THE AVALANCHE PRINCIPLE AND OTHER ESTIMATES ON GRASSMANN MANIFOLDS

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Abstract. The main result of this paper, called the Avalanche Principle (AP), relates the expansion of a long product of matrices with the product of expansions of the individual matrices. This principle was introduced by M. Goldstein and W. Schlag in the context of SL(2, C) matrices. Besides extending the AP to matrices of arbitrary dimension, possibly non-invertible, the geometric approach we use here provides a relation between the most expanding (singular) directions of such a long product of matrices and the corresponding singular directions of the first and last matrices in the product. The AP along with other estimates on the action of matrices on Grassmann manifolds will play a fundamental role in [6] to establish the continuity the Lyapunov exponents and of the Oseledets decomposition for linear cocycles.

This is the draft of a chapter in our forthcoming research monograph [6].

1. Grassmann Geometry

Grassmann geometry is the geometric study of manifolds of linear subspaces of an Euclidean space, and the action of linear groups (and algebras) on them. Its foundations lie on the masterpiece ‘Die lineale Ausdehnungslehre’ of Hermann Grassmann, whose full geniality is still miscomprehended (see [2]).

1.1. Projective spaces. The projective space is the simplest compact model to study the action of a linear map. Given a n-dimensional Euclidean space V, consider the equivalence relation defined on V \ {0} by u ≡ v if and only if u = λ v for some λ ≠ 0. For v ∈ V \ {0}, the set ˆv := { λ v : λ ∈ R \ {0} } is the equivalence class of the vector v by this relation. The projective space of V is the quotient P(V) := { ˆv : v ∈ V \ {0} } of V \ {0} by this equivalence relation. It is a compact topological space when endowed with the quotient topology.

The unit sphere S(V) := { v ∈ V : ∥v∥ = 1 } is a compact Riemannian manifold of constant curvature 1 and diameter π. The natural projection ˆπ : S(V) → P(V), ˆπ(v) = ˆv, is a (double) covering map. Hence the projective space P(V) has a natural smooth Riemannian structure for which the covering map ˆπ is a local isometry. Thus P(V) is a compact Riemannian manifold with constant curvature 1 and diameter π/2.

Given a linear map g ∈ L(V) define P(g) := { ˆv ∈ P(V) : g v ≠ 0 }. We refer to the linear map φg : P(g) ⊂ P(V) → P(V), φg( ˆv ) := ˆπ( g ˆv ∥g ˆv∥ ), as the projective action of g on P(V). If g is invertible then φg : P(V) → P(V) is a diffeomorphism with inverse φg⁻¹ : P(V) → P(V). Through these maps, the group GL(V), of all linear automorphisms on V, acts transitively on the projective space P(V).
We will consider three different metrics on the projective space \( \mathbb{P}(V) \). The Riemannian distance, \( \rho \), measures the length of an arc connecting two points in the sphere. More precisely, given \( u, v \in S(V) \),

\[
\rho(\hat{u}, \hat{v}) := \min\{ \angle(u, v), \angle(u, -v) \}
\]

(1.1)

The second metric, \( d \), corresponds to the Euclidean distance measured in the sphere. More precisely, given \( u, v \in S(V) \),

\[
d(\hat{u}, \hat{v}) := \min\{ \|u - v\|, \|u + v\| \}
\]

(1.2)

measures the smallest chord of the arcs between \( u \) and \( v \), and between \( u \) and \( -v \). The third metric, \( \delta \), measures the sine of the arc between two points in the sphere. More precisely, given \( u, v \in S(V) \),

\[
\delta(\hat{u}, \hat{v}) := \frac{\|u \wedge v\|}{\|u\|\|v\|} = \sin(\angle(u, v))
\]

(1.3)

The fact that \( \delta \) is a metric on \( \mathbb{P}(V) \) follows from the sine addition law, which implies that \( \sin(\theta + \theta') \leq \sin \theta + \sin \theta' \), for all \( \theta, \theta' \in [0, \frac{\pi}{2}] \).

These three distances are equivalent. For all \( \hat{u}, \hat{v} \in \mathbb{P}(V) \),

\[
\delta(\hat{u}, \hat{v}) = \sin \rho(\hat{u}, \hat{v}) \quad \text{and} \quad d(\hat{u}, \hat{v}) = \text{chord} \rho(\hat{u}, \hat{v})
\]

(1.4)

The inequalities

\[
\frac{2\theta}{\pi} \leq \sin \theta \leq \text{chord} \theta = 2 \sin(\theta/2) \leq \theta \quad \forall 0 \leq \theta \leq \frac{\pi}{2}
\]

imply that

\[
\frac{2}{\pi} \rho(\hat{u}, \hat{v}) \leq \delta(\hat{u}, \hat{v}) \leq d(\hat{u}, \hat{v}) \leq \rho(\hat{u}, \hat{v})
\]

(1.5)

Because of (1.4), these three metrics determine the same group of isometries on the projective space.

1.2. Exterior algebra. Exterior Algebra was introduced by H. Grassmann in the ‘Ausdehnungslehre’. We present here an informal description of some of its properties. See the book of Shlomo Stenberg [9] for a rigorous treatment of the subject.

Let \( V \) be a finite \( n \)-dimensional Euclidean space. Given \( k \) vectors \( v_1, \ldots, v_k \in V \), their \( k \)-th exterior product is a formal skew-symmetric product \( v_1 \wedge \ldots \wedge v_k \), in the sense that for any permutation \( \sigma = (\sigma_1, \ldots, \sigma_k) \in S_k \),

\[
v_{\sigma_1} \wedge \ldots \wedge v_{\sigma_k} = (-1)^{\text{sgn}(\sigma)} v_1 \wedge \ldots \wedge v_k .
\]

These formal products are elements of an anti-commutative and associative graded algebra \( (\wedge V, +, \wedge) \), called the exterior algebra of \( V \). Formal products \( v_1 \wedge \ldots \wedge v_k \) are called simple \( k \)-vectors of \( V \). The \( k \)-th exterior power of \( V \), denoted by \( \wedge_k V \), is the linear span of all simple \( k \) vectors of \( V \). Elements of \( \wedge_k V \) are called \( k \)-vectors.

An easy consequence of this formal definition is that \( v_1 \wedge \ldots \wedge v_k = 0 \) if and only if \( v_1, \ldots, v_k \) are linearly dependent. Another simple consequence is that given two bases
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Let \( v_1, \ldots, v_k \) and \( w_1, \ldots, w_k \) of the same \( k \)-dimensional linear subspace of \( V \), if for some real matrix \( A = (a_{ij}) \) we have \( w_i = \sum_{j=1}^k a_{ij} v_j \) for all \( i = 1, \ldots, k \), then
\[
w_1 \wedge \ldots \wedge w_k = (\det A) v_1 \wedge \ldots \wedge v_k.
\]

More generally, two families \( \{v_1, \ldots, v_k\} \) and \( \{w_1, \ldots, w_k\} \) of linearly independent vectors span the same \( k \)-dimensional subspace if and only if for some real number \( \lambda \neq 0 \), \( w_1 \wedge \ldots \wedge w_k = \lambda v_1 \wedge \ldots \wedge v_k \). Hence we identify the line spanned by a simple \( k \)-vector \( v = v_1 \wedge \ldots \wedge v_k \), i.e., the projective point \( \hat{v} \in \mathbb{P}(\wedge_k V) \) determined by \( v \), with the \( k \)-dimensional subspace spanned by the vectors \( \{v_1, \ldots, v_k\} \), denoted hereafter by \( \langle \langle v_1 \wedge \ldots \wedge v_k \rangle \rangle \).

The subspaces \( \wedge_k V \) induce the grading structure of the exterior algebra \( \Lambda V \), i.e., we have the direct sum decomposition \( \Lambda^* V = \bigoplus_{k=0}^{\dim V} \wedge_k V \) with \( (\wedge_k V) \cap (\wedge_{k'} V) \subset \wedge_{k+k'} V \) for all \( 0 \leq k, k' \leq \dim V \). Geometrically, the exterior product operation \( \wedge : \wedge_k V \times \wedge_k V \to \wedge_{k+k'} V \) corresponds to the algebraic sum of linear subspaces, in the sense that given families \( \{v_1, \ldots, v_k\} \) and \( \{w_1, \ldots, w_k\} \) of linearly independent vectors such that \( \langle \langle v_1 \wedge \ldots \wedge v_k \rangle \rangle \cap \langle \langle w_1 \wedge \ldots \wedge w_k \rangle \rangle = 0 \), then
\[
\langle \langle v_1 \wedge \ldots \wedge v_k \wedge w_1 \wedge \ldots \wedge w_k \rangle \rangle = \langle \langle v_1 \wedge \ldots \wedge v_k \rangle \rangle + \langle \langle w_1 \wedge \ldots \wedge w_k \rangle \rangle.
\]

Let \( \Lambda_k^n \) be the set of all \( k \)-subsets \( I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\} \), with \( i_1 < \ldots < i_k \), and order it lexicographically. Given a basis \( \{e_1, \ldots, e_n\} \) of \( V \), define for each \( I \in \Lambda_k^n \), the \( k \)-th exterior product \( e_I = e_{i_1} \wedge \ldots \wedge e_{i_k} \). Then the ordered family \( \{e_I : I \in \Lambda_k^n\} \) is a basis of \( \wedge_k V \). In particular \( \dim \wedge_k V = \binom{n}{k} \).

The exterior algebra \( \Lambda V \) inherits an Euclidean structure from \( V \). More precisely, there is a unique inner product on \( \Lambda V \) such that for any orthonormal basis \( \{e_1, \ldots, e_n\} \) of \( V \), the family \( \{e_I : I \in \Lambda_k^n, 0 \leq k \leq n\} \) is an orthonormal basis of the exterior algebra \( \Lambda V \).

A simple \( k \)-vector \( v_1 \wedge \ldots \wedge v_k \) of norm one will be called a unit \( k \)-vector. From the previous considerations the correspondence \( v_1 \wedge \ldots \wedge v_k \mapsto \langle \langle v_1 \wedge \ldots \wedge v_k \rangle \rangle \) is one-to-one, between the set of unit \( k \)-vectors in \( \wedge_k V \) and the set of oriented \( k \)-dimensional linear subspaces of \( V \). In particular, if \( V \) is an oriented Euclidean space then the 1-dimensional space \( \wedge_n V \) has a canonical unit \( n \)-vector, denoted by \( \omega \), and called the volume element of \( \wedge_n V \). In this case there is a unique operator, called the Hodge star operator, \( * : \Lambda^n V \to \wedge_n V \) defined by
\[
v \wedge (*w) = \langle v, w \rangle \omega, \quad \text{for all } v, w \in \wedge_n V.
\]

The Hodge star operator maps \( \wedge_k V \) isomorphically, and isometrically, onto \( \wedge_{n-k} V \), for all \( 0 \leq k \leq n \). Geometrically it corresponds to the orthogonal complement operation on linear subspaces, i.e., for any simple \( k \)-vector,
\[
\langle \langle * (v_1 \wedge \ldots \wedge v_k) \rangle \rangle = \langle \langle v_1 \wedge \ldots \wedge v_k \rangle \rangle^\perp.
\]

A dual product operation \( \vee : \Lambda^n V \times \Lambda^n V \to \wedge_n V \) can be defined by
\[
v \vee w := ((*v) \wedge (*w)), \quad \text{for all } v, w \in \wedge_n V.
\]

This operation maps \( \wedge_k V \times \wedge_{k'} V \) to \( \wedge_{k+k'-n} V \), and describes the intersection operation on linear subspaces, in the sense that given families \( \{v_1, \ldots, v_k\} \) and \( \{w_1, \ldots, w_k\} \) of linearly independent vectors,
independent vectors with $\langle v_1 \wedge \ldots \wedge v_k \rangle + \langle w_1 \wedge \ldots \wedge w_{k'} \rangle = V$, then

$$\langle (v_1 \wedge \ldots \wedge v_k) \vee (w_1 \wedge \ldots \wedge w_{k'}) \rangle = \langle v_1 \wedge \ldots \wedge v_k \rangle \cap \langle w_1 \wedge \ldots \wedge w_{k'} \rangle .$$

By duality, this interpretation of the $\vee$-operation reduces to the previous ones on sums ($\wedge$) and complements ($\ast$).

Any linear map $g : V \to V$ induces a linear map $\wedge_k g : \wedge_k V \to \wedge_k V$, called the $k$-th exterior power of $g$, such that for all $v_1, \ldots, v_k \in V$,

$$\wedge_k g(v_1 \wedge \ldots \wedge v_k) = g(v_1) \wedge \ldots \wedge g(v_k) .$$

This construction is functorial in the sense that for all linear maps $g, g' : V \to V$,

$$\wedge_k g' \circ \wedge_k g = \wedge_k (g' \circ g) \quad \text{and} \quad \wedge_k (g^\ast) = (\wedge_k g)^\ast ,$$

where $g^\ast : V \to V$ denotes the adjoint operator.

A clear consequence of these properties is that if $g : V \to V$ is an orthogonal automorphism, i.e., $g^\ast g = \text{id}_V$, then so is $\wedge_k g : \wedge_k V \to \wedge_k V$.

Consider a matrix $A \in \text{Mat}_n(\mathbb{R})$. Given $I, J \in \Lambda^n_0$, we denote by $A_{I \times J}$ the square submatrix of $A$ indexed in $I \times J$. If a linear map $g : V \to V$ is represented by $A$ relative to a basis $\{e_1, \ldots, e_n\}$, then the $k$-th exterior power $\wedge_k g : \wedge_k V \to \wedge_k V$ is represented by the matrix $\wedge_k A := (\det A_{I \times J}) I, J$ relative to the basis $\{e_I : I \in \Lambda^n_0\}$. The matrix $\wedge_k A$ is called the $k$-th exterior power of $A$. Obviously, matrix exterior powers satisfy the same functorial properties as linear maps, i.e., for all $A, A' \in \text{Mat}_n(\mathbb{R})$,

$$\wedge_k I_n = I_{(n)}, \quad \wedge_k (A' A) = (\wedge_k A')(\wedge_k g) \quad \text{and} \quad \wedge_k A^\ast = (\wedge_k A)^\ast ,$$

where $A^\ast$ denotes the transpose matrix of $A$.

1.3. Grassmann manifolds. Grassmannians, like projective spaces, are compact Riemannian manifolds which stage the action of linear maps. For each $0 \leq k \leq n$, the Grassmannian $\text{Gr}_k(V)$ is the space of all $k$-dimensional linear subspaces of $V$. Notice that the projective space $\mathbb{P}(V)$ and the Grassmannian $\text{Gr}_1(V)$ are the same object if we identify each point $\hat{v} \in \mathbb{P}(V)$ with the line $\langle v \rangle = \{ \lambda v : \lambda \in \mathbb{R} \}$. The full Grassmannian $\text{Gr}(V)$ is the union of all Grassmannians $\text{Gr}_k(V)$ with $0 \leq k \leq n$. Denote by $\mathcal{L}(V)$ the algebra of linear endomorphisms on $V$, and consider the map $\pi : \text{Gr}(V) \to \mathcal{L}(V)$, $E \mapsto \pi_E$, that assigns the orthogonal projection $\pi_E$ onto $E$, to each subspace $E \in \text{Gr}(V)$. This map is one-to-one, and we endow $\text{Gr}(V)$ with the unique topology that makes the map $\pi : \text{Gr}(V) \to \pi(\text{Gr}(V))$ a homeomorphism. With it, $\text{Gr}(V)$ becomes a compact space, and each Grassmannian $\text{Gr}_k(V)$ is a closed connected subspace of $\text{Gr}(V)$.

The group $\text{GL}(V)$ acts transitively on each Grassmannian. The action of $\text{GL}(V)$ on $\text{Gr}_k(V)$ is given by $\cdot : \text{GL}(V) \times \text{Gr}_k(V) \to \text{Gr}_k(V), (g, E) \mapsto g E$. The special orthogonal group $\text{SO}(V)$, of orientation preserving orthogonal automorphisms, acts transitively on Grassmannians too. All Grassmannians are compact homogeneous spaces.

For each $0 \leq k \leq n$, the Plücker embedding is the map $\psi : \text{Gr}_k(V) \to \mathbb{P}(\wedge_k V)$ that to each subspace $E$ in $\text{Gr}_k(V)$ assigns the projective point $\hat{v} \in \mathbb{P}(\wedge_k V)$, where $v = v_1 \wedge \ldots \wedge v_k$.
Proof. Consider the unit vector $u$ and (c) follows taking the maximum over all unit vectors for any unit vector $u$ which are preserved by orthogonal linear maps in $SO(V)$.

We also define the minimum distance between any two subspaces $E, F \in Gr_k(V)$,

$$\delta_{\min}(E, F) := \min_{u \in E \setminus \{0\}, v \in F \setminus \{0\}} \delta(u, v),$$

and the Hausdorff distance between subspaces $E, F \in Gr_k(V)$,

$$\delta_H(E, F) := \max \left\{ \max_{u \in E \setminus \{0\}} \delta_{\min}(u, F), \max_{v \in F \setminus \{0\}} \delta_{\min}(v, E) \right\}.$$

**Definition 1.1.** Given $E, F \in Gr(V)$, we denote by $\pi_F : V \to V$ the orthogonal projection onto $F$, and by $\pi_{E,F} : E \to F$ the restriction of $\pi_F$ to $E$.

**Proposition 1.1.** Given $E, F \in Gr_k(V)$,

(a) $\delta(E, F) = \sqrt{1 - \det(\pi_{E,F})^2} = \sqrt{1 - \det(\pi_{F,E})^2}$,

(b) $\delta_H(E, F) = \|\pi_{E,F}^\perp\| = \|\pi_{F,E}^\perp\|$

(c) $\delta_H(E, F) \leq \delta(E, F)$.

**Proof.** Consider the unit $k$-vectors $e = \Psi(E)$ and $f = \Psi(F)$.

For (a) notice first that $\delta(E, F) = \delta(e, f) = \sqrt{1 - \langle e, f \rangle^2}$. Since the exterior power $\wedge_k \pi_{F,E} : \wedge_k F \to \wedge_k E$ is also an orthogonal projection we have $\langle e, f \rangle = \langle e, \wedge_k \pi_{F,E}(f) \rangle = \|\wedge_k \pi_{F,E}\| = |\det(\pi_{F,E})|$.

Given an orthogonal map $g \in SO(V)$ such that $g(F) = E$, we have $g^{-1}(E^\perp) = F^\perp$ and $\pi_{E,F^\perp} = g^{-1} \circ \pi_{F,E^\perp} \circ g$. Therefore $\|\pi_{E,F^\perp}\| = \|\pi_{F,E^\perp}\|$

Item (b) follows because for any unit vector $u \in \hat{u}$, with $\hat{u} \in \mathcal{P}(E)$,

$$\|\pi_{E,F^\perp}(u)\| = \min_{v \in F \setminus \{0\}} \delta(\hat{u}, v).$$

Since $\pi_{E,F}$ is an orthogonal projection all its singular values are in the range $[0, 1]$. Hence, for any unit vector $u \in E$, $\|\pi_{E,F}(u)\| \geq m(\pi_{E,F}) \geq \det(\pi_{E,F})$. Thus

$$\|\pi_{E,F^\perp}(u)\|^2 = 1 - \|\pi_{E,F}(u)\|^2 \leq 1 - \det(\pi_{E,F})^2,$$

and (c) follows taking the maximum over all unit vectors $u \in E$. \qed
Given \( k, k' \geq 0 \) such that \( k + k' \geq n = \dim V \), the intersection of subspaces is an operation \( \cap : \Gr_{k,k'}(\cap) \subset \Gr_k(V) \times \Gr_{k'}(V) \to \Gr_{k+k'-n}(V) \) where

**Definition 1.2.** The domain is defined by

\[
\Gr_{k,k'}(\cap) := \{ (E, E') \in \Gr_k(V) \times \Gr_{k'}(V) : E + E' = V \}.
\]

Similarly, given \( k, k' \geq 0 \) such that \( k + k' \leq n = \dim V \), the algebraic sum of subspaces is operation \( + : \Gr_{k,k'}(+ \cap) \subset \Gr_k(V) \times \Gr_{k'}(V) \to \Gr_{k+k'-n}(V) \) where

**Definition 1.3.** The domain is defined by

\[
\Gr_{k,k'}(+ \cap) := \{ (E, E') \in \Gr_k(V) \times \Gr_{k'}(V) : E \cap E' = \{0\} \}.
\]

The considerations in subsection [2] show that the Plücker embedding satisfies the following relations:

**Proposition 1.2.** Given \( E \in \Gr_k(V), E' \in \Gr_{k'}(V) \), consider unit vectors \( v \in \Psi(E) \) and \( v' \in \Psi(E') \).

(a) If \( (E,E') \in \Gr_{k,k'}(\cap) \) then \( \psi(E \cap E') = v \lor v' \).

(b) If \( (E,E') \in \Gr_{k,k'}(+ \cap) \) then \( \psi(E + E') = v \land v' \).

A duality between sums and intersections stems from these facts.

**Proposition 1.3.** The orthogonal complement operation \( E \mapsto E^\perp \) is a d-isometric involution on \( \Gr(V) \) which maps \( \Gr_{k,k'}(+ \cap) \) to \( \Gr_{n-k,n-k'}(\cap) \) and satisfies for all \( (E, E') \in \Gr_{k,k'}(+ \cap) \),

\[
(E + E')^\perp = (E^\perp) \cap (E'^\perp).
\]

The composition semigroup \( \mathcal{L}(V) \) has two partial actions on Grassmannians, called the push-forward action and the pull-back action. Before introducing them a couple facts is needed.

**Definition 1.4.** Given \( g \in \mathcal{L}(V) \), we denote by \( Kg := \{ v \in V : g v = 0 \} \) the the kernel of \( g \), and by \( Rg := \{ g v : v \in V \} \) the range of \( g \).

**Lemma 1.4.** Given \( g \in \mathcal{L}(V) \) and \( E \in \Gr(V) \),

1. if \( E \cap (Kg) = \{0\} \) then the linear map \( g|_E : E \to g(E) \) is an isomorphism, and in particular \( \dim g(E) = \dim E \).
2. if \( E + (Rg) = V \) then the linear map \( g^*|_{E^\perp} : E^\perp \to g^{-1}(E)^\perp \) is an isomorphism, and in particular \( \dim g^{-1}(E) = \dim E \).

**Proof.** The first statement is obvious because if \( E \cap (Kg) = \{0\} \) then \( K(g|_E) = \{0\} \). If \( E + (Rg) = V \) then, since \( Kg^* = (Rg)^{\perp} \), we have \( E^\perp \cap Kg^* = E^\perp \cap (Rg)^{\perp} = (E + Rg)^{\perp} = \{0\} \). Hence by 1, the linear map \( g^*|_{E^\perp} : E^\perp \to g^*(E^\perp) \) is an isomorphism. It is now enough to remark that \( g^*(E^\perp) = g^{-1}(E)^\perp \). In fact, the inclusion \( g^*(E^\perp) \subset g^{-1}(E)^\perp \) is clear. Since
$g^*|E^\perp$ is injective, $\dim g^*(E^\perp) = \dim(E^\perp)$. On the other hand, by the transversality condition, $g^{-1}(E)$ has dimension
\[
\dim g^{-1}(E) = \dim ((g|_{(Kg)^\perp})^{-1}(E \cap Rg)) + \dim(Kg) \\
= \dim(E \cap Rg) + \dim(Kg) \\
= \dim(E) + \dim(Rg) - n + \dim(Kg) = \dim(E).
\]
Hence both $g^*(E^\perp)$ and $g^{-1}(E)^\perp$ have dimension equal to $\dim(E^\perp)$, and the equality follows. 

Given $g \in \mathcal{L}(V)$ and $k \geq 0$ such that $k + \dim(Kg) \leq n = \dim V$, the push-forward by $g$ is the map $\varphi_g : \text{Gr}_k(g) \subset \text{Gr}_k(V) \to \text{Gr}_k(V)$, $E \mapsto gE$, where

**Definition 1.5. the domain is defined by**
\[
\text{Gr}_k(g) := \{ E \in \text{Gr}_k(V) : E \cap (Kg) = \{0\}\}.
\]

Similarly, given $k \geq 0$ such that $k + \dim(Rg) \geq n = \dim V$, the pull-back by $g$ is the map $\varphi_g^{-1} : \text{Gr}_k(g^{-1}) \subset \text{Gr}_k(V) \to \text{Gr}_k(V)$, $E \mapsto g^{-1}E$, where

**Definition 1.6. the domain is defined by**
\[
\text{Gr}_k(g^{-1}) := \{ E \in \text{Gr}_k(V) : E + (Rg) = V \}.
\]

From the proof of proposition 1.4 we obtain a duality between push-forwards and pull-backs which can be expressed as follows.

**Proposition 1.5.** Given $g \in \mathcal{L}(V)$ and $k \geq 0$ such that $k + \dim(Rg) \geq n = \dim V$, we have $\text{Gr}_k(g^{-1}) = \text{Gr}_{n-k}(g^*)^\perp$ and for all $E \in \text{Gr}_k(g^{-1})$,
\[
(g^{-1}E)^\perp = g^*(E^\perp).
\]

In section 3 we will derive modulus of Lipschitz continuity, w.r.t. the metric $\delta$, for the sum, intersection, push-forward and pull-back operations.

1.4. Flag manifolds. Let $V$ be a finite $n$-dimensional Euclidean space. Any strictly increasing sequence of linear subspaces $F_1 \subset F_2 \subset \ldots \subset F_k \subset V$ is called a flag in the Euclidean space $V$. Formally, flags are denoted as lists $F = (F_1, \ldots, F_k)$. The sequence $\tau = (\tau_1, \ldots, \tau_k)$ of dimensions $\tau_j = \dim F_j$ is called the signature of the flag $F$. The integer $k$ is called the length of the flag $F$, and the length of the signature $\tau$. Let $\mathcal{F}(V)$ be the set of all flags in $V$, and define $\mathcal{F}_\tau(V)$ to be the space of flags with a given signature $\tau$. Two special cases of flag spaces are the projective space $\mathbb{P}(V) = \mathcal{F}_1(V)$, when $\tau = (1)$, and the Grassmannian $\text{Gr}_k(V) = \mathcal{F}_k(V)$, when $\tau = (k)$.

The general linear group $\text{GL}(V)$ acts naturally on $\mathcal{F}(V)$. Given $g \in \text{GL}(V)$ the action of $g$ on $\mathcal{F}_\tau(V)$ is given by the map $\varphi_g : \mathcal{F}_\tau(V) \to \mathcal{F}_\tau(V)$, $\varphi_g F = (gF_1, \ldots, gF_k)$. The special orthogonal subgroup $\text{SO}(V) \subset \text{GL}(V)$ acts transitively on $\mathcal{F}_\tau(V)$. Hence, all flag manifolds $\mathcal{F}_\tau(V)$ are compact homogeneous spaces. Each of them is a compact connected...
Riemannian manifold where the group \( \text{SO}(V) \) acts by isometries. Since \( \mathcal{F}_\tau(V) \subset \text{Gr}_{\tau_1}(V) \times \text{Gr}_{\tau_2}(V) \times \ldots \times \text{Gr}_{\tau_k}(V) \), the product distances

\[
\rho_\tau(F, F') = \max_{1 \leq j \leq k} \rho(F_j, F'_j) \quad (1.10)
\]

\[
d_\tau(F, F') = \max_{1 \leq j \leq k} d(F_j, F'_j) \quad (1.11)
\]

\[
\delta_\tau(F, F') = \max_{1 \leq j \leq k} \delta(F_j, F'_j) \quad (1.12)
\]

are equivalent to the Riemannian distance on \( \mathcal{F}_\tau(V) \). With these metrics, the flag manifold \( \mathcal{F}_\tau(V) \) has diameter \( \frac{\pi}{2} \sqrt{2} \) and 1, respectively. The group \( \text{SO}(V) \) acts isometrically on \( \mathcal{F}_\tau(V) \) with respect to these distances.

Given a signature \( \tau = (\tau_1, \ldots, \tau_k) \), if \( n = \dim V \), we define

\[
\tau^\perp := (n - \tau_k, \ldots, n - \tau_1).
\]

When \( \tau = (\tau_1, \ldots, \tau_k) \) we will write \( \tau^\perp = (\tau_1^\perp, \ldots, \tau_k^\perp) \), where \( \tau_j^\perp = n - \tau_{k+1-i} \).

**Definition 1.7.** Given a flag \( F = (F_1, \ldots, F_k) \in \mathcal{F}_\tau(V) \), its orthogonal complement is the \( \tau^\perp \)-flag \( F^\perp := (F_1^\perp, \ldots, F_k^\perp) \).

The map \( \cdot^\perp : \mathcal{F}(V) \to \mathcal{F}(V) \) is an isometric involution on \( \mathcal{F}(V) \), mapping \( \mathcal{F}_\tau(V) \) onto \( \mathcal{F}_{\tau^\perp}(V) \). The involution character, \( (F^\perp)^\perp = F \) for all \( F \in \mathcal{F}(V) \), is clear. As explained in section 1.2, the Hodge star operator \( * : \wedge_k V \to \wedge_{n-k} V \) is an isometry between these Euclidean spaces. By choice of metrics on the Grassmannians, see [1.8], the Plücker embeddings are isometries. Finally, the Plücker embedding conjugates the orthogonal complement map \( \cdot^\perp : \text{Gr}_k(V) \to \text{Gr}_{n-k}(V) \) with the Hodge star operator. Hence for each \( 0 \leq k \leq n \), the map \( \cdot^\perp : \text{Gr}_k(V) \to \text{Gr}_{n-k}(V) \) is an isometry. The analogous conclusion for flags follows from the definition of distance \( d_\tau \).

Given \( g \in \mathcal{L}(V) \) and a signature \( \tau \) such that \( \tau_i + \dim(Kg) \leq n \) for all \( i \), the push-forward by \( g \) on flags is the map \( \varphi_g : \mathcal{F}_\tau(g) \subset \mathcal{F}_\tau(V) \to \mathcal{F}_\tau(V) \), \( \varphi_g F := (g F_1, \ldots, g F_k) \), where

**Definition 1.8.** the domain is defined by

\[
\mathcal{F}_\tau(g) := \{ F \in \mathcal{F}_\tau(V) : F_k \cap (Kg) = \{0\} \}.
\]

Similarly, given a signature \( \tau \) such that \( \tau_i + \dim(Rg) \geq n \) for all \( i \), the pull-back by \( g \) on flags is the map \( \varphi_{g^{-1}} : \mathcal{F}_\tau(g^{-1}) \subset \mathcal{F}_\tau(V) \to \mathcal{F}_\tau(V) \), \( \varphi_{g^{-1}} F := (g^{-1} F_1, \ldots, g^{-1} F_k) \), where

**Definition 1.9.** the domain is defined by

\[
\mathcal{F}_\tau(g^{-1}) := \{ F \in \mathcal{F}_\tau(V) : F_1 + (Rg) = V \}.
\]

The duality between duality between push-forwards and pull-backs is expressed as follows.

**Proposition 1.6.** Given \( g \in \mathcal{L}(V) \), \( \mathcal{F}_\tau(g^{-1}) = \mathcal{F}_{\tau^\perp}(g^*) \) and for all \( F \in \mathcal{F}_\tau(g^{-1}) \),

\[
(\varphi_{g^{-1}} F)^\perp = \varphi_{g^*}(F^\perp).
\]
2. Singular Value Geometry

Singular value geometry refers here to the geometry of the singular value decomposition (SVD) of a linear endomorphism $g : V \to V$ on some Euclidean space $V$. It also refers to some geometric properties of the action of $g$ on Grassmannians and flag manifolds related to the singular value decomposition of $g$.

2.1. Singular value decomposition. Let $V$ be an Euclidean space of dimension $n$.

**Definition 2.1.** Given $g \in \mathcal{L}(V)$, the singular values of $g$ are the square roots of the eigenvalues of the quadratic form $Q_g : V \to \mathbb{R}$, $Q_g(v) = \|gv\|^2 = \langle gv, gv \rangle$, i.e., the eigenvalues of the positive semi-definite self-adjoint operator $\sqrt{g^*g}$.

Given $g \in \mathcal{L}(V)$, let

$$s_1(g) \geq s_2(g) \geq \ldots \geq s_n(g) \geq 0,$$

denote the sorted singular values of $g$. The adjoint $g^*$ has the same singular values as $g$ because the operators $\sqrt{g^*g}$ and $\sqrt{gg^*}$ are conjugate.

The largest singular value, $s_1(g)$, is the square root of the maximum value of $Q_g$ over the unit sphere, i.e., $s_1(g) = \max_{\|v\|=1} \|gv\| = \|g\|$ is the operator norm of $g$. Likewise, the least singular value, $s_n(g)$, is the square root of the minimum value of $Q_g$ over the unit sphere, i.e., $s_n(g) = \min_{\|v\|=1} \|gv\|$. This number also denoted by $m(g)$ is called the least expansion of $g$. If $g$ is invertible $m(g) = \|g^{-1}\|^{-1}$, while otherwise $m(g) = 0$.

**Definition 2.2.** The eigenvectors of the quadratic form $Q_g$, i.e., of the positive semi-definite self-adjoint operator $\sqrt{g^*g}$, are called the singular vectors of $g$.

By the spectral theory of self-adjoint operators, for any $g \in \mathcal{L}(V)$ there exists an orthonormal basis consisting of singular vector of $g$.

**Proposition 2.1.** Given $g \in \mathcal{L}(V)$ and $v \in V$ be a unit singular vector of $g$ such that $g^*gv = \lambda^2 v$, there exists a unit vector $w \in V$ such that

(a) $gv = \lambda w$,

(b) $gg^*w = \lambda^2 w$, i.e., $w$ is a singular vector of $g^*$.

**Proof.** Let $v \in V$ be a unit singular vector of $g$. Then $g^*gv = \lambda^2 v$ and $\lambda^2 = \langle \lambda^2 v, v \rangle = \langle g^*gv, v \rangle = \|gv\|^2$, which implies that $\lambda = \|gv\|$. Since $(gg^*) (gv) = g (g^*g)v = \lambda^2 gv$, if $\lambda \neq 0$ then setting $w = gv/\|gv\| = \lambda^{-1} gv$, we have $(gg^*) w = \lambda^2 w$, which proves that $w$ is a singular vector of $g^*$. By definition $gv = \lambda w$. When $\lambda = 0$, take $w$ to be any unit vector in $Kg^*$. Notice that dim($Kg$) = dim($Kg^*$). In this case $v$ and $w$ are singular vectors of $g$ and $g^*$, respectively, such that $gv = 0 = \lambda w$. \qed

By the previous proposition, given $g \in \mathcal{L}(V)$ there exist two orthonormal singular vector basis of $V$, $\{v_1(g), \ldots, v_n(g)\}$ and $\{v_1(g^*), \ldots, v_n(g^*)\}$ for $g$ and $g^*$, respectively, such that

$$gv_j(g) = s_j(g) v_j(g^*)$$

for all $1 \leq j \leq n$.

Denote by $D_g$ the diagonal matrix with diagonal entries $s_j(g)$, $1 \leq j \leq n$, seen as an operator $D_g \in \mathcal{L}(\mathbb{R}^n)$. Define the linear maps $U_g, U_{g^*} : \mathbb{R}^n \to V$ by $U_g(e_j) = v_j(g)$ and
The number of gaps and most expanding directions.

Definition 2.3. Given \( g \in \mathcal{L}(V) \), we call singular basis of \( g \) to any orthonormal basis \( \{v_1, \ldots, v_m\} \) of \( V \) ordered such that \( \| g v_i \| = s_i(g) \), for all \( i = 1, \ldots, m \).

2.2. Gaps and most expanding directions. Consider a linear map \( g \in \mathcal{L}(V) \) and a number \( 0 \leq k \leq \dim V \).

Definition 2.4. The \( k \)-th gap ratio of \( g \) is defined to be

\[
\text{gr}_k(g) := \frac{s_k(g)}{s_{k+1}(g)} \geq 1.
\]

We will write \( \text{gr}(g) \) instead of \( \text{gr}_1(g) \).

Definition 2.5. We say that \( g \) has a first singular gap when \( \text{gr}(g) > 1 \). More generally, we say that \( g \) has a \( k \)-singular gap when \( \text{gr}_k(g) > 1 \).

In some occasions it is convenient to work with the inverse quantity, denoted by

\[
\sigma_k(g) := \text{gr}_k(g)^{-1} \leq 1. \tag{2.1}
\]

Proposition 2.2. For any \( 1 \leq k \leq \dim V \), \( \| \wedge_k g \| = s_1(g) \ldots s_k(g) \).

Proof. Let \( n = \dim V \). Consider orthonormal singular vector basis \( \{v_1, \ldots, v_n\} \) and \( \{v_1^*, \ldots, v_n^*\} \) for \( g \) and \( g^* \), respectively, such that

\[
g v_j = s_j v_j^*, \text{ where } s_j = s_j(g) \text{ for all } 1 \leq j \leq n.
\]

Given \( I = \{i_1, \ldots, i_k\} \in \Lambda_k^n \), with \( i_1 < \ldots < i_k \), we have

\[
(\wedge_k g)(v_{i_1} \wedge \ldots \wedge v_{i_k}) = (s_{i_1} \ldots s_{i_k}) (v_{i_1}^* \wedge \ldots \wedge v_{i_k}^*).
\]

Therefore, the \( k \)-vectors \( v_I = v_{i_1} \wedge \ldots \wedge v_{i_k} \) and \( v_I^* = v_{i_1}^* \wedge \ldots \wedge v_{i_k}^* \) form two orthonormal singular vector basis for \( \wedge_k g \) and \( \wedge_k g^* \), respectively, while the products \( s_I = s_{i_1} \ldots s_{i_k} \) are the singular values of both \( \wedge_k g \) and \( \wedge_k g^* \). Since the largest singular value is attained with \( I = \{1, \ldots, k\} \), \( \| \wedge_k g \| = s_1 \ldots s_k \).

\[\square\]

Corollary 2.3. For any \( 1 \leq k < \dim V \),

\[
\text{gr}_k(g) = \frac{\| \wedge_k g \|^2}{\| \wedge_{k-1} g \| \| \wedge_{k+1} g \|}.
\]

Given \( g \in \mathcal{L}(V) \), if \( \text{gr}(g) > 1 \) then the singular value \( s_1(g) = \| g \| \) is simple.
Definition 2.6. We denote by \( v(g) \in \mathbb{P}(V) \) the associated singular direction, and refer to it as the \( g \)-most expanding direction.

By definition we have
\[
\varphi_g v(g) = v(g^*) .
\]
(2.2)

More generally, given \( 1 \leq k \leq \dim V \), if \( \text{gr}_k(g) > 1 \)

Definition 2.7. we define the \( g \)-most expanding \( k \)-subspace to be
\[
\mathfrak{v}_k(g) := \Psi^{-1}(\mathfrak{v}(\wedge_k g)) ,
\]
where \( \Psi \) stands for the Plücker embedding defined in subsection 1.3.

The subspace \( \mathfrak{v}_k(g) \) is the direct sum of all singular directions associated with the singular values \( s_1(g), \ldots, s_k(g) \). We have
\[
\varphi_g \mathfrak{v}_k(g) = \mathfrak{v}_k(g^*) .
\]
(2.3)

Analogously, let \( n = \dim V \) and assume \( \text{gr}_{n-k}(g) > 1 \).

Definition 2.8. We define the \( g \)-least expanding \( k \)-subspace as
\[
\mathfrak{v}_k(g) := \mathfrak{v}_{n-k}(g)^\perp .
\]

The subspace \( \mathfrak{v}_k(g) \) is the direct sum of all singular directions associated with the singular values \( s_{n-k+1}(g), \ldots, s_n(g) \). Again we have
\[
\varphi_g \mathfrak{v}_k(g) = \mathfrak{v}_k(g^*) .
\]
(2.4)

Let \( \tau = (\tau_1, \ldots, \tau_k) \) be a signature with \( 1 \leq \tau_1 < \ldots < \tau_k \leq \dim V \).

Definition 2.9. We define the \( \tau \)-gap ratio of \( g \) to be
\[
\text{gr}_\tau(g) := \min_{1 \leq j \leq k} \text{gr}_{\tau_j}(g) .
\]

When \( \text{gr}_\tau(g) > 1 \) we say that \( g \) has a \( \tau \)-gap pattern.

Note that \( \text{gr}_\tau(g) > 1 \) means that \( g \) has a \( \tau_j \) singular gap for \( 1 \leq j \leq k \). Recall that \( \mathcal{F}_\tau(V) \) denotes the space of all \( \tau \)-flags, i.e., flags \( F = (F_1, \ldots, F_k) \) such that \( \dim(F_j) = \tau_j \) for \( j = 1, \ldots, k \).

Definition 2.10. If \( \text{gr}_\tau(g) > 1 \) the most expanding \( \tau \)-flag is defined to be
\[
\mathfrak{v}_\tau(g) := (\mathfrak{v}_{\tau_1}(g), \ldots, \mathfrak{v}_{\tau_k}(g)) \in \mathcal{F}_\tau(V) .
\]

Given \( g \in \mathcal{L}(V) \) the domain of its push-forward action on \( \mathcal{F}_\tau(V) \) is

Definition 2.11. \( \mathcal{F}_\tau(g) := \{ F \in \mathcal{F}_\tau(V) : F_k \cap K_k = 0 \} . \)

The push-forward of a flag \( F \in \mathcal{F}_\tau(g) \) by \( g \) is defined to be
\[
\varphi_g F = g F := (g F_1, \ldots, g F_k) .
\]

Proposition 2.4. Given \( g \in \mathcal{L}(V) \) such that \( \text{gr}_\tau(g) > 1 \), the push-forward induces a map
\[
\varphi_g : \mathcal{F}_\tau(g) \to \mathcal{F}_\tau(g^*) \text{ such that } \varphi_g \mathfrak{v}_\tau(g) = \mathfrak{v}_\tau(g^*) .
\]
If \[ \text{Definition 2.13.} \]
\[ \{ F \in \mathcal{F}_\tau(g) : F_1 + R_g = V \} . \]

The **pull-back** of a flag \( F \in \mathcal{F}_\tau(g) \) by \( g \) is defined to be
\[ \varphi_g F = g^{-1} F := (g^{-1} F_1, \ldots, g^{-1} F_k) . \]

**Definition 2.13.** If \( \text{gr}_{\tau^+}(g) > 1 \) the **least expanding \( \tau \)-flag** is defined as
\[ \mathfrak{v}_\tau(g) := (\mathfrak{v}_{\tau_1}(g), \ldots, \mathfrak{v}_{\tau_k}(g)) \in \mathcal{F}_\tau(V) . \]

**Proposition 2.5.** If \( \text{gr}_\tau(g) > 1 \) then \( \mathfrak{v}_{\tau^+}(g) = \mathfrak{v}_\tau(g)^\bot \).

**Proof.** Let \( \{v_1, \ldots, v_n\} \) be a singular basis of \( g \). Since this basis is orthonormal,
\[ \mathfrak{v}_{n-k}(g) = \langle v_{k+1}, \ldots, v_n \rangle = \langle v_1, \ldots, v_k \rangle^\bot = \mathfrak{v}_k(g)^\bot . \]

Hence
\[ \mathfrak{v}_{\tau^+}(g) = (\mathfrak{v}_{n-\tau_1}(g), \ldots, \mathfrak{v}_{n-\tau_k}(g)) = (\mathfrak{v}_{\tau_1}(g), \ldots, \mathfrak{v}_{\tau_k}(g))^\bot = \mathfrak{v}_\tau(g)^\bot . \]

**Proposition 2.6.** Given \( g \in \mathcal{L}(V) \) such that \( \text{gr}_{\tau^+}(g) > 1 \), the pull-back induces a map
\[ \varphi_g^{-1} : \mathcal{F}_{\tau^+}(g) \to \mathcal{F}_{\tau^+}(g^*) \text{ such that } \varphi_g^{-1} \mathfrak{v}_\tau(g) = \mathfrak{v}_\tau(g^*). \]

**Proof.** Given \( F \in \mathcal{F}_{\tau^+}(g) \), we have \( F_j + R_g = V \) for all \( j = 1, \ldots, k \). Hence \( \dim g^{-1} F_j = \dim F_j = \tau_j \) for all \( j \), which proves that \( \varphi_g^{-1} F \in \mathcal{F}_\tau(V) \). To check that \( \varphi_g^{-1} F \in \mathcal{F}_{\tau^+}(g^*) \) just notice that \( g^{-1} F_1 + R_g^* \supseteq K_g + K_\bot = V \).

The second statement follows from \([2.4]\) and proposition \([2.5]\).

We end this subsection proving that the orthogonal complement involution conjugates the push-forward action by \( g \in \mathcal{L}(V) \) with the pull-back action by the adjoint map \( g^* \).

**Proposition 2.7.** Given \( g \in \mathcal{L}(V) \) such that \( \text{gr}_{\tau^+}(g) > 1 \), the action of \( \varphi_g^{-1} \) on \( \mathcal{F}_\tau(V) \) is conjugate to the action of \( \varphi_{g^*} \) on \( \mathcal{F}_{\tau^+}(V) \) by the orthogonal complement involution. More precisely, we have \( \mathcal{F}_{\tau^+}(g) = \mathcal{F}_{\tau^+}(g^*)^\bot \) and \( \mathcal{F}_{\tau^+}(g^*) = \mathcal{F}_{\tau^+}(g)^\bot \), and the following diagram commutes
\[
\begin{array}{ccc}
\mathcal{F}_{\tau^+}(g) & \xrightarrow{\varphi_g^{-1}} & \mathcal{F}_{\tau^+}(g^*) \\
\downarrow{\perp} & & \downarrow{\perp} \\
\mathcal{F}_{\tau^+}(g) & \xrightarrow{\varphi_{g^*}} & \mathcal{F}_{\tau^+}(g^*)
\end{array}
\]
Proof. To see that $\mathcal{F}_\tau^{-1}(g) = \mathcal{F}_{\tau^\perp}(g^*)^\perp$, notice that the following equivalences hold:

\[
F \in \mathcal{F}_\tau^{-1}(g) \iff F_1 + R_g = V \iff F_1^\perp \cap K_{g^*} = 0 \iff F^\perp \in \mathcal{F}_{\tau^\perp}(g^*) .
\]

Exchanging the roles of $g$ and $g^*$ we obtain the relation $\mathcal{F}_\tau^{-1}(g^*) = \mathcal{F}_{\tau^\perp}(g)^\perp$.

Finally, notice it is enough to prove the diagram’s commutativity at the Grassmannian level. For that we use proposition 1.5. □

2.3. Angles and expansion. Throughout this subsection let $\hat{\mathcal{p}}, \hat{\mathcal{q}} \in \mathbb{P}(V)$, and $p \in \hat{\mathcal{p}}$, $q \in \hat{\mathcal{q}}$ denote representative vectors. The projective distance $\delta(\hat{\mathcal{p}}, \hat{\mathcal{q}})$ was defined by

\[
\delta(\hat{\mathcal{p}}, \hat{\mathcal{q}}) := \sqrt{1 - \frac{\langle p, q \rangle^2}{\|p\|^2\|q\|^2}} = \frac{\|p \wedge q\|}{\|p\|\|q\|} = \sin \rho(\hat{\mathcal{p}}, \hat{\mathcal{q}}) .
\]

The complementary quantity plays a special role in the sequel.

Definition 2.14. The $\alpha$-angle between $\hat{\mathcal{p}}$ and $\hat{\mathcal{q}}$ is defined to be

\[
\alpha(\hat{\mathcal{p}}, \hat{\mathcal{q}}) := \frac{|\langle p, q \rangle|}{\|p\|\|q\|} = \cos \rho(\hat{\mathcal{p}}, \hat{\mathcal{q}}) .
\]

In order to give a geometric meaning to this angle we define the projective orthogonal complement of $\hat{\mathcal{p}} \in \mathbb{P}(V)$ as

\[
\Sigma(\hat{\mathcal{p}}) := \{ \hat{x} \in \mathbb{P}(V) : \langle x, p \rangle = 0 \quad \text{for} \quad x \in \hat{x} \} .
\]

The number $\alpha(\hat{\mathcal{p}}, \hat{\mathcal{q}})$ is the sine of the minimum angle between $\hat{\mathcal{p}}$ and $\Sigma(\hat{\mathcal{q}})$.

Proposition 2.8. For any $\hat{\mathcal{p}}, \hat{\mathcal{q}} \in \mathbb{P}(V)$,

\[
\alpha(\hat{\mathcal{p}}, \hat{\mathcal{q}}) = \sin \rho_{\min}(\hat{\mathcal{p}}, \Sigma(\hat{\mathcal{q}})) = \delta_{\min}(\hat{\mathcal{p}}, \Sigma(\hat{\mathcal{q}})) \quad (2.5)
\]

\[
\alpha(\hat{\mathcal{p}}, \hat{\mathcal{q}}) = 0 \iff \delta(\hat{\mathcal{p}}, \hat{\mathcal{q}}) = 1 \iff p \perp q . \quad (2.6)
\]

These concepts extend naturally to Grassmannians and flag manifolds.

Definition 2.15. Given $E, F \in \text{Gr}_k(V)$, we define the $\alpha$-angle between them

\[
\alpha(E, F) = \alpha_k(E, F) := \alpha(\Psi(E), \Psi(F)) ,
\]

where $\Psi : \text{Gr}_k(V) \to \mathbb{P}(\Lambda_k V)$ denotes the Plücker embedding (see subsection 1.3).

Definition 2.16. We say that two $k$-subspaces $E, F \in \text{Gr}_k(V)$ are orthogonal, and we write $E \perp F$, iff $\alpha(E, F) = 0$.

The Grassmannian orthogonal complement of $F$ is defined as

\[
\Sigma(F) := \{ E \in \text{Gr}_k(V) : \alpha(E, F) = 0 \} .
\]

As before, the number $\alpha(E, F)$ measures the sine of the minimum angle between $E$ and $\Sigma(F)$. 
Given Proposition 2.10. Given $E, F \in \text{Gr}_k(V)$, 

(a) $\alpha(E, F) = \alpha(E^\perp, F^\perp)$, 
(b) $\alpha(E, F) = |\det(\pi_{E,F})| = |\det(\pi_{F,E})|$, 
(c) $E \perp F$ iff there exists a pair $(e, f)$ of unit vectors such that $e \in E \cap F^\perp$ and $f \in F \cap E^\perp$, 
(d) $\delta_{\text{min}}(E, F^\perp) \geq \alpha(E, F)$.

Proof. Given $E, F \in \text{Gr}_k(V)$, take orthonormal basis $\{u_1, \ldots, u_k\}$ and $\{v_1, \ldots, v_k\}$ of $E$ and $F$, respectively, and consider the associated unit $k$-vectors $u = u_1 \wedge \ldots \wedge u_k$ and $v = v_1 \wedge \ldots \wedge v_k$, so that $u \in \Psi(E)$ and $v \in \Psi(F)$.

Using the Hodge star operator we obtain unit vectors $*u \in \Psi(E^\perp)$ and $*v \in \Psi(F^\perp)$. Hence

$$\alpha(E^\perp, F^\perp) = |\langle *u, *v \rangle| = |\langle u, v \rangle| = \alpha(E, F) ,$$

which proves (a). Also

$$\alpha(E, F) := |\langle u_1 \wedge \ldots \wedge u_k, v_1 \wedge \ldots \wedge v_k \rangle| = |\det \begin{pmatrix} \langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle & \ldots & \langle u_1, v_k \rangle \\ \langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle & \ldots & \langle u_2, v_k \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_k, v_1 \rangle & \langle u_k, v_2 \rangle & \ldots & \langle u_k, v_k \rangle \end{pmatrix}| = |\det(\pi_{E,F})| .$$

For the second equality write $u_i = w_i + \sum_{j=1}^k \langle u_i, v_j \rangle v_j$ with $w_i \in F^\perp$ and use the anti-symmetry of the exterior product. For the third equality remark that the matrix with entries $\langle u_i, v_j \rangle$ represents $\pi_{E,F}$ w.r.t. the given orthonormal basis. By symmetry, $\alpha(E, F) = |\det(\pi_{F,E})|$. This proves (b).

From these relations, $\alpha(E, F) = 0 \Leftrightarrow K(\pi_{E,F}) \neq \{0\} \Leftrightarrow K(\pi_{F,E}) \neq \{0\}$, which explains (c).

By proposition [1.1](b), and because all singular values of $\pi_{E,F}$ are in $[0,1]$, 

$$\delta_{\text{min}}(E, F^\perp) = \|\pi_{E,F}\| \geq |\det(\pi_{E,F})| = \alpha_k(E, F) ,$$

which proves (d).

Finally, we extend $\alpha$-angle to flags. Consider a signature $\tau$ of length $k$.

Definition 2.17. Given flags $F, G \in \mathcal{F}_\tau(V)$, define

$$\alpha(F, G) = \alpha_\tau(F, G) := \min_{1 \leq j \leq k} \alpha(F_j, G_j) .$$
**Definition 2.18.** We say that two \( \tau \)-flags \( F, G \in \mathcal{F}_\tau(V) \) are orthogonal, and we write \( F \perp G, \) iff \( F_j \perp G_j \) for some \( j = 1, \ldots, k \).

Comparing the two definitions, for all \( F, G \in \mathcal{F}_\tau(V) \)

\[
\alpha_\tau(F, G) = 0 \Leftrightarrow G \perp F.
\]

Hence, the **orthogonal flag hyperplane** of \( F \) is defined as

\[
\Sigma(F) := \{\Sigma(F) := \{ G \in \mathcal{F}_\tau(V) : \alpha(G, F) = 0 \} \}.
\]

As in the previous cases, the number \( \alpha_\tau(F, G) \) measures the sine of the minimum angle between \( F \) and \( \Sigma(G) \).

**Proposition 2.11.** For any \( F, G \in \mathcal{F}_\tau(V) \),

\[
\alpha(E, F) = \sin \rho_{\min}(F, \Sigma(G)) = \delta_{\min}(F, \Sigma(G)).
\]

Consider a sequence of linear maps \( g_0, g_1, \ldots, g_{n-1} \in \mathcal{L}(V) \). The following quantities, called **expansion rifts**, measure the break of expansion in the composition \( g_{n-1} \ldots g_1 g_0 \) of the maps \( g_j \).

**Definition 2.19.** The first expansion rift of the sequence above is the number

\[
\rho(g_0, g_1, \ldots, g_{n-1}) := \frac{\|g_{n-1} \ldots g_1 g_0\|}{\|g_{n-1}\| \ldots \|g_1\| \|g_0\|} \in [1, +\infty).
\]

Given \( 1 \leq k \leq \dim V \), the \( k \)-th expansion rift is

\[
\rho_k(g_0, g_1, \ldots, g_{n-1}) := \rho(\wedge_k g_0, \wedge_k g_1, \ldots, \wedge_k g_{n-1}).
\]

Given a signature \( \tau = (\tau_1, \ldots, \tau_k) \), the \( \tau \)-expansion rift is defined as

\[
\rho_\tau(g_0, g_1, \ldots, g_{n-1}) := \min_{1 \leq j \leq k} \rho_{\tau_j}(g_0, g_1, \ldots, g_{n-1}).
\]

The key concept of this section is that of angle between linear maps. The quantity \( \alpha(g, g') \), for instance, is the sine of the angle between \( \varphi_g(\overline{b}(g)) = \overline{b}(g^*) \) and \( \Sigma(\overline{b}(g')) \). As we will see, this angle is a lower bound on the expansion rift of two linear maps \( g \) and \( g' \).

**Definition 2.20.** Given \( g, g' \in \mathcal{L}(V) \), we define

\[
\begin{align*}
\alpha(g, g') &:= \alpha(\overline{b}(g^*), \overline{b}(g')) \quad \text{if } g \text{ and } g' \text{ have a first gap ratio} \\
\alpha_k(g, g') &:= \alpha(\overline{b}_k(g^*), \overline{b}_k(g')) \quad \text{if } g \text{ and } g' \text{ have a } k \text{ gap ratio} \\
\alpha_\tau(g, g') &:= \alpha(\overline{b}_\tau(g^*), \overline{b}_\tau(g')) \quad \text{if } g \text{ and } g' \text{ have a } \tau \text{ gap pattern.}
\end{align*}
\]

The following exotic operation is introduced to obtain an upper bound on the expansion rift \( \rho(g, g') \). Consider the algebraic operation \( a \oplus b := a + b - a b \) on the set \( [0, 1] \). Clearly \( ([0, 1], \oplus) \) is a commutative semigroup isomorphic to \( ([0, 1], \cdot) \). In fact, the transformation \( \Phi : ([0, 1], \oplus) \to ([0, 1], \cdot), \Phi(x) := 1 - x, \) is a semigroup isomorphism. We summarize some properties of this operation.

**Proposition 2.12.** For any \( a, b, c \in [0, 1] \),
which proves (a).

(1) \(0 \oplus a = a\),
(2) \(1 \oplus a = 1\),
(3) \(a \oplus b = (1 - b) a + b = (1 - a) b + a\),
(4) \(a \oplus b < 1 \iff a < 1 \text{ and } b < 1\),
(5) \(a \leq b \Rightarrow a \oplus c \leq b \oplus c\),
(6) \(b > 0 \Rightarrow (ab^{-1} \oplus c) b \leq a \oplus c\),
(7) \(a c + b \sqrt{1 - a^2} \sqrt{1 - c^2} \leq \sqrt{a^2 \oplus b^2}\).

**Proof.** Items (1)-(6) are left as exercises. For the last item consider the function \(f : [0, 1] \rightarrow [0, 1]\) defined by \(f(c) := a c + b \sqrt{1 - a^2} \sqrt{1 - c^2}\). A simple computation shows that

\[f'(c) = a - \frac{b c \sqrt{1 - a^2}}{\sqrt{1 - c^2}}\]

The derivative \(f'\) has a zero at \(c = a/\sqrt{a \oplus b}\), and one can check that this zero is a global maximum of \(f\). Since \(f(a/\sqrt{a \oplus b}) = \sqrt{a^2 \oplus b^2}\), item (7) follows. \(\Box\)

**Definition 2.21.** Given \(g, g' \in \mathcal{L}(V)\) with \(\tau\)-gap patterns, the upper \(\tau\)-angle between \(g\) and \(g'\) is defined to be

\[
\beta_\tau(g, g') := \sqrt{\mathcal{G}'(g)^{-2} \oplus \alpha_\tau(g, g')^2 \oplus \mathcal{G}'(g')^{-2}}.
\]

We will write \(\beta_k(g, g')\) when \(\tau = (k)\), and \(\beta(g, g')\) when \(\tau = (1)\).

Next proposition relates norm expansion by \(g\) and distance contraction by \(\varphi_g\) with angles and gap ratios.

**Proposition 2.13.** Given \(g \in \mathcal{L}(V)\) with \(\sigma(g) < 1\), a point \(\hat{w} \in \mathbb{P}(V)\) and a unit vector \(w \in \hat{w}\),

\begin{enumerate}
\item \(\alpha(\hat{w}, \overline{\mathcal{B}}(g)) \|g\| \leq \|g w\| \leq \|g\| \sqrt{\alpha(\hat{w}, \overline{\mathcal{B}}(g))^2 \oplus \sigma(g)^2}\),
\item \(\delta(\varphi_g(\hat{w}), \overline{\mathcal{B}}(g^*)) \leq \frac{\sigma(g)}{\alpha(\hat{w}, \overline{\mathcal{B}}(g))} \delta(\hat{w}, \overline{\mathcal{B}}(g))\).
\end{enumerate}

**Proof.** Let us write \(\alpha = \alpha(\hat{w}, \overline{\mathcal{B}}(g))\) and \(\sigma = \sigma(g)\). Take a unit vector \(v \in \overline{\mathcal{B}}(g)\) such that \(\angle(v, w)\) is non obtuse. Then \(w = \alpha v + u\) with \(u \perp v\) and \(\|u\| = \sqrt{1 - \alpha^2}\). Choosing a unit vector \(v^* \in \overline{\mathcal{B}}(g^*)\), we have \(g w = \alpha \|g\| v^* + gu\) with \(gu \perp v^*\) and \(\|gu\| \leq \sqrt{1 - \alpha^2} s_2(g) = \sqrt{1 - \alpha^2} \sigma \|g\|\). We define the number \(0 \leq \kappa \leq \sigma\) so that \(\|gu\| = \sqrt{1 - \alpha^2} \kappa \|g\|\). Hence

\[
\alpha^2 \|g\|^2 \leq \alpha^2 \|g\|^2 + \|gu\|^2 = \|gw\|^2,
\]

and also

\[
\|gw\|^2 = \alpha^2 \|g\|^2 + \|gu\|^2 = \|g\|^2 \left(\alpha^2 + (1 - \alpha^2)\kappa^2\right) = \|g\|^2 \left(\alpha^2 + \kappa^2\right) = \|g\|^2 \left(\alpha^2 + \sigma^2\right),
\]

which proves (a).
Item (b) follows from

$$
\delta(\varphi_g(\hat{w}), \overline{v}(g^*)) = \frac{||g \vee gw||}{||gw||} = \frac{||g \vee gu||}{||g||} = \frac{||v^* \wedge gu||}{||gw||} \\
= \frac{||gu||}{||gw||} \leq \frac{\sigma \sqrt{1 - \alpha^2} ||g||}{\alpha ||g||} = \frac{\sigma \delta(\hat{w}, \overline{v}(g))}{\alpha}.
$$

\[\square\]

Next proposition relates the expansion rift $\rho(g, g')$ with the angle $\alpha(g, g')$ and the upper angle $\beta(g, g')$.

**Proposition 2.14.** Given $g, g' \in L(V)$ with a (1)-gap pattern,

$$
\alpha(g, g') \leq \frac{||g' g||}{||g'|| ||g||} \leq \beta(g, g')
$$

**Proof.** Let $\alpha := \alpha(g, g') = \alpha(\overline{v}(g^*), \overline{v}(g'))$ and take unit vectors $v \in \overline{v}(g)$, $v^* \in \overline{v}(g^*)$ and $v' \in \overline{v}(g')$ such that $\langle v^*, v' \rangle = \alpha > 0$ and $g v = ||g|| v^*$.

Since $\varphi_g(\overline{v}(g)) = \overline{v}(g^*)$, $w = \frac{g v}{||g v||}$ is a unit vector in $\hat{w} = \overline{v}(g^*)$. Hence, applying proposition 2.13 (a) to $g'$ and $\hat{w}$, we get

$$
\alpha(g, g') ||g'|| = \alpha(\hat{w}, \overline{v}(g')) ||g'|| \leq \frac{||g' g v||}{||g v||} \leq \frac{||g' g||}{||g||},
$$

which proves the first inequality.

For the second, consider $\hat{w} \in \mathbb{P}(g)$ and a unit vector $w \in \hat{w}$ such that $a := \langle w, v \rangle = \alpha(\hat{w}, \overline{v}(g)) \geq 0$. Then $w = a v + \sqrt{1 - a^2} u$, where $u$ is a unit vector orthogonal to $v$. It follows that $g w = a ||g|| v^* + \sqrt{1 - a^2} g u$ with $g u \perp v^*$, and $||g u|| = \kappa ||g||$ for some $0 \leq \kappa \leq \sigma(g)$. Therefore

$$
\frac{||g w||^2}{||g||^2} = a^2 + (1 - a^2) \kappa^2 = a^2 \oplus \kappa^2.
$$

and

$$
\frac{g w}{||g w||} = \frac{a}{\sqrt{a^2 \oplus \kappa^2}} v^* + \frac{\sqrt{1 - a^2}}{\sqrt{a^2 \oplus \kappa^2}} g u.
$$
The vector \( v' \) can be written as \( v' = \alpha v^* + w' \) with \( w' \perp v^* \) and \( \|w'\| = \sqrt{1 - \alpha^2} \). Set now \( b := \alpha (\varphi_g(w), \overline{v}(g')) \). Then

\[
\begin{align*}
\langle g, w' \rangle \leq \frac{\alpha a}{\sqrt{a^2 + \kappa^2}} + \frac{\sqrt{1 - \alpha^2}}{\sqrt{a^2 + \kappa^2}} \frac{\langle g, v' \rangle}{\|g\|} \\
\leq \frac{\alpha a}{\sqrt{a^2 + \kappa^2}} + \frac{\kappa \sqrt{1 - \alpha^2}}{\sqrt{a^2 + \kappa^2}} \frac{\langle g, u \rangle}{\|g\|} \|w'\| \\
\leq \frac{\alpha a}{\sqrt{a^2 + \kappa^2}} + \frac{\kappa \sqrt{1 - \alpha^2}}{\sqrt{a^2 + \kappa^2}} \|w'\| \\
\leq \frac{\alpha a}{\sqrt{a^2 + \kappa^2}} + \frac{\kappa \sqrt{1 - \alpha^2}}{\sqrt{a^2 + \kappa^2}} \sqrt{\frac{1 - \alpha^2}{a^2 + \kappa^2}} \leq \sqrt{\frac{a^2 + \kappa^2}{a^2 + \kappa^2}}.
\end{align*}
\]

We use Lemma 2.12 (7) on the last inequality. Finally, by proposition 2.13 (a)

\[
\begin{align*}
\|g' w\| \leq \|g'\| \sqrt{b^2 + \sigma(g')^2} \|g\| \\
\leq \|g'\| \|g\| \sqrt{b^2 + \sigma(g')^2} \sqrt{a^2 + \kappa^2} \\
\leq \|g'\| \|g\| \sqrt{\kappa^2 + \alpha^2 + \sigma(g')^2} \leq \beta(g, g') \|g'\| \|g\|,
\end{align*}
\]

where on the two last inequalities use items (6) and (5) of lemma 2.12.

We use Lemma 2.12 (7) on the last inequality. Finally, by proposition 2.13 (a)

\[
\begin{align*}
\beta(g, g') \leq \alpha(g, g') \leq \sqrt{1 + \frac{\text{gr}(g)^{-2} + \text{gr}(g')^{-2}}{\alpha(g, g')^2}}.
\end{align*}
\]

Proof. Just notice that

\[
\frac{\sqrt{\kappa^2 + \alpha^2 + (\kappa')^2}}{\alpha} \leq \frac{\alpha^2 + (\kappa^2 + (\kappa')^2)}{\alpha^2} = \sqrt{1 + \frac{\kappa^2 + (\kappa')^2}{\alpha^2}}.
\]
Proposition 2.17. Given \( g, g' \in \mathcal{L}(V) \) with a (1)-gap pattern
\[
\alpha(g, g') \geq \rho(g, g') \sqrt{1 - \frac{\text{gr}(g)^{-2} + \text{gr}(g')^{-2}}{\rho(g, g')^2}}.
\]
Proof. By proposition 2.14
\[
\rho(g, g')^2 \leq \beta(g, g')^2 \leq \alpha(g, g')^2 + \sigma(g)^2 + \sigma(g')^2,
\]
which implies the claimed inequality.

These inequalities then imply the following more general fact.

Proposition 2.18. Given \( g_0, g_1, \ldots, g_{n-1} \in \mathcal{L}(V) \), if for all \( 1 \leq i \leq n-1 \) the linear maps \( g_i \) and \( g^{(i)} = g_{i-1} \ldots g_0 \) have (1)-gap patterns, then
\[
\prod_{i=1}^{n-1} \alpha(g^{(i)}, g_i) \leq \frac{\|g_{n-1} \ldots g_1 g_0\|}{\|g_{n-1}\| \ldots \|g_1\| \|g_0\|} \leq \prod_{i=1}^{n-1} \beta(g^{(i)}, g_i).
\]
Proof. By definition \( g^{(n-1)} = g_{n-1} \ldots g_1 g_0 \), and by convention \( g^{(0)} = \text{id}_V \). Hence \( \|g_{n-1} \ldots g_1 g_0\| = \prod_{i=0}^{n-1} \frac{\|g^{(i+1)}\|}{\|g^{(i)}\|} \). This implies that
\[
\frac{\|g_{n-1} \ldots g_1 g_0\|}{\|g_{n-1}\| \ldots \|g_1\|} = \left( \prod_{i=0}^{n-1} \frac{1}{\|g_i\|} \right) \left( \prod_{i=0}^{n-1} \frac{\|g^{(i+1)}\|}{\|g^{(i)}\|} \right)
= \prod_{i=0}^{n-1} \frac{\|g_i g^{(i)}\|}{\|g_i\| \|g^{(i)}\|}.
\]
It is now enough to apply proposition 2.14 to each factor.

3. Lipschitz Estimates

In this section we will derive some inequalities describing quantities such as the contracting behaviour of a linear endomorphism on the projective space, the Lipschitz dependence of a projective action on the acting linear endomorphism, the continuity of most expanding directions as functions of a linear map, and the Lipschitz modulus of continuity for sum and intersection operations on flag manifolds.

3.1. Projective action.

Proposition 3.1. Given \( p, q \in V \setminus \{0\}, \)
\[
\left\| \frac{p}{\|p\|} - \frac{q}{\|q\|} \right\| \leq \max\left\{ \frac{1}{\|p\|}, \frac{1}{\|q\|} \right\} \|p - q\|.
\]
Proof. Given to vectors \( u, v \in V \) with \( \| u \| \geq \| v \| = 1 \) we have
\[
\| \frac{u}{\| u \|} - \frac{v}{\| v \|} \| \leq \| u - v \|. 
\]
Assume for instance that \( \| p \| \geq \| q \| \), so that
\[
\max\{\| p \|^{-1}, \| q \|^{-1}\} = \| q \|^{-1}.
\]
Applying the previous inequality with \( u = \frac{p}{\| p \|} \) and \( v = \frac{q}{\| q \|} \), we get
\[
\| \frac{p}{\| p \|} - \frac{q}{\| q \|} \| = \| \frac{u}{\| u \|} - \frac{v}{\| v \|} \| \leq \| u - v \| = \| \frac{p}{\| q \|} - \frac{q}{\| q \|} \|
\]
\[
= \| q \|^{-1} \| p - q \| = \max\{\| p \|^{-1}, \| q \|^{-1}\} \| p - q \|. 
\]
\( \square \)

Given a linear map \( g \in \mathcal{L}(V) \), the projective action of \( g \) is given by the map \( \varphi_g : \mathbb{P}(g) \to \mathbb{P}(g^*) \), \( \varphi_g(\hat{p}) := \hat{q} \).

For any non-collinear vectors \( p, q \in V \) with \( \| p \| = \| q \| = 1 \), define
\[
v_p(q) := \frac{q - \langle p, q \rangle p}{\| q - \langle p, q \rangle p \|} 
\]
to be the versor of the orthogonal projection of \( q \) onto \( p^\perp \).

**Proposition 3.2.** Given \( g \in \mathcal{L}(V) \), and points \( \hat{p} \neq \hat{q} \) in \( \mathbb{P}(V) \),
\[
\delta(\varphi_g(\hat{p}), \varphi_g(\hat{q})) = \frac{\| gp \wedge g v_p(q) \|}{\| gp \| \| g q \|}. 
\]

**Proof.** Let \( p \in \hat{p} \) and \( q \in \hat{q} \) be unit vectors such that \( \theta = \angle(p, q) \in [0, \frac{\pi}{2}] \). We can write
\[
q = (\cos \theta) p + (\sin \theta) v_p(q). 
\]
Hence
\[
\delta(\hat{p}, \hat{q}) = \| p \wedge q \| = (\sin \theta) \| p \wedge v_p(q) \| = \sin \theta, 
\]
and
\[
\delta(\varphi_g(\hat{p}), \varphi_g(\hat{q})) = \frac{\| gp \wedge g q \|}{\| gp \| \| g q \|} = (\sin \theta) \frac{\| gp \wedge g v_p(q) \|}{\| gp \| \| g q \|}. 
\]
\( \square \)

Given a unit vector \( v \in V \), \( \| v \| = 1 \), denote by \( \pi_v, \pi_v^\perp : V \to V \) the orthogonal projections \( \pi_v(x) := \langle v, x \rangle v \), respectively \( \pi_v^\perp(x) := x - \langle v, x \rangle v \).

**Lemma 3.3.** Given \( u, v \in V \) non-collinear with \( \| u \| = \| v \| = 1 \), denote by \( P \) the plane spanned by \( u \) and \( v \). Then
\begin{enumerate}
\item is \( \pi_v - \pi_u \) a self-adjoint endomorphism,
\item \( K(\pi_v - \pi_u) = P^\perp \),
\item the restriction \( \pi_v - \pi_u : P \to P \) is anti-conformal with similarity factor \( \| \sin \angle(u, v) \| \),
\item \( \| \pi_v^\perp - \pi_u^\perp \| = \| \pi_v - \pi_u \| \leq \| v - u \| \).
\end{enumerate}
Proof. Item (a) follows because orthogonal projections are self-adjoint operators.

Given \( w \in P^\perp \), we have \( \pi_u(w) = \pi_v(w) = 0 \), which implies \( w \in K(\pi_u - \pi_v) \). Hence \( P^\perp \subset K(\pi_u - \pi_v) \). Since \( u \) and \( v \) are non-collinear, \( \pi_u - \pi_v \) has rank 2. Thus \( K(\pi_u - \pi_v) = P^\perp \), which proves (b).

For (c) we may assume that \( V = \mathbb{R}^2 \) and consider \( u = (u_1, u_2) \), \( v = (v_1, v_2) \), with \( u_1^2 + u_2^2 = v_1^2 + v_2^2 = 1 \). The projections \( \pi_u \) and \( \pi_v \) are represented by the matrices

\[
U = \begin{pmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} v_1^2 & v_1 v_2 \\ v_1 v_2 & v_2^2 \end{pmatrix}
\]

w.r.t. the canonical basis. Hence \( \pi_v - \pi_u \) is given by

\[
V - U = \begin{pmatrix} v_1^2 - u_1^2 & v_1 v_2 - u_1 u_2 \\ v_1 v_2 - u_1 u_2 & v_2^2 - u_2^2 \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ \alpha & -\beta \end{pmatrix}
\]

where \( \alpha = v_1 v_2 - u_1 u_2 \) and \( \beta = v_1^2 - u_1^2 = -(v_2^2 - u_2^2) \). This proves that the restriction of \( \pi_v - \pi_u \) to the plane \( P \) is anti-conformal. The similarity factor of this map is

\[
\|\pi_v - \pi_u\| = \|\pi_v(u) - u\| = \|\pi_v^+(u)\| = |\sin \angle(u, v)|
\]

Finally, since \( u - \langle v, u \rangle v \perp v \),

\[
\|\pi_v^+ - \pi_u^+\|^2 = \|\pi_v - \pi_u\|^2 = \|\pi_v^+(u)\|^2 = \|u - \langle v, u \rangle v\|^2 = \|u - v\|^2 - \|v - \langle v, u \rangle v\|^2 \leq \|u - v\|^2.
\]

\( \square \)

Given a point \( \hat{p} \in \mathbb{P}(V) \), we identify the tangent to the projective space at \( \hat{p} \) as \( T_{\hat{p}}\mathbb{P}(V) = p^\perp \), for any representative \( p \in \hat{p} \).

**Proposition 3.4.** Given \( g \in \mathcal{L}(V) \), \( \hat{x} \in \mathbb{P}(g) \), and a representative \( x \in \hat{x} \), the derivative of the map \( \varphi_g : \mathbb{P}(g) \to \mathbb{P}(g^*) \) at \( \hat{x} \) is given by

\[
(D\varphi_g)_{\hat{x}} v = \frac{g v - \langle \frac{g x}{\|g x\|}, g v \rangle \frac{g x}{\|g x\|}}{\|g x\|} = \frac{1}{\|g x\|} \pi_g^\perp \pi_x g x/\|g x\| (g v)
\]

**Proof.** The sphere \( S(V) := \{ v \in V : \|v\| = 1 \} \) is a double covering space of \( \mathbb{P}(V) \), whose covering map is the canonical projection \( \hat{\pi} : S(V) \to \mathbb{P}(V) \). With the identification \( T_{\hat{p}}\mathbb{P}(V) = p^\perp \), the derivative of \( \hat{\pi} \), \( D\hat{\pi}_x : T_x S(V) \to T_{\hat{p}}\mathbb{P}(V) \), is the identity linear map. The map \( \varphi_g \) lifts to the map defined on the sphere by \( \tilde{\varphi}_g(x) := \frac{g x}{\|g x\|} \). Hence we can identify the derivatives \((D\tilde{\varphi}_g)_x\) and \((D\varphi_g)_x\). The explicit expression for \((D\tilde{\varphi}_g)_x v\) follows by a simple calculation.

\( \square \)

We will use the following closed ball notation

\[
B^{(d)}(\hat{p}, r) := \{ \hat{x} \in \mathbb{P}(V) : d(\hat{x}, \hat{p}) \leq r \},
\]
where the superscript emphasizes the distance in matter. Given a projective map $f : X \subset \mathbb{P}(V) \to \mathbb{P}(V)$, we denote by $\text{Lip}_d(f)$ the least Lipschitz constant of $f$ with respect to the distance $d$. Next proposition refers to the projective metrics $\delta$ and $\rho$ defined in subsection [1.1]

**Proposition 3.5.** Given $0 < \kappa < 1$ and $g \in \mathcal{L}(V)$ such that $\text{gr}(g) \geq \kappa^{-1}$,

1. $\varphi_g(B(\delta(\overline{V}(g), r)) \subset B(\delta(\overline{V}(g^*), \kappa r/\sqrt{1-r^2})), \text{ for any } 0 < r < 1$,
2. $\varphi_g(B(\rho(\overline{V}(g), a)) \subset B(\rho(\overline{V}(g^*), \kappa \tan a), \text{ for any } 0 < a < \frac{\pi}{2}$,
3. $\text{Lip}_\rho(\varphi_g|_{B(\delta(\overline{V}(g), r))}) \leq \kappa \frac{r + \sqrt{1-r^2}}{1-r^2}, \text{ for any } 0 < r < 1$.

**Proof.** Item (1) of this proposition follows from proposition [2.13](b), because

$$\delta(\hat{w}, \overline{V}(g)) < r \quad \text{implies} \quad \alpha(\hat{w}, \overline{V}(g)) = \sqrt{1 - \delta(\hat{w}, \overline{V}(g))^2} \geq \sqrt{1 - r^2}.$$  

Item (2) reduces to (1), because we have $\delta(\hat{u}, \hat{v}) = \sin \rho(\hat{u}, \hat{v})$, which implies that $B(\rho)(\hat{v}, a) = B(\delta)(\hat{v}, \sin a)$.

To prove (3), take unit vectors $v \in \overline{V}(g)$ and $v^* \in \overline{V}(g^*)$ such that $g v = \|g\| v^*$. Because $v$ is a $g$-most expanding vector, $\|\pi_\perp \circ g\| = \|g \circ \pi_\perp\| \leq s_2(g) \leq \kappa \|g\|$. Given $\hat{x}$ such that $\delta(\hat{x}, \overline{V}(g)) < r$, and a unit vector $x \in \hat{x}$, by proposition [2.13](a)

$$\frac{\|g\|}{\|gx\|} \leq \frac{1}{\alpha(\hat{x}, \overline{V}(g))} \leq \frac{1}{\sqrt{1 - r^2}}.$$  

Using item (b) of the same proposition we get

$$\|\varphi_g(x) - v^*\| \leq \delta(\varphi_g(\hat{x}), \overline{V}(g^*)) \leq \frac{\sigma(g)}{\alpha(\hat{x}, \overline{V}(g))} \delta(\hat{x}, \overline{V}(g)) \leq \frac{\kappa r}{\sqrt{1 - r^2}}.$$  

By proposition [3.4] we have

$$(D\varphi_g)_x v = \frac{1}{\|g x\|} \pi_\perp^g (g v) + \frac{1}{\|g x\|} \left( \pi_\perp^g (x) - \pi_\perp^g \right) (g v).$$  

Thus, by lemma [3.3](d),

$$\|(D\varphi_g)_x\| \leq \frac{\kappa \|g\|}{\|gx\|} + \frac{\|\varphi_g(x) - v^*\| \|g\|}{\|gx\|} \leq \frac{\kappa}{\sqrt{1 - r^2}} + \frac{\kappa r}{1 - r^2} = \frac{\kappa (r + \sqrt{1-r^2})}{1 - r^2}.$$  

Since $B(\delta)(\overline{V}(g), r)$ is a convex Riemannian disk, by the mean value theorem $\varphi_g|_{B(\delta)(\overline{V}(g), r)}$ has Lipschitz constant $\leq \frac{\kappa (r + \sqrt{1-r^2})}{1 - r^2}$ with respect to distance $\rho$. □
3.2. Operations on flag manifolds. As before let $V$ be a $n$-dimensional Euclidean space. Recall that the Grassmann manifold $\text{Gr}_k(V)$ identifies through the Plücker embedding with a submanifold of $\mathbb{P}(\wedge_k V)$. Up to a sign, $E \in \text{Gr}_k(V)$ is identified with the unit $k$-vector $e = e_1 \wedge \ldots \wedge e_k$ associated to any orthonormal basis $\{e_1, \ldots, e_k\}$ of $E$. Recall that the Grassmann distance \((1.8)\) on $\text{Gr}_k(V)$ can be characterized by

$$d(E_1, E_2) := \min\{\|e_1 - e_2\|, \|e_1 + e_2\|\},$$

where $e_j$ is a unit $k$-vector of $E_j$, for $j = 1, 2$.

**Definition 3.1.** Given $E, F \in \text{Gr}(V)$, we say that $E$ and $F$ are $(\cap)$ transversal iff $E + F = V$. Analogously, we say that $E$ and $F$ are $(\cup)$ transversal iff $E \cap F = \{0\}$.

The following numbers quantify the transversality of two linear subspaces.

**Definition 3.2.** Given $E \in \text{Gr}_r(V)$ and $F \in \text{Gr}_s(V)$, consider a unit $r$-vector $e$ of $E$, a unit $s$-vector $f$ of $F$, a unit $(n-r)$-vector $e^\perp$ of $E^\perp$, and a unit $(n-s)$-vector $f^\perp$ of $F^\perp$. We define

$$\theta_+(E, F) := \|e \wedge f\|,\quad \theta_-(E, F) := \|e^\perp \wedge f^\perp\|.$$

Since the chosen unit vectors are unique up to a sign, these quantities are well-defined.

**Remark 3.1.** If $r + s > n$ then $\theta_+(E, F) = 0$. Similarly, if $r + s < n$ then $\theta_-(E, F) = 0$.

**Remark 3.2.** Given $E, F \in \text{Gr}(V)$, $\theta_-(E, F) = \theta_+(E^\perp, F^\perp)$.

Next proposition establishes a Lipschitz modulus of continuity for the sum and intersection operations on Grassmannians in terms of the previous quantities.

**Proposition 3.6.** Given $r, s \in \mathbb{N}$ and $E, E' \in \text{Gr}_r(V)$, $F, F' \in \text{Gr}_s(V)$,

1. $d(E + F, E' + F') \leq \max \left\{ \frac{1}{\theta_+(E, F)}, \frac{1}{\theta_+(E', F')} \right\} (d(E, E') + d(F, F'))$,  
2. $d(E \cap F, E' \cap F') \leq \max \left\{ \frac{1}{\theta_-(E, F)}, \frac{1}{\theta_-(E', F')} \right\} (d(E, E') + d(F, F'))$.

**Proof.** (1) Consider unit $r$-vectors $e$ and $e'$ representing the subspaces $E$ and $E'$ respectively. Consider also unit $s$-vectors $f$ and $f'$ representing the subspaces $F$ and $F'$ respectively. By Proposition 3.1

$$d(E + F, E' + F') = \frac{\|e \wedge f\|}{\|e \wedge f\|} - \frac{\|e' \wedge f'\|}{\|e' \wedge f'\|} \leq K \|e \wedge f - e' \wedge f'\| \leq K (\|e \wedge (f - f')\| + \|(e - e') \wedge f'\|) \leq K (\|e - e'\| + \|f - f'\|)$$
where $K = \max\{\|e \wedge f\|^{-1}, \|e' \wedge f'\|^{-1}\} = \max\{\theta_+(E, F)^{-1}, \max\{\theta_+(E', F')^{-1}\}$. (2) reduces to (1) by duality (see Proposition 1.3).

Next proposition gives an alternative characterization of the transversality measurements $\theta_+(E, F)$ and $\theta_\cap(E, F)$. Let, as before, $\pi_E : V \to E$ denote the orthogonal projection onto a subspace $E \subset V$, and define the restriction $\pi_{E,F} := \pi_F|_E : E \to F$.

**Proposition 3.8.** Given $E \in \text{Gr}_r(V)$ and $F \in \text{Gr}_s(V)$,

1. $\theta_+(E, F) = |\det(\pi_{E,F})| = |\det(\pi_{F,E})|$.
2. $\theta_\cap(E, F) = |\det(\pi_{E,F})| = |\det(\pi_{F,E})|$.

**Proof.** Notice that $E \cap F = K(\pi_{E,F}) = K(\pi_{F,E})$. If $E \cap F \neq \emptyset$ then the three terms in (1) vanish. Otherwise $\pi_{E,F}$ and $\pi_{F,E}$ are isomorphisms. Take an orthonormal basis $\{f_1, \ldots, f_s, f_{s+1}, \ldots, f_r\}$ such that $\{f_1, \ldots, f_s\}$ spans $F$ and the family of vectors $\{f_1, \ldots, f_r, f_{s+1}, \ldots, f_r\}$ spans $E + F$. Consider the unit $s$-vector $f = f_1 \wedge \ldots \wedge f_s$ of $F$, and a unit $r$-vector $e = e_1 \wedge \ldots \wedge e_r$ of $E$. Then

$$\theta_+(E, F) = \|(e_1 \wedge \ldots \wedge e_r) \wedge (f_1 \wedge \ldots \wedge f_s)\| = |\pi_{E,F}(e_1) \wedge \ldots \wedge \pi_{E,F}(e_r) \wedge f_1 \wedge \ldots \wedge f_s| = |\det(\pi_{E,F})| \|f_{s+r} \wedge f_1 \wedge \ldots \wedge f_s\| = |\det(\pi_{E,F})|.$$

Reversing the roles of $E$ and $F$, and because $\|e \wedge f\|$ is symmetric in $e$ and $f$, we obtain $\theta_+(E, F) = |\det(\pi_{F,E})|$, which proves (1).

By duality and remark [3.2] item (2) reduces to (1).
Lemma 3.10. Given $E, E' \in \text{Gr}_r(V)$, $\theta_\cap(E', E^\perp) = \alpha_r(E', E)$.

Proof. Take orthonormal basis $\{v_1, \ldots, v_r\}$ of $E$, and $\{v'_1, \ldots, v'_r\}$ of $E'$. Then
\[
\theta_\cap(E', E^\perp) = |\det(\pi_{E', E})| \\
= |\langle \wedge r \pi_{E', E'}(v_1 \wedge \ldots \wedge v_r), v'_1 \wedge \ldots \wedge v'_r \rangle| \\
= |\langle \pi_{E'}(v_1) \wedge \ldots \wedge \pi_{E'}(v_r), v'_1 \wedge \ldots \wedge v'_r \rangle| \\
= |\langle v_1 \wedge \ldots \wedge v_r, v'_1 \wedge \ldots \wedge v'_r \rangle| = \alpha_r(E', E') .
\]

Next proposition gives a modulus of lower semi-continuity for the transversality measurement $\theta_\cap$.

Proposition 3.11. Given $E, E_0 \in \text{Gr}_r(V)$ and $F, F_0 \in \text{Gr}_s(V)$,
\[
\theta_\cap(E, F) \geq \theta_\cap(E_0, F_0) - d(E, E_0) - d(F, F_0) .
\]

Proof. Consider unit vectors $e \in \Psi(E^\perp)$, $f \in \Psi(F^\perp)$, $e_0 \in \Psi(E_0^\perp)$ and $f_0 \in \Psi(F_0^\perp)$, chosen so that
\[
d(E, E_0) = d(E^\perp, E_0^\perp) = \|e - e_0\| , \\
d(F, F_0) = d(F^\perp, F_0^\perp) = \|f - f_0\| .
\]
Hence
\[
\theta_\cap(E, F) = \|e \wedge f\| \geq \|e_0 \wedge f_0\| - \|e \wedge f - e_0 \wedge f_0\| \\
\geq \theta_\cap(E_0, F_0) - \|e \wedge (f - f_0)\| - \|(e - e_0) \wedge f_0\| \\
\geq \theta_\cap(E_0, F_0) - \|f - f_0\| - \|e - e_0\| \\
\geq \theta_\cap(E_0, F_0) - d(F, F_0) - d(E, E_0) .
\]

The exterior product is a continuous operation. A lower bound on its modulus of continuity can be expressed in terms of the angle between the arguments.

Proposition 3.12. Given $E, F \in \text{Gr}_k(V)$, and families of vectors $\{u_1, \ldots, u_k\} \subset E$ and $\{u_{k+1}, \ldots, u_{k+i}\} \subset F^\perp$ with $1 \leq i \leq m - k$,
\[
\text{(a) } \|u_1 \wedge \ldots \wedge u_k \wedge u_{k+1} \wedge \ldots \wedge u_{k+i}\| \leq \|u_1 \wedge \ldots \wedge u_{k+1}\| \|u_{k+1} \wedge \ldots \wedge u_{k+i}\| , \\
\text{(b) } \|u_1 \wedge \ldots \wedge u_k \wedge u_{k+1} \wedge \ldots \wedge u_{k+i}\| \geq \alpha(E, F) \|u_1 \wedge \ldots \wedge u_k\| \|u_{k+1} \wedge \ldots \wedge u_{k+i}\| .
\]

Proof. Since $\pi_{F^\perp, E^\perp}$ is an orthogonal projection, all its singular values are in $[0, 1]$. Thus, because $|\det(\pi_{F^\perp, E^\perp})|$ is the product of all singular values, while $\mathbf{m}(\wedge_i \pi_{F^\perp, E^\perp})$ is the product of the $i$ smallest singular values, we have
\[
|\det(\pi_{F^\perp, E^\perp})| \leq \mathbf{m}(\wedge_i \pi_{F^\perp, E^\perp}) \leq \|\wedge_i \pi_{F^\perp, E^\perp}\| \leq 1 .
\]
Hence
\[ \|u_1 \wedge \ldots \wedge u_k \wedge u_{k+1} \wedge \ldots \wedge u_{k+i}\| = \|u_1 \wedge \ldots \wedge u_k \wedge \pi_{F^\perp,E^\perp}(u_{k+1}) \wedge \ldots \wedge \pi_{F^\perp,E^\perp}(u_{k+i})\| \]
\[ = \|u_1 \wedge \ldots \wedge u_k\|\|\pi_{F^\perp,E^\perp}(u_{k+1}) \wedge \ldots \wedge \pi_{F^\perp,E^\perp}(u_{k+i})\| \]
\[ \leq \|\wedge_i \pi_{F^\perp,E^\perp}\| \|u_1 \wedge \ldots \wedge u_k\| \|u_{k+1} \wedge \ldots \wedge u_{k+i}\| \]
\[ \leq \|u_1 \wedge \ldots \wedge u_k\| \|u_{k+1} \wedge \ldots \wedge u_{k+i}\| , \]
which proves (a). By proposition 2.10 we have
\[ \alpha(E,F) = \alpha(F^\perp,E^\perp) = |\det(\pi_{F^\perp,E^\perp})| \leq m(\wedge_i(\pi_{F^\perp,E^\perp})) . \]
Thus
\[ \|u_1 \wedge \ldots \wedge u_k \wedge u_{k+1} \wedge \ldots \wedge u_{k+i}\| = \|u_1 \wedge \ldots \wedge u_k \wedge \pi_{F^\perp,E^\perp}(u_{k+1}) \wedge \ldots \wedge \pi_{F^\perp,E^\perp}(u_{k+i})\| \]
\[ = \|u_1 \wedge \ldots \wedge u_k\|\|\pi_{F^\perp,E^\perp}(u_{k+1}) \wedge \ldots \wedge \pi_{F^\perp,E^\perp}(u_{k+i})\| \]
\[ \geq m(\wedge_i \pi_{F^\perp,E^\perp}) \|u_1 \wedge \ldots \wedge u_k\| \|u_{k+1} \wedge \ldots \wedge u_{k+i}\| \]
\[ \geq \alpha(E,F) \|u_1 \wedge \ldots \wedge u_k\| \|u_{k+1} \wedge \ldots \wedge u_{k+i}\| , \]
which proves (b).

Of course the angle function $\alpha$ is Lipschitz continuous.

**Proposition 3.13.** Given $u,u',v,v' \in \mathbb{P}(V)$,
\[ |\alpha(u,v) - \alpha(u',v')| \leq d(u,u') + d(v,v') . \]

**Proof.** Exercise. \( \square \)

The intersection of complementary flags satisfying the appropriate transversality conditions determines a decomposition of the Euclidean space $V$. We end this subsection proving a modulus of continuity for this intersection operation.

Consider a signature $\tau = (\tau_1, \ldots, \tau_k)$ of length $k$ with $\tau_k < \dim V$. We make the convention that $\tau_0 = 0$ and $\tau_{k+1} = \dim V$.

**Definition 3.3.** A $\tau$-decomposition is a family of linear subspaces $E_i = \{E_i\}_{1 \leq i \leq k+1}$ in $\text{Gr}(V)$ such that $V = \oplus_{i=1}^{k+1} E_i$ and $\dim E_i = \tau_i - \tau_{i-1}$ for all $1 \leq i \leq k+1$.

Let $\mathcal{D}_\tau(V)$ denote the space of all $\tau$-decompositions, which is a metric space with the distance
\[ d_\tau(E,E') = \max_{1 \leq i \leq k+1} d_{\tau_i-\tau_{i-1}}(E_i, E'_i) , \]
and where $d_{\tau_i-\tau_{i-1}}$ stands for the distance (1.8) in $\text{Gr}_{\tau_i-\tau_{i-1}}(V)$.

Given two flags $F \in \mathcal{F}_\tau(V)$ and $F' \in \mathcal{F}_{\tau'}(V)$, we will define a decomposition, denoted by $F \cap F'$, formed out of intersecting the components of these flags. For that we introduce the following a measurement.

Definition 3.4. Given two flags $F \in \mathcal{F}_T(V)$ and $F' \in \mathcal{F}_{T^\perp}(V)$, let
\[ \theta_T(F, F') := \min_{1 \leq i \leq k} \theta_T(F_i, F'_{k-i+1}) . \]

Notice that $\dim F_i = \tau_i$ and $\dim F'_{k-i+1} = \tau^\perp_{k-i+1} = \dim V - \tau_i$, i.e., the subspaces $F_i$ and $F'_{k-i+1}$ have complementary dimensions. We will refer to this quantity as the measurement of the transversality between the flags $F$ and $F'$.

In the next proposition we complete $F$ and $F'$ to full flags of length $k+1$ setting $F_{k+1} = F'_{k+1} = V$. Assume also that $\tau_0 = 0$ and $\tau_{k+1} = \dim V$.

Proposition 3.14. If $\theta_T(F, F') > 0$ then the following is a direct sum decomposition in the space $D_{T'}(V)$,
\[ V = \bigoplus_{i=1}^{k+1} F_i \cap F'_{k-i+2} , \]
with $\dim(F_i \cap F'_{k-i+2}) = \tau_i - \tau_{i-1}$ for all $1 \leq i \leq k+1$.

Proof. Since the subspaces $F_i$ and $F'_{k-i+1}$ have complementary dimensions, the relation $\theta_T(F_i, F'_{k-i+1}) > 0$ implies that
\[ V = F_i \oplus F'_{k-i+1} . \]
(3.1)

By lemma 3.9 \[ \theta_T(F_i, F'_{k-i+2}) \geq \theta_T(F_i, F'_{k-i+1}) > 0. \] Therefore $F_i + F'_{k-i+2} = V$ and
\[ \dim(F_i \cap F'_{k-i+2}) = \tau_i + \tau^\perp_{k-i+2} - \dim V = \tau_i + (\dim V - \tau_{i-1}) - \dim V = \tau_i - \tau_{i-1} . \]

We prove by finite induction in $i = 1, \ldots, k+1$ that
\[ F_i = \bigoplus_{j \leq i} F_j \cap F'_{k-j+2} . \]
(3.2)

Since $F_{k+1} = V$ the proposition will follow from this relation at $i = k+1$.

For $i = 1$, (3.2) reduces to $F_1 = F_1 \cap V$. The induction step follows from
\[ F_{i+1} = F_i \oplus (F_{i+1} \cap F'_{k-i+1}) . \]

Since the following dimensions add up
\[ \dim F_{i+1} = \tau_{i+1} = \tau_i + (\tau_{i+1} - \tau_i) = \dim F_i + \dim(F_{i+1} \cap F'_{k-i+1}) , \]
it is enough to see that
\[ F_i \cap (F_{i+1} \cap F'_{k-i+1}) = F_i \cap F'_{k-i+1} = \{0\} \]
which holds because of (3.1).

Hence, by the previous proposition we can define
We denote by $\Gamma_{\tau}(F,F')$ the gap between $F$ and $F'$.

Next proposition gives a modulus of lower semi-continuity for the transversality measure $\theta_{\tau}$.

**Proposition 3.5.** Given flags $F \in \mathcal{F}_\tau(V)$ and $F' \in \mathcal{F}_\tau(V)$ such that $\theta_{\tau}(F,F') > 0$ we define $\Gamma\cap F' := \{F_i \cap F'_{k-i+1}\}_{1 \leq i \leq k+1}$ and call it the intersection decomposition of the flags $F$ and $F'$.

Proof. Apply proposition 3.11.

The modulus of continuity for the intersection map $\cap : \mathcal{F}_\tau(V) \times \mathcal{F}_\tau(V) \to D_\tau(V)$ is established below.

**Proposition 3.16.** Given flags $F_1, F_2 \in \mathcal{F}_\tau(V)$ and $F'_1, F'_2 \in \mathcal{F}_\tau(V)$,

$$d_{\tau}(F_1 \cap F'_1, F_2 \cap F'_2) \leq \max\left\{\frac{1}{\theta_{\tau}(F_1,F'_1)}, \frac{1}{\theta_{\tau}(F_2,F'_2)}\right\} (d_{\tau}(F_1,F_2) + d_{\tau}(F'_1,F'_2)).$$

Proof. The proof reduces to apply proposition 3.6.

Any two linear maps $g_0, g_1 \in \mathcal{L}(V)$ having $\tau$-gap ratios, and such that $\alpha_{\tau}(g_0, g_1) > 0$, determine a $\tau$-decomposition of $V$ as intersection of the image by $\varphi_{\tau}$ of the $g_0$ most expanding $\tau$-flag with the $g_1$ least expanding $\tau^\perp$-flag. Recall definitions 2.10 and 2.13. The corresponding intersection decomposition is bounded below by the angle $\alpha_{\tau}(g_0, g_1)$.

**Proposition 3.17.** Given $g_0, g_1 \in \mathcal{L}(V)$, if $\text{gr}_{\tau}(g_0) > 1$ and $\text{gr}_{\tau}(g_1) > 1$ then

$$\theta_{\tau}(\bar{\nu}_{\tau}(g_0^*), \bar{\nu}_{\tau}(g_1^*)) \geq \alpha_{\tau}(g_0, g_1).$$

In particular, if $\alpha_{\tau}(g_0, g_1) > 0$ the flags $\bar{\nu}_\tau(g_0^*)$ and $\bar{\nu}_\tau(g_1^*)$ determine the decomposition $\bar{\nu}_\tau(g_0^*) \cap \bar{\nu}_\tau(g_1^*) \subseteq D_\tau(V)$.

Proof. Let $n = \dim V$. Consider the flags $F = \bar{\nu}_{\tau}(g_0^*)$ and $F' = \bar{\nu}_{\tau}(g_1^*)$. We have $F_i = \bar{\nu}_{\tau}(g_0^*)$ and $F_{k-i+1} = \bar{\nu}_{\tau}(g_1^*)$. Hence by lemma 3.10

$$\theta_{\tau}(F_i, F'_{k-i+1}) = \theta_{\tau}(\bar{\nu}_{\tau}(g_0^*), \bar{\nu}_{\tau}(g_1^*)) = \alpha_{\tau}(\bar{\nu}_{\tau}(g_0^*), \bar{\nu}_{\tau}(g_1^*)) = \alpha_{\tau}(g_0, g_1),$$

and taking the minimum, $\theta_{\tau}(F, F') \geq \alpha_{\tau}(g_0, g_1)$.

3.3. Dependence on the linear map. We establish a modulus of Lipschitz continuity for the most expanding direction of a linear endomorphism with a gap between its first and second singular values. For any $0 < \kappa < 1$, consider the set $\mathcal{L}_\kappa := \{g \in \mathcal{L}(V) : \text{gr}(g) \geq \frac{1}{\kappa}\}$. We denote by $\bar{\nu} : \mathcal{L}_\kappa \to \mathbb{P}(V)$ the map that assigns the $g$-most expanding direction to each $g \in \mathcal{L}_\kappa$. 
The relative distance between linear maps \( g, g' \in \mathcal{L}(V) \setminus \{0\} \) is defined as

\[
d_{\text{rel}}(g, g') := \frac{\|g - g'\|}{\max\{\|g\|, \|g'\|\}}.
\]

Notice that this relative distance is not a metric. It does not satisfy the triangle inequality. We introduce it just to lighten the notation.

**Proposition 3.18.** The map \( \bar{v} : \mathcal{L}_\kappa \to \mathbb{P}(V) \) is locally Lipschitz.

More precisely, given \( 0 < \kappa < 1 \) there exists \( \varepsilon_0 > 0 \) such that for any \( g_1, g_2 \in \mathcal{L}_\kappa \) satisfying \( d_{\text{rel}}(g_1, g_2) \leq \varepsilon_0 \),

\[
d(\bar{v}(g_1), \bar{v}(g_2)) \leq \frac{16}{1 - \kappa^2} d_{\text{rel}}(g_1, g_2).
\]

**Proof.** Let \( g \in \mathcal{L}_\kappa \) and \( \lambda > 0 \). The singular values (resp. singular vectors) of \( g \) are the eigenvalues (resp. eigenvectors) of \( \sqrt{g^* g} \). Hence \( s_j(\lambda g) = \lambda s_j(g) \), for all \( j \). We also have \( \bar{v}(\lambda g) = \bar{v}(g) \) and \( \text{gr}(\lambda g) = \text{gr}(g) \).

Consider the subspace \( \mathcal{L}_\kappa(1) := \{ g \in \mathcal{L}_\kappa : \|g\| = 1 \} \). The projection \( g \mapsto g/\|g\| \) takes \( \mathcal{L}_\kappa \) to \( \mathcal{L}_\kappa(1) \). It also satisfies \( \bar{v}(g/\|g\|) = \bar{v}(g) \) and

\[
\| \frac{g_1}{\|g_1\|} - \frac{g_2}{\|g_2\|} \| \leq 2 d_{\text{rel}}(g_1, g_2).
\]

Hence we can focus our attention on the restricted map \( \bar{v} : \mathcal{L}_\kappa(1) \to \mathbb{P}(V) \).

Let \( \mathcal{L}_\kappa^+(1) \) denote the subspace of \( g \in \mathcal{L}_\kappa(1) \) such that \( g = g^* \geq 0 \), i.e., \( g \) is positive semi-definite.

Given \( g \in \mathcal{L}_\kappa(1) \), we have \( \|g^* g\| = 1 = \|g\| \), \( \text{gr}(g^* g) = \text{gr}(g)^2 \) and \( \bar{v}(g^* g) = \bar{v}(g) \). Also, for all \( g_1, g_2 \in \mathcal{L}_\kappa(1) \),

\[
\|g_1^* g_1 - g_2^* g_2\| \leq \|g_1^*\| \|g_1 - g_2\| + \|g_1^* - g_2^*\| \|g_2\|
= (\|g_1^*\| + \|g_2\|) \|g_1 - g_2\| \leq 2 \|g_1 - g_2\|.
\]

Hence, the mapping \( g \mapsto g^* g \) takes \( \mathcal{L}_\kappa(1) \) to \( \mathcal{L}_\kappa^+(1) \) and has Lipschitz constant \( 2 \). Therefore, it is enough to prove that the restricted map \( \bar{v} : \mathcal{L}_\kappa^+(1) \to \mathbb{P}(V) \) has (locally) Lipschitz constant \( 4 (1 - \kappa^2)^{-1} \).

Let \( \delta_0 \) be a small positive number and take \( 0 < \varepsilon_0 < \frac{\delta_0^2}{4} \). The size of \( \delta_0 \) will be fixed throughout the rest of the proof according to necessity. Take \( h_1, h_2 \in \mathcal{L}_\kappa^+(1) \) such that \( \|h_1 - h_2\| < \varepsilon_0 \) and set \( \hat{p}_0 := \bar{v}(h_1) \). By Proposition 3.5 we have

\[
\varphi_{h_1}(B(\hat{p}_0, \delta_0)) \subset B \left( \hat{p}_0, \frac{\kappa^2 \delta_0}{\sqrt{1 - \delta_0^2}} \right) \subset B(\hat{p}_0, \delta_0),
\]

where all balls refer to the projective sine-metric \( \delta \) defined in (1.3). The second inclusion holds if \( \delta_0 \) is chosen small enough. Take any \( \hat{p} \in B(\hat{p}_0, \delta_0) \) and choose unit vectors \( p \in \hat{p} \) and \( p_0 \in \hat{p}_0 \) such that \( \langle p, p_0 \rangle > 0 \). Then \( p = \langle p, p_0 \rangle p_0 + w \), with \( w \in p_0^\perp \), \( h_1(p_0) = p_0 \) and
Thus, by Lemma 3.23 below, for all fixed point $Lip(\kappa)$.

Given Lemma 3.20.

Lemma 3.19. Let $T$ be a complete metric space, $T$ a Lipschitz contraction with $\text{Lip}(T_1) < \kappa < 1$, $x_1^* = T_1(x_1^*)$ a fixed point, and $T_2 : X \to X$ any other map with a fixed point $x_2^* = T_2(x_2^*)$. Then

$$d(x_1^*, x_2^*) \leq \frac{1}{1 - \kappa} d(T_1, T_2),$$

where $d(T_1, T_2) := \sup_{x \in X} d(T_1(x), T_2(x))$.

Proof.

$$d(x_1^*, x_2^*) = d(T_1(x_1^*), T_2(x_2^*))$$

$$\leq d(T_1(x_1^*), T_1(x_2^*)) + d(T_1(x_2^*), T_2(x_2^*))$$

$$\leq \kappa d(x_1^*, x_2^*) + d(T_1, T_2),$$

which implies that

$$d(x_1^*, x_2^*) \leq \frac{1}{1 - \kappa} d(T_1, T_2).$$

Lemma 3.20. Given $g_1, g_2 \in \mathcal{L}(V)$, for any $1 \leq i \leq \dim V$,

$$\|\wedge_i g_1 - \wedge_i g_2\| \leq \max(1, \|g_1\|, \|g_2\|) \sum_{j=1}^{i-1} \|g_1 - g_2\|.$$
Proposition 3.22. Given any unit $i$-vector $v_1 \wedge \ldots \wedge v_i \in \wedge_i V$, determined by an orthonormal family of vectors \{v_1, \ldots, v_i\},

$$\| (\wedge_i g_1)(v_1 \wedge \ldots \wedge v_i) - (\wedge_i g_2)(v_1 \wedge \ldots \wedge v_i) \| = \| (g_1 v_1) \wedge \ldots \wedge (g_1 v_i) - (g_2 v_1) \wedge \ldots \wedge (g_2 v_i) \|$$

$$\leq \sum_{j=1}^{i} \| (g_1 v_1) \wedge \ldots \wedge (g_1 v_{j-1}) \wedge (g_1 v_j - g_2 v_j) \wedge (g_2 v_{j+1}) \wedge \ldots \wedge (g_2 v_i) \|$$

$$\leq \sum_{j=1}^{i} \| g_1 \|^{j-1} \| g_2 \|^{i-j} \| g_1 v_j - g_2 v_j \|$$

$$\leq i \max \{1, \| g_1 \|, \| g_2 \| \}^{i-1} \| g_1 - g_2 \| .$$

\[ \square \]

Given a dimension $1 \leq l \leq \text{dim } V$ and $0 < \kappa < 1$, consider the set

$$\mathcal{L}_{l, \kappa} := \{ g \in \mathcal{L}(V) : \text{gr}_l(g) \geq \kappa^{-1} \} ,$$

and define

$$C_l(g_1, g_2) := \frac{l \max \{1, \| g_1 \|, \| g_2 \| \}^{l-1}}{\max \{ \| \wedge_l g_1 \|, \| \wedge_l g_2 \| \} } .$$

Corollary 3.21. The map $\overline{\delta} : \mathcal{L}_{l, \kappa} \to \text{Gr}_l(V)$ is locally Lipschitz.

More precisely, given $0 < \kappa < 1$ there exists $\varepsilon_0 > 0$ such that for any $g_1, g_2 \in \mathcal{L}_{l, \kappa}$ such that $\| g_1 - g_2 \| \leq \varepsilon_0 C_l(g_1, g_2)^{-1}$, we have

$$d(\overline{\delta}_l(g_1), \overline{\delta}_l(g_2)) \leq \frac{16}{1 - \kappa^2} C_l(g_1, g_2) \| g_1 - g_2 \| .$$

Proof. By lemma 3.20, $d_{\text{rel}}(\wedge_l g_1, \wedge_l g_2) \leq C_l(g_1, g_2) \| g_1 - g_2 \|$. Apply proposition 3.18 to the linear maps $\wedge_l g_j : \wedge_l V \to \wedge_l V$, $j = 1, 2$.

\[ \square \]

Given $g \in \mathcal{L}(V)$ having $k$ and $k + r$ gap ratios, if a subspace $E \in \text{Gr}_k(V)$ close to the $g$ most expanding subspace $\overline{\delta}_k(g)$ then the restriction $g|_{E^\perp}$ has a $r$-gap ratio and the most expanding $r$-dimensional subspace of $g|_{E^\perp}$ is close to the intersection of $\overline{\delta}_{k+r}(g)$ with $E^\perp$. Next proposition expresses this fact in a quantitative way.

Proposition 3.22. Given $\varkappa > 0$ small enough, and integers $1 \leq k < k + r \leq \text{dim } V$, there exists $\delta_0 > 0$ such that for all $g \in \mathcal{L}(V)$ and $E \in \text{Gr}_k(V)$, if

(a) $\sigma_k(g) < \varkappa$ and $\sigma_{k+r}(g) < \varkappa$,

(b) $\delta(E, \overline{\delta}_k(g)) < \delta_0$

then

(1) $\sigma_r(g|_{E^\perp}) \leq 2 \varkappa$,

(2) $\delta (\overline{\delta}_r(g|_{E^\perp}), \overline{\delta}_{k+r}(g) \cap E^\perp) \leq \frac{20}{1 - 4 \varkappa^2} \delta(E, \overline{\delta}_k(g))$. 


Proof. Consider the compact space \( \mathcal{K}_r = \{ h \in \mathcal{L}(V) : \|h\| \leq 1 \text{ and } \sigma_r(h) \leq \varkappa \} \).

By uniform continuity of \( \sigma_r \) on \( \mathcal{K}_r \) there exists \( \delta_0 > 0 \) such that for all \( h \in \mathcal{L}(V) \) if there exists \( h_0 \in \mathcal{K}_r \) with \( \|h - h_0\| < \delta_0 \) then \( \sigma_r(h) \leq 2 \varkappa \).

Recall that \( \pi_F \) denotes the orthogonal projection onto a linear subspace \( F \subset V \).

Given \( h \in \mathcal{L}(V) \) such that (a) holds, consider the map \( h = \rho_{\pi_k(g)^\perp} \circ \pi_k(g)^\perp \). We have \( h \in \mathcal{K}_r \) because \( \sigma_r(h) = \sigma_r(g \circ \pi_k(g)^\perp) = \sigma_{k+r}(g) < \varkappa \).

Given \( E \in \mathcal{G}_k(V) \) such that (b) holds, we define \( h_E = \rho_{\pi_k(g)^\perp} \circ \pi_{E^\perp} \). Then
\[
\|h - h_E\| \leq \|\pi_k(g)^\perp - \pi_{E^\perp}\| \lesssim \delta(\pi_k(g)^\perp, E^\perp) = \delta(E, \pi_k(g)) < \delta_0,
\]
which implies that \( \sigma_r(g|_{E^\perp}) = \sigma_r(h_E) \leq 2 \varkappa \), and hence proves (1).

For (2) we use the following triangle inequality
\[
\delta(\pi_{k+r}(g) \cap E^\perp), \pi_{k+r}(g) \cap E^\perp) \leq \delta(\pi_{k+r}(g), \pi_{k+r}(g) \cap E^\perp) \leq \delta(E, \pi_k(g)) \leq \frac{20}{1 - 4 \varkappa^2} \delta(E, \pi_k(g)).
\]
The bound on the first distance is obtained through corollary 3.21 with \( C_r(h_E, h) = r \). The second distance is zero. Finally the bound on the third distance comes from proposition 3.6 (2), using that \( \theta \cap (\pi_{k+r}(g), \pi_k(g)) = 1 \), because \( \pi_k(g) \subset \pi_{k+r}(g) \).

\[
\text{Lemma 3.23.} \text{ Given } g_1, g_2 \in \mathcal{L}(V), \hat{p} \in \mathbb{P}(g_1) \cap \mathbb{P}(g_2) \text{ and any unit vector } p \in \hat{p},
\]
\[
d(\varphi_{g_1}(\hat{p}), \varphi_{g_2}(\hat{p})) \leq \max\{1/\|g_1 p\|, 1/\|g_2 p\|\} \|g_1 - g_2\|.
\]

**Proof.** Assume \( p \in V \) is a unit vector such that \( \hat{p} \in \mathbb{P}(g_1) \cap \mathbb{P}(g_2) \). Applying proposition 3.1 to the non-zero vectors \( g_1 p \) and \( g_2 p \), we get
\[
d(\varphi_{g_1}(\hat{p}), \varphi_{g_2}(\hat{p})) \leq \frac{\|g_1 p\|}{\|g_1 p\|} - \frac{\|g_2 p\|}{\|g_2 p\|} \|g_1 - g_2\|
\leq \max\{|\|g_1 p\|^{-1}, \|g_2 p\|^{-1}\} \|g_1 - g_2\|
\leq \max\{|\|g_1 p\|^{-1}, \|g_2 p\|^{-1}\} \|g_1 - g_2\|.
\]

The final four lemmas of this subsection apply to invertible linear maps in \( \mathcal{G}(V) \). They express the continuity of the map \( g \mapsto \varphi_g \) with values in the space of Lipschitz or Hölder continuous maps on the projective space. These facts will be useful in [6].
Lemma 3.24. Given $g_1, g_2 \in \text{GL}(V)$, and $\hat{p} \neq \hat{q}$ in $\mathbb{P}(V)$,
\[
\left| \frac{\delta(\varphi_{g_1}(\hat{p}), \varphi_{g_1}(\hat{q}))}{\delta(\hat{p}, \hat{q})} - \frac{\delta(\varphi_{g_2}(\hat{p}), \varphi_{g_2}(\hat{q}))}{\delta(\hat{p}, \hat{q})} \right| \leq C(g_1, g_2) \|g_1 - g_2\|,
\]
where $C(g_1, g_2) := (\|g_1^{-1}\|^2 + \|g_2\|^2 \|g_1^{-1}\|^2 \|g_2^{-1}\|^2) (\|g_1\| + \|g_2\|)$.

Proof. Given $p \in \hat{p}$ and $q \in \hat{q}$, by proposition 3.2
\[
\left| \frac{\delta(\varphi_{g_1}(\hat{p}), \varphi_{g_1}(\hat{q}))}{\delta(\hat{p}, \hat{q})} - \frac{\delta(\varphi_{g_2}(\hat{p}), \varphi_{g_2}(\hat{q}))}{\delta(\hat{p}, \hat{q})} \right| = \left| \frac{\|g_1 p \wedge g_1 v_p(q)\| - \|g_2 p \wedge g_2 v_p(q)\|}{\|g_1 p\||\|g_1 q\|} \right| \\
\leq \frac{1}{\|g_1 p\||\|g_1 q\|} \left( \|g_1 p \wedge g_1 v_p(q)\| - \|g_2 p \wedge g_2 v_p(q)\| \right) \\
\leq \|g_1^{-1}\|^2 \|g_1 p \wedge (g_1 v_p(q) - g_2 v_p(q))\| + \|g_1^{-1}\|^2 \|(g_1 p - g_2 p) \wedge g_2 v_p(q)\| \\
+ \|g_1^{-1}\|^2 \|g_2^{-1}\|^2 \left( \|g_1 p\||\|g_1 q\| - \|g_2 q\| \right) + \|g_2 q\||\|g_1 p\| - \|g_2 p\| \|g_2\|^2 \\
\leq \|g_1^{-1}\|^2 \left( \|g_1\| + \|g_2\| \right) \|g_1 - g_2\| \\
+ \|g_2\|^2 \|g_1^{-1}\|^2 \|g_2^{-1}\|^2 \left( \|g_1\| + \|g_2\| \right) \|g_1 - g_2\| \\
= (\|g_1^{-1}\|^2 + \|g_2\|^2 \|g_1^{-1}\|^2 \|g_2^{-1}\|^2) \left( \|g_1\| + \|g_2\| \right) \|g_1 - g_2\|.
\]

Lemma 3.25. Given $g \in \text{GL}(V)$ and $\hat{p} \neq \hat{q}$ in $\mathbb{P}(V)$,
\[
\frac{1}{\|g\|^2 \|g^{-1}\|^2} \leq \frac{\delta(\varphi_{g}(\hat{p}), \varphi_{g}(\hat{q}))}{\delta(\hat{p}, \hat{q})} \leq \|g\|^2 \|g^{-1}\|^2.
\]

Proof. Given $\hat{p} \neq \hat{q}$ in $\mathbb{P}(V)$ consider unit vectors $p \in \hat{p}$, $q \in \hat{q}$ and set $v = v_p(q)$. We have $\|p\| = \|q\| = \|v\| = 1$ and $\langle p, v \rangle = 0$. This last relation implies $\|p \wedge v\| = 1$. Hence
\[
\|gp \wedge gv\| = \|(\wedge g)(p \wedge v)\| \leq \|\wedge g\| \leq \|g\|^2.
\]
Analogously
\[
\|gp \wedge gv\| = \|(\wedge g)(p \wedge v)\| \leq \|\wedge g\| \leq \|g\|^2.
\]
We also have
\[
\|g^{-1}\|^2 \leq \|gp\||\|g\|| \leq \|g\|^2.
\]
To finish the proof combine these inequalities with proposition 3.2.

Given $g \in \text{GL}(V)$, we define
\[
\ell(g) := \max \{ \log \|g\|, \log \|g^{-1}\| \}.
\]
Lemma 3.26. For every $g \in \text{GL}(V)$ and $\hat{p} \neq \hat{q}$ in $\mathbb{P}(V)$,

$$-4 \ell(g) \leq \log \left[ \frac{\delta(\varphi_g(\hat{p}), \varphi_g(\hat{q}))}{\delta(\hat{p}, \hat{q})} \right] \leq 4 \ell(g).$$

Proof. Follows from lemma 3.25. □

Lemma 3.27. Given $g_1, g_2 \in \text{GL}(V)$, $0 < \alpha \leq 1$ and $\hat{p} \neq \hat{q}$ in $\mathbb{P}(V)$,

$$\left| \left( \frac{\delta(\varphi_{g_1}(\hat{p}), \varphi_{g_1}(\hat{q}))}{\delta(\hat{p}, \hat{q})} \right)^\alpha - \left( \frac{\delta(\varphi_{g_2}(\hat{p}), \varphi_{g_2}(\hat{q}))}{\delta(\hat{p}, \hat{q})} \right)^\alpha \right| \leq C_1(g_1, g_2) \|g_1 - g_2\|,$$

where $C_1(g_1, g_2) = \alpha \max\{\|g_1\| \|g_1^{-1}\|, \|g_2\| \|g_2^{-1}\|\}^{2(1-\alpha)} C(g_1, g_2)$, and $C(g_1, g_2)$ stands for the constant in lemma 3.24.

Proof. Setting $\Delta_1 := \frac{\delta(\varphi_{g_1}(\hat{p}), \varphi_{g_1}(\hat{q}))}{\delta(\hat{p}, \hat{q})}$ and $\Delta_2 := \frac{\delta(\varphi_{g_2}(\hat{p}), \varphi_{g_2}(\hat{q}))}{\delta(\hat{p}, \hat{q})}$, from lemmas 3.24 and 3.25 we get

$$\left| \Delta_1^\alpha - \Delta_2^\alpha \right| \leq \alpha \max\{\Delta_1^{\alpha-1}, \Delta_2^{\alpha-1}\} \left| \Delta_1 - \Delta_2 \right|$$

$$\leq \alpha \max\{\|g_1\| \|g_1^{-1}\|, \|g_2\| \|g_2^{-1}\|\}^{2(1-\alpha)} \left| \Delta_1 - \Delta_2 \right|$$

$$\leq \alpha \max\{\|g_1\| \|g_1^{-1}\|, \|g_2\| \|g_2^{-1}\|\}^{2(1-\alpha)} C(g_1, g_2) \|g_1 - g_2\|. \quad \square$$

4. Avalanche Principle

Consider a long chain of $n$ linear maps $g_0 : V_0 \to V_1$, $g_1 : V_1 \to V_2$, etc, between Euclidean spaces $V_i$ of the same dimension $m$. The AP relates the expansion $\|g_{n-1} \ldots g_1 g_0\|$ of the composition $g_{n-1} \ldots g_1 g_0$ with the product of the individual expansions $\|g_{n-1}\| \ldots \|g_1\| \|g_0\|$. Given two quantities $M_n$ and $N_n$ depending on a large number $n \in \mathbb{N}$, we say in rough terms that they are $\varepsilon$-asymptotic, and write $M_n \varepsilon \sim N_n$, when $e^{-n} \leq M_n/N_n \leq e^{-n}$. In general it is not true that $\|g_{n-1} \ldots g_1 g_0\| \varepsilon \sim \|g_{n-1}\| \ldots \|g_1\| \|g_0\|$ for some small $\varepsilon > 0$, unless some atypically sharp alignment of the singular directions of the linear maps $g_j$ occurs. Given the chain of linear maps $g_0, g_1, \ldots, g_{n-1}$, its rift $\rho(g_0, \ldots, g_{n-1}) := \frac{\|g_{n-1} \ldots g_0\|}{\|g_{n-1}\| \ldots \|g_0\|} \in [0, 1]$ measures the break of expansion in the composition $g_{n-1} \ldots g_1 g_0$. The AP says that given any such chain $g_0, g_1, \ldots, g_{n-1}$, where the gap ratio\(^1\) of each map $g_j$ is large, and the rift of any pair of consecutive maps is never too small, then the rift of the composition behaves multiplicatively, in the sense that for some small number $\varepsilon > 0$,

$$\rho(g_0, g_1, \ldots, g_{n-1}) \varepsilon \rho(g_0, g_1) \rho(g_1, g_2) \ldots \rho(g_{n-2}, g_{n-1}),$$

or, equivalently,

$$\frac{\|g_{n-1} \ldots g_1 g_0\|}{\|g_1 g_0\|} \frac{\|g_1\|}{\|g_0\|} \ldots \frac{\|g_{n-1} g_{n-2}\|}{\|g_{n-2}\|} \varepsilon 1.$$\(^1\)

\(^1\)ratio between the first and second singular value

The AP was introduced by M. Goldstein and W. Schlag \cite[proposition 2.2]{7} as a technique to obtain Hölder continuity of the integrated density of states for quasi-periodic Schrödinger
c cocycles. In the original version, the AP applies to chains of unimodular matrices in \( \text{SL}(2, \mathbb{C}) \), and the length of the chain is assumed to be bounded by some lower bound on the norms of the matrices. Notice that for unimodular matrices, the gap ratio and the norm are two equivalent measurements. Still in this unimodular setting, for matrices in \( \text{SL}(2, \mathbb{R}) \), J. Bourgain and S. Jitomirskaya [11, lemma 5] have greatly relaxed the constraint on the length of the chain of matrices, and later J. Bourgain [3, lemma 2.6] has completely removed it, at the cost of slightly weakening the conclusion of the AP.

Later, W. Schlag [8, lemma 1] has generalized the AP to invertible matrices in \( \text{GL}(m, \mathbb{C}) \). Recently, C. Sadel has shared with the authors an earlier draft of [1], containing his version of the AP for \( \text{GL}(m, \mathbb{C}) \) matrices. Both these higher dimensional APs assume some bound on the length of the chains of matrices. A higher dimensional AP without this assumption was proven by the authors [5, theorem 3.1] for invertible real matrices.

We present here the proof of a more general AP, that holds for (possibly non-invertible) matrices in \( \text{Mat}(m, \mathbb{R}) \). As a by-product of the geometric approach used in the proof, we also obtain a quantitative control on the most expanding directions of the product matrix, something essential to prove the continuity of the Oseledets decomposition.

4.1. Contractive shadowing. We prove here a shadowing lemma saying that under some conditions a loose pseudo-orbit of a chain of contracting maps is shadowed by a true orbit of the mapping sequence. In particular, a closed pseudo-orbit is shadowed by a periodic orbit of the mapping chain.

Given a metric space \((X, d)\), denote the closed \(r\)-ball around \(x \in X\) by

\[
B(x, \varepsilon) := \{ z \in X : d(z, x) \leq \varepsilon \}.
\]

Given an open set \(X^0 \subset X\), define

\[
X^0(\varepsilon) := \{ x \in X^0 : d(x, \partial X^0) \geq \varepsilon \},
\]

where \(\partial X^0\) denotes the topological boundary of \(X^0\) in \((X, d)\).

**Lemma 4.1** (shadowing lemma). Consider \(\varepsilon > 0\) and \(0 < \delta < \kappa < 1\) such that \(\delta/(1-\kappa) < \varepsilon < 1/2\).

Given a family \(\{(X_j, d_j)\}_{0 \leq j \leq n}\) of compact metric spaces with diameter 1, a chain of continuous mappings \(\{g_j : X_j^0 \to X_{j+1}^0\}_{0 \leq j \leq n-1}\) defined on open sets \(X_j^0 \subset X_j\), and a sequence of points \(x_j \in X_j\), assume that for every \(0 \leq j \leq n-1\):

(a) \(x_j \in X_j^0\) and \(d(x_j, \partial X_j^0) = 1\),
(b) \(g_j\) has Lipschitz constant \(\leq \kappa\) on \(X_j^0(\varepsilon)\),
(c) \(g_j(x_j) \in X_{j+1}^0(2\varepsilon)\),
(d) \(g_j(X_j^0(\varepsilon)) \subset B(g_j(x_j), \delta)\).

Then, setting \(g^{(n)} := g_{n-1} \circ \ldots \circ g_1 \circ g_0\), the following hold:

1. the composition \(g^{(n)}\) is defined on \(B(x_0, \varepsilon)\) and \(\text{Lip}(g^{(n)}|_{B(x_0, \varepsilon)}) \leq \kappa^n\),
2. \(d(g_{n-1}(x_{n-1}), g^{(n)}(x_0)) \leq \frac{\delta}{1-\kappa}\),
3. if \(x_0 = g_{n-1}(x_{n-1})\) then \(g^{(n)}(B(x_0, \varepsilon)) \subset B(x_0, \varepsilon)\) and there is a point \(x^* \in B(x_0, \varepsilon)\) such that \(g^{(n)}(x^*) = x^*\) and \(d(x_0, x^*) \leq \frac{\delta}{(1-\kappa)(1-\kappa^n)}\).
**Proof.** The proof’s inductive scheme is better understood with the help of figure 1 where we set \( z^i_j := (g_{j-1} \circ \ldots \circ g_{i+1} \circ g_i)(x_i) \) for \( i \leq j \leq n \). Of course we have to prove that all points \( z^i_j \) are well-defined.

| \( X_0 \) | \( X_1 \) | \( X_2 \) | \( X_3 \) | \ldots | \( X_{n-1} \) | \( X_n \) |
|---|---|---|---|---|---|---|
| \( z_0 \) | \( g_0 \) | \( z_1 \) | \( g_1 \) | \( z_2 \) | \( g_2 \) | \ldots | \( g_{n-2} \) | \( z_{n-1} \) | \( g_{n-1} \) | \( z_n \) |
| \( \delta \) | \( \kappa \delta \) | \( \delta \) | \( \kappa \delta \) | \( \delta \) | \( \kappa \delta \) | \( \delta \) | \( \kappa \delta \) | \( \delta \) | \( \kappa \delta \) |

![Figure 1](image_url)  

**Figure 1.** Family of orbits for the chain of mappings \( \{g_j : X_j^0 \to X_{j+1}\}_j \).

The boxed expressions represent upper bounds on the distance between the points respectively above and below the box. The \( i \)-th row represents the orbit of \( x_i \in X_i \) by the chain of mappings \( \{g_j\}_{j \geq i} \). All points in the \( j \)-th column belong to the space \( X_j \).

To explain the last upper bound at the bottom of each column, first notice that \( z^i_j = x_i \). By (a), \( z^{i-1}_i = g_{i-1}(x_{i-1}) \) is well-defined, and by (c), \( z^{i-1}_i \in X^0_{i-1}(\varepsilon) \subset X^0_i(\varepsilon) \). Likewise \( z^{i-2}_i \in X^0_{i-1}(\varepsilon) \), and \( z^{i-2}_i = g_{i-1}(g_{i-2}(x_{i-2})) \) is well-defined. Then by (d) we have

\[
 d(z^{i-1}_i, z^{i-2}_i) = d(g_{i-1}(x_{i-1}), g_{i-1}(g_{i-2}(x_{i-2}))) \leq \delta. \tag{4.1}
\]

All other bounds are obtained applying (b) inductively. More precisely, we prove by induction in the column index \( j \) that

(i) all points \( z^i_j \) in the \( j \)-th column are well-defined and belong to \( X^0_j(\varepsilon) \),

(ii) distances between between consecutive points in the column \( j \) are bounded by the expressions in figure 1 i.e., for all \( 1 \leq i \leq j - 1 \),

\[
 d(z^{i-1}_j, z^i_j) \leq \kappa^{j-i-1} \delta. \tag{4.2}
\]

The initial inductive steps, \( j = 0, 1, 2 \), follow from (a), (c) and (4.1). Assume now that the points \( z^i_j \) in \( j \)-th column satisfy (i) and (ii). Then their images \( z^i_{j+1} = g_j(z^i_j) \) are
This proves (i) for the column well-defined. By (b) we have for all $1 \leq i \leq j - 1$, 
\[ d(z_{j+1}^i, z_{j+1}^i) = d(g_j(z_{j+1}^i), g_j(z_j^i)) \leq \kappa d(z_{j+1}^{i-1}, z_j^i) \leq \kappa^{i-1} \delta. \]
Together with (4.1) this proves (ii) for the column $j+1$. To prove (i) consider any $1 \leq i \leq j$.

By (c) and the triangle inequality, 
\[ \begin{align*}
  d(z_{j+1}^i, \partial X_{j+1}^0(\varepsilon)) & \geq d(z_{j+1}^i, \partial X_{j+1}^0(\varepsilon)) - d(z_{j+1}^i, z_{j+1}^i) \\
  & \geq d(g_j(x_j), \partial X_{j+1}^0(\varepsilon)) - \sum_{l=i+1}^j d(z_{j+1}^{l-1}, z_{j+1}^l) \\
  & \geq 2 \varepsilon - \sum_{l=i+1}^j \kappa^{j-l} \delta \geq 2 \varepsilon - \frac{\delta}{1 - \kappa} \geq \varepsilon.
\end{align*} \]

This proves (i) for the column $j + 1$, and concludes the induction.

Conclusion (1) follows from (b) and the following claim, to be proved by induction in $i$.

For every $i = 0, 1, \ldots, n - 1$, $g^{(i)}(B(x_0, \varepsilon)) \subset X_{i+1}^0(\varepsilon)$, where $g^{(i)} = g_{i-1} \circ \ldots \circ g_0$.

Consider first the case $i = 0$. Given $x \in B(x_0, \varepsilon)$,
\[ d(x, \partial X_0^0) \geq d(x, \partial X_0^0) - d(x, x_0) \geq 1 - \varepsilon > \varepsilon. \]

This implies that $d(g_0(x), g_0(x_0)) \leq \kappa d(x, x_0)$. Thus
\[ \begin{align*}
  d(g_0(x), \partial X_0^0) & \geq d(g_0(x), \partial X_0^0) - d(g_0(x), g_0(x)) \geq 2 \varepsilon - d(g_0(x_0), g_0(x)) \\
  & \geq 2 \varepsilon - \kappa d(x_0, x) \geq 2 \varepsilon - \kappa \varepsilon > \varepsilon
\end{align*} \]

which proves that $g_0(B(x_0, \varepsilon)) \subset X_1^0(\varepsilon)$.

Assume now that for every $l \leq i - 1$,
\[ (g_l \circ \ldots \circ g_0)(B(x_0, \varepsilon)) \subset X_{i+1}^0(\varepsilon). \]

By (b), $g^{(i)}$ acts as a $\kappa^i$ contraction on $B(x_0, \varepsilon)$ and $g^{(i)}(B(x_0, \varepsilon)) \subset X_i^0(\varepsilon)$. Thus for every $x \in B(x_0, \varepsilon)$,
\[ \begin{align*}
  d(g^{(i+1)}(x), \partial X_{i+1}^0) & \geq d(g_i(x_i), \partial X_{i+1}^0) - d(g_i(x_i), g^{(i+1)}(x)) \\
  & \geq 2 \varepsilon - d(z_{i+1}^i, z_{i+1}^i) - d(z_{i+1}^0, g^{(i+1)}(x)) \\
  & \geq 2 \varepsilon - \sum_{l=0}^{i-1} d(z_{i+1}^l, z_{i+1}^{l+1}) - d(g^{(i+1)}(x_0), g^{(i+1)}(x)) \\
  & \geq 2 \varepsilon - (\delta + \kappa \delta + \ldots + \kappa^{i-1} \delta) - \kappa^i d(x_0, x) \\
  & \geq 2 \varepsilon - (\delta + \kappa \delta + \ldots + \kappa^{i-1} \delta) - \kappa^i \varepsilon \\
  & \geq 2 \varepsilon - (1 - \kappa) \varepsilon (1 + \kappa + \ldots + \kappa^{i-1}) - \kappa^i \varepsilon = \varepsilon
\end{align*} \]

which proves that $g^{(i+1)}(B(x_0, \varepsilon)) \subset X_{i+1}^0(\varepsilon)$, and establishes the claim above.

Thus $g^{(n)}$ is well-defined on $B(x_0, \varepsilon)$, and, because of assumption (b), $g^{(n)}$ is a $\kappa^n$ Lipschitz contraction on this ball. This proves (1).
Item (2) follows by \([4,2]\). In fact
\[
d(g_{n-1}(x_{n-1}), g^{(n)}(x_0)) = d(z^{n-1}_n, z^n_0) \leq \sum_{l=1}^{n-1} d(z^l_n, z^{l-1}_n) \leq \sum_{l=1}^{n-1} \kappa^{n-l-1} \delta \leq \frac{\delta}{1 - \kappa}.
\]

Finally we prove (3). Assume \(x_0 = g_{n-1}(x_{n-1})\).
It is enough to see that \(g^{(n)}(B(x_0, \varepsilon)) \subset B(x_0, \varepsilon)\), because by (1) \(g^{(n)}\) acts as a \(\kappa^n\)-contraction in the closed ball \(B(x_0, \varepsilon)\). The conclusion on the existence of a fixed point, as well as the proximity bound, follow from the classical fixed point theorem for Lipschitz contractions.

Given \(x \in B(x_0, \varepsilon)\), we know from the previous calculation that
\[
d(x_0, g^{(n)}(x_0)) < \delta + \kappa \delta + \ldots + \kappa^{n-2} \delta.
\]

Hence
\[
d(g^{(n)}(x), x_0) \leq d(g^{(n)}(x), g^{(n)}(x_0)) + d(g^{(n)}(x_0), x_0) \\
\leq \kappa^{n-1} d(x, x_0) + \delta + \kappa \delta + \ldots + \kappa^{n-2} \delta \\
\leq \delta + \kappa \delta + \ldots + \kappa^{n-2} \delta + \kappa^{n-1} \varepsilon \\
\leq (1 - \kappa) \varepsilon (1 + \kappa + \ldots + \kappa^{n-2}) + \kappa^{n-1} \varepsilon \\
\leq (1 - \kappa) \varepsilon \frac{1 - \kappa^{n-1}}{1 - \kappa} + \kappa^{n-1} \varepsilon = \varepsilon.
\]

Thus \(g^{(n)}(x) \in B(x_0, \varepsilon)\), which proves that \(g^{(n)}(B(x_0, \varepsilon)) \subset B(x_0, \varepsilon)\).

\[\square\]

4.2. **Statement and proof of the AP.** In the AP’s statement and proof we will use the notation introduced in subsection [2.3]. Given a chain of linear mappings \(\{g_j : V_j \to V_{j+1}\}_{0 \leq j \leq n-1}\) we denote the composition of the first \(i\) maps by \(g^{(i)} := g_{i-1} \ldots g_1 g_0\).

**Theorem 4.1** (Avalanche Principle). There exists a constant \(c > 0\) such that given \(0 < \varepsilon < 1, 0 < \kappa \leq c \varepsilon^2\) and a chain of linear mappings \(\{g_j : V_j \to V_{j+1}\}_{0 \leq j \leq n-1}\) between Euclidean spaces \(V_j\), if
\begin{enumerate}
\item[(a)] \(\sigma(g_i) \leq \kappa\), for \(0 \leq i \leq n - 1\), and
\item[(b)] \(\alpha(g_{i-1}, g_i) \geq \varepsilon\), for \(1 \leq i \leq n - 1\),
\end{enumerate}
then

1. \( d(\overline{v}(g^{(n)}), \overline{v}(g_0)) \lesssim \kappa \varepsilon^{-1} \),
2. \( d(\overline{v}(g^{(n)}), \overline{v}(g_{n-1}^*)) \lesssim \kappa \varepsilon^{-1} \),
3. \( \sigma(g^{(n)}) \leq \left( \frac{\kappa(4+2\varepsilon)}{\varepsilon^2} \right)^n \),
4. \( |\log \|g^{(n)}\| + \sum_{i=1}^{n-2} \log \|g_i\| - \sum_{i=1}^{n-1} \log \|g_i g_{i-1}\| | \lesssim n \frac{\kappa}{\varepsilon^2} \).

Remark 4.1 (On the assumptions). Assumption (a) says that the (first) gap ratio of each \( g_j \) is large, \( \text{gr}(g_j) \geq \kappa^{-1} \). By propositions 2.14 and 2.17, assumption (b) is equivalent to a condition on the rift, \( \rho(g_{j-1}, g_j) \geq \varepsilon \) for all \( j = 1, \ldots, n-1 \).

Remark 4.2 (On the conclusions). Conclusions (1) and (2) say that the most expanding direction \( \overline{v}(g^{(n)}) \) of the product \( g^{(n)} \), and its image \( \varphi_{g^{(n)}} \overline{v}(g^{(n)}) \), are respectively \( \kappa/\varepsilon \)-close to the most expanding direction \( \overline{v}(g_0) \) of \( g_0 \), and to the image \( \varphi_{g_{n-1}} \overline{v}(g_{n-1}) \) of the most expanding direction of \( g_{n-1} \). Conclusion (3) says that the composition map \( g^{(n)} \) has a large gap ratio. Finally, conclusion (4) is equivalent to

\[ e^{-n C \kappa \varepsilon^{-2}} \leq \frac{\|g_{n-1} \cdots g_1 g_0\| \|g_1\| \cdots \|g_{n-2}\|}{\|g_1 g_0\| \cdots \|g_{n-1} g_{n-2}\|} \leq e^{n C \kappa \varepsilon^{-2}}, \]

for some universal constant \( C > 0 \). These inequalities describe the asymptotic almost multiplicative behavior of the rifts

\[ \rho(g_0, g_1, \ldots, g_{n-1}) ^ {C \kappa \varepsilon^{-2}} \sim \rho(g_0, g_1) \rho(g_1, g_2) \cdots \rho(g_{n-2}, g_{n-1}). \]

Proof. The strategy of the proof is to look at the contracting action of linear mappings \( g_j \) on the projective space.

For each \( j = 0, 1, \ldots, n \) consider the compact metric space \( X_j = \mathbb{P}(V_j) \) with the normalized Riemannian distance, \( d(\hat{u}, \hat{v}) = \frac{2}{\pi} \rho(\hat{u}, \hat{v}) \), and define for \( 0 \leq j < n \)

\[ X_j^0 := \{ \hat{v} \in X_j : \alpha(\hat{v}, \overline{v}(g_j)) > 0 \}, \]
\[ Y_j^0 := \{ \hat{v} \in X_j : \alpha(\hat{v}, \overline{v}(g_{j-1}^*)) > 0 \}. \]

The domain of the projective map \( \varphi_{g_j} : \mathbb{P}(g_j) \subset X_j \to X_{j+1} \) clearly contains the open set \( X_j^0 \). Analogously, the domain of \( \varphi_{g_{j-1}^*} : \mathbb{P}(g_{j-1}^*) \subset X_j \to X_{j-1} \) contains \( Y_j^0 \). We will apply lemma 4.1 to chains of projective maps formed out of the mappings \( \varphi_{g_j} : X_j^0 \to X_{j+1} \) and their adjoints \( \varphi_{g_{j-1}^*} : Y_j^0 \to X_{j-1} \).
Take positive numbers $\varepsilon$ and $\kappa$ such that $0 < \kappa \ll \varepsilon^2$, let $r := \sqrt{1 - \varepsilon^2}/4$, and define the following input parameters for the application of lemma 4.1,

\begin{align*}
\varepsilon_{\text{sh}} & := \frac{1}{\pi} \arcsin \varepsilon, \\
\kappa_{\text{sh}} & := \frac{\kappa + \sqrt{1 - r^2}}{1 - r^2} \approx \frac{4\kappa}{\varepsilon^2}, \\
\delta_{\text{sh}} & := \frac{\kappa r}{\sqrt{1 - r^2}} \approx \frac{2\kappa}{\varepsilon}.
\end{align*}

A simple calculation shows that there exists $0 < c < 1$ such that for any $0 < \varepsilon < 1$ and $0 < \kappa \leq c\varepsilon^2$, the pre-conditions $0 < \delta_{\text{sh}} < \kappa_{\text{sh}} < 1$ and $\frac{\delta_{\text{sh}}}{1 - \kappa_{\text{sh}}} < \varepsilon_{\text{sh}} < 1/2$ of the shadowing lemma are satisfied.

Define $x_j = \overline{s}(g_j)$ and $x_j^* = \overline{s}(g_j^* - 1)$. This lemma is going to be applied to the following chains of maps and sequences of points

(A) $\varphi_{g_0}, \ldots, \varphi_{g_{n-1}}, \varphi_{g_n^*}, \ldots, \varphi_{g_0^*}, x_0, \ldots, x_{n-1}, x^*_n, \ldots, x^*_1$,

(B) $\varphi_{g_{n-1}^*}, \ldots, \varphi_{g_0^*}, \varphi_{g_0}, \ldots, \varphi_{g_{n-1}}, x^*_n, \ldots, x^*_1, x_0, \ldots, x_{n-1}$,

from which we will infer the conclusions (1) and (2). Let us check now that assumptions (a)-(d) of lemma 4.1 hold in both cases (A) and (B).

By definition $\partial X_j^0 := \{ \hat{v} \in X_j : \alpha(\hat{v}, x_j) = 0 \} = \{ \hat{v} \in X_j : \hat{v} \perp x_j \}$. Hence, if $\hat{v} \in \partial X_j^0$ then $d(x_j, \hat{v}) = 1$, which proves that $d(x_j, \partial X_j^0) = 1$. Analogously, $\partial Y_j^0 = \{ \hat{v} \in X_j : \hat{v} \perp x_j^* \}$ and $d(x_j^*, \partial Y_j^0) = 1$. Therefore assumption (a) holds.

By definition of $X_j^0(\varepsilon)$,

$$
\hat{v} \in X_j^0(\varepsilon) \iff d(\hat{v}, \partial X_j^0) \geq \varepsilon \iff \rho(\hat{v}, \partial X_j^0) \geq \frac{\pi}{2} \varepsilon
$$

$$
\iff \delta(\hat{v}, \partial X_j^0) = \alpha(\hat{v}, x_j) \geq \sin \left( \frac{\pi}{2} \varepsilon \right)
$$

$$
\iff \delta(\hat{v}, x_j) \leq \cos \left( \frac{\pi}{2} \varepsilon \right).
$$

Similarly, by definition of $Y_j^0(\varepsilon)$,

$$
\hat{v} \in Y_j^0(\varepsilon) \iff \delta(\hat{v}, x_j^*) \leq \cos \left( \frac{\pi}{2} \varepsilon \right).
$$

Thus, because

$$
\cos \left( \frac{\pi}{2} \varepsilon_{\text{sh}} \right) = \cos \left( \frac{1}{2} \arcsin \varepsilon \right) \leq \sqrt{1 - \frac{\varepsilon^2}{4}} = r,
$$

we have $X_j^0(\varepsilon_{\text{sh}}) \subset B^{(d)}(x_j, r)$ and $Y_j^0(\varepsilon_{\text{sh}}) \subset B^{(d)}(x_j^*, r)$, and assumption (b) holds by proposition 3.5 (3).

By the gap assumption,

$$
\alpha(\varphi_{g_j}(x_j), x_{j+1}) = \alpha(\overline{s}(g_j^*), \overline{s}(g_{j+1})) = \alpha(g_j, g_{j+1}) \geq \varepsilon.
$$
Therefore
\[ d(\varphi_{g_j}(x_j), \partial X^0_{j+1}) = \frac{2}{\pi} \arcsin \delta(\varphi_{g_j}(x_j), \partial X^0_{j+1}) = \frac{2}{\pi} \arcsin \alpha(\varphi_{g_j}(x_j), x_{j+1}) \]
\[ \geq \frac{2}{\pi} \arcsin \varepsilon = 2 \varepsilon_{sh} . \]

Similarly, by the gap assumption,
\[ \alpha(\varphi_{g^*_j-1}(x^*_j), x^*_{j-1}) = \alpha(\overline{\varphi}(g^*_{j-1}), \overline{\varphi}(g^*_{j-1})) = \alpha(g^*_{j+1}, g^*_j) = \alpha(g_j, g_{j+1}) \geq \varepsilon , \]
and in the same way we infer that
\[ d(\varphi_{g^*_j-1}(x^*_j), \partial Y^0_{j-1}) \geq \frac{2}{\pi} \arcsin \varepsilon = 2 \varepsilon_{sh} . \]

This proves that (c) of the shadowing lemma holds. Notice that in both cases (A) and (B), the assumption (c) holds trivially for the middle points, because \( \varphi_{g^*_{n-1}}(x_{n-1}) = x^*_n \in Y^0_n(2 \varepsilon_{sh}) \) and \( \varphi^*_{g^*_{0}}(x^*_1) = x_0 \in X^0_0(2 \varepsilon_{sh}) \).

It was proved above that \( X^0_j(\varepsilon_{sh}) \subset B^{(\delta)}(x^*_j, r) \) and \( Y^0_j(\varepsilon_{sh}) \subset B^{(\delta)}(x^*_j, r) \). By (1.5) we have \( d(\hat{u}, \hat{v}) \leq \delta(\hat{u}, \hat{v}) \). Thus by proposition 3.5 (1),
\[ \varphi_{g_j}(X^0_j(\varepsilon_{sh})) \subset B^{(\delta)}(x^*_j, \delta_{sh}) \subset B^{(\delta)}(x^*_j, \delta_{sh}) \]
with \( x^*_j = \varphi_{g_j}(x_j) \),
and analogously,
\[ \varphi_{g^*_j-1}(Y^0_j(\varepsilon_{sh})) \subset B^{(\delta)}(x^*_j-1, \delta_{sh}) \subset B^{(\delta)}(x^*_j-1, \delta_{sh}) \]
with \( x_{j-1} = \varphi_{g^*_j-1}(x^*_j) \).

Hence, (d) of lemma 4.1 holds.

Therefore, because \( \varphi_{g^*_{0}}(x^*_1) = x_0 \) and \( \varphi_{g^*_{n-1}}(x_{n-1}) = x^*_n \), conclusion (2) of lemma 4.1 holds for both chains (A) and (B). The projective points \( \overline{\varphi}(g^{(n)}) \) and \( \overline{\varphi}(g^{(n)*}) \) are the unique fixed points of the chains of mappings (A) and (B), respectively. Hence, by the shadowing lemma both distances \( d(x_0, \overline{\varphi}(g^{(n)})) \) and \( d(x^*_n, \overline{\varphi}(g^{(n)*})) \) are bounded above by
\[ \frac{\delta_{sh}}{(1 - \kappa_{sh}) \left( 1 - \kappa_{sh}^2 \right)} \leq \frac{\kappa_{sh}}{\varepsilon} . \]

This proves conclusions (1) and (2) of the AP.

From proposition 3.4 we infer that for any \( g \in L(V) \),
\[ \| (D\varphi_g)_{\overline{\varphi}(g)} \| = \frac{s_2(g)}{\| g \|} = \sigma(g) . \]

Hence, by (1) of the shadowing lemma,
\[ \sigma(g^{(n)}) = \| (D\varphi_{g^{(n)}})_{\overline{\varphi}(g^{(n)})} \| \leq \text{Lip}(\varphi_{g^{(n)}}|_{B(\overline{\varphi}(g_0), \varepsilon_{sh})}) \]
\[ \leq (\kappa_{sh})^n \leq \left( \frac{\kappa (4 + 2 \varepsilon)}{\varepsilon^2} \right)^n . \]

This proves conclusion (3) of the AP.

Before proving (4), notice that applying (3) to the chain of linear maps \( g_0, \ldots, g_{i-1} \) we get that \( g^{(i)} := g_{i-1} \ldots g_0 \) has a first gap ratio for all \( i = 1, \ldots, n \).
We claim that
\[ |\alpha(g^{(i)}, g_i) - \alpha(g_{i-1}, g_i)| \lesssim \kappa \varepsilon^{-1}. \] (4.3)

By (2) of the AP, on the chain of linear maps \( g_0, \ldots, g_{i-1}, \)
\[ d(\overline{g}(g^{(i)}), \overline{g}(g^{(i)*})) \leq \frac{\delta_{sh}}{(1 - \kappa_{sh})(1 - \kappa_{sh}^2)} \lesssim \kappa \varepsilon^{-1}. \]

Hence, by proposition 3.13
\[ |\alpha(g^{(i)}, g_i) - \alpha(g_{i-1}, g_i)| = |\alpha(\overline{g}(g^{(i)}), \overline{g}(g_i)) - \alpha(\overline{g}(g^{(i)*}), \overline{g}(g_i))| \leq d(\overline{g}(g^{(i)}), \overline{g}(g^{(i)*})) \lesssim \kappa \varepsilon^{-1}. \]

For any \( i \), the logarithm of any ratio between the four factors \( \alpha(g^{(i)}, g_i), \beta(g^{(i)}, g_i), \alpha(g_{i-1}, g_i) \) and \( \beta(g_{i-1}, g_i) \) is of order \( \kappa \varepsilon^{-2} \). In fact, by (4.3)
\[ |\log \frac{\alpha(g^{(i)}, g_i)}{\alpha(g_{i-1}, g_i)}| \leq \frac{1}{\varepsilon} |\alpha(g^{(i)}, g_i) - \alpha(g_{i-1}, g_i)| \lesssim \kappa \varepsilon^{-2}. \]

By Lemma 2.16, and since \( \sigma_{\gamma_j}(g_i) \leq \kappa \), the other ratios are of the same magnitude as this one. Thus, for some universal constant \( C > 0 \), each of these four ratios is inside the interval \([e^{-C\kappa \varepsilon^{-2}}, e^{C\kappa \varepsilon^{-2}}]\).

Finally, applying proposition 2.18 to the rifts \( \rho(g_0, \ldots, g_{n-1}), \rho(g_0, g_1), \rho(g_1, g_2) \), etc, we have
\[ e^{-nC\kappa \varepsilon^{-2}} \leq \prod_{i=1}^{n-1} \frac{\alpha(g^{(i)}, g_i)}{\beta(g_{i-1}, g_i)} \leq \frac{\rho(g_0, \ldots, g_{n-1})}{\prod_{i=1}^{n-1} \rho(g_{i-1}, g_i)} \leq \prod_{i=1}^{n-1} \frac{\beta(g^{(i)}, g_i)}{\alpha(g_{i-1}, g_i)} \leq e^{nC\kappa \varepsilon^{-2}}, \]
which by remark 4.2 is equivalent to (4). \( \square \)

Next proposition is a practical reformulation of the Avalanche Principle.

**Proposition 4.2.** There exists \( c > 0 \) such that given \( 0 < \epsilon < 1, 0 < \kappa \leq c \epsilon^2 \) and \( g_0, g_1, \ldots, g_{n-1} \in \text{Mat}(m, \mathbb{R}) \), if
\[ (\text{gaps}) \: \text{gr}(g_i) > \frac{1}{\kappa} \quad \text{for all} \quad 0 \leq i \leq n-1 \]
\[ (\text{angles}) \: \frac{||g_i \cdot g_{i-1}||}{||g_i|| \cdot ||g_{i-1}||} > \epsilon \quad \text{for all} \quad 1 \leq i \leq n-1 \]
then
\[ \max \left\{ d(\overline{g}(g^{(n)}), \overline{g}(g^{(n-1)})), d(\overline{g}(g^{(n)}), \overline{g}(g_0)) \right\} \lesssim \kappa \epsilon^{-1} \]
\[ \log ||g^{(n)}|| + \sum_{i=1}^{n-2} \log ||g_i|| - \sum_{i=1}^{n-1} \log ||g_i \cdot g_{i-1}|| \lesssim n \cdot \frac{\kappa}{\epsilon^2}. \]
Proof. Consider the constant $c > 0$ in theorem 4.1 let $c' := c (1 - 2 c^2)$ and assume $0 < \kappa \leq c' \epsilon^2$.

Assumption (gaps) here is equivalent to assumption (a) of theorem 4.1. By proposition 2.17, the assumption (angles) here implies

$$\alpha(g_{i-1}, g_i) \geq \rho(g_{i-1}, g_i) \sqrt{1 - \frac{2 \kappa^2}{\rho(g_{i-1}, g_i)^2}}$$

$$\geq \epsilon \sqrt{1 - \frac{2 \kappa^2}{\epsilon^2}} \geq \epsilon \sqrt{1 - 2 \epsilon^2 c^2} =: \epsilon' ,$$

Since $0 < \kappa \leq c' \epsilon^2$, and $c' \epsilon^2 \leq c (1 - 2 \epsilon^2 c^2) \epsilon^2 = c (\epsilon')^2$ we have $0 < \kappa \leq c (\epsilon')^2$. Thus, because $\epsilon \approx \epsilon'$, this proposition follows from conclusions (1), (2) and (4) of theorem 4.1.

4.3. Consequences of the AP. Given a chain of linear maps $\{g_j : V_j \rightarrow V_{j+1}\}_{0 \leq j \leq n-1}$ between Euclidean spaces $V_i$, and integers $0 \leq i < j \leq n$ we define

$$g^{(j,i)} := g_{j-1} \circ \ldots \circ g_{i+1} \circ g_i .$$

With this notation the following relation holds for $0 \leq i < k < j \leq n,$

$$g^{(j,i)} = g^{(j,k)} \circ g^{(k,i)} .$$

Next proposition states, in a quantified way, that the most expanding directions $\overline{v}(g^{n,i}) \in \mathbb{P}(V_i)$ are almost invariant under the adjoints of the chain mappings.

Proposition 4.3. Under the assumptions of theorem 4.1, where $0 < \kappa \ll \epsilon^2$,

$$d(\varphi_{g_i}^* \overline{v}(g^{(n,i+1)}), \overline{v}(g^{(n,i)})) \leq \frac{\kappa (4 + 2 \epsilon)}{\epsilon^2} n - i .$$

Proof. Consider $\kappa$, $\epsilon$, $\kappa_{sh}$ and $\epsilon_{sh}$ as in theorem 4.1. From the proof of item (3) of the AP, applied to the chain of mappings $g^{n-1}_{n-1}, \ldots, g^n_{n}$, we conclude that the composition $g^{(n,i)} = g^n_{i} \circ \ldots \circ g^1_{n-1}$ is a $(\kappa_{sh})^{n-i}$-Lipschitz contraction on the ball $B(\overline{v}(g^n_{n-1}), \epsilon_{sh})$. On the other hand, by (2) of the AP we have $d(\overline{v}(g^{(n,i+1)}), \overline{v}(g^{n-1}_{n-1})) \leq \kappa \epsilon^{-1}$ and $d(\overline{v}(g^n_{n-1}), \overline{v}(g^{(n,i+1)})) \leq \kappa \epsilon^{-1}$. Since $\kappa \epsilon^{-1} \ll \epsilon \approx \epsilon_{sh}$, both projective points $\overline{v}(g^{(n,i)})$ and $\overline{v}(g^{(n,i+1)})$ belong to the ball $B(\overline{v}(g^n_{n-1}), \epsilon_{sh})$. Thus,

$$d(\varphi_{g_i}^* \overline{v}(g^{(n,i+1)}), \overline{v}(g^{(n,i)})) =$$

$$= d(\varphi_{g_i}^* \varphi_{g^{(n,i+1)}}, \overline{v}(g^{(n,i+1)}), \overline{v}(g^{(n,i)}))$$

$$\leq d(\varphi_{g^{(n,i)}}, \overline{v}(g^{(n,i+1)}), \overline{v}(g^{(n,i)}))$$

$$\leq \epsilon^{-i} d(\overline{v}(g^{(n,i+1)}), \overline{v}(g^{(n,i)}))$$

$$\leq \frac{\kappa (4 + 2 \epsilon)}{\epsilon^2} n - i .$$
which proves the proposition.

Most expanding directions and norms of products of chains matrices under an application of the AP admit the following modulus of continuity.

**Proposition 4.4.** Let \( c > 0 \) be the universal constant in theorem \( 4.1 \). Given numbers \( 0 < \epsilon < 1 \) and \( 0 < \kappa < c\epsilon^2 \), and given two chains of matrices \( g_0, \ldots, g_{n-1} \) and \( g'_0, \ldots, g'_{n-1} \) in \( \text{Mat}(m, \mathbb{R}) \), both satisfying the assumptions of the AP for the given parameters \( \kappa \) and \( \epsilon \), if \( d_{\text{rel}}(g_i, g'_i) < \delta \) for all \( i = 0, 1, \ldots, n - 1 \), then

(a) \( d((\bar{\mathbf{g}}(g_{n-1} \ldots g_0), \bar{\mathbf{g}}'(g'_{n-1} \ldots g'_0)) \leq \frac{\kappa}{\epsilon^2} + 8\delta \),

(b) \( \left| \log \frac{\|g_n \|g_{n-1} \ldots g_0\|}{\|g'_n \|g'_{n-1} \ldots g'_0\|} \right| \leq n \left( \frac{\kappa}{\epsilon^2} + \frac{\delta}{\epsilon} \right) \).

**Proof.** Item (a) follows from conclusion (1) of theorem \( 4.1 \) and proposition \( 3.18 \)

\[
d((\bar{\mathbf{g}}(g_{n-1} \ldots g_0), \bar{\mathbf{g}}'(g'_{n-1} \ldots g'_0)) \leq d((\bar{\mathbf{g}}(g_{n-1} \ldots g_0), \bar{\mathbf{g}}(g_0)))
+ d(\bar{\mathbf{g}}(g_0), \bar{\mathbf{g}}(g'_0)) + d(\bar{\mathbf{g}}(g'_0), \bar{\mathbf{g}}'(g'_{n-1} \ldots g'_0))
\leq 2 \frac{\kappa}{\epsilon} + 16 \delta \frac{\kappa}{1 - \kappa^2} \leq \frac{\kappa}{\epsilon} + 8\delta .
\]

Assuming \( \|g_i\| \geq \|g'_i\| \), we have

\[
\frac{\|g_i\|}{\|g'_i\|} \leq 1 + \frac{\|g_i - g'_i\|}{\|g'_i\|} \leq 1 + \frac{\|g_i\|}{\|g'_i\|} d_{\text{rel}}(g_i, g'_i) \leq 1 + \delta \frac{\|g_i\|}{\|g'_i\|}
\]
which implies

\[
\frac{\|g_i\|}{\|g'_i\|} \leq \frac{1}{1 - \delta} .
\]

Because the case \( \|g_i\| \leq \|g'_i\| \) is analogous, we conclude that

\[
\left| \log \frac{\|g_i\|}{\|g'_i\|} \right| \leq \log \left( \frac{1}{1 - \delta} \right) \leq \frac{\delta}{1 - \delta} \leq \delta .
\]

Since the two chains of matrices satisfy the assumptions of the AP we have

\[
\frac{\|g_i g_{i-1}\|}{\|g_i\| \|g_{i-1}\|} \geq \alpha(g_{i-1}, g_i) \geq \epsilon \quad \text{and} \quad \frac{\|g'_i g'_{i-1}\|}{\|g'_i\| \|g'_{i-1}\|} \geq \alpha(g'_{i-1}, g'_i) \geq \epsilon .
\]

A simple calculation gives

\[
d_{\text{rel}}(g_i g_{i-1}, g'_i g'_{i-1}) \leq \frac{2}{(1 - \delta)^2} \frac{\delta}{\epsilon} \leq \frac{\delta}{\epsilon} .
\]

Therefore

\[
\left| \log \frac{\|g_i g_{i-1}\|}{\|g'_i g'_{i-1}\|} \right| \leq \frac{\delta}{\epsilon} .
\]
Hence, by conclusion (4) of the AP we have
\[
\left| \log \frac{\|g_{n-1} \cdots g_0\|}{\|g'_{n-1} \cdots g'_0\|} \right| \leq \left| \log \frac{\|g_{n-1} \cdots g_0\|}{\|g_1 g_0\| \cdots \|g_{n-1} g_{n-2}\|} \right|
\]
\[
+ \left| \log \frac{\|g'_{n-1} \cdots g'_0\|}{\|g'_1 \| \cdots \|g'_{n-2}\|} \right|
\]
\[
+ \sum_{i=1}^{n-2} \left| \log \frac{\|g'_i\|}{\|g_i\|} \right| + \sum_{i=1}^{n-1} \left| \log \frac{\|g_i g_{i-1}\|}{\|g'_i g'_{i-1}\|} \right|
\]
\[
\lesssim 2n \frac{\kappa}{\varepsilon^2} + (n-2) \delta + (n-1) \frac{\delta}{\varepsilon}
\]
\[
\lesssim n \left( \frac{\kappa}{\varepsilon^2} + \frac{\delta}{\varepsilon} \right),
\]
which proves (b).

Next proposition is a flag version of the AP.
Let \( \tau = (\tau_1, \ldots, \tau_k) \) be a signature with \( 0 < \tau_1 < \tau_2 < \ldots < \tau_k < m \).
We call \( \tau \)-block product to any of the functions \( \pi_{\tau,j} : \text{Mat}(m, \mathbb{R}) \to \mathbb{R} \),
\[
\pi_{\tau,j}(g) := s_{\tau_{j-1}+1}(g) \cdots s_{\tau_j}(g), \quad 1 \leq j \leq k,
\]
where by convention \( \tau_0 = 0 \). A \( \tau \)-singular value product, abbreviated \( \tau \)-s.v.p., is any product of distinct \( \tau \)-block products. By definition, \( \tau \)-block products are \( \tau \)-singular value products. Other examples of \( \tau \)-singular value products are the functions
\[
p_{\tau,j}(g) = s_1(g) \cdots s_{\tau_j}(g) = \|\wedge_{\tau_j} g\|.
\]
Note that for every \( 1 \leq j \leq k \) we have:
\[
\pi_{\tau,j}(g) = \frac{p_{\tau,j}(g)}{p_{\tau,j-1}(g)},
\]
and
\[
p_{\tau,j}(g) = \pi_{\tau,1}(g) \cdots \pi_{\tau,j}(g).
\]

**Proposition 4.5.** Let \( c > 0 \) be the universal constant in theorem \([4.1] \). Given numbers \( 0 < \varepsilon < 1, 0 < \kappa \leq c \varepsilon^2 \) and a chain of matrices \( g_j \in \text{Mat}(m, \mathbb{R}) \), with \( j = 0, 1, \ldots, n-1 \), if

(a) \( \sigma_\tau(g_i) \leq \kappa, \) for \( 0 \leq i \leq n-1 \), and
(b) \( \alpha_\tau(g_{i-1}, g_i) \geq \varepsilon, \) for \( 1 \leq i \leq n-1 \),

then

1. \( d(\overline{v}_\tau(g^{(n)*}), \overline{v}_\tau(g'_n)) \lesssim \kappa \varepsilon^{-1} \)
2. \( d(\overline{v}_\tau(g^{(n)}), \overline{v}_\tau(g_0)) \lesssim \kappa \varepsilon^{-1} \)
3. \( \sigma_\tau(g^{(n)}) \leq \left( \frac{\kappa (4+2\varepsilon)}{\varepsilon^2} \right)^n \)
(4) For any $\tau$-s.v.p. function $\pi$,

$$|\log \pi(g^{(n)}) + \sum_{i=1}^{n-2} \log \pi(g_i) - \sum_{i=1}^{n-1} \log \pi(g_i g_{i-1})| \lesssim n \frac{K}{\varepsilon^2}.$$ 

Proof. For each $j = 1, \ldots, k$, consider the chain of matrices $\wedge_{\tau,j}g_0, \wedge_{\tau,j}g_1, \ldots, \wedge_{\tau,j}g_{n-1}$. Assumptions (a) and (b) here imply the corresponding assumptions of theorem 4.1 for all these chains of exterior power matrices. Hence, by (1) of the AP

$$d(\bar{\nu}_{\tau,j}(g^{(n)}), \bar{\nu}_{\tau,j}(g^{(n)}_{n-1})) = d(\Psi(\bar{\nu}_{\tau,j}(g^{(n)})), \Psi(\bar{\nu}_{\tau,j}(g^{(n)}_{n-1}))) = d(\bar{\nu}(\wedge_{\tau,j}g^{(n)}), \bar{\nu}(\wedge_{\tau,j}g^{(n)}_{n-1})) \lesssim \kappa \varepsilon^{-1}.$$ 

Thus, taking the maximum in $j$ we get $d(\bar{\nu}_\tau(g^{(n)}), \bar{\nu}_\tau(g^{(n)}_{n-1})) \lesssim \kappa \varepsilon^{-1}$, which proves (1).

Conclusion (2) follows in the same way.

Similarly, from (3) of theorem 4.1 we infer the corresponding conclusion here

$$\sigma_{\tau}(g^{(n)}) = \max_{1 \leq j \leq k} \sigma_{\tau,j}(g^{(n)}) = \max_{1 \leq j \leq k} \sigma(\wedge_{\tau,j}g^{(n)}) \leq \left(\frac{\kappa (4 + 2\varepsilon)}{\varepsilon^2}\right)^n.$$ 

Let us now prove (4).

For the $\tau$-s.v.p. $\pi(g) = p_{\tau,j}(g) = \|\wedge_{\tau,j}g\|$ conclusion (4) is a consequence of the corresponding conclusion of theorem 4.1.

For the $\tau$-block product $\pi = \pi_{\tau,j}$, since

$$\log \pi(g) = \log \|\wedge_{\tau,j}g\| - \log \|\wedge_{\tau,j-1}g\|,$$

collection (4) follows again from theorem 4.1 (4).

Finally, since any $\tau$-s.v.p. is a finite product of $\tau$-block products we can reduce (4) to the previous case.

We finish this section with a version of the AP for complex matrices.

The singular values of a complex matrix $g \in \text{Mat}(m, \mathbb{C})$ are defined to be the eigenvalues of the positive semi-definite hermitian matrix $g^* g$, where $g^*$ stands for the transjugate of $g$, i.e., the conjugate transpose of $g$. Similarly, the singular vectors of $g$ are defined as the eigenvectors of $g^* g$. The sorted singular values of $g \in \text{Mat}(m, \mathbb{C})$ are denoted by $s_1(g) \geq s_2(g) \geq \ldots \geq s_m(g)$. The top singular value of $g$ coincides with its norm, $s_1(g) = \|g\|.$

The (first) gap ratio of $g$ is the quotient $\sigma(g) := s_2(g)/s_1(g) \leq 1$. We say that $g \in \text{Mat}(m, \mathbb{C})$ has a (first) gap ratio when $\sigma(g) < 1$. When this happens the complex eigenspace

$$\{ v \in \mathbb{C}^m : g^* g v = \|g\| v \} = \{ v \in \mathbb{C}^m : \|g v\| = \|g\| \|v\| \}$$

has complex dimension one, and determines a point in $\mathbb{P}^m$, denoted by $\bar{\nu}(g)$ and referred as the $g$-most expanding direction.
Given points \( \hat{v}, \hat{u} \in \mathbb{P}(\mathbb{C}^m) \), we set
\[
\alpha(\hat{v}, \hat{u}) := \frac{|\langle v, u \rangle|}{\|v\| \|u\|} \quad \text{where} \quad v \in \hat{v}, \ u \in \hat{u}.
\] (4.4)

Given \( g, g' \in \text{Mat}(m, \mathbb{C}) \), both with (first) gap ratios, we define the angle between \( g \) and \( g' \) to be
\[
\alpha(g, g') := \alpha(\mathfrak{B}(g^*), \mathfrak{B}(g')).
\]

With these definitions, the real version of the AP leads in a straightforward manner to a slightly weaker complex version, stated and proved below. However, adapting the original proof to the complex case, replacing each real concept by its complex analog, would lead to the same stronger estimates as in theorem 4.1.

**Proposition 4.6 (Complex AP).** Let \( c > 0 \) be the universal constant in theorem 4.1. Given numbers \( 0 < \varepsilon < 1, \ 0 < \kappa \leq c \varepsilon^4 \) and a chain of matrices \( g_j \in \text{Mat}(m, \mathbb{C}) \), with \( j = 0, 1, \ldots, n - 1 \), if
(a) \( \sigma(g_i) \leq \kappa, \ \text{for} \ 0 \leq i \leq n - 1, \ \text{and} \)
(b) \( \alpha(g_{i-1}, g_i) \geq \varepsilon, \ \text{for} \ 1 \leq i \leq n - 1, \)
then
(1) \( d(\mathfrak{B}(g^{(n)}), \mathfrak{B}(g^{(n-1)})) \lesssim \kappa \varepsilon^{-2} \)
(2) \( d(\mathfrak{B}(g^{(n)}), \mathfrak{B}(g_0)) \lesssim \kappa \varepsilon^{-2} \)
(3) \( \sigma(g^{(n)}) \leq \left( \frac{\kappa (4+2 \varepsilon^2)^2}{\varepsilon^4} \right)^n \)
(4) \( \left| \log \|g^{(n)}\| + \sum_{i=1}^{n-2} \log \|g_i\| - \sum_{i=1}^{n-1} \log \|g_i g_{i-1}\| \right| \lesssim \frac{n \kappa}{\varepsilon^4} \).

**Proof.** Make the identification \( \mathbb{C}^m \equiv \mathbb{R}^{2m} \), and given \( g \in \text{Mat}(m, \mathbb{C}^m) \) denote by \( g^R \in \text{Mat}(2m, \mathbb{R}) \) the matrix representing the linear operator \( g : \mathbb{R}^{2m} \rightarrow \mathbb{R}^{2m} \) in the canonical basis.

We make explicit the relationship between gap ratios and angles of the complex matrices and \( g, g' \in \text{Mat}(m, \mathbb{C}) \), and the gap ratios and angles of their real analogues \( g^R \) and \( (g')^R \).

Given \( g \in \text{Mat}(m, \mathbb{C}) \), for each eigenvalue \( \lambda \) of \( g \), the matrix \( g^R \) has a corresponding pair of eigenvalues \( \lambda, \overline{\lambda} \). Since \( g \mapsto g^R \) is a \( C^* \)-algebra homomorphism, we have \( (g^*)^R = (g^R)^* (g^R) \). Therefore, for all \( i = 1, \ldots, m \), \( s_i(g) = s_{2i-1}(g^R) = s_{2i}(g^R) \). In particular, considering the signature \( \tau = (2) \),
\[
\sigma_{(2)}(g^R) = \frac{s_3(g^R)}{s_1(g^R)} = \frac{s_2(g)}{s_1(g)} = \sigma(g).
\] (4.5)

The \( g \)-most expanding direction \( \mathfrak{B}(g) \in \mathbb{P}(\mathbb{C}^m) \) is a complex line which we can identify with the real 2-plane \( \mathfrak{B}_{(2)}(g^R) \). This identification, \( \mathfrak{B}(g) \equiv \mathfrak{B}_{(2)}(g^R) \), comes from a natural isometric embedding \( \mathbb{P}(\mathbb{C}^m) \hookrightarrow \text{Gr}_2(\mathbb{R}^{2m}) \).

Consider two points \( \hat{v}, \hat{u} \in \mathbb{P}(\mathbb{C}^m) \) and take unit vectors \( v \in \hat{v} \) and \( u \in \hat{u} \). Denote by \( U, V \subset \mathbb{C}^m \) the complex lines spanned by these vectors, which are planes in \( \text{Gr}_2(\mathbb{R}^{2m}) \). Consider the complex orthogonal projection onto the complex line \( V, \pi_{u,v} : U \rightarrow V \),
defined by \( \pi_{u,v}(x) := \langle x, v \rangle v \). By (4.4) we have \( \alpha(\hat{v}, \hat{u}) = \| \pi_{u,v} \| \). On the other hand, since \( \pi_{u,v} \circ \pi_{u,v} = \pi_{u,v} \) and \( \langle x - \pi_{u,v}(x), v \rangle = 0 \) for all \( x \in U \), it follows that \( \pi_{u,v} \) is the restriction to \( U \) of the (real) orthogonal projection onto the 2-plane \( V \). Thus, by proposition 2.10(b),

\[
\alpha_2(U, V) = \left| \det \mathbb{R}(\pi_{u,v}) \right| = \left| \det \mathbb{C}(\pi_{u,v}) \right|^2 = \| \pi_{u,v} \|^2 = \alpha(\hat{v}, \hat{u})^2.
\]

In particular,

\[
\alpha_2(g^R, (g')^R) = \alpha_2(\mathbf{\overline{v}}((g^R)^*), \mathbf{\overline{v}}((g')^R)) = \alpha(\mathbf{\overline{v}}(g^*), \mathbf{\overline{v}}(g'))^2 = \alpha(g, g')^2. \tag{4.6}
\]

Take \( \kappa, \varepsilon > 0 \) such that \( \kappa < c \varepsilon^4 \), \( 0 < \varepsilon < 1 \), and consider a chain of matrices \( g_j \in \text{Mat}(m, \mathbb{C}) \), \( j = 0, 1, \ldots, n - 1 \) satisfying the assumptions (a) and (b) of the complex AP. By (4.5) and (4.6), the assumptions (a) and (b) of proposition 4.5 hold for the chain of real matrices \( g_j^R \in \text{Mat}(2m, \mathbb{R}) \), \( j = 0, 1, \ldots, n - 1 \), with parameters \( \kappa \) and \( \varepsilon^2 \), and with \( \tau = (2) \). Therefore conclusions (1)-(4) of the complex AP follow from the corresponding conclusions of proposition 4.5. In conclusion (4) we use the \((2)\)-singular value product \( \pi(g) := \| g \|^2 = \| \wedge_2 g^R \| \).

\[\square\]

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