Quantum Double Models coupled with matter: 
an algebraic dualisation approach

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Abstract

In this paper, we constructed a new generalization of a class of discrete bidimensional models, the so called Quantum Double Models, by introduce matter qunits to the faces of the lattice that supports these models. This new generalization can be interpreted as the algebraic dual of a first, where we introduce matter qunits to the vertices of this same lattice. By evaluating the algebraic and topological orders of these new models, we prove that, as in the first generalization, a new phenomenon of quasiparticle confinement may appear again: this happens when the co-action homomorphism between matter and gauge groups is non-trivial. Consequently, this homomorphism not only classifies the different models that belong to this new class, but also suggests that they can be interpreted as a 2-dimensional restriction of the 2-lattice gauge theories.

1. Introduction

Quantum Double Models ($D(G)$) \cite{1, 2, 3} is the name given to a class of models that, since they are defined on two-dimensional lattices, have a topological order \cite{4} that allows to perform some fault-tolerant quantum computation \cite{1, 5}. This topological order is due to the fact these models are constructed by associating qunits to edges of a lattice $L_2$ that, in general, discretizes some 2-dimensional compact orientable manifold $M_2$. In the case of a $D(G)$ where $G$ is not a Abelian group, part of this fault-tolerant quantum computation power is justified, for instance, due to presence of non-Abelian anyons among its low energy excitations \cite{6}.

Since there is no qunit associated with other lattice elements, some works were published recently in order to evaluate $D(G)$ generalizations where new qunits are associated to lattice vertices. These generalizations were denoted as Quantum Double Models plus matter.

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(\(D_M (G)\)) \([7, 8, 9]\), although this term matter does not necessarily have to be thought of in the same way as elementary particle physics. Among the main properties of these generalizations, we can highlight the presence of algebraic and topological orders, as well as the presence of non-Abelian fusion rules, even when the gauge group is cyclic Abelian \([9]\).

However, as unlike the \(D (G)\) \([10]\), this generalization is not self-dual, one question that arises is how to use the \(D_M (G)\) as the basis for defining a self-dual generalization that associates matter qunits on both faces and vertices of \(\mathcal{L}_2\). By the way, does the construction of a new generalization, purposely defined as the algebraic dual of the \(D_M (G)\), show us if it is possible? In order to answer these questions, in this work we analyse a class of models that can be interpreted as the algebraic dual of the \(D_M (G)\); this new generalization \(D_K (G)\) has the same gauge structure of the \(D (G)\), but its matter qunits are associated only to the centroids of the faces of \(\mathcal{L}_2\), since these centroids can be interpreted as the vertices of a dual lattice \(\mathcal{L}_2^*\) \([11]\).

2. A brief review about the Quantum Double Models plus matter

As we previously mentioned in the Introduction, the \(D_M (G)\) is a class of two-dimensional lattice models that was purposely constructed to be interpreted as a generalization of the \(D (G)\) \([9]\). This construction is done:

(i) by taking an oriented lattice \(\mathcal{L}_2\) that discretizes a 2-dimensional compact orientable manifold \(\mathcal{M}_2\);
(ii) by assigning gauge and matter qunits

\[
|\varphi_j\rangle = a_0^{(e)} |0\rangle + \ldots + a_{N-1}^{(e)} |N-1\rangle \quad \text{and} \quad |\chi_v\rangle = a_0^{(x)} |0\rangle + \ldots + a_{M-1}^{(x)} |M-1\rangle
\]

to edges and vertices of \(\mathcal{L}_2\) respectively; and
(iii) by defining a Hamiltonian operator

\[
H_{D_M(G)} = -\sum_v A_v^{(G,S)} - \sum_p B_p^{(G,S)} - \sum_j C_j^{(G,S)}
\]

such that \(D_M (G)|_{M=1} = D (G)\).

The operators that make up the Hamiltonian are

\[
A_v^{(G,S)} = \frac{1}{|G|} \sum_{g \in G} A_v^g, \quad B_p^{(G,S)} = B_p^0 \quad \text{and} \quad C_j^{(G,S)} = C_j,
\]

whose components are given by the Figure 1. These operators \((2)\) act effectively in the subspaces that are associated with the edges subsets which, as shown in Figure 2, give structure to the \(v\)-th vertex, the \(p\)-th face and the \(j\)-th edge of \(\mathcal{L}_2\) respectively.

This reduction \(D_M (G)|_{M=1} = D (G)\) can be easily understood if we analyse each of the operators in \((2)\) individually by noting that, as \(|\varphi_j\rangle\) and \(|\chi_v\rangle\) need to be related to each other, these qunits belong to Hilbert subspaces \(H_N\) and \(H_M\) that are a group algebra \(\mathbb{C} (G)\) and a
\[ A_v^g \left| \begin{array}{c} a \\ \in \alpha \\ \in c \\ \rightarrow b \\ \end{array} \right\rangle = \sum_{\gamma} \delta \left( \theta(g, \alpha), \gamma \right) \left| \begin{array}{c} ga \\ \in \gamma \\ \in cg^{-1} \rightarrow gb \\ \end{array} \right\rangle \]

\[ B_p^b \left| \begin{array}{c} a \\ \in d \\ \in c \\ \rightarrow b \\ \end{array} \right\rangle = \delta \left( h, a^{-1}bcd^{-1} \right) \left| \begin{array}{c} a \\ \in d \\ \in c \\ \rightarrow b \\ \end{array} \right\rangle \]

\[ C_j^\alpha \left| \begin{array}{c} \in \alpha \\ \in a \\ \rightarrow \beta \\ \end{array} \right\rangle = \delta \left( \theta(a, \alpha), \beta \right) \left| \begin{array}{c} \in a \\ \in a \\ \rightarrow \beta \\ \end{array} \right\rangle \]

Figure 1: Definition of how the components \( A_v^g, B_p^b \) and \( C_j^\alpha \), which define the vertex, face and edge operators mentioned in (2) respectively, act on the Hilbert space that is associated to \( L_2 \). Here, in the same way that the symbol \( a \) is indexing a basis element of the gauge Hilbert subspace \( H_N \), the symbol \( \alpha \) indexes a basis element of the matter Hilbert subspace \( H_M \).

The \( CG \)-module \( [12] \) respectively. In the case of the vertex operator \( A_v^{(G,S)} \), this reduction comes from the fact that it is a modified operator (in relation to the \( D(G) \) vertex operator) that performs gauge transformations due to the presence of matter qudits at lattice vertices \( [9] \). After all, since \( B_j = \{ |g\rangle : g \in G \} \) and \( B_v = \{ |\alpha\rangle : \alpha \in S \} \) are two bases for \( H_N \) and \( H_M \) respectively, the multiplication \( \theta : G \times S \rightarrow S \) that defines \( H_M \) as a left \( CG \)-module automatically defines how the gauge group acts on these matter qudits.

In relation to the face operator \( B_p^{(G,S)} \) there is nothing new to be said: it is exactly the same as the \( D(G) \) face operator since it does not act on the matter qudits. It measures only flat connections, i.e., the trivial holonomies characterised by \( h = 0 \) along the faces. However, the novelty of the \( D_M(G) \) is the presence of an edge operator \( C_j^{(G,S)} \) in the Hamiltonian (1) that, together with the other operators, allows to state that its ground state \( |\Psi_0\rangle \) is such that

\[ A_v^{(G,S)} |\xi_0\rangle = |\xi_0\rangle , \quad B_p^{(G,S)} |\xi_0\rangle = |\xi_0\rangle \quad \text{and} \quad C_j^{(G,S)} |\xi_0\rangle = |\xi_0\rangle , \] (3)

is valid for all values of \( v, p \) and \( j \). This edge operator works literally as a comparator; i.e., \( C_j^{(G,S)} \) compares two neighbouring matter qudits by checking whether they are aligned\(^1\) by according to the \( \theta \) perspective \([9] \).

2.1. General \( D_M(G) \) properties

By virtue of the gauge structure of the \( D_M(G) \) is exactly the same as that of the \( D(G) \), the \( D_M(G) \) supports the same \( D(G) \) quasiparticles. However, although all the \( D(G) \) fusion rules are preserved in the \( D_M(G) \), some \( D(G) \) quasiparticles that are detectable by \( B_v^{(G,S)} \) acquire confinement properties when \( \theta \) is not a trivial action: that is, transporting these quasiparticles always increases the system energy and this energy increases as a function of the number of edges involved in this transport.

\(^1\)Since they can be interpreted as vectors that define a vector field.
One of the consequences of this quasiparticle confinement is that the $D_M(G)$ ground state degeneracy no longer depends on the order of the fundamental group $\pi_1$ associated with $\mathcal{M}_2$. In the case of the cyclic Abelian $D_M(\mathbb{Z}_N)$, they have an algebraic order and, implicitly, a topological order too: this algebraic order is characterized by the fact that this degeneracy is at least a function of the number of cycles that the action $\theta$ defines; this implicit topological order is consequence of the fact that the $D_M(\mathbb{Z}_N)$ ground state degeneracy depends on the second group of homology.

Another notable property of the $D_M(\mathbb{Z}_N)$ is the presence of quasiparticles with non-Abelian fusion rules. These quasiparticles are always necessary when this action of the gauge group is represented by

$$
\Theta(g) = \begin{pmatrix} \mathcal{A}(g) & 0 \\ 0^T & 1 \end{pmatrix}
$$

(4)

so that the lattice system can go from one vacuum state to another and vice versa. Here, $\mathcal{A}$ is a block diagonal representation of $\mathbb{Z}_N$ expressed by shift matrices, whereas $1$ is an identity matrix. In this fashion, since we can always define a $D_M(\mathbb{Z}_N)$ with this action representation when $M > N \geq 2$, there will always be a particular case where these non-Abelian fusion rules are present. In particular, when $M$ and $N$ are coprime natural numbers, the only way to represent this action is by (4).
3. Quantum Double Models plus matter via a dualisation procedure

One notable advantage of having already constructed the $D_M (G)$ is that it can be used as the basis for new generalizations, where, for example, new qunits can be assigned to the elements of the lattice $L_2$ that support it. And one of these generalizations is what we will denote by $D^K(G)$, where gauge and new matter qunits

$$|φ_j⟩ = a_0^{(φ)} |0⟩ + \ldots + a_{N-1}^{(φ)} |N - 1⟩ \quad \text{and} \quad |χ_v⟩ = a_0^{(χ)} |0⟩ + \ldots + a_{K-1}^{(χ)} |K - 1⟩$$

are allocated only to edges and face centroids of $L_2$ respectively.

In order to understand how this new allocation of qunits leads to a class of models other than $D_M (G)$, it is worth remembering that the $D (G)$ has a property that the $D_M (G)$ does not have: the $D (G)$ is self dual [10]. From the physical point of view, this means that for each excitation detectable by a vertex operator in the $D (G)$ there is always another, with the same properties, that is detectable by a face operator and vice versa. The reason for this is that, when we take a lattice $L_2$ that discretizes some 2-dimensional compact orientable manifold, each vertex (face) operator acting on $L_2$ can be identified as a face (vertex) operator that acts on the dual lattice $L_2^*$. Based on this finding, it is interesting to realise a dualisation procedure on the $D_M (G)$ in order to evaluate the aim features of the class $D^K(G)$ thus obtained, which is at least based on the existence of a correspondence between the faces in $L_2 (L_2^*)$ and the vertices in $L_2^* (L_2)$ [11]. This correspondence implies that the $D^K(G)$ Hamiltonian operator must be defined as

$$H_{D^K(G)} = -\sum_v A_v^{(G,\tilde{S})} - \sum_p B_p^{(G,\tilde{S})} - \sum_j D_j^{(G,\tilde{S})},$$

where, as suggested by Figure 3, its vertex ($A_v^{(G,\tilde{S})}$), face ($B_p^{(G,\tilde{S})}$) and edge ($D_j^{(G,\tilde{S})}$) operators can be purposely identified as a dualisation of the $D_M (G)$ face, vertex and edge operators respectively. These operators are specifically defined as

$$A_v^{(G,\tilde{S})} = \frac{1}{|G|} \sum_{g \in G} \tilde{A}_g^v, \quad B_p^{(G,\tilde{S})} = \tilde{B}_p^0 \quad \text{and} \quad D_j^{(G,\tilde{S})} = \frac{1}{|\tilde{S}|} \sum_{\lambda \in \tilde{S}} \tilde{D}_j^\lambda,$$

whose components are defined in Figure 4. Here, $\tilde{S}$ must be interpreted at least as the index set for the basis $B_p = \{ |\tilde{a}⟩ : \tilde{a} \in \tilde{S} \}$ analogously to what happens to basis $B_v$.

3.1. Solvability requirements

However, for this dualisation procedure to be consistent, it is necessary that $D^K (G)$ be a class of solvable models, i.e., that the vertex, face and edge operators of each of these models have to commute between them. And by analysing these commutation rules, we conclude that, for this to happen, it is necessary that $G$ and $\tilde{S}$ are at least two groups. After all, as this dualisation procedure implies that $|χ_\tilde{a}⟩$ and $|φ_\tilde{j}⟩$ are related by a co-action $\tilde{a} \mapsto \mathcal{F} (\tilde{a}) = \tilde{a} \otimes f (\tilde{a})$, where $f : \tilde{S} \rightarrow G$ needs be such that

$$f (1) = 1, \quad (f (\tilde{a}))^{-1} = f (\tilde{a}^{-1}) = f^{-1} (\tilde{a}) \quad \text{and} \quad f (\tilde{a}_1) \cdot f (\tilde{a}_2) = f (\tilde{a}_1 \ast \tilde{a}_2),$$
Figure 3: Piece of $\mathcal{L}_2$ superimposed on its dual lattice $\mathcal{L}_2^*$ (dashed). Here, the rose and baby blue sectors represent the same ones already mentioned in Figure 2, whereas the new green sector corresponds to the $j$-th edge comprised between two adjacent faces that now support matter qubits.

the double action of an edge operator (as the one that is present in Figure 5) requires that

$$a'' = f(\tilde{\lambda}') \cdot a' = f(\tilde{\lambda}^*) \cdot f(\tilde{\lambda}) \cdot a = f(\tilde{\lambda}^* \cdot \tilde{\lambda}^*) \cdot a,$$

$$\tilde{\alpha}'' = \tilde{\alpha}' \cdot \tilde{\lambda}' = \tilde{\alpha}' \cdot (\tilde{\lambda}^* \cdot \tilde{\lambda}') \quad \text{and} \quad \tilde{\beta}'' = (\tilde{\lambda}')^{-1} \cdot \tilde{\beta}' = (\tilde{\lambda}^*)^{-1} \cdot \tilde{\lambda}^{-1} \cdot \tilde{\beta} = (\tilde{\lambda}^* \cdot \tilde{\lambda})^{-1} \cdot \tilde{\beta},$$

and therefore

$$\tilde{\alpha} \cdot \tilde{\beta} = \tilde{\beta} \cdot \tilde{\alpha} \quad \Rightarrow \quad f(\tilde{\alpha} \cdot \tilde{\beta}) = f(\tilde{\beta} \cdot \tilde{\alpha}) \quad \Rightarrow \quad f(\tilde{\alpha}) \cdot f(\tilde{\beta}) = f(\tilde{\beta}) \cdot f(\tilde{\alpha}).$$ (9)

That is, since $f$ is a homomorphism that satisfies (9), $\tilde{S}$ and $\text{Im}(f) \subset G$ must be two Abelian groups. However, as the Figures 7 and 8 show that the only way to cancel $[A_v, D_j]$ and $[B_p, D_j]$ is by taking

$$f(\tilde{\gamma}) \cdot g = g \cdot f(\tilde{\gamma}),$$ (10)

we conclude that $\text{Im}(f)$ must be the centre of group $G$ [13].

3.2. About the dualisation of the quasiparticles properties

Since the conditions above guarantee that the $D^K(G)$ is solvable, many things can already be said about this model. And one of the standard things that can be said is that its ground state $|\tilde{\xi}_0\rangle$ can be characterized by the following relations:

$$A_v^{(g,s)}|\tilde{\xi}_0\rangle = |\tilde{\xi}_0\rangle, \quad B_p^{(g,s)}|\tilde{\xi}_0\rangle = |\tilde{\xi}_0\rangle \quad \text{and} \quad D_j^{(g,s)}|\tilde{\xi}_0\rangle = |\tilde{\xi}_0\rangle.$$ (11)
Figure 4: Definition of the components $\bar{A}_g$, $\bar{B}_h$ and $\bar{D}_j$ that define the vertex, face and edge operators mentioned in (7) respectively. As well as in the $D^K(G)$ case, the symbol $\tilde{\alpha}$ represents the $\tilde{\alpha}$-th basis element of the dual matter Hilbert subspace $B_p$. Here, $a' = f(\tilde{\lambda}) \cdot a$, $\tilde{\alpha}' = \tilde{\alpha}^{\ast} \tilde{\lambda}$ and $\tilde{\beta}' = \tilde{\lambda}^{\ast} \tilde{\beta}$, where $f : \tilde{S} \rightarrow G$.

However, the first non-standard comment we can make about the $D^K(G)$ concerns the comparison between the $D_M(Z_N)$ and $D^K(Z_N)$ quasiparticles. This duality stems from the fact that the $D^K(Z_N)$ contains the same $D(Z_N)$ quasiparticles, but, when $f$ is a non-trivial homomorphism, those that are detected by the vertex operator acquire confinement properties. In other words, transporting these latter quasiparticles by using an operator like

$$O^{(q)}_{\gamma} = \prod_{j \in \gamma} Z_j^{\pm g},$$

where $\gamma$ is a path composed by two by two adjacent edges, always increases the system energy and this energy increases as a function of the number of edges involved in this transport.
\[ D^\lambda_j \left( \alpha_1 \cdots \alpha_d, \beta_1 \cdots \beta_n \right) = \sum_\lambda \left( \alpha_1' \cdots \alpha_d', \beta_1' \cdots \beta_n' \right) \Rightarrow D^\lambda \circ D^\lambda_j \left( \alpha_1 \cdots \alpha_d, \beta_1 \cdots \beta_n \right) = \sum_{\lambda, \lambda'} \left( \alpha_1'' \cdots \alpha_d'', \beta_1'' \cdots \beta_n'' \right) \]

Figure 5: Scheme related to the double action of the edge operator \( D^\lambda_j \), which is used to help us conclude that \( \hat{S} \) must be an Abelian group.

### 3.3. Some examples

Although this confinement property is completely analogous to what happens in the \( D_M(\mathbb{Z}_N) \), there are some facts that seem to "break" this dual aspect related to these two classes. And the first fact is related to the impossibility of constructing a \( D^K(G) \) substantially different from a \( D(G) \) when \( K \) and \( N \) are coprime numbers. In the case of a cyclic Abelian \( D^K(Z_N) \) the following proposition is relevant [14]:

**Proposition 1.** Every group homomorphism \( f : \mathbb{Z}_K \rightarrow \mathbb{Z}_N \) can be completely determined by

\[
f([x]) = [nx] \quad ,
\]

where \( n \) is a natural number that assumes values other than zero if, and only if, \( N \) is a natural number divisible by \( nK \).

#### 3.3.1. Example: \( G = \mathbb{Z}_2 \) and \( \hat{S} = \mathbb{Z}_2 \)

In order to understand how the possibility of defining these several homomorphisms influences in the definition of the \( D^K(Z_N) \), we will take some simple examples. And the first one is the \( D^2(\mathbb{Z}_2) \) whose gauge and matter groups are \( \mathbb{Z}_2 \).

According to the Proposition 1 above, there are two ways of constructing this \( D^2(\mathbb{Z}_2) \): one where \( f \) is the trivial homomorphism and, consequently, the representations (12) are reduced to\(^2\)

\[
A_{v,1} = \frac{1}{2} (1 \otimes 1 \otimes 1 \otimes 1 + \sigma^x_a \otimes \sigma^y_b \otimes \sigma^z_c \otimes \sigma^z_d) \quad ,
\]

\[
B_{p,1} = \frac{1}{2} (1 \otimes 1 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes 1 + 1 \otimes \sigma^z_a \otimes \sigma^z_b \otimes \sigma^z_c \otimes \sigma^z_d) \quad \text{and}
\]

\[
D_{j,1} = \frac{1}{2} (1 \otimes 1 \otimes 1 \otimes 1 + \sigma^z_{p_1} \otimes \sigma^z_{j} \otimes \sigma^z_{p_2}) \quad ;
\]

\(^2\)In these examples, we will omit the super indexies \((G, \hat{S})\) associated with these operators in favour of a lighter notation that will become very useful later on. From now on, we will also index the vertex, face and edge operators that compose the Hamiltonian (6) with a “1” for a reason that will be clear later.
\[ A_v \circ B_p \left| \begin{array}{cccc} d & a & b & c \\
 & 1 & 0 & 1 \\
 s & 0 & 1 & 0 \\
g & 0 & 1 & 0 \\
 \end{array} \right\rangle = \delta \left( f \left( \tilde{\alpha} \right) bg^{-1}s^{-1}c, h \right) A_v \left| \begin{array}{cccc} d & a & b & c \\
 & 1 & 0 & 1 \\
 s & 0 & 1 & 0 \\
g & 0 & 1 & 0 \\
 \end{array} \right\rangle \]

\[ = \sum_r \delta \left( f \left( \tilde{\alpha} \right) bg^{-1}s^{-1}c, h \right) \left| \begin{array}{cccc} d & a & b & c \\
 r & 1 & 0 & 1 \\
 c & 0 & 1 & 0 \\
r & 0 & 1 & 0 \\
g & 0 & 1 & 0 \\
 \end{array} \right\rangle \]

\[ B_p \circ A_v \left| \begin{array}{cccc} d & a & b & c \\
 & 1 & 0 & 1 \\
 s & 0 & 1 & 0 \\
g & 0 & 1 & 0 \\
 \end{array} \right\rangle = \sum_r B_p \left| \begin{array}{cccc} d & a & b & c \\
 r & 1 & 0 & 1 \\
 c & 0 & 1 & 0 \\
r & 0 & 1 & 0 \\
g & 0 & 1 & 0 \\
 \end{array} \right\rangle \]

\[ = \sum_r \delta \left( f \left( \tilde{\alpha} \right) bg^{-1}s^{-1}c, h \right) \left| \begin{array}{cccc} d & a & b & c \\
 r & 1 & 0 & 1 \\
 c & 0 & 1 & 0 \\
r & 0 & 1 & 0 \\
g & 0 & 1 & 0 \\
 \end{array} \right\rangle \]

Figure 6: Result of the action of the operators \( A_v \circ B_p \) and \( B_p \circ A_v \) in a site \((v, p)\), from which all the commutativity between \( A_v \) and \( B_p \) is clear.

and the other where \( F_p \left( \tilde{\alpha} : g \right) = \sigma^z_p \) and \( F_j \left( \tilde{\alpha} : g \right) = \sigma^x_j \), and therefore

\[ A_{v,1} = \frac{1}{2} \left( \mathbf{1}_a \otimes \mathbf{1}_b \otimes \mathbf{1}_c \otimes \mathbf{1}_d + \sigma^x_a \otimes \sigma^x_b \otimes \sigma^x_c \otimes \sigma^x_d \right) \]

\[ B_{p,1} = \frac{1}{2} \left( \mathbf{1}_p \otimes \mathbf{1}_r \otimes \mathbf{1}_s \otimes \mathbf{1}_t \otimes \mathbf{1}_u + \sigma^x_p \otimes \sigma^x_r \otimes \sigma^x_s \otimes \sigma^x_t \otimes \sigma^x_u \right) \quad \text{and} \quad (17) \]

\[ D_{j,1} = \frac{1}{2} \left( \mathbf{1}_{p_1} \otimes \mathbf{1}_j \otimes \mathbf{1}_{p_2} + \sigma^x_{p_1} \otimes \sigma^x_j \otimes \sigma^x_{p_2} \right) . \]

Both possibilities lead to a model that houses the same quasiparticles already related to the Toric Code \( D(Z_2) \), which are produced in pairs by the action of the operators [15].

\[ \sigma^x_j, \sigma^z_j \quad \text{and} \quad \sigma^y_j = \sigma^x_j \circ \sigma^z_j = \sigma^z_j \circ \sigma^x_j . \]

However, it is worth noting that only the model with (17) leads to a \( D_2(Z_2) \) substantially different from its correspondent \( D(Z_2) \). After all, in this modified Toric Code \( D(Z_2) \), the only quasiparticles that can be moved without increase the system energy are those detectable only by the face operators, i.e., the quasiparticles \( m \) that are produced by the action of \( \sigma^x_j \).

Regardless of the topological features of the \( D^2(Z_2) \) ground states, it is also important to note that the different choices we have made for \( f \) also imply another kind of ground state degeneracy.
\[ A_v \circ D_j \left| d \rightarrow \frac{a}{c} \begin{array}{c} \hat{a} \\ \hat{b} \end{array} \right\rangle = \sum_{\lambda} A_v \left| d \rightarrow \frac{a}{c} f(\hat{\lambda}) \right\rangle \]
\[ = \sum_{k, \lambda} dk^{-1} \begin{array}{c} \frac{ka}{ck} \\\ \hat{\beta} \end{array} k f(\hat{\lambda}) b \]

\[ D_j \circ A_v \left| d \rightarrow \frac{a}{c} \begin{array}{c} \hat{a} \\ \hat{b} \end{array} \right\rangle = \sum_{\lambda} D_j \left| d \rightarrow \frac{a}{c} f(\hat{\lambda}) \right\rangle \]
\[ = \sum_{k, \lambda} dk^{-1} \begin{array}{c} \frac{ka}{ck} \\\ \hat{\beta} \end{array} k f(\hat{\lambda}) b \]

Figure 7: Result of action of the operators \( A_v \circ D_j \) and \( D_j \circ A_v \) on the lattice, which make it clear that \( [A_v, D_j] = 0 \) when \( f(\hat{\lambda}) \) belongs to the centre of \( G \).

I. In the case of the former \( D^2(Z_2) \) with (16), its ground state is two-fold degenerate and given by

\[ |\tilde{\xi}^{(1)}_0\rangle = \frac{1}{\sqrt{2}} \prod_{v'} A_{v'} \prod_{j'} D_{j'} \left( \otimes_j |0\rangle \otimes \left( \otimes_p |0\rangle \right) \right) \text{ and } \]

\[ |\tilde{\xi}^{(2)}_0\rangle = \frac{1}{\sqrt{2}} \prod_{v'} A_{v'} \prod_{j'} D_{j'} \left( \otimes_j |0\rangle \otimes \left( \otimes_{p\neq p'} |0\rangle \right) \otimes |1\rangle_{p'} \right). \]  

(18)

This two-fold degeneracy is justified as a result of (i) none of the operators in (16) is able to detect any change \( |0\rangle_{p'} \leftrightarrow |1\rangle_{p'} \) and (ii) the operator \( \sigma^x_{p'} \) executing it cannot be expressed as a product involving the operators (16).

II. In the case of the latter \( D^2(Z_2) \) with (17), the ground state is non-degenerate and given by (18) because the face operator in (17) can detect a change \( |0\rangle_{p'} \leftrightarrow |1\rangle_{p'} \). In this regard, in addition to the quasiparticles inherited from the \( D(Z_2) \), this \( D^2(Z_2) \) also admits other quasiparticles \( Q^{(J,K)} \) arising by effect of some \( \tilde{W}^{(J,K)} \) operators such that

\[ B_{p,J} \circ \tilde{W}^{(J,K)} = \tilde{W}^{(J,K)} \circ B_{p,1} \text{ and } D_{j,K} \circ \tilde{W}^{(J,K)} = \tilde{W}^{(J,K)} \circ D_{j,1}, \]  

(20)

where \( B_{p,J} \) and \( D_{j,K} \) are the elements that define the respective projector sets \( \mathfrak{B}_p \) and \( \mathfrak{D}_j \). Here, these two sets are given by

\[ \mathfrak{B}_p = \{B_{p,1}, B_{p,2}\} \text{ and } \mathfrak{D}_j = \{D_{j,1}, D_{j,2}\}, \]
\[ B_p \circ D_j \left| \begin{array}{c} a \\ b \\ c \\ d \end{array} \right. = \sum_{\lambda} B_p \left| \begin{array}{c} a \\ f(\lambda) b \\ c \\ d \end{array} \right. \]

\[ = \sum_{\lambda} \delta(f(\tilde{\alpha} \ast \tilde{\lambda}) ab^{-1} f^{-1}(\tilde{\lambda}) c^{-1} d, h) \left| \begin{array}{c} a \\ f(\tilde{\lambda}) b \\ c \\ d \end{array} \right. \]

\[ D_j \circ B_p \left| \begin{array}{c} a \\ b \\ c \\ d \end{array} \right. = \sum_{\lambda} \delta(f(\tilde{\alpha}) ab^{-1} c^{-1} d, h) D_j \left| \begin{array}{c} a \\ f(\tilde{\lambda}) b \\ c \\ d \end{array} \right. \]

\[ = \sum_{\lambda} \delta(f(\tilde{\alpha}) ab^{-1} c^{-1} d, h) \left| \begin{array}{c} a \\ f(\tilde{\lambda}) b \\ c \\ d \end{array} \right. \]

Figure 8: Result of action of the operators \( B_p \circ D_j \) and \( D_j \circ B_p \) on the lattice, which only reinforces the need for \( f(\tilde{\lambda}) \) to belong to the centre of \( G \) so that this model is solvable.

where

\[ B_{p,2} = \frac{1}{2} \left( I_p \otimes I_r \otimes I_s \otimes I_t \otimes I_u - \sigma_p^z \otimes \sigma_r^z \otimes \sigma_s^z \otimes \sigma_t^z \otimes \sigma_u^z \right) \quad \text{and} \]

\[ D_{j,2} = \frac{1}{2} \left( I_{p_1} \otimes I_j \otimes I_{p_2} - \sigma_{p_1}^x \otimes \sigma_j^x \otimes \sigma_{p_2}^x \right) . \]

According to these expressions, the only satisfactory solution of (20) is

\[ \tilde{W}_p^{(1,1)} = I_p , \quad \tilde{W}_p^{(1,2)} = \sigma_p^z , \quad \tilde{W}_p^{(2,1)} = \sigma_p^x \quad \text{and} \quad \tilde{W}_p^{(2,2)} = “ \sigma_p^y ” . \] (21)

Note that, by the point of view of the face operator \( B_{p,1} \), the quasiparticle \( Q^{(2,1)} \) produced by \( \tilde{W}_p^{(2,1)} \) behaves effectively as a monopole \( m \).

3.3.2. A comment on the triviality of the homomorphism

Although we have not made any comment on the quasiparticles that can be produced by operators acting only on the faces centroids in the \( D^2(\mathbb{Z}_2) \) with (16), these quasiparticles exist: they are the same ones produced by the operators \( \tilde{W}_p^{(J,K)} \) mentioned in (21). But as the operators \( B_{p,1} \) and

\[ B_{p,2} = \frac{1}{2} \left( I_p \otimes I_r \otimes I_s \otimes I_t \otimes I_u - I_p \otimes \sigma_r^z \otimes \sigma_s^z \otimes \sigma_t^z \otimes \sigma_u^z \right) \]

completing \( \mathfrak{B}_p \), in this \( D^2(\mathbb{Z}_2) \) where \( f \) is trivial, are such that \( F(\tilde{\alpha} : g) = 1 \), these quasiparticles cannot be completely distinguished from each other. Sicilicet, the quasiparticles \( \tilde{Q}^{(2,1)} \) and \( \tilde{Q}^{(2,2)} \) are interpreted effectively as equal to \( \tilde{Q}^{(1,1)} \) and \( \tilde{Q}^{(1,2)} \) respectively.
An entirely analogous comment applies to more general models where $G = \mathbb{Z}_N$ and $\tilde{S} = \mathbb{Z}_K$, since all these models support a case where $f$ is trivial in accordance with the Proposition 1. And one general characteristic of these $D^K(\mathbb{Z}_N)$, where $f(\tilde{\alpha}) = 0$ for all $\tilde{\alpha} \in \mathbb{Z}_N$, is that the quasiparticles that are produced in pairs by the action of

$$X_j^g, \quad Z_j^h \quad \text{and} \quad 'Y^{(g,h)} = X_j^g \circ Z_j^h = Z_j^h \circ X_j^g,$$ (22)

on the lattice edges are insensitive of those that are produced by operators that act exclusively on the face centroids. This quasiparticle insensitivity can be evidenced in both the $D^2(\mathbb{Z}_2)$ with (16) and the $D^3(\mathbb{Z}_2)$, whose orthonormal sets of operators $\mathfrak{A}_v$, $\mathfrak{B}_p$, and $\mathfrak{D}_j$ are uniquely defined by

$$A_{v,1} = \frac{1}{2} (1_a \otimes 1_b \otimes 1_c \otimes 1_d + \sigma^x_a \otimes \sigma^y_b \otimes \sigma^z_c \otimes \sigma^z_d),$$

$$A_{v,2} = \frac{1}{2} (1_a \otimes 1_b \otimes 1_c \otimes 1_d - \sigma^x_a \otimes \sigma^y_b \otimes \sigma^z_c \otimes \sigma^z_d),$$

$$B_{p,1} = \frac{1}{2} (1_p \otimes 1_r \otimes 1_s \otimes 1_t \otimes 1_u + 1_p \otimes \sigma^x_r \otimes \sigma^y_s \otimes \sigma^z_t \otimes \sigma^z_u),$$

$$B_{p,2} = \frac{1}{2} (1_p \otimes 1_r \otimes 1_s \otimes 1_t \otimes 1_u - 1_p \otimes \sigma^x_r \otimes \sigma^y_s \otimes \sigma^z_t \otimes \sigma^z_u),$$ (23)

$$D_{j,1} = \frac{1}{2} (1_{p_1} \otimes 1_j \otimes 1_{p_2} + X_{p_1} \otimes 1_j \otimes X_{p_2}^2 + X_{p_1}^2 \otimes 1_j \otimes X_{p_2}),$$

$$D_{j,2} = \frac{1}{2} (1_{p_1} \otimes 1_j \otimes 1_{p_2} + iX_{p_1} \otimes 1_j \otimes X_{p_2}^2 - iX_{p_1}^2 \otimes 1_j \otimes X_{p_2}) \quad \text{and}$$

$$D_{j,3} = \frac{1}{2} (1_{p_1} \otimes 1_j \otimes 1_{p_2} - iX_{p_1} \otimes 1_j \otimes X_{p_2}^2 + iX_{p_1}^2 \otimes 1_j \otimes X_{p_2})$$

because $N = 2$ and $K = 3$ are coprime numbers. In these cases where $f(\tilde{\alpha}) = 0$ for all $\tilde{\alpha} \in \mathbb{Z}_N$, if we leave aside the topological aspects related to the manifold $\mathcal{M}_2$, the $D^K(\mathbb{Z}_N)$ ground states are $N$-fold degenerate and given by

$$|\tilde{\xi}^{(\tilde{\alpha})}\rangle = \frac{1}{\sqrt{2}} \prod_v A_v \prod_{j,j'} D_{j'} \left( \bigotimes_j |0\rangle \right) \otimes \left( \bigotimes_{p \neq p'} |0\rangle \right) \otimes |\tilde{\alpha}\rangle_{p'}$$ (24)

because $\ker(f) = \mathbb{Z}_N$.

3.4. The ground state degeneracy and the classifiability of the $D^K(\mathbb{Z}_N)$

Note that this insensitivity mentioned above can be “broken” gradually as the $D^K(\mathbb{Z}_N)$ supports other homomorphisms beyond the trivial. This is the case of the $D^2(\mathbb{Z}_4)$ and $D^3(\mathbb{Z}_4)$ that exhibit a 4-fold degenerate ground state when $f$ is trivial, but that, by taking

$$f(0) = f(2) = 0 \quad \text{and} \quad f(1) = f(3) = 1,$$ (25)
have a 2-fold degenerate ground state given by

\[ |\tilde{\xi}_0^{(1)}\rangle = \frac{1}{\sqrt{2}} \prod_{v'} A_{v'} \prod_{j'} D_{j'} \left( \bigotimes_j |0\rangle \right) \otimes \left( \bigotimes_{p \neq p'} |0\rangle \right) \] (26)

\[ |\tilde{\xi}_0^{(2)}\rangle = \frac{1}{\sqrt{2}} \prod_{v'} A_{v'} \prod_{j'} D_{j'} \left( \bigotimes_j |0\rangle \right) \otimes \left( \bigotimes_{p \neq p'} |0\rangle \right) \otimes |2\rangle_{p'} . \] (27)

After all, in these cases with (25), the operators performing transitions between (26) and (27) cannot be expressed as a product involving the vertex, face and edge operators that define the \( D^2(\mathbb{Z}_4) \) and \( D^4(\mathbb{Z}_4) \) Hamiltonians.

However, in the case of the \( D^4(\mathbb{Z}_4) \), for instance, when \( f : \mathbb{Z}_4 \to \mathbb{Z}_4 \) is taken as a faithful homomorphism, its ground state is reduced only to (26). This follows because, similarly to what happens in the case \( D^2(\mathbb{Z}_2) \) with (17), the face operator that define the \( D^4(\mathbb{Z}_4) \) Hamiltonian can detect any changes \( |0\rangle_{p'} \leftrightarrow |\tilde{\alpha}\rangle_{p'} \), where \( \tilde{\alpha} \neq 0 \).

In general, what these examples make clear is that, in order for the quasiparticles produced by (22) to be sensitive to those produced by operators that act exclusively on the face centroids (and consequently the quasiparticles detectable by the vertex operator acquire confinement properties), it is necessary to consider non-trivial homomorphisms, which end up decreasing the algebraic degeneracy of the \( D^K(\mathbb{Z}_N) \) ground states. This algebraic degeneracy is given by

\[ d_{\text{alg}} = | \ker (f) | , \]

since all elements belonging to the \( \ker (f) \) cause the “fake holonomy”

\[ h' = f(\bar{\gamma})ab^{-1}c^{-1}d = f(\bar{\gamma})h \] (28)

(which is measured by the face operator shown in Figure 4) to correspond to the true holonomy \( h \). Based on this, we conclude that all these \( D^K(\mathbb{Z}_N) \) can be classified in terms of an ordered 3-tuple \((N, K, n)\) as follows:

(a) \((N, K, 0)\) are those whose algebraic degeneracy is maximal and where, on each one of their vacuum states (24), there is a \( D(\mathbb{Z}_N) \) that can support new quasiparticles \( Q^{(J,K)} \). These quasiparticles, which are produced by operators \( W_p^{(J,K)} \) that act exclusively on the face centroids when \( K > 0 \), are insensitive to those produced by (22).

(b) \((N, N, N)\) are those that, due to their minimal algebraic degeneracy, are identified as a modified \( D(\mathbb{Z}_N) \). Their quasiparticles \( e^g \) (produced by operators \( Z_{j'}^g \)) and \( \epsilon^{(g,h)} \) (produced by operators \( X_j^{h} \) and “ \( Y_{j'}^{(g,h)} \)” respectively), are sensitive to \( Q^{(J,K)} \).

(c) \((N, K, n)\) have intermediate properties to those mentioned in items (a) and (b) when \( n \) is a natural number such that \( 0 < n < N \) and \( N|nK \). That is, on each one of their vacuum states (24), there is a modified \( D(\mathbb{Z}_N) \) where only the quasiparticles \( e^{g'}, m^{h'} \) and \( \epsilon^{(g',h')} \), with \( g' = (g + 1) \mod k \) and \( h' = (h + 1) \mod k \), have the properties mentioned in item (b).
Note that the classification \((N, 1, 0)\) deleted from \((a)\) is associated with the identification of \(D^1(Z_N)\) as \(D(Z_N)\), since

\[
H_{D^1(G)} = -\sum_v A^{(G,0)}_v - \sum_p B^{(G,0)}_p - \sum_j D^{(G,0)}_j
\]

\[
= -\sum_v A^{(G)} - \sum_p B^{(G)} - \sum_j 1_j = H_{D(G)} + \text{cte} .
\]

3.4.1. A topological comment

In addition to all these algebraic considerations, we also need to take into account something important: as the quasiparticles \(m^h\) can be moved (by using an operator like

\[
O^{\gamma(g)} = \prod_{j \in \gamma^*} X_j^{\pm g} ,
\]

where \(\gamma^*\) is any path composed by two by two adjacent dual edges) without increase the system energy, the \(D^K(Z_N)\) ground state degeneracy also depends on the order of the fundamental group \(\pi_1\) associated with \(M_2\). This additional degeneracy \(d_{\text{top}}\), which is specifically of topological origin, must be taken into account to characterize, for instance, the fact that the number of vacuum states increases as the order of the gauge group \(Z_N\) increases.

In order to understand this number increasing, it should be noted that the choice of a non-trivial \(f\) causes some quasiparticles \(\tilde{Q}(J,K)\) are interpreted effectively as monopoles \(m^h\). That is, this effective equivalence between these quasiparticles must be discarded in the calculation of the independent vacuum states. This allows us to affirm that, if the \(D^K(Z_N)\) is defined in a discretization of a manifold \(M_2\) with genus \(g\), its total ground state degeneracy is given by

\[
d_{(N,K,n)} = d_{\text{alg}} \cdot d_{\text{top}} = |\ker(f)| \cdot |Z_K / \text{Im} f|^{2g} .
\]

3.5. The behaviour of the edge operator as a comparator

Besides the fact that we are unable to construct a \(D^K(Z_N)\) with a classification other than \((N, K, 0)\) when \(N\) and \(K\) are coprime numbers, another point deserves attention in this dualisation procedure. After all, while the \(D_M(Z_N)\) edge operator behaves like a comparator (i.e. as an operator that can check the alignment of two adjacent matter qunits), the \(D^K(Z_N)\) edge operator does something that seems different from a comparison and seems more like a kind of gauge transformation.

Although it is not incorrect to think that \(D^{(G,\bar{s})}_j\) may actually be performing some kind of gauge transformation, one of the ways to understand what this operator does is to see how it acts on a diagonal basis. For this, besides taking into account that

\[
D_j|\hat{\alpha}, g, \hat{\beta}\rangle = \frac{1}{|S|} \sum_{\lambda \in \bar{s}} |\hat{\alpha} \ast \hat{\lambda}, f(\hat{\lambda}) \cdot g, \hat{\lambda}^{-1} \ast \hat{\beta}\rangle ,
\]
we must note that this basis is obtained through the unitary transformations
\[ |g'\rangle = \frac{1}{|G|} \sum_{g \in G} \omega_{g'}(g) |g\rangle \quad \text{and} \quad |\tilde{\alpha}'\rangle = \frac{1}{|\tilde{S}|} \sum_{\tilde{\alpha} \in \tilde{S}} \chi_{\tilde{\alpha}'}(\tilde{\alpha}) |\tilde{\alpha}\rangle \quad \text{and} \quad (32) \]
where \( \omega_{g'}(g) \) and \( \chi_{\tilde{\alpha}'}(\tilde{\alpha}) \) are the characters of \( G \) and \( \tilde{S} \) respectively. The substitution of relations (32) into (31) shows that
\[ D_j|\tilde{\alpha}', g', \tilde{\beta}'\rangle = \frac{1}{|\tilde{S}|} \sum_{\lambda \in \tilde{S}} \chi_{\tilde{\alpha}'}(\tilde{\lambda}) \omega_{g'}(f(\tilde{\lambda})) \tilde{\chi}_{\tilde{\beta}'}(\tilde{\lambda}) |\tilde{\alpha}, g, \tilde{\beta}\rangle \quad (33) \]

Given this result, it is important to note that, since \( \tilde{S} \) and \( \text{Im}(f) \subset G \) are two finite Abelian groups, the Fourier transform \( \hat{f} \in L(S^*) \) is such that
\[ \hat{f}(\chi) = \sum_{\lambda \in \tilde{S}} f(\tilde{\lambda}) \chi(\tilde{\lambda}) \quad \text{and} \quad f(\tilde{\lambda}) = \frac{1}{|\tilde{S}|} \sum_{\chi \in \tilde{S}^*} \hat{f}(\chi) \chi(\tilde{\lambda}) , \]
where the dual group \( \tilde{S}^* \) is isomorphic to \( \tilde{S} \) [13, 16, 17, 18]. After all, by noting that an expression of the sort \( \chi_{\tilde{\alpha}'}(\tilde{\lambda}) \tilde{\chi}_{\tilde{\beta}'}(\tilde{\lambda}) = \chi_{\{\tilde{\alpha}', \tilde{\beta}'\}}(\tilde{\lambda}) \) is always a character, the substitution of these relations into (33) yields
\[ D_j|\tilde{\alpha}', g', \tilde{\beta}'\rangle = \frac{1}{|\tilde{S}|} \sum_{\chi \in \tilde{S}^*} [\omega_{g'} \circ f](\chi) \left( \frac{1}{|\tilde{S}|} \sum_{\lambda \in \tilde{S}} \chi_{\{\tilde{\alpha}', \tilde{\beta}'\}}(\lambda) \tilde{\chi}_{\tilde{\beta}'}(\lambda) \right) |\tilde{\alpha}, g, \tilde{\beta}\rangle \]
\[ = \frac{1}{|\tilde{S}|} \sum_{\chi \in \tilde{S}^*} [\omega_{g'} \circ f](\chi) \cdot \delta \left( \chi_{\{\tilde{\alpha}', \tilde{\beta}'\}}, \chi \right) |\tilde{\alpha}, g, \tilde{\beta}\rangle \]
\[ = \frac{1}{|\tilde{S}|} \omega_{g'} \cdot f(\chi_{\{\tilde{\alpha}', \tilde{\beta}'\}}) |\tilde{\alpha}, g, \tilde{\beta}\rangle . \]

That is, although the exact form of the index \( \{\tilde{\alpha}', \tilde{\beta}'\} \) depends on the nature of \( \tilde{S} \), we conclude that the \( D_j^{(G, \tilde{S})} \) can also be interpreted as an operator that compares matter qubits differently, which only becomes clear when this operator acts on a diagonal basis. This different way of comparing rests on the Pontryagin duality, which ensures that there is a one-to-one correspondence between the characters \( \chi_{\lambda} \) and the elements of \( \tilde{S} \) [19].

4. Final remarks

According to what we saw above, it is perfectly possible to perform a dualisation procedure on the \( D_M(G) \) and, thus, obtain another class \( D^K(G) \) of solvable models that can also be interpreted as a generalization of the \( D(G) \). However, this algebraic dual class \( D^K(G) \), when superimposed on the \( D_M(G) \) to define a more general new class with Hamiltonian
\[ H_{\text{total}} = H_{D_M(G)} + H_{D^K(G)} , \]
does not necessarily define self-dual models. After all, unlike the $D_M(Z_N)$ where $M$ and $N$ are coprime numbers, which supports a non-trivial case where quasiparticles with non-Abelian fusion rules are required, it is impossible to create a $D^K(Z_N)$ different from the trivial when $N$ and $K$ are coprime numbers. That is, from the physical point of view, this means that for each excitation detectable by the face or/and edge operators in (2) there will not necessarily be another, with the same properties, that is detectable by the vertex or/and edge operators in (7) respectively and vice versa.

In any case, it is important to emphasize that this construction of the $D^K(Z_N)$, which was based on its recognition as an algebraic dual of the $D^M(Z_N)$, really allows us to recognize some dual traces between these two classes. Note that all this construction, when made by using $G = Z_N$ and $\tilde{S} = Z_K$ where $N$ and $K$ are not coprime numbers, leads to models in which the quasiparticles $m^k$ are free and the $e^g$ are confined: that is, while the first quasiparticles can be transported by an operator as (29) without increasing the system energy, the latter, when transported by an operator as (14), increase this energy.

Despite this confinement observation about $e^g$ was first made on page 7 as a result of an analysis of the $D^K(Z_N)$, it is worth noting that this confinement extends to the class $D^K(G)$ as a whole, provided that $f$ is not a trivial homomorphism. That is, whenever it is possible to define a model with non-trivial $f$, at least a part of the quasiparticles detectable by the vertex operators will be confined. Another interesting aspect of the $D^K(Z_N)$, which can also be extended to the $D^K(G)$ as a whole, is associated with the possibility of classifying them as presented in items (a), (b) and (c) on page 13. This is a complete classification because, once the ordered 3-tuple $(N, K, n)$ of a model is identified, it is possible to identify not only all the properties of its quasiparticles, but also to calculate its ground state degeneracy. Note that the $D^K(G)$ has algebraic and topological orders: the algebraic order is due to the co-action of the gauge group on the matter quanta; the topological order is due to the fact that, as in the $D(G)$, the $D^K(G)$ ground state degeneracy depends on the order of the fundamental group $\pi_1$ associated with $M_2$.

Although this generalization $D^K(G)$ has been successful, there is no impediment, a priori, to construct others without the artifice of the dualisation. So, one question we can ask about these other generalizations is whether one of them bring the same results from a different point of view. One of the possibilities we can explore is that in which $f$ defines a crossed module [20]: that is, $f$ is a homomorphism that, together with an action $\theta : G \times \tilde{S} \to \tilde{S}$, respects two conditions

$$f(\theta(g, \tilde{\alpha})) = g f(\tilde{\alpha}) g^{-1} \quad \text{and} \quad \theta(f(\tilde{\alpha}), \tilde{\beta}) = \tilde{\alpha} \tilde{\beta} \tilde{\alpha}^{-1}$$

where the second is known as the Peiffer condition [21, 22]. Note that the homomorphisms that define the $D^K(G)$ satisfy these two conditions when this action is trivial because $G$ and $\tilde{S}$ are Abelian groups. And the possible advantage of taking $f$ as the homomorphism that completes a crossed module lies in the fact that it seems possible to recover the $D^K(G)$ as a particular case of the higher lattice gauge theories [23], which are based on the higher-dimensional category theory [24, 25, 26]. A good example of this is in Ref. [27], where a 2-lattice gauge theory is defined by using a three-dimensional lattice to which we can measure 1- and 2-holonomies: after all, while 1-holonomy is identified as the same “fake
holonomy” (28) that is preserved by the action of the operator $A_v^{(G,\tilde{S})}$ that performs gauge transformations, the 2-holonomy [28] is preserved by the action of the operator

$$A^D_v = \prod_{j \in S_v} D_j^{(G,\tilde{S})},$$

which corroborates with the perception of $D_j^{(G,\tilde{S})}$ as the component of an operator that performs another kind of gauge transformations. Note that, if $f$ is the homomorphism that defines the crossed module $\mathcal{G} = (G, \tilde{S}; f, \theta)$, the first and second homotopy groups of this crossed module can be defined as $\pi_1(\mathcal{G}) = G / \text{Im } f = \text{coker } (f)$ and $\pi_2(\mathcal{G}) = \ker (f) = \pi_2(\mathcal{G})$ respectively [29], whose orders define the result (30). We will return to this topic in a future work.

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