Nonlinear enhanced dissipation in viscous Burgers type equations II

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Abstract

In this follow up but self contained paper, we focus on the viscous Burgers equation. There, using the Hopf-Cole transformation, we compute the long time behavior of solutions for some classes of infinite mass initial datas. We show that an enhanced dissipation effect occurs generically, that is the decay rate in time is better than if we considered instead the heat equations for the same initial value. We also show the existence of a kind of global attractor per class.

1 Introduction and presentation of the results

We are interested in this paper on the long time behavior of solutions to either the heat or the viscous Burgers equation on the real line. It is well known that for the heat equation \( \partial_t u - \partial^2_x u = 0 \), for an initial data \( u_0 \in L^1(\mathbb{R}) \) we have the asymptotic profile

\[
\sqrt{t} u\left(z\sqrt{t}, t\right) \to \int_{\mathbb{R}} u_0(z) e^{-\frac{z^2}{4t}}
\]

when \( t \to +\infty \), uniformly in \( z \in \mathbb{R} \).

A similar result holds for the viscous Burgers equation \( \partial_t v - \partial^2_x v + v \partial_x v = 0 \) for initial data \( v_0 \in L^1(\mathbb{R}) \), (see [9], [15], [17]), as we have

\[
\sqrt{t} v\left(z\sqrt{t}, t\right) \to \frac{2 \left(e^{M^2} - 1\right) e^{-z^2/4}}{e^{M^2} \sqrt{4\pi} + \left(1 - e^{M^2}\right) \int_{-\infty}^{\infty} e^{-s^2/4} ds}
\]

when \( t \to +\infty \), uniformly in \( z \in \mathbb{R} \), with \( M = \int_{\mathbb{R}} v_0 \).

In both case, the \( L^\infty \) norm of the solution decays like \( t^{-\frac{1}{2}} \). Other asymptotic behavior results have been established in other convection-diffusion equations for initial datas in \( L^1(\mathbb{R}) \), we refer to [18] and references therein, as well as [7], [16].

Our first goal is to show that, for some class of initial datas that are not in \( L^1(\mathbb{R}) \), the decay rate for the viscous Burgers equation is better than the one of the heat equation.

1.1 Nonlinear enhanced dissipation

Enhanced dissipation results are well known for the heat equation with an additional linear transport term (see for instance [1], [4], [5], and references therein) or for Navier-Stokes on \( \mathbb{T} \times \mathbb{R} \) (see [2], [6], [12], [14]). It has also been established in [8] for some convection-diffusion equations on the real line.

The main result of this subsection is an estimation of the decay in time for solutions to the viscous Burgers equation in a particular class of initial datas, on which the decay is better than the one for the heat equation.

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Theorem 1.1 Consider the problem

$$\partial_t f - \partial_x^2 f + f \partial_x f = 0$$

(1.1)

for an initial data $f_0 \in C^0(\mathbb{R}, \mathbb{R})$, and suppose that there exists $\kappa_1, \kappa_2 > 0$ and $\alpha \in [0, 1]$ such that

$$\frac{\kappa_2}{(1 + |y|)^\alpha} \geq f_0(y) \geq \frac{\kappa_1}{(1 + |y|)^\alpha}.$$ 

Then, there exists $K_1, K_2 > 0$ depending on $\kappa_1, \kappa_2, \alpha$, such that, for all $t \geq 0$,

$$\frac{K_1}{(1 + t)^{\frac{\alpha}{2}}} \leq \|f(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \frac{K_2}{(1 + t)^{\frac{\alpha}{2}}}.$$ 

This result is to be compared with the decay estimates for solutions to the heat equation for the same initial conditions.

Proposition 1.2 Consider the problem

$$\partial_t f - \partial_x^2 f = 0$$

for an initial data $f_0 \in C^0(\mathbb{R}, \mathbb{R})$ satisfying the conditions of Theorem 1.1. Then, there exists $K_1, K_2 > 0$ depending on $\kappa_1, \kappa_2, \alpha$ such that, for $t \geq 0$,

$$\frac{K_1}{(1 + t)^{\frac{\alpha}{2}}} \leq \|f(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \frac{K_2}{(1 + t)^{\frac{\alpha}{2}}}.$$ 

We see here that the dissipation is enhanced by adding the Burgers term $f \partial_x f \left(\frac{\alpha + 2}{2}\right)$ for $\alpha \in [0, 1]$. It was proven in [11] that for a generalized version of the viscous Burgers equation, this effect appears for a particular function (and small perturbations of it) behaving like $\kappa_+ |x|^{-\alpha}$ when $x \to \pm \infty$. Theorem 1.1 shows that this is in fact true for all functions with this behavior at infinity for the viscous Burgers equations.

If we suppose some decay on derivatives of the initial data, it is possible to estimate the derivatives of the solution of the viscous Burgers equation (1.1).

Proposition 1.3 Consider the problem

$$\partial_t f - \partial_x^2 f + f \partial_x f = 0$$

for an initial data $f_0 \in C^j(\mathbb{R}, \mathbb{R})$ with $j \geq 1$, and suppose that there exists $\kappa_1, \kappa_2 > 0$ and $\alpha \in [0, 1]$ such that

$$\frac{\kappa_2}{(1 + |y|)^\alpha} \geq f_0(y) \geq \frac{\kappa_1}{(1 + |y|)^\alpha}.$$ 

Furthermore, if for any $i \in \mathbb{N}, i \leq j$, there exists $\lambda_i > 0$ such that

$$|f_0^{(i)}(y)| \leq \frac{\lambda_i}{(1 + |y|)^{\alpha + 1}},$$ 

then for any $n, k$ such that $2n + k \leq j$, there exists $K_{n, k} > 0$ depending on $\kappa_1, \kappa_2, (\lambda_i), \alpha, n$ and $k$ such that, for $t \geq 0$,

$$\|\partial_t^n \partial_x^k f(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \frac{K_{n, k}}{(1 + t)^{\frac{\alpha}{2} + n + \frac{k}{2}}}.$$ 

This also has to be compared with the similar result for the heat equation, where derivatives adds more decay than for viscous Burgers.

Proposition 1.4 Consider the problem

$$\partial_t f - \partial_x^2 f = 0$$

for an initial data $f_0 \in C^j(\mathbb{R}, \mathbb{R}), j \geq 1$ satisfying the conditions of Proposition 1.3. Then for any $n, k \in \mathbb{N}$, there exists $K_{n, k} > 0$ depending on $\kappa_1, \kappa_2, \alpha, n, k$ such that, for $t \geq 0, n + k \leq j$,

$$\|\partial_t^n \partial_x^k f(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \frac{K_{n, k}}{(1 + t)^{\frac{\alpha}{2} + n + \frac{k}{2}}}.$$ 

Remark that for the heat equation, adding a derivative in time or in position add respectively an additional decay like $t^{-1}$ and $t^{-2}$, but in the viscous Burgers equation (1.1), these decays are respectively $t^{-\frac{\alpha}{2} + \frac{1}{2}}$ and $t^{-\frac{\alpha}{2} + 1}$. It is possible that these rates are not optimal. See section 1.4 for the strategy of the proof.
1.2 Computation of the asymptotic profile

We are interested in computing the limit when $t \to +\infty$ of $t^{\frac{\alpha}{1+\alpha}} f \left( zt^{\frac{1}{1+\alpha}}, t \right)$ for some fixed $z \in \mathbb{R}$ where $f$ is the solution of the viscous Burgers equation (1.1) with initial data $f_0$ similar to the one of Theorem 1.1. In [11], we computed the asymptotic profile in one particular case, and we will see here that it is in fact, in some sense, a global attractor. This will involve two particular functions defined by an implicit equation. Consider the function

$$g(y) := y + \kappa |y|^\alpha.$$ 

We are interested in solving the equation $z = g(y(z))$ for the unknown function $y$. Here is the plot of $\mathcal{C} := \{(z, y) \in \mathbb{R}^2, z = g(y)\}$ (in the case $\kappa = 1, \alpha = 1/3$):

![Plot of the set $\mathcal{C}$ for $\kappa = 1, \alpha = \frac{1}{3}$. The two black axes are $\{z = 0\}$ for the horizontal one and $\{y = 0\}$ for the vertical one.](image)

We will show in section 3.1 that $\mathcal{C}$ has this shape for any $\kappa > 0, \alpha \in [0, 1]$. In particular, there exists a point $A$ (the red dot) where a cusp happens on the curve, and its coordinates are

$$A = (g(y_0), y_0)$$
with \( y_0 := (\kappa \alpha)^{\frac{1}{1+\alpha}} \). Remark that \( g(y_0) = \kappa^{\frac{1}{1+\alpha}} (\alpha^{\frac{1}{1+\alpha}} + \alpha^{-\frac{1}{1+\alpha}}) > 0 \). By the implicit function theorem, we construct the functions

\[
y^*_+ : g(y_0), +\infty \to y_0, +\infty \quad \text{(in dark blue)}, \quad y^*_- : \mathbb{R} \to y_0, +\infty \quad \text{(in cyan)}
\]
as the respective inverses of \( g : y_0, +\infty \to g(y_0), +\infty \) and \( g : \mathbb{R} \to \mathbb{R} \). We can now describe the first order of the solution.

**Theorem 1.5** For \( \kappa > 0, \alpha \in [0, 1] \), there exists \( z_c > g(y_0) \) such that the following holds. Consider \( f \) the solution to the viscous Burgers equation \( \partial_t f - \partial_x^2 f + f \partial_x f = 0 \) for an initial condition \( f_0 \in C^1(\mathbb{R}, \mathbb{R}) \) with

\[
f_0(y) = \frac{\kappa (1 + o_y \to \pm\infty(1))}{|y|^{\alpha}}
\]
and

\[
f_0'(y) = -\alpha \frac{\kappa (1 + o_y \to \pm\infty(1))}{y |y|^{\alpha}}.
\]

Then, for any \( z \in \mathbb{R} \setminus \{z_c\} \), we have the convergence

\[
t^{\frac{\alpha}{1+\alpha}} f \left( z t^{\frac{1}{1+\alpha}}, t \right) \to p(z)
\]
when \( t \to +\infty \), where the profil \( p \) is defined by

\[
p(z) := \begin{cases} 
|y^*_+(z)|^{-\alpha} & \text{if } z > z_c \\
|y^*_-(z)|^{-\alpha} & \text{if } z < z_c.
\end{cases}
\]

Furthermore, for any \( \varepsilon > 0 \), the convergence is uniform on \( \mathbb{R} \setminus [z_c - \varepsilon, z_c + \varepsilon] \), and

\[
\left\| t^{\frac{\alpha}{1+\alpha}} f \left( z t^{\frac{1}{1+\alpha}}, t \right) - p \right\|_{L^\infty(\mathbb{R} \setminus [z_c - \varepsilon, z_c + \varepsilon])} = O_{t \to +\infty} \left( t^{-\frac{1}{2(1+\alpha)}} \right).
\]
We can show that the profile \( p \) is never continuous at \( z = z_c \). The calculation of \( z_c \) is rather intricate. We have it by an implicit equation: it is the unique solution of \( \mathcal{H}(y^+_c(z_c)) = \mathcal{H}(y^-_c(z_c)) \) where

\[
\mathcal{H}(y) := \frac{\kappa^2}{4|y|^{2\alpha}} - \frac{\kappa(1-\alpha)}{2|y|^{1-\alpha}}
\]

in the set \( \{ y \in ]y_0, +\infty[ \} \).

We can check that \( y^+_c(z) - z \to 0 \) when \( z \to +\infty \) and \( y^-_c(z) - z \to 0 \) when \( z \to -\infty \). This means that if \( |z| \gg 1 + t \), we have \( p(z) \simeq \frac{\kappa}{1+2\alpha} \simeq t^{\frac{\alpha}{1+\alpha}} f_0 \left( \frac{zt^{\frac{\alpha}{1+\alpha}}}{\sqrt{\alpha}} \right) \).

Remark that if \( f(x, t) \) is solution of the viscous Burgers equation, then so is \(-f(-x, t)\). We can therefore consider negative initial data as well in Theorems 1.1 and 1.5.

This result does not say what is happening at \( z_c \), but away from it, the profile \( p \) is a global attractor. However, from [11], we know that there are no generic profile near \( z_c \). A consequence of Theorem 1.4 there is that in a vicinity of \( z_c \), we can construct functions \( f_0 \) such that \( t^{\frac{\alpha}{1+\alpha}} f \left( \frac{zt^{\frac{\alpha}{1+\alpha}}}{\sqrt{\alpha}}, t \right) \) converges to different profiles near \( z_c \).

As previously, it is interesting to compare this result with the first order for the heat equation for the same initial data.

**Proposition 1.6** For \( \kappa > 0, \alpha \in ]0, 1[ \), consider \( f \) the solution of the heat equation \( \partial_t f - \partial_x^2 f = 0 \) for an initial condition

\[
\partial_t f_0(y) = \frac{\kappa(1 + o_{y \to +\infty}(1))}{(1 + |y|)^\alpha}.
\]

Then, uniformly in \( z \in \mathbb{R} \), we have the convergence

\[
t^{\alpha/2} f \left( \sqrt{\alpha} z, t \right) \to \frac{\kappa}{\sqrt{4\pi}} \int_{\mathbb{R}} \frac{1}{|y|^{\alpha}} e^{-\frac{(z-y)^2}{4}} dy
\]

when \( t \to +\infty \).

This result has been shown in [11] (see Proposition 1.1 there). Remark in particular that here, the limit profile is continuous. This also means that the long time behaviour is highly nonlinear, as for instance has been shown in [8] in another context.

### 1.3 Other examples of discontinuous asymptotic profiles

In this section, we compute the first order of solutions to the viscous Burgers equation, but for other non integrable initial datas. Our first example is the case where

\[
f_0(y) = \frac{\kappa(1 + o_{y \to +\infty}(1))}{|y|^{\alpha}},
\]

where we get a similar profile.

**Proposition 1.7** For \( \kappa > 0, \alpha \in ]0, 1[ \), there exists \( z_d > 0 \) such that the following holds. Consider \( f \) the solution to the viscous Burgers equation \( \partial_t f - \partial_x^2 f + f \partial_x f = 0 \) for an initial condition \( f_0 \in C^1(\mathbb{R}, \mathbb{R}) \) with

\[
f_0(y) = \frac{-\kappa(1 + o_{y \to +\infty}(1))}{|y|^{\alpha}}, \quad f_0(y) = \frac{\kappa(1 + o_{y \to -\infty}(1))}{|y|^{\alpha}}
\]

and

\[
f_0(y) = \frac{\alpha \kappa(1 + o_{y \to +\infty}(1))}{y|y|^{\alpha}}, \quad f_0(y) = \frac{-\alpha \kappa(1 + o_{y \to -\infty}(1))}{y|y|^{\alpha}}.
\]

Then, for any \( z \in \mathbb{R} \setminus \{z_d\} \), we have the convergence

\[
t^{\frac{\alpha}{1+\alpha}} f \left( \frac{zt^{\frac{\alpha}{1+\alpha}}}{\sqrt{\alpha}}, t \right) \to p(z)
\]
when \( t \to +\infty \) to some profile \( p \). Furthermore, for any \( \varepsilon > 0 \), the convergence is uniform on \( \mathbb{R}\setminus[z_d - \varepsilon, z_d + \varepsilon] \), and

\[
\left\| t^{1/2} f \left( t^{1/2}, t \right) - p \right\|_{L^\infty(\mathbb{R}\setminus[z_d - \varepsilon, z_d + \varepsilon])} = O(t^{-\frac{1}{2} + \varepsilon} ).
\]

The profile \( p \) will have a similar definition as the one in Theorem 1.5, and presents a discontinuity at some point \( z_d \), but they are not identical. We can check that \( p \) is the unique entropic solution in the sense of section 1.1.3 of [11] that behaves like \( \frac{\pm \kappa}{|y|^{\alpha}} \) at \( \pm \infty \).

Interestingly, we cannot show a similar result for the last remaining case, that is

\[
f_0(y) = \frac{\pm \kappa (1 + o(y, \pm \infty)(1))}{|y|^{\alpha}} ,
\]
as our approach for the proof fails in that case. We believe that if the convergences occurs, the profile will be of different type than the previous ones (in particular, it will have at least two discontinuities). See subection 3.3.1 for more details about these two cases, as well as subsection 1.1.3 of [11].

We now consider other type of decay, namely the case

\[
f_0(y) = \frac{\kappa (1 + o(y, \pm \infty)(1))}{|y|^{\alpha} \ln^\beta(|y|)}
\]

for \( \kappa, \beta > 0, \alpha \in ]0, 1[ \). It turns out that the functions \( y_\pm^* \) defined for Theorem 1.5 will also appear here.

**Proposition 1.8** For \( \kappa > 0, \alpha \in ]0, 1[ , \beta > 0 \), there exists \( z_c > g(y_\alpha) \) such that the following holds. Consider \( f \) the solution to the viscous Burgers equation \( \partial_t f - \partial_x^2 f + \partial_x f = 0 \) for an initial condition \( f_0 \in C^1(\mathbb{R}, \mathbb{R}) \) with

\[
f_0(y) = \frac{\kappa (1 + o(y, \pm \infty)(1))}{|y|^{\alpha} \ln^\beta(|y|)}
\]

and

\[
f_0'(y) = \frac{-\alpha \kappa (1 + o(y, \pm \infty)(1))}{y|y|^{\alpha} \ln^\beta(|y|)} .
\]

Then, for any \( z \in \mathbb{R}\setminus\{z_c\} \), we have, with \( \mu(t) \) the solution for \( t \) large enough to

\[
\mu(t)^{1+\alpha} \ln^\beta(\mu(t)) = t,
\]

the convergence

\[
\frac{t}{\mu(t)} f(\mu(t), t) \to p(z)
\]

when \( t \to +\infty \), where the profile \( p \) is the same as Theorem 1.5. Furthermore, for any \( \varepsilon > 0 \), the convergence is uniform on \( \left[-\frac{1}{2}, \frac{1}{2}\right]\setminus[z_c - \varepsilon, z_c + \varepsilon] \).

Here, the limite profile \( p \) is the same as in the case \( \beta = 0 \) (that is Theorem 1.5), but the scalings are different. A similar result can be proven for a larger class of asymptotics on the initial condition, but writing a general result is still difficult at this point.

Remark that at fixed \( \alpha \) but for different values of \( \beta \), the initial datas in Proposition 1.8 are in the same \( L^p(\mathbb{R}) \) space, and not in the same \( L^p(\mathbb{R}) \), but the scaling \( \mu(t) \) is different. This means that we can not expect the asymptotique profile to only depends in which \( L^p(\mathbb{R}) \) spaces the initial data is or is not to deduce the asymptotic profile, we need more informations than that.

More generically, our approach should work for initial datas \( f_0 \) such that there exists \( \mu(t) \) and \( f \) a \( C^1 \) function that never cancels such that

\[
\frac{t}{\mu(t)} f_0(\mu(t), y) \to f(y), t f_0'(\mu(t), z) \to f'(y)
\]
for \( y \neq 0 \) when \( t \rightarrow +\infty \), and with \( y \rightarrow y + f(y) : \mathbb{R}^* \rightarrow \mathbb{R} \) surjective. The asymptotic profile will then depend only on \( f \). Remark that for either \( f_0(x) \sim \frac{\kappa}{|x|^\alpha} \) and \( \mu(t) = t^{1+\alpha} \) or \( f_0(x) \sim \frac{\kappa}{|y|^{1+\alpha} \ln^\beta(|y|)} \) and \( \mu(t) \) solution of \( \mu(t)^{1+\alpha} \ln^\beta(\mu(t)) = t \), we have

\[
\frac{t}{\mu(t)} f_0(\mu(t)y) \rightarrow \frac{\kappa}{|y|^{\alpha}}
\]

for \( y \neq 0 \), and this is why both profiles are identical, only the scaling has changed.

Finally, as a last exemple, we look at the non symmetric case, when the decay rate is not the same at \( \pm \infty \). We require there more precisions on the behavior of \( f_0 \). This is because we are in a case where \( y + f(y) \) will not be surjective.

**Proposition 1.9** For \( \kappa > 0, \alpha \in [0,1] \) there exists \( z_c > 0 \) such that the following holds. Given \( 1 > \beta > \alpha \), consider \( f \) the solution to the viscous Burgers equation \( \partial_t f - \partial_y^2 f + \partial_x f = 0 \) for an initial condition \( f_0 \in C^1(\mathbb{R}, \mathbb{R}) \) with

\[
f_0(y) = \frac{\kappa(1 + o_{y \rightarrow +\infty}(1))}{|y|^\alpha}, f_0(y) = \frac{\kappa(1 + o_{y \rightarrow -\infty}(1))}{|y|^\beta}
\]

and

\[
f'_0(y) = -\frac{\alpha \kappa(1 + o_{y \rightarrow +\infty}(1))}{y|y|^\alpha}, f'_0(y) = \frac{-\beta \kappa(1 + o_{y \rightarrow -\infty}(1))}{y|y|^\beta}.
\]

Suppose also that \( f_0 > 0 \) and \( f'_0 > 0 \) on \( -\infty, 0[ \) and \( f'_0 < 0 \) on \( ]0, +\infty[ \). Then, for any \( z \in \mathbb{R} \backslash \{0, z_c\} \), we have the convergence

\[
t^{\frac{\alpha}{1+\alpha}} f\left(zt^{\frac{1}{1+\alpha}}, t\right) \rightarrow p(z)
\]

when \( t \rightarrow +\infty \), where the profile \( p \) is defined by

\[
p(z) := \begin{cases} 
\kappa|y_c^*(z)|^{-\alpha} & \text{if } z > z_c \\
0 & \text{if } 0 < z < z_c \\
0 & \text{if } z < 0.
\end{cases}
\]

Remark that here also, the profile has a discontinuity at \( z_c \), and its derivative also has a discontinuity at \( 0 \). Here, the branch \( y_c^* \) is the same as in Theorem 1.5, but for the same value of \( \alpha \) and \( \kappa \), we do not have \( z_c = z_c \). The fact that the identity appears on some interval is consistent with the entropic arguments of section 1.1.3 of [11].

In all these exemples, we have taken the same value of \( \kappa \) at \( +\infty \) and \( -\infty \), but in fact this is not necessary for our proofs. It is taken to simplify some computations and notations, but similar results should hold in the more general setting. See subsection 1.4 for the main steps of the proof of these results.

### 1.4 Plan of the proofs

The proofs of Theorem 1.1 and Proposition 1.3 rely on the Hopf-Cole transformation ([3], [13]), and they are done in section 2. By this transformation we show that the solution of the viscous Burgers equation is

\[
f(x,t) = \frac{\int_{\mathbb{R}} x-y \exp\left(-\frac{(x-y)^2}{4t} - \frac{1}{2} \int_0^y f^y_0(z) dz\right) dy}{\int_{\mathbb{R}} \exp\left(-\frac{(x-y)^2}{4t} - \frac{1}{2} \int_0^y f^y_0(z) dz\right) dy}.
\]

We check that this quantity makes sense even with the slow decay of \( f_0 \). After an integration by parts, we write it

\[
f(x,t) = \frac{\int_{\mathbb{R}} f_0(y) e^{H(y)} dy}{\int_{\mathbb{R}} e^{H(y)} dy}
\]

where \( H(y) := -\frac{(x-y)^2}{4t} - \frac{1}{2} \int_0^y f^y_0(z) dz \). The core of our proof is to show that, although it is difficult to estimate \( \int_{\mathbb{R}} e^{H(y)} dy \), most of its mass (up to an almost exponentially small error in time) comes from the integral on
for some small $\varepsilon > 0$. Therefore, since $|f_0(y)|$ decays like $|y|^{-\alpha}$ and does not cancel, we have

$$\int_{\mathbb{R}} |f_0(y)| e^{H(y)} \, dy \lesssim \left( \sup_{\mathbb{R} \setminus [-\varepsilon t^\frac{1}{4\alpha}, \varepsilon t^\frac{1}{4\alpha}]} |f_0| \right) \int_{\mathbb{R}} e^{H(y)} \, dy \lesssim \frac{K}{t^\frac{4}{4\alpha}} \int_{\mathbb{R}} e^{H(y)} \, dy,$$

leading to the estimate on $f$. To show Proposition 1.3, we differentiate equation (1.2) and we check that $\partial_t \partial_x^\varepsilon f$ can be estimated by terms of the form $\frac{g_0}{t^\frac{1}{4\alpha}} e^{H(y)} \, dy$, where $g_0$ is a sum of derivatives and powers of $f_0$. The proof follows from similar arguments as the one of Theorem 1.1.

The proof of Theorem 1.5 follows similar ideas, but with the additional precision on $f_0$ at infinity, we can compute exactly where $H \left( t^\frac{1}{4\alpha} \right)$ reaches its maximum. It turns out that at most two particular functions of $z$, that we will call $y_-(z, t)$ and $y_+(z, t)$, can reach it. When $t \to +\infty$ they both converge nicely to $y_+(z)$ and $y_+(z)$ respectively, at least where they have a chance to reach the maximum. For $z$ close to $-\infty$ it will be reached only by $y_-$, and close to $+\infty$ only by $y_+$. We show that there exists only one value of $z$ (which is $z_e$) such that the maximum is reached by both. Then, if $z \neq z_e$, most of the mass of $\int_{\mathbb{R}} e^{H(y)} \, dy$ is coming from a small neighborhood of the point where $H \left( t^\frac{1}{4\alpha} \right)$ reaches its maximum, and thus

$$\int_{\mathbb{R}} f_0(y) e^{H(y)} \, dy \simeq f_0 \left( y_{\text{max}} t^\frac{1}{4\alpha} \right) \int_{\mathbb{R}} e^{H(y)} \, dy \simeq -\kappa t^{-\frac{\alpha}{4\alpha}} |y_{\text{max}}|^{-\alpha} \int_{\mathbb{R}} e^{H(y)} \, dy$$

where $y_{\text{max}}$ is the value reaching the maximum. This result is proven in section 3. Proposition 1.7 to 1.9 are proven using similar arguments, see section 3.3.

Propositions 1.2 and 1.4 follow from computations on the explicit solution of the heat equation $\partial_t f - \partial_x^2 f = 0$ (see [10]):

$$f(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f_0(y) e^{-\frac{(x-y)^2}{4t}} \, dy.$$

Although their proofs are not difficult, for the sake of completeness, they are done in annex A.

### 1.5 Open problems

Our approach does not work if the initial data satisfies

$$f_0(y) = \frac{\pm \kappa (1 + o(y \to \pm \infty))}{|y|^{\alpha}}.$$

We can show that then, $t^\frac{1}{4\alpha} f \left( \varepsilon t^\frac{1}{4\alpha}, t \right)$ converges to some profile $p$ outside of a compact interval, but we have no informations on what is happening on it. It is likely that the solution is unstable and depends also on $f_0$ and not simply on its behavior at $\pm \infty$.

It would also be interesting to understand how much of the results is still true if we consider initial datas satisfying

$$\frac{\kappa_2}{(1 + |y|)^{\alpha}} \geq f_0(y) \geq \frac{\kappa_1}{(1 + |y|)^{\beta}}$$

if $1 > \alpha > \beta > 0$, or for other type of behavior at infinity of $f_0$.

Concerning similar problem on other equations, since the proofs here use the Hopf-Cole transform, it is unlikely that they will hold in a more general setting. In [11] we show the existence of a local attractor (instead of a global one) for the equation $\partial_t u - \partial_x^2 u + \partial_x \left( \frac{u^2}{2} + J(u) \right) = 0$ if $|J(u)| \leq C|u|^3$, see Proposition 1.5 there. A natural generalisation would be to look at similar equations in dimension 2 or higher.

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2 Estimates on the viscous Burgers equation

This section is devoted to the proofs of Theorem 1.1 and Proposition 1.3. We recall that the quantity

\[ f(x, t) = \frac{\int_{\mathbb{R}} \frac{x-y}{t} \exp \left(-\frac{(x-y)^2}{4t} - \frac{1}{2} \int_0^y f_0(z)dz \right) dy}{\int_{\mathbb{R}} \exp \left(-\frac{(x-y)^2}{4t} - \frac{1}{2} \int_0^y f_0(z)dz \right) dy} \]  

(2.1)
is well defined for all \( t \geq 0, x \in \mathbb{R} \) and is the solution to the viscous burgers equation \( \partial_t f - \partial_x^2 f + f \partial_x f = 0 \) with initial data \( f_0 \). We compute, by integration by parts, that

\[
\int_{\mathbb{R}} \frac{x-y}{t} \exp \left(-\frac{(x-y)^2}{4t} - \frac{1}{2} \int_0^y f_0(z)dz \right) dy = \int_{\mathbb{R}} 2\partial_y \left( \exp \left(-\frac{(x-y)^2}{4t} \right) \right) \exp \left(-\frac{1}{2} \int_0^y f_0(z)dz \right) dy
\]

= \int_{\mathbb{R}} f_0(y) \exp \left(-\frac{(x-y)^2}{4t} - \frac{1}{2} \int_0^y f_0(z)dz \right) dy.

This integration by part is justified for any \( t > 0 \) and \( x \in \mathbb{R} \) by the fact that

\[
\left| \int_0^y f_0(z)dz \right| \leq K|y|^{1-\alpha} \leq \frac{(x-y)^2}{8t}
\]

for \(|y|\) large enough. We define the quantity

\[ H(y) = -\frac{(x-y)^2}{4t} - \frac{1}{2} \int_0^y f_0(z)dz, \]

so that

\[ f(x, t) = \frac{\int_{\mathbb{R}} f_0(y)e^{H(y)}dy}{\int_{\mathbb{R}} e^{H(y)}dy}. \]

(2.2)

In particular we have the inequality

\[ |f(x, t)| \leq \|f_0\|_{L^\infty}. \]

We study here the case \( \frac{-\kappa_1}{(1+|y|)^\alpha} \leq f_0(y) \leq \frac{\kappa_1}{(1+|y|)^\alpha} \). If \( f(x, t) \) solves the viscous Burgers equation, then so does \(-f(-x, t)\), so this is equivalent as considering \( \frac{-\kappa_1}{(1+|y|)^\alpha} \geq f_0(y) \geq \frac{\kappa_1}{(1+|y|)^\alpha} \). We do so because some estimates will now be in a more usual direction. We consider the more general quantity

\[ F(x, t) := \frac{\int_{\mathbb{R}} g(y)e^{H(y)}dy}{\int_{\mathbb{R}} e^{H(y)}dy}, \]

where we still take \( H(y) = -\frac{(x-y)^2}{4t} - \frac{1}{2} \int_0^y f_0(z)dz \). For now, we simply suppose that \( g \in C^0(\mathbb{R}) \) with \( \|g\|_{C^0(\mathbb{R})} \leq \kappa \) where \( \kappa \) only depends on \( \kappa_1, \kappa_2 \) and \( \alpha \), so that in particular the quantity \( \int_{\mathbb{R}} g(y)e^{H(y)}dy \) is well defined. Let us first show the following result.

**Proposition 2.1** There exists \( T^*, K, \nu > 0, \mu_1 < 0, \mu_2 > 0 \) depending on \( \kappa_1, \kappa_2, \alpha \) such that, if we define the set \( D := \left[-\infty, \mu_1 t \right] \cup \left[ \mu_2 t \frac{1}{1+\nu}, +\infty \right] \), then for \( x \in \mathbb{R}, t \geq T^* \),

\[ |F(x, t)| \leq \sup_D |g| + Ke^{-\nu t^{1-\alpha}}. \]
2.1 Proof of Proposition 2.1

We first estimate

\[ |F(x, t)| \leq \sup_D |g| \int_D e^{H(y)} dy + \|g\|_{L^\infty} \int_{\mathbb{R}\setminus D} e^{H(y)} dy \leq \sup_D |g| + \kappa, \]

therefore the result hold on \([0, T^\ast]\) for any constant \(T^\ast > 0\) depending on \(\kappa_1, \kappa_2, \alpha\).

We now suppose that \(t \geq T^\ast\).

We compute

\[ H'(y) = \frac{1}{2t}(x - y) - \frac{1}{2}f_0(y) = \frac{1}{2} \left( \frac{x - y}{t} - f_0(y) \right). \]

Since \(H(y) \to -\infty\) when \(y \to \pm\infty\), the function \(H\) must reach its maximum at a point where \(H'(y) = 0\). We consider any such \(y^\ast \in \mathbb{R}\) solution of \(H'(y^\ast) = 0\). Let us give some estimates on \(y^\ast\). We have

\[ \frac{1}{2t}(x - y^\ast) + \frac{\kappa_1}{2(1 + |y^\ast|)^\alpha} \leq \frac{1}{2t}(x - y^\ast) - \frac{1}{2}f_0(y^\ast) = 0 \leq \frac{1}{2t}(x - y^\ast) + \frac{\kappa_2}{2(1 + |y^\ast|)^\alpha}, \]

therefore

\[ -\kappa_2 t \leq (1 + |y^\ast|)^\alpha(x - y^\ast) \leq -\kappa_1 t. \]

We will use different arguments to show the estimate on \(F\) depending on the value of \(\frac{x}{t^{1+\alpha}}\). We decompose the problem into four different cases.

In the rest of the proof, we will use \(\varepsilon > 0\) a small constant depending on \(\kappa_1, \kappa_2, \alpha\), and then we will take \(T^\ast\) large, depending on \(\kappa_1, \kappa_2, \alpha\) and \(\varepsilon\). In this section, a generic constant \(K\) can depend on \(\kappa_1, \kappa_2\) and \(\alpha\), but not on \(\varepsilon\) or \(T^\ast\), except if it is explicitly stated.

2.1.1 Estimate on \(F\) in the case \(\frac{|x|}{t^{1+\alpha}} \leq \varepsilon\)

We recall that we take \(t \geq T^\ast > 0\) with \(T^\ast\) large. Define \(z^\ast = \frac{y^\ast}{t^{1+\alpha}}\). By (2.4), it satisfies

\[ -\kappa_2 \leq \left( \frac{1}{t^{1+\alpha}} + |z^\ast| \right)^\alpha \left( \frac{x}{t^{1+\alpha}} - z^\ast \right) \leq -\kappa_1. \]

First, let us show that for \(t\) large enough, we have \(z^\ast > \varepsilon\). Indeed, if it’s not true, then \(z^\ast \leq \varepsilon\) and by (2.5) since

\[ \frac{x}{t^{1+\alpha}} - z^\ast \leq 0, \]

we have \(z^\ast \geq \frac{x}{t^{1+\alpha}}\), therefore \(|z^\ast| \leq \max(\varepsilon, \frac{|x|}{t^{1+\alpha}}) \leq \varepsilon\). But then,

\[ \left| \left( \frac{1}{t^{1+\alpha}} + |z^\ast| \right)^\alpha \left( \frac{x}{t^{1+\alpha}} - z^\ast \right) \right| \leq \left( \frac{1}{t^{1+\alpha}} + \varepsilon \right)^\alpha 2\varepsilon < \kappa_1 \]

for \(\varepsilon\) small enough and \(T^\ast\) large enough (depending on \(\varepsilon, \kappa_1, \kappa_2\)), leading to a contradiction.

Secondly, there exists \(\kappa > 1\) large enough (depending only on \(\kappa_2, \varepsilon\)) such that \(z^\ast \leq \kappa\).

Indeed, otherwise \(z^\ast \geq \kappa\) and \(\frac{x}{t^{1+\alpha}} - z^\ast \leq \varepsilon - \kappa < 0\), leading to

\[ -\kappa_2 \leq \left( \frac{1}{t^{1+\alpha}} + |z^\ast| \right)^\alpha \left( \frac{x}{t^{1+\alpha}} - z^\ast \right) \leq (2\kappa)^\alpha(-\kappa + \varepsilon) < -\kappa_2 \]
if $\kappa$ and $T^*$ are large enough.

We summarize. There exists $\Lambda_2 > \Lambda_1 > 0$ two constants depending on $\kappa_1, \kappa_2, \varepsilon$ such that if $\frac{|y|}{t^{1+\alpha}} \leq \varepsilon$, then any $y^* \in \mathbb{R}$ solution of $H'(y^*) = 0$ is in the set $[\Lambda_1 t^{\frac{1}{1+\alpha}}, \Lambda_2 t^{\frac{1}{1+\alpha}}]$. Since $H(y) \to -\infty$ when $y \to \pm \infty$, its maximum is reached in this set. For such a $y^* \in \mathbb{R}$, since $f_0(y) \leq \frac{-\kappa_1}{(1+|y|)^\alpha}$ we have

$$-\frac{1}{2} \int_0^{y^*} f_0(z) dz \geq \frac{\kappa_1}{4(1-\alpha)}(1+|y^*|)^{1-\alpha} \geq \frac{\kappa_1 \Lambda_1}{4(1-\alpha)} t^{\frac{1}{1+\alpha}}$$

and

$$(x-y^*)^2 \leq (\varepsilon - \Lambda_2)^2 t^{\frac{2}{1+\alpha}},$$

leading to

$$-\frac{(x-y^*)^2}{4t} \geq \frac{-(\varepsilon - \Lambda_2)^2}{4} t^{\frac{1}{1+\alpha}}.$$

We deduce that

$$H(y^*) \geq \left( \frac{\kappa_1 \Lambda_1}{4(1-\alpha)} - \frac{(\varepsilon - \Lambda_2)^2}{4} \right) t^{\frac{1}{1+\alpha}}.$$

We define

$$C_0 := t^{-\frac{1}{1+\alpha}} \max_{y \in [\Lambda_1, \Lambda_2]} H(y)$$

which therefore satisfies $C_0 \geq \frac{\kappa_1 \Lambda_1}{4(1-\alpha)} - \frac{(\varepsilon - \Lambda_1)^2}{4}$. Remark that $C_0$ depends on time and it is possible that $C_0 = 0$ or $C_0 < 0$. We check similarly that $C_0 \leq K(\varepsilon)$, thus $C_0$ is uniformly bounded in time.

Now, we estimate for $\gamma > 0$ that

$$H' \left( \gamma t^{\frac{1}{1+\alpha}} \right) \geq \frac{t^{\frac{1}{1+\alpha}}}{2} \left( \frac{x}{t^{\frac{1}{1+\alpha}}} - \gamma + \frac{\kappa_1}{t^{\frac{1}{1+\alpha}} + \gamma} \right).$$

We deduce that if $T^*$ is large enough, there exists $\lambda_1 > 0$ with $\lambda_1 < \Lambda_1$ such that $\frac{\gamma}{t^{\frac{1}{1+\alpha}}} - \gamma + \frac{\kappa_1}{t^{\frac{1}{1+\alpha}} + \gamma} \geq 2$ for $\gamma \in \left[ \frac{\lambda_1}{2}, \lambda_1 \right]$ and thus $H'(y) \geq t^{-\frac{\alpha}{2}}$ for $y \in \left[ \frac{\lambda_1}{2} t^{\frac{1}{1+\alpha}}, \lambda_1 t^{\frac{1}{1+\alpha}} \right]$. Similarly, we check that for $\gamma > 0$,

$$H' \left( \gamma t^{\frac{1}{1+\alpha}} \right) \leq \frac{t^{\frac{1}{1+\alpha}}}{2} \left( \frac{x}{t^{\frac{1}{1+\alpha}}} - \gamma + \frac{\kappa_2}{t^{\frac{1}{1+\alpha}} + \gamma} \right),$$

and that there exists $\lambda_2 > \Lambda_2$ such that $\frac{\gamma}{t^{\frac{1}{1+\alpha}}} - \gamma + \frac{\kappa_2}{t^{\frac{1}{1+\alpha}} + \gamma} \leq -2$ if $\gamma > \lambda_2$. This implies that $H'(y) \leq -t^{-\frac{\alpha}{2}}$ for $y \geq \lambda_2 t^{\frac{1}{1+\alpha}}$.

Since $H'$ only has zeros in $[\Lambda_1 t^{\frac{1}{1+\alpha}}, \Lambda_2 t^{\frac{1}{1+\alpha}}]$ and $H(y) \to -\infty$ when $y \to -\infty$, we have that $H' \geq 0$ on $]-\infty, \Lambda_1 t^{\frac{1}{1+\alpha}}[$. Therefore, for $y \in ]-\infty, \Lambda_1 t^{\frac{1}{1+\alpha}}[$, we have

$$H(y) = H \left( \lambda_1 t^{\frac{1}{1+\alpha}} \right) - \int_y^{\lambda_1 t^{\frac{1}{1+\alpha}}} H'(z) dz$$

$$\leq \max H - \int_{\lambda_1 t^{\frac{1}{1+\alpha}}}^{\lambda_1 t^{\frac{1}{1+\alpha}}} H'(z) dz$$

$$\leq C_0 t^{-\frac{\alpha}{2}} - \frac{\lambda_1}{2} t^{\frac{1}{1+\alpha}} + \frac{\alpha}{2}$$

$$\leq \left( C_0 - \frac{\lambda_1}{2} \right) t^{\frac{1}{1+\alpha}}.$$
Furthermore, for \( y \in \left[ 2\lambda_2 t^{\frac{1}{\alpha}}, +\infty \right] \) we have

\[
H(y) \leq \max H + \int_{\lambda_2 t^{\frac{1}{\alpha}}}^{2\lambda_2 t^{\frac{1}{\alpha}}} H'(z) dz \\
\leq (C_0 - \lambda_2) t^{\frac{1}{\alpha}}.
\]

We define the domain

\[ D_1 := \left[ \frac{\lambda_1}{2} t^{\frac{1}{\alpha}}, 2\lambda_2 t^{\frac{1}{\alpha}} \right]. \]

We have shown that there exists \( 1 > \nu > 0 \) a small constant depending on \( \kappa_1, \kappa_2, \varepsilon, T^* \) such that, for \( y \in \mathbb{R} \setminus D_1 \), \( H(y) \leq (C_0 - \nu) t^{\frac{1}{\alpha}} \). Now we estimate

\[
\left| \int_{\mathbb{R}} g(y) e^{H(y)} dy \right| \leq \max_{D_1}(|g|) \int_{D_1} e^{H(y)} dy + \|g\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R} \setminus D_1} e^{H(y)} dy
\]

and thus

\[
|F(x, t)| \leq \max_{D_1}(|g|) + K \int_{\mathbb{R}} e^{H(y)} dy.
\]

Since \( H'(y) = \frac{1}{2} \left( \frac{x-y}{t} - f_0(y) \right) \), for \( |x| \leq \varepsilon t^{\frac{1}{\alpha}} \) and \( y \in \left[ \frac{\lambda_1}{2} t^{\frac{1}{\alpha}}, \lambda_2 t^{\frac{1}{\alpha}} \right] \), if \( T^* \) is large enough we have \( |H'(y)| \leq K \). Therefore, there exists \( \rho > 0 \) depending only on \( \kappa_1, \kappa_2 \) such that, for a point \( y^* \) with \( H(y^*) = \max H \) and \( y \in [y^* - \rho, y^* + \rho] \), we have \( H(y) \geq (\max H) - 1 \). We deduce that

\[
\int_{\mathbb{R}} e^{H(y)} dy \geq \int_{[y^* - \rho, y^* + \rho]} e^{H(y)} dy \geq K(\kappa_1, \kappa_2) e^{\max H} \geq K(\kappa_1, \kappa_2) e^{C_0 t^{\frac{1}{\alpha}}}
\]

We recall that

\[
H(y) = -\frac{(x - y)^2}{4t} - \frac{1}{2} \int_0^y f_0(z) dz,
\]

and since

\[
\left| \int_0^y f_0(z) dz \right| \leq K(1 + |y|)^{1-\alpha},
\]

there exists \( \Omega_1 < \frac{\lambda_1}{2}, \Omega_2 > 2\lambda_2 \) (with eventually \( \Omega_1 < 0 \)) depending on \( \kappa_1, \kappa_2, \varepsilon, T^* \), such that, outside of \( \left[ \Omega_1 t^{\frac{1}{\alpha}}, \Omega_2 t^{\frac{1}{\alpha}} \right] \), we have

\[
H(y) \leq -\frac{(x - y)^2}{8t} + (C_0 - \nu) t^{\frac{1}{\alpha}}
\]

(we recall that here \( \frac{|x|}{t^{\frac{1}{\alpha}}} \leq \varepsilon \) and \( C_0 \) is uniformly bounded in time). We deduce that

\[
\int_{\mathbb{R} \setminus \left[ \Omega_1 t^{\frac{1}{\alpha}}, \Omega_2 t^{\frac{1}{\alpha}} \right]} e^{H(y)} dy \leq e^{(C_0 - \nu) t^{\frac{1}{\alpha}}} \int_{\mathbb{R} \setminus \left[ \Omega_1 t^{\frac{1}{\alpha}}, \Omega_2 t^{\frac{1}{\alpha}} \right]} e^{-\frac{(x - y)^2}{8t}} dy
\]

\[
\leq e^{(C_0 - \nu) t^{\frac{1}{\alpha}}} \int_{\mathbb{R}} e^{-\frac{y^2}{8t}} dy
\]

\[
\leq K e^{(C_0 - \nu) t^{\frac{1}{\alpha}}} \sqrt{t},
\]

and we compute

\[
\int_{\Omega_1 t^{\frac{1}{\alpha}}, \Omega_2 t^{\frac{1}{\alpha}}} e^{H(y)} dy
\]

\[
\leq e^{(C_0 - \nu) t^{\frac{1}{\alpha}}} \text{Vol} \left( \left[ \Omega_1 t^{\frac{1}{\alpha}}, \Omega_2 t^{\frac{1}{\alpha}} \right] \setminus D_1 \right)
\]

\[
\leq K(\varepsilon, T^*) t^{\frac{1}{\alpha}} e^{(C_0 - \nu) t^{\frac{1}{\alpha}}}.
\]
We deduce that
\[
\frac{\int_{\mathbb{R}\setminus D_1} e^{H(y)} dy}{\int_{\mathbb{R}} e^{H(y)} dy} \leq K(\epsilon, T^*) \left( \sqrt{t} e^{(C_0 - \nu) t^{1+\alpha} } + t \frac{1}{1+\alpha} e^{(C_0 - \nu) t^{1+\alpha} } \right) \leq K(\epsilon, T^*) e^{-\nu t^{1+\alpha}/2}.
\]

This completes the proof of
\[
|F(x, t)| \leq \max_{D_1}(|y|) + K \frac{\int_{\mathbb{R}\setminus D} e^{H(y)} dy}{\int_{\mathbb{R}} e^{H(y)} dy} \leq \max_{D}(|y|) + K(\epsilon, T^*) e^{-\frac{\nu t^{1+\alpha}}{2}}
\]
with \( D_1 = \left[ \frac{\lambda_1 t^{1+\alpha}}{1+\alpha}, 2\lambda_2 t^{1+\alpha} \right] \) and \( \lambda_1 > 0 \) in the case \( \frac{|x|}{t^{1+\alpha}} \leq \epsilon. \)

### 2.1.2 Estimate on \( F \) in the case \( \frac{x}{t^{1+\alpha}} \geq \epsilon \)

Remark that here, \( x \geq \epsilon t^{1+\alpha} \geq 2 \) for \( T^* \) large enough. From equation (2.4), we have for any \( y^* \in \mathbb{R} \) such that \( H'(y^*) = 0 \) that \( x - y^* \leq 0 \), hence \( y^* \geq x. \) Furthermore, there exists \( \Lambda > 1 \) such that \( y^* \leq \Lambda x. \) Indeed, if \( y^* \geq \Lambda x, \) then equation (2.4) implies that
\[
-\kappa_2 t \leq (1 + |y^*|) \alpha(x - y^*) \leq -(\Lambda - 1) x^{1+\alpha},
\]
which is in contradiction with \( t \leq \frac{\nu t^{1+\alpha}}{2} \) for \( \Lambda \) large enough depending on \( \epsilon \) (and \( T^* \) large enough).

In summary, in that case, \( \Lambda x \geq y^* \geq x \) for some constant \( \Lambda > 1 \) depending on \( \epsilon \), since \( x > 0 \) we compute that for some constant \( K > 0, \n\)
\[
H(x) = -\frac{1}{2} \int_0^x f_0(z) dz \geq K x^{1-\alpha}.
\]

This implies that \( \max H \geq H(x) \geq K x^{1-\alpha} > 0. \)

Remark that for \( y \leq 0 \), since \( f_0 \leq 0 \), we have \( -\frac{1}{2} \int_0^y f_0(z) dz \leq 0, \) hence
\[
H(y) \leq -\frac{(x - y)^2}{4t}.
\]
(2.7)
Furthermore, we have in that case \( t \leq \left( \frac{x}{\epsilon} \right)^{1+\alpha} \), which implies the inequality
\[
-\frac{1}{t} \leq -\left( \frac{\epsilon}{x} \right)^{1+\alpha},
\]
and we deduce that for \( \gamma > 0, \n\)
\[
H(\gamma x) \leq -\frac{1}{4t} x^2 (1 - \gamma)^2 + K x^{1-\alpha} \gamma^{1-\alpha} \leq x^{1-\alpha} \left( -\frac{\epsilon^{1+\alpha}}{4} (1 - \gamma)^2 + K \gamma^{1-\alpha} \right).
\]

Remark that \( -\frac{\epsilon^{1+\alpha}}{4} (1 - \gamma)^2 + K \gamma^{1-\alpha} \leq 0 \) if \( \gamma > 0 \) is close to 0 or \( \gamma \) is large (depending on \( \epsilon \)). We deduce that there exists \( \lambda_2 > \lambda_1 > 0 \) depending on \( \epsilon \) such that for \( y \) outside of \( D_2 := [\lambda_1 x, \lambda_2 x], \)
\[
\frac{H(y)}{\max H} \leq \frac{1}{4},
\]
(2.8)
(we recall that here $\max H \geq K x^{1-\alpha} > 0$). We compute as in the previous case that

$$|F(x, t)| \leq \max_{D_2}(|g|) + K \int_{\mathbb{R} \setminus D_2} e^{H(y)}dy,$$

Here, $x \geq \varepsilon t^{\frac{1}{1+\alpha}}$, therefore $D_2 \subset [\lambda_1 \varepsilon t^{\frac{1}{1+\alpha}}, +\infty]$ with $\lambda_1 \varepsilon > 0$. Furthermore, we check as in the previous case that for $z \in [-1, 1]$ and any $x^*$ such that $H(x^*) = \max H$, $H(x^* + z) - H(x^*) \leq K$,

therefore

$$\int_{\mathbb{R}} e^{H(y)}dy \geq K e^{\max H}.$$

With (2.7), we check that

$$\int_{|y| \leq 0} e^{H(y)}dy \leq \int_{|y| \leq 0} e^{-\frac{(x-y)^2}{8t}}dy \leq \int_{\mathbb{R}} e^{-\frac{x^2}{8t}}dy \leq K \sqrt{t}.$$

Now, with $H(y) = -\frac{(x-y)^2}{4t} - \frac{1}{2} \int_0^y f_0(z)dz$, we check that there exists $\Lambda_2 > \lambda_2$ depending on $\varepsilon$ such that if $y \geq \Lambda_2 x$, then

$$H(y) \leq \frac{-(x-y)^2}{8t}.$$

We deduce that

$$\int_{\Lambda_2 x}^{+\infty} e^{H(y)}dy \leq \int_{\Lambda_2 x}^{+\infty} e^{-\frac{(x-y)^2}{8t}}dy \leq \int_{\mathbb{R}} e^{-\frac{x^2}{8t}}dy \leq K \sqrt{t}.$$

Finally, with (2.8) we check that

$$\int_{[0, \Lambda_2 x] \setminus D_2} e^{H(y)}dy \leq e^{\frac{x}{4} \max H} \left[0, \Lambda_2 x\right] \setminus D_2 \leq K(\varepsilon, T^*) e^{\frac{x}{4} \max H},$$

where $\text{vol}(A)$ is the Lebesgue measure of the set $A$. Combining these estimates and $\max H \geq K x^{1-\alpha} \geq K \varepsilon t^{\frac{1}{1+\alpha}}$, we infer that

$$\int_{\mathbb{R} \setminus D_2} e^{H(y)}dy \leq \int_{\mathbb{R}} e^{H(y)}dy \leq K(\varepsilon, T^*) e^{\frac{x}{4} \max H}$$

This concludes the proof of

$$|F(x, t)| \leq \max_{D_2}(|g|) + K \int_{\mathbb{R} \setminus D} e^{H(y)}dy \leq \max_D(|g|) + K(\varepsilon, T^*) e^{-\frac{x}{2} t^{\frac{1}{1+\alpha}}}$$

for some small $\nu > 0$ with $D_2 \subset [\lambda_1 \varepsilon t^{\frac{1}{1+\alpha}}, +\infty]$, $\lambda_1 \varepsilon > 0$ in the case $\frac{x}{t^{\frac{1}{1+\alpha}}} \geq \varepsilon$.

### 2.1.3 Estimate on $F$ in the case $-\frac{1}{\varepsilon} \leq \frac{x}{t^{\frac{1}{1+\alpha}}} \leq -\varepsilon$

In that case, for $y^* \in \mathbb{R}$ solution of $H'(y^*) = 0$ and $z^* = \frac{y^*}{t^{\frac{1}{1+\alpha}}}$, we recall equation (2.5):

$$-\kappa_2 \leq \left(\frac{1}{t^{\frac{1}{1+\alpha}}} + |z^*|^{\alpha} \left(\frac{x}{t^{\frac{1}{1+\alpha}}} - z^*\right) \leq -\kappa_1.$$
We deduce that \(|z^*| \geq \varepsilon^{\frac{1}{\alpha} + 1}\), since otherwise \(|z^*| \leq \varepsilon^{\frac{1}{\alpha} + 1}\) and
\[
\kappa_1 \leq \left| \left( \frac{1}{t^{\frac{1}{\alpha}}} + |z^*| \right)^\alpha \left( \frac{x}{t^{\frac{1}{\alpha}}} - z^* \right) \right| \leq \left( \frac{1}{t^{\frac{1}{\alpha}}} + \varepsilon^{1 + \frac{1}{\alpha}} \right)^\alpha \left( \frac{1}{\varepsilon} + \varepsilon^{1 + \frac{1}{\alpha}} \right) \leq \left( 1 + \frac{1}{t^{\frac{1}{\alpha}}} \varepsilon^{1 + \frac{1}{\alpha}} \right) \left( \varepsilon^\alpha + \varepsilon^{2 + \frac{1}{\alpha} + \alpha} \right)
\]
which is impossible if \(\varepsilon\) is small enough and \(T^*\) large enough.

We check also that \(|z^*| \leq \frac{2}{\varepsilon}\). Otherwise,
\[
\left| \frac{x}{t^{\frac{1}{\alpha}}} - z^* \right| \geq \frac{1}{\varepsilon}
\]
and \(\left( \frac{1}{t^{\frac{1}{\alpha}}} + |z^*| \right)^\alpha \geq \left( \frac{2}{\varepsilon} \right)^\alpha\), thus
\[
\kappa_2 \geq \left| \left( \frac{1}{t^{\frac{1}{\alpha}}} + |z^*| \right)^\alpha \left( \frac{x}{t^{\frac{1}{\alpha}}} - z^* \right) \right| \geq 2^{\alpha} \varepsilon^{1 + \alpha},
\]
which is impossible if \(\varepsilon\) is small enough. We define as previously
\[
C_0 = t^{-\frac{1}{\alpha}} \max H,
\]
and we just showed that for \(y^* \in \mathbb{R}\) solution of \(H'(y^*) = 0\), we have
\[
|y^*| \in \left[ \varepsilon^{\frac{1}{\alpha} + 1} t^{\frac{1}{\alpha^2}}, \frac{2}{\varepsilon} \frac{1}{t^{\frac{1}{\alpha}}} \right].
\]
We can show as previously that \(C_0\) is bounded uniformly in time. Now, we compute that for \(\gamma \in \mathbb{R}\),
\[
H' \left( \gamma t^{\frac{1}{\alpha}} \right) \geq t^{-\frac{\alpha}{\alpha^2}} \frac{1}{2} \left( \frac{x}{t^{\frac{1}{\alpha}}} - \gamma + \frac{\kappa_1}{t^{\frac{1}{\alpha}} + |\gamma|} \right)
\]
and
\[
H' \left( \gamma t^{\frac{1}{\alpha}} \right) \leq t^{-\frac{\alpha}{\alpha^2}} \frac{1}{2} \left( \frac{x}{t^{\frac{1}{\alpha}}} - \gamma + \frac{\kappa_2}{t^{\frac{1}{\alpha}} + |\gamma|} \right).
\]
In particular, there exists \(\lambda > 0\) small enough (depending on \(T^*\) and \(\varepsilon\)) such that for \(\gamma \in [-\lambda, \lambda]\) and \(T^*\) large enough (depending on \(\varepsilon\)), we have
\[
H' \left( \gamma t^{\frac{1}{\alpha}} \right) \geq t^{-\frac{\alpha}{\alpha^2}}. \tag{2.9}
\]
This is because \(\frac{\kappa_1}{t^{\frac{1}{\alpha}} + |\gamma|} \to +\infty\) when \(|\gamma| \to 0\) and \(t \to +\infty\), while \(\frac{x}{t^{\frac{1}{\alpha}}} - \gamma\) stays bounded.

In the limit \(|\gamma| \to \infty\), the dominating term is \(-\gamma\), and therefore there exists \(\Lambda_1 < 0, \Lambda_2 > 0\) (depending on \(\varepsilon\)) such that for all \(\gamma < \Lambda_1\),
\[
H' \left( \gamma t^{\frac{1}{\alpha}} \right) \geq t^{-\frac{\alpha}{\alpha^2}} \tag{2.10}
\]
and for all \(\gamma > \Lambda_2\),
\[
H' \left( \gamma t^{\frac{1}{\alpha}} \right) \leq -t^{-\frac{\alpha}{\alpha^2}}. \tag{2.11}
\]
We recall that the maximum of \(H\) is reached either in \(\left[ \varepsilon^{\frac{1}{\alpha} + 1} t^{\frac{1}{\alpha^2}}, \frac{2}{\varepsilon} \frac{1}{t^{\frac{1}{\alpha}}} \right]\) or in \(\left[ -\frac{2}{\varepsilon} t^{\frac{1}{\alpha^2}}, -\varepsilon^{\frac{1}{\alpha} + 1} t^{\frac{1}{\alpha^2}} \right]\). Let us show that (2.9) implies that for \(y \in \left[ -\frac{2}{\varepsilon} t^{\frac{1}{\alpha^2}}, \frac{1}{2} t^{\frac{1}{\alpha^2}} \right]\), we have
\[
H(y) \leq \left( C_0 - \frac{\lambda}{4} \right) t^{\frac{1}{\alpha^2}}. \tag{2.12}
\]
Indeed, if it’s not true for some \( y \), then

\[
H(\lambda t^{1/\alpha}) = H(y) + \int_y^{\lambda t^{1/\alpha}} H'(z)dz
\]

\[
\geq \left( C_0 - \frac{\lambda}{4} \right) t^{1/\alpha} + \int_{\frac{1}{4}t^{1/\alpha}}^{\lambda t^{1/\alpha}} t^{-\frac{1}{\alpha}} dz
\]

\[
\geq \left( C_0 - \frac{\lambda}{4} \right) t^{1/\alpha} + \frac{\lambda}{2} t^{1/\alpha}
\]

\[
\geq \left( C_0 + \frac{\lambda}{4} \right) t^{1/\alpha}
\]

\[
> \max H
\]

which is a contradiction. We then define the domain

\[
D_3 := \left( (A_1 - 1)t^{1/\alpha}, (A_2 + 1)t^{1/\alpha} \right] \setminus \left( -\frac{\lambda}{2} t^{1/\alpha}, \frac{\lambda}{2} t^{1/\alpha} \right]
\]

and we estimate as the other steps that

\[
|F(x, t)| \leq \left| \int_{\mathbb{R}} g_0(y)e^{H(y)}dy \right| \leq \max_{D_3} |g_0(y)| + K\|f_0\|_{L^\infty} e^{-C_0 t^{1/\alpha}} \int_{\mathbb{R}\setminus D_3} e^{H(y)}dy.
\]

We have \( \mathbb{R}\setminus D_3 = \left[ -\frac{\lambda}{2} t^{1/\alpha}, \frac{\lambda}{2} t^{1/\alpha} \right] \cup \left( \mathbb{R} \setminus \left( (A_1 - 1)t^{1/\alpha}, (A_2 + 1)t^{1/\alpha} \right) \right) \) and by (2.12),

\[
\int_{\left[ -\frac{\lambda}{2} t^{1/\alpha}, \frac{\lambda}{2} t^{1/\alpha} \right]} e^{H(y)}dy \leq e^{(C_0 - \frac{\lambda}{4}) t^{1/\alpha}} \int_{\left[ -\frac{\lambda}{2} t^{1/\alpha}, \frac{\lambda}{2} t^{1/\alpha} \right]} e^{\lambda t^{1/\alpha} e^{(C_0 - \frac{\lambda}{4}) t^{1/\alpha}}}. 
\]

Furthermore, by equations (2.10) and (2.11), and with similar argument as the previous steps, we show that for some \( \nu > 0 \) small and depending on \( \varepsilon, T^\ast \),

\[
\int_{\mathbb{R}\setminus \left( (A_1 - 1)t^{1/\alpha}, (A_2 + 1)t^{1/\alpha} \right]} e^{H(y)}dy \leq K\varepsilon, T^\ast e^{(C_0 - \nu) t^{1/\alpha}}. 
\]

We deduce that

\[
e^{-C_0 t^{1/\alpha}} \int_{\mathbb{R}\setminus D_3} e^{H(y)}dy
\]

\[
\leq K\varepsilon e^{-\left( \min\left( \nu, \frac{\lambda}{4} \right) \right) t^{1/\alpha}} \left( 1 + \sqrt{t} \right)
\]

\[
\leq K\varepsilon e^{-\left( \min\left( \nu, \frac{\lambda}{4} \right) \right) t^{1/\alpha}},
\]

which concludes the proof of

\[
|F(x, t)| \leq \max_{D_3} |g| + Ke^{-\left( \min\left( \nu, \frac{\lambda}{4} \right) \right) t^{1/\alpha}}
\]

with \( D_3 \subset \left] -\infty, -\frac{\lambda}{2} t^{1/\alpha} \right] \cup \left[ \frac{\lambda}{2} t^{1/\alpha}, +\infty \right] \) in the case \(-\frac{1}{\varepsilon} \leq \frac{\nu}{t^{1/\alpha}} \leq -\varepsilon \).

**2.1.4 Estimate on \( F \) in the case \( \frac{\nu}{t^{1/\alpha}} \leq \frac{1}{\varepsilon} \)**

In that case, \( x < -1 \) for \( T^\ast \) large enough and we compute that

\[
H(x) = -\frac{1}{2} \int_0^x f_0(z)dz \geq -C_1|x|^{1-\alpha}
\]
for some $C_1 > 0$. Now, take $\gamma > 0$, and we estimate, for $T^*$ large enough, that

$$-rac{(1 - \gamma)^2}{4t} x^2 - K\kappa_2 \gamma^{1-\alpha}|x|^{1-\alpha} \geq H(\gamma x) \geq -\frac{(1 - \gamma)^2}{4t} x^2 - K\kappa_2 \gamma^{1-\alpha}|x|^{1-\alpha}.$$

We check that if $\gamma \leq \varepsilon$ or $\gamma \geq \frac{1}{\varepsilon}$ for $\varepsilon$ small enough, then

$$H(\gamma x) \leq -2C_1 K|x|^{1-\alpha}$$

(using that $\frac{x^2}{t^{1+\alpha}} \leq \frac{1}{\varepsilon}$). We deduce that the maximum of $H$ is reached in $D_4 := \left[\frac{\varepsilon}{T} \varepsilon x\right]$, and with similar arguments as the second case, we deduce that for some $\nu > 0$

$$|F(x, t)| \leq \max_{D_4}|g| + K(\varepsilon, T^*) e^{-\nu t \frac{1}{1+\alpha}}$$

with $D_4 \subset \left[-\infty, -\frac{x}{t^{1+\alpha}}\right]$ in the case $\frac{x}{t^{1+\alpha}} \leq -\frac{1}{\varepsilon}$.

### 2.2 Proof of Theorem 1.1 and Proposition 1.3

**Proof** [of Theorem 1.1] We recall that the solution of the viscous Burgers equation is

$$f(x, t) = \int_{\mathbb{R}} f_0(y) e^{H(y)} dy \int_{\mathbb{R}} e^{H(y)} dy$$

where $H(y) = -\frac{(y-x)^2}{4t} - \frac{1}{2} \int_0^y f_0(z) dz$. The estimate

$$\|f(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \frac{K_2}{(1+t)^\frac{1}{1+\alpha}}$$

in Theorem 1.1 is a consequence of Proposition 2.1 for $g = f_0$ since

$$\sup_{D} |f_0| \leq \frac{K}{(1+t)^\frac{1}{1+\alpha}}.$$

To show the lower bound, let us compute $f(0, t) < 0$. We have defined in the first case (which contains the specific value $x = 0$) of the proof of Proposition 2.1 the set $D_1 = \left[\frac{\lambda_1}{2} t^{1+\alpha}, 2\lambda_2 t^{1+\alpha}\right]$ for some $\lambda_1, \lambda_2 > 0$. We estimate

$$|f(0, t)| \geq \inf_{D_1}(|f_0|) \frac{\int_{D_1} e^{H(y)} dy}{\int_{\mathbb{R}} e^{H(y)} dy} \|f_0\|_{L^\infty} \frac{\int_{\mathbb{R}} e^{H(y)} dy}{\int_{D_1} e^{H(y)} dy}$$

$$\geq \inf_{D_1}(|f_0|) - (\inf_{D_1}(|f_0|) + \|f_0\|_{L^\infty}) \frac{\int_{\mathbb{R}} e^{H(y)} dy}{\int_{D_1} e^{H(y)} dy}$$

and we have shown in the first case that

$$\frac{\int_{\mathbb{R}\setminus D_1} e^{H(y)} dy}{\int_{\mathbb{R}} e^{H(y)} dy} \leq Ke^{-\nu t \frac{1}{1+\alpha}}$$

for some constants $K, \nu > 0$. Since $\inf_{D_1}(|f_0|) \geq \frac{K}{t^{1+\alpha}}$, for $t$ large enough,

$$|f(0, t)| \geq \frac{K_1}{t^{1+\alpha}}.$$
This completes the proof of Theorem 1.1. \[\square\]

**Proof** [of Proposition 1.3] Take a function \( g \in C^2(\mathbb{R}, \mathbb{R}) \) with \( \|g\|_{C^2} \leq \kappa \) and define

\[ A_g(x,t) := \int_{\mathbb{R}} g(y)e^{H(y)}dy. \]

We compute \( \partial_x H = \frac{-1}{2t}(x-y) \) and \( \frac{1}{2t}(x-y)e^{-\frac{(x-y)^2}{4t}} = -\partial_y \left(e^{-\frac{(x-y)^2}{4t}}\right) \), therefore

\[
\partial_x A_g = -\int_{\mathbb{R}} g(y)\partial_y \left(e^{-\frac{(x-y)^2}{4t}}\right) e^{-\frac{1}{2}f_0^i f_0(z)dz} dy = \int_{\mathbb{R}} \left(g'(y) - \frac{1}{2}g(y)f_0(y)\right) e^{H(y)}dy = A_g' - \frac{1}{2}g f_0.
\]

Similarly, we have \( \partial_t H = \frac{(x-y)^2}{4t^2} \) and therefore

\[
\partial_t A_g = \int_{\mathbb{R}} \frac{(x-y)}{2t} g(y)\partial_y \left(e^{-\frac{(x-y)^2}{4t}}\right) e^{-\frac{1}{2}f_0^i f_0(z)dz} dy = \int_{\mathbb{R}} \frac{g(y)}{2t} e^{H(y)}dy - \int_{\mathbb{R}} \left(g'(y) - \frac{1}{2}g(y)f_0(y)\right) \frac{(x-y)}{2t} e^{H(y)}dy.
\]

We continue,

\[
-\int_{\mathbb{R}} \left(g'(y) - \frac{1}{2}g(y)f_0(y)\right) \frac{(x-y)}{2t} e^{H(y)}dy = \int_{\mathbb{R}} \left(g''(y) - \frac{1}{2}g''(y)f_0(y) - \frac{1}{2}g'(y)f_0(y) + \frac{1}{4}g(y)f_0^2(y)\right) e^{H(y)}dy,
\]

therefore

\[
\partial_t A_g = \int_{\mathbb{R}} \left(g(y) + g''(y) - g'(y)f_0(y) - \frac{1}{2}g(y)f_0'(y) + \frac{1}{4}g(y)f_0^2(y)\right) e^{H(y)}dy = A_{\frac{f_0}{g} + g'' - g'f_0 - \frac{1}{2}g'f_0 + \frac{1}{4}g f_0^2}.
\]

Remark that

\[
f(x,t) = \frac{A_{f_0}}{A_1},
\]

and therefore \( \partial_t^n \partial_x^k f \) can be written as a sum of products of terms of the form \( \frac{A_k}{A_1} \) for functions \( g \) that are polynomials on the variables \( f_0^{(i)} \) for \( i \leq 2n + k \). By Proposition 2.1, we have

\[ \left| \frac{A_k}{A_1} \right| \leq \sup_D |g| + Ke^{-\nu t^{\frac{1}{1+\alpha}}} \]

for \( D = [-\infty, \mu_1 t^{-\frac{1}{\alpha}}] \cup \left[\mu_2 t^{-\frac{1}{\alpha}}, +\infty\right] \) where \( \mu_1 < 0, \mu_2 > 0, \nu, K > 0 \) depend on \( n, k \). We recall that \( \partial_x A_g = A_{g' - \frac{1}{2}g f_0} \) and \( \partial_t A_g = A_{\frac{f_0}{g} + g'' - g'f_0 - \frac{1}{2}g'f_0 + \frac{1}{4}g f_0^2} \). With the hypothesis on \( f_0 \) in Proposition 1.3, we check that for a function \( g \) that is polynomial on the variables \( f_0^{(i)} \) for \( i \leq 2n + k - 1 \), then we will gain in \( \sup_D |g' - \frac{1}{2}g f_0| \) an additional factor \( t^{-\frac{1}{1+\alpha}} \) compared to an estimation on \( \sup_D |g| \). Similarly, if \( i \leq 2n + k - 1 \), we will gain in \( \sup_D \left[\frac{2}{g} + g'' - g'f_0 - \frac{1}{2}g f_0 + \frac{1}{4}g f_0^2\right] \) an additional factor \( t^{-\frac{1}{1+\alpha}} \) compared to \( \sup_D |g| \) (since \( 1 > \frac{2n}{1+\alpha} \) for \( 0 < \alpha < 1 \)).

This concludes the proof of Proposition 1.3. \[\square\]
3 Computation of the first order

This section is devoted to the proofs of Theorem 1.5 and Propositions 1.7, 1.8 and 1.9. All these proofs are similar in their approach. We start with the proof of Theorem 1.5 in subsection 3.2, and then we will explain what are the differences to get the propositions in subsection 3.3.

3.1 Definition of the functions \( y^*_\pm \) and \( y^*_m \)

We recall the definition for \( \alpha \in ]0,1[ \) and \( \kappa > 0 \) of the function
\[
g(y) = y + \frac{\kappa}{|y|^\alpha}.
\]

We are interested in the solutions of \( z = g(y) \) (see the figure in the introduction). First, remark that \( g(y) \to +\infty \) when \( y \to 0^\pm \), \( g(y) \to \pm \infty \) when \( y \to \pm \infty \) and \( g'(y) \to 1 \) when \( y \to \pm \infty \). We compute for \( y \neq 0 \) that
\[
g'(y) = 1 - \frac{\kappa \alpha}{y |y|^\alpha}.
\]

In particular, \( g' > 0 \) on \( ]-\infty,0[ \). We have \( g'(y) = 0 \) if and only if \( y = y_0 = (\kappa \alpha)^{1/\alpha} \). This implies that on \( [y_0, +\infty[ \), we have \( g'(y) > 0 \). We compute easily that
\[
g(y_0) = \kappa^{1/\alpha} \left( \alpha^{1/\alpha} + \alpha - \frac{2}{\alpha} \right) > 0.
\]

By the implicit function theorem, we construct two particular branches of functions. First, a smooth function \( y^*_+ : ]-\infty,0[ \to R \), solution of \( z = g(y^*_+(z)) \) for any \( z \in \mathbb{R} \), defined as the inverse of the invertible function \( g : ]-\infty,0[ \to \mathbb{R} \), and another smooth function
\[
y^*_+ : ]g(y_0), +\infty[ \to ]y_0, +\infty[ \]

solution of \( z = g(y^*_+(z)) \), defined as the inverse of
\[
g : ]y_0, +\infty[ \to ]g(y_0), +\infty[ \]

There is also a third branch defined as the inverse of
\[
g : ]0^+, y_0[ \to ]g(y_0), +\infty[ \]

that we denote \( y^*_m \) (it is the one in black in the figure on the introduction).

3.2 Proof of Theorem 1.5

3.2.1 Rescaling in the Hopf-Cole formula

We recall that the solution of the viscous Burgers equation can be written as
\[
f(x, t) = \frac{\int_R f_0(y) e^{H(y)} dy}{\int_R e^{H(y)} dy}
\]

with
\[
H(y) = -\frac{(x - y)^2}{4t} - \frac{1}{2} \int_0^y f_0(u) du.
\]

We introduce for \( t > 0 \) the change of variable \( x = z t^{1/\alpha} \), and we have
\[
f \left( z t^{1/\alpha}, t \right) = \frac{\int_R f_0 \left( y t^{1/\alpha} \right) e^{H(y)} \int_R e^{H(y)} dy}{\int_R e^{H(y)} dy} H_t(y, z) dy.
\]
where
\( \tilde{H}_t(y, z) := -\frac{(z - y)^2}{4} - \frac{1}{2} t^{-\frac{1}{1+\alpha}} \int_0^{yt^{\frac{1}{1+\alpha}}} f_0(u)du. \)

We compute
\[ \partial_y \tilde{H}_t(y, z) = \frac{1}{2} \left( z - y - t^{\frac{\alpha}{1+\alpha}} f_0 \left( yt^{\frac{1}{1+\alpha}} \right) \right) \]
and
\[ \partial_y^2 \tilde{H}_t(y, z) = \frac{1}{2} \left( -1 - tf'_0 \left( yt^{\frac{1}{1+\alpha}} \right) \right). \]

Remark that \( \partial_y^2 \tilde{H}_t \) no longer depends on \( z \).

### 3.2.2 Construction of the approximate branches \( y_{\pm}(\cdot, t) \) and \( y_m(\cdot, t) \)

We introduce the function
\[ g_t(y) := y + t^{\frac{1}{1+\alpha}} f_0 \left( yt^{\frac{1}{1+\alpha}} \right). \]
The zeros of \( \partial_y \tilde{H}_t \) are the solutions of \( z = g_t(y) \). We compute that
\[ g'_t(y) = 1 + tf'_0 \left( yt^{\frac{1}{1+\alpha}} \right). \]

First, we remark that for any \( \varepsilon > 0 \), we have
\[ \|g_t - g\|_{C^1([\varepsilon, 1-\varepsilon])} \to 0 \]
when \( t \to +\infty \), where \( g = y + \frac{\kappa}{\mu\alpha} t \). Therefore, we expect some zeros of \( \partial_y \tilde{H}_t \) for large time, i.e. solutions of \( z = g_t(y) \), to be close to \( y_{\pm}^*(z) \) or \( y_m^*(z) \) defined in subsection 3.1. However, we will not be able to construct them on the same domains of definition.

Take some small \( \nu > 0 \) (depending on \( \alpha \) and \( \kappa \) but independent of \( t > 0 \)). Then, for \( t \) large enough, we have that \( tf'_0 \left( yt^{\frac{1}{1+\alpha}} \right) \geq 0 \) if \( y \leq -\nu \). In particular, on \( -\infty, -\nu \), we have \( g'_t \geq 1 \). We compute that
\[ g_t(-\nu) = -\nu + \frac{\kappa}{\mu\alpha} (1 + o_{t \to \infty}(1)) \to +\infty \]
when \( \nu \to 0 \), and, for \( \nu \) small enough and \( t \) large enough, we have \( g_t(-\nu) > 0 \). By the implicit function theorem, we deduce that for \( t \) large enough, there exists \( y_{-}(z, t) \) a function solution of \( z = g_t(y_{-}(z, t)) \), which is the inverse of \( g_t : -\infty, -\nu \to -\infty, -\nu \), \( g_t(-\nu) \) defined in
\[ y_{-}(\cdot, t) : -\infty, -\nu \to -\infty, -\nu \].
Since \( \|g_t - g\|_{C^1([\varepsilon, 1-\varepsilon])} \to 0 \) for any \( \varepsilon > 0 \) and \( g'(y) > 0 \) if \( y > y_0 = (\kappa\alpha)^{\frac{1}{1+\alpha}} > 0 \), for any \( \nu > 0 \) small and \( t \) large enough, we have that \( g'_t(y) > 0 \) if \( y > y_0 + \nu \). We therefore construct \( y_{+}(z, t) \) as the inverse of \( g_t : [y_0 + \nu, +\infty] \to [y_t(y_0 + \nu), +\infty] \). It is solution of \( z = g_t(y_{+}(z, t)) \) and
\[ y_{+}(\cdot, t) : [y_0 + \nu, +\infty] \to [y_0 + \nu, +\infty]. \]
By a similar arguments, if we take \( \nu > 0 \) small enough and \( t \) large enough (depending on \( \nu \)), we can construct the middle branch
\[ y_m(\cdot, t) : g_t(y_0 + \nu), \frac{1}{\nu} \to y_0 + \nu, g_t^{-1} \left( \frac{1}{\nu} \right). \]
3.2.3 Properties of the branches $y_{\pm}(., t)$ and $y_m(., t)$

We define the set

$$C_t := \{(z, y) \in \mathbb{R}^2, z = g_t(y)\},$$

that is the set of the zeros of $\partial_y \tilde{H}_t$. We want to prove here that there exists $\delta, \mu, \varepsilon > 0$ small, $T^* > 0$ large and $C_1, C_2 > 0$, all of them depending only on $\kappa$ and $\alpha$ such that, for $t \geq T^*$,

1. The set $C_t$ converges to $C = \{(z, y) \in \mathbb{R}^2, z = g(y)\}$ outside of a vicinity of $\{(z, y) \in \mathbb{R}^2, y = 0\}$ in the following sense:

$$\forall w, \varepsilon > 0, \exists T^* > 0, \forall (y, z) \in C_t, |y| \geq \varepsilon \quad \text{and} \quad t \geq T^* \to \exists X \in C, |X - (y, z)| \leq w.$$

2. For any $(z, y) \in C_t, \{y \leq t - \frac{1}{|x| + \varepsilon}\}$, we have

$$|\partial_y \tilde{H}_t(y, z)| \leq 1 + |z| + \|f_0\|_{L^\infty} t^{1 + \mu}.$$

4. Outside of $\{y \leq t - \frac{1}{|x| + \varepsilon}\}$ we have that $\partial_y \tilde{H}_t(y, z) \to \frac{1}{2}(z - g(y))$ locally uniformly.

5. The set $C_t := \{(z, y) \in \mathbb{R}^2, \partial_y \tilde{H}_t(y, z) = 0\}$ cut the plane in two parts. In the one containing $(0, 1)$, we have $\partial_y \tilde{H}_t < 0$ and in the other one, we have $\partial_y \tilde{H}_t > 0$.

6. The set $C_t \cap \{z \leq g(y_0) + \mu\}$ contains parts of the branch $y_-$, and any other elements must be in $\{y \leq t - \frac{1}{|x| + \varepsilon}\}$

or in $B(A, \varepsilon)$ where $A = (g(y_0), y_0)$.

7. The set $C_t \cap \{z \geq \frac{1}{\mu}\}$ contains parts of the branch $y_+$, and any other elements must be in $\{y \leq 1\}$.

8. The set $C_t \cap \{g(y_0) + \mu \leq z \leq \frac{1}{\mu}\}$ contains parts of the branches $y_+, y_-, y_m$, and any other elements must be in

$$\{y \leq t - \frac{1}{|x| + \varepsilon}\}.$$

9. For $z \in [g(y_0) + \mu, \frac{1}{\mu}]$ and $y \in \left[\frac{1}{\delta}, -\delta\right] \cup [y_0 + \delta, \frac{1}{\delta}]$, then

$$-C_2 \leq \partial_y^2 \tilde{H}_t(y, z) \leq -C_1.$$

These properties can be summarized in the following way:
Plot in the plane \((z, y) \in \mathbb{R}^2\) of \(C_t\) for \(\kappa = 1, \alpha = \frac{1}{3}, t = 0.03\). The red dot is \(A = (g(y_0), y_0)\), the dotted blue line is \(\{z = g(y_0) + \mu\}\) and the dotted red line is \(\{z = \frac{1}{\mu}\}\).

Let us explain what these properties means on this graph. The set \(C_t\), that is the set of the zeros of \(\partial_y \tilde{H}_t\), in black on the graph, converges in a sense to the first figure in subsection 1.2 when \(t \to +\infty\), but not uniformly near \(\{y = 0\}\). In fact, \(C_t\) has only one branch while \(C\) is two separated branches (we can check that \(C \cap \{y = 0\} = \emptyset\)). However, if we avoid \(\{|y| \leq \varepsilon\}\) for any \(\varepsilon > 0\), then the convergence will be uniform (see property 1.).

As such, the three branches \(y_+^*, y_-^*, y_m^*\), that are parts of \(C_t\) and that can be defined outside of \(\{|y| \leq \varepsilon\}\) for any \(\varepsilon > 0\) and outside of a neighborhood of \(A\), the red point, converges to their respective limits \(y_+^*, y_-^*, y_m^*\) outside of this domain (see property 2.). However, the set \(C_t\) can be very complicated near \(\{y = 0\}\) and near \(A\). Its description there does not only depend on the equivalents at \(\pm \infty\) of \(f_0\), but on the full function itself. Luckily, we will not need a good description of \(C_t\) there.

More generally than its set of zero, the function \(\partial_y \tilde{H}_t\) converges for large time to its limit \(\frac{1}{2}(z - g(y))\) uniformly, except in a shrinking neighborhood of \(\{y = 0\}\). In it, we can still bound its values for large time (see properties 3., 4.). We can also compute its sign on both sides of the plane cut by \(C_t\) (see property 5.).
Now, in \( \{ z \leq g(y_0) + \mu \} \), that is left of the dotted blue line, we see that the set \( C_t \) contains the branch \( y_- \) there, but also possibly elements near the red point \( A \) or near \( \{ y = 0 \} \) (see property 6.). Indeed, although \( C_t \) converges to \( C \) near \( A \), it is not obvious that there is only one smooth cusp there.

In the set \( \{ z > \frac{1}{\mu} \} \), that is right of the dotted red line, the curve \( C_t \) has not yet fully converges to \( C \) near \( \{ y = 0 \} \). However, it contains the branch \( y_+ \), which is far away from \( \{ y = 0 \} \) (at distance of size \( \frac{1}{\mu} \) for small \( \mu \)), and any other element must be in a neighborhood of size 1 around \( \{ y = 0 \} \) (see property 7.).

Finally, in the set \( \{ g(y_0) + \mu \leq z \leq \frac{1}{\mu} \} \), that is between the two dotted lines, the set \( C_t \) contains the three branches \( y_+, y_-, y_m \) that are all three uniformly far away from \( \{ y = 0 \} \), and any other elements must be in a shrinking neighborhood of \( \{ y = 0 \} \) (see property 8.). We conclude with property 9., stating that in a vicinity of the branches \( y_+, y_- \) and \( y_m \), between the two dotted lines, the quantity \( \partial_y^2 \tilde{H}_t \) is uniformly strictly negative.

Let us now prove all these properties.

Properties 1. and 2. are a direct consequence of \( \| g_t - g \|_{C^1(\mathbb{R} \setminus [-\varepsilon, \varepsilon])} \to 0 \) for any \( \varepsilon > 0 \) and the fact that \( |g'_t| + |g'| \leq K(\varepsilon) \) on \( \mathbb{R} \setminus [-\varepsilon, \varepsilon] \). Property 3. follows from the computation

\[
|\partial_y \tilde{H}_t(y, z)| = \frac{1}{2} \left( z - y - t \frac{\alpha}{1 + t \frac{\alpha}{|y|}} f_0 \left( y t \frac{1}{1 + t \frac{\alpha}{|y|}} \right) \right) \leq \left( |y| + |z| + t \frac{\alpha}{1 + t \frac{\alpha}{|y|}} \right) \| f_0 \|_{L^\infty(\mathbb{R})}.
\]

Property 4. is a consequence of the fact that if \( |y| \geq t^{-1/n} + \varepsilon \), then \( |y t \frac{1}{1 + t \frac{\alpha}{|y|}}| \) is large when \( t \to +\infty \), and thus \( f_0 \left( y t \frac{1}{1 + t \frac{\alpha}{|y|}} \right) \) behaves like its equivalent when the time becomes large.

Property 5. follows from the continuity of \( g_t \) and the computations

\[
\partial_y \tilde{H}_t(0, 1) = \frac{-1 - t \frac{\alpha}{1 + t \frac{\alpha}{|y|}} f_0 \left( t \frac{1}{1 + t \frac{\alpha}{|y|}} \right)}{2} < 0
\]

for \( t \) large enough or \( \partial_y \tilde{H}_t(z, 0) \to -\infty \) when \( z \to -\infty \).

Properties 6., 7., and 8. are then a consequence of properties 1., 2., and 4. For the last property, we recall that

\[
\partial_y^2 \tilde{H}_t(y, z) = \frac{1}{2} \left( -1 - t f'_0 \left( y t \frac{1}{1 + t \frac{\alpha}{|y|}} \right) \right).
\]

In \( \{|y| \geq \delta\} \), we have \( \frac{1}{2} \left( -1 - t f'_0 \left( y t \frac{1}{1 + t \frac{\alpha}{|y|}} \right) \right) \to \frac{1}{2} \left( -1 + \frac{\alpha \delta}{y |y|} \right) \) uniformly when \( t \to +\infty \). We check easily that \(-1 + \frac{\alpha \delta}{y |y|} \) is only happens if \( y = y_0 \), and that \(-1 + \frac{\alpha \delta}{y |y|} < K(\delta) < 0 \) if \( y \in [y_0 + \delta, \frac{1}{\delta}] \).

This completes the proof of the nine properties.

### 3.2.4 Position of the maximum of \( \tilde{H}_t \)

Take \( z \leq g(y_0) + \mu \). Then, the maximum of \( y \to \tilde{H}_t(y, z) \) must be reached at a zero of \( \partial_y \tilde{H}_t(\cdot, z) \). By 6., this is either at \( y = y_-(z, t) \), or possibly in \( \{|y| \leq t^{-1/n} + \varepsilon \} \) or in \( B(A, \varepsilon) \). Now, by 3. and 4., remark that \( y \to \partial_y \tilde{H}_t(\cdot, z) \) is

- Strictly increasing on \( ]-\infty, y_-(z, t)[ \)
- Strictly decreasing on \( ]y_-(z, t), -t^{-1/n} + \varepsilon[ \)
- Bounded by \( 1 + |z| + \| f_0 \|_{L^\infty} t \frac{1}{1 + t \frac{\alpha}{|y|}} \) on \( ]-t^{-1/n} + \varepsilon, t^{-1/n} + \varepsilon[ \)
Still by 3., there exists $\nu > 0$ depending only on $\mu, \kappa, \alpha$ such that

$$\tilde{H}_t(y_-(z, t), z) - \tilde{H}_t\left(-t^{-\frac{1}{1+\kappa}} + \epsilon, z\right) \geq \nu$$

and we check easily that

$$\left|\tilde{H}_t\left(-t^{-\frac{1}{1+\kappa}} + \epsilon, z\right) - \tilde{H}_t\left(t^{-\frac{1}{1+\kappa}} + \epsilon, z\right)\right| \leq 2t^{-\epsilon}\|f_0\|_{L^\infty} + 2(1 + |z|)t^{-\frac{1}{1+\kappa}} \to 0$$

when $t \to +\infty$.

We deduce that for $z \leq g(y_0) + \mu$, the maximum of $y \to \tilde{H}_t(y, z)$ is reached at $y_-(z, t)$, and that there exists a constant $\nu_0 > 0$ depending only on $\alpha, \kappa$ such that

$$\tilde{H}_t(y_-(z, t), z) \geq \tilde{H}_t(y, z) + \nu_0$$

for any $y \in \{y \leq t^{-\frac{1}{1+\kappa}} + \epsilon\} \cup B(A, \epsilon)$.

Indeed, the maximum cannot be near $\{y = 0\}$ because there $\tilde{H}_t$ cannot increase by $\nu > 0$, and it cannot be near $A$ if we take $\epsilon > 0$ small enough.

A similar argument can be made for $z \geq \frac{1}{\mu}$ if we take $\mu > 0$ large enough. In that case, the maximum is reached either at $y_+(z, t)$ or in $\{|y| \leq 1\}$, but using the above properties we can show that in $\{|y| \leq 1\}$,

$$\tilde{H}_t(y, z) \leq -\frac{(z + 1)^2}{4} - K$$

for some universal constant $K > 0$. Now, we estimate

$$\tilde{H}_t(y_+(z, t), z) \geq -\frac{(z - y_+(z, t))^2}{4} - K|y_+(z, t)|^{1-\alpha},$$

and since $|y_+(z, t) - z| \to 0$ when $z \to +\infty$ (uniformly in $t$), for $z \geq \frac{1}{\mu}$ and $\mu$ small enough we have

$$\tilde{H}_t(y_+(z, t), z) \geq \tilde{H}_t(y, z) + \nu_0$$

for a universal constant $\nu_0 > 0$, for any $y \in \mathbb{R}$ such that $|y| \leq 1$.

We now focus on the case $g(y_0) + \mu \leq z \leq \frac{1}{\mu}$. By similar arguments on the sign of $\partial_y \tilde{H}_t(., z)$ in different regions, we can show that the maximum is reached at either $y_+(z, t)$ or $y_-(z, t)$, with a margin $\nu_0 > 0$ compared to $y_m(z, t)$ (it is a local minimum of $\tilde{H}_t$) and $\{|y| \leq t^{-\frac{1}{1+\kappa}} + \epsilon\}$.

We introduce the quantities

$$h_+(z) := \tilde{H}_t(y_+(z, t), z), h_-(z) := \tilde{H}_t(y_-(z, t), z).$$

Since $\partial_y \tilde{H}(y_\pm(z, t), z) = 0$ by construction, we have

$$h_\pm'(z) = \partial_z \tilde{H}(y_\pm(z, t), z) = -\frac{1}{2}(z - y_\pm(z, t)) = -t^{\frac{\kappa}{1+\kappa}} f_0(y_\pm(z, t) t^{\frac{1}{1+\kappa}}) < 0.$$

Also,

$$h'_-(z) - h'_+(z) = \frac{1}{2}(-y_+(z, t) + y_-(z, t)) > 0$$

since $y_-(z, t) < 0$ and $y_+(z, t) > 0$. We deduce that $h'_+(z) < h'_-(z)$ on $\left[g(y_0) + \mu, \frac{1}{m}\right]$. 

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Since $h_+(1/\mu) > h_-(1/\mu)$ and $h_+(g(y_0) + \mu) < h_-(g(y_0) + \mu)$, by continuity there exists $z_c(t) \in \left[ g(y_0) + \mu, \frac{1}{\mu} \right]$ such that $h_+(z_c(t)) = h_-(z_c(t))$. That is, $\max \tilde{H}_t(., z_c(t))$ is reached at two points, $y_+(z_c(t), t) > 0$ and $y_-(z_c(t), t) < 0$. But since $h'_+(z) < h'_-(z)$ on $\left[ g(y_0) + \mu, \frac{1}{\mu} \right]$, there is only one such point $z_c(t)$.

By the uniform convergence $\|g_t - g\|_{C^1(R \setminus [-\varepsilon, \varepsilon])} \to 0$ when $t \to +\infty$ (the branches $y_+(z, t)$ and $y_-(z, t)$ are uniformly far away from 0 for $z \in \left[ g(y_0) + \mu, \frac{1}{\mu} \right]$), the quantity $z_c(t)$ converges to a limit $z_c$ when $t \to +\infty$, define as the unique solution of $\mathcal{H}(y^*_c(z_c)) = \mathcal{H}(y_+(z_c))$ where

$$\mathcal{H}(y) = -\frac{\kappa^2}{4|y|^{2\alpha}} - \frac{\kappa(1-\alpha)}{2} |y|^{1-\alpha}.$$ 

Indeed, the equation satisfied by $z_c(t)$ is $\tilde{H}_t(y_+(z_c(t), t), z_c(t)) = \tilde{H}_t(y_-(z_c(t), t), z_c(t))$, and we get the limit equation by taking $t \to +\infty$ using

$$y_\pm(z_c(t), t) - g_t(y_\pm(z_c(t), t)) = -t^{\frac{\alpha}{1+\alpha}} f_0 \left( y_\pm(z_c(t), t) t^{\frac{1}{1+\alpha}} \right)$$

and the fact that $y_\pm(z_c(t), t)$ are far from 0 uniformly in time.

From now on, we consider any small $\varepsilon > 0$ and $z \in \mathbb{R} \setminus [z_c - \varepsilon, z_c + \varepsilon]$. Then, for $t \geq T^*$ large enough (depending on $\varepsilon$), there exists $\nu(\varepsilon) > 0$ a small constant such that

+ if $z \leq z_c - \varepsilon$, then $\max \tilde{H}_t(., z)$ is reached only at $y_- (z, t)$, and $\max \tilde{H}_t(., t) = \tilde{H}_t(y_-(z, t), t) \geq \tilde{H}_t(y_+(z, t), t) + \nu(\varepsilon)$
+ if $z \geq z_c + \varepsilon$, then $\max \tilde{H}_t(., z)$ is reached only at $y_+(z, t)$, and $\max \tilde{H}_t(., t) = \tilde{H}_t(y_+(z, t), t) \geq \tilde{H}_t(y_-(z, t), t) + \nu(\varepsilon)$.

To simplify the notations, we define for $z \in \mathbb{R} \setminus \{z_c\}$

$$y_{\max}(z, t) := \begin{cases} y_+(z, t) & \text{if } z > z_c \\ y_-(z, t) & \text{if } z < z_c \end{cases}$$

as well as $y^*_{\max}(z)$, its limit when $t \to +\infty$, and

$$\tilde{H}_{\max}(z, t) := \tilde{H}_t(y_{\max}(z, t), t).$$

Remark that $|y_{\max}(z, t)| \geq C_0 > 0$ where $C_0$ is a constant independent of $z$ and $t$.

Now, we infer that there exists small constants $\gamma, \nu > 0$ depending on $\alpha, \kappa, \varepsilon$ such that for $t \geq T^*$ large enough, if $y \in \mathbb{R} \setminus [y_{\max}(z, t) - \gamma, y_{\max}(z, t) + \gamma]$ and $z \in \mathbb{R} \setminus [z_c - \varepsilon, z_c + \varepsilon]$, then

$$\tilde{H}_{\max}(z, t) \geq \tilde{H}_t(y, t) + \nu.$$

This is a consequence of 3.4. and the comparison between $\tilde{H}_t(y_-(z, t), t)$ and $\tilde{H}_t(y_+(z, t), t)$ when $g(y_0) + \mu \leq z \leq \frac{1}{\mu}$.

### 3.2.5 End of the proof of Theorem 1.5

We are now equipped to estimate

$$f \left( z t^{\frac{1}{1+\alpha}}, t \right) = \frac{\int_{\mathbb{R}} f_0 \left( y t^{\frac{1}{1+\alpha}} \right) e^{t^{\frac{1-\alpha}{1+\alpha}} \tilde{H}_t(y, z) dy} \right) \int_{\mathbb{R}} e^{t^{\frac{1-\alpha}{1+\alpha}} \tilde{H}_t(y, z)} dy.$$

We define

$$A_t(z, t) := \int_{\mathbb{R}} f \left( y t^{\frac{1}{1+\alpha}} \right) e^{t^{\frac{1-\alpha}{1+\alpha}} \tilde{H}_t(y, z)} dy.$$
for \( j \in \{1, f_0\} \), as well as

\[
\mathcal{D}_{z, t, \gamma} := [y_{\max}(z, t) - \gamma, y_{\max}(z, t) + \gamma].
\]

We decompose

\[
A_j(z, t) = \int_{\mathcal{D}_{z, t, \gamma}} f(yt^{\frac{1}{1+\alpha}}) e^{t \frac{1}{1+\alpha} H_t(y, z)} dy + \int_{\mathbb{R} \setminus \mathcal{D}_{z, t, \gamma}} f(yt^{\frac{1}{1+\alpha}}) e^{t \frac{1}{1+\alpha} H_t(y, z)} dy.
\]

On \( \mathbb{R} \setminus \mathcal{D}_{z, t, \gamma} \) we have shown that \( \tilde{H}_t(z, t) \leq H_{\max}(z, t) - \nu \), and using arguments similar as for the proof of (2.6) to deal with the integrability, we check that

\[
\left| \int_{\mathbb{R} \setminus \mathcal{D}_{z, t, \gamma}} f(yt^{\frac{1}{1+\alpha}}) e^{t \frac{1}{1+\alpha} H_t(y, z)} dy \right| 
\leq K \| f \|_{L^\infty(\mathbb{R})} e^{t \frac{1}{1+\alpha} \tilde{H}_{\max}(z, t) - \frac{\nu}{1+\alpha}} 
\leq Ke^{t \frac{1}{1+\alpha} \tilde{H}_{\max}(z, t) - \frac{\nu}{1+\alpha}}.
\]

We continue with

\[
\int_{\mathcal{D}_{z, t, \gamma}} f_0(yt^{\frac{1}{1+\alpha}}) e^{t \frac{1}{1+\alpha} H_t(y, z)} dy 
= f_0(\max(y, z) t^{\frac{1}{1+\alpha}}) \int_{\mathcal{D}_{z, t, \gamma}} e^{t \frac{1}{1+\alpha} H_t(y, z)} dy 
+ \int_{\mathcal{D}_{z, t, \gamma}} f_0(yt^{\frac{1}{1+\alpha}}) - f_0(\max(y, z) t^{\frac{1}{1+\alpha}}) \right) e^{t \frac{1}{1+\alpha} H_t(y, z)} dy.
\]

From 9., we have for \( y \in \mathcal{D}_{z, t, \gamma} \) (and \( \gamma \) small enough) that

\[
\tilde{H}_t(y, z) \geq H_{\max}(z, t) - C_2(y - y_{\max}(z, t))^2,
\]

hence

\[
\int_{\mathcal{D}_{z, t, \gamma}} e^{t \frac{1}{1+\alpha} H_t(y, z)} dy 
\geq e^{\tilde{H}_{\max}(z, t) t^{\frac{1}{1+\alpha}}} \int_{\mathcal{D}_{z, t, \gamma}} e^{-C_2 t \frac{1}{1+\alpha} (y - y_{\max}(z, t))^2} dy 
\geq e^{\tilde{H}_{\max}(z, t) t^{\frac{1}{1+\alpha}}} \int_{-\gamma}^{\gamma} e^{-C_2 t \frac{1}{1+\alpha} y^2} dy 
\geq K t^{-\frac{1}{2(1+\alpha)}} e^{\tilde{H}_{\max}(z, t) t^{\frac{1}{1+\alpha}}}.
\]

We continue. Still from 9., we have for \( y \in \mathcal{D}_{z, t, \gamma} \) (and \( \gamma \) small enough) that

\[
\tilde{H}_t(y, z) \leq H_{\max}(z, t) - C_1(y - y_{\max}(z, t))^2,
\]

hence

\[
\left| \int_{\mathcal{D}_{z, t, \gamma}} \left( f_0(yt^{\frac{1}{1+\alpha}}) - f_0(\max(y, z) t^{\frac{1}{1+\alpha}}) \right) e^{t \frac{1}{1+\alpha} H_t(y, z)} dy \right| 
\leq e^{t \frac{1}{1+\alpha} \tilde{H}_{\max}(z, t)} \int_{\mathcal{D}_{z, t, \gamma}} \left| f_0(yt^{\frac{1}{1+\alpha}}) - f_0(\max(y, z) t^{\frac{1}{1+\alpha}}) \right| e^{-C_2 t \frac{1}{1+\alpha} (y - y_{\max}(z, t))^2} dy 
\leq e^{t \frac{1}{1+\alpha} \tilde{H}_{\max}(z, t) \sup_{\mathcal{D}_{z, t, \gamma}} f_0(yt^{\frac{1}{1+\alpha}}) \left| \int_{\mathcal{D}_{z, t, \gamma}} t^{\frac{1}{1+\alpha}} |y - y_{\max}| e^{-C_2 t \frac{1}{1+\alpha} (y - y_{\max}(z, t))^2} dy 
\leq Ke^{t \frac{1}{1+\alpha} \tilde{H}_{\max}(z, t) t^{-1+\frac{1}{1+\alpha}}} \int_{-\gamma}^{\gamma} |y| e^{-C_2 t \frac{1}{1+\alpha} y^2} dy 
\leq Ke^{t \frac{1}{1+\alpha} \tilde{H}_{\max}(z, t) t^{-1+\frac{1}{1+\alpha}}}.
\]

(3.3)
We define
\[
\mathcal{I}(z, t) := \int_{D_{t, \gamma}} e^{\frac{1}{t+\alpha}} R_t(y, z) \, dy
\]
and \(\mathcal{J}(z, t), \mathcal{K}(z, t)\) by
\[
\int_{\mathbb{R}} f_0 \left( y t^{\frac{1}{\alpha+1}} \right) e^{\frac{1}{t+\alpha}} R_t(y, z) \, dy = f_0 \left( y_{\text{max}}(z, t) t^{\frac{1}{\alpha+1}} \right) \mathcal{I}(z, t) + \mathcal{J}(z, t)
\]
and
\[
\mathcal{K}(z, t) := \int_{\mathbb{R}} e^{\frac{1}{t+\alpha}} R_t(y, z) \, dy - \mathcal{I}(z, t).
\]
We have
\[
f(z t^{\frac{1}{\alpha+1}}, t) = f_0 \left( y_{\text{max}}(z, t) t^{\frac{1}{\alpha+1}} \right) + \frac{\mathcal{J}(z, t) - f_0 \left( y_{\text{max}}(z, t) t^{\frac{1}{\alpha+1}} \right)}{\mathcal{I}(z, t) + \mathcal{K}(z, t)} \mathcal{K}(z, t).
\]
From (3.2) we have
\[
\mathcal{I}(z, t) \geq K t^{-\frac{1}{\alpha+1}} e^{\mathcal{K}(z, t) t^{\frac{1}{\alpha+1}}}
\]
and from (3.1) we have
\[
|\mathcal{K}(z, t)| \leq K e^{-\frac{1}{2} t^{\frac{1}{\alpha+1}}} e^{\mathcal{K}(z, t) t^{\frac{1}{\alpha+1}}}.
\]
Therefore, for \(t\) large enough, \(\mathcal{I}(z, t) + \mathcal{K}(z, t) \geq \mathcal{I}(z, t)/2\). From (3.1) and (3.3), for \(t\) large enough we have
\[
|\mathcal{J}(z, t)| \leq K e^{\frac{1}{t+\alpha}} \mathcal{K}(z, t) t^{-\frac{1}{\alpha+1}}
\]
hence
\[
\left| \frac{\mathcal{J}(z, t)}{\mathcal{I}(z, t) + \mathcal{K}(z, t)} \right| \leq K t^{-\frac{1}{2} - \frac{1}{\alpha+1}} \leq K t^{-\frac{1}{2}}.
\]
We check similarly that
\[
\left| f_0 \left( y_{\text{max}}(z, t) t^{\frac{1}{\alpha+1}} \right) \mathcal{K}(z, t) \right| \leq K e^{-\frac{1}{2} t^{\frac{1}{\alpha+1}}}.
\]
Combining these estimates, we have
\[
\left| t^{\frac{1}{\alpha+1}} f \left( z t^{\frac{1}{\alpha+1}}, t \right) - t^{\frac{1}{\alpha+1}} f_0 \left( y_{\text{max}}(z, t) t^{\frac{1}{\alpha+1}} \right) \right| \leq K t^{-\frac{1}{2} + \frac{1}{\alpha+1}}.
\]
Since \(|y_{\text{max}}(z, t)| \geq C_0 > 0\) where \(C_0\) depends only on \(\alpha\) and \(\kappa\), we have
\[
\lim_{t \to +\infty} t^{\frac{1}{\alpha+1}} f_0 \left( y_{\text{max}}(z, t) t^{\frac{1}{\alpha+1}} \right) = \kappa |y^*_{\text{max}}(z)|^{-\alpha}.
\]
This concludes the proof of Theorem 1.5.

About the case \(z = z_c\). The difficulty is now that
\[
\int_{y_+(z, t) + \gamma}^{y_+(z, t) - \gamma} f \left( y t^{\frac{1}{\alpha+1}} \right) e^{\frac{1}{t+\alpha}} R_t(y, z) \, dy \text{ and } \int_{y_-(z, t) + \gamma}^{y_-(z, t) - \gamma} f \left( y t^{\frac{1}{\alpha+1}} \right) e^{\frac{1}{t+\alpha}} R_t(y, z) \, dy
\]
are comparable in size (the rest of the integral is small compared to these two terms). We can show with similar computations as previously, with
\[
\mathcal{A}_+(t) := \int_{y_+(z, t) - \gamma}^{y_+(z, t) + \gamma} e^{\frac{1}{t+\alpha}} R_t(y, z) \, dy, \mathcal{A}_-(t) := \int_{y_-(z, t) - \gamma}^{y_-(z, t) + \gamma} e^{\frac{1}{t+\alpha}} R_t(y, z) \, dy,
\]

that at leading order we have
\[
\frac{t^\frac{1}{\alpha}}{f_0(y)} = \kappa \left| y_+(z_c, t) \right|^{-\alpha} A_+(t) + \left| y_-(z_c, t) \right|^{-\alpha} A_-(t).
\]

Although \( \tilde{H}_t(y_+(z_c, t), z_c) \) and \( \tilde{H}_t(y_-(z_c, t), z_c) \) converges to the same constant, it is not clear if
\[
e^{-\frac{(\tilde{H}_t(y_+(z_c, t), z_c) - \tilde{H}_t(y_-(z_c, t), z_c))}{t^\frac{1}{\alpha}}}
\]
converges or not. We can only show that for any subsequence of time such that \( t_n^\frac{1}{\alpha} f \left( z t_n^\frac{1}{\alpha}, t_n \right) \) converges, the limit must be between \( \kappa |y_+(z_c)|^{-\alpha} \) and \( \kappa |y_-(z_c)|^{-\alpha} \).

3.3 The other asymptotic profiles

This subsection is devoted to the proofs of Proposition 1.7 to 1.9. They follow a similar path than the proof of Theorem 1.5. As such, we will only give the main arguments of their proof.

3.3.1 Proof of Proposition 1.7

The key difference here, compared to the proof of Theorem 1.5, is that the functions \( g \) is different. We define, in this subsection,
\[
g(y) = \begin{cases} 
  y - \frac{\zeta}{|y|} & \text{if } y > 0 \\
  y + \frac{\zeta}{|y|} & \text{if } y < 0.
\end{cases}
\]

Then, with \( g_t(y) = y + t^{\frac{1}{\alpha}} f_0 \left( y t^{\frac{1}{\alpha}} \right) \), we still have
\[\|g_t - g\|_{L^\infty(\mathbb{R}|< -\varepsilon, \varepsilon|)} \to 0\]
when \( t \to +\infty \). Here, we have \( g'(y) > 0 \) everywhere on \( \mathbb{R^*} \), and we thus define
\[y^*_+ : \mathbb{R} \to ]-\infty, 0[ \] and \( y^*_- : \mathbb{R} \to ]0, +\infty[ \)
as the respective inverses of \( g : ]-\infty, 0[ \to \mathbb{R} \) and \( g : ]0, +\infty[ \to \mathbb{R} \).

Then, we take \( \mu > 0 \) small and we decompose in three cases : \( z \leq -\frac{1}{\mu} \), \( z \geq \frac{1}{\mu} \) and \( z \in \left[ -\frac{1}{\mu}, \frac{1}{\mu} \right] \). To find where the maximum of \( \tilde{H}_t(., z) \) is reached, the first two cases can be treated like the case \( z \geq \frac{1}{\mu} \) in the proof of Theorem 1.5, where the maximum will be reached respectively at \( y_+(z, t) \) and \( y_-(z, t) \), and in the middle this can be treated as the case \( g(y_0) + \mu \leq z \leq \frac{1}{\mu} \) in the proof of Theorem 1.5. We conclude with computations similars as the ones of subsection 3.2.5.

Remark that we did not infer any result in the case
\[f_0(y) = \frac{\mp \kappa (1 + o_{y \to \pm\infty} (1))}{|y|^{\alpha}}\]
and this is because our approach does not work here. The difficulty there is that, with the function
\[
g(y) = \begin{cases} 
  y + \frac{\zeta}{|y|} & \text{if } y > 0 \\
  y - \frac{\zeta}{|y|} & \text{if } y < 0,
\end{cases}
\]
there exists values of \( z \) close to 0 such that the problem \( z = g(y) \) does not admit any solution. This means that the solutions of \( z = g_t(y) \) are only for values of \( y \) close to 0 when \( t \to +\infty \) in that case.

As such, we can show the convergence of \( t^{\frac{1}{\alpha}} f \left( z t^{\frac{1}{\alpha}}, t \right) \) for large \(|z|\), but not in the middle, where it is unclear if the profile converge, explode or oscillate.
3.3.2 Proof of Proposition 1.8

We consider here the problem
\[
\begin{cases}
\partial_t f - \partial^2_x f + f \partial_x f = 0 \\
f_0(x) = \frac{\kappa}{|y|^{\alpha} \ln^{\beta}(|y|)} (1 + O_y \rightarrow \pm \infty (1)).
\end{cases}
\]

We write
\[
\lambda(t)f(\mu(t)z,t) = \frac{\int_{\mathbb{R}} \lambda(t)f_0(\mu(t)z)e^{\frac{z^2(t)}{4\mu_2(t)}} \tilde{H}_t(y,z)dy}{\int_{\mathbb{R}} \lambda e^{\frac{z^2(t)}{4\mu_2(t)}} \tilde{H}_t(y,z)dy}
\]
where \( \mu(t), \lambda(t) > 0 \) are functions going to \(+\infty\) when \( t \rightarrow +\infty \), and
\[
\tilde{H}_t(y,z) = -\frac{(z-y)^2}{4} - \frac{t}{2\mu_2(t)} \int_0^{y\mu(t)} f_0(u)du.
\]

We have
\[
\partial_y \tilde{H}_t(y,z) = \frac{1}{2} \left( z - y - \frac{t}{\mu(t)} f_0(y\mu(t)) \right).
\]

For a fixed \( y > 0 \), we have
\[
\frac{t}{\mu(t)} f_0(y\mu(t)) \sim \frac{\kappa t}{\mu(t)^{1+\alpha}|y|^{\alpha} \ln^{\beta}(y\mu(t))} \sim \frac{\kappa t}{\mu(t)^{1+\alpha} \ln^{\beta}(y\mu(t)) |y|^{\alpha}}
\]
when \( t \rightarrow +\infty \). We therefore take \( \mu(t) \) such that
\[
\mu(t)^{1+\alpha} \ln^{\beta}(\mu(t)) = t
\]
and \( \lambda(t) = \frac{t}{\mu(t)} \). Remark that then,
\[
\mu(t) \sim t^{\frac{1}{1+\alpha}} \left( \frac{1 + \alpha}{\ln(t)} \right)^{\frac{\beta}{1+\alpha}}
\]
when \( t \rightarrow +\infty \).

Now, we define as in the other cases
\[
g(y) = y + \frac{\kappa}{|y|^\alpha}
\]
and
\[
g_t(y) = y + \frac{t}{\mu(t)} f_0(y\mu(t)).
\]

We check that for any \( \varepsilon > 0 \),
\[
\|g_t - g\|_{C^1([-\frac{1}{2}, \frac{1}{2}] \setminus [-\varepsilon, \varepsilon])} \rightarrow 0.
\]
when \( t \rightarrow +\infty \). Remark that here the convergence is uniform on \([-\frac{1}{2}, \frac{1}{2}] \setminus [-\varepsilon, \varepsilon]\) and not on \( \mathbb{R} \setminus [-\varepsilon, \varepsilon] \) as in the other cases, because for \( |y| \geq \varepsilon \),
\[
f_0(y\mu(t)) \sim \frac{\kappa}{|y|^\alpha} \mu(t)^{-\alpha}(\ln(|y|) + \ln(\mu(t)))^{-\beta} \sim \frac{\kappa}{|y|^\alpha} \mu(t)^{-\alpha} \ln^{-\beta}(\mu(t))
\]
when \( t \rightarrow +\infty \), but not uniformly in \( |y| \) large.

We define \( y^*_\pm \) to be the same branch as in subsection 3.1. The steps of the proof are the same, the only difference is that we are working on \([-\frac{1}{2}, \frac{1}{2}]\) from the start instead than on \( \mathbb{R} \).
3.3.3 Proof of Proposition 1.9

We give more details for the proof of this proposition, because it diverges at some point to the proof of Theorem 1.5. This is because for some values of $z$, the maximum of $\partial_y \tilde{H}_t(y, z)$ will be reached for $y$ close to 0, where we have to be careful about the convergences.

We recall that

$$
t^{\frac{\nu}{1+\alpha}} f \left( zt^{\frac{1}{1+\alpha}}, t \right) = \frac{\int_{\mathbb{R}} t^{\frac{\nu}{1+\alpha}} f_0 \left( yt^{\frac{1}{1+\alpha}} \right) e^{t^{\frac{\nu}{1+\alpha}} \tilde{H}_t(y, z)} dy}{\int_{\mathbb{R}} e^{t^{\frac{\nu}{1+\alpha}} \tilde{H}_t(y, z)} dy},
$$

$$
\tilde{H}_t(y, z) = -\frac{(z - y)^2}{4} - \frac{1}{2} t^{\frac{\nu}{1+\alpha}} \int_{0}^{yt^{\frac{1}{1+\alpha}}} f_0(u) du,
$$

and

$$
\partial_y \tilde{H}_t(y, z) = \frac{1}{2} \left( z - y - t^{\frac{\nu}{1+\alpha}} f_0 \left( yt^{\frac{1}{1+\alpha}} \right) \right),
$$

we define

$$
g(y) = \begin{cases} 
y + \frac{\nu}{1+\alpha} & \text{if } y > 0 \\
y & \text{if } y < 0
\end{cases}
$$

and

$$
g_t(y) = y + t^{\frac{\nu}{1+\alpha}} f_0 \left( yt^{\frac{1}{1+\alpha}} \right).
$$

For any $\varepsilon > 0$, as in the previous cases, we have

$$\|g_t - g\|_{C^1([0, \varepsilon])} \to 0$$

when $t \to +\infty$. We define $y_0 = (\kappa \alpha)^{\frac{1}{1+\alpha}} > 0$, the only point where $g'(y_0) = 0$ on $\mathbb{R}^+$. On $|y_0, +\infty|$ we have $g' > 0$, hence we define $g^*_+ : [g(y_0), +\infty[ \to [0, +\infty]$ as the inverse of $g : ]y_0, +\infty[ \to ]g(y_0), +\infty[$. It is the exact same definition as in subsection 3.1.

For $y < 0$, since $g(y) = y$, its inverse is simply the identity there.

Let us start with the case $z < 0$. Remark that if $y \geq 0$, then by the hypotheses of Proposition 1.9 we have $g(y) \geq 0$. Furthermore, on $] -\infty, 0[$, we have

$$g_t(y) = 1 + tf_0' \left( yt^{\frac{1}{1+\alpha}} \right) \geq 1,$$

and since $g_t(0) = t^{\frac{\nu}{1+\alpha}} f_0(0) \to +\infty$ when $t \to +\infty$, $g_t(y) \to -\infty$ when $y \to -\infty$, the problem $z = g_t(y)$ admits exactly one solution for $t$ large enough (depending on $z$). The maximum, denoted $y_{t, z} < 0$. Given $\nu > 0$ small enough, this branch continues up to $z = \frac{1}{\nu}$ if $t$ is large enough (depending on $\nu$). For $z < 0$, it satisfies $y_{t, z} \to z$ when $t \to +\infty$.

Indeed, for $y < 0$ we have

$$g_t(y) \sim y + \kappa t^{\frac{\alpha - \beta}{1+\alpha}} |y|^{-\beta} \to y$$

when $t \to +\infty$ since $\beta > \alpha$. Now,

$$\tilde{H}_t(y_{t, z}, z) \to 0$$

when $t \to +\infty$. Therefore there exists $\nu(z) > 0$ such that, for $y \in \mathbb{R} \setminus \left[ z - \frac{|z|}{2}, z + \frac{|z|}{2} \right]$ we have

$$\tilde{H}_t(y, z) < -\nu(z).$$
Since for $y \in \left[ z - \frac{\alpha}{2}, z + \frac{\alpha}{2} \right]$ we have

$$t^{\frac{\alpha}{\alpha+\beta}} f_0 \left( yt^{\frac{1}{\alpha+\beta}} \right) \leq K(z) t^{\frac{\alpha}{\alpha+\beta}} \to 0$$

when $t \to +\infty$, we deduce, with arguments similar as in subsection 3.2.5, that

$$t^{\frac{\alpha}{\alpha+\beta}} f \left( zt^{\frac{1}{\alpha+\beta}}, t \right) \to 0$$

when $t \to +\infty$, uniformly on $]-\infty, -\varepsilon[$ for any $\varepsilon > 0$.

We now focus on the case $0 < z < g(y_0) + \nu$ for some small $\nu > 0$. There, the problem $g_t(y) = z$ admits exactly one solution in $] - \infty, 0[$ (it might have others in $[0, +\infty[$), that we still denote $y_{t,z} < 0$ (as it is the continuation of it). Here, we have that $t^{\frac{\alpha}{\alpha+\beta}} y_{t,z} \to -\infty$ when $t \to +\infty$, since otherwise, $y_{t,z}$ is bounded and

$$0 < z - y_{t,z} = t^{\frac{\alpha}{\alpha+\beta}} f_0 \left( y_{t,z} t^{\frac{1}{\alpha+\beta}} \right) \to +\infty$$

when $t \to +\infty$, which is a contradiction. This leads to

$$y_{t,z} t^{\frac{-\alpha - \beta}{(\alpha+\beta)\alpha}} \to - \left( \frac{z}{K} \right)^{\frac{1}{\gamma}}$$

when $t \to +\infty$. In particular $y_{t,z} \to 0$ when $t \to +\infty$.

Now, for $0 < z < g(y_0) + \nu$, the maximum is reached either at $y_{t,z}$, or near $A$. It cannot be reached near $A$ because of arguments similar as in subsection 3.2.4.

Remark that

$$t^{\frac{\alpha}{\alpha+\beta}} f_0 \left( y_{t,z} t^{\frac{1}{\alpha+\beta}} \right) = z - y_{t,z} \to z$$

when $t \to +\infty$, and for $y \in [y_{t,z} - t^{-\lambda}, y_{t,z} + t^{-\lambda}]$, $\lambda > 0$, we have

$$\left| t^{\frac{\alpha}{\alpha+\beta}} f_0 \left( yt^{\frac{1}{\alpha+\beta}}, t \right) - t^{\frac{\alpha}{\alpha+\beta}} f_0 \left( y_{t,z} t^{\frac{1}{\alpha+\beta}} \right) \right| \leq K t^{-\lambda} \left| f_0 \left( y_{t,z} t^{\frac{1}{\alpha+\beta}} \right) \right| \leq K t^{1-\lambda} (\frac{1+\beta}{1+\alpha})^{\frac{\alpha}{\alpha+\beta}}$$

hence if $\lambda > \frac{\beta - \alpha}{1+\alpha} > 0$ then $1 - \lambda - \frac{(1+\beta)\alpha}{(1+\alpha)\beta} < 0$ and

$$\left| t^{\frac{\alpha}{\alpha+\beta}} f_0 \left( yt^{\frac{1}{\alpha+\beta}}, t \right) - z \right| \leq K t^{1-\lambda} (\frac{1+\beta}{1+\alpha})^{\frac{\alpha}{\alpha+\beta}} + \left| t^{\frac{\alpha}{\alpha+\beta}} f_0 \left( y_{t,z} t^{\frac{1}{\alpha+\beta}} \right) + y_{t,z} \right| \to 0$$

when $t \to +\infty$. We therefore define

$$D_{z,t,\varepsilon} := \left[ y_{t,z} - t^{\frac{-\alpha - \beta}{(\alpha+\beta)\alpha}} + \varepsilon, y_{t,z} + t^{\frac{-\alpha - \beta}{(\alpha+\beta)\alpha}} - \varepsilon \right].$$

For $y \in D_{z,t,\varepsilon}$, with arguments similar as in subsection 3.2.4, there exists $C_1, C_2 > 0$ such that

$$\tilde{H}_t(y,z) \leq \tilde{H}_{t,\max} - C_1 t^{\frac{\beta - \alpha}{(\alpha+\beta)\alpha}} (y - y_{t,z})^2$$

and

$$\tilde{H}_t(y,z) \geq \tilde{H}_{t,\max} - C_2 t^{\frac{\beta - \alpha}{(\alpha+\beta)\alpha}} (y - y_{t,z})^2.$$

We compute that

$$\int_{D_{z,t,\varepsilon}} e^{t^{\frac{\alpha}{\alpha+\beta}} \tilde{H}_t(y,z)} dy \geq e^{t^{\frac{\alpha}{\alpha+\beta}} \tilde{H}_{t,\max}} \left[ \int_{D_{z,t,\varepsilon}} e^{-C_1 t^{\frac{\beta - \alpha}{(\alpha+\beta)\alpha}} (y - y_{t,z})^2} dy \right]$$

$$\geq e^{t^{\frac{\alpha}{\alpha+\beta}} \tilde{H}_{t,\max}} \left[ \int_{D_{z,t,\varepsilon}} e^{t^{\frac{-\alpha - \beta}{(\alpha+\beta)\alpha}} - \varepsilon} e^{-C_1 t^{\frac{\beta - \alpha}{(\alpha+\beta)\alpha}} y^2} dy \right]$$

$$\geq t^{\gamma(\alpha,\beta,\varepsilon)} e^{t^{\frac{\alpha}{\alpha+\beta}} \tilde{H}_{t,\max}}$$
for some function $\gamma(\alpha, \beta, \varepsilon) > 0$ and $t$ large enough.

We continue, for $y \in \mathbb{R} \setminus D_z$, we have $|y - \gamma_{t,z}| \geq t^{1/(1+\sigma)}$, therefore

$$H_t(y,z) \leq H_{t,\max} - C_1 t^{1/(1+\sigma)} f^2(y^{1/(1+\sigma)} - \varepsilon),$$

leading to (with arguments similar as for the proof of (2.6) to deal with the integrability)

$$\int_{\mathbb{R} \setminus D_{z,t,\varepsilon}} e^{t^{1/(1+\sigma)} H_t(y,z)} dy 
\leq \int_{\mathbb{R} \setminus D_{z,t,\varepsilon}} e^{t^{1/(1+\sigma)} H_t(y,z)} dy 
\leq C_t^{1/(1+\sigma)} + C_{\alpha,\beta,\varepsilon} - 2(1+\beta)$$

for some function $\gamma_1(\alpha, \beta, \varepsilon) > 0$, and

$$\frac{1 - \alpha}{1 + \alpha} + \frac{\beta - \alpha}{(1+\alpha)\beta} + 2 \left( \frac{\alpha - \beta}{(1+\alpha)\beta} - \varepsilon \right) = \frac{\alpha(1-\beta)}{(1+\alpha)\beta} - 2\varepsilon > 0$$

if $\beta < 1$ and $\varepsilon > 0$ small enough.

We deduce that

$$\left| t^{1/(1+\sigma)} f \left( zt^{1/(1+\sigma)} , t \right) - \int_{D_{z,t,\varepsilon}} t^{1/(1+\sigma)} f_0 \left( y^{1/(1+\sigma)} \right) e^{t^{1/(1+\sigma)} H_t(y,z)} dy \right| \to 0$$

when $t \to +\infty$, and since on $D_{z,t,\varepsilon}$ we have $|t^{1/(1+\sigma)} f_0 \left( y^{1/(1+\sigma)} , t \right) - z| \to 0$ when $t \to +\infty$. We have

$$\int_{D_{z,t,\varepsilon}} t^{1/(1+\sigma)} f_0 \left( y^{1/(1+\sigma)} \right) e^{t^{1/(1+\sigma)} H_t(y,z)} dy 
\leq \int_{D_{z,t,\varepsilon}} e^{t^{1/(1+\sigma)} H_t(y,z)} dy 
\leq \int_{D_{z,t,\varepsilon}} e^{t^{1/(1+\sigma)} H_t(y,z)} dy 
\leq \int_{\mathbb{R} \setminus D_{z,t,\varepsilon}} e^{t^{1/(1+\sigma)} H_t(y,z)} dy 
\leq C_t^{1/(1+\sigma)} + C_{\alpha,\beta,\varepsilon} - 2(1+\beta)$$

when $t \to +\infty$. Here, we have

$$\tilde{H}_{t,\max} = -\frac{(z - \gamma_{t,z})^2}{4} - \frac{1}{2} t^{1/(1+\sigma)} \int_0^{y_{t,z}} t^{1/(1+\sigma)} f_0(u) du$$

and since $\gamma_{t,z} t^{-1/(1+\sigma)} \to -\left( \frac{y_{t,z}}{\kappa} \right)^\frac{1}{\sigma}$ when $t \to +\infty$, we have

$$\tilde{H}_{t,\max} \to \frac{-z^2}{4}$$

when $t \to +\infty$.

Let us now look at the case $\frac{1}{\nu} > z > g(y_0) + \nu$. There, there are three solutions of $z = g_t(y)$ for $t$ large enough (depending on $\nu$). One is the continuation of the branch $y_{t,z}$, and the other two are $y_{+}(z, t)$ and $y_{m}(z, t)$, defined as in the proof of Theorem 1.5. We check, as previously, that $y_{m}$ is a local minimizer of $\tilde{H}_t(\cdot, z)$, so the maximum is either reached at $y_{t,z}$ or $y_{+}(z, t)$. 

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Defining here
\[ b_+(z) = H_t(y_+(z,t), z) \quad \text{and} \quad b_-(z) = H_t(y_{t,z}, z), \]
we have
\[ b'_+(z) = -\frac{1}{2}(z - y_+(z,t)), \quad b'_-(z) = -\frac{1}{2}(z - y_{t,z}), \]
hence
\[ b'_+(z) - b'_-(z) = \frac{y_+(z,t) - y_{t,z}}{2} > 0 \]
if \( t \) is large enough since \( y_+(z,t) > C(\nu) \) and \( y_{t,z} \to 0 \) when \( t \to +\infty \). Therefore, following the proof of Theorem 1.5, there exists exactly one point \( z_c(t) > g(y_0) \), converging when \( t \to +\infty \) such that the maximum is reached at \( y_{t,z} \) for \( z < z_c(t) \), and is reached at \( y_+(z,t) \) for \( z > z_c(t) \).

Finally, the last case, namely \( z > \frac{1}{\nu} \), can be treated as in the proof of Theorem 1.5.

### A Proof of Propositions 1.2 and 1.4

**Proof** We check easily these two propositions if \( t \leq 1 \). We suppose now that \( t \geq 1 \). We recall that the solution of the heat equation is
\[ f(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f_0(y)e^{-\frac{(x-y)^2}{4t}} dy. \]
We supposed here that \( \frac{\sigma^2}{(1+|y|)^{\alpha}} \leq f_0(y) \leq \frac{\sigma^2}{(1+|y|)^{\alpha}} \). We estimate
\[ \frac{1}{\sqrt{4\pi t}} \int_{|y| \leq \sqrt{t}} f_0(y)e^{-\frac{(x-y)^2}{4t}} dy \leq \frac{K}{\sqrt{1+t}} \int_{|y| \leq \sqrt{t}} \frac{1}{(1+|y|)^{\alpha}} \leq \frac{K}{(1+t)^{\alpha/2}} \]
and
\[ \frac{1}{\sqrt{4\pi t}} \int_{|y| \geq \sqrt{t}} f_0(y)e^{-\frac{(x-y)^2}{4t}} dy \leq \frac{K}{\sqrt{t}} \sup_{|y| \geq \sqrt{t}} (f_0) \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} dy \leq \frac{K}{\sqrt{t}} (1+t)^{-\alpha/2} \sqrt{t} \leq \frac{K}{(1+t)^{\alpha/2}}, \]
leading to the upper bound on \( \|f(., t)\|_{L^\infty} \). Finally,
\[ f(0,t) \geq \frac{K}{\sqrt{4\pi t}} \int_{|y| \leq \sqrt{t}} e^{-\frac{x^2}{4t}} dy \geq \frac{K}{\sqrt{4\pi t}} \int_{|y| \leq \sqrt{t}} \frac{1}{(1+|y|)^{\alpha}} \geq \frac{K}{(1+t)^{\alpha/2}}. \]
This completes the proof of Proposition 1.2. Now, if we define
\[ G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \]
by standard computations, we have that for any \( n,k \in \mathbb{N} \), there exists \( K_{n,k} > 0 \) such that
\[ |\partial_t^n \partial_x^k G(x,t)| \leq \frac{K_{n,k}}{t^{\frac{n}{2} + 1 + \frac{k}{2}}} e^{-\frac{x^2}{4t}}. \]
Therefore,
\[ |\partial_t^n \partial_x^k f(x,t)| \leq \frac{K_{n,k}}{t^{\frac{n}{2} + 1 + \frac{k}{2}}} \int_{\mathbb{R}} f_0(y)e^{-\frac{(x-y)^2}{4t}} dy \]
and Proposition 1.4 follows from the same upper bound estimate as previously.
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