SL(2,R)-DUALITY COVARIANCE
OF KILLING SPINORS IN
AXION-DILATON BLACK HOLES

Tomás Ortín
Department of Physics
Queen Mary and Westfield College
Mile End Road, London E1 4NS, U.K.

Abstract

Under SL(2,R) electric-magnetic duality transformations the Bogomolnyi bound of dilaton-axion black holes is known to be invariant. In this paper we show that this invariance corresponds to the covariance of the $N = 4$ supersymmetry transformation rules and their parameters. In particular this implies that Killing spinors transform covariantly into Killing spinors. As an example, we work out completely the case of the largest known family of axion-dilaton black holes which is $SL(2,R)$-invariant, finding the Killing spinors with the announced properties.

---

1E-mail address: ortin@qmph.cern.ch
Introduction

The \(SL(2,R)\)-duality invariance of the low-energy effective string theory equations of motion \([1, 2, 3]\) \((N = 4\) supergravity \([4]\)) is a fascinating symmetry that interchanges the strong and weak coupling regimes. Quantum effects (instantons \([1]\) or dyon charge quantization \([5, 6]\)) effectively break this symmetry to \(SL(2, Z)\).

It was conjectured in \([7]\) and more recently in \([2]\) that \(SL(2, Z)\) could be an exact, non-perturbative, symmetry of superstring theory. This idea is supported by the fact that the spectrum of dyon charges \([6]\) is \(SL(2, Z)\)-invariant. It is also extremely interesting the interplay between the \(SL(2, Z)\)-duality symmetry and other symmetries of compactified string theory: the \(O(6,22; Z)\) symmetry, target-space duality and the string-fivebrane duality that has been explored in \([8]\) and \([9]\).

In this letter we are going study the behavior of some supersymmetry properties under general \(SL(2, R)\)-duality transformations. In particular we will show in Sec. 1 the covariance of the \(N = 4\) supergravity transformation laws of the gravitino and dilatino fields under \(SL(2, R)\)-duality transformations. As a consequence, Killing spinors transform covariantly and the number of unbroken supersymmetries of \(SL(2, R)\)-related on-shell field configurations is invariant. Furthermore, the positivity bounds associated to the supersymmetry algebra are also invariant. Some of these results were hinted or proven in special cases in Ref. \([10]\) and in a different context in \([11]\).

In Sec. 2 we study the particular case of the \(SL(2, R)\)-invariant family of dilaton-axion black holes recently found in Ref. \([6]\). We find the Killing spinors and check that their behavior under duality agrees with the results of Sec. 1. The invariance of the Bogomolnyi-Gibbons-Hull-type bound for axion-dilaton black holes is built in the solutions and it is, anyway, a straightforward generalization of the results of Ref. \([10]\). Therefore we will nos discuss it.

Our conclusions are in the last section.

Throughout all this paper we use the conventions and notations of Refs. \([12, 6]\), but our definitions of the charges differ slightly. The Appendix contains, for the sake of completeness, a description of the axion-dilaton black holes and the definitions of charges we are using.
1 \ SL(2, R)-covariant Killing spinors

In the \( SU(4) \) version of \( N = 4 \) supergravity, the bosonic part of the super-symmetry rules of the gravitino and dilatino fields are respectively

\[
\frac{1}{2} \delta \epsilon \Psi_{\mu I} = \nabla_\mu \epsilon_I - \frac{i}{4} e^{2\phi} \partial_\mu \epsilon_I - \frac{1}{8} \sigma^{\rho \sigma} T^{+}_{\rho \sigma I J} \gamma_\rho \epsilon^J ,
\]

\[
\frac{1}{2} \delta \epsilon \Lambda_I = -\frac{i}{2} (e^{2\phi} \partial \lambda) \epsilon_I + \frac{1}{4} \sigma^{\rho \sigma} T^{-}_{\rho \sigma I J} \epsilon^J ,
\]

where

\[
\lambda = a + i e^{-2\phi} ,
\]

is a complex scalar field built out of \( \phi \), the dilaton field, and \( a \), the axion field. Also

\[
T_{\rho \sigma I J} = 2 \sqrt{2} e^{-\phi} [ F_{\rho \sigma} \alpha_{I J} + i G_{\rho \sigma} \beta_{I J}] ,
\]

where \( F_{\rho \sigma} = \partial_\rho A_\sigma - \partial_\sigma A_\rho \) and \( G_{\rho \sigma} = \partial_\rho B_\sigma - \partial_\sigma B_\rho \) are the field strength tensors of two \( U(1) \) vector fields \( A_\mu \) and \( B_\mu \). Finally, \( \alpha_{I J} \) and \( \beta_{I J} \) are two \( SO(4) \) matrices with \( SO(4) \) indices \( I, J \).

Given any \( SL(2, R) \) matrix \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \), \( \alpha \delta - \beta \gamma = 1 \) and the complex scalar field \( \lambda \) of the original field configuration we can define two functions

\[
R = \alpha \lambda + \beta ,
\]

\[
S = \gamma \lambda + \delta .
\]

In terms of \( R \) and \( S \) and the old field configurations, the transformed fields can be conveniently written in this way:

\[
\lambda' = R/S ,
\]

\[
F^{\pm}_{\rho \sigma} = SF^{\pm}_{\rho \sigma} ,
\]

and the analogous equation for \( G \). In the Einstein frame (which we are using) \( SL(2, R) \) does not act on the metric.

These transformations rotate continuously the “Maxwell law” into the Bianchi identity, hence the name electric-magnetic duality.

It is a well known fact (Refs. [4, 2]) that the equations of motion of this theory are invariant under the above transformations while the action is not.
In addition to this, it is easy to see that the equations (1) and (2) conserve their form (i.e. transform covariantly under $SL(2,R)$) if we assume the following transformation laws for the supersymmetry parameters and variations

$$\epsilon_I' = e^{\frac{i}{2} \text{Arg}(S)} \epsilon_I,$$

$$\left(\delta_{\epsilon} \Psi_{\mu I}\right)' = e^{\frac{i}{2} \text{Arg}(S)} \delta_{\epsilon} \Psi_{\mu I},$$

$$\left(\delta_{\epsilon} \Lambda_I\right)' = e^{-3i \frac{\text{Arg}(S)}{2}} \delta_{\epsilon} \Lambda_I.$$

There is no need to use the equations of motion to prove this result and, thus, it holds for any on- or off-shell field configuration.

Although this result may not be too surprising we would like to stress that it is far from being trivial. $SL(2,R)$ is only an invariance of the classical equations of motion and so it transforms on-shell configurations into on-shell configurations. There is no apparent reason why arbitrary field configurations should behave nicely under $SL(2,R)$. Furthermore, the supersymmetry parameter $\epsilon$ is not even a field of the theory.

Observe that the transformation (7) is consistent with the invariance of the metric and (accordingly) of the Killing vector $\xi^\mu$ that can be built out of a (commuting) set of Killing spinors $\epsilon_I$:

$$\xi^\mu = \bar{\epsilon} I \gamma^\mu \epsilon_I.$$  

Now let us study some consequences of this result.

Let us assume that for a particular field configuration the equations $\delta_{\epsilon} \Psi_{\mu I} = \delta_{\epsilon} \Lambda_I = 0$ are satisfied by some set of spinors $\epsilon_I$, some (perhaps all) of them trivial (i.e. vanishing). We call this set of spinors “Killing spinors”. Then the above result implies that in the duality-transformed field configurations these equations are also satisfied by another set of Killing spinors, the number of trivial (i.e. vanishing) and non-trivial Killing spinors being the same as in the original field configuration.

On-shell configurations admitting non-trivial asymptotically constant Killing spinors are said to have unbroken supersymmetries. Hence, the number of unbroken supersymmetries is the same for any two $SL(2,R)$-related classical solutions.
For off-shell configurations the situation is subtly different. They obey equations of the form

\[
\begin{align*}
\frac{\delta S_{\text{Class}}}{\delta g^{\mu\nu}} &= J_{\mu\nu}, \\
\frac{\delta S_{\text{Class}}}{\delta A_\mu} &= J_A^\mu, \\
\frac{\delta S_{\text{Class}}}{\delta B_\mu} &= J_B^\mu, \\
\frac{\delta S_{\text{Class}}}{\delta \lambda} &= J,
\end{align*}
\]

(11)

where \(S_{\text{Class}}\) is the classical action

\[
S_{\text{Class}} = \int d^4x \sqrt{-g} \{-R - 2 \frac{\partial_\mu \lambda \partial^{\mu} \lambda}{(\lambda - \bar{\lambda})^2} + [i\lambda(F^{+2} + G^{+2}) + \text{c.c.}]\} 
\]

(12)

and \(J_{\mu\nu}, J_A^\mu, J_B^\mu, J\) are non-vanishing functions (otherwise the configurations would be on-shell) that may describe additional matter sources or quantum corrections, for instance. The fact that a field configuration admits Killing spinors does not automatically imply that it obeys the classical equations of motion\(^2\) It does not imply, either, that the sources in the right-hand side of eqs. (11) are coupled in a way consistent with supersymmetry.

In Ref. [14] a set of consistency conditions for these sources to be consistent with supersymmetry was derived. In our case, these “Killing Spinor Identities” (KSI) take the form

\[
\begin{align*}
\frac{e^\phi}{\sqrt{2}} [J_A^\mu \alpha^{IJ} + i\gamma_5 J_B^\mu \beta^{IJ}] \tau_I \gamma_\mu + 2iJ e^{-2\phi} \epsilon^I &= 0, \\
2J^{\mu\nu} \tau^I \gamma_\nu + \frac{e^\phi}{\sqrt{2}} [J_A^\mu \alpha^{IJ} + i\gamma_5 J_B^\mu \beta^{IJ}] \tau_I &= 0,
\end{align*}
\]

(13)

where \(\epsilon^I\) is a set of Killing spinors.

\(^2\)It does not even imply that we have a “good” field configuration. A set of fields \(\lambda, F_{\mu\nu}, G_{\mu\nu}, g_{\mu\nu}\) may admit Killing spinors and still it may not exist a vector fields \(A_\mu, B_\mu\) (which are the true dynamical fields) such that \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) and \(G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\). See Ref. [13] for clear examples. In what follows we are going to exclude this possibility, so we will always have \(\nabla^\mu F_{\mu\nu} = \nabla^\mu G_{\mu\nu} = 0\).
Under an $SL(2, R)$ rotation of the field configuration the sources transform as follows:

\[
\begin{align*}
(J_{\mu \nu})' &= J_{\mu \nu}, \\
(J_{A,B}^{\mu})' &= \alpha J_{A,B}^{\mu}, \\
J' &= S^2 J.
\end{align*}
\]

It is obvious that the KSI will generally be violated after a general $SL(2, R)$ rotation, unless the sources vanish. This result is obviously related to the fact that a non-vanishing electric current $J$ produces a non-vanishing magnetic current after the rotation.

Thus, in general, only the unbroken supersymmetries of on-shell field configurations will be preserved by an electric-magnetic $SL(2, R)$ duality transformation.

This result can be read from a slightly different point of view, as in Ref. [15]. If the sources come from quantum corrections and we select only supersymmetric configurations, then only those with no quantum corrections will survive. In our case ($N = 4$) there is no trace anomaly and we don’t know what the quantum corrections would be like, but this is clearly a subject worth studying.

Finally, one is tempted to conjecture that in a supersymmetric theory with manifest $SL(2, R)$ invariant couplings the number of unbroken supersymmetries would always be invariant, but such a theory does not yet exist.

2  The Killing spinors of axion-dilaton black holes

Recently a quite general family of static black-hole solutions of the low-energy string theory equations of motion has been presented in [6]. The main feature of this family, which contains many already known solutions [16, 1, 12], is that it constitutes a representation of the whole $SL(2, R)$-duality group in the sense that by applying any $SL(2, R)$ transformation to any solution in the family we get another member of the family.\footnote{\textit{SL}(2, R) does not act irreducibly on this family. In particular, as we are going to see, this family contains backgrounds with different numbers of unbroken supersymmetries,}

---

5
Our immediate goal is to find the non-trivial Killing spinors of this family of configurations and check explicitly Eq. (7). We already know that the non-extreme black-hole solutions in this class do not have non-trivial Killing spinors [12], so we will study only the solutions describing several extreme black holes in equilibrium, which can be found in the Appendix.

Note that only 6 of the $LU(1)$ fields of the general solution fit into pure $N = 4$ supergravity and we will take only 2 to be non-vanishing as in the previous section: $A^{(1)}_{\mu} \equiv A_{\mu}, A^{(2)}_{\mu} \equiv B_{\mu}$.

We are looking for time-independent ($\partial_t \epsilon_I = 0$) solutions of the equations

\begin{align}
\delta_\epsilon \Psi_{\mu I} &= 0, \quad (15) \\
\delta_\epsilon \Lambda_I &= 0. \quad (16)
\end{align}

First we contract Eq. (15) with $\gamma_\mu$ and obtain

$$\nabla \epsilon_I - \frac{i}{4} e^{2\phi} \partial_a \epsilon_I = 0. \quad (17)$$

For time-independent Killing spinors of the above backgrounds this equation reduces to

$$\gamma^i \partial_i [e^{-\frac{i}{2}(U+i\theta)} \epsilon_I] = 0, \quad (18)$$

where

$$\mathcal{H}_2 / \mathcal{H}_2 = e^{2i\theta}, \quad (19)$$

and we have used the property of these solutions

$$\partial_t \ln(\mathcal{H}_2 / \mathcal{H}_2) = ie^{2\phi} \partial_a. \quad (20)$$

If we apply the operator $\gamma^i \partial_i$ to Eq. (18) we get the integrability condition

$$\partial_i \partial_t [e^{-\frac{i}{2}(U+i\theta)} \epsilon_I] = 0, \quad (21)$$

which means that the combination in brackets is a harmonic spinor. If we substitute now in Eq. (18) a general harmonic spinor we see that only a constant one satisfies it, and thus

$$\epsilon_I = e^{-\frac{i}{2}(U+i\theta)} \epsilon_I(0), \quad (22)$$

and, as a consequence of the results of the previous section, the corresponding subfamilies provide smaller representations of the duality group. Further subdivisions are labeled by non-duality equivalent values of parameters like the asymptotic value of asymptotic of $\lambda$. We won’t pursue this issue farther in this letter.
where the $\epsilon_{I(0)}$s are constant spinors.

This result allows us to check partially the main result in the previous section. To do this, first observe that under $SL(2, R)$ the functions $\mathcal{H}_1$ and $\mathcal{H}_2$ a transform in this way:

\[
\mathcal{H}_1' = \frac{\alpha \mathcal{H}_1 + \beta \mathcal{H}_2}{\gamma \lambda_0 + \delta}, \\
\mathcal{H}_2' = \frac{\gamma \mathcal{H}_1 + \delta \mathcal{H}_2}{\gamma \lambda_0 + \delta}.
\]

Then, according to Eq. (22)

\[
\epsilon_I' = e^{\frac{i}{2}(U + i\theta')}\epsilon_{I(0)}
\]

\[
= e^{\frac{i}{2}U} \left( \frac{\mathcal{H}_2'}{\mathcal{H}_2} \right)^{\frac{1}{4}} \left( \frac{\gamma \lambda_0 + \delta}{\gamma \lambda_0 + \delta} \right)^{\frac{1}{4}} \epsilon_{I(0)}
\]

\[
= e^{\frac{i}{2} \text{Arg}(S)} e^{\frac{i}{2}(U + i\theta')} \epsilon_{I(0)}
\]

and, looking at Eqs. (7) and (22) we see that we only need to check that the asymptotic value of the Killing spinors transforms as follows:

\[
\epsilon_{I(0)}' = e^{\frac{i}{2} \text{Arg}(S)} \epsilon_{I(0)}
\]

where $S_0$ is the value of $S$ at infinity.

We only need to find the constant spinors. It is enough to consider the equations $\delta_i \Psi_{I} = \delta_i \Lambda_I = 0$. A long but straightforward calculation leads to the following two equations for each black hole:

\[
\frac{ie^{\phi_0}}{\sqrt{2}} [\lambda_0 M_i + \bar{\lambda}_0 \Upsilon_i] \epsilon_{I(0)} - \{\bar{Q}_i\}_{IJ} \gamma^0 \epsilon_{J(0)}' = 0,
\]

\[
\frac{ie^{\phi_0}}{\sqrt{2}} [M_i + \Upsilon_i] \epsilon_{I(0)} - \{P_i\}_{IJ} \gamma^0 \epsilon_{J(0)}' = 0,
\]

where we have used abbreviated notation

\[
\{X\}_{IJ} = \{X^A \alpha_{IJ} + X^B \beta_{IJ}\},
\]

for any quantity $X$ associated to the vector fields $A_\mu$ and $B_\mu$, the only ones we are considering here.
The algebraic constraints Eqs. (26) tell us that the whole solution will have at most as many unbroken supersymmetries as the single black hole with less unbroken supersymmetries. Note that Eqs. (35), (36) and (37) guarantee that both Eqs. (26) are always compatible for all black holes at once.

After using the explicit form of the $SO(4)$ matrices $\alpha_{IJ}$ and $\beta_{IJ}$ (see Ref. [12]) we can describe the final result as follows. First we define the central charges

$$z_\pm = \sqrt{2} e^{-\phi_0} (\Gamma^A \pm i \Gamma^B).$$

There are three different cases

1. One unbroken supersymmetry in the 1, 2 sector with the following relations holding for each black hole and the following spinors:

$$M_i = |z_{-i}|,$$
$$|\Upsilon_i| = |z_{+i}|,$$
$$\mu_- = -i\sqrt{2} e^{-\phi_0} \frac{Q^A - i\tilde{Q}^B}{\lambda_0 M + \bar{\lambda}_0 \Upsilon} = -i\sqrt{2} e^{-\phi_0} \frac{P^A - iP^B}{M + \Upsilon},$$
$$\epsilon_{3(0)} = \epsilon_{4(0)} = 0,$$
$$\epsilon_{1(0)} = \mu - i\gamma^0 \epsilon_{2(0)}.$$

2. One unbroken supersymmetry in the 3, 4 sector with the following relations holding for each black hole and the following spinors:

$$M_i = |z_{+i}|,$$
$$|\Upsilon_i| = |z_{-i}|,$$
$$\mu_+ = -i\sqrt{2} e^{-\phi_0} \frac{Q^A + i\tilde{Q}^B}{\lambda_0 M + \bar{\lambda}_0 \Upsilon} = -i\sqrt{2} e^{-\phi_0} \frac{P^A + iP^B}{M + \Upsilon},$$
$$\epsilon_{1(0)} = \epsilon_{2(0)} = 0,$$
$$\epsilon_{3(0)} = \mu + i\gamma^0 \epsilon_{2(0)}.$$

3. Two unbroken supersymmetries, one in the 1, 2 sector and one in the 3, 4 sector with the following relations holding for each black hole and
the following spinors:

\[ M_i = |\Upsilon_i| = |z_{+i}| = |z_{-i}|, \]

\[ \epsilon_{1(0)} = \mu_{+i} \epsilon_{2(0)} , \]

\[ \epsilon_{3(0)} = \mu_{+i} \gamma^0 A_i \epsilon_{(0)}. \]  (31)

For just one black hole, we always have supersymmetry. When there is more than just one, the situation is not so clear. To have supersymmetry the complex constants \( \mu_{+i} \) or \( \mu_{-i} \) must have the same value for the \( N \) black holes. That this is indeed possible is clear: just take for one of them a set of charges \((M_i, \Sigma_i, \Delta_i, \varphi_i^A, Q_i^B, P_i^A, P_i^B)\) that satisfies the Bogomolnyi-Gibbons-Hull bound Eq. (35) and take for the rest of the black holes sets of charges proportional to this one. This subclass of configurations has at least one unbroken supersymmetry. However, due to the large amount of charges and the involved relations between them we haven’t been able to prove that all of them are supersymmetric\(^4\) although this seems likely.

Back to our main problem, it’s easy to see that the constants \( \mu_{\pm} \) transform exactly in the form required by the result of the previous section, or, equivalently, by Eq. (25)

\[ \mu'_{\pm} = e^{i \text{Arg}(S_0)} \mu_{\pm}. \]  (32)

This illustrates the main result of this paper.

### 3 Conclusions

In this letter we have proven that if a field configuration admits Killing spinors, then all of its \( SL(2, R) \) images do do, and that the Killing spinors of the images are related very simply to those of the original. This property is related to the \( SL(2, R) \) invariance of the Bogomolnyi-Gibbons-Hull bound of

\(^4\)One might naively think that it would suffice to use the result of the first section, performing a duality transformation that would take us to purely electric or magnetic multi-black-hole solutions. However, these solutions were found through a duality rotation of multi-black-hole solutions with 2 electric and 2 magnetic charges but no axion. Thus, we could get rid of the axion charge at most. That simplification is not enough, and perhaps, a better expression for these solutions is need here.
axion-dilaton black holes. Since this symmetry does not act on the (Einstein-frame) metric, this result provides further examples of supersymmetry acting as a cosmic censor, although one needs to check some consistency conditions (the Killing Spinor Identities) to see whether supersymmetry is preserved when quantum corrections are taken into account.

There is a number of other duality symmetries in String Theory, and it would be interesting to find whether similar properties hold for them. Target-space duality, for instance, can be seen also as a symmetry of the low-energy String Theory equations of motion, which can be embedded in a theory with local supersymmetry. Although it looks very different from $SL(2,R)$ duality, there are some examples in which unbroken supersymmetries were preserved by it \cite{[17]}. Note that $SL(2,R)$ does transform the stringy metric too. We hope to report on results on this problem soon.

**Acknowledgements**

The author wish to thank R. Kallosh for many discussions and her support. This work has been partially supported by a Spanish Government MEC postdoctoral grant and by a postdoctoral European Communities Human Capital and Mobility program grant.

**A Axion-dilaton black-hole solutions**

Here we briefly describe the extreme axion-dilaton multi-black-hole solutions found in Ref. \cite{[6]}. Our definitions of the charges differ slightly from those of that reference.

Apart from the metric and complex scalar field these solutions have $L U(1)$ fields $F_{\mu\nu}^{(n)} = \partial_\mu A_\nu^{(n)} - \partial_\nu A_\mu^{(n)}$.

We also use the auxiliary fields (the $SL(2,R)$-duals) $\tilde{F}_{\mu\nu}^{(n)} = e^{-2\phi} F_{\mu\nu}^{(n)} - ia F_{\mu\nu}^{(n)}$. Owing to the “Maxwell law”, which can be written $\nabla^\mu \tilde{F}_{\mu\nu} = 0$, there exist locally $L$ real vector fields $A_\mu^{(n)}$ such that $\tilde{F}_{\mu\nu} = i(\partial_\mu A_\nu^{(n)} - \partial_\nu A_\mu^{(n)})$. They simplify the description of the solutions, which are given by

$$ds^2 = e^{2U} dt^2 - e^{-2U} d\vec{x}^2 ,$$
\begin{align}
e^{-2U}(\vec{x}) &= 2 \text{Im} \left( \mathcal{H}_1(\vec{x}) \overline{\mathcal{H}_2(\vec{x})} \right), \\
\lambda(\vec{x}) &= \frac{\mathcal{H}_1(\vec{x})}{\mathcal{H}_2(\vec{x})}, \\
A^{(n)}(\vec{x}) &= e^{2U}(k^{(n)}\mathcal{H}_2(\vec{x}) + \text{c.c.}), \\
\tilde{A}^{(n)}(\vec{x}) &= -e^{2U}(k^{(n)}\mathcal{H}_1(\vec{x}) + \text{c.c.}),
\end{align}

where \( \mathcal{H}_1(\vec{x}), \mathcal{H}_2(\vec{x}) \) are two complex harmonic functions with \( N \) poles corresponding to \( N \) labeled by \( i = 1, \ldots, N \)

\begin{align}
\mathcal{H}_1(\vec{x}) &= e^{\phi_0} \sqrt{2} \left\{ \lambda_0 + \sum_{i=1}^{N} \frac{\lambda_i M_i + \overline{\lambda}_0 \Upsilon_i}{|\vec{x} - \vec{x}_i|} \right\}, \\
\mathcal{H}_2(\vec{x}) &= e^{\phi_0} \sqrt{2} \left\{ 1 + \sum_{i=1}^{N} \frac{M_i + \Upsilon_i}{|\vec{x} - \vec{x}_i|} \right\}, \\
\partial_i \partial_i \mathcal{H}_1 &= \partial_i \partial_i \mathcal{H}_1 = 0.
\end{align}

The different constants, which are defined later, have to satisfy several identities. First, they have to satisfy Bogomolnyi-Gibbons-Hull-type identities for each black hole and for the global solution:

\begin{align}
M_i^2 + |\Upsilon_i|^2 + 2i(\lambda_0 - \overline{\lambda}_0) \sum_{n=1}^{L} |\Gamma^{(n)}_i|^2 &= 0, \\
M^2 + |\Upsilon|^2 + 2i(\lambda_0 - \overline{\lambda}_0) \sum_{n=1}^{L} |\Gamma^{(n)}|^2 &= 0.
\end{align}

Secondly, the consistency of the solution requires

\begin{align}
k^{(n)} &= -\sqrt{2} e^{-\phi_0} \frac{\Gamma(n) M + \overline{\Gamma}(n) \Upsilon}{M^2 - |\Upsilon|^2} = k^{(n)}_i, \\
\text{Arg}(\Upsilon) &= \text{Arg}(\Upsilon_i),
\end{align}

for each black hole \( i \).

Finally, \( \Upsilon \) and \( \Gamma \) are related by

\begin{align}
\Upsilon = i(\lambda_0 - \overline{\lambda}_0) \sum_{n=1}^{L} \frac{\Gamma^{(n)}_i^2}{M}.
\end{align}
for each black hole and for the global solutions.

The different constants are defined by the asymptotic expansions in the “upper sheet” (global charges) and in the $i^{th}$ black hole sheet:

$$g_{tt} \sim 1 - \frac{2M}{r}.$$  

$$F^{(n)+} \sim \frac{\Gamma^{(n)}}{r^2},$$  

$$\tilde{F}^{(n)+} \sim \frac{\tilde{\Gamma}^{(n)}}{r^2},$$  

$$\lambda \sim \lambda_0 - (\lambda_0 - \bar{\lambda}_0) \frac{\Upsilon}{r}.$$  

(38)

We can write these complex constants in terms of real constants: electric ($Q$) and magnetic ($P$) charges, dilaton ($\Sigma$) and axion ($\Delta$) charges and dilaton $\phi_0$ and axion $a_0$ asymptotic values:

$$\Gamma^{(n)} = \frac{1}{2}(Q^{(n)} + iP^{(n)}),$$  

$$\Upsilon = \Delta - i\Sigma,$$  

$$\lambda_0 = a_0 + ie^{-2\phi_0}.$$  

(39)

The charges in the upper sheet (the global charges) are the sum of the charges of the $N$ black holes.

References

[1] A. Shapere, S. Trivedi and F. Wilczek, Mod. Phys. Lett. A6, 2677 (1991).

[2] A. Sen, Nucl. Phys. B404, 109 (1993).

[3] J. Maharana and J.H. Schwarz, Nucl. Phys. B390, 3 (1993); J.H. Schwarz, preprint CALT-68-1815, hepth/9209125, presented at the International Workshop on String Theory, Quantum Gravity and the Unification of Fundamental Interactions, Rome, Italy, 21-26 Sep. 1992.
[4] S. Ferrara, J. Scherk and B. Zumino, Nucl. Phys. B121, 393 (1977); E. Cremmer, J. Scherk and S. Ferrara, Phys. Lett. 74B, 61 (1978); B. de Wit, Nucl. Phys. B158, 189 (1979); E. Cremmer and B. Julia, Nucl. Phys. B159, 141 (1979); M.K. Gaillard and B. Zumino, Nucl. Phys. B193, 221 (1981).

[5] A. Sen, Phys. Lett. 303B, 22 (1993).

[6] R. Kallosh and T. Ortín, Phys. Rev. D48 742-747 (1993).

[7] A. Font, L. Ibáñez, D. Lüst and F. Quevedo, Phys. Lett. 249B, 35 (1990); S.J. Rey, Phys. Rev. D43, 526 (1991).

[8] J.H. Schwarz and A. Sen, Phys. Lett. 312B, 105 (1993); Nucl. Phys. B411, 35 (1994).

[9] P. Binétruy, Phys. Lett. 315B, 80 (1993).

[10] T. Ortín, Phys. Rev. D47, 3136 (1993).

[11] A. Sen, Mod. Phys. Lett. A8, 2023 (1993).

[12] R. Kallosh, A. Linde, T. Ortín, A. Peet, and A. Van Proeyen, Phys. Rev. D46, 5278 (1992).

[13] K.P. Tod, Phys. Lett. 121B, 241 (1983).

[14] R. Kallosh and T. Ortín, *Killing spinor identities*, Stanford University Report SU-ITP-93-16, hepth/9306085.

[15] R. Kallosh and T. Ortín, *Supersymmetry, trace anomaly and naked singularities*, Isaac Newton Institute Report NI94002, Queen Mary and Westfield College Report QMW-PH-94-7, Stanford University Report SU-ITP-94-8, hepth/9404006.

[16] G. W. Gibbons, Nucl. Phys. B207, 337 (1982); G. W. Gibbons and K. Maeda, Nucl. Phys. B298, 741 (1988); D. Garfinkle, G. Horowitz and A. Strominger, Phys. Rev. D43, 3140 (1991).
[17] E. Bergshoeff, I. Entrop and R. Kallosh, *Exact duality in string effective action*, University of Groningen Report UG-8/93, Stanford University Report SU-ITP-93-37, and hep-th/9401025.