ON CLOSED UNBOUNDED SETS CONSISTING OF FORMER REGULARS

MOTI GITIK

Abstract. A method of iteration of Prikry type forcing notions as well as a forcing for adding clubs is presented. It is applied to construct a model with a measurable cardinal containing a club of former regulars, starting with \( o(\kappa) = \kappa + 1 \). On the other hand, it is shown that the strength of above is at least \( o(\kappa) = \kappa \).

Suppose that \( \kappa \) is an inaccessible cardinal. We wish to find a generic extension (usually cardinal preserving) such that \( \{ a < \kappa \mid a \text{ is regular in } V \} \) contains a club. Radin introduced a basic method to do this. Simply start with a measurable \( \kappa \) with \( o(\kappa) = \kappa^+ \) and then force with the Radin forcing constructed from \( o(\kappa) = \kappa^+ \). If one wishes to keep \( \kappa \) a measurable in the extension, then a weak repeat point suffices. Both facts are proved by Mitchell [6].

We show how to reduce assumptions rendering the above possible.

A method of iteration generalizing those of [2] is presented. Then a variant of it is used to iterate forcing for shooting clubs. We think that this method of iteration can be applied to other distributive forcings as well.

We like to thank the referee for pointing out that the proof of Section 2 gives only \( o(\kappa) = \kappa \) and not \( o(\kappa) = \kappa + 1 \) as was claimed in the previous version, for long and detailed list of corrections and for his requests on explaining certain parts of the paper.

§1. Forcing construction. We will now prove the following two theorems.

THEOREM 1.1. Suppose that \( \kappa \) is an inaccessible cardinal such that for every \( \delta < \kappa \) the set of \( \alpha \)'s below \( \kappa \) with \( o(\alpha) \geq \delta \) is stationary. Then there is a cardinal preserving generic extension such that the set \( \{ \alpha < \kappa \mid \alpha \text{ is regular in } V \} \) contains a club.

THEOREM 1.2. Suppose that \( \kappa \) is a measurable cardinal with \( o(\kappa) = \kappa + 1 \). Then there is a cardinal preserving extension satisfying the following:

1. \( \kappa \) is a measurable,
2. \( \{ \alpha < \kappa \mid \alpha \text{ is a regular in } V \} \) contains a club.

The proofs of these theorems use an iteration Prikry type forcing notion that was introduced in [2].

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Basically for every \( \alpha < \kappa \) with \( \alpha > o(\alpha) > 0 \) we are forcing a Prikry or Magidor sequence to \( \alpha \) without adding new bounded subsets. The order type of the sequence is \( \omega^{o(\alpha)} \), where the exponentiation is the ordinal one. The forcing used for this is \( \langle \mathcal{P}(\alpha, o(\alpha)), \leq, \leq^{*} \rangle \) which was introduced in [2]. The definition of it is quite long and will not be used explicitly here. The only facts used in connection with this are stated below.

**Fact 1.** \( \langle \mathcal{P}(\alpha, o(\alpha)), \leq, \leq^{*} \rangle \) satisfies Prikry condition and it is \( \alpha \)-weakly closed, i.e.,

(a) \( \leq \leq^{*} \);
(b) for every \( p \in \mathcal{P}(\alpha, o(\alpha)) \) and every statement \( \sigma \) of the forcing \( \langle \mathcal{P}(\alpha, o(\alpha)), \leq \rangle \) there is \( q \geq p \) deciding \( \sigma \);
(c) \( \leq^{*} \) is \( \alpha \)-closed.

**Fact 2.** \( \langle \mathcal{P}(\alpha, o(\alpha)), \leq \rangle \) satisfies \( \alpha^{+} \)-c.c. and has cardinality \( 2^{\alpha} \).

**Fact 3.** \( \langle \mathcal{P}(\alpha, o(\alpha)), \leq \rangle \) adds a closed cofinal in \( \alpha \) sequence consisting of regular in \( V \) cardinals of order type \( \omega^{o(\alpha)} \) and almost containing in every club of \( \alpha \) of \( V \).

We are going to iterate \( \mathcal{P}(\alpha, o(\alpha)) \)'s using the iteration of [2]. For the benefit of the reader let us give a precise definition.

Let \( A \) be a set consisting of \( \alpha \)'s such that \( \alpha < \kappa \) and \( \alpha > o(\alpha) > 0 \). Denote by \( A^{\ell} \) the closure of the set \( \{ \alpha + 1 \mid \alpha \in A \} \cup A \). For every \( \alpha \in A^{\ell} \) define by induction \( \mathcal{R}_{\alpha} \) to be the set of all elements \( p \) of the form \( \langle p_{y} \mid y \in g \rangle \) where

(1) \( g \) is a subset of \( \alpha \cap A \).
(2) \( g \) has an Easton support, i.e., for every inaccessible \( \beta \leq \alpha \), \( \beta > |\text{dom} \ g \cap \beta| \);
(3) \( p \upharpoonright y = \langle p_{\beta} \mid \beta \in y \cap g \rangle \in \mathcal{R}_{y} \) and \( p \upharpoonright y \models \mathcal{P}(y, o(y)) \) for every \( y \in \text{dom} \ g \).

Let \( p = \langle p_{y} \mid y \in g \rangle \), \( q = \langle q_{y} \mid y \in f \rangle \) be elements of \( \mathcal{R}_{\alpha} \). Then \( p \geq q \) (\( p \) is stronger than \( q \)) if the following holds:

(1) \( g \supseteq f \).
(2) For every \( y \in f \) \( p \upharpoonright y \models \mathcal{P}(y, o(y)) \).
(3) There exists a finite subset \( b \) of \( f \) so that for every \( y \in f \setminus b \), \( p \upharpoonright y \Vdash \langle p_{y} \upharpoonright y \rangle \upharpoonright y \models \mathcal{P}(y, o(y)) \).

If \( b = \emptyset \), then we say \( p \geq^{*} q \).

By [2], such iteration preserves cardinals. The final forcing \( \mathcal{P}_{\kappa} \) satisfies \( \kappa \)-c.c. We refer to [2] for more details.

**Proof of Theorem 1.1.** Let \( G_{\kappa} \) be a generic subset of \( \mathcal{P}_{\kappa} \). We force over \( V[G_{\kappa}] \) with the forcing \( P[E] = \{ d \mid d \text{ is closed, bounded subset of } E \} \) ordered by end extension, where

\[
E = \{ \alpha < \kappa \mid \alpha \text{ is a regular in } V \},
\]

i.e., the usual forcing for adding a club via a stationary set.

By Avraham-Shelah [1], if \( E \) is fat then this forcing is \( (\kappa, \infty) \)-distributive. Where fatness means that for every \( \delta < \kappa \) and every closed unbounded subset \( C \) of \( \kappa \) there is a closed subset \( s \subseteq C \cap E \) of order type \( \delta \).
So it is enough to show that $E$ is fat in $V[G_\kappa]$. Let $\delta < \kappa$ and $C \subseteq \kappa$ be a club in $V[G_\kappa]$. By \(\kappa\)-c.c. of the forcing $\mathcal{P}_\kappa$, there is $C^* \subseteq C$ a club which belongs to $V$. Then pick an $\alpha^*$ a limit point of $C^* \cap \{\alpha < \kappa \mid \sigma(\alpha) \geq \delta\}$. By Fact 3 there will be a closed subset of $C^* \cap \alpha^* \cap E$ of order type $\delta$. This completes the proof.

We now turn to the Proof of Theorem 1.2 which takes up the rest of this section.

PROOF OF THEOREM 1.2. Denote by $U$ the measure concentrating on $\alpha$’s with $o(\alpha) = \alpha$. We preserve the notation of Theorem 1.1.

Let us explain the idea of the proof. We like to force with $P[E]$ a club through the set $E$ of regular cardinals below $\kappa$. So lots of changes of cofinalities are needed. An additional task is to preserve measurability of $\kappa$. So for a set of $\alpha$’s in $U$ we need to use $P[E \cap \alpha]$. The problem with it is that this forcing is not closed or weakly closed but only distributive. Iteration of $P[E \cap \alpha]$’s in Easton fashion even $\omega$-times collapses cardinals. The key idea will be to embed $P[E \cap \alpha]$ into $\mathcal{P}(\alpha, \beta) \ast \beta$-closed, where $\beta < \alpha$ and we will allow to change $\beta$ from time to time in order to increase the degree of closeness. Then the scheme of iteration of [2] will be generalized to make possible the iteration of less closed forcing notions.

Let us start with a generalization of the iteration process of [2].

We describe a general scheme of iteration. A specific forcing used for the proof of Theorem 1.2 will be defined later.

Let $A$ be a set consisting of inaccessible cardinals. Denote by $A^\dagger$ the closure of the set $A \cup \{\alpha + 1 \mid \alpha \in A\}$. We define an iteration $\langle \mathcal{P}_\alpha, Q_\alpha \mid \alpha \in A^\dagger \rangle$. For every $\alpha \in A^\dagger$ define by induction $\mathcal{P}_\alpha$ to be the set of all elements $p$ of the form $\langle p_\gamma \mid \gamma \in g \rangle$, where

1. $g$ is a subset of $\alpha \cap A$;
2. $g$ has an Easton support, i.e., for every inaccessible $\beta \leq \alpha$ $\beta > |\text{dom } g \cap \beta|$;
3. for every $\gamma \in \text{dom } g$

$$p \upharpoonright \gamma = \langle p_\beta \mid \beta < \gamma \rangle \in \mathcal{P}_\gamma$$

and $p \upharpoonright \gamma \models \mathcal{P}_\gamma$ “$p_\gamma$ is a condition in the forcing $\langle Q_\gamma, \leq_\gamma, \leq_\gamma^* \rangle$ satisfying the Prikry condition and of cardinality below the next element of $A$ above $\gamma$”.

The difference here is that we do not require that $\langle Q_\gamma, \leq_\gamma^* \rangle$ is $\gamma$-closed.

Let $p = \langle p_\gamma \mid \gamma \in g \rangle$ and $q = \langle q_\gamma \mid \gamma \in f \rangle$ be elements of $\mathcal{P}_\alpha$. Then $p \geq q$ ($p$ is stronger than $q$) if the following holds:

1. $g \supseteq f$.
2. for every $\gamma \in f$

$$p \upharpoonright \gamma \models \mathcal{P}_\gamma$ “$p_\gamma \geq_\gamma q_\gamma$ in the forcing $Q_\gamma$”.

3. there exists a finite subset $b$ of $f$ so that for every $\gamma \in f \setminus b$ $p \upharpoonright \gamma \models \mathcal{P}_\gamma$ “$p_\gamma \geq_\gamma^* q_\gamma$ in the forcing $Q_\gamma$”.

If the set $b$ in (3) is empty we call $p$ a direct extension of $q$ and denote this by $p \geq^* q$.

LEMMA 1.3. Let $\alpha \in A^\dagger$, $p \in \mathcal{P}_\alpha$, and $\sigma$ be a statement of the forcing language appropriate for $\mathcal{P}_\alpha$. Then there is a direct extension $p^*$ of $p$ deciding $\sigma$. 

Suppose otherwise. Let \( p = \langle p_y \mid y \in g \rangle \) and \( \beta \) be the minimal element of \( g \). We assume that \( g \neq \emptyset \), otherwise any extension of it is direct.

Let \( G \) be a generic subset of \( \mathcal{P}_{\beta+1} \), so that \( p \upharpoonright \beta + 1 \in G \). We shall mean by \( p^* = \langle p^*_y \mid y \in g \setminus (\beta + 1) \rangle \) the interpretation of it in \( V[G] \), i.e., an element of the forcing \( \mathcal{P}_\alpha / G \). Define now \( p^* \in \mathcal{P}_\alpha / G \). If there exists some \( q \in \mathcal{P}_\alpha / G \) a direct extension of \( p \setminus (\beta + 1) \) deciding \( \sigma \), then set \( p^* \) to some such \( q \). Otherwise, set \( p^* = p \setminus (\beta + 1) \). Let \( p^* = \langle p^*_y \mid y \in g^* \rangle \). Then \( g^* \in V[G] \).

CLAIM 1.4.1. There is \( p^*_p \) such that \( \phi \vdash_{\mathcal{P}_\beta} \langle p^*_p \rangle \geq p^*_p \) and \( \langle p^*_p \rangle \vdash_{\mathcal{P}_{\beta+1}} \langle p^*_p \rangle = p \setminus (\beta + 1) \).

PROOF. Let \( G \subseteq \mathcal{P}_\beta \) be generic. Work in \( V[G] \). Let \( p^*_\beta \) be a direct extension of \( p \) (in \( Q_\beta \)) deciding the statement "\( p^* = p \setminus (\beta + 1) \)". Suppose for a moment that the decision is negative. Then \( p^*_\beta \vdash_{Q_\beta} \langle p^*_\beta \rangle \) is a direct extension of \( p \setminus (\beta + 1) \) deciding \( \sigma \), by the choice of \( p^* \). Now pick a direct extension \( p^* \) of \( p^*_\beta \) so that

\[
p^* \vdash_{Q_\beta} \langle p^*_p \rangle \vdash_{\mathcal{P}_{\beta+1}} \langle p^*_p \rangle.
\]

where \( ^0 \sigma = \sigma \) and \( ^1 \sigma = \neg \sigma \). Pick some \( r \in G \) so that

\[
r \cap \langle p^*_\beta \rangle \vdash_{\mathcal{P}_{\beta+1}} \langle p^*_p \rangle \vdash_{\mathcal{P}_{\beta+1}} \langle p^*_p \rangle.
\]

Then \( r \cap \langle p^*_\beta \rangle \cap p^* \) will be a direct extension of \( p \) forcing \( ^1 \sigma \). Which contradicts our assumption.

Now we replace \( p_{\beta} \) in \( p \) by \( p^*_{\beta} \). Denote the resulting condition by \( p(\{\beta\}) \). Let \( \beta_1 \) be the second element of \( g \). We proceed as above replacing \( \beta \) by \( \beta_1 \) and \( p \) by \( p(\{\beta\}) \). This will define \( p_{\beta_1} \). Then set \( p(\{\beta, \beta_1\}) = \langle p_{\beta_1}^*, p_{\beta_1}^* \rangle \cap \langle p^*_y \mid y \in g \setminus \{\beta, \beta_1\} \rangle \). We continue in the same fashion at successor stages. Unions are taken at limit ones, using that at \( \gamma \langle Q_\gamma, \leq_\gamma \rangle \), is \( \gamma \)-closed. Finally, after going through all the elements of \( g \) we will obtain a condition \( p(g) = \langle p^*_y \mid y \in g \rangle \) which is a direct extension of \( p \).

Now let \( q \geq p(g) \) be a condition deciding \( \sigma \). By the definition of extension, there is a maximal \( \gamma \in g \) such that \( q \upharpoonright \gamma \vdash_{\mathcal{P}_{\gamma}} \langle q^*_y \rangle \) is not a direct extension of \( p^*_\gamma \). But this will contradict the choice of \( p(\gamma \cap (\gamma + 1)) \), since above \( \gamma \), \( q \setminus \gamma \) will be a direct extension of \( p \setminus \gamma \) deciding \( \sigma \). Contradiction.

Now let us define the iteration needed for the Proof of Theorem 1.2.

Let \( \alpha < \kappa \) be an ordinal with \( o(\alpha) = \alpha \). We like to define \( Q_\alpha \). But first, let us consider the forcing \( \mathcal{P}(\alpha, \beta) \ast [E \cap \alpha] \), where \( \beta < \alpha \). Define a \( * \)-ordering on \( \mathcal{P}(\alpha, \beta) \ast [E \cap \alpha] \) by setting \( p = (b, g) \leq^* p' = (b', g') \) if \( p = p' \) or (i) \( p \preceq p' \) and (ii) \( b \preceq^* b' \) in \( \mathcal{P}(\alpha, \beta) \).

LEMMA 1.4. The forcing \( \langle \mathcal{P}(\alpha, \beta) \ast [E \cap \alpha] \rangle, \leq^* \) satisfies the Prikry condition.

The proof follows easily from Fact 1.
LEMMA 1.5. Every condition in $\mathcal{P}(\alpha, \beta) * P[E \cap \alpha]$ can be extended to a condition of the form $(b, \check{c})$.

It follows by Fact 1 since $\mathcal{P}(\alpha, \beta)$ does not add new bounded subsets to $\alpha$.

Let $D_{\alpha, \beta} = \{(b, a) \in \mathcal{P}(\alpha, \beta) * P[E \cap \alpha] | b \Vdash_{\mathcal{P}(\alpha, \beta)} "\sup a \in B" \}$, where $B$ is the canonical name of the closed cofinal in $\alpha$ sequence added by $\mathcal{P}(\alpha, \beta)$.

LEMMA 1.6. For every $p \in \mathcal{P}(\alpha, \beta) * P[E \cap \alpha]$ there exists $q \in D_{\alpha, \beta}$ such that $q \geq^* p$.

PROOF. Obviously, since $b \Vdash_{\mathcal{P}(\alpha, \beta)} \sup a < \alpha$, where $p = \langle b, a \rangle$. \(\square\)

LEMMA 1.7. The order $\leq^*$ is $\beta'$-closed over $D_{\alpha, \beta}$ and hence $\mathcal{P}(\alpha, \beta) * P[E \cap \alpha]$ does not add new bounded subsets of $\beta$.

REMARK. Notice that all cardinals $\leq \alpha$ are collapsed to $\beta$, since the forcing with $P[E \cap \alpha]$ over $V^{\mathcal{P}(\beta)}$ is isomorphic to the Levy collapse $Col(\beta, \alpha)$.

PROOF. Let $\langle b_i, a_j \rangle | i < \tau < \beta$ be a $\leq^*$-increasing sequence of elements of $D_{\alpha, \beta}$. Since $(\mathcal{P}(\alpha, \beta), \leq^*)$ is $\alpha$-closed, there is $b \in \mathcal{P}(\alpha, \beta)$ such that $b^* \geq b_i$ for all $i < \tau$. Let $\check{a}$ be a name of the union of $a_i$'s. Since for every $i < \tau$, $b \Vdash \mathcal{P}(\alpha, \beta) \sup(a_i) \in B$ where $B$ is a canonical name of the generic closed cofinal in $\alpha$ sequence, $b \Vdash_{\mathcal{P}(\alpha, \beta)} "\sup a \in B"$ and in particular $b \Vdash_{\mathcal{P}(\alpha, \beta)} "\sup a \text{ is regular in } V"$. Hence $\langle b, \check{a} \rangle$ is a condition, it belongs to $D_{\alpha, \beta}$ and it is $\leq^*$ stronger than each $\langle b_i, a_j \rangle (i < \tau)$. \(\square\)

DEFINITION 1.8. (1) $Q_\alpha = P[E \cap \alpha] \cup \bigcup_{\beta < \alpha} (\mathcal{P}(\alpha, \beta) * P[E \cap \alpha])$.

(2) The ordering of $Q_\alpha$:

(a) The ordering of $P[E \cap \alpha]$ and of $\mathcal{P}(\alpha, \beta) * P[E \cap \alpha] (\beta < \alpha)$ is the usual one.

(b) Let $c \in P[E \cap \alpha]$ and $(b, a) \in \mathcal{P}(\alpha, \beta) * P[E \cap \alpha]$, for some $\beta < \alpha$. Define

$$\langle b, a \rangle \geq c \text{ iff } b \Vdash_{\mathcal{P}(\alpha, \beta)} \sup(a_i) \in B$$

$$c \geq \langle b, a \rangle \text{ iff } b \Vdash_{\mathcal{P}(\alpha, \beta)} \sup a \in B.$$ 

(c) Let $\langle b_i, a_j \rangle \in \mathcal{P}(\alpha, \beta) * P[E \cap \alpha]$ where $i = 0, 1, \beta_0 \neq \beta_1 < \alpha$. Define

$$\langle b_0, a_0 \rangle \leq \langle b_1, a_1 \rangle \text{ iff } c \in P[E \cap \alpha] \text{ such that } \langle b_0, a_0 \rangle \leq c \leq \langle b_1, a_1 \rangle.$$ 

We also define $\ast$-ordering of $Q_\alpha$.

DEFINITION 1.9. Let $p, q \in Q_\alpha$. Set $p \leq^* q$ iff $p = q$ or for some $\beta < \alpha$, $p = \langle b_1, a_1 \rangle$, $q = \langle b_2, a_2 \rangle \in \mathcal{P}(\alpha, \beta) * P[E \cap \alpha]$ and (i) $p \leq q$ (ii) $b_1 \leq^* b_2$ in $\mathcal{P}(\alpha, \beta)$.

LEMMA 1.10. (1) $P[E \cap \alpha]$ is dense in $Q_\alpha$.

(2) The forcing $(Q_\alpha, \leq)$ is equivalent to $(P[E \cap \alpha], \leq)$ and hence preserves cofinalities of cardinals.

PROOF. $P[E \cap \alpha]$ is dense in $Q_\alpha$ by the definition of the order. Also conditions incompatible in $P[E \cap \alpha]$ remain so in $Q_\alpha$. \(\square\)

LEMMA 1.11. Let $\beta < \alpha$. Suppose that $\langle b, a \rangle \in \mathcal{P}(\alpha, \beta) * P[E \cap \alpha]$ and $\sigma$ is a statement of the forcing language of $Q_\alpha$. Then there is a direct extension (in $Q_\alpha$) of $\langle b, a \rangle$ deciding $\sigma$ in the forcing with $Q_\alpha$. 


REMARK. The lemma actually shows the Prikry condition above $P(\alpha, \beta) \ast P[E \cap \alpha]$ part of $Q_\alpha$.

The lemma will follow from the following statement.

**LEMMA 1.12.** For every $\beta < \alpha$ $Q_\alpha$ is a projection of $P(\alpha, \beta) \ast P[E \cap \alpha]$.

**PROOF.** It is enough to project a dense subset of $P(\alpha, \beta) \times P[E \cap \alpha]$ onto a dense subset of $Q_\alpha$. Consider $D \subseteq P(\alpha, \beta) \times P[E \cap \alpha]$, which is dense in $P(\alpha, \beta) \ast P[E \cap \alpha]$. There is such $D$ since $P(\alpha, \beta)$ does not add new bounded subsets of $\alpha$. For $(b, a) \in D$ set $\pi((b, a)) = a$. By Lemma 10, $\text{rng} \pi = P[E \cap \alpha]$ is dense in $Q_\alpha$. It is trivial that $\pi$ is a projection map.

We are ready now to define the iteration. The definition will be as above only $\beta$'s for $Q_\alpha$ will be picked generically more carefully. This is needed for cardinals preservation.

Let $A$ and $A^{+}$ be as above. For every $\alpha \in A^{+}$ we define $P_\alpha$ by induction.

**DEFINITION 1.13.** A forcing notion $P_\alpha$ consists of all elements $p$ of the form $(p_y | y \in g)$ where

1. $g$ is a subset of $\alpha \cap A$;
2. $g$ has an Easton support;
3. for every $y \in \text{dom } g$ $p \upharpoonright y = (p_\beta | \beta < y) \in P_y$ and $p \upharpoonright y \Vdash \forall \gamma \ "p_\gamma/â€”$ is a condition in either the forcing $Q_y$, if $o(y) = y$ or in the forcing $P(y, o(y))$, if $o(y) < y$;
4. for every $\tau \leq \alpha$ the set $\{y < \alpha | p_y \in P[E \cap y] \text{ or } y \geq \tau \text{ and for some } \beta < \tau \}$ $p_y \in P(\gamma, \beta) \ast P[E \cap y]$ is finite.

The ordering on $P_\alpha$ is defined without changes.

The definition and Lemma 1.10 insures that for every $y$ with $o(y) = y$ the actual forcing used over $y$ is $P[E \cap y]$. But in every separate condition $p \in P_\alpha$ only finitely many $y$'s with $p_y \in P[E \cap y]$ are allowed (the condition (4)). The reason for this is to insure that the iteration preserves cardinals. Intuitively, finite iteration of forcings $P[E \cap y]$'s does no harm. In order to do infinite iterations (even of the length $\omega$), we like to have in advance some information about closed pieces of $E$. Forcings $P(\gamma, \beta)$'s are actually used for this purpose. Namely canonical generic sequences produced by such forcings.

**LEMMA 1.14.** $(P_\alpha, \leq, \leq^*)$ satisfies the Prikry condition.

The proof repeats the Proof of Lemma 1.13. The additional condition (4) has no effect on it.

**LEMMA 1.15.** Let $\gamma < \alpha$. Suppose that $G$ is a generic subset of $P_{\gamma+1}$. Then the forcing $P_\alpha / G$ does not add new subsets of $\gamma$.

**PROOF.** Suppose that some $p \Vdash \check{\gamma} \subseteq \check{\gamma}$. It is enough to show the following.

**CLAIM 1.15.1.** There are $b \in V[G]$ and $\bar{p} \geq p$ such that $\bar{p} \Vdash \check{a} = \check{b}$.
**Remark.** \( \bar{p} \) is not required to be a direct extension of \( p \). The reason for this is the finite set of \( y \)'s in \( p \) satisfying (4).

**Proof.** Using Lemma 1.14, we define by induction a \(*\)-increasing sequence \( \langle p_i \mid i \leq \gamma \rangle \) of extensions of \( p \) so that for each \( i \)

(a) \( p_i \) decides the statement "\( i \in \mathbb{g} \)."

(b) if for some \( \delta \) with \( o(\delta) = \delta \delta \in g_i \), then \( p_i \upharpoonright \delta \vDash \# \beta > \gamma \) "\( p_{\beta \delta} \in \mathcal{P}(\delta, \beta) \) for some \( \beta > \gamma \)"

where \( p_i = (p_i v \mid v \in g_i) \).

(c) if \( j > i \), \( j \leq \gamma \), then for every \( \delta \in g_j \setminus g_i \) with \( o(\delta) = \delta \)

\[
p_j \upharpoonright \delta \vDash \# \beta > \gamma \] \( p_{\beta \delta} \in \mathcal{P}(\delta, \beta) \) * \( \mathcal{P}[E \cap \delta] \)

for some \( \beta \) which is above \( \sup(g_i \cap \delta) \).

First we extend \( p \) to a condition \( p' \) satisfying (b). By (4) of Definition 1.13 it is always possible. But \( p' \) need not be a direct extension of \( p \). Now, by Lemma 1.9, find \( p'' \geq^* p' \) deciding "\( 0 \in \mathbb{g} \)." Let \( p_0 \) be an extension of \( p'' \) obtained as follows. We replace each \( p_{y''} \) in \( p'' \) for \( \delta \) with \( o(\delta) = \delta \) which is not in \( p' \) by a stronger condition in \( \mathcal{P}(\delta, \beta) \) * \( \mathcal{P}[E \cap \delta] \) where \( \beta \) is picked to be above every coordinate of \( p' \) below \( \delta \). By (3), (4) of Definition 1.13, only finitely many coordinates \( \delta \) in \( p'' \) should be fixed this way. So \( p_0 \) will be a condition stronger than \( p'' \) and a direct extension of \( p' \) deciding "\( 0 \in \mathbb{g} \)."

We continue by induction. On successor stages we proceed as above. Suppose now that \( i \leq \gamma \) is a limit ordinal and a sequence \( \langle p_{\rho} \mid \rho < i \rangle \) is defined and satisfies the conditions (a)-(c) above. Let us argue that there is \( p' \) a direct extension of all of \( p_{\rho} \)'s \( (\rho < i) \). Let \( p' \) be obtained by taking direct extensions in each coordinate separately. This is possible by (b). It is enough to show that such an obtained \( p' \) is a condition. The only problematic point is (4) of Definition 1.13. By (b), \( \{ \gamma < \alpha \mid p'_\gamma \in P[E \cap \gamma] \} \) is empty. So it remains to show that for every \( \tau < \alpha \) the set \( \{ \gamma < \alpha \mid \tau \leq \gamma \}, \) for some \( \beta < \tau \) \( p'_\gamma \in \mathcal{P}(\gamma, \beta) \) * \( \mathcal{P}[E \cap \gamma] \) is finite. Suppose otherwise. Let \( \tau, \tau \leq \gamma_0 < \gamma_1 < \cdots < \gamma_n < \cdots < \alpha \) be witnessing this. Then for each \( n < \omega \) there is \( \beta_n < \tau \) such that \( p'_{\gamma_n} \in \mathcal{P}(\gamma_n, \beta_n) \) * \( \mathcal{P}[E \cap \gamma_n] \). For each \( n < \omega \)

let \( i_n < i \) be the least such that the coordinate \( \gamma_n \) appears in \( p_{i_n} \). Shrinking the set of indexes if necessary, we assume that the sequence \( \langle i_n \mid n < \omega \rangle \) is strictly increasing. But this is impossible by (c). Since \( \gamma_1 > \gamma_0, \gamma_1 \) is in \( p_{i_1} \) but not in \( p_{i_0} \), \( \gamma_0 \) is in \( p_{i_0} \)

but \( p'_{\gamma_1} \in \mathcal{P}(\gamma_1, \beta_1) \) * \( \mathcal{P}[E \cap \gamma_1] \) where \( \beta_1 < \tau < \gamma_0 \). Contradiction. So \( p' \) is a condition. Now we continue as in successor stages. ~d of the claim.

**Lemma 1.16.** \( \mathcal{P}_\kappa \) satisfies \( \kappa\)-c.c. and preserves the cardinals.

Follows from Definition 1.13 and Lemma 1.15.

Let \( G_\kappa \) be a generic subset of \( \mathcal{P}_\kappa \). Force with \( P[E] \) over \( V[G_\kappa] \). Let \( C \) be a generic club.

**Lemma 1.17.** \( \kappa \) is a measurable cardinal in \( V[G_\kappa, C] \).

**Proof.** Let \( U \) be the measure in \( V \) concentrating over \( \{ \alpha < \kappa \mid o(\alpha) = \alpha \} \).

Denote by \( i_U : V \rightarrow N_U \simeq \text{Ult}(V, U) \) the corresponding elementary embedding.
Since $\mathcal{P}_\kappa$ satisfies $\kappa$-c.c. and using Claim 1.15.1 it is routine to extend $i_U$ (in $V[G_\kappa, C]$) to an embedding

$$i : V[G_\kappa] \rightarrow N[G_{i_U(\kappa)}]$$

where $N$ is an iterated ultrapower of $N_U$ moving only ordinals in the interval $(\kappa^+, i_U(\kappa))$. We sketch the proof. See [2, 3] or [5] for the details.

Working in $V[G_\kappa, C]$ we define a normal $V[G_\kappa]$-ultrafilter over $\kappa$ (i.e., a normal ultrafilter over $\mathcal{P}^{V[G_\kappa]}(\kappa)$). Proceed as follows: pick in $V$ an enumeration $(D_a | a < K)$ such that in $N_U$ $D_a$ is a $*$-dense open subset of $i_U(\mathcal{P}_\kappa)/\mathcal{P}_{\kappa+1}$ and if for some $D_0$ $D$ is a $*$-dense open subset of $i_U(\mathcal{P}_\kappa)/\mathcal{P}_{\kappa+1}$ then for some $a < \kappa^+$ $D \subseteq D_a$.

Then define a master condition sequence $\langle r_\alpha | \alpha < \kappa^+ \rangle \in V[G_\kappa, C]$ such that

(a) $r_\alpha \leq r_\beta$ for every $\alpha \leq \beta < \kappa^+$.
(b) $r_\alpha \in D_\alpha[G_\kappa, C]$ for every $\alpha < \kappa^+$.
(c) $\langle r_\alpha | \alpha < \beta \rangle \in N[U[G_\kappa, C]]$ for every $\beta < \kappa^+$.

Using $\langle r_\alpha | \alpha < \kappa^+ \rangle$ we define now $U^* \supseteq U$ by setting $X \in U^*$ iff for some $p \in G_\kappa * C$, $\alpha < \kappa^+$, a $\mathcal{P}_\kappa$-name $X$ of $X$ in $N_U$.

$$\langle p, r_\alpha \rangle \models_{i_U(\mathcal{P}_\kappa)} \langle X \rangle$$

It is easy to check that $U^*$ is a normal $V[G_\kappa]$-ultrafilter. Moreover, since $P[E]$ does not add new sequences of the length less than $\kappa$ to $V[G_\kappa]$, $U^*$ is $\kappa$-complete in $V[G_\kappa, C]$. Hence $\text{Ult}(V[G_\kappa], U^*)$ is well-founded. Let $N^*$ be its transitive collapse and $i : V[G_\kappa] \rightarrow N^*$ be the corresponding elementary embedding. By elementarity $N^*$ is of the form $N[G_{i(\kappa)}]$ where $G_{i(\kappa)}$ is $\kappa$-generic subset of $\mathcal{P}_{i(\kappa)}$. Examining the structure of $N$ it is possible to show that it is in fact an iterated ultrapower of $N_U$ moving only ordinals in interval $(\kappa^+, i_U(\kappa))$. But this is not needed for further argument.

Since $P[E]$ does not add new $< \kappa$-sequences, $N[G_{i(\kappa)}]$ is closed under $< \kappa$-sequences of its elements. But really more is true:

**Claim 1.17.1.** $\kappa N[G_{i(\kappa)}] \subseteq N[G_{i(\kappa)}]$.

*Proof:* Let $\langle t_\alpha | \alpha < \kappa \rangle$ be a sequence of elements of $N[G_{i(\kappa)}]$. Without loss of generality, we may assume that each $t_\alpha$ is an ordinal. Let $t_\alpha$ be a canonical $\mathcal{P}_\kappa \ast P[E]$ name of this sequence. Then for each $\alpha$ $t_\alpha(\kappa)$ is a set of cardinality at most $\kappa$ consisting of pairs $\langle p, \delta \rangle$ where $p \in \mathcal{P}_\kappa \ast P[E]$ and $\delta$ is an ordinal. For every $\alpha, \gamma < \kappa$ such that $\sigma(\gamma) = \gamma$ let $t_\alpha(\kappa) \upharpoonright \gamma$ denotes the set of $\gamma$ first pairs $\langle p, \delta(\gamma) \rangle$ in $t_\alpha(\kappa)$ such that $p \in \mathcal{P}_\gamma \ast P[E \cap \gamma]$ where $\delta$ represents $\delta$ in $N_\kappa$. Set $t_\alpha(\kappa) \upharpoonright \gamma = (t_\alpha(\kappa) \upharpoonright \gamma | \alpha < \gamma)$. Then, $i_U(t_\alpha(\kappa) \upharpoonright \gamma | \gamma < \kappa)(\kappa) = t_\alpha(\kappa) \upharpoonright \gamma(\kappa)$. Now define a function $g \in V[G_\kappa]$ representing $\langle t_\alpha | \alpha < \kappa \rangle$ in $N[G_{i(\kappa)}]$. Set $g(\gamma) = t_\alpha(\kappa) \upharpoonright \gamma | \gamma \neq \kappa \rangle$, where $G_{\kappa} \ast C_\gamma = G_{\kappa+1} = G_\kappa \cap \mathcal{P}_\gamma \ast P[E \cap \gamma]$.

Now, in $V[G_\kappa]$ there is a Rudin-Kiesler increasing commutative sequence of ultrafilters $\langle U_\alpha | \alpha < \kappa \rangle$ over $\kappa$. Thus for each $\alpha < \kappa$ the measure concentrating on $\{\beta < \kappa | \sigma(\beta) = \alpha\}$ in $V$ extends to $U_\alpha$ in $V[G_\kappa]$. Actually, such extensions are used to define $\mathcal{P}(\gamma, \delta)$'s. Then, in $N[G_{i(\kappa)}]$ we will have such a sequence over $i(\kappa)$ of the length $i(\kappa)$. Form the direct limit. Let $k : N[G_{i(\kappa)}] \rightarrow M[G_\kappa]$ be the
corresponding embedding, where $\lambda = k(i(\kappa))$. Let $j = k \circ i : V[G_\kappa] \to M[G_\lambda]$. Notice that since the length of the sequence used to form the direct limit is $i(\kappa)$ and $\text{cf}(i(\kappa)) = \kappa^+$, $^*M[G_\lambda] \subseteq M[G_\lambda]$. An additional point here is that $\lambda$ is a limit of critical points of embeddings used in the direct limit. They are singular cardinals in $M[G_\lambda]$ but are regular in $M$. Since $M$ is just an iterated ultrapower of $\mathcal{H}$, by [8]. Hence, in $V[G_\kappa, C]$, $j(E)$ contains a club. Now we use this in the standard fashion to diagonalize over $\kappa^+$ dense subset of $(P[j(E)])^{M[G_\lambda]}$. It will produce a club $C^* \supset C$ which is $M[G_\lambda]$-generic. Notice that by Lemma 1.10(2), $\mathcal{Q}_\kappa$ is equivalent to $P[E \cap \kappa]$. So $C$ is $\mathcal{Q}_\kappa$ generic. So we obtain $j \subseteq j^* : V[G_\kappa, C] \to M[G_\lambda, C^*]$. Hence $\kappa$ is measurable in $V[G_\kappa, C]$.

§2. On the strength of the existence of a club of “regulars”. In this section we will show that the hypothesis used in Theorems 1.1 are the best possible and those of Theorem 1.2 are close to this.

The next theorem is basically due to Mitchell.

THEOREM 2.1. Suppose that $\kappa$ is an inaccessible and the set $E = \{\alpha < \kappa \mid \alpha$ is regular in $\mathcal{H}\}$ contains a club. Then for every $\delta < \kappa$ the set $\{\alpha < \kappa \mid o^\mathcal{H}(\alpha) \geq \delta\}$ is stationary.

PROOF. Let $\delta < \kappa$. We show that $\{\alpha < \kappa \mid o^\mathcal{H}(\alpha) \geq \delta\} = A_\delta$ is stationary. Let $C$ be a club contained in $E$. Choose some $\alpha \in C$ of cofinality $\delta$. Then $\text{cf}^\mathcal{H}(\alpha) = \alpha$ since $\alpha \in E$. So its cofinality changed to $\delta$. Then by Mitchell [8], $o^\mathcal{H}(\alpha) \geq \delta$. Hence $C \cap A_\delta \neq \emptyset$. 

THEOREM 2.2. Let $\kappa$ be a measurable cardinal and a set $E = \{\alpha < \kappa \mid \alpha$ is regular in $\mathcal{H}\}$ contain a club. Then $o^\mathcal{H}(\kappa) \geq \kappa$.

PROOF. First note that by Mitchell [7], $\kappa^+ = (\kappa^+)^\mathcal{H}$. Let $U$ be a normal measure over $\kappa$. Consider its elementary embedding $j_U : V \to N \simeq \text{Ult}(V, U)$. By Mitchell [7], $j_U \upharpoonright \mathcal{H} = i$ is an iterated ultrapower of $\mathcal{H}$ by its measures. Suppose that $o^\mathcal{H}(\kappa) < \kappa$. Then $i(\kappa)$ is not a limit point of iteration. Since $^*N \subseteq N$, cf $i(\kappa) \geq \kappa^+$. That is, there is a last measure in which the ultrapower reached $i(\kappa)$, (or more precisely, the image of its critical point). Denote this critical point by $\lambda$. Then for cofinally many $\alpha < i(\kappa)$ there is $f : [\kappa]^n \to \kappa (n < \omega)$ in $\mathcal{H}$ and $\mathcal{H}$ holds that $i(f)(\beta_1, \ldots, \beta_n) = \alpha$.

Using coding of $n$-tuples, we can replace $n$-placed functions by a 1-placed ones. Then, for every $\alpha < i(\kappa)$ (or $g : \kappa \to \kappa$) there will be $\beta \leq \lambda$ and $f : \kappa \to \kappa$ in $\mathcal{H}$ such that $i(f)(\beta) \geq \alpha$ (or $i(f)(\beta) = j_U(g)(\kappa)$).

CLAIM 1. For every $\alpha < i(\kappa)$ there is $f \in ^*\kappa \cap \mathcal{H}$ such that $\alpha \leq i(f)(\lambda)$.

PROOF. Let $\alpha < i(\kappa)$ and $g \in ^*([\kappa]^n) \cap \mathcal{H}$ be such that $i(g)(\beta_1, \ldots, \beta_n, \lambda) \geq \alpha$, where $\beta_1 < \cdots < \beta_{n-1} < \lambda$. Define $f \in ^*\kappa \cap \mathcal{H}$ as follows:

$$f(v) = \sup\{g(v_1, \ldots, v_{k-1}, v) \mid v_1 < \cdots < v_{n-1} < v\}.$$ 

CLAIM 2. The set 

$$A = \{\beta < i(\kappa) \mid \exists f \in ^*\kappa \cap \mathcal{H}, \quad i(f)(\lambda) = \beta\}$$
is $\kappa$-closed.

**Proof.** First suppose that the embedding $i$ is definable in $\mathcal{H}$. Let $\{\beta_v \mid v < \rho\}$ be a subset of $A$ for some $\rho < \kappa$. Denote by $\beta$ the sup$\{\beta_v \mid v < \rho\}$. Since there is no measurable cardinals of $\mathcal{H}$ between $\kappa$ and $2^\kappa$, there will be a set $B \subseteq \mathcal{H}$, $|B| \leq \kappa$, covering $\{\beta_v \mid v < \rho\}$, by Mitchell [8]. Notice that since $\kappa^+ = (\kappa^+)^\mathcal{H}$, $|B|^\mathcal{H} \leq \kappa$.

Using the regularity of $\kappa$, we can find $B^* \subseteq B \cap A$, $B^* \in \mathcal{H}$ of cardinality $< \kappa$ cofinal in $\beta$.

Now it is obvious that for a function $f \in {}^\kappa \cap \mathcal{H}$, $i(f)(\lambda) = \beta$. So $\beta \in A$.

Let us now deal with the general case, i.e., $i$ is not necessarily definable in $\mathcal{H}$. We replace $i$ by a definable in $\mathcal{H}$ iteration $i^*$. Proceed as follows. Iterate $\mathcal{H}$ using every measure over $\kappa$ as well as the new ones appearing in the process $(2^\kappa)^\mathcal{H}$-many times. Let $i^*$ be such an iteration. Obviously, there is an embedding $k : i(\mathcal{H}) \to i^*(\mathcal{H})$ so that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{i} & i(\mathcal{H}) \\
& \searrow & \downarrow k \\
& & i^*(\mathcal{H})
\end{array}
\]

Notice that $k \upharpoonright \kappa^+ = id$. So, $k(A) = \{\beta < i^*(\kappa) \mid \exists f \in {}^\kappa \cap \mathcal{H}, i^*(f)(k(\lambda)) = \beta\} = k^"(A)$.

Now, the argument used above for $i$, $A$ works for $i^*$, $k(A)$.

Let us now change the cofinality of $\kappa$ to $\omega$ by the Prikry forcing with $U$. Let $\langle \kappa_n \mid n < \omega \rangle$ be the Prikry sequence and $\langle \lambda_n \mid n < \omega \rangle$ the sequence corresponding to $\lambda$, i.e., $\langle f(\kappa_n) \mid n < \omega \rangle$ for $f \in {}^\kappa$ representing $\lambda$ in the ultrapower by $U$.

Choose an elementary submodel $M$, of large enough portion of the universe $|M| < \kappa$, $\omega M \subseteq M$ containing all relevant information. Let $\alpha = sup(M \cap i(\kappa))$ and $\alpha_n = sup(M \cap \kappa_{n+1})$ for $n < \omega$. Notice that $cf \alpha < \kappa$, since $|M| < \kappa$ and $cf \alpha > \omega$, since $\omega M \subseteq M$.

Since $A = \{\beta < i(\kappa) \mid \exists f \in {}^\kappa \cap \mathcal{H}, i(f)(\lambda) = \beta\}$ is $\kappa$-closed and unbounded in $i(\kappa)$ in $V$, it contains its limit points of cofinality $\delta, \omega < \delta < \kappa$, in $V[\langle \kappa_n \mid n < \omega \rangle]$. Hence, $\alpha \in A$, since $A \subseteq M$ and $\omega < cf \alpha < \kappa$. Let $f_\alpha \in {}^\kappa \cap \mathcal{H}$ be such that $i(f_\alpha)(\lambda) = \alpha$.

**Claim 3.** For all but finitely many $n$'s, $f_\alpha(\lambda_n) = \alpha_n$.

**Proof.** Let $g_v$ be a function in $V$ such that $v = i(g_v)(\kappa)$. Then it is enough to show that $g_\alpha(\kappa_n) = \alpha_n$ for almost all $n$, for then

\[
i(f_\alpha)(\lambda) = \alpha \iff \{v : f_\alpha(g_\alpha(v)) = g_\alpha(v)\} \in U \\
\iff f_\alpha(g_\alpha(\kappa_n)) = g_\alpha(\kappa_n) \text{ for sufficiently large } n \\
\iff f_\alpha(\lambda_n) = \alpha_n \text{ for sufficiently large } n.
\]

Now we show that $g_\alpha(\kappa_n) = \alpha_n$ for almost every $n$. Let $g \in M \cap \prod_{n<\omega} \kappa_{n+1}$. By standard argument on Prikry forcing used inside $M$, we can find $\beta \in M \cap \alpha$ such that $g_\beta(\kappa_n) > g(n)$ for all but finitely many $n$'s. So $\alpha_n \leq g_\alpha(\kappa_n)$. Now
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\[ \omega < \text{cf}(\alpha) < \kappa \text{ in } V \text{ since } \omega < \text{cf}(\alpha) < \kappa \text{ in } [\kappa_n : n \in \omega] \]. Choose an increasing sequence \((\xi_v : v < \rho) \in V\) which is cofinal in \(\alpha\), for some \(\rho < \kappa\). Then

\[ \left\{ \eta : g_\alpha(\eta) = \sup_{v < \rho} g_{\xi_v}(\eta) \right\} \in U. \]

If \(\alpha_n < g_\alpha(\kappa_n)\) for infinitely many \(n\) then there is \(v < \rho\) such that \(\alpha_n < g_{\xi_v}(\kappa_n)\) for infinitely many \(n\) (using the fact that \(\text{cf}(\alpha) > \omega\)). This is impossible, since if we pick \(\xi \in M\) with \(\xi_v < \xi < \alpha\) then \(g_{\xi_v}(\kappa_n) < g_{\xi}(\kappa_n) < \alpha_n\) for almost all \(n\). \(\neg\)

Let us now use the club \(E \in V\) consisting of regular in \(\mathcal{H}\) cardinals. For every \(n < \omega\), \(E \cap \kappa_{n+1}\) will be such a club in \(\kappa_{n+1}\). Hence the same is true in \(M\). So, \(\alpha_n = M \cap \kappa_{n+1} \in E\) for every \(n < \omega\). We argue that this is impossible.

Briefly we apply Mitchell’s analysis of \(M\), see [8]. It implies that for all but finitely many \(n\)’s, \(\alpha_n\) is a limit of indiscernibles. Since \(\kappa_{n+1}\) is indiscernible for \(\kappa\), there will be a club subset of \(\alpha_n \cap E\) consisting of indiscernibles for at least \(\kappa\). Select a sequence \(\langle c_n : n < \omega \rangle\) of such indiscernibles so that \(\lambda_n < c_n < \alpha_n\). Then \(\langle c_n : n < \omega \rangle \in M\) and by Claim 3, \(f_\alpha(\lambda_n) > c_n (n < \omega)\), which is impossible. A contradiction.

Let us provide more details.

We apply the technique of the second section of [8]. We will use \(M\) as the set \(N\) of that paper, i.e., we are covering the set \(M\).

Write \(C(\alpha) = \bigcup_{\beta} \mathcal{B}(\alpha, \beta)\) and \(C = \bigcup_{\alpha} C(\alpha)\). It follows from what is given there that there are \(h^M \in \mathcal{H}\) and \(\xi < \kappa\) such that:

\[ (1) (a) \forall \gamma \in M \cap \kappa \gamma \in h^M(\xi \cup (C \cap \gamma + 1)). \]

(b) If for \(c \in C\) we write \(\tau(c)\) for the largest ordinal \(\tau\) such that \(c \in \mathcal{B}(\tau)\) then \(\tau(c)\) always exists and is in \(h^M(\xi \cup (C \cap c))\).

Since \(\alpha_n\) is regular in \(\mathcal{H}\), it follows that \(C\) is unbounded in \(\alpha_n\) and \(\kappa_n \in C\) for sufficiently large \(n\). Since \(\langle \kappa_n : n \in \kappa \rangle\) is a Prikry sequence it follows that \(\tau(\kappa_n) = \kappa\) for sufficiently large \(n\). Also it follows that for \(c\) in a closed unbounded subset of \(\alpha_n \cap M\) we have \(\tau(c) \notin \alpha_n\). Since \(\tau(c) \in M\) it follows that \(\tau(c) > \alpha_n\).

An additional fact from [8] that we are using:

\[ (2) \text{ For every } f \in \mathcal{H}\text{ there is } \rho < \kappa \text{ such that if } c > \rho \text{ then } (f''c) \cap \tau(c) \subseteq c. \]

However we have \(f_\alpha(\lambda_n) = \alpha_n\) for all sufficiently large \(n\), and this is impossible because for all sufficiently large \(n\) there are \(c_n\) such that \(\lambda_n < c_n < \alpha_n\) and \(\tau(c_n) > \alpha\). \(\neg\)

We do not know if it is possible to improve the above to \(o(\kappa) = \kappa + 1\).

**Question.** Is \(o(\kappa) = \kappa\) enough for a model with a measurable containing a club of former regular cardinals?

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SCHOOL OF MATHEMATICAL SCIENCES
SACKLER FACULTY OF EXACT SCIENCES
TEL AVIV UNIVERSITY
RAMAT-AVIV 69978 ISRAEL

E-mail: gitik@math.tau.ac.il