Toric Degenerations of Fano Varieties and Constructing Mirror Manifolds

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Abstract

For an arbitrary smooth $n$-dimensional Fano variety $X$ we introduce the notion of a small toric degeneration. Using small toric degenerations of Fano $n$-folds $X$, we propose a general method for constructing mirrors of Calabi-Yau complete intersections in $X$. Our mirror construction is based on a generalized monomial-divisor mirror correspondence which can be used for computing Gromov-Witten invariants of rational curves via specializations of GKZ-hypergeometric series.

1 Introduction

Recent progress in understanding the mirror symmetry phenomenon using explicit mirror constructions for Calabi-Yau hypersurfaces and complete intersections in toric varieties [2, 4, 5, 9] leads to the following natural question:

Is it possible to extend the mirror constructions for Calabi-Yau complete intersections in toric Fano varieties to the case of Calabi-Yau complete intersections in nontoric Fano varieties?

The first progress in this direction has been obtained for Grassmannians [4] and, more generally, for partial flag manifolds [8]. The key idea in both examples is based on a degeneration of Grassmannians (resp. partial flag manifolds) to some singular Gorenstein toric Fano varieties. These degenerations have been introduced and investigated by Sturmfels, Gonciulea and Lakshmibai in [16, 17, 21, 29, 30].

The present paper is aimed to give a short systematic overview of our method for constructing mirror manifolds and to formulate some naturally arising questions and open problems.

In Section 2 we start with a review of a method for constructing degenerations of unirational varieties $X$ to toric varieties $Y$ using canonical subalgebra bases.
This method has been discovered by Kapur & Madlener [19] and independently by Robbiano & Sweedler [28]. Further results on this topic have been obtained in [27, 23, 29] (see also [30] for more details).

In Section 3 we introduce the notion of a small toric degeneration of a Fano manifold and discuss some examples. Finally, in Section 4 we explain our generalized mirror construction which uses small toric degenerations.

2 Canonical subalgebra bases

Let $A$ be a finitely generated subalgebra of the polynomial ring

$$K[u] := K[u_1, \ldots, u_n],$$

i.e., $X = \text{Spec } A$ is an unirational affine algebraic variety together with a dominant morphism $\mathbb{A}^n \to X$. We choose a weight vector $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^n$ and set

$$\text{wt}(u^a) = \text{wt}(u_1^{a_1} \cdots u_n^{a_n}) := \sum_{i=1}^{n} a_i \omega_i.$$

The number $\text{wt}(u^a)$ will be called the weight of the monomial $u^a$. We define a partial order on the set of all monomials in $K[u]$ as follows:

$$u^a \prec u^a' \iff \text{wt}(u^a) \leq \text{wt}(u^a').$$

If $f \in K[u]$ is a polynomial, then $\text{in}_<(f)$ denotes the initial part of $f$, i.e., the sum of those monomials in $f$ whose weight is maximal. By definition, one has $\text{in}_<(fg) = \text{in}_<(f)\text{in}_<(g)$. For sufficiently general choice of the weight vector $\omega \in \mathbb{R}^n$ the initial part of a polynomial $f \in K[u]$ is a single monomial.

**Definition 2.1** The $K$-vector space spanned by initial terms of elements $f \in A$ is called the initial algebra and is denoted by

$$\text{in}_<(A) := \{\text{in}_<(f) : f \in A\}.$$

**Definition 2.2** A subset $\mathcal{F} \subset A$ is called a canonical basis of the subalgebra $A \subset K[u]$, if the initial subalgebra $\text{in}_<(A)$ is generated by the elements

$$\{\text{in}_<(f) : f \in \mathcal{F}\}.$$

Fix a set of polynomials $\mathcal{F} = \{f_1, \ldots, f_m\} \subset A$. We set $K[v] := K[v_1, \ldots, v_m]$. Let $I$ be the kernel of the canonical epimorphism

$$\varphi : K[v] \to A$$

$$v_i \mapsto f_i$$

and $I_<$ the kernel of the canonical epimorphism

$$\varphi_0 : K[v] \to \text{in}_<(A)$$

$$v_i \mapsto \text{in}_<(f_i)$$

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Remark 2.3 It is easy to show that the ideal $I_{\prec}$ is generated by binomials (see [12] for general theory of binomial ideals). Hence, the spectrum of $in_{\prec}(A)$ is an affine toric variety (possibly not normal).

Now we assume that $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{Z}^n$ an integral weight vector. If the set of polynomials $F = \{f_1, \ldots, f_m\} \subset A$ form a canonical basis of the subalgebra $A \subset K[u]$ with respect to the partial order defined by $\omega$, then we can define a 1-parameter family of subalgebras

$$A_t := \{f(t^{-\omega_1}u_1, \ldots, t^{-\omega_n}u_n) \mid f(u_1, \ldots, u_n) \in A\}, \quad t \in K \setminus \{0\}.$$  

Setting $A_0 := in_{\prec}(A)$, we obtain a flat family of subalgebras $A_t \subset K[u]$ such that $A_t \cong A$ for $t \neq 0$ and $A_0 \cong K[v]/I_{\prec}$. This allows us to consider the affine toric variety $Spec A_0$ as a flat degeneration of $Spec A$.

Remark 2.4 It is important to remark that the above method for constructing toric degenerations strongly depends on the choice of the coordinates $u_1, \ldots, u_n$ on $A^n$ and on the choice of a weight vector $\omega$.

Example 2.5 Let $A(r, s) \subset K[X] := K[X_{ij}] (1 \leq i \leq r, 1 \leq j \leq s)$ be the subalgebra of the polynomial algebra $K[X]$ generated by all $r \times r$ minors of a generic $r \times s$ matrix ($r \leq s$), i.e., $A(r, s)$ is the homogeneous coordinate ring of the Plücker embedded Grassmannian $G(r, s) \subset \mathbb{P}^{(s)}$. Define the weights of monomials as follows

$$wt(X_{ij}) := (j - 1)s^{i-1}, \quad i, j \geq 1.$$  

In particular, one has

$$wt(X_{1,i_1} \cdots X_{r,i_r}) = (i_1 - 1) + (i_2 - 1)s + \cdots + (i_r - 1)s^{r-1}$$  

and therefore the initial term of each $(i_1, \ldots, i_r)$-minor ($1 \leq i_1 < \cdots < i_r \leq s$) is exactly the product of terms on the main diagonal:

$$X_{1,i_1} \cdots X_{r,i_r}.$$  

The following result is due to Sturmfels [29, 30]:

Theorem 2.6 The set of all $s \times s$-minors form a canonical base of the subalgebra $A(r, s) \subset K[X]$ with respect to the partial order defined by the above weight vector. In particular, one obtains a natural toric degeneration of the Grassmannian $G(r, s)$.  

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3 Small toric degenerations of Fano varieties

Definition 3.1 Let $X \subset \mathbb{P}^m$ be a smooth Fano variety of dimension $n$. A normal Gorenstein toric Fano variety $Y \subset \mathbb{P}^m$ is called a small toric degeneration of $X$, if there exists a Zariski open neighbourhood $U$ of $0 \in \mathbb{A}^1$ and an irreducible subvariety $\mathcal{X} \subset \mathbb{P}^m \times U$ such that the morphism $\pi : \mathcal{X} \to U$ is flat and the following conditions hold:

(i) the fiber $X_t := \pi^{-1}(t) \subset \mathbb{P}^m$ is smooth for all $t \in U \setminus \{0\}$;
(ii) the special fiber $X_0 := \pi^{-1}(0) \subset \mathbb{P}^m$ has at worst Gorenstein terminal singularities (see [20]) and $X_0$ is isomorphic to $Y \subset \mathbb{P}^m$;
(iii) the canonical homomorphism
$$\text{Pic}(\mathcal{X}/U) \to \text{Pic}(X_t)$$
is an isomorphism for all $t \in U$.

Remark 3.2 It is well-known that if $Y$ has at worst terminal singularities, then the codimension of the singular locus of $Y$ is at least 3. On the other hand, it is easy to show that the only possible toric Gorenstein terminal singularities in dimension 3 are ordinary double points (or nodes): $x_1x_2 - x_3x_4 = 0$. So, if $Y$ is a small toric degeneration of $X$, then the singular locus of $Y$ in codimension 3 must consist of nodes.

Example 3.3 Let $Y := P(r, s) \subset \mathbb{P}^{(r,s) - 1}$ be the toric degeneration of the Grassmannian $X := Gr(r, s) \subset \mathbb{P}^{(r,s) - 1}$ (see Example 2.5). Then $Y$ is a small toric degeneration of $X$ [7].

Example 3.4 Let $X := F(n_1, \ldots, n_k, n) \subset \mathbb{P}^m$ be the partial flag manifold it is Plücker embedding. It is proved in [8] that the toric degenerations introduced and investigated by Gonciulea and Lakshmibai in [16, 17, 21] are small toric degenerations of $X$.

Example 3.5 Let $V_{d,n} \subset \mathbb{P}^{n + 1}$ be a Gorenstein toric Fano hypersurface of degree $d$ $(d \geq 2)$ in projective space of dimension $n \geq 2d - 2$ defined by the homogeneous equation
$$z_1 \cdots z_d = z_{d+1} \cdots z_{2d}.$$ 
It is easy to check that irreducible components of the singular locus of $V_{d,n}$ are
d\frac{d^2(d - 1)^2}{4}
codimension-3 linear subspaces
$$z_i = z_j = z_k = z_l = 0,$$
$$\{i, j\} \subset \{1, \ldots, d\}, \{k, l\} \subset \{d + 1, \ldots, 2d\}, i \neq j, k \neq l.$$
consisting of nodes.
**Theorem 3.6** \( V_{d,n} \subset \mathbb{P}^{n+1} \) is a small toric degeneration of a smooth Fano hypersurface \( X_{d,n} \subset \mathbb{P}^n \) of degree \( d \).

**Proof.** Let us first consider the case \( n = 2d - 2 \). In this case the \( 2(d-1) \)-dimensional fan \( \Sigma_d \) defining the toric variety \( V_{d,2(d-1)} \) can be constructed as follows:

Let \( e_1, \ldots, e_{d-1}, f_1, \ldots, f_{d-1} \) be a \( \mathbb{Z} \)-basis of the lattice \( \mathbb{Z}^{2(d-1)} \). We set \( e_d := -e_1 - \cdots - e_{d-1} \) and \( f_d := -e_1 - \cdots - f_{d-1} \). We denote by \( h_{i,j} \) the sum \( e_i + f_j \) \( (i, j \in \{1, \ldots, n\}) \). If \( \Delta^*_d \) denotes the convex hull of \( d \)-points \( h_{i,j} \), then the fan \( \Sigma_d \subset N_{\mathbb{R}} \) consists of cones over faces of the reflexive polyhedron \( \Delta^*_d \), where the integral lattice \( N \subset \mathbb{Z}^{2(d-1)} \) is generated by all \( d \)-lattice vectors \( h_{i,j} \) (the sublattice \( N \subset \mathbb{Z}^{2(d-1)} \) coincides with \( \mathbb{Z}^{2(d-1)} \) unless \( d = 2 \)).

Using the combinatorial characterisations of terminal toric singularities [20], one immediately obtains that all singularities of \( V_{d,2(d-1)} \) are terminal, since the only \( N \)-lattice points on the faces of \( \Delta^*_d \) are their vertices. If \( d \geq 3 \), then the Picard group of \( V_{d,2(d-1)} \) is generated by the class of the hyperplane section, i.e., \( \text{Pic}(V_{d,2(d-1)}) \cong \mathbb{Z} \) and the anticanonical class of \( V_{d,2(d-1)} \) is \( d \)-th multiple of the generator of \( \text{Pic}(V_{d,2(d-1)}) \). The latter can be show as follows:

Consider a \( (2d-3) \)-dimensional face of \( \Delta^*_d \) having vertices

\[ h_{i,j}, \ i \in \{1, \ldots, d-1\}, \ j \in \{1, \ldots, d\}. \]

Then every \( \Sigma_d \)-piecewise linear function \( \varphi : N_{\mathbb{R}} \rightarrow \mathbb{R} \), up to summing a linear function, can be normalized by the condition

\[ \varphi(h_{i,j}) = 0, \ \forall i \in \{1, \ldots, d-1\}, \forall j \in \{1, \ldots, d\}. \]

On the other hand, for any \( j \neq j', j, j' \in \{1, \ldots, d\} \) four lattice points

\[ h_{d,j}, h_{1,j}, h_{d,j'}, h_{1,j'} \]

generate a 3-dimensional cone in \( \Sigma_d \). Hence

\[ \varphi(h_{d,j}) = \varphi(h_{d,j'}) \ \forall j, j' \in \{1, \ldots, d\}. \]

This means that the space of all \( \Sigma_d \)-piecewise linear functions modulo linear functions is 1-dimensional. The anticanonical class is represented by the \( \Sigma_d \)-piecewise linear function \( \varphi_1 \) taking values 1 on each vector \( h_{i,j} \) \( i, j \in \{1, \ldots, d\} \). Considering the difference

\[ \varphi'_1 := \varphi_1 - \lambda, \]

where \( \lambda \) is a linear function on \( N_{\mathbb{R}} \) satisfying the conditions

\[ \lambda(e_1) = \cdots = \lambda(e_{d-1}) = 1, \ \lambda(e_d) = -(d-1), \ \lambda(f_1) = \cdots = \lambda(f_d) = 0, \]

we obtain a \( \Sigma_d \)-piecewise linear function having the properties

\[ \varphi'_1(h_{i,j}) = 0, \ \forall i \in \{1, \ldots, d-1\}, \forall j \in \{1, \ldots, d\}. \]
and 

\[ \varphi(h_{d,j}) = d \quad \forall j \in \{1, \ldots, d\}. \]

So the class of \( \varphi_1 \) modulo linear functions is a \( d \)-th multiple of a generator of \( \text{Pic}(V_{d,2(d-1)}) \).

The general case \( n > 2(d - 1) \) can be obtained by similar arguments using the fact that \( V_{d,n} \) is a projective cone over \( V_{d,2(d-1)} \). In order to construct the required flat 1-parameter family \( X \) (cf. 3.1), it suffices to consider a pencil of hypersurfaces of degree \( d \) in \( \mathbb{P}^{n+1} \) joining \( X_{d,n} \) and \( V_{d,n} \).

\[ \Box \]

**Theorem 3.7** Let \( X_d \subset \mathbb{P}^{n+1} \) be a smooth Fano hypersurface of degree \( d \). Then \( X_d \) admits a small toric degeneration if and only if \( n \geq 2d - 2 \).

**Proof.** By 3.6, it suffices to show that \( X_d \) does not admit a small toric degeneration if \( n < 2d - 2 \). Assume that \( X_d \) admits a small toric degeneration \( Y_d \). Then \( Y_d \) is a toric hypersurface defined by a binomial equation \( M_1 = M_2 \) where \( M_1 \) and \( M_2 \) are monomials in \( z_0, \ldots, z_{n+1} \) of degree \( d \) (\( z_0, \ldots, z_{n+1} \) are homogeneous coordinates on \( \mathbb{P}^{n+1} \)). If \( n < 2d - 2 \), then at least one of the monomials \( M_1 \) and \( M_2 \) must be divisible by \( z_i^2 \) for some \( i \in \{0, \ldots, n + 1\} \). We can assume that for instance \( z_0^2 \) divides \( M_1 \). If \( z_k \) and \( z_l \) are two variables appearing in \( M_2 \), then \( n - 2 \)-dimensional linear subspace

\[ z_0 = z_k = z_l = 0 \]

is contained in \( \text{Sing}(Y_d) \). This contradicts the fact that terminal singularities on \( Y_d \) could appear only in codimension \( \geq 3 \) (see 3.2). \[ \Box \]

Using 3.2, one immediately obtains:

**Proposition 3.8** If \( X \) is a smooth Del Pezzo surface, then \( X \) admits a small toric degeneration if and only if \( X \) is itself a toric variety (i.e. \( K_X^2 \geq 6 \)).

As we have seen from 3.6, a smooth quadric 3-fold in \( \mathbb{P}^4 \) is an example of nontoric smooth Fano variety which admits a small toric degeneration. By 3.7, cubic and quartic 3-folds do not admit small toric degenerations. The complete classification of smooth Fano 3-folds has been obtained in [10, 18, 24, 25, 26]. It is natural to ask the following:

**Question 3.9** Which 3-dimensional nontoric smooth Fano varieties do admit small toric degenerations?

## 4 The mirror construction

For our convenience, we assume \( K = \mathbb{C} \).

Let \( X \) be a smooth Fano \( n \)-fold over \( \mathbb{C} \) and \( Y \) is its small toric degeneration. The toric variety \( Y \) is defined by some complete rational polyhedral fan \( \Sigma \subset N_\mathbb{R} \), where
$N_R = N \otimes \mathbb{R}$ is the real scalar extension of a $N \cong \mathbb{Z}^n$. We denote by $Cl(Y)$ (resp. by $Pic(Y)$) the group of Weil (resp. Cartier) divisors on $Y$ modulo the rational equivalence. One has a canonical embedding

$$\alpha : Pic(Y) \hookrightarrow Cl(Y).$$

If $\{e_1, \ldots, e_k\} \subset N$ is the set of integral generators of 1-dimensional cones in $\Sigma$, then $Cl(Y)$ is a finitely generated abelian group of rank $k - n$ and the convex hull of $e_1, \ldots, e_k$ is a reflexive polyhedron $\Delta^*$ (for definition of reflexive polyhedra see [2]). Assume that there exists a partition of the set $I = \{e_1, \ldots, e_k\}$ into $r$ disjoint subsets $J_1, \ldots, J_r$ such that the union $D_i$ of toric strata in $Y$ corresponding to elements of $J_i$ is a semiample Cartier divisor on $Y$ for each $i \in \{1, \ldots, r\}$. Denote by $Z \subset Y$ a Calabi-Yau complete intersection of $r$ hypersurfaces $Z_i \subset Y$ defined by vanishing of generic global sections of $O_Y(D_i)$. By [4] (see also [9]), the mirrors $Z^*$ of Calabi-Yau complete intersections $Z \subset Y$ are birationally isomorphic to affine complete intersections in $(\mathbb{C}^*)^n = Spec \mathbb{C}[t_{i1}^{\pm 1}, \ldots, t_{in}^{\pm 1}]$ defined by $r$ equations

$$1 = \sum_{e_j \in J_i} a_j t^{e_j}, \quad i \in \{1, \ldots, r\},$$

where $(a_1, \ldots, a_k) \in \mathbb{C}^k$ is a general complex vector and $t^{e_1}, \ldots, t^{e_k}$ are Laurent monomials in variables $t_1, \ldots, t_n$ with the exponents $e_1, \ldots, e_k$.

**Definition 4.1** A complex vector $(a_1, \ldots, a_k) \in \mathbb{C}^k$ is called $\Sigma$-admissible, if there exists a $\Sigma$-piecewise linear function

$$\varphi : N_R \rightarrow \mathbb{R},$$

(i.e., a continuous function such that $\varphi|_{\sigma}$ is linear for every $\sigma \in \Sigma$) having the property

$$\varphi(e_i) = \log |a_i|, \quad \forall i \in \{1, \ldots, k\}.$$ 

The set of all $\Sigma$-admissible vectors will be denoted by $A(\Sigma)$.

**Remark 4.2** It is easy to show that $A(\Sigma) \subset \mathbb{C}^k$ is an irreducible closed subvariety which is isomorphic to an affine toric variety of dimension $rk \Pic(Y) + n \leq k$.

Now our generalization of the mirror construction from [4] to the case of Calabi-Yau complete intersections in a nontoric Fano variety $X$ can be formulated as follows:

**Generalized mirror construction:** Mirrors $W^*$ of generic Calabi-Yau hypersurfaces $W \subset X$ are birationally isomorphic to the affine complete intersections

$$1 = \sum_{i=1}^{k} a_i t^{e_i},$$
where \( a := (a_1, \ldots, a_k) \) is a general point of \( A(\Sigma) \).

**Monomial-divisor correspondence:** Let us explain the monomial-divisor mirror correspondence for this mirror construction (cf. [1]). By 3.1(iii), the group \( Pic(Y) \) can be canonically identified with \( Pic(X) \). The image of the restriction homomorphism \( Pic(X) \to Pic(W) \) defines a subgroup \( G \subset Pic(W) \), whose elements correspond to monomial deformations of the complex structure on mirrors:

if \( \psi \) is an integral \( \Sigma \)-piecewise linear function representing an element \( \gamma \in G \), then the 1-parameter family of hypersurfaces

\[
1 = \sum_{i=1}^{k} t_i^{\varphi(e_i)} e_i, \quad t_0 \in \mathbb{C}
\]

defines the corresponding 1-parameter deformation of the complex structure on \( W^* \) via the deformation of the coefficients \( a_i = t_i^{\varphi(e_i)} \).

**The main period:** Let \( R(\Sigma) \) the group of all vectors \((l_1, \ldots, l_k) \in \mathbb{Z}^k\) satisfying the condition \( \sum_{i=1}^{k} l_i e_i = 0 \) and \( L(\Sigma) \subset R(\Sigma) \) be the semigroup consisting of vectors \((l_1, \ldots, l_k) \in R(\Sigma)\) with nonnegative coordinates \( l_i \) \((i = 1, \ldots, k)\). There exists a canonical pairing \( \langle \ast, \ast \rangle : R(\Sigma) \times Pic(Y) \to \mathbb{Z} \) which is the intersection pairing between 1-dimensional cycles and Cartier divisors on \( Y \). According to [4], we can compute the main period in the family of mirrors \( W^* \) in our generalized mirror construction as follows

\[
\Phi_0(a) = \sum_{1=(l_1,\ldots, l_k) \in L(\Sigma)} \frac{\langle l, D_1 + \cdots + D_r \rangle!}{\langle l, D_1 \rangle! \cdots \langle l, D_r \rangle!} \prod_{i=1}^{k} a_i^{l_i}, \quad a \in A(\Sigma).
\]

The condition \( a \in A(\Sigma) \) can be interpreted as a specialization of GKZ-hypergeometric series from [4].

Some evidences in favor of our generalized mirror construction were presented in [7, 8]. For our next examples confirming the proposed generalized mirror construction we use the following simple combinatorial statement:

**Proposition 4.3** Let \( S_d(m) \) be the set of all \( d \times d \)-matrices \( K = (k_{ij}) \) with nonnegative integral coefficients \( k_{ij} \) satisfying the equations

\[
\begin{pmatrix}
k_{11} & \cdots & k_{1d} \\
. & \cdots & . \\
. & \cdots & . \\
k_{d1} & \cdots & k_{dd}
\end{pmatrix}
(1 \quad \cdots \quad 1)
= (m \quad \cdots \quad m)
\]

and

\[
\begin{pmatrix}
k_{11} & \cdots & k_{1d} \\
. & \cdots & . \\
. & \cdots & . \\
k_{d1} & \cdots & k_{dd}
\end{pmatrix}
(1)
= (m)
\]

and

\[
\begin{pmatrix}
m \\
. \\
. \\
m
\end{pmatrix}
\]
Then
\[
\sum_{K \in S_d(m)} \frac{(m!)^d}{\prod_{i,j=1}^{d} (k_{ij})!} = \frac{(dm)!}{(m!)^d}.
\]

**Proof.** Let \( A \) be the set \( \{1, 2, \ldots, dm\} \) of first \( dm \) natural numbers. We fix a splitting \( A \) into the disjoint union of \( d \) subsets
\[
A_i := \{(i-1)m+1, (i-1)m+2, \ldots, im\}, \quad i = 1, \ldots, d
\]
consisting of \( m \) elements. Let \( \beta : A = B_1 \cup \cdots \cup B_d \) be an arbitrary representation of \( A \) as a disjoint union of the subsets \( B_1, \ldots, B_d \) with the property \( |B_1| = \cdots = |B_d| = m \). Then every such a representation defines a matrix \( K(\beta) = (k_{ij}(\beta)) \in S_d(m) \) as follows:
\[
k_{ij}(\beta) := |A_i \cap B_j|, \quad i, j \in \{1, \ldots, d\}.
\]
For a fixed matrix \( K \in S_d(m) \) there exist exactly
\[
\prod_{j=1}^{d} \frac{(m!)^{d}}{\prod_{i=1}^{d} (k_{ij})!}
\]
ways to construct a representation \( \beta \) of \( A \) as a disjoint union of \( m \)-element subsets \( B_1, \ldots, B_d \) such that \( K = K(\beta) \). Therefore,
\[
\sum_{K \in S_d(m)} \frac{(m!)^d}{\prod_{i,j=1}^{d} (k_{ij})!}
\]
is the total number of ways to split \( A \) into a disjoint union of \( m \)-element subsets \( B_1, \ldots, B_d \). On the other hand, this number is equal to the multinomial
\[
\frac{(dm)!}{(m!)^d}.
\]
\( \Box \)

**Example 4.4** Let \( W \) be a generic Calabi-Yau complete intersection of two hypersurfaces \( V_d, V'_d \) in \( \mathbb{P}^{2d-1} \). By 3.6, we can construct a small toric degeneration of one smooth hypersurface \( V'_d \) to the \( 2(d-1) \)-dimensional toric variety \( Y_d \subset \mathbb{P}^{2d-1} \)
\[
z_0 z_1 \cdots z_{d-1} = z_d z_{d+1} \cdots z_{2d-1}.
\]
Using an explicit description of the Picard group \( Pic(Y_d) \) from the proof of 3.6, our generalized mirror construction suggests that mirrors \( W^* \) for \( W \) are birationally isomorphic to the affine hypersurfaces \( Z_F \) in the algebraic torus
\[
Spec \mathbb{C}[t_1^{\pm 1}, \ldots, t_{d-1}^{\pm 1}, u_1^{\pm 1}, \ldots, u_{d-1}^{\pm 1}]
\]
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defined by the 1-parameter family of the equations

\[ 1 = F(t_1, \ldots, t_{d-1}, u_1, \ldots, u_{d-1}, z) = \sum_{i=1}^{d-1} \sum_{j=1}^{d-1} t_i u_j + (u_1 \cdots u_{d-1})^{-1} \left( \sum_{i=1}^{d-1} t_i \right) + z(t_1 \cdots t_{d-1})^{-1} \left( u_1 \cdots u_{d-1}^{-1} + \sum_{i=1}^{d-1} u_i \right), \quad z \in \mathbb{C} \]

On the other hand, it is known via a toric mirror construction for Calabi-Yau complete intersection \( W = V_d \cap V_d' \) (see [4]) that the power series

\[ \Phi_0(z) = \sum_{m \geq 0} \frac{(dm)!^2}{(m!)^{2d}} z^m \]

generates the Picard-Fuchs \( D \)-module describing the quantum differential system. Now we compare our generalized mirror construction with the known one from [4] computing the main period of the family \( Z_F \) by the Cauchy residue formula:

\[ \Psi_F(z) := \frac{1}{(2\pi \sqrt{-1})^{2(d-1)}} \int_{\Gamma} \frac{1}{1 - F(t, u, z)} \frac{dt}{t} \wedge \frac{du}{u} = 1 + a_1 z + a_2 z^2 + \cdots, \]

where the coefficients \( a_m \) of the power series \( \Psi_F(z) \) can be computed by the formula

\[ a_m = \sum_{K \in S_d(m)} \frac{(dm)!}{\prod_{i,j=1}^{d} (k_{ij})!}. \]

Using [4] we obtain that

\[ a_m = \frac{(dm)!^2}{(m!)^{2d}}, \]

i.e., the power series \( \Psi_F(z) \) coincides with \( \Phi_0(z) \) and therefore our generalized mirror construction agrees with the already known one from [4].

For the special case \( d = 3 \), we obtain a description for mirrors \( W^* \) of complete intersections \( W \) of two cubics in \( \mathbb{P}^5 \) as smooth compactifications of hypersurfaces in the 4-dimensional algebraic torus

\[ Spec \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, u_1^{\pm 1}, u_2^{\pm 1}] \]

defined by the 1-parameter family of the equations

\[ 1 = F(t_1, t_2, u_1, u_2, \lambda) = t_1 u_1 + t_1 u_2 + t_1 (u_1 u_2)^{-1} + t_2 u_1 + t_2 u_2 + t_2 (u_1 u_2)^{-1} + z(t_1 t_2)^{-1} (u_1 + u_2 + (u_1 u_2)^{-1}), \quad z \in \mathbb{C}. \]

This description of mirrors is different from the one proposed by Libgober and Teitelbaum in [22], but it seems that both constructions are equivalent to each other.
Now we want to suggest some problem which naturally arise from the proposed generalized mirror construction.

**Problem 4.5** Check the topological mirror duality test

\[ E_{st}(W^*; u, v) = (-u)^n E_{st}(W; u^{-1}, v) \]

for the above generalized mirror construction. Here \( E_{st} \) is the stringy \( E \)-function introduced in [3].

**Remark 4.6** The main difficulty of this checking arises from the fact that the affine complete intersections in the above mirror construction are not generic. For \( \Delta^* \)-regular affine hypersurfaces there exists explicit combinatorial formula for their \( E \)-polynomials (see [4]). However, the affine hypersurfaces in our mirror construction are not \( \Delta^* \)-regular and no explicit formula for their \( E \)-polynomials (or Hodge-Deligne numbers) is known so far.

**Problem 4.7** Generalize the method of Givental [13, 14, 15] for computing Gromov-Witten invariants of complete intersections in smooth Fano varieties \( X \) admitting small toric degenerations.

**Remark 4.8** If \( X \) is a smooth Fano \( n \)-fold admitting a small toric degeneration \( Y \), then one can not expect that there exists a \( \mathbb{C}^* \)-action on \( X \). So the equivariant arguments from [13] can not be applied directly to \( X \). However, one could try to use equivariant Gromov-Witten theory for the ambient projective space \( \mathbb{P}^m \) containing both \( X \) and \( Y \) and to show that the virtual fundamental classes corresponding to \( Y \) and \( X \) are the same. It seems that small quantum cohomology of \( Y \) carry complete information about the subring in the small quantum cohomology ring \( QH^*(X) \) generated by the classes of divisors. This would give an explicit description of such a subring (see [14]) as well as of its gravitational version via Lax operators (see [11]).

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