Two-Dimensional Klein–Gordon Oscillator in the Presence of a Minimal Length

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Abstract—Minimal length of a two-dimensional Klein–Gordon oscillator is investigated and illustrates the wave functions in the momentum space. The eigensolutions are found and the system is mapping to the well-known Schrödinger equation in a Pöschl–Teller potential.

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1. INTRODUCTION

Recently, there have been growing interest in obtaining exact solutions of relativistic wave equations. In particular exact solutions of the Klein–Gordon equation with various vector and scalar potentials.

The Dirac relativistic oscillator is an important potential both for theory and application. It was for the first time studied by Ito et al. [1]. They considered a Dirac equation in which the momentum \( \hat{p} \) is replaced by \( -i \beta \omega \pi m \hat{r} \), with \( \hat{r} \) being the position vector, \( m \) the mass of particle, and \( \omega \) the frequency of the oscillator. The interest in the problem was revived by Moshinsky and Szczepaniak [2], who gave it the name of Dirac oscillator (DO) because, in the non-relativistic limit, it becomes a harmonic oscillator with a very strong spin-orbit coupling term. Physically, it can be shown that the (DO) interaction is a physical system, which can be interpreted as the interaction of the anomalous magnetic moment with a linear electric field [3, 4]. The electromagnetic potential associated with the DO has been found by Benitez et al. [5]. The Dirac oscillator has attracted a lot of interest both because it provides one of the examples of the Dirac’s equation exact solvability and of its numerous physical applications (see [6] and reference therein). Recently, Franco-Villafane et al. [7] exposed the proposal of the first experimental microwave realization of the one-dimensional DO. The experiment relies on a relation of the DO to a corresponding tight-binding system. The experimental results obtained, concerning the spectrum of the one-dimensional DO with and without the mass term, are in good agreement with those obtained in the theory. In addition, Quimbay et al. [8, 9] show that the Dirac oscillator can describe a naturally occurring physical system. Specifically, the case of a two-dimensional Dirac oscillator can be used to describe the dynamics of the charge carriers in graphene, and hence its electronic properties. Also, the exact mapping of the DO in the presence of a magnetic field with a quantum optics leads to consider the DO as a theory of an open quantum systems coupled to a thermal bath (see [6] and references therein).

The unification between the general theory of relativity and the quantum mechanics is one of the most important problems in theoretical physics. This unification predicts the existence of a minimal measurable length on the order of the Planck length. All approaches of quantum gravity show the idea that near the Planck scale, the standard Heisenberg uncertainty principle should be reformulated. The minimal length uncertainty relation has appeared in the context of the string theory, where it is a consequence of the fact that the string cannot probe distances smaller than the string scale \( h\sqrt{\beta} \), where \( \beta \) is a small positive parameter called the deformation parameter. This minimal length can be introduced as an additional uncertainty in position measurement, so that the usual canonical commutation relation between position and momentum operators becomes \([\hat{x}, \hat{p}] = i\hbar(1 + \beta p^2)\). This commutation relation leads to the standard Heisenberg uncertainty relation \( \Delta \hat{x} \Delta \hat{p} \geq \hbar(1 + \beta(\Delta p)^2) \), which clearly implies the existence of a non-zero minimal length \( \Delta x_{min} = \hbar \sqrt{\beta} \). This modification of the uncertainty relation is usually termed the generalized uncertainty principle (GUP) or the minimal length uncertainty principle [11–14].

Nowadays, the reconsideration of the relativistic quantum mechanics in the presence of a minimal
measurable length have been studied extensively. In this context, many papers were published where a different quantum system in space with Heisenberg algebra was studied. They are: the Abelian Higgs model [15], the thermostatics with minimal length [16], the one-dimensional Hydrogen atom [17], the casimir effect in minimal length theories [18], the effect of minimal lengths on electron magnetism [19], the Dirac oscillator in one and three dimensions [20, 21], the solutions of a two-dimensional Dirac equation in presence of an external magnetic field [22], the non-commutative phase space Schrödinger equation [23], Schrödinger equation with Harmonic potential in the presence of a Magnetic Field [24].

The purpose of this work is to investigate the formulation of a two-dimensional Klein–Gordon oscillator by solving fundamental equations in the framework of relativistic quantum mechanics with minimal length. The problem describes a relativistic particle moving in the relativistic harmonic oscillator called the Dirac oscillator.

The paper is organized as follows. In Section II, we exposed the solutions of our problem within habitual quantum mechanics, using the new method developed by Menculini et al. [10] and Jana et al. [25]. Then, Section III will be devoted to the our case, i.e., the solution of a two dimensional Klein–Gordon oscillator in the framework of relativistic quantum mechanics with minimal length. Finally, in Section V, we present the conclusion.

2. THE SOLUTIONS WITHIN HABITUAL QUANTUM MECHANICS

A two-dimensional Klein–Gordon oscillator is

\[
\left\{ (p_x + im_0\omega x)(p_x - im_0\omega x) + (p_y + im_0\omega y)(p_y - im_0\omega y) - \frac{E^2 - m_0^2c^4}{c^2} \right\} \psi_{KG} = 0,
\]

with

\begin{align*}
\cup &= (p_x + im_0\omega x)(p_x - im_0\omega x) = p_x^2 + m_0^2\omega^2x^2 - m_0m_0, \\
\cap &= (p_y + im_0\omega y)(p_y - im_0\omega y) = p_y^2 + m_0^2\omega^2y^2 - m_0m_0.
\end{align*}

Now, for the sake of simplicity, we bring the problem into the momentum space. Recalling that

\[
\begin{align*}
&x = ih\frac{\partial}{\partial p_x}, \quad y = ih\frac{\partial}{\partial p_x}, \\
&\hat{p}_x = p_x, \quad \hat{p}_y = p_y,
\end{align*}
\]

and passing onto polar coordinates with the following definition [10]

\[
p_x = p\cos\theta, \quad p_y = p\sin\theta,
\]

with \(p^2 = p_x^2 + p_y^2\),

\[
\hat{x} = ih\frac{\partial}{\partial p_x} = ih\left(\cos\theta\frac{d}{dp} - \sin\theta\frac{d}{dp}\right),
\]

\[
\hat{y} = ih\frac{\partial}{\partial p_y} = ih\left(\sin\theta\frac{d}{dp} + \cos\theta\frac{d}{dp}\right).
\]

Equations (2) and (3) transform into

\[
\cup = p^2\cos^2\theta - m_0^2\omega^2\frac{\partial^2}{\partial p_x^2} - 2hm_0\omega - \zeta
\]

\[
\cap = p^2\sin^2\theta - m_0^2\omega^2\frac{\partial^2}{\partial p_y^2} - 2hm_0\omega - \zeta.
\]

In this case, Eq. (1) becomes

\[
\left( p^2 - m_0^2\omega^2\frac{\partial^2}{\partial p_x^2} + \frac{\partial^2}{\partial p_y^2} - 2hm_0\omega - \zeta \right)\psi_{KG} = 0.
\]

By using Eqs. (6), (7) and (8), we have

\[
\frac{\partial^2}{\partial p_x^2} + \frac{\partial^2}{\partial p_y^2} = \frac{\partial^2}{\partial p_x^2} + \frac{1}{p}\frac{\partial^2}{\partial p_x^2} + \frac{1}{p}\frac{\partial^2}{\partial p_y^2},
\]

and consequently, we obtain

\[
\left( p^2 - \lambda^2\frac{\partial^2}{\partial p_x^2} + \frac{1}{p^2}\frac{\partial^2}{\partial p_x^2} + \frac{1}{p^2}\frac{\partial^2}{\partial p_y^2} - 2\lambda - \zeta \right)\psi_{KG} = 0.
\]

with \(\lambda = m_0\omega\) and

\[
\zeta = \frac{E^2 - m_0^2c^4}{c^2}.
\]

Now, when we choose

\[
\psi_{KG}(p, \theta) = f(p)e^{i\theta},
\]

Equation (13) is transformed into

\[
\left( \frac{df(p)}{dp} + \frac{1}{p}\frac{df(p)}{dp} - \frac{f(p)}{p} \right) + (\kappa^2 - k^2p^2)f(p) = 0,
\]

with

\[
\kappa^2 = 2\lambda + \zeta, \quad k^2 = \frac{1}{\lambda^2}.
\]

Now, putting [25]

\[
f(p) = p^\alpha e^{-\frac{\kappa^2}{2p}}F(p),
\]

the differential equation

\[
F'' + \left(\frac{2|\kappa|^2 + 1}{p} - 2kp\right)F' - [2k(|\kappa| + 1) - \kappa^2]F = 0.
\]
is obtained for $F(p)$ which by using, instead of $p$, the
variable $xt = k p^2$, is transformed into the Kummer
equation
\begin{equation}
  t \frac{d^2 F}{dt^2} + \left[|l + 1 - t| \right] \frac{d F}{dt} - \frac{1}{2} \left[\left|l + 1 - \frac{K^2}{4kt}\right| \right] F = 0, \quad (20)
\end{equation}
whose solution is the confluent series $F_l(a;|l + 1; t)$, with
\begin{equation}
  a = \frac{1}{2}(|l + 1| - \frac{K^2}{4kt}). \quad (21)
\end{equation}
The confluent series becomes a polynomial if and only
if $a = -n$, $(n = 0, 1, 2, \ldots)$. We then have the solutions
\begin{equation}
  \psi_{KG}(p, \theta) = C \kappa \eta p^j e^{-\frac{\kappa}{2} p^2} F_l(-n;|l + 1; k p^2) e^{i\eta}, \quad (22)
\end{equation}
\begin{equation}
  E_n = \pm nc_6 \sqrt{1 + 2rN}. \quad (23)
\end{equation}
with $N = 2n + |l|$ is the principal quantum number, and $r = \frac{\hbar \kappa}{m c^2}$. This form of energy is in a good agree-
ment with that obtained in the [11].

3. THE SOLUTIONS IN THE PRESENCE
OF A MINIMAL LENGTH

In the minimal length formalism, the Heisenberg
algebra is given by [12–24]
\begin{equation}
  [\hat{x}_i, \hat{p}_j] = i \hbar \delta_{ij}(1 + \beta p^2), \quad (24)
\end{equation}
where $\beta > 0$ is the minimal length parameter. A representa-
tion of $\hat{x}_i$ and $\hat{p}_j$ which satisfies Eq. (24), may be taken as
\begin{equation}
  \hat{x} = i \hbar (1 + \beta p^2) \frac{d}{dx}, \quad \hat{p}_x = p_x, \quad (25)
\end{equation}
\begin{equation}
  \hat{y} = i \hbar (1 + \beta p^2) \frac{d}{dp_y}, \quad \hat{p}_y = p_y. \quad (26)
\end{equation}
In this case, the KG oscillator equation is
\begin{equation}
  \left\{ p_x^2 + p_y^2 + m^2 \omega^2 (x^2 + y^2) + \text{im} \alpha [x, p_x] \right\} \psi_{KG} = 0. \quad (27)
\end{equation}
By using the Eqs. (25) and (26), Eq. (27) becomes
\begin{equation}
  \left\{ \frac{\hbar^2}{2} \left[ \frac{\partial^2}{\partial p_x^2} + \frac{\partial^2}{\partial p_y^2} + \frac{1}{p_x^2} \frac{\partial}{\partial p_x} + \frac{1}{p_y^2} \frac{\partial}{\partial p_y} \right] - 2\beta \lambda^2 (1 + \beta p^2) p \frac{\partial}{\partial p} - 2\lambda (1 + \beta p^2) - \xi \right\} \psi_{KG} = 0. \quad (28)
\end{equation}
Now, when we put that
\begin{equation}
  \psi_{KG} = \psi(p)e^{i\eta}, \quad (29)
\end{equation}
with $j = 0, \pm 1, \pm 2, \ldots$, then Eq. (28) transformed into
\begin{equation}
  \left\{ -a(p) \frac{\partial^2}{\partial p^2} + b(p) \frac{\partial}{\partial p} + c(p) - \xi \right\} \psi(p) = 0, \quad (30)
\end{equation}
with
\begin{align*}
  a(p) &= \lambda^2 (1 + \beta p^2)^2, \\
  b(p) &= \frac{\lambda^2 (1 + \beta p^2)^2}{p} - 2\beta \lambda^2 (1 + \beta p^2)p \\
  c(p) &= \frac{\lambda^2 (1 + \beta p^2)^2}{p^2} - 2\lambda (1 + \beta p^2) \\
  p &= \frac{1}{\sqrt{3}} \tan(\beta \lambda \sqrt{\beta}), \quad (31)
\end{align*}
Now, according to the following substitution [25]
\begin{equation}
  h(p) = \rho(p) \rho(p), \quad q = \int \frac{1}{\sqrt{a(p)}} dp, \quad (32)
\end{equation}
\begin{equation}
  \rho(p) = \exp \left( \int \chi(p) dp \right), \quad \chi(p) = \frac{2b + a'}{4a} = -\frac{1}{2p}, \quad (33)
\end{equation}
we have
\begin{equation}
  \left\{ -\frac{d^2 \varphi(p)}{dp^2} + V(p) \right\} \varphi(p) = \zeta \varphi(p), \quad (34)
\end{equation}
with
\begin{equation}
  V(p) = p^2 - 2\lambda (1 + \beta p^2) \\
  + \beta \lambda^2 (1 + \beta p^2) + \frac{\lambda^2 (1 + \beta p^2)^2}{p^2} \left( f - \frac{1}{4} \right). \quad (35)
\end{equation}
We note here that the function $\rho(p) = p^{-\frac{1}{2}}$.
Now, if we put
\begin{equation}
  p = \frac{1}{\sqrt{3}} \tan(\beta \lambda \sqrt{\beta}), \quad (36)
\end{equation}
the term $V(p)$ becomes
\begin{equation}
  V(p) = -\frac{1}{\beta} \lambda^2 \frac{\lambda^2}{\beta} \frac{1}{\cos^2 \alpha q} \left( \frac{f + \frac{3}{4} - \frac{2}{\beta \lambda^2} + \frac{1}{\beta^2 \lambda^2}}{\cos^2 \alpha q} + \frac{f - \frac{1}{4}}{\cos^2 \alpha q} \right), \quad (37)
\end{equation}
and consequently, the final form of our differential equation is
\begin{equation}
  \left\{ -\frac{d^2 \varphi(p)}{dp^2} + U_0 \frac{1}{2} \left( \frac{f + \frac{3}{4} - \frac{2}{\beta \lambda^2} + \frac{1}{\beta^2 \lambda^2}}{\cos^2 \alpha q} + \frac{f - \frac{1}{4}}{\cos^2 \alpha q} \right) \right\} \times \varphi(p) = \zeta \varphi(p), \quad (38)
\end{equation}
where $\zeta = \zeta + \frac{1}{\beta}$. Equation (38) brings into
\begin{equation}
  \left\{ -\frac{d^2 \varphi(p)}{dp^2} + U_0 \left( \frac{\zeta_1 (\zeta_1 - 1)}{\cos^2 \alpha q} + \frac{\zeta_2 (\zeta_2 - 1)}{\sin^2 \alpha q} \right) \right\} \times \varphi(p) = \zeta \varphi(p), \quad (39)
\end{equation}
with
\[ V(p) = -\frac{1}{\beta} + \beta\lambda^2 \left( \frac{\zeta_1(\zeta_1 - 1)}{\sin^2(\alpha q)} + \frac{\zeta_2(\zeta_2 - 1)}{\cos^2(\alpha q)} \right), \quad (40) \]

where
\[ \zeta_1(\zeta_1 - 1) = j^2 - \frac{1}{4}, \quad (41) \]
\[ \zeta_2(\zeta_2 - 1) = j^2 + \frac{3}{4} - \frac{2}{\beta\lambda} + \frac{1}{\beta^2\lambda^2}. \quad (42) \]

Thus, we have
\[ \left( -\frac{d^2}{dq^2} + \frac{1}{2} U_0 \left( \frac{\zeta_1(\zeta_1 - 1)}{\sin^2(\alpha q)} + \frac{\zeta_2(\zeta_2 - 1)}{\cos^2(\alpha q)} \right) \right) \varphi(p) = \xi^2\varphi(p), \quad (43) \]

where $U_0 = \alpha^2$ with $\alpha = \lambda\sqrt{\beta}$. Equation (43) is the well-known Schrödinger equation in a Pöschl–Teller potential with [26]
\[ U = \frac{1}{2} U_0 \left( \frac{\zeta_1(\zeta_1 - 1)}{\sin^2(\alpha q)} + \frac{\zeta_2(\zeta_2 - 1)}{\cos^2(\alpha q)} \right), \quad (44) \]

and with the following conditions $\zeta_1 > 1$ and $\zeta_2 > 1$.

By comparison Eq. (38) with Eq. (43), we have
\[ \zeta_1 = |j| \pm \frac{1}{2}, \quad (45) \]
\[ \zeta_2 = \frac{1}{2} \pm \left( \frac{1}{\beta\lambda} - 1 \right) \sqrt{1 + \frac{j^2}{\left( \frac{1}{\beta\lambda} - 1 \right)^2}}. \quad (46) \]

Now, in order to solve Eq. (38), we introduce the new variable
\[ z = \sin^2(\alpha q). \quad (47) \]

In this case, Eq. (38) can be written by
\[ z(1 - z)\varphi'' + \left( \frac{j}{z} - 1 \right)\varphi' + \frac{1}{4}\left( \frac{\xi^2}{\alpha^2} - \frac{\zeta_1(\zeta_1 - 1)}{z} - \frac{\zeta_2(\zeta_2 - 1)}{1 - z} \right)\varphi = 0. \quad (48) \]

With the new wave function $\varphi$, defined by
\[ \varphi = z^2(1 - z)^2\varphi(z), \quad (49) \]

we arrive at
\[ z(1 - z)\Psi'' + \left( \frac{\zeta_1 + \frac{1}{2}}{2} - z(\zeta_1 + \zeta_2 + 1) \right)\Psi' + \frac{1}{4}\left( \frac{\xi}{\alpha} - \zeta_1 - \zeta_2 \right)^2 \Psi = 0. \quad (50) \]

The general solution of this equation is
\[ \Psi = C_1 F_1(a';b';c';z) + C_2 F_1(a' - 1 + c';b' + 1 - c';2 - c;z), \quad (51) \]

with
\[ a' = \frac{1}{2} \left( \zeta_1 + \zeta_2 + \frac{\xi}{\alpha} \right), \]
\[ b' = \frac{1}{2} \left( \zeta_1 + \zeta_2 - \frac{\xi}{\alpha} \right), \quad c' = \zeta_1 + \frac{1}{2}. \quad (52) \]

With the condition $a' = -n$, we obtain
\[ \xi^2 = \alpha^2(\zeta_1 + \zeta_2 + 2n^2). \quad (53) \]

In order to obtain the energy spectrum, it should be to note that in the limit $\beta \to 0$, the energy spectrum should regenerate to the no-GUP result.

Thus, the exact form of $\zeta_1$ and $\zeta_2$ are
\[ \zeta_1 = |j| + \frac{1}{2}, \quad (54) \]
\[ \zeta_2 = \frac{1}{2} \pm \frac{1}{2} \left( \frac{1}{\beta\lambda} - 1 \right) \sqrt{1 + \frac{j^2}{\left( \frac{1}{\beta\lambda} - 1 \right)^2}}, \quad (55) \]

where $j \neq 0$.

With the aid of Eqs. (45), (46) and (53), we obtain the final form of the spectrum of energy, which is given by
\[ \frac{E}{mc^2} = \pm \sqrt{1 - 2r + \frac{\beta}{\beta_0} r^2 + 2\Sigma (N + 1) \left( r - \frac{\beta}{\beta_0} r^2 \right) + \frac{\beta}{\beta_0} r^2 N^2}, \quad (56) \]

with
\[ \Sigma = \sqrt{1 + \frac{j^2}{\left( \frac{1}{\beta\lambda} - 1 \right)^2}}. \quad (57) \]

The corresponding wave function is
\[ \Psi_{KG} = \psi_{K}\psi_{\psi}(1 - z)^2F_1(-n; b'; c'; z), \quad (58) \]

with $N$ is the constant of normalization.

4. CONCLUSIONS

In this paper, we have exactly solved the Klein–Gordon oscillator in two dimensions in the framework...
of relativistic quantum mechanics with minimal length. The eigensolutions are obtained using a method developed in [10, 25] to solve a two-dimensional Dirac equation and Klein–Gordon equations. We firstly consider the case of the Klein–Gordon oscillator within the ordinary quantum mechanics: our results are in good agreement with those obtained in [11]. Then we have extended it in the case of the presence of a minimal length. The energy levels, for both cases, show a dependence on $\beta^2$ in the presence of the minimal length, which describes a hard confinement. In the limit where $\beta \to 0$, we recover the energy spectrum of no-GUP.

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