Asymptotic analysis of the Hermite polynomials from their differential-difference equation

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February 2, 2008

Abstract

We analyze the Hermite polynomials $H_n(x)$ and their zeros asymptotically, as $n \to \infty$. We obtain asymptotic approximations from the differential-difference equation which they satisfy, using the ray method. We give numerical examples showing the accuracy of our formulas.

Keywords: Hermite polynomials, asymptotic analysis, ray method, orthogonal polynomials, differential-difference equations, discrete WKB method.

MSC-class: 33C45 (Primary) 34E05, 34E20 (Secondary)

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1 Introduction

It would be difficult to find a more ubiquitous polynomial family than the Hermite polynomials $H_n(x)$, defined by the Rodrigues formula

$$H_n(x) = (-1)^n \exp \left( x^2 \right) \frac{d^n}{dx^n} \exp \left( -x^2 \right), \quad n = 0, 1, 2, \ldots.$$  

They appear in several problems of mathematical physics [23], the most important probably being the solution of the Schrödinger equation [6], [14]. Being the limiting case of several families of classical orthogonal polynomials [16], they are of fundamental importance in asymptotic analysis [24], [33].

The Hermite polynomials satisfy the orthogonality condition

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = \sqrt{\pi} 2^n n! \delta_{mn},$$

the differential-difference equation

$$H_{n+1} + H'_n = 2x H_n, \quad (1)$$

and the reflection formula

$$H_n(-x) = (-1)^n H_n(x). \quad (2)$$

The zeros of the Hermite polynomials have several applications, notably in Gauss’ quadrature formula for numerical integration [13], [29], [30]. Several properties and their asymptotic behavior were studied in [11], [21], [3], [4], [7], [26], [28], [31] and [32].

The asymptotic behavior of $H_n(x)$ was studied by M. Plancherel and W. Rotach in [27] using the method that now bears their name. F. W. J. Olver [25] obtained asymptotic expansions for the Hermite polynomials as a consequence of his WKB analysis of the differential equation satisfied by the Parabolic Cylinder function $D_\nu(z)$, related to $H_n(x)$ by

$$H_n(x) = 2^{\frac{n}{2}} \exp \left( \frac{x^2}{2} \right) D_n \left( \sqrt{2}x \right).$$

A similar analysis using perturbation techniques was carried on by A. Voznyuk in [34].
As an application of the results from his doctoral thesis on the multiplication-interpolation method, L. Heflinger [15] established asymptotic series for the Hermite polynomials. In [39], M. Wyman derived asymptotic formulas for $H_n(x)$ based on one of their integral representations.

In this paper we shall take a different approach and analyze the differential-difference equations that the Hermite polynomials satisfy (1) using the techniques presented in [10]. A similar method (which we may call the discrete WKB method) has been applied to the solution of difference equations [5], [8], [12], [38] and it is currently being extended [11], [35], [36], [37], to include difference equations with turning points. Another type of analysis, based on perturbation techniques, was considered by C. Lange and R. Miura in [17], [18], [19], [20], [21], and [22].

2  Asymptotic analysis

We consider the approximation

$$H_n(x) \sim \exp\left[f(x,n) + g(x,n)\right], \quad n \to \infty$$

(3)

where

$$g = o(f), \quad n \to \infty.$$  

(4)

Note that since $H_0(x) = 1$, we must have

$$f(x,0) = 0$$  

(5)

and

$$g(x,0) = 0.$$  

(6)

Using (3) in (1), we have

$$\exp\left(f + \frac{\partial f}{\partial n} + \frac{1}{2} \frac{\partial^2 f}{\partial n^2} + g + \frac{\partial g}{\partial n}\right)$$

$$+ \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right) \exp(f + g) = 2x \exp(f + g),$$

(7)

where we have used

$$f(x,n+1) = f(x,n) + \frac{\partial f}{\partial n}(x,n) + \frac{1}{2} \frac{\partial^2 f}{\partial n^2}(x,n) + \cdots.$$
Simplifying (7) and taking (4) into account we obtain, to leading order,

\[
\exp \left( \frac{\partial f}{\partial n} \right) + \frac{\partial f}{\partial x} = 2x. \quad (8)
\]

Using (8) in (7) we get

\[
\exp \left( \frac{1}{2} \frac{\partial^2 f}{\partial n^2} + \frac{\partial g}{\partial n} \right) + \frac{\partial g}{\partial x} \exp \left( -\frac{\partial f}{\partial n} \right) = 1,
\]

or, to leading order,

\[
\frac{1}{2} \frac{\partial^2 f}{\partial n^2} + \frac{\partial g}{\partial n} + \frac{\partial g}{\partial x} \exp \left( -\frac{\partial f}{\partial n} \right) = 0. \quad (9)
\]

### 2.1 The ray expansion

To solve (8) we use the method of characteristics, which we briefly review. Given the first order partial differential equation

\[
F(x, n, f, p, q) = 0,
\]

where

\[
p = \frac{\partial f}{\partial x}, \quad q = \frac{\partial f}{\partial n},
\]

we search for a solution \( f(x, n) \) by solving the system of “characteristic equations”

\[
\begin{align*}
\frac{dx}{dt} &= \frac{\partial F}{\partial p}, \quad \frac{dn}{dt} = \frac{\partial F}{\partial q}, \\
\frac{dp}{dt} &= -\frac{\partial F}{\partial x} - p \frac{\partial F}{\partial f}, \quad \frac{dq}{dt} = -\frac{\partial F}{\partial n} - q \frac{\partial F}{\partial f}, \\
\frac{df}{dt} &= p \frac{\partial F}{\partial p} + q \frac{\partial F}{\partial q},
\end{align*}
\]

where we now consider \( \{x, n, f, p, q\} \) to all be functions of the variables \( t \) and \( s \).

For (8), we have

\[
F(x, n, f, p, q) = e^q + p - 2x \quad (10)
\]
and therefore the characteristic equations are
\[
\frac{dx}{dt} = 1, \quad \frac{dn}{dt} = e^q, \quad \frac{dp}{dt} = 2, \quad \frac{dq}{dt} = 0, \quad \text{(11)}
\]

and
\[
\frac{df}{dt} = p + qe^q. \quad \text{(12)}
\]

Solving (11) subject to the initial conditions
\[
x(0, s) = s, \quad n(0, s) = 0, \quad q(0, s) = A(s), \quad \text{(13)}
\]

we obtain
\[
x = t + s, \quad n = te^A, \quad p = 2t + 2s - e^A, \quad q = A, \quad \text{(14)}
\]

where we have used
\[
0 = F|_{t=0} = e^A + p(0, s) - 2s.
\]

From (5) and (13) we have
\[
f(0, s) = 0, \quad \text{(15)}
\]

which implies
\[
0 = \frac{df}{ds}f(0, s) = \left[ \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial n} \frac{\partial n}{\partial s} \right]_{t=0}
\]
\[= p(0, s) \times 1 + q(0, s) \times 0 = 2s - e^A.
\]

Thus, \(A(s) = \ln(2s)\) and (14) becomes
\[
x = t + s, \quad n = 2ts, \quad p = 2t, \quad q = \ln(2s), \quad \text{(16)}
\]

with \(t \geq 0\) and \(s > 0\). Since \(s > 0\), we shall consider only the region \(x > 0\) for now. Using (16) in (12) and taking (15) into account, we obtain
\[
f(t, s) = t^2 + 2s \ln(2s)t. \quad \text{(17)}
\]

Solving for \(t\) and \(s\) in terms of \(x\) and \(n\) in (16), we get
\[
t = \frac{x}{2} \pm \frac{1}{2} \sigma, \quad s = \frac{x}{2} \pm \frac{1}{2} \sigma \quad \text{(18)}
\]
with
\[ \sigma = \sqrt{x^2 - 2n}. \]  (19)

For \( \sigma \) to be a real number, we shall impose the condition \( x > \sqrt{2n} \). Since (for a fixed value of \( n \)) we have \( t \to 0 \) as \( x \to \infty \), we consider the solution
\[ t = \frac{x}{2} - \frac{1}{2} \sigma, \quad s = \frac{x}{2} + \frac{1}{2} \sigma. \]  (20)

Replacing (20) in (17) we obtain
\[ f(x, n) = \frac{x^2 - \sigma x - n}{2} + n \ln (x + \sigma), \quad x > \sqrt{2n}. \]  (21)

We shall now find \( g(x, n) \). Using (21) in (9), we get
\[- \frac{1}{2 \sigma (x + \sigma)} + \frac{\partial g}{\partial n} + \frac{\partial g}{\partial x} (x + \sigma) = 0,\]
or
\[ (x + \sigma) \frac{\partial g}{\partial n} + \frac{\partial g}{\partial x} = \frac{1}{2 \sigma}. \]  (22)

Solving (22), we obtain
\[ g(x, n) = \frac{1}{2} \ln \left( -2 \frac{x^2 - n + x \sigma}{x^2 - 2n + x \sigma} \right) + C(x + \sigma), \]
where \( C(x) \) is a function to be determined. Imposing the condition (5), we have
\[ 0 = g(x, 0) = \frac{1}{2} \ln(-2) + C(2x). \]

Thus,
\[ g(x, n) = \frac{1}{2} \ln \left[ \frac{1}{2} \left( \sigma + x \right) \right]. \]  (23)

We summarize our results in the following theorem.

**Theorem 1** In the region \( x > \sqrt{2n} \), the Hermite polynomials admit the asymptotic representation
\[ H_n(x) \sim \Phi_1(x, n) = \exp \left[ \frac{x^2 - \sigma x - n}{2} + n \ln (\sigma + x) \right] \times \sqrt{\frac{1}{2} \left( 1 + \frac{x}{\sigma} \right)}, \quad n \to \infty, \]  (24)

where \( \sigma(x, n) \) was defined in (14).
Using the reflection formula we can extend our result to the region $-x > \sqrt{2n}$ and obtain:

**Corollary 2** In the region $x < -\sqrt{2n}$, the Hermite polynomials admit the asymptotic representation

$$H_n(x) \sim \Phi_2(x, n) = (-1)^n \exp \left[ \frac{x^2 + \sigma x - n}{2} + n \ln (\sigma - x) \right]$$

$$\times \sqrt{\frac{1}{2} \left( 1 - \frac{x}{\sigma} \right)}, \quad n \to \infty. \quad (25)$$

To illustrate the accuracy of our results, in Figure we graph $H_4(x)$ and the asymptotic approximations $\Phi_1(x, 4)$ and $\Phi_2(x, 4)$. 

Figure 1: A comparison of $H_4(x)$ (solid curve) and the asymptotic approximations $\Phi_1(x, 4)$ and $\Phi_2(x, 4)$ (ooo).
2.2 The transition layer

We shall now find an asymptotic approximation for $|x| \approx \sqrt{2n}$. We will consider the case $x \approx \sqrt{2n}$ and find the corresponding result for $x \approx -\sqrt{2n}$ by using (2). From (24) we have

$$
\Phi_1(x, n) \sim \exp \left[ \frac{n}{2} \ln (2n) - \frac{3}{2} n + \sqrt{2nx} \right], \quad x \rightarrow \sqrt{2n}^+.
$$

We define the function $G_n(x)$ by

$$
H_n(x) = \exp \left[ \frac{n}{2} \ln (2n) - \frac{3}{2} n + \sqrt{2nx} \right] G_n(x). \quad (26)
$$

Using (26) in (21) we get

$$
\exp \left[ \frac{n+1}{2} \ln (2n+2) - \frac{n}{2} \ln (2n) - \frac{3}{2} + \left( \sqrt{2(n+1)} - \sqrt{2n} \right) x \right] G_{n+1}
$$

$$
+ \sqrt{2n} G_n + G'_n = 2xG_n(x). \quad (27)
$$

We introduce the stretch variable $\beta > 0$ defined by

$$
x = \sqrt{2n} + \frac{\beta}{n^\frac{1}{6}} \quad (28)
$$

and the function $\Lambda(\beta)$ defined by

$$
G_n(x) = \Lambda \left[ \left( x - \sqrt{2n} \right) n^{\frac{1}{6}} \right]. \quad (29)
$$

From (28) we have

$$
\exp \left[ \frac{n+1}{2} \ln (2n+2) - \frac{n}{2} \ln (2n) - \frac{3}{2} + \left( \sqrt{2(n+1)} - \sqrt{2n} \right) x \right] \sim \sqrt{2n} + \beta n^{-\frac{1}{3}}, \quad n \to \infty. \quad (30)
$$

Using (28) in (29) we obtain

$$
G_{n+1}(x) = \Lambda \left[ \left( \sqrt{2n} - \sqrt{2n+1} + \frac{\beta}{n^\frac{1}{6}} \right) (n+1)^{\frac{1}{6}} \right]
$$

$$
\sim \Lambda(\beta) - \frac{1}{\sqrt{2}} \Lambda'(\beta) n^{-\frac{1}{3}} + \frac{1}{4} \Lambda''(\beta) n^{-\frac{2}{3}}, \quad n \to \infty, \quad (31)
$$
and
\[
\sqrt{2n}G_n + G'_n - xG_n(x) = -\sqrt{2n}\Lambda (\beta) + \Lambda' (\beta) n^{\frac{1}{2}} - 2\beta\Lambda (\beta) n^{-\frac{1}{2}}. \tag{32}
\]
Using (30), (31) and (32) in (27) we obtain, to leading order, the Airy equation
\[
\Lambda'' (\beta) = 2\sqrt{2}\beta\Lambda (\beta). \tag{33}
\]
Thus,
\[
\Lambda (\beta) = C_1 \text{Ai}\left(\sqrt{2}\beta\right) + C_2 \text{Bi}\left(\sqrt{2}\beta\right), \tag{34}
\]
where Ai (·) and Bi (·) denote the Airy functions and C_1, C_2 are to be determined. Replacing (28) and (34) in (26) we have
\[
H_n(x) \sim \exp\left[n^2 \ln (2ne) + \sqrt{2}\beta n^{\frac{1}{2}}\right] \left[C_1 \text{Ai}\left(\sqrt{2}\beta\right) + C_2 \text{Bi}\left(\sqrt{2}\beta\right)\right]. \tag{35}
\]
To find C_1, C_2 we shall match (35) with (24). Using (28) in (24) we get
\[
\Phi_1(x, n) \sim \exp\left[n^2 \ln (2ne) + \sqrt{2}\beta n^{\frac{1}{2}} - 2\frac{2\pi}{3}\beta x - \frac{4}{3}n\right] 2^{-\frac{x}{4}}\beta^{-\frac{1}{4}}n^{\frac{1}{2}}, \tag{36}
\]
as \beta \to 0. Using (28) and the well known asymptotic expansions of the Airy functions
\[
\text{Ai} (x) \sim \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{2}{3}x^\frac{3}{2}\right) x^{-\frac{1}{4}}, \quad x \to \infty
\]
\[
\text{Bi} (x) \sim \frac{1}{\sqrt{\pi}} \exp\left(\frac{2}{3}x^\frac{3}{2}\right) x^{-\frac{1}{4}}, \quad x \to \infty,
\]
in (36) we have
\[
\exp\left[n^2 \ln (2n) - \frac{3}{2}n + \sqrt{2nx}\right] = \exp\left\{n^2 \left[1 + \ln (2n)\right] + \sqrt{2}\beta n^{\frac{1}{2}}\right\} \tag{37}
\]
and
\[
\frac{C_1}{\sqrt{\pi} 2^{\frac{1}{4}} \beta^{\frac{7}{4}}} \exp\left(-\frac{2\pi}{3} \beta^{\frac{3}{2}}\right) + \frac{C_2}{\sqrt{\pi} 2^{\frac{1}{4}} \beta^{\frac{7}{4}}} \exp\left(\frac{2\pi}{3} \beta^{\frac{3}{2}}\right), \beta \to \infty. \tag{38}
\]
Matching (36) to (37) and (38), we conclude that

\[ C_1 = \sqrt{2\pi n^{\frac{1}{6}}}, \quad C_2 = 0. \] (39)

This completes the analysis. Combining the results above, we have the following result:

**Theorem 3** For \( x \approx \sqrt{2n} \), the Hermite polynomials have the asymptotic representation

\[
H_n(x) \sim \Phi_3(x, n) = \exp \left[ \frac{n}{2} \ln (2n) - \frac{3}{2} n + \sqrt{2n} x \right] \times \sqrt{2\pi n^{\frac{1}{6}}} \text{Ai} \left[ \sqrt{2} \left( x - \sqrt{2n} \right) n^{\frac{1}{6}} \right], \quad n \to \infty.
\] (40)

Use of the reflection formula (2) provides the corresponding result for \( x \approx -\sqrt{2n} \).

**Corollary 4** For \( x \approx -\sqrt{2n} \), the Hermite polynomials have the asymptotic representation

\[
H_n(x) \sim \Phi_4(x, n) = (-1)^n \exp \left[ \frac{n}{2} \ln (2n) - \frac{3}{2} n - \sqrt{2n} x \right] \times \sqrt{2\pi n^{\frac{1}{6}}} \text{Ai} \left[ -\sqrt{2} \left( x + \sqrt{2n} \right) n^{\frac{1}{6}} \right], \quad n \to \infty.
\] (41)

### 2.3 The oscillatory region

We now study the region bounded by the curve \( n = \frac{x^2}{2} \), where the zeros of \( H_n(x) \) are located. In this region, the solution is a linear combination of (24) and (25)

\[
H_n(x) \sim \Phi_5(x, n) \equiv K_1 \Phi_1(x, n) + K_2 \Phi_2(x, n), \quad n \to \infty
\]

with \( |x| < \sqrt{2n} \) and \( K_1, K_2 \) are constants to be determined. We shall require \( \Phi_5(x, n) \) to match \( \Phi_3(x, n) \) asymptotically in the local variable \( \beta \), i.e., it must satisfy the limiting condition

\[
\lim_{\beta \to 0} \Phi_5(\beta, n) = \lim_{\beta \to -\infty} \Phi_3(\beta, n).
\]
Writing \( (40) \) in terms of \( \beta \), we have
\[
\Phi_3(\beta, n) = \exp \left[ \frac{n}{2} \ln (2ne) + \sqrt{2} \beta n^{\frac{1}{3}} \right] \sqrt{2\pi n^{\frac{1}{6}}} \text{Ai} \left( \sqrt{2} \beta \right). \tag{42}
\]

Using the asymptotic formula
\[
\text{Ai} (x) \sim \frac{1}{\sqrt{\pi}} \sin \left[ \frac{2}{3} (-x)^{\frac{3}{2}} + \frac{\pi}{4} \right] (-x)^{-\frac{1}{4}}, \quad x \to -\infty
\]
in \( (42) \) we get
\[
\Phi_3(\beta, n) \sim \exp \left[ \frac{n}{2} \ln (2ne) + \sqrt{2} \beta n^{\frac{1}{3}} \right] 2^{\frac{2}{3}} n^{\frac{1}{6}}
\]
\[
\times \sin \left[ \frac{1}{3} 2^{\frac{2}{3}} (-\beta)^{\frac{3}{2}} + \frac{\pi}{4} \right] (-\beta)^{-\frac{1}{4}}, \quad \beta \to -\infty,
\]
which can be rewritten as
\[
\Phi_3(\beta, n) \sim 2^{-\frac{2}{3}} n^{\frac{1}{6}} \beta^{-\frac{1}{4}} \exp \left[ \frac{n}{2} \ln (2ne) + \sqrt{2} \beta n^{\frac{1}{3}} \right]
\]
\[
\times \left[ \exp \left( -\frac{1}{3} 2^{\frac{2}{3}} \beta^{\frac{3}{2}} \right) + i \exp \left( \frac{1}{3} 2^{\frac{2}{3}} \beta^{\frac{3}{2}} \right) \right], \quad \beta \to -\infty. \tag{43}
\]

Using \( (28) \) in \( (24) \), we have
\[
\Phi_1(\beta, n) \sim \exp \left[ \frac{n}{2} \ln (2ne) + \sqrt{2} \beta n^{\frac{1}{3}} - \frac{1}{3} 2^{\frac{2}{3}} \beta^{\frac{3}{2}} \right] 2^{-\frac{2}{3}} n^{\frac{1}{6}} \beta^{-\frac{1}{4}}, \quad \beta \to 0. \tag{44}
\]

Similarly, using \( (28) \) in \( (25) \), we obtain
\[
\Phi_2(\beta, n) \sim \exp \left[ \frac{n}{2} \ln (2ne) + \sqrt{2} \beta n^{\frac{1}{3}} + \frac{1}{3} 2^{\frac{2}{3}} \beta^{\frac{3}{2}} \right] 2^{-\frac{2}{3}} n^{\frac{1}{6}} \beta^{-\frac{1}{4}}, \quad \beta \to 0, \tag{45}
\]

where we have used
\[
(-1)^n \exp \left[ \frac{x^2 + \sigma x - n}{2} + n \ln (\sigma - x) \right] = \exp \left[ \frac{x^2 + \sigma x - n}{2} + n \ln (x - \sigma) \right].
\]

Comparing \( (43) \) with \( (44) \) and \( (45) \) we conclude that \( K_1 = 1 = K_2 \) and therefore
\[
\Phi_5(x, n) = \Phi_1(x, n) + \Phi_2(x, n). \tag{46}
\]
Since $-\sqrt{2n} < x < \sqrt{2n}$, we set
\[ x = \sqrt{2n} \sin (\theta), \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}. \] (47)

Using (47) in (19), we have
\[ \sigma = \sqrt{2n} \cos (\theta) i. \] (48)

Replacing (48) in (24), we get
\[ \exp \left[ x^2 - \sigma x - n \frac{\ln (\sigma + x)}{2} \right] = \exp \left\{ \frac{n}{2} \left[ \ln (2n) - \cos (2\theta) \right] - n \left[ \frac{1}{2} \sin (2\theta) + \theta - \frac{\pi}{2} \right] i \right\} \] (49)

and
\[ \sqrt{\frac{1}{2} \left( 1 + \frac{x}{\sigma} \right)} = \frac{\exp \left( -\frac{\theta}{2} i \right)}{\sqrt{2 \cos (\theta)}}. \] (50)

Similarly, replacing (48) in (25), we obtain
\[ (-1)^n \exp \left[ x^2 + \sigma x - n \frac{\ln (\sigma - x)}{2} \right] = \exp \left\{ \frac{n}{2} \left[ \ln (2n) - \cos (2\theta) \right] + n \left[ \frac{1}{2} \sin (2\theta) + \theta - \frac{\pi}{2} \right] i \right\} \] (51)

and
\[ \sqrt{\frac{1}{2} \left( 1 - \frac{x}{\sigma} \right)} = \frac{\exp \left( \frac{\theta}{2} i \right)}{\sqrt{2 \cos (\theta)}}. \] (52)

Using (49)–(52) in (46), we have
\[ \Phi_5 \left[ \sqrt{2n} \sin (\theta), n \right] = \sqrt{\frac{2}{\cos (\theta)}} \exp \left\{ \frac{n}{2} \left[ \ln (2n) - \cos (2\theta) \right] \right\} \] (53)
\[ \times \cos \left\{ n \left[ \frac{1}{2} \sin (2\theta) + \theta - \frac{\pi}{2} \right] + \frac{\theta}{2} \right\}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}. \]

Thus, we have proved the following:
Theorem 5 In the region $|x| < \sqrt{2n}$, the Hermite polynomials have the asymptotic representation

\[
H_n \left[ \sqrt{2n} \sin (\theta) \right] \sim \sqrt{\frac{2}{\cos (\theta)}} \exp \left\{ \frac{n}{2} \left[ \ln (2n) - \cos (2\theta) \right] \right\} \times \cos \left\{ n \left[ \frac{1}{2} \sin (2\theta) + \theta - \frac{\pi}{2} \right] + \frac{\theta}{2} \right\}, \quad n \to \infty,
\]

with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

In Figure 2 we graph

\[
H_n \left[ \sqrt{2n} \sin (\theta) \right] \exp \left\{ -\frac{n}{2} \left[ \ln (2n) - \cos (2\theta) \right] \right\}
\]

and

\[
\sqrt{\frac{2}{\cos (\theta)}} \cos \left\{ n \left[ \frac{1}{2} \sin (2\theta) + \theta - \frac{\pi}{2} \right] + \frac{\theta}{2} \right\},
\]

with $n = 20$. We only include the range $0 \leq \theta < \frac{\pi}{2}$, since both functions are even.

The same results obtained in this section were derived in [9] using a different method, based on the limit relation between the Charlier and Hermite polynomials [16].

3 Zeros

We shall now find asymptotic formulas for the zeros of the Hermite polynomials using the results from the previous section. Let’s denote by $\zeta_1^n > \zeta_2^n > \cdots > \zeta_n^n$ the zeros of $H_n (x)$, enumerated in decreasing order. Then, it follows from (54) that

\[
\zeta_k^n \sim \sqrt{2n} \sin (\tau_k^n), \quad n \to \infty
\]

where $\tau_k^n$ is a solution of the equation

\[
n \left[ \frac{1}{2} \sin (2\tau_k^n) + \tau_k^n - \frac{\pi}{2} \right] + \frac{\tau_k^n}{2} = (1 - 2k) \frac{\pi}{2}, \quad 1 \leq k \leq n.
\]
Figure 2: A comparison of the exact (solid curve) and asymptotic (ooo) values of $H_{20}(x)$ in the oscillatory region.
Solving (56) numerically and using (55) we get very good approximations of \( \zeta^2_k \). One could also solve (56) exactly (as we did in [9]) and obtain a Kapteyn series expansion for \( \tau^2_k \)

\[
\tau^2_k = \frac{\pi}{2} - \frac{\pi}{2} (4k - 1) N^{-1} - \sum_{j=1}^{\infty} \frac{1}{j} J_j \left[ (1 - N^{-1}) j \right] \sin \left( \frac{4k - 1}{N} j \pi \right), \quad (57)
\]

where \( N = 2n + 1 \) and \( J_j (\cdot) \) denotes the Bessel function of the first kind. However, (57) is difficult to analyze asymptotically. Hence, we will take a different approach and find an approximation for \( \tau^2_n \) from (56) through perturbation techniques.

We will consider two cases: \( k = O(1) \) which corresponds to the largest zeros of \( H_n (x) \) and \( k = O \left( \frac{n}{2} \right) \), related to the zeros close to \( x = 0 \).

### 3.1 Case I: \( k = O(1) \)

Replacing

\[
\tau^2_k = \frac{\pi}{2} - \sum_{i \geq 1} a_i (k) n^{-\frac{1}{4}} \quad (58)
\]

in (56) we obtain, as \( n \to \infty \)

\[
\begin{align*}
a_1 & = \frac{1}{2} \kappa^\frac{1}{3}, & a_2 & = -\frac{1}{2} \kappa^{-\frac{1}{3}}, & a_3 & = \frac{\kappa}{120}, & a_4 & = -\frac{\kappa^{-\frac{4}{3}}}{30} (\kappa^2 - 5) \\
a_5 & = \frac{\kappa^{-\frac{7}{3}}}{8400} (3\kappa^4 + 350\kappa^2 + 1400), & a_6 & = -\frac{43}{16800} \kappa, \\
a_7 & = \frac{\kappa^{-\frac{11}{3}}}{50400} (\kappa^6 + 350\kappa^4 - 980\kappa^2 - 11200), \\
a_8 & = -\frac{\kappa^{-\frac{13}{3}}}{63000} (13\kappa^6 + 475\kappa^4 + 1400\kappa^2 + 17500), \\
a_9 & = \frac{\kappa^{-\frac{15}{3}}}{67200} \kappa + \frac{43}{3494600} \kappa^3, \\
a_{10} & = -\frac{\kappa^{-\frac{17}{3}}}{1397088000} (23817\kappa^8 + 2608760\kappa^6 - 4592280\kappa^4 \\
& \quad - 51744000\kappa^2 + 664048000),
\end{align*}
\]

with

\[
\kappa (k) = 3\pi (4k - 1). \quad (60)
\]
Using (58)-(59) in (55), we get

\[ \zeta_n^k \sim \sqrt{2} \left( n^{\frac{1}{2}} - \frac{\kappa^2}{8} n^{-\frac{1}{4}} + \frac{1}{4} n^{-\frac{3}{4}} - \frac{\kappa^2 + 80}{640\kappa^2} n^{-\frac{5}{4}} \right) \]

\[ - \frac{11\kappa^2 + 3920}{179200} n^{-\frac{1}{2}} + \frac{5\kappa^4 + 96\kappa^2 + 640}{7680\kappa^2} n^{-\frac{3}{2}} \]

\[ - \frac{823\kappa^6 + 647200\kappa^4 - 2464000\kappa^2 - 2508800}{258048000\kappa^8} n^{-\frac{5}{2}} \]

\[ + \frac{3064 + 33\kappa^2}{716800} n^{-\frac{7}{2}} \right), \quad n \to \infty. \]  

\[ (61) \]

3.2 Case II: \( k = O \left( \frac{n}{2} \right) \)

We now set

\[ k = \left\lfloor \frac{n}{2} \right\rfloor + 1 - j = \frac{n}{2} - \alpha + 1 - j, \]

where \( \alpha = \frac{\text{frac}(\frac{n}{2})}{2} \) (the fractional part of \( \frac{n}{2} \)) and \( j = 0, 1, 2, \ldots \). Using (62) and

\[ \tau_n^k = \sum_{i\geq 1} b_i(j) n^{-i} \]

in (56) we obtain, as \( n \to \infty \)

\[ b_1 = \xi, \quad b_2 = -\frac{\xi}{4}, \quad b_3 = \frac{\xi}{48} \left( 3 + 16\xi^2 \right), \]

\[ b_4 = -\frac{\xi}{192} \left( 3 + 64\xi^2 \right), \quad b_5 = \frac{\xi}{3840} \left( 15 + 800\xi^2 + 1024\xi^4 \right), \]

\[ b_6 = -\frac{\xi}{15360} \left( 15 + 1600\xi^2 + 7424\xi^4 \right), \]

with

\[ \xi(j) = \frac{\pi}{4} (2j + 2\alpha - 1). \]

(65)

Using (63) - (64) in (55), we obtain

\[ \zeta_n^k \sim \sqrt{2}\xi \left( n^{-\frac{1}{2}} - \frac{1}{4} n^{-\frac{3}{2}} + \frac{3 + 8\xi^2}{48} n^{-\frac{5}{2}} - \frac{3 + 80\xi^2}{192} n^{-\frac{7}{2}} \right), \quad n \to \infty. \]

(66)

In Table 1 we compare the exact value of the positive zeros of \( H_{20}(x) \) with the approximations given by solving (56) numerically and formulas (61) and
Table 1: A comparison of the exact and approximate values for the positive zeros of $H_{20}(x)$.

| $\zeta_k^n$ | (56) | (66) | (61) |
|-------------|------|------|------|
| .24534      | .24536 | .24536 | -    |
| .73747      | .73751 | .73750 | -    |
| 1.2341      | 1.2342  | 1.2340  | -    |
| 1.7385      | 1.7387  | 1.7376  | -    |
| 2.2550      | 2.2552  | 2.2512  | 2.2592 |
| 2.7888      | 2.7892  | 2.7779  | 2.7912 |
| 3.3479      | 3.3486  | -       | 3.3492 |
| 3.9448      | 3.9456  | -       | 3.9460 |
| 4.6037      | 4.6056  | -       | 4.6055 |
| 5.3875      | 5.3939  | -       | 5.3937 |

Note that the biggest error corresponds to the larger zero, where the asymptotic approximation (54) almost breaks down.

We summarize the results of this section in the following theorem.

**Theorem 6** Letting $\zeta_1^n > \zeta_2^n > \cdots > \zeta_n^n$ be the zeros of $H_n(x)$, enumerated in decreasing order, we have:

1. 
   
   $\zeta_k^n \sim \sqrt{2} \left( n^{\frac{1}{2}} - \frac{k^2}{8} n^{\frac{1}{6}} + \frac{1}{4} n^{\frac{1}{2}} - \frac{k^2 + 80}{640k_2^3} n^{\frac{5}{6}} \right), \quad n \to \infty,$

   where $k = O(1)$ and $\kappa(k)$ was defined in (60).

2. 

   $\zeta_k^n \sim \sqrt{2} \xi \left( n^{-\frac{1}{2}} - \frac{1}{4} n^{-\frac{3}{2}} + \frac{3 + 8\xi^2}{48} n^{-\frac{5}{2}} - \frac{3 + 80\xi^2}{192} n^{-\frac{7}{2}} \right), \quad n \to \infty,$

   where $k = \frac{n}{2} - \alpha + 1 - j$, $\alpha = \frac{n}{2}$ and $\xi(j)$ was defined in (63).
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