New spaces and Continuity via $\hat{\Omega}$-closed sets

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ABSTRACT: In this paper, we introduce new spaces like $\hat{\Omega}T_\delta$ and $\omega T_{\hat{\Omega}}$. It turns out that the space $\hat{\Omega}T_\delta$ coincide with semi-$T_1$ and in $\omega T_{\hat{\Omega}}$-space every closed set is $\hat{\Omega}$-closed set and in semi-$T_1$ every $\hat{\Omega}$-closed set is closed in a topological space. Also we introduce some kinds of generalized continuity such as $\hat{\Omega}$-continuity, $\hat{\Omega}$-irresolute and weakly $\hat{\Omega}$-continuity.

Key Words: $\hat{\Omega}$-closed sets, $\hat{\Omega}T_\delta$-spaces, $\omega T_{\hat{\Omega}}$-spaces, $\hat{\Omega}$-continuity and $\hat{\Omega}$-irresolute.

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1. Introduction

In the recent years, many new separation axioms were introduced by using generalized closed sets such as $[3]$ $T_b$, $[12]$ $T_d$, $[2]$ semi-$T_1$, semi-$T_4$ and $[6]$ $T_3$. The class of $\hat{\Omega}$-closed sets $[8]$ has been introduced recently by using the $\delta$-closure operator. As the class of $\hat{\Omega}$-closed sets are independent of the class of closed sets, our interest is to find the spaces in which they are interrelated. With these motivation, new separation axioms via $\hat{\Omega}$-closed sets such as $\hat{\Omega}T_\delta$ which coincide with semi-$T_1$ and $\omega T_{\hat{\Omega}}$ in which every $\hat{\Omega}$-closed set is closed set are introduced and studied. Moreover, we introduce and study various types of the continuity via $\hat{\Omega}$-closed sets. Also we investigate the role of new spaces and it’s impacts on all these continuities.

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2. Preliminaries

Throughout this paper \((X, \tau)\) (or briefly \(X\)) represent a topological space on which no separation axioms are assumed unless explicitly stated. For a subset \(A\) of \((X, \tau)\), we denote the closure of \(A\), the interior of \(A\) and the complement of \(A\) as \(\text{cl}(A)\), \(\text{int}(A)\) and \(A^c\) respectively. The family of all open sets in \((X, \tau)\) are denoted by \(O(X)\) and \(O(X, x) = \{ U \in X : x \in U \in O(X) \}\) Let us recall the following definitions, which are useful in the sequel.

**Definition 2.1.** [7] A subset \(A\) of \(X\) is called \(\delta\)-closed in a topological space \((X, \tau)\) if \(A = \delta\text{cl}(A)\), where \(\delta\text{cl}(A) = \{ x \in X : \text{int}(\text{cl}(U)) \cap A \neq \emptyset, U \in O(X, x) \}\). The complement of \(\delta\)-closed set in \((X, \tau)\) is called \(\delta\)-open set in \((X, \tau)\).

The following notations are used in this paper. The family of all \(\delta\)-open (resp. \(\hat{\Omega}\)-open) sets on \(X\) are denoted by \(\delta O(X)\) (resp. \(\hat{\Omega}O(X)\)).

- \(\delta O(X, x) = \{ U \in X : x \in U \in \delta O(X) \}\)
- \(\hat{\Omega}O(X, x) = \{ U \in X : x \in U \in \hat{\Omega}O(X) \}\)

**Definition 2.2.** A subset \(A\) of a topological space \((X, \tau)\) is called

(i) semi open [9] if \(A \subseteq \text{cl}(\text{int}(A))\).

(ii) pre open [11] if \(A \subseteq \text{int}(\text{cl}(A))\).

(iii) regular open [15] if \(A = \text{int}(\text{cl}(A))\).

(iv) a \(\delta\) generalised closed (briefly \(\delta g\)-closed) set [5] if \(\delta\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).

(v) \(\alpha\)-closed set [8] if \(\alpha\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\alpha\) open in \((X, \tau)\).

(vi) \(\tilde{\gamma}\) (or) \(\omega\)-closed set [17] if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi open in \((X, \tau)\).

(vii) \(\hat{\Omega}\)-closed set [8] if \(\delta\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi open in \((X, \tau)\).

The complement of semi open (resp. pre open, regular open \(\delta g\)-closed \(\omega\)-(or \(\tilde{\gamma}\))-closed) set is called semi closed (resp. pre closed, regular closed, \(\delta g\)-open, \(\omega\)-(or \(\tilde{\gamma}\))-open,\(\hat{\Omega}\)-open).

**Definition 2.3.** A function \(f : (X, \tau) \to (Y, \sigma)\) is called a

(i) contra continuous [5] if the inverse image of open set in \(Y\) is closed set in \(X\).

(ii) perfectly continuous [14] if the inverse image of open set in \(Y\) is clopen set in \(X\).
(iii) completely continuous [1] if the inverse image of open set in \( Y \) is regular open set in \( X \).

(iv) super continuous [10] if the inverse image of open set in \( Y \) is \( \delta \) open set in \( X \).

(v) semi continuous [9] if the inverse image of open set in \( Y \) is semi open set in \( X \).

(vi) pre continuous [11] if the inverse image of open set in \( Y \) is pre open set in \( X \).

(vi) \( \delta g \)-continuous [6] if the inverse image of open set in \( Y \) is \( \delta g \)-open set in \( X \).

(vii) \( \omega \)-continuous [17] if the inverse image of open set in \( Y \) is \( \omega \)-open set in \( X \).

(viii) \( g \alpha \)-continuous [4] if the inverse image of open set in \( Y \) is \( g \alpha \)-open set in \( X \).

(ix) \( \delta \)-closed mapping [13] if the image of every \( \delta \)-closed set in \( X \) is \( \delta \)-closed in \( Y \).

**Definition 2.4.** A space \((X, \tau)\) is said to be \( T_{\delta} \)-space if every \( \hat{\Omega} \)-open set is \( \delta \)-open set in \( X \).

**Definition 2.5.** A space \((X, \tau)\) is said to be weakly Hausdorff [6] if for every two different point \( x \) and \( y \), there exists a regular open set \( U \) such that \( x \in U \), \( y \notin U \).

### 3. \( \hat{\Omega} \)-\( T_{\delta} \)-spaces and \( \omega \hat{\Omega} \)-spaces

In this section, we introduce two new spaces in which we get the relationship between the class of \( \hat{\Omega} \)-open sets and that of open sets. These spaces provide a new path to class of \( \hat{\Omega} \)-open sets to travel along with open sets.

**Definition 3.1.** A space \((X, \tau)\) is called \( \hat{\Omega} T_{\delta} \)-space if every \( \hat{\Omega} \)-closed set in \( X \) is \( \delta \)-closed in \( X \).

**Example 3.2.** \( X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \). Here, \( \hat{\Omega}O(X) = \tau = \delta O(X) \).

**Theorem 3.3.** Every \( T_{\hat{\Omega}} \)-space is \( \hat{\Omega} T_{\delta} \)-space.

**Proof:** Suppose that \((X, \tau)\) is a \( T_{\hat{\Omega}} \)-space and \( A \) is any \( \hat{\Omega} \)-closed set in \((X, \tau)\). By [8] Theorem 3.5, \( A \) is \( \delta g \)-closed set in \( X \). Since \( X \) is \( T_{\hat{\Omega}} \)-space, \( A \) is \( \delta \)-closed set in \( X \).

Thus, \( X \) is \( \hat{\Omega} T_{\delta} \)-space.

**Remark 3.4.** Converse is not always possible from the Example 3.2.

**Example 3.5.** By [6] the digital line is \( T_{\hat{\Omega}} \). By Theorem 3.3, the digital line is \( \hat{\Omega} T_{\delta} \)-space.
Let us give some characterizations of $\hat{\Omega}T_\delta$-space.

**Theorem 3.6.** A topological space $(X, \tau)$ is $\hat{\Omega}T_\delta$-space if and only if every singleton set is semi closed in $(X, \tau)$.

**Proof:** Necessity- Let $x \in X$. By [8] Proposition 4.7, $\{x\}$ is semi closed or $\{x\}^c$ is $\Omega$-closed in $X$. Therefore, if $\{x\}$ is not semi closed then $\{x\}^c$ is $\Omega$-closed in $X$.

By hypothesis, $\{x\}^c$ is $\delta$-closed set in $(X, \tau)$ and hence $\{x\}$ is $\delta$-open in $(X, \tau)$. By [6] Lemma 4.2, $\{x\}$ is regular open and hence $\{x\}$ is $\delta$-closed in $X$.

Sufficiency- Suppose that $A$ is any $\hat{\Omega}$-closed set in $(X, \tau)$ and $x \in \delta cl(A) \setminus A$.

Since $A$ is $\Omega$-closed set in $(X, \tau)$, [8] by Theorem 4.4, $\delta cl(A) \setminus A$ does not contain any non-empty semi closed set in $X$. Therefore, $\{x\}$ is not a semi closed in $(X, \tau)$, a contradiction.

Therefore, $\delta cl(A) = A$ and hence $A$ is $\delta$-closed in $(X, \tau)$.

**Theorem 3.7.** In a topological space $(X, \tau)$, the following statements are equivalent.

(i) $(X, \tau)$ is semi-$T_1$-space.

(ii) $\{x\}$ is semi closed in $X$ for every $x \in X$.

(iii) $X$ is $\hat{\Omega}T_\delta$.

**Proof:** By [12], (i)$\Leftrightarrow$ (ii) holds and by Theorem 3.6, (iii)$\Leftrightarrow$ (ii) holds.

**Theorem 3.8.** Every $\hat{\Omega}T_\delta$-space is semi-$T_{1\frac{1}{2}}$-space.

**Proof:** Since every semi-$T_1$-space is semi-$T_{1\frac{1}{2}}$ and by Theorem 3.6, every $\hat{\Omega}T_\delta$-space is semi-$T_{1\frac{1}{2}}$-space.

**Remark 3.9.** Converse is not true in general as seen from the following example.

**Example 3.10.** $X = \{a, b, c\}, \sigma = \{\emptyset, \{a\}, X\}$ is semi-$T_{1\frac{1}{2}}$ but not $\hat{\Omega}T_\delta$-space.

**Theorem 3.11.** A topological space $(X, \tau)$ is weakly Hausdorff if and only if $(X, \tau)$ is $\hat{\Omega}T_\delta$-space and every singleton set is $\Omega$-closed in $(X, \tau)$.

**Proof:** Necessity- By [6] Lemma 4.17, every $\{x\}$ is $\delta$-closed and hence semi closed in $X$. By Theorem 3.7, $X$ is $\hat{\Omega}T_\delta$-space. By [8] Theorem 3.2, $\{x\}$ is $\Omega$-closed in $(X, \tau)$.

Sufficiency- Let $x \in X$. By hypothesis, $\{x\}$ is $\delta$-closed in $X$ and again by [6] Lemma 4.17, $(X, \tau)$ is weakly Hausdorff.

**Definition 3.12.** A space $(X, \tau)$ is called $\omega T_{\Omega\delta}$-space if every $\omega$-closed set in $X$ is $\Omega$-closed in $X$. 


Example 3.13. Let \( X = \{a, b, c, d\} \), \( \tau = \{\emptyset, \{a\}, \{bcd\}, X\} \). Then \( \tau \) is \( \omega T_{\hat{\Omega}} \)-space.

Let us characterize \( \omega T_{\hat{\Omega}} \)-space via closed sets.

Theorem 3.14. A space \( (X, \tau) \) is \( \omega T_{\hat{\Omega}} \)-space if and only if every closed set is \( \hat{\Omega} \)-closed in \( (X, \tau) \).

Proof: Necessity- Suppose that \( A \) is any closed set in \( (X, \tau) \). Then by [16], \( A \) is \( \omega \)-closed set in \( X \). By hypothesis, \( A \) is \( \hat{\Omega} \)-closed set in \( X \).

Sufficiency- Suppose that \( A \) is \( \omega \)-closed set in \( X \). Then by [16], \( cl(A) \subseteq sker(A) \). Then \( sker(cl(A)) \subseteq sker(sker(A)) = sker(A) \). Since \( cl(A) \) is closed set in \( X \), by hypothesis, \( cl(A) \) is \( \hat{\Omega} \)-closed set in \( X \). By [8] Theorem 4.10, \( \delta cl(cl(A)) \subseteq sker(cl(A)) \subseteq sker(A) \). But \( \delta cl(A) \subseteq \delta cl(cl(A)) \). Therefore, \( \delta cl(A) \subseteq sker(A) \).

Again by [8] Theorem 4.10, \( A \) is \( \hat{\Omega} \)-closed in \( (X, \tau) \).

Remark 3.15. From the above discussion, a space \( (X, \tau) \) is \( \omega T_{\hat{\Omega}} \)-space if and only if every open set is \( \hat{\Omega} \)-open in \( (X, \tau) \).

Let us characterize semi-\( T_{\frac{1}{2}} \)-space via \( \hat{\Omega} \)-closed sets.

Theorem 3.16. Every \( \hat{\Omega} \)-closed set in topological space \( (X, \tau) \) is closed in \( X \) if and only if every singleton set is either open or semi closed in \( (X, \tau) \).

Proof: Necessity- Assume that every \( \hat{\Omega} \)-closed set in a topological space \( (X, \tau) \) is closed set in \( X \) and suppose that there exists \( x \in X \) such that \( \{x\} \) is not semi closed subset of \( X \). Then \( \{x\}^c \) is not semi open subset of \( X \). Here, \( X \) is the only semi open set containing \( \{x\} \) and \( \delta cl(\{x\}) \subseteq \{x\}^c \subseteq X \). Therefore, \( \{x\}^c \) is \( \hat{\Omega} \)-closed subset of \( X \) and by hypothesis, \( \{x\}^c \) is closed in \( X \). Thus, \( \{x\} \) is open in \( X \).

Sufficiency- Suppose that \( A \) is \( \hat{\Omega} \)-closed set in \( X \) and there exists \( x \in cl(A) \setminus \{A\} \). Case(i)- \( \{x\} \) is open in \( X \). Then \( A \cap \{x\} \neq \emptyset \), a contradiction to \( x \in cl(A) \setminus \{A\} \).

Case(ii)- \( \{x\} \) is semi closed in \( X \). Since \( cl(A) \setminus \{A\} \subseteq \delta cl(A) \setminus \{A\} \), \( \{x\} \) is a semi closed subset of \( \delta cl(A) \setminus \{A\} \), a contradiction to Theorem 4.4 in [8]. In both cases, we arrive at a contradiction. Therefore, \( cl(A) = A \) and hence \( A \) is closed in \( X \). \( \square \)

Theorem 3.17. In a topological space \( (X, \tau) \), the following statements are equivalent.

(i) \( (X, \tau) \) is semi-\( T_{\frac{1}{2}} \)-space.

(ii) \( \{x\} \) is either open or semi closed in \( X \) for every \( x \in X \).

(iii) Every \( \hat{\Omega} \)-closed set in \( (X, \tau) \) is closed set in \( (X, \tau) \).

(iv) Every \( \omega \)-closed set is closed in \( X \).
Proof: (i) ⇒ (ii) By [16] Theorem 4.8, it follows.
(ii) ⇒ (iii) By Theorem 3.16, it follows.
(iii) ⇒ (iv) Suppose that \( F \) is \( \omega \)-closed set in \( X \). By hypothesis and by Theorem 3.16, every \( \{x\} \) is either open or semi closed in \( X \). By [17] Theorem 5.01, \( F \) is closed in \( X \).
(iv) ⇒ (i) By [17] Theorem 5.01, it holds.

Remark 3.18. From the above discussion, a space \((X, \tau)\) is semi-\(T_2\) if and only if every \(\hat{\Omega}\)-open set is open in \((X, \tau)\).

4. \(\hat{\Omega}\)-continuity

Definition 4.1. A function \( f : (X, \tau) \to (Y, \sigma) \) is called \(\hat{\Omega}\)-continuous if the inverse image of every open set in \((Y, \sigma)\) is \(\hat{\Omega}\)-open in \((X, \tau)\).

Example 4.2. \( X \equiv Y = \{a, b, c\}, \tau = \{\emptyset, \{a, b\}, X\}, \sigma = \{\emptyset, \{a\}, \{a, b\}, X\} \). If \( f : (X, \tau) \to (Y, \sigma) \) is defined by \( f(a) = a, f(b) = b, f(c) = b \). Then, \( f \) is \(\hat{\Omega}\)-continuous.

Theorem 4.3. Let \( f : (X, \tau) \to (Y, \sigma) \) be any function. Then, the following statements hold.
(i) Every super continuous function is \(\hat{\Omega}\)-continuous function.
(ii) Every \(\hat{\Omega}\)-continuous function is \(\delta g\)-continuous function.
(iii) Every \(\hat{\Omega}\)-continuous function is \(\omega\)-continuous function.

Proof: It follows from the Definition 4.1 and by [8] Theorems 3.2 and 3.5.

Remark 4.4. The converse is not always true from the following example.

Example 4.5. Example 4.2 shows that \( f \) is \(\hat{\Omega}\)-continuous but not super continuous. \( X = Y = \{a, b, c, d\}, \tau = \{\emptyset, \{a, b\}, \{a, b, c\}, X\}, \sigma = \{\emptyset, \{b, c\}, \{a, b, c\}, \{b, c, d\}, Y\} \). If \( f : (X, \tau) \to (Y, \sigma) \) is defined by \( f(a) = b, f(b) = c, f(c) = f(d) = a \). Then \( f \) is \( g, \delta g \) and \(\omega\)-continuous but not \(\hat{\Omega}\)-continuous.

Remark 4.6. The notion of \(\hat{\Omega}\)-continuous and continuous are independent from the examples 4.2 and 4.5.

Remark 4.7. The pictorial representation of the above discussions and the existing results are given below. The reversible implication is never possible.
The following Theorem gives the conditions under which reversible implication of Theorem 4.3 holds.

**Theorem 4.8.** In a topological space \((X, \tau)\), the following holds.

(i) If \((X, \tau)\) is \(\hat{\Omega}T_\delta\)-space, then every \(\hat{\Omega}\)-continuous function is super continuous function.

(ii) If \((X, \tau)\) is \(T_{\hat{\delta}}\)-space, then every \(\delta g\)-continuous function is \(\hat{\Omega}\)-continuous function.

(iii) If \((X, \tau)\) is \(\omega T_{\hat{\Omega}}\)-space, then every \(\omega\)-continuous function is \(\hat{\Omega}\)-continuous function.

**Proof:**

(i) Since in a space \(\hat{\Omega}T_\delta\) every \(\hat{\Omega}\)-open set is \(\delta\)-open, it follows.

(ii) In a space \(T_{\hat{\delta}}\), every \(\delta g\)-open set is \(\delta\)-open. Therefore, by [8] Theorem 3.2, it follows.

(iii) In a space \(\omega T_{\hat{\Omega}}\), every \(\omega\)-open set is \(\hat{\Omega}\)-open. Therefore, it follows.

\[\blacksquare\]

**Theorem 4.9.** If a space \((X, \tau)\) is \(\omega T_{\hat{\Omega}}\)-space, then every continuous function is \(\hat{\Omega}\)-continuous function.

**Proof:** It follows from the Definition 4.1 and Theorem 3.14. \[\blacksquare\]

**Theorem 4.10.** If a space \((X, \tau)\) is semi-\(T_{\hat{\delta}}\)-space, then every \(\hat{\Omega}\)-continuous function is continuous function.

**Proof:** It follows from the Definition 4.1 and Theorem 3.17. \[\blacksquare\]

Let us decompose perfectly continuous functions via \(\hat{\Omega}\)-continuous mappings as follows.
Theorem 4.11. If \( f: (X, \tau) \to (Y, \sigma) \) is a function, then the followings are equivalent.

(i) \( f \) is perfectly continuous.

(ii) \( f \) is completely continuous and contra continuous.

(iii) \( f \) is super continuous and contra continuous.

(iv) \( f \) is \( \hat{\Omega} \)-continuous and contra continuous.

(v) \( f \) is \( \omega \)-continuous and contra continuous.

(vi) \( f \) is \( g\alpha \)-continuous and contra continuous.

(vii) \( f \) is pre-continuous and contra continuous.

Proof:

(i) Since a clopen set is regular closed and open, it holds.

(ii) Since a regular closed set is \( \delta \)-closed, it holds.

(iii) Since a \( \delta \)-closed set is \( \hat{\Omega} \)-closed, it holds.

(iv) Since a \( \hat{\Omega} \)-closed set is \( \omega \)-closed, it holds.

(v) Since a \( \omega \)-closed set is \( g\alpha \)-closed, it holds.

(vi) Since a \( g\alpha \)-closed set is pre-closed, it holds.

(vii) If \( A \) is pre-closed and open set in \( (X, \tau) \), then \( \text{cl}(\text{int}(A)) \subseteq A \), and \( A = \text{int}A \).

Thus \( \text{cl}(A) \subseteq A \) and hence \( A \) is closed. Thus \( A \) is clopen in \( X \). Therefore, every pre-continuous and contra continuous function is perfectly continuous.

\[ \square \]

Remark 4.12. The composition of \( \hat{\Omega} \)-continuous functions is not always \( \hat{\Omega} \)-continuous function as seen from the following example.

Example 4.13. Let \( X = Y = Z = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{a, b\}, X\}, \sigma = \{\emptyset, \{a\}, \{b, c\}, Y\}, \eta = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, Z\} \). If \( f: (X, \tau) \to (Y, \sigma) \) is defined by \( f(a) = b, f(b) = b, f(c) = c \). Then \( f \) is \( \hat{\Omega} \)-continuous function. Define \( g: (Y, \sigma) \to (Z, \eta) \) as \( g(a) = b, g(b) = c, g(c) = a \). Then \( g \) is \( \hat{\Omega} \)-continuous function. But \( g \circ f: (X, \tau) \to (Z, \eta) \), defined by \( (g \circ f)(x) = g(f(x)) \) for all \( x \in X \) is not a \( \hat{\Omega} \)-continuous function.

Theorem 4.14. The following statements are true.

(i) If \( f: (X, \tau) \to (Y, \sigma) \) is \( \hat{\Omega} \)-continuous and \( g: (Y, \sigma) \to (Z, \eta) \) is continuous, then \( g \circ f: (X, \tau) \to (Z, \eta) \) is \( \hat{\Omega} \)-continuous.
(ii) If \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) are continuous, then \( g \circ f : (X, \tau) \to (Z, \eta) \) is \( \Omega \)-continuous provided \((X, \tau)\) is \( \omega_\Omega \)-space.

(iii) If \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) are \( \Omega \)-continuous, then \( g \circ f : (X, \tau) \to (Z, \eta) \) is \( \Omega \)-continuous whenever \((Y, \sigma)\) is \( \text{semi-} T_{\frac{1}{2}} \)-space.

(iv) If \( f : (X, \tau) \to (Y, \sigma) \) is continuous and \( g : (Y, \sigma) \to (Z, \eta) \) is \( \Omega \)-continuous, then 
\[
g \circ f : (X, \tau) \to (Z, \eta) \text{ is } \Omega \text{-continuous provided } (X, \tau) \text{ is } \omega_\Omega \text{-space and } (Y, \sigma) \text{ is semi-} T_{\frac{1}{2}} \text{-space.}
\]

(v) If \( f : (X, \tau) \to (Y, \sigma) \) is continuous and \( g : (Y, \sigma) \to (Z, \eta) \) is \( \Omega \)-continuous then,
\[
g \circ f : (X, \tau) \to (Z, \eta) \text{ is continuous whenever } (Y, \sigma) \text{ is semi-} T_{\frac{1}{2}} \text{-space.}
\]

(vi) If \( f : (X, \tau) \to (Y, \sigma) \) is \( \hat{\Omega} \text{-continuous and } g : (Y, \sigma) \to (Z, \eta) \) is continuous, then 
\[
g \circ f : (X, \tau) \to (Z, \eta) \text{ is continuous whenever } (X, \tau) \text{ is semi-} T_{\frac{1}{2}} \text{-space.}
\]

Proof: They follow straight forward from their definitions.

\[\square\]

**Theorem 4.15.** \( f : (X, \tau) \to (Y, \sigma) \) is a surjective function such that \( f(U) \) is \( \hat{\Omega} \)-open set in \( Y \) for any \( \Omega \)-open set in \( U \) in \( X \) and \( g : (Y, \sigma) \to (Z, \eta) \) is any function. 
If \( g \circ f : (X, \tau) \to (Z, \eta) \) is \( \hat{\Omega} \)-continuous, then \( g \) is \( \hat{\Omega} \)-continuous function.

Proof: Suppose that \( y \in Y \) and \( W \) is any open set in \( Z \) containing \( g(y) \). Since \( f \) is surjective, there exists \( x \in X \) such that \( f(x) = y \). Then \( (g \circ f)(x) = g(f(x)) = g(y) \in W \). Since \( g \circ f \) is \( \hat{\Omega} \)-continuous, there exists an \( \hat{\Omega} \)-open set \( U \) in \( X \) containing \( x \) such that \( (g \circ f)(U) \subseteq W \). If we take \( V = f(U) \), then by our assumption, \( V \) is a \( \hat{\Omega} \)-open set in \( Y \) containing \( f(x) \) and \( (g \circ f)(U) = g(V) \subseteq W \). Thus, \( g \) is \( \hat{\Omega} \)-continuous function.

Let us characterize \( \hat{\Omega} \)-continuous function.

**Theorem 4.16.** If \( f : (X, \tau) \to (Y, \sigma) \) is any function, then the following statements are equivalent.

(i) \( f \) is \( \hat{\Omega} \)-continuous function.

(ii) For each \( x \in X \) and each \( V \in O(Y, f(x)) \), there exists \( U \in \hat{\Omega}O(X, x) \) such that \( f(U) \subseteq V \).

(iii) The inverse image of every closed set in \((Y, \sigma)\) is \( \hat{\Omega} \)-closed set in \((X, \tau)\).

(iv) \( f(\hat{\Omega}cl(A)) \subseteq cl(f(A)) \) for each subset \( A \) of \((X, \tau)\).

(v) \( \hat{\Omega}cl(f^{-1}(B)) \subseteq f^{-1}(cl(B)) \) for each subset \( B \) of \((Y, \sigma)\).
Proof: (i) ⇒ (ii) Suppose that \( x \in X \) and \( V \) is any open set in \((Y, \sigma)\) containing \( f(x) \). Then \( x \in f^{-1}(V) \). Define \( U = f^{-1}(V) \). By hypothesis, \( U \) is \( \hat{\Omega} \)-open in \((X, \tau)\) containing \( x \). Also \( f(U) \subseteq V \).

(ii) ⇒ (i) Suppose that \( V \) is any open set in \((Y, \sigma)\) and \( x \in f^{-1}(V) \) be arbitrary. By hypothesis, there exists \( U_x \in \hat{\Omega}O(X, x) \) such that \( f(U_x) \subseteq V \). Then \( x \in U_x \subseteq f^{-1}(V) \) implies that \( f^{-1}(V) = \bigcup \{U_x : x \in f^{-1}(V)\} \). By [8] Theorem 4.16, \( f^{-1}(V) \) is \( \hat{\Omega} \)-open in \((X, \tau)\).

(i) ⇒ (iii) Let \( F \) be any closed set in \((Y, \sigma)\). Then \( Y \setminus F \) is open in \((Y, \sigma)\). By hypothesis, \( f^{-1}(Y \setminus V) = X \setminus f^{-1}(V) \) is \( \hat{\Omega} \)-open in \((X, \tau)\). Therefore, \( f^{-1}(V) \) is \( \hat{\Omega} \)-closed in \((X, \tau)\).

(iii) ⇒ (iv) Let \( A \) be any subset of \( X \). Then \( A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(cl(f(A))) \). By hypothesis, \( f^{-1}(cl(f(A))) \) is a \( \hat{\Omega} \)-closed in \((X, \tau)\) containing \( A \). By the definition of \( \hat{\Omega} \)-closure, \( \hat{\Omega}cl(A) \subseteq f^{-1}(cl(f(A))) \). Therefore, \( f(\hat{\Omega}cl(A)) \subseteq cl(f(A)) \).

(iv) ⇒ (i) Suppose that \( F \) is any closed set in \((Y, \sigma)\). By hypothesis, \( f(\hat{\Omega}cl(f^{-1}(F))) \subseteq cl(f(f^{-1}(F))) \subseteq cl(F) \subseteq F \). Then \( \hat{\Omega}cl(f^{-1}(F)) \subseteq f^{-1}(F) \). Thus \( f^{-1}(F) \) is \( \hat{\Omega} \)-closed in \((X, \tau)\).

(v) ⇒ (iv) Let \( B \) be any subset of \((Y, \sigma)\). By hypothesis, \( f(\hat{\Omega}cl(f^{-1}(B))) \subseteq cl(f(f^{-1}(B))) \subseteq cl(B) \). Then \( \hat{\Omega}cl(f^{-1}(B)) \subseteq f^{-1}(cl(B)) \).

Proof: Let \( F \) be any open subset of \((Y, \sigma)\). By hypothesis, \( f^{-1}(F) \) is \( \hat{\Omega} \)-open subset of \((X, \tau)\). By [8] Theorem 6.10, \( f^{-1}(F) \cap A \) is \( \hat{\Omega} \)-open in \((A, \tau|A)\). Therefore, \( (f|A)^{-1}(F) \) is \( \hat{\Omega} \)-open in \((A, \tau|A)\). Thus, \( f|A \) is \( \hat{\Omega} \)-continuous.

Theorem 4.18. Let \( \{A_\alpha, \alpha \in \Lambda\} \) be a cover of \( X \) by open and preclosed subsets of \((X, \tau)\). If \( f|A_\alpha : (A_\alpha, \tau|A_\alpha) \rightarrow (Y, \sigma) \) is \( \hat{\Omega} \)-continuous function for each \( \alpha \in \Lambda \), then \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \hat{\Omega} \)-continuous function.

Proof: Let \( V \) be any open subset of \((Y, \sigma)\). Since \( f|A_\alpha \) is \( \hat{\Omega} \)-continuous function, \( (f|A_\alpha)^{-1}(V) \) is \( \hat{\Omega} \)-open subset of \((A_\alpha, \tau|A_\alpha)\). By [8] Theorem 6.9, for each \( \alpha \in \Lambda \), \( (f|A_\alpha)^{-1}(V) \) is \( \hat{\Omega} \)-open subset of \((X, \tau)\). Since \( f^{-1}(V) = \bigcup \{(f|A_\alpha)^{-1}(V) : \alpha \in \Lambda\} \) and since \( \hat{\Omega}O(X) \) is closed under arbitrary union, \( f^{-1}(V) \) is \( \hat{\Omega} \)-open subset of \((X, \tau)\). Thus, \( f \) is \( \hat{\Omega} \)-continuous function.

Let us prove a significant theorem as follows.
Theorem 4.19. "The Pasteing lemma" for \( \hat{\Omega} \)-continuous function.
Let \( A \) and \( B \) be any two both open and pre-closed subsets of \( (X, \tau) \) such that \( X = A \cup B \). Let \( f: (A, \tau|A) \to (Y, \sigma) \) and \( g: (B, \tau|B) \to (Y, \sigma) \) be \( \hat{\Omega} \)-continuous functions such that \( f(x) = g(x) \) for every \( x \in A \cap B \). Then the combination \((f \updownarrow g)(x) = f(x)\) for every \( x \in A \) and \((f \updownarrow g)(y) = g(y)\) for every \( y \in B \) is \( \hat{\Omega} \)-continuous function.

Proof: Let \( F \) be any closed set in \( (Y, \sigma) \). Then \((f \updownarrow g)^{-1}(F) = ((f \updownarrow g)^{-1}(F) \cap A) \cup ((f \updownarrow g)^{-1}(F) \cap B) = f^{-1}(F) \cup g^{-1}(F) = C \cup D \), where \( C = f^{-1}(F) \) and \( D = g^{-1}(F) \). Since \( f \) and \( g \) are \( \hat{\Omega} \)-continuous, \( C \) and \( D \) are \( \Omega \)-closed sets in \((A, \tau|A)\) and \((B, \tau|B)\) respectively. By [8], Theorem 4.12, \((f \updownarrow g)^{-1}(F)\) is \( \Omega \)-closed sets in \((X, \tau)\). Thus, the combination \((f \updownarrow g)\) is \( \hat{\Omega} \)-continuous function.

Theorem 4.20. If \( f: (X, \tau) \to (Y, \sigma) \) is semi-continuous and \( \delta \)-closed mapping and if \( A \) is \( \hat{\Omega} \)-closed set in \((X, \tau)\), then \( f(A) \) is \( \hat{\Omega} \)-closed in \((Y, \sigma)\).

Proof: Suppose that \( f(A) \subseteq U \), where \( U \) is any semi-open subset of \( Y \). Then \( A \subseteq f^{-1}(U) \). Since \( f \) is semi-continuous, \( f^{-1}(U) \) is semi-open set in \( X \). Since \( A \) is \( \hat{\Omega} \)-closed set in \((X, \delta cl) \subseteq f^{-1}(U) \) and hence \( f(\delta cl(A)) \subseteq U \). By [7] Corollary 4, \( \delta cl(A) \) is \( \delta \)-closed set and since \( f \) is a \( \delta \)-closed mapping, \( \delta cl(f(A)) \subseteq \delta cl(f(\delta cl(A))) \subseteq f(\delta cl(A)) \subseteq U \). Thus, \( f(A) \) is \( \hat{\Omega} \)-closed in \((Y, \sigma)\).

5. \( \hat{\Omega} \)-irresolute mappings

Definition 5.1. A function \( f : (X, \tau) \to (Y, \sigma) \) is said to be \( \hat{\Omega} \)-irresolute if \( f^{-1}(V) \) is \( \hat{\Omega} \)-open set in \((X, \tau)\) for every \( \hat{\Omega} \)-open set \( V \) in \((Y, \sigma)\).

Example 5.2. Let \( X = Y = \{a, b, c, d\} \), \( \tau = \{\emptyset, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\} \) and \( \sigma = \{\emptyset, \{a\}, \{a, b\}, Y\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) as \( f(a) = b, f(b) = a, f(c) = c, f(d) = d \). Here \( f \) is \( \hat{\Omega} \)-irresolute function.

Let us characterize \( \hat{\Omega} \)-irresolute function in the following theorems

Theorem 5.3. A function \( f : (X, \tau) \to (Y, \sigma) \) is \( \hat{\Omega} \)-irresolute if and only \( f^{-1}(F) \) is \( \hat{\Omega} \)-closed set in \((X, \tau)\) for every \( \hat{\Omega} \)-closed set \( F \) in \((Y, \sigma)\).

Proof: Necessity- Suppose that \( f \) is \( \hat{\Omega} \)-irresolute and \( F \) is any \( \hat{\Omega} \)-closed set in \((Y, \sigma)\). Then \((Y \backslash F) \) is \( \hat{\Omega} \)-open set in \((Y, \sigma)\) and by hypothesis, \( f^{-1}(Y \backslash F) \) is \( \hat{\Omega} \)-open set in \((X, \tau)\). Then \((X \backslash f^{-1}(F)) \) is \( \hat{\Omega} \)-open set in \((X, \tau)\). Thus, \( f^{-1}(F) \) is \( \hat{\Omega} \)-closed set in \((X, \tau)\).

Sufficiency- Suppose \( V \) is any \( \hat{\Omega} \)-open set in \((Y, \sigma)\). Then \((Y \backslash V) \) is \( \hat{\Omega} \)-closed set in \((Y, \sigma)\). By hypothesis, \( f^{-1}(Y \backslash V) \) is \( \hat{\Omega} \)-closed set in \((X, \tau)\) and hence \((X \backslash f^{-1}(V)) \) is \( \hat{\Omega} \)-closed set in \((X, \tau)\). Thus, \( f^{-1}(V) \) is \( \hat{\Omega} \)-open set in \((X, \tau)\).

Theorem 5.4. A function \( f : (X, \tau) \to (Y, \sigma) \) is \( \hat{\Omega} \)-irresolute if and only if \( f(\hat{\Omega}cl(A)) \subseteq \hat{\Omega}cl(f(A)) \) for every subset \( A \) of \( X \).
Proof: Necessity- Suppose that \( f \) is \( \hat{\Omega} \)-irresolute and \( A \) is any subset of \( X \). Let \( x \in \hat{\Omega}cl(A) \) and \( f(x) \notin \hat{\Omega}cl(f(A)) \). By [8] Theorem 5.11, there exists a \( \Omega \)-open set \( V \) in \( Y \) containing \( f(x) \) such that \( V \cap f(A) = \emptyset \). Then \( f^{-1}(V) \cap A = \emptyset \). Since \( f \) is \( \hat{\Omega} \)-continuous, \( f^{-1}(V) \) is \( \hat{\Omega} \)-open set in \( X \) containing \( x \). Since \( x \in \hat{\Omega}cl(A) \) and by [8] Theorem 5.11, \( f^{-1}(V) \cap A \neq \emptyset \), a contradiction. Therefore, \( f(\hat{\Omega}cl(A)) \subseteq \hat{\Omega}cl(f(A)) \).

Sufficiency- Let \( F \) be any \( \hat{\Omega} \)-closed set in \( Y \). If \( x \in \hat{\Omega}cl(f^{-1}(F)) \), then \( f(x) \in f(\hat{\Omega}cl(f^{-1}(F))) \). By hypothesis, \( f(x) \in \hat{\Omega}cl(f(f^{-1}(F))) \subseteq \hat{\Omega}cl(F) = F \). Then \( x \notin f^{-1}(F) \). By [8] Theorem 5.3 (v), \( f^{-1}(F) \) is \( \hat{\Omega} \)-closed set in \( X \) and hence \( \hat{\Omega} \)-continuous.

\[ \square \]

**Theorem 5.5.** A function \( f: (X, \tau) \to (Y, \sigma) \) is \( \hat{\Omega} \)-irresolute iff \( \hat{\Omega}cl(f^{-1}(B)) \subseteq f^{-1}(\hat{\Omega}cl(B)) \) for each subset \( B \) of \( (Y, \sigma) \).

**Proof: Necessity-** For any set \( B \) in \( Y \) by Theorem 5.4, \( \hat{\Omega}cl(f^{-1}(B)) \subseteq \hat{\Omega}cl(f(f^{-1}(B))) \subseteq \hat{\Omega}cl(B) \). Therefore, \( \hat{\Omega}cl(f^{-1}(B)) \subseteq f^{-1}(\hat{\Omega}cl(B)) \).

**Sufficiency-** It suffices to show that inverse image of a closed subset of \( Y \) is \( \hat{\Omega} \)-closed subset of \( X \). Suppose that \( F \) is any closed set in \( X \). Then by [8] Theorem 5.3, \( F = \hat{\Omega}cl(F) \). By hypothesis, \( \hat{\Omega}cl(f^{-1}(F)) \subseteq f^{-1}(\hat{\Omega}cl(F)) = f^{-1}(F) \). Again by [8] Theorem 5.3, \( f^{-1}(F) \) is \( \hat{\Omega} \)-closed in \( X \).

\[ \square \]

**Theorem 5.6.** \( f: (X, \tau) \to (Y, \sigma) \) is \( \hat{\Omega} \)-continuous function if and only if for every \( x \in X \) and for every \( V \in \Omega O(Y, f(x)) \), there exists \( U \in \hat{\Omega}O(X, x) \) such that \( f(U) \subseteq V \).

**Proof: Necessity-** Suppose that \( x \in X \) and \( V \) is any \( \hat{\Omega} \)-open set in \( (Y, \sigma) \) containing \( f(x) \). Then \( x \in f^{-1}(V) \). Define \( U = f^{-1}(V) \). By hypothesis, \( U \) is \( \hat{\Omega} \)-open in \( (X, \tau) \) containing \( x \). Also \( f(U) \subseteq V \).

**Sufficiency-** Suppose that \( V \) is any \( \hat{\Omega} \)-open set in \( (Y, \sigma) \) and \( x \in f^{-1}(V) \). By hypothesis, there exists \( U_x \in \Omega O(X, x) \) such that \( f(U_x) \subseteq V \). Then \( x \in U_x \subseteq f^{-1}(V) \) implies that \( f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x \). By [8] Theorem 4.16, \( f^{-1}(V) \) is \( \hat{\Omega} \)-open in \( (X, \tau) \).

\[ \square \]

**Remark 5.7.** From the following examples, it is shown that the notion of \( \hat{\Omega} \)-irresolute and \( \hat{\Omega} \)-continuous functions are independent. Moreover, the concept of \( \hat{\Omega} \)-irresolute and continuous functions are independent.

**Example 5.8.** \( X = Y = \{a, b, c, d\}, \tau = \{\emptyset, \{a, b, c\}, \{a, b, c, d\}, X\}, \sigma = \emptyset, \{a\}, \{a, b, Y\}. \) If \( f: (X, \tau) \to (Y, \sigma) \) is defined by \( f(a) = b, f(b) = a, f(c) = c, f(d) = d \). Then \( f \) is \( \hat{\Omega} \)-irresolute but not continuous.

**Example 5.9.** \( X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, X\}, \sigma = \emptyset, \{a\}, \{b, c\}, Y\}. \) If \( f: (X, \tau) \to (Y, \sigma) \) is defined by \( f(a) = b, f(b) = c, f(c) = b \). Then \( f \) is \( \hat{\Omega} \)-continuous and continuous but not \( \hat{\Omega} \)-irresolute.
Example 5.10. $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, \{c\}, \{c, d\}, X\}$, $\sigma = \{\emptyset, \{a\}, \{a, b\}, Y\}$. If $f: (X, \tau) \to (Y, \sigma)$ is defined by $f(a) = c$, $f(b) = d$, $f(c) = a$, $f(d) = b$. Then $f$ is $\hat{\Omega}$-irresolute but not $\hat{\Omega}$-continuous.

Theorem 5.11. Let $(Y, \sigma)$ be a $\omega T_{\hat{\Omega}}$-space. If $f: (X, \tau) \to (Y, \sigma)$ is $\hat{\Omega}$-irresolute then it is $\hat{\Omega}$-continuous. Moreover, if $(X, \tau)$ is semi-$T_{\hat{\Omega}}$-space, then $f$ is continuous.

Proof: Suppose that $V$ is open in $Y$. Since $Y$ is $\omega T_{\hat{\Omega}}$, $V$ is $\hat{\Omega}$-open in $Y$. Since $f$ is $\hat{\Omega}$-irresolute, $f^{-1}(V)$ is $\hat{\Omega}$-open in $X$. Therefore, $f$ is $\hat{\Omega}$-continuous. If $X$ is semi-$T_{\hat{\Omega}}$, then $f^{-1}(V)$ is open in $X$ and hence $f$ is continuous.

Theorem 5.12. Let $(Y, \sigma)$ be a semi-$T_{\hat{\Omega}}$-space. If $f: (X, \tau) \to (Y, \sigma)$ is $\hat{\Omega}$-continuous then it is $\hat{\Omega}$-irresolute.

Proof: It follows from their definitions.

Theorem 5.13. Suppose $(Y, \sigma)$ is a semi-$T_{\hat{\Omega}}$-space and $(X, \tau)$ is $\omega T_{\hat{\Omega}}$-space. If $f: (X, \tau) \to (Y, \sigma)$ is continuous then it is $\hat{\Omega}$-irresolute.

Proof: It follows from their definitions.

Composition theorems.

Theorem 5.14. The following statements are true.

(i) If $f: (X, \tau) \to (Y, \sigma)$ is $\hat{\Omega}$-irresolute and $g: (Y, \sigma) \to (Z, \eta)$ is $\hat{\Omega}$-irresolute, then $g \circ f: (X, \tau) \to (Z, \eta)$ is $\hat{\Omega}$-irresolute.

(ii) If $f: (X, \tau) \to (Y, \sigma)$ is continuous and $g: (Y, \sigma) \to (Z, \eta)$ is $\hat{\Omega}$-irresolute function, then $g \circ f: (X, \tau) \to (Z, \eta)$ is $\hat{\Omega}$-irresolute whenever $(Y, \sigma)$ is semi-$T_{\hat{\Omega}}$-space and $(X, \tau)$ is $\omega T_{\hat{\Omega}}$-space.

(iii) If $f: (X, \tau) \to (Y, \sigma)$ is $\hat{\Omega}$-continuous and $g: (Y, \sigma) \to (Z, \eta)$ is $\hat{\Omega}$-irresolute, then $g \circ f: (X, \tau) \to (Z, \eta)$ is $\hat{\Omega}$-irresolute whenever $(Y, \sigma)$ is semi-$T_{\hat{\Omega}}$-space.

(iv) If $f: (X, \tau) \to (Y, \sigma)$ is $\hat{\Omega}$-irresolute and $g: (Y, \sigma) \to (Z, \eta)$ is continuous function, then $g \circ f: (X, \tau) \to (Z, \eta)$ is $\hat{\Omega}$-continuous whenever $(Y, \sigma)$ is $\omega T_{\hat{\Omega}}$-space.

(v) If $f: (X, \tau) \to (Y, \sigma)$ is $\hat{\Omega}$-irresolute and $g: (Y, \sigma) \to (Z, \eta)$ is $\hat{\Omega}$-continuous, then $g \circ f: (X, \tau) \to (Z, \eta)$ is $\hat{\Omega}$-continuous.
If \( f : (X, \tau) \to (Y, \sigma) \) is \( \hat{\Omega} \)-irresolute and \( g : (Y, \sigma) \to (Z, \eta) \) is \( \hat{\Omega} \)-continuous, then
\[
g \circ f : (X, \tau) \to (Z, \eta) \text{ is continuous whenever } (X, \tau) \text{ is semi-} T_{\frac{1}{2}} \text{-space}.
\]

**Proof:** They follow straightforward from their definitions.

Let us prove an application of \( \hat{\Omega} \)-irresolute functions as follows.

**Theorem 5.15.** "The Pasting lemma" for \( \hat{\Omega} \)-irresolute function.

Let \( A \) and \( B \) be any two both open and pre-closed sets in \( (X, \tau) \) such that \( X = A \cup B \). Let \( f : (A, \tau |_A) \to (Y, \sigma) \) and \( g : (B, \tau |_B) \to (Y, \sigma) \) be \( \hat{\Omega} \)-irresolute functions such that \( f(x) = g(x) \) for every \( x \in A \cap B \). Then the combination \( (f \vee g)(x) = f(x) \) for every \( x \in A \) and \( (f \vee g)(y) = g(y) \) for every \( y \in B \) is \( \hat{\Omega} \)-irresolute function.

**Proof:** It is similar to that of 4.18.

**Theorem 5.16.** Let \( (X, \tau) \) be a Hausdorff space and if \( f : (X, \tau) \to (Y, \sigma) \) is \( \hat{\Omega} \)-continuous, then the set \( A = \{ x \in X : f(x) = x \} \) is \( \hat{\Omega} \)-closed set in \( (X, \tau) \).

**Proof:** Suppose that \( x \in \hat{\text{cl}}(A) \setminus A \). Then \( x \neq f(x) \). Since \( X \) is Hausdorff, there exists two disjoint open sets \( U \) and \( V \) in \( X \) such that \( x \in U \) and \( f(x) \in V \). Since \( f \) is \( \hat{\Omega} \)-continuous and by [8] Theorem 4.12, \( f^{-1}(V) \cap U \) is \( \hat{\Omega} \)-open set containing \( x \). Since \( x \in \hat{\text{cl}}(A) \), by [8] Theorem 5.11, \( f^{-1}(V) \cap U \cap A \neq \emptyset \). Therefore, \( U \cap V \neq \emptyset \) a contradiction. Thus \( A \) is \( \hat{\Omega} \)-closed set in \( (X, \tau) \).

**Corollary 5.17.** Let \( (X, \tau) \) be a Hausdorff and \( \omega T_{\hat{\Omega}} \)-space and if \( f : (X, \tau) \to (Y, \sigma) \) is \( \hat{\Omega} \)-irresolute, then the set \( A = \{ x \in X : f(x) = x \} \) is \( \hat{\Omega} \)-closed set in \( (X, \tau) \).

**Theorem 5.18.** If \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (X, \tau) \to (Y, \sigma) \) are \( \hat{\Omega} \)-irresolute from \( X \) into a Hausdorff space \( Y \), then the set \( A = \{ x \in X : f(x) = g(x) \} \) is \( \hat{\Omega} \)-closed set in \( (X, \tau) \).

**Proof:** Let \( x \in (X \setminus A) \). Then \( f(x) \neq g(x) \). Since \( Y \) is Hausdorff space, there exists two disjoint open sets \( U \) and \( V \) in \( Y \) such that \( f(x) \in U \) and \( g(x) \in V \). Since \( f \) is \( \hat{\Omega} \)-continuous and by [8] Theorem 4.12, \( f^{-1}(U) \cap g^{-1}(V) \) is \( \hat{\Omega} \)-open set containing \( x \). Moreover, \( f^{-1}(U) \cap g^{-1}(V) \cap A = \emptyset \). Therefore, \( x \in f^{-1}(U) \cap g^{-1}(V) \subseteq (X \setminus A) \). This implies that \( (X \setminus A) \) is \( \hat{\Omega} \)-open in \( (X, \tau) \) and hence \( A \) is \( \hat{\Omega} \)-closed set in \( (X, \tau) \).

**Corollary 5.19.** Let \( (Y, \sigma) \) be a \( \hat{\Omega} \)-\( T_2 \)-space and if \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (X, \tau) \to (Y, \sigma) \) are \( \hat{\Omega} \)-irresolute, then the set \( A = \{ x \in X : f(x) = g(x) \} \) is \( \hat{\Omega} \)-closed set in \( (X, \tau) \).

**Theorem 5.20.** Let \( (X, \tau) \) be a semi-\( T_{\frac{1}{2}} \)-space. If \( f : (X, \tau) \to (Y, \sigma) \) is \( \hat{\Omega} \)-continuous injective and \( Y \) is Hausdorff, then \( X \) is Hausdorff.
Definition 5.21. A space is said to be \( \hat{\Omega}-T_1 \) if such that \( x, y \in X \) there exists two disjoint \( \Omega \)-open sets \( U \) and \( V \) in \( Y \) such that \( f(x) \in U \) and \( f(y) \in V \). Since \( f \) is \( \hat{\Omega} \)-continuous, \( f^{-1}(U) \) and \( f^{-1}(V) \) are disjoint \( \Omega \)-open sets in \( X \) containing \( x \) and \( y \) respectively. Since \( X \) is semi-\( T_2 \)-space, \( f^{-1}(U) \) and \( f^{-1}(V) \) are disjoint open sets in \( X \) containing \( x \) and \( y \) respectively. Thus, \((X, \tau)\) is Hausdorff.

Corollary 5.22. Let \((Y, \sigma)\) be Hausdorff space. If \( f: (X, \tau) \to (Y, \sigma) \) is \( \hat{\Omega} \)-continuous injective, then \( X \) is \( \hat{\Omega}-T_2 \).

Corollary 5.23. Let \((X, \tau)\) be semi-\( T_2 \)-space and \((Y, \sigma)\) be \( \hat{\Omega}-T_2 \)-space. If \( f: (X, \tau) \to (Y, \sigma) \) is \( \hat{\Omega} \)-irresolute injective, then \( X \) is Hausdorff.

Corollary 5.24. If \( f: (X, \tau) \to (Y, \sigma) \) is \( \hat{\Omega} \)-irresolute injective and \( Y \) is \( \hat{\Omega}-T_2 \), then \( X \) is \( \hat{\Omega}-T_2 \).

6. weakly \( \hat{\Omega} \)-continuous

Definition 6.1. A function \( f: (X, \tau) \to (Y, \sigma) \) is weakly \( \hat{\Omega} \)-continuous at \( x \in X \) if for each \( G \in O(Y, f(x)) \), there exists \( U \in \hat{\Omega}O(X, x) \) such that \( f(U) \subseteq \text{cl}(G) \). If \( f \) is weakly \( \hat{\Omega} \)-continuous at each \( x \in X \), then \( f \) is weakly \( \hat{\Omega} \)-continuous on \( X \).

Example 6.2. \( X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{a, b\}, X\}, \sigma = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, Y\} \). Define \( f: (X, \tau) \to (Y, \sigma) \) as an identity function. Then \( f \) is weakly \( \hat{\Omega} \)-continuous.

Theorem 6.3. Every \( \hat{\Omega} \)-continuous function is weakly \( \hat{\Omega} \)-continuous function.

Proof: Let \( x \in X \) and \( V \in O(Y, f(x)) \). By hypothesis, their exists \( U \in \hat{\Omega}O(X, x) \) such that \( f(U) \subseteq V \subseteq \text{cl}(V) \). Thus, \( f \) is weakly \( \hat{\Omega} \)-continuous function.

Remark 6.4. The converse is not possible in general, as seen from the example 6.2.

The following theorem gives the conditions under which the reversible implication of Theorem 6.3 is true.

Theorem 6.5. If \((Y, \sigma)\) is regular space, then every weakly \( \hat{\Omega} \)-continuous function \( f: (X, \tau) \to (Y, \sigma) \) is \( \Omega \)-continuous function.

Proof: Let \( x \in X \) and \( V \) be any open set in \( Y \) containing \( f(x) \). Since \( Y \) is regular, there exists an open set \( W \) in \( Y \) such that \( f(x) \in W \subseteq \text{cl}(W) \subseteq V \). Since \( f \) is weakly \( \hat{\Omega} \)-continuous, there exists \( \Omega \)-open set in \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq \text{cl}(W) \subseteq V \). Therefore, \( f \) is \( \hat{\Omega} \)-continuous function.

The following theorem states the characterization of weakly \( \hat{\Omega} \)-continuous function.
Theorem 6.6. The following statements are equivalent.

(i) \( f : (X, \tau) \to (Y, \sigma) \) is weakly \( \hat{\Omega} \)-continuous function.

(ii) \( \hat{\Omega}cl(f^{-1}(int(V))) \subseteq f^{-1}(cl(V)) \) for every subset \( V \) of \( Y \).

(iii) \( \hat{\Omega}cl(f^{-1}(int(F))) \subseteq f^{-1}(F) \) for every regular closed subset \( F \) of \( Y \).

(iv) \( \hat{\Omega}cl(f^{-1}(U)) \subseteq f^{-1}(cl(U)) \) for every open subset \( U \) of \( Y \).

(v) \( f^{-1}(U) \subseteq \hat{\Omega}int(f^{-1}(cl(V))) \) for every open subset \( U \) of \( Y \).

Proof:

(i) \( \Rightarrow \) (ii) Suppose that \( V \) is any subset of \( Y \) and \( x \notin f^{-1}(cl(V)) \). Then \( f(x) \notin cl(V) \) implies that there exists an open set \( U \) in \( Y \) containing \( f(x) \) such that \( U \cap V = \emptyset \). Therefore \( U = int(U) \subseteq int(X \setminus V) = (X \setminus cl(V)) \). Then \( cl(U) \cap int(cl(V)) = \emptyset \) and hence \( f^{-1}(int(cl(V))) \subseteq (X \setminus f^{-1}(cl(U))) \). Since \( f \) is weakly \( \hat{\Omega} \)-continuous, there exists \( \hat{\Omega} \)-open set \( W \) in \( X \) containing \( x \) such that \( f(W) \subseteq cl(U) \) and then \( W \subseteq f^{-1}(cl(U)) \). Therefore, \( f^{-1}(int(cl(V))) \subseteq (X \setminus W) \) where \( (X \setminus W) \) is a \( \hat{\Omega} \)-closed set in \( X \). By [8] the definition of \( \Omega \) closure, \( \hat{\Omega}cl(f^{-1}(int(cl(V)))) \subseteq (X \setminus W) \). Then \( W \cap \hat{\Omega}cl(f^{-1}(int(cl(V)))) = \emptyset \). Since \( x \in W, x \notin \hat{\Omega}cl(f^{-1}(int(cl(V)))) \).

(ii) \( \Rightarrow \) (iii) Suppose that \( F \) is any regular closed set in \( Y \). By hypothesis, \( \hat{\Omega}cl(f^{-1}(int(cl(F)))) \subseteq f^{-1}(cl(int(F))) \). Then, \( \hat{\Omega}cl(f^{-1}(int(F))) \subseteq f^{-1}(F) \).

(iii) \( \Rightarrow \) (iv) Suppose that \( U \) is any open set in \( Y \) and then \( cl(U) \) is a regular closed set in \( Y \). By hypothesis, \( \hat{\Omega}cl(f^{-1}(int(cl(U)))) \subseteq f^{-1}(cl(U)) \). Since \( U \) is open, \( \hat{\Omega}cl(f^{-1}(U)) \subseteq \hat{\Omega}cl(f^{-1}(int(cl(U)))) \subseteq f^{-1}(cl(U)) \). Thus, \( \hat{\Omega}cl(f^{-1}(U)) \subseteq f^{-1}(cl(U)) \).

(iv) \( \Rightarrow \) (v) Suppose that \( U \) is open in \( Y \) and then \( (Y \setminus cl(U)) \) is open in \( Y \). By hypothesis, \( \hat{\Omega}cl(f^{-1}(cl(Y \setminus cl(U)))) \subseteq f^{-1}(cl(Y \setminus cl(U))) \). That is, \( \hat{\Omega}cl(f^{-1}(cl(U))) \subseteq f^{-1}(cl(Y \setminus cl(U))) \). By [8] Theorem 5.15 (i), \( (X \setminus \hat{\Omega}int(f^{-1}(cl(U)))) \subseteq f^{-1}(cl(Y \setminus cl(U))) \). Therefore, \( f^{-1}(U) \subseteq \hat{\Omega}int(f^{-1}(cl(U))) \).

(v) \( \Rightarrow \) (i) Suppose that \( x \in X \) and \( U \) be any open set in \( Y \) containing \( f(x) \). By hypothesis, \( f^{-1}(U) \subseteq \hat{\Omega}int(f^{-1}(cl(U))) \). By [8] Remark 5.13, \( \hat{\Omega}int(f^{-1}(cl(U))) \) is a \( \hat{\Omega} \) open set in \( X \) such that \( \hat{\Omega}int(f^{-1}(cl(U))) \subseteq f^{-1}(cl(U)) \). Therefore, \( f(\hat{\Omega}int(f^{-1}(cl(U)))) \subseteq cl(U) \). If we take \( W = \hat{\Omega}int(f^{-1}(cl(U))) \), then \( W \) is a \( \hat{\Omega} \)-open set in \( X \) containing \( x \) such that \( f(W) \subseteq cl(U) \).

Theorem 6.7. \( f : (X, \tau) \to (Y, \sigma) \) is weakly \( \hat{\Omega} \)-continuous function if and only if \( \hat{\Omega}cl(f^{-1}(int(F))) \subseteq f^{-1}(F) \) for every closed subset \( F \) of \( Y \).

Proof: Necessity- Suppose that \( F \) is any closed set in \( Y \) and then \( (Y \setminus F) \) is open set in \( Y \). By Theorem 6.6, (v), \( f^{-1}(Y \setminus F) \subseteq \hat{\Omega}int(f^{-1}(cl(Y \setminus F))) \). Then, \( (X \setminus f^{-1}(F)) \subseteq \hat{\Omega}int(f^{-1}(Y \setminus int(F))) = \hat{\Omega}int([f^{-1}(Y) \setminus f^{-1}(int(F))] = \hat{\Omega}int([X \setminus f^{-1}(int(F)) = (X \setminus \hat{\Omega}cl(f^{-1}(int(F)))) \). Therefore, \( \hat{\Omega}cl(f^{-1}(int(F))) \subseteq f^{-1}(F) \).

Sufficiency- Suppose that \( x \in X \) and \( U \) be any open set in \( Y \) containing \( f(x) \).
Then, \((Y \setminus U)\) is closed in \(Y\) not containing \(f(x)\). By hypothesis, \(\hat{\Omega}(f^{-1}(\text{int}(Y \setminus U))) \subseteq f^{-1}(Y \setminus U)\). Since \(x \notin f^{-1}(\text{int}(Y \setminus U))\), \(x \notin \hat{\Omega}(f^{-1}(\text{int}(Y \setminus U)))\). By [8] Theorem 5.11, there exists \(\hat{\Omega}\)-open set \(V\) in \(X\) such that \(V \cap f^{-1}(\text{int}(Y \setminus U)) = \emptyset\). Also \(f(V) \cap \text{int}(Y \setminus U) = \emptyset\) implies \(f(V) \subseteq cl(U)\). Thus \(f\) is \(\hat{\Omega}\)-continuous function.

\(\square\)

**Remark 6.8.** If \(f\) and \(g\) are weakly \(\hat{\Omega}\)-continuous functions, then their composition is not always weakly \(\hat{\Omega}\)-continuous function from the following example. However, composition is possible if we consider \(g\) as a continuous function.

**Example 6.9.** \(X = Y = Z = \{a, b, c, d\}\) and \(\tau = \{\emptyset, \{a\}, \{a, b\}, X\}, \sigma = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, Z\}\). Define \(f: (X, \tau) \to (Y, \sigma)\) and \(g: (Y, \sigma) \to (Z, \eta)\) by \(f(a) = b, f(b) = c, f(c) = d, f(d) = b, g(a) = a, g(b) = a, g(c) = d, g(d) = b\). Then \(f\) and \(g\) are weakly \(\hat{\Omega}\)-continuous functions where as \(g \circ f: (X, \tau) \to (Z, \eta)\), defined by \((g \circ f)(x) = g(f(x))\) for all \(x \in X\) is not a weakly \(\hat{\Omega}\)-continuous function.

**Theorem 6.10.** If \(f: (X, \tau) \to (Y, \sigma)\) is weakly \(\hat{\Omega}\)-continuous function and \(g\) is continuous,then \(g \circ f: (X, \tau) \to (Z, \eta)\) is weakly \(\hat{\Omega}\)-continuous function.

**Proof:** Suppose that \(x \in X\) and \(W\) is any open set in \(Z\) containing \((g \circ f)(x)\).
Since \(g\) is continuous, \(g^{-1}(W)\) is open set in \(Y\) containing \(f(x)\). Since \(f\) is weakly \(\hat{\Omega}\)-continuous, there exists \(\hat{\Omega}\)-open set \(U\) in \(X\) containing \(x\) such that \(f(U) \subseteq cl(g^{-1}(W))\). Since \(g\) is continuous, \((g \circ f)(U) \subseteq g(f(U)) \subseteq g(cl(g^{-1}(W))) \subseteq cl(gg^{-1}(W)) \subseteq cl(W)\). Therefore, \(g \circ f: (X, \tau) \to (Z, \eta)\) is weakly \(\hat{\Omega}\)-continuous function.

\(\square\)

**Theorem 6.11.** \(f: (X, \tau) \to (Y, \sigma)\) is a surjective function such that \(f(U)\) is \(\hat{\Omega}\)-open set in \(Y\) for any \(\hat{\Omega}\)-open set in \(U\) in \(X\) and \(g: (Y, \sigma) \to (Z, \eta)\) is any function. If \(g \circ f: (X, \tau) \to (Z, \eta)\) is weakly \(\hat{\Omega}\)-continuous,then \(g\) is weakly \(\hat{\Omega}\)-continuous function.

**Proof:** Suppose that \(y \in Y\) and \(W\) is any open set in \(Z\) containing \(g(y)\). Since \(f\) is surjective, there exists \(x \in X\) such that \(f(x) = y\). Then \((g \circ f)(x) = g(f(x)) = g(y) \in W\). Since \(g \circ f\) is weakly \(\hat{\Omega}\)-continuous, there exists an \(\hat{\Omega}\)-open set \(U\) in \(X\) containing \(x\) such that \((g \circ f)(U) \subseteq cl(W)\). If we take \(V = f(U)\), then by our assumption, \(V\) is a \(\hat{\Omega}\)-open set in \(Y\) containing \(f(x)\) and \((g \circ f)(U) = g(V) \subseteq cl(W)\). Thus, \(g\) is weakly \(\hat{\Omega}\)-continuous function.

\(\square\)

**Theorem 6.12.** If \(f: (X, \tau) \to (Y, \sigma)\) is \(\hat{\Omega}\)-irresolute function and \(g\) is weakly \(\hat{\Omega}\)-continuous, then \(g \circ f: (X, \tau) \to (Z, \eta)\) is weakly \(\hat{\Omega}\)-continuous function.

**Proof:** Suppose that \(x \in X\) and \(W\) is any open set in \(Z\) containing \((g \circ f)(x)\). Since \(g\) is weakly \(\hat{\Omega}\)-continuous,there exists \(\hat{\Omega}\)-open set \(V\) in \(Y\) containing \(f(x)\) such that \(g(V) \subseteq cl(W)\). Since \(f\) is \(\hat{\Omega}\)-irresolute,there exists \(\hat{\Omega}\)-open set \(U\) in \(X\) containing \(x\) such that \(f(U) \subseteq V\). Then \((g \circ f)(U) \subseteq g(f(U)) \subseteq g(V) \subseteq cl(W)\). Therefore, \(g \circ f: (X, \tau) \to (Z, \eta)\) is weakly \(\hat{\Omega}\)-continuous function.

\(\square\)
Theorem 6.13. If $f: (X, \tau) \to (Y, \sigma)$ is weakly $\hat{\Omega}$-continuous and $Y$ is Hausdorff, then $f$ has $\hat{\Omega}$-closed points inverses.

Proof: Suppose that $y \in Y$ and $x \in \{x \in X : f(x) \neq y\}$. Since $f(x) \neq y$ in a Hausdorff space $Y$, there exists two disjoint open sets $U$ and $V$ in $Y$ such that $f(x) \not\in U$ and $y \in V$. Moreover, $cl(U) \cap V = \emptyset$ implies that $y \not\in cl(U)$. Since $f$ is weakly $\hat{\Omega}$-continuous, there exists $\hat{\Omega}$-open set $W$ in $X$ containing $x$ such that $f(W) \subseteq cl(U)$. If $W \not\subseteq \{x \in X : f(x) \neq y\}$, then choose a point $z \in W$ such that $f(z) = y$. Then $y = f(z) \in f(W) \subseteq cl(U)$, a contradiction. Hence $x \in W \subseteq \{x \in X : f(x) \neq y\}$ and hence $\{x \in X : f(x) \neq y\}$ is $\hat{\Omega}$-open in $X$. Therefore, $f^{-1}(y) = \{x \in X : f(x) = y\}$ is $\hat{\Omega}$-closed in $X$. \hfill $\blacksquare$

Definition 6.14. If $A \subseteq X$, weakly $\hat{\Omega}$-continuous retraction of $X$ onto $A$ is a weakly $\hat{\Omega}$-continuous function $f: X \to A$ such that $f(a) = a$ for each $a \in A$.

Theorem 6.15. If $A \subseteq X$, and $f: X \to A$ is a weakly $\hat{\Omega}$-continuous retraction of a Hausdorff space $X$ onto $A$, then $A$ is $\hat{\Omega}$-closed in $X$.

Proof: Suppose that there exists $x \in \hat{\Omega}cl(A) \setminus A$. Then $x \neq f(x)$. Since $X$ is Hausdorff, there exists two disjoint open sets $U$ and $V$ in $X$ such that $x \in U$ and $f(x) \in V$. Also $U \cap cl(V) = \emptyset$. Let $W$ be an arbitrary $\hat{\Omega}$-open set in $X$ containing $x$. By [8] Theorem 4.12, $U \cap W$ is a such that $\hat{\Omega}$-open set in $X$ containing $x$. Since $x \in \hat{\Omega}cl(A)$, by [8] Theorem 5.11, $U \cap W \cap A \neq \emptyset$. If we choose $y \in U \cap W \cap A$, then $y \in A$ implies that $y = f(y) \in U$ and hence $f(y) \not\in cl(V)$. That is, there exists $y \in W$ such that $f(y) \not\in cl(V)$. Therefore, $f(W) \not\subseteq cl(V)$ for any $\hat{\Omega}$-open set in $X$ containing $x$, a contradiction. Thus, $A$ is $\hat{\Omega}$-closed in $X$. \hfill $\blacksquare$

Definition 6.16. A space $(X, \tau)$ is called Urysohn space if for every pair of distinct points $x$ and $y$ in $X$, there exists open sets $U$ and $V$ in $X$ such that $x \in U$, $y \in V$ and $cl(U) \cap cl(V) = \emptyset$.

Theorem 6.17. If $f: (X, \tau) \to (Y, \sigma)$ is injective weakly $\hat{\Omega}$-continuous from $X$ into a Urysohn space $Y$, then $X$ is $\hat{\Omega}$-$T_2$-space.

Proof: Suppose that $x, y \in X$ such that $x \neq y$. Since $f$ is injective, $f(x) \neq f(y)$ and since $Y$ is Urysohn space, there exists two open sets $U$ and $V$ in $X$ such that $f(x) \in U$, $f(y) \in V$ and $cl(U) \cap cl(V) = \emptyset$. Since $f$ is weakly $\hat{\Omega}$-continuous, there exists $\hat{\Omega}$-open set $U_1$ and $V_1$ in $X$ containing $x$ and $y$ respectively such that $f(U_1) \subseteq cl(U)$, $f(V_1) \subseteq cl(V)$. Moreover, $U_1 \cap V_1 = \emptyset$. Thus, $X$ is $\hat{\Omega}$-$T_2$-space. \hfill $\blacksquare$

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