Charm mass corrections to the bottomonium mass spectrum

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The one-loop corrections to the bottomonium mass spectrum due to the finite charm mass are evaluated in the framework of the relativistic quark model. The obtained corrections are compared with the results of perturbative QCD.

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Recently effects of the finite mass of the charm quark in determining the bottomonium mass spectrum and the $b$ quark mass were considered within perturbative QCD [1, 2, 3, 4]. Here we study these effects in the framework of the relativistic quark model taking into account the confining quark interaction. Since the $b\bar{b}$ bound system is rather far from being a Coulombic one (even in the ground state) we expect the size of these effects to differ noticeably from that predicted within perturbative QCD, where they were found to be substantial [4]. This report may be considered as completing our previous paper [5], where the mass spectra of heavy quarkonia were calculated with the account of the one-loop radiative corrections. Our model was also successfully applied for describing the heavy-light meson mass spectra and radiative and weak decays of heavy mesons [6, 7].

In papers [6, 7] a meson is described by the wave function of the bound quark-antiquark state, which satisfies the quasipotential equation of the Schrödinger type in the center-of-
mass frame:

\[
\left(\frac{b^2(M)}{2\mu_R} - \frac{P^2}{2\mu_R}\right) \Psi_M(p) = \int \frac{d^3q}{(2\pi)^3} V(p, q; M) \Psi_M(q),
\]

(1)

where the relativistic reduced mass is

\[
\mu = \frac{M^4 - (m_1^2 - m_2^2)^2}{4M^2},
\]

(2)

and \(b^2(M)\) denotes the on-mass-shell relative momentum squared

\[
b^2(M) = \frac{[M^2 - (m_1 + m_2)^2][M^2 - (m_1 - m_2)^2]}{4M^2}.
\]

(3)

Here \(m_1, m_2\) and \(M\) are quark masses and a meson mass, respectively. For the \(b\bar{b}\) bound system (bottomonium) \(m_1 = m_2 = m_b\) and Eqs. (2), (3) take the form

\[
\mu = \frac{M}{4}, \quad b^2(M) = \frac{M^2}{4} - m_b^2.
\]

(4)

The kernel \(V(p, q; M)\) in Eq. (1) is the quasipotential operator of the quark-antiquark interaction. It is constructed with the help of the off-mass-shell scattering amplitude, projected onto the positive energy states. An important role in this construction is played by the Lorentz-structure of the confining quark-antiquark interaction in the meson. In constructing the quasipotential of the quark-antiquark interaction we have assumed that the effective interaction is the sum of the usual one-gluon exchange term and the mixture of vector and scalar linear confining potentials. The quasipotential is then defined by

\[
V(p, q; M) = \bar{u}_1(p)\bar{u}_2(-p)V(p, q; M)u_1(q)u_2(-q),
\]

(5)

with

\[
V(p, q; M) = \frac{4}{3} \alpha_s D_{\mu\nu}(k)\gamma_{1\mu}\gamma_{2\nu} + V_{\text{conf}}^V(k)\Gamma_{1\mu}\Gamma_{2\nu} + V_{\text{conf}}^S(k),
\]

(6)

where \(\alpha_s\) is the QCD coupling constant, \(D_{\mu\nu}\) is the gluon propagator in the Coulomb gauge and \(k = p - q\); \(\gamma_\mu\) and \(u(p)\) are the Dirac matrices and spinors. The effective long-range vector vertex is given by

\[
\Gamma_{\mu}(k) = \gamma_{\mu} + \frac{ik}{2m_b}\sigma_{\mu\nu}k^\nu, \quad k^\nu = (0, k),
\]

(7)

where \(\kappa\) is the Pauli interaction constant characterizing the nonperturbative anomalous chromomagnetic moment of quarks. Vector and scalar confining potentials in the nonrelativistic limit reduce to

\[
V_{\text{conf}}^V(r) = (1 - \varepsilon)Ar + B, \quad V_{\text{conf}}^S(r) = \varepsilon Ar,
\]

(8)
reproducing
\[ V_{\text{conf}}(r) = V_{\text{conf}}^S(r) + V_{\text{conf}}^V(r) = Ar + B, \]  
(9)

where \( \varepsilon \) is the mixing coefficient.

The quasipotential for the heavy quarkonia, including retardation, one-loop radiative corrections and expanded in \( p^2/m^2 \), can be found in Ref. \[3\]. All the parameters of our model, such as quark masses, parameters of the linear confining potential, mixing coefficient \( \varepsilon \) and anomalous chromomagnetic quark moment \( \kappa \), were fixed from the analysis of heavy quarkonium spectra \[4\] and radiative decays \[7\]. The quark masses \( m_b = 4.88 \text{ GeV}, m_c = 1.55 \text{ GeV}, m_s = 0.50 \text{ GeV}, m_{u,d} = 0.33 \text{ GeV} \) and the slope of the linear potential \( A = 0.18 \text{ GeV}^2 \) in our model have the fixed values which agree with the usually accepted. The constant term \( B = -0.16 \text{ GeV} \) may be adjusted producing the overall level shift. In Ref. \[8\] we have considered the expansion of the matrix elements of weak heavy quark currents between pseudoscalar and vector meson ground states up to the second order in inverse powers of the heavy quark masses. It has been found that the general structure of the leading, first, and second order \( 1/m_Q \) corrections in our relativistic model is in accord with the predictions of heavy quark effective theory.

The heavy quark symmetry and QCD impose rigid constraints on the parameters of the long-range potential in our model. The analysis of the first order corrections fixes the value of the Pauli interaction constant \( \kappa = -1 \) \[8\]. The same value of \( \kappa \) was found previously from the fine splitting of heavy quarkonia \( ^3P_J \) states \[3\]. The value of the parameter characterizing the mixing of vector and scalar confining potentials, \( \varepsilon = -1 \), was found from the analysis of the \( 1/m_Q^2 \) corrections \[8\] and from considering radiative decays of heavy quarkonia \[7\].

The perturbative heavy quark potential to two loops including the effects of massive loop quarks can be found in Ref. \[3\]. In our model we take into account only one-loop contributions since the uncertainty brought by the confining interaction is larger then the two-loop contributions. The one-loop correction with the finite c quark mass to the Coulomb potential can be included in the form \[3\]:
\[
V(r, m_c) = -C_F \frac{\alpha_V(r, m_c)}{r},
\]
\[
\alpha_V(r, m) = \alpha_s(\mu) \left[ 1 + v_1(r, m, \mu) \frac{\alpha_s(\mu)}{\pi} \right],
\]
\[ v_1(r, m, \mu) = \frac{C_A}{4} \left\{ \frac{31}{9} + \frac{22}{3} \ln(\mu r) + \gamma_E \right\} - \frac{5}{9} T_F + \frac{T_F}{3} \left[ \ln \frac{a_0 m^2}{\mu^2} + 2E_1(\sqrt{a_0 mr}) \right], \quad (10) \]

where
\[ E_1(x) = \int_x^\infty e^{-t} \frac{dt}{t} = -\gamma_E - \ln x - \sum_{n=1}^\infty \frac{(-x)^n}{n \cdot n!}, \]

the Euler constant \( \gamma_E \approx 0.5772 \) and \( C_A = 3, \quad C_F = 4/3, \quad T_F = 1/2 \) in QCD. A simple representation for the vacuum polarization operator with \( a_0 = 5.2 \) was used \[3\]. Subtracting from \( v_1(r, m, \mu) \) its value at \( m = 0 \), namely
\[ v_1(r, 0, \mu) = \frac{31}{36} C_A - \frac{5}{9} T_F + \left( \frac{11}{6} C_A - \frac{2}{3} T_F \right) \ln(\mu r) + \gamma_E, \]

we obtain
\[ \Delta v_1(r, m, \mu) = v_1(r, m, \mu) - v_1(r, 0, \mu) = \frac{2}{3} T_F \left[ \ln(\sqrt{a_0 mr}) + \gamma_E + E_1(\sqrt{a_0 mr}) \right]. \quad (11) \]

Thus the one-loop correction to the static \( Q\bar{Q} \) potential in QCD due to the finite \( c \) quark mass reads as:
\[ \Delta V(r, m_c) = -\frac{4}{3} \frac{\alpha_s^2(\mu)}{\pi r} \Delta v_1(r, m_c, \mu) = -\frac{\alpha_s^2(\mu)}{\pi r} \frac{4}{9} \left[ \ln(\sqrt{a_0 m_c r}) + \gamma_E + E_1(\sqrt{a_0 m_c r}) \right]. \quad (12) \]

The charm mass correction to the bottomonium mass spectrum is then given by
\[ \Delta M = \langle \Delta V_{m_c} \rangle, \quad \alpha_s(m_b) = 0.22, \quad m_c = 1.55 \text{ GeV}. \quad (13) \]

For averaging we use both the wave functions obtained in calculating the heavy quarkonium mass spectra \[5\] and the Coulomb wave functions. The Coulomb averaging is carried out also for \( \alpha_s = 0.3 \) sometimes used in the perturbative QCD description of bottomonium. The numerical values of this spin-independent correction for different \( \bar{b}\bar{b} \) states are presented in the Table \[4\].

The Table \[4\] shows that for a fixed value of \( \alpha_s \) the averages with and without confining potential substantially differ especially for the excited states. For growing \( n = n_r + L + 1 \) the values of \( \langle \Delta V_{m_c} \rangle \) slowly decrease in our model whereas for the Coulomb potential they fall rapidly. The growth of \( \langle \delta E_{\bar{b}\bar{b}} \rangle^{(1)}_{m_c} = \langle \Delta V_{m_c} \rangle_{\text{Coul}} \) in Ref. \[4\] (Table I) is evoked by fast increasing values of \( \alpha_s(\mu) \). For these values of \( \alpha_s(\mu) \) we also reproduce all the corrections \( \langle \delta E_{\bar{b}\bar{b}} \rangle^{(1)}_{m_c} \) obtained in \[4\]. The values of the scale \( \mu \) were fixed in Ref. \[4\] from the stability conditions containing the corrections \( \delta m_b \) to the \( b \) quark mass which are absent in our model.
TABLE I: Charm mass corrections to the masses of bottomonium \( \{n_r + 1\}L(n = n_r + L + 1) \) states (in MeV).

| State | 1S(1) | 1P(2) | 2S(2) | 1D(3) | 2P(3) | 3S(3) |
|-------|-------|-------|-------|-------|-------|-------|
| \( \langle \Delta V_{mc} \rangle \) | -12   | -9.3  | -8.7  | -7.6  | -7.5  | -7.2  |
| \( \langle \Delta V_{mc} \rangle_{\alpha_s=0.22} \alpha_s \) | -9.5  | -4.2  | -3.8  | -2.3  | -2.2  | -2.1  |
| \( \langle \Delta V_{mc} \rangle_{\alpha_s=0.3} \alpha_s \) | -20.7 | -9.7  | -8.8  | -5.5  | -5.2  | -4.9  |
| \( \langle \Delta V_{mc} \rangle_{Coul} \) | -14.3 | -22.1 | -21.9 | -49   | -40.5 |
| \( \alpha_s(\mu) \) | 0.277 | 0.437 | 0.452 | 0.733 | 0.698 |

We estimate the uncertainty in calculating the spin-averaged mass spectrum of the bottomonium to be few MeV. Some of its sources are the uncertainties in the quark masses and confining potential as well as higher order relativistic and radiative corrections. The present knowledge of the spin-independent part of the heavy quark potential is rather uncertain since it strongly depends on the Lorentz structure of the confining potential. The calculated corrections partly may be absorbed in the definition of the constant term in the static potential \([8]\). Nevertheless they may become essential in future when the long-range \(Q\bar{Q} \) interaction will be known better. Then the heavy quark mass should be treated more self-consistently not as a fixed phenomenological parameter but as a scale-dependent one normalized from the heavy quarkonium mass spectrum.

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