Dark matter candidate induced by Horndeski theory: 
dark matter halo and cosmological evolution

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Abstract

We study spherically symmetric solutions with a scalar field in the shift-symmetric subclass of
the Horndeski theory. Constructing an effective energy-momentum tensor of the scalar field based
on the two-fluid model, we decompose the scalar field into two components: dark matter and dark
energy. We find the dark-matter fluid is pressure-less, and its distribution of energy density obeys
the inverse-square law. We show the scalar field dark matter can explain the galaxy rotation curve
and discuss the time evolution of the dark matter in the cosmic background.

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I. INTRODUCTION

The evidences for dark matter (DM) in our Universe have been accumulated so far by independent observations such as galaxy rotation curves, gravitational lensing on cluster scale, temperature fluctuations in Cosmic Microwave Background (CMB) [1–4]. The DM is usually considered a non-baryonic matter which mainly interacts with gravity, and DM emits little or no radiation since interactions between DM and ordinary matters are weak. A recent study has revealed the coldness of DM through the whole cosmic history, which implies that the DM is well-described in terms of the pressure-less fluid [5]. Although the DM occupies about 30% of the whole Universe, which plays an important role in the cosmological evolution, it has not been discovered in non-gravitational experiments yet. Therefore, we have not identified the particle nature of DM, while various candidates are hypothesized in the models beyond the standard model of particle physics.

As an illustration, let us consider the issue of the rotational curves of the galaxies. The rotational velocity of an object at a radial distance \( r \) is given by \( v \propto \sqrt{M(r)/r} \) in the Newtonian gravity, where \( M(r) \) denotes the total mass enclosed by the object’s orbit. However, the observations imply that the rotational velocity is approximately constant at a large distance \( r \sim \mathcal{O}(\text{kpc}) \) or the larger, where there is almost no luminous matter. It requires a DM halo with mass density \( \rho(r) \propto 1/r^2 \) to be introduced other than the visible matters, and the mass of the DM halo dominates the total mass of the galaxy.

Instead of the new particles beyond the standard model of particle physics, a possible way to account for the DM is to modify the general relativity. The modified gravity theories can introduce more degrees of freedom into the general relativity, and the role of DM particles can be replaced with those extra degrees of freedom, to explain the galactic rotation curve [6–10]. Generally speaking, the additional degrees of freedom, which are often rewritten in terms of new dynamical fields, change the gravitational interaction. Such new fields directly affect and contribute to the spacetime as new matters, literally as the DM. In other words, the modified gravity can allow us to discuss the DM as a gravitational phenomenon which is unexplainable in the framework of the general relativity and ordinary matters. In this work, we cast a scalar field induced by the modified gravity as the DM fluid and investigate its nature.

We consider a simple case that the DM is rephrased with the scalar field in the scalar-
tensor theory. For instance, k-essence theory has been studied in [11] to explain the rotation curve. On the other hand, static and inhomogeneous configurations of the scalar field usually lead to the anisotropic pressure in the scalar-tensor theory. Actually, several works have suggested that a fluid with pressure can generate a halo and result in a flat rotation curve [12, 13]. However, these results are contradictory with the cosmological observation [5], which implies that the DM fluid is pressure-less.

Now we take a look at a wider framework of the scalar-tensor theory, the generalized Galileon theory [14–16], which was found to be identical to the Horndeski theory [17] established in 1974. As a new scalar-tensor theory, the Horndeski theory introduces four arbitrary functions of scalar $\phi$, and the scalar field generates the fifth force which modifies the dynamics of the general relativity. Although many works on Horndeski theory have revealed the cosmological solutions (see [18] for a review), it is necessary to check the applicability of Horndeski theory to the sub-cosmological scale physics, which help us comprehend the Horndeski theory as the theory of gravitation in the Universe. In 2012, the first nontrivial static black hole solution was found in a subclass of Horndeski theory [19], and later, a time-dependent black hole solution was done by [20]. Further studies on the Horndeski theory at local scale, including new spacetime solutions [21–27], self-tuning issues [28–30], scalar hair [31–35], and so on have been under discussion.

In this paper, we will improve the model in [19] to study new black hole solutions and discuss an application to describe the DM halo. We employ a spherically symmetric solution to study the rotational curve. In order to address the anisotropic pressure of the scalar field, we make use of the two-fluid model [36–38] and study the cosmic evolution. Dividing the energy-momentum tensor of the scalar field into two perfect fluids, we show that one fluid corresponding to DM can be pressure-less and another works as Dark Energy (DE). This paper is organized as follows. In Sec. II, we provide an action possessing the shift symmetry of the scalar field $\phi$ and utilize this action to study a spherically symmetric black hole solution. Moreover, we consider the application of this solution to explain the galactic rotation curve. In Sec. III, we apply the two-fluid model to study the anisotropic fluid composed by the scalar field and obtain a pressure-less DM fluid. Next in Sec. IV, we discuss the behavior of the DM fluid in cosmological evolution, to find that it is consistent with the result in galaxy scale. We will also face our model to the constraint on the sound speed squared of the gravitational waves given by GW170817 and GRB170817A. Finally, we conclude our
result in Sec. V.

II. SHIFT-SYMMETRIC HORNDESKI THEORY AND ITS BLACK HOLE SOLUTION

In this section, we briefly review the Horndeski theory and introduce a particular subclass by imposing symmetries. Assuming specific forms of the arbitrary functions in the Horndeski theory, we demonstrate the existence of a spherically symmetric black hole solution. Moreover, we define the energy-momentum tensor of the scalar field and analyze the energy density and pressure of the scalar field based on the fluid description.

A. The Horndeski theory

The Horndeski theory is described by the following action:

\[
S = \sum_{i=2}^{5} \int d^4x \sqrt{-g} \mathcal{L}_i ,
\]

where \( \mathcal{L}_i \) are defined as follows:

\[
\mathcal{L}_2 = G_2(\phi, X),
\]

\[
\mathcal{L}_3 = -G_3(\phi, X) \Box \phi ,
\]

\[
\mathcal{L}_4 = G_4(\phi, X) R + G_{4X} \left[ (\Box \phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2 \right],
\]

\[
\mathcal{L}_5 = G_5(\phi, X) G^{\mu\nu} \nabla_\mu \nabla_\nu \phi - \frac{G_{5X}}{6} \left[ (\Box \phi)^3 - 3(\Box \phi) (\nabla_\mu \nabla_\nu \phi)^2 + 2(\nabla_\mu \nabla_\nu \phi)^3 \right].
\]

\( G_i(\phi, X) \) are arbitrary functions of the scalar field \( \phi \) and its kinetic term \( X = -(\partial_\mu \phi)^2 / 2 \), \( R \) is the Ricci scalar, and \( G_{\mu\nu} \) is the Einstein tensor. We choose the convention such that \( M_p^{-2} \equiv 8\pi G = 1 \), and use the notation such as \( f_X \equiv \partial f / \partial X \), \( f_\phi \equiv \partial f / \partial \phi \) to describe the derivatives of a function \( f(\phi, X) \) with respect to \( \phi \) and \( X \). Note that by choosing specific forms of the above functions \( G_i(\phi, X) \), the Horndeski theory turns to coincide with the various modified gravity theories, including the general relativity.

Varying the action (1), we obtain

\[
\delta S = \sqrt{-g} \left[ \sum_{i=2}^{5} G^i_{\mu\nu} \delta g^{\mu\nu} + \sum_{i=2}^{5} (P^i_\phi - \nabla_\mu J^i_\mu) \delta \phi \right],
\]
up to the total derivatives. The concrete expressions of $\mathcal{G}^i_{\mu
u}$, $\mathcal{P}^i_\phi$, and $\mathcal{J}^i_\mu$ are listed in the appendix of Ref. [23]. Then, the equation of motion for both metric and the scalar field in the Horndeski theory are symbolically expressed as

$$\sum_{i=2}^{5} \mathcal{G}^i_{\mu
u} = 0, \quad (7)$$

and

$$\sum_{i=2}^{5} \mathcal{P}^i_\phi - \sum_{i=2}^{5} \nabla^\mu \mathcal{J}^i_\mu = 0, \quad (8)$$

respectively. It is known that the equations of motion Eqs. (7) and (8) are of the second order of derivatives, which allows us to avoid the Ostrogradsky instability.

**B. Shift-symmetric model**

We consider the following subclass of the Horndeski theory:

$$\mathcal{L}_2 = G_2(X), \quad \mathcal{L}_4 = G_4(X)R + G_{4X} \left[\left(\Box \phi\right)^2 - \left(\nabla_\mu \nabla_\nu \phi\right)^2\right], \quad \mathcal{L}_3 = \mathcal{L}_5 = 0. \quad (9)$$

The above model possesses the shift symmetry, $\phi \rightarrow \phi + c$ where $c$ is a constant, and the $Z_2$ symmetry, $\phi \rightarrow -\phi$. These two symmetries allow only $G_2$ and $G_4$ in the Lagrangian, and these functions depend only on $X$. The above subclass was introduced and examined by earlier works [21–25].

In this case, Eq. (7) is written as

$$\mathcal{E}_{\mu\nu} \equiv \mathcal{G}^{(2)}_{\mu\nu} + \mathcal{G}^{(4)}_{\mu\nu} = 0. \quad (10)$$

Here, each part is given as (also see, [16])

$$\mathcal{G}^{(2)}_{\mu\nu} = -\frac{1}{2} G_{2X} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} G_{2}\nabla_\mu \nabla_\nu \phi, \quad (11)$$

$$\mathcal{G}^{(4)}_{\mu\nu} = G_{4} G_{\mu\nu} - \frac{1}{2} G_{4X} R \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} G_{4X} \left[\left(\Box \phi\right)^2 - \left(\nabla_\alpha \nabla_\beta \phi\right)^2\right] \nabla_\mu \phi \nabla_\nu \phi$$

$$- G_{4X} \Box \phi \nabla_\mu \nabla_\nu \phi + G_{4X} \nabla_\lambda \nabla_\mu \phi \nabla^\lambda \nabla_\nu \phi + 2 \nabla_\lambda G_{4X} \nabla^\lambda \nabla_\mu \nabla_\nu \phi - \nabla_\mu G_{4X} \nabla^\lambda \phi \nabla_\nu \phi - \nabla_\nu G_{4X} \nabla^\lambda \phi \nabla_\mu \phi$$

$$+ g_{\mu\nu} \left\{ G_{4XX} \nabla_\alpha \nabla_\lambda \phi \nabla_\beta \nabla^\lambda \phi \nabla_\alpha \nabla_\beta \phi + \frac{1}{2} G_{4X} \left[\left(\Box \phi\right)^2 - \left(\nabla_\alpha \nabla_\beta \phi\right)^2\right]\right\}$$

$$+ 2 \left[ G_{4X} R_\lambda (\mu \nabla_\nu) \phi \nabla^\lambda \phi - \nabla_\phi (G_{4X} \nabla_\mu \nabla_\nu) \phi \Box \phi \right] - g_{\mu\nu} \left[ G_{4X} R^{\alpha\beta}_\nu \nabla_\alpha \phi \nabla_\beta \phi - \nabla_\phi G_{4X} \nabla^\lambda \phi \Box \phi \right]$$

$$+ G_{4X} R_{\mu\nu\alpha\beta} \nabla^\alpha \phi \nabla^\beta \phi - G_{4XX} \nabla^\alpha \phi \nabla_\alpha \nabla_\mu \phi \nabla^\beta \phi \nabla_\beta \nabla_\nu \phi, \quad (12)$$
where the parenthesis () expresses the symmetric part with two indices swapped, \( A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}) \). Corresponding to the shift symmetry \( \phi \rightarrow \phi + c \), we have a Noether current \( J^\mu \),

\[
J^\mu = \frac{\delta (\mathcal{L}_2 + \mathcal{L}_4)}{\delta \nabla_\mu \phi} = -G_{2X} \nabla^\mu \phi + 2G_{4X} G^{\mu\nu} \nabla_\nu \phi - G_{4XX} \left[ (\Box \phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2 \right] \nabla^\mu \phi
-2G_{4XX} \nabla^\mu X \Box \phi + 2G_{4XX} \nabla^\nu X \nabla^\mu \nabla_\nu \phi .
\] (13)

Note that the equation of motion with respect to the scalar field is rephrased as the current conservation law:

\[ \nabla_\mu J^\mu = 0. \] (14)

In the following analysis, we employ a toy model with the specific choice of \( G_2(X) \) and \( G_4(X) \):

\[
G_2(X) = -2\Lambda + 2\eta X^p ,
\]

\[
G_4(X) = \zeta + \beta X^p ,
\] (15)

where \( \Lambda, \eta, \zeta, \beta, \) and \( p \) are parameters. Note that for \( p = 1 \), the above model reproduces the so-called non-minimal derivative coupling \( G^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \), as in [24]

\[
S = \int d^4x \sqrt{-g} \left[ -2\Lambda + 2\eta X + (\zeta + \beta X) R + \beta \left[ (\Box \phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2 \right] \right]
= \int d^4x \sqrt{-g} \left[ \zeta R - 2\Lambda + 2\eta X + \beta G^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right] ,
\] (16)

where we have used the following equation defining Riemann curvature tensor,

\[ \nabla_\mu \nabla_\nu \nabla^\rho \phi - \nabla_\mu \nabla^\rho \phi \nabla_\nu \phi = R_{\rho \nu}^\rho \nabla_\mu \phi . \] (17)

The spherically symmetric black hole solutions have been studied by [19–22] in the case where \( p = 1 \), and thus, our model can be considered a generalization of the model which the earlier works have employed.

C. Spherically symmetric system

In order to analyze the galaxy rotation curve, we first consider the static and spherically symmetric spacetime solutions in the shift-symmetric model Eq. (15), which gives us approximated descriptions of the spacetime around the galaxy. We start with the general form
of the static and spherically symmetric spacetime:
\[ ds^2 = -h(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega^2. \] (18)

For the scalar field \( \phi \), we also assume the scalar field is static and the spherically symmetric, that is, \( \phi = \phi(r) \).

The nonvanishing components of \( E_{\mu\nu} \) and \( J^\mu \) are given by

\[
E_{tt} = \frac{8hf XX'G_{4XX}}{r} + \frac{4fhG_{4X}}{r^2} + \frac{4f'hXG_{4X}}{r} + \frac{4fh'X'G_{4X}}{r} - \frac{2fhG_4}{r^2} + G_2 - \frac{2f'hG_4}{r^2} + \frac{2G_4h}{r^2},
\]

\[
E_{rr} = -\frac{8G_{4XX}X^2}{r^2} - \frac{8h'G_{4XX}X^2}{hr} - \frac{8G_{4XX}}{r^2} + \frac{8h'G_{4XX}X}{hr} + \frac{2XG_{2X}}{f} + \frac{4G_{4XX}}{fr^2},
\]

\[
E_{\theta\theta} = -\frac{2XX'G_{4XX}f'h'^2r^2}{h} - \frac{2XG_{4XX}fh'^r^2}{h} + \frac{XG_{4XX}f'h'^2r^2}{h} - 4XX'G_{4XX}fr - \frac{2XG_{4XX}fh'r}{h} + \frac{XG_{4XX}f'h'^r}{h} - 2XG_{4XX}f'r
- \frac{X'G_{4XX}fh'^r}{h} + \frac{G_4fh'^r}{h} - \frac{G_4fh'^2r^2}{2h^2}
- \frac{2X'G_{4XX}fr + G_4fh'r}{h} + \frac{G_4fh'^r}{2h} + G_4f'r - G_2r^2,
\]

\[
E_{\phi\phi} = \sin^2 \theta E_{\theta\theta},
\]

\[
J^r = \frac{f\phi'}{r^2 h} \left\{ -(r^2G_{2X} + 2G_{4X})h + 2(G_{4X} + 2XG_{4XX})(rh)'f \right\},
\]

where the prime expresses the derivative with respect to \( r \). Note that we have the Noether current \( J^\mu = (0, J^r, 0, 0) \) under the assumption of static field configuration. One can find that the conservation law leads to \( \partial_r (\sqrt{h/f}J^r r^2) = 0 \), and a solution is given by \( \sqrt{h/f}J^r r^2 = C \) where \( C \) is an integration constant. If we want to get a black hole solution, there should exist a horizon \( r = r_h \), where \( f(r_h) = 0 \). Therefore from Eq. (23), one can find that this will lead to a vanishing current, namely \( J^r = 0 \). This can be viewed as a specific solution of the continuity equation (14), although not necessary at the horizon. [39]

Imposing the vanishing current in terms of Eq. (14) and using Eq. (10), we have four equations

\[
J^\mu = 0, \ E_{tt} = 0, \ E_{rr} = 0, \ E_{\theta\theta} = 0
\] (24)
where a parameter $\mu$ is a specific solution of the continuity equation (14). For the case where $\eta \beta > 0$, Eqs. (24) give the following solutions of $h(r), f(r),$ and $\phi(r)$:

\[
\begin{align*}
  h(r) &= \frac{\beta(2p - 1)^2(\Lambda \beta + \zeta r^2)^2}{4\zeta^2 p^2 \sqrt{\eta \beta}} \arctan\left(\frac{r \eta}{\sqrt{\eta \beta}}\right) - \frac{(2p - 1)(\eta r^2 + 6 \beta p - 3 \beta)}{6 \zeta \eta r^2} \Lambda \\
  f(r) &= \frac{h(r)}{w(r)}, \\
  w(r) &= \frac{(2p - 1)(2 \Lambda \beta p r^2 - \Lambda \beta r^2 - \zeta \eta r^2 - 2 \zeta \beta p)^2}{4 \zeta^2 p^2(\eta r^2 + \beta)^2}, \\
  \phi'(r)^2 &= -2 \left(\frac{2 \Lambda \beta p r^2 - \Lambda \beta r^2 + \zeta \eta r^2 - 2 \zeta \beta p + 2 \zeta \beta}{2(2p - 1)(\eta r^2 + \beta)}\right)^2 f(r)^{-1},
\end{align*}
\]

(25)

where a parameter $\mu$ in Eq. (25) is an integration constant. If $\eta \beta < 0$, one can find a similar solution by replacing $\sqrt{\eta \beta} \rightarrow \sqrt{-\eta \beta}$ and $\arctan(\eta r / \sqrt{\eta \beta}) \rightarrow \arctanh(\eta r / \sqrt{-\eta \beta})$ in Eq. (25) while keeping Eqs. (26), (27), and (28) unchanged. As an example, our generalized solutions can surely reproduce the black hole solution derived in the earlier work [19] when we choose a specific set of parameters, $p = 1, \Lambda = 0, \eta = 1/2, \zeta = 1/2$, and $\beta = \zeta_0/2$:

\[
\begin{align*}
  h(r) &= \frac{3}{4} + \frac{r^2}{12 \zeta_0} - \frac{\mu}{r} + \frac{\sqrt{\zeta_0}}{4 r} \arctan\left(\frac{r}{\sqrt{\zeta_0}}\right), \\
  f(r) &= \frac{4(r^2 + \zeta_0)^2 h(r)}{(r^2 + 2 \zeta_0)^2}, \\
  \phi'(r)^2 &= -\frac{r^2(r^2 + 2 \zeta_0)^2}{4 \zeta_0 (r^2 + \zeta_0)^2 h(r)}.
\end{align*}
\]

(29)

(30)

(31)

As a side remark, we note that this solution cannot be applied to the specific parameter choice $p = 1/2$, since $\phi'(r)^2$ diverges in Eq. (28). Using the original continuity equation $\nabla_\mu J^\mu = 0$ one can get another class of solution:

\[
\begin{align*}
  h(r) &= \frac{\Sigma}{(\eta r^2 + \beta)^2}, \\
  f(r) &= \frac{(\eta r^2 + \beta)(\Lambda r^2 - \zeta)}{\zeta(3 \eta r^2 - \beta)}, \\
  \phi'(r)^2 &= \frac{8 \zeta r^4(-9 \Lambda \eta r^2 + 4 \Lambda \beta \eta r^2 + 6 \zeta \eta r^2 + \Lambda \beta^2 - 6 \zeta \beta \eta)^2}{(\eta r^2 + \beta)^3(-3 \eta r^2 + \beta)^3(\Lambda r^2 - \zeta)^2}.
\end{align*}
\]

(32)

(33)

(34)

where $\Sigma$ represents an integration constant. It is obvious that the solution gives the horizon at $r_h = \sqrt{\zeta / \Lambda}$, but there is no infinite redshift surface which causes $h(r) = 0$, so it shows significant difference from a black hole solution.
Next, we look further into the solutions Eqs. (25), (26), (27), and (28). Using the definition $X = -f(r)\phi'(r)^2/2$, one finds

$$X^p = \frac{2\Lambda \beta pr^2 - \Lambda \beta r^2 + \zeta \eta r^2 - 2\zeta \beta p + 2\zeta \beta}{2(2p-1)(\eta r^2 + \beta)}.$$  \quad (35)

For simplicity, we focus on the specific case where $\beta \Lambda + \zeta \eta = 0$, therefore the first term of Eq. (25) could be canceled. Defining $z \equiv \beta/\eta = -\zeta/\Lambda$, the black hole solution is expressed by the following simple form:

$$h(r) = 1 - \frac{\mu}{r} - \frac{r^2}{3(-z)},$$  \quad (36)

$$f(r) = (2p-1)^{-1}h(r),$$  \quad (37)

$$X^p = \frac{(p-1)\Lambda}{(2p-1)\eta}.$$  \quad (38)

It is remarkable that our model produces a constant solution for $X$. Hence, as in the equation of motion (12), the coefficient $G_4$ in front of the Einstein tensor $G_{\mu\nu}$ is constant, which would be absorbed into the redefinition of the Planck scale. This result implies that our solution does not break Einstein’s equivalence principle.

Hereafter, we assume $\zeta, \eta, \Lambda > 0$, $\beta < 0$, and $p > 1$. In those parameter regions, $h(r)$ takes the similar form to the Schwarzschild-de Sitter solution with positive cosmological constant $1/(-z) = \Lambda/\zeta > 0$ as $X > 0$ and $X^p > 0$. However, it does not mean that our model is as simple as GR plus a cosmological constant. This is because the nontrivial coupling between the scalar field and gravity still appears in the Einstein Equations, and the energy-momentum tensor, as we will show soon, has $r$-dependence and can play the role of DM. A similar case can be found in the vacuum solution of ghost condensate field in cosmology [40, 41].

### D. Energy-Momentum Tensor of Scalar Field

The equation of motion (10) can be rewritten in a similar form of the Einstein equation, given as

$$G_{\mu\nu} = T_{\mu\nu}^{\text{eff}},$$  \quad (39)

Here, $T_{\mu\nu}^{\text{eff}}$ represents the energy-momentum tensor of the scalar field. Since for spherical symmetric solutions one has $G^r_r \neq G^\theta_\theta \neq G^\phi_\phi$, namely $G^i_i$ is anisotropic in these three directions, one will expect the energy-momentum $T_{\mu\nu}^{\text{eff}}$ to be the same. Therefore we write the
effective energy momentum tensor $T_{\mu\nu}^{\text{eff}}$ as [42]:

$$T_{\mu\nu}^{\text{eff}} = \rho e^0_\mu e^0_\nu + p_\parallel e^1_\mu e^1_\nu + p_\perp e^2_\mu e^2_\nu + p_\perp e^3_\mu e^3_\nu.$$  \hspace{1cm} (40)

Here, the tetrads $e^a_\mu$ satisfy $g_{\mu\nu} e^a_\mu e^b_\nu = \eta^{ab}$, where $\eta^{ab} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski spacetime metric, and $e^a_\mu = \text{diag}(\sqrt{-g_{00}}, \sqrt{g_{11}}, \sqrt{g_{22}}, \sqrt{g_{33}})$. The energy density and the pressures of scalar field are defined as follows,

$$\rho \equiv -T^0_0 = \frac{2(p-1)}{(2p-1)r^2} + \frac{1}{(2p-1)(-z)},$$  \hspace{1cm} (41)

$$p_\parallel \equiv T^1_1 = -\frac{2(p-1)}{(2p-1)r^2} - \frac{1}{(2p-1)(-z)},$$  \hspace{1cm} (42)

$$p_\perp \equiv T^2_2 = T^3_3 = -\frac{1}{(2p-1)(-z)}.$$  \hspace{1cm} (43)

One can find that the second term in Eq. (41) and (42), as well as the term in Eq. (43), are behaving like a cosmological constant with isotropic pressure. So, this part could be naively viewed as dark energy (DE). Moreover, the first term in Eq. (41) is proportional to $r^{-2}$. According to the analysis in [11], this behavior of DM can explain the gravitational rotation curve. However, it is well-known that the cold dark matter needs vanishing pressure in all directions. Although the corresponding terms in $p_\perp$ can be viewed as vanishing, that in $p_\parallel$ is obviously non-vanishing, making this term fail to be explained as DM. To make out of this dilemma, in the next section, we will take a different view of this model.

### III. ANISOTROPIC FLUID AND TWO-FLUID MODEL

#### A. DM and DE fluids

Due to the anisotropy of the energy momentum tensor, $p_\parallel - p_\perp \neq 0$, in Eq.(40), it can be decomposed into the two non-interacting perfect fluids sourcing the spacetime structure of solution Eq.(36), expressed as

$$T_{\mu\nu} = (p_1 + \rho_1)u_\mu u_\nu + p_1 g_{\mu\nu} + (p_2 + \rho_2)v_\mu v_\nu + p_2 g_{\mu\nu}.$$  \hspace{1cm} (44)

Here $\rho_i$ and $p_i$ ($i = 1, 2$) denote the energy density and pressure of each perfect fluid, $v^\mu$, $u^\mu$ are timelike 4-velocities of each component of two-fluid system, i.e., $u_\mu u^\mu = v_\mu v^\mu = -1$, and the anisotropy vanishes if $u^\mu = v^\mu$. 

10
Using Eqs. (40) and (44), the energy density $\rho$ and the pressures $p_\parallel, p_\perp$ are given by [36–38]

$$
\rho = \frac{1}{2}(\rho_1 - p_1 + \rho_2 - p_2) + \frac{1}{2}\sqrt{(\rho_1 + p_1 + \rho_2 + p_2)^2 + 4(\rho_1 + p_1)(\rho_2 + p_2)(K^2 - 1)},
$$
$$
p_\parallel = \left(-\frac{1}{2}(\rho_1 - p_1 + \rho_2 - p_2) + \frac{1}{2}\sqrt{(\rho_1 + p_1 - \rho_2 - p_2)^2 + 4(\rho_1 + p_1)(\rho_2 + p_2)K^2}\right),
$$

$$
p_\perp = p_1 + p_2,
$$

where $K \equiv v_\mu u^\mu < 0$ since $v^\mu, u^\mu$ are timelike vectors. From equations (45), we obtained the analytical solution:

$$
\rho_1 = \frac{(xn + 1)r^2 - 4(p - 1)z}{(x - 1)(2p - 1)r^2z},
$$
$$
\rho_2 = -\frac{x[(n + 1)r^2 - 4(p - 1)z]}{(x - 1)(2p - 1)r^2z},
$$

$$
p_1 = \frac{\rho_1}{n},
$$
$$
p_2 = \frac{\rho_2}{xn},
$$

$$
K = \pm \frac{r^2(n + 1)(xn + 1) - 2(p - 1)z(xn + n + 2)}{\sqrt{(n + 1)(xn + 1)[(n + 1)r^2 - 4(p - 1)z]r^2(xn + 1) - 4(p - 1)z}}.
$$

Here, since there are less variables than the equations in (45) and thus the solution cannot be uniquely determined, we introduce two free parameters $x$ and $n$. Moreover, if we set $n \to -1$ and $x \to \pm \infty$, the solutions are given by

$$
\rho_1 \to \frac{1}{(2p - 1)(-z)}, \quad \rho_2 \to \frac{4(p - 1)}{(2p - 1)r^2}, \quad p_1 \to -\rho_1, \quad p_2 \to 0, \quad K \to \pm \infty.
$$

In Eqs. (47), $\rho_1$ can be read off as the DE density, while $\rho_2$ can be the DM density since the equation-of-state parameter $w_{DM} \equiv p_2/\rho_2 = 0$. That is, we can regard the first fluid ($\rho_1, p_1$) as the DE fluid as the second one ($p_2, p_2$) as the DM fluid in this scenario. Note that the total mass enclosed by the orbit is symbolically written as $m(r) = \int [\rho_2(r) + \rho_e(r)]dV$, where $\rho_e$ is the average density of environment of the ordinary matter rather than the scalar field, and $\rho_1$ should be so small compared to $\rho_e$ that it can be negligible.

We further explain the physical meaning of $K$ reaching infinity. In general, the 4-velocity can be decomposed like $u^\mu = (Z^\mu + \bar{u}^\mu)/\sqrt{1 - \bar{u}^\mu\bar{u}_\mu}$, where $Z^\mu$ is the 4-velocity of observer at rest in the coordinate system, and $\bar{u}^\mu$ is spacelike satisfying $Z_\mu\bar{u}^\mu = 0$. Here the 4-velocity of observer can be $v^\mu$, thus this observer of 4-velocity $v^\mu$ measures a particle of 4-velocity $u^\mu = \gamma_\lambda v^\mu + \gamma_\lambda \lambda^i \delta^\mu_i, \ i = 1, 2, 3$, where $\gamma_\lambda = 1/\sqrt{1 - \lambda^i \lambda_i} \geq 1$ is the Lorentz factor.
Thus $K = v_\mu u^\mu$ can be rewritten by the 3-velocity $\lambda^i$ which is measured by the observer of 4-velocity $v^\mu$, that is

$$K = v_\mu u^\mu = -\frac{1}{\sqrt{1 - \lambda^i \lambda_i}}. \quad (48)$$

The right-hand side of the Eq. (48) can be regarded as the Lorentz factor $\gamma_\lambda = 1/\sqrt{1 - \lambda^i \lambda_i} \geq 1$. Therefore, if $n \to -1$ and $K \to -\infty$, the relative velocity between the DM and DE fluids is the speed of light.

**B. Facing observational constraints on DM fluids**

*Galaxy rotation curve.* Without the loss of generality of our model, we can choose $\zeta = 1$ for simplicity. To study the rotation curve, we need to know the velocity of object travelling around the galaxy, which is given by the geodesic equation in the Newtonian limit,

$$v^2 \equiv r \Gamma^r_{00} \simeq \frac{G_N m(r)}{r} - \frac{\Lambda}{3(2p - 1)} r^2, \quad (49)$$

where $G_N = G/(2p - 1)$. $\Gamma^r_{00}$ is the $(t, t, r)$ component of Christoffel symbol $\Gamma^\lambda_{\mu\nu}$, and the parameter $m(r)$ is related to the mass of DM and ordinary matter enclosed by the orbit.

Based on the previous analysis in the two-fluid model, when we consider $\Lambda$ in Eq. (15) as the cosmological constant, it is so small that we can neglect its effect at the galactic scales. The Eq. (49) can approximate

$$v = \sqrt{\frac{G_N m(r)}{r}}, \quad m(r) = \int \rho dV = \frac{4\pi}{3} (3\rho_2 + \rho_e) r^3, \quad (50)$$

and we have two cases of the velocity of test particle:

$$\left\{ \begin{array}{l}
    v \propto r \quad \text{when} \quad \rho_2 \ll \rho_e, \\
    v \sim \text{constant} \quad \text{when} \quad \rho_2 \gg \rho_e.
\end{array} \right. \quad (51)$$

Assuming the environmental density has a power-law form of $\rho_e \sim 10^{-n} \text{[g/cm}^3\text{]}$, we can estimate the critical radius $r_c$ for the tangential velocity $v$, given by the condition $3\rho_2 = \rho_e$:

$$r_c \simeq 2 \times 10^{n/2 - 5} v \text{[pc]}. \quad (52)$$

The tangential velocity is usually of $O(10^{-3}) \sim O(10^{-4})$ in natural unit, or $O(10) \sim O(10^2)\text{[km/s]}$, according to observations on galaxies (see [43, 44] for more details). For
instance, for $n = 23$, namely $\rho_e \sim 10^{-114} M_p^4$ while the tangential velocity of test particle $v \simeq 3 \times 10^{-4}$ in the galaxy, we have

$$r_c \sim 2 \text{[kpc]}.$$  \hfill (53)

Another example is to have $n = 17, v \simeq 3 \times 10^{-4}$ for the solar system, one can obtain $r_c = 1.9 \text{[pc]}$. It is acceptable that $r_c$ is much larger than the size of the object as a gravitational source in order to maintain the Newtonian gravity.

Moreover, in the non-relativistic limit, Eq. (50) implies

$$|p - 1| \simeq v^2/2,$$  \hfill (54)

and for a typical value of $v^2 \sim 10^{-6}$, one has

$$|p - 1| \sim 10^{-6}.$$  \hfill (55)

**Post-Newtonian parameter.** The post Newtonian parameter $\gamma$ describes how the current theory of gravity deviates from GR. For a massive object whose gravity causes deflection of the light-ray passing it, the deflection angle $\varphi$ is expressed as [45]:

$$\varphi \propto \frac{(1 + \gamma) M}{2d},$$  \hfill (56)

where $\gamma$ is the post-Newtonian parameter, $M$ is the mass of the object, and $d$ is the distance between the center of the gravity potential and the light ray (see Ref. [46] for detailed analysis based on the Gauss-Bonnet theorem). In our model, we find that $\gamma$ relates with $p$ as:

$$\frac{1 + \gamma}{2} = \frac{1}{\sqrt{2p - 1}},$$  \hfill (57)

The detection of the deflection angle puts severe constraints on $\gamma$ as [45]:

$$|\gamma - 1| < 2.3 \times 10^{-5}.$$  \hfill (58)

which implies

$$|p - 1| \simeq \frac{1}{2} |\gamma - 1| < 1.15 \times 10^{-5}.$$  \hfill (59)

The constraints Eq.(55) and Eq.(59) on $p$ given by the two experiments are therefore consistent with each other.
IV. DARK MATTER BEHAVIOR IN COSMOLOGICAL EVOLUTION

In the last sections, we use the model (15) to generate a spherically symmetric black hole solution, and use the two-fluid description to give rise to the behaviors of DM and DE (in form of cosmological constant) in local frame. However, [47] shows that our model with $p = 1$ breaks the consistency between the DM behavior in cosmological evolution and the constraints on the speed of the gravitational wave. This inconsistency also happens as $p \simeq 1 + v^2/2$ in our model. Therefore, we need to improve our model to get consistent predictions with other physics. In the following, we consider a simple extension of our toy model in which we can keep the black hole solution obtained in the previous section.

A. Improved Model for Cosmology

We choose the new choice of functions in the following form:

$$
G_2 = -2\Lambda + 2\eta X^p + 2\alpha (X - X_1)^2(X - X_0)^q, \\
G_4 = 1 + \beta X^p,
$$

where $G_2$ is corrected by the third term $\tilde{G}_2(X) \equiv 2\alpha (X - X_1)^2(X - X_0)^q$ where $X_1$ is an additional parameter, and $\zeta = 1$ is assumed in the original model Eq. (15). The corrected term is inspired by the interesting model dubbed as purely kinetic k-essence [48, 49] where the Lagrangian is written as $L = -\Lambda + \alpha (X - X_0)^q$ with $q \geq 2$, which is claimed to give an almost pressure-less DM fluid.

Imposing $\beta \Lambda + \eta = 0$, we can still obtain the black hole solution in vacuum Eq. (36) when we choose $X_1 = \sqrt{(p-1)\Lambda/(2p-1)\eta}$ thanks to $(X - X_1)^2$. Indeed, the additional term to original $G_2$ function yields to the condition $\tilde{G}_2(X = X_1) = \tilde{G}_2X(X = X_1) = 0$. $(X - X_0)^q$ makes the scalar field behave like DM in cosmological evolution as in the purely kinetic k-essence. Now we will investigate the cosmological evolution in the improved model Eq. (60).

In flat FRW metric, $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$ where $a(t)$ is the cosmic scale factor, the Hubble parameter is defined as $H = \dot{a}/a$. The Friedmann equation takes the following
form,

\[ E \equiv 3H^2 \]

\[ = \tilde{\rho}_\phi + \tilde{\rho}_r + \tilde{\rho}_b \]

\[ = \frac{1}{M_*^2} \left[ -G_2 + 2XG_{2X} + 12H^2(XG_{4X} + 2X^2G_{4XX}) \right] + \tilde{\rho}_r + \tilde{\rho}_b, \tag{61} \]

where the effective Planck mass squared \( M_*^2 \) is

\[ M_*^2 = 2(G_4 - 2XG_{4X}), \tag{62} \]

and \( \tilde{\rho}_r, \tilde{\rho}_b, \tilde{\rho}_b = \tilde{\rho}_{DM} + \tilde{\rho}_{DE} \) respectively denote the induced energy densities of radiation, baryon and scalar field containing DM and the DE, with

\[ \tilde{\rho}_{DE} = \Lambda = 3H_0^2\Omega_{\Lambda 0}, \tag{63} \]

\[ M_*^2\tilde{\rho}_b = 3H_0^2\Omega_{b0}a^{-3}, \tag{64} \]

\[ M_*^2\tilde{\rho}_r = 3H_0^2\Omega_{r0}a^{-4}. \tag{65} \]

Here, we set the scale factor \( a_0 = 1 \) at present and \( \Omega_{i0} \) is the current value of the density fraction. Another equation of motion about the total pressure \( \tilde{P} \) including the baryon, radiation, and the scalar field is given by

\[ \tilde{P} \equiv -(3H^2 + 2\dot{H}) \]

\[ = \tilde{\rho}_\phi + \tilde{\rho}_r + \tilde{\rho}_b \]

\[ = \frac{1}{2(G_4 - 2XG_{4X})}(G_2 - 8HX\dot{X}G_{4XX} - 4H\dot{X}G_{4X}) + \tilde{\rho}_r + \tilde{\rho}_b, \tag{66} \]

where \( \tilde{\rho}_b = 0, \tilde{\rho}_r = \tilde{\rho}_r/3 \) and \( \tilde{\rho}_\phi = \tilde{\rho}_{DM} + \tilde{\rho}_{DE} \), with \( \tilde{\rho}_{DE} = -\Lambda \). The equation of motion for scalar field in the FRW Universe is then written as

\[ \dot{X} = \frac{3}{2}XH^{-1}a_K^{-1}(2\tilde{P}\alpha_B - 8G_{2X}X), \tag{67} \]

where the braiding parameter \( \alpha_B \) is defined as

\[ \alpha_B = \frac{8(XG_{4X} + 2X^2G_{4XX})}{M_*^2}, \tag{68} \]

and the the kineticity parameter \( \alpha_K \) is defined as

\[ \alpha_K = \frac{1}{M_*^2H^2}[12H^2(4X^3G_{4XXX} + 8X^2G_{4XX} + XG_{4X}) + 4X^2G_{2XX} + 2XG_{2X}] \tag{69} \].
From Eqs. (61) and (66), we define the effective pressure and energy density of scalar dark matter field

\[
M_*^2 \tilde{\rho}_{DM} = -G_2 - M_*^2 \Lambda + 2XG_{2X} + \frac{3}{2} M_*^2 \alpha_B H^2 ,
\]

\[
M_*^2 \tilde{p}_{DM} = G_2 + M_*^2 \Lambda - M_*^2 \alpha_B H \frac{\ddot{\phi}}{\dot{\phi}} .
\]

In the model (60), when the correction term \( \tilde{G}_2(X) \) is dominant in the cosmological evolution compared to the \( \eta X^p \), we can evaluate the equation of state of DM as

\[
w_{DM} \equiv \frac{\tilde{p}_{DM}}{\tilde{\rho}_{DM}} = \frac{(X - X_0)(X - X_1)}{(3 + 2q)X^2 - X_0X_1 + X(-3X_0 + X_1 - 2qX_1)} .
\]

In order to solve Eq. (67), we assume \( \tilde{P} \alpha_B \ll G_2X \) and \( G_4 \simeq \text{constant} \), which is very reasonable for small \( \beta \). Thus we can obtain the following solution [48, 49]

\[
XG_{2X}^2 = ka^{-6} ,
\]

where \( k \) is a positive constant. Furthermore, if we define \( \varepsilon \equiv (X - X_0)/X_0 \ll 1 \), from above equation we have

\[
\varepsilon = \left( \frac{a}{a_q} \right)^{-3/(q-1)} ,
\]

where \( a_q^{-3} = \alpha(X_1 - X_0)^2qX_0^q(X_0k)^{-1/2} \). And then, the equation of state of the scalar field approximates

\[
w_{DM} \simeq \frac{\varepsilon}{2q} ,
\]

and the energy density is evaluated as

\[
\rho_{DM} \equiv M_*^2 \tilde{\rho}_{DM} \simeq 2q\alpha(X_1 - X_0)^2X_0^q \left( \frac{a}{a_q} \right)^{-3} .
\]

Because the scalar field should behave like the DM before the epoch of the matter-radiation equality we need the condition \( \varepsilon \ll 1 \) at that epoch. Using Eq. (74), the necessary condition is \( a_{eq} \gg a_q \) where \( a_{eq} \) is the scale factor at the epoch of the matter-radiation equality, given by \( a_{eq} = 3 \times 10^{-4} \). Notice the energy densities of DE and DM in the current universe has a relation \( \rho_{DM0}/\Lambda = \Omega_{DM0}/\Omega_{\Lambda0} \). Combining with Eq. (76), one can obtain

\[
a_q^3 = \frac{\Lambda \Omega_{DM0}}{2q\Omega_{\Lambda0}\alpha(X_1 - X_0)^2X_0^q} \ll a_{eq}^3 \simeq 3 \times 10^{-11} .
\]
To satisfy the above constraint, $\alpha$ can be so large that it makes $\tilde{G}_2(X)$ dominated in cosmological evolution. Furthermore, the equation of state of DM has been constrained by observations \cite{5}, i.e. $|w_{DM}| \ll 1$, which is surely guaranteed by Eq. (75).

Moreover, when $\varepsilon \gg 1$, from Eq. (72) the equation of state becomes

$$w_{DM} \simeq \frac{1}{2q + 3}. \quad (78)$$

since this case correspond to the very early universe when DM has not become important yet, we will not discuss it anymore.

B. Instability and Speed of Gravitational Wave

We have discussed the cosmological evolution of the scalar-field DM in the previous subsection, to find that we can make the equation of state parameter $w_\phi$ smaller than the unity, which realizes the almost pressure-less DM in the cosmic history. In the following, we consider other aspects of the scalar field: the sound speed of the scalar field and the speed of the gravitational waves. The sound speed $c_s$ is related to the instability of scalar perturbation. The sound speed has to be small enough to generate the cosmic large-scale structure formation and CMB temperature anisotropies \cite{49}. For instance, it should be extremely small $c_s^2 < 10^{-10.7}$ at present constrained by observations \cite{50}. Meanwhile, the speed of gravitational wave $c_T$ is also tightly constrained by the observation of GW170817 and GRB 170817A \cite{51}.

The action at quadratic order of a scalar field $\zeta$ and tensor modes $h_{ij}$ are given by

$$S_2 = \int dtd^3xa^3 \left[ Q_s \left( \dot{\zeta}^2 - \frac{c_s^2}{a^2} (\partial_i \zeta)^2 \right) + Q_T \left( \dot{h}_{ij}^2 - \frac{c_T^2}{a^2} (\partial_k h_{ij})^2 \right) \right], \quad (79)$$

where (see appendix A)

$$Q_s = \frac{2M_*^2 D}{(2 - \alpha_B)^2}, \quad (80)$$

$$c_s^2 = - \frac{(2 - \alpha_B) \left( \dot{H} - \frac{1}{2} H^2 \alpha_B (1 + \alpha_T) - H^2 (\alpha_M - \alpha_T) \right) - H \dot{\alpha}_B + \bar{\rho}_m + \bar{\rho}_m}{H^2 D}, \quad (81)$$

$$D \equiv \alpha_K + \frac{3}{2} \alpha_B, \quad (82)$$
and

\[ M_*^2 \alpha_T \equiv 4XG_{4X}, \quad (83) \]
\[ HM_*^2 \alpha_M \equiv \frac{d}{dt} M_*^2, \quad (84) \]
\[ Q_T = \frac{M_*^2}{8}, \quad (85) \]
\[ c_T^2 = 1 + \alpha_T. \quad (86) \]

Here \( \alpha_M \) is the rate of running of the Planck mass and \( \alpha_T \) is the tensor speed excess. To avoid the theoretical instabilities, we should impose the conditions \( Q_s > 0, c_s^2 > 0, Q_T > 0 \) and \( c_T^2 > 0 \). If the tensor speed excess is small \( \alpha_T \ll 1 \) and \( \alpha_M, \alpha_B, \dot{\alpha}_B \ll 1 \), namely \( G_4 \) is almost a constant, the speed of the scalar perturbation mode is given by

\[ c_s^2 \simeq \frac{1}{H^2 \alpha_K} (\tilde{\rho}_\phi + \tilde{p}_\phi), \quad (87) \]

utilizing Eq. (72) and \( \varepsilon \ll 1 \), the speed of scalar perturbation is given by

\[ c_s^2 \simeq \frac{\varepsilon}{2(q - 1)}, \quad (88) \]

thus we have a small sound speed of scalar perturbation which behaves like dark matter.

For the tensor perturbation, from Eq. (83), the deviation of the speed of gravitational wave from that of light in the low-redshift era can be evaluated as

\[ \alpha_T \simeq \frac{2p \beta X_0^p}{1 - (2p - 1) \beta X_0^p}. \quad (89) \]

According to the observation constraint from GW170817 and GRB 170817A, the difference between speeds of the gravitational wave and light is should be suppressed [51],

\[ -3 \times 10^{-15} < c_T - 1 < 7 \times 10^{-16}. \quad (90) \]

Thus, by combining Eqs. (86), (89), and (90), \( p \simeq 1 \) and \( \beta < 0 \) gives the constraint on \( X_0 \):

\[ -3 \times 10^{-15} < \beta X_0 < 0. \quad (91) \]

V. CONCLUSION AND DISCUSSION

We have investigated the shift-symmetric subclass of the Horndeski theory, where \( G_2 \) and \( G_4 \) are the power functions of the kinetic term \( X^p \) as in Eq. (15), to find the spherically
symmetric solutions with the vanishing Noether current of the scalar field. We have also found that the scalar-field fluid becomes anisotropic for $p \neq 1$. Based on the two-fluid model, we have decomposed the scalar field fluid into two parts corresponding to the DM and DE.

We have found the DM part of energy density scales as the inverse square of the radial coordinate in the spherically symmetric solution which we have derived, and we can explain the observed galaxy rotation curve in our scenario.

Moreover, by calculating the gravitational mass $m(r)$ including both baryonic matter and scalar field, we have confirmed that the constant velocity appears in the galaxy rotation curve at $\mathcal{O}(\text{kpc})$ scale and that the parameter $p$ is related to the velocity $v$ by $p \simeq 1 + v^2/2$ in the non-relativistic limit. Using the observational constraint on the velocity, $v \simeq \mathcal{O}(10^{-3}) \sim \mathcal{O}(10^{-4})$, we can put constraint on parameter $p$. Furthermore, $p$ can also be related to the post-Newtonian parameter $\gamma$ in terms of $p \simeq (3 - \gamma)/2$, where $|\gamma - 1| < 2.3 \times 10^{-5}$ is given by the deflect angle of light passing the massive object. One can find that the two constraints on $p$ are consistent with each other.

Since the above model (15) suffers from the inconsistency between the cosmic evolution and the gravitational wave speed, we have studied the correction term $\alpha(X - X_1)^2(X - X_0)^\delta$ added to this model while keeping the black hole solution. Based on this corrected model, we have evaluated the effective energy density and pressure of the scalar field, showing that the corrected model gives a DM behavior in cosmic history when the additional term is dominant and the constraint Eq. (77) is satisfied. Regarding the speed of the gravitational wave, although it is not exactly equal to the speed of light, we can obtain the consistent result with the observation if $-3 \times 10^{-15} < \beta X_0 < 0$.

Before closing, let us make several comments for future works. First of all, we have assumed a specific combination of the parameters $\beta \Lambda + \zeta \eta = 0$ for the simplicity, and our analysis has demonstrated in a specific black hole solution. It would be interesting to investigate the other parameter regions to keep $X$ or $X^\rho$ analytically well-defined. Moreover, regarding the two-fluid model, we have observed the product of 4-velocities of two fluids $K = v_\mu u^\mu \to -\infty$ in our calculation. The above divergence shows up because we have assumed cosmological constant $w = -1$ for the DE fluid. Thus, if we admit a slight change of the equation-of-state parameter, that is, a small deviation from the cosmological constant for the DE, $K$ does not diverge, and the relative speed is finite.

It is also of great importance to reconsider our assumptions about the symmetry and
model-building. For instance, we can generalize our work to include time-dependence for the scalar field $\phi$ as in $[23]$ and to refine our study on the galaxy rotation curve in the stationary spacetime solution. Regarding the tuning of the parameter for the speed of gravitational waves, it would be milder if we invoke the disformal transformation $\tilde{g}_{\mu\nu} = g_{\mu\nu} + D(X, \phi)\phi_{\mu}\phi_{\nu}$ with an appropriate expression of function $D(X, \phi)$ $[30, 52]$. By starting from the study on the scalar-field DM demonstrated in this paper, it is necessary to apply it to the other sub-classes of the Horndeski theory or beyond-Horndeski theories.

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**Appendix A: Cosmological evolution and perturbation analysis in Horndeski theory**

In Sec. IV, we follow the $\alpha$ parametrization introduced by Bellini and Sawacki $[53, 54]$, which is given as follows:

\[
M_*^2 \equiv 2(G_4 - 2XG_{4X} + XG_{5\phi} - \dot{\phi}HXG_{5X}), \quad (A1)
\]

\[
HM_*^2 \alpha_M \equiv \frac{d}{dt}M_*^2, \quad (A2)
\]

\[
H^2M_*^2 \alpha_K \equiv 2X(G_{2X} + 2XG_{2XX} - 2G_{3\phi} - 2XG_{3\phi X}) \\
+ 12\dot{\phi}XH(G_{3X} + XG_{3XX} - 3G_{4\phi X} - 2XG_{4\phi XX}) \\
+ 12XH^2(G_{4X} + 8XG_{4XX} + 4X^2G_{4XXX}) \\
- 12XH^2(G_{5\phi} + 5XG_{5\phi X} + 2X^2G_{5XX}) \\
+ 4\dot{\phi}XH^3(3G_{5X} + 7XG_{5XX} + 2X^2G_{5XXX}), \quad (A3)
\]

\[
HM_*^2 \alpha_B \equiv 2\dot{\phi}(XG_{3X} - G_{4\phi} - 2XG_{4\phi X}) \\
+ 8XH(G_{4X} + 2XG_{4XX} - G_{5\phi} - XG_{5\phi X}) \\
+ 2\dot{\phi}XH^2(3G_{5X} + 2XG_{5XX}), \quad (A4)
\]

\[
M_*^2 \alpha_T \equiv 2X(2G_{4X} - 2G_{5\phi} - (\ddot{\phi} - \dot{\phi}H)G_{5X}). \quad (A5)
\]
The the Friedman equations in the Horndeski theory are given by

\begin{align}
3H^2 &= \tilde{\rho}_m + \tilde{\rho}_\phi, \quad (A6) \\
2\dot{H} + 3H^2 &= -\tilde{p}_m - \tilde{p}_\phi, \quad (A7)
\end{align}

where \( \tilde{\rho}_m \equiv \rho_m / M_*^2 \) and \( \tilde{p}_m \equiv p_m / M_*^2 \). Then \( \tilde{\rho}_\phi \) and \( \tilde{p}_\phi \) are given by

\begin{align}
M_*^2\tilde{\rho}_\phi &= -G_2 + 2X(G_{2X} - G_{3\phi}) \\
&\quad + 6\dot{\phi}H(XG_{3X} - G_{4\phi} - 2XG_{4\phi X}) \\
&\quad + 12H^2X(G_{4X} + 2XG_{4XX} - G_{5\phi} - XG_{5\phi X}) \\
&\quad + 4\dot{\phi}H^3X(G_{5X} + XG_{5XX}), \quad (A8)
\end{align}

\begin{align}
M_*^2\tilde{p}_\phi &= G_2 - 2X(G_{3\phi} - 2G_{4\phi\phi}) \\
&\quad + 4\dot{\phi}H(G_{4\phi} - 2XG_{4\phi X} + XG_{5\phi\phi}) \\
&\quad - M_*^2\alpha_B H\dot{\phi} - 4H^2X^2G_{5\phi X} + 2\dot{\phi}H^3XG_{5X}. \quad (A9)
\end{align}

The action at quadratic order of a scalar field \( \zeta \) and tensor modes \( h_{ij} \) are given by

\begin{align}
S_2 &= \int dt d^3x a^3 \left[ Q_s \left( \dot{\xi}^2 - \frac{c_s^2}{a^2}(\partial_i \xi)^2 \right) + Q_T \left( \dot{h}_{ij}^2 - \frac{c_T^2}{a^2}(\partial_k h_{ij})^2 \right) \right], \quad (A10)
\end{align}

where

\begin{align}
Q_s &= \frac{2M_*^2D}{(2 - \alpha_B)^2}, \quad (A11) \\
c_s^2 &= \left( 2 - \alpha_B \right) \frac{\dot{H} - \frac{1}{2}H^2\alpha_B \left( 1 + \alpha_T \right) - H^2(\alpha_M - \alpha_T)}{H^2D}, \quad (A12) \\
D &\equiv \alpha_K + \frac{3}{2}\alpha_B^2,
\end{align}

while

\begin{align}
Q_T &= \frac{M_*}{8}, \quad (A13) \\
c_T^2 &= 1 + \alpha_T, \quad (A14)
\end{align}

To avoid the ghost and gradient instability, we should impose the condition that \( Q_s > 0, c_s^2 > 0, Q_T > 0 \) and \( c_T^2 > 0 \).

[1] E. Corbelli and P. Salucci, Mon. Not. Roy. Astron. Soc. 311, 441 (2000), arXiv:astro-ph/9909252 [astro-ph].
[2] A. Vikhlinin, A. Kravtsov, W. Forman, C. Jones, M. Markevitch, S. S. Murray, and L. Van Speybroeck, Astrophys. J. **640**, 691 (2006), arXiv:astro-ph/0507092 [astro-ph].

[3] D. Clowe, M. Bradac, A. H. Gonzalez, M. Markevitch, S. W. Randall, C. Jones, and D. Zaritsky, Astrophys. J. **648**, L109 (2006), arXiv:astro-ph/0608407 [astro-ph].

[4] P. A. R. Ade et al. (Planck), Astron. Astrophys. **594**, A13 (2016), arXiv:1502.01589 [astro-ph.CO].

[5] M. Kopp, C. Skordis, D. B. Thomas, and S. Ilić, Phys. Rev. Lett. **120**, 221102 (2018), arXiv:1802.09541 [astro-ph.CO].

[6] M. Rinaldi, Phys. Dark Univ. **16**, 14 (2017), arXiv:1608.03839 [gr-qc].

[7] Á. O. F. de Almeida, L. Amendola, and V. Niro, JCAP **1808**, 012 (2018), arXiv:1805.11067 [astro-ph.GA].

[8] A. P. Naik, E. Puchwein, A.-C. Davis, and C. Arnold, Mon. Not. Roy. Astron. Soc. **480**, 5211 (2018), arXiv:1805.12221 [astro-ph.CO].

[9] S. Panpanich and P. Burikham, Phys. Rev. **D98**, 064008 (2018), arXiv:1806.06271 [gr-qc].

[10] L. Sebastiani, S. Vagnozzi, and R. Myrzakulov, Adv. High Energy Phys. **2017**, 3156915 (2017), arXiv:1612.08661 [gr-qc].

[11] C. Armendariz-Picon and E. A. Lim, JCAP **0508**, 007 (2005), arXiv:astro-ph/0505207 [astro-ph].

[12] U. Nucamendi, M. Salgado, and D. Sudarsky, Phys. Rev. **D63**, 125016 (2001), arXiv:gr-qc/0011049 [gr-qc].

[13] S. Bharadwaj and S. Kar, Phys. Rev. **D68**, 023516 (2003), arXiv:astro-ph/0304504 [astro-ph].

[14] C. Deffayet, G. Esposito-Farese, and A. Vikman, Phys. Rev. **D79**, 084003 (2009), arXiv:0901.1314 [hep-th].

[15] C. Deffayet, S. Deser, and G. Esposito-Farese, Phys. Rev. **D80**, 064015 (2009), arXiv:0906.1967 [gr-qc].

[16] T. Kobayashi, M. Yamaguchi, and J. Yokoyama, Prog. Theor. Phys. **126**, 511 (2011), arXiv:1105.5723 [hep-th].

[17] G. W. Horndeski, Int. J. Theor. Phys. **10**, 363 (1974).

[18] T. Kobayashi, Rept. Prog. Phys. **82**, 086901 (2019), arXiv:1901.07183 [gr-qc].

[19] M. Rinaldi, Phys. Rev. **D86**, 084048 (2012), arXiv:1208.0103 [gr-qc].

[20] E. Babichev and C. Charmousis, JHEP **08**, 106 (2014), arXiv:1312.3204 [gr-qc].
[21] M. Minamitsuji, Phys. Rev. D89, 064017 (2014), arXiv:1312.3759 [gr-qc].
[22] A. Anabalon, A. Cisterna, and J. Oliva, Phys. Rev. D89, 084050 (2014),
arXiv:1312.3597 [gr-qc].
[23] T. Kobayashi and N. Tanahashi, PTEP 2014, 073E02 (2014), arXiv:1403.4364 [gr-qc].
[24] A. Maselli, H. O. Silva, M. Minamitsuji, and E. Berti, Phys. Rev. D92, 104049 (2015),
arXiv:1508.03044 [gr-qc].
[25] E. Babichev, C. Charmousis, and M. Hassaine, JCAP 1505, 031 (2015),
arXiv:1503.02545 [gr-qc].
[26] E. Babichev, C. Charmousis, and A. Lehébel, JCAP 1704, 027 (2017),
arXiv:1702.01938 [gr-qc].
[27] L. Sebastiani, Int. J. Geom. Meth. Mod. Phys. 15, 1850152 (2018), arXiv:1807.06592 [gr-qc].
[28] S. Appleby, JCAP 1505, 009 (2015), arXiv:1503.06768 [gr-qc].
[29] E. Babichev and G. Esposito-Farese, Phys. Rev. D95, 024020 (2017),
arXiv:1609.09798 [gr-qc].
[30] E. Babichev, C. Charmousis, G. Esposito-Farèse, and A. Lehébel,
Phys. Rev. Lett. 120, 241101 (2018), arXiv:1712.04398 [gr-qc].
[31] H. Ogawa, T. Kobayashi, and T. Suyama, Phys. Rev. D93, 064078 (2016),
arXiv:1510.07400 [gr-qc].
[32] L. Hui and A. Nicolis, Phys. Rev. Lett. 110, 241104 (2013), arXiv:1202.1296 [hep-th].
[33] T. P. Sotiriou and S.-Y. Zhou, Phys. Rev. Lett. 112, 251102 (2014), arXiv:1312.3622 [gr-qc].
[34] O. J. Tattersall, P. G. Ferreira, and M. Lagos, Phys. Rev. D97, 084005 (2018),
arXiv:1802.08606 [gr-qc].
[35] E. Babichev, C. Charmousis, and A. Lehébel, Class. Quant. Grav. 33, 154002 (2016),
arXiv:1604.06402 [gr-qc].
[36] P. S. Letelier, Phys. Rev. D22, 807 (1980).
[37] S. Bayin, Astrophys. J. 303, 101 (1986).
[38] D. Dey, K. Bhattacharya, and T. Sarkar, Phys. Rev. D87, 103505 (2013),
arXiv:1304.2598 [astro-ph.GA].
[39] For non-rotating black hole solution, \( h(r_h) = f(r_h) = 0 \) while \( h(r_h)/f(r_h) \) is still finite, so
our solution does not change. A well-known example is the Schwarzschild solution where
\( h(r) = f(r) = 1 - 2GM/r \).
[40] N. Arkani-Hamed, H.-C. Cheng, M. A. Luty, and S. Mukohyama, JHEP 05, 074 (2004), arXiv:hep-th/0312099 [hep-th].
[41] N. Arkani-Hamed, P. Creminelli, S. Mukohyama, and M. Zaldarriaga, JCAP 0404, 001 (2004), arXiv:hep-th/0312100 [hep-th].
[42] R. M. Wald, General Relativity (Chicago Univ. Pr., Chicago, USA, 1984).
[43] Y. Sofue and V. Rubin, Ann. Rev. Astron. Astrophys. 39, 137 (2001), arXiv:astro-ph/0010594 [astro-ph].
[44] F. Lelli, S. S. McGaugh, and J. M. Schombert, Astron. J. 152, 157 (2016), arXiv:1606.09251 [astro-ph.GA].
[45] C. M. Will, Living Rev. Rel. 17, 4 (2014), arXiv:1403.7377 [gr-qc].
[46] K. Jusufi, A. Ovgün, and A. Banerjee, Phys. Rev. D96, 084036 (2017), [Addendum: Phys. Rev.D96,no.8,089904(2017)], arXiv:1707.01416 [gr-qc].
[47] A. Casalino and M. Rinaldi, Phys. Dark Univ. 23, 100243 (2019), arXiv:1807.01995 [gr-qc].
[48] R. J. Scherrer, Phys. Rev. Lett. 93, 011301 (2004), arXiv:astro-ph/0402316 [astro-ph].
[49] D. Bertacca, N. Bartolo, and S. Matarrese, Adv. Astron. 2010, 904379 (2010), arXiv:1008.0614 [astro-ph.CO].
[50] M. Kunz, S. Nesseris, and I. Sawicki, Phys. Rev. D94, 023510 (2016), arXiv:1604.05701 [astro-ph.CO].
[51] B. P. Abbott et al. (LIGO Scientific, Virgo, Fermi GBM, INTEGRAL, IceCube, AstroSat Cadmium Zinc Telluride Imager Team, IPN, Insight-Hxmt, ANTARES, Swift, AGILE Team, 1M2H Team, Dark Energy Camera GW-EM, DES, DLT40, GRAWITA, Fermi-LAT, ATCA, ASKAP, Las Cumbres Observatory Group, OzGrav, DWF (Deeper Wider Faster Program), AST3, CAASTRO, VINROUGE, MASTER, J-GEM, GROWTH, JAGWAR, CaltechNRAO, TTU-NRAO, NuSTAR, Pan-STARRS, MAXI Team, TZAC Consortium, KU, Nordic Optical Telescope, ePESSTO, GROND, Texas Tech University, SALT Group, TOROS, BOOTES, MWA, CALET, IKI-GW Follow-up, H.E.S.S., LOFAR, LWA, HAWC, Pierre Auger, ALMA, Euro VLBI Team, Pi of Sky, Chandra Team at McGill University, DFN, ATLAS Telescopes, High Time Resolution Universe Survey, RIMAS, RATIR, SKA South Africa/MeerKAT), Astrophys. J. 848, L12 (2017), arXiv:1710.05833 [astro-ph.HE].
[52] J. M. Ezquiaga and M. Zumalacárregui, Phys. Rev. Lett. 119, 251304 (2017), arXiv:1710.05901 [astro-ph.CO].
[53] E. Bellini and I. Sawicki, JCAP 1407, 050 (2014), arXiv:1404.3713 [astro-ph.CO].

[54] S. Arai and A. Nishizawa, Phys. Rev. D97, 104038 (2018), arXiv:1711.03776 [gr-qc].