High-$T_c$ Superconductivity and Finite-Temperature Phase Diagram of the t-$t'$-$J$ Model

Dai-Wei Qu,1,2 Bin-Bin Chen,3 Xin Lu,4 Qiaoyi Li,4,5 Yang Qi,6,7,* Shou-Shu Gong,4,† Wei Li,5,2,8,† and Gang Su1,2,§

1Kavli Institute for Theoretical Sciences, University of Chinese Academy of Sciences, Beijing 100190, China
2CAS Center of Excellence in Topological Quantum Computation, University of Chinese Academy of Sciences, Beijing 100190, China
3Department of Physics and HKU-UCAS Joint Institute of Theoretical and Computational Physics, The University of Hong Kong, Pokfulam Road, Hong Kong, China
4School of Physics, Beihang University, Beijing 100191, China
5CAS Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China
6State Key Laboratory of Surface Physics and Department of Physics, Fudan University, Shanghai 200433, China
7Collaborative Innovation Center of Advanced Microstructures, Nanjing 210093, China
8Peng Huanwu Collaborative Center for Research and Education, Beihang University, Beijing 100191, China

The robust d-wave superconductive (SC) ground state was recently identified in the square-lattice t-$t'$-$J$ model with both the nearest-neighbor ($t$) and next-nearest-neighbor ($t'$) hoppings. However, its finite-temperature phase diagram remains elusive. It is therefore highly motivated to explore the proposed $d$-wave SC phase and possible unusual normal states at elevated temperatures. Here we exploit state-of-the-art thermal tensor network approach with implemented $U(1)_{\text{charge}} \times SU(2)_{\text{spin}}$ symmetry to accurately simulate the finite-temperature properties of this $t$-$t'$-$J$ model on cylindrical lattices. Below a temperature scale of $T^*_{\alpha}$, we find prominent SC fluctuations intertwined with charge order, where the $d$-wave pairing susceptibility diverges as $\chi_{SC} \sim 1/T^*_{\alpha}$ (with $\alpha \approx 1$). Above $T^*_c$ yet below a much higher crossover temperature $T^*$, the magnetic susceptibility is suppressed, resembling the pseudogap behavior observed in the underdoped cuprate superconductors. Remarkably, the SC onset temperature $T^*_c$ is found firstly enhanced and then decreased as the hole doping increases, which gives rise to a dome-like structure that reaches the highest value of $T^*_c \approx 0.05 \, t$ near the optimal doping, implying a high-temperature SC in the 2D limit.

Introduction.— Since the discovery of cuprate superconductors, understanding the exotic states in the phase diagram has become one of the major challenges of modern condensed matter and quantum physics. Besides the long-term quest for the pairing mechanism of high-temperature superconductivity (SC) [1–6], the overall understanding on the unusual normal states including the pseudogap (PG), charge density wave (CDW), and strange metal is believed to be even more fundamental, which contains not only the essential clue to unveil the pairing glue but also the new paradigm of strongly correlated systems [4]. Theoretically, the two-dimensional (2D) square-lattice Hubbard model [7, 8] and the related $t$-$J$ model [9, 10] are believed to capture certain essence of the many-electron properties in cuprates [11, 12]. However, lacking of well-controlled analytical solutions in 2D, our understandings of the possible $d$-wave SC, the exotic normal states, and their interplay in the Hubbard and $t$-$J$ models are far from clear.

Recently, there are intensive efforts and great progresses in the numerical simulations of these models [13–33]. For the nearest-neighbor (NN) Hubbard model with a large Hubbard $U$, the ground state simulations suggest the absence of long-range SC order even though the system has a strong SC response [25, 27]. Instead, a stripe order is revealed near $1/8$ doping [13, 14, 19, 25–27]. This stripe phase can persist to a finite temperature and a PG or PG-like phase has been suggested in the intermediate-temperature regime [34]. Interestingly, the recent large-scale density matrix renormalization group (DMRG) studies of the $t$-$J$ model find that the increased ratio of the next-nearest-neighbor (NNN) to the NN hopping $t'/t > 0$ can quickly suppress the stripe order and lead to a robust $d$-wave SC order that is likely intertwined with spin and charge orders depending on the tuning couplings and doping ratios [30–32, 35–39].

Despite the promising discovery at zero temperature, the finite-temperature properties of the superconducting $t$-$t'$-$J$ model are largely unexplored. It is of great interest to examine whether this canonical model contains the essential physics in the renowned phase diagram of cuprates. There are several fundamental questions to address: How do the SC and intertwined orders evolve and interplay as temperature increases? What is the characteristic temperature $T^*_{\alpha}$ of the (fluctuating) SC order, and does it suggest a high-temperature SC in 2D limit? Above $T^*_c$, is there an unusual normal state similar to the PG phase of the cuprates? To address these intriguing questions, efficient and unbiased finite-temperature calculations are requisite. While the thermodynamic Lanczos method [41–43] is limited within small system sizes [44] and the quantum Monte Carlo approaches [21–24] suffer from the sign problem at finite doping, the thermal tensor networks provide a powerful framework to simulate correlated systems at large scale, which are witnessing a rapid development in recent years [34, 45–62].

In this work, we study the finite-temperature properties of the square-lattice $t$-$t'$-$J$ model on the 2-leg and 4-leg systems [Figs. 1(a,d)] using the state-of-the-art fermion exponential tensor renormalization group (XTRG) approach [54, 61]. With tuning doping level and temperature, we establish a phase diagram with two temperature scales. Upon cooling,
both the magnetic susceptibility and specific heat curves exhibit maxima around the temperature that is close to the energy scale of the spin gap, which represents a possible onset of the PG regime. This upper temperature scale \( T^* \) decreases with increasing the doping ratio, resembling the observations in cuprates [63]. By further lowering the temperature, we identify a fluctuating SC phase below the estimated \( T^* \), which forms a dome-like regime with intertwined charge order and spin correlation. Our phase diagram obtained by the accurate XTRG calculations indicates that the \( t-t'-J \) model not only serves as a viable platform for exploring the \( d \)-wave SC ground state but also sheds light on the astonishingly complex cuprate phase diagram.

**Model and methods.**— The Hamiltonian of the square-lattice \( t-t'-J \) model considered in this work reads

\[
H = -\sum_{i,j,\sigma} t_{ij} (c_{i\sigma}^\dagger c_{j\sigma} + h.c.) + \sum_{i,j} J_{ij} (S_i \cdot S_j - \frac{1}{4} n_i n_j),
\]

where \( c_{i\sigma}^\dagger \) (\( c_{i\sigma} \)) is the electron creation (annihilation) operator with spin \( \sigma = \uparrow, \downarrow \), \( S_i \) denotes the spin-1/2 operator, and \( n_i = n_{i\uparrow} + n_{i\downarrow} \) is the particle number. We consider the NN hopping \( t \), the NNN hopping \( t' \) and also include the NNN hopping \( J' \) as \( J/J' = (t/t')^2 \) [c.f. Fig. 1(a)]. We project out the double occupancy in the local Hilbert space, which can be implemented conveniently via the tensor network library QSPACE [64, 65]. The \( U(1)_{\text{charge}} \times SU(2)_{\text{spin}} \) symmetry is implemented, and we set the NN coupling \( J = 1 \) as the energy scale in our XTRG simulation. The NN hopping \( t/J = 3 \) is chosen to mimic the large \( U/t = 12 \) in the original Hubbard model.

We denote the lattice as \( W \times L \) with \( W(L) \) being the width(length) of system, and below we focus on the 2-leg ladder [up to \( 2 \times 64 \), Fig. 1(d)] and 4-leg cylinder [\( 4 \times 20 \), Fig. 1(a)], whose ground-state properties have been systematically studied [36–40, 66–69]. We increase temperature from the ground state in the Luther-Emery liquid (LEL) phase with coexisting quasi-long-range charge density wave (CDW) order and \( d \)-wave SC order [70]. Typical model parameters with \( t'/t > 0 \) located in the LEL phase are selected based on the ground state phase diagrams [31, 40], i.e., \( J'/J = 0.05 \) \((t'/t \simeq 0.22)\) for the 2-leg ladder and \( J'/J = 0.0289 \) \((t'/t = 0.17)\) for the 4-leg cylinder. The latter system has an ordinary \( d \)-wave pairing instead of the special plaquette \( d \)-wave type [38, 39].

In XTRG simulations, we take the grand canonical ensemble with a chemical potential term \( -\mu \sum n_i \) for the hole doping. We keep up to 1000 \( U(1)_{\text{charge}} \times SU(2)_{\text{spin}} \) multiplets (\( \sim 2600 \) individual states, with truncation errors \( \epsilon \lesssim 10^{-4} \)) for 2-leg ladder and 1600 multiplets (\( \sim 4600 \) equivalent states, \( \epsilon \lesssim 10^{-3} \)) for 4-leg cylinder. In finite-\( T \) calculations, \( \epsilon \) directly reflects the estimated error of the free energy, and the obtained thermodynamic properties are found to converge well with the retained bond states (c.f., Fig. 2 and Supplementary Materials [71]). For computing the magnetic (pairing) susceptibilities, a small magnetic (pairing) field is applied, and only \( U(1)_{\text{charge}} \times U(1)_{\text{spin}} \times SU(2)_{\text{spin}} \) symmetry is implemented. Besides finite-\( T \) calculations, we also employ DMRG to study the corresponding ground states, with 4096 \( U(1)_{\text{charge}} \times SU(2)_{\text{spin}} \) multiplets (\( \sim 11,000 \) equivalent states) retained to guarantee high accuracies (\( \epsilon \lesssim 10^{-6} \)).

**Finite-temperature phase diagram.**— We first demonstrate our main findings in the temperature-doping phase diagram in Figs. 1(b) and (c), where \( \delta = 1 - n \) measures the hole doping (with \( n \) the electron density). We characterize SC by computing the pairing correlation \( \Phi_{\alpha\beta}(r) = \langle \Delta^\dagger_\alpha(r_0) \Delta_\beta(r_0 + r) \rangle \).
with $\Delta_{\alpha}(r) = (c_{r,\uparrow}c_{r+\alpha,\downarrow} + c_{r,\downarrow}c_{r+\alpha,\uparrow})/\sqrt{2}$ being the (singlet) pairing field operator ($\alpha$ is the unit vector $\hat{x}$ or $\hat{y}$). We show the color contours of the pairing correlations $\Phi_{yy}$, from which we observe a dome-like regime with prominent SC fluctuations above the LEL ground state. Despite the fact that a true long-range order is absent in quasi-1D system, the SC dome-like regime with strong SC fluctuations is quite robust on both the 2-leg and 4-leg systems. There exists a characteristic SC temperature scale $T^*_c \sim 0.15 \ J$ (i.e., $0.05 \ t$), below which the pairing susceptibility falls into an algebraic divergence behavior [Figs. 2(b1) and (b2)]. Such a strong pairing instability can be clearly seen in the snapshots [Figs. 1(a) and (d)], just like in a quantum gas microscope of ultra-cold fermions [72–78]. Such snapshots are taken through perfect samplings based on the matrix product density operator [71]. From Figs. 1(a) and (d), one can see that the holes tend to bound as real-space pairs, constituting an icon of superconductivity [78]. As $\delta$ increases, both pairing correlation $\Phi_{yy}$ and $T^*_c$ get enhanced until an optimal doping $\delta_c \sim 0.15$ (2-leg ladder) and $\delta_c \sim 0.085$ (4-leg cylinder) is reached. With further increased $\delta$, $T^*_c$ is found to decline. Although we only present $\Phi_{yy}(r = 3)$ data here at a fixed distance $r = 3$ along the $\hat{x}$ direction, the situation is very similar for other distances [71].

Even more intriguingly, above the SC dome we find an intermediate regime below a higher temperature scale of the spin gap, which resembles the magnetic susceptibility $\delta > \delta_c$ which is possibly a metal phase whose properties should also be strongly influenced by electron correlations yet deformed by the constraint of no double occupancy. Moreover, in the larger-doping regime $\delta > \delta_c$, we find that the Tomonaga-Luttinger liquid (TLL) behavior sets in at much higher temperature than $T^*_c$, i.e., $T/J \sim 1$ [71].

**Two-temperature scale scenario.**—We first compute the specific heat normalized by site number $N$ with a constant $\mu$, i.e., $c_{\mu} = - (T^2/N) \langle \partial^2 \Psi / \partial T^2 \rangle_{\mu}$, where $\Psi$ is the grand potential $\Psi = -T \ln [\text{Tr} e^{-(H - \mu N_e)/T}]$ with $N_e$ the total number of electrons. In the square-lattice Heisenberg model (i.e., $t$-$J$ model at half filling), the specific heat exhibits a single round peak at $T \sim 0.6 J$ [55]. In our doped systems, a two-temperature scale scenario appears. For the 2-leg ladder [Fig. 2(a1)], $c_{\mu}$ clearly shows two peaks at $T_1 \sim 0.15 J$ and $T_0 \sim 0.55 J$, respectively. On the 4-leg cylinder, although the obtained results still show small quantitative change as the bond dimension increases [see the shaded regime in Fig. 2(a2)], the feature of two-temperature scale is robust. Compared with the 2-leg system, the lower peak becomes a steplike shoulder in the 4-leg system, which has also been observed in the doped 4-leg Hubbard model [34].

**Superconductivity dome and pseudogap regime.**—To unveil the physics related to the two-temperature scale, we compute the $d$-wave pairing susceptibility $\chi_{SC} = (1/N) \langle \partial (\Delta_{tot}) / \partial h \rangle$ and magnetic susceptibility $\chi_{m} = (1/N) \langle \partial m / \partial h \rangle$. $\chi_{SC}$ is computed by applying a pairing field term $-h_p \Delta_{tot}$, with a small pairing strength $h_p \approx 0.01$ and $\Delta_{tot} = \sum_{r,\alpha} f_{\alpha} [\Delta_{\alpha}(r) + \Delta^*_\alpha(r)]/2$. $f_{\alpha} = -1$ and $f_{\beta} = 1$. Remarkably, we find $\chi_{SC}$ diverges algebraically below $T^*_c \sim T_1$ [Figs. 2(b1) and (b2)], characterizing $T^*_c$ as the onset temperature scale of the prominent SC correlations [71].

The insets show good power-law fittings $\chi_{SC} \sim 1/T^\alpha$ with the exponent $\alpha$ close to 1, consistent with the LEL theory. On the other hand, as shown in Figs. 2(c1) and (c2) the magnetic susceptibility $\chi_{m}$ exhibits a maximum at a temperature $T^{*} \sim T_{h}$, resembling the unusual suppression of the Knight shift in the cuprate superconductors [79] and may indicate the formation of the spin-gapped PG below $T^{*}$. Moreover, $\chi_{m}$ keeps to decline towards zero as $T$ lowers, which agrees with the spin gapped LEL ground state.

Furthermore, we compute the spin gap $\Delta_{s} = E(N_{e},1) - E(N_{e},0)$ and single-particle gap $\Delta^{(1)}_{tot} = [E(N_{e}+1,1/2) + E(N_{e}-1,1/2)]/2 - E(N_{e},0)$ using DMRG, where $E(N_{e},S)$ is the lowest energy in the sector with electron number $N_e$ and total spin $S$ [Figs. 2(a1) and (a2)]. The spin gaps $\Delta_{s}$ in both the $W = 2, 4$ systems are close to the PG onset temperature.
T*. For the single-particle gap $\Delta^{(1)}$, we find $\Delta^{(1)} \approx \Delta_0$ in the 2-leg system but smaller than $\Delta_0$ in the 4-leg system, which agree with the DMRG result that the single-particle correlations decay much slower in the 4-leg system [38], which may be ascribed to the nodal structure of d-wave pairing as well as the cylinder geometry of finite width. In Figs. 1(b,c), $T^*$ is found to lower as doping level increases, which also resembles the PG boundary in the phase diagram of cuprates. As doping ratio $\delta$ exceeds a certain value, $\chi_m$ does not vanish as $T \to 0$ [71], which suggests the close of spin gap in the overdoped regime with TLL states.

**Intertwined spin and charge correlations.** The intertwined spin, charge, and pairing orders are ubiquitous in cuprate superconductors [80], which can also be observed in the ground states of Hubbard and t-J models [13, 14, 19, 25–27, 36–38, 40, 67]. For the 2-leg ladder, there exists a quasi-long-range CDW with the wave vector $k_{CDW} = (2\pi \delta, 0)$. Besides, there exists a $\pi$ phase shift in the spin correlation with $k_{SDW} = ((1 - \delta)\pi, \pi)$ [40]. For the 4-leg cylinder, on the other hand, the CDW is characterized by the wave vector $k_x = 4\delta \pi$ and the spin correlations are short-ranged and with a fixed $k = (\pi, \pi)$, i.e., without the magnetic $\pi$ phase shift.

Now we explore the intertwined correlations at finite temperature. In Figs. 3(a) and (e), we compute the spin structure factor $S_m(k) = \sum_{i,j} e^{i k (r_i - r_j)} (S_i \cdot S_j)$ of the 2-leg ladder, and observe that the peak of $S_m$ shifts from $(\pi, \pi)$ to $((1 - \delta)\pi, \pi)$ for $T \leq T_1$. For the 4-leg cylinder, the dominant peak of $S_m$ is found always located at $(\pi, \pi)$ [Figs. 3(c) and (g)]. We also compute the charge structure factor $S_q(k) = \sum_{i,j} e^{i k (r_i - r_j)} \langle n_i n_j \rangle - \langle n_i \rangle \langle n_j \rangle$. As the particle number changes vs. $T$ even with a fixed $\mu$, in Fig. 3 we show the normalized $S_q(k_x, k_y) = S_q/(1 - n)$ at $k_y = 0$, where $(1 - n) = \delta$ is the hole doping. From Figs. 3(b) and (d) we see that while in the PG regime $\bar{S}_c$ is featureless, the peaks at $k_{CDW}$ emerge for $T < T_1$ for both geometries. To see the results more clearly, we show the lines in Figs. 3(f) and (h) and find that $\bar{S}_c(k_{CDW})$ starts to develop at the temperature $T \sim \Delta^{(1)}_c$ and becomes prominent for $T < T_1 \sim T^*_c$. Besides correlations, the charge distribution $n(x)$ is found to oscillate in real space below $T^*_c$, which well converges to the ground-state DMRG results [71].

**Discussion and outlook.** It has long been conjectured that the Hubbard and t-J models can capture the quintessence of the high-temperature superconductivity. In this work, beyond ground state properties [30–32], we explore the phase diagram of the square-lattice $t$-$t'$-$J$ model. A diverging d-wave pairing susceptibility $\chi_{SC} \sim 1/T^{3+}$ with dominating pairing correlations is revealed in a dome regime with a high onset temperature $T^*_c/t \sim 0.5$. Considering the hopping $t$ in the order of 5, 000 K as in a typical cuprate [81], $T^*_c$ can be estimated to be in the order of 10$^2$ K, belonging to the high-$T_c$ SC.

Above $T^*_c$ yet below $T^*$, we uncover an intriguing undoped regime resembling the PG regime in the cuprate phase diagram. The intertwined SC and charge orders are also “fermented” in this PG regime and become prominent in the dome below $T^*_c$. The different onset temperatures of the fluctuating SC ($T^*_c$) and the PG regime ($T^*$) suggests a two-step establishment of the SC order in the system. The SC phase in the ground state appears to be robust with increased system width [30–32, 38, 39]. Given the finite thermal correlation length in our finite-$T$ calculations here, we believe the obtained phase diagram would be very suggestive for the 2D limit.

Our findings make the frustrated $t$-$t'$-$J$ model a very appealing platform for the further explorations of not only the SC in the ground state but also the stimulating finite-$T$ phase.
diagram. For example, it is informative to search for optimized Hubbard or t-J model parameters with enhanced $T_c$. Above $T'_c$, the PG regime shows very exotic properties when we compare the finite temperature states above the optimally doped LEL states with those above the overdoped TLL states (see Supplementary Fig. S5 [71]). The onset temperature of TLL behaviors is $T/J \simeq 1 \gg T'_c$, much higher than the SC onset temperature. The distinct difference between the superconductive LEL and “metallic” TLL phases not only in the ground state but also in the finite-$T$ PG regime again recalls us of the cuprate superconductors, which worthy further investigations.

Recently, the ultra-cold fermion gas quantum simulations of the 2-leg Fermi-Hubbard [77] and mix-dimensional t-J ladder [78] have been realized with specifically designed hopping amplitudes. Our findings based on the thermal tensor network simulations of $t$-$t'$-$J$ model suggest the strong influence of $t'$ to the finite-$T$ phase diagram, and also provide useful guide for the quantum simulation of correlated electrons.

Acknowledgments.— W.L. and D.-W.Q are indebted to Tao Shi, Zi-Xiang Li, Wei Wu, and Lei Wang for helpful discussions. This work was supported by the National Natural Science Foundation of China (Grant Nos. 12222412, 11974036, 11834014, 12047503, 12174386, 12274014, and 11874115), National Key R&D Program of China (Grant No. 2018YFA0305800), Strategic Priority Research Program of CAS (Grant No. XDB28000000), and CAS Project for Young Scientists in Basic Research (Grant No. YSBR-057). We thank the HPC-ITP for the technical support and generous allocation of CPU time.

[1] J. G. Bednorz and K. A. Müller, Possible high $T_c$ superconductivity in the Ba-La-Cu-O system, Z. Physik B - Condensed Matter 64, 189 (1986).
[2] C. C. Tsuei and J. R. Kirtley, Pairing symmetry in cuprate superconductors, Rev. Mod. Phys. 72, 969 (2000).
[3] P. A. Lee, N. Nagaosa, and X.-G. Wen, Doping a mot insulator: Physics of high-temperature superconductivity, Rev. Mod. Phys. 78, 17 (2006).
[4] B. Keimer, S. A. Kivelson, M. R. Norman, S. Uchida, and J. Zaanen, From quantum matter to high-temperature superconductivity in copper oxides, Nature 518, 179 (2015).
[5] C. Proust and L. Taillefer, The remarkable underlying ground states of cuprate superconductors, Annu. Rev. Condens. Matter Phys. 10, 409 (2019).
[6] D. P. Arovas, E. Berg, S. A. Kivelson, and S. Raghu, The Hubbard model, Annu. Rev. Condens. Matter Phys. 13, 239 (2022).
[7] J. Hubbard, Electron correlations in narrow energy bands, Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences 276, 238 (1963).
[8] M. C. Gutzwiller, Effect of Correlation on the Ferromagnetism of Transition Metals, Phys. Rev. Lett. 10, 159 (1963).
[9] F. C. Zhang and T. M. Rice, Effective Hamiltonian for the superconducting Cu oxides, Phys. Rev. B 37, 3759 (1988).
[10] J. Spalek, t-J model and then and now: a personal perspective from the pioneering times, Acta Physica Polonica A 111, 409 (2007).
[11] F. W. Anderson, The resonating valence bond state in La$_2$CuO$_4$ and superconductivity, Science 235, 1196 (1987).
[12] F. W. Anderson, P. A. Lee, M. Randeria, T. M. Rice, N. Trivedi, and F. C. Zhang, The physics behind high-temperature superconducting cuprates: the ‘plain vanilla’ version of RVB, J. Phys.: Condens. Matter 16, R755 (2004).
[13] S. R. White and D. J. Scalapino, Density matrix renormalization group study of the striped phase in the 2D t-J model, Phys. Rev. Lett. 80, 1272 (1998).
[14] S. R. White and D. J. Scalapino, Stripes on a 6-leg Hubbard ladder, Phys. Rev. Lett. 91, 136403 (2003).
[15] A. Georges, G. Kotliar, W. Krauth, and M. J. Rozenberg, Dynamical mean-field theory of strongly correlated fermion systems and the limit of infinite dimensions, Rev. Mod. Phys. 68, 13 (1996).
[16] T. Maier, M. Jarrell, T. Pruschke, and M. H. Hettler, Quantum cluster theories, Rev. Mod. Phys. 77, 1027 (2005).
[17] G. Kotliar, S. Y. Savrasov, K. Haule, V. S. Oudovenko, O. Parcollet, and C. A. Marianetti, Electronic structure calculations with dynamical mean-field theory, Rev. Mod. Phys. 78, 865 (2006).
[18] G. Knizia and G. K.-L. Chan, Density matrix embedding: A simple alternative to dynamical mean-field theory, Phys. Rev. Lett. 109, 186404 (2012).
[19] P. Corboz, T. M. Rice, and M. Troyer, Competing states in the t-J model: Uniform d-wave state versus stripe state, Phys. Rev. Lett. 113, 046402 (2014).
[20] Z.-C. Gu, F. Verstraete, and X.-G. Wen, Grassmann tensor network states and its renormalization for strongly correlated fermionic and bosonic states (2010), arXiv:1004.2563 [cond-mat.str-el].
[21] F. Assaad and H. Evertz, World-line and determinantal quantum monte carlo methods for spins, phonons and electrons, in Computational Many-Particle Physics, edited by H. Fehske, R. Schneider, and A. Weiße (Springer Berlin Heidelberg, Berlin, Heidelberg, 2008) p. 277–356.
[22] C.-C. Chang and S. Zhang, Spin and charge order in the doped hubbard model: Long-wavelength collective modes, Phys. Rev. Lett. 104, 116402 (2010).
[23] W. Wu, M. Ferrero, A. Georges, and E. Kozik, Controlling feynman diagrammatic expansions: Physical nature of the pseudogap in the two-dimensional hubbard model, Phys. Rev. B 96, 041105 (2017).
[24] Y.-Y. He, M. Qin, H. Shi, Z.-Y. Lu, and S. Zhang, Finite-temperature auxiliary-field quantum monte carlo: Self-consistent constraint and systematic approach to low temperatures, Phys. Rev. B 99, 045108 (2019).
[25] B.-X. Zheng, C.-M. Chung, P. Corboz, G. Ehlers, M.-P. Qin, R. M. Noack, H. Shi, S. R. White, S. Zhang, and G. K.-L. Chan, Stripe order in the underdoped region of the two-dimensional Hubbard model, Science 358, 1155 (2017).
[26] B. Ponsioen, S. S. Chung, and P. Corboz, Period 4 stripe in the extended two-dimensional Hubbard model, Phys. Rev. B 100, 195141 (2019).
[27] M. Qin, C.-M. Chung, H. Shi, E. Vitali, C. Hubig, U. Schollwöck, S. R. White, and S. Zhang (Simons Collaboration on the Many-Electron Problem), Absence of superconductivity in the pure two-dimensional Hubbard model, Phys. Rev. X 10, 031016 (2020).
S. Jiang, D. J. Scalapino, and S. R. White, Ground-state phase
H.-C. Jiang and S. A. Kivelson, High temperature superconductivity in a lightly doped quantum spin liquid, Phys. Rev. Lett. 127, 097002 (2021).
S. Gong, W. Zhu, and D. N. Sheng, Robust $d$-wave superconductivity in the square-lattice $t$-$t'$ model, Phys. Rev. Lett. 127, 097003 (2021).
S. Jiang, D. J. Scalapino, and S. R. White, Ground-state phase diagram of the $t$-$t'$-$J$ model, Proc. Natl. Acad. Sci. 118, e2109978118 (2021).
M. Qin, T. Schäfer, S. Andergassen, P. Corboz, and E. Gull, The Hubbard model: A computational perspective, Annu. Rev. Condens. Matter Phys. 13, 275 (2022).
A. Wietek, Y.-Y. He, S. R. White, A. Georges, and E. M. Stoudenmire, stripes, antiferromagnetism, and the pseudogap in the doped Hubbard model at finite temperature, Phys. Rev. X 11, 031007 (2021).
J. F. Dodaro, H.-C. Jiang, and S. A. Kivelson, Intertwined order in a frustrated four-leg $t$-$J$ cylinder, Phys. Rev. B 95, 155116 (2017).
H.-C. Jiang, Z.-Y. Weng, and S. A. Kivelson, Superconductivity in the doped $t$-$J$ model: Results for four-leg cylinders, Phys. Rev. B 98, 140505 (2018).
H. C. Jiang and T. P. Devreese, Superconductivity in the doped Hubbard model and its interplay with next-nearest hopping $t''$, Science 365, 1424 (2019).
Y.-F. Jiang, J. Zaanen, T. P. Devreese, and H.-C. Jiang, Ground state phase diagram of the doped Hubbard model on the four-leg cylinder, Phys. Rev. Research 2, 033073 (2020).
C.-M. Chung, M. Qin, S. Zhang, U. Schollwöck, and S. R. White (The Simons Collaboration on the Many-Electron Problem), Plaquette versus ordinary $d$-wave pairing in the $t$'-Hubbard model on a width-$4$ cylinder, Phys. Rev. B 102, 041106 (2020).
X. Lu, D.-W. Qu, Y. Qi, W. Li, and S.-S. Gong, Ground state phase diagram of the extended two-leg $t$-$J$ ladder (2022), arXiv:2211.03778 [cond-mat.str-el].
J. Jaklič and P. Prelovšek, Lanczos method for the calculation of finite-temperature quantities in correlated systems, Phys. Rev. B 49, 5065 (1994).
J. Jaklič and P. Prelovšek, Finite-temperature properties of doped antiferromagnets, Adv. Phys. 49, 1 (2000).
A. Avella and F. Mancini, eds., Strongly Correlated Systems (Springer Berlin Heidelberg, 2013).
J. Jaklič and P. Prelovšek, Thermodynamic properties of the planar $t$-$J$ model, Phys. Rev. Lett. 77, 892 (1996).
S. R. White, Minimally entangled typical quantum states at finite temperature, Phys. Rev. Lett. 102, 190601 (2009).
E. M. Stoudenmire and S. R. White, Minimally entangled typical state algorithms, New J. Phys. 12, 055026 (2010).
A. Wietek, R. Rossi, F. Šimkovic, M. Klett, P. Hansmann, M. Ferrero, E. M. Stoudenmire, T. Schäfer, and A. Georges, Mott insulating states with competing orders in the triangular lattice hubbard model, Phys. Rev. X 11, 041013 (2021).
W. Li, S.-J. Ran, S.-S. Gong, Y. Zhao, B. Xi, F. Ye, and G. Su, Linearized tensor renormalization group algorithm for the calculation of thermodynamic properties of quantum lattice models, Phys. Rev. Lett. 106, 127202 (2011).
P. Czarnik and J. Dziarmaga, Fermionic projected entangled pair states at finite temperature, Phys. Rev. B 90, 035144 (2014).
P. Czarnik, M. M. Rams, and J. Dziarmaga, Variational tensor network renormalization in imaginary time: Benchmark results in the Hubbard model at finite temperature, Phys. Rev. B 94, 235142 (2016).
P. Czarnik, J. Dziarmaga, and P. Corboz, Time evolution of an infinite projected entangled pair state: An efficient algorithm, Phys. Rev. B 99, 035115 (2019).
P. Czarnik and P. Corboz, Finite correlation length scaling with infinite projected entangled pair states at finite temperature, Phys. Rev. B 99, 245107 (2019).
A. Sinha, M. M. Rams, P. Czarnik, and J. Dziarmaga, Finite temperature tensor network study of the Hubbard model on an infinite square lattice (2022), arXiv:2209.00985 [cond-mat.str-el].
B.-B. Chen, L. Chen, Z. Chen, W. Li, and A. Weichselbaum, Exponential thermal tensor network approach for quantum lattice models, Phys. Rev. X 8, 031082 (2018).
H. Li, B.-B. Chen, Z. Chen, J. von Delft, A. Weichselbaum, and W. Li, Thermal tensor renormalization group simulations of square-lattice quantum spin models, Phys. Rev. B 100, 045110 (2019).
L. Chen, D.-W. Qu, H. Li, B.-B. Chen, S.-S. Gong, J. von Delft, A. Weichselbaum, and W. Li, Two-temperature scales in the triangular-lattice Heisenberg antiferromagnet, Phys. Rev. B 99, 140404(R) (2019).
H. Li, Y. D. Liao, B.-B. Chen, X.-T. Zeng, X.-L. Sheng, Y. Qi, Z. Y. Meng, and W. Li, Kosterlitz-Thouless melting of magnetic order in the triangular quantum Ising material $\text{TmMgGaO}_4$, Nat. Commun. 11, 11110 (2020).
H. Li, D.-W. Qu, H.-K. Zhang, Y.-Z. Jia, S.-S. Gong, Y. Qi, and W. Li, Universal thermodynamics in the Kitaev fractional liquid, Phys. Rev. Research 2, 043015 (2020).
H. Li, H.-K. Zhang, J. Wang, H.-Q. Wu, Y. Gao, D.-W. Qu, Z.-X. Liu, S.-S. Gong, and W. Li, Identification of magnetic interactions and high-field quantum spin liquid in $\alpha$-RuCl$_3$, Nat. Commun. 12, 4007 (2021).
S. Yu, Y. Gao, B.-B. Chen, and W. Li, Learning the effective spin Hamiltonian of a quantum magnet, Chinese Phys. Lett. 38, 097502 (2021).
B.-B. Chen, C. Chen, Z. Chen, J. Cui, Y. Zhai, A. Weichselbaum, J. von Delft, Z. Y. Meng, and W. Li, Quantum many-body simulations of the two-dimensional Fermi-Hubbard model in ultracold optical lattices, Phys. Rev. B 103, 040117 (2021).
X. Lin, B.-B. Chen, W. Li, Z. Y. Meng, and T. Shi, Exciton proliferation and fate of the topological Mott insulator in a twisted bilayer graphene lattice model, Phys. Rev. Lett. 128, 157201 (2022).
D. C. Johnston, Magnetic susceptibility scaling in $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$-y, Phys. Rev. Lett. 62, 957 (1989).
A. Weichselbaum, Non-abelian symmetries in tensor networks: A quantum symmetry space approach, Ann. Phys. 327, 2972 (2012).
A. Weichselbaum, X-symbols for non-Abelian symmetries in tensor networks, Phys. Rev. Research 2, 023385 (2020).
L. Balents and M. P. A. Fisher, Weak-coupling phase diagram of the two-chain Hubbard model, Phys. Rev. B 53, 12133 (1996).
S. R. White, I. Affleck, and D. J. Scalapino, Friedel oscillations
and charge density waves in chains and ladders, Phys. Rev. B 65, 165122 (2002).

[68] Y. Gannot, Y.-F. Jiang, and S. A. Kivelson, Hubbard ladders at small U revisited, Phys. Rev. B 102, 115136 (2020).

[69] H.-C. Jiang, S. Chen, and Z.-Y. Weng, Critical role of the sign structure in the doped mott insulator: Luther-Emery versus Fermi-liquid-like state in quasi-one-dimensional ladders, Phys. Rev. B 102, 104512 (2020).

[70] A. Luther and V. J. Emery, Backward scattering in the one-dimensional electron gas, Phys. Rev. Lett. 33, 589 (1974).

[71] In the Supplementary Materials, we show data convergence check of the finite-temperature simulations (Sec. I), the perfect sampling of thermal tensor networks (Sec. II), and the supplementary data of pairing correlations (Sec. III), magnetic susceptibility (Sec. IV), and charge order (Sec. V).

[72] M. F. Parsons, F. Huber, A. Mazurenko, C. S. Chiu, W. Setiawan, K. Wooley-Brown, S. Blatt, and M. Greiner, Site-resolved imaging of fermionic $^6$Li in an optical lattice, Phys. Rev. Lett. 114, 213002 (2015).

[73] M. F. Parsons, A. Mazurenko, C. S. Chiu, G. Ji, D. Greif, and M. Greiner, Site-resolved measurement of the spin-correlation function in the Fermi-Hubbard model, Science 353, 1253 (2016).

[74] D. Greif, M. F. Parsons, A. Mazurenko, C. S. Chiu, S. Blatt, F. Huber, G. Ji, and M. Greiner, Site-resolved imaging of a fermionic Mott insulator, Science 351, 953 (2016).

[75] A. Mazurenko, C. S. Chiu, G. Ji, M. F. Parsons, M. Kanász-Nagy, R. Schmidt, F. Grusdt, E. Demler, D. Greif, and M. Greiner, A cold-atom Fermi–Hubbard antiferromagnet, Nature 545, 462 (2017).

[76] C. S. Chiu, G. Ji, A. Bohrdt, M. Xu, M. Knap, E. Demler, F. Grusdt, M. Greiner, and D. Greif, String patterns in the doped Hubbard model, Science 365, 251 (2019).

[77] P. Sompet, S. Hirthe, D. Bourgund, T. Chalopin, J. Bibo, J. Koepsell, P. Bojović, R. Verresen, F. Pullmann, G. Salomon, C. Gross, T. A. Hilker, and I. Bloch, Realizing the symmetry-protected Haldane phase in Fermi-Hubbard ladders, Nature 606, 484 (2022).

[78] S. Hirthe, T. Chalopin, D. Bourgund, P. Bojović, A. Bohrdt, E. Demler, F. Grusdt, I. Bloch, and T. A. Hilker, Magnetically mediated hole pairing in fermionic ladders of ultracold atoms (2022), arXiv:2203.10027 [cond-mat.quant-gas].

[79] H. Alloul, T. Ohno, and P. Mendels, $^{89}$Y NMR evidence for a Fermi-liquid behavior in YBa$_2$Cu$_3$O$_{6+x}$, Phys. Rev. Lett. 63, 1700 (1989).

[80] E. Fradkin, S. A. Kivelson, and J. M. Tranquada, Colloquium: Theory of intertwined orders in high temperature superconductors, Rev. Mod. Phys. 87, 457 (2015).

[81] M. Hirayama, T. Misawa, T. Ohgoe, Y. Yamaji, and M. Imada, Effective hamiltonian for cuprate superconductors derived from multiscale ab initio scheme with level renormalization, Phys. Rev. B 99, 245155 (2019).

[82] A. J. Ferris and G. Vidal, Perfect sampling with unitary tensor networks, Phys. Rev. B 85, 165146 (2012).

[83] M. Buser, U. Schollwöck, and F. Grusdt, Snapshot-based characterization of particle currents and the Hall response in synthetic flux lattices, Phys. Rev. A 105, 033303 (2022).

[84] H. Schlömer, A. Bohrdt, L. Pollet, U. Schollwöck, and F. Grusdt, Robust stripes in the mixed-dimensional t – J model (2022), arXiv:2208.07366 [cond-mat.quant-gas].

[85] T. Giamarchi, Quantum Physics in One Dimension (Clarendon Press, 2004).

[86] C. A. Hayward and D. Poilblanc, Luttinger-liquid behavior and superconducting correlations in t-J ladders, Phys. Rev. B 53, 11721 (1996).
FIG. S1. Convergence check of the free energy per site. (a) shows the results of 2-leg ladder ($W = 2, L = 32, \mu = 4.7$) obtained with bond dimension $D^* = 600$ (equivalent to $D \simeq 1600$) and $D^* = 1000$ ($D \simeq 2600$). (b) shows the results of 4-leg cylinder ($W = 4, L = 20, \mu = 5.34$) with bond dimension $D^* = 1000$ ($D \simeq 2800$) and $D^* = 1600$ ($D \simeq 4600$). The insets show the relative difference $|\delta f/f|$ which indicate a very good convergence.

Supplementary Materials for High-$T_c$ Superconductivity and Finite-Temperature Phase Diagram of the $t$-$t'$-$J$ Model

I. CONVERGENCE OF THE FREE ENERGY

Here we show the convergence check of the free energy results in our exponential tensor renormalization group (XTRG) calculations on the 2-leg ladder and 4-leg cylinder systems.

In Fig. S1(a) we show the free energy per site of the 2-leg ladder ($W = 2, L = 32$) with chemical potential $\mu = 4.7$, which are obtained by keeping $D^* = 600$ (multiplets, equivalently $D \simeq 1600$ individual states) and $D^* = 1000$ ($D \simeq 2600$), respectively. The relative difference of $f$ is within $10^{-4}$, showing a very good convergence. For the 4-leg cylinder, in Fig. S1(b) we find the $f$ results of the $4 \times 20$ cylinder (with $\mu = 5.34$) show relative difference within $10^{-3}$ for $D^* = 1000$ ($D \simeq 2800$) and $D^* = 1600$ ($D \simeq 4600$). The results in Fig. S1 guarantee good convergence of the thermodynamic quantities (such as the specific heat $c_\mu$) as derivatives of the free energy.

II. PERFECT SAMPLING IN THE THERMAL TENSOR NETWORKS

We present the snapshot sampling algorithm from the obtained XTRG density matrix as follows. Such a sampling scheme is similar to that in METTS [45, 46], and recently also used in the generating snapshots in other thermal tensor network methods [82–84].

1. As shown in Fig. S2(a), firstly we calculate the reduced density matrix of the first site $\hat{\rho}_1$ by fully contracting all the indices except for the physical index of the 1st site, i.e.,

\[
\hat{\rho}_1 = \sum_{\{\sigma_2 \ldots \sigma_n\}} \langle \sigma_1 \sigma_2 \ldots \sigma_n | \hat{\rho}(\beta) | \sigma_1 \sigma_2 \ldots \sigma_n \rangle.
\] (S1)

2. Meanwhile, $\hat{\rho}_1 = p(0) \ket{0}\bra{0} + p(\downarrow) \ket{\downarrow}\bra{\downarrow} + p(\uparrow) \ket{\uparrow}\bra{\uparrow}$, where the probability of the local state $|\tilde{\sigma}_1\rangle$ is,

\[
p(\tilde{\sigma}_1) = \sum_{\{\sigma_2 \ldots \sigma_n\}} \langle \tilde{\sigma}_1 \sigma_2 \ldots \sigma_n | \hat{\rho}(\beta) | \tilde{\sigma}_1 \sigma_2 \ldots \sigma_n \rangle.
\] (S2)

To collapse to the local state $|\tilde{\sigma}_1\rangle$ with the same probability, one can generate a random number $n_{\text{rand}}$ and choose the local state following

\[
\tilde{\sigma}_1 = \begin{cases} 
0 & \text{if } n_{\text{rand}} \leq p(0), \\
\downarrow & \text{elseif } n_{\text{rand}} \leq p(0) + p(\downarrow), \\
\uparrow & \text{otherwise}.
\end{cases}
\] (S3)
FIG. S2. Perfect sampling algorithm for snapshots. The reduced density matrix of (a) the 1st site $\rho_1$ and (b) 3rd site $\rho_3$, and so on so forth.

3. To collapse the rest part, e.g., 2nd local state, one similarly calculates its reduced density matrix

$$\hat{\rho}_2 = \frac{1}{p(\tilde{\sigma}_1)} \sum_{\{\sigma_3 \cdots \sigma_n\}} \langle \tilde{\sigma}_1 \sigma_2 \cdots \sigma_n | \hat{\rho}(\beta) | \tilde{\sigma}_1 \sigma_2 \cdots \sigma_n \rangle,$$

by fixing the 1st local state to be $|\tilde{\sigma}_1\rangle$ as shown in Fig. S2(b). Note that $\hat{\rho}_2$ is normalized by $1/p(\tilde{\sigma}_1)$ in Eq. (S4). Following the same line as in Eq. (S3), one can collapse to the local state $\tilde{\sigma}_2$ with probability

$$p(\tilde{\sigma}_2 | \tilde{\sigma}_1) = \frac{1}{p(\tilde{\sigma}_1)} \sum_{\{\sigma_3 \cdots \sigma_n\}} \langle \tilde{\sigma}_1 \sigma_2 \cdots \sigma_n | \hat{\rho}(\beta) | \tilde{\sigma}_1 \sigma_2 \cdots \sigma_n \rangle.$$

4. By repeating step 3 until the last site $n$ is reached, one finally obtains a classical product state (CPS) $|\tilde{\sigma}_1 \tilde{\sigma}_2 \cdots \tilde{\sigma}_n\rangle$, whose probability is the product of a sequence of conditional probabilities, i.e.,

$$p(\tilde{\sigma}_1) \cdot \frac{p(\tilde{\sigma}_1 \tilde{\sigma}_2)}{p(\tilde{\sigma}_1)} \cdots \frac{p(\tilde{\sigma}_1 \tilde{\sigma}_2 \cdots \tilde{\sigma}_n)}{p(\tilde{\sigma}_1 \tilde{\sigma}_2 \cdots \tilde{\sigma}_{n-1})},$$

which is exactly the Boltzmann weight of this CPS

$$p(\tilde{\sigma}_1 \tilde{\sigma}_2 \cdots \tilde{\sigma}_n) = \langle \tilde{\sigma}_1 \tilde{\sigma}_2 \cdots \tilde{\sigma}_n | \hat{\rho}(\beta) | \tilde{\sigma}_1 \tilde{\sigma}_2 \cdots \tilde{\sigma}_n \rangle.$$

III. SUPPLEMENTAL DATA OF SUPERCONDUCTIVITY PAIRING CORRELATIONS

In this section, we provide more data on the superconductivity pairing correlations. Firstly, we recap some analytical results on the 2-leg ladder, to facilitate the analysis of our numerical data. For the parameters we consider: $t/J = 3$, $J'/J = 0.05$, $t' = t\sqrt{J'/J}$, and doping $\delta \leq \delta_c \approx 0.34$, the ground state is in the Luther-Emery liquid (LEL) phase with quasi-long range SC correlations. While for $\delta > \delta_c$ (except for $\delta = 1/2$), the ground state is in the Tomonaga-Luttinger liquid (TLL) phase. At doping $\delta = 1/2$, there exists a special SDW point where the system opens up a small charge gap [40], while the antiferromagnetic spin-spin correlations are still quasi-long-ranged. Bosonization analysis shows that the pairing correlation functions in LEL has two leading terms in the long-range scaling [85, 86], i.e.,

$$\Phi_{\alpha\beta}(r) = \frac{C_0}{r^{1/(2K_{\nu})}} + C_1 \frac{\cos(2k_{F}r)}{r^{2K_{\nu}+1/(2K_{\nu})}},$$

(S5)
FIG. S3. Pairing correlation functions for 2-leg ladder at various temperatures. Here we show the simulated data in an \( L = 32 \) system. The reference point of correlation is at \( x = L/4 + 1 \), and the distance is denoted by \( r \). (a,b) show respectively the \( y-y \) and \( y-x \) pairing correlation functions \( \Phi_{yy}(r) \) and \( \Phi_{yx}(r) \) with \( \mu = 4.7 \) (near optimal doping, LEL phase). We find in (a) that \( \Phi_{yy}(r) \) is always positive, while the \( y-x \) pairing is negative [hence we show \( -\Phi_{yx}(r) \) in (b)]. (c) shows the absolute value \( |\Phi_{yx}| \) for \( \mu = 2.3 \) (overdoped, TLL phase). The dashed lines are the ground-state DMRG results with 10 doped holes in (a,b) and 26 holes in (c).

FIG. S4. Pairing correlation length results \( \xi_{yy} \) for \( y-y \) pairing and \( \xi_{yx} \) for \( y-x \) pairing are plotted vs. \( T \) for the 2-leg ladder with \( \mu = 4.7 \) and length \( L = 64 \). The constant-\( \mu \) specific heat \( c_\mu \) is also plotted (see right \( y \)-axis), and here we show the pairing correlation length: \( \xi_{yy} \) and \( \xi_{yx} \) increase rapidly for \( T < T_1 \sim T_c \), where \( T_1(T_c) \) is the characteristic temperature scale for the SC correlations.

where \( K_\rho \) is the Luttinger parameter, \( k_F \) is the Fermi momentum estimated from the particle density, and \( C_0, C_1 \) are non-universal constants. As \( K_\rho > 0 \) the first uniform term dominates in LEL, regardless of the specific values of \( C_0, C_1 \), and the conclusion holds for different directions of pairings. On the other hand, the scaling behavior in TLL is [85, 86]

\[
\Phi_{\alpha \beta}(r) = \frac{C_2}{r^{1+1/(2K_\rho)}} + \frac{C_3 \cos(2k_F r)}{r^{2K_\rho+1/(2K_\rho)}},
\]

where instead the second \( 2k_F \)-oscillation term dominates, as we always find \( K_\rho < 1/2 \) in the 2-leg ladder [40].

In Figs. S3(a,b) we show \( \Phi_{yy}(r) \) and \( -\Phi_{yx}(r) \) at various temperatures, for the 2-leg ladder with \( \mu = 4.7 \) (LEL ground state). According to the \( d \)-wave paring symmetry, for \( \Phi_{yy} \) we have \( C_0 > 0 \) while for \( \Phi_{yx} \) the constant is \( C_0 < 0 \), and indeed here \( \Phi_{yy} > 0 \) and \( \Phi_{yx} < 0 \) are observed. Although the pairing correlations are found always decaying exponentially at finite temperature, the correlation length \( \xi_{SC} \) increase rapidly as temperature decreases and becomes rather long compared to the system length below the estimated \( T_c^* \) (see Fig. S4). On the other hand, in Fig. S3(c) we show \( |\Phi_{yx}(r)| \) at various \( T \) in the TLL phase (with \( \mu = 2.3 \)), where the pairing correlations show clear oscillation behaviors vs. \( r \). In both phases, LEL and TLL, the pairing correlations obtained by thermal tensor networks converge to the ground-state DMRG results at sufficiently low temperatures.

In the main text, we have shown the superconducting dome by the contour of pairing correlation \( \Phi_{yy}(r) \) at a typical distance...
FIG. S5. (a-d) Contour plot of the $y$-$y$ pairing correlation $\Phi_{yy}(r)$ ($r = 1-4$) on a 2-leg ladder, as a function of doping $\delta$ and temperature $T$. The dashed lines denote the LEL-TLL critical point $\delta_c \sim 0.34$ in ground state, and the dotted dashed lines denote the SDW point at $\delta_{SDW} = 1/2$.

FIG. S6. (a-d) Contour plot of the $y$-$y$ pairing correlation $\Phi_{yy}(r)$ ($r = 1-4$) on a 4-leg cylinder, as a function of doping $\delta$ and temperature $T$. 
$r = 3$. Here we show that such a dome structure can also be observed in $\Phi_{yy}(r)$ at other distances $r$. Figs. S5(a-d) shows the contour of $\Phi_{yy}(r)$ as a functions of $\delta$ and $T$ for $r = 1$ and a wide doping range $0 < \delta < 0.8$. We find that except for the nearest-neighbor $r = 1$, there is always a dome structure with prominent and positive $\Phi_{yy}$ correlations. The specific boundaries on the right side of the dome can nevertheless change slightly for different distances $r$, which is possibly influenced by the nearby TLL phase.

For $\delta > \delta_c$, $\Phi_{yy}$ does not vanish but exhibits a vertical bar-like structure with alternating signs as $\delta$ changes. This observation can be well explained with the dominant $2\pi F$-oscillation in Eq. (S6), where $k_F = \pi n = \pi(1 - \delta)$. As expected, in Figs. S5 we find $\Phi_{yy}$ oscillates like $-\cos(2\pi r \delta)$ for a fixed $T$ and $r$. The special SDW point $\delta = 1/2$ can also be seen in the finite-temperature calculations, where the oscillation is just a $\pi$-shift, $(-1)^{r-1}$, from panel (a) to (d). The amplitudes of pairing correlations at $\delta = 1/2$ decay exponentially even in the ground state, due to the presence of a finite charge gap. Unlike the SC dome for $\delta < \delta_c$, these bars extent to quite high temperature $T/J \approx 1$. This indicates the paring correlation described in Eq. (S6), particularly the oscillating term, has already been established at relatively high temperature in the TLL phase, much higher than the SC correlation in the superconductive LEL phase. However, from Fig. S5 we see that the paring correlations in the TLL phase decay very rapidly as $r$ increases, while they are much more robust against increasing distance $r$ in the LEL regime under the dome.

For the $W = 4$ cylinder, in Fig. S6 we find a slightly different scenario. Although the pairing correlations also exhibit a sign change as doping ration $\delta \gtrsim 0.25$, the $\Phi_{yy}(r)$ correlations in the highly doped regime decay much more rapidly with $r$. For $r \gtrsim 4$ we can hardly see the pairing correlation in the overdoped regime. The bright dome regime with prominent pairing correlations, on the other hand, is persistent and the overall shape changes only slightly as $r$ increases.

IV. SUPPLEMENTAL DATA OF MAGNETIC SUSCEPTIBILITY

In Fig. 1 of the main text, we have shown the suppression temperature $T^*$ of the magnetic susceptibility $\chi_m$ versus doping ratio $\delta$. In Fig. S7 we show $\chi_m$ as a function of $T$ for various chemical potentials $\mu$ on the 2-leg ladder in (a). Considering that the electron density varies with $T$ even with a fixed $\mu$, in Fig. S7(b) we also plot the normalized $\chi_m/n$, with the electron density $n$ shown in Fig. S7(c). We find as $\mu$ decreases (accordingly $\delta$ increases), $\chi_m$ is enhanced and the peak is pushed to lower temperature. For sufficiently high dopings $\delta \gtrsim \delta_c$, $\chi_m$ is no longer suppressed to zero in the low temperature limit, reflecting the absence of a spin gap. Moreover, as the hole doping further increases, the peak location of $\chi_m$ move towards higher temperature again, indicating the enhancement of antiferromagnetic correlations. This may be ascribed to a proximity effect near the SDW point $\delta_{SDW} = 1/2$, a special parameter point in the 2-leg ladder ground state phase diagram [40] that resembles a Heisenberg spin-chain system.

V. INTERTWINED CHARGE ORDER IN THE SUPERCONDUCTIVE REGIME

The charge order is found intertwined with the superconductive order in the dome below $T^*_c$. Here we show in Fig. S8 more data of the real-space charge density distribution $n(x) = \frac{1}{W} \sum_y n(x, y)$, where the emergence of CDW order can be observed. As a quasi-long-range order in the ground state, the CDW correlations oscillate strongly in the presence of open boundaries, i.e., there exists the prominent Friedel oscillations that penetrates deeply into the bulk. In Fig. S8 we show $n(x)$ for 2-leg ladder
FIG. S8. Real-space charge density distribution $n(x)$ at various temperatures for (a) 2-leg ladder and (b) 4-leg cylinder. In (a) we plot only $n(x)$ with $x \in [1, 20]$ for the sake of clarity. The dashed red lines are the ground-state DMRG results with 20 doped holes for the 2-leg ladder in (a) and 8 holes for the 4-leg cylinder in (b). We find the finite-temperature simulations smoothly connect to the ground-state DMRG data in the low-temperature limit.

(a) and 4-leg cylinder (b) at various temperatures. As the system cools down, below $T_c^*/J \approx 0.15$ the system leaves the PG regime and enter the SC dome. We find the charge distribution oscillations appear also below $T_c^*$ and approach the ground-state modulations in the low temperature limit.