of corresponding vectors of \( \mathbb{R}^3 \). We define the content of a matrix \( w \in \text{Mat}_{2 \times 2}(\mathbb{Z}) \) or of a point \( y \in \mathbb{Z}^3 \) as the greatest common divisor of their coefficients. We say that such a matrix or point is primitive if its content is 1.

**Notation.** In this section, \( (w_i)_{i \geq 0} \) is an admissible \( \psi \)-Sturmian sequence. We denote by \( N \in M \) and \( (y_j)_{j \geq -2} \) the matrices defined in Definition 3.5. Moreover let \( J \) be the matrix \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

**Definition 4.2.** We define the sequence \( (z_j)_{j \geq -1} \) in the following way:

\[
z_{tk+l} = \frac{1}{\det(w_k)} y_{\psi(tk+1)} \wedge y_{tk+l}, \quad 0 \leq k \text{ and } 0 \leq l < s_{k+1}.
\]  \hspace{1cm} (4.1)

Proposition 4.3\footnote{\textsuperscript{14}} below and Corollary 4.4 extend results of Proposition 4.1 and of Corollary 4.2 in \cite{14}.

**Proposition 4.3.**

(a) Recurrence of sequences \((\text{Tr}(w_i)), (\det(w_i))\):

\[
\text{Tr}(w_{k}^l w_{k-1}) = \text{Tr}(w_k)\text{Tr}(w_{k-1}^{l-1}) - \det(w_k)\text{Tr}(w_{k-1}^{l-2}) \quad (4.2)
\]

where \( k \geq 1 \) and \( l \geq 2 \). In particular, we have

\[
\text{Tr}(w_k) \equiv \text{Tr}(w_k^{l-1}) \text{Tr}(w_{k-1}) \mod \det(w_k) \quad (4.3)
\]

(b) Recurrence of the sequence \((y_i)_i\):

\[
y_{tk+l+1} = \text{Tr}(w_k) y_{tk+l} - \det(w_k) y_{\psi(tk+l)}, \quad 0 \leq l < s_{k+1}, \quad (4.4)
\]

with \( k \geq 1 \). In particular, the case \( k - 1 \) applied with \( l = s_k - 1 \) gives

\[
y_{tk} = \text{Tr}(w_{k-1}) y_{tk-1} - \det(w_k) y_{\psi(tk-1)}, \quad k \geq 2, \quad (4.5)
\]

(c) Recurrence of the sequence \((z_i)_i\):

\[
\begin{cases}
z_{tk+l+1} = \text{Tr}(w_k)z_{tk+l} - y_{\psi(tk+l)} \wedge y_{\psi(tk+l)} \quad (0 \leq l < s_{k+1} - 1) \\
z_{tk+1} = \text{Tr}(w_{k-1})z_{tk-1} - y_{\psi(tk)} \wedge y_{\psi(tk-1)} \quad (k \geq 2),
\end{cases}
\]

the first relation remaining valid for \( k \geq 1 \).

(d) For \( k \geq 0 \) we have

\[
\det(y_{tk-1}, y_{tk}, y_{tk+1}) = -\det(w_k) \det(y_k) \text{Tr}(JN_{k+1}),
\]

in particular, if \( N \) is not symmetric (which is equivalent to \( \text{Tr}(JN) \neq 0 \)), then \( y_{tk-1}, y_{tk} \) and \( y_{tk+1} \) are linearly independent for all \( k \geq 0 \).

(e) For \( k \geq 0 \) we have

\[
z_{tk+1} \wedge z_{tk+l} = \det(N_k) \text{Tr}(JN_{k+1}) y_{tk+l} \quad (0 \leq l < s_{k+1}). \quad (4.8)
\]
Proof Let $k \geq 1$. Following Roy’s idea and using Cayley-Hamilton Theorem for $w_k$, one finds
\[
\mathbf{w}_k^2 = \text{Tr}(\mathbf{w}_k)\mathbf{w}_k - \det(\mathbf{w}_k)\mathbf{I}_{d_2},
\]
where $\mathbf{I}_{d_2}$ is the identity matrix of $\text{Mat}_{2 \times 2}(\mathbb{R})$. Then, we prove \([4.2]\) by multiplying each side of the previous equality on the left by $\mathbf{w}_k^{l-2}$ and on the right by $\mathbf{w}_{k-1}$ (with $l \geq 2$), then by taking the trace. One gets Equation \([4.3]\) by induction by taking \([4.2]\) modulo $\det(\mathbf{w}_k)$.

For recurrence \([4.4]\), one starts with \([4.9]\), and by multiplying each side of this equality on the left by $\mathbf{w}_k^l$ (with $k \geq 1, 0 \leq l < s_{k+1}$) and on the right by $\mathbf{w}_{k-1}$, one finds
\[
\mathbf{w}_k^{l+2}\mathbf{w}_{k-1} = \text{Tr}(\mathbf{w}_k)\mathbf{w}_k^{l+1}\mathbf{w}_{k-1} - \det(\mathbf{w}_k)\mathbf{w}_k^l\mathbf{w}_{k-1},
\]
which by the definition of the $y_i$ may be rewritten as
\[
y_{t_k+l+1} = \text{Tr}(\mathbf{w}_k)y_{t_k+l} - \det(\mathbf{w}_k)\mathbf{w}_k^l y_{\psi(t_k)}.
\]
To conclude it suffices to note that for each $0 < l < s_{k+1}$, we have $y_{\psi(t_k+l)} = y_{t_k+l-1} = \mathbf{w}_k^l y_{\psi(t_k)}$. Also note that \([4.4]\) is equivalent to \((19)\) of \([13]\).

We find the first relation of \([4.6]\) by using \([4.4]\) together with the definition of $z_{t_k+l+1}$ and $y_{t_k+l+1}$. For the second relation, using \([4.4]\), we obtain by induction on $l$, $0 \leq l < s_{k+1}$,
\[
y_{t_k+l} \land y_{t_k+l+1} = \det(\mathbf{w}_k)^l y_{t_k} \land y_{t_k+1} = \det(\mathbf{w}_k)^{l+1} y_{\psi(t_k)} \land y_{t_k}.
\]
Similarly, we may show that for each point $x \in \mathbb{R}^3$,
\[
\det(x, y_{t_k+l}, y_{t_k+l+1}) = \det(\mathbf{w}_k)^l \det(x, y_{t_k}, y_{t_k+1}).
\]
Equation \((4.10)\) implies that
\[
\det(\mathbf{w}_{k+1})z_{t_{k+1}} = y_{t_{k+s_{k+1}}-1} \land y_{t_{k+s_{k+1}}} = \det(\mathbf{w}_k)^{k+1} y_{\psi(t_k)} \land y_{t_k}.
\]
If $k \geq 2$, Equation \((4.5)\) provides the relation
\[
y_{\psi(t_k)} \land y_{t_k} = \text{Tr}(\mathbf{w}_{k-1})\mathbf{z}_{t_{k+1}} - \det(\mathbf{w}_{k-1})y_{\psi(t_k)} \land y_{\psi(t_k-1)}.
\]
Finally, by using $\det(\mathbf{w}_{k+1}) = \det(\mathbf{w}_k)^{k+1} \det(\mathbf{w}_{k-1})$ we obtain
\[
z_{t_{k+1}} = \frac{\det(\mathbf{w}_k)^{k+1} \det(\mathbf{w}_{k-1})}{\det(\mathbf{w}_{k+1})} \times (\text{Tr}(\mathbf{w}_{k-1})\mathbf{z}_{t_{k+1}} - y_{\psi(t_k)} \land y_{\psi(t_k-1)}).
\]
In order to prove \((4.7)\), we use the formula $\det(x, y, z) = \text{Tr}(JxJyJz)$ (valid for all $x, y, z \in \mathbb{R}^3$; see formula (2.1) of \([13]\)). If $k \geq 1$, by noticing that $JxJx = -\det(x)\mathbf{I}_{d_2}$ for all $x \in \mathbb{R}^3$ (cf \([13]\)) we get
\[
\det(y_{t_k}, y_{t_k+1}, y_{t_k-1}) = -\det(y_{t_k+1})\text{Tr}(Jy_{t_k}y_{t_k-1}^{-1}y_{\psi(t_k+1)})
\]
\[
= -\det(\mathbf{w}_k)\text{Tr}(Jy_{t_k}(\mathbf{w}_k y_{t_k})^{-1}(\mathbf{w}_k N_{k+1}))
\]
\[
= -\det(\mathbf{w}_k)\text{det}(y_{t_k})\text{Tr}(JN_{k+1}).
\]
We must eventually show \((4.8)\). Let $k, l$ be integers, with $k \geq 0$ and $0 \leq l < s_{k+1}$. Equation \((4.10)\) (for the two cases $s_{k+1} - 1$ and $l$) provides the identity
\[
y_{t_{k+1}} \land y_{t_{k+1}} = \det(\mathbf{w}_k)^{k+1-l-1}y_{t_k+l} \land y_{t_{k+l+1}},
\]
and by using identity $(x \land y) \land (y \land z) = \det(x, y, z)y$ (valid for all points $x, y, z$ of $\mathbb{R}^3$), we get
\[
\det(\mathbf{w}_{k+1})\det(\mathbf{w}_k)z_{t_{k+1}} \land z_{t_{k+1}} = (y_{t_{k+1}} \land y_{t_{k+1}}) \land (y_{t_{k+1}} \land y_{t_{k+1}})
\]
\[
= -\det(\mathbf{w}_k)^{k+1-l-1} \det(y_{t_{k+1}}, y_{t_k+l+1}y_{t_k+1})y_{t_k+l}
\]
\[
= -\det(\mathbf{w}_k)^{k+1-l-1} \det(y_{t_{k+1}}, y_{t_k+1}y_{t_k+l})y_{t_k+l},
\]
using \((4.11)\) with $(x = y_{t_{k+1}})$ for the last equality. Finally we end the proof for \((4.8)\) by using \((4.7)\) and identity $\det(\mathbf{w}_{k+1}) = \det(\mathbf{w}_k)^{k+1} \det(\mathbf{w}_{k-1})$. □
Corollary 4.4. Suppose that $\text{Tr}(w_i^l w_0)$ and $\det(w_i^l w_0)$ are relatively prime for $i = 0, 1, \ldots, s_2+1$, as well as $\text{Tr}(w_i)$ and $\det(w_i)$. Then

(a) for all $k \geq 1$, $0 \leq l \leq s_{k+1} + 1$, $\text{Tr}(w_k^l w_{k-1})$ and $\det(w_k^l w_{k-1})$ are relatively prime,

(b) for all $k \geq 1$, $0 \leq l \leq s_{k+1} + 1$, the matrix $w_k^l w_{k-1}$ is primitive,

(c) the content of $y_j$ divides $\det(N)$ $(j \geq -2)$,

(d) the point $\det(w_2)z_j$ belongs to $\mathbb{Z}^3$ $(j \geq -1)$, and its content divides $\det(w_2)^2 \det(N)^2 \text{Tr}(JN)$.

Proof We adapt the structure of [14, proof of Corollary 4.2] to our context. We show $[a]$ by induction. Let $\mathcal{P}(k)$ be the property “ $\text{Tr}(w_k^l w_{k-1})$ and $\det(w_k^l w_{k-1})$ are relatively prime for $l = 0, \ldots, s_{k+1} + 1$, as well as $\text{Tr}(w_k)$ and $\det(w_k)$ ”. By hypothesis $\mathcal{P}(1)$ is true. Suppose that $\mathcal{P}(k_0)$ is true for a fixed $k_0 \geq 1$. Then equations provided by $\mathcal{P}(k_0)$ for cases $l = s_{k_0} + 1$ and $l = s_{k_0} + 1$ ensure that $\text{Tr}(w_{k_0}^l w_{k_0})$ and $\det(w_{k_0}^l w_{k_0})$ are relatively prime, as well as $\text{Tr}(w_{k_0} w_{k_0})$ and $\det(w_{k_0} w_{k_0})$. By [4.3], we have

$$\text{Tr}(w_{k_0}^l w_{k_0}) \equiv \text{Tr}(w_{k_0} w_{k_0})^{l-1} \text{Tr}(w_{k_0+1} w_{k_0}) \mod \det(w_{k_0+1}) \quad (l \geq 2).$$

Now it suffices to notice that for each $l \geq 2$, $\det(w_{k_0+1} w_{k_0})$ has the same prime factors than $\det(w_{k_0+1})$ (and than $\det(w_l)$), therefore the previous equation ensures that $\text{Tr}(w_{k_0+1}^l w_{k_0})$ and $\det(w_{k_0+1}^l w_{k_0})$ are relatively prime. Thus $\mathcal{P}(k_0 + 1)$ is true, which completes the induction step.

The assertion [b] results directly from [a] since the content of a matrix $w$ always divides its trace and its determinant.

Let $k, l$ be integers with $k \geq 0$ and $0 \leq l < s_{k+1}$. By definition, we have $y_{t_{k+l}} = w_{k+1}^l w_{k+1-1} N_k$, thus

$$y_{t_{k+l}} \text{Adj}(N_k) = \det(N)w_{k+1}^{l+1} w_{k+1-1},$$

where $\text{Adj}(N_k)$ denotes the transpose of the comatrix of $N_k$. In particular, since $w_{k+1}^{l+1} w_{k+1-1}$ is primitive, the content of $y_{t_{k+l}}$ divides $\det(N)$ (the same argument enables us to deal with cases $y_{l-1}$ and $y_{l-2}$) which prove [c].

Finally we have to prove $[d]$. The fact that $\det(w_2)z_{t_{k+l}}$ belongs to $\mathbb{Z}^3$ for $k = 0, 1, 2$ and $0 \leq l < s_{k+1}$ is immediate by the definition of $z_{t_{k+l}}$ and because $\det(w_1)$ and $\det(w_0)$ divide $\det(w_2)$. The recurrences of [4.6] imply that $\det(w_2)z_j \in \mathbb{Z}^3$ for each $j \geq 0$. Moreover, the content of $\det(w_2)z_{t_{k+l}}$ divides that of $\det(w_2)^2 z_{t_{k+l}} \wedge z_{t_{k+l}}$, thus by [4.8] and [c] it divides $\det(w_2)^2 \det(N)^2 \text{Tr}(JN)$.

$\square$

5 $\psi$-Sturmian sequences with multiplicative growth

In this section we show that under a multiplicative growth hypothesis, there exists a real number $\delta \geq 0$ such that $|\det(\omega_k)| \leq ||\omega_k||^\delta$ (see Proposition 5.6). This number $\delta$ will play a crucial role to determine the Diophantine exponents associated with $\xi$ and it is needed to define proper $\psi$-number.

Definition 5.1. We denote by $||w||$ the norm of a matrix $w \in \text{Mat}_{2 \times 2}(\mathbb{R})$, defined as the maximum of the absolute values of its coefficients.

Let $(w_i)_{i \geq 0}$ be a $\psi$-Sturmian sequence in $\text{GL}_2(\mathbb{R})$. We state that $(w_i)_{i}$ has a multiplicative growth if there exist two positive constants $c_1$ and $c_2$ such that for each integer $k \geq 1$ and for each $1 \leq l \leq s_{k+1} + 1$, we have

$$c_1 ||w_k|| \times ||w_k^{-1} w_{k-1}|| \leq ||w_k^l w_{k-1}|| \leq c_2 ||w_k|| \times ||w_k^{-1} w_{k-1}||,$$

which could be more concisely rewritten as

$$||w_k^l w_{k-1}|| \asymp ||w_k|| \times ||w_k^{-1} w_{k-1}||$$

for $k \geq 1$ and $1 \leq l \leq s_{k+1} + 1$. 
The next Lemma is implied by the proof of [14 Lemma 5.1].

**Lemma 5.2.** Let $w_0, w_1 \in \text{GL}_d(\mathbb{R})$ be two matrices of the form \[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}
\] with $1 \leq a \leq \min\{b, c\} \leq \max\{b, c\} \leq d$. Then, the $\psi$-Sturmian sequence defined by $w_0$ and $w_1$ has multiplicative growth and we may take $c_1 = 1$ and $c_2 = 2$ in \eqref{5.1}.

**Lemma 5.3** (Bugeaud-Laurent (2005)). Let $(X_n)_{n \geq 0}$ be a sequence of positive real numbers which tends to infinity and $(s_n)_{n \geq 1}$ be a sequence of positive integers. Suppose that there exists $c \geq 1$ such that
\[
 c^{-s_{n+1}} X_n^{s_{n+1}} X_{n-1} \leq X_{n+1} \leq c^{s_{n+1}} X_n^{s_{n+1}} X_{n-1} \quad (n \geq 1).
\]
For $n \geq 0$, we define $\mu_n$ by the formula $X_{n+1} = X_n^{\mu_n}$. Then, as $n$ tends to infinity we have
\[
\mu_n = [s_{n+1}; s_n, \ldots, s_1] (1 + o(1)).
\]

**Proof** Cf [5 Lemma 4.1].

The next lemma is a technical result that we will use in order to get asymptotical information on $\psi$-Sturmian sequences with multiplicative growth. It will be used in particular for the proof of Proposition \ref{5.5}.

**Lemma 5.4.** Let $(X_n)_{n \geq 0}$ be a sequence of positive real numbers which satisfies the hypothesis of Lemma \ref{5.3}. Then
\[
\begin{align*}
(a) \quad & \text{The sequence } (\log(X_n))_n \text{ strictly increases for } n \text{ large enough.} \\
(b) \quad & \text{We have } \liminf_{n \to +\infty} \frac{\log(X_{n+2})}{\log(X_n)} > 1. \\
(c) \quad & \text{There exists } \lambda > 1 \text{ such that } \log(X_n) \gg \lambda^n. \\
(d) \quad & \text{The sequence } (1/\log(X_n))_n \text{ is a summable sequence and as } n \text{ tends to infinity}
\sum_{k \geq n} \frac{1}{\log(X_k)} \asymp \frac{1}{\log(X_n)}.
\end{align*}
\]

**Proof** First we prove \[a\]. Since the sequence $(\log(X_n))_n$ tends to $+\infty$, there exists $N \geq 1$ such that for each $n \geq N$, $\log(X_{n-1}) + \log(c) > 0$, where $c$ is the constant involved in Lemma \ref{5.3}. For all $n \geq N$, by the growth hypothesis on $(X_n)_n$ we have
\[
\log(X_{n+1}) \geq s_{n+1} (\log(X_n) + \log(c)) + \log(X_{n-1}) > \log(X_n),
\]
with the choice of $N$. Thus the sequence $(\log(X_n))_n$ strictly increases starting index $N$. For the assertion \[b\] we define $\mu_n$ by the formula $\log(X_{n+1}) = \mu_n \log(X_n)$. According to Lemma \ref{5.3} we have $\mu_n = (1 + o(1))[s_{n+1}; s_n, \ldots, s_1]$. Then, we conclude by noticing that
\[
\frac{\log(X_{n+2})}{\log(X_n)} = \mu_{n+1} \mu_n \sim [s_{n+2}; s_{n+1}, \ldots, s_1][s_{n+1}; s_n, \ldots, s_1] \geq \frac{3}{2}
\]
We may deduce from this the existence of $\lambda > 1$ such that
\[
\liminf_{n \to +\infty} \frac{\log(X_{n+2})}{\log(X_n)} > \lambda^2 > 1,
\]
and such a real $\lambda$ satisfies $\log(X_n) \gg (\lambda^2)^{n/2} = \lambda^n$, hence \[c\]. In particular, the sequence $(1/\log(X_n))_n$ is summable. Set
\[
R_n = \sum_{k \geq n} \frac{1}{\log(X_k)} < +\infty.
\]
For $n$ large enough we have the trivial lower bound $R_n \geq 1/\log(X_n)$. On the other hand, for $n$ large enough and choosing $\lambda$ satisfying (5.2) we have

$$R_n \leq \frac{1}{\log(X_n)} + \frac{1}{\log(X_{n+1})} + \sum_{k \geq n} \frac{1}{\log(X_k)} \leq \frac{2}{\log(X_n)} + \frac{1}{\lambda^2} R_n,$$

hence $(1 - \frac{1}{\lambda^2}) R_n \leq 2/\log(X_n)$, which concludes the proof of the assertion (d). □

The following property is crucial to build the 3-system of Section 7 (cf Figure 2) and to compute the exponents $\hat{\lambda}_2, \lambda_2, \hat{\omega}_2$ and $\omega_2$ associated to a $\psi$-Sturmian number defined in Proposition 6.1.

**Proposition 5.5.** Let $(w_k)_{k \geq 0}$ be a $\psi$-Sturmian sequence in $\text{GL}_2(\mathbb{R})$ with multiplicative growth such that $(\|w_k\|)_{k}$ tends to infinity, and let $(D_k)_{k \geq 0}$ be a sequence of real numbers $\geq 1$ satisfying the recurrence

$$D_{k+1} = D_k^{s_{k+1}} D_{k-1} \quad \text{for all } k \geq 1. \quad (5.3)$$

Then, there exists a real number $\delta \geq 0$ such that

$$D_k \asymp \|w_k\|^\delta.$$

In particular there exists a sequence $(\hat{W}_k)_{k \geq 0}$ satisfying (5.3) and such that

$$\hat{W}_k \asymp \|w_k\|.$$

**Proof** For each $k \geq 0$, we write $W_k = \|w_k\|$ and we choose a sequence $(D_k)_{k \geq 0}$ of real numbers $\geq 1$ satisfying the recurrence (5.3). Omitting a finite number of initial terms if necessary, we may suppose that $W_k \geq 2$ for each $k$. By multiplicative growth, the sequence $(W_k)_k$ satisfies the hypothesis of Lemma 5.4. First, we prove that

$$\log(D_k) = O(\log(W_k)). \quad (5.4)$$

By multiplicative growth, there is a constant $M > 0$ such that

$$s_{k+1}(\log(W_k) - M) + \log(W_{k-1}) \leq \log(W_{k+1}), \quad (5.5)$$

for each $k$ large enough. According to the assertion (e) of Lemma 5.4 there exists $k_0 \geq 1$ such that

$$(1 - \frac{1}{k}) \log(W_k) \leq (1 - \frac{1}{k+1}) \log(W_k) - M, \quad (5.6)$$

for all $k \geq k_0$. Now, let $C > 0$ be such that for $k = k_0 - 1, k_0$

$$C \log(D_k) \leq (1 - \frac{1}{k}) \log(W_k). \quad (5.7)$$

By induction we show that (5.7) remains valid for each $k \geq k_0$. Suppose that it is true for $k - 1$ and $k$. Then

$$C \log(D_{k+1}) = s_{k+1} C \log(D_k) + C \log(D_{k-1})$$

$$\leq s_{k+1} \left(1 - \frac{1}{k}\right) \log(W_k) + \left(1 - \frac{1}{k-1}\right) \log(W_{k-1})$$

$$\leq s_{k+1} \left(1 - \frac{1}{k+1}\right) \log(W_k) - M + \left(1 - \frac{1}{k+1}\right) \log(W_{k-1})$$

$$\leq \left(1 - \frac{1}{k+1}\right) \log(W_{k+1});$$

indeed, the first inequality is given by the recurrence hypothesis (for $k$ and $k - 1$). The second one is provided by (5.6) and the third one by (5.5). Therefore (5.7) is true for each $k \geq k_0$; this proves in particular Equation (5.4).

Now, we define $\varepsilon_k$ by

$$\varepsilon_k = \log(W_k) \times \left(\frac{\log(D_{k+1})}{\log(W_{k+1})} - \frac{\log(D_k)}{\log(W_k)}\right).$$
and we shall prove that $\varepsilon_k = O(1)$. For all $k \geq 1$, by using $\log(W_{k+1}) = s_{k+1} \log(W_k) + \log(W_{k-1}) + O(s_{k+1})$, we obtain

$$
\varepsilon_k = \frac{\log(W_k) \log(D_{k+1}) - \log(D_k) \log(W_{k-1})}{\log(W_{k+1})} + O\left(\frac{s_{k+1} \log(D_k)}{\log(W_{k+1})}\right)
$$

$$
= -\frac{\log(W_k)}{\log(W_{k+1})} \varepsilon_{k-1} + O(1),
$$

since $s_{k+1} \log(D_k)/\log(W_{k+1}) \leq \log(D_{k+1})/\log(W_{k+1}) = O(1)$ by \((5.4)\). This directly implies $\varepsilon_k = \log(W_k)/\log(W_{k+1}) \varepsilon_{k-1} + O(1)$. By the assertion \((b)\) of Lemma \(5.4\), there are $\mu > 0$ and $k_0 \geq 1$ such that $\log(W_k)/\log(W_{k+1}) < \mu < 1$ for each $k \geq k_0$. We deduce from the previous estimates that there is a constant $C > 0$ such that $|\varepsilon_{k+2}| \leq \mu |\varepsilon_k| + C$, for all $k$. This implies that $\varepsilon_k = O(1)$ since $\mu < 1$.

Finally, since $1/\log(W_k)$ is summable by the assertion \((d)\) of the Lemma \(5.4\) the same goes for

$$
\frac{\varepsilon_k}{\log(W_k)} = \frac{\log(D_{k+1})}{\log(W_{k+1})} - \frac{\log(D_k)}{\log(W_k)},
$$

and we obtain the existence of a real number $\delta \geq 0$ such that $\log(D_k)/\log(W_k)$ tends to $\delta$ as $k$ tends to infinity. Furthermore we have

$$
\left|\delta - \frac{\log(D_k)}{\log(W_k)}\right| \leq \left|\sum_{j \geq k} \frac{\varepsilon_j}{\log(W_j)}\right| \leq O\left(\sum_{j \geq k} \frac{1}{\log(W_j)}\right) \leq O\left(\frac{1}{\log(W_k)}\right).
$$

Finally, we have shown that $\log(D_k) = \delta \log(W_k) + O(1)$, which proves the first part of this Lemma.

For the second assertion, it suffices to take a sequence $(D_k)_k$ satisfying the recurrence \((5.3)\) with first terms $D_0, D_1 > 1$. Such a sequence tends to $+\infty$, and so the real number $\delta \geq 0$ previously defined cannot be zero. Then, it suffices to set $W_k := D_k^{1/\delta}$.

**Proposition 5.6.** Let $(w_k)_{k \geq 0}$ be a $\psi$-Sturmian sequence in $GL_2(\mathbb{R})$ with multiplicative growth such that $(\|w_k\|)_k$ tends to infinity. Then, there exists $0 \leq \delta \leq 2$ such that

$$
|\det(w_k)| \asymp \|w_k\|^{\delta}.
$$

Let $c_1, c_2$ be the implicit constants of the multiplicative growth such that \((5.1)\) is satisfied. Suppose that there exists $\alpha, \beta \geq 0$ such that the relation

$$
(c_2 \|w_k\|^\alpha) \leq |\det(w_k)| \leq (c_1 \|w_k\|^\beta),
$$

is true for $k = 0, 1$. Then this relation remains true for all $k \geq 0$; in particular we have the estimates

$$
\alpha \leq \delta \leq \beta.
$$

**Proof** Proposition 5.5 applied with the sequence $(D_k)_{k \geq 0} = (|\det(w_k)|)_{k \geq 0}$ provides the existence of $\delta \geq 0$ such that $|\det(w_k)| \asymp \|w_k\|^{\delta}$. Moreover, since $|\det(w_k)| \leq 2 \|w_k\|^2$, we always have $\delta \leq 2$.

In order to prove the second assertion, we follow Roy’s arguments (cf [14, Proposition 5.3]) and we proceed by induction, \((5.8)\) being true for $k = 0, 1$. Let us recall that the multiplicative growth gives

$$
(c_1 \|w_{k+1}\|^{x_{k+2}} \|w_k\| \leq \|w_{k+2}\| \leq (c_2 \|w_{k+1}\|^{x_{k+2}} \|w_k\|) \quad (k \geq 0).
$$

Suppose that \((5.8)\) is true for $k = j$ and $k = j + 1$ (for an index $j \geq 0$). Then we have

$$
|\det(w_{j+2})| = |\det(w_{j+1})|^{x_{j+2}} |\det(w_{j+1})| \leq \left(c_1 \|w_{j+1}\|^{x_{j+2}} (c_1 \|w_{j+1}\|)\right)^\beta \leq (c_1 \|w_{j+2}\|)^\beta,
$$

and similarly $|\det(w_{j+2})| \geq (c_2 \|w_{j+2}\|)^\alpha$. Thus \((5.8)\) is valid for $k = j + 2$, which concludes the induction.

**Remark.** In section \(8.1\) we will investigate Roy’s examples (cf [14, Example 5.4]) and show that they satisfy \((5.8)\) (which corresponds to (12) of [14]).
6 Construction of Sturmian type numbers

The main goal of this section is to establish the existence of the $\psi$-number $\xi$ (Proposition 6.1) and to estimate precisely quantities $|x|, |y|, |y^\perp y|, |y :: z|$ (where $y = (1, \xi, \xi^2)$) in Proposition 6.5. These estimates will be used to construct the 3-system $P$ of Section 7.2 (see Figure 2). The Diophantine exponents associated with $\xi$ are determined thanks to this 3-system. The definition of Sturmian type number is given in Definition 6.2.

Notation. In this section, $(w_i)_{i \geq 0}$ is an admissible $\psi$-Sturmian sequence in $\mathcal{M}$ with multiplicative growth such that $(\|w_i\|)_i$ tends to $+\infty$. We denote by $N$, $(y_i)_{i > 2}$ and $(z_i)_{i > 2}$ the matrix in $\text{GL}_2(\mathbb{R})$ and the two sequences of symmetric matrices which are associated to $(w_i)_{i \geq 0}$ by Definition 3.5 and by 1.1. We assume that $\text{Tr}(JN) \neq 0$ (i.e. $N$ is not symmetric). Finally $\delta \geq 0$ is the exponent given by Proposition 6.6 and satisfying

$$|\det(w_i)| = \|w_i\|^\delta.$$  

Proposition 6.1. Assume $\delta < 2$. Then there exists $y \in \mathbb{R}^3 \setminus \{0\}$ such that $\det(y) = 0$ and

$$\|y_i \wedge y\| \leq \frac{|\det(y_i)|}{\|y_i\|}.$$  

Moreover, if $\delta < 1$, the coordinates of $y$ are linearly independent over $\mathbb{Q}$ and we may assume $y = (1, \xi, \xi^2)$ for a real number $\xi$ satisfying $[\mathbb{Q}(\xi) : \mathbb{Q}] > 2$. Such a number $\xi$ is called a $\psi$-Sturmian number. If $\delta < \sigma/(1 + \sigma)$, where $\sigma$ is defined by

$$\sigma = \frac{1}{\lim \sup_{k \to \infty} [s_{k+1}; s_k, \ldots, s_1]},$$

and if the content of $y_i$ is bounded, then we say that $\xi$ is a proper $\psi$-Sturmian number.

Definition 6.2. We say that a real number $\xi$ is of Sturmian type if there exists $\psi \in \mathcal{F}_{\text{star}}$ such that $\xi$ is a proper $\psi$-Sturmian number.

Proof We adapt the proof of [14] Proposition 6.1] to our context. Let $i \geq 0$ be an integer and let us write $i = t_k + l$ with $k \geq 0$ and $0 \leq l < s_{k+1}$. First, we have

$$\|y_i \wedge y_{i+1}\| \leq |\det(y_i)| \times \|w_k\|.$$  

(6.1)

Indeed, as Roy noticed, one has

$$\|x \wedge z\| \leq 2\|xJz\| \quad \forall x, z \in \mathbb{R}^3,$$  

(6.2)

since the coefficients of the diagonal of $xJz$ coincide with the first and third coefficients of $x \wedge z$, while the sum of the coefficients of $xJz$ outside of the diagonal is the middle coefficient of $x \wedge z$ multiplied by $-1$. But according to Proposition 3.6(b) and using the fact that $y_{i+1}$ and $y_i$ are symmetric, we have

$$\|y_i J y_{i+1}\| = \|y_i J (y_i^t w_k)\| = |\det(y_i) J^t w_k| = |\det(y_i)| \|w_k\|$$

(recall the formula $xJx = \det(x)J$ for each point $x \in \mathbb{R}^3$). By using both the previous estimate and (6.2) with $x = y_i$ and $z = y_{i+1}$, we find [6.4], which one may compare with (16) of [14]. We define $\delta_i = |\det(y_i)|/\|y_i\|^2$. We claim that there exists $c > 0$ such that

$$\text{dist}(y_i, y_{i+1}) \leq c \delta_i$$  

(6.3)

(one may compare this estimate with (19) of [14]). Indeed, by (3.2), by multiplicative growth and with $c_1, c_2 > 0$ denoting the constants of (5.1), we have

$$\|y_{i+1}\| = \|w_k^{l+2} w_{k-1} N_k\| \leq 2 \|N\| \|w_k^{l+2} w_{k-1}\|$$

$$\leq 2c_2 \|N\| \|w_k\| \|w_k^{l+1} w_{k-1}\|$$

$$\leq (4c_2 \|N\| \|N^{-1}\|) \|w_k\| \|w_k^{l+1} w_{k-1} N_k\|$$

$$= (4c_2 \|N\| \|N^{-1}\|) \|w_k\| \|y_i\|,$$
Therefore the sequence \( t \) tends to infinity, since we complete the proof of (6.3) by applying (6.1). Furthermore, the ratio \( \xi \) denoting the second coordinate of \( y \) for \( i \) large enough. This implies that \( t \) tends to infinity; we deduce by continuity that \( \delta + 1/\delta \) tends to 0 as \( i \) tends to infinity, since

\[
\frac{\delta + 1}{\delta} = \left( \frac{\|y\|}{\|y\|_1} \right)^2 \left| \frac{\det(y_{i+1})}{\det(y_i)} \right| = \left| \frac{\det(w_k)}{\|w_k\|^2} \right| \rightarrow \|w_k\|^{-2} \quad (k \rightarrow \infty) \text{ (because } \delta < 2). 
\]

We follow Roy’s argument: by the above, there is an index \( i_0 \) such that \( \delta + 1/\delta \) tends to 0 for each \( i \geq i_0 \). Then we find for each \( j > i > i_0 \)

\[
\text{dist} (y_i, y_j) \leq \sum_{m=i}^{j-1} \text{dist} (y_m, y_{m+1}) \leq c \sum_{m=i}^{j-1} \delta_m \leq 2c \delta_i.
\]

Therefore the sequence \( (\|y_i\|)_{i \geq 0} \) converges in \( \mathbb{R}^2 \) to a point \( [y] \) for some non-zero \( y \in \mathbb{R}^3 \). Additionally, the ratio \( \delta \) is \( |\det(y_i)|/\|y_i\|^2 \) and depends only on the class \( [y_i] \) of \( y_i \) in \( \mathbb{R}^2 \) (and tends to 0 as \( t \) tends to infinity; we deduce by continuity that \( |\det(y)|/\|y\|^2 \) is essentially a consequence of Proposition 6.1 which provides the estimate \( \|y_i \wedge y\| \leq \|\det(y_i)\|/\|y_i\| \). As for \( \delta \), if we set \( \delta_i = |\det(y_i)|/\|y_i\| \), then we have \( \delta_{i+1}/\delta_i = |\det(w_k)|/\|w_k\| \|w_k\|^{-1} \), which tends to 0 as \( i \) tends to infinity. Thus the same goes for \( \delta_i \).

\[
2\|u\|/\|y_i \wedge y\| \geq \|u, y_i \|^2 - (u, y_i) y_i = \|u, y_i \|/\|y\|
\]

for each \( i \geq 0 \). Yet, by the above, the left side tends to 0 and the integer \( (u, y_i) \) is necessarily zero for \( i \) large enough. This implies that \( u = 0 \) since by hypothesis \( \text{Tr}(JN) \neq 0 \) implies that \( y_{t_k} \) and \( y_{t_k+1} \) are linearly independent for each \( k \) (thanks to (4.3) of Proposition 4.3). Thus, the coordinates of \( y \) are linearly independent over \( Q \). In particular, the first coordinate of \( y \) is not zero, and by dividing \( y \) by this coordinate, we may assume that it is equal to 1. In this case, by denoting \( \xi \) the second coordinate of \( y \), the condition \( \det(y) = 0 \) implies \( y = (1, \xi, \xi^2) \). The linearly independence over \( Q \) of these three numbers ensures that \( |Q(\xi) : Q| > 2 \).

\[
\Box
\]

**Corollary 6.3.** Assume \( \delta < 1 \) and let \( y = (1, \xi, \xi^2) \) be the vector given by Proposition 6.1 Then

\[
\|y_i \wedge y\| \geq \frac{|\det(y_i)|}{\|y_i\|}.
\]

Moreover

\[
\|y_{i+1}\| \asymp \|w_k\| \|y_i\| \text{ and } \|y_{i+1} \wedge y\| \asymp \|w_k\|^\delta \|y_i \wedge y\|, 
\]

with \( i \geq 0 \) and \( k \) such that \( t_k \leq i < t_{k+1}. \) In particular, as \( i \) tends to infinity we have \( \|y_i\| = o(\|y_{i+1}\|) \) and \( \|y_{i+1} \wedge y\| = o(\|y_i \wedge y\|). \)

**Proof** The estimate of \( |y_i \wedge y| \) is essentially a consequence of Proposition 6.1 which provides the estimate \( |\det(y_i)| \gg \|y_i\| \|y \wedge y\| \). Write \( y_i = (y_i, y_{i+1}, y_{i+2}) \). We conclude in a classical manner that

\[
\det(y_i) = \begin{vmatrix}
  y_{i,0} & y_{i,1} & y_{i,2} \\
  y_{i,0} & y_{i,1} - \xi y_{i,0} & y_{i,1} - \xi y_{i,2} \\
  y_{i,1} & y_{i,2} & y_{i,1} - \xi y_{i,0}
\end{vmatrix} = y_{i,0}(y_{i,2} - \xi y_{i,1}) - y_{i,1}(y_{i,1} - \xi y_{i,0}),
\]

which gives \( |\det(y_i)| \ll \|y_i\| \|y_i \wedge y\| \), hence the announced result.

Equation (6.3) yields directly \( \|y_{i+1}\| \gg \|w_k\| \|y_i\| \), and using the relations \( \det(y_{i+1}) = \det(w_k) \det(y_i) \), \( \det(w_k) \approx \|w_k\|^\delta \) and the first part of the corollary, we find the last estimate: \( \|y_{i+1} \wedge y\| \asymp \|w_k\|^\delta \|y_i \wedge y\|. \)

\[
\Box
\]
Lemma 6.4. Assume that the sequence \((y_i)_{i \geq 2}\) tends in \(\mathbb{P}^2(\mathbb{R})\) to \(y = (1, \xi, \xi^2)\) \((\xi \neq 0)\) and that\[
\|y_i\| = o(\|y_{i+1}\|) \quad \text{and} \quad \|y_{i+1} \wedge y\| = o(\|y_i \wedge y\|)\]as \(i\) tends to infinity. For each \(i, j\) large enough, \(i > j\), we have\[
\frac{\|y_i \wedge y_j\|}{\det(y_j)} \leq \|y_j^{-1}\| \leq \frac{\|y_j \wedge y\|}{\det(y_j)} \leq \frac{\|y_i\|}{\|y_j\|}.
\] (6.7)

**Proof.** The last estimate of (6.7) is a direct consequence of Corollary 6.3. For each \(i \geq -2\), write \(y_i = (y_{0,i}, y_{1,i}, y_{2,i})\) and set\[
L_i^{(1)} = y_{0,i} \xi - y_{1,i} \quad \text{and} \quad L_i^{(2)} = y_{0,i} \xi^2 - y_{2,i}.
\]
Note that \(y_{i,\xi} - y_{2,i} = L_i^{(2)} - \xi L_i^{(1)}\) and that \(\|y_i \wedge y\| \approx \max(\|L_i^{(1)}\|, \|L_i^{(2)}\|)\). Write \(w := \det(y_j) y_j^{-1} = \begin{pmatrix} w_{0,0} & w_{0,1} \\ w_{1,0} & w_{1,1} \end{pmatrix}\). We have\[
w = \begin{pmatrix} y_{0,i} y_{1,j} - y_{1,i} y_{0,j} - y_{0,j} y_{1,j} \\ y_{1,i} y_{2,j} - y_{2,i} y_{1,j} - y_{2,j} y_{1,i} \end{pmatrix}
= \begin{pmatrix} y_{0,i} (\xi L_j^{(1)} - L_j^{(2)}) + y_{1,j} L_j^{(1)} \\ y_{1,i} (\xi L_j^{(1)} - L_j^{(2)}) + y_{0,i} L_j^{(1)} - y_{0,j} (L_j^{(2)} - \xi L_j^{(1)}) \end{pmatrix},\]the last equality being obtained by rewriting the coefficients of \(w\) using determinants, for instance\[
w_{0,0} = \begin{vmatrix} y_{0,i} & y_{1,i} \\ y_{1,i} & y_{2,i} \end{vmatrix} = \begin{vmatrix} y_{0,i} & y_{1,i} \\ y_{1,i} - \xi y_{0,i} & y_{2,i} - \xi y_{1,i} \end{vmatrix} = \begin{vmatrix} y_{0,i} & y_{1,j} \\ -L_j^{(1)} & -L_j^{(2)} \end{vmatrix} = y_{0,i} (\xi L_j^{(1)} - L_j^{(2)}) + y_{1,j} L_j^{(1)}\]
and we proceed in a similar manner to express the three remaining coefficients of the matrix \(w\). Since \(\|y_j\| \|y_i \wedge y\| = o(\|y_j\| \|y_i \wedge y\|)\) as \(j\) tends to infinity uniformly with respect to \(i\) such that \(i > j\), Equation (6.8) directly implies that\[
\|w\| \ll \|y_j\| \|y_i \wedge y\|
\]for \(i > j\) with \(i, j\) large enough.

In the view of the form of the coefficient \(w_{0,1}\) in (6.8) and the growth hypothesis given by (6.6), we may show that if \(\|\xi\| L_j^{(1)} \geq |L_j^{(2)}|/2\), then there exists \(c' > 0\), a constant which depends only on \(\xi\), such that for each \(j\) large enough, we have\[
\|w\| \geq |w_{0,1}| \geq c' \|y_j\| \|y_i \wedge y\|.
\]

Similarly, if \(|\xi\| L_j^{(1)} \leq |L_j^{(2)}|/2\), considering this time the coefficient \(w_{1,0}\) of (6.8) we may show that the existence of \(c'' > 0\), a constant which depends only on \(\xi\), such that\[
\|w\| \geq |w_{1,0}| \geq c'' \|y_i\| \|y_j \wedge y\|.
\]
This ensures \(\|w\| \gg \|y_i\| \|y_j \wedge y\|\) as \(j\) tends to infinity uniformly with respect to \(i > j\), and thus \(\|w\| \gg \|y_i\| \|y_j \wedge y\|\) under the same conditions. This is equivalent to the mid estimation of (6.7). Moreover according to the previous consideration, either \(|w_{0,1}| \geq c' \|y_i\| \|y_j \wedge y\|\), or \(|w_{1,0}| \geq c'' \|y_i\| \|y_j \wedge y\|\). Thus \(\|w\| \approx \max(|w_{0,1}|, |w_{1,0}|)\). But we have the identity\[
y_i \wedge y_j = (w_{0,0}, w_{1,1} - w_{0,0}, -w_{0,1}),
\]thus, since up to a sign \(|w_{0,1}|\) and \(|w_{1,0}|\) are two coefficients of \(y_i \wedge y_j\), this ensures that \(\|y_i \wedge y_j\| \gg \|w\|\). On the other hand, the previous formula of \(\|y_i \wedge y_j\|\) directly provides estimate \(\|y_i \wedge y_j\| \ll \|w\|\). Finally \(\|y_i \wedge y_j\| \ll \|w\| = |\det(y_j)| \|y_j^{-1}\|\) for \(j\) large enough, \(i > j\), which ends the proof of this lemma.
Estimates of the next proposition were established by Roy for the Fibonacci case, although they were formulated in a slightly different way (see Proposition 6.1 of [14] and its proof). They are at the center of the construction of a 3-system representing a \( \psi \)-Sturmian number (cf Figure 2).

**Proposition 6.5.** Assume \( \delta < 1 \) and denote by \( y = (1, \xi, \xi^2) \) the vector given by Proposition 6.1. We thus have the following estimates

\[
\begin{align*}
(a) \quad \|y_1 \wedge y\| & = \frac{|\det(y_1)|}{|y_1|}, \\
(b) \quad \|y_{i+1}\| & = \|y_i\|^2 \|y_\psi(i)\|^{-1}, \\
(c) \quad \|z_i\| & \approx \|y_\psi(i)\|, \\
(d) \quad |\langle z_i, y_{i+1}\rangle| & \approx |\det(y_i)|, \\
(e) \quad |\langle z_i, y_i\rangle| & \approx \frac{|\det(y_i)|}{\|y_{i+1}\|}.
\end{align*}
\]

**Proof.** The assertion (a) is provided by Corollary 6.3. For (b), considering (3.4) it suffices to apply Lemma 6.4 with \((i + 1, i)\) and \((i, \psi(i))\):

\[
\|y_{i+1}\| \|y_i\|^{-1} = \|y_{i+1}y_i^{-1}\| = \|y_iy_\psi(i)^{-1}\| \approx \|y_i\| \|y_\psi(i)\|^{-1}.
\]

Now, fix \( k \geq 0 \) and \( 0 \leq l < s_k+1 \). For (c), by (6.7) (by taking the transpose), we have

\[
|\det(w_k)| \|z_{t_k+l}\| \geq |\det(y_\psi(t_{k+1}))| \|y_\psi(t_{k+1})^{-1}y_{t_k+l}\|
\]

\[
\approx |\det(y_\psi(t_{k+1}))| N_{k+1}^{-1}w_{k+1}w_k^{-1}N_k
\]

\[
\approx |\det(w_k)| \|w_kw_{k-1}N_k\|.
\]

The second estimate is obtained using (3.3) and (3.2). We can conclude noticing that \( w_kw_{k-1}N_k = y_{t_k+l} = y_\psi(t_{k+1}) \) if \( l \geq 1 \) and \( w_kw_{k-1}N_l = y_\psi(t_{k+1}) \) if \( l = 0 \).

Let us show (d) By (4.11) with \( x = y_{t_k-l} \) and by using identity \( \langle x \wedge y, z\rangle = \det(x, y, z) \) (valid for all \( x, y, z \in \mathbb{R}^3 \)) we have by (4.7)

\[
|\langle z_{t_k+l}, y_{t_k+l}\rangle| = |\det(w_k)|^{-1}|\det(y_{t_k-l}, y_{t_k+l}, y_{t_k+l+1})|
\]

\[
\approx |\det(w_k)|^{l-1}|\det(y_{t_k-l}, y_{t_k}, y_{t_k+l+1})|
\]

Finally, for (e) we use Property (2.1) of the wedge product. By (a) and (c) we thus obtain

\[
\|\langle z_{t_k+l}, y\rangle y_{t_k+l+1} - \langle z_{t_k+l}, y_{t_k+l+1}\rangle y\| \ll \|z_{t_k+l}\| \|y\| \ll \|y_\psi(t_{k+1})\| \frac{|\det(y_{t_k+l+1})|}{\|y_{t_k+l+1}\|}
\]

\[
\ll \frac{|\det(y_{t_k+l+1})|}{\|w_k\|^2},
\]

since by multiplicative growth we have \( \|y_{t_k+l+1}\| \geq \|w_ky_{t_k+l}\| \geq \|w_k\|^2\|y_{t_k+l}\| \). Yet, by (d) we have \( \frac{|\det(y_{t_k+l+1})|}{\|w_k\|^2} = o\left( \frac{|\det(y_{t_k+l})|}{\|z_{t_k+l}, y_{t_k+l+1}\|} \right) \), thus necessarily, \( \|\langle z_{t_k+l}, y\rangle y_{t_k+l+1}\| \sim \|\langle z_{t_k+l}, y_{t_k+l+1}\rangle y\| \sim |\det(y_{t_k+l})| \) as \( k \) tends to infinity. \( \square \)

**7 Partial 3-system representing a \( \psi \)-Sturmian number**

Section 7.1 is devoted to some reminders on Schmidt and Summerer’s parametric geometry of numbers and on the notion of 3-system (see Definition 7.8). In Section 7.2, under the assumption that the sequence \( (s_k)_k \) associated with our Sturmian recurrence is bounded (note that all previous results remain valid without this hypothesis), we construct a 3-system which partially represents a...
ψ-number $\xi$ (see Figure 2). Propositions 7.14 and 7.16 play a central role in this paper, their proof are presented in Section 7.4. This allows us to determine the six parametric Diophantine exponents associated with $\xi$: see Theorems 7.3 and 7.4, our main result. For two exponents (namely $\psi_q$ and $\bar{\psi}_q$) we need an additional assumption on $\delta$ in order to determine their precise value (by ignoring the gray areas of the 3-system representing $\xi$). Finally in Section 7.3 we construct some particular integer points whose trajectory goes through the gray areas. They require strong conditions for the gray areas to be maximal.

### 7.1 Parametric geometry of numbers

Let $n \geq 2$ be an integer and let $u \in \mathbb{R}^n$ be a point whose coordinates are linearly independent over $\mathbb{Q}$ (in the following we will have $n = 3$ and $u = (1, \xi, \xi^2)$). In this section we quickly present Schmidt and Summerer’s tools from the parametric geometry of numbers (cf [21] and [22]) and the main result due to Roy [15], [16] (namely Theorem 7.2 below which settles a conjecture of Schmidt and Summerer). We recall the notion of $n$-systems in Definition 7.8.

**Notation.** Let $I$ be an unbounded set of non negative real numbers, and let $(K_q)_{q \in I}$, $(C_q)_{q \in I}$ be two families of convex bodies of $\mathbb{R}^n$ indexed by $I$. We note $K_q \asymp C_q$ and we say that $(K_q)_{q \in I}$ and $(C_q)_{q \in I}$ are equivalent if there exists $c > 0$ such that for each index $q$ large enough we have

$$
\frac{1}{c} K_q \subset C_q \subset cK_q.
$$

In the following, the letter $q$ will always denotes a positive real number. For all points $x, y \in \mathbb{R}^n$ we denote by $x \cdot y = \langle x, y \rangle$ the standard scalar product of $x$ and $y$, and in this section $\| \cdot \|$ denotes the Euclidean norm associated with the scalar product.

**Remark.** Since all norms are equivalent in finite dimension and since estimates of the type $\asymp$ remain valid up to a multiplicative constant, we may suppose that the norm of Proposition 6.5 is the norm of our choice (although the calculations were made with the norm of Definition 5.1).

One of the founding ideas of the parametric geometry of numbers is to consider a family of convex bodies parameterized by a real number $q$ and to study the successive minima associated to this family. The choice of the family may differ according to the context. In this paper we choose to consider the convex bodies family of [10].

**Definition 7.1.** We set

$$
C_u(e^q) := \{ x \in \mathbb{R}^n ; \| x \| \leq 1, |x \cdot u| \leq e^{-q} \}
$$

and

$$
C_n^*(e^q) := \{ x \in \mathbb{R}^n ; \| x \| \leq e^q, \| x \wedge u \| \leq 1 \}.
$$

For $j = 1, \ldots, n$, $\lambda_j(q)$ (resp. $\lambda_j^*(q)$) denotes the $j$-th successive minimum of the convex body $C_u(e^q)$ (resp. $C_n^*(e^q)$) with respect to the lattice $\mathbb{Z}^n$. We also define

$$
L_j(q) = \log \lambda_j(q) \quad \psi_j(q) = \frac{L_j(q)}{q},
$$

$$
\bar{\psi}_j = \limsup_{q \to \infty} \psi_j(q) \quad \underline{\psi}_j = \liminf_{q \to \infty} \psi_j(q),
$$

as well as the analogous quantities $L_j^*(q)$, $\psi_j^*(q)$, $\bar{\psi}_j^*$, $\underline{\psi}_j^*$ associated to $\lambda_j^*(q)$. We group these successive minima $L_j$ (resp. $L_j^*$) into a single map $L_u = (L_1, \ldots, L_n)$ (resp. $L_n^* = (L_1^*, \ldots, L_n^*)$).

**Definition 7.2.** We set for each $N \geq 1$

$$
\Delta_N := \{ (x_1, \ldots, x_N) \in \mathbb{R}^N ; x_1 \leq \cdots \leq x_N \}
$$

and

$$
\Phi_N : \mathbb{R}^N \to \Delta_N
$$

the continuous map which lists the coordinates of a point in monotone non-decreasing order.
Definition 7.3. We follow [22, §3] and we define the combined graph of a set of real valued functions defined on an interval \( I \) to be the union of their graphs in \( I \times \mathbb{R} \). For a map \( P : [c, +\infty) \to \Delta_n \) and an interval \( I \subset [c, +\infty) \), we also defined the combined graph of \( P \) on \( I \) to be the combined graph of its components \( P_1, \ldots, P_n \) restricted to \( I \).

Definition 7.4. In order to draw the combined graph of the map \( L_u \), it is useful to define for each point \( x \in \mathbb{R}^n \setminus \{0\} \) the quantity \( \lambda_x(q) \) (resp. \( \lambda^*_x(q) \)) to be the smallest real number \( \lambda > 0 \) such that \( x \in \lambda C_u(e^q) \) (resp. \( x \in \lambda C^*_u(e^q) \)). Then we set

\[
L_x(q) = \log(\lambda_x(q)) \quad \text{and} \quad L^*_x(q) = \log(\lambda^*_x(q)).
\]

Roy calls the graph of \( L_x \) (or of \( L^*_x \)) the trajectory of \( x \). Locally, the combined graph of \( L_u \) is included in the combined graph of a finite set of \( L_x \), and for each \( x \neq 0 \) we have

\[
\begin{align*}
L_x(q) &= \max \{ \log(\|x\|), \log(\|x \cdot u\|) + q \}, \\
L^*_x(q) &= \max \{ \log(\|x \wedge u\|), \log(\|x\|) - q \}.
\end{align*}
\]

Note that

\[
L_1(q) = \min_{x \neq 0} L_x(q) \quad \text{and} \quad L^*_1(q) = \min_{x \neq 0} L^*_x(q).
\]

Definition 7.5. Following [7] and [8], we say that a point \( x \neq 0 \) is a minimal point (with respect to the family \( \{C_u(e^q)\}_{q \geq 0} \), resp. to the family \( \{C^*_u(e^q)\}_{q \geq 0} \)) if \( x \) is such that there is \( q \geq 0 \) for which

\[
L_1(q) = L_x(q) \quad \text{(resp. } L^*_1(q) = L^*_x(q))\text{.}
\]

If there is no ambiguity we may not specify the considered family of convex bodies.

Proposition 7.6 (Mahler). If \( K_q \) denotes the dual convex body of \( C_u(e^q) \), then \( K_q = C^*_u(e^q) \), and this implies that for each \( j = 1, \ldots, n \), we have

\[
L_j(q) = -L^*_{n+1-j}(q) + O(1)
\]
as \( q \) tends to infinity.

Proposition 7.7. Functions \( L_j \) are continuous, piecewise linear with slopes 0 and 1, and satisfy the following properties

(a) \( L_1(q) \leq \cdots \leq L_n(q) \).

(b) \( L_1(q) + \cdots + L_n(q) = q + O(1) \) as \( q \) tends to infinity.

In a dual manner, functions \( L^*_j \) are continuous, piecewise linear with slopes 0 and \(-1\), and satisfy

(a) \( L^*_1(q) \leq \cdots \leq L^*_n(q) \).

(b) \( L^*_1(q) + \cdots + L^*_n(q) = -q + O(1) \) as \( q \) tends to infinity.

Schmidt and Summerer describe precisely the behavior of components \( L_j \) by introducing the model of \((n, 0)\)-systems in [22]. In [10] Roy gives the following equivalent definition.

Definition 7.8. Fix a real number \( q_0 \geq 0 \). A \((n, 0)\)-system (or \((n, 0)\)-system) on \([q_0, +\infty)\) is a continuous piecewise linear map \( P = (P_1, \ldots, P_n) : [q_0, +\infty) \to \mathbb{R}^n \) with the following properties:

(a) For each \( q \geq q_0 \), we have \( 0 \leq P_1(q) \leq \cdots \leq P_n(q) \) and \( P_1(q) + \cdots + P_n(q) = q \).

(b) If \( H \) is a non-empty open subinterval of \([q_0, +\infty)\) on which \( P \) is differentiable, then there is an integer \( r \) (\( 1 \leq r \leq n \)), such that \( P_r \) has slope 1 on \( H \) while the other components \( P_j \) of \( P \) (\( j \neq r \)) are constant on \( H \).
(c) If \( q > q_0 \) is a point at which \( P \) is not differentiable and if the integers \( r \) and \( s \) for which \( P_r \) has slope 1 on \( (q - \varepsilon, q) \) and \( P_s \) has slope 1 on \( (q, q + \varepsilon) \) (for \( \varepsilon > 0 \) small enough) satisfy \( r < s \), then we have \( P_r(q) = P_{r+1}(q) = \cdots = P_s(q) \).

Given a subset \( A \) of \( \mathbb{R} \), we call \textit{interval of} \( A \) any interval of \( \mathbb{R} \) included in \( A \). Here, the condition “\( P \) is piecewise linear” means that for all bounded intervals \( J \subset [q_0, +\infty) \), the intersection of \( J \) with the set \( D \) of points in \([q_0, +\infty)\) at which \( P \) is not differentiable is finite, and that the derivative of \( P \) is locally constant on \([q_0, +\infty) \setminus D \). The slope of a component \( P_j \) of \( P \) on a non-empty open interval \( H \) of \([q_0, +\infty) \setminus D \) is the constant value of its derivative on \( H \), or equivalently the slope of its graph over \( H \).

**Definition 7.9.** Fix a real number \( q_0 \geq 0 \). A dual \( n \)-system on \([q_0, +\infty)\) is a map \( P : [q_0, +\infty) \to \mathbb{R}^n \) such that \(-P\) is a \( n \)-system on \([q_0, +\infty)\).

**Theorem 7.1** (Schmidt and Summerer, 2013). For each non-zero point \( u \in \mathbb{R}^n \), there exist \( q_0 > 0 \) and a \( n \)-system \( P \) on \([q_0, +\infty)\) such that \( \|L_u - P\|_\infty \) is bounded over \([0, +\infty)\).

**Theorem 7.2** (Roy, 2015). For each \( n \)-system \( P \) on an interval \([q_0, +\infty)\), there exists a non-zero point \( u \in \mathbb{R}^n \) such that \( \|L_u - P\|_\infty \) is bounded on \([0, +\infty)\).

**Definition 7.10.** Let \( \xi \) be a real number and write \( u = (1, \xi, \ldots, \xi^n) \). We say that a \( n \)-system \( P \) on \([q_0, +\infty)\) (resp. a dual \( n \)-system \( P \) on \([q_0, +\infty)\)) represents \( \xi \), or is a representant of \( \xi \), if it satisfies that \( \|L_u - P\|_\infty \) (resp. \( \|L_u - P\|_\infty \)) is bounded on \([q_0, +\infty)\).

Theorem 7.1 of Schmidt and Summerer ensures that there always exists a \( n \)-system which represents \( \xi \).

### 7.2 Construction of a partial 3-system representing \( \xi \)

In this section, under the assumption that the sequence \((s_i)\), associated with our Sturmian recurrence is bounded (note that all previous results remain valid without this hypothesis), we construct a 3-system which partially represents a \( \psi \)-Sturmian number \( \xi \) (see Figure 3). Propositions 7.14 and 7.18 play a central role in this paper, their proof are presented in Section 7.4. We deduce from Proposition 7.18 our main result: Theorem 7.3

**Notation.** We keep notations (and hypotheses) of Section 6. \( (w_i)_{i \geq 0} \) is an admissible \( \psi \)-Sturmian sequence in \( M \) with multiplicative growth such that \( \det(w_i) \) tends to \(+\infty\). We denote by \( N \), \( (y_i)_{i \geq -2} \) and \( (z_i)_{i \geq -1} \) the matrix in \( \text{GL}_2(\mathbb{R}) \) and the two sequences of symmetric matrices which are associated to \( (w_i)_{i \geq 0} \) by Definition 3.5 and by (4.1). We assume that \( \text{Tr}(JN) \neq 0 \) (i.e. \( N \) is not symmetric). Finally \( \delta \geq 0 \) is the exponent given by Proposition 5.6 and satisfying

\[
\det(w_i) \geq \|w_i\|^\delta.
\]

In this section we additionally assume that \( \delta < 1 \).

Let \( y = (1, \xi, \xi^2) \) denote the vector given by Proposition 6.1. In the following, \( L_1, L_2, L_3 \) and \( L_u = (L_1, L_2, L_3) \) denote the functions defined in Section 7.1 for the point \( u = y \in \mathbb{R}^3 \). Recall that we also assume that the sequence \((s_i)_{i \geq 0}\) associated to \( \psi \) is bounded, so that by multiplicative growth, Proposition 5.6 directly provides the following estimate

\[
\det(y_i) \geq \|y_i\|^\delta. \tag{7.1}
\]

We set

\[
\sigma = \frac{1}{\limsup_{k \to +\infty} |s_{k+1} : s_k, \ldots, s_1|} = \liminf_{k \to +\infty} \frac{\log(\|w_k\|)}{\log(\|w_{k+1}\|)}. \tag{7.2}
\]

Let \((\hat{W}_k)_{k \geq 0}\) be a sequence (provided by Proposition 5.3) of positive real numbers satisfying

(a) \( \hat{W}_{k+1} = \hat{W}s_{k+1} \hat{W}_{k-1} \) for \( k \geq 1 \),

(b) \( \hat{W}_k \geq \|w_k\| \).
Let $k_0 \geq 1$ be an integer such that for each $k \geq k_0 - 1$ we have $\hat{W}_k > 1$ (in particular $(\log(\hat{W}_k))_k$ is increasing for $k \geq k_0$).

**Definition 7.11.** Let $k, l$ be integers with $0 \leq k$ and $0 \leq l < s_{k+1}$. We define $\hat{Y}_{t_k+l}, \hat{E}_{t_k+l}^*, \hat{Z}_{t_k+l}$ and $\hat{E}_{t_k+l}^*$ by

\[
\hat{Y}_{t_k+l} = \frac{\hat{W}_{k+l}^{(l+1)}}{\hat{W}_{k-1}^{(l)}} \\
\hat{E}_{t_k+l}^* = \left(\frac{\hat{W}_{k+l}^{(l+1)}}{\hat{W}_{k-1}^{(l)}}\right)^{\delta - 1} \\
\hat{Z}_{t_k+l} = \frac{\hat{W}_{k}^{(l+1)}}{\hat{W}_{k-1}^{(l)}} \\
\hat{E}_{t_k+l}^* = \frac{\hat{W}_{k}^{(\delta - 1)(l+1) - 1}}{\hat{W}_{k-1}^{\delta - 1}}.
\]

Note that these formulas for $\hat{Y}_{t_k+l}$ and $\hat{E}_{t_k+l}^*$ remain valid in the case $l = s_{k+1}$. We also have $\hat{Z}_i = \hat{Y}_{\psi(i)}$.

**Proposition 7.12.** Let $i \geq 0$ be an integer. As $i$ tends to infinity we have:

(a) $\log(\|y_i\|) = \log(\hat{Y}_i) + O(1)$.

(b) $\log(\|y_i \wedge y\|) = \log(\hat{E}_i^*) + O(1)$.

(c) $\log(\|z_i\|) = \log(\hat{Z}_i) + O(1)$.

(d) $\log(\|z_i \cdot y\|) = \log(\hat{E}_i) + O(1)$.

**Proof** Since $(s_i)_i$ is bounded, we obtain by multiplicative growth and by (3.2)

\[
\log(\|y_{t_k+l}\|) = (l + 1)\log(\hat{W}_k) + \log(\hat{W}_{k-1}) + O(1)
\]

for $0 \leq k$ tending to infinity, with $0 \leq l \leq s_{k+1}$. All assertions may be deduced from the estimates of Proposition 6.5. \qed

**Definition 7.13.** Let $k, l$ be integers with $0 \leq k$ and $0 \leq l < s_{k+1}$. Let us write $i = t_k + l$ and for $q \geq 0$ let us set

\[
\hat{L}_i(q) = \max\{\log(\hat{Z}_i), \log(\hat{E}_i) + q\}, \\
\hat{L}_i^*(q) = \max\{\log(\hat{E}_i^*), \log(\hat{Y}_i) - q\},
\]

and

\[
q_i = \log(\hat{Z}_i) - \log(\hat{E}_i) = \log(\hat{Y}_i) - \log(\hat{E}_i^*) = (2 - \delta)\log(\hat{Y}_i).
\]

We also define

\[
c_i = \log(\hat{Y}_{t_k+1}) - \log(\hat{E}_i^*) = q_i + \log(\hat{W}_k),
\]

the intersection point abscissa of $\hat{L}_i^*$ and $\hat{L}_{t_k+1}^*$, which is also the intersection point abscissa of $\hat{L}_i$ and $\hat{L}_{\psi^{-1}(i)}$ (recall that by the definition of $\psi$, we have $\psi^{-1}(i) = i + 1$ if $i < t_{k+1} - 1$, and $\psi^{-1}(t_{k+1} - 1) = t_{k+2}$). Also note that

\[
-\hat{L}_i^*(c_i) = -\hat{L}_{t_k+1}^*(c_i) = -\log(\hat{E}_i^*) \quad \text{and} \quad \hat{L}_i(c_i) = \hat{L}_{\psi^{-1}(i)}(c_i) = \log(\hat{Z}_{\psi^{-1}(i)}).
\]  

**Proposition 7.14.** We have the following properties:

(a) For each $i \geq t_{k_0}$, we have $q_i < c_i < q_{i+1} < c_{i+1}$.

(b) There exists a constant $C > 0$ such that for any $i$ and for any $q > 0$ we have

\[
|\hat{L}_i(q) - L_i(q)| \leq C \quad \text{and} \quad |\hat{L}_i^*(q) - L_i^*(q)| \leq C,
\]

i.e. the graph of $\hat{L}_i$ (resp. $\hat{L}_i^*$) approximates the trajectory of $z_i$ (resp. of $y_i$) within $O(1)$, uniformly with respect to $i$.  

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(c) The function \( \hat{L}_i \) is continuous piecewise linear, constant on the interval \([0, q_i]\), and increasing with slope 1 on the interval \([q_i, +\infty)\).

(d) The function \(-\hat{L}_i^*\) is continuous piecewise linear, increasing with slope 1 on the interval \([0, q_i]\), and constant on the interval \([q_i, +\infty)\).

(e) Assume that \( \delta < \frac{\sigma}{1+\sigma} \). For \( i = t_k + l \) with \( 0 \leq l < s_{k+1} - 1 \) and \( k \) large enough, the form of the combined graph of \( \hat{L}_{t_k+l}, -\hat{L}_{t_k+l}^*, \hat{L}_{t_k+l+1} \) on \([c_i, c_{i+1}]\) is similar to the drawing on the left of Figure 2. The form of the combined graph of \( \hat{L}_{t_k+l}, -\hat{L}_{t_k+l}^*, \hat{L}_{t_k+1} \) on \([c_{t_k-1}, c_{t_k}]\) is similar to the drawing on the right of Figure 2.

(f) Assume \( \delta < \frac{\sigma}{1+\sigma} \). As \( i = t_k + l \) (with \( 0 \leq l < s_{k+1} \)) tends to infinity, we have
\[
-\hat{L}_i^*(q_i) - \max(\hat{L}_i(q_i), \hat{L}_{t_k+l}(q_i)) \to +\infty. \quad (7.4)
\]

Figure 1: Combined graph of \( \hat{L}_{t_k+l}, -\hat{L}_{t_k+l}^*, \hat{L}_{t_k+l+1} \) on \([c_i, c_{i+1}]\)

**Definition 7.15.** We define \( \mathbf{P} = (P_1, P_2, P_3) \) by setting for \( t_k - 1 \leq i < t_{k+1} - 1 \) (with \( k_0 \leq k \)), and \( c_i \leq q < c_{i+1} \):
\[
P(q) = \Phi_3(\hat{L}_{t_k+l}(q), -\hat{L}_{t_k+l}^*(q), \hat{L}_{t_k+l+1}(q)),
\]
and we denote by \( I_{i+1} = [a_{i+1}, b_{i+1}] \subset [c_i, c_{i+1}] \) the interval on which \( P_3 = -\hat{L}_{t_k+l}^* \) (see Figure 1). We also set \( I'_{i+1} = [b_i, a_{i+1}] \).

**Proposition 7.16.** Recall that \( \mathbf{L}_u = (L_1, L_2, L_3) \) denotes the map of successive minima (see Definition 2.18 with \( u = (1, \xi, \xi^2) \)). Assume that \( \delta < \frac{\sigma}{1+\sigma} \) holds and that the content of \( y_i \) is bounded (which is indeed the case if the hypotheses of Corollary 4.4 are satisfied). Then \( \mathbf{P} \) is a 3-system on \([y_{t_ko}, +\infty)\) and its combined graph is of the type of the one given in Figure 3. Moreover, there exists \( C > 0 \) depending only on \( \xi \) such that

(a) \( |L_1(q) - P_1(q)| \leq C \) for any \( q \) large enough,

(b) \( |L_2(q) - P_2(q)| \leq C \) and \( |L_3(q) - P_3(q)| \leq C \) for each \( q \in I_j \) with \( j \) large enough,

(c) \( P_2(q) - C \leq L_2(q) \leq L_3(q) \leq P_3(q) + C \) for each \( q \in I'_j \) with \( j \) large enough.

Roughly speaking we may say that up to a bounded difference the combined graph of \( \mathbf{L}_u \) coincides with that of \( \mathbf{P} \) outside the shaded areas on Figure 2, and for each \( j \) large enough, the combined graph of \( L_2 \) and \( L_3 \) on the interval \( I'_j \) is included – within \( \mathcal{O}(1) \) – in the corresponding shaded area (we also know that in this “gray” area the graphs of \( L_2 \) and \( L_3 \) will zigzag around a line with slope \( 1/2 \) but here it is not important for our purpose).
The following proposition states a classical result in parametric geometry of numbers and establishes a relation between standard diophantine exponents \( \hat{\lambda}_2(\xi), \lambda_2(\xi), \omega_2(\xi), \hat{\omega}_2(\xi) \) and the diophantine exponents \( \psi_1, \overline{\psi}_1, \psi_3, \overline{\psi}_3 \) (attached to the vector \( u = (1, \xi, \xi^2) \)) which come from parametric geometry of numbers. Cf [21] and [15]; note that \( \lambda_2(\xi), \hat{\lambda}_2(\xi), \omega_2(\xi), \hat{\omega}_2(\xi) \) are respectively denoted by \( \lambda(u), \tilde{\lambda}(u), \tau(u), \tilde{\tau}(u) \) in [15].

Proposition 7.17. We have

\[
(\psi_1, \overline{\psi}_1, \psi_3, \overline{\psi}_3) = \left( \frac{1}{\omega_2(\xi) + 1}, \frac{1}{\tilde{\omega}_2(\xi) + 1}, \frac{\hat{\lambda}_2(\xi)}{\lambda_2(\xi) + 1}, \frac{\lambda_2(\xi)}{\lambda_2(\xi) + 1} \right).
\] (7.5)

In particular, Jarník’s relation (1.1) may be rewritten as

\[
2\psi_1(\xi) + 2\overline{\psi}_1(\xi) - 3\psi_3(\xi)\overline{\psi}_3(\xi) - 1 = 0.
\] (7.6)

Theorem 7.3. Assume that \( \delta < \frac{\sigma}{1 + \sigma} \) and that the content of \( y_i \) is bounded. Then we have

\[
\begin{align*}
\psi_1 &= \frac{\sigma}{(2 - \delta)(1 + \sigma)}, & \overline{\psi}_1 &= \frac{1}{(1 - \delta)(1 + \sigma) + 2}, \\
\psi_2 &= \frac{1}{2 + \sigma}, & \overline{\psi}_2 &= \frac{1}{1 + \sigma}, \\
\psi_3 &= \frac{(1 - \delta)(1 + \sigma)}{1 + 2(1 - \delta)(1 + \sigma)}, & \overline{\psi}_3 &\leq \frac{1}{2 - \delta} \leq \max \left( \frac{1 - \delta}{2 - \delta}, \frac{1}{2 - \delta + \sigma} \right),
\end{align*}
\]

where \( \sigma \) is defined by (7.2) and the considered exponents are as in Definition 7.1 (for \( u = (1, \xi, \xi^2) \)). In particular, if \( \delta \) satisfies the condition \( \delta \leq h(\sigma) \) with

\[
h(\sigma) = \frac{\sigma}{2} + 1 - \sqrt{\left( \frac{\sigma}{2} \right)^2 + 1} \leq \frac{\sigma}{1 + \sigma},
\] (7.7)

then we have

\[
\overline{\psi}_3 = \frac{1 - \delta}{2 - \delta}.
\]

Using (7.5) and Roy’s examples (see Section 8.1), Theorem 7.3 directly implies Theorem 1.1.
Remark 7.18. If the hypotheses of Corollary 4.4 are fulfilled, then the content of $y_i$ is bounded. Also note that if $\delta > \frac{\sigma}{1-\sigma}$, then $\mathbf{P}$ (see Figure 2) is no longer a 3-system (see the proof of the assertion (6) of Proposition 7.14). Actually, for $\delta > \frac{\sigma}{1-\sigma}$, the expression which gives the value of $\hat{\lambda}_2(\xi)$ becomes $< 1/2$, and such value of $\hat{\lambda}_2(\xi)$ is clearly forbidden. Finally, we think that our condition $\sigma \leq h(\sigma)$ is not the best possible and may be improved for particular $\psi$-Sturmian numbers. See Section 7.3 for some clues supporting this conjecture.

For $\overline{\psi}_3$ (which is the only parametric exponent not given by Theorem 7.3) the situation is more complicated and we need to introduce new quantities, see Theorem 7.4.

Proof of Theorem 7.3. To the end of calculating the exponents of Theorem 7.3, we use Proposition 7.16 together with Proposition 7.14 (cf Figures 1 and 2 to which we will often refer). For the exponent $\psi_1$, it suffices to calculate

$$\psi_1 = \liminf_{q \to +\infty} \frac{L_1(q)}{q} = \liminf_{q \to +\infty} \frac{P_1(q)}{q} = \liminf_{k \to \infty} \frac{P_1(q_{tk})}{q_{tk}}.$$

Yet, in view of Definitions 7.13 and 7.11, we have

$$P_1(q_{tk}) = \frac{\log(\hat{Z}_{tk})}{q_{tk}} = \frac{\log(\hat{W}_{k-1})}{q_{tk}} = \frac{1}{(2-\delta) + \frac{\log(W_k)}{\log(W_{k-1})}},$$

which decreases with $\frac{\log(W_k)}{\log(W_{k-1})}$, and by taking the infimum it follows $\psi_1 = \frac{1}{(2-\delta)(1+\frac{1}{2})}$.

For $\overline{\psi}_1$ we have

$$\overline{\psi}_1 = \limsup_{q \to +\infty} \frac{P_1(q)}{q} = \limsup_{k \to \infty} \frac{P_1(d_k)}{d_k},$$

where $d_k = (3-\delta) \log(\hat{W}_k) + (1-\delta) \log(\hat{W}_{k-1})$ is the intersection point abscissa of $\hat{L}_{tk}$ and $\hat{L}_{tk+1}$ (cf Figure 1). Then we have

$$\frac{P_1(d_k)}{d_k} = \frac{\log(\hat{Z}_{tk+1})}{d_k} = \frac{1}{3 - \delta + (1 - \delta) \frac{\log(W_{k-1})}{\log(W_k)}},$$

(7.8)

and by taking the supremum we find $\overline{\psi}_1 = \frac{1}{3 - \delta + (1 - \delta) \sigma}$.

For $\overline{\psi}_3$, it may be shown that it suffices to calculate $\psi_3 = \liminf_{k \to \infty} \frac{P_3(b_{ki})}{b_{ki}}$. We may also directly use Jarník’s identity (7.6) with the previous expression of $\overline{\psi}_1$.

The calculation of $\overline{\psi}_3$ is a bit more delicate because we have to show that the shaded areas of Figure 2 may be ignored under the condition (7.7). In the view of the form of the 3-system $\mathbf{P}$ (see Figure 2), we have

$$\limsup_{q \to +\infty} \frac{P_3(q)}{q} = \max \left( \limsup_{i \to \infty} \frac{P_3(c_i)}{c_i}, \limsup_{i \to \infty} \frac{P_3(q_i)}{q_i} \right).$$

On the one hand we have

$$\frac{P_3(q_i)}{q_i} = \frac{-\hat{L}_i^*(q_i)}{q_i} = \frac{-\log(\xi_i^*)}{q_i} = \frac{1 - \delta}{2 - \delta}.$$

On the other hand we claim that the maximum value of $P_3(c_i)/c_i$ for $i = t_k + l$ with $0 \leq l \leq s_{k+1} - 1$ is reached for $l = s_{k+1} - 1$. Indeed, in the view of Figure 1 for $i = t_k + s_{k+1} - 1 = t_{k+1} - 1$, we have

$$\frac{P_3(c_{t_{k+1} - 1})}{c_{t_{k+1} - 1}} = \frac{\hat{L}_{t_{k+2}}(c_{t_{k+1} - 1})}{c_{t_{k+1} - 1}} = \frac{\log(\hat{Z}_{t_{k+2}})}{q_{t_{k+1} - 1} + \log(W_k)} = \frac{\log(\hat{W}_{k+1})}{(2 - \delta) \log(W_{k+1}) + \log(W_k)}.$$
whereas for \( i < t_k + s_{k+1} - 1 \) we have:

\[
\psi_2 = \limsup_{k \to +\infty} \frac{\log(Z_{t_k+1})}{q_k + \log(W_k)} - \frac{(l + 1) \log(W_k) + \log(W_{k-1})}{(2 - \delta)(l + 1) \log(W_k) + \log(W_{k-1}) + \log(W_k)}.
\]

and the right hand side of the last inequality increases with \( l \), therefore it reaches its maximal value for \( l = s_{k+1} - 1 \). Finally we have

\[
\limsup_{q \to +\infty} \frac{\psi_3}{q} = \max \left( \frac{1}{2 - \delta + \sigma}, \frac{1 - \delta}{2 - \delta} \right).
\]

Thus

\[
\limsup_{q \to +\infty} \frac{\psi_3}{q} = \max \left( \frac{1}{2 - \delta + \sigma}, \frac{1 - \delta}{2 - \delta} \right).
\]

Now, note that if \( \delta \leq h(\sigma) \), then \( \delta^2 - (\sigma + 2)\delta + \sigma \geq 0 \), so that we have max \( \left( \frac{1}{2 - \delta + \sigma}, \frac{1 - \delta}{2 - \delta} \right) = \frac{1 - \delta}{2 - \delta} \).

This ends the proof of the fact that under the condition (7.7) we have

\[
\psi_2 = \limsup_{k \to +\infty} \frac{L_2(a_{t_k})}{a_{t_k}}.
\]

This follows from the fact that \( \psi_2 \leq \psi_3 \leq 1/2 \) and \( L_2 \) is (within \( O(1) \)) always under the line with slope \( 1/2 \) passing through the diagonals of the shaded areas located between \( q_k \) and \( q_{t_k+1} \).

See Figure 2 (it is a consequence of Proposition 7.16 and of assertion [10] of Proposition 7.7).

Recall that \( a_{t_k} \) is the intersection point abscissa of \( L_{t_k+1} \) and \( -L_{t_k}^c \) (cf figure 1), therefore

\[
a_{t_k} = \log(Z_{t_{k+1}}) + \log(\psi_{t_k}) = 2 \log(W_k) + \log(W_{k-1}).
\]

Moreover we have

\[
P_2(a_{t_k}) = L_{t_{k+1}}(a_{t_k}) = \log(Z_{t_{k+1}}).
\]

It thus yields

\[
\psi_2 = \limsup_{k \to +\infty} \frac{L_2(a_{t_k})}{a_{t_k}} = \limsup_{k \to +\infty} \frac{P_2(a_{t_k})}{a_{t_k}} = \limsup_{k \to +\infty} \frac{1}{2 + \frac{\log(W_{k-1})}{\log(W_k)}} = \frac{1}{2 + \sigma}.
\]

\[\square\]

**Theorem 7.4.** Assume that \( \delta < \frac{\sigma}{1 + \sigma} \) and that the content of \( y_i \) is bounded. Let us define

\[
\tau = \limsup_{k \to \infty} \frac{1}{[sk_k; sk_{k-1}, \ldots, s_1]} \quad \text{and} \quad \sigma' = \liminf_{k \to \infty} \frac{1}{sk_k; sk_{k-1}, \ldots, s_1},
\]

(with the convention that \( \sigma' = +\infty \) if \( s_{k+1} = 1 \) for each \( k \) large enough). Let us set

\[
\theta(\delta) = \min \left( \frac{1 + \sigma'}{(2 - \delta)(2 + \sigma')}, \frac{1}{2 + (1 - \delta)(1 + \tau)} \right).
\]

Then we have

\[
\min \left( \frac{(1 - \delta)(1 + \sigma)}{(2 - \delta)(1 + \sigma) + 1}, \theta(\delta) \right) \leq \psi_2 \leq \theta(\delta).
\]

In particular, there exists a constant \( c > 0 \), which depends only on the Sturmian function \( \psi \), such that if \( \delta \leq c \), then

\[
\psi_2 = \theta(\delta).
\]
Proof We claim that a point at which the function $P_2(q)/q$ reaches its minimum over the interval $[q_{k+1}, q_{k+1}]$ belongs to $\{d_k, c_k\}$ if $s_{k+1} = 1$ and belongs to $\{d_k, c_k, q_{k+1}\}$ if $s_{k+1} > 1$ (see Figures 1 and 2).

Indeed, if $q$ is a minimum of $P_2(q)/q$ over $[q_{k+1}, q_{k+1}]$, then $q$ is necessarily either $d_k$, or one of the $c_{k+1}$ (with $0 \leq l < s_{k+1}$), or one of the $q_{k+l}$ (with $0 < l < s_{k+1}$).

For points $c_{k+1}$ (with $0 \leq l < s_{k+1}$) coming from the gray areas, we have

$$
\frac{P_2(c_{k+1})}{c_{k+1}} = - \frac{\log(\widehat{E}_{k+1})}{q_{k+1} + \log(W_k)} = \frac{(1 - \delta)(l + 1) \log(W_k) + \log(W_{k-1})}{(2 - \delta)(l + 1) \log(W_k) + \log(W_{k-1}) + \log(W_{k+1})},
$$

which increases with $l$, thus being always greater than (or equal to)

$$
\frac{P_2(c_k)}{c_k} = \frac{(1 - \delta)(1 + \frac{\log(W_{k-1})}{\log(W_k)})}{(2 - \delta)(1 + \frac{\log(W_{k-1})}{\log(W_k)}) + 1}.
$$

(7.9)

For points $q_{k+l}$, with $0 < l < s_{k+1}$ and $k$ such that $s_{k+1} > 1$ (there are infinitely many such $k$ if and only if $\xi$ is not of the Fibonacci type), we have

$$
\frac{P_2(q_{k+l})}{q_{k+l}} = \frac{\log(\widehat{\mathcal{T}}_{k+l})}{q_{k+l}} = \frac{(l + 1) \log(W_k) + \log(W_{k-1}) - \log(W_k)}{(2 - \delta)(l + 1) \log(W_k) + \log(W_{k-1}) + \log(W_{k+1})}
= \frac{1}{2 - \delta} \left( 1 + \frac{\log(W_k)}{(l + 1) \log(W_k) + \log(W_{k-1})} \right)
$$

which increases with $l$, thus reaching its minimal value for $l = 1$. This yields

$$
\frac{P_2(q_{k+l})}{q_{k+l}} \geq \frac{P_2(q_{k+1})}{q_{k+1}} = \frac{1}{2 - \delta} \left( 1 - \frac{1}{2 + \frac{\log(W_{k-1})}{\log(W_k)}} \right).
$$

(7.10)

For points $d_k$, by (7.8), we have

$$
\frac{P_2(d_k)}{d_k} = \frac{P_1(d_k)}{d_k} = \frac{1}{3 - \delta + \frac{1}{\frac{\log(W_{k-1})}{\log(W_k)}}}.
$$

(7.11)

We may conclude by taking the infimum of (7.9), (7.10) and (7.11).

If $\delta = 0$, then we have

$$
\frac{(1 - \delta)(1 + \sigma)}{(2 - \delta)(1 + \sigma) + 1} = \frac{1 + \sigma}{3 + 2\sigma} > \theta(0) = \min \left( \frac{1 + \sigma'}{4 + 2\sigma'}, \frac{1}{3 + \tau} \right),
$$

by using $\sigma', \tau \geq \sigma$. Thus, by continuity, there exists $c > 0$, which depends only on $\sigma, \sigma', \tau$, such that

$$
\min \left( \frac{(1 - \delta)(1 + \sigma)}{(2 - \delta)(1 + \sigma) + 1}, \theta(0) \right) = \theta(0) \quad (0 \leq \delta \leq c).
$$

\[\square\]

7.3 Discussing the gray areas

Here we discuss the condition $\delta \leq h(\sigma)$ of Theorems 7.3 and 1.1 which may possibly be improved, at least for some class of $\psi$-Sturmian numbers, as suggested by the constructions presented below. Indeed we construct some particular integer points $x_{n}^{(i)}$ (see Definition 7.19) whose trajectory goes through the gray areas (see Figure 3). Proposition 7.22 implies that they require strong conditions for the gray areas to be maximal.

The question is: what does the combined graph of $L_{\sigma}$ look like in the gray areas of the 3-system $P$? (see Figure 3). The optimality of condition $\delta \leq h(\sigma)$ (with $h(\sigma)$ as defined in Theorem
is equivalent to the existence of \( \psi \)-Sturmian numbers for which the combined graph of the corresponding successive minima map \( L_u \) coincides with the combined graph of \( P \) at an infinite number of suitable gray areas within \( o(q) \) (in other words, at the level of these suitable gray areas, the picture of the combined graph of \( L_u \) matches precisely the outlines of the corresponding gray area within \( o(q) \)).

We consider in this section the dual 3-system \(-P\) (and the map \( L_u^* \)) rather than \( P \) (and \( L_u \)). We will now construct some points \( x^{(i)}_m \) whose trajectory intersects with the \( i \)-th gray area and which demand at least two conditions (believed to be hard to satisfy) on the sequence \( (w_i)_i \), so that the combined graph of \( L_u^* \) matches precisely the outlines of the \( i \)-th gray area. Proposition 7.22 formulates this idea.

For an easier approach, let us focus on the Fibonacci case (although our constructions may be used to deal with the general case). In this setting, we have \( s_k = 1 \) for each \( k \geq 1 \), thus \( t_k = k - 1 \) for each \( k \geq 0 \). Let us fix an admissible \( \psi \)-Sturmian sequence \( (w_i)_i \) with multiplicative growth and such that \( \langle \|w_i\| \rangle \) tends to infinity. Then, by definition \( (w_i)_i \), satisfies

\[
w_{i+1} = w_i w_{i-1} \quad (i \geq 1),
\]

and its associated sequences \( (y_i)_i \), \( (z_i)_i \) are defined by

\[
\begin{align*}
y_i &= w_{i+2} \nu_i \quad \text{for } i \geq -2, \\
z_i &= \frac{1}{\det(w_{i+1})} y_{i-1} \wedge y_i \quad \text{for } i \geq -1.
\end{align*}
\]

Note that up to an index shifting, these sequences are exactly the same as Roy’s in [14]. For each \( i \geq 0 \), set

\[
t_i = \text{Tr}(w_i) \quad \text{and} \quad d_i = \det(w_i).
\]

Assume that \( \delta < \frac{\sigma - \gamma}{1 - \sigma} = \frac{1}{\gamma} \), where \( \delta \) is given by Proposition 5.6 (recall that in the Fibonacci case we have \( \sigma = \frac{1}{2} \), where \( \gamma \) denotes the golden ratio). Denote by \( y = (1, \xi, \xi^2) \) the vector given by Proposition 6.1. By virtue of Proposition 1.3, the sequence \( (y_i)_i \) satisfies the following recurrence

\[
y_{i+1} = t_{i+1}y_i - d_{i+1}y_{i-2} \quad (i \geq 0).
\]

**Definition 7.19.** Let \( \left( p^{(i)}_m/q^{(i)}_m \right)_{-1 \leq m \leq r_i} \) be the (finite) convergents of the rational \( t_{i+1}/d_{i+1} \). By convention we set \( p^{(i)}_{-1} = 1 \) and \( q^{(i)}_{-1} = 0 \). We denote by \( a^{(i)}_0 = \left[ t_{i+1}/d_{i+1} \right] , a^{(i)}_1 , \ldots \) the partial quotients of \( t_{i+1}/d_{i+1} \). For \( -1 \leq m \leq r_i \) we define the integer point

\[
x^{(i)}_m = p^{(i)}_m y_1 - q^{(i)}_m y_{i-2} \in \mathbb{Z}^3.
\]

Note that this sequence begins with \( x^{(i)}_{-1} = y_i \) and ends with \( x^{(i)}_{r_i} = y_{i+1} \). It is known that \( (p^{(i)}_m), (q^{(i)}_m) \) satisfy the recurrence \( u_{m+1} = a^{(i)}_{m+1} u_m + u_{m-1}, (m \geq 0) \) (see for instance [20, Chapter I]). Therefore the sequence \( (x^{(i)}_m)_m \) satisfies

\[
x^{(i)}_{m+1} = a^{(i)}_{m+1} x^{(i)}_m + x^{(i)}_{m-1} \quad (m \geq 0).
\]

By 7.12 and using the identity \( d_{i+2} = d_{i+1} d_i \), Equation (7.13) may be rewritten as

\[
x^{(i)}_m = \frac{\alpha^{(i)}_m y_i + \beta^{(i)}_m y_{i+1}}{d_{i+2}},
\]

with \( \alpha^{(i)}_m = d_i \left( d_{i+1} p^{(i)}_m - t_{i+1} q^{(i)}_m \right) \) and \( \beta^{(i)}_m = d_i q^{(i)}_m \). Furthermore, the theory of continued fractions ensures that for \( m < r_i \),

\[
|\alpha^{(i)}_m| \leq \left| \frac{d_{i+2}}{q^{(i)}_{m+1}} \right| \ll |d_{i+2}|.
\]

On the other hand we have

\[
|\beta^{(i)}_m| = |d_i q^{(i)}_m| \leq |d_{i+2}|.
\]
since \(q_m^{(i)} \leq |d_{i+1}|\). Thus, in view of the growth of \(\|y_i\|\) we deduce from (7.15) and the previous estimates that
\[
\|x_m^{(i)}\| \approx \frac{\|x_m^{(i)}\|\|y_{i+1}\|}{|d_{i+2}|} \leq \|y_{i+1}\| \quad \text{and} \quad \|x_m^{(i)} \land y\| \approx \frac{\|x_m^{(i)}\|\|y_{i+1}\land y\|}{|d_{i+2}|} \leq \|y_{i+1}\land y\|,
\]
which precisely means that the trajectory of \(x_m^{(i)}\) goes through the gray area associated with \(y_i\) and \(y_{i+1}\). In fact, thanks to (7.16) and (7.17), we may prove that the trajectories of \(x_m^{(i)}\), \(m = -1, \ldots, r_i\), will zigzag around a line with slope \(-1/2\) (which is the vertical translate of the decreasing gray area diagonal by \(\frac{1}{2} \log |d_i|\); see Figure 3 below.

![Figure 3: Combined graph of the \(x_m^{(i)}\) in the \(i\)-th gray area](image)

Note that the graphs of \(L_1^*\) and \(L_2^*\) will zigzag around the decreasing \(i\)-th gray area diagonal. Therefore, the combined graph of the points \(x_m^{(i)}\) does not match that of \(L_1^*\) and \(L_2^*\).

**Proposition 7.20.** Let \(c_m^{(i)}\) denote the content of \(x_m^{(i)}\). Then

(a) The product \(c_m^{(i)}c_{m+1}^{(i)}\) divides \(d_i\)

(b) The gcd of \(c_m^{(i)}\) and \(c_{m+1}^{(i)}\) divides the content of \(y_i\).

**Remark.** In particular, if there exists a prime \(p\) such that \(d_i = p^{a_i}\) for each \(i\) (which is possible by taking \(a = p\) with Roy’s examples; see Section 8.1), then at least half of the \(x_m^{(i)}\) are primitive up to a bounded factor.

**Proof** The recurrence (7.12) together with the identity \(d_{i+2} = d_{i+1}d_i\) imply \(y_{i-2} \land y_i = d_iy_{i+1}\).

Also note that
\[
\begin{bmatrix}
p_m^{(i)} \\
q_m^{(i)}
\end{bmatrix}
\begin{bmatrix}
p_{m+1}^{(i)} \\
q_{m+1}^{(i)}
\end{bmatrix}
= \pm 1 \quad \text{by the theory of continued fractions. Hence} \quad \|x_m^{(i)} \land x_{m+1}^{(i)}\| = \pm d_iy_{i+1},
\]

which proves assertion \((a)\).

Assertion \((b)\) is a direct consequence of (7.14). If \(d\) divides both the contents of \(x_m^{(i)}\) and \(x_{m+1}^{(i)}\), then it divides the content of all the \(x_m^{(i)}\), especially that of \(x_{i-1}^{(i)} = y_i\).

The following proposition shows that any minimal point \(v\) which goes near the \(i\)-th gray area bottom outline is necessarily proportional to a \(x_m^{(i)}\). Note that \(\{y_i, y_{i+1}\}\) forms a basis of the linear subspace of \(\mathbb{R}^3\) generated by all the minimal points whose trajectory intersects with the \(i\)-th gray area.

**Proposition 7.21.** Let \(v\) be a minimal point whose trajectory intersects with the \(i\)-th gray area. Let us write
\[
v = \frac{\alpha y_i + \beta y_{i+1}}{d_{i+2}}.
\]

Let us set \(\lambda = \det(w_2)^2 \det(N)^2 |\text{Tr}(JN)|\) and suppose that \(|\alpha \beta| < \frac{|d_{i+1}|}{2^\lambda}\). Then \(v\) is proportional to a point \(x_m^{(i)}\) (with \(-1 \leq m \leq r_i\)).
Proof We have $\alpha z_{i+1} = v \land y_{i+1} \in \mathbb{Z}^3$ and $\beta z_{i+1} = v \land y_i \in \mathbb{Z}^3$. According to assertion (3) of Corollary 4.4, we have $\lambda_\alpha, \lambda_\beta \in \mathbb{Z}$. Let us write $v = (a y_i - b y_{i-2})/d_i$. Similarly, by considering $-b z_{i+1} = v \land y_i \in \mathbb{Z}^3$ and $a z_{i+1} = v \land y_{i-2} \in \mathbb{Z}^3$, we may show that $\lambda_\alpha, \lambda_\beta \in \mathbb{Z}$. Also note that, following from (7.12), we have

$$v = \frac{\alpha y_i + \beta y_{i+1}}{d_{i+2}},$$

thus $\alpha = ad_{i+1} - bt_{i+1}$ and $\beta = b$. Since $v$ is a minimal point we have $1 \ll |\alpha\beta| \ll |d_{i+2}|$. Now assume that $|\alpha\beta| < \frac{|d_{i+1}|}{2d}$. Also suppose that $b \neq 0$ (otherwise $v$ is proportional to $y_i$). Under these hypothesis we have

$$\left|\frac{\alpha'}{b'} - \frac{t_{i+1}}{d_{i+1}}\right| < \frac{1}{2b^2},$$

where $\alpha' = \lambda_a, b' = \lambda b \in \mathbb{Z}$. This proves that $\alpha'/b'$ is a convergent $p_m/q_m$ of $t_{i+1}/d_{i+1}$, and $v$ is proportional to $x_m^{(i)}$.

\[\square\]

Proposition 7.22. Suppose that up to a bounded difference the combined graph of $L_u^*$ matches the outlines of the $i$-th gray area (see Figure 3). Then

(a) There exists $0 \leq m < r_i$ such that $x_m^{(i)}/c_m^{(i)}$ is a minimal point and its trajectory matches the $i$-th gray area bottom outline up to a bounded difference (where $c_m^{(i)}$ denotes the content of $x_m^{(i)}$). Moreover the $(m+1)$-th partial quotient $a_{m+1}^{(i)}$ satisfies the estimate $a_{m+1}^{(i)} \approx |d_{i+1}|$.

(b) The integer $c_m^{(i)}$ satisfies the estimate $c_m^{(i)} \approx |d_i|$ (it is thus maximal).

Proof Suppose that the combined graph of $L_u^*$ matches precisely the outlines of the $i$-th gray area up to a bounded difference. This is equivalent to the existence of a minimal point $v$ whose trajectory matches precisely the outlines of the $i$-th gray area up to a bounded difference. If we write

$$v = \frac{\alpha y_i + \beta y_{i+1}}{d_{i+2}},$$

then we have $\alpha \approx \beta \approx 1$. According to Proposition 7.21, there exists $0 \leq m < r_i$ such that $v$ is proportional to $x_m^{(i)}$ if $i$ is large enough. More precisely, $v = \pm x_m^{(i)}/c_m^{(i)}$, where $c_m^{(i)}$ denotes the content of $x_m^{(i)}$. In view of the trajectory of $x_m^{(i)}$ (see Figure 3) and by Proposition 7.20, the content of $x_m^{(i)}$ has to be $\approx |d_i|$. Moreover by using (7.15) we may deduce that

$$\left|\frac{a_m^{(i)}}{d_i}\right| = \left|\frac{t_{i+1} p_m^{(i)} - t_{i+1} q_m^{(i)}}{q_m^{(i)}}\right| \approx |\alpha| \approx 1,$$

and

$$\left|\frac{\beta_m^{(i)}}{d_i}\right| = \left|\frac{q_m^{(i)} p_m^{(i)}}{q_m^{(i)}}\right| \approx |\beta| \approx 1.$$ 

Finally by using the classical estimate

$$\left|\frac{p_m^{(i)}}{q_m^{(i)}} - \frac{t_{i+1}}{d_{i+1}}\right| \approx \frac{1}{(q_m^{(i)})^2} a_{m+1}^{(i)},$$

we may conclude that $a_{m+1}^{(i)} \approx |d_{i+1}|$.

\[\square\]

Numerical tests with Roy’s matrices (see Section 8.1) seem to indicate that the partial quotients $a_m^{(i)}$ are small, just as the content of $x_m^{(i)}$ (therefore the combined graph of $L_u$ does not match the outlines of the gray area). A more precise study of these quantities could lead us to a better comprehension of the combined graph of $L_u$ in the gray areas.
7.4 Proof of Propositions 7.14 and 7.16

In this section we essentially prove Propositions 7.14 and 7.16 of Section 7.2.

 Lemma 7.23. We have the following inequality

\[
\limsup_{k \to +\infty} \frac{1}{[s_{k+1}; s_k, \ldots, s_1]} \leq \frac{1}{1 + \sigma}, \tag{7.18}
\]

which implies the inequality

\[
\frac{\sigma}{1 + \sigma} \leq 1 - \limsup_{k \to +\infty} \frac{\log(W_{k-1})}{\log(W_k)}. \tag{7.19}
\]

Note that (7.18) and (7.19) are equalities if the sequence \((s_k)_k\) is equal to the constant sequence \((1,1,\ldots)\) (Fibonacci case).

**Proof** By the definition of \(\sigma\) we have

\[
\frac{1}{[s_{k+1}; s_k, \ldots, s_1]} \leq \frac{1}{1 + \frac{1}{[s_k; s_{k-1}, \ldots, s_1]}} \leq \frac{1}{1 + \sigma} + o(1),
\]

and we find the first inequality of our Lemma by taking the supremum of the left hand side. In the case \((s_n)_n = (1,1,\ldots)\) a quick computation shows that equality is reached by using the formula

\[
\limsup_{k \to +\infty} \frac{1}{[s_{k+1}; s_k, \ldots, s_1]} = \liminf_{k \to +\infty} \frac{1}{[s_{k+1}; s_k, \ldots, s_1]} = \frac{1}{\gamma}
\]

(recall that \(\gamma\) denotes the golden ratio).

The second inequality may be deduced from the first one using Lemma 5.3.

**Proof** of Proposition 7.14

Let \(k \geq k_0\) be an integer and let us write \(i = t_k + l\) with \(0 \leq l < s_{k+1}\).

Since \(\log(W_k) > 0\) and \(c_l = q_l + \log(W_k)\) and \(q_{l+1} = q_l + (2 - \delta) \log(W_k)\) (with \(\delta < 1\) by hypothesis), we have the estimates \(q_l < c_l < q_{l+1}\), proving immediately [a] The assertion [b] is a direct consequence of estimates of Proposition 7.12 and of the definition of functions \(L_{z_i}\) and \(L_{\gamma_i}^*\).

Assertions [c] and [d] directly result from the definitions of functions \(\bar{L}_i\) and \(\bar{L}_i^*\).

Now let us prove [c] First assume that \(0 \leq l < s_{k+1} - 1\) and let us show that the form of the combined graph of \(L_{t_{k+1}}\), \(\hat{L}_{t_{k+1}}\) on \([c_l, c_{l+1}]\) is similar to the one on the left of Figure 4.

By [c] and [d] applied with \(i + 1\) and since the inequality \(c_{l+1} < q_{t_{k+1}}\) implies that \(\hat{L}_{t_{k+1}}\) is constant on \([c_l, c_{l+1}]\), it suffices to prove that

\[
\hat{L}_{t_{k+1}}(q_{l+1}) < -\hat{L}_{t_{k+1}}^*(q_{l+1}), \tag{7.20}
\]

\[
-\hat{L}_{t_{k+1}}^*(c_l) \leq \hat{L}_{t_{k+1}}(c_l) \quad \text{and} \quad -\hat{L}_{t_{k+1}}^*(c_{l+1}) \leq \hat{L}_{t_{k+1}}(c_{l+1}), \tag{7.21}
\]

\[
\hat{L}_{t_{k+1}}(c_l) < -\hat{L}_{t_{k+1}}^*(c_l). \tag{7.22}
\]

We claim that the inequality (7.20) holds when \(\delta < \sigma\) (implied by our hypothesis \(\delta < \sigma/\gamma\)).

Inequalities (7.21) are satisfied under the single hypothesis \(\delta \geq 0\) (with equality if and only if \(\delta = 0\)). The last inequality (7.22) holds under the hypothesis \(\delta < 1/\gamma + \sigma\).

In order to prove (7.20) we consider the following equivalences:

\[
-\hat{L}_{t_{k+1}}^*(q_{l+1}) > \hat{L}_{t_{k+1}}(q_{l+1}) \Leftrightarrow (1 - \delta)(l + 2) \log(W_{k+1}) > (l + 1) \log(W_k) + \log(W_{k-1})
\]

\[
\Leftrightarrow \delta < \frac{\log(W_k)}{(l + 2) \log(W_k) + \log(W_{k-1})}.
\]

The right hand side of the last inequality decreases with \(l\), thus it is always greater than (or equal to) \(\log(W_{k+1}) / \log(W_k)\) because \(l \leq s_{k+1} - 2\). Yet according to Lemma 5.3 and by the definition of \(\sigma\), we have for any \(\varepsilon > 0\)

\[
\frac{\log(W_k)}{\log(W_{k+1})} = \frac{1}{[s_{k+1}; s_k, \ldots, s_1]}(1 + o(1)) \geq \sigma - \varepsilon
\]
for any $k$ large enough, proving \((7.20)\) by the equivalences we have just showed (and by using $\delta < \sigma$). Moreover, one may deduce from the calculations that
\[
-\hat{L}^*_t(q_t) - \hat{L}_t(q_t) \to +\infty
\]
as $i$ tends to infinity with $i = t_k + l$ and $0 < l < s_k+1$.

For \((7.21)\), it suffices to note that for each $t_k \leq i < t_{k+1} - 1$ we have
\[
-\hat{L}^*_{t+1}(c_i) = (1 - \delta)\hat{L}_{i+1}(c_i),
\]
\[
-\hat{L}^*_{t+1}(c_{i+1}) = (1 - \delta)\hat{L}_{i+1}(c_{i+1}),
\]
the equality \((7.25)\) remaining valid for $i = t_k - 1$.

Finally, let us prove that the hypothesis $\delta < \frac{\sigma}{1+\sigma}$ implies \((7.22)\). Since $c_i$ is the intersection point abscissa of $\hat{L}^*_t$ and $\hat{L}^*_{t+1}$, we have
\[
-\hat{L}^*_{t+1}(c_i) > \hat{L}_{t+1}(c_i) \iff -\hat{L}^*_t(c_i) > \hat{L}_{t+1}(c_i)
\]
\[
\iff \log(E^*_t) > \log(W_k)
\]
\[
\iff 1 - \frac{1}{(l + 1) + \frac{\log(W_{k-1})}{\log(W_k)}} > \delta.
\]
The left hand side of the last equality decreases with $l$, therefore it reaches its minimum for $l = 0$.

Yet, by the definition of $\sigma$, for any $\varepsilon > 0$ small enough we have for $k$ large enough
\[
1 - \frac{1}{1 + \frac{\log(W_{k-1})}{\log(W_k)}} > 1 - \frac{1}{1 + \sigma} - \varepsilon > \delta,
\]
thus allowing \((7.22)\) to hold according to the previous equivalences. This ends the proof that the form of the combined graph of $\hat{L}_{t+1}, \hat{L}^*_t, \hat{L}_t$ on $[c_i, c_{i+1}]$ is similar to the drawing on the left of Figure 1 for the case $t_k \leq i < t_{k+1} - 1$.

Let us determine the form of the combined graph of $\hat{L}_{t+1}, -\hat{L}^*_t, \hat{L}_t$ on $[c_{t_k-1}, c_{t_k}]$. Thanks to \((c)\) and \((d)\) if we prove that
\[
\hat{L}_{t_k}(c_{t_k-1}) \leq -\hat{L}^*_t(c_{t_k-1}) \leq \hat{L}_{t+k}(c_{t_k-1})
\]
and
\[
\hat{L}_{t+k}(c_{t_k}) \leq -\hat{L}^*_t(c_{t_k}) \leq \hat{L}_t(c_{t_k})
\]
hold, then it shows that the form of the combined graph is similar to the drawing on the right of Figure 1. If $t_k + 1 < t_{k+1}$, since $\hat{L}^*_{t_k}(c_{t_k}) = -\hat{L}^*_{t+1}(c_{t_k})$ and $\hat{L}_{t_k}(c_{t_k}) = \hat{L}_{t+k}(c_{t_k})$, then inequalities \((7.27)\) come from the form of the combined graph on $[c_i, c_{i+1}]$ for $i = t_k$, which has already been proved. If $t_k + 1 = t_{k+1}$, since $\hat{L}^*_{t_k}(c_{t_k}) = -\hat{L}^*_{t+1}(c_{t_k})$ and $\hat{L}_{t_k}(c_{t_k}) = \hat{L}_{t+k+1}(c_{t_k})$, then inequalities \((7.27)\) are in fact inequalities \((7.26)\) for $k + 1$. Let us prove \((7.26)\). We have
\[
-\hat{L}^*_{t_k}(c_{t_k-1}) = -\log(Y_{t_k}) + \log(Y_{t_k}) - \log(E^*_{t_k-1}) = (1 - \delta)\log(W_k),
\]
from which we directly deduce $-\hat{L}^*_t(c_{t_k-1}) \leq \hat{L}_{t+k}(c_{t_k-1}) = \log(W_{k-1})$, with equality if and only if $\delta = 0$. Furthermore, we have $\hat{L}_{t_k}(c_{t_k-1}) = \log(W_{k-1})$, therefore the relation \((7.28)\) shows that \((7.26)\) results from the upper bound
\[
\delta < 1 - \frac{\log(W_{k-1})}{\log(W_k)},
\]
which is provided by the hypothesis $\delta < \frac{\sigma}{1+\sigma}$ and by inequality \((7.19)\). This ends the proof of \((7.26)\), therefore the combined graph of $\hat{L}_{t+k}, -\hat{L}^*_t, \hat{L}_t$ on $[c_{t_k-1}, c_{t_k}]$ has indeed the announced form. Moreover, inequalities \((7.26)\) and \((7.27)\) are strict inequalities if $\delta > 0$. 

Finally let us prove the assertion [e]. According to the assertion [e] (cf Figure 1) for \( i = t_k + l \) with \( 0 < l < s_{k+1} \), we have
\[
\max (\hat{L}_i(q_i), \hat{L}_{t_{k+1}}(q_i)) = \hat{L}_i(q_i),
\]
and by (7.28), we have the estimate [f] if \( i \) is of the form \( i = t_k + l \) with \( 0 < l < s_{k+1} \). Let us fix \( \varepsilon > 0 \). For \( i = t_k \) we have
\[
\max (\hat{L}_i(q_i), \hat{L}_{t_{k+1}}(q_i)) = \hat{L}_{t_{k+1}}(q_i),
\]
and for \( k \) large enough
\[
\frac{-\hat{L}_{t_k}^*(q_{t_k})}{\hat{L}_{t_{k+1}}(q_{t_k})} = \frac{(1 - \delta)(\log(W_k) + \log(W_{k-1}))}{\log(W_k)} \geq (1 - \delta)(1 + \sigma - \varepsilon).
\]
The hypothesis \( \delta < \frac{\sigma}{1+\sigma} \) is equivalent to \( (1 - \delta)(1 + \sigma) > 1 \), thus by choosing \( \varepsilon \) small enough, we obtain the existence of a constant \( \mu \) such that \( -\hat{L}_{t_k}^*(q_{t_k})/\hat{L}_{t_{k+1}}(q_{t_k}) > \mu > 1 \) for each \( k \) large enough. This proves (7.3) for \( i \) of the form \( i = t_k \), which ends the proof of [e].

**Proof of Proposition 7.16**

First let us prove that \( \mathbf{P} \) is a 3-system. We start with the continuity of \( \mathbf{P} \). We are reduced to verifying that \( \mathbf{P} \) is continuous at abscissas \( c_i \). Recall that for \( t_k \leq i < t_{k+1} - 1 \), we have
\[
-\hat{L}_i^*(c_i) = -\hat{L}_{i+1}^*(c_i) \quad \text{and} \quad \hat{L}_i(c_i) = \hat{L}_{i+1}(c_i).
\]
This directly ensures the continuity of \( \mathbf{P} \) on the whole interval \([c_{t_k-1}, c_{t_{k+1}-1}]\). Taking \( i = t_{k+1} - 2 \), at point \( c_{i+1} \) we have
\[
-\hat{L}_{i+1}^*(c_{i+1}) = -\hat{L}_{t_{k+1}}^*(c_{i+1}) \quad \text{and} \quad \hat{L}_{i+1}(c_{i+1}) = \hat{L}_{t_{k+2}}(c_{i+1}).
\]
Yet, by the definition of \( \mathbf{P} \), we have
\[
\mathbf{P}(c_{i+1}) = \Phi_3 \left( \hat{L}_{t_{k+1}}(c_{i+1}), \hat{L}_{t_{k+2}}(c_{i+1}), -\hat{L}_{t_{k+1}}^*(c_{i+1}) \right),
\]
finally leading to \( \mathbf{P} \) being continuous at this point. So the map \( \mathbf{P} : [q_{t_{k+1}}, +\infty) \) is continuous and its components \( P_i \) are piecewise linear with slopes 1 or 0. In view of assertion [e] of Proposition 7.14 and of the combined graph of Figure 1, the form of the combined graph of \( \mathbf{P} \) is that of Figure 2.

Let \( k, i \) be integers with \( k_0 \leq k \) and \( t_k - 1 \leq i < t_{k+1} - 1 \). Recall that \( c_i < q_{k+1} < c_{i+1} \) and monotone increasing with slope 1 on \([q_{k+1}, c_{i+1}]\), while \( -\hat{L}_{t_{k+1}}^* \) is monotone decreasing with slope 1 on \([c_i, q_{k+1}]\), and constant on \([q_{k+1}, c_{i+1}]\). Since \( c_{i+1} < q_{k+1}, \hat{L}_{t_{k+1}} \) is constant on \([c_i, c_{i+1}]\). We may infer from this that \( \mathbf{P} \) satisfies the point (a) of Definition 7.8. In particular, \( P_1 + P_2 + P_3 \) is affine with slope 1.

Since by the definition of \( \mathbf{P} \) we have \( P_1 \leq P_2 \leq P_3 \), showing that \( \mathbf{P} \) fulfills the point (a) of Definition 7.8 requires to prove that \( P_1(q) + P_2(q) + P_3(q) = q \) for \( q \geq q_{t_{k+1}} \). And since \( P_1 + P_2 + P_3 \) is affine with slope 1 from the above, it suffices to verify that it is fulfilled at a particular point \( q \).

Yet for \( k \geq k_0 \), we have
\[
P_1(q_{t_k}) + P_2(q_{t_k}) + P_3(q_{t_k}) = \hat{L}_{t_k}(q_{t_k}) + \hat{L}_{t_{k+1}}(q_{t_k}) - \hat{L}_{t_k}^*(q_{t_k}) = \log(W_{k-1}) + \log(W_k) - (\delta - 1)(\log(W_k) + \log(W_{k-1})) = q_{t_k}.
\]
Finally, in view of the form of its combined graph given in Figure 2 it is clear that \( \mathbf{P} \) fulfills the point (c) of Definition 7.8. As we claimed, \( \mathbf{P} \) is therefore a 3-system.

Now let us prove the assertions on \( \mathbf{L} \) and \( \mathbf{P} \). First note that if \( i \neq j \) (with \( i \) and \( j \) large enough) then \( z_i \) and \( z_j \) are linearly independent. If not, then there would exist a constant \( \lambda \) such that \( L_{z_i} = L_{z_j} + \lambda \) and it is clear in view of Figure 2 that this never happens. Let \( i = t_k + l \) be an
integer with $0 \leq l < s_{k+1}$ and $k$ large enough. By the definition of the successive minima, we deduce by the above and by assertion (b) of Proposition 7.14 that we have for $q \in I_i = [a_i, b_i]$

$$L_1(q) \leq \min(L_{a_i}(q), L_{a_{i+1}}(q)) = P_1(q) + O(1), \quad (7.29)$$
$$L_2(q) \leq \max(L_{a_i}(q), L_{a_{i+1}}(q)) = P_2(q) + O(1). \quad (7.30)$$

On the other hand, we also have $L_1^*(q) \leq L_{y_i}^*(q)$. Let us prove that $y_i$ is proportional to a minimal point and that there exists a constant $c > 0$ depending only on $\xi$ such that for each $i$ large enough and for each $q \in I_i$, we have:

$$[L_1^*(q) - \hat{L}_i^*(q)] \leq c, \quad (7.31)$$

i.e. $y_i$ realizes $L_1^*$ on $I_i$ within $O(1)$. Indeed, let $q$ belong to $I_i$ and $x \neq 0$ be a point (which depends at first sight on $q$) such that $L_1^*(q) = L_{y_i}^*(q)$, and suppose that $y_i$ is not proportional to $x$. Then, necessarily $L_2^*(q) \leq L_{y_i}^*(q)$, and by the definition of the successive minima we obtain $L_2^*(q) \leq L_{y_i}^*(q)$. Thus, since $L_2 = -L_2^* + O(1)$ by Proposition 7.6, we have the inequality

$$-L_{y_i}^*(q) + O(1) \leq L_2(q).$$

This inequality combined with inequality (7.30) and with assertion (b) of Proposition 7.14 yields:

$$P_3(q) = -\hat{L}_i^*(q) = -L_{y_i}^*(q) + O(1) \leq P_2(q) + O(1) = \max(\hat{L}_i(q), \hat{L}_{i+1}(q)) + O(1), \quad (7.32)$$

for $q \in I_i$. According to assertion (e) of Proposition 7.14 (cf Figure 1) and by (7.4), there exists $c' > 0$ which does not depend on $i$, such that inequality (7.32) does not hold if $q \in [a_i + c', b_i - c']$. If $q$ belongs to this interval, we infer that $y_i$ is proportional to the point $x$ (which is a minimal point), and since the content of $y_i$ is bounded by hypothesis, it follows that $|L_1^*(q) - L_{y_i}^*(q)| \leq O(1)$. Then, with the choice of $x$, this may be rewritten as

$$|\hat{L}_i^*(q) - L_1^*(q)| \leq O(1).$$

This inequality is valid on $[a_i + c', b_i - c']$ and by 1-Lipschitz character of $\hat{L}_i^*$ and $L_1^*$ it directly extends to $I_i$, which proves (7.31). By Proposition 7.6 this implies

$$L_3(q) + O(1) \leq -\hat{L}_i^*(q) = P_3(q),$$

for $q \in I_i$. From (7.29), (7.30) and from the previous inequality we deduce

$$L_j(q) \leq P_j(q) + O(1) \quad \text{for } j = 1, 2, 3.$$

Moreover by assertion (b) of Proposition 7.7 and point (a) of Definition 7.8 of a 3-system, this implies

$$L_j(q) = P_j(q) + O(1) \quad \text{for } j = 1, 2, 3, \quad (7.33)$$

for $q \in I_i$. This proves assertion (b) of Proposition. Assertion (a) follows from the growth of $L_1$ and from (7.33). Indeed, for $i = t_k + l$ with $0 < l \leq s_{k+1}$ we have (cf Figures 1 and 2)

$$L_1(b_{i-1}) + O(1) = P_1(b_{i-1}) = \hat{L}_{i+1}(b_{i-1}) = \log(\hat{W}_k),$$

and

$$L_1(a_i) + O(1) = P_1(a_i) = \hat{L}_{i+1}(a_i) = \log(\hat{W}_k) = L_1(b_{i-1}) + O(1),$$

thus

$$L_1(q) + O(1) = P_1(q)$$

for $q \in [b_i, a_{i+1}] = I_i$, and since families $(I_j)_j$ and $(I'_j)_j$ cover $[M, +\infty)$ for $M$ large enough, this implies (a)

The assertion (c) may also be deduced from (7.33). Indeed these inequalities for $q = b_j, a_{j+1}$ imply $L_2(q) + O(1) = L_3(q) + O(1) = P_2(q) = P_3(q)$, and we conclude noticing that $L_2, L_3$ are continuous piecewise linear with slopes 0 or 1.

\[\square\]
8 Examples of proper $\psi$-numbers

In Section 8.1 we show that Roy’s matrices [14] Examples 3.3 give $\psi$-Sturmian sequences and proper $\psi$-Sturmian numbers. Thanks to these examples we may prove Corollary 1.4. In Section 8.2 we present Bugeaud-Laurent’s examples. They give proper $\psi$-Sturmian numbers for which the associated $\delta$ given by Proposition 5.6 satisfies $\delta = 0$.

8.1 Roy’s matrices

Let $\psi \in \mathcal{F}_{\text{stur}}$ be a Sturmian function and let $\sigma$ denote the real number defined by (7.2). In this section we show that Roy’s examples (see Examples 3.3, 4.3 and 5.4 of [14]) provide examples of proper $\psi$-Sturmian numbers. The main result due to Roy’s work [14] is Proposition 8.3.

**Example 8.1** (Roy, 2007). Example 3.3 of [14] remains valid in our case: if we fix an integer triple $A = (a, b, c)$ such that $a \geq 2$ and $c \geq b \geq 1$, then the $\psi$-Sturmian sequence $(w_i)_{i \geq 0}$ defined by

$$w_0 = \begin{pmatrix} 1 \\ a \\ a(b+1) \end{pmatrix}, \quad w_1 = \begin{pmatrix} 1 \\ a \\ a(c+1) \end{pmatrix}$$

is admissible with the matrix

$$\psi = \begin{pmatrix} -1 + a(b+1)(c+1) & -a(b+1) \\ -a(c+1) & a \end{pmatrix}.$$ 

With these matrices we have the following result.

**Proposition 8.2.** The $\psi$-Sturmian sequence $(w_i)_i$ has a multiplicative growth (we may choose the constants $c_1 = 1$ and $c_2 = 2$ in (5.1)) and $(||w_i||)_i$ tends to infinity. Moreover for each $i \geq 0$ the matrix $w_i$ is primitive, and $\text{Tr}(w_i)$ and $\text{det}(w_i)$ are relatively prime. In particular, $(w_i)_i$ gives a $\psi$-Sturmian number $\xi_A$.

**Remark.** The hypotheses of Corollary 1.4 are fulfilled. As Roy points out [14] Example 5.4], since $\text{det}(w_0) = \text{det}(w_1) = a$, the condition (5.8) is true for $i = 0, 1$ with

$$\alpha = \frac{\log(a)}{\log(2a(c+1))} \quad \text{and} \quad \beta = \frac{\log(a)}{\log(2a(b+1))}.$$ 

**Proof** The multiplicative growth with the announced implicit constants is implied by Lemma 5.2. The fact that $(||w_i||)_i$ tends to infinity follows from $c_1 = 1$ and from $||w_0||, ||w_1|| > 1$ (since $a > 1$).

Considering the matrices $w_i$ modulo $a$, Roy shows [14] Example 4.3] that $\text{Tr}(w_i)$ and $\text{det}(w_i)$ are relatively prime, which implies that $w_i$ is primitive.

We denote by $\delta_A \geq 0$ the real number provided by Proposition 5.6. We have the inequalities

$$\alpha \leq \delta_A \leq \beta. \quad (8.1)$$

**Proposition 8.3.** The set $\{\delta_A \mid A = (a, b, c) \text{ with } a \geq 2 \text{ and } c \geq b \geq 1\}$ is dense in $[0, \frac{\sigma}{1+\sigma}]$.

**Proof** Cf [14] Proof of Corollary 7.2]. Fix $\delta \in (0, \frac{\sigma}{1+\sigma})$ and $\varepsilon > 0$. Then, there are two integers $k$ and $l$ with $0 < l < k$ and

$$\delta - \varepsilon \leq \frac{l}{k+2} \leq \frac{l}{k} \leq \delta.$$ 

Consider the integer triple $A = (a, b, c)$ with $a = 2^{l+1}, b = 2^{k-1} - 1$ and $c = 2^{k-1}$. With this choice we have $\alpha = l/(k+2)$ and $\beta = l/k$. Thus by (8.1) $\delta_A$ satisfies $|\delta - \delta_A| \leq \varepsilon$. 

□

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8.2 Bugeaud and Laurent’s examples

In this section we briefly present Bugeaud and Laurent’s examples and show that their numbers \( \xi_{\psi'} \) are (proper) \( \psi \)-Sturmian numbers for which the associated \( \delta \) satisfies \( \delta = 0 \).

Let \( \xi' = (s'_k)_{k \geq 1} \) be a sequence of positive integers. Fix \( a \) and \( b \) two distinct positive integers. We define inductively a sequence of words \((m_k)_{k \geq 0}\) by the formulas

\[
m_0 = b, \quad m_1 = b^{s'_1-1}a \quad \text{and} \quad m_{k+1} = m_k s'_k m_{k-1} \quad (k \geq 1).
\]

This sequence converges to the infinite word

\[
m_\psi = \lim_{k \to +\infty} m_k = b^{s'_1-1}a \ldots,
\]

which is usually called the \textit{Sturmian characteristic word of angle} (or of slope) \( \phi := [0; s'_1, s'_2, s'_3, \ldots] \) constructed on the alphabet \( \{a,b\} \). Denote by \( \xi_{\psi'} = [0; m_{\psi'}] \) the real number whose partial quotients are successively \( 0 \), and the letters of the infinite word \( m_{\psi'} \).

Now, set \( \xi = (s_k)_{k \geq 0} \) be the sequence defined by \( s_0 = -1, \ s_1 = 1 \) and \( s_k = s'_k \) \( (k \geq 2) \). We denote by \( \psi \) the Sturmian function associated with \( \xi \). By Lemma 5.3 of [5], the sequence \((w_i)_{i} \) is admissible with the matrix

\[
N = \left[ \begin{array}{cc} a & 1 \\ l_1 & 1 \\
\end{array} \right]^{-1}.
\]

Proposition 8.4. The sequence \((w_i)_{i} \) satisfies the hypotheses of Section 6 and the associated (proper) \( \psi \)-Sturmian number is \( \xi_{\psi'} \). Moreover, the associated \( \delta \) (given by Proposition 5.6) is equal to \( 0 \).

Proof First note that \((w_i)_{i} \) has multiplicative growth (each \( w_i \) belongs to \( \mathcal{M} \)), the sequence \( (|w_i|)_i \) tends to infinity and \( |\det(w_i)| = 1 \) for each \( i \) (so that the associated \( \delta \) is equal to \( 0 \)). We denote by \( (y_i)_i \) the sequence of symmetric matrices associated with \((w_i)_{i} \) and by \( l_i, l_2, \ldots \) the letters of the word \( m_{\psi'} \). Let \( p_n/q_n \) denote the \( n \)-th convergent of \( \xi_{\psi'} \). The classical properties of continued fraction expansions give (see for instance [20] Chapter I)

\[
\left( \begin{array}{cc} q_n & q_{n-1} \\ p_n & p_{n-1} \end{array} \right) = \left( \begin{array}{cc} l_1 & 1 \\ 1 & 1 \end{array} \right) \cdots \left( \begin{array}{cc} l_n & 1 \\ 1 & 1 \end{array} \right).
\]

Since each matrix \( y_i \) is of the last form for a particular \( n \) (by Lemma 5.3 of [5]), this implies directly that the \( \psi \)-number given by Proposition 6.1 is equal to \( \xi_{\psi'} \).

\( \square \)

9 Proofs of Theorem 1.1 and Corollary 1.3

Let \( \psi \in \mathcal{S} \) be a Sturmian function. We denote by \( \sigma \) the real number defined by (7.2). In this section we prove Theorem 1.1 and Corollary 1.3 stated in the introduction.

Proof of Theorem 1.1

Let us set

\[
\Delta_\psi = \{ \delta_\Delta \mid \Delta = (a,b,c) \text{ with } a \geq 2, c \geq b \geq 1 \text{ and } \delta_\Delta < \frac{\sigma}{1+\sigma} \} \cup \{0\}.
\]
Proposition 8.3 implies that $\Delta_\psi$ is dense in $[0, \sigma/(1 + \sigma)]$. By virtue of Propositions 8.2 and 8.4, for each $\delta \in \Delta_\psi$, there exists a proper $\psi$-Sturmian number $\xi$ such that $\delta$ is the number associated to $\xi$ by Proposition 7.6. We conclude by Theorem 7.3 together with Proposition 7.17.

**Proof** (of Corollary 1.3)

In order to establish his results, Cassaigne [6] shows that

$$S = \{1/b \mid b \in (\mathbb{Z}_{\geq 1})^\mathbb{N}\} \text{ such that for each } k \geq 0 \text{ we have } |b| \geq |T^k b|, \tag{9.1}$$

where $|b|$ is the continued fraction $[b_0; b_1, \ldots]$ and $T$ is the right shift operator ($|T^k b|$ is the real number $b_0; b_{k+1}, \ldots$).

According to Theorem 1.1 for each $\sigma \in S$ the set of $\omega_2(\xi)$ with $\xi$ a proper $\psi$-Sturmian number and $\sigma(\psi) = \sigma$ is dense in the interval $[2/\sigma, 1 + 2/\sigma]$. It is therefore sufficient to show that

$$\bigcup_{\sigma \in S} \left[\frac{2}{\sigma}, \frac{2}{\sigma} + 1\right] = [1 + \sqrt{5}, 2 + \sqrt{5}] \cup [2 + 2\sqrt{2}, 3 + 2\sqrt{3}] \cup [3 + \sqrt{13}, +\infty).$$

For $n \geq 1$ and $1 \leq a \leq n$ we define the sequence $u_{a,n} = (s_k)_{k \geq 0}$ by

$$\begin{cases} s_{2k} = n, \\ s_{2k+1} = a, \end{cases}$$

for each $k \geq 0$. It is clear that for each $k \geq 0$ we have $u_{a,n} \geq |T^k u_{a,n}|$, thus $1/u_{a,n} \in S$, and

$$[u_{a,n}] = \left[n; a, \frac{an + \sqrt{(an)^2 + 4an}}{2a}\right].$$

Then we define

$$\delta_{a,n} = 2|u_{a,n}| = n + n\sqrt{1 + \frac{4}{an}}.$$

Let us study the union of intervals $[\delta_{a,n}, \delta_{a,n} + 1]$ for $n \geq 1$ and $1 \leq a \leq n$.

We remark that

$$0 \leq \delta_{a,n} - \delta_{a+1,n} \leq 1 \quad (n \geq 2 \text{ and } 1 \leq a < n), \tag{9.2}$$

and

$$|\delta_{n+1,n+1} - \delta_{1,n}| < 1 \quad (n \geq 3). \tag{9.3}$$

We may deduce from (9.2) and from (9.3) that

$$\bigcup_{1 \leq a \leq n} [\delta_{a,n}, \delta_{a,n} + 1] = [\delta_{1,1} , \delta_{1,1} + 1] \cup [\delta_{2,1}, \delta_{2,1} + 1] \cup [\delta_{3,1}, +\infty)$$

$$= [1 + \sqrt{5}, 2 + \sqrt{5}] \cup [2 + 2\sqrt{2}, 3 + 2\sqrt{3}] \cup [3 + \sqrt{13}, +\infty).$$

As pointed out by Bugeaud and Laurent in [5], the two greatest values of $\sigma \in S$ are $(1 + \sqrt{5})/2$ and $-1 + \sqrt{5}$. This implies that the two smallest values of $2/\sigma$ for $\sigma \in S$ is $\delta_{1,1} = 2 + \sqrt{5} = 3.23\ldots$ and $\delta_{2,2} = 2 + 2\sqrt{2} = 4.82\ldots$. To conclude, it is enough to show that there does not exist any $\sigma \in S$ such that $\delta_{1,2} < 2/\sigma < \delta_{3,3}$. By (9.1), it suffices to show that if $b \in (\mathbb{Z}_{\geq 1})^\mathbb{N}$ is such that $|b| \geq |T^k b|$ for each $k \geq 0$ and if

$$[2; T^2] \leq [b_0; b_1, \ldots] \leq [3; 3],$$

then, either $[b_0; b_1, \ldots] = [2; T^2]$ or $[b_0; b_1, \ldots] = [3; 3].$

Note that if $a = (a_k)_{k \geq 0}$ and $a' = (a'_{k})_{k \geq 0}$ are two distinct sequences of positive integers and if $k$ is the smallest index for which $a_k \neq a'_k$, then $[a] > [a']$ if and only if $a_k > a'_k$ if $k$ is even or $a_k < a'_k$ if $k$ is odd.

Set $x = [b_0; b_1, \ldots]$. The integer part of $x$ is $b_0$ and is equal to 2 or 3. Since $x \geq |T^k b|$ for each $k \geq 0$, we have $1 \leq b_k \leq b_0$ for all $k$. Suppose that $b_0 = 2$. Then, with the previous remark we may show by induction that $x = [2; T^2]$. Similarly, if $b_0 = 3$ we show that $x = [3; 3].$
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