Minimal inference from incomplete $2 \times 2$-tables

Li-Chun Zhang

*University of Southampton, UK*

and Raymond L. Chambers

*University of Wollongong, Australia*

**Summary.** Estimates based on $2 \times 2$ tables of frequencies are widely used in statistical applications. However, in many cases these tables are incomplete in the sense that the data required to compute the frequencies for a subset of the cells defining the table are unavailable. Minimal inference addresses those situations where this incompleteness leads to target parameters for these tables that are interval, rather than point, identifiable. In particular, we develop the concept of corroboration as a measure of the statistical evidence in the observed data that is not based on likelihoods. The corroboration function identifies the parameter values that are the hardest to refute, i.e., those values which, under repeated sampling, remain interval identified. This enables us to develop a general approach to inference from incomplete $2 \times 2$ tables when the additional assumptions required to support a likelihood-based approach cannot be sustained based on the data available. This minimal inference approach then provides a foundation for further analysis that aims at making sharper inference supported by plausible external beliefs.

**Keywords:** Identification region; Likelihood; Assurance; Observed power of rejection; Missing data; Ecological inference

1 Introduction

Incomplete $2 \times 2$ tables are often encountered in statistical analysis. Table 1 illustrates the two cases that we pay special attention to in this paper. Both tables correspond to the cross-classification of two binary variables. To the left, $X = 1, 0$ is the outcome variable of interest, and $R = 1, 0$ indicates whether an observation is missing or not. The two-way table is incomplete since $X$ is only observed if $R = 1$. We refer to it as the *missing data setting*. The two-way table on the right shows the joint distribution of two binary variables $X$ and $Y$. This table is completely unobserved. Instead, one has observations on two independent samples of $n_1$ values of $X$ and $n_2$ values of $Y$, respectively. We refer to it as the *matched data setting*. For either case, we assume that the complete data corresponding to the unobserved $2 \times 2$ table follow a multinomial distribution $P_\lambda$, with parameter $\lambda = (\lambda_{11}, \lambda_{10}, \lambda_{01}, \lambda_{00})$ referring to the probabilities of observing each of the four...
possible configurations of two binary variables. Under this assumption the parameter \( \lambda \) is \textit{point-identifiable} given the complete data, in the sense that \( \lambda = \lambda' \) whenever \( P_\lambda = P_{\lambda'} \); and so is the parameter of interest \( \theta = \Pr(X = 1) \) in the missing data setting and \( \theta = \Pr(X = 1, Y = 1) \) in the matched data setting.

Table 1: Two cases of incomplete 2 \( \times \) 2-table. Left: binary variable subjected to missing data, sample size \( n \); Right: statistical matching of two binary variables from separate samples of sizes \( n_1 \) and \( n_2 \), respectively. Unobserved hypothetical complete sample counts marked by ‘–’.

| Hypothetical Complete Sample Data: \((n_{11}, n_{01}, n_{10}, n_{00}) \sim \text{multinomial}(n, \lambda_{11}, \lambda_{01}, \lambda_{10}, \lambda_{00})\) | \begin{tabular}{llllll}
| \( R = 1 \) | \( R = 0 \) | Total | \( Y = 1 \) | \( Y = 0 \) | Total |
|---|---|---|---|---|---|
| \( X = 1 \) | \( n_{11} \) | \( n_{01} \) | \( n \) | \( n_x \) | \( n_1 - n_x \) |
| \( X = 0 \) | \( n_{10} \) | \( n_{00} \) | \( n \) | \( n_y \) | \( n_2 - n_y \) |
| Total | \( n_{+1} \) | \( n_{+0} \) | \( n \) | Total | Observation: Independent \((n_x, n_1), (n_y, n_2)\) |
| Sampling | Observation: Independent \((n_x, n_1), (n_y, n_2)\) |
| Distribution: \(\text{multinomial}(n, \lambda_{11}, \lambda_{01}, \lambda_{10}, \lambda_{00})\) | Distribution: \(\text{binomial}(n, \lambda_{1+})\) |
| \( \lambda_{11}, \lambda_{01}, \lambda_{10}, \lambda_{00} \) | \( \lambda_{1+} = \lambda_{11} + \lambda_{01} \) |
| Parameter of Interest: \( \theta = \lambda_{1+} \) | Parameter of Interest: \( \theta = \lambda_{11} \) |
| Identification Region: \( \lambda_{11} \leq \theta \leq \lambda_{11} + \lambda_{10} \) | Identification Region: \( \theta \leq \min(\lambda_{1+}, \lambda_{11}) \cap \theta \geq \max(\lambda_{1+}, \lambda_{11}) - 1, 0 \) |

Table 1 also shows the sampling distribution of the observed data for each setting. Point-identification for \( \theta \) based on the observed data is only achievable if additional assumptions are made. In this table these assumptions are independence of \( X \) and \( R \) in the missing data setting, which means missing-completely-at-random (MCAR, Rubin, 1976), and independence of \( X \) and \( Y \) in the statistical matching setting, which is a special case of the conditional independence assumption (Okner, 1972). But such additional assumptions are often contentious. It therefore seems reasonable ask ‘what the data say’ about \( \theta \) given the \textit{accepted} sampling distribution of the observed data, \textit{without} the additional “esoteric” (Tamer, 2010) assumptions that enable point-identification of this parameter. The aim of this paper is to describe a general approach to inference based on incomplete \( 2 \times 2 \) tables given such a setting.

To illustrate, consider a missing data example discussed by Zhang (2010). The observed data from the Obstructed Coronary Bypass Graft Trials (OCBGT, see Hollis, 2002) are \((n_{11}, n_{01}, n_{+0}) = (32, 54, 24)\), with the sampling distribution parameter \( \psi = (\lambda_{11}, \lambda_{01}, \lambda_{+0}) \). The likelihood of \( \psi \) is proportional to \( \lambda_{11}^{n_{11}} \lambda_{01}^{n_{01}} \lambda_{+0}^{n_{+0}} \). This yields the profile likelihood of the parameter of interest \( \theta = \lambda_{1+} \), denoted by \( L_p(\theta) \), which is the dashed curve in Figure 1. It is seen that \( L_p(\theta) \) is flat over \([n_{11}/n, (n_{11} + n_{+0})/n]\), which we call the \textit{maximum likelihood region}, denoted by \( \tilde{\Theta} \), with all values of \( \theta \) in \( \tilde{\Theta} \) equally likely based on the observed data. Asymptotically, as \( n \to \infty \), \( \tilde{\Theta} \) tends to the \textit{identification region} of \( \theta \), i.e. \( \lambda_{11} \leq \theta \leq \lambda_{11} + \lambda_{+0} \), which is a function of the identifiable...
parameter $\psi$. This identification region is the asymptote of ‘what the data say’ about $\theta$ under the setting here. The dotted curve gives the standardised likelihood under the additional MCAR assumption that enables point-identification of $\theta$. It peaks at the maximum likelihood estimate (MLE) $\hat{\theta}_{MCAR} = n_{11}/n_{+1}$, which converges to $\lambda_{11}/\lambda_{+1}$ in probability. Clearly, the MLE derived from the MCAR likelihood will be inconsistent as long as $\lambda_{1+} \neq \lambda_{11}/\lambda_{+1}$.

Figure 1: Observed corroboration and standardised likelihoods (with peak value 1) based on OCBGT data: observed corroboration (solid), profile likelihood (dashed), likelihood under MCAR assumption (dotted); maximum likelihood region marked by vertical dashed lines.

The fact that the profile likelihood shown in Figure 1 is constant within the observed $\hat{\Theta}$ does not mean that all the values of $\theta$ in it are equally likely to be in a $\hat{\Theta}$ that could be observed given a random draw from the sampling distribution of the observed data. In Section 2 we develop the concept of corroboration, noting that values of $\theta$ that are more likely to appear in a $\hat{\Theta}$ on repeated sampling are better corroborated by the observed data than values of $\theta$ that only infrequently appear in a $\hat{\Theta}$. The solid curve in Figure 1 shows how the estimated corroboration varies with $\theta$ for the OCBGT data. The computation of the estimated corroboration is explained in Section 2. The key point to note here is that the corroboration varies for the points within $\hat{\Theta}$, where the profile likelihood is constant. This allows us to construct high corroboration level sets within $\hat{\Theta}$. It will be shown that asymptotically the set of values with the maximum observed corroboration becomes indistinguishable from the identification region except for its bounds. Unlike the MLE that aims at the most likely parameter value, the maximum corroboration set identifies those parameter values that are the hardest to refute based on the observed data. In effect, these are the points in which we have the highest confidence. We develop a Corroboration Test in Section 5 for the settings of Table 1 where the Likelihood Ratio Test is inapplicable insofar as the parameter.
of interest is not point identifiable. The test will be applied the OCBGT data.

There are several related approaches within the matched data setting. In ecological inference (Goodman, 1953; King, 1997), the observed data are the margins of the unobserved complete $2 \times 2$ table. See Wakefield (2004) for a comprehensive review. It is clearly recognised that critical but untestable assumptions are needed to arrive at a point estimate in this context, and that there is a fundamental difficulty associated with choosing between different models based only on the observed data; see Greenland and Robins (1994), Freedman (2001) and Gelman et al. (2001). Statistical matching deals with the same setting, where the set of multinomial distributions $P_\lambda$ compatible with the sampling distribution of the observed data is referred to as the uncertainty space. Evaluation of the uncertainty space has received much attention (Kadane, 1978; Moriarity and Scheuren, 2001; D’Orazio et al., 2006; Kiesel and Rässler, 2006; Conti et al. 2012; Zhang, 2015; Conti et al., 2015). The concept of uncertainty space is closely related to that of identification uncertainty (Koopmans, 1949; Tamer, 2010). The “partial identification” framework (Manski, 1995, 2003, 2007) recognises situations where, due to the structure of the data, even a hypothetical infinite number of observations may only constrain the parameter of interest without being able to point-identify it. It is important in this context to distinguish between the study of identification, provided an infinite amount of data under the given structure, and statistical inference from finite samples. Partial identification in econometrics can be traced back to Frisch (1934) and Marschak and Andrews (1944), and there is a growing literature on the construction of confidence regions of the identified parameter set. See e.g. Imbens and Manski (2004), Chernozhukov et al. (2007), Beresteanu and Molinari (2008), and Ramano and Shaikh (2010).

It is clear that all the aforementioned approaches aim at inference based on an identifiable sampling distribution that is acceptable to all, no matter which untestable additional assumptions an analyst may or may not introduce in order to resolve the identification issue. As seen in Figure 1, the novelty of the approach proposed in this paper is that it achieves this objective via a measure of the statistical evidence in the observed data that is not based on comparing likelihoods.

2 Corroboration

Denote by $f(d_n; \psi)$ the identifiable sampling distribution of the observed data $d_n$ with generic sample size $n$, and with parameter $\psi$. Denote by $P_\lambda$ the distribution of the hypothetical complete data, which is characterised by the parameter $\lambda$ with parameter space $\Lambda$. Denote by $\theta = \theta(\lambda)$ a scalar parameter of interest, and by $\Theta$ the parameter space of $\theta$. For any given $\psi$ let $\Lambda(\psi)$ be the constrained parameter space defined by $\psi$. That is, $\Lambda(\psi)$ consists of all $\lambda$ that are consistent with $\psi$. Let $\Theta(\psi)$ be the induced parameter space of $\theta$, which contains all $\theta(\lambda)$ where $\lambda \in \Lambda(\psi)$. For inference under a minimal setting in this paper, we then require both conditions below to hold.

(M1) The induced parameter space $\Theta(\psi)$ is a closed interval. In particular, it is not a singleton $\Theta(\psi) = \theta(\psi)$, nor is it invariant towards $\psi$ in the sense that $\Theta(\psi) = \Theta(\psi')$ for all $\psi \neq \psi'$. 


(M2) The parameter $\psi$ of the sampling distribution is point-identifiable, and the MLE $\hat{\psi}$ is such that $\hat{\psi} \xrightarrow{p} \psi_0$, asymptotically as $n \to \infty$, where $\psi_0$ is the true parameter value.

Under a minimal setting, $\Theta(\psi) = [L(\psi), U(\psi)]$, where $L(\psi)$ is the lower bound of $\theta$ induced by $\psi$, and $U(\psi)$ the upper bound. The identification region is $\Theta_0 = \Theta(\psi_0) = [L_0, U_0]$, where $L_0 = L(\psi_0)$ and $U_0 = U(\psi_0)$. Thus, for the missing data setting in Table 1 we have $\psi_0 = (\lambda_{11}^0, \lambda_{01}^0, \lambda_{+0}^0)$, with

$$\Theta_0 = [L_0, U_0] = [\lambda_{11}^0, \lambda_{01}^0 + \lambda_{+0}^0].$$

For the matched data setting, we have $\psi_0 = (\lambda_{1+}^0, \lambda_{+1}^0)$, and the Fréchet bounds (Fréchet, 1951) define the identification region

$$\Theta_0 = [L_0, U_0] = [\max(\lambda_{1+}^0 + \lambda_{+1}^0 - 1, 0), \min(\lambda_{1+}^0, \lambda_{+1}^0)].$$

Let $\hat{L} = L(\hat{\psi})$ and $\hat{U} = U(\hat{\psi})$ be the MLEs of $L_0$ and $U_0$, respectively, and let $\hat{\Theta} = \Theta(\hat{\psi}) = [\hat{L}, \hat{U}]$ denote the maximum profile likelihood estimator of $\theta$. The points inside $\hat{\Theta}$ can all be considered as equally most likely, i.e. best supported according to the likelihood based on $d_n$ under the observed data model. We define the corroboration function of $\theta$, for $\theta \in \Theta$, to be

$$c(\theta; \psi) = \Pr(\theta \in \hat{\Theta}; \psi),$$

i.e. the probability for the given value of $\theta$ to be covered by $\hat{\Theta}$, where the probability is evaluated with respect to $f(d_n; \psi)$. Let the actual corroboration be

$$c_0(\theta) = c(\theta; \psi_0),$$

i.e. evaluated over the true sampling distribution. In particular, $c(\theta_0; \psi_0)$ is the confidence level of $\hat{\Theta}$ as an interval estimator of $\theta_0$. Let the observed corroboration be

$$\hat{c}(\theta) = c(\theta; \hat{\psi}).$$

Since $\hat{c}(\theta)$ is the MLE of $c_0(\theta)$, one may then define the observed corroboration as the most likely level of corroboration for $\theta$ given the observed data. As illustrated in Figure 1 for the OCBGT data, if one treats the observed corroboration as a function of $\theta$ then this function can generally vary over $\hat{\Theta}$, as opposed to the profile likelihood which is flat over the same region. Note that in this case in order to calculate $\hat{c}(\theta)$, where $(\hat{\lambda}_{11}, \hat{\lambda}_{+0}) = (n_{11}/n, n_{11}/n + n_{+0}/n)$, we employ the bivariate normal approximation $(\hat{\lambda}_{11}, \hat{\lambda}_{+0}) \sim N_2(\mu, \Sigma)$, where $\mu = (\lambda_{11}, \lambda_{+0})$ and the distinctive elements of $\Sigma$ are $V(\hat{\lambda}_{11}) = \lambda_{11}(1 - \lambda_{11})/n$, $V(\hat{\lambda}_{+0}) = \lambda_{+0}(1 - \lambda_{+0})/n$ and $Cov(\hat{\lambda}_{11}, \hat{\lambda}_{+0}) = -\lambda_{11}\lambda_{+0}/n$.

More generally, the observed corroboration can be calculated via simulation as follows.

**Bootstrap for $\hat{c}(\theta)$** For given $\theta$ and the MLE $\hat{\psi}$, repeat for $b = 1, \ldots B$: 

5
- generate $d_n^{(b)}$ from $f(d_n; \hat{\psi})$ to obtain $\hat{\psi}^{(b)}$ and the corresponding $[L(\hat{\psi}^{(b)}), U(\hat{\psi}^{(b)})]$;
- set $\delta^{(b)} = 1$ if $\theta \in [L(\hat{\psi}^{(b)}), U(\hat{\psi}^{(b)})]$, and 0 otherwise.

Put $\bar{c}(\theta) = \sum_{b=1}^B \delta^{(b)}/B$ as the bootstrap estimate of the observed corroboration for $\theta$. □

3 Maximum corroboration set

Let the level-$\alpha$ corroboration set be given by

$$A_\alpha(\psi) = \{\theta : c(\theta; \psi) \geq \alpha\},$$

provided there exists some $\theta \in A_\alpha(\psi)$ where $c(\theta; \psi) = \alpha$. Thus, by definition we have $c(\theta; \psi) < \alpha$, for any $\theta \notin A_\alpha(\psi)$, whilst we cannot have $c(\theta; \psi) > \alpha$ for all $\theta \in A_\alpha(\psi)$. Some properties of $A_\alpha(\psi)$ are given below, with proofs in the Appendix. Notice that we use $c(\theta)$ as a short-hand for $c(\theta; \psi)$ and $A_\alpha$ that of $A_\alpha(\psi)$, where it is not necessary to emphasise their dependence on $\psi$.

**Theorem 1** Suppose that a minimal inference setting applies, i.e. provided conditions (M1) and (M2) hold. Then:

(i) Let $A_{\alpha_1} = [L_1, U_1]$ and $A_{\alpha_2} = [L_2, U_2]$. If $\alpha_1 > \alpha_2$, then $[L_1, U_1] \subset [L_2, U_2]$.

(ii) Let $\theta_L < \theta_U$, where $c(\theta_L) = c(\theta_U) = \alpha$. Then $c(\theta) \geq \alpha$ for any $\theta \in (\theta_L, \theta_U)$.

**Theorem 2** Given a minimal inference setting, there exists a maximum corroboration value denoted by $\theta^{\text{max}}$, such that $c(\theta^{\text{max}}) \geq c(\theta)$ for any $\theta \neq \theta^{\text{max}}$.

Denote by $A^{\text{max}} = A^{\text{max}}(\psi_0)$ the maximum corroboration set, such that $c_0(\theta) > c_0(\theta')$ for any $\theta \in A^{\text{max}}$ and $\theta' \notin A^{\text{max}}$, and $c_0(\theta) = c_0(\theta')$ for any $\theta \neq \theta' \in A^{\text{max}}$. It follows from (1) that these are the points for which $\hat{\Theta}$ implies the highest confidence, in which sense one may consider these to be the parameter values that are the hardest to refute. Replacing $\psi_0$ by $\hat{\psi}$, we obtain the MLE of $A^{\text{max}}$ or the observed maximum corroboration set

$$\hat{A}^{\text{max}} = A^{\text{max}}(\hat{\psi}).$$

Figure 2 illustrates corroboration in the matched data setting, where $\theta = \lambda_{11}$. The true sampling distribution parameters $(\lambda_{1+}, \lambda_{+1})$ are $(0.1, 0.9)$ for the left plot and $(0.3, 0.3)$ to the right. The sample sizes are $(n_1, n_2) = (1000, 500)$ to the left and $(200, 300)$ to the right. The identification region $\Theta_0$ is the interval between the vertical dashed lines, and the solid curve shows how the actual corroboration (denoted $c_{\text{value}}$ in the plots) varies with $\theta$. The corroboration of some interior points of $\Theta_0$ can be 1, whereas it can be 0 for many $\theta \notin \Theta_0$. In the left plot, both $c_0(L_0)$ and $c_0(U_0)$ are about 0.5; in the right plot, we have $c_0(L_0) = 1$ and $c_0(U_0) \approx 0.25$. 

6
Let $\tilde{c}(\theta; \psi) = \lim_n c(\theta; \psi) = \lim_n \Pr(\theta \in \hat{\Theta}_n; \psi)$ be the asymptotic corroboration of $\theta$ evaluated at $\psi$, where $\lim_n$ stands for $\lim_{n \to \infty}$ and $\hat{\Theta}_n$ makes explicit the dependence on sample size. Table 2 summarises the asymptotic actual corroboration $\tilde{c}_0(\theta) = \tilde{c}(\theta; \psi_0)$ for both data settings. Let $\hat{A}_{\text{max}}$ be the asymptotic maximum actual corroboration set based on $\tilde{c}_0(\theta)$. Lemma 1 states that, apart from the bounds $L_0$ and $U_0$, $\hat{A}_{\text{max}}$ is indistinguishable from $\Theta_0$ and $\tilde{c}_0(\theta)$ is an indicator function on $\Theta_0$. Theorem 3 states that the interior of the observed maximum corroboration set $\hat{A}^\text{max}_n$ converges to the interior of $\Theta_0$ in probability. The proofs are given in the Appendix.

**Lemma 1** Given a minimal inference setting, $\theta \in \hat{A}_{\text{max}}$ and $\tilde{c}_0(\theta) = 1$ if $\theta \in \text{Int}(\Theta_0) = (L_0, U_0)$, i.e. if $\theta$ belongs to the interior of $\Theta_0$, then $\theta \notin A_{\text{max}}^\text{max}$ and $\tilde{c}_0(\theta) = 0$ for any $\theta \notin [L_0, U_0]$.

**Theorem 3** Given a minimal inference setting, we have $\text{Int}(\hat{A}^\text{max}_n) \xrightarrow{\text{Pr}} \text{Int}(\Theta_0)$; that is, $\lim_n \Pr(\theta \in \hat{A}^\text{max}_n) = 1$ if $\theta \in \text{Int}(\Theta_0)$ and $\lim_n \Pr(\theta \in \hat{A}^\text{max}_n) = 0$ if $\theta \notin \Theta_0$. 

### Table 2: Asymptotic actual corroboration $\tilde{c}_0(\theta)$ in missing and matched data settings

| Data Setting | $\theta \notin [L_0, U_0]$ | $\theta = L_0$ | $\theta \in (L_0, U_0)$ | $\theta = U_0$ |
|--------------|-----------------------------|----------------|--------------------------|----------------|
| Missing      | 0                           | 0.5 if $L_0 > 0$ | 1                        | 0.5 if $U_0 < 1$ |
| Matching     | 0                           | 0.5 if $\lambda_{1+} + \lambda_{+1} \geq 1$ | 1                        | 0.5 if $\lambda_{1+} \neq \lambda_{+1}$ |
|              | 1                           | 1 if $\lambda_{1+} + \lambda_{+1} < 1$       |                          | 0.25 if $\lambda_{1+} = \lambda_{+1}$ |

Figure 2: Illustration of corroboration in matched data setting. Left: $(\lambda_{1+}, n_1) = (0.1, 1000)$ and $(\lambda_{+1}, n_2) = (0.9, 500)$. Right: $(\lambda_{1+}, n_1) = (0.3, 200)$ and $(\lambda_{+1}, n_2) = (0.3, 300)$. 

Let $\hat{\Theta}_n$ be the asymptotic maximum actual corroboration set based on $\tilde{c}_0(\theta)$. Lemma 1 states that, apart from the bounds $L_0$ and $U_0$, $\hat{A}_{\text{max}}$ is indistinguishable from $\Theta_0$ and $\tilde{c}_0(\theta)$ is an indicator function on $\Theta_0$. Theorem 3 states that the interior of the observed maximum corroboration set $\hat{A}^\text{max}_n$ converges to the interior of $\Theta_0$ in probability. The proofs are given in the Appendix.
4 High assurance estimation of $\Theta_0$

Given a minimal inference setting, a confidence region $C_n$ for $\Theta_0$ (which is an interval) has the confidence level $\Pr(\Theta_0 \subseteq C_n)$; see e.g. Chernozhukov et al. (2007). Given a high confidence level, the probability that $C_n$ contains points that do not belong to $\Theta_0$ must also be high, due to sampling variability, and so $C_n$ asymptotically contracts towards $\Theta_0$ from ‘outside’ of it. In contrast, any point in $\Theta_0$ is irrefutable, and $\hat{A}^{\text{max}}$ identifies those parameter values that are the hardest to refute given the observed data. We thus define the assurance of $\hat{A}^{\text{max}}$ to be

$$\tau_0 = \Pr(\hat{A}^{\text{max}} \subseteq \Theta_0),$$

where the probability is evaluated with respect to $f(d_n; \psi_0)$. That is, this is the probability that the points in the observed $\hat{A}^{\text{max}}$ are indeed all irrefutable. If $\hat{A}^{\text{max}}$ has a high assurance, there will be a low probability that it contains points outside of $\Theta_0$. As the sample size increases, a high assurance estimator of $\Theta_0$ should therefore grow towards $\Theta_0$ from ‘inside’ of it. In light of Theorem 1, for some small constant $h \geq 0$, a high assurance estimator of $\Theta_0$ can therefore be defined as

$$\hat{A}_h = \{\theta : c(\theta; \hat{\psi}) \geq \max_{\theta} c(\theta; \hat{\psi}) - h\},$$

The following bootstrap can be used to estimate $\hat{A}_h$, including $\hat{A}_0 = \hat{A}^{\text{max}}$.

**Bootstrap for $\hat{A}_h$** Given the MLE $\hat{\psi}$ and the corresponding $[\hat{L}, \hat{U}]$, repeat for $b = 1, \ldots, B$:

1. generate $d_n^{(b)}$ from $f(d_n; \hat{\psi})$, and obtain $\hat{\psi}^{(b)}$;

2. for any given $h$, where $0 \leq h < 1$, obtain $\hat{A}_h^{(b)}$ at $\hat{\psi}^{(b)}$ in the same way as $\hat{A}_h$ at $\hat{\psi}$, and the corresponding $L^{(b)} = L(\hat{A}_h^{(b)})$ and $U^{(b)} = U(\hat{A}_h^{(b)})$;

3. set $\delta^{(b)} = 1$ if $\hat{L} \leq L^{(b)} < U^{(b)} \leq \hat{U}$, and $\delta^{(b)} = 0$ otherwise.

Calculate the bootstrap estimate of assurance as $\hat{\tau}(\hat{A}_h; \psi_0) = \sum_{b=1}^B \delta^{(b)}/B$, with corresponding bootstrap estimate of the lower end of $\Theta_0$ given by $L(\hat{A}_h) = \sum_{b=1}^B L^{(b)}/B$ and of the upper end of $\Theta_0$ given by $U(\hat{A}_h) = \sum_{b=1}^B U^{(b)}/B$. □

For small $h$, $\hat{A}_h$ can have higher assurance than $\hat{\Theta}$, whereas it can be ‘closer’ to $\hat{A}^{\text{max}}$ than $\hat{A}_0$ by Theorem 1, since $\hat{A}_0 \subset \hat{A}_h$. Setting $h < 0.25$ makes $\text{Int}(\hat{A}_h)$ asymptotically indistinguishable from $\text{Int}(\Theta_0)$ for the two settings depicted in Table 2. In a finite-sample situation, one may calculate $\hat{A}_h$ and its assurance for several different choices of $h$. Since the length of $\hat{A}_h$ increases with $h$ while its assurance decreases, one may choose the longest $\hat{A}_h$ as an estimator of $\Theta_0$ subject to an acceptable level of assurance.
5 A Corroboration Test

Consider testing the null hypothesis \( H_A : \theta^* \in (L_0, U_0) \) against \( H_B : \theta^* \notin \Theta_0 \). A minimal inference setting for this test is nonstandard because, under both \( H_A \) and \( H_B \), the set of possible distributions of the observed data are exactly the same, i.e. \( f(d_n; \psi) \). The Likelihood Ratio Test is inapplicable. Let instead the test statistic be \( T_n = 1 \) if \( \theta^* \in \text{Int}(\widehat{\Theta}_n) \) and \( T_n = 0 \) if \( \theta^* \notin \widehat{\Theta}_n \). Suppose we reject \( H_A \) if \( T_n = 0 \). The power function of this testing procedure is then \( \beta_n(\theta^*) = \Pr(T_n = 0; \psi_0) \), and is such that

\[
\overline{\beta}(\theta^*) \equiv \lim_{n \to \infty} \beta_n(\theta^*) = 1 - \lim_{n \to \infty} \Pr(T_n = 1; \psi_0) = 1 - \overline{c}_0(\theta^*).
\]

If \( H_A \) is true, but \( T_0 = 0 \) and we reject \( H_A \), by Lemma 1 the probability of Type-I error converges to zero since \( \overline{c}_0(\theta^*) = 1 \) if \( \theta \in \text{Int}(\Theta_0) \). Similarly, if \( H_B \) is true, but \( T = 1 \) and we do not reject \( H_A \), the Type-II error probability also asymptotes to zero since \( \overline{c}_0(\theta^*) = 0 \) if \( \theta^* \notin \Theta_0 \).

| \( T_n = 1 \) | Low Power \( \beta_n(\theta^*) \) | High Power \( \beta_n(\theta^*) \) |
|----------------|-----------------------------------|---------------------------------|
| Support \( H_A \) | Support neither, improbable event | Support \( H_B \) |
| \( T_n = 0 \) | Support neither, improbable event | Support \( H_B \) |

Let the \textit{observed power} be \( \hat{\beta}_n(\theta^*) = 1 - \hat{c}_n(\theta^*) \), which is a consistent estimator of \( \overline{\beta}(\theta^*) \). While \( \hat{c}_n(\theta^*) \) is a consistent estimator of the Type-II error probability, we cannot use it to estimate the Type-I error probability. The reason is that \( c_0(\theta^*) \) is the same under \( H_A \) or \( H_B \), due to the minimal inference setting, so that it cannot be related to \textit{both} types of errors. We shall therefore define the Corroboration Test to have observed power \( \beta \), where \( \beta = \hat{\beta}_n(\theta^*) \in (0, 1) \), if \( H_A \) is rejected when \( T_n = 0 \). As summarised in Table 3, a Corroboration Test of high observed power would lead one to reject \( \theta^* \) if it is outside of \( \widehat{\Theta}_n \) and have a low observed corroboration. By the consistency of \( \hat{A}_n^{\max} \) established in Theorem 3, we have

\[
\lim_n \Pr(\text{Reject } H_A \text{ when } H_A \text{ is true}) = 0 < \lim_n \Pr(\text{Reject } H_A \text{ when } H_B \text{ is true}) = 1.
\]

That is, the Corroboration Test is strongly Chernoff-consistent, since \( T_n \) has limiting size 0 and the Type-II error probability converges to 0, for any \( \theta^* \) specified in \( H_A \).

**Theorem 4** Given a minimal inference setting, the Corroboration Test of observed power \( \beta = \hat{\beta}_n(\theta^*) \), for \( \beta \in (0, 1) \), is strongly Chernoff-consistent.
6 Application: Missing OCBGT data

Consider the OCBGT data \( n = (n_{11}, n_{01}, n_{+0}) = (32, 54, 24) \). The profile likelihood is

\[
L_p(\theta) = \begin{cases} 
\frac{\hat{\lambda}_{11} \hat{\lambda}_{i0}^\theta \hat{\lambda}_{+0} \hat{\lambda}_{i+0}^\theta}{n_{11} n_{01} n_{+0} n_{i+0}} & \text{if } \theta < \hat{\lambda}_{11} \\
\frac{\hat{\lambda}_{11} \hat{\lambda}_{i0}^\theta \hat{\lambda}_{i+0} \hat{\lambda}_{+0}^\theta}{n_{11} n_{01} n_{+0} n_{i+0}} & \text{if } \hat{\lambda}_{11} \leq \theta \leq \hat{\lambda}_{11} + \hat{\lambda}_{+0} \\
\frac{n_{11} n_{01} n_{+0} n_{i+0}}{n_{11} n_{01} n_{+0} n_{i+0}} & \text{if } \theta > \hat{\lambda}_{11} + \hat{\lambda}_{+0}
\end{cases}
\]

(Zhang, 2010). The likelihood is \( L_{MCAR}(\theta) \propto n_{11}^{-\theta} n_{01}^{1-\theta} \), under the additional assumption of independent \((X, R)\). Figure 4 plots both, as well as the observed corroboration \( c(\theta) \).

The likelihood \( L_{MCAR} \) does not vary with \( n_{+0} \), e.g. whether this value is 4, 24 or 104. Accordingly \( n_{+0} \) is not part of the available statistical evidence. Clearly, such insensitiveness towards the observed data requires some external belief to sustain. Next, consider the relative plausibility of \( \theta^* = 0.2, 0.3, 0.5, 0.6 \) against \( \theta_1 = 0.4 \) based on the profile likelihood ratio, denoted by \( LR_p(\theta^*, \theta_1) \) in the left part of Table 4. The values 0.3 and 0.5 cannot be distinguished from 0.4, since all are inside \( \hat{\Theta} = [0.29, 0.51] \); the negative evidence of 0.2 and 0.6 against 0.4 is “moderate” according to Royall (1997), as they fall in the range \( 1/32 - 1/8 \). Nevertheless, as noted before, the Likelihood Ratio Test is inapplicable here.

Table 4: Left, profile likelihood ratio \( LR_p(\theta^*, \theta_1) \) with \( \theta_1 = 0.4 \), observed corroboration \( c(\theta^*) \) based on OCBGT data. Right, assurance \( \hat{\tau}(\hat{A}_h; \psi_0) \) of \( \hat{A}_h \), expected left end \( L(\hat{A}_h) \) and right end \( U(\hat{A}_h) \), with values obtained by bootstrap with \( B = 5000 \). In addition, \( \hat{\Theta} : [\hat{L}, \hat{U}] = [0.29, 0.51] \), \( \hat{\tau}(\hat{\Theta}) = 0.19 \).

| \( \theta^* \) | \( LR_p(\theta^*, \theta_1) \) | \( c(\theta^*) \) | \( h \) | \( \hat{\tau}(\hat{A}_h; \psi_0) \) | \( L(\hat{A}_h), U(\hat{A}_h) \) |
|---|---|---|---|---|---|
| 0.2 | 0.076 | 0.018 | 0 | 0.99 | [0.40, 0.40] |
| 0.3 | 1 | 0.583 | 0.01 | 0.95 | [0.38, 0.41] |
| 0.4 | 1 | 0.985 | 0.06 | 0.84 | [0.36, 0.44] |
| 0.5 | 1 | 0.576 | 0.40 | 0.25 | [0.30, 0.50] |
| 0.6 | 0.156 | 0.028 | 0.80 | 0.00 | [0.25, 0.55] |

Now, based on the observed corroboration \( c(\theta^*) \) in Table 4, one may reject the null hypothesis \( H_0 : 0.2 \in \Theta_0 \) on the basis of the Corroboration Test with observed power 0.982. Similarly for \( H_0 : 0.6 \in \Theta_0 \), with observed power 0.972. Meanwhile, 0.3 and 0.5 are just inside \( \hat{\Theta} \), with \( \hat{c}(0.3) \) and \( \hat{c}(0.5) \) slightly below 0.6, and so cannot be rejected with high observed power. The Corroboration Test thus allows us to reject an unlikely value of \( \theta \) with a high observed power.

Finally, five observed corroboration level sets \( \hat{A}_h \) are illustrated in the right part of Table 4 where the estimated assurance \( \hat{\tau}(\hat{A}_h; \psi_0) \) and expected end points \( L(\hat{A}_h) \) and \( U(\hat{A}_h) \) are calculated using the bootstrap described in Section 4. As an estimator of \( \Theta_0 \), \( \hat{A}_0 \) is very narrow but has 99% assurance; \( \hat{A}_{0.01} \) has 95% assurance and is expected to span from 0.38 to 0.41. Using \( \hat{\Theta} \)
as an estimator of $\Theta_0$ would perform comparably to $\tilde{A}_0$, but with low assurance. The observed corroboration level sets thus allow us to identify true irrefutable points in $\Theta_0$ with a high assurance.

References

[1] Beresteanu, A. and Molinari, F. (2008) Asymptotic properties of a class of partially identified models. Econometrica, 76, 763-814.

[2] Chambers, R.L. and Steel, D. (2001). Simple methods for ecological inference in $2 \times 2$ tables. J. R. Statist. Soc. A, 164, 175-192.

[3] Chao, A. (1987). Estimating the population size for capture-recapture data with unequal catchability. Biometrics, 43, 783-791.

[4] Chernozhukov, V., Hong, H., Tamer, E. (2007) Estimation and confidence regions for parameter sets in econometric models. Econometrica, 75, 1243-1284.

[5] Conti, P.L., Marella, D. and Scanu, M. (2015). Statistical matching analysis for complex survey data with applications. J. Am. Statist. Ass., DOI: 10.1080/01621459.2015.1112803

[6] Conti, P.L., Marella, D. and Scanu, M. (2012) Uncertainty analysis in statistical matching. J. Off. Statist., 28, 69-88.

[7] D’Orazio, M., Di Zio, M. and Scanu, M. (2006) Statistical Matching: Theory and Practice. Chichester: Wiley.

[8] Freedman, D. A. (2001) Ecological inference and the ecological fallacy. In International Encyclopaedia of the Social and Behavioural Sciences (eds N. J. Smelser and P. B. Baltes), vol. 6, pp. 40274030. New York: Elsevier.

[9] Fréchet, M. (1951) Sur les tableaux de correlation dont les marges sont données. Ann. Univ. Lyon A, 3, 53-77.

[10] Frisch, R. (1934) Statistical Confluence Analysis. Publ. No. 5. Oslo: Univ. Inst. Econ.

[11] Gelman, A., Park,D. K., Ansolabehere, S., Price, P.N. and Minnite, L. C. (2001) Models, assumptions and model checking in ecological regressions. J. R. Statist. Soc. A, 164, 101-118.

[12] Goodman, L. (1953) Ecological regressions and the behavior of individuals. Am. Sociol. Rev., 18, 663-666.

[13] Greenland, S. and Robins, J. (1994) Ecological studies biases, misconceptions and counterexamples. Am. J. Epidem., 139, 747-760.
[14] Hollis, S. (2002). A graphical sensitivity analysis for clinical trials with nonignorable missing binary outcome. *Stat. Med.*, 21, 3823-3834.

[15] Imbens, G. and Manski, C.F. (2004) Confidence intervals for partially identified parameters. *Econometrica*, 72, 1845-1857.

[16] Kadane, J.B. (1978). Some Statistical Problems in Merging Data Files. In *1978 Compendium of Tax Research*, pp. 159-171. U.S. Department of Treasury. (Reprinted in J. Off. Statist., 17, 423-433.)

[17] Kapadia, A.S., Chan, W. and Moyé, L. (2005) *Mathematical Statistics with Applications*. Chapman & Hall/CRC.

[18] Kiesl, H. and Raessler, S. (2006) *How valid can data fusion be?* Institut fur Arbeitsmarkt- und Berufsforschung (IAB) Discussion Paper 15/2006.

[19] King, G. (1997) *A Solution to the Ecological Inference Problem: Reconstructing Individual Behavior from Aggregate Data*. Princeton: Princeton University Press.

[20] Koopmans, T. (1949) Identification problems in economic model construction. *Econometrica*, 17, 125-144.

[21] Manski, C.F. (1995) *Identification Problems in the Social Sciences*. Harvard University Press.

[22] Manski, C.F. (2003) *Partial Identification of Probability Distributions*. New York: Springer.

[23] Manski, C.F. (2007) *Identification for Prediction and Decision*. Cambridge, MA: Harvard Univ. Press.

[24] Marschak, J. and Andrews, W.H. (1944) Random simultaneous equations and the theory of production. *Econometrica*, 12, 143-203.

[25] Moriarity, C. and Scheuren, F. (2001) Statistical matching: A paradigm for assessing the uncertainty in the procedure. *J. Off. Statist.*, 17, 407-422.

[26] Nadarajah, S. and Kotz, S. (2008). Exact distribution of the max/min of two Gaussian random variables. *IEEE Transactions on very large scale integration (VLSI) systems*, 16(2), 210-212.

[27] Okner, B.A. (1972) Constructing a new microdata base from existing microdata sets: the 1966 merge file. *Ann. Econ. Soc. Mea.*, 1, 325-342.

[28] Romano, J. and Shaikh, A. (2010). Inference for the identified set in partially identified econometric models. *Econometrica*, 78, 169-211.
A Appendix

A.1 Proof of Theorem 1

(i) On the one hand, we have $A_{\alpha_1} \setminus A_{\alpha_2} = \emptyset$ because, otherwise, there must exist some $\theta \in A_{\alpha_1} \setminus A_{\alpha_2}$ such that $c(\theta) \geq \alpha_1$ (because $\theta \in A_{\alpha_1}$) and $c(\theta) < \alpha_2$ (because $\theta \notin A_{\alpha_2}$) at the same time, contradictory to $\alpha_1 > \alpha_2$ as stipulated. On the other hand, the set $A_{\alpha_2} \setminus A_{\alpha_1}$ is non-empty because, otherwise, every $\theta \in A_{\alpha_2}$ must belong to $A_{\alpha_1}$ and, thus, $c(\theta) \geq \alpha_1$, so that there exists no $\theta \in A_{\alpha_2}$ such that $c(\theta) = \alpha_2 < \alpha_1$, contradictory to the definition of $A_{\alpha_2}$.

(ii) Each $\Theta$ can be classified into 4 distinct types, denoted by (a) $\hat{\Theta}_{L\hat{L}}$ where $\theta_L \notin \hat{\Theta}$ and $\theta_U \notin \hat{\Theta}$, (b) $\hat{\Theta}_{LU}$ where $\theta_L \in \hat{\Theta}$ and $\theta_U \in \hat{\Theta}$ and, thus, $\theta \in \hat{\Theta}_{LU}$, (c) $\hat{\Theta}_L$ where $\theta_L \in \hat{\Theta}$ and $\theta_U \notin \hat{\Theta}$, (d) $\hat{\Theta}_U$ where $\theta_L \notin \hat{\Theta}$ and $\theta_U \in \hat{\Theta}$. Type (c) can be further classified into (c.1) $\hat{\Theta}_{L1}$ where $\theta \in \hat{\Theta}_{L1}$ and (c.2) $\hat{\Theta}_{L2}$ where $\theta \notin \hat{\Theta}_{L2}$, i.e. depending on whether or not $\theta$ appears in $\hat{\Theta}$. Similarly, type (d) into (d.1) $\hat{\Theta}_{U1}$ where $\theta \in \hat{\Theta}_{U1}$ and (d.2) $\hat{\Theta}_{U2}$ where $\theta \notin \hat{\Theta}_{U2}$. We have

$$c(\theta_L) = \Pr(\hat{\Theta}_{LU}) + \Pr(\hat{\Theta}_L) = \Pr(\hat{\Theta}_{LU}) + \Pr(\hat{\Theta}_{L1}) + \Pr(\hat{\Theta}_{L2})$$
$$c(\theta_U) = \Pr(\hat{\Theta}_{LU}) + \Pr(\hat{\Theta}_U) = \Pr(\hat{\Theta}_{LU}) + \Pr(\hat{\Theta}_{U1}) + \Pr(\hat{\Theta}_{U2})$$
$$c(\theta) \geq \Pr(\hat{\Theta}_{LU}) + \Pr(\hat{\Theta}_{L1}) + \Pr(\hat{\Theta}_{U1}).$$

Thus, if $\Pr(\hat{\Theta}_{U1}) \geq \Pr(\hat{\Theta}_{L2})$, then $c(\theta) \geq c(\theta_L)$, or if $\Pr(\hat{\Theta}_{U1}) \leq \Pr(\hat{\Theta}_{L2})$, then $\Pr(\hat{\Theta}_{L1}) \geq \Pr(\hat{\Theta}_{U2})$ since $c(\theta_L) = c(\theta_U)$, such that $c(\theta) \geq c(\theta_U)$. Similarly on comparison between $\Pr(\hat{\Theta}_{L1})$ and $\Pr(\hat{\Theta}_{U2})$. □

A.2 Proof of Theorem 2

Take any initial level-$\alpha_1$ corroborated set $A_{\alpha_1} = [L_{\alpha_1}, U_{\alpha_1}]$. Without losing generality, one of the end points must have corroborated $\alpha_1$ by Theorem 1.i; suppose $c(L_{\alpha_1}) \geq c(U_{\alpha_1}) = \alpha_1$. By
definition $c(\theta) \geq \alpha_1$ for all $\theta \in A_{\alpha_1}$. If $c(\theta) = c(L_{\alpha_1})$ for all $L_{\alpha_1} < \theta < U_{\alpha_1}$, then $\theta_{\text{max}} = L_{\alpha_1}$, since $c(\theta) < \alpha_1 \leq c(L_{\alpha_1})$ for any $\theta \notin A_{\alpha_1}$. Otherwise, there exists $L_{\alpha_1} < \theta < U_{\alpha_1}$, where $c(\theta) = \alpha_2 > c(L_{\alpha_1}) \geq \alpha_1$, and the corresponding level-$\alpha_2$ corroboration set, denoted by $A_{\alpha_2} = [L_{\alpha_2}, U_{\alpha_2}]$. By Theorem 1.1, we have $[L_{\alpha_2}, U_{\alpha_2}] \subset [L_{\alpha_1}, U_{\alpha_1}]$. Since $\alpha \leq 1$, iteration of the argument must terminate at some maximum level-$\alpha$. □

A.3 Proof of Lemma 1

Let $\delta(\theta; \hat{\psi}_n) = 1$ if $\theta \in \text{Int}(\hat{\Theta}) = (\hat{L}_n, \hat{U}_n)$, and 0 otherwise, where $\hat{\psi}_n$ is the MLE. Without losing generality, for any $\theta = U_0 - \epsilon$, where $0 < 2\epsilon < U_0 - L_0$, we have $\delta(\theta; \hat{\psi}_n) = 1$ if $|\hat{U}_n - U_0| < \epsilon$ and $|\hat{L}_n - L_0| < \epsilon$, the probability of which tends to 1, since $\hat{\psi}_n \xrightarrow{\text{Pr}} \psi_0$. Thus, $\delta(\theta; \hat{\psi}_n) \xrightarrow{\text{Pr}} 1$, i.e. $\bar{c}_0(\theta) = 1$ and $\theta \in \hat{A}_{\text{max}}$. Similarly, it can be shown that $\bar{c}_0(\theta) = 1$, for $\theta \notin \Theta_0$, i.e. $\theta \notin \hat{A}_{\text{max}}$. □

A.4 Proof of Theorem 3

By the general form of Slutsky’s Theorem (e.g. Theorem 7.1, Kapadia et al., 2005), we have $\bar{c}(\theta; \hat{\psi}_n) \xrightarrow{\text{Pr}} \bar{c}(\theta; \psi_0)$, since $\hat{\psi}_n \xrightarrow{\text{Pr}} \psi_0$ and $\bar{c}(\theta; \psi)$ is a bounded for all $\psi$. Thus, if $\theta \in (L_0, U_0)$, such that $\bar{c}(\theta; \psi_0) = 1$ by Lemma 1, we have $\bar{c}(\theta; \hat{\psi}_n) \xrightarrow{\text{Pr}} \bar{c}(\theta; \psi_0) = 1$, meaning $\lim_n \text{Pr}(\theta \in \hat{A}_{\text{max}}^n) = 1$. Similarly, it can be shown that $\lim_n \text{Pr}(\theta \in \hat{A}_{\text{max}}^n) = 0$, for $\theta \notin \Theta_0$. □