Constructions of the soluble potentials for the non-relativistic quantum system by means of the Heun functions

Shishan Dong\textsuperscript{1}, G. Yáñez-Navarro\textsuperscript{2}, M. A. Mercado Sánchez\textsuperscript{3}, C. Mejía García\textsuperscript{2} 
Guo-Hua Sun\textsuperscript{4}, and Shi-Hai Dong\textsuperscript{3}\textsuperscript{*}

\textsuperscript{1}Information and Engineering College, Dalian University, Dalian 116622, P. R. China
\textsuperscript{2}Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional, Edificio 9, Unidad Profesional ALM, CDMX C. P. 07738, Mexico
\textsuperscript{3}Laboratorio de Información Cuántica, CIDETEC, Instituto Politécnico Nacional, Unidad Profesional ALM, CDMX 07700, Mexico
\textsuperscript{4}Catedrática CONACyT, CIC, Instituto Politécnico Nacional, Unidad Profesional ALM, CDMX 07700, Mexico

\textsuperscript{*}E-mail address: dongsh2@yahoo.com

Abstract

The Schrödinger equation $\psi''(x) + \kappa^2 \psi(x) = 0$ where $\kappa^2 = k^2 - V(x)$ is rewritten as a more popular form of a second order differential equation through taking a similarity transformation $\psi(z) = \phi(z)u(z)$ with $z = z(x)$. The Schrödinger invariant $I_S(x)$ can be calculated directly by the Schwarzian derivative \{z, x\} and the invariant $I(z)$ of the differential equation $u_{zz} + f(z)u_z + g(z)u = 0$. We find an important relation for moving particle as $\nabla^2 = -I_S(x)$ and thus explain the reason why the Schrödinger invariant $I_S(x)$ keeps constant. As an illustration, we take the typical Heun differential equation as an object to construct a class of soluble potentials and generalize the previous results through choosing different $\rho = z'(x)$ as before. We get a more general solution $z(x)$
through integrating \((z')^2 = \alpha_1 z^2 + \beta_1 z + \gamma_1\) directly and it includes all possibilities for those parameters. Some particular cases are discussed in detail.

**Key words**: Schrödinger invariant, Schwarzian derivative, Heun differential equation, Soluble potentials.

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1 Introduction

The exact solution of the Schrödinger equation with physical potentials has played an important role in quantum mechanics. Generally speaking, for a given external field, one of our main tasks is to show how to solve the differential equation through choosing suitable variables and then find its solutions can be expressed by some special functions. Here we focus on how to construct a class of the solvable potentials within the framework of the non-relativistic Schrödinger equation. Similar works have been carried out [1, 2, 3, 4, 5], but it is worth pointing out that Ishkhanyan and his co-authors took \(\rho^2 \propto (z - a_1)^{m_1}(z - a_2)^{m_2}(z - a_3)^{m_3}\), where the parameters \(a_{1,2,3}\) are three singularity points, to construct the soluble potentials with some constraints on the parameters \(-1 \leq m_{1,2,3} \leq 1\) and \(1 \leq m_1 + m_2 + m_3 \leq 3\) [4, 5]. By choosing different values of these parameters which satisfy these constraints, some interesting results have been obtained. However, the approaches which were taken by Natanzon [2], who constructed a class of the soluble potentials related to the hypergeometric functions and Bose [1], who discussed the Riemann and Whittaker differential equations are different from Ishkhanyan et al. Nevertheless, in Bose’s classical work [1], he only studied a few special cases for the differential equation \((z')^2 = \alpha_1 z^2 + \beta_1 z + \gamma_1\). Its general solutions were not presented at that time due to the limit on the possible computation condition. In this work our aim is to construct the soluble potentials within the framework of the Schrödinger invariant \(I_S(z)\) through solving \(z'(x)\) differential equation directly and then obtaining its more general solutions but not only considering several special cases for the parameters \(\alpha_1, \beta_1\) and \(\gamma_1\).

The rest of this work is organized as follows. In Section 2 we present the Schwarzian derivative \(\{z, x\}\) and the invariant \(I(z)\) of the differential equation \(u_{zz} + f(z)u_z + g(z)u = 0\) through acting the similarity transformation \(\psi(z) = \phi(z)u(z)\) on the Schrödinger equation. In
Section 3 as an illustration we take the Heun differential equation as a typical example but with different approach taken by Ishkhanyan et al. The all soluble potentials are obtained completely in Section 4. Some concluding remarks are given in Section 5.

2 Similarity transformations to the Schrödinger equation

As we know, the Schrödinger equation has the form

$$\frac{d^2}{dx^2} \psi(x) + [k^2 - V(x)] \psi(x) = 0.$$  \hfill(1)

where we call $k^2$ an energy term and $V(x)$ an external potential.

Through choosing a similarity transformation $\psi(z) = \phi(z) u(z)$ where $z = z(x)$, we are able to obtain the following differential equation

$$u_{zz}(z) + \left( \frac{\rho_z}{\rho} + \frac{2 \phi_z}{\phi} \right) u_z(z) + \left[ \frac{\phi_{zz}}{\phi} + \frac{\rho_z \phi_z}{\rho \phi} + \frac{k^2 - V(x)}{\rho^2} \right] u(z) = 0, \quad \rho(x) = \frac{dz(x)}{dx}, \hfill(2)$$

which can be rewritten as

$$u_{zz} + f(z) u_z + g(z) u = 0, \hfill(3)$$

which implies that

$$f(z) = \left( \frac{\rho_z}{\rho} + \frac{2 \phi_z}{\phi} \right), \quad g(z) = \frac{\phi_{zz}}{\phi} + \frac{\rho_z \phi_z}{\rho \phi} + \frac{k^2 - V(x)}{\rho^2}. \hfill(4)$$

Integrating the first differential equation allows us to obtain

$$\phi(z) = \rho^{-\frac{1}{2}} e^{\int f(z) dz}. \hfill(5)$$

Substitution of this into the second differential equation of Eqs. (4) yields

$$g - \frac{1}{2} f_z - \frac{1}{4} f^2 = -\frac{1}{2} \left( \frac{\rho_z}{\rho} \right)_z - \frac{1}{4} \left( \frac{\rho_z}{\rho} \right)^2 + \frac{(E - V)}{\rho^2}, \hfill(6)$$

from which we define the expression [6]

$$I(z) = g - \frac{f_z}{2} - \frac{f^2}{4}. \hfill(7)$$
as the invariant of Eq. (3). Using Schwarzian derivative
\[
\{z, x\} = \frac{d^2 \log z'(x)}{dx^2} - \frac{1}{2} \left( \frac{d \log z'(x)}{dx} \right)^2 = \left( \frac{z''(x)}{z'(x)} \right)' - \frac{1}{2} \left[ \frac{z''(x)}{z'(x)} \right]^2
\]
where we have used the relation \( z''(x) = \rho^2 \frac{d^2}{dx^2} = \rho \rho_z \) and considering equation (7), then equation (6) can be rewritten as
\[
\rho^2 I(z) + \frac{1}{2} \{z, x\} = k^2 - V(x) \equiv I_S(x),
\]
where \( I_S(x) \) is defined as the Schrödinger invariant \([1, 2]\). Thus, the problem of the construction of the soluble potentials for the original Schrödinger equation (1) is solvable on the basis of the functions corresponding to a given \( I(z) \) (7) becomes a problem of deciding transformations \( z(x) \) such that the relation \( \rho^2 I(z) + \frac{1}{2} \{z, x\} = z''(x)^2 \rho^2 I(z) + \frac{1}{2} \{z, x\} = I_S(x) \) holds. The Schrödinger invariant \( I_S(x) \) is thus characterized by two elements, i.e. \( I(z) \) and the Schwarzian derivative \( \{z, x\} \), which is directly related to the function \( z(x) \).

3 Application to Heun differential equation

The Heun differential equation is given by \([7, 8, 9, 10, 11]\)
\[
u_{zz} + \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a} \right) u_z + \frac{\alpha \beta z - q}{z(z-1)(z-a)} u = 0,
\]
where the parameters satisfy the Fuchsian relation
\[
\alpha + \beta + 1 = \gamma + \delta + \epsilon.
\]

For this equation, the Heun Invariant \( I_h \) can be calculated as
\[
I_h(z) = \frac{\alpha \beta z - q}{z(z-1)(z-a)} - \frac{1}{4} \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a} \right)^2 + \frac{1}{2} \left( \frac{\gamma}{z^2} + \frac{\delta}{(1-z)^2} + \frac{\epsilon}{(a-z)^2} \right)
\]
\[
= A z^4 + B z^3 + C z^2 + D z + F
\]
\[
z^2(z-1)^2(z-a)^2.
\]

1Such a process is known as the normal form of the equation. Equations which have the same normal form are equivalent.
where the parameters $A, B, C, D, F$ depend on the parameters $a$ and $\alpha, \beta, \gamma, \delta, \epsilon$, i.e.

\[
A = \frac{1}{4}[4\alpha\beta - (\gamma + \delta + \epsilon - 2)(\gamma + \delta + \epsilon)],
\]

\[
B = \frac{1}{2}\{-2(a + 1)\alpha\beta + (\gamma + \delta + \epsilon - 2)(a\gamma + a\delta + \gamma + \epsilon) - 2q\}
\]

\[
C = \frac{1}{4}\left\{a^2[-(\gamma + \delta - 2)](\gamma + \delta) + a[4\alpha\beta - 2\epsilon(2\gamma + \delta) - 4\gamma(\gamma + \delta - 2)]
+ 4(a + 1)q - (\gamma + \epsilon - 2)(\gamma + \epsilon)\right\},
\]

\[
D = \frac{1}{2}a[\gamma(a(\gamma - 2) + \gamma + \epsilon - 2) - 2q],
\]

\[
F = -\frac{1}{8}a^2\gamma(\gamma - 2).
\]

4 Soluble potentials constructed by Heun invariant transferred to Schrödinger invariant

Now, let us determine functions $z'(x)$ that can be used to transform $I_h$ to $I_S$ in order to calculate the Schwarzian derivative $\{z, x\}$. The form of the present invariant $I_h(z)$ given in Eq. (12) suggests us to take the class of functions defined by

\[
\rho^2 = z'(x)^2 = \alpha_1 z(x)^2 + \beta_1 z(x) + \gamma_1,
\]

(14)

where $\alpha_1, \beta_1, \gamma_1$ are arbitrary constants. Such a choice is to make the $\rho^2 I_h(z)$ generate a constant to cancel the energy level term $k^2$. Otherwise, the expansion terms for $\rho^2 I_h(z)$ without including a constant will make the $k^2 = 0$, which means the particle moving in a free field. In terms of this equation (14), one has

\[
\{z, x\} = -\frac{\alpha_1}{2} - \frac{3}{8} \frac{\beta_1^2 - 4\alpha_1\gamma_1}{\alpha_1 z^2 + \beta_1 z + \gamma_1},
\]

(15)

where we have used the relation $\{z, x\} = \rho\rho_{zz} - \frac{1}{2}\rho^2$ and $\rho = z'(x)$.

To solve equation (14), we first obtain its general solutions for arbitrary parameters. In this

\footnote{It should be pointed out that present choice is different from previous one \cite{5}, in which the $\rho^2 \propto (z - a_1)^{m_1}(z - a_2)^{m_2}(z - a_3)^{m_3}$ is chosen in order to adapt the mathematical character of the Heun invariant \cite{12}.}
case, one has

\[ z(x) = \frac{e^{\sqrt{-\alpha}(c_1 \pm x)}}{4\alpha_1} \left[ \beta_1^2 - 4\alpha_1 \gamma_1 - 2\beta_1 e^{\sqrt{-\alpha}(c_1 \pm x)} + e^{2\sqrt{-\alpha}(c_1 \pm x)} \right] \]

\[ = \frac{-\beta_1}{2\alpha_1} + \frac{e^{\pm\sqrt{\alpha}x}(\beta_1^2 - 4\alpha_1 \gamma_1) + e^{\pm\sqrt{\alpha}x}}{4\alpha_1}, \]  

\( (16) \)

where \( c_1 \) is an integral constant and we take \( c_1 = 0 \) for simplicity.

Let us study it in various options for those parameters. If choosing the constants \( \alpha_1, \beta_1, \gamma_1 \) and the constants of integration suitably, in terms of above result \( (16) \) we can obtain their solutions but ignore unimportant integral constants as follows:

1) When \( \beta_1 = \gamma_1 = 0, \alpha_1 = 4a^2 \), we have

\[ z_1^\pm (x) = g \exp(\pm 2a x), \quad g \in \text{constant}. \]  

\( (17) \)

2) When \( \alpha_1 = -\beta_1, \gamma_1 = 0, \alpha_1 = 4a^2 \), we have

\[ z_2^a = \cosh^2(a x), \quad z_2^b = -\sinh^2(a x). \]  

\( (18) \)

3) When \( \alpha_1 = -\beta_1, \gamma_1 = 0, \alpha_1 = -4b^2 \), we have

\[ z_3^a = \cos^2(b x), \quad z_3^b = \sin^2(b x). \]  

\( (19) \)

4) When \( \alpha_1 = \gamma_1 = 0, \beta_1 = 4c \), we have

\[ z_4 = c x^2, \quad c \in \text{constant}. \]  

\( (20) \)

5) When \( \alpha_1 = \beta_1 = 0, \gamma_1 = \sigma^2 \), we have

\[ z_5^\pm = \pm \sigma x. \]  

\( (21) \)

It is not difficult to find that the cases 2) and 3) can be obtained each other by considering the relations \( \sin(i x) = i \sinh(x) \) and \( \cos(i x) = \cosh(x) \) when \( 2a \) is replaced by \( 2ib \).

On the other hand, it was recalled that \( [1, 3] \)

\[ \{z_t, x\} = \{z, x\}, \quad z_t = \frac{A_1 z + B_1}{C_1 z + D_1}, \]  

\( (22) \)

where \( A_1, B_1, C_1, D_1 \) are constants but \( A_1D_1 - B_1C_1 \neq 0 \). From Eq. \( (22) \), we have

\[ z = \frac{D_1 z_t - B_1}{A_1 - C_1 z_t}. \]  

\( (23) \)
If differentiating \( z_t \) given in Eq. (22) with respect to \( x \) and eliminating the variable \( z \), one has
\[
\frac{dz_t}{dx} = \frac{d^2z_t}{dx^2} = -\frac{(C_1 z_t - A_1)^2}{B_1 C_1 - A_1 D_1} z_t(x),
\]
\[
\left( \frac{dz_t}{dx} \right)^2 = \frac{(C_1 z_t - A_1)^2}{(B_1 C_1 - A_1 D_1)^2} \left[ \alpha_1 (B_1 - D_1 z_t)^2 + \beta_1 (B_1 - D_1 z_t)(C_1 z_t - A_1) + \gamma_1 (C_1 z_t - A_1)^2 \right]
\]
\[
= \frac{(A_1 - C_1 z_t)^2 (D_1 z_t - B)[\beta_1 (A_1 - C_1 z_t) - \alpha_1 B_1 + \alpha_1 D_1 z_t] + \gamma_1 (A_1 - C_1 z_t)^4}{(B_1 C_1 - A_1 D_1)^2},
\]
(24)
where \( z_t(x) \) is given by Eq. (14). It is not difficult to see that the solutions of Eq. (24) are also possible transformations since it is a generalization of Eq. (14). Up to now, we have found a class of functions for transforming \( I_h \) to \( I_s \). It is worth noting that this class of functions can be characterized differently. We are going to give a useful remark on the \( z_t(x) \) given in Eq. (24). If we use this to calculate the Schrödinger invariant \( I_s(x) \), then we will find that the soluble potentials would become rather complicated and do not consider this for simplicity, but it should be recognized that the variable \( z_t \) is just \( z(x) \) as given in Eq. (14).

We are now in the position to construct the simple Schrödinger invariants corresponding to the general Heun differential equation Invariant with the aid of the transformation (15) we obtained above. First, let us consider the simpler transform (14). Substituting equations (12) and (15) into Eq. (9) allows us to obtain the following useful Schrödinger invariant
\[
I_s = \rho^2 I_h + \frac{1}{2} \{ z, x \}
\]
\[
= (\alpha_1 z^2 + \beta_1 z + \gamma_1) \left\{ \frac{\alpha \beta z - q}{z(z-1)(z-a)} - \frac{1}{4} \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a} \right)^2 + \frac{1}{2} \left[ \frac{\gamma}{z^2} + \frac{\delta}{(z-1)^2} + \frac{\epsilon}{(z-a)^2} \right] \right\} - \left( \frac{\alpha_1}{4} + \frac{3}{16} \frac{\beta_1^2 - 4 \alpha_1 \gamma_1}{\alpha_1 z^2 + \beta_1 z + \gamma_1} \right)
\]
(25)
\[
\]
\[
= (\alpha_1 z^2 + \beta_1 z + \gamma_1) \left[ \frac{A z^4 + B z^3 + C z^2 + D z + F}{z^2(z-1)(z-a)^2} \right] - \left( \frac{\alpha_1}{4} + \frac{3}{16} \frac{\beta_1^2 - 4 \alpha_1 \gamma_1}{\alpha_1 z^2 + \beta_1 z + \gamma_1} \right)
\]
Let us write down the Schrödinger invariants (essentially related to potentials) based on Eqs. (17), (18), (19), (20) and (21).
\]
\[
i)
I_s(H 1^+) = a^2 \left\{ \frac{4 \left[ g e^{2ax} (g e^{2ax} (A g e^{2ax} + B) + C) + D \right] + F}{(a - g e^{2ax})^2 (g e^{2ax} - 1)^2} - 1 \right\},
\]
(26)
\[
I_s(H 1^-) = a^2 \left\{ \frac{4 \left[ B g^3 e^{2ax} + C g^2 e^{2ax} + D g e^{6ax} + F g e^{8ax} + A g^4 \right]}{(e^{2ax} - g)^2 (g - ae^{2ax})^2} - 1 \right\}.
\]
ii)  
\[ I_S(H \ 2^a) = -\frac{a^2 \text{csch}^2(2ax)}{\cos^2(ax) - a^2} \left\{ 3a^2 + (4 - 16A) \cosh^8(ax) - 4(2a + 4B + 1) \cosh^6(ax) \right. \\
+ [4a(a + 2) - 16C + 3] \cosh^4(ax) - 2[a(2a + 3) + 8D] \cosh^2(ax) - 16F \right\}, \]

\[ I_S(H \ 2^b) = -\frac{a^2 \sinh^4(ax) \tanh^2(ax)}{4[\sinh^2(ax) + a^2]} \left\{ (3a^2 - 16F) \cosh^8(ax) + 4(2a + 4B + 1) \cosh^2(ax) \right. \\
+ [4a(a + 2) - 16C + 3] \cosh^4(ax) + 2[a(2a + 3) + 8D] \cosh^6(ax) - 16A + 4 \right\}. \]  

(27)

iii)  
\[ I_S(H \ 3^a \ \{3^b\}) = -\frac{b^2 \csc^2(2bx)}{\cos^2(bx) \sin^2(bx) - a^2} \left\{ 3a^2 + (4 - 16A) \cos^8(bx) \left\{ \sin^8(bx) \right\} \\
- 4(2a + 4B + 1) \cos^6(bx) \left\{ \sin^6(bx) \right\} + [4a(a + 2) - 16C + 3] \cos^4(bx) \left\{ \sin^4(bx) \right\} \\
- 2[a(2a + 3) + 8D] \cos^2(bx) \left\{ \sin^2(bx) \right\} - 16F \right\}. \]  

(28)

iv)  
\[ I_S(H \ 4) = \frac{16 \left\{ cx^2 \left[ cx^2 (Acx^2 + B) + C) + D \right] + F \right\}}{4x^2 (cx^2 - 1)^2 (a - cx^2)^2} - \frac{3}{4x^2}. \]  

(29)

v)  
\[ I_S(H \ 5^{\pm}) = \frac{\sigma x [\sigma x(A \sigma x \pm B) + C \pm D] + F}{x^2 (x \mp \sigma x)^2 (a \mp \sigma x)^2}. \]  

(30)

Here, we have used the symbol \((H \ n^{(\pm,a,b)})\) to denote the above invariants, \(H\) referring to \(I_h\) and \(n^{(\pm,a,b)}\) to \(\sigma n^{(\pm,a,b)}\). Let us analyze these potentials through expanding them as follows:

For the i) case, we have

\[ I_S(H \ 1^+) = a^2(4A - 1) + \frac{A_1^+}{(ge^{2ax} - 1)^2} + \frac{B_1^+}{(ge^{2ax} - a)} + \frac{C_1^+}{(ge^{2ax} - a)} + \frac{D_1^+}{(ge^{2ax} - a)^2} \]  

(31)

where

\[ A_1^+ = \frac{4(a^2A + a^2B + a^2C + a^2D + a^2F)}{(a - 1)^2}, \]

\[ B_1^+ = \frac{4(4a^3A + 3a^3B + 2a^3C + a^3D - 2a^2A - a^2B + a^2D + 2a^2F)}{(a - 1)^3}, \]

\[ C_1^+ = \frac{4(2a^6A - 4a^5A + a^5B - 3a^4B - 2a^3C - a^3D - a^2D - 2a^2F)}{(a - 1)^3}, \]

\[ D_1^+ = \frac{4(a^6A + a^5B + a^4C + a^3D + a^2F)}{(a - 1)^2}. \]  

(32)
For the ii) case, we have

\[
I_S(H 2^a) = \frac{A_2^a \csch^2(2ax)}{[\cosh^2(ax) - a]^2} + \frac{B_2^a \csch^2(ax)}{[\cosh^2(ax) - a]^2} - \frac{C_2^a \coth^2(ax)}{[\cosh^2(ax) - a]^2} + \frac{D_2^a \cosh^2(ax) \coth^2(ax)}{[\cosh^2(ax) - a]^2} + \frac{E_2^a \cosh^4(ax) \coth^2(ax)}{[\cosh^2(ax) - a]^2}
\]

(33)

where

\[
A_2^a = a^2(16F - 3a^2), \quad B_2^a = \frac{a^2[a(2a + 3) + 8D]}{2},
\]

\[
C_2^a = \frac{a^2[4a(a + 2) - 16C + 3]}{4}, \quad D_2^a = a^2(2a + 4B + 1), \quad E_2^a = a^2(4A - 1).
\]

For the iii) case, one has

\[
I_S(H 3^a) = \frac{A_3^a \csc^2(2bx)}{[\cos(bx)^2 - a]^2} + \frac{B_3^a \csc^2(bx)}{[\cos(bx)^2 - a]^2} - \frac{C_3^a \cot^2(bx)}{[\cos(bx)^2 - a]^2} + \frac{D_3^a \cos^2(bx) \cot^2(bx)}{[\cos(bx)^2 - a]^2} + \frac{E_3^a \cos^4(bx) \cot^2(bx)}{[\cos(bx)^2 - a]^2}
\]

(35)

where

\[
A_3^a = b^2(16F - 3a^2), \quad B_3^a = \frac{b^2[a(2a + 3) + 8D]}{2},
\]

\[
C_3^a = \frac{b^2[4a(2 + a) - 16C + 3]}{4}, \quad D_3^a = b^2(2a + 4B + 1), \quad E_3^a = b^2(4A - 1).
\]

For the special case iv), we have

\[
I_S(H 4) = \frac{A_4}{x^2} + \frac{B_4}{(cx^2 - a)^2} + \frac{C_4}{(cx^2 - a)^2} + \frac{D_4}{(cx^2 - 1)^2} + \frac{E_4}{(cx^2 - 1)^2}
\]

(37)

where

\[
A_4 = \frac{16F - 3a^2}{4a^2}, \quad B_4 = \frac{4c(a^4A + a^3B + a^2C + aD + F)}{(a - 1)^2a},
\]

\[
C_4 = \frac{4c(a^5A - 3a^4A - 2a^3B - a^2C - a^2C - 2a^2D - 3aF + F)}{(a - 1)^3a^2},
\]

\[
D_4 = \frac{4c(\frac{A + B + C + D + F)}{(a - 1)^2}},
\]

\[
E_4 = \frac{4c(3aA + 2aB + aC - aF - A + C + 2D + 3F)}{(a - 1)^3}.
\]

For the v) case, one has

\[
I_S(H 5^-) = \frac{A_5^-}{x} + \frac{B_5^-}{x^2} + \frac{C_5^-}{(a + \sigma x)^2} + \frac{D_5^-}{(a + \sigma x)^2} + \frac{E_5^-}{(\sigma x + 1)^2} + \frac{F_5^-}{(\sigma x + 1)^2}
\]

(39)
where
\[ A_5^- = -\frac{\sigma(aD + 2aF + 2F)}{a^3}, \quad B_3^- = \frac{F}{a^2}, \]
\[ C_5^- = \frac{\sigma^2(a^4A + a^3B + a^2C + aD + F)}{(a-1)^2a^2}, \]
\[ D_5^- = \frac{\sigma^2(2a^4A + a^4B + a^3B + 2a^3C + 3a^2D - aD + 4aF - 2F)}{(a-1)^3a^3}, \]
\[ E_5^- = \frac{\sigma^2(A + B + C + D + F)}{(a-1)^2}, \]
\[ F_5^- = \frac{\sigma^2(-2aA - aB + aD + 2aF - B - 2C - 3D - 4F)}{(a-1)^3}. \]  

(40)

Obviously, the potential given in case i) is more complicated than the usual Eckart potential. The potentials discussed in cases ii) and iii) are more complicated than the first and second type Pöschl-Teller potentials. The potential studied in case iv) is more like the \( x^{-2} + x^{-4} \) while the potential given in v) case essentially is the sum of the Coulomb potential plus a centrifugal term. The other cases such as \( I_S(H^{-1}) \), \( I_S(H^{-2}) \), \( I_S(H^{-3}) \), \( I_S(H^{-5}) \) have similar properties to their partners.

Now, let us study the wave function. In terms of Eqs. (5) and the function
\[ f(z) = \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a} \right), \]

(41)
given in (10), one has the following form
\[ \phi(z) = \frac{1}{\sqrt{\rho}} z^{\frac{\gamma}{2}} (z-1)^{\frac{\delta}{2}} (z-a)^{\frac{\epsilon}{2}}, \]

(42)
where \( \rho \) given in Eq. (14) depends on the solutions (16), while those particular cases given in Eqs. (17) to (21). The partial wave function \( u(x) \) involved in the whole wave function \( \psi(z) = \phi(z)u(z) \) is given by the Heun functions \( H_l(a, q, \alpha, \beta, \gamma, \delta, \epsilon; z) \).

5 Concluding remarks

The Schrödinger equation is rewritten as a more popular form of a second order differential equation through taking a similarity transformation. We find that this classical equation is closely related to the Schwarzian derivative and the invariant identity of the differential equation \( u_{zz} + f(z)u_z + g(z)u = 0 \). As a typical differential equation, the corresponding mathematical
properties of the Heun differential equation are studied. Before ending this work, we give a useful remark on the Schrödinger invariant $I_S(x)$. First, let us consider the Schrödinger equation (1) and equation (9). We find that the Schrödinger equation can also be rewritten as $\nabla^2 \psi(x) = -I_S(x)\psi(x)$. Since $\nabla^2$ represents the kinetic term $T$ of the moving particle, it should keep invariant for the same particle. This is also reflection of the conservation of energy $T + V = E$.

Competing Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

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