Abstract. In this paper we use 3-manifold techniques to illuminate the structure of the category of tangles. In particular, we show that every idempotent morphism \( A \) in such a category naturally splits as \( A = B \circ C \) such that \( C \circ B \) is an identity morphism.

1. Introduction

An *idempotent* of a category is a morphism that is idempotent with respect to composition, i.e., a morphism \( f \) such that \( f = f \circ f \). Idempotents have significance to quantum observations or measurements \([8]\), can reflect self-replication in biology (such as DNA) \([6]\), and can form building blocks for numerous algebraic structures \([5]\). An idempotent \( f \) splits if there are morphisms \( g \) and \( h \) such that \( f = g \circ h \), but \( h \circ g \) is the identity morphism. By direct inspection, one can see that any morphism \( f \) with such a property is idempotent (if \( h \circ g \) is the identity, then \((g \circ h) \circ (g \circ h) = g \circ (h \circ g) \circ h = g \circ h\)); but in many categories, not all idempotents split. A category where every idempotent splits is called *Karoubi complete*. Idempotent splitting may adopt significance from various interpretations of the categories involved. For example, Selinger studied idempotents of dagger categories, and described in \([8, \text{Remark 3.5}]\) how the splitting of idempotents may clarify data types. In \([6]\), Kauffman related idempotents to DNA replication, and saw the idempotents in a Karoubi complete category as appealing models for self-replicators.

We show that the category of unoriented tangles up to isotopy is Karoubi complete. Objects of this category are points in the disc \( D^2 \), morphisms are properly embedded 1-manifolds in \( D^2 \times I \) (these are the tangles), and the morphism composition is achieved via a stacking operation. Categories of tangles were studied in \([9]\) to understand the combinatorial structure of tangle composition, and various categories of tangles are classified in \([4]\) as certain types of braided pivotal categories. A related category, called the Temperley-Lieb category, can be described similarly, but with \( D^2 \) replaced by \( I \), and hence the category
of tangles we consider is a natural generalization of the Temperley-Lieb category. It was shown in [1] that the Temperley-Lieb category is Karoubi complete.

Although the main result of this paper extends that in [1], the techniques in this paper are different and are rather inspired by the proof of the prime decomposition theorem for string links provided in [2]. The main technical tool in the current paper, as well as in [2], is a bound, established in [3], on the number of non-parallel essential surfaces in a compact 3-manifold. The paper is organized as follows. In Section 3 we precisely define the tangle category. In Section 4 we review incompressible punctured surfaces and their properties. In Section 5 we adapt to tangles the notion of braid-equivalence for string links in [2] and apply this idea to factoring morphisms as the composition of two morphisms. In Section 6 we prove that all idempotents in the category of tangles split.

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3. Idempotents in the Category of Tangles

Let $\mathcal{T}$ be the category of tangles. The definition of $\mathcal{T}$ that we give here is essentially equivalent to the definition of the category of unoriented tangles up to isotopy, denoted $\text{TANG}$, in [4]. The objects of $\mathcal{T}$ are the natural numbers. Each natural number $n$ is identified with $n$ distinct fixed points $\{x_1, x_2, ..., x_n\}$ in $D^2$. The morphisms of $\mathcal{T}$ are tangles. A tangle is a pair $(D^2 \times I, A)$ such that $A$ is a properly embedded compact 1-manifold in $D^2 \times I$ with the following conditions: the boundary of each arc (connected component with non-trivial boundary) of $A$ is contained in $(D^2 \times \{0\}) \cup (D^2 \times \{1\})$; the intersection of $A$ with the lower and upper boundaries of $D^2 \times I$ are $A \cap (D^2 \times \{0\}) = \{(x_1, 0), (x_2, 0), ..., (x_n, 0)\}$ and $A \cap (D^2 \times \{1\}) = \{(x_1, 1), (x_2, 1), ..., (x_m, 1)\}$; and for each boundary point $(x_i, 0)$ or $(x_i, 1)$ of $A$ and each sufficiently small neighborhood of that point, the derivatives of all orders of the embedding agree with the maps $t \mapsto (x_i, t)$. For simplicity, we will occasionally refer to the tangle $(D^2 \times I, A)$ as the morphism $A$. We denote $D^2 \times \{1\}$ by $\partial_+(D^2 \times I)$ and $D^2 \times \{0\}$ by $\partial_-(D^2 \times I)$.

Definition 3.1 (Tangle equivalence). Tangles $(D^2 \times I, A)$ and $(D^2 \times I, B)$ are equivalent if there is an isotopy of $D^2 \times I$ fixing $\partial(D^2 \times I)$.
that takes $A$ to $B$. In this case, (by abuse of notation) we will write $A = B$.

Let $h : D^2 \times I \to I$ be the projection map onto the second coordinate. A \textit{braid} is a tangle that is equivalent to a tangle $(D^2 \times I, A)$ with the property that for every component $\alpha$ of $A$, the restriction of $h$ to $\alpha$ is a smooth, one-to-one and onto function with no critical points in its domain. Given tangle $(M, A)$ from $k$ to $l$ and tangle $(N, B)$ from $l$ to $m$, with $M = N = D^2 \times I$, denote the \textit{composition} of these morphisms by $(D^2 \times I, A \circ B)$ which is the quotient of $M \cup N$ achieved by gluing $\partial_+(M)$ to $\partial_-(N)$ via the map $(x, 1) \mapsto (x, 0)$, and $A \circ B$ is the properly embedded 1-manifold in the quotient which is the image of $A \cup B$ under this identification. The resulting quotient of $M \cup N$ is again homeomorphic to $D^2 \times I$ and we choose to identify the image of $M$ under the quotient map with $D^2 \times [0, 1/2]$ and the image of $N$ with $D^2 \times [1/2, 1]$ in the obvious ways. Again, by abuse of notation, we will write the resulting tangle as the morphism $A \circ B$ and consider it the composition of morphisms $A$ and $B$. See Figure 1.

We illustrate in Figure 2 an \textit{idempotent} in the category of tangles, i.e. a morphism $A$ of $T$ such that $A \circ A = A$. Note that if $A$ is a braid and an idempotent then $A$ is an identity morphism, since braids form a group under composition. Recall that a category is \textit{Karoubi complete} if all of its idempotents split, where an idempotent $A$ \textit{splits} if there exist morphisms $C$ and $B$ such that $A = C \circ B$ and $B \circ C = \text{Id}_n$ is an identity morphism. Our main theorem is the following.

\textbf{Theorem 3.2.} The category $T$ of tangles is Karoubi complete.

\section{Incompressible punctured surfaces}

Our primary tool in the classification of idempotents in $T$ will be the study of punctured surfaces up to transverse isotopy. Unless otherwise stated all manifolds are compact. Suppose $\alpha$ is a 1-manifold properly embedded in a 3-manifold $M$. If $F$ is a properly embedded surface in $M$ which meets $\alpha$ transversely in $k$ points, we say $F$ is $k$\textit{-punctured}. An isotopy $\phi_t$ of $F$ in $M$ is \textit{proper} if its restriction to $\partial F \times I$ is an isotopy of $\partial F$ in $\partial M$. Moreover, the isotopy $\phi_t$ is \textit{transverse} to $\alpha$ if the embedding $\phi_t$ is transverse to $\alpha$ for all fixed values of $t$. Isotopies of surfaces in this paper will always be proper isotopies that are transverse to the relevant 1-manifolds.

If $\alpha$ is a 1-manifold properly embedded in $M \cong D^2 \times I$ and $F$ is a properly embedded $k$-punctured surface in $M$, $F$ is \textit{boundary-parallel} either if $F$ is a 2-punctured 2-sphere bounding a 3-ball that meets
α in an unknotted arc or if there is a proper, transverse isotopy of $F$ in $M$ which takes $F$ to a punctured subsurface contained in $\partial M$. Otherwise, we say $F$ is non-boundary parallel. A loop $\gamma$ embedded in $F$ is essential if it does not bound a 0-punctured disk in $F$. The $k$-punctured surface $F$ is compressible in $(M, \alpha)$ (or just compressible when context is understood) if $F$ is a 0-punctured 2-sphere bounding a 3-ball or if there exists a disk $D$ embedded in $M$ such that $D \cap F = \partial D$, $\partial D$ is essential in $F$ and $D$ is disjoint from $\alpha$. Such a disk is called a compressing disk. Otherwise, we say $F$ is incompressible. A punctured surface $F$ is essential if $F$ is incompressible and non-boundary parallel.

Given a 1-manifold $\alpha$ properly embedded in $M \cong D^2 \times I$ and $F$ a properly embedded $k$-punctured surface with compressing disk $D$, we can compress $F$ along $D$ to form a new embedded $k$-punctured surface $F^*$. See Figure 3. Let $D^2 \times I$ be a fibered neighborhood of $D$ in $M$ such that $D = D^2 \times \{\frac{1}{2}\}$, $D^2 \times I$ is disjoint from $\alpha$, and $\partial(D^2) \times I$ is an embedded annulus in $F$ that is disjoint from the punctures of $F$. Then
we define $F^*$ to be a surface transversely isotopic to $(F \setminus (\partial(D^2) \times I)) \cup (D^2 \times \{0, 1\})$. 

Figure 2. An idempotent morphism in the category of tangles

Figure 3. An example of compressing a 4-punctured sphere to obtain two boundary-parallel 2-punctured spheres.
Although we take the point of view of incompressible and non-boundary parallel punctured surfaces in this paper, we could have equivalently adopted the point of view of studying incompressible and non-boundary parallel meridional surfaces properly embedded in the exterior of $\alpha$ in $M$. In particular, if $F$ is a properly embedded non-boundary parallel punctured surface in $(D^2 \times I, \alpha)$ and $\eta(\alpha)$ is a small open regular neighborhood of $\alpha$ in $D^2 \times I$, then $F \setminus \eta(\alpha)$ is non-boundary parallel in $(D^2 \times I) \setminus \eta(\alpha)$. Similarly, if $F$ is a properly embedded incompressible punctured surface in $(D^2 \times I, \alpha)$ and $\eta(\alpha)$ is a small open regular neighborhood of $\alpha$ in $D^2 \times I$, then $F \setminus \eta(\alpha)$ is incompressible in $(D^2 \times I) \setminus \eta(\alpha)$. In particular, we will make use of the following Theorem of Freedman and Freedman.

**Theorem 4.1.** [3] Let $M$ be a compact 3-manifold with boundary and $b$ an integer greater than zero. There is a constant $c(M, b)$ so that if $F_1, \ldots, F_k, k > c$, is a collection of incompressible surfaces such that all the Betti numbers $b_i(F_i) < b$, $1 \leq i \leq k$, and no $F_i$, $1 \leq i \leq k$, is a boundary parallel annulus or a boundary parallel disk, then at least two members $F_i$ and $F_j$ are parallel.

Note that Freedman and Freedman define two disjoint properly embedded surfaces $F_i$ and $F_j$ in a compact 3-manifold $M$ to be parallel if $F_i \cup F_j$ cobound a product $F \times I$ in $M$ such that $\partial F \times I \subset \partial M$, $F_i = F \times \{0\}$ and $F_j = F \times \{1\}$.

## 5. Decomposing Disks and Braid Equivalence

Decomposing a morphism in $T$ as a composition of two morphisms involves some amount of choice. This choice can be captured via the notion of braid-equivalence.

**Definition 5.1.** Two tangles $(D^2 \times I, A)$ and $(D^2 \times I, B)$ are braid-equivalent if there exist braids $C_1$ and $C_2$ such that $A = C_1 \circ B \circ C_2$.

**Proposition 5.2.** Tangles $(D^2 \times I, T_1)$ and $(D^2 \times I, T_2)$ are braid-equivalent if and only if there is an isotopy of $D^2 \times I$ which fixes $(\partial D^2) \times I$ and which takes $T_1$ to $T_2$.

**Proof.** This follows from a nearly identical adaptation of the proof of Proposition 3.6 of [2]. $\square$

**Definition 5.3.** A decomposing disk for a tangle $(D^2 \times I, A)$ is a punctured disk which is properly embedded in $D^2 \times I$, whose boundary is isotopic in $\partial(D^2 \times I)$ to $\partial(\partial_+(D^2 \times I))$. See the disk $F$ in Figure [2].
Figure 4. A minimal decomposing disk for an idempotent tangle.

A decomposing disk \( F \) for a tangle \((D^2 \times I, A)\) separates \(D^2 \times I\) into two connected components, one containing \(\partial_- (D^2 \times I)\) and the other containing \(\partial_+ (D^2 \times I)\). The closure of each component is homeomorphic to \(D^2 \times I\), so \(F\) decomposes \((D^2 \times I, A)\) into two tangles, each of which is well-defined up to composition with braids (cf. braid-equivalence in Definition 5.1). If \((D^2 \times I, B)\) is the tangle resulting from restricting \(A\) to the side of \(F\) in \(D^2 \times I\) that contains \(\partial_- (D^2 \times I)\) and \((D^2 \times I, C)\) is the tangle resulting from restricting \(A\) to the side of \(F\) in \(D^2 \times I\) that contains \(\partial_+ (D^2 \times I)\), we say that \(F\) decomposes \((D^2 \times I, A)\) as \((D^2 \times I, B \circ C)\), or, more simply, \(A = B \circ C\).

6. The Proof

Definition 6.1. A decomposing disk \( F \) for a tangle \((D^2 \times I, A)\) is minimal if there is no decomposing disk \( G \) such that \(|F \cap A| > |G \cap A|\). See Figure 4.

Lemma 6.2. If \((D^2 \times I, A)\) is an idempotent, then either \((D^2 \times I, A)\) is an identity morphism or any minimal decomposing disk is essential.

Proof. Assume that \((D^2 \times I, A)\) is an idempotent. If \((D^2 \times I, A)\) is a braid, then, as braids form a group, \(A\) is an identity morphism. So, we may assume that \((D^2 \times I, A)\) is not a braid. If \(A\) contains \(l\) distinct closed loops, then, since \(A\) is idempotent, \(A\) must contain \(2l\) distinct closed loops, a contradiction unless \(l = 0\). Hence, we can assume that \(A\) contains no closed loops.

Since \((D^2 \times I, A)\) is an idempotent, then \((D^2 \times I, A)\) is a morphism from \(n\) points to \(n\) points for some \(n\). Since \(A\) is non-empty, \(n \geq 1\). By Theorem 4.1 there is an integer \(c\) such that if \(F_1, ..., F_k\) is a collection of disjoint incompressible decomposing disks in \(D^2 \times I\) that each meet \(A\)
in at most \( n \) points (hence the first Betti number of each \( F_i \) is bounded above by \( n \)) and \( k > c \), then at least two members \( F_i \) and \( F_j \) are parallel or one of the surfaces \( F_1, \ldots, F_k \) is a boundary parallel 0-punctured disk or a boundary parallel 1-punctured disk.

Claim: \( \partial_+(D^2 \times I) \) or \( \partial_-(D^2 \times I) \) is compressible in \((D^2 \times I, A)\).

Proof of claim: Since \((D^2 \times I, A)\) is equivalent to \((D^2 \times I, A \circ A)\), then \((D^2 \times I, A)\) is equivalent to \((D^2 \times I, A^{c+2})\). Hence, we can find \( c+1 \) pairwise decomposing disks, \( F_1, \ldots, F_{c+1} \), for \((D^2 \times I, A)\) that decompose \((D^2 \times I, A)\) into \( c+2 \) copies of \((D^2 \times I, A)\). If both \( \partial_+(D^2 \times I) \) and \( \partial_-(D^2 \times I) \) are incompressible in \((D^2 \times I, A)\), then each of \( F_1, \ldots, F_{c+1} \) are incompressible in \((D^2 \times I, A)\). Since \( n \geq 1 \), then none of the surfaces \( F_1, \ldots, F_{c+1} \) is a 0-punctured disk. Moreover, if any of the surfaces was a boundary parallel once punctured disk (i.e. a boundary parallel annulus in the exterior of \( A \)), then, by the isotopy extension theorem \([7]\), \( A \) would be a trivial braid on one strand. Thus, the collection \( F_1, \ldots, F_{c+1} \) meets the hypothesis of Theorem \([1]\) and there exist two members \( F_i \) and \( F_j \) that are parallel. The tangle between \( F_i \) and \( F_j \) in \( D^2 \times I \) is braid-equivalent to \((D^2 \times I, A^l)\) for some \( l \geq 1 \); however, since \( F_i \) is parallel to \( F_j \), then \( A^l \) and, thus, \( A \) is a braid, a contradiction. Hence, one of \( \partial_+(D^2 \times I) \) or \( \partial_-(D^2 \times I) \) is compressible in \((D^2 \times I, A)\). □

Without loss of generality, suppose that \( \partial_+(D^2 \times I) \) is compressible in \((D^2 \times I, A)\). Compressing \( \partial_+(D^2 \times I) \) once results in a surface with two connected components. One component is a decomposing disk that meets \( A \) in strictly fewer than \( n \) points and the other component is a punctured sphere. Note that any boundary parallel decomposing disk would be properly, transversely isotopic to \( \partial_+(D^2 \times I) \) or \( \partial_-(D^2 \times I) \) in \((D^2 \times I, A)\), and hence meets \( A \) in \( n \) points. Since we have found a decomposing disk that meets \( A \) in strictly fewer than \( n \) points, then a minimal decomposing disk cannot be boundary parallel.

Let \( F \) be a minimal decomposing disk for \((D^2 \times I, A)\). By the above argument, \( F \) is non-boundary parallel. Next, we show that \( F \) is incompressible. If \( F \) is compressible, then compressing \( F \) once results in a surface with two connected components. One component is a decomposing disk that meets \( A \) in fewer points than \( F \) does. This is a contradiction to \( F \) being minimal. Hence, \( F \) is incompressible. Since \( F \) is both incompressible and non-boundary parallel, then \( F \) is essential. □

We now restate and prove our main theorem (Theorem 3.2).

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\(^1\)Observe that in Figure 3 \( \partial_+(D^2 \times I) \) is compressible.
Theorem 6.3. If $(D^2 \times I, A)$ is an idempotent, then there exist $B$ and $C$, such that $A = B \circ C$ and $C \circ B$ is an identity morphism.

Proof. Let $F$ be a minimal decomposing disk for $(D^2 \times I, A)$. By Lemma 6.2, $F$ is essential. Denote the tangles that $F$ decomposes $(D^2 \times I, A)$ into by $(D^2 \times I, B)$ and $(D^2 \times I, C)$ so that $A = B \circ C$. Note that since $A$ is an idempotent, then $A$ is a morphism from $n$ points to $n$ points for some $n$. Moreover, since $\partial_+ (D^2 \times I)$ is a decomposing disk for every $(D^2 \times I, A)$, then $F$ meets $A$ in at most $n$ points.

By Theorem 4.1, there is an integer $c$ such that if $F_1, ..., F_k$ is a collection of disjoint essential decomposing disks in $D^2 \times I$ that each meet $A$ in at most $n$ points (hence the first Betti number of each $F_i$ is bounded above by $n$) and $k > c$, then at least two members $F_i$ and $F_j$ are parallel (note that none of the $F_i$ are boundary parallel since each is essential).

![Diagram](image)

**Figure 5.** An example of how three copies of an idempotent $A = B \circ C$ can be decomposed into one copy of $B$, two copies of $C \circ B$, and one copy of $C$.

Since we have established that $F$ is an essential punctured surface in $(D^2 \times I, A)$ and since $(D^2 \times I, A)$ is equivalent to $(D^2 \times I, A^{c+1})$, then we can find $c + 1$ disjoint minimal decomposing disks, $F_1, ..., F_{c+1}$, for $(D^2 \times I, A)$ each representing the copy of $F$ in each copy of $A$.
in $A^{c+1}$. Each of $F_1, \ldots, F_{c+1}$ is essential in $(D^2 \times I, A)$ and together they decompose $(D^2 \times I, A)$ into one copy of $(D^2 \times I, B)$, $c$ copies of $(D^2 \times I, C \circ B)$, and one copy of $(D^2 \times I, C)$. See Figure 5. By Theorem 4.1 there exist two members $F_i$ and $F_j$ that are parallel. The tangle between $F_i$ and $F_j$ in $D^2 \times I$ is equivalent to $(D^2 \times I, (C \circ B)^l)$ for some $l \geq 1$, however, since $F_i$ is parallel to $F_j$, then $(C \circ B)^l$ and, thus, $C \circ B$ is a braid.

Since $A^2 = A$, then $B \circ C \circ B \circ C = B \circ C$. We can compose on the left by $C$ and the right by $B$ to obtain $(C \circ B)^3 = (C \circ B)^2$. However, since braids on $n$ strands form a group under composition, $(C \circ B)^3 = (C \circ B)^2$ implies $C \circ B$ is an identity morphism. □

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