ON KERNELS OF TOEPLITZ OPERATORS

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Abstract. We apply the theory of de Branges-Rovnyak spaces to describe kernels of some Toeplitz operators on the classical Hardy space $H^2$. In particular, we discuss the kernels of the operators $T_{\bar{f}/f}$ and $T_{I\bar{f}/f}$, where $f$ is an outer function in $H^2$ and $I$ is inner such that $I(0) = 0$. We also obtain results on de Branges-Rovnyak spaces generated by nonextreme functions.

1. Introduction

Let $H^2$ denote the standard Hardy space on the unit disk $\mathbb{D}$. For $\varphi \in L^\infty(\partial \mathbb{D})$ the Toeplitz operator on $H^2$ is given by $T_\varphi f = P_+ (\varphi f)$, where $P_+$ is the orthogonal projection of $L^2(\partial \mathbb{D})$ onto $H^2$. We will denote by $\mathcal{M}(\varphi)$ the range of $T_\varphi$ equipped with the range norm, that is, the norm that makes the operator $T_\varphi$ a coisometry of $H^2$ onto $\mathcal{M}(\varphi)$. For a nonconstant function $b$ in the unit ball of $H^\infty$ the de Branges-Rovnyak space $\mathcal{H}(b)$ is the image of $H^2$ under the operator $(1 - T_b T_{\bar{b}})^{1/2}$ with the corresponding range norm. The norm and the inner product in $\mathcal{H}(b)$ will be denoted by $\| \cdot \|_b$ and $\langle \cdot, \cdot \rangle_b$. The space $\mathcal{H}(b)$ is a Hilbert space with the reproducing kernel

$$k^b_w(z) = \frac{1 - b(w)b(z)}{1 - wz} \quad (z, w \in \mathbb{D}).$$

In the case when $b$ is an inner function the space $\mathcal{H}(b)$ is the well-known model space $K_b = H^2 \ominus bH^2$.

If the function $b$ fails to be an extreme point of the unit ball in $H^\infty$, that is, when $\log(1 - |b|) \in L^1(\mathbb{T})$, we will say simply that $b$ is nonextreme. In this case one can define an outer function $a$ whose modulus on $\partial \mathbb{D}$ equals $(1 - |b|^2)^{1/2}$. Then we say that the functions $b$ and $a$ form a pair $(b, a)$. By the Herglotz representation theorem there exists a positive measure $\mu$ on $\partial \mathbb{D}$ such that

$$\frac{1 + b(z)}{1 - b(z)} = \int_{\partial \mathbb{D}} \frac{1 + e^{-i\theta}z}{1 - e^{-i\theta}z} d\mu(e^{i\theta}) + i \text{Im} \frac{1 + b(0)}{1 - b(0)}, \quad z \in \mathbb{D}. \tag{1}$$

Moreover the function $|a/b|^2$ is the Radon-Nikodym derivative of the absolutely continuous component of $\mu$ with respect to the normalized Lebesgue measure. If the measure $\mu$ is absolutely continuous the pair $(b, a)$ is called special.
Recall that a function $f \in H^1$ is called rigid if and only if no other functions in $H^1$, except for positive scalar multiples of $f$ have the same argument as $f$ a.e. on $\partial \mathbb{D}$.

If $(b, a)$ is a pair, then $\mathcal{M}(a)$ is contained contractively in $\mathcal{H}(b)$. If a pair $(b, a)$ is special and $f = \frac{a}{1 - b}$, then $\mathcal{M}(a)$ is dense in $\mathcal{H}(b)$ if and only if $f^2$ is rigid (see [18]). Spaces $\mathcal{H}(b)$ for nonextreme $b$ have been studied in [1], [2], [3], [13], [14], [20], and [21].

It is known that the kernel of a Toeplitz operator $T_\varphi$ is a subspace of $H^2$ of the form $\ker T_\varphi = fK_I$, where $K_I = H^2 \ominus IH^2$ is the model space corresponding to the inner function $I$ such that $I(0) = 0$ and $f$ is an outer function of unit $H^2$ norm that acts as an isometric multiplier from $K_I$ onto $fK_I$. Moreover, $f$ can be expressed as $f = \frac{a}{1 - b}$, where $(b_0, a)$ is a special pair and $(\frac{a}{1 - b_0})^2$ is a rigid function in $H^1$. Then we also have $\ker T_\frac{a}{1 - b} = fK_I$. In the recent paper [5], the authors considered the Toeplitz operator $T_\frac{a}{1 - b}$ where $g \in H^\infty$ is outer. Among other results, they described all outer functions $g$ such that $\ker T_\frac{a}{1 - b} = K_I$. In Section 2 we describe all such functions $g$ for which $\ker T_\frac{a}{1 - b} = fK_I$.

If $(b, a)$ is a special pair, $f = \frac{a}{1 - b}$ and $b = Ib_0$, where $I$ as above, then $fK_I \subset \ker T_\frac{a}{1 - b}$. In the next two sections we study the space $\ker T_\frac{a}{1 - b} \ominus fK_I$ and show that it is isometrically isomorphic to the orthogonal complement of $\mathcal{M}(a)$ in the de Branges-Rovnyak space $\mathcal{H}(b_0)$. We also give an example of a function $f$ for which the space $\ker T_\frac{a}{1 - b} \ominus fK_I$ is one dimensional. In the last section we discuss the orthogonal complement of $\mathcal{M}(a)$ in $\mathcal{H}(b)$ and get a generalization of results obtained in [13] and [4] for the case when pairs are rational.

2. The kernel of $T_\frac{a}{1 - b}$

It is known that if $g$ is an outer function in $H^2$, then the kernel of $T_\frac{a}{1 - b}$ is trivial if and only if $g^2$ is rigid (see e.g. [16]).

The finite dimensional kernels of Toeplitz operators were described by Nakazi [15]. Nakazi’s theorem says that $\dim \ker T_\varphi = n$ if and only if there exists an outer function $f \in H^2$ such that $f^2$ is rigid and $\ker T_\varphi = \{fp: p \in \mathcal{P}_{n-1}\}$, where $\mathcal{P}_{n-1}$ denotes the set of all polynomials of degree at most $n - 1$.

Consider the following example.

**Example.** For $\alpha > -\frac{1}{2}$ set $g(z) = (1 - z)^\alpha$, $z \in \mathbb{D}$. Then the kernel of $T_\frac{a}{1 - b}$ is trivial for $\alpha \in (-\frac{1}{2}, \frac{1}{2}]$ and dimension of the kernel of $T_\frac{a}{1 - b}$ is $n$ for $\alpha \in (n - \frac{1}{2}, n + \frac{1}{2}]$, $n = 1, 2, \ldots$ , and

$$
\ker T_{\frac{a}{1 - b}}(\frac{1 - z}{1 - \bar{z}}) = (1 - z)^{\alpha - n}K_{z^n}.
$$

In the general case the kernels of Toeplitz operators are characterized by Hayashi’s theorem. To state this theorem we need some notation. We note that an outer function $f$ having unit norm in $H^2$ ($\|f\|_2 = 1$) can be written as

$$
f = \frac{a}{1 - b},
$$

where $(b_0, a)$ is a special pair and $(\frac{a}{1 - b_0})^2$ is a rigid function in $H^1$. Then we also have $\ker T_\frac{a}{1 - b} = fK_I$. In the recent paper [5], the authors considered the Toeplitz operator $T_\frac{a}{1 - b}$ where $g \in H^\infty$ is outer. Among other results, they described all outer functions $g$ such that $\ker T_\frac{a}{1 - b} = K_I$. In Section 2 we describe all such functions $g$ for which $\ker T_\frac{a}{1 - b} = fK_I$.
where $a$ is an outer function, $b$ is a function from the unit ball of $H^\infty$ such that $|a|^2 + |b|^2 = 1$ a.e. on $\partial\mathbb{D}$. Following Sarason [18, p. 156] we call $(b, a)$ the pair associated with $f$. Note also that $b$ is a nonextreme point of the closed unit ball of $H^\infty$ and is given by

\begin{equation}
1 + b(z) \over 1 - b(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + e^{-i\theta}z}{1 - e^{-i\theta}z} |f(e^{i\theta})|^2 d\theta, \quad z \in \mathbb{D}.
\end{equation}

Let $S$ denote the unilateral shift operator on $H^2$, i.e. $S = T_z$. A closed subspace $M$ of $H^2$ is said to be nearly $S^*$-invariant if for every $f \in M$ vanishing at 0, we also have $S^* f \in M$. It is known that the kernels of Toeplitz operators are nearly $S^*$-invariant.

Nearly $S^*$-invariant spaces are characterized by Hitt’s theorem [12].

**Hitt’s Theorem.** The closed subspace $M$ of $H^2$ is nearly $S^*$-invariant if and only if there exists a function $f$ of unit norm and a model space $K_I = H^2 \ominus IH^2$ such that $M = T_f K_I$, where $I$ is an inner function vanishing at the origin, and $T_f$ acts isometrically on $K_I$.

It has been proved by D. Sarason [16] that $T_f$ acts isometrically on $K_I$ if and only if $I$ divides $b$ (the first function in the pair associated with $f$). Consequently, the function $f$ in Hitt’s theorem can be written as

\[ f = \frac{a}{1 - Ib_0}. \]

The function $\frac{1 + b(z)}{1 - b(z)}$ has a positive real part and is the Herglotz integral of a positive measure on $\partial\mathbb{D}$ up to an additive imaginary constant,

\begin{equation}
\frac{1 + b(\zeta)}{1 - b(\zeta)} = \int_{\partial\mathbb{D}} \frac{1 + e^{-i\theta}z}{1 - e^{-i\theta}z} d\mu(e^{i\theta}) + ic.
\end{equation}

Clearly $b_0$ is also a nonextreme point of the closed unit ball of $H^\infty$ and $|a|^2 + |b_0|^2 = 1$ a.e. on $\partial\mathbb{D}$.

We remark that in view of (2) the pair $(b, a)$ associated with an outer function $f \in H^2$ is special, while the pair $(b_0, a)$ need not to be special. Under the above notations Hayashi’s theorem reads as follows:

**Hayashi’s Theorem.** The nearly $S^*$-invariant space $M = T_f K_I$ is the kernel of a Toeplitz operator if and only if the pair $(b_0, a)$ is special and $f_0^2 = \left( \frac{a}{1 - b_0} \right)^2$ is a rigid function.

Moreover, it follows from Sarason’s proof of Hayashi’s theorem that if $M = T_f K_I$ is the kernel of a Toeplitz operator then it is the kernel of $T_{\bar{f}}$.

Recently E. Fricain, A. Hartmann and W. T. Ross [3] considered the Toeplitz operators $T_g$ where $g \in H^\infty$ is outer. If $\ker T_g$ is non-trivial, then by Hayashi’s theorem there exist the outer function $f$ and the inner function $I$, $I(0) = 0$, such that

\[ \ker T_g = f K_I. \]

In the above mentioned paper [3] the authors described all outer functions $g \in H^\infty$ for which

\[ \ker T_g = K_I, \]
where $I$ is an inner function not necessarily satisfying $I(0) = 0$.

We prove the following

**Theorem 1.** Assume that $g \in H^2$ is outer and $M = T_f K_I$ is the nearly $S^*$-invariant space, where $I$ is an inner function such that $I(0) = 0$, $(b_0, a)$ is the pair associated with the outer function $f$, $(b_0, a)$ is special, and $f_0^2 = \left(\frac{a}{1-b_0}\right)^2$ is rigid. Then $\ker T_\varphi = M$ if and only if

$$g = i \frac{I_1 + I_2}{I_1 - I_2} (1 + I) f,$$

where $I_1$ and $I_2$ are inner and $I_1 - I_2$ is outer.

Recall that the Smirnov class $\mathcal{N}^+$ consists of those holomorphic functions in $D$ that are quotients of functions in $H^\infty$ in which the denominators are outer functions.

In the proof of Theorem 1, similarly to [5], we use the following result due to H. Helson [11].

**Helson’s Theorem.** The functions $f \in \mathcal{N}^+$ that are real almost everywhere on $\partial D$ can be written as

$$f = i \frac{I_1 + I_2}{I_1 - I_2},$$

where $I_1$ and $I_2$ are inner and $I_1 - I_2$ is outer.

We also apply a description of kernels in terms of $S^*$-invariant subspaces $K_\varphi^2(\|f\|^2)$ of weighted Hardy spaces (in the case when $p = 2$) considered by A. Hartmann and K. Seip in their paper [8] (see also [6]). For an outer function $f$ in $H^2$ the weighted Hardy space is defined as follows

$$H^2(\|f\|^2) = \{g \in \mathcal{N}^+: \|g\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} |g(e^{it})|^2 |f(e^{it})|^2 dt < \infty\}$$

and, for an inner function $I$, $K_I^2(\|f\|^2) = K_I(\|f\|^2)$ is given by

$$K_I(\|f\|^2) = \{g = I \psi \in H^2(\|f\|^2): \psi \in H_0^2(\|f\|^2)\},$$

where $H_0^2(\|f\|^2) = z H^2(\|f\|^2)$.

Then $K_I(\|f\|^2)$ is $S^*$-invariant and $f K_I(\|f\|^2) = \ker T_{\overline{f}}$ (see [8]).

**Proof of Theorem 1.** Assume that $\ker T_{\overline{f}} = f K_I$. Then

$$f K_I = \ker T_{\overline{f}} = \ker T_{\overline{\varphi}}.$$
constant. Replacing $cI_0$ by $I_0$, we get

$$\bar{g} = \bar{T}_0 \bar{f}. \quad (4)$$

It then follows

$$fK_I = \ker T_\bar{g} = \ker T_{\bar{\tau}_f} = fK_{I_0}(\lvert f \rvert^2),$$

which implies $I = I_0$ up to a unimodular constant. Indeed, these equalities imply that an analytic function $h$ can be written in the form $h = fI_0\psi_0$, where $\psi_0 \in H_0^2(\lvert f \rvert^2)$, if and only if $h = f\bar{I}\psi$, where $\psi \in H_0^2$. Since $\lvert \psi_0 \rvert = \lvert \psi \rvert$ a.e. on $\lvert z \rvert = 1$ and $\psi_0 \in N^+$, we see that also $\psi_0 \in H_0^2$. Hence $K_I = K_{I_0}$.

Consequently, equality (4) can be written as

$$\bar{g} = \frac{f(1 + I)}{f(1 + I)} \quad \text{a.e. on } \partial \mathbb{D},$$

which means that the function $\frac{g}{f(1 + I)}$ is real a.e. on $\partial \mathbb{D}$. Since this function is in the Smirnov class $N^+$, our claim follows from Helson’s theorem. To prove the other implication it is enough to observe that if

$$g = i\frac{I_1 + I_2}{I_1 - I_2}(1 + I)f,$$

then

$$\bar{g} = \frac{\bar{T}_f}{\bar{f}}.$$

\[\square\]

3. The complement of $fK_I$ in $\ker T_{\bar{\tau}_{\bar{f}}}$

It was noticed in [3, vol. 2, Cor. 30.21] that if $f$ is an outer function of the unit norm, $(b, a)$ is the pair associated with $f$, and $I$ is an inner function vanishing at the origin that divides $b$, then

$$fK_I \subset \ker T_{\bar{\tau}_{\bar{f}}}$$

and, according to Hayashi’s theorem, the equality holds if and only if the pair $(b_0, a)$ is special and $f_0^2$ is rigid.

Recall that $\mathcal{M}(a)$ is dense in $\mathcal{H}(b_0)$ if and only if the pair $(b_0, a)$ is special and $f_0^2$ is a rigid function.

**Theorem 2.** Assume that $(Ib_0, a)$, where $I$ is inner, and $I(0) = 0$, is the pair associated with an outer function $f$. If the pair $(b_0, a)$ is not special or the function $f_0^2 = \left(\frac{a}{1 - b_0}\right)^2$ is not rigid, then for a positive integer $k$,

$$\dim(\ker T_{\bar{\tau}_{\bar{f}}} \ominus fK_I) = k$$

if and only if the codimension of $\overline{\mathcal{M}(a)}$ in the de Branges-Rovnyak space $\mathcal{H}(b_0)$ is $k$. 

In the proof of this theorem we use some ideas from Sarason’s proof of Hayashi’s theorem. If a positive measure $\mu$ on the unit circle $\partial \mathbb{D}$ is as in (11) and $H^2(\mu)$ is the closure of the polynomials in $L^2(\mu)$, then an operator $V_b$ given by

\[
(5) \quad (V_bq)(z) = (1 - b(z)) \int_{\partial \mathbb{D}} \frac{q(e^{i\theta})}{1 - e^{-i\theta}z} \, d\mu(e^{i\theta}).
\]

is an isometry of $H^2(\mu)$ onto $\mathcal{H}(b)$ (19, 3). Furthermore, if $(b, a)$ is a pair and $f = \frac{a}{1-b}$, then the operator $T_{1-b}T_f$ is an isometry of $H^2$ into $\mathcal{H}(b)$. Its range is all of $\mathcal{H}(b)$ if and only if the pair $(b, a)$ is special.

**Proof of Theorem 2.** Since the pair $(b, a)$ is special, the operator $T_{1-b}T_f$ is an isometry of $H^2$ onto $\mathcal{H}(b)$. Moreover, since $I$ divides $b$, $T_f$ acts as an isometry on $K_I$ and $T_{1-b}T_f(fK_I) = K_I$ [18]. Hence

\[
\mathcal{H}(b) = T_{1-b}T_f(H^2) = T_{1-b}T_f((H^2)\oplus (H^2)^\perp) = \overline{IM(a)^b + T_{1-b}T_f(\ker T_f)} = \overline{IM(a)^b + T_{1-b}T_f(fK_I) \oplus T_{1-b}T_f(\ker T_f \ominus fK_I)} = \overline{IM(a)^b + K_I \oplus T_{1-b}T_f(\ker T_f \ominus fK_I)},
\]

where $\overline{T_f(H^2)}$ denotes the closure of $T_f(H^2)$ in $H^2$ and $\overline{IM(a)^b}$ denotes the closure of $IM(a)$ in $\mathcal{H}(b)$. On the other hand,

\[
\mathcal{H}(b) = \mathcal{H}(b_0I) = K_I \oplus \mathcal{H}(b_0) = K_I \oplus \mathcal{H}(b_0) \ominus \overline{IM(a)^{b_0}} \oplus \overline{IM(a)^{b_0}}.
\]

Since $T_f : \mathcal{H}(b_0) \to \mathcal{H}(b_0I)$ is an isometry ([18, Prop. 4]), $\overline{IM(a)^{b_0}} = \overline{IM(a)^{b}}$. It then follows,

\[
(6) \quad T_{1-b}T_f(\ker T_f \ominus fK_I) = \mathcal{H}(b_0) \ominus \overline{IM(a)^{b_0}}.
\]

\[\square\]

We remark that the orthogonal complement of $\mathcal{M}(a)$ in $\mathcal{H}(b)$ is discussed in Section 5.

### 4. The Example

Let, as in the previous sections, $f$ be an outer function in $H^2$ and let $(b, a)$ be the pair associated with $f$. Let $b = I b_0$, where $I$ is an inner function such that $I(0) = 0$ and $f_0 = \frac{a}{1 - b_0}$. Then $fK_I \subset \ker T_f$ and equality holds if and only if the pair $(b_0, a)$ is special and $f_0^2$ is rigid. Moreover, if the pair $(b_0, a)$ is special and $f_0^2$ is rigid, then $(b, a)$ is special and $f^2$ is rigid but the converse implication fails ([16, p. 158]).

In [3, vol. 2, pp. 541–542] the authors constructed a function $h$ in $\ker T_{1-b}$ which is not in $fK_I$ under the assumption that $f^2$ is not rigid. Here we consider the function $f$
such that $f^2$ is rigid, the pair $(b_0, a)$ is special but $f_0^2$ is not rigid, and describe the space $\ker T \vec{I} \ominus f K_I$. 

Our example is a slight modification of the one given in [17], see also [3], vol. 2, p. 494. The corresponding functions $f$ and $f_0$ are defined by taking $a(z) = \frac{1}{2}(1 + z)$, $b_0(z) = \frac{1}{2}z(1 - z)$, and $I(z) = zB(z)$, where $B(z)$ is a Blaschke produkt with zero sequence $\{r_n\}^\infty_{n=1}$ lying in $(-1, 0)$ and converging to $-1$. It has been proved in [17] (see also [3], vol. 2, pp. 494–496) that $f^2$ is rigid while $f_0^2$ is not. Notice that the pair $(b_0, a)$ is rational and the point $-1$ is the only zero of the function $a$. It then follows from [13, Thm. 4.1.] (see also [4]) that $M(a)$ is a closed subspace of $H(b_0)$ and

$$H(b_0) = M(a) \oplus \mathbb{C}k_{b_0}^{-1},$$

where

$$k_{b_0}^{-1}(z) = \frac{1 - \overline{b_0}(-1)b_0(z)}{1 + z} = \frac{2 - z}{2}.$$ 

Thus we see that

$$H(b_0) \ominus M(a) = \mathbb{C}k_{b_0}^{-1}.$$ 

Moreover [6] implies that

$$T_{1-b}T_f(\ker T \vec{I} \ominus f K_I) = \mathbb{C}k_{b_0}.$$

Our aim is to prove that

$$\ker T \vec{I} \ominus f K_I = \mathbb{C}g,$$

where the function $g \in H^2$ is given by $g = f k_{-1}(I + 1)$, with $k_{-1}(z) = (1 + z)^{-1}$, $z \in \mathbb{D}$.

For $\lambda \in \mathbb{D}$ let $k_\lambda$ denote the kernel function in $H^2$ for the functional of evaluation at $\lambda$, $k_\lambda(z) = (1 - \overline{\lambda}z)^{-1}$. In the proof of (7) we will apply the following

**Lemma ([7]).**

(i) $P_+ (|f|^2 I k_\lambda) = \frac{Ik_\lambda}{1 - b} + \frac{b_0(\overline{\lambda})k_\lambda}{1 - b(\lambda)}.$

(ii) $P_+ (|f|^2 k_\lambda) = \frac{k_\lambda}{1 - b} + \frac{b(\lambda)k_\lambda}{1 - b(\lambda)}.$

Since $I(r_n) = 0$, (i) and (ii) in the Lemma yield

$$T_{1-b}T_f(f I k_{r_n}) = Ik_{r_n}(1 - b_0(r_n)b_0) + b_0(r_n)k_{r_n},$$

$$T_{1-b}T_f(f k_{r_n}) = k_{r_n}.$$ 

Hence

$$T_{1-b}T_f(f k_{r_n}(I - b_0(r_n))) = Ik_{r_n}(1 - b_0(r_n)b_0) = Ik_{b_0}.$$ 

It follows from [3], vol. 2, Thm. 21.1 that

$$\|k_{r_n} - k_{b_0}\|_{b_0} \rightarrow 0.$$
Next, since $T_f: \mathcal{H}(b_0) \rightarrow \mathcal{H}(Ib_0) = \mathcal{H}(b)$ is an isometry and $T_{1-b}T_f$ is an isometry of $H^2$ onto $\mathcal{H}(b)$, we see that $\{fk_{r_n}(I - \overline{b_0(r_n)})\}_{n \in \mathbb{N}}$ is a bounded sequence in $H^2$. So it contains a subsequence that converges weakly, say, to a function $g \in H^2$. Without loss of generality, we may assume that the sequence $\{fk_{r_n}(I - \overline{b_0(r_n)})\}$ itself converges weakly to $g$. Then for any point $z \in \mathbb{D}$,

$$g(z) = \langle g, k_z \rangle = \lim_{n \rightarrow \infty} \langle fk_{r_n}(I - \overline{b_0(r_n)}), k_z \rangle = \lim_{n \rightarrow \infty} \frac{f(z)(I(z) - \overline{b_0(r_n)})}{1 - r_nz} = \frac{f(z)(I(z) + 1)}{1 + z}.$$ 

Now observe that since

$$\|fk_{r_n}(I - \overline{b_0(r_n)})\|_2 = \|k_{b_0}\|_{b_0} \quad \text{and} \quad \|fk_{-1}(I + 1)\|_2 = \|k_{b_1}\|_{b_0},$$

$\{fk_{r_n}(I - \overline{b_0(r_n)})\} \rightarrow fk_{-1}(I + 1)$ in $H^2$ strongly. Finally, passing to the limit in (8) gives

$$T_{1-b}T_f(fk_{-1}(I + 1)) = Ik_{-1}(1 + b_0) = Ik_{b_0},$$

which proves (7).

**Remark.** One can check directly that the function $g = fk_{-1}(I + 1)$ is in $\ker T_f \ominus fK_I$.

Indeed, we have

$$T_f(fk_{-1}(I + 1)) = P_+ \left( \frac{fI + 1}{1 + z} \right) - P_+ \left( \frac{f\overline{z}(I + 1)}{\overline{z} + 1} \right) = P_+ (zfk_{k-1}(I + 1)) = 0.$$

To see that the functions $\{fk_{r_n}I - \overline{b_0(r_n)}fk_{r_n}\}$ are orthogonal to $fK_I$ note that a function $h \in H^2$ is in $K_I$ if and only if $h = h - IP_+(Ih)$. So, we have to check that for any $h \in H^2$,

$$\langle fk_{r_n}I - \overline{b_0(r_n)}fk_{r_n}, f(h - IP_+(Ih)) \rangle = 0.$$

Since the functions $\{k_{\lambda}, \lambda \in \mathbb{D}\}$ are dense in $H^2$, it is enough to show that for any $\lambda \in \mathbb{D}$,

$$\langle fk_{r_n}I - \overline{b_0(r_n)}fk_{r_n}, f(k_{\lambda} - IP_+(I\lambda)) \rangle = \langle fk_{r_n}I - \overline{b_0(r_n)}fk_{r_n}, f(k_{\lambda} - I\overline{\lambda})Ik_{\lambda}) \rangle = 0.$$

Finally, the last equality follows from

$$\langle fk_{r_n}I, fk_{\lambda} \rangle = \frac{I(\lambda)k_{r_n}(\lambda)}{1 - b(\lambda)} + b_0(r_n)k_{r_n}(\lambda),$$

$$\langle fk_{r_n}I, -I(\overline{\lambda})Ik_{\lambda} \rangle = -\frac{I(\lambda)k_{r_n}(\lambda)}{1 - b(\lambda)},$$

$$\langle -b_0(r_n)fk_{r_n}, fk_{\lambda} \rangle = -\frac{b_0(r_n)k_{r_n}(\lambda)}{1 - b(\lambda)},$$

and

$$\langle -b_0(r_n)fk_{r_n}, -I(\overline{\lambda})Ik_{\lambda} \rangle = b_0(r_n)I(\lambda)\frac{b_0(\lambda)k_{r_n}(\lambda)}{1 - b(\lambda)} = \frac{b_0(r_n)b(\lambda)k_{r_n}(\lambda)}{1 - b(\lambda)}.$$
5. A REMARK ON ORTHOGONAL COMPLEMENT OF $\mathcal{M}(a)$ IN $\mathcal{H}(b)$

In this section we continue to assume that $b$ is nonextreme. Let $\mathcal{H}_0(b)$ denote the orthogonal complement of $\mathcal{M}(a)$ in $\mathcal{H}(b)$. Let $Y$ be the restriction of the shift operator $S$ to $\mathcal{H}(b)$ and let $Y_0$ be the compression of $Y$ to the subspace $\mathcal{H}_0(b)$. Necessary and sufficient conditions for $\mathcal{H}_0(b)$ to have finite dimension are given in Chapter X of [19]. The space $\mathcal{H}_0(b)$ depends on the spectrum of the restriction of the operator $Y^*$ to $\mathcal{H}_0(b)$ which actually equals $Y_0^*$. The spectrum of $Y_0^*$ is contained in the unit circle. More exactly, if $|z_0| = 1$ and $k$ is a positive integer, then $\ker(Y - z_0)^k \subset \mathcal{H}_0(b)$ and the dimension of $\mathcal{H}_0(b)$ is $N$ if and only if the operator $Y_0^*$ has eigenvalues $z_1, z_2, \ldots, z_s$ with their algebraic multiplicities $n_1, \ldots, n_s$ and $N = n_1 + n_2 + \cdots + n_s$.

In this section we consider the case when the eigenspaces corresponding to eigenvalues $z_1, z_2, \ldots, z_s$ are one dimensional.

For $|\lambda| = 1$ let $\mu_\lambda$ denote the measure for which equality in (1) holds when $b$ is replaced by $\bar{\lambda}b$. If we put $F_\lambda(z) = \frac{a}{1 - \bar{\lambda}b}$, then the Radon-Nikodym derivative of the absolutely continuous component of $\mu_\lambda$ is $|F_\lambda|^2$.

The following theorem is proved in Chapter X of [19].

**Sarason’s Theorem.** Let $z_0$ be a point of $\partial \mathbb{D}$ and $\lambda$ a point of $\partial \mathbb{D}$ such that the measure $\mu_\lambda$ is absolutely continuous. The following conditions are equivalent.

(i) $\bar{z}_0$ is an eigenvalue of $Y^*$.

(ii) The function $\frac{F_\lambda(z)}{1 - \bar{z}_0 z}$ is in $H^2$.

(iii) The function $b$ has an angular derivative in the sense of Carathéodory at $z_0$.

The space $\mathcal{H}_0(b)$ is described by means of an operator $A_\lambda$ on $H^2$ that intertwines $T_{1 - \bar{\lambda}b}T_{\bar{\tau}_\lambda}$ with the operator $Y^*$, i.e.

\[
T_{1 - \bar{\lambda}b}T_{\bar{\tau}_\lambda}A_\lambda = Y^*T_{1 - \bar{\lambda}b}T_{\bar{\tau}_\lambda}.
\]

The operator $A_\lambda$ is given by

\[
A_\lambda = S^* - F_\lambda(0)^{-1}(S^*F_\lambda \otimes 1).
\]

It follows from the proof of Sarason’s theorem that if one of conditions (i) – (iii) holds true, then the space $\ker(A_\lambda - \bar{z}_0)$ is one dimensional and is spanned by the function

\[
g(z) = \frac{F_\lambda(z)}{1 - \bar{z}_0 z} = F_\lambda(z)k_{z_0}(z).
\]

We also note that since $b$ has the angular derivative in the sense of Carathéodory at $z_0$, the function

\[
k_{z_0}^b(z) = \frac{1 - \bar{b(z_0)b(z)}}{1 - \bar{z}_0 z},
\]

where $b(z_0)$ is the nontangential limit of $b$ in $z_0$, is in $\mathcal{H}(b)$.

Using Sarason’s theorem cited above we show the following
Theorem 3. If the assumptions of Sarason’s theorem are satisfied and \( \bar{z}_0 \) is an eigenvalue of \( Y^* \), then ker\((Y^* - \bar{z}_0)\) is spanned by \( k_{\bar{z}_0}^b \).

Proof. In view of (11) and (9) the space ker\((Y^* - \bar{z}_0)\) is spanned by
\[
h = T_{1-\bar{z}_0}T_{F_\lambda}g = T_{1-\bar{z}_0}T_{F_\lambda}(F_\lambda k_{\bar{z}_0}).
\]
It is known that if \( V_b \) is given by (5), then for a fixed \( w \in \mathbb{D} \),
\[
V_b((1 - \bar{b}(w))k_w) = k_w^b.
\]

Hence we get
\[
V_{\bar{b}}((1 - \lambda \bar{b}(w))k_w)(z) = (1 - \lambda \bar{b}(z))(1 - \lambda \bar{b}(w)) \int_{\partial \mathbb{D}} \frac{|F_\lambda(e^{i\theta})|^2 d\theta}{(1 - \bar{w}e^{i\theta})(1 - z e^{-i\theta})} = (1 - \lambda \bar{b}(z))T_{F_\lambda}((1 - \lambda \bar{b}(w))F_k k_w)(z) = k_w^b(z).
\]

Let \( \{z_n\} \) be a sequence in \( \mathbb{D} \) converging nontangentially to \( z_0 \). Then
\[
T_{1-\bar{z}_0}T_{F_\lambda}(1 - \lambda \bar{b}(z_n))F_k k_{z_n} = k_{z_n}^b.
\]

Observe also that since \( \mu_\lambda \) is absolutely continuous, \( b(z_0) \neq \lambda \) [19] VI-7, VI-9. Moreover, we know that \( k_{z_n}^b \) tends to \( k_{z_0}^b \) weakly and \( \|k_{z_n}^b\| \) tends to \( \|k_{z_0}^b\| \) as \( z \) tends nontangentially to \( z_0 \) [19] VI-5. Clearly this implies that \( k_{z_n}^b \) tends to \( k_{z_0}^b \) in norm as \( z \) tends to \( z_0 \) nontangentially. It then follows that the sequence \( \{(1 - \lambda \bar{b}(z_n))F_k k_{z_n}\} \) converges in \( H^2 \), which in turn implies compact and pointwise convergence. Hence passing to the limit in the last equality yields
\[
T_{1-\bar{z}_0}T_{F_\lambda}(F_k k_{z_0}) = Ck_{z_0}^b,
\]
where \( C = (1 - \lambda \bar{b}(z_0))^{-1} \). \( \Box \)

Finally we remark that the next Corollary generalizes results obtained in [4] and in [13] for the case when pairs \((b,a)\) are rational.

Corollary. If \( z_1, z_2, \ldots, z_s \) are the only eigenvalues of \( Y_0 \) and each of them is of multiplicity one, then \( \mathcal{H}_0(b) \) is spanned by the function \( k_{z_1}^b, k_{z_2}^b, \ldots, k_{z_s}^b \).

Acknowledgements. The third named author would like to thank the Institute of Mathematics of the Maria Curie-Skłodowska University for supporting his visit to Lublin where part of this paper was written.

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