ALMOST GLOBAL EXISTENCE FOR 4-DIMENSIONAL QUASILINEAR WAVE EQUATIONS IN EXTERIOR DOMAINS

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Abstract. This article focuses on almost global existence for quasilinear wave equations with small initial data in 4-dimensional exterior domains. The nonlinearity is allowed to depend on the solution at the quadratic level as well as its first and second derivatives. For this problem in the boundaryless setting, Hörmander proved that the lifespan is bounded below by \( \exp(c/\varepsilon) \) where \( \varepsilon > 0 \) denotes the size of the Cauchy data. Later Du, the second author, Sogge, and Zhou showed that this inequality also holds for star-shaped obstacles. Following up on the authors’ work in the 3-dimensional case, we weaken the hypothesis on the geometry and only require that the obstacle allow for a sufficiently rapid decay of local energy for the linear homogeneous wave equation. The key innovation of this paper is the use of the boundary term estimates of the second author and Sogge in conjunction with a variant of an estimate of Klainerman and Sideris, which will be obtained via a Sobolev inequality of Du and Zhou.

1. Introduction. In this article, we establish a lower bound of \( \exp(c/\varepsilon) \) on the lifespan of small-data solutions to quasilinear wave equations in 4-dimensional exterior domains with Dirichlet boundary conditions. Here \( \varepsilon \) denotes the size of the Cauchy data in a suitably chosen Sobolev norm. As the lifespan grows exponentially as the size of the initial data shrinks, the solution is said to exist almost globally. The nonlinearities that we are considering may depend on the solution \( u \) in addition to its first and second derivatives at all levels. The lifespan bound established in this article was first proved in [6] for boundaryless wave equations. A relatively recent paper [1] established the same lifespan bound for the exterior of star-shaped domains. This article relaxes the geometric assumptions to allow for domains in which there is a sufficiently rapid decay of local energy with a possible loss in \( L^2 \) regularity.

We now introduce the problem at hand. Let \( \mathcal{K} \subset \mathbb{R}^4 \) be a bounded domain with smooth boundary. Note that we shall not assume that \( \mathcal{K} \) is connected. We then examine the following quasilinear wave equation exterior to \( \mathcal{K} \):

\[
\begin{align*}
\square u(t, x) &= Q(u, u', u''), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^4 \setminus \mathcal{K}, \\
u(t, \cdot)|_{\partial \mathcal{K}} &= 0, \\
u(0, \cdot) &= f, \quad \partial_t u(0, \cdot) &= g.
\end{align*}
\]

(1.1)

Here \( \square = \partial_t^2 - \Delta \) is the d’Alembertian, and \( u' = \partial u = (\partial_t u, \nabla_x u) \) denotes the space-time gradient. Throughout this paper, \( u \) will refer to the solution to (1.1) with initial data \( f, g \). The nonlinear term \( Q \) vanishes to second-order at the origin and is linear in \( u'' \). Due to the fact that the wave equation is invariant under scaling and translations, we shall

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take $0 \in K \subseteq \{|x| < 1\}$, without a loss of generality, throughout the paper. While we shall state the lifespan bound for the scalar equation \([14]\), the methods we shall use can be easily adapted to prove almost global existence for systems of wave equations, even with multiple wave speeds.

The nonlinearity $Q$ can be expanded as

$$Q(u, u', u'') = A(u, u') + B^{\alpha\beta}(u, u')\partial_\alpha \partial_\beta u,$$

where $A(u, u')$ vanishes to second order at the origin and $B^{\alpha\beta}$ are functions which are symmetric in $\alpha, \beta$ and vanish to first order at $(0, 0)$. Here we are using the summation convention where repeated indices are implicitly summed from $0$ to $4$, $x_0 = t$, $\partial_0 = \partial_t$, and $\partial_\alpha = \partial_{x_\alpha}$ for $1 \leq \alpha \leq 4$. We will also frequently use multi-index notation, setting e.g. $\partial^\mu = \partial_{\mu_0} \partial_{\mu_1} \cdots \partial_{\mu_4}$ where $\mu = (\mu_0, \ldots, \mu_4)$.

Since we are working with small Cauchy data, the arguments used to control the quadratic terms can be easily adapted to handle the higher order terms in the Taylor expansion of $Q$ about $(0, 0, 0)$. Thus, we shall truncate $Q$ at the quadratic level. We may write

$$Q(u, u', u'') = A(u, u') + b^{\alpha\beta} u \partial_\alpha \partial_\beta u + b^{\alpha\beta}_\gamma \partial_\alpha \partial_\gamma u \partial_\beta u,$$

where $b^{\alpha\beta}$ and $b^{\alpha\beta}_\gamma$ are real constants which are symmetric in $\alpha, \beta$ and $A(u, u')$ is a quadratic form.

Solving \([14]\) requires the Cauchy data to satisfy compatibility conditions. Formally, for a solution $u \in H^m$, we write $\partial_t^k u(0, \cdot) = \psi_k(J_k f, J_k g)$, $0 \leq k \leq m$ where $J_k u = \{\partial_t^k u : 0 \leq |\mu| \leq k\}$. For $(f, g) \in H^m \times H^{m-1}$, the compatibility condition requires that the compatibility functions, $\psi_k$, vanish on $\partial K$ for all $0 \leq k \leq m - 1$. For smooth data, we require the compatibility conditions to hold for all $m$. See \([12]\) for a more detailed description of the compatibility conditions.

Our only geometric assumption on $K$ requires the local energy for solutions to the linear homogeneous wave equation with compactly supported data to decay at a sufficiently rapid, fixed, algebraic rate. We allow for a loss of $D$ derivatives. Our proof permits a loss of any fixed number of derivatives, though we do require that the local energy decay at a rate faster than $t^{-2}$. More specifically, we assume that there are fixed constants $\sigma > 0$ and $D \geq 0$ such that if $\Box u = 0$ and if the Cauchy data $u(0, \cdot), \partial_t u(0, \cdot)$ are supported on the set $\{|x| < 10\}$, then the following inequality holds

$$\|u'(t, \cdot)\|_{L^2((x \in \mathbb{R} \setminus \{K : |x| < 10\})} \lesssim (t)^{-2-\sigma} \sum_{|\mu| \leq D} \|\partial^\mu u'(0, \cdot)\|_2. \tag{1.2}$$

Here $(t) = (1 + t^2)^{1/2}$. The notation $A \lesssim B$ indicates that there is a positive unspecified constant $C$, which may change from line to line, so that $A \leq CB$. Moreover, this $C$ will implicitly be independent of any important parameters in our problem.

Local energy decay estimates such as \([1.2]\) have an extensive history. We shall only briefly describe some past results that directly relate to the problem at hand. For non-trapping obstacles, it was initially shown in $n = 3$ that no loss ($D = 0$) in the right hand side is necessary in order to obtain exponential decay. See, e.g., \([32]\). Moreover, \([34]\) showed that a loss, e.g. $D \neq 0$, must occur in \([1.2]\) when there is trapping. In the other direction, \([9, 10]\) first established examples in 3 spatial dimensions of geometries with trapping for which \([1.2]\) holds, and there have been many subsequent works in this
direction in odd dimensions. In even dimensions \(n\), it was shown in \([33]\) that the local energy decays at a rate of \(O(t^{-(n-1)})\) when there is no trapping \((D = 0)\). See also \([20]\) and \([42]\). Thus, the assumption \((1.2)\) is a weaker assumption than was made in \([1]\).

We may now state our main theorem, which shows that for Cauchy data of size \(\varepsilon\) solutions to \((1.1)\) must exist up to \(T_\varepsilon = \exp(c/\varepsilon)\) for some small constant \(c\).

**Theorem 1.1.** Let \(\mathcal{K}\) be a smooth, bounded domain for which \((1.2)\) holds, and let \(Q\) be as above. Suppose that the Cauchy data \(f, g \in C^\infty(\mathbb{R}^4 \setminus \mathcal{K})\) are compactly supported and satisfy the compatibility conditions to infinite order. Then there exist constants \(N\) and \(c\) so that if \(\varepsilon\) is sufficiently small and

\[
\sum_{|\mu| \leq N} \|\partial_\mu^N f\|_2 + \sum_{|\mu| \leq N-1} \|\partial_\mu^N g\|_2 \leq \varepsilon,
\]

then \((1.1)\) has a unique solution \(u \in C^\infty([0,T_\varepsilon] \times \mathbb{R}^4 \setminus \mathcal{K})\) where

\[
T_\varepsilon = \exp(c/\varepsilon).
\]

We are assuming here that the Cauchy data are compactly supported. It is likely that it would suffice to take the data to be small in certain weighted Sobolev norms.

The lifespan in Theorem 1.1 was first proved in \([6]\) for boundaryless wave equations in 4 dimensions. However, \([17]\) and \([19]\) demonstrate that this bound for the boundaryless case can be improved to \(T_\varepsilon \gtrsim \exp(c/\varepsilon^2)\). Thus, it may be possible that the lifespan bound presented in this paper can be improved. In the other direction, it was shown in \([36]\) and later in \([45]\) and \([44]\) that solutions to \((1.1)\) must blow up in finite time. See also \([43]\) for related blow up results for semilinear wave equations. When \(Q\) only depends on \(u'\) and \(u''\), it is well-known that solutions corresponding to sufficiently small data exist globally. See, e.g., \([14]\), \([7]\), \([41]\), \([4]\). The dependence of \(Q\) on the solution \(u\) itself instead of only its first and second order derivatives hinders many of the energy methods that are commonly employed in proving lifespan bounds.

For exterior domains, \([26]\) proved small data global existence under similar conditions on the geometry when \(n \geq 4\) and \(Q\) is independent of \(u\). The first paper establishing lifespan bounds for exterior domain problems in case \(Q\) depends on \(u\) at the quadratic level is \([2]\). They established an analog of the lifespan bound \(T_\varepsilon \gtrsim \varepsilon^{-2}\) of \([18]\) for 3 dimensional wave equations outside of a star-shaped obstacle. A subsequent paper \([1]\) proved an exterior domain analog of the almost global existence theorem of \([6]\) for star-shaped obstacles in 4 spatial dimensions. It is precisely this geometric condition that we are seeking to relax. Moreover \([28]\) shows global existence exterior to star-shaped obstacles provided \(Q_{uu}(0,0,0) = 0\), which is an exterior domain analog of another result of \([6]\).

The previous paper of the authors \([3]\) extended the result of \([2]\) to exterior domains that contain trapped rays. This paper is a follow-up to our previous work and proves the lifespan bound of \([6]\) and \([1]\) under weaker geometric assumptions.

The proof shall utilize the method of invariant vector fields \([15]\), which was adapted to the exterior domain setting in \([11]\), \([13]\) and \([24]\), \([25]\). Although this paper primarily concerns solutions in 4 spatial dimensions, many of the estimates we shall state also hold
in other dimensions. In $\mathbb{R} \times \mathbb{R}^n$, we set
\begin{equation}
Z = \{ \partial_\alpha, \Omega_{ij} = x_i \partial_j - x_j \partial_i : 0 \leq \alpha \leq n, 1 \leq i < j \leq n \},
\end{equation}
and $L = t \partial_t + r \partial_r$, where $r = |x|$. The $\Omega_{ij}$ are the generators of spatial rotations in $\mathbb{R}^n$, and $L$ is the scaling vector field. We shall frequently make use of the fact that
\[ [\square, Z] = 0, \quad [\square, L] = 2\square. \]
These vector fields are regarded as invariant since they share the property that if $\square u = 0$, then $\square Zu = 0$ and $\square Lu = 0$. In earlier works such as [14], Lorentz boosts $\Omega_{0j} = t \partial_j + x_j \partial_t$ were also included in the collection (1.5) when dealing with single-speed wave equations in the case that there is no boundary. We omit them since they seem to be compatible with neither Dirichlet boundary value problems nor with systems of wave equations with multiple wave speeds. In particular, boosts are not favorable for these sorts of problems since they have an unbounded normal component on $\partial K$ as well as an associated wave speed: $[(\partial_t^2 - c^2 \Delta), \Omega_{0j}]u \neq 0$ if $c \neq 1$. While all members of $Z$ commute nicely with $\square$, only the generators of time translations $\partial_t$ also preserve the Dirichlet boundary conditions. However, [11] demonstrated how to adapt the methods to the remaining vector fields in $Z$ using that they almost preserve the Dirichlet boundary conditions in the sense that their normal components are uniformly bounded on $\partial K$. In particular, this allows them to be handled using elliptic regularity and localized energy estimates. Although the scaling vector field has coefficients that are unbounded for large $t$, its normal component is also bounded on $\partial K$. Thus, while it may be used, we will need to employ few $L$ relative to the number of vector fields from $Z$. This method of proof was started in [13] and was further developed in [24], [22, 23] and [3].

The previous result [1] assumed that $K$ is star-shaped in order to prove useful energy and localized energy estimates for variable coefficient wave equations. The techniques used in their paper are reminiscent of [25], which reproved the earlier result of [13] without using the scaling vector field $L$. It is not clear that one can obtain analogous localized energy estimates under the weaker geometric assumption (1.2) that we are using. To make up for the lack of such localized estimates when $K$ is not star-shaped, in [3] the authors used Hörmander’s $L^1 - L^\infty$ estimate [7] that was adapted to the exterior domain setting in [13]. A higher dimensional analogue of this estimate was proven in [8], but that estimate involves Lorentz boosts and does not seem easily applicable to the problem at hand since, at this point in time, the authors are not aware of a proof that eliminates the boosts.

The main innovation of this paper is to combine the methods of [2], [24] and [16] to prove almost global existence. This includes using the estimates of [2] and [1] to obtain a useful pointwise estimate for $u$ rather than $u'$ (cf. [26] Lemma 4.2), whose role in our proof will be analogous to the role of Hörmander’s $L^1 - L^\infty$ estimate [7] in previous results such as [13], [24], [2], and [3]. Just as with Hörmander’s estimate, our new estimate also necessitates the use of the scaling vector field in our proof. In previous papers, such as [2] and [13], the star-shaped assumption allowed one to use methods similar to those of Morawetz [30] to show that the worst boundary term resulting from $L$ has a beneficial sign and, therefore, can be ignored. We shall control norms that include $L$ in a manner similar to [24, 26] by using boundary term estimates that use our local energy decay assumption (1.2).
The remainder of the article is organized as follows. In Section 2, we state our main energy and localized energy estimates. For the most part, these consist of energy and localized energy estimates combined with the Sobolev estimate used in [2] and [1] as well as the main estimates of [24, 26], which enable the use of the scaling vector field in exterior domains where $K$ is not star-shaped. In Section 3, we state our main pointwise estimates. These include a well-known application of the Sobolev embedding theorem on annuli (see [15]). In this section we also combine the estimate of [24] with an estimate that is similar to those of [16] to establish our main pointwise dispersive estimate. This is reminiscent of the estimates appearing in e.g. [15], [16], [37], [39], [4] and [5]. In Section 4, we prove almost global existence of small-data solutions to (1.1) as stated in Theorem 1.1.

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2. $L^2$ estimates.

In this section, we state the main $L^2$ estimates that we shall need in our iteration argument. These estimates largely consist of well-known energy and localized energy estimates of the solution when vector fields from the collection $\{L, Z\}$ are being applied. The most basic energy and localized energy estimate for wave equations on $\mathbb{R} \times \mathbb{R}^4$ is the following.

\begin{equation}
(2.1) \quad \sup_{t \in [0,T]} \|v'(t, \cdot)\|_2 + \sup_{R \geq 1} R^{-1/2} \|v''\|_{L^2_{t,x}([0,T] \times \{|x| < R\})} \lesssim \|v'(0, \cdot)\|_2 + \inf_{\Box v = f + g} \left( \int_0^T \|f(s, \cdot)\|_2 \, ds + \sum_{j \geq 0} \|\langle x \rangle^{1/2} g\|_{L^2_{t,x}([0,T] \times \{|x| \approx 2^j\})} \right).
\end{equation}

Most of the $L^2$ and weighted $L^2_{t,x}$ estimates for $v'$ presented in this paper will be variants of the above estimate. A version of (2.1) was first proved in [31] and was applied in many subsequent papers. See, e.g., [25, 27, 29] for versions of the estimate that are more reminiscent of (2.1) and for a more complete history. In [11, 13], such localized energy estimates were applied to prove long time existence in the exterior domain setting. Since then nearly every long time existence result for obstacle problems such as [11] has employed localized energy estimates similar to (2.1). An important consequence of (2.1) that we will apply throughout this paper is the following.

**Lemma 2.1.** Let $v \in C^\infty(\mathbb{R} \times \mathbb{R}^4)$, and assume that $v$ vanishes for large $|x|$ for every $t$. Then for $T \geq 1$, we have

\begin{equation}
(2.2) \quad \|\langle x \rangle^{-3/4} v'\|_{L^2_{t,x}([0,T] \times \mathbb{R}^4)} + \log(2 + T)^{-1/2} \|\langle x \rangle^{-1/2} v'\|_{L^2_{t,x}([0,T] \times \mathbb{R}^4)} \lesssim \|v'(0, \cdot)\|_2 + \inf_{\Box v = f + g} \left( \int_0^T \|f(s, \cdot)\|_2 \, ds + \sum_{j \geq 0} \|\langle x \rangle^{1/2} g\|_{L^2_{t,x}([0,T] \times \{|x| \approx 2^j\})} \right).
\end{equation}

The proof of this lemma is fairly straightforward. In the case where the spatial norms in the left hand side of (2.2) are taken over $\{|x| > T\}$, the left hand side of (2.2) is bounded by the first term in the left hand side of (2.1). For $|x| < T$, we decompose...
Proposition 2.3. and \[1\]. The mixed norm appearing in (2.4) is defined as
\[ (2.4) \parallel v(t, \cdot) \parallel_{L^2_x(\{x \in \mathbb{R}^d : |x| < 2\})} \lesssim \parallel v'(t, \cdot) \parallel_{L^2_x(\{x \in \mathbb{R}^d : |x| < 2\})}, \]
which allows us to control the $L^2$ norm of the solution in terms of the energy in this remaining region.

We now state the following Sobolev-type estimate of \[2\]. Ignoring the behavior near the origin, the estimate is roughly dual to estimates appearing in \[37\]. For detailed proofs the reader should consult \[2\] and \[1\].

Lemma 2.2. Let $n \geq 3$ and $h \in C_0^\infty(\mathbb{R}^n)$. Then it follows that
\[ (2.4) \parallel h \parallel_{H^{−1}} \lesssim \parallel h \parallel_{L^2_x(\{x \in \mathbb{R}^d : |x| < 2\})} + \parallel |x|^{−(n−2)/2}h \parallel_{L^1_xL^2(|x| > 1)}. \]
The mixed norm appearing in (2.4) is defined as
\[ \parallel h \parallel_{L^p_xL^q(|x| > 1)} = \left( \int_1^\infty \left[ \int_{\mathbb{R}^{n−1}} |h(r\omega)|^q \, d\omega \right]^{p/q} r^{n−1} \, dr \right)^{1/p}. \]

Using this lemma, one obtains the following proposition, which first appeared in \[2\] and \[1\].

Proposition 2.3. Let $v \in C_\infty(\mathbb{R} \times \mathbb{R}^d)$, and assume that $v$ vanishes for large $|x|$ for every $t$. Then for any $T \geq 1$, we have
\[ (2.5) \parallel v \parallel_{L^p_xL^q(|[0,T] \times \mathbb{R}^d)} + \log(2+T)^{−1/2} \parallel \langle x \rangle^{−1/2}v \parallel_{L^2_x([0,T] \times \mathbb{R}^d)} \lesssim \parallel v(0, \cdot) \parallel_{2} + \parallel \partial_t v(0, \cdot) \parallel_{H^{−1}} + \int_0^T \parallel \langle x \rangle^{−1} \Box v(s, \cdot) \parallel_{L^1_xL^2(|x| > 1)} \, ds \quad \text{and} \quad \int_0^T \parallel \Box v(s, \cdot) \parallel_{L^{4/3}(|x| < 1)} \, ds. \]

To prove Proposition 2.3 one needs only to apply (2.2) to a combination of the Riesz transforms of $u$. After doing so, applying Lemma 2.2 finishes the proof.

To handle the commutator terms that result from applying a spatial cutoff to the solution, we argue similarly but instead measure the forcing term using the last term in the right of (2.2). Indeed, by arguing as above, if $v$ solves
\[ (2.6) \begin{cases} \Box v = G, & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ v(t, x) = 0, & t \leq 0, \end{cases} \]
Proposition 2.4. Let $v(t,x)$ be a solution to (2.5) where $G(t,x) = 0$ for $|x| > 3$. Then it follows that

$$
\|v\|_{L^\infty_t L^2_x([0,T] \times \mathbb{R}^4)} + \log(2 + T)^{-1/2} \|\langle x \rangle^{-1/2} v\|_{L^\infty_t L^4_x([0,T] \times \mathbb{R}^4)} \lesssim \sum_{j=1}^{4} \|\langle x \rangle^{1/2+\delta} (\Delta^{-1} \partial_j G)\|_{L^2([0,T] \times \mathbb{R}^4)},
$$

for any $\delta > 0$. As the kernel of the operator $\Delta^{-1} \partial_j$ is $O(|x-y|^{-3})$ for $j = 1, \ldots, 4$, it follows from Young’s inequality that

$$
\|v\|_{L^\infty_t L^2_x([0,T] \times \mathbb{R}^4)} \lesssim \|G(t,\cdot)\|_2,
$$

if $G(t,x) = 0$ for $|x| > 3$ and $0 < \delta < 1/2$. Combining this inequality with (2.7), we obtain the following estimate, which appeared in [1].

Proposition 2.5. Let $v \in C^\infty(\mathbb{R} \times \mathbb{R}^4)$ be a solution to (2.5) where $G(t,x) = 0$ for $|x| > 3$. Then it follows that

$$
\|v\|_{L^\infty_t L^2_x([0,T] \times \mathbb{R}^4)} + \log(2 + T)^{-1/2} \|\langle x \rangle^{-1/2} v\|_{L^\infty_t L^4_x([0,T] \times \mathbb{R}^4)} \lesssim \|G\|_{L^\infty_t L^2_x([0,T] \times \{ |x| < 3 \})}.
$$

We shall need to take advantage of better bounds than those provided by (2.5) when the forcing term is in divergence form. The following estimate appeared in [28] and was inspired by the similar estimates of [6] and [18]. See [3] for a similar application in the 3-dimensional setting.

Proposition 2.5. Let $v \in C^\infty(\mathbb{R} \times \mathbb{R}^4)$ solve

$$
\begin{aligned}
\Box v(t,x) &= \sum_{j=0}^{4} a_j \partial_j G(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^4, \\
v(0,x) &= 0, \quad t \leq 0,
\end{aligned}
$$

where $a_j \in \mathbb{R}$. Also suppose that $v(t,x)$ vanishes for large $|x|$ for every $t$. Then it follows that

$$
\|v\|_{L^\infty_t L^2_x([0,T] \times \mathbb{R}^4)} + \log(2 + T)^{-1/2} \|\langle x \rangle^{-1/2} v\|_{L^\infty_t L^4_x([0,T] \times \mathbb{R}^4)} \lesssim \|G(0,\cdot)\|_2 + \int_0^T \|G(s,\cdot)\|_2 \, ds.
$$

Here we need only observe that $v = \sum_{j=0}^{4} a_j \partial_j v_1 - a_0 v_2$, where $\Box v_1 = G$ with vanishing initial data and $\Box v_2 = 0$ with initial data $v_2(0,x) = 0, \partial_t v_2(0,x) = G(0,x)$, and apply (2.2) and (2.5).

2.2. Energy estimates on $\mathbb{R} \times \mathbb{R}^4 \setminus \partial \mathcal{K}$. We will need pointwise in time $L^2$ energy estimates for the proof of Theorem 1.1. Since we are concerned with solutions to quasilinear wave equations, we will need estimates for solutions to perturbed wave equations. When the solution satisfies the Dirichlet boundary conditions, these estimates are well-known for the solution $v$ itself. Indeed, as the Dirichlet boundary conditions imply that $\partial_t v(t,x) = 0$ for any $x \in \partial \mathcal{K}$, such estimates follow from the standard energy integral calculations that can be found in texts such as [10] and [11].
However, when one applies a vector field from our collection \( \{ L, Z \} \), the Dirichlet boundary conditions are only preserved by time translations \( \partial_t \). Hence, it is more difficult to control the boundary terms that result from integrating by parts. A hierarchy of vector fields results. From an estimate involving only \( \partial_t \) vector fields, using elliptic regularity, an estimate for any \( \partial_{t,x} \) vector field results. Using that the coefficients of \( Z \) are \( O(1) \) on the boundary of the compact obstacle, the standard argument yields an estimate up to an error term that can be controlled by localized energy estimates for \( \partial^\mu v \). Here, however, we must permit more vector fields of the form \( \partial_t \) rather than \( Z \). Due to the large coefficient, particular care must be paid to controlling \( L \) on the boundary of \( \mathcal{K} \). Here we shall apply a cutoff to the vector field \( L \) so that the Dirichlet boundary conditions are preserved by this new vector field. The resulting commutator will only involve vector fields that are higher in the hierarchy, and the resulting error term is controlled by pointwise estimation of the solution. This yields estimates for \( L \partial^\mu v \), and estimates for \( LZ^\mu \) follow, again, via a localized energy estimate.

This adaptation of the vector field method to exterior domains originates in \([11, 12]\). In \([12]\), star-shapedness was used to control the worst boundary term that resulted when applying the scaling vector field. The method saw particular further development in \([24]\) where the idea of applying a cutoff to the scaling vector field was first used to permit more general geometries. The estimate for the resulting commutator term of \([24]\) in 3-dimensions relied on Huygens’ principle, and this method was adapted in \([26]\) to generic dimension \( n \geq 4 \).

Here we are merely gathering the estimates of \([26, 24]\), and unless otherwise specified we refer the reader there for detailed proofs.

In particular, we are interested in solutions to

\[
\begin{aligned}
\Box \gamma v(t, x) &= G(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^4 \setminus \mathcal{K}, \\
v(t, x) &= 0, \quad x \in \partial \mathcal{K}, \\
v(0, x) &= f(x), \quad \partial_t v(0, x) = g(x),
\end{aligned}
\]

where

\( \Box \gamma = (\partial_t^2 - \Delta) + \gamma^{\alpha\beta}(t, x) \partial_\alpha \partial_\beta. \)

The perturbation terms \( \gamma^{\alpha\beta} \) satisfy \( \gamma^{\alpha\beta} = \gamma^{\beta\alpha} \) as well as

\[
\| \gamma^{\alpha\beta}(t, \cdot) \|_\infty \leq \frac{\delta}{1 + t}, \quad 0 < \delta \ll 1.
\]

We shall also use the notation

\[
\| \gamma'(t, \cdot) \|_\infty = \sum_{\alpha, \beta, \mu=0}^4 \| \partial_\mu \gamma^{\alpha\beta}(t, \cdot) \|_\infty
\]

and the perturbation will be chosen so that

\[
\| \gamma'(t, \cdot) \|_{L^\infty} \leq \frac{\delta}{1 + t}, \quad 0 < \delta \ll 1.
\]

We set \( e_0(v) \) to be the energy form

\[
e_0(v) = |v|^2 + 2\gamma^{0\alpha} \partial_\alpha v \partial_\alpha v - \gamma^{\alpha\beta} \partial_\alpha v \partial_\beta v.
\]
The first estimate involves the quantity

\[ E_M(t) = E_M(v)(t) := \int_{\mathbb{R}^4 \setminus \mathcal{K}} e_0(\partial_t^M v)(t, x) \, dx. \]

Due to the fact that the vector field \( \partial_t \) preserves the Dirichlet boundary conditions specified in (2.11), standard energy methods yield the following estimate.

**Lemma 2.6.** Fix \( M = 0, 1, 2, \ldots \) and assume that the perturbation terms \( \gamma^\alpha \beta \) are as above. Suppose also that \( v \in C^\infty(\mathbb{R} \times \mathbb{R}^4 \setminus \mathcal{K}) \) solves (2.11) and vanishes for large \( |x| \) for every \( t \). Then

\[ \partial_t E_M^{1/2}(t) \lesssim \sum_{j=0}^M \left\| \Box_j \partial_t^j v(t, \cdot) \right\|_2 + \| \gamma'(t, \cdot) \|_\infty E_M^{1/2}(t). \] (2.14)

In the proof of Theorem 1.1 we shall frequently make use of the fact that

\[ e_0(v) \approx |v'|^2 \]

provided that (2.12) holds for \( \delta \) sufficiently small. From the class of estimates provided by Lemma 2.6 one can then use elliptic regularity to control \( L^2 \) norms involving \( \partial^\mu_{t,x} v' \). This approach was used in [11, 13] and [24] amongst others. See also, e.g., [35]. Specifically, we will make use of the following estimate, which as stated is from [24].

**Lemma 2.7.** For \( M, N = 0, 1, 2, \ldots \) fixed and for \( v \in C^\infty(\mathbb{R} \times \mathbb{R}^4 \setminus \mathcal{K}) \) solving (2.11) and vanishing for large \( |x| \) for every \( t \), it follows that

\[ \sum_{|\mu| \leq N} \left\| L^M \partial^\mu v'(t, \cdot) \right\|_2 \lesssim \sum_{m+j \leq N+M} \left\| L^m \partial_t^j v'(t, \cdot) \right\|_2 + \sum_{m \leq M} \| L^m \partial^\mu \Box v(t, \cdot) \|_2. \] (2.15)

To obtain useful estimates involving \( L \), a modified version of this operator, \( \tilde{L} \), shall be introduced. We set

\[ \tilde{L} = t\partial_t + \chi(x) r \partial_r, \] (2.16)

where \( \chi(x) = 0 \) for \( x \in \mathcal{K} \) and \( \chi(x) = 1 \) for \( |x| > 1 \). It should be obvious that \( \tilde{L} \) preserves the Dirichlet boundary conditions. However, due to the fact that \( \tilde{L} \) fails to commute with \( \Box \), we must be able to control the commutator terms that arise in the next estimate. We fix the quantity

\[ X_{j,M}(t, x) = X_{j,M}(v)(t, x) = \int_{\mathbb{R}^4 \setminus \mathcal{K}} e_0(\tilde{L}^M \partial_t^j v)(t, x) \, dx. \]

Using this modified energy quantity, we state the following lemma, which is essentially from [24]. See, also, [3].
Lemma 2.8. Let \( v \in C^\infty(\mathbb{R} \times \mathbb{R}^4 \setminus \mathcal{K}) \) solve (2.11) where \( \gamma^{\alpha\beta} \) are as above. If \( v(t, x) \) vanishes for large \( |x| \) for each fixed \( t \), then it follows that

\[
\partial_t X_{j,M}^{1/2}(t) \lesssim X_{j,M}^{1/2}(t) \|\gamma'(t, \cdot)\|_{\infty} + \|\tilde{L}^M \partial_t \square \gamma v(t, \cdot)\|_2 + \|L^M \partial_t \gamma^{\alpha\beta} \partial_\alpha \partial_\beta v(t, \cdot)\|_2
\]

\[
+ \sum_{m \leq M-1} \|L^m \partial_t \square v(t, \cdot)\|_2 + \sum_{m+|\mu| \leq M+j, m \leq M-1} \|L^m \partial^\mu v'(t, \cdot)\|_{L^2(|x|<1)}.
\]

To shorten notation, we have written \( L^2(\{x \in \mathbb{R}^4 \setminus \mathcal{K} : |x| < 1\}) \) as \( L^2(|x| < 1) \).

We now state our final energy estimate which involves the full collection of admissible vector fields: scaling, rotations and translations. This estimate follows from the same proof as Lemma 2.6 except that one applies the trace theorem to the resulting boundary terms. We will control the terms that arise from the boundary using (1.2) and localized energy estimates, which will appear in the next section of this paper.

Proposition 2.9. For fixed \( N, M \), set

\[
Y_{N,M}(t) = \sum_{m+|\mu| \leq N+M \atop m \leq M} \int_{\mathbb{R}^4 \setminus \mathcal{K}} c_0(L^m Z^\mu \partial^\nu v)(t, x) \, dx.
\]

Suppose that (2.12) holds for \( \delta \) sufficiently small. Also suppose that \( v(t, x) \) vanishes for large \( |x| \) for every \( t \). Then it follows that

\[
\partial_t Y_{N,M}(t) \lesssim Y_{N,M}^{1/2}(t) \sum_{m+|\mu| \leq N+M \atop m \leq M} \|\gamma L^m Z^\mu \partial^\nu v(t, \cdot)\|_2
\]

\[
+ \|\gamma'(t, \cdot)\|_{\infty} Y_{N,M}(t) + \sum_{m+|\mu| \leq N+M+2 \atop m \leq M} \|L^m \partial^\mu v'(t, \cdot)\|_{L^2(|x|<1)}^2.
\]

Just as in [3], we note that this estimate slightly deviates from the versions appearing in earlier papers [24, 26]. The important difference is that in [24, 26], one did not need to distinguish between \( Z \) and a derivative \( \partial \) in the definition of \( Y_{N,M} \). Despite this subtle difference, the proof of the above proposition is identical to the proof presented in [24].

2.3. Localized energy estimates and boundary term estimates on \( \mathbb{R} \times \mathbb{R}^4 \setminus \mathcal{K} \). This section concerns solutions to the Dirichlet-wave equation

\[
\begin{cases}
\square v(t, x) = G(t, x), & (t, x) \in \mathbb{R} \times \mathbb{R}^4 \setminus \mathcal{K}, \\
v(t, x) = 0, & x \in \partial \mathcal{K}, \\
v(t, x) = 0, & t \leq 0.
\end{cases}
\]

Here we seek to provide estimates to handle the boundary terms that arise in Lemma 2.8 and Proposition 2.9.

The first is a variant of the localized energy estimate (2.1) that holds in our exterior domains.
Proposition 2.10. Suppose that \( K \subset \{ x \in \mathbb{R}^4 \setminus \mathcal{K} : |x| < 1 \} \) satisfies (1.2) and suppose that \( v \in C^\infty(\mathbb{R} \times \mathbb{R}^4 \setminus \mathcal{K}) \) solves (2.10). Then for any fixed \( N \) and \( 0 \leq M \leq 1 \), if \( v(t,x) \) vanishes for large \(|x|\) for every \( t \), then

\[
\sum_{|\mu|+m \leq N+M \quad m \leq M} \| L^m \partial^\mu v' \|_{L^2_{t,x}([0,T] \times \{ x \in \mathbb{R}^4 \setminus \mathcal{K} : |x| < 5 \})} \lesssim \int_0^T \sum_{|\mu|+m \leq N+M+D \quad m \leq M} \| \Box L^m \partial^\mu v(s,) \|_{L^2} \, ds
\]

\[
+ \sum_{|\mu|+m \leq N+M-1 \quad m \leq M} \| \Box L^m \partial^\mu v \|_{L^2_{t,x}([0,T] \times \mathbb{R}^4 \setminus \mathcal{K})}.
\]

Proof. Using cutoffs, we split our analysis into two cases: (1) \( \Box v(t,x) = 0 \) when \(|x| < 10\), and (2) \( \Box v(t,x) = 0 \) when \(|x| > 6\).

In the former case, we apply elliptic regularity (2.15) and local energy decay (1.2) to see that

\[
\sum_{|\mu|+m \leq N+M \quad m \leq M} \| L^m \partial^\mu v' \|_{L^2_{t,x}([0,T] \times \{ x \in \mathbb{R}^4 \setminus \mathcal{K} : |x| < 5 \})} \lesssim \left( \int_0^T \sum_{|\mu|+m \leq N+M+D \quad m \leq M} \{ t-s \}^{-2-\sigma+m} \| \Box L^m \partial^\mu v(s,) \|_{L^2_{t,x}([x \in \mathbb{R}^4 \setminus \mathcal{K} : |x| < 10])} \, ds + \sum_{|\mu|+m \leq N+M-1 \quad m \leq M} \| \Box L^m \partial^\mu v(t,) \|_{L^2_{t,x}([x \in \mathbb{R}^4 \setminus \mathcal{K} : |x| < 10])} \right)^2.
\]

See [2] Lemma 2.8 for more details. Applying Young’s inequality to the first term in the right hand side of (2.21) establishes (2.20) for case (1).

To handle case (2), fix a cutoff \( \rho \in C^\infty(\mathbb{R}^4) \) where \( \rho(x) = 1 \) when \(|x| < 5\) and \( \rho(x) = 0 \) when \(|x| > 6\). Let \( w = \rho v_0 + v_r \) where \( v_0 \) solves the boundaryless wave equation \( \Box v_0 = \Box v \) with vanishing Cauchy data. We see that \( w \) solves

\[
\begin{cases}
\Box w(t,x) = -2 \nabla_x \rho(x) \cdot \nabla_x v_0(t,x) - (\Delta \rho(x))v_0(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^4 \setminus \mathcal{K}, \\
w(t,x) = 0, \quad x \in \partial \mathcal{K}, \\
w(0,x) = 0, \quad t \leq 0.
\end{cases}
\]

Note that \( v(t,x) = w(t,x) \) for \(|x| < 5\). Applying (2.21) to \( w \), integrating both sides over \([0,T]\), and using Young’s inequality, we see that
Proposition 2.11. Suppose that \( v \in C^\infty(\mathbb{R} \times \mathbb{R}^4 \setminus \mathcal{K}) \) solves (2.19) and vanishes for each \( x \in \partial \mathcal{K} \). Also suppose that \( \mathcal{K} \subset \{ |x| < 1 \} \) satisfies (1.2) and that \( v(t,x) \) vanishes for large \( |x| \) for every \( t \). Then it follows that if \( N \geq 0 \) and \( 0 \leq M \leq 1 \) are fixed, then we have the estimate

\[
\int_0^t \sum_{|\mu|+m \leq N+M} \| L^m \partial^\mu v'(s,\cdot) \|_{L^2(|x|<2)} \, ds \lesssim \int_0^t \sum_{|\mu|+m \leq N+M+D} \| L^m \partial^\mu v(s,\cdot) \|_2 \, ds
\]

\[+ \int_0^t \int_{\mathbb{R}^4 \setminus \mathcal{K}} \sum_{|\mu|+m \leq N+M+D+4} |L^m Z^t \partial^\mu v(s,y)| \, dy \, ds \frac{1}{|y|^{3/2}}.
\]

3. Pointwise Estimates. We now state the main pointwise estimates that will be used to prove Theorem 1.1. We first need a version of the Sobolev embedding theorem for annuli. See [13] for more details.

Lemma 3.1. Suppose that \( h \in C^\infty(\mathbb{R}^n) \). Then it follows that for \( R \geq 1 \),

\[
\| h \|_{L^\infty(R/2<|x|<R)} \lesssim R^{-(n-1)/2} \sum_{|\mu| \leq \frac{n}{2} + 1} \| Z^\mu h \|_{L^2(R/4<|x|<2R)}.
\]

After localizing to an annulus using a cutoff, these estimates follow from applying Sobolev embedding on \( \mathbb{R}_+ \times S^{n-1} \) and the fact that the volume element in \( \mathbb{R}^n \) in polar coordinates is \( r^{n-1} dr \).

We will also need estimates similar to those originally from [16]. Similar estimates have also appeared in [14, 37, 38, 39, 4, and 5]. Most versions of these estimates...
involves controlling $u'$ whereas our current estimate uses Lemma 2.2 to obtain a pointwise dispersive estimate for the solution $u$ itself.

Our main pointwise estimate will be a combination of the estimates of [10] and [1]. We begin with an estimate that is essentially from [4] and is strongly rooted in the preceding results of [10] and [3]. As opposed to what appeared in [4], by merely having a spatial derivative on the right, we can eliminate the $x$ dependence with the weight on the first term on the right.

**Proposition 3.2.** Suppose $n \geq 3$. Let $v \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ such that for each fixed $t$, $v(t, x)$ vanishes for $|x|$ sufficiently large. Then it follows that

$$\langle r \rangle \frac{n}{2} - 1 \langle t - r \rangle |\nabla_x v(t, x)| \lesssim \sum_{|\mu| \leq n/2 + 1} \|Z^\mu \Box v(t, \cdot)\|_2 + \sum_{|\mu| + m \leq n/2 + 1 \atop m \leq 1} \|L^m Z^\mu v'(t, \cdot)\|_2.$$  

In order to prove Proposition 3.2, we state a couple of preliminary lemmas.

**Lemma 3.3 ([10 Lemma 2.3]).** Let $v \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$. Then it follows that

$$\langle t - r \rangle |\Delta v(t, x)| \lesssim \sum_{|\alpha| + |\mu| \leq 1} |\partial^\mu Z^\alpha v(t, x)| + t \|\Box v(t, x)\|.$$  

The next estimate appeared in [4]. We state a variant of the original estimate in which only spatial derivatives are being applied to the solution in the left hand side.

**Lemma 3.4 ([4 Lemma 4.1]).** Let $v \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$. Then for $n \geq 3$, it follows that

$$\langle r \rangle^{n/2 - 1} \langle t - r \rangle |\nabla_x v(t, x)| \lesssim \sum_{|\mu| \leq n/2 + 1 \atop |\nu| = 2} \|\partial^\mu Z^\alpha v(t, x)\|_2 + \sum_{|\mu| \leq n/2 + 1} \|Z^\mu \nabla_x v(t, \cdot)\|_2.$$  

We are now ready to prove Proposition 3.2.

**Proof of Proposition 3.2.** Applying Lemma 3.3 one can see that it suffices to show that the first term in the right hand side of (3.4) is controlled by the right hand side of (3.2).

Fixing $\mu$ and letting $w = Z^\mu v$, We note that integration by parts gives

$$\sum_{i,j=1}^n \|\langle t - r \rangle \partial_i \partial_j w(t, \cdot)\|_2^2 = \sum_{i,j=1}^n \int \langle t - r \rangle^2 \partial_i \partial_j w \partial_i \partial_j w \, dx$$

$$= \sum_{i,j=1}^n \int \langle t - r \rangle^2 \partial_i^2 w \partial_j^2 w \, dx$$

$$+ 2 \sum_{i,j=1}^n \int \langle t - r \rangle \partial_i \langle t - r \rangle \partial_i w \partial_j^2 w \, dx$$

$$- 2 \sum_{i,j=1}^n \int \langle t - r \rangle \partial_j \langle t - r \rangle \partial_i w \partial_i \partial_j w \, dx.$$  

(3.5)
Applying Cauchy-Schwarz and the inequality \( ab \leq (a^2 + b^2)/2 \), we see that the right hand side of (3.7) is bounded by

\[
C \| (t-r) \Delta w(t,\cdot) \|^2_2 + C \| \nabla_x w(t,\cdot) \|^2_2 + \frac{1}{4} \sum_{i,j=1}^n \| \langle t-r \rangle \partial_i \partial_j w(t,\cdot) \|^2_2;
\]

where \( C \) is sufficiently large. Applying Lemma 3.3 to the first term in the above expression, we see that it is controlled by the right hand side of (3.2). The second term is controlled by the second term in the right hand side of (3.2). The last term can be bootstrapped back into the left hand side of (3.7).

Now we will combine Lemma 2.2 and Proposition 3.2 to prove our main pointwise estimate in \( \mathbb{R} \times \mathbb{R}^n \).

**Proposition 3.5.** Let \( v \in C^\infty(\mathbb{R} \times \mathbb{R}^n) \) where \( n \geq 3 \) such that for each fixed \( t \), \( v(t,x) \) vanishes for \( |x| \) sufficiently large. Then it follows that

\[
(r)^{n/2-1} \langle t-r \rangle |v(t,x)| \lesssim \sum_{|\mu|+m \leq n/2+1} \| L^m Z^\mu v(0,\cdot) \|_{L_x^2} + \sum_{|\mu|+m \leq n/2+1} \| L^m Z^\mu \partial_t v(0,\cdot) \|_{H^{-1}}
\]

\[
+ \langle t \rangle \sum_{|\mu| \leq n/2+1} \| \partial^\mu \Box v(t,\cdot) \|_{L^2(|x|<1)} + \langle t \rangle \sum_{|\mu| \leq n/2+1} \| |x|^{-(n-2)/2} Z^\mu \Box v(t,\cdot) \|_{L^1_t L^2_x(|x|>1)}
\]

\[
+ \int_0^t \sum_{|\mu|+m \leq n/2+1} \| L^m \partial^\mu \Box v(s,\cdot) \|_{L^2(|x|<1)} \ ds
\]

\[
+ \int_0^t \sum_{|\mu|+m \leq n/2+1} \| |x|^{-(n-2)/2} L^m Z^\mu \Box v(s,\cdot) \|_{L^1_t L^2_x(|x|>1)} \ ds.
\]

**Proof.** We proceed in a manner similar to the beginning of the proof of Theorem 2.3 in [1]. We first note that for \( 1 \leq i,j,k \leq n \) and any \( h \in C_0^\infty(\mathbb{R} \times \mathbb{R}^n) \) the following inequalities hold:

\[
\| \Delta^{-1} \partial_i h \|_2 \lesssim \| h \|_{H^{-1}(\mathbb{R}^n)},
\]

\[
\| \Omega_{ij}, \Delta^{-1} \partial_k h \|_2 \lesssim \sum_{\ell=1}^n \| \Delta^{-1} \partial_\ell h \|_2,
\]

\[
\| [L, \Delta^{-1} \partial_j] h \|_2 \lesssim \| \Delta^{-1} \partial_j h \|_2.
\]

We define

\[
v_j(t,x) = (2\pi)^{-n/2} \int \frac{\xi_j}{|\xi|^2} e^{ix \cdot \xi} \hat{\nu}(t,\xi) \ d\xi
\]
where $\hat{v}$ is the Fourier transform of $v$ in the $x$ variable. One can see that $iv = \sum_{j=1}^{n} \partial_j v_j$. We then apply Proposition 3.2 and the energy inequality to see that

\begin{equation}
\langle r \rangle^{n/2-1} \langle t-\tau \rangle |\partial_j v_j(t,x)| \lesssim \sum_{|\mu|+m \leq n/2+1 \atop m \leq 1} \|L^m Z^\mu v_j(0,\cdot)\|_2 + \langle t \rangle \sum_{|\mu| \leq n/2+1} \|Z^\mu \Box v_j(t,\cdot)\|_2
\end{equation}

\[ + \int_0^t \sum_{|\mu|+m \leq n/2+1 \atop m \leq 1} \|L^m Z^\mu \Box v_j(s,\cdot)\|_2 \ ds. \]

By our earlier observations (3.7), we can apply Lemma 2.2 to see that the right hand side of (3.8) is controlled by the right hand side of (3.6). This completes the proof. \qed

We will now extend Proposition 3.2 to wave equations in exterior domains. Although this is stated for 4-dimensional domains, it is evident that similar arguments will yield estimates for higher dimensions.

**Proposition 3.6.** Let $v \in C^\infty(\mathbb{R} \times \mathbb{R}^4 \setminus \mathcal{K})$ solve

\begin{equation}
\left\{ \begin{array}{ll}
\Box v(t,x) = G(t,x), & (t,x) \in \mathbb{R} \times \mathbb{R}^4 \setminus \mathcal{K}, \\
v(t,x) = 0, & x \in \partial \mathcal{K}, \\
v(t,x) = 0, & t \leq 0.
\end{array} \right.
\end{equation}

Also suppose that for each fixed $t$, $v(t,x)$ vanishes for $|x|$ sufficiently large. Then, for $M$ and $|\mu| = N$ fixed, the following estimate holds

\begin{equation}
\langle r \rangle \langle t-r \rangle \|L^M Z^\mu v(t,x)\|_2 \lesssim \langle t \rangle \sum_{|\nu| \leq N+M+3 \atop m \leq M} \|\langle x \rangle^{-1} L^m Z^\nu \Box v(t,\cdot)\|_{L^1_t L^2_x(|x|>1)}
\end{equation}

\[ + \int_0^t \sum_{|\nu|+m \leq N+M+3 \atop m \geq M+1} \|\langle x \rangle^{-1} L^m Z^\nu \Box v(s,\cdot)\|_{L^1_t L^2_x(|x|>1)} \ ds \]

\[ + \int_0^t \sum_{|\nu|+m \leq N+M+4 \atop m \geq M+1} \|L^m \partial^\nu v(s,\cdot)\|_{L^2_x(|x|<4)} \ ds. \]

**Proof.** When $|x| < 4$, one can apply Sobolev embedding to see that the left hand side of (3.10) is controlled by

\begin{equation}
(1+t) \sum_{|\nu|+m \leq N+M+3 \atop m \leq M} \|L^m \partial^\nu v(t,\cdot)\|_{L^2_x(|x|<4)}.
\end{equation}

For $|x| > 4$, fix a cutoff $\eta \in C^\infty(\mathbb{R}^n)$ such that $\eta(x) = 1$ for $|x| > 4$ and vanishes for $|x| < 3$. If $v_0 = \eta v$, then it follows that $v_0$ solves the boundaryless wave equation

\begin{equation}
\left\{ \begin{array}{ll}
\Box v_0 = \eta G - 2\nabla_x \eta \cdot \nabla_x v - (\Delta \eta)v, \\
v_0(t,x) = 0, & t \leq 0.
\end{array} \right.
\end{equation}
Applying the analogous Minkowski space estimate provided in (3.6), we see that we obtain the inequality

\[
\langle r \rangle \langle t - r \rangle | L^M Z^n v_0(t, x) | \lesssim \langle t \rangle \sum_{|\nu| \leq N + M + 3} \left\| \left\| x \right\|^{-1} L^m Z^n \Box v(t, \cdot) \right\|_{L^1_t L^2_x(|x| > 1)}
\]

\[
+ \int_0^t \sum_{|\nu| + m \leq N + M + 3} \left\| \left\| x \right\|^{-1} L^m Z^n \Box v(s, \cdot) \right\|_{L^1_t L^2_x(|x| > 1)} ds
\]

\[
+ \langle t \rangle \sum_{|\nu| + m \leq N + M + 4} \left\| L^m \partial^n v(t, \cdot) \right\|_{L^2(|x| < 4)}
\]

\[
+ \int_0^t \sum_{|\nu| + m \leq N + M + 4} \left\| L^m \partial^n v(s, \cdot) \right\|_{L^2(|x| < 4)} ds.
\]

Applying the Fundamental Theorem of Calculus and (2.3), we see that (3.11) and the last two terms in (3.13) are controlled by

\[
\int_0^t \sum_{|\nu| + m \leq N + M + 4} \left\| L^m \partial^n v(s, \cdot) \right\|_{L^2(|x| < 4)} ds.
\]

This completes the proof. \(\square\)

4. Proof of Theorem 1.1. We will now prove Theorem 1.1 using an iteration argument. Using standard local existence theory (see [12, Theorems 9.4, 9.5] for more details), we first note that we have a local solution on a fixed timestrip.

**Theorem 4.1.** If \( \varepsilon \) in (1.3) is sufficiently small and \( f, g \) are as in Theorem 1.1 with \( N \geq 9 \), then there is a local in time solution \( u \in C^\infty([0, 2] \times \mathbb{R}^4 \setminus \mathcal{K}) \) to (1.1) that satisfies

\[
\sup_{0 \leq t \leq 2} \sum_{|\nu| \leq N} \left\| \partial^\nu u(t, \cdot) \right\|_2 \leq C\varepsilon.
\]

Using the fact that we have a local solution on \([0, 2] \times \mathbb{R}^4 \setminus \mathcal{K}\), we will now reduce to the case where we have vanishing initial data at the expense of an additional forcing term in our nonlinear equation. This will enable us to avoid dealing with the compatibility conditions on the initial data \((f, g)\) that were mentioned earlier. To do this, we first fix a cutoff in time \( \eta \in C^\infty(\mathbb{R}) \) such that \( \eta(t) = 1 \) for \( t < 1/2 \) and \( \eta(t) = 0 \) for \( t > 1 \). If we set \( u_0 = \eta u \), then \( u_0 \) solves

\[
\Box u_0 = \eta Q(u, u', u'') + [\Box, \eta]u.
\]

Letting \( w = u - u_0 \), it is clear that solving our original equation (1.1) is equivalent to showing that \( w \) solves

\[
\begin{cases}
\Box w = (1 - \eta)Q(u_0 + w, (u_0 + w)', (u_0 + w)'') - [\Box, \eta]u, \\
w(t, x) = 0, & x \in \partial\mathcal{K}, \\
w(0, x) = \partial_t w(0, x) = 0.
\end{cases}
\]
We solve our new equation (4.2) using an iteration argument. We set the initial term
\[ w_0 \equiv 0. \]
We then recursively define \( w_k \) to solve
\[
\begin{cases}
\square w_k = (1 - \eta)Q(u_0 + w_{k-1}, (u_0 + w_{k-1})', (u_0 + w_{k})'') - \square \eta u, \\
w_k(t, x) = 0, \quad x \in \partial \Omega, \\
w_k(0, x) = \partial_t w_k(0, x) = 0.
\end{cases}
\]

We first show that our solution is bounded in an appropriate norm. To construct this
norm, we fix an integer \( N_0 \) with the property that
\[ N_0 \geq \frac{N_0 + 6D + 61}{2} + 10, \]
where \( D \) is the integer appearing in (4.1). This inequality will be used implicitly throughout
the iteration argument to control the lower order terms that result from applying the
product rule. For each \( k \), we set
\[
(3.4)
M_k(T) = \sup_{0 \leq t \leq T} t \sum_{|\mu| \leq N_0 + 6D + 60} \| \partial^\mu w_k(t, \cdot) \|_2 + \sum_{|\mu| \leq N_0 + 5D + 50} \left\| \langle x \rangle^{-3/4} \partial^\mu w_k \right\|_{L^2_t(S_T)}
+ \sup_{0 \leq t \leq T} \sum_{|\mu| \leq N_0 + 4D + 40} \| Z^\mu \partial w_k(t, \cdot) \|_2 + \log(2 + T)^{1/2} \sum_{|\nu| \leq N_0 + 1} \left\| \langle x \rangle^{-1/2} Z^\nu \partial w_k \right\|_{L^2_t(S_T)}
+ \sup_{0 \leq t \leq T} \sum_{|\mu| \leq N_0 + 3D + 30} \| L \partial^\mu w_k(t, \cdot) \|_2 + \sum_{|\nu| \leq N_0 + 2D + 20} \left\| \langle x \rangle^{-3/4} L \partial^\nu w_k \right\|_{L^2_t(S_T)}
+ \sup_{0 \leq t \leq T} \sum_{|\mu| \leq N_0 + D + 10} \| LZ^\mu \partial w_k(t, \cdot) \|_2 + \log(2 + T)^{-1/2} \sum_{|\nu| \leq N_0 + 1} \left\| \langle x \rangle^{-1/2} LZ^\nu \partial w_k \right\|_{L^2_t(S_T)}
+ \sup_{0 \leq t \leq T} \sum_{|\mu| \leq N_0} \| \langle r \rangle \| (t - r) \| \partial^\mu u_k(t, \cdot) \|_{L^\infty(S_T)}.
\]

We label the terms in \( M_k(T) \) by \( I, II, \ldots, IX \). Letting \( M_0(T) \) equal the above quantity
with \( w_k \) replaced by \( u_0 \), if we apply (4.4) with \( N = N_0 + 6D + 62 \), then it follows that
there is a uniform constant \( C_0 \) such that
\[
(4.4)
M_0(T) \leq C_0 \varepsilon.
\]
Moreover, it also follows that
\[
(4.5)
\sup_{0 \leq t \leq T} \sum_{|\mu| \leq N_0 + 6D + 62} \| \partial^\mu u_0(t, \cdot) \|_2 \leq C_0 \varepsilon.
\]
We wish to show via induction that there is a uniform constant \( C_1 \) such that
\[
(4.6)
M_k(T) \leq C_1 \varepsilon.
\]
Letting
\[
(4.7)
M_{k-1}(T) \leq C_1 \varepsilon
\]
be our induction hypothesis, we will show that this implies (4.6) for \( \varepsilon \) taken to be
sufficiently small. By induction, this will prove our desired uniform bound.
Bound for $I$: We apply (2.14) and (2.15) with $M = 0$ where the perturbation terms are set as

$$\gamma^\alpha_\beta = -(1 - \eta) \left[ b^{\alpha\beta}(u_0 + w_{k-1}) + b^{\alpha\beta\gamma}\partial_\gamma(u_0 + w_{k-1}) \right].$$

By the induction hypothesis (4.7) and the bound given to us by the local existence theorem (4.1), we know that (2.12) and (2.13) hold with $\delta$. By the induction hypothesis (4.7) and the bound given to us by the local existence theorem (4.1), we know that (2.12) and (2.13) hold with $\delta = C_1 \varepsilon$. Applying Gronwall's inequality to (2.14) and applying (1.4), we see that an application of (2.15) shows that controlling $I$ reduces to proving bounds for

$$\int_0^T \sum_{j \leq N_0 + 6D + 60} \left\| \Box \partial_j^I w_k(t, \cdot) \right\|_2 dt + \sup_{0 \leq t \leq T} \sum_{|\mu| \leq N_0 + 6D + 59} ||\partial^\mu \Box w_k(t, \cdot)||_2.$$

It is clear that

$$\sum_{j \leq N_0 + 6D + 60} \left| \Box \partial_j^I w_k \right| \lesssim \sum_{|\mu| \leq N_0} \left| \partial^\mu (u_0 + w_{k-1}) \right|$$

$$\times \left[ \sum_{|\mu| \leq N_0 + 6D + 60} \left| \partial^\mu (u_0 + w_k) \right| + \sum_{|\mu| \leq N_0 + 6D + 62} \left| \partial^\mu u_0 \right| \right]$$

$$+ \sum_{|\mu| \leq N_0 - 1} \left| \partial^\mu (u_0 + w_k) \right| \sum_{|\mu| \leq N_0 + 6D + 60} \left| \partial^\mu (u_0 + w_{k-1}) \right|$$

$$+ \sum_{|\mu| \leq N_0 - 1} \left| \partial^\mu (u_0 + w_{k-1}) \right| \sum_{|\mu| \leq N_0 + 6D + 60} \left| \partial^\mu (u_0 + w_{k-1}) \right|$$

$$+ \left| u_0 + w_{k-1} \right|^2 + \sum_{|\mu| \leq N_0 + 6D + 60} \left| \partial^\mu [\Box, \eta] u \right|.$$

By (4.10) and (4.5), it follows that the first term in (4.9) can be controlled using terms $I, III$ and $IX$ in (1.8). It follows from (4.11) that

$$\int_0^T \sum_{j \leq N_0 + 6D + 60} \left\| \Box \partial_j^I w_k(t, \cdot) \right\|_2 dt \lesssim (M_0(T) + M_k(T))(M_0(T) + M_{k-1}(T)) \int_0^T (1 + t)^{-1} dt$$

$$+ (M_0(T) + M_{k-1}(T))^2 \int_0^T (1 + t)^{-1} dt + \varepsilon.$$

A similar argument shows that the second term in (4.9) is also controlled by the right hand side of (4.11). It follows that

$$I \leq C(M_0(T) + M_{k-1}(T) + M_k(T))(M_0(T) + M_{k-1}(T)) \log(2 + T) + C_2 \varepsilon,$$

where $C_2$ is a constant that is chosen to be sufficiently large relative to the size of the constant $C$ in (4.1). For the remainder of this proof, we will allow $C_2$ to vary from line to line, but it will be clear from the proof that this constant is independent of important parameters such as $k, \varepsilon, T, C_1$.

Bound for $II$: Fix a cutoff $\chi \in C_\infty_c (\mathbb{R}^4)$ such that $\chi(x) = 1$ for $|x| < 3$ and zero for $|x| > 4$. Fixing the multi-index $\mu$, we will first consider $(1 - \chi) \partial^\mu w_k$. We see that this solves the boundaryless wave equation

$$\Box(1 - \chi) \partial^\mu w_k = (1 - \chi) \partial^\mu \Box w_k - [\Box, \chi] \partial^\mu w_k.$$
with vanishing Cauchy data. We apply (2.2) to \((1 - \chi)\partial^\mu w_k\) to see that

\[
(4.14) \quad II \lesssim \int_0^T \sum_{|\mu| \leq N_0 + 6D + 50} \|\partial^\mu \Box w_k(s, \cdot)\|_2 \, ds + \sum_{|\mu| \leq N_0 + 5D + 50} \|\partial^\mu w_k\|_{L^2_{\tau,x}(S_T \cap \{|x| < 4\})},
\]

where we have also applied (2.3) to the commutator term in (4.13). If we apply (2.20) to the second term on the right hand side of (4.14), we see that controlling the second term in (4.15) can be controlled using a simpler argument, it follows that the first term in (4.15) can be controlled using (4.1) and I, III and IX in (4.3):

\[
(4.15) \quad \int_0^T \sum_{|\mu| \leq N_0 + 6D + 50} \|\partial^\mu \Box w_k(s, \cdot)\|_2 \, ds + \sum_{|\mu| \leq N_0 + 5D + 49} \|\partial^\mu \Box w_k\|_{L^2_{\tau,x}(S_T)}.
\]

Using the fact that

\[
\sum_{|\mu| \leq N_0 + 6D + 50} \|\Box \partial^\mu w_k\| \lesssim \sum_{|\mu| \leq N_0} |\partial^\mu (u_0 + w_{k-1})| \sum_{|\mu| \leq N_0 + 6D + 50} |\partial^\mu (u_0 + w_k)'|
\]

\[
+ \sum_{|\mu| \leq N_0 - 1} |\partial^\mu (u_0 + w_{k-1})'| \sum_{|\mu| \leq N_0 + 6D + 50} |\partial^\mu (u_0 + w_k)'|
\]

\[
+ \sum_{|\mu| \leq N_0 - 1} |\partial^\mu (u_0 + w_{k-1})'| \sum_{|\mu| \leq N_0 + 6D + 50} |\partial^\mu (u_0 + w_k)'|
\]

\[
+ |u_0 + w_{k-1}|^2 + \sum_{|\mu| \leq N_0 + 5D + 50} |\partial^\mu [\Box, \eta] u|,
\]

it follows that the first term in (4.15) can be controlled using (4.1) and I, III and IX in (4.3):

\[
(4.16) \quad \sum_{|\mu| \leq N_0 + 6D + 50} \|\partial^\mu \Box w_k(s, \cdot)\|_2 \lesssim \sum_{|\mu| \leq N_0} |\partial^\mu (u_0 + w_{k-1})| \sum_{|\mu| \leq N_0 + 6D + 50} |\partial^\mu (u_0 + w_k)'|
\]

\[
+ \sum_{|\mu| \leq N_0 - 1} |\partial^\mu (u_0 + w_{k-1})'| \sum_{|\mu| \leq N_0 + 6D + 50} |\partial^\mu (u_0 + w_k)'|
\]

\[
+ \sum_{|\mu| \leq N_0 - 1} |\partial^\mu (u_0 + w_{k-1})'| \sum_{|\mu| \leq N_0 + 6D + 50} |\partial^\mu (u_0 + w_k)'|
\]

\[
+ |u_0 + w_{k-1}|^2 + \sum_{|\mu| \leq N_0 + 5D + 50} |\partial^\mu [\Box, \eta] u|,
\]

Since the second term in (4.15) can be controlled using a simpler argument, it follows that

\[
(4.17) \quad \int_0^T \sum_{|\mu| \leq N_0 + 6D + 51} \|\partial^\mu \Box w_k(s, \cdot)\|_2 \, ds \lesssim (M_0(T) + M_k(T))(M_0(T) + M_{k-1}(T)) \int_0^T (1 + t)^{-1} \, dt
\]

\[
+ (M_0(T) + M_{k-1}(T))^2 \int_0^T (1 + t)^{-1} \, dt + \varepsilon.
\]

Since the second term in (4.15) can be controlled using a simpler argument, it follows that

\[
(4.18) \quad II \leq C(M_0(T) + M_{k-1}(T) + M_k(T))(M_0(T) + M_{k-1}(T)) \log(2 + T) + C_2 \varepsilon.
\]

**Bound for III and IV**, \(|\nu| = 0\): To control most of these terms, we will be applying Proposition 2.3. The only case in which this method will not work is the instance where the full number of vector fields is applied to the terms with second-order derivatives, which does not fit nicely with the weighted \(L^2_{\tau,x}\) spaces that are specified in (4.3). We overcome this difficulty by expressing these terms in divergence form and applying Proposition 2.5.

Due to (2.3) and the fact that

\[
\|Z w_k(t, \cdot)\|_{L^2_{\tau,x}(\{x \in \mathbb{R}^4 \setminus \mathcal{K} : |x| < 4\})} \lesssim \|w'_k(t, \cdot)\|_2,
\]

on the set \(|x| < 4\), terms III and IV are controlled by I and II. Thus, to complete the bound for terms III and IV, it suffices to control \((1 - \chi)Z^\mu w_k\), where \(|\mu| \leq N_0 + 4D + 40\) is
fixed and \( \chi \) is the same cutoff function that was defined earlier. We note that \( (1 - \chi)Z^\mu w_k \) solves the boundaryless wave equation,

\[
(4.19) \quad \Box (1 - \chi)Z^\mu w_k = (1 - \chi)Z^\mu \Box w_k - [\Box, \chi]Z^\mu w_k,
\]

where we note that the second term in the right hand side is supported on the set \( \{|x| < 4\} \). We further rewrite the first term as

\[
(4.20) \quad (1 - \chi)Z^\mu \Box w_k = (1 - \eta)(1 - \chi)\partial_\nu [(1 - \eta)(1 - \chi)(b^{\alpha\beta}(u_0 + w_{k-1})Z^\mu \partial_\beta(u_0 + w_k) + b^{\alpha\gamma}_\nu \partial_\nu (u_0 + w_{k-1})Z^\mu \partial_\gamma (u_0 + w_k)) + G_{k,\mu}(t, x),
\]

where \( G_{k,\mu}(t, x) \) contains no terms that have more than a total of \(|\mu| + 1\) derivatives and vector fields applied to them. Applying (2.10) and (2.5) to the first and second terms in (4.20), respectively, and (2.8) to the commutator term in the right hand side of (4.19) we see that

\[
(4.21) \quad \sup_{t \in [0, T]} \sum_{|\mu| \leq N_0 + 4D + 40} \| (1 - \chi)Z^\mu w_k(t, \cdot) \|_2^2 + \sum_{|\mu| \leq N_0 + 4D + 40} \log(2 + T)^{-1/2} \| \langle x \rangle^{-1/2} (1 - \chi)Z^\mu w_k(t, \cdot) \|_{L^2_x(S_T)} \leq \int_0^T \sum_{|\mu| \leq N_0 + 4D + 40} \| \langle x \rangle^{-1} G_{k,\mu}(s, \cdot) \|_{L^1_x L^2_y(|x| > 3)} ds
\]

\[
+ \int_0^T \sum_{|\mu| \leq N_0 + 4D + 40} \sum_{|\nu| \leq 1} \| \partial_\nu^\mu (u_0 + w_{k-1}) \|_{L^2_y} \| Z^\mu (u_0 + w_k) \|_2 ds + \sum_{|\mu| \leq N_0 + 4D + 41} \| \partial^\mu w_k^\nu \|_{L^2_x(S_T \cap \{|x| < 4\})}.
\]

The last term in the right hand side of (4.21) is controlled by term II, whose bounds we established earlier. To control the first term in the right hand side of (4.21), we see that

\[
(4.22) \quad \sum_{|\mu| \leq N_0 + 4D + 40} |G_{k,\mu}| \lesssim \sum_{|\mu| \leq N_0} |Z^\mu (u_0 + w_{k-1})| \sum_{|\mu| \leq N_0 + 4D + 40} |Z^\mu (u_0 + w_k)|
\]

\[
+ \sum_{|\mu| \leq N_0} |Z^\mu (u_0 + w_k)| \sum_{|\nu| \leq N_0 + 4D + 40} |Z^\nu \partial_\nu (u_0 + w_{k-1})|
\]

\[
+ \sum_{|\mu| \leq N_0} |Z^\mu (u_0 + w_{k-1})| \sum_{|\nu| \leq N_0 + 4D + 40} |Z^\nu \partial_\nu (u_0 + w_{k-1})|
\]

\[
+ \sum_{|\mu| \leq N_0 + 4D + 40} |\partial^\mu [\Box, \eta] u|.
\]
Applying (4.1), Cauchy-Schwarz, and Sobolev embedding on $S^3$, we see that

\begin{equation}
(4.23) \quad \int_0^T \sum_{|\alpha| \leq N_0+4D+40} \left\| \langle x \rangle^{-1} G(x, \cdot) \right\|_{L^2_t L^2_x(|x|>2)} \, ds \lesssim \varepsilon
\end{equation}

\begin{align*}
&+ \sum_{|\theta| \leq N_0+2} \left\| \langle x \rangle^{-1/2} Z^\theta(u_0 + w_{k-1}) \right\|_{L^2_t(L^2_x(S_t))} \sum_{|\theta| \leq N_0+4D+40} \left\| \langle x \rangle^{-1/2} Z^\theta(u_0 + w_k) \right\|_{L^2_t(L^2_x(S_t))} \\
&+ \sum_{|\theta| \leq N_0+2} \left\| \langle x \rangle^{-1/2} Z^\theta(u_0 + w_k) \right\|_{L^2_t(L^2_x(S_t))} \sum_{|\theta| \leq N_0+4D+40} \left\| \langle x \rangle^{-1/2} Z^\theta \partial^\lambda(u_0 + w_{k-1}) \right\|_{L^2_t(L^2_x(S_t))} \\
&+ \sum_{|\theta| \leq N_0+2} \left\| \langle x \rangle^{-1/2} Z^\theta(u_0 + w_{k-1}) \right\|_{L^2_t(L^2_x(S_t))} \sum_{|\theta| \leq N_0+4D+40} \left\| \langle x \rangle^{-1/2} Z^\theta \partial^\lambda(u_0 + w_k) \right\|_{L^2_t(L^2_x(S_t))}.
\end{align*}

Each factor in the right hand side can be bounded in terms of $\log(2 + T)^{1/2} IV$. This implies that the right hand side of (4.23) is controlled by

$$C(M_0(T) + M_{k-1}(T) + M_k(T))(M_0(T) + M_{k-1}(T)) \log(2 + T) + C_2 \varepsilon.$$}

The second term in the right hand side of (4.21) can be bounded using simpler arguments similar to those used to bound term $I$. Hence it follows that

$$III|_{|\nu|=0} + IV|_{|\nu|=0} \leq C(M_0(T) + M_{k-1}(T) + M_k(T))(M_0(T) + M_{k-1}(T)) \log(2 + T) + C_2 \varepsilon.$$}

**Bound for III and IV, $|\nu| = 1$:** As with the previous section, we will only concern ourselves with controlling these terms away from the boundary. We fix the same cutoff $\chi$ and observe that for $\{|x| < 4\}$, the coefficients of $Z$ are bounded. Thus, near the obstacle, we see that these terms can be controlled by terms $I$ and $II$, and the previously established bounds for these pieces can be applied. It then suffices to consider $(1 - \chi)Z^\mu \partial w_k$, where the multi-index $\mu$ is fixed. Applying (2.1) and (2.2), we see that

\begin{equation}
(4.24) \quad \sup_{t \in [0, T]} \left\| Z^\mu \partial w_k(t, \cdot) \right\|_{L^2(|x|>3)} + \log(2 + T)^{-1/2} \left\| \langle x \rangle^{-1/2} Z^\mu \partial w_k \right\|_{L^2_t(L^2_x(S_t \cap \{|x|>3\}))} \lesssim \sum_{|\theta| \leq N_0+4D+40} \int_0^T \left\| Z^\theta \Box w_k(s, \cdot) \right\|_2 \, ds + \sum_{|\theta| \leq N_0+4D+40} \left\| \partial^\theta w_k \right\|_{L^2_t(L^2_x(S_t \cap \{|x|<4\}))}.
\end{equation}
As stated earlier, the last term in (4.24) is controlled by term $II$. To control the first term, we apply an argument similar to the one used to bound term $I$ to see that

$$(4.25) \int_0^T \sum_{|\theta| \leq N_0} \|Z^\theta \square w_k(s, \cdot)\|_2 \, ds \lesssim$$

$$\int_0^T \sum_{|\theta| \leq N_0} \|Z^\theta (u_0 + w_{k-1})(s, \cdot)\|_\infty \sum_{\lambda \leq N_0+4D+40} \|Z^\theta \partial^\lambda (u_0 + w_k)(s, \cdot)\|_2 \, ds$$

$$+ \int_0^T \sum_{|\theta| \leq N_0-1} \|Z^\theta (u_0 + w_k)(s, \cdot)\|_\infty \sum_{\lambda \leq N_0+4D+40 \lambda \neq \lambda_k} \|Z^\theta (u_0 + w_{k-1})(s, \cdot)\|_2 \, ds$$

$$+ \int_0^T \sum_{|\theta| \leq N_0} \|Z^\theta (u_0 + w_{k-1})(s, \cdot)\|_\infty \sum_{\lambda \leq N_0+4D+40 \lambda \neq \lambda_k} \|Z^\theta \partial^\lambda (u_0 + w_{k-1})(s, \cdot)\|_2 \, ds$$

$$+ \sup_{t \in [0,2]} \sum_{|\theta| \leq N_0+4D+43} \|\partial^\theta \square [\eta] u(t, \cdot)\|_2.$$

Using (4.1) and terms $III$ and $IX$ in (4.13), we see that the right hand side is controlled by

$$C(M_0(T) + M_{k-1}(T) + M_k(T))(M_0(T) + M_{k-1}(T)) \log(2 + T) + C_2 \varepsilon.$$

**Bound for $III$, $|\nu| = 2$:** Just as in controlling term $I$, we choose $\gamma$ as before so that (2.12) and (2.13) are satisfied with $\delta = C_1 \varepsilon$. We then apply (2.18) and integrate in the $t$ variable. After applying (2.18), (1.4) and Gronwall’s inequality, we see that it suffices to control

$$(4.26) \int_0^T \sum_{|\mu| \leq N_0+4D+40} \|\partial^\mu \square Z^\gamma \partial^\lambda w_k(s, \cdot)\|_2 \, ds + \sum_{|\mu| \leq N_0+4D+42} \|\partial^\mu \square w_k\|_{L^2_{t,x}(S_T \cap \{|x| < 2\})}.$$
The second term in (4.26) is controlled using the same argument used to bound term \( II \). Thus, it remains to control the first term. We see that

\[
(4.27) \quad \sum_{|\mu| \leq N_0 + 4D + 40, |\lambda| = 1} |\Box_\gamma Z^\mu \partial^\lambda w_k| \lesssim \sum_{|\mu| \leq N_0} |Z^\mu (u_0 + w_{k-1})| \\
\times \left( \sum_{|\mu| \leq N_0 + 4D + 40, |\lambda| \leq 1} |Z^\mu \partial^\lambda (u_0 + w_k)'| + \sum_{|\mu| \leq N_0 + 4D + 43} |Z^\mu u_0| \right) \\
+ \sum_{|\mu| \leq N_0 - 1} |Z^\mu (u_0 + w_k)'| \sum_{|\mu| \leq N_0 + 4D + 40, |\lambda| \leq 2} |Z^\mu \partial^\lambda (u_0 + w_{k-1})| \\
+ \sum_{|\mu| \leq N_0} |Z^\mu (u_0 + w_{k-1})| \sum_{|\mu| \leq N_0 + 4D + 40, |\lambda| \leq 2} |Z^\mu \partial^\lambda (u_0 + w_{k-1})| \\
+ \sum_{|\mu| \leq N_0 + 4D + 41} |\partial^\mu [\Box, \eta] u|.
\]

By arguments similar to those that were used to obtain (4.11), we can control the first term in (4.26) with (4.27), (4.1) and terms \( III \) and \( IX \) from (4.3). This gives us the bound

\[
(4.28) \quad \int_0^T \sum_{|\mu| \leq N_0 + 4D + 40, |\lambda| = 1} \left\| \Box_\gamma Z^\mu \partial^\lambda w_k(s, \cdot) \right\|_2^2 ds \\
\lesssim (M_0(T) + M_k(T) + M_{k-1}(T))(M_0(T) + M_{k-1}(T)) \int_0^T (1 + t)^{-1} dt + \varepsilon.
\]

From this, it follows that

\[
(4.29) \quad III|_{|\mu|=2} \leq C(M_0(T) + M_{k-1}(T) + M_k(T))(M_0(T) + M_{k-1}(T)) \log(2 + T) + C_2 \varepsilon.
\]

**Bound for \( V \):** In previous papers, such as [11] and [28], the authors were able to avoid using \( L \) in their iteration arguments. By assuming \( K \) is star-shaped, the authors of [11] and [28] were able to employ localized energy estimates for variable coefficient wave equations enabled them to avoid using \( L \) in their existence arguments. However, the proof of these localized estimates relies explicitly on \( K \) being star-shaped to control the boundary terms that arise from integrating by parts. At this time, the authors are not aware of a proof of these estimates that relies solely on weaker geometric assumptions, such as (1.2). Instead we follow an approach similar to [9] and [24, 26], which relies on (2.23).
We use our elliptic regularity estimate (2.15) to see that controlling $V$ reduces to proving bounds for

$$
\sum_{j+m \leq N_0+3D+31 \atop m \leq 1} \left\| L^m \partial^j_t w_k(t, \cdot) \right\|_2 + \sum_{|\mu|+m \leq N_0+3D+30 \atop m \leq 1} \left\| L^m \partial^\mu \Box w_k(t, \cdot) \right\|_2,
$$

for $0 \leq t \leq T$. Since the energy estimates we have stated in this paper involve the modified scaling vector field defined in (2.16), we note that the first term is controlled by

$$
\sum_{j \leq N_0+3D+30} \left\| \tilde{L} \partial^j_t w_k(t, \cdot) \right\|_2 + \sum_{|\mu| \leq N_0+3D+31} \left\| \partial^\mu w_k(t, \cdot) \right\|_2.
$$

The bounds for $I$ can be cited to handle the second term in (4.30). It follows that we can further reduce controlling term $V$ to estimating

$$
\sum_{j \leq N_0+3D+30} \left\| \tilde{L} \partial^j_t w_k(t, \cdot) \right\|_2 + \sum_{|\mu|+m \leq N_0+3D+30 \atop m \leq 1} \left\| L^m \partial^\mu \Box w_k(t, \cdot) \right\|_2.
$$

We then let $\gamma$ be as in (4.8) and apply (2.17) to the first term. After integrating over the timestrip $[0, T]$ and applying (2.13) and (1.4), we have (4.31) bounded by

$$
\int_0^T \sum_{j+m \leq N_0+3D+31 \atop m \leq 1} \left[ \left\| \tilde{L}^m \Box w_k(s, \cdot) \right\|_2 + \left\| \tilde{L}^m \partial^j_t, \gamma^\alpha \beta \partial_\alpha \partial_\beta \right\| w_k(s, \cdot) \right\|_2 \right] ds
$$

$$
+ \int_0^T \left[ \sum_{j \leq N_0+3D+31} \left\| \partial^j_t \Box w_k(s, \cdot) \right\|_2 + \sum_{|\mu| \leq N_0+3D+31} \left\| \partial^\mu w_k(s, \cdot) \right\|_{L^2(|x|<1)} \right] ds
$$

$$
+ \sup_{t \in [0, T]} \sum_{|\mu|+m \leq N_0+3D+30 \atop m \leq 1} \left\| L^m \partial^\mu \Box w_k(t, \cdot) \right\|_2.
$$
The third term in (4.32) is controlled using the same arguments used to control term $I$. Using the product rule, we see that

$$ (\sum_{j+m \leq N_0 + 3D + 31} \left| L^m \partial^j \gamma w_k \right| + \left| \tilde{L}^m \partial^j \gamma \partial_{\alpha} \partial_{\beta} w_k \right|) \lesssim $$

$$ \left( \sum_{|\mu| + m \leq N_0 + 3D + 31} |L^m \partial^\mu (w_k + u_0)'| + \sum_{|\mu| + m \leq N_0 + 3D + 33} |L^m \partial^\mu u_0| \right) \sum_{|\mu| \leq N_0} |\partial^\mu (w_{k-1} + u_0)| $$

$$ + \sum_{|\mu| + m \leq N_0 + 3D + 31} |L^m \partial^\mu (w_{k-1} + u_0)'| \sum_{|\mu| \leq N_0} \left( |\partial^\mu (w_{k-1} + u_0)'| + |\partial^\mu (w_k + u_0)'| \right) $$

$$ + \left( \sum_{|\mu| \leq N_0 + 3D + 31} \left( |\partial^\mu (w_{k-1} + u_0)'| + |\partial^\mu (w_k + u_0)'| \right) + \sum_{|\mu| \leq N_0 + 3D + 33} |\partial^\mu u_0| \right) $$

$$ \times \sum_{|\mu| + m \leq N_0 + 3D + 31} |L^m \partial^\mu (w_{k-1} + u_0)| $$

$$ + \sum_{m \leq 1} |L^m (w_{k-1} + u_0)'| \left| (w_{k-1} + u_0) \right| + \sum_{|\mu| + m \leq N_0 + 3D + 31} |L^m \partial^\mu [\square, \eta] u| $$

Due to the fact that the initial data are compactly supported, it follows from finite propagation speed that

$$ \sum_{|\mu| + m \leq N_0 + 3D + 31} |L^m \partial^\mu [\square, \eta] u| \lesssim \sum_{|\mu| \leq N_0 + 3D + 31} |\partial^\mu [\square, \eta] u| $$

since the coefficients of $L$ are uniformly bounded on the support of $[\square, \eta] u$. A similar statement holds for $u_0$.

The third and fourth terms in the right hand side of (4.33) can be controlled by summing over dyadic intervals and applying Lemma 3.1 to the lower order terms. For more details on this computation, the reader can see [11]. Taking the $L^2_x$-norm of these
It follows that the last term in (4.32) can be controlled using terms without a logarithmic loss. The third-to-last term in (4.32) can be bounded in the same way. Arguing in this manner, we see that this term can be controlled (4.36) that it is bounded by V.

The first term can be controlled in a manner similar to the argument that was used to control term I. Integrating in the $t$ variable and applying Cauchy–Schwarz, we see that these terms are controlled using (4.3), II, and log$(2 + T)^{1/2}$ VIII in (4.3). The first two terms in the right hand side of (4.33) can be controlled using terms V and IX. The second-to-last term can be controlled using terms VII and IX. It follows that (4.35)

$$
\sum_{|\mu|+\nu+m \leq N_0+3D+31} \left\| \langle x \rangle^{-3/4} L^m \partial^\mu(w_k + u_0) \right\|_2 + \sum_{|\mu|+\nu+m \leq N_0+3D+31} \left\| \langle x \rangle^{-3/4} \partial^\mu(w_{k-1} + u_0) \right\|_2 
\times 
\left( \sum_{|\mu| \leq N_0+3D+31} \left( \left\| \langle x \rangle^{-3/4} \partial^\mu(w_k - 1 + u_0) \right\|_2 + \left\| \langle x \rangle^{-3/4} \partial^\mu(w_k + u_0) \right\|_2 \right) + \sum_{|\mu| \leq N_0+3D+33} \left\| \partial^\mu u_0 \right\|_2 \right).$

Integrating in the $t$ variable and applying Cauchy–Schwarz, we see that these terms are controlled using (4.3), II, and log$(2 + T)^{1/2}$ VIII in (4.3). The first two terms in the right hand side of (4.33) can be controlled using terms V and IX. The second-to-last term can be controlled using terms VII and IX. It follows that (4.35)

$$
\sum_{|\mu|+\nu+m \leq N_0+3D+30} \left\| \langle x \rangle^{-3/4} L^m \partial^\mu \square w_k(s, \cdot) \right\|_2 + \left\| \langle x \rangle^{-3/4} \partial^\mu \square w_k(s, \cdot) \right\|_2 \right) ds 
\leq C(M_0(T) + M_{k-1}(T) + M_k(T))(M_0(T) + M_{k-1}(T)) log(2 + T)^{1/2} + C_2 \varepsilon.
$$

Note that

$$
\sum_{|\mu|+\nu+m \leq N_0+3D+30} |L^m \partial^\mu \square w_k| is also controlled by the right hand side of (4.33).
$$

It follows that the last term in (4.32) can be controlled using terms I and V in (4.3) via Sobolev embedding. Arguing in this manner, we see that this term can be controlled without a logarithmic loss. The third-to-last term in (4.32) can be bounded in the same manner as term I. To handle the second-to-last term in (4.32), we apply (4.36) to see that it is bounded by (4.36)

$$
\sum_{|\mu| \leq N_0+4D+32} \left\| \partial^\mu \square w_k(s, \cdot) \right\|_2 ds + \int_0^T \left\| \partial^\mu \square w_k(s, y) \right\|_2 dy ds
\leq C(M_0(T) + M_{k-1}(T) + M_k(T))(M_0(T) + M_{k-1}(T)) log(2 + T)^{1/2} + C_2 \varepsilon.
$$

The first term can be controlled in a manner similar to the argument that was used to bound term I. Applying the product rule, we see that (4.37)

$$
\sum_{|\mu| \leq N_0+3D+36} |Z^\mu \square w_k(s, y)| \leq \sum_{|\mu| \leq N_0} |Z^\mu(w_{k-1} + u_0)| \times 
\left( \sum_{|\mu| \leq N_0+3D+36} \left| Z^\mu \partial^\nu(w_{k-1} + u_0) \right| + \sum_{|\mu| \leq N_0+3D+36} \left| Z^\mu \partial^\nu(w_k + u_0) \right| \right)
\times
\sum_{|\mu| \leq N_0+3D+36} \left| \partial^\mu \square \eta \right| |u|,
$$
The last term can be controlled using \((4.35)\). Thus, by \((4.37)\) and Cauchy-Schwarz, we see that the second term in \((4.36)\) is controlled by
\[
\sum_{|\mu| \leq N_0} \left\| (x)^{-1/2} Z^\mu (w_{k-1} + u_0) \right\|_{L^2_t \times (S_T)} \left( \sum_{|\mu| \leq N_0+3D+36} \left\| (x)^{-1/2} Z^\mu \partial^\nu (w_{k-1} + u_0) \right\|_{L^2_t \times (S_T)} \right) + \sum_{|\mu| \leq N_0+3D+36} \left\| (x)^{-1/2} Z^\mu \partial^\nu (w_k + u_0) \right\|_{L^2_t \times (S_T)} + \varepsilon.
\]

Since each factor in \((4.38)\) is controlled by \(\log(2 + T)^{1/2} IV\) in \((4.3)\), we see that
\[
(4.39) \quad \int_0^T \int_{\mathbb{R}^4 \setminus \mathcal{K}} \sum_{|\mu| \leq N_0+3D+36} |Z^\mu \Box w_k(s, y)| \frac{dy \, ds}{|y|^{3/2}} \leq C(M_0(T) + M_{k-1}(T) + M_k(T))(M_0(T) + M_{k-1}(T)) \log(2 + T) + C_2 \varepsilon.
\]

From this and previous arguments, we conclude that
\[
V \leq C(M_0(T) + M_{k-1}(T) + M_k(T))(M_0(T) + M_{k-1}(T)) \log(2 + T) + C_2 \varepsilon.
\]

**Bound for VI:** Controlling terms \(VI, VII\) and \(VIII\) will be conducted in a manner similar to the analogous terms where no \(L\) is being applied. In the case of term \(VI\), we apply a cutoff that is supported away from the obstacle \(\mathcal{K}\) to \(L \partial^\mu w_k\) and apply \((2.2)\). For the remaining term that is supported near \(\mathcal{K}\) and the commutator term that results from the cutoff we apply \((2.20)\). Thus, it suffices for us to control
\[
(4.40) \quad \int_0^T \sum_{|\mu| + m \leq N_0+3D+21} \left\| L^m \partial^\mu \Box w_k(s, \cdot) \right\|_{L^2_t \times (S_T)}^2 \, ds + \sum_{m \leq 1, \mu + m \leq N_0+2D+20} \left\| L^m \partial^\mu w_k \right\|_{L^2_t \times (S_T)}.
\]

One only needs to demonstrate control for the first term in the above expression since the second term can be controlled in a simpler manner by applying Sobolev embedding.
to the lower order terms. Applying the product rule, we see that

\[(4.41) \quad \sum_{|\mu|+m \leq N_0+3D+21} \left| L^m \partial^\mu \Box w_k \right| \lesssim \sum_{|\mu|+m \leq N_0+3D+22} \left| L^m \partial^\mu (w_k + u_0) \right| \sum_{|\mu| \leq N_0} \left| \partial^\mu (w_{k-1} + u_0) \right|
+ \sum_{|\mu|+m \leq N_0+3D+21} \left| L^m \partial^\mu (w_{k-1} + u_0) \right| \sum_{m \leq 1} \left( \left| L^m \partial^\mu (w_k + u_0) \right| + \left| L^m \partial^\mu (w_{k-1} + u_0) \right| \right)
+ \sum_{|\mu| \leq N_0+3D+22} \left| \partial^\mu (w_k + u_0) \right| \sum_{|\mu|+m \leq N_0} \left| L^m \partial^\nu (w_{k-1} + u_0) \right|
+ \left| w_{k-1} + u_0 \right| \sum_{m \leq 1} \left| L^m (w_{k-1} + u_0) \right| + \sum_{|\mu| \leq N_0+3D+21} \left| \partial^\mu [\Box, \eta] u \right|.
\]

Upon taking the $L^1([0, T]; L^2(\mathbb{R}^4 \setminus K))$-norm, we bound the corresponding terms in a manner that is reminiscent of the above arguments. Where $L$ is being applied to the higher order terms, we use terms $V$ and $IX$ of (4.33). The second-to-last term in (4.41) can be controlled using terms $VII$ and $IX$. The last term in (4.41) can be controlled using (4.1). The remaining terms can be controlled by decomposing dyadically in the $x$ variable, applying (4.1), and applying Cauchy-Schwarz in $t$ and in the dyadic summation variable.

This results in quantities that can be controlled by $\log(2+T)^{1/2} IV$ and $\log(2+T)^{1/2} VIII$. Therefore, we see that

\[VI \leq C(M_0(T) + M_k(T) + M_{k-1}(T))(M_0(T) + M_{k-1}(T)) \log(2 + T) + C_2 \varepsilon.\]

**Bound for VII and VIII, $|\nu| = 0$:** We closely follow the preceding argument that was used to control $III$ and $IV$, $|\nu| = 0$. Fix $\chi$ just as before. The Dirichlet boundary conditions and the boundedness of the coefficients of $Z$ on the boundary of $K$ demonstrate that $\chi ZX^\mu w_k$ is bounded by terms $V$ and $VI$. It, thus, suffices to prove bounds for $(1 - \chi) ZX^\mu w_k$ for $|\mu| \leq N_0 + D + 10$ fixed. We write

\[(4.42) \quad \Box (1 - \chi) ZX^\mu w_k = (1 - \eta)(1 - \chi) \partial_\alpha \left[ (1 - \eta)(1 - \chi) (b^{\alpha\beta}(u_0 + w_{k-1}) ZX^\mu \partial_\beta (u_0 + w_k) + b^{\alpha\beta\gamma} \partial_\gamma (u_0 + w_{k-1}) ZX^\mu \partial_\beta (u_0 + w_k)) \right] - [\Box, \chi] ZX^\mu w_k + \tilde{G}_{k, \mu}(t, x).\]
Applying (2.10), (2.8) and (2.5), we see that

\[
\sup_{0 \leq t \leq T} \left\| (1 - \chi) L Z^\mu w_k(t, \cdot) \right\|_{L^2_t(S_T)} + \log(2 + T)^{-1/2} \left\| (1 - \chi) (x)^{-1/2} L Z^\mu w_k \right\|_{L^2_t(S_T)} 
geq \int_0^T \left\| (x)^{-1} \tilde{G}_{k, \mu}(s, \cdot) \right\|_{L^1_t L^2_x(|x| > 2)} ds 
+ \int_0^T \sum_{|\theta| \leq N_0 + D + 10} \sum_{|\lambda| \leq 1} \left\| L Z^\theta (w_k + u_0)' \right\|_{L^2_t(S_T)} ds 
+ \sum_{|\theta| + m \leq N_0 + D + 11} \sum_{m \leq 1} \left\| L \theta^\mu w_k' \right\|_{L^2_t(S_T \cap \{|x| < 4\})}.
\]

The last term is controlled by term VI. Using terms VII and IX, the second-to-last term is controlled by

\[(M_0(T) + M_{k-1})(M_0(T) + M_k(T)) \log(2 + T).\]

The first term in (4.43) can be controlled using a rough application of the product rule where we allow for at most one occurrence of the scaling vector field on each term. Applying Sobolev embedding on $S^3$ and Cauchy-Schwarz, we see that

\[
\int_0^T \left\| (x)^{-1} \tilde{G}_{k, \mu}(s, \cdot) \right\|_{L^1_t L^2_x(|x| > 2)} ds \lesssim \varepsilon 
+ \sum_{m \leq 1} \left\| (x)^{-1/2} L^m Z^\theta (w_k + u_0)' \right\|_{L^2_t(S_T)} \sum_{m \leq 1} \left\| (x)^{-1/2} L^m Z^\theta (w_k - 1 + u_0) \right\|_{L^2_t(S_T)} 
+ \sum_{m \leq 1} \left\| (x)^{-1/2} L^m Z^\theta (w_k + u_0)' \right\|_{L^2_t(S_T)} \sum_{m \leq 1} \left\| (x)^{-1/2} L^m Z^\theta (w_k - 1 + u_0) \right\|_{L^2_t(S_T)}.
\]

Using terms IV and VIII, it follows that

\[
\int_0^T \left\| (x)^{-1} \tilde{G}_{k, \mu}(s, \cdot) \right\|_{L^1_t L^2_x(|x| > 2)} ds \lesssim \varepsilon + (M_0(T) + M_{k-1}(T) + M_k(T))(M_0(T) + M_{k-1}(T)) \log(2 + T),
\]

and therefore

\[VII_{|\nu| = 0} + VIII_{|\nu| = 0} \leq C(M_0(T) + M_{k-1}(T) + M_k(T))(M_0(T) + M_{k-1}(T)) \log(2 + T) + C_2 \varepsilon.\]

**Bound for VII and VIII, $|\nu| = 1$:** Again, we need only provide bounds for $(1 - \chi)L Z^\mu w_k'$ since $\chi L Z^\mu w_k'$ can be controlled using terms V and VI. Applying (2.1)
and \([2.2]\), we see that

\[
(4.45) \quad \sup_{0 \leq t \leq T} \sum_{|\mu| \leq N_0 + D + 10} \| (1 - \chi) LZ^\mu w_k'(t, \cdot) \|_2^+ + \sum_{|\mu| \leq N_0 + D + 10} \left\| (1 - \chi) (\cdot)^{-3/4} LZ^\mu w_k \right\|_{L^2_t(\mathbb{R}^n)} \\
\lesssim \int_0^T \sum_{|\mu| + m \leq N_0 + D + 11} m \sum_{m \leq 1} \left( L^m_2 w_k(s, \cdot) \right) \| \mu \|_2 ds + \sum_{|\mu| + m \leq N_0 + D + 11} \| L^m \partial^\mu w_k \|_{L^2_t(\mathbb{R}^n \times \{|x| < 4\})}.
\]

The second term in the right hand side is controlled by term \(VI\). The first term can be handled by carefully applying the product rule. We see that

\[
(4.46) \quad \sum_{|\mu| + m \leq N_0 + D + 11 \atop m \leq 1} \left| L^m Z^\mu \Box w_k \right| \lesssim \sum_{|\mu| + m \leq N_0 + D + 11 \atop \theta \in [0, \frac{1}{2}]} \left| L^m Z^\mu \partial^\theta (w_k + u_0) \right| \sum_{|\mu| \leq N_0} \left| Z^\mu (w_{k-1} + u_0) \right|
\]

\[
+ \sum_{|\mu| \leq N_0 + D + 11 \atop \theta \in [1/2, 1]} \left| Z^\mu \partial^\theta (w_k + u_0) \right| \sum_{m \leq 1} \left| L^m Z^\mu (w_{k-1} + u_0) \right|
\]

\[
+ \sum_{|\mu| + m \leq N_0 + D + 11 \atop \theta \in [1/2, 1]} \left| L^m Z^\mu \partial^\theta (w_{k-1} + u_0) \right| \sum_{m \leq 1} \left( \left| L^m Z^\mu (w_{k-1} + u_0) \right| + \left| L^m Z^\mu (w_k + u_0) \right| \right)
\]

\[
+ \sum_{|\mu| + m \leq N_0 + D + 11 \atop \theta \in [1/2, 1]} \left| L^m Z^\mu \Box \eta \right| |u|.
\]

By finite propagation speed and the fact that the initial data are compactly supported the coefficients of \(L\) and \(Z\) are bounded uniformly on the support of \(\Box, \eta \Box u\). Thus, the last term in \((4.46)\) is handled using \((4.1)\). Taking the \(L^1([0, T]; L^2(\mathbb{R}^4, \mathbb{R}^4))\)-norm of the first term of the right hand side, we see that it is controlled by terms \(III, VII,\) and \(IX\):

\[
(4.47) \quad \int_0^T \sum_{|\mu| + m \leq N_0 + D + 11 \atop m \in [1/2]} \left\| L^m Z^\mu \partial^\theta (w_k + u_0)(s, \cdot) \right\|_2 \sum_{|\mu| \leq N_0} \| Z^\mu (w_{k-1} + u_0)(s, \cdot) \|_\infty \; ds
\]

\[
\leq C(M_k(T) + M_0(T))(M_{k-1}(T) + M_0(T)) \log(2 + T).
\]
To handle the remaining terms, we decompose dyadically and apply (3.1) to the lower order pieces. We see that they can be controlled by

\[
\sum_{|\mu| \leq N_0 + D + 11} \left\| \langle x \rangle^{-3/4} Z^\mu \partial^\theta (w_k + u_0) \right\|_{L^2_t L^\infty_x (S_T)} \sum_{|\mu| \leq N_0 + 3} \left\| \langle x \rangle^{-1/2} L^m Z^\mu (w_{k-1} + u_0) \right\|_{L^2_t L^\infty_x (S_T)} 
\]

\[
+ \sum_{|\mu| + m \leq N_0 + D + 11} \left\| \langle x \rangle^{-1/2} L^m Z^\mu \partial^\theta (w_{k-1} + u_0) \right\|_{L^2_t L^\infty_x (S_T)} \sum_{m \leq 1} \left\| \langle x \rangle^{-1/2} L^m Z^\mu (w_{k-1} + u_0) \right\|_{L^2_t L^\infty_x (S_T)} 
\]

\[
+ \sum_{|\mu| + m \leq N_0 + 3} \left\| \langle x \rangle^{-3/4} L^m Z^\mu (w_{k} + u_0) \right\|_{L^2_t L^\infty_x (S_T)} \sum_{|\mu| + m \leq N_0 + D + 11} \left\| \langle x \rangle^{-1/2} L^m Z^\mu \partial^\theta (w_{k-1} + u_0) \right\|_{L^2_t L^\infty_x (S_T)}. 
\]

Since all the factors in the above expression are controlled by either \( \log(2 + T)^{1/2} IV \) or \( \log(2 + T)^{1/2} VIII \), these terms are bounded by

\[
C(M_{k-1} + M_k(T) + M_0(T))(M_{k-1} + M_0(T)) \log(2 + T). 
\]

Therefore, we see that

\[
VII|_{|\nu|=1} + VIII|_{|\nu|=1} \leq C(M_{k-1} + M_k(T) + M_0(T))(M_{k-1} + M_0(T)) \log(2 + T) + C_2 \varepsilon. 
\]

**Bound for VII, |\nu| = 2:** We let \( \gamma \) be as before in (4.8). Letting \( \delta = C_1 \varepsilon \), we apply (2.13). Integrating the resulting inequality over the timestrip \([0, T] \) and applying (2.13) and (4.4), we see that \( VII|_{|\nu|=2} \) is controlled by

\[
\int_0^T \sum_{|\mu| + m \leq N_0 + D + 11} \left\| \Box \gamma L^m Z^\mu \partial^\lambda w_k(s, \cdot) \right\|_2 ds + \sum_{|\mu| + m \leq N_0 + D + 13} \left\| L^m \partial^\mu \Box w_k \right\|_{L^2_t L^\infty_x (S_T \cap \{|x| < 2\})}. 
\]

The second term in (4.49) is controlled using an argument similar to the one used to bound term VI. To control the integrand inside the first term, we apply the product rule
to see that
\[
(4.50) \sum_{|\mu|+m \leq N_0+D+11 \atop m \leq 1 \atop |\lambda|=1} |\Box, L^m Z^\nu \partial^\lambda w_k|
\lesssim \left( \sum_{|\mu|+m \leq N_0+D+11 \atop m \leq 1 \atop |\lambda|=1} |L^m Z^\nu \partial^\lambda (w_k + u_0)| + \sum_{|\mu| \leq N_0+D+14 \atop |\mu| \leq N_0} |\partial^\mu u_0| \sum_{|\mu| \leq N_0} |Z^\mu (w_{k-1} + u_0)| \right)
+ \left( \sum_{|\mu| \leq N_0+D+11 \atop |\lambda| \leq 2} |Z^\mu \partial^\lambda (w_k + u_0)| + \sum_{|\mu| \leq N_0+D+14 \atop |\mu| + m \leq N_0 \atop m \leq 1} |\partial^\mu u_0| \sum_{|\mu| + m \leq N_0} |L^m Z^\mu (w_{k-1} + u_0)| \right)
+ \left( \sum_{|\mu| \leq N_0+D+11 \atop |\lambda| \leq 2} |Z^\mu \partial^\lambda (w_{k-1} + u_0)| \sum_{|\mu| + m \leq N_0 \atop m \leq 1} \left( |L^m Z^\mu (w_k + u_0)| + |L^m Z^\mu (w_{k-1} + u_0)| \right) \right)
+ \sum_{|\mu| + m \leq N_0+D+12 \atop m \leq 1} |L^m Z^\mu [\Box, \eta] u|.
\]

By finite propagation speed and the fact that the Cauchy data are compactly supported, the coefficients of \( L \) and \( Z \) are bounded on the support of \([\Box, \eta] u\).

The \( L^1((0,T]; L^2(\mathbb{R}^+ \\setminus K))\)-norm of the second term in (4.50) is controlled by
\[
(4.51) \int_0^T \sum_{|\mu|+m \leq N_0+D+11 \atop m \leq 1 \atop |\lambda|=2} \|L^m Z^\nu \partial^\lambda (w_{k-1} + u_0)(s, \cdot)\|_2 \times \sum_{|\mu| \leq N_0} \left( \|Z^\mu (w_k + u_0)(s, \cdot)\|_\infty + \|Z^\mu (w_{k-1} + u_0)(s, \cdot)\|_\infty \right) \, ds,
\]
which can be handled using \(III, VII \) and \(IX\). The first term in the right hand side of (4.50) can be handled in an analogous manner. The third and fourth terms can be handled using a dyadic decomposition in the \( x \) variable followed by an application of Lemma 3.1. We will illustrate this method by bounding the fourth term. Taking the \( L^2_x\)-norm of this term, we see that they can be controlled by
\[
(4.52) \sum_{|\mu| \leq N_0+D+11 \atop |\lambda| \leq 2} \left\| \langle x \rangle^{-1/2} Z^\mu \partial^\lambda (w_{k-1} + u_0) \right\|_2 \times \left( \sum_{|\mu| + m \leq N_0+3 \atop m \leq 1} \left\| \langle x \rangle^{-3/4} L^m Z^\mu (w_k + u_0) \right\|_2 + \sum_{|\mu| + m \leq N_0+3 \atop m \leq 1} \left\| \langle x \rangle^{-1/2} L^m Z^\mu (w_{k-1} + u_0) \right\|_2 \right).
\]

Integrating in time and applying Cauchy-Schwarz, we see that this is controlled by \( \log(2+T)^{1/2} IV \) and \( \log(2+T)^{1/2} VII \). The third term in the right hand side of (4.50) can be
controlled in a similar manner. From these arguments, we see that
\[
\int_0^T \sum_{|\mu|+m \leq N_0+D+11} \| \square \lambda \Delta \partial^\lambda \psi_k (s, \cdot) \|_2 \, ds
\]
\[
\leq C(M_k(T) + M_{k-1}(T) + M_0(T))(M_{k-1}(T) + M_0(T)) \log (2 + T) + C_2 \varepsilon.
\]
Therefore, we conclude that
\[
\mathcal{I} \leq 0 \leq 0 \leq C(M_k(T) + M_{k-1}(T) + M_0(T))(M_{k-1}(T) + M_0(T)) \log (2 + T) + C_2 \varepsilon.
\]

**Bound for IX:** Applying (3.10) and (2.23), we see that
\[
(4.53) \quad IX \lesssim \sup_{t \in [0,T]} \left( \sum_{|\mu| \leq N_0+3} \| |x|^{-1} \Delta \psi_k (t,\cdot) \|_{L^1_t L^2_x (|x| > 2)} \right)
\]
\[
+ \int_0^T \sum_{|\mu|+m \leq N_0+D+4} \| \partial^\mu \psi_k (s, \cdot) \|_2 \, ds
\]
\[
+ \int_0^T \sum_{|\mu|+m \leq N_0+3} \| |x|^{-1} \Delta \psi_k (s, \cdot) \|_{L^1_t L^2_x (|x| > 2)} \, ds
\]
\[
+ \int_0^T \int_{\mathbb{R}^4 \setminus K} \sum_{|\mu| \leq N_0+D+8} \| \Delta \psi_k (s, y) \| \, du \, ds.
\]

We see that the integrand in the second term in the right hand side of (4.53) is controlled by the right hand side of (4.40). Thus, it follows from the same argument used to control (4.40) that this term is bounded by \(C(M_k(T) + M_{k-1}(T) + M_0(T))(M_{k-1}(T) + M_0(T)) \log (2 + T) + C_2 \varepsilon\). We also see that the third and fourth terms are controlled by the right hand sides of (4.44) and (4.39), respectively. Therefore, it suffices to demonstrate how to control the first term in the right hand side of (4.53).

To deal with the first term, we see that it is controlled by
\[
(4.54) \quad \sup_{t \in [0,T]} \left( \sum_{|\mu| \leq N_0+3} \int_0^{\max (2, \sqrt{2})} \left( \int_{S^3} \| \Delta \psi_k (t, t_\omega) \|_{L^2} \omega \| \, d\omega \right)^{1/2} \, r^2 \, dr
\]
\[
+ \sup_{t \in [0,T]} \sum_{|\mu| \leq N_0+3} \int_0^{\max (2, \sqrt{2})} \left( \int_{S^3} \frac{1}{r^2} \| \Delta \psi_k (t, t_\omega) \|_{L^2} \omega \| \, d\omega \right)^{1/2} \, r^3 \, dr.
\]

Due to the fact that \(t \leq r\) and \(r > 2\) on the region of integration for the second integral in (4.54), we see that this term is controlled by
\[
(4.55) \quad \sup_{t \in [0,T]} \sum_{|\mu| \leq N_0+3} \| \Delta \psi_k (t, \cdot) \|_{L^1_t L^2_x (|x| > 2)}.
\]

Applying Cauchy-Schwarz and Sobolev embedding on \(S^3\), the above quantity can be controlled using term III and (4.14). Thus, we see that (4.55) is bounded by
\[
C(M_k(T) + M_{k-1}(T) + M_0(T))(M_{k-1}(T) + M_0(T)) + C_2 \varepsilon.
\]
To control the first integral in (4.54), we apply Cauchy-Schwarz in the $r$ variable to see that

$$
\sup_{t \in [0,T]} \langle t \rangle \sum_{|\mu| \leq N_0 + 3} \int_{S^3} ( \int_{S^3} |Z^\mu \Box w_k(s, r \omega)|^2 \, d\omega )^{1/2} r^2 \, dr
\lesssim \log(2 + T)^{1/2} \sup_{t \in [0,T]} \langle t \rangle \sum_{|\mu| \leq N_0 + 3} \| r Z^\mu \Box w_k(t, \cdot) \|_{L^2_t \left( \{ x \in \mathbb{R}^4 : 2 < |x| < \max(2, t/2) \} \right)}.
$$

Note that for $v_1, v_2 \in C^\infty(\mathbb{R} \times \mathbb{R}^4 \backslash \mathcal{K})$,

$$
\langle t \rangle \|rv_1(t, \cdot)v_2(t, \cdot)\|_{L^2_t(\{ x \in \mathbb{R}^4 : 2 < |x| < \max(2, t/2) \})} \lesssim \| \langle r \rangle \langle t - r \rangle v_1(t, \cdot) \|_{\infty} \| v_2(t, \cdot) \|_2.
$$

From this observation and the fact that

$$
\sum_{|\mu| \leq N_0 + 3} |Z^\mu \Box w_k| \lesssim \sum_{|\mu| \leq N_0} |Z^\mu(w_{k-1} + u_0)| \sum_{|\mu| \leq N_0 + 3} |Z^\mu \partial^\nu(w_k + u_0)|
+ \sum_{|\mu| \leq N_0 - 1} |Z^\mu(u_0 + w_k)| \sum_{|\mu| \leq N_0 + 3} |Z^\mu(w_{k-1} + u_0)|
+ \sum_{|\mu| \leq N_0} |Z^\mu(u_0 + w_k-1)| \sum_{|\mu| \leq N_0 + 4} |Z^\mu(w_{k-1} + u_0)|
+ \sum_{|\mu| \leq N_0 + 3} |Z^\mu[\Box, \eta]u|,
$$

we use terms $III$ and $IX$ in (4.3) to see that the right hand side of (4.54) is controlled by

$$(M_0(T) + M_k(T) + M_{k-1}(T))(M_0(T) + M_{k-1}(T)) \log(2 + T)^{1/2} + \varepsilon.$$

Hence, it follows that

$$IX \leq C(M_k(T) + M_{k-1}(T) + M_0(T))(M_{k-1}(T) + M_0(T)) \log(2 + T) + C_2 \varepsilon.$$

**Boundness of $M_k(T)$:** Here we show that (4.6) implies (4.7) with the same uniform constant $C_1$. Combining the estimates that we have obtained for $I, \ldots, IX$, we see that

$$M_k(T) \leq C(M_k(T) + M_{k-1} + M_0(T))(M_{k-1} + M_0(T)) \log(2 + T) + C_2 \varepsilon.$$

If we pick $C_1$ such that $C_1 > 2C_2$, then applying (4.4) and (4.7) yields the inequality

$$M_k(T) \leq C(M_k(T) + \varepsilon) \log(2 + T) + \frac{C_1}{2} \varepsilon.$$

If $\varepsilon, c$ in (1.3) and (1.4) are sufficiently small, then (4.6) follows.

**Convergence of $\{w_k\}$:** We shall now show that uniform boundedness of each $M_k(T)$ implies that the sequence $\{w_k\}$ is Cauchy. Standard results show that this implies that $\{w_k\}$ converges to a solution to (4.2), which implies Theorem 1.1. Setting

$$A_k(T) = \sup_{t \in [0,T]} \sum_{|\mu| \leq D + 20} \| \partial^\mu(w_k - w_{k-1})(t, \cdot) \|_2
+ \sum_{|\mu| \leq 2} \log(2 + T)^{-1/2} \left\| \langle x \rangle^{-1/2} \partial^\mu(w_k - w_{k-1}) \right\|_{L^2_t \langle \cdot \rangle(S_x)},$$
similar arguments to those used to bound $I, \ldots, IX$ along with \((4.6)\) imply that

$$A_k(T) \leq \frac{1}{2} A_{k-1}(T)$$

for $T \leq T_\varepsilon$, provided that $\varepsilon, c$ are sufficiently small. This immediately yields that $\{w_k\}$ is Cauchy in the space $X_T$ whose norm is given by

$$\|v\|_{X_T} = \sup_{t \in [0,T]} \sum_{|\mu| \leq D+20} \|\partial^\mu v(t,\cdot)\|_2.$$

This completes the proof.

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