The Buffon needle problem for randomly spaced points

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Abstract. What is the probability that a needle dropped at random on a set of points scattered on a line segment does not fall on any of them? We compute the exact scaling expression of this hole probability when the spacings between the points are independent identically distributed random variables with a power-law distribution of index less than unity. This question is related to the study of some correlation functions of simple models of statistical physics.

1. Introduction

In the classical geometric probability problem devised by Buffon, a needle of length $r$ is dropped at random on a plane on which a set of parallel lines separated by a distance $d$ ($d > r$) have been drawn. As is well known, the probability that the needle does not intersect a line is equal to $1 - (2r)/(\pi d)$, where the geometrical factor $2/\pi$ comes from the possible orientations of the needle [1, 2, 3]. This factor is no longer present in the one-dimensional Buffon problem where the lines are replaced by equidistant points, in which case the probability that the needle does not hit a point simplifies to $1 - r/d$.

The goal of the present work is to generalise the one-dimensional Buffon problem with equidistant points to the situation where the points are randomly spaced—the spacings between the points are independent, identically distributed (iid) random variables—with particular focus on the case where their common distribution is heavy-tailed, with tail index $\theta < 1$. The question posed is: what is the chance for an interval of length $r$ (representing the needle) dropped at random on this set of points not to cover any point (see figures 1 and 2), in short, what is the probability of a point-free interval of length $r$? As will be explained below, this question naturally arises in some particular models of statistical physics.

This very same question can be translated in the temporal domain. A customer is waiting for the arrival of a taxi at the airport. What is his chance of waiting for more than $r$ units of time before he gets one?

A prerequisite for answering this question consists in determining the chance for the needle, with left endpoint located at $j$, not to cover a point, i.e., for the interval $(j, j+r)$ to be empty. Transcribed in the temporal domain, the question is to determine the chance for a customer arriving at time $j$ to wait more than $r$ units of time before getting a taxi. Solving the Buffon problem then consists in determining the same hole probability when the interval of length $r$ is placed uniformly at random on the line (see figures 1 and 2).
Whenever the distribution of spacings between the points equilibrates at large distance (or large time)—which requires its first moment to be finite—the answers to these questions are simple, as will be recalled below. However, when this distribution does not possess a finite first moment, the problem is more subtle. Studies on this topic can be found in [4] for the ‘free’ process, i.e., infinite to the right, as in figure 1 and in [5, 6, 7]‡ when the process is conditioned to a fixed (spatial or temporal) length \(L\) as in figure 2; that is, when the segment of length \(L\) is partitioned by a random number of iid intervals (see figure 2).

The present work improves and completes these previous studies. For free renewal processes, the central result is the scaling form of the Buffon probability (3.29). For tied-down renewal processes, the central results are the universal scaling function (5.5) from which the Buffon probability (5.10) ensues. The study made in [5] only predicted the form of the scaling function in the two regimes of interest analysed in §5 and given by (5.8) and (5.9). Together with [7] the present study supersedes the analysis made in [6].

The paper is structured as follows. Section 2 gives the definitions of the renewal process under study, with free or tied-down boundary conditions. Section 3 gives a survey of known results on the probability of an empty interval for a free renewal process, from which the Buffon probability ensues. Section 4 derives the expression of the probability of an empty interval for a tied-down renewal process, then illustrates the formalism by examples of systems converging to equilibrium. Section 5, which is the central section of the present work, gives the asymptotic analysis of the probability of an empty interval for distributions with power-law fall-off and tail index \(\theta<1\), for a tied-down renewal process. Section 6 is devoted to a variant of the formalism. Section 7 complements the body of the text. Section 8 provides a discussion. Other complements can be found in the two appendices.

![Figure 1](image-url)

Figure 1. Point events (or renewals) are figured by red ticks on the (spatial or temporal) coordinate axis. Spacings between them are iid random variables. A needle of size \(r\) is dropped at random on the semi-infinite line. The first question is to determine the chance for the needle, with left endpoint located at \(j\), not to cover a point, i.e., for the interval \((j, j+r)\) to be empty. Transcribed in the temporal domain, the question is to determine the chance for a customer arriving at time \(j\) to wait more than \(r\) units of time before getting a taxi. Solving the Buffon problem then consists in determining the same hole probability when the interval of length \(r\) is placed uniformly at random on the line. In order to give a precise meaning to this question, the system is put in a box of size \(L\).

‡ The context around [6, 7] is discussed in §8.
2. Definition of the process

We consider the following point process whose definition can be indifferently given in the temporal or spatial domains. Events occur at the random epochs of time (or space coordinates) \( S_1, S_2, \ldots \), from some origin \( S_0 = 0 \). When the intervals of time (space) between events, \( X_1 = S_1, X_2 = S_2 - S_1, \ldots \), are independent and identically distributed random variables, the process thus formed is a renewal process \([2, 8, 9]\).

Hereafter we shall use synonymously the denominations events or renewals. We take the origin of time (space) on a renewal. The sum \( S_n \) therefore reads

\[
S_n = X_1 + \cdots + X_n. \tag{2.1}
\]

The common distribution of the iid random variables \( X_1, X_2, \ldots \) is denoted by \( f(k) = P(X = k) \) when \( X \) is discrete, and we keep the notation \( f(k) \) to designate the density when \( X \) is a continuous random variable. In the sequel we shall use both the discrete and continuum formalisms. Thus the variables \( j, k, r, L \) used below will stand either for integers or for real numbers. The transcription of one formalism to another is easy.

The renewal process can be free, i.e., unbounded to the right as in figure 1, or tied-down \([5, 10, 11]\), i.e., conditioned by the presence of a renewal at \( L \), as in figure 2.

In the first case, denoting by \( N_j \) the random number of renewals between 0 and \( j \), the epoch of the last renewal before \( j \) is

\[
S_{N_j} = X_1 + \cdots + X_{N_j}.
\]

The backward recurrence time \( B_j \) and forward recurrence time \( E_j \) (or excess time) are defined respectively as

\[
B_j = j - S_{N_j}, \quad E_j = S_{N_j+1} - j, \tag{2.2}
\]

and the interval straddling \( j \) is made of their sum,

\[
X_{N_j+1} = B_j + E_j = S_{N_j+1} - S_{N_j}. \tag{2.3}
\]

These definitions are illustrated in figure 3.
In the second case, i.e., if the intervals $X_i$ are conditioned to sum up to the fixed length $L$, we have
\[ S_{N_L} = X_1 + \cdots + X_{N_L} = L, \] (2.4)
i.e., $B_L = 0$. The simplest realisation of such a process (in the temporal domain) is the simple random walk, with steps $\pm 1$, starting and ending at the origin. The $X_i$ are the time intervals between two returns of the walk at the origin. This walk is dubbed the tied-down random walk because it starts and ends at the origin [12, 8], or else the random walk bridge (see §4). Its continuous limit is the Brownian bridge, where now the intervals $X_i$ are continuous random variables.

Such systems, made of a random number of iid intervals conditioned by the value of their sum, dubbed tied-down renewal processes, are naturally encountered in statistical physics. The model introduced in [13, 14] is an example, corresponding to a particular choice of distribution $f(k)$ with power-law tail (2.5), (see [13, 14, 15, 7, 11] for details). In the language of balls in boxes used in [13], the system is made of a random number of boxes, $N_L$, with a fixed total number $L$ of balls in them. The occupations of the boxes are the lengths of the intervals $X_1, X_2, \ldots, X_{N_L}$. These models belong to the broader class of linear models defined by Fisher in [16], such as the Poland-Scherraga model [17, 18] or models of wetting. As explained at the end of this paper, the probability of an empty interval is a key quantity to discuss correlations in such models.

We complete the definition of the process under study by some more details on the distribution of intervals $f(k)$. In what follows, $f(k)$ will be either a narrow distribution with finite moments, or a broad distribution characterised by a power-law tail with index $\theta$ and parameter $c$,
\[ f(k) \underset{k \to \infty}{\approx} \frac{c}{k^{1+\theta}}, \] (2.5)
If $\theta < 1$ all moments of $f(k)$ are divergent, while if $1 < \theta < 2$, the first moment $\langle X \rangle$ is finite but higher moments are divergent, and so on. In Laplace space, where $s$ is conjugate to $k$, for a narrow distribution we have
\[ \mathcal{L}_k f(k) = \hat{f}(s) = 1 - \langle X \rangle s + \frac{1}{2} \langle X^2 \rangle s^2 + \cdots \] (2.6)
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For a broad distribution, we have

\[
\hat{f}(s) \approx \begin{cases} 
1 - |a|s^\theta & \theta < 1 \\
1 - s(X) + \cdots + as^\theta & \theta > 1,
\end{cases}
\]

with

\[ a = c\Gamma(-\theta). \]

The parameter \( a \) is negative if \( 0 < \theta < 1 \), positive if \( 1 < \theta < 2 \), and so on.

3. Probability of an empty interval for a free renewal process

We start with a brief survey of some known results on the probability of an empty interval for a free renewal process (see figure 1). To this end we give the main structure of the reasoning made in [4], without repeating all the details of the calculations. We then compute the integrated Buffon probability, both at equilibrium and for nonequilibrium distributions with tail index \( \theta < 1 \) (see (3.29)). This material will allow to highlight the correspondences between the free and tied-down processes.

3.1. Number of renewals between two arbitrary times

Consider the number of events \( n_j(r) = N_{j+r} - N_j \) occurring between \( j \) and \( j + r \), and its probability distribution, denoted as

\[ p_n(j,r) = \mathbb{P}(N(j,r) = n). \]

The time of occurrence of the \( n \)-th event, counted from time \( j \), is denoted by \( T_n \), with, by convention, \( T_0 = 0 \). By definition of the forward recurrence time \( E_j \) (see (2.2)), the first event after time \( j \) occurs at time \( T_1 = E_j \), when counted from time \( j \). Hence the time of occurrence of the \( n \)-th event between \( j \) and \( j + r \) reads

\[ T_n = E_j + X_2 + \cdots + X_n. \]

Therefore,

\[ p_n(j,r) = \mathbb{P}(T_n < r < T_{n+1}). \]

In particular, for \( n = 0 \), we have

\[ p_0(j,r) = \mathbb{P}(E_j > r) = \int_r^\infty \mathcal{d}\ell f_E(j,\ell), \]

where \( f_E(j,\ell) \) is the density of \( E_j \) (see Appendix A for notations). In other words, the probability of an empty interval \( p_0(j,r) \) is the survival probability up to time \( j + r \), counted from time \( j \).

In Laplace space, where \( u \) is conjugate to \( r \), (3.2) and (3.3) lead to [4]

\[
\mathcal{L}_r p_n(j,r) = \hat{p}_n(j,u) = \hat{f}_E(j,u) \hat{f}(u)^n - 1 - \hat{f}(u) \]

\[
\mathcal{L}_r p_0(j,r) = \hat{p}_0(j,u) = 1 - \hat{f}_E(j,u), \]

where \( \hat{f}_E(j,u) \) is the Laplace transform of \( f_E(j,\ell) \) with respect to \( \ell \). The Laplace transform of the latter with respect to \( j \) reads [4]

\[
\hat{f}_E(s,u) = \frac{\hat{f}(u) - \hat{f}(s)}{s - u} \frac{1}{1 - \hat{f}(s)}.
\]

The implications of the results above are now discussed according to the nature of the distribution of intervals \( f(k) \).
3.2. Probability of an empty interval at equilibrium

If the first moment $\langle X \rangle$ of the distribution $f(k)$ is finite, i.e., for narrow distributions of intervals or distributions with power-law tail (2.5) of index $\theta > 1$, the process reaches equilibrium at large distances (or long times) [4], in particular,

$$ \lim_{j \to \infty} p_0(j, r) = (p_0)_{eq}(r). \quad (3.6) $$

The answer to the Buffon problem ensues. Indeed, consider a large (time or space) interval $L$ (see figure 1), then the probability of an empty interval placed at random on the line is defined as the limit, when $L \to \infty$ of

$$ P_0(r, L) = \frac{1}{L - r} \int_0^{L - r} dk p_0(j, r). \quad (3.7) $$

Using (3.6), we obtain

$$ P_0(r) = \lim_{L \to \infty} P_0(r, L) = (p_0)_{eq}(r). $$

At equilibrium, the two quantities $(p_0)_{eq}(r)$ and $P_0(r)$ are the same. This holds whether the process is free or conditioned by (2.4) since both $j$ and $L$ are sent to infinity.

Let us start with the simple example of an exponential distribution of time (space) intervals with $f(k) = \lambda e^{-\lambda k}$. Then (3.4) and (3.5) imply that

$$ f_E(j, \ell) = e^{-\lambda \ell}, $$

$$ p_n(j, r) = e^{-\lambda r} \frac{(\lambda r)^n}{n!} \quad (n \geq 0), \quad (3.8) $$

which are independent of $j$, showing that the Poisson point process is at equilibrium at all times.

The explicit expression of $(p_0)_{eq}(r)$ is given in (3.12). In order to derive this expression we proceed as follows. For the forward recurrence time, taking the limit of (3.5) for $s \to 0$, we get

$$ (\hat{f}_E)_{eq}(u) = \lim_{s \to 0} s \hat{f}_E(s, u) = \frac{1 - \hat{f}(u)}{\langle X \rangle u}, $$

hence

$$ (f_E)_{eq}(\ell) = \frac{1}{\langle X \rangle} \int_\ell^\infty dk f(k). \quad (3.9) $$

Therefore, from (3.4),

$$ (\hat{p}_n)_{eq}(u) = (\hat{f}_E)_{eq}(u) \hat{f}(u)^{n-1} \frac{1 - \hat{f}(u)}{u} \quad (n \geq 1), \quad (3.10) $$

$$ (\hat{p}_0)_{eq}(u) = \frac{1 - (\hat{f}_E)_{eq}(u)}{u}. \quad (3.11) $$

Thus, at equilibrium, the distribution of the random variable $N(j, r)$ no longer depends on $j$. In particular, its average, $\langle N(j, r) \rangle$ is equal to $r/\langle X \rangle$. By inversion of (3.11), we get

$$ (p_0)_{eq}(r) = \int_r^\infty d\ell (f_E)_{eq}(\ell) = \int_r^\infty d\ell \frac{1}{\langle X \rangle} \int_\ell^\infty dk f(k). $$
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Changing the order of the integrations, one finally finds

$$(p_0)_{eq}(r) = \frac{1}{\langle X \rangle} \int_r^\infty dk \, f(k)(k-r). \quad (3.12)$$

This result can be recovered by the following alternative reasoning. We first note that, at equilibrium, the density of $X_{N_j+1}$, the interval straddling $j$ (see figure 3), reads

$$(f_{X_{N_j+1}})_{eq}(k) = \frac{k f(k)}{\langle X \rangle}. \quad (3.13)$$

The fact that the distribution of this interval is not the same as the distribution $f(k)$ of a generic interval $X_1, X_2, \ldots$, is related to the inspection paradox [2]. The result (3.13), whose intuitive content is clear, has a simple derivation (see §7.1). Thus, taking the average, with this distribution, of the ratio of the available space for the needle to the total length interval, we infer that

$$(p_0)_{eq}(r) = \int_r^\infty dk \, (f_{X_{N_j+1}})_{eq}(k) \frac{k-r}{k}. \quad (3.14)$$

Using now (3.13) in (3.14) reproduces (3.12).

The transcription of these expressions for discrete random variables is straightforward, in particular,

$$(p_0)_{eq}(r) = \frac{1}{\langle X \rangle} \sum_{k \geq r} f(k)(k-r). \quad (3.15)$$

Let us illustrate the preceding analysis on simple examples.

(a) Coming back to the exponential distribution $f(k) = \lambda e^{-\lambda k}$, using (3.12) we instantly get

$$(p_0)_{eq}(r) = e^{-\lambda r},$$
in accord with (3.8).

(b) Let us now consider a geometric distribution of points, which is the discrete version of the previous example. Starting from some origin, let us mark the integer points as renewal events with probability $p$. Let $X$ be the number of unmarked points before the first marked point. Its distribution is

$$f(k) = \mathbb{P}(X = k) = p q^{k-1},$$

where $q = 1-p$ and $k = 1, 2, \ldots$. Thus $\langle X \rangle = 1/p$. Applying (3.15) leads to

$$(p_0)_{eq}(r) = q^r, \quad (3.16)$$

which is simply the probability that $r$ consecutive points are unmarked.

(c) For a uniform distribution $U(0, 2d)$, such that $\langle X \rangle = d$, (3.12) gives

$$(p_0)_{eq}(r) = \left(1 - \frac{r}{2d}\right)^2 = 1 - \frac{r}{d} + \frac{r^2}{4d^2}. \quad (3.16)$$

(d) We now consider broad distributions of intervals (2.5) with tail index $\theta > 1$. First, (3.9) implies

$$(f_E)_{eq}(\ell) \approx \frac{c}{\ell \langle X \rangle} \frac{1}{\ell \theta}. \quad (3.17)$$
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Then, in the stationary regime $1 \ll r \ll j$, (3.12) yields

$$ (p_0)_{eq}(r) \approx \frac{c}{\theta(\theta - 1)\langle X \rangle} \frac{1}{r^{\theta - 1}}. \quad (3.18) $$

In the scaling regime where $j$ and $r$ are large and comparable, using (3.4) and (3.5), we obtain

$$ \hat{p}_0(s, u) \approx \frac{a}{\langle X \rangle} \frac{s^{\theta - 1} - u^{\theta - 1}}{s(s - u)}, $$

where $a$ is defined in (2.8), which by inversion yields [4],

$$ p_0(j, r) \approx \frac{c}{\theta(\theta - 1)\langle X \rangle} \left( \frac{1}{r^{\theta - 1}} - \frac{1}{(j + r)^{\theta - 1}} \right). \quad (3.19) $$

For $1 \ll r \ll j$ we recover (3.18), while for $1 \ll j \ll r$ we obtain

$$ p_0(j, r) \approx \frac{c}{\theta(\langle X \rangle)} \frac{j}{r^{\theta - 1}}. $$

(e) As can be seen on the three examples (a), (b), (c), for $r$ small enough, $(p_0)_{eq}(r) \approx 1 - r/\langle X \rangle$, which is the same behaviour as for equally spaced points (with $d = \langle X \rangle$). Example (d) does not allow to decide on the matter since only the tail of $f(k)$ is given. Let us therefore consider an explicit example where the distribution $f(k)$ of the random spacing $X$ is known for all values of its argument. We take $X = U^{-1/\theta}$, where $U$ is a uniform random variable $\mathcal{U}(0, 1)$, so, for $k \geq 1$ ($k \in \mathbb{R}$),

$$ f(k) = \frac{\theta}{k^{1+\theta}}, $$

with mean $\langle X \rangle = \theta/\theta - 1$. Using (3.12), we get

$$ (p_0)_{eq}(r) = \begin{cases} 1 - \frac{r}{\langle X \rangle} & 0 \leq r \leq 1 \\ \frac{1}{\theta r^{\theta - 1}} & r \geq 1. \end{cases} $$

This result confirms the general behaviour of $(p_0)_{eq}(r)$ for $r$ small. The last line reproduces the right side of (3.18).

3.3. Broad distributions of intervals with index $\theta < 1$

For such distributions the system is never at equilibrium. This is manifested by the fact that at long distances the system becomes self-similar.

In the scaling regime where $j$, $r$ and $\ell$ are large and comparable, we have, from (3.5) (see Appendix A for notations),

$$ \lim_{j \to \infty} f_{j-1,E}(x) = \frac{\sin \pi \theta}{\pi} \frac{1}{x^{\theta}(1 + x)} \quad (0 < x < \infty), \quad (3.20) $$

thus, according to (3.3),

$$ p_0(j, r) \approx \frac{\sin \pi \theta}{\pi} \int_{r/j}^{\infty} \frac{dx}{x^{\theta}(1 + x)} = \int_0^{j/(j+r)} dx \beta_{\theta,1-\theta}(x) $n$$

$$ = \frac{\sin \pi \theta}{\pi} B \left( \frac{j}{j + r}; \theta, 1 - \theta \right) $$

$$ = 1 - \frac{\sin \pi \theta}{\pi} B \left( \frac{r}{j + r}; 1 - \theta, \theta \right), \quad (3.21) $$

This result confirms the general behaviour of $(p_0)_{eq}(r)$ for $r$ small. The last line reproduces the right side of (3.18).
where the beta distribution is defined as
\[ \beta_{a,b}(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \]
and \( B(\cdot) \) is the incomplete beta function
\[ B(z; a,b) = \int_0^z dx x^{a-1}(1-x)^{b-1}. \tag{3.22} \]

For example, for \( \theta = \frac{1}{2} \),
\[ p_0(j,r) \approx 1 - \frac{2}{\pi} \arcsin \sqrt{\frac{r}{j+r}}. \tag{3.23} \]

Moreover, in the same scaling regime \( 1 \ll j \sim r \), the mean number of events occurring between \( j \) and \( j+r \) reads \[ 4 \],
\[ \langle N(j,r) \rangle \approx \frac{\sin \pi \theta}{\pi c} \frac{1}{j^{1-\theta}}. \tag{3.24} \]

A consequence of (3.21) and (3.24) is that, in the short separations regime \( 1 \ll r \ll j \), the probability of finding an event between \( j \) and \( j+r \) goes to zero, i.e., \( p_0(j,r) \to 1 \). In other words, in order to have a chance to observe a renewal, one has to wait a duration \( r \) of order \( j \). The intuitive explanation is that, as \( j \) is growing, larger and larger intervals of time \( k \) may appear. The density of events at large times is therefore decreasing (see \[ 2 \], vol 1 p. 322]). These observations can be made more precise by noting, firstly, that in the regime of short separations (3.21) yields
\[ p_0(j,r) \approx 1 - \frac{\sin \pi \theta}{\pi (1-\theta)} \left( \frac{r}{j} \right)^{1-\theta}, \tag{3.25} \]
and, secondly, that the mean density of events between \( j \) and \( j+r \) has, in the same regime, the expression
\[ \frac{\langle N(j,r) \rangle}{r} \approx \frac{\theta \sin \pi \theta}{\pi c} \frac{1}{j^{1-\theta}}. \tag{3.26} \]

On the other hand, in the opposite regime of large separations between \( j \) and \( j+r \) \( (1 \ll j \ll r) \), (3.21) yields the aging form
\[ p_0(j,r) \approx \frac{\sin \pi \theta}{\pi \theta} \left( \frac{j}{r} \right)^{\theta}. \tag{3.27} \]

To summarise,
\[ \lim_{r \to \infty} p_0(j,r) = 0 \quad \forall j \text{ finite}, \]
\[ \lim_{j \to \infty} p_0(j,r) = 1 \quad \forall r \text{ finite}. \]

This behaviour was dubbed \emph{weak ergodicity breaking} in \[ 19, 20 \], in the sense that ‘true ergodicity breaking only sets in after infinite waiting time’, i.e., \( j \to \infty \) in the present context.

Finally the answer to the Buffon problem consists in estimating the integral (3.7) in the regime \( 1 \ll r \sim L \). Introducing the scaling variables
\[ x = \frac{j}{L}, \quad y = \frac{r}{L}, \]
and using (3.21), we obtain the scaling form

\[ P_0(r, L) \approx C(y) = \frac{\sin \pi \theta}{\pi} \frac{1}{1 - y} \int_0^{1-y} dx \, B \left( \frac{x}{x+y} ; \theta, 1-\theta \right), \]  

(3.28)

which, after some algebra, yields the universal result

\[ C(y) = \frac{\sin \pi \theta}{\pi} \frac{1}{1 - y} \left( B(1 - y; \theta, 1 - \theta) - \frac{y^\theta(1 - y)^{1-\theta}}{\theta} \right). \]  

(3.29)

For instance, for \( \theta = 1/2 \), we have

\[ C(y) = 1 - \frac{2}{\pi} \left( \frac{\arcsin \sqrt{y}}{1 - y} + \sqrt{\frac{y}{1 - y}} \right). \]

For \( y \to 0 \), (3.29) yields

\[ C(y) \approx 1 - \frac{\sin \pi \theta}{\pi \theta (1 - \theta)} y^{1-\theta}, \]

while for \( y \to 1 \), it yields

\[ C(y) \approx \frac{\sin \pi \theta}{\pi \theta (1 + \theta)}. \]

Both limit expressions could have been obtained by integrations of the limit expressions of \( p_0(j, r) \) in the same regime, i.e., respectively (3.25) and (3.27).

4. Probability of an empty interval for a tied-down renewal process

We now consider the tied-down renewal process, with condition (2.4) at \( L \). Consider the interval of length \( r \), with left endpoint located on site \( j \) (see figure 2). The aim of this section is to give the general expression of the probability \( p_0(j, r, L) \) for the interval \((j, j + r)\) to be empty, which is the quantity of central interest. Here we shall describe the process using a discrete formalism (see also [11]). This framework is appropriate for the description of the tied-down random walk; it also complements and enriches the viewpoint based on the continuum formalism used in [5, 10].

4.1. Weight of configurations

A configuration of the process, \( \mathcal{C} = \{X_1 = k_1, X_2 = k_2, \ldots, X_{N_L} = k_n, N_L = n\} \), has weight [5, 10, 11]

\[ \mathbb{P}(\mathcal{C}) = \frac{1}{Z(L)} f(k_1) \cdots f(k_n) \delta \left( \sum_{i=1}^n k_i, L \right), \]  

(4.1)

where the partition function,

\[ Z(L) = \sum_{\mathcal{C}} \mathbb{P}(\mathcal{C}) = \sum_{n \geq 0} \sum_{\{k_i\}} f(k_1) \cdots f(k_n) \delta \left( \sum_{i=1}^n k_i, L \right) \]

\[ = \sum_{n \geq 0} \mathbb{P}(S_n = L) = \mathbb{P}(S_{N_L} = L) = \langle \delta(S_{N_L}, L) \rangle, \]  

(4.2)

is the probability that a renewal occurs at \( L \) (with the notation \( S_n = X_1 + \cdots + X_n \), see (2.1)). Its generating function ensues from (4.2),

\[ \hat{Z}(z) = \sum_{L \geq 0} z^L Z(L) = \frac{1}{1 - f(z)}, \quad \hat{f}(z) = \sum_{k \geq 1} z^k f(k). \]  

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Consider for instance the case where
\[ \tilde{f}(z) = 1 - \sqrt{1 - z}, \] (4.4)
corresponding to
\[ f(k) = \frac{(2k - 2)!}{2^{2k-1} (k-1)! k!} \approx \frac{1}{2\sqrt{\pi} k^{3/2}}, \] (4.5)
Thus
\[ \tilde{Z}(z) = \frac{1}{\sqrt{1 - z}}, \] (4.6)
and
\[ Z(L) = \frac{1}{2L} \left( \frac{2L}{L} \right) \approx \frac{1}{\sqrt{\pi L}}. \] (4.7)
The first values of these quantities read
\[ Z(0) = 1, Z(1) = 1/2, \ldots, \]
\[ f(1) = 1/2, f(2) = 1/8, \ldots. \] These quantities can be interpreted in terms of the tied-down random walk [5, 10, 11, 12]. The partition function \( Z(L) \) corresponds to the probability of return at the origin of the walk at time \( 2L \), while \( f(k) \) is the probability that the first return to the origin occurs at time \( 2k \) [2].

4.2. Probability for the interval \((j, j+r)\) to be empty

The probability \( p_0(j, r, L) \) for the interval \((j, j+r)\) to be empty reads
\[ p_0(j, r, L) = \frac{1}{Z(L)} \sum_{k_1=0}^{j} \sum_{k_2=j+r+1}^{L} Z(k_1) f(k_2 - k_1) Z(L - k_2). \] (4.8)
This expression can easily be understood. The summand \( Z(k_1) f(k_2 - k_1) Z(L - k_2) \) is the probability to find a renewal on the left of the needle \((k_1 \leq j)\), followed by an empty interval \( k_2 - k_1 \), where \( k_2 \) is the first renewal encountered on the right of the needle \((k_2 > j + r)\)\(^\dagger\). This expression is left-right symmetric, as can be seen by changing \( k \) to \( L - k \).

In order to answer the Buffon problem, we have to compute the probability \( P_0(r, L) \) for an interval of length \( r \) placed uniformly at random on \((0, L)\) to be empty. This Buffon probability is obtained by summation of (4.8) on all the possible positions of the left end of the needle, \( 0 \leq j \leq L - r \), so
\[ P_0(r, L) = \frac{1}{L - r + 1} \sum_{j=0}^{L-r-1} p_0(j, r, L). \] (4.9)
The upper bound \( L - r - 1 \) takes into account the fact that \( p_0(j, r, L) \) vanishes for \( j \geq L - r \) since there is a renewal at \( j = L \).

\(^\dagger\) We count a renewal located at \( j \), i.e., on the left end of the needle, as being outside the needle, while a renewal located at \( j + r \), the right end, is counted as being inside. This convention has no consequence for the sequel.
4.3. Probability of an empty interval at equilibrium

We first illustrate (4.8) and (4.9) on two examples of systems converging to equilibrium.

(a) Consider first the simple example of the geometric distribution of points defined in §3.2. Simple calculations lead to
\[ P(S_n = L) = \binom{L-1}{n-1} p^n q^{L-n}, \]
so
\[ Z(L) = \sum_{n \geq 0} P(S_n = L) = p, \]
as it should, since, as said above, \( Z(L) \) is the probability of finding a renewal (i.e., a marked point) at \( L \). As a result of (4.8), one gets
\[ p_0(j, r, L) = (p_0)_{eq}(r) = q^r \quad (0 \leq j \leq L - r - 1), \]
hence
\[ P_0(r, L) = \frac{L - r - c}{L - r + 1} q^r \quad (L - r - 1) \rightarrow L \rightarrow \infty P_0(r) = q^r. \quad (4.10) \]
The constant value of \( Z(L) = 1/\langle X \rangle = p \) is in line with the fact that the process is at equilibrium.

(b) For a distribution of the intervals \( X_1, X_2, \ldots \) with power-law tail (2.5) and index \( \theta > 1 \), the first moment \( \langle X \rangle \) of the distribution \( f(k) \) is finite, and (see, e.g., [10, 11]),
\[ Z(L) \approx \frac{1}{\langle X \rangle} + \frac{c}{\theta(\theta - 1)\langle X \rangle^2} L^{1-\theta}, \quad (4.11) \]
which shows that asymptotically the process becomes stationary, with a density of renewals equal to \( 1/\langle X \rangle \) at large \( L \). Keeping only the dominant term in (4.11) yields (3.19), which for \( 1 \ll r \ll j \) leads to (3.18) and therefore to
\[ P_0(r, L) \approx P_0(r) = (p_0)_{eq}(r), \]
by summation on \( j \) according to (4.9).

5. Asymptotics for distributions with power-law tail (\( \theta < 1 \))

This is the central section of the present work. As for free renewal processes, when the distribution (2.5) has tail index \( \theta < 1 \), the system is never at equilibrium and becomes self-similar at long distances. The scaling analysis of (4.8) proceeds as follows.

If \( \theta < 1 \), it is known, and easy to infer from (4.3), that (see, e.g., [10, 11]),
\[ Z(L) \approx \frac{\theta \sin \pi \theta}{\pi c} L^{1-\theta}, \quad (5.1) \]
which means that renewals become rarer and rarer as \( L \) increases (their density decays as \( 1/L^{1-\theta} \))—a manifestation of the non-stationarity of the process. Note that (5.1) is the same as (3.26) on replacing \( j \) by \( L \). Equations (4.5) and (4.7) are particular cases of (2.5) and (5.1).

In the continuum limit, with \( 1 \ll j \sim r \sim L \), (4.8) can be recast as
\[ p_0(j, r, L) \approx \frac{1}{Z(L)} \int_0^j dk_1 Z(k_1) \int_{j+r}^L dk_2 f(k_2 - k_1) Z(L - k_2). \]
Introducing the scaling variables
\[ x = \frac{j}{L}, \quad y = \frac{r}{L}, \quad a = \frac{k_1}{L}, \quad b = \frac{k_2}{L}, \]
and using (5.1), the probability of interest takes the scaling form
\[ p_0(j, r, L) \approx c(x, y), \]
where
\[ c(x, y) = \frac{\theta \sin \pi \theta}{\pi} \int_0^x da \int_{x+y}^1 db \frac{1}{a^{1-\theta} (b-a)^{1+\theta} (1-b)^{1-\theta}}. \tag{5.2} \]
The second integral can be performed explicitly,
\[ \int_z^1 db \frac{1}{(b-a)^{1+\theta} (1-b)^{1-\theta}} = \frac{(1-z)^\theta}{\theta (1-a)(z-a)^\theta}. \tag{5.3} \]
We are thus left with
\[ c(x, y) = \frac{\sin \pi \theta}{\pi} (1-x-y)^\theta \int_0^x da \frac{1}{(1-a)^{1-\theta} (x+y-a)^\theta}. \tag{5.4} \]
For \( y = 0 \) this expression gives \( c(x, 0) = 1 \), as it should. As we now show, (5.4) has the more explicit expression
\[ c(x, y) = 1 - \frac{\sin \pi \theta}{\pi} \text{B} \left( \frac{y}{(1-x)(x+y)}; 1-\theta, \theta \right), \tag{5.5} \]
where \( B(\cdot) \) is the incomplete beta function (see (3.22)),
\[ B(z; 1-\theta, \theta) = \int_0^z dx x^{-\theta} (1-x)^{\theta-1}. \]
This expression—which is the main result of this section—is universal since it only depends on the tail index \( \theta \) and not on the details of the distribution \( f(k) \). For \( \theta = 1/2 \), (5.5) reduces to the known result for the Brownian bridge (see (B.1)),
\[ c^{\text{bridge}}(x, y) = 1 - \frac{2}{\pi} \arccos \sqrt{\frac{x(1-x-y)}{(1-x)(x+y)}}. \tag{5.6} \]
In order to derive (5.5), the key point is to notice that the scaling function (5.4) only depends on a single variable, namely the cross-ratio of the four points \( x_1 = 0, x_2 = x, x_3 = x+y \) and \( x_4 = 1 \),
\[ \zeta = \frac{(x_1-x_4)(x_2-x_3)}{(x_1-x_3)(x_2-x_4)} = \frac{y}{(1-x)(x+y)}. \tag{5.7} \]
This can already be noticed on the expression (5.6) of \( c^{\text{bridge}}(x, y) \), which satisfies \( \cos(\pi c^{\text{bridge}}/2)^2 = \zeta \). More generally, in order to obtain (5.5), it suffices to make the following change of variable in (5.4),
\[ t = \frac{x+y-a}{(1-a)(x+y)}. \]
In the regime of short separations, such that \( 1 \ll r \ll j \sim L \), hence \( y \to 0 \), the expression (5.5) simplifies to
\[ c(x, y) \approx 1 - \frac{\sin \pi \theta}{\pi (1-\theta)} \left( \frac{y}{x(1-x)} \right)^{1-\theta}. \tag{5.8} \]
In the regime of large separations, such that \( 1 \ll j \ll r \sim L \), hence \( x \to 0 \), the expression (5.5) gives

\[
c(x, y) \approx \frac{\sin \pi \theta}{\pi \theta} \left( \frac{x(1-y)}{y} \right)^\theta.
\] (5.9)

These two last results were found by another method in [5] (see the remark in section 6).

Remark. It should be noted that (5.5) generalises the expression (3.21) found for the free renewal process with \( \theta < 1 \), in the regime \( 1 \ll j \sim r \). Conversely, (5.5) taken in the regime \( 1 \ll j \ll r \ll L \) reduces to

\[
p_0(j, r) \approx 1 - \sin \frac{\pi \theta}{\pi} B \left( \frac{y}{x+y}; 1-\theta, \theta \right),
\]

which is precisely (3.21). Accordingly, (5.8) and (5.9) generalise (3.25) and (3.27).

The last step consists in deriving the integrated scaling function. By integration of (5.5) on \( x \), the probability (4.9) takes the scaling form

\[
P_0(r, L) \approx C \left( y = \frac{r}{L} \right) = \frac{1}{1-y} \int_0^{1-y} dx c(x, y),
\]

yielding, after some algebra,

\[
C(y) = \frac{\sin \pi \theta \Gamma(\theta)}{2\sqrt{\pi} \Gamma(2+\theta) \Gamma(2-\theta)} \left( \frac{1+y}{1+y^2} \right)^{1-y} \cdot B \left( \frac{(1-y)^2}{1+y^2}; 1/2 + \theta, 1-\theta \right).
\] (5.10)

This expression gives the answer to the Buffon problem for distributions with power-law tails of index \( \theta < 1 \). It is universal, as was (5.5).

In the regime of short separations, \( y \to 0 \), (5.10) yields

\[
C(y) \approx 1 - \frac{\Gamma(\theta)}{\Gamma(2\theta) \Gamma(2-\theta)} y^{1-\theta},
\] (5.11)

which can also be obtained by integration of (5.8) on \( x \) [6, 7]. In the other regime of interest, \( y \to 1 \) (large separations), we have

\[
C(y) \approx \frac{(1-\theta)\Gamma(\theta)}{2(1+\theta)\Gamma(2\theta) \Gamma(2-\theta)} (1-y)^{2\theta}.
\] (5.12)

Finally, for \( \theta = 1/2 \), the expression (5.10) simplifies to

\[
C(y) = \frac{1-\sqrt{y}}{1+\sqrt{y}}.
\] (5.13)

which coincides with the result given in (B.2) for \( C^{\text{bridge}}(y) \), as it should.

### 6. A variant

The probability of an empty interval \( p_0(j, r, L) \) can also be expressed in terms of the distribution \( p_E(j, \ell, L) = \mathbb{P}(E_j = \ell) \) of the forward recurrence time (or excess time) \( E_j \), starting at \( j \), (see figure 3). Indeed,

\[
p_0(j, r, L) = \mathbb{P}(E_j > r) = \sum_{\ell=r+1}^{L-j} p_E(j, \ell, L).
\] (6.1)
Comparing to (4.8), we have, setting \( k_2 - j = \ell \),

\[
p_E(j, \ell, L) = \frac{Z(L - j - \ell)}{Z(L)} \sum_{k_1 \leq j} Z(k_1) f(j + \ell - k_1).
\] (6.2)

The scaling analysis of this quantity—still focusing on the case where \( \theta < 1 \)—leads to

\[
p_E(j, \ell, L) \approx \frac{1}{L} g_E(x, \xi),
\] (6.3)

with \( x = j/L \) and \( \xi = \ell/L \) and where

\[
g_E(x, \xi) = \frac{\theta \sin \pi \theta}{\pi} \int_0^x da \frac{(1 - x - \xi)\theta - 1}{a^{1-\theta}(x + \xi - a)^{1+\theta}}
\] (6.4)

\[
= \frac{\sin \pi \theta}{\pi} \frac{x^\theta}{(x + \xi)\xi^{1-\theta}(1 - x - \xi)^{1-\theta}}.
\] (6.5)

Summing this expression upon \( \xi \) in the interval \((y, 1 - x)\) gives back (5.5).

Remark

The numerator of the generating function of \( p_E(j, \ell, L) \) with respect to all its arguments (with conjugate variables \( u, v, z \)) is given by

\[
\tilde{p}_E(u, v, z)|_{\text{num}} = \sum_{L \geq 0} z^L \sum_{j=0}^L u^j \sum_{\ell=1}^L v^\ell p_E(j, \ell, L)|_{\text{num}}
\]

\[
= \hat{Z}(z) \tilde{Z}(zu) \frac{v f(zu)}{v - u}.
\] (6.6)

This expression is the discrete counterpart of the expression given in [5] for the density of the forward recurrence time in Laplace space. According to (6.1) it is related to the generating function of \( p_0(j, r, L) \) with respect to its arguments \( j, r, L \) (with conjugate variables \( u, v, z \)) as

\[
\tilde{p}_0(u, v, z)|_{\text{num}} = \sum_{L \geq 0} z^L \sum_{j=0}^L u^j \sum_{r=0}^L v^r p_0(j, r, L)|_{\text{num}}
\]

\[
= \hat{p}_E(u, 1, z)|_{\text{num}} - \tilde{p}_E(u, v, z)|_{\text{num}}.
\] (6.7)

The asymptotic analysis of this expression was made in [5] in the regimes of short and large separations, yielding respectively (5.8) and (5.9). The direct approach based on (6.2) is more convenient for the derivation of the complete expression of the scaling function (5.5), valid in all regimes.

7. Complementary remarks

This section brings some complements to the body of the text.
7.1. Straddling interval and inspection paradox

In Laplace space, for a free renewal process, the density of the straddling interval (2.3) reads

\[ L_{j,k} f_{X_{N+j-1}}(s) = \frac{\hat{f}(u) - \hat{f}(s + u)}{s} \cdot \frac{1}{1 - \hat{f}(s)} \]  \hspace{1cm} (7.1)

This expression can easily be demonstrated by the methods of [4]. We have

\[ f_{X_{N_j+1},N_j}(j,k,n) = \langle \delta(k - S_{n+1} + S_n)I(S_n < j < S_{j+1}) \rangle, \]

where \(I(\cdot)\) is the indicator function of the event inside the parenthesis. Laplace transforming with respect to \(j\) and \(k\) and summing upon \(n\) leads to (7.1).

Hence, at equilibrium, we have, using (7.1),

\[ (\hat{f}_{X_{N_j+1}})_{eq}(u) = \lim_{s \to 0} s \hat{f}_{X_{N_j+1}}(s,u) = -\frac{1}{\langle X \rangle} \frac{d\hat{f}(u)}{du}, \]

we thus recover (3.13),

\[ (f_{X_{N_{j+1}}})_{eq}(k) = \frac{k f(k)}{\langle X \rangle}. \]

The inspection paradox, which states that \(\langle X_{N_j+1} \rangle\) is larger than \(\langle X \rangle\), ensues immediately. Indeed, if \(f(k)\) is a narrow distribution or a broad distribution with tail index \(\theta > 2\), one has, at equilibrium,

\[ \langle X_{N_{j+1}} \rangle_{eq} = \frac{1}{\langle X \rangle} \int_0^{\infty} dk k^2 f(k) = \frac{\langle X^2 \rangle}{\langle X \rangle} > \langle X \rangle. \]  \hspace{1cm} (7.2)

In contrast, if \(1 < \theta < 2\), this second moment diverges. The inspection paradox is all the more true. One can compute the asymptotic behaviour [4]

\[ \langle X_{N_{j+1}} \rangle \approx (\theta - 1)(2 - \theta)\langle X \rangle j^{2-\theta}. \]

7.2. Behaviour of the hole probability \(p_0\) when \(r \to 0\)

It was noted on examples in §3.2 that for microscopic values of \(r\) the behaviour of the hole probability at equilibrium \((p_0)_{eq}(r) \approx 1 - r/\langle X \rangle\). Let us prove this result in all generality and extend it to the case of broad laws with tail index \(\theta < 1\).

When \(r\) is microscopic the asymptotic behaviours of the free and tied-down processes are the same, so we can restrict the discussion to the case of the free process. We start from (3.4) and (3.5). If \(r\) is microscopic, the conjugate Laplace variable \(u\) is large. For large \(j, s \to 0\). Firstly, for narrow distributions or broad laws with tail index \(\theta > 1\), we have

\[ \hat{f}_E(s,u) \approx \frac{1}{u} \frac{1}{s\langle X \rangle}, \]

which, by Laplace inversion, yields

\[ (p_0)_{eq}(r) \approx 1 - \frac{r}{\langle X \rangle}, \]  \hspace{1cm} (7.3)

which confirms the generality of the results found in §3.2. Secondly, for broad laws with tail index \(\theta < 1\), the same reasoning yields

\[ \hat{f}_E(s,u) \approx \frac{1}{u} \frac{1}{as^{\theta}}, \]
from which ensues that in the regime \(1 \sim r \ll j\),
\[ p_0(r) \approx 1 - \frac{r}{\langle X_j \rangle}, \]
where we introduced the notation \(\langle X_j \rangle\) for the typical interval length
\[ \langle X_j \rangle \approx \frac{\pi c}{\theta \sin \pi \theta} j^{1-\theta}. \]

The beauty of this result is that one recognizes the asymptotic expression of \(1/Z(j)\) (see (5.1)). Now, for distributions with a finite first moment, \(Z(j)\) has the interpretation of the inverse of the typical interval \(\langle X_j \rangle\) for \(j\) large. The present discussion extends this result to the case of distributions without finite first moment. This interpretation was actually already pointed out in the comment below (5.1) in view of (3.26).

Let us finally mention that (7.3) is more general and also holds if the intervals \(X_1, X_2, \ldots\) are not independent (see [21]).

### 7.3. Buffon problem on a ring

Instead of considering the probability of an empty interval on a segment \((0, L)\), this probability can also be considered on a ring of size \(L\). However, this requires giving a precise definition to the process. If one keeps the definition of the configurations of the process given in §4.1, with weights (4.1), meaning that the two endpoints 0 and \(L\) are identified into a single point, the only change to be made in the definition of the Buffon probability given by (4.9) is the interval of integration. The latter is changed into
\[ P_0(r, L) = \frac{1}{L} \sum_{j=0}^{L-r-1} p_0(j, r, L), \] (7.4)
or into
\[ P_0(r, L) = \frac{1}{L} \int_0^{L-r} dj p_0(j, r, L), \] (7.5)
in the continuum formalism.

### 7.4. Correlation function

We defined the probability of having \(n\) renewals in the interval \((j, j+r)\) as, (see (3.1)),
\[ p_n(j, r, L) = \mathbb{P}(N(j, r) = n), \]
where \(N(j, r)\) is the number of renewals in \((j, j+r)\). Knowing this probability allows the determination of the correlation function
\[ \langle (-1)^{N(j, r)} \rangle = \sum_{n \geq 0} (-1)^n p_n(j, r, L). \] (7.6)

This correlation is a natural quantity to consider if the intervals represent ±1 spin domains. The study of this spin-spin correlation function was performed in [4] for the case of a semi-infinite one-dimensional system, that is, for a free renewal process as in figure 1. Reference [5] gave an extension of this former study to the case of a system of fixed length \(L\), conditioned by (2.4), that is, for a tied-down renewal process as in figure 2.

It turns out that, for both free and tied-down renewal processes, when the distribution \(f(k)\) is of the form (2.5), with \(0 < \theta < 2\), this correlation function is
asymptotically dominated by its first term, namely \( p_0(j,r) \) or \( p_0(j,r,L) \) respectively, and therefore the knowledge of this hole probability suffices to determine the correlation (7.6) [4, 5].

8. Discussion

In this paper the focus was on the probability of an empty interval—or hole probability—both for free and tied-down renewal processes. In the first case the hole probability \( p_0(j,r) \) is a function of \( j \), the position of the left end of the needle, and \( r \), the length of the needle, while in the second case \( p_0(j,r,L) \) also depends on the length \( L \) of the system.

The present work is a completion of [4] and [5]. The main novelty with respect to [4] is the determination of the Buffon probability for a distribution of intervals \( f(k) \) with a power-law tail (2.5) and tail index \( \theta < 1 \) in the scaling regime \( 1 \ll r \sim L \) (see (3.28) and (3.29)). As for [5], the main novelties are the complete expression (5.5) of the universal scaling function \( c(x,y) \), as well as its integrated expression, the Buffon probability (5.10), in the regime where all the arguments of \( p_0(j,r,L) \approx c(x = j/L, y = r/L) \) are large and comparable, again for broad laws with tail index \( \theta < 1 \). The results given in [5] were limited to the expressions of \( c(x,y) \) in the two regimes of short and large separations, i.e., respectively (5.8) and (5.9) and to the expression (5.6) of \( c(x,y) \) for the case of the Brownian bridge. For a large system, the expression (5.5) still keeps a dependence in the ratio \( x = j/L \), which demonstrates the non stationarity of the process. In contrast, if \( \theta > 1 \), the process becomes stationary, when \( j \sim L \to \infty \), i.e., no longer depends on \( j \).

As mentioned in the Introduction, this kind of systems, made of a random number of iid intervals conditioned by the value of their sum, are naturally encountered in statistical physics, e.g., the tied-down random walk [12, 10], the model introduced in [13, 14], or more generally the linear models defined by Fisher [16], such as the Poland-Scherraga model [17, 18] or models of wetting.

The critical spin-spin correlation function of the model considered in [14], where the iid intervals represent positive or negative spin domains partitioning the segment \((0, L)\), with a power-law distribution (2.5) and tail index \( \theta < 1 \), identifies to (7.6) [7]. Its expression in the scaling regime is therefore given by (5.5), or by (5.10) after summation on \( x \). The computation of this correlation function, restricted to the regime of short separations, was revisited in [15, 22], using heuristic methods which boil down to considering the system at equilibrium. It was however pointed out in [6, 7] that applying the equilibrium formalism of §3.2 or §4.3 to a situation where the tail exponent \( \theta < 1 \) is incorrect, irrespective of whether the geometry of the system is a line segment [22] or a ring [15].

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\[ \parallel \text{Taking periodic boundary conditions in the context of a spin model with Boltzmann weights given by (4.1), as done in [15], brings the additional difficulty of having to select configurations such that the parity of the number of domains is compatible with periodicity.} \]
Appendix A. A word on notations

Asymptotic equivalence
The symbol $\approx$ stands for asymptotic equivalence; the symbol $\sim$ is weaker and means ‘of the order of’.

Probability densities, Laplace transforms, limiting distributions
The probability density function of the continuous random variable $X$ is denoted by $f_X(x)$, with

$$f_X(x) = \frac{d}{dx} \mathbb{P}(X < x).$$

In the case of the forward recurrence time $E_j$, the density $f_{E}(j, \ell) = \frac{d}{d\ell} \mathbb{P}(E_j < \ell)$ has a dependence in the (time or space) coordinate $j$. Its Laplace transform with respect to $\ell$ is

$$\hat{f}_E(j, u) = \mathcal{L}_\ell f_E(j, \ell) = \langle e^{-uE_j} \rangle = \int_0^{\infty} d\ell e^{-u\ell} f_E(j, \ell),$$

and its double Laplace transform with respect to $j$ and $\ell$ is denoted by

$$\hat{f}_E(s, u) = \mathcal{L}_j,\ell f_E(j, \ell) = \mathcal{L}_j \langle e^{-uE_j} \rangle = \int_0^{\infty} dj e^{-sj} \int_0^{\infty} d\ell e^{-u\ell} f_E(j, \ell).$$

When $\theta < 1$, $E_j$ scales asymptotically as $j$. As $j \to \infty$ the rescaled variable $E_j/j$ converges to a limit, denoted by

$$X = \lim_{j \to \infty} j^{-1} E_j,$$

with limiting density $f_X(x)$, (see (3.20)).

Appendix B. Correlation function for the Brownian bridge

This appendix is a reminder of results given in [5, 6]. The continuum limit of the tied-down random walk defined in §2 (see (4.7), (4.5)) is the Brownian bridge. A direct computation, based on the fact that the Brownian bridge is a Gaussian process, gives the correlation function defined in (7.6) [5],

$$\langle (-1)^N(j, r) \rangle = 1 - \frac{2}{\pi} \arccos \sqrt{\frac{j(L-j-r)}{(j+r)(L-j)}}, \quad (B.1)$$

yielding (5.6) for $c^\text{bridge}(x, y)$. A comparison between (4.8) for the tied-down random walk and (B.1) for $L = 400, r = 40$ is given in figure B1, with excellent agreement. This figure gives an illustration of the asymptotic dominance of $p_0(j, r, L)$ mentioned in §8.

Integrating (B.1) on $j$ gives

$$C^\text{bridge}(y) = \frac{1 - \sqrt{y}}{1 + \sqrt{y}}, \quad (B.2)$$

in agreement with (5.13) which was obtained from (5.10) for $\theta = 1/2$. 
The Buffon needle problem for randomly spaced points

Figure B1. Exact probability \( p_0(j, r, L) \) given by (4.8) as a function of \( j/L \) for the tied-down random walk (red dots). This probability is compared to the analytical expression (B.1) of the correlation function of the sign of the position for the Brownian bridge (continuous curve). Here \( r = 40, L = 400. \)

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