An Osgood’s criterion for a semilinear stochastic differential equation

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Abstract

The purpose of this paper is to give an Osgood’s criterion for solutions of semilinear stochastic differential equations of the form
\[ X_t = \xi + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s)X_s dW_s, \quad t \geq 0. \]
Here, \( b \) is a non-negative, non-decreasing by components and continuous random field and \( \sigma \) is a predictable and continuous process. Also we present a generalization of the so-called Feller’s test whenever \( \sigma \equiv 1 \).

Keywords and phrases: explosion time, semilinear stochastic differential equations, Feller’s test, Osgood’s criterion

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1 Introduction

In the case of an ordinary differential equation (ODE), the explosion in finite time is a very old and well-known subject. In fact, in 1898, W.F. Osgood (see [9]) established that the solution $y$ of

$$y'(t) = b(y(t)), \quad t > 0,$$

$$y(0) = \xi,$$

with $b > 0$, blows up in finite time if and only if

$$\int_{\xi}^{\infty} \frac{ds}{b(s)} < \infty.$$

Moreover, (2) is the explosion time of the equation (1).

For the case of partial differential equations (PDE) the study of the phenomenon of explosion was originated with the works of S. Kaplan [6] and H. Fujita [4] and it is currently an area of very fruitful research, see for instance [5], [11].

On the other hand, within the stochastic framework, since Feller’s test [3] for determining the explosion time of the autonomous stochastic differential equation (SDE)

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad t > 0,$$

$$X_0 = \xi,$$

only few results have been developed, see for instance [8]. However, the growing interest in the application of the explosion of SDE has motivated its study. For example, when $b$ and $\sigma$ are power functions the equation (3) can be used for modeling the crack failure of some materials (see for example [12]). Also some numerical schemes have been analyzed in order to approximate the time of explosion (consult Dávila et al. [11]).

For the non-autonomous case, Feller’s test and Osgood’s criterion are not useful anymore. The contribution of this work is to deal with an extension of the Osgood’s criterion for the blow up in finite time of the solution $X$ of the semilinear stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t)X_t dW_t, \quad t > 0,$$

$$X_0 = \xi.$$
This paper is organized as follows. In Section 2 we precise the Osgood’s criterion and then present some necessary results in order to prove an extension of Osgood’s criterion for non-autonomous equations. The criterion of blow up in finite time for the equation (4) is stated and proved in Section 3.

2 Preliminaries

In this section for the convenience of the reader we briefly introduce three well-known topics: the Feller’s test, a comparison lemma and the Osgood’s criterion. We also present an extension of the latter.

Conditions necessary to determine whether or not the solution \( X \) of an equation as (3) explodes in finite time with probability 1 have been developed, most notably by William Feller [3] (with \( \xi \) a real number). The Feller’s explosion test just needs to know the coefficients \( b \) and \( \sigma \) of the equation.

**Theorem 1 (Feller’s test)** Suppose that \( b, \sigma : (\ell, r) \to \mathbb{R} \), with \(-\infty \leq \ell < r \leq \infty\), are continuous functions and \( \sigma^2 > 0 \) in \((\ell, r)\). The explosion time \( \tau \) of the solution \( X \) of equation (3) is finite with probability 1 if and only if one of the following conditions holds:

(i) \( v(r-) < \infty \) and \( v(\ell+) < \infty \),

(ii) \( v(r-) < \infty \) and \( p(\ell+) = -\infty \), or

(iii) \( v(\ell+) < \infty \) and \( p(r-) = \infty \),

where

\[
\begin{align*}
 p(x) &= \int_{\zeta}^{x} \exp \left( -2 \int_{\zeta}^{s} \frac{b(r)dr}{\sigma^2(r)} \right) ds, \\
v(x) &= \int_{\zeta}^{x} p'(y) \int_{\zeta}^{y} \frac{2dz}{p'(z)\sigma^2(z)}dy,
\end{align*}
\]

(5) (6)

here \( \zeta \in (\ell, r) \) is a constant.

**Proof.** See in Chapter 5 of [7] the Proposition 5.32.

In the case where the coefficient \( b \) is non-negative and the term \( \sigma \) is zero, the equation (3) becomes an ordinary differential equation for which the
criterion of explosion is known as Osgood’s criterion. To establish this result we introduce the following function

\[ B_\xi(x) = \int_\xi^x \frac{ds}{b(s)}, \quad \xi \leq x \leq \infty. \]

**Proposition 2 (Osgood’s criterion)** Let \( b : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \( b > 0 \) in \((c, \infty)\) and \( \xi > c \in \mathbb{R} \). If \( y \) is a solution of the integral equation

\[ y(t) = \xi + \int_0^t b(y(s)) ds, \quad t \geq 0, \]

then the explosion time \( T_e := \sup\{t > 0 : |y(t)| < \infty\} \) of \( y \) is finite if and only if \( B_\xi(\infty) < \infty \). Moreover the solution must be

\[ y(t) = B^{-1}_\xi(t), \quad 0 \leq t < B_\xi(\infty) = T_e. \]

We use the following notation, if \( X \) is the solution of certain equation with initial condition \( x_0 \), then by \( T^X_{x_0} \) we will denote the time of explosion of \( X \).

The following comparing result will be essential in our study of the behavior of semilinear SDE.

**Lemma 3** Let \( b : \mathbb{R} \to \mathbb{R} \) be a continuous non-negative function. Also assume that \( b \) is non-decreasing and positive in \((c, \infty)\), \( \xi > c \in \mathbb{R} \) and \( x, y : [0, T] \to \mathbb{R} \) are two continuous functions:

(i) If

\[ y(t) \geq \xi + \int_0^t b(y(s)) ds, \quad t \in [0, T], \quad (7) \]

and

\[ x(t) = \xi + \int_0^t b(x(s)) ds, \quad t \in [0, T], \]

then \( y(t) \geq x(t) \), for all \( t \in [0, T] \).

(ii) Moreover, if we assume \( y > c \) on \([0, T]\),

\[ y(t) \leq \xi + \int_0^t b(y(s)) ds, \quad t \in [0, T], \]
and
\[ x(t) = \xi + \int_0^t b(x(s))ds, \quad t \in [0, T], \]
then \( y(t) \leq x(t) \), for all \( t \in [0, T] \).

**Proof. Case (i):** Let \( 0 < r < \xi - c \) and consider the solution \( x_r \) of
\[ x_r(t) = \xi - r + \int_0^t b(x_r(s))ds, \quad t \in [0, T]. \]
Define \( N_r \) as the set \( \{ t \in [0, T] : x_r(s) \leq y(s), \forall s \in [0, t] \} \). Observe that \( N_r \neq \emptyset \) (indeed \( 0 \in N_r \)). Since \( b \geq 0 \), then \( x_r \geq \xi - r \) on \( [0, T] \). Using that \( b \) is non-decreasing in \((c, \infty)\) we have that \( b(y(s)) \geq b(x_r(s)) \) if \( y(s) \geq x_r(s) > c \).

Let us see that \( T_r := \sup N_r < T \) is not possible. In fact, since
\[
\lim_{\varepsilon \downarrow 0} (y(T_r + \varepsilon) - x_r(T_r + \varepsilon)) \geq r + \int_0^{T_r} [b(y(s)) - b(x_r(s))]ds \\
+ \lim_{\varepsilon \downarrow 0} \int_{T_r}^{T_r+\varepsilon} [b(y(s)) - b(x_r(s))]ds \\
\geq r > 0
\]
then the continuity of \( y - x_r \) implies \( T_r < \sup N_r \). Therefore \( x_r \leq y \) on \([0, T]\).

On the other hand, by Proposition 2 we have
\[ B_{\xi}^{-1}(t) = \lim_{r \downarrow 0} B_{\xi-r}^{-1}(t) = \lim_{r \downarrow 0} x_r(t) \leq y(t), \quad t \in [0, T]. \]

To get the first equality, in the above expression, we have used that the continuity of \( B(t) \) implies the continuity of \( B^{-1}(t) \).

**Case (ii):** The proof is like the previous case, but now it is convenient consider the solution \( x_r \) of
\[ x_r(t) = \xi + r + \int_0^t b(x_r(s))ds, \quad t \in [0, T], \]
where \( r > 0 \).

In Theorem 2.2.4 of \([10] \) the interested reader can see other version of Lemma 3 (in \([10] \) the function \( b \) is supposed to be monotone throughout its domain).

Using the last two results we can state the following extension of Osgood’s criterion for non-autonomous equations, which is important in itself.

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**Proposition 4** Let $b : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function. We also assume that $b$ is positive and non-decreasing by components on $[0, \infty) \times (c, \infty)$, with $c \in \mathbb{R}$. Then, a solution of the equation

$$
y'(t) = b(t, y(t)), \quad t > 0,
$$

$$
y(0) = \xi,
$$

where $\xi > c$, explodes in finite time if and only if

$$
\int_{\xi}^{\infty} \frac{ds}{b(a, s)} < \infty,
$$

for some $a > 0$.

**Proof.** Suppose that $y$ is a solution of (8) and that explodes at time $T_\xi^y < \infty$. Inasmuch as $b(\cdot, x)$ is non-decreasing we obtain

$$
y(t) \leq \xi + \int_0^t b(T_\xi^y, y(s)) ds, \quad t < T_\xi^y.
$$

Using that $b \geq 0$ we see, from (8), that $y$ is non-decreasing. Therefore $y(0) = \xi$ imply $y > c$ on $[0, \infty)$. Hence applying Lemma 3 (ii) we can deduce that the solution of the equation

$$
v(t) = \xi + \int_0^t b(T_\xi^y, v(s)) ds, \quad t \geq 0,
$$

explodes in finite time. Thus, by Osgood’s criterion we can conclude that

$$
\int_{\xi}^{\infty} (b(T_\xi^y, s))^{-1} ds < \infty.
$$

Reciprocally, suppose that the solution $y$ of (8) does not blow up in finite time. Let $a > 0$, using that $b \geq 0$ we have

$$
y(t) \geq \xi + \int_a^t b(s, y(s)) ds \geq \xi + \int_a^t b(a, y(s)) ds, \quad t \geq a.
$$

Consequently, the solution of

$$
u(t) = \xi + \int_0^t b(a, u(s)) ds, \quad t \geq 0,
$$

does not explode in finite time, because by Lemma 3 (i) we can deduce that $u(t) \leq y(t + a)$, for each $t \geq 0$. Therefore $\int_{\xi}^{\infty} (b(a, s))^{-1} ds = \infty$ due to Osgood’s criterion. 

\[\blacksquare\]
3 Semilinear stochastic differential equations

The purpose of this section is to study the semilinear stochastic differential equation (SDE)

$$X_t = \xi + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s)X_s dW_s, \quad t \geq 0.$$  \hspace{1cm} (9)

Hereinafter $W = \{W_t : t \geq 0\}$ is a one-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is complete and $(\mathcal{F}_t)_{t \geq 0}$ is supposed to satisfy the usual conditions. The initial condition $\xi$ is a $\mathcal{F}_0$-measurable random variable and the coefficients $b$ and $\sigma$ satisfy the following assumptions:

**H1:** $b : (\Omega \times [0, \infty) \times \mathbb{R}, \mathcal{P} \otimes \mathcal{B}(\mathbb{R})) \to \mathbb{R}$ is a continuous non-negative random field with probability one. Here $\mathcal{P}$ is the predictable $\sigma$-algebra and $\mathcal{B}(\mathbb{R})$ is the Borel $\sigma$-algebra on $\mathbb{R}$.

**H2:** $\sigma : \Omega \times [0, \infty) \to \mathbb{R}$ is a predictable and continuous process.

Note that these assumptions, on the coefficients in (9), will be assumed on the rest of the section.

3.1 A particular case of Feller’s test

As a first step, in order to have a better understanding of the remainder of the article, in this subsection we analyze the autonomous case of the equation (9) when $\sigma \equiv 1$, specifically we study the SDE

$$Z_t = \xi + \int_0^t b(Z_s)ds + \int_0^t Z_s dW_s, \quad t \geq 0,$$  \hspace{1cm} (10)

where $\xi > 0$ is a real number and $b : \mathbb{R} \to \mathbb{R}$ is a continuous non-negative function. We begin with the next result for which we use the following notation

$$\bar{b}(x) = \frac{b(x)}{x}, \quad x > 0.$$
Theorem 5  Suppose that $\bar{b} : (0, \infty) \to \mathbb{R}$ is a non-decreasing function such that $\bar{b} > 1/2$. Then, the explosion time $T_\xi^Z$ of the solution $Z$ of (10) is finite with probability 1 if and only if
\[
\int_\xi^\infty \frac{ds}{2\bar{b}(s) - s} < \infty.
\]

Proof. Applying Itô’s formula, to the process $R$ defined as
\[
R_t = Z_t \exp \left( -W_t + \frac{t}{2} \right), \quad t < T_\xi^Z,
\]
we obtain
\[
R_t = \xi + \int_0^t e^{-W_s + \frac{t}{2} \bar{b}(e^{W_s - \frac{t}{2}} R_s)} ds, \quad t < T_\xi^Z. 
\]  \hspace{1cm} (11)
If one use that $\bar{b}$ is non-negative then one see that
\[
Z_t = R_t \exp \left( W_t - \frac{t}{2} \right) > 0, \quad t \geq 0. 
\]  \hspace{1cm} (12)
Then we will be able to prove the result using the Feller’s test for explosions with $l = 0$, $r = \infty$ and $\zeta = \xi$ (see the case $(ii)$ of Theorem 1). The monotonicity of $\bar{b}$ turns out
\[
p(0) = - \int_0^\xi \exp \left( 2 \int_s^\xi \bar{b}(r) \frac{dr}{r} \right) ds 
\leq - \int_0^\xi \exp \left( 2\bar{b}(s) \int_s^\xi \frac{dr}{r} \right) ds 
= - \int_0^\xi \left( \frac{\xi}{s} \right)^{2\bar{b}(s)} ds.
\]
Using that $\xi/s \geq 1$ and $2\bar{b}(s) - 1 > 0$ we have
\[
p(0) \leq - \int_0^\xi \frac{\xi}{s} ds = -\infty.
\]
Therefore from Feller’s test, it is enough to show that $v(\infty) < \infty$ if and only if $\int_\xi^\infty (2\bar{b}(s) - s)^{-1} ds < \infty$. By the definition (6) of $v$ we deduce
\[
v(\infty) = \int_\xi^\infty \int_\xi^y \frac{2}{z^2} \exp \left( -2 \int_z^y \bar{b}(r) \frac{dr}{r} \right) dz dy
\]
\[ \geq \int_{\xi}^{\infty} \int_{\xi}^{y} \frac{2}{z^2} \exp \left( -2b(y) \int_{z}^{y} \frac{dr}{r} \right) dzdy \]
\[ = 2 \int_{\xi}^{\infty} y^{-2b(y)} \int_{\xi}^{y} \frac{dz}{z^{2-2b(y)}} dy \]
\[ = 2 \int_{\xi}^{\infty} \left\{ \frac{1}{2b(y) - y} - \frac{\xi^{2b(y)-1}y^{1-2b(y)}}{2b(y) - y} \right\} dy. \quad (13) \]

Since the function \( 2\bar{b} - 1 \) is non-decreasing we obtain
\[ \int_{\xi}^{\infty} \frac{\xi^{2b(y)-1}y^{1-2b(y)}}{2b(y) - y} dy = \xi^{-1} \int_{\xi}^{\infty} \frac{1}{2b(y) - 1} \left( \frac{\xi}{y} \right)^{2b(y)} dy \]
\[ \leq \frac{\xi^{-1}}{2b(\xi) - 1} \int_{\xi}^{\infty} \left( \frac{\xi}{y} \right)^{2b(y)} dy. \]

Hence, the facts that \( \xi/y < 1 \) and \( \bar{b} \) is non-decreasing lead us to
\[ \int_{\xi}^{\infty} \frac{\xi^{2b(y)-1}y^{1-2b(y)}}{2b(y) - y} dy \leq \frac{\xi^{-1}}{2b(\xi) - 1} \int_{\xi}^{\infty} \left( \frac{\xi}{y} \right)^{2b(\xi)} dy \]
\[ = \frac{1}{(2b(\xi) - 1)^2}. \]

Thus, (13) implies that \( \int_{\xi}^{\infty} (2b(s) - s)^{-1} ds < \infty \) if \( v(\infty) < \infty \).

On the other hand, by Fubini’s theorem
\[ v(\infty) \leq \int_{\xi}^{\infty} \int_{\xi}^{y} \frac{2}{z^2} \exp \left( -2b(z) \int_{z}^{y} \frac{dr}{r} \right) dzdy \]
\[ = \int_{\xi}^{\infty} y^{-2b(z)} \int_{\xi}^{y} \frac{dy}{z^2} \]
\[ = 2 \int_{\xi}^{\infty} z^{2b(z)-2} \int_{z}^{\infty} y^{-2b(z)} dydz \]
\[ = 2 \int_{\xi}^{\infty} \frac{z^{-1}}{2b(z) - 1} dz \]
\[ = 2 \int_{\xi}^{\infty} \frac{dz}{2b(z) - z}. \]

Consequently, \( v(\infty) < \infty \) if \( \int_{\xi}^{\infty} (2b(s) - s)^{-1} ds < \infty. \)
3.2 A generalization of Feller’s test

Now we deal with the non-autonomous stochastic differential equation

\[ Y_t = \xi + \int_0^t b(s, Y_s)ds + \int_0^t Y_s dW_s, \quad t \geq 0, \quad (14) \]

where \( \xi \) is defined as in equation (9) and remember that the function \( b \) satisfies the condition \( H1 \). Henceforth we will use the notation \( \tilde{b}(\omega, t, x) = b(\omega, t, e^x), (\omega, t, x) \in \Omega \times [0, \infty) \times \mathbb{R} \).

**Theorem 6** Let \( c \geq 0 \) and suppose that with probability one the function \( \tilde{b} : \Omega \times [0, \infty) \times \mathbb{R} \to \mathbb{R} \) satisfy

(i) for each \( x \in \mathbb{R} \), \( \tilde{b}(\cdot, x) : \Omega \times [0, \infty) \to \mathbb{R} \) is non-decreasing (in the time component),

(ii) for each \( t \in [0, \infty) \), \( \tilde{b}(t, \cdot) : \Omega \times (c, \infty) \to \mathbb{R} \) is non-decreasing (in the space component),

(iii) for each \( (t, x) \in [0, \infty) \times \mathbb{R} \), \( \tilde{b}(t, x) \geq 1/2 \) and for each \( (t, x) \in [0, \infty) \times (c, \infty) \), \( \tilde{b}(t, x) > 1/2 \).

Then, for almost all \( \omega_0 \) in

\[ \tilde{\Omega} = \{ \omega \in \Omega : W(\omega) \text{ is continuous and } b(\omega, \cdot), \tilde{b}(\omega, \cdot) \text{ satisfy the above hypotheses, } \xi(\omega) > 0 \}, \]

the solution \( Y(\omega_0) \) of equation (14) explodes in finite time if and only if

\[ \int_{\theta}^{\infty} \frac{ds}{2b(\omega_0, a, s) - s} < \infty, \quad \forall \theta > e^c, \quad (15) \]

for some \( a > 0 \).

**Remark 7** (a) Note that \( a \) depends on \( \omega_0 \).

(b) If with probability one \( \tilde{b}(t, x) > 1/2 \) for each \( (t, x) \in [0, \infty) \times \mathbb{R} \), then (15) is equivalent to

\[ \int_{\xi}^{\infty} \frac{ds}{2b(\omega_0, a, s) - s} < \infty. \]
Proof of Theorem 6. Applying Itô’s formula as in (11) to
\[ R_t = Y_t \exp \left( -W_t + \frac{t}{2} \right), \quad 0 \leq t < T^Y_\xi, \]  
and using that \( b \geq 0 \) we obtain a non-decreasing process \( R \) given by
\[ R_t = \xi + \int_0^t e^{-W_s + \frac{s}{2}} b(s, e^{W_s - \frac{s}{2} R_s}) ds, \quad 0 \leq t < T^Y_\xi. \]  
Then (16) implies that \( Y > 0 \), thus the process \( Z_t = \log(Y_t), \quad 0 \leq t < T^Y_\xi \) is well defined and \( T^{Z_\log \xi} = T^Y_\xi \). We can apply again Itô’s formula to obtain
\[ Z_t = \log \xi + \int_0^t \tilde{b}(s, Z_s) ds + W_t - \frac{t}{2}, \quad 0 \leq t < T^{Z_\log \xi}_\xi. \]  
Now fix \( \omega_0 \in \bar{\Omega} \), for which the expression (19) is satisfied.

Necessity: Let us suppose that \( T^{Y_\xi}_\xi(\omega_0) < \infty \). Because \( Y(\omega_0) > 0 \), then \( Y(\omega_0) \) explodes to \(+\infty\), hence \( T^{Z_\log \xi}_\xi(\omega_0) < \infty \) and \( Z(\omega_0) \) explodes to \(+\infty\). Therefore we can find a \( T \in (0, T^{Z_\log \xi}_\xi(\omega_0)) \), such that
\[ Z_t(\omega_0) > c, \quad \forall t \in [T, T^{Z_\log \xi}_\xi(\omega_0)]. \]  
We rewrite equation (19) as
\[ Z_{T+t}(\omega_0) = Z_T(\omega_0) + \int_T^{T+t} \left\{ \tilde{b}(\omega_0, T_s, Z_s) - \frac{1}{2} \right\} ds, \quad 0 \leq t < T^{Z_\log \xi}_\xi(\omega_0) - T. \]  
Setting \( y(t) = Z_{T+t}(\omega_0) \) we have
\[ y(t) = Z_T(\omega_0) + \int_0^t \left\{ \tilde{b}(\omega_0, T + s, y(s)) - \frac{1}{2} \right\} ds, \quad 0 \leq t < T^{Z_\log \xi}_\xi(\omega_0) - T. \]  
The fact that \( \tilde{b} \) is non-decreasing in the time variable bring about the inequality
\[ y(t) \leq M + \int_0^t \left\{ \tilde{b}(\omega_0, T^{Z_\log \xi}_\xi(\omega_0), y(s)) - \frac{1}{2} \right\} ds, \quad 0 \leq t < T^{Z_\log \xi}_\xi(\omega_0) - T, \]
with
\[ M = Z_T(\omega_0) + 2 \sup \{|W_t(\omega_0)| : t \in [0, T^Z_{\log \xi}(\omega_0)]\} + c. \]

Consider the integral equation
\[ x(t) = M + \int_0^t \left\{ \tilde{b}(\omega_0, T^Z_{\log \xi}(\omega_0), x(s)) - \frac{1}{2} \right\} ds, \quad t \geq 0. \]

Lemma 3 (ii) yields \( T^x_M \leq T^Z_{\log \xi}(\omega_0) \). Since \( M > c \) the Proposition 2 implies
\[ 2 \int_M^\infty \frac{ds}{2\tilde{b}(\omega_0, T^Z_{\log \xi}(\omega_0), s) - 1} < \infty \]
and therefore the continuity of \( b \) gives
\[ \int_\theta^\infty \frac{ds}{2b(\omega_0, T^Z_{\log \xi}(\omega_0), s) - s} < \infty, \quad \forall \theta > c. \]

**Sufficiency:** Now let us assume \( T^Y_\xi(\omega_0) = \infty \) and take \( a > 0 \) fix. As before, \( Y_t(\omega_0) > 0 \), for each \( t \geq 0 \), and (18) turns out, \( T^Z_{\log \xi}(\omega_0) = \infty \). By the law of iterated logarithm (see for instance Theorem 4.3 in León and Villa [8]) we can find a sequence \( \{t_n : n \in \mathbb{N}\} \) such that \( a \leq t_n \uparrow \infty \) and
\[ \inf\{W_{h+t_n}(\omega_0) : 0 \leq h \leq 1\} \uparrow \infty, \quad n \to \infty. \quad (20) \]

Hence from equation (19) and using the hypothesis that \( \tilde{b} \) satisfies, we obtain
\[ Z_{t+t_n}(\omega_0) \geq \tilde{m}_n + \int_0^t \left\{ \tilde{b}(\omega_0, a, Z_{s+t_n}(\omega_0)) - \frac{1}{2} \right\} ds, \quad t \in [0, 1], \quad (21) \]
where
\[ \tilde{m}_n = \log \xi(\omega_0) + \inf\{W_{h+t_n}(\omega_0) : 0 \leq h \leq 1\}. \]

Observe that (20) implies that \( \tilde{m}_n > c \), for all \( n \) large enough. Then Lemma 3 (i) implies that the explosion time \( T^u_{\tilde{m}_n} \) of
\[ u(t) = \tilde{m}_n + \int_0^t \left\{ \tilde{b}(\omega_0, a, u(s)) - \frac{1}{2} \right\} ds, \quad t \geq 0, \]
is bigger or equal than 1. Hence
\[ 2 \int_{\tilde{m}_n}^\infty \frac{ds}{2\tilde{b}(\omega_0, a, s) - 1} \geq 1. \]
Then \((20)\) necessary gives
\[
\int_{\theta}^{\infty} \frac{ds}{2b(\omega_0, a, s)} = \infty, \quad \forall \theta > \theta_0.
\]
Thus the proof is complete.

Now we present other Osgood type criteria.

**Proposition 8** Let \(c \in \mathbb{R}\) and assume that with probability one the function \(b : \Omega \times [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}\) satisfy
(i) \(b\) is non-decreasing by components,
(ii) for each \((t, x) \in [0, \infty) \times (c, \infty)\), \(b(t, x) > 0\).
For almost all \(\omega_0\) in
\[
\tilde{\Omega} = \{\omega \in \Omega : W(\omega) \text{ is continuous and } b(\omega, \cdot) \text{ satisfy}
\]
the above hypotheses, \(\xi(\omega) > c\},
if the solution \(Y(\omega_0)\) of \((14)\) explodes in finite time, then
\[
\int_{\theta}^{\infty} \frac{ds}{b(\omega_0, a, s)} < \infty, \quad \forall \theta > c,
\]
for some \(a > 0\).

**Proof.** By hypothesis \(T^Y_\xi(\omega_0) < \infty\). The continuity of \(W(\omega_0)\) implies that
\[
m = \inf \{e^{W_s(\omega_0)-\frac{s}{2}} : s \in [0, T^Y_\xi(\omega_0)]\} > 0.
\]
Since \(R(\omega_0)\) explodes to \(+\infty\), then there exists a \(0 < T < T^Y_\xi(\omega_0)\) such that
\[
R_s(\omega_0) > \frac{c}{m}, \quad T \leq s < T^Y_\xi(\omega_0).
\]
From \((17)\) we see that
\[
R_{T+t}(\omega_0) = R_T(\omega_0) + \int_{T}^{t} e^{-W_{T+s}(\omega_0)+\frac{1}{\xi}b(\omega_0, T+s, e^{W_{T+s}(\omega_0)}-\frac{T}{\xi}R_{T+s})} ds,
\]
\[0 \leq t < T^Y_\xi(\omega_0) - T.\]
The condition (22) implies that
\[ MR_{T+s}(\omega_0) \geq e^{W_{T+s}(\omega_0)} - \frac{R_T}{R_{T+s}} R_{T+s} \geq mR_{T+s} > c, \quad 0 \leq s < T_{\xi}^Y(\omega_0) - T, \quad (23) \]
where
\[ M = R_T + \exp \left( \sup_{t \in [0,T_{\xi}^Y(\omega_0)]} |W_t(\omega_0)| + \frac{R_{T+s}(\omega_0)}{2} \right) + c + 1. \]

Now, using the hypothesis (i) we get
\[ R_{T+t}(\omega_0) \leq M + \int_0^t MB(\omega_0, T_{\xi}^Y(\omega_0), MR_{T+s}(\omega_0)) ds, \quad 0 \leq t < T_{\xi}^Y(\omega_0) - T. \]

Let us define \( y(t) = MR_{T+t}(\omega_0) \), then the previous inequality leads to
\[ y(t) \leq M^2 + \int_0^t M^2b(\omega_0, T_{\xi}^Y(\omega_0), g(s)) ds, \quad 0 \leq t < T_{\xi}^Y(\omega_0) - T. \]

We consider the integral equation
\[ x(t) = M^2 + \int_0^t M^2b(\omega_0, T_{\xi}^Y(\omega_0), x(s)) ds, \quad t \geq 0. \]

It is clear that \( M^2 > c \) and (23) yields \( y > c \) on \([0,T_{\xi}^Y(\omega_0) - T)\), hence by Lemma 3 (ii) we deduce that \( x(t) \geq MR_{T+t}(\omega_0) \), for all \( 0 \leq t < T_{\xi}^Y(\omega_0) - T \), therefore by Osgood criterion’s we obtain
\[ \int_{M^2}^{\infty} ds \frac{ds}{M^2b(\omega_0, T_{\xi}^Y(\omega_0), s)} < \infty. \]

The result follows from the continuity of \( b \) and hypothesis (ii).

**Example 9** Consider the equation
\[ Y_t = 1 + \frac{1}{2} \int_0^t Y_s^2 ds + \int_0^t Y_s dW_s, \quad t \geq 0. \quad (24) \]

Proceeding as in (12) we deduce that \( Y > 0 \). Therefore we can use Feller’s test (Theorem 1) to see the explosive behavior of \( Y \) in \((0, \infty]\). In this case, by equation (5) we see that
\[ p(0) = - \int_0^1 \exp \left( \int_s^1 dr \right) ds \]

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and by Fubini’s theorem equation (6) can be written as

\[ v(0) = 2 \int_0^1 \int_y^1 \frac{\exp (z - y)}{z^2} dz dy \]
\[ \geq 2 \int_0^1 \int_y^1 \frac{1}{z^2} dz dy \]
\[ = 2 \int_0^1 \frac{1}{z} dz = \infty. \]

Thus the solution \( Y \) of (24) does not blow up in finite time with positive probability. However note that

\[ \int_\theta^\infty \frac{dr}{r^2} < \infty, \quad \forall \theta > 0, \]

hence we do not have the converse of Proposition 8. As we shall see in Proposition 10 the reason of this singularity is that

\[ \int_0^\infty \frac{dr}{r^2} = \infty. \quad (25) \]

We have the converse of Proposition 8 if the corresponding integral is divergent, that is nothing similar to case (25).

**Proposition 10** Assume that with probability 1 the function \( \tilde{b} : \Omega \times [0, \infty) \times \mathbb{R} \to \mathbb{R} \) satisfy

(i) \( \tilde{b} \) is non-decreasing by components,
(ii) for each \((t, x) \in [0, \infty) \times (0, \infty), b(t, x) > 0. \)

Then the solution \( Y(\omega_0) \) of (14) explodes in finite time if

\[ \int_0^\infty \frac{ds}{b(\omega_0, a, s)} < \infty \]

for some \( a > 0. \) Here \( \omega_0 \) is in the set, of probability 1,

\[ \tilde{\Omega} = \{ \omega \in \Omega : W(\omega) \text{ is continuous and } \tilde{b}(\omega, \cdot), b(\omega, \cdot) \text{ satisfy the above hypotheses, } \xi(\omega) > 0 \}. \]
Proof. Suppose that $T^Y_\xi(\omega_0) = \infty$. As in (21) we obtain, for $a > 0$,

$$Z_{t+a}(\omega_0) \geq \log \xi(\omega_0) + W_{t+a}(\omega_0) - \frac{t + a}{2} + \int_0^t \bar{b}(\omega_0, a, Z_{s+a}(\omega_0))ds, \quad t \geq 0.$$ 

Renaming $X_t = Z_{t+a}$ we obtain

$$X_t(\omega_0) \geq m_n + \int_0^t \bar{b}(\omega_0, a, X_s(\omega_0))ds, \quad t \in [0, n],$$

where

$$m_n = \log \xi(\omega_0) - \sup_{t \in [0, n]} |W_{t+a}(\omega_0)| - \frac{n + a}{2}.$$

In a similar fashion as in previous results we take into account the equation

$$x(t) = m_n + \int_0^t \bar{b}(\omega_0, a, x(s))ds, \quad t \geq 0.$$ 

From the comparison Lemma 3 (i) and Osgood’s criterion (see Proposition 2) we can establish the inequality

$$\int_0^\infty \frac{ds}{b(\omega_0, a, s)} \geq \int_{\exp(m_n)}^\infty \frac{ds}{b(\omega_0, a, s)} \geq n.$$ 

The result is obtained by letting $n \to \infty$. 

3.3 Main results

Now we are ready to state the main results of this article.

Theorem 11 Assume that with probability one

(i) $b$ is non-decreasing by components,
(ii) for each $(t, x) \in [0, \infty) \times (0, \infty)$, $b(t, x) > 0$.

Let $X$ be the solution of equation (9) and

$$\tilde{\Omega} = \{\omega \in \Omega : W.(\omega) \text{ is continuous and } b(\omega, \cdot) \text{ satisfy}
\text{ the above hypotheses, } \xi(\omega) > 0\}.$$ 

For almost all $\omega_0$ in $\tilde{\Omega}$, if $X.(\omega_0)$ explodes in finite time then

$$\int_0^\infty \frac{ds}{b(\omega_0, a, s)} < \infty, \quad \forall \theta > 0,$$

for some $a > 0$. 

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Proof. Set
\[ g(t) = \exp \left( - \int_0^t \sigma(s) dW_s + \frac{1}{2} \int_0^t \sigma^2(s) ds \right), \quad t \geq 0. \] (26)

So, using Itô’s formula, we have
\[ Y_t = \xi + \int_0^t g(s)b(s, f(s)Y_s)ds, \quad t \geq 0, \] (27)

where
\[ Y_t = g(t)X_t \quad \text{and} \quad f(t) = \frac{1}{g(t)}, \quad t \geq 0. \]

Consequently, for \( \omega_0 \in \bar{\Omega} \) such that satisfies (27), the continuity of \( g \) and \( b(\omega_0, \cdot) \) imply that (27) can be written as
\[ y'(t) = g(\omega_0, t)b(\omega_0, t, f(\omega_0, t)y(t)), \quad t > 0, \] (28)
\[ y(0) = \xi(\omega_0), \]

where \( y(t) = Y_t(\omega_0), \quad t \geq 0. \) Since \( b \geq 0 \) then \( f(\omega_0, t)y(t) \geq f(\omega_0, t)\xi > 0. \) Therefore (28) and hypothesis (ii) turns out
\[ \int_{\xi(\omega_0)}^{Y_t(\omega_0)} \frac{ds}{b(\omega_0, y^{-1}(s), sf(\omega_0, y^{-1}(s)))} = \int_0^t g(\omega_0, s)ds \]
\[ := G(\omega_0, t). \] (29)

Suppose now that \( T_{\xi(\omega_0)}^{Y(\omega_0)} < \infty, \) where \( T_{\xi(\omega_0)}^{Y(\omega_0)} \) is defined as in the Proposition
\footnote{Hence, \( y^{-1}(t) < T_{\xi(\omega_0)}^{Y(\omega_0)}, \quad t \geq \xi(\omega_0), \) and therefore hypothesis (i) implies
\[ \int_{\xi(\omega_0)}^{\infty} \frac{ds}{b(\omega_0, T_{\xi(\omega_0)}^{Y(\omega_0)}, sM)} \leq \int_{\xi(\omega_0)}^{Y_{T_{\xi(\omega_0)}^{Y(\omega_0)}}} \frac{ds}{b(\omega_0, y^{-1}(s), sf(\omega_0, y^{-1}(s)))}, \]
\[ = G(\omega_0, T_{\xi(\omega_0)}^{Y(\omega_0)}) < \infty, \]

with
\[ M = \sup\{f(\omega_0, y^{-1}(s)) : s \geq \xi(\omega_0)\} \leq \sup\{f(\omega_0, r) : r \in [0, T_{\xi(\omega_0)}^{Y(\omega_0)}]\}. \]

Thus, the proof is complete \footnote{\textbf{\textcircled{1}}}{17}
In the remainder of this paper we will need the following notation

\[ \Lambda(t) = \int_0^t \sigma^2(s) ds, \quad t \geq 0. \]

The following two results are in certain sense the converse of Theorem 11.

**Theorem 12** Let \( X \) be the solution of (9). Assume that hypotheses of Theorem 11 are true. Let \( \omega_0 \in \tilde{\Omega} \) be such that \( \Lambda(\omega_0, \infty) < \infty \) and \( X_t(\omega_0) \) is finite for all \( t \geq 0 \). Then,

\[ \int_\theta^\infty \frac{ds}{b(\omega_0, a, s)} = \infty, \quad \forall \theta > 0, \]

for all \( a > 0 \).

**Remark 13** Theorem 3.4.9 in [2], implies that

\[ \left\{ \omega \in \Omega : \int_0^\infty \sigma(s) dW_s \text{ is bounded on } \mathbb{R}_+ \right\} \]

coincides with the set \( \{ \omega \in \Omega : \Lambda(\omega, \infty) < \infty \} \) by redefining \( \sigma \) on a set of probability zero.

**Proof.** Let \( \omega_0 \) be as in the statement of the theorem. Then (26) and (29) lead us to

\[
G(\omega_0, \infty) \geq \int_0^\infty \exp \left( - \left( \int_0^t \sigma(r) dW_r \right)(\omega_0) \right) ds \\
\geq \int_0^\infty \exp \left( - \sup_{t \geq 0} \left| \left( \int_0^t \sigma(r) dW_r \right)(\omega_0) \right| \right) ds = \infty, \quad (30)
\]

where we have used the Remark 13 in the last equality. Inasmuch as \( b \geq 0 \) we see that \( Y(\omega_0) \) is increasing, then \( \lim_{t \to \infty} Y_t(\omega_0) \) exists. On the other hand, (29) and (30) implies

\[
\int_{\xi(\omega_0)}^{\lim_{t \to \infty} Y_t(\omega_0)} \frac{ds}{b(\omega_0, y^{-1}(s), sf(\omega_0, y^{-1}(s)))} = \infty. \quad (31)
\]
Since \( b(\omega_0, \cdot) \) is continuous and \( b(\omega_0, \cdot) > 0 \) on \([0, \infty) \times (0, \infty)\) we can deduce that \( \lim_{t \to \infty} Y_t(\omega_0) = \infty \). Let \( a > 0 \), by (31) one obtains

\[
\infty = \int_{Y_a(\omega_0)}^\infty \frac{ds}{b(\omega_0, y^{-1}(s), sf(\omega_0, y^{-1}(s)))} \leq \int_{Y_a(\omega_0)}^\infty \frac{ds}{b(\omega_0, a, sm)};
\]

where

\[
m = \inf\{f(\omega_0, r) : r \geq 0\} \geq \exp\left(-\frac{1}{2}\Lambda(\omega_0, \infty)\right) \times \exp\left(-\sup_{t \geq 0} \left| \left( \int_0^t \sigma(r) dW_r \right)(\omega_0) \right| \right).
\]

Using again the Remark 13 we deduce that \( m > 0 \), from which allows us to conclude the result.

**Theorem 14** Assume that with probability one

(i) \( \sigma^2 > 0 \) in \((0, \infty)\) and \( \Lambda(\infty) = \infty \),

(ii) the function \( \tilde{b} : \Omega \times [0, \infty) \times \mathbb{R} \to \mathbb{R} \), defined as

\[
\tilde{b}(\omega, t, x) = \frac{b(\omega, \Lambda^{-1}(t), e_x)}{\sigma^2(\Lambda^{-1}(t)) e^x},
\]

is non-decreasing by components,

(iii) for each \((t, x) \in [0, \infty) \times (0, \infty), b(t, x) > 0\).

For almost all \( \omega_0 \) in

\[
\tilde{\Omega} = \{\omega \in \Omega : W(\omega) \text{ is continuous and } \tilde{b}(\omega, \cdot), b(\omega, \cdot) \text{ satisfy the above hypotheses, } \xi(\omega) > 0\}
\]

the solution \( X(\omega_0) \) of (9) explodes in finite time if

\[
\int_0^\infty \frac{ds}{b(\omega_0, a, s)} < \infty
\]

for some \( a > 0 \).

**Proof.** From Theorem 3.4.4 in [2] we know that there exists a Brownian motion \( \tilde{B} = \{\tilde{B}_t : t \geq 0\} \) such that

\[
\int_0^t \sigma(s) dW_s = \tilde{B}_{\Lambda(t)}, \quad t \geq 0.
\]

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This leads us to write \((27)\) as
\[
Y_t = \xi + \int_{0}^{t} e^{-\tilde{B}(s) + \frac{1}{2}\Lambda(s)} b(s, e^{\tilde{B}(s) - \frac{1}{2}\Lambda(s)} Y_s) ds, \quad t \geq 0.
\]
Moreover, making the change of variable \(u = \Lambda(s)\) and setting \(Z_t = Y_{\Lambda^{-1}(t)}\) we consider the equation
\[
Z_t = \xi + \int_{0}^{t} \frac{e^{-\tilde{B}(s) + \frac{1}{2}s}}{\sigma^2(\Lambda^{-1}(s))} b(\Lambda^{-1}(s), e^{\tilde{B}(s) - \frac{1}{2}s} Z_s) ds, \quad t \geq 0.
\]
Finally, if \(\tilde{Z}_t = Z_t e^{\tilde{B}_t - t/2}\), by Itô’s formula we have
\[
\tilde{Z}_t = \xi + \int_{0}^{t} \frac{b(\Lambda^{-1}(s), \tilde{Z}_s)}{\sigma^2(\Lambda^{-1}(s))} ds + \int_{0}^{t} \tilde{Z}_s d\tilde{B}_s, \quad t \geq 0,
\]
and the result follows from Proposition [10] because \(b\) and \(\tilde{b}\) meet the respective assumptions of such proposition.

**Remark 15** Let \(c \geq 0\) and suppose that with probability one the function \(\tilde{b}\), defined in [32], satisfies hypothesis (i) — (iii) in Theorem [7]. Then, for all \(\omega_0 \in \tilde{\Omega}\) the solution \(X(\omega_0)\) of equation [7] explodes in finite time if and only if
\[
\int_{\theta}^{\infty} \frac{ds}{2b(\omega_0, a, s) - \sigma^2(a)s} < \infty, \quad \forall \theta > e^c,
\]
for some constant \(a > 0\).

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