Small $x$ behaviour of parton distributions with soft initial conditions

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Abstract

We present an analytical parametrization of the QCD description of the small $x$ behaviour of parton distribution functions in the leading twist approximation of the Wilson operator product expansion, in the case of soft initial conditions. The results are in very good agreement with deep inelastic scattering experimental data from HERA.

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1 Introduction

The measurements of the deep-inelastic scattering (DIS) structure function (SF) \( F_2 \) in HERA [1, 2, 3] have permitted the access to a very interesting kinematical range for testing the theoretical ideas on the behavior of quarks and gluons carrying a very low fraction of momentum of the proton, the so-called small \( x \) region. In this limit one expects that nonperturbative effects may give essential contributions. However, the reasonable agreement between HERA data and the NLO approximation of perturbative QCD has been observed for \( Q^2 > 1 \text{GeV}^2 \) (see the recent review in [4]) and, thus, perturbative QCD could describe the evolution of structure functions up to very low \( Q^2 \) values, traditionally explained by soft processes. It is of fundamental importance to find out the kinematical region where the well-established perturbative QCD formalism can be safely applied at small \( x \).

The standard program to study the small \( x \) behavior of quarks and gluons is carried out by comparison of data with the numerical solution of the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations [5, 6] by fitting the parameters of the \( x \) profile of partons at some initial \( Q^2_0 \) and the QCD energy scale \( \Lambda \) [8]-[11]. However, if one is interested in analyzing exclusively the small \( x \) region, there is the alternative of doing a simpler analysis by using some of the existing analytical solutions of DGLAP in the small \( x \) limit [12]-[18]. This was done so in Ref. [12, 13] where it was pointed out that the HERA small \( x \) data can be interpreted in terms of the so-called doubled asymptotic scaling phenomenon related to the asymptotic behaviour of the DGLAP evolution discovered in [5, 19] many years ago.

On the other hand, various groups have been able to fit the available data (essentially at large \( Q^2 \)) using a hard input at small \( x \): \( x^{-\lambda}, \lambda > 0 \). In some sense, it is not very surprising, because the modern HERA data cannot distinguish yet between the behavior of a steep input parton distribution and the quite steep dynamical evolution from a soft initial condition. Moreover, for the full set of anomalous dimensions (AD) obtained at \( x \to 0 \) in Ref. [20] based on BFKL, the results weakly depend on the form of the initial condition (see [21]), preserving the hard ones and changing the soft ones. In the case of working with fixed order AD, the initial conditions are important when the data are considered in a wide range of \( Q^2 \) and it is necessary to choose the form of the PD asymptotics at some \( Q^2_0 \). In this work we use a soft initial condition in agreement with the experimental situation: low-\( Q^2 \) data [22, 23] are well described at \( Q^2 \leq 0.4 \text{ GeV}^2 \) by Regge theory with Pomeron intercept \( \varepsilon_P \equiv \lambda + 1 = 1.08 \), closed to the standard (\( \varepsilon_P = 1 \)) one. Moreover, HERA data [23] with \( Q^2 > 1 \text{ GeV}^2 \) are in good agreement with GRV predictions [4, 11] which support our aim to develop an analytical form for the parton densities at small \( x \) because, at least conceptually, it is very closed to the GRV approach.

Thus, the purpose of this article is to obtain the small \( x \) asymptotic form of parton distributions (PD) in the framework of the DGLAP equation starting at some \( Q^2_0 \) with the flat function:

\[
 f_a(Q^2_0) = A_a \quad (a = q, g),
\]

where \( f_a \) are the parton distributions multiplied by \( x \) and \( A_a \) are unknown parameters.
that have to be determined from data. Through this work at small \( x \) we neglect the non-singlet quark component.

The article is organized as follows. Sections 2 and 3 contain LO analyses: in Sect. 2 we consider the simple but important case without quarks, while in Sect. 3 the quarks distributions are taken into account. The NLO analysis, which is the main result of this article, is performed in Sect. 4. In Sect. 5 we present the fits to experimental data and some discussions of the obtained results. In the Appendix we illustrate the method [24] of replacing at small \( x \) the convolution of two functions by a simple product. The method is used in the present work for the correct incorporation of the non-singular part of parton distributions to our formulae.

## 2 Leading order without quarks

First of all, we consider the leading order (LO) approximation without quarks as a pedagogical example of the more cumbersome calculations below. This case is at the same time very simple and very closed to the real situation, because gluons give the basic contribution at small \( x \).

At the momentum space, the solution of the DGLAP equation in this case has the form

\[
M_g(n, Q^2) = M_g(n, Q_0^2)e^{-d_{gg}(n)s},
\]

where \( M_g(n, Q^2) \) are the moments of the gluon distribution,

\[
s = \ln \left( \frac{\alpha(Q_0^2)}{\alpha(Q^2)} \right) \quad \text{and} \quad d_{gg} = \frac{\gamma_{gg}(n)}{2\beta_0}
\]

The terms \( \gamma_{gg}(n) \) and \( \beta_0 \) are respectively the LO coefficients of the gluon-gluon AD and the QCD \( \beta \)-function. Through this work we use the short notation \( \alpha(Q^2) = \alpha_s(Q^2)/(4\pi) \).

At LO, \( s \) can be written in terms of the QCD scale \( \Lambda \) as:

\[
s_{\text{LO}} = \ln \left( \frac{\ln(Q^2/\Lambda_{\text{LO}}^2)}{\ln(Q_0^2/\Lambda_{\text{LO}}^2)} \right)
\]

For any perturbatively calculable variable \( K(n) \), it is very convenient to separate the singular part when \( n \to 1 \) (denoted by \( \tilde{K} \)) and the regular part (marked as \( K \)). Then, Eq. (3) can be represented by the form

\[
M_g(n, Q^2) = M_g(n, Q_0^2)e^{-d_{gg}s_{\text{LO}}/(n-1)}e^{-\tilde{d}_{gg}(n)s_{\text{LO}}},
\]

with \( \tilde{\gamma}_{gg} = -8C_A \) and \( C_A = N \) for \( SU(N) \) group.

Because we are only interested in the small \( x \) behaviour and the initial conditions are given by the soft (\( x \)-independent) functions in Eq. (4), we use permanently the variable.
\[ z = x/x_0 \] with values \( 0 < z < 1 \) with some arbitrary \( x_0 \leq 1 \), i.e.

\[ M_a(n, Q^2) = \int_0^1 dz z^{n-2} f_a(z, Q^2) \]

Finally, if one takes the soft boundary conditions given by Eq. (4), the coefficient in Eq. (3) becomes

\[ M_a(n, Q_0^2) = \frac{A_a}{n-1} \tag{5} \]

### 2.1 Classical double-logarithmic case

As a first step, we consider the classical double-logarithmic case which corresponds to use only the AD singular part, i.e. \( d_{gg}(n) = 0 \) in Eq. (3).

Then, expanding the second exponential in the r.h.s. of Eq. (3)

\[ M_{cdl}^g(n, Q^2) = A_g \sum_{k=0}^{\infty} \frac{1}{k!} \left( -\hat{d}_{gg}s_{LO} \right)^k \]

and using the Mellin transformation for \( \left( \ln(1/z) \right)^k \):

\[ \int_0^1 dz z^{n-2} \left( \ln(1/z) \right)^k = \frac{k!}{(n-1)^{k+1}} \]

we immediately obtain the well known double-logarithmic behavior \[5, 19\]:

\[ f_{cdl}^g(z, Q^2) = A_g \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left( -\hat{d}_{gg}s_{LO} \right)^k \left( \ln(1/z) \right)^k = A_g I_0(\sigma_{LO}) \tag{6} \]

where \( I_0(\sigma_{LO}) \) is the modified Bessel function with argument \[\sigma_{LO} = 2 \sqrt{\hat{d}_{gg}s_{LO} \ln(z)}\].

### 2.2 The more general case

For a regular kernel \( \tilde{K}(z) \) (see \[24\] and Appendix), having Mellin transform

\[ K(n) = \int_0^1 dz z^{n-2} \tilde{K}(z) \]

and the PD \( f_a(z) \) in the form \( I_\nu(\sqrt{d \ln(1/z)}) \) we have the following equation

\[ \tilde{K}(z) \otimes f_a(z) = K(1) f_a(z) + O\left( \frac{\sqrt{\hat{d}}}{\ln(1/z)} \right) \tag{7} \]

---

4 The correct incorporation of the regular part of the parton distributions can be done only for small values of \( z \) (see Eq. (3)). Thus, \( x_0 \) should be restricted to the range \( 0.1 \leq x_0 \leq 1 \) in order to keep the correctness of our formulae at \( x \leq 10^{-2} \).

5 Hereafter we follow the popular Ball-Forte variables \( \sigma_{LO} \) and \( \rho_{LO} \). Below, they are generalized to be used beyond the LO approximation (\( \sigma \) and \( \rho \)).
The Eq. (7) has been obtained for the nonsingular (at \( n \to 1 \)) functions \( K(n) \) and Regge-like PD many years ago by the Madrid group \cite{25}. It has been expanded for arbitrary kernel \( \tilde{K}(z) \) in \cite{24} and it was already used in \cite{26} and \cite{27} to extract the gluon distribution and the longitudinal structure function \( F_L \) from \( F_2 \) and \( dF_2/d\ln(Q^2) \) data. For arbitrary \( \tilde{K}(z) \) the ability of the method \cite{24} has been checked numerically in Refs. \cite{26} and \cite{27} using MRS sets of PD.

With the help of Eq. (7) one can find the general solution for the LO gluon density without the influence of quarks

\[
 f_g(z, Q^2) = A_g I_0(\sigma_{LO}) e^{-\tilde{d}_{gg}(1)s_{LO}} + O(\rho_{LO}),
\]

(8)

where

\[
 \rho_{LO} = \sqrt{\frac{\tilde{d}_{gg}s_{LO}}{2 \ln(1/z)}} = \frac{\sigma_{LO}}{2\ln(1/z)}, \quad \overline{\tau}_{gg}^{(0)}(1) = 22 + \frac{4}{3}f \quad \text{and} \quad \tilde{d}_{gg}(1) = 1 + \frac{4f}{3\beta_0}
\]

with \( f \) as the number of active quarks.

\[\text{3 Leading order (complete)}\]

At the momentum space, the solution of the DGLAP equation at LO has the form

\[
 M_a(n, Q^2) = M_a^+(n, Q^2) + M_a^-(n, Q^2) \quad \text{and} \quad M_a^+(n, Q^2) = M_a^+(n, Q^2)^o e^{-\hat{d}_+(n)s} = M_a^+ e^{-\tilde{d}_+(s/(n-1))} e^{-\tau_{gg}(n)s},
\]

(9)

where

\[
 M_a^\pm(n, Q^2) = \varepsilon_{ab}^\pm(n) M_b(n, Q^2), \quad d_{ab} = \frac{\gamma_{ab}^{(0)}(n)}{2\beta_0},
\]

\[
 d_\pm(n) = \frac{1}{2} \left[ (d_{gg}(n) + d_{qg}(n)) \pm (d_{gg}(n) - d_{qq}(n)) \right] \sqrt{1 + \frac{4d_{gg}(n)d_{qg}(n)}{(d_{gg}(n) - d_{qq}(n))^2}}
\]

\[
 \varepsilon_{\pm qq}(n) = \varepsilon_{\pm gg}(n) = \frac{1}{2} \left( 1 + \frac{d_{qq}(n) - d_{gg}(n)}{d_\pm(n) - d_\mp(n)} \right), \quad \varepsilon_{\pm ab}(n) = \frac{d_{ab}(n)}{d_\pm(n) - d_\mp(n)}(a \neq b),
\]

(10)

As the singular (when \( n \to 1 \)) part of the + component of the anomalous dimension is \( \hat{d}_+ = \hat{d}_{gg} = -4C_A/\beta_0 \) while the - component does not exist \( (\hat{d}_- = 0) \), we consider below both cases separately.

\[\text{6We use a non-standard definition (see \cite{28}) of the projectors \( \varepsilon_{ab}^\pm(n) \), which is very convenient beyond LO (see Eq. (22)). The connection with the more usual definition \( \alpha, \tilde{\alpha} \) and \( \varepsilon \) in ref. \cite{29,30} is given by: \( \varepsilon_{qq}(n) = \alpha(n), \varepsilon_{gg}(n) = \tilde{\alpha}(n) \) and \( \varepsilon_{gg}(n) = \varepsilon(n) \).} \]
3.1 The “+” component

The analysis of the “+” component is practically identical to the case studied in section 2. The only difference lies in the appearance of new terms $\varepsilon_{ab}^{+}(n)$. If they are expanded in the vicinity of $n = 1$ in the form $\varepsilon_{ab}^{+}(n) = \tilde{\varepsilon}_{ab}^{+} + (n - 1)\tilde{\varepsilon}_{ab}^{+}$, then for the terms $\tilde{\varepsilon}_{ab}^{+}$ multiplying $M_b(n, Q^2)$, we have the same results as in previous section:

$$\tilde{\varepsilon}_{ab}^{+} M_b(n, Q^2) \xrightarrow{M^{-1}} \tilde{\varepsilon}_{ab}^{+} A_b I_0(\sigma_{LO}) e^{-\tilde{d}_{+}(1)s_{LO}} + O(\rho_{LO}),$$

where the symbol $\xrightarrow{M^{-1}}$ denotes the inverse Mellin transformation. The values of $\sigma$ and $\rho$ coincide with those defined in the previous section because $\tilde{d}_{+} = \tilde{d}_{gg}$.

The terms $\tilde{\varepsilon}_{ab}^{+}$ that come with the additional factor $(n - 1)$ in front, lead to the following results

$$(n - 1)\tilde{\varepsilon}_{ab}^{+} \frac{A_b}{(n - 1)} e^{-\tilde{d}_{+} s_{LO}/(n-1)} = \tilde{\varepsilon}_{ab}^{+} A_b \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(n - 1)^k} (-\tilde{d}_{+} s_{LO})^k (\ln(1/z))^{k-1} = \tilde{\varepsilon}_{ab}^{+} A_b \rho_{LO} I_1(\sigma_{LO}),$$

i.e. the additional factor $(n - 1)$ in momentum space leads to replacing the Bessel function $I_0(\sigma_{LO})$ by $\rho_{LO} I_1(\sigma_{LO})$ in $z$-space.

Thus, we obtain that the term $\varepsilon_{ab}^{+}(n) M_b(n, Q^2)$ leads to the following contribution in $z$ space:

$$\left( \tilde{\varepsilon}_{ab}^{+} I_0(\sigma_{LO}) + \tilde{\varepsilon}_{ab}^{+} \rho_{LO} I_1(\sigma_{LO}) \right) A_b e^{-\tilde{d}_{+}(1)s_{LO}} + O(\rho_{LO}) \quad (11)$$

Because the Bessel function $I_\nu(\sigma)$ has the $\nu$-independent asymptotic behavior $e^{(\sigma)/\sqrt{\sigma}}$ at $\sigma \to \infty$ (i.e. $z \to 0$), the second term in Eq. (11) is $O(\rho)$ and must be kept only when $\tilde{\varepsilon}_{ab}^{+} = 0$. This is the case for the quark distribution at the LO approximation.

Using the concrete AD values, one has

$$f_{g}^{+}(z, Q^2) = \left( A_g + \frac{4}{9} A_q \right) I_0(\sigma_{LO}) e^{-\tilde{d}_{+}(1)s_{LO}} + O(\rho_{LO}) \quad \text{and}$$

$$f_{q}^{+}(z, Q^2) = \frac{f}{9} \left( A_g + \frac{4}{9} A_q \right) \rho_{LO} I_1(\sigma_{LO}) e^{-\tilde{d}_{+}(1)s_{LO}} + O(\rho_{LO}) \quad (12)$$

where $\tilde{d}_{+}(1) = 1 + 20 f/(27 \beta_0)$.

3.2 the “−” component

In this case the anomalous dimension is regular and using Eq. (10) one has

$$\varepsilon_{ab}^{−}(n) A_b e^{-d_{−}(n)s} \xrightarrow{M^{-1}} \varepsilon_{ab}^{−}(1) A_b e^{-d_{−}(1)s_{LO}} + O(z)$$

for PD.

\footnote{In the Regge-like case (see [24] and Appendix) $f_{a}(x) \sim x^{-\lambda} \sim z^{-\lambda}$ the Eq. (11) has $K(1 + \lambda)$ in its r.h.s. with the accuracy $O(z)$ for an arbitrary $\lambda$, including the case $\lambda = 0$.}
Using the concrete AD values, we have

\[
\begin{align*}
    f_g^-(z, Q^2) &= -\frac{4}{9} A_q e^{-d_-(1)_{sLO}} + O(z) \quad \text{and} \\
    f_q^-(z, Q^2) &= A_q e^{-d_-(1)_{sLO}} + O(z),
\end{align*}
\]

where \( d_-(1) = 16 f/(27 \beta_0) \).

Finally we present the full small \( x \) asymptotic results for PD and \( F_2 \) structure function at LO of perturbation theory:

\[
\begin{align*}
    f_a(z, Q^2) &= f_a^+(z, Q^2) + f_a^-(z, Q^2) \quad \text{and} \\
    F_2(z, Q^2) &= e \cdot f_q(z, Q^2)
\end{align*}
\]

where \( f_q^+, f_g^+, f_q^- \) and \( f_g^- \) are given by Eqs. (12) and (13) and \( e = \sum e_i^2/f \) is the average charge square of the \( f \) active quarks.

Let us now describe the main conclusions that follow Eq. (14).

- Our LO results for PD coincide with the ones obtained by Mankievi\'t\'s et al. in Ref. [17] for \( f = 4 \) active quarks. The “+” component part coincides with the Ball and Forte LO results [12].

- The “+” and “−” components are presented explicitly separated. The “−” component \( \sim Const \) is negligible at small \( x \) (and large \( Q^2 \)) in comparison with \( \rho_{LO} I_1(\sigma_{LO}) \) (as it has been already observed in [28, 22, 17]) and the LO quark distribution is “driven” by the influence of the gluons: \( f_q^+(z, Q^2) \approx (f/9) \rho_f^+(z, Q^2) \) (see also [28, 22, 13, 17]). However, at intermediate \( Q^2 \), the “−” component is essential (as it was discussed in [13, 17]). Thus, in order to give the more general result valid for a wide \( Q^2 \) range, we consider PD (see Eq. (14)) as the combinations of the “+” and “−” components, where every component evolves independently.

- The separation of the singular and regular parts of the AD performed above leads to the possibility of avoiding complicated methods for evaluating the inverse Mellin convolution or special analyses of DGLAP equations (see [13] for a review on these methods). In our case, we use the exact solution to get the moments of the PD. The simple form of the singular part of this exact solution is easily transformed to the \( z \)-space. The non-singular part is added by the method of replacing Mellin convolution by usual product [24]. In this case the non-singular part in the \( z \)-space is equal to the corresponding contribution for the first moment \( n = 1 \).

In the following we resume the steps we have followed to reach the small \( x \) approximate solution of DGLAP shown above:

- Use the \( n \)-space exact solution.

- Expand the perturbatively calculated parts (AD and coefficient functions) in the vicinity of the point \( n = 1 \).
• The singular part with the form

$$A_a(n-1)^k e^{-\hat{d}s_{LO}/(n-1)}$$  \(15\)

leads to Bessel functions in the \(z\)-space in the form

$$A_a\left(\frac{\hat{d}s_{LO}}{lnz}\right)^{(k+1)/2} I_{k+1}\left(2\sqrt{\hat{d}s_{LO}/lnz}\right)\)  \(16\)

• The regular part \(B(n) \exp\left(-\hat{d}(n)s_{LO}\right)\) leads to the additional coefficient (see [24] and Appendix)

$$B(1) \exp(-\hat{d}(1)s_{LO}) + O(\sqrt{\hat{d}s_{LO}/lnz})$$

behind of the Bessel function (16) in the \(z\)-space. Because the accuracy is \(O(\sqrt{\hat{d}s_{LO}/lnz})\), it is necessary to use only the basic term of Eq. (16), i.e. all terms \((n-1)^k\) in front of \(\exp(-\hat{d}/(n-1))\), with the exception of one with the smaller \(k\) value, can be neglected.

• If the singular part at \(n \to 1\) is absent, i.e. \(\hat{d} = 0\) in (13), the result in the \(z\)-space is determined by \(B(1) \exp(-\hat{d}(1)s_{LO})\) with accuracy \(O(z)\).

We would like to stress that the applicability of the above recipe is not limited by the order in perturbation theory but by the form of the singular part of the anomalous dimensions. At the first two orders of perturbation theory the singular part is proportional to \(\sim (n - 1)^{-1}\) but this behaviour does not remain at higher orders. The most singular terms have been calculated in [20, 31]. For example, the gluon-gluon AD has the form

$$\gamma_{gg}(n, \alpha) = \gamma(n, \alpha) + O(\frac{\alpha}{n-1})^k$$  \(17\)

where the terms \(\sim O(\alpha/(n-1))^k\) have been evaluated [32] very recently.

The BFKL anomalous dimension \(\gamma(n, \alpha)\) is obtained by solving the implicit equation

$$1 = \frac{4C_A\alpha}{n-1} \chi(\gamma(n, \alpha)),$$

where the characteristic function \(\chi(\gamma)\) has the following expression in terms of the Euler \(\Psi\)-function:

$$\chi(\gamma) = 2\Psi(1) - \Psi(\gamma) - \Psi(1 - \gamma)$$

The expansion of \(\gamma(n, \alpha)\) in powers of \(\bar{s}_s = 4C_A\alpha\) gives:

$$\gamma(n, \alpha) \simeq \frac{\bar{s}_s}{n-1} + 2.404\left(\frac{\bar{s}_s}{n-1}\right)^4 + 2.074\left(\frac{\bar{s}_s}{n-1}\right)^6 + O\left(\left(\frac{\bar{s}_s}{n-1}\right)^k\right)$$  \(18\)

which explicitly shows the of the the term \(\sim (n-1)^{-4}\) in the fourth order of the expansion. Moreover, the dependence \(\sim (n-1)^{-2}\) has been found [32] when it is considered the third
order terms proportional to \( (\alpha/\langle n-1 \rangle)^k \) in the r.h.s. of Eq. (14). Note that the regular part of this third order coefficient in the perturbative expansion of the AD is only partially known, although there is the possibility to calculate it completely [34].

Thus, an extension of our recipe beyond the NLO approximation requires the evaluation of singular terms \( (n-1)^{-k} \) \((k > 1)\) which is not easy and out of the scope of this work. We restrict ourselves to the first two orders in perturbation theory.

4 Next-to-leading order

At the momentum space, the solution of the DGLAP equation has the form

\[
M_a(n, Q^2) = M_{a+}(n, Q^2) + M_{a-}(n, Q^2) \quad \text{and}
\]

\[
M_{a+}^+(n, Q^2) = \tilde{M}_{a+}^+(n, Q^2, Q_0^2) \exp(-d_+^s(n) s - D_+^p(n))
\]

\[
= \tilde{M}_{a+}^+(n, Q^2, Q_0^2) \exp(-(\tilde{d}_+^s(n) s + \tilde{D}_+^p(n)))(n-1)) \exp(-\tilde{d}_+^s(n) s - \tilde{D}_+^p(n)p),
\]

where the new variable \( p = \alpha(Q_0^2) - \alpha(Q^2) \).

In comparison with the LO expressions, the NLO AD leads to the following additional factors in the moments:

- The term proportional to \( p \) contributes to the evolution part. It can be represented by

\[
D_\pm(n) = d_\pm^\pm(n) - \frac{\beta_1}{\beta_0} d_\pm(n)
\]

and analogously for its singular and regular parts.

- The terms proportional to \( \alpha(Q^2) \) and \( \alpha(Q_0^2) \) change the normalization factor \( M_{a+}^+(n, Q_0^2) \rightarrow \tilde{M}_{a+}^+(n, Q^2, Q_0^2) \):

\[
\tilde{M}_{a+}^+(n, Q^2, Q_0^2) = \left(1 - d_{\pm+}^a(n) \alpha(Q^2)\right) M_{a+}^+(n, Q_0^2) + d_{\pm+}^a(n) \alpha(Q_0^2) M_{a+}^+(n, Q_0^2)
\]

The NLO components of the r.h.s. of Eqs. (20, 21) have the form:

\[
d_{\pm\pm}(n) = \sum_{a,b=q,g} \varepsilon_{ba}^\pm d_{ab}^{(1)}
\]

\[
d_{\pm}(n) = \sum_{a=q,g} \varepsilon_{ag}^\pm \delta_{d^{(1)}g} - \varepsilon_{qq}^\pm d_{qq}^{(1)} + (\varepsilon_{gg}^\pm - \varepsilon_{qq}^\pm / \varepsilon_{ag}^\pm) d_{gg}^{(1)}
\]

\[
d_{\mp}(n) = \frac{d_{\mp+}(n)}{1 + d_{\pm}(n) - d_{\mp}(n)}, \quad d_{\pm+}(n) = d_{\pm+}^q(n) \varepsilon_{qq}^\mp / \varepsilon_{qq}^\pm.
\]

---

8 It is known the \( n=2, 4, 6, 8 \) and 10 Mellin moments [33]

9 The representation (19) for the moments \( M_{a+}^+(n, Q^2) \) is slightly different from the one in Ref. [30] (see Eqs. (2.139)-(2.144) in [30]) because we prefer to separate the “+” and “−” evolutions exponents exactly at NLO. The sum \( M_{a+}^+(n, Q^2) + M_{a-}^-(n, Q^2) \) coincides with the one of [30].
where \( d_{ab}^{(1)}(n) = \gamma_{ab}^{(1)}(n)/2 \beta_0 \) and \( \gamma_{ab}^{(1)}(n) \) are the NLO AD.

We would like to stress that the exponent in Eq. (19) contains the NLO contribution proportional to Eq. (20). We have checked this result by direct calculation of NNLO corrections to the solution of DGLAP equation (following the review of Buras [30]), arriving to terms with the form \( \sim D_{\mp}^2(n)/2 \) \(^\text{10}\).

The exponential representation of the term proportional to Eq. (20) is very important in the case of the + component because it contains a singular part when \( n \to 1 \) that could make it of the order of the LO contribution, spoiling the perturbative convergence if one attempts to expand the NLO part of the exponent.

Following the steps given in the previous section, one can easily obtain the small \( x \) behaviour of the PD and \( F_2 \) at NLO. It has the form:

\[
\begin{align*}
\alpha_s(z, Q^2) & = f_a^+(z, Q^2) + f_a^-(z, Q^2) \quad \text{and} \\
f_a^+(z, Q^2) & = A^+_g(Q^2, Q^2_0) \exp(-d_-(1)s - D_-(1)p) + O(z) \\
f_a^-(z, Q^2) & = A^-_g(Q^2, Q^2_0) I_0(\sigma) \exp(-\bar{D}_+(1)s - D_+(1)p) + O(\rho) \\
f_q^+(z, Q^2) & = A^+_q(Q^2, Q^2_0) \left[ (1 - \bar{D}_+(1)\alpha(Q^2))\rho I_1(\sigma) + 20\alpha(Q^2)I_0(\sigma) \right] \\
& \quad \cdot \exp(-\bar{D}_+(1)s - D_+(1)p) + O(\rho) \\
F_2(z, Q^2) & = e \cdot \left( f_q(z, Q^2) + \frac{2}{3} f\alpha(Q^2)f_g(z, Q^2) \right)
\end{align*}
\]

where

\[
\begin{align*}
\sigma & = 2\sqrt{(\bar{d}_+ + \bar{D}_+ p) \ln z} \quad , \quad \rho = \sqrt{\frac{(\bar{d}_+ + \bar{D}_+ p) \ln z}{\ln z}} = \frac{\sigma}{2\ln(1/z)} \\
A^+_g(Q^2, Q^2_0) & = \left[ 1 - \frac{80}{81} f\alpha(Q^2) \right] A_g + \frac{4}{9} \left[ 1 + 3(1 + \frac{1}{81} f)\alpha(Q^2_0) - \frac{80}{81} f\alpha(Q^2) \right] A_q \\
A^-_g(Q^2, Q^2_0) & = A_g - A^+_g(Q^2, Q^2_0) \\
A^+_q & = \frac{f}{9} \left( A_g + \frac{4}{9} A_q \right) \quad , \quad A^-_q = A_q - 20\alpha(Q^2_0)A^+_q
\end{align*}
\]

The components of the singular and regular parts of \( D_{\pm} \) have the form:

\[
\begin{align*}
\bar{d}_+ & = \frac{412}{27\beta_0} f \quad , \quad \bar{d}_- = -20 \quad , \quad \bar{d}_{++} = 0 \\
\bar{D}_+(1) & = \frac{8}{\beta_0} \left[ 36\zeta_3 + 33\zeta_2 - \frac{1643}{12} + \frac{2}{9} f \left[ \frac{68}{9} - 4\zeta_2 - \frac{13}{243} f \right] \right] \\
\bar{D}_-(1) & = \frac{134}{3} - 12\zeta_2 - \frac{13}{81} f \quad , \quad \bar{D}_{++} = \frac{80}{81} f \\
d_-(1) & = \frac{16}{9\beta_0} \left[ 2\zeta_3 - 3\zeta_2 + \frac{13}{4} + f \left[ 4\zeta_2 - \frac{23}{18} + \frac{13}{243} f \right] \right] \\
d_{++} & = 0 \quad , \quad d_{++}(1) = -3 \left( 1 + \frac{f}{81} \right)
\end{align*}
\]

\(^{10}\) These corrections are also needed to extend the NNLO analysis of structure functions \([35]\) from non-singlet to singlet behaviour.
The numerical values of these coefficients are listed in Tab. 1 for a different number of active quarks.

The Eqs. (23)-(29) together with the recipes in the end of the section 3 are the main results of this article.

Looking carefully Eqs. (23)-(26), we arrive to the following conclusions:

- Our NLO results coincide with the corresponding of Ball and Forte in Ref. [13] if one neglects the “−” component, expands our NLO singular terms \((\rho)^k I_{k+1}(\sigma)\) in the vicinity of the point \(\sigma = \sigma_{LO}\) and ignores the NLO regular terms (i.e. put \(\exp(-D_+(1)p) = \exp(-D_-(1)p) = 1\) and cancel the terms proportionals to \(\alpha(Q_0^2)\) into the normalization factors \(A_q^+\) and \(A_q^-\)). We think, however, that this expansion is not so correct because it generates NLO corrections of the order of the LO terms.

- The negative sign of the NLO correction in \(\sigma\) (see Eq. (27)) makes excellent the agreement of our result with the parametrization of \(F_2\) obtained by De Roeck and De Wolf [30]. Their result is very similar to our LO form of \(f_q^+\) in Eq. (12) if one replaces \(s_{LO} \rightarrow s_{LO}^r\) in the definition (3) of \(s_{LO}\). The value \(\delta = 0.708\) has been obtained in the fit to H1 and ZEUS data. Due to \(\delta < 1\), it shows less \(Q^2\)-dependence than it is predicted by perturbative QCD at LO. This slower \(Q^2\)-dependence may be explained naturally by the negative NLO corrections to \(\sigma\) obtained here.

- The behaviour of eqs. (23)-(26) can mimic a power law shape over a limited region of \(x, Q^2\).

\[
f_a(x, Q^2) \sim x^{-\lambda_a^{\text{eff}}(x, Q^2)} \quad \text{and} \quad F_2(x, Q^2) \sim x^{-\lambda_{F_2}^{\text{eff}}(x, Q^2)}
\]

The quark and gluon effective slopes \(\lambda_a^{\text{eff}} = -\frac{d}{d\ln z} \ln f_a(z, Q^2)\) are reduced by the NLO terms that leads to the decreasing of the gluon distribution at small \(z\). For the quark case it is not the case, because the normalization factor \(A_q^+\) of the “+” component produces an additional contribution undampening as \(\sim (\ln z)^{-1}\).

- The gluon effective slope \(\lambda_g^{\text{eff}}\) is larger than the quark slope \(\lambda_q^{\text{eff}}\), which is in excellent agreement with a recent MRS and GRV analyses [11, 14]. Indeed, because \(d/d\ln x = d/d\ln z\), the effective slopes have the form,

\[
\lambda_a^{\text{eff}}(z, Q^2) = \frac{f_q^+(z, Q^2)}{f_g(z, Q^2)} \cdot \rho \cdot \frac{I_1(\sigma)}{I_0(\sigma)} \\
\lambda_q^{\text{eff}}(z, Q^2) = \frac{f_q^+(z, Q^2)}{f_q(z, Q^2)} \cdot \rho \cdot \frac{I_2(\sigma)(1 - D_{\mu+}(1)\alpha(Q^2)) + 20\alpha(Q^2)I_1(\sigma)/\rho}{I_1(\sigma)(1 - D_{\mu+}(1)\alpha(Q^2)) + 20\alpha(Q^2)I_0(\sigma)/\rho} \\
\lambda_{F_2}^{\text{eff}}(z, Q^2) = \frac{\lambda^{\text{eff}}(z, Q^2) \cdot f_q^+(z, Q^2) + (2f)/3\alpha(Q^2) \cdot \lambda_q^{\text{eff}}(z, Q^2) \cdot f_g(z, Q^2)}{f_q(z, Q^2) + (2f)/3\alpha(Q^2) \cdot f_g(z, Q^2)}
\]

The effective slopes \(\lambda_a^{\text{eff}}\) and \(\lambda_{F_2}^{\text{eff}}\) depend on the magnitudes \(A_a\) of the initial PD and also on the chosen input values of \(Q_0^2\) and \(\Lambda\). At quite large values of \(Q^2\), where
the “−” component is not relevant, the dependence on the magnitudes of the initial PD disappear, having in this case for the asymptotic values:

\[
\begin{align*}
\lambda_{q}^{\text{eff,as}}(z, Q^2) &= \frac{I_1(\sigma)}{I_0(\sigma)} \approx \frac{1}{4 \ln(1/z)} \\
\lambda_{q}^{\text{eff,as}}(z, Q^2) &= \frac{I_2(\sigma)(1 - \frac{\alpha}{1+1(1+\alpha)} + 2\alpha(\alpha^2) + 20\alpha(\alpha^2)I_1(\sigma)/\rho}{I_1(\sigma)(1 - \frac{\alpha}{1+1(1+\alpha)} + 2\alpha(\alpha^2)) + 20\alpha(\alpha^2)I_0(\sigma)/\rho} \\
&\approx \left(\rho - \frac{3}{4 \ln(1/z)}\right)\left(1 - \frac{10\alpha(\alpha^2)}{(d_s + d_p)}\right) \\
\lambda_{F_2}^{\text{eff,as}}(z, Q^2) &= \lambda_{q}^{\text{eff,as}}(z, Q^2)^{1 + 6\alpha(\alpha^2)\lambda_{q}^{\text{eff,as}}(z, Q^2)^{-1} + 6\alpha(\alpha^2)\lambda_{q}^{\text{eff,as}}(z, Q^2)^{-1} + O(\alpha^2(\alpha^2))} \\
&\approx \lambda_{q}^{\text{eff,as}}(z, Q^2) + \frac{3\alpha(\alpha^2)}{\ln(1/z)},
\end{align*}
\]

where simbol \(\approx\) marks approximations obtained by expansions of modified Bessel functions \(I_n(\sigma)\). These approximations should be correct only at very large \(\sigma\) values (i.e. at very large \(Q^2\) and/or very small \(x\)).

We would like to note that at LO, where \(\rho = \rho_{LO}\), the slope \(\lambda_{F_2}^{\text{eff,as}}(z, Q^2) = \lambda_{q}^{\text{eff,as}}(z, Q^2)\) coincides at very large \(\sigma\) with one obtained in [37] (see also [4]) in the case of flat input. At the NLO approximation the slope \(\lambda_{F_2}^{\text{eff,as}}(z, Q^2)\) lies between quark and gluon ones but closely to quark slope \(\lambda_{q}^{\text{eff,as}}(z, Q^2)\) (see also Fig. 3).

• Both slopes \(\lambda_{q}^{\text{eff}}\) decrease with decreasing \(z\). A \(z\) dependence of the slope should not appear for a PD with a Regge type asymptotic \((x^{-\lambda})\) and precise measurement of the slope \(\lambda_{q}^{\text{eff}}\) may lead to the possibility to verify the type of small \(x\) asymptotics of parton distributions.

## 5 Results of the fits

With the help of the results obtained in the previous section we have analyzed \(F_2\) HERA data at small \(x\) from the H1 [1] and ZEUS [2] collaborations separately. Initially our solution of the DGLAP equations depends on five parameters, i.e. \(Q_0^2, x_0, A_q, A_g\) and \(\Lambda_{\overline{\text{MS}}}^{\text{MS}}(n_f = 4)\). In order to keep the analysis as simple as possible we have fixed \(\Lambda_{\overline{\text{MS}}}^{\text{MS}}(n_f = 4) = 250 \text{ MeV}\) which is a reasonable value extracted from the traditional (higher \(x\)) experiments and that has also been used by others [17]. The initial scale of the PD was also fixed into the fits to \(Q_0^2 = 1 \text{ GeV}^2\), although later it was released to study the sensitivity of the fit to the variation of this parameter. The analyzed data region was restricted to \(x < 0.01\) to remain within the kinematical range where our results are accurate. Finally, the number of active flavors was fixed to \(f = 4\).

Tab. 2 contains the results of the fits to H1 data using Eqs. (14) at LO and (26) at NLO. The errors used in the calculation of \(\chi^2\) are statistical and systematic added in quadrature.
Fig. 1 shows $F_2$ calculated from the fit with $Q^2 > 1 \text{ GeV}^2$ given in table 1 in comparison with H1 data. Only the lower $Q^2$ bins are shown. One can observe that the NLO result lies closer to the data than the LO curve. The lack of agreement between data and lines observed at the lowest $x$ and $Q^2$ bins suggests that the flat behavior should occur at $Q^2$ lower than $1 \text{ GeV}^2$.

In order to study this point we have done the analysis considering $Q^2_0$ as a free parameter. In exchange, we have fixed $x_0$ to 1. In this case, our formulas must be used with caution because during the fit $Q^2_0$ could eventually reach a very small value such that the argument of the square root in equation (26) becomes negative when the NLO term proportional to $p$ becomes larger than the LO term proportional to $s$. The kinematic region where this problem happens depends on $Q^2_0$ ($Q^2_0 < Q^2 < Q^2_2$), where the upper limit $Q^2_2$ also depends on $Q^2_0$. For example for the sequence $Q^2_0 = 0.5/0.4/0.3/0.2 \text{ GeV}^2$ $Q^2_2 = 0.7/1.2/3.180 \text{ GeV}^2$. Thus, in the fits one should be out of this region where more terms in the perturbative expansion (NNLO) are needed.

Comparing the results of the fits in table 3 with those in table 2 one can notice a significant reduction in the value of $A_g$, $Q^2_0$ and the $\chi^2$. In Fig. 1 the better agreement with the experiment of the NLO curve is apparent at the lowest kinematical bins.

Tab. 3 also contains the results of the combined fit to 3 different data set of ZEUS [2]. The LO and NLO results are compared with data in Fig. 2. The main conclusion is that the parameters extracted from fits to H1 and ZEUS are very similar.

The observed decreasing of the gluon density at small $Q^2$ agrees with the traditional assumption (see [23]) that the main contribution at small $Q^2$ comes from valence quarks. Thus, this non-zero contribution of singlet quark distribution at small $Q^2$ seems to mimic the valence quark distribution. Indeed, in our approach the nonsinglet quark distribution is omitted and this approximation maybe is not so correct at very small $Q^2$. As it was shown in the analysis of Ref. [33] based on BFKL dynamics, the nonsinglet quark distributions has the following small $x$ asymptotic behavior:

$$xf_{NS}(x, Q^2) \sim x^{1-a_{NS}} \quad \text{with} \quad a_{NS} \sim \sqrt{\frac{32}{3} \alpha_{LO}},$$

(32)

where $\alpha_{LO}$ is a $Q^2$-independent coupling constant. The numerical estimations with the running coupling (see [33]) lead to the behaviour similar to (32) with $\alpha_{LO}(Q^2)$ having some effective energy scale $\Lambda$ proportional $\Lambda_{\text{MS}}$.

At very small $Q^2_0$ (and $x \sim 10^{-2}$) the value of $a_{NS}$ may reach the large value $a_{NS} \sim 1$, and the NS quark distributions may start to contribute together with the gluon and the singlet distributions. Unfortunately, it is very difficult to add the NS quark distribution to our analysis based on DGLAP evolution. Indeed, in agreement with DGLAP dynamics, the non-singlet quark distribution with $a_{NS} \sim 1$ should be $Q^2$-independent which seems contrary to its functional form (32) with the running coupling $\alpha_{LO}(Q^2)$.

Finally with the help of Eq. (30) we have estimated the $F_2$ effective slope using the value of the parameters extracted from NLO fits to data. For H1 data we found $0.05 < \lambda_{F_2}^{eff} < 0.30 - 0.37$ and for ZEUS $0.07 - 0.09 < \lambda_{F_2}^{eff} < 0.31 - 0.34$. The lower (upper) limits corresponds to $Q^2 = 1.5 \text{ GeV}^2$ ($Q^2 = 400 \text{ GeV}^2$). The dispersion in some of the limits is due to the $x$ dependence. Fig. 3 shows that the three types of asymptotical slopes have similar values. The NLO values of $\lambda_{F_2}^{eff,as}$ lie between the quark and the
gluon ones but closer to the quark slope $\lambda_{q}^{eff,as}$. These results are in excellent agreement with those obtained by others (see references [1, 10, 11, 37] and also the review [1] and references therein).

6 Conclusions

We have presented the rules to construct the small $x$ form of parton distributions having soft initial conditions at the first two orders of perturbation theory. The rules are based on the exact $n$-space solution and lead to the possibility of avoiding complicated methods for evaluating the inverse Mellin convolution or special analyses of DGLAP equations (see [13] for a review on this methods). We have presented here the PD form at LO and NLO approximations of perturbation theory, where the corresponding exact solutions in $n$-space are fully known. Our expressions have quite simple form and reproduce many properties of parton distributions at small $x$, that have been known from global fits.

We found the very good agreement between our approach based on QCD at NLO approximation and HERA data, as it has been observed earlier with other approaches (see the review [1]). Thus, the nonperturbative contributions as shadowing effects [39], higher twist effects [10] and others seems to be quite small or seems to canceled between them and/or with $ln(1/x)$ terms containing by higher orders of perturbative theory. To clear up the correct contributions of nonperturbative dynamics and higher orders containing strong $ln(1/x)$ terms, it is necessary as more precise data and further efforts in developing of theoretical approaches.

It is very useful to evaluate the derivatives of $F_2$ and the parton distributions with respect to the logarithms of $1/x$ and $Q^2$ directly from eqs.(23)-(26). We have presented the result for the $x$ logarithmic dependence. The calculation of $dF_2/dln(Q^2)$ and $df_a/dln(Q^2)$ is also extremely important in view of the recent claim of disagreement between data and predictions in perturbative QCD [11] (see, however, new set GRV analysis [11], where this disagreement decreases essentially). We are considering to present this work and also predictions for SF $F_L$ in a forthcoming article [42].

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7 Appendix

For reader convenience we present here the illustration[11] of the method to replace the convolution of two functions by a simple product at small $x$. We limite ourselves to the

11 Contrary to Ref. [24] we use here the variable $z = x/x_0$. 
case of regular behavior of kernel moments at $n \to 1$. More detailed analysis can be found in [24] and in [12].

Let us consider the set of PD with have the different forms:

- Regge-like form $f_R(z) = z^{-\delta} \tilde{f}(z),$
- Logarithmic-like form $f_L(z) = z^{-\delta} \ln(1/z) \tilde{f}(z),$  
- Bessel-like form $f_I(z) = z^{-\delta} I_k(2 \sqrt{d \ln(1/z)}) \tilde{f}(z),$

where $\tilde{f}(z)$ and its derivative $\tilde{f}'(z) \equiv d \tilde{f}(z) / dz$ are smooth at $z = 0$ and both are equal to zero at $z = 1$:

$$\tilde{f}(1) = \tilde{f}'(1) = 0$$

1. Consider the basic integral with even $n > 1$:

$$J_{\delta,i}(n, z) = z^n \otimes f_i(z) \equiv \int_1^z \frac{dy}{y} y^n f_i \left( \frac{z}{y} \right), \quad i = R, L, I$$

a) Regge-like case. Expanding $\tilde{f}(z)$ near $\tilde{f}(0)$, we have

$$J_{\delta,R}(n, z) = z^{-\delta} \int_1^z \frac{dy}{y} y^{n+\delta-1} \left[ \tilde{f}(0) + \frac{z}{y} \tilde{f}'(0) + \ldots + \frac{1}{k!} \left( \frac{z}{y} \right)^k \tilde{f}^{(k)}(0) + \ldots \right]$$

$$= z^{-\delta} \left[ \frac{1}{n+\delta} \tilde{f}(0) + O(z) \right]$$

b) Logarithmic-like case. Using the simple relation $z^{-\delta} \ln(1/z) = (d/d\delta) z^{-\delta}$ we immediately obtain

$$J_{\delta,L}(n, z) = z^{-\delta} \ln(1/z) \left[ \frac{1}{n+\delta} \left( 1 - \frac{1}{(n+\delta) \ln(1/x)} \right) \tilde{f}(0) + O(z) \right]$$

$$= \frac{1}{n+\delta} \left( 1 - \frac{1}{(n+\delta) \ln(1/z)} \right) f_L(z) + O(z^{1-\delta})$$

$$= \frac{1}{n+\delta} f_L(z) + O(1/\ln(1/z))$$
c) Bessel-like case. Representing Bessel function in the form

\[ I_k(2\sqrt{\hat{d}\ln(1/z)}) = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n + k + 1)} \left( \frac{\hat{d}}{\hat{d}\delta} \right)^n z^{-\delta} \bigg|_{\delta=0} \]

and repeating the above analysis, we have

\[ J_{\delta,I}(n, z) = \frac{1}{n + \delta} f_I(z) + O\left( \sqrt{\frac{\hat{d}}{\ln(1/z)}} \right) \]

2. Consider the integral

\[ I_\delta(z) = \tilde{K}(z) \otimes f(z) \equiv \int_z^1 \frac{dy}{y} \tilde{K}(y) f\left(\frac{z}{y}\right) \]

and define the moments of the kernel \( \tilde{K}(y) \) in the following form

\[ K_n = \int_0^1 dy y^{n-2} \tilde{K}(y) \]

In analogy with subsection 1 we have for the Regge-like case:

\[
I_{\delta,R}(z) = z^{-\delta} \int_z^1 dy y^{\delta-1} \tilde{K}(y) \left[ \tilde{\varphi}(0) + \frac{z}{y} \tilde{\varphi}^{(1)}(0) + \ldots + \frac{1}{k!} \left( \frac{z}{y} \right)^k \tilde{\varphi}^{(k)}(0) + \ldots \right] \\
= z^{-\delta} \left[ K_{1+\delta} \tilde{\varphi}(0) + O(z) \right] \\
- \left[ N_{1+\delta}(x) \tilde{\varphi}(0) + N_\delta(z) \tilde{\varphi}^{(1)}(0) + \ldots + \frac{1}{k!} N_{1+\delta-k}(z) \tilde{\varphi}^{(k)}(0) + \ldots \right],
\]

where

\[ N_{\eta}(z) = \int_0^1 dy y^{\eta-2} \tilde{K}(zy) \]

The case \( K_{1+\delta} = 1/(n + \delta) \) corresponds to \( \tilde{K}(y) = y^n \) and has been already considered in subsection 1. In the more general cases (for example, \( K_{1+\delta} = \Psi(1 + \delta) + \gamma \)) we can represent the ”moment” \( K_{1+\delta} \) as series of the sort \( \sum_{m=1}^{\infty} 1/(n + \delta + m) \).

So, for the initial integral at small \( x \) we get the simple equation:

\[ I_{\delta,R}(z) = z^{-\delta} K_{1+\delta} \tilde{f}(z) + O(z^1 - \delta) = K_{1+\delta} f_R(z) + O(z^{1-\delta}) \]

Repeating the analysis of the subsections (1b) and (1c), one easily obtains

\[ I_{\delta,L}(n, z) = K_{1+\delta} f_L(z) + O\left( \frac{1}{\ln(1/z)} \right) \text{ and} \]

\[ I_{\delta,I}(n, z) = K_{1+\delta} f_I(x) + O\left( \sqrt{\frac{\hat{d}}{\ln(1/z)}} \right) \]

Thus, in the nonsingular case the results do not depend on the shape of the PD but the accuracy of the method decreases for a \( \ln(1/z) \)-dependent PD.
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Tables

| $f$ | $d_+$ | $D_+$ | $d_+(1)$ | $D_+(1)$ | $d_-(1)$ | $D_-(1)$ |
|-----|-------|-------|--------|--------|--------|--------|
| 3   | -4/3 | 1180/81 | 101/81 | -43.37 | 16/81 | 1.974  |
| 4   | -36/25 | 91096/5625 | 61/45 | -45.49 | 64/225 | 3.108  |
| 5   | -36/23 | 84964/4761 | 307/207 | -47.73 | 80/207 | 4.675  |
| 6   | -12/7 | 8576/441 | 103/63 | -50.05 | 32/63 | 6.864  |

Table 1. The values of the parameters used in the calculation of the parton distributions as a function of the number of flavors

| $Q^2$ | $A_q$ | $A_g$ | $x_0$ | $\chi^2/n.o.p.$ |
|-------|------|------|------|----------------|
| LO (H1) | 1   | $1.06\pm0.07$ | $2.46\pm0.19$ | $0.11\pm0.02$ | $114/104$ |
| 3     | $0.96\pm0.08$ | $2.53\pm0.19$ | $0.12\pm0.02$ | $88/92$ |
| 5     | $0.83\pm0.09$ | $2.47\pm0.18$ | $0.15\pm0.02$ | $40/83$ |
| 8.5   | $0.80\pm0.10$ | $2.30\pm0.18$ | $0.18\pm0.03$ | $23/67$ |
| NLO (H1) | 1   | $0.97\pm0.08$ | $1.30\pm0.11$ | $0.20\pm0.03$ | $63/104$ |
| 3     | $0.91\pm0.10$ | $1.31\pm0.11$ | $0.22\pm0.03$ | $50/92$ |
| 5     | $0.81\pm0.10$ | $1.28\pm0.11$ | $0.26\pm0.04$ | $27/83$ |
| 8.5   | $0.84\pm0.11$ | $1.22\pm0.11$ | $0.28\pm0.05$ | $21/67$ |

Table 2. The result of the LO and NLO fits to H1 (1994) data for different low $Q^2$ cuts. In the fits $Q_0^2$ is fixed to 1 GeV$^2$

| Aprox. | $A_q$ | $A_g$ | $Q_0^2$ | $\chi^2/n.o.p.$ |
|--------|------|------|--------|----------------|
| LO (H1) | $1.10\pm0.08$ | $0.35\pm0.06$ | $0.55\pm0.02$ | $60/104$ |
| NLO (H1) | $0.83\pm0.09$ | $0.21\pm0.07$ | $0.55\pm0.03$ | $45/104$ |
| LO (ZEUS) | $1.13\pm0.07$ | $0.28\pm0.05$ | $0.55\pm0.02$ | $174/126$ |
| NLO (ZEUS) | $0.85\pm0.08$ | $0.18\pm0.05$ | $0.56\pm0.02$ | $143/126$ |

Table 3. The results of the fits to H1 and ZEUS (1994) data at LO and NLO with $Q_0^2$ free.
Figure captions

Figure 1. The structure function $F_2$ as a function of $x$ for different $Q^2$ bins. The experimental points are from H1 [1]. The inner error bars are statistic while the outer bars represents statistic and systematic errors added in quadrature. The dashed and dot-dashed curves are obtained from fits at LO and NLO respectively with fixed $Q_0^2 = 1$ GeV$^2$ (see table 2). The solid line is from the fit at NLO giving $Q_0^2 = 0.55$ GeV$^2$ (see table 3).

Figure 2. The structure function $F_2$. Experimental points are from ZEUS (squares from Ref. [2] and diamonds and crosses from two different types of measurements reported in Ref. [3]). The error bars are displayed as in Fig. 1. Solid (dashed) lines are calculated with the parameters given in table 1 from fits at NLO (LO).

Figure 3. The asymptotical values of effective slopes $\lambda_{q,as}^{eff}$, $\lambda_{g,as}^{eff}$ and $\lambda_{F_2}^{eff,as}$ calculated at NLO with the parameters from a NLO fit to H1 with $x_0 = 1$ (see Tab. (3)). Lower curves correspond to $x = 3 \times 10^{-5}$ while the upper ones are for $x = 10^{-2}$. The experimental points are from H1 [1]. The error bars are displayed as in Fig. 1.
Fig. 1
