EXPOSED CIRCUITS, LINEAR QUOTIENTS, AND CHORDAL CLUTTERS

ANTON DOCHTERMANN

ABSTRACT. A graph $G$ is said to be chordal if it has no induced cycles of length four or more. In a recent preprint Culbertson, Guralnik, and Stiller give a new characterization of chordal graphs in terms of sequences of what they call ‘edge-erasures’. We show that these moves are in fact equivalent to a linear quotient ordering on $I_G$, the edge ideal of the complement graph. Known results imply that $I_G$ has linear quotients if and only if $G$ is chordal, and hence this recovers an algebraic proof of their characterization. We investigate higher-dimensional analogues of this result, and show that in fact linear quotients for more general circuit ideals of $d$-clutters can be characterized in terms of removing exposed circuits in the complement clutter. Restricting to properly exposed circuits can be characterized by a homological condition. This leads to a notion of higher dimensional chordal clutters which borrows from commutative algebra and simple homotopy theory. The interpretation of linear quotients in terms of shellability of simplicial complexes also has applications to a conjecture of Simon regarding the extendable shellability of $k$-skeleta of simplices. Other connections to combinatorial commutative algebra, chordal complexes, and hierarchical clustering algorithms are explored.

1. INTRODUCTION

Chordal graphs are a widely studied class of combinatorial objects, with connections to various algorithmic and structural questions and generalizations in a variety of directions. Perhaps a major reason for their wide appeal is their various characterizations in terms of seemingly unrelated properties, incorporating topological, combinatorial, and algebraic notions. For instance the clique complex of a chordal graph has collapsible components, whereas the independence complex of a chordal graph is known to be vertex decomposable [10], which in particular implies that is has the homotopy type of a wedge of spheres.

Recently in [9] a new characterization of chordal graphs was given in terms of performing a series of ‘edge-erasures’ on a complete graph. For this we say that an edge $e$ is a graph $G$ is exposed if $e$ is contained in a unique maximal clique $K$ of $G$. We say that $e$ is properly exposed if $|K| > 2$ (i.e. $e$ is contained in some triangle). If $G$ is a graph and $e \in G$ is properly exposed we say that $G - \{e\}$ is obtained from $G$ via an edge erasure. With this notation the authors of [9] prove the following.

Theorem 1.1 ([9]). A connected graph $G$ is chordal if and only if $G$ can be obtained from a complete graph by a sequence of edge erasures.

As the authors point out, this description of a chordal graph in terms of sequences of edges has a different flavor than other characterizations in terms of simplicial neighborhoods of vertices, etc. In [9] this characterization is used to give a new algorithm for finding a minimum spanning tree in a finite metric space, a modified version of the greedy algorithm due to Kruskal. The notion of an exposed edge is reminiscent of the elementary collapses from simple homotopy theory and in

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particular makes sense in the more general context of hypergraphs and d-clutters. We discuss this further below.

Chordal graphs also make an appearance in the context of combinatorial commutative algebra. Suppose $G = (V, E)$ is a graph on vertex set $V = [n] = \{1, 2, \ldots, n\}$. For a fixed field $\mathbb{K}$ one can construct the edge ideal $I_G$ in the polynomial ring $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$. By definition $I_G$ is the monomial ideal generated by quadratic monomials corresponding to edges of the $G$:

$$I_G = \langle x_i x_j : ij \in E(G) \rangle.$$ 

We note that any squarefree quadratic monomial ideal $I$ can be realized as the edge ideal of some graph. Typical research questions involve studying how algebraic properties of $I_G$ relate to combinatorial properties of the underlying graph $G$.

A well-known theorem of Fröberg [13] characterizes chordal graphs in terms of an algebraic property of the associated edge ideal: a graph $G$ is chordal if and only if $\text{t } I_G$, the edge ideal of the complement graph, has a linear resolution. The property of having a linear resolution describes a particularly low 'complexity' in the relations among the generators of $I_G$ (along with the relations among the relations, etc.). More recently in [16] it has been shown that if an edge ideal $I_G$ has a linear resolution then in fact it has linear quotients, a condition that is stronger in the case of more general ideals. To say that an ideal $I$ has linear quotients means that there exists an ordering of the generators $I = \langle m_1, m_2, \ldots, m_k \rangle$ of the ideal such that each colon ideal $(I_j : m_{j+1})$ is generated by a collection of linear forms. Here $I_j = \langle m_1, m_2, \ldots, m_j \rangle$. More details regarding these algebraic concepts are given in the next section.

The notion of an edge ideal of a graph generalizes to the context of d-clutters (uniform hypergraphs where the edge set consists of d-subsets of $[n]$), where generators again correspond to circuits (the 'edges' of the d-clutter). If $e$ is a circuit of a d-clutter $C$ we use $x_e$ to denote the squarefree monomial that it defines. The notion of an exposed edge also generalizes: if $e \in C$ is a circuit of some d-clutter $C$ on vertex set $[n] = \{1, 2, \ldots, n\}$ we say that $e$ is exposed if it is uniquely contained in some maximal d-clique $K$. We say that $e$ is properly exposed if $|K| > d$. Our main result says that removing an exposed circuit from a d-clutter corresponds to adding a generator to the circuit ideal of the complement clutter that satisfies a particular algebraic property.

**Theorem 3.1.** Suppose $C$ is a d-clutter and let $e \in C$ be a circuit in $C$. Then $e$ is an exposed circuit if and only if $x_e$ is a linear divisor for the ideal $I_C$, where $\overline{C}$ is the complement of $C$. Moreover $e$ is contained in a unique maximal clique $K$ if and only if the colon ideal $(I_C : x_e)$ is generated by variables corresponding to vertices in the complement of $K$.

As a consequence we obtain an algebraic proof of one part of Theorem 1.1, namely that a graph $G$ is obtained from a complete graph through a sequence of removing exposed edges if and only if $I_G$, the edge ideal of the complement graph, has linear quotients. The result of [16] then implies that this is the case if and only if $G$ is chordal. In addition, it is not hard to show that a chordal graph $G$ obtained from a sequence of exposed edges is connected if and only if each edge in the sequence is properly exposed. The property of a graph $G$ being connected also has an algebraic interpretation in terms of the Betti table of the underlying edge ideal $I_G$. This generalizes to the context of circuit ideals of clutters where we establish the following higher dimensional analogue of Theorem 1.1.
Proposition 3.8. A $d$-clutter $C$ can be obtained from the complete $d$-clutter $K^d_n$ through a sequence of circuit erasures if and only if $I_C$ has linear quotients and $\text{pdim}(I_C) < n - d$.

Our result also provides a formula for the Betti numbers of an ideal with linear quotients (generated in a fixed degree) in terms of the combinatorics of the exposed faces removed in the complement, see Corollary 3.9.

It is known that squarefree ideals with linear quotients are strongly related (via Alexander duality) to the notion of shellability for a simplicial complex. A shellable simplicial complex $\Delta$ is said to be extendably shellable if every shelling of a subcomplex of $\Delta$ can be continued to a shelling of $\Delta$. Not all shellable complexes are extendably shellable (for instance certain $d$-dimensional simplicial spheres for $d \geq 3$, as discussed in [24]) but a conjecture of Simon [22] says that all $k$-skeleta of a simplex on $[n]$ are extendably shellable. In Section 4.1 we show how our results lead to a proof of this conjecture for the case $k \geq n - 3$ (which was also obtained recently in [6] using other methods).

Corollary 4.4. For all $k \geq n - 3$, the $k$-skeleton of a simplex on vertex set $[n]$ is extendably shellable.

As we have seen, a sequence of deleting (properly) exposed edges from a complete graph gives a characterization of (connected) chordal graphs. Hence Theorem 3.1 provides a candidate for a notion of a higher dimensional `chordal complex` which borrows from simple homotopy and combinatorial commutative algebra. In recent years several authors have introduced (mostly independent) notions of chordal complexes which generalize the various characterizations of chordal graphs to higher dimensions. As far as we know the direct connection to free faces and elementary collapses has not been considered, although the recent preprint [3] explores similar territory. We briefly discuss these approaches in Section 4.2 where we also offer a conjectural connection to the constructions discussed here.

The rest of the paper is organized as follows. In Section 2 we review relevant definitions, including basic notions from clutter theory and combinatorial commutative algebra. In Section 3 we prove the results mentioned above and discuss some further corollaries and examples. In Section 4.2 we discuss applications to shellability and higher dimensional notions of chordal complexes. We end with some discussion regarding connections to data clustering (the original motivation for [9]), as well as some open problems.

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2. Definition and objects of study

2.1. Clutters and simplicial complexes. We begin by recalling some revelant combinatorial notions. Recall that a $d$-clutter (or $d$-uniform hypergraph) on vertex set $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$, each of size $d$. Note that a (simple) graph is the same as a 2-clutter. In this context the elements of $C$ are called circuits. The complement of a $d$-clutter $C$, denoted $\overline{C}$, is the $d$-clutter on
the same vertex set \([n]\), where a \(d\)-subset \(S \subset [n]\) is a circuit in \(\mathcal{C}\) if and only if \(S \notin \mathcal{C}\). An independent set of \(\mathcal{C}\) is a subset of \([n]\) containing no circuit. For any integer \(d \geq 2\) we use \(K_n^d\) to denote the complete \(d\)-clutter on vertex set \([n]\), which by definition consists of all \(d\)-subsets of \([n]\). It is customary to use \(K_n\) to denote \(K_n^2\), the complete graph.

If \(\mathcal{C}\) is a \(d\)-clutter then a \(d\)-clique (or just clique if the context is clear) is a nonempty collection of vertices \(S \subset [n]\) with the property that \(|S| < d\) or if \(|S| \geq d\) every \(d\) subset of \(S\) is a circuit of \(\mathcal{C}\). The clique is said to be maximal if \(S\) is maximal with this property. If \(\mathcal{C}\) is a \(d\)-clutter and \(e \in \mathcal{C}\) is a circuit we say that \(e\) is exposed if \(e\) is contained in a unique maximal clique \(S\) in \(\mathcal{C}\). We say that \(e\) is properly exposed if \(|S| > d\).

A simplicial complex \(\Delta\) on vertex set \([n] = \{1, 2, \ldots, n\}\) is a collection of subsets of \([n]\) (called the faces of \(\Delta\)) with the property that any subset of a face of \(\Delta\) is itself a face of \(\Delta\). If a face \(F \in \Delta\) has cardinality \(d + 1\) we say that has dimension \(d\) (and call it a \(d\)-face). A subset \(\sigma \subset [n]\) is a minimal non-face of \(\Delta\) if \(\sigma \notin \Delta\), but any proper subset of \(\sigma\) is a face of \(\Delta\). The maximal faces of \(\Delta\) (under inclusion of sets) are called facets, and \(\Delta\) is said to be pure if all facets have the same dimension. The Alexander dual of \(\Delta\) is the simplicial complex \(\Delta^*\) on vertex set \([n]\) with faces given by

\[
\Delta^* = \{\sigma \subset [n] : [n] \setminus \sigma \notin \Delta\}.
\]

In particular the facets of \(\Delta^*\) are given by the complements of minimal non-faces of \(\Delta\).

A pure \(d\)-dimensional simplicial complex \(\Delta\) is said to be shellable if there is an ordering of the facets \(F_1, F_2, \ldots, F_s\) such that for all \(k = 2, 3, \ldots, n\) the simplicial complex induced by

\[
(\bigcup_{i=1}^{k-1} F_i) \cap F_k
\]

is pure of dimension \(d - 1\).

Note that clutters and simplicial complexes are related via the following constructions (we follow the conventions of [23]). For any clutter \(\mathcal{C}\) on vertex set \([n]\) let

\[
I(\mathcal{C}) = \{\sigma \subset [n] : \sigma \text{ is an independent set of } \mathcal{C}\}
\]

denote the independence complex of \(\mathcal{C}\). For any simplicial complex \(\Delta\) let \(\mathcal{C}(\Delta)\) denote the clutter consisting of all minimal non-faces of \(\Delta\). Then one can check that

\[
\mathcal{C}(I(\mathcal{C})) = \mathcal{C} \quad \text{and} \quad I(\mathcal{C}(\Delta)) = \Delta.
\]

2.2. Circuit ideals and linear quotients. Next we recall some relevant notions from commutative algebra. We will fix a field \(K\) and let \(R = K[x_1, x_2, \ldots, x_n]\) denote the polynomial ring on \(n\) variables. A \(d\)-clutter \(\mathcal{C}\) on vertex set \([n] = \{1, 2, \ldots, n\}\) naturally gives rise to a monomial ideal in \(R\). For this if \(e = \{v_1, v_2, \ldots, v_d\} \subset [n]\) is any subset of the vertex set we let

\[
x_e = x_{v_1} x_{v_2} \cdots x_{v_d}
\]

denote the corresponding monomial in \(R\). We then let \(I_\mathcal{C}\) denote the circuit ideal of \(\mathcal{C}\), generated by all such monomials corresponding to circuits of \(\mathcal{C}\):

\[
I_\mathcal{C} = \langle x_e : e \in \mathcal{C} \rangle.
\]
When \( d = 2 \) we often say that \( I_C \) is the edge ideal of the underlying graph \( C \). We note that any squarefree monomial ideal generated in degree \( d \) can be thought of as the circuit ideal of a \( d \)-clutter (and vice versa) so these concepts are equivalent. Quadratic squarefree monomial ideals are precisely the edge ideals of (simple) graphs.

We will be interested in homological properties of circuit ideals. Given a graded ideal (or more generally a graded \( R \)-module) \( I \), a free resolution of \( I \) is an exact sequence

\[
0 \leftarrow I \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_p,
\]

where each

\[
F_i = \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{i,j}}
\]

is a free \( R \)-module and each map is a homogeneous module homomorphisms. Here \( R(-j) \) indicates the ring \( R \) with the shifted grading, so that for all \( a \in \mathbb{Z} \) we have

\[
R(-j)_a = R_{a-j}.
\]

Note that replacing the last two maps in Equation 1 with \( 0 \leftarrow R/I \leftarrow R \leftarrow F_0 \) provides a resolution of the quotient ring \( R/I \), so we will sometimes move between the two notions.

The resolution is said to be minimal if the rank of each \( F_i \) is minimum among all resolutions of \( I \). In this case we have \( \beta_{i,j} = \beta_{i,j}(I) = Tor^R_i(I, R) \), and these integers are called the graded Betti numbers of \( I \). The ordinary \( i \)th Betti number is given by \( \beta_i = \sum_{j \in \mathbb{Z}} \beta_{i,j} \). For each \( i = 2, 3, \ldots p \), we can think of the maps \( \partial_i : F_i \to F_{i-1} \) as matrices with entries in \( R \), and the ideal \( I \) is said to have a linear resolution if all entries are linear forms. If \( I \) is generated in degree \( d \) this is equivalent to having if \( \beta_{i,j} = 0 \) whenever \( i, j \) satisfy \( j \neq i + d \). We will often think of homological properties of an ideal that are preserved as we add one generator at a time. For this we need the following notion.

**Definition 2.1.** Suppose \( I_C \subset R \) is the circuit ideal associated to a \( d \)-clutter \( C \), and suppose \( x_e \) is a squarefree monomial of degree \( d \) that is not a generator (so that \( e \) is not an element of \( C \)). Then we say \( x_e \) is a linear divisor for \( I_C \) if the colon ideal

\[
(I_C : x_e) = \{ r \in R : rx_e \in I_C \}
\]

is generated by a subset of the variables \( \{x_1, x_2, \ldots, x_n\} \).

**Definition 2.2.** A circuit ideal (or more generally any monomial ideal) \( I \) is said to have linear quotients if there exists an ordering of its generators \( (m_1, m_2, \ldots, m_g) \) such that \( m_{j+1} \) is a linear divisor for \( I_j \) for all \( j = 1, 2, \ldots g - 1 \). Here for \( j = 1, \ldots, n \) we use the notation \( I_j = \langle m_1, m_2, \ldots, m_j \rangle \).

The notion of an ideal with linear quotients was introduced by Herzog and Takayama in [18]. The concept makes sense for arbitrary monomials ideals but here we will restrict ourselves to those that are squarefree and generated in a fixed degree (arising as the circuit ideal of some \( d \)-clutter \( C \)). Examples of such ideals include squarefree stable ideals as well as ideals generated by a collection of monomials whose support form the bases of a matroid.
Example 2.3. For a specific example consider the graph (2-clutter) $G$ depicted in Figure 1. The edge ideal of the complement graph $\overline{G}$ is given by

$$I_{\overline{G}} = \langle x_1x_3, x_1x_4, x_1x_5, x_2x_3, x_2x_4 \rangle.$$ 

One can check that this ordering of the generators is in fact a linear quotient ordering for $I_{\overline{G}}$. For instance we have

$$I_4 = \langle x_1x_3, x_1x_4, x_1x_5, x_2x_3 \rangle, \quad m_5 = x_2x_4, \quad \text{and} \quad (I_4 : m_5) = \langle x_1, x_3 \rangle.$$ 

Squarefree monomial ideals with linear quotients are closely related to shellable simplicial complexes, as the next observation indicates. Here for a face $F \subset [n]$ we use $\overline{F}$ to denote the complement set, so that $\overline{F} = [n] \setminus F$.

Proposition 2.4 ([17]). Suppose $\Delta$ is a shellable simplicial complex on the vertex set $[n]$. Then $F_1, F_2, \ldots, F_s$ is a shelling order for $\Delta$ if and only if the ideal $\langle x_{F_1}, x_{F_2}, \ldots, x_{F_s} \rangle$ has linear quotients with respect to the given order.

One can check that the ideal $\langle x_{F_1}, x_{F_2}, \ldots, x_{F_s} \rangle$ is in fact the Alexander dual of the Stanley-Reisner ideal of $\Delta$.

In [18] Herzog and Takayama study minimal resolutions of monomial ideals with linear quotients. To describe their construction suppose that $I$ is a monomial ideal with linear quotients for some ordering $(m_1, m_2, \ldots, m_k)$ of the generators. A minimal resolution of $I$ is obtained by iteratively constructing mapping cones as follows. Each time we add a generator $m_{j+1}$ we have a short exact sequence of $\mathbb{R}$-modules

$$0 \rightarrow \mathbb{R}/(I_j : m_{j+1}) \rightarrow \mathbb{R}/I_j \rightarrow \mathbb{R}/I_{j+1} \rightarrow 0,$$

where $I_j = \langle m_1, m_2, \ldots, m_j \rangle$ is the ideal generated by the first $j$ monomials in our (ordered) generating set. By assumption the colon ideal $(I_j : m_{j+1})$ is generated by some subset of the variables, say $\{x_{i_1}, x_{i_2}, \ldots, x_{i_t} \}$. Hence $\mathbb{R}/(I_j : m_{j+1})$ has a minimal resolution given by a Koszul complex $K_t$ on $t$ generators. Assuming we have a minimal resolution $F$ for the ideal $I_j$ we obtain a minimal resolution of $I_{j+1}$ by constructing the mapping cone $C(f)$ for the map of complexes $f : K_t \rightarrow F$ induced by the short exact sequence above. Recall that the complex $C(f)$ is given by

$$C(f) = K_t[1] \oplus F,$$

so that the module in the $i$th homology degree of $C(f)$ has rank given by

$$\text{rank } F_i = \binom{t}{i}$$ (2)



Figure 1. A chordal graph G.
where $F_i$ is the module in the $i$th homology degree of the complex $F$. A cellular realization for this mapping cone construction (under some further conditions on the ideal) was described in [11].

**Example 2.5.** Returning to the ideal discussed in Example 2.3 we have $I_4 = \langle x_1x_3, x_1x_4, x_1x_5, x_2x_3 \rangle$, which has a minimal resolution given by

$$F = 0 \leftarrow I_4 \leftarrow R^4 \leftarrow R^4 \leftarrow R \leftarrow 0.$$ 

If we add the generator $m_5 = x_2x_4$ we see that the colon ideal $(I_4 : m_5) = \langle x_1, x_3 \rangle$ is generated by two variables and hence has a minimal resolution given by a Koszul complex

$$K = 0 \leftarrow (I_4 : m_5) \leftarrow R^2 \leftarrow R \leftarrow 0.$$ 

Taking the mapping cone of the map of complexes $K \to F$ induced by the inclusion $m_5 \to I_5$ we obtain a minimal resolution of $I_5 = I_\sigma$ given by

$$0 \leftarrow I_\sigma \leftarrow R^3 \leftarrow R^6 \leftarrow R^2 \leftarrow 0.$$ 

3. **Main results and discussion**

In this section we provide proofs of the results discussed in the introduction. Recall that a $d$-clutter $C$ on vertex set $[n]$ gives rise to a monomial ideal $I_\sigma \subset K[x_1, x_2, \ldots, x_n]$, generated by all squarefree monomials of degree $d$ not appearing in $C$. Removing a circuit from $C$ corresponds to adding a generator to $I_\sigma$. We then have the following.

**Theorem 3.1.** Suppose $C$ is a $d$-clutter for $d \geq 1$ and let $e \in C$ be a circuit in $C$. Then $e$ is an exposed circuit if and only if $x_e$ is a linear divisor for the ideal $I_\sigma$, where $\sigma$ is the complement of $C$. Moreover $e$ is contained in a unique maximal clique $K$ if and only if the colon ideal $(I_\sigma : x_e)$ is generated by variables corresponding to the vertices $[n] \setminus K$.

Proof. Suppose $C$ is a clutter on vertex set $[n] = \{1, 2, \ldots, n\}$, and let $I_\sigma \subset R = K[x_1, x_2, \ldots, x_n]$ denote the circuit ideal of its complement. Suppose $e = \{v_1, v_2, \ldots, v_d\}$ is a circuit in $C$.

For one direction of the theorem suppose $e$ is an exposed circuit, so that $e$ is contained in a unique maximal $d$-clique of $G$. Without loss of generality suppose the $d$-clique consists of the vertices $K = \{1, 2, \ldots, k\}$. Let $I = I_\sigma$ denote the edge ideal of the complement clutter $\sigma$, and let $I_e = \langle I, x_e \rangle$ denote the ideal obtained by adding the monomial $x_e$ as another generator. We then have the inclusion $I \to I_e$ and the induced short exact sequence

$$0 \to R/(I : x_e) \to R/I \to R/I_e \to 0.$$ 

We claim that the colon ideal

$$\langle I : x_e \rangle = \{r \in R : rx_e \in I\}$$

is generated by the variables $X = \{x_{k+1}, \ldots, x_n\}$, corresponding to variables not in the clique $K$. To see this first note that if $x_\ell \in X$ then $\ell \cup (K \setminus \{i\})$ must not be a circuit of $C$ for some $i = 1, 2, \ldots, d$ (otherwise since $e$ is a circuit we would obtain a clique $e \cup \{\ell\}$ in $C$ of size $d + 1$, meaning that we could either add $\ell$ to $K$ to obtain a larger clique or else have that $e$ is contained in two distinct maximal cliques - either way a contradiction). Without loss of generality suppose $\{\ell, v_2, v_3, \ldots, v_d\}$ is missing from $C$, so that $x_1x_{v_2}x_{v_3} \cdots x_{v_d} \in I$ and hence $x_1x_e \in I$. We conclude that $x_\ell \in (I : x_e)$, and hence $\langle x_{k+1}, \ldots, x_n \rangle \subset (I : x_e)$. 


Next suppose \( m \in \langle I : x_e \rangle \), so that \( mx_e \in I \). We claim that \( m \) is contained in the ideal generated by the variables \( X = \{x_{k+1}, \ldots, x_n\} \). For this we will use the fact that since \( I \) and \( \langle x_e \rangle \) are both monomial, the colon ideal \( \langle I : x_e \rangle \) is also monomial (see [17]). In fact a (possibly redundant) set of generators of \( \langle I : x_e \rangle \) is given by the collection
\[
\{u / \gcd(u, x_e) : u \in G(I)\}.
\]
Here \( G(I) \) denotes a set of generators of \( I \). To show that \( m \) is contained in the desired ideal it’s enough to show that each such generator of \( \langle I : x_e \rangle \) contains some variable from among \( \{x_{k+1}, \ldots, x_n\} \). But recall that \( K = \{1, 2, \ldots, k\} \) is a \( d \)-clique and every generator \( u \in G(I) \) is given by noncircuits. Hence every generator \( u \in G(I) \) contains some variable among \( \{x_{k+1}, \ldots, x_n\} \). Since \( \{v_1, \ldots, v_d\} \subseteq \{1, 2, \ldots, k\} \) we have that \( u / \gcd(u, x_e) \) must also contain that variable. We conclude that every generator of \( \langle I : x_e \rangle \) contains some element among the variables \( X \), and hence \( \langle I : x_e \rangle \subseteq \langle x_{k+1}, \ldots, x_n \rangle \).

For the other direction suppose \( x_e = x_{v_1}x_{v_2} \cdots x_{v_d} \) is a linear divisor for the ideal \( I = I_G \), so that the colon ideal \( \langle I : x_e \rangle \) is generated by a subset of the variables. Without loss of generality suppose
\[
\langle I : x_e \rangle = \langle x_1, x_2, \ldots, x_k \rangle.
\]
We claim that the vertex set \( S = \{k + 1, k + 2, \ldots, n\} \) forms a maximal \( d \)-clique in the clutter \( C \), and that the circuit \( e = \{v_1, \ldots, v_d\} \) is uniquely contained in this clique. For this suppose \( W = \{w_1, w_2, \ldots, w_d\} \subseteq S \) and for a contradiction suppose \( W \) did not form a \( d \)-clique. Then \( x_W \) would be a generator of \( I \). So then \( x_Wx_e \in I \) and hence \( x_W \in \langle I : x_e \rangle = \langle x_1, x_2, \ldots, x_k \rangle \), a contradiction.

To show that \( S \) is maximal suppose \( a \leq k \) with the property that every \( d \)-subset of \( \{a\} \cup S \) forms a circuit of \( C \). Then we have that \( x_a x_W \) is not a generator of \( I = I_G \) for all \( W \subseteq S \) with \( |W| = d - 1 \). But since \( x_a \) is a generator of \( \langle I : x_e \rangle \) we have \( x_a x_e \in I \) so that \( x_a x_W \) is a generator of \( I \) for some \( W \subseteq \{v_1, \ldots, v_d\} \subseteq S \) with \( |W| = d - 1 \), a contradiction.

Finally we show that \( e \) is uniquely contained in \( S \). For this suppose that \( e = \{v_1, \ldots, v_k\} \) is contained in some other maximal clique \( T \), distinct from \( S \). Then there must be some vertex \( t \in T \) (so that \( t \leq k \)) such that every \( d \)-subset of \( \{t\} \cup \{v_1, \ldots, v_d\} \) is a circuit in \( C \). But \( x_t \) is a generator of \( \langle I : x_e \rangle \) so that \( x_t x_W \) is a generator of \( I \) for some for some subset \( W \subseteq S \), again a contradiction. The result follows.

\[\square\]

**Remark 3.2.** As mentioned in the introduction, it is known that if \( \Delta \) is a simplicial complex then \( I_\Delta \), the Stanley-Reisner ideal of \( \Delta \), has linear quotients if and only the Alexander dual \( \Gamma^* \) has a shellable Stanley-Reisner complex. Presumably one can use this characterization to give another more combinatorial proof of Theorem 3.1 but we prefer the direct algebraic argument since it says a bit more (and we found it first).

Note that in the case of a linear quotient ordering we are building the ideal one generator at a time, and in the complement this corresponds to deleting circuits from a complete \( d \)-clutter on vertex set \( [n] \). For the case of \( d = 2 \) we explicitly state the corollary.

**Corollary 3.3.** Suppose \( G \) is a graph and let \( e = ij \) be an edge in \( G \). Then \( e = ij \) is an exposed edge if and only if \( x_i x_j \) is a linear divisor for the ideal \( I_\overline{G} \), where \( \overline{G} \) is the complement of \( G \).
Combining this with the results of [16] we obtain another proof of the result from [9] mentioned in the introduction. Recall that by definition an edge erasure is the result of removing an edge \( e \) that is \emph{properly} exposed. For the case of the graphs this characterizes (complements of) chordal graphs that are connected.

**Corollary 3.4 ([9]).** A graph \( G \) can be obtained from a complete graph through a sequence of edge erasures if and only if \( G \) is a connected chordal graph.

**Proof.** From Theorem 3.1 we have that removing an exposed edge \( e = ij \) from a graph \( G \) corresponds to adding the generator \( x_i x_j \) that is a linear divisor in \( I_G \). Hence performing a sequence of edge erasures on a complete graph results in an ideal with linear quotients. An arbitrary (monomial) ideal with linear quotients has a linear resolution, and hence in this case \( G \) is chordal by Fröberg’s Theorem. On the other hand we know that if \( G \) is chordal we have that \( I_G \) has a linear resolution. In [16] is it shown that edge ideals with linear resolutions in fact have linear quotients. Hence from above we know that the underlying graph \( G \) is obtained by a sequence of removing exposed edges, starting from a complete graph.

We next claim that if a graph \( G \) if obtained from the complete graph \( K_n \) via a sequence of removing exposed edges, then it is connected if and only if each exposed edge was in fact \emph{properly} exposed. For one direction, note that if \( G \) is disconnected then at some point in the process of removing exposed edges the graph became disconnected, which can only happen if the edge was not properly exposed. For the other direction suppose one of the edges \( e = ij \) in the deletion sequence was not properly exposed. We claim that removing this edge results in a disconnected graph. If not, there must be another path in the graph that connects the vertices \( i \) and \( j \), say \( i = v_1, v_2, \ldots, v_k = j \). Choose this path to be of minimum length. If \( k = 3 \), the the set \( \{i, v_2, j\} \) forms a clique, a contradiction to the assumption that \( e \) was not properly exposed. But if \( k > 4 \) then the vertices \( v_1, v_2, \ldots, v_k \) forms a \( k \)-cycle, which must have a chord since the underlying graph is chordal. This chord provides a shorter path from \( i \) to \( j \), a contradiction to the choice of \( v_1, \ldots, v_k \). We conclude that the graph in fact became disconnected by removing \( e \). The result follows. \( \square \)

![Figure 2](image_url)

**Figure 2.** A sequence of erasures resulting in the graph \( G \). For example in the last step, the edge 24 is contained in the maximal clique 245, and the relevant colon ideal is given by \( (I_4 : x_2 x_4) = \langle x_1, x_3 \rangle \).

We wish to generalize the connectivity condition for graphs to the context of \( d \)-clutters for \( d > 2 \). For this we use the following algebraic characterization of connectivity. Here \( \text{pdim} \) refers to the \emph{projective dimension} of the underlying module.

**Lemma 3.5.** A graph \( G \) on vertex set \( [n] \) is connected if and only if \[ \text{pdim}(I_G) < n - 2. \]
Proof. We first observe that $I_{\Sigma}$ is the Stanley-Reisner ideal of the simplicial complex $\Delta(G)$, where $\Delta(G)$ is the clique complex of $G$ (the simplicial complex whose faces are complete subgraphs of $G$). We then employ Hochster’s formula (see for instance [20]), which describes the Betti numbers of a

\[ \beta_{i,j} = \sum \tilde{H}_{j-i-2}(\Delta(S); \mathbb{K}), \]

where the sum is over all $j$-subsets $S \subset [n] = V(G)$, and $\Delta(S)$ denotes the clique complex of the graph induced on the vertex set $S$.

The projective dimension of $I_{\Sigma}$ is the largest $i$ such that $\beta_{i,j} \neq 0$ for some $j$. If $G$ is disconnected then we have $H_0(\Delta(G), \mathbb{K}) \neq 0$, so that $\beta_{n-2,n} \neq 0$ and hence $\text{pdim}(I_{\Sigma}) \geq n - 2$. On the other hand if $\text{pdim}(I_{\Sigma}) \geq n - 2$ then by Hochster’s formula we must have $\beta_{n-2,n} \neq 0$ so that $\tilde{H}_0(\Delta(G)) \neq 0$, which implies that $G$ is disconnected.

□

For general $d$-clutters we have an an analogous statement. We begin with a definition.

**Definition 3.6.** Suppose $C$ is a $d$-clutter with complement circuit ideal $I_{\Sigma}$. Suppose $e \in C$ has the property that $x_e$ is a linear divisor for $I_{\Sigma}$, and let $I_e = \langle I_{\Sigma} \cup \{x_e\} \rangle$. Define the Betti contribution of $x_e$ to be the set

\[ \{i \in \mathbb{N} : \beta_i(I_{\Sigma}) \neq \beta_i(I_e)\}. \]

From Equation 2 we have that the Betti contribution has the form $\{0, 1, \ldots, k\}$ for some integer $k$. We say that the Betti contribution is small if $k < n - d$.

**Proposition 3.7.** Suppose $C$ is a $d$-clutter and $e \in C$ is an exposed circuit. Then $e$ is properly exposed if and only if the Betti contribution of $x_e$ is small.

Proof. Let $I_{\Sigma}$ denote the circuit ideal of the complement of $C$, and let $I_e$ denote the ideal obtained from adding the generator $x_e$. Suppose the colon ideal $(I_{\Sigma} : x_e)$ is given by $\langle x_{i_1}, x_{i_2}, \ldots, x_{i_\ell} \rangle$. As discussed in Section 2, a minimal resolution of $I_e$ is obtained by taking the mapping cone of $f : K_\ell \to F$, where $K_\ell$ is a Koszul resolution on $\ell$ generators and $F$ is a minimal resolution of $I_{\Sigma}$. From Equation 2 we see that the largest element in the Betti contribution of $x_e$ is $\ell$.

Suppose the edge $e$ is uniquely contained in the maximal clique $K$. From Theorem 3.1 we have that $x_{i_j} \in (I_{\Sigma} : x_e)$ if only if the vertex $i_j$ satisfies $i_j \notin K$. If $e$ is itself a maximal clique of the $d$-clutter $C$ then we have $n - d$ vertices in the complement and hence by Equation 2 we have that the Betti contribution of $x_e$ has value $n - d$. On the other hand if $|K| > d$ (so that $e$ is strictly contained in $K$) then we have at most $n - d - 1$ vertices in the complement, in which case the Betti contribution is small.

□

From this we get the desired analogue of Corollary 3.4 in the setting of $d$-clutters. Once again recall that a circuit erasure is the removal of a circuit that is properly exposed.

**Proposition 3.8.** A $d$-clutter $C$ can be obtained from a complete $d$ clutter $K_n^d$ through a sequence of circuit erasures if and only if $I_{\Sigma}$ has linear quotients and

\[ \text{pdim}(I_{\Sigma}) < n - d. \]
If \( C \) is a \( d \)-clutter obtained by removing exposed circuits from \( K^d_n \), we see from Proposition 3.7 that \( \beta_{n-d+1} \) counts the number of circuits that were not properly exposed in the removal process. The other Betti numbers are similarly controlled by the cardinalities of cliques in the removal process, as the next result spells out (see also [17] for a similar observation in the purely algebraic setting).

**Corollary 3.9.** Suppose \( C \) is a \( d \)-clutter on vertex set \([n]\) obtained from \( K^d_n \) by removing a sequence of exposed circuits \((e_1, e_2, \ldots, e_r)\). By definition each \( e_j \) is contained in a unique maximal clique \( K_j \), let \( k_j = n - |K_j| \).

Then the Betti numbers of \( I_C \) are given by

\[
\beta_i = \sum_{j=1}^{r} \left( \begin{array}{c} k_j \\ i \end{array} \right).
\]

**Proof.** From 3.1 we have that each time we add the generator \( x_{e_i} \) we glue on a Koszul resolution on \( k_i \) generators to the desired minimal resolution. The result then follows from Equation 2. \( \square \)

**Remark 3.10.** The multiset \( \{k_j\}_{j=1}^{n} \) described in Corollary 3.9 is an invariant of the \( d \)-clutter \( C \), and in fact can be seen to coincide with the \( h \)-vector of a certain simplicial complex obtained from \( C \). Namely, let \( S(C) \) denote the simplicial complex whose facets are given by \([n] - e : e \in C\), the complements of the circuits in the \( d \)-clutter \( C \). One can then show that

\[
h_i(S(C)) = |\{k_j : k_j = i\}|.
\]

Our result is similar in spirit to a formula for the chromatic polynomial of a chordal graph obtained from its description as a sequence of simplicial vertices. To recall this connection suppose \((v_1, v_2, \ldots, v_n)\) is an ordering of the vertices of \( G \) with the property that \( N_i(v_i) \) is a complete graph, where \( N_i(v_i) \) denotes the neighborhood of \( v_i \) in the subgraph of \( G \) induced by vertex set \( \{v_i, v_{i+1}, \ldots, v_n\} \). For each \( i \) let \( d_i \) denote the number \( |N_i(v_i)| - 1 \). Then one can show that the chromatic polynomial of \( G \) is given by

\[
\chi_t(G) = \prod_{i=1}^{n} (t - d_i).
\]

We do not know if there are other combinatorial interpretations of the \( k_i \).

We note that there is a geometric interpretation of the Betti numbers of (certain) ideals with linear quotients in terms of the face numbers of certain polyhedral complexes supporting a cellular resolution. We refer to [11] for details but note that in our running example (Example 2.3 from above) a minimal resolution of \( I_G \)

\[
0 \leftarrow I_G \leftarrow \mathbb{R}^3 \leftarrow \mathbb{R}^6 \leftarrow \mathbb{R}^2 \leftarrow 0.
\]

is supported on the polyhedral complex depicted below. The complex has five vertices, six edges, and two 2-cells. Also note that \( \text{pdim}(I_G) = 2 < 5 - 2 \).

We discuss one more higher-dimensional example to illustrate our constructions.

**Example 3.11.** Let \( K^3_5 \) denote the complete 3-clutter on 5 vertices. We will remove circuits \( K^3_5 \) in the following order. Here we suppress set brackets, so that \( 125 = \{1, 2, 5\} \).

1. Remove the circuit 125, uniquely contained in the clique 12345 and giving \( k_1 = 0 \).
2. Remove 135, uniquely contained in the clique 1345 so that \( k_2 = 1 \).
(3) Remove 145, itself a clique and giving $k_3 = 2$.

At this point we have a 3-clutter $C$ that is geometrically a bipyramid over a triangle. The corresponding complement circuit ideal is

$$I_3 = \langle x_1x_2x_5, x_1x_3x_5, x_1x_4x_5 \rangle,$$

which indeed has a linear resolution (see below). From Corollary 3.9 we can compute Betti numbers

$$\beta_0 = \binom{0}{0} + \binom{1}{0} + \binom{2}{0} = 3$$

$$\beta_1 = \binom{0}{1} + \binom{1}{1} + \binom{2}{1} = 3$$

$$\beta_2 = \binom{0}{2} + \binom{1}{2} + \binom{2}{2} = 1.$$
Also note that in Step (3) we removed a circuit that was exposed but not properly exposed. This is reflected by the fact that the corresponding ideal has projective dimension 2. If in Step (3) we instead remove 123 (which is uniquely contained in the clique 1234) we obtain the ‘connected’ 3-clutter depicted below. This clutter has the property that if we include the complete 1-skeleton the resulting simplicial complex has vanishing first homology.

![Diagram of a 'connected' chordal 3-clutter D, with the Betti table for R/D. The clutter consists of all 3-subsets of [5] except 123, 125, and 135.]

3.1. **Relation to simple homotopy.** The concepts of exposed circuits and circuit erasures are reminiscent of certain constructions from the study of simple homotopy theory (see for example [7]). Here if $\Delta$ is a simplicial complex, a face $\tau \in \Delta$ is called a free face if it is contained in a unique facet $\sigma$. The removal of $\tau$ along with all simplices $\gamma$ such that $\tau \subset \gamma \subset \sigma$ is called an elementary collapse. Since elementary collapses preserve (simple) homotopy type, in particular any such complex obtained this way from a simplex will be contractible.

If $C$ is a $d$-clutter on $[n]$ we define $\Delta(C)$ to be the simplicial complex on the same vertex set with

- a complete $(d-2)$-skeleton
- $(d-1)$-dimensional faces corresponding to the circuits of $C$
- for $k \geq d$, all $k$-faces $\sigma$ such that all $d$-subsets of $\sigma$ are circuits in $C$.

This is the analogue of the clique complex of a (connected) graph. One can check that an exposed circuit $e$ in a $d$-clutter $C$ corresponds to a free face in the simplicial complex $\Delta(C)$. The wrinkle here is that in the context of simple homotopy theory, the removal of the simplex $\tau$ includes removing all of its subsets (including the underlying vertices). In the algebraic context, however, we only remove the $d$-subset (which corresponds to adding a monomial generator of degree $d$). However, the condition that $\tau$ is properly contained in $\sigma$ guarantees that the two constructions agree. We note that a further connection between chordality and simple homotopy theory (in the context of $d$-collapsibility) in the non-pure case has been explored by Bigdeli and Faridi in the recent preprint [3].

4. **Applications: Simon’s Conjecture and Higher Chordality**

Next we discuss other applications and corollaries of our results. We also relate our study to other constructions of chordal complexes from the literature.
4.1. Extendably shellable complexes. From Theorem 3.1 and Proposition 2.4 we see that the process of removing exposed circuits from a d-clutter on vertex set [n] is closely related to shellings of (pure) simplicial complexes of dimension n − d − 1. We now discuss how our results from above can be applied in this context. We begin with a definition.

**Definition 4.1.** A shellable complex $\Delta$ is said to be **extendably shellable** if any shelling of a subcomplex of $\Delta$ can be extended to a shelling of $\Delta$.

Here a subcomplex of $\Delta$ is a simplicial complex $\Gamma$ on the same vertex set, whose set of facets consists of a subset of the facets of $\Delta$. Ziegler [24] has shown that there exist simple and simplicial polytopes whose boundary complexes are not extendably shellable. Simon [22] has conjectured that every k-skeleton of a simplex is extendably shellable. Our interpretation of results of [9] leads to a proof of the conjecture in some special cases, also leads to generalizations. Recall that if $\Delta$ is pure k-dimensional simplicial complex $\Delta$ with shelling order of its facets $(F_1, F_2, \ldots, F_k)$, the **restricted set** of the facet $F_i$ is the set of $(k-1)$ dimensional faces in the intersection of the facet $F_i$ with the subcomplex $F_1 \cup F_2 \cup \cdots \cup F_{i-1}$.

**Lemma 4.2.** A sequence $e_1, e_2, \ldots, e_k$ of removing exposed edges from the complete graph $K_n$ corresponds to a shelling sequence $F_1, F_2, \ldots, F_k$ of the $(n-3)$-dimensional complex on vertex set $[n]$ whose facets are given $F_i = [n] \setminus e_i$. An edge $e_i$ is properly exposed if and only if the restricted set of $F_i$ consists of less than $n-2$ elements.

**Proof.** Theorem 3.1 implies that each $x_{e_i}$ is a linear divisor for the ideal $I_{i-1} = \langle x_{e_1}, x_{e_2}, \ldots, x_{e_{i-1}} \rangle$. Proposition 2.4 then implies that $F_1, F_2, \ldots, F_k$ is a shelling order for the simplicial complex it defines, where $F_i = \overline{e_i} = [n] \setminus e_i$. The restricted set of $F_i$ corresponds to variables generating the colon ideal $\langle I_{i-1} : x_{e_i} \rangle$, which by Theorem 3.1 is given by $[n] \setminus K$, where $K$ is the unique maximal clique containing the edge $e_i$. Hence $|K| > 2$ if and only if the restricted set of $F_i$ consists of less than $n-2$ elements. \[\square\]

**Corollary 4.3.** The $(n-3)$-skeleton of a simplex on vertex set $[n]$ is extendably shellable.

**Proof.** Let $\Delta^{(n-3)}_n$ denote the $(n-3)$-skeleton of the simplex on $[n]$. Suppose $H$ is a shellable proper subcomplex of $\Delta^{(n-3)}_n$, with shelling order $F_1, F_2, \ldots, F_h$. Note that each $F_i$ is a subset of $[n]$ of size $n-2$. Lemma 4.2 implies that the graph on vertex set $[n]$ with edges $e_i = [n] \setminus F_i$ is obtained from the complete graph $K_n$ by removing exposed edges; hence it is chordal. In [9] it is shown that if $G$ is any chordal graph then $G$ contains an exposed edge $e$ (in fact if $G$ is connected then this edge can be taken to be properly exposed). Hence we can extend the shelling sequence with the facet $F_{h+1} = [n] \setminus e$. Induction on $\binom{n}{2} - h$ implies that any shelling of a subcomplex of $\Delta^{(n-3)}_n$ can be extended to an entire shelling. \[\square\]

Note that the $(n-2)$-skeleton of the simplex on vertex set $[n]$ is the boundary of a simplex, which is clearly extendably shellable (in fact any sequence of facets constitutes a shelling). The $(n-1)$-skeleton is the simplex itself which consists of a single facet. Hence we have the following corollary.

**Corollary 4.4.** For all $k \geq n-3$, the $k$-skeleton of a simplex on $[n]$ is extendably shellable.
This result was also obtained in [6] using other methods related to another notion of chordal clutters (see the next section). A more careful analysis of the results from [9] leads to another class of \((n - 3)\)-dimensional simplicial complexes that are extendably shellable.

**Proposition 4.5.** Suppose \(\Delta\) is an \((n - 3)\)-dimensional shellable simplicial complex on vertex set \([n]\). If \(\Delta\) is contractible and has \(\binom{n}{2} - n + 1\) facets then it is extendably shellable.

**Proof.** By Lemma 4.2 a sequence \(e_1, e_2, \ldots, e_k\) of removing properly exposed edges from \(K_n\) corresponds to a shelling sequence \(F_1, F_2, \ldots, F_k\) of an \((n - 3)\)-dimensional complex, where at each step the restricted set consists of less than \(n - 2\) elements. It is known that a shellable complex \(\Delta\) is contractible if and only if any shelling of \(\Delta\) there are no restricted sets of size \(n - 2\) (in fact \(\Delta\) is either contractible or has the homotopy type of \(\ell\) spheres of dimension \(n - 3\), where \(\ell\) is the number of restricted sets of size \(n - 2\)).

Hence a contractible shellable complex with \(\binom{n}{2} - n + 1\) facets corresponds to a connected graph with \(n - 1\) edges— in other words a tree \(T\) on vertex set \([n]\). Any shelling of a subcomplex \(\Gamma\) of \(\Delta\) corresponds to a connected chordal graph \(G\) containing the tree \(T\). Results from [9] imply that if \(H\) is an edge-weighted connected chordal graph then a minimal spanning tree of \(H\) can be obtained by a sequence of edge erasures (removing properly exposed edges). Hence if we assign weights to the edges of our graph \(G\) so that \(T\) is the only minimal spanning tree of \(G\), we can obtain \(T\) from \(G\) via such a sequence. This implies that we can extend the shelling of \(\Gamma\) to a shelling of \(\Delta\), as desired. \(\square\)

**Remark 4.6.** The method of finding a minimal spanning tree from [9] (and discussed above) is a variation of Kruskal’s algorithm [19]. We have used this to show that any spanning tree of a connected chordal graph \(G\) can be obtained via a sequence of deleting properly exposed edges. One wonders if similar arguments can be employed to show that any chordal subgraph \(H\) of \(G\) can be obtained from \(G\) via sequence of removing exposed edges. If true this would imply that any \((n - 3)\)-dimensional shellable complex on vertex set \([n]\) is extendably shellable. We do not know of any counterexamples to this statement.

Finally we end this section with a reformulation of Simon’s conjecture in terms of exposed circuits.

**Conjecture 4.7** (reformulation of Simon’s conjecture). Suppose \(C\) is a \(d\)-clutter obtained from the complete \(d\)-clutter \(K^n_d\) by a sequence of removing exposed circuits. Then \(C\) contains an exposed circuits.

### 4.2. Chordal complexes in higher dimensions

As we have seen the process of removing exposed edges from a complete \(d\)-clutter gives rise to a circuit ideal \(I_C\) that has linear quotients. For the case \(d = 2\) this in fact characterizes chordal graphs. Hence our constructions give rise to a natural candidate for what might be considered a ‘chordal \(d\)-clutter’ (or at least the complement of one). In recent years several authors have studied various generalizations of chordal graphs in the setting of hypergraphs/clutters/simplicial complexes. Many of these are inspired by Fröberg’s Theorem in an attempt to give a combinatorial characterization of squarefree monomial ideals having a \(d\)-linear resolution over any field (a property that is strictly weaker than having linear quotients). By Hochster’s formula this is equivalent to restricting the topology of induced subcomplexes, although one hopes for a more global description. For the reader’s convenience we briefly review some of these approaches below.

In one attempt to define a chordal complex, the notion of a ‘chordless cycle’ is generalized to the higher-dimensional setting. This is the approach taken by Connon and Faridi [8] in which they give
a combinatorial description of a \( (d-1) \)-dimensional cycle as a \( d \)-clutter \( C \) that is strongly connected (for each pair of circuits \( e \) and \( f \) in \( C \) there exists a sequence of circuits \( e = e_1, e_2, \ldots, e_k = f \) such that \( e_1 \cap e_{i+1} \) has cardinality \( d-1 \)) and such that the ‘degree’ of each ridge is even. The authors introduce notions of ‘chordless’ cycles and for instance show that if \( C \) is a \( d \)-clutter such that \( I_C \) admits a linear resolution over any field, then \( C \) is ‘orientally-cycle-complete’. As a partial converse, they show that the clutter ideal of the complement of a \( d \)-tree (a clutter with no cycles) has a linear resolution over any field of characteristic 2. In [1] a more homological approach is taken to study notions of higher chordality.

In other attempts to generalize chordal graphs the notion of a ‘simplicial vertex’ is taken as the starting point. This is the approach taken by Emtander in [12] where a vertex \( v \in C \) is said to have a complete-neighborhood if the induced subclutter on \( S = \{ x \in [n] : (x,v) \subset e, e \in C \} \) is the complete \( d \)-uniform clutter \( K^d_{[n]} \), consisting of all possible \( d \) subsets of \( S \). Recall that if \( S \) is a subset of the vertices of \( C \) then the induced subclutter on \( S \) consists of all circuits of \( C \) whose vertices are contained in \( S \). A \( d \)-uniform clutter is then ‘chordal’ in this context if every induced subclutter admits a vertex with a complete neighborhood (or else has no circuits). Woodroofe [23] takes a related but independent approach, defining a vertex \( v \in C \) to be simplicial if for every pair of circuits \( e \) and \( f \) of \( C \) that contains \( v \) there exists a circuit \( g \) such that \( g \subset (e \cup f) - \{v\} \). A clutter is then said to be ‘chordal’ in this context if every minor of \( C \) admits a simplical vertex. This definition is reminiscent of the circuit characterization of matroids and in fact the collection of circuits of a matroid provide a (possibly non-uniform) example of a chordal complex in this setting.

In yet another direction Bigdeli, Yazdan Pour, and Zaare-Nahandi [5] use the notion of a simplicial ridge to provide a definition of a chordal clutter. Recall that a ridge in a \( d \)-clutter \( C \) is a set \( R \) of vertices of size \( d-1 \) such that \( R \subset e \) for some circuit \( e \in C \) (note that in the setting of connected graphs a vertex is also a ridge). From [5] a ridge is said to be simplicial if the induced subclutter on \( R \cup \{v \in [n] : R \cup \{v\} \in C \} \) is the complete \( d \)-uniform clutter. A clutter \( C \) is then ridge-chordal if there exists a sequence of ridges \( R_1, R_2, \ldots, R_k \) of \( C \) such that \( R_i \) is simplicial in the clutter \( C - (R_1 \cup \cdots \cup R_{i-1}) \), and \( C - (R_1 \cup \cdots \cup R_k) = \emptyset \). In [4] it is shown that if \( C \) is ridge-chordal then the ideal \( I_C \) has a linear resolution over every field, and in fact this notion of a ridge-chordal \( d \)-clutter includes all other constructions that satisfy this property. There do, however, exist monomial squarefree ideals with a linear resolution over every field that do not arise as complements of ridge-chordal clutters (for example the clutter ideal coming from a certain triangulation of a dunce hat). As far as we know the following question is still open.

**Conjecture 4.8.** If \( C \) is a \( d \)-clutter with the property that \( I_C \) has linear quotients, then \( C \) is ridge-chordal.

Our constructions are also related to ridge-chordality as follows. If \( C \) is a \( d \)-clutter on vertex set \([n]\), let \( C^{d+1} \) denote the \( (d+1) \)-clutter on the same vertex set, with circuits given by all cliques in \( C \) of size \( d+1 \). For example if \( d = 2 \), so that \( C \) is a graph, then \( C^{d+1} \) consists of all triangles in the underlying graph. One can then show that \( e \) is an exposed edge of \( C \) if and only if \( e \) is a simplicial ridge of \( C^{d+1} \). We thank Mina Bigdeli for pointing this out to us [2]. In [21] it is shown that for a graph \( G \), a sequence of edges \( e_1, \ldots, e_t \) is a simplicial sequence of ridges in \( G^3 \) if and only if the ideal \( I_{e_1} \) has linear quotients. Hence this observation provides alternative proof for their result.
5. Final Remarks

As mentioned above, the study of edge erasures in chordal graphs developed in [9] was originally motivated by questions involving clustering algorithms and in particular generalizations of single-linkage clustering. Finding a minimal spanning tree of a weighted complete graph (finite metric space) provides the basis for single-linkage clustering and the hope was that minimal chordal graphs may serve a similar role for more general clustering algorithms that allow overlaps. It is not clear if the chordal d-clutters discussed here might have any relevance to these constructions. For this one might want a generalization of a metric space where d-tuples of points are assigned a ‘distance’.

In the process of generalizing Kruskal’s algorithm for finding minimal spanning trees, the authors of [9] use the following property of properly exposed edges in a chordal graph.

**Theorem 5.1 ([9]).** Suppose G is a chordal graph, and let ∂G denote the edge-induced subgraph of G determined by the properly exposed edges of G. Then every connected component of ∂G is 2-edge connected.

We do not know if something similar holds in the context of higher-dimensional d-clutters. Is it the case that properly exposed circuits in a chordal d-clutter are also contained in some version of higher-dimensional cycles? This would potentially lead to progress on Conjecture 4.8. One also wonders if there is an interpretation of Theorem 5.1 in the context of commutative algebra.

Another natural question to ask is if the generalization of Kruskal’s algorithm that relies on Theorem 5.1 can be generalized to the context of higher dimensional complexes or more general matroids. In particular given a circuit-weighted d-clutter C can one find a minimal ‘spanning tree’ by removing properly exposed faces? Note that the connectivity of such a spanning tree is not simply vanishing of the top (d − 1)-homology of the simlicial complex ∆ defined by the circuits (thought of as facets of ∆), but also vanishing of the (d − 2) homology of the complex obtained by also including its complete (d−2)-skeleton (see Figure 3.11).

Finally, one wonders what role of edge/circuit weighted graphs and d-clutters might play in combinatorial commutative algebra. For instance if we assign weights to all quadratic squarefree monomials $x_i x_j$ in the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ our discussion above implies that among all ‘minimal’ quadratic monomial ideals I with $\binom{n}{2} - n + 1$ generators satisfying $\text{pdim} I < n - 2$, we can find such an ideal with the property that I has linear quotients. Here the weight of a monomial ideal is the sum of the weights of its generators (in its minimal set of generators).

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DEPARTMENT OF MATHEMATICS, TEXAS STATE UNIVERSITY, SAN MARCOS

E-mail address: dochtermann@txstate.edu