Factorization of Dyck words and the distribution of the divisors of an integer

José Manuel Rodríguez Caballero
Université du Québec à Montréal, Montréal, QC, Canada
rodriguez_caballero.josemanuel@uqam.ca

Abstract. In [2], we associated a Dyck word ⟨⟨n⟩⟩λ to any pair (n, λ) consisting of an integer n ⩾ 1 and a real number λ > 1. The goal of the present paper is to show a relationship between the factorization of ⟨⟨n⟩⟩λ as the concatenation of irreducible Dyck words and the distribution of the divisors of n. In particular, we will provide a characterization of λ-densely divisible numbers (these numbers were introduced in [1]).

Keywords: Dyck word, concatenation, densely divisible numbers.

1 Introduction

Zhang [7] established the first finite bound on gaps between prime numbers. In order to refine Zhang’s result, the polymath8 project led by Tao [1] introduced the so-called densely divisible numbers, which are a weak version of the classical smooth numbers. An integer n ⩾ 1 is λ-densely divisible, where λ > 1 is a real number, if for all R ∈ [1, n], there is at least one divisor of n on the interval [λ−1R, R].

Let L be a finite set of real numbers. Consider the set

\[ T(L; t) := \bigcup_{\ell \in L} [\ell, \ell + t], \]

endowed with the topology inherited from \( \mathbb{R} \), where t > 0 is an arbitrary real number. It is natural to associate any integer n ⩾ 1 with the topological space

\[ T_\lambda(n) := T(L; t), \]

where \( L := \{\ln d : d|n\} \) and t := ln λ. It follows that an integer n ⩾ 1 is λ-densely divisible if and only if \( T_\lambda(n) \) is connected (see Proposition 22).

In this paper, we will show a relationship between the number of connected components of \( T(L; t) \) and the factorization of the Dyck word \( \langle\langle S\rangle\rangle_\lambda \) introduced in [2], provided that \( L = \{\ln s : s \in S\} \) and t = ln λ. From this general result, we will derive a characterization of λ-densely divisible numbers in terms of the Dyck word \( \langle\langle n\rangle\rangle \), also introduced in [2]. We recall the definitions of \( \langle\langle S\rangle\rangle_\lambda \) and \( \langle\langle a\rangle\rangle \) given in [2].

Definition 1. Consider a real number λ > 1 and a 2-letter alphabet \( \Sigma = \{a, b\} \).
(i) Given a finite set of positive real numbers $S$, the $\lambda$-class of $S$ is the word

$$\langle \langle S \rangle \rangle_{\lambda} := w_0 w_1 w_2 ... w_{k-1} \in \Sigma^*,$$  \hspace{1cm} (2)

such that each letter is given by

$$w_i := \begin{cases} a & \text{if } \mu_i \in S, \\ b & \text{if } \mu_i \in \lambda S, \end{cases}$$  \hspace{1cm} (3)

for all $0 \leq i \leq k-1$, where $\mu_0, \mu_1, ..., \mu_{k-1}$ are the elements of the symmetric difference $S \triangle \lambda S$ written in increasing order, i.e.

$$\lambda S := \{ \lambda s : s \in S \},$$

$$S \triangle \lambda S = \{ \mu_0 < \mu_1 < ... < \mu_{k-1} \}. \hspace{1cm} (4)$$

(ii) If $S$ is the set of divisors of $n$, then we will write $\langle \langle n \rangle \rangle_{\lambda} := \langle \langle S \rangle \rangle_{\lambda}$. The word $\langle \langle n \rangle \rangle_{\lambda}$ will be called the $\lambda$-class of $n$.

The proof that $\langle \langle n \rangle \rangle_{\lambda}$ and $\langle \langle S \rangle \rangle_{\lambda}$ are Dyck words was given in [2]. Also, the height of the Dyck path associated to $\langle \langle n \rangle \rangle_{\lambda}$ coincides with the generalized Hoo-ley’s $\Delta_{\lambda}$-function

$$\Delta_{\lambda}(n) := \max_{R > 0} \# \{ d \mid n : d \in [\lambda^{-1} R, R] \},$$

where $R$ runs over the positive real numbers (see [2]).

The main result in the present paper is the following theorem.

**Theorem 2.** Let $\lambda > 1$ be a real number.

(i) For any integer $n \geq 1$, the number of connected components of $T_{\lambda}(n)$ is precisely $\Omega(\langle \langle n \rangle \rangle_{\lambda})$.

(ii) An integer $n \geq 1$ is $\lambda$-densely divisible if and only if $\langle \langle n \rangle \rangle_{\lambda}$ is an irreducible Dyck word.

The function $\Omega(w)$, formally defined using diagram [5], is just the number of irreducible Dyck words needed to obtain the Dyck word $w$ as a concatenation of them [1]. We will derive Theorem 2 taking $S$ to be the set of divisors of $n$ in the following more general result.

**Proposition 3.** Let $\lambda > 1$ be a real number. Consider a finite set of positive real numbers $S$. Define $L := \{ \ln s : s \in S \}$ and $t := \ln \lambda$. The number of connected components of $T(L; t)$ is $\Omega(\langle \langle S \rangle \rangle_{\lambda})$.  

\footnote{We use the notation $\Omega(w)$ in analogy to the arithmetical function $\Omega(n)$ which is equal to the number of prime factors of $n$ counting their multiplicities.}
2 Preliminaries

Consider a 2-letter alphabet \( \Sigma = \{a, b\} \). The \textit{bicyclic semigroup} \( \mathcal{B} \) is the monoid given by the presentation

\[
\mathcal{B} := \langle a, b \mid ab = \varepsilon \rangle,
\]

where \( \varepsilon \) is the empty word.

Let \( \pi : \Sigma^* \to \mathcal{B} \) be the canonical projection. The \textit{Dyck language} \( \mathcal{D} \) is the kernel of \( \pi \), i.e.

\[
\mathcal{D} := \pi^{-1}(\pi(\varepsilon)).
\]

Interpreting the letters \( a \) and \( b \) as the displacements \( 1 + \sqrt{-1} \) and \( 1 - \sqrt{-1} \) in the complex plane \( \mathbb{C} \), we can represent each word \( w \in \mathcal{H} \) by means of a Dyck path, i.e. a lattice path from \( 0 \) to \( |w| \), using only the above-mentioned steps and always keeping the imaginary part on the upper half-plane \( \{ z \in \mathbb{C} : \ \text{Im} \ z \geq 0 \} \). For an example of Dyck path, see Fig 1. It is easy to check that \( \mathcal{D} \) can be described as the language corresponding to all possible Dyck paths.

The language of \textit{reducible Dyck words} is the submonoid

\[
\tilde{\mathcal{D}} := \{ \varepsilon \} \cup \{ uv : \ u, v \in \mathcal{D} \setminus \{\varepsilon\} \}
\]

of \( \mathcal{D} \). The elements of the complement of \( \tilde{\mathcal{D}} \) in \( \mathcal{D} \), denoted

\[
\mathcal{P} := \mathcal{D} \setminus \tilde{\mathcal{D}}
\]

are called \textit{irreducible Dyck words}.

It is well-known that \( \mathcal{D} \) is freely generated by \( \mathcal{P} \), i.e. every word in \( \mathcal{D} \) may be formed in a unique way by concatenating a sequence of words from \( \mathcal{P} \). So, there is a unique morphism of monoids \( \Omega : \mathcal{D} \to \mathbb{N} \), where \( \mathbb{N} \) is the monoid of non-negative integers endowed with the ordinary addition, such that the diagram

\[
\begin{array}{ccc}
\mathcal{D} & \to & \mathcal{P}^* \\
\Omega \downarrow & & \downarrow \\
\mathbb{N} & & 
\end{array}
\]

commutes, where \( \mathcal{D} \to \mathcal{P}^* \) is the identification of \( \mathcal{D} \) with the free monoid \( \mathcal{P}^* \) and \( \mathcal{P}^* \to \mathbb{N} \) is just the length of a word in \( \mathcal{P}^* \) considering each element of the set \( \mathcal{P} \) as a single letter (of length 1). In other words, \( \Omega(w) \), with \( w \in \mathcal{D} \), is the number of irreducible Dyck words that we need to obtain \( w \) as a concatenation of them.

We will use the following result proved in [2].

\textbf{Proposition 4.} Let \( S \) be a finite set of positive real numbers. For any real number \( \lambda > 1 \) we have that \( \langle \langle S \rangle \rangle_\lambda \in \mathcal{D} \), i.e. \( \langle \langle S \rangle \rangle_\lambda \) is a Dyck word.

\footnote{In this paper, the bicyclic semigroup is not just a semigroup, but also a monoid. We preserved the word “semigroup” in the name for historical reasons.}
3 Generic case

Given a finite set of positive real numbers $S$, we say that a real number $\lambda > 1$ is regular (with respect to $S$) if $S$ and $\lambda S$ are disjoint. Otherwise, we say that $\lambda > 1$ is singular (with respect to $S$). This notion was already introduced in [2].

It is easy to check that the number of singular values (corresponding to a finite set $S$) is finite. In this section we will prove Proposition 3 under the additional hypothesis that $\lambda$ is regular. The proof that this proposition also holds true for singular values of $\lambda$ will be deduced from the case for regular values in next section.

Lemma 5. Let $\lambda > 1$ be a real number. Consider a finite set of positive real numbers $S$. Suppose that $\lambda$ is regular. Define $L := \{\ln s : s \in S\}$ and $t := \ln \lambda$. The space $T(L; t)$ is disconnected if and only if $\langle S \rangle_\lambda$ is a reducible Dyck word.

Proof. Define $L + t := \{\ell + t : \ell \in L\}$. We have $L \cup (L + t) = \{\ln \mu_i : 0 \leq i \leq k - 1\}$ because $\lambda$ is regular. Let $\mu_0, \mu_1, ..., \mu_{k-1}$ be the numbers appearing in (4). Consider the word $\langle S \rangle_\lambda = w_0 w_1 ... w_{k-1}$ as defined in (2).

Suppose that $T(L; t)$ is disconnected. In virtue of (4), for some $0 \leq j < k - 1$, we have $\ln \mu_j + t < \ln \mu_{j+1}$, i.e. $\lambda \mu_j < \mu_{j+1}$. So, the list $\mu_0, \mu_1, ..., \mu_j$ contains as many elements from $S$ as elements from $\lambda S$. It follows from (3) that $u := w_0 w_1 ... w_j$ satisfies $|u|_a = |u|_b$. So, $u$ is Dyck word. Therefore, $\langle S \rangle_\lambda$ is a reducible Dyck word, because its nonempty proper prefix $u$ is a Dyck word.

By Proposition 4 $\langle S \rangle_\lambda$ is a Dyck word. Suppose that $\langle S \rangle_\lambda$ is reducible. For some $0 \leq j < k - 1$ we have that the nonempty proper prefix $u := w_0 w_1 ... w_j$ of $\langle S \rangle_\lambda$ is a Dyck word. The relation $|u|_a = |u|_b$ and (3) imply that the list $\mu_0, \mu_1, ..., \mu_j$ contains as many elements from $S$ as elements from $\lambda S$. So, $\lambda \mu_j < \mu_{j+1}$, i.e. $\ln \mu_j + t < \ln \mu_{j+1}$. Using (1) we conclude that $T(L; t)$ is disconnected. $\square$

Lemma 6. Let $\lambda > 1$ be a real number. Consider a finite set of positive real numbers $S$. Suppose that $\lambda$ is regular. Define $L := \{\ln s : s \in S\}$ and $t := \ln \lambda$. The number of connected components of $T(L; t)$ is $\Omega(\langle S \rangle_\lambda)$.

Proof. Let $\mu_0, \mu_1, ..., \mu_{k-1}$ be the numbers appearing in (4). Consider the word $\langle S \rangle_\lambda = w_0 w_1 ... w_{k-1}$ as defined in (2). By Proposition 3 $\langle S \rangle_\lambda$ is a Dyck word. We proceed by induction on the number $c \geq 1$ of connected components of $T(L; t)$.

Consider the case $c = 1$. Suppose that $T(L; t)$ is connected. By Lemma 5 $\langle S \rangle_\lambda$ is irreducible. Then $c = \Omega(\langle S \rangle_\lambda) = 1$.

Suppose that the number of connected components of $T(L; t)$ is $\Omega(\langle S \rangle_\lambda)$, provided that $T(L; t)$ has at most $c - 1$ connected components for some $c > 1$. Assume that $T(L; t)$ has precisely $c$ connected components. By Lemma 5 $\langle S \rangle_\lambda$ is reducible. Let $p_1, p_2, ..., p_h$ be irreducible Dyck words satisfying $\langle S \rangle_\lambda = p_1 p_2 ... p_h$.

For some $0 \leq j < k - 1$ we have $P_j = w_0 w_1 ... w_j$. Notice that $\lambda \mu_j < \mu_{j+1}$. Setting $R = \{\mu_0, \mu_1, ..., \mu_j\}$, it follows that $\langle S \rangle_\lambda = p_2 p_3 ... p_h$. 


The space $T(L \setminus \ln(R); t)$, where $\ln(R) := \{\ln s : s \in R\}$, has precisely $c-1$ connected components, because $\ln \mu_j + \ln \lambda < \ln \mu_{j+1}$. Applying the induction hypothesis, $c - 1 = \Omega(\langle \langle S \setminus R \rangle \rangle_0) = h - 1$. Hence, $c = \Omega(\langle \langle S \setminus R \rangle \rangle_0) = h$.

By the principle of induction, we conclude that the number of connected components of $T(L; t)$ is $\Omega(\langle \langle S \setminus R \rangle \rangle_0)$.

4 General case
Consider a 3-letter alphabet $\Gamma = \{a, b, c\}$. We define the Hooley monoid $C$ to be the monoid given by the presentation

$$C := \langle a, b, c \mid ab = \varepsilon, a \cdot cb = ab, cc = c \rangle.$$  

Let $\varphi : \Gamma^* \longrightarrow C$ be the canonical projection. The Hooley-Dyck language $H$ is the kernel of $\varphi$, i.e.

$$H := \varphi^{-1}(\varphi(\varepsilon)).$$

Associating the letters $a$, $b$ and $c$ to the displacements $1 + \sqrt{-1}$, $1 - \sqrt{-1}$ and $1$, respectively, in the complex plane $C$, it follows that each word $w \in H$ can be represented by Schröder path, i.e. a lattice paths from 0 to $|w|$, using only the above-mentioned steps and always keeping the imaginary part on the upper half-plane $\{z \in C : \text{Im } z \geq 0\}$. For an example of Schröder path, see Fig 2.

Notice that the language $H$ corresponds to all possible Schröder paths having all the horizontal displacements (corresponding to $c$) strictly above the real axis.

The language of reducible Hooley-Dyck words is the submonoid $\widetilde{H} := \{\varepsilon\} \cup \{uv : u, v \in H \setminus \{\varepsilon\}\}$ of $H$. The elements of the complement of $\widetilde{H}$ in $H$, denoted $Q := H \setminus \widetilde{H}$, are called irreducible Hooley-Dyck words.

It is easy to check that $Q$ freely generates $H$. So, there is a unique morphism of monoids $\Theta : H \longrightarrow \mathbb{N}$, where $\mathbb{N}$ is the monoid of non-negative integers endowed with the ordinary addition, such that the diagram

$$\begin{array}{ccc}
H & \longrightarrow & Q^* \\
\Theta \downarrow & & \downarrow \\
\mathbb{N} & \longrightarrow & \\
\end{array}$$

commutes, where $H \longrightarrow Q^*$ is the identification of $Q$ with the free monoid $Q^*$ and $Q^* \longrightarrow \mathbb{N}$ is just the length of a word in $Q^*$ considering each element of the set $Q$ as a single letter (of length 1).

**Lemma 7.** Let $\gamma : \Gamma^* \longrightarrow \Sigma^*$ be the morphism of monoids given by $a \mapsto a$, $b \mapsto b$ and $c \mapsto \varepsilon$. We have that $\gamma(H) \subseteq D$. 
Proof. Notice that the diagram
\[
\begin{array}{ccc}
\Gamma^* & \xrightarrow{\psi} & C \\
\gamma & \downarrow & \downarrow \psi \\
\Sigma^* & \xrightarrow{\pi} & B
\end{array}
\] (7)
commutes, where \( \psi \) is the morphism of monoids given by \( \psi(C) := \gamma(C) \), for each equivalence class \( C \in \mathcal{C} \).

Take \( w \in \gamma(\ker \varphi) \). By definition, \( w = \gamma(v) \) for some \( v \in \ker \varphi \). Using the equalities
\[
\begin{align*}
\pi(w) &= \pi(\gamma(v)) \\
&= \psi(\varphi(v)) \\
&= \psi(\varphi(\varepsilon)) \\
&= \pi(\varepsilon),
\end{align*}
\]
we obtain that \( w \in \ker \pi \). Hence, \( \gamma(\ker \varphi) \subseteq \ker \pi \), i.e. \( \gamma(H) \subseteq D \).

**Lemma 8.** The morphism \( \gamma \) defined in Lemma 7 satisfies \( \gamma(Q) \subseteq P \).

**Proof.** Take \( q \in Q \). By Lemma 7 we have \( \gamma(q) \in D \). Also, we have \( \gamma(q) \neq \varepsilon \), because \( c^* \) and \( Q \) are disjoint, where \( c^* := \{\varepsilon, c, cc, ccc, \ldots\} \).

Suppose that \( \gamma(q) = uv \), for some \( u, v \in D \setminus \{\varepsilon\} \). It follows that \( q = \hat{u} \hat{v} \) for some \( \hat{u}, \hat{v} \in I^* \) satisfying \( \gamma(\hat{u}) = u \) and \( \gamma(\hat{v}) = v \). Using the commutative diagram 7 the fact that \( \psi \) is an isomorphism and the equalities,
\[
\begin{align*}
\varphi(\hat{u}) &= \psi^{-1}(\pi(\gamma(\hat{u}))) \\
&= \psi^{-1}(\pi(u)) \\
&= \psi^{-1}(\pi(\varepsilon)) \\
&= \varphi(\varepsilon),
\end{align*}
\]
we obtain that \( \hat{u} \in \ker \varphi = H \). Similarly, \( \hat{v} \in \ker \varphi = H \). Hence, \( q \notin Q \), contrary to our hypothesis. By reductio ad absurdum, \( \gamma(Q) \subseteq P \).

**Lemma 9.** Given \( w \in H \), we have \( \Theta(w) = \Omega(\gamma(w)) \), where \( \gamma \) is the morphism defined in Lemma 7, \( \Theta \) is given by diagram 7 and \( \Omega \) is given by diagram 5.

**Proof.** Notice that the diagram
\[
\begin{array}{ccc}
H & \rightarrow & Q^* \\
\downarrow & & \downarrow \\
D & \rightarrow & P^*
\end{array}
\]
commutes, where \( D \rightarrow P^* \) is the identification of \( D \) with the free monoid \( P^* \), \( H \rightarrow Q^* \) is the identification of \( H \) with the free monoid \( Q^* \), \( Q^* \rightarrow P^* \) is
the morphism of monoids given by $w \mapsto \gamma(w)$ for all $w \in Q$ (this function is well-defined in virtue of Lemma 8) and $H \rightarrow D$ is given by $w \mapsto \gamma(w)$ (this function is well-defined in virtue of Lemma 7). It follows that $\Theta(w) = \Omega(\gamma(w))$ holds for each $w \in H$.

**Lemma 10.** Let $\alpha : \Gamma^* \rightarrow \Sigma^*$ be the morphism of monoids given by $a \mapsto a$, $b \mapsto b$ and $c \mapsto ab$. We have that $\alpha(H) \subseteq D$.

**Proof.** Notice that the diagram

$$
\begin{array}{ccc}
\Gamma^* & \xrightarrow{\varphi} & C \\
\alpha \downarrow & & \downarrow \chi \\
\Sigma^* & \xrightarrow{\pi} & B
\end{array}
$$

commutes, where $\chi$ is the morphism of monoids given by $\chi(C) := \alpha(C)$, for each equivalence class $C \in C$.

Take $w \in \alpha(ker \varphi)$. By definition, $w = \alpha(v)$ for some $v \in ker \varphi$. Using the equalities

$$
\begin{align*}
\pi(w) &= \pi(\alpha(v)) \\
&= \chi(\varphi(v)) \\
&= \chi(\varphi(\varepsilon)) \\
&= \pi(\varepsilon),
\end{align*}
$$

we obtain that $w \in ker \pi$. Hence, $\alpha(ker \varphi) \subseteq ker \pi$, i.e. $\alpha(H) \subseteq D$.

**Lemma 11.** The morphism $\alpha$ defined in Lemma 10 satisfies $\alpha(Q) \subseteq P$.

**Proof.** Take $q \in Q$. By Lemma 10 we have $\alpha(q) \in D$. Using the fact that $\alpha$ does not decrease length, we have that $\alpha(q) \neq \varepsilon$, because $\varepsilon \notin Q$.

Suppose that $\alpha(q) = uv$, for some $u, v \in D \setminus \{\varepsilon\}$. It follows that $q = \hat{u} \hat{v}$ for some $\hat{u}, \hat{v} \in \Gamma^*$ satisfying $\alpha(\hat{u}) = u$ and $\alpha(\hat{v}) = v$. Using the commutative diagram 8 the fact that $\chi$ is an isomorphism and the equalities

$$
\begin{align*}
\varphi(\hat{u}) &= \chi^{-1}(\pi(\alpha(\hat{u}))) \\
&= \chi^{-1}(\pi(u)) \\
&= \chi^{-1}(\pi(\varepsilon)) \\
&= \varphi(\varepsilon),
\end{align*}
$$

we obtain that $\hat{u} \in ker \varphi = H$. Similarly, $\hat{v} \in ker \varphi = H$. Hence, $q \notin Q$, contrary to our hypothesis. By reductio ad absurdum, $\alpha(Q) \subseteq P$.

**Lemma 12.** Given $w \in H$, we have $\Theta(w) = \Omega(\alpha(w))$, where $\alpha$ is the morphism defined in Lemma 10, $\Theta$ is given by diagram 6 and $\Omega$ is given by diagram 5.
Proof. Notice that the diagram
\[
\begin{array}{c}
\mathcal{H} \\
\downarrow \\
\mathcal{D}
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\mathcal{Q}^* \\
\longrightarrow \\
\mathcal{P}^*
\end{array}
\]
commutes, where \( \mathcal{D} \longrightarrow \mathcal{P}^* \) is the identification of \( \mathcal{D} \) with the free monoid \( \mathcal{P}^* \), \( \mathcal{H} \longrightarrow \mathcal{Q}^* \) is the identification of \( \mathcal{H} \) with the free monoid \( \mathcal{Q}^* \), \( \mathcal{Q}^* \longrightarrow \mathcal{P}^* \) is the morphism of monoids given by \( w \mapsto \alpha(w) \) for all \( w \in \mathcal{Q} \) (this function is well-defined in virtue of Lemma 11) and \( \mathcal{H} \longrightarrow \mathcal{D} \) is given by \( w \mapsto \alpha(w) \) (this function is well-defined in virtue of Lemma 10). It follows that \( \Theta(w) = \Omega(\alpha(w)) \) holds for each \( w \in \mathcal{H} \).

The following construction was previously used in [2].

Definition 13. Given a finite set of positive real numbers \( S \), let \( \nu_0, \nu_1, \ldots, \nu_{r-1} \) be the elements of the union \( S \cup \lambda S \) written in increasing order, i.e.
\[
S \cup \lambda S = \{ \nu_0 < \nu_1 < \ldots < \nu_{r-1} \}.
\]

Consider the word
\[
[S]_{\lambda} := u_0 u_1 u_2 \ldots u_{r-1} \in \Gamma^*,
\]
where each letter is given by
\[
u_i := \begin{cases} 
a & \text{if } \nu_i \in S \setminus (\lambda S), 
b & \text{if } \nu_i \in (\lambda S) \setminus S, 
c & \text{if } \nu_i \in S \cap \lambda S,
\end{cases}
\]
for all \( 0 \leq i \leq r-1 \).

Example 14. The Dyck path corresponding to \( \langle 126 \rangle_2 = aabaababbabb \) is shown in Fig 1. The Schröder path corresponding to \( [126]_2 = acabcaabccabbcabcb \) is shown in Fig 2.

Lemma 15. Consider a finite set of positive real numbers \( S \). For any real number \( \lambda > 1 \) we have \( [S]_{\lambda} \in \mathcal{H} \).

Proof. We proceed by induction on the number of elements of \( S \), denoted \( m := \#S \).

For \( m = 0 \), we have \( [S]_{\lambda} = \varepsilon \in \mathcal{H} \).

Given \( m > 0 \), suppose that for each finite set of positive real numbers \( S \), we have \( [S]_{\lambda} \in \mathcal{H} \), provided that \( \#S < m \). Take an arbitrary finite set of real numbers \( S \) having precisely \( \#S = m \) elements. Denote \( \nu_0, \nu_1, \nu_2, \ldots, \nu_{r-1} \) the elements of \( S \cup \lambda S \) written in increasing order. Consider the word \( [S]_{\lambda} := u_0 u_1 u_2 \ldots u_{r-1} \) as given in Definition 13.

The inequality \( \lambda > 1 \) implies that there exists at least one integer \( i \) satisfying \( u_i \neq a \) and \( 1 \leq i \leq r-1 \). Define \( j := \min \{ i : u_i \neq a \text{ and } 1 \leq i \leq r-1 \} \).
Suppose that $u_j = b$. Setting $S' := S \setminus \{v_0\}$, we have

$$[S']_\lambda = u_0 u_1 u_2 \ldots u_{j-2} \widehat{u}_{j-1} \ u_j \ u_{j+1} \ldots u_{r-1},$$

where the hat indicates the corresponding letter is suppressed. By induction hypothesis, $[S']_\lambda \in \mathcal{H}$. Hence, $[S]_\lambda \in \mathcal{H}$, because it can be transformed into $[S']_\lambda \in \mathcal{H}$ using the relation $\varepsilon$ from $\mathcal{C}$.

Suppose that $u_j = c$ and $u_{j+1} = b$. Setting $S' := S \setminus \{v_0\}$, we have

$$[S']_\lambda = u_0 u_1 u_2 \ldots u_{j-2} \widehat{u}_{j-1} \ u_j \ u_{j+1} \ldots u_{r-1}.$$

By induction hypothesis, $[S']_\lambda \in \mathcal{H}$. Hence, $[S]_\lambda \in \mathcal{H}$, because it can be transformed into $[S']_\lambda \in \mathcal{H}$ using the relation $ac = ab$.

Suppose that $u_j = c$ and $u_{j+1} = c$. Setting $S' := S \setminus \{v_0\}$, we have

$$[S']_\lambda = u_0 u_1 u_2 \ldots u_{j-2} \widehat{u}_{j-1} \ u_j \ u_{j+1} \ldots u_{r-1}.$$

By induction hypothesis, $[S']_\lambda \in \mathcal{H}$. Hence, $[S]_\lambda \in \mathcal{H}$, because it can be transformed into $[S']_\lambda \in \mathcal{H}$ using the relation $cc = c$.

Finally, suppose that $u_j = c$ and $u_{j+1} = a$. Setting $S' := S \setminus \{v_0\}$, we have

$$[S']_\lambda = u_0 u_1 u_2 \ldots u_{j-2} \widehat{u}_{j-1} \ u_j \ u_{j+1} \ldots u_{r-1}.$$

By induction hypothesis, $[S']_\lambda \in \mathcal{H}$. Then using the rewriting rules from $\mathcal{C}$, the word

$$u_0 \ u_1 \ u_2 \ldots u_{j-2} \ u_{j-1} \ u_j \ u_{j+1} \ldots u_{r-1}$$

can be reduced to

$$u_0 \ u_1 \ u_2 \ldots u_{j-2} \ u_{j-1} \ u_j \ u_{i_1} \ u_{i_2} \ldots u_{i_h},$$

where $u_{i_1} = b$, and the word obtained after the reduction $u_{j-1} \ u_{i_1} = \varepsilon$,

$$u_0 \ u_1 \ u_2 \ldots u_{j-2} \ u_{i_2} \ldots u_{i_h},$$

can be reduced to the empty word using the rewriting rules from $\mathcal{C}$. So, using the rewriting rules from $\mathcal{C}$, the original word $[S]_\lambda$ can be reduced to

$$u_0 \ u_1 \ u_2 \ldots u_{j-2} \ u_{j-1} \ u_j \ u_{i_1} \ u_{i_2} \ldots u_{i_h},$$

and the word obtained after the reduction $u_{j-1} \ u_{j} \ u_{i_1} = acb = ab = \varepsilon$, can be reduced to the empty word as we mentioned above. Hence, $[S]_\lambda \in \mathcal{H}$.

By the principle of induction, we conclude that $[S]_\lambda \in \mathcal{H}$ for any finite set of positive real numbers $S$.

**Lemma 16.** Consider a finite set of positive real numbers $S$. For any real number $\lambda > 1$, we have $\gamma([S]_\lambda) = (S)_{\lambda}$, where $\gamma$ is the morphism defined in Lemma 9.
Proof. In virtue of the identity \((S \cup \lambda S) \setminus (S \cap \lambda S) = S \triangle \lambda S\), the result follows just combining Definition 1 and Definition 13.

Example 17. Lemma 16 can be illustrated by means of Fig 1 and Fig 2.

Lemma 18. Consider a finite set of positive real numbers \(S\). For any real number \(\lambda > 1\), the equality \(\alpha ([S]_\lambda) = \langle \langle S \rangle \rangle_{\lambda'}\) holds for all \(\lambda' \in \lambda, +\infty\) near enough to \(\lambda\), where \(\alpha\) is the morphism defined in Lemma 10.

Proof. For any \(\lambda' \in \lambda, +\infty\), the change from \(S \cup \lambda S\) to \(S \cup \lambda' S\) keeps fixed the points in \(S\) and it displaces the points in \(\lambda S\) to the right. This displacement to the right can be made as small as we want just setting \(\lambda'\) near enough to \(\lambda\). In particular, any point in \(S \cap \lambda S\), after this transformation, becomes a pair of different points, one stays at the original position and the other one displaces to the right an arbitrary small distance. Notice that \(S \cap \lambda' S = \emptyset\) for all \(\lambda' \in \lambda, +\infty\) near enough to \(\lambda\) (this guarantees that \(\lambda'\) will be regular). Combining Definition 1 and Definition 13, we conclude that \(\alpha ([S]_\lambda) = \langle \langle S \rangle \rangle_{\lambda'}\) provided that \(\lambda' \in \lambda, +\infty\) is near enough to \(\lambda\).

Example 19. Lemma 18 can be illustrated by means of Fig 2 and Fig 3.

Fig. 1. Representation of \(\langle \langle 126 \rangle \rangle_2 = aabaababbabb\).

Fig. 2. Representation of \([126]_2 = acabaabccabcabc\).

Fig. 3. Representation of \([126]_{2,001} = \langle \langle 126 \rangle \rangle_{2,001} = aababaababababbababbababb\).
Lemma 20. Let $S$ be a finite set of positive real numbers. The step function $\|1, +\infty[ \rightarrow \mathbb{N}$, given by $\lambda \mapsto \Omega(\langle \langle S \rangle \rangle_\lambda)$, is continuous from the right, i.e. given a real number $\lambda > 1$, for each real number $\lambda' \in ]\lambda, +\infty[$, we have $\Omega(\langle \langle S \rangle \rangle_\lambda) = \Omega(\langle \langle S \rangle \rangle_{\lambda'})$, provided that $\lambda'$ is near enough to $\lambda$.

Proof. By Lemma 14, $\|S\|_\lambda \in \mathcal{H}$. By Lemma 16, $\gamma(\langle \langle S \rangle \rangle_\lambda) = \langle \langle S \rangle \rangle_\lambda$, where $\gamma$ is the morphism defined in Lemma 7. Using Lemma 9 we obtain $\Theta(\langle \langle S \rangle \rangle_\lambda) = \Omega(\langle \langle S \rangle \rangle_{\lambda'})$ for all $\lambda' \in ]\lambda, +\infty[$ near enough to $\lambda$, where $\alpha$ is the morphism defined in Lemma 10. Using Lemma 13 we obtain $\Theta(\langle \langle S \rangle \rangle_\lambda) = \Omega(\langle \langle S \rangle \rangle_{\lambda'})$ for all $\lambda' \in ]\lambda, +\infty[$ near enough to $\lambda$. Therefore, $\Omega(\langle \langle S \rangle \rangle_\lambda) = \Omega(\langle \langle S \rangle \rangle_{\lambda'})$ for all $\lambda' \in ]\lambda, +\infty[$ near enough to $\lambda$.

Lemma 21. Let $L$ be a finite set of real numbers. Consider the step function $f : [0, +\infty[ \rightarrow \mathbb{N}$ such that $f_L(t)$ is the number of connected components of $T(L; t)$. The function $f_L(t)$ is continuous from the right, i.e. given a real number $t > 0$ we have $f_L(t') = f_L(t)$ for all $t' \in ]t, +\infty[$ near enough to $t$.

Proof. Let $\ell_0, \ell_1, \ell_2, \ldots, \ell_{k-1}$ be the elements of $L$ written in increasing order, i.e. $L = \{\ell_0 < \ell_1 < \ell_2 < \ldots < \ell_{k-1}\}$.

Denote $c := f_L(t)$. In virtue of (1), we can write $T(L; t)$ as the union

$$T(L; t) = [\ell_1, \ell_2 + t] \cup [\ell_3, \ell_4 + t] \cup \ldots \cup [\ell_{2c-1}, \ell_{2c} + t]$$

of the pairwise disjoint sets $[\ell_1, \ell_2 + t], [\ell_3, \ell_4 + t], \ldots, [\ell_{2c-1}, \ell_{2c} + t]$, for some subsequence $i_1 < i_2 < i_3 < i_4 < \ldots < i_{2c-1} < i_{2c}$ of $0, 1, 2, \ldots, k-1$. So, for all $t' \in ]t, +\infty[$, the set $T(L; t')$ can be expressed as the union

$$T(L; t') = [\ell_1, \ell_2 + t'] \cup [\ell_3, \ell_4 + t'] \cup \ldots \cup [\ell_{2c-1}, \ell_{2c} + t']$$

where some of sets in the list $[\ell_1, \ell_2 + t'], [\ell_3, \ell_4 + t'], \ldots, [\ell_{2c-1}, \ell_{2c} + t']$ may overlap among them. Assuming that $t'$ is near enough to $t$, we guarantee that the sets $[\ell_1, \ell_2 + t'], [\ell_3, \ell_4 + t'], \ldots, [\ell_{2c-1}, \ell_{2c} + t']$ are pairwise disjoint. Hence, $f_L(t') = f_L(t)$ for all $t' \in ]t, +\infty[$ near enough to $t$. Therefore, $f_L(t)$ is continuous from the right.

Using the previous auxiliary results, we can prove Proposition 3.

Proof (of Proposition 3). By Lemma 20, the step function $[1, +\infty[ \rightarrow \mathbb{N}$, given by $\lambda \mapsto \Omega(\langle \langle S \rangle \rangle_\lambda)$, is continuous from the right. By Lemma 21, the step function $f_L : [0, +\infty[ \rightarrow \mathbb{N}$ is continuous from the right, where $f_L(t)$ is the number of connected components of $T(L; t)$. Notice that the step function $[1, +\infty[ \rightarrow \mathbb{N}$, given by $\lambda \mapsto f_L(\ln \lambda) - \Omega(\langle \langle S \rangle \rangle_\lambda)$, is continuous from the right, because the natural logarithm is continuous on $]0, +\infty[$. By Lemma 6, $f_L(\ln \lambda') - \Omega(\langle \langle S \rangle \rangle_{\lambda'}) = 0$ for all $\lambda' \in ]\lambda, +\infty[$ near enough to $\lambda$ (this guarantees that $X'$ is regular). Hence, $f_L(\ln \lambda) - \Omega(\langle \langle S \rangle \rangle_\lambda) = 0$ follows by continuity from the right. Therefore, the space $T(L; t)$ has precisely $\Omega(\langle \langle S \rangle \rangle_\lambda)$ connected components.
Proposition 22. Given a real number \( \lambda > 1 \), an integer \( n \geq 1 \) is \( \lambda \)-densely divisible if and only if \( \mathcal{T}_\lambda(n) \) is connected.

Proof. Suppose that \( n \) is \( \lambda \)-densely divisible and \( \mathcal{T}_\lambda(n) \) is disconnected. In virtue of (1), there are two divisors of \( n \), denoted \( d d' \), satisfying

\[
\ln d + \ln \lambda < \ln d'
\]

and there is no divisor of \( n \) on the interval \([d, d']\). Using the fact that \( n \) is \( \lambda \)-densely divisible, there is a divisor of \( n \) on the interval \([\lambda^{-1} R, R] \), with \( 1 \leq R := \lambda (d + \epsilon) < d' \leq n \), for all \( \epsilon > 0 \) small enough. Notice that \([\lambda^{-1} R, R] \not\subseteq [d, d']\). So, there is a divisor of \( n \) on the interval \([d, d']\). By reductio ad absurdum, if \( n \) is \( \lambda \)-densely divisible then \( \mathcal{T}_\lambda(n) \) is connected.

Now, suppose that \( \mathcal{T}_\lambda(n) \) is connected and \( n \) is not \( \lambda \)-densely divisible. Then there is some \( R \in [1, n] \) such that there is no divisor of \( n \) on the interval \([\lambda^{-1} R, R] \). It follows that \( R > \lambda > 1 \), because 1 is a divisor of \( n \). Let \( d \) be the largest divisor of \( n \) satisfying \( d \leq \lambda^{-1} R \). It follows that \( d < n \), because \( \lambda^{-1} R \leq \lambda^{-1} n < n \). Let \( d' \) be the smallest divisor of \( n \) satisfying \( \lambda^{-1} R < d' \). Notice that \( \lambda^{-1} R < d' \), \( \lambda d \leq R \) and there is no divisor of \( n \) on the interval \([d, d']\).

Using the fact that \( \mathcal{T}_\lambda(n) \) is connected, we have that

\[
[\ln d, \ln d + \ln \lambda] \cap [\ln d', \ln d' + \ln \lambda] \neq \emptyset.
\]

It follows that \( \ln d' < \ln d + \ln \lambda \), i.e. \( d' \leq \lambda d \). So, \( \lambda^{-1} R < d' \leq \lambda d \leq R \). In particular, \( d' \in [\lambda^{-1} R, R] \). By reductio ad absurdum, if \( \mathcal{T}_\lambda(n) \) is connected then \( n \) is \( \lambda \)-densely divisible.

We proceed now with the proof of the main result of this paper.

Proof (of Theorem 4). Statement (i) follows by Proposition 3 taking \( S \) to be the set of divisors of \( n \).

Take an integer \( n \geq 1 \). By Proposition 22 \( n \) is \( \lambda \)-densely divisible if and only if \( \mathcal{T}_\lambda(n) \) is connected. By Proposition 4, the space \( \mathcal{T}_\lambda(n) \) is connected if and only if \( \langle n \rangle_\lambda \) is irreducible. Hence, \( n \) is \( \lambda \)-densely divisible if and only if \( \langle n \rangle_\lambda \) is irreducible. Therefore, statement (ii) holds.

5 Final remarks

Consider the finite field with \( q \) elements, denoted \( \mathbb{F}_q \). Let \( \mathbb{Z} \oplus \mathbb{Z} \) be the free abelian group of rank 2. For each integer \( n \geq 1 \), there is a unique polynomial \( P_n(q) \) such that for any prime power \( q \), the value of \( P_n(q) \) is precisely the number of ideals \( I \) of \( \mathbb{F}_q \mathbb{Z} \oplus \mathbb{Z} \) satisfying that \( \mathbb{F}_q \mathbb{Z} \oplus \mathbb{Z} / I \) is an \( n \)-dimensional vector space over \( \mathbb{F}_q \). It was already observed in [2] that, as a consequence of an explicit formula for \( P_n(q) \) due to Kassel and Reutenauer (see [4] and [5]), the non-zero coefficients of the polynomial \( (1 - q) P_n(q) \) are determined by the Dyck word \( \langle n \rangle_2 \). Combining these results with Theorem 3, it follows that an integer \( n \geq 1 \) is 2-densely divisible if and only if all the coefficients of \( P_n(q) \) are non-zero.
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