MINIMISERS AND KELLOGG’S THEOREM

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ABSTRACT. We extend the celebrated theorem of Kellogg for conformal mappings to the minimizers of Dirichlet energy. Namely we prove that a diffeomorphic minimizer of Dirichlet energy of Sobolev mappings between double connected domains having $C^{1,\alpha}$ boundary is $C^{1,\alpha}$ up to the boundary. It is crucial that, every diffeomorphic minimizer of Dirichlet energy has a very special Hopf differential and this fact is used to prove that every diffeomorphic minimizer of Dirichlet energy can be locally lifted to a certain minimal surface near an arbitrary point inside and at the boundary.

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1. INTRODUCTION AND BACKGROUND

Throughout this text $D$ and $\Omega$ will be double connected domains in the complex plane $\mathbb{C}$. By $D$ we denote the unit disk and by $T$ its boundary. If $R > r > 0$ then we define the annulus $A(r, R) = \{z : r < |z| < R\}$. The Dirichlet energy of a diffeomorphism $f : D \to \Omega$ is defined and denoted by

$$\mathcal{E}[f] = \int_D \|Df\|^2 = 2 \int_D (|\partial f|^2 + |\bar{\partial} f|^2)$$

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where $\|Df\|$ is the Hilbert-Schmidt norm of the differential matrix of $f$. The primary goal of this paper is to establish the boundary behaviors of a diffeomorphism $f: D \mapsto \Omega$ of smallest (finite) Dirichlet energy, provided it exists and the boundary is smooth. A change of variables in (1.1) yields

$$E[f] = 2 \int_D J(z, f) \, dz + 4 \int_D |\bar{\partial}f|^2 \geq 2|\Omega|$$

where $J(z, f)$ is the Jacobian determinant and $|\Omega|$ is the measure of $\Omega$. In this paper we will assume that diffeomorphisms as well as Sobolev homeomorphisms are orientation preserving. This means that $J(z, f) > 0$. A conformal mapping of $D$ onto $\Omega$, would be an obvious minimizer of (1.2), because $\bar{\partial}f = 0$, provided it exists. Since we study the boundary character of minimisers, by the famous Kellogg theorem (Proposition 1.2), the minimiser is in this case smooth if the boundary is smooth.

For double connected domains there is no such mapping if they have different conformal modulus.

We have the following extension of the Kellogg’s theorem, which is the main result of the paper.

**Theorem 1.1.** Let $\alpha \in (0, 1)$. Assume that $D$ and $\Omega$ are two double connected domains in the complex plane with $C^{1,\alpha}$ boundaries. Assume that $f$ is a diffeomorphic minimiser of energy (1.1) throughout the class of all diffeomorphisms between $D$ and $\Omega$. Then $f$ has a $C^{1,\alpha}$ extension up to the boundary.

We continue this section with some background on the topic. We will reformulate Theorem 1.1 in Section 2, where we will describe the key point of the proof. In Section 3 we prove that such diffeomorphisms are Hölder continuous at the boundary components. This is needed to prove the global Lipschitz continuity of such diffeomorphisms, as in [20]. Further by following an approach in the proof of corresponding result in [20] we obtain the desired conclusion. Namely section 4 contains the proof of the main results. The last section is devoted to an open problem.

1.1. **Background.** The starting point of this paper is the following classical result.

**Proposition 1.2** (Kellogg ($n = 1$) see [6] and Warshawski ($n > 1$), [25][26]). Let $n \in \mathbb{N}$, $0 < \alpha < 1$. If $D$ and $\Omega$ are Jordan domains having $C^{n,\alpha}$ boundaries and $\omega$ is a conformal mapping of $D$ onto $\Omega$, then $\omega^{(n)} \in C^{\alpha}(D)$ and $(\omega^{-1})^{(n)} \in C^{\alpha}(\Omega)$.

For a function $\xi \in \mathcal{C}^{\alpha}(D)$ i.e. a function $\xi: D \rightarrow \mathbb{C}$ satisfying the condition

$$\sup_{z \neq w, z, w \in D} \frac{|\xi(z) - \xi(w)|}{|z - w|^\alpha} < \infty$$

we say that is a uniformly $\alpha-$Hölder continuous function. From now one, instead of $\omega^{(n)} \in C^{\alpha}(D)$ we write $\omega \in C^{n,\alpha}(D)$. In similar way we define the class $C^{n,\alpha}(D)$ of non-necessarily conformal mappings. The theorem of Kellogg and of Warshawski has been extended in various directions, see for example the extension to conformal minimal parametrization of minimal surfaces by Nitsche [20], and to
q.c. harmonic mappings w.r. hyperbolic metric by Tam and Wan [22, Theorem 5.5.]. For some other extensions and quantitative Lipschitz constants we refer to the paper [18].

1.2. **Deformations.** In this part we borrow same notation, definitions and statements from [8]. The natural setting for our minimization problem is the Sobolev space $W^{1,2}(\Omega)$. Let us choose the notation $H^{1,2}(D,\Omega)$ for the set of all sense-preserving $W^{1,2}$-homeomorphisms $h: D \rightarrow \Omega$. When this set is nonempty, we define

$$
E_H(D,\Omega) = \inf \{ E[h]: h \in H^{1,2}(D,\Omega) \}.
$$

Because of the density of diffeomorphisms in $H^{1,2}(D,\Omega)$, the minimization of energy among sense-preserving diffeomorphisms leads to the same value $E_H(D,\Omega)$.

A homeomorphism $h \in H^{1,2}(D,\Omega)$ is called energy-minimal if it attains the infimum in (1.3). The set $H^{1,2}(D,\Omega) \subset W^{1,2}(D)$ is unbounded. Due to lacking of compactness of $H^{1,2}(D,\Omega)$, in [8] there were introduced the deformations.

The deformations which will be defined in the sequel are shown to be good setting of the problem of minimising of energy. These are sense-preserving surjective mappings of the Sobolev class $W^{1,2}$ that can be approximated by homeomorphisms in a certain way.

A homeomorphism of a planar domain is either sense-preserving or sense-reversing. For homeomorphisms of the Sobolev class $W^{1,1}_{loc}(D)$ this implies that the Jacobian determinant does not change sign.

Let $D$ and $\Omega$ be bounded domains in $\mathbb{C}$. For a mapping $f: D \rightarrow \overline{\Omega}$ it is defined a boundary distance function $\delta_f(z) = \text{dist}(f(z), \partial \Omega)$ which is set to 0 on the boundary of $D$.

The following concept, which lies between $c$-uniform (i.e., uniform on compact subsets) and uniform convergence, proves to be effective.

**Definition 1.3.** A sequence of mappings $h_j: D \rightarrow \overline{\Omega}$ is said to converge $c\delta$-uniformly to $h: D \rightarrow \overline{\Omega}$ if $h_j \rightarrow h$ uniformly on compact subsets of $D$ and $\delta_{h_j} \rightarrow \delta_h$ uniformly on $D$.

We denote it as $h_j \xrightarrow{c\delta} h$.

**Definition 1.4.** A mapping $h: D \rightarrow \overline{\Omega}$ is called a deformation if $h \in W^{1,2}(D)$, the Jacobian $J_h = \det Dh$ is nonnegative a.e. in $D$, $\int_D J_h \leq |\Omega|$ and there exist sense-preserving homeomorphisms $h_j: D \rightarrow \overline{\Omega}$, called an approximating sequence, such that $h_j \xrightarrow{c\delta} h$ on $D$.

The set of deformations $h: D \rightarrow \overline{\Omega}$ is denoted by $\mathcal{H}(D,\Omega)$.

A important thing is that $H^{1,2}(D,\Omega) \subset \mathcal{H}(D,\Omega)$.

Define

$$
E(D,\Omega) = \inf \{ E[h]: h \in \mathcal{H}(D,\Omega) \}
$$

where $E[h]$ is as in (1.1). A deformation that attains the infimum in (1.4) is called energy-minimal. It is obvious that $E_H(D,\Omega) \geq E(D,\Omega)$, but whether the equality holds is not clear. We now state the existence result proved in [8].
Proposition 1.5. Suppose that $D$ and $\Omega$ are bounded doubly connected domains in $\mathbb{C}$ such that $\text{Mod} \ D \leq \text{Mod} \ \Omega$. There exists a diffeomorphism $h \in H^{1,2}(D, \Omega)$ that minimizes the energy among all deformations; that is, $E[h] = E(D, \Omega)$ and hence, $E_{H}(D, \Omega) = E(D, \Omega)$. Moreover, $h$ is harmonic and it is unique up to a conformal automorphism of $D$.

The most important issue in proving Theorem 1.5 was to establish some key properties of stationary deformations formulated in the next subsection.

1.3. Noether harmonic mappings (cf. [4]). We call a mapping $h : D \to \Omega$ Noether harmonic if

\[
\frac{d}{dt} \bigg|_{t=0} E[h \circ \phi_t^{-1}] = 0
\]

for every family of diffeomorphisms $t \to \phi_t : \Omega \to \Omega$ which depend smoothly on the parameter $t \in \mathbb{R}$ and satisfy $\phi_0 = \text{id}$. The latter mean that the mapping $\mathbb{Y} \times [0, \epsilon_0] \ni (t, z) \to \phi_t(z) \in \Omega$ is a smooth mapping for some $\epsilon_0 > 0$. Not every Noether harmonic mapping $h$ is a harmonic mapping, however if the mapping $h$ is a diffeomorphism, then it is harmonic, i.e. it satisfies the equation $\Delta h = 0$.

1.4. Some key properties of Noether harmonic diffeomorphisms. The following key properties of the Noether harmonic mappings are derived in the proof of [10, Lemma 1.2.5]:

1. The function $\varphi := g_z g_{\bar{z}}$, which is called the Hopf differential, a priori in $L^1(D)$, is holomorphic.
2. If $\partial D$ is $C^\alpha$-smooth then $\varphi$ extends continuously to $\overline{D}$, and the quadratic differential $\varphi \ dz^2$ is real on each boundary curve of $D$.

Further by using those key properties in [8] (and in [11]) it is shown the following statement. Let $D = A(r, R)$ be a circular annulus centered at origin, $0 < r < R < \infty$, and $\Omega$ a doubly connected domain. If $g$ is a stationary deformation, then

\[
g_z g_{\bar{z}} \equiv \frac{c}{z^2} \quad \text{in } D
\]

where $c \in \mathbb{R}$ is a constant.

For the constant $c$ obtained in (1.6) we have

Proposition 1.6. [11, Corollary 5.2]. If $g$ is a stationary deformation, then we have

- if $\text{Mod} \ D < \text{Mod} \ \Omega$, then $c > 0$,
- if $\text{Mod} \ D = \text{Mod} \ \Omega$, then $c = 0$,
- if $\text{Mod} \ D > \text{Mod} \ \Omega$ then $c < 0$.

A sense preserving mapping $w$ of class ACL between two planar domains $X$ and $Y$ is called $(K, K')$-quasi-conformal if

\[
\|Dw\|^2 \leq 2KJ(z, w) + K',
\]

for almost every $z \in X$. Here $K \geq 1$, $K' \geq 0$, $J(z, w)$ is the Jacobian of $w$ in $z$ and $\|Dw\|^2 = |w_x|^2 + |w_y|^2 = 2|w_x|^2 + 2|w_y|^2$. For a related definition for the mappings between surfaces we refer to the paper [23].
In [12] it was proved that the minimizer belongs to the class of $(K, K')$ quasiconformal mappings.

**Lemma 1.7. [12]** Every sense-preserving Noether harmonic map $g : \mathbb{H}(\rho, 1) \to \Omega$ is $(K, K')$ quasiconformal, where

$$K = 1 \text{ and } K' = \frac{2|c|}{\rho^2},$$

and $c$ is the constant from (1.6). The result is sharp and for $c = 0$ the Noether harmonic map is $(1, 0)$ quasiconformal, i.e. it is a conformal mapping. In this case $\Omega$ is conformally equivalent with $\mathbb{H}(\rho, 1)$.

Assume that $\gamma : [0, l] \to \Gamma$ is the arc-length parametrization of the rectifiable Jordan curve $\Gamma$. Here $l = |\Gamma|$ is the length of $\Gamma$. We say that a continuous mapping $f : \mathbb{T} \to \Gamma$ of the unit circle onto a rectifiable Jordan curve is monotone, if there exists a monotone function $\phi : [0, 2\pi] \to [0, l]$ so that $f(e^{it}) = \gamma(\phi(s))$. In a similar way we define a monotone function between $\rho \mathbb{T} := \{z : |z| = \rho\}$ and $\Gamma$. In view of [16, Proposition 5] and Proposition 3.1 below we can formulate the following simple lemma

**Lemma 1.8.** Assume that $f$ is a diffeomorphic minimiser of Dirichlet energy between the annuli $\mathbb{H}_\rho$ and $\Omega$, where $\Omega$ is double connected bounded by the outer boundary $\Gamma$ and inner boundary $\Gamma_1$. Then $f$ has continuous extension up to the boundary and boundary mapping is monotone in both boundary curves.

## 2. THE MAIN RESULTS

**Theorem 2.1.** Let $\alpha \in (0, 1)$. Assume that $D$ and $\Omega$ are two double connected domains in the complex plane with $C^{1, \alpha}$ boundaries. Assume that $f$ is a diffeomorphic minimiser of energy (1.1) throughout the class $\mathcal{D}(D, \Omega)$. Then $f$ has a $C^{1, \alpha}$ extension up to the boundary.

Theorem 2.1 and Proposition 1.5 imply the following result:

**Corollary 2.2.** Assume that $D$ and $\Omega$ are two double connected domains in $C$ with $C^{1, \alpha}$ boundary. Assume also that $\text{Mod}(D) \leq \text{Mod}(\Omega)$. Then there exists a minimiser $h$ of Dirichlet energy $E$ and it has a $C^{1, \alpha}$ extension up to the boundary. Moreover it is unique up to the conformal change of $D$.

The minimiser of Dirichlet energy is not always a diffeomorphism when $\text{Mod}(D) \geq \text{Mod}(\Omega)$. Moreover it fails to be smooth in the domain if the boundary is not smooth [2].

**Remark 2.3.** By using Lemma 1.7, the first author in [12] proved that, a minimiser of $\rho-$energy between double connected domains having $C^2$ boundary is Lipschitz continuous. The $\rho-$energy, is a certain generalization of Euclidean energy, and we will omit details in this paper.
2.1. **Minimizing mappings and minimal surfaces.** Since \( D \) is conformally equivalent to \( \mathbb{A}_\rho = \{ z : \rho < |z| < 1 \} \), for some \( \rho \in (0, 1) \), we can assume that \( D = \mathbb{A}_\rho \).

Since \( f \) is harmonic, for every \( p \in \partial \mathbb{A}_\rho \), there is a Jordan domain \( D_p \), containing a Jordan arc \( T_p \) in \( \partial \mathbb{A}_\rho \), whose interior contains \( p \). Moreover in view of Lemma 1.8, enlarging \( T_p \) if necessary, we can assume that \( \delta_p := f(T_p) \) is a Jordan arc containing \( q = f(p) \) in its interior in \( \partial D \). Moreover we can assume that \( D_p \) has a \( C^\infty \) boundary. Assume now that \( \Phi_p \) is a conformal mapping of the unit disk \( \mathbb{D} \) onto \( D_p \). Then \( f_p = f \circ \Phi_p \) has the representation

\[
f_p(z) = g(z) + h(z),
\]

where \( g(z) = g_p(z) \) and \( h(z) = h_p(z) \) are holomorphic mappings defined on the unit disk. Moreover \( f_p \) is a sense preserving diffeomorphism and this means that

\[
J(z, f_p) = |g'(z)|^2 - |h'(z)|^2 > 0.
\]

From (1.6) we have

\[
f_z f_{\bar{z}} = \frac{c}{z^2}, \quad z \in \mathbb{A}_\rho.
\]

It follows from (2.2) and (2.1) that

\[
h_p' g_p' = c \frac{\Phi_p'(z)^2}{\Phi_p^2(z)}.
\]

Then it defines locally the minimal surface by its conformal minimal coordinates, \( \varphi_p = (\varphi_1, \varphi_2, \varphi_3) \), and this is crucial for our approach:

\[
\varphi_1(z) = \Re(g + h), \quad \varphi_2(z) = \Im(g - h), \quad \varphi_3(z) = \Re(2i\sqrt{c} \log \Phi_p(z)).
\]

This can be written

\[
\varphi_1(z) = \varphi_1(z_0) + \Re \int_{z_0}^z (g'(z) + h'(z))dz
\]

\[
\varphi_2(z) = \varphi_2(z_0) + \Re \int_{z_0}^z i(h'(z) - g'(z))dz
\]

\[
\varphi_3(z) = \varphi_3(z_0) + \Re \int_{z_0}^z 2i \sqrt{h'(z)g'(z)}dz.
\]

Thus the Weierstrass–Enneper parameters are

\[
p(z) = g'(z), \quad q(z) = \sqrt{\frac{h'(z)}{g'(z)}}.
\]

The first fundamental form is given by \( ds^2 = \lambda(z)|dz|^2 \), where

\[
\lambda(z) = \frac{1}{2} \sum_{j=1}^3 |k_j|^2.
\]
Here
\[ k_1(z) = g'(z) + h'(z), \quad k_2(z) = i(h'(z) - g'(z)), \quad k_3(z) = 2i \sqrt{h'(z)g'(z)}. \]

Then as in [3, Chapter 10], we get
\[ \lambda(z) = |p|^2 (1 + |q|^2)^2 = |g'(z)|^2 \left(1 + \frac{|g'(z)|}{|h'(z)|}\right)^2 = (|g'(z)| + |h'(z)|)^2. \]

Let us note the following important fact, the boundary curve of the minimal surface defined in (2.4) is
\[ \varphi_p(e^{is}) = (\varphi_1(e^{is}), \varphi_2(e^{is}), \varphi_3(e^{is})), s \in [0, 2\pi), \]
\[ p \in \partial A_\rho. \] Its trace is not smooth in general. However the trace of curve
\[ z_p(e^{is}) = (\varphi_1(e^{is}), \varphi_2(e^{is})) \]
is smooth as well as the function \( k_3 \) is smooth in a small neighborhood of \( p \). This will be crucial in proving our main results.

We will prove certain boundary behaviors of \( f \) near the boundary by using the representation (2.1), and this is why we do not need global representation. The idea is to prove that \( f \) is Lipschitz and has smooth extension up to the boundary locally. And this will imply the same behaviour on the whole boundary. The conformal mapping \( \Phi_p \) is a diffeomorphism and it is \( \mathcal{C}^{\infty}(\overline{D_p}) \), provided the boundary of \( D_p \) belongs to the same class. So we will go back to the original mapping easily.

In the previous part we have showed that every minimizing mapping can be lifted locally to a certain minimal surface. In the following part we show that in certain circumstances the lifting is global.

Every harmonic mapping \( f \) defined on the annulus \( \mathbb{A}_\rho \) can be expressed (see e.g. [7]) as
\[ f(z) = a_0 \log |z| + b_0 + \sum_{k \neq 0} (a_k z^k + \overline{b_k} \bar{z}^k). \]

Assume now that \( f \) is a diffeomorphic minimiser between \( \mathbb{A}_\rho \) and \( \Omega \) and that \( c < 0 \), i.e. \( \text{Mod}(\mathbb{A}_\rho) > \text{Mod}(\Omega) \) (see Proposition 1.6). If \( a_0 = 0 \), then we have the following decomposition \( f(z) = g(z) + h(z) \), where
\[ g(z) = \sum_{k \neq 0} a_k z^k, \]
and
\[ h(z) = \sum_{k \neq 0} b_k z^k. \]

Then we get the following conformal parametrization of a minimal surface \( \Sigma \), \( k : \mathbb{A}_\rho \to \Sigma \), defined by
\[ k(z) = \left( \Re(g + h), \Im(g - h), 2\sqrt{-c} \log \frac{1}{|z|} \right). \]

Let us close this subsection with the following explicit example. Let
\[ f(z) = \frac{r(R - r)}{(1 - r^2)^2} \bar{z} + \frac{(1 - rR)z}{1 - r^2}. \]
Then $f(z)$ is a harmonic mapping of the annuli $A_r$ onto $A_R$ that minimizes the Dirichlet energy ([1]). Further, under notation of this subsection we have

$$p(z) = \frac{1 - rR}{1 - r^2}$$

and

$$q(z) = \frac{\sqrt{r(r-R)(1-rR)}}{(1-r^2)z}.$$  

Put $k_1 = \Re f(z)$, $k_2(z) = \Im f(z)$ and assume that $\text{Mod}(A_r) > \text{Mod}(A_R)$, i.e. $R > r$. Then we have from (2.10) that

$$k_3(z) = \Re \int 2i g(z) dz = \Re \int \frac{2iz^2 \sqrt{r(r-r)(1-rR)}}{(1-r^2)z} dz$$

$$= 2 \frac{\sqrt{r(r-r)(1-rR)}}{(1-r^2)} \log \frac{1}{|z|}.$$

Here $\int g(z) dz$ stands for the primitive function of $g(z)$. It follows that (2.11) defines a global minimal surface by its conformal minimal coordinates $k(z) = (k_1(z), k_2(z), k_3(z))$. This minimal graph is a part of the upper slab of catenoid. (see Figure 1).

![Figure 1. A part of catenoid over an annulus. Here $R = 2/3$ and $r = 1/2$.](image)

We finish this section with a lemma needed in the sequel

**Lemma 2.4.** a) Assume that $\Phi$ is a holomorphic mapping of the unit disk into itself so that $\Phi(1) = 1$ and $\Phi$ has the derivative at 1. Then

$$\Phi'(1) \geq \frac{1 - |\Phi(0)|}{1 + |\Phi(0)|} > 0.$$  

b) Assume that $\Phi$ is a holomorphic mapping of the unit disk into the exterior of the disk $rU$ with $\Phi(1) = r$. Then

$$\Phi'(1) < r \frac{r - |\Phi(0)|}{|\Phi(0)| + r} < 0.$$
Proof of Lemma 2.4 Consider
\[ F(z) = \frac{(1 - \Phi(0))(\Phi(0) - \Phi(z))}{(1 - \Phi(0))(1 - \Phi(0))}. \]
Then
\[ F'(1) = 1 + \frac{1}{|\Phi(0)|} |\Phi'(1)|. \]
Since \( F'(0) = 0, F(1) = 1 \), it follows that \( F \) satisfies the boundary Schwarz lemma, and therefore \( F'(0) \) is a real positive number bigger or equal to 1. This implies a).

In order to prove b), consider the auxiliary function \( g(z) = \frac{r}{\Phi(z)} \). By applying a) to \( g \) we get
\[ g'(1) \geq \frac{1 - |g(0)|}{1 + |g(0)|}. \]
Since
\[ \Phi'(z) = -rf'(z)\Phi^2(z), \]
we get
\[ -r\Phi'(1) \geq \frac{1 - |g(0)|}{1 + |g(0)|} \]
and so
\[ -\Phi'(r) \geq r \frac{1 - \frac{r}{|\Phi(0)|}}{1 + \frac{r}{|\Phi(0)|}} = r \frac{|\Phi(0)| - r}{|\Phi(0)| + r}. \]
This finishes the proof. \( \square \)

3. Hölder property of minimisers

In this section we prove that the minimisers of the energy are Hölder continuous on the boundary provided that the boundary is \( C^{1,\alpha} \).

We first formulate the following result

Proposition 3.1 (Carathéodory theorem for \((K, K')\) mappings). [13] Let \( \Omega \) be a simply connected domain in \( \mathbb{C} \) whose boundary has at least two boundary points such that \( \infty \notin \partial D \). Let \( f : D \to D \) be a continuous mapping of the unit disk \( D \) onto \( \Omega \) and \((K, K')\) quasiconformal near the boundary \( \Gamma \).

Then \( f \) has a continuous extension up to the boundary if and only if \( \partial D \) is locally connected.

For \( a \in \mathbb{C} \) and \( r > 0 \), put \( D(a, r) := \{ z : |z - a| < r \} \) and define \( \Delta_r = \Delta_r(z_0) = U \cap D(z_0, r) \). Denote by \( k_\rho \) the circular arc whose trace is \( \{ \zeta \in U : |\zeta - \zeta_0| = \rho \} \).
Lemma 3.2 (The length-area principle). Assume that $f$ is a $(K, K')$–q.c. on $\Delta_r$, $0 < r < r_0 \leq 1$, $z_0 \in \mathbb{T}$. Then

$$F(r) := \int_0^r \frac{l^2}{r} d\tau \leq \pi KA(r) + \frac{\pi}{2} K'r^2,$$

where $l_\tau = |f(k_\tau)|$ denote the length of $f(k_\tau)$ and $A(r)$ is the area of $f(\Delta_r)$.

Let $\Gamma \in \mathcal{C}^{1,\mu}$, $0 < \mu \leq 1$, be a Jordan curve and let $g$ be the arc length parameterization of $\Gamma$ and let $l = |\Gamma|$ be the length of $\Gamma$. Let $d_\Gamma$ be the distance between $g(s)$ and $g(t)$ along the curve $\Gamma$, i.e.

$$d_\Gamma(g(s), g(t)) = \min\{|s-t|, (l-|s-t|)|\}.$$  

A closed rectifiable Jordan curve $\Gamma$ enjoys a $b$–chord-arc condition for some constant $b > 1$ if for all $z_1, z_2 \in \Gamma$ there holds the inequality

$$d_\Gamma(z_1, z_2) \leq b|z_1 - z_2|.$$  

It is clear that if $\Gamma \in \mathcal{C}^{1,\alpha}$ then $\Gamma$ enjoys a chord-arc condition for some $b = b_\alpha > 1$. In the following lemma we use the notation $\Omega(\Gamma)$ for a Jordan domain bounded by the Jordan curve $\Gamma$. Similarly, $\Omega(\Gamma, \Gamma_1)$ denotes the double connected domain between two Jordan curves $\Gamma$ and $\Gamma_1$, such that $\Gamma_1 \subset \Omega(\Gamma)$.

The following lemma is a $(K, K')$–quasiconformal version of Lemma 1. Moreover, here we give an explicit Hölder constant $L_\Gamma(K, K')$.

Lemma 3.3. Assume that the Jordan curves $\Gamma, \Gamma_1$ are in the class $\mathcal{C}^{1,\alpha}$. Then there is a constant $B > 1$ depending on $\Gamma$ and $\Gamma_1$ with the following property: for every $(K, K')$–q.c. mapping $f$ between the annulus $A_\rho$ and the double connected domain $\Omega = \Omega(\Gamma, \Gamma_1)$ there holds

$$|f(z_1) - f(z_2)| \leq L|z_1 - z_2|^\beta$$

for $z_1, z_2 \in \mathbb{T}$ and $z_1, z_2 \in r\mathbb{T}$ for $\beta = \frac{1}{K(1+2B)^2}$ and $L = L_\Gamma(K, K', B, \beta, \rho, f)$.

Proof: Let $\Phi$ be a conformal mapping of $\Omega(\Gamma)$ onto the unit disk, where $\Omega(\Gamma)$ is the Jordan domain bounded by $\Gamma$, so that $\Phi(f(1)) = 1$, $\Phi(f(e^{\pm i\frac{2\pi}{3}})) = e^{\pm i\frac{2\pi}{3}}$.

Then $\Phi \circ f$ is a normalized $(K_1, K'_1)$ quasiconformal mapping near $\mathbb{T} \subset \partial A_\rho$. For $a \in \mathbb{C}$ and $r > 0$, put $D(a, r) := \{z : |z-a| < r\}$. Since $\Phi$ is a diffeomorphism near $\mathbb{T}$, the inequality (3.4) will be proved for $f$ if we prove it for $\Phi \circ f$.

It is clear that if $z_0 \in \mathbb{T}$, then, because of normalization, $f(\mathbb{T} \cap \overline{D(z_0, 1)})$ has common points with at most two of three arcs $w_0w_1, w_1w_2$ and $w_2w_0$. (Here $w_0, w_1, w_2 \in \Gamma$ divide $\Gamma$ into three arcs with the same length such that $f(1) = w_0$, $f(e^{2\pi i/3}) = w_1$, $f(e^{4\pi i/3}) = w_2$, and $\mathbb{T} \cap \overline{D(z_0, 1)}$ do not intersect at least one of three arcs defined by $1, e^{2\pi i/3}$ and $e^{4\pi i/3}$).

Let $\kappa_\tau = \{t \in [0, 2\pi] : z_0 + \tau e^{\pm it} \in k_\tau\}$. Let $l_\tau = |f(k_\tau)|$ denotes the length of $f(k_\tau)$. Let $\Gamma_\tau := f(\mathbb{T} \cap D(z_0, \tau))$ and let $|\Gamma_\tau|$ be its length. Assume $w$ and $w'$ are the endpoints of $\Gamma_\tau$, i.e. of $f(k_\tau)$. Then $|\Gamma_\tau| = d_\Gamma(w, w')$ or $|\Gamma_\tau| = |\Gamma| - d_\Gamma(w, w')$. If the first case holds, then since $\Gamma$ enjoys the $B$–chord-arc condition,
it follows $|\Gamma_\tau| \leq B|w - w'| \leq B l_\tau$. Consider now the last case. Let $\Gamma'_\tau = \Gamma \setminus \Gamma_\tau$. Then $\Gamma'_\tau$ contains one of the arcs $w_0w_1$, $w_1w_2$, $w_2w_0$. Thus $|\Gamma_\tau| \leq 2|\Gamma'_\tau|$, and therefore

$$|\Gamma_\tau| \leq 2 B l_\tau.$$  

Using the first part of the proof, it follows that the length of boundary arc $\Gamma_\tau$ of $f(\Delta_\tau)$ does not exceed $2 B l_\tau$ which, according to the fact that $\partial f(\Delta_\tau) = \Gamma_\tau \cup f(k_\tau)$, implies

$$|\partial f(\Delta_\tau)| \leq l_\tau + 2 B l_\tau.$$  

Therefore, by the isoperimetric inequality

$$A(r) \leq \frac{|\partial f(\Delta_\tau)|^2}{4\pi} \leq \frac{(l_\tau + 2 B l_\tau)^2}{4\pi} = l_\tau^2 \frac{(1 + 2 B)^2}{4\pi}.$$  

Employing now (3.1) we obtain

$$F(r) := \int_0^r \frac{l_\tau^2}{\tau} \, d\tau \leq K l_\tau^2 \frac{(1 + 2 B)^2}{4} + \frac{\pi K'}{2} r^2.$$  

Observe that for $0 < r \leq 1 - \rho$ there holds $r F'(r) = l_\tau^2$. Thus

$$F(r) \leq K r F'(r) \frac{(1 + 2 B)^2}{4} + \frac{\pi K'}{2} r^2.$$  

Let $G$ be the solution of the equation

$$F(r) = K r F'(r) \frac{(1 + 2 B)^2}{4} + \frac{\pi K'}{2} r^2$$  

defined by

$$G(r) = \frac{\pi K'}{2 K (1 + 2 B)^2 + 1} r^2 = \frac{2 \pi K'}{K (1 + 2 B)^2 + 4} r^2.$$  

It follows that for

$$\beta = \frac{2}{K (1 + 2 B)^2}$$  

there holds

$$\frac{d}{dr} \log([F(r) - G(r)] \cdot r^{-2\beta}) \geq 0,$$

i.e. the function $[F(r) - G(r)] \cdot r^{-2\beta}$ is increasing. This yields

$$[F(r) - G(r)] \leq [F(1 - \rho) - G(1 - \rho)] (r/(1 - \rho))^{2\beta} \leq C(K, K', B, \beta, \rho, f) r^{2\beta}.$$  

Now for every $r \leq 1 - \rho$ there exists an $r_1 \in [r/\sqrt{2}, r]$ such that

$$F(r) = \int_0^r \frac{l_\tau^2}{\tau} \, d\tau \geq \int_{r/\sqrt{2}}^r \frac{l_\tau^2}{\tau} \, d\tau = l_{r_1}^2 \log \sqrt{2}.$$  

Hence,

$$l_{r_1}^2 \leq \frac{C_1(K, K', B, \beta, \rho, f)}{\log 2} r^{2\beta}.$$
If $z$ is a point with $|z| \leq 1$ and $|z - z_0| = r/\sqrt{2}$, then by (3.5)

$$|f(z) - f(z_0)| \leq (1 + 2B)l_1.$$ 

Therefore

$$|f(z) - f(z_0)| \leq H|z - z_0|\beta,$$

where

$$H = H(K, K', B, \beta, \rho, f).$$

Now for $z_1, z_2 \in T$, then the arch $(z_1, z_2)$ can be divided into $Q = Q(\rho)$ equal arcs by points $w_0, \ldots, w_Q$, so that $|w_i - w_{i+1}| \leq \rho$. Then we get the inequality

$$|f(z_1) - f(z_2)| \leq \sum_{j=1}^{Q} |f(w_j) - f(w_{j-1})| \leq QH|w_1 - w_2|\alpha \leq Q^2H|z_1 - z_2|\alpha.$$ 

Thus

$$|f(z_1) - f(z_2)| \leq L(K, K', B, \beta, \rho, f)|z_1 - z_2|\beta.$$ 

In order to deal with the inner boundary, we take the composition

$$F(z) = \frac{1}{f(\rho/z) - a},$$

which maps the annulus $\Lambda_\rho$ into $\Omega' = \{1/(z - a) : z \in \Omega\}$. Here $a$ is a point inside the inner Jordan curve. Then $\Omega' = \Omega'(\Gamma', \Gamma'_1)$ is a double connected domain with $C^{1,\alpha}$ boundary.

Now we construct a conformal mapping $\Phi_1$ between the domain $\Omega(\Gamma')$ and the unit disk and repeat the previous case in order to get that the inequality (3.4) does hold in both boundary components.

4. **Proof of the Main Theorem**

By repeating the proofs of corresponding result in [20] we can formulate the following result.

**Theorem 4.1.** Assume that $\Gamma$ is a Jordan curve in $\mathbb{R}^3$ and assume that $\vec{X}(z) = (X_1, X_2, X_3) : D \to \mathbb{R}^3$ is a minimal graph so that $\vec{X}(\Gamma) = \Gamma$. Assume that $\vec{X}$ is Hölder continuous in an arc $T_p \subset T$ containing $p$ in its interior. If the arc $T_p$ of $T$ is mapped onto the arc $\Gamma'_p \subset \Gamma$ so that $\Gamma'_p \in \mathcal{C}^{1,\alpha}$, then $\vec{X}$ is $\mathcal{C}^{1,\alpha}$ in a small neighborhood of $p$. i.e. in a domain $D_{p,\delta} = \{z : |z - p| < \delta, |z| \leq 1\}$.

The proof of Theorem 4.1 depends deeply on the proof of a similar statement in [20]. We observe that, almost all results proved in [20] are of local nature (see [20, Lemma 5, Lemma 6, Lemma 7]), thus we will not write the details here.

Since the minimising property is preserved under composing by a conformal mapping, in view of the the original Kellogg theorem [6], we can assume that the domain is $\Lambda_\rho = \{z : \rho < |z| < 1\}$.

On the other hand, the minimising harmonic mapping has the local representation (2.4). Here $\Phi_p$ is a $C^\infty$ diffeomorphism, and it does not cause any difficulty.
Let \( p \in \partial \mathbb{H}_\rho \) be arbitrary, say \(|p| = 1\) (the other possibility is \(|p| = \rho\)). Because the boundary mapping is continuous and monotone, in view of Lemma 1.8 it follows that, there is a neighborhood \( T_p \) which is mapped onto the arc \( \Gamma_p \subset \partial \mathbb{D} \). Therefore by Theorem 4.1 having in mind the notation from subsection 2.1 the mapping

\[
\tilde{X}(z) = \tilde{X}_p(z) = \{ F_p(z), \exists f_p(z), \Re(2i\sqrt{\log \Phi_p(z)}) \}
\]

is \( C^{1,\alpha} \) in a neighborhood of \( p \), provided the boundary arc is of the same class. But we do not know that \( \tilde{X}(T_p) \in \mathcal{C}^{1,\alpha} \). We only know that \( \Phi_p \) is a priory in \( C^\infty(\mathbb{D}) \) and \( \delta_p = f_p(T_p) \in \mathcal{C}^{1,\alpha} \). This will be enough for the proof.

4.1. Proof of Lipschitz continuity. We use the notation from Subsection 2.1. Assume \( f \) is a minimiser of the energy between two double connected domains \( \mathbb{H}_\rho \) and \( \Omega \). For \( p \in \partial \mathbb{H}_\rho \), let \( \Phi = \Phi_p \) be a conformal mapping of the unit disk onto a smooth domain \( D_p \) in \( \mathbb{H}_\rho \), containing a Jordan arc \( T_p \) and assume that \( p \) is an inner point of \( T_p \). Let \( f_p(z) = g(z) + \overline{h(z)} = f(\Phi_p(z)) \). Enlarging \( T_p \) if necessary, we can assume that \( \delta_p = f(T_p) \) is a Jordan arc containing \( q = f(p) \) in its interior part. Let \( \gamma : [-l/2, l/2] \to \delta_p \) be the arc-length parametrization of \( \delta_p \). Then there is a function \( \psi : [0, l_0] \to [-l/2, l/2] \), where \( l_0 = |T_p| \), so that

\[
f_p(e^{it}) = f \circ \Phi_p(e^{it}) = \gamma(\psi(t)).
\]

because \( \delta_p \in C^{1,\alpha} \) we have as in \([20]\) eq. 3, the following relation

\[
|\gamma(s) - \gamma(t)| \leq C|s-t|\{\min\{|s|^{\alpha}, |t|^{\alpha}\} + |t-s|^{\alpha}\}, \quad |t| < l/2, |s| < l/2.
\]

Assume that \( p = 1 \) and \( f(1) = 0 \). By using translations and rotations in the domain and image domain, we will obtain this property, and therefore we do not loose the generality. Now, an important is the following, which follows from (4.2) and Lemma 3.3

\[
|f_p(e^{it}) - f_p(1)| = |f_p(e^{it})| = |\gamma(\psi(t))| \leq C|\psi(t)|^{1+\alpha} \leq C'|t|^{\beta(1+\alpha)}
\]

and

\[
|F(e^{it}) - F(e^{is})| = |\gamma(\psi(t)) - \gamma(\psi(s))|
\leq C|\psi(s) - \psi(t)|\{\min\{|\psi(s)|^{\alpha}, |\psi(t)|^{\alpha}\} + |\psi(t) - \psi(s)|^{\alpha}\}.
\]

and so

\[
|f_p(e^{it}) - f_p(e^{is})| = |\gamma(\psi(t)) - \gamma(\psi(s))|
\leq CL^{1+\alpha}|s-t|^{\beta}\{\min\{|s|^{\alpha\beta}, |t|^{\beta\alpha}\} + |t-s|^{\beta\alpha}\}.
\]

By repeating the proof of \([20]\) Lemma 7 we have:

**Lemma 4.2.** Assume that \( f = P[f^*] \), is a bounded harmonic minimiser defined in the unit disk. Denote \( f^* \) by \( f \). Assume that \( f \) is defined on the unit circle \( T \)
and is bounded by $M$. Further assume that for some $t_0, \eta, \mu \leq \pi/2$ so that for $-t_0 \leq t, s \leq t_0$ we have

$$|f(t) - f(s)| \leq M|t - s|^\mu \{\min\{|t|^\eta, |s|^\eta\} + |t - s|^\eta\}.$$ 

Then we have the estimates

$$|f(z)| \leq \begin{cases} M_1|z|^\eta(1 - \rho)^{\mu - 1} + M_2(1 - \rho)^{\mu + \eta - 1} + M_3, & \text{if } \mu + \eta < 1; \\ M_1|z|^\eta(1 - \rho)^{\mu - 1} + M_2 \log \frac{1}{1 - \rho} + M_3, & \mu + \eta = 1; \\ M_1|z|^\eta(1 - \rho)^{\mu - 1} + M_2, & \mu < 1 \text{ and } \mu + \eta > 1; \\ M_1|z|^\eta \log \frac{1}{1 - \rho} + M_3, & \mu = 1; \\ M_1, & \mu > 1. \end{cases}$$

(4.7)

$$|f(\rho) - f(1)| \leq \begin{cases} N(1 - \rho)^{\mu + \eta}, & \mu + \eta < 1; \\ N(1 - \rho) \log \frac{1}{1 - \rho}, & \mu + \eta = 1; \\ N(1 - \rho), & \mu + \eta > 1. \end{cases}$$

(4.5)

and

$$|f(e^{i\theta}) - f(1)| \leq \begin{cases} N|\rho|^{\mu + \eta}, & \mu + \eta < 1; \\ N|\rho| \log \frac{1}{|\rho|}, & \mu + \eta = 1; \\ N|\rho|, & \mu + \eta > 1. \end{cases}$$

(4.6)

Here $N, M_1, M_2, M_3$ depends on $K, K', \eta, \mu$

Now we get

$$|Df_{p}(\rho)| \leq C_2(1 - \rho)^{(1 + \alpha)\beta - 1},$$

for $1/2 \leq \rho < 1$. Since $w = 1$ is not a special point, we get that

$$|Df_{p}(z)| \leq C_2(1 - |z|)^{(1 + \alpha)\beta - 1},$$

for all $z$ near the boundary $|z| = 1$. A similar inequality does hold for the points $z$ so that $|z| = \rho$. By repeating the proof of the theorem of Hardy and Littlewood, [6, Theorem 3, p 411], we can state the following.

**Theorem 4.3.** Assume that $f$ is a harmonic mapping defined in the unit disk so that

$$|Df(z)| \leq M(1 - |w|)^{\mu - 1},$$

where $0 < \mu < 1$ and $z \in D_\epsilon$. Then the radial limit

$$\lim_{\rho \rightarrow 1^{-}} f(\rho e^{i\theta}) = f(e^{i\theta})$$

exists for every $\theta \in (-\epsilon, \epsilon)$ and we have there the inequality

$$|f(w) - f(w')| \leq N|w - w'|^\mu, \quad w, w' \in U_\epsilon,$$

where $N$ depends on $M$ and $\mu$.

We now reformulate a result of Privalov [6, p. 414, Theorem 5] in its local form (w.r.t. the boundary).
Theorem 4.4. Assume that \( f = u + iv \) is a holomorphic bounded function defined on the unit disk \( D \) and assume that \( u \) is Hölder continuous on \( |\theta - \theta_0| \leq 2\epsilon \), i.e.
\[
|u(e^{it}) - u(e^{is})| \leq M|e^{it} - e^{is}|^\alpha,
\]
for \( |s - \theta_0| < 2\epsilon \) and \( |t - \theta_0| < 2\epsilon \), then there is a small neighborhood \( U_\epsilon \) of \( e^{i\theta_0} \) in \( D \) so that \( |f(z) - f(w)| \leq M|z - w|^\alpha \) for \( z, w \in U_\epsilon. \)

Proof of Theorem 4.4 From Schwarz formula we have
\[
f(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) \frac{e^{it} + \zeta}{e^{it} - \zeta} dt + iC.
\]
Thus
\[
f'(\zeta) = \frac{2}{2\pi} \int_0^{2\pi} \frac{u(e^{it}) e^{it} dt}{(e^{it} - \zeta)^2} = \frac{1}{\pi} \int_0^{2\pi} \frac{u(e^{it}) - u(e^{is})}{(e^{it} - \zeta)^2} e^{it} dt,
\]
\( \zeta = re^{is} \).

Let
\[
U_\epsilon = \{ z = re^{is} : 1 - \epsilon \leq r \leq 1, \ |s - \theta_0| \leq \epsilon \}.
\]

Then we get
\[
|f'(\zeta)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|u(e^{is+it}) - u(e^{is})|}{1 - 2r \cos t + r^2} dt.
\]

Now for \( \zeta \in D_\epsilon \), we write \( [-\pi, \pi] = [-\pi, \pi] \setminus (-\epsilon, \epsilon) \cup (-\epsilon, \epsilon) \). If \( t \in [-\pi, \pi] \),
\[
1 - 2r \cos t + r^2 \geq (1 - r)^2 + \frac{4r}{\pi^2} t^2.
\]

Further, if \( s \in (\theta_0 - \epsilon, \theta_0 + \epsilon) \), \( t \in (-\epsilon, \epsilon) \) then we get
\[
|u(e^{is+it}) - u(e^{is})| \leq K|t|^\alpha.
\]

If \( t \in [-\pi, \pi] \setminus (-\epsilon, \epsilon) \),
\[
|u(e^{is+it}) - u(e^{is})| \leq 2M = \frac{2M}{\epsilon^\alpha}$
\[
|f'(\zeta)| \leq \frac{N}{(1 - |\zeta|)^{1 - \alpha}},
\]
for \( \zeta \in U_\epsilon \). Then from Theorem 4.3 we get the desired result.

4.1.1. Proof of Lipschitz continuity. Then from (4.3) and Theorem 4.3 we get that \( f_p \) is \( C^0,(1+\alpha)^\beta \) in a small neighborhood of \( \Phi_p(p) \) which is subset of \( T_p \subset T \). Chose \( \beta < 1 \) so that none of numbers \( (1 + \alpha)^k \beta \) is equal to 1 for every \( k \). Let \( n \) be so that \( (1 + \alpha)^n \beta < 1 < (1 + \alpha)^{n+1} \beta \). Then by successive application of the previous procedure we get
\[
|Df_p(z)| \leq M(1 - |z|)^{(1+\alpha)^n \beta - 1}, \rho_1 < |z| < 1,
\]
where \( \rho_1 = (1 + \rho)/2 \). Then we get
\[
|f_p(w) - f_p(w')| \leq N|w - w'|^{(1+\alpha)^n \beta}, \ w, w' \in D_\epsilon,
\]
where $N$ depends on $M$ and $\mu$ and so

$$|f_p(e^{it}) - f_p(e^{is})| \leq N|s - t|(1+\alpha)^n \beta,$$

for $|s - t_0| < \epsilon$, $|s - t_0| < \epsilon$ and

$$|f_p(e^{it}) - f_p(e^{is})| \leq C L^{1+\alpha}|s - t|(1+\alpha)^n \beta \{\min\{|s|(1+\alpha)^n \alpha \beta, |t|(1+\alpha)^n \alpha \beta\} + |t - s|(1+\alpha)^n \alpha \beta\}.$$  

From Lemma 4.2, for $\mu = (1 + \alpha)^n \beta$ and $\eta = (1 + \alpha)^n \alpha \beta$ we get

$$|f_\zeta(\rho)| \leq M_1.$$  

Since

$$\partial_r f_p = e^{i\theta} \partial_\zeta f_p + e^{-i\theta} \partial_{\overline{\zeta}} f_p,$$

by taking $s = 0$ in the previous relation we get

$$|\partial_r f_p(re^{it})| \leq M.$$  

Since

$$|Df_p(z)| \leq K |\partial_r f_p(re^{it})| + \sqrt{K'},$$

we get that $f$ is Lipschitz near $p$ say in $T_p$.  

Since

$$\partial \Omega = \bigcup_{p \in \partial \Lambda} \delta_p,$$

there is a finite family of $C^{1,\alpha}$ Jordan domains

$$\{D_k = D_{pk}, \ k = 1, \ldots, m\},$$

so that

$$\partial \Lambda \subset \bigcup_{k=1}^m T_{pk},$$

and

$$\partial \Omega \subset \bigcup_{k=1}^m \delta_{pk}.$$  

Here $\delta_{pk} = f(T_{pk})$ (see subsection 2.1). Since $f \circ \Phi_{pk}$ is $C^{1,\alpha}(\overline{T_p})$, by compactness property of $\partial \Lambda$, we get that $f$ is Lipschitz in a neighborhood of $\partial \Lambda$ w.r.t. $\Lambda$.  

Since $f \in C^\infty(\Lambda)$ we get $f \in C^{0,1}(\overline{\Lambda})$ as claimed.

5. THE MINISER IS $C^{1,\alpha}$ UP TO THE BOUNDARY

The constant $C$ that appear in the proof is not the same and its value can vary from one to the another appearance. Assume that $f = u + iv : \Lambda \to \Omega$ is a minimiser, where $\Lambda = \{z : \rho < |z| < 1\}$. Then it is $C^{1,\alpha}$, provided that $\partial \Omega \in C^{1,\alpha}$. Assume that $q \in \Gamma = \partial \Omega$, and assume without loss of generality that $q = 0$, and the tangent line at $q$ is the real axis. Then in a small neighborhood of $q$, $\Gamma$ has the following parametrisation $G(x) = (x, \phi(x))$, $x \in (-\epsilon, \epsilon)$, so that $\phi(0) = \phi'(0) = 0$. Assume as we may that, $p = 1$ and $f(1) = q = 0$. And assume that for a small angle $\Lambda : |e^{i\theta} - 1| < \delta$ we have $f(\Lambda) = G(-\epsilon, \epsilon)$. We want to
localize the problem. We only need to prove that $f$ is $C^{1,\alpha}$ in a small neighborhood of 1. We also work with $f_p = f \circ \Phi_p : U \to U_p$ instead of $f$, where $\Phi_p(1) = 1$ as in the previous part of the paper. We will show that $f_p \in C^{1,\alpha}(U_p)$, where $U_p$ is a small neighborhood of $p$. We will from time to time use notation $f$ instead of $f_p$, since they behave in the same way in a small neighborhood of $p$, because $\Phi_p$ is a priori in $C^\infty$.

Thus, there exists a function $x : \alpha \to \mathbb{R}$ so that $f(e^{it}) = G(x(e^{it}))$.

We can also assume that

$$\partial_t x(e^{it}) \geq 0$$

for almost every $t$, because $f$ is a restriction of a harmonic diffeomorphism between domains and consequently it is monotone at the boundary.

We proved that $f$ is Lipschitz continuous. We know as well that $\phi \in C^{1,\alpha}$. Thus we have $x$ is Lipschitz continuous.

Since

$$\frac{|\phi(x) - \phi(0) - \phi'(0)x|}{|x|^{1+\alpha}} = \frac{|\phi'(\theta x) - \phi'(0)|}{|x|^\alpha} \leq C,$$

where $\theta \in (0, 1)$. Since

$$v(\theta) = v(e^{i\theta}) = \phi(x(e^{i\theta}))$$

we get

$$|v(e^{i\theta}) - v(1)| = |\phi(x(e^{i\theta}))| \leq C|x(e^{i\theta})|^{1+\alpha}.$$  

It follows from (5.3) that $v$ is differentiable with respect to $\theta$ for $\theta = 0$, i.e. in 1 and

$$\partial_\theta v(1) = \partial_\theta v(e^{i\theta})|_{\theta=0} = 0.$$  

Therefore

$$|\partial_\theta v(e^{i\theta}) - \partial_\theta v(1)| = |\partial_\theta v(e^{i\theta})|$$

$$= |\phi'(x(e^{i\theta}))| \cdot |\partial_\theta x(e^{i\theta})|$$

$$\leq C|(x(e^{i\theta}))| \leq C|\theta|^\alpha$$

Now we have $v = \Im(f) = \Im(g + \bar{h}) = \Im(g - h)$ and therefore,

$$v_\theta = \Im(iz(g' - h')) = \Re(z(h' - g')).$$

From (5.6), by repeating the proof of Theorem 4.4 we conclude that

$$|((h' - g')'(\rho))| \leq C(1 - \rho)^{\alpha-1}, \quad 1/2 \leq \rho \leq 1.$$  

Let $k_1(z) = (g'(z) + h'(z)), \ k_2(z) = i(h'(z) - g'(z)).$
In view of (2.3)

\[ k_3(z) = 2i \sqrt{h'(z)g'(z)} = 2i \sqrt{e^{\left(\Phi_p'(z)\right)^2/\Phi_p(z)}}. \]

Then from (5.7) we have that following limit

\[ k_2(1) := \lim_{\rho \to 1} k_2(\rho) \]

exists. Moreover we have

(5.8) \[ |k_2(1) - k_2(\rho)| \leq C(1 - \rho)^\alpha, \quad 1/2 \leq \rho \leq 1. \]

We conclude that

\[ k_3(1) = 2i \lim_{r \to 1} \sqrt{h'(r)g'(r)} \]

exists and

(5.9) \[ |k_3(1) - k_3(\rho)| \leq C|1 - \rho|^\alpha, \quad 1/2 \leq \rho < 1. \]

Further since

\[ x(e^{i\theta}) = \Re(f(e^{i\theta})) = \Re(g + h), \]

we get

\[ u'(0) = \partial_\theta x(1) = \Re(i(g'(1) + h'(1))) \geq 0. \]

Then the following equality is crucial in our approach

(5.10) \[ k_1^2 + k_2^2 + k_3^2 = 0. \]

We now proceed as J. C. C. Nitsche did in [20]. So \( k_1^2(\rho) = -k_2^2(\rho) - k_3^2(\rho) \).

It follows that the following limit

\[ k_1^2(1) := \lim_{\rho \to 1} k_1^2(\rho) = -k_2^2(1) - k_3^2(1), \]

exists. Therefore we get

\[ |k_1(1)^2 - k_1(\rho)^2| = |k_2^2(1) - k_2^2(\rho) + k_3^2(1) - k_3^2(\rho)|. \]

Then from (5.8) and (5.9) we get

(5.11) \[ |k_1(\rho)^2 - k_1(1)^2| \leq C|\rho - 1|^\alpha \equiv \epsilon, \quad 1/2 \leq \rho < 1. \]

From (5.4) we get

\[ \Re(k_2(1)) = 0. \]

Further, from (5.10), we have

\[ \Re(k_1(1))\Im(k_1(1)) + \Re(k_2(1))\Im(k_2(1)) + \Re(k_3(1))\Im(k_3(1)) = 0 \]

and

(5.12) \[ \Re^2(k_1(1)) + \Re^2(k_2(1)) + \Re^2(k_3(1)) = \Im^2(k_1(1)) + \Im^2(k_2(1)) + \Im^2(k_3(1)). \]

From Lemma 2.4 it follows that \( k_3(1) \) is a real or an imaginary number. Therefore we have \( \Re(k_3(1))\Im(k_3(1)) = 0 \), and thus \( \Re(k_1(1))\Im(k_1(1)) = 0 \). But it cannot be \( \Im(k_1(1)) \neq 0 \), because in that case \( \Re(k_1(1)) = 0 \), and therefore by (5.12), we get
where \( t \) (5.14) \(|\alpha| < 1\) we get the inequalities we have a priori this inequality for \( k \) \varepsilon > 0\).

From (5.8), (5.9) and (5.13), and remembering that (5.15) \(|k| = 0\) we get the next estimate

\[
\begin{align*}
|k_3(e^{it}) - k_3(e^{is})| &\leq C|s - t|^{\alpha/2},
\end{align*}
\]

where \( t \in (-t_0, t_0) \).

Further as in (20) we obtain that

\[
|k_j(e^{it}) - k_j(e^{is})| \leq C|s - t|^{\alpha/2}, j = 1, 2
\]

and \( t, s \in (-t_0, t_0) \). The same behavior has \( k_3 \) a priori. From this it follows that

\[
\begin{align*}
k_j &\in C^{0, \alpha/2} (D_w),
\end{align*}
\]

Since

\[
v(e^{i\theta}) = \phi(x(e^{i\theta})),
\]

we get

\[
v_\theta(e^{i\theta}) = \phi'(x(e^{i\theta}))x_\theta(e^{i\theta})
\]

and thus for \( t, s \in (-t_0, t_0) \)

\[
\begin{align*}
|v_\theta(e^{is}) - v_\theta(e^{it})| &= |\phi'(x(e^{is}))x_\theta(e^{is}) - \phi'(x(e^{it}))x_\theta(e^{it})|
\end{align*}
\]

\[
\leq |\phi'(x(e^{is})) - \phi'(x(e^{it}))| \cdot |x_\theta(e^{is})| + |\phi'(x(e^{it}))| |x_\theta(e^{is}) - x_\theta(e^{it})|
\]

(5.16)

and

\[
\begin{align*}
|v_\theta(e^{it}) - v_\theta(e^{is})| &= |\phi'(x(e^{it}))x_\theta(e^{it}) - \phi'(x(e^{is}))x_\theta(e^{is})|
\end{align*}
\]

\[
\leq |\phi'(x(e^{it})) - \phi'(x(e^{is}))| \cdot |x_\theta(e^{it})| + |\phi'(x(e^{is}))| |x_\theta(e^{it}) - x_\theta(e^{is})|
\]

(5.17)

Therefore by using (5.16), (5.17) and (5.15) we get

\[
|v_\theta(e^{is}) - v_\theta(e^{it})| \leq C \left( |s - t|^{\alpha} + |s - t|^{\alpha/2} \min \{|t|^{\alpha}, |s|^{\alpha} \} \right).
\]

By using Lemma 7, once for \( \mu = \alpha, \eta = 0 \), and the second time for \( \mu = \alpha/(\alpha + 2) \) and \( \eta = \alpha \) we get

\[
|k_2(e^{is}) - k_2(1)| \leq C|s|^{\alpha}, \quad |s| \leq s_0/2
\]

We have a priori this inequality for \( k_3 \).

As in (20) we get the next estimate
\[ |k_1^2(e^{is}) - k_1^2(1)| \leq \epsilon^2 + 2\epsilon|k_1(1)|, \]

for \( \epsilon = C|s|^\alpha \).

Now we put \( w_1 = k_1(e^{is}) \) and \( w_2 = k_1(1) \) in the following lemma.

**Lemma 5.2.** [20] Let \( w_1 = a + ib \) and \( w_2 = c + id \) be complex numbers satisfying the inequalities \( a \geq 0 \) and \( c \geq 0 \) and \( |w_1^2 - w_2^2| \leq \epsilon^2 + 2\epsilon c \) for some \( \epsilon > 0 \). Then
\[
|w_1 - w_2| \leq 5\epsilon.
\]

In view of (5.19), Lemma 5.2 implies
\[
|k_1(e^{is}) - k_1(1)| \leq C|s|, |s| \leq s_0/2.
\]

Since \( w = 1 \) is not a point with a special geometric character, we conclude that \( k_1, k_2 \in C^{0,\alpha}(\mathbb{D}_w) \). Thus
\[
h(z) = \frac{1}{2} \int_0^z (k_1(\zeta) + ik_2(\zeta))d\zeta
\]
and
\[
g(z) = \frac{1}{2} \int_0^z (k_1(\zeta) - ik_2(\zeta))d\zeta
\]
belongs to the class \( C^{1,\alpha} \) in a small neighborhood of \( p \), say \( T_p \subset \partial \mathcal{A}_p \) containing \( p \) in its interior. By using compactness property as in the proof of Lipschitz continuity, we get that \( f \in C^{1,\alpha}(\mathcal{A}_p) \) as claimed.

Thus we have finished the proof of Theorem 2.1.

6. CONCLUDING REMARK

We expect that the following statement is true:

**Conjecture 6.1.** If \( \text{Mod}(D) \leq \text{Mod}(\Omega) \) then the diffeomorphic minimiser of Dirichlet energy has a \( C^{1,\alpha} \) diffeomorphic extension up to the boundary, provided \( D \) and \( \Omega \) have \( C^{1,\alpha} \) boundary.

This conjecture is motivated by the existing result described in Proposition 1.5 and the example presented in [21] of the unique minimiser (up to the rotation) of Dirichlet energy between annuli \( \mathcal{A}_r \) and \( \mathcal{A}_R \), that maps the outer boundary onto the outer boundary (see [1] for details). The mapping is a a diffeomorphism between \( \overline{\mathcal{A}_r} \) and \( \overline{\mathcal{A}_R} \), provided that
\[
R < \frac{2r}{1 + r^2}.
\]

If \( R = \frac{2r}{1 + r^2} \), and \( 0 < r < 1 \), then the mapping \( w(z) = \frac{r^2 + |z|^2}{2(1 + r^2)} \) is a harmonic minimiser (see [1]) of the Euclidean energy of mappings between \( \mathcal{A}(r, 1) \) and \( \mathcal{A}(\frac{2r}{1 + r^2}, 1) \), however \( |w_z| = |w_{\overline{z}}| = \frac{1}{1 + r^2} \) for \( |z| = r \), and so \( w \) is not bi-Lipschitz.

Note that (6.1) is satisfied provided that \( \text{Mod} \mathcal{A}_r \leq \text{Mod} \mathcal{A}_R \). The inequality (6.1) (with \( \leq \) instead of \( < \)) is necessary and sufficient for the existence of a harmonic diffeomorphism between \( \mathcal{A}_r \) and \( \mathcal{A}_R \) a conjecture raised by J. C. C. Nitsche.
in [21] and proved by Iwaniec, Kovalev and Onninen in [7], after some partial results given by Lyzzaik [19], Weitsman [27] and Kalaj [14]. If
\[ R > \frac{2r}{1 + r^2}, \]
then the minimiser of Dirichlet energy throughout the deformations \( D(A_r, A_R) \) is not a diffeomorphism (see [1] and [2, Example 1.2]).

We want to refer to one more interesting behavior that minimisers of energy share with conformal mappings. Namely, if \( f \) is a diffeomorphic minimiser of Dirichlet energy between the domains \( A_\rho \) and \( \Omega(\Gamma, \Gamma_1) \) so that \( \Gamma \) and \( \Gamma_1 \) are convex, then \( f(t^T) \) is convex for \( t \in (\rho, 1) \) [16]. Further if \( \Gamma \) and \( \Gamma_1 \) are circles, then \( f(t^T) \) is a circle [17].

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