Modification to the Luminosity distance redshift relation in modified gravity theories

Éanna É. Flanagan, Eran Rosenthal, and Ira M. Wasserman

Center for Radiophysics and Space Research,
Cornell University, Ithaca, New York, 14853

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Abstract

We derive an expression for the luminosity distance as a function of redshift for a flat Robertson-
Walker spacetime perturbed by arbitrary scalar perturbations possibly produced by a modified
gravity theory with two different scalar perturbation potentials. Measurements of the luminosity
distance as function of redshift provide a constraint on a combination of the scalar potentials and
so they can complement weak lensing and other measurements in trying to distinguish among the
various alternative theories of gravity.
I. INTRODUCTION

General relativity (GR) is in good agreement with all astrophysical observations of binary pulsars and solar system tests [1]. These observations provide tight constraints on deviations from GR on scales that are smaller or comparable with our solar system. Observations on larger scales are less restrictive, so it is possible that gravity is substantially different from GR on these scales. In recent years there has been a considerable effort to construct theories that modify GR on large scales, partly for the purpose of explaining the current phase of accelerated expansion of the Universe without introducing a dark energy component. Among these theories are scalar tensor theories (see e.g. [2, 3]), f(R) theories [4] and DGP gravity [5]. Many studies have discussed comparing observables in these theories with present and future observations, and in this way constraining and sometimes refuting these alternative theories of gravity (see e.g. [6, 7, 8, 9, 10, 11]). In this paper, we focus on one observable – the luminosity distance as function of redshift for a perturbed Robertson-Walker (RW) Universe. We calculate this observable for a class of metric theories of gravity including GR. Measurements of the luminosity distance as function of redshift for type Ia supernovae (SNe) provide evidence that the Universe expands at an accelerating rate [12, 13]. These studies would be extended by the planned joint dark energy mission (JDEM). Observations of the luminosity distance as function of redshift may be able to constrain the various alternatives to GR.

For any metric theory of gravity the RW metric provides a good description of the Universe on large scales. The construction of this metric is based on observations that on large scales the Universe is homogeneous and isotropic (see e.g. [14, 15]), and also on the assumption that the Copernican principle holds, namely that we are not in any special location in the Universe. Since these general considerations are independent of the theory of gravity the construction of the RW metric remains valid for a large class of modified GR theories. Assuming the RW metric, one can normally adjust the parameters of a given modified GR theory so that it would produce the observed expansion history of the Universe. However, small deviations from homogeneity and isotropy which give rise to metric perturbations do depend on the particular theory of gravity in use. This makes the perturbed RW metric an ideal framework for studying observables that could distinguish among theories of gravity.

Many previous studies have suggested using cosmic microwave measurements, weak lens-
ing measurements and other observations to distinguish among the various modified GR theories (see e.g. [7, 8, 9, 10, 11]). Recently Bonvin, Durrer, and Gasparini have suggested that measurements of the luminosity distance power spectrum induced by cosmological perturbations may be used to determine cosmological parameters [16]. Cooray, Holz, and Huterer have showed that two-point angular correlation function of SNe can provide useful data to study the foreground of large scale structure [17]. SNe surveys normally have smaller data sets then weak lensing surveys so they typically have a larger statistical uncertainty. Therefore, with regard to sensitivity to cosmological fluctuations the SNe surveys are not competitive with weak lensing surveys. Nevertheless, they provide an independent measurement of a different physical quantity (SNe surveys are sensitive to the luminosity of SNe while weak lensing surveys are sensitive to the distortion of galaxy images) and so they can be used to complement weak lensing measurements.

Another motivation to study the luminosity distance fluctuations is that they degrade the accuracy of the determination of cosmological parameters from SNe data [18, 19, 20, 21]. Estimation of the systematic error produced by cosmological perturbations is therefore necessary for SNe luminosity distance data analysis.

The lack of tight constraints on the theory of gravity on large scales together with the sensitivity of future SNe surveys to the foreground of cosmological perturbations motivates the calculation of luminosity distance as function of redshift for a perturbed RW Universe in modified GR theories.

In this paper we shall consider a flat RW metric with linear scalar perturbations. In a Newtonian gauge this metric takes the form

$$ds^2 = a^2[-d\eta^2(1 + 2\psi) + (1 - 2\phi)\mathbf{dx} \cdot \mathbf{dx}]. \quad (1.1)$$

Here $a(\eta)$ denotes the scale factor as function of conformal time, and the potentials $\phi$ and $\psi$ satisfy $|\phi(x, \eta)|, |\psi(x, \eta)| << 1$. GR in the absence of anisotropic stresses gives $\psi = \phi$. The luminosity distance for a perturbed RW Universe have been studied before by several authors. Thus Sasaki [22] has studied the luminosity distance as function of redshift for a general perturbed spacetime. While Sasaki’s analysis is very general, it gives an explicit expression for the luminosity distance only for the case of an Einstein-deSitter Universe with $\psi = \phi$. For the case of $\phi = \psi$ an explicit expression for the luminosity distance was derived by Pyne and Birkinshaw [23] and was later corrected by Hui and Greene [20]. An
equivalent expression was recently derived by Bonvin, Durrer, and Gasparini [16]. In this paper we calculate the luminosity distance as function of redshift for the metric (1.1). We use a different method of calculation than the methods used in Refs. [16, 20, 22, 23], and our result generalizes the expressions in these references since we allow for possibly different scalar potentials $\phi \neq \psi$. Our expression (6.1) reduces to the corresponding expression in Ref. [20] for the special case of $\phi = \psi$ [see Eq. (C21) in this reference].

Our complete expression for the luminosity distance $D_L$ as function of redshift for the metric (1.1) is given by a somewhat cumbersome formula (6.1). Fortunately, in practice most SNe surveys are sensitive only to subhorizon density perturbations. For these surveys, we may drop the terms that are subdominant for subhorizon perturbations, and obtain a simpler expression for the (subhorizon) luminosity distance $D_{L}^{\text{sub}}$ reading

$$D_{L}^{\text{sub}}(z, \mathbf{n}) \approx (\chi_s - \chi_o)(1 + z) \left\{ 1 + \mathbf{v}_s \cdot \mathbf{n} - \frac{(\mathbf{v} \cdot \mathbf{n})_{\chi_o}}{(\chi_s - \chi_o)H_s} \left( \frac{(\phi + \psi)(\chi - \chi_o)(\chi_s - \chi)}{\chi_s - \chi_o} d\chi \right) \right\}. \quad (1.2)$$

Here the luminosity distance is expressed in terms of the observed redshift $z$, and the direction to the source, where $\mathbf{n}$ (also denoted as $n^a$) denotes a unit spatial vector from the observer to the source. The notation $\approx$ denotes an approximate equality accurate up to first-order in the potentials $\phi$, $\psi$, and the peculiar velocities (in the conformal spacetime) $\mathbf{v}$. The subscripts $s$ and $o$ refer to the source and the observer, respectively. The conformal Hubble rate is denoted $H = \frac{da}{d\eta} a^{-1}$. The potentials $\phi$ and $\psi$, which by definition are functions of the spacetime coordinates, are considered here to be functions of an affine parameter $\chi$ defined on the zeroth order photon null geodesics. The affine parameter $\chi$ is an implicit function of the observed redshift $z$. This function is determined by the unperturbed RW background, and is given by

$$\chi_s = \int_0^z \frac{1}{H(z)} dz + \chi_o,$$

where $H(z) \equiv H/a$ is the Hubble rate, $\chi_o$ denotes an arbitrary initial value for the affine parameter, and we assume that the background spacetime is either expanding or contracting.

We now briefly discuss Eq. (1.2) and how it may be used to distinguish among various theories of gravity. Notice that this formula has two types of term: terms that are proportional to the peculiar velocities of the observer and the source, these terms represent kinematical Doppler-shift, and a term depending on the Laplacian of the potentials that
represents gravitational lensing. This lensing term depends only on the combination $\psi + \phi$ of the two potentials, and so it cannot be used to differentiate the GR case $\psi = \phi$ from the more general case $\phi \neq \psi$. Fortunately, the velocity terms in Eq. (1.2) provide additional information that breaks this degeneracy. The dependence of the peculiar velocities on the potentials follows from the equations of motion of the observer and the source. We assume that the observer and the source feel no interaction other than gravity, and furthermore we assume that the total energy momentum tensor is covariantly conserved. It now follows that the observer and the source move along geodesics of the perturbed spacetime, and therefore their peculiar velocities satisfy

$$v_{,\eta} + a_{,\eta} a^{-1} v + \nabla \psi \approx 0. \quad (1.3)$$

Since the peculiar velocities depend on $\psi$, but are independent of $\phi$, they provide information that breaks the degeneracy. It is also useful to examine the dependence on the redshift of the various terms in Eq. (1.2). Terms that depend on peculiar velocities are expected to be bounded (the peculiar velocity of a host galaxy is normally of order $500 \text{km sec}^{-1}$) and therefore the ratio of a typical peculiar velocity to the Hubble flow is larger at low redshift and smaller at high redshift. We might also expect the RMS effect due to lensing to increase with redshift. Therefore, the lensing term is expected to be dominant at high redshift, while the peculiar velocity terms are expected to become dominant at low redshift.

Information about the scalar perturbations can be extracted from the data by calculating the correlation function $[\bar{D}_L(z')\bar{D}_L(z)]^{-1} \langle D_L(z', \mathbf{n}') D_L(z, \mathbf{n}) \rangle$, where $\bar{D}_L(z)$ denotes the average over angles, and $\langle \ldots \rangle$ denotes an ensemble average. We anticipate that if we specialize to either low or high redshifts, this correlation function would be sensitive to either the peculiar velocity terms or the lensing term. This means that by combining information from both low and high redshifts one can overcome the above mentioned degeneracy. Such measurements may be able to provide information that could distinguish among various theories of gravity.

To calculate the correlation function from theory, one must have some knowledge about the underlying theory of gravity. In particular one must know the relation between the overdensity and the gravitational potentials $\phi$ and $\psi$. Instead of specializing to a particular modified theory of gravity it is possible to employ a parameterized framework where the relation between the two potentials and the relations among the potentials and the overden-
sity are parameterized, such that various modified gravity theories produce different values for the parameters (see e.g. Ref. [29, 30]). In such a framework one can use Eq. (1.2) in combination with the power spectrum of density fluctuations to obtain specific predictions for the power spectrum of the luminosity distance, which can then be compared with observations. In this paper we keep our assumptions about the theory of gravity at minimum, and we defer the more detailed parameterized analysis to future work. Nevertheless, we can still make a rough order of magnitude estimate of the correlation function. It reasonable to expect that even in a modified gravity theory the potentials $\psi$ and $\phi$ while not precisely equal, should be of the same order of magnitude. This means that while the correlation function should differ from the one in GR, it is likely to be of the same order of magnitude. Ref. [16] shows that for GR and a CDM Universe the contribution to the correlation function form lensing alone can be as large as $10^{-5}$ for $z = z' = 2$ at $l \approx 300$. Ref. [17] estimates that this lensing contribution to the cross-correlation function could be detected with a signal to noise ratio of 10 with a survey of 10,000 SNe over 10deg$^2$ between redshifts of 0.1 and 1.7. While the signature of modified gravity is likely to have a smaller signal to noise, it may still be detectable in future SNe surveys.

This paper is organized as follows. In Sec. II we present the framework for calculating luminosity distance in a general spacetime, in Sec. III we specialize to the metric (1.1) and simplify the calculation by introducing a transformation to a conformal spacetime, in Sec. IV we impose initial conditions, and finally in Sec. VI we obtain our final expression (6.1).

II. THE LUMINOSITY DISTANCE IN A GENERAL SPACETIME

There is a well known general relation [25, 26] between the observed luminosity distance $D_L(z)$ and the observed angular diameter distance $D_A(z)$ which reads

$$D_L(z) = (1 + z)^2 D_A(z).$$

This relation is valid in any metric theory provided that the linear momentum of photons is conserved, and so in particular it holds for the metric (1.1). Below we calculate $D_A(z)$ which is a purely geometrical quantity, and substitute our result back into Eq. (2.1) to obtain the luminosity distance $D_L(z)$.

The angular diameter distance $D_A(z)$ is defined in the following manner. Suppose that
an observer views a sizable distant object (e.g. a distant galaxy or a structure of the CMB anisotropy) that subtends a small solid angle $\Delta \Omega$. Using the geometric optics approximation we can describe the electromagnetic radiation using a congruence of null geodesics. We consider geodesics that emanate from a vertex at the observer and propagate backwards in time towards the source. The angular diameter distance $D_A$ is given by

$$D_A(\lambda) \equiv \sqrt{\frac{\Delta A(\lambda)}{\Delta \Omega}}. \quad (2.2)$$

Here $\lambda$ denotes the affine parameter along the congruence, and $\Delta A(\lambda)$ denotes the transverse cross sectional area of the congruence at a fixed $\lambda$.

We pick a representative null geodesic from the thin congruence and denote its worldline with $x^\alpha(\lambda)$, and denote its tangent null vector field with $k^\alpha = \frac{dx^\alpha}{d\lambda}$, where throughout Greek indices run from 0 to 3. In general the evolution of a thin null congruence is completely described by an expansion parameter $\theta$, a shear tensor $\sigma_{\alpha\beta}$, and a rotation tensor $\omega_{\alpha\beta}$, where the expansion parameter takes the form of

$$\theta = \frac{1}{\Delta A} \frac{d}{d\lambda} \Delta A. \quad (2.3)$$

We assume that the rotation tensor $\omega_{\alpha\beta}$ vanishes at the observer, since this tensor satisfies a homogeneous transport equation (see e.g. [27]); this initial condition forces it to vanish everywhere. Therefore, Raychaudhuri’s equation for the null congruence reads

$$\frac{d\theta}{d\lambda} = -\frac{1}{2} \theta^2 - R_{\alpha\beta} k^\alpha k^\beta - \sigma_{\alpha\beta} \sigma^{\alpha\beta}. \quad (2.4)$$

Eqs. (2.2,2.3,2.4) gives the focusing equation (see e.g. [28])

$$\frac{1}{D_A} \frac{d^2 D_A}{d\lambda^2} = -\frac{1}{2} (R_{\alpha\beta} k^\alpha k^\beta + \sigma_{\alpha\beta} \sigma^{\alpha\beta}), \quad (2.5)$$

which we solve to obtain $D_A$.

### III. CONFORMAL ANGULAR DIAMETER DISTANCE

We now specialize the calculation of $D_A$ to the metric (1.1). This metric allows us to factor out the dependence of $D_A$ on the scale factor, and thereby simplify the calculation. To this end we consider the following conformal transformation

$$ds^2 = a^2 d\tilde{s}^2, \quad \delta \Omega = \delta \tilde{\Omega}, \quad D_A = a \tilde{D}_A, \quad d\lambda = a^2 d\tilde{\lambda}, \quad k^\alpha = a^{-2} \tilde{k}^\alpha. \quad (3.1)$$
Here quantities with and without tildes denote the conformal space and the real space, respectively. Notice that the form of Eq. (2.5) is invariant under this conformal transformation, this means that the focusing equation in the conformal spacetime is obtained by adding tildes to all the quantities in Eq. (2.5). The focusing equation in the conformal spacetime can be transformed into an integral equation of the form
\[ \tilde{D}_A = -\frac{1}{2} \int_{\tilde{\lambda}_o}^{\tilde{\lambda}_s} (\tilde{R}_{\alpha\beta} \tilde{k}^\alpha \tilde{k}^\beta + \tilde{\sigma}_{\alpha\beta} \tilde{\sigma}^{\alpha\beta}) \tilde{D}_A(\tilde{\lambda})(\tilde{\lambda}_s - \tilde{\lambda}) d\tilde{\lambda} + \tilde{C}_1 + \tilde{C}_2 \tilde{\lambda}_s. \] (3.2)
Here \( \tilde{C}_1 \) and \( \tilde{C}_2 \) are constants which depend on the initial conditions, and all the quantities in the first brackets on the right hand side are evaluated at \( \tilde{\lambda}_s \). So far we have not used a perturbative approximation and Eq. (3.2) is accurate to all orders in the perturbation potentials. Below we calculate \( \tilde{D}_A \) to the first order in the potentials \( \phi \) and \( \psi \).

Expanding the null geodesic worldline \( x^\alpha(\tilde{\lambda}) \) in a perturbation series gives
\[ x^\alpha(\tilde{\lambda}) \approx \bar{x}^\alpha(\tilde{\lambda}) + \delta \bar{x}^\alpha(\tilde{\lambda}), \]
\[ \tilde{k}^\alpha(\tilde{\lambda}) \approx \bar{k}^\alpha(\tilde{\lambda}) + \delta \bar{k}^\alpha(\tilde{\lambda}), \]
where an overbar denotes an unperturbed quantity, and a \( \delta \) preceding a quantity denotes the first-order perturbation to that quantity. In this perturbation scheme the affine parameter \( \tilde{\lambda}_s \) characterizes the location of the source on the null geodesic. The same \( \tilde{\lambda}_s \) is used both for the perturbed quantity and for the unperturbed quantity. Notice, however, that due to the perturbation potentials the source is characterized by different coordinates in the background spacetime and in the full spacetime \( x^\alpha(\tilde{\lambda}_s) \neq \bar{x}^\alpha(\tilde{\lambda}_s) \). Using the above notation we also have
\[ \tilde{D}_A \approx \bar{D}_A + \delta \bar{D}_A, \quad \tilde{R}_{\alpha\beta} \approx \delta \bar{R}_{\alpha\beta}, \quad \tilde{\sigma}_{\alpha\beta} \approx \delta \bar{\sigma}_{\alpha\beta}, \quad \tilde{C}_1,2 \approx \bar{C}_1,2 + \delta \bar{C}_1,2. \] (3.4)
Notice that at the leading order the conformal spacetime is flat so that the expansions for \( \tilde{R}_{\alpha\beta} \) and \( \tilde{\sigma}_{\alpha\beta} \) start at first-order. Using Eq. (3.2) together with Eqs. (3.3,3.4) we obtain
\[ \tilde{D}_A(\tilde{\lambda}_s) = \bar{C}_1 + \bar{C}_2 \tilde{\lambda}_s, \] (3.5)
\[ \delta \tilde{D}_A(\tilde{\lambda}_s) = \delta \bar{C}_1 + \delta \bar{C}_2 \tilde{\lambda}_s - \frac{1}{2} \int_{\tilde{\lambda}_o}^{\tilde{\lambda}_s} \delta \bar{R}_{\alpha\beta} \tilde{k}^\alpha \tilde{k}^\beta \bar{D}_A(\tilde{\lambda})(\tilde{\lambda}_s - \tilde{\lambda}) d\tilde{\lambda}. \] (3.6)
The flatness of the background spacetime implies that the background null vector \( \bar{k}^\mu \) is a constant four-vector. By rescaling \( \bar{\lambda} \) we set this vector to be
\[ \bar{k}^\mu = \frac{d\bar{x}^\mu}{d\bar{\lambda}} = (-1, \mathbf{n}), \]
where \( \mathbf{n} \) is a unit vector by virtue of the nullity of \( \vec{k}^\mu \). Locating the observer at the origin  
\( \bar{x}_o^a = 0 \), where \( a, b = 1, 2, 3 \), we have

\[
\vec{n}(\bar{\lambda}) = \mathbf{n}^o (\bar{\lambda} - \bar{\lambda}_o).
\]  

(3.7)

**IV. INITIAL CONDITIONS**

We now supplement equations (3.5, 3.6) with initial conditions at \( \bar{\lambda} = \bar{\lambda}_o \), and thereby determine the constants \( \bar{C}_1 \) and \( \bar{C}_2 \). First we demand that \( \bar{D}_A(\bar{\lambda}_o) = 0 \), which gives

\[
\bar{C}_1 = -\bar{\lambda}_o \bar{C}_2 , \quad \delta \bar{C}_1 = -\bar{\lambda}_o \delta \bar{C}_2.
\]

We therefore have at the leading order \( \bar{D}_A(\bar{\lambda}_s) = \bar{C}_2(\bar{\lambda}_s - \bar{\lambda}_o) \). Eq. (3.7) implies that \( \bar{C}_2 = 1 \).

With the above initial condition Eqs. (3.5, 3.6) read

\[
\bar{D}_A(\bar{\lambda}_s) = \bar{\lambda}_s - \bar{\lambda}_o
\]

(4.1)

\[
\delta \bar{D}_A(\bar{\lambda}_s) = \delta \bar{C}_2 \bar{D}_A(\bar{\lambda}_s) - \frac{1}{2} \int_{\bar{\lambda}_o}^{\bar{\lambda}_s} \delta \bar{R}_{ab} \bar{R}^{ab} \bar{D}_A(\bar{\lambda})(\bar{\lambda}_s - \bar{\lambda}) d\bar{\lambda}
\]

(4.2)

The constant \( \delta \bar{C}_2 \) is determined from the value of the derivative \( \frac{d\bar{D}_A}{d\bar{\lambda}} \) at \( \bar{\lambda}_o \). To determine this constant it is instructive to first calculate \( \frac{d\bar{D}_A}{d\bar{\lambda}} \), where \( \bar{D}_A \) is the conformal angular diameter distance for a comoving observer, ignoring for the moment the observer’s peculiar velocity. It is then possible to correct the expression and account for the observer’s peculiar velocity.

In the vicinity of a comoving observer we can Taylor expand the potentials \( \phi \) and \( \psi \) by treating the distance from the observer as the small parameter. It follows from the metric (1.1) that the angular diameter distance takes the form of

\[
\bar{D}_A(\bar{\lambda}) = R(\bar{\lambda}) \sqrt{1 - 2\phi(\bar{\lambda})} + O(R^{3/2}).
\]

where \( R = (\delta_{ab} x^a x^b)^{1/2} \). Imposing initial conditions \( \delta x^a(\bar{\lambda}_o) = \frac{dx^a}{d\bar{\lambda}}(\bar{\lambda}_o) = 0 \) gives

\[
\frac{d\bar{D}_A}{d\bar{\lambda}}(\bar{\lambda}_o) = 1 - \phi_o,
\]

(4.3)

which gives rise to a constant \( \delta \bar{C}_2 = -\phi_o \) for a comoving observer. To correct for the observer’s peculiar velocity let us consider a transformation to a realistic reference frame moving in a velocity \( \mathbf{v}_o \) with respect to the static observer in the conformal space, where
\( v_o = |v_o| = O(\psi) = O(\phi) \). We maintain the notation \( \tilde{D}_A \) for the angular diameter distance in the realistic frame, which is given by

\[
\tilde{D}_A(\tilde{\lambda}) \equiv \sqrt{\frac{\Delta A}{\Delta \Omega}} = \sqrt{\frac{\Delta \hat{A}[1 + O(v_o^2)]}{\Delta \hat{\Omega}[1 - 2v_o \cdot n + O(v_o^2)]}} = \hat{D}_A(\tilde{\lambda})[1 + v_o \cdot n + O(v_o^2)].
\] (4.4)

Using Eq. (4.2) together with Eqs. (4.3, 4.4) we obtain

\[
\delta \tilde{D}_A(\tilde{\lambda}_s) = \hat{D}_A(\tilde{\lambda}_s)(v_o \cdot n - \phi_o) - \frac{1}{2} \int_{\tilde{\lambda}_o}^{\tilde{\lambda}_s} \delta \hat{R}_{\alpha\beta\tilde{k}^\alpha\tilde{k}^\beta} \hat{D}_A(\tilde{\lambda})(\tilde{\lambda}_s - \tilde{\lambda})d\tilde{\lambda}.
\] (4.5)

Evaluating the expression inside the integral gives

\[
\delta \tilde{D}_A(\tilde{\lambda}_s) = \hat{D}_A(\tilde{\lambda}_s)(v_o \cdot n - \phi_o)
\]

\[
- \frac{1}{2} \int_{\tilde{\lambda}_o}^{\tilde{\lambda}_s} \left[ \nabla^2(\phi + \psi) + 2\phi_{,\eta\eta} - 4\nabla\phi_{,\eta} \cdot n + (\phi_{,ab} - \psi_{,ab})n^a n^b \right] \hat{D}_A(\tilde{\lambda})(\tilde{\lambda}_s - \tilde{\lambda})d\tilde{\lambda}.
\] (4.6)

V. REDSHIFT

So far we have calculated the dependence of \( \hat{D}_A \) and \( \delta \hat{D}_A \) on the affine parameter \( \tilde{\lambda} \). The goal of this section is to use these relations to express the luminosity distance as function of the observed redshift \( z \). Using Eq. (3.4) and Eq. (3.1) together with Eq. (2.1) we find that

\[
D_L(\tilde{\lambda}_s) \approx (1 + z)^2 a(\tilde{\lambda}_s)[\hat{D}_A(\tilde{\lambda}_s) + \delta \hat{D}_A(\tilde{\lambda}_s)].
\] (5.1)

By definition the observed redshift is given by

\[
1 + z = \frac{(g_{\mu\nu}k^{\mu}u^{\nu})_{\text{source}}}{(g_{\alpha\beta}k^{\alpha}u^{\beta})_{\text{observer}}}.
\] (5.2)

We also introduce a conformal redshift \( 1 + \tilde{z} \), defined by adding tildes to all the quantities in Eq. (5.2). Using Eqs. (5.2, 3.1) together with \( u^\alpha = a^{-1}\tilde{u}^\alpha \) we find that

\[
1 + z = \frac{1 + \tilde{z}}{a},
\] (5.3)
where for convenience we set $a(\tilde{\lambda}_o) = 1$ at the observer and use the implicit notation $a \equiv a(\tilde{\lambda}_s)$. Substituting Eq. (5.3) into Eq. (5.1) gives

$$D_L(\tilde{\lambda}_s) \approx (1 + z)[(1 + \tilde{z})\tilde{D}_A(\tilde{\lambda}_s) + \delta\tilde{D}_A(\tilde{\lambda}_s)].$$  

(5.4)

To obtain the deviation of the luminosity distance as function of redshift due to the metric perturbations we need to compare the perturbed luminosity distance and the background luminosity distance at the same redshift. Following Hui and Greene [20] we calculate $D_L(\tilde{\lambda}_s + \delta\tilde{\lambda})$ the luminosity distance at a shifted affine parameter where the shift $\delta\tilde{\lambda}$ is defined by the relation

$$1 + z(\tilde{\lambda}_s + \delta\tilde{\lambda}) = 1 + \tilde{z}(\tilde{\lambda}_s),$$  

(5.5)

where $1 + \tilde{z}(\tilde{\eta}) = a^{-1}(\tilde{\eta})$ is the standard redshift in the RW background spacetime. These definitions would allow us later to substitute the standard RW relation $\tilde{\lambda}_s(\tilde{z})$ into Eq. (5.4). Using Eq. (5.4) and recalling that $\tilde{D}_A = \tilde{\lambda}_s - \tilde{\lambda}_o$ we find that

$$D_L(\tilde{\lambda}_s + \delta\tilde{\lambda}) \approx (1 + z)[\tilde{D}_A(\tilde{\lambda}_s)(1 + \tilde{z}) + \delta\tilde{\lambda} + \delta\tilde{D}_A(\tilde{\lambda}_s)].$$  

(5.6)

We now calculate $\tilde{z}$ and $\delta\tilde{\lambda}$ and substitute their expressions into Eq. (5.6).

First, we consider $\tilde{z}$. Using the relation $\tilde{u}^\mu \approx (1 - \psi, v)$ together with the metric (1.1) and the redshift definition (5.2) applied to the conformal space, we find that

$$\tilde{z} \approx \left(\psi + v \cdot n - \delta\tilde{k}^0\right)_{\tilde{\lambda}_o}^{\tilde{\lambda}_s}.$$  

To calculate the quantity $(\delta\tilde{k}^0)_{\tilde{\lambda}_o}^{\tilde{\lambda}_s}$ we integrate the null geodesic equation. After integration by parts we obtain

$$\tilde{z} \approx (-\psi + v \cdot n)_{\tilde{\lambda}_o}^{\tilde{\lambda}_s} - \int_{\tilde{\lambda}_o}^{\tilde{\lambda}_s} (\phi, \eta + \psi, \eta)d\tilde{\lambda}.$$  

(5.7)

Next, we consider $\delta\tilde{\lambda}$. Combining Eq. (5.3) with Eq. (5.5) we find that

$$1 + z[\tilde{\eta}(\tilde{\lambda}_s)] \approx \frac{1 + \tilde{z}(\tilde{\lambda}_s)}{a(\eta_{shift})},$$  

(5.8)

where

$$\eta_{shift} = \tilde{\eta}(\tilde{\lambda}_s + \delta\tilde{\lambda}) + \delta\eta(\tilde{\lambda}_s).$$

From which we find that

$$\delta\tilde{\lambda} = \delta\eta - \frac{\tilde{z}}{H_s},$$  

(5.9)

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where we introduced the notation $\mathcal{H} \equiv a_\eta a^{-1}$ and $\delta \eta \equiv \delta x^0$ [$\delta x^0$ was defined in Eq. (3.3)].

Finally, we calculate $\delta \eta$ by integrating the null geodesic equation which gives

$$\delta \eta(\tilde{\lambda}) = (\phi_o - \psi_o) \tilde{D}_A + \int_{\tilde{\lambda}_o}^{\tilde{\lambda}} (\phi_{,\eta} - \psi_{,\eta})(\tilde{\lambda}_a - \tilde{\lambda})d\tilde{\lambda} + 2 \int_{\tilde{\lambda}_o}^{\tilde{\lambda}} \psi d\tilde{\lambda} + 2 \int_{\tilde{\lambda}_o}^{\tilde{\lambda}} \psi_{,\eta}(\tilde{\lambda}_a - \tilde{\lambda})d\tilde{\lambda}. \quad (5.10)$$

To derive this expression we used the initial conditions $\delta \eta(\tilde{\lambda}_o) = 0$ and $\delta \tilde{k}_0(\tilde{\lambda}_o) = \phi_o + \psi_o$, where the last equation is obtained from the requirements $\tilde{k}^\mu \tilde{k}^\nu \tilde{g}_{\mu\nu} = \tilde{k}_0 \tilde{\eta}_{\mu\nu} = 0$ together with the initial condition $\delta \tilde{k}(\tilde{\lambda}_o) = 0$

**VI. RESULTS**

We now substitute Eqs. (4.1,4.6) into Eq. (5.6) and use Eqs.( 5.7,5.9,5.10) after some integrations by parts and a change of notation, $\chi \equiv \tilde{\lambda}$, we finally obtain

$$D_L(z, \mathbf{n}) \approx (\chi_s - \chi_o)(1 + z) \left\{ 1 - \frac{1}{\mathcal{H}_s} \int_{\chi_o}^{\chi_s} (\phi_{,\eta} + \psi_{,\eta}) d\chi + \frac{1}{2} \int_{\chi_o}^{\chi_s} \nabla^2 (\phi + \psi) (\chi - \chi_o)(\chi_s - \chi) d\chi + \int_{\chi_o}^{\chi_s} (\phi_{,\eta} + \psi_{,\eta}) (\chi - \chi_o)(\chi_s - \chi) d\chi ight\}.$$  \quad (6.1)

This expression gives the luminosity distance as function of the observed redshift and the direction to the source for the perturbed RW metric (1.1). Notice that the last three terms in the curly brackets vanish for the case of GR where $\phi = \psi$, but may differ from zero for modified GR theories. Roughly speaking, the various terms in Eq. (6.1) can be interpreted as representing the following physical phenomena. Terms depending on the velocities represent kinematic Doppler shift arising from the peculiar velocities of the observer and the source, terms depending on the potentials (without derivatives) represent gravitational redshifts, and terms depending on a single time derivative of the potentials are analogous to the integrated Sachs-Wolfe (ISW) effect. In fact the ISW effect appears in the final term of Eq. (5.7). It has been previously shown [8] that the ISW effect in modified GR theories produces a modification to the low multipoles of the CMB anisotropy power spectra.

Consider next the first term in the third line of Eq. (6.1), which depends on the Laplacian of the potentials. This term represents gravitational lensing and it agrees with an existing
expression for the convergence in the context of modified gravity theories \[24\]. Consider next the second term in the third line, which contains two time derivatives. The fact that this term is integrated with the standard lensing weight function \((\chi - \chi_o)(\chi_s - \chi)/(\chi_s - \chi_o)\) signals that this term is also a lensing term. However, it is smaller than the leading lensing term by \(\sim (aH/k)^2\), where \(k\) denotes the comoving wave number of a perturbation, and \(H\) denotes the Hubble rate. Therefore, it is subdominant for subhorizon perturbations.

There is some ambiguity in the above classification of the various terms in Eq. (6.1), and some of the terms may be classified differently. For example, we may use integration by parts to replace some of the boundary terms with terms containing integrals over derivatives of the potentials. The ambiguity in the classification originates from the fact that only the entire luminosity distance is observable, and the different individual terms do not correspond to a gauge invariant expression.

As mentioned in the introduction, in many cases Eq. (6.1) can be simplified by specializing to subhorizon perturbations. Under this approximation, terms that have a time derivative of the potentials are smaller by \(\sim aH/k\) with respect to the term containing spatial derivatives. Furthermore, under this approximation Eq. (1.3) implies that the potential \(\psi\) is smaller by \(\sim aH/k\) with respect to the velocity terms. While the potential \(\phi\) is unconstrained by relation (1.3), it is reasonable to assume that its magnitude is not much larger then the magnitude of \(\psi\). For this reason we neglect terms that depend on \(\phi\) and \(\psi\) (with no derivatives). The above considerations gives the approximate expression

\[
D_{L}^{\text{sub}}(z, \mathbf{n}) \approx (\chi_s - \chi_o)(1 + z)\left\{ 1 + \mathbf{v}_s \cdot \mathbf{n} - \frac{(\mathbf{v} \cdot \mathbf{n})\chi_s}{(\chi_s - \chi_o)H_s} \right\}
\]

that was discussed in the introduction.

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