Classical tests of multidimensional gravity: negative result

Maxim Eingorn and Alexander Zhuk

Astronomical Observatory and Department of Theoretical Physics, Odessa National University,
Street Dvoryanskaya 2, Odessa 65082, Ukraine
E-mail: maxim.eingorn@gmail.com and ai-zhuk2@rambler.ru

Received 31 March 2010, in final form 6 July 2010
Published 14 September 2010
Online at stacks.iop.org/CQG/27/205014

Abstract

In the Kaluza–Klein model with toroidal extra dimensions, we obtain the metric coefficients in a weak-field approximation for delta-shaped matter sources. These metric coefficients are applied to calculate the formulas for frequency shift, perihelion shift, deflection of light and parameterized post-Newtonian (PPN) parameters. In the leading order of approximation, the formula for frequency shift coincides with the well-known general relativity expression. However, for perihelion shift, light deflection and the PPN parameter $\gamma$ we obtain formulas $D\pi r_g/[\{(D - 2)a(1 - e^2)]$, $(D - 1)r_g/[\{(D - 2)\rho]$ and $1/(D - 2)$, respectively, where $D$ is the total number of spatial dimensions. These expressions demonstrate good agreement with experimental data only in the case of ordinary three-dimensional ($D = 3$) space. This result does not depend on the size of the extra dimensions. Therefore, in the considered multidimensional Kaluza–Klein models the point-like masses cannot produce the gravitational field which corresponds to the classical gravitational tests.

PACS numbers: 04.25.Nx, 04.50.Cd, 04.80.Cc, 11.25.Mj

1. Introduction

The idea of the multidimensionality of our Universe demanded by the theories of unification of the fundamental interactions is one of the most breathtaking ideas of theoretical physics. It takes its origin from the pioneering papers by Kaluza and Klein [1] and now the most self-consistent modern theories of unification such as superstrings, supergravity and M-theory are constructed in spacetime with extra dimensions [2]. Different aspects of the idea of the multidimensionality are intensively used in numerous modern articles. Therefore, it is very important to suggest experiments which can reveal the extra dimensions. For example, one of the aims of the Large Hadronic Collider consists in detecting of Kaluza–Klein particles which correspond to excitations of the internal spaces (see e.g. [3]). On the other hand, if
we can show that the existence of the extra dimensions is in contrast to observations, then these theories are prohibited. This important problem is extensively discussed in the recent scientific literature (see e.g. [4–10]).

It is well known that classical gravitational tests such as frequency shift, perihelion shift, deflection of light and time delay of radar echoes (the Shapiro time delay effect) are crucial tests of any gravitational theory. For example, there is the significant discrepancy for Mercury between the measurement value of the perihelion shift and its calculated value using Newton’s formalism [11]. It indicates that non-relativistic Newton’s theory of gravity is not complete. This problem was resolved with the help of general relativity, which is in good agreement with observations. A similar situation happened with deflection of light [12]. The Shapiro time delay effect is used to get an upper limit for the parameterized post-Newtonian parameter \( \gamma \) [13]. Obviously, multidimensional gravitational theories should also be in concordance with these experimental data. To check it, the corresponding estimates were carried out in a number of papers. For example, in [8], the well-known multidimensional black hole solution [14] was investigated and the authors obtained a negative result. However, this result was clear from the very beginning because the solution [14] does not have any non-relativistic Newtonian limit in the case of extra dimensions. Definitely, in the solar system such solutions lead to results which are far from the experimental data. The 5D soliton metrics [15–17] were explored in [4–7]. In [5] and [6], the range of parameters was found for which classical gravitational tests for these metrics satisfy the observational values. The black string (see e.g. [18]) is a particular limiting case of such solutions with a trivial metric coefficient for the extra dimension. However, it can be easily shown that such solutions do not correspond to point-like matter sources.

In the 5D non-factorizable brane world model, classical gravitational tests were investigated in [19]. Here, the model contains one free parameter associated with the bulk Weyl tensor. For appropriate values of this parameter, the perihelion shift in this model does not contradict observations. Certainly, this result is of interest, and it is necessary to examine this model carefully to verify the naturalness of the conditions imposed.

In our paper, we consider classical gravitational tests in Kaluza–Klein models (factorizable geometry) with an arbitrary number of spatial dimensions \( D \geq 3 \). We suppose that in the absence of gravitating masses the metric is a flat one. Gravitating point-like masses (moving or at rest) perturb this metric, and we consider these perturbations in a weak-field approximation. In this approximation, we obtain the asymptotic form of the metric coefficients. Then we admit that, first, the extra dimensions are compact and have the topology of tori and, second, gravitational potential far away from gravitating masses tends to the non-relativistic Newtonian limit. In the case of a gravitating mass at rest, the obtained metric coefficients are used to calculate the frequency shift, perihelion shift, deflection of light and parameterized post-Newtonian (PPN) parameters. We demonstrate that for the frequency shift type experiment it is hardly possible to observe the difference between the usual four-dimensional general relativity and multidimensional Kaluza–Klein models. However, the situation is quite different for the perihelion shift, deflection of light and PPN parameters. In these cases, we get formulas which generalize the corresponding ones in general relativity. We show that formulas for the perihelion shift, deflection of light and PPN parameter \( \gamma \) depend on the total number of spatial dimensions, and they are in good agreement with observations only in ordinary three-dimensional space. It is important to note that this result does not depend explicitly on the size of the extra dimensions\(^1\). So, we cannot avoid the problem with classical gravitational tests in

\(^1\) In the leading order of approximation, our formulas do not depend on sizes of the extra dimensions. All correction terms, where the sizes of the extra dimensions appear, are exponentially suppressed.
a limit of arbitrary small (but non-zero!) size of the extra dimensions. It is worth noting that in [9] the authors arrived at the same conclusions in spite of the fact that they used a different approach.

Therefore, our results show that in the considered multidimensional Kaluza–Klein models point-like gravitating masses cannot produce the gravitational field which corresponds to the classical gravitational tests.

The paper is organized as follows. In section 2, we get the asymptotic metric coefficients in the weak-field limit for the delta-shaped matter gravitating source. These metric coefficients are applied to calculate the formulas of the frequency shift, perihelion shift, deflection of light and PPN parameters in section 3. The main results are summarized in section 4.

2. Weak gravitational field approximation

To start with, we consider the general form of the multidimensional metric:

\[
d s^2 = g_{ik} \, dx^i \, dx^k = g_{00}(dx^0)^2 + 2g_{0a} \, dx^0 \, dx^a + g_{ab} \, dx^a \, dx^b,
\]

(2.1)

where the Latin indices \(i, k = 0, 1, \ldots, D\) and the Greek indices \(\alpha, \beta = 1, \ldots, D\). \(D\) is the total number of spatial dimensions. We make the natural assumption that in the case of the absence of matter sources the spacetime is the Minkowski spacetime:

\[
g_{00} = \eta_{00} = 1, \quad g_{0a} = \eta_{0a} = 0, \quad g_{ab} = \eta_{ab} = -\delta_{ab}.
\]

At the same time, the extra dimensions may have the topology of tori. In the presence of matter, the metric is not a Minkowskian one, and we will investigate it in the weak-field limit. This means that the gravitational field is weak, and the velocities of the test bodies are small compared to the speed of light \(c\). In this case, the metric is only slightly perturbed from its flat spacetime value:

\[
g_{ik} \approx \eta_{ik} + h_{ik},
\]

(2.2)

where \(h_{ik}\) are corrections of the order of \(1/c^2\). In particular, \(h_{00} \equiv 2\phi/c^2\). Later we will demonstrate that \(\phi\) is the non-relativistic gravitational potential. The same conclusion with respect to \(\phi\) can be easily obtained from the comparison of the non-relativistic action of a test mass moving in a gravitational field with its relativistic action. To get the other correction terms up to the same order \(1/c^2\), we should consider the multidimensional Einstein equation

\[
R_{ik} = \frac{2S_D \tilde{G}_D}{c^4} \left( T_{ik} - \frac{1}{D - 1} g_{ik} T \right),
\]

(2.3)

where \(S_D = 2\pi^{D/2} / \Gamma(D/2)\) is the total solid angle (the surface area of the \((D-1)\)-dimensional sphere of unit radius) and \(\tilde{G}_D\) is the gravitational constant in the \((D = D + 1)\)-dimensional spacetime. We are going to investigate the weak-field approximation where the gravitational field is generated by \(N\) moving point masses. Therefore, the energy–momentum tensor is

\[
T^{ik} = \sum_{p=1}^{N} m_p (-1)^D g^{1/2} \frac{dx^i}{dt} \frac{dx^k}{dt} \frac{cdt}{ds} \delta(r - r_p),
\]

(2.4)

where \(m_p\) is the rest mass and \(r_p\) is the radius vector of the \(p\)th particle, respectively. All radius vectors \(r\) and \(r_p\) are \(D\) dimensional, e.g. \(r = (x^1, x^2, \ldots, x^D)\), where \(x^\alpha\) are coordinates in the metric (2.1). The rest mass density is

\[
\rho \equiv \sum_{p=1}^{N} m_p \delta(r - r_p).
\]

(2.5)
### 2.1. $1/c^2$ correction terms

Obviously, to hold on the right-hand side of (2.3) the terms up to the order $1/c^2$, the components of the energy–momentum tensor (2.4) are approximated as

$$T_{00} \approx \rho c^2, \quad T_{0\alpha} \approx 0, \quad T_{\alpha\beta} \approx 0 \Rightarrow T = T_i \approx \rho c^2.$$

Taking into account that $h_{ik}$ are of the order of $1/c^2$, the covariant components of the Riemann and Ricci tensors

$$R_{iklm} = \frac{1}{2} \left( \frac{\partial^2 g_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{km}}{\partial x^i \partial x^l} \right) + \gamma_{up} \left( \Gamma^n_{kl} \Gamma^p_{im} - \Gamma^n_{km} \Gamma^p_{il} \right), \quad R_{km} = g^{il} R_{iklm}$$

up to the same order read correspondingly

$$R_{iklm} \approx \frac{1}{2} \left( \frac{\partial^2 h_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 h_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 h_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 h_{km}}{\partial x^i \partial x^l} \right),$$

$$R_{km} \approx \frac{1}{2} \eta^{il} \left( \frac{\partial^2 h_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 h_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 h_{kl}}{\partial x^i \partial x^m} - \eta^{il} \frac{\partial^2 h_{km}}{\partial x^i \partial x^l} \right).$$

where $h_{ik} = \eta^{im} h_{mk}$. With the help of the gauge conditions

$$\frac{\partial}{\partial x^k} \left( h_{ik} - \frac{1}{2} \frac{\partial h_i^k}{\partial t} \right) = 0,$$

formula (2.9) can be written in the form

$$R_{km} \approx -\frac{1}{2} \eta^{il} \frac{\partial^2 h_{km}}{\partial x^i \partial x^l}.$$

Taking into account that the derivatives with respect to $x^0 = ct$ are much smaller than the derivatives with respect to $x^a$, we obtain from (2.11)

$$R_{00} \approx -\frac{1}{2} \eta^{0\beta} \frac{\partial^2 h_{00}}{\partial x^0 \partial x^\beta} = \frac{1}{2} \delta^{0\beta} \frac{\partial^2 h_{00}}{\partial x^0 \partial x^\beta} = \frac{1}{2} \Delta h_{00},$$

$$R_{0\alpha} \approx \frac{1}{2} \Delta h_{0\alpha}, \quad R_{\alpha\beta} \approx \frac{1}{2} \Delta h_{\alpha\beta},$$

where $\Delta = \delta^{0\beta} \partial^2/\partial x^0 \partial x^\beta$ is the $D$-dimensional Laplace operator. It is worth noting that for condition (2.10) up to the order $1/c^2$ holds

$$\frac{\partial}{\partial x^\beta} \left( h_{a}^\beta - \frac{1}{2} \frac{\partial h_i^\beta}{\partial x^i} \right) = 0 + O(1/c^3), \quad \frac{\partial h_0^\beta}{\partial x^\beta} = 0 + O(1/c^3).$$

Therefore, keeping the left- and right-hand sides of (2.3) terms up to the order $1/c^2$ we obtain the following equations:

$$\Delta h_{00} = \frac{2 S_D G_D}{c^2} \rho, \quad \Delta h_{0\alpha} = 0, \quad \Delta h_{a\beta} = \frac{1}{D - 2} \cdot \frac{2 S_D G_D}{c^2} \rho \delta_{a\beta}. $$

4
where \( G_D = [2(D - 2)/(D - 1)] \tilde{G}_D \). Substitution of \( h_{00} = 2\varphi/c^2 \) into the above equation for \( h_{00} \) demonstrates that \( \varphi \) satisfies the \( D \)-dimensional Poisson equation:

\[
\Delta \varphi = S_D G_D \rho. \tag{2.16}
\]

Therefore, \( \varphi \) is the non-relativistic gravitational potential. From (2.15) we obtain

\[
h_{0\alpha} = 0, \quad h_{\alpha\beta} = \frac{1}{D-2} h_{00}\delta_{\alpha\beta} = \frac{1}{D-2} \frac{2\varphi}{c^2} \delta_{\alpha\beta}. \tag{2.17}
\]

It can be easily seen that in this approximation the spacial coordinates of the metric (2.1) are the isotropic ones, i.e. the spatial part of the metric is conformally related to the Euclidean one. It is also worth noting that the relation \( h_{\alpha\beta}/h_{00} = \left[1/(D-2)\right] \delta_{\alpha\beta} \) can also be obtained from the corresponding equations in papers [14, 20].

2.2. \( 1/c^3 \) and \( 1/c^4 \) correction terms

Now, we want to keep in the metric (2.1) the terms up to the order \( 1/c^2 \). Because the coordinate \( x^0 = ct \) contains \( c \), this means that in \( g_{00} \) and \( g_{0\alpha} \) we should keep correction terms up to the order \( 1/c^4 \) and \( 1/c^3 \), respectively, and to leave \( g_{\alpha\beta} \) without changes in the form \( g_{\alpha\beta} \approx \eta_{\alpha\beta} + h_{\alpha\beta} \) with \( h_{\alpha\beta} \) from (2.17).

First, we investigate the energy–momentum tensor (2.4) which we split into three expressions:

\[
T^{00} = \sum_{p=1}^{N} m_p c^2 \left[ (-1)^D g \right]^{-1/2} \frac{c}{ds} \delta(r - r_p), \tag{2.18}
\]

\[
T^{0\alpha} = \sum_{p=1}^{N} m_p c \left[ (-1)^D g \right]^{-1/2} v^\alpha_p \frac{c}{ds} \delta(r - r_p), \tag{2.19}
\]

\[
T^{\alpha\beta} = \sum_{p=1}^{N} m_p \left[ (-1)^D g \right]^{-1/2} v^\alpha_p v^\beta_p \frac{c}{ds} \delta(r - r_p), \tag{2.20}
\]

where \( v^\alpha_p = \frac{dx^\alpha_p}{dt} \). From (2.20) we obtain up to order \( 1 \) (in units \( c \)) the covariant components

\[
T_{\alpha\beta} \approx \sum_{p=1}^{N} m_p v^\alpha_p v^\beta_p \delta(r - r_p). \tag{2.21}
\]

Thus, taking into account the prefactor \( 1/c^4 \) on the right-hand side of (2.3), these components can contribute to \( g_{\alpha\beta} \) terms of the order of \( 1/c^4 \) which is not of interest for us. For \( T^{0\alpha} \) we find from (2.19)

\[
T^{0\alpha} \approx - \sum_{p=1}^{N} m_p c v^\alpha_p \delta(r - r_p). \tag{2.22}
\]

Hence, these components can give in \( g_{0\alpha} \) terms of the order of \( 1/c^3 \) which is of interest for us. Finally, for \( T_{00} \) we get from (2.18)

\[
T_{00} = g_{00} g_{0k} T^{0k} \approx (g_{00})^2 T^{00} \approx \left( 1 + \frac{2\varphi}{c^2} \right)^2 \left[ (-1)^D \left( 1 + \frac{2\varphi}{c^2} \right) \left( -1 + \frac{1}{D-2} \frac{2\varphi}{c^2} \right)^D \right]^{-1/2} \times \sum_{p=1}^{N} m_p c^2 \left[ \left( 1 + \frac{2\varphi}{c^2} \right)^2 - \frac{v^2_p}{c^2} \right]^{-1/2} \delta(r - r_p) \approx \left( 1 + \frac{4\varphi}{c^2} \right) \left( 1 + \frac{1}{D-2} \frac{2\varphi}{c^2} \right). \tag{2.23}
\]
× \sum_{p=1}^{N} m_p c^2 \left( 1 - \frac{\varphi}{c^2} + \frac{v_p^2}{2c^2} \right) \delta(\mathbf{r} - \mathbf{r}_p) \\
\approx \sum_{p=1}^{N} m_p c^2 \left( 1 + \frac{3D - 4 \varphi}{D - 2} c^2 + \frac{v_p^2}{2c^2} \right) \delta(\mathbf{r} - \mathbf{r}_p) \\
= \sum_{p=1}^{N} m_p c^2 \mathbf{r} - \mathbf{r}_p \right) + \sum_{p=1}^{N} m_p \left( \frac{3D - 4}{D - 2} \varphi_p + \frac{1}{2} v_p^2 \right) \delta(\mathbf{r} - \mathbf{r}_p). \quad (2.23)

where \( \varphi_p \) is the potential of the gravitational field at a point with radius vector \( \mathbf{r}_p \). At the moment, we do not care about the fact that \( \varphi_p \) contains the infinite contribution of the \( p \)th particle. Thus, up to order 1 we get

\[
T = g^{ik} T_{ik} \approx g^{00} T_{00} + g^{ab} T_{ab} \approx \sum_{p=1}^{N} m_p c^2 \delta(\mathbf{r} - \mathbf{r}_p)
\]

\[
+ \sum_{p=1}^{N} m_p \left( \frac{D}{D - 2} \varphi_p - \frac{1}{2} v_p^2 \right) \delta(\mathbf{r} - \mathbf{r}_p). \quad (2.24)
\]

With the help of (2.23) and (2.24) we obtain up to the order 1/c^4

\[
2SD \tilde{G}_D \left( T_{00} - \frac{1}{D - 1} g_{00} T \right) \approx \frac{S_D G_D}{c^2} \sum_{p=1}^{N} m_p \delta(\mathbf{r} - \mathbf{r}_p)
\]

\[
+ \frac{S_D G_D}{c^4} \sum_{p=1}^{N} m_p \left( \frac{3D - 4}{D - 2} \varphi_p + \frac{D}{2(D - 2)} v_p^2 \right) \delta(\mathbf{r} - \mathbf{r}_p). \quad (2.25)
\]

Similarly, from (2.22) and (2.24) we get up to the order 1/c^3:

\[
2SD \tilde{G}_D \left( T_{0a} - \frac{1}{D - 1} g_{0a} T \right) \approx - \frac{D - 1}{D - 2} \frac{S_D G_D}{c^3} \sum_{p=1}^{N} m_p v_p a \delta(\mathbf{r} - \mathbf{r}_p). \quad (2.26)
\]

Now, we shall work out the left-hand side of (2.3) up to an appropriate order of 1/c. As we wrote above, we are looking for the corrections of the order of 1/c^4 and 1/c^3 to the metric components \( g_{00} \) and \( g_{0a} \), respectively. To this end, it is convenient to present \( g_{ik} \) as follows:

\[
g_{ik} \approx n_{ik} + h_{ik} + f_{ik}, \quad (2.27)
\]

where \( f_{00} \) and \( f_{0a} \) are of the order of 1/c^4 and 1/c^3, respectively. Then, the Riemann curvature tensor (2.7) reads

\[
R_{iklm} \approx \frac{1}{2} \left( \frac{\partial^2 h_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 h_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 h_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 h_{km}}{\partial x^i \partial x^l} \right)
\]

\[
+ \frac{1}{2} \left( \frac{\partial^2 f_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 f_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 f_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 f_{km}}{\partial x^i \partial x^l} \right) + \eta^p q \left( \Gamma_{n,kl} \Gamma_{p,lm} - \Gamma_{n,lm} \Gamma_{p,kl} \right)
\]

\[
\approx \frac{1}{2} \left( \frac{\partial^2 h_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 h_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 h_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 h_{km}}{\partial x^i \partial x^l} \right)
\]

\[
+ \frac{1}{2} \left( \frac{\partial^2 f_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 f_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 f_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 f_{km}}{\partial x^i \partial x^l} \right)
\]
\[
\frac{1}{4} \eta^{pp} \left( \frac{\partial h_{nk}}{\partial x^l} + \frac{\partial h_{nl}}{\partial x^k} - \frac{\partial h_{kl}}{\partial x^n} \right) \left( \frac{\partial h_{pi}}{\partial x^m} + \frac{\partial h_{pm}}{\partial x^i} - \frac{\partial h_{im}}{\partial x^p} \right)
\]
\[
\frac{1}{4} \eta^{np} \left( \frac{\partial h_{nk}}{\partial x^m} + \frac{\partial h_{nm}}{\partial x^k} - \frac{\partial h_{km}}{\partial x^n} \right) \left( \frac{\partial h_{pi}}{\partial x^l} + \frac{\partial h_{pl}}{\partial x^i} - \frac{\partial h_{il}}{\partial x^p} \right).
\]

(2.28)

From this formula we obtain the Ricci tensor
\[
R_{km} \approx -\frac{1}{2} \eta^{ij} \left( \frac{\partial^2 h_{km}}{\partial x^i \partial x^j} + \frac{\partial^2 h_{mk}}{\partial x^k \partial x^i} - \frac{\partial^2 h_{ik}}{\partial x^i \partial x^k} \right)
\]
\[
+ \frac{1}{4} \eta^{np} \left( \frac{\partial h_{nk}}{\partial x^m} + \frac{\partial h_{nm}}{\partial x^k} - \frac{\partial h_{km}}{\partial x^n} \right) \left( \frac{\partial h_{pi}}{\partial x^l} + \frac{\partial h_{pl}}{\partial x^i} - \frac{\partial h_{il}}{\partial x^p} \right).
\]

(2.29)

where we introduced the notations
\[
H_{km} = \eta^{ij} \left( \frac{\partial^2 h_{km}}{\partial x^i \partial x^j} + \frac{\partial^2 h_{mk}}{\partial x^k \partial x^i} - \frac{\partial^2 h_{ik}}{\partial x^i \partial x^k} \right)
\]
\[
= \frac{\partial^2 h^0_{km}}{\partial x^k \partial x^m} + \frac{\partial^2 h^0_{mk}}{\partial x^m \partial x^k} - \frac{\partial^2 h^0_{ik}}{\partial x^i \partial x^k} - \frac{\partial^2 h^i_{km}}{\partial x^k \partial x^m} \quad \text{(2.30)}
\]

and
\[
F_{km} = \eta^{ij} \left( \frac{\partial^2 f_{km}}{\partial x^i \partial x^j} + \frac{\partial^2 f_{mk}}{\partial x^k \partial x^i} - \frac{\partial^2 f_{ik}}{\partial x^i \partial x^k} \right)
\]
\[
= \frac{\partial^2 f^0_{km}}{\partial x^k \partial x^m} + \frac{\partial^2 f^0_{mk}}{\partial x^m \partial x^k} + \frac{\partial^2 f^0_{ik}}{\partial x^i \partial x^k} - \frac{\partial^2 f^i_{km}}{\partial x^k \partial x^m} \quad \text{(2.31)}
\]

Taking into account that \( h^0_{\alpha} = h^0_{\alpha} \equiv 0 \), we get
\[
H_{00} = \frac{\partial^2 h^0_{00}}{\partial (x^0)^2}
\]
\[
H_{0x} = \frac{\partial^2 h^0_{0x}}{\partial x^0 \partial x^x} - \frac{\partial^2 h^x_{0x}}{\partial x^0 \partial x^x} \quad \text{(2.32)}
\]

and
\[
H_{xx} = \frac{\partial^2 h^x_{xx}}{\partial (x^x)^2} - \frac{\partial^2 h^0_{xx}}{\partial x^0 \partial x^x},
\]

(2.33)

which are of the order of \( 1/c^4 \) and \( 1/c^3 \), respectively. For the components \( F_{km} \) we obtain
\[
F_{00} \approx \frac{\partial^2 f^0_{00}}{\partial (x^0)^2} = \frac{\partial^2 h^0_{00}}{\partial (x^0)^2} = \frac{1}{2} \frac{\partial^2 h^0_{00}}{\partial x^0 \partial x^x} \quad \text{(2.34)}
\]

which are defined up to the order \( 1/c^4 \) and \( 1/c^3 \), respectively. To obtain these expressions, we use the gauge condition
\[
\frac{\partial f^0_{00}}{\partial x^0} = \frac{1}{2} \frac{\partial h^0_{00}}{\partial x^0} \quad \text{(2.35)}
\]

Therefore, the 00-component of the Ricci tensor reads
\[
R_{00} \approx \frac{1}{2} \triangle h_{00} + \frac{1}{2} \frac{\partial^2 h_{00}}{\partial (x^0)^2} + \frac{1}{2} \frac{\partial^2 h^0_{00}}{\partial (x^0)^2} - \frac{1}{2} \frac{\partial^2 h^0_{00}}{\partial (x^0)^2} + \frac{1}{2} \eta^{ij} \eta^{kp} \frac{\partial^2 h_{00}}{\partial x^i \partial x^j}
\]

(7)
\[
\begin{align*}
\frac{1}{4} \eta^{il} \eta^{np} \left( \frac{\partial h_{pl}}{\partial x^i} - \frac{\partial h_{il}}{\partial x^p} \right)
&- \frac{1}{4} \eta^{il} \eta^{np} \left( \frac{\partial h_{il}}{\partial x^i} - \frac{\partial h_{pl}}{\partial x^p} + \frac{\partial h_{il}}{\partial x^i} - \frac{\partial h_{il}}{\partial x^p} \right).
\end{align*}
\]

With the help of the following relations (which are correct up to the order \(1/c^4\)):

\[
\frac{1}{2} \eta^{ij} \eta^{np} \frac{\partial^2 h_{00}}{\partial x^i \partial x^j} \approx \frac{1}{2} \Delta h_{11} = \frac{1}{D-2} \cdot \frac{2}{c^4} \psi \Delta \psi,
\]

\[
\frac{1}{4} \eta^{ij} \eta^{np} \left( \frac{\partial h_{il}}{\partial x^i} - \frac{\partial h_{pl}}{\partial x^p} + \frac{\eta_{ij} \eta_{lp} \partial h_{ij}}{\partial x^l} \right) \approx \frac{1}{2} \eta^{ij} \partial h_{00} \partial h_{00} \approx - \frac{2}{c^4} (\nabla \psi)^2.
\]

and

\[
\eta^{il} \eta^{np} \frac{\partial h_{il}}{\partial x^i} \approx \frac{1}{2} \partial h_{00} \partial h_{00} \approx - \frac{2}{c^4} (\nabla \psi)^2.
\]

where condition (2.14) was used in the latter expression, the 00-component (2.36) of the Ricci tensor takes the form

\[
R_{00} \approx \frac{1}{c^4} \Delta \psi + \frac{1}{2} \Delta f_{00} + \frac{1}{D-2} \cdot \frac{2}{c^4} \psi \Delta \psi = \frac{2}{c^4} (\nabla \psi)^2.
\]

The 0\(\alpha\)-component of the Ricci tensor (2.29) up to the order \(1/c^4\) reads

\[
R_{0\alpha} \approx \frac{1}{2} \eta^{\alpha \beta} \partial^2 h_{0\alpha} + \frac{1}{2} H_{0\alpha} + \frac{1}{2} \eta^{\alpha \beta} \partial^2 f_{0\alpha} + \frac{1}{2} F_{0\alpha} + \frac{1}{2} \partial^2 \psi + \frac{1}{2} \Delta f_{0\alpha} + \frac{1}{2} \partial \Delta \psi + \frac{1}{2} \partial \Delta \psi,
\]

where we used formulas (2.33) and (2.34).

Now, we come back to the Einstein equation (2.3). Substituting (2.25) and (2.40) into (2.3) and taking into account (2.5) and (2.16), we get the following equation for \(f_{00}\):

\[
\Delta f_{00} + \frac{1}{D-2} \cdot \frac{4}{c^4} \psi \Delta \psi - \frac{4}{c^4} (\nabla \psi)^2 = \frac{2 S_D G D}{c^4} \sum_{p=1}^N m_p \left( \frac{3D-4}{2(D-2)} v_p^2 + \frac{D}{2(D-2)} \psi_p \right) \delta(\mathbf{r} - \mathbf{r}_p).
\]

With the help of the auxiliary equation:

\[
4(\nabla \psi)^2 = 2 \Delta (\psi^2) - 4 \psi \nabla \psi
\]

and equations (2.5) and (2.16), equation (2.42) takes the form

\[
\Delta \left( f_{00} - \frac{2}{c^4} \psi^2 \right) = \frac{2 S_D G D}{c^4} \sum_{p=1}^N m_p \left( \psi_p + \frac{D}{2(D-2)} v_p^2 \right) \delta(\mathbf{r} - \mathbf{r}_p).
\]

Here, \(\psi_p\) is the potential of the gravitational field at a point with radius vector \(\mathbf{r}_p\) produced by all particles, except the pth. Subtraction of the infinite contribution of the gravitational field of the pth particle corresponds to the renormalization of its mass (see [21]). The solution of (2.44) is

\[
f_{00} = \frac{2}{c^4} \psi^2(\mathbf{r}) + \frac{1}{c^4} \sum_{p=1}^N \psi_p \psi(\mathbf{r} - \mathbf{r}_p) + \frac{D}{2(D-2)} \cdot \frac{1}{c^4} \sum_{p=1}^N v_p^2 \psi(\mathbf{r} - \mathbf{r}_p).
\]

where \(\psi(\mathbf{r} - \mathbf{r}_p)\) is the potential of the gravitational field of the pth particle which satisfies the Poisson equation

\[
\Delta \psi = \delta^{\alpha \beta} \frac{\partial^2 \psi}{\partial x^\alpha \partial x^\beta} = S_D G m_p \delta(\mathbf{r} - \mathbf{r}_p).
\]
It can be easily verified with the help of (2.5) and (2.16) that \( \psi'(r - r_p) \) satisfies the condition
\[
\psi(r) = \sum_{p=1}^{N} \psi'(r - r_p).
\] (2.47)

Therefore, substituting \( h_{00} = 2\psi/c^2 \) and \( f_{00} \) into (2.27), we obtain \( g_{00} \) up to the order \( 1/c^4 \):
\[
g_{00} \approx 1 + \frac{2}{c^2} \psi(r) + \frac{2}{c^4} \psi'(r) + \frac{2}{c^4} \sum_{p=1}^{N} \psi_p \psi'(r - r_p) + \frac{D}{D - 2} \frac{1}{c^4} \sum_{p=1}^{N} v_p^2 \psi'(r - r_p).
\] (2.48)

We should mention that the radius vectors \( r_p \) of the moving gravitating masses depend on time. In this case, potential \( \psi(r) \) in (2.47) also depends on time.

The equation for \( f_{00} \) can be obtained by substituting (2.26) and (2.41) into the Einstein equation (2.3):
\[
\Delta f_{00} = -\frac{2(D - 1)}{D - 2} \frac{S_D G_D}{c^3} \sum_{p=1}^{N} m_p v_{p\alpha} \delta(r - r_p) - \frac{1}{c^3} \frac{\partial^2 \psi}{\partial t \partial x^\alpha},
\] (2.49)
whose solution is
\[
f_{00} = -\frac{2(D - 1)}{D - 2} \frac{1}{c^3} \sum_{p=1}^{N} m_p v_{p\alpha} \psi'(r - r_p) + \frac{1}{c^3} \frac{\partial^2 f}{\partial t \partial x^\alpha}.
\] (2.50)

where the function \( f \) satisfies the following equation:
\[
\Delta f = \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} = \psi(r).
\] (2.51)

Therefore, substituting \( h_{00} = 0 \) and \( f_{00} \) into (2.27), we get \( g_{00} \) up to the order \( 1/c^3 \):
\[
g_{00} \approx \frac{2}{c^4} \psi(r) + \frac{2}{c^4} \psi'(r) + \frac{D}{(D - 2)c^4} \psi(r)
\] (2.52)

It is necessary to note that in the three-dimensional case \( D = 3 \) (2.48) and (2.52) exactly coincide with (106.13) and (106.14) in [21] if we take into account that \( \psi'(r - r_p) = -G_N m_p/|r - r_p| \).

From now on we shall consider the case of one gravitating particle of mass \( m_1 \equiv m \) at rest in our 3D space but, for generality, moving with constant speed in extra dimensions. That is \( p = 1 \Rightarrow \psi_1 = 0 \) and \( v^\alpha = dx^\alpha/dt = (0, 0, 0, v_4, v_5, \ldots, v_D) \), where \( v_4, v_5, \ldots, v_D \) are constants. In this case (2.48) and (2.52) are reduced correspondingly to
\[
g_{00} \approx 1 + \frac{2}{c^4} \psi(r) + \frac{D}{(D - 2)c^4} \psi(r)
\] (2.53)
and
\[
g_{00} \approx -\frac{2(D - 1)v_0}{(D - 2)c^4} \psi(r) + \frac{1}{c^3} \frac{\partial^2 f}{\partial t \partial x^\alpha},
\] (2.54)

where \( \psi(r) \) satisfies the Poisson equation
\[
\Delta \psi = \frac{\partial^2 \psi}{\partial x^\alpha \partial x^\beta} = S_D G_D m \delta(r)
\] (2.55)
and \( v^2 = -g_{\alpha\beta} v^\alpha v^\beta = v_4^2 + v_5^2 + \cdots + v_D^2 + O(1/c^2) \) (at the same accuracy \( v_0 = -v^\alpha \)). Obviously, the transition to the case where the gravitating mass is at rest both in our three-dimensional space and in the extra dimensions corresponds to the limit \( v = 0 \Rightarrow v^2 = 0 \). In
this case the potential $\phi(r)$ as well as the function $f$ does not depend on time $t$. We recall that the covariant components $g_{\alpha\beta}$ read (see (2.17))

$$g_{\alpha\beta} \approx -\left(1 - \frac{1}{D-2} \cdot \frac{2}{c^2} \phi(r)\right) \delta_{\alpha\beta}. \quad (2.56)$$

To obtain all the above results, we did not use any concrete form of topology. The only things we used were assumptions of the flatness of the metric in the absence of gravitating masses and the weakness of the gravitational field and velocities of gravitating masses which perturb the flat metric. Now, to solve (2.55) we should specify the topology of space and the boundary conditions. We suppose that the $(D = 3 + d)$-dimensional space has the factorizable geometry of a product manifold $M_D = \mathbb{R}^3 \times T^d$. Here $\mathbb{R}^3$ describes the three-dimensional flat external (our) space, and $T^d$ is a torus which corresponds to a $d$-dimensional internal space with volume $V_d$. For this topology, and with the boundary condition that at infinitely large distances from the gravitating body the potential must go to the Newtonian expression, we can find the exact solution of the Poisson equation (2.55) [22, 23]. The boundary condition requires that the multidimensional $G_D$ and Newtonian $G_N$ gravitational constants are connected by the following condition: $S_D G_D / V_d = 4\pi G_N$. Assuming that we consider the gravitational field of a gravitating mass $m$ at distances much greater than periods of tori, we can restrict ourselves to the zero Kaluza–Klein mode. For example, this approximation is very well satisfied for the planets of the solar system because the inverse square law experiments show that the extra dimensions in Kaluza–Klein models should not exceed submillimeter scales [24] (see however [22, 23] for models with smeared extra dimensions where Newton’s law preserves its shape for arbitrary distances). Then, the gravitational potential reads

$$\phi(r) \approx -\frac{G_N m}{r_3} = -\frac{r_3 c^2}{2 r_3}, \quad (2.57)$$

where $r_3$ is the length of a radius vector in three-dimensional space, and we introduce the three-dimensional Schwarzschild radius $r_3 = 2G_N m / c^2$. As we mentioned above, the gravitating mass $m$ is at rest in our three-dimensional space but may move in the extra dimensions. In this case, the extra dimensional components of the $D$-dimensional radius vector of the gravitating particle depend on time. The exact formulas for the non-relativistic gravitational potential (see [22, 23]) show that this dependence ‘nests’ only in nonzero Kaluza–Klein modes which are exponentially suppressed in the considered approximation. Therefore, in this approximation the potential $\phi(r)$ in (2.57) does not depend on time.

It is worth noting that all the previous analysis works also in the case when the gravitating masses are uniformly smeared over some or all extra dimensions. Let us take for simplicity one $(p = 1)$ gravitating mass $m_1 \equiv m$ which is smeared over all extra dimensions. Obviously, this mass can move only in our usual three dimensions: $v_i^d = dx_i^d / dt = (v_1^t, v_1^3, 0, \ldots, 0)$ and its rest mass density (2.5) now reads

$$\rho = \left(\frac{m}{\prod_{a=1}^d a_a}\right) \delta(r_3 - r_{1(3)}), \quad (2.58)$$

where $a_a$ are the periods of tori. Then, the solution of the Poisson equation (2.16) exactly coincides with the Newton potential if the multidimensional $G_D$ and Newtonian $G_N$ gravitational constants are connected as $S_D G_D / \prod_{a=1}^d a_a = 4\pi G_N$ [22, 23]. Therefore, in this case the approximate formula (2.57) becomes the exact equality:

$$\phi(r) = \phi(r_3) = -\frac{G_N m}{r_3} = -\frac{r_3 c^2}{2 r_3}. \quad (2.59)$$
In approximation (2.57) (or with (2.59) for ‘smeared’ extra dimensions), the covariant components (2.53), (2.54) and (2.56) take the form

\[ g_{00} \approx 1 - \frac{r_g}{r_3} + \frac{r_g^2}{2r_3^2} - \frac{Dv^2}{2(D-2)c^2 r_3}, \]

\[ g_{0\alpha} \approx \frac{(D-1)v_\alpha}{(D-2)c} r_3, \quad g_{\alpha\beta} \approx -\left(1 + \frac{1}{D-2} \cdot \frac{r_g}{r_3}\right) \delta_{\alpha\beta}. \]

(2.60)

For the contravariant components we obtain

\[ g^{00} \approx 1 + \frac{r_g}{r_3} + \frac{r_g^2}{2r_3^2} + \frac{Dv^2}{2(D-2)c^2 r_3}, \]

\[ g^{0\alpha} \approx -\frac{(D-1)v^\alpha}{(D-2)c} r_3, \quad g^{\alpha\beta} \approx -\left(1 - \frac{1}{D-2} \cdot \frac{r_g}{r_3}\right) \delta^{\alpha\beta}. \]

(2.61)

It is not difficult to verify that these components satisfy the condition

\[ g_{ik} g^{kj} = \left(1 + O(1/c^6)0 + O(1/c^5)\right) \left(0 + O(1/c^5)\delta_{\alpha\beta} + O(1/c^4)\right). \]

(2.62)

The metric components (2.60) demonstrate that in this approximation the spatial section \( t = \text{const} \) is conformal to the Euclidean metric. Hence, the spatial coordinates are isotropic.

It is convenient to use the three-dimensional spherical coordinates \( r_3, \theta, \psi \) instead of the Cartesian coordinates \( x_1 \equiv x, x_2 \equiv y, x_3 \equiv z \). In these coordinates the metric (2.61) reads

\[ ds^2 \approx \left(1 - \frac{r_g}{r_3} + \frac{r_g^2}{2r_3^2} - \frac{Dv^2}{2(D-2)c^2 r_3}\right) c^2 dr_3^2 + \frac{2(D-1)}{(D-2)c} r_3 D r_3 \sum_{a=1}^D v_a \, dx^a \]

\[ - \left(1 + \frac{1}{D-2} \cdot \frac{r_g}{r_3}\right) \left(dr_3^2 + r_3^2 d\theta^2 + r_3^2 \sin^2 \theta \, d\psi^2\right) \]

\[ = \left(1 + \frac{1}{D-2} \cdot \frac{r_g}{r_3}\right) \left((dx^4)^2 + (dx^5)^2 + \cdots + (dx^D)^2\right). \]

(2.63)

As we mentioned above, this metric corresponds to a gravitating mass at rest in our three-dimensional space. If the mass is smeared over extra dimensions, the appropriate velocity components vanish.

3. Classical gravitational tests

Now, we want to check the above-obtained multidimensional metric (2.63) from the point of its consistency with the famous classical tests: frequency shift, perihelion shift, deflection of light and time delay of radar echoes (the Shapiro time delay effect). We also want to calculate the parameterized post-Newtonian (PPN) parameters for obtained metric coefficients. It is well known that four-dimensional general relativity is in good agreement with these experiments and observed PPN parameters. Can the considered Kaluza–Klein models with point-like sources also be in concordance with observations?

3.1. Frequency shift

To investigate the gravitational redshift formula in the spacetime (2.63), we can use the famous expression for the relation between the frequency \( \omega_1 \) of a light signal emitted at point 1 with the
metric component \(g_{00}|_1\) and the frequency \(\omega_2\) received at point 2 with the metric component \(g_{00}|_2\):

\[
\omega_1[(g_{00})^{1/2}]_1 = \omega_2[(g_{00})^{1/2}]_2.
\]

(3.1)

Therefore, up to the order \(1/c^2\) we get

\[
\omega_2 \approx \omega_1 \left(1 + \frac{\varphi_1 - \varphi_2}{c^2}\right),
\]

(3.2)

where the non-relativistic potential \(\varphi\) is given by (2.57). In the considered approximation, this formula exactly coincides with the one from general relativity. Therefore, for this type of experiment it is hardly possible to observe the difference between the usual four-dimensional general relativity and multidimensional Kaluza–Klein models.

### 3.2. Perihelion shift

Let us consider now the motion of a test body of mass \(m'\) in the gravitational field described by the metric (2.63). The Hamilton–Jacobi equation

\[
g^{ik} \partial S \frac{\partial S}{\partial x^i} - m'^2c^2 = 0
\]

(3.3)

for this test body moving in the orbital plane \(\theta = \pi/2\) reads

\[
\frac{1}{c^2} \left(1 + \frac{r_g}{r_3} + \frac{r_g^2}{2r_3^2} + \frac{Dv^2}{2(D - 2)c^2 r_3} \right) \frac{\partial S}{\partial t} ^2 - \frac{2(D - 1)v^a r_g}{(D - 2)c^2 r_3} \frac{\partial S}{\partial x^a}
\]

\[
- \left(1 - \frac{1}{D - 2} r_3 \right) \frac{\partial S}{\partial r_3} ^2 - \frac{1}{r_3^2} \left(1 - \frac{1}{D - 2} r_3 \right) \frac{\partial S}{\partial \psi} ^2
\]

\[
- \left(1 - \frac{1}{D - 2} r_3 \right) \left[ \left( \frac{\partial S}{\partial x^4} \right) ^2 + \cdots + \left( \frac{\partial S}{\partial x^D} \right) ^2 \right] - m'^2c^2 \approx 0.
\]

(3.4)

We investigate this equation by the separation of variables considering the action in the form

\[
S = -E't + M\psi + S_3(r_3) + S_4(x^4) + \cdots + S_D(x^D).
\]

(3.5)

Here, \(E' \approx m'c^2 + E\) is the energy of the test body, which includes the rest energy \(m'c^2\) and non-relativistic energy \(E\) and \(M\) is the angular momentum. Substituting this expression for the action \(S\) in formula (3.4), we obtain an expression for \((dS_3/dr_3)\)\(^2\) holding there the terms up to the order \(1/c^2\):

\[
\left( \frac{dS_3}{dr_3} \right) ^2 \approx \frac{E'^2}{c^2} \left(1 - \frac{1}{D - 2} r_3 \right) ^{-1} \left(1 + \frac{r_g}{r_3} + \frac{r_g^2}{2r_3^2} + \frac{Dv^2}{2(D - 2)c^2 r_3} \right)
\]

\[
- \frac{M^2}{r_3^2} + E' \left(1 - \frac{1}{D - 2} r_3 \right) ^{-1} \frac{2(D - 1)v^a r_g}{(D - 2)c^2 r_3} \frac{\partial S}{\partial x^a}
\]

\[
- \left( \frac{dS_4}{dx^4} \right) ^2 - \cdots - \left( \frac{dS_D}{dx^D} \right) ^2 - m'^2c^2 \left(1 - \frac{1}{D - 2} r_3 \right) ^{-1}
\]

\[
\approx \left(2m'E - \left(p_2^2 + \cdots + p_D^2\right) + \frac{E^2}{c^2} \right) - \frac{1}{r_3^2} \left(M^2 - \frac{Dm'^2c^2r_g^2}{2(D - 2)} \right)
\]

\[
+ \frac{1}{r_3} \left(m'^2c^2r_g + \frac{2(D - 1)}{D - 2} m'E r_g + \frac{D}{2(D - 2)} m'^2r_g v^2 \right)
\]

\[
+ \frac{2}{D - 2} m'r_g \sum_{a=4}^{D} v^a p_a \right).\]

(3.6)
where \( p_\alpha = \partial S / \partial x^\alpha = dS_\alpha / dx^\alpha \) \((\alpha = 4, \ldots, D)\) are the components of the momentum of the test body in the extra dimensions. If the gravitating and test masses are localized on the same brane, then these components are equal to zero. Integrating the square root of this expression with respect to \( r_3 \), we get \( S_{r_3} \) in the following form:

\[
S_{r_3} \approx \int \left[ \left( 2m'E - \left( p_4^2 + \cdots + p_D^2 \right) + \frac{E^2}{c^2} \right) + \frac{1}{r_3} \left( m^2 c^2 r_\xi + \frac{2(D-1)}{D-2} m'E r_\xi + \frac{D}{2(D-2)} m^2 r_\xi v^2 \right) + \frac{2(D-1)}{(D-2)} m'r_3 \sum_{\alpha=4}^D v^\alpha p_\alpha \right]^{-1/2} dr_3. \tag{3.7}
\]

It is well known (see e.g. section 47 in [25]) that for any integral of motion \( I \) of a system with action \( S \) the following equation should hold:

\[
\frac{\partial S}{\partial I} = \text{const}. \tag{3.8}
\]

Because the angular momentum \( M \) is the integral of motion, the trajectory of the test body is defined by the equation

\[
\frac{\partial S}{\partial M} = \psi + \frac{\partial S_{r_3}}{\partial M} = \text{const}, \tag{3.9}
\]

where we use (3.5).

Let now the Sun be the gravitating mass and the planets of the solar system be the test bodies. Then, the change of the angle during one revolution of a planet on an orbit is

\[
\Delta \psi = -\frac{\partial}{\partial M} \Delta S_{r_3}, \tag{3.10}
\]

where \( \Delta S_{r_3} \) is the corresponding change of \( S_{r_3} \). It is well known that the perihelion shift originates due to the small relativistic correction \( \varepsilon \) to \( M^2 \) in \( S_{r_3} \): \( M^2/\varepsilon^2 \Rightarrow (M^2 - \varepsilon)/\varepsilon^2 \).

Equation (3.7) shows that in our case \( \varepsilon = Dm^2 c^2 r_\xi^2/[2(D-2)] \). Expanding \( S_{r_3} \) in powers of this correction:

\[
S_{r_3} = S_{r_3}(M^2 - \varepsilon) \approx S_{r_3}(0) - \varepsilon \frac{\partial S_{r_3}(0)}{\partial M^2},
\]

\[
= S_{r_3}(0) - \frac{\varepsilon}{2M} \frac{\partial S_{r_3}(0)}{\partial M} = S_{r_3}(0) - \frac{Dm^2 c^2 r_\xi^2}{4(D-2)M} \frac{\partial S_{r_3}(0)}{\partial M}, \tag{3.11}
\]

where \( S_{r_3}(0) \equiv S_{r_3}(M) \), we obtain

\[
\Delta S_{r_3} \approx \Delta S_{r_3}(0) - \frac{Dm^2 c^2 r_\xi^2}{4(D-2)M} \frac{\partial \Delta S_{r_3}(0)}{\partial M}. \tag{3.12}
\]

Differentiating this equation with respect to \( M \) we get

\[
\Delta \psi \approx 2\pi + \frac{D\pi m^2 c^2 r_\xi^2}{2(D-2)M^2}, \tag{3.13}
\]

where we took into account \(-\partial \Delta S_{r_3}(0)/\partial M = \Delta \psi(0) = 2\pi\). Therefore, the second term in (3.13) gives the required formula for the perihelion shift in our multidimensional case:

\[
\delta \psi = \frac{D\pi m^2 c^2 r_\xi^2}{2(D-2)M^2} = \frac{D\pi r_\xi}{(D-2)a(1-e^2)}, \tag{3.14}
\]

\( 13 \)
where in this equation we used the well-known relation $M^2 = m^2 r_g c^2 a(1 - e^2)/2$, with $a$ and $e$ being the semi-major axis and the eccentricity of the ellipse, respectively. For the three-dimensional case $D = 3$, this equation exactly coincides with formula (101.7) in [21]. It can be easily seen that the result (3.14) does not depend on the motion of the gravitating and test masses in the extra dimensions.

It makes sense to apply this formula to Mercury because in the solar system it has the most significant discrepancy between the measurement value of the perihelion shift and its calculated value using Newton’s formalism. The observed discrepancy is $43.11 \pm 0.21$ arcsec per century. This missing value is usually explained by the relativistic effects of the form (3.14). However, only in the three-dimensional case, $D = 3$ (3.14) gives the satisfactory result $42.94''$ which is within the measurement accuracy. For $D = 4$ and $D = 9$ models we obtain $28.63''$ and $18.40''$, respectively, which are very far from the observable value.

### 3.3. Deflection of light

Let us consider now the propagation of light in the gravitational field with the metric (2.63). In the case of massless particles, the Hamilton–Jacobi equation (3.3) is reduced to the eikonal equation

$$g^{ik} \frac{\partial \Psi}{\partial x^i} \frac{\partial \Psi}{\partial x^k} = 0,$$

(3.15)

which for the metric (2.63) reads

$$\frac{1}{c^2} \left( 1 + \frac{r_g}{r_3} + \frac{r_g^2}{2r_3^2} + \frac{Dv^2}{2(D - 2)c^2 r_3} \right) \left( \frac{\partial \Psi}{\partial t} \right)^2 = \frac{2(D - 1)e^\omega r_g}{(D - 2)c^2 r_3} \frac{\partial \Psi}{\partial x^a} - \left( 1 - \frac{1}{D - 2 r_3} \right) \left( \frac{\partial \Psi}{\partial r} \right)^2 - \left( 1 - \frac{1}{D - 2 r_3} \right) \left( \frac{\partial \Psi}{\partial \psi} \right)^2 \approx 0,$$

(3.16)

where we take into account that light propagates in the orbital plane $\theta = \pi/2$. The eikonal function $\Psi$ can be written in the form

$$\Psi = -\omega_0 t + \frac{\rho \omega_0}{c} \psi + \Psi_2(r_3) + \Psi_4(x^4) + \Psi_5(x^5) + \cdots + \Psi_D(x^D),$$

(3.17)

where $\omega_0 = -\partial \Psi/\partial t$ is the frequency of light and $\rho$ is a constant. Later we will show that $\rho$ is the impact parameter, i.e., the distance of the closest approach of the ray’s path to the gravitating mass. Taking into account that $k = \omega_0/c$ is the absolute value of the wave-vector, it is clear that $M \equiv \rho k = \rho \omega_0/c$ plays the role of angular momentum for the light beam.

Now we consider the natural case when the light propagates in our three-dimensional space and does not have components of momentum in the extra dimensions, that is $p_\alpha = d\Psi_\alpha/dx^\alpha = 0, \quad \alpha = 4, \ldots, D$. Then from (3.16), using (3.17), we obtain up to the order $O(1/c^4)$ the following formula:

$$\left( \frac{d\Psi_2}{dr_3} \right)^2 \approx \frac{\omega_0^2}{c^2} \left( 1 - \frac{1}{D - 2 r_3} \right)^{-1} \left( 1 + \frac{r_g}{r_3} + \frac{r_g^2}{2r_3^2} + \frac{Dv^2}{2(D - 2)c^2 r_3} \right) - \frac{\rho^2 \omega_0^2}{c^2 r_3^2} \left( 1 + \frac{D - 1}{D - 2 r_3} - \frac{\rho^2}{r_3^2} \right),$$

(3.18)
Integrating this expression we get
\[ \psi_{r_3} \approx \frac{\alpha_0}{c} \int \left( 1 + \frac{D - 1}{\rho} \right)^{1/2} dr_3. \]
Equation (3.19)

Considering the term with \( r_g / r_3 \) as a small relativistic correction, we expand the integrand up to the order \( O(1/c^3) \):
\[ \psi_{r_3} \approx \psi_{r_3}^{(0)} + \frac{D - 1}{2(D - 2)} \frac{r_g \alpha_0}{c} \int (r_3^2 - \rho^2)^{-1/2} dr_3
\]
\[ = \psi_{r_3}^{(0)} + \frac{D - 1}{2(D - 2)} \frac{r_g \alpha_0}{c} \arccosh \frac{r_3}{\rho}. \]
where the non-relativistic (i.e. gravity is absent: \( r_g \equiv 0 \)) eikonal function is
\[ \psi_{r_3}^{(0)} = \frac{\alpha_0}{c} \int \left( 1 - \frac{\rho^2}{r_3^2} \right)^{1/2} dr_3 \equiv \int \left( \frac{\alpha_0}{c} \right)^2 - \frac{M^2}{r_3^2} \right)^{1/2} dr_3. \]
Equation (3.21)

For this non-relativistic approximation the trajectory of the light beam is a straight line. Indeed, in this case (by full analogy with (3.9)) we have
\[ \frac{\partial \psi_{r_3}^{(0)}}{\partial M} = \psi_{r_3}^{(0)} = - \arccos(\rho/r_3) = 0, \]
Equation (3.22)

where the constant is taken in such a way that \( \psi_{r_3}^{(0)} \to \pi/2 \) for \( r_3 \to \infty \). Thus, the trajectory \( \rho = r_3 \cos \psi_{r_3}^{(0)} \) is a straight line. Obviously, in the non-relativistic case the total change of the angle \( \psi_{r_3}^{(0)} \) is \( \Delta \psi_{r_3}^{(0)} = - \partial \Delta \psi_{r_3}^{(0)}/\partial M = \pi \).

Coming back to the relativistic case (3.20), for the light beam travelling from some distance \( r_3 = R \) to the closest approach to the gravitating mass at \( r_3 = \rho \) and again to the distance \( r_3 \) = \( R \), the change of the eikonal function is
\[ \Delta \psi_{r_3} \approx \Delta \psi_{r_3}^{(0)} + \frac{D - 1}{D - 2} \frac{r_g \alpha_0}{c} \arccosh \frac{R}{\rho}. \]
Equation (3.23)

The corresponding change of the polar angle \( \psi \) is
\[ \frac{\partial \psi}{\partial M} = \psi + \frac{\partial \psi_{r_3}}{\partial M} = \text{const} \]

\[ \Delta \psi = - \frac{\partial \Delta \psi_{r_3}}{\partial M} \approx - \frac{\partial \Delta \psi_{r_3}^{(0)}}{\partial M} + \frac{D - 1}{D - 2} \frac{r_g R}{\rho} (R^2 - \rho^2)^{-1/2}. \]
Equation (3.24)

Thus in the limit \( R \to +\infty \) we finally get
\[ \Delta \psi \approx \pi + \frac{D - 1}{D - 2} \frac{r_g}{\rho}. \]
Equation (3.25)

Therefore, the second term in (3.25) gives the required formula for the deflection of light in our multidimensional case:
\[ \delta \psi = \frac{D - 1}{D - 2} \frac{r_g}{\rho}. \]
Equation (3.26)

For the three-dimensional case \( D = 3 \), this equation exactly coincides with formula (101.9) in [21].

Now we apply this formula to the Sun. Obviously, the radius \( R_{\text{Sun}} \) of the Sun is much greater than the size of the extra dimensions and approximation (2.57) works well on the distances \( r_3 \geq R_{\text{Sun}} \). For general relativity and for a ray that grazes the Sun’s limb (i.e. \( \rho \approx R_{\text{Sun}} \)) \( \delta \psi \approx 1.75 \) arcsec which is in very good agreement with observational data [12]. Equation (3.26) shows that we get this value of \( \delta \psi \) only for the usual three-dimensional space. In the cases \( D = 4 \) and \( D = 9 \), we obtain correspondingly \( \delta \psi \approx 1.31^\circ \) and \( \delta \psi \approx 1.00^\circ \) which are very far from the observable value.
3.4. Parameterized post-Newtonian parameters and gravitational tests

It is well known (see e.g. [26, 27]) that in PPN formalism the static, spherically symmetric metric in isotropic coordinates reads

\[ ds^2 = \left(1 - \frac{r_g}{r} + \frac{\beta}{2} \frac{r_g}{r} \right) c^2 dt^2 - \left(1 + \gamma \frac{r_g}{r} \right) \sum_{i=1}^{3} (dx^i)^2. \]  

(3.27)

In general relativity, \( \beta = \gamma = 1 \). However, simple comparison of equations (3.27) and (2.63) shows that the PPN parameters \( \beta \) and \( \gamma \) in our case are

\[ \beta = 1, \quad \gamma = \frac{1}{D - 2}. \]  

(3.28)

The latter expression shows that the parameter \( \gamma \) coincides with the corresponding value in general relativity if \( D = 3 \). Only in this case \( \gamma = 1 \). According to the experimental data, \( \gamma \) should be very close to 1. The tightest constraint on \( \gamma \) comes from the Shapiro time-delay experiment using the Cassini spacecraft: \( \gamma - 1 = (2.1 \pm 2.3) \times 10^{-5} \) [13, 28, 29]. On the other hand, for \( D = 4, 9 \) we get from (3.28) that \( \gamma - 1 = -1/2, -6/7 \) respectively, which is very far from the experimental data.

The formulas of the gravitational tests can be expressed via the PPN parameters [26, 28]. For example, the perihelion shift and the deflection of light read correspondingly

\[ \delta \psi = \frac{1}{3} \left( 2 + 2 \gamma - \beta \right) \frac{3\pi r_g}{a(1 - e^2)}, \]  

(3.29)

\[ \delta \psi = (1 + \gamma) \frac{r_p}{\rho}. \]  

(3.30)

Now, if we substitute in these expressions the values from equation (3.28) for \( \beta \) and \( \gamma \), then we exactly restore our formulas (3.14) and (3.26).

It makes sense to present the expression for the time delay of radar echoes (the Shapiro time delay effect) via the PPN parameters. This effect consists in time difference of the propagation of electromagnetic signals between two points (or for a round trip) in the curved and flat spaces. Usually, a signal transmits from the Earth through a region near the Sun to another planet or satellite and then reflects back to the Earth. If the planet (or satellite) is on the far side of the Sun from the Earth (superior conjunction), then the formula for the time delay reads [26, 28]

\[ \delta t = (1 + \gamma) \frac{r_g}{c} \ln \left( \frac{4r_{\text{Earth}}r_{\text{planet}}}{R_{\text{Sun}}^2} \right), \]  

(3.31)

where \( r_{\text{Earth}} \) and \( r_{\text{planet}} \) are the distances from the Sun to the Earth and to the planet, respectively. If we put into this formula the value of \( \gamma \) from equation (3.28), we get

\[ \delta t = \frac{D - 1}{D - 2} \frac{r_g}{c} \ln \left( \frac{4r_{\text{Earth}}r_{\text{planet}}}{R_{\text{Sun}}^2} \right). \]  

(3.32)

Obviously, this formula coincides with general relativity only for \( D = 3 \) (see e.g. [27]). For all others values of \( D \) the time delay differs from the general relativity by the factor of \( O(1) \).

4. Conclusion

In our paper, we investigated classical gravitational tests (frequency shift, perihelion shift, deflection of light and time delay of radar echoes) for multidimensional models with compact
internal spaces in the form of tori. We supposed that in the absence of gravitating masses the metric is a flat one. Gravitating point-like masses (moving or at rest) perturb this metric and we considered these perturbations in a weak-field approximation. In this approximation, we obtained the asymptotic form of the metric coefficients. Until this point, we did not require the compactness of the extra dimensions. This approach is valid for any number of spatial dimensions \( D \geq 3 \) and generalizes well-known calculations [21] in four-dimensional spacetime. Then, we admitted that, first, the extra dimensions are compact and have the topology of tori and, second, gravitational potential far away from gravitating masses tends to the non-relativistic Newtonian limit. It gave us a possibility of specifying the non-relativistic gravitational potential for considered models. In turn, it enabled us to specify the metric coefficients. In the case of a gravitating delta-shaped body at rest, we used these metric coefficients to calculate frequency shift, perihelion shift, deflection of light and parameterized post-Newtonian parameters \( \beta \) and \( \gamma \). With the help of the PPN parameter \( \gamma \), we also obtain the formula for the time delay of radar echoes. We demonstrated that for the frequency shift type experiment, it is hardly possible to observe the difference between the usual four-dimensional general relativity and multidimensional Kaluza–Klein models. However, the situation is quite different for the perihelion shift, deflection of light and the Shapiro time delay effect. In these three cases, we obtained formulas which generalize the corresponding ones in general relativity. We showed that all of these formulas depend on the total number of spatial dimensions \( D \), and they are in good agreement with observations only in the ordinary three-dimensional space \( D = 3 \). This result does not depend explicitly on the size of the extra dimensions. Therefore, it is impossible to avoid the problem with classical gravitational tests in a limit of arbitrary small sizes of the extra dimensions.

Therefore, our results show that in considered multidimensional Kaluza–Klein models the point-like gravitating masses cannot produce the gravitational field which corresponds to the classical gravitational tests. Moreover, it is not difficult to show (see our forthcoming paper) that a similar problem arises in the case of a compact static spherically symmetric perfect fluid with the following conditions for the energy–momentum tensor:

\[
T_{00} \gg T_{0\alpha}, T_{\alpha\beta}, \quad \alpha, \beta = 1, \ldots, D.
\]

To avoid this problem, it is necessary to break a symmetry between our three usual spatial dimensions and the extra dimensions. The branes are among the most natural candidates for solving this problem. Our results work in favor of the brane-world models. It is of interest also to check models with a nonlinear action \( f(R) \). However, to prove viability of these models it is necessary to perform similar investigations.

Acknowledgments

We want to thank Uwe Günther and Ignatios Antoniadis for useful comments. This work was supported in part by the ‘Cosmomicrophysics’ program of the Physics and Astronomy Division of the National Academy of Sciences of Ukraine. AZh acknowledges the hospitality of the Theory Division of CERN and the High Energy, Cosmology and Astroparticle Physics Section of the ICTP during final preparation of this work.

References

[1] Kaluza Th 1921 Zum Unitätsproblem der Physik Sitzungsber. Preuss. Akad. Wiss. Berlin, Math. Phys. K1 966
Klein O 1926 Quantentheorie und fünfdimensionale relativitätstheorie Z. Phys. 37 895
[2] Polchinski J 1998 String Theory, Volume 2: Superstring Theory and Beyond (Cambridge: Cambridge University Press)
[3] Bhattacharyya G, Datta A, Majee S K and Raychaudhuri A 2009 Nucl. Phys. B 821 48 (arXiv:hep-ph/0904.0937)
[4] Kalligas D, Wesson P S and Everitt C W F 1995 Astrophys. J. 439 548
[5] Liu H and Overduin J M and Wesson P S 1995 J. Math. Phys. 36 6907
[6] Liu H and Overduin J 2000 Astrophys. J. 538 386 (arXiv:gr-qc/0003034)
[7] Liko T, Overduin J M and Wesson P S 2004 Space Sci. Rev. 110 337 (arXiv:gr-qc/0311054)
[8] Rahaman F, Ray S, Kalam M and Sarker M 2009 Int. J. Theor. Phys. 48 3124 (arXiv:gr-qc/0707.0951)
[9] Xu P and Ma Y 2007 Phys. Lett. B 656 165 (arXiv:gr-qc/0710.3677)
[10] Poplawski N J 2010 Einstein–Cartan gravity excludes extra dimensions arXiv:hep-th/1001.4324
[11] Shapiro I I, Smith W B, Ash M E and Herrick S 1971 Astron. J. 76 588
[12] Shapiro S S, Davis J L, Lebach D E and Gregory J S 2004 Phys. Rev. Lett. 92 121101
[13] Bertotti B, Iess L and Tortora P 2003 Nature 425 574
[14] Myers R C and Perry M J 1986 Ann. Phys. 172 304
[15] Kramer D 1970 Acta Phys. Polon. B 2 807
[16] Gross D J and Perry M J 1983 Nucl. Phys. B 226 29
[17] Davidson A and Owen D 1985 Phys. Lett. 155 247
[18] Gregory R and Laflamme R 1993 Phys. Rev. Lett. 70 2837 (arXiv:hep-th/9301052)
[19] Boehmer C G, Harko T and Lobo F S N 2008 Class. Quantum Grav. 25 045015 (arXiv:gr-qc/0801.1375)
[20] Chodos A and Detweiler S 1982 Gen. Rel. Grav. 14 879
[21] Landau L D and Lifshitz E M 2000 The Classical Theory of Fields Volume 2 (Course of Theoretical Physics Series) 4th edn (Oxford: Pergamon)
[22] Eingorn M and Zhuk A 2009 Phys. Rev. D 80 124037 (arXiv:hep-th/0907.5371)
[23] Eingorn M and Zhuk A 2010 Class. Quantum Grav. 27 055002 (arXiv:gr-qc/0910.3507)
[24] Kapner D J, Cook T S, Adelberger E G, Gundlach J H, Heckel B R, Hoyle C D and Swanson H E 2007 Phys. Rev. Lett. 98 021101 (arXiv:hep-ph/0611184)
[25] Landau L D and Lifshitz E M 2000 Mechanics: Volume 1 (Course of Theoretical Physics Series) 3rd edn (Oxford: Pergamon)
[26] Will C M 2000 Theory and Experiment in Gravitational Physics (Cambridge: Cambridge University Press)
[27] Straumann N 1984 General Relativity and Relativistic Astrophysics (Berlin: Springer)
[28] Will C M 2005 Was Einstein right? Testing relativity at the century 100 Years of Relativity: Spacetime Structure—Einstein and Beyond ed A Ashtekar (Singapore: World Scientific) p 205 (arXiv:gr-qc/0504086)
[29] Jain Bh and Khoury J 2010 Cosmological tests of gravity arXiv:astro-ph/1004.3294