A combinatorial problem about binary necklaces and attractors of Boolean automata networks

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Abstract. It is known that there are no more Lyndon words of length \( n \) than there are periodic necklaces of same length. This paper considers a similar problem where, additionally, the necklaces must be without some forbidden factors. This problem relates to a different context, concerned with the behaviours of particular discrete dynamical systems, namely, Boolean automata networks. A formal argument supporting the following idea is provided: addition of cycle intersections in network structures causes exponential reduction of the networks' number of attractors.

Keywords: Interaction networks, attractors, cycle interactions, combinatorics on words

1 Introduction and informal motivation

Generally, I aim at understanding the "clockworks" of interaction networks (a.k.a. sets of related things) through a study of formal prototypes called Boolean automata networks (BANs). These mathematical models are a generalisation of the neural networks introduced by McCulloch and Pitts [15] in 1943. They are still widely studied as models of biological networks. Considerable effort has thereby already been invested into understanding and describing their dynamics [2, 5, 9, 16, 21, 27, 31].

More precisely and informally, here, I especially aim at understanding what structural properties can be considered fundamentally responsible for diversity and variety in the asymptotic behaviour of a network, and conversely, which ones can be considered responsible for a lesser degree of "asymptotic freedom" (cf. Section 7). This aim leads to addressing some crucial lingering problems about BANs through a new, elementary stance. In particular, this raises a combinatorial problem about the asymptotic dynamics of particular instances of BANs (cf. Section 8) which translates directly into a combinatorial word theoretic problem (cf. Section 10).

Before describing these two equivalent combinatorial problems and how they relate (in Sections 6 to 10), in Sections 2 to 5, we give basic definitions about BANs.

1 Commonly, BANs are studied through their dynamics, often with the aim of relating their dynamical properties to their other (structural) properties. Notably, my present aim doesn't exactly coincide with this. It is based on a general approach that is essentially constructive. To start, I isolate features of networks (e.g. cycles, intersections, non-monotony) in order to study their role in conditions that favour their decisiveness. Then, the complexity of problems addressed can gradually be increased by adding and combining features that have already been studied separately.
their structures, and their (asymptotic) behaviours. Section 11 sketches the (lengthy) proof of the main result in the word theoretic setting (the full proof is detailed in Appendix A). Finally, Sections 12 and 13 derive and discuss implications of this result.

2 Boolean Automata Networks

Let $\mathbb{B} = \{0, 1\}$. A **Boolean automata network** (BAN) of size $n \in \mathbb{N}$ is a set of $n$ Boolean functions $\mathcal{N} = \{f_i : \mathbb{B}^n \to \mathbb{B}, \ i < n\}$. Index $i < n$ represents **automaton** $i$. Here, the word **automaton** is to be taken as referring to a computing unit regarded as a black box (our interest here is in how networks of automata work, rather than in how automata work) [4]. The computation that is made by automaton $i$ in configuration $x \in \mathbb{B}^n$ of $\mathcal{N}$ is: $x \mapsto f_i(x)$. In principle, $f_i$ can be any Boolean function. In practice, for the sake of convenience and in consistence with our general approach to these networks, we restrict the $f_i(x)$’s to fully locally monotone functions: in the CNF or DNF of any $f_i(x)$, no literal $x_j$ ($j < n$) can appear both negated and un-negated (typically, this excludes the XOR function).

3 Structure of a Boolean automata network

Let $\mathcal{V} = \{i < n\}$ denote the set of automata of $\mathcal{N}$. Interactions between automata of $\mathcal{N}$ are represented in its **interaction digraph** – also called **structure** – $\mathcal{G} = (\mathcal{V}, \mathcal{A})$, where $\mathcal{A} \subset \mathcal{V} \times \mathcal{V}$ is defined by: $(j, i) \in \mathcal{A} \iff \exists x = (x_0, \ldots, x_n) \in \mathbb{B}^n, f_i(x_0, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_{n-1}) \neq f_i(x_0, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_{n-1})$. In $\mathcal{G}$, arc $(j, i)$ is said to be **negative** (resp. **positive**) if:

$$\forall x \in \mathbb{B}^n, \quad f_i(x_0, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_{n-1}) (\text{resp.} \geq) f_i(x_0, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_{n-1}).$$

Because of the assumption on the local monotony of the $f_i$’s, all arcs can be signed. Naturally, we let $s_{j,i} \in \{+, -\}$ denote the **sign** of arc $(j, i) \in \mathcal{A}$. Then, $\forall x \in \mathbb{B}^n, \forall (j, i) \in \mathcal{A}$, we can introduce the following notation to denote the input that $i$ receives from $j$ in configuration $x$: $s_{j,i}(x_j) = x_j$ if $s_{j,i} = +$ and $s_{j,i}(x_j) = \neg x_j$ if $s_{j,i} = -$.

![Fig. 1](image_url)

Fig. 1. A Boolean automata network $\mathcal{N}$ of size $n = 4$. Left: The defining local functions of $\mathcal{N} = \{f_i, i < n\}$, i.e. the graph of function $F : x \in \mathbb{B}^4 \mapsto (f_0(x), \ldots, f_3(x)) \in \mathbb{B}^4$ revealing one 1-attractor, one 2-attractor and one 6-attractor so that the order of $\mathcal{N}$ is $\omega = \text{lcm}\{1, 3, 6\} = 6$.

All walks and cycles mentioned in this paper are considered to be directed. The sign of a walk or cycle is the product of the signs of its arcs: a negative (resp. positive) walk or cycle in $\mathcal{G}$ is comprised of an odd (resp. even) number of negative arcs.

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2 We take (local) monotony as a reference and aim at understanding monotone BANs first so that we can then aim at understanding the role of non-monotony per se by studying how a little, localised addition of it impacts on the network’s behaviour.
4 Behaviour of a Boolean automata network

Assuming a parallel update of each automaton state in each network configuration, \( \mathcal{N} \) undergoes transitions of the form \( x \rightarrow F(x) = (f_0(x), f_1(x), \ldots, f_{n-1}(x)) \) (as each automaton \( i \) undergoes change \( x_i \rightarrow f_i(x) \)). Given a configuration \( x \in \mathbb{B}^n \), if we settle that \( x = x(0) \), then \( \forall t \in \mathbb{N}, x(t) \) denotes configuration \( F^t(x) \). The graph \( \mathcal{T} \) of function \( F \) is called the transition graph of \( \mathcal{N} \). It represents the behaviour of \( \mathcal{N} \) under the parallel updating.

5 Asymptotic behaviour of a Boolean automata network

In the present deterministic case, terminal strongly connected components of \( \mathcal{T} \) are directed cycles. To avoid confusion with the structural cycles of \( G \), a cycle of length \( p \) in \( \mathcal{T} \) is rather called an attractor of primitive period \( p \) or a \( p \)-attractor (abusing language since an attractor need not attract anything in this setting).

We introduce the order \( \omega \) of \( \mathcal{N} \) as the least common multiple of all of its attractor periods. Equivalently, with \( X \subset \mathbb{B}^n \) denoting the set of recurrent configurations of \( \mathcal{N} \) (those belonging to its attractors), \( \omega \) is defined by: \( \omega = \min \{ p \in \mathbb{N} \mid \forall x \in X, F^p(x) = x \} \).

We let \( X(p) = \{ x \in \mathbb{B}^n, F^p(x) = x \} \) denote the set of configurations of period \( p \). In particular, \( \forall p \in \mathbb{N}, X(p) \subset X(0) = X \), and \( X(p) \neq \emptyset \implies p \mid \omega \). Let us introduce here the following notation: \( \forall x \in X, \forall t \in \mathbb{Z}, x(t) = F^t \mod \omega(x) = x(t \mod \omega) \).

The primitive period of any \( x \in X \), is \( \min \{ p \mid F^p(x) = x \} \). We let \( \overline{X}(p) \) be the set of configurations with primitive period \( p \): \( \overline{X}(p) = X(p) \setminus \bigcup_{q | p, q < p} X(q) \).

Let us abuse language and notations to confuse attractors with orbits \( \{ F^k(x), k < p \} \) of configurations \( x \) inducing them. We let \( \overline{A}(p) = \bigcup_{x \in X(p)} \{ F^k(x), k < p \} \) denote the set of \( p \)-attractors of \( \mathcal{N} \), and we let \( A(p) = \bigcup_{q | p} \overline{A}(q) \) denote its set of attractors with period \( p \). In particular, \( A(\omega) \) is the set of all attractors of \( \mathcal{N} \).

6 Preliminary combinatorial notations and relations

Let us specify notations for cardinals of the sets introduced above: \( \forall p, X(p) = |X(p)|, \overline{X}(p) = |X(p)|, A(p) = |A(p)|, \) and \( \overline{A}(p) = |\overline{A}(p)| \).

Provided a characterisation of attractor periods yielding \( \omega \), and a characterisation of \( X \) yielding \( X(\omega) \), one can immediately derive \( \overline{X}(p), \overline{A}(p) \) and \( A(p) \), by exploiting the following relationships, where \( \star \) is the Dirichlet convolution operator, \( \mathbb{1} : n \in \mathbb{N} \mapsto \frac{1}{n} \), \( \text{inv} : n \in \mathbb{N} \mapsto \frac{1}{n} \), \( \mu \) is the Möbius function, and \( \phi \) is the Euler totient:

\[
X = \overline{X} \star \mathbb{1} \\
\overline{X} = X \star \mu \\
\overline{A} = \text{inv} \times (X \star \mu) \\
A = \overline{A} \star \mathbb{1} = \text{inv} \times (X \star \phi).
\] (1)

The 3rd relation above corresponds to the Witt formula counting the number of Lyndon words [3, 8, 11, 13, 14]. The last equality comes from Burnside’s orbit-counting Lemma.

Let us note that the total number of attractors of a BAN is never greater than what it would be if all attractors had the largest possible period \( \omega \): Thus:

\[
A(\omega) \geq X(\omega) / \omega.
\] (2)
7 Cycles, tangent cycles, and a more formal motivation

Our general, informal motivation described in the introduction leads to taking (formal) interest in the order $\omega$ (intuitively accounting for a form of “asymptotic diversity”), in the distributions of a network’s configuration and attractor periods, and in the total number of attractors $K(\omega)$ (intuitively accounting for a form of “asymptotic variety”). More precisely, we are interested in how all of these relate to the cycles in $G$, to their signs, and to their interactions.

It is commonly accepted and has been supported by formal arguments in several frameworks more or less related to $\mathcal{N}$ decisively impact on its (asymptotic) behaviour [20, 21, 23, 25, 30]. Having had so much attention, cycles are now rather well understood. The specific way that cycle intersections per se impact on the overall network behavioural possibilities, however, is not at all. Our need to increment understanding of cycles with some primary insight on this, drives us to taking interest in “tangent cycles”.

We call $\mathbf{BAC}$ (Boolean Automata Cycle) any $\mathbf{BAN}$ that is structured as a simple cycle (cf. Table 1). We call $\mathbf{BAD}$ (Boolean Automata Double-cycle) a $\mathbf{BAN}$ structured as two tangent cycles (cf. Table 1). There are 2 types of $\mathbf{BACS}$ and 3 types of $\mathbf{BADS}$ (cf. Table 1).

In [6, 17], the (asymptotic) behaviours (as defined in Section 5) of all these 5 types of $\mathbf{BAN}$s has been characterised, and explicit formulae have been derived for all the quantities introduced in Section 6 relative to them. These results are based (non-exclusively) on results stating that in all five cases attractor periods divide positive cycle lengths without dividing negative cycle lengths, on results summed up in Table 1, and on some results that can be derived from them using (1).

8 The combinatorial problem relative to Boolean automata networks

Through its implications (cf. Sections 12 and 13), our main result falls in line with motivations presented above. In this theorem, the lower bound of (3) follows from (2).

Theorem 1. The total number of attractors of any $\mathbf{BAC}$ and almost\(^3\) any $\mathbf{BAD}$ of order $\omega$ satisfies:

$$X(\omega)/\omega \leq K(\omega) \leq 2 \cdot \bar{X}(\omega) = 2 \cdot X(\omega)/\omega \leq 2 \cdot X(\omega)/\omega. \tag{3}$$

The least upper bound of (3) equivalently means that the expected value of an attractor period is big: $\sum_{p \in \mathcal{A}, \mathcal{A}, p \neq p} \bar{X}(p) \geq \omega/2$. Thus, almost all periodic configurations of $\mathbf{BACS}$ and $\mathbf{BADS}$ have the greatest possible primitive period $\omega$ and $K(\omega) = \Theta(X(\omega)/\omega)$.

In the case of $\mathbf{BACS}$, the set of $p$-attractors $\mathcal{A}(p)$ is isomorphic to the set of (unlabelled) Lyndon words of length $p$ [3, 7, 8, 10–14, 26, 29]. The existence of an injective map $\bigcup_{p \in \mathcal{P}, p \subset \omega} \mathcal{A}(p) \rightarrow \mathcal{A}(\omega)$ [24], implies that $\mathbf{BACS}$ satisfy: $K(\omega) = \sum_{p \in \mathcal{P}, p \subset \omega} \mathcal{A}(p) + \bar{K}(\omega) \leq 2\mathcal{A}(\omega) \leq 2\bar{X}(\omega)/\omega$ and thus (3). Moreover, [17] proves that positive $\mathbf{BADS}$ behave as positive $\mathbf{BACS}$ of same order: the asymptotic (strongly connected) part of their transition graphs are isomorphic. Equation (3) therefore holds for all $\mathbf{BACS}$ and positive $\mathbf{BADS}$.

\(^3\) As the detailed proof in Appendix A reveals, with the notations introduced further on in this paper, Theorem 1 holds for all $\mathbf{BADS}$ except those satisfying $(K = 10 \land \Delta = 1) \lor (K = 6 \land \Delta = 2)$. That is, Theorem 1 holds for all $\mathbf{BADS}$ except the 3 types of negative $\mathbf{BADS}$ such that either $(f, r)$ or $(r, f)$ belongs to $\{(1, 9), (3, 7), (2, 10)\}$.

4
Although we are now going to focus on BADs, in order to clarify how things work, let us first note that in a \( \text{BAD} \) of size \( n \) and sign \( s \), identifying \( V \) with \( \mathbb{Z}/n\mathbb{Z} \), we have: \( \forall i \in V, f_i(x) = s_i \cdot (x_{i-1}) \). It follows that any configuration \( x = x(t) \) in \( \mathbb{B}^n \) satisfies \( x(t+n) = x(t) \) if \( s = + \) and \( x(t+n) = (\neg x_0 \ldots \neg x_{n-1}) \) if \( s = - \).

In a \( \text{BAD} \) \( \mathcal{D} \) of size \( n = \ell + r - 1 \), all automata with in-degree 1 also satisfy \( f_i(x) = s_i \cdot (x_{i-1}) \) and the intersection automaton \( i = 0 \) satisfies: \( f_0(x) = s_{\ell-1,0} \circ s_{r-1,0} \circ s_{n-1,0} \circ s_{n-1,0} \), where \( \circ \in \{\lor, \land\} \) (this is because of the assumed local monotony of \( f_0 \)). We are going to concentrate on the case where \( \circ = \lor \). The case where \( \circ = \land \) is similar. Let \( s^L \) (resp. \( s^R \)) be the sign of the left (resp. right) cycle of \( \mathcal{D} \). Let \( d = \ell \) mod \( p \) and \( d' = r \) mod \( p \). Importantly, if \( p \) is a network period (\( X(p) \neq \emptyset \)), and \( s^L = - \), then (cf. end of Section 7) \( \neg p \) and consequently \( d > 0 \).

Any recurrent configuration \( x = x(t) \in X(p) \) of the \( \text{BAD} \) satisfies:

\[
x_0(t) = \begin{cases} 
\neg x_0(t-d) \lor x_0(t) & \text{when } s^L = - \text{ and } s^R = + \\
\neg x_0(t-d) \lor \neg x_0(t+d) & \text{when } s^L = s^R = - 
\end{cases}
\]  

\( \textbf{Table 1.} \) Summary of results concerning the behaviour of \( \text{BACS} \) and \( \text{BADS} \). \( \forall k, m \in \mathbb{N}, \neg (k|m) \) equals 0 if \( k|m \) and 1 otherwise. For \( \text{BADS} \), \( \Delta = \gcd(\omega, \ell), K = \omega/\Delta, \Delta_p = \gcd(p, \ell) = \gcd(p, d) = \gcd(\Delta, p) \) and \( K_p = p/\Delta_p \). \( (\text{L}(n))_{n \in \mathbb{N}} \) is the \textbf{Lucas Sequence} [19, 22] (sequence \( \text{A294 of the OEIS} \) [28]) and \( (P(n))_{n \in \mathbb{N}} \) is the \textbf{Perrin Sequence} [1] (sequence \( \text{A1608 of the OEIS} \)).
Now, consider the orbit of automaton 0 of \( W \) class of words under iterates of the rotation \( W \). We then define the following set that will play an essential role in the sequel:

\[
\text{Lemma 1. Let } p \in \mathbb{N}, \text{ let } x \in X(\omega) \text{ be a recurrent configuration of a bad of order } \omega \text{ with negative left cycle of length } \ell \equiv d \text{ mod } p, \text{ and let } w \in \mathbb{B}^n \text{ be the orbit of intersection automaton } 0 \text{ that is initiated in } x. \text{ Then:}
\]

\[
\begin{cases}
0 \omega & \text{if } s^L \neq s^R \\
0 \omega, 1w1v1 \text{ if } s^L = s^R = -.
\end{cases}
\]

We then define the following set that will play an essential role in the sequel:

\[
W^n_d = \{ w \in \mathbb{B}^n, \text{ w contains no factor in } F_d \} \subset \mathbb{B}^n.
\]

Importantly, let us note that \( \forall d > 0, W^n_d = \emptyset \). Also, \( \text{mathbf{W}}^n_d = \text{mathbf{W}}_n^{W_n-d} \). Thus, (4) implies part of the following Lemma which is proven in [17], and which can also be stated for <bacs> using the corresponding sets \( F_d \) defined in Table 1.

**The combinatorial problem relative to binary necklaces**

Henceforth, we concentrate on binary words \( w \in \mathbb{B}^n \) of arbitrary length \( n \in \mathbb{N} \), with letters indexed from 0 to \( n-1 \). We abuse notations so as to let \( w_k, \forall k \in \mathbb{Z} \), refer to letter \( w_k = w_k \mod n \) of word \( w \). A necklace of length \( n \in \mathbb{N} \) represents an equivalence class of words under iterates of the rotation \( : w \mapsto w_{n-1}w_0 \ldots w_{n-2} \). The necklace (conjugacy class) representing (containing) word \( w \in \mathbb{B}^n \) and all of its rotations (conjugates) \( p^k(w), k \in \mathbb{Z}/n\mathbb{Z} \) is denoted by \( \langle w \rangle \) and we write \( w \equiv w' \) when \( \langle w \rangle = \langle w' \rangle \).

Additionally to set \( W^n_d \) defined in (5) above, \( \forall p|n \) and \( \forall d < n \), we define the following sets:

\[
\begin{align*}
W^n_d(p) &= \{ w \in W^n_d \text{ has period } p \} = \{ u^p \in W^n_d, u \in \mathbb{B}^p \} = \{ u^p, u \in \mathbb{W}^n_d \} \\
\bar{W}^n_d(p) &= \{ w \in W^n_d \text{ has primitive period } p \} = W^n_d(p) \setminus \bigcup_{q|p, q < p} W^n_d(q)
\end{align*}
\]

In particular, \( \bar{W}^n_d = \bar{W}^n_d(n) = \{ w \in W^n_d \text{ is primitive (aperiodic)} \} \)

\[
\begin{align*}
C^n_d &= \{ \langle w \rangle, w \in W^n_d \} \\
C^n_d(p) &= \{ \langle w \rangle, w \in W^n_d(p) \} \\
\bar{C}^n_d(p) &= \{ \langle w \rangle, w \in \bar{W}^n_d(p) \} \quad \text{and in particular, } \bar{C}^n_d = \bar{C}^n_d(n) = \{ \langle w \rangle \in C^n_d \text{ is primitive} \}.
\end{align*}
\]
We let \( W_n^d, W_{d'}^n(p), W_d^n(p), W_d^n, C_d^n(p), C_{d'}^n(p), \) and \( \tilde{C}_d^n \) straightforwardly denote the cardinals of all these sets.

The last equality of (6) holds because \( w = u^\nu \in W_d^n(p) \) implies that \( u \in W_d^n \). Consequences of this are the following, where we write \( rations \) and attractors (relative to a negative cycle of length \( d \)).

As a result of Lemma 1, we also have the following relations between sets of configurations and attractors (relative to a negative cycle of length \( \ell \equiv d \ mod \ p \)), and sets of words and necklaces:

\[
W_d^n(p) = W_d^n, \quad \tilde{W}_d^n(p) = \tilde{W}_d^n, \quad C_d^n(p) = C_d^n, \quad \tilde{C}_d^n(p) = \tilde{C}_d^n. \quad (7)
\]

As a result of (7) and (8), proving the least upper bound of Theorem 1 is equivalent to proving the following key proposition:

**Proposition 1.** Let \( n, d \in \mathbb{N}, d < n \). Let \( F_d \) be any of the sets of forbidden word factors defined in the last line of Table 1. The cardinals of the sets of binary necklaces defined above relatively to \( F_d \) almost always satisfy\(^4\):

\[
C_d^n \leq 2 \cdot \tilde{C}_d^n. \quad (9)
\]

In other terms, to prove that \( BADs \), just like \( BACs \), have no more small attractors (\( p \)-attractors s.t. \( p < \omega \)) than big ones (\( \omega \)-attractors), we want to show that there are no more periodic necklaces without the forbidden factors of \( F_d \) than there are aperiodic ones. Explicit formulae for cardinals of all sets introduced above are known [17]. However, comparing these formulae turns out to be very tricky. Thus, here, we propose to build and injective map \( \gamma : \bigcup_{p \mid n, p < n} \tilde{C}_d^p \rightarrow \tilde{C}_d^n \).

The existence of this map will prove Proposition 1 and thereby Theorem 1.

### 11 Proof of the main result

We prove (9) in the case where \( F_d = \{000, 1u1v1 \mid u, v \in B^{d-1}, \} \), i.e. we prove (3) for negative \( BACs \). The case where \( F_d = \{000 \mid u \in B^{d-1} \} \) corresponding to mixed \( BACs \) is proven in a very similar but much easier manner\(^5\).

Throughout this section, \( n, d, \Delta, K \) denote integers satisfying: \( 0 < d < n, \Delta = \gcd(n, d) \) and \( K = n/\Delta > 1 \) (\( n \) corresponds to the integer \( \omega \) of the previous sections on \( BACs \)), and \( (\Delta, K) \not\in \{(1, 10), (2, 6)\} \).

\( F_1 = \{00111\} \) is easier to manipulate than \( F_d, d > 1 \). For this reason, given a word \( w \in W_d^n \), the baseline idea is going to be to see \( w \) as an interleaving \( L \) of a certain

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\(^4\) They always do if \((d, n), (n - d,) \not\in \{(1, 10), (3, 10), (2, 12)\}\); cf. Footnote 3.

\(^5\) The mixed \( BAD \) case involves less forbidden factors than the negative \( BAD \) case. Also, while the latter relates to the Perrin integer sequence which has a rugged beginning, the former relates to the much smoother Lucas sequence (cf. Table 1). For these reasons, the mixed \( BAD \) case induces no special sub-cases, especially none of the sort of \( K = 6 \) which is involved in the negative \( BAD \) case (cf. Appendix A.2 and A.6).
number \(m\) of smaller words of length \(k\), \(L(0), \ldots, L(m-1)\) such that \(\forall 0 \leq j < m\), letters of sub-word \(L(j)\) appear every \(d\) position in \(w\), and \(L(j) \in W_1^K\) (cf. Fig. 2). As we are about to see, \(m = \Delta\) and \(k = K\).

Fig. 2. A word \(w \in \mathbb{B}^n, n = 15\), represented as an interleaving of \(\Delta = \text{gcd}(6, n) = 3\) words \(L(1)\) (light grey), \(L(2)\) (dark grey) and \(L(3)\) (white) of length \(K = 5\).

Generally, \(\forall w \in \mathbb{B}^n\), we define the list \(L(w, d) = \{L(0), L(1), \ldots, L(\Delta - 1)\} \in (\mathbb{B}^K)^\Delta\) by: \(\forall j < \Delta, \forall k < K, L(j)_k = w_{j+kd}\). And conversely, given a list \(L = \{L(0), \ldots, L(\Delta - 1)\} \in (\mathbb{B}^K)^\Delta\), we define the word \(w(L, d)\) by: \(\forall i < n, i = j + q\Delta \equiv j \mod \Delta, w(L, d)_i = L(j)_q \equiv \frac{j}{\Delta}\) Note that by definition of \(\Delta\) there exists Bezout integers \(a, b \in \mathbb{Z}\) such that \(\Delta = an + bd\). This implies that \(\frac{1}{\Delta} \equiv b \mod n\) so \(q \times \frac{1}{\Delta} \in \mathbb{Z} / K\). The following lemma can easily be checked.

**Lemma 2.** For any word \(w \in \mathbb{B}^n\) and any list \(L \in (\mathbb{B}^K)^\Delta\), \(w \equiv w(L, d) \iff L = L(w, d)\).

In the sequel, for any divisor \(p\) of \(n\), \(\Delta_p\) and \(K_p\) respectively denote the divisors of \(\Delta\) and of \(K\) defined by \(\Delta_p = \text{gcd}(p, d) = \text{gcd}(\Delta, p)\) and \(K_p = \frac{p}{\Delta_p}\), where \(d = \ell \mod p, d > 0\) (cf. Lemma 1) so that necessarily \(K_p > 1\). The first part of the following result may be checked using the definitions of \(\mathbf{F}_d\) and \(\mathbf{F}_3\):

**Lemma 3.** For any word \(w \in \mathbb{B}^n\), \(w \in W_d^n \iff L(w, d) \in (W_1^K)^\Delta\). Consequently:

\[
W_d^n \cong (W_1^K)^\Delta \cong W_\Delta^n \quad \text{and} \quad W_d^n(p) \cong W_d^n(p) \equiv (W_1^K)^{\Delta_p} \cong W_\Delta^n(p). \quad (10)
\]

Equation (10) allows to concentrate on the case where \(\Delta = d\), w.l.g. The rest of the proof of Proposition 1 consists of the following steps, detailed in Appendix A:

1. Relate the period \(p = K_p\Delta_p\) of a word \(w \in W_d^n(p)\) to the interleaving \(L = L(w, \Delta)\) representing it. More precisely, show that \(K_p\) equals the least common multiple of the primitive periods of \(L\)’s sub-words \(L(j)\) (cf. Lemma 4), and that \(\forall j < \Delta, L(j) \equiv L(j + \Delta_p)\) (cf. Lemma 6).

2. Define a unique representative list \(L((w), \Delta)\) for every conjugacy class \(\langle w \rangle\). Introduce set \(\tilde{L}^{K, \Delta} = \{L((w), \Delta), \langle w \rangle \in C^{K, \Delta}\}\). As follows, define a map \(\Gamma : \cup_{p|n, p < n} \tilde{L}^{K_p, \Delta_p} \rightarrow \tilde{L}^{K, \Delta}\) (cf. A.3) from which \(\gamma\) can be derived directly (cf. item 7 below).

3. Let \(L \in \tilde{L}^{K_p, \Delta_p}\) \((p|n, p < n)\) be an arbitrary list for which we must define an image by \(\Gamma\). First, as follows, define \(L' \in (W_1^n)^\Delta\) so that \(w = w(L', \Delta)\) is primitive.
(a) Define $L'$ in the case where $\Delta_p = \Delta$ (cf. A.4). To do this, first elongate one of the words $L(j) \in W^K$ of list $L$ so as to turn it into a primitive word $L(j') \in W^K$ (cf. the elongation map: $u \in W^K \rightarrow \alpha(K, u) \in W^K$ defined in A.4). Except in (many) special cases, this elongation can be done by concatenating a primitive word of length $K - K_p$ to $L(j)$. Next, repeat all other words to make them longer: $\forall j < \Delta_p, j \neq j'$, define $L'(j) = L(j) \frac{K_j}{K_p}$.

(b) Define $L'$ in the case where $\Delta_p < \Delta$ (cf. A.5). To do this, first, if necessary (it could be that $K_p = K$ in this case), lengthen all $L(j)$'s by repeating them: $\forall j < \Delta_p, L'(j) = L(j) \frac{K_p}{K_p}$. Then, add a series of $\Delta - \Delta_p \geq \Delta/2$ consecutive, identical primitive words $w \in W^p$, in a way that this series will not be confused with the rest of $L'$.

4. In both cases 3a and 3b, primitivity of $w = w(L', \Delta)$ follows from item 1 above and from: (i) at least one of the $L'(j)$'s is primitive, and (ii) $L'$ is aperiodic itself by construction (in the general cases, a series of 1 periodic and $\Delta - 1$ aperiodic subwords cannot be periodic, nor can a series in which at least half of the sub-words are identical, consecutive and distinct from the non-empty rest of the series).

5. Define $\Gamma(L) = L(\langle u \rangle, \Delta)$ where $w = w(L', \Delta)$.

6. The injectivity of $\Gamma$ comes from the effort made in the construction of $L'$ to encode non-ambiguously all information of $L$ into $L''$, and from the fact that the domain of $\Gamma$ only contains one list $L = L(\langle u \rangle, \Delta)$ per conjugacy class $\langle u \rangle, u \in W^K_N$.

7. Define map $\gamma : \bigcup_{p<n} C^p_{\Delta_p} \rightarrow \check{C}^p_N$ by $\gamma(\langle u \rangle) = \langle w \rangle$ where $w = w(\Gamma(L(\langle u \rangle, \Delta_p)), \Delta)$, $\forall \langle u \rangle \in C^p_{\Delta_p}$, $p|n$, $p < n$. The injectivity of $\gamma$ follows from that of $\Gamma$.

12 Back to Boolean automata networks

With Theorem 1 informing on the behaviours of BACS and BADS, it still remains to gain insight on the role played specifically by the cycle intersections in the defining of network (asymptotic) behavioural possibilities. To do this, comparisons between BACS and BADS need to be made. Our last formal result below (whose yet unpublished proof is given in Appendix B) exploits Theorem 1 to go further in this direction. We let $A_+^+(\omega), A_+^-(\omega), A_+^{++}(\omega)$, $A_+^{+-}(\omega)$ and $A_+^{-+}(\omega)$ respectively denote the total numbers of attractors of a positive BAC, of a negative BAC, of a positive BAD, of a mixed BAC and of a negative BAD of order $\omega$ (and such that $\Delta = \gcd(\ell, r)$).

**Theorem 2.** The numbers of attractors of negative BACS (resp. BADS) are (resp. exponentially w.r.t. $\omega$) smaller than that of positive BACS of same order $\omega \in \mathbb{N}$:

$$
A_+^-(\omega) \leq \frac{1}{2} A_+^+(\omega) \quad \quad \quad \quad A_+^{+-}(\omega) \leq 2 \left(\frac{\omega^2}{2}\right)^\omega A_+^+(\omega)
$$

$$
A_+^{++}(\omega) = A_+^+(\omega) \quad \quad \quad \quad A_+^{-+}(\omega) \leq 2 \left(\frac{\omega^2}{2}\right)^\omega A_+^+(\omega) \text{ if } K = 3
$$

$$
A_+^{-+}(\omega) \leq 2 \left(\frac{\omega^2}{2}\right)^\omega A_+^+(\omega) \text{ if } K \neq 3
$$

This effort and the many special cases that need to be taken into account separately are the explanation for the great length of the full proof given in Appendix A.
where, for a BAD, $\Delta$ denotes the gcd of its underlying cycle lengths, and $\omega = K\Delta$.

13 Discussion

Informal insights and scope of results.

By Theorem 1, the largest attractors of a BAC or BAD are the most numerous. Let $U(x) = \{i < n, x_i \neq f_i(x)\}$ denote the set of local instabilities in configuration $x$. Intuitively, amongst the large attractors of a BAC or BAD must feature the most stable, i.e. those involving configurations with small values of $\#U(x)$. The idea is that this sort of attractor, induced by little momentum, corresponds to a small number $\#U(x)$ of local instabilities, circulating on the cycles of $G$, punctually destabilising each automaton one after the other, before returning to their initial locations. This agrees with the fact that the order of a BAC or BAD has the order of its size (cf. Table 1).

Let us imagine turning BACs into BADs, for instance by forcing two automata to work as one, either adjoining two simple cycles, or pursing one large cycle into two smaller ones. This way, the order $\omega$ of the overall network, and, much more significantly, by Theorem 2, the total number $\mathcal{A}(\omega)$ of attractors decrease. Backed up with simulation relations between BACs and BADs established in [18], Theorem 2 serves as as a base case supporting the following informal idea: networks tend to lose degrees of freedom as their underlying structural cycles become more intricately intersected. Put in other terms, this just means that cycles that are forced to interact tend to hinder themselves rather than the contrary, and Theorem 2 represents a first formal argument in this direction and in the context of BANs.

The intersection automaton $0$ of a BAD can receive at most $2 = \deg^-(0)$ local instabilities as inputs from automata $\ell - 1$ and $n - 1$. It can output at most $2 = \deg^+(0)$. Examining the different cases reveals that, $\#U(x)$ is less often increased than it is maintained or decreased. It seems that all in all, BAD intersections tend to synchronise local instabilities and reduce the number of them.

Now, let us use the ratio $\xi(\mathcal{N}) = \frac{\mathcal{T}(\omega)}{\omega}$ to pinpoint formally a general notion of degree of freedom (or propensity to behave in numerous, various ways) of a BAN $\mathcal{N}$. As we turn BACs into BADs, the size $n$ of the overall network hardly changes at all. The order $\omega$ doesn’t change much either since it still has the order of $n$. But on the contrary, by Theorem 2, $\xi(\mathcal{N})$ is very significantly decreased. We can build on all the previous remarks of this section, assuming that small attractors are induced by greater numbers of local instabilities, and that cycle intersections filter out both global and local instabilities. Thus, comparing BACs and BADs, the substantial difference in $\xi(\mathcal{N})$ that comes with no substantial change of $\omega$ can be interpreted as follows: by getting rid of local instabilities, cycle intersections induce larger attractors and get rid of the smaller, less stable ones. This would mean that even under the parallel updating which is the best at entertaining local instabilities on uninterrupted paths of $G$, cycle intersections “force” asynchrony in the sense that they reduce the number of possible changes that are possible at once (note that this is a sort of asynchrony that is inherent to the system rather than an assumption of the practitioner), thereby reducing the overall network asymptotic degree of freedom, and increasing its overall stability.
**Perspectives.**

First of all, of course, all the semantic remarks made right above call for a proper formalisation and a verification. One way or the other, I believe these remarks to be noteworthy since at the very least they can serve as very tangible (and new) guidelines for further researches. Moreover, one practical purpose can be expected to be drawn from the results of this paper, in the lines of these informal remarks: to yield a constructive method for approximating networks, based on elementary operations (including elementary operations on digraphs $G$, not dissimilar to the contractions underlying the definition of a graph minor) that “simplify” the networks (structures), and as a consequence, in a controlled manner, add noise in the description of their behaviours. Besides the pertinence of this with regards to modelling considerations where complexity is especially limiting, this would allow to derive bounds on the numbers of attractors of arbitrary networks.

As for the technical aspects of this paper, we have built an injective map from a set of periodic binary necklaces satisfying certain conditions to the set of primitive binary necklaces satisfying the same conditions. This raises the problem of specifying more generally what are the types of conditions on necklaces that allow this to remain true (just like it was already known to be true for necklaces satisfying no conditions at all [24]). Perhaps tightly related to this but with a different viewpoint is the following problem which arises from Lemma 7 and Equation (21) in Appendix A. All five cases considered in this paper and shown to fall under the scope of Theorem 1 are based on integer sequences $(X(n))_{n \in \mathbb{N}}$ whose value in $n \in \mathbb{N}$ either equals or is very close and asymptotically equivalent to the $n^{th}$ power $a^n$ of some value $a \in \mathbb{R}$, $1 < a$. We have found no mention in the literature of a more general result characterising a larger class of integer sequences $(X(n))_{n \in \mathbb{N}}$ satisfying Theorem 1. This is one the most immediate sequels to the present work.

**References**

1. W. Adams and D. Shanks. Strong primality tests that are not sufficient. *Mathematics of Computation*, 39:255–300, 1982.
2. J. Aracena, J. Demongeot, and E. Goles. Positive and negative circuits in discrete neural networks. *IEEE Transactions on Neural Networks*, 15:77–83, 2004.
3. J. Berstel and D. Perrin. The origins of combinatorics on words. *European Journal of Combinatorics*, 28:996–1022, 2007.
4. C. Choffrut. An introduction to automata network theory. In *Automata Networks*, volume 316 of *Lecture Notes in Computer Science*, pages 1–18. Springer Berlin Heidelberg, 1988.
5. J. Demongeot, C. Jézéquel, and S. Sené. Boundary conditions and phase transitions in neural networks. theoretical results. *Neural Networks*, 21:971–979, 2008.
6. J. Demongeot, M. Noual, and S. Sené. Combinatorics of Boolean automata circuits dynamics. *Discrete Applied Mathematics*, 160:398–415, 2012.
7. S. W. Golomb. *Shift register sequences*. Holden-Day, 1967.
8. R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete mathematics: a foundation for computer science*. Addison-Wesley, 1989.
9. S. A. Kauffman. Metabolic stability and epigenesis in randomly constructed genetic nets. *Journal of Theoretical Biology*, 22:437–467, 1969.
10. M. Lothaire. *Combinatorics on words*. Addison-Wesley, 1983.
11. É. Lucas. *Le calcul des nombres entiers. Le calcul des nombres rationnels. La divisibilité arithmétique*. Gauthier-Villars, 1891.
12. R. C. Lyndon. On Burnside's problem. *Transactions of the American Mathematical Society*, 77:202–215, 1954.

13. P. A. MacMahon. Application of a theory of permutations in circular procession to the theory of numbers. *Proceedings of the London Mathematical Society*, 23:305–313, 1892.

14. W. Magnus, A. Karass, and D. Solitar. *Combinatorial group theory: presentation of groups in terms of generators and relations*. Interscience Publishers, 1966.

15. W. S. McCulloc and W. H. Pitts. A logical calculus of the ideas immanent in nervous activity. *Bulletin of Mathematical Biophysics*, 5:115–133, 1943.

16. L. Mendoza, D. Thieffry, and E. R. Alvarez-Buylla. Genetic control of flower morphogenesis in Arabidopsis thaliana: a logical analysis. *Bioinformatics*, 15:593–606, 1999.

17. M. Noual. Dynamics of circuits and intersecting circuits. In *Proceedings of LATA*, volume 7183 of *LNCS*, pages 433–444, 2012.

18. M. Noual. [http://perso.ens-lyon.fr/mathilde.noual/candidatures_CR2_CNRS/manuscrit.pdf](http://perso.ens-lyon.fr/mathilde.noual/candidatures_CR2_CNRS/manuscrit.pdf) Updating Automata Networks. PhD thesis, Ecole normale supérieure de Lyon, 2012.

19. Y. Puri and T. Ward. A dynamical property unique to the Lucas sequence. *The Fibonacci Quarterly*, 39:398–402, 2001.

20. É. Remy, B. Mossé, C. Chaouiya, and D. Thieffry. A description of dynamical graphs associated to elementary regulatory circuits. *Bioinformatics*, 19:ii172–ii178, 2003.

21. É. Remy and P. Ruet. From minimal signed circuits to the dynamics of Boolean regulatory networks. *Bioinformatics*, 24:i220–i226, 2008.

22. P. Ribenboim. *The new book of prime number records*. Springer-Verlag, 1996.

23. A. Richard. Local negative circuits and fixed points in non-expansive Boolean networks. *Discrete Applied Mathematics*, 159:1085–1093, 2011.

24. J. Riordan. *An Introduction to Combinatorial Analysis*. Princeton University Press, 1962.

25. F. Robert. *Les systèmes dynamiques discrets*. Springer Verlag, 1995.

26. F. Ruskey. Combinatorial generation. Book preliminary working draft, 2003.

27. H. Siebert. Dynamical and structural modularity of discrete regulatory networks. In *Proceedings of COMPMOD*, volume 6 of *Electronic Proceedings in Theoretical Computer Science*, page 109, 124.

28. N. J. A. Sloane. The on-line encyclopedia of integer sequences. [https://oeis.org/](https://oeis.org/).

29. N. J. A. Sloane. On single-deletion-correcting codes. In *Codes and designs*, pages 273–291. de Gruyter, 2002.

30. R. Thomas. On the relation between the logical structure of systems and their ability to generate multiple steady states or sustained oscillations. In *Numerical methods in the study of critical phenomena*, volume 9 of *Springer Series in Synergetics*, pages 180–193. Springer-Verlag, 1981.

31. L. Tournier and M. Chaves. Uncovering operational interactions in genetic networks using asynchronous Boolean dynamics. *Journal of Theoretical Biology*, 260:196–209, 2009.

32. H. Van der Laan. *Le nombre plastique, quinze leçons sur l'ordonnance architectonique*. E.J. Brill, 1960.
Appendix A

A Proof of Proposition 1

A.1 Lists and words (sequel)

We let integers \( n, \Delta, K, p, \Delta_p, K_p > 1 \) be as before. In particular \( (\Delta, K) \notin \{(1, 10), (2, 6)\} \).

Lemmas 4 and 6 show how the (primitive) period \( p \) of a word \( w \in \mathcal{W}_\Delta^n(p) \) translates in terms of the interleaving \( L = L(w, \Delta) \). First, Lemma 4 relates \( p \) to the primitive period of the interleaved words \( L(j) \):

**Lemma 4.** If \( p = K_p \Delta_p \) (\( \Delta_p = \gcd(\Delta, p) \)) is the primitive period of word \( w \in \mathcal{W}_\Delta^n(p) \), then \( K_p = \text{lcm}_{j < \Delta}(K_j) \) where, \( \forall j < \Delta, K_j \) denotes the primitive period of word \( L(j) \in \mathcal{W}_\Delta^n \) of list \( L(w, \Delta) \).

**Proof.** Let \( K' = \text{lcm}_{j < \Delta}(K_j) \). On the one hand, \( \forall j < \Delta, \forall k < K, L(j)_{k+K_p} = w_{j+(k+K_p)\Delta} = w_{j+k\Delta+p\frac{\Delta}{\Delta_p}} = w_{j+k\Delta} = L(j)_k \). Thus, \( K_p \) is a common period of all \( L(j) \)'s and \( K' \) \( \mid K_p \). On the other hand, \( \forall i < n, i = j + q \Delta \equiv j \mod \Delta, w_{i+K'\Delta} = L(j)_{q+K'\Delta} = L(j)_q = w \); so \( K' \Delta \) is a period of \( w \) and \( p \mid K' \Delta \) which implies \( K_p \mid K \) because \( \gcd(K_p, \Delta/\Delta_p) = 1 \). \( \square \)

Further, Lemma 6 relates \( p \) to the interleaving of words \( L(j) \). To do that, it exploits Lemma 5 which relates the list \( L^q = L(\rho^q(w), \Delta) \) associated to an arbitrary conjugate \( \rho^q(w) \) of a word \( w \in \mathcal{B}_n \), with the list \( L = L(w, \Delta) \):

**Lemma 5.** The words in the list representing an arbitrary conjugate \( w' = \rho^q(w) \) of \( w \) are rotations of the words in the list representing \( w \), i.e. \( w' \) and \( w \) are interleavings (w.r.t. \( \Delta \)) of rotations of the same sub-words: if \( q = m\Delta + \delta \equiv \delta \mod \Delta, \) then:

\[
L^q = L(w', \Delta) = \left\{ \rho^{m+1}(L(\Delta - \delta)), \ldots, \rho^{m+1}(L(\Delta - 1)), \rho^m(L(0)), \ldots, \rho^m(L(\Delta - \delta - 1)) \right\}.
\]

**Proof.** If \( j < \delta \), then \( 0 < \Delta + j - \delta < \Delta \) so \( \forall k < K, L^q(j)_k = \rho^q(w)_{j+k\Delta} = w_{j+k\Delta-q} = w_{(j-\delta)\Delta+(k-1)\Delta} = L(j-\delta)_k = \rho^m(L(j-\delta))_k \). If \( \delta \leq j < \Delta \), then \( 0 < \delta - \delta < \Delta \) so \( \forall k < K, L^q(j)_k = L(j-\delta)_k = \rho^m(L(j-\delta))_k \). \( \square \)

**Lemma 6.** \( \forall w \in \mathcal{W}_\Delta^n(p), L(j + \Delta_p) \equiv L(j) \). More precisely, \( L(j + \Delta_p) = \rho^{\frac{\Delta_p}{\Delta}}(L(j)) \) if \( j < \Delta - \Delta_p \) and \( L(j + \Delta_p) = \rho^{\frac{\Delta_p}{\Delta}-1}(L(j)) \) otherwise.

**Proof.** By definition, there exists Bezout integers \( a, b \in \mathbb{Z} \) s.t. \( \Delta_p = a \Delta + b p \). Lemma 6 is proven by using Lemma 5, noting that \( w \in \mathcal{W}_\Delta^n \Leftrightarrow \rho^{kp}(w) = w, \forall k \in \mathbb{N}, \) and taking \( k = b \). \( \square \)
A.2 Primitive words

In this section we look closer at some primitive words of \(\overline{W}_a^b\), depending on \(a\) and \(b\). First let us recall that \(\forall a, \overline{W}_a^a = W_a^a = \varnothing\). Next, let us consider the cases where \(K\) equals either 4 or 6.

Since \(W_1^4 = (0101) = W_1^4(2)\), it holds that \(\overline{W}_1^4 = \varnothing\), and, by Lemma 3 and Equation (7), that \(W_{1,3}^4 \sim (W_1^2)^2 \sim (W_1^2)^2 \sim W_{2,3}^4 \sim W_{1,2}^4(2\Delta)\) so there are no primitive words in \(W_{2,3}^4\) when \(n = 4\Delta\). \(\forall \Delta, W_{1,2}^4(2\Delta) = \varnothing\).

Since \(W_1^6 = (010101) \cup (011011) = \overline{W}_1^6(2) \cup \overline{W}_1^6(3)\), it holds that \(\forall \Delta, \overline{W}_1^6 = \varnothing\). And if \(w \in \overline{W}_1^6\), then \(L(w, \Delta)\) is composed of words of period 2 or 3 and contains at least one of each.

We recall that \(K > 1\) necessarily holds because \(\omega = n = K\Delta\) cannot divide \(\ell\) and thus nor can it divide \(\Delta\) (cf. Lemma 1). From now on, \(K \in \{1, 4\}\) is assumed and the case \(K = 6\) will be treated separately. \(\forall \Delta \in \{1, 4, 6\}\), let us define the following four canonical words of \(W_1^K\) that can be checked to be primitive:

\[
\begin{align*}
\upsilon(K) &= (01)^a(011)^b & \text{where } a \text{ is maximal so that } K = 2a + 3b, \text{ with } b \leq 2 \\
x(K) &= \upsilon(K - 5)u(5), \forall K \geq 12, \\
v(K) &= (01)^a(011)^b & \text{where } b \text{ is maximal such that } K = 2a + 3b, \text{ with } a \leq 3 \\
y(K) &= v(K - 7)v(7), \forall K > 14.
\end{align*}
\]

Let us call macro-letter any of the two factors 01 and 011. Any word of \(W_1^K\) is a word on alphabet \(\{0, 1\}\).

Comparing \(a\) and \(b\) in the writing of \(u\) and \(v\) and also, comparing the number of alternations of macro-letters, the following may be proven easily and will serve as a basis to the map \(\Gamma\) on lists built in the next paragraphs:

\[
\begin{align*}
\forall K \leq 10, \overline{W}_1^K &= (\upsilon(K)) \; \text{where } \upsilon(11) \neq v(11) \\
W_{1,1}^4 &= (\upsilon(11)) \cup (v(11)) \\
W_{1,2}^4 &= (\upsilon(12)) \cup (v(12)) & \text{where } \upsilon(12) \equiv v(12) \neq x(12) \\
\forall K > 12, \overline{W}_1^K = (\upsilon(K)) \cup (v(K)) \cup (x(K)) & \text{and } \upsilon(K) \neq v(K) \neq x(K) \neq \upsilon(K) \\
\forall K > 14, \overline{W}_1^K = (\upsilon(K)) \cup (v(K)) \cup (x(K)) \cup (y(K)) & \text{and } y \notin \{(\upsilon(K)), (v(K)), (x(K))\} \\
& \forall u \in \overline{W}_1^K, K_p \in \{2, 3\}, \upsilon(\upsilon(K - K_p)u) \in \{(\upsilon(K)), (v(K)), (x(K))\}.
\end{align*}
\]

A.3 The maps \(\Gamma\) and \(\Gamma'\) on lists

If we were to map primitive words \(u \in \overline{W}_1^p \subset \mathbb{P}\), with \(p < n\), injectively onto primitive words \(w \in W_\Delta^n \subset \mathbb{W}\), we could do this through the construction of a one-to-one map belonging to \(\bigcup_{p < n} \overline{W}_1^p \simeq \mathbb{P}\) on lists representing primitive words. Here, to prove (9), we just want to map primitive necklaces \(\langle u \rangle \in C_\Delta^n\) injectively onto primitive necklaces \(\langle w \rangle \in \overline{C}_\Delta^n\). To do this, we are going to build a map on lists that represent primitive necklaces. Thus, we must define the representative list \(L(\langle u \rangle, \Delta)\) of an arbitrary necklace \(\langle u \rangle\).

For that purpose, we order words and list of words in \(W_1^K\) lexicographically (i.e. w.r.t. order \(\prec\) on letters defined by \(0 < 1\), or w.r.t. to order \(\prec'\) on macro-letters defined by \(01 \prec' 011\)). For any \(u \in W_1^K\), let us denote by \(\hat{u}\) the smallest word in \(\langle u \rangle\).
We define $\tilde{L}(\langle w \rangle, \Delta)$ to be the smallest of the lists in \{L(w', \Delta), \, w' \equiv w \} such that:

$$L(0) = \tilde{L}(0) \text{ and } \forall \, j < \delta, \, L(0) \preceq L(\tilde{j})$$

(13)

(where $\preceq$ is the lexicographical order on words induced by $\prec$). It follows from Lemma 5 that such a list exists indeed. Then, we can naturally introduce the following sets of lists:

$$\mathcal{K}^\Delta = \{L(\langle w \rangle, \Delta), \langle w \rangle \in \mathcal{C}_\Delta^\Delta\} \quad \text{and} \quad \tilde{\mathcal{K}}^\Delta = \{L(\langle w \rangle, \Delta), \langle w \rangle \in \tilde{\mathcal{C}}^\Delta_\Delta\}.$$ 

And now we aim at defining a map to define a map $L$ necessary representative) lists associated to primitive words of $\tilde{\mathcal{W}}_\Delta^p$. From the definition of this map $\Gamma'$, the definition of $\Gamma$ will immediately be given by:

$$\Gamma(L) = \tilde{L}(\langle w \rangle, \Delta) \text{ where } w = w(\Gamma'(L), \Delta).$$

(14)

We thus need to turn a list $L$ of $\Delta_p$ words of length $K_p$ into a list $L'$ of $\Delta$ words of length $K$ (and take the representative $\tilde{L}'$ of the latter).

Notably, this list $L$ we want to build images $L'$ and $\tilde{L}'$ by $\Gamma'$ and $\Gamma$, need not be any list: it must satisfy (13) and be a representative list $L = \tilde{L}(\langle w \rangle, K_p)$, representing a primitive necklace $\langle w \rangle$ of length $p$, where, importantly, $p|n$, $p < n$ is a proper divisor of the length $n$ of the necklace $\langle w \rangle = \langle w(\tilde{L}', \Delta) \rangle$. And we recall that as before, $p$ does not divide $\Delta$, implying that $K_p > 1$.

To build $L'$ from $L$, we want to add $K - K_p$ letters to the sub-words $L(j) \in \tilde{\mathcal{W}}_1^K$ of $L$, and add $\Delta - \Delta_p$ new words. In the general case, $L(j)$ will just be repeated to create $L'(j) = L(j)^{K/K_p} \in \tilde{\mathcal{W}}_1^K(K_p)$. As for the $\Delta - \Delta_p$ added words, we will choose them to be primitive in order to ensure the primitivity of the resulting word $w = w(L', \Delta)$ (cf. Lemma 4). Of course this only works when $\Delta_p < \Delta$. When $\Delta_p = \Delta$, by Lemma 4, words in $L'$ cannot keep on having common period $K_p$, otherwise $w$ would have period $K_p \Delta < n$. So in this case, we still must ensure the primitivity of $w$, but we must do it without adding any words. For this reason, when $\Delta_p = \Delta$ (implying $K_p < K$) we are going to extend exactly one of the $L(j)$'s into a primitive word $a(K, L(j)) = L'(j) \in \tilde{\mathcal{W}}_1^K$. And we are going to do it in a way that, given only $L'(j)$, it is still possible to derive what $L(j)$ was $L'(j)$ constructed from (to ensure injectivity of $\Gamma$). So we want there to exist a map $\beta: a(K, L(j)) \in \tilde{\mathcal{W}}_1^K \mapsto L(j) \in \bigcup_{K_p|K} \tilde{\mathcal{W}}_1^K$. The baseline idea of this extension is to concatenate word $u(K - K_p)$ or word $v(K - K_p)$ – cf. (11) – to word $L(j)$ so that the elongated version of $L(j)$ looks like: $L'(j) = a(K, L(j)) \equiv (01)^a(01)^b L(j)$ and is primitive. However, the injectivity of $\Gamma$ and primitivity of $w(L', \Delta)$ require that this idea be adjusted carefully in some cases.

A.4 Case $\Delta_p = \Delta$

In this case, $K_p|K > K_p$ so in particular, $K$ is not prime. Also, we assume that $K \notin \{1, 4, 6\}$. We want to elongate a word $u = L(j) \in \tilde{\mathcal{W}}_1^K$ of length $K_p$ into a primitive word $a(K, u) = L'(j) \in \tilde{\mathcal{W}}_1^K$ of length $K$. Equation (12) limits the choice for the latter: $\forall K \leq 10$, there is only one primitive word of length $K$, and for $K = 12$, there only are
two. However, as a result of assumptions, we are only considering the following cases (otherwise, \( K \) is prime or belongs to \( \{1, 4, 6\} \)):

- \( K = 8 \) implying that \( K_p = 2 \) (by A.2),
- \( K = 9 \) and \( K_p = 3 \),
- \( K = 10 \) and \( K_p \in \{2, 5\} \),
- \( K = 12 \) and \( K_p \in \{2, 3, 6\} \),
- \( K = 14 \) and \( K_p \in \{2, 7\} \), and
- \( K > 14 \).

Thus, \( \forall u \in W_1^{K_p} \), we define \( a(K, u) \in W_1^K \) as follows.

When \( K_p \in \{2, 3, 6\} \):

\[
\forall \hat{u} \in W_1^{K_p}, a(K, \hat{u}) = u(K - K_p) \hat{u} \quad \text{if} \; \hat{u} \neq u(K)
\]

\[
a(K, u(K)) = v(K) \quad \text{(noting that for} \; u(K) \text{to exist, it must hold that} \; K \neq 12, \text{i.e.} \; K = 10 \lor K > 12)
\]

When \( K_p \in \{2, 3, 6\} \):

\[
a(K, 01) = a(K, 010101) = u(K)
\]

\[
a(K, 011) = a(K, 011011) = \begin{cases} u(K) & \text{if } K < 12 \\ x(K) & \text{if } K \geq 12 \end{cases}
\]

\[
\text{and } 3|K, \text{ i.e. } K = 9 \quad \text{and } 3|K, \text{ i.e. } K = 12 \lor K > 14
\]

Generally, \( \forall k \in \mathbb{Z} / K_p \mathbb{Z}, \forall u = \rho^k(\hat{u}) \in W_1^{K_p} \), we let: \( a(K, u) = \rho^k(\alpha(K, \hat{u})) \).

It follows from the definitions of \( u(k), v(k) \) and \( x(k) \) in (11) and from (12) that \( \alpha(K, u) \) is primitive \( \forall K \in \{1, 4, 6\}, \forall u \in W_1^{K_p} \).

Let \( K_j \) denote again the primitive period of word \( L(j) \) of \( L \), and let \( K_j = \text{lcm}(K_i, i \neq j) \). If \( K_p \in \{2, 3, 6\} \), let \( j^* < \Delta = \Delta_p \) be such that \( K_{j^*} \notin \{2, 3\} \). Otherwise let \( j^* = 0 \). We define \( L' = \Gamma'(L) \) by:

\[
\begin{cases}
L'(j^*) = \alpha(K, L(j^*)) \\
L'(j) = L(j)^{K_p}, \; \forall j \neq j^*.
\end{cases}
\]

Now, as follows we define \( \beta \) (for the general case where \( K \neq 10 \)) and \( \beta' \) (for case \( K = 10 \)) to retrieve \( L(j) \) from \( L'(j) = \alpha(K, L(j)) \):

\[
\forall K \geq 12, \beta(u(8)) = \beta(u(K)) = 01 \quad \beta'(2, u(10)) = 01
\]

\[
\beta(9)) = \beta(K(\hat{K})) = 011 \quad \beta'(5, u(10)) = 010111
\]

\[
\forall K > 12, \beta(u(K)) = u(\frac{K}{2}) \quad \text{and } \forall u = \rho^q(\hat{u}), \beta'(k, u) = \rho^q(\beta(k, \hat{u})).
\]

\[
\forall K > 12, \forall \hat{u}, \beta(u(K') u) = \hat{u} \quad \beta'(k, w) = \rho^q(\beta(k, \hat{u})).
\]

\[
\text{and } \forall u = \rho^q(\hat{u}), \beta(w) = \rho^q(\beta(\hat{u})).
\]

(16)

Unless \( u \in W_1^{K_p} \) has period 2 or 3, \( \beta(a(K, u)) = u \), and \( \beta'(|u|, a(10, u)) = u \).
When $A.5$ Case

Examining $L' = \Gamma(L)$, $K_p$ can be derived in each case as follows (which will be useful in $A.8$):

- $K = 10 \implies K_p \in \{2, 5\}$ and $K_p = K_j, \forall j < \Delta$
- $\overline{K'} \notin \{2, 3, 6\} \implies K_p = \text{lcm}(K_p, \overline{K'}) \notin \{2, 3, 6\}
- $\overline{K'} = 6 \implies K_p = 6$
- $\overline{K'} \in \{2, 3\} \implies \text{either } \overline{K'} \neq K_p \text{ and thus } K_p = \text{lcm}(K_p, \overline{K'}) = 6 (\implies K \geq 12)$

$$\implies K_p = K_j \in \{2, 3\}$$

(17)

A.5 Case $\Delta_p < \Delta$

Here, we will not elongate any word of $L$. In the main case, the idea is to insert a series of $\Delta - \Delta_p$ new identical primitive words $z \in W^K_{\overline{K'}}$ at a certain index $j^* < \Delta_p$ of the list that will guarantee the primitivity of $w(L', \Delta)$. Note that unless $\Delta \leq 2$, we will be adding more than 1 primitive word. This way, $L'$ constructed in this case ($\Delta_p < \Delta$) cannot be confused with a $L'$ constructed in the previous ($\Delta_p = \Delta$) which only contains 1 primitive word (cf. $A.4$). The only possible ambiguity is when $\Delta = 2$ and $L'$ contains one primitive sub-word and one imprimite sub-word. The resolution of this ambiguity will come later.

In the general case, we want $z$ and $j^*$ to be such that the range of added $z$'s is distinguishable from the rest of $L'$ made from words of $L$. Given such a $z$ and such a $j^*$, we define $L' = \Gamma'(L)$ by:

$$
\begin{cases}
L'(j) = L(j)^{\overline{K'}} & \forall j < j^* \\
L'(j) = z & \forall j^* \leq j < j^* + \Delta - \Delta_p \\
L'(j) = L(j - \Delta + \Delta_p)^{\overline{K'}} & \forall j \geq j^* + \Delta - \Delta_p
\end{cases}
$$

(18)

When $K > 14$, we let $j^* = 1$ and $z = \rho^q(y(K))$ (word $y(K)$ is defined in (11)) where $q$ is such that $z \notin (L'(0), L'(1))$. This is always possible because since $y(K)$ is primitive, it holds that $|y(K)| = K > 14 > 2 = ||L'(0), L'(1)||$.

If $K \leq 14$ and $\forall K \in \{1, 4, 6\}$, we let $j^*$ and $z$ be such that (cf. Lemma 5):

$$
\begin{cases}
\text{if } j^* > 0 \text{ and } z \notin (L(\Delta_p - 1), L(\Delta)) & \text{or} \\
\text{if } j^* = 0 \text{ and } z \notin (\rho(L(\Delta_p - 1), L(0))
\end{cases}
$$

(19)

This is always possible. Indeed, if not, i.e. if there is no such $j^*$ and $z$, then $K_p = K$ must hold since $z$ has length $K$ and the $L(j)$'s have length $K_p$. Also, we must have $\forall 0 < j < \Delta_p, W^K_{\overline{K'}} = (L(j), L(j + 1)) = (L(0), \rho(L(\Delta_p - 1)))$, and thus $|W^K_{\overline{K'}}| \leq 2$. As a result, $K = K_p = 2$ holds and $\Delta_p$ has to be odd (so that $L(0) = L(\Delta_p - 1) = \rho(L(\Delta_p - 1))$. But in this case, we can show that:

$$u = W(\Delta_p) \equiv \begin{pmatrix}
\Delta_p - 1 \\
0 \\
\Delta_p - 1
\end{pmatrix}
\begin{pmatrix}
1^{\Delta_p - 1} \\
2^{\Delta_p - 1} \\
1^{\Delta_p - 1}
\end{pmatrix}
\equiv (01)^{\Delta_p} \notin W_{\overline{K'}}^{2\Delta_p}$$

$1^{\text{st}}$ letters of words $z$ and last letters $L(j), 0 \leq j < \Delta_p$ of words $L(j), 0 \leq j < \Delta_p$

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has period 2 rather than $2\Delta$. It is imprimitive and like all imprimitive words, its associated list $L$ needs no image by $\Gamma$. Thus, when $K \leq 14$, there always are $j^*$ and $z$ satisfying (19).

Let $w = w(L', \Delta)$. By Lemma 5, if $q = m\Delta + \Delta - j^* \equiv \Delta - j^* \mod \Delta$, then $w' = \rho^q(w)$ is characterised by a list starting with a series of at least $\frac{\Delta}{\Delta_p} - \frac{2}{\Delta}$ identical primitive words:

$$L^q = (\ldots \rho^{m+1}(z) \ldots \rho^{m+1}(L(j) \ \Gamma^e) \ldots \rho^m(L(j) \ \Gamma^e) \ldots).$$

$L^{q+1}$, however, does not start with such a series. Indeed, $L^{q+1}(0) = \rho(L^q(\Delta - 1)) = \rho^{m+1}(L(j)^{\rho(K_p)}) \neq L^{q+1}(1) = \rho^{m+1}(z)$. Moreover, in $L^q$ (as in any $L^q$) this longest series of identical words remains well bounded from the right since $L^q(\Delta - \Delta_p - 1) = \rho^{m+1}(z) \neq L^q(\Delta - \Delta_p) = \rho^{m+1}(L(j)^{\Gamma})$. As a result of this and of its long length, this series can be identified non-ambiguously in the list $L(v, \Delta)$ of any conjugate $v$ of $w$.

A.6 Case $K = 6$

Importantly, in this case, $\Delta > 1$ necessarily holds. In addition, we assume that $\Delta \neq 2$ so that $\Delta > 2$. This case is strongly inspired from the general case (A.4 and A.5). The main difficulty lies in that since $W_1^6 = \phi$, $L' = \Gamma'(L) \in (W^6)^\Delta$ must be composed of imprimitive words of period 2 and 3.

Let $u(6) = 010101$ and $v(6) = 011011$ so that $W_1^6 = (u(6)) \cup (v(6))$.

1. If $\Delta_p = \Delta > 2$ (and $K = 6 > K_p \in \{2, 3\}$), rather than elongating a word of $L$ into a primitive word as before, we replace one of period $K_p$ with one of period $\frac{\Delta}{\Delta_p}$: we define $L' = \Gamma'(L)$ by:

$$L'(0) = v(6) \text{ if } K_p = 2 \text{ and } L(0) = 01$$

$$L'(0) = u(6) \text{ if } K_p = 3 \text{ and } L(0) = 011 \quad (20)$$

$$L'(j) = L(j) \ \Gamma^e \quad \forall j > 0.$$

Thus, $L'$ contains $\Delta - 1$ sub-words of period 2 (or 3) and one sub-word of period 3 (resp. 2).

2. Otherwise, if $\Delta_p > 1$ and $D = \Delta - \Delta_p > 2$, we add a series of $D$, identical words of period $z$ of period $K_p \in \{2, 3\}$. To do so, we let $j^*$ and $z \in W_1^6$ be such that (19) holds. This is trivially possible if $K_p = 2$ (resp. 3) because we can just take $z = v(6)$ (resp. $z = u(6)$). Otherwise is also possible because $|W_1^6| > 2$. We define $L' = \Gamma'(L)$ as in (18). Let us note that if $\Delta_p = \Delta/2$, then $K_p \in \{3, 6\}$. Indeed, if $K_p = 2$, then $p = \Delta/\Delta_p$ which is impossible. As a consequence, only one list $L$ is mapped onto list $L' = \{v(6)^{\Delta/2}, u(6)^{\Delta/2}\}$, that is list $L = \{v(6)^{\Delta/2}\}$.

Since $\Delta_p > 1$, the series of words that we add has length $D < \Delta - 1$. The resulting list $L'$ cannot be confused with a list defined as in the previous case. Also, since we add more than two words, this case cannot be confused with the next ones either.

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A.7 Primitivity of word \( w \)

Let \( \text{primitivity of at least one of the bounded series of consecutive identical primitive words when } L \neq \emptyset \). In this case, we define \( L' = \Gamma' (L) = (u(6), v(6), u(6), v(6)) = L(0^41^40^31^40^41^5, 4) \). If \( K_p = 6 \) holds, then \( L = (u(6), v(6)) = L(001101100111, 2) \). In this case, we define \( L' = \Gamma' (L) = (v(6), v(6), u(6), v(6)) = L(0^41^30^21^30^31^4, 4) \).

In the second case, \( K_p = 2 \) (otherwise \( p|\Delta \)), and \( L = \{01\} \). We define (consistently with the next item) \( L' = \Gamma' (L) = (u(6), u(6), v(6), v(6)) = L(01101100111, 2) \).

In both cases, \( L' \) has at most two consecutive sub-words of same period.

**A.7 Primitivity of word \( w(\Gamma(L), \Delta) \)**

Let \( w = w(L', \Delta) \) and let \( q = K_q \Delta_q \) be the primitive period of \( w \). In all cases, the primitivity of at least one of the \( L'(j) \)'s guarantees that \( K_q = K \) by Lemma 4. The unicity of the primitive word in \( L' \) when \( \Delta_p = \Delta \), and the long length \( (\geq \Delta/2) \) of the added well bounded series of consecutive identical primitive words when \( \Delta_p < \Delta \), guarantee by Lemma 6 that \( \Delta_q = \gcd(q, \Delta) = \Delta \). Thus, the primitivity of \( w \) is ensured in all cases.

**A.8 Injectivity of \( \Gamma \)**

Let \( \langle u \rangle \in C_n^\Delta \) be a primitive necklace of length \( n \). Let \( L' = \Gamma(\langle u \rangle, \Delta) \in \mathbb{L}^{K, \Delta} \) be its representative list. We assume that \( K \in [1,4] \). Algorithm 1 shows that there is at most one divisor \( p = K_p \Delta_p < n \) of \( n \) and one necklace \( \langle u \rangle \in C_n^\Delta \) such that \( L' = \Gamma(\langle L(\langle u \rangle, \Delta_p) \rangle) \in \Gamma(\bigcup_{p|\Delta} \mathbb{L}^{K_p, \Delta_p}) \).  

\[ \Box \]
if $K \neq 6$ then
  if the number $\pi$ of primitive words $\hat{L}'(j)$ in $\hat{L}'$ equals $\pi = 1$ then
    Let $j^*$ be the index of the only primitive word $\hat{L}'(j^*) \in \hat{W}_1^{K_p}$ of $\hat{L}'$.
    if the number $\iota$ of imprimitive words $\hat{L}'(j)$ in $\hat{L}'$ equals $\iota = 0$ then
      $\Delta_p = 1 = \Delta$ and thus $L(0) = \alpha(K_p, u)$ for some $K_p < K$, $K_p \neq 6$, and some
      $u = L(0) \in \hat{W}_1^{K_p}$. Recalling that $(\Delta, K) \neq (1, 10)$ is assumed, $L(0)$ can only be
      $\beta(\hat{L}'(0))$ and $K_p = |L(0)|$ (cf. A.4 and Equation (16)).
    if $1 > \pi = 1$ or if $\Delta = 2 \land \hat{L}'(j^*) \neq y(K)$ then
      $\Delta_p = \Delta$ must hold and both $K_p$ and $L(\hat{L}'(j^*))$ can be retrieved
      non-ambiguously using $\beta$ or $\beta^*$ and (17). Then, $\forall j \neq j^*$,
      $L(j) = \hat{L}'(j_0 \ldots j_k)|_{j_p - 1}$ is given by the first $K_p$ letters of $\hat{L}'(j)$.
  if $\pi = 1 = \iota, \Delta = 2$, and $\hat{L}'(j^*) = y(K)$ then
    $\Delta_p = 1 < \Delta$ and $K_p = K_p^{(i+1)}$ and $L = (L(0))$ where
    $L(0) = \hat{L}'(j^* + 1) \ldots \hat{L}'(j^* + 1)_{\hat{L}'(j^*)}$ is given by the first $K_p$ letters of $\hat{L}'(j^* + 1)$.
if $\pi > 1$ then
  $\Delta_p < \Delta$ must hold and there exists a rotation $\rho^w_v(u)$ so that $\hat{L}'^w_{\Delta_p}$ starts
  with a series of identical words. Let $v \equiv u$ be such that the length $D$ of this
  series in $L(v, \Delta) = L''$ is the longest (cf. A.5). $D$ satisfies $D \geq \Delta/2$. By definition of
  $\Gamma$ in this case, $\Delta_p = \Delta - D$ must hold, as well as $\Delta_p = \Gamma \Delta = \Delta^D_p$ where $\Delta^D_p$
  is the primitive period of $L''(j)$. Thus, we let $j \in (W_1^{K_p})^{\Delta_p}$ be the list s.t.
  $\forall j < \Delta_p, j(j) = L''(j + D) \ldots L''(j + D)_{j_p - 1}$ is defined by the first $K_p$ letters of
  $L''(j + D)$. Then, $L = \hat{L}'^\Delta_p \neq \hat{L}'$ must hold.
if $K = 6$ then
  if $\Delta > 2$ and $\hat{L}'$ contains one sub-word $\hat{L}'(j^*)$ that has a different period from all the
  $\Delta - 1$ other sub-words: $K_{j^*} \neq K_j = 6/K_{j^*}, \forall j \neq j^*$ then
    (cf. Item 1 in A.6) $K_p = K_j, \forall j \neq j^*$, $\Delta_p = \Delta$, and list $L$ can be retrieved from the
    knowledge that in this case, (20) is satisfied by $\hat{L}'$.
  if $\hat{L}'$ contains $D$ identical, consecutive sub-words, where $\Delta - 1 > D \geq 2$ then
    Let $z$ be the value of these. Let $K_z \in (2, 3)$ be its period. Let $K_p = \delta_z$ be the
    common period of the remaining sub-words of $\hat{L}'$, and let $\Delta_p = \Delta - D$. List $L$
    can be retrieved from the knowledge that this case corresponds to Item 2 of
    A.6.
else
  $\hat{L}'$ contains at most two consecutive sub-words of same period and $L$ can be
  retrieved non-ambiguously from the knowledge that this case corresponds to
  Items 3 and 4 of A.6.

Algorithm 1: Proof of the injectivity of $\Gamma$.
Appendix B

Comparing the behaviours of BACs and BADS: proof of Theorem 2

Lemma 7 below relies on the results given in Table 1 as well as on some properties satisfied by the Lucas [22] and Perrin [1] sequences in relation, respectively, to the two roots of $x^2 - x - 1 = 0$, i.e. the golden ratio $\varphi = \frac{1 + \sqrt{5}}{2} \approx 1.61803399$ and $\varphi = \frac{1 - \sqrt{5}}{2} \approx -0.61803399$, and to the three roots of $x^3 - x - 1 = 0$ which are the plastic number $\pi \approx 1.32471796 \in \mathbb{R}$, $\nu = \frac{1}{2}(-\pi + i \cdot \sqrt{3} - 1)$ and its complex conjugate $\overline{\nu}$.

**Lemma 7.** For a divisor $p = K_p \Delta_p (\Delta_p = \gcd(\omega, p))$ of the order $\omega$ of a mixed BAD and of a negative BAD, the number of configurations of period $p$ are bounded respectively as follows:

$$g^p \sim X^{-} \leq \sqrt{3}^p \quad \text{and} \quad \pi^p \sim X^{-} \leq \begin{cases} \frac{3}{\pi} & \text{if } K_p = 3 \\ \sqrt{2}^p & \text{if } K_p \neq 3 \end{cases}$$

where $\Delta$ is the gcd of cycle lengths.

**Proof.** First, the Lucas sequence satisfies [22]: $\forall n \in \mathbb{N}, L(n) = g^n + \overline{g}^n = g^n + (-\frac{1}{g})^n$.

Consequences of this and of Table 1 are: $X^{-}(p) = L(K_p)\Delta_p = (g^{K_p} + \overline{g}^{K_p})\Delta_p \xrightarrow{K_p \to \infty} g^p$

proving $X^{-}(p) \sim g^p$. Moreover, using $\varphi^2 = 1 + \varphi$ and $\overline{\varphi} = -\frac{1}{\varphi}$ and the binomial formula, we derive:

$$X^{-}(p) = \sum_{k=\Delta_p}^{p} \binom{p}{k} (-g^2)^{K_p \cdot k} \cdot \overline{\varphi}^p = (-g^2)^{K_p \cdot 1} \overline{\varphi}^p = \begin{cases} \overline{\varphi}^p \cdot (g^{2K_p} - 1)^\Delta_p & \text{if } K_p \text{ is odd} \\ \overline{\varphi}^p \cdot (g^{2K_p} + 1)^\Delta_p & \text{if } K_p \text{ is even} \end{cases}$$

Let us note that if $p$ is odd (and necessarily so are $K_p$ and $\Delta_p$), then $X^{-}(p)$ is maximal when $\Delta_p$ is minimal, i.e. when $\Delta_p = 1$. And if $p$ is even then, on the contrary, $X^{-}(p)$ is maximal when $\Delta_p$ is maximal, i.e. when $\Delta_p = p/2$. In both cases:

$$X^{-}(p) \leq |\overline{\varphi}^p (g^{2K_p} + 1)^\Delta_p | \leq |\overline{\varphi}^p (g^4 + 1)^{\frac{p}{2}} | = \frac{(3+\varphi)^{\frac{p}{2}}}{g^p} \leq 3^{\frac{p}{2}},$$

which proves the first inequality of Lemma 7.

The rest of Lemma 7 derives from Table 1 and from the following relation that is satisfied by the Perrin sequence [1]: $\forall n \geq 2$, $P(n) = \pi^n + \nu^n + \overline{\nu}^n$. Indeed, this yields $X^{-}(p) = (\pi^{K_p} + 2 \cos(\arg(\nu^{K_p})) \cdot |\nu|^{K_p})\Delta_p$ where $|\nu| = 1/\sqrt{\overline{\varphi}} < 1$, and thus:

$$(\pi^{K_p} - 2 |\nu|^{K_p})\Delta_p \leq X^{-}(p) \leq (\pi^{K_p} + 2 |\nu|^{K_p})\Delta_p.$$ 

Since $(\pi^{K_p} \pm 2 |\nu|^{K_p})/\pi^P = \left[ 1 \pm 2/\pi^{\frac{1}{2}K_p} \right] \Delta_p \xrightarrow{K_p \to \infty} 1$, we have: $\lim_{p \to \infty} \frac{X^{-}(p)}{\pi^P} - 1 = 0$. Now, if $K_p = 3$, then $X^{-}(p) = P(3)^{\Delta_p} = 3^{\frac{p}{2}}$. Generally, by the definition of $\pi$, $\forall a \geq$
\( \pi \), it holds that \( a + 1 \leq a^3 \). As a consequence, if, for some \( b \in \mathbb{R}, P(n) \leq ba^n, \forall n \leq m + 1 \), then: \( P(m + 3) = P(m + 1) + P(m) \leq ba^m(a + 1) \leq ba^{m+3} \) and by induction on \( m, \forall n, P(n) \leq ba^n \). Therefore, to prove the last inequality of Lemma 7, it suffices to check that it is satisfied for the base cases of the corresponding induction of this form, where \( b = 1 \) and \( a = \sqrt[3]{2} \).

Finally, for any network \( \mathcal{N} \) that is either a \textsc{bac} or a \textsc{bad} of order \( \omega \) (where \( \omega = K\Delta \) as before in the case of a \textsc{bad}), let:

\[
a = \begin{cases} 
2 & \text{if } \mathcal{N} \text{ is a positive } \textsc{bac} \text{ or } \textsc{bad} \\
\sqrt[3]{3} & \text{if } \mathcal{N} \text{ is a mixed } \textsc{bad} \\
\sqrt[3]{2} & \text{if } \mathcal{N} \text{ is a negative } \textsc{bac} \text{ or a negative } \textsc{bad} \text{ s.t. } K \mod 3 \neq 0 \\
3^{1/3} & \text{if } \mathcal{N} \text{ is a negative } \textsc{bad} \text{ s.t. } K \mod 3 = 0.
\end{cases}
\] (22)

Then, using Table 1, the formulation of \( A(\omega) \) in terms of Dirichlet convolutions (cf. Section 6), and Lemma 7 above, the following can be drawn immediately by noting that both \( X \) and the Euler totient \( \varphi \) are non-negative:

\[
A(\omega) \leq \frac{(\varphi \ast Y)(\omega)}{\omega} \text{ where } \forall p|\omega, Y(p) = a^p.
\] (23)

This combined with Theorem 1 directly yields Theorem 2. \( \square \)