Hall magnetohydrodynamics of partially ionized plasmas

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ABSTRACT
The Hall effect arises in a plasma when electrons are able to drift with the magnetic field but ions cannot. In a fully-ionized plasma this occurs for frequencies between the ion and electron cyclotron frequencies because of the larger ion inertia. Typically this frequency range lies well above the frequencies of interest (such as the dynamical frequency of the system under consideration) and can be ignored. In a weakly-ionized medium, however, the Hall effect arises through a different mechanism – neutral collisions preferentially decouple ions from the magnetic field. This typically occurs at much lower frequencies and the Hall effect may play an important role in the dynamics of weakly-ionised systems such as the Earth’s ionosphere and protoplanetary discs.

To clarify the relationship between these mechanisms we develop an approximate single-fluid description of a partially ionized plasma that becomes exact in the fully-ionized and weakly-ionized limits. Our treatment includes the effects of ohmic, ambipolar, and Hall diffusion. We show that the Hall effect is relevant to the dynamics of a partially ionized medium when the dynamical frequency exceeds the ratio of ion to bulk mass density times the ion-cyclotron frequency, i.e. the Hall frequency. The corresponding length scale is inversely proportional to the ion to bulk mass density ratio as well as to the ion-Hall beta parameter. In a weakly ionized medium, the critical frequency becomes small enough that Hall MHD is an accurate representation of the dynamics. More generally, ohmic and ambipolar diffusion may also be important.

We show that both ambipolar and Hall diffusion depend upon the fractional ionization of the medium. However, unlike ambipolar diffusion, Hall diffusion may also be important in the high fractional ionization limit. The wave properties of a partially-ionized medium are investigated in the ambipolar and Hall limits. We show that in the ambipolar regime wave damping is dependent on both fractional ionization and ion-neutral collision frequencies. In the Hall regime, since the frequency of a whistler wave is inversely proportional to the fractional ionization, and bounded by the ion-neutral collision frequency it will play an important role in the Earth’s ionosphere, solar photosphere and astrophysical discs.

Key words: Earth, Sun:atmosphere, Stars:Formation, MHD, plasmas, waves.

1 INTRODUCTION
In ideal magnetohydrodynamics (MHD), ions and electrons are both tied to the magnetic field. When ions are decoupled from the field and electrons are not, the magnetic field and electrons drift together through the ions and the generalised Ohm’s law is modified by the Hall electric field, proportional to $J \times B$, where $J$ and $B$ are the current density and magnetic field. This modification of ideal MHD is called Hall magnetohydrodynamics.

Hall MHD plays a crucial role in a variety of astrophysical, space and laboratory environments, often providing the dominant mechanism for plasma drift against the magnetic field, from flux expulsion in neutron star crusts (Goldrich & Reisenegger 1992) to angular momentum transport in weakly ionized protoplanetary discs (Wardle, 1999; Balbus and Terquem, 2001; hereafter W99 and BT01 respectively). The formation of intensive flux tubes in the solar atmosphere (Khodachenko & Zaïtsev 2002), waves in the solar wind (Zhelyankov et al 1996; Miteva et al. 2003), propagation of whistlers in the Earth’s ionosphere (Aburjania at al. 2005) and sub-Alfvénic plasma expansion (Huba 1995) are but a few examples where Hall MHD appears to play significant role. In fusion plasmas, the Hall effect can play an important role in describing various discharge behaviour (Kappraff et al. 1981; Wang & Bhattacharjee 1993). For example, it can significantly enhance the non-Ohmic current drive in tokamaks (Pandey et al. 1995).

Two mechanisms may decouple the ions from the magnetic field under different physical conditions. This has led
to distinct approaches being adopted to investigate the role of the Hall effect in the dynamics of laboratory (Kappraff et al. 1981; Wang & Bhattacharjee 1993; Pandey et al. 1995), space (Huba 1995, 2003; Aburjania et al. 2005; Richmond & Thayer 2000; Zhelyankov et al 1996) and astrophysical (W99, BT01, Goldreich and Reisenegger, 1992) plasmas.

In a highly ionized plasma the Hall effect arises because of the difference in electron and ion inertia: ions are unable to follow magnetic fluctuations at frequencies higher than their cyclotron frequency, whereas electrons remain coupled to the magnetic field. The corresponding physical scale, the ion skin depth, is typically much smaller than the scale of the system. In this case the Hall effect has typically been incorporated by explicitly including the ion-electron drift in the induction equation.

In a partially ionized plasma the Hall effect may instead arise because neutral collisions more easily decouple ions from the magnetic field than electrons. In this case, the Hall scale can become comparable to the size of the system itself. Its effects are typically incorporated through a second-rank conductivity tensor appearing in a generalized Ohm’s law (Cowling 1957; Mitchner and Kruger 1979).

The Hall dynamics of highly ionized and weakly ionized plasmas are similar, but occur on very different frequency ranges and spatial scales due to the different mechanisms responsible for the underlying symmetry breaking in ion and electron dynamics. This has led to some confusion in the literature, where estimates of the fully-ionized Hall length scale have been applied to the ionized component of partially ionized media to conclude that the Hall effect is irrelevant in circumstances when it is, in fact, crucial (Huba 1995, 2003; Bacciotti et al. 1997; Rudakov 2001).

The purpose of this paper is to clarify the relationship between the fully ionized and weakly ionized limits by developing a unified single-fluid framework for the dynamics of plasmas of arbitrary ionisation. Our treatment is of necessity approximate in the intermediate case, but has the correct behaviour in the highly- or weakly-ionized limits and is not strongly limited in applicability in the intermediate ionisation regime. This allows us to explore the change of scale in the Hall effect in moving from fully to partially ionized plasmas and gain a deep physical understanding of the nature of the transition between the two ionisation regimes. Furthermore, this formulation is useful in gaining insight into the behaviour of plasmas that are neither fully ionized nor weakly ionized (e.g. near a tokamak wall or the surface of a white dwarf), when neutral collisions and ionized plasma inertia may both be important.

The paper is organised in the following fashion. In section 2 we derive a set of fluid equation in the bulk frame suitable for the weakly ionized medium and the characteristic scales on which the Hall effect manifests, are discussed. In section 3, waves in a partially ionized plasma are described and the dependence of the wave damping on fractional ionization in the ambipolar regime is discussed. The very low frequency ion-cyclotron and high frequency collisional whistler is shown to be the two branches in the Hall regime. In section 4 we discuss the potential wide ranging applications of this work to laboratory, space, and astrophysical plasmas. A brief summary of the results is given in the final section.

2 FORMULATION

Space and astrophysical plasmas are generally partially ionized consisting of electrons, ions, neutrals, and charged and neutral dust grains. We shall neglect grains in the present formulation and consider a partially ionized plasma consisting of electrons, ions, and neutrals. The dynamics of such a plasma is complex but depending upon the physical conditions pertaining to the problem at hand, reasonable simplifying assumptions can be made. For example, the dynamics of a protoplanetary disc has been investigated by assuming that the neutrals provide the inertia of the bulk fluid and plasma particles carry the current (W99). This approach is reasonable as in a cold protostellar disc, the ionization fraction (i.e. the ratio of electron to the neutral number density) is very low ($\sim 10^{-8} - 10^{-13}$) and the relative drift between ions and neutrals are small. Therefore, such a description is not only economical but also captures the essential physics of the protoplanetary discs. However, the inertia of the ionized components may in general play an important role, e.g. near the wall of a tokamak, in the lower part of Earth’s F-region, at the base of the solar chromosphere, in the outer part of AGN discs, in the discs around the dwarf novae etc., when the ionization fraction is small and yet not negligible. In neutron star crusts too, neutron and proton densities are comparable and a multi-component description of the strongly magnetized fluid is desirable. In the solar chromosphere, utilizing three component description, Alfvén wave damping have been studied in the context of spicule dynamics (Pontieu & Haerendel 1998). In the solar photosphere, the effect of ion-neutral damping on the propagation of waves has also been recently studied (Kumar & Roberts 2003). Our aim therefore is to develop an approximate single fluid like description of a multi-component, partially ionized plasma and demand that it reduces to the fully and weakly-ionized descriptions in different fractional ionization limits. This approximate formulation will permit us to explore the relationship between the onset of the Hall effect due to the ion inertia or due to the ion-neutral collisions. Furthermore, such a description will provide us the freedom to investigate the effect of fractional ionization in various limits on the MHD wave modes.

2.1 A single-fluid model for partially-ionized plasma

We start with the three-component (ions, electrons and neutrals) description of a partially ionized plasma and reduce it to a single fluid description. The continuity equation is

$$\frac{\partial \rho_j}{\partial t} + \nabla \cdot (\rho_j \mathbf{v}_j) = 0,$$

(1)

where $\rho_j = n_i n_j$ is the mass density, $\mathbf{v}_j$ is the velocity, and $n_i$ and $n_j$ are the number density and particle mass of the various components for $j = i, e, n$. We shall assume that the ions are singly charged and adopt charge neutrality, so that $n_i = n_e$. The momentum equations for the electrons, ions and neutrals are

$$\frac{d\mathbf{v}_e}{dt} = -\frac{\nabla P_e}{\rho_e} - \frac{c}{m_e} \left( \mathbf{E} + \frac{\mathbf{v}_e}{c} \times \mathbf{B} \right) - \sum_{j=i,n} \nu_{ej} (\mathbf{v}_e - \mathbf{v}_j)$$

(2)
The momentum equation can be derived by adding equations (2)-(3) on the right hand side pressure gradient, Lorentz force and collisional momentum exchange terms where $P_i$ is the pressure, $E$ and $B$ are the electric and magnetic field, $c$ is the speed of light, and $\nu_{ij}$ is the collision frequency for species $i$ with species $j$. The electron-ion collision frequency $\nu_{ei}$ can be expressed in terms of the fractional ionization $x_e = n_e/n_n$ and the plasma temperature $T_e = T_i = T$ as

$$\nu_{ei} = 51 \times n_e n_n T^{-1.5} \text{s}^{-1},$$

where $T$ and $n_n$ are in K and cm$^{-3}$ respectively. The plasma-neutral collision frequency $\nu_{jn}$ is

$$\nu_{jn} = \gamma_{jn} \rho_n = \frac{\sigma v}{m_e + m_j} \rho_n.$$

Here $\langle \sigma v \rangle_j$ is the rate coefficient for the momentum transfer by collision of the $j$th particle with the neutrals. The ion-neutral and electron-neutral rate coefficients are (Draine et al. 1983)

$$<\sigma v>_j = 1.9 \times 10^{-9} \text{ cm}^3 \text{s}^{-1},$$

$$<\sigma v>_e = 8.28 \times 10^{-10} T^{3/2} \text{ cm}^3 \text{s}^{-1}.$$ (7)

The density of the bulk fluid is

$$\rho = \rho_e + \rho_i + \rho_n \approx \rho_i + \rho_n.$$ (8)

Then defining the neutral density fraction

$$D = \frac{\rho_n}{\rho},$$

the bulk velocity is $v = (\rho_i v_i + \rho_n v_n)/\rho$ can be written as $v = (1-D) v_i + D v_n$. (9)

Note that we are implicitly neglecting the electron inertia in (10), and therefore in the momentum equation (13) below.

The continuity equation for the bulk fluid is obtained by summing up equation (1) for each species:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0.$$ (11)

The momentum equation can be derived by adding equations (2) - (4) to obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left( \frac{\rho \rho_n}{\rho} v_D v_D \right) = -\nabla P + \frac{J \times B}{c},$$

where $P = P_e + P_i + P_n$ is the total pressure, $v_D = v_i - v_n$ is the ion-neutral drift velocity, and $J = n_e c (v_i - v_e)$ is the current density.

Defining $\nu_A = B/\sqrt{4 \pi \rho}$ as the Alfvén speed in the bulk fluid and $c_A = \sqrt{\gamma p/\rho}$ as the acoustic speed, we note that if $\rho_i \rho_n v_{ei}^2 \ll \rho^2 (v_{ei}^2 + c_A^2)$ then we may neglect the $v_D v_D$ term in Eqn (12) and recover the single-fluid momentum equation

$$\frac{\partial \rho}{\partial t} = -\nabla P + \frac{J \times B}{c}. $$ (13)

To derive a criterion for this, we estimate $v_D$ by rewriting the ion and neutral equations of motion (3) and (4) as

$$(\rho_i \nu_{in} + \rho_e \nu_{en}) \nu_D = -\nu_{in} \frac{\partial \nu}{\partial t} - \nabla (P_i + P_n)$$

and

$$(\rho_i \nu_{in} + \rho_e \nu_{en}) \nu_D = \rho_n \frac{\partial \nu}{\partial \rho} + \nabla P_n + \frac{m_e \nu_{en} e}{c} J,$$

respectively. Multiplying Eq. (14) by $\rho_n$ and Eq. (15) by $\rho_i$ and then adding

$$(\rho_i \nu_{in} + \rho_e \nu_{en}) \nu_D = D \frac{J \times B}{c} + \nabla P_n - D \nabla P + \rho_n \frac{\partial \nu}{\partial \rho} - (\nu \cdot \nabla) \nu + (\nu \cdot \nabla) \nu D + \frac{m_e \nu_{en} e}{c} J.$$ (16)

The term in the square bracket can be neglected if

$$\omega \lesssim \frac{P}{\rho_n \nu_{ni}}.$$ (17)

Then equation (16) can be written as,

$$\nu_D = D \frac{J \times B}{c} + \nabla P_n - D \nabla P + \rho_n \frac{\partial \nu}{\partial \rho} - (\nu \cdot \nabla) \nu + (\nu \cdot \nabla) \nu D + \frac{m_e \nu_{en} e}{c} J,$$

where

$$\beta_j = \frac{\omega_{cj}}{\nu_j},$$

is the ratio of the cyclotron frequency of the $j$th particle $\omega_{cj} = eB/m_j c$ (where $e$, $B$, $m_j$, $c$ denotes electron charge, magnetic field, mass and speed of light respectively) to the sum of the plasma-plasma, and plasma-neutral, $\nu_{jn}$ collision frequencies. For electrons $\nu_e = \nu_{en} + \nu_{ie}$ and for ions $\nu_i = \nu_{in} + \nu_{ic}$. While writing (18), we have used $\rho_i \nu_{in} \lesssim \rho_i \nu_{in}$. In the weakly ionized limit, when $D \rightarrow 1$, neglecting plasma pressure terms and assuming $\beta_e \gg 1$, Eq. (18) reduces to the strong coupling approximation, i.e. $v_D \approx (J \times B)/(e \rho n_v \nu_{in})$ (Shu 1983).

Equation (18) implies that for gradients with a length scale $L$, and signal speed $s$, $v_D \sim \rho_s s^2 (1 + 1/D \beta_i)/(\rho_i \nu_{in} L)$. The associated dynamical frequency is $\omega \sim s/L$, so the requirement $\rho_i \rho_n v_{ei}^2 \ll \rho^2 (v_{ei}^2 + c_A^2)$ means that the $v_D v_D$ term in (12) can be neglected for dynamical frequencies satisfying

$$\omega \lesssim \frac{\rho}{\sqrt{\rho_s \rho_n}} \left( \frac{D \beta_i}{1 + D \beta_e} \right) \nu_{ni}.$$ (20)

At higher frequencies the single-fluid approximation (13) breaks down. Note that this frequency constraint is much weaker in the highly-ionized and weakly-ionized limits, for which $\rho \approx \rho_n (D \rightarrow 1)$. In the appendix we show that Eq. (20) is a conservative bound on the dynamical frequency. Further, we also show in the appendix that (17) is implied by Eq. (20).

To obtain an equation for the evolution of the magnetic field, we need to derive an expression for the electric field $E$ in terms of the fluid properties to insert into Faraday’s law

$$\frac{\partial B}{\partial t} = -\nabla \times E.$$ (21)

We start with the electron momentum equation (2), which in the zero electron inertia limit yields an expression for the electric field in the rest frame of the ions:

$$E + \frac{\nu_{in}}{c} \times B = -\nabla P_i + \frac{J \times B}{c e n_e} + \frac{m_e \nu_{en} e}{c} \nu_D$$ (22)

where
The relative importance of the various terms in the induction equation (25) can be easily estimated. The ratio of the Hall (H) and the Ohm (O) terms gives $H/O \sim \beta_e$, the electron Hall parameter. The ratio between ambipolar ($A$) and Hall ($H$) terms is $A/H \sim D^2 \beta_e$. In the weak ionization ($D \to 1$) limit, $A/H \sim \beta_e$ i.e. ion Hall parameter. Hall Hall diffusion does not disappear in the high fractional ionization limit. Unlike ambipolar diffusion, Hall diffusion is not negligible in the high fractional ionization limit.

The ambipolar diffusion terms in (25) arise from $D \nabla \times B$ in the $\mathbf{v}_i \times B$ term in (22) since $\nabla \times B = \mathbf{v} \times B + D \nabla \times B$. The terms due to pressure gradients $\nabla P \times B$ are negligible compared to the inductive term $\mathbf{v} \times B$ when

$$\omega \lesssim \left( \frac{v_A}{c^2} \right) \frac{\rho^2}{\rho_i \rho_n} \nu_{ni} ,$$

where $c_s$ is some effective sound speed. We note that for $D \beta_e \sim 1$, Eq. (20) guarantees (26) when $v_A \lesssim c_s$. In the opposite limit, when $v_A > c_s$, (26) is not implied by (20). Our final induction equation without $\nabla P \times B$ term becomes:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[ (\mathbf{v} \times B) - \frac{4 \pi \eta}{c} J \times B \right] + \frac{4 \pi \eta_H}{c} J \times \mathbf{B} ,$$

(27)

where $\mathbf{B} = B \mathbf{B}$, and the Ohmic ($\eta$), ambipolar ($\eta_A$) and Hall ($\eta_H$) diffusivity are

$$\eta = \frac{e^2}{4 \pi} , \eta_A = \frac{D^2 B^2}{4 \pi \rho_i \nu_{ni}} \equiv \frac{v_A^2}{\nu_{ni}} , \eta_H = \frac{e B}{4 \pi e \nu_{ni}} .$$

Equation (27) is identical to the known expression for a weakly-ionized medium (e.g. Königl 1989) apart from the appearance of the factor $D^2$ in $\eta_A$, which suppresses ambipolar diffusion if the ionisation of the plasma is significant. The dependence of ambipolar term on the $D^2$ factor was first noted by Cowling (1957).

To summarize, the single fluid equations have been derived neglecting electron inertia in the low frequency limit given by (20). Pressure gradient terms have been neglected in the induction equation (27), which is valid if inequality (20) is satisfied. Then equations (11), (13), and (27) along with prescriptions for determining $P$ and $n_e$ describe the dynamics of a plasma of arbitrary ionization. For example, when the plasma is fully ionized, (i.e. $D \to 0$), $\mathbf{v} = \mathbf{v}_i$ and (11), (13), and, (27) reduces to the fully ionized Hall resistive MHD description. In the other extreme limit $D \to 1$, the equations reduce to those describing weakly ionized MHD (W99, BT01).

2.2 The Hall scale

In fully ionized plasmas the Hall effect becomes important for frequencies in excess of the ion gyrofrequency. In natural systems the associated time scales are usually much shorter than those of interest and Hall dynamics can be safely neglected. However, in partially ionised plasmas the Hall effect becomes important on longer length and time scales, and in weakly ionised plasmas these may even become comparable to the dynamical time scale of the system.

This behaviour is easily inferred from the fluid equations derived in the previous section. If diffusion is unimportant, the characteristic lengthscale of a gradient in the fluid associated with frequency $\omega$ is $L \sim v_A/\omega$ where $v_A$ is the Alfvén speed in the total fluid (not just the ionized component). Then comparing the magnitudes of the advective and Hall diffusion terms in the induction equation (27), we find that the Hall term becomes important for frequencies in excess of the Hall frequency

$$\omega_H = \frac{e B}{m_i c} = \frac{\rho_i}{\rho} \omega_{ci} \equiv \frac{v_A^2}{\eta_H} ,$$

(29)

where the effective ion mass is

$$m_i^* = \rho/\rho_n .$$

(30)

The corresponding Hall length scale is

$$L_H = \frac{v_{A}}{\omega_H} = \left( \frac{\rho_i}{\rho} \right)^{1/2} \delta_i = \left( \frac{\rho_i}{\rho} \right) \left( \frac{v_A}{\nu_{in}} \right)^{1/2} .$$

(31)

where $\delta_i = v_{Ai}/\omega_{ci}$ is the ion skin depth with $v_{Ai} = B/\sqrt{4 \pi \rho_i}$ as the Alfvén speed in the ion fluid.

The Hall effect arises because through an asymmetry in the ability of positive and negative charge carriers to drift in response to the instantaneous electric field. In the fully-ionized limit, for frequencies $\omega_{ci} \ll \omega \ll \omega_{ce}$ electrons are able to attain a drift velocity in instantaneous balance between electric, magnetic and collisional stresses, whereas the inertia of the ions prevents them from doing so. In the single-fluid approximation, the ions are tightly coupled to the neutrals by collisions so that they are unable to drift through them but must carry them along also. Thus they pick up the neutral inertia, gaining an effective mass $m_i^*$, and are unable to fully respond to changes with frequencies.

1. The effects associated with $\omega \gtrsim \omega_{ce}$ are absent in our estimate (29) because electron inertia was explicitly neglected in our development of the single-fluid equations.
in excess of $\omega_H$ (cf. Pandey & Wardle 2006a, 2006b). Implicit in this is the requirement that collisions are able to provide the strong coupling between ions and neutrals, as noted in eq. (20).

The condition $\omega \gtrsim \omega_H$ implies that the Hall term dominates the inductive term in (27), but does not guarantee that it is the dominant diffusion mechanism. As noted earlier the ratio of the Hall and Ohmic diffusion terms is $\sim \beta_e$, whereas the ratio of the ambipolar and Hall diffusion terms is $\sim D^2 \beta_i$, so for Hall diffusion to dominate the other mechanisms, we require $D^2 \beta_i \ll 1 < \beta_e$.

Note that in the weakly-ionized limit we recover the standard requirement $\beta_i \ll 1 < \beta_e$. In the fully-ionized limit $D^2 \beta_i \rightarrow 0$ and the first inequality is guaranteed.

### 2.3 Magnetovorticity

In the Hall-dominated regime, if the effective ion mass $m_i^* = \rho/\eta_i$ is constant (in space and time) then the concept of flux freezing can be generalised to the freezing of magnetovorticity

$$\omega_M = \omega_H \vec{B} + \nabla \times \vec{v}$$

into the fluid flow (see e.g., the review by Polynyaannakis & Moussas 2000). To show this we take the curl of the momentum equation (13) and use the identity $(\nabla \times \vec{v}) \times \vec{v} = -\frac{1}{2} \nabla \delta \rho + (\vec{v} \cdot \nabla) \vec{v}$ to obtain

$$\frac{\partial (\nabla \times \vec{v})}{\partial t} = \nabla \times (\nabla \times \vec{v}) + \nabla \times \left( \frac{\mathbf{J} \times \mathbf{B}}{\rho c_s} \right),$$

where $\nabla \rho \times \nabla P$ has been neglected. In the absence of magnetic forces, this is the equation for conservation of the vorticity $\nabla \times \vec{v}$. The magnetic term is directly proportional to the Hall term in the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\nabla \times \mathbf{B}) + \nabla \times \left( \frac{\mathbf{J} \times \mathbf{B}}{\rho c_s} \right),$$

and eliminating this term between the two yields

$$\frac{\partial \omega_M}{\partial t} = \nabla \times (\vec{v} \times \omega_M)$$

which shows that the magnetovorticity is frozen into the fluid.

In the limit $|\nabla \times \vec{v}| \ll \omega_H$, this reduces to magnetic flux freezing, because Hall diffusion is not significant in the induction equation. In the opposite limit, $|\nabla \times \vec{v}| \gg \omega_H$, $\omega_M$ reduces to the usual fluid vorticity $\nabla \times \vec{v}$ because the magnetic term is unimportant in the momentum equation.

### 3 WAVES IN A PARTIALLY IONIZED MEDIUM

In this section we examine the wave modes supported by a partially ionized plasma satisfying eqs (11), (13) and (27).

We shall assume a homogeneous, uniform background with zero flow and investigate the wave properties of the medium in various ionization limits. We assume that the medium is isothermal, i.e. that $P = \rho c_s^2$ with constant sound speed $c_s$. The linearized equations for the perturbations $\delta \rho$, $\delta \vec{v}$, $\delta \mathbf{B}$, and $\delta \mathbf{J}$ are

$$\frac{\partial \delta \rho}{\partial t} + \nabla \cdot (\rho \delta \vec{v}) = 0, \quad (37)$$

$$\rho \frac{\partial \delta \vec{v}}{\partial t} = -c_s^2 \delta \rho + \frac{\delta \mathbf{J} \times \mathbf{B}}{c}. \quad (38)$$

$$\frac{\partial \delta \mathbf{B}}{\partial t} = \nabla \times \left[ (\delta \vec{v} \times \mathbf{B}) - \frac{4\pi \eta_i}{c} \delta \mathbf{J} - \frac{4\pi \eta_n}{c} \delta \mathbf{J} \times \mathbf{B} + \frac{4\pi \eta_n}{c} (\delta \mathbf{J} \times \mathbf{B}) \times \mathbf{B} \right], \quad (39)$$

and

$$\delta \mathbf{J} = \frac{c}{4\pi} \nabla \times \delta \mathbf{B}. \quad (40)$$

Note that we do not need an explicit prescription for $\rho_i$ or $\eta_n$ as their perturbations do not appear in the linearized induction equation. Assuming that the perturbations have the form $\exp(i \omega t - i \mathbf{k} \cdot \mathbf{x})$, and using (40), equations (37) and (38) become

$$\omega \delta \rho - \rho k \cdot \delta \vec{v} = 0, \quad (41)$$

$$\omega \delta \vec{v} = c_s^2 \left( \frac{k \cdot \delta \vec{v}}{\omega} \right) \mathbf{k} - \frac{(k \cdot \mathbf{B}) \delta \mathbf{B}}{4\pi \rho} + \frac{(\delta \mathbf{B} \cdot \mathbf{B}) k}{4\pi \rho} \quad (42)$$

We define $\omega^2 = \omega^2 - k^2 c_s^2$, dot equation (42) with $\mathbf{k}$, and use $k \cdot \delta \mathbf{B} = 0$ to write

$$k \cdot \delta \vec{v} = \frac{\omega}{4\pi \rho \omega^2} (\mathbf{B} \cdot \delta \mathbf{B}). \quad (43)$$

We see that in the incompressible limit, both $\delta \vec{v}$ and $\delta \mathbf{B}$ are transverse to the background magnetic field $\mathbf{B}$. Making use of equation (43), equation (42) can be written as

$$\omega \delta \vec{v} = -\frac{1}{4\pi \rho} \left[ (k \cdot \mathbf{B}) \delta \mathbf{B} - \frac{\omega^2}{\omega^2} (\mathbf{B} \cdot \delta \mathbf{B}) \mathbf{k} \right]. \quad (44)$$

Defining $\mathbf{k} \cdot \mathbf{B} = \cos \theta$ and $\omega^2_k = k^2 v_A^2$, and eliminating $\delta \vec{v}$ and $k \cdot \delta \vec{v}$ from equation (39) we get an equation in terms of $\delta \mathbf{B}$ only

$$\omega^2 - (v_A^2 + i \eta_A \omega) k^2 \cos^2 \theta - i \eta_k k^2 \omega \delta \mathbf{B} = \left[ \frac{\omega^2}{\omega^2} \omega^2_A + i \eta_A k^2 \omega \right] (\delta \mathbf{B} \cdot \mathbf{B}) \left( \mathbf{B} \cdot \frac{k \cos \theta}{k} \right) - i \eta_k k^2 \omega \cos \theta \left( \mathbf{k} \times \delta \mathbf{B} \right), \quad (45)$$

where $\omega_A = k v_A$ is the Alfvén frequency. After some straightforward algebra, following dispersion relation can be derived from equation (45)

$$\left[ \omega^2 - (v_A^2 + i \eta_A \omega) k^2 \cos^2 \theta - i \eta_k k^2 \omega \right] \times$$

$$\left\{ \omega^2 - (v_A^2 + i \eta_A \omega) k^2 \cos^2 \theta - i \eta_k k^2 \omega \right\} - k^2 \sin^2 \theta \left[ \omega^2_A + i \eta_A \omega \right] - \eta_k k^4 \omega^2 \cos^2 \theta = 0. \quad (46)$$

In the following sections, we shall investigate the dispersion relation, Eq. (46) in various limits.

### 3.1 No Hall limit

In the absence of Hall (i.e. $\omega \ll \omega_H$), the last term in dispersion relation (46) can be ignored. We note that the modes related to the $\delta \mathbf{B} \parallel \mathbf{B}$ and to $\delta \mathbf{B} \parallel \mathbf{k} \times \mathbf{B}$ are mixed. For example, when the magnetic field perturbation is parallel to
In the absence of Ohmic dissipation, modes with frequency but also on the ratio of the neutral to the bulk ionization, the cut off wavelength decreases.

The normal mode behaviour of the waves are similar in both $\delta B \cdot B$ and $\delta B \cdot \hat{k} \times \hat{B}$ cases except for the $\cos \theta$ reduction factor in the later case. We note that equation (53) is identical to equation (17) of Desch (2004). The damping of the magnetic fluctuations along $\hat{n}$ is reduced by the $\cos \theta$ factor in the transverse direction.

In the limit $c_n^2 \to \infty$, we obtain the dispersion relation found by Desch (2004)) in the Boussinesq approximation ($\delta p = 0, \delta P \neq 0$).

### 3.2 Hall limit

This dispersion relation (46), acquires a familiar form (cf. Wardle & Ng 1999, Eq. 25) when wave is propagating along the ambient magnetic field ($\theta = 0$)

$$\omega^2 - i \eta F k^2 \omega - \omega_A^2 = \pm \eta_H k^2 \omega,$$

(54)

where $\eta_F = \eta_A + \eta$. In the low frequency limit $\omega \ll \omega_A$, neglecting $\omega^2$ in (54), we get

$$Re[\omega] = \pm \omega_H \frac{1}{1 + (\beta_c^{-1} + D^2 \beta_h)^2},$$

$$Im[\omega] = \omega_H \frac{\beta_c^{-1} + D^2 \beta_h}{1 + (\beta_c^{-1} + D^2 \beta_h)^2}.$$  

The modified ion-cyclotron mode, Eq. (55) has very low threshold of excitation in a weakly ionized medium since $Re[\omega] \approx \omega_H \approx 0$. The ratio of imaginary and real part of the frequency only in the presence of ambipolar diffusion ($\eta = 0$) gives

$$\frac{Im[\omega]}{Re[\omega]} = D^2 \beta_h,$$

(56)

which is same as the ratio of ambipolar to Hall term in the induction Eq. (27). Above expression can also be written in terms of Hall and Pedersen conductivities (Wardle & Ng 1999). We note that when $\beta_h \ll 1$, damping of the waves will be insignificant and system can support very low frequency ion-cyclotron modes.

In the presence of Ohmic diffusion only, above ratio is

$$\frac{Im[\omega]}{Re[\omega]} = \frac{1}{\beta_e},$$

(57)

Recall that $H/O = \beta_e$ in the induction equation Eq. (27). Therefore when $\beta_e \gg 1$, i.e. when Hall dominates Ohm, the damping of the ion-cyclotron mode is insignificant. We may conclude that in the Hall regime, i.e. when $D^2 \beta_h \ll 1 \ll \beta_e$, weakly ionized plasma can easily excite very low frequency ion-cyclotron mode which will propagate undamped in the medium.

Since the excitation threshold of modified ion-cyclotron wave is close to zero this mode will always be present in the medium, except when the direction of wave propagation is almost transverse to the ambient magnetic field, since for oblique propagation $Re[\omega] \approx \omega_H \cos \theta$. We note that the excitation of the very low frequency modified ion cyclotron mode in a weakly ionized medium is a novel feature of the Hall MHD. This feature makes it different from a highly ionized case. Therefore, in the weakly ionized medium such as dark clouds and protoplanetary discs, where $\omega_H \approx 0$, the ion-cyclotron mode is likely to exist in the medium.

In the high frequency limit $\omega_A \ll \omega$, neglecting $\omega^2$ in (54), and assuming $\eta = 0$, we get

$$Re[\omega] = \pm \frac{\omega_A^2}{\omega_H},$$
and the ratio of imaginary and real part of the frequency gives $D^2\beta_i$ implying that, the system excites low frequency ion-cyclotron and high frequency whistler waves in the system when $D^2\beta_i \ll 1$.

We note that the nature of the whistler wave in a partially ionized medium is different from the whistler in a fully ionized medium. In $\omega_H \to 0$ limit, the whistler frequency can become very high but the present single fluid description is valid only for whistler frequencies satisfying equation (20). In terms of wavelength this constraint becomes

$$\lambda \gtrsim 2\pi \left( \frac{\rho_i}{\rho_n} \right)^{1/4} \left[ \frac{1 + D\beta_i}{D\beta_e} \right]^{1/2} \left( \frac{\eta_H \cos \theta}{\nu_{ni}} \right)^{1/2}. \quad (59)$$

We note that the above expression provides a lower bound on the wavelength. In a medium such as molecular clouds, taking $D = 1$ and calculating $\eta_H$ for a mGauss field with $n_e \sim 0.1$ cm$^{-3}$, $m_i = 30m_p$, $n_n = 2.35m_p$ and $\nu_{ni} = 2.8 \times 10^{-12}$ s$^{-1}$ for $n_n \sim 10^6$ cm$^{-3}$, we get $\lambda \gtrsim 10^3$ cm. This suggests that single fluid description permits the excitation of very small wavelength fluctuations in the cloud. We may conclude that the weakly ionized interstellar medium is capable of exciting high frequency, short ($k \to \infty$) whistlers in the medium.

To summarize, both the modified ion-cyclotron and whistler waves correspond to the short wavelength limit of Eq. (54), i.e. $\omega_H \ll \omega_A$. In the long wavelength ($\omega_A \ll \omega_H$) limit, we get familiar Alfvén wave $\omega^2 \sim \omega_A^2$.

## 4 APPLICATIONS

In this section we consider the relevance of the Hall effect in fusion, space, and astrophysical plasmas. We do this by adopting typical parameters for the plasmas and examining the Hall length and time scales, and the relative magnitudes of Hall, ambipolar and Ohmic diffusivities. We also discuss its likely implications.

### 4.1 Fusion plasmas

It is well known that in the fusion devices Hall can play an important role in discharge behaviour such as the sawtooth collapse of the tokamak discharge (Wang & Bhattacharjee 1993), or, in non-Ohmic current drive schemes (Pandey et al. 1995). The Hall effect is also potentially important near the partially ionized wall region of a tokamak. Since $n_n/n_i \sim (10^{-3} - 10^{-4})$ near the wall region (Filip et al. 2001), it is clear from Eq. (29)-(31) that $\omega_H \approx \omega_{ci}$ and $L_H \approx \delta_i$, where we have assumed $m_i = m_i$. Clearly, Hall scaling is similar in both fully ionized core and partially ionized wall region of the tokamak. Therefore, Hall effect near the wall region will be important for $\omega_{ci} \lesssim \omega$ and scale comparable to the ion-inertial scale.

For typical densities $\sim 10^{14}$ cm$^{-3}$ and 10 kG field in fusion plasmas, assuming $m_i = m_p$ we get $\omega_{ci} = 10^8$ s$^{-1}$. The ion Alfvén speed is $v_{Ai} = 2.25 \times 10^8$ cm, and, the ion skin depth is $\delta_i = v_{Ai}/\omega_{ci} \sim 2.23$ cm. Adopting $10^4$ cm as the major radius of the tokamak plasma, the Alfvén frequency is $\omega_A \equiv R^{-1} V_A \sim 10^7$ s$^{-1}$. Therefore, the Hall scale $\sim$ few cm.

### 4.2 Ionospheric plasmas

An important question for magnetosphere-ionosphere coupling is the interaction between the collisionless magnetospheric plasma and the collisional ionospheric plasma. The magnetosphere is well described by the ideal or Hall MHD equations whereas the ionosphere is described by the fluid equations along with the inertial plasma determining the relationship between the current and the electric field through a generalized Ohms law. The transition between the two regions is not easily facilitated using these different approaches. In particular they mask why Hall operates at large scales in the ionosphere, and shrinks to the ion-inertial scale in the magnetosphere. The unified set of equations presented here treats the ionosphere and magnetosphere in the same framework and describes the dynamics of the transition region in a consistent fashion. The added bonus of this approach is that it explains why the Hall scale shrinks as one moves from ionosphere to the magnetosphere.

To illustrate this points and to gauge the relative importance of ambipolar, Hall, and Ohmic diffusion in the lower ionosphere, we present representative neutral mass density, collision frequencies (Akasofu & Chapman 1972; Song et al. 2001), and the corresponding Hall-beta parameters, the Hall frequency and Hall length scale in Table 1. Molecular nitrogen and oxygen are the dominant components of the lower atmosphere, thus we have adopted a mean neutral mass $m_n = 16 m_p$. Hall and Ohmic diffusion are dominant in the lower E-layer of the ionosphere. Above $\sim 100$ km, Hall diffusion is dominant $\omega_H$ is very low and the corresponding Hall scale is very large (Table 1). With increasing height, the density ratio $\rho_i/\rho$ increases and Hall length shrinks becoming of the order of ion-inertial scale when $\rho_i/\rho \sim 1$ in the magnetosphere.

We see from Table 1 that since ratio of the ambipolar (A) and Hall (H) terms are, $A/H \equiv \eta_A/\eta_H = D^2\beta_i \sim 1$, both these diffusion will operate on an equal footing towards the upper E-layer ($\sim 130$ km) and lower F-layer ($\gtrsim 150$ km), whereas Ohmic diffusion will be unimportant. Observations of the partially ionized D and E regions close to the lower boundary of the Earth’s ionosphere ($\sim 70-140$ km), reveal the permanent presence of ULF waves. In the E-region of the ionosphere, these waves have slow and fast components with phase velocities between $1-100$ m s$^{-1}$ and $2-20$ km s$^{-1}$ and frequencies between $10^{-1} - 10^{-4}$ Hz and $10^{-4} - 10^{-6}$ Hz respectively, with wavelength $\gtrsim 10^3$ km and a period of variation ranging between few days to tens of days (Zhou et al 1997; Bauer et al. 1995). Day and night time observations give an order of magnitude difference in the phase velocity.

The slow ULF waves have been identified as Alfven waves, which due to presence of neutrals, converts to whistler waves in the E-layer (Aburjania et al. 2005). We note that in the lower layer of the ionosphere (D, E and lower F-layers, $\sim 70-140$ km), the ionization mass fraction could be as low as $\sim 10^{-12}$ (Table 1). Hall MHD is applicable in this region, and for $B = 0.3$ G and an ion mass $m_i \approx m_{O^+} \sim 10^{-22}$ g, the Hall criterion is $\omega \gtrsim \omega_H = 10^{-9}$ Hz rather than the requirement $\omega \gtrsim \omega_{ci} \sim 10^8$ s$^{-1}$ that would hold if the medium were fully ionized. Thus, waves in the $10^{-1} - 10^{-4}$ Hz range are most likely whistler waves. Identifying the observed wave speed ($1-2$ km) (Aburjania at al. 2005) with the whistler, we obtain the characteristic
wavelength $\lambda \sim 10^3$ km at 130 km. Therefore, what is being identified as slow Alfvén mode converting to whistler could be just low frequency whistler mode without any mode conversion.

In fully-ionized plasmas the Hall effect considerably modifies the classical Kelvin-Helmholtz (KH) and Rayleigh-Taylor (RT) instabilities on the ion-inertial scale (Talwar & Kalra 1965). Both these instabilities grow faster in the presence of Hall effect. We anticipate that similar modifications to the KH and RT instabilities will be effective on much longer scales and that the Hall effect may facilitate the energy cascade from large to small scales in the weakly-ionized regions of the ionosphere.

### 4.3 Solar atmosphere

The potential role of the Hall effect in the solar atmosphere has escaped the attention of solar community owing to the confusion about the Hall scaling in partially ionized plasmas. The interaction between the partially ionised solar atmosphere and fully ionized corona has important consequences for the wave heating of the corona. MHD waves can be easily excited in the medium by e.g. convective gas motions. A tiny fraction of this energy carried by the waves to higher altitude would suffice to heat the corona to high temperatures (Priest 1987). However, the dynamics of the weakly ionized photosphere is dominated by collisional effects whereas highly ionized corona is described by ideal MHD. It is unclear how does one makes a transition between the collisional photosphere and collisionless corona. The present formulation not only allows us to investigate the dynamics of the collisional lower photosphere but also allows taking the proper limit to the fully ionized coronal region. As we shall see below, Hall may indeed become important in the solar photosphere. To show this, we give typical collision frequencies and Hall parameters for the solar atmosphere in Table 2 for standard solar photosphere and chromosphere models (Vennazza et al. 1981; Cox 2000). The proton-hydrogen ($H^+ - H$) elastic collision cross section is temperature-dependent and at $0.5\,\text{eV}$ is typically $2 \cdot 10^{-14}\,\text{cm}^2$ (Krstic & Schultz 1999). The electron-hydrogen ($e^- - H$) collision cross-section is also temperature-dependent and is $3.5 \cdot 10^{-15}\,\text{cm}^2$ at $0.5\,\text{eV}$ (Bedersen & Kieffer 1971; Zecca et al. 1996). Table 2 somewhat underestimates the collision frequencies as charge exchange between ionized and neutral hydrogen in the solar atmosphere has a large collision cross section $\sim 5.6 \cdot 10^{-15}\,\text{cm}^2$ (Krstic & Schultz 1999). To calculate the Hall parameters, we have assumed a magnetic field $B = 100\,\text{G}$. We note here that in the solar photosphere electron - ion collisions are as significant as plasma - neutral collisions. This could be accounted for by multiplying the plasma - neutral collision frequencies in Table 2 by two. However, this does not change the overall conclusions of this work.

Ohmic diffusion dominates in the quiet-Sun photosphere. However, given the uncertainty about the collision frequencies and strength of the magnetic field, Ohmic and Hall diffusion may be comparable at the base of the chromosphere. Then the Hall effect may significantly affect the excitation and propagation of waves in the photosphere and chromosphere. As is seen from Table 2, near the surface of the Sun, the Hall effect will operate over few meters whereas near the base of the chromospheres, the typical Hall scale may be of the order of few hundred kilometres. We note that for stronger magnetic fields the Hall scale will be considerably modified. For example, for a 1 kG field, the Hall scale ranges from tens of km near the surface of the Sun to hundreds of km as approaching the chromosphere.

Heating of the solar corona by MHD surface waves is a plausible mechanism for explaining the high coronal temperature (Priest 1987). Waves in the corona are thought to have emerged from the lower photosphere, where they may have been excited by footpoint motion of the magnetic field lines. Alfvén waves propagate to the corona and lose energy by resonance damping, and heat the plasma. However, the lower solar photosphere is comparatively cold ($T \sim 6000\,\text{K}$) and weakly ionized, and ion-neutral collisional dynamics may play an important role. We see from the values of $\beta_i$ in Table 2 that except at $h = 0$ where Ohm probably competes with Hall, Hall is the dominant diffusion mechanism in the medium. For a field $> 10^2\,\text{G}$ (a field probably present in the network, internetwork, plaque and sunspot Berger & Title (2001); Dominguez et al. (2003)) even at $h = 0$ Hall will be the dominant mechanism. Ambipolar diffusion is unimportant in the lower photosphere ($\lesssim 1000\,\text{km}$), where $\beta_i \ll 1$. At higher altitudes, typically between (1000 – 3000) km, ambipolar diffusion will be important, above this the role of ambipolar diffusion diminishes as the neutral number density plummets rapidly. We infer that Hall effect is present over the entire solar atmosphere although with shrinking spatial scales with increasing altitude and will modify the large scale wave motion in the medium.

Granulations or convective motions are believed to generate Alfvén waves in the photosphere. The Alfvén wave is a promising candidate for heating and acceleration of the solar plasma from coronal holes. High-frequency ($\sim 10^4\,\text{Hz}$) ion-cyclotron waves have been proposed as a candidate for preferential heating of the heavy ions (Kohl et al. 1998). The power spectra of horizontal photospheric motions suggest that waves with frequencies ($10^{-5} - 0.1\,\text{Hz}$) are present at a few solar radii (Cranmer & Ballegooijen 2005). Thus, it

| $h$ (km) | $\rho$ (g cm$^{-3}$) | $\rho_i/\rho_n$ | $\nu_i$ (Hz) | $\nu_n$ (Hz) | $\nu_{en}$ (Hz) | $D_i^2 \beta_i / \beta_e$ | $\omega_H$ (Hz) | $L_H$ (km) |
|----------|----------------------|----------------|-------------|-------------|----------------|----------------|-------------|--------|
| 80       | $10^{-8}$            | $10^{-12}$     | $7 \cdot 10^5$ | $10^7$      | $10^{-5}$      | $0.07$        | $10^{-10}$   | $10^7$  |
| 100      | $10^{-9}$            | $10^{-10}$     | $7 \cdot 10^3$ | $10^5$      | 0.7           | 70            | $10^{-8}$    | $10^5$  |
| 130      | $10^{-10}$           | $10^{-7}$      | $10^2$       | $3 \cdot 10^3$ | 1             | $10^3$        | $10^{-5}$    | $10^4$  |
| 150      | $10^{-11}$           | $10^{-4}$      | 30           | $8 \cdot 10^2$ | $3 \cdot 10^4$ | $10^{-2}$     | $10^2$       | $10$    |
4.4 Protoplanetary Discs

The role of Hall drift on the dynamics of protoplanetary discs (PPDs) has been investigated in several papers (W99, BT01, Sano & Stone (2002a,b); Salmeron & Wardle (2003, 2005); Pandey & Wardle (2006a)). Table 3 gives the collision frequencies, plasma Hall parameters and Hall length and time scales in a protostellar disc, for a minimum-solar-mass nebula model at 5 AU from the central star in the presence of a $10^{-2}$ G field, using the ionization fraction calculated by Wardle (2007). For these parameters, Hall diffusion is important for $\omega \gtrsim \omega_H \sim 10^{-6} - 10^{-8} \text{s}^{-1}$ comparable to the orbital frequency. Near the mid-plane of the PPDs Hall diffusion will be important whereas towards the surface of the disc, ambipolar diffusion becomes dominant. The Hall scale $L_H \sim 10^3 \text{km}$ is comparable to the disc thickness, suggesting that Hall operates over the large part of the disc.

5 SUMMARY

Weakly and fully ionized plasmas often exist side by side in nature. The transition from weakly to fully ionized plasma is typically not abrupt but occurs smoothly, and the transition from one to the other has posed a considerable theoretical difficulty, requiring separate treatments of Ohm’s law for weakly (with zero plasma inertia) and fully ionized plasmas. In this paper we have developed a consistent MHD framework for fully, partially, and weakly ionized plasmas. Our main results are as follows:

(i) Our single fluid description encompasses the continuity, momentum and induction equations (11), (13), and (27), along with simple expressions for the Ohmic, Hall and ambipolar diffusivities specified by eqs. (28). These, along with prescriptions for determining $P$ and $n_e$, specify the dynamics of a partially ionized plasma. The equations neglect electron inertia limit, and are restricted to low frequencies, i.e. $\omega \lesssim \sqrt{\rho_i/\rho_e} \beta_i \nu_{ni}/(1 + D \beta_i)$.

(ii) In a partially ionized plasma the frequency $\omega_H$ above which the Hall effect becomes important and the corresponding spatial scale $L_H$ depend on the fractional ionization, i.e.

$$\omega_H \sim \frac{\rho_i}{\rho} \omega_{ci}$$

$$L_H \sim \sqrt{\frac{\rho}{\rho_i}} \delta_i \equiv \frac{v_A}{\omega_H}.$$  \hspace{1cm} (60)

This occurs because the ions are effectively coupled to the neutrals by collisions, so that the effective ion mass is $(\rho/\rho_i) m_i$. This behaviour explains why the Hall scale is negligibly small in fully ionized plasmas and yet is large in protoplanetary discs, and the Earth’s ionosphere. It also predicts that the Hall effect plays an important role in the dynamics of the solar photosphere.

(iii) We derived a general dispersion relation and showed that when ambipolar dominates both Hall ($D^2 \beta_i \gg 1$) and Ohmic ($D^2 \beta_e \gg 1$) diffusion, only waves with wavelength larger than a cutoff value (Eq. (51)) can propagate in the medium. This cutoff wavelength depends on the ratio $\rho_i/\rho$. Low frequency modified ion-cyclotron modes ($\omega = \omega_H$) and high-frequency whistler ($\omega = \omega_A/\omega_H$) are the normal modes of the medium in the Hall regime. A weakly ionized medium, with $\beta_i << 1$ will always support these waves owing to negligible damping ($\sim D^2 \beta_i << 1$).

Finally, we emphasise that the single-fluid formulation here will prove useful in studying the coupling between the Earth’s ionosphere and magnetosphere and also between the different solar layers of solar atmosphere.

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Table 2. Same as in Table 1, but for the solar atmosphere. A 100 G magnetic field has been assumed.

| h (km) | $\rho (g \text{cm}^{-3})$ | $\rho_i/\rho_n$ | $\nu_{ci}$ (Hz) | $\nu_{ce}$ (Hz) | $D^2 \beta_i$ | $\beta_e$ | $\omega_H$ (Hz) | $L_H$ (km) |
|-------|---------------------|-----------------|-----------------|----------------|----------------|--------|---------------|-----------|
| 0     | $2.77 \times 10^{-7}$ | $10^{-4}$       | $1.6 \times 10^9$ | $1.3 \times 10^{10}$ | $10^{-4}$      | $10^{-1}$ | 10            | 10        |
| 525   | $4.87 \times 10^{-9}$ | $10^{-4}$       | $2.2 \times 10^7$ | $2 \times 10^6$   | $10^{-2}$      | 1      | 10            | 10        |
| 1000  | $5.07 \times 10^{-11}$ | $10^{-3}$       | $2.2 \times 10^5$ | $2 \times 10^6$   | 1              | 10     | 10            | 10        |
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APPENDIX A:
In this section we derive a general criteria for the validity of the single fluid description, i.e. condition under which terms $\sim v_D^2$ in Eq. (12) can be neglected. Further, we show that the validity condition of the single fluid description ensures the validity of equation (17) which allows us to write the induction equation (25). We note from Eq. (16)

$$v_D \sim \frac{\omega \rho_{iA}}{\nu_{ni} + \frac{B_0^2}{\rho}} \left(1 + D \frac{\beta_e}{\beta_i}\right), \quad (A1)$$

and the single fluid description, Eq. (13) is valid provided $\rho_i \nu_{ni} v_D^2 \ll \rho^2 \left(\nu_A^2 + c_s^2\right)$, i.e.

$$\frac{\omega}{\nu_{ni}} \leq \frac{\rho}{\sqrt{\rho_i \nu_{ni}}} \left(1 + \frac{\rho_i \nu_{ni}}{\rho} \frac{\omega}{\nu_{ni}} \right) \left(\frac{D \beta_i}{1 + D \beta_e}\right). \quad (A2)$$

In the following discussion, without loss of generality, we shall assume $D \beta_e/(1 + D \beta_e) \sim 1$. The argument is easily generalized for arbitrary values of this ratio. It is clear that condition (A2) is less restrictive than Eq. (20). However, it is not clear if induction Eq. (25) which has been derived by assuming an expression for $v_D$, Eq. (18), under condition (17) is consistent with Eq. (A2). In order to check the consistency of condition (A2) with induction Eq. (25), let us assume that condition (17) is not valid, i.e.

$$\frac{\omega}{\nu_{ni}} \geq \frac{\rho}{\rho_i}. \quad (A3)$$

Writing Eq. (22) in the bulk frame,

$$E + D \frac{v_D \times B}{c} = -\frac{\nu \times B}{\sigma} - \frac{\nabla P_e}{\rho_e} + \frac{J \times B}{\sigma} + \frac{J \times B}{c_e e_n} - \frac{m_e \nu_{en}}{e} \psi_D. \quad (A4)$$
We note that if $Dv_D \gtrsim v_A$ then equation (A4) is not valid implying invalidity of equation (25). Condition $Dv_D \gtrsim v_A$ implies
\begin{equation}
\frac{\omega}{\nu_{ni}} \gtrsim \frac{\rho_i}{\rho_n} \left( \frac{\omega}{\nu_{ni}} + \frac{\rho}{\rho_i} \right), \tag{A5}
\end{equation}
Clearly under condition (A5), we are not allowed to write the induction equation (25). We consider two cases: (i) $\rho_i \gtrsim \rho_n$ and (ii) $\rho_i < \rho_n$ and show that condition (A5) is incompatible with single fluid condition (A2). When $\rho_i \gtrsim \rho_n$, it is straightforward to see that condition (A5) is never satisfied. Thus induction Eq. (25) is valid when under a general single fluid condition, Eq. (A2). When $\rho_i < \rho_n$, we note that
\begin{equation}
\frac{\omega}{\nu_{ni}} \gtrsim \frac{\rho_i}{\rho_n} \frac{\omega}{\nu_{ni}}, \tag{A6}
\end{equation}
and
\begin{equation}
\frac{\omega}{\nu_{ni}} \gtrsim \frac{\rho_i}{\rho_n} \frac{\omega}{\nu_{ni}}. \tag{A7}
\end{equation}
Adding Eqs. (A6) and (A7), we get
\begin{equation}
\frac{\omega}{\nu_{ni}} \gtrsim \frac{\rho_i}{\rho_n} \left( \frac{\omega}{\nu_{ni}} + \frac{\rho}{\rho_i} \right). \tag{A8}
\end{equation}
Clearly, condition (A3) with $\rho_i < \rho_n$ implies that Eq. (A2) is violated. Thus we have reached a contradiction. Therefore, the induction equation (25) is valid under a more general condition, Eq. (A2).

In order to graphically sketch condition (A2), we write Eq. (A2) as an equality by multiplying right hand side by a small factor $f$. Assuming $f = 0.1 \frac{D\beta_e}{1 + D\beta_e}$, $\alpha = \sqrt{\rho_i/\rho_n}$ and $y = \omega/\nu_{ni}$, Eq. (A2) becomes
\begin{equation}
y = \frac{f \alpha (1 + \alpha^{-2})}{1 - f \alpha}. \tag{A9}
\end{equation}
We can rewrite Eq. (A2) in the following form
\begin{equation}
\frac{\omega}{\nu_{ni}} \leq \frac{f \rho_i}{\nu_{ni} \rho_n} \left( \frac{D\beta_e}{1 + D\beta_e} \right) \sqrt{\rho_i}. \tag{A10}
\end{equation}
In fig. 1, we plot $\omega/\nu_{ni}$ against $\log(\rho_i/\rho_n)$ using Eq. (A9) for $f = 0.1$ and $f = 0.5$. Both in weakly ($\rho_i \ll \rho_n$), and highly ($\rho_n \ll \rho_i$) ionized limits we see from fig. 1 that $\omega \gg \nu_{ni}$. In the weakly ionized limit, $\omega/\nu_{ni} \sim \sqrt{\rho_n/\rho_i}$ and thus when $\rho_i/\rho_n \rightarrow 0$, $\omega$ becomes arbitrary large (fig. 1). In the highly ionized limit, $\omega/\nu_{ni} \sim 1/(1 - f \rho_i/\rho_n)$. This results in arbitrary large $\omega$ when $\rho_n/\rho_i \rightarrow f$. Therefore, the dynamical frequency of a partially ionized plasma is bounded between the frequencies of weakly and fully ionized plasmas.