ASYMPTOTIC REPETITIVE THRESHOLD OF BALANCED SEQUENCES

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Abstract. The critical exponent $E(u)$ of an infinite sequence $u$ over a finite alphabet expresses the maximal repetition of a factor in $u$. By the famous Dejean’s theorem, $E(u) \geq 1 + \frac{1}{d-1}$ for every $d$-ary sequence $u$. We define the asymptotic critical exponent $E^*(u)$ as the upper limit of the maximal repetition of factors of length $n$. We show that for any $d > 1$ there exists a $d$-ary sequence $u$ having $E^*(u)$ arbitrarily close to 1. Then we focus on the class of $d$-ary balanced sequences. In this class, the values $E^*(u)$ are bounded from below by a threshold strictly bigger than 1. We provide a method which enables us to find a $d$-ary balanced sequence with the least asymptotic critical exponent for $2 \leq d \leq 10$.

1. Introduction

The concatenation of $e \in \mathbb{N}$ copies of a non-empty word $u$ is usually abbreviated as $u^e$. In 1972, Dejean extended this exponential notation to rational exponents. If $u$ is a non-empty word of length $\ell$ and $e$ is a positive rational number of the form $n/\ell$, then $u^e$ denotes the prefix of length $n$ of the infinite periodic sequence $uuu \cdots = u^\omega$. For instance, a Czech word starosta (mayor) can be written in this formalism as $(staro)^8/5$. The rational exponent $e$ describes the repetition rate of $u$ in the string $u^e$.

The critical exponent $E(u)$ of an infinite sequence $u = u_0u_1u_2\cdots$ captures the maximal possible repetition rate of factors occurring in $u$, formally,

$$E(u) = \sup \{e \in \mathbb{Q} : u^e \text{ is a factor of } u \}.$$ 

The number $E(u)$ can be rational or irrational. Krieger and Shallit [20] have shown that any positive real number larger than one is a critical exponent of some sequence over a finite alphabet. If the size of the alphabet is a fixed number $d \in \mathbb{N}, d \geq 2$, then the critical exponent of any sequence $u$ over this alphabet cannot be smaller than a threshold larger than one. This bound is denoted in [7] as $RT(d)$ and called the repetitive threshold, i.e.,

$$RT(d) = \inf \{E(u) : u \text{ is a sequence over a } d\text{-ary alphabet} \}.$$ 

For instance, if $u$ is a binary sequence over $\{0, 1\}$, then 00 or 11 or 0101 appears in $u$, and thus $E(u) \geq 2$. The existence of an infinite binary sequence $u$ with $E(u) = 2$ was demonstrated by Thue in 1912. The binary sequence for which $E(u) = 2$ is nowadays known as the Thue-Morse sequence (or the Prouhet-Thue-Morse sequence). Therefore $RT(2) = 2$. Dejean [9] proved that $RT(3) = \frac{7}{4}$ and this bound is attained. In the same paper she also conjectured:

- $RT(4) = 7/5$;
- $RT(d) = 1 + \frac{1}{d-1}$ for $d \geq 5$.

The conjecture had been proved step by step by many people [5, 7, 21, 23, 24, 27].

Recently, Rampersad, Shallit and Vandome asked the same question for $d$-ary balanced sequences. Let us recall that a sequence over a finite alphabet is balanced if, for any two of its factors $u$ and $v$ of the same length, the number of occurrences of each letter in $u$ and $v$ differs by at most 1. Binary

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balanced aperiodic sequences coincide with Sturmian sequences and the least critical exponent is equal to \(2 + \frac{\sqrt{5}}{2}\) and it is reached by the Fibonacci sequence \([6]\). Let us denote the repetitive threshold of balanced sequences over a \(d\)-ary alphabet by \(RTB(d)\), i.e.,

\[
RTB(d) = \inf\{E(\mathbf{v}) : \mathbf{v} \text{ is a balanced sequence over a } d\text{-ary alphabet}\}.
\]

The following results have been proved so far:

- \(RTB(3) = 2 + \frac{\sqrt{5}}{2}\) and \(RTB(4) = 1 + \frac{\sqrt{5}}{2}\) \([20]\);
- \(RTB(d) = 1 + \frac{\sqrt{d}}{2}\) for \(5 \leq d \leq 10\) \([2, 3, 12]\);
- \(RTB(d) = 1 + \frac{\sqrt{d}}{2}\) for \(d = 11\) and all even numbers \(d \geq 12\) \([15]\).

It remains as an open problem to prove the conjecture \(RTB(d) = 1 + \frac{1}{\sqrt{d}}\) also for all odd numbers \(d \geq 13\).

In this paper we focus on the asymptotic critical exponent \(E^*(\mathbf{u})\) of \(\mathbf{u}\). It is defined to be \(+\infty\) if \(E(\mathbf{u}) = +\infty\) and

\[
E^*(\mathbf{u}) = \limsup_{n \to \infty} \{e \in \mathbb{Q} : u^e \text{ is a factor of } \mathbf{u} \text{ for some } u \text{ of length } n\},
\]

otherwise. Obviously, \(E^*(\mathbf{u}) \leq E(\mathbf{u})\) and the equality holds true whenever \(E(\mathbf{u})\) is irrational. It is for instance the case of the Fibonacci sequence. Nevertheless, \(E^*(\mathbf{u})\) and \(E(\mathbf{u})\) can coincide even if \(E(\mathbf{u})\) is rational: it is the case of the Thue-Morse sequence.

While the value \(E(\mathbf{u})\) takes into account repetitions of all factors of \(\mathbf{u}\), \(E^*(\mathbf{u})\) considers only repetitions of factors of length tending to infinity. There is a huge literature devoted to questions situated between these two extremes. For example, Shur and Tuncer \([31]\) construct a \(d\)-ary sequence \(\mathbf{u}\) whose all factors (except the trivial repetition of one letter in factors of the form \(a_1a_2\cdots a_{q-1}a_1\)) have the exponent \(< 1 + \frac{1}{\sqrt{d}} = RT(\mathbf{d})\). Other results of this flavour can be found in \([1, 10, 29]\).

We provide a simple construction showing that for any \(d \in \mathbb{N}, d \geq 2,\) and any \(\epsilon > 0\), there exists a \(d\)-ary sequence \(\mathbf{u}\) with \(E^*(\mathbf{u}) < 1 + \epsilon\). Therefore, if we denote

\[
RT^*(d) = \inf\{E^*(\mathbf{u}) : \mathbf{u} \text{ is a sequence over a } d\text{-ary alphabet}\},
\]

we have \(RT^*(d) = 1\) for all \(d \geq 2\). Then we restrict our study to balanced sequences and look for the threshold

\[
RTB^*(d) = \inf\{E^*(\mathbf{v}) : \mathbf{v} \text{ is a balanced sequence over a } d\text{-ary alphabet}\}.
\]

This threshold is bounded from below by \(1 + \frac{\sqrt{d}}{2}\), see Corollary \([15]\). It is known that \(RTB^*(2) = \sqrt{5}\) and \(RTB^*(2) = 2 + \frac{\sqrt{5} + 1}{2} = 3.618\) and it is reached by the Fibonacci sequence. We introduce a new tool – graphs of admissible tails – for computation of the exact value of \(RTB^*(d)\). Using it we obtain at once \(RTB^*(d)\) for \(d \leq 10\), see Table \([\ref{tab:results}]\). Let us point out that a similar result for \(RTB(d)\) had been done step by step by several research teams.

Comparing the results with the minimal critical exponent of balanced sequences, we can see that \(RTB^*(d) = RTB(d)\) for \(d \in \{2, 3, 4, 5\}\), but \(RTB^*(d) < RTB(d)\) for larger \(d\). Moreover, the precise values \(RTB^*(d)\) we have found for \(d \leq 10\) suggest that \(RTB^*(d) < 1 + q^2\) for some positive \(q < 1\), whereas \(RTB(d) > RT(d) = 1 + \frac{1}{\sqrt{d}}\). Looking into Table \([\ref{tab:results}]\) we can see that

\[
RTB^*(6) < 1 + \frac{1}{\sqrt{6}}, \quad RTB^*(7) < 1 + \frac{1}{\sqrt{7}}, \quad RTB^*(8) < 1 + \frac{1}{\sqrt{8}}, \quad RTB^*(9) < 1 + \frac{1}{\sqrt{9}}, \quad RTB^*(10) < 1 + \frac{1}{\sqrt{10}}.
\]

The time and space complexity of our algorithm allowed us to determine \(RTB^*(d)\) only for \(d \leq 10\). It seems that a new idea is needed to extend Table \([\ref{tab:results}]\) for \(d \geq 11\).

2. Preliminaries

An alphabet \(\mathcal{A}\) is a finite set of symbols called letters. A word over \(\mathcal{A}\) of length \(n\) is a string \(u = u_0u_1\cdots u_{n-1}\), where \(u_i \in \mathcal{A}\) for all \(i \in \{0, 1, \ldots, n-1\}\). The length of \(u\) is denoted by \(|u|\). The set of all finite words over \(\mathcal{A}\) together with the operation of concatenation forms a monoid, denoted \(\mathcal{A}^*\). Its neutral element is the empty word \(\varepsilon\) and we denote \(\mathcal{A}^+ = \mathcal{A}^* \setminus \{\varepsilon\}\). If \(u = xyz\) for some \(x, y, z \in \mathcal{A}^*\), then \(x\) is a prefix of \(u\), \(z\) is a suffix of \(u\) and \(y\) is a factor of \(u\). To any word \(u\) over \(\mathcal{A}\) with cardinality
Theorem 2. The number of occurrences of a factor \( y \) of a sequence \( u \) is denoted by \( R_y(u) \). In particular, if \( i \) is not eventually periodic, then it is recurrent. Moreover, if for each factor the distances between its consecutive occurrences are bounded, \( u \) is uniformly recurrent.

The language \( \mathcal{L}(u) \) of a sequence \( u \) is the set of all its factors. A factor \( w \) of \( u \) is right special if \( wa,wb \) are in \( \mathcal{L}(u) \) for at least two distinct letters \( a,b \in A \). A left special factor is defined symmetrically.

A sequence \( u \in \mathcal{A}^N \) is balanced if for every letter \( a \in A \) and every pair of factors \( u,v \in \mathcal{L}(u) \) with \( |u| = |v| \), we have \( |u|_a - |v|_a \leq 1 \). Every recurrent balanced sequence over any alphabet is uniformly recurrent, see [11].

A morphism over \( A \) is a mapping \( \psi : A^* \to A^\ast \) such that \( \psi(wv) = \psi(w)\psi(v) \) for all \( w,v \in A^\ast \). Morphisms can be naturally extended to \( \mathcal{A}^N \) by setting \( \psi(uq_1u_2\cdots) = \psi(u_0)\psi(u_1)\psi(u_2)\cdots \). A fixed point of a morphism \( \psi \) is a sequence \( u \) such that \( \psi(u) = u \).

Example 1. The most famous Sturmian sequence is the Fibonacci sequence
\[
\mathbf{u}_f = \text{babbababbababbabab},
\]
defined as the fixed point of the morphism \( f : b \mapsto ba, a \mapsto b \). The critical exponent of \( \mathbf{u}_f \) is \( 2 + \frac{1+\sqrt{5}}{2} \). As shown by Carpi and de Luca [6], it is the least critical exponent for Sturmian sequences, i.e., \( RTB(2) = E(\mathbf{u}_f) \).

Consider a factor \( w \) of a recurrent sequence \( u = u_0u_1u_2\cdots \). Let \( i < j \) be two consecutive occurrences of \( w \) in \( u \). Then the word \( u_iu_{i+1}\cdots u_{j-1} \) is a return word to \( w \) in \( u \). The set of all return words to \( w \) in \( u \) is denoted by \( \mathcal{R}_u(w) \). If \( u \) is uniformly recurrent, the set \( \mathcal{R}_u(w) \) is finite for each factor \( w \). If \( w \) is a prefix of \( u \), then \( u \) can be written as a concatenation \( u = w_1w_2\cdots w_n \) of return words to \( w \). The sequence \( d_u(w) = dd_1d_2\cdots \) over the alphabet of cardinality \( \#\mathcal{R}_u(w) \) is called the derived sequence of \( u \) to \( w \). If \( w \) is not a prefix, then we construct the derived sequence in an analogous way starting from the first occurrence of \( w \) in \( u \). The concept of derived sequences was introduced by Durand [13].

### 3. Asymptotic critical exponent and its relation to return words

In [12], a handy formula for computation of the critical exponent and asymptotic critical exponent of uniformly recurrent sequences is deduced. It uses the notion of return words to a factor of a sequence.

**Theorem 2** ([12]). Let \( u \) be a uniformly recurrent aperiodic sequence. Let \( (u_n) \) be a sequence of all bispecial factors ordered by their length. For every \( n \in \mathbb{N} \), let \( r_n \) be a shortest return word to \( u_n \) in \( u \). Then
\[
E(u) = 1 + \sup_{n \in \mathbb{N}} \left\{ \frac{|u_n|}{|r_n|} \right\} \quad \text{and} \quad E^*(u) = 1 + \limsup_{n \to \infty} \frac{|u_n|}{|r_n|}.
\]

**Theorem 3.** Let \( A \) be a finite alphabet of size at least 2. Then
\[
\inf \{ E^*(u) : u \in \mathcal{A}^N, u \text{ uniformly recurrent} \} = 1.
\]

Consequently, \( RT^*(d) = 1 \) for every \( d \geq 2 \).

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1Arseny M. Shur in private communication pointed out to us that the same formula remains valid for recurrent aperiodic sequences.
Proof. It is enough to prove the statement for the alphabet $A = \{0, 1\}$. For every Fibonacci number $F_k$, with $k \geq 7$, we construct a binary sequence $u = u^{(k)}$ such that every bispecial factor $u$ of length at least $3(k+1)$ and every return word $r$ to $u$ in $u$ satisfy $|u| < \frac{2}{\sqrt{5}}$. For the sequence $u^{(k)}$, the second formula of Theorem 2 gives $E^\ast(u^{(k)}) \leq 1 + \frac{|u|}{|r|}$, which implies the theorem. Construction of $u^{(k)}$ follows.

Let $\mathcal{D} = \{0, 1, \ldots, d-1\}$, where $d = 2\left[\frac{1}{2}F_k\right] \in \{F_k-1, F_k\}$. By [19] there exists a balanced (and hence uniformly recurrent) $d$-ary sequence $v$ having $E(v) = \frac{\phi - 1}{2}$. Zeckendorf’s theorem [35] says that every $i \in \mathcal{D}$ can be written in the form $i = \sum_{n=2}^{k-1} c_n F_n$, where $c_{k-1} c_{k-2} \cdots c_2$ is a word over the alphabet $\{0, 1\}$, which does not contain two consecutive 1’s. We denote the $(k-2)$-tuple $c_{k-1} c_{k-2} \cdots c_2$ representing $i$ by $(i)_{Fib}$ and define a morphism $\psi : \mathcal{D}^* \mapsto \{0, 1\}^*$ and the binary sequence $u$ as follows:

$$\psi(i) = 110(i)_{Fib} \text{ for every } i \in \mathcal{D} \text{ and } u = u^{(k)} = \psi(v).$$

The morphism $\psi$ is uniform since the image of any letter $i$ by $\psi$ has length $|\psi(i)| = k + 1$. Moreover, $u$ is uniformly recurrent as $v$ is uniformly recurrent. $\psi$ is a coding since the factor 110 occurs only as a prefix of each $\psi(i)$. Hence any factor $u \in \mathcal{L}(u)$ longer than $k + 2$ can be written uniquely in the form $u = u^{(L)} \psi(v) u^{(R)}$, where $v \in \mathcal{L}(v)$, $u^{(L)}$ is a proper suffix of $\psi(i)$ and $u^{(R)}$ is a proper prefix of $\psi(j)$ for some $i, j \in \mathcal{D}$. Obviously, if $u$ is a left special factor of $u$, i.e., $0u$ and $1u$ belong to the language of $u$, then $v$ is a left special factor of $v$. An analogous statement is true for right special factors of $u$.

Let $u$ be a bispecial factor of $u$ with $|u| \geq 3(k+1)$ and $r$ be a return word to $u$ in $u$. Then there exist a bispecial factor $v \in \mathcal{L}(v)$ of length $|v| \geq 2$ and $s \in \mathcal{L}(v)$ such that $u = u^{(L)} \psi(v) u^{(R)}$ and $ru = u^{(L)} \psi(sv) u^{(R)}$. Obviously, $s$ is a return word or concatenation of several return words to $v$ in $v$ and $|r| = |\psi(s)| = (k + 1)|s|$. As $E(v) = 1 + \frac{1}{\sqrt{5}}$, Theorem [2] implies $\frac{|u|}{|r|} \leq \frac{2}{\sqrt{5}} < \frac{1}{\sqrt{5}}$. Hence

$$\frac{|u|}{|r|} = \frac{|u^{(L)}| + (k+1)|v| + |u^{(R)}|}{(k+1)|s|} \leq \frac{2k + (k+1)|v|}{(k+1)|s|} < \frac{2|v|}{|s|} < \frac{2}{F_k - 3},$$

as we wanted to show. \hfill \Box

4. STURMIAN SEQUENCES

Sturmian sequences, i.e., aperiodic balanced sequences over a binary alphabet, are a principal tool in the study of balanced sequences over arbitrary alphabets. In the sequel, we will restrict our consideration to standard sequences. Let us recall that a Sturmian sequence $u$ is called a standard sequence if both sequences $au$ and $bu$ are Sturmian. For each Sturmian sequence there exists a unique standard sequence having the same language and thus the same critical exponent and the asymptotic critical exponent. Sturmian sequences have well defined frequencies of letters. Let us recall that the frequency of a letter $a$ in a sequence $u$ is the limit $\rho_a(u) = \lim_{n \to \infty} \frac{|uw_{n-1}u|}{n}$ if it exists.

We use the characterization of standard sequences by their directive sequences. To introduce them, we recall two morphisms

$$G = \begin{cases} a \to a \\ b \to ab \end{cases} \text{ and } D = \begin{cases} a \to ba \\ b \to b \end{cases}.$$

Proposition 4 ([19]). For every standard sequence $u$ there is a uniquely given directive sequence $\Delta = \Delta_0 \Delta_1 \Delta_2 \cdots \in \{G, D\}^\infty$ of morphisms and a sequence $(u^{(n)})$ of standard sequences such that $u = \Delta_0 \Delta_1 \cdots \Delta_{n-1} \left(u^{(n)}\right)$ for every $n \in \mathbb{N}$.

Both $G$ and $D$ occur in the sequence $\Delta$ infinitely often.

If $\Delta_0 = D$, then $b$ is the most frequent letter in $u$. Otherwise, $a$ is the most frequent letter in $u$. We adopt the convention that $\rho_b(u) > \rho_a(u)$ and thus the directive sequence of $u$ starts with $D$. Let us write this sequence in the run-length encoded form $\Delta = D^{a_1} G^{a_2} D^{a_3} G^{a_4} \cdots$, where all integers $a_n$ are positive. Then the number $\theta$ having the continued fraction expansion $\theta = [0, a_1, a_2, a_3, \ldots]$ equals the ratio $\frac{\rho_b(u)}{\rho_a(u)}$ (see [4]) and $\theta$ is called the slope of $u$.

2The Fibonacci sequence is defined recursively: $F_0 = 0, F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for every $n \in \mathbb{N}$.
The convergents to the continued fraction of $\theta$, usually denoted $\frac{p_N}{q_N}$, have a close relation to return words in a Sturmian sequence. Recall that the sequences $(p_N)$ and $(q_N)$ both satisfy the recurrence relation
\begin{equation}
X_{N+1} = a_{N+1}X_N + X_{N-1} \tag{1}
\end{equation}
with initial conditions $p_1 = 1$, $p_0 = 0$ and $q_1 = 0$, $q_0 = 1$. Two consecutive convergents satisfy
\[p_Nq_{N-1} - p_{N-1}q_N = (-1)^{N+1}\] for every $N \in \mathbb{N}$.

Vuillon [22] showed that an infinite recurrent sequence $u$ is Sturmian if and only if each of its factors has exactly two return words. Moreover, the derived sequence of a Sturmian sequence to any of its factors is also Sturmian.

All bispecial factors of any standard sequence $u$ are its prefixes. So, one of the return words to a bispecial factor of $u$ is a prefix of $u$.

**Remark 6.** The formulae for computation of $E(u)$ and $E^*(u)$ for a Sturmian sequence $u$ with slope $\theta = [0, a_1, a_2, a_3, \ldots]$ were provided by Daminik and Lenz in [8], see also [15].

\[E(u) = 2 + \sup_{N \in \mathbb{N}} \left\{ a_{N+1} + \frac{q_{N-2}}{q_N} \right\} \quad \text{and} \quad E^*(u) = 2 + \limsup_{N \to \infty} \left\{ a_{N+1} + \frac{q_{N-1}}{q_N} \right\}.
\]

These formulae can be deduced easily using Proposition 5 and Theorem 2.

In the sequel, it will be necessary to recognize which vectors are Parikh vectors of some factors in a Sturmian sequence. The answer follows.

**Lemma 7 ([12]).** Let $u$ be a Sturmian sequence with slope $\theta = [0, a_1, a_2, a_3, \ldots]$ be such that $|w|_b = k$ and $|w|_a = \ell$ if and only if $|\ell \Delta - k| < \delta + 1$.

\[|\ell \Delta - k| < \delta + 1. \tag{2}\]

5. **Balanced sequences**

In 2000 Hubert [13] characterized balanced sequences over alphabets of cardinality bigger than 2 in terms of Sturmian sequences, colourings, and constant gap sequences.

**Definition 8.** Let $u$ be a sequence over $\{a, b\}$, $y$ and $y'$ be arbitrary sequences. The *colouring of $u$ by $y$* and $y'$ is the sequence $v = \text{colour}(u, y, y')$ obtained from $u$ by replacing the subsequence of all $a$'s with $y$ and the subsequence of all $b$'s with $y'$.

**Definition 9.** A sequence $y$ is a constant gap sequence if for each letter $a$ occurring in $y$ the distance between any consecutive occurrences of $a$ in $y$ is constant.

There is a rich literature on a notion equivalent to constant gap sequences, the so-called exact covering systems, see [16,25,34].
Example 10. The periodic sequences \( y = (34)^\omega \) and \( y' = (0102)^\omega \) are constant gap sequences over binary and ternary alphabet, respectively. The sequence \( v = \text{colour}(u_f, y, y') \), where \( u_f \) is the Fibonacci sequence defined in Example 1 looks as follows:

\[
\begin{align*}
\theta_0 &= \text{babbababababababababababababab}\cdots \\
v &= 0310423014023041032401302403104\cdots
\end{align*}
\]

Theorem 11 (15). A recurrent aperiodic sequence \( v \) is balanced if and only if \( v = \text{colour}(u, y, y') \) for some Sturmian sequence \( u \) and constant gap sequences \( y, y' \) over two disjoint alphabets.

Let \( \mathcal{A}, \mathcal{B} \) be two disjoint alphabets. The “discolouration map” \( \pi \) is defined for any word or sequence over \( \mathcal{A} \cup \mathcal{B} \); it replaces all letters from \( \mathcal{A} \) by a and all letters from \( \mathcal{B} \) by b. If \( v = \text{colour}(u, y, y') \), where \( y \in \mathcal{A}^N, y' \in \mathcal{B}^N \), then \( \pi(v) = u \) and \( \pi(v) \in \mathcal{L}(u) \) for every \( v \in \mathcal{L}(v) \).

Corollary 12 (12). Let \( v = \text{colour}(u, y, y') \) and \( u \in \mathcal{L}(u) \). For any \( i, j \in \mathbb{N} \), the word \( v \) obtained by colouring \( u \) with shifted constant gap sequences \( \sigma^i(y) \) and \( \sigma^j(y') \) is in \( \mathcal{L}(v) \). In particular, if a Sturmian sequence \( \tilde{u} \) has the same language as \( u \), then, \( E(v) = E(\tilde{v}) \) and \( E^*(v) = E^*(\tilde{v}) \).

Example 13. Let \( u, v, y \) and \( y' \) be as in Example 10. Let \( u = \text{bab} \). The reader is invited to check that all words 031, 130, 032, 230, 041, 140, 042, 240 are factors of \( v \).

As a consequence of the previous corollary when studying the asymptotic critical exponent of balanced sequences, we can limit our consideration to colourings of standard Sturmian sequences.

Proposition 14 (12). Let \( v = \text{colour}(u, y, y') \) and \( \theta = [0, a_1, a_2, a_3, \ldots] \). One has:

1. \( E(v) \geq E^*(v) \geq 1 + \frac{1}{\text{lcm}(y, y')} \).
2. \( E'(v) \) depends on \( \text{Per}(y) \) and \( \text{Per}(y') \), not on the structure of \( y \) and \( y' \).
3. \( E^*(v) \) is finite \( \iff \text{Per}(u) \) is finite \( \iff (a_n) \) is a bounded sequence.

Hoffman et al. proved that any \( d \)-ary constant gap sequence satisfies \( \text{Per}(y) \leq 2^{d-1} \) (see Theorem 2 in [17] based on Corollary 2 from [30]). Therefore, the first item of Proposition 14 leads to the following corollary.

Corollary 15. \( RTB^*(d) \geq 1 + \frac{1}{\text{lcm}(y, y')} \) for every positive integer \( d \).

The main task we solve in the paper is the following: for given \( P \) and \( P' \) find a balanced sequence \( v \) such that its asymptotic critical exponent \( E^*(v) \) has the least value among all balanced sequences which arise as colouring of Sturmian sequences by two constant gap sequences \( y \) and \( y' \) with \( P = \text{Per}(y) \) and \( P' = \text{Per}(y') \).

Having a method for solving this task, we are able to determine \( RTB^*(d) \) for a fixed \( d \). We apply the method to all pairs \( P, P' \) such that \( P = \text{Per}(y) \) for some constant gap sequence \( y \) over \( d_\mathbb{N} \)-ary alphabet and \( P' = \text{Per}(y') \) for some constant gap sequence \( y' \) over \( d_\mathbb{N} \)-ary alphabet, where \( d_\mathbb{N} = d \).

For a fixed \( d \) there are only finitely many pairs \( P, P' \) with the described property. It seems that to find the periods of all constant gap sequences over a \( d \)-ary alphabet is a difficult problem. Nevertheless, it is known (see [28]) that constant gap sequences over \( d \) letters with \( d \leq 12 \) may be obtained by interlacing.

Definition 16. Let \( \mathcal{A}^{(0)}, \mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(k-1)} \) be mutually disjoint alphabets and let \( y^{(i)} = y_0^{(i)} y_1^{(i)} y_2^{(i)} \cdots \) be a constant gap sequence over the alphabet \( \mathcal{A}^{(i)} \) for every \( i \in \{0, 1, \ldots, k - 1\} \). The \textbf{interlacing} of \( y^{(0)}, y^{(1)}, \ldots, y^{(k-1)} \) is the sequence \( y = y_0 y_1 y_2 \cdots \), where \( y_{kn+j} = y_n^{(j)} \) for every \( n \in \mathbb{N} \) and \( j \in \{0, 1, \ldots, k - 1\} \).

In other words, the interlacing of \( y^{(0)}, y^{(1)}, \ldots, y^{(k-1)} \) is a sequence obtained by listing step by step the first letters of \( y^{(0)}, y^{(1)}, \ldots, y^{(k-1)} \), then the second letters, the third letters etc.

The interlacing \( y \) of \( y^{(0)}, y^{(1)}, \ldots, y^{(k-1)} \) satisfies

\[
\text{Per}(y) = k \cdot \text{lcm}\{\text{Per}(y^{(0)}), \text{Per}(y^{(1)}), \ldots, \text{Per}(y^{(k-1)})\}.
\]
Example 17. The interlacing of \( y^{(0)} = (0102)^\omega \) and \( y^{(1)} = (34)^\omega \) equals \( (03140324)^\omega \). The reader may easily verify that it is again a constant gap sequence and its period equals \( 2 \cdot \text{lcm}\{\text{Per}y^{(0)}, \text{Per}y^{(1)}\} = 8 \).

Remark 18. Using the fact that all constant gap sequences with at most 12 letters are obtained by interlacing, it is not difficult to verify that the periods of constant gap sequences over alphabet of size \( d \) are as follows:

| \( d \) | period |
|---|---|
| 1 | 1 |
| 2 | 2 |
| 3 | 3, 4 |
| 4 | 4, 6, 8 |
| 5 | 5, 6, 8, 9, 12, 16 |
| 6 | 6, 8, 10, 12, 16, 18, 24, 32 |
| 7 | 7, 8, 9, 10, 12, 15, 16, 18, 20, 24, 30, 32, 36, 48, 64 |
| 8 | 8, 10, 12, 14, 16, 18, 20, 24, 30, 32, 36, 40, 48, 54, 64, 72, 96, 128 |
| 9 | 9, 10, 12, 14, 15, 16, 18, 20, 21, 24, 25, 27, 28, 30, 32, 36, 40, 45, 48, 54, 60, 64, 72, 80, 81, 96, 108, 128, 144, 192, 256 |

6. Formula for the asymptotic critical exponent of balanced sequences

In this section, \( \theta = [0, a_1, a_2, a_3, \ldots] \) denotes the continued fraction of the slope of a standard Sturmian sequence \( u, P = \text{Per}y \) and \( P' = \text{Per}y' \) are periods of two constant gap sequences and \( v = \text{colour}(u, y, y') \). The computation of the asymptotic critical exponent of \( v \) is based on the knowledge of bispecial factors of the Sturmian sequence \( u \) and their return words. To provide an explicit formula let us first fix some notation.

Convention: Given positive \( c, d \in \mathbb{N} \).
- If \( a_1 = b_1 \mod c \) and \( a_2 = b_2 \mod d \), we write 
  \[ (a_1^d) = (b_1^d) \mod (d^c) \].
- If \( (a_i^d) = (b_i^d) \mod (d^c) \) for \( i = 1, 2 \), we write 
  \[ (a_{i_1}^{d_1}, a_{i_2}^{d_2}) = (b_{i_1}^{d_1}, b_{i_2}^{d_2}) \mod (d_i^c) \].

A formula for computation of \( E^*(v) \) is deduced in [12]. To keep this paper self-contained, we provide a sketch of its proof. It is based on the following simple observation.

Observation 19. Let \( v \) be a factor of \( v \) and \( u = \pi(v) \), where \( |u|_a \geq P, |u|_b \geq P' \). Then
1. \( v \) is bispecial in \( v \) if and only if \( u \) is bispecial in \( u \);
2. if \( i \in \mathbb{N} \) is an occurrence of \( v \) in \( v \), then \( i \) is an occurrence of \( u \) in \( u \).

Denote \( A_N = (\frac{P_N - 1}{Q_N}, \frac{P_N}{Q_N}) \), \( \delta_N = [a_{N+1}, a_{N+2}, \ldots] \).

Let \( (N, m) \) be a pair associated in Proposition 5 to a bispecial factor of \( u \). We assign to \( (N, m) \) the sets:
- \( S_1(N, m) = \{(\frac{r}{m} \mod (P'_{\delta_N}) \}; \)
- \( S_2(N, m) = \{(\frac{r}{m} \mod (P'_{\delta_N} - m) - k < \delta_N - m + 1 \text{ and } k + \ell > 0 \}; \)
- \( S(N, m) = S_1(N, m) \cap S_2(N, m) \).

Proposition 20. Let \( u \) be a Sturmian sequence with slope \( \theta = [0, a_1, a_2, a_3, \ldots] \) and \( y \) and \( y' \) be two constant gap sequences. Put
\[
\Phi_N := \max \left\{ 1 + m + \frac{Q_N - 1}{Q_N} k + \ell m + \frac{Q_N - 1}{Q_N} k + \ell m : \left( \frac{r}{m} \right) \in S(N, m) \text{ and } 0 \leq m < a_{N+1} \right\}.
\]

Then the asymptotic critical exponent of the balanced sequence \( v = \text{colour}(u, y, y') \) equals \( E^*(v) = 1 + \limsup_{N \to \infty} \Phi_N \).
By Proposition 5, $\ell \vec{V}$ Parikh vector of the derived sequence $d$ $\ell$ $[0$ sequences when searching for balanced sequences with the minimal asymptotic critical exponent. Let us a pair be computed explicitly. The algorithm is explained in details in [12]. We implement it and throughout $N,m$ possible pairs $(\theta$ $\theta$ $θ$ $θ$ being the most frequent letter in $v$ $v$ a shortest return word to $u$ $w$. Let $(N,m)$ be the pair associated by Proposition 5 with $u$ and $r$ and $s$ be two return words to $u$ in $u$. By Item 2 of Observation 19 the projection $\pi(w)$ is concatenated from $\ell$ return words $s$ and $k$ return words $r$ for some $\ell,k \in \mathbb{N}, \ell + k > 0$. Obviously, the vector $\left( \frac{\ell}{k} \right)$ is the Parikh vector of the derived sequence $[0,a_{N+1} - m,a_{N+2},a_{N+3},\ldots]$. The inverse of $\theta'$ is $\delta_N - m = [a_{N+1} - m,a_{N+2},a_{N+3},\ldots]$. By Lemma 7 the pair $\ell,k$ satisfies the inequality $|\ell(\delta_N - m) - k| < \delta_N - m + 1$. Hence $\left( \frac{\ell}{k} \right)$ belongs to $\mathcal{S}_2(N,m)$.

Since $i$ and $j$ are occurrences of $v$ in $v$, the number of letters $a$, resp. $b$ in $\pi(w)$ is a multiple of $P$, resp. $P'$. Using the corresponding Parikh vectors we have $\tilde{V}(\pi(w)) = \ell \vec{V}(s) + k \vec{V}(r) = (\frac{\ell}{k}) \mod (\frac{P'}{P})$.

By Proposition 5 $\ell \vec{V}(s) + k \vec{V}(r) = \alpha_N \left( \frac{1}{m} \right) \left( \frac{\ell}{k} \right)$, hence $\left( \frac{\ell}{k} \right)$ belongs to $\mathcal{S}_1(N,m)$ and thus to $\mathcal{S}(N,m)$.

Let us evaluate the ratio $\frac{|v|}{|w|}$. We abbreviate $x_N = \frac{Q_{N-1}}{Q_N}$. By Proposition 5

$$\frac{|v|}{|w|} = \frac{|r| + |s| - 2}{k|s| + \ell|s|} = \frac{(1 + m)Q_N + Q_{N-1} - 2}{(k + \ell m)Q_N + kQ_{N-1}} = \frac{1 + m + x_N}{k + \ell m + \ell x_N} - \frac{2}{|w|}.$$ 

The previous equality is valid for each factor $w$ occurring between two occurrences of $v$ in $v$. If $w$ is a shortest return word to $v$, then

$$\frac{|v|}{|w|} = \max \left\{ \frac{1 + m + x_N}{k + \ell m + \ell x_N} : \left( \frac{\ell}{k} \right) \in \mathcal{S}(N,m) \right\} - \frac{2}{|w|}.$$ 

Hence $\Phi_N$ expresses (up to the subtracted fraction $\frac{2}{|w|}$) the maximal value of the ratio $|v|/|w|$ among all possible pairs $(N,m)$ with a fixed $N$. Since the length $|w|$ tends to infinity with growing $N$, Theorem 2 concludes the proof.

If the slope of a Sturmian sequence is quadratic irrational, then the asymptotic critical exponent can be computed explicitly. The algorithm is explained in details in [12]. We implemented it and throughout this paper we use our computer program which for a given eventually periodic continued fraction $\theta$ and a pair $P = \text{Per } y$, $P' = \text{Per } y'$ finds the exact value of $E^*(v)$.

7. Colouring with linked parameters

In this section we study how to construct the periods $P,P'$ of constant gap sequences when searching for balanced sequences with the minimal asymptotic critical exponent. Let us recall that in the whole paper we work without loss of generality with Sturmian sequences $u$ having the slope $\theta = \frac{a_n(a_n)}{\pi_n(a_n)} \in (0,1)$, i.e., the letter $b$ is the most frequent letter in $u$.

Proposition 21. Let $y$ and $y'$ be two constant gap sequences and $u \in \{a,b\}^\mathbb{N}$ be a Sturmian sequence with $b$ being the most frequent letter in $u$. Then there exists a Sturmian sequence $\tilde{u} \in \{a,b\}^\mathbb{N}$ with $b$ being the most frequent letter in $\tilde{u}$ such that

$$E^*(\text{colour}(\tilde{u},y',y)) = E^*(\text{colour}(u,y,y')).$$

Proof. Let $\theta = [0,a_1,a_2,a_3,\ldots]$ be the slope of $u$ and $\frac{a_n}{\pi_n}$ be the $N^{th}$ convergent to $\theta$. Define the slope $\tilde{\theta} = [0,b_1,b_2,\ldots]$ of the Sturmian sequence $\tilde{u}$ by $b_1 = P = \text{Per } y$ and $b_{N+1} = a_N$ for each $N \in \mathbb{N}$. For the slope $\theta$, we use the notation $\mathcal{S}(N,m)$ and $\Phi_N$ as introduced in Section 5 and $Q_N = p_N + q_N$. Analogously, for the slope $\tilde{\theta}$, we use the notation $\tilde{\mathcal{S}}(N,m)$, $\tilde{\Phi}_N$ and $\tilde{Q}_N$.

Using the recurrent relation (14), we get $\tilde{q}_{N+1} = q_N$, $\tilde{Q}_{N+1} = p_N + q_N P$ and $\tilde{Q}_{N+1} = Q_N + q_N P$ for each $N \in \mathbb{N}$. It follows then immediately for each $N \in \mathbb{N}$ and $m < a_{N+1} = b_{N+2}$ that

1. $\tilde{\mathcal{S}}(N+1,m) = \mathcal{S}(N,m)$;
2. $|\tilde{Q}_{N+1} - \frac{Q_{N-1}}{Q_N}| \leq \frac{P}{Q_N^2}$.
Items 1 and 2 imply that \( \lim_{N \to \infty} \left( \Phi_{N+1} - \Phi_N \right) = 0 \). Hence by Proposition 20

\[
E^* \left( \text{colour}(\tilde{u}, y', y) \right) = 1 + \limsup_{N \to \infty} \Phi_N = 1 + \limsup_{N \to \infty} \Phi_N = E^* \left( \text{colour}(u, y, y') \right).
\]

\[\square\]

In the statement of Proposition 24 the asymptotic critical exponent cannot be replaced by the critical exponent. The following example demonstrates this fact.

**Example 22.** Baruwal and Shallit show in [3] that the minimal critical exponent of 5-ary balanced sequences equals \( \frac{3}{2} \) and it is reached by the sequence \( v = \text{colour}(u, y, y') \), where \( u \in \{a, b\}^\mathbb{N} \) is a Sturmian sequence with slope \([0, 1, \frac{2}{3}]\) and \( a \)'s are coloured by \( y = (01)^\mathbb{N} \) and \( b \)'s are coloured by \( y' = (2324)^\mathbb{N} \).

Now we colour \( a \)'s by \( y' = (2324)^\mathbb{N} \) and \( b \)'s by \( y = (01)^\mathbb{N} \). We will explain that for each sequence \( \tilde{v} = \text{colour}(\tilde{u}, y', y) \), where \( \tilde{u} \) is a Sturmian sequence with slope \([0, 1, \frac{2}{3}]\) and \( a \)'s are coloured by \( y = (01)^\mathbb{N} \) and \( b \)'s are coloured by \( y' = (2324)^\mathbb{N} \).

- If \( a_1 \geq 2 \), then by Proposition 4 and the definition of slope, \( D_{a_1} (aa) = b^a_1 b^{a_2} a \) is a factor of \( \tilde{u} \).
- If \( a_1 = 1 \) and \( a_2 = 1 \), then \( DG(bba) = b abbaba \) is a factor of \( \tilde{u} \). In both cases, \( babb \in L(\tilde{u}) \).

Hence by Corollary 12 the factor 01201 \( \in L(\tilde{v}) \) and thus \( E(\tilde{v}) \geq \frac{5}{3} > \frac{3}{2} \).

- If \( a_1 = 1 \) and \( a_2 \geq 2 \), then \( DG_{a_2} (ab) = ba(ba)^{a_2} b \) is a factor of \( \tilde{u} \), in particular, \( bababab \in L(\tilde{u}) \).

Hence 023021 \( \in L(\tilde{u}) \) and thus \( E(\tilde{v}) \geq \frac{7}{4} > \frac{3}{2} \).

**Lemma 23.** Let \( v = \text{colour}(u, y, y') \) and \( \tilde{v} = \text{colour}(\tilde{u}, y', y') \). If \( \text{Per} y \) is divisible by \( \text{Per} y \) and \( \text{Per} y' \) is divisible by \( \text{Per} y' \), then \( E^*(v) \geq E^*(\tilde{v}) \).

**Proof.** Let us denote by \( S(N, m) \) the set corresponding to \( v \) defined in Section 6 and similarly \( \tilde{S}(N, m) \) for \( \tilde{v} \). Since we colour the same Sturmian sequence \( u \), we have \( S(N, m) \subseteq S(N, m) \). Applying Proposition 20 we obtain \( E^*(v) \geq E^*(\tilde{v}) \).

\[\square\]

8. **Equivalence on unimodular matrices**

By Proposition 14 the asymptotic critical exponent depends only on the periods of constant gap sequences \( y \) and \( y' \), not on their structure. In the sequel we use for a fixed pair \( P = \text{Per} y \) and \( P' = \text{Per} y' \) the notation \( H, L, Y, Y' \) introduced by the following relations:

\[
H = \gcd(P, P'), \quad P = HY, \quad P' = HY' \quad \text{and} \quad L = \text{lcm}(P, P').
\]

Obviously, \( Y \) and \( Y' \) are coprime. We always consider \( L > 1 \) since for \( \text{Per} y = \text{Per} y' = 1 \), the sequences \( v = \text{colour}(u, y, y') \) and \( u \) are the same and the minimal asymptotic critical exponent for Sturmian sequences is known.

In this section we will study the form of the set

\[
\mathcal{S}_1(N, m) = \left\{ \left( \ell \right) : A_N \left( \begin{smallmatrix} m & 0 \\ 1 & 1 \end{smallmatrix} \right) \left( \ell \right) = \left( \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right) \text{ mod } \left( \begin{smallmatrix} P & \ast \\ P' & \ast \end{smallmatrix} \right) \right\},
\]

which plays an essential role in the definition of \( \Phi_N \) (see Proposition 20), and consequently in the computation of the asymptotic critical exponent of balanced sequences. Note that the determinant of \( A_N \left( \begin{smallmatrix} m & 0 \\ 1 & 1 \end{smallmatrix} \right) \) equals \( \pm 1 \), i.e., the matrix is unimodular. A solution \( \left( \ell \right) \) depends only on entries of the matrix counted mod \( P \) in the first row and mod \( P' \) in the second row. Hence it is possible to group matrices into classes of the same behaviour with respect to the form of \( \mathcal{S}_1(N, m) \). The following lemma prepares such grouping.

**Lemma 24.** Let \( A \in \mathbb{Z}^{2 \times 2} \) be unimodular and \( \ell, k \in \mathbb{Z} \). Then \( A \left( \begin{smallmatrix} \ell \\ k \end{smallmatrix} \right) = \left( \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right) \mod \left( \begin{smallmatrix} P & \ast \\ P' & \ast \end{smallmatrix} \right) \) if and only if there exist \( \lambda, \kappa \in \mathbb{Z} \) such that

\[
\left( \begin{smallmatrix} \ell \\ k \end{smallmatrix} \right) = H \left( \begin{smallmatrix} \lambda \\ \kappa \end{smallmatrix} \right) \quad \text{and} \quad A \left( \begin{smallmatrix} \lambda \\ \kappa \end{smallmatrix} \right) = \left( \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right) \mod \left( \begin{smallmatrix} P' & \ast \\ \ast & \ast \end{smallmatrix} \right).
\]

**Proof.** \( (\Rightarrow) \) We have \( A \left( \begin{smallmatrix} \ell \\ k \end{smallmatrix} \right) = \left( \begin{smallmatrix} a & b' \\ b & \ast \end{smallmatrix} \right) = \left( \begin{smallmatrix} P & \ast \\ \ast & \ast \end{smallmatrix} \right) \mod \left( \begin{smallmatrix} P & \ast \\ \ast & \ast \end{smallmatrix} \right) \) for some \( a, b \in \mathbb{Z} \). Since \( A \) is unimodular, it is invertible in \( \mathbb{Z} \), hence

\[
\left( \begin{smallmatrix} \lambda \\ \kappa \end{smallmatrix} \right) := H^{-1} \left( \begin{smallmatrix} \ell \\ k \end{smallmatrix} \right) = A^{-1} \left( \begin{smallmatrix} aY' \\ bY' \end{smallmatrix} \right) \in \mathbb{Z}^2.
\]

Moreover, \( A \left( \begin{smallmatrix} \ell \\ k \end{smallmatrix} \right) = H \left( \begin{smallmatrix} aY' \\ bY' \end{smallmatrix} \right) \) implies \( A \left( \begin{smallmatrix} \lambda \\ \kappa \end{smallmatrix} \right) = \left( \begin{smallmatrix} aY' \\ bY' \end{smallmatrix} \right) \mod \left( \begin{smallmatrix} P' & \ast \\ \ast & \ast \end{smallmatrix} \right) \).

\( (\Leftarrow) \) The reasoning is analogous.

\[\square\]
Remark 25. To solve the equation \( A(\lambda) = (0) \mod (Y) \), where \( A \) is a unimodular matrix, we just need to know the remainders of division by \( Y \), resp. \( Y' \), of the first row, resp. of the second row of \( A \). Therefore, we consider instead of a matrix \( A = \left( \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) \) the so-called \((Y,Y')\)-name of \( A \) defined as \( MA = \left( \begin{array}{cc} a_{11} \mod Y & a_{12} \mod Y' \\ a_{21} \mod Y' & a_{22} \mod Y' \end{array} \right) \).

Obviously, \( A(\lambda) = (0) \mod (Y) \) if and only if \( MA(\lambda) = (0) \mod (Y') \). Hence, matrices with the same \((Y,Y')\)-name have the same solutions \( \lambda, \kappa \). By Lemma 27, the values \( H, m \) and the \((Y,Y')\)-name of \( AN \) capture all pieces of information we need to decide whether \( \left( \begin{array}{c} \lambda \\ \kappa \end{array} \right) \) belongs to the set \( S_1(N,m) \).

Nevertheless, matrices with distinct \((Y,Y')\)-names can have the same solutions as well. The following example illustrates it.

Example 26. Let \( Y = 3 \) and \( Y' = 4 \). Consider the unimodular matrices \( A = \left( \begin{array}{cc} 5 & 31 \\ 21 & 130 \end{array} \right) \) and \( B = \left( \begin{array}{cc} 1 & 11 \\ 3 & 31 \end{array} \right) \). Their \((3,4)\)-names are \( MA = \left( \begin{array}{cc} 1 & 2 \\ 1 & 2 \end{array} \right) \) and \( MB = \left( \begin{array}{cc} 1 & 2 \\ 2 & 2 \end{array} \right) \). By the previous remark, a pair \( \lambda, \kappa \) solves \( A(\lambda) = (0) \mod (\lambda) \) if and only if
\[
\begin{align*}
2\lambda+\kappa &= 0 \mod 3 & \iff & & 2(2\lambda+\kappa) &= 0 \mod 3 \\
\lambda+2\kappa &= 0 \mod 4 & \iff & & 3(\lambda+2\kappa) &= 0 \mod 3 \\
\end{align*}
\]
which is equivalent to \( B(\lambda) = (0) \mod (\lambda) \).

To group unimodular matrices into classes with the same pairs \( \lambda, \kappa \) of solutions, we introduce an equivalence.

Definition 27. Let \( A \) and \( B \) be unimodular matrices in \( \mathbb{Z}^{2\times 2} \). We say that \( A \) is equivalent to \( B \), and write \( A \equiv B \), if there exist \( c \in \mathbb{Z} \) coprime with \( Y \) and \( c' \in \mathbb{Z} \) coprime with \( Y' \) such that
\[
\left( \begin{array}{cc} c & 0 \\ 0 & c' \end{array} \right) A = B \mod \left( \begin{array}{cc} Y & 0 \\ 0 & Y' \end{array} \right).
\]
The equivalence class containing a matrix \( A \) will be denoted \( [A]_\equiv \).

Remark 28. The relation \( \equiv \) is an equivalence. Evidently \( A \equiv A \). For a fixed \( z \in \mathbb{Z} \), the set of elements coprime with \( z \) is closed under inverse modulo \( z \) and multiplication modulo \( z \), which implies that \( \equiv \) is symmetric and transitive. Obviously, if \( A \) and \( B \) satisfy \( (5) \), then the \((Y,Y')\)-names \( MA \) and \( MB \) satisfy \( \left( \begin{array}{cc} c & 0 \\ 0 & c' \end{array} \right) MA = MB \mod \left( \begin{array}{cc} Y & 0 \\ 0 & Y' \end{array} \right) \).

Lemma 29. Let \( k, l \in \mathbb{N} \) and let \( A \) and \( B \) be unimodular matrices in \( \mathbb{Z}^{2\times 2} \) such that \( A \equiv B \). Then
\[
\begin{align*}
(1) & \quad A(\lambda) = (0) \mod (Y) \quad \text{if and only if} \quad B(\lambda) = (0) \mod (Y'), \\
(2) & \quad AC \equiv BC \quad \text{for any unimodular matrix} \quad C \in \mathbb{Z}^{2\times 2}.
\end{align*}
\]

Proof. Let \( A \) and \( B \) satisfy \((5)\) and \( \lambda, \kappa \in \mathbb{N} \). Then
\[
\left( \begin{array}{cc} c & 0 \\ 0 & c' \end{array} \right) A(\lambda) = B(\lambda) \mod \left( \begin{array}{cc} Y & 0 \\ 0 & Y' \end{array} \right) \mod \left( \begin{array}{cc} Y & 0 \\ 0 & Y' \end{array} \right).
\]
Let us point out a simple fact: If \( c \) is coprime with \( Y \), then \( c x = 0 \mod Y \) if and only if \( x = 0 \mod Y \) for every \( x \in \mathbb{Z} \) and analogously for the coprime values \( c' \) and \( Y' \). Hence
\[
\left( \begin{array}{cc} c & 0 \\ 0 & c' \end{array} \right) A(\lambda) = (0) \mod (Y') \quad \text{if and only if} \quad A(\lambda) = (0) \mod (Y'),
\]
Equations \((6)\) and \((7)\) imply
\[
A(\lambda) = (0) \mod (Y') \quad \text{if and only if} \quad B(\lambda) = (0) \mod (Y').
\]
Lemma 27 together with Equation \((8)\) prove Item 1.

Item 2 is a direct consequence of Equation \((6)\) in which \( A(\lambda) \) represents the first column of the matrix \( C \) and then its second column. \( \square \)
9. A lower bound on the asymptotic critical exponent for fixed periods of constant gap sequences

We associate to \( \theta = [0, a_1, a_2, a_3, \ldots] \) a sequence of classes of equivalent matrices \( ([A_N]_\equiv) \) with representatives \( A_N = \left( \frac{PN-1}{QN-1} \right) \) and a sequence of \( (\delta_N) \) with \( \delta_N = [a_{N+1}, a_{N+2}, \ldots] \). Let us stress that \( A_N \) depends only of the first \( N \) coefficients of the continued fraction of \( \theta \), whereas \( \delta_N \) depends only on the remaining coefficients of \( \theta \). Representatives of two consecutive classes satisfy

\[
A_{N+1} = A_N \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{a_{N+1}}.
\]

**Definition 30.** Let \( A \in \mathbb{Z}^{2 \times 2} \) be unimodular and \( \beta > 0 \). We say that \( \delta > 1 \) is \((1+\beta)\)-forcing for the class \([A]_\equiv\) if there exist \( m, k, \ell \in \mathbb{N} \), \( \ell + k > 0 \), such that

**\( \mathfrak{P}_1 \):**

\( A \left( \begin{array}{cc} 1 & 0 \\ m & 1 \end{array} \right) \left( \begin{array}{c} \ell \\ k \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \mod \left( \begin{array}{c} P \\ P^r \end{array} \right) \);

**\( \mathfrak{P}_2 \):**

\[ m + 1 < \delta \quad \text{and} \quad |\ell(\delta - m) - k| < \delta - m + 1; \]

**\( \mathfrak{P}_3 \):**

- if \( k = \ell \), then \( \frac{k}{\ell} > \beta; \)
- if \( k > \ell \), then \( \frac{1+m}{k+m} \geq \beta; \)
- if \( k < \ell \), then \( \frac{2+m}{k+m} \geq \beta. \)

The set of \((1+\beta)\)-forcing \( \delta \)'s for the class \([A]_\equiv\) is denoted \( \mathcal{F}(\beta, A) \).

Note that the definition is correct because it does not depend on the choice of the representative \( A \) from the class of equivalence. Indeed, only \( \mathfrak{P}_1 \) depends on \( A \) and by Lemma 20 the solutions \( \ell, k \) are the same for each matrix in \([A]_\equiv\).

**Theorem 31.** Let \( \mathbf{v} = \text{colour}(\mathbf{u}, \mathbf{y}, \mathbf{y}') \), where \( \mathbf{u} \) is a Sturmian sequence with slope \( \theta = [0, a_1, a_2, a_3, \ldots] \), and let \( \beta \) be a fixed positive number.

Assume that there exist infinitely many \( N \in \mathbb{N} \) such that \( \delta_N \) is \((1+\beta)\)-forcing for the class \([A_N]_\equiv\). Then \( E^*(\mathbf{v}) > 1 + \beta \).

**Proof.** First assume that the sequence \((a_n)\) of coefficients in the continued fraction expansion of \( \theta \) is unbounded, then by Proposition 14 \( E^*(\mathbf{v}) = +\infty \). In the sequel, assume that \((a_n)\) is bounded, say by \( K \in \mathbb{N} \).

Let \( N \in \mathbb{N} \) be such that \( \delta_N \) is \((1+\beta)\)-forcing for the class \([A_N]_\equiv\). We point out that \( \mathfrak{P}_1 \) and \( \mathfrak{P}_2 \) of Definition 30 together mean that \( m < a_{N+1} \) and \( \left( \begin{array}{c} \ell \\ k \end{array} \right) \) belongs to \( S(N, m) \) – the set used in the definition of \( \Phi_N \) in Proposition 20. Hence \( \Phi_N \) can be rewritten as

\[
\Phi_N = \max \left\{ \frac{1+m+\varepsilon}{k+m+2m}: m, k, \ell \text{ satisfy } \mathfrak{P}_1 \text{ and } \mathfrak{P}_2 \text{ for } A = A_N \text{ and } \delta = \delta_N \right\};
\]

where we abbreviate notation putting \( x_N = \frac{Q_{N-1}}{Q_N} \in [0, 1] \). Using \( \mathfrak{P}_3 \), we get

\[
\frac{1+m+\varepsilon}{k+m+2m} \geq \min \left\{ \frac{1+m+x}{k+m+2m}: x \in [0, 1] \right\} = \begin{cases} \frac{1}{\ell} & \text{if } k = \ell; \\ \frac{1+m}{k+\ell} \geq \beta & \text{if } k > \ell; \\ \frac{2+m}{k+\ell(m+1)} \geq \beta & \text{if } k < \ell. \end{cases}
\]

Hence \( \Phi_N \geq \beta \) for infinitely many \( N \). It implies \( E^*(\mathbf{v}) = 1 + \limsup_{N \to \infty} \Phi_N \geq 1 + \beta \).

To prove the strict inequality \( E^*(\mathbf{v}) > 1 + \beta \), we need to show more, namely, that there exists a positive number, say \( \mu > 0 \), such that \( \Phi_N \geq \mu + \beta \) for each \( N \), where \( \delta_N \) is \((1+\beta)\)-forcing. Existence of such \( \mu \) follows from the three facts:

1) \( x_N \in \left[ \frac{1}{K+1}, \frac{K+1}{K+2} \right] \subset [0, 1] \). Indeed, the recurrence relation \( Q_N = a_N Q_{N-1} + Q_{N-2} \) and the inequalities \( 1 \leq a_n \leq K \) imply on one hand

\[
x_N = \frac{Q_{N-1}}{Q_N} = \frac{Q_{N-1}}{a_N Q_{N-1} + Q_{N-2}} \geq \frac{Q_{N-1}}{K Q_{N-1} + Q_{N-2}} = \frac{1}{K+1}
\]
and on the other hand
\[ x_N = \frac{1}{a_N + \frac{Q_{N-1}}{Q_{N-2}}} \leq \frac{1}{1 + \frac{1}{K+1}} = \frac{K+1}{K+2}. \]

2) If \( k \neq \ell \), the function \( f_{m,k,\ell}(x) = \frac{1+m+\ell}{k+tm+\ell} \) is strictly monotonous and thus the minimum of \( f_{m,k,\ell} \)
on the interval \([\frac{1}{K}, \frac{K+1}{K+2}]\) is strictly bigger than the minimum on \([0,1] \). If \( k = \ell \), the strict inequality is required by \( \mathfrak{P}3 \).

3) There are only finitely many triplets \( m, k, \ell \in \mathbb{N} \) satisfying \( \mathfrak{P}2, \mathfrak{P}3 \) and \( m < K \). \( \square \)

Example 32. Using the previous proposition we show that \( P = \text{Per} \ y = 1 \) and \( P' = \text{Per} \ y' = 3 \) implies \( E^*(v) \geq 2 \) for every \( v = \text{colour}(u, y, y') \). In particular, we show that for any \( \beta \in (0,1) \) every sequence of \( (\delta_N) \) contains infinitely many \((1+\beta)-forcing\) \( \delta_N \).

There are four classes of equivalent matrices with \((Y, Y')-names:\)

\[ M_1 = (\begin{smallmatrix} 0 & 0 \\ \ell & m \end{smallmatrix}), M_2 = (\begin{smallmatrix} 0 & 1 \\ \ell & m \end{smallmatrix}), M_3 = (\begin{smallmatrix} 1 & 0 \\ \ell & m \end{smallmatrix}), M_4 = (\begin{smallmatrix} 1 & 1 \\ \ell & m \end{smallmatrix}). \]

If \([A]_\equiv \) has the name

- \( M_1 \), then every \( \delta > 2 \) is \((1+\beta)-forcing\) for \([A]_\equiv \), as \( \delta \) and \( A \) satisfy Properties \( \mathfrak{P}1, \mathfrak{P}2 \) and \( \mathfrak{P}3 \) with the triplet \( m = k = \ell = 1 \).
- \( M_2 \), then every \( \delta > 1 \) is \((1+\beta)-forcing\) for \([A]_\equiv \), the corresponding triplet is \( m = 0, k = \ell = 1 \).
- \( M_3 \), then every \( \delta > 1 \) is \(2\)-forcing for \([A]_\equiv \), the corresponding triplet is \( m = 0, k = 1, \ell = 0 \).
- \( M_4 \), then every \( \delta > 1 \) is \(2\)-forcing for \([A]_\equiv \), the corresponding triplet is \( m = 0, k = 0, \ell = 1 \).

Therefore, if \( E^*(v) \) is smaller than \( 2 \) for some \( v \), then necessarily \( \delta_N < 2 \) and \( M_1 \) is the name of \([A_N]_\equiv \) for all \( N > N_0 \). In particular, \( a_{N+1} = \delta_N = 1 \) for all \( N > N_0 \). However, if \([A_N]_\equiv \) has the name \( M_1 \), then the class \([A_{N+1}]_\equiv \) containing the matrix \( A_{N+1} = A_N \left( \begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix} \right) \) has the name \( M_3 \) - a contradiction.

Example 33. Now we apply Theorem 31 to balanced sequences obtained by colouring with two constant gap sequences both having the period 2, i.e., \( P = P' = 2 \). Using notation \([1] \), we have \( H = 2, Y = 1, Y' = 1 \). According to Definition 24 all integer unimodular matrices belong to the same class of equivalence. By Lemma 23, the triplet \( m = 1, k = 2 \) and \( \ell = 0 \) satisfies Property \( \mathfrak{P}1 \) for any unimodular matrix \( A \).

If we fix \( \beta = 1 \), then every \( \delta > 2 \) with the same triplet satisfies \( \mathfrak{P}2 \) and \( \mathfrak{P}3 \). In other words, \( \delta > 2 \) is \(2\)-forcing. Therefore, the only candidates for \( v \) with \( E^*(v) < 2 \) are colourings of Sturmian sequences with slope \( \theta = [0, w, \overline{1}] \), where \( w \) is any finite preperiod. For every such \( \theta \), the formula \( \Phi_N = 1 + \frac{\delta_N - 1}{2\delta_N - 1} \) gives \( \Phi_N \) for sufficiently large \( N \), where \( (Q_N) \) fulfils the recurrence relation \( Q_{N+1} = Q_N + Q_{N-1} \). Consequently, \( E^*(v) = 1 + \lim \Phi_N = 1 + \frac{\delta_N - 1}{2\delta_N - 1} = 1.809 \) is the minimum value of the asymptotic critical exponent for \( P = P' = 2 \) and it is attained if and only if \( v \) is a colouring of a Sturmian sequence with slope \( \theta = [0, w, \overline{1}] \) for a finite preperiod \( w \).

10. Admissible tails of continued fractions

In the previous section, we have associated with a continued fraction \( \theta = [0, a_1, a_2, a_3, \ldots] \) a sequence of classes of equivalent matrices \(([A_N]_\equiv \). Since the number of classes is finite, there exists \( N_0 \) such that any class of equivalence either occurs in \(([A_N]_\equiv \) \( N > N_0 \) infinitely many times or does not occur at all in it. To find a balanced sequence \( v = \text{colour}(u, y, y') \) with \( E^*(v) \leq 1 + \beta \), we should at least guarantee that \( \delta_N \) is not \((1+\beta)-forcing\) for the class \([A_N]_\equiv \) for each \( N > N_0 \). Formally, \( \delta_N \notin \mathcal{F}(\beta, A_N) \).

Remark 34. In Definition 30 only Property \( \mathfrak{P}2 \) depends on \( \delta \). Therefore, for each triplet \( m, k, \ell \) satisfying \( \mathfrak{P}1 \) and \( \mathfrak{P}3 \), we add to \( \mathcal{F}(\beta, A) \) the interval of \( \delta \)'s satisfying \( \mathfrak{P}2 \), i.e., the interval

\[
\begin{align*}
(m + k - 1, +\infty) \cap (m + 1, +\infty) & \quad \text{if } \ell = 0; \\
(m + \frac{k-1}{2}, +\infty) \cap (m + 1, +\infty) & \quad \text{if } \ell = 1; \\
(m + \frac{k-1}{2}, m + \frac{k+1}{2}) \cap (m + 1, +\infty) & \quad \text{if } \ell \geq 2.
\end{align*}
\]
The set \( F(\beta, A) \) is a union of several open intervals. Boundaries of these intervals are rational. Rational \( \delta \)'s have finite continued fractions and do not occur as tails of the continued fraction expansion of slopes of Sturmian sequences.

**Definition 35.** Let \( \beta > 0 \) and \( A \in \mathbb{Z}^{2 \times 2} \) be unimodular. We denote

\[
D(\beta, A) = \{ \delta > 1 : \delta \text{ is NOT in the closure of } F(\beta, A) \}.
\]

Using the notation of \( D(\beta, A) \), Theorem 31 can be rephrased as the following corollary.

**Corollary 36.** Let \( \theta = [0, a_1, a_2, a_3, \ldots] \) be the slope of a Sturmian sequence \( u \) and \( \beta > 0 \). If \( E^*(v) \leq 1 + \beta \), then there exists \( N_0 \) such that for every \( N > N_0 \) the set \( D(\beta, AN) \) contains \( \delta_N \).

**Lemma 37.** Let \( \beta > 0 \) and \( L = \text{lcm}(P, P') > 1 \). Then the set \( D(\beta, A) \) is a subset of \( (1, \lceil L(1 + \beta) \rceil - 2) \). In particular \( D(\beta, A) \) is bounded for each equivalence class \([A]_\equiv \).

**Proof.** Note that the pair \( k = L \) and \( \ell = 0 \) fulfills \( \mathcal{P}1 \) of Definition 30 independently on \( A \) and \( m \in \mathbb{N} \). If we take \( m = [L \beta] - 1 \), then \( \frac{1 + m}{\ell} \geq \beta \) and \( \mathcal{P}3 \) of Definition 30 is satisfied as well. By Remark 34 the set \( (m + k - 1, +\infty) = ([L(1 + \beta)] - 2, +\infty) \) belongs to \( F(\beta, A) \).

To compute easily \( D(\beta, A) \), we collect several practical observations on triplets \( m, k, \ell \in \mathbb{N} \) that may influence the form of \( D(\beta, A) \). We will apply these rules in examples worked out in hand. They follow immediately from Definitions 30 and 36 from Remark 34 and Lemmas 24 and 37.

**Remark 38.** Let \( A \) be a unimodular matrix, \( M \) denotes its \((Y, Y')\)-name. The following statements hold true.

1. The number of triplets \( m, k, \ell \) that may affect \( D(\beta, A) \) is bounded. In particular, by Definition 36 and Lemma 37 it holds \((1, m + 1) \subset D(\beta, A) \subset (1, \lceil L(1 + \beta) \rceil - 2) \). It together with Property \( \mathcal{P}3 \) forces \( m, k, \ell \) to satisfy \( 0 \leq m < \lceil L(1 + \beta) \rceil - 2 \) and \( k + \ell m \leq \frac{1}{\ell}(m + 2) \). The number of such triplets is finite.

   Each triplet adds to the set \( F(\beta, A) \) an interval as described in Remark 34. In fact, only few of the triplets add really new elements to \( F(\beta, A) \), or equivalently, erase some elements of \( D(\beta, A) \).

   Hence, when we present in examples the set \( D(\beta, A) \), we list only the triplets which determine the set. It means that no other triplet reduces the set \( D(\beta, A) \) any more.

2. \( \mathcal{P}2 \) is satisfied whenever \( m + 1 < \delta \) and

   \[
   \left\lceil \frac{m}{\ell} \right\rceil \in \{(0), (1), (1), (2), (2), (3)\}.
   \]

3. \( \left\lfloor \frac{m}{\ell} \right\rfloor \) with \( \ell \geq k + 2 \) does not fulfill \( \mathcal{P}2 \) for any \( m \in \mathbb{N} \) and \( \delta > m + 1 \).

4. If \( H \geq 2 \), then only \( k \geq \ell \) may satisfy \( \mathcal{P}1 \) and \( \mathcal{P}2 \) for some \( m \in \mathbb{N} \) and \( \delta > m + 1 \).

5. Let \( \beta < 1 \) and \( \mathcal{P}1 \) be fulfilled for some \( m \in \mathbb{N} \) and \( \left\lfloor \frac{m}{\ell} \right\rfloor \in \{(0), (1), (1)\} \). Then \( \mathcal{P}3 \) is fulfilled, too. Moreover, \( \mathcal{P}2 \) is satisfied for all \( \delta > m + 1 \).

Consequently,

\[
D(\beta, A) = \emptyset \text{ for } m = 0 \quad \text{and} \quad D(\beta, A) \subset (1, m + 1) \text{ for } m \geq 1.
\]

6. If a triplet \( m, k, \ell \) satisfies Properties \( \mathcal{P}1 \) and \( \mathcal{P}2 \), then the triplet \( m, T k, T \ell \) with \( T \in \mathbb{N}, T \geq 2 \), need not be taken into account when constructing \( D(\beta, A) \).

   Indeed, if \( m, T k, T \ell \) satisfies \( \mathcal{P}1 \) and \( \mathcal{P}2 \), then

   \[
   (m + T k - 1, +\infty) \subset (m + k - 1, +\infty)
   \]

   \[
   (m + \frac{T k - 1}{2}, +\infty) \subset (m + \frac{k - 1}{2}, +\infty)
   \]

   \[
   (m + \frac{T k - 1}{T + 1}, m + \frac{T k + 1}{T + 1}) \subset (m + \frac{k - 1}{T + 1}, m + \frac{k + 1}{T - 1}) \text{ for } \ell \geq 2.
   \]

The statement follows by Remark 34.
Example 39. Let $P = 2$, $P' = 4$ and $\beta = \frac{1}{2}$. In this case $H = 2$, $L = 4$, $Y = 1$, $Y' = 2$ and there are three classes of equivalent matrices with $(Y, Y')$-names:

\[ M_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}. \]

To find $\mathcal{D}(\beta, M_i)$ we will apply Remark 38.

- $M_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
  Consider the triplet $m = 0$, $k = 2$, $\ell = 0$. $\mathfrak{P}1$ is satisfied by Lemma 24 $\mathfrak{P}2$ is satisfied for $\delta > 1$, by Item 2. As $2 = k > \ell = 0$, $\mathfrak{P}3$ says $\frac{1}{2} = \frac{1 + m}{k + m} \geq \beta = \frac{1}{2}$.

  Hence, any $\delta > 1$ is $(1 + \frac{1}{2})$-forcing and thus $\mathcal{D}(\beta, M_1) = \emptyset$.

- $M_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
  Due to Lemma 24 we consider only triplets with $k = 2\kappa$, $\ell = 2\lambda$, where $M_2 \left( \begin{pmatrix} \kappa \\ \lambda \end{pmatrix} \right) = \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \text{ mod } (\begin{pmatrix} 1 \\ 0 \end{pmatrix})$ forces $\kappa$ to be even. Item 4 gives the restriction $\kappa \geq \lambda$. Hence by Items 1 and 6, we need to work only with $(\begin{pmatrix} k \\ \ell \end{pmatrix}) \in \{ (\begin{pmatrix} 0 \\ 0 \end{pmatrix}), (\begin{pmatrix} 1 \\ 0 \end{pmatrix}), (\begin{pmatrix} 0 \\ 1 \end{pmatrix}) \}$ and $m < 4$. The inequality $\frac{1 + m}{k + m} \geq \frac{1}{2}$ forces $\mathfrak{P}3$ is fulfilled only if $k = 4$, $\ell = 0$ and $1 \leq m < 4$.

  By Remark 38 the set of $(1 + \frac{1}{2})$-forcing $\delta$'s is $(4, +\infty)$ and thus $\mathcal{D}(\beta, M_2) = (1, 4)$.

- $M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$
  By Lemma 24 we need to check only the values $k = 2\kappa$, $\ell = 2\lambda$, where $\lambda + \kappa$ is even. Item 4 gives $\kappa \geq \lambda$. Items 1 and 6 restrict $(\begin{pmatrix} k \\ \ell \end{pmatrix}) \in \{ (\begin{pmatrix} 0 \\ 0 \end{pmatrix}), (\begin{pmatrix} 1 \\ 0 \end{pmatrix}), (\begin{pmatrix} 0 \\ 1 \end{pmatrix}) \}$ and $m < 4$. The inequality in $\mathfrak{P}3$ is fulfilled only for $k = 4$, $\ell = 0$ and $1 \leq m < 4$.

  Analogously to the previous case, $\mathcal{D}(\beta, M_3) = (1, 4)$.

Note an interesting fact. If we replace the value $\beta = \frac{1}{2}$ by some $\beta < \frac{1}{2}$, the triplets $m = 0$, $k = 2$, $\ell = 2$ and $m = 2$, $k = 2$, $\ell = 2$ extend the set of $(1 + \frac{1}{2})$-forcing $\delta$'s and $\mathcal{D}(\beta, M_3) = \emptyset$.

11. Graphs of admissible tails

In this section we create the main tool that enables us to find balanced sequences with a small value of $E^*$. More precisely, for given $P = \text{Per} \theta$ and $P' = \text{Per} \theta'$ and $\beta > 0$, we will be able to identify candidates for suffixes (we call them “admissible tails”) of the continued fraction expansion of the slope $\theta$ of a Sturmian sequence $u$ whose colouring $v = \text{colour}(u, y, y')$ satisfies $E^*(v) \leq 1 + \beta$.

Definition 40. Given $\beta > 0$ and $P$, $P' \in \mathbb{N}$. An oriented graph $(V, E)$ is called the graph of $(1 + \beta)$-admissible tails, and denoted $\Gamma_\beta$, if

- the set of vertices $V$ consists of classes of equivalence $\equiv$;
- a pair $([A]_\equiv, [B]_\equiv)$ labeled by $a$ belongs to the set $E$ of oriented edges if $a = [\delta]$ for some $\delta \in \mathcal{D}(\beta, A)$ and $B \in [A \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)]_\equiv$.

The graph $\Gamma_\beta$ is finite: it has a finite number of vertices as the number of classes of equivalence is finite and a finite number of edges as by Lemma 67 the set $\mathcal{D}(\beta, A)$ is bounded for each class $[A]_\equiv$. Moreover, if $\beta_1 > \beta_2$, then $\mathcal{D}(\beta_2, A) \subset \mathcal{D}(\beta_1, A)$ for every unimodular matrix $A$. Hence $\Gamma_{\beta_2}$ is a subgraph of $\Gamma_{\beta_1}$.

Let us recall that an oriented infinite path in a graph $(V, E)$ is a sequence $v_0, v_0, v_1, v_1, v_2, v_2, \ldots$ such that $v_i \in V, e_i \in E$ and $(v_0, v_{i+1}) = e_i$ for every $i \in \mathbb{N}$.

The graph terminology allows us to rephrase Corollary 39 into the following theorem.

Theorem 41. Let $\beta > 0$ and $v = \text{colour}(u, y, y')$, where $u$ is a Sturmian sequence with slope $\theta = [0, a_1, a_2, a_3, \ldots]$. Assume that $E^*(v) \leq 1 + \beta$.

Then there exists an infinite oriented path $v_0, e_0, v_1, e_1, v_2, e_2, \ldots$ in $\Gamma_\beta$ and $N_0 \in \mathbb{N}$ such that for every $N \in \mathbb{N}$

(1) $[A_{N+N_0}]_\equiv = x_N$;
(2) $a_{N+N_0+1}$ is the label of the edge $e_N$.

In the graph $\Gamma_\beta$ of admissible tails we are interested in infinite paths on which vertices occur infinitely many times. Therefore it is enough to consider strongly connected components, i.e., subgraphs with an oriented path from each vertex to each vertex. In particular, if a component has only one vertex, it has to have a loop.
Corollary 42. Let \( \beta > 0 \) and \( P, P' \in \mathbb{N} \). If \( \Gamma_\beta \) contains no oriented cycle, then \( E^*(v) > 1 + \beta \) for every colouring \( v \) of a Sturmian sequence by constant gap sequences of periods \( P \) and \( P' \).

Example 43. Let us construct the graph of \( (1 + \beta) \)-admissible tails \( \Gamma_\beta \) for parameters \( P = 2, P' = 4 \) and \( \beta = \tfrac{1}{2} \). By Example 39 the graph has three vertices with \((Y,Y')\)-names

\[
M_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

and the corresponding sets

\[
D(\beta, M_1) = \emptyset, \quad D(\beta, M_2) = (1, 4), \quad D(\beta, M_3) = (1, 4).
\]

The graph \( \Gamma_\beta \) and its only strongly connected component are depicted in Figure 1.

By Theorem 41 the suffix \( \mathbb{A} \) is the only candidate for the suffix of \( \theta \) associated with a Sturmian sequence \( u \) such that \( v = \text{colour}(u,y,y') \) with \( \text{Per} y = 2 \), \( \text{Per} y' = 4 \) has \( E^*(v) \leq \tfrac{3}{2} \). And indeed, \( E^*(v) = \tfrac{3}{2} \) for \( \theta = [0,1,\mathbb{A}] \). The reader is invited to check that \( \Phi_N = \tfrac{1}{2} \) in Proposition 20 (the maximum is attained for \( m = 0, k = \ell = 2 \)).

As we have noticed in Example 39 if we choose instead of \( \beta = \tfrac{1}{2} \) the value \( \beta_- < \tfrac{1}{2} \), then \( D(\beta_-, M_1) = \emptyset \) and the corresponding graph contains no oriented cycle. In other words, we can see immediately that there is no \( v = \text{colour}(u,y,y') \) with \( E^*(v) < \tfrac{3}{2} \) for \( \text{Per} y = 2 \), \( \text{Per} y' = 4 \).

Example 44. The authors of [26] show that the least critical exponent \( E(v) \) on 4 letters – in our notation \( RTB(4) \) – equals \( 1 + \sqrt{5} + \tfrac{3}{2} \) and it is reached for the colouring of the Fibonacci sequence by constant gap sequences with \( P = P' = 2 \). Let us deduce that \( RTB^*(4) = RTB(4) \).

Thanks to Proposition 21 Examples 22 and 23 and Remark 18 it remains to inspect only the case \( P = 1 \) and \( P' = 4 \).

There are six classes of equivalent matrices with \((Y,Y')\)-names

\[
M_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

Let us write down for each \( M_i \) the triplets \( m,k,\ell \) that influence the set \( D(\beta, M_i) \) (the other triplets satisfying Property \( \mathcal{P}1 \) and \( \mathcal{P}3 \) do not reduce \( D(\beta, M_i) \) any more):

\[
\begin{align*}
M_1 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & m &= 0, k &= 1, \ell &= 0 & D(\beta, M_1) &= \emptyset \\
M_2 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & m &= 0, k &= 0, \ell &= 1 & D(\beta, M_2) &= \emptyset \\
M_3 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & m &= 2, k &= 1, \ell &= 1 & D(\beta, M_3) &= (1, 3) \\
M_4 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & m &= 1, k &= 1, \ell &= 1 & D(\beta, M_4) &= (1, 2) \\
M_5 &= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} & m &= 1, k &= 2, \ell &= 0 & D(\beta, M_5) &= (1, 2) \\
M_6 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & m &= 0, k &= 1, \ell &= 1 & D(\beta, M_6) &= \emptyset
\end{align*}
\]
In the graph $\Gamma_\beta$ with parameters $P = 1$, $P' = 4$ and $\beta = \frac{\sqrt{5}+1}{2}$ (see Figure 2) there is no strongly connected component. This completes the proof that $RTB^*(4) = 1 + \frac{\sqrt{5}+1}{2}$.

12. Asymptotic repetitive threshold for binary to quinary balanced sequences

Using graphs of admissible tails we are able to list the least asymptotic critical exponent of $d$-ary balanced sequences for $2 \leq d \leq 5$.

- It is known that $RTB^*(2) = RTB(2) = 2 + \frac{\sqrt{5}+1}{2} = 3.618$ and it is reached for the Fibonacci sequence.

- $RTB^*(3) = 2 + \frac{1}{\sqrt{2}}$. It suffices to consider $P = 1, P' = 2$ and $\beta = 1 + \frac{1}{\sqrt{2}}$. There are three classes of equivalent matrices with $(Y, Y')$-names:

$$M_1 = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right), M_2 = \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right), M_3 = \left( \begin{array}{cc} 0 & 0 \\ 1 & 1 \end{array} \right).$$

Let us write down for each $M_i$ the triplets $m, k, \ell$ that influence the set $D(\beta, M_i)$:

- $M_1 = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)$: $m = 1, k = 1, \ell = 0$ and $D(\beta, M_1) = (1,2)$.
- $M_2 = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)$: $m = 0, k = 0, \ell = 1$ and $D(\beta, M_2) = \emptyset$.
- $M_3 = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)$: $m = 3, k = 2, \ell = 0$ and $D(\beta, M_3) = (1,4)$.

There is a unique strongly connected component in the graph $\Gamma_\beta$, which is the same as the one depicted in Figure 1. By Theorem 11 the suffix $\overline{3}$ is the only candidate for the suffix of the continued fraction $\theta$ associated with a Sturmian sequence $u$ such that $v = \text{colour}(u, y, y')$ with $\text{Per}_1 = 1, \text{Per}_2 = 2$ has $E^*(v) \leq 2 + \frac{1}{\sqrt{2}}$. And indeed, $E^*(v) = 2 + \frac{1}{\sqrt{2}}$ for $\theta = [0,1,\overline{3}]$. The reader is invited to check that $\Phi_N = \frac{2 + \sqrt{5}}{1 + \sqrt{5}}$ (the maximum in 3) is attained for $m = 1$, $k = 0$, $\ell = 1$ and $Q_{N+1} = 2Q_N + Q_{N-1}$ for $N \geq 1$. Using Proposition 20 we have $E^*(v) = 1 + \lim_{N \to \infty} \Phi_N = 2 + \frac{1}{\sqrt{2}}$.

- $RTB^*(4) = 1 + \frac{\sqrt{5}+1}{2}$: It was shown in Example 44.

- $RTB^*(5) = \frac{3}{2}$: We set $\beta = \frac{1}{2}$. By Remark 15, Proposition 24 and Lemma 28 we have to inspect the pairs $(P, P') \in \{(2,3), (2,4), (1,6), (1,8)\}$. In all cases besides $(2,4)$ there are no strongly connected components in the graph $\Gamma_\beta$. The case of $P = 2, P' = 4$ was examined in Example 49 where it was shown that the unique admissible tail is $\overline{2}$ and $E^*(v) = \frac{3}{2}$ for $\theta = [0,1,\overline{2}]$.

The method used to determine $RTB^*(d)$ for $d \leq 5$ does not work for $d = 6$. Let us explain why. To find $RTB^*(6)$ we have to inspect the pairs $(P, P') \in \{(3,3), (3,4), (2,6), (2,8), (1,5), (1,9), (1,12), (1,16)\}$. In the sequel, we will show that $RTB^*(6) = 1 + \frac{3\sqrt{5}+5}{80} \approx 1.2398$. If we construct the graph $\Gamma_\beta$ with the optimal value $\beta = \frac{3\sqrt{5}+5}{80}$, we find out that in all cases besides $(P, P') = (1,16)$ there are no strongly connected components in $\Gamma_\beta$.

Hence we focus on the case $(P, P') = (1,16)$. For this pair, the strongly connected component of $\Gamma_\beta$ is depicted in Figure 8 (we will show later that the bold cycle corresponds to the unique $(1 + \beta)$-admissible tail).
We always apply the forward and backward reductions together with searching for strongly connected components as long as the graph changes. This method enables us to find a small value of $E$.

Let us illustrate both kinds of reductions on the following example.

Although $\overline{1, 1, 3}$ is the label of an infinite path in $\Gamma_\beta$, the asymptotic critical exponent $E^*(\mathbf{v}) \geq 1.41 > 1 + \beta$ for any colouring of a Sturmian sequence with slope $\theta$ having eventually periodic continued fraction with the period $\overline{1, 1, 3}$. In other words, the implication in Theorem 41 cannot be reversed. Our next goal is to reduce the graph $\Gamma_\beta$ in order to exclude paths corresponding to the asymptotic critical exponent $> 1 + \beta$.

13. Reduction of the graphs

We have seen that the existence of an infinite path in the graph of admissible tails does not guarantee a small value of $E^*$. There may be even uncountably many paths in the graph $\Gamma_\beta$ corresponding to $E^*(\mathbf{v}) > 1 + \beta$. However, we will show how to reduce the graph so that the number of unsuitable paths is diminished. This method enables us to find $RTB^*(d)$ for $6 \leq d \leq 10$.

We describe two reasons why the implication in Theorem 41 cannot be reversed.

1. Let $d_1, d_2, d_3, \ldots$ be labels of edges in an infinite path $v_0, v_1, e_1, v_2, e_2, \ldots$ in $\Gamma_\beta$ and all vertices of the path occurs in it infinitely many times. If the path corresponds to an asymptotic critical exponent $\leq 1 + \beta$, then for each $N \in \mathbb{N}$, the continued fraction $[d_{N+1}, d_{N+2}, \ldots]$ belongs to $\mathcal{D}(\beta, A)$, where $A \in v_N$. But the construction of the graph requires only $d_{N+1} = [\delta]$ for some $\delta \in \mathcal{D}(\beta, A)$.

Hence, if it is not possible to extend the edge starting in $v_N$ and labeled by $d_{N+1}$ to an infinite path whose labels make the continued fraction expansion of some $\delta \in \mathcal{D}(\beta, A)$, then this edge may be erased without influencing the validity of Theorem 41.

We call this process forward reduction.

2. The terms used to describe Property $\mathfrak{P}3$ in Definition 30 represent the minimal value of the function $f_{m, k, \ell}(x) = \frac{1 + m x + \ell x}{k + (m + \ell) x}$ for $x \in [0, 1]$, see (11). The actual value of $x$ we use in evaluation of $\Phi_N$ in (11) is $x_N = Q_{N-1}/Q_N = [0, a_N, a_{N-1}, a_{N-2}, \ldots, a_1]$. It may happen that $f_{m, k, \ell}(x_N) > \beta$ even if the minimum of $f_{m, k, \ell}$ on $[0, 1]$ is smaller than $\beta$.

To construct $\Gamma_\beta$ we use only the fact that $x_N \in [0, 1]$. As $x_N = [0, a_N, a_{N-1}, a_{N-2}, \ldots, a_1]$, the value $x_N$ is given by the labels of edges of a path of length $N$ ending in $v_N = [A]_\equiv$. Assume that the structure of $\Gamma_\beta$ enables to deduce that all paths entering the vertex $[A]_\equiv$ have $x_N \in J \subset [0, 1]$. If for some $m, k, \ell$ satisfying Property $\mathfrak{P}1$ the following inequality holds true

$$\min\{f_{m, k, \ell}(x) : x \in J\} > \beta,$$

then $\delta$ satisfying Property $\mathfrak{P}2$ is $(1 + \beta)$-forcing, i.e., all such $\delta$’s should be deleted from the set $\mathcal{D}(\beta, A)$. The new set $\mathcal{D}'(\beta, A)$ may cause that some edges starting in the vertex $[A]_\equiv$ are deleted as well.

We call this process backward reduction.

We always apply the forward and backward reductions together with searching for strongly connected components as long as the graph changes.

Let us illustrate both kinds of reductions on the following example.
Example 45. Let us construct $\Gamma_\beta$ for $P = 3$, $P' = 4$, $\beta = \frac{1}{3}$. Then $H = 1$ and $L = 12$. There are 24 classes of equivalence with the following $(Y, Y')$-names:

$$
\begin{align*}
(\frac{0}{1}, \frac{1}{1}), (\frac{0}{1}, \frac{2}{1}), (\frac{1}{1}, \frac{0}{1}), (\frac{1}{1}, \frac{2}{1}), (\frac{2}{0}, \frac{1}{1}), (\frac{2}{0}, \frac{2}{1}), (\frac{1}{2}, \frac{0}{1}), (\frac{1}{2}, \frac{2}{1}), (\frac{2}{2}, \frac{0}{1}), (\frac{2}{2}, \frac{2}{1}), (\frac{1}{3}, \frac{0}{1}), (\frac{1}{3}, \frac{2}{1}), (\frac{2}{3}, \frac{2}{1}), (\frac{2}{3}, \frac{2}{1}).
\end{align*}
$$

By Lemma 37 $\mathcal{D}(\beta, M_i) \subset (1, 14)$. We write down for each class the triplets $m, k, \ell$ that influence the set $\mathcal{D}(\beta, M_i)$:

| Triplet | Set of Influences |
|---------|-------------------|

| $M_1 = (0, 1)$ | $\mathcal{D}(\beta, M_1) = \emptyset$ |
| $M_2 = (1, 0)$ | $\mathcal{D}(\beta, M_2) = \emptyset$ |
| $M_3 = (1, 2)$ | $\mathcal{D}(\beta, M_3) = \emptyset$ |
| $M_4 = (2, 1)$ | $\mathcal{D}(\beta, M_4) = \emptyset$ |
| $M_5 = (1, 3)$ | $\mathcal{D}(\beta, M_5) = \emptyset$ |
| $M_6 = (0, 1)$ | $\mathcal{D}(\beta, M_6) = \emptyset$ |
| $M_7 = (0, 2)$ | $\mathcal{D}(\beta, M_7) = (1, 2)$ |
| $M_8 = (1, 0)$ | $\mathcal{D}(\beta, M_8) = (1, 2)$ |
| $M_9 = (1, 2)$ | $\mathcal{D}(\beta, M_9) = (1, 2)$ |
| $M_{10} = (0, 3)$ | $\mathcal{D}(\beta, M_{10}) = (1, 2)$ |
| $M_{11} = (1, 3)$ | $\mathcal{D}(\beta, M_{11}) = (1, 4)$ |
| $M_{12} = (1, 3)$ | $\mathcal{D}(\beta, M_{12}) = (1, 3)$ |
| $M_{13} = (0, 4)$ | $\mathcal{D}(\beta, M_{13})(1, 3) \cup (\frac{2}{3}, 4)$ |
| $M_{14} = (0, 5)$ | $\mathcal{D}(\beta, M_{14})(1, 2) \cup (\frac{2}{3}, 4)$ |
| $M_{15} = (1, 1)$ | $\mathcal{D}(\beta, M_{15})(1, 2) \cup (\frac{2}{3}, 4)$ |
| $M_{16} = (1, 3)$ | $\mathcal{D}(\beta, M_{16}) = (1, 2)$ |
| $M_{17} = (0, 1)$ | $\mathcal{D}(\beta, M_{17}) = (1, 2) \cup (4, \frac{9}{7})$ |
| $M_{18} = (1, 2)$ | $\mathcal{D}(\beta, M_{18}) = \emptyset$ |
| $M_{19} = (1, 3)$ | $\mathcal{D}(\beta, M_{19}) = (2, 3)$ |
| $M_{20} = (1, 2)$ | $\mathcal{D}(\beta, M_{20}) = (1, 2)$ |
| $M_{21} = (3, 4)$ | $\mathcal{D}(\beta, M_{21}) = (3, 4)$ |
| $M_{22} = (1, 1)$ | $\mathcal{D}(\beta, M_{22}) = (1, 3)$ |
| $M_{23} = (1, 3)$ | $\mathcal{D}(\beta, M_{23}) = (1, 3)$ |
| $M_{24} = (1, 4)$ | $\mathcal{D}(\beta, M_{24}) = (1, 4)$ |

A unique strongly connected component of the graph $\Gamma_\beta$ is depicted in Figure 4. Using our computer program we can see that for the Sturmian sequence $u$ associated to $\theta = [0, 3, 1, 1, 1, 2]$, the balanced sequence $v = \text{colour}(u, y, y')$ with $\text{Per } y = 3$, $\text{Per } y' = 4$ has $E^*(v) = \frac{4}{3}$.

If we search for $v = \text{colour}(u, y, y')$ such that $E^*(v) < \frac{4}{3}$, i.e., if we choose $\beta_\beta < \frac{1}{3}$, then by Theorem 37 we have to exclude also the solution $m = 0$, $k = \ell = 3$, which reduces $\mathcal{D}(\beta_\beta, M_{10})$ to $\emptyset$, therefore no strongly connected component remains in the graph. To summarize, there is no $v = \text{colour}(u, y, y')$ with $E^*(v) < \frac{4}{3}$ for $\text{Per } y = 3$, $\text{Per } y' = 4$.

Let us apply first the forward reduction on the graph from Figure 4. Consider the vertex $M_{17} = (\frac{1}{3})$ and the outgoing edge labeled by 4. Each prolongation to an infinite path has the next edge label equal to 1. Therefore the corresponding $\delta = [4, 1, \ldots] \in (\frac{2}{3}, 5)$, i.e., $\delta \notin \mathcal{D}(\beta, M_{17}) = (1, 2) \cup (4, \frac{9}{7})$. Consequently, the edge labeled by 4 may be erased.

Next, we apply the backward reduction. Consider the vertex $M_{21} = (\frac{1}{3})$. The sequence of edge labels of each ingoing path ends in 3, 1, 1. For $m = 1$, $k = 6$, $\ell = 1$ satisfying $M_{21}(\frac{1}{3}) (\frac{1}{3}) = (\frac{6}{1})$ we have $[0, 1, 1, 3, \ldots] \in J = (\frac{2}{3}, \frac{5}{3})$ and $\min\{f_{m, k, \ell}(x) : x \in J\} = \frac{2 + 5\beta}{2 + 5\beta} = \frac{2 + 5\beta}{2 + 5\beta} > \beta = \frac{1}{3}$. Therefore, by Remark 54 the triplet $m = 1$, $k = 6$, $\ell = 1$ reduces $\mathcal{D}(\beta, M_{21})$ to $\mathcal{D}^*(\beta, M_{21}) = (3, 4) \cap (1, \frac{7}{2}) = (3, \frac{7}{2})$.

Finally, we apply once more the forward reduction. Consider again the vertex $M_{21}$, now with $\mathcal{D}^*(\beta, M_{21}) = (3, \frac{7}{2})$, and its unique outgoing edge labeled by 3. Each prolongation to an infinite
be the value of the minimal critical exponent since by definition $E$ and Lemma 23.

balanced sequence. For this task we use Remark 18. Then some pair $s$ are eliminated due to Proposition 21

\[ \delta \in D(\beta, M_{21}) = (3, \frac{7}{2}). \]

Thus, the edge labeled by 3 may be erased. Consequently, the graph is reduced

so that it contains a unique cycle labeled by $1, 1, 1, 2$, which proves that it is the only $(1 + \frac{1}{2})$-admissible suffix of $\theta$.

14. ASYMPTOTIC REPETITIVE THRESHOLD FOR SENARY TO DENARY BALANCED SEQUENCES

In order to find $RTB^*(d)$, we first detect possible periods $P = \text{Per} y, P' = \text{Per} y'$ which give a $d$-ary balanced sequence. For this task we use Remark $18$. Then some pairs are eliminated due to Proposition $21$ and Lemma $23$.

The starting parameter $\beta$ for the construction of graphs of $(1 + \beta)$-admissible tails can be chosen to be the value of the minimal critical exponent since by definition $E(\mathbf{v}) \geq E^*(\mathbf{v})$ for every sequence $\mathbf{v}$.

By Proposition $14$ we have $E^*(\mathbf{v}) \geq 1 + \frac{1}{P + P'}$ for every balanced sequence $\mathbf{v} = \text{colour}(u, y, y')$. Therefore when searching for $E^*(\mathbf{v}) \leq 1 + \beta$ we have to consider only the pairs $(P, P')$ such that $\frac{1}{\beta} \leq P P'$.

As soon as we find $\mathbf{v}$ with $E^*(\mathbf{v}) < 1 + \beta$, we lower $\beta$ to $\beta' = E^*(\mathbf{v}) - 1$ and construct graphs of $(1 + \beta')$-admissible tails for the remaining pairs $(P, P')$.

Using the above described procedure, we have found the minimal asymptotic critical exponent of $d$-ary balanced sequences up to $d = 10$. They are listed in Table $1$.

Let us comment properties of the graphs $\Gamma_\beta$ with the optimal value $\beta = RTB^*(d) - 1$.

$2 \leq d \leq 5$ : It was sufficient to use the graphs of admissible tails without reduction, as explained in Section $12$.

$6 \leq d \leq 9$ : The reduction was necessary. After reduction of the graph $\Gamma_\beta$ constructed for the pair $(P, P')$ from Table $1$ there is always a unique strongly connected component in the form of a cycle, i.e., the period of the continued fraction was determined uniquely by the graph. For all other pairs $(P, P')$ of admissible periods the graph has no strongly connected component. Let us point out that for 6 letters the forward reduction suffices, while for more letters both the forward and the backward reduction is needed. For 6 letters, the unique $(1 + \beta)$-admissible tail corresponds to the bold cycle in Figure $3$.

$d = 10$ : For $(P, P') \neq (4, 64)$ the graph has no strongly connected component. For $P = 4, P' = 64$, a unique strongly connected component of $\Gamma_\beta$ after reduction is depicted in Figure $5$. Thus, there are two oriented cycles sharing two vertices. Hence a “human intervention” is needed to pick up the suitable continued fraction. Using our computer program we obtain: $E^*(\tilde{\mathbf{v}}) = RTB^*(10)$ for the balanced sequence $\tilde{\mathbf{v}} = \text{colour}(\tilde{u}, \tilde{y}, y')$, where $\tilde{\theta} = \{0, 2, 1, 1, 1, 1, 1, 1, 1, 1\}$. In Figure $5$ $\tilde{\theta}$ corresponds to an infinite path going alternatively through the right hand cycle, then the left hand cycle, and again the right hand cycle, then the left hand cycle, and so on. The following two arguments show that if $\tilde{\theta}$ does not have a suffix corresponding to such alternation of the right.
and left hand cycle, then $E^*(\nu) > 1 + \beta$. First we observe, that every path in the graph from Figure 5 uses infinitely many times the vertices named $(0,0)$ and $(\frac{0}{1}, \frac{0}{1})$.

- Let us explain that any $\theta$ corresponding to an infinite path going infinitely many times twice consecutively through the left hand cycle gives rise to a colouring $\nu$ with $E^*(\nu) > 1 + \beta$.

Since $P = 4$, $P' = 64$, we have $H = 4$ and $Y = 1$, $Y' = 16$. Assume $\theta = [0, a_1, a_2, a_3, \ldots]$ and there exist infinitely many $N$ such that $A_N$ belongs to the class with $(Y, Y')$-name $(\frac{0}{1}, \frac{0}{1})$, where we arrive to $A_N$ using the left hand cycle and we leave $A_N$ using again the left hand cycle. In this case $\delta_N = [a_{N+1}, a_{N+2}, \ldots] = [1, 2, 1, 1, 1, 1, \ldots] \in (\frac{20}{76}, \frac{18}{76})$ and $\frac{Q_{N-1}}{Q_N} = [0, a_N, \ldots, a_2, a_1] = [0, 1, 1, 1, 2, 1, 1, \ldots] \in (\frac{12}{76}, \frac{7}{76})$. Then $(\frac{0}{1}, \frac{0}{1}) (\frac{13}{16}) = (\frac{0}{1})$ mod $(\frac{16}{16})$, therefore $k = 4 \cdot 19 = 76$, $\ell = 4 \cdot 14 = 56$ belongs to $S_1(N, 0)$. It belongs to $S_2(N, 0)$ as the inequality $|56\delta_N - 76| < \delta_N + 1$ is satisfied as well. Using (3) we obtain

$$\Phi_N \geq \frac{1 + \frac{Q_{N-1}}{Q_N}}{76 + \frac{Q_{N-1}}{Q_N}} \geq \frac{1 + \frac{12}{76}}{76 + \frac{12}{76}} > 1 + \beta.$$ 

By Proposition 20 $E^*(\nu) = 1 + \limsup \Phi_N > 1 + \beta$.

![Figure 5](image.png)

Figure 5. The strongly connected component of the graph of admissible tails for $\beta = \frac{60 - 21\sqrt{7}}{304}$ and $P = 4$, $P' = 64$.

| $d$ | $RTB^*(d)$ | $\theta$ | $P$ | $P'$ |
|-----|-------------|-----------|-----|-----|
| 2   | $2 + \frac{\sqrt{5} + 1}{2}$ | $[0, 1]$ | 1   | 1   |
| 3   | $2 + \frac{1}{\sqrt{2}}$ | $[0, 1, 2]$ | 1   | 2   |
| 4   | $1 + \frac{\sqrt{5} + 1}{4}$ | $[0, 1]$ | 2   | 2   |
| 5   | $\frac{3}{2} = 1.5$ | $[0, 1, 2]$ | 2   | 4   |
| 6   | $\frac{75 + 3\sqrt{65}}{80} \leq 1.239835$ | $[0, 4, 1, 2, 1, 1, 1]$ | 1   | 16  |
| 7   | $\frac{99 + 5\sqrt{77}}{64} \leq 1.140950$ | $[0, 5, 1, 1, 1, 5, 2]$ | 1   | 32  |
| 8   | $1 + \frac{3 - \sqrt{5}}{16} \leq 1.047746$ | $[0, 1]$ | 8   | 8   |
| 9   | $\frac{21 - \sqrt{20}}{16} \leq 1.032992$ | $[0, 1, 4]$ | 8   | 16  |
| 10  | $\frac{364 - 21\sqrt{7}}{304} \leq 1.0146027$ | $[0, 6, 1, 1, 1, 1, 2, 1, 1, 1, 1]$ | 4   | 64  |
• Let us first make some comments on complexity of computation of the asymptotic repetitive threshold. Time complexity is not the only obstacle to revealing the value $1 + \beta$. For example, for every $d$ we have to consider $P = 1$ and $P' = 2d^2$. The number of vertices in the corresponding graph $I_\beta = \sum_{\beta} \frac{1}{Q}$ is equal to $3 \cdot 2d^3$.

The closer the value $1 + \beta$ to $RTB^*(d)$ the more time consuming determination of $D(\beta, A)$. The reason is the fact that more triplets $m, k, \ell$ satisfy Properties 31 and 33. Hence, it would be useful to have a good upper bound on $RTB^*(d)$ so that we do not have to repeat the computation for more values $\beta$.

Time complexity is not the only obstacle to revealing $RTB^*(d)$ for $d \geq 11$. We are afraid that for larger $d$ our method will not give a unique period of the continued fraction corresponding to the minimal $E^*(v)$ and that “human intervention” will be needed similarly as in the case $d = 10$.

Let us conclude with some other open problems connected to the topic.

• We conjecture that $RTB^*(d) < 1 + q^d$ for some positive $q < 1$. But from the obtained values $RTB^*(d)$, we are not able to derive a conjecture on the precise value $RTB^*(d)$. We observe at least that if $(P, P')$ is an optimal pair of periods for $d$ even, then $(P, 2P')$ is optimal for $d + 1$. It works for $d = 2, 4, 6, 8$.

• We believe that $RTB^*(d)$ always belongs to a quadratic field, but we have no proof of this fact.

• $RTB^*(d)$ is defined to be $\inf \{E^*(u) : u$ d-ary balanced $\}$. We can see for $d \leq 10$ that $RTB^*(d)$ is minimum of the set and it is not its accumulation point. The question is whether $RTB^*(d)$ may be an accumulation point. And if not, what is the second, third, etc. smallest element of the set.

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