THE PROJECTIVE UNITARY
IRREDUCIBLE
REPRESENTATIONS
OF THE GALILEI GROUP IN 1+2
DIMENSIONS

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Abstract
We give an elementary analysis of the multiplicator group of the Galilei
group in 1+2 dimensions $G^+$. For a non-trivial multiplicator we give
a list of all the corresponding projective unitary irreducible representa-
tions of $G^+$. 

Shortened title: Galilei group in 1+2 dimensions

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1 Introduction

Recently we have determined a complete list of all projective unitary irreducible representations of the Poincaré group in $1+2$ dimensions [1]. In this paper we intend to provide a similar analysis for the Galilei group.

The main technical obstacle seems to be the rather complicated structure of the multiplicator group (see [2], Appendix A). In Section 2 we give an elementary analysis of the second cohomology group of the universal covering group $\tilde{G}^\uparrow_+$ of the Galilei group $G^\uparrow_+$ in $1+2$ dimensions.

In Section 3 we construct for every non-trivial multiplicator a certain extension of $\tilde{G}^\uparrow_+$ which exhibits a semi-direct product group structure. Then we are able to apply Mackey induced representations method to determine the desired representations. In this paper only non-trivial multiplicators are considered. The case of true representation (i.e. trivial multiplicators) is elementary to analyse and raises no problems [1].

2 The Galilei group in 1+2 dimensions

A. Notations

By definition the ortochronous Galilei group in $1+2$ dimensions $G^\uparrow_+$ is set-theoretically $O(2) \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$ with the composition law:

\[(R_1, v_1, \eta_1, a_1) \cdot (R_2, v_2, \eta_2, a_2) = (R_1 R_2, R_1 v_2 + v_1, \eta_1 + \eta_2, R_1 a_2 + a_1 + \eta_2 v_1).\]  

(2.1)

We organize $\mathbb{R}^2$ as column vectors, $O(2)$ as the $2 \times 2$ real orthogonal matrices and we use consistently matrix notations. This group acts naturally on $\mathbb{R} \times \mathbb{R}^2$:

\[(R, v, \eta, a) \cdot (T, X) = (T + \eta, R X + T v + a).\]  

(2.2)

We will also consider the proper ortochronous Galilei group $G^\uparrow_+$ defined as:

\[G^\uparrow_+ \equiv \{(R, v, \eta, a)|det(R) = 1\}\]  

(2.3)

and the universal covering group $\tilde{G}^\uparrow_+$ of $G^\uparrow_+$. We can take $\tilde{G}^\uparrow_+ = \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$ with the composition law:

\[(x_1, v_1, \eta_1, a_1) \cdot (x_2, v_2, \eta_2, a_2) = (x_1 + x_2, v_1 + R(x_1) v_2, \eta_1 + \eta_2, a_1 + R(x_1) a_2 + \eta_2 v_1),\]  

(2.4)

where

\[R(x) \equiv \begin{pmatrix} \cos(x) & sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix}\]  

(2.5)
The covering homomorphism \( \delta : \widetilde{G}_+^4 \to G_+^4 \) is:

\[
\delta(x, v, \eta, a) = (R(x), v, \eta, a).
\] (2.6)

Finally we describe the Lie algebra of \( \widetilde{G}_+^4 \), \( \text{Lie}(\widetilde{G}_+^4) \cong \text{Lie}(G_+^4) \) using the fact that \( G_+^4 \) can be organized as a matrix group. Indeed one has the group isomorphism:

\[
G_+^4 \ni (R, v, \eta, a) \leftrightarrow \begin{pmatrix} R & v & a \\ 0 & 1 & \eta \\ 0 & 0 & 1 \end{pmatrix} \in M_{\mathbb{R}}(4, 4). \] (2.7)

Then \( \text{Lie}(G_+^4) \) can be identified with the linear space of \( 4 \times 4 \) real matrices of the form:

\[
(\alpha, u, t, x) \equiv \begin{pmatrix} \alpha A & u & x \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix}.
\] (2.8)

Here \( u, x \in \mathbb{R}^2, t, \alpha \in \mathbb{R}, A \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), and the exponential map is the usual matrix exponential. One can easily obtains the Lie bracket as:

\[
[(\alpha_1, u_1, t_1, x_1), (\alpha_2, u_2, t_2, x_2)] =
(0, A(\alpha_1 u_2 - \alpha_2 u_1), 0, A(\alpha_1 x_2 - \alpha_2 x_1) + t_2 u_1 - t_1 u_2).
\] (2.9)

**B. Computation of \( H^2(\text{Lie}(\widetilde{G}_+^4), \mathbb{R}) \)**

As it is well known, to classify all multipliers of a Lie group \( G \) one has to compute first the second cohomology group of \( \text{Lie}(G) \) with real coefficients [3]. As we have said in the Introduction, we provide here an elementary derivation of this group. If \( \xi \in Z^2(\text{Lie}(\widetilde{G}_+^4), \mathbb{R}) \) the cocycle equation writes as:

\[
\xi([(\alpha_1, u_1, t_1, x_1), (\alpha_2, u_2, t_2, x_2)], (\alpha_3, u_3, t_3, x_3)) + \text{circular permutations} = 0.
\] (2.10)

Here \( \xi : \text{Lie}(\widetilde{G}_+^4) \times \text{Lie}(\widetilde{G}_+^4) \to \mathbb{R} \) is, by definition, bilinear and antisymmetric. Using (2.9), we can explicitate (2.10):

\[
\xi((0, A(\alpha_1 u_2 - \alpha_2 u_1), 0, A(\alpha_1 x_2 - \alpha_2 x_1) + t_2 u_1 - t_1 u_2), (\alpha_3, u_3, t_3, x_3)) +
\text{circular permutations} = 0.
\] (2.11)

We have to consider some distinct cases of this equation:

(i) \( t_i = 0, x_i = 0 \) (\( i = 1, 2, 3 \))

One obtains:

\[
\xi((0, A(\alpha_1 u_2 - \alpha_2 u_1), 0, 0)) + \text{circular permutations} = 0.
\] (2.12)
From bilinearity we have:

$$\xi((0, u, 0, 0), (0, u', 0, 0)) = u'Cu'$$

where $C$ is a $2 \times 2$ real matrix. From antisymmetry we find $C^t = -C$ so necessarily $C = \frac{1}{2} FA$ ($F \in \mathbb{R}$). So we have:

$$\xi((0, u, 0, 0), (0, u', 0, 0)) = \frac{1}{2} F < u, u'>$$  \hspace{1cm} (2.13)

where $<\cdot, \cdot>$ is the sesquilinear form on $\mathbb{R}^2$ given by:

$$<u, v> \equiv u^tAv$$  \hspace{1cm} (2.14)

It is easy to see that (2.12) becomes an identity.

(ii) $\alpha_i = 0, \ u_i = 0 \ (i = 1, 2, 3)$

Equation (2.11) becomes an identity.

(iii) $t_1 = 0, \ x_1 = 0, \ \alpha_i = 0, \ u_i = 0 \ (i = 1, 2)$

One easily obtains from (2.11):

$$\xi((0, 0, t, x), (0, 0, t, x')) = 0. \hspace{1cm} (2.15)$$

(iv) $t_i = 0, \ x_i = 0 \ (i = 1, 2), \ \alpha_3 = 0, \ u_3 = 0$

Equation (2.11) becomes:

$$\xi((0, A(\alpha_1u_2 - \alpha_2u_1), 0, 0), (0, 0, t_3, x_3)) +$$

$$\xi((0, 0, 0, 0, 0, 0), (0, 0, 0, 0)) +$$

$$\xi((0, 0, -\alpha_1Ax_3 - t_3u_2), (0, 0, 0, 0)) = 0. \hspace{1cm} (2.16)$$

From bilinearity one has:

$$\xi((0, u, 0, 0), (0, 0, 0, x)) = x^tDu$$  \hspace{1cm} (2.17)

where $D$ is a $2 \times 2$ real matrix.

If we take in (2.16) $\alpha_2 = 0, \ t_3 = 0$ and we insert the expression (2.17) we get

$$[D, A] = 0 \iff D = -\tau \times id + cA \ (\tau, c \in \mathbb{R})$$

so, (2.17) takes the form:

$$\xi((0, u, 0, 0), (0, 0, 0, x)) = \tau x \cdot u + c < x, u >. \hspace{1cm} (2.18)$$

If we insert (2.18) into (2.16) we get $c = 0$ and:

$$\xi((0, A'u, 0, 0), (0, 0, 1, 0)) + \xi((0, 0, 0, u), (1, 0, 0, 0)) = 0$$
Because of linearity we have:
\[
\xi((0, 0, 1, 0), (0, u, 0, 0)) = P \cdot u \quad (P \in \mathbb{R}^2)
\]  
(2.19)

and the preceding relation gives:
\[
\xi((0, 0, 0, x), (1, 0, 0, 0)) = < P, x >
\]  
(2.20)

Finally (2.18) reduces to:
\[
\xi((0, u, 0, 0), (0, 0, 0, x)) = \tau x \cdot u.
\]  
(2.21)

We denote:
\[
S \equiv \xi((1, 0, 0, 0), (0, 0, 1, 0))
\]  
(2.22)

and use linearity to obtain:
\[
\xi((1, 0, 0, 0), (0, u, 0, 0)) = < G, u > \quad (G \in \mathbb{R}^2).
\]  
(2.23)

Finally we can reconstruct the cocycle $\xi$ from (2.13), (2.15), (2.17) and (2.19)-(2.23):
\[
\xi(\alpha_1, u_1, t_1, x_1), (\alpha_2, u_2, t_2, x_2)) = < G, \alpha_1 u_2 - \alpha_2 u_1 > + \frac{1}{2} F < u_1, u_2 > + S(\alpha_1 t_2 - \alpha_2 t_1) - < P, A(\alpha_1 x_2 - \alpha_2 x_1) + t_2 u_1 - t_1 u_2 > + \tau(x_1 \cdot u_2 - x_2 \cdot u_2) \]  
(2.24)

If we take the generic element of $C^1(\text{Lie}(\tilde{G}_+^1), \mathbb{R}) \simeq (\text{Lie}(\tilde{G}_+^1))^*$ to be $(\beta, G, E, P)$ defined by:
\[
< (\beta, G, E, P), (\alpha, u, t, x) > = -\beta \alpha - G \cdot u - Et + P \cdot x
\]  
(2.25)

then it is elementary to see that (2.24) rewrites as:
\[
\xi = \tau \xi_0 + F\xi_1 + S\xi_2 + \partial (0, G, 0, P)
\]  
(2.26)

where $\xi_0, \xi_1, \xi_2 \in Z^2(\text{Lie}(\tilde{G}_+^1), \mathbb{R})$ are:
\[
\xi_0((\alpha_1, u_1, t_1, x_1), (\alpha_2, u_2, t_2, x_2)) = x_1 \cdot u_2 - x_2 \cdot u_1
\]  
(2.27)
\[
\xi_0((\alpha_1, u_1, t_1, x_1), (\alpha_2, u_2, t_2, x_2)) = \frac{1}{2} < u_1, u_2 >
\]  
(2.28)
\[
\xi_0((\alpha_1, u_1, t_1, x_1), (\alpha_2, u_2, t_2, x_2)) = \alpha_1 t_2 - \alpha_2 t_1
\]  
(2.29)

So every cocycle $\xi \in Z^2(\text{Lie}(\tilde{G}_+^1), \mathbb{R})$ is cohomologous with a cocycle of the form $\tau \xi_0 + F\xi_1 + S\xi_2$.

It is easy to establish now that this cocycle is not a coboundary. So we can summarize the preceding discussion as:

**Proposition 1:** $H^2(\text{Lie}(\tilde{G}_+^1), \mathbb{R})$ is a three dimensional real space. In every cohomology class there exists exactly one cocycle of the type $\tau \xi_0 + F\xi_1 + S\xi_2$. 

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C. Computation of $H^2(\tilde{G}^1_+, \mathbb{R})$

As in [3] one uses the fact that for $G$ a connected and simply connected Lie group, $H^2(G, \mathbb{R})$ is isomorphic to $H^2(Lie(G), \mathbb{R})$. So, to determine $H^2(G, \mathbb{R})$ one should determine for $\xi_0, \xi_1, \xi_2$ some corresponding group cocycles. One easily determines (see [2]):

\[
\omega_0(g, g') = \frac{1}{2} [a \cdot R(x)v' - v \cdot R(x)a' + t'v \cdot R(x)\nu'] \tag{2.30}
\]

\[
\omega_1(g, g') = \frac{1}{2} \langle v, R(x)v' \rangle \tag{2.31}
\]

\[
\omega_2 = \eta x' \tag{2.32}
\]

where: $g = (x, v, \eta, a)$, $g' = (x', v', \eta', a')$

So we have:

**Corollary 1:** $H^2(\tilde{G}^1_+, \mathbb{R})$ is a three dimensional real linear space. In every cohomology class there exists exactly one cocycle of the type $\tau \omega_0 + F \omega_1 + S \omega_2$.

Finally, applying again Thm. 7.37 of [3] we have:

**Corollary 2:** Every multiplier of $G^1_+$ is equivalent to a multiplier of the form $m_\tau m_F m_S$ where:

\[
m_\tau(g, g') \equiv \exp \left\{ \frac{i}{2} [a \cdot R(x)v' - v \cdot R(x)a' + t'v \cdot R(x)\nu'] \right\} \tag{2.33}
\]

\[
m_F(g, g') \equiv \exp \left\{ \frac{i}{2} \langle v, R(x)v' \rangle \right\} \tag{2.34}
\]

\[
m_S(g, g') \equiv e^{i\eta x'} \tag{2.35}
\]

**Remark 1:** In 1+3 dimensions it is well known that only a multiplier of type (2.33) survives. So, $m_F$ and $m_S$ are characteristic to 1+2 dimensions. According to the general theory of the projective unitary representations, one has to consider the most general multiplier, which in this case is $m_\tau m_F m_S$. In [4] only the multiplier $m_\tau$ was considered, so the statement of Thm. 2 is rather careless.

3 The projective unitary irreducible representations of the Galilei group in 1+2 dimensions

A. The method

As anticipated, we classify here the unitary irreducible $m_\tau m_F m_S$ representations of $G^1_+$. We try to mimick the method form [3] which consists of two steps:
a) One first applies a simple result, namely a generalization of Thm. 7.16 from [3].

**Proposition 2:** Let $m_0, \ldots, m_p$ be multipliers for the group $G$ and let $U : g \mapsto U_g$ be an $m_0 \cdots m_p$-representation of $G$ in a (separable) Hilbert $H$ space over the complex numbers. Let $G_{m_0, \ldots, m_p} \equiv G \times T \times \cdots \times T$ (where $T$ is the set of complex numbers of modulus 1 considered as a multiplicative group) with the composition law:

$$(g; \zeta_0, \ldots, \zeta_p) \cdot (g'; \zeta'_0, \ldots, \zeta'_p) = (gg'; \zeta_0 \zeta'_0 m_0(g, g'), \ldots, \zeta_p \zeta'_p m_p(g, g')) \quad (3.1)$$

Then $G_{m_0, \ldots, m_p}$ is a group, and:

$$(g; \zeta_0, \ldots, \zeta_p) \mapsto V_{g; \zeta_0, \ldots, \zeta_p} \equiv \zeta_0^{-1} \cdots \zeta_p^{-1} U_g \quad (3.2)$$

is a representation of $G_{m_0, \ldots, m_p}$ in $H$ such that:

$$V_{e; \zeta_0, \ldots, \zeta_p} = \zeta_0^{-1} \cdots \zeta_p^{-1} \times id \quad (3.3)$$

Conversely, if $V$ is a representation of $G_{m_0, \ldots, m_p}$ in $H$ such that (3.3) is verified, then if we write:

$$U_g \equiv V_g; 1, \ldots, 1 \quad (p+1)\text{-times} \quad (3.4)$$

$g \mapsto U_g$ is an $m_0 \cdots m_p$-representation $G$ in $H$ and the connection (3.2) between $U$ and $V$ is true.

The proof is elementary and Thm. 7.16 of [3] is the particular case $p = 0$.

Is is clear that we will apply Prop. 2 with $p = 2$, and $m_0 = m_{\tau}$, $m_1 = m_F$, $m_2 = m_S$. We will denote the the corresponding group $G_{m_0 m_1 m_2}$ by $G^{\tau,F,S}$.

b) We remember that in [3], where one has to study only $G^{\tau,0,0}$ (see ch. IX, section 8), a semi-direct product group structure is exhibited, so one can apply the induction procedure. The same procedure works for $G^{\tau,F,S}$. Indeed one has an group isomorphism $G^{\tau,F,S} \simeq H^F \times_{\tau,S} A^{\tau,S}$ where:

- Let us define the group $H \equiv \mathbb{R} \times \mathbb{R}^2$ with the composition law:

$$(x, \mathbf{v}) \cdot (x', \mathbf{v'}) = (x + x', \mathbf{v} + R(x)\mathbf{v'}) \quad (3.5);$$

then $H^F \equiv H \times T$ with the composition law:

$$(h_1; \zeta_1) \cdot (h_2; \zeta_2) = (h_1 h_2; \zeta_1 \zeta_1 m_F(h, h')) \quad (3.6)$$

The notation $m_F(h, h')$ makes sense because in the right hand side of (2.34) only the variables $x$ and $\mathbf{v}$ appear.

- $A^{\tau,S} \equiv \mathbb{R} \times \mathbb{R}^2 \times T \times T$ with the composition law:

$$(\eta, a; \zeta_0, \zeta_2) \cdot (\eta', a'; \zeta'_0, \zeta'_2) = (\eta + \eta', a + a'; \zeta_0 \zeta'_0, \zeta_2 \zeta'_2) \quad (3.7)$$
\(- t_{r,S} : H^F \to \text{Aut} \left( A^{r,S} \right)\) is given by:

\[
t^\tau_{h,\zeta_1} \left( a; \zeta_0, \zeta_2 \right) = \left( t_h(a); \zeta_0 \exp \left\{ - \frac{it}{2} \left[ 2v \cdot R(x)a + \eta v^2 \right] \right\}, \zeta_2 e^{-iS\eta x} \right)
\]

(3.8)

Here \(a \equiv (\eta, a)\) and

\[
t_h(\eta, a) = (\eta, R(x)a + \eta v)
\]

(3.9)

The semi-direct product structure follows from the homomorphism property of \(t_{r,S}\). Then the isomorphism is:

\[
H^F \times t_{r,S} A^{r,S} \ni \left\{ (x, v; \zeta_1), (\eta, a; \zeta_0, \zeta_2) \right\} \leftrightarrow
\left( x, v, \eta, a; \zeta_0 \exp \left( \frac{it}{2} a \cdot v \right), \zeta_1, \zeta_2 e^{iS\eta x} \right) \in G^{r,F,S}
\]

(3.10)

We note that \(A^{r,S}\) is Abelian, so we will be able to apply Mackey induction procedure.

**Remark 2:** One may wonder why we did not apply Prop. 2 with \(p = 0\) and \(m_0 = m_r m_F m_S\). The reason is that we did not succeed to find a convenient semi-direct product structure for \(G_{m_0}\) in this case.

Taking into account (3.10) the representations of \(G^{r,F,S}\) follow from representations of \(H^F \times t_{r,S} A^{r,S}\) according to:

\[
W(x_v, \eta, a; \zeta_0, \zeta_2) = W(x_v, \eta, a; \zeta_0 \exp \left\{ - \frac{it}{2} a \cdot v \right\}, \zeta_1, \zeta_2 e^{iS\eta x})
\]

(3.11)

According to Prop. 2 we are looking for unitary irreducible representations of \(G^{r,F,S}\) verifying:

\[
W_{e; \zeta_0, \zeta_1, \zeta_2} = \zeta_0^{-1} \zeta_1^{-1} \zeta_2^{-1} \times \text{id}
\]

(3.12)

Now we will follow the method of induced representations as presented in section II C of [1].

**B. Computation of the orbits**

It is clear that every character of \(A^{r,S}\) has the form:

\[
\chi_{p_0, p; n_0, n_2}(\eta, a; \zeta_0, \zeta_2) = \zeta_0^{n_0} \zeta_2^{n_2} \exp \left\{ i(\eta p_0 + a \cdot p) \right\}
\]

(3.13)

where \(p_0 \in \mathbb{R}, p \in \mathbb{R}^2\) and \(n_0, n_2 \in \mathbb{Z}\). So, \(\hat{A}^{r,S} \equiv \mathbb{R} \times \mathbb{R}^2 \times \mathbb{Z} \times \mathbb{Z}\) with the generic element denoted by \([p_0, p; n_0, n_2]\). One easily computes the dual action of \(t^\tau_{h,S}\), namely:

\[
(x_v; \zeta_1) \cdot [p_0, p; n_0, n_2] = [p_0 - v \cdot R(x)p - 1/2 n_0 \tau v^2 + Sn_2 x, R(x)p + n_0 \tau v; n_0, n_2]
\]

(3.14)
The classification of the orbits of this action is elementary. We distinguish
three cases which must be studied separately:

a) \( \tau \neq 0, \ S = 0 \)

The orbits are:

\[
Z^1_{\nu,0} \equiv \{[p_0,0;0,n_2]; \ p_0 \in \mathbb{R}, \ n_2 \in \mathbb{Z} \}
\]

\[
Z^2_{\nu,0} \equiv \{[p_0,p;0,n_2]|p^2 = \tau^2, \ p_0 \in \mathbb{R}]; \ r \in \mathbb{R}_+, \ n_2 \in \mathbb{Z} \}
\]

\[
Z^3_{\nu,0,n_2,\rho} \equiv \{[p_0,p;n_0,n_2]|p^2 + 2n_0 \tau p_0 = \rho]; \ \rho \in \mathbb{R}, \ n_0 \in \mathbb{Z}^*, \ n_2 \in \mathbb{Z} \}
\]

b) \( \tau = 0, \ S \neq 0 \)

\[
Z^4_{\nu,0} \equiv \{[p_0,0;0,n_0]; \ p_0 \in \mathbb{R}, \ n_0 \in \mathbb{Z} \}
\]

\[
Z^5_{\nu,0,n_2} \equiv \{[p_0,0;n_0,n_2]|p_0 \in \mathbb{R}]; \ n_0 \in \mathbb{Z}, \ n_2 \in \mathbb{Z}^* \}
\]

\[
Z^6_{\nu,0,n_2,\rho} \equiv \{[p_0,p;n_0,n_2]|p^2 = \tau^2, \ p_0 \in \mathbb{R}]; \ r \in \mathbb{R}_+, \ n_0,n_2 \in \mathbb{Z} \}
\]

c) \( \tau \neq 0, \ S \neq 0 \)

\[
Z^7_{\nu} \equiv \{[p_0,0;0,0]; \ p_0 \in \mathbb{R} \}
\]

\[
Z^8_{\nu} \equiv \{[p_0,0;0,n_2]|p_0 \in \mathbb{R}]; \ n_2 \in \mathbb{Z}^* \}
\]

\[
Z^9_{\nu,\rho} \equiv \{[p_0,p;0,n_0]|p^2 + 2n_0 \tau p_0 = \rho]; \ \rho \in \mathbb{R}, \ n_0 \in \mathbb{Z}^* \}
\]

\[
Z^{10}_{\nu,n_2} \equiv \{[p_0,p;n_0,n_2]|p \in \mathbb{R}^2, p_0 \in \mathbb{R}]; \ n_0,n_2 \in \mathbb{Z}^* \}
\]

It is not hard to see that condition (3.12) cannot be fulfilled by the induced
representations corresponding to the orbits: \( Z^1, Z^2, Z^4, Z^7, Z^8 \) and \( Z^9 \). So, we
have to analyse only the cases \( Z^3, Z^5, Z^6 \) and \( Z^{10} \). We analyse one by one
these four cases.

C. The representations

a) \( \tau \neq 0, \ S = 0 \)

In this case, \( Z = Z^3_{\nu,0,n_2,\rho} \). A reference point is \( \left[ \frac{\nu}{2n_0 \tau}, 0; n_0, n_2 \right] \) and we have:

\[
H^F_{\nu \cdot 0; n_0, n_2} \simeq \{ (x,0; \zeta_1|x \in \mathbb{R}, \ \zeta_1 \in T) \} \simeq \mathbb{R} \times T
\]

The unitary irreducible representations of this Abelian subgroup are of the
form \( \pi^{(s,n_1)}(s \in \mathbb{R}, \ n_1 \in \mathbb{Z}) \); they are one-dimensional, are acting in \( \mathbb{C} \) as follows:

\[
\pi^{(s,n_1)}(x,0; \zeta_1) = e^{isx} \zeta_1^{n_1}
\]

(3.15)

In this case one can find explicitly a corresponding cocycle \( \phi^\pi \), namely:

\[
\phi^{(s,n_1)}((x,v; \zeta_1), [p_0,p;n_0,n_2]) = e^{isx} \zeta_1^{n_1} \exp \left\{ \frac{im_1 x}{2n_0 \tau} < v, R(x) p > \right\}
\]

(3.16)
As in [3] we identify \( Z^3 \simeq \mathbb{R}^2 \) according to:

\[
[(\rho - \mathbf{p}^2)/2n_0\tau, \mathbf{p}; n_0, n_2] \leftrightarrow \mathbf{p}
\]

and we consider the strictly invariant Lebesgue measure \( d\mathbf{p} \) on \( \mathbb{R}^2 \).

Applying (3.11), the corresponding induced representation is acting in \( \mathcal{H} = L^2(\mathbb{R}^2, d\mathbf{p}) \) according to:

\[
(W_{x,v,\eta,a}(\zeta_0, \zeta_1))f)(\mathbf{p}) =
\]

\[
\zeta_0^n_0\zeta_1^n_1\zeta_2^n_2\exp \left\{ i \left[ -\frac{n_0\tau}{2} \mathbf{a} \cdot \mathbf{v} + \eta \rho - \frac{\mathbf{p}^2}{2n_0\tau} + \mathbf{a} \cdot \mathbf{p} + sx + \frac{n_1F}{2n_0\tau} < \mathbf{v}, \mathbf{p} > \right] \right\}
\]

\[
f(R(x)^{-1}(\mathbf{p} - n_0\tau\mathbf{v})
\]

The condition (3.12) imposes \( n_0 = n_1 = n_2 = -1 \). The factor \( \exp \left( -\frac{i\mathbf{p}^2}{2} \right) \) can be dropped because we are looking for projective representations. We get the projective representations \( V^{\tau,s} \) \((\tau \in \mathbb{R}^*, s \in \mathbb{R})\) acting in \( L^2(\mathbb{R}^2, d\mathbf{p}) \) as follows:

\[
(V^{\tau,s}_{x,v,\eta,a}f)(\mathbf{p}) =
\]

\[
\exp \left\{ i \left( sx + \mathbf{a} \cdot \mathbf{p} + \frac{\eta\mathbf{p}^2}{2\tau} + \frac{\tau}{2} \mathbf{a} \cdot \mathbf{v} + \frac{F}{2\tau} < \mathbf{v}, \mathbf{p} > \right) \right\} f(R(x)^{-1}(\mathbf{p} + \tau\mathbf{v})).
\]

(3.17)

\[\]

b) \( \tau = 0, \ S \neq 0 \)

b1) \( Z = Z^5_{n_0,n_2} \).

A reference point is \([0,0; n_0, n_2]\) and we have:

\[
H^F_{[0,0;n_0,n_2]} = \{(0, \mathbf{v}; \zeta_1) | \mathbf{v} \in \mathbb{R}^2, \zeta_1 \in \mathbb{T}\}
\]

i.e. a central extension of the Abelian group \( \mathbb{R}^2 \). Let \( \pi \) be an unitary irreducible representation of this group. Because of (3.12) we must have:

\[
\pi_{(0,0;\zeta_1)} = \zeta_1^{-1} \times \text{id}
\]

(3.18)

This easily implies that we have:

\[
\pi_{(0,v;1)}\pi_{(0,v';1)} = \exp \left\{ -\frac{iF}{2} < \mathbf{v}, \mathbf{v}' > \right\} \pi_{(0,v+v';1)}
\]

(3.19)

i.e. \( \mathbf{v} \mapsto \pi_{(0,v;1)} \) is an unitary irreducible representation of the canonical commutation relations in Weyl form. According to Stone-von Neumann theorem there exists (up to unitary equivalence) exactly one such representation, denoted \( \pi^{CCR} \) and acting in the Hilbert space \( \mathcal{K} \) (for an explicit expression see e.g. [4], ch. 3.1). So, the representations of \( H^F_{[0,0;n_0,n_2]} \) we are looking for are of the form:

\[
\pi_{(0,v;\zeta_1)} = \zeta_1^{-1}\pi^{CCR}_{\mathbf{v}}.
\]

(3.20)
A corresponding cocycle is clearly:

$$\phi^{CCR}(x, v; \zeta_1) = \zeta_1^{-1} \pi^v_{CCR}$$  \hspace{1cm} (3.21)

If we identify naturally $\mathbb{Z}^5 \simeq \mathbb{R}$ with the strictly invariant measure $dp_0$, then the corresponding induced representation is acting in $\mathcal{H} = L^2(\mathbb{R}, \mathcal{K}, dp_0)$ as follows:

$$\left( W_{x,v;\eta,a}(p) \right)(p_0) = \zeta_0^n \left( z e^{-is\eta x} \right)^{n_2} e^{inp_0} \zeta_1^{-1} \left( \pi^v_{CCR} f \right)(p_0 - Sn_2x)$$  \hspace{1cm} (3.22)

Again (3.12) imposes $n_0 = n_2 = -1$ and we are left with the projective representations $V^{CCR}$ acting in $\mathcal{H} = L^2(\mathbb{R}, \mathcal{K}, dp_0)$ according to:

$$\left( V^{CCR}_{x,v;\eta,a} f \right)(p_0) = e^{in(p_0 + Sx)} \left( \pi^v_{CCR} f \right)(p_0 + Sx)$$  \hspace{1cm} (3.23)

b2) $Z = Z^6_{n_0,n_2,p}$.

A reference point is $[0, re_1; n_0, n_2]$ and one easily obtains:

$$H^{F}_{[0,re_1;n_0,n_2]} = \left\{ (2\pi n, \frac{2\pi nSn_2}{r} e_1 + \alpha e_2; \zeta_1) \mid n \in \mathbb{Z}, \alpha \in \mathbb{R}, \zeta \in \mathbb{T} \right\}$$

If we denote:

$$(n, \alpha; \zeta_1) \equiv \left( 2\pi n, \frac{2\pi nSn_2}{r} e_1 + \alpha e_2; \zeta_1 \right)$$

then the composition law is:

$$(n, \alpha; \zeta_1) \cdot (n', \alpha'; \zeta_1') = (n + n', \alpha + \alpha'; \zeta_1\zeta_1'exp\{i(k(\alpha n' - \alpha' n))\})$$  \hspace{1cm} (3.24)

i.e. the little group is in this case a central extension of the Abelian group $\mathbb{Z} \times \mathbb{R}$. Here $k = \frac{2\pi FSn_2}{r}$. Let $(n, \alpha; \zeta_1) \mapsto \pi_{(n,\alpha;\zeta_1)}$ be an unitary irreducible representation fulfilling:

$$\pi_{(n_0,0;\zeta_1)} = \zeta_1^{-1} \times id$$  \hspace{1cm} (3.25)

(The argument leading to this relation is similar to the argument leading to (3.18)).

Then we have for $\pi_{n,\alpha} \equiv \pi_{n,\alpha;1}$:

$$\pi_{n,\alpha} \pi_{n',\alpha'} = exp\{i(k(\alpha n' - \alpha' n))\} \pi_{n+n',\alpha+\alpha'}$$  \hspace{1cm} (3.26)

The resemblance of this relation to the Weyl system (3.19) suggests us to associate to (3.26) a sort of imprimitivity system $[\mathfrak{3}]$. We denote: $\pi_{1,0} \equiv V$ and $\pi_{\alpha} = \pi_{0,\alpha}$. It is clear that (3.26) is equivalent to:

$$\pi_{n,0} = V^n; \ V\pi_{\alpha}V^{-1} = e^{2\pi k}\pi_{\alpha}; \ \pi_{\alpha}\pi_{\alpha'} = \pi_{\alpha+\alpha'}$$  \hspace{1cm} (3.27)
So, it is sufficient to find an irreducible couple \((V, \pi_\alpha)\) fulfilling the last two relations of (3.27). According to GNS theorem \(\pi_\alpha\) is of the form:

\[
\pi_\alpha = \int_{\mathbb{R}} e^{i\lambda \alpha} dP(\lambda)
\]

where \(\Delta \mapsto P_\Delta\) is a projection valued measure in the Hilbert space \(\mathcal{K}\). The second relation (3.27) is equivalent to:

\[
VP_\Delta V^{-1} = P_{\Delta-2k}
\]

Applying the same idea as in [3] Lemma 6.10, we can prove that \(P\) is homogeneous i.e. \(\mathcal{K} = L^2(\mathbb{R}, C^n, \beta)\) where \(\beta\) is measure on \(\mathbb{R}\) quasi-invariant with respect to the transformation \(\lambda \mapsto \lambda + 2k\) and

\[
(P_\Delta f)(\lambda) = \chi_\Delta(\lambda)f(\lambda)
\]

Moreover, like in [3] Thm. 6.12 one can show that \(V\) has the ”diagonal” expression:

\[
(Vf)(\lambda) = r_V(\lambda + 2k)^{1/2}v(\lambda)(\lambda + 2k)
\]

(here \(r_V\) is a Radon-Nycodim derivative).

It is easy to see that for \(n > 1\) the system \((V, P_\Delta)\) is not irreducible. So, \(n = 1\). Moreover one can show that \(supp(\beta)\) must be discrete with period \(2k\) i.e. \(supp(\beta) = \{\lambda + 2km | m \in \mathbb{Z}\}\), \(\lambda \in \mathbb{R}\).

It is clear that we can take \(\beta = \sum_{m \in \mathbb{Z}} \delta_{\lambda+2km}\). In this case one can also take \(\mathcal{K} = l^2\), (i.e. sequences \(\{f_m\}_{m \in \mathbb{Z}}\) such that \(\sum_{m \in \mathbb{Z}} |f_m|^2 < \infty\) and \(P = \beta\). It follows that we have:

\[
(\pi_\alpha f)_m = e^{i(\lambda+2km)\alpha} f_m; \quad (Vf)_m = v_m f_{m+1} \quad (\forall m \in \mathbb{Z})
\]

(Here \(v_m \in T\)).

The system \((V, \pi_\alpha)\) is irreducible. So, the unitary irreducible representations of the little group \(H_{[0, re_1, re_2]}\) verifying (3.25) are of the form \(\pi^\lambda (\lambda \in \mathbb{R})\) acting in \(l^2\) according to:

\[
(\pi^\lambda_{n, \alpha, \xi_1} f)_m = \zeta_1^{-1} e^{i[\alpha + k(2m + n)]} f_{m+n}
\]

We denote by \(\phi^\lambda\) a cocycle corresponding to \(\pi^\lambda\). (We have not been able to find a simple expression for this cocycle). Then, if we identify \(Z^6 \simeq \mathbb{R} \times S^1\) with the strictly invariant measure \(d\mu_0 d\Omega\), we get in the end the projective representation \(V^{r,\lambda}\) acting in \(L^2(\mathbb{R} \times S^1, l^2, d\mu_0 d\Omega)\) according to:

\[
(V^{r,\lambda}_{x, v, \eta, a} f) (p_0, p) = \exp\{i[\eta(p_0 + Sx) + a \cdot p]\} \times \\
\phi^\lambda((x, v; 1), [p_0 + v \cdot p + Sx, R(x)^{-1}p]) f(p_0 + v \cdot p + Sx, R(x)^{-1}p)
\]

(3.32)
c) $\tau \neq 0$, $S \neq 0$

Only the orbit $Z_{n_0, n_2}^{10}$ is involved. A reference point is $[0, 0; n_0, n_2]$ and we have:

$$H_{[0, 0; n_0, n_2]}^F = \{(0, 0; \zeta_1) | \zeta_1 \in T \} \simeq T$$

with the unitary irreducible representations of the form $\pi_{n_1}^n$ acting in $C$ according to:

$$\pi_{(0, 0; \zeta_1)}^{n_1} = \zeta_1^{n_1} \quad (3.33)$$

A corresponding cocycle can be found:

$$\phi_{n_1}^n((x, v; \zeta_1), [p_0, p; n_0, n_2]) = \zeta_1^{n_1} \exp \left\{ \frac{iF_{n_1}}{2n_0\tau} < v, R(x)p > \right\} \quad (3.34)$$

We identify $Z^{10} \simeq \mathbb{R} \times \mathbb{R}^2$ with the strictly invariant measure $dp_0dp$. The induced representations obey (3.12) iff $n_0 = n_1 = n_2 = -1$ and we get projective representations $V^\tau$ acting in $L^2(\mathbb{R} \times \mathbb{R}^2, dp_0dp)$ according to:

$$(V^\tau_{x,v,\eta,a}f)(p_0, p) = \exp \left\{ i \left[ \eta(p_0 + Sx) + \frac{\tau}{2} a \cdot v + a \cdot p + \frac{F}{2\tau} < v, p > \right] \right\}$$

$$f(p_0 + Sx + v \cdot p + \frac{1}{2} \tau v^2, R(x)^{-1}(p + \tau v)). \quad (3.35)$$

So every projective unitary irreducible representation of $G_+^T$ with a non-trivial multiplicator is unitary equivalent to a representation of the type $V^{\tau,s}$, $V^{CCR}$, $V^{\tau,\lambda}$, or $V^\tau$ described by (3.17), (3.23), (3.32) and (3.35) respectively.

### D. Infinitesimal generators

The representations (3.17) obtained in case a) are the analogues of the representations obtained in 1+3 dimensions (see [3], ch. IX, subsec. 8). The infinitesimal generators of this representation are:

$$(Hf)(p) = \frac{p_0^2}{2\tau} f \quad (3.36)$$

$$(Pf)(p) = pf \quad (3.37)$$

$$(Jf)(p) = i \left( p_1 \frac{\partial f}{\partial p_2} - p_2 \frac{\partial f}{\partial p_1} \right) + sf \quad (3.38)$$

$$(Kf)(p) = -i \tau \frac{\partial f}{\partial p} + \frac{F}{2\tau} Apf \quad (3.39)$$

So $\tau$ and $s$ must be interpreted as the mass and the spin of the system respectively. The interpretation of $F$ is not clear. Because $F$ appears from a central extension of a translation group (see (3.19)), it is tempting to associate...
$F$ with some kind of magnetic force. By restricting to the covering group of the Euclidean group $SE(2)$ we obtain:

$$(V^\tau_{x,a} f)(p) = e^{i(sx+a \cdot p)} f(R(x)^{-1} p)$$

(3.40)

so performing a Fourier transform we can conclude that the system is localisable on $\mathbb{R}^2$:

$$(V^\tau_{x,a} f)(X) = e^{isx} f(R(x)^{-1}(X-a))$$

(3.41)

The representations (3.23) and (3.32) from case b) are not localisable on $\mathbb{R}^2$ (the argument is similar to the one in [3] and is based on the existence of the constrain $p^2 = r^2$).

In case c) the infinitesimal generators are:

$$H f = p_0 f$$

(3.42)

$$P f = p f$$

(3.43)

$$J f = i \left( p_1 \frac{\partial f}{\partial p_2} - p_2 \frac{\partial f}{\partial p_1} - S \frac{\partial f}{\partial p_0} \right)$$

(3.44)

$$K f = -i \left( \tau \frac{\partial f}{\partial p} + p \frac{\partial f}{\partial p_0} \right) + \frac{F}{2\tau} \Delta p f$$

(3.45)

So in this case $S$ can be interpreted as the spin of the system. For $\tau$ and $F$ it is natural to keep the interpretations from case a).

By restricting to $SE(2)$ we obtain:

$$(V^\tau_{x,a} f)(p_0, p) = e^{ia \cdot p} (p_0 + Sx, R(x)^{-1} p))$$

(3.46)

By performing a three-dimensional Fourier transform we get:

$$(V^\tau_{x,a} f)(X_0, X) = e^{iSx_0} f(X_0, R(x)^{-1}(X-a))$$

(3.47)

The formula shows that the system is localisable on $\mathbb{R}^2$.

The appearance of the representation $V^\tau$ is rather intriguing. Indeed in the relativistic case [1], there exists a single class of projective representation of the Poincaré group corresponding to non-zero mass systems and one would expect that in the non-relativistic limit we obtain only the representations from case a) (like in 1+3 dimensions).

Finally, we note that the true representations of $G^+_\perp$ are easy to find (see [1] Thm. 2) but they do not correspond to localisable systems.
References

[1] D. R. Grigore, ”The Projective Unitary Irreducible Representations of the Poincaré Group in 1+2 Dimensions”, Journ. Math. Phys. 34 (1993) 4172-4189

[2] J. Mund, R. Schrader, ”Hilbert Space for Non-relativistic and Relativistic “Free” Plektons (Particles with Braid Group Statistics)”, preprint Univ. Berlin, SFB 288 no. 74

[3] V. S. Varadarajan, “Geometry of Quantum Theory” (second edition), Springer, New York, 1985

[4] W. Thirring, ”Quantum Mechanics of Atoms and Molecules”, Springer, New York, 1979