Vacuum structure in supersymmetric Yang–Mills theories with any gauge group.

V. G. Kac

IHÉS, Le Bois–Marie, 35 route de Chartres, F-91440, Bures-sur-Yvette, France
and
Department of Mathematics, M.I.T., Cambridge, Massachusetts 02139, USA

and A.V. Smilga

Université de Nantes, 2, Rue de la Houssinière, BP 92208, F-44322, Nantes CEDEX 3, France
and
ITEP, B. Cheremushkinskaya 25, Moscow 117218, Russia

Abstract

We consider the pure supersymmetric Yang–Mills theories placed on a small 3-dimensional spatial torus with higher orthogonal and exceptional gauge groups. The problem of constructing the quantum vacuum states is reduced to a pure mathematical problem of classifying the flat connections on $T^3$. The latter problem is equivalent to the problem of classification of commuting triples of elements in a connected simply connected compact Lie group which is solved in this paper. In particular, we show that for higher orthogonal $SO(N), N \geq 7$, and for all exceptional groups the moduli space of flat connections involves several distinct connected components. The total number of vacuum states is given in all cases by the dual Coxeter number of the group which agrees with the result obtained earlier with the instanton technique.

1 Introduction

It was shown by Witten long time ago \cite{1} that, in a pure $N = 1$ supersymmetric gauge theory with any simple gauge group, the supersymmetry is not broken spontaneously. Placing the theory in a finite spatial box, the number of supersymmetric vacuum states [the Witten index $\text{Tr}(-1)^F$] was calculated to be $\text{Tr}(-1)^F = r + 1$ where $r$ is the rank of the gauge group. This result conforms with other estimates for $\text{Tr}(-1)^F$ for unitary and symplectic groups. For higher orthogonal and exceptional groups, it disagrees, however, with the general result \cite{1}

$$\text{Tr}(-1)^F = h^\vee,$$

(1.1)

where $h^\vee$ is the dual Coxeter number of the group (see e.g. \cite{3}, Chapt. 6; it coincides with the Casimir $T^aT^a$ in the adjoint representation when a proper normalization is chosen.). For $SO(N \geq 7)$, $h^\vee = N - 2 > r + 1$. Also for exceptional groups $G_2, F_4, E_6, 7, 8$, the index \cite{1} is larger than Witten’s original estimate (see Table 1).

\footnote{which follows e.g. from the counting of gluino zero modes on the instanton background and also from the analysis of weakly coupled theories with additional matter supermultiplets \cite{1, 2}.}
This paradox persisting for more than 15 years has been recently resolved by Witten himself in the case of orthogonal groups \([4]\). He has found a flaw in his original arguments and shown that, for \(SO(N \geq 7)\), vacuum moduli space is richer than it was thought before so that the total number of quantum vacua is \(N - 2\) in accordance with the result (1.1). In recent paper \([5]\), this result was confirmed, and also the theory with the \(G_2\) gauge group was analyzed where an extra vacuum state has been found. The aim of our paper is to develop a general method to construct the moduli space of flat connections for any gauge group including higher exceptional groups \(F_4, E_{6,7,8}\). As was anticipated in \([5]\), the moduli space for these groups involves not one and not two, but many distinct components (up to 12 components for \(E_8\)). The total number of quantum vacuum states coincides with \(h^\vee\) in all cases.

Let us first recall briefly Witten’s original reasoning.

- Put our theory on the spatial 3D torus and impose periodic boundary conditions on the gauge fields \([3]\). Choose the gauge \(A_0^a = 0\). A classical vacuum is defined as a gauge field configuration \(A_i^a(x, y, z)\) with zero field strength (a flat connection in mathematical language).

- For any flat periodic connection, we can pick out a particular point in our torus \((0, 0, 0) \equiv (L, 0, 0) \equiv \ldots\) and define a set of holonomies (Wilson loops along non-trivial cycles of the torus)

\[
\Omega_1 = P \exp \left\{ i \int_0^L A_1(x, 0, 0) dx \right\},
\]

\[
\Omega_2 = P \exp \left\{ i \int_0^L A_2(0, y, 0) dy \right\},
\]

\[
\Omega_3 = P \exp \left\{ i \int_0^L A_3(0, 0, z) dz \right\},
\]

(\(A_i = A_i^a T^a\) where \(T^a\) are the group generators in a given representation). \(\text{Tr}\{\Omega_i\}\) are invariant under periodic gauge transformations.

- A necessary condition for the periodic connection to be flat is that all the holonomies \((1.2)\) commute \([\Omega_i, \Omega_j] = 0\). For a simply connected group with \(\pi_1(G) = 1\), it is also a sufficient condition.

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\(^2\)For unitary groups, one can perform the counting also with ‘t Hooft twisted boundary conditions, but for the orthogonal and exceptional groups where the mismatch in the Witten index calculations was observed this method does not work.
A sufficient condition for the group matrices to commute is that their logarithms belong to a Cartan subalgebra of the corresponding Lie algebra. For unitary and simplectic groups, this happens to be also a necessary condition. In other words, any set of commuting group matrices $\Omega_i$ with $[\Omega_i, \Omega_j] = 0$ can be presented in the form

$$\Omega_i = \exp\{iC_i\}, \quad [C_i, C_j] = 0.$$  \hspace{1cm} (1.3)

A flat connection with the holonomies $\Omega_i$ is then just $A_i = C_i/L$. The moduli space of all such connections presents (up to factorization over the discrete Weyl group) the product $T_C \times T_C \times T_C$ where $T_C$ is the Cartan torus whose dimension coincides with the rank $r$ of the group.

The Witten’s original assumption which came out not to be true is that this is also the case for all other groups. Assuming this, Witten constructed an effective Born–Oppenheimer hamiltonian for the slow variables $A_a^i$ where the index $a$ runs only over the Cartan subalgebra. It involves $3r$ bosonic degrees of freedom and their fermionic counterparts $\lambda_a^\alpha$ ($\alpha = 1, 2$) so that the gluino field $\lambda_a^\alpha$ is the Weyl spinor. Imposing further the condition of the Weyl symmetry (a remnant of the original gauge symmetry) for the eigenstates of this hamiltonian, one finds $r + 1$ supersymmetric quantum vacuum states:

$$\Psi(A_a^i, \lambda_a^\alpha) \sim 1, \epsilon^{\alpha\beta\gamma\alpha\beta} \lambda_a^a \lambda_b^b, \ldots, \left(\epsilon^{\alpha\beta\gamma\alpha\beta} \lambda_a^a \lambda_b^b\right)^r, \hspace{1cm} (1.4)$$

where $\gamma^{ab} = \delta^{ab}$ if an orthonormal basis in the Cartan subalgebra is chosen.

However, for the groups $Spin(N)$ and for all exceptional groups, there are some triples of the group elements $(\Omega_1, \Omega_2, \Omega_3)$ which commute with each other but which cannot be conjugated (gauge–transformed) to the maximal Cartan torus.

In the case of $Spin(7)$, there is a unique (up to conjugation $\Omega_i \rightarrow g^{-1}\Omega_i g$, $g \in Spin(7)$) non–trivial triple. It can be chosen in the form

$$\Omega_1 = \exp\left\{\pi/2 (\gamma_1 \gamma_2 + \gamma_3 \gamma_4)\right\} = \gamma_1 \gamma_2 \gamma_3 \gamma_4,$$

$$\Omega_2 = \exp\left\{\pi/2 (\gamma_1 \gamma_2 + \gamma_5 \gamma_6)\right\} = \gamma_1 \gamma_2 \gamma_5 \gamma_6,$$

$$\Omega_3 = \exp\left\{\pi/2 (\gamma_1 \gamma_3 + \gamma_5 \gamma_7)\right\} = \gamma_1 \gamma_3 \gamma_5 \gamma_7. \hspace{1cm} (1.5)$$

Obviously, $[\Omega_i, \Omega_j] = 0$. On the other hand, as we will shortly be convinced, the triple (1.5) cannot be conjugated to the maximal torus. This isolated triple corresponds to

$\footnote{See Appendix B for a rigorous proof.}$

$\footnote{Spin(N) is the simply connected universal covering of SO(N). It is represented by the matrices $\exp\{\omega_{\mu\nu} \gamma_\mu \gamma_\nu/2\}$ where $\gamma_\mu$ are the gamma–matrices of the corresponding dimension forming the Clifford algebra $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}$, $\mu = 1, \ldots, N$. We have already mentioned and will see later that the condition $\pi_1(G) = 0$ is very important in the whole analysis. It what follows, we will discuss mostly the groups Spin(N) rather than their orthogonal counterparts.}$

$\footnote{This fact has actually been noticed by topologists long time ago. See e.g. Refs. 6, 7 where some examples of such non–trivial triples were constructed.}$


an isolated flat connection on $T^3$ which brings about the extra quantum vacuum state. Thereby,

$$\text{Tr}(-1)^F = r_{\text{Spin}(7)} + 1 + 1 = h^\vee_{\text{Spin}(7)} .$$

(1.6)

For $\text{Spin}(8)$ and $G_2$, there is also only one extra isolated triple and one extra vacuum state. For $\text{Spin}(N \geq 9)$, there is a family of such triples with each element lying on the "small" torus which can be interpreted as the Cartan torus of the centralizer of the triple $(1.3)$. The continuous part of the latter is $\text{Spin}(N-7)$. This brings about $r_{\text{Spin}(N-7)} + 1$ extra vacuum states. The total counting is

$$\text{Tr}(-1)^F = r_{\text{Spin}(N)} + 1 + r_{\text{Spin}(N-7)} + 1 = N - 2 = h^\vee_{\text{Spin}(N)} .$$

(1.7)

Before proceeding further, let us make a simple remark. For a group which is not simply connected, there are always non-trivial commuting pairs of the elements which cannot be conjugated on the maximal torus. Consider for example $SO(3)$. The elements $\text{diag}(-1, -1, 1)$ and $\text{diag}(-1, 1, -1)$ obviously commute, but their logarithms proportional to the generators of the corresponding rotations $\sim T_3$ and $\sim T_2$ do not. In the covering $SU(2)$ group, the corresponding elements $i\sigma_3$ and $i\sigma_2$ anticommute. Such a pair of the elements of $SU(2)$ is usually called the Heisenberg pair. Generally, we will call the Heisenberg pair a pair of the elements $\tilde{b}, \tilde{c}$ which satisfy the relations

$$\tilde{b}\tilde{c} = z\tilde{c}\tilde{b}$$

(1.8)

where $z$ is the element of the center of the group. An example of the Heisenberg pair for $SU(3)$ with $z = e^{2\pi i/3}I$ is

$$\tilde{b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{-2\pi i/3} \end{pmatrix}, \quad \tilde{c} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} .$$

(1.9)

A permutation of $\tilde{b}$ and $\tilde{c}$ gives the pair with $z = e^{-2\pi i/3}I$. As we will prove later, all Heisenberg pairs in $SU(3)$ are equivalent by conjugation to (1.9) or its permutation. Heisenberg pairs will play the central role in the whole analysis.

For a simply connected group, a pair of commuting elements $g, h$ can always be conjugated to the maximal torus. That follows from the fact that one element can always be put on the torus and from the so called Bott’s theorem: the centralizer $G_g$ of any element $g$ in a connected simply connected compact Lie group $G$ is always connected. We

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6 The centralizer of the element $g$ of the group $G$ is defined as a subgroup of $G$ involving all elements $h$ which commute with $g$. The centralizer of a triple is the subgroup of $G$ commuting with each element of the triple.

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7 In the case $N = 9$, there is a subtlety: the Witten’s index associated with the abelian $\text{Spin}(2) = U(1)$ gauge group turns out to be zero, not $r_{U(1)} + 1 = 2$. Still, the counting (1.7) is correct due to the presence of certain discrete factors in the centralizer which will be discussed in more details in Appendix C. Moreover, we will show in Appendix A that, for odd $N$, one can choose the triple in such a way that the continuous part of its centralizer would be $\text{Spin}(N - 6)$ rather than $\text{Spin}(N - 7)$. For $N = 9$ that gives $r_{\text{Spin}(3)} + 1 = 2$ extra vacuum states. Also for $N = 11, 13, \ldots$, the counting (1.7) holds due to the fact that the ranks of $\text{Spin}(N - 6)$ and $\text{Spin}(N - 7)$ coincide for odd $N$. 

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first put \( g \) on the torus of \( G \) and then conjugate \( h \) to the torus of the centralizer which coincides with the large torus (obviously, once \( g \) is put on the torus, any element of the torus belongs to the centralizer).

Consider an arbitrary simply connected group \( G \). For a commuting triple of elements \( g_1, g_2, g_3 \), the elements \( g_2, g_3 \) belong to the centralizer \( G_{g_1} \) of \( g_1 \). Now if \( G_{g_1} \) presents a product of a simply connected group and some number of \( U(1) \) factors as is usually the case (e.g. the centralizer of an element of \( SU(3) \) can be \( SU(3), SU(2) \times U(1) \) or \([U(1)]^2\) ), one can put \( g_2 \) and \( g_3 \) on the torus of \( G_{g_1} \) and hence the whole triple on the torus of \( G \). But if \( G_{g_1} \) involves a factor whose fundamental group is finite, one cannot do it in a general case.

Once can prove (we will do it later) that the centralizer of any element of a unitary or a simplectic group is always simply connected up to some number of \( U(1) \) factors and hence these groups do not admit non–trivial triples. A central statement is that it is not so for higher orthogonal and exceptional groups. And this is the origin of non–trivial triples.

An expert can skip it and continue reading from the beginning of the next section.

The several following paragraphs are addressed for the reader who is not an expert in compact Lie group theory with a hope to give him a feeling what is going on here. An expert can skip it and continue reading from the beginning of the next section.

The Dynkin diagram for \( Spin(7) \) is depicted in Fig. 1. \( \alpha, \beta, \) and \( \gamma \) are the simple roots. The term "root" means that the corresponding generators \( T_{\alpha,\beta,\gamma} \) are eigenvectors of the (complex) Lie algebra of \( Spin(7) \) with respect to its Cartan subalgebra. In other words, they satisfy the relations

\[
[e_i, T_{\alpha}] = \alpha_i T_{\alpha}, \quad [e_i, T_{\beta}] = \beta_i T_{\beta}, \quad [e_i, T_{\gamma}] = \gamma_i T_{\gamma},
\]  

(1.11)
where $e_i$ is the normalized orthogonal basis in the Cartan subalgebra. A convenient choice is $e_1 = T_{12}, e_2 = T_{34}, e_3 = T_{56}$ where $T_{ij} = iγ_iγ_j/2$ are the generators of $Spin(7)$. In that case

$$
\begin{align*}
T_α &= \frac{1}{2}(T_{13} + T_{24} - i(T_{23} + T_{41})) , \quad T_β = \frac{1}{2}(T_{35} + T_{46} - i(T_{45} + T_{63})) \\
T_γ &= T_{57} - iT_{67} ,
\end{align*}
$$

(1.12)

$[T_{α,β},γ$ multiplied by any factor would also satisfy the commutation relations (1.11), but the normalization chosen in Eq.(1.12) is the most convenient one]. The root vectors are $α = (1, -1, 0), β = (0, 1, -1)$ and $γ = (0, 0, 1)$. One can define the scalar product $⟨.,.⟩$ in a natural way. The squares of the lengths of the roots are equal to 2 for $α, β$ and to 1 for $γ$. Thereby, $γ$ is a "short root" and it is denoted with the small circle. A single line between the roots $α, β$ means that $⟨α, β⟩ = -1$ so that the angle between these roots $θ_{α,β} = \arccos[⟨α, β⟩/\sqrt{⟨α, α⟩⟨β, β⟩}] = 120^°$. The double line between $β$ and $γ$ means that $θ_{γ,β} = 135^°$. The absence of the line between $α$ and $γ$ means that these roots are orthogonal and the corresponding generators in the Lie algebra commute. Any other positive root of the algebra can be presented as a linear combination of the simple roots $a_αα + a_ββ + a_γγ$ with positive integer $a_α, a_β, a_γ$. The sum $a_α + a_β + a_γ$ is called the weight of the root. The root with the highest weight $θ = α + 2β + 2γ$ [the corresponding coroot $θ^\vee$ is $T_{12} + T_{34}$] plays a special role. It (or rather the corresponding negative root $-θ$) is denoted by the dashed circle in Fig. [I]. The Dynkin labels $a_i = (1, 2, 2)$ seen on the diagram are just the coefficients of the expansion of the highest root via the simple roots.

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$^8$ For any such root $κ$, there is also a negative root $-κ$. The commutator $[T_κ, T_{-κ}]$ lies in the Cartan subalgebra. In the following, we will need also the notion of coroot which is defined as the element $κ^\vee$ of the Cartan subalgebra proportional to $[T_κ, T_{-κ}]$ such that $[κ^\vee, T_{±κ}] = ±2T_{±κ}$. In other words, the generators $κ^\vee, T_{±κ}$ form the $A_1$ [or $su(2)$] subalgebra in the same way as the matrices $σ_3, σ_±$ do. For any coroot $κ^\vee$, the property $\exp\{2πiκ^\vee\} = 1$ holds. The ratio of the lengths of some coroots $κ^\vee, ρ^\vee$ is always inverse compared to the ratio of the lengths of the corresponding roots $κ, ρ$ (so that a short root corresponds to a long coroot and the other way round). For the $Spin(7)$ group, the coroots corresponding to the simple roots $(αβγ)$ are $α^\vee = T_{12} - T_{34}, β^\vee = T_{34} - T_{56}, γ^\vee = 2T_{56}$. 
It is convenient to ascribe the label 1 for the negative highest root. We see that 
length 2, is orthogonal to \( \alpha \) and \( \gamma \), and form the angle 120° with the root \( \beta \).

We will prove later that whenever the algebra involve a simple root with the property 
\( a_i^\gamma = a_i(\alpha_i, \alpha_i)/2 > 1 \), an element of the group exist whose centralizer, with its \( U(1) \) factors being stripped off, (that is called the semi-simple part of the centralizer) is not simply connected. And if not, then not. That is why such exceptional elements are absent for unitary and simplectic groups where \( a_i^\gamma = 1 \) for all simple roots.

For \( Spin(7) \), \( a_i^\gamma = a_i^\gamma = 1 \) and \( a_i^\beta = 2 \). Thereby, the root \( \beta \) provides us with the element whose centralizer is simply connected (no \( U(1) \) factors here and the centralizer coincides with its semi–simple part). The explicit form of this element is

\[
\sigma = \exp\{\pi i \omega_\beta\} ,
\]

where \( \omega_\beta \) is the fundamental coweight — the element of the Cartan algebra satisfying the property \([\omega_\beta, T_\alpha] = [\omega_\beta, T_\gamma] = 0, \ [\omega_\beta, T_\beta] = T_\beta \). Indeed, in our case, \( \omega_\beta = \theta^\vee \) and \( \sigma \) coincides with \( \Omega_1 \) in Eq. (1.3). The centralizer of \( \sigma \) has been constructed explicitly above, but it can also be calculated on the basis of the Dynkin diagram. Obviously, \( \sigma \) commutes with the generators corresponding to the simple roots \( \alpha, \gamma \). But it also commutes with the generator \( T_\theta \) corresponding to the highest root \( \theta \). That follows from the fact that \([\omega_\beta, T_\theta] = 2T_\theta \). Thereby, the centralizer of \( \sigma \) is formed by the commuting generators \( T_\alpha, T_\gamma, T_\theta \) and is hence \([SU(2)]^3 \). The fact that this product has to be factorized over the common center \( \mathbb{Z}_2 \) is not seen right away now, but we have seen it explicitly for \( Spin(7) \) and it will be proven in a general manner in the next section.

Consider now higher groups \( Spin(N \geq 9) \). As is seen from Table 2, the corresponding Dynkin diagrams involve several nodes with \( a_i^\gamma = 2 \). Each element \( \sigma_j = \exp\{\pi i \omega_j\} \) has a centralizer which is semi-simple and is not simply connected. Moreover, in this case we have a family of the elements

\[
\sigma_s = \exp\left\{2\pi i \sum_j s_j \omega_j\right\}
\] (1.14)

where \( s_i \) are some real numbers with \( 2 \sum_i s_i = 1 \) and the sum runs over all the nodes with \( a_i^\gamma = 2 \). If all \( s_i \) are nonzero, the element (1.14) commutes with two (for odd \( N \)) and three (for even \( N \)) simple roots with \( a_i^\gamma = 1 \) and with the highest root \( \theta \). In this general case, the centralizer of the element (1.14) is \([SU(2)]^3 / \mathbb{Z}_2 \times [U(1)]^{r-3} \) for \( N = 2r + 1 \) and \([SU(2)]^4 / \mathbb{Z}_2 \times [U(1)]^{r-4} \) for \( N = 2r \). The family (1.14) spans a \( d \)-dimensional torus in \( Spin(N) \) (\( d = r - 3 \) or \( r - 4 \) depending on whether \( N \) is odd or even. It coincides with the Cartan torus of a subgroup \( Spin(N - 7) \) (or \( Spin(N - 6) \) for odd \( N \)) as is not difficult to see when substituting the explicit expressions for the fundamental weights: \( \omega_1 = i(\gamma_1 \gamma_2 + \gamma_3 \gamma_4)/2, \ \omega_2 = i(\gamma_1 \gamma_2 + \gamma_3 \gamma_4 + \gamma_5 \gamma_6)/2, \) etc.

We will see later that one can restrict all \( s_i \) in Eq. (1.14) to be positive and the elements (1.14) involving some negative \( s_i \) can be reduced by conjugation to the same canonical form (1.14) with all \( s_i \) positive. Thereby, the moduli space of exceptional elements (the space of exceptional gauge orbits) presents the torus \( T^d = [U(1)]^d \) factorized over the action of a certain discrete group which plays the role of the Weyl group for this small torus. For each such element (including also the case when some of \( s_i \) vanish), the moduli
| Group      | Algebra | extended Dynkin diagrams |
|------------|---------|--------------------------|
| SU(r+1)    | A<sub>r</sub> | ![Dynkin diagram for SU(r+1)] |
| Spin(2r+1) | B<sub>r</sub> | ![Dynkin diagram for Spin(2r+1)] |
| Sp(2r)     | C<sub>r</sub> | ![Dynkin diagram for Sp(2r)] |
| Spin(2r)   | D<sub>r</sub> | ![Dynkin diagram for Spin(2r)] |
| G<sub>2</sub> | G<sub>2</sub> | ![Dynkin diagram for G<sub>2</sub>] |
| F<sub>4</sub> | F<sub>4</sub> | ![Dynkin diagram for F<sub>4</sub>] |
| E<sub>6</sub> | E<sub>6</sub> | ![Dynkin diagram for E<sub>6</sub>] |
| E<sub>7</sub> | E<sub>7</sub> | ![Dynkin diagram for E<sub>7</sub>] |
| E<sub>8</sub> | E<sub>8</sub> | ![Dynkin diagram for E<sub>8</sub>] |

Table 2: Dynkin diagrams and Dynkin labels for all compact Lie groups. Smaller circles denote short simple roots.
space of nonequivalent Heisenberg pairs in the centralizer presents a subset of \( T^d \times T^d \) (or rather its quotient by a finite group). Thereby, the moduli space of all nonequivalent triples lies in \( T^d \times T^d \times T^d \) in agreement with the previous results.

As is seen from the Table 2, the exceptional groups also involve a number of nodes with \( a_i^\vee > 1 \). For example, \( G_2 \) involves only one such node with \( a_2^\vee = 2 \) which gives rise to only one non–trivial isolated triple. Other exceptional groups involve several such nodes, and the moduli space is somewhat more complicated but, as we will see, it can be rather easily constructed just by the form of the corresponding Dynkin diagram.

The three following sections present the hard core of the paper. In the next section, after some preliminary remarks, we establish the families of the exceptional elements whose centralizer is not simply connected. The structure of the Heisenberg pairs in these centralizers is the subject of Sect. 3. In Sect. 4 we construct the moduli space of all commuting triples. In Sect. 5, we spell out the results for higher exceptional groups in a more explicit form. In Appendix A, we calculate the centralizers of the Heisenberg pairs and of non–trivial commuting triples for all gauge groups. In Appendix B, we give a rigourous proof that each distinct component in the moduli space of triples associated with the torus \( T^d \times T^d \times T^d \) gives exactly \( d + 1 \) vacuum states. Appendix C is devoted to calculation of one particular object — the group \( \Gamma \) which is defined as the intersection of the relevant Cartan sub–torus with its centralizer (this group affects the global structure of the triples’ moduli space). We also take the opportunity there to write down some further explicit illustrative formulae.

2 Conjugacy classes and centralizers in compact Lie groups.

Let \( G \) be a compact Lie group and let \( \text{Lie} \ G \) be its Lie algebra. Fix a \( G \)-invariant symmetric positive definite bilinear form \( \langle ., . \rangle \) on \( \text{Lie} \ G \). Choose a maximal torus \( T \) in \( G \) and let \( \mathfrak{h} = \mathfrak{i} \text{Lie} \ T \). The bilinear form \( \langle ., . \rangle \) defines an isomorphism of real vector spaces \( \nu : \mathfrak{h} \rightarrow \mathfrak{h}^* \) given by \( h \rightarrow \nu(h) : \nu(h)(h') = \langle h, h' \rangle, \ h, h' \in \mathfrak{h} \), and hence a bilinear form \( \langle ., . \rangle \) on \( \mathfrak{h}^* \). Given a non-zero vector \( \alpha \in \mathfrak{h}^* \), the vector \( \alpha^\vee = 2\nu^{-1}(\alpha)/\langle \alpha, \alpha \rangle \in \mathfrak{h} \) is independent of the choice of \( \langle ., . \rangle \).

Let \( \Delta \subset \mathfrak{h}^* \) be the set of (non-zero) roots of \( \text{Lie} \ G \) and let \( \Delta^\vee = \{ \alpha^\vee | \alpha \in \Delta \} \subset \mathfrak{h} \) be the set of coroots. Denote by \( Q(G) \) and \( Q^\vee(G) \) the \( \mathbb{Z} \)-span of the sets \( \Delta \) and \( \Delta^\vee \), respectively. These are called the root and coroot lattices and actually depend only on \( \text{Lie} \ G \). Introduce the coweight lattice

\[
P^\vee(G) = \{ h \in \mathfrak{h} | e^{2\pi i h} = 1 \} .
\]  

This lattice contains the lattice \( Q^\vee(G) \). It is well known that if \( G \) is a connected compact Lie group, then its fundamental group is given by the following formula

\[
\pi_1(G) = P^\vee(G)/Q^\vee(G) .
\]  

Suppose now that \( G \) is a connected simply connected (almost) simple compact Lie group of rank \( r \). Choose a set of simple roots \( \Pi = \{ \alpha_1, \ldots, \alpha_r \} \subset \Delta \), then \( \Pi^\vee = \)
\( \{\alpha_1^\vee, \ldots, \alpha_r^\vee\} \subset \Delta^\vee \) is a set of simple coroots. Let
\[
\alpha_0 = -\theta = -\sum_{j=1}^{r} a_j \alpha_j, \quad \alpha_0^\vee = -\sum_{j=1}^{r} a_j^\vee \alpha_j^\vee
\tag{2.3}
\]
be the lowest root [i.e. \( \alpha_0 - \alpha_j \) are not roots for all \( j = 1, \ldots, r \)] and its coroot. The numbers \( a_j \) and \( a_j^\vee \) are (well-known) positive integers. Recall that \( G \) of type \( A-D-E \) is called \textit{simply laced} (or \( 1 \)-laced), \( G \) of type \( B-C-F \) is called \textit{2–laced} and the group \( G_2 \) is \textit{3–laced}. If \( \alpha_j \) is a long root, one has: \( a_j^\vee = a_j \) (that happens for all \( j \) in the \( A-D-E \) cases). If \( \alpha_j \) is a short root, one has \( a_j^\vee = a_j/l \) if \( G \) is \( l \)-laced. Let \( a_0^\vee = a_0 = 1 \). The set of vectors \( \{\alpha_0, \alpha_1, \ldots, \alpha_r\} \) along with the integers \( a_j \) are depicted by the extended Dynkin diagrams \( \hat{D}(G) \) which are listed in Table 3. The Dynkin diagram \( D(G) \) of \( G \) is obtained from \( \hat{D}(G) \) by removing \( \alpha_0 \)-th node.

Define fundamental coweights \( \omega_1, \ldots, \omega_r \in \mathfrak{h} \) by \( \alpha_i(\omega_j) = \delta_{ij}, \ i, j = 1, \ldots, r \). Note that their \( Z \)-span is the coweight lattice of the adjoint group \( Ad \ G \).

**Theorem 1.** Let \( G \) be a connected simply connected (almost) simple compact Lie group. For a set of \( r + 1 \) real numbers \( \vec{s} = (s_0, s_1, \ldots, s_r) \) such that
\[
s_j \geq 0, \ j = 0, \ldots, r; \quad \sum_{j=0}^{r} a_j s_j = 1
\tag{2.4}
\]
consider the element
\[
\sigma_{\vec{s}} = \exp \left\{ 2\pi i \sum_{j=1}^{r} s_j \omega_j \right\} \in T.
\tag{2.5}
\]

(a) Any element of \( G \) can be conjugated to a unique element \( \sigma_{\vec{s}} \).

(b) The centralizer of \( \sigma_{\vec{s}} \) (in \( G \)) is a connected compact Lie group which is a product of \( [U(1)]^{n-1} \) where \( n \) is the number of non-zero \( s_i \), and a connected semi-simple group whose Dynkin diagram is obtained from the extended Dynkin diagram \( \hat{D}(G) \) by removing the nodes \( i \) for which \( s_i \neq 0 \).

c) The fundamental group of the centralizer of \( \sigma_{\vec{s}} \) is isomorphic to a direct product of \( \mathbb{Z}^{n-1} \) and a cyclic group of order \( a_{\vec{s}} \) where
\[
a_{\vec{s}} = \gcd\{a_j^\vee | s_j \neq 0, j = 0, \ldots, r\}. \tag{2.6}
\]

In particular, the fundamental group of the semi-simple part of the centralizer of \( \sigma_{\vec{s}} \) is the cyclic group of order \( a_{\vec{s}} \).

**Corollary 1.** The connected subgroup \( H \) of \( G \) corresponding to a subdiagram of \( D(G) \) is simply connected.

**Definitions.** If \( g \in G \) is conjugate to \( \sigma_{\vec{s}} \), the real numbers \( s_0, \ldots, s_r \) are called \textit{coordinates} of \( g \). We will call the element \( g \) \textit{m–exceptional} whenever \( m = a_{\vec{s}} > 1 \). The set of all possible coordinates for the \( m \)-\textit{exceptional} elements will be called \textit{fundamental alcove} of type \( m \). A positive integer \( m \) is called \textit{G–exceptional} if it divides one of the \( a_j^\vee \) \( (j = 0, \ldots, r) \).
A look at Table 2 gives the following possibilities for $G$-exceptional integers:

- $SU(N)$, $Sp(2N)$ : 1
- $Spin(N)$, $G_2$ : 1, 2
- $F_4$, $E_6$ : 1, 2, 3
- $E_7$ : 1, 2, 3, 4
- $E_8$ : 1, 2, 3, 4, 5, 6

(2.7)

**Proof of Theorem 1.** Statement $a)$ is well known. In this form for finite order elements it can be found e.g. in Ref.[3], Chapter 8. The connectedness of the centralizer of any element of $G$ is usually attributed to Bott (see e.g. [4], §3). A proof of the second part of $b)$ can also be found in [3], Chapter 8.

Finally, the proof of $c)$ is based on the formula (2.2). If $\tilde{G}$ is a subgroup of $G$ containing $T$, then $P^\vee(\tilde{G}) = P^\vee(G) = Q^\vee(G)$ and $Q^\vee(\tilde{G}) \subset Q^\vee(G)$, hence

$$\pi_1(\tilde{G}) = Q^\vee(G)/Q^\vee(\tilde{G})$$

(2.8)

Note that by $b)$, $Q^\vee(G_{\sigma x})$ is spanned over $\mathbb{Z}$ by those $\alpha_i^\vee$ for which $s_j = 0$ ($j = 0, \ldots, r$). The proof of $c)$ now is completed by the following simple lemma.

**Lemma 1.** Let $Q = \oplus_{i=1}^r \mathbb{Z}\alpha_i$ be a free abelian group of rank $r$ and let $\alpha_0 = \sum_{i=1}^r a_i\alpha_i$ be a non-zero element of $Q$. Let $I$ be a subset of $\{0, 1, \ldots, r\}$ and let $Q_I = \oplus_{i \in I} \mathbb{Z}\alpha_i$. Then the group $Q/Q_I$ is isomorphic to a direct product of $\mathbb{Z}^{r-\#I}$ and a cyclic group of order $\gcd\{a_i | i \notin I\}$.

**Proof.** If $0 \notin I$, the lemma is obvious. If $0 \in I$, we may choose a basis of $Q_I$ of the form $\alpha_0, \alpha_{i_1}, \alpha_{i_2}, \ldots$ where $i_1, i_2, \ldots \geq 1$. After reordering of the set $\{1, \ldots, r\}$, the matrix expressing this basis in terms of the basis of $Q$ takes the form

$$M = \begin{pmatrix} a_1 & \cdots & a_s \\ \vdots & & \vdots \\ a_{s+1} & \cdots & a_r \\ \mbox{diag}(1, \ldots, 1) & & 0 \end{pmatrix}$$

(2.9)

It follows that all elementary divisors of $M$ are $a := \gcd\{a_{s+1}, \ldots, a_r\}, 1, \ldots, 1$. Hence the maximal finite subgroup of $Q/Q_I$ is a cyclic group of order $a$.

**Remark 1.** Recall that the element $g \in G$ is called non–ad–exceptional if it can be written in the form $g = e^{2\pi i\beta}$, where $(Ad g)x = x$, iff $[\beta, x] = 0$ ($\beta, x \in i \ Lie G$). It was shown in Ref.[4] that $g$ is non–ad–exceptional iff the set of integers $a_i$ for which the coordinates $s_i$ of $g$ are non–zero are relatively prime. It follows from our Theorem 1c that, for a non–ad–exceptional $g$, $\pi_1(G_g)$ is a free abelian group (or, equivalently, the semi-simple part of $G_g$ is simply connected) and that for a simply laced $G$ the converse holds as well.
Figure 2: Coroot lattices for $G = G_2$ and its subgroup $\tilde{G} = [SU(2)]^2/\mathbb{Z}_2$. Blank circles mark out the nodes of $Q^\vee(G)$ not belonging to $Q^\vee(\tilde{G})$.

Fig. 2 provides an illustration for the Theorem 1 in the simplest non-trivial case of $G_2$. The lattice $Q^\vee(G = G_2)$ is formed by the coroots $\alpha^\vee$ and $\beta^\vee$ and the lattice $Q^\vee(\tilde{G})$ for the centralizer is formed by the (orthogonal) coroots $\beta^\vee$ and $\alpha^\vee_0 = 2\alpha^\vee + \beta^\vee$. We see that $Q^\vee(G)$ involves, indeed, extra nodes compared to $Q^\vee(\tilde{G})$ (the blank circles in Fig. 2) and that $Q^\vee(G)/Q^\vee(\tilde{G}) \equiv \pi_1(\tilde{G}) = \mathbb{Z}_2$. (In this simple case, $\pi_1(\tilde{G})$ does not involve extra $\mathbb{Z}$ factors associated with the $U(1)$ factors in the centralizer.)

3 Heisenberg pairs in compact Lie groups.

Given a central element $z$ in a compact Lie group $G$, a Heisenberg pair with center $z$ is a pair of elements $\tilde{b}, \tilde{c} \in G$ satisfying the commutation relation (1.8). If in addition $z$ has order $m$, and one has

$$\tilde{b}^m = \tilde{z}^m = \begin{cases} 1 & \text{if } m \text{ is odd} \\ z^{m/2} & \text{if } m \text{ is even} \end{cases}$$

then $(\tilde{b}, \tilde{c})$ is called a standard Heisenberg pair with center $z$. A Heisenberg pair $(\tilde{b}, \tilde{c})$ is called lowest dimensional if its orbit $G \cdot (\tilde{b}, \tilde{c})$ under conjugation has minimal dimension (among Heisenberg pairs with center $z$).

Example 1. Let $(\tilde{b}, \tilde{c})$ be an irreducible Heisenberg pair in $U(N)$ (i.e. there are no non-trivial subspace invariant with respect to $\tilde{b}$ and $\tilde{c}$). Then $z = \epsilon I_N$ where $\epsilon$ is a primitive $m$-th root of 1. Let $v_1$ be an eigenvector for $\tilde{b}$: $\tilde{b}v_1 = \beta v_1$, and let $v_{j+1} = \tilde{c}^{j+1}v_1$, $j = 1, \ldots, N - 1$. Since the pair $(\tilde{b}, \tilde{c})$ is irreducible, the vectors $v_j$, $j = 1, \ldots, N$ form a basis of $\mathbb{C}^N$. Due to (1.8), we have

$$\tilde{b}v_j = \epsilon^{j-1}\beta v_j, \quad j = 1, \ldots, N.$$
Since, due to (1.8), $\tilde{c}$ permutes cyclically the eigenspaces of $\tilde{b}$, it follows from irreducibility of the pair $(\tilde{b}, \tilde{c})$ that all eigenspaces of $\tilde{b}$ are 1-dimensional. Hence, we have
\[
\tilde{c}v_j = v_{j+1}, \quad j = 1, \ldots, N - 1
\]
\[
\tilde{c}v_N = \gamma v_1 .
\] (3.3)

Since $\tilde{c}^N v_1$ is a multiple of $v_1$, we get by (1.8) that $\epsilon^N = 1$. Therefore
\[
N = m .
\] (3.4)

Thus, all irreducible Heisenberg pairs $(\tilde{b}, \tilde{c})$ in $U(N)$ with center $z = \epsilon I_N$, where $\epsilon$ is a primitive $m$-th root of 1, are given up to conjugation by $(3.2) - (3.4)$, where $\beta$ and $\gamma$ are arbitrary constants of modulus 1.

If $(\tilde{b}, \tilde{c})$ is a standard irreducible Heisenberg pair in $U(m)$ with center $\epsilon I, \epsilon$ being a primitive $m$-th root of 1, then clearly
\[
\beta^m = \gamma = (-1)^{m+1} .
\] (3.5)

It follows that there exists a unique up to conjugation such pair and it lies in $SU(m)$. [For $m = 2$, the matrices $(3.2), (3.3)$ with the condition $(3.3)$ are reduced to $i\sigma_3, -i\sigma_2$ and, for $m = 3$, $\epsilon = e^{2\pi i/3}$, they coincide with $(1.3)$.]

Due to complete reducibility, we obtain that there exists in $U(N)$ a Heisenberg pair with center $z = \epsilon I_N$, $\epsilon$ being a primitive $m$-th root of 1, if $m$ divides $N$ and that any such pair is conjugate to a direct sum of $N/m$ such irreducible pairs. Moreover, in this case there exists a unique up to conjugation standard Heisenberg pair, say $(\tilde{b}, \tilde{c})$, which is a direct sum of irreducible such pairs, and any Heisenberg pair in $U(N)$ or $SU(N)$ can be conjugated to a pair obtained by multiplying $(\tilde{b}, \tilde{c})$ by a direct sum of matrices $\beta_i I_m, i = 1, \ldots, N/m$, where $|\beta_i| = 1$ and, in the $SU(N)$ case, $\prod_i \beta_i^m = 1$.

Note that the centralizer of the standard Heisenberg pair $(\tilde{b}, \tilde{c})$ is isomorphic to the direct product of $\langle z \rangle$ and $SU(N/m)$ (here and further $\langle z \rangle$ denotes the cyclic group generated by $z$). It is also clear that this is a lowest dimensional Heisenberg pair with center $\epsilon I_N$, where $\epsilon$ is a primitive $m$-th root of 1.

**Proposition 1.** Let $G$ be as in Theorem 1, let $z$ be a central element of $G$ and let $(\tilde{b}, \tilde{c})$ be a Heisenberg pair with center $z$. Let $T^0$ be a maximal torus in the centralizer of this pair in $G$. Then any Heisenberg pair with center $z$ can be conjugated to a Heisenberg pair of the form $(\tilde{b}t, \tilde{c}t')$, where $t, t' \in T^0$, and the rank of its centralizer always equals $\dim T^0$. Two pairs from $P := \tilde{b}T^0 \times \tilde{c}T^0$ can be conjugated to each other iff they can be conjugated by an element from the normalizer of $P$ in $G$.

**Proof.** Consider the group $G_{\tilde{b},(z)} = \{ x \in G| x^{-1} \tilde{b} x \in \tilde{b} \langle z \rangle \}$. Its connected components are $z'G_{\tilde{b}}$, where $z' \in \langle z \rangle$. The connected component containing $\tilde{c}$ is $zG_{\tilde{b}}$; it consists of all $\tilde{c}'$ such that $(\tilde{b}, \tilde{c}')$ is a Heisenberg pair with center $z$. The element $\tilde{c}$ acts on $G_{\tilde{b},(z)}$ by conjugation preserving the component $zG_{\tilde{b}}$. Recall that, by Gantmacher’s theorem [4], (i) all conjugacy classes in a compact Lie group $K$ contained in a connected component of its element $g$ intersect the set $gT^0$, where $T^0$ is a maximal torus in the centralizer of $g$ in $K$, (ii) up to conjugacy, $T^0$ depends only on the connected component, (iii) two elements from $gT^0$ can be conjugated by an element of $K$ iff they can be conjugated by an element
from the normalizer of $gT^0$ in $K$. Hence, by the first part of Gantmacher’s theorem, all conjugacy classes of the group $G_{\hat{b}(z)}$ contained in $zG_{\hat{b}}$ intersect the set $\tilde{c}T^0$.

Let $(\tilde{b}', \tilde{c}')$ be a Heisenberg pair with center $z$. By above, we may assume that $\tilde{c}' = \tilde{c}T'$, $t' \in T^0$. But $\tilde{b} \in G_{\tilde{c}T', (z)}$, and arguing as above we see that $\tilde{b}'$ can be conjugated in the group $G_{\tilde{c}T', (z)}$ by the conjugation action of $G_{\tilde{c}T'}$ into the set $\tilde{b}T^0$. The fact that the rank of the centralizer of $(\tilde{b}', \tilde{c}')$ equals $\dim T^0$ follows from the second part of Gantmacher’s theorem. The last part of the proposition follows from the last part of Gantmacher’s theorem.

Now we will construct standard Heisenberg pairs $(\tilde{b}_m, \tilde{c}_m)$ in the universal covers of semi-simple parts of centralizers of exceptional elements. This will provide us with a collection of commuting pairs $(b_m, c_m)$ in the centralizers themselves.

Given a $G$–exceptional integer $m > 1$, denote by $D_m$ the subdiagram of $\hat{D}(G)$ consisting of the nodes $a_j$ such that $a_j^\vee$ is divisible by $m$, and let $\tilde{D}_m$ be the complement of $D_m$ in $\hat{D}(G)$. Denote by $T'_m$ the subtorus of $T$ consisting of elements of the form $\exp \left\{ 2\pi i \sum_{j \in D_m} s_j \omega_j \right\}$ where $s_j \in \mathbb{R}$, and define the following (connected) subsets of $T'_m$:

$$T_m = \left\{ \exp \left( 2\pi i \sum_{j \in D_m} s_j \omega_j \right) \bigg| \sum_j a_j s_j = 0 \right\},$$

$$T_m^{alc} = \left\{ \exp \left( 2\pi i \sum_{j \in D_m} s_j \omega_j \right) \bigg| \sum_j a_j s_j = 1, s_j \geq 0 \right\}.$$

Note that $T_m^{alc}$ is the intersection of $T_m$ with the fundamental alcove [consisting of the elements $2\mathbb{Z}$ satisfying $2\mathbb{Z}$] and $T_m$ is a subtorus in $T'_m$ of codimension 1. Note also that $\dim T_m = \dim T_m^{alc} = r_m$, where $r_m$ is the number of nodes of $D_m$ minus 1.

It is clear that the centralizer of $T'_m$ in $G$ is $T_m L_m$, where $L_m$ is a semi-simple subgroup. Due to Theorem 1b,c, $L_m$ is a connected semi-simple compact Lie group with Dynkin diagram $\tilde{D}_m$ and its fundamental group is a cyclic group of order $m$. Let $\tilde{L}_m$ be the universal cover of $L_m$, so that $L_m = \tilde{L}_m / \mathbb{Z}_m$, where $\mathbb{Z}_m \subset \tilde{L}_m$ is a cyclic group of order $m$. Let $z_m^{(1)}, \ldots, z_m^{(u_m)}$ be all elements of $\mathbb{Z}_m$ such that $\langle z_m^{(i)} \rangle = \mathbb{Z}_m$. Note that $u_m$ is the number of positive integers $\leq m$ relatively prime to $m$. A look at Table 2 gives the following possibilities for $\tilde{L}_m$ (provided that $m$ is $G$–exceptional), for an $l$–laced $G$:

| $m$ | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|
| $L_m$ | $[SU(2)]^{5-1}$ | $[SU(3)]^{4-1}$ | $[SU(4)]^2 \times SU(2)$ | $[SU(5)]^2$ | $SU(6) \times SU(3) \times SU(2)$ |

Table 3: The groups $\tilde{L}_m$.

It follows from Corollary 1 that $\mathbb{Z}_m$ is embedded diagonally in $\tilde{L}_m$. Since each $\tilde{L}_m$ is a product of special unitary groups, using Example 1, one immediately constructs for each $z_m^{(i)}$, $i = 1, \ldots, u_m$, a standard Heisenberg pair $(\tilde{b}_m^{(i)}, \tilde{c}_m^{(i)})$ in $\tilde{L}_m$. Their images $b_m^{(i)}$ and $c_m^{(i)}$ in $L_m$ form a commuting pair.

**Theorem 2.** Let $G$ be as in Theorem 1 and let $m \geq 2$ be a $G$–exceptional integer.

(a) Let $Q_m$ denote the connected semi-simple subgroup of $G$ obtained by removing only some ($\geq 1$) of the nodes $j$ from $\hat{D}(G)$ such that $a_j^\vee$ is divisible by $m$ and let $L_m \subset Q_m$ be...
the inclusion corresponding to the inclusion of the Dynkin diagrams. Then \( Z_m \) is central in \( \tilde{Q}_m \) and \( Q_m = \tilde{Q}_m/Z_m \).

(b) Let \( b, c \in Q_m \) be a commuting pair such that \( \tilde{b} \tilde{c} = z_m^{(i)} \tilde{c} \tilde{b} \) for (any) preimages of \( b \) and \( c \) in \( Q_m \). Then the pair \( (b, c) \) can be conjugated by \( Q_m \) to \( b_m^{(i)} T_m \times c_m^{(i)} T_m \). The centralizer of all pairs from this subset has rank \( r_m \).

\textbf{Proof.} (a) is established by a direct verification for maximal \( \tilde{Q}_m \)’s. (b) follows from Proposition 1.

4 Classification of commuting triples .

Denote by \( T \subset G \times G \times G \) the set of all commuting triples. For each \( G \)-exceptional integer \( m \geq 2 \), we construct the following \( u_m \) families of commuting triples \( (i = 1, \ldots, u_m)\):

\[
F_m^{(i)} = T_m^{alc} \times b_m^{(i)} T_m \times c_m^{(i)} T_m \subset T
\]

where \( b_m^{(i)}, c_m^{(i)} \in L_m \subset G \) are commuting pairs constructed in Sect. 3. It is clear from the construction that \( T_m \) is a maximal torus in the centralizer of each such triple.

Let \( F_1^{(i)} = T_1^{alc} \times T \times T \) and let

\[
T_m^{(i)} = G \cdot F_m^{(i)} \subset T ,
\]

where \( m \) is a \( G \)-exceptional integer and \( i = 1, \ldots, u_m \). Note that each \( T_m^{(i)} \) is a connected component in the set of all commuting triples in \( G \). Since \( \dim F_m^{(i)} = 3r_m \) \( (r_1 = r) \) and the centralizer in \( G \) of a generic triple of \( F_m^{(i)} \) is a torus of dimension \( r_m \), we have

\[
\dim T_m^{(i)} = \dim G + 2r_m .
\]

Triples from \( \cup_i T_m^{(i)} \) are called \( m \)-exceptional.

\textbf{Theorem 3.} Let \( G \) be the same as in Theorem 1.

(a) The set \( T \) of commuting triples in \( G \) is a disjoint union of the subsets \( T_m^{(i)} \), where \( m \) runs over all \( G \)-exceptional integers and \( i = 1, \ldots, u_m \).

(b) Rank of the centralizer of a triple from \( T_m^{(i)} \) is \( r_m \).

(c) The sum over all components of \( T \) of the numbers \( r_m + 1 \) is equal to \( h^\vee \).

\textbf{Proof.} Let \( (a, b, c) \) be a triple of commuting elements in \( G \). If \( a \) is not exceptional, then by Theorem 1c, \( \pi_1(G_a) \) is a free abelian group, hence \( G_a \) is a direct product of a torus \( T_1 \subset T \) and the semi-simple part which is a connected simply connected group. But then the commuting pair \( (b, c) \subset G_a \) can be conjugated to \( T \) in \( G_a \) (since the centralizer of \( b \) in \( G_a \) is connected, the element \( c \) can be conjugated in \( G_a \cap G_b \) to \( T \)). Thus, in this case \( (a, b, c) \) is a trivial triple (i.e. it can be conjugated to \( T \)).

Let now \( a \) be \( m \)-exceptional for \( m \geq 2 \). We may assume that \( a \in T_m^{alc} \) (see Theorem 1a). Then the group \( G_a \) is a product of one of the groups \( Q_m \) (see Theorem 2) and a torus, say \( T_1 \). Let \( G_a = T_1 \times Q_m \) so that \( G_a = \tilde{G}_a/Z_m \) and let \( \tilde{b}, \tilde{c} \) denote some preimages of \( (b, c) \in G_a \) in \( \tilde{G}_a \). If \( \tilde{b} \tilde{c} = \tilde{c} \tilde{b} \), then the pair \( (b, c) \) can be conjugated in \( \tilde{G}_a \) to the preimage of \( T \) and the triple \( (a, b, c) \) is trivial. Finally, if \( \tilde{b} \tilde{c} = z \tilde{c} \tilde{b} \) for a non-trivial element \( z = z_d^{(i)} \) of \( Z_m \) of order \( d \), where \( d \) divides \( m \), then by Theorem 2, the pair \( (b, c) \) can be conjugated in \( G_a \) to the family \( \tilde{b}_d^{(i)} T_d \times \tilde{c}_d^{(i)} T_d \). It is also clear that, for all commuting triples from a
connected component, $z$ is the same and that $T_d \supset T_m$, $T_{d}^{\text{alc}} \supset T_{m}^{\text{alc}}$. This completes the proof of (a). (b) follows from Theorem 2.

Of course, one can check directly that in all cases one has

$$\sum_{m}(r_{m} + 1)u_{m} = h^{\vee}. \quad (4.4)$$

Here is a unified proof which uses the classical fact that for any positive integer $a$ one has

$$a = \sum_{j|a} u_{j}. \quad (4.5)$$

Indeed, it follows from this formula applied to each $a_{i}^{\vee}$ that

$$h^{\vee} := \sum_{i=0}^{r} a_{i}^{\vee} = \sum_{j \geq 1} u_{j}N_{j}^{\vee}, \quad (4.6)$$

where $N_{j}^{\vee} = \#\{a_{s}^{\vee} | j \text{ divides } a_{s}^{\vee} (s = 0, \ldots, r)\}$. On the other hand, $N_{j}^{\vee} = r_{j} + 1$.

The values of $r_{m}$ and $u_{m}$ for $m > 1$ are listed in Table 4. The group $W_{m}$ in this table is the Weyl group of the torus $T_{m}$, i.e. the group of linear transformations of $i\text{Lie} T_{m}$ generated by conjugations of $G$ which leave $T_{m}$ invariant ( $T_{1}$ is just a maximal torus and $W_{1}$ is the Weyl group $W$ of $G$). We will exploit it in the following section and in the Appendix B where the calculation of $W_{m}$ for $m > 1$ is also explained (see Remark B2). The group $C_{m}$ is the semi-simple part of the centralizer of $T_{m}$ in $G$. It will be discussed and calculated below where we will give an explicit construction of the moduli space for the commuting triples. The same refers to the finite group $G_{m}$ in the last column of Table 4.

Remark 2. Recall that, due to Theorem 1c, an element $g$ of $G$ can be conjugated to $T_{m}^{\text{alc}}$ iff the order of the maximal finite subgroup of $\pi_{1}(G_{g})$ is divisible by $m$. Pick an element $a_{m} \in T_{m}^{\text{alc}}$. Then the centralizer of any generic element $g \in a_{m}T_{m}$ is $T_{m}^{\text{alc}}$, and therefore $g$ can be conjugated to $T_{m}^{\text{alc}}$ by $W$. It follows that any element from $a_{m}T_{m}^{\text{alc}}$ can be conjugated (by $W$) to $T_{m}^{\text{alc}}$. Hence we have

$$E_{m}^{(i)} : = a_{m}T_{m} \times b_{m}^{(i)}T_{m} \times c_{m}^{(i)}T_{m} \subset T_{m}^{(i)}. \quad (4.7)$$

We proceed now to explicitly describe the moduli space $M_{m}^{(i)}$ of triples from $T_{m}^{(i)}$ as a quotient of $E_{m}^{(i)}$ by an action of a finite group.

A triple from $T$ is called isolated if its centralizer is finite. It follows from Theorem 3 that the following are equivalent conditions for a triple to be isolated:

- $r_{m} = 0$
- $T_{m}^{(i)}$ is a single $G$–orbit.

It is easy to see that the centralizer of an isolated triple $(a, b, c)$ is a product of three cyclic groups of order $m$ (generated by $a, b$ and $c$). A look at Table 4 provides a complete list of isolated triples given by Table 5dvips triples (for each $G$ and $m$ there are $u_{m}$ of them):
Consider the centralizer $C'_m$ of $T_m$ and let $C_m$ denotes the semi–simple part of $C'_m$. Due to Corollary 1, $C_m$ is a connected simply connected semi–simple subgroup of $G$. Note that $C_m \supset L_m$, hence $\text{rank}(C_m) \geq r - r_m$, but since $T_m$ commutes with $C_m$, we get also the reverse equality. Hence

$$\text{rank } (C_m) = r - r_m$$  \hspace{1cm} (4.8)$$

and $C'_m = C_mT_m$. It is clear that $E^{(i)}_m \subset C'_m \times C'_m \times C'_m$ and therefore $E^{(i)}_m \cap (C_m \times C_m \times C_m)$ is not empty. In other words, $C_m$ contains an $m$–exceptional triple, say $(a, b, c)$. This triple is isolated in $C_m$, since in the contrary case the rank of its centralizer in $G$ would be greater than $r_m$, in contradiction with Theorem 3b.

Thus $C_m$ is a semi-simple subgroup of $G$ containing an isolated $m$–exceptional triple, and its Dynkin diagram is a subdiagram of $D(G)$ which consists of $r - r_m$ nodes. A look at Table 5 establishes that the only possibilities for $C_m$ are those listed in Table 4.

**Remark 3.** We have obtained a slightly more canonical construction of the sets $E^{(i)}_m$, $i = 1, \ldots, u_m$. Let $k_1, \ldots, k_{u_m}$ be all integers between 1 and $m$ relatively prime to $m$. Pick an isolated triple $(a_m, b_m, c_m)$ in the subgroup $C_m$. Then

$$E^{(i)}_m = a_mT_m \times b_mT_m \times c_m^{k_i}T_m$$  \hspace{1cm} (4.9)$$

Denote by $N_m$ the normalizer of the set $E^{(i)}_m$ in $G$ (clearly, it is independent of $i$), i.e. $N_m = \{ g \in G | gE^{(i)}_mg^{-1} \subset E^{(i)}_m \}$. Note that the centralizer of $E^{(i)}_m$ in $G$ is $T_m$. Let
\( R_m = N_m/T_m \); then the moduli space \( \mathcal{M}_m^{(i)} \) of triples from \( T_m^{(i)} \) is the quotient

\[
\mathcal{M}_m^{(i)} = E_m^{(i)}/R_m ,
\]

where \( R_m \) is a finite group that we are about to compute. This follows from the last part of Proposition 1. For example, in the case of trivial triples one has: \( \mathcal{M}_1^{(i)} = (T \times T \times T)/W \) (with the diagonal action of \( W \)).

Let \( N(T_m) = \{ g \in G | gT_mg^{-1} \subset T_m \} \). Recall that \( W_m = N(T_m)/(C_m T_m) \) is the Weyl group of \( T_m \). One, clearly, has

\[
N_m = \{ g \in N(T_m) | g a_m g^{-1} \in a_m T_m, \ g b_m g^{-1} \in b_m T_m, \ g c_m g^{-1} \in c_m T_m \} . \tag{4.10}
\]

Since there always exists a triple in \( E_m^{(i)} \) for which the Weyl group of the centralizer is \( W_m \) (cf. Table 7), we obtain:

\[
R_m = W_m A_m , \tag{4.11}
\]

where \( W_m \) normalizes \( A_m \) and

\[
A_m = \frac{\{ g \in C_m | g(a_m, b_m, c_m) g^{-1} = (a_m t_1, b_m t_2, c_m t_3) \text{ for some } t_1, t_2, t_3 \in T_m \}}{\text{centralizer of } (a_m, b_m, c_m) \text{ in } C_m} \tag{4.12}
\]

But \( t_1, t_2, t_3 \) lie also in \( C_m \), hence they are central in \( C_m \). Note that multiplying each element of the triple \( (a_m, b_m, c_m) \) by a central element of \( C_m \) again gives an isolated triple of \( C_m \). Hence we have:

\[
A_m = \Gamma_m \times \Gamma_m \times \Gamma_m , \tag{4.13}
\]

where \( \Gamma_m = \text{Center}(C_m) \cap T_m \). We thus have proven the following theorem:

**Theorem 4.** The connected component \( \mathcal{M}_m^{(i)} \) of the moduli space of \( m \)-exceptional triples in \( G \) looks as follows:

\[
\mathcal{M}_m^{(i)} = \left( \frac{a_m T_m}{\Gamma_m} \times \frac{b_m T_m}{\Gamma_m} \times \frac{c_m}{\Gamma_m} \right)/W_m \tag{4.14}
\]

Here \( (a_m, b_m, c_m) \) is an isolated \( m \)-exceptional triple in \( C_m \), \( k_i \) are all integers between 1 and \( m \) relatively prime to \( m \), \( \Gamma_m = \text{Center}(C_m) \cap T_m \) acting by multiplication, and \( W_m \) is the Weyl group of \( T_m \) acting diagonally. The groups \( W_m \) are listed in the corresponding column of Table 4. (They happen to coincide with the Weyl group of the centralizer of a lowest-dimensional \( m \)-exceptional triple; see Remark B2 and Corollary B2.)

The only remaining question is what are the groups \( \Gamma_m \). We have resolved it by explicit calculations described in Appendix C. The results are given in the last column of Table 4. It turns out that in almost all cases \( \Gamma_m \) coincides with the center of \( C_m \). The only exception are the groups \( Spin(2r), \ r > 4 \); the center of \( C_2 = Spin(8) \) in this case is \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), but the group \( \Gamma_2 \) is just \( \mathbb{Z}_2 \).

**Remark 4.** Let \( G \) be as in Theorem 1 and let \( \nu \) be a diagram automorphism of \( G \) of order \( k = 2 \) (for types \( A, D, E_6 \)) or 3 (for type \( D_4 \)). Let \( G' \) be a connected component of the semidirect product of \( \langle \nu \rangle \) and \( G \), containing \( \nu \). Then Theorems 1-4 hold if instead of \( G \) one takes \( G' \) and instead of the extended Dynkin diagram (denoted by \( X_r^{(1)} \) in [3]), one takes the "twisted diagrams" \( X_r^{(k)} \), \( k = 2, 3 \). In particular, the sum \( \sum_m (r_m + 1)u_m \) equals the dual Coxeter number of \( X_r^{(k)} \) which is equal to the Coxeter number of the group \( G \).
5 Counting quantum vacua. Examples.

Consider a connected component $\mathcal{M}_m^{(i)}$ in the moduli space of all commuting triples. As we have seen, it presents a subset of the product of three (shifted) tori $T^{r_m}$ of dimension $r_m$ (or rather a quotient of this product by action of a finite group), where $r_1$ is the rank of the group and $r_m$ for all non–trivial components for all groups are listed in Table 4. In the full analogy with (1.4), $\mathcal{M}_m^{(i)}$ gives rise to $r_m + 1$ quantum vacuum states

$$
\Psi \sim 1, \sum_{i}^{r_1} e^{i \alpha \lambda^i_\alpha \lambda^i_\beta}, \ldots, \left( \sum_{i}^{r_m} e^{\alpha \beta \lambda^i_\alpha \lambda^i_\beta} \right)^{r_m},
$$

(5.1)

where $i$ mark the Cartan generators forming an orthonormal basis on the small torus $T^{r_m}$ and $\sum_{i}^{r_m} q_i q_i^\dagger$ is the quadratic invariant of the Weyl group $W_m$ of $T^{r_m}$ (see Table 4, last column). Note that the powers higher than $r_m$ in Eq. (5.1) would give just zero due to anticommuting nature of $\lambda^i_\alpha$. The Appendix B is devoted to the proof of the fact that all linear independent elements of the Grassmann algebra with the base $\lambda^i_\alpha$, $\alpha = 1, 2$, invariant under the action of the Weyl group are reduced to the powers of quadratic invariant which means that all quantum vacuum states associated with the component $\mathcal{M}_k$ are, indeed, listed in Eq. (5.1). We have all together

$$
\text{Tr}(-1)^F = \sum_{k}^{(r_m + 1)k}
$$

(5.2)

where the sum runs over all components $\mathcal{M}_k$. From Table 4, we have $(1+1) \cdot 1 + (0+1) \cdot 2 = 4$ extra states (associated with the components with $m > 1$) for $F_4$, $3 \cdot 1 + 1 \cdot 2 = 5$ extra states for $E_6$, $4 \cdot 1 + 2 \cdot 2 + 1 \cdot 2 = 10$ extra states for $E_7$, and $5 \cdot 1 + 3 \cdot 2 + 2 \cdot 2 + 1 \cdot 4 + 1 \cdot 2 = 21$ extra states for $E_8$. As is seen from Table 1, adding these states to $r_G + 1$ gives exactly $h^\vee$ in all cases.

This latter fact has been established by Theorem 3c. Another way to understand it is the following. We have seen that the extra components $\mathcal{M}_k$ are associated with the nodes with $a_i^\vee > 1$ on the Dynkin diagrams. Just by construction, the sum (5.2) over the components with $m > 1$ coincides with the sum $\sum_{j}^{r_m} (a_j^\vee - 1)$ over all such nodes. Consider, e.g. the node 6 with $a_6^\vee = 6$ on the Dynkin diagram for $E_8$. It gives rise to two different isolated triples with $m = 6$ (and, correspondingly, to two different quantum states). Together with two nodes with $a_6^\vee = 3$, it gives rize to the component $\mathcal{M}_k$ based on the torus $T^{r_m=6}$ (and if the node 6 were not present in the superposition, the dimension of the torus would be one unit less, and we would have not 3, but just 2 vacuum states ). Finally, together with the nodes with $a_6^\vee = 2, 4$, it forms the torus $T^4$. As we have here $m = 3$, there are two different components $\mathcal{M}_k$ involving such torus and, correspondingly, two extra quantum states brought about by the node 6. All together, this node gives rise to $2 + 1 + 2 = 5 = a_6^\vee - 1$ extra states. The same counting works for all other nodes.

9Choosing a particular embedding $T^{r_m} \subset G$ amounts to fixing a particular gauge. After that, one should still require the invariance with the respect to a discrete group of Weyl transformations which is a remnant of the original gauge group after the embedding is fixed. The relevant groups $W_m$ are listed in Table 4.

10The wave functions in Eq. (5.1) depend only on the elements of the algebra and do not know about the global structure of the triples’ moduli space studied carefully in the previous section. Therefore, this structure does not affect the counting (5.2).
But the sum \( \sum_j (a_j^\vee - 1) \) is just the difference \( h^\vee - r_G - 1 \). Indeed, \( h^\vee \) coincides with the sum of all labels \( a_i^\vee \) in the extended Dynkin diagram \( \mathfrak{G} \) while \( r_G + 1 \) is just the number of its nodes.

At the end, we list again our results for higher exceptional groups. Further explicit formulae can be found in Appendix C.

- **\( F_4 \).**
  - i) There is a root \( 3 \) with \( a_3^\vee = 3 \). According to Theorem 1, there is only one associated element \( g = \exp\{2\pi i/3) \omega_3\} \). The centralizer of this element is \( SU(3) \times SU(3) \) (the Dynkin diagram for the centralizer is obtained from the extended Dynkin diagram of \( F_4 \) by removing the node \( 3 \)) factorized over the common center \( \mathbb{Z}_3 \). \( \mathbb{Z}_3 \) has two non-trivial elements and, correspondingly, there are two different non-trivial Heisenberg pairs in \( SU(3) \): the pair in Eq. (1.3) and its permutation. We have proven earlier \([\text{see the remark after Eq.}(3.5)]\) that all other Heisenberg pairs can be reduced to \([1.3]\) by conjugation. From this, we have two different isolated triples \( \{\Omega_i\} \) and two different corresponding quantum vacuum states. Note that, in this case, the second triple is obtained from the first one just by reordering \( \Omega_2 \leftrightarrow \Omega_3 \), but they are still two different triples from our viewpoint. They describe two different flat connections on \( T^3 \) which cannot reduced one to another by gauge transformation.

  - ii) There are two nodes (the long root \( 2 \) and the short root \( 4 \)) with \( a_i^\vee = 2 \). The centralizer of \( g = \exp\{2\pi i(s_2 \omega_2 + s_4 \omega_4)\} \), \( 2s_2 + 4s_4 = 1 \), is non-trivial. If \( s_2, s_4 \neq 0 \), it is \([SU(2)]^3 / \mathbb{Z}_2 \times U(1) \). Correspondingly, there is a family \( T^1 \times T^1 \times T^1 \) [up to a factorization over a certain discrete group — see Eq.(1.14) and also Appendix C for more details] of non-trivial commuting triples which gives rise to two extra quantum vacua.

- **\( E_6 \).**
  - i) The element \( \exp\{2\pi i/3) \omega_3\} \) associated with the node with \( a_i = 3 \) has a centralizer \([SU(3)]^3 / \mathbb{Z}_3 \). It gives two different isolated triples and two corresponding vacua.

  - ii) Three nodes with \( a_i = 2 \) produce a family \( \subset T^2 \times T^2 \times T^2 \) of non-trivial commuting triples. There are 3 quantum vacuum states associated with that.

- **\( E_7 \).**
  - i) The centralizer of the element \( \exp\{\pi i/2) \omega_4\} \) associated with the node with \( a_i = 4 \) is \( G_4 = \tilde{G}_4 / Z_4 \) where \( \tilde{G}_4 = [SU(4)]^2 \times SU(2) \) and the factorization over \( Z_4 \) implies the following identification of the elements of the center of \( \tilde{G}_4 \):

    \[ 1 \equiv (\pm I_4, \pm I_4, -I_2) \equiv (-I_4, -I_4, I_2) . \]  \tag{5.3}

The elements \( (\pm I_4, \pm I_4, -I_2) \) are of the fourth order. For each such element, there is unique up to conjugation Heisenberg pair is \( \tilde{G}_4 \): \( (\tilde{b}_4^{(4)}, \tilde{b}_4^{(4)}, i\sigma_3) \) and \( (\tilde{c}_4^{(4)}, \tilde{c}_4^{(4)}, i\sigma_2) \), where \( \tilde{b}_4^{(4)}, \tilde{c}_4^{(4)} \) are the standard Heisenberg pairs with center \( \pm I_4 \) in \( SU(4) \). The corresponding elements in the centralizer commute. Thereby, we have two different isolated triples and two new vacua.

\[ ^{11}\text{Note that the element } g = q^2 \text{ has also this property, but it gives nothing new because it is equivalent to } q \text{ by conjugation. This is guaranteed by Theorem 1a.} \]
\[ ^{12}\text{All the roots for the groups } E_{6,7,8} \text{ are long and there is no distinction between } a_i \text{ and } a_i^\vee. \]
ii) The centralizer $G_4$ admits also Heisenberg pairs based on the 2–order element of $\mathbb{Z}_4$. A standard Heisenberg pair in $\tilde{G}_4$ associated with that has the form $(\tilde{b}^{(2)}, \tilde{b}^{(2)}, 1); (\tilde{c}^{(2)}, \tilde{c}^{(2)}, 1)$, where $(\tilde{b}^{(2)}, \tilde{c}^{(2)})$ is the standard Heisenberg pair with center $-I_4$ in $SU(4)$. The centralizer of the triple formed by a canonical exceptional element $\exp\{i\pi/2 \omega_4\}$ and the commuting elements in $G_4$ corresponding to this Heisenberg pair is $[SU(2)]^3$ where first two $SU(2)$ factors are the centralizers of the pair $(\tilde{b}^{(2)}, \tilde{c}^{(2)})$ in each $SU(4)$ and the third one is the whole $SU(2)$ factor in $\tilde{G}_4$. The $3 + 3 –$ parametric family of triples following from that is actually a subset of a larger family lying in $T^3 \times T^3 \times T^3$ which is based on a generic exceptional element formed by the node 4 together with three other nodes with $a_i = 2$. The centralizer of a generic such element is $[SU(2)]^4/\mathbb{Z}_2 \times [U(1)]^3$ and the centralizer of a generic such triple is $[U(1)]^3$. From this, we obtain 4 extra quantum vacua.

iii) There is a one-parametric family of exceptional elements associated with the nodes with $a_i = 3$. The centralizer of a generic such element is $[SU(3)]^3/\mathbb{Z}_3 \times U(1)$. Correspondingly, there are two different components $\subset T^1 \times T^1 \times T^1$ in the moduli space of the triples $\mathcal{M}$, each of them giving 2 extra vacua.

- $E_8$. i) The element $\exp\{\pi i/3 \omega_6\}$ associated with the node with $a_i = 6$ has the centralizer $G_6 = \tilde{G}_6/\mathbb{Z}_6$ where $\tilde{G}_6 = SU(6) \times SU(3) \times SU(2)$ and the factorization over $\mathbb{Z}_6$ implies

$$1 \equiv (e^{\pi i/3}I_6, e^{2\pi i/3}I_3, -I_2) \equiv (e^{2\pi i/3}I_6, e^{4\pi i/3}I_3, I_2) \equiv \cdots \quad (5.4)$$

The center of $\tilde{G}_6$ has two elements of the 6-th order which are identified to 1 after factorization. They give rise to two different Heisenberg pairs in $\tilde{G}_6$ which correspond to commuting elements in $G_6$. We have two isolated triples and two extra vacua. Besides, $\mathbb{Z}_6$ has 2 elements of order 3 and an element of order 2. As was also the case for $E_7$, Heisenberg pairs associated with that are ”absorbed“ in the families iii) and iv) below, formed by a combination of several nodes.

ii) The element $\exp\{2\pi i/5 \omega_5\}$ associated with the node with $a_i = 5$ has the centralizer $[SU(5)]^2/\mathbb{Z}_5$. There are four different Heisenberg pairs in $SU(5)$ with the property $bc = \epsilon cb$, $bc = \epsilon^2 cb$, $bc = \epsilon^3 cb$ and $bc = \epsilon^4 cb$, $\epsilon = \exp\{2\pi i/5\}$. Correspondingly: 4 isolated triples and 4 extra vacua.

iii) There are two distinct components $\subset T^1 \times T^1 \times T^1$ associated with a 1–parametric mixture of the 4–exceptional elements $\exp\{\pi i/2 \omega_4\}$ and $\exp\{\pi i/2 \omega_4\}$ associated with two different nodes with $a_i = 4$. Each such component gives 2 vacua.

iv) There are two distinct components $\subset T^2 \times T^2 \times T^2$ formed by the nodes 6 and two nodes 3, each of them giving 3 vacua.

v) The component lying in $T^4 \times T^4 \times T^4$ formed by the nodes 6, two nodes 4 and two nodes 2 gives 5 extra vacua.

Note. A few days before we sent the first version of this paper to the archive, another paper on this subject appeared [11] where an explicit construction of the set of non–trivial triples with $m = 2$ for all exceptional groups has been done. It was followed by the paper
where the structure of triples with $m = 3, 4, 5, 6$ was analyzed. The results agree with ours. The question of completeness of this classification was not studied, however, in these papers. The theorems proven here guarantee that, indeed, all non–trivial triples lie in one of the families discussed above and the total number of quantum states always coincides, indeed, with the dual Coxeter number of the group.

Acknowledgements: V. K. wishes to thank E.B. Vinberg for very useful correspondence and remarks. A. S. is indebted to V. Rubtsov for useful discussions.

Appendix A: Heisenberg pairs in all connected simply connected compact Lie groups.

Here is a general way to construct a Heisenberg pair $(\tilde{b}, \tilde{c})$ with center $z$ in a simply connected simple compact Lie group $G$ (for semi-simple $G$ we just take the product of the $(\tilde{b}_i, \tilde{c}_i)$ over all simple factors $G_i$).

Recall that all non–trivial central elements of $G$ are of the form $e^{2\pi i \omega_j}$ where $j$ is such that $a_j = 1$. Pick such a $j$ and consider the set of roots $\{\alpha_0, \alpha_1, \ldots, \alpha_r\} \backslash \alpha_j$. This is another set of simple roots, hence there exists a unique element $w_j$ of the Weyl group $W$ of $G$ which transforms it to the set $\alpha_1, \ldots, \alpha_r$, hence $w_j \alpha_j = \alpha_0$. This gives us a canonical embedding of the center of $G$ in $W$: $e^{2\pi i \omega_j} \to w_j$, and, at the same time, in the group of symmetries of the extended Dynkin diagram $\tilde{D}(G)$.

Let $z = e^{2\pi i \omega_j}$ be a central element of $G$ of order $m \geq 2$. Consider the corresponding element of the Weyl group $w_j$, and denote by $\langle w_j \rangle$ a Let $I_{\langle w_j \rangle} = \langle w_j \rangle \cdot \alpha_0$ (the notation $\langle w_j \rangle$ stands like before for the cyclic subgroup of $W$ generated by the element $w_j$) be the subset of the set of nodes of $\tilde{D}(G)$ obtained by the action of the group $\langle w_j \rangle$ on the node $\alpha_0$ ($a_s = 1$ for all nodes from $I_{\langle w_j \rangle}$).

Let $\tilde{b} = \sigma_x$ as in Eq. (2.5), where $\sigma_x$ is defined by $s_i = 1/m$ if $i \in I_{\langle w_j \rangle}$ and $s_i = 0$ otherwise. Write $w_j = r_{\gamma_1} \cdots r_{\gamma_k}$ as a shortest product of reflections with respect to some (not necessarily simple) roots $\gamma_1, \ldots, \gamma_k$ and let

$$\tilde{c} = q_{\gamma_1} \cdots q_{\gamma_k},$$

where $q_{\gamma_t}$ is the Pauli matrix $i \sigma_2$ in the $SU(2)$ corresponding to the root $\gamma_t$ [or in other words, $q_{\gamma_t} = \exp \left\{ \frac{i}{2} (T_{\gamma_t} - T_{-\gamma_t}) \right\}$. The conjugation of any element $h$ of the Cartan subalgebra $\mathfrak{h}$ with the group element $q_{\gamma_t}$ presents the reflection with respect to the root $\gamma_t$:

$$(q_{\gamma_t})^{-1} h q_{\gamma_t} = h - \frac{2\langle h, \gamma_t^\vee \rangle}{\langle \gamma_t^\vee, \gamma_t^\vee \rangle} \gamma_t^\vee$$

Thereby, Eq. (A.2) presents a particular lifting of $w_j$ in $G$: for the elements of the maximal torus, $\tilde{c}^{-1} e^{ih} \tilde{c} = e^{i w_j h}$, and the action of this conjugation can of course be also defined.
We have \( \tilde{a} \) a diagram automorphism of \( SU(3) \) matrices:

For corresponding elements of the Weyl group present the cyclic permutations of the eigenvalues of the diagonal element from a maximal torus from the centralizer of \((\tilde{c})\). This proves \( \langle \rangle \sum_{\gamma_i=1}^{m} \gamma_i \) for all other elements of \( G \). The choice \( (A.2) \) is convenient, but one could take for \( \tilde{c} \) any other such lifting.

**Theorem A1.** Let \( G \) be as in Theorem 1 and let \( z = e^{-2\pi i \omega_j} \) be a central element of \( G \) of order \( m \geq 2 \).

(a) The pair \((\tilde{b}, \tilde{c})\) given by \((A.1)\) and \((A.2)\) is a Heisenberg pair with center \( z \).

(b) Any Heisenberg pair with center \( z \) in \( G \) can be conjugated to a Heisenberg pair of the form \((\tilde{b}t, \tilde{c}t')\), where \((\tilde{b}, \tilde{c})\) is given by \((A.1)\) and \((A.2)\) and \( t, t' \) lie in a fixed maximal torus \( T^{0} \) of the centralizer of \((\tilde{b}, \tilde{c})\). Two pairs from \( \mathcal{P} := (\tilde{b}T^{0}, \tilde{c}T^{0}) \) can be conjugated to each other if they are conjugated by the normalizer of \( \mathcal{P} \).

(c) The centralizer of a lowest dimensional Heisenberg pair of \( G \) with center \( z \) is the direct product of \( \langle \rangle \sum_{\delta} \) and a connected subgroup \( K \) listed in Table 6.

**Proof.** It is easy to see that \((\tilde{b}, \tilde{c})\) is a Heisenberg pair with center \( z \). Indeed, \( \tilde{c}^{-1} \tilde{b} \tilde{c} = e^{i w_j \lambda} \), where \( \lambda = (2\pi / m) \sum_{s \in I(w_j) \setminus 0} \omega_s \) and one can check that \( e^{i(w_j \lambda - \lambda)} = z \), hence \( \tilde{b} \tilde{c} = z \tilde{b} \tilde{c} (b) \) follows from Proposition 1. Finally, the centralizer of \( \tilde{b} \) in \( G \) is obtained by removing all nodes \( j \) with \( a_j \equiv 1 \) from \( \tilde{D}(G) \). Multiplying, if necessary, \( \tilde{c} \) by an element from a maximal torus from the centralizer of \((\tilde{b}, \tilde{c})\), we may assume that \( \tilde{c} \) induces a diagram automorphism of \( G_{\tilde{b}} \) for which the fixed point sets are well-known (see \([3\), Chapter 8\). This proves (c).

---

\(^{13}\)Let us illustrate how this general construction works in the simplest non-trivial case of \( G = SU(3) \). Fundamental coweights are \( \omega_1 = (2\alpha_1 + \alpha_2)/3 = (1/3) \text{diag}(2, -1, -1), \phantom{.} \omega_2 = (2\alpha_2 + \alpha_1)/3 = (1/3) \text{diag}(1, 1, -2) \). We have \( \exp\{-2\pi i \omega_1\} = e^1 \) and \( \exp\{-2\pi i \omega_2\} = e^21 \) (\( e = \exp(2\pi i / 3) \)). The corresponding elements of the Weyl group present the cyclic permutations of the eigenvalues of the diagonal \( SU(3) \) matrices: \( w_1 := (231) \) and \( w_2 := (312) \). In this case, \( \langle w_1 \rangle = \langle w_2 \rangle, I(w_{j}) \) coincides with the whole extended Dynkin diagram, and \( \tilde{b} = \exp\{2\pi i (\omega_1 + \omega_2)/3\} = \text{diag}(\epsilon, 1, \epsilon^5) \). Pick up \( w_1 = r_{\alpha_2} r_{\alpha_1} \). Eq. (A.2) is reduced to

\[
\tilde{c} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
\]

We have \( \tilde{b} \tilde{c} = e \tilde{c} \tilde{b} \) and the pair \((\tilde{b}, \tilde{c})\) thus constructed is equivalent by conjugation to the pair \((1.3)\).

\(^{14}\)Example: Consider the group \( E_7 \). The center of \( E_7 \) is \( Z_2 \) and the extended Dynkin diagram of \( E_7 \) has \( Z_2 \) mirror symmetry. The centralizer of \( \tilde{b} \) in \( E_7 \) is \( E_6 \). The centralizer of the pair \((\tilde{b}, \tilde{c})\) is a subgroup of
Table 7: Centralizers of exceptional triples associated with a given node, \( m = a_j \).

**Remark A1.** Theorem A1 gives a classification of all up to conjugacy commuting pairs in connected compact (not necessarily simply connected) Lie groups \( G \). In particular, this theorem shows that connected components of the set of pairs of commuting elements of \( G \) are in one-to-one correspondence with elements of the maximal finite subgroup of \( \pi_1(G) \).

Using Table 6, one immediately gets the list of centralizers of \( m \)-exceptional triples \((a, b, c)\), where \( a = \exp\{2\pi i\omega_j/a_j\}, \ m = a_j^\vee \) and \((b, c)\) is the lowest dimensional pair in \( G \).

\( E_6 \) invariant under the conjugation corresponding this mirror symmetry. The Dynkin diagram for such a subgroup is obtained from the Dynkin diagram of \( E_6 \) by gluing together two pairs of the (co)roots. This is the Dynkin diagram of \( F_4 \) [the embedding \( F_4 \subset E_6 \) thus constructed will be discussed in some more details in Appendix C: see Eq. (C.10)]. That is how the result quoted in the last line of the last column of Table 6 is obtained, and all other cases can be treated in a similar way. To avoid confusion, note that, in contrast to the centralizer of a single element, the centralizer of a Heisenberg pair need not, and generally is not a regular subgroup of \( G \).
| $G$ | node | $m$ | $G_{\text{triple}}$ |
|-----|------|-----|------------------|
| $E_7$ | 4    | 2   | $[SU(2)]^3$      |
| $E_8$ | 6    | 2   | $SU(3)^2$        |
| $E_8$ | 4    | 2   | $SU(2) \times Spin(7)$ |
| $E_8$ | 4'   | 2   | $SU(2) \times SU(4)$ |
| $E_8$ | 6    | 3   | $[SU(2)]^2$      |

Table 8: The same as in Table 7 with $m < a_j^\vee$. The node 4 for $E_8$ is placed on the ”long arm” and the node 4’ — on the ”short arm” of the Dynkin diagram.

In the case of $Spin(N)$ groups, these centralizers are as follows:

$$N = 2r + 1 : G_{\text{triple}}^{(k)} = Spin(2k-1) \times Spin(2r-2k-3), \ k = 1, \ldots, r-2$$

$$N = 2r : G_{\text{triple}}^{(k)} = Spin(2k-1) \times Spin(2r-2k-5), \ k = 1, \ldots, r-3 \quad (A.4)$$

where $k$ labels the nodes with $a_j^\vee = 2$ on the Dynkin diagram of the corresponding group. Note that in the case of even $N$ the maximal centralizer is $G_{\text{triple}}^{(1)} = G_{\text{triple}}^{(r-3)} = Spin(N-7)$ as was noticed before in \[4, 5\]. However, for $N = 2r + 1$, the maximal centralizer is $Spin(N-6)$. An example of the corresponding non–trivial triple for $Spin(9)$ is $(\gamma_1 \gamma_2 \gamma_3 \gamma_4, \gamma_1 \gamma_2 \gamma_5, \gamma_1 \gamma_3 \gamma_6)$. It can be embedded in $Pin(6)$, but not in $Spin(6)$.

For all exceptional groups these centralizers are listed in Table 7. The groups $E_{7,8}$ involve also the nodes with not prime $a_j^\vee = a_j$. These nodes admit triples with $m < a_j^\vee$, $m|a_j^\vee$. As was explained in some details in Sect. 5, the centralizer of the 2–exceptional triple in $E_7$ associated with the node 4 is $[SU(2)]^3$. The calculations for $E_8$ are done in a similar way and the results are listed in Table 8. Note that the centralizers of the triples with a given $m$ associated with different nodes might be different, but their rank is always the same as it of course should be in virtue of Proposition 1.

### Appendix B: Grassmann invariants of Weyl Groups.

Let $V = i \text{ Lie } T$ be the euclidean $r$–dimensional vector space on which the Weyl group $W$ acts via the reflection representation. We shall assume in this section that the group $G$ is (almost) simple, hence the representation of $W$ in $V$ is irreducible. The group $W$ acts diagonally on $V \oplus V$ and this action induces an action on the Grassmann algebra

$$\Lambda = \Lambda(V \oplus V).$$

Let $\Omega = \sum_i e_i^{(1)} \otimes e_i^{(2)}$ be the obvious quadratic invariant of $W$ in $V$, where $\{e_i^{(s)}\}_{i=1}^r$, $s = 1$ or 2, is the same orthonormal basis in both copies of $V$. Then $\Omega^j, \ j = 0, \ldots, r$, are linearly independent $W$–invariant elements of $\Lambda$.

**Theorem B1.** Elements $\Omega^j, \ j = 0, \ldots, r$, form a basis of the space $\Lambda^W$ of $W$–invariants in $\Lambda$.

**Proof.** It is a simple and well–known fact that, for any representation of a finite group $\Gamma$ in a finite–dimensional vector space $U$, the dimension of the space of invariants of $\Gamma$ in
the Grassmann algebra \( \Lambda(U) \) over \( U \) is given by the formula

\[
\dim \Lambda(U)^\Gamma = \frac{1}{\# \Gamma} \sum_{g \in \Gamma} \det(1 + g) .
\]  

(B.1)

Applying this to \( \Gamma = W, U = V \oplus V \), we see that Theorem B1 is equivalent to the following formula:

\[
\sum_{w \in W} [P_w(-1)]^2 = (r + 1) \# W ,
\]  

(B.2)

where \( P_w(x) \) stands for the characteristic polynomial \( \det(x - w) \) of \( w \) acting on \( V \). Fortunately, there is in Ref. [12] a complete table of characteristic polynomials for all Weyl groups which allows one to check (B.2) in all cases.

E. Vinberg pointed out that a conceptual proof of this theorem has been given by L. Solomon [13] long ago: Note that \( \dim \Lambda^W = \dim \text{End}_W[\Lambda(V)] \). But all \( \Lambda^j(V), j = 0, \ldots, r \), are irreducible and inequivalent representations of \( W \). Hence the right-hand side of the last equality equals \( r + 1 \).

**Remark B1.** In fact, one has a refinement of (B.1):

\[
\sum_{j \geq 0} \dim \Lambda^j(U)^\Gamma x^j = \frac{1}{\# \Gamma} \sum_{g \in \Gamma} \det(x + g) .
\]  

(B.3)

Hence, Theorem B1 is equivalent to a refinement of (B.2):

\[
1 + x^2 + x^4 + \ldots + x^{2r} = \frac{1}{\# W} \sum_{w \in W} [\det(x + g)]^2 .
\]  

(B.4)

Therefore, letting \( x = -1 \), we have:

\[
r + 1 = \frac{1}{\# W} \sum_{w \in W} \det(1 - w)^2 .
\]  

(B.5)

The number \( \det(1 - w) \) is called the defect of \( w \). The defect of the Coxeter elements is equal to \( \# \text{Center}(G) \). It follows from (B.3) that in the \( A_n \) case (and only in this case) all other elements have defect 0 (cf. [12]).

**Corollary B1.** The dimension of the space of the invariants of the group \( W_m \) on the Grassmann algebra over \( i \text{ Lie T}_m \oplus i \text{ Lie T}_m \) is \( r_m + 1 \).

**Proof.** In order to apply Theorem B1, we need to show that \( W_m \) contains an irreducible finite reflection group. To show this, remark that if \( T_m \) is a maximal torus of a compact subgroup, say \( K \), of \( G \), then \( W_m \) contains the Weyl group of \( K \). From the list of groups given in Table 7, we conclude that \( W_m \) contains the subgroup listed in the \( W_m \) column of Table 4.

**Remark B2.** It is not difficult to show that the group listed in the \( W_m \) column of Table 4 (denote this group by \( W'_m \)) coincides with \( W_m \). Indeed, we have:

\[
W'_m \subseteq W_m \subseteq W \cap GL(i \text{ Lie T}_m) ,
\]  

(B.6)
and one can check that in all cases the first group coincides with the last. In the most difficult $E_8$ case, for $m = 2$ (resp. 3), this follows from the fact that $W_{F_4}$ (resp. $W_{G_2}$) is a maximal finite subgroup of $GL(m, \mathbb{Z})$. This argument works in all cases except for two: i) $G = Spin(15)$ or $Spin(16)$ with $W_m = W_{B_4}$ which can in principle be embedded in $W_{F_4}$ and ii) $G = E_6$, $m = 2$ with $W_m = W_{A_2} \subset W_{G_2}$. However, the explicit calculations of Appendix C show that in all cases $\#W_m = \#W'_m$. Together with Eq.(B.6), that means that $W_m$ and $W'_m$ coincide.

**Corollary B2.** If $W_m = W_{X_r}$ in Table 4, then the centralizer of a lowest-dimensional $m$-exceptional triple in $G$ is a compact Lie group of type $X_r$. 

**Appendix C: Tori, fundamental alcoves and triples’ moduli space.**

We present here some explicit formulae for the fundamental weights $\omega_j$ entering the definition of the alcoves (2.4), (2.5), study the corresponding tori (3.6) and find the intersections $\Gamma_m$ of $T_m$ with the semi-simple parts of their centralizers $C_m$. We do it both by illustrative purposes and also because we need the groups $\Gamma_m$ to construct explicitly the moduli space (4.14) of the commuting triples.

We will not discuss here the groups $G_2$, $Spin(7)$, $Spin(8)$ where the 2-exceptional triples are isolated. That was done in enough details in the main text.

i) $Spin(2r+1)$, $r > 3$. The Dynkin diagram is drawn anew in Fig. 3. The fundamental alcove (2.4), (2.3) involves the weights $\omega_1, \ldots, \omega_{r-2}$ which can be conveniently written in this case in the orthonormal basis $e_j = i\gamma_{2j-1}\gamma_{2j}/2$:

$$
\begin{align*}
\omega_1 &= e_1 + e_2 = \theta^\vee \\
\ldots \\
\omega_{r-2} &= e_1 + e_2 + \ldots + e_{r-1}
\end{align*}
$$

(C.1)

A general expression (2.5) for the exceptional element can be thus rewritten in the form $\sigma_s = \exp\{\pi i(e_1 + e_2)\}t$, where

$$
   t = \exp\{\pi i(q_{r-3}e_3 + \ldots + q_{1}e_{r-1})\}
$$

(C.2)

with the condition $0 \leq q_1 \leq \ldots q_{r-3} \leq 1$. The volume of the fundamental alcove is $V_{dom} = 1/(r - 3)!$. Taking $a_2 = \exp\{\pi i\theta^\vee\}$ as the first element of the triple, we can pick
up a Heisenberg pair in \( \tilde{L}_2 = [SU(2)]^3 \) formed by the roots \((\alpha, \beta, \theta)\). The canonical triple \((a_2, b_2, c_2)\) thus constructed lies in \( L_2 \). It can be conjugated to the form

\[
a_2 = \gamma_1 \gamma_2 \gamma_3 \gamma_4, \quad b_2 = \gamma_1 \gamma_2 \gamma_{r-1} \gamma_{2r}, \quad c_2 = \gamma_1 \gamma_3 \gamma_{r-1} \gamma_{2r+1}
\]

(C.3)

which coincides with Eq. (1.3) up to the change \(5 \to 2r - 1, 6 \to 2r, 7 \to 2r + 1\). The fact that it can be embedded in \( Spin(7) \) is seen quite directly, but an universal way to demonstrate this which works also for exceptional groups is to trade the root \( \theta \) for

\[
\zeta = \frac{\theta - \alpha - 2\beta}{2} = \epsilon_1 + \ldots + \epsilon_{r-2}
\]

(C.4)

The roots \((\alpha, \zeta, \beta)\) present a system of simple roots for \( Spin(7) \).

The element \((C.2)\) lies in the torus \( T_2 \) which commutes with this \( Spin(7) \). The torus presents a subset of the domain \( \{0 \leq q_j \leq 4\} \) (indeed, \(\exp\{4\pi i e_j\} = 1\)). For \( r > 4 \), this domain (one may call it large torus) should, however, be factorized by identifications \(\{q_j\}: (0, \ldots, 0) \equiv (2, 2, 0, \ldots, 0) \equiv \ldots \) (indeed, \(\exp\{2\pi i (e_3 + e_4)\} = \ldots = 1\)). As is not difficult to see, the corresponding factor group involves \(2^{r-4}\) elements and the volume of the torus is \(V_{tor} = 4^{r-3}/2^{r-4} = 2^{r-2}\). Thereby, the fundamental alcove \((C.2)\) is obtained from the torus by factorization with a finite group with

\[
\frac{V_{tor}}{V_{dom}} = (r - 3)! \cdot 2^{r-2}
\]

(C.5)

elements. In the first place, this factorization is due to the action of the Weyl group \(W_m\) of the centralizer of our canonical triple. The continuous part of the latter is \(Spin(2r-6)\).

The corresponding Weyl group \(W_{D_{r-3}}\) involves \((r-3)! \cdot 2^{r-4}\) elements (the permutations of \(e_j, \ j = 3, \ldots, r-1\) and the reflection of a pair \(e_{j_1} \to -e_{j_1}, \ e_{j_2} \to -e_{j_2}\)). However, a centralizer of our canonical triple involves also an additional discrete \(Z_2\) factor. Indeed, the triple \((C.3)\) commutes with the element \(\gamma_2 \gamma_4 \gamma_5 \gamma_{2r}\) of the \(Spin(2r + 1)\) group. The conjugation of a generic triple by such an element amounts to the reflection with respect to \(e_5: \ q_{r-3} \to -q_{r-3}\). All together we obtain the group with \((r-3)! \cdot 2^{r-3}\) elements involving all possible permutations and reflections of \(e_j, \ j = 3, \ldots, r-1\). This is the group \(W_{B_{r-3}}\). Thus \(W_m\) coincides with the Weyl group of the maximal centralizer of the triple \([Spin(2r - 5)\) in this case] and we will see that this is also true in all other cases.

Besides, as was explained in Sect. 4, there might be conjugations corresponding to a multiplication by a non–trivial element of the center of \(Spin(7)\) which belongs also to the torus. Indeed, the element \(\exp\{2\pi ie_3\} \equiv \exp\{2\pi ie_4\} \equiv \ldots = -1\) of the torus belongs to the center of \(Spin(7)\) which is \(Z_2\) [in this simple case, it belongs also to the center of the large group \(Spin(2r + 1)\), but this will not be so for the exceptional groups].

The elements of \(W_{B_{r-3}}\) act on all elements of the triple simultaneously while the elements of \(Z_2\) act on each component separately. We arrive thereby at the result (4.14).

\[ \text{ii) } Spin(2r), \ r > 4. \] All calculations are exactly parallel to the calculations for \(Spin(2r + 1)\). One only has to substitute \(r \to r - 1\) in all formulae. The canonical triple

\[ \text{\footnote{This is so for the triple (C.3). As was mentioned before, one can also choose a triple whose centralizer is larger = Spin(2r - 5). Such a triple cannot, however, be embedded in } C_2 = Spin(7) \text{ and is not convenient for our purposes.} } \]

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is conveniently embedded in $Spin(8)$ (and further in $Spin(7)$). There is an element of the torus $T_2$ which coincides with the element $-1$ of the center of $Spin(8)$. Another non-trivial element of $\text{Center}(Spin(8))$, $\gamma_9 = \gamma_1 \cdots \gamma_8$, does not belong to the torus, however. Hence, $\Gamma_2[Spin(2r)] = Z_2$. In this case, the canonical triple has a maximal centralizer $Spin(2r - 7)$ and $W_2 = W_{B_{r-4}}$.

iii) $F_4$. The Dynkin diagram is depicted in Fig. 4. First, there are two isolated 3–exceptional triples with the first element $a_3 = \exp\left\{\frac{2\pi i}{3}\omega_3\right\}$, $\omega_3 = 3\alpha + 6\beta + 4\gamma + 2\delta^\vee$. Let us study the structure of the fundamental alcove and of the torus $T_2$ associated with the nodes with $a_j^\vee = 2$. We have

$$\begin{align*}
\omega_\alpha &= 2\alpha^\vee + 3\beta^\vee + 2\gamma^\vee + 6\delta^\vee = \theta^\vee \\
\frac{1}{2}\omega_\gamma &= 2\alpha^\vee + 4\beta^\vee + 3\gamma^\vee + \frac{3}{2}\delta^\vee
\end{align*}
$$

A general 2–exceptional element lying in the fundamental alcove can be written in the form

$$\sigma = \exp\{\pi i\theta^\vee\} \exp\{\pi i s\lambda\}, \quad (C.6)$$

where

$$\lambda = \frac{1}{2}\omega_\gamma - \omega_\alpha = \beta^\vee + \gamma^\vee + \frac{1}{2}\delta^\vee$$

and $0 \leq s \leq 1$.

The canonical triple with the first element $\exp\{\pi i\theta^\vee\}$ can be embedded in $[SU(2)]^3/Z_2$ based on the roots $(\theta\beta\delta)$ and further in $Spin(7)$ with the simple roots $(\beta\zeta\delta)$, where

$$\zeta = \frac{\theta - \beta - 2\delta}{2} = \alpha + \beta + 2\gamma \quad (C.8)$$

Note that the highest root $\theta$ for $F_4$ is also the highest root for its $C_2$ subgroup $Spin(7)$. This property also holds for $Spin$ groups [cf. Eq.(C.4)] and for higher exceptional groups.

The full torus $\exp\{\pi i s\lambda\}$ corresponds to the domain $(0 \leq s \leq 4)$ ($\exp\{4\pi i\lambda\} = 1$ due to the property $\exp\{2\pi i\kappa^\vee\} = 1$ for any coroot $\kappa^\vee$) and is 4 times larger than the fundamental alcove (C.7). The factorization over $Z_2$ comes from the Weyl group of our 1–dimensional torus $W_{A_1} = Z_2$ (with the non–trivial element acting as $s \to -s$). Another $Z_2 = \Gamma_2(F_4)$ corresponds to the shift $s \to s + 2$. Indeed, $\exp\{2\pi i\lambda\} = \exp\{\pi i\delta^\vee\} = -1$ as far as the $Spin(7)$ with the simple roots $(\beta\zeta\delta)$ is concerned.

iv) $E_6$. The Dynkin diagram is depicted in Fig. 4. There are two isolated 3–exceptional triples associated with the node $\gamma$ with the first element

$$a_3 = \exp\left\{\frac{2\pi i}{3}\omega_\gamma\right\}, \quad \omega_\gamma = 2(\alpha^\vee + \epsilon^\vee) + (\beta^\vee + \delta^\vee) + 6\gamma^\vee + 3\rho^\vee \quad (C.9)$$
Note that they can be embedded further in $F_4$. Indeed, the embedding $F_4 \subset E_6$ is realized by gluing the nodes of the Dynkin diagram for $E_6$ to its "mirror images": the root $\alpha$ is glued together with the root $\epsilon$ and the root $\beta$ together with the root $\delta$. To be quite exact, the coroots of the subgroup $F_4$ are expressed via the coroots of $E_6$ as follows:

$$(\alpha^\vee, \beta^\vee, \gamma^\vee, \delta^\vee)_{\text{notation as in Fig.4}} = (\rho^\vee, \gamma^\vee, \beta^\vee + \delta^\vee, \alpha^\vee + \epsilon^\vee)_{\text{notation as in Fig.5}} \quad (C.10)$$

One can see directly that the 3–exceptional element (C.9) for $E_6$ is also the 3–exceptional element for $F_4$. Two other elements of the $E_6$ triple present the Heisenberg pairs in the $SU(3)$ factors formed by the nodes $(\rho, -\theta)$, $(\alpha, \beta)$ and $(\delta, \epsilon)$. When going from $E_6$ down to $F_4$, two latter subgroups are glued together so that the second and the third element of the $E_6$ triple coincide with the second and the third element of the $F_4$ triple.

Let us study now 2–exceptional triples associated with the nodes $\beta, \delta, \rho$ in $E_6$. We have

$$\begin{align*}
\omega_\beta &= \frac{5}{3} (\alpha + 2\beta) + 4\gamma + 2\rho + \frac{4}{3} (\epsilon + 2\delta) \\
\omega_\delta &= \frac{4}{3} (\alpha + 2\beta) + 4\gamma + 2\rho + \frac{5}{3} (\epsilon + 2\delta) \\
\omega_\rho &= \alpha + \epsilon + 2(\beta + \delta + \rho) + 3\gamma = \theta \quad (C.11)
\end{align*}$$

(as the groups $E_{6,7,8}$ are simply laced, we will not bother to mark coroots with the symbol "^\vee" anymore).

The fundamental alcove of 2–exceptional elements is

$$\sigma = \exp\{\pi i \theta\} \exp\{\pi i (s_1 \lambda_1 + s_2 \lambda_2)\}, \quad s_1 + s_2 \leq 1, \ s_{1,2} \geq 0, \quad (C.12)$$

where

$$\begin{align*}
\lambda_1 &= \omega_\beta - \omega_\rho = \frac{1}{3} (2\alpha + 4\beta + 3\gamma + 2\delta + \epsilon) \\
\lambda_2 &= \omega_\delta - \omega_\rho = \frac{1}{3} (\alpha + 2\beta + 3\gamma + 4\delta + 2\epsilon) \quad (C.13)
\end{align*}$$
Figure 6: Torus and fundamental alcove for $E_6$.

We have
\[ \langle \lambda_1, \lambda_1 \rangle = \langle \lambda_2, \lambda_2 \rangle = 4/3, \quad \langle \lambda_1, \lambda_2 \rangle = 2/3 \]
so that the weights $\lambda_1, \lambda_2$ form the angle $\pi/3$.

The canonical triple with the first element $\exp\{\pi i \theta^\vee\}$ can be embedded in $[SU(2)]^4/\mathbb{Z}_2$ based on the roots $(\theta, \alpha, \gamma, \epsilon)$ and further in $Spin(8)$ with the simple roots $(\alpha \gamma \epsilon)$ and
\[ \zeta = \frac{\theta - \alpha - \gamma - \epsilon}{2} = \beta + \gamma + \delta + \rho \] (C.14)
The root $\zeta$ lies in the center of the Dynkin diagram for $Spin(8)$ for which $\theta$ serves as the highest root.

The full torus corresponds to the rhombus in Fig. 6. Indeed,
\[ \exp\{2\pi i (\lambda_1 + \lambda_2)\} = \exp\{2\pi i (2\lambda_1 - \lambda_2)\} = \exp\{2\pi i (2\lambda_2 - \lambda_1)\} = \ldots = 1 \]
( with the property $\exp\{2\pi i \alpha_j\} = 1$ repeatedly used). It is 24 times larger than the fundamental alcove (the small triangle in Fig. 6 ). The factorization by 24 comes first from the Weyl group of the small torus $W_{A_2}$ ( $\# W_{A_2} = 6$) and, second, from the group $\Gamma_2$ which coincides in this case with the full center of $Spin(8)$ which is $\mathbb{Z}_2 \times \mathbb{Z}_2$. Indeed, the elements
\[ \exp\{\pi i (\lambda_1 + \lambda_2)\}, \exp\{\pi i (2\lambda_1 - \lambda_2)\}, \exp\{\pi i (2\lambda_2 - \lambda_1)\} \] (C.15)
of the torus (marked out by the blobs in Fig. 6) depend only on the roots $\alpha, \gamma, \epsilon$ and belong therefore to our subgroup $C_2(E_6) = Spin(8)$. As the whole torus commutes with this $Spin(8)$, the elements (C.15) present 3 different non–trivial elements of its center.

v) $E_7$. The Dynkin diagram is depicted in Fig. 7.

- $m = 2$ . We have
\[
\begin{align*}
\omega_\kappa &= \alpha + 2\beta + 3\gamma + 4\delta + 3\epsilon + 2\kappa + 2\rho = \theta \\
\omega_\beta &= 2\alpha + 4\beta + 5\gamma + 6\delta + 4\epsilon + 2\kappa + 3\rho \\
\omega_\rho &= \frac{3}{2}\alpha + 3\beta + \frac{9}{2}\gamma + 6\delta + 4\epsilon + 2\kappa + \frac{7}{2}\rho \\
\frac{1}{2}\omega_\delta &= \frac{3}{2}\alpha + 3\beta + \frac{9}{2}\gamma + 6\delta + 4\epsilon + 2\kappa + 3\rho 
\end{align*}
\] (C.16)
Figure 7: Dynkin diagram for $E_7$.

The fundamental alcove is

$$\sigma = \exp\{\pi i \theta\} \exp\{\pi i (s_1 \lambda_1 + s_2 \lambda_2 + s_3 \lambda_3)\}, \quad s_1 + s_2 + s_3 \leq 1, \quad s_{1,2,3} \geq 0,$$

(C.17)

where

$$\lambda_1 = \omega_{\beta} - \omega_{\kappa} = \alpha + 2 \beta + 2 \gamma + 2 \delta + \epsilon + \rho,$$

$$\lambda_2 = \omega_{\rho} - \omega_{\kappa} = \frac{1}{2} \alpha + \beta + \frac{3}{2} \gamma + 2 \delta + \epsilon + \frac{3}{2} \rho,$$

$$\lambda_3 = \frac{1}{2} \omega_{\delta} - \omega_{\kappa} = \frac{1}{2} \alpha + \beta + \frac{3}{2} \gamma + 2 \delta + \epsilon + \rho$$

(C.18)

The canonical triple with the first element $\exp\{\pi i \theta\}$ can be embedded in $[SU(2)]^4/\mathbb{Z}_2$ and further in $Spin(8)$ with the simple roots $(\alpha \gamma \epsilon)$ and

$$\zeta = \frac{\theta - \alpha - \gamma - \epsilon}{2} = \beta + \gamma + 2 \delta + \epsilon + \kappa + \rho$$

(C.19)

The torus corresponds to the domain $(0 \leq s_1 \leq 2, \quad 0 \leq s_{2,3} \leq 4)$ (without further factorizations). The ratio

$$\frac{V_{tor}}{V_{alc}} = \frac{32}{1/6} = 4 \# W_{C_3}$$

(C.20)

is brought about by the factorization over the Weyl group of $Sp(6)$, the maximal centralizer of the triple (see Table 7; the Weyl group of $Sp(6)$ is the same as for the dual $Spin(7)$ and consists of $2^3 \cdot 3! = 48$ elements), and the factorization over $\Gamma_2(E_7) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Three non–trivial elements of $\Gamma_2(E_7)$ are the following elements of the torus:

$$\exp\{2 \pi i \lambda_3\} = \exp\{\pi i (\alpha + \gamma)\}$$

$$\exp\{\pi i (\lambda_1 + 2 \lambda_2)\} = \exp\{\pi i (\gamma + \epsilon)\}$$

$$\exp\{\pi i (\lambda_1 + 2 \lambda_2 + 2 \lambda_3)\} = \exp\{\pi i (\alpha + \epsilon)\}$$

(C.21)

- $m = 3$. The relevant nodes are $\gamma$ and $\epsilon$. We have

$$\omega_{\gamma} = \frac{5}{2} \alpha + 5 \beta + \frac{15}{2} \gamma + 9 \delta + 6 \epsilon + 3 \kappa + \frac{9}{2} \rho$$

$$\omega_{\epsilon} = 2 \alpha + 4 \beta + 6 \gamma + 8 \delta + 6 \epsilon + 3 \kappa + 4 \rho$$

(C.22)

\textsuperscript{16}We hope that the universal notation $s_j, \lambda_j$ for different tori considered will bring about no confusion.
The fundamental alcove is
\[ \sigma = \exp\{(2\pi i/3)\omega_\gamma\} \exp\{(2\pi i/3)s\lambda\}, \quad 0 \leq s \leq 1, \] (C.23)
where
\[ \lambda = \omega_\gamma - \omega_\epsilon = \frac{1}{2}\alpha + \beta + \frac{3}{2}\gamma + \delta + \frac{1}{2}\rho \] (C.24)
The canonical triple with the first element \(\exp\{(2\pi i/3)\omega_\gamma\}\) can be embedded in \([SU(3)]^3/\mathbb{Z}_3\) and further in \(E_6\) with the simple roots \((\alpha\beta, \rho\delta, \kappa)\) and the root
\[ \zeta = \frac{\theta - \alpha - \delta - 2(\beta + \rho + \kappa)}{3} = \gamma + \delta + \epsilon \] (C.25)
with nonzero projections \(\langle \zeta, \beta \rangle = \langle \zeta, \rho \rangle = \langle \zeta, \kappa \rangle = -1\).
The torus corresponds to the domain \((0 \leq s \leq 6)\). It gives the fundamental alcove after factorization over \(W_{A_1} = \mathbb{Z}_2\) and over the group \(\Gamma_3(E_7) = \mathbb{Z}_3\). One can see, indeed, that the elements \(\exp\{(4\pi i/3)\lambda\}\) and \(\exp\{(8\pi i/3)\lambda\}\) of the torus in Eq.(C.23) belong to \(E_6\) constructed above and to its center.

• There are also two isolated triples with \(m = 4\).

vi) \(E_8\). The Dynkin diagram is depicted in Fig. 8.

• \(m = 2\) . This is the most rich and beautiful case. There are 5 relevant nodes: \(\alpha, \gamma, \epsilon, \kappa, \mu\). We have
\[
\begin{align*}
\omega_\alpha &= 2\alpha + 3\beta + 4\gamma + 5\delta + 6\epsilon + 4\kappa + 2\mu + 3\rho = \theta \\
\frac{1}{2}\omega_\gamma &= 2\alpha + 4\beta + 6\gamma + \frac{15}{2}\delta + 9\epsilon + 6\kappa + 3\mu + \frac{9}{2}\rho \\
\frac{1}{2}\omega_\kappa &= 2\alpha + 4\beta + 6\gamma + 8\delta + 10\epsilon + 7\kappa + \frac{7}{2}\mu + 5\rho \\
\frac{1}{3}\omega_\epsilon &= 2\alpha + 4\beta + 6\gamma + 8\delta + 10\epsilon + \frac{20}{3}\kappa + \frac{10}{3}\mu + 5\rho \\
\omega_\mu &= 2\alpha + 4\beta + 6\gamma + 8\delta + 10\epsilon + 7\kappa + 4\mu + 5\rho \\
\end{align*}
\] (C.26)
The fundamental alcove is
\[ \sigma = \exp\{\pi i\theta\} \exp\{\pi i(s_1\lambda_1 + s_2\lambda_2 + s_3\lambda_3 + s_4\lambda_4)\}, \quad s_1 + s_2 + s_3 + s_4 \leq 1, \ s_{1,2,3,4} \geq 0, \] (C.27)
where

\[
\begin{align*}
\lambda_1 &= \frac{1}{2} \omega_\gamma - \omega_\alpha = \beta + 2\gamma + 5\delta + 3\epsilon + 2\kappa + \mu + \frac{3}{2} \rho \\
\lambda_2 &= \frac{1}{3} \omega_\epsilon - \omega_\alpha = \beta + 2\gamma + 3\delta + 4\epsilon + \frac{8}{3} \kappa + \frac{4}{3} \mu + 2\rho \\
\lambda_3 &= \frac{1}{2} \omega_\kappa - \omega_\alpha = \beta + 2\gamma + 3\delta + 3\epsilon + \frac{3}{2} \mu + 2\rho \\
\lambda_4 &= \omega_\mu - \omega_\alpha = \beta + 2\gamma + 3\delta + 4\epsilon + 3\kappa + 2\mu + 2\rho
\end{align*}
\]  

(C.28)

The canonical triple with the first element \(\exp\{\pi i \theta\}\) can be embedded in \(\left[ SU(2)^4 / \mathbb{Z}_2 \right]\) and further in \(Spin(8)\) with the simple roots \((\beta \delta \rho)\) and the central root

\[
\zeta = \frac{\theta - \beta - \delta - \rho}{2} = \alpha + \beta + 2\gamma + 2\delta + 3\epsilon + 2\kappa + \mu + \rho
\]  

(C.29)

The torus corresponds to the domain \((0 \leq s_{1,3} \leq 4, 0 \leq s_2 \leq 6, 0 \leq s_4 \leq 2)\). The ratio

\[
\frac{V_{\text{tor}}}{V_{\text{alc}}} = \frac{192}{1/24} = 4\#W_{F_4}
\]  

(C.30)

is brought about by the factorization over the Weyl group of the maximal centralizer of the triple \(F_4\) \((\#W_{F_4} = 1152)\) and over \(\Gamma_2(E_8) = \mathbb{Z}_2 \times \mathbb{Z}_2\). Three non–trivial elements of \(\Gamma_2(E_8)\) are the following elements of the torus:

\[
\begin{align*}
\exp\{2\pi i \lambda_1\} &= \exp\{\pi i (\delta + \rho)\} \\
\exp\{3\pi i \lambda_2\} &= \exp\{\pi i (\beta + \delta)\} \\
\exp\{\pi i (2\lambda_1 + 3\lambda_2)\} &= \exp\{\pi i (\beta + \rho)\}
\end{align*}
\]  

(C.31)

\[m = 3.\] The relevant nodes are \(\beta, \epsilon\) and \(\rho\). The corresponding fundamental weights are

\[
\begin{align*}
\omega_\beta &= 3\alpha + 6\beta + 8\gamma + 10\delta + 12\epsilon + 8\kappa + 4\mu + 6\rho \\
\frac{1}{2} \omega_\epsilon &= 3\alpha + 6\beta + 9\gamma + 12\delta + 15\epsilon + 10\kappa + 5\mu + \frac{15}{2} \rho \\
\omega_\rho &= 3\alpha + 6\beta + 9\gamma + 12\delta + 15\epsilon + 10\kappa + 5\mu + 8\rho
\end{align*}
\]  

(C.32)

The fundamental alcove is

\[
\sigma = \exp\{(2\pi i/3)\omega_\beta\} \exp\{(2\pi i/3)(s_1 \lambda_1 + s_2 \lambda_2)\},
\]  

\[s_{1,2} \geq 0, \quad s_1 + s_2 \leq 1, \quad \]  

(C.33)

where

\[
\begin{align*}
\lambda_1 &= \frac{1}{2} \omega_\epsilon - \omega_\beta = \gamma + 2\delta + 3\epsilon + 2\kappa + \mu + \frac{3}{2} \rho \\
\lambda_2 &= \omega_\rho - \omega_\beta = \gamma + 2\delta + 3\epsilon + 2\kappa + \mu + 2\rho
\end{align*}
\]  

(C.34)
The canonical triple with the first element \(\exp\{(2\pi i/3)\omega_\beta\}\) can be embedded in \([SU(3)]^3/\mathbb{Z}_3\) and further in \(E_6\) with the simple roots \((\alpha, \gamma, \delta, \mu, \kappa)\) and the root
\[
\zeta = \frac{\theta - \gamma - \kappa - 2(\alpha + \delta + \mu)}{3} = \beta + \gamma + \delta + 2\epsilon + \kappa + \rho \tag{C.35}
\]
with nonzero projections \(\langle \zeta, \alpha \rangle = \langle \zeta, \delta \rangle = \langle \zeta, \mu \rangle = -1\).

The torus corresponds to the domain \((0 \leq s_1 \leq 6, \ 0 \leq s_2 \leq 3)\). Its volume is 36 times larger than the volume of the fundamental alcove \((C.33)\). The latter is obtained from the torus after factorization over \(W_{G_2}\) (the Weyl group for \(G_2\) has 12 elements involving besides the elements from \(W_{A_2}\) also 6 elements \(-W_{A_2}\)) and further over the group \(\Gamma_3(E_8) = \mathbb{Z}_3\). Indeed, the elements \(\exp\{(4\pi i/3)\lambda_1\}\) and \(\exp\{(8\pi i/3)\lambda_1\}\) of the torus belong to \(E_6\).

- \(m = 4\). The relevant nodes are \(\gamma\) and \(\kappa\). We have
\[
\omega_\gamma = 4\alpha + 8\beta + 12\gamma + 15\delta + 18\epsilon + 12\kappa + 6\mu + 9\rho \\
\omega_\kappa = 4\alpha + 8\beta + 12\gamma + 16\delta + 20\epsilon + 14\kappa + 7\mu + 10\rho \tag{C.36}
\]
The fundamental alcove is
\[
\sigma = \exp\{(\pi i/2)\omega_\gamma\} \exp\{(\pi i/2)s\lambda\}, \quad 0 \leq s \leq 1, \tag{C.37}
\]
where
\[
\lambda = \omega_\kappa - \omega_\gamma = \delta + 2\epsilon + 2\kappa + \mu + \rho \tag{C.38}
\]
The canonical triple with the first element \(\exp\{(\pi i/2)\omega_\gamma\}\) can be embedded in \([SU(4) \times SU(4) \times SU(2)]/\mathbb{Z}_4\) and further in \(E_7\) with the simple roots \((\alpha, \beta, \mu, \rho, \delta)\) and the root
\[
\zeta = \frac{\theta - \delta - 2(\alpha + \epsilon + \mu) - 3(\beta + \rho)}{4} = \gamma + \delta + \epsilon + \kappa \tag{C.39}
\]
with nonzero projections \(\langle \zeta, \beta \rangle = \langle \zeta, \rho \rangle = \langle \zeta, \mu \rangle = -1\).

The torus corresponds to the domain \((0 \leq s \leq 4)\). It is four times larger than the alcove in Eq.\((C.37)\). The latter is obtained from the torus after factorization over \(W_{A_1} = \mathbb{Z}_2\) and further over the group \(\Gamma_4(E_8) = \mathbb{Z}_2\). Indeed, \(\exp\{\pi i\lambda\} = \exp\{\pi i(\delta + \mu + \rho)\} \in E_7\).

- And there are, of course, also "genuine" \(E_8\) triples with \(m = 5\) and \(m = 6\) which are isolated.

**Appendix D: Minimal non-trivial \(n\)-tuples.**

The main subject of this paper were the non-trivial commuting triples. But one can equally well be interested in non-trivial quadruples, quintuples etc. If we define a non-trivial \(n\)-tuple as an (ordered) set of \(n\) commuting elements which cannot be simultaneously conjugated to the maximal torus, they are abundant in any group which admits a
nontrivial triple: just pick up such a triple and complete it with arbitrary $n - 3$ commuting elements from the centralizer of this triple.

It is clear that the classification of non-trivial commuting $n$-tuples is reduced to the classification of $n$-tuples of a special kind which we will call minimal:

**Definition.** A minimal commuting $n$-tuple is a non-trivial commuting $n$-tuple such that its any $m$-element subset with $m < n$ is a trivial $m$-tuple.

**Example D1 [14]**. Consider the following set of 4 elements of $SO(16)$:

\[
\begin{align*}
\Omega_1 &= \text{diag}(-1, -1, -1, -1, -1, -1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) \\
\Omega_2 &= \text{diag}(-1, -1, -1, 1, 1, 1, 1, -1, -1, -1, 1, 1, 1, 1, 1, 1) \\
\Omega_3 &= \text{diag}(-1, -1, 1, 1, -1, -1, 1, 1, -1, 1, 1, -1, 1, 1, 1, 1) \\
\Omega_4 &= \text{diag}(-1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, 1, 1, 1, 1) \\
\end{align*}
\]

This quadruple is a natural generalization of the non-trivial triple in $Spin(8)$ written in [13]: write a $4 \times 16$ matrix whose columns are all possible 4-vectors with coordinates $\pm 1$ taken once; then the 4 rows of this matrix viewed as diagonal matrices in $SO(16)$ is the quadruple (D.1).

Obviously, $[\Omega_i, \Omega_j] = 0$, and also the liftings of $\Omega_i$ up to $Spin(16)$:

\[
\begin{align*}
\tilde{\Omega}_1 &= \gamma^1\gamma^2\gamma^3\gamma^4\gamma^5\gamma^6\gamma^7\gamma^8 \\
\tilde{\Omega}_2 &= \gamma^1\gamma^2\gamma^3\gamma^4\gamma^9\gamma^{10}\gamma^{11}\gamma^{12} \\
\tilde{\Omega}_3 &= \gamma^1\gamma^2\gamma^5\gamma^6\gamma^9\gamma^{10}\gamma^{13}\gamma^{14} \\
\tilde{\Omega}_4 &= \gamma^1\gamma^3\gamma^5\gamma^7\gamma^9\gamma^{11}\gamma^{13}\gamma^{15}
\end{align*}
\]

[a direct generalization of Eq.(1.5)] commute.

The quadruple (D.2) is non-trivial. Indeed, consider the centralizer $G_{\tilde{\Omega}_i} = [Spin(8) \times Spin(8)]/\mathbb{Z}_2$. In this case, the presence of the $\mathbb{Z}_2$ factor over which the product $Spin(8) \times Spin(8)$ is factorized is irrelevant. What is relevant is that each $Spin(8)$ factor admits a non-trivial triple. It is not difficult to see that $(\Omega_2, \tilde{\Omega}_3, \tilde{\Omega}_4)$ present indeed a non-trivial triple in each $Spin(8)$ factor. Therefore, $(\tilde{\Omega}_2, \tilde{\Omega}_3, \tilde{\Omega}_4)$ cannot be conjugated to the maximal torus in $G_{\tilde{\Omega}_i}$ and the whole quadruple (D.2) cannot be conjugated to the maximal torus in $Spin(16)$.

On the other hand, any pair like $(\tilde{\Omega}_2, \tilde{\Omega}_3)$ can be conjugated to the maximal torus in $G_{\tilde{\Omega}_i}$, and hence the triple $(\tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3)$ can be conjugated to the maximal torus in $G$. Also the triple $(\tilde{\Omega}_2, \tilde{\Omega}_3, \tilde{\Omega}_4)$ is trivial. One can be directly convinced that it can be conjugated to the maximal torus in $Spin(16)$ involving the generators

\[
T_{1,9, \ldots, T_{8,16}}.
\]

This torus kind of “mixes” two $Spin(8)$ factors in $G_{\tilde{\Omega}_i}$.

Thereby, the quadruple (D.2) is a minimal quadruple. Since $Spin(16)$ involves a unique up to conjugacy element whose centralizer involves two $Spin(8)$ factors and since there is a unique up to conjugacy triple $(\Omega_2, \tilde{\Omega}_3, \tilde{\Omega}_4)$ in $Spin(8) \times Spin(8)$ which is non-trivial in both factors, it follows that all minimal quadruples in $Spin(16)$ are equivalent.
by conjugation to Eq. (D.2): the quadruple (D.2) is isolated. The quadruple (D.2) does not involve $\gamma^{16}$ and presents obviously also a minimal isolated quadruple in $Spin(15)$ [its image in $SO(15)$ coincides with Eq. (D.1) with the last column crossed out].

**Example D2.** The construction of Example D1 is easily generalized to minimal $n$–tuples with arbitrary $n \geq 4$. Write a $4 \times 2^n$ matrix whose columns are all possible $n$–vectors with coordinates $\pm 1$ taken once and treat the rows as a set of $n$ diagonal $SO(2^n)$ matrices. It is not difficult to see that the set of the corresponding $\tilde{\Omega}_1, \ldots, \tilde{\Omega}_n$ of remaining $n - 1$ elements of our set presents the minimal $(n - 1)$–tuple in each $Spin(2^{n-1})$ factor. Therefore this set is not conjugable to the maximal torus in the centralizer and the whole set of $n$ elements in not conjugable to the maximal torus in $Spin(2^n)$. This non–trivial $n$–tuple is minimal: any set of $n - 1$ elements including $\tilde{\Omega}_1$ is conjugable to the maximal torus due to the fact that the $(n - 1)$–tuple $\tilde{\Omega}_2, \ldots, \tilde{\Omega}_n$ is minimal in each simple factor of $\tilde{G}_{\tilde{\Omega}_1}$, and also the set $\tilde{\Omega}_2, \ldots, \tilde{\Omega}_n$ is conjugable to the maximal torus in $Spin(2^n)$: this torus is formed by the generators $T_{1,2^{n-1}+1}, \ldots, T_{2^n-1,2^n}$.

The constructed $n$–tuple in $Spin(2^n)$ is isolated and presents obviously also an isolated $n$–tuple in $Spin(2^n - 1)$.

The minimal $n$–tuples exist also for $Spin(N)$ groups with $N = 2^n + 2s$ and $N = 2^n + 2s - 1, s > 0$, but they are not isolated anymore, a moduli space appears. Consider a subgroup $Spin(2^n) \subset Spin(N)$ and pick up the minimal $n$–tuple $\tilde{\Omega}_1, \ldots, \tilde{\Omega}_n$ in $Spin(2^n)$. Its centralizer is a group of rank $s$. Let its maximal torus be $T^s$. The set of elements $\tilde{\Omega}_1 t_1, \ldots, \tilde{\Omega}_n t_n, t_j \in T^s$ presents also a minimal $n$–tuple. Thus, for $N > 2^n$ the moduli space of minimal $n$–tuples is a subset of $\tilde{\Omega}_1 T^s \times \cdots \times \tilde{\Omega}_n T^s$. Moreover, one can easily generalize the analysis of Section 4 and Appendix C to this case and find the moduli space exactly. We have

$$\mathcal{M}_{n-\text{tuple}}^{Spin(2^n + 2s - 1)} = \mathcal{M}_{n-\text{tuple}}^{Spin(2^n + 2s)} = \left[ \frac{\tilde{\Omega}_1 T^s}{\mathbb{Z}_2} \times \cdots \times \frac{\tilde{\Omega}_n T^s}{\mathbb{Z}_2} \right] / W_{B_s}, \quad (D.4)$$

where $W_{B_s}$ is the Weyl group of the maximal centralizer of a minimal $n$–tuple in $Spin(N)$ [this centralizer is $Spin(2s + 1)$] and $\mathbb{Z}_2$ is the intersection of $T^s$ with the center of $Spin(2^n)$ or $Spin(2^n - 1)$ for even and odd $N$, respectively.

**Example D3.** Consider the group $E_8$ and take the element $g = \exp\{i \pi \omega_n\}$ [the notations are as in Fig. 8]. By Theorem 1, its centralizer is $Spin(16)/\mathbb{Z}_2$. Take the minimal quadruple (D.2) in $Spin(16)$. Denote by $Q$ the corresponding set of $E_8$ elements. The same reasoning as before shows that the quintuple $(g, Q)$ is non–trivial. On the other hand, we will see later that $E_8$ does not admit minimal quadruples which means that $Q$ is trivial in $E_8$. It follows then that our quintuple is minimal. Note that, as is also the case for the minimal quadruple (D.2) and all non–trivial triples and minimal $n$–tuples considered before, the first element $g$ in the quintuple $(g, Q)$ does not play a special role: all other elements of this quintuple are equivalent to $g$ by conjugation.

The groups $Spin(2^n)$ or $Spin(2^n - 1)$ are the centralizers of $T^s$ in large $Spin(N)$ group and play the role of the subgroups $C_m$ listed in Table 4 which were relevant when discussing the moduli space of non–trivial commuting triples.
Theorem D1. (a) $SU(N)$ and $Sp(N)$ contain no non–trivial $n$–tuples.

(b) $Spin(N)$ for $N < 15$ and the groups $G_2$, $F_4$, $E_6$, $E_7$ contain no minimal $n$–tuples for $n > 3$.

(c) $E_8$ contains no minimal $n$–tuples for $n = 4$ and $n > 5$. $E_8$ contains a unique up to conjugacy minimal quintuple and it is described in Example D3.

(d) All minimal $n$–tuples with $n > 3$ in $Spin(N)$, $N \geq 15$ are described in Example D2.

Proof. (a) is well known (and follows from Theorem 1c). Next, if $a_i$ is an element of a minimal $n$–tuple $(a_1, \ldots, a_n)$ in $G$ with $n > 3$, then at least one of the simple components of $G_{a_1}$ should not be of $A$ or $C$ type. Indeed, otherwise either the preimage $(\tilde{a}_2, \ldots, \tilde{a}_n)$ in $\tilde{G}_{a_1}$ is a trivial $(n - 1)$–tuple in which case the whole $n$–tuple is trivial in $G$ [due to (a)] or some pair $\tilde{a}_i, \tilde{a}_j$ ($1 < j < j \leq n$) does not commute in which case the triple $(a_1, a_i, a_j)$ is non–trivial contradicting the minimality assumption.

(b) The centralizer of any element of $Spin(7)$ or $Spin(8)$ not belonging to the center presents a product of simple factors of $A$ or $C$ type and the remark above applies. Consider $Spin(9)$ and suppose that it contains a minimal $n$–tuple $(a_1, \ldots, a_n)$ with $n \geq 4$. By the same remark, the only possibilities for the semi–simple part of $G_{a_1}$ (and for all $G_{a_i}$ as well) to be discussed are $Spin(7)$ or $Spin(8)$ (see Theorem 1) and $(a_2, \ldots, a_n)$ should be a non–trivial $(n - 1)$–tuple in $G_{a_1}$. We have just shown that $Spin(7)$ and $Spin(8)$ do not admit minimal $n$–tuples with $n \geq 4$, hence $G_{a_1}$ should contain a non–trivial triple. A non–trivial triple of $Spin(8)$ presents also a non–trivial triple in its subgroup $Spin(7)$. But it follows from Appendix C and Section 4 that the isolated triple $(a, b, c)$ in $Spin(7)$ remains non–trivial in $Spin(9)$ and that all non–trivial triples in $Spin(9)$ are up to conjugacy of the form

$$(at_1, bt_2, ct_3),$$

where $t_i$ lie in a maximal torus of the centralizer of $Spin(7)$ in $Spin(9)$. Hence our $n$–tuple is not minimal.

Consider $Spin(9)$, $9 < N < 15$. The centralizer of any element of such a group can involve only one simple factor which is not of $A$ or $C$ type, and this factor is of $Spin(M < N)$ (i.e. $B$ or $D$) type. For an $n$–tuple $(a_1, \ldots, a_n)$ to be non–trivial, the $(n - 1)$–tuple $(a_2, \ldots, a_n)$ should be non–trivial in $G_{a_1}$. Using the previous results and inductive reasoning, we deduce that $G_{a_1}$ and its $Spin(M)$ factor should contain a non–trivial triple. Again, the analysis of Appendix C and Section 4 shows that any such triple is of the form $[D, 3]$ where $(a, b, c)$ is a canonical triple built up according to the rules of Appendix C which can be embedded in $Spin(7)$, and $t_i$ lie on the maximal torus of the centralizer of the canonical triple [and $Spin(7)$] in $Spin(M)$. The same applies to $Spin(N)$. These maximal tori can be embedded one into another and hence a non–trivial triple of $Spin(M)$ is also a non–trivial triple in $Spin(N)$.

Let now $G$ be one of the exceptional groups. A centralizer of any element in $G_2$ involves only semi–simple factors of $A$ or $C$ type and hence $G_2$ does not contain a non–trivial $n$–tuple with $n > 3$. Consider $F_4$. There are elements with centralizer involving the factor $Spin(7)$ or $Spin(9)$. Again, the construction of the Appendix C and the reasoning above can be used to show that any non–trivial triple of $Spin(7)$ or $Spin(9)$ remains also non–trivial in $F_4$. In the $E_6$ case, the only possible semi–simple factors in the centralizer
which are not of A or C type are $\text{Spin}(8)$ or $\text{Spin}(10)$, and we argue in the same way as for $\text{Spin}(9)$ and $F_4$. In the $E_7$ case, the only possible semi–simple factors in the centralizer which are not of A or C type are $\text{Spin}(8)$, $\text{Spin}(10)$, $\text{Spin}(12)$ or $E_6$, and we argue in the same way as above.

To prove $\langle c \rangle$, note first that the only “interesting” possibilities for the simple factors of (the universal covering of) the centralizer of an element $g \in E_8$ (the factors which are not of A or C type) are the following: (i) $E_7$, $E_6$, $\text{Spin}(12)$, $\text{Spin}(10)$, $\text{Spin}(8)$; (ii) $\text{Spin}(16)$, and there can be only one such “interesting” factor. Arguing as above, we show that any non–trivial triple of $G_{a_1}$ remains non–trivial in $E_8$, hence $E_8$ has no minimal quadruples. In case (i), $G_{a_1}$ contains no minimal $n$–tuples for $n \geq 4$ and hence $E_8$ has no non–trivial $n$–tuples for $n \geq 5$. In the remaining case (ii), $G_{a_1} = \text{Spin}(16)/\mathbb{Z}_2$ contains a unique up to conjugacy minimal quintuple which gives us the unique up to conjugacy minimal quintuple in $E_8$ constructed in Example D3.

Also $E_8$ contains no minimal $n$–tuples with $n > 5$ since $G_{a_1}$ contains no minimal $n$–tuples with $n \geq 5$ (see Example D2 and also the final part of the proof).

Consider finally the groups $\text{Spin}(N)$, $N \geq 15$. Let us explain first why the reasoning used for the groups with $N < 15$ does not work in this case. The assumption that $\text{Spin}(N)$ contains a non–trivial quadruple $(a_1, a_2, a_3, a_4)$ allows one to conclude that the triple $(a_2, a_3, a_4)$ is non–trivial in the centralizer $G_{a_1}$. If $G_{a_1}$ involves only one simple factor admitting a non–trivial triple, we can argue as above to show that the triple $(a_2, a_3, a_4)$ remains non–trivial in $\text{Spin}(N)$, etc. But for $N \geq 15$, there is at least one element $g \in \text{Spin}(N)$ such that the semi–simple part of its centralizer $G_g$ has two simple factors admitting a non–trivial triple. Let $a_1$ be such an element and let $K$ and $H$ be such factors. By Theorem 1, $K \equiv \text{Spin}(M_1)$ and $H \equiv \text{Spin}(M_2)$ with $M_{1,2} \geq 7$ and $M_1 + M_2 \leq N$. A non–trivial triple in $G_{a_1}$ can in this case be of different types. In can be non–trivial in only one such factor and be trivial in another factor. In that case, the triple $(a_2, a_3, a_4)$ can be presented in the form (D.5), i.e. it remains non–trivial in $\text{Spin}(N)$.

But a triple which is non–trivial both in $K$ and in $H$ becomes trivial in the large group. Indeed, a non–trivial triple in $K$ has the form $(a_K t_1^K, b_K t_2^K, c_K t_3^K)$ where $(a_K, b_K, c_K) \in \text{Spin}(7) \subseteq K$ and $t_1^K$ lie on the maximal torus $T^K$ of the centralizer of the canonical triple $(a_K, b_K, c_K)$ in $K$. The same concerns the triple in $H$. One can choose now a maximal torus in $\text{Spin}(N)$ which involves $T^K$, $T^H$, and also the torus formed by the generators like in Eq. (D.3) which “mix” two $\text{Spin}(7)$ factors embedding the canonical triples $(a_K, b_K, c_K)$ and $(a_H, b_H, c_H)$. It is clear that this maximal torus contains the triple $(a_2, a_3, a_4)$ which is thereby trivial and the quadruple $(a_1, a_2, a_3, a_4)$ is minimal.

In the range $15 \leq N < 31$, the $\text{Spin}(N)$ groups admit minimal quadruples, but not minimal quintuples, 6–tuples etc. The reason is that the centralizer of any element of such $\text{Spin}(N)$ group contains at most one simple factor admitting a minimal quadruple and the reasoning which proves the clause $\langle b \rangle$ above works in this case. Starting from $\text{Spin}(31)$, the elements whose centralizer involves two simple factors admitting a minimal quadruple appear. Let $a_1$ be such element and let $(a_2, a_3, a_4, a_5)$ presents a minimal quadruple in each such factor. Arguing as above we show that the quadruple $(a_2, a_3, a_4, a_5)$ is trivial in $\text{Spin}(N)$ and the quintuple $(a_1, \ldots, a_5)$ is minimal. Likewise, the groups $\text{Spin}(N)$ admit minimal 6–tuple starting from $N = 63$, etc.

For $N > 2^n$, minimal $n$–tuples form the moduli space (D.4).
The generators | change sign when conjugated by | are invariant when conjugated by
---|---|---
$T_{15}, T_{26}, T_{37}$ | $\Omega_1$ | $\Omega_2, \Omega_3$
$T_{13}, T_{24}, T_{57}$ | $\Omega_2$ | $\Omega_1, \Omega_3$
$T_{12}, T_{34}, T_{56}$ | $\Omega_3$ | $\Omega_1, \Omega_2$
$T_{17}, T_{35}, T_{46}$ | $\Omega_1, \Omega_2$ | $\Omega_3$
$T_{16}, T_{25}, T_{47}$ | $\Omega_1, \Omega_3$ | $\Omega_2$
$T_{14}, T_{23}, T_{67}$ | $\Omega_2, \Omega_3$ | $\Omega_1$
$T_{27}, T_{36}, T_{45}$ | $\Omega_1, \Omega_2, \Omega_3$ | —

Table 9: $\mathbb{Z}_2^3$–gradation of $Spin(7)$.

**Remark D1.** The minimal quintuple of $E_8$ is equivalent to that constructed in Ref. [13], which defines via the adjoint representation a $\mathbb{Z}_2^5$–gradation of the Lie algebra of $E_8$ such that the eigenspaces attached to all non–trivial characters of $\mathbb{Z}_2^5$ are 8–dimensional (and that with a trivial character is trivial). Likewise, the isolated triples consisting of $m$–exceptional elements with prime $m$ of $G_2$, $Spin(7)$, $Spin(8)$, $F_4$, $E_6$, and $E_8$ define, respectively, $\mathbb{Z}_2^3, \mathbb{Z}_2^3, \mathbb{Z}_3^3, \mathbb{Z}_3^3, \mathbb{Z}_3^5$ – gradations of the Lie algebras such that the eigenspaces attached to all non–trivial characters are 2–, 3–, 4–, 2–, 3–, and 2–dimensional, respectively [15].

Similarly, the isolated quadruples for $Spin(15)$ [resp. $Spin(16)$] define $\mathbb{Z}_2^4$–gradations of the corresponding Lie algebra with eigenspaces of dimensions 7 (resp. 8), the isolated quintuples for $Spin(31)$ [resp. $Spin(32)$] define $\mathbb{Z}_2^5$–gradations of the corresponding Lie algebra with eigenspaces of dimensions 15 (resp. 16), etc.

Let us illustrate the $Spin(7)$ example. As was noted in Ref. [5], the standard generators $T_{ij} = i\gamma_i\gamma_j/2$ of the $Spin(7)$ Lie algebra either are invariant or change sign under conjugation by the elements of the triple (1.5). The $\mathbb{Z}_2^3$–gradation subdivides 21 generators of $Spin(7)$ in 7 classes involving each 3 commuting generators as in Table 9.

**Remark D2.** In a connected, but not simply connected $G$, the classification of commuting pairs follows from Theorem A1, and the classification of minimal commuting $n$–tuples for $n > 2$ follows from that in the simply connected cover of $G$. If $G$ is not connected, non–trivial “1–tuples” appear which are classified by Gantmacher’s theorem [1]. Thereby, we have the classification of minimal commuting $n$–tuples in any compact Lie group $G$.

The problem of classification the non–trivial triples is related to the problem of counting the vacuum states in 4–dimensional supersymmetric Yang–Mills theory. We are not aware about possible physical applications of the problem of classification of minimal $n$–tuples, but it does not mean that such applications do not exist.

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