Hilbert’s Incompleteness, Chaitin’s Ω number and Quantum Physics

Tien D Kieu *
Centre for Atom Optics and Ultrafast Spectroscopy,
Swinburne University of Technology, Hawthorn 3122, Australia

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Abstract

To explore the limitation of a class of quantum algorithms originally proposed for the Hilbert’s tenth problem, we consider two further classes of mathematically non-decidable problems, those of a modified version of the Hilbert’s tenth problem and of the computation of the Chaitin’s Ω number, which is a representation of the Gödel’s Incompleteness theorem. Some interesting connection to Quantum Field Theory is pointed out.

Introduction

Quantum physical processes have been considered for the purpose of computation for some time, and more recently some are proposed in [1] for certain class of mathematical non-computables associated with the Hilbert’s tenth problem and, equivalently, the Turing halting problem [2]. There is, however, a hierarchy of non-computable/undecidable problems [3] of which the Turing halting problem is at the lowest level. Another well-known non-decidable result is the Gödel’s Incompleteness Theorem about the incompatibility of consistency and completeness of Arithmetics, see [4] for a readable account. As the mathematical property of being Diophantine, which is at the heart of the Turing halting problem, is not a sufficient condition for that of being arithmetic [5], which concerns the Gödel’s, the quantum algorithm cited above has no direct consequence on the Gödel’s theorem.

Notwithstanding this, in order to explore the limitation of our algorithm we consider two further classes of mathematically non-decidable problems. We find some connection between Quantum Field Theory and the Chaitin’s Ω number which intimately links to the Gödel’s theorem. Along the way we also consider some modified versions of the Hilbert’s

*email: kieu@swin.edu.au
tenth problem where the question about a Diophantine equation is now concerning the finiteness of the number of its integer solutions.

In the next section we recall the proposed quantum algorithm, and its physical principles, for the Hilbert’s tenth problem to set the scene.

Quantum algorithm for the Hilbert’s tenth problem

All the considerations in this work are related through the Diophantine equation which is a polynomial equation with integer coefficients and many unknowns,

\[ P(x_1, \cdots, x_K) = 0, \tag{1} \]

of which it suffices to consider only non-negative integer solutions. The Hilbert’s tenth problem asks for a universal algorithm to determine whether such an equation has any solution or not.

Instead of looking for the zeroes of the polynomial in (1) over the non-negative integers, which may not exist, we will search for the absolute minimum of its square, \((P(x_1, \cdots, x_K))^2\), which always exists and finite. Quantum mechanics facilitates this search through the ground state of the corresponding Hamiltonian

\[ H_P = (P(a_1^\dagger a_1, \cdots, a_K^\dagger a_K))^2, \tag{2} \]

where \((a, a^\dagger)\) commute for different indices and are the creation and annihilation operators acting on some Fock space. The ground state has the energy which is exactly the searched-for absolute minimum. It also carries some information, which can be revealed through measurements, about the set of some integers \((n_1, \cdots, n_K)\) at which the minimum is obtained. In [1] we consider the processes to simulate the desired Hamiltonian, and to obtain the ground state through some adiabatic evolution starting from some readily constructed ground state of some initial Hamiltonian. One such initial Hamiltonian is

\[ H_I = \sum_{i=1}^{K}(a_i^\dagger - \alpha_i^*)(a_i - \alpha_i), \]

where \(\alpha_i\) are complex numbers. This Hamiltonian has discrete spectrum. Its ground state is \(\otimes_{i=1}^{K}|\alpha_i\rangle\), a direct product of the coherent state \(|\alpha_i\rangle\),

\[ a_i|\alpha_i\rangle = \alpha_i|\alpha_i\rangle, \]

which can be produced by stabilised lasers in the case quantum optical systems are used for computation. Note that all the Hamiltonians and observables involved have discrete eigenvalues. In a generic case, it can be argued [4] that there should be no level crossing in the interpolating time, except at the end point \(t = T\) where the adiabatic process ends with the Hamiltonian [4] which has some obvious symmetry. This finiteness of the gap results in the finiteness of the running time \(T\).
Another decision problem

We can also modify the Hilbert’s tenth problem and ask a different question whether the equation (1) has a finite number of non-negative integer solutions (including the case it has no solution) or an infinite number. In general, we cannot tell the degree of degeneracy of the ground state, so we will consider the most direct application of the algorithm of the last section as follows.

We first note that if the Diophantine equation has only a finite number of solutions then the shifting in the first argument, say, of the Diophantine polynomial will render the equation

\[
P(x_1 + L, \ldots, x_K) = 0
\]

has no more solutions for any \( L \) larger than the maximum integer value of \( x_1 \) of all the solutions of the original, unshifted Diophantine equation. This trick works because the \( x_i \)'s are restricted to non-negative integer numbers. With this observation, we could apply the quantum algorithm of the last section for

\[
P(x_1 + i, \ldots, x_K) = 0
\]

for \( i = 0, 1, 2, \ldots \) in turn. If for some \( i = L \) the algorithm tells us that the equation has no solution we could conclude that it has only a finite number of solutions. Otherwise, we will go on forever when there is an infinite number of solutions. This is thus a kind of halting problem of different flavour.

To proceed further, we observe that

\[
\sum_{i=0}^{\infty} (P(x_1 + i, \ldots, x_K))^2
\]

will have zero absolute minimum value if and only if equation (1) has an infinite number of non-negative integer solutions.

It is clear that the absolute minimum of the sum (6) diverges if the equation has only a finite number of solutions. To regularise the absolute minimum of the sum and make it convergent we can introduce some coefficients \( \beta_i \),

\[
\sum_{i=0}^{\infty} \beta_i (P(x_1 + i, \ldots, x_K))^2.
\]

For example,

\[
\beta_i = 1/i!
\]

will suffice because, for large \( i \), the factorial function increases much faster than any polynomial

\[
\min (P(x_1 + i, x_2, \ldots, x_K))^2 \leq (P(i, 0, \cdots, 0))^2 << i!.
\]
The first inequality follows from the fact that the point \((i, 0, \cdots, 0)\) belongs to the boundary of the domain of the square of the Diophantine polynomial.

We could now apply the quantum algorithm of the last section to form the Hamiltonian

\[
H_P = \sum_{i=0}^{\infty} \beta_i \left(P\left(a_1^\dagger a_1 + i, \cdots, a_K^\dagger a_K\right)\right)^2
\]  

and try to obtain its ground state as before. As with regularisation in Quantum Field Theory, we eventually remove the regulator through some dependence on some parameter \(0 < s < 1\)

\[
\lim_{s \to 0} \beta_i(s) = 1,
\]

for instance, \(\beta_i(s) = (1/i!)^s\). If we could tell that the ground state energy is zero as \(s \to 0\), starting from \(\beta_i(1)\), throughout then the equation has an infinite number of solutions. However, implementing such limit may not be achievable since the coefficients on the right hand side of (9), unlike before, are no longer integers but a convergent series for each of them. We may not be able in general to evaluate these series, even though they are made convergent, in order to simulate the Hamiltonian to the desired precision. (There should be no problem of the unbounded amount of energy that would be required in the regulator-removal limit, \(s \to 0\) if it could be implemented, because as soon as the ground state energy is distinguishable from zero then we know that the equation has a finite number of solutions.)

Godöel’s Incompleteness and Chaitin’s \(\Omega\) number

The Gödel’s Second Incompleteness Theorem \([4]\) is the negative answer for the Hilbert’s second problem about formalisation of mathematics. Gödel’s theorem stipulates that Arithmetics cannot be both consistent and complete. Interpretation of this theorem tells us that if a formal system, with a finite set of axioms and inference rules, is consistent and could include Arithmetics then there exist unprovable statements within that system; incompleteness thus entails. (Interestingly, it may be precisely the statement about consistency of the system that is unprovable.) Our quantum algorithm for the Hilbert’s tenth problem only deals with the mathematical property of being Diophantine, which is a sub-property of arithmetic \([5]\), and thus cannot resolve the Gödel’s decision problem which concerns with the property of being arithmetic in general.

Chaitin has approached this problem from the perspectives of Algorithmic Information Theory \([7, 8]\) and shown that there exist many unprovable statements in Arithmetics simply because they have irreducible algorithmic contents, measurable in bits, that are more than the complexity, also measurable in bits, of the finite set of axioms and inference rules for the system. That is, there is randomness even in pure mathematics. And more frustratingly, we can never prove randomness since we could only ever deal with finite axiomatic complexity.

Chaitin, to illustrate the point, has introduced the number \(\Omega\) as the halting probability for a random program, with some random input, being emulated by a particular Turing.
machine. It is also an average measure over all programs run on the universal Turing machine. This number has many interesting properties, and has been generalised to a quantum version \[9\], but we only make use here the linkage between this number and polynomial Diophantine equations. When expressed in binary, the value of the \( k \)-th bit of \( \Omega \) is 0 or 1 depending on some Diophantine equation corresponding to the Turing machine in consideration,

\[
C(k, N, x_1, \cdots, x_K) = 0.
\]  

(11)

For a given \( k > 0 \), the \( k \)-th bit is determined by whether there are finitely or infinitely many values for the parameter \( N > 0 \) for which the equation above has solutions in non-negative integers \( (x_1, \ldots, x_K) \).

We now try to adapt our quantum algorithm for the Hilbert's tenth problem for this kind of equations which represent the Gödel's theorem in a different guise. But then we could only come up with the hamiltonian

\[
H_P = \sum_{N=0}^{\infty} \beta_N(s) \left( C(k, N, a_{i+NK}^1 a_{1+NK}, \cdots, a_{(1+N)K}^1 a_{(1+N)K}) \right)^2,
\]  

(12)

where we have appealed to the framework of Quantum Field Theory for the appearance of a countably infinite number of pairs of operators \( (a, a^\dagger) \). Theoretically, the ground state of (12) has relatively zero energy in the regulator-removal limit, \( s \to 0 \), if and only if the equation (11) has solutions for an infinitely many values for the parameter \( N \).

The connection to Quantum Field Theory is interesting but it makes the whole exercise somewhat academic because we yet know how to create QFT hamiltonians on demand.

**Concluding remarks**

If, as we hope, the quantum algorithm for the Hilbert’s tenth problem can be implemented then all is the better. However, there might be some fundamental physical principles, not those of practicality, which prohibit the implementation. Or, there might be not enough physical resources (ultimately limited by the total energy and the lifetime of the universe) to satisfy the execution of the algorithm. In either cases, the exercise is still very interesting as the unsolvability of those problems and the limit of mathematics itself are also dictated by physical principles and/or resources.

Generalised noncomputability and undecidability set the boundary for computation carried out by mechanical (including quantum mechanical) processes, and in doing so help us to understand much better what can be so computed. With this in mind, we consider some adaptations of our quantum algorithm for the Hilbert’s tenth problem to some modified version of the Hilbert’s tenth problem. We also consider the computation of Chaitin’s \( \Omega \) number. These problems are all inter-related through questions about existence of solutions of Diophantine equations. Among the implementation difficulties, some interesting connections to Quantum Field Theory are pointed out along the way but we do not know how to create QFT hamiltonians on demand.
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