BRST, anti-BRST and their geometry

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Abstract
We continue the comparison between the field theoretical and geometrical approaches to the gauge field theories of various types, by deriving their Becchi–Rouet–Stora–Tyutin (BRST) and anti-BRST transformation properties and comparing them with the geometrical properties of the bundles and gerbes. In particular, we provide the geometrical interpretation of the so-called Curci–Ferrari conditions that are invoked for the absolute anticommutativity of the BRST and anti-BRST symmetry transformations in the context of non-Abelian one-form gauge theories as well as the Abelian gauge theory that incorporates a two-form gauge field. We also carry out the explicit construction of the three-form gauge fields and compare it with the geometry of 2-gerbes.

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1. Introduction

In a gauge-fixed quantum gauge field theory, the local gauge symmetry is traded with the nilpotent Becchi–Rouet–Stora–Tyutin (BRST) symmetry. If the gauge-fixing is chosen in a particularly symmetric fashion, the BRST symmetry is accompanied by a twin symmetry, called the anti-BRST. In this paper, we continue the work started in [11] which consists in comparing the language of the quantum field theory (QFT) with that of the underlying geometry. In particular, in our present work, we concentrate primarily on two topics. The first one is a geometrical interpretation of the BRST and anti-BRST transformations in the twin cases of the non-Abelian one-form gauge theories and theories incorporating the Abelian two-form gauge potentials. For the second topic, we focus on the Abelian three-form gauge field theory and compare it with the 2-gerbe formalism. The above gauge theories, as is well known, are always endowed with the first-class constraints in the language of Dirac’s prescription for the classification scheme [1, 2].
As far as the first topic is concerned, we would like to understand why in QFT one needs the constraint relations (referred to as Curci–Ferrari (CF) relations) among the ghost fields and auxiliary fields in order to close the BRST and anti-BRST algebra. We start by re-examining an old problem: in the non-Abelian one-form gauge field theories, the requirement that the BRST and anti-BRST transformations must anti-commute imposes a constraint on the (auxiliary) fields of the theory (the so-called CF condition). We explain, first, how this fact can be given a geometrical interpretation. This type of constraints, however, are characteristic, not only of the non-Abelian one-form gauge theories but also of the higher form Abelian (and perhaps non-Abelian) gauge theories whose field contents are based on the concept of gerbes. We wish to give a geometrical interpretation also of these constraints, which we keep referring to as the CF constraints.

Finally, as for our second topic, we introduce a gauge theory based on the Abelian three-form gauge potential and define its BRST and anti-BRST transformations. Furthermore, we obtain the CF-type conditions for their absolute anti-commutativity. This is to be compared with the geometry of 2-gerbes. To this end we first introduce the latter, find their gauge transformations and show that they can be reduced to the ones determined by the field theoretical methods. Thus, we establish a connection between the field theory of the Abelian three-form gauge theory with the geometry associated with the 2-gerbes.

2. An old problem: the CF constraints in non-Abelian gauge theories

Let us consider a non-Abelian one-form gauge theory with a gauge group \( G \). We write the gauge potential one-form as \( A = A_\mu(x)T^a d\mathbf{x}^\mu \), where \( T^a \) are the anti-Hermitian generators of \( \text{Lie}(G) \). The appropriate geometry for such a theory is well known to be a principal fiber bundle \( P(M,G) \), the base space being the spacetime manifold \( M \) and with the structure group \( G \). A gauge transformation with the infinitesimal parameter \( \lambda = \lambda^a(x)T^a \) is given by

\[
\delta_\lambda A = D\lambda \equiv d\lambda + [A, \lambda].
\]  

(1)

Quantizing the theory requires gauge fixing, which, via the Faddeev–Popov procedure, replaces the classical gauge invariance with the BRST invariance. A BRST transformation is analogous to a gauge transformation, except that the gauge parameter is replaced by an anticommuting field \( \lambda \rightarrow c = c^a(x)T^a \). The origin of this transmutation was explained in [8]: \( c \) represents not a single gauge transformation but the whole set of gauge transformations. It is, in fact, an alias of the Maurer–Cartan form \( \omega \) in the group of gauge transformations. The fact that \( \omega \) is a one-form accounts for the anticommutativity of \( c \). Moreover, contracting it with left-invariant vector fields in \( P \) gives rise to all the gauge transformations. For this reason, we say that \( c \) represents the whole set of gauge transformations. In other words, the ghost field \( c \) is the heuristic and compact form that QFT adopts to express the geometric set-up of the non-Abelian one-form gauge theories.

The BRST transformation (1) is not nilpotent unless we endow \( c \) itself with a BRST transformation. Since we have \( s^2 A = D(sc) + \frac{1}{2} D([c, c])_v \), the nilpotent BRST transformations must be as follows:

\[
sA = Dc, \quad sc = -\frac{1}{2}[c, c].
\]  

(2)

Nilpotency is simply a translation, in the language of the QFT, of the fact that the gauge transformations form a Lie algebra. In fact this implies that

\[
(\delta_1, \delta_2) A = \delta_{\{1,2\} A} = 0,
\]  

(3)

and we see that (2) exactly mimics this. In particular, the transformation \( sc \) mimics the last term on the LHS of (3). This is important for the sequel: the nilpotency of the BRST
transformation is the quantum equivalent of the Lie algebra (or Lie group) product law. In other words, if the BRST transformations were not nilpotent, they could not be derived from a classical Lie group product law.

If, in the quantization process, we use the Lorentz gauge fixing, then the quantum theory turns out to be more symmetric. In this case, we have an additional symmetry, the anti-BRST symmetry. The parameter of this new symmetry is the anti-ghost field $\bar{c}$, which turns out to be more symmetric. In this case, we have an additional symmetry, the anti-BRST transformation is the quantum equivalent of the Lie algebra (or Lie group) product law. Therefore, it seems that the natural geometrical setting for a BRST and anti-BRST-symmetric theory is $P(M, G \times G)$, that is, a principal fiber bundle with a structure group $G \times G$, with $c$ taking values in the Lie algebra of the first group and $\bar{c}$ in the Lie algebra of the second group. But this cannot be the case since $\{c, \bar{c}\} \neq 0$.

We propose the following geometrical set-up. Starting from the two isomorphic principal fiber bundles $P(M, G)$ and $Q(M, G')$ on the same base space, we can easily construct the bundle $(P + Q)(M, G \times G')$ on $M$ with a structure group $G \times G'$. By pulling back the product $P(M, G) \times Q(M, G')$ via the diagonal map $\Delta(x) = (x, x)$. The fibers of $P + Q$ are couples of fibers of $P$ and $Q$, i.e. they are $(p, q)$ such that $\pi_P(p) = x = \pi_Q(q)$. The transition functions $\psi_{ab}$ of $P + Q$ are given by

$$\psi_{ab} \equiv (\psi_{a\beta}, \psi'_{a\beta}) : U_a \cap U_\beta \rightarrow G \times G',$$

where $\psi_{a\beta}$ and $\psi'_{a\beta}$ are the transition functions of $P$ and $Q$, respectively.

Now, if $G \equiv G'$, in $\psi_{ab}$ we can choose, in particular, $\psi_{a\beta} \equiv \psi'_{a\beta}$. This means that $P + Q$ is reducible to a bundle $R(M, G_d)$, with a structure group $G_d = \text{Diag}(G \times G)$ (see [10], proposition 5.3). It is obvious that the fibers of $R$ are the diagonal fibers of $P + Q$. $R$ represents the geometric set-up we are looking for. It is isomorphic to $P(M, G)$, as it should be, but what matters is the fine structure of this isomorphism. In fact, it hosts two gauge groups, two copies of $G$, and this is what we need to accommodate the above-cited two different types of gauge transformations.
Let us consider the relation between $R$ and $P + Q$. The reduction is specified by a homomorphism $R \to P + Q$, defined by a function $f$ such that $\forall u \in R$, $f(u) \in P + Q$ which reduces to the identity on the base $M$, and a group homomorphism $f_\ast : G \to \text{Diag}(G \times G)$, such that $f(ug) = f(u)f_\ast(g)$ for any $u \in R$ and $g \in G$. A connection $A$ in $P + Q$ reduces to a connection $\tilde{A}$ in $R$ and the two are related by [10],

$$f^*A = f_\ast \cdot A,$$

where $f^*$ denotes the pull-back and $(f_\ast \cdot A)(X) = df_\ast(A(X)) \equiv (A(X), A(X))$ for any vector field $X$ in $R$. The correspondence between $A$ and $\tilde{A}$ is one-to-one (see, e.g., [10] chapter II.6).

Let us now consider finite gauge transformations. They are given by the vertical automorphisms $\psi$ of a principal bundle, i.e. by bundle morphisms that do not affect the basis. In the case of $R$, the fibers get transformed as $\psi((p, p)) = (\psi(p), \psi(p))$, where $\psi$ is an automorphism of $P = Q$. Now let us consider the same construction of $P + Q$ as above, but with $P$ replaced by $\psi_1^*P$, where $\psi_1$ is now an automorphism of $P$ (as opposed to an automorphism of $Q$), i.e. we consider $\psi_1^*P + Q$ where $\psi_1^*P$ is the gauge transformed form of $P$. Naturally we would get a version of $R$ with fibers $(\psi_1(p), q)$, where $q = \psi_1(q)$. Therefore, trivially $\psi_1((p, q)) = (\psi_1(p), q)$, which defines an automorphism $\psi_1$ of $R$ originating from an automorphism of $P$. In the same way, we can get the automorphisms $\psi_2$ originated from the automorphisms of $Q$.

Taking a connection $A$ in $R$, we get, therefore, the two types of gauge transformations $\psi_1^*A$ and $\psi_2^*A$. We link them to $\lambda$ and $\bar{\lambda}$, respectively. In this way $\lambda$ ($\bar{\lambda}$) is associated with the Maurer–Cartan form of the first (second) factor in $G \times G$, respectively. But they are projected to the diagonal group. As a consequence, the anticommutator $[c, \bar{c}]_\lambda$ is non-vanishing. This seems to be the most appropriate geometrical set-up for the BRST and anti-BRST.

Next, let us consider two infinitesimal gauge transformations of the first type ($\psi_1$) and call them $\lambda_1$ and $\lambda_2$, and two of the second type $\bar{\lambda}_1$ and $\bar{\lambda}_2$. It is easy to prove that

$$((\delta_{\lambda_1} + \delta_{\bar{\lambda}_1})(\delta_{\lambda_2} + \delta_{\bar{\lambda}_2}) - (\delta_{\lambda_2} + \delta_{\bar{\lambda}_2})(\delta_{\lambda_1} + \delta_{\bar{\lambda}_1}))A = \delta_{[\lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2]}A = 0. \tag{9}$$

This is the geometrical meaning of (4). However, in the anticommuting language, we cannot reproduce it without introducing the auxiliary fields $B$ and $\bar{B}$, for we have

$$(s\bar{s} + \bar{s}s)A = D(s\bar{c} + \bar{s}c + [c, \bar{c}]_\lambda), \tag{10}$$

which motivates the definition of $B$ and $\bar{B}$ in (4).

We stress again that (4) and (6) express, in the language of quantum field theory, the simple Lie algebra rule (9).

Finally, let us make a comment concerning the Abelian case. When the gauge group $G$ is Abelian, we can, of course, repeat everything word by word. In equations (3) and (4) the Lie brackets vanish, so BRST and anti-BRST transformations are disconnected. The CF constraint is replaced by $B + \bar{B} = 0$. These auxiliary fields are, in fact, decoupled and one can do without them. We would like to point out, however, that the one-form Abelian bundle case is the only one in which the CF constraint is superfluous. In the next more complicated case of the Abelian two-form, which corresponds to the geometry of 1-gerbes, the CF constraint is essential.

3. CF constraints for 1-gerbes

The BRST and anti-BRST transformations for the 1-gerbes were worked out in [11]. Here we like to give the relevant geometrical interpretation.

Let us recall some basic definitions. A 1-gerbe (for the mathematical properties of gerbes see [12, 13] and for previous applications in physics see [15]) may be characterized by a triple

\[ (\mathcal{X}, \mathcal{F}, \mathcal{R}) \]
(B, A, f), formed by the two-form B, one-form A and zero-form f, respectively. These are related in the following way. Given a covering \{U_i\} of the manifold M, we associate with each U_i a two-form B_i. On a double intersection U_i \cap U_j, we have B_i - B_j = dA_{ij}. On the triple intersections U_i \cap U_j \cap U_k, we must have A_{ij} + A_{ik} + A_{ki} = d f_{ijk}. Finally, on the quadruple intersections U_i \cap U_j \cap U_k \cap U_l, the following integral cocycle condition must be satisfied:

\[ f_{ijl} - f_{ijk} + f_{jkl} - f_{ikl} = 2\pi n, \quad n = 0, 1, 2, 3, \ldots \]  

This integrality condition does not concern us in our Lagrangian formulation but it has to be imposed as an external condition.

The two triples, represented by (B, A, f) and (B', A', f'), respectively, are gauge equivalent if they satisfy the following relations:

\[ B'_i = B_i + dC_i \quad \text{on} \quad U_i, \]  

\[ A'_{ij} = A_{ij} + C_i - C_j + d\lambda_{ij} \quad \text{on} \quad U_i \cap U_j, \]  

\[ f'_{ijk} = f_{ijk} + \lambda_{ij} + \lambda_{ik} + \lambda_{jk} \quad \text{on} \quad U_i \cap U_j \cap U_k, \]  

for the one-form C and the zero-form \( \lambda \).

We now define the BRST and anti-BRST transformations corresponding to these geometrical transformations. As shown in [11], one can proceed in two different ways. Either one defines an action for the triple of local fields (B, A, f), quantizes it by adding all the ghost and auxiliary fields that are needed and verifies that the quantum action has the two BRST and anti-BRST symmetries below (this is what we do in section 4 for the three-form gauge field). Or, more heuristically, by analogy with the one-form gauge theory, one starts from the (known) gauge transformations of the (B, A, f) fields and constructs the BRST and anti-BRST transformations by simply relying on nilpotency and consistency. The two procedures lead to the same results up to minor ambiguities (see below). It should be recalled that while the above geometric transformations are defined on (multiple) neighborhood overlaps, the BRST and anti-BRST transformations in quantum field theory are defined on a single local coordinate patch. These (local, field-dependent) transformations are the means that QFT uses to record the underlying geometry.

The appropriate BRST and anti-BRST transformations are\(^5\)

\[
\begin{align*}
\bar{s}B &= d\bar{C}, & sB &= dC, & sA &= C + d\lambda, & sf &= \lambda + \mu, \\
\bar{s}C &= -d\bar{\beta}, & s\lambda &= \beta, & s\mu &= -\bar{\beta}, \\
\bar{s}\bar{C} &= -\bar{K}, & s\bar{K} &= d\rho, & s\bar{\mu} &= -g, \\
\bar{s}\bar{\beta} &= -\bar{\rho}, & s\bar{\lambda} &= g, & s\bar{g} &= \rho,
\end{align*}
\]  

\[ s[\rho, \bar{\rho}, \lambda, \bar{\lambda}] = 0, \]  

\[ \bar{s}[\bar{\rho}, \bar{\lambda}, \lambda, \bar{\lambda}] = 0, \]  

\[ \bar{s}[\bar{\lambda}, \bar{\rho}, \rho, \bar{\rho}] = 0. \]  

It can be easily verified that \((s + \bar{s})^2 = 0\) if the following constraint is satisfied:

\[ \bar{K} - K = d\bar{g} - dg. \]  

\(^5\) As done in [11], we take into account here also the scalar field \( f \).
This condition is both BRST and anti-BRST invariant. It is the analog of the Curci–Ferrari condition in non-Abelian one-form gauge theories and we refer to it with the same name.

The field content and the BRST and anti-BRST structure for 1-gerbe field theories are shown schematically in figure 1.

Before we proceed with the discussion, we note that the above realization of the BRST and anti-BRST algebra is not the only possibility. In general, it may be possible to augment it by the addition of a sub-algebra of elements which are all in the kernel of both $s$ and $\bar{s}$ or, if it contains such a sub-algebra, the latter could be moded out. For instance, in equations (15) and (16), $\rho$ and $\bar{\rho}$ form an example of this type of subalgebra. It is easy to see that $\rho$ and $\bar{\rho}$ can be consistently set equal to 0.

Now we suggest a geometrical setting for these transformations and the relevant CF constraints. We have already noted that for Abelian gauge bundles, we can carry out a construction similar to that of the previous section. The Lie bracket terms in (4) are trivial and we can, in fact, dispense with the auxiliary fields $B$ and $\bar{B}$. That is to say the BRST and anti-BRST transformations are independent of any interference term and the CF condition is unnecessary. The gauge transformations, underlying a 1-gerbe, are of the Abelian type too, but, contrary to the one-form gauge theory (i.e. 0-gerbe), they produce a non-trivial CF condition (17).

The geometric set-up for the BRST and anti-BRST 1-gerbe transformations is similar to the one that is true for the gauge bundles. One can define both Cartesian products and pullbacks of 1-gerbes, (see, e.g., [13]). We can, therefore, take the two copies of the same 1-gerbe and make their product. Then we pull the result back by the diagonal map (see above). So far, the construction is parallel to the one in the previous section. Now we have to do the analog of the mapping onto the diagonal of $G \times G$. This is not so easy in the above formulation of the 1-gerbes. But there is another formulation due to Hitchin [14] which we now recall and dwell upon. With respect to a covering $\{U_i\}$, a 1-gerbe is specified by the following data:

- a line bundle $L_{ij}$ for any intersection $U_i \cap U_j$,
- an isomorphism between $L_{ij}$ and $L_{ji}^{-1}$,
• a trivialization of $L_{ij} L_{jk} L_{ki}$, that is a map $\theta_{ijk} : U_i \cap U_j \cap U_k \rightarrow U(1)$,
• this map satisfies the cocycle condition $\theta_{jkl} \theta_{-1}^{-1} \theta_{ijl} \theta_{-1}^{-1} = 1$.

In this definition, the line bundles are associated with the $U(1)$ principal bundles: $L^{-1}$ represents the bundle dual to $L$, and the products of the bundles are tensor products. It is now possible to redo the construction of the previous section (including the diagonal mapping for fibers) locally, i.e. for each line bundle and each map $\theta$ (by replacing each local $P$ with a corresponding isomorphic $R$) while satisfying the conditions of the 1-gerbe definition. This means that we can think of two copies of the transformations (12)–(14) with the parameters $C_i, \lambda_{ij}$ and $\bar{C}_i, \bar{\lambda}_{ij}$, respectively.

This is the classical set-up we propose in order to accommodate the BRST and anti-BRST 1-gerbe transformations.

From the two copies of (12)–(14), it is easy to recognize the origin of many of the transformations in (15) and (16). For instance from (12), we see that $C_i$ is not uniquely defined, we could choose $C'_i = C_i + d\beta_i$ (or $\bar{C}'_i = \bar{C}_i + d\bar{\beta}_i$), obtaining in this way, the classical analog of $sC$ (or $\bar{s}\bar{C}$). Plugging this into (13), we get $\lambda'_{ij} = \lambda_{ij} - \beta_i + \beta_j$ and obtain the analog of $s\lambda$, etc. Denoting now, for simplicity, the classical infinitesimal transformations $\delta$ and $\bar{\delta}$ corresponding to the two copies of (12)–(14), we get the following:

$$0 = (\delta\bar{\delta} - \bar{\delta}\delta) B = \delta(d\bar{C}) - \bar{\delta}(dC) = d(\delta\bar{C} - \bar{\delta}C),$$

which is nothing but the geometric origin of the CF constraint.

4. Abelian three-form gauge theories and 2-gerbes

In this section, we give a more complex example, that of the Abelian three-form gauge theory, which has not been explicitly worked out as yet. In fact, we derive the proper BRST and anti-BRST symmetry transformations within a field theoretical approach. In the next section, we compare it with the geometric setting of 2-gerbes.

The nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations for the Abelian three-form theory have been derived [7] by exploiting the geometrical superfield formalism [5]. Section 4.1 is devoted to the discussion of the (anti-)BRST invariance of the coupled Lagrangian densities of the theory. Section 4.2 deals with the discrete and ghost scale symmetry transformations for the ghost part of the Lagrangian densities. We obtain the algebraic structures, satisfied by the conserved charges (corresponding to the above continuous symmetries), in section 4.3.

4.1. Preliminary: nilpotent (anti-)BRST symmetries in superfield formulation

By exploiting the superfield approach to the BRST formalism [5], the nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations for the free Abelian three-form gauge theory have been derived in [7] by exploiting the geometrical superfield formalism [5]. In a field theoretic approach, it is more convenient to use the component notation rather than the synthetic differential form language.

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6 In [7], the theory was considered in 4D, but the results hold in a $D$-dimensional Minkowski spacetime. We take here the Greek indices $\mu, \nu, \eta, \ldots = 0, 1, \ldots, D - 1$ to correspond to the spacetime directions of the flat Minkowski spacetime manifold endowed with a metric with signatures $(+1, -1, \ldots, -1)$. For algebraic convenience, we have changed $\bar{B} \rightarrow B, B^{(2)}_{\mu
u} \rightarrow B_{\mu\nu}, B^{(3)}_{\mu
u\tau} \rightarrow B_{\mu\nu\tau}$ and exchanged $F^{(2)}_{\mu\nu} \leftrightarrow \bar{F}^{(2)}_{\mu\nu}$ in the notations of [7].

7 In a field theoretic approach, it is more convenient to use the component notation rather than the synthetic differential form language.
two of the CF-type restrictions in (22) are fermionic in nature and one of them is bosonic

(iv) the CF-type restriction is one of the key properties of any arbitrary

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sB_{\mu\nu\eta} = \tilde{\partial}_\mu C_{\nu\eta} + \partial_\eta C_{\mu\nu} + \partial_\nu C_{\mu\eta}, \quad sC_{\mu\nu} = \partial_\mu \beta_\nu - \partial_\nu \beta_\mu,
\[ sC_{\mu\nu} = B_{\mu\nu}, \quad sB_{\mu\nu} = \partial_\mu f_\nu - \partial_\nu f_\mu, \quad s\tilde{\beta}_\mu = \tilde{F}_\mu, \quad s\beta_\mu = \tilde{\partial}_\mu C_2, \]
\[ sF_\mu = -\partial_\mu B, \quad s\tilde{f}_\mu = \partial_\mu B_1, \quad s\tilde{C}_2 = B_2, \quad sC_1 = -B, \]
\[ s\phi_\mu = f_\mu, \quad s\tilde{C}_1 = B_1, \quad s[C_2, f_\mu, B, B_1, B_2, B_{\mu\nu}] = 0, \]
\[ s\bar{B}_{\mu\nu\eta} = \bar{\partial}_\mu C_{\nu\eta} + \partial_\eta C_{\mu\nu} + \partial_\nu C_{\mu\eta}, \quad \bar{s}C_{\mu\nu} = \partial_\mu \beta_\nu - \partial_\nu \beta_\mu, \]
\[ \bar{s}B_{\mu\nu\eta} = \bar{\partial}_\mu C_{\nu\eta} + \partial_\eta C_{\mu\nu} + \partial_\nu C_{\mu\eta}, \quad \bar{s}C_{\mu\nu} = \partial_\mu \beta_\nu - \partial_\nu \beta_\mu, \]
\[ \bar{s}\tilde{F}_\mu = -\partial_\mu B_2, \quad \bar{s}f_\mu = -\partial_\mu B_1, \quad \bar{s}C_2 = B, \quad \bar{s}C_1 = -B_1, \]
\[ \bar{s}\phi_\mu = \tilde{f}_\mu, \quad \bar{s}\tilde{C}_1 = -B_2, \quad \bar{s}[\tilde{C}_2, \tilde{f}_\mu, B, B_1, B_2, \tilde{B}_{\mu\nu}] = 0, \]

where \( B_{\mu\nu\eta} \) is the totally antisymmetric tensor gauge field, \( (\tilde{C}_{\mu\nu})C_{\mu\nu} \) are the fermionic antisymmetric (anti-)ghost fields with ghost number \((-1) + 1\), \( (\tilde{\beta}_\mu)\beta_\mu \) are the Lorentz vector bosonic ghost-for-ghost (anti-)ghost fields with ghost number \((-2) + 2\) and \( (\tilde{C}_2)C_2 \) are the fermionic ghost-for-ghost (anti-)ghost Lorentz scalar fields with ghost number \((-3) + 3\). The vector field \( \phi_\mu \) and the auxiliary fields \( B, B_1, B_2 \) are bosonic in nature and the fermionic fields \( (\bar{f}_\mu)f_\mu \) and \( (\bar{B}_\mu)F_\mu \) are the auxiliary (anti-)ghost fields with ghost numbers \((-1) + 1\), respectively. Furthermore, it will be noted that the bosonic auxiliary fields \( B \) and \( B_2 \) carry the ghost numbers \(+2\) and \(-2\), respectively.

The above off-shell nilpotent \( \{s, \bar{s}\} = 0 \) (anti-)BRST symmetry transformations are not absolutely anticommuting in nature because

\[ \{s, \bar{s}\} B_{\mu\nu\eta} \neq 0, \quad \{s, \bar{s}\} C_{\mu\nu\eta} \neq 0, \quad \{s, \bar{s}\} \tilde{C}_{\mu\nu} \neq 0. \]  

It is interesting, however, to mention that the superfield formalism [7] yields the CF-type restrictions

\[ f_\mu + F_\mu = \partial_\mu C_1, \quad \tilde{f}_\mu + \tilde{F}_\mu = \partial_\mu \tilde{C}_1, \]
\[ B_{\mu\nu} + \bar{B}_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu \]  

which ensure the absolute anticommutativity (i.e. \( ss + \bar{s}s \equiv \{s, \bar{s}\} = 0 \)) of the above nilpotent (anti-)BRST symmetry transformations.

We wrap up this section with the following comments:

(i) the nilpotency and absolute anticommutativity of the (anti-)BRST symmetry transformations are the indispensable consequences of the superfield approach to the BRST formalism [5];

(ii) there are three CF-type restrictions (cf (22)) for the 4D Abelian three-form gauge theory whereas there is one each for the 4D Abelian two-form [11] and 4D (non-)Abelian one-form gauge theories [6];

(iii) two of the CF-type restrictions in (22) are fermionic in nature and one of them is bosonic (whereas for the Abelian two-form and (non-)Abelian one-form gauge theories only bosonic-type CF restriction exists). For the Abelian one-form gauge theory, the CF-type restriction is trivial;

(iv) the CF-type restriction is one of the key properties of any arbitrary \( p \)-form gauge theory described in the framework of BRST formalism, and

(v) the CF-type restrictions, that emerge from superfield formalism, are always (anti-)BRST invariant relationships (e.g. \( s, \bar{s} \) \( \{f_\mu + F_\mu - \partial_\mu C_1\} = 0 \), \( s, \bar{s} \) \( \{\tilde{f}_\mu + \tilde{F}_\mu - \partial_\mu \tilde{C}_1\} = 0 \), \( s, \bar{s} \) \( \{B_{\mu\nu} + \bar{B}_{\mu\nu} - \partial_\mu \phi_\nu - \partial_\nu \phi_\mu\} = 0 \).
4.2. Lagrangian densities: nilpotent (anti-)BRST symmetry transformations

With the help of the (anti-)BRST symmetry transformations (19) and (20), one can derive the (anti-)BRST invariant Lagrangian density for the Abelian three-form gauge theory as

\[ \mathcal{L}_B = \frac{1}{23} H_{\mu \nu \rho} H^{\mu \nu \rho} + s \left\{ \frac{1}{2} \bar{C}_2 C_2 - \frac{1}{2} \bar{C}_1 C_1 - \frac{1}{2} \bar{C}_\mu C^{\mu \nu} - \bar{B}_\mu B^{\mu \nu} - \frac{1}{2} \phi_\mu \phi^{\mu} - \frac{1}{6} B_{\mu \nu \rho} B^{\mu \nu \rho} \right\}, \]

\[ \bar{L}_B = \frac{1}{23} H_{\mu \nu \rho} H^{\mu \nu \rho} - s \left\{ \frac{1}{2} \bar{C}_2 C_2 - \frac{1}{2} \bar{C}_1 C_1 - \frac{1}{2} \bar{C}_\mu C^{\mu \nu} - \bar{B}_\mu B^{\mu \nu} - \frac{1}{2} \phi_\mu \phi^{\mu} - \frac{1}{6} B_{\mu \nu \rho} B^{\mu \nu \rho} \right\}. \]

(23)

where \( H_{\mu \nu \rho} = \partial_\mu B_{\nu \rho} - \partial_\nu B_{\rho \mu} + \partial_\rho B_{\mu \nu} \) is the totally antisymmetric curvature tensor derived from the four-form \( H^{(4)} = dB^{(3)}\equiv [(dx^\mu \wedge dx^\nu \wedge dx^\delta \wedge dx^\gamma)]/(4!)H_{\mu \nu \rho}. \)

Here \( d = dx^\mu \partial_\mu \) (with \( d^2 = 0 \)) is the exterior derivative and the three-form \( B^{(3)} = [(dx^\mu \wedge dx^\nu \wedge dx^\delta)]/(3!)B_{\mu \nu \rho} \) defines the totally antisymmetric tensor gauge field \( B_{\mu \nu \rho} \).

It is noted that, within the square brackets of (23), we have taken the combination of terms like (26) and (27).

Modulo some total spacetime derivatives, the explicit computation of the square bracketed terms yields the following:

\[ s \left\{ \frac{1}{2} \bar{C}_2 C_2 - \frac{1}{2} \bar{C}_1 C_1 - \frac{1}{2} \bar{C}_\mu C^{\mu \nu} - \bar{B}_\mu B^{\mu \nu} - \frac{1}{2} \phi_\mu \phi^{\mu} - \frac{1}{6} B_{\mu \nu \rho} B^{\mu \nu \rho} \right\} \]

\[ = (\partial_\mu B^{\mu \nu}) B_{\nu \rho} + \frac{1}{2} B_{\mu \nu} B^{\mu \nu} + (\partial_\mu \bar{C}_\nu + \partial_\nu \bar{C}_\mu + \partial_\rho \bar{C}_\mu)(\partial^\mu C^{\nu \rho}) \]

\[ - (\partial_\mu \bar{B}_\nu - \partial_\nu \bar{B}_\mu)(\partial^\mu B^{\nu \rho}) - B B_2 - \frac{1}{2} B_2^2 + (\partial_\mu \bar{C}^{\mu \nu}) F_\nu - (\partial_\mu C^{\mu \nu}) \bar{F}_\nu \]

\[ + \partial_\mu \bar{C}_2 \partial^\mu C_2 + \bar{F}_\mu f_\mu - \bar{F}_\mu F_\mu + (\partial \cdot \beta) B_2 + (\partial \cdot \phi) B_1 - (\partial \cdot \bar{B}) B. \]

(24)

\[-s \left\{ \frac{1}{2} \bar{C}_2 C_2 - \frac{1}{2} \bar{C}_1 C_1 - \frac{1}{2} \bar{C}_\mu C^{\mu \nu} - \bar{B}_\mu B^{\mu \nu} - \frac{1}{2} \phi_\mu \phi^{\mu} - \frac{1}{6} B_{\mu \nu \rho} B^{\mu \nu \rho} \right\} \]

\[ = -(\partial_\mu B^{\mu \nu}) \bar{B}_{\nu \rho} + \frac{1}{2} B_{\mu \nu} \bar{B}^{\mu \nu} + (\partial_\mu \bar{C}_\nu + \partial_\nu \bar{C}_\mu + \partial_\rho \bar{C}_\mu)(\partial^\mu C^{\nu \rho}) \]

\[ - (\partial_\mu \bar{B}_\nu - \partial_\nu \bar{B}_\mu)(\partial^\mu \bar{B}^{\nu \rho}) - B B_2 - \frac{1}{2} B_2^2 - (\partial_\mu \bar{C}^{\mu \nu}) F_\nu + (\partial_\mu C^{\mu \nu}) \bar{F}_\nu \]

\[ + \partial_\mu \bar{C}_2 \partial^\mu C_2 + \bar{F}_\mu f_\mu - \bar{F}_\mu F_\mu + (\partial \cdot \beta) B_2 + (\partial \cdot \phi) B_1 - (\partial \cdot \bar{B}) B. \]

(25)

The difference in the above explicit computations is due to the fact that the (anti-)BRST symmetry transformations are anticommuting only on the constrained surface defined by the CF-type conditions (22).

With the help of the CF-type restrictions (22), we can re-express the Lagrangian density \( \bar{L}_B \) in an appropriate form as

\[ \mathcal{L}_B = \frac{1}{23} H_{\mu \nu \rho} H^{\mu \nu \rho} + B^{\mu \nu}(\partial^\mu B_{\mu \nu} + \frac{1}{2}[\partial_\mu \phi_\nu - \partial_\nu \phi_\mu]) \]

\[ = \frac{1}{2} B_{\mu \nu} B^{\mu \nu} + (\partial_\mu \bar{C}_\nu + \partial_\nu \bar{C}_\mu + \partial_\rho \bar{C}_\mu)(\partial^\mu C^{\nu \rho}) - (\partial \cdot \bar{B}) B \]

\[ = (\partial_\mu \bar{B}_\nu - \partial_\nu \bar{B}_\mu)(\partial^\mu \bar{B}^{\nu \rho}) - B B_2 - \frac{1}{2} B_2^2 + (\partial_\mu \bar{C}^{\mu \nu}) F_\nu + (\partial_\mu C^{\mu \nu}) \bar{F}_\nu \]

\[ - (\partial_\mu C^{\mu \nu}) \bar{F}_\nu + \partial_\mu \bar{C}_2 \partial^\mu C_2 + (\partial \cdot \beta) B_2 + (\partial \cdot \phi) B_1. \]

(26)

It should be noted that, in the above, we have taken expression (24) but have replaced \( \bar{B}_{\mu \nu}, \bar{F}_\mu \) by exploiting the CF-type conditions in (22). In an exactly similar fashion, the

8 In fact, there are, in total, six more possibilities of expressing the equivalent Lagrangian densities with the help of CF-type restrictions (22). It is, however, only the equivalent forms like (26) and (27) that have perfect transformations like (28) and (29).
Lagrangian density $\mathcal{L}_B$ can be rewritten as

$$\mathcal{L}_B = \frac{1}{2} \bar{H}^{\mu \nu \rho} (\bar{B}^{\mu \nu} - \bar{B}^{\nu \mu} - \frac{1}{2} \partial_{\mu} \partial_{\nu} \partial_{\rho}) + \frac{1}{2} \bar{B}_{\mu \nu} \bar{B}^{\mu \nu} - (\bar{B}_{\mu \nu} - \bar{B}_{\nu \mu}) \partial_{\mu} \partial_{\nu} \phi \partial_{\rho} - \frac{1}{2} \partial_{\mu} \partial_{\nu} \partial_{\rho} [\bar{C}_{\mu \nu} + \bar{C}_{\nu \mu}] - \frac{1}{2} \partial_{\mu} \partial_{\nu} \partial_{\rho} [\bar{C}_{\mu \nu} + \bar{C}_{\nu \mu}]$$

Therefore, we have taken expression (25) but have substituted for $B_{\mu \nu}, f_{\mu}, \nabla_{\mu}$ by exploiting the CF-type conditions in (22).

Thus, we note that both the above-coupled Lagrangian densities are equivalent with respect to the BRST transformations (19) on the submanifold defined by the CF-type field equations (22).

In exactly the above fashion, we note that, under the anti-BRST symmetry transformations (20), the coupled Lagrangian densities transform as

$$sL_B = \partial_{\mu} [(\partial^{\mu} C^{\nu} + \partial^{\nu} C^{\mu} + \partial^{\mu} \bar{C}^{\nu} - \partial^{\nu} \bar{C}^{\mu}) B_{\nu \eta} + B^{\mu \nu} f_{\nu} - (\partial^{\mu} \bar{B}^{\nu} - \partial^{\nu} \bar{B}^{\mu}) \bar{F}_{\nu}$$

$$+ B_{1} \bar{F}^{\mu} - B_{2} B^{\mu \nu} C_{2}],$$

$$s\bar{L}_B = -\partial_{\mu} [(\partial^{\mu} C^{\nu} + \partial^{\nu} C^{\mu} + \partial^{\mu} \bar{C}^{\nu} - \partial^{\nu} \bar{C}^{\mu}) \bar{B}_{\nu \eta} + B^{\mu \nu} f_{\nu} - (\partial^{\mu} B^{\nu} - \partial^{\nu} B^{\mu}) f_{\nu} - B_{1} f^{\mu}$$

$$+ B \bar{F}^{\mu} - B_{2} B^{\mu \nu} C_{2} + B^{\mu \nu}(\partial_{\nu} f_{\eta} - \partial_{\eta} f_{\nu} + \bar{C}^{\mu \nu} \partial_{\nu} B + C^{\mu \nu} \partial_{\nu} B) + X, \quad (29)$$

where the extra piece $X$, in the above, is given below

$$X = (\partial^{\mu} C^{\nu} + \partial^{\nu} C^{\mu} + \partial^{\mu} \bar{C}^{\nu} - \partial^{\nu} \bar{C}^{\mu}) \bar{B}_{\nu \eta} + B^{\mu \nu} f_{\nu} - (\partial^{\mu} \bar{B}^{\nu} - \partial^{\nu} \bar{B}^{\mu}) \bar{F}_{\nu}$$

$$+ B_{1} \bar{F}^{\mu} - B_{2} B^{\mu \nu} C_{2} \right],$$

$$s\bar{L}_B = \partial_{\mu} [(\partial^{\mu} \bar{C}^{\nu} + \partial^{\nu} \bar{C}^{\mu} + \partial^{\mu} C^{\nu} - \partial^{\nu} C^{\mu}) B_{\nu \eta} + \bar{B}^{\mu \nu} \bar{f}_{\nu} - (\partial^{\mu} \bar{B}^{\nu} - \partial^{\nu} \bar{B}^{\mu}) \bar{F}_{\nu}$$

$$+ B_{1} \bar{F}^{\mu} + B_{2} B^{\mu \nu} C_{2}],$$

$$s\bar{L}_B = \partial_{\mu} [(\partial^{\mu} \bar{C}^{\nu} + \partial^{\nu} \bar{C}^{\mu} + \partial^{\mu} C^{\nu} - \partial^{\nu} C^{\mu}) B_{\nu \eta} + \bar{B}^{\mu \nu} \bar{f}_{\nu} - (\partial^{\mu} \bar{B}^{\nu} - \partial^{\nu} \bar{B}^{\mu}) \bar{F}_{\nu}$$

$$+ B_{1} \bar{F}^{\mu} + B_{2} B^{\mu \nu} C_{2} + (B^{\mu \nu})(\partial_{\nu} \bar{f}_{\eta} - \partial_{\eta} \bar{f}_{\nu} + \bar{C}^{\mu \nu} \partial_{\nu} B + C^{\mu \nu} \partial_{\nu} B) + Y, \quad (32)$$

where the extra piece $Y$, in the above equation, is

$$Y = (\partial^{\mu} C^{\nu} + \partial^{\nu} C^{\mu} + \partial^{\mu} \bar{C}^{\nu} - \partial^{\nu} \bar{C}^{\mu}) \partial_{\mu} [\bar{B}_{\nu \eta} + B_{\nu \eta} - (\partial_{\nu} \partial_{\eta} \partial_{\mu} f_{\eta}) + B^{\mu \nu} \partial_{\mu} [\bar{F}_{\nu} + F_{\nu} - \partial_{\nu} \bar{C}_{1}]$$

$$- (\bar{B}_{\mu \nu} + B_{\mu \nu} - (\partial_{\nu} \partial_{\mu} \partial_{\eta} f_{\eta}) + B^{\mu \nu} \partial_{\mu} [\bar{F}_{\nu} + F_{\nu} - \partial_{\nu} \bar{C}_{1}])$$

$$- (\partial^{\mu} \bar{B}^{\nu} - \partial^{\nu} \bar{B}^{\mu}) \partial_{\mu} [\bar{F}_{\nu} + F_{\nu} - \partial_{\nu} \bar{C}_{1}],$$

(33)

It is gratifying to note that both the above-coupled Lagrangian densities respect the off-shell nilpotent (anti-)BRST symmetry transformations on the submanifold of the spacetime that is described by the constrained CF-type field conditions (22). These features, under the (anti-)BRST symmetry transformations, are exactly like the ones we come across in the context of the BRST approach to 4D non-Abelian one-form gauge theory (see, e.g., [9]).
4.3. Ghost Lagrangian density: global scale and discrete symmetry transformations

The ghost parts of the Lagrangian densities (26) and (27), even though they look quite different in their appearance, are actually equal on the submanifold defined by the CF-type restrictions (22). Thus, let us take one of them as

\[
L^{(B)}_{g} = (\partial_\mu \bar{\partial}_\nu C_{\nu \eta} + \partial_\nu \bar{\partial}_\mu C_{\mu \nu}) (\partial^\mu \bar{\partial}^\nu C_{\eta \mu} - (\partial_\mu \bar{\partial}_\nu - \partial_\nu \bar{\partial}_\mu) (\partial^\mu \bar{\partial}^\nu C_{\eta \mu} - B B_2 \\
+ (\partial_\mu \bar{\partial}_\nu + \partial^\nu \bar{\partial}_\mu)f_\mu - (\partial_\mu C_{\mu \nu} - \partial^\nu C_1) F_\nu + \partial_\mu \bar{\partial}_\nu \partial^\mu \bar{\partial}^\nu C_{\eta \mu} + (\partial \cdot \bar{\partial} C_{\eta \mu} - B B_2).)
\]

The above Lagrangian density respects the following continuous global scale transformations for the ghost fields:

\[
\begin{align*}
C_{\mu \nu} &\rightarrow e^{\Omega \bar{C}_{\mu \nu}}, \\
\bar{C}_{\mu \nu} &\rightarrow e^{-\Omega \bar{C}_{\mu \nu}}, \\
C_1 &\rightarrow e^{i \Omega C_1}, \\
\bar{C}_1 &\rightarrow e^{-i \bar{C}_1}, \\
f_\mu &\rightarrow e^{\Omega f_\mu}, \\
F_\mu &\rightarrow e^{-\Omega F_\mu}, \\
\bar{f}_\mu &\rightarrow e^{i \bar{f}_\mu}, \\
\bar{F}_\mu &\rightarrow e^{-i \bar{F}_\mu}, \\
\beta_\mu &\rightarrow e^{2 \Omega \beta_\mu}, \\
\bar{\beta}_\mu &\rightarrow e^{-2 \bar{\beta}_\mu}, \\
B &\rightarrow e^{2 \Omega B}, \\
\bar{B} &\rightarrow e^{-2 \bar{B}}, \\
C_2 &\rightarrow e^{3 \Omega C_2}, \\
\bar{C}_2 &\rightarrow e^{-3 \bar{C}_2},
\end{align*}
\]

where \( \Omega \) is a global scale infinitesimal parameter and the numbers, present in the exponentials, denote the ghost numbers of the corresponding dynamical and/or auxiliary (anti-)ghost fields.

In addition to the above global continuous symmetry transformations, the above Lagrangian density \( L^{(B)}_{g} \) also respects the following discrete symmetry transformations amongst the (anti-)ghost fields:

\[
\begin{align*}
C_{\mu \nu} &\rightarrow \pm i \bar{C}_{\mu \nu}, \\
\bar{C}_{\mu \nu} &\rightarrow \pm i C_{\mu \nu}, \\
C_1 &\rightarrow \mp i C_1, \\
\bar{C}_1 &\rightarrow \mp i \bar{C}_1, \\
f_\mu &\rightarrow \pm i F_\mu, \\
F_\mu &\rightarrow \pm i f_\mu, \\
\beta_\mu &\rightarrow \mp i \beta_\mu, \\
\bar{\beta}_\mu &\rightarrow \mp i \bar{\beta}_\mu, \\
B &\rightarrow \mp i B, \\
\bar{B} &\rightarrow \mp i \bar{B}, \\
C_2 &\rightarrow \pm i \bar{C}_2, \\
\bar{C}_2 &\rightarrow \pm i C_2.
\end{align*}
\]

The above symmetry transformations enable us to go from the BRST symmetry transformations to the anti-BRST symmetry transformations and vice versa. The above transformations have been written for the ghost part of the Lagrangian density \( L_\beta \). However, similar kind of transformations can be written out for the ghost part of the Lagrangian density \( L_{\bar{\beta}} \) where, in addition to the above transformations, we require

\[
\begin{align*}
\bar{f}_\mu &\rightarrow \pm i F_\mu, \\
F_\mu &\rightarrow \pm i \bar{f}_\mu.
\end{align*}
\]

We close this section with the remark that the ghost part of the coupled Lagrangian densities remains invariant under (35)–(37). Furthermore, it can be checked that the CF-type restrictions also remain invariant under the transformations (35)–(37) because \( f_\mu + F_\mu = \pm \partial_\mu C_1 \) and \( \bar{f}_\mu + \bar{F}_\mu = \pm \partial_\mu \bar{C}_1 \) are allowed by the anticommutativity property. It is trivial to state that \( B_{\mu \nu} \rightarrow B_{\mu \nu}, \bar{B}_{\mu \nu} \rightarrow B_{\mu \nu}, \bar{B}_{\mu \nu} \rightarrow B_{\mu \nu}, \phi_\mu \rightarrow \phi_\mu \) under the ghost transformations because these fields carry a ghost number equal to zero.

4.4. Conserved charges: algebraic structures

It is clear from equations (28) and (29) that the Lagrangian densities \( L_\beta \) and \( L_{\bar{\beta}} \) transform precisely to the total spacetime derivative under the BRST and anti-BRST transformations.
(19) and (20). According to the Noether theorem, we have the following expressions for the conserved currents:

\[ J^{\mu}_{(B)} = s\Phi_i \frac{\partial L_B}{\partial (\partial_\mu \Phi_i)} - Z^\mu, \]
\[ J^{\mu\nu}_{(B)} = s\Phi_i \frac{\partial L_B}{\partial (\partial_\mu \partial_\nu \Phi_i)} - S^{\mu\nu}, \]

where the generic dynamical field \( \Phi_i = B_{\mu\nu}, C_{\mu}, \tilde{C}_{\mu}, \beta_\mu, \tilde{\beta}_\mu, \phi_\mu, \tilde{\phi}_\mu, \tilde{C}_1, C_1, C_2, \tilde{C}_2 \) and the explicit expression for \( Z^\mu \) and \( S^{\mu\nu} \) are (cf (28) and (31))

\[ Z^\mu = (\partial^\mu \Phi + \partial^\nu C^{\mu\nu} + \partial^\nu \tilde{C}^{\mu\nu}) B_{\nu\eta} + B^{\nu\mu} f_\nu + (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) F_\nu + B_1 f^\mu - B F^\mu + B_2 \partial^\mu C_2, \]
\[ S^{\mu\nu} = (\partial^\mu \tilde{C}^{\nu\eta} + \partial^\mu \tilde{C}^{\mu\eta} + \partial^\nu \tilde{C}^{\mu\eta}) B_{\nu\eta} + B^{\nu\mu} f_\nu - (\partial^\mu \tilde{\beta}^\nu - \partial^\nu \tilde{\beta}^\mu) F_\nu + B_1 \tilde{f}^\mu + B_2 F^\mu - B \partial^\mu \tilde{C}_2. \]

The conserved currents that emerge from the above equations are as follows:

\[ J^{\mu}_{(B)} = H^{\mu\nu\rho}(\partial_\rho \tilde{C}) + (\partial^\mu \Phi + \partial^\nu C^{\mu\nu} + \partial^\nu \tilde{C}^{\mu\nu}) B_{\nu\eta} + B^{\nu\mu} f_\nu + B_1 f^\mu + B_2 \partial^\mu C_2, \]
\[ J^{\mu\nu}_{(B)} = H^{\mu\nu\rho}(\partial_\rho \tilde{C}) + (\partial^\mu \tilde{C}^{\nu\eta} + \partial^\mu \tilde{C}^{\mu\eta} + \partial^\nu \tilde{C}^{\mu\eta}) B_{\nu\eta} + B^{\nu\mu} f_\nu + B_1 \tilde{f}^\mu + B_2 \partial^\mu \tilde{C}_2. \]

where all the derivatives of (38) have been calculated from \( L_B \) and \( \mathcal{L}_B \).

The conservation law (i.e., \( \partial_\mu J^\mu_{(B)} = 0 \)), \( i = B, \tilde{B} \) can be proven by taking into account the following equations of motion that are derived from \( L_B \):

\[ \Box B_{\mu\nu} = 0, \quad B_{\mu\nu} = (\partial^\rho B_{\rho\mu\nu}) + \frac{1}{2}(\partial_\rho \phi_\nu - \partial_\nu \phi_\rho), \quad \Box \beta_\mu = 0, \]
\[ \Box \tilde{\beta}_\mu = 0, \quad \partial \cdot F = 0, \quad \partial \cdot \tilde{F} = 0, \quad \partial \cdot f = 0, \quad \partial_\mu B^{\mu\nu} + \partial^\nu B_1 = 0, \]
\[ \partial \cdot \tilde{f} = 0, \quad \partial_\mu C^{\mu\nu} = \partial^\nu C_1, \quad \partial_\mu \tilde{C}^{\mu\nu} = -\partial^\nu \tilde{C}_1, \]
\[ \Box C_1 = 0, \quad \Box \tilde{C}_1 = 0, \quad \Box C_2 = 0, \quad \Box \tilde{C}_2 = 0, \]
\[ B = \partial \cdot \beta, \quad B_1 = \partial \cdot \phi, \quad B_2 = - (\partial \cdot \tilde{\beta}), \quad \phi_\mu + \partial_\mu (\partial \cdot \phi) = 0, \]
\[ \Box C_{\mu\nu} + \frac{1}{2}(\partial_\mu f_\nu - \partial_\nu f_\mu) = 0, \quad \Box \tilde{C}_{\mu\nu} + \frac{1}{2}(\partial_\mu \tilde{f}_\nu - \partial_\nu \tilde{f}_\mu) = 0, \]
\[ \partial_\mu H^{\mu\nu\rho} + (\partial^\gamma B^{\nu\rho} + \partial^\gamma B^{\rho\nu} + \partial^\rho B^{\gamma\nu}) = 0. \]

The Euler–Lagrange equations of motion, that emerge from \( \mathcal{L}_B \), are the same as the above but for the following differences:

\[ \tilde{B}_{\mu\nu} = -(\partial^\rho B_{\rho\mu\nu}) + \frac{1}{2}(\partial_\rho \phi_\nu - \partial_\nu \phi_\rho), \quad \partial_\mu \tilde{B}^{\mu\nu} + \partial^\nu B_1 = 0, \]
\[ \Box \tilde{C}_{\mu\nu} = - \frac{1}{2}(\partial_\mu F_\nu - \partial_\nu F_\mu) = 0, \quad \Box \tilde{C}_{\mu\nu} = - \frac{1}{2}(\partial_\mu \tilde{F}_\nu - \partial_\nu \tilde{F}_\mu) = 0, \]
\[ \partial_\mu H^{\mu\nu\rho} - (\partial^\gamma \tilde{B}^{\nu\rho} + \partial^\gamma \tilde{B}^{\rho\nu} + \partial^\rho \tilde{B}^{\gamma\nu}) = 0. \]

It is elementary to state that (42) and (43) are exploited, in a judicious manner, for the proof of the conservation laws.
The continuous ghost symmetry transformations of (35) lead to the derivation of the ghost conserved current as given below:

\[
J^\mu_{(g)} = (\partial^\mu \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^\mu_{\nu\eta} + \partial^\eta \bar{C}^{\nu\mu}) C_{\nu\eta} + (\partial^\mu C^{\nu\eta} + \partial^\nu C^\mu_{\nu\eta} + \partial^\eta C^{\mu\nu}) \bar{C}_{\nu\eta} - 2(\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \beta_\eta + 2(\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \bar{\beta}_\eta + 2B \beta^\mu + 2B_2 \beta^\mu + 3(\partial^\mu \bar{C}_2) C_2 - 3\bar{C}_2 \partial^\mu C_2 - \bar{C}^{\mu\nu} F_\nu + C_1 F^\mu - \bar{C}_1 f^\mu. \tag{44}
\]

The conservation law (i.e. \( \partial_\mu J^\mu_{(g)} = 0 \)) can be proven by taking the help of the equations of motion (42) and (43). The conserved (i.e. \( Q(r) = 0 \), \( r = B, \bar{B}, g \)) charges (i.e. \( Q(r) = \int d^3x J_{(r)}^\mu, r = B, \bar{B}, g \)), that emerge from the conserved currents (40), (41) and (44), are as follows:

\[
Q_{(B)} = \int d^3x [F^{0jk}(\partial_j C_{k}) + (\partial^0 C^{\nu\eta} + \partial^\nu C^0_{\nu\eta}) B_{\nu\eta} + B_1 f^0 - (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) \partial_i C_2 - (\partial^0 \beta^i - \partial^i \beta^0) \bar{F}_1 + B_0^0 f_1 + B_2 C_2 - (\partial^0 C^\nu_{\nu\eta} + \partial^\nu C_{\nu\eta} + \partial^\eta C^\nu_{\nu\mu})(\partial_i \beta_\eta - \partial_\eta \beta_i) - B \bar{F}^0], \tag{45}
\]

\[
Q_{(\bar{B})} = \int d^3x [F^{0jk}(\partial_j \bar{C}_{k}) - (\partial^0 C^\nu_{\nu\eta} + \partial^\nu C^0_{\nu\eta}) B_{\nu\eta} + B_1 f^0 - (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) \partial_i \bar{C}_2 - (\partial^0 \beta^i - \partial^i \beta^0) \bar{F}_1 + B_0^0 \bar{f}_1 - B \bar{C}_2 + (\partial^0 C^\nu_{\nu\eta} + \partial^\nu C_{\nu\eta} + \partial^\eta C^\nu_{\nu\mu})(\partial_i \bar{\beta}_\eta - \partial_\eta \bar{\beta}_i) + B_2 F^0], \tag{46}
\]

\[
Q_{(g)} = \int d^3x [3\bar{C}_2 C_2 - 3\bar{C}_2 \bar{C}_2 + (\partial^0 \bar{C}^{\nu\eta} + \partial^\nu \bar{C}^0_{\nu\eta}) C_{\nu\eta} - \bar{C}_1 f^0 + 2(\partial^0 \beta^i - \partial^i \beta^0) \bar{F}_i - 2(\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) \beta_i + C_0 f_i - C_0 \bar{F}_i + 2B \beta^0 + 2B_2 \beta^0 + C_1 \bar{F}^0 + (\partial^0 C^{\nu\eta} + \partial^\nu C^0_{\nu\eta} + \partial^\eta C^0_{\nu\mu}) \bar{C}_{\nu\eta}] \tag{47}.
\]

The above conserved charges are the generators of the nilpotent and continuous (anti-)BRST transformations as well as continuous ghost scale transformations. The application of the continuous symmetry transformations on the above charges produces the following algebra:

\[
s Q_{(B)} = -i[Q_{(B)}, Q_{(B)}] = 0 \Rightarrow Q_{(B)}^2 = 0,
\]

\[
\bar{s} Q_{(B)} = -i[Q_{(B)}, \bar{Q}_{(B)}] = 0 \Rightarrow Q_{(B)} = 0,
\]

\[
s Q_{(\bar{B})} = -i[Q_{(\bar{B}), \bar{Q}_{(\bar{B})}}] = 0 \Rightarrow \{Q_{(\bar{B})}, \bar{Q}_{(\bar{B})}\} = 0,
\]

\[
s Q_{(g)} = -i[Q_{(g)}, Q_{(g)}] = -Q_{(g)} \Rightarrow i[Q_{(g)}, Q_{(g)}] = +Q_{(g)},
\]

\[
\bar{s} Q_{(g)} = -i[Q_{(g)}, \bar{Q}_{(g)}] = +Q_{(g)} \Rightarrow i[Q_{(g)}, \bar{Q}_{(g)}] = -Q_{(g)}. \tag{48}
\]

The above algebra is the standard algebra in the BRST formalism. In the above, we have not written some more transformations on charges. However, those are implied from the above (e.g. \( \bar{s} Q_{(B)} = \{Q_{(B)}, Q_{(B)}\} = 0, \) etc). The above algebra ensures that the ghost number of the BRST charge is (+1) and that of the anti-BRST charge is (−1).

5. BRST and anti-BRST for a 2-gerbe field theory

A 2-gerbe can be described by a quadruple \((C, B, A, f)\), defined by a three-form \(C\), a two-form \(B\), a one-form \(A\) and a scalar \(f\) with the following relations. Given a covering \(\{U_i\}\) of \(M\), we associate with each \(U_i\) a three-form \(C_i\). On a double intersection \(U_i \cap U_j\), we have
Figure 2. A schematic view of BRST and anti-BRST transformations for 2-gerbes.

\[ C_i - C_j = dB_{ij}, \]

On the triple intersections \( U_i \cap U_j \cap U_k \), we must have \( B_{ij} + B_{jk} + B_{ki} = dA_{ijk} \).

On quadruple intersections \( U_i \cap U_j \cap U_k \cap U_l \), we must have \( A_{ijl} - A_{ijk} + A_{jkl} - A_{ikl} = df_{ijkl} \)

and on quintuple intersection \( U_i \cap U_j \cap U_k \cap U_l \cap U_m \) the following integral cocycle condition must be satisfied:

\[ f_{ijlm} - f_{ijkm} + f_{ijkl} - f_{iklm} + f_{jklm} = 2\pi n, \quad n = 0, 1, 2, 3, \ldots \] (49)

The two quadruples \((C, B, A, f)\) and \((C', B', A', f')\) are gauge equivalent if they satisfy

\[ B'_{ij} = B_{ij} + \Gamma_i - \Gamma_j + db_{ij}, \] (50)

\[ A'_{ijk} = A_{ijk} + b_{ij} + b_{jk} + b_{ki} + d\gamma_{ijk}, \] (51)

\[ f'_{ijkl} = f_{ijkl} + \gamma_{ijl} - \gamma_{ijk} + \gamma_{jkl} - \gamma_{ikl}. \] (52)

Starting from these transformation properties, we can define the BRST and anti-BRST transformations appropriate for the field contents of a 2-gerbe field theory. We have to introduce a whole lot of auxiliary fields, even more than in the previous cases, in order to capture its ample gauge freedom in the language of the absolute anticommutativity property. The overall field content is summarized in figure 2.
The BRST and anti-BRST transformations, for the boundary faces of the figure, are as follows:

\[ sC = d\Gamma, \quad s\Gamma = db, \quad sb = de, \]
\[ sB = \Gamma + d\gamma, \quad s\gamma = -b + dt, \quad st = e = st', \]
\[ sA = \gamma + \eta + d\xi, \quad s\eta = b - dt', \quad s\xi = t - t' = -s\zeta, \]
\[ sf = \xi + \zeta, \]

\[ (53) \]

\[ \bar{s}C = d\bar{\Gamma}, \quad \bar{s}\Gamma = d\bar{b}, \quad \bar{sb} = d\bar{e}, \]
\[ \bar{s}B = \bar{\Gamma} + d\bar{\gamma}, \quad \bar{s}\gamma = -\bar{b} + \bar{d}t, \quad \bar{s}t = \bar{\epsilon} = \bar{st}' = \bar{s}\zeta, \]
\[ \bar{s}f = \bar{\xi} + \bar{\zeta}. \]

\[ (54) \]

The remaining transformations are

\[ s\bar{\gamma} = \bar{\Xi} + \bar{d}\bar{g}, \quad s\bar{\eta} = -\bar{\Xi} - \bar{d}\bar{g}, \quad s\bar{g} = -\lambda, \]
\[ s\bar{X} = -\rho, \quad s\bar{X} = -d\lambda, \quad s\bar{g} = \lambda, \]
\[ s\bar{\xi} = \bar{s}\bar{\zeta} = \frac{1}{2} (g + \bar{g}), \quad s\bar{t} = s\bar{t}' = -\bar{\lambda}, \]

\[ (55) \]

and

\[ \bar{s}\Gamma = \bar{K}, \quad \bar{s}b = -\bar{\rho}, \quad \bar{s}K = d\rho, \]
\[ \bar{s}\bar{\gamma} = \bar{\Xi} + \bar{d}\bar{g}, \quad \bar{s}\bar{\eta} = -\bar{\Xi} - \bar{d}\bar{g}, \quad \bar{s}\bar{g} = -\bar{\lambda}, \]
\[ \bar{s}\bar{X} = -\bar{\rho}, \quad \bar{s}\bar{X} = d\bar{\lambda}, \quad \bar{s}\bar{g} = \bar{\lambda}, \]
\[ \bar{s}\bar{\xi} = \bar{s}\bar{\zeta} = -\frac{1}{2} (g + \bar{g}), \quad \bar{s}\bar{t} = \bar{s}\bar{t}' = \lambda. \]

\[ (56) \]

Moreover, we also have \( s [\epsilon, \bar{\epsilon}, K, \rho, \bar{\rho}, \lambda, \bar{\lambda}] = 0 \) and \( \bar{s} [\epsilon, \bar{\epsilon}, \bar{K}, \rho, \bar{\rho}, \lambda, \bar{\lambda}] = 0 \).

With these transformation properties, the anticommutator of the BRST and anti-BRST transformations \( (s\bar{s} + \bar{s}s) \) annihilates all the above fields, except for \( C \) and \( A \), which require the CF condition

\[ K + \bar{K} + d(X + \bar{X}) = 0. \]

\[ (57) \]

Once again, we believe, observing the above BRST and anti-BRST algebra, that a geometric set-up, similar to the one for 1-gerbes, could be constructed. However, since a mathematical formulation (analogous to Hitchin’s for 1-gerbes) is still missing for 2-gerbes, we leave this problem open.

### 5.1. Correspondence with the field theory model

Let us now compare the formulas of this section with the field-theoretical derivation of section 4. It is clear that in section 4 we considered a reduced model, where the only non-ghost field is the three-form field. The correspondence with that Abelian three-form gauge
theory can be seen as follows. Suppress the two-form, one-form and zero-form (non-ghost) fields of the previous subsection. For the remaining fields the correspondence is as follows:

\[ \begin{align*} 
B_{\mu\nu} &\leftrightarrow C, & C_{\mu\nu} &\leftrightarrow \Gamma, & \bar{C}_{\mu\nu} &\leftrightarrow \bar{\Gamma}, \\
\beta_{\mu} &\leftrightarrow b, & \bar{\beta}_{\mu} &\leftrightarrow \bar{b}, & B_{\mu\nu} &\leftrightarrow K, \\
\bar{B}_{\mu\nu} &\leftrightarrow \bar{K}, & f_{\mu} &\leftrightarrow \rho, & F_{\mu} &\leftrightarrow -\rho, \\
\bar{f}_{\mu} &\leftrightarrow \bar{\rho}, & \bar{F}_{\mu} &\leftrightarrow \bar{\rho}, & C_{2} &\leftrightarrow \epsilon, \\
\bar{C}_{2} &\leftrightarrow \bar{\epsilon}, & C_{1} &\leftrightarrow \lambda, & \bar{C}_{1} &\leftrightarrow \bar{\lambda}, \\
B &\leftrightarrow 0, & B_{1} &\leftrightarrow 0, & B_{2} &\leftrightarrow 0, 
\end{align*} \] (58)

and

\[ \phi_{\mu} \leftrightarrow -\bar{X} - \bar{\bar{X}}. \] (59)

The ghost fields \( B, B_{1} \) are \( B_{2} \) are an example of the subalgebra that can be moded out. Upon moding it out, the first two CF constraints in equation (22) turn out to be trivial.

6. Conclusion

In this paper, we have proposed a geometrical interpretation of the BRST and anti-BRST algebra in an ordinary non-Abelian one-form gauge field theory. To be more precise, we have provided a geometrical interpretation of the CF constraints, which are needed in order to close the algebra. For this purpose, we have slightly modified the well-known geometric setting of the gauge theories based on the principal fiber bundles to allow for the non-trivial anti-commutator between ghosts and anti-ghosts fields.

We have remarked that if the gauge group is Abelian, the CF conditions are pointless. If an Abelian structure, however, involves a two-form gauge field, the non-trivial CF constraints turn up, once again. The geometry of such theories is that of 1-gerbes. This geometry dictates the BRST and anti-BRST structure separately, but a modification is necessary in order to interpret the absolute anticommutativity of the joint BRST and anti-BRST transformations. Based on the previous example of one-form gauge field theories, we have suggested how this modification can be implemented.

Finally, we have studied the case of Abelian gauge field theories involving three-form fields. We have derived the relevant BRST and anti-BRST transformations in the framework of quantum field theory (essentially by using the superfield method) and, in particular, we have worked out the specific CF conditions. The geometry relevant to this kind of theories is based on the 2-gerbe structure. We have derived the BRST and anti-BRST transformations from the 2-gerbe geometry and shown that they are compatible with those obtained via field theory methods. The geometrical interpretation of the CF conditions, in this context, has still to be spelled out in detail.

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