Minimal conductivity and signatures of quantum criticality in ballistic graphene bilayer

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Abstract – We study the ballistic conductivity of graphene bilayer in the presence of next-nearest neighbor hopping between the layers. An undoped and unbiased system was found in Rut G. and Rycerz A., Phys. Rev. B, 89 (2014) 045421, to show a nonuniversal (length-dependent) conductivity \( \sigma(L) \), approaching the value of \( \sigma_\ast = 3/\pi \simeq 0.95 \) for large \( L \). Here we demonstrate one-parameter scaling and determine the scaling function \( \beta(\sigma) = d \ln \sigma/d \ln L \). The scaling flow has an attractive fixed point \( (\beta(\sigma_\ast) = 0, \beta'(\sigma_\ast) < 0) \) reproducing the scenario predicted for random impurity scattering of Dirac fermions with Coulomb repulsion, albeit the system considered is perfectly ballistic and interactions are not taken into account. The role of electrostatic bias between the layers is also briefly discussed.

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Introduction. – One of the most unexpected properties of graphene—a two-dimensional form of carbon discovered in 2005 [1]—is the pseudodiffusive nature of charge transport via undoped ballistic samples, manifesting itself by the fact that dc conductance obeys Ohm’s law for classical conductors characterized by the universal quantum value of the conductivity [2–5], namely

\[
G = \sigma_0 \frac{W}{L}, \quad \sigma_0 = \frac{1}{\pi} \left[ \frac{se^2}{\hbar} \right],
\]

where \( W \) is the sample width, \( L \) is the length [6], and the units of \( se^2/\hbar \) with \( s = 4 \) are chosen to account the spin and valley degeneracies. Such a macroscopic quantum phenomenon has a remarkable high-frequency analog, i.e., the visible light opacity of graphene also takes quantized values [7]. Although the opacity directly scales with the number of graphene layers, such an additive property usually does not apply for dc conductance [8].

Early theoretical works on ballistic graphene bilayers [9,10] showed that the minimal conductivity at zero bias situation changes abruptly as a function of the next-nearest neighbor interlayer hopping integral \( t' \), taking the value of \( \sigma_0 = 1/\pi \) for \( t' = 0 \), or \( \sigma_\ast = 3\sigma_0 \) for any \( t' \neq 0 \) [11], provided that \( s = 8 \) in eq. (1) due to the additional layer degeneracy. The appearance of such a quantum-critical behavior was attributed to the topological transition of the Fermi surface at low energies [8]. Experimental values of the minimal conductivity are generally lower than \( \sigma_\ast \), covering the range from \( \sim \sigma_0 \) [12] up to \( 2.5\sigma_0 \) [13].

We have recently shown, employing the Landauer-Büttiker formalism, that the minimal conductivity of finite, ballistic samples is not universal but length dependent [14], and can be rationalized, for large \( L \), as

\[
\sigma(L) \simeq \sigma_\ast \left[ 1 - \left( \frac{\lambda}{L} \right)^\gamma \right],
\]

where the characteristic length \( \lambda = \lambda(t') \), and \( \gamma < 1 \) is the parameter-independent exponent. In turn, the predictions of refs. [9,10] are restored for \( L \to \infty \), whereas in the opposite limit \( L \to 0 \) one gets \( \sigma(L) \to \sigma_0 \) regardless of \( t' = 0 \) or \( t' \neq 0 \). It is also shown in ref. [14] that the universal conductivity is restored for resonances with the Landau levels at high magnetic fields [15].

In this paper, we point out that the scaling function

\[
\beta(\sigma) = \frac{d \ln \sigma}{d \ln L},
\]

which plays a central role in the conceptual understanding of the metal insulator transition [16] and is widely considered in the context of disordered Dirac or spin-orbit systems [17–19] (see fig. 1), also unveils an intriguing analogy between interaction-induced quantum criticality in disordered Dirac systems [19] and transport properties of...
ballistic graphene bilayer with skew interlayer hoppings. The paper is organized as follows: In the next section we present the mode-matching analysis for transport of Dirac fermions via finite samples of ballistic bilayer. In the third section we discuss the functions $\sigma(L)$ for different values of $t'$ and demonstrate one-parameter scaling. Possible effects of nonzero bias between the layers are summarized in the fourth section. A brief overview of the results given in the last section.

Mode-matching for Dirac fermions. – The analysis starts from the four-band effective Hamiltonian for low-energy excitations [8], which can be written as

$$H = \xi \begin{pmatrix} -V/2 & e^{-i\theta}\pi & \xi t_\perp & 0 \\ e^{i\theta}\pi & -V/2 & 0 & \nu e^{-i\theta}\pi \\ \xi t_\perp & 0 & V/2 & e^{i\theta}\pi \\ 0 & \nu e^{i\theta}\pi & e^{-i\theta}\pi & V/2 \end{pmatrix} ,$$

(4)

where the valley index $\xi = 1 (-1)$ for the $K' (K)$ valley, $\theta$ denotes the angle between the $x$-axis and the armchair direction (see fig. 2), $\pi = h v_F (i\partial_x + \partial_y)$ with $v_F = \sqrt{3} t_0 a/2 \approx 10^6 \text{m/s}$ being the Fermi velocity (in a monolayer) defined via the interlayer hopping $t_0 = 3.16 \text{eV}$ [20] and the lattice parameter $a = 0.246 \text{nm}$, the nearest-neighbor interlayer hopping $t_\perp = 0.38 \text{eV}$, $\nu = t'/t_0$ with $t'$ being the next-nearest-neighbor interlayer hopping, and $V$ is the electrostatic bias between the layers. We further consider solutions of the Dirac equation $H\Psi = E\Psi$ in the form

$$\Psi(x, y) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \\ \phi_4(x) \end{pmatrix} e^{i k_y y} \exp(i k_x x) ,$$

(5)

due to the translational invariance in the $y$-direction.

We focus here on a zero bias case $V = 0$ (for the discussion of $V \neq 0$ case see the fourth section), for which a general solution for $E = 0$ (the sample area) and the $K$ valley reads

$$\begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \\ \phi_4(x) \end{pmatrix} = c_1 \begin{pmatrix} -\alpha_+ f_+^+ \\ 0 \\ 0 \\ -\alpha_+ f_+^- \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -\alpha_- f_-^- \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ e^{-i\theta} f_-^+ \\ -\alpha_- f_-^- \\ -\alpha_+ f_+^- \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ -e^{-i\theta} f_-^- \\ -\alpha_- f_-^- \\ -\alpha_+ f_+^- \end{pmatrix} ,$$

(6)

where $\alpha_+ = \nu \exp(i 3\theta) [i + \xi \sqrt{8 k_y |\exp((3i\theta) t_\perp)| + |1]$, $f_+^\pm = \exp((\pm k_y + \alpha_+^\pm i \tilde{t}) x)$, $\tilde{t} = t_\perp / (h v_F)$, and the coefficients $c_1, \ldots, c_4$ are to be determined later. In the opposite limit of $E \to \infty$ (heavily doped leads) we obtain

$$\begin{pmatrix} \phi_{1,s}(x) \\ \phi_{2,s}(x) \\ \phi_{3,s}(x) \\ \phi_{4,s}(x) \end{pmatrix} = \mathcal{N}_s(\nu) \begin{pmatrix} - (\mu_+/2) \exp(-2i\theta) \\ s \eta_+ (\mu_+ / \sqrt{2}) \exp(-i\theta) \\ s \sqrt{2} \eta_\perp \exp(i\theta) \\ 1 \end{pmatrix} ,$$

(7)

where $s = \text{sgn}(k_y)$, $\mu_\pm = \nu \pm \sqrt{\nu^2 + 4}$, $\eta_\perp = 1 / \sqrt{2 + \nu \mu_\perp}$, and the factors $\mathcal{N}_s(\nu) = \sqrt{\mu_+/ \mu_- / 2 (\mu_+ + 2)}$ are chosen to normalize the current. Matching the solutions given by

Fig. 1: (Colour on-line) Schematic scaling functions $\beta(\sigma)$ (3) for two-dimensional disordered Dirac (solid lines) and spin-orbit (dashed lines) systems. Left: noninteracting case [17]; right: Coulomb interaction included [19]. Arrows indicate the flows of the dimensionless conductivity $\sigma$ with increasing $L$. (Adapted from ref. [19].)

Fig. 2: A strip of bilayer graphene (shaded area) of width $W$, contacted by two electrodes (white rectangles) at a distance $L$. A voltage source drives the electric current through the device. Separate top- and bottom-gate electrodes (not shown) allow both the Fermi energy $E$ and the bias between the layers $V$ to be controlled electrostatically. A magnified view exhibits the crystallographic orientation, with the angle $\theta$ between an armchair direction (dashed line in the main plot) and $x$-axis of the coordinate system (see bottom left).
Minimal conductivity and signatures of quantum criticality in ballistic graphene bilayer

\[
\begin{pmatrix}
\phi_{1,-1}^+ & \phi_{1,-1}^- & \alpha_{1}^+ & \alpha_{1}^-
\phi_{2,-1}^+ & \phi_{2,-1}^- & 0 & 0 & -e^{-i\theta} & -e^{-i\theta} & 0 & 0 & 0 & 0
\phi_{3,-1}^+ & \phi_{3,-1}^- & 0 & 0 & \alpha_{1}^- & \alpha_{1}^+ & 0 & 0 & 0 & 0
\phi_{4,-1}^+ & \phi_{4,-1}^- & -e^{-i\theta} & -e^{-i\theta} & 0 & 0 & \phi_{1,1}^+ & \phi_{1,1}^- & 0 & 0
0 & 0 & \alpha_{1}^+ f_1^+ & \alpha_{1}^- f_1^- & 0 & 0 & \phi_{3,1}^+ & \phi_{3,1}^- & t_p^+ & t_n^+
0 & 0 & 0 & 0 & -e^{-i\theta} f_1^- & -e^{-i\theta} f_1^- & c_{1}^+ & c_{1}^- & 0 & 0
0 & 0 & 0 & 0 & \alpha_{1}^- f_1^- & \alpha_{1}^+ f_1^- & c_{3}^+ & c_{3}^- & 0 & 0
0 & 0 & -e^{i\theta} f_1^+ & -e^{i\theta} f_1^- & 0 & 0 & \phi_{3,1}^+ & \phi_{3,1}^- & t_p^+ & t_n^+
\end{pmatrix}
= \begin{pmatrix}
-\phi_{1,1}^+
-\phi_{1,1}^-
c_{1}^+
c_{1}^-
c_{3}^+
c_{3}^-
t_p^+
t_n^+
0
0
\end{pmatrix},
\]
(8)

Table 1: Least-square fitted parameters \(\sigma_*, \lambda, \gamma\) of the function \(\sigma(L)\), defined by eq. (2), corresponding to the lines in fig. 3. The values of \(L_{0.01}\), such that for \(L \geq L_{0.01}\) the function \(\sigma(L)\) matches the actual conductivity with accuracy better that 1%, are given in the last column.

| \(t'(eV)\) | \(\sigma_*(8e^2/h)\) | \(\lambda/l_\|\) | \(\gamma\) | \(L_{0.01}/l_\|\) |
|-----------|-----------------|-----------------|--------|----------------|
| 0.1       | 0.96            | 16.8            | 0.50   | 935            |
| 0.2       | 0.98            | 12.6            | 0.55   | 457            |
| 0.3       | 0.99            | 6.1             | 0.53   | 278            |

\(K^\prime\) is preserved, and each transmission eigenvalue from one valley has a copy in the other valley [21].

**Conductivity and one-parameter scaling.** – Next, the dimensionless conductivity is determined from the Landauer-Büttiker formula [22]

\[
\sigma(L) = \frac{L}{W} \sum_{(k_y)} \text{Tr} \left[ t(k_y) t^\dagger(k_y) \right],
\]
(10)

where we have assumed periodic boundary conditions at \(y\)-direction, leading to the quantization of the transverse wave number \(k_y = 0, \pm 2\pi/W, \pm 4\pi/W, \ldots\). Our numerical results for \(E = V = 0\) are summarized in fig. 3 and table 1. For demonstrative purposes, we have chosen \(W/L = 20\) and \(\theta = \pi/4\). For any \(t' \neq 0\), the conductivity given by eq. (10) slowly grows with \(L\), taking the values from the interval \(1/\pi \leq \sigma(L) \leq 3/\pi\), with the upper (lower) bound approached for \(L \to 0\) (\(L \to \infty\)). The best-fitted approximating functions \(\sigma(L)\) of the form given by eq. (2) (lines in fig. 3) rationalize the numerical results for large \(L\) (datapoints). The least-square fitted parameters \(\sigma_*\) and \(\gamma\) (see table 1) weakly depend on \(t'\), taking the values close to \(\sigma_* \approx 3/\pi\) and \(\gamma \approx 1/2\) for small \(t'\). These scaling characteristics appear generically for other crystallographic orientations \(\theta\), except for \(\theta = \pi/6 + n\pi/3\) (with integer \(n\)), corresponding to the propagation along a zigzag direction, for which a lower value of \(\gamma \approx 1/4\) was found (see also ref. [14]).

The scaling function \(\beta(\sigma)\) (3) can be obtained by numerical differentiation of \(\sigma(L)\) given by eq. (10) (see inset in fig. 3). For the asymptotic range, eq. (2) leads to

\[
\beta(\sigma) \simeq \gamma (1 - \sigma_*/\sigma),
\]
(11)

Fig. 3: (Colour on-line) Minimal conductivity of an unbiased graphene bilayer as a function of the sample length \(L\) (specified in units of \(l_\| = h v_F/\pi \approx 1.60\) nm). Different datapoints correspond to different values of the next-nearest neighbor interlayer hopping: \(t' = 0.1\) eV (\(\triangle\)), \(0.2\) eV (\(\circ\)), and \(0.3\) eV (\(\bullet\)). Lines depict best-fitted approximating functions \(\sigma(L)\) (2) (see table 1 for further details). The inset shows the scaling function \(\beta(\sigma)\) (3) obtained by numerical differentiation of the data. Notice that the approximating function is replotted for \(t' = 0.1\) eV only, as the three lines overlap for the variables used in the inset.) The sample aspect ratio is fixed at \(W/L = 20\); the crystallographic orientation is \(\theta = \pi/4\).

Eqs. (6), (7) at \(x = 0\) and \(x = L\) leads to

\[
\text{see eq. (8) above}
\]

where we have further defined \(\phi_{\pm}^\pm \equiv \phi_{\pm}^\pm(0), f_{\pm}^\pm = f_{\pm}^\pm(L)\).

Solving the linear system of equations (8) one obtains the transmission and reflection matrices for a given transverse wave number

\[
t(k_y) = \begin{pmatrix} t_p^+ & t_n^- \\ t_p^- & t_n^+ \end{pmatrix}, \quad r(k_y) = \begin{pmatrix} r_p^+ & r_n^- \\ r_p^- & r_n^+ \end{pmatrix},
\]
(9)

where the internal structure arises from the presence of two subbands in the dispersion relation. At zero magnetic field, time-reversal symmetry coupling the valleys \(K\) and \(\overline{K}\)
with $\sigma = \sigma_* \simeq 3/\pi$ being an attractive fixed point ($\beta'(\sigma_*) < 0$ for $\gamma \simeq 1/2 > 0$) of the renormalization group flow. Such a scenario, earlier predicted for disordered Dirac systems with Coulomb interaction [19], is reproduced by our results for graphene bilayer. The values of $\beta(\sigma)$ obtained numerically become $t'$-independent and follow eq. (11) for $\sigma \gtrsim 0.8$.

This surprising coincidence (it is worth stressing here that the system we consider is ballistic and no interactions are taken into account) seems difficult to understand in terms of the existing symmetry-based theory of localization [18,19]. Particular features of the results suggest that next-nearest neighbor interlayer hoppings, apart from breaking the rotational symmetry of the Hamiltonian in a single valley (a phenomenon known as trigonal warping [8]), may also induce corrections to $\beta(\sigma)$ of the Altshuler-Aronov type [23], destroying the supermetallic phase in graphene. A further clarification of the above-mentioned issue requires a numerical study of charge transport through the disordered graphene bilayer, which is beyond the scope of this paper.

Effects of finite bias between the layers. – Probably, the most intriguing property of a graphene bilayer is the possibility to convert it from semimetal to narrow-gap semiconductor by applying a perpendicular electrostatic field [24–28], leading to a finite bias between the layers $V$ in the effective Hamiltonian $H$ (4). Also, some experimental works showed that the energy gap may also appear spontaneously, due to electron-electron interactions, for bilayer samples close to the charge-neutrality point [29,30]. For these reasons, the extension of our discussion on the $V \neq 0$ case is desirable.

In such a case, the effective Dirac equation $H \Psi = E \Psi$, with $\Psi(x, y)$ in the form given by eq. (5), is integrated numerically for the sample area $(0 < x < L)$, separately for each value of the transverse wave number $k_y$. The obtained solutions are then matched with wave functions in the leads (see eq. (7)), in analogy with the procedure presented in the second section.

The resulting conductivity spectra, for selected values of $V$ and $t'$, are displayed in figs. 4(a)–(d). For demonstrating purposes the sample dimensions are fixed at $W/L = 20$, $L = 48$ $\hbar v_F / t_{\perp}$. For the unbiased sample case (see fig. 4(a)) the conductivity systematically grows with increasing $t'$ in the small vicinity of the Dirac point, the width of which can be roughly approximated by $|E| \lesssim E_L$, with

$$E_L = \frac{1}{4} t_{\perp} (t' / t_0)^2$$

being the Lifshitz energy [8], reaching the value of $E_L \simeq 1$ meV for $t' = 0.32$ eV. For $V = 0$ and higher Fermi energies, $\sigma$ is weakly affected by $t'$. For $V > 0$, the conductivity is strongly suppressed in the range of $|E| < V/2$ for any $t'$ (see figs. 4(b)–(d)), provided that $V \gg E_L$. Again, for sufficiently high energies the conductivity is almost unaffected by either the value of $V$ or $t'$.

![Fig. 4](Image)

**Fig. 4.** (Colour on-line) Conductivity of a graphene bilayer as a function of the Fermi energy. The value of bias between the layers $V$ is varied between the panels. Different lines at each panel depict the data obtained numerically for different values of the next-nearest neighbor interlayer hopping: $t' = 0$ (red dashed line), $t' = 0.16$ eV (green dash-dotted line), and $t' = 0.32$ eV (blue solid line). The vertical dashed line marks the value of $E = V/2$, the horizontal solid line corresponds to $\sigma_0 = 1/\pi$. The sample length is fixed at $L = 48 l_\perp \simeq 77$ nm; the remaining system parameters are the same as in fig. 3.

Probably, the most interesting feature of the results presented in fig. 4 is that the dimensionless conductivity at its first maximum as a function of $E$ reaches the value close to $\sigma \simeq 1/\pi$ for $V \gg E_L$ and arbitrary $t'$, while it is significantly higher for $V = 0$. For this reason, the measurements of the conductivity spectra of ballistic samples at zero magnetic field, and different biases between the layers, may constitute an alternative experimental method for detecting the Lifshitz transition in a graphene bilayer, supplementing the recent study focusing on the anomalies in the sequence of Landau levels [31], at least in principle.

A brief overview. – We have investigated, by means of analytical mode-matching for the effective Dirac equation, the length-dependent minimal conductivity $\sigma(L)$ of unbiased graphene bilayer with the nearest ($t_\perp$) and the next-nearest neighbor ($t'$) interlayer hoppings included. The scaling function $\beta(\sigma) = d \ln \sigma / d \ln L$ was found i) to be insensitive to the precise value of $t'$ and to the crystallographic orientation of the sample, provided that the physical dimensions are in the asymptotic range, i.e., that $W \gg L \gg \hbar v_F / t_{\perp}$ (with $v_F$ being the energy-independent Fermi velocity in a monolayer), and ii) to have an attractive fixed point at $\sigma_0 \simeq 3/\pi$. These features closely resemble the quantum-critical behavior predicted theoretically for disordered Dirac systems with Coulomb interaction [19], although the system we consider is ballistic and interactions are not taken into account. Our results show that the well-known correspondence between charge-transfer characteristics of a classical diffusive conductor and perfectly clean monolayer graphene [3,32] is accompanied by another, probably more surprising, analogy between chaotic impurity scattering of interacting Dirac
fermions and ballistic transport via bilayer samples, which gets unveiled when one-parameter scaling is demonstrated.

The actual effects of electron-electron interaction in bilayer graphene are generally beyond the scope of this paper. Nevertheless, it is worth pointing out that possible effects primarily include the gap opening due to spontaneous breaking of the symmetry between the layers [29,30]. We show, by numerically analysing the transport through a biased bilayer, that opening a few meV gap \( V \) (i.e., larger than the Lifshitz energy) leads to the appearance of conductivity peaks at Fermi energies \( E \simeq \pm V/2 \), where \( \sigma \simeq 1/\pi \), reproducing the dimensionless conductivity of a ballistic monolayer. These are the reasons for which an extensive experimental study of size-dependent conductance for clean bilayer samples (with their lengths \( L > 1 \mu m \) and widths \( W \gg L \)), which is missing so far, seems crucial to determine the significance of the factors such as the trigonal warping and the electron-electron interaction in the effective description of bilayer-based graphene nanodevices.

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[11] We further note that the result of ref. [14], reporting the conductivity quite close to \( \sigma_\ast \), suggests that the influence of electron-electron interactions on charge transport in large ballistic bilayer samples may be negligible.