PEL MODULISPAECES WITHOUT C-VALEED POINTS

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To my parents, Heinrich and Karola

ABSTRACT. We give several new moduli interpretations of the fibers of certain Shimura varieties over several prime numbers. As a corollary we obtain that for every prescribed odd prime characteristic $p$ every bounded symmetric domain possesses quotients by arithmetic groups whose models have good reduction at a prime divisor of $p$.

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1. Introduction

Let \((G, X)\) be a Shimura datum in the sense of all five axioms [6, (2.1.1.1)-(2.1.1.5)] of Deligne. One knows that there exists a canonical variation \(\text{Fib}_{\text{Hodge}}(\rho)\) of pure Hodge structures over the quasi-projective algebraic variety

\[
K_M(G, X) = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}^\infty) / K)
\]

for every \(\mathbb{Q}\)-rational linear representation \(\rho : G \to \text{GL}(V/\mathbb{Q})\), where we tacitly assume that \(K\) is a neat compact open subgroup. In rare cases these yield so-called “moduli interpretations”, more specifically if there exists an injective map \(\rho : G \to \text{GSp}_{2g}\) with \(\rho(X) \subset \mathfrak{h}_g\), then \(K_M(G, X)\) is a moduli space of \(g\)-dimensional abelian varieties with additional structure, i.e. equipped with additional Hodge cycles (on suitable powers). This result is not only theoretically significant, but it also has an enormous practical meaning, because it is this particular class of Shimura varieties of Hodge type which are, at least in principle, amenable to the methods of arithmetic algebraic geometry. For instance, it is this way that Milne has reobtained Deligne’s canonical models over the reflex field \(E \subseteq \mathbb{C}\) in [17]. Outside this class of Shimura varieties these methods fail, but the work of many people (e.g. [11], [1]) has culminated in more general results. Canonical models are finally shown to exist unconditionally in [18]. However, the proof uses a very amazing reduction to the GL(2)-cases, and again these are treated by means of abelian varieties whose endomorphism rings have a suitable structure.

Fix a rational prime \(p\) and write \(\mathcal{O}_{E_p}\) for the complete valuation rings corresponding to prime divisors \(p\mid p\). The Shimura variety [11] is expected to possess a certain smooth integral \(\mathcal{O}_{E_p}\)-model \(K\mathcal{M}\), provided that \(K\) can be written as \(K^p \times K_p\), where \(K^p\) is a compact open subgroup of \(G(\mathbb{A}^{\infty, p})\), and \(K_p\) is a hyperspecial subgroup of \(G(\mathbb{Q}_p)\). Apart from being canonical in the sense of Milne-Vasiu one wants this model
to satisfy a number of other nice properties, for example every $\mathbb{Q}_p$-rational representation $\rho : G \times \mathbb{Q}_p \to \text{GL}(V/\mathbb{Q}_p)$ should determine a $F$-isocrystal $\text{Fib}_{\text{Cris}}(\rho)$ over the reduction $K\mathcal{M} \times_{O_K} \mathcal{O}_E / p =: K\overline{M}$, and a variant should exist for the category of Hodge-$F$-isocrystals over the $p$-adic completion of $K\mathcal{M}$ of $K\mathcal{M}$. In either case $K_p$-invariant lattices in $V$ should determine interesting lattices in $\text{Fib}_{\text{Cris}}(\rho)$. For Shimura varieties of Hodge type see [12] and the references therein. Even without any of the above additional requirements the existence of these models is highly conjectural, in fact the following weakening is already interesting: From now on we fix an embedding $W(F_{\text{ac}}) \hookrightarrow \mathbb{C}$, we fix the hyperspecial subgroup $K_p$, and we write $T$ for the set of compact open subgroups $K_p \subset G(A_{\infty},p)$ such that $K_p \times K_p$ is neat. Write $X'$ for one of the connected components of $X$, and let us say that a discrete and cocompact subgroup $\tilde{\Delta} \subset G^{\text{ad}}(\mathbb{R}) \times G(A_{\infty},p)$ is a lattice of good reduction if there exists a family of smooth and projective $W(F_{\text{ac}})$-schemes $\{K_p\mathcal{M}\}_{K_p \in T}$ together with biholomorphic bijections

\begin{equation}
\tilde{\Delta} \backslash (X' \times G(A_{\infty},p)/K_p) \xrightarrow{\cong} K_p\mathcal{M}(\mathbb{C}),
\end{equation}

such that every inclusion of the form $K_1 \subset \text{Int}(\gamma)K_2$ with $K_1, K_2 \in T$, gives rises to a commutative diagram

\begin{equation}
\begin{array}{ccc}
\tilde{\Delta} \backslash (X' \times G(A_{\infty},p)/K_1^p) & \xrightarrow{\cong} & K_1^p\mathcal{M}(\mathbb{C}) \\
\gamma_p \downarrow & & \downarrow \\
\tilde{\Delta} \backslash (X' \times G(A_{\infty},p)/K_2^p) & \xrightarrow{\cong} & K_2^p\mathcal{M}(\mathbb{C})
\end{array}
\end{equation}

in which the right-hand side vertical arrow is the complexification of some étale covering map $K_1^p\mathcal{M} \to K_2^p\mathcal{M}$ while the left-hand vertical arrow is multiplication by the group element $\gamma_p \in G(A_{\infty},p)$, from the right. If $X'$ does not contain factors of rank one, i.e. irreducible bounded symmetric domains of type $I_{n,1}$, then the left hand side of (2) is a finite union of arithmetic varieties in the sense of [11], simply by one of Margulis’ arithmeticity theorems, namely [15, Theorem(1.11), chapter IX]. The focus of the present paper is on domains of type $V$ and $VI$, and its outcome is the following result:

**Theorem 1.1.** Let $W(F_{\text{ac}}) \hookrightarrow \mathbb{C}$ be as above, and assume in addition that $p$ is odd. Let $D$ be an irreducible bounded symmetric domain, which is not of type $II_n$ for any $n \geq 5$. Let $z \geq 1$ and $r \geq 4z + 1$ be integers. Then there exists at least one Shimura datum $(G, X)$ having a $\mathbb{Q}$-simple adjoint group $G^{\text{ad}}$ and fulfilling:

- The connected components of $X$ decompose into a product of $z$ copies of $D$. 
The complexification of $G^{\text{ad}}$ decomposes into a product of $r$ simple factors.

There exists a lattice of good reduction for $(G, X)$.

Among other things, those requirements on $(G, X)$ imply the anisotropicity of $G$, in which case one can view the smoothness and the projectivity as utmost minimal requirements in the pursuit of whatever notion of integral models to \([1]\). In the projective case it turns out, that one can use deformation theory to deduce the existence of $K_p \mathcal{M}$ from the existence of its special fiber $\overline{K_p \mathcal{M}}$, which is a particular smooth projective algebraic variety over a particular finite extension, say $\mathbb{F}_{p^f} \supset \mathcal{O}_{E_p}/p$. The construction of $\overline{K_p \mathcal{M}}$, which we give in this paper, is more involved and relies on discoveries which go a little bit beyond the world of good reduction: Under the assumptions of theorem \([1.1]\) we introduce a (projective) moduli-space $K_p \mathcal{M}$ parametrizing abelian varieties with a certain kind of additional structure, which is a slightly more elegant variant of a notion that appeared already in \([3, \text{Definition 5.3}]\). The very same kind of additional structure poses an analogous moduli problem in the category of $p$-divisible groups, which is used to introduce a certain fpqc-stack \(\mathcal{B}\). There exists an important (formally étale) 1-morphism $K_p \mathcal{M} \to \mathcal{B}$, which is easily defined by the ‘passage to the underlying $p$-divisible group’. Now our search for $\overline{K_p \mathcal{M}}$ is narrowed down by a certain hypothetical morphism:

\[(3) \quad K_p \mathcal{M} \to \overline{K_p \mathcal{M}}.\]

This morphism is radicial and finite, but it fails to have an extension to $K_p \mathcal{M}$, actually the generic fiber of $K_p \mathcal{M}$ has a strong tendency to be the empty scheme, and in fact it is only its fiber over (one single prime divisor of) $p$, that forms the starting point of several subsequent investigations.

In order to complete the construction of \((3)\) we make a systematic use of displays \([30]\), in fact we make use of a certain, modest generalization, the basic motivating idea is this: At least over a local ring a display $(P, Q, F, V^{-1})$ possesses a normal decomposition $P = T \oplus L$ and bases $e_1, \ldots, e_d$ of $T$ and $e_{d+1}, \ldots, e_{h}$ of $L$ relative to which there are structural equations $\sum_{i=1}^{h} \alpha_{i,j} e_i = \begin{cases} F e_j & j \leq d \\ V^{-1} e_j & j > d \end{cases}$, and hence an invertible $h \times h$-display matrix $(\alpha_{i,j}) = U$, cf. \([30, \text{Lemma 9}]\). Suppose that $U'$ is the matrix of another display $(P', Q', F, V^{-1})$ of possibly different height $h'$ and dimension $d'$, again taken with respect to a normal decomposition, and a choice of its bases. A linear map from $P'$ to $P$ can be visualized as $h \times h'$- matrix over the ring of Witt vectors with a
block decomposition $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with the upper left block having $d$ rows and $d'$ columns. This map is a morphism $P' \to P$ in the category of displays if and only if:

(i) the entries in $B$ are Witt vectors with vanishing 0th ghost component, and

(ii) $U^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} U' = \begin{pmatrix} F_A & V^{-1}B \\ p^tC & F_D \end{pmatrix}$ holds.

It is enough to check the commutation of $k$ with the maps $V^{-1}$ on $Q$ and $Q'$ and these are given by $U$ and $U'$ precomposed with $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} V^{-1}x \\ Fy \end{pmatrix}$. In particular the isomorphism classes of displays of height $h$ and dimension $d$ are simply the Witt vector-valued elements of $\text{GL}(h)$ modulo the equivalence relation

$$k^{-1} U^\Phi k \sim U,$$

here we write $U^\Phi k$ for the right hand side of (ii) which is well defined if the condition (i) on the upper right block of $k$ is valid. In the body of the text we will show that a similar map $\Phi$ exists on the inverse image of every minuscule parabolic subgroup, which gives rise to a definition of a stack $\mathcal{B}(\mathcal{G}, \mu)$ of $3n$-display with $\mathcal{G}$-structure. Slightly less obvious is, that a working notion of display with $\mathcal{G}$-structure can be defined by recourse to the structure of the crystalline realization in the adjoint representation. Once this has been done one can easily introduce an explicit map

$$\text{Flex} : \mathcal{B}(\mathcal{G}, \mu) \times_{\mathcal{O}_{p^{t}}} \mathbb{F}_{p^{t}} \to \mathfrak{B}$$

which is, roughly speaking a formal analog of the map (3). This adds a lot of content to the whole picture, and it gives a clue to interpret $K_{p^{t}}M$ as the scheme which represents the upper left entry in the 2-cartesian diagram

$$\begin{array}{ccc}
K_{p^{t}}\text{Mot} & \longrightarrow & K_{p^{t}}\mathfrak{M} \\
\downarrow & & \downarrow \\
\mathcal{B}(\mathcal{G}, \mu) \times_{\mathcal{O}_{p^{t}}} \mathbb{F}_{p^{t}} & \longrightarrow & \mathfrak{B}
\end{array}$$

de of fpqc-stacks. The bulk of this paper is devoted to the proof of partial representability properties of $K_{p^{t}}\text{Mot}$. We start by proving that it is radicial over $K_{p^{t}}\mathfrak{M}$, i.e. its diagonal morphism is relatively representable by dominant closed immersions. It follows that $K_{p^{t}}\text{Mot}(R)$ possesses at most one element if $R$ is a reduced noetherian $\mathcal{O}_{K_{p^{t}}\mathfrak{M}}$-algebra. The next
step is devoted to the study of the image of $K_p \text{Mot}$: Outside a certain Zariski closed subset which we call $K_p \mathcal{M}_{\text{can}}$ we can show that $K_p \text{Mot}$ is always empty, and every point in $K_p \mathcal{M}_{\text{can}}$ gives rise to a non-empty $K_p \text{Mot}$ when regarded over a certain finite purely inseparable extensions of the residue field. In this manner, the generic point of $K_p \mathcal{M}_{\text{can}}$ should yield the function field of $K_p \overline{M}$, and the normalisation of $K_p \mathcal{M}_{\text{can}}$ in this particular field should yield the variety $K_p \overline{M}$, at least heuristically. However, additional complications arise because there could be several candidates for the function field of $K_p \overline{M}$. The details can be found at the end of section 7.2 and eventually we do obtain a variety $K_p \overline{M}$, which represents the restriction of $K_p \text{Mot}$ on the category of test schemes that are of finite type over an algebraic extension of $\mathbb{F}_{p'}$, and the existence of (3) is then clear.

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2. 3n-Displays with additional structure

In this section we introduce displays with additional structure. We prefer to work with arbitrary smooth, not necessarily reductive group schemes. This gives great conceptual clarity and could be of some use in other contexts.

2.1. Preliminaries. Write $\mathbb{F}_p$ for the prime field of characteristic $p$ and $\mathbb{F}_{p'}$ for its extension of degree $f$. Recall the ringscheme $W$ of Witt vectors along with its truncated version $W_n$ for each finite length $n$. There are canonical maps:

$$F_n : W_{n+1} \rightarrow W_n$$

and

$$V_n : W_n \rightarrow \overline{I_n} \subset W_{n+1},$$

where the ideal $I_n$ is defined to be the kernel of $w_0$, the 0th coordinate of the ghost map, and, as usual, we shall suppress “$n$” in the notation.
if we mean "\(n = \infty\)". We need to make a number of remarks on set-valued covariant functors \(F\) on the category of \(W(\mathbb{F}_{p^f})\)-algebras: Let us write \(\text{Res}_{W_n} F\) for the functor taking any \(S/W(\mathbb{F}_{p^f})\) to \(F(W_n(S))\), where the \(W(\mathbb{F}_{p^f})\)-algebra structure on \(W_n(S)\) is induced by the natural map

\[
\Delta_n : W(\mathbb{F}_{p^f}) \to W_n(W(\mathbb{F}_{p^f}))
\]

which one gets by truncating the usual functorial diagonal \(\Delta\) from \(W\) to \(W \circ W\). Let us also write \(^F\mathcal{F}\) for the pull-back via the Frobenius \(F : W(\mathbb{F}_{p^f}) \to W(\mathbb{F}_{p^f})\), i.e. \(^F\mathcal{F}(S) = \mathcal{F}(S[F])\), here the subscripted \([F]\) indicates the change of \(W(\mathbb{F}_{p^f})\)-algebra structure on the ring \(S\) by means of \(F\). Observe that for every ring \(S\) there exist functorially truncated Frobenii: \(F_n : W_{n+1}(S) \to W_n(S)\) which give rise to natural maps

\[
(6) \quad F_n : \text{Res}_{W_{n+1}} \mathcal{F} \to \text{Res}_{W_n} ^F\mathcal{F}.
\]

This is due to the commutativity of the diagram:

\[
\begin{array}{ccc}
W_{n+1}(W(\mathbb{F}_{p^f})) & \xrightarrow{F_n} & W_n(W(\mathbb{F}_{p^f})) \\
\Delta_{n+1} \uparrow & & \Delta_n \uparrow \\
W(\mathbb{F}_{p^f}) & \xrightarrow{F} & W(\mathbb{F}_{p^f})
\end{array}
\]

which is merely a special case of the remarkable compatibility between \(\Delta\) and \(F\), a fact that holds for just any commutative ring. In the special case of a functor \(\mathcal{F}\) that "descends" (i.e. extends) to a functor over \(W(\mathbb{F}_p)\) we may certainly drop the "\(F\)" in the right hand side of \((6)\), however do notice that we can always write \(^F\text{Res}_{W_n} \mathcal{F}\) instead of \(\text{Res}_{W_n} ^F\mathcal{F}\), because of the commutativity of the diagram

\[
\begin{array}{ccc}
W_n(W(\mathbb{F}_{p^f})) & \xrightarrow{W_n(F)} & W_n(W(\mathbb{F}_{p^f})) \\
\Delta_n \uparrow & & \Delta_n \uparrow \\
W(\mathbb{F}_{p^f}) & \xrightarrow{F} & W(\mathbb{F}_{p^f})
\end{array}
\]

which, this time, does use that \(\mathbb{F}_{p^f}\) has an absolute Frobenius endomorphism which becomes \(F\) if one applies \(W\). Finally the ghost maps give rise to a canonical:

\[
(7) \quad w : \text{Res}_{W_n} \mathcal{F} \to \prod_{i=0}^{n-1} F^i \mathcal{F}.
\]

Let us set \(w_0\) for its 0th component. Finally we denote the functor \(S \mapsto M \otimes_{W(\mathbb{F}_{p^f})} S\) by \(M\), provided that \(M\) is a torsionfree. Notice that
there is a natural short exact sequence:

$$0 \to \text{Res}_{W_n} FM \xrightarrow{V_n} \text{Res}_{W_{n+1}} M \xrightarrow{w_0} M \to 0$$

We will write $\text{Res}_{I_{n+1}} M$ for the kernel of the right-hand side map and

$$V_n^{-1} : \text{Res}_{I_{n+1}} M \to \text{Res}_{W_n} FM$$

for the inverse of the left-hand side one. Needless to say that the functor $FM$ is no other than $W(F_{p^f}) \otimes_{F,W(F_{p^f})} M$, finally observe $F_n \circ V_n = p$.

2.2. $\Phi$-data. Let $S$ be a scheme, let $G$ be a group-valued contravariant functor on the category of $S$-schemes, and let $P$ be a set-valued contravariant functor on the category of $X$-schemes, where $X$ is a scheme over $S$. If $P$ is endowed with a right $G$-action, such that each non-empty $P(Y)$ becomes a principal homogeneous $G(Y)$-space, then we will say that $P$ is a formal principal homogeneous space for $G$ over $X$. We will say that $P$ is a locally trivial principal homogeneous space for $G$ over $X$ if the following holds in addition:

- $P$ satisfies the sheaf axiom for fpqc (i.e. faithfully flat and quasicompact) morphisms.
- $P(Y)$ is non-empty for at least one fpqc map $Y \to X$.

Notice that locally trivial principal homogeneous spaces for affine group schemes are affine schemes, by descent theory (and [10, Proposition (2.7.1.xiii)]). In this case it is interesting to consider the groupoid $\text{Tors}_X(G)$ of all locally trivial principal homogeneous spaces for $G$ over $X$. An obvious notion of pull-back defines a fibration over the category of $S$-schemes, and we obtain the stack $\text{Tors}(G)$ if we think of the base category as the big fpqc site to $S$. In case $G$ is flat and affine over $S = \text{Spec} B$, where $B$ is a Dedekind ring, we will utilize the additive rigid $\otimes$-category $\text{Rep}_0(G)$ of 'representations', i.e. $G-B$-modules which are finitely generated and projective over $B$, this has been introduced and studied in [23, II.4.1.2.1], along with various functors defined thereon: Write $\text{Vec}_X$ for the additive rigid $\otimes$-category of vector bundles on $X$, i.e. of locally free, finitely generated $\mathcal{O}_X$-modules. Then there is a natural forgetful fiber functor

$$\varpi^G_0 : \text{Rep}_0(G) \to \text{Vec}_S,$$

which takes a $G-B$-module to its underlying $B$-module, furthermore for every $P \in \text{Tors}_X(G)$ there is also a twisted fiber functor

$$\varpi^G_P : \text{Rep}_0(G) \to \text{Vec}_X,$$

which takes a representation $(V, \rho)$ to the locally free $\mathcal{O}_X$-module $P \times^G V$, which is obtained by the usual extension of structure group (the
map $\rho$ takes $\text{Tors}_X(\mathcal{G})$ naturally to $\text{Tors}_X(\text{GL}(\mathcal{V}/B))$, which in turn is naturally contained in $\text{Vec}_X$, cf. [23 II.3.2.3.4]).

To fix ideas further consider a smooth group scheme over $W(\mathbb{F}_{p'})$, together with a cocharacter $\mu : \mathbb{G}_m \times W(\mathbb{F}_{p'}) \to \mathcal{G}$ and observe that $\varpi^G_0$ inherits a graduation of type $\mathbb{Z}$ from it, [23 Corollaire IV.1.2.2.2]. Let $\mathcal{G}^-$ be the group of ascendingly filtered $\otimes$-automorphisms of $\varpi^G_0$, and let $\mathcal{G}^+$ be the group of descendingly filtered $\otimes$!-automorphisms of $\varpi^G_0$, in the sense of [23 IV.2.1.3]. These group functors are representable by (and will be identified with) closed subschemes of $\mathcal{G}$ and these are smooth over $W(\mathbb{F}_{p'})$, by [23 Proposition IV.2.1.4.1]. Let $\text{Ad}^-$ and $\text{Ad}^+$ stand for the natural adjoint representations of $\mathcal{G}$ and $\mathcal{G}^-$ on their respective Lie algebras, which we denote by $\mathfrak{g}$ and $\mathfrak{g}^-$. We need notations for two more $\mathcal{G}^-$-representations of some significance: Observe that $\text{Ad}^-$ is a subrepresentation of $\text{Ad}|_{\mathcal{G}^-}$, and write $\text{Ad}_-$ for the quotient of these two $\mathcal{G}^-$-representations. Its underlying $W(\mathbb{F}_{p'})$-module may be identified with the Lie algebra of $\mathcal{G}^+$, which we denote by $\mathfrak{g}^+$, this is due to $\mathfrak{g} \cong \mathfrak{g}^- \oplus \mathfrak{g}^+$ (one should be cautioned to not confuse $\text{Ad}_-$ with the adjoint representation of $\mathcal{G}^+$). By passage to the determinant one obtains a character $\text{ch} : \mathcal{G}^- \to \mathbb{G}_m$, or equivalently a one-dimensional $\mathcal{G}^-$-representation on the invertible module $\mathfrak{f} := (\det \mathfrak{g}) \otimes W(\mathbb{F}_{p'}) (\det \mathfrak{g}^-)$.

It also follows that the group multiplication identifies $\mathcal{G}^- \times_{W(\mathbb{F}_{p'})} \mathcal{G}^+$ with an open subscheme $\mathcal{G}^\times \subset \mathcal{G}$ (use the implication "b$\Rightarrow$a") in [10 Théorème (17.9.1)] followed by the "d$\Rightarrow$b")-one in the result [10 Théorème (17.11.1)]). There is a canonical homomorphism

$$\mu^{-\text{Int}} : \mathbb{A}^1_{W(\mathbb{F}_{p'})} \to \text{End}(\mathcal{G}^-),$$

of multiplicative monoids which extends the interior $\mathbb{G}_m \times W(\mathbb{F}_{p'})$-action $\text{Int}_{\mathcal{G}^-} \circ \mu^{-1}$ on $\mathcal{G}^-$, and $\mu$ will be called a $\Phi$-datum for $\mathcal{G}$ if and only if the following holds:

(M1) The only $\mu$-weight of $\mathfrak{g}^+$ is 1.
(M2) $\mathcal{G}$ has connected fibers.

If (M1) holds, then there exists an isomorphism

$$e^+ : \frac{\mathfrak{g}^+}{\mathfrak{g}^-} \cong \mathcal{G}^+,$$

such that the interior $\mathbb{G}_m \times W(\mathbb{F}_{p'})$-operation of $\mu$ on $\mathcal{G}^+$ becomes the scalar multiplication of this vector group scheme.

We write $\mathcal{H}^\pm_n$ and $\mathcal{H}_n$ for the affine group schemes $\text{Res}_{W_n} \mathcal{G}^\pm$ and $\text{Res}_{W_n} \mathcal{G}$, which are smooth of relative dimensions $n \dim \mathcal{G}^\pm / W(\mathbb{F}_{p'})$.
and \( n \dim \mathcal{G}/W(F_{p^{f}}) \). Clearly its non-finite type affine proalgebraic pendants will be denoted by \( \mathcal{H}^{\pm} \) and \( \mathcal{H} \).

The closed subgroup \( W(F_{p^{f}}) \)-scheme \( \mathcal{I}_n \subset \mathcal{H}_n \) is defined to be the inverse image of \( \mathcal{G}^{-} \) by means of the 0th component \( w_0 \) of the ghost map \( w \) and the subgroup \( \mathcal{J}_n \subset \mathcal{H}_n^{+} \) is defined to be the inverse image of the trivial group. We will need the following:

**Lemma 2.1.** Suppose that \( R \) is a commutative ring.

(i) If \( p \in \text{rad}(R) \) then \( \text{rad}(W_n(R)) \) is equal to the inverse image of \( \text{rad}(R) \) via the 0th ghost coordinate.

(ii) If \( n \) is finite and if \( p \in \sqrt{0_R} \), then \( I_n(R) \) is nilpotent and \( \sqrt{0_{W_n(R)}} \) is equal to the inverse image of \( \sqrt{0_R} \) via the 0th ghost coordinate.

**Proof.** By a limit process it is enough to check the assertion (i) for \( n \neq \infty \). By induction it suffices to check that \( V^{n-1}W_n(R) \) is contained in \( \text{rad}(W_n(R)) \), which is implied by

\[
1 - (1 - V^{n-1})^{-1} = V^{n-1} \left( \frac{x_0}{p^{n-1}x_0 - 1} \right),
\]

where \( x_0 \) is the 0th ghost coordinate of \( x \). The assertion (ii) is trivial. \( \square \)

**Corollary 2.2.** Let \( R \) be a \( W(F_{p^{f}}) \)-algebra whose radical contains \( p \), then \( I_n(R) \) is the product of \( \mathcal{H}_n^{-} \) and \( \mathcal{J}_n(R) \) (in any order for any \( n \)).

**Proof.** The diagram

\[
\begin{array}{ccc}
\text{Spec } W_n(R) & \xrightarrow{g} & \mathcal{G} \\
\text{Spec } w_0 \uparrow & & \uparrow \\
\text{Spec } R & \xrightarrow{g_0} & \mathcal{G}^{-}
\end{array}
\]

assures us of \( g_0 \in \mathcal{G}^{-}(R) \) and \( g \in \mathcal{G}^{-}(W_n(R)) \), by lemma 2.1. The equation \( \mathcal{G}^{x} = \mathcal{G}^{-} \times_{W(F_{p^{f}})} \mathcal{G}^{+} \) finishes the proof. \( \square \)

For example it now follows that the natural group multiplication identifies the \( W_m(F_{p^{f}}) \)-scheme \( \mathcal{I}_n \times_{W(F_{p^{f}})} W_m(F_{p^{f}}) \) with the cartesian product of the \( W_m(F_{p^{f}}) \)-schemes \( \mathcal{J}_n \times_{W(F_{p^{f}})} W_m(F_{p^{f}}) \) and \( \mathcal{H}_n^{-} \times_{W(F_{p^{f}})} W_m(F_{p^{f}}) \).

**2.3. Twisted Frobenius maps, part I.** Let us denote the natural Frobenius groupscheme homomorphisms from \( \mathcal{H}_n^{\pm} \) and \( \mathcal{H}_{n+1} \) to \( \mathcal{F}^{\mathcal{H}_n^{\pm}} \), and \( \mathcal{F}^{\mathcal{H}_n} \) by \( F_n^{\pm} \) and \( F_n \). There are yet several more interesting homomorphisms \( \Phi_n^{\pm} \) and \( \Phi_n \) with targets one of the groups \( \mathcal{F}^{\mathcal{H}_n^{\pm}} \) and
Notice that \( \mu^{-\text{Int}} \) induces a functorial action of the multiplicative monoid \( W(\mathbb{F}_{p'}) \)-schemes \( W_n \) on the group \( W(\mathbb{F}_{p'}) \)-schemes \( H_n \). When evaluated at the element \( p = p[1_{W(\mathbb{F}_{p'})}]_{n} \in W_{n}(W(\mathbb{F}_{p'})) \) one obtains a map

\[
\mu^{-\text{Int}}(p) : H_n \to H^{-}_{n}; g \mapsto \mu(p)^{-1}g\mu(p),
\]

which gives us further

\[
\Phi_{n}^{-} : H^{-}_{n+1} \to F\mathcal{H}^{-}_{n}
\]

ones, when composed with \( F_{n}^{-} \). Finally Zink’s “\( V^{-1} \)” makes sense as an \( F \)-morphism, of group \( W(\mathbb{F}_{p'}) \)-schemes i.e.

\[
\Phi_{n}^{+} : J_{n+1} \to F\mathcal{H}^{+}_{n},
\]

as in [9]. Now we are in a position to introduce the twisted Frobenius map

\[
\Phi_{n} : \mathcal{I}_{n+1} \to F\mathcal{H}_{n}
\]

that will appear to have pivotal importance: Write \( A \) for the \( W(\mathbb{F}_{p'}) \)-algebra of which the spectrum is \( \mathcal{I}_{n+1} \), and consider the faithfully flat quasicompact covering \( \text{Spec}(1+pA)^{-1}A \cup \text{Spec} A[\frac{1}{p}] \). On the first chart the corollary 2.2 on factorisations allows us to define \( \Phi_{n} \) to be the product of \( \Phi_{n}^{-} \) and \( \Phi_{n}^{+} \), and on the second chart we can most easily define it by the same formula as \( \Phi_{n}^{-} \), namely by \( F_{n} \circ \text{Int}_{\mathcal{I}_{n+1} \times W(\mathbb{F}_{p'})} K(\mathbb{F}_{p'}) (\mu(p)^{-1}) \).

The ghost coordinates may be used to check that these definitions coincide on \( \text{Spec}(1+pA)^{-1}A[\frac{1}{p}] \), so that we may descent.

By a \( \Phi \)-datum we mean a pair \((\mathcal{G}, \mu)\) where \( \mathcal{G} \) is a \( W(\mathbb{F}_{p'}) \)-group, and \( \mu \) is a \( \Phi \)-datum for some scalar extension \( \mathcal{G} \times_{W(\mathbb{F}_{p'})} W(\mathbb{F}_{p'}) \). If a \( \Phi \)-datum is given we continue to denote \( \text{Res}_{W_n} \mathcal{G} \) by \( \mathcal{H}_n \), however do notice that the aforementioned twisted Frobenius map reads \( \Phi_{n} : \mathcal{I}_{n+1} \to \mathcal{H}_{n} \times_{W(\mathbb{F}_{p'})} W(\mathbb{F}_{p'}) \), so that the following definition is meaningful:

**Definition 2.3.** Let \( \mu : \mathbb{G}_{m} \times W(\mathbb{F}_{p'}) \to \mathcal{G} \times_{W(\mathbb{F}_{p'})} W(\mathbb{F}_{p'}) \) be a \( \Phi \)-datum, and let \( X \) be a \( W(\mathbb{F}_{p'}) \)-scheme. A length \( n \)-truncated \( 3n \)-display over \( X \) with \( \mathcal{G} \)-structure is a quadruple \( P = (P_n, P_{n+1}, Q_{n+1}, \Phi_n) \) consisting of the following data:

1. **(D1)** \( X \)-schemes \( P_n \) and \( P_{n+1} \) having the structure of a locally trivial principal homogeneous space over \( X \) for the groups \( \mathcal{H}_n \) and \( \mathcal{H}_{n+1} \), and being equipped with a \( \mathcal{H}_{n+1} \)-equivariant forgetful \( X \)-map \( P_{n+1} \to P_n \)

2. **(D2)** A closed subscheme \( Q_{n+1} \hookrightarrow P_{n+1} \) that is \( \mathcal{I}_{n+1} \)-invariant and acquires the structure of a locally trivial principal homogeneous space over \( X \) for this group.
(D3) A $X$-morphism $\Phi_n$ from $Q_{n+1}$ to the length $n$ truncation $P_n$ such that the diagram

\[
\begin{array}{c}
Q_{n+1} \times W(F_{p'}) I_{n+1} \\
\Phi_n \times W(F_{p'}) \\
P_n \times W(F_p) H_n
\end{array} \longrightarrow \begin{array}{c}
Q_{n+1} \\
\Phi_n \\
P_n \times W(F_p) H_n
\end{array}
\]

of $X$-schemes commutes.

A $3n$-display over $X$ with $G$-structure is defined to be an equivariant closed immersion $Q \leftarrow P$ of locally trivial principal homogeneous spaces under $H \times W(F_{p'}) X$ and $I \times W(F_{p'}) X$, together with an additional map $\Phi : Q \rightarrow P$ satisfying an appropriate version of (D3).

Adopting the convention of [30] we denote the $\Phi$ (resp. $\Phi_n$)-operation on elements of the group $I$ (resp. $I_{n+1}$) by $"\Phi g"$ (resp. $"\Phi_n g"$) in order to distinguish it from the map $\Phi$ which determines the display. Isomorphisms between the possibly truncated $3n$-displays over $X$ with $G$-structure are defined in the obvious way and give rise to groupoids $B^n(G, \mu)$, which are fibered over the base category of $W(F_{p'})$-schemes. One more piece of notation will prove useful: In case that $\epsilon : G_m \times W(F_{p'}) \rightarrow G \times W(F_{p}) W(F_{p'})$ is a cocharacter which lies in the center of $G$, there is another $\Phi$-datum $(G, \mu \epsilon)$ giving rise to the same twisted Frobenius map and hence to an absolutely identical notion of length $n$-truncated $3n$-display over $X$ with $G$-structure. The (slightly tautological) 1-morphism $B^n(G, \mu) \rightarrow B^n(G, \mu \epsilon)$, thus defined shall be denoted by $P \mapsto P(\epsilon)$, and is called twist.

For all $n$ including $\infty$ one may interpret $B^n(G, \mu)$ as the unique fibered category rendering the diagram

\[
\begin{array}{c}
B^n(G, \mu) \\
\Phi_n \times \Psi_n
\end{array} \longrightarrow \begin{array}{c}
\text{Tors}(I_{n+1}) \\
\text{Tors}(H_n) \times \text{Tors}(H_n)
\end{array}
\]

2-cartesian, here $\Psi_n$ is the composition of the inclusion of $I_{n+1}$ into $H_{n+1}$ and the truncation $H_{n+1} \rightarrow H_n$. It follows immediately that $B^n(G, \mu)$, being the fibered product of stacks over a (pre)stack is indeed a stack for the fpqc topology.

Furthermore, it is noteworthy that the fiber of the upper horizontal arrow over any $S$-valued point of $\text{Tors}(I_{n+1})$ is just the fiber of $\Delta_{\text{Tors}(H_n)}$ over that point, in particular we obtain a morphism

\[
b^n(G, \mu) := c(I_{n+1}) \times_{\text{Tors}(I_{n+1})} B^n(G, \mu) : \text{Spec } W(F_{p'}) \times W(F_p) H_n \rightarrow B^n(G, \mu).
\]
Remark 2.4. Later on we need some facts on the structure of the underlying locally trivial principal homogeneous spaces: If \((P_n, P_{n+1}, Q_{n+1}, \Phi_n)\) is as in definition 2.3, then the natural truncations \(Q_{n+1} \to Q_n\) and \(P_{n+1} \to P_n\) are smooth morphisms of smooth \(X\)-schemes for all finite \(n\), this follows from [10, Proposition (2.7.1.iv)] in conjunction with [10, Proposition (17.7.1)]. If \(P = (P, Q, \Phi)\) is a \(3n\)-display with \(G\)-structure over \(X\), then neither \(Q\) nor \(P\) are of finite type, however the natural truncations \(Q \to Q_n\) and \(P \to P_n\) are still formally smooth and flat \(X\)-schemes: This follows from an elementary argument as \(Q \cong \lim_{\leftarrow} Q_n\) and \(P \cong \lim_{\leftarrow} P_n\) (use [10, Proposition (2.7.1.viii)] in conjunction with [10, Proposition (8.2.5)]).

2.4. The fibered groupoid of twisted conjugacy classes, part I.

An object in \(B_n^X(G, \mu)\) will be called banal if the underlying principal homogeneous space \(Q_{n+1}\) is the trivial one. The fibered full subcategory of banal displays will be denoted by \(B_n^X(G, \mu)\), it is a prestack, and it allows a very concrete description: The objects are maps

\[ \Phi_n : I_{n+1} \times W(F_{pf}) X \to H_n \times W(F_{pf}) X, \]

of the form \(g \mapsto U^{\Phi_n} g\), and thus indexed by the group elements \(\Phi_n(1) = U \in H_n(X)\). Consider a morphism \(\psi_{n+1}\) from the object that has \(\Phi_{n+1}(1) = U'\) to the object that has \(\Phi_n(1) = U\). In the first place \(\psi_{n+1}\) must be an automorphism of \(I_{n+1} \times W(F_{pf}) X\) (regarded as a trivial principal homogeneous space), hence equal to multiplication by some element \(k \in I_{n+1}(X)\). This morphism preserves the \(\Phi_n\)-structures if and only if the diagram

\[
\begin{array}{ccc}
Q'_{n+1} & \xrightarrow{\Phi_n} & P'_{n+1} \\
\psi_{n+1} \downarrow & & \downarrow \psi_n \\
Q_{n+1} & \xrightarrow{\Phi_n} & P_n
\end{array}
\]

commutes. Here \(\psi_n\) is the truncation of \(\psi_{n+1}\), hence it is equal to the multiplication by the same element \(k\). However \(Q'_{n+1} = Q_{n+1} = I_{n+1} \times W(F_{pf}) X\), and therefore one must have

\[ U' = k^{-1} U^{\Phi_n} k, \]

because the upper horizontal arrow is given by \(g \mapsto U'^{\Phi_n} g\) and the lower one by \(g \mapsto U^{\Phi_n} g\). Our explicit description entails an important equivalence of categories:

\[(11) \quad B^n_X(G, \mu) \overset{\cong}{\to} B^n_{\Gamma(X, \mathcal{O}_X)}(G, \mu).\]

For later use we note two more lemmas:
Lemma 2.5. Let $R$ be a $W(\mathbb{F}_p)$-algebra, and let $\mathcal{P}$ be a length $n$-truncated $3n$-display with $G$-structure over $\text{Spec} R$.

(i) If $I \subset R$ is a nilpotent ideal, then $\mathcal{P}$ is banal if and only if $\mathcal{P} \times_{\text{Spec} R} \text{Spec} R/I$ is banal.

(ii) If $pR$ is nilpotent then there exists an étale and faithfully flat $R$-algebra over which $\mathcal{P}$ becomes banal.

Proof. The first of these two assertions follows from remark 2.4, and in the finite length case both of them are trivial, due to [10] Corollaire (17.16.3.ii)]. In order to prove the infinite length case of the remaining assertion we may assume $p1_R = 0$, due to [10] Théorème 18.1.2]. Now we can clearly find an étale and faithfully flat $R$-algebra $R'$ over which the level 0-truncation of $\mathcal{P}' := \mathcal{P} \times_R R'$ is banal. However, notice that there exists a short exact sequence

$$0 \to g \otimes \mathbb{F}_p \to I_{n+1} \times \mathbb{F}_p \to I_n \times \mathbb{F}_p \to 1.$$ 

By induction on $n \in \mathbb{N}$ we deduce the banality of all of the level $n$-truncations of $\mathcal{P}'$, due to [16] Chapter III, Proposition 3.7]. Finally it follows that $\mathcal{P}'$ is banal.

Lemma 2.6. Let $k$ be a perfect field that contains $\mathbb{F}_p$, and let $\mathcal{P}$ and $\mathcal{P}'$ be length $n$-truncated $3n$-display with $G$-structure over $\text{Spec} k$, where $n \in \mathbb{N} \cup \{\infty\}$. Then there exists a finite Galois extension $l/k$, a polynomial $h \in l[t]$ with $h(0) \neq 0 \neq h(1)$, and a length $n$-truncated $3n$-display $\tilde{\mathcal{P}}$ with $G$-structure over $\text{Spec} l[t]_{\mathbb{F}_p}$, such that $\mathcal{P} \times_k l \cong \tilde{\mathcal{P}}|_{t=0}$ and $\mathcal{P}' \times_k l \cong \tilde{\mathcal{P}}|_{t=1}$.

Proof. Pick a smooth map $\text{Spec} W(\mathbb{F}_p)[x_1, \ldots, x_m] \xrightarrow{r} G$, where $f$ is a polynomial which is not contained in $pW(\mathbb{F}_p)[x_1, \ldots, x_m]$, and let $\text{Spec} W(\mathbb{F}_p)[x_1, \ldots, x_m]_{\mathbb{F}_p} := S \xrightarrow{g} G$ be defined by $g(x_1, \ldots, x_m) = r(x_1, \ldots, x_m)(x_{m+1}, \ldots, x_{2m})$, where $g(x_1, \ldots, x_m)$ is the polynomial defined by $f(x_1, \ldots, x_m)(x_{m+1}, \ldots, x_{2m})$. The map $g$ is smooth and surjective. Now choose a finite Galois extension over which $\mathcal{P}$ and $\mathcal{P}'$ become banal, and write $U$ and $U'$ for representatives. Over a possibly further finite Galois extension $k'$, these can be lifted to $S$, in order to achieve $U = g(x_1, \ldots, x_{2m})$, and $U' = g(x'_1, \ldots, x'_{2m})$, where $x_1, \ldots, x_{2m}, x'_1, \ldots, x'_{2m} \in W_n(l)$. Put $z_i := x_i + [t](x'_i - x_i) \in W_n(l[t])$, and write $z \in l[t]$ for the (mod $p$) reduction of the polynomial $h(t) := g(x_1 + t(x'_1 - x_1), \ldots, x_{2m} + t(x'_{2m} - x_{2m})) \in W_n(l[t])$. Observe that $z := (z_1, \ldots, z_{2m})$ is a $W_n(l[t]_{\mathbb{F}_p})$-valued point of $S$ (the 0th ghost coordinate of $g(z_1, \ldots, z_{2m})$ is equal to $h$). Note $h(0) \neq 0 \neq h(1)$, and let $\tilde{\mathcal{P}}$ be the length $n$-truncated $3n$-display with $G$-structure over $\text{Spec} l[t]_{\mathbb{F}_p}$ which is represented by the element $\tilde{U} := g(z) \in G(W_n(l[t]_{\mathbb{F}_p}))$. □
2.5. **Separatedness, part I.** Let $\phi$ be an automorphism of a length $n$-truncated $3n$-display $\mathcal{P}$ over $S$ with $\mathcal{G}$-structure. In the sequel we need to know that the condition ”$\phi = \text{id}$” defines a closed subscheme of $S$. If $S = \text{Spec} \, A$, for instance, then there should exist an ideal $I \subset A$ such that $\phi \times_A B = \text{id}_{\mathcal{P} \times_A B}$ holds if and only if $IB$ vanishes. The following proposition achieves this:

**Proposition 2.7.** Suppose that $\mathcal{P}'$ and $\mathcal{P}$ are $n$-truncated $3n$-displays over $S$ with $\mathcal{G}$-structure. Then there exists a relatively affine $S$ scheme which represents the set-valued contravariant functor $X \mapsto \text{Hom}(\mathcal{P}' \times_S X, \mathcal{P} \times_S X)$ on the category of $S$-schemes.

**Proof.** The functor in question satisfies the sheaf axiom for fpqc morphisms. Descent theory implies that one may assume $\mathcal{P}'$ and $\mathcal{P}$ to be banal, hence represented by display matrices $U', U \in \mathcal{H}_n(\Gamma(S, \mathcal{O}_S))$. In this case the morphisms from $\mathcal{P}' \times_S X$ to $\mathcal{P} \times_S X$ are the solutions of the equation $U' = h^{-1} U^\Phi \cdot h$ in $H_{n+1}(\Gamma(X, \mathcal{O}_X))$. This is obviously representable by a closed subscheme of $H_{n+1}(\mathcal{F}_{\mathcal{P}'}) S$.

**Remark 2.8.** The result may be also be interpreted by saying that the diagonal 1-morphism $\Delta_{\mathcal{B}^n(\mathcal{G}, \mu)} : \mathcal{B}^n(\mathcal{G}, \mu) \to \mathcal{B}^n(\mathcal{G}, \mu) \times_{\mathcal{F}_{\mathcal{P}'}} \mathcal{B}^n(\mathcal{G}, \mu)$ is schematic and affine, hence separated and quasicompact. Moreover, for finite $n$ one may regard $\mathcal{B}^n(\mathcal{G}, \mu)$ as an algebraic stack over $\mathcal{F}_{\mathcal{P}'}$, basically because the 1-morphism $b^n(\mathcal{G}, \mu) : \text{Spec}(\mathbb{F}_{\mathcal{P}'}) \times_{\mathcal{F}_{\mathcal{P}'}} \mathcal{H}_n \to \mathcal{B}^n(\mathcal{G}, \mu)$ is a smooth presentation. We refrain from doing that, due to our (non-standard) choice of Grothendieck topology being the fpqc one. It should also be noticed, that the prime object of this work is the limit stack $\mathcal{B}(\mathcal{G}, \mu) := \mathcal{B}^\infty(\mathcal{G}, \mu)$, which is by no means algebraic.

2.6. **Compatibility with limits.**

**Lemma 2.9.** Let $R$ be a complete local noetherian $W(\mathbb{F}_{\mathcal{P}'})$-algebra with maximal ideal $m$. Then the natural forgetful functor

$$
\mathcal{B}^n_R(\mathcal{G}, \mu) \to 2 - \lim_{\text{max}} \mathcal{B}^n_{R/m^\nu}(\mathcal{G}, \mu)
$$

is an equivalence of categories, for all $n$ including $\infty$.

**Proof.** The computation of the essential image is the only issue, so consider an object in the right hand side category, which is a sequence $\mathcal{P}^{(\nu)} \in \mathcal{B}^n_{R/m^\nu}(\mathcal{G}, \mu)$ endowed with transition maps $\psi^{(\nu)} : \mathcal{P}^{(\nu)} \to \mathcal{P}^{(\nu+1)} \times_{R/m^{\nu+1}} R/m^\nu$. Under the additional assumption that every $\mathcal{P}^{(\nu)}$ is banal one can easily construct a preimage: Let $U^{(\nu)} \in \mathcal{H}_n(R/m^\nu)$ represent $\mathcal{P}^{(\nu)}$, so that giving $\psi^{(\nu)}$ boils down to the equation $U^{(\nu)} = (h^{(\nu)})^{-1} U^{(\nu+1)} \Phi h^{(\nu)}$ where $U^{(\nu+1)}$ stands for the image of $U^{(\nu+1)}$ in $\mathcal{H}_n(R/m^\nu)$,
and \( h^{(v)} \in \mathcal{I}_{n+1}(R/m^\nu) \). Next we choose lifts \( \tilde{h}^{(v)} \in \mathcal{I}_{n+1}(R) \) of the \( h^{(v)} \)'s, and we define sequences of elements \( m^{(v)} := \tilde{h}^{(v-1)}, \ldots, \tilde{h}^{(1)} \in \mathcal{I}_{n+1}(R) \) and \( O^{(v)} := (\overline{m}^{(v)})^{-1} \)\( U^{(v)} \Phi m^{(v)} \in \mathcal{H}_n(R/m^\nu) \), where \( \overline{m}^{(v)} \) stands for the image of \( m^{(v)} \) in \( \mathcal{I}_{n+1}(R/m^\nu) \). It is clear that the sequence of \( O^{(v)} \in \mathcal{H}_n(R/m^\nu) \) together with neutral transitions lies in the image of the functor in question, and it is also clear that the sequence of \( \overline{m}^{(v)} \in \mathcal{I}_{n+1}(R/m^\nu) \) defines an isomorphism \( \lim_{\nu} O^{(v)} \rightarrow \lim_{\nu} U^{(v)} \).

In the general case there still exists a finite Galois extension \( k := R/m \) such that \( \mathcal{P}^{(1)} \times_k l \) is banal, according to part (ii) of lemma 2.5. Write \( B \) for the unique finite Galois extension over \( R \) of which the special fiber is \( l \), by [10, Proposition (18.3.2)]. For every index \( \nu \) we can deduce the banality of \( \mathcal{P}^{(v)} \times_{R/m^\nu} B/Bm^\nu \), according to part (i) of lemma 2.5. This completes the proof because one can use descent along \( R \rightarrow B \).

\[ \square \]

2.7. Example. We give a brief sketch of an example: Fix a \( p \)-adically separated and complete ring \( R \), and let us coin the term \( n \)-truncated \( 3n \)-display over \( R \) for the following class of quadruples \((M, N, F, V^{-1})\):

- \( M \) is a finitely generated projective \( W_{n+1}(R) \)-module
- \( N \subseteq M \) is a submodule which contains \( I_{n+1}(R)P \)
- \( M/N \) is projective, when regarded as a module over \( R \)
- \( F : M \rightarrow W_n(R) \otimes W_{n+1}(R)M \) and \( V^{-1} : N \rightarrow W_n(R) \otimes W_{n+1}(R)M \) are \( F_n \)-linear maps, satisfying \( V^{-1}(V a x) = a V^{-1}(x) \) for all \( x \in M \) and \( a \in W_n(R) \) (of which the Verschiebung is an element in \( W_{n+1}(R) \)).
- the image of \( V^{-1} \) generates \( M \) as a \( W_{n+1}(R) \)-module.

An morphism between the \( n \)-truncated \( 3n \)-displays \((M', N', F, V^{-1})\) and \((M, N, F, V^{-1})\) is a \( W_{n+1}(R) \)-linear mapping \( M' \rightarrow M \) sending \( N' \) to \( N \) and such that the maps \( F \) and \( V^{-1} \) are preserved. We will write \( \text{Dis}_R^n \) for the additive category of \( n \)-truncated \( 3n \)-displays. We will write \( \text{Dis}_R^n(d, h) \) for the full subcategory made up of quadruples \((M, N, F, V^{-1})\) with \( \text{rank}_{W_{n+1}(R)} M = h \) and \( \text{rank}_R M/N = d \). When discarding all morphisms other than the isomorphisms in \( \text{Dis}_R^n(d, h) \) we obtain a groupoid: \( \text{Dis}_R^n(d, h)^* \). Consider the cocharacter

\[ \mu_{d,h} : \mathbb{G}_m \rightarrow \text{GL}(h); z \mapsto \text{diag}(z, \ldots, z, 1, \ldots, 1), \]

then there is a natural equivalence of categories

\[ \text{Co}_{d,h} : \mathcal{B}_R^n(\text{GL}(h), \mu_{d,h}) \xrightarrow{\cong} \text{Dis}_R^n(d, h)^*. \]
By the theory of Witt-descent it suffices to construct the restriction of \( \text{C}^d,h \) to the subcategory \( \text{B}^n_R(\text{GL}(h),\mu_{d,h}) \) of banal displays. Whenever some object of \( \text{B}^n_R(\text{GL}(h),\mu_{d,h}) \) is represented by a matrix \( U \in \text{GL}(h,W_n(R)) \) the effect of \( \text{C}^d,h \) on it is defined as follows: Write \( U^\text{C}^d,h \mapsto (M,N,F,V^{-1}) \) where:

\[
M := W_{n+1}(R)^{\oplus h} \\
F : M \to W_n(R) \otimes_{W_{n+1}(R)} M; \begin{pmatrix} x \\ y \end{pmatrix} \mapsto U \begin{pmatrix} Fx \\ p^Fy \end{pmatrix} \\
N := I_{n+1}(R)^{\oplus d} \oplus W_{n+1}(R)^{\oplus h-d} \\
V^{-1} : N \to W_n(R) \otimes_{W_{n+1}(R)} M; \begin{pmatrix} x \\ y \end{pmatrix} \mapsto U \begin{pmatrix} V^{-1}x \\ Fy \end{pmatrix}
\]

Whenever \( k \) represents a morphism between two banal displays which are represented by the matrices \( U' = k^{-1}U^\Phi \) and \( U \), then the effect of \( \text{C}^d,h \) on it is defined as follows: Write \( k \) as a \( h \times h \)-block matrix \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) with \( B \equiv 0 \mod I_{n+1}(R) \), and let \( \text{C}^d,h(k) \) act on \( \text{C}^d,h(U') = W_{n+1}(R)^{\oplus h} \) by plain matrix multiplication.

The functor \( \text{C}^d,h \) is fully faithful. In order to determine the set of \( n \)-truncated \( 3n \)-displays \((M,N,F,V^{-1})\) which lie in the essential image one has to observe that the ring \( W_{n+1}(R) \) is separated and complete with respect to the \( I_{n+1}(R) \)-adic topology, so that the modules \( N \) can always be written as \( I_{n+1}(R)T \oplus L \) for suitable \( (n\)-truncated) normal decompositions \( M = T \oplus L \). This allows one to consider the map

\[
(12) \quad F \oplus V^{-1} : M \to W_n(R) \otimes_{W_{n+1}(R)} M; x + y \mapsto F(x) + V^{-1}(y)
\]

where \( x \in T \), and \( y \in L \). Following the ideas in, and using the language of [30, Lemma 9] one proves (12) to be a \( F_n \)-linear isomorphism. Over some fpqc extension there exist \( W_{n+1}(R) \)-bases of \( T \) and of \( L \), with respect to which (12) can be written as \( F_n \) composed with some \( U \in \text{GL}(h,W_n(R)) \) and we are done.

3. Graded \((A,\tau)\)-modules

In the sequel we need a number of rather harmless constructions taking place within the category of graded Frobenius modules. We collect them in this section, primarily in order to introduce some notation. Let \( A \) be a \( p \)-adically separated and complete ring and let \( \tau \) be an endomorphism on \( A \), which agrees with the absolute Frobenius on \( A/pA \). By a \( \mathbb{Z}/r\mathbb{Z} \)-graded \((A,\tau)\)-module we mean:
\begin{itemize}
  \item a finitely generated projective \( \mathbb{Z}/r\mathbb{Z} \)-graded \( A \)-module 
  \( M = \bigoplus_{\sigma \in \mathbb{Z}/r\mathbb{Z}} M_{\sigma} \),
  \item a pair of \( A \)-linear maps \( F^\natural : A \otimes_{\tau, A} M \to M \) and 
  \( V^\natural : M \to A \otimes_{\tau, A} M \) of degrees \(-1\) and \(1\), and
  \item a \( r \)-tuple of non-negative integers \( w_{\sigma} \), such that 
  \( V^\natural \circ F^\natural |_{A \otimes_{\tau, A} M_{\sigma}} = p^{w_{\sigma}} \text{id}_{A \otimes_{\tau, A} M_{\sigma}} \),
  \end{itemize}

and

\[ F^\natural \circ V^\natural |_{M_{\sigma}} = p^{w_{\sigma}+1} \text{id}_{M_{\sigma}} \]

too.

The invariant \( w \) is called the graded width and should be considered to
be part of the structure. By a homomorphism between \( \mathbb{Z}/r\mathbb{Z} \)-graded
\((A, \tau)\)-modules \( N \) and \( M \) of the same width \( w \) we mean \( A \)-linear homo-
morphisms \( f_{\sigma} : N_{\sigma} \to M_{\sigma} \) for each \( \sigma \in \mathbb{Z}/r\mathbb{Z} \), such that the diagrams:

\[
\begin{array}{ccc}
A \otimes_{\tau, A} M_{\sigma+1} & \xrightarrow{F^\natural} & M_{\sigma} \\
\tau(f_{\sigma+1}) \uparrow & & \uparrow f_{\sigma} \\
A \otimes_{\tau, A} N_{\sigma+1} & \xrightarrow{F^\natural} & N_{\sigma}
\end{array}
\quad
\begin{array}{ccc}
A \otimes_{\tau, A} M_{\sigma} & \xleftarrow{V^\natural} & M_{\sigma+1} \\
\tau(f_{\sigma+1}) \uparrow & & \uparrow f_{\sigma} \\
A \otimes_{\tau, A} N_{\sigma} & \xleftarrow{V^\natural} & N_{\sigma+1}
\end{array}
\]

are commutative. These form abelian groups \( \text{Hom}^w_{\mathbb{Z}/r\mathbb{Z}}(N, M) \), so that
the class of \( \mathbb{Z}/r\mathbb{Z} \)-graded \((A, \tau)\)-modules of some width \( w \) forms an
additive category \( (A, \tau) - \text{Mod}^w_{\mathbb{Z}/r\mathbb{Z}} \). Observe that there is a natural
duality defined on it, moreover the graded tensor product defines bi-
additive functors:

\[
(A, \tau) - \text{Mod}^w_{\mathbb{Z}/r\mathbb{Z}} \times (A, \tau) - \text{Mod}^{w'}_{\mathbb{Z}/r\mathbb{Z}} \xrightarrow{\hat{\otimes}} (A, \tau) - \text{Mod}^{w+w'}_{\mathbb{Z}/r\mathbb{Z}}.
\]

\textbf{Remark 3.1.} If one precomposes the map \( F^\natural \) with \( x \mapsto 1 \otimes_{\tau, A} x \), then
one obtains a semilinear endomorphism, i.e. a map \( F^\flat : M \to M \), that
satisfies

\[ F^\flat(ax) = \tau(a) F^\flat(x) \]

(and sends \( M_{\sigma+1} \) into \( M_{\sigma} \)). Dually there exists a semilinear endomor-
phism \( \tilde{F}^\flat \) on the dual \( A \)-module \( \tilde{M} = \text{Hom}(M, A) \), which (sends \( \tilde{M}_{\sigma+1} 
\)
to \( \tilde{M}_{\sigma} \) and) satisfies

\[ \tilde{F}^\flat(ay) = \tau(a) \tilde{F}^\flat(y) \]

and

\[ (F^\flat x, \tilde{F}^\flat y) = p^{w_{\sigma}+1} \tau(x, y), \]

for \( x \in M_{\sigma+1} \) and \( y \in \tilde{M}_{\sigma+1} \). If there exists an integer \( n \) such that
\( (F^\flat)^{\text{on}}(M) \subset pM \), then \( M \) is called \( F \)-nilpotent, and it is called \( V \)-nilpotent, if \( (\tilde{F}^\flat)^{\text{on}}(\tilde{M}) \subset p\tilde{M} \).
Let us also introduce the additive, rigid $\otimes$-category $(A, \tau) - \text{Mod}_Z/rZ$ consisting of finitely generated projective $Z/rZ$-graded $A$-modules together with a pair of mutually inverse $Z/rZ$-graded maps $F^\sharp : Q \otimes A \otimes_{\tau,A} M \to Q \otimes M$ and $V^\sharp : Q \otimes M \to Q \otimes A \otimes_{\tau,A} M$ of degrees $-1$ and $1$. Finally notice, that whenever $A$ is a $W(F_p)$-algebra, then the grading can be interpreted as an action of $W(F_p) \to \text{End}(M)$, by declaring $M_\sigma$ to be the $F^{-\sigma}$-eigenspace.

3.1. **Flexibility, part I.** By a mod $r$-multidegree we mean a map $d : Z \to Z$ with the following properties:

- $d(\omega) + r = d(\omega + r)$
- $\omega \leq d(\omega)$
- $d$ is monotone

Let us denote $\max\{\omega \mid d(\omega) \leq \sigma\} =: d^*(\sigma)$, and $\max\{d(\omega) - \omega \mid \omega \in Z\} = \max\{\sigma - d^*(\sigma) \mid \sigma \in Z\} =: |d|$. Let $\Sigma \subset Z$ be the image of $d$, and let $\sigma_1 < \ldots < \sigma_z (=: \sigma_0 + r)$ be the elements of a ($z$-element) set of representatives for the (mod $r$) congruence classes of $\Sigma$. Now write $[\omega_{j-1} + 1, \ldots, \omega_j]$ for the intervals $d^{-1}(\{\sigma_j\})$, so that $d(\omega_j) = \sigma_j$ and $d^*(\sigma_j) = \omega_j$. In this context we declare $\omega_{j+1}$ to be $\omega_j + r$, and $\sigma_{j+1}$ to be $\sigma_j + r$ for $j \in \{1, \ldots, z\}$. The purpose of this subsection is to define a certain functor

$$\text{Flex}^d : (A, \tau) - \text{Mod}_Z/rZ \to (A, \tau) - \text{Mod}_Z/rZ,$$

where $w^d_\omega := \sum_{\sigma = d(\omega - 1) + 1}^{d(\omega)} w_\sigma$, let us start out by putting:

$$\text{Flex}^d(M)_\omega = A \otimes_{\tau,A, \omega-\omega, A} M_{d(\omega)}.$$

For any $d$-stationary index $\omega$, i.e. one with $d(\omega + 1) = d(\omega)$ we have $A \otimes_{\tau,A} \text{Flex}^d(M)_{\omega+1} \cong \text{Flex}^d(M)_\omega$ and thus we may take this very isomorphism to constitute the $\omega$’th map $F^\sharp$ on $\text{Flex}^d(M)$. Otherwise we have to come up with a map from $A \otimes_{\tau,A} \text{Flex}^d(M)_{\omega_{j+1}}$ to $\text{Flex}^d(M)_{\omega_j}$ which boils down to:

$$A \otimes_{\tau,A, \omega_{j+1}-\omega_j, A} M_{\sigma_{j+1}} \to A \otimes_{\tau, \omega_{j+1}-\omega_j, A} M_{\sigma_{j}}$$

Our definition here is the image under $\tau_{\omega_{j+1}-\omega_j}$ of the map

$$A \otimes_{\tau, \sigma_{j+1}-\sigma_{j}, A} M_{\sigma_{j+1}} \to M_{\sigma_{j}}$$

which arises naturally from composition of $\sigma_{j+1} - \sigma_j$ many maps $F^\sharp$.

One proceeds analogously for the $V^\sharp$’s, use duality for example. The upshot is a graded $(A, \tau)$-module $\text{Flex}^d(M)$ of width equal to $w^d_\omega$. There are natural isomorphisms

$$\text{Flex}^d(M \otimes M') \cong \text{Flex}^d(M) \otimes \text{Flex}^d(M')$$
(in the category \((A, \tau) - \operatorname{Mod}_{\mathbb{Z}/r\mathbb{Z}}^{\text{mod}}(w + w')\)), whenever \(M\) and \(M'\) are \(\mathbb{Z}/r\mathbb{Z}\)-graded \((A, \tau)\)-modules of widths \(w\) and \(w'\) and analogously for duality.

**Lemma 3.2.** Write \(u_\omega := \sum_{\sigma = \omega + 1}^{d(\omega)} w_\sigma\) (this only depends on the \((\text{mod } r)\) congruence class of \(\omega\)) and write \(u := \max\{u_\omega | \omega \in \mathbb{Z}/r\mathbb{Z}\}\). Then there exists a canonical map

\[
\operatorname{Hom}^w_{\mathbb{Z}/r\mathbb{Z}}(\operatorname{Flex}^d(N), \operatorname{Flex}^d(M)) \to \operatorname{Hom}^w_{\mathbb{Z}/r\mathbb{Z}}(N, M)
\]

of which the composition with \(\operatorname{Flex}^d : \operatorname{Hom}^w_{\mathbb{Z}/r\mathbb{Z}}(N, M) \to \operatorname{Hom}^w_{\mathbb{Z}/r\mathbb{Z}}(\operatorname{Flex}^d(N), \operatorname{Flex}^d(M))\) is equal to multiplication by \(p^u\), for any \(\mathbb{Z}/r\mathbb{Z}\)-graded \((A, \tau)\)-module \(M\) of graded width \(w\).

**Proof.** Assume that \(\tilde{f}\) is some endomorphism of \(\tilde{M} = \operatorname{Flex}^d(M)\) with components \(\tilde{f}_\omega\). We simply define a map \(f_\omega\) by the commutative diagram:

\[
M_\omega \xrightarrow{(V^1)^{d(\omega) - \omega}} A \otimes_{\tau, d(\omega) - \omega, A} M_{d(\omega)} = \tilde{M}_\omega
\]

\[
f_\omega \quad p^u f_\omega
\]

\[
M_\omega \xrightarrow{(F^1)^{d(\omega) - \omega}} A \otimes_{\tau, d(\omega) - \omega, A} M_{d(\omega)} = \tilde{M}_\omega
\]

where \(v_\omega = u - u_\omega\). Now the whole point is the equality \(w^d - w_\omega = v_{\omega - 1} - u_\omega\). The proof is completed by writing down the diagrams:

\[
\begin{array}{ccc}
N_\omega & \xrightarrow{p^u (V^1)^{d(\omega) - \omega}} & \tilde{N}_\omega & \xrightarrow{\tilde{f}_\omega} & \tilde{M}_\omega \\
F^1 \uparrow & & \tilde{F}^1 \uparrow & & \tilde{F}^1 \\
A \otimes_{\tau, A} N_{\omega + 1} & \xrightarrow{p^u (V^1)^{d(\omega + 1) - \omega - 1}} & A \otimes_{\tau, A} \tilde{N}_{\omega + 1} & \xrightarrow{\tilde{f}_{\omega + 1}} & A \otimes_{\tau, A} \tilde{M}_{\omega + 1}
\end{array}
\]

and

\[
\begin{array}{ccc}
\tilde{N}_\omega & \xrightarrow{f_\omega} & \tilde{M}_\omega & \xrightarrow{p^u (F^1)^{d(\omega) - \omega}} & M_\omega \\
\tilde{V}^1 \downarrow & & \tilde{V}^1 \downarrow & & \tilde{V}^1 \\
A \otimes_{\tau, A} \tilde{N}_{\omega + 1} & \xrightarrow{\tilde{f}_{\omega + 1}} & A \otimes_{\tau, A} \tilde{M}_{\omega + 1} & \xrightarrow{p^u (F^1)^{d(\omega + 1) - \omega - 1}} & A \otimes_{\tau, A} M_{\omega + 1}
\end{array}
\]

which are commutative because the outer rectangles are commutative. Notice that this proof is free of any assumptions on the \(p\)-torsion. □

**Remark 3.3.** One has \(u \leq |d| \max\{w_\sigma | \sigma \in \mathbb{Z}\}\).
3.2. Windows with additional structure, part I. If $A$ and $\tau$ are as above then some triple $(A, J, \tau)$ is called a frame over $W(F_{p'})$ if and only if $A$ is a torsionfree $W(F_{p'})$-algebra and $J$ is a pd-ideal that contains some power of $p$. Notice that these conditions imply $J \subset \rad(A)$ and $\tau(J) \subset pA$. Now let $(A, J, \tau)$ be as above and consider the $\Phi$-datum $\mathbb{G}_m \to \mathbb{G} \times_{W(F_p)} W(F_{p'})$. It is easy to see that the product of $\mathbb{G}^-(A)$ and the kernel of $\mathbb{G}^+(A) \to \mathbb{G}^+(A/J)$ is a group which will be denoted by $\hat{\mathbb{G}}^-_J(A)$. It follows that we obtain a map

$$\hat{\Phi}^-_J : \hat{\mathbb{G}}^-_J(A) \to \mathbb{G}(A),$$

gotten from the restriction of the map $\tau \circ \Int_{\mathbb{G} \times_{W(F_p)} A[1/p]}(\mu(p)^{-1})$, which sends $\mathbb{G}(A[1/p])$ to itself. On the set $\mathbb{G}(A)$ we introduce the structure of a category $\hat{\mathbb{B}}_{A,J}(\mathbb{G}, \mu)$ by the constraint $U' \overset{h}{\to} U$ whenever $U' = h^{-1}U \hat{\Phi}^-_J h$ with $h \in \hat{\mathbb{G}}^-_J(A)$. Clearly all morphisms are invertible, i.e. $\hat{B}_{A,J}(\mathbb{G}, \mu)$ is a groupoid. If we suppress the ideal in the notation it shall be understood that $J = pA$, in particular we simply write $\hat{B}_{A}(\mathbb{G}, \mu)$ if we mean $\hat{B}_{A,pA}(\mathbb{G}, \mu)$, and we also set $B_{A}(\mathbb{G}, \mu)$ for the faithful subcategory of $\hat{B}_{A}(\mathbb{G}, \mu)$ in which the morphisms are elements of $\mathbb{G}^-(A)$. For this paper it is vital to have various notions of realizations in $\mathbb{G}$-spaces. To this end consider a finitely generated torsionfree $W(F_{p'})$-module $V$ where $r$ divides $f$ and suppose that $\rho : \mathbb{G} \times W(F_{p'}) \to \GL(V/W(F_{p'}))$

is a group homomorphism. Pulling back with the $-\sigma$ fold iterates of Frobenius results in modules:

$$W(F_{p'}) \otimes_{F^{-\sigma}, W(F_{p'})} V = V_{\sigma},$$

and let us write $\rho_{\sigma}$ for the natural representation of $\mathbb{G} \times W(F_{p'})$ on that space. Finally we set $v_{\sigma}$ for the image of $\mu$ under $\rho_{\sigma}$. From now on we assume that the weights of $v_{\sigma}$ are contained in an interval of integers $[a_{\sigma}, b_{\sigma}]$. We define

$$\alpha_{\sigma} := \frac{v_{\sigma}}{z^{a_{\sigma}}} : \mathbb{A}^1 \times W(F_{p'}) \to \End_{W(F_{p'})}(V_{\sigma})$$

and

$$\beta_{\sigma} := \frac{z^{b_{\sigma}}}{v_{\sigma}} : \mathbb{A}^1 \times W(F_{p'}) \to \End_{W(F_{p'})}(V_{\sigma}),$$

note that $(\alpha_{\sigma}\beta_{\sigma})(z) = z^{b_{\sigma} - a_{\sigma}}$. We want to set up a functor

$$\Fil^\rho : \hat{B}_{A,J}(\mathbb{G}, \mu) \to (A, \tau) - \Mod_{\mathbb{Z}/r\mathbb{Z}}.$$
Let $U \in \mathsf{Ob}_{B_{A,J}(G,\mu)} = \mathcal{G}(A)$ be given. We start by defining $\text{Fib}^{ab}_\sigma(\rho) := A \otimes_{F^{-\sigma},W(F)} V = A \otimes_{W(F)} \Sigma \nu, \alpha \otimes_{W(F)} \nu$, and we set $F^\sigma_\alpha : A \otimes_{F,W(F)} \Sigma \nu_{+1} \to \alpha$ equal to $\rho_\sigma(U)F^{\sigma+1}(p)$, and we set $V^\sigma_\nu : A \otimes_{W(F)} \nu \to \alpha \otimes_{F,W(F)} \nu_{+1}$ equal to $F^{\sigma+1}(p)\rho_\sigma(U)^{-1}$. We decree the effect of $\text{Fib}^{ab}$ on a morphism $U' \xrightarrow{h} U$ in $\mathsf{Mor}_{B_{A,J}(G,\mu)}$ as follows: The diagrams:

$$
\begin{array}{c}
A \otimes_{W(F)} \nu \xrightarrow{\rho_\sigma(U)^F \beta_{\sigma+1}(p)} A \otimes_{W(F)} \nu \\
\rho_\sigma(U)^F \beta_{\sigma+1}(p) \uparrow \quad \rho_\sigma(U)^F \beta_{\sigma+1}(p) \\
A \otimes_{F,W(F)} \nu_{+1} \xleftarrow{F^{\rho_{\sigma+1}(h)}_{\rho_\sigma(U)^F \beta_{\sigma+1}(p)}} A \otimes_{F,W(F)} \nu_{+1}
\end{array}
$$

are commutative, as can be seen from $U' = h^{-1}U\hat{\Phi}^J_1 h$ in conjunction with

\begin{align}
(13) & \quad F^{\rho_\sigma(h)}(p) = \alpha_{\sigma+1}(p)\rho_\sigma(h) \\
(14) & \quad \beta_{\sigma+1}(p)^F \rho_\sigma(h) = \rho_\sigma(h)\beta_{\sigma+1}(p),
\end{align}

and this exhibits an isomorphism $\text{Fib}^{ab}_\sigma(\rho, U') \cong \text{Fib}^{ab}_\sigma(\rho, U)$, namely the multiplication by $\rho_\sigma(h)$-map on the $\sigma$-eigenspace $\text{Fib}_\sigma(\rho, U')$. There are natural isomorphisms

\begin{align}
(15) & \quad \text{Fib}^{ab+ab',b+U}_\sigma(\rho \otimes_{W(F)} \rho', U) \cong \text{Fib}^{ab}_\sigma(\rho, U) \otimes A \text{Fib}^{ab',b+U}_\sigma(\rho', U),
\end{align}

whenever the terms are well-defined and analogously for duality. Now we turn to the category of $n$-truncated $3n$-displays over $R$ with $G$-structure, where we assume $pR = 0$. In this case there exists a canonical absolute Frobenius $F : W_n(R) \to W_n(R)$ and again there is a canonical realization functor

$$
\text{Fib}^{ab}_\sigma : \mathcal{B}_R^n(G, \mu) \to (W_n(R), F) \xrightarrow{\text{Mod}_{\mathbb{Z}/r\mathbb{Z}}\text{-}}
$$

whose existence is justified in a similar manner: Use Witt descent ([30, Proposition 33]) to reduce to the banal situation and then argue as before.

The sole reason for the introduction of the regularized cocharchers $\alpha_\sigma$ and $\beta_\sigma$, is to make the above functor sensitive to torsion phenomena. If one is willing to invert the prime $p$ and restrict to $n = \infty$, then there is no need to keep track of $a$ and $b$ and one could much more easily just work with $\alpha_\sigma := \nu_\sigma$ and $\beta_\sigma := \nu_\sigma^{-1}$, to arrive at the completely self-explanatory functor

$$
\text{Fib}(\sigma) : \mathcal{B}_R(G, \mu) \to (W(R), F) \xrightarrow{\text{Mod}_{\mathbb{Z}/r\mathbb{Z}}\text{-}}.
$$
which has the evident advantage of being a $W(F_{p^r})$-linear rigid $\otimes$-functor in the variable $\rho$.

Unitarity becomes incorporated as follows: Let $r$ be even and let $\chi: G \times W(F_{p^r}) \to \mathbb{G}_m$ be a character such that $\chi_\sigma \circ \mu$ is of weight $c_\sigma$, where

$$(a_\sigma + \frac{r}{2}, b_\sigma + \frac{r}{2}) = (c_\sigma - b_\sigma, c_\sigma - a_\sigma).$$

Suppose there is a pairing $\Psi: \mathcal{V} \times \mathcal{V} \to W(F_{p^r})$, which satisfies

$$-\Psi(x, y) = \tau^r(\Psi(y, x))$$

and is $W(F_{p^r})$-linear in the first variable. The representation $\rho$ is called unitary if $\Psi(\gamma(x), \gamma(y)) = \chi(\gamma) \Psi(x, y)$ holds (for all $\gamma \in G(W(F_{p^r}))$ and likewise functorially). We can think of the pairing as a morphism $\Psi_\sigma: \rho_{a+\frac{r}{2}} \to \chi_\sigma \otimes \tilde{\rho}_\sigma$ of $G \times W(F_{p^r})$-representations. This endows the $(A, \tau)$-module $\text{Fib}^{\sigma}_{a, b}(\rho)$ with maps

$$\text{Fib}^{\sigma}_{a, b}_{a+\frac{r}{2}}(\rho) \to \text{Fib}^{\sigma}_{a, b}_{a}(\rho).$$

Remark 3.4. If there exists at least one $\sigma_0 \in \mathbb{Z}$ such that the weights of $\nu_{\sigma_0}$ are actually contained in the smaller interval $[a_{\sigma_0}, b_{\sigma_0} - 1]$ (resp. $[a_{\sigma_0} + 1, b_{\sigma_0}]$), then $\text{Fib}^{a, b}_{\sigma}(\rho)$ is $F$-nilpotent (resp. $V$-nilpotent). This follows rather formally from [15] if $\rho'$ is taken to be the trivial representation.

4. THE FIBERED GROUPOID OF TWISTED CONJUGACY CLASSES, PART II

Suppose that $\mathcal{P} = (P_n, P_{n+1}, Q_{n+1}, \Phi_n)$ is an object of $B_{\chi}^+(G, \mu)$. By slight abuse of notation we will write $\varpi_\mathcal{P}(\rho)$ for $\varpi_{Q_1}(\rho)$, where $\rho$ is a representation of $G^-$ and $Q_1$ is the level-0 truncation of $\mathcal{P}$. One of the aims in this section is to show that $T_\mathcal{P} := \varpi_{\mathcal{P}}(\text{Ad}_-)$ behaves like a tangent bundle, provided that $n = \infty$. We have to start with a weak substitute for the result [30, Theorem 44] in the language of our stack $B(G, \mu)$.

Definition 4.1. Let $X$ be a $W(F_{p^r})$-scheme. An (infinite length) $3n$-display with $G$-structure over $X$ is called a display with $G$-structure over $X$, if and only if $X \times_{W(F_{p^r})} W(F_{p^r})$ has an affine open covering $\bigcup U_l$, such that

$$\text{Fib}^{a, 1}(\mathcal{P} \times_X U_l, \text{Ad}),$$

is $F$-nilpotent for all $l$, where $\text{Ad}$ is the adjoint representation on the Lie algebra of $G$, and $a$ is a sufficiently small integer.
The full fibered subcategory of $\mathcal{B}(\mathcal{G}, \mu)$ whose objects are the displays will be denoted by $\mathcal{B}'(\mathcal{G}, \mu)$. The variants $\mathcal{B}'(\mathcal{G}, \mu)$ and $\mathcal{B}'_{A,J}(\mathcal{G}, \mu)$ are defined completely analogously.

For every ideal $a$ in an $W(\mathbb{F}_p)$-algebra $S$ we will write $\hat{\mathcal{T}}_{a,n}$ (resp. $\hat{\mathcal{J}}_{a,n}$) for the covariant set-valued functors that take any $S$-algebra $R$ to the inverse image of $\mathcal{T}_n(R/aR)$ in $\mathcal{H}_n(R)$ (resp. of $\mathcal{J}_n(R/aR)$ in $\mathcal{H}_n^+(R)$). The functors $\hat{\mathcal{T}}_{a,n}$ and $\hat{\mathcal{J}}_{a,n}$ are sheaves for the fpqc topology. Notice also that $\hat{\mathcal{T}}_{a,n}(R)$ is the product of $\mathcal{H}_n(R)$ and $\hat{\mathcal{J}}_{a,n}(R)$ in any order for any $n$, if $pR + aR \subset \text{rad } R$. If we suppress the ideal in the notation, then it shall be understood that $a = pS$, e.g. $\hat{\mathcal{T}}_n(R)$ is the inverse image of $\mathcal{I}_n(R/pR)$ in $\mathcal{H}_n(R)$. There exist Lie-theoretic analogs for these functors: Let us write $i_n$ for the functor (on $W(\mathbb{F}_p)$-algebras $R$) $\text{Res}_{I_n} g^+ \oplus \text{Res}_{W_n} g^-$ and $h_n$ for the functor (on $W(\mathbb{F}_p)$-algebras) $\text{Res}_{W_n} g$. Furthermore let $\hat{i}_{a,n}(R) \subset h_n(R)$ stand for the subfunctor obtained as the inverse image of $g^- \otimes_{W(\mathbb{F}_p)} R/aR$, equivalently it is the sum of $i_n(R)$ and $h_n(aR)$ (for variable $R$). For every ideal $b \subset R$ of vanishing square there exists an exact sequence:

$$0 \to \hat{i}_{a,n}(b) \hookrightarrow \hat{\mathcal{T}}_{a,n}(R) \to \hat{\mathcal{T}}_{a,n}(R/b) \to 0,$$

which follows from analogous sequences for the groups $\mathcal{H}_n$ and $\mathcal{I}_n$, moreover the functor $\hat{\mathcal{T}}_{a,n}$ acts on $i_{a,n}$ by conjugation, basically because of analogous properties of $\mathcal{H}_n$ and $\mathcal{I}_n$.

4.1. Twisted Frobenius maps, part II. We start with the simple observation that $\Phi_n$ has a Lie-theoretic analog

$$\phi_n : i_{n+1} \to h_n \times_{W(\mathbb{F}_p)} W(\mathbb{F}_p),$$

which is defined by $V_n^{-1}$ (as in [8]) on the positive summand, and by $F_n$ precomposed with the derived action of the endomorphism $\mu^{-1\text{Int}}(p) \in \text{End}(\mathcal{G}^-)$ on the negative summand $g^- = \text{Lie}(\mathcal{G}^-)$ (as in [8]). In two instances we need to extend the maps $\Phi_n$ to maps $\hat{\Phi}_{a,n}$, first assume that $a^2 + pa = 0$. This implies that $F$ vanishes on $W(a)$ and that there exists a canonical splitting

$$W_n(a) = a \oplus I_n(a),$$

which was introduced by Norman in the slightly different context of [20, Section 1]. The group $\hat{\mathcal{T}}_{a,n}(R)$ is consequently the product of $\mathcal{T}_n(R)$ and $g \otimes_{W(\mathbb{F}_p)} aR$ and the latter subgroup is a normal one. Now we come to the definition of our functorial maps $\hat{\Phi}_{a,n} : \hat{\mathcal{T}}_{a,n+1}(R) \to \mathcal{H}_n(R)$ by

$$\hat{\Phi}_{a,n}|_{\mathcal{I}_{n+1}(R)} = \Phi_n$$

and

$$g \otimes_{W(\mathbb{F}_p)} R a \subset \ker(\hat{\Phi}_{a,n}).$$
This is consistent with $g^- \otimes_{W(F_p)} Ra \subset \ker(\Phi_n)$ and the same procedure applies to get an extension of $\phi_n$ to a Lie-theoretic map $\hat{\phi}_{a,n}$ on $\hat{\mathfrak{i}}_{a,n}$. Second, assume $n = \infty$ and that $a$ is a pd-ideal in a $p$-adically separated and complete torsionfree ring $S$. However, this implies that $(W(S), W(a) + I(S), F)$ is a frame over $W(F_{p'})$, so that we merely have to appeal to our definition

$$\hat{\Phi}_{W(a) + I(S)} =: \hat{\Phi}_a,$$

which was already introduced in the earlier subsection 3.2. We need the following version of the Grothendieck-Messing crystalline functoriality:

Lemma 4.2. Let $S$ be a $W(F_{p'})$-algebra, and let $a$ be an ideal that contains some power of $p$. Suppose that one of the following three assumptions is in force:

(i) $a^2 + pa = 0$.
(ii) $S$ is $p$-adically separated and complete, torsionfree and $a$ is a pd-ideal.
(iii) $g^+ = 0$ and there exists a number $N$ such that $x^N = 0$ for all $x \in a$.

Recall the group homomorphism $\hat{\Phi}_a : \hat{\mathfrak{i}}_a(S) \to \mathcal{H}(S)$, which is well-defined in each of these cases (in case that (ii) holds this is the restriction of $F \circ \text{Int}_{G \times W(F_p)W(S)[\frac{1}{p}]}(\mu(p)^{-1})$, sending $G(W(S)[\frac{1}{p}])$ to itself, and in case (iii) holds one has $\hat{\Phi}_a = \Phi = \Phi^-$ as $\hat{\mathfrak{i}}_a = \mathcal{I} = \mathcal{H} = \mathcal{H}^-$). Now suppose that $O, U \in \mathcal{H}(S)$ satisfy the nilpotence condition and fulfil

$$O \equiv h^{-1}U^{\hat{\Phi}_a}h \mod a$$

for some $h \in \hat{\mathfrak{i}}_a(S)$. Then there exists a unique $k \in \hat{\mathfrak{i}}_a(S)$ such that

(18) $$O = k^{-1}U^{\hat{\Phi}_a}k$$
(19) $$h \equiv k \mod a$$

Proof. Without loss of generality we assume that $h = 1$. In the first case we set $C$ for the map on the abelian group $\hat{\mathfrak{i}}_a(a) = g \otimes_{W(F_p)} W(a)$ that sends some random lift $\hat{\mathfrak{i}}_a(S) \ni k \equiv 1 \mod a$ to $C(k) = U^{\hat{\Phi}_a}kO^{-1}$. It is easy to see that every element $x \in g \otimes_{W(F_p)} W(a)$ satisfies $C(k + x) = C(k) + c(x)$, where $c$ is the map given by $\text{Ad}(U) \circ \hat{\phi}_a = \text{Ad}(O) \circ \hat{\phi}_a$, and where $\hat{\phi}_a$ is the previously introduced derivative of $\hat{\Phi}_a$. (N.B.: On the whole of $g \otimes_{W(F_p)} W(S)$ the maps $\text{Ad}(U)$ and $\text{Ad}(O)$ may not necessarily agree but their restrictions to $W(a)$ do agree). It suffices to prove that $c$ is nilpotent on $g \otimes_{W(F_p)} W(a)$! The logarithmic
ghost map \( w' \) (along with the inclusion \( a \subset S \)) gives rise to the diagram

\[
\begin{array}{c}
g \otimes_{W(\mathbb{F}_p)} W(a) \xrightarrow{w'} g \otimes_{W(\mathbb{F}_p)} (S^{\mathbb{N}_0}) \\
\overset{\hat{\phi}_a}{\downarrow} \quad \overset{\phi^*}{\uparrow}
\end{array}
\]

where \( \phi^* \) is defined using \([x'_0, \ldots] \mapsto [x'_1, \ldots] \) in place of \( V^{-1} \). Now, given that the former map is exactly the effect of the absolute Frobenius on the usual, non-logarithmic ghost coordinates, we see immediately that the topological nilpotence of \( \text{Ad}(U) \circ \phi^* \) is implied by the \( F \)-nilpotence of \( \text{Fib}^{a-1}(\mathcal{P}, \text{Ad}) \). In the torsionfree case the proof is completely analogous. For the third variant one may assume that \( pa = 0 \) and \( N = p \) hold to then argue as in the proof of [30, Lemma 42].

As a consequence we get the following result on the rigidity of automorphisms, it can be regarded as an analog of [30, Proposition 40]:

**Corollary 4.3.** Let \( A \) be an \( a \)-adically separated \( W(\mathbb{F}_p) \)-algebra, where \( a \) is an ideal that contains some power of \( p \), and let \( \phi \) be an automorphism of a display with \( G \)-structure \( \mathcal{P}/A \). Then \( \phi \times_A A/a \) is the identity if and only if \( \phi \) is the identity.

**Proof.** By proposition 2.7 and by an induction argument we can assume that \( a^2 + pa = 0 \). After passage to some affine fpqc covering we can also assume that \( \mathcal{P} \) is banal. In this case, however, the result follows immediately from lemma 4.2.

Suppose \( a \) is an ideal in a \( W(\mathbb{F}_p) \)-algebra \( S \) such that \( p \) and \( a \) are nilpotent, and fix \( \mathcal{P}_0 \in \mathcal{B}'_{S/A}(G, \mu) \). For a scheme \( X \) over \( \text{Spec} \ S \) we write \( X_0 \) for \( X \times_{\text{Spec} \ S} \text{Spec} \ S/a \). By a lift of \( \mathcal{P}_0 \) over \( X \) we mean a pair \((\mathcal{P}, \delta)\) with \( \mathcal{P} \in \mathcal{B}'_X(G, \mu) \) and \( \delta : \mathcal{P}_0 \times_{\text{Spec} \ S/a} X_0 \to \mathcal{P} \times_X X_0 \). Isomorphisms between lifts are expected to preserve the respective \( \delta \)'s, in particular no lift has any automorphisms other than the identity, by corollary 4.3. Let \( \hat{T}_{\mathcal{P}_0}(X) \) be the set of isomorphism classes of lifts over \( X \). By pull-back of lifts this is a contravariant functor on the category of schemes over \( \text{Spec} \ S \).

**Corollary 4.4.** Fix a display \( \mathcal{P}_0 \) over the ring \( S/aS \) with \( G \)-structure, where \( a^2 + pa = 0 \):

(i) The functor \( \hat{T}_{\mathcal{P}_0} \) possesses the structure of a locally trivial principal homogeneous space for the fpqc-sheaf

\[
X \mapsto T_{\mathcal{P}_0} \otimes_{S/aS} a\Gamma(X, \mathcal{O}_X) = \text{Hom}_{S/aS}(\hat{T}_{\mathcal{P}_0}, a\Gamma(X, \mathcal{O}_X)).
\]

(ii) \( \hat{T}_{\mathcal{P}_0}(\text{Spec} \ S) \neq \emptyset \)
Proof. Let us check that \( \hat{T}_{\mathcal{P}_0} \) satisfies the sheaf axiom for a fpqc map \( Y \to X \): If the pull-backs \( \text{pr}_1^*(\mathcal{P}) \) and \( \text{pr}_2^*(\mathcal{P}) \) of some \((\mathcal{P}, \delta) \in \hat{T}_{\mathcal{P}_0}(Y)\) agree for the two projections \( \text{pr}_1, \text{pr}_2 : Y \times_X Y \to Y \), then this means that there exists \( \alpha : \text{pr}_1^*(\mathcal{P}) \cong \text{pr}_2^*(\mathcal{P}) \) which restricts to the identity on \( Y_0 \times_{X_0} Y_0 \). We easily deduce the cocycle condition for \( \alpha \), because any equality of isomorphisms of displays over \( Y \times_X Y \times_X Y \) can be checked over \( Y_0 \times_{X_0} Y_0 \times_{X_0} Y_0 \), by corollary \( \text{[4.3]} \). This shows that \( \mathcal{P} \) (resp. \( \delta \)) descent to \( X \) (resp. \( X_0 \)).

We are now allowed to assume that \( \mathcal{P}_0 \) is banal, so choose a representing element \( U_0 \in \mathcal{H}(S/\mathfrak{a}) \), indeed any lift of \( \mathcal{P}_0 \) over any \( X/\text{Spec} \, A \) is going to be banal too, so we may restrict to the affine case \( X = \text{Spec} \, A \) from now on (cf. \( \text{[11]} \)), more specifically: Every lift over \( R \) is determined by a pair \( (U, h) \in \mathcal{H}(R) \times \mathcal{I}(R/\mathfrak{a}R) \), with \( U_0 = h^{-1}U^0h \), and another pair \( (U', h') \) determines the same element in \( \hat{T}_{\mathcal{P}_0}(\text{Spec} \, R) \) if and only if \( U' = k^{-1}U^0k \) holds for some \( k \in \mathcal{I}(R) \) with \( S(U') = h \) (where \( U' \in \mathcal{H}(R/\mathfrak{a}R) \) and \( S \in \mathcal{I}(R/\mathfrak{a}R) \) stand for the reductions mod \( \mathfrak{a}R \)).

Recall that we defined a natural embedding of \( g \otimes_{W(\mathcal{P}_{\mathfrak{p}})} \mathfrak{a}R \) into \( \hat{T}_a(R) \subset \mathcal{H}(R) \). It is now evident that for \( N \in g \otimes_{W(\mathcal{P}_{\mathfrak{p}})} \mathfrak{a}R \) the assignment \( (U, h) \mapsto ((\text{ad}(h))N)U, h) \) defines an action on \( \hat{T}_{\mathcal{P}_0}(\text{Spec} \, R) \), and that for \( N \in g^- \otimes_{W(\mathcal{P}_{\mathfrak{p}}_{\mathfrak{p}})} \mathfrak{a}R \) that action is the trivial one.

Now suppose that \( U_0 \) is the mod \( \mathfrak{a} \)-reduction of some fixed \( U \in \mathcal{H}(S) \). Then lemma \( \text{[4.2]} \) tells us that every lift of \( \mathcal{P}_0 \) over \( R \) can be written as \( U' = NU \), for a unique \( N \in g \otimes_{W(\mathcal{P}_{\mathfrak{p}_{\mathfrak{p}}})} \mathfrak{a}R \). This finishes the proof of the part (i), and the final supplement is a straightforward consequence of \( \text{[11]}, \text{Chapter III, Proposition 3.7} \). \( \square \)

Definition 4.5. Let \( \mathcal{P} \) be a display with \( \mathcal{G} \)-structure over a \( \mathbb{F}_{p_f} \)-algebra \( A \), and let \( d : A \to \Omega^1_{A/\mathbb{F}_{p_f}} \) be the universal derivation into the \( A \)-module of Kaehler-differentials. The map

\[
\theta_A : A \to A \oplus \Omega^1_{A/\mathbb{F}_{p_f}} ; \, x \mapsto x + d(x),
\]

is a ring homomorphism, as the augmentation ideal \( \Omega^1_{A/\mathbb{F}_{p_f}} \) is given the trivial multiplication. Consequently there exists a unique \( K_{\mathcal{P}} \in \text{Hom}_A(\hat{T}_{\mathcal{P}}, \Omega^1_{A/\mathbb{F}_{p_f}}) \) which measures the difference between the elements \( \mathcal{P} \times_{A, \theta_A} A \oplus \Omega^1_{A/\mathbb{F}_{p_f}} \) and \( \mathcal{P} \) of \( \hat{T}_{\mathcal{P}}(\text{Spec} \, A \oplus \Omega^1_{A/\mathbb{F}_{p_f}}) \). We will call \( K_{\mathcal{P}} \) the Kodaira-Spencer element of \( \mathcal{P} \). We will call \( \mathcal{P} \) formally smooth (resp. formally etale) if and only if \( A \) is a formally smooth \( \mathbb{F}_{p_f} \)-algebra, and \( K_{\mathcal{P}} \) is a split injection (resp. bijection).
Remark 4.6. The methods of this section have the following consequence: Let $k$ be a perfect field extension of $\mathbb{F}_p'$, and let $P_0$ be a display with $\mathcal{G}$-structure over $k$. Let $P_1$ be the unique display with $\mathcal{G}$-structure over the augmented $k$-algebra $k \oplus \tilde{T}_{P_0}$, such that the difference between $P_1$ and $P_0$ is measured by the element $N = id_{\tilde{T}_{P_0}}$. Let $P_{uni}$ be an arbitrary lift of $P_1$ to the power series $k$-algebra $k[[\tilde{T}_{P_0}]]$, the existence of such a lift follows from the part (ii) of corollary 4.4 together with lemma 2.9, but it might fail to be a display. Still, it is easy to see that $P_{uni}$ is the universal formal deformation of $P_0$. This is completely analogous to [30, 2.4]. See also [27, 3.2.7 Remarks 8b)] for yet another setting.

One can enliven this a bit more, suppose that $P$ is a display with $\mathcal{G}$-structure over an algebraic variety $X/k$. Then $U = \{x \in X | P \times_X O_{X,x} \text{ is formally } \acute{e}tale\}$ is Zariski open. Moreover, for all closed points $x \in X$ we have $x \in U$ if and only if the display $P \times_X \hat{O}_{X,x}$ is a universal formal deformation of its special fiber $P \times_X k(x)$ (as usual $k(x) := O_{X,x}/m_x$ and $\hat{O}_{X,x} := \lim_{\leftarrow} O_{X,x}/m_x^{\nu}$).

Corollary 4.7. Suppose that $A$ is a sub-$W(\mathbb{F}_p')$-algebra of $B$ and that $a$ is an ideal of $A$ containing some power of $p$. Assume that one of the following holds:

(i) $a$ is nilpotent.
(ii) $B$ is finite over $A$, and $A$ is a $a$-adically separated and complete noetherian ring.

Then for each pair $P$ and $T$ of displays over $\text{Spec} A$ with $\mathcal{G}$-structure every two compatible isomorphisms $\bar{\phi} : P \times_A A/a \to T \times_A A/a$ and $\psi : P \times_A B \to T \times_A B$ are scalar extensions of some unique $\phi : P \to T$.

Proof. For the first assertion we may also assume $a^2 + pa = 0$, so that the map $\bar{\phi}$ allows to view $T$ as a lift of $P_0 := P \times_A A/a$. Let $N \in T_{P_0} \otimes_{A/a} a$ measure the difference between the elements $T$ and $P$ of $\tilde{T}_{P_0}(\text{Spec} A)$. Its image in $T_{P_0} \otimes_{A/a} aB$ has to vanish, according to the existence of $\psi$. Now the injectivity of $T_{P_0} \otimes_{A/a} a \to T_{P_0} \otimes_{A/a} aB$ yields $N = 0$.

Due to the Artin-Rees lemma and proposition 2.7 one gets the second assertion as a consequence of the first. \qed

4.2. Newton points and their specializations. Let us assume that the generic fiber $G := \mathcal{G} \times_{W(\mathbb{F}_p')} K(\mathbb{F}_p)$ is a reductive algebraic group, and let $k$ be a perfect field of characteristic $p$. We have to recall and apply some of the key-concepts in [22]: On the set of $K(k)$-valued points of $G$ one defines the important equivalence relation of $F$-conjugacy by
requiring that \( b' \sim b \) if and only if \( b' = g^{-1}bFg \), for some \( K(k) \)-valued point \( g \), write \( B_k(G) := G(K(k))/\sim \) for the set of \( F \)-conjugacy classes, and \( B(G) := B_{\mathbb{F}_p}(G) \). Recall that any \( \mathcal{P} \in \mathfrak{Ob}_{\mathcal{B}_{\text{Spec}}(\mathcal{G}, \mu)} \) is represented by the twisted conjugacy class of an element \( U \in \mathcal{G}(W(k)) \), and observe that the \( F \)-conjugacy class of \( U^F\mu(p)^{-1} \) defines therefore a canonical element \( \overline{b}_\mathcal{P} \in B_k(G) \). From now onwards we assume the algebraic closedness of \( k \), so that \( \mathcal{B}_{\text{Spec}}(\mathcal{G}, \mu) = \mathcal{B}_{\text{Spec}}(\mathcal{G}, \mu) \) holds. In this case matters are simplified even more as \( B_k(G) = B(G) \) by [22, Lemma 1.3], and the same independence result is valid for the set of Newton points that we introduce next: Write \( \mathbb{Q} \otimes \mathbb{G}_m \) for the pro-algebraic torus whose character group is \( \mathbb{Q} \) and put

\[
\mathcal{N}(G) := (G(K(k))\backslash \text{Hom}(\mathbb{Q} \otimes \mathbb{G}_m \times K(k), G \times_{K(\mathbb{F}_p)} K(k)))^{<F>},
\]

where the left \( G(K(k)) \)-action is defined by composition with interior automorphisms, and where the action of the infinite cyclic group \( <F> \) is defined by \( \nu \mapsto F\nu \). Every element \( b \in G(K(k)) \) gives rise to a so-called slope homomorphism \( \nu_b : \mathbb{Q} \otimes \mathbb{G}_m \times K(k) \rightarrow G \times_{K(\mathbb{F}_p)} K(k) \), for which we refer the reader to loc.cit. Its formation is canonical in the sense that \( g^{-1}\nu_bg = \nu_{g^{-1}Fg} \). Consequently, one is in a position to introduce the Newton-map:

\[
\nu : B(G) \rightarrow \mathcal{N}(G); \overline{b} \mapsto \nu_b := \overline{\nu_b},
\]

here notice that the fractional cocharacters \( F\nu_b \) and \( \nu_b \) are lying in the same conjugacy class.

Now fix a maximal torus \( T \subset G \times K(\mathbb{F}_p)^{ac} \). Consider its associated Weyl group \( \Omega \), and its associated lattice of cocharacters \( X_s(T) \), which is equipped with a left \( \Omega \rtimes \text{Gal}(K(\mathbb{F}_p)^{ac}/K(\mathbb{F}_p)) \)-action. There are natural inclusions

\[
\mathcal{N}(G) \subset (\Omega \backslash X_s(T)_{\mathbb{Q}})^{\text{Gal}(K(\mathbb{F}_p)^{ac}/K(\mathbb{F}_p))} \subset \Omega \backslash X_s(T)_{\mathbb{Q}},
\]

moreover, the set \( \Omega \backslash X_s(T)_{\mathbb{R}} \) is partially ordered in a very natural way, as one puts \( \Omega x \preceq \Omega x' \) if and only if \( x \) lies in the convex hull of the \( \Omega \) orbit of \( x' \), cf. [22, Lemma 2.2(i)]. The same notation is used for the partial orders on \( \mathcal{N}(G) \) and \( B(G) \) which are immediately inherited via the Newton map.

Now let \( \mathcal{P} \) be an infinite length \( 3n \)-display with \( \mathcal{G} \)-structure over an \( \mathbb{F}_p \)-algebra \( R \). Consider the function \( \text{Spec} R \rightarrow B(G) \) defined by \( \overline{\mathfrak{p}}_{\mathcal{P}}(\mathfrak{p}) := \overline{b}_{\mathcal{P} \times_{kR}(\mathfrak{p})^{ac}} \). The following variant of Grothendieck’s specialization theorem (cf. [22, Theorem 3.6(ii)]) will be used in the sequel:

**Theorem 4.8.** If \( \overline{b}_0 \in B(G) \), and if \( \mathcal{P} \) is as above, then there exists a finitely generated ideal whose zero set is \( \{ \mathfrak{p} \in \text{Spec} R | \overline{b}_{\mathcal{P}}(\mathfrak{p}) \prec \overline{b}_0 \} \).
We would also like to adjust the inequality of Mazur to the needs of our category \( \mathcal{B}(G, \mu) \). So assume that \( G \) is reductive, and write \( B \) for a \( K(\mathbb{F}_p) \)-rational Borel group of \( G \), and write \( T \) for a maximal torus of \( B \). Pick an element \( \gamma \in G(K(\mathbb{F}_{pc})) \) such that \( \gamma \mu \gamma^{-1} \) lies in \( T \) and is dominant with respect to \( B \). Pick a factorization \( \gamma = \gamma_1 \gamma_2 \), where \( \gamma_1 \in B(K(\mathbb{F}_{pc})) \), and \( \gamma_2 \in G(W(\mathbb{F}_p)) \), and define an infinite length banal \( 3n \)-display with \( G \)-structure \( \mathcal{P}_{ord}/\mathbb{F}_{pc} \) by the representative \( \gamma^{-1} \gamma_2 \in G(W(\mathbb{F}_{ac})) \), then we have (cf. [22, Theorem 4.2(ii)]):

**Theorem 4.9.** Under the above hypotheses on \( G \) we have \( \mathcal{P}_p < \mathcal{P}_{ord} \), for all \( \mathcal{P} \in \mathcal{Ob}_{\mathcal{B}(\text{Spec} K(G, \mu))} \).

Finally we arrive at the main result of this subsection:

**Corollary 4.10.** Let \( k \) be a perfect field containing \( \mathbb{F}_{p'} \), and let \( \mathcal{P}_0 \) be a display with \( G \)-structure over \( k \). Let \( \mathcal{P}_{uni}/k[[t_1, \ldots, t_d]] \), be its universal formal deformation (\( d \) being the \( W(\mathbb{F}_{pc}) \)-rank of \( \mathfrak{g}^+ \), see remark 4.6). Then the function \( b_{\mathcal{P}_{uni}} \) attains the value \( b(\mathcal{P}_{ord}) \) at the generic point of \( \text{Spec} k[[t_1, \ldots, t_d]] \).

**Proof.** Lemma [2.6] provides us with a family \( \mathcal{P} \in \mathcal{Ob}_{\mathcal{B}(G, \mu)} \) specializing to \( \mathcal{P}_0 \) and \( \mathcal{P}_{ord} \), when evaluated at the closed points 0 and 1 lying in an open subscheme \( U \subset \text{Spec} l[[t]] \), for some finite extension \( l \). The theorems [4.8] and [4.9] imply that \( b_{\mathcal{P}} \) attains the value \( b(\mathcal{P}_{ord}) \) at the generic point, clearly the same is true for \( \mathcal{P} \times_U \text{Spec} l[[t]] \). Finally notice that the latter object descents to \( \text{Spec}(k + tl[[t]]) \), and thus provides a morphism to \( \text{Spec} k[[t_1, \ldots, t_d]] \) showing that the function \( b_{\mathcal{P}_{uni}} \) attains \( b(\mathcal{P}_{ord}) \) at some, or equivalently at the generic point of \( \text{Spec} k[[t_1, \ldots, t_d]] \). \( \Box \)

**Remark 4.11.** We may deduce a simple criterion for whether or not \( \mathcal{P}_{uni} \) is a display, namely according to whether or not \( \mathcal{P}_{ord} \) is a display. This happens if and only if every \( K(\mathbb{F}_p) \)-rational simple factor of \( G \) contains a \( K(\mathbb{F}_{pc}) \)-rational simple factor in which \( \mu \) is trivial.

5. Flexibility, part II

5.1. **Gauges.** Let \( \mathcal{V} \) be a finitely generated torsionfree \( W(\mathbb{F}_{p^2}) \)-module. Suppose that the pairing \( \Psi : \mathcal{V} \times \mathcal{V} \to W(\mathbb{F}_{p^2}) \) is \( W(\mathbb{F}_{p^2}) \)-linear in the first variable and satisfies:

\[
\Psi(x, y)^* = -\Psi(y, x)
\]

\[
\mathcal{V} = \{ y \in \mathbb{Q} \otimes \mathcal{V} | \Psi(\mathcal{V}, y) \subset W(\mathbb{F}_{p^2}) \}.
\]

Let \( \text{GU}(\mathcal{V}/W(\mathbb{F}_{p^2})), \Psi) \subset \text{Res}_{W(\mathbb{F}_{p^2})/W(\mathbb{F}_{p^2})} \text{GL}(\mathcal{V}/W(\mathbb{F}_{p^2})) \) be the \( W(\mathbb{F}_{p^2}) \)-subgroup whose sets of \( R \)-valued points are defined by the condition
\( \Psi(\gamma(x), \gamma(y)) = m \Psi(x, y) \) for all \( x, y \in R \otimes_{W(F_{p^r})} V \), and let us write \( \chi : GU(V/W(F_{p^r}), \Psi) \to \mathbb{G}_m; \gamma \mapsto m \) for the multiplier character. Let \( f \) be a fixed multiple of \( 2r \). In this subsection we introduce a certain averaging processes for cocharacters:

\[
u : \mathbb{G}_m \times W(F_{p^r}) \to \text{Res}_{W(F_{p^r})/W(F_p)} GU(V/W(F_{p^r}), \Psi).
\]

First of all notice that giving \( \nu \) is equivalent to giving a family of cocharacters \( \nu_\sigma : \mathbb{G}_m \times W(F_{p^r}) \to \text{GL}(V/W(F_{p^r})) \) each of whose duals \( \check{\nu}_\sigma \) (cocharacters of \( \text{GL}(V/W(F_{p^r})) \)) coincides with \( \Psi_\sigma = \nu_\sigma \circ \nu_{\sigma+r} \) up to a homothety. Here it is taken for granted that \( \Psi_\sigma \) and \( \Psi_\sigma : \mathcal{V}_{\sigma+r} \cong \mathcal{V}_\sigma \) are defined as in subsection 3.2. For any \( l \in \mathbb{Z} \) we denote

\[
H_0(l) := \begin{cases} 0 & l \leq 0, \\ 1 & l \geq 1, \end{cases}
\]

and for any \( \nu : \mathbb{G}_m \times W(F_{p^r}) \to \text{GL}(V/W(F_{p^r})) \) we will write \( H_0(\nu) \) for the cocharacter acting with weight \( H_0(\nu) \) on every direct summand of \( \mathcal{V}_\sigma \) on which \( \nu \) would act with weight \( \nu \).

Now fix a \((\text{mod } r)\) multidegree \( d \), and let us retain the notation \( \sigma_j \) and \( \omega_j \) for the indices in the images of \( d \) and \( d^* \). Let us write \( a_\sigma \) for the minimal, and \( b_\sigma \) for the maximal weight of \( \nu_\sigma \) on \( \mathcal{V}_\sigma \). From now on we wish to assume that \( \text{Res}_{W(F_{p^r})/W(F_p)} \chi \) \( \circ \nu \) is equal to the diagonal cocharacter \((z, \ldots, z)\) of \( \text{Res}_{W(F_{p^r})/W(F_p)} \mathbb{G}_m \), which is equivalent to \( a_\sigma + b_\sigma = 1 \) for all \( \sigma \). Finally we denote \( b_\sigma - a_\sigma \) by \( w_\sigma \).

**Definition 5.1.** By a gauge \( j \) of multidegree \( d \) for \( \nu \) we mean a function \( j : \mathbb{Z} \to \mathbb{Z} \) such that:

- For each \( \sigma \in \mathbb{Z} \) and \( l \in [a_\sigma, b_\sigma - 1] \) there exists a unique \( \omega \in \mathbb{Z} \) such that \( d(\omega) = \sigma \) and \( j(\omega) = l \).
- \( j(\omega + r) = -j(\omega) \)

Notice that the first of the two conditions implies \( d(\mathbb{Z}) \supset \{ \sigma \mid w_\sigma \neq 0 \} \). This allows us to define a cocharacter

\[
\check{\nu}_\omega(z) = H_0\left( \frac{F^{d(\omega) - \omega} u_d(\omega)}{z^j(\omega)} \right)
\]

for all \( \omega \). Clearly, the weights of \( \check{\nu}_\omega \) are contained in \([a_\omega, b_\omega]\), where

\[
\begin{align*}
\check{a}_\omega &= H_0(a_d(\omega) - j(\omega)), \\
\check{b}_\omega &= H_0(b_d(\omega) - j(\omega)), \\
\check{w}_\omega &= \check{b}_\omega - \check{a}_\omega = \begin{cases} 1 & j(\omega) \in [a_d(\omega), b_d(\omega) - 1], \\
0 & \text{otherwise}, \end{cases}
\end{align*}
\]
Notice that \( \tilde{a}_{\omega+r} = 1 - \tilde{b}_{\omega} \), due to \( H_0(1-l) = 1 - H_0(l) \). Furthermore, one sees that
\[
\sum_{\omega = \omega_j+1}^{\omega_j} \tilde{w}_\omega = \sum_{\sigma = \sigma_j+1}^{\sigma_j} w_\sigma = w_{\sigma_j}
\]
holds. Let us also define two “error homotheties” for the \( W(\mathbb{F}_p) \)-group 
\[ \text{Res}_{W(\mathbb{F}_p')/W(\mathbb{F}_p)} \] 
GU(\( \mathcal{V}/W(\mathbb{F}_p'') \), \( \Psi \)) by \( \epsilon_\sigma(z) = z^{a_\sigma} \) and \( \tilde{\epsilon}_\omega(z) = z^{\tilde{a}_\omega} \), one has
\[
\prod_{\omega = \omega_j+1}^{\omega_j} F_{\omega-\sigma_j} (\tilde{v}_\omega \tilde{\epsilon}_\omega^{-1}) = \prod_{\sigma = \sigma_j+1}^{\sigma_j} F_{\sigma-\sigma_j} (v_\sigma \epsilon_\sigma^{-1}) = v_{\sigma_j} \epsilon_{\sigma_j}^{-1}
\]
moreover, the set of cocharacters occurring in the above product do mutually commute.

5.2. Faithfulness of Flex, part I. Consider a homomorphism \( \rho : \mathcal{G} \to \text{GU}(\mathcal{V}/W(\mathbb{F}_{p''}), \Psi) \) of group schemes over \( W(\mathbb{F}_{p'}) \). Notice that:
\[
\text{Res}_{W(\mathbb{F}_{p''})/W(\mathbb{F}_p)} \mathcal{G} \times W(\mathbb{F}_{p'}) = \bigoplus_{\sigma = 0}^{r-1} \mathcal{G}_\sigma
\]
where \( \mathcal{G}_\sigma = F^{-\sigma} \mathcal{G} = \mathcal{G} \times_{W(\mathbb{F}_{p''}), F^{-\sigma}} W(\mathbb{F}_{p'}) \) (here \( F = \text{Frobenius on } W(\mathbb{F}_{p'}) \)). In order to give a \( \Phi \)-datum \( \mu \) of \( \text{Res}_{W(\mathbb{F}_{p''})/W(\mathbb{F}_p)} \mathcal{G} \) over \( W(\mathbb{F}_{p'}) \) it is necessary and sufficient to give \( \Phi \)-data \( \mu_\sigma \) for each of the groups \( \mathcal{G}_\sigma \). It will not cause confusion to write \( \mathcal{B}^n(\mathcal{G}, \mu) \) instead of \( \mathcal{B}^n(\text{Res}_{W(\mathbb{F}_{p''})/W(\mathbb{F}_p)} \mathcal{G}, \mu) \). Let \( j \) be a gauge of multidegree \( d \) for \( v := (\text{Res}_{W(\mathbb{F}_{p''})/W(\mathbb{F}_p)} \rho) \circ \mu \), and let \( \tilde{v} \) be as in (20), recall that we regarded it as a family of cocharacters \( \tilde{v}_\sigma \) for the groups \( \tilde{\mathcal{G}}_\sigma := \text{GL}(\mathcal{V}_\sigma/W(\mathbb{F}_{p'})) \), where \( \sigma \in \mathbb{Z}/2r\mathbb{Z} \). Recall the group schemes \( \text{Res}_{W_n} \tilde{\mathcal{G}}_\sigma = \mathcal{H}_{\sigma,n} \supset \tilde{\mathcal{I}}_{\sigma,n} \) that are associated to \( \tilde{v}_\sigma \) according to subsection (2.2). In this subsection our aim is to introduce a family of fibered functors:
\[
\text{Flex}^d_X(\rho) : \mathcal{B}_X^{n+w}(\mathcal{G}, \mu, \tilde{v}) \to \mathcal{B}_X^n(\text{GU}(\mathcal{V}/W(\mathbb{F}_{p''}), \Psi), \tilde{v}),
\]
where \( X \) is a \( \mathbb{F}_{p'} \)-scheme and where \( w := \max\{w_\sigma\} < p \). We begin with a version of Flex for the fibered groupoid of \( \Phi \)-conjugacy classes over an affine scheme \( X = \text{Spec } R \). So consider some \( U_\sigma \in \mathcal{G}_\sigma(W_{n+w}(R)) = \mathcal{H}_{\sigma,n+w}(R) \) and define
\[
\tilde{U}_\omega := \rho_\omega \left( \prod_{d^*(\sigma) = \omega} F_{-\omega} U_\sigma \right) = \begin{cases} \rho_\omega (F_{s_j}^{-\omega_j} U_{s_j} \cdots F_{s_{j+1}^{-\omega_{j+1}}-1} U_{s_{j+1}-1}) & \omega = \omega_j \\
1 & \omega \notin \Omega \end{cases}
\]
The order of multiplication is the one indicated and does matter. Next we consider a family of elements \( k_\sigma \in \mathcal{I}_{\sigma,n+w+1}(R) \), say, representing
an isomorphism between $U_\sigma$ and $O_\sigma = k_{\sigma}^{-1}U_\sigma(\Phi_{n+1}^{\omega})k_{\sigma+1}$. Before we proceed to an analogous definition of elements $k_\omega$ we need a lemma:

**Lemma 5.2.** Suppose $k \in \mathcal{I}_\sigma(R)$, then

$$F_{\omega_j - \omega_j - 1}^{-1} \rho_{\sigma_j - 1}(\Phi_{\mu_\sigma_j}^\omega k) = \Phi_{\omega_j - 1}^{\omega_j - 1} \circ \cdots \circ \Phi_{\omega_j}^\omega \rho_{\sigma_j}(k)$$

holds. Here it shall be understood that the right hand side is well defined in the following iterative sense: For every index $\omega \in \{\omega_j - 1 + 1, \ldots, \omega_j\}$ the element $\Phi_{\omega_j - 1}^{\omega_j - 1} \circ \cdots \circ \Phi_{\omega_j}^\omega \rho_{\sigma_j}(k)$ lies in the subgroup $\tilde{\mathcal{I}}_\omega(R)$, so that one may apply $\Phi^{\tilde{\omega}}$ to it.

**Proof.** We begin with elements $k \in g_{\sigma_j}^{\mathfrak{g}^{+}} \otimes W(\mathfrak{p}_{\rho_j}) I(R) = J_{\sigma_j}(R)$. Let us exploit that they have a Lie theoretic meaning too, and thus could be exposed to the derived operation

$$g_{\sigma_j}^{\mathfrak{g}^{+}} \otimes W(\mathfrak{p}_{\rho_j}) \mathcal{V}_{j,l} \to \mathcal{V}_{j,l+1},$$

where $\bigoplus_{l = a_{\sigma_j}}^{b_{\sigma_j}} \mathcal{V}_{j,l} = \mathcal{V}_{a_{\sigma_j}}$ is the decomposition according to $\mu_{\sigma_j}$-weights, $g_{\sigma_j}^{\mathfrak{g}^{+}}$ being of weight one. This results in elements

$$t_l \in \text{Hom}_{W(\mathfrak{p}_{\rho_j})}(\mathcal{V}_{j,l}, \mathcal{V}_{j,l+1}) \otimes W(\mathfrak{p}_{\rho_j}) I(R),$$

which completely determine the image:

$$\begin{pmatrix}
1 & t_{b_{\sigma_j} - 1} & \cdots & \frac{t_{b_{\sigma_j} - 1} \cdots t_{a_{\sigma_j}}}{w_{\sigma_j}!}
0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & t_{a_{\sigma_j}}
0 & \cdots & 0 & 1
\end{pmatrix}
$$

of $k$ under $\rho_{\sigma_j}$. On the one hand the effect of $\Phi^{\tilde{\omega}}$ on block matrices can be visualised as

$$\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \mapsto \begin{pmatrix}
FA & V^{-1}B \\
pFC & FD
\end{pmatrix}
$$

on the other hand all interesting cuts $l \in [a_{\sigma_j}, b_{\sigma_j} - 1]$ occur exactly once (along with $\omega_j - \omega_j - 1 - w_{\sigma_j}$ many less interesting ones). This patches to give a matrix of the form:

$$\begin{pmatrix}
1 & s_{b_{\sigma_j} - 1} & \cdots & \frac{s_{b_{\sigma_j} - 1} \cdots s_{a_{\sigma_j}}}{w_{\sigma_j}!}
0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & s_{a_{\sigma_j}}
0 & \cdots & 0 & 1
\end{pmatrix},$$
where we have used the identity \( F^{n-1}(\prod_{\nu=1}^{n} V_{\nu}) = V^{-n}(\prod_{\nu=1}^{n} x_{\nu}) \) which follows for arbitrary elements \( x_{\nu} \in I(R) \) from \( V(p^{n-1} \prod_{\nu=1}^{n} x_{\nu}) = \prod_{\nu=1}^{n} V_{\nu} \) together with \( FV = VF = p \), which is valid in characteristic \( p \) rings, \( s_{t} \) means \( t^{\omega_{j} - \omega_{j-1} - 1} \). In the case of elements \( k \) in \( H_{\sigma_{j}}(R) \) one can utilize

\[
\rho_{\sigma_{j}} \circ \mu_{\sigma_{j}}^{-\Int}(p) = \nu_{\sigma_{j}}^{-\Int}(p) \circ \rho_{\sigma_{j}},
\]

because \( \rho_{\sigma_{j}}(k) \) is a \( W(R) \)-valued point of the parabolic subgroup of \( \tilde{G}_{\sigma_{j}} \) that is defined by the ascending filtration \( V_{j,a_{\sigma_{j}}} \subset \cdots \subset \bigoplus_{l \neq b_{\sigma_{j}}} V_{j,l} \).

This parabolic is easily seen to be preserved by the pairwise commuting endomorphisms

\[
H_{0}(\frac{\nu_{\sigma_{j}}}{z_{\sigma_{j}}})^{-\Int}(p), \ldots, H_{0}(\frac{\nu_{\sigma_{j}}}{z_{\sigma_{j}}-1})^{-\Int}(p),
\]

of which the composition is equal to \( \nu_{\sigma_{j}}^{-\Int}(p) \). For a general \( k \in I_{\sigma_{j}}(R) \) one applies the corollary 2.2 on factorizations.

The lemma allows us to define

\[
(27) \quad \tilde{k}_{\omega} := \begin{cases}  
\rho_{\sigma_{j}}(k_{\sigma_{j}}) & \omega = \omega_{j} \\ 
(\Phi_{\omega+1})^{-1} \tilde{k}_{\omega+1} & \omega \notin \Omega 
\end{cases},
\]

and it proves the required compatibility

\[ O_{\omega} = \tilde{k}_{\omega}^{-1} U_{\omega} (\Phi_{\omega+1}) \tilde{k}_{\omega+1} \]

too. In the non-banal case one completes the definition of (25) with a simple sheafification argument, which we leave to the reader. With a little bit of extra work one sees that the assignments in equation (27) come from actual algebraic group homomorphisms:

\[
(28) \quad \tilde{\rho}_{\omega} : I_{d(\omega)} \times \mathbb{F}_{p} \to \tilde{I}_{\omega} \times \mathbb{F}_{p},
\]

Notice that the diagrams

\[
\begin{array}{ccc}
I_{d(\omega+1)} \times \mathbb{F}_{p} & \xrightarrow{\tilde{\rho}_{\omega+1}} & \tilde{I}_{\omega+1} \times \mathbb{F}_{p} \\
\Phi^{\nu d(\omega+1)} \downarrow & & \Phi^{\nu_{\omega+1}} \downarrow \\
H_{d(\omega+1)-1} \times \mathbb{F}_{p} & \xrightarrow{F_{d(\omega+1)-1} \circ \Phi^{\nu d(\omega+1)-1}} & \tilde{H}_{\omega} \times \mathbb{F}_{p}
\end{array}
\]

and

\[
\begin{array}{ccc}
I_{d(\omega)} \times \mathbb{F}_{p} & \xrightarrow{\tilde{\rho}_{\omega}} & \tilde{I}_{\omega} \times \mathbb{F}_{p} \\
\downarrow & & \downarrow \\
H_{d(\omega)} \times \mathbb{F}_{p} & \xrightarrow{F_{d(\omega)} \circ \Phi^{\nu_{d(\omega)}}} & \tilde{H}_{\omega} \times \mathbb{F}_{p}
\end{array}
\]
are commutative, provided that \( d(\omega) \neq d(\omega + 1) \), and otherwise

\[
\begin{array}{ccc}
\mathcal{I}_{d(\omega)} \times \mathbb{F}_p f & \xrightarrow{\tilde{\rho}_\omega} & \tilde{\mathcal{I}}_{\omega} \times \mathbb{F}_p f \\
\tilde{\mathcal{I}}_{\omega+1} \times \mathbb{F}_p f & \xrightarrow{\tilde{\mathcal{H}}_{\omega} \times \mathbb{F}_p f}
\end{array}
\]

is a commutative diagram (the arrows without label stand for the inclusions \( \mathcal{I}_{d(\omega)} \times \mathbb{F}_p f \subset \mathcal{H}_{d(\omega)} \times \mathbb{F}_p f \) and \( \tilde{\mathcal{I}}_{\omega} \times \mathbb{F}_p f \subset \tilde{\mathcal{H}}_{\omega} \times \mathbb{F}_p f \)).

5.2.1. Compatibility with realizations. We need to point out a compatibility with the realization functor of subsection 3.2. Let \( \mathcal{P} \) be a \( n + w \)-truncated display with \( G \)-structure over a \( \mathbb{F}_p f \)-algebra \( R \), and let us write \( a = (\ldots, a_\sigma, \ldots) \) and \( b = (\ldots, b_\rho, \ldots) \) for the invariants introduced in subsection 5.1, in the unitary case these are \( 2r \)-tuples, and \( r \)-tuples otherwise. For any \( \mathcal{P} \) we have a canonical isomorphism

\[
W_n(R) \otimes W_{n+w}(R)(\text{Flex}_d(Fib(\rho, \mathcal{P}))) \cong \text{Flex}_d^{*d}(\text{Fib}_{\tilde{\mathcal{D}}}(\text{std}, \text{Flex}_R^d(\rho, \mathcal{P}))),
\]

which follows from \( d = d \circ d^* \circ d \). Applying the functor \( \text{Flex}_d^{*d} \) will be called the "rectification procedure". In case \( n = \infty \), we have a canonical isogeny:

\[
\mathbb{Q} \otimes \text{Fib}(\rho, \mathcal{P}) \cong \mathbb{Q} \otimes \text{Fib}(\text{std}, \text{Flex}_R^d(\rho, \mathcal{P})(\epsilon)),
\]

which follows along the same lines.

5.2.2. Compatibility with the modulus character. We also need to point out a useful compatibility with the determinant of the tangent bundle, which was introduced in section 4 if

\[
\tilde{\rho}_{\omega}^- := (\prod_{\sigma=\omega+1}^d(\text{Int}(p) \circ (\rho_{d(\omega)})(\varphi_{d(\omega)})) : \mathcal{G}_{d(\omega)}^- \to \mathcal{F}^{\omega-d(\omega)}_{\omega^-} \tilde{\mathcal{G}}_{\omega}^- \subset \tilde{\mathcal{G}}_{d(\omega)},
\]

then the diagram

\[
\begin{array}{ccc}
\mathcal{I}_{d(\omega)} \times \mathbb{F}_p f & \xrightarrow{\tilde{\rho}_\omega} & \tilde{\mathcal{I}}_{\omega} \times \mathbb{F}_p f \\
\mathcal{G}_{d(\omega)}^- \times \mathbb{F}_p f & \xrightarrow{\mathcal{F}^{d(\omega)-\omega \circ \tilde{\rho}_\omega}} & \tilde{\mathcal{G}}_{\omega}^- \times \mathbb{F}_p f
\end{array}
\]

commutes (notice that the higher levels of truncation are not at all preserved by \( \tilde{\rho}_\omega \), but they are rather shifted down by \( \sum_{\sigma=\omega+1}^d \tilde{w}_\sigma \)).
Now observe that there is a canonical $G_{d(\omega)}$-action on the $W(\mathbb{F}_p)$-module $F^{\omega-d(\omega)}\tilde{g}_\omega$, simply by means of the composition
\[
G_{d(\omega)} \rightarrow GL(F^{\omega-d(\omega)}\tilde{g}_\omega/W(\mathbb{F}_p)),
\]
in which the last arrow is the scalar extension of the adjoint action of $\tilde{g}_\omega$ on the $W(\mathbb{F}_p)$-module $\tilde{g}_\omega/\tilde{g}_\omega = \tilde{g}_\omega^+$. Consider the peculiar invertible $W(\mathbb{F}_p)$-module
\[
\chi^d_J := \left(\det W(\mathbb{F}_p)^{d(\omega)-\omega}\right),
\]
in order to put $ch^d_J$ for the $\prod_{i=0}^{r-1} G_{\sigma}$-character which is defined by it. The upshot are isomorphisms
\[
\omega^{\text{Flex}^d_J}(\chi) \cong \bigotimes_{\omega=0}^{r-1} \omega^P(ch^d_J),
\]
for any $w$-truncated $3n$-display $P$ with $G$-structure over any $\mathbb{F}_q$-scheme $X$.

5.2.3. **Compatibility with multipliers.** There exists a canonical 2-commutative diagram
\[
\begin{array}{ccc}
\mathcal{B}_X^{n+w}(G, \mu) & \xrightarrow{\chi^0_P} & \mathcal{B}_X^{n+w}(G_m \times W(\mathbb{F}_p'), \mu_1) \\
\downarrow \text{Flex}^d_J(\rho) & & \downarrow \\
\mathcal{B}_X^n(\text{GU}(\mathcal{V}/W(\mathbb{F}_p'), \Psi), \bar{\nu}) & \xrightarrow{\chi} & \mathcal{B}_X^n(G_m \times W(\mathbb{F}_p'), \mu_1),
\end{array}
\]
where the right vertical arrow is the truncation to the appropriate level.

5.2.4. **Faithfulness.** From now onwards we will always assume that $d$ is not a translation, i.e. $d \neq |d|$. In view of remark 3.4 this implies $\mathcal{B}(G, \mu) = \mathcal{B}'(G, \mu)$. In the rest of this subsection we study a first faithfulness property of the product functor $\prod_{i \in \Lambda} \text{Flex}^d_J(\rho_i, \ldots)$ where $\rho_i : G \rightarrow \text{GU}(\mathcal{V}_i/W(\mathbb{F}_p'), \Psi_i)$ is a family of (possibly) unitary representations each of which is equipped with a gauge $j_i$. Let us begin with the following triviality:

**Lemma 5.3.** Let $A$ be a $\mathbb{F}_q$-algebra which is separated with respect to the $M$-adic topology where $M = \{x \in A | x^p = 0\}$. Suppose that the product representation $\prod_{i \in \Lambda} \rho_i : G \rightarrow \prod_{i \in \Lambda} \text{GU}(\mathcal{V}_i/W(\mathbb{F}_p'), \Psi_i)$ is a monomorphism. Then the product functor $\prod_{i \in \Lambda} \text{Flex}^d_J(\rho_i, \ldots)$ is faithful (i.e. injective on morphisms).
Proof. In view of corollary 4.3 this is clear, because a close inspection of the formula (27) shows that the kernel of \(\prod_{i \in \Lambda} \text{Flex}^{d_i}(\rho_i, \ldots)\) is killed by a power of Frobenius. \(\square\)

From now on we do assume that \(\prod_{i \in \Lambda} \rho_i\) is a monomorphism, we have a very important corollary:

**Corollary 5.4.** If \(k\) is any field extension of \(\mathbb{F}_p^f\), then the product functor \(\prod_{i \in \Lambda} \text{Flex}^{d_i}(\rho_i, \ldots)\) is faithful over the ring \(R = k^{\text{perf}} \otimes_k k^{\text{perf}}\).

**Proof.** We must prove that \(R\) satisfies the criterion of the previous lemma. In order to control the situation we use a \(p\)-basis \(\{x_i|i \in I\}\), so that every element of \(k\) has a unique representation as a sum \(x = \sum_n a^n x^n\) where \(n = (\ldots, n_i, \ldots)\) runs through the set of multiindices with \(p-1 \geq n_i \geq 0\) an \(n_i = 0\) for almost all \(i\). It is easy to see that \(k^{\text{perf}}\) is the quotient of the polynomial algebra \(k[[t_{i,e}|i \in I, e = 1, \ldots]]\) by the ideal which is generated by \(t_{i,e+1} - t_{i,e}\) and \(t_{1,1} - x_i\). It follows that \(k^{\text{perf}} \otimes_k k^{\text{perf}}\) is the quotient of the polynomial algebra \(k^{\text{perf}}[[s_{i,e}|i \in I, e = 1, \ldots]]\) by the ideal which is generated by \(s_{i,e+1} - s_{i,e}\) and \(s_{p,1}\), so that \(M\) is generated by the set \(\{s_{i,1}\}\). We deduce that there exist ring endomorphisms \(\theta_{I_0}: R \to R\), defined by

\[
{s_{i,e} \mapsto \begin{cases} s_{i,e} & i \in I_0 \\ 0 & i \notin I_0 \end{cases}}
\]

for every finite subset \(I_0 \subset I\). Now \(M'\) is generated by the set \(\{s_{1,\nu}|\sum_i \nu_i = \nu\}\). The only multiindices which give rise to non-zero products are bounded by \(\nu_i < p\), and these are seen to involve factors indexed by at least \(\frac{\nu}{p-1}\) many elements of \(I\). Consequently \(\theta_{I_0}(M') = 0\) if \(\text{Card}(I_0) < \frac{\nu}{p-1}\), and this shows \(\bigcap_\nu M' = 0\). \(\square\)

**Remark 5.5.** The results of this subsection apply mutatis mutandis to the case of linear groups:

\[
\text{Flex}^{d_i}_{R}(\rho): \mathcal{B}^n_{R}(G, \mu) \to \mathcal{B}^n_{R}(G \times \text{GL}(V/W(\mathbb{F}_{p'})), \tilde{\nu}),
\]

where \(V\) is a finitely generated torsion free \(W(\mathbb{F}_{p'})\)-module. Do notice however that even in this case we are actually working with a highly non-split \(W(\mathbb{F}_p)\)-group namely: \(\text{Res}_{W(\mathbb{F}_{p'})/W(\mathbb{F}_p)}(G \times \text{GL}(V/W(\mathbb{F}_{p'})))\). All further results in this paper will apply simultaneously in the unitary and in the linear situation (although some of our notation indicates emphasis to the unitary case).

There is only one notable aspect, in which the unitary situation is very
slightly different: Notice that
\[ \sum_{\sigma \in \mathbb{Z}/2\mathbb{Z}} a_{\sigma} - \sum_{\omega \in \mathbb{Z}/2\mathbb{Z}} \tilde{a}_{\omega} = \sum_{\sigma \in \mathbb{Z}/2\mathbb{Z}} b_{\sigma} - \sum_{\omega \in \mathbb{Z}/2\mathbb{Z}} \tilde{b}_{\omega} = \sum_{\omega \in \mathbb{Z}/2\mathbb{Z}} \tilde{a}_{\omega + r} - \sum_{\sigma \in \mathbb{Z}/2\mathbb{Z}} a_{\sigma + r}, \]
and therefore \( \prod_{\sigma \in \mathbb{Z}/r\mathbb{Z}} F_{\sigma} e_{\sigma} = \prod_{\omega \in \mathbb{Z}/r\mathbb{Z}} F_{\omega} \tilde{e}_{\omega}. \) Contrarily, the cocharacters \( \prod_{\sigma \in \mathbb{Z}/r\mathbb{Z}} F_{\sigma} e_{\sigma} \) and \( \prod_{\omega \in \mathbb{Z}/r\mathbb{Z}} F_{\omega} \tilde{e}_{\omega} \) may well be different in the linear case.

5.3. Separatedness, part II. We are now able to state our first serious result:

**Proposition 5.6.** Let \( \mathbb{F}_{p^{f}} \subset A \subset B \) be a ring extension. Let \( \mathcal{P} \) and \( \mathcal{T} \) be displays with \( \mathcal{G} \)-structure over \( \text{Spec} A \) and let \( \psi_{i} \) be isomorphisms between \( \text{Flex}^{d_{j_{i}}} (\rho_{i}, \mathcal{P}) \) and \( \text{Flex}^{d_{j_{i}}} (\rho_{i}, \mathcal{T}) \). Assume that one of the following two assumptions is in force:

(i) The nilradical of \( B \) is trivial and
\[ A = B \cap \sqrt[2]{A} \] holds, the intersection taking place in \( \sqrt[2]{B} \).

(ii) The nilradical of \( B \) is nilpotent and \( A \) is noetherian.

Then the family of isomorphisms \( (\ldots, \psi_{i}, \ldots) \) has a preimage under the product functor \( \prod_{i \in \Lambda} \text{Flex}^{d_{j_{i}}} (\rho_{i}, \ldots) \) if and only if the same statement holds for the scalar extensions \( (\ldots, \psi_{i} \times_{A} B, \ldots) \).

**Proof.** The functor \( \prod_{i \in \Lambda} \text{Flex}^{d_{j_{i}}} (\rho_{i}, \ldots) \) is faithful over \( B \) and over \( A \), in both cases (i) and (ii), in particular a preimage over \( A \) is always unique and it exists if and only if the given preimage \( \phi_{B} \in \text{Hom}(\mathcal{P} \times_{A} B, \mathcal{T} \times_{A} B) \) descends to an isomorphism between \( \mathcal{P} \) and \( \mathcal{T} \). It seems to be elementary to check the case (i) of the assertion directly from the formula (27). More general, let us say that the \( \psi_{i} \)'s become flexed over some \( A \)-algebra \( B' \) if one has found a preimage of \( (\ldots, \psi_{i} \times_{A} B', \ldots) \).

We do the proof of (ii) in several steps:

**Step 1.** \( A \) is a product of finitely many fields.

It is clear that we can reduce the situation to the case of a single field, in which case \( B \) can be assumed to be a field as well, in fact \( B \) can be assumed to be equal to the perfection of \( A \), according to part (i) of our proposition. Now we merely have to prove that the scalar extensions by means of the two maps \( B \rightarrow B \otimes_{A} B \) give the same answer when applied to the isomorphism \( \phi_{B} \). As these are still preimages of the \( \psi_{i} \times_{A} (B \otimes_{A} B) \)'s we conclude by using corollary 5.4.

**Step 2.** \( A \) is a reduced complete local noetherian ring.

By the previous step and part (i) of proposition 5.6 we know that the
ψ_i’s become flexed over the normalization A’. Now part (ii) of corollary 4.7 and Théorème (23.1.5) tell us that we only need to know that the ψ_i’s become flexed over A/\text{rad}(A) which is dealt with by another application of the previous step.

Step 3. A is a complete local noetherian ring.
This case follows from the previous step and part (i) of corollary 4.7.

Step 4. A is a reduced local noetherian ring.
By step 1 we do know that the ψ_i’s become flexed over K(A). By the previous step we have the same result over the completion \hat{A}. Due to A = \hat{A} \cap K(A) this is sufficient.

Step 5. A is a general noetherian ring.
As before we can assume reducedness. By (flatness of A_p and) the previous step we have that the ψ_i’s become flexed over the local rings A_p. Now use A = \bigcap_{p \in \text{Spec} A} A_p, the intersection taking place in K(A).

□

Here is a separatedness property of (5), as promised in the title of this subsection:

**Theorem 5.7.** Let A be a noetherian \mathbb{F}_p[t]-algebra and let \mathcal{P} and \mathcal{T} be displays with \mathcal{G}-structure over Spec A and let \psi_i be isomorphisms between Flex_{d_j}(\rho_i, \mathcal{P}) and Flex_{d_j}(\rho_i, \mathcal{T}). Then there exists an ideal I \subset A, such that the following two assertions are equivalent for all A-algebras B with nilpotent nilradical:

(i) There exists an isomorphism \phi_B : \mathcal{P} \times_A B \to \mathcal{T} \times_A B such that Flex_{d_j}(\rho_i, \phi_B) = \psi_i \times_A B

(ii) BI = 0

Proof. It is clear that the set I = \{I \subset A|the \psi_i’s are flexed \ (\text{mod} \ I)\} is stable under intersections of finitely many ideals, and it is also clear that \bigcap I =: I_0 \in I is all we need to know. The ring \prod_{I \in I} A/\sqrt{I} =: B_1 is reduced, and so we know \sqrt{I_0} \in I. The nilradical of \prod_{I \ni I \subset \sqrt{I_0}} A/I =: B_2 is nilpotent and so we know that I_0 \in I.

5.4. Faithfulness of Flex, part II. Let \{\rho_i\}_{i \in \Lambda} be a family of (possibly unitary) representations. If one has fixed gauges \mathbf{j}_i for \psi_i = (\text{Res}_{W(\mathbb{F}_{p^r})/W(\mathbb{F}_p)} \rho_i) \circ \mu of some common multidegree \mathbf{d} then we will say that the family is gauged. Recall that we have constructed a specific cocharacter

\bar{\psi}_i : \mathbb{G}_m \times W(\mathbb{F}_{p^r}) \to \text{Res}_{W(\mathbb{F}_{p^r})/W(\mathbb{F}_p)} \text{GU}(V_i/W(\mathbb{F}_{p^r}), \Psi_i)
from \(v_i\) and \(j_i\). For each subset \(\pi \subset \Lambda\) of odd cardinality we need to introduce a further unitary group cocharacter

\[
\mathbb{G}_m \times W(F_p) \to \text{Res}_{W(F_{p^r})/W(F_p)} \text{GU}(\mathcal{V}_\pi/W(F_{p^{2r}}), \Psi_\pi)
\]

where \((\mathcal{V}_\pi, \Psi_\pi)\) is defined to be the skew-Hermitian \(W(F_{p^{2r}})\)-module \(\bigotimes_{i \in \pi} \mathcal{V}_i\), and where \(\tilde{\nu}_\pi\) is defined to be the product of the central cocharacter \(z \mapsto z^{-\text{Card}(\pi)}\) with the image of the diagonal cocharacter \((\ldots, \tilde{\nu}_1, \ldots)\) under the natural homomorphism

\[
g_{\pi} : \prod_{i \in \Lambda} \text{GU}(\mathcal{V}_i/W(F_{p^{2r}}), \Psi_i) \to \text{GU}(\mathcal{V}_\pi/W(F_{p^{2r}}), \Psi_\pi),
\]

that is given by the tensor product of the factors. Furthermore \(\rho_\pi\) will stand for the natural representation \(g_{\pi} \circ \prod_{i \in \Lambda} \rho_i\), finally write

\[
a = \sum_i \omega_i
\]

and ditto for \(b, w, \tilde{a}, \tilde{b}, \tilde{w}\), and let \(w\) be \(\max\{w_i, \sigma | i \in \Lambda, \sigma \in \mathbb{Z}\}\).

**Definition 5.8.** Fix a gauged family \(\{\rho_i\}_{i \in \Lambda}\) as above. A subset \(\pi \subset \Lambda\) of odd cardinality is called multiplyable if the weights of \(\tilde{\nu}_\pi\) are contained in \(\{0, 1\}\). A set \(\Pi\) of subsets is called strict if the following statements are valid:

1. **(S1)** Each element of \(\Pi\) is multiplyable
2. **(S2)** \(\prod_{i \in \Lambda} \rho_i : \mathcal{G} \to \prod_{i \in \Lambda} \text{GU}(\mathcal{V}_i/W(F_{p^{2r}}), \Psi_i)\) is a closed immersion
3. **(S3)** The generic fiber \(\mathcal{G}\) is the stabilizer of \(\bigcup_{\pi \in \Pi} \text{End}_G(\mathcal{V}_\pi)\)

**Remark 5.9.** If one assumes that \(p\) is odd and that \(\mathcal{G}\) is reductive, then \(\prod_{i \in \Lambda} \rho_i\) is a closed immersion if and only if its generic fiber has that property ([21 Corollary 1.3]). Hence in this case the condition \((\text{S2})\) is a consequence of \((\text{S3})\).

Furthermore \((\text{S2})\) implies that \(\mathcal{G}\) agrees with the Zariski-closure in \(\prod_{i \in \Lambda} \text{GU}(\mathcal{V}_i/W(F_{p^{2r}}), \Psi_i)\) of its generic fiber, and in turn this implies that one can pin down a constant \(c\) such that \(\mathcal{G}\) is the stabilizer of \(\mathcal{O}_H = \text{End}_G(\bigotimes_{i \in \Lambda} \mathcal{V}_i^\otimes c)\), cf. [12 Proposition(1.3.2)].

For every \(n+w\)-truncated \(3n\)-display \(\mathcal{P}\) with \(\mathcal{G}\)-structure over \(R/F_p\), and for every multiplyable \(\pi\) we obtain:

\[
W_n(R) \otimes_{W_n(F_p)} (\text{Flex}^d(Y\mathcal{H}(\rho_\pi, \mathcal{P}))) \cong \text{Flex}^{d_{\text{od}}} \circ \text{Fib}^{2\mathcal{H}}(\text{std}, \bigotimes_{i \in \pi} \text{Flex}^{d_{j_i}}(\rho_i, \mathcal{P})),
\]

from subsubsection 5.2.1. According to lemma 3.2 and remark 3.3 it follows that we obtain a canonical functorial homomorphism

\[
p^{d_{\text{od}}} \text{End}_G(\mathcal{V}_\pi) \to \text{End}_{\mathbb{Z}/2r\mathbb{Z}}(\text{Flex}^{2\mathcal{H}}(\text{std}, \bigotimes_{i \in \pi} \text{Flex}^{d_{j_i}}(\rho_i, \mathcal{P}))),
\]
whence it follows the existence of a canonical ring homomorphism:

\[(33)\]

\[W(F_{p^2}) + p^{|d^* \cdot d| + 1} \text{End}_G(V_\pi) =: \mathcal{O}_{L_\pi} \cong \text{End} (\bigotimes_{i \in \pi} \text{Flex}^{d_j}(\rho_i, \mathcal{P})).\]

The right-hand side graded tensor product is constructed along the lines of [3, Proposition 4.3], equivalently observe that there is a natural functoriality

\[B(g_{\pi}) : \prod_{i \in \Lambda} B(\text{GU}(V_i/W(F_{p^2}), \Psi_i), \tilde{\nu}_i) \rightarrow B(\text{GU}(V_\pi/W(F_{p^2}), \Psi_\pi), \tilde{\nu}_\pi),\]

because the interior actions of \(g_{\pi}(\ldots, \tilde{\nu}_i, \ldots)\) and \(\tilde{\nu}_\pi\) agree. Now we return to the situation in theorem 5.7 and study some necessary and sufficient conditions on the \(\psi_i\)'s to arise from an isomorphism \(\phi\):

**Corollary 5.10.** Let the assumptions be as in theorem 5.7. Let \(\Pi\) be the set of multiplyable subsets \(\pi \subset \Lambda\) such that the isomorphisms

\[\bigotimes_{i \in \pi} \psi_i : \bigotimes_{i \in \pi} \text{Flex}^{d_j}(\rho_i, \mathcal{P}) \xrightarrow{\cong} \bigotimes_{i \in \pi} \text{Flex}^{d_j}(\rho_i, \mathcal{T})\]

are \(\mathcal{O}_{L_\pi}\)-linear. If \(\Pi\) is strict, then the ideal \(I\) which was proven to exist in theorem 5.7 is nilpotent.

**Proof.** By what we have shown already we can assume that \(A\) is an algebraically closed field, in particular \(\mathcal{P}\) and \(\mathcal{T}\) are banal, and hence represented by graded display matrices \(U_\sigma, O_\sigma \in \mathcal{G}_\sigma(W(A))\). We are allowed to assume that \(U_\sigma = O_\sigma = 1\) for all \(\sigma\) outside the image of \(d\), so that formula (26) reads:

\[\tilde{U}_{i,\omega} = \begin{cases} \rho_i(f^{\sigma_j \omega_j} U_{\sigma_j}) & \omega = \omega_j \\ 1 & \omega \notin \Omega \end{cases}\]

and similarly for \(\tilde{O}_{i,\omega}\). Now assume that we have

\[\tilde{O}_{i,\omega} = \tilde{k}_{i,\omega}^{-1} \tilde{U}_{i,\omega} F(\tilde{\nu}_{i,\omega+1} p^{-1} \tilde{k}_{i,\omega+1} \tilde{\nu}_{i,\omega+1} p)).\]

It is straightforward to check that the families \((\ldots, \tilde{k}_{i,\omega_j} \ldots) = \tilde{k}_{\omega_j}\) constitute elements in \(\mathcal{G}_{\omega_j}\) and thus can be written in the form \(F^{\sigma_j \omega_j} \rho_i(k_{\sigma_j})\). The difficulty lies in checking that indeed \(k_{\sigma_j} \in \mathcal{I}_{\sigma_j}(A)\). Notice however, that \(\tilde{U}_{i,\omega} = \tilde{O}_{i,\omega} = 1\) for all \(\omega\) outside the image of \(d^*\), and
consequently
\[
\tilde{U}_{i,\omega_{j-1}}^{-1}\tilde{k}_{i,\omega_{j-1}}\tilde{O}_{i,\omega_{j-1}} = \\
(\prod_{\omega=\omega_{j-1}+1}^{\omega_j} \tilde{v}_{i,\omega}(p)^{-1})(\prod_{\omega=\omega_{j-1}+1}^{\omega_j} \tilde{k}_{i,\omega})(\prod_{\omega=\omega_{j-1}+1}^{\omega_j} \tilde{v}_{i,\omega}(p))
\]
\[
= \tilde{\rho}_i(\tilde{\varphi}_{\omega_{j}} k_{\sigma_j})
\]
where we have used lemma 5.2. Given that \(\tilde{U}^{-1}_{i,\omega_{j-1}}, \tilde{k}_{i,\omega_{j-1}}, \) and \(\tilde{O}_{i,\omega_{j-1}}\) are elements in \(\mathcal{H}_{\omega_{j-1}}(A)\) we get the integrality of \(\tilde{\varphi}_{\omega_{j}} k_{\sigma_j}\) which was sought for.

For elements \(\omega\) outside the image of \(d^*\) it is important to understand that the corresponding families \(\tilde{k}_{\omega} = (\ldots, \tilde{k}_{i,\omega}, \ldots)\) might not constitute elements in \(\mathcal{G}_{\omega}\), in fact they are of no significance. \(\Box\)

We have to work with two more fibered categories, assuming that a gauged family and a set \(\Pi\) of multiplyable subsets of \(\Lambda\) are fixed: For every \(W(\mathbb{F}_{p^r})\)-scheme \(X\) we define \(\mathfrak{B}^n_X\) to be the groupoid of data \((\{S_i\}_{i\in\Lambda}, \{s_{\pi}\}_{\pi\in\Pi})\) where

- \(S_i\) is a \(n\)-truncated \(3n\)-display with \(\text{GU}(V_i/W(\mathbb{F}_{p^r}), \Psi_i)\)-structure over \(X\), for every \(i \in \Lambda\).
- \(s_{\pi}: \mathcal{O}_{L_{\pi}} \to \text{End}(\bigotimes_{i\in\pi} S_i)\) is a *-preserving ring homomorphism for every \(\pi \in \Pi\), where \(\mathcal{O}_{L_{\pi}}\) is as in (33).

This is a again a stack over \(W(\mathbb{F}_{p^r})\) for the fpqc topology (the isomorphisms in \(\mathfrak{B}^n_X\) are tuples \(\zeta_i: S_i^i \cong S_i\) such that the diagrams:

\[
\begin{array}{ccc}
\bigotimes_{i\in\pi} S_i & \xleftarrow{s_{\pi}(a)} & \bigotimes_{i\in\pi} S_i^i \\
\downarrow s_{\pi}(a) & & \downarrow s_{\pi}(a) \\
\bigotimes_{i\in\pi} S_i & \xleftarrow{s_{\pi}(a)} & \bigotimes_{i\in\pi} S_i^i
\end{array}
\]

commute for all \(\pi \in \Pi\) and \(a \in \mathcal{O}_{L_{\pi}}\). Analogously we define \(\mathfrak{C}_X\) to be the groupoid of data \((\{M_i\}_{i\in\Lambda}, \{m_{\pi}\}_{\pi\in\Pi})\) where

- \(M_i \in \text{Vec}_X\) and \(V_i \in \text{Vec}_{\text{Spec} W(\mathbb{F}_{p^r})}\) have the same constant rank for every \(i \in \Lambda\).
- \(m_{\pi}: \mathcal{O}_{L_{\pi}} \to \text{End}_{\mathcal{O}_X}(\bigotimes_{i\in\pi} M_i)\) is a \(W(\mathbb{F}_{p^r})\)-linear ring homomorphism for every \(\pi \in \Pi\), where \(\mathcal{O}_{L_{\pi}}\) is as in (33).

The significance of \(\mathfrak{C}\) stems from the fact, that for every \(p\)-adically separated and complete ring \(R\), there are natural forgetful functors

\[
\mathfrak{Fib}^n_{\pi,R}: \mathfrak{B}^n_R \to \mathfrak{C}_{W_{n+1}(R)},
\]
which are induced by passage to the $\sigma$-eigenspaces of the underlying modules (in the sense of subsection 2.7) for all $n$ including $\infty$. The significance of $B^n$ stems from the product functor 

$$\prod_{i \in \Lambda} \text{Flex}^{d_j}(\rho_i) : B^{n+w}(G, \mu) \times_{W(\mathbb{F}_{p^f})} \mathbb{F}_{p'} \to B^n \times_{W(\mathbb{F}_{p^{2v}})} \mathbb{F}_{p'},$$

which is naturally induced according to (33). Let us now fix a $\mathbb{F}_{p^f}$-scheme $X$ and an object $\left(\{S_i\}_{i \in \Lambda}, \{s_\pi\}_{\pi \in \Pi}\right)$ of $\mathcal{B}^*_X$. By a flexibilisator for $\left(\{S_i\}_{i \in \Lambda}, \{s_\pi\}_{\pi \in \Pi}\right)$ over an $X$-scheme $X$ we mean a 2-commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & X \\
\downarrow & & \downarrow \\
B^{n+w}(G, \mu) \times_{W(\mathbb{F}_{p^f})} \mathbb{F}_{p'} & \longrightarrow & B^n \times_{W(\mathbb{F}_{p^{2v}})} \mathbb{F}_{p'}
\end{array}
$$

and we will simply say that $\left(\{S_i\}_{i \in \Lambda}, \{s_\pi\}_{\pi \in \Pi}\right)$ becomes flexed over $X$ if there exists at least one such diagram. If $X = \text{Spec} R$ and $n = \infty$, then the following holds:

- If $R$ is separated with respect to the $M$-adic topology, where $M = \{x \in R | x^p = 0\}$, then no flexibilisator has non-trivial automorphisms,
- whenever $R$ is perfect or noetherian and reduced then flexibilisators are unique up to unique isomorphism.

It is practical to have the following principle as a reference:

**Corollary 5.11.** As above, we fix a gauged family of representations $\{\rho_i\}_{i \in \Lambda}$ and a strict set of subsets $\Pi$. Let $\mathbb{F}_{p^f} \subset A \subset B$ be an extension of reduced noetherian rings and let $C$ be a faithfully flat noetherian reduced $A$-algebra. Assume that $\left(\{S_i\}_{i \in \Lambda}, \{s_\pi\}_{\pi \in \Pi}\right)$ is an object of $\mathcal{B}_{\text{Spec} A}$ which becomes flexed over $B$ and over $C$. Suppose that one of the following two assumptions is in force:

(i) The condition (32) is satisfied, $B$ is essentially finitely generated over $A$, and

$$\text{(35)} \quad C \otimes_A \sqrt{A} = \sqrt{C}$$

holds (as $C$-algebras).

(ii) Every maximal point of $\text{Spec} B$ is mapped to a maximal point of $\text{Spec} A$ and belongs to a separable residue field extension, and $C$ is essentially finitely generated over $A$.

Then $\left(\{S_i\}_{i \in \Lambda}, \{s_\pi\}_{\pi \in \Pi}\right)$ is already flexed over $A$. 
Proof. Choose preimages $\mathcal{P}/B$ and $\mathcal{T}/C$ of $\mathcal{S}_i \times_A B$ and $\mathcal{S}_i \times_A C$ under the product functor $\prod_i \text{Flex}^d_{\mathcal{S}_i}(\rho_i, \ldots)$. Notice that in the (i)-case the maximal points of $\text{Spec} \ C$ are mapped to the maximal points of $\text{Spec} \ A$ and belong to separable residue field extensions, by a theorem of MacLane, hence the $A$-algebra $B \otimes_A C$ is noetherian and reduced in both cases, so that $\mathcal{P} \times_B (B \otimes_A C) \cong \mathcal{T} \times_C (B \otimes_A C)$ canonically. Now use $\mathcal{P}/B$ to construct a descent datum for $\mathcal{T} \times_C (B \otimes_A C)$ relative to the fpqc morphism $B \to (B \otimes_A C)$ and use proposition 5.6 to restrict it to a descent datum for $\mathcal{T}$ relative to the fpqc morphism $A \to C$. □

Remark 5.12. In the same vein one would obtain a similarly satisfactory variant of the part (i) of corollary 5.11 if $A \to C$ was required to be a normal morphism with $A$ integrally closed in $B$. See [10, Proposition (6.14.4)].

Suppose a ring $B$ was reduced and faithfully flat, essentially finitely generated and generically separable (in the sense of part (ii) of the previous corollary) over some noetherian subring $A$. It follows that every flexibilisator over $B$ comes from $A$.

We have used several times that property (32) is preserved under flat base change to $A$-algebras $C$ with property (35). The same is true for the morphism $A \to \prod_m A_m$ (which may or may not satisfy (35)). This leads to the following analog of part (i) of 5.11:

Corollary 5.13. Suppose that $A$ is a reduced noetherian $\mathbb{F}_p$-algebra which happens to be equal to $K(A) \cap \sqrt{A}$. Then the previous assumptions on $\{\rho_i\}_{i \in A}$ and $\Pi$ imply that an object $\{\{S_i\}_{i \in A}, \{s_{\xi}\}_{\xi \in \Pi}\}$ of $\mathcal{B}_{\text{Spec} \ A}$ is flexed over $A$ if and only if the same holds over all localisations $A_m$.

Proof. The whole point is that $C = \prod_m A_m$ is faithfully flat over $A$ and that

$$(\prod_m A_m) \otimes_A (\prod_m A_m) \subset (\prod_m K(A)) \otimes_K(A) \otimes_K(A)$$

satisfies condition (32): Given that the inclusion $\prod_m A_m \subset \prod_m K(A)$ satisfies this already by assumption, this fact is easily implied by the fact that the base change by means of the morphism $A \to \prod_m A_m$ preserves condition (32), as does $A \to \prod_m K(A)$. □

6. Windows with additional structure, part II

Fix a $\Phi$-datum $(\mathcal{G}, \mu)$. The following is an analog of the main result in [29], for our category $\mathcal{B}'(\mathcal{G}, \mu)$:
Lemma 6.1. Let \((A, J, \tau)\) be a frame over \(W(\mathbb{F}_p)\). The canonical functor
\[
\hat{\delta} : \hat{B}'_{A,J}(G, \mu) \to \hat{B}'_{W(A),W(J)+I(A)}(G, \mu) \cong B'_{A/J}(G, \mu)
\]
that is induced from Cartier’s diagonal map \(A \to W(A)\) is an equivalence of categories.

Proof. In the torsionfree situation the ghost map is injective, and it is well known that one can use the Frobenius lift to give a precise description of the image:
\[
w : W(A) \cong \{(w_0, \ldots) \in A^{\mathbb{N}_0} | w_{i+1} \equiv \tau(w_i) \pmod{p^{i+1}}\},
\]
This implies the congruence \(F^n x \equiv \tau(x) \pmod{W(pA)}\). Now consider some \(x, x' \in W(A)\) have the same image in \(W(A/p^s A)\) if and only if the components of their respective ghost images \(w, w' \in A^{\mathbb{N}_0}\) satisfy \(w_i \equiv w'_i \pmod{p^{i+s}}\). Observe that \(F U \equiv \tau(U) \pmod{W(pA)}\) holds. Part (ii) of lemma 4.2 implies the existence of some \(\tilde{\tau}_pA(A) \ni h \equiv 1 \pmod{W(pA)}\) with
\[
FU = h^{-1} \tau(U)^{\hat{\phi}_{pA}} h.
\]
By a recursion argument one easily finds \(h = \tau(\hat{m}) \hat{m}^{-1} \), for some \(\hat{m} \in \tilde{\tau}_pA(A)\) and thus
\[
F(h^{-1}U \hat{\phi}_{pA} \hat{m}) = \tau(h^{-1}U \hat{\phi}_{pA} \hat{m}).
\]
According to the introductory remarks of section 4 and the generalization of the canonical splitting \([17]\) due to \([30, 1.4]\) one can find a factorization \(\hat{m} = mm^+ \), where \(m \in \mathcal{I}(A)\), and \(m^+ \in g^+ \otimes_{W(\mathbb{F}_p)} pA\).
Write \(U' := m^{-1}U \hat{\phi} \hat{m} \in \mathcal{O}_{B_{A,J}(G, \mu)}\); this is the required preimage of \(U\), as \(m^+\) lies in the image of \(\delta\). \(\square\)

Remark 6.2. The same argument shows that there exists an equivalence \(B'_A(G, \mu) \to B'_{W(A),I(A)}(G, \mu) \cong B'_A(G, \mu)\).

6.1. Algebraic Connections. Let \(X\) be a smooth \(S\)-scheme with sheaf of Kähler differentials \(\Omega^1_{X/S}\). As usual we extend the differentation \(d : \mathcal{O}_X \to \Omega^1_{X/S}\) to an endomorphism (still denoted by \(d\)) on the exterior algebra \(\bigoplus_m \Omega^m_{X/S}\) in such a way that the Leibniz rule \(d(\eta_1 \wedge \eta_2) = d(\eta_1) \wedge \eta_2 + (-1)^{\deg(\eta_1)} \eta_1 \wedge d(\eta_2)\) holds, so that in particular \(d \circ d = 0\). By a connection (relative to \(S\)) on some \(\mathcal{F} \in \text{Vec}_X\) one means a map \(\nabla : \mathcal{F} \to \Omega^1_{X/S} \otimes \mathcal{O}_X \mathcal{F}\) satisfying an analogous Leibniz
Suppose, for instance, that \( F \in \text{Vec}_X \) is isomorphic to \( \mathcal{O}_X^h \). The connections on \( F \) can be written in the form \( d + D \), for some \( D \in \Gamma(X, \Omega^1_{X/S} \otimes_{\mathcal{O}_X} F \otimes_{\mathcal{O}_X} \hat{F}) \), furthermore the horizontal isomorphisms from some \( (\mathcal{O}_X^h, d + E) \) to \( (\mathcal{O}_X^h, d + D) \) are those \( u \in \text{GL}(h, \Gamma(X, \mathcal{O}_X)) \), such that

\[
(36) \quad uE u^{-1} - d(u) u^{-1} = D
\]

Finally, notice that \( R = d(D) + D \wedge D \), furthermore the above implies

\[
(37) \quad uQ u^{-1} = R
\]

where \( Q \) is the curvature of \( d + E \). From now on we assume that \( S \) is a scheme over a Dedekind ring \( B \), and that \( \mathcal{G} \) is a smooth affine group scheme over \( B \). By an integrable connection \( \nabla \) on some locally trivial principal homogeneous space \( P \) over \( X \), we simply mean a \( \text{Sys}_{X/S} \)-valued rigid \( \otimes \)-category structure on \( \text{Sys}_{X/S} \) (and so they would without the integrability condition).

Suppose, for instance that \( \mathcal{F} \in \text{Vec}_X \) is isomorphic to \( \mathcal{O}_X^h \). The connections on \( \mathcal{F} \) can be written in the form \( d + D \), for some \( D \in \Gamma(X, \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \hat{\mathcal{F}}) \), furthermore the horizontal isomorphisms from some \( (\mathcal{O}_X^h, d + E) \) to \( (\mathcal{O}_X^h, d + D) \) are those \( u \in \text{GL}(h, \Gamma(X, \mathcal{O}_X)) \), such that

\[
(36) \quad uE u^{-1} - d(u) u^{-1} = D
\]

Finally, notice that \( R = d(D) + D \wedge D \), furthermore the above implies

\[
(37) \quad uQ u^{-1} = R
\]

where \( Q \) is the curvature of \( d + E \). From now on we assume that \( S \) is a scheme over a Dedekind ring \( B \), and that \( \mathcal{G} \) is a smooth affine group scheme over \( B \). By an integrable connection \( \nabla \) on some locally trivial principal homogeneous space \( P \) over \( X \), we simply mean a \( \text{Sys}_{X/S} \)-valued rigid \( \otimes \)-functor factorizing the twisted fiber functor \( \varpi_P \) (see e.g. [23 VI.1.2.3.1]). Suppose, for instance that \( P \) is trivial, so that \( \varpi_P \cong \mathcal{O}_X \otimes_B \mathcal{O}_0 \). It is immediate that every \( D \in \Omega^1_{X/S} \otimes_B \mathfrak{g} \) gives rise to a connection on each \( \varpi_P(\rho) \), namely

\[
(38) \quad \nabla_D(\rho) := d + \rho^{\text{der}}(D),
\]

clearly one has \( \nabla_D(\rho_1 \otimes_B \rho_2) = \nabla_D(\rho_1) \otimes_{\mathcal{O}_X} \nabla_D(\rho_2) \) for any \( \rho_1, \rho_2 \in \text{Rep}_0(\mathcal{G}) \), and similarly for duality, furthermore the curvature of \( \nabla_D(\rho) \) is equal to \( \rho^{\text{der}}(d(D) + \frac{[D,D]}{2}) \), and in the sequel we will refer to the expression \( d(D) + \frac{[D,D]}{2} \) as the curvature of the \( \mathfrak{g} \)-valued 1-form \( D \). Now assume that \( E \) and \( D \) are both \( \mathfrak{g} \)-valued 1-forms with vanishing curvature, and consider some element \( u \in \mathcal{G}(X) \). The resulting automorphism of \( \varpi_P(\rho) \) is an isomorphism \( (\varpi_P(\rho), \nabla_E(\rho)) \to (\varpi_P(\rho), \nabla_D(\rho)) \) in \( \text{Sys}_{X/S} \) if and only if

\[
\rho^{\text{der}}(\text{Ad}(u)(E) - D) = d((\rho(u)) \rho(u)^{-1}.
\]

The right-hand side of this equation can be rewritten in an aesthetically pleasing way: Consider \( \eta_\rho = d(f_1) \wedge f_2 \) where \( f_1 \) is the inclusion of
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GL(\(\varpi_G^0(\rho)/B\)) into \(\text{End}_B(\varpi_G^0(\rho))\), and \(f_2(g) = f_1(g^{-1})\). It is worthy of remark that the 1-form \(\eta_\rho\) is right-invariant, so let \(\eta_G\) be the canonical, so called Cartan-Maurer, right-invariant \(g\)-valued 1-form on \(G\). One has \(\rho^{\text{der}}(\eta_G) = \eta_\rho \circ \rho\). Hence \(u\) is a horizontal, provided that

\[
\text{Ad}(u)E - D = \eta_G \circ u
\]

(where \(\circ\) denotes the pull-back of a differential form by means of a morphism).

Connections are related to displays:

**Proposition 6.3.** Let \(\mathcal{P}\) be a display with \(G\)-structure over a smooth \(W_\nu(k)\)-scheme \(X\), where \(\nu\) is a positive integer and \(k\) is a perfect field containing \(\mathbb{F}_{p_f}\). Then there exists a canonical connection \(\nabla_\mathcal{P}\) on the composite of the \(\otimes\)-functors:

\[
\text{Rep}_0(G \times W(\mathbb{F}_{p_f})) \to \text{Rep}_0(G^-) \xrightarrow{\varpi_p} \text{Vec}_X.
\]

The formation of \(\nabla_\mathcal{P}\) is functorial in \(\mathcal{P}\) and it commutes with base change in the following sense: Whenever there is a diagram

\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec } W_{\nu'}(k') & \longrightarrow & \text{Spec } W_\nu(k)
\end{array}
\]

where \(\nu' \leq \nu\) and \(k'\) is a perfect field extension of \(k\), the canonical isomorphism \(O_X \otimes_{O_X} \varpi_\mathcal{P} \cong \varpi_{\mathcal{P}'}\) is a horizontal one, if the left-hand side is endowed with \(\nabla_\mathcal{P}\) and the right-hand side is endowed with \(\nabla_{\mathcal{P}'}\), where \(\mathcal{P}' := \mathcal{P} \times_X X'\).

**Proof.** It clearly suffices to consider an affine basis \(X = \text{Spec } R\), and in fact the part (ii) if lemma 2.5 tells us that it is sufficient to consider a banal display, so that the \(\otimes\)-functor in question is just \(R \otimes_{W(\mathbb{F}_{p_f})} \omega_0^G\).

So choose a \(p\)-adically formally smooth, separated and complete ring \(A\) with \(A/pA \cong R\), such a ring is unique up to non-unique isomorphism. Let \(\hat{\Omega}_A^1\) be the formal Kähler differentials of \(A\). Now one can reduce the construction of \(\nabla_{\mathcal{P}}\) from the following, slightly stronger problem: Given any \(U \in \mathcal{G}(W(A))\) (regarded as a banal display with an auxiliary lift to \(A\)), find an associated element \(D_U \in \hat{\Omega}_A^1 \otimes g\), such that the following holds:

(i) For all \(h \in \hat{I}(A)\) we have \(\text{Ad}(w_0(h))(D_{U'}) - D_U = \eta_G \circ w_0(h)\),

where \(w_0\) denotes the 0th component of the ghost map and \(U'\) stands for \(h^{-1}U^\# h\)

(ii) \(d(D_U) + \frac{[D_U, D_U]}{2} = 0\)

Let \(\theta_A : A \to A \oplus \hat{\Omega}_A^1\) be the map \(x \mapsto x + d(x)\), and let \(U'\) be the image of \(U\) under the map \(W(\theta_A)\). Notice that \(\hat{\Omega}_A^1\) is a pd-ideal in a
\textit{p}-adically separated and complete ring, so that the part (ii) of lemma 4.2 grants the existence of a unique \( k \in \mathcal{H}(A \oplus \hat{\Omega}_A^1) \) with \( k^{-1}U^{\hat{\Phi}}k = U' \) and \( 1 \equiv k \mod \hat{\Omega}_A^1 \) (this particular pd-ideal contains no powers of \( p \), so that one might rather look at the pd-ideals \( a := p^n A \oplus \hat{\Omega}_A^1 \) in the first place, and afterwards use the uniqueness of \( k \) in order to pass to their intersection). That equation can be rewritten as:

\[ U^{\hat{\Phi}}kU^{-1} = k(U'U^{-1}) , \]

and given that all of the elements \( U^{\hat{\Phi}}kU^{-1}, k, \) and \( U'U^{-1} \) are \( \equiv 1 \mod \hat{\Omega}_A^1 \) the above problem is solved by \( D_U := w_0(k) \), which can be regarded as an element of \( \hat{\Omega}_A^1 \otimes \mathfrak{g} \). The vanishing of the curvature is postponed until lemma 6.4. □

Suppose that \( A \) is a \( \textit{p} \)-adically separated, complete, and formally smooth \( W(F_p) \)-algebra with module of formal Kähler differentials \( \hat{\Omega}_A^1 \). If \( \tau \) is a Frobenius lift, then so is the map

\[ A \oplus \hat{\Omega}_A^1 \to A \oplus \hat{\Omega}_A^1; x + \sum x_i dy_i \mapsto \tau(x) + \sum \tau(x_i)d\tau(y_i) , \]

which we continue to denote by \( \tau \) (here notice that \( py_i^{p-1}dy_i = dy_i^p \equiv d\tau(y_i) \mod \hat{\Omega}_A^1 \)). We would like to translate some of the previous techniques into the language of windows, so suppose that \( U \in \mathcal{G}(A) \) satisfies the nilpotence condition, define \( U' \in \mathcal{G}(A \oplus \hat{\Omega}_A^1) \) as above, and suppose for a while that the equation

\[ U^{\hat{\Phi}}kU^{-1} = k(U'U^{-1}) \]

held for some element \( \hat{G}_a(A \oplus \hat{\Omega}_A^1) \ni k \equiv 1 \mod \hat{\Omega}_A^1 \), where \( a = pA \oplus \hat{\Omega}_A^1 \). Plugging \( U^{\hat{\Phi}}kU^{-1}, k, \) and \( U'U^{-1} \) into Cartier’s diagonal map \( \hat{\delta} \), would reveal that the construction of the above \( \mathfrak{g} \)-valued 1-form reads:

\[ D_{\hat{\delta}(U)} = w_0(\hat{\delta}(k)) = k \]

(regarded as an element in the kernel \( \ker(\hat{G}_a(A \oplus \hat{\Omega}_A^1) \to \hat{G}_a(A)) = \hat{\Omega}_A^1 \otimes \mathfrak{g} \)). The existence and uniqueness of that \( \mathfrak{g} \)-valued formal 1-form, can be justified independently of lemma 4.2.

\textbf{Lemma 6.4.} Fix \( U \in \mathcal{G}(A) \), where \((A,pA,\tau)\) is a formally smooth frame over \( W(F_{p^n}) \). Let us write \( \phi^* \) for the \( \tau \)-linear map on \( \bigoplus_{m \neq 0} \hat{\Omega}_A^m \otimes \mathfrak{g} \) which is given by the composition of \( \tau \otimes \text{id} \) with \( \text{id} \otimes \text{Ad}(\mu(p)^{-1}) \) (notice that “\( m = 0 \)” is omitted). If \( U \) satisfies the nilpotence condition, then there exists a unique element \( D_U \in \hat{\Omega}_A^1 \otimes \mathfrak{g} \) with \( \text{Ad}(U)(\phi^* D_U) = D_U = \).

implies that $D_p$ vanishes.

Proof. This is analogous to [3, Lemma 2.8]. The nilpotence condition implies that $D \mapsto \text{Ad}(U)(\phi(D)) - \eta_G \circ U$ is a contractive map for the $p$-adic topology, so there exists a unique fixedpoint $D_U$. We have $R_U = \text{Ad}(U)(\phi^* R_U)$, again this has a unique solution, namely $R_U = 0$. □

6.2. Formal Connections. We want to look at $A := W(k_0)[[t_1, \ldots, t_d]]$, where the field $k_0$ is an algebraically closed or an algebraic extension of $\mathbb{F}_p$, and $d := \text{rank}_{W(\mathbb{F}_p)} \mathfrak{g}^+$. Let us fix the Frobenius lift determined by $\tau(t_i) := t_i^p$. Any display $\mathcal{P}_0$ with $\mathcal{G}$-structure over $k_0$ is automatically banal, and hence represented by some $U_0 \in \mathcal{G}(W(k_0))$. Choose a $W(\mathbb{F}_p)$-basis $\{N_1, \ldots, N_d\}$ of $\mathfrak{g}^+$. Assume that $\mathcal{G}$ is reductive and that $\mu$ fulfills the criterion in remark [4.11]. Then we have $U_1 := e^+(t_1 N_1 + \cdots + t_d N_d) U_0 \in \mathcal{Ob}_{\mathcal{G}}(\mathcal{P}_0)$, so that the element $U_{uni} := \hat{\delta}(U_1) \in \mathcal{Ob}_{\mathcal{G}}(\mathcal{P}_0)$ represents the universal formal mixed characteristic deformation $\mathcal{P}_{uni}$ of $\mathcal{P}_0$. This object has a rich amount of symmetry: For every $\gamma \in \text{Aut}(\mathcal{P}_0)$, there exists a unique pair $(h_0, s_0) \in \mathcal{I}(A) \rtimes \text{Aut}(A/W(k_0))$ with:

- $s_0(U_{uni}) = h_0^{-1} U_{uni} s_0 h_0$
- $h_0$ is a lift of $\gamma \in \mathcal{I}(k_0)$

We will be particularly interested in the truncated version $(u_0, s_0) \in \mathcal{G}^-(A) \rtimes \text{Aut}(A/W(k_0))$, where $u_0 := w_0(h_0)$. Let $D_1$ be the 1-form that corresponds to $U_1$, as has been pointed out before. Dwork’s trick provides us with an element $\Theta \in \mathcal{G}(K(k_0)\{t_1, \ldots, t_d\})$ satisfying:

\[
\begin{aligned}
\Theta(0, \ldots, 0) &= 1 \\
\Theta^{-1} U_1^F \mu(p)^{-1} \tau(\Theta) &= U_0^F \mu(p)^{-1} \\
\eta_G \circ \Theta &= -D_1
\end{aligned}
\]

(the last equation has to be interpreted in $K(k_0)\{t_1, \ldots, t_d\} \otimes_A \hat{\mathcal{O}}_A^1 := \hat{\mathcal{O}}_A^1(K(k_0)\{t_1, \ldots, t_d\})$). Note the following consequence: An arbitrary element $(u, s) \in \mathcal{G}(K(k_0)\{t_1, \ldots, t_d\}) \rtimes \text{Aut}(A/W(k_0))$ lies in $\text{Int}(\Theta)(\mathcal{G}(K(k_0)) \rtimes \text{Aut}(A/W(k_0)))$ if and only if

\[
\text{Ad}(u) s(D_1) - D_1 = \eta_G \circ u,
\]

here the right-hand side is again the image of $\eta_G$ under the map $\hat{\mathcal{O}}_A^1 \rightarrow \hat{\mathcal{O}}_A^1(K(k_0)\{t_1, \ldots, t_d\})$, and the left-hand side utilizes the natural action of $\text{Aut}(A/W(k_0))$ on $\hat{\mathcal{O}}_A^1(K(k_0)\{t_1, \ldots, t_d\})$, from the left. Let us call the element $(u, s)$ horizontal with respect to $D_1$ if (39) holds. The canonicity of $D_1$ shows that the elements $(u_0, s_0)$ are indeed horizontal with respect to
\[ D_1, \text{ for all } \gamma \in \text{Aut}(\mathcal{P}_0). \]
After these preparatory remarks we are able to state and prove the important technical fact that \( D_1 \) tends to be complicated as can be. More specifically, write \( G \) and \( G^+ \) for the generic fibers of \( \mathcal{G} \) and \( \mathcal{G}^+ \) and let \( Z \subset G \) be the center, and let \( G^{\text{spec}} \subset G \) be the smallest algebraic subgroup containing the commutator of the \( K(\mathbb{F}_p) \)-groups \( G^+ \) and \( G \times K(\mathbb{F}_p) K(\mathbb{F}_p') \), then we have \( G^+ \subset G^{\text{spec}} \times_{K(\mathbb{F}_p)} K(\mathbb{F}_p') \), and \( G^{\text{spec}} \lhd G \).

In the following result we call any \( \eta \) by means of the formal analogue of \( \eta \) and \( \nabla \) holds:
\[ \eta \circ U_1 = \eta \circ e_1(t_N N_1 + \cdots + t_d N_d) \in \hat{\Omega}_A^1 \otimes_{K(\mathbb{F}_p)} \text{Lie}(G^{\text{spec}}). \]
Hence the \( p \)-adically contractive map \( D \mapsto \text{Ad}(U_1)(\nabla D) - \eta \circ U_1 \) preserves \( \hat{\Omega}_A^1 \otimes_{K(\mathbb{F}_p)} \text{Lie}(G^{\text{spec}}) \), because \( \text{Lie}(G^{\text{spec}}) \) is a Lie ideal.

It does no harm to replace \( k_0 \) by a finite extension, and hence we can assume \( \text{Aut}(\mathcal{P}_0 \times k_0^{\text{ac}}) = \text{Aut}(\mathcal{P}_0) \), and that \( Z \) splits over \( K(k_0) \), i.e.

there exists a finitely generated \( \text{Gal}(K(k_0)/K(\mathbb{F}_p)) \)-module \( X \), whose Cartier-dual is \( Z \). Now let \( F \) be a \( \nabla \)-stable submodule contained in \( M = \mathbb{Q} \otimes A \otimes_{W(\mathbb{F}_p)} \mathcal{V} \). The \( Z \)-action on \( \mathbb{Q} \otimes \mathcal{V} \) induces a decomposition \( M = \bigoplus_{\lambda \in X} M_{\lambda} \), we will proceed with the study of the \( \nabla \)-stable projections of \( \mathcal{F} \) onto each \( M_{\lambda} \), which we denote by \( \mathcal{F}_{\lambda} \). The assumptions on \( \mathcal{P}_0 \) and \( k_0 \) imply that there exists some \( s \in \mathbb{N} \) such that the \( s \)-fold iteration of \( F^{\text{sp}} \) preserves \( M_{\lambda} \) and acts on \( W(k_0) \otimes A M_{\lambda} \) by multiplication with a scalar. It follows that \( (F^{\text{sp}})^s \), being horizontal, must preserve \( \mathcal{F}_{\lambda} \), simply because it does that at the specific point \( t_1 = \cdots = t_d = 0 \). We can deduce that each non-zero \( \mathcal{F}_{\lambda} \) contains a non-zero horizontal section.

The proof of the lemma is complete if we show the vanishing of the \( K(k_0) \)-vector space \( H = \ker(\nabla) \). Let \( k_1 \) be any algebraically closed
field containing $A/pA$. Cartier’s diagonal map identifies $A$ with a subring of $W(k_1)$, so let us write $b_1 \in G(K(k_1))$ for the image under that map of the element $U_1 F \mu(p)^{-1}$ and let us write $\nu_1 : \mathbb{Q} \otimes \mathbb{G}_m \times K(k_1) \to G \times_{K(p)} K(k_1)$ for the slope homomorphism of $b_1$. Every $x \in K(k_1) \otimes_{K(k_0)} H$ is surely fixed by $\nu$, as the inclusion $A \hookrightarrow W(k_1)$ preserves the Frobenius lifts.

The automorphism group $\text{Aut}(A/W(k_0))$ acts from the left on both $G(A)$ and $M$, so the semidirect product $G(A) \rtimes \text{Aut}(A/W(k_0))$ acts thereon too. The horizontal elements preserve $H$, moreover $\text{Aut}(P_0)$ is a (rather small) compact open subgroup in an inner form of $G$, by [22, Proposition 1.12], and consequently Zariski-dense. We deduce that $K(k_1) \otimes_{K(k_0)} H$ must be invariant under the whole conjugacy class of $\nu_1$, and therefore it is invariant under all elements of $G^{\text{spec}}(K(k_1))$, by corollary 4.10.

We finish this subsection with a quantitative version of corollary 4.4

**Corollary 6.6.** Let $N$ be a discretely valued complete field containing $K(\mathbb{F}_p)$, and let the residue field $k$ of its ring of integers $\mathcal{O}_N$ be perfect of characteristic $p$. Let $P$ be a banal display with $G$ structure over $\mathcal{O}_N$, and let $P_0 := P \otimes_{\mathcal{O}_N} k$ be its special fiber. Then there exists an element $m \in G(N)$ such that every $h \in \text{Aut}(P) \hookrightarrow \mathcal{I}(\mathcal{O}_N)$ satisfies the equation

$$w_0(h) = \text{Int}(m)(h_0),$$

of elements in $G(N)$, where $h_0 \in \text{Aut}(P_0) \hookrightarrow \mathcal{I}(k) \subset G(W(k))$ stands for the special fiber of $h$ (N.B.: if $N = W(k)$ the element $m = 1$ satisfies this, however in the general case $m$ might not even be in $G(\mathcal{O}_N)$).

**Proof.** We may assume that $k$ is algebraically closed. Notice that $m$ is independent of the choice of the banality, even though the two inclusions $\text{Aut}(P) \hookrightarrow \mathcal{I}(\mathcal{O}_N)$ and $\text{Aut}(P_0) \hookrightarrow \mathcal{I}(k)$ do depend on it. Now recall the aforementioned universal formal mixed characteristic deformation $P_{uni} \in \mathcal{B}_{W(k)[[t_1, \ldots, t_d]]}(G, \mu)$ of $P_0$, and let $\kappa : W(k)[[t_1, \ldots, t_d]] \rightarrow \mathcal{O}_N$ be the classifying morphism to $P$. Recall the canonical embedding of $\text{Aut}(P_0)$ into the intersection:

$$\mathcal{G}^-(W(k)[[t_1, \ldots, t_d]]) \times \text{Aut}(W(k)[[t_1, \ldots, t_d]])/W(k)$$

$$\cap \text{Int}(\Theta)(G(K(k)) \times \text{Aut}(W(k)[[t_1, \ldots, t_d]])/W(k)),$$

of subgroups in $G(K(k)\{\{t_1, \ldots, t_d\}\}) \times \text{Aut}(W(k)[[t_1, \ldots, t_d]])/W(k))$. Notice also that the image of any element in $\text{Aut}(P_0)$ in the ambient group $\mathcal{I}(k) \subset \mathcal{G}(K(k))$ can be recovered as the $\mathcal{G}(K(k))$-component in the second of the two groups above, for example by applying the lemma [6.4] to the specific lift $t_1 = \cdots = t_d = 0$. Moreover, the subgroup
\[ \text{Aut}(\mathcal{P}) \subset \text{Aut}(\mathcal{P}_0) \text{ is mapped into:} \]
\[ \mathcal{G}^{-}(W(k)[[t_1, \ldots, t_d]]) \rtimes \text{Aut}_{\mathcal{O}_N}(W(k)[[t_1, \ldots, t_d]]/W(k)), \]

because it preserves the augmentation \( \kappa \). It remains to look at the homomorphic image of \( \text{Aut}(\mathcal{P}) \) under the canonical homomorphism
\[ \mathcal{G}(K(k)\{\{t_1, \ldots, t_d\}\}) \rtimes \text{Aut}_{\mathcal{O}_N}(W(k)[[t_1, \ldots, t_d]]/W(k)) \rightarrow \mathcal{G}(\mathcal{N}), \]
to find that the image of \( \Theta \) is the group element \( m \) that we have sought for.  \( \square \)

7. **Moduli spaces of abelian varieties with mock \( G^0 \)-structure**

Let \( (G, X) \) be a Shimura datum in the sense of the axioms \cite[2.1.1.1)-(2.1.1.5)], let us assume that the rational weight homomorphism \( w_X : \mathbb{G}_m \rightarrow G \) is injective, write \( G^1 \subset G \) for the complement to its image, and assume that this group can be written as \( \text{Res}_{L^+/\mathbb{Q}} G^0 \), and fix a totally imaginary quadratic extension \( L/L^+ \). It is convenient to fix a specific element \( h \in X \) and we also want to choose \( \sqrt{-1} \in S(\mathbb{R}) \) once and for all. Notice that \( h(\sqrt{-1}) \in G^1(\mathbb{R}) \) and \( w_X(-1) \in G^0(L^+) \).

By a polarization on some \((V, \rho) \in \text{Rep}_0(G^0 \times L^+)\), we mean a \( G^0 \)-equivariant \( \mathbb{L} \)-linear map \( \Psi : \mathbb{L} \otimes V^* \rightarrow \tilde{V} \) such that:

- \((\text{tr}_{L/\mathbb{Q}} \Psi)(\rho(h(\sqrt{-1}))x, y) \) is positive definite on \( \mathbb{R} \otimes V \)
- \( \Psi(\rho(w_X(-1))x, y)^* = \Psi(y, x) \).

By slight abuse of notation we will also write \( \rho : G^0 \rightarrow U(V/L, \Psi) \) to denote the corresponding group homomorphism, here it is understood that \( U(V/L, \Psi) \) represents the group functor
\[ R \mapsto \{ \gamma \in \text{GL}_{L \otimes_{L^+} R}(V \otimes_{L^+} R) | \Psi(\gamma x, \gamma y) = \Psi(x, y) \forall x, y \in V \otimes_{L^+} R \}. \]

Notice that the tensor product of polarized representations carries a natural polarization. Let us say that \( \rho \) is even (resp. odd) if it maps the group element \( w_X(-1) \) to 1 (resp. to \(-1\)). We call \((G, X)\) \('poly-unitary' if there exist:

- a family of odd representation \((V_i, \rho_i)\), indexed by some set \( \Lambda \), and each endowed with a polarization \( \Psi_i \), and
- a family of \( G^0 \)-invariant \( L\ast \)-subalgebras \( L_\pi \subset \text{End}_L(\bigotimes_{i \in \pi} V_i) \), indexed by some set \( \Pi \) of index subsets \( \pi \subset \Lambda \) of odd cardinality, such that
- \( G^0 \) is the stabilizer of \( \bigcup_{\pi \in \Pi} L_\pi \).

See part \( A \) of the appendix for a couple of examples. In this scenario we write \( G_i \) for the subgroup generated by \( G^1_i := \text{Res}_{L^+/\mathbb{Q}} U(V_i/L, \Psi_i) \) and \( \mathbb{G}_m \) (i.e. the centre of \( \text{GL}(V_i/\mathbb{Q}) \)), and we write \( \varrho_i : G \rightarrow G_i \) for
the extension of Res_{L^+ / \mathbb{Q}} \rho_i$ with $\varrho_i(w_X(z)) = z^{-1}$. We call our poly-unitary structure 'unramified at $p'$ if $L$ is unramified at $p$, and if there exist hyperspecial subgroups $K_{i,p}$ of $G_i(\mathbb{Q}_p)$ such that

\begin{equation}
K_p = \bigcap_{i \in \Lambda} \varrho_i^{-1}(K_{i,p})
\end{equation}

is a hyperspecial subgroup of $G(\mathbb{Q}_p)$. Let us denote the composition $\mathbb{S} \xrightarrow{h} G \times \mathbb{R} \xrightarrow{\varrho_i} G_i \times \mathbb{R}$ by $h_i$, it gives rise to Hodge decompositions $V_{i,\lambda} = \bigoplus_{p+q=-1} V_{i,p}^{p,q}$, where $V_{i,\lambda} = \mathbb{C} \otimes_{\mathbb{Q},L} V$ stands for the eigenspace, and where $\lambda$ runs through the set $L^\text{an} = \text{Spec} L(\mathbb{C}) = \text{Hom}(L, \mathbb{C})$ of embeddings of $L$ into $\mathbb{C}$. Let us set $[a_{i,t}, b_{i,t}]$ for the smallest intervals of integers such that these decompositions are of type $\{(-b_{i,t}, b_{i,t} - 1), \ldots, (-a_{i,t}, a_{i,t} - 1)\}$. The lengths $w_{i,t} := b_{i,t} - a_{i,t}$ of these intervals are particularly important invariants, and their maximum $w = \max\{w_{i,t} | i \in \Lambda, t \in L^\text{an}\}$ will be called the width of the poly-unitary structure. Finally notice that $\nabla_{i,t}^{p,q} = V_{i,\lambda \circ \psi}^{p,q}$, so that $b_{i,t} = 1 - a_{i,t}$. We also need to introduce certain combinatorial data: Observe that $L^\text{an}$ carries a natural left $\text{Gal}(R/\mathbb{Q})$-action commuting with the complex conjugation which could be viewed as acting from the right, the subfield $R$ stands for the normal closure. Fix an element $\psi \in \text{Gal}(R/\mathbb{Q})$. In this global context we proceed with the following variant of definition 5.1:

**Definition 7.1.** By a $\psi$-multidegree we mean a function $d^+: L^\text{an} \to \mathbb{N}_0$ with $d^+(\psi \circ \iota) \leq d^+(\iota) + 1$ and $d^+(\iota \circ \ast) = d^+(\iota)$. By a $\psi$-poly-gauge for $\{h_i\}_{i \in \Lambda}$ we mean a family of functions $j_i : L^\text{an} \to \mathbb{Z}$, together with a $\psi$-multidegree $d^+$ such that the function $d(\iota) := \psi^{-d^+(\iota)} \circ \iota$ satisfies the following properties:

1. (G1) For every $\iota \in L^\text{an}$ and $l \in [a_{i,t}, b_{i,t} - 1]$ there exists a unique $\kappa \in L^\text{an}$ such that $d(\kappa) = \iota$ and $j_i(\kappa) = l$.
2. (G2) $j_i(\iota \circ \ast) = -j_i(\iota)$
3. (G3) For every given $\pi$ and $\iota$, the cardinality of both sets $\{i \in \pi | j_i(\iota) < a_{i,d(\iota)}\}$, and $\{i \in \pi | j_i(\iota) \geq b_{i,d(\iota)}\}$ is at least $\frac{\text{Card}(\pi) - 1}{2}$.
4. (G4) For every given $\pi$ each $\psi$-orbit of $L^\text{an}$ contains at least one element $\iota$ which satisfies $j_i(\iota) \notin [a_{i,d(\iota)}, b_{i,d(\iota)} - 1]$ for all $i \in \pi$. 

\begin{thebibliography}{10}

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\end{thebibliography}
These properties are rather restrictive and they can only be satisfied if the inequalities
\[
\lim_{N \to \infty} \sum_{k=1}^{N} \frac{\max\{\sum_{i \in \pi} w_{i, \vartheta^k, \iota} | \pi \in \Pi\}}{N} \leq 1
\]
\[
\max_{i \in \pi} \lim_{N \to \infty} \sum_{k=1}^{N} \frac{w_{i, \vartheta^k, \iota}}{N} | \pi \in \Pi < 1
\]
hold for every \(\iota\). Conversely, in the key case \(\Pi = \{\Lambda\} \cup \{\{i\} | i \in \Lambda\}\) it is not difficult to prove that \(\vartheta\)-poly-gauges exist if and only if one has
\[
\sum_{i \in \Lambda} \lim_{N \to \infty} \sum_{k=1}^{N} \frac{w_{i, \vartheta^k, \iota}}{N} < 1.
\]
We call a closed point \(r \in \text{Spec} \mathcal{O}_R\) to be of type \(\vartheta\), if \(\vartheta\) fixes \(r\) and induces the absolute Frobenius automorphism on \(\mathcal{O}_R/r\). If a \(\vartheta\)-poly-gauge is fixed, then there exist Hodge structures \(\tilde{V}_i\), which are endowed with \(L\)-operations and \(L\)-linear polarizations \(\tilde{\Psi}_i\) such that:

(i) \(\dim \tilde{V}_{i, \tilde{\psi}} = \begin{cases} 
\sum_{p < -j_i} \dim V_{i, d_i}^{p, q} (\tilde{p}, \tilde{q}) = (-1, 0) \\
\sum_{p \geq -j_i} \dim V_{i, d_i}^{p, q} (\tilde{p}, \tilde{q}) = (0, -1) \\
0 \\
\end{cases}\)

(ii) There exists a similarity

\[\mathbb{A}^\infty \otimes \tilde{V}_i \xrightarrow{\tilde{\psi}_i} \mathbb{A}^\infty \otimes V_i\]

of skew-Hermitian \(L \otimes \mathbb{A}^\infty\)-modules,

(iii) When calculating in the \(\otimes_L\)-category of Hodge structures with coefficients in \(L\) (see [3, section 3.1]), one finds that \(\tilde{V}_\pi := (\bigotimes_{i \in \pi} \tilde{V}_i)(\frac{1-\text{Card}(\pi)}{2})\) is an object of type \((-1, 0), (0, -1)\) for every \(\pi \in \Pi\), and by means of \(\tilde{\Psi}_\pi := \bigotimes_{i \in \pi} \tilde{\Psi}_i\) it is canonically \(L\)-linearly polarized as well.

See [3, section 6.2] for a related consideration, in this scenario we write \(\tilde{G}_\pi\) for the subgroup generated by \(\tilde{G}_{\pi}^1 := \text{Res}_{L^+/Q} U(\tilde{V}_\pi/L, \tilde{\Psi}_\pi)\) and \(G_m\) (i.e. the centre of \(\text{GL}(\tilde{V}_\pi/Q)\)). This sets up a family of canonical PEL-type Shimura data \((\tilde{G}_\pi, \tilde{X}_\pi)\), where \(\tilde{X}_\pi\) is the \(\tilde{G}_\pi(\mathbb{R})\)-conjugacy class of Hodge-structures on \(\tilde{V}_\pi\) specified by (iii) above. Notice that their reflex fields are contained in \(R\). Moreover, regarding \(\tilde{V} := \bigoplus_{i \in \Lambda} \tilde{V}_i\) as a skew-Hermitian module over the \(*\)-algebra \(L^\Lambda = \bigoplus_{i \in \Lambda} \mathcal{O}_L^\Lambda\) yields yet another canonical PEL-type Shimura datum \((\tilde{G}, \tilde{X})\), where \(\tilde{G}\) is the \(Q\)-group of \(L^\Lambda\)-linear similitudes of \(\tilde{V}\), and \(\tilde{X}\) is the \(\tilde{G}(\mathbb{R})\)-conjugacy class of Hodge-structures on \(\tilde{V}\) specified by (i) above. For later reference we also put \(e_i \subset \mathcal{O}_L^\Lambda\) for the ideal generated by the
idempotent \((1, \ldots, 1, 0, 1, \ldots, 1)\) with the “0” in the \(i\)th position. Finally observe that there exist canonical morphisms of Shimura data \(g_\pi : (\overline{G}, \overline{X}) \to (\overline{G}_\pi, \overline{X}_\pi)\), and hence canonical morphisms of Shimura varieties:

\[(42) \quad \overline{K} M(\overline{G}, \overline{X}) \xrightarrow{g_\pi} \overline{K}_\pi M(\overline{G}_\pi, \overline{X}_\pi),\]

under the assumption \(g_\pi(\overline{K}) \subset \overline{K}_\pi\) of course. We need to collect further facts on integrality, first notice that a decomposition \(\overline{K} = \overline{K}_p \times \overline{K}_p\) for some hyperspecial \(\overline{K}_p \subset \overline{G}(\mathbb{Q}_p)\) implies an analogous decomposition for \(\overline{K}_\pi\) basically by (iii) above, and second we have a canonical extension (cf. [3, Theorem 4.8])

\[(43) \quad \overline{K} U \xrightarrow{g_\pi} \overline{K}_\pi U,\]

of \((42)\), where \(\overline{K} U\) and \(\overline{K}_\pi U\) are the usual \(\mathcal{O}_R \otimes \mathbb{Z}(p)\)-models of these unitary group Shimura varieties, which are smooth (cf. [14]) and proper, by the above property (G4). We write \(Y_\pi\) for the pull-back of the universal abelian scheme on \(\overline{K}_\pi U\) to \(\overline{K} U\) and tacitly omit the mentioning of level structures. Now and again we need to invoke the projective limit \(\overline{K}_p U = \lim_{\overrightarrow{\rho}} \overline{K}_\rho U\), which is a scheme with a right \(\overline{G}(\mathbb{A}^{\infty,p})\)-action. We will now consider a prime \(p \subset \mathcal{O}_{ER}\) which restricts to a prime of type \(\vartheta\). Once and for all we also fix a set \(\mathcal{R}\) of extensions to \(L\) of the primes of \(L^+\) over \(p\), and for each \(q \in \mathcal{R}\) we let \(r_q\) be the degree of \(q^+ := q \cap \mathcal{O}_{L^+}\), and we let \(r\) be the degree of \(p \cap \mathcal{O}_{R}\), and \(f\) the degree of \(p\). For each \(q \in \mathcal{R}\) we fix an embedding \(\iota_q : L \hookrightarrow R\) with \(\iota_q(q) \subset p\), so that

\[\tau^{r_q} \circ \iota_q = \begin{cases} \iota_q \circ \ast & q^* = q \\ \iota_q & \text{otherwise} \end{cases}\]

(* is the conjugation, and \(\tau\) is the Frobenius acting on \(K(\mathbb{F}_p^f)\), which contains \(R\)). In the local settings the function \(d^+\) translates into (mod \(r_q\)) multidegrees

\[d_q(\omega) := \omega + d^+(\vartheta^{-\omega} \circ \iota_q).\]

One can find selfdual \(\mathbb{Z}(p)\)-lattices \(\mathcal{V}_i\) whose stabilizers in the \(\mathbb{Q}_p\)-points of \(\overline{G}_i\), are the hyperspecial groups \(K_{i,p}\). Set \(\mathcal{V}_{i,q}\) for the modules:

\[\mathcal{V}_i \otimes \mathcal{O}_{L^+} \quad \begin{cases} W(\mathbb{F}_p^2) & q^* = q \\ W(\mathbb{F}_p) & \text{otherwise} \end{cases},\]

on which we have perfect pairings with \(\Psi_{i,q}(x, y) = -\Psi_{i,q}(y, x)^*\) in case \(q^* = q\), and ditto for \(\mathcal{V}_{\pi,q}\). Let \(\overline{G}_d/W(\mathbb{F}_p^\omega)\) (resp. \(\overline{G}_d^1/W(\mathbb{F}_p^\omega)\)) be the
Zariski closure of \((\mathbb{G}_m \times_L^+ G^0)/\{\pm 1\}\) (resp. \(G^0\)) in
\[
\prod_{i \in \Lambda} \left\{ \begin{array}{ll}
\text{GU}(V_{i,q}/W(\mathbb{F}_p^{r_q}), \Psi_{i,q}) & q^* = q \\
\mathbb{G}_m \times GL(V_{i,q}/W(\mathbb{F}_p^{r_q})) & \text{otherwise},
\end{array} \right.
\]
and denote \((\mathbb{G}_m \times \prod_{q \in \mathcal{R}} \text{Res}_{W(\mathbb{F}_p^{r_q})/W(\mathbb{F}_p)} G^1_q)/\{\pm 1\}\) by \(G/W(\mathbb{F}_p)\), these groups are reductive, as \(K_p\) was assumed to be hyperspecial. Choose any \(*\)-invariant \(O_L\)-order of \(L_\pi\) with:
\[
O_{L_\pi} \subset O_L + (L_\pi \cap \bigoplus_{q \in \mathcal{R}} p|d_i|^\epsilon_\pi + 1 \text{End}_{L_\pi}(V_{\pi,q})).
\]
Now consider the family of compact open subgroups \(K \subset G(\mathbb{A}^\infty)\) which satisfy
\begin{align}
(44) & \quad K = K^p \times K_p \\
(45) & \quad K^p = \bigcap_{i \in \Lambda} \theta_i^{-1}(K_i^p),
\end{align}
for suitable \(K_i^p \subset G_i(\mathbb{A}^{\infty,p})\) and for fixed choices of hyperspecial subgroups \(K_{i,p}\) and \(K_p\) satisfying \((40)\). Finally choose \(\tilde{K} = \hat{K}^p \times \hat{K}_p\) compatible with the groups \(\tilde{K}_i = K_i^p \times K_{i,p}\) with respect to \((41)\). Let the scheme \(\mathcal{M}/W(\mathbb{F}_p^r)\) represent the functor that sends a connected pointed base scheme \((S, s)\) to the set of \(4 + \text{Card}(\Pi)\)-tuples \((Y, \lambda, \iota, \pi, y)\) with the following properties:
(i) \((Y, \lambda, \iota, \pi)\) is a \(\mathcal{Z}_p\)-isogeny class of: homogeneously \(p\)-principally polarized abelian schemes \((Y, \lambda)\) together with a \(*\)-invariant action \(\iota : O_L^\lambda \to \mathbb{Z}_p \otimes \text{End}(Y)\) satisfying the determinant condition with respect to the skew-Hermitian \(L^\lambda\)-module \(\hat{V}\), and a \(\pi_1(S, s)\)-invariant \(\tilde{K}^p\)-orbit \(\pi\) of \(O_L^\lambda\) linear similitudes
\[
\tilde{\eta}^p : A^{\infty_p} \otimes \hat{V} \xrightarrow{\cong} H_1^{\tilde{\iota}}(Y \times_S s, \mathbb{A}^{\infty,p}).
\]
(ii) \(y_\pi : O_{L_\pi} \to \mathcal{Z}_p \otimes \text{End}_{L_\pi}(Y_\pi)\) is a \(O_L\)-linear \(*\)-preserving homomorphism such that \(\eta\) contains at least one element \(\tilde{\eta}^p\) rendering the diagrams
\[
A^{\infty_p} \otimes \bigotimes_{i \in \pi} \text{End}_L(V_i) \xrightarrow{\otimes_{i \in \pi} \eta_i^p} \bigotimes_{i \in \pi} \text{End}_L(V_i)
\]

\[
\text{End}(\bigotimes_{i \in \pi} H_1^{\tilde{\iota}}(Y[\eta_i] \times_S s, \mathbb{A}^{\infty,p})(1 - \text{Card}(\pi)/2)) \xrightarrow{H_1^{\tilde{\iota}}} \text{End}_L(Y_\pi \times_S s)
\]
commutative, simultaneously for all \(\pi \in \Pi\) where \(\eta_i^p\) stands for the unique \(\pi_1(S, s)\)-invariant \(O_L\)-linear similitude \(A^{\infty_p} \otimes V_i \xrightarrow{\cong} H_1^{\tilde{\iota}}(Y[\eta_i] \times_S s, \mathbb{A}^{\infty,p})\) with \(\tilde{\eta}^p = \sum_{i \in \Lambda} \eta_i^p \circ \xi_i.\)
Notice that the projective limit \( K_p^\infty \mathcal{M} = \lim_{\longrightarrow} K_p^\infty \mathcal{M} \) is equipped with a right \( G(\mathbb{A}_k^\infty) \)-action and with a \( G(\mathbb{A}_k^\infty) \)-equivariant morphism to \( \tilde{k}_p^\infty \mathcal{U} \times_{\mathcal{O}_R} W(\mathbb{F}_p) \), which at the finite levels recovers the tautological forgetful morphism \( K_\mathcal{M} \to \tilde{k}_p^\infty \mathcal{U} \times_{\mathcal{O}_R} W(\mathbb{F}_p) \); \( (Y, \lambda, \iota, V, y) \mapsto (Y, \lambda, \iota, V) \).

We write \( \mathcal{O}_{H_c} = \text{End}_G((\otimes \nu_i)_{i \in C} \mathcal{L}_c) \), where \( c \) is a fixed constant which is sufficiently large for all of the \( \mathcal{G}_q \)'s in the sense of remark 5.9. Let us also fix cocharacters \( \mu : \mathbb{G}_m \times W(\mathbb{F}_p) \to \mathcal{G}_q \), and let us write \( \mathcal{B}_{i,q} \) for the cocharacters that go with it, according to subsection 5.1. For future reference we will write \( \mathcal{B}_{i,q} \) for the weight invariants that go with it, according to subsection 5.1. We have shown that the gauges \( j_i(q) := j_i(\vartheta^i \circ \iota_q) \) defined cocharacters \( v_i,q \) and functors

\[
\text{Flex}^{d_i,j_i,q}(\mu_i) : \mathcal{B}(\mathcal{G}_q, \mu_q) \times \mathbb{F}_p^m \to \mathcal{B}_{i,q}
\]

\[
\mathcal{B}_{i,q} := \begin{cases} 
\mathcal{B}(\mu_i, \mathbb{F}_p^m, \mathcal{G}_q, q^* = q) & \text{q is a fixed constant which is sufficiently large for all of the } \mathcal{G}_q \text{'s in the sense of remark 5.9.} \\
\mathcal{B}(\mathcal{B}_{i,q}, \mathbb{F}_p^m, \mathcal{G}_q, q^* = q) & \text{q is a fixed constant which is sufficiently large for all of the } \mathcal{G}_q \text{'s in the sense of remark 5.9.} 
\end{cases}
\]

And recall also, that there are associated invariants \( \mathcal{D}_{i,q} \) and \( \mathcal{D}_{i,q} \), again according to subsection 5.1. For future reference we will write \( \mathcal{S}_{i,q} : K_\mathcal{M} \to \mathcal{B}_{i,q} \) for the classifying 1-morphisms of the graded displays of the universal families \( \mathcal{B}_{i,q} \) over the moduli space \( \mathcal{K}_\mathcal{M} \) (unitary ones if \( q^* = q \)). The tuple \( \{ \mathcal{S}_{i,q} \}_{i \in \Lambda} \) will be denoted by \( \mathcal{S}_q \) and we will think of it as a 1-morphism

\[
\mathcal{K}_\mathcal{M} \to \mathcal{B}_q,
\]

of stacks, where the \( \mathcal{B}_q \)’s are defined as in subsection 5.3.

7.1. The canonical locus. A morphism of \( \mathbb{F}_p^m \)-schemes \( X \to \mathcal{K}_\mathcal{M} \times_{\mathcal{O}_R} W(\mathbb{F}_p) \) \( \mathbb{F}_p \) will be called canonical if \( X \) is noetherian and reduced, and (16) becomes flexed over \( X \) for every \( q \in \mathcal{R} \). Our first result in this section deals with fields:

**Proposition 7.2.** Let \( k \) be a finitely generated field extension of \( \mathbb{F}_p \), and let \( l \) be an algebraically closed extension of \( k \). If the image of \( \xi \in \mathcal{K}_\mathcal{M}(k) \) in \( \mathcal{K}_\mathcal{M}(l) \) is canonical, then the following is true too:

- There exists a finitely generated subfield \( k_0 \subset l \), containing \( k \) and such that the image of \( \xi \) in \( \mathcal{K}_\mathcal{M}(k_0) \) is canonical.
- The image of \( \xi \) in \( \mathcal{K}_\mathcal{M}(k^{ac}_0) \) is canonical.

**Proof.** Fix a flexibilisator

\[
\zeta_{i,q} : \text{Flex}^{d_i,j_i,q}(\mathcal{V}_{i,q}, \mathcal{P}_q) \cong \mathcal{S}_{i,q} \times_{\mathcal{K}_\mathcal{M}} (\mathcal{P}_q).
\]

The displays with \( \mathcal{G}_q \)-structure \( \mathcal{P}_q \) are banal, and thus we are allowed to choose representing group elements \( U_{q,\sigma} \in \mathcal{G}_q(\mathcal{P}_q) \). Analogously
we write $\tilde{M}_{i, q}$ for the graded windows of $Y_{i, q}[\mathbb{Q}[\tau]]$ relative to a choice of Frobenius lift $\tau : C \to C$ on the Cohen $W(\mathbb{F}_{p^l})$-frame for $k$. Let $\tilde{N}_{i, q}$ be $\text{Fib}^{d_1 d_2, q, \mathcal{L}}(V_{i, q}, \mathcal{P}_{q}^c)$, and let $M_{i, q}$ be $\text{Flex}^{d_1 d_2, q}(\tilde{M}_{i, q})$. For varying $i$ the family

$$N_{i, q, \sigma} := \text{Flex}^{d_1}(\tilde{N}_{i, q})_{\sigma} \overset{\zeta_{i, q, \sigma}}{\rightarrow} W(l) \otimes_{C} M_{i, q, \sigma},$$

of isomorphisms is easily seen to be a $W(l)$-valued point in the moduli scheme $\mathcal{S}_{q, \sigma}$ of the $\mathcal{O}_{H_{\tau}}$-linear families of isomorphisms between $V_{i, q, \sigma} = C \otimes_{\mathbb{F}_{p^l}} \mathcal{L} \otimes_{\mathcal{O}_{\mathcal{V}_i}} V_i$ and $M_{i, q, \sigma}$. As $W(l)$ is faithfully flat over $C$ we get the local triviality of this principal homogeneous space for $\mathcal{G}_{q, \sigma}$. This gives us the existence of a point $\zeta_{i, q, \sigma}$ over some finite unramified subextension $C' \subset W(l)$. Consider the unique $W(l)$-valued group element $h'_{q, \sigma}$ turning the family $(\ldots, \zeta_{i, q, \sigma}, \ldots)$ into $(\ldots, \zeta_{i, q, \sigma}', \ldots)$, and choose further elements $\mathcal{G}_{q, \sigma}(W(l)) \ni h_{q, \sigma} \equiv 1 \pmod{p}$ such that $h''_{q, \sigma} := h^{-1}_{q, \sigma} \circ h'_{q, \sigma}$ are contained in $\mathcal{G}_{q, \sigma}(W(k''))$, where $k''$ is a finitely generated field extension of $k' = C'/pC'$ in $l$:

$$W(l) \otimes_{C} N_{i, q, \sigma} \xrightarrow{W(l) \times_{C} \zeta_{i, q, \sigma}} W(l) \otimes_{C} M_{i, q, \sigma}$$

and in case $q' = q$ it does no harm to assume $h_{q, \sigma} q_{q} \tau$ proportional to $h_{q, \sigma}$. A suitable twisted conjugation endows the family $\zeta_{i, q, \sigma} \circ \rho_{i}(h_{q, \sigma})$ with the structure of a flexibilisator, and we merely need to show its rationality over a finitely generated field extension! To this end one checks that some $O_{q, \sigma} \in \mathcal{G}_{q, \sigma}(W(l))$ which possesses $W(k'')$-rational images under the functors $\text{Flex}^{d_1 d_2, q}(V_{i, q})$ is surely $W(\sqrt{d}k'')$-rational for some $d$, at least if one arranges $O_{q, \sigma} = 1$ for $\sigma$ outside the image of $d$.

The second assertion of the proposition follows easily from the first one in conjunction with remark 5.12 please notice that one cannot apply remark 5.12 directly to $l$. \qed

A two-fold application of this proposition yields the significant strengthening that $k_0$ may be taken to be finite over $k$. We are now in a position to give a set theoretic definition for what will later turn out to be our Shimura variety: We set $\mathcal{K}_{\text{can}} \subset \mathcal{K} \times_{W(\mathbb{F}_{p^l})} \mathbb{F}_{p^l}$ for the locus of points that are canonical over some finite (or equivalently over any algebraically closed) extension of the residue field. The following proposition proves that the set $\mathcal{K}_{\text{can}}$ is stable under specialization:
Proposition 7.3. Consider an algebraically closed field extension \( k \) of \( \mathbb{F}_p \). Then for every \( \eta \in k\mathfrak{M}(k[[t]]) \), the following are equivalent.

- The image of \( \eta \) in \( k\mathfrak{M}(R) \) is canonical for some finite extension \( k[[t]] \subset R \).
- There exists a finite extension of \( k((t)) \) with that property.

Proof. Throughout the whole proof we are working with the Frobenius lift \( \tau : t \mapsto t^p \) on the \( W(\mathbb{F}_p) \)-frames \( W(k)[[t]] \) and \( W(k)((t)) \). For the non-trivial direction we may assume that there exist flexibilisators:

\[
\zeta_{i,\eta}^\circ : \text{Flex}^{d_1} \left( \mathcal{V}_{i,\eta}, \mathcal{P}_q \right) \xrightarrow{\simeq} \mathcal{S}_{i,\eta} \times_{k\mathfrak{M},\eta} k((t)),
\]

and we may assume that the \( \mathcal{P}_q/k((t))'s \) are banal, and hence represented by the graded window matrices \( U_{q,\sigma} \in \mathcal{G}_{q,\sigma}(W(k)((t))) \) by means of lemma 6.1. We start by defining the graded \( (W(k)((t)), \tau) \)-module

\[
\tilde{N}_{i,\eta} = \text{Fil}_{\mathcal{V}_{i,\eta}} \mathcal{W}_q \left( \mathcal{V}_{i,\eta}, \mathcal{P}_q \right),
\]

and analogously we write \( \tilde{M}_{i,\eta} \) for the graded \( W(k)[[t]] \)-windows of \( Y_{i,\eta}[q^\infty] \) and \( \tilde{M}_{i,\eta} \) for its generic fiber. We have canonically induced isomorphisms:

\[
N_{i,\eta,\sigma} := \text{Flex}^{d_1} \left( \tilde{N}_{i,\eta} \right) \xrightarrow{\zeta_{i,\eta}^\circ} \text{Flex}^{d_1} \left( \tilde{M}_{i,\eta} \right) =: M_{i,\eta,\sigma},
\]

and clearly we want to denote \( \text{Flex}^{d_1} \left( \tilde{M}_{i,\eta} \right) \) by \( M_{i,\eta,\sigma} \). Now we should look at the moduli scheme \( \mathcal{G}_{q,\sigma}/W(k)[[t]] \) of \( \mathcal{O}_{H,-} \)-preserving isomorphisms

\[
\cdots \otimes \mathcal{V}_{i,\eta,\sigma} \xrightarrow{\zeta_{i,\eta,\sigma}} \cdots \otimes \mathcal{M}_{i,\eta,\sigma},
\]

where \( \mathcal{V}_{i,\eta,\sigma} \) means \( W(k)[[t]] \otimes_{F^{-q}_{\text{et},q},\mathcal{O}_k} \mathcal{V}_i \), and where the \( \mathcal{O}_{H,-} \)-structure on the right arises from the \( \mathcal{O}_{H,-} \)-structure on \( M_{i,\eta,\sigma} \). It is important to understand that the lattice families \( \tilde{M}_{i,\eta,\sigma} \) and \( M_{i,\eta,\sigma} \) may not have a \( \mathcal{O}_{H,-} \)-structure and they have the wrong Hodge numbers too. By Dwork's trick \( \mathcal{G}_{q,\sigma} \) has a point over \( K(k)\{\{t\}\} \), so that the pull-back of \( \mathcal{G}_{q,\sigma} \) to \( W(k)[[t]][\frac{1}{p}] \) is an honest locally trivial principal homogeneous space. Now we can follow the ideas in [12], given the existence of \( \zeta_{i,\eta,\sigma} \) we see immediately that the same is true over \( W(k)((t)) \), i.e. the pull back of \( \mathcal{G}_{q,\sigma} \) to \( W(k)((t))_{(p)} \) is again a locally trivial principal homogeneous space under \( \mathcal{G}_{q,\sigma} \). This glues to a principal homogeneous space over the scheme \( \text{Spec} W(k)[[t]] \rightarrow \text{Spec} k \), and by [13] we get a principal homogeneous space over \( W(k)[[t]] \), in fact a trivial one, because the basis is strictly henselian. In particular we are allowed to choose isomorphisms:

\[
\mathcal{V}_{i,\eta,\sigma} \xrightarrow{\zeta_{i,\eta,\sigma}} \mathcal{M}_{i,\eta,\sigma}.
\]
Consider the unique $W(k)((t))$-valued group element $h'_{q^\sigma}$ turning the family $(\ldots, \zeta_{i,q^\sigma}, \ldots)$ into $(\ldots, \zeta_{i,q^\sigma}, \ldots)$. By adjusting with an element in $G_{q^\sigma}(W(k)[[t]])$ we may assume the $(\text{mod } p)$-reductions of $h'_{q^\sigma}$ are contained in $G_{q^\sigma}(W(k)((t)))$, given that these subgroups are parabolic ones. The rest of the proof runs similar to the arguments in proposition 7.2 and again we arrange $h'_{q^\sigma + r^q}$ to be proportional to $h'_{q^\sigma}$, in case $q^* = q$. When working over a large finite covering of the form $k[[\sqrt[p]{\tau}]]$ there exists one and only one graded window matrix $O_{q^\sigma} \in G_{q^\sigma}(W(k)((\sqrt[p]{\tau})))$ and a family of elements $k_{q^\sigma} \in G_{q^\sigma}(W(k)((\sqrt[p]{\tau})))$ describing a morphism from $\{O_{q^\sigma}\}$ to $\{U_{q^\sigma}\}$, such that:

- If $\sigma$ lies in the image of $d_q$, then $k_{q^\sigma} = \tau_{d_q}^{-\sigma} h'_{q^\sigma} d_q^{-\sigma}$.
- If $\sigma$ lies outside the image of $d_q$, then $O_{q^\sigma} = 1$.

The associated change of coordinates gives rise to $W(k)[[\sqrt[p]{\tau}]]$-rational maps $\zeta_{i,q^\sigma}$, and finally guarantees the existence of the requested integral models of $\mathcal{P}_q$ and $\zeta_{i,q}$.

Now consider the scheme

$$K \mathcal{M} \times \prod_{q \in \mathbb{Q}} \mathbb{A}^q_\mathbb{Q} \times \prod_{q \in \mathbb{Q}} \prod_{\sigma \in \mathbb{Z}/r_q\mathbb{Z}} \mathcal{H}_{q,\sigma, n+w},$$

where the structural morphism of the second factor is given by:

$$\prod_{q \in \mathbb{Q}} \prod_{i \in \Lambda} \text{Flex}^{d_q, j_i, s}(\rho_i) \circ b^{n+w}(G_q, \mu_q),$$

and write $K \mathcal{M}_{can}^n$ for its image under the projection to

$$K \mathcal{M} \times W(F_{pr}) \mathbb{F}_{p^f},$$

i.e. the first factor. For each finite $n$ this is a constructible subset, according to Chevalley’s theorem ([10 Théorème(1.8.4)]). The constructibility of $K \mathcal{M}_{can}$ is harder to get by and requires some digressions: For every perfect field $k$ we denote the algebraic closure of $K(k)$ by $K(k)^{ac}$ and we denote the ring of integers in $K(k)^{ac}$ by $\mathcal{O}_{K(k)^{ac}}$ and we denote the $p$-adic completion of $\mathcal{O}_{K(k)^{ac}}$ by $\hat{\mathcal{O}}_{K(k)^{ac}}$, and we denote the fraction field of $\hat{\mathcal{O}}_{K(k)^{ac}}$ by $\hat{K}(k)^{ac}$. An object $(\{M_i\}_{i \in \Lambda}, \{m_{i\pi}\}_{\pi \in \Pi})$ of $\mathfrak{C}_{W(k)}$ is called nearly trivial for $q$ if there exists a sequence of families of $\hat{K}(k)^{ac}$-linear isomorphisms $\alpha_{i,\nu} : \hat{K}(k)^{ac} \otimes_{\mathbb{Q}, \mathcal{O}_L} \mathcal{V}_i \to \hat{K}(k)^{ac} \otimes_{W(k)} M_i$ such that:

$$\lim_{\nu \to \infty} (g_{\pi}(\ldots, \alpha_{i,\nu}, \ldots) \circ a \circ g_{\pi}(\ldots, \alpha_{i,\nu}, \ldots)^{-1}) = m_{\pi}(a)$$
holds for all \( \pi \in \Pi \) and \( a \in \mathcal{O}_{L_\pi} \). An object of \( \mathfrak{B}_{q,k} \) is called nearly trivial its image under \( \tilde{\mathfrak{b}}_{q,\sigma,k} \) is nearly trivial for \( \sigma \). Let us write \( K \mathcal{M}_{\text{triv}} \) for the locus of points \( x \in K \mathcal{M} \) such that the \( \mathfrak{B}_{q,k(x)} \)-objects \( \mathcal{S}_{q,x} \times_{K \mathcal{M}} k(x)_{ac} \) are nearly trivial for all \( q \). It is easy to see that \( K \mathcal{M}_{\text{triv}} \) is Zariski-closed, hence constructible.

**Lemma 7.4.** Let \( k \) be an algebraically closed extension of \( \mathbb{F}_p \). For every integer \( m \geq 0 \) there exists an integer \( \overline{m} \geq m \) such that the following is true: Suppose that \( \{ \{ M_\pi \} \}_{\pi \in \Pi}, \{ m_\pi \} \in \mathfrak{Ob}_W(k) \) is nearly trivial for \( q \). Then to any isomorphism

\[
\overline{\pi}_i : W_{\overline{m}}(k) \otimes_{k_{ac}, \mathcal{O}_L} \mathcal{V}_i \to W_{\overline{m}}(k) \otimes_{W(k)} M_i
\]

(whose source and target are regarded as objects of \( C_{W(\overline{m})} \)) there exists another isomorphism

\[
\alpha_i : W(k) \otimes_{k_{ac}, \mathcal{O}_L} \mathcal{V}_i \to M_i
\]

(between objects of \( C_{W(k)} \)) such that \( \{ \overline{\pi}_i \} \) and \( \{ \alpha_i \} \) have the same pull-backs to \( W_m(k) \).

**Proof.** Fix \( q \) and write \( V_i^{ac} := \hat{K}(k)_{ac} \otimes_{k_{ac}, \mathcal{O}_L} \mathcal{V}_i \). By a \( p \)-adic analytic consideration there exists a nondecreasing function \( N_0 \to \mathbb{N}_0; m \mapsto \overline{m} \) with the following property:

For all families \( \gamma_i \in \text{GL}_{\hat{K}(k)}(V_i^{ac}) \) fulfilling:

\[
g_\pi(\ldots, \gamma_i, \ldots) \circ a \circ g_\pi(\ldots, \gamma_i, \ldots)^{-1} \equiv m_\pi(a) \mod p^{\overline{m}}
\]

(\( \forall \pi \in \Pi, a \in \mathcal{O}_{L_\pi} \)) there exist families \( \tilde{\gamma}_i \in \text{GL}_{\hat{K}(k)}(V_i^{ac}) \) fulfilling:

\[
g_\pi(\ldots, \tilde{\gamma}_i, \ldots) \circ a \circ g_\pi(\ldots, \tilde{\gamma}_i, \ldots)^{-1} =
\]

\[
g_\pi(\ldots, \gamma_i, \ldots) \circ a \circ g_\pi(\ldots, \gamma_i, \ldots)^{-1}
\]

\[
\tilde{\gamma}_i \equiv 1 \mod p^m
\]

(\( \forall \pi \in \Pi, a \in \mathcal{O}_{L_\pi} \)). The isomorphisms \( \overline{\pi}_i \) allow lifts \( \alpha'_i \), which however might not respect the tensors in \( \mathcal{O}_{L_\pi} \). By induction on \( \nu \) we will now define a sequence of families of \( \hat{K}(k)_{ac} \)-linear isomorphisms such that:

\[
(i) \quad g_\pi(\ldots, \tilde{\alpha}_{i,\nu}, \ldots) \circ a \circ g_\pi(\ldots, \tilde{\alpha}_{i,\nu}, \ldots)^{-1} \equiv m_\pi(a) \mod p^{\overline{m}}
\]

\[
(ii) \quad \tilde{\alpha}_{i,\nu+1} \equiv \tilde{\alpha}_{i,\nu} \mod p^{\nu'}
\]

\[
(iii) \quad \tilde{\alpha}_{i,m} = \alpha'_i,
\]

in fact all of these families are automatically integral, i.e. already defined over \( \hat{O}_{\hat{K}(k)} \), simply because \( \tilde{\alpha}_{i,m} \) is. Suppose that the family \( \tilde{\alpha}_{i,\nu} \) has been found. The existence of a family, say \( \alpha_{i,\nu+1} \) that satisfies property (i) alone is clear, applying the above to \( \gamma_i := \tilde{\alpha}_{i,\nu}^{-1} \circ \alpha_{i,\nu+1} \) gives a family of isomorphisms \( \tilde{\gamma}_i = \tilde{\alpha}_{i,\nu}^{-1} \circ \alpha_{i,\nu+1} \), and we have \( \tilde{\alpha}_{i,m} \equiv \tilde{\alpha}_{i,\nu} \mod p^{\nu'} \). Consider the family \( \tilde{\alpha}_i = \lim_{\nu \to \infty} \tilde{\alpha}_{i,\nu} \). One can establish a
canonical $O_{H_c}$-action on $\bigotimes_{i \in \Lambda} M_i^{\otimes c}$, just use the isomorphism $\{\tilde{\alpha}_i\}_{i \in \Lambda}$ for transport of structure and observe that the left $G_0$-coset of this element is $K(k)$-rational. The existence of $\{\tilde{\alpha}_i\}_{i \in \Lambda}$ shows, that the $O_{H_c}$-linear isomorphisms from $W(k) \otimes_{q_0, O_L} V_i$ to $M_i$ form a locally trivial principal homogeneous space for $\mathcal{G}_q^1$. As the latter one is smooth so are its locally trivial principal homogeneous spaces, and consequently there exists a lift of the mod $p^m$-reduction of $\{\tilde{\alpha}_i\}_{i \in \Lambda}$.

\[ \square \]

**Corollary 7.5.** There exists a constant $b$ with:

\[(47) \quad K \mathcal{M}_{can} = (K \mathcal{M}_{triv} \times \mathbb{F}_{p^r} \times \mathbb{F}_{p^s}) \cap K \mathcal{M}_{can}^b \]

**Proof.** Write $V_{i,q,\sigma} := W(k) \otimes_{F^{-\sigma}, O_L} V_i$. By the crystalline boundedness principle ([28]) there exists a positive integer $m$ with the following property: Given any congruence $F^{(1)}_{i,q,\sigma} \equiv F^{(2)}_{i,q,\sigma} \mod p^m$ between families of $\tau$-linear maps $F^{(1)}_{i,q,\sigma}, F^{(2)}_{i,q,\sigma} : V_{i,q,\sigma+1} \rightarrow V_{i,q,\sigma}$, such that:

(i) there exist $\tilde{F}^{(\nu)}_{i,q,\sigma} : \tilde{V}_{i,q,\sigma+1} \rightarrow \tilde{V}_{i,q,\sigma}$ with $(\tilde{F}^{(\nu)}_{i,q,\sigma} x, \tilde{F}^{(\nu)}_{i,q,\sigma} y) = p\tau(x, y)$ for $x \in V_{i,q,\sigma+1}$ and $y \in \tilde{V}_{i,q,\sigma+1}$,

(ii) the diagrams:

\[
\begin{array}{ccc}
\mathcal{V}_{\pi,q,\sigma+1} & \xrightarrow{\otimes_{i \in \pi} F^{(\nu)}_{i,q,\sigma}} & \mathcal{V}_{\pi,q,\sigma} \\
v_{\pi}(a) \uparrow & & \uparrow v_{\pi}(a) \\
\mathcal{V}_{\pi,q,\sigma+1} & \xrightarrow{\otimes_{i \in \pi} F^{(\nu)}_{i,q,\sigma}} & \mathcal{V}_{\pi,q,\sigma}
\end{array}
\]

commute ($\forall \pi \in \Pi, a \in \mathcal{O}_L, \nu \in \{1, 2\}$), and so do:

\[
\begin{array}{ccc}
\tilde{\mathcal{V}}_{i,q,\sigma+1} & \xrightarrow{\tilde{F}^{(\nu)}_{i,q,\sigma}} & \tilde{\mathcal{V}}_{i,q,\sigma} \\
\Psi_{i,q,\sigma+1} \uparrow & & \uparrow \Psi_{i,q,\sigma} \\
\tilde{\mathcal{V}}_{i,q,\sigma+r_q+1} & \xrightarrow{\tilde{F}^{(\nu)}_{i,q,\sigma}} & \tilde{\mathcal{V}}_{i,q,\sigma+r_q}
\end{array}
\]

in case $q^* = q$ ($\forall i \in \Lambda, \nu \in \{1, 2\}$),
one always obtains further commutative diagrams:

\[
\begin{array}{ccc}
V_{i,q,\sigma+1} & \xrightarrow{F^{(2)}_{i,q,\sigma}} & V_{i,q,\sigma} \\
\gamma_{i,q,\sigma+1} & \uparrow & \gamma_{i,q,\sigma} \\
V_{i,q,\sigma+1} & \xrightarrow{F^{(1)}_{i,q,\sigma}} & V_{i,q,\sigma}
\end{array}
\]

for a suitable family of elements \(\{\gamma_{i,q,\sigma}\}_{i \in \Lambda} = \gamma_{q,\sigma} \in G_q(W(k))\). The whole point of lemma 7.4 is to yield another positive integer \(b := m\) for which we are going to verify our corollary, so pick any \(\xi \in K M(k)\) lying in the right-hand side of (47). Choose a \(m\)-truncated flexibilisator

\[\overline{\zeta}_{i,q} : \text{Flex}^{\Lambda_{d_{i,q}}}(V_{i,q}, \overline{P}_q) \xrightarrow{\simeq} S_{i,q} \times_{K M(k)} \xi\]

where \(S_{i,q}\) is the truncation of \(S_{i,q}\) at level \(\overline{m}\). Fix a banalisation of \(P\) with a corresponding \(m + w\)-truncated graded display matrix \(\{U_{q,\sigma \overline{m} + w}\}_{\sigma \in \mathbb{Z}/r_q \mathbb{Z}}\). Lift it by \(U_{q,\sigma} \in G_q(W(k))\), let \(P_q\) be the banal display with matrix \(\{U_{q,\sigma}\}_{\sigma \in \mathbb{Z}/r_q \mathbb{Z}}\), and consider \(S'_{i,q} := \text{Flex}^{\Lambda_{d_{i,q}}}(V_{i,q}, P_q)\), and notice that the underlying eigenspaces of this object are canonically identified with \(S'_{i,q,\sigma} = V_{i,q,\sigma}\). Write \(S_{i,q,\sigma}\) for the underlying eigenspaces of the displays \(S_{i,q} \times_{K M(k)} \xi\). The previous lemma finds us a \(\nu_{i,q}\)-preserving family of isometries \(\zeta_{i,q,\sigma} : V_{i,q,\sigma} \rightarrow S_{i,q,\sigma}\) such that \(\zeta_{i,q,\sigma} \equiv \overline{\zeta}_{i,q,\sigma} \mod p^m\). The crystalline boundedness principle tells us that \(\{S'_{i,q}\}_{i \in \Lambda} \cong \{S_{i,q} \times_{K M(k)} \xi\}_{i \in \Lambda}\) and we are done.

It follows immediately that \(K M_{\text{can}}\) is constructible, hence closed, because it is also stable under specialization. This enables us to endow it with the reduced subscheme structure that is induced by the ambient scheme, namely \(K M \times W(\mathbb{F}_{p'}) \mathbb{F}_{p'}\). One should appreciate that \(K M_{\text{can}}\) does not depend on the choice of the orders \(O_{L_q}\).

7.2. Construction of \(K M_p\). Consider the stack rendering the diagram:

\[
\begin{array}{ccc}
K Mot_p & \longrightarrow & K M \times W(\mathbb{F}_{p'}) \mathbb{F}_{p'} \\
\downarrow & & \downarrow \\
\prod_{q \in \mathcal{R}} \mathcal{B}(G_q, \mu_q) \times W(\mathbb{F}_{p'}) \mathbb{F}_{p'} & \longrightarrow & \prod_{q \in \mathcal{R}} \mathcal{B}_q \times W(\mathbb{F}_{p'}) \mathbb{F}_{p'}
\end{array}
\]

2-cartesian, where the lower horizontal morphism is given by

\[\prod_{q \in \mathcal{R}} \left(\prod_{i \in \Lambda} \text{Flex}^{\Lambda_{d_{i,q}}}(\rho_i)\right),\]

one could expect that \(K Mot_p\) is representable (in particular discrete), here is a result in that direction:
Theorem 7.6. There exists a 2-commutative diagram

\[
\begin{array}{c}
\prod_{q \in \mathcal{R}} \mathcal{B}(G_q, \mu_q) \times W(\mathbb{F}_p) \mathbb{F}_p \\
\downarrow \\
\prod_{q \in \mathcal{R}} B_q \times W(\mathbb{F}_p) \mathbb{F}_p
\end{array}
\]

with a finite, surjective, and radicial upper horizontal morphism, and \(\prod_{q \in \mathcal{R}} \mathcal{P}_q\) formally étale (in the sense of definition 4.5).

Proof. For every closed point \(x \in \mathcal{K}_{\text{can}}\) there exists a flexibilisator which is defined over the residue field \(k(x)\), because it is perfect. Let us write \(\tilde{A}_x\) for the complete local ring which represents the product of the universal formal deformation spaces of the underlying displays with \(G_q\)-structure, and write \(\tilde{P}_q, x\) for the universal objects over either Spf \(\tilde{A}_x\) or Spec \(\tilde{A}_x\), which according to lemma 2.9 makes no difference. Clearly \(\tilde{A}_x\) is a power series algebra over \(k(x)\). The Serre-Tate theorem yields naturally a finite map

\[
\text{Spec } \tilde{A}_x \to \text{Spec } \hat{O}_{\mathcal{K}_{\text{can}}, x},
\]

which is also radicial, the diagonal being dominant (corollary 5.10). We will need the dominance of this very map, which can be established as follows: Every minimal prime ideal is the kernel of some local homomorphism \(\hat{O}_{\mathcal{K}_{\text{can}}, x} \to k[[t]]\), and possibly after going to a finite extension one obtains a canonical one, again by proposition 7.3 and by the definition of \(\mathcal{K}_{\text{can}}\). The morphism \(\tilde{A}_x \to k[[t]]\), which we can deduce from that, tells us that the given minimal prime comes from a minimal prime of \(\tilde{A}_x\).

Consequently it is also clear that there exists a regular scheme \(\tilde{U}_x\) together with a finite and surjective map onto an affine open neighbourhood \(U_x \subset \mathcal{K}_{\text{can}}\) of \(x\), such that the completion at \(x\) gives us back \(\tilde{A}_x\). Just look at the fraction fields \(K_x, L_x, \tilde{L}_x\) of the rings \(O_{\mathcal{K}_{\text{can}}, x}, \hat{O}_{\mathcal{K}_{\text{can}}, x} = A_x, \tilde{A}_x\), observe that there exists a unique finite purely inseparable field extension \(\tilde{K}_x/K_x\) with \(\tilde{K}_x \otimes_{K_x} L_x = \tilde{L}_x\), and consider the normalization \(\tilde{U}_x\) in \(\tilde{K}_x\) of some affine open neighbourhood \(U_x\). This reasoning is justified by applying the result [10, Proposition (6.14.4)] to the normal morphism \(O_{\mathcal{K}_{\text{can}}, x} \to A_x\). Passage to the completion in \(x\) shows that this scheme is at least regular in \(x\), so that a further shrinkage of \(U_x\) enforces the regularity of the whole of \(\tilde{U}_x\). Moreover, the existence of the rings \(\tilde{A}_x\) tells us that the stalk \(O_{\tilde{U}_x, x}\) is radicial over \(U_x\) by [10, Proposition(6.15.3.1)(ii)].
Now we come to the canonicity of the maps
\[ \text{Spec } \tilde{K}_x \rightarrow K\mathcal{M}_{\text{can}}. \]
This is easy, for by very definition of the canonical locus and by the lemma \[7.2\] one can find a flexibilisator over some finitely generated field extension \( C \) of \( \tilde{K}_x \), and then one applies part (ii) of corollary \[5.11\] with \( \tilde{A}_x \) playing the role of the ring \( B \).

We next study the canonicity of \( \tilde{U}_x \): It is easy to see that the stalks
\[ \text{Spec } \mathcal{O}_{\tilde{U}_x,x} \rightarrow K\mathcal{M}_{\text{can}} \]
are canonical, as this follows immediately from part (i) of corollary \[5.11\] applied to \( B = \tilde{K}_x \) and \( C = \tilde{A}_x \). Let us write \( \mathcal{P}_{q,x} \) and \( \zeta_{i,q,x} \) for the display with \( \mathcal{G}_q \)-structure and the flexibilisator over \( \mathcal{O}_{\tilde{U}_x,x} \). Remember that our aim was to prove that \( \tilde{U}_x \rightarrow K\mathcal{M}_{\text{can}} \) is canonical! To this end we continue to argue stalkwise: For every closed point \( y \in U_x \) the stalk \( \mathcal{O}_{\tilde{U}_y,y} \) is by definition equal to the normalization of the local ring \( \mathcal{O}_{K\mathcal{M}_{\text{can}},y} \) in the field extension \( \tilde{K}_x \), and the stalk \( \mathcal{O}_{\tilde{U}_y,y} \) is by definition equal to the normalization of the local ring \( \mathcal{O}_{K\mathcal{M}_{\text{can}},y} \) in the field extension \( \tilde{K}_y \). Unfortunately we must not use that the fields \( \tilde{K}_x \) and \( \tilde{K}_y \) are equal, but given they are both finite purely inseparable field extensions of \( K_x = K_y \) we can find an integer \( n \) such that \( \tilde{K}_y \subset \sqrt[n]{\tilde{K}_x} \), and this gives rise to inclusions:
\[ \mathcal{O}_{\tilde{U}_y,y} \subset \sqrt[n]{\mathcal{O}_{\tilde{U}_x,y}} \supset \mathcal{O}_{\tilde{U}_y,y}, \]
which show us the canonicity of the map
\[ \text{Spec } \sqrt[n]{\mathcal{O}_{\tilde{U}_x,y}} \rightarrow K\mathcal{M}_{\text{can}}. \]

Now apply part (ii) of corollary \[5.11\] to the rings \( B = \tilde{A}_x \) and \( C = \sqrt[n]{\mathcal{O}_{\tilde{U}_x,y}} \), which is faithfully flat because \( \mathcal{O}_{\tilde{U}_x,y} \) is regular. The last step towards the canonicity over \( \tilde{U}_x \) is accomplished by the corollary \[5.13\] (whose proof is morally yet another application of part (i) of corollary \[5.11\]). By slight abuse of notation we will denote the extensions of \( \mathcal{P}_{q,x} \) and \( \zeta_{i,q,x} \) over the whole of \( \tilde{U}_x \) by the same symbols. Remark \[4.6\] provides a further shrinkage of \( \tilde{U}_x \) that gains the formal étaleness of \( \prod_{q \in \mathcal{R}} \mathcal{P}_{q,x} \), so that the pull-backs to all of the equicharacteristic completed local rings (at the closed points other than \( x \)) are universal formal deformations. It follows that these agree canonically with the \( \prod_{q \in \mathcal{R}} \tilde{\mathcal{P}}_{q,y} \)'s, and as a by-product we do obtain that indeed \( \tilde{K}_x = \tilde{K}_y \).

Surely all those \( \tilde{U}_x \)'s glue coherently and we finished the construction of a smooth and projective \( \mathbb{F}_p \)-scheme \( \bar{K} \bar{M} \) that is finite and radicial.
over $K\mathcal{M}_{can}$ and over which there is a formally étale flexibilisator, alternatively $\overline{K\mathcal{M}}$ is (componentwise) the normalization of $K\mathcal{M}_{can}$ in the function fields $\mathcal{K}_x$, which only depend on the irreducible component that passes through $x$.

**Corollary 7.7.** If $X$ is a scheme of finite type over an algebraic extension of $\mathbb{F}_p$, then the natural map from the set of $X$-valued points of $\overline{K\mathcal{M}}_p$ to the fiber of $K\text{Mot}_p$ over $X$ is an equivalence of discrete groupoids (i.e. a bijection of sets).

Let $\nu$ be a positive integer, let $\bigcup_l U_l$ be an open affine covering of $\overline{K\mathcal{M}}_p$, and let $\mathcal{P}_{l,q}$ be the restriction of $\mathcal{P}_q$ to $U_l$. Choose smooth affine $W_{\nu}(\mathbb{F}_{p'})$-schemes $U^{(\nu)}_l$ lifting the smooth affine $\mathbb{F}_{p'}$-schemes $U_l$, and write $U^{(\nu)}_{l,m} \subset U^{(\nu)}_l$ for the open subscheme whose underlying point set is $U_l \cap U_m$. Pick lifts $\mathcal{P}^{(\nu)}_{l,q} \in \check{T}_{p,q}(U^{(\nu)}_l)$, and write $\mathcal{P}^{(\nu)}_{l,m,q}$ for the restriction of $\mathcal{P}^{(\nu)}_{l,q}$ to $U^{(\nu)}_{l,m}$. Let finally be $\psi^{(\nu)}_{l,m} : U^{(\nu)}_{l,m} \rightarrow U^{(\nu)}_{m,l}$ the unique lift of the identity which pulls-back $\mathcal{P}^{(\nu)}_{m,l,q}$ to $\mathcal{P}^{(\nu)}_{l,m,q}$. The transition maps $\psi^{(\nu)}_{l,m}$ satisfy the cocycle condition and define a formally smooth lift $K\mathcal{M}^{(\nu)}_p/W_{\nu}(\mathbb{F}_{p'})$ along with a family of displays $\mathcal{P}^{(\nu)}_q$ with $\mathcal{G}_q$-structure over it. The locally trivial principal homogeneous spaces for $\mathcal{G}^{-}_q$ over $K\mathcal{M}^{(\nu)}_p$ which arise by truncating $\mathcal{P}^{(\nu)}_q$ at level 0 shall be denoted by $K\mathcal{F}^{(\nu)}_q = \prod_{\sigma=0}^{q-1} K\mathcal{F}^{(\nu)}_{q,\sigma}$. Naturally these give rise to invertible sheaves

$$L^{(\nu)}_{i,\omega} := \varinjlim_{\sigma} \mathcal{F}^{(\nu)}_{q,\sigma} (\text{ch}^d_{q,\omega})^{\otimes -1}.$$ 

Let $L^{(\nu)}$ be their product. The line bundle $L^{(1)}$ is just the pull-back of the canonical sheaf on the ambient Shimura variety $\overline{K\mathcal{M}}_p$, according to [30], so it is ample ([19]). By [9, Théorème (5.4.5)], this completes the construction of a smooth projective $W(\mathbb{F}_{p'})$-scheme $K\mathcal{M}_p$ together with an isomorphism of formal schemes

$$K\mathcal{M}_p \simeq \lim_{\nu \rightarrow \infty} K\mathcal{M}^{(\nu)}_p,$$

where the left-hand side is the $p$-adic completion. Let $L_{i,\omega}$ and $L$ be the projective limits of the $L^{(\nu)}_{i,\omega}$’s and $L^{(\nu)}$’s. We need a limit of the $K\mathcal{F}^{(\nu)}_{q,\omega}$’s too, which can be justified as follows: Write

$$K\mathcal{F}^{(\nu)}_{i,\omega} := \varinjlim_{\nu} K\mathcal{F}^{(\nu)}_{i,\omega}(\rho_i)$$

and let $K\mathcal{F}_{i,\omega}$ be the formal principal homogeneous space of ascended filtered $\mathcal{O}_{H_i}$-preserving maps from $\mathcal{V}_{i,\omega} = \mathcal{O}_{K\mathcal{M}_p} \otimes_{\mathcal{A}_{\omega,L} \mathcal{O}_L} \mathcal{V}_i$ to $K\mathcal{F}_{i,\omega}$. 


In order to verify the local triviality of $K\mathcal{F}_{q,\sigma}$ it suffices to construct points over the completed local rings at each minimal point of $K\mathcal{M}_p$, which one can do using the smoothness of the $K\mathcal{F}_{q,\sigma}^{(v)}$’s. Another limit process yields connections $\nabla_{q,\sigma}$ on the composite of the $\otimes$-functors

$$Rep_0(\mathcal{G}_{q,\sigma}) \rightarrow Rep_0(\mathcal{G}_{q,\sigma}^-) \xrightarrow{\omega_{K\mathcal{F}_{q,\sigma}}} Vec_{K\mathcal{M}_p},$$

i.e. connections on $K\mathcal{F}_{q,\sigma} \times \mathcal{G}_{q,\sigma} \mathcal{G}_{q,\sigma}$.

7.3. Stabilizers. Notice that the display of the multiplicative $p$-divisible group $\mu_{p^\infty}$ is canonically identified with the multipliers of all of the $\mathcal{S}_{i,q}$’s, being the displays of the $p$-divisible groups $Y[q^\infty][\epsilon_i]$. One sees immediately that the diagram in 5.2.3 gives rise to a canonical display $\mathcal{P}$ with $\mathcal{G}$-structure over $K\mathcal{M}_p$ that maps to the tuple $\{P_q\}_{q \in R}$, under the upper horizontal map in the 2-cartesian diagram below:

$$
\begin{align*}
\mathcal{B}(\mathcal{G}, \mu) & \longrightarrow \prod_{q \in R} \mathcal{B}(\mathcal{G}_q, \mu_q) \\
\downarrow & \downarrow \\
\mathcal{B}(\mathcal{G}_m, \mu_1) & \longrightarrow \prod_{q \in R} \mathcal{B}(\mathcal{G}_m \times W(F_{p^q}), \mu_1)
\end{align*}
$$

Analogously, one sees that the there is a canonical locally trivial principal homogeneous space $K\mathcal{F}$ over $K\mathcal{M}_p$ yielding $K\mathcal{F}_{q,\sigma}$ upon an extension of structure groups from $\mathcal{G}$ to $\mathcal{G}_{q,\sigma}^-$. From now onwards it is more comfortable to work with $\mathcal{P}$ and $K\mathcal{F}$ rather then $P_q$ and $K\mathcal{F}_{q,\sigma}$.

Let $k_0$ be a perfect field. Let $\mathbb{D}(Y)$ denote the covariant Grothendieck-Messing crystalline Dieudonné theory of an abelian variety $Y/k_0$, and let $\mathbb{D}^0(Y)$ denote the (non-effective) $F$-isocrystal whose underlying $K(k_0)$-space is $\mathbb{Q} \otimes \mathbb{D}(Y)$, and whose Frobenius operator $K(k_0) \otimes F; K(k_0) \mathbb{D}^0(Y) \rightarrow \mathbb{D}^0(Y)$ is the inverse of $\mathbb{D}(F Y/k_0)$, where $F Y/k_0; Y \rightarrow Y \times_{k_0,F} k_0; y \mapsto y^p$ is the relative Frobenius.

Let $(Y, \lambda, t)$ be a homogeneously $p$-principally polarized $\mathbb{Z}(p)$-isogeny class of abelian $\mathbb{F}^{ac}_p$-varieties with a $*$-invariant action $t : \mathcal{O}_L^A \rightarrow \text{End}(Y)$ satisfying the determinant condition with respect to the skew-Hermitian $L^A$-module $\hat{V}$. Consider the eigenspaces $Y[\epsilon_i]$. Any choice of a full $\hat{K}^p$-level structure completes this to a PEL-quadruple, say $(Y, \lambda, t, \tilde{\eta}^p) \in \hat{K}^p \mathcal{U}(\mathbb{F}^{ac}_p)$. From now onwards we will write $\bigotimes_{i \in I} Y[\epsilon_i]$ to denote the homogeneously $p$-principally polarized $\mathbb{Z}(p)$-isogeny class of abelian $\mathbb{F}^{ac}_p$-varieties derived by discarding the level structure from its image quadruple $g_\pi(Y, \lambda, t, \tilde{\eta}^p) \in \hat{K}_\pi \mathcal{U}_\pi(\mathbb{F}^{ac}_p)$, under the map \((\ref{iso-mult})\). The formation of $\bigotimes_{i \in I} Y[\epsilon_i]$ is independent of the choice of $\tilde{\eta}^p$, and the canonical $*$-invariant action $t_\pi : \mathcal{O}_L \rightarrow \mathbb{Z}(p) \otimes \text{End}(\bigotimes_{i \in I} Y[\epsilon_i])$ satisfies the determinant condition with respect to the skew-Hermitian $L$-module $\hat{V}_\pi$, \(\square\).
moreover we have canonical isomorphisms
\[ \bigotimes_{L \otimes \mathbb{A}^{\infty,p}} H_1^{\text{et}}(Y[\epsilon_i], \mathbb{A}^{\infty,p}) \cong H_1^{\text{et}}(\bigotimes_{i \in \pi} Y[\epsilon_i], \mathbb{A}^{\infty,p})(\frac{\text{Card}(\pi) - 1}{2}), \]

and similarly for \( \mathbb{D}^0 \) (Sketch: once one chooses a \( \mathbb{C} \)-lift \( Y' \) of \( Y \) one obtains an isomorphism of the following Hodge structures
\[ \bigotimes_{\mathcal{O}_L} H_1(Y'[\epsilon_i](\mathbb{C}), \mathbb{Z}) \cong H_1(\bigotimes_{i \in \pi} Y'[\epsilon_i](\mathbb{C}), \mathbb{Z})(\frac{\text{Card}(\pi) - 1}{2}) \]

with coefficients in \( \mathcal{O}_L \), and \( \bigotimes_{i \in \pi} Y'[\epsilon_i] \) turns out to be a \( \mathbb{C} \)-lift of \( \bigotimes_{i \in \pi} Y[\epsilon_i] \). We need two more functoriality properties which are mediate consequences of Deligne’s theory of absolute Hodge cycles:

**Fact 1.** To every quasi-isogeny \( \gamma : Y_1 \dashrightarrow Y_2 \) there is a canonical quasi-isogeny \( g_{\pi}(\gamma) : \bigotimes_{i \in \pi} Y_1[\epsilon_i] \dashrightarrow \bigotimes_{i \in \pi} Y_2[\epsilon_i] \), such that application of \( \mathbb{D}^0 \) and \( H_1^{\text{et}}(\ldots, \mathbb{A}^{\infty,p}) \) recovers the usual tensor-products (with coefficients in \( L \)).

**Fact 2.** There is a canonical \( \ast \)-preserving and \( L \)-linear homomorphism of algebras:
\[ \bigotimes_{L} \text{End}_L^0(Y[\epsilon_i]) \rightarrow \text{End}_L^0(\bigotimes_{i \in \pi} Y[\epsilon_i]); (\ldots, f_i, \ldots) \mapsto \bigotimes_{i \in \pi} f_i, \]

such that application of \( \mathbb{D}^0 \) and \( H_1^{\text{et}}(\ldots, \mathbb{A}^{\infty,p}) \) recovers the usual tensor-products (with coefficients in \( L \)).

For a situation, which is slightly different from the one at hand the proofs have been carried out in full detail (cf. [3, Theorem 4.10] and [3 Proposition 5.1]). Let \( (Y, \lambda, \iota) \) be a homogeneously polarized \( \mathbb{Q} \)-isogeny class of abelian \( \mathbb{F}_{ac} \)-varieties with a \( \ast \)-invariant action \( \iota : L^A \rightarrow \text{End}^0(Y) \). By fact \( \mathbb{III} \) the homogeneously polarized \( \mathbb{Q} \)-isogeny class of \( \bigotimes_{i \in \pi} Y[\epsilon_i] \) has a well-defined meaning, provided only that the homogeneously polarized \( \mathbb{Q} \)-isogeny class of \( Y \) contains a member for which

- the polarization \( \lambda : Y \rightarrow \breve{Y} \) is \( p \)-principal
- the action \( L^A \rightarrow \text{End}^0(Y) \) is integral and satisfies the determinant condition with respect to the skew-Hermitian \( L^A \)-module \( \breve{V} \).

From now on we assume that the \( \partial \)-poly-gauge \( \{ j_i \}_{i \in A} \) is ‘normalized’ in the sense that \( \partial_{\mathfrak{a}, q} = \bar{\partial}_{\mathfrak{a}, q} \) whenever \( q^* \neq q \in \mathcal{R} \). There is a \( G(\mathbb{A}^{\infty,p}) \)-equivariant bijection between the \( G(\mathbb{A}^{\infty,p}) \)-set \( \kappa_p \mathcal{M}_p(\mathbb{F}_{ac}^p) \) and tuples \( (Y, \lambda, \iota, y_p, \eta_p, P_0) \) where:
(i) \((Y, \lambda, \iota)\) is a homogeneously polarized \(\mathbb{Q}\)-isogeny class of abelian \(\mathbb{F}_p^{ac}\)-varieties with \(*\)-invariant action \(\iota : L^\Lambda \to \text{End}^0(Y)\) containing at least one member for which \(\lambda\) is \(p\)-principal and for which \(\iota\) is integral and satisfies the determinant condition with respect to the skew-Hermitian \(L^\Lambda\)-module \(\tilde{V}\).

(ii) \(y_\pi : L_\pi \to \text{End}^0_L(\bigotimes_{i \in \pi} Y[\epsilon_i])\) is a homomorphism which commutes with \(L\) and preserves \(*\), for every \(\pi \in \Pi\).

(iii) \(P_0\) is a display with \(G\)-structure over \(\mathbb{F}_p\) and \(\eta_p\) is a \(L^\Lambda\)-linear isomorphism \(\bigoplus_{i \in \pi} \eta_i^p : \mathbb{Q} \otimes \text{Fib}(\bigoplus_{i \in \pi} \mathcal{G}_i, P_0) \xrightarrow{\cong} D^0(Y)\) of \(F\)-isocrystals, preserving the pairings on either side up to a factor in \(K(\mathbb{F}_p^{ac})\) and such that the diagrams

\[
\begin{array}{ccc}
\text{End}^0(\text{Fib}(\bigotimes_{i \in \pi} \mathcal{G}_i, P)) & \xleftarrow{y_\pi} & L_\pi \\
\otimes_{i \in \pi} \eta_i^p \downarrow & & \downarrow y_\pi \\
\text{End}_L(\bigotimes_{i \in \pi} D^0(Y[\epsilon_i])(\frac{1 - \text{Card}(\pi)}{2})) & \xleftarrow{\cong} & \text{End}^0_L(\bigotimes_{i \in \pi} Y[\epsilon_i])
\end{array}
\]

are commutative for all \(\pi \in \Pi\).

(iv) \(\eta^p = \bigoplus_{i \in \Lambda} \eta_i^p : \mathbb{A}^{\infty,p} \otimes V \xrightarrow{\cong} H^{\dagger t}_1(Y, \mathbb{A}^{\infty,p})\) is a \(L^\Lambda\)-linear similitude such that the diagrams

\[
\begin{array}{ccc}
\text{End}(\bigotimes_{i \in \pi} H^{\dagger t}_1(Y_i, \mathbb{A}^{\infty,p})(\frac{1 - \text{Card}(\pi)}{2})) & \xleftarrow{\cong} & \text{End}^0_L(\bigotimes_{i \in \pi} Y_i) \\
\otimes_{i \in \pi} \eta_i \downarrow & & \downarrow y_\pi \\
\text{End}_L(\bigotimes_{i \in \pi} Y_i)(\frac{1 - \text{Card}(\pi)}{2}) & \xleftarrow{y_\pi} & \text{End}^0_L(\bigotimes_{i \in \pi} Y_i)
\end{array}
\]

are commutative for all \(\pi \in \Pi\).

Let \(y = (Y, \iota, \lambda, \iota_\pi)\) be data consisting of (i) and (ii). Write \(I_y/\mathbb{Q}\) (resp. \(J_y/\mathbb{Q}_p\)) for the algebraic group representing the functor:

\[
R \mapsto \{(f, \ldots, f_i, \ldots) \in R^\times \times \prod_{i \in \Lambda} R \otimes \text{End}^0_L(Y[\epsilon_i])^\times | \\
\forall \pi \in \Pi : \bigotimes_{i \in \pi} f_i \in R \otimes \text{End}^0_L(\bigotimes_{i \in \pi} Y[\epsilon_i])^\times \\
\forall i \in \Lambda : f_i^* f_i = f\} \quad \text{(resp.)}
\]

\[
R \mapsto \{(f, \ldots, f_i, \ldots) \in R^\times \times \prod_{i \in \Lambda} R \otimes_{\mathbb{Q}_p} \text{End}_L(\bigotimes_{i \in \pi} Y[\epsilon_i])^\times | \\
\forall \pi \in \Pi : \bigotimes_{i \in \pi} f_i \in R \otimes_{\mathbb{Q}_p} \text{End}_L(\bigotimes_{i \in \pi} Y[\epsilon_i])^\times \\
\forall i \in \Lambda : f_i^* f_i = f\} \}
Notice that every choice of full $K_p$-level structure $\eta^p$ as in (iv) furnishes $I_y \times A^{\infty,p}$ with a group homomorphism, say $i_{y,\eta^p}$, to $G \times A^{\infty,p}$ and that one has $i_{y,\eta^p} \circ i_{y,\eta^p}$ for all $\gamma^p \in G(\mathbb{A}^{\infty,p})$. Notice also that there is a natural embedding $I_y \times \mathbb{Q}_p \subset J_y$, and that choices of $\eta_p$ and $\mathcal{P} \in \mathcal{B}_{\text{finite}}(G, \mu)$ as in (iii) do in fact identify the group $\text{Aut}(\mathcal{P}_0)$ with a subgroup of $J_y(\mathbb{Q}_p)$, so let us eventually write $\Gamma_{y,\eta_p,\mathcal{P}} \subset I_y(\mathbb{Q})$ for the inverse image of $\text{Aut}(\mathcal{P}_0)$ under this embedding. Whenever $\mathcal{P}_0$ is lifted by a display $\mathcal{P}_1$ with $\mathcal{G}$-structure over a complete local noetherian $W(\mathbb{F}_p)$-algebra $A$, then we will also write $\Gamma_{y,\eta_p,\mathcal{P}} \subset I_y(\mathbb{Q})$ for the inverse image of $\text{Aut}(\mathcal{P}_1)$ under that embedding (N.B.: the lifts of $\mathcal{P}_0$ are in one-one correspondence with lifts of the $\mathbb{F}_p$-valued point of $K_p \mathcal{M}_p$ that corresponds to the tuple $(Y, \lambda, t, y_\pi, \eta^p, \eta_p, \mathcal{P}_0)$).

Once $\mathcal{P}_0$ is represented by a concrete element $U \in \mathcal{G}(W(\mathbb{F}_p^{\text{ac}}))$ there exists a map $\tilde{j}_{y,\eta_p,\mathcal{U}} : \tilde{J} \times \tilde{K}(\mathbb{F}_p^{\text{ac}}) \to \tilde{G} \times \tilde{K}(\mathbb{F}_p^{\text{ac}})$, which is analogous to $i_{y,\eta^p}$. However a change of coordinates of the form $h^{-1}U^\Phi h = U'$ will alter $\tilde{j}_{y,\eta_p,\mathcal{U}}$ and turn it into: $\text{Int}(h)^{-1} \circ \tilde{j}_{y,\eta_p,\mathcal{U}} = \tilde{j}_{y,\eta_p,\mathcal{U}}$.

We need one more concept: Let us call $\mathcal{P}_0 \in \mathcal{B}_{\text{finite}}(G, \mu)$ elliptic if it possesses a representative $U \in \mathcal{G}(W(\mathbb{F}_p^{\text{ac}}))$ which can be written in the form $U = g^{-1}Fg$, where $g \in G(K(\mathbb{F}_p^{\text{ac}}))$ is such that the image of $\text{Int}(g) \circ \mu$ lies in a maximal $\mathbb{Q}_p$-elliptic torus $T \subset G \times \mathbb{Q}_p$. If this holds then $\text{Aut}(\mathcal{G}(W(\mathbb{F}_p^{\text{ac}})) \cap \text{Int}(g^{-1})T(K(\mathbb{F}_p))$ are automorphisms of $\mathcal{P}_0$. There is a very nice lift $\mathcal{P}_1 \in \mathcal{B}_{\text{finite}}(G, \mu)$ for example by regarding the very same representative $U$ as an object in the category $\mathcal{B}_{\mathcal{F}_p^{\text{ac}}}(G, \mu)$ (use remark 6.2), $\mathcal{G}(W(\mathbb{F}_p^{\text{ac}})) \cap \text{Int}(g^{-1})T(K(\mathbb{F}_p))$ acts on $\mathcal{P}_1$ too.

**Proposition 7.8.** Let $\xi \in K_p \mathcal{M}_p(\mathbb{F}_p^{\text{ac}})$ correspond to $(y, \eta^p, \eta_p, \mathcal{P}_0)$. The stabilizer of $\xi$ in $G(\mathbb{A}^{\infty,p})$ agrees with $i_{y,\eta^p}(\Gamma_{y,\eta_p,\mathcal{P}_0})$.

**Suppose that $\mathcal{P}_0$ is elliptic. Then there exists a display $\mathcal{P}_1$ with $\mathcal{G}$-structure over $W(\mathbb{F}_p^{\text{ac}})$ which lifts $\mathcal{P}_0$ such that $\Gamma_{y,\eta_p,\mathcal{P}}$ contains the $\mathbb{Z}_p$-points of a $\mathbb{Z}_p$-model of a $\mathbb{Q}_p$-elliptic, maximal torus in $I_y$.**

**Proof.** The first assertion is trivial. To prove the second recall that giving a lift is effectively equivalent to giving a pair $(\mathcal{P}_1, \delta)$ with $\mathcal{P}_1 \in \text{Ob} \mathcal{B}_{W(\mathbb{F}_p^{\text{ac}})}(G, \mu)$ and $\delta \in \text{Hom}(\mathcal{P}_0, \mathcal{P}_1 \times W(\mathbb{F}_p^{\text{ac}}))$. Let $g^{-1}Fg = U \in \mathcal{G}(W(\mathbb{F}_p^{\text{ac}}))$ stand for a representative that qualifies $\mathcal{P}_0$ as an elliptic display, so that $\text{Aut}(\mathcal{P}_0)$ reads $\{h \in I(\mathbb{F}_p^{\text{ac}}) | h^{-1}U^\Phi h\}$ and contains $\mathcal{G}(W(\mathbb{F}_p^{\text{ac}})) \cap \text{Int}(g^{-1})T(K(\mathbb{F}_p))$. It causes no harm to denote the endomorphism $\Phi \circ \text{Int}_{i}(\mu(p)^{-1}) : \mathcal{G}(K(\mathbb{F}_p^{\text{ac}})) \to \mathcal{G}(K(\mathbb{F}_p^{\text{ac}}))$ by $\Phi$ (so-named as it extends the twisted Frobenius map $I(\mathbb{F}_p^{\text{ac}}) \to \mathcal{H}(\mathbb{F}_p^{\text{ac}})$ as introduced at the beginning of subsection 2.3). In this language one sees...
that

\[ \text{Aut}(\mathcal{P}_0) \subset J_y(\mathbb{Q}_p) \cong \{ h \in G(K(F_p^{ac}))|h^{-1}U^\Phi h \} \subset G(K(F_p^{ac})) \].

If \( T^\times \subset T(\mathbb{Q}_p) \) stands for the subset of regular elements, then the subset \( \bigcup_{j \in \text{Aut}(\mathcal{P}_0)} \text{Int}(j^{-1}g^{-1})T^\times \) is open in the \( p \)-adic topology of \( I_y(\mathbb{Q}_p) \), so that it contains a rational element \( \text{Int}(j^{-1}g^{-1})t \in I_y(\mathbb{Q}) \). It follows that the image of \( T(\mathbb{Q}_p) \) under \( \text{Int}(j^{-1}g^{-1}) \) is equal to \( S(\mathbb{Q}_p) \) for some \( \mathbb{Q} \)-rational torus \( S \subset I_y \) (namely the centralizer of \( \text{Int}(j^{-1}g^{-1})t \)). The lift \((\mathcal{P}_1, j)\) solves our problem, as \( \mathcal{P}_1 \) has already been exhibited before. \( \square \)

From now on we study exclusively the union of those connected components of \( K_\mathfrak{p}\mathcal{M}_\mathfrak{p} \) which contain at least one elliptic point. We write \( K_\mathfrak{p}\mathcal{M}_\mathfrak{p}^* \) for this open and closed \( \text{G}(\mathbb{A}^{\infty, p}) \)-invariant subscheme, and \( K_\mathfrak{p}\mathcal{F}_\mathfrak{p}^* \) is defined similarly, as are their analogs \( K_\mathfrak{p}\mathcal{M}_\mathfrak{p}^* \) and \( K_\mathfrak{p}\mathcal{F}_\mathfrak{p}^* \) for finite level structures \( K \subset G(\mathbb{A}^{\infty, p}) \). Whether one actually has \( K_\mathfrak{p}\mathcal{M}_\mathfrak{p} = K_\mathfrak{p}\mathcal{M}_\mathfrak{p}^* \) is a difficult question whose answer has no significance for the present paper (the answer is probably: 'yes').

Let \( G^-/K(F_{p,l}) \) be the generic fiber of \( G^- \), and let \( G'/K(F_{p,l}) \) be the quotient of \( G \times K(F_{p,l}) \) by the largest normal subgroup that is entirely contained in \( G^- \), and let \( G^-'/K(F_{p,l}) \) be the image of \( G^- \) under the canonical projection \( G \times K(F_{p,l}) \to G' \). Choose an embedding \( \iota_\infty : W(F_{p,l}) \hookrightarrow \mathbb{C} \). It is known that \( K_\mathfrak{p}\mathcal{M}_\mathfrak{p}^*(\mathbb{C}) \) can be written as a disjoint union

\[ \coprod_t \tilde{\Delta}_t \smallsetminus (\tilde{M}_t \times G(\mathbb{A}^{\infty, p})) \]

for certain connected and simply connected complex manifolds \( \tilde{M}_t \) with certain discrete cocompact subgroups \( \tilde{\Delta}_t \subset \text{Aut}(\tilde{M}_t) \times G(\mathbb{A}^{\infty, p}) \). The locally trivial principal homogeneous space \( K\mathcal{F}_\mathfrak{p}^* \) together with the connection on \( K\mathcal{F}_\mathfrak{p}^* \times G^- \) induce homomorphisms \( \phi_t : \tilde{\Delta}_t \to G(\mathbb{C}) \) and locally biholomorphic \( \tilde{\Delta}_t \)-equivariant maps \( \rho_t : \tilde{M}_t \to G/G^- (\mathbb{C}) \). The composition of \( \phi_t \) with the canonical projection \( G \times K(F_{p,l}) \to G' \) shall be denoted by \( \phi'_t \). For every \( l \) one obtains a bounded period morphism \((\tilde{\Delta}_l, \tilde{M}_l, G', G'^-, \phi'_l, \rho_l)\) in the sense that all properties (P1), (P2), (P3), and (P4) of the part [B] of the appendix are valid. The reasons are as follows:

- property (P1) holds because the holonomy group is already maximal in the formal category by lemma [6.3]. To utilize this we only need a single basic point, for example an elliptic one.
• property (P2) holds because the corollary 6.6 allows to describe the stabilizers in characteristic 0 in terms of stabilizers in characteristic \( p \), which are given by the \( \mathbb{Q} \)-groups self-isogenies, by the first part of proposition 7.8.

• property (P3) holds because of the second part of the proposition 7.8.

• property (P4) holds because \( \mathcal{L} \) is ample.

The fact \( \mathfrak{2} \) implies that \( \mathcal{K}\mathcal{M}_p^* \neq \emptyset \). We have shown the following result:

**Theorem 7.9.** Fix a Shimura-datum \((G, X)\) with a poly-unitary structure of width \( w \) relative to a CM field \( L \) whose normal closure is \( R \subset \mathbb{C} \). Fix \( \vartheta \in \text{Gal}(R/\mathbb{Q}) \) and a \( \vartheta \)-poly-gauge as in the beginning of this section. Let \( p \subset \mathcal{O}_{ER} \) be a prime of norm \( p^f \). Assume that \( p \cap \mathcal{O}_R \) is of type \( \vartheta \), that the poly-unitary structure is unramified at \( p \), and that \( p > \max\{2, w\} \). There exist data \( \mathcal{K}\mathcal{M}_p^*, \mathcal{K}\mathcal{F}^*, \nabla \) that are canonically associated to each of the compact open subgroups in the family outlined in (14) and (15) such that:

- \( \mathcal{K}\mathcal{M}_p^* \) is a non-empty projective and smooth \( W(\mathbb{F}_{p^f}) \)-scheme, of which the special fiber is a finite \( \tilde{\mathcal{K}}\mathcal{U} \times_{\mathcal{O}_R} \mathbb{F}_{p^f} \)-scheme in a canonical way, where \( \tilde{\mathcal{K}}\mathcal{U} \) is the integral model of the unitary group Shimura variety described at the beginning of this section.

- \( \mathcal{K}\mathcal{F}^* \) is a locally trivial principal homogeneous space for \( \mathcal{G}^- \) over \( \mathcal{K}\mathcal{M}_p^* \).

- \( \nabla \) is a connection on the composite of the \( \otimes \)-functors:
  \[
  \text{Rep}_0(\mathcal{G} \times W(\mathbb{F}_{p^f})) \to \text{Rep}_0(\mathcal{G}^-) \otimes_{\mathcal{K}\mathcal{F}^*} \text{Vec}_{\mathcal{K}\mathcal{M}_p^*}
  \]

- The association of \( \mathcal{K}\mathcal{M}_p^*, \mathcal{K}\mathcal{F}^*, \nabla \) is functorial in the sense that each pair of compact open subgroups \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) satisfying \( \mathcal{K}_1 \subset \text{Int}(\gamma^p)\mathcal{K}_2 \) for some \( \gamma^p \in G(\mathbb{A}^\infty_p) \), gives rise to a commutative diagram:
  \[
  \begin{array}{cccc}
  \mathcal{K}_1\mathcal{F}^* & \to & \mathcal{K}_1\mathcal{M}_p^* & \to & \mathcal{K}_1\mathcal{U} \times_{\mathcal{O}_R} \mathbb{F}_{p^f} \\
  \downarrow & & \downarrow & & \downarrow \\
  \mathcal{K}_2\mathcal{F}^* & \to & \mathcal{K}_2\mathcal{M}_p^* & \to & \mathcal{K}_2\mathcal{U} \times_{\mathcal{O}_R} \mathbb{F}_{p^f}
  \end{array}
  \]

where the vertical arrow on the left is a \( \mathcal{G}^- \)-equivariant étale covering which is compatible with the connections on \( \mathcal{K}_1\mathcal{F}^* \times^{\mathcal{G}^-} \mathcal{G} \) and \( \mathcal{K}_2\mathcal{F}^* \times^{\mathcal{G}^-} \mathcal{G} \) while the vertical arrow on the right is the mod \( p \cap \mathcal{O}_R \)-reduction of the canonical action of the group element \((\ldots, \varepsilon_i^{-1} \circ g_i(\gamma^p) \circ \varepsilon_i, \ldots) \in \hat{G}(\mathbb{A}^\infty_p) \).
• Let \( X' \) be a connected component of \( X \), and let \( \text{Aut}(X') \) be the real Lie group of its biholomorphic transformations. Then there exists a finite family of discrete cocompact lattices \( \tilde{\Delta}_i \subset \text{Aut}(X')^\circ \times G(\mathbb{A}^{\infty,p}) \) and a biholomorphic map between the disjoint union:

\[
\coprod_{\gamma \neq \bar{\gamma}} \coprod_{\gamma \in \tilde{\Delta}_i} (\tilde{\Delta}_i \cap \text{Int}(\gamma^p)K^p) / X'
\]

and the complexification of \( K.M_p^* \).

**Appendix A. Existence of poly-unitary Shimura data**

Let \( k \) be an algebraically closed field of characteristic 0, and let \( g \) be a semi-simple Lie algebra over \( k \). We call a finite-dimensional representation \( \rho : g \to \text{End}_k(U) \) asymmetrical if for every non-inner automorphism \( \alpha : g \to g \) one can find some sub-representation \( \sigma \) of \( \rho \) which is nonequivalent to \( \sigma \circ \alpha \). The raw material for sample poly-unitary Shimura data is supplied by:

**Proposition A.1.** Let \( \rho_0 : G_0 \to \text{GL}(U_0/k) \) be a linear representation of the semi-simple \( k \)-group \( G_0 \) and write \( \rho_0 \) for the direct sum of \( \rho_0 \) and \( \text{Ad} \) on the representation space \( U'_0 = U_0 \oplus g_0 \). Assume that \( \rho'_0 \) is faithful and has an asymmetrical derived representation \( \rho_0^{\text{der}} : g_0 \to \text{End}_k(U'_0) \). Write \( U' := U \oplus g_0 \), where \( U \) is the trivial one-dimensional representation of \( G_0 \), and fix an element \( e \in U \). Consider the sub-\( k \)-algebras \( k[a] \subset \text{End}_{G_0}(U \otimes_k U' \otimes_k U'_0) \), and \( k[a'] \subset \text{End}_{G_0}(U') \) that are generated by the single endomorphisms

\[
x_1 \otimes (x + x') \otimes (x_0 + x'_0) \mapsto^\alpha x_1 \otimes (x'_0 \otimes x' + e \otimes \rho_0^{\text{der}}(x')(x_0 + x'_0)),
\]

and

\[
x + x' \mapsto^{a'} x,
\]

where \( x_0 \in U_0, x, x_1 \in U, \) and \( x'_0, x' \in g_0 \). Write \( G^0 \) for the stabilizer in \( \text{GL}(U/k) \times_k \text{GL}(U'/k) \times_k \text{GL}(U'_0/k) \) of \( k[a], k[a'] \) and \( \text{End}_{G_0}(U'_0) \), and let \( T^0 \) be the center of \( G^0 \). Then one has:

\[
T^0G_0 = G^0
\]

More specifically, write \( T'_0 \) for the center of \( \text{End}_{G_0}(U'_0)^\times \), and write \( T' \subset T'_0 \) for the sub-torus consisting of elements whose action on \( g_0 \) is a scalar. Then \( T' \) is naturally contained in \( T^0 \), and indeed one has \( \mathbb{G}_m^2 \times T' = T^0 \), where the two copies of \( \mathbb{G}_m \) act as scalars on \( U \) and \( U' \). The rank of \( T^0 \) is equal to

\[
3 + \text{Card}\{\text{isotypic components of } U'_0\} - \text{Card}\{\text{simple factors of } g_0\},
\]

and it is connected.
Proof. The proposition and its proof are both similar to [3, Lemma 7.3]. Fix an element of $G^0$, according to the presence of $\text{End}_{G^0}(U'_0)$ and $a'$ we can write \( \begin{pmatrix} g_0 & 0 \\ 0 & g'_0 \end{pmatrix} \), \( \begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix} \), and $g_1$, for the induced maps on $U'_0$, $U'$, and $U$. Notice that $g'_0$ and $g'$ are proportional, and that:

\[
\frac{g_0}{g} \rho^\text{der}(x') x_0 = \rho^\text{der}(\frac{g'}{g} x')(\frac{g_0}{g} x_0)
\]

\[
\frac{g'_0}{g} [x', x_0] = \left[ \frac{g'}{g} x', \frac{g_0}{g} x_0 \right]
\]

according to the presence of $a$. This means that $\alpha := \frac{g'}{g}$ is an automorphism of the Lie algebra $g_0$, and that $\beta := \frac{1}{g} \begin{pmatrix} g_0 & 0 \\ 0 & g'_0 \end{pmatrix}$ intertwines $\rho^\text{der}_0$ and $\rho^\text{der}_0 \circ \alpha$. Using the asymmetry of $\rho^\text{der}_0$, and again the presence of $\text{End}_{G_0}(U'_0)$, we see that $\alpha$ must be an inner automorphism, so that it is induced from an element in $G_0(k)$. Upon an adjustment we are allowed to assume $\alpha = 1$, so that $g'$ is equal to the multiplication by the scalar $g$. Consequently $\beta$ lies in the center of $\text{End}_{G_0}(U'_0)^\times$. Upon a further adjustment we are allowed to assume that each of $g_0$, $g'_0$ and $g'$ is equal to the multiplication by the scalar $g$. The remaining degrees of freedom are $g$, $g_1 \in k^\times$. $\square$

In the special case of a simply connected algebraic group $G_0$, the condition on the asymmetry of the derived representation $\rho^\text{der}_0$ may be removed from the assumptions of the previous proposition, because this is an automatic consequence of the faithfulness of $\rho_0$. In fact, if $G_0$ is simply connected and semisimple without simple factors of type $E_8$, $F_4$ or $G_2$, then the faithfulness of $\rho_0$ implies already the asymmetry of $\rho^\text{der}_0$ (Sketch: every automorphism of $g_0$ comes from an automorphism of $G_0$, which is inner if and only if it restricts to the identity on the center).

In the special case of an algebraic group of adjoint type over an algebraically closed field, one can easily give a specific example of a representation $\rho_0$ such that $\rho^\text{der}_0$ is faithful and asymmetrical: Let $G_0 = \prod_{i=1}^d G_i$ be the decomposition into simple factors. Let us write $\rho_i : G_i \to \text{GL}(U_i/k)$ for the following asymmetrical representations:

- If $G_i \cong \text{PGL}(n)$, then $U_i := \text{sym}^n(\text{std}) \otimes \bigwedge^n(\tilde{\text{std}})$,

- If $G_i \cong \text{SO}(2n)/\{\pm 1\}$, then $U_i := \begin{cases} \bigwedge^n(\text{std}) & n \equiv 0 \pmod{2} \\ \text{sym}^2(\bigwedge^n(\text{std})) & n \equiv 1 \pmod{2} \end{cases}$,

- If $G_i$ is of type $E_6$, then $U_i := \text{sym}^3(\mathbb{J})$,
• $U_i := 0$ in all remaining cases,

and let us write $\rho_0 : G_0 \to \text{GL}(U_0/k)$ for the natural representation on the (exterior) direct sum $U_0 := \bigoplus_{i=1}^{d} U_i$ (here std means standard representation and $\mathbb{J}$ is the 27-dimensional exceptional Jordan algebra).

In general $U_0$ need not be faithful nor asymmetrical, but it is easy to see that $U'_0 = U_0 \oplus g_0$ does indeed possess both of these properties. In the sequel this is our prime example, we also need the following $\mathbb{N}_0$-valued function on the set of isomorphism classes of connected groups of adjoint type, which we call the ”radius”:

• $r(G_1 \times \cdots \times G_d) := \max\{r(G_1), \ldots, r(G_d)\}$, if $G_1, \ldots, G_d$ are simple.

• $r(\text{PGL}(n)) := n - 1$.

• $r(\text{SO}(2n)/\{\pm 1\}) := \begin{cases} n & n \equiv 0 \pmod{2} \\ n & n \equiv 1 \pmod{2} \end{cases}$.

• If $G_0$ is of type $E_6$, then $r(G_0) := 4$.

• If $G_0$ is of type $E_7$, $E_8$ or $B_1$, then $r(G_0) := 1$.

• If $G_0$ is of type $E_8$, $F_4$ or $G_2$, then $r(G_0) := 0$.

The number $r(G_0)$ has the following significance: If $U'_0$ is the representation above, then no minuscule cocharacter of $G_0$ possesses a weight on $U'_0$ which is strictly greater than $r(G_0)$ (or strictly smaller than $-r(G_0)$). For a totally imaginary quadratic extension $L$ of a totally real field $L^+$ we have to introduce the following algebraic tori:

$$C^0 := \ker(\text{Res}_{L/L^+} \mathbb{G}_m^N \to \mathbb{G}_m)$$

$$C^1 := \text{Res}_{L^+/Q} C^0$$

$$C := (\mathbb{G}_m \times C^1)/\{\pm 1\},$$

Every CM-type for $L$ endows $C$ with the structure of a Shimura datum.

Now we turn to our poly-unitary examples:

**Proposition A.2.** Fix an $\mathbb{R}$-algebra homomorphism $c : \mathbb{C} \to \mathbb{R} \otimes L$, i.e. a CM type. Let $G_0/L^+$ be an algebraic group of adjoint type, where $L^+$ is as above. Let $(\text{Res}_{L^+/Q} G_0, X_0)$ be a Shimura datum. Then there exists an embedding of connected $L^+$-tori $C^0 \hookrightarrow T^0$ together with a poly-unitary Shimura datum of the form

$$(G, X) = ((\mathbb{G}_m \times \text{Res}_{L^+/Q} T^0)/\{\pm 1\}, \{c\}) \times (\text{Res}_{L^+/Q} G_0, X_0),$$

with the following additional features:

• The torus $T^0$ splits over the composite of $L$ with the smallest field over which $G_0$ becomes an inner form of a split form, and $\text{Res}_{L^+/Q} T^0$ is compact over $\mathbb{R}$. 

The poly-unitary structure is unramified outside the set of primes for which at least one of \( L \) or \( \text{Res}_{L^+/Q} G_0 \) is ramified, and its width is \( 2r \), where \( r \) is the radius of \( G_0 \).

Finally, this particular poly-unitary structure allows a \( \vartheta \)-poly-gauge if and only if the automorphism \( \vartheta \) in the Galois group of the normal closure \( R^+ \subset \mathbb{R} \) of \( L^+ \) satisfies:

\[
\frac{2r + 1}{2r + 2} < \lim_{N \to \infty} \frac{\text{Card}\{k \in \{1, \ldots, N\} | G_0 \times_{L^+, \vartheta} \mathbb{R} \text{ is compact} \}}{N}
\]

for all embeddings \( \iota : L^+ \to \mathbb{R} \).

**Proof.** We begin with any representation \( \rho_0 : G_0 \to \text{GL}(W_0/L) \) whose scalar extension \( L^{ac} \otimes_L W_0 \) is a direct sum of any number of copies of \( \text{Gal}(L^{ac}/L) \)-conjugates of the previously described \( U_0 \). Let \( \rho_1 : G_0 \to \text{GL}(W_1/L) \) be the trivial one-dimensional representation. Pick polarizations \( \Psi_i : L \otimes_{\ast} L W_i \to \tilde{W}_i \) for \( i \in \{0, 1\} \), notice that \( \Psi_i(x, y)^* = \Psi_i(y, x) \), as the \( \rho_i \)'s are even. Consider the prolongation of the Killing form on \( g_0 \) to a sesquilinear form \( \Psi' \) on \( W' := L \otimes_{L^+} g_0 \). The triple of polarized \( G_0 \times_{L^+} L \)-representations, that we wish to work with is:

\( W_1, W_2 := W_1 \oplus W', \) and \( W_3 := W_0 \oplus W' \). The \( G_0 \)-invariant subalgebras \( L_s \subset \text{End}_L(V_s) \) are constructed along the lines of proposition [A.1] and \( \Pi := \{\{1\}, \{2\}, \{3\}, \{1, 2, 3\}\} \). The same proposition exhibits a connected \( L^+ \)-torus \( T^0 \) such that \( T^0 \times_{L^+} G_0 \) is the stabilizer in \( U(W_1/L, \Psi_1) \times_{L^+} U(W_2/L, \Psi_2) \times_{L^+} U(W_3/L, \Psi_3) \) of \( \bigcup_{\pi \in \Pi} L_{\pi} \), moreover the proof thereof shows that \( T^0 \times_{L^+} L \) splits over the splitting field of \( G_0 \times_{L^+} L \). Write \( T^1 := \text{Res}_{L^+/Q} T^0 \) and \( T := (G_m \times T^1)/\{\pm 1\} \), and let us note in passing that \( C \) is embedded diagonally into \( T \), as \( C^1 \), the complement in \( C \) to \( G_m \), is embedded diagonally into \( T^1 \). Notice that \( w_{\mathcal{X}} = w_{\mathcal{Y}^c} \) maps \(-1\) to \(-1\), and thus we obtain a poly-unitary structure if we only divide out each of the pairings \( \Psi_i \) by \( \sqrt{-1} \), for \( i \in \{1, 2, 3\} \).

In order to obtain an unramified poly-unitary structure we have to work a little bit harder: To fix ideas pick a compact and open subgroup of the form \( K = K_S \times \prod_{p \notin S} K_p \subset \text{Res}_{L^+/Q} G_0(\mathbb{A}^\infty) \) such that every \( K_p \) is a hyperspecial subgroup of \( \text{Res}_{L^+/Q} G_0(\mathbb{Q}_p) \). Sufficiently small \( K \)-invariant \( \mathcal{O}_L[\prod_{p \in S} p^{-1}] \)-lattices \( \mathcal{W}_i \subset \mathcal{W}_i \) for \( i \in \Lambda \), and a sufficiently large integer \( d \) would satisfy: \( \mathcal{W}_i \subset \mathcal{W}_i^d \subset \mathcal{W}_i \). Now recall Zarhin’s trick: Choose a lattice \( Z \) of rank 8 containing a direct factor \( D \) of rank 4, such that:

- \( Z^\perp = Z \)
- \( D^\perp \supset \frac{D}{d} \).
Replace $W_i$ by $V_i := Z \otimes W_i$. The $L$-algebras $\text{End}(Z) \otimes \text{Card}(\pi) \otimes L_\pi$ are evidently $*$-invariant subalgebras of $\text{End}_L(\otimes_{i \in \pi} V_i)$ for $\pi \in \Pi$, and the lattices $V_i := Z \otimes W_i + D \otimes W_i^\perp \subset V_i$ are $K$-invariant and unimodular for $i \in \Lambda$. The groups of $Z_p$-points in the unitary similitude groups of these lattices are hyperspecial for $p \notin S$, and we are done. \hfill \Box

Let us round off the discussion with another family of examples:

**Proposition A.3.** Let $G_0/L^+$ be an absolutely simple connected, algebraic group, where $L^+$ is as above. Let $(\text{Res}_{L^+/Q} G_0, X_0)$ be a Shimura datum without factors of type $D_l^H$ for any $l \geq 5$. Then there exists a certain poly-unitary Shimura datum $(G, X)$ of width 2, relative to a certain totally imaginary quadratic extension $L$, such that both of them are unramified outside the set of odd primes for which at least one of $L^+$ or $\text{Res}_{L^+/Q} G_0$ is ramified, and they enjoy the following properties:

- There exist a certain embedding of connected $L^+$-tori $C_0 \hookrightarrow T^0$ together with a certain central extension of algebraic $L^+$-groups

$$1 \to T^0 \to C_0 \to G_0 \to 1,$$

where $\text{Res}_{L^+/Q} T^0$ is unramified outside the set of odd primes for which at least one of $L^+$ or $\text{Res}_{L^+/Q} G_0$ is ramified, and this $\mathbb{Q}$-torus is compact over $\mathbb{R}$.

- The quotient of $G_0 \times \text{Res}_{L^+/Q} G_0$ by the subgroup $\{\pm 1\} \subset G_0 \times C^1$ agrees with $G$, the canonical epimorphism $G \twoheadrightarrow \text{Res}_{L^+/Q} G_0$ which is derived from that sends $X$ to $X_0$, and $w \times$ agrees with the canonical inclusion $G_0 \hookrightarrow G$.

Finally, this particular poly-unitary structure allows a $\vartheta$-poly-gauge if and only if the automorphism $\vartheta$ in the Galois group of the normal closure $R^+ \subset \mathbb{R}$ of $L^+$ satisfies:

$$\frac{3}{4} < \lim_{N \to \infty} \frac{\text{Card}\{k \in \{1, \ldots, N\}|G_0 \times_{L^+, \vartheta^k \alpha_0} \mathbb{R} \text{ is compact}\}}{N}$$

for all embeddings $\iota : L^+ \to \mathbb{R}$.

**Proof.** In the cases of type $A_l$ there is nothing to prove. In the cases of type $E_l$, $C_l$, $B_l$ we can just apply the proposition [A.2] to the quadratic extension $L := L^+ [\sqrt{-1}]$. In the $E_6$ or $D_l$-situations our proof starts with a description of a provisional central extension: Let $\overline{G}_0$ be the simply connected covering group of $G_0$ and let $\overline{C}_0$ be its center. This group is of order 3 or 4 and it is cyclic, except for the $D_l$-case with $l \equiv 0 \pmod{2}$. Excluding this for while one may observe that there is one and only one choice of a totally imaginary quadratic extension $L$, such that there exists a rational embedding $\overline{C}_0 \hookrightarrow C_0$, because in those cases
the Weyl-opposition is a non-inner automorphism, and there are no other ones. Given that $G_0 := (C^0 \times \overline{C}_0)/\overline{C}_0$ has connected center there exists a Shimura datum $(\text{Res}_{L^+/\mathbb{Q}} \overline{G}^0, \overline{X})$ which lifts $X_0$ and has a trivial weight homomorphism. Let $\rho_0 : \overline{G}^0 \times_{L^+} L \to \text{GL}(W_0/L)$ be a $L$-rational representation which is a direct sum of minuscule representations each of whose restrictions to the center $C^0 \times_{L^+} L = \mathbb{G}_m \times L$ agree with scalar multiplication. The latter one is unique so that $L \otimes_{\mathbb{Q}} W_0$ is equivalent to $\overline{W}_0$. The sets of $\mu$-weights of each eigenspace (to the embeddings $L \hookrightarrow \mathbb{C}$) can be read off from [6, Table 1.3.9], so these are translates of $\{-\frac{2}{3}, \frac{1}{3}, \frac{4}{3}\}$ or $\{\pm \frac{1}{2}\}$, depending on whether or not we are in the $E_6$-situation. Notice that $X$ can be multiplied with an arbitrary homomorphism $S : \mathbb{C} \to \mathbb{C}$ without changing $X_0$, so upon a careful adjustment the $\mu$-weights of all eigenspaces are in $\{0, \pm 1\}$. The proof is completed by following the arguments in the proof of proposition A.2, notice that the $L^+$-structure of the group $G^0$ (or equivalently of $T^0$) may depend on the choice of the polarizations which in turn assumes a choice of $\overline{X}$. Finally it remains to do the $D_l$-case with $l \equiv 0 \pmod{2}$, in this case $\overline{C}_0$ is a 2-dimensional $\mathbb{F}_2$-vector space scheme and $\text{Res}_{L^+/\mathbb{Q}} \overline{G}_0$ splits over $\mathbb{R}$. Let $C^+$ be a connected $L^+$-torus satisfying:

- $C^+[2] \cong \overline{C}_0$
- The $\mathbb{Q}$-torus $\text{Res}_{L^+/\mathbb{Q}} C^+$ splits over $\mathbb{R}$.

Such a torus exists because $\text{GL}(2, \mathbb{Z}) \to \text{GL}(2, \mathbb{F}_2)$ is a split epimorphism. Write $\overline{C}$ for the connected $L^+$-torus $C^0 \otimes X^*(C^+)$, where we use $L := L^+[\sqrt{-1}]$ again for the above definition of $C^0$. Notice that the group scheme of its 2-torsion points satisfies $\overline{C}[2] \cong \overline{C}_0$, so let us put $G^0 := (\overline{C} \times \overline{C}_0)/\overline{C}_0$ and let us choose a Shimura datum $(\text{Res}_{L^+/\mathbb{Q}} \overline{G}^0, \overline{X})$ which lifts $X_0$ and has a trivial weight homomorphism (notice that $\text{Res}_{L^+/\mathbb{Q}} \overline{G}$ is compact over $\mathbb{R}$). Now pick a selfdual, faithful, $L$-rational representation $\rho_0 : \overline{G}^0 \times_{L^+} L \to \text{GL}(W_0/L)$ decomposing into any direct sum of minuscule ones. In case $l \neq 4$, assume in addition that $\rho_0$ does not contain the unique representation of dimension $2l$, i.e. the standard representation. The proof is again completed by the arguments in the proof of proposition A.2.  

**Appendix B. Varshavsky’s characterization method**

The paper [26] deals with characterizations of Shimura varieties. In this work we want to be slightly more basic and confine to characterizations of symmetric Hermitian spaces, here are the objects under consideration:
• Let $\tilde{\Delta}$ be a group and let $\tilde{M}$ be a separated, non-empty, connected, complex manifold with a holomorphic left $\tilde{\Delta}$-action.
• Let $G_1, \ldots, G_r$ be connected, reductive, simple algebraic groups over $\mathbb{C}$, and let $P_i$ be a proper minuscule parabolic subgroup of $G_i$. Denote the associated irreducible symmetric Hermitian domains of compact type by $X_i := G_i(\mathbb{C})/P_i(\mathbb{C})$, and write $G := \prod_{i=1}^r G_i$ and $P := \prod_{i=1}^r P_i$ and $X := \prod_{i=1}^r X_i$.
• Let $\phi : \tilde{\Delta} \to G(\mathbb{C})$ be a group homomorphism, and let $\rho_0 : \tilde{M} \to X$ be a locally biholomorphic $\tilde{\Delta}$-equivariant map (use $\phi$ to define a left action of $\tilde{\Delta}$ on $X$).

A sextuple $(\tilde{\Delta}, \tilde{M}, G_i, P_i, \phi, \rho_0)$ as above is called a period map if:

(P1) The image of $\tilde{\Delta}$ in $G(\mathbb{C})$ is Zariski-dense.
(P2) The image in $G(\mathbb{C})$ of the stabilizer in $\tilde{\Delta}$ of any element in $\tilde{M}$ possesses a compact closure.
(P3) There exists some $y_0 \in \tilde{M}$ whose stabilizer in $\tilde{\Delta}$ contains some subgroup $\Delta_0$ the closure of whose image in $G(\mathbb{C})$ is equal to $\prod_{i=1}^r T_i(\mathbb{R})$, where each $T_i$ is a maximal compact torus in $\text{Res}_{\mathbb{C}/\mathbb{R}} G_i$ (i.e. the $\mathbb{R}$-rank of $T_i$ is equal to the $\mathbb{C}$-rank of $G_i$).

Without any attempt of originality we wish to give a slight reformulation of [25, p.89,p.92-94] in the above axiomatic setup:

**Theorem B.1** (Varshavsky). Suppose that $(\tilde{\Delta}, \tilde{M}, G_i, P_i, \phi, \rho_0)$ is a period map. Then there exist real forms $J_i$ of $G_i$ for every $i \in \{1, \ldots, r\}$, such that $\phi(\tilde{\Delta}) = J(\mathbb{R})^\circ$, where $J := \prod_{i=1}^r J_i$. The locally biholomorphic map $\rho_0$ is actually an injection and the $\tilde{\Delta}$-action on $\tilde{M}$ can be extended to a continuous and transitive $J(\mathbb{R})^\circ$-action thereon.

Pick a base point $\tilde{y} \in \tilde{M}$, and assume for notational convenience that $\rho_0(\tilde{y})$ is the canonical base point of $\prod_{i=1}^r X_i$, i.e. equal to $(1, \ldots, 1)$. Write $U_i := P_i \cap \overline{P}_i$, where $\overline{P}_i$ denotes the complex conjugate of $P_i$ with respect to the real form $J_i$. Consider the homogeneous spaces $\tilde{M}_i := J_i(\mathbb{R})^\circ/U_i(\mathbb{R})$, so that and $\tilde{M} = \prod_{i=1}^r \tilde{M}_i$. Then for each $i \in \{1, \ldots, r\}$ one and only one of the following alternatives hold:

• $J_i$ is compact, and $U_i$ is a maximal proper connected subgroup,
• or $J_i$ is not compact, and $U_i$ is a maximal compact subgroup,

and in any case $U_i$ has indiscrete center so that $\tilde{M}_i$ is a symmetric Hermitian domain of compact or of non-compact type.

We give a synopsis of the proof: The solution to the 5th problem of Hilbert implies that $J := \overline{\phi(\Delta)}$ is a Lie-group. Note that $\mathbb{C} \otimes_{\mathbb{R}} \text{Lie} J$ is semisimple, because $\text{Lie} G$ is semisimple and both of $\text{Lie} J \cap \sqrt{-1} \text{Lie} J$
and $\text{Lie } J + \sqrt{-1} \text{Lie } J$, being $G$-invariant $\mathbb{C}$-subspaces of $\text{Lie } G$ in view of (P1), are semisimple too. Moreover, there exists a semi-simple real algebraic group $J \subset \text{Res}_{\mathbb{C}/\mathbb{R}} G$ such that $\text{Lie } J = \text{Lie } J$, as semisimple Lie-algebras are algebraic. Finally the existence of $T$ tells us that $J = \prod_{i=1}^{r} J_{i}$ where each single $J_{i}$ contains $T_{i}$ and it is either a real form of $G_{i}$ or it is equal to $\text{Res}_{\mathbb{C}/\mathbb{R}} G_{i}$. Let us write $\text{Aut}(\tilde{M})$ for the homeomorphism group of $\tilde{M}$, and let us endow it with the compact-open topology. Let $\tilde{J}$ (resp. $\tilde{T}$) denote the closure of the image of $\Delta$ (resp. $\Delta_{0}$) in $\text{Aut}(\tilde{M})$. The group $\tilde{T}$ is compact because it fixes a point and preserves a suitable Riemannian metric, [13, II, Theorem 1.2]. It is straightforward to see that $\phi$ extends to a continuous group homomorphism, say $\tilde{\phi}: \tilde{J} \rightarrow J$, the kernel of which is a discrete subgroup of $\tilde{J}$, cf. [25, Lemma 3.1]. The following argument shows that $\tilde{\phi}$ is surjective: We clearly have $\tilde{\phi}(J) = J$ and note also that $\tilde{\phi}(\tilde{T}) = T(\mathbb{R})$ because $\tilde{T}$ is compact. Now there exist elements $\gamma_{1}, \ldots, \gamma_{n} \in \tilde{\Delta}$ such that $\text{Ad}(\gamma_{1}) \text{Lie } T + \cdots + \text{Ad}(\gamma_{n}) \text{Lie } T = \text{Lie } J$. It follows that the product $\tilde{\phi}(\gamma_{1} \tilde{T} \gamma_{1}^{-1}) \cdots \tilde{\phi}(\gamma_{n} \tilde{T} \gamma_{n}^{-1})$ contains an open neighborhood of the identity in $J$ and whence it follows that $\tilde{\phi}(\tilde{J}) = J$. One shows the openness of $\tilde{\phi}$ along the same lines.

Remark B.2. Let Grass$_{d,h}$ be the scheme whose $S$-valued points are $h+1$-tuples $(F, e_{1}, \ldots, e_{h})$ where $F \in \text{Vec}_{S}$ is of rank $d$ and $e_{1}, \ldots, e_{h} \in \Gamma(S, F)$ generate $F$ as an $\mathcal{O}_{S}$-module. This $\mathbb{Z}$-scheme is projective and smooth of relative dimension $d(h-d)$. Write $\omega_{d,h} := \Omega_{\text{Grass}_{d,h}/\mathbb{Z}}^{d(h-d)}$ for its
canonical line bundle. Let us say that a period map \((\tilde{\Delta}, \tilde{M}, G_i, P_i, \phi, \rho_0)\) is bounded if

(P4) there exists a finite family of projective representations \(\rho_j : G \to \text{PGL}(h_j) \times \mathbb{C}\), together with points \(y_j \in \text{Grass}_{d_j, h_j}(\mathbb{C})\) such that the intersection of the stabilizers in \(G\) of the points \(y_j\) is equal to \(P\), and there exist positive integers \(p_j\) such that the pull-back of some line bundle of the form \(\bigotimes_j \omega_j^{\otimes p_j}\) by means of \(\rho_0\) is generated by its holomorphic global sections, where \(\omega_j\) denotes the pull-back of \(\omega_{d_j, h_j}\) by means of the map \(G/P \to \text{Grass}_{d_j, h_j} \times \mathbb{C}; gP \mapsto g(y_j)\).

It is clear that bounded period maps give rise to bounded symmetric Hermitian domains \(\tilde{M}\).

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