ON EXCEPTIONAL MAASS FORMS

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Abstract. We prove certain relations between Satake parameters of cuspidal representations of $GL_2(\mathbb{A}_Q)$ at finite and archimedean places. Consequently, we show that the Ramanujan-Petersson conjecture at a fixed prime $p \mid N$ for non-exceptonal Maass forms of level $N$ implies the conjecture at $p$ for all Maass forms of level $N$ and the Selberg’s 1/4-eigenvalue conjecture simultaneously. As an application, we improve Kim and Sarnak’s $7/64$-bound towards the Satake parameters at all $p \mid N$ for exceptional Maass forms.

1. Introduction

Let $\Gamma$ be a congruence subgroup of $SL_2(\mathbb{Z})$. Then $\Gamma$ acts on the upper half plane $\mathbb{H}$ by linear fractional transforms. Let $L^2(\Gamma\backslash \mathbb{H})$ be the space of square integrable functions on the quotient space $\Gamma\backslash \mathbb{H}$. Let $0 = \lambda_0(\Gamma) < \lambda_1(\Gamma) < \lambda_2(\Gamma) < \cdots$ be the eigenvalues of the non-euclidean Laplacian operator on $L^2(\Gamma\backslash \mathbb{H})$. Selberg [Sel65] conjectures that the smallest nonzero eigenvalue $\lambda_1(\Gamma) \geq 1/4$, and proves $\lambda_1(\Gamma) \geq 3/16$ is admissible. When $\Gamma = \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}$, the Hecke congruence subgroup of level $N$, it has been known that the conjecture holds for sufficiently small values of $N$ (e.g., cf. [DHS3] or [Vig81]). However, Selberg’s eigenvalue conjecture remains one of the fundamental unsolved questions in the theory of modular forms.

Langlands [Lan70] interprets the Selberg conjecture as a Ramanujan-Petersson conjecture "at infinity" and thus puts both conjectures on an equal conceptual footing. This adelic viewpoint has roots in Satake’s earlier work. In this unified statement the conjectures are conventionally called the Ramanujan conjecture. We then proceed to give a brief description of the Ramanujan conjecture along this perspective.

Let $\pi = \otimes'_p \pi_p$ be a unitary cuspidal automorphic representation of $GL_2(\mathbb{A}_Q)$. Suppose $\pi_\infty$ is spherical, then the Gamma factor attached to $\pi_\infty$ is $L_\infty(s, \pi_\infty) = \Gamma_2(s - \mu_1, \infty)\Gamma_2(s - \mu_2, \infty)$ for certain $\mu_1, \mu_2 \in \mathbb{C}$, where $\Gamma_2(s) = \pi^{-s/2}\Gamma(s/2)$. Then Selberg’s eigenvalue conjecture amounts to that $\text{Re}(\mu_{j, \infty}) = 0$, $j = 1, 2$.

Let $p$ be a prime such that $\pi_p$ is unramified. Then the local $L$-factor attached to $\pi_p$ is $L_p(s, \pi_p) = (1 - \alpha_1, p^{-s})^{-1}(1 - \alpha_2, p^{-s})^{-1}$, where $\alpha_1, p$ and $\alpha_2, p$ are the Satake parameters. Then the Ramanujan-Petersson conjecture asserts that $|\alpha_1, p| = 1$, i.e., $\log_p |\alpha_j, p| = 0$, $1 \leq j \leq 2$.

So far the best known result towards the Ramanujan conjecture is due to Kim and Sarnak [KS03], where they obtain bounds for cuspidal representations of $GL_n(\mathbb{A}_Q)$, and their result on $GL_2$ (together with the functoriality of symmetric fourth power lifting by Kim and Shahidi [KS02]) is

$$|\text{Re}(\alpha_{j, \infty})| \leq 7/64, \quad \text{and} \quad |\text{Re}(\log_p |\alpha_{j, p}|)| \leq 7/64, \quad j = 1, 2,$$

where $p$ is a prime such that $\pi_p$ is unramified. Also, Blomer and Brumley [BB11] generalize [KS03] to number fields.

Note that the proofs of inequalities in (1) are parallel. Relations between Selberg’s conjecture (i.e., at $\infty$) and Ramanujan-Petersson conjecture (i.e., at $p$) is
not clear, although they both follow from the Linnik-Selberg conjecture (cf. [Lin63] and [Sel65]) concerning cancellations due to signs of Kloosterman sums; or they can be deduced from the hypothetical functorial lifting of all high enough symmetric powers of $\pi$ by the methods developed in [DI83], [LRS95] and techniques in [BB11] for $\pi$ over number fields. Nevertheless, we establish a new approach to detect relations between $\mu_{j,\infty}$ and $\mu_{j,p}$ for families of cuspidal representations, from which we obtain an improvement of Kim and Sarnak’s result in $\text{GL}_2(\mathbb{A}_Q)$ case:

**Theorem A.** Let notation be as before. Then

$$|\text{Re}(\alpha_{j,\infty})| + |\text{Re}(\log p|\alpha_{j,p}|)| \leq 7/64, \quad j = 1, 2.$$  

In fact, the inequality (2) is a consequence of our Theorem B below, which is a more refined result. By Deligne [Del72] the Ramanujan-Petersson conjecture is known for holomorphic modular forms. Henceforth we only consider $\pi$ corresponding to a Maass form and use classical language.

Let $N$ be a positive integer. Let $\mathcal{F}(N) = \{\rho_1, \rho_2, \cdots\}$ be an orthonormal basis of Hecke-Maass forms of level $N$, such that the first Fourier coefficient of a new form $\rho \in \mathcal{F}(N)$ is equal to $(p, \rho)^{-1/2}$. Moreover, since $\rho$ is a new form, the first Fourier coefficients $a_{\rho}(1) \neq 0$. Then Selberg’s conjecture amounts to that $t \in \mathbb{R}$. On the other hand, let $p \nmid N$ be a prime. Let $\{\alpha_{\rho,p}, \beta_{\rho,p}\}$ be the normalized spherical parameters of $\rho$ at $p$. Let $r_p(\rho) = \max\{|\text{Re}(\log p|\alpha_{\rho,p}|)|, |\text{Re}(\log p|\beta_{\rho,p}|)|\}$. Then Ramanujan-Petersson conjecture asserts that $r_p(\rho) = 0$. Unconditionally, the best result is the second inequality in (1), i.e., $r_p(\rho) \leq 7/64$, $\forall \rho \in \mathcal{F}(N)$. Moreover, for general $\Gamma$, the best known result is $\lambda_\Gamma \geq 1/4 - (7/64)^2 = 0.238...$ by the first inequality in (1).

In this paper, we show, by studying poles of local $L$-functions at $p$ and $\infty$, that the Ramanujan-Petersson conjecture at a prime $p \nmid N$ implies $\lambda_\Gamma(\Gamma_0(N)) \geq 1/4$. This may provide a new way to attack Selberg’s eigenvalue conjecture by studying the Ramanujan-Petersson conjecture at just one prime $p \nmid N$. When $N = 1$, Bruggeman [Bru78] proves a vertical Sato-Tate law at $p$ for Maass forms of level 1. As a consequence, almost all (i.e., with density 1) Maass forms of level 1 satisfies the Ramanujan-Petersson conjecture at $p$. Sarnak [Sar87] also shows this from another perspective. A highbrow generalization of this phenomenon can be found in [ST16].

1.1. Statement of Main Results. Let $\mathcal{F}_\infty(N)$ be the set of forms in $\mathcal{F}(N)$ which fail the Selberg’s $1/4$-eigenvalue conjecture. Conventionally elements in $\mathcal{F}_\infty(N)$ are called exceptional Maass forms. Note that $\mathcal{F}_\infty(N)$ is a finite set by the Weyl law. We may assume that $\mathcal{F}_\infty(N) = \{\rho_1, \cdots, \rho_l\}$, with $|t_1| \geq |t_2| \geq \cdots \geq |t_l|$, where $\lambda_j = 1/4 - |t_j|^2$ is the eigenvalue of $\rho_j$.

Let $1 \leq j \leq l$. Denote by $R_p^{(j)} = |t_j| + r_p(\rho_j)$, where $\rho_j \in \mathcal{F}_\infty(N)$. Also, let

$$R_p^\infty := \sup_{\rho \in \mathcal{F}(N) - \mathcal{F}_\infty(N)} r_p(\rho).$$

Then $R_p^\infty$ is well defined and $0 \leq R_p^\infty \leq 7/64$. Our main result is

**Theorem B.** Let notation be as before. Let $p$ be a prime coprime to $N$. Then

$$\max_{1 \leq j \leq l} R_p^{(j)} \leq R_p^\infty.$$  

In particular, the Ramanujan-Petersson conjecture at $p$ for all non-exceptional Maass forms implies the Selberg’s eigenvalue conjecture and the Ramanujan-Petersson conjecture at $p$ for all $\rho \in \mathcal{F}(N)$.

As a corollary, we obtain from (3), in conjunction with (1), a refined lower bound for the least eigenvalue:
Corollary 3. Let notation be as before. Suppose $r_p(\rho) \leq \theta_p$ for all $\rho \in \mathcal{F}(N)$ -- $\mathcal{F}_\infty(N)$. Then $\lambda_1(\Gamma_0(N)) \geq 1/4 - (\theta_p - r_p(\rho))^2$, where $\rho_1$ is a Hecke Maass form in $\mathcal{F}_\infty(N)$ with eigenvalue $\lambda_1(\Gamma_0(N))$. In particular, one has

\begin{equation}
\lambda_1(\Gamma_0(N)) \geq \frac{1}{4} - \left( \frac{7}{64} - \sup_{p \nmid N} r_p(\rho_1) \right)^2.
\end{equation}

Remark: Note that (4) is an improvement of the lower bound $1/4 - (7/64)^2$ in [KS03] when $\rho_1$ does not satisfy the Ramanujan-Petersson conjecture.

Corollary 4. Let notation be as before. Let $\rho_1 \in \mathcal{F}_\infty(N)$ be an exceptional Maass form with eigenvalue $\lambda = 1/4 - |t|^2$ and $|t_1| > 0$. Then

\begin{equation}
\sup_{p \nmid N} \sup_{\rho \in \mathcal{F}(N) - \mathcal{F}_\infty(N)} r_p(\rho) \leq 7/64 - |t_1|.
\end{equation}

Remark: Note that for the exceptional form $\rho_1$, the parameter $t_1 \neq 0$. Hence (5) improves Kim and Sarnak’s bound $7/64$.

1.2. Idea of Proofs. Recall that Selberg’s original approach in [Sel65] was to relate the eigenvalue problem to Kloosterman sums, which can be handled by Weil’s bound. Iwaniec [Iwa89] provides a simple proof of Selberg’s 3/16-bound [Sel65], while still being along the lines of Kloosterman sums, avoids appealing to Weil’s bounds. However, to cross the natural barrier at 3/16 one needs to detect cancellations among Kloosterman sums, and arithmetic geometry offers nothing in this direction. Our approach also relies on Kloosterman sums, but our treatment is different from [Sel65] or [Iwa89].

We consider a family of variants of the Petersson formula parameterized by $n \in \mathbb{Z}_{>0}$ (see Proposition 12). Then we construct certain double Dirichlet series whose Mellin inversion, when integrated over a vertical segment, gives the contribution from Kloosterman sums and Bessel $J$-functions in the geometric side of the Petersson formula. Suppose the inequality (3) fails, then one can shift the contour and get certain residues, which involves both archimedean parameters $t_j$’s and nonarchimedean parameters $\alpha_{p,p}$ and $\beta_{p,p}$’s. Then we do estimate for all these terms to see that the residues have a lower bound of the form $p^{n\delta}$ for some $\delta > 0$ independent of $n$; and other terms can be bounded from above by some terms irrelevant to $n$. Hence, we obtain a contradiction by taking $n \to \infty$. Here is the arrangement of this paper:

In Section 2, we introduce a double Dirichlet series with coefficients being certain Kloosterman sum zeta functions. We study its analytic properties, obtaining its spectral expansion, distribution of poles and computing the residues.

In Section 3, we establish an integral form of the Petersson formula, which is flexible enough so that one can do more analytic manipulations. In conjunction with spectral expansion of the double Dirichlet series, we can detect poles of local $L$-functions at $p$ and $\infty$ by shifting contours.

In Section 4, we prove our main theorems. Further discussions about an extreme case of Theorem B are provided in Section 5.

It is plausible that our main results can be generalized to totally real fields without essential new input.

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2. A Double Dirichlet Series Involving Kloosterman Sum Zeta

In this section, we introduce a double Dirichlet series $Z(s_1, s_2)$ (see (12) below for the definition), which can be viewed as a Dirichlet series with coefficients being certain Kloosterman sum zeta functions. Then we will study its spectral expansions, residues and certain integral transforms. These results will be gathered into a limited Petersson formula developed in Section 4.

2.1. A Double Dirichlet Series. In this subsection, we shall consider a double Dirichlet series $Z(s_1, s_2)$ (see (12) below) with Kloosterman sums being its coefficients. To start with, we briefly recall Fourier expansion of Eisenstein series and classical Kloosterman sum zeta functions.

Let $\Gamma_0(N)$ be the Hecke congruence subgroup of level $N$. Let $a$ be an equivalent class of cusps of $\Gamma_0(N)$. We can explicitly express $a$ as $a = u/v$ with $u \geq 1$, $v | N$, $(u, v) = 1$ and $u \mod (v, N/v)$, for some integers $u$ and $v$.

Let $\Gamma_a := \{ \gamma \in \Gamma_0(N) : \gamma a = a \}$ be the stabilizer of $a$. Let $\sigma_a$ be a scaling matrix, i.e., $\sigma_a \in \Gamma_0(N) \sigma_a = \Gamma_\infty$, where $\Gamma_\infty$ is the stabilizer of the cusp $\infty$. Let $U := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z} \right\}$. Let

$$E_a(z, s) := \sum_{\gamma \in \Gamma_a \backslash \Gamma_0(N)} (\text{Im} \sigma_a^{-1}\gamma z)^s, \quad \text{Re}(s) > 1.$$ 

Then the Eisenstein series $E_a(z, s)$ is well defined, and is absolutely convergent, uniformly on compact sets, for $\text{Re}(s) > 1$. Moreover, it admits a meromorphic continuation to the whole complex plane, with a possible simple pole at $s = 1$. Moreover, $E_a(z, s)$ has a Fourier expansion at the cusp $\infty$:

$$E_a(z, s) = \delta_{a, \infty} y^s + \varphi_{a, \infty} y^{1-s} + y^{1/2} \sum_{n \neq 0} \varphi_{a, \infty} (n, s) K_{s-1/2} (2\pi |n| y) e(nz),$$

where

$$\varphi_{a, \infty} (n, s) = \frac{2\pi^s |n|^{s-1/2}}{\Gamma(s)} \sum_{c} e^{2\pi i c^s} \sum_{(d, m_a) = 1} e \left( \frac{dn}{c} \right) \sum_{(c, v m_a) = 1} \sum_{\chi \mod (v m_a)} \sum_{\delta \mod (v m_a)} \sum_{\gamma \in U \backslash \Gamma_a(N) \sigma_a / U} \chi(\delta) \varphi(\gamma z),$$

Lemma 5. Let notation be as above. Let $a$ be a cusp. Assume $a = u/v$ with $v \geq 1$, $v | N$, $(u, v) = 1$ and $u \mod (v, N/v)$. Let $n$ be a positive integer. Then $\varphi_{a, \infty} (n, s)$ is equal to

$$\frac{2\pi^s |n|^{s-1/2} v^{-2s}}{\Gamma(s) m_a^s \varphi(v m_a)} \sum_{r \mid n} r^{1-2s} \sum_{a \mod (v m_a)} \delta \equiv a_r (\mod \delta) \sum_{\chi \mod (v m_a)} \chi(\delta) \sum_{\delta \mod (v m_a)} \frac{\varphi(\gamma z)}{\chi(\delta)},$$

where $w = N/v$, $m_a = N/(N, v^2)$ is the weight of the cusp $a$, $t$ is a fixed integer coprime to $N$, such that $ut \equiv 1 (\mod v)$; and $\varphi(\cdot)$ is the Euler's totient function. In particular, $\varphi_{a, \infty} (n, s)$ is holomorphic in the right half plane $\text{Re}(s) \geq 1/2$.

Proof. Since $(u, v) = 1$, there are integers $s, t$ such that $ut - sv = 1$ and $(t, N) = 1$. Fix such a pair $(s, t)$. Let

$$\tau_a = \begin{pmatrix} u & s \\ v & t \end{pmatrix} \in \text{SL}(2, \mathbb{Z}), \quad \rho_a = \left( \sqrt{m_a} \quad 1/\sqrt{m_a} \right).$$
Then $\sigma_a = \tau_a \rho_a$. Therefore, we have
\[
\Gamma_0(N)\sigma_a = \left\{ \left( \frac{\alpha \sqrt{m_a}}{\gamma \sqrt{m_a}}, \frac{*}{\delta} \right) : \alpha \delta \equiv 1 \pmod{\gamma}, \ \delta \equiv \gamma t \pmod{w} \right\}.
\]

Note that if $\delta \equiv \gamma t \pmod{w}$ and $(\delta, \gamma v) = 1$, then $(\delta, m_a) = 1$. Otherwise, there exists some $d'$ such that $d' | \delta$ and $d' | m_a$. Note that $m_a | w$ and $(t, N) = 1$, $w | N$. So $d' \nmid t$. Hence, $d' | \gamma$, contradicting to $(\delta, \gamma v) = 1$. Hence,
\[
\sum_{* \equiv d \pmod{c} \atop (c, d) \in U \cap \Gamma_0(N)\sigma_a / U} e\left( \frac{\delta n}{\gamma v m_a} \right) = \sum_{\delta \equiv \gamma t \pmod{w} \atop (\delta, \gamma v m_a) = 1} \sum_{\delta \equiv \gamma t \pmod{w} \atop (\delta, v m_a) = 1} e\left( \frac{\delta n}{\gamma v m_a} \right).
\]

Let $c_\gamma(n)$ be the Ramanujan sum of modulo $\gamma$, namely,
\[
(7) \quad c_\gamma(n) = \sum_{x (\mod \gamma)} e\left( \frac{x n}{\gamma} \right) = \sum_{r(\gamma, n)} \mu\left( \frac{\gamma}{r} \right),
\]
where $\mu$ is the M"obius function. Then by Chinese reminder theorem one obtains the multiplicity
\[
(8) \quad \sum_{\delta \equiv \gamma t \pmod{w} \atop (\delta, v m_a) = 1} e\left( \frac{\delta n}{\gamma v m_a} \right) = c_\gamma(n) \sum_{\delta \equiv \gamma t \pmod{w} \atop (\delta, v m_a) = 1} e\left( \frac{\delta n}{v m_a} \right).
\]

Substituting (7) and (8) into (6) we then have
\[
\varphi_{a, \infty}(n, s) = \frac{2\pi^s |n|^{s-1/2}}{\Gamma(s)} \sum_{\gamma = 1}^{\infty} \frac{1}{\gamma^{2s} \gamma^{2s} m_a} \sum_{r(\gamma, n)} \mu\left( \frac{\gamma}{r} \right) \sum_{\delta \equiv \gamma t \pmod{w} \atop (\delta, v m_a) = 1} e\left( \frac{\delta n}{v m_a} \right).
\]

Note that $w = v m_a \cdot (N, v^2) / v^2$. So $m_a | w$ and $w | v m_a$. Then
\[
\varphi_{a, \infty}(n, s) = \frac{2\pi^s |n|^{s-1/2}}{\Gamma(s) v^{2s} m_a^s} \sum_{r | n} r^{1-2s} \sum_{a (\mod v m_a) \delta \equiv \gamma t \pmod{w} \atop (a, v m_a) = 1} \sum_{\delta \equiv \gamma t \pmod{w} \atop (\delta, v m_a) = 1} e\left( \frac{\delta n}{v m_a} \right) D(2s; a, v m_a),
\]
where
\[
D(s; a, v m_a) := \sum_{\gamma \equiv a (\mod v m_a)} \mu(\gamma) \frac{1}{\varphi(v m_a)} \sum_{\chi (\mod v m_a)} \frac{\chi(a)}{L(s, \chi)}.
\]

Therefore, $\varphi_{a, \infty}(n, s)$ is equal to
\[
\frac{2\pi^s |n|^{s-1/2}}{\Gamma(s) v^{2s} m_a^s} \sum_{r | n} r^{1-2s} \sum_{a (\mod v m_a) \delta \equiv \gamma t \pmod{w} \atop (a, v m_a) = 1} \sum_{\delta \equiv \gamma t \pmod{w} \atop (\delta, v m_a) = 1} e\left( \frac{\delta n}{v m_a} \right) \chi (\mod v m_a) \sum_{\chi (\mod v m_a)} \frac{\chi(a)}{L(2s, \chi)},
\]
from which one deduces that $\varphi_{a, \infty}(n, s)$ is holomorphic in the region $\text{Re}(s) \geq 1/2$ by noting that $L(s, \chi) \neq 0$ when $\text{Re}(s) = 1$. \qed
Lemma 7. Let notation be as before. Let \( L_{a,p}(s; t) = \sum_{l=0}^{\infty} \tau_a(p^l, t) \cdot p^{-ls} \). Then

\[
\sum_{a} \int_{-\infty}^{\infty} \frac{\Gamma(s_2 - \frac{1}{2} - it) \Gamma(s_2 - \frac{1}{2} + it)}{\cosh \pi t} \cdot \tau_a(1, t) L_{a,p}(s_1; t) dt
\]

converges absolutely when \( \text{Re}(s_2) > 1/2 \) and \( \text{Re}(s_1) > 0 \).

Proof. Let \( a = u/v \) with \( v \geq 1, v \mid N, (u, v) = 1 \) and \( u \mod (v, N/v) \). By Lemma 5, \( \varphi(vm_a) \Gamma(\frac{1}{2} + it) \tau_a(p^l, t) \) is equal to

\[
\frac{2\pi e^{\frac{1}{2} + it} p^{\frac{1}{2} + it}}{v^{1+2it} m_a^{\frac{1}{2} + it}} \sum_{r \mid p} r^{-2it} \sum_{a \equiv \alpha r t \mod (vma)} \sum_{\delta \equiv \alpha \mod (vma), \delta \equiv \alpha \mod (vma)} e^{\frac{\delta p^l}{vm_a}} \chi \left( \frac{a}{(vma)} \right) \frac{\chi(\mod vm_a)}{L(1 + 2it, \chi)}. \]

Moreover, \( L(1 + 2it, \chi) \gg (vm_a(|t| + 1))^{-1/2} \), with the implied constant being absolute (cf. e.g., Theorems 11.4 and 11.11 of [MV07]). Hence, in conjunction with
triangle inequality, we have

\[
\tau_a(p^l, t) \ll \frac{\sqrt{|t| + 1}}{|\Gamma(1/2 + it)|}\sqrt{v_{\mathcal{P}}(vm_{\alpha})} \sum_{\ell | p^l} \sum_{a \equiv ( \text{mod } vm_{\alpha})} \sum_{\delta \equiv \text{art } ( \text{mod } w)} \chi_{\delta}(mod \; vm_{\alpha}) 1.
\]

Consequently, we obtain, for \( l \geq 0 \), that

\[
\tau_a(p^l, t) \ll \frac{v^{3/2} m^2 \sqrt{|t| + 1}}{|\Gamma(1/2 + it)|} , (l + 1),
\]

where the implied constant is absolute. Then it follows from (11) and Stirling formula that (10) is convergent absolutely if \( \text{Re}(s_2) > 1/2 + |t_1| \) and \( \text{Re}(s_1) > 0 \). \( \square \)

Let notation be as before. We then denote (at least formally) by

\[
Z(s_1, s_2) = (1 - p^{-2s_1}) \cdot \sum_{l \geq 0} \frac{Z_1,p(s_2/2)}{p^{ls_1}}, \quad (s_1, s_2) \in \mathbb{C}^2.
\]

By Proposition 6 the function \( Z_1,p(s_2/2) \) is well defined when \( \text{Re}(s_2) > 1 + 2|t_1| \). By Lemma 10 and the bound for Ramanujan-Petersson parameter we then see that \( Z(s_1, s_2) \) is well defined when \( \text{Re}(s_2) > 1 + 2|t_1| \) and \( \text{Re}(s_1) > 7/64 \). Then we have a spectral expansion of \( Z(s_1, s_2) \):

**Lemma 8.** Let notation be as before. Let \( \text{Re}(s_2) > 1 + 2|t_1| \) and \( \text{Re}(s_1) > 7/64 \). Then

\[
Z(s_1, s_2) = (1 - p^{-2s_1}) \sin \frac{\pi s_2}{2} \sum_{j=1}^{\infty} \frac{\Gamma(\frac{s_2}{2} - \frac{1}{2} - it_j) \Gamma(\frac{s_2}{2} - \frac{1}{2} + it_j)}{\cosh \pi t_j} \tau_j(1)L_p(s_1, \rho_j)
\]

\[
+ \frac{4\pi (1 - p^{-2s_1})}{(2\pi^{s_2} p^{s_1 - 1})} \sum_{k \in \mathbb{Z}_{>0}} \frac{(k-1)! \Gamma(\frac{s_2}{2} + \frac{1}{2} - 1)}{(4\pi)^k \Gamma(\frac{k}{2} + 1 - \frac{s_2}{2})} \sum_j a_f(1)L_p(s_1, \phi_{jk})
\]

\[
+ \frac{1 - p^{-2s_1}}{4\pi} \int_{-\infty}^{\infty} \frac{\Gamma(\frac{s_2}{2} - \frac{1}{2} - it)}{\cosh \pi t} \cdot |\tau_a(1,t)L_{a,\rho}(s_1,t)| dt,
\]

where \( L_p(s_1, \rho_j) \) is the local \( L \)-function of the cusp form \( \rho_j \) at \( p \); \( L_p(s_1, \phi_{jk}) \) is the local \( L \)-function of the cusp form \( \phi_{jk} \) at \( p \); and \( L_{a,\rho}(s_1,t) = \sum_{j=0}^{\infty} \tau_a(p^j,t) \cdot p^{-ls_1} \) is the local zeta function of the Eisenstein series \( E_{a}(z, 1/2 + it) \) at \( p \).

**Proof.** By Proposition 6, we have \( Z_1,p(s_2/2) = S_1(s_2;l) + S_2(s_2;l) + S_3(s_2;l) \), where

\[
S_1(s_2;l) = \frac{\sin \frac{\pi s_2}{2}}{2(2\pi^{s_2} p^{s_1 - 1})} \sum_{j=1}^{\infty} \frac{\Gamma(\frac{s_2}{2} + \frac{1}{2} - it_j) \Gamma(\frac{s_2}{2} + \frac{1}{2} + it_j)}{\cosh \pi t_j} \tau_j(1)L_p(s_1, \rho_j)
\]

\[
S_2(s_2;l) = \frac{4\pi p^{l/2}}{(2\pi^{s_2} p^{s_1 - 1})} \sum_{k \in \mathbb{Z}_{>0}} \frac{(k-1)! \Gamma(\frac{s_2}{2} + \frac{1}{2} - 1)}{(4\pi)^k \Gamma(\frac{k}{2} + 1 - \frac{s_2}{2})} \sum_{f \in S_k(N)} a_f(1)L_p(s_1, \phi_{jk})
\]

\[
S_3(s_2;l) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\Gamma(\frac{s_2}{2} + \frac{1}{2} - it) \Gamma(\frac{s_2}{2} + \frac{1}{2} + it)}{\cosh \pi t} \cdot |\tau_a(1,t)L_a(p^l,t)| dt.
\]
Let $Z_i(s_1, s_2) = (1 - p^{-2s_1}) \cdot \sum_{i \geq 0} S_i(s_2; l) p^{-is_1}$, $1 \leq i \leq 3$. Then

$$Z_1(s_1, s_2) = \frac{(1 - p^{-2s_1}) \sin \frac{\pi s_2}{s_2}}{2(2\pi p^{1/2})^{s_2-1}} \sum_{j=1}^{\infty} \frac{\Gamma\left(\frac{\pi}{2} - \frac{1}{2} - it_j\right) \Gamma\left(\frac{\pi}{2} + \frac{1}{2} + it_j\right)}{\cosh \pi t_j} a_{p, s}(1) \sum_{l=0}^{\infty} a_{p, l}(p^j)\,$$

$$= \frac{(1 - p^{-2s_1}) \sin \frac{\pi s_2}{s_2}}{2(2\pi p^{1/2})^{s_2-1}} \sum_{j=1}^{\infty} \frac{\Gamma\left(\frac{\pi}{2} - \frac{1}{2} - it_j\right) \Gamma\left(\frac{\pi}{2} + \frac{1}{2} + it_j\right)}{\cosh \pi t_j} \eta_j(1) L_p(s_1, \rho);$$

$$Z_2(s_1, s_2) = \frac{4\pi (1 - p^{-2s_1})}{(2\pi)^2 p^{1/2}} \sum_{k \in \mathbb{Z}_{>0}} \frac{(k - 1)! \Gamma\left(\frac{\alpha}{2} + \frac{1}{2} - 1\right)}{(4\pi i)^k \Gamma\left(\frac{\alpha}{2} + 1 - \frac{\gamma}{2}\right)} \sum_{j} \frac{\eta_j(1)}{p^j} \phi_{jk}(p^j);$$

$$Z_3(s_1, s_2) = \frac{1 - p^{-2s_1}}{4\pi} \sum_{g} \int_{-\infty}^{\infty} \frac{\Gamma\left(\frac{s-1}{2} - it\right) \Gamma\left(\frac{s-1}{2} + it\right)}{\cosh \pi t} \eta_{g}(1, t) \sum_{l=0}^{\infty} \tau_{a, l}(p^j, t) \frac{dt}{p^{is_1}}$$

$$= \frac{1 - p^{-2s_1}}{4\pi} \sum_{g} \int_{-\infty}^{\infty} \frac{\Gamma\left(\frac{s-1}{2} - it\right) \Gamma\left(\frac{s-1}{2} + it\right)}{\cosh \pi t} \eta_{g}(1, t) L_{a, p}(s_1, t);$$

Then Lemma 8 follows. □

Denote by

$$R_p = \max \left\{ \max_{1 \leq i \leq 3} r_p(\rho_j), R_p^\infty \right\} = \sup_{\rho \in \mathcal{F}(\mathbb{N})} r_p(\rho).$$

As a direct consequence of Lemma 8, we have the following:

**Corollary 9.** Let notation be as before. Then $Z(s_1, s_2)$ defines a holomorphic function when $\text{Re}(s_1) > R_p$ and $\text{Re}(s_2) > 1 + |t_1|$. Now we consider $Z(s + \lambda, 1 - 2\lambda)$, as a function of $s$ and $\lambda$. Then by Corollary 9, $Z(s + \lambda, 1 - 2\lambda)$ is holomorphic when $\text{Re}(\lambda) < -|t_1|$ and $\text{Re}(s) + \text{Re}(\lambda) > R_p$.

**Lemma 10.** Let notation be as before. Let $s$ be such that $\text{Re}(s) > 1 + |t_1|$. Then

$$\text{Res}_{\lambda = -|t_j|} Z(s + \lambda, 1 - 2\lambda) = -\frac{(1 - p^{-2(s - |t_j|)/2}) \Gamma(2|t_j|)}{4(2\pi)^2 |t_j|} \sum_{\rho \in \mathcal{F}(\mathbb{N})} a_{p, 1} L_p(s - |t_j|, \rho),$$

where $\lambda_\rho$ is the eigenvalue of $\rho$.

**Remark.** Note that when $\rho$ is an old form, then $a_{p, 1} = 0$. However, there exists at least one new form $\rho \in \mathcal{F}(\mathbb{N})$ with $\lambda_\rho = 1/4 - |t_j|^2$. Hence, the summands in Lemma 10 are not vanishing simultaneously.

### 2.2. An Integral Transform of $Z(s_1, s_2)$

Let $\kappa = (k - 1)/2$. Denote by

$$\hat{Z}(s) = \frac{1}{2\pi i} \int_{(-2)} \frac{\left(2\pi\right)^{-2\lambda} \Gamma(\kappa + \lambda)}{\Gamma(1 + \kappa - \lambda)} \cdot Z(s + \lambda, 1 - 2\lambda) d\lambda.$$

Let $n \geq 0$ be an integer. Set

$$\varphi_n(s) = \varphi_{\rho, n}(s) = \sum_{0 \leq k \leq (n-1)/2} \left[ p^{(n-2k)s} + p^{(n-2k)s} \right] + \frac{1 + (-1)^n}{2},$$

if $n \geq 1$; and set $\varphi_n(s) \equiv 1$ if $n = 0$. 
Lemma 11. Let notation be as before. Let $n \geq 0$. Let $A \geq 2$. Then

$$\sum_{c \geq 1 \atop N \mid c} \frac{S(1, p^{n}; c)}{c} \cdot J_{k-1} \left( \frac{4\pi p^{n/2}}{c} \right) = \frac{\log p}{2\pi i} \int_{A - \frac{\pi}{i \log p}}^{A + \frac{\pi}{i \log p}} \varphi_n(s) \hat{Z}(s) ds,$$

where $\hat{Z}(s)$ is defined in (13).

Proof. Let $c \geq 1$ be an integer. Set

$$D(s; c) = (1 - p^{-2s}) \cdot \sum_{l \geq 0} \frac{S(1, p^{l}; c)}{p^{ls}}, \quad \text{Re}(s) > 0.$$

Then by Weil’s bound $S(1, p^{l}; c) \leq \tau(c) \sqrt{c}$, we conclude that $D(s; c)$ is well defined for all fixed $c$. Let $\lambda \in \mathbb{C}$. Then by unfolding and the following orthogonality

$$\frac{\log p}{2\pi i} \int_{A - \frac{\pi}{i \log p}}^{A + \frac{\pi}{i \log p}} p^{1s} p^{-ls} ds = \delta_{l_1, l_2},$$

where $l_1$ and $l_2$ are arbitrary integers, we then conclude that

$$\frac{\log p}{2\pi i} \int_{A - \frac{\pi}{i \log p}}^{A + \frac{\pi}{i \log p}} \varphi_n(s) \cdot D(s + \lambda; c) ds = \frac{S(p^n, 1; c)}{p^{n\lambda}}, \quad \text{Re}(s) > - \text{Re}(\lambda).$$

To handle the Bessel function, we apply a Mellin–Barnes type integral expansion for $J$-function (e.g., see [Bat53], vol. II, p.21):

$$J_{k-1}(x) = \frac{1}{2\pi i} \int_{\sigma - \infty}^{\sigma + \infty} \frac{\Gamma(k + \lambda)}{\Gamma(1 + \kappa - \lambda)} \cdot \left( \frac{x}{2} \right)^{-2\lambda} d\lambda,$$

where $\kappa = (k - 1)/2$, and $x > 0$; and $-\kappa < \sigma < 1/2$. Moreover, we have

$$Z(s_1, s_2) = (1 - p^{-2s_1}) \cdot \sum_{c \geq 1 \atop N \mid c} \frac{S(1, p^{n}; c)}{p^{s_1c^{s_2}}} = \sum_{c \geq 1 \atop N \mid c} \frac{D(s_1; c)}{c^{s_2}}.$$

Then (15) follows from (17), (18) and (19). \hfill \Box

3. A Variant of Petersson Formula

Let $S_k(N)$ be an orthogonal basis of the space of holomorphic cusp forms of weight $k$ level $N$. Then the Petersson formula, which is a kind of orthogonality relation between coefficients of cusp forms in $S_k(N)$:

$$\sum_{f \in S_k(N)} \frac{a_f(m_1)a_f(m_2)}{c_{m_1, m_2}} \cdot (f, f) = \delta_{m_1, m_2} + \frac{2\pi}{k} \sum_{c \geq 1 \atop N \mid c} \frac{S(m_1, m_2; c)}{c} J_{k-1} \left( \frac{4\pi \sqrt{m_1m_2}}{c} \right),$$

where $c_{m_1, m_2} = (4\pi \sqrt{m_1m_2})^{k-1} \cdot \nu_N/(k - 2)!$ and $a_f(m_j)$ is the $m_j$-th Fourier coefficient of $f$, $1 \leq j \leq 2$.

We then move forward to prove a variant of Petersson formula:

Proposition 12. Let $p \nmid N$ be a prime. Let $(p^{\rho}, p^{-\theta})$ be the Langlands parameter of $f$ at $p$. Let $A \geq 2$ and $B > 0$. Then

$$\frac{(k - 2)!}{(4\pi \nu_N)^{k-1}} \sum_{f \in S_k(N)} \frac{1}{(f, f)} \int_{(\nu_N - B)}^{(\nu_N + B)} \left[ \phi_n(s + \theta_f) + \phi_n(s - \theta_f) \right] \cdot \zeta_p(s + B) ds$$

$$= \int_{-\frac{\pi}{i \log p}}^{\frac{\pi}{i \log p}} \phi_n(s) ds + \frac{2\pi}{ik} \int_{A - \frac{\pi}{i \log p}}^{A + \frac{\pi}{i \log p}} \phi_n(s) \hat{Z}(s + B) ds,$$

where $\phi_n(s) = p^{ns} + p^{-ns}$, $n \geq 0$; and $\zeta_p(s) = (1 - p^{-s})^{-1}$. 

Proof. By Hecke relation we have $\varphi_n(\theta_f) = a_f(p^n)p^{-nu}$, where $\varphi_n$ is defined in (14). Substituting this identity into Petersson formula (20) we then obtain

\[(21) \quad \frac{(k-2)!}{(4\pi)^{k-1}N} \sum_{f \in S_k(N)} a_f(1)\varphi_n(\theta_f) = \delta_n,0 + \frac{2\pi}{k} \sum_{c \geq 1, \delta \geq 1} S(1, p^n; c) J_{k-1} \left( \frac{4\pi p^{n/2}}{c} \right).\]

Now substituting (15) into the trace formula (21), one thus deduces

\[(22) \quad \frac{(k-2)!}{(4\pi)^{k-1}N} \sum_{f \in S_k(N)} a_f(1)\varphi_n(\theta_f) = \delta_n,0 + \log p \int_{\mathfrak{A}} \varphi_n(s) \tilde{Z}(s) ds.\]

On the other hand, we have, by unfolding and (16), that

\[(23) \quad \left\{ \begin{array}{l}
\varphi_n(\theta_f) = \log p \int_{\mathfrak{A} - \frac{4\pi}{2\pi}} \varphi_n(s + \theta_f) + \varphi_n(s - \theta_f) \cdot \zeta_p(ds), \\
\delta_n,0 = \log p \int_{\mathfrak{A} - \frac{4\pi}{2\pi}} \varphi_n(s) ds,
\end{array} \right.
\]

where $A_1$ and $A_2$ are arbitrary positive numbers. Plugging (23) into (22) gives

\[\int_{\mathfrak{A} - \frac{4\pi}{2\pi}} \varphi_n(s) ds + \frac{2\pi}{k} \int_{\mathfrak{A} - \frac{4\pi}{2\pi}} \varphi_n(s) \tilde{Z}(s) ds.\]

Let $\tilde{\varphi}_n(s) = p^{n_\pi} + p^{n\tau}$. Then one can write $\phi_n(s)$ as a linear combination of $\tilde{\varphi}_n(s)$, $0 \leq m \leq n$. We note, in the first equation of (23), that $\tilde{\theta}_f = -\theta_f$ by the work of Deligne [Del72]. Then we may replace $\varphi_n(s)$ with $\tilde{\varphi}_n(s)$ in the above formula, obtaining the following identity:

\[\int_{\mathfrak{A} - \frac{4\pi}{2\pi}} \tilde{\varphi}_n(s) ds + \frac{2\pi}{k} \int_{\mathfrak{A} - \frac{4\pi}{2\pi}} \tilde{\varphi}_n(s) \tilde{Z}(s) ds.\]

We then take $A_1 = A_2 = A_3 = B \geq 0$ to conclude that

\[\int_{\mathfrak{A} - \frac{4\pi}{2\pi}} \tilde{\varphi}_n(s) ds + \frac{2\pi}{k} \int_{\mathfrak{A} - \frac{4\pi}{2\pi}} \tilde{\varphi}_n(s) \tilde{Z}(s) ds.\]

Then Proposition 12 follows from the fact

\[\int_{\mathfrak{A} - \frac{4\pi}{2\pi}} \tilde{\varphi}_n(s) \tilde{Z}(s + B) ds = \int_{\mathfrak{A} - \frac{4\pi}{2\pi}} \phi_n(s) \tilde{Z}(s + B) ds,
\]

assuming $A \geq 0$ and $B > 0$. \hfill \Box

4. Proof of the Main Theorems

Proof of Theorem B. Let $\varphi$ be a locally integrable function on $\mathbb{C}$. Consider the following distribution:

\[I(\varphi; x) = \int_{2 - \frac{4\pi}{2\pi}} \varphi(s) \tilde{Z}(s + x) ds, \quad x > 0.\]
Recall that
\begin{equation}
\tilde{Z}(s) = \frac{1}{2\pi i} \int_{(-2)} (2\pi)^{-2\lambda} \cdot \Gamma(\kappa + \lambda) \Gamma(1 + \kappa - \lambda) \cdot Z(s + \lambda, 1 - 2\lambda) d\lambda.
\end{equation}

Then by Proposition 6 we have $Z(s + \lambda, 1 - 2\lambda) = Z_1(s + \lambda, 1 - 2\lambda) + Z_2(s + \lambda, 1 - 2\lambda) + Z_3(s + \lambda, 1 - 2\lambda)$, where

\begin{align*}
Z_1(s + \lambda, 1 - 2\lambda) &= \frac{(1 - p^{-2(s+\lambda)}) \cos \lambda \pi}{2(2\pi)^{-2\lambda} \cosh \pi t} \sum_{j=1}^{\infty} \frac{\Gamma(-\lambda - it_j) \Gamma(-\lambda + it_j) \tilde{\eta}_j(1) L_p(s + \lambda, \rho_j)}{\cosh \pi t_j}; \\
Z_2(s + \lambda, 1 - 2\lambda) &= \frac{4\pi(1 - p^{-2(s+\lambda)})}{(2\pi)^{1-2\lambda} \pi^{2\lambda}} \sum_{k \in \mathbb{Z}_{>0}} \frac{(k-1)! \Gamma(k+1 - \lambda)}{(4\pi i)^k \Gamma(k+1 + \lambda)} \sum_{j \in \mathbb{Z}_{>0}} \tilde{\eta}_j(1) L_p(s + \lambda, \phi_{jk}); \\
Z_3(s + \lambda, 1 - 2\lambda) &= \frac{1}{4\pi} \sum_{a} \int_{-\infty}^{\infty} \Gamma(-\lambda - it) \Gamma(-\lambda + it) \cdot \tau_a(1) L_{a,p}(s + \lambda; t) dt.
\end{align*}

Let $R_p^{(j_1)} \geq R_p^{(j_2)} \cdots \geq R_p^{(j_5)}$ be a rearrangement of $R_p^{(j)}$. Suppose $R_p^{(j_1)} > R_p^{\infty}$. Let $1 \leq m \leq l$ be the largest integer such that $R_p^{(j_m)} > R_p^{\infty}$. Let

$$r = \min \left\{ \left\lfloor \frac{\log 2}{2} \frac{R_p^{(j_m)} - R_p^{\infty}}{2} \right\rfloor > 0. \right.$$

Note that the integral of $\varphi(s) \tilde{Z}(s + x)$ along paths $[\frac{\pi i}{\log p}, 2 - \frac{\pi i}{\log p}]$ and $[2 + \frac{\pi i}{\log p}, \frac{\pi i}{\log p}]$ have opposite signs and same absolute value. So they cancel with each other. Then one can shift the contour to obtain

$$\tilde{Z}(s) = \frac{1}{2\pi i} \int_{(-r)} (2\pi)^{-2\lambda} \cdot \Gamma(\kappa + \lambda) \cdot Z(s + \lambda, 1 - 2\lambda) d\lambda + R(s),$$

where

$$R(s) = \sum_{j=1}^{l} \frac{(2\pi)^{2|t_j|} \cdot \Gamma(\kappa - |t_j|)}{\Gamma(1 + \kappa + |t_j|)} \cdot \text{Res}_{\lambda=-|t_j|} Z_1(s + \lambda, 1 - 2\lambda).$$

Let $x = (R_p^{(j_m)} + R_p^{\infty} + r)/2$. Then $r + R_p^{\infty} < x < R_p^{(j_m)}$. We can shift the contour from $\text{Re}(s) = 2$ to $\text{Re}(s) = 0$, obtaining

$$I_1(\phi_n; x) = \frac{1}{2\pi i} \int_{\frac{\pi i}{\log p}}^{\frac{\pi i}{\log p}} \phi_n(s) \int_{(-r)} (2\pi)^{-2\lambda} \cdot \Gamma(\kappa + \lambda) \cdot Z(s + \lambda, 1 - 2\lambda) d\lambda ds,$$

$$I_2(\phi_n; x) = \int_{\frac{\pi i}{\log p}}^{\frac{\pi i}{\log p}} \phi_n(s) R(s + x) ds + \sum_{m=1}^{m} \phi_n(R_p^{m} - x) \cdot \text{Res}_{s=\alpha_p^{m}} R(s + x),$$

where $\alpha_p^{m}$ is the pole of $L_p(s, \rho_j)$ such that $\text{Re}(\alpha_p^{m}) = R_p^{m}$.

Therefore, from Proposition 12 we conclude that

$$J(\phi_n; x) = \int_{\frac{\pi i}{\log p}}^{\frac{\pi i}{\log p}} \phi_n(s) ds + I_1(\phi_n; x) + I_2(\phi_n; x),$$

where the function $J(\phi_n; x)$ is defined by

$$J(\phi_n; x) = \int_{\frac{\pi i}{\log p}}^{\frac{\pi i}{\log p}} \phi_n(s + \theta_f) + \phi_n(s - \theta_f) \cdot \zeta_p(s + x) ds.$$

Since $f$ is holomorphic, $\theta_f \in i\mathbb{R}$. We then have

$$J(\phi_n; x) \ll \frac{(k - 2)!}{(4\pi)^{k-1}} \sum_{f \in \mathcal{A}_{k}(N)} \frac{1}{|f|} \cdot \frac{1}{1 - p^{-s}}.$$
where the right hand side is independent of \( n \). Also, we have

\[
(28) \quad I_1(\phi_n; x) \ll \sup_{|t| \leq \pi / \log p} \left| \int_{(-r)} \frac{(2\pi)^{-2\lambda} \cdot \Gamma(\kappa + \lambda)}{\Gamma(1 + \kappa - \lambda)} \cdot Z(it + \lambda + x, 1 - 2\lambda) d\lambda \right|
\]

On the other hand, applying Lemma 10 (and the Remark thereafter) and the fact \(|t_j| \leq 7/64\) we then have, for all \( 1 \leq k \leq m \), that

\[
(29) \quad \frac{(2\pi)^{2|t_j|} \cdot \Gamma(\kappa - |t_j|)}{\Gamma(1 + \kappa + |t_j|)} \text{Res}_{s = \alpha_j^0 - x} \text{Res}_{\lambda} Z_1(s + \lambda, 1 - 2\lambda) \leq 0,
\]

where the equality holds if and only if for all \( \rho \in F(N) \) such that \( \lambda_\rho = 1/4 - |t_j|^2 \), \( \rho(1) = 0 \). Note that, for each \(|t_j|\), there exists a Hecke-Maass new form \( \rho \in F_\infty(N) \) with eigenvalue \( \lambda_\rho = 1/4 - |t_j|^2 \). So (29) is a strict inequality. Therefore, we have

\[
\text{Res}_{s = R_\rho^1 - x} R(s + x) = \sum_{j=1}^l \frac{(2\pi)^{2|t_j|} \cdot \Gamma(\kappa - |t_j|)}{\Gamma(1 + \kappa + |t_j|)} \text{Res}_{s = \alpha_j^0 - x} \text{Res}_{\lambda} Z_1(s + \lambda, 1 - 2\lambda) < 0,
\]

for all \( 1 \leq k \leq m \). As a consequence, we find a lower bound for \( I_2(\phi_n; x) \):

\[
(30) \quad - I_2(\phi_n; x) \geq -p^{n(R_\rho^1 - x)} \text{Res}_{s = \alpha_j^0 - x} R(s + x) - \sup_{|t| \leq \pi / \log p} |R(it + x)|,
\]

where the implied constant is independent of \( n \).

Substituting (27), (28) and (30) into (26) one then gets

\[
p^{n(R_\rho^1 - x)} \text{Res}_{s = R_\rho^1 - x} R(s + x) \ll 1,
\]

where the implied constant depends only on \( N, x \); in particular, it is independent of \( n \). Then we encounter a contradiction by taking \( n \) to be large enough. Thus the inequality (3) follows. \( \square \)

**Proof of Theorem A.** We may assume \( \pi \) corresponding to a Maass form \( \rho \) of level \( N \). Without loss of generality, we may also assume \( \rho \) has trivial nebentypus. Then apply (17) we have \( R_\rho^\infty \leq 7/64 \). Hence by Theorem B we have

\[
|t_j| + r_\rho(p_j) \leq \max_{1 \leq j \leq l} R_\rho^{(j)} \leq R_\rho^\infty \leq 7/64,
\]

for \( 1 \leq j \leq l \). Then (2) holds. \( \square \)

**5. Further Discussions**

In this section we provide further discussion on Theorem B. According to the Ramanujan conjecture, both side of (3) should be vanishing. Hence it is reasonable to expect the equality to hold in (3), i.e., \( \max_{1 \leq j \leq l} R_\rho^{(j)} = R_\rho^\infty \). If this is true, then the exploring of the Ramanujan conjecture boils down to studying exceptional Maass forms.

Suppose the inequality (15) is strict, namely, \( \max_{1 \leq j \leq l} R_\rho^{(j)} < R_\rho^\infty \). Then we can take certain suitable parameter \( x \) to detect Maass forms \( \rho \) such that \( r_\rho(\rho) \). Let \( x \in (2 + \max_{1 \leq j \leq l} R_\rho^{(j)}, 2 + R_\rho^\infty) \) be such that \( x \neq 2 + r_\rho(\rho) \) for \( \rho \in F(N) \). Let

\[
E_\rho(s; \rho) := \text{Res}_{s = \alpha_\rho(s)} \frac{\phi_\rho(s) Z(s + x + \lambda, 1 - 2\lambda)}{(2\pi)^{2\lambda}},
\]

with \( \alpha_\rho(p_j) \) being a pole of the local \( L \)-factor \( L_\rho(s, \rho_j) \) and \( \text{Re} \left( \log_p |\alpha_\rho(p_j)| \right) = r_\rho(\rho) \).
Substituting (25) into (24) and inverting the sums, we then shift the contour and appeal to Cauchy theory to obtain

\[(31) \quad I(\phi_n; x) = \int_{-\frac{\pi i}{\cos \pi t_j}}^{\frac{\pi i}{\cos \pi t_j}} \phi_n(s) \tilde{Z}(s + x) ds + \mathcal{E}_n(x),\]

where

\[\mathcal{E}_n(x) = \sum_{\rho_j \in \mathcal{F}(N) - \mathcal{F}_\infty(N)} \int_{\mathbb{C}} \frac{\Gamma(\kappa + \lambda)}{\Gamma(1 + \kappa + \lambda)} \cdot E_n(\rho_j; \lambda) d\lambda,\]

By a straightforward computation using Proposition 6 we see \(E_n(\rho_j; \lambda)\) becomes

\[\phi_n(\alpha_p(\rho_j) - x - \lambda)(1 - p^{-2\alpha_p(\rho_j)}) \overline{\mathcal{P}_j(1) \Gamma(-\lambda - it_j) \Gamma(-\lambda + it_j) \cos \pi t_j}.\]

Hence \(\mathcal{E}_n(x)\) converges absolutely due to the exponential growth of \(\cosh \pi t_j\), observing \(t_j > 0\) since \(\rho_j \in \mathcal{F}(N) - \mathcal{F}_\infty(N)\). Denote by

\[I_n^\pm(\rho_j) := \int_{\mathbb{C}} \frac{\Gamma(\kappa + \lambda) \Gamma(\lambda - it_j) \Gamma(\lambda + it_j)}{\Gamma(1 + \kappa + \lambda)} \cdot p^{\pm \lambda n} \cos \pi t_j d\lambda.\]

It is easy to see that \(I_n(\rho_j)\) converges absolutely by Stirling formula and \(2 \cos \lambda \pi = e^{i\lambda \pi} + e^{-i\lambda \pi}\). Note that \(\cos \lambda \pi = e^{-1a\pi - b\pi} + e^{ia\pi + b\pi} = e^{\lambda \pi} + e^{-\lambda \pi} = \cos \lambda \pi\). So \(I_n(\rho_j) \in \mathbb{R}\). Moreover, from the above discussion, we obtain

**Lemma 13.** Let notation be as before. Then

\[\mathcal{E}_n(x) = \frac{1}{2 \log p} \sum_{\rho_j \in \mathcal{F}(N) - \mathcal{F}_\infty(N)} I_n^\pm(\rho_j) \overline{\mathcal{P}_j(1)} p^{n(\alpha_p(\rho_j) - x)}(1 - p^{-2\alpha_p(\rho_j)}),\]

\[+ \frac{1}{2 \log p} \sum_{\rho_j \in \mathcal{F}(N) - \mathcal{F}_\infty(N)} I_n^\pm(\rho_j) \overline{\mathcal{P}_j(1)} p^{-n(\alpha_p(\rho_j) - x)}(1 - p^{-2\alpha_p(\rho_j)}).\]

Let \(c_k = (-1)^{k/2} \in \{1, -1\}\). Then

\[\cos \lambda \pi = \sin(1/2 - \lambda)\pi = \frac{\pi}{\Gamma(1/2 + \lambda) \Gamma(1/2 - \lambda)} = \frac{c_k \pi}{\Gamma(1 - \kappa + \lambda) \Gamma(1 - \kappa - \lambda)}.\]

Hence we can rewrite \(I_n^\pm(\rho_j)\) as

\[I_n^\pm(\rho_j) = \frac{c_k}{2i} \int_{\mathbb{C}} \frac{\Gamma(\lambda - it_j) \Gamma(\lambda + it_j)}{\Gamma(\lambda + 1 + \kappa) \Gamma(\lambda + 1 - \kappa)} \cdot p^{\pm \lambda n} d\lambda.\]

Then by Mellin inversion we obtain

\[I_n^\pm(\rho_j) = \frac{c_k}{\pi} \cdot G^{0, 2}_{2, 2} \left(1 + it_j, 1 - it_j \mid p^n \right),\]

where \(G^{0, 2}_{2, 2} \left(1 + it_j, 1 - it_j \mid p^n \right)\) is the Meijer G-function. Then by Slater [Sla66] the residue \(I_n^\pm(\rho_j)\) is related to Gauss's hypergeometric functions.

Applying to the trivial bound \(I_n(\rho_j) \ll p^{-2n}\) we then obtain from (31) and Lemma 13 that

**Proposition 14.** Let notation be as before. Suppose \(\max_{1 \leq j \leq l} R_p^{(j)} < R_p^\infty\). Then

\[(32) \sum_{\rho_j \in \mathcal{F}(N) - \mathcal{F}_\infty(N)} G^{0, 2}_{2, 2} \left(1 + it_j, 1 - it_j \mid p^n \right) \overline{\mathcal{P}_j(1)} p^{n(\alpha_p(\rho_j) - x)}(1 - p^{-2\alpha_p(\rho_j)}) \ll 1,\]

where the implied constant is independent of \(n\).
Remark. Estimate (32) gives a relation between Satake parameters of non-exceptional Maass forms at $\infty$ and a finite place $p$. Note that the Ramanujan conjecture predicts that $\alpha_p(\rho_j) = 0$.

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