TRAVELING WAVES IN A NONLOCAL DISPERAL EPIDEMIC MODEL WITH SPATIO-TEMPORAL DELAY

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Abstract. In this paper, we investigate the existence and nonexistence of traveling wave solutions in a nonlocal dispersal epidemic model with spatio-temporal delay. It is shown that this model admits a nontrivial positive traveling wave solution when the basic reproduction number $R_0 > 1$ and the wave speed $c \geq c^*$ ($c^*$ is the critical speed) and this model has no traveling wave solutions when $R_0 \leq 1$ or $c < c^*$. This indicates that $c^*$ is the minimal wave speed.

1. Introduction and main results. In diffusive epidemic models, traveling waves can describe the state that a disease spreads geographically with a constant speed. The existence of traveling waves in these models has become one of the important issues in mathematical epidemiology [1, 5, 7–9, 11, 12, 14, 20, 27–33, 35, 36, 38, 40–46, 48–50]. For example, Wang et al. [30] considered a reaction-diffusion epidemic model

\[
\begin{aligned}
\partial_t S(x,t) &= d_1 \partial_{xx} S(x,t) - \frac{\beta S(x,t) I(x,t)}{S(x,t) + I(x,t) + R(x,t)}, \\
\partial_t I(x,t) &= d_2 \partial_{xx} I(x,t) + \frac{\beta S(x,t) I(x,t)}{S(x,t) + I(x,t) + R(x,t)} - (\gamma + \delta) I(x,t), \\
\partial_t R(x,t) &= d_3 \partial_{xx} R(x,t) + \gamma I(x,t),
\end{aligned}
\]

(1.1)

where $S(x,t)$, $I(x,t)$ and $R(x,t)$ refer to the densities of susceptible, infected and recovered individuals at location $x$ and time $t$, respectively. The coefficients $d_i > 0$ ($i = 1, 2, 3$) denote the diffusion rates of each class, $\beta > 0$ represents the transmission rate, $\gamma > 0$ is the recovery rate and $\delta \geq 0$ is the disease-induced death rate. They proved that if $R_0 = \beta/(\gamma + \delta) > 1$, $c > c^* = 2\sqrt{d_2(\beta - \gamma - \delta)}$ and $d_3 < 2d_2$, then

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(1.1) has a nonnegative traveling wave solution in the form of \((S(x + ct), I(x + ct), R(x + ct))\) such that
\[
\begin{array}{l}
S(-\infty) = S_1 > S(\infty) = S_2, \quad I(\pm \infty) = 0, \\
R(-\infty) = 0, \quad R(\infty) = \gamma(S_1 - S_2)/(\gamma + \delta),
\end{array}
\] (1.2)
where \(S_1 > 0\) is a given constant and \(S_2 \geq 0\) is a constant. On the other hand, they obtained that when \(R_0 \leq 1\) or \(c < c^*\), (1.1) admits no traveling waves solutions. Their results [12] improved the corresponding results in [30]. Considering this biological fact that infected individuals often take time to move in space and are usually not in the same position at previous time, Wang et al. [31] introduced a spatio-temporal delay term
\[
K \ast I(x,t) = \int_{-\infty}^{t} \int_{\mathbb{R}} K(x - y, t - s)I(y,s)dyds
\]
into a diffusive epidemic model
\[
\begin{align*}
\partial_t S(x,t) &= d_1 \partial_{xx} S(x,t) - \beta S(x,t) I(x,t) - S(x,t) K \ast I(x,t), \\
\partial_t I(x,t) &= d_2 \partial_{xx} I(x,t) + \beta S(x,t) K \ast I(x,t) - \gamma I(x,t), \\
\partial_t R(x,t) &= d_3 \partial_{xx} R(x,t) + \gamma I(x,t),
\end{align*}
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(1.3)
where \(\tau \geq 0\) is the time delay. They showed that there exists a constant \(c^* > 0\) such that (1.3) admits a nonnegative traveling wave solution \((S(x + ct), I(x + ct), R(x + ct))\) satisfying boundary conditions (1.2) when \(R_0 = \beta/(\gamma + \delta) > 1\) and \(c \geq c^*\). Meanwhile, they also proved that if \(R_0 \leq 1\) or \(c < c^*\), then (1.3) admits no traveling wave solutions. Their results [12] improved the corresponding results in [30]. Considering this biological fact that infected individuals often take time to move in space and are usually not in the same position at previous time, Wang et al. [31] introduced a spatio-temporal delay term
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\partial_t R(x,t) &= d_3 \partial_{xx} R(x,t) + \gamma I(x,t),
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0 < c < c*. For the study of other diffusive epidemic models with spatio-temporal delay, we refer to [6, 23, 46, 50].

It is known that the Laplacian operator describes the local range process, while the nonlocal dispersal operator is better to represent the long range movements of each class [2, 15–17, 25, 26, 40]. Yang et al. [38, 41] considered a nonlocal dispersal epidemic model

\[ \begin{align*}
\partial_t S(x, t) &= d_1 \left[ J * S(x, t) - S(x, t) \right] - \beta S(x, t)I(x, t), \\
\partial_t I(x, t) &= d_2 \left[ J * I(x, t) - I(x, t) \right] + \beta S(x, t)I(x, t) - \gamma I(x, t), \\
\partial_t R(x, t) &= d_3 \left[ J * R(x, t) - R(x, t) \right] + \gamma I(x, t),
\end{align*} \]

(1.5)

where \( J * U(x, t) = \int_{\mathbb{R}} J(y)U(y, t)dy \) (\( U \) can be either \( S, I \) or \( R \)). The kernel function \( J \) was assumed to satisfy that \( J \in C^1(\mathbb{R}) \), \( J(x) = J(-x) \geq 0 \), \( \int_{\mathbb{R}} J(y)dy = 1 \) and \( J \) is compactly supported. The nonlocal dispersal terms \( J * S(x, t) - S(x, t) \), \( J * I(x, t) - I(x, t) \) and \( J * R(x, t) - R(x, t) \) describe that the rate of susceptible, infected and recovered individuals at position \( x \) and time \( t \) depend on the influence of neighboring \( S, I \) and \( R \) in all other positions \( y \). For any given \( S_0 > 0 \), they obtained that if \( R_0 = \beta S_0/\gamma > 1 \) and \( c \geq c^* \) (\( c^* \) is the critical speed), (1.5) has a nontrivial positive traveling wave solution. On the other hand, they proved that (1.5) has no traveling wave solutions if \( R_0 \leq 1 \) or \( c < c^* \). More recently, Yang et al. [40] studied the following epidemic model

\[ \begin{align*}
\partial_t S(x, t) &= d_1 \left[ J * S(x, t) - S(x, t) \right] - \frac{\beta S(x, t)I(x, t)}{S(x, t) + I(x, t) + R(x, t)}, \\
\partial_t I(x, t) &= d_2 \left[ J * I(x, t) - I(x, t) \right] + \frac{\beta S(x, t)I(x, t)}{S(x, t) + I(x, t) + R(x, t)} - (\gamma + \delta)I(x, t), \\
\partial_t R(x, t) &= d_3 \left[ J * R(x, t) - R(x, t) \right] + \gamma I(x, t),
\end{align*} \]

(1.6)

With the aid of the upper-lower solutions method, they constructed an invariant cone with initial functions defined on a large bounded interval \([-X, X]\). Then they introduced a initial value problem and applied Schauder’s fixed point theorem to establish the existence of a nonnegative traveling wave solution on that cone. Subsequently, they extended the existence of a nonnegative traveling wave solution to the whole real line by a limiting argument and derived its asymptotic boundary. Finally, they obtained the nonexistence of traveling wave solutions by the two-sided Laplace transform and contradictory arguments. The results in [40] are summarised as follows. When \( R_0 = \beta/(\gamma + \delta) > 1 \), there exists a constant \( c^* > 0 \) such that for each \( c > c^* \), model (1.6) has a nonnegative traveling wave solution \( (S(x + ct), I(x + ct), R(x + ct)) \) satisfying \( S(-\infty) = S_1 > S(\infty) = S_2 \) \( (S_1 > 0 \) is a given constant and \( S_2 \geq 0 \) is some constant), \( I(\pm\infty) = 0 \) and \( R(-\infty) = 0 \). Moreover, \( R(\infty) = \gamma(S_1 - S_2)/(\gamma + \delta) \) if \( R(z) \) is bounded in \( \mathbb{R} \). When \( R_0 \leq 1 \) or \( 0 < c < c^* \), there exist nontrivial nonnegative traveling wave solutions. For more investigation of nonlocal dispersal models, see [10, 20, 21, 24, 26, 29, 40, 46].

In the present paper, motivated by [12, 40, 47], we propose the spatio-temporal delayed counterpart of (1.6)
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Throughout this paper, the kernel functions $J(x)$ and $K(x,t)$ in (1.7) satisfy the following hypotheses.

(H1) $J(x) \in C(\mathbb{R})$, $J(x) = J(-x) \geq 0$, $\int_{\mathbb{R}} J(x)dx = 1$ and $J$ is compactly supported with $R_1$ the radius of $\text{supp} J$.

(H2) $K(x,t) \in C(\mathbb{R} \times [0, \infty))$, $\int_{0}^{\infty} \int_{\mathbb{R}} K(x,t)dxdt = 1$ and $K(x,t) = K(-x,t) \geq 0$ for $(x,t) \in \mathbb{R} \times [0, \infty)$.

(H3) $K(x,t)$ is compactly supported with respect to spatial variable $x$ with $R_2$ the radius of $\text{supp} K$ for variable $x$ and $K(x,t)$ is compactly supported with respect to time variable $t$ with $T$ the length of $\text{supp} K$ for variable $t$.

System (1.7) can model the recent global epidemic outbreak of SARS, H1N1 and avian influenza [3,4]. The aim of this paper is to establish the existence and nonexistence of traveling wave solutions to (1.7). A traveling wave solution of (1.7) is a wave of the form of $(S(x,t), I(x,t), R(x,t)) = (S(z), I(z), R(z)) \in C^3(\mathbb{R}, \mathbb{R}^3)$, $z = x + ct$, which satisfies

\[
\begin{align*}
\partial_t S(x,t) &= d_1 [J * S(x,t) - S(x,t)] - \frac{\beta S(x,t) K * I(x,t)}{S(x,t) + K * I(x,t) + R(x,t)}, \\
\partial_t I(x,t) &= d_2 [J * I(x,t) - I(x,t)] + \frac{\beta S(x,t) K * I(x,t)}{S(x,t) + K * I(x,t) + R(x,t)} - (\gamma + \delta)I(x,t), \\
\partial_t R(x,t) &= d_3 [J * R(x,t) - R(x,t)] + \gamma I(x,t),
\end{align*}
\]

where

\[
K * I(x,t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} K(x-y,t-s)I(y,s)dyds.
\]

with the asymptotic boundary

\[
\begin{align*}
S(-\infty) := S_1 > S(\infty) := S_2 \geq 0, \quad I(\pm \infty) = 0, \\
R(-\infty) = 0, \quad R(\infty) = \gamma(S_1 - S_2)/(\gamma + \delta),
\end{align*}
\]

where

\[
J * U(z) = \int_{-\infty}^{\infty} J(y)U(z-y)dy, \quad (U \text{ can be either } S, I \text{ or } R),
\]

\[
K * I(z) = \int_{0}^{\infty} \int_{-\infty}^{\infty} K(y,s)I(z-y-cs)dyds,
\]

$S_1$ and $S_2$ are the densities of the initial and eventual susceptible individuals, respectively.

Compared with the work on the existence of super-critical traveling waves [39–41], the method we will apply is different. To be specific, firstly we’ll construct a pair of upper-lower solutions $(S_{\pm}(z), I_{\pm}(z), R_{\pm}(z))$ of (1.8)–(1.10) on $\mathbb{R}$; secondly, we’ll introduce a functional space $B_{\mu}(\mathbb{R}, \mathbb{R}^3)$ with the decay norm and an invariant cone $\Gamma$ of $B_{\mu}(\mathbb{R}, \mathbb{R}^3)$, whose elements sandwich between the upper solution and the lower solution; thirdly, we’ll define a nonlinear operator $F$ on $\Gamma$ and prove that $F: \Gamma \mapsto \Gamma$
is a completely continuous with respect to the decay norm in $B_{µ}(\mathbb{R},\mathbb{R}^{2})$; fourthly, we’ll use Schauder’s fixed point theorem to obtain the existence of a fixed point of $F$, then the positiveness of the fixed point of $F$ is deduced to ensure the existence of a positive solution of (1.8)-(1.10); finally, we’ll utilize a subtle analysis to derive the asymptotic boundary of the solution.

As stressed in [4, 19, 38], the critical epidemic waves play a vital role in the evolution of epidemic diseases, but this issue is very challenging. In [40], Yang et al. didn’t obtain the existence of critical traveling waves for (1.6). As for (1.5), Yang et al. [38] derived the existence of critical traveling waves via a limiting argument. Their argument depends heavily on the bilinear term in (1.5). While the nonlinear term in (1.7) is a rational function and the method in [38] seems not to be applicable to (1.7). In this paper, inspired by [11], we will apply the upper-lower solution method together with Schauder’s fixed point theorem to investigate the existence of critical traveling waves for (1.7). This upper-lower solution method is concise and explicit, and has been used subsequently to other diffusive models [12, 22, 48]. We note that the research on the existence of critical traveling wave solutions to nonlocal dispersal epidemic models with spatio-temporal delay have not been seen in the literature.

In the viewpoint of mathematical biology, a nontrivial positive traveling wave solution can formulate the disease invasion process [13]. In [38, 39, 41], the authors used the contradictory arguments to obtain the positiveness of traveling wave solutions for their models. Herein we will use a different approach to prove the positiveness of the traveling wave solutions in (1.7) (see the detailed discussions in Proposition 4.1). This approach has been applied to obtain the positiveness of traveling waves for a reaction-diffusion epidemic model [18].

Now we are in a position to state our results.

**Theorem 1.1.** For any given $S_{1} > 0$, if $R_{0} := β/(γ + δ) \leq 1$ and $c \in \mathbb{R}$ or $R_{0} > 1$ and $c < c^{*}$ ($c^{*} > 0$ is defined in Lemma 2.1), then system (1.8)-(1.10) admits no nontrivial positive solutions $(S(z), I(z), R(z))$ satisfying

$$ S(−∞) = S_{1}, \sup_{z \in \mathbb{R}} S(z) \leq S_{1}, \ I(±∞) = 0, \ R(−∞) = 0, \ \sup_{z \in \mathbb{R}} R(z) < ∞. \ (1.12) $$

**Remark 1.1.** Theorem 1.1 shows that (1.7) has no nontrivial positive traveling waves in the case of $R_{0} > 1$ and $c \leq 0$, while Yang et al. [40] didn’t obtained the similar result for (1.6).

**Theorem 1.2.** For any given $S_{1} > 0$, if $R_{0} > 1$ and $c \geq c^{*}$, then system (1.8)-(1.10) admits a solution $(S(z), I(z), R(z))$ such that:

(i) $0 < S(z) < S_{1}$, $0 < I(z) < (β − γ − δ)S_{1}/(γ + δ)$, $R(z) > 0$ for $z \in \mathbb{R}$;
(ii) $(S(−∞), I(−∞), R(−∞)) = (S_{1}, 0, 0)$; if $z \to −∞$, then $I(z) = O(e^{λ_{1}z})$ for $c > c^{*}$ and $I(z) = O(−ze^{λ_{1}z})$ for $c = c^{*}$, where $λ_{1} > 0$ and $λ^{*} > 0$ are defined in Lemma 2.1.
(iii) $I(∞) = 0$; the limit $S(∞)$ exists and $S_{2} < S_{1}$;
(iv) If $R(z)$ is bounded in $\mathbb{R}$, then $R(∞) = γ(S_{1} − S_{2})/(γ + δ)$ and $S′(±∞) = I′(±∞) = R′(±∞) = 0$;
(v) $(γ + δ) \int_{\mathbb{R}} I(z)dz = β \int_{\mathbb{R}} S(z)K(z)*I(z)dz/R(z) = c(S_{1} − S_{2})$.

**Remark 1.2.** To obtain their existence and nonexistence theorems, Yang et al. [40] need the kernel function $J$ belongs to $C^{1}$ class, while in Theorems 1.1 and 1.2 we only need $J(x) \in C(\mathbb{R})$. 


Remark 1.3. Although $I(\infty) = 0$ for both $c > c^*$ and $c = c^*$, the decay rates of $I(z)$ as $z \to -\infty$ are quite different in these two cases. $I(\infty) = 0$ means that the infected individuals will vanish after a long time. Due to the deficiency of monotonicity of $S(z)$ in $\mathbb{R}$, the exact boundary of $S(\infty)$ is hard to obtain. We leave it for future investigation.

Remark 1.4. To obtain the asymptotic boundary of $R$ at plus infinity, we add an extra condition, i.e., $R(z)$ is bounded in $\mathbb{R}$. From the viewpoint of biology, this condition is reasonable. However, we can’t prove it mathematically at present. In fact, when the wave speed is large enough, one can prove the boundedness of $R(z)$ in $\mathbb{R}$, see the similar arguments in [39, Corollary 2.10].

Remark 1.5. In view of

\[
\int_\mathbb{R} I(z)dz = c(S_1 - S_2)/(\gamma + \delta),
\]

one can get that

\[
\int_\mathbb{R} I(x,t)dx = c(S_1 - S_2)/(\gamma + \delta),
\]

and

\[
\int_\mathbb{R} I(x,t)dt = (S_1 - S_2)/(\gamma + \delta).\]

These two equalities show that the total number of infected individuals at any fixed time $t \in \mathbb{R}$ depends on the wave speed, the initial and eventual states of the susceptible individuals and the recovery and death rates; but it is independent of wave speed at any fixed location $x \in \mathbb{R}$. In these two cases, the larger recovery and death rates yield the less observed infected individuals.

The rest of the paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we establish the nonexistence of nontrivial positive traveling wave solutions for (1.7) based on the contradictory arguments and the two-sided Laplace transform. In Section 4, we prove the existence of super-critical traveling wave solutions for (1.7) by the upper-lower solution method and Schauder’s fixed point theorem. In Section 5, we construct a delicately designed pair of upper and lower solutions of (1.8)-(1.10). Then we obtain the existence of critical traveling waves in (1.7) by the similar way as that for the super-critical traveling waves.

2. Preliminaries. In this section, we state some lemmas which are crucial for proving our main results.

Lemma 2.1. Assume that $R_0 > 1$ and denote

\[
\Theta(\lambda, c) := d_2 \int_{-\infty}^\infty J(y)e^{-\lambda y}dy - d_2 - c\lambda
\]

\[
+ \beta \int_0^\infty \int_{-\infty}^\infty K(y,s)e^{-\lambda(y+cs)}dyds - \gamma - \delta, (2.1)
\]

then there exists a positive pair $(\lambda^*, c^*)$ such that

\[
\Theta(\lambda^*, c^*) = -d_2 \int_{-\infty}^\infty yJ(y)e^{-\lambda^* y}dy - d_2 - c^*\lambda^*
\]

\[
+ \beta \int_0^\infty \int_{-\infty}^\infty K(y,s)e^{-\lambda^*(y+cs)}dyds - \gamma - \delta = 0 (2.2)
\]

and

\[
\Theta(\lambda, c^*) = -d_2 \int_{-\infty}^\infty yJ(y)e^{-\lambda y}dy - c^*
\]

\[
- \beta \int_0^\infty \int_{-\infty}^\infty (y + c^*s)K(y,s)e^{-\lambda(y+cs)}dyds = 0. (2.3)
\]

Furthermore,

(i) if $0 < c < c^*$, then $\Theta(\lambda, c) > 0$ for $\lambda \in [0, \infty)$;
Lemma 2.3. Let

\[ \int_{-\infty}^{\infty} sK(y, s) e^{-\lambda(y+cs)} dy < 0, \quad \forall \lambda > 0, \]

\[ \Theta(\lambda, 0) = d_2 \int_{-\infty}^{\infty} J(y) (e^{-\lambda y} - 1) dy + \beta \int_{-\infty}^{\infty} K(y, s) e^{-\lambda y} dy - \gamma - \delta \]

\[ \Theta_{\lambda}(0, c) = -c - \beta \int_{-\infty}^{\infty} (y + cs) K(y, s) dyds \]

\[ \Theta_{\lambda\lambda}(\lambda, c) = d_2 \int_{-\infty}^{\infty} y^2 J(y) e^{-\lambda y} dy + \beta \int_{-\infty}^{\infty} (y + cs)^2 K(y, s) e^{-\lambda(y+cs)} dyds > 0. \]

The assertions of this lemma follow immediately from the above computations. 

Lemma 2.2. Let

\[ \Delta(\lambda, c) := c\lambda - d_1 \int_{-\infty}^{\infty} J(y) e^{-\lambda y} dy + d_3. \quad (2.4) \]

Then for each fixed \( c > 0 \), there exists some \( \hat{\lambda} > 0 \) such that \( \Delta(\lambda, c) > 0 \) for any \( \lambda \in (0, \hat{\lambda}) \).

Proof. A direct computation yields that

\[ \Delta(0, c) = 0, \quad \Delta_{\lambda}(0, c) = c > 0, \quad \Delta_{\lambda\lambda}(\lambda, c) = -d_3 \int_{-\infty}^{\infty} y^2 J(y) e^{\lambda y} dy < 0 \]

and

\[ \Delta(\infty, c) = -\infty \]

for each fixed \( c > 0 \). Then the result follows. 

Lemma 2.3. Assume that \( (S(z), I(z), R(z)) \in C^1(\mathbb{R}, \mathbb{R}^3) \) is a nontrivial positive solution of (1.8)-(1.10) satisfying

\[ S(-\infty) = S_1, \quad \sup_{z \in \mathbb{R}} S(z) \leq S_1, \quad I(\pm \infty) = 0, \quad R(-\infty) = 0, \quad \sup_{z \in \mathbb{R}} R(z) < \infty. \quad (2.5) \]

Then

\[ \int_{-\infty}^{\infty} \frac{S(z) K * I(z)}{S(z) + K * I(z) + R(z)} dz < \infty, \quad (2.6) \]

\[ \int_{-\infty}^{\infty} I(z) dz < \infty, \quad (2.7) \]

\[ \int_{-\infty}^{\infty} [J * I(z) - I(z)] dz < \infty \quad (2.8) \]
and
\[ \int_{-\infty}^{\infty} K \ast I(z) \, dz < \infty. \] (2.9)

Proof. Integrating (1.8) from \( \xi \) to \( \eta \) \((\eta > \xi)\) yields
\[
\int_{\xi}^{\eta} \frac{S(z)K \ast I(z)}{S(z) + K \ast I(z) + R(z)} \, dz =
\begin{align*}
\beta & \int_{\xi}^{\eta} J(y)[S(z - y) - S(z)] \, dy \, dz + cS(\xi) - cS(\eta) \\
& = -d_1 \int_{\xi}^{\eta} \int_{-R_1}^{R_1} yJ(y) \left[ S'(z - \theta y) \, d\theta \right] dy \, dz + cS(\xi) - cS(\eta) \quad \text{(by (H1))}
\end{align*}
\]
(by Fubini’s theorem)
\[
\leq 2d_1 S_1 \int_{-R_1}^{R_1} |y|J(y) \, dy + 2|c|S_1, \quad \text{(by (2.5))},
\]
which together with the positiveness of \((S(z), I(z), R(z))\) implies that
\[ \int_{-\infty}^{\infty} \frac{S(z)K \ast I(z)}{S(z) + K \ast I(z) + R(z)} \, dz < \infty. \] (2.10)

Since \(I(\pm \infty) = 0\) and \(I(z) \in C^1(\mathbb{R})\) is nontrivial and positive, there exists a positive constant \(M\) such that \(|I(z)| \leq M\) for \(z \in \mathbb{R}\). Integrating (1.9) from \( \tau \) to \( \zeta \) \((\zeta > \tau)\) gives
\[
(\gamma + \delta) \int_{\tau}^{\zeta} I(z) \, dz = d_2 \int_{-R_1}^{R_1} yJ(y) \int_{0}^{1} [I(\tau - \theta y) - I(\zeta - \theta y)] \, d\theta \, dy \\
\quad + \int_{\tau}^{\zeta} \beta S(z)K \ast I(z) \, dz + cI(\tau) - cI(\zeta) \\
\leq 2d_2 M \int_{-R_1}^{R_1} |y|J(y) \, dy + \int_{-\infty}^{\infty} \beta S(z)K \ast I(z) \, dz + 2|c|M.
\]
This combined with (2.10) leads to
\[ \int_{-\infty}^{\infty} I(z) \, dz < \infty. \] (2.11)

It follows from (2.10), (2.11), (1.9) and \(I(\pm \infty) = 0\) that
\[ \int_{-\infty}^{\infty} [J \ast I(z) - I(z)] \, dz < \infty. \] (2.12)
Note that
\[ \int_{0}^{\infty} [K * I(z) - I(z)] dz \]
\[ = \int_{0}^{\infty} \int_{0}^{\infty} K(y, s) [I(z - y - cs) - I(z)] dy ds dz \]
\[ = \int_{0}^{\infty} \int_{0}^{T} \int_{R_{z}}^{R_{z}} (y + cs) K(y, s) \int_{0}^{1} \frac{I'(z - \theta(y + cs)) d\theta dy ds dz}{dy ds dz} \quad (\text{by (H3)}) \]
\[ = \int_{0}^{\infty} \int_{R_{z}}^{R_{z}} (y + cs) K(y, s) \int_{0}^{1} \left[ I(\theta - \theta(y + cs)) - I(\zeta - \theta(y + cs)) \right] d\theta dy ds \]
(by Fubini’s theorem)
\[ \leq 2M \int_{0}^{T} \int_{R_{z}}^{R_{z}} |y + cs| K(y, s) dy ds \]
for any \( \theta, \zeta \in \mathbb{R} \), which ensures that \( \int_{\mathbb{R}} [K * I(z) - I(z)] dz < \infty \). Then we obtain from (2.11) that \( \int_{\mathbb{R}} K * I(z) dz < \infty \). The proof of this lemma is finished. \( \square \)

3. Nonexistence of traveling waves. In this section, we will prove the Theorem 1.1 by the contradictory arguments. Suppose that \((S(z), I(z), R(z)) \in C^{1}(\mathbb{R}, \mathbb{R}^{3})\) is a nontrivial positive solution of (1.8)-(1.10) satisfying (1.12).

**Case 1:** \( R_{0} \leq 1 \) and \( c \in \mathbb{R} \). Integrating (1.9) over \( \mathbb{R} \), we have from Lemma 2.3 and \( I(\pm \infty) = 0 \) that
\[ (\gamma + \delta) \int_{-\infty}^{\infty} I(z) dz = \]
\[ = d_{2} \int_{-\infty}^{\infty} [J * I(z) - I(z)] dz + \int_{-\infty}^{\infty} \frac{\beta S(z) K * I(z)}{S(z) + K * I(z) + R(z)} dz \]
\[ = \int_{-\infty}^{\infty} \frac{\beta S(z) K * I(z)}{S(z) + K * I(z) + R(z)} dz \quad (\text{by Fubini’s theorem and (H1)}) \]
\[ < \beta \int_{-\infty}^{\infty} K * I(z) dz \quad (\text{since } S(z) > 0, I(z) > 0 \text{ and } R(z) > 0 \text{ in } \mathbb{R}) \]
\[ \leq (\gamma + \delta) \int_{-\infty}^{\infty} I(z) dz, \quad (\text{by Fubini’s theorem, (H2) and } R_{0} \leq 1), \]
which leads to a contradiction.

**Case 2:** \( R_{0} > 1 \) and \( 0 < c < c^{*} \). From (1.12) and (H3), we have
\[ \lim_{z \to -\infty} \frac{\beta S(z)}{S(z) + K * I(z) + R(z)} = \beta. \] \( (3.2) \)
Due to \( R_{0} > 1 \) and (3.2), there exists a constant \( z^{*} \ll 0 \) such that
\[ \frac{\beta S(z)}{S(z) + K * I(z) + R(z)} > \frac{\beta + \gamma + \delta}{2} \quad \text{for } z < z^{*}. \] \( (3.3) \)
Then for \( z < z^{*} \), it follows from (1.9) and (3.3) that
\[ cI'(z) \geq d_{2}[J * I(z) - I(z)] + \frac{\beta + \gamma + \delta}{2}[K * I(z) - I(z)] + \frac{\beta - \gamma - \delta}{2} I(z). \] \( (3.4) \)
By (2.7), we define $Q(z) := \int_{-\infty}^{\xi} I(\eta)d\eta$ in $\mathbb{R}$. Integrating (3.4) from $-\infty$ to $z$ with $z < z^*$, we get

$$\frac{\beta - \gamma - \delta}{2} Q(z) \leq cI(z) - d_2 \int_{-\infty}^{\xi} [J * Q(z) - Q(z)] d\eta - \frac{\beta + \gamma + \delta}{2} \int_{-\infty}^{\xi} [K * Q(z) - Q(z)] d\eta,$$

(3.5)

where we have used (2.8), (2.9), Fubini’s theorem and $I(-\infty) = 0$. By Lebesgue’s dominated convergence theorem, Fubini’s theorem and $Q(-\infty) = 0$, we derive

$$\int_{-\infty}^{\xi} [J * Q(\eta) - Q(\eta)] d\eta = \lim_{\xi \to -\infty} \int_{\xi}^{\xi} \int_{-\infty}^{\infty} J(y)[Q(\eta - y) - Q(\eta)] dyd\eta$$

$$= \lim_{\xi \to -\infty} - \int_{\xi}^{\xi} \int_{-\infty}^{\infty} yJ(y) \int_{0}^{1} Q'(\eta - \theta y) d\theta dyd\eta$$

$$= \lim_{\xi \to -\infty} - \int_{\xi}^{\xi} \int_{-\infty}^{\infty} yJ(y) \int_{0}^{1} [Q(\eta - y) - Q(\xi - \theta y)] d\theta dy$$

$$= - \int_{-\infty}^{\xi} yJ(y) \int_{0}^{1} Q(z - \theta y) d\theta dy$$

(3.6)

and

$$\int_{-\infty}^{\xi} [K * Q(\eta) - Q(\eta)] d\eta$$

$$= \lim_{\xi \to -\infty} \int_{\xi}^{\xi} \int_{0}^{1} \int_{-\infty}^{\infty} K(y,s)[Q(\eta - y - cs) - Q(\eta)] dyd\eta dsd\eta$$

$$= \lim_{\xi \to -\infty} - \int_{\xi}^{\xi} \int_{0}^{1} \int_{-\infty}^{\infty} (y + cs)K(y,s) \int_{0}^{1} Q'(\eta - \theta(y + cs)) d\theta dyd\eta ds d\eta$$

$$= - \int_{0}^{\xi} \int_{-\infty}^{\infty} (y + cs)K(y,s) \int_{0}^{1} Q(z - \theta(y + cs)) d\theta dy ds,$$

(3.7)

which implies that $J * Q(z) - Q(z)$ and $K * Q(z) - Q(z)$ are integrable on $(-\infty, z]$ for any $z \in \mathbb{R}$. Then integrating (3.5) from $-\infty$ to $z$ with $z < z^*$, we have

$$\frac{\beta - \gamma - \delta}{2} \int_{-\infty}^{\xi} Q(\eta)d\eta$$

$$\leq cQ(z) - d_2 \int_{-\infty}^{\xi} [J * Q(\eta) - Q(\eta)] d\eta - \frac{\beta + \gamma + \delta}{2} \int_{-\infty}^{\xi} [K * Q(\eta) - Q(\eta)] d\eta$$

$$= cQ(z) + d_2 \int_{-\infty}^{\xi} yJ(y) \int_{0}^{1} Q(z - \theta y) d\theta dy$$

$$+ \frac{\beta + \gamma + \delta}{2} \int_{0}^{\xi} \int_{-\infty}^{\infty} (y + cs)K(y,s) \int_{0}^{1} Q(z - \theta(y + cs)) d\theta dy ds.$$

(3.8)
Since $xQ(z - \theta x)$ is nonincreasing with respect to $\theta \in [0, 1]$, we deduce from (3.8) that
\[
\frac{\beta - \gamma - \delta}{2} \int_{-\infty}^z Q(\eta)d\eta \\
\leq \left[ c + d_2 \int_{-\infty}^\infty y J(y)dy + \frac{\beta + \gamma + \delta}{2} \int_0^\infty \int_{-\infty}^\infty (y + cs)K(y, s)dyds \right] Q(z) \\
= \left[ c + \frac{\beta + \gamma + \delta}{2} \int_0^T \int_{-R_2}^{R_2} sK(y, s)dyds \right] Q(z) \quad \text{(by (H1)-(H3))} \\
= C_0 Q(z) \quad \text{for } z < z^*,
\]
where $C_0 := c + \frac{\beta + \gamma + \delta}{2} \int_0^T \int_{-R_2}^{R_2} sK(y, s)dyds$. Due to $Q(\cdot)$ is nondecreasing, we obtain for $z < z^*$ and any $\eta > 0$ that $\frac{\beta - \gamma - \delta}{2} \eta Q(z - \eta) \leq C_0 Q(z)$. Hence there exist a large enough $\eta_0 > 0$ and a constant $\kappa \in (0, 1)$ such that
\[
Q(z - \eta) \leq \kappa Q(z) \quad \text{for } z < z^*.
\]
Define $\mu_0 := \frac{1}{\eta_0} \ln \frac{1}{\kappa} > 0$ and $L(z) := Q(z)e^{-\mu_0 z}$. Then it gives
\[
L(z - \eta_0) = Q(z - \eta_0)e^{-\mu_0(z - \eta_0)} \leq \kappa Q(z)e^{-\mu_0 z}e^{\mu_0 \eta_0} = L(z) \quad \text{for } z < z^*.
\]
This together with $L(z) \geq 0$ in $\mathbb{R}$ implies that $L(-\infty)$ exists. It follows from (2.7) that
\[
\lim_{z \to \infty} L(z) = \lim_{z \to \infty} Q(z)e^{-\mu_0 z} = 0.
\]
So there exists some positive constant $L_0$ such that
\[
\sup_{z \in \mathbb{R}} \{Q(z)e^{-\mu_0 z}\} \leq L_0. \quad (3.9)
\]
Then a direct computation yields that
\[
[J * Q(z)]e^{-\mu_0 z} = \int_{-\infty}^\infty J(y)e^{-\mu_0 y}Q(z - y)e^{-\mu_0 (z - y)}dy \\
\leq L_0 \int_{-R_1}^{R_1} J(y)e^{-\mu_0 y}dy \quad \text{(by (H1) and (3.9))} \\
< \infty
\]
and
\[
[K * Q(z)]e^{-\mu_0 z} \\
= \int_0^\infty \int_{-\infty}^\infty K(y, s)e^{-\mu_0 (y + cs)}Q(z - y - cs)e^{-\mu_0 (z - y - cs)}dyds \\
\leq L_0 \int_0^T \int_{-R_2}^{R_2} K(y, s)e^{-\mu_0 (y + cs)}dyds \quad \text{(by (H3) and (3.9))} \\
< \infty \quad \text{for } z \in \mathbb{R}.
\]
It follows from (1.9) that
\[
cI'(z) \leq d_2 [J * I(z) - I(z)] + \beta K * I(z) - (\gamma + \delta)I(z). \quad (3.12)
\]
Integration (3.12) from $-\infty$ to $z$, utilizing Fubini’s theorem and $I(-\infty) = 0$, we have
\[
cI(z) \leq d_2 [J * Q(z) - Q(z)] + \beta K * Q(z) - (\gamma + \delta)Q(z). \quad (3.13)
\]
Thus we infer from (3.9)-(3.11) and (3.13) that
\[
\sup_{z \in \mathbb{R}} \{I(z)e^{-\mu_0 z}\} < \infty. \tag{3.14}
\]

From (3.14), we deduce
\[
[J * I(z)]e^{-\mu_0 z} = \int_{-\infty}^{\infty} J(y)e^{-\mu_0 y} I(z - y)e^{-\mu_0 (z-y)} dy \\
\leq \sup_{z \in \mathbb{R}} \{I(z)e^{-\mu_0 z}\} \int_{-R_1}^{R_1} J(y)e^{-\mu_0 y} dy < \infty
\]
and
\[
[K * I(z)]e^{-\mu_0 z} = \int_{0}^{\infty} \int_{-\infty}^{\infty} K(y, s)e^{-\mu_0 (y+cs)} I(z - y - cs)e^{-\mu_0 (z-y-cs)} dy ds \\
\leq \sup_{z \in \mathbb{R}} \{I(z)e^{-\mu_0 z}\} \int_{0}^{T} \int_{-R_2}^{R_2} K(y, s)e^{-\mu_0 (y+cs)} dy ds < \infty,
\]
which ensures that
\[
\sup_{z \in \mathbb{R}} \{[J * I(z)]e^{-\mu_0 z}\} < \infty \quad \text{and} \quad \sup_{z \in \mathbb{R}} \{[K * I(z)]e^{-\mu_0 z}\} < \infty. \tag{3.15}
\]

By (3.12), (3.14) and (3.15), we obtain
\[
\sup_{z \in \mathbb{R}} \{I'(z)e^{-\mu_0 z}\} < \infty. \tag{3.16}
\]

Let \(\nu(z) \in C^\infty(\mathbb{R}, [0, 1])\) be a nondecreasing function satisfying \(\nu(z) = 0\) on \((-\infty, -2]\) and \(\nu(z) = 1\) on \([-1, \infty)\). For \(N \in \mathbb{N}\), set \(\nu_N(z) = \nu(\frac{z}{N})\). Multiplying (1.10) by \(e^{-\nu z} \nu_N(z)\) and integrating the resultant equation over \(\mathbb{R}\), we have
\[
c \int_{-\infty}^{\infty} R'(z)e^{-\nu z} \nu_N(z) dz = c \int_{-\infty}^{\infty} [J * R(z) - R(z)]e^{-\nu z} \nu_N(z) dz \\
+ \gamma \int_{-\infty}^{\infty} I(z)e^{-\nu z} \nu_N(z) dz. \tag{3.17}
\]

An elementary computation yields
\[
\int_{-\infty}^{\infty} J * R(z)e^{-\nu z} \nu_N(z) dz \\
= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} J(z - y)R(y)dy \right] e^{-\nu z} \nu_N(z) dz \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(y)e^{-\nu y} J(z)e^{-\nu z} \nu_N(z + y) dy dz \\
\leq \int_{-\infty}^{\infty} I(z)e^{-\nu z} dz \int_{-\infty}^{\infty} R(y)e^{-\nu y} dy \quad \text{(since } \nu_N(z) \leq 1 \text{ on } \mathbb{R}) \tag{3.18}
\]
and
\[
\int_{-\infty}^{\infty} R'(z)e^{-\nu z} \nu_N(z) dz = \nu \int_{-\infty}^{\infty} R(z)e^{-\nu z} \nu_N(z) dz - \int_{-\infty}^{\infty} R(z)e^{-\nu z} \nu_N'(z) dz. \tag{3.19}
\]
It follows from (3.17)-(3.19) that
\[
(cv + d_3) \int_{-\infty}^{\infty} R(z)e^{-\nu z}v_N(z)dz - c \int_{-\infty}^{\infty} R(z)e^{-\nu z}v'_N(z)dz - d_3 \int_{-\infty}^{\infty} J(y)e^{-\nu y}dy \int_{-\infty}^{\infty} R(z)e^{-\nu z}dz \leq \gamma \int_{-\infty}^{\infty} I(z)e^{-\nu z}v_N(z)dz.
\]
(3.20)

In view of
\[
\Delta(\nu, c) = cv - d_3 \int_{-\infty}^{\infty} J(y)e^{-\nu y}dy + d_3 > 0, \quad \nu \in (0, \lambda)
\]
and passing to the limits in (3.20) as \(N \to \infty\), we obtain
\[
\int_{-\infty}^{\infty} R(z)e^{-\nu z}dz \leq \frac{\gamma}{\Delta(\nu, c)} \int_{-\infty}^{\infty} I(z)e^{-\nu z}dz \quad \text{for} \ \nu \in (0, \lambda).
\]
Hence we derive that
\[
\int_{-\infty}^{\infty} R(z)e^{-\nu z}dz < \infty
\]
(3.21)
for some \(\nu \in (0, \bar{\mu})\), where \(\bar{\mu} := \min\{\mu_0, \lambda\}\). Then we obtain from (3.15) and (3.21) that
\[
\int_{-\infty}^{\infty} e^{-\lambda z} \beta K * I(z)[K * I(z) + R(z)] dz < \infty
\]
for \(\lambda \in (0, \mu_0 + \bar{\mu})\). Moreover, (1.9) is equivalent to
\[
d_2[J * I(z) - I(z)] - cI'(z) + \beta K * I(z) - (\gamma + \delta)I(z) = \frac{\beta[K * I(z) + R(z)]K * I(z)}{S(z) + K * I(z) + R(z)}.
\]
(3.22)

For \(\lambda \in \mathbb{C}\) with \(0 < \text{Re}\lambda < \mu_0\), define the two-sided Laplace transform of \(I(z)\) by \(\mathcal{L}(\lambda) := \int_{\mathbb{R}} e^{-\lambda z}I(z)dz\). Noting the facts that
\[
\int_{-\infty}^{\infty} e^{-\lambda z}J * I(z)dz = \mathcal{L}(\lambda) \int_{-\infty}^{\infty} e^{-\lambda y}J(y)dy,
\]
\[
\int_{-\infty}^{\infty} e^{-\lambda z}K * I(z)dz = \mathcal{L}(\lambda) \int_{0}^{\infty} \int_{-\infty}^{\infty} e^{-\lambda(y+cs)}K(y, s)dyds
\]
and taking the two-sided Laplace transform on (3.22), we derive
\[
\Theta(\lambda, c)\mathcal{L}(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda z} \frac{\beta[K * I(z) + R(z)]K * I(z)}{S(z) + K * I(z) + R(z)} dz
\]
(3.23)
for \(\lambda \in \mathbb{C}\) with \(0 < \text{Re}\lambda < \mu_0\).

The property of Laplace transform [34] asserts that either there exists a real number \(\mu_0\) such that \(\mathcal{L}(\lambda)\) is analytic for \(\lambda \in \mathbb{C}\) with \(0 < \text{Re}\lambda < \mu_0\) and \(\lambda = \mu_0\) is singular point of \(\mathcal{L}(\lambda)\), or for \(\lambda \in \mathbb{C}\) with \(\text{Re}\lambda > 0\), \(\mathcal{L}(\lambda)\) is well defined. For (3.23), we claim that the two Laplace integrals can be analytically continued to the whole right half-plane. Otherwise the integral on the left-hand side of (3.23) has a singularity at \(\lambda = \mu_0\) and it is analytic for all \(\lambda\) with \(\text{Re}\lambda < \mu_0\). However, the integral on the right-hand side of (3.23) is actually analytic for all \(\lambda\) with \(\text{Re}\lambda < \mu_0 + \bar{\mu}\), a contradiction. Thus (3.23) holds for all \(\lambda\) with \(\text{Re}\lambda > 0\). While Lemma 2.1 implies that \(\Theta(\infty, c) = 0\) for \(R_0 > 1\) and \(c \in (0, c^*)\). Another contradiction appears.
Case 3: $R_0 > 1$ and $c \leq 0$. Integrating (3.4) from $-\infty$ to $z \, (z < z^*)$ twice, we deduce that
\[0 \geq cQ(z) \quad \text{(since } c \leq 0 \text{ and } Q(z) > 0 \text{ on } \mathbb{R})\]
\[\geq d_2 \int_{-\infty}^{z} [J \ast Q(\eta) - Q(\eta)]d\eta + \frac{\beta + \gamma + \delta}{2} \int_{-\infty}^{z} [K \ast Q(\eta) - Q(\eta)]d\eta\]
\[+ \frac{\beta - \gamma - \delta}{2} \int_{-\infty}^{z} Q(\eta)d\eta \quad \text{(by Fubini’s theorem and } I(-\infty) = 0)\]
\[= -d_2 \int_{-\infty}^{\infty} yJ(y) \int_{0}^{1} Q(z - \theta y)d\theta dy\]
\[- \frac{\beta + \gamma + \delta}{2} \int_{0}^{1} \int_{-\infty}^{\infty} (y + cs)K(y, s) \int_{0}^{1} Q(z - \theta(y + cs))d\theta ds\]
\[+ \frac{\beta - \gamma - \delta}{2} \int_{-\infty}^{z} Q(\eta)d\eta \quad \text{(by (3.6) and (3.7))}\]
\[\geq \left[ -d_2 \int_{-\infty}^{\infty} yJ(y)dy - \frac{\beta + \gamma + \delta}{2} \int_{0}^{1} \int_{-\infty}^{\infty} (y + cs)K(y, s)d\theta ds \right] Q(z)\]
\[+ \frac{\beta - \gamma - \delta}{2} \int_{-\infty}^{z} Q(\eta)d\eta \quad \text{(since } -xQ(z - \theta x) \text{ is nondecreasing with respect to } \theta \in [0, 1])\]
\[= -c^2 \frac{\beta + \gamma + \delta}{2} Q(z) \int_{0}^{T} \int_{-R_2}^{R_2} sK(y, s)dy ds + \frac{\beta - \gamma - \delta}{2} \int_{-\infty}^{z} Q(\eta)d\eta\]
\[\text{(by (H1)-(H3))}\]
\[> 0, \quad \text{(since } Q(z) > 0 \text{ on } \mathbb{R}, R_0 > 1 \text{ and } c \leq 0),\]
which leads to a contradiction. Combining the above three cases, we obtain the assertions in Theorem 1.1.

4. **Super-critical traveling waves.** This section is devoted to proving the existence result for the case of $R_0 > 1$ and $c > c^*$ in Theorem 1.2. For this, we define the following nonnegative continuous functions on the real line.

\[S_+ (z) := S_1, \quad I_+ (z) := \begin{cases} e^{\lambda_1 z}, & z < z_1, \\ \frac{(\beta - \gamma - \delta)S_1}{\gamma + \delta}, & z \geq z_1, \end{cases}\]
\[R_+ (z) := L_1 e^{\epsilon_1 z}, \quad S_- (z) := \begin{cases} S_1 - \epsilon_2^{-1} e^{\epsilon_2 z}, & z < z_2, \\ 0, & z \geq z_2, \end{cases}\]
\[I_- (z) := \begin{cases} e^{\lambda_2 z} - L_2 e^{(\lambda_2 + \epsilon_1) z}, & z < z_3, \\ 0, & z \geq z_3, \end{cases} \quad R_- (z) := 0,\]
where
\[z_1 = \frac{1}{\lambda_1} \ln \left( \frac{\beta - \gamma - \delta}{\gamma + \delta} S_1 \right), \quad z_2 = \frac{1}{\epsilon_2} \ln (\epsilon_2 S_1), \quad z_3 = \frac{1}{\epsilon_1} \ln \frac{1}{L_2},\]
$\lambda_1$ is defined in Lemma 2.1, $L_1, L_2, \epsilon_1$ and $\epsilon_2$ are positive constants to be determined in the following lemma.
Lemma 4.1. For given small enough $c_1 > 0$ and large enough $L_i > 1$ ($i = 1, 2$), the functions $S_\pm(z), I_\pm(z)$ and $R_\pm(z)$ satisfy
\[
d_1[j * S_+(z) - S_+(z)] \leq cS'_+(z) \leq \beta S_+(z)K * I_-(z) / S_+(z) + K * I_+(z) + R_+(z) \quad (4.1)
\]
\[
d_2[j * I_+(z) - I_+(z)] \leq cI'_+(z) + \beta S_+(z)K * I_+(z) / S_+(z) + K * I_+(z) + R_+(z) \quad (4.2)
\]
\[
d_3[j * R_+(z) - R_+(z)] \leq cR'_+(z) + \beta S_+(z)K * I_+(z) / S_+(z) + K * I_+(z) + R_+(z) \quad (4.3)
\]
\[
d_1[j * S_-(z) - S_-(z)] \leq cS'_-(z) \leq \beta S_-(z)K * I_+(z) / S_-(z) + K * I_+(z) + R_+(z) \quad (4.4)
\]
\[
d_2[j * I_-(z) - I_-(z)] \leq cI'_-(z) + \beta S_-(z)K * I_+(z) / S_-(z) + K * I_+(z) + R_+(z) \quad (4.5)
\]
\[
d_3[j * R_-(z) - R_-(z)] \leq cR'_-(z) + \beta S_-(z)K * I_+(z) / S_-(z) + K * I_+(z) + R_+(z) \quad (4.6)
\]

Proof. Proof of (4.1) and (4.6). By the definitions of $S_+(z), I_-(z)$ and $R_\pm(z)$ in $\mathbb{R}$, one can see that (4.1) and (4.6) hold obviously.

Proof of (4.2). From the definition of $I_+(z)$ and the assumptions (H1), (H2), we get for $z \in \mathbb{R}$ that
\[
J * I_+(z) \leq \min \left\{ e^{\lambda_1z} \int_{-\infty}^{\infty} J(y) e^{-\lambda_1 y} dy, \frac{\beta - \gamma - \delta)S_1}{\gamma + \delta} \right\} \quad (4.7)
\]
and
\[
K * I_+(z) \leq \min \left\{ e^{\lambda_1z} \int_{0}^{\infty} \int_{-\infty}^{\infty} K(y, s) e^{-\lambda_1(y + cs)} dy ds, \frac{\beta - \gamma - \delta)S_1}{\gamma + \delta} \right\} \quad (4.8)
\]
If $z < z_1$, then $I_+(z) = e^{\lambda_1z}$. We have for $z < z_1$ that
\[
d_2[j * I_+(z) - I_+(z)] \leq cI'_+(z) + \beta S_+(z)K * I_+(z) / S_+(z) + K * I_+(z) + R_+(z) \quad (4.9)
\]
\[
\leq d_2 e^{\lambda_1z} \left[ \int_{-\infty}^{\infty} J(y) e^{-\lambda_1 y} dy - 1 \right] + \beta e^{\lambda_1z} \int_{0}^{\infty} \int_{-\infty}^{\infty} K(y, s) e^{-\lambda_1(y + cs)} dy ds
\]
\[
- c\lambda_1 e^{\lambda_1z} - (\gamma + \delta) e^{\lambda_1z} \quad (by (4.7) and (4.8))
\]
\[
= e^{\lambda_1z} \Theta_1(\Lambda_1, c) = 0, \quad (by Lemma 2.1).
\]
If $z > z_1$, then $I_+(z) = \frac{\beta - \gamma - \delta)S_1}{\gamma + \delta}, S_+(z) = S_1$ and $R_-(z) = 0$. Using (4.7) and (4.8), we get for $z > z_1$ that
\[
d_2[j * I_+(z) - I_+(z)] \leq cI'_+(z) + \beta S_+(z)K * I_+(z) / S_+(z) + K * I_+(z) + R_+(z) \quad (4.10)
\]
\[
\leq \frac{\beta S_1}{S_1 + \frac{\beta - \gamma - \delta)S_1}{\gamma + \delta} - (\gamma + \delta) \frac{\beta - \gamma - \delta)S_1}{\gamma + \delta} = 0.
\]
Lemma 2.2 that obtain from Lemma 2.2 that

If \( z < z_1 \), we derive from (4.9) that

\[
d_3[J * R_+(z) - R_+(z)] - \gamma I_+(z)
= d_3 \left[ L_1 e^{\epsilon_1 z} \int_{-\infty}^{\infty} J(y) e^{-\epsilon_1 y} dy - L_1 e^{\epsilon_1 z} \right] - cL_1 e^{\epsilon_1 z} + \gamma e^{\lambda_1 z}
= L_1 e^{\epsilon_1 z} \left[ d_3 \int_{-\infty}^{\infty} J(y) (e^{-\epsilon_1 y} - 1) dy - \gamma c e^{(\lambda_1 - \epsilon_1) z} \right]
\leq L_1 e^{\epsilon_1 z} \left[ \frac{\gamma}{L_1} e^{(\lambda_1 - \epsilon_1) z} - \Delta(\epsilon_1, c) \right] \quad \text{(by choosing 0 < \( \epsilon_1 < \min\{\lambda_1, \hat{\lambda}\} \)}
\leq 0, \quad \text{(by selecting large enough} \quad L_1 > 1).}

If \( z \geq z_1 \), then \( I_+(z) = \frac{(\beta - \gamma - \delta)S_1}{\gamma + \delta} \) and \( R_+(z) = L_1 e^{\epsilon_1 z} \). For \( z \geq z_1 \), we derive from Lemma 2.2 that

\[
d_3[J * R_+(z) - R_+(z)] - cR_+(z) + \gamma I_+(z)
= d_3 \left[ L_1 e^{\epsilon_1 z} \int_{-\infty}^{\infty} J(y) e^{-\epsilon_1 y} dy - L_1 e^{\epsilon_1 z} \right] - cL_1 e^{\epsilon_1 z} + \gamma \left( \frac{\beta - \gamma - \delta)S_1}{\gamma + \delta} \right)
= L_1 e^{\epsilon_1 z} \left[ d_3 \int_{-\infty}^{\infty} J(y) (e^{-\epsilon_1 y} - 1) dy - \gamma c e^{(\lambda_1 - \epsilon_1) z} \right]
\leq L_1 e^{\epsilon_1 z} \left[ \frac{\gamma(\beta - \gamma - \delta)S_1}{L_1(\gamma + \delta)} e^{-\epsilon_1 z} - \Delta(\epsilon_1, c) \right] \quad \text{(by selecting 0 < \( \epsilon_1 < \min\{\lambda_1, \hat{\lambda}\} \)}
\leq 0 \quad \text{(by choosing large enough} \quad L_1 > 1).}

Proof of (4.4). By the definition of \( S_-(z) \) and (H1), we obtain for \( z \in \mathbb{R} \) that

\[
J * S_-(z) \geq \max \left\{ S_1 - \epsilon_2^{-1} e^{\epsilon_2 z} \int_{-\infty}^{\infty} J(y) e^{-\epsilon_2 y} dy, 0 \right\}.
\]

Select \( \epsilon_2 \in (0, \lambda_1) \) small enough such that

\[
d_1 \epsilon_2^{-1} \int_{-\infty}^{\infty} J(y)(1 - e^{-\epsilon_2 y}) dy + c
- \beta e^{(\lambda_1 - \epsilon_2) z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(y, s) e^{-\lambda_1(y+s)} dy ds \geq 0, \quad z < z_2.
\]

If \( z < z_2 \), then

\[
S_-(z) = S_1 - e^{\epsilon_2 z}.
\]

For \( z < z_2 \), it follows that

\[
d_1[J * S_-(z) - S_-(z)] - cS'_-(z) - \frac{\beta S_-(z) K * I_+(z)}{S_-(z) + K * I_+(z) + R_-(z)}
\geq d_1[J * S_-(z) - S_-(z)] - cS'_-(z) - \beta K * I_+(z)
\geq d_1 \left[ \epsilon_2^{-1} e^{\epsilon_2 z} - \epsilon_2^{-1} e^{\epsilon_2 z} \int_{-\infty}^{\infty} J(y) e^{-\epsilon_2 y} dy \right] + c e^{\epsilon_2 z}
- \beta e^{\lambda_1 z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(y, s) e^{-\lambda_1(y+s)} dy ds \quad \text{(by (4.8), (4.9) and (4.11))}
\]
\[ \begin{align*}
&= e^{\epsilon z} \left[ d_1 e_2^{-1} \int_{-\infty}^{\infty} J(y)(1 - e^{-\epsilon z})dy + c \right. \\
&\left. - \beta e^{(\lambda_2 - \epsilon_2)} \int_{0}^{\infty} \int_{-\infty}^{\infty} K(y, s)e^{-\lambda_1(y+cs)}dyds \right] \geq 0 \text{ (by (4.10)).}
\end{align*} \]

If \( z > z_2 \), then \( S_-(z) = 0 \) and (4.4) holds trivially.

Proof of (4.5). From (H1), (H2) and the definition of \( I_-(z) \), we deduce for \( z \in \mathbb{R} \) that

\[ J \ast I_-(z) \geq \max \left\{ e^{\lambda_1 z} \int_{-\infty}^{\infty} J(y)e^{-\Lambda_1 y}dy - L_2 e^{(\Lambda_1 + \epsilon_1) z} \int_{-\infty}^{\infty} J(y)e^{-\Lambda_1 y}dy, 0 \right\} \]

and

\[ e^{\lambda_1 z} \int_{0}^{\infty} \int_{-\infty}^{\infty} K(y, s)e^{-\Lambda_1(y+cs)}dyds \]

\[ - L_2 e^{(\Lambda_1 + \epsilon_1) z} \int_{0}^{\infty} \int_{-\infty}^{\infty} K(y, s)e^{-(\Lambda_1 + \epsilon_1)(y+cs)}dyds \]

\[ \leq K \ast I_-(z) \leq K \ast I_+(z) \leq m_0 e^{\lambda_1 z}, \tag{4.12} \]

where \( m_0 := \int_{0}^{T} \int_{-R_2}^{R_2} K(y, s)e^{-\Lambda_1(y+cs)}dyds \). Choose small enough \( \epsilon_1 \in (0, \min \{ \epsilon_2, \lambda_2 - \lambda_1 \}) \) and large enough \( L_2 > 1 \) such that \( z_3 < z_2 \) and \( S_1 - \epsilon_2^{-1} e^{\epsilon_2 z_3} \geq S_1 / 2 \). Then we have for \( z < z_3 \) that

\[ I_-(z) = e^{\lambda_1 z} - L_2 e^{(\Lambda_1 + \epsilon_1) z}, \quad S_-(z) = S_1 - \epsilon_2^{-1} e^{\epsilon_2 z} \geq S_1 / 2, \quad R_+(z) = L_1 e^{\epsilon_1 z}. \tag{4.14} \]

Since \( \epsilon_1 < \lambda_1 \), we get for \( z < z_3 < 0 \) that

\[ e^{(\lambda_1 - \epsilon_1) z} < 1. \tag{4.15} \]

Notice that

\[ \Theta(\lambda_1, c) = 0 \quad \text{and} \quad \Theta(\lambda_1 + \epsilon_1, c) < 0 \quad \text{for} \quad \lambda_1 < \lambda_1 + \epsilon_1 < \lambda_2. \tag{4.16} \]

It follows from (4.13) and (4.14) that

\[ \begin{align*}
&- \beta K \ast I_-(z) + \frac{\beta S_-(z)K \ast I_-(z)}{S_-(z) + K \ast I_-(z) + R_+(z)} \\
&= - \beta \left[ K \ast I_-(z) + R_-(z) \right] + \frac{\beta K \ast I_-(z)R_+(z)}{S_-(z) + K \ast I_-(z) + R_+(z)} \\
&\geq - \frac{2\beta}{S_1} \left[ m_0^2 e^{2\lambda_1 z} + m_0 L_1 e^{(\lambda_1 + \epsilon_1) z} \right].
\end{align*} \tag{4.17} \]
We derive for \( z < z_3 \) that
\[
d_2[J \ast I_-(z) - I_-(z)] - cI_-'(z) + \frac{\beta S_-(z)K \ast I_-(z)}{S_-(z) + K \ast I_-(z) + R_+(z)} - (\gamma + \delta)I_-(z)
\]
\[
= d_2[J \ast I_-(z) - I_-(z)] - cI_-'(z) + \beta K \ast I_-(z) - (\gamma + \delta)I_-(z)
\]
\[
- \beta K \ast I_-(z) + \frac{\beta S_-(z)K \ast I_-(z)}{S_-(z) + K \ast I_-(z) + R_+(z)}
\]
\[
\geq e^{\lambda_1 z} \Theta(\lambda_1, c) - L_2 e^{(\lambda_1 + \epsilon_1)z} \Theta(\lambda_1 + \epsilon_1, c) - \frac{2\beta}{S_1} [m_0 e^{2\lambda_1 z} + m_0 L_1 e^{(\lambda_1 + \epsilon_1)z}]
\]
(by (4.12)-(4.14), (4.17) and (2.1))
\[
= -e^{(\lambda_1 + \epsilon_1)z} \Theta(\lambda_1 + \epsilon_1, c) \left[ L_2 - \frac{2\beta m_0^2 e^{(\lambda_1 - \epsilon_1)z} + 2\beta m_0 L_1}{-S_1 \Theta(\lambda_1 + \epsilon_2, c)} \right]
\]
\[
\geq -e^{(\lambda_1 + \epsilon_1)z} \Theta(\lambda_1 + \epsilon_1, c) \left[ L_2 - \frac{2\beta m_0^2 + 2\beta m_0 L_1}{-S_1 \Theta(\lambda_1 + \epsilon_2, c)} \right] \quad \text{(by (4.15) and (4.16))}
\]
\[
\geq 0 \quad \text{for large enough } L_2 > 1.
\]

If \( z > z_3 \), then \( I_-(z) = 0 \) and (4.5) follows immediately. \( \square \)

Now we introduce a functional space
\[
B_\mu(\mathbb{R}, \mathbb{R}^3) := \left\{ \phi = (\phi_1, \phi_2, \phi_3) \in C(\mathbb{R}, \mathbb{R}^3) : \left| \phi_i \right|_\mu < \infty, \ i = 1, 2, 3 \right\}
\]
endowed with the norm
\[
\left| \phi \right|_\mu := \max \left\{ \sup_{z \in \mathbb{R}} \left| \phi_i(z) \right| e^{-\mu|z|}, \ i = 1, 2, 3 \right\},
\]
where
\[
\epsilon_1 < \mu \leq \min \left\{ \frac{\sigma + d_1}{c}, \frac{\gamma + \delta + d_2}{c}, \frac{d_3}{c} \right\} \quad \text{and} \quad \sigma > \beta.
\]
Set
\[
\Gamma := \left\{ (S, I, R) \in B_\mu(\mathbb{R}, \mathbb{R}^3) \ : \ S_-(z) \leq S(z) \leq S_+(z), \right. \quad \left. I_-(z) \leq I(z) \leq I_+(z), \quad R_-(z) \leq R(z) \leq R_+(z) \right\}.
\]
Then \( \Gamma \) is nonempty, bounded, closed and convex cone in \( B_\mu(\mathbb{R}, \mathbb{R}^3) \). For any \( (S, I, R) \in \Gamma \), we define an operator \( F = (F_1, F_2, F_3) : \Gamma \mapsto C^1(\mathbb{R}, \mathbb{R}^3) \) by
\[
F_1(S, I, R)(z) := \frac{1}{c} \int_{-\infty}^{z} e^{-\frac{\sigma + d_1}{c}(z-\eta)} H_1(S, I, R)(\eta) \, d\eta,
\]
\[
F_2(S, I, R)(z) := \frac{1}{c} \int_{-\infty}^{z} e^{-\frac{\gamma + \delta + d_2}{c}(z-\eta)} H_2(S, I, R)(\eta) \, d\eta,
\]
\[
F_3(S, I, R)(z) := \frac{1}{c} \int_{-\infty}^{z} e^{-\frac{d_3}{c}(z-\eta)} H_3(S, I, R)(\eta) \, d\eta,
\]
where
\[
H_1(S, I, R)(z) := d_1 J \ast S(z) + \sigma S(z) - A(S, I, R)(z),
\]
\[
H_2(S, I, R)(z) := d_2 J \ast I(z) + A(S, I, R)(z),
\]
\[
H_3(S, I, R)(z) := d_3 J \ast R(z) + \gamma I(z)
\]
and
\[
A(S, I, R)(z) := \begin{cases} 
\frac{\beta S(z)K + I(z)}{S(z) + K + I(z) + R(z)}, & S(z)K + I(z) \neq 0, \\
0, & S(z)K + I(z) = 0.
\end{cases}
\]

Lemma 4.2. \( F(\Gamma) \subseteq \Gamma \).

Proof. By the monotonicity of \( H_i \) \((i = 1, 2, 3)\) with respect to its variables, it suffices to show that for any \((S, I, R) \in \Gamma\),
\[
S_-(z) \leq F_1(S_-, I_+, R_-)(z) \leq F_1(S_+, I_-, R_+)(z) \leq S_+(z), \quad (4.18)
\]
\[
I_-(z) \leq F_2(S_-, I_-, R_+)(z) \leq F_2(S_+, I_+, R_-)(z) \leq I_+(z), \quad (4.19)
\]
\[
R_-(z) \leq F_3(S_-, I_-, R_-)(z) \leq F_3(S_+, I_+, R_+)(z) \leq R_+(z). \quad (4.20)
\]
Firstly, we infer from (4.1) that
\[
F_1(S_+, I_-, R_+)(z) = \frac{1}{c} \int_{-\infty}^{z} e^{-\frac{\tau_1}{c}(z-\eta)} H_1(S_+, I_-, R_+)(\eta)d\eta \\
\leq \frac{1}{c} \int_{-\infty}^{z} e^{-\frac{\tau_1}{c}(z-\eta)} [cS_+(\eta) + (\sigma + d_1)S_+(\eta)]d\eta \\
= S_+(z) \quad \text{for} \ z \in \mathbb{R}.
\]
On the other hand, we obtain from (4.4) that
\[
F_1(S_-, I_+, R_-)(z) = \frac{1}{c} \int_{-\infty}^{z} e^{-\frac{\tau_1}{c}(z-\eta)} H_1(S_-, I_+, R_-)(\eta)d\eta \\
\geq \frac{1}{c} \int_{-\infty}^{z} e^{-\frac{\tau_1}{c}(z-\eta)} [cS_-(\eta) + (\sigma + d_1)S_-(\eta)]d\eta \\
= S_-(z) \quad \text{for} \ z \neq z_2.
\]
Then utilizing the continuity of both \( F_1(S_-, I_+, R_-)(z) \) and \( S_-(z) \) at the point \( z_2 \) gives \( F_1(S_-, I_+, R_-)(z) \geq S_-(z) \) for \( z \in \mathbb{R} \). Secondly, it follows from (4.2) that
\[
F_2(S_+, I_+, R_-)(z) = \frac{1}{c} \int_{-\infty}^{z} e^{-\frac{\gamma + \delta + \frac{\tau_1}{c}}{c}(z-\eta)} H_2(S_+, I_+, R_-)(\eta)d\eta \\
\leq \frac{1}{c} \int_{-\infty}^{z} e^{-\frac{\gamma + \delta + \frac{\tau_1}{c}}{c}(z-\eta)} [cI_+(\eta) + (\gamma + \delta + d_2)I_+(\eta)]d\eta \\
= I_+(z) \quad \text{for} \ z \neq z_1.
\]
Then using the continuity of \( F_2(S_+, I_+, R_-)(z) \) and \( I_+(z) \) at the point \( z_1 \) yields \( F_2(S_+, I_+, R_-)(z) \leq I_+(z) \) for \( z \in \mathbb{R} \). On the other hand, we have from (4.5) that
\[
F_2(S_-, I_-, R_+)(z) = \frac{1}{c} \int_{-\infty}^{z} e^{-\frac{\gamma + \delta + \frac{\tau_1}{c}}{c}(z-\eta)} H_2(S_-, I_-, R_+)(\eta)d\eta \\
\geq \frac{1}{c} \int_{-\infty}^{z} e^{-\frac{\gamma + \delta + \frac{\tau_1}{c}}{c}(z-\eta)} [cI_-(\eta) + (\gamma + \delta + d_2)I_-(\eta)]d\eta \\
= I_-(z) \quad \text{for} \ z \neq z_3.
\]
Applying the continuity of both \( F_2(S_-, I_-, R_+)(z) \) and \( I_-(z) \) at the point \( z_3 \), we get \( F_2(S_-, I_-, R_+)(z) \geq I_-(z) \) for \( z \in \mathbb{R} \). Lastly, from (4.3) and (4.6), we deduce
that
\[ F_3(S, I_+, R_+)(z) = \frac{1}{c} \int_{-\infty}^{z} e^{-\frac{d_3}{c}(z-\eta)} H_3(S, I_+, R_+)(\eta) d\eta \]
\[ \leq \frac{1}{c} \int_{-\infty}^{z} e^{-\frac{d_3}{c}(z-\eta)} [cR'_+({\eta}) + d_3 R_+({\eta})] d\eta \]
\[ = R_+(z) \quad \text{for } z \in \mathbb{R} \]
and
\[ F_3(S, I_-, R_-)(z) = \frac{1}{c} \int_{-\infty}^{z} e^{-\frac{d_3}{c}(z-\eta)} H_3(S, I_-, R_-)(\eta) d\eta \]
\[ \geq \frac{1}{c} \int_{-\infty}^{z} e^{-\frac{d_3}{c}(z-\eta)} [cR'_-({\eta}) + d_3 R_-({\eta})] d\eta \]
\[ = R_-(z) \quad \text{for } z \in \mathbb{R}. \]

The proof of this lemma is completed. \[ \square \]

**Lemma 4.3.** The operator \( F = (F_1, F_2, F_3) : \Gamma \mapsto \Gamma \) is completely continuous with respect to the decay norm \(| \cdot |_\mu\) in \( B_\mu(\mathbb{R}, \mathbb{R}^3)\).

**Proof.** (i) **Continuity.** For any \((\hat{\mathcal{S}}_1, \hat{\mathcal{I}}_1, \hat{\mathcal{R}}_1) \in \Gamma\) and \((\hat{\mathcal{S}}_2, \hat{\mathcal{I}}_2, \hat{\mathcal{R}}_2) \in \Gamma\), we derive that
\[ |H_1(\hat{\mathcal{S}}_1, \hat{\mathcal{I}}_1, \hat{\mathcal{R}}_1)(z) - H_1(\hat{\mathcal{S}}_2, \hat{\mathcal{I}}_2, \hat{\mathcal{R}}_2)(z)| e^{-\mu|z|} \]
\[ \leq d_1 |J * \hat{\mathcal{S}}_1(z) - J * \hat{\mathcal{S}}_2(z)| e^{-\mu|z|} + (\sigma + 2\beta)|\hat{\mathcal{I}}_1(z) - \hat{\mathcal{S}}_1(z)| e^{-\mu|z|} \]
\[ + 2\beta |K * \hat{\mathcal{I}}_1(z) - K * \hat{\mathcal{S}}_2(z)| e^{-\mu|z|} + \beta|\hat{\mathcal{R}}_1(z) - \hat{\mathcal{R}}_2(z)| e^{-\mu|z|} \]
\[ \leq (\sigma + 2\beta)|\hat{\mathcal{S}}_1(z) - \hat{\mathcal{S}}_2(z)| e^{-\mu|z|} + d_1 e^{-\mu|z|} \int_{-R_1}^{R_1} J(y)|\hat{\mathcal{S}}_1(z-y) - \hat{\mathcal{S}}_2(z-y)| dy \]
\[ + 2\beta e^{-\mu|z|} \int_0^{T} \int_{-R_2}^{R_2} K(y, s)|\hat{\mathcal{I}}_1(z-y) - \hat{\mathcal{I}}_2(z-y)| ds dy \]
\[ \leq (\sigma + 2\beta)|\hat{\mathcal{S}}_1(z) - \hat{\mathcal{S}}_2(z)| e^{-\mu|z|} + d_1 |\hat{\mathcal{S}}_1(z) - \hat{\mathcal{S}}_2(z)| e^{-\mu|z|} \int_{-R_1}^{R_1} J(y) dy \]
\[ + 2\beta |\hat{\mathcal{I}}_1(z) - \hat{\mathcal{I}}_2(z)| e^{-\mu|z|} |e^{\mu|z|} + R_2 + cT| \int_0^{T} \int_{-R_2}^{R_2} K(y, s) ds dy \]
\[ = (\sigma + 2\beta + d_1 e^{R_1})|\hat{\mathcal{S}}_1(z) - \hat{\mathcal{S}}_2(z)| + 2\beta e^{R_2+cT}|\hat{\mathcal{I}}_1(z) - \hat{\mathcal{I}}_2(z)| + \beta|\hat{\mathcal{R}}_1(z) - \hat{\mathcal{R}}_2(z)|. \]

Similarly, one can deduce that
\[ |H_2(\hat{\mathcal{S}}_1, \hat{\mathcal{I}}_1, \hat{\mathcal{R}}_1)(z) - H_2(\hat{\mathcal{S}}_2, \hat{\mathcal{I}}_2, \hat{\mathcal{R}}_2)(z)| e^{-\mu|z|} \]
\[ \leq 2\beta |\hat{\mathcal{S}}_1(z) - \hat{\mathcal{S}}_2(z)| + \beta|\hat{\mathcal{R}}_1(z) - \hat{\mathcal{R}}_2(z)| + [d_2 e^{R_1} + 2\beta e^{R_2+cT}]|\hat{\mathcal{I}}_1(z) - \hat{\mathcal{I}}_2(z)| \]
and
\[ |H_3(\hat{\mathcal{S}}_1, \hat{\mathcal{I}}_1, \hat{\mathcal{R}}_1)(z) - H_3(\hat{\mathcal{S}}_2, \hat{\mathcal{I}}_2, \hat{\mathcal{R}}_2)(z)| e^{-\mu|z|} \leq d_3 e^{R_1}|\hat{\mathcal{R}}_1(z) - \hat{\mathcal{R}}_2(z)| + \gamma|\hat{\mathcal{I}}_1(z) - \hat{\mathcal{I}}_2(z)|, \]
which implies that there exists a positive constant \( C_0 \) such that
\[ |H_i(\hat{\mathcal{S}}_1, \hat{\mathcal{I}}_1, \hat{\mathcal{R}}_1) - H_i(\hat{\mathcal{S}}_2, \hat{\mathcal{I}}_2, \hat{\mathcal{R}}_2)| \leq C_0(|\hat{\mathcal{S}}_1(z) - \hat{\mathcal{S}}_2(z)| + |\hat{\mathcal{I}}_1(z) - \hat{\mathcal{I}}_2(z)| + |\hat{\mathcal{R}}_1(z) - \hat{\mathcal{R}}_2(z)|), \quad i = 1, 2, 3. \]
Then we obtain that
\[
|F_1(\hat{S}_1, \hat{I}_1, \hat{R}_1) - F_1(\hat{S}_2, \hat{I}_2, \hat{R}_2)|_\mu \\
= |F_1(\hat{S}_1, \hat{I}_1, \hat{R}_1)(z) - F_1(\hat{S}_2, \hat{I}_2, \hat{R}_2)(z)|e^{-\mu|z|} \\
\leq \frac{1}{c} |H_1(\hat{S}_1, \hat{I}_1, \hat{R}_1) - H_1(\hat{S}_2, \hat{I}_2, \hat{R}_2)|_\mu \\
\quad \cdot \int_{-\infty}^{\infty} e^{-\frac{\varsigma + d_1}{c}(z-n)e^{\mu|\eta|}e^{-\mu|z|}} d\eta \\
\leq \frac{1}{c} |H_1(\hat{S}_1, \hat{I}_1, \hat{R}_1) - H_1(\hat{S}_2, \hat{I}_2, \hat{R}_2)|_\mu \\
\quad \cdot \int_{-\infty}^{\infty} e^{-\frac{\varsigma + d_1}{c}(z-n)e^{\mu|\eta-z|}} d\eta \\
= \frac{C_0}{\sigma + d_1 - c\mu}(|\hat{S}_1 - \hat{S}_2|_\mu + |\hat{I}_1 - \hat{I}_2|_\mu + |\hat{R}_1 - \hat{R}_2|_\mu).
\]

Analogously, one can have that
\[
|F_2(\hat{S}_1, \hat{I}_1, \hat{R}_1) - F_2(\hat{S}_2, \hat{I}_2, \hat{R}_2)|_\mu \leq \frac{C_0}{\gamma + \delta + d_2 - c\mu}(|\hat{S}_1 - \hat{S}_2|_\mu + |\hat{I}_1 - \hat{I}_2|_\mu + |\hat{R}_1 - \hat{R}_2|_\mu)
\]
and
\[
|F_3(\hat{S}_1, \hat{I}_1, \hat{R}_1) - F_3(\hat{S}_2, \hat{I}_2, \hat{R}_2)|_\mu \leq \frac{C_0}{d_3 - c\mu}(|\hat{I}_1 - \hat{I}_2|_\mu + |\hat{R}_1 - \hat{R}_2|_\mu),
\]
which proves that the operator $F$ is continuous with respect to the decay norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R}, \mathbb{R}^3)$.

(ii) **Compactness.** For any $(S, I, R) \in \Gamma$, we get that
\[
H_1(S, I, R)(z) \leq (d_1 + \sigma)S_1,
\]
\[
H_2(S, I, R)(z) \leq \beta S_1 + d_2 \frac{(\beta - \gamma - \delta)S_1}{\gamma + \delta}
\]
and
\[
H_3(S, I, R)(z) \leq d_3 L_1 e^{\epsilon_1 z} \int_{-R_1}^{R_1} J(y) e^{-\epsilon_1 y} dy + \gamma \frac{(\beta - \gamma - \delta)S_1}{\gamma + \delta} \leq \rho e^{\epsilon_1 z} + \gamma \frac{(\beta - \gamma - \delta)S_1}{\gamma + \delta}
\]
for $z \in \mathbb{R}$, where $\rho = d_3 L_1 e^{\epsilon_1 R_1}$. Then for $z \in \mathbb{R}$, we have
\[
\left| \frac{dF_1(S, I, R)(z)}{dz} \right| \\
= \left| -\frac{\sigma + d_1}{c^2} \int_{-\infty}^{\infty} e^{-\frac{\varsigma + d_1}{c}(z-n)H_1(S, I, R)(\eta)} d\eta + \frac{1}{c} H_1(S, I, R)(z) \right| \quad (4.21)
\]
\[
\left| \frac{dF_2(S, I, R)(z)}{dz} \right| \\
= \left| -\frac{\gamma + \delta + d_2}{c^2} \int_{-\infty}^{\infty} e^{-\frac{\varsigma + d_1}{c}(z-n)H_2(S, I, R)(\eta)} d\eta + \frac{1}{c} H_2(S, I, R)(z) \right| \quad (4.22)
\]
\[
\leq 2S_1 \left( \beta + d_2 \frac{\beta - \gamma - \delta}{\gamma + \delta} \right)
\]
Using (4.21)-(4.23) and Arzelà-Ascoli theorem, we can select finite elements in investigate some properties concerning the fixed point \( (\Gamma) \) on \([-Z, Z]\) with the supremum norm. Together with (4.24), we conclude that it is also a finite \( \varepsilon \)-net of \( F(\Gamma) \) in \( \mathbb{R} \) with the decay norm \(|\cdot|_\mu\). Hence \( F \) is compact with respect to the decay norm \(|\cdot|_\mu\) in \( B_\mu(\mathbb{R}, \mathbb{R}^3) \). This ends the proof.

Applying Lemmas 4.2, 4.3 and Schauder’s fixed point theorem, we obtain that the operator \( F \) admits a fixed point \((S, I, R) \in \Gamma \). In the rest of this section, we investigate some properties concerning the fixed point \((S, I, R) \) of \( F \).

**Proposition 4.1.** Assume that \( R_0 > 1 \) and \( c > c^* \). Then the fixed point \((S, I, R) \) of \( F \) is the solution of (1.8)-(1.10) and satisfies the following assertions.

(i) \( 0 < S(z) < S_1, 0 < I(z) < (\beta - \gamma - \delta)S_1/(\gamma + \delta), R(z) > 0 \) for \( z \in \mathbb{R} \).

(ii) \((S(-\infty), I(-\infty), R(-\infty)) = (S_1, 0, 0); I(z) = O(e^{\lambda z}) \) as \( z \to -\infty \).

(iii) \( I(\infty) = 0; \) the limit \( S(\infty) \) exists and \( S_2 < S_1 \).

(iv) \((\gamma + \delta) \int_{\mathbb{R}} I(z)dz = \beta \int_{\mathbb{R}} \frac{S(z)K + I(z)}{S(z) + K + I(z) + R(z)}dz = c(S_1 - S_2) \).

(v) If \( R(z) \) is bounded in \( \mathbb{R} \), then \( R(\infty) = \gamma(S_1 - S_2)/(\gamma + \delta) \) and \( S'(\pm\infty) = I'(\pm\infty) = R'(\pm\infty) = 0 \).

**Proof.** (i) Noting that \( \sigma > \beta \) and \((S, I, R) \in \Gamma \), we obtain for \( z \in \mathbb{R} \) that

\[
S(z) \geq \frac{1}{c} \int_{-\infty}^{z} e^{-\frac{\sigma + \beta}{\gamma + \delta}(z - \eta)} \left[ d_1 J * S(\eta) + \sigma S(\eta) - \frac{\beta S(\eta)K + I(\eta)}{S(\eta) + K + I(\eta) + R(\eta)} \right] d\eta
\]

\[
\geq \frac{1}{c} \int_{-\infty}^{z} e^{-\frac{\sigma + \beta}{\gamma + \delta}(z - \eta)} \left[ d_1 J * S_-(\eta) + (\sigma - \beta)S_-(\eta) \right] d\eta > 0,
\]
and
\[ R(z) \geq \frac{\gamma}{c} \int_{-\infty}^{z} e^{-\frac{\gamma+\alpha}{c}(z-\eta)} I_-(\eta) d\eta > 0. \]

This together with the definition of operator \( F \) implies that the fixed point \((S, I, R)\) of \( F \) is the positive solution of (1.8)-(1.10). Assume that there exists some \( \hat{z} \in \mathbb{R} \) such that \( S(\hat{z}) = S_1 \), then \( S'(\hat{z}) = 0 \). By (H1) and (H2), we deduce from (1.8) that

\[
0 = d_1[J * S(\hat{z}) - S(\hat{z})] - cS(\hat{z}) - \frac{\beta S(\hat{z}) K * I(\hat{z})}{S(\hat{z}) + K * I(\hat{z}) + R(\hat{z})}
\]

\[
= d_1[J * S(\hat{z}) - S_1] - \frac{\beta S_1 K * I(\hat{z})}{S_1 + K * I(\hat{z}) + R(\hat{z})} \quad \text{(since } S(\hat{z}) = S_1 \text{ and } S'(\hat{z}) = 0 \text{)}
\]

\[
\leq - \frac{\beta S_1 K * I(\hat{z})}{S_1 + K * I(\hat{z}) + R(\hat{z})} \quad \text{(since } S(z) \leq S_1 \text{ in } \mathbb{R} \text{)}
\]

\[
< 0 \quad \text{(since } K * I(z) > 0 \text{ and } R(z) > 0 \text{ in } \mathbb{R},
\]

which yields a contradiction. Thus \( S(z) < S_1 \) for \( z \in \mathbb{R} \). Suppose that there is a \( \xi \in \mathbb{R} \) such that \( I(\xi) = \frac{(\beta - \gamma - \delta)S_1}{\gamma + \delta} \), then \( I'(\xi) = 0 \). By (H1) and (H2), we obtain from (1.9) that

\[
0 = d_2[J * I(\xi) - I(\xi)] - cI'(\xi) + \frac{\beta S(\xi) K * I(\xi)}{S(\xi) + K * I(\xi) + R(\xi)} - (\gamma + \delta)I(\xi)
\]

\[
< \frac{\beta S_1 (\beta - \gamma - \delta)S_1}{S_1 + (\beta - \gamma - \delta)S_1} - (\gamma + \delta)(\beta - \gamma - \delta)S_1
\]

\[
\quad \quad \quad \frac{\gamma + \delta}{\gamma + \delta}
\]

\[
\left(\text{since } I(z) \leq \frac{(\beta - \gamma - \delta)S_1}{\gamma + \delta}, R(z) > 0 \text{ and } S(z) < S_1 \text{ in } \mathbb{R}\right)
\]

\[
= 0,
\]

which leads to a contradiction. Hence \( I(z) < \frac{(\beta - \gamma - \delta)S_1}{\gamma + \delta} \) for \( z \in \mathbb{R} \).

\( \text{(ii)} \) Notice that

\[
S_-(z) \leq S(z) \leq S_+(z), \quad I_-(z) \leq I(z) \leq I_+(z), \quad R_-(z) \leq R(z) \leq R_+(z) \quad (4.25)
\]

for \( z \in \mathbb{R} \). Applying sandwich rule in (4.25) gives that \((S(-\infty), I(-\infty), R(-\infty)) = (S_1, 0, 0)\) and \( I(z) = O(e^{\lambda z}) \) as \( z \to -\infty \).

\( \text{(iii)} \) Using a similar argument as in Lemma 2.3, we have that there exist two sequence \( \{\xi_n\} \) and \( \{\eta_n\} \) satisfying \( \xi_n, \eta_n \to \infty \) as \( n \to \infty \) such that

\[
\lim_{n \to \infty} S(\xi_n) = \limsup_{z \to \infty} S(z) = \rho_1, \quad S'(\xi_n) = 0, \quad (4.26)
\]

\[
\lim_{n \to \infty} S(\eta_n) = \liminf_{z \to \infty} S(z) = \rho_2 < \rho_1 \quad \text{and} \quad S'(\eta_n) = 0. \quad (4.27)
\]

Utilizing the similar arguments in \([40, 46, 49]\), one can deduce that \( S(\xi_n + y) \to \rho_1 \) and \( S(\eta_n + y) \to \rho_2 \) as \( n \to \infty \) for arbitrary \( y \in [-R_1, R_1] \). In view of (2.6), we
have
\[ \lim_{n \to \infty} \int_{\eta_n}^{\xi_n} \frac{\beta S(z)K \ast I(z)}{S(z) + K \ast I(z) + R(z)} \, dz = 0. \] (4.28)

Integrating (1.8) from \( \eta_n \) to \( \xi_n \) gives
\[
0 < c(\rho_1 - \rho_2) \quad \text{(by (4.26) and (4.27))}
\]
\[
= c \lim_{n \to \infty} \left[ S(\xi_n) - S(\eta_n) \right]
\]
\[
d_1 \lim_{n \to \infty} \int_{\eta_n}^{\xi_n} \int_{-R_1}^{R_1} J(y) |S(z - y) - S(z)| \, dy \, dz
\]
\[
- \lim_{n \to \infty} \int_{\xi_n}^{\eta_n} \frac{\beta S(z)K \ast I(z)}{S(z) + K \ast I(z) + R(z)} \, dz \quad \text{(by (H1))}
\]
\[
d_1 \lim_{n \to \infty} \int_{\eta_n}^{\xi_n} \int_{-R_1}^{R_1} J(y) |S(z - y) - S(z)| \, dy \, dz \quad \text{(by (4.28))}
\]
\[
= -d_1 \lim_{n \to \infty} \int_{\eta_n}^{\xi_n} \int_{-R_1}^{R_1} y J(y) \int_0^1 S'(z - \theta y) \, d\theta \, dy \, dz
\]
\[
= -d_1 \lim_{n \to \infty} \int_{-R_1}^{R_1} y J(y) \int_0^1 \left[ S(\eta_n - \theta y) - S(\xi_n - \theta y) \right] \, d\theta \, dy
\]
\[
\quad \text{(by Fubini’s theorem)}
\]
\[
= 0, \quad \text{(by Lebesgue’s dominated convergence theorem)}
\]
a contradiction appears. This implies that \( \limsup_{z \to \infty} S(z) = \liminf_{z \to \infty} S(z) \), i.e., then
\( S(\infty) := S_2 \) exists. We are now in a position to show that \( S_2 < S_1 \). Due to
\( S(z) < S_1 \) in \( \mathbb{R} \), we have \( S_2 \leq S_1 \). Assume that \( S_2 = S_1 \), then integrating (1.8) over
\( \mathbb{R} \), we obtain
\[
0 = c(S_2 - S_1)
\]
\[
= d_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J(y) |S(z - y) - S(z)| \, dy \, dz - \int_{-\infty}^{\infty} \frac{\beta S(z)K \ast I(z)}{S(z) + K \ast I(z) + R(z)} \, dz
\]
\[
= d_1 \left[ \int_{-\infty}^{\infty} J(y) \int_{-\infty}^{\infty} S(z - y) \, dy \, dz - \int_{-\infty}^{\infty} S(z) \, dz \right]
\]
\[
- \int_{-\infty}^{\infty} \frac{\beta S(z)K \ast I(z)}{S(z) + K \ast I(z) + R(z)} \, dz \quad \text{(by Fubini’s theorem)}
\]
\[
= -\int_{-\infty}^{\infty} \frac{\beta S(z)K \ast I(z)}{S(z) + K \ast I(z) + R(z)} \, dz < 0,
\]
which yields a contradiction. Thus \( S_2 < S_1 \).
(iv) Integrating (1.8) over \( \mathbb{R} \) yields
\[
0 = d_1 \int_{-\infty}^{\infty} [J \ast S(z) - S(z)] \, dz - c \int_{-\infty}^{\infty} S'(z) \, dz
\]
\[
- \int_{-\infty}^{\infty} \frac{\beta S(z)K \ast I(z)}{S(z) + K \ast I(z) + R(z)} \, dz
\]
\[
= c(S_1 - S_2) - \int_{-\infty}^{\infty} \frac{\beta S(z)K \ast I(z)}{S(z) + K \ast I(z) + R(z)} \, dz
\]
(4.29)
\[
\quad \text{(by Fubini’s theorem, } S(-\infty) = S_1 \text{ and } S(\infty) = S_2).\]
Integrating (1.9) over \( \mathbb{R} \) gives
\[
0 = d_2 \int_{-\infty}^{\infty} [J * I(z) - I(z)]dz - c \int_{-\infty}^{\infty} I'(z)dz \\
+ \int_{-\infty}^{\infty} \beta S(z) K * I(z) \frac{dz}{S(z) + K * I(z) + R(z)} - (\gamma + \delta) \int_{-\infty}^{\infty} I(z)dz
\]
\[= \int_{-\infty}^{\infty} \beta S(z) K * I(z) \frac{dz}{S(z) + K * I(z) + R(z)} - (\gamma + \delta) \int_{-\infty}^{\infty} I(z)dz \] (4.30)

Combining (4.29) and (4.30), we obtain that
\[
(\gamma + \delta) \int_{-\infty}^{\infty} I(z)dz = \beta \int_{-\infty}^{\infty} \frac{S(z)K * I(z)}{S(z) + K * I(z) + R(z)}dz = c(S_1 - S_2). \quad (4.31)
\]

(v) Assume for the contrary that \( \lim \sup R(z) > \lim \inf R(z) \), then one can prove that the existence of \( R(\infty) \) in a similar manner as (iii). Integrating (1.10) over \( \mathbb{R} \) and using Fubini’s theorem, we deduce that \( cR(\infty) = \gamma \int_{\mathbb{R}} I(z)dz \), which together with (4.31) implies that \( R(\infty) = \frac{\gamma(S_1 - S_2)}{\gamma + \delta} \).

Applying Lebesgue’s dominated convergence theorem, (H1) and (H2), we have that
\[
\lim_{z \to -\infty} J * S(z) = S_1, \quad \lim_{z \to +\infty} J * S(z) = S_2, \quad \lim_{z \to -\infty} J * I(z) = 0, \\
\lim_{z \to +\infty} K * I(z) = 0, \quad \lim_{z \to -\infty} J * R(z) = 0, \quad \lim_{z \to +\infty} J * R(z) = \frac{S_1 - S_2}{\gamma + \delta}.
\]

Together with (1.8)-(1.10), we conclude that \( S'(\pm \infty) = I'(\pm \infty) = R'(\pm \infty) = 0 \).

The proof is completed. \( \Box \)

5. Critical traveling waves. Motivated by [11, 12, 22, 48], in this section we will prove the existence result for \( R_0 > 1 \) and \( c = c^* \) in Theorem 1.2. For our purpose, we choose \( L_3 > 1 \) to be suitable large such that the equation \( -L_3 z e^{\lambda z} = \frac{(\beta - \gamma - \delta)S_1}{\gamma + \delta} \) admits two negative roots \( z_4 \) and \( z^* \) satisfying
\[
z^* - z_4 > \max \{R_1, R_2\} := \bar{R}, \quad (5.1)
\]

where \( R_1 \) and \( R_2 \) are the radii of \( \text{supp} J \) and \( \text{supp} K \) for spatial variable, respectively.

Now we define the following nonnegative continuous functions in \( \mathbb{R} \).

\[
S_+^*(z) := S_1, \quad I_+^*(z) := \begin{cases}
-L_3 z e^{\lambda z}, & z < z_4, \\
\frac{(\beta - \gamma - \delta)S_1}{\gamma + \delta}, & z \geq z_4,
\end{cases}
\]

\[
R_+^*(z) := L_4 e^{\epsilon_3 z}, \quad S_+^*(z) := \begin{cases}
S_1 - \epsilon_4^{-1} e^{\epsilon_4 z}, & z < z_5, \\
0, & z \geq z_5,
\end{cases}
\]

\[
I_+^*(z) := \begin{cases}
-L_3 z e^{\lambda z} - L_5 (-z)^{1/2} e^{\lambda z}, & z < z_6, \\
0, & z \geq z_6,
\end{cases}
\]

where \( \lambda \) is defined in Lemma 2.1, \( z_4 \) is in (5.1), \( z_5 = \epsilon_4^{-1} \ln(\epsilon_4 S_1) \), \( z_6 = \frac{-L_5^2}{L_3} \), \( \epsilon_3, \epsilon_4, L_3, L_4 \) and \( L_5 \) are positive constants to be clarified later. To illustrate that the parameters are admissible, we present Figure 1.
Lemma 5.1. For given small enough \( \epsilon_3, \epsilon_4 > 0 \) and large enough \( L_3, L_4, L_5 > 1 \), the functions \( S^*_\pm(z), I^*_\pm(z) \) and \( R^*_\pm(z) \) satisfy

\[
d_1[J * S^*_\pm(z) - S^*_\pm(z)] - c^*(S^*_\pm)'(z) - \frac{\beta S^*_\pm(z)K * I^*_\pm(z)}{S^*_\pm(z) + K * I^*_\pm(z) + R^*_\pm(z)} \leq 0, \quad z \in \mathbb{R}, \quad (5.2)
\]

\[
d_2[J * I^*_\pm(z) - I^*_\pm(z)] - c^*(I^*_\pm)'(z) + \frac{\beta S^*_\pm(z)K * I^*_\pm(z)}{S^*_\pm(z) + K * I^*_\pm(z) + R^*_\pm(z)} \leq 0, \quad z \neq z_4, \quad (5.3)
\]

\[
d_3[J * R^*_\pm(z) - R^*_\pm(z)] - c^*(R^*_\pm)'(z) + \gamma I^*_\pm(z) \leq 0, \quad z \in \mathbb{R}, \quad (5.4)
\]

\[
d_1[J * S^-\pm(z) - S^-\pm(z)] - c^*(S^-\pm)'(z) - \frac{\beta S^-\pm(z)K * I^-\pm(z)}{S^-\pm(z) + K * I^-\pm(z) + R^-\pm(z)} \geq 0, \quad z \neq z_5, \quad (5.5)
\]

\[
d_2[J * I^-\pm(z) - I^-\pm(z)] - c^*(I^-\pm)'(z) + \frac{\beta S^-\pm(z)K * I^-\pm(z)}{S^-\pm(z) + K * I^-\pm(z) + R^-\pm(z)} \geq 0, \quad z \neq z_6, \quad (5.6)
\]

\[
d_3[J * R^-\pm(z) - R^-\pm(z)] - c^*(R^-\pm)'(z) + \gamma I^-\pm(z) \geq 0, \quad z \in \mathbb{R}. \quad (5.7)
\]

Proof. Proof of (5.2) and (5.7). By the definitions of \( S^*_\pm(z), I^*_\pm(z) \) and \( R^*_\pm(z) \), we find that (5.2) and (5.7) hold naturally.

Proof of (5.3). For \( z < z_4 \), we derive that

\[
I^*_\pm(z) = -L_3 z e^{\lambda^* z}, \quad (I^*_\pm)'(z) = -L_3 e^{\lambda^* z}(1 + \lambda^* z), \quad (5.8)
\]

\[
J * I^*_\pm(z) = \int_{-\infty}^{\infty} J(y)I^*_\pm(z - y)dy
\]

\[
= \int_{-\infty}^{z-z^*} J(y)I^*_\pm(z - y)dy + \int_{z-z^*}^{\infty} J(y)I^*_\pm(z - y)dy
\]

\[
= \int_{z-z^*}^{\infty} J(y)I^*_\pm(z - y)dy \quad \text{(by (5.1) and (H1))} \quad (5.9)
\]

\[
\leq -L_3 \int_{z-z^*}^{\infty} J(y)(z - y)e^{\lambda^*(z-y)}dy
\]

\[
= -L_3 \int_{-\infty}^{\infty} J(y)(z - y)e^{\lambda^*(z-y)}dy
\]
where and

\[ n_1 := \int_{-\infty}^{\infty} J(y) e^{\lambda^* y} dy, \quad n_2 := \int_{-\infty}^{\infty} J(y) y e^{\lambda^* y} dy, \]

\[ m_1 := \int_{0}^{\infty} \int_{-\infty}^{\infty} K(y, s) e^{\lambda^* (y - c^* s)} dy ds, \]

\[ m_2 := \int_{0}^{\infty} \int_{-\infty}^{\infty} K(y, s) (y - c^* s) e^{\lambda^* (y - c^* s)} dy ds. \]

In the sequel, we retain the notations: \( n_1, n_2, m_1 \) and \( m_2 \). Then by (5.8)-(5.10), we obtain for \( z < z_4 \) that

\[
d_2[J * I_+^*(z) - I_+^*(z)] - c^* (I_+^*)'(z) + \frac{\beta S_+^*(z) K * I_+^*(z)}{S_+^*(z) + K * I_+^*(z) + R_+^*(z)} - (\gamma + \delta) I_+^*(z)
\]
\[
\leq d_2[J * I_+^*(z) - I_+^*(z)] - c^* (I_+^*)'(z) + \beta K * I_+^*(z) - (\gamma + \delta) I_+^*(z)
\]
By L'Hospital rule and (H1), we deduce that
\begin{align*}
\beta \leq \gamma + \delta \leq \beta S^*(z) = \frac{K \ast I^*_+(z)}{S^*_+(z)} - (\gamma + \delta) I^*_+(z) \\
= \frac{\beta S^*_+(z)}{\gamma + \delta} - (\gamma + \delta) \left( \frac{S^*_+(z)}{\gamma + \delta} + K \ast I^*_+(z) + R^*_+(z) \right) = 0, \quad \text{(by Lemma 2.1)}.
\end{align*}
Using the definition of $I^*_+(z)$ and the assumptions (H1), (H2), we have for $z > z_4$ that
\begin{align*}
J \ast I^*_+(z) \leq \frac{(\beta - \gamma - \delta)S_1}{\gamma + \delta} \quad \text{and} \quad K \ast I^*_+(z) \leq \frac{(\beta - \gamma - \delta)S_1}{\gamma + \delta}. \quad (5.11)
\end{align*}
This implies that
\begin{align*}
d_2J \ast I^*_+(z) - I^*_+(z) - c^*(I^*_+)'(z) + \frac{\beta S^*_+(z)K \ast I^*_+(z)}{S^*_+(z) + K \ast I^*_+(z) + R^*_+(z)} - (\gamma + \delta) I^*_+(z) \\
\leq \frac{\beta S_1(\beta - \gamma - \delta)S_1}{\gamma + \delta} - (\gamma + \delta) \left( \frac{\beta - \gamma - \delta)S_1}{\gamma + \delta} \right) = 0 \quad \text{for} \quad z > z_4.
\end{align*}
Proof of (5.4). If $z < z_4$, then $I^*_+(z) = -L_3c^* e^{\lambda^* z}$ and $R^*_+(z) = L_4 e^{\epsilon_3 z}$. Then for $z < z_4$, it follows that
\begin{align*}
d_3J \ast R^*_+(z) - R^*_+(z) - c^*(R^*_+)'(z) + \gamma I^*_+(z) \\
= d_3L_4 e^{\epsilon_3 z} \int_{-\infty}^{\infty} J(y) e^{-\epsilon_3 y} dy - d_3L_4 e^{\epsilon_3 z} - c^*L_4 e^{\epsilon_3 z} - \gamma L_3 e^{\lambda^* z} \\
= L_4 e^{\epsilon_3 z} \left[ d_3 \int_{-\infty}^{\infty} J(y) e^{-\epsilon_3 y} dy - d_3 - c^* e^3 - \frac{\gamma L_3}{L_4} e^{(\lambda^* - \epsilon_3) z} \right] \\
= L_4 e^{\epsilon_3 z} \left[ - \Delta(\epsilon_3, c^* - \frac{\gamma L_3}{L_4} e^{(\lambda^* - \epsilon_3) z}) \right] \quad \text{(by Lemma 2.2)} \\
\leq 0, \quad \text{(by choosing} \epsilon_3 \in (0, \min\{\lambda, \lambda^*\}) \text{and large enough} L_4 > 1). \quad \text{If} \quad z \geq z_4, \text{then} \quad I^*_+(z) = \frac{\beta - \gamma - \delta)S_1}{\gamma + \delta} \text{and} \quad R^*_+(z) = L_4 e^{\epsilon_3 z}. \text{Then for} \quad z \geq z_4, \text{we deduce that}
\begin{align*}
d_3J \ast R^*_+(z) - R^*_+(z) - c^*(R^*_+)'(z) + \gamma I^*_+(z) \\
= d_3L_4 e^{\epsilon_3 z} \int_{-\infty}^{\infty} J(y) e^{-\epsilon_3 y} dy - d_3L_4 e^{\epsilon_3 z} - c^*L_4 e^{\epsilon_3 z} + \frac{\gamma (\beta - \gamma - \delta)S_1}{\gamma + \delta} \\
= L_4 e^{\epsilon_3 z} \left[ d_3 \int_{-\infty}^{\infty} J(y) e^{-\epsilon_3 y} dy - d_3 - c^* e^3 + \frac{\gamma (\beta - \gamma - \delta)S_1}{L_4} e^{-\epsilon_3 z} \right] \\
\leq L_4 e^{\epsilon_3 z} \left[ - \Delta(\epsilon_3, c^*) + \frac{\gamma (\beta - \gamma - \delta)S_1}{L_4} e^{-\epsilon_3 z} \right] \quad \text{(by Lemma 2.2)} \\
\leq 0, \quad \text{(by selecting} \epsilon_3 \in (0, \min\{\lambda, \lambda^*\}) \text{and large enough} L_4 > 1). \quad \text{Proof of (5.5). From (H1) and the definition of} S^*_+(z), \text{we get}
\begin{align*}
J \ast S^*_+(z) \geq S_1 - e^{\epsilon_3 z} \int_{-\infty}^{\infty} J(y) e^{-\epsilon_3 y} dy \quad \text{for} \quad z \in \mathbb{R}. \quad (5.12)
\end{align*}
By L'Hospital rule and (H1), we deduce that
\begin{align*}
d_1 e^{\epsilon_3 z} \int_{-\infty}^{\infty} J(y)(1 - e^{-\epsilon_3 y}) dy \to 0 \quad \text{as} \quad \epsilon_4 \to 0^+.
\end{align*}
For $\epsilon_4 \in (0, \lambda^*)$ and a given number $L_3 > 1$, it follows that
\[
\beta L_3 m_1 e^{(\lambda^* - \epsilon_4)z} + \beta L_3 m_2 e^{(\lambda^* - \epsilon_4)z} \to 0 \quad \text{as} \quad z \to -\infty.
\]
Since $z_5 = \epsilon_4^{-1} \ln(\epsilon_4 S_1) \to -\infty$ as $\epsilon_4 \to 0^+$, we then select sufficiently small $\epsilon_4 \in (0, \lambda^*)$ such that $z_5 < z_4$ and
\[
d_1 \epsilon_4^{-1} \int_{-\infty}^{\infty} J(y)(1 - e^{-\epsilon_4 y})dy + c^* + \beta L_3 m_1 e^{(\lambda^* - \epsilon_4)z} + \beta L_3 m_2 e^{(\lambda^* - \epsilon_4)z} \geq 0
\]
for $z < z_5$. Then by (5.10), we have for $z < z_5$ that
\[
K * I^*_+(z) \leq -L_3 m_1 e^{\lambda^* z} - L_3 m_2 e^{\lambda^* z}.
\]
If $z < z_5$, then
\[
S^*_-(z) = S_1 - \epsilon_4^{-1} e^{\epsilon_4 z}.
\]
For $z < z_5$, we obtain that
\[
d_1 |J * S^*_+(z) - S^*_-(z)| = c^*(S^*_+(z)) - \frac{\beta S^*_-(z)K * I^*_+(z)}{S^*_-(z) + K * I^*_+(z) + R^*_-(z)}
\]
\[
\geq d_1 |J * S^*_+(z) - S^*_-(z)| - c^*(S^*_+(z)) - \beta K * I^*_+(z)
\]
\[
\geq d_1 \left[ \epsilon_4^{-1} e^{\epsilon_4 z} - \epsilon_4^{-1} e^{\epsilon_4 z} \int_{-\infty}^{\infty} J(y) e^{-\epsilon_4 y}dy \right] + c^* e^{\epsilon_4 z}
\]
\[
- \beta (-L_3 m_1 e^{\lambda^* z} - L_3 m_2 e^{\lambda^* z}) \quad \text{(by (5.12), (5.14) and (5.15))}
\]
\[
= e^{\epsilon_4 z} \left[ d_1 \epsilon_4^{-1} \int_{-\infty}^{\infty} J(y)(1 - e^{-\epsilon_4 y})dy + c^* + \beta L_3 m_1 e^{(\lambda^* - \epsilon_4)z} + \beta L_3 m_2 e^{(\lambda^* - \epsilon_4)z} \right]
\]
\[
\geq 0, \quad \text{(see (5.13)).}
\]
If $z > z_5$, then $S^*_-(z) = 0$ and (5.5) holds.

Proof of (5.6). For given numbers $L_3, L_4 > 1$, we have that
\[
-\frac{2\beta}{S_1} L_2^2 (-z)^{\frac{3}{2}} (z m_1 + m_2)^2 e^{\lambda^* z} + \frac{2\beta}{S_1} L_3 L_4 (-z)^{\frac{3}{2}} (z m_1 + m_2) e^{\epsilon_4 z} \to 0
\]
and
\[
\frac{1}{z} \int_{-R_1}^{R_1} J(y) y^3 e^{-\lambda^* y}dy \to 0 \quad \text{as} \quad z \to -\infty.
\]
Since $z_6 = \frac{L_2^2}{L_3} \to -\infty$ as $L_5 \to +\infty$, we then choose sufficiently large $L_5 > 1$ such that $z_6 < z_5$, such that
\[
\frac{1}{16} d_2 L_5 \int_{-R_1}^{R_1} J(y) y^2 e^{-\lambda^* y}dy - \frac{2\beta}{S_1} L_3^2 (-z)^{\frac{3}{2}} (z m_1 + m_2)^2 e^{\lambda^* z}
\]
\[
+ \frac{2\beta}{S_1} L_3 L_4 (-z)^{\frac{3}{2}} (z m_1 + m_2) e^{\epsilon_4 z} \geq 0, \quad z < z_6,
\]
\[
\int_{-R_1}^{R_1} J(y) y^2 e^{-\lambda^* y}dy + \frac{1}{z} \int_{-R_1}^{R_1} J(y) y^3 e^{-\lambda^* y}dy \geq 0, \quad z < z_6
\]
and
\[
S^*_+(z) = S_1 - \epsilon_4^{-1} e^{\epsilon_4 z} \geq S_1/2, \quad z < z_6.
\]
Recalling that $z_5 < z_4$, we have $I^*_-(z) \leq I^*_+(z)$ for $z \in \mathbb{R}$. Then

$$K * I^*_-(z) = \int_0^\infty \int_{-\infty}^\infty K(y,s)I^*_-(z-y-c^s s)dyds$$

$$\leq \int_0^\infty \int_{-\infty}^\infty K(y, s)I^*_+(z-y-c^s s)dyds$$

$$\leq -L_3 e^{\lambda^* z}m_1 - L_3 e^{\lambda^* z}m_2 \quad \text{for } z < z_6.$$  (5.19)

We infer from (5.18) and (5.19) that

$$-\beta K * I^*_-(z) + \frac{\beta S^*_-(z)K * I^*_-(z)}{S^*_-(z) + K * I^*_-(z) + R^*_+(z)}$$

$$= \frac{\beta[K * I^*_-(z)]^2 + \beta K * I^*_-(z)R^*_+(z)}{S^*_-(z) + K * I^*_-(z) + R^*_+(z)}$$

$$\geq \frac{2\beta S^*_-(z)}{S^*_-(z)} \left[ L_3^2 e^{2\lambda^* z}(zm_1 + m_2)^2 - L_3 L_4 e^{(\lambda^* + \epsilon)z}(zm_1 + m_2) \right] \quad \text{for } z < z_6.$$  (5.20)

Note that

$$(I^*_-)'(z) = -L_3 e^{\lambda^* z} - L_3 \lambda^* e^{\lambda^* z} + \frac{1}{2}L_5(-z)^{-\frac{1}{2}}e^{\lambda^* z} - L_5\lambda^*(-z)^{\frac{1}{2}}e^{\lambda^* z} \quad \text{(5.21)}$$

for $z < z_6$. Using Taylor's theorem, we have for $z < z_6$ that

$$(-z + y + c^s s)^{\frac{1}{2}} \leq (-z)^{\frac{1}{2}} + \frac{1}{2}(-z)^{-\frac{1}{2}}(y + c^s s)$$

and

$$(-z + y)^{\frac{1}{2}} \leq (-z)^{\frac{1}{2}} + \frac{1}{2}(-z)^{-\frac{1}{2}}y - \frac{1}{8}(-z)^{-\frac{3}{2}}y^2 + \frac{1}{16}(-z)^{-\frac{5}{2}}y^3.$$  

This implies that

$$J * I^*_-(z) = \int_{-\infty}^\infty J(y)I^*_-(z-y)dy = \int_{-R_1}^{R_1} J(y)I^*_-(z-y)dy$$

$$\geq \int_{-R_1}^{R_1} J(y)\left[ -L_3(z-y)e^{\lambda^*(z-y)} - L_5(-z+y)^{\frac{1}{2}}e^{\lambda^*(z-y)} \right]dy$$

$$\geq -L_3 \int_{-R_1}^{R_1} J(y)(z-y)e^{\lambda^*(z-y)}dy - L_5 \int_{-R_1}^{R_1} J(y)\left[ (-z)^{\frac{1}{2}} + \frac{y}{2}(-z)^{-\frac{1}{2}} \right]e^{\lambda^*(z-y)}dy$$

$$- L_5 \int_{-R_1}^{R_1} J(y)\left[ -\frac{1}{8}(-z)^{-\frac{3}{2}}y^2 + \frac{1}{16}(-z)^{-\frac{5}{2}}y^3 \right]e^{\lambda^*(z-y)}dy$$

$$= -L_3 e^{\lambda^* z}n_1 - L_3 e^{\lambda^* z}n_2 - L_5(-z)^{\frac{1}{2}}e^{\lambda^* z}n_1 + \frac{1}{2}L_5(-z)^{-\frac{1}{2}}e^{\lambda^* z}n_2$$

$$+ \frac{e^{\lambda^* z}}{8}L_5(-z)^{-\frac{3}{2}}\int_{-R_1}^{R_1} J(y)yg^2e^{-\lambda^* y}dy - \frac{e^{\lambda^* z}}{16}L_5(-z)^{-\frac{5}{2}}\int_{-R_1}^{R_1} J(y)y^3e^{-\lambda^* y}dy$$

(5.22)

and

$$K * I^*_-(z) = \int_0^\infty \int_{-\infty}^\infty K(y,s)I^*_-(z-y-c^s s)dyds$$
\[
\begin{align*}
&= \int_0^\infty \int_{-R_2}^{R_2} K(y, s) I_1^*(z - y - c^*s) \, dy \, ds \\
\geq &\int_0^\infty \int_{-R_2}^{R_2} K(y, s) \left[ - L_3(z - y - c^*s) e^{\lambda^*(z-y-c^*)} \\
&- L_5(z - y + c^*s)^{1/2} e^{\lambda^*(z-y-c^*)} \right] \, dy \, ds \\
\geq &- L_3 \int_0^\infty \int_{-R_2}^{R_2} K(y, s)(z - y - c^*s) e^{\lambda^*(z-y-c^*)} \, dy \, ds \\
&- L_5 \int_0^\infty \int_{-R_2}^{R_2} K(y, s) \left[ (-z)^{1/2} + \frac{1}{2}(-z)^{-1/2}(y + c^*) \right] e^{\lambda^*(z-y-c^*)} \, dy \, ds \\
= &- L_3 e^{\lambda^*} m_1 - L_3 e^{\lambda^*} m_2 - L_5(-z)^{1/2} e^{\lambda^*} m_1 + \frac{1}{2} L_5(-z)^{-1/2} e^{\lambda^*} m_2.
\end{align*}
\]
\[= -L_3 e^{λ z}[zΘ(λ^*, c^*) + Θ_λ(λ^*, c^*)]
\]
\[\quad - \frac{1}{2} L_5(-z)^{-\frac{3}{2}} e^{λ z} \Theta(λ^*, c^*) + \frac{1}{2} L_5(-z)^{-\frac{3}{2}} e^{λ z} Θ_λ(λ^*, c^*)\]
\[\quad + (-z)^{-\frac{3}{2}} e^{λ z} \left[ \frac{1}{16} d_2 L_5 \int_{-R_1}^{R_1} J(y) y^2 e^{-λ^* y} dy - \frac{2β}{S_1} L_5(-z)^{-\frac{3}{2}} (zm_1 + m_2)^2 e^{λ z} \right]\]
\[\quad + \frac{2β}{S_1} L_4(-z)^{\frac{3}{2}} (zm_1 + m_2) e^{λ z} \left[ \int_{-R_1}^{R_1} J(y) y^2 e^{-λ^* y} dy + \frac{1}{z} \int_{-R_1}^{R_1} J(y) y^3 e^{-λ^* y} dy \right]\]
\[\geq 0\]
for sufficiently large \(L_5 > 1\). If \(z > z_0\), then \(I^*_+(z) = 0\) and (5.6) follows immediately.

The proof of this lemma is finished. \(\square\)

By the upper and lower solutions \(S^*_\pm(z)\), \(I^*_+(z)\) and \(R^*_\pm(z)\) constructed above and the similar arguments in Section 4, we can deduce that model (1.7) with (1.11) admits a nontrivial positive traveling wave solution with critical speed. Particularly, as \(z \to -∞\), \(I(z) = O(-ze^{λ z})\) for \(R_0 > 1\) and \(c = c^*\). Combining Sections 4 and 5, we complete the proof of Theorem 1.2.

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