Production of heralded pure single photons from imperfect sources using cross phase modulation

Thomas Konrad\textsuperscript{1,2}, Michael Nock\textsuperscript{1}, Artur Scherer\textsuperscript{3}, and Jürgen Audretsch\textsuperscript{1}

\textsuperscript{1} Fachbereich Physik, Universität Konstanz, D-78457 Konstanz, Germany
\textsuperscript{2} School of Pure and Applied Physics, University of KwaZulu-Natal, Durban 4000, South Africa

Realistic single-photon sources do not generate single photons with certainty. Instead they produce statistical mixtures of photons in Fock states $|1\rangle$ and vacuum (noise). We describe how to eliminate the noise in the output of the sources by means of another noisy source or a coherent state and cross phase modulation (XPM). We present a scheme which announces the production of pure single photons and thus eliminates the vacuum contribution. This is done by verifying a XPM related phase shift with a Mach-Zehnder interferometer.

PACS numbers: 03.67.-a, 03.67.Lx, 03.67.Hk, 42.50.Gy.

I. INTRODUCTION

Light interacts in general weakly with its environment. Therefore on the one hand photonic states are durable and can be optimally employed as carrier of quantum information. On the other hand it is more difficult to process the information encoded in the state of light. To circumvent this obstacle elaborated schemes have been invented to emulate such interaction by means of linear optics and conditioned photodetection. The outstanding schemes of Knill, Laflamme and Milburn \cite{1} as well as teleportation and projection synthesis \cite{2,3} are useful tools for optical quantum information processing. Yet all these schemes require pure single photons, i.e. Fock states $|1\rangle$, on demand which in turn impedes their realization.

In recent years a variety of implementations for single-photon sources has been investigated. Among them are schemes based on single molecule or atom excitation \cite{4,5,6}, single ions trapped in cavities \cite{7}, color centers in diamonds \cite{8,9}, quantum dots \cite{10,11} and parametric down conversion (PDC) \cite{12,13}. These sources differ in the wavelength and purity of the state of the emitted photons, their repetition rate and whether they produce a photon on demand or heralded, i.e. announced by an event. The latter is for example the case with PDC-sources. PDC produces randomly photon pairs and the presence of one photon is indicated by the detection of the other.

None of the existing single-photon sources, however, emits a pure single photon at a given time with certainty. The emission of multiple photons is negligible for most single-photon sources, cf. for example \cite{7}. Therefore their output in a certain mode can be modeled by a mixture of a Fock state $|1\rangle$ and vacuum $|0\rangle$:

$$\rho = p|1\rangle\langle 1| + (1-p)|0\rangle\langle 0|,$$  \hspace{0.5cm} (1)

where $p$ is called the efficiency of the single-photon source ($0 < p < 1$). We refer to such mixed states as noisy photons while we call Fock states $|1\rangle$ pure photons. Good sources have efficiencies of $p \approx 0.6$. To the best of our knowledge the highest efficiencies reached so far are $p = 0.83$ \cite{13} and $p = 0.86$ \cite{14}. The challenge is to construct a set-up by which the efficiency $p$ can be increased in the ideal case up to one.

Investigations so far indicate that the efficiency may not be improved by means of linear optics without adding multi-photon components. It has also been shown \cite{14,15} that the enhancement of $p$ is limited under these circumstances. The efficiency $p$ cannot reach 1 given a finite number of imperfect sources by means of linear optics and photodetection. In addition, three or less noisy photons are not enough to obtain an improvement at all. Employing homodyne detection and one noisy photon is not sufficient either \cite{16}. At least an enhancement of $p$ has been achieved in \cite{17} but at the expense of adding a two-photon component.

In this article we present how to purify and herald a noisy photon as given in \cite{11}, by means of nonlinear optics. The scheme we employ has formerly been used by Milburn \cite{17}, Imoto \cite{18} and others for alternative purposes and in different manners. It consists of a medium in which a signal and a probe mode experience cross phase modulation. The resulting phase shift of the probe mode is verified by means of a Mach-Zehnder interferometer. Similar features like the use of XPM and coherent states can also be found in \cite{19}. There a QND measurement of the photon number is proposed. However, the actual scheme can distinguish between one photon and vacuum only up to a small non-vanishing error probability. In \cite{20} we have proposed a method based on two-photon absorption which grants vanishing error probability. Thus it enables the generation of a pure photon. The scheme proposed in this article also possesses this property. The advantage of the present scheme proposed in this article is that it allows in principle to detect and announce a single photon emitted by the source with arbitrarily high probability.

This article is organized as follows. First we discuss cross phase modulation (Sec. \ref{sect:CPM}) and explain the functional principle of our set-up (Sec. \ref{sect:setup}). In Sec. \ref{sect:design} we study the case in which two noisy photons are used as inputs. Then we turn to the more realistic case in which we use one noisy photon in the signal mode and a coherent state in the probe mode (Sec. \ref{sect:real}). Eventually, in Sec. \ref{sect:realization} we discuss a potential realization for large cross phase modulation, which is desirable in our scheme. An appendix contains transparency conditions for a Mach-Zehnder interferometer and extended schemes to generate single photons.

*the corresponding author: thomas.konrad@uni-konstanz.de
II. CROSS PHASE MODULATION

Cross phase modulation (XPM), which is also referred to as cross Kerr interaction, is an interaction between two modes $A$ and $B$ of a light field governed by the Hamiltonian

$$H_{\text{XPM}} = -\chi a^\dagger ab^\dagger b ,$$

(2)

cf. [15]. Here, $\chi$ is a real constant which is related to the third-order nonlinear susceptibility coefficient usually denoted by $\chi^{(3)}$, and $a$, $b$ represent the annihilation operators of a photon in mode $A$ and mode $B$, respectively.

Two light modes which undergo a cross phase modulation during time $\Delta t$ acquire a phase shift which depends on the product of the number of photons the two modes contain. In the Schrödinger picture this effect shows up, e.g., in the evolution of a Fock state with the number of photons the two modes contain. In the evolution of a Fock state with $n$ photons in Mode $A$ and $m$ photons in Mode $B$:

$$|n^A, m^B\rangle \xrightarrow{\text{XPM}} \exp(i\chi \Delta t a^\dagger ab^\dagger b)|n^A, m^B\rangle = \exp(i\phi_{\chi} nm)|n^A, m^B\rangle$$

(3)

with $\phi_{\chi} := 2 \chi \Delta t$. We exclude the case of a non-working XPM ($\phi_{\chi} = 2k\pi$, $k \in \mathbb{N}$). Eq. (3) can be interpreted as if light in mode $B$ (probe mode) experiences a phase shift due to XPM depending on the number of photons entering in mode $A$ (signal mode), cf. Fig. 1.

On the other hand no phase shift occurs, if mode $A$ contains zero photons. We employ this effect in the following to detect and announce a single photon in mode $A$ without absorbing it. This is done by verifying a phase shift of light in mode $B$ by means of a Mach-Zehnder interferometer followed by a photodetector.

XPM naturally occurs in non-linear media (Kerr media) where the index of refraction depends on the intensity of incoming light. Since we require vanishing absorption rates in order not to loose incoming photons, we are confronted mostly with media which also possess very low cross phase modulation rates ($\chi \ll 1$). This problem can be tackled by arranging long interaction times either via electromagnetically induced transparency (EIT) or due to co-propagation of two modes over long distances in optical fibers. We postpone the discussion of possible realizations of XPM to Sec. [VI].

III. DETECTION OF XPM-PHASE SHIFT BY MEANS OF A MACH-ZEHNDER INTERFEROMETER

Before we explain the functioning principle of our scheme let us briefly introduce our beam splitter convention. A general beam splitter with two input modes $A_1$ and $A_2$ and two output modes $A_1'$ and $A_2'$ is depicted in Fig. [2].

Like in the case of cross phase modulation we represent the action of such a beam splitter on the electromagnetic field in the Schrödinger picture (cf. [21]). A pure input state given by $f(a_1^\dagger, a_2^\dagger)|0, 0\rangle$ is transformed due to the beam splitter into the output state $f(\tilde{a}_1^\dagger, \tilde{a}_2^\dagger)|0, 0\rangle$.

![Beam splitter with two input modes A1 and A2 and two output modes A1' and A2'](image)

FIG. 2: Beam splitter with two input modes $A_1$ and $A_2$ and two output modes $A_1'$ and $A_2'$.

Thereby the creation operators $a_1^\dagger$ and $a_2^\dagger$ corresponding to the input modes $A_1$ and $A_2$ are replaced as arguments of the unmodified function $f$ by $\tilde{a}_1^\dagger$ and $\tilde{a}_2^\dagger$ according to:

$$a_1^\dagger \xrightarrow{\text{BS}} \tilde{a}_1^\dagger = \cos(\theta) a_1^\dagger + e^{-i\phi} \sin(\theta) a_2^\dagger$$

$$a_2^\dagger \xrightarrow{\text{BS}} \tilde{a}_2^\dagger = -e^{i\phi} \sin(\theta) a_1^\dagger + \cos(\theta) a_2^\dagger ,$$

(4)

cf. [21]. Here $a_1^\dagger$, $a_2^\dagger$ are the creation operators of the output field modes $A_1'$ and $A_2'$, respectively. $\phi$ represents a relative phase shift, $\cos^2(\theta)$ and $\sin^2(\theta)$ are the reflectivity and transmittivity of the beam splitter. For the sake of simplicity, however, in what follows we will omit the prime labels and denote the output field modes and the corresponding operators by the same letters as the input modes.

Our scheme to detect the production of a single photon is depicted in Fig. 3. An imperfect single photon source emits in the upper arm. A click of detector in mode $C$ announces a pure single photon Fock state in mode $A$.

![Mach-Zehnder interferometer with cross phase modulation in the upper arm](image)

FIG. 3: Mach-Zehnder interferometer with cross phase modulation in the upper arm. A click of detector in mode $C$ announces a pure single photon Fock state in mode $A$. 

mode $A$ either a single-photon state $|1\rangle$ with probability $p_A$ or vacuum $|0\rangle$ with probability $1 - p_A$. The resulting statistical mixture is denoted by

$$\rho^A = p_A|1^A\rangle\langle1^A| + (1 - p_A)|0^A\rangle\langle0^A|.$$  \hfill (5)

$\rho^A$ enters the set-up in mode $A$. The input states of the auxiliary modes $B$ and $C$ of the Mach-Zehnder interferometer, which is composed of beam splitters $BS_1$ and $BS_2$, are given by $\rho^B$ and the vacuum $|0^C\rangle$, respectively. In the upper arm of the interferometer a medium is placed which exerts a cross phase modulation (XPM) between mode $A$ and mode $B$.

In case mode $A$ contains only vacuum, the light in the upper arm of the interferometer (mode $B$) does not experience a phase shift relative to the light in the lower arm (mode $C$). It is important that the beam splitters of the Mach-Zehnder interferometer are adjusted such, that in this case no photons leave the interferometer in mode $C$. Therefore the detector placed in output $C$ cannot respond if mode $A$ does not contain a photon. On the other hand, if the single-photon source emits a photon in mode $A$, the phase of the light in mode $B$ is shifted due to XPM relative to the phase in Mode $C$. This results in a non-vanishing probability for a click of the detector in output $C$. Hence, the detection of light in mode $C$ implies the presence of a photon in mode $A$. A selection according to the clicks of the detector yields the preparation of a single-photon Fock state $|1\rangle$ in mode $A$. Therefore single photons are heralded by the response of the photodetector in mode $C$.

We will address two main issues:

(i) The condition that a detection event in mode $C$ announces with certainty a pure single photon in mode $A$.

(ii) Provided that condition (i) is fulfilled, what is the probability to detect a photon present in the signal mode $A$, i.e., what is the detection efficiency?

The latter depends on the strength of the XPM and on the state $\rho^B$ of the auxiliary mode $B$. We will discuss two possible inputs in $B$: a second noisy photon and a coherent state of light.

IV. HERALDING PURE SINGLE PHOTONS BY MEANS OF NOISY PHOTONS

Let us discuss the case that both states $\rho^A$ and $\rho^B$ which enter the set-up are noisy photons:

$$\rho^A = p_A|1^A\rangle\langle1^A| + (1 - p_A)|0^A\rangle\langle0^A|,$$
$$\rho^B = p_B|1^B\rangle\langle1^B| + (1 - p_B)|0^B\rangle\langle0^B|.$$  \hfill (6) \hfill (7)

The initial state of the signal mode $A$ and the probe mode $B$ thus reads

$$\rho^A \otimes \rho^B = p_A p_B|1^A1^B\rangle\langle1^A1^B| + p_A(1 - p_B)|1^A0^B\rangle\langle1^A0^B| + (1 - p_A)p_B|0^A1^B\rangle\langle0^A1^B| + (1 - p_A)(1 - p_B)|0^A0^B\rangle\langle0^A0^B|.$$  \hfill (8)

With respect to both issues (i) and (ii) above we only have to consider the case that a photon enters in probe mode $B$, which happens with probability $p_B$. Otherwise, if mode $B$ contains only vacuum, no photon is present in the Mach-Zehnder interferometer and thus no heralding of a photon in mode $A$ is possible.

We turn to condition (i). In order to herald pure single photons in the signal mode the detector must not respond if the signal mode contains vacuum. Whether this condition is fulfilled can be checked by propagating an input state which contains vacuum in the signal mode, one photon in the probe mode and no photon in mode $C$. Its evolution due to the Mach-Zehnder interferometer and XPM is as a consequence of transformations $\rho^A$ and $\rho^B$ of the form:

$$|0^A1^B0^C\rangle \xrightarrow{BS_2 \circ XPM \circ BS_1} c_{101}|1^A0^B1^C\rangle + c_{001}|0^A1^B0^C\rangle.$$  \hfill (9)

The detector in mode $C$ cannot respond if the amplitude $c_{001}$ vanishes, i.e.,

$$c_{001} = e^{-i\phi_2}\cos(\theta_1)\sin(\theta_2) + e^{-i\phi_1}\sin(\theta_1)\cos(\theta_2) = 0.$$  \hfill (10)

Here $\cos^2(\theta_{1,2})$ and $\sin^2(\theta_{1,2})$ are the reflectivity and transmittivity of beam splitters $BS_1$ and $BS_2$, respectively (cf. Eq. (4)). This leads to one of the following two constraints:

$$\phi_1 - \phi_2 = 2k\pi \quad \text{and} \quad \theta_1 + \theta_2 = l\pi, \quad k, l \in \mathbb{Z}, \quad (11)$$
$$\phi_1 - \phi_2 = (2k + 1)\pi \quad \text{and} \quad \theta_1 - \theta_2 = l\pi, \quad k, l \in \mathbb{Z}. \quad (12)$$

It is shown in the Appendix VIII A that if one of these constraints is satisfied, any state $|0^A\rangle |\psi\rangle^BC$ entering the interferometer does not change.

We choose beam splitters $BS_1$ and $BS_2$ according to either constraint (11) or (12) and assume that the detector has no dark counts. Then the conditional probability for a click given that the signal mode $A$ contains just vacuum is zero, i.e., $p(\text{click}|0^A) = 0$, which is equivalent to saying that, provided a click occurs there must be a photon in mode $A$. Hence, the conditional probability to find one photon outgoing in mode $A$ if the detector clicks is equal to one:

$$p(1^A_{\text{out}}|\text{click}) = 1.$$  \hfill (13)

Thus condition (i) is fulfilled. We have obtained the following result: Selecting the cases in which the detector clicks amounts to the preparation of the pure one-photon state $|1^A\rangle$ in mode $A$. This is independent of the parameters $\phi_X$ of the XPM and the efficiency $p_B$ of $\rho^B$, respectively.

However, with regard to the practical use of the set-up the question of issue (ii) remains, namely, how probable it is that a photon in the signal mode $A$ is detected, i.e., causes a click of the detector. This depends on the strength $\phi_X$ of the XPM and on the efficiency $p_B$ of the source feeding probe mode $B$.

The action of the Mach-Zehnder interferometer combined with the XPM medium on the input state $|1^A1^B0^C\rangle$ is given by the transformation (cf. Eq. (3) and (4)):

$$|1^A1^B0^C\rangle \xrightarrow{BS_2 \circ XPM \circ BS_1} c_{101}|1^A1^B0^C\rangle + c_{001}|0^A0^B1^C\rangle.$$  \hfill (14)
with
\[ c_{110} = e^{i\phi_1} \cos(\theta_1) \cos(\theta_2) - e^{-i(\phi_1 - \phi_2)} \sin(\theta_1) \sin(\theta_2), \]
\[ c_{101} = e^{i(\phi_1 - \phi_2)} \cos(\theta_1) \sin(\theta_2) + e^{-i\phi_1} \sin(\theta_1) \sin(\theta_2). \]
(15)

Inserting either constraint (11) or (12) leads to the probability for a click of the detector given by:
\[ p(\text{click}|1^A_{\text{in}}0^B_{\text{in}}) = |c_{101}|^2 = \sin^2(\frac{\phi_1}{2}) \sin^2(2\theta_1). \]
(16)
Its maximal value
\[ p(\text{click}|1^A_{\text{in}}0^B_{\text{in}}) = \sin^2(\frac{\phi_1}{2}) \]
(17)
is thus achieved for \( \theta_1 = \frac{\pi}{4} \) or \( \theta_1 = \frac{3\pi}{4} \). In both cases we are free to choose \( \phi_1 - \phi_2 = 0 \) or \( \phi_1 - \phi_2 = \pi \). The corresponding \( \theta_2 \) follows from Eq. (11) or (12), respectively. In all these cases beam splitters BS\(_1\) and BS\(_2\) are symmetric. Please note, that the optimal choice of beam splitters does not depend on the exerted phase shift \( \phi_1 \). Since we assume \( \phi_1 \neq 2k\pi \) (with \( k \in \mathbb{N} \)), the detector responds in some of the cases when a single photon enters in the signal mode, while it does not so if this mode is empty. We have thus obtained a heralded single-photon source with efficiency \( p = 1 \).

The probability \( P_E \) that a photon present in the signal mode is successfully detected and announced by a click of the detector in output \( C \) (i.e., the detection efficiency, cf. question (ii)) amounts to
\[ P_E = p(\text{click}|1^A_{\text{in}}0^B_{\text{in}})P_B = \sin^2(\frac{\phi_1}{2})P_B, \]
(18)
where \( P_B \) is the probability that one photon enters in mode \( B \). \( P_E \) is the detection efficiency. This leads to the total probability \( P_T \) to produce a heralded single photon from the output of two imperfect single photon sources. It is given by
\[ P_T = P_E P_A, \]
where \( P_A \) is the efficiency of the source feeding mode \( A \).

Although it is in principle possible to generate a heralded pure photon from two imperfect sources using our scheme, the detection efficiency \( P_E \propto \sin^2(\frac{\phi_1}{2}) \) is low if the phase shift \( \phi_1 \) corresponding to the cross phase modulation is small (cf. also Sec. VI). This disadvantage can be partly compensated by using intensive laser light instead of a noisy photon as input of mode \( B \).

V. HERALDING PURE SINGLE PHOTONS BY MEANS OF COHERENT STATES

In this section we explore the possibility to produce a heralded pure photon from a mixture \( \rho^A = p_A|1^A\rangle\langle 1^A| + (1 - p_A)|0^A\rangle\langle 0^A| \) with the set-up described in Sec. [11]and a coherent state \( \rho^B = |\beta\rangle\langle \beta| \) as input of mode \( B \).

For this purpose we have to ensure that the detector in mode \( C \) cannot click if mode \( A \) contains just vacuum. This leads to the same constraints (11) and (12) for the beam splitters BS\(_1\) and BS\(_2\), which we obtained in the previous section (cf. the transparency conditions for a Mach-Zehnder interferometer in Appendix VIII A).

Hence, provided beam splitters BS\(_1\) and BS\(_2\) are chosen according to either Eq. (11) or Eq. (12) and the detector has no dark counts, the conditional probability to find one photon outgoing in mode \( A \) if the detector clicks is again one
\[ p(1^A_{\text{out}}|\text{click}) = 1. \]
(19)

As in the previous section [15] selection according to the clicks of the detector amounts to the preparation of the pure single-photon Fock state \( |1^A\rangle \) (cf. issue (i)). But what is the probability of such a preparation? We now calculate the detection efficiency (cf. issue (iii)).

If a photon is present in the signal mode \( A \) we obtain the following state transition
\[ |\beta\rangle^B C \xrightarrow{BP_{12}} |\beta\cos(\theta_1)\rangle^B |\beta e^{-i\phi_1} \sin(\theta_1)\rangle^C \]
\[ \xrightarrow{XP_{PM}} |e^{i\phi_2} \beta \cos(\theta_1)\rangle^B |\beta e^{-i\phi_1} \sin(\theta_1)\rangle^C \]
\[ \xrightarrow{BS_{12}} |e^{i\phi_2} \beta \cos(\theta_1)\rangle^B |\beta e^{-i\phi_1} \sin(\theta_1)\sin(\theta_2)\rangle^C \
\[ + |\beta e^{i(\phi_2 - \phi_1)} \cos(\theta_1)\sin(\theta_2) + \beta e^{i\phi_1} \sin(\theta_1)\cos(\theta_2)\rangle^C \
\[ = |\beta\rangle^B \cos^2(\theta_1) + \sin^2(\theta_1)\rangle^C \]
\[ \propto |\beta e^{-i\phi_2} \frac{1}{2} \sin(2\theta_1)(1 - e^{i\phi_1})\rangle^C. \]
(20)

In order to obtain the last line we have inserted constraint (11). The outgoing state is separable being a product of coherent states in modes \( B \) and \( C \) as expected for classical fields which pass through an interferometer.

Based on this outgoing state we calculate the probability \( p(\text{click}|1^A_{\text{in}}0^B_{\text{in}}) \) for a response of the detector given that one photon entered the setup in mode \( A \). We assume here that the response of the detector corresponds to the effect operator \( \hat{P}_{\text{c.det.}} = \sum_{k=1}^{\infty} |k^C\rangle\langle k^C| \). Thus the related probability amounts to:
\[ p(\text{click}|1^A_{\text{in}}0^B_{\text{in}}) = C\langle \gamma' | \hat{P}_{\text{c.det.}} |\gamma'\rangle^C \]
\[ = 1 - |\langle 0^C|\gamma\rangle|^2 \]
\[ = 1 - e^{-|\beta|^2 \sin^2(2\theta_1) \sin^2(\frac{\phi_1}{2})}. \]
(21)
This is the detection efficiency \( P_E \) to successfully detect a photon present in the signal mode. Constraint (12) leads to the same expression for \( P_E \). It assumes its maximal value for the same choice of beam splitters as in Sec. [14].
\[ P_E = p(\text{click}|1^A_{\text{in}}0^B_{\text{in}}) = 1 - e^{-|\beta|^2 \sin^2(\frac{\phi_1}{2})}. \]
(22)
This result is to be compared with \( P_E \) of Eq. (18). Since the detector does not click if no photon is present in the signal mode, but does respond with a finite probability \( P_E \) (provided that \( \phi_1 \neq 2k\pi \) if the signal mode contains a single photon, it is possible to generate pure heralded photons with our set-up using a coherent state in mode \( B \). The resulting single-photon source has the efficiency \( p = 1 \). The total probability \( P_T \) to
obtain a heralded pure photon from a source with efficiency $p_A$ then amounts to $P_F = P_{EPA}$.

The ability to produce pure photons from an imperfect source heralded by a photodetection distinguishes our scheme among others. Its quality depends on the probability $P_E$ to announce a single photon present in the signal mode. It crucially depends on the product $|\beta|^2 \sin^2(\pi \phi)$. For any phase shift $\phi$, the detection efficiency $P_E$ can be increased arbitrarily close to 1 by choosing a sufficiently high mean photon number $|\beta|^2$ (cf. Fig. 4). It can be seen that $P_E$ increases rapidly already for small values of $|\beta|^2$.

**VI. REALIZATION OF XPM**

The objective of every possible realization scheme of our proposal is to produce giant Kerr nonlinearities so as to make XPM as large as possible, even for field intensities corresponding to that of a single photon. In the ideal case we would like to be able to choose phase shifts $\phi$, on the order of $\pi$. In the following we report on a promising XPM scheme [22, 23] that makes much huge phase shifts feasible, even if light pulses of ultra-small energies, i.e. single photons, are involved. It is based on the electromagnetically induced transparency (EIT) phenomenon [24] which makes possible to resonantly enhance the Kerr nonlinearity $\chi^{(3)}$ along with simultaneous elimination of absorption losses due to vanishing linear susceptibilities [22]. The principle of the method is illustrated in Fig. 5.

The Kerr medium consists of two atomic species $1$ and $2$ which resonantly interact with the propagating fields $E_A$ and $E_B$ of modes $A$ and $B$ as depicted in Fig. 5. $E_B$ is tuned to resonance with the atomic transition $b_1 \leftrightarrow a_1$ of species $1$ whereas $E_A$ with the atomic transition $b_2 \leftrightarrow a_2$ of species $2$. Both atomic species are in addition resonantly driven by strong classical fields $\Omega_1$ and $\Omega_2$, which couple the atomic transitions $c_1 \leftrightarrow a_1$ and $c_2 \leftrightarrow a_2$, respectively. The quantum interferences created by the classical driving fields involve sharp transmission resonances for the corresponding quantum fields $E_A$ and $E_B$. By this means EIT is established for both fields. We get double EIT (DEIT). As a consequence, both $E_A$ and $E_B$ propagate without absorption losses or refraction, and their group velocities are considerably reduced. Large Kerr nonlinearities leading to cross phase modulation between the fields $E_A$ and $E_B$ are obtained via Stark effect. The signal field $E_A$ is non-resonantly coupled to another optically allowed transition $c_1 \leftrightarrow d_1$ with a detuning $\Delta$ within the atoms of species 1. This results in a Stark shift of level $c_1$, thus involving a change of the refractive index of field $E_B$. Since the refractive index dispersion is very strong near resonances, relatively small Stark shifts are sufficient to induce a large index change. The Kerr nonlinearities accomplished in this way have been shown to yield $\chi^{(3)}$-values that are orders of magnitude higher than in conventional systems [22]. Moreover, the resulting XPM of the fields $E_A$ and $E_B$ can be sustained for a very long interaction time. As already mentioned, due to DEIT both fields propagate without absorption losses and with strongly reduced group velocities. Furthermore, the experimental conditions can be arranged such that their group velocities become equal [23]. This in turn involves a potentially very long interaction time between the two fields $E_A$ and $E_B$ thus making possible very large conditional nonlinear phase shifts. As shown by Lukin and Imamoglu in [23] this realization scheme makes feasible XPM phase shifts of the order of $\pi$, even if single-photon fields are involved. The requirement to be fulfilled is $\tau_g \Delta \omega_{\text{max}} \gg 1$. Here $\tau_g$ is the group delay and $\Delta \omega_{\text{max}}$ the bandwidth of the EIT resonance. More details can be found in [22, 23]. Finally, it has recently

![Graph](image-url)
been shown [25] that DEIT and large XPM between slowly co-propagating weak fields may also be obtained using only one atomic species. Unfortunately, however, the XPM phase shift between two photons as estimated by the authors of [25] is considerably smaller than the estimates obtained by Lukin and Imamoğlu in [23].

An experimental implementation of our scheme by means of DEIT would require an analysis that allows for unavoidable losses due to spontaneous emission processes as well as pump field fluctuations. These losses being small, they nevertheless might limit to some extent the ability of the proposed scheme to improve the efficiency of single photon sources. A full analysis of DEIT that takes into account all the various loss mechanisms including spontaneous emission noise and pump field fluctuations is beyond the scope of the present paper. Moreover, to the best of our knowledge, such a comprehensive analysis does not exist in the literature. Here we take into account losses phenomenologically, in terms of a finite probability of absorption of the single-photon and a corresponding attenuation of the amplitude of the coherent state, respectively. Absorption losses are mainly caused by spontaneous decay of the excited states. Another source of absorption is introduced by the decay of coherence between the ground state levels, i.e. decoherence of the dark state. We examine the extent of negative impact these absorption losses entail on our scheme. In particular, we estimate an upper bound on the amount of spontaneous emission noise that can be tolerated, so as our scheme still works. The heuristic analysis below refers to the purification scheme that uses coherent states, as discussed in Sec. V.

Without expanding on the various loss mechanisms let us take into consideration losses in terms of the following sensible heuristic assumption. We assert that there is a finite probability \( \phi_0 \) that a photon is absorbed. In particular, we make the following heuristic ansatz which is reasonable with regard to the light field evolution equations suggested in [23]:

\[
\langle \hat{n}_{\text{out}} \rangle = (1 - \phi_0) \langle \hat{n}_{\text{in}} \rangle.
\]

(23)

In the above equation, \( \langle \hat{n}_{\text{in}} \rangle \) is the initial mean photon number of the light pulse when it enters the medium, while \( \langle \hat{n}_{\text{out}} \rangle \) denotes the mean photon number of the outgoing field. The mean photon number decreases when the light pulses propagate through the Kerr medium. It is assumed to be attenuated by the factor \( 1 - \phi_0 \), where \( \phi_0 \) is a probability \( 0 \leq \phi_0 \leq 1 \).

In order to allow for absorption losses according to Eq. (23) we modify the usual XPM state transformation by the following heuristic equations:

\[
|1\rangle^A |\beta\rangle^B \xrightarrow{\text{XPM}} \sqrt{(1 - \phi_0)}|1\rangle^A |\beta'\sqrt{1 - \phi_0} e^{i\phi_0}\rangle^B \times \text{h.c.}
\]

\[
+ \sqrt{\phi_0}|0\rangle^A |\beta\sqrt{1 - \phi_0}\rangle^B \times \text{h.c.}
\]

\[
|0\rangle^A |\beta\rangle^B \xrightarrow{\text{XPM}} \sqrt{1 - \phi_0} \langle \hat{n}_{\text{in}} \rangle
| \beta' \rangle^B.
\]

(24)

These state transformations comprise both the usual XPM interaction leading to a phase shift of the coherent state in mode \( B \) if a photon is present in mode \( A \) and the required absorption losses in mode \( A \) as well as in mode \( B \). The absorption of photons from the coherent state in mode \( B \) is described by the attenuation \( |\beta'| \rightarrow \sqrt{1 - \phi_0} |\beta'| \), which is reasonable because of \( \langle \hat{n}_{\text{in}} \rangle = |\beta|^2 \). Please note that there is an attenuation of the coherent state in mode \( B \) even if there is no photon in mode \( A \). In the above model we have made the approximation that either a full phase shift \( e^{i\phi_0} \) is acquired or no phase shift at all.

The existence of absorption losses involves an imperfect operation of our scheme. The mere possibility of absorption leads eventually to faulty clicks of the detector, i.e. detector clicks even if there is no photon leaving the setup in mode \( A \). As a consequence, the efficiency \( p_A' \) of the resulting heralded single-photon source becomes less than one. Yet, we have examined the conditions under which an improvement of the efficiency, i.e. \( p_A' > p_A \), is still possible. Using the above heuristic assumptions and methods of Sections III and V we have calculated an upper bound on \( \phi_0 \) up to which absorption losses can be tolerated, so as our scheme to improve the efficiency of single photon sources still works. This upper bound depends on the XPM phase shift \( \phi_0 \), as well as on \( |\beta|^2 \), i.e. the mean photon number in mode \( B \). The dependence on \( \phi_0 \) is appreciable only for low mean photon numbers, it becomes less important with increasing value of \( |\beta|^2 \). Being not too optimistic, let us assume \( \phi_0 = 10 \text{ mrad} \), the phase shift which has been estimated by the authors of Ref. [25]. Using this value the following upper bounds on \( \phi_0 \) have been obtained, depending on the mean photon number of the coherent state: \( \phi_0^{\text{max}} \approx 0.008 \) in case \( |\beta|^2 = 10^4 \), \( \phi_0^{\text{max}} \approx 0.020 \) in case \( |\beta|^2 = 10^2 \) and \( \phi_0^{\text{max}} \approx 0.021 \) in case \( |\beta|^2 = 10^2 \). Thus it appears that for \( \phi_0 = 10 \text{ mrad} \) absorption losses of not more than about 2% are acceptable. Larger absorption losses could be tolerated if low mean photon numbers are employed and higher XPM phase shifts were feasible. In the ideal case \( \phi_0 \approx \pi \) the following upper bounds have been attained with our heuristic model: \( \phi_0^{\text{max}} \approx 0.06 \) in case \( |\beta|^2 = 10^4 \), \( \phi_0^{\text{max}} \approx 0.35 \) in case \( |\beta|^2 = 10^2 \) and \( \phi_0^{\text{max}} \approx 0.80 \) in case \( |\beta|^2 = 1 \).

We conclude that XPM schemes based on DEIT provide promising realizations to implement the heralded single-photon generator via Kerr effect as proposed in this paper.

VII. CONCLUSION

We have suggested a scheme to produce a pure single photon from the output of an imperfect single-photon source given by the mixed state \( |\phi/1\rangle = (|1| + (1 - p)|0\rangle) |0\rangle \) with finite efficiency \( p \) which may be arbitrarily small. Heralded production of a single photon has been achieved in two respects. First of all we have derived the interferometer adjustments which ensure that a detector click indicates a single photon with certainty. Conditioning on the clicks of the detector leads already to \( p = 1 \). Secondly we have shown that the detection efficiency can be made arbitrarily high.

Using noisy photons in both signal and probe mode causes a low detection efficiency for a single photon in the signal mode. This problem can be overcome by enforcing higher interaction via DEIT or other techniques. This disadvantage has been shown to be naturally attenuated by using a coherent state in the probe mode. In this case the detection effi-
ciency $P_E$ depends on the product $|β|^2 \sin^2(\phi_2)$. By choosing the mean photon number $|β|^2$ sufficiently high $P_E$ can be increased arbitrarily close to one.

**Acknowledgment:** We would like to thank Christian Kasztelan, Rudolf Bratschitsch, Karl-Peter Marzlin and Alfred Leitenstorfer for helpful discussions. This work was supported by the “Center for Applied Photonics” (CAP) at the University of Konstanz.

### VIII. APPENDIX

#### A. Transparency conditions for a Mach-Zehnder interferometer

We consider a Mach-Zehnder interferometer which is composed of the two beam splitters $BS_1$ and $BS_2$. Here we show how these beam splitters have to be adjusted in order that arbitrary ingoing states of light do not change under the interferometer’s action.

Any pure state of light entering two modes $B$ and $C$ (cf. Fig. 4, including an entangled state can be expressed as a function $f(b^\dagger, c^\dagger)$ of the creation operators $b^\dagger$ and $c^\dagger$ acting on the vacuum $|0, 0\rangle$. As pointed out in Sec. 11 the action of both beam splitters in the Schrödinger picture can then be written as

$$f(b^\dagger, c^\dagger)|0, 0\rangle \xrightarrow{BS_2} f(\tilde{b}^\dagger, \tilde{c}^\dagger)|0, 0\rangle \xrightarrow{BS_1} f(b^\dagger, c^\dagger)|0, 0\rangle.$$  \hspace{1cm} (25)

Hereby the function $f$ does not change, but its operator-valued arguments are transformed according to the transformation rules

$$\left(\begin{array}{c} \tilde{b}^\dagger \\ \tilde{c}^\dagger \end{array}\right) = U_2 \left(\begin{array}{c} b^\dagger \\ c^\dagger \end{array}\right) = U_2 U_1 \left(\begin{array}{c} b^\dagger \\ c^\dagger \end{array}\right).$$  \hspace{1cm} (26)

$U_1$ and $U_2$ are unitary transformations which can be read off from Eq. 4.

Input state and output state in transformation (25) are equal for arbitrary functions $f$ if $\tilde{b}^\dagger = b^\dagger$ and $\tilde{c}^\dagger = c^\dagger$. As can be seen from Eq. (26) this transparency condition is fulfilled if $U_2 U_1 = \mathbb{1}$, or equivalently

$$U_2 = U_1^\dagger.$$  \hspace{1cm} (27)

This means that beam splitter $BS_2$ reverses the action of $BS_1$. Comparing the matrix elements of $U_2$ and $U_1$, we obtain the following two sets of constraints

$$\phi_1 - \phi_2 = 2k\pi \quad \text{and} \quad \theta_1 + \theta_2 = l\pi, \quad k, l \in \mathbb{Z},$$  \hspace{1cm} (30)

$$\phi_1 - \phi_2 = (2k + 1)\pi \quad \text{and} \quad \theta_1 - \theta_2 = l\pi, \quad k, l \in \mathbb{Z}. \hspace{1cm} (31)

Note that we already have obtained the same conditions in Sec. III for a special input state.

#### B. Extended schemes

Starting from our basic scheme as discussed in this article, we now construct extended schemes.

The setup of Sec. III constitutes a basic building block, cf. Fig. 4. By a suitable combination of several such basic modules, we can increase the probability to generate a single photon. The combined set-ups discussed below do not process the noisy photons and the coherent state independently. Rather, they use resources more efficiently. And by considering more noisy photons it becomes obvious how fast the probability to purify at least one of them increases with their number.

![FIG. 6: The dashed triangle symbol is used as a replacement for the solid drawn single set-up.](image)

The first combined scheme is depicted in Fig. 4. It is intended to purify one noisy photon with a higher probability than a single set-up does. This is efficiently achieved by reusing the outgoing coherent state again in a second set-up and so on.

![FIG. 7: A noisy photon and a coherent state enter the first set-up like in the basic scheme. Its outputs are reused as inputs of the second set-up, and so on.](image)

Let the input state be

$$\rho = p|1\rangle\langle 1| + (1 - p)|0\rangle\langle 0|.$$  \hspace{1cm} (32)

In the case that no photon enters from left the coherent state will not cause a detection. But if there is a photon entering the first set-up it will not necessarily cause a click. Instead there is still a chance that the coherent state leaving mode $C$ is projected onto vacuum by detection. Nonetheless the amplitude of the coherent state leaving Mode $B$ would be decreased by a factor of $\cos(\frac{\phi_2}{2})$. The probability $p_n$ for the first click to
We obtain the probability \( P_n \) for the first click to happen in the \( n \)-th set-up can be expressed as

\[
p_n = \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-k-1} \times \exp \left[ -|\alpha|^2 \sin^2 \left( \frac{\phi_k}{2} \right) \cos^{2k} \left( \frac{\phi_k}{2} \right) - 1 \right] \times p(1 - \exp \left[ -|\alpha|^2 \cos^{2k} \left( \frac{\phi_k}{2} \right) \sin^2 \left( \frac{\phi_k}{2} \right) \right] ) .
\]

The probability \( P_T \) for heralding at least one photon is obtained by summing over all single-photon sources. It tends to 1 for \( |\alpha|, N \gg 1 \),

\[
P_T = \sum_{n=1}^{N} p_n \rightarrow 1 .
\]

In the second combined scheme, which is illustrated in Fig. 8, several single-photon sources are to be processed with one coherent state. Similarly to the first set-up we look for the probability that at least one photon is heralded. The probability \( p_n \) for the first click to happen in the \( n \)-th set-up can be expressed as

\[
p_n = \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-k-1} \times \exp \left[ -|\alpha|^2 \sin^2 \left( \frac{\phi_k}{2} \right) \cos^{2k} \left( \frac{\phi_k}{2} \right) - 1 \right] \times p(1 - \exp \left[ -|\alpha|^2 \cos^{2k} \left( \frac{\phi_k}{2} \right) \sin^2 \left( \frac{\phi_k}{2} \right) \right] ) .
\]