Fuzzy triangular numbers in - Sierpinski triangle and right angle triangle

T Sudha¹ and G Jayalalitha²
¹²Department of Mathematics, Vels Institute of Science Technology and Advanced studies (VISTAS), Chennai, Tamil Nadu, India.
E.Mail: ¹psudha.thakur@gmail.com, ²g.jayalalithamaths.sbs@velsuniv.ac.in

Abstract—Sierpinski Triangle or Sierpinski Gasket or Sierpinski Sieve is a fractal and attractive fixed set with the overall shape of an equilateral triangle, subdivided recursively into smaller equilateral triangles. In this paper the number theoretical aspect of Fuzzy Triangular Numbers (FTN) in Sierpinski Triangle and Right Angle Triangle have been established and some arithmetic operations of α – Cut, Centroid of the Triangle, Approximation of the Triangular Fuzzy Numbers are explained.

Keywords— Fuzzy Numbers, Fractal, Sierpinski triangle, Fuzzy Triangular Numbers, Centroid of the Triangle.

1. Introduction
1.1 Fuzzy numbers
The notion of Fuzzy number’s as being convex and normal fuzzy set of some referential set was introduced by Zadeh.[1]

1.2 Fractal
Fractal was coined by Mandelbrot in his fundamental essay from the Latin fractus meaning broken, to describe the objects that were too irregular to fit into a traditional geometrical setting [2]. Many fractals have same degree of self-similarity- it made up parts that resemble the whole in some way. Sometimes, the resemblance may be weaker than strict geometrical similarity for example the similarity may be approximate or statistical. Sierpinski Triangle, Cantor Set and Von Koch Curve, Menger sponge, Dragon Curve, Julia Set, Mandelbrot Set are the examples of Fractals.

1.3 Sierpinski triangle [2, 9]
Sierpinski triangle is a fractal described in 1915 by Waclaw Sierpinski. It is a self-similar structure that occurs at different levels of iterations, or magnifications. The Sierpinski Triangle has all the properties of a Fractal.

For triangular area, with each iteration, the side of the inside triangle reduces by a factor of 2. The numbers of these little triangles, on the other had increases not by 4 but by factor of 3. The dimension of self-similar object is then (log3/ log 2) = 1.58 approximately [3]. In this paper in Section 2, Definitions are explained. In Section 3, the operations of Fuzzy Triangular Numbers are discussed using Sierpinski triangle with side 1unit, 2unit, 3unit and 12 unit and Right angle triangle with side 6,8,10 units.[4] Michael Voskoglou explained Fuzzy Triangular Number as a tool for student assessment. And [1] Nagoor Gani discussed a New Operation on Triangular fuzzy Number for solving Fuzzy Linear Programming Problem, Karpagam and Sumathi explained New Approach to solve Fuzzy Linear Programming Problems by the Ranking Function. [8]. Senthilkumar explained about Triangular
Approximation of Fuzzy Numbers- a new Approach[5], they all discussed with randomly chosen Triangular fuzzy numbers. But in this paper Triangular Fuzzy Numbers are chosen from the self-similarity Fractal set Sierpinski triangle and Right angle Triangle explained the basic operations of addition, subtraction, .Approximation of Two fuzzy Triangular Numbers multiplication and division including a – Cut operations with graphical representation.

2. Preliminaries

2.1 Fuzzy set [6].

Any set which allows its members to have membership of different grades in the interval [0, 1] is a Fuzzy Set.

2.2 α-cut or α-level set [6].

For any Fuzzy Set \( \tilde{A} \), \( \tilde{A}_\alpha = \{x \mid \mu_A(x) \geq \alpha \} \) \( \alpha \) is arbitrary, \( \alpha \) belongs to \([0, 1]\) is called \( \alpha \)-Cut or \( \alpha \)-Level Set.

2.3 Support of fuzzy set

The support of a fuzzy set \( F \) is a crisp set of all points in the Universe of Discourse \( U \) (range of all possible values for an input to a Fuzzy system) such that the membership function of \( F \) is non-zero [6].

2.4 Fuzzy number [1]

A fuzzy set \( A \) on \( R \) must possess at least the following three properties to qualify as a fuzzy number,

- \( A \) must be a normal fuzzy set;
- \( A_\alpha \) must be closed interval for every \( \alpha \, [0,1] \)
- The support of \( A \), \( A_0 \), must be bounded

2.5 Triangular fuzzy numbers (TFN) [4]

Let \( a, b, \) and \( c \) be real numbers with \( a \leq b \leq c \). Then the Triangular Fuzzy Number (TFN) \( A = (a, b, c) \) is a Fuzzy Number with membership function:

\[
m(x) = \begin{cases} 
\frac{x-a}{b-a}, & a \leq x \leq b \\
\frac{c-x}{c-b}, & b \leq x \leq c \\
0, & x < a, x > c 
\end{cases}
\]

2.6 Operation of triangular fuzzy number using function principle [1]

The following are the four operations that can be performed on triangular fuzzy numbers:

Let \( C = (a_1, a_2, a_3) \) and \( D = (b_1, b_2, b_3) \) then,

(i) Addition: \( C + D = (a_1+b_1, a_2+b_2, a_3+b_3) \).

(ii) Subtraction: \( C - D = (a_1-b_3, a_2-b_2, a_3-b_1) \).

(iii) Multiplication: \( CD = [\min (a_1b_1, a_2b_2, a_3b_3), a_2b_1, a_1b_3)] \).

(iv) Division: \( C/D = [\min (a_1/b_1, a_2/b_2, a_3/b_3), a_2/b_1, a_1/b_3] \).

2.7 α-cuts of a TFN [4]

The \( \alpha \)-cuts \( A^\alpha \) of a TFN \( A = (a, b, c) \) are calculated by the formula

\[ A^\alpha = [A_1^\alpha, A_2^\alpha, A_3^\alpha] = [a + \alpha (b - a), c - \alpha (c - b)] \].

2.8 Centroid of the triangle [4]

The coordinates \((X, Y)\) of the COG of the triangle forming the graph of the TFN \((a, b, c)\) are calculated by the formulas

\[ X = \frac{a+b+c}{3}, \quad Y = \frac{1}{2} \]
3. Operations on fuzzy triangular numbers

3.1 Equilateral triangle

Figure 1 represents Fuzzy Triangular Numbers of Equilateral Triangle of side 1 unit.

In Figure 2, \( G \) represents the Centroid of the Fuzzy Triangular numbers triangle \( A(0,0), B(1/2,1), \) and \( C(1,0) \) and \( M \) \((1/2, 0)\) is the midpoint of \( A \) and \( C \), and \( N(3/4, 1/2) \) is the midpoint of \( B \) and \( C \). Based on the definition 2.5 and from the Figure 2.

Equation of the straight line \( A (0, 0) \) and \( N(3/4, 1/2) \) on which \( AN \) lies in
\[
6y - 4x = 0 \tag{1}
\]

From the Figure 2 based on the definition 2.5 Equation of the straight line on which \( B (1/2, 1) \) and \( M (1/2, 0) \)
\[
x + 1/2 = 0 \tag{2}
\]
The linear system of equations (1), and (2) has a unique solution with the respect to the variables \( x \) and \( y \), determining the coordinates of the triangle \( COG \), Centroid of the triangle after observing the following
\[
D = \begin{bmatrix} 4 & -6 \\ 1 & 0 \end{bmatrix} \neq 6
\]
\[
D_x = \begin{bmatrix} 0 & -6 \\ 1/2 & 0 \end{bmatrix} \neq 3
\]
\[
D_y = \begin{bmatrix} 4 & 0 \\ 1/2 & 1/2 \end{bmatrix} \neq 2
\]

Therefore \( x = \frac{1}{2}, y = \frac{1}{3} \).

Solving equation (1) and (2), based on the definition 2.8 the coordinates of Centroid of the triangle \( G \) are \((6, 1/3)\) which is equal to \( x = \frac{0 + 1 + 1/2}{3}, y = \frac{1 + 0 + 0}{3} \) that is \((\frac{1}{2}, \frac{1}{3})\).

3.1.1 Approximation of Triangular Fuzzy Numbers [7]

**Case I** For two positive fuzzy numbers
\[
\begin{align*}
\mathbf{\tilde{A}} &= \langle a_1, b_1, c_1 \rangle = \lfloor (b_1-c_1) \alpha + a_1, -(c_1-b_1) \alpha + c_1 \rfloor = (S1, S1) \\
\mathbf{\tilde{B}} &= \langle a_2, b_2, c_2 \rangle = \lfloor (b_2-c_2) \alpha + a_2, -(c_2-b_2) \alpha + c_2 \rfloor = (S2, S2)
\end{align*}
\]
The product can be calculated,
\[
P = \mathbf{\tilde{A}} \ast \mathbf{\tilde{B}} = [\min(s_{12}, s_{13}, s_{14}, s_{23}, s_{24}, s_{34}), \max(s_{12}, s_{13}, s_{14}, s_{23}, s_{24}, s_{34})], \alpha \in (0, 1)
\]

**Case II** [7]

When any \( \tilde{A} \) and \( \tilde{B} \) is partial negative and other is positive fuzzy number, the product of \( \mathbf{\tilde{A}} \ast \mathbf{\tilde{B}} \) can not obtain according to the equation (6). The interval of \( \alpha \in (0, 1) \) will be divided into two parts, according
to the intersection point of the two minimum expression in the α cut of \( P \). Let us this intersection point is \( \alpha_s \). Then

\[
P = \bar{A} \times \bar{B} = \begin{cases} 
\min(s_1, s_2, \bar{s}_1, \bar{s}_2), & \alpha \in [0, \alpha_s] \\
\max(s_1, s_2, \bar{s}_1, \bar{s}_2), & \alpha \in (\alpha_s, 1] 
\end{cases}
\]

(7)

**Case III**

When \( \bar{A} \) is negative and \( \bar{B} \) is positive Fuzzy number. Then the multiplication of \( \bar{A} \) and \( \bar{B} \) can be found as equation (6).

**Case IV**

When \( \bar{A} \) and \( \bar{B} \) both are negative fuzzy number. Then the multiplication of \( \bar{A} \) and \( \bar{B} \) can also be found as equation (6). The equation (6) and (7) are known as analytical method for fuzzy arithmetic operation.

In actual product the lines connecting the ends points are parabolic and in standard approximation lines connecting the ends points are triangular form.

From the Figure.1, the two positive Fuzzy Triangular Numbers \( A(0, 1/2, 1) \), \( B(0, 1/4, 1/2) \) it can derive [5] the approximation of Fuzzy Triangular Numbers shown in the Figure.3

---

**Figure 3.** Approximation of Fuzzy Triangular Numbers

3.1.2 *Equilateral Triangle with side 12 units*

Let us consider the equilateral triangle of side 12 cm. the following are the iterations of equilateral triangle

---

**Figure 4.** Iteration of Sierpinski Triangle From stage 0 to 3

| Table 1. Iteration of Sierpinski Triangle |
|------------------------------------------|
| Iteration | Scaling | TFNs | number |
|-----------|---------|------|--------|
|           |         |      |        |
From the Figure 5 The Triangular Fuzzy numbers are (0,6,12), (0,3,6), (0,3,6), (0,6,9), (3,6,9)……and so on.

| $I_1$ | $\frac{1}{2}$ | (0,6,12) |
|-------|---------------|-----------|
| $I_2$ | $\frac{1}{4}$ | (0,3,6), (6,9,12), (0,1,5,3), (3,4,5,6) |
|       |               |           |
| $I_n$ | $(1/2)^n$ n-numbers |

Fuzzy Triangular Numbers (0, 6, 12), (0, 3, 6), (0, $\frac{3}{2}$ 3)……

From the Figure 5 $G$ (0, $\frac{3}{2}$ 3) and $F$ (0, 3, 6) are the Two TFN

Based on the definition 2.3 (i), (ii)

$G + F = (0 + 0, \frac{3}{2} + 3, 3 + 6) = (0, \frac{9}{2}, 9)$ (8)

$G - F = (0 - 6, \frac{3}{2} - 3, 3 - 3) = (-6, \frac{3}{2}, 0)$ (9)

Based on the definition 2.4 $\alpha$ - Cut,

**Operation by $G + F$ $\alpha$-Cut**

- on $G$ (0, $\frac{3}{2}$ 3)

$G^\alpha = [a_1^\alpha, a_3^\alpha] = (\frac{3}{2} a_\alpha - \frac{3}{2} (a + 3))$ (10)

- If $\alpha = 0$
  - $= (0, 3)$
- If $\alpha = 1$
  - $= (\frac{3}{2}, \frac{3}{2})$

**Operations by $\alpha$ - cut based on the definition 2.7**

- $F$ (0, 3, 6),

$F^\alpha = [a_1^\alpha, a_3^\alpha] = (3a_\alpha - 3\alpha + 6)$ (11)

$F^\alpha + G^\alpha = (\frac{9}{2}a_\alpha, 9 - \frac{9}{2}a_\alpha)$ (12)

- If $\alpha = 0$
  - $= (0, 9)$
- If $\alpha = 1$
  - $= (\frac{9}{2}, \frac{9}{2})$

$G^\alpha + F^\alpha = (0, 9/2, 9)$ (13)

From these equations we observe that (8) = (13)

Therefore $G^\alpha + F^\alpha = G + F$

From the equation (8) and (13) shows that the sum and the difference of two TFN are also TFN.

$G$ (0, $\frac{3}{2}$ 3), $F$ (0,3,6) are the Two TFN
Case I Approximation for Multiplication
From the equation (10), (11)
\[ G^a = [a_1^a, a_3^a] = \left( \frac{3}{2} \alpha, -\frac{3}{2} \alpha + 3 \right) \]
\[ F^a = [a_1^a, a_3^a] = \left( 3 \alpha - 3 \alpha + 6 \right) \]
\[ G^a \cdot F^a = \left( \frac{3}{2} \alpha, 3\alpha, -\frac{3}{2} \alpha + 3, -3 \alpha + 6 \right) \]
\[ \alpha = 0 \Rightarrow (0, 18) \]
\[ \alpha = 1 \Rightarrow \left( \frac{9}{2}, 9 \right) \]
\[ G^a(\cdot) F^a \cong \left( 0, \frac{9}{2}, 18 \right) \]

Case II Approximation for Division
\[ F(0, 3, 6), H(3, 6, 9) \text{ are the Two TFN} \]
\[ F^a = [a_1^a, a_3^a] = \left( 3 \alpha, -\frac{3}{2} \alpha + 6 \right) \]
\[ H^a = [a_1^a, a_3^a] = \left( 3 \alpha + 3, -3 \alpha + 9 \right) \]
\[ F^a / H^a = \left( \frac{3\alpha}{9-3\alpha}, \frac{6-3\alpha}{3\alpha+3} \right) \]
\[ \alpha = 0 \Rightarrow (0, 2) \]
\[ \alpha = 1 \Rightarrow \left( \frac{1}{2}, \frac{1}{2} \right) \]
\[ F^a / H^a \cong \left( 0, \frac{1}{2}, 2 \right) \]

Therefore multiplication and Division of two TFN, although they are FNs, they are not always TFNs.

In Figure 6, G represents the Centroid of the Fuzzy Triangular numbers triangle A (0, 0), B(6,1), and C(12,0) and M (6, 0) is the midpoint of A and C, and N (9/2,0) is the midpoint of B and C.

Based on the definition 2.5 and from the Figure 6.

**Figure 6.** Fuzzy Triangular numbers triangle with Centroid and median

From the Figure 6 Equation of the straight line A (0, 0) and N (9/2, 1/2) on which AN lies in
\[ X-18y = 0 \] (16)

From the Figure 6 based on the definition 2.5 Equation of the straight line on which B (6, 1) and M (6, 0) lies in
\[ X + 0y = 6 \] (17)

The linear system of equations (16), and (17) has a unique solution with the respect to the variables x and y determining the coordinates of the triangle COG in figure.6, Centroid of the triangle after observing the following
\[ D = \begin{bmatrix} 1 & -18 \\ 1 & 0 \end{bmatrix} = 18 \] (18)
\[ \neq 0 \]
\[ D_6 = \begin{bmatrix} 0 & -18 \\ 6 & 0 \end{bmatrix} = 108 \] (19)
\[ D_7 = \begin{bmatrix} 11 & 0 \\ 1 & 6 \end{bmatrix} = 6 \] (20)
Therefore $x = 6$, $y = \frac{1}{3}$ Solving equation (16) and (17), based on the definition 2.8 the coordinates of Centroid of the triangle $G$ are $(6, \frac{1}{3})$ which is equal to $x = \frac{0+6+12}{3}$, $y = \frac{1+0+3}{3}$ i.e. $(6, \frac{1}{3})$

Based on Equilateral triangle of side 1 unit and 12 units it is easy to get Fuzzy Triangular Numbers, Approximation of Fuzzy Triangular Numbers, Centroid and Median of the Fuzzy Triangular numbers Triangle for Equilateral Triangle of side 2 units and 3 units.

The following Figures 7, 8, 9 represents side 2 units and Figure.10, 11, 12 represents side 3 units

3.2. Right angle triangle

Let us consider the right triangle with sides 6cm, 8cm, 10cm respectively. The following are iterations of the right triangle in 5 stages $s(0)$, $s(1)$, $S(2)$, $s(3)$, $s(4)$, $s(5)$ respectively.
Figure 13. Iteration of Right Angle Triangle from stage 0 to 5

Table 2. Iteration of Right angle triangle

| Iteration number | Scaling | TFNs          |
|------------------|---------|---------------|
| I₁               | 1/2     | (0,4,8)       |
| I₂               | 1/4     | (0,2,4),(0,1,2), |
| .                | .       | (2,3,4),(3,4,5) |
| .                | .       | .             |
| .                | .       | .             |
| Iₙ               | (1/2)ⁿ  | n-numbers    |

Multiplication of two fuzzy numbers

Based on the Figure 13 Consider two Triangular Fuzzy numbers A = (0, 4, 8) and B = (3, 4, 5) Then their respective α-cuts will be $A^α = (4α, -4α+8)$ and $B^α = (3 + a, 5 - α)$

$A^α \cdot B^α = (4 \cdot a \cdot a + 3, a + 3, - α + 5)$

Based on lemma[5]

$U(α) = 4α^2 + 12α$ (21)

And $U^2(α) = 4α^2 - 28α + 40$ (22)

Here $U(α)$ is a parabola meeting α-axis at (0,0) and (0,-3).

$U^2(α)$ is a parabola meeting the same α-axis at (0,2) and (0,5) and meeting u-axis at (40,0). These two curves meet at (16,1). It is obvious that $U(1) = U^2(1) = 16$. Now $m=16$, $U(0) = 0$, $U^2(0) = 40$. Hence the approximate triangular fuzzy from Figure 14 is (8,16, 28) (23)

Figure 14. Approximate Fuzzy Triangular numbers

4. Conclusion

Basic operations such as Addition, Subtraction, Multiplication and Division of Fuzzy Triangular Numbers are discussed in Self-Similarity of Sierpinski triangle. This triangle satisfied Fractal property. From this it concludes that Addition and Subtraction of two Fuzzy Triangular Numbers are Fuzzy Triangular Numbers. But Multiplication and Division of two Fuzzy Triangular Numbers are Approximation of Fuzzy Triangular Numbers. Median, Centroid derived from the Triangle formed by
Fuzzy Triangular Numbers triangle are same as normal triangle. Similar discussions are also in Right Angle Triangle.

References
[1] Nagoor Gani A and Mohamed Assarudeen S N 2012 A new operation on triangular fuzzy number for solving fuzzy linear programming problem Applied Mathematical Sciences 6(11) pp 525-532
[2] Mandelbrot B B and Freeman W H 1982 The Fractal Geometry of Nature,(San Francisco).
[3] Nithya R, Kamali R and Jayalalitha G 2017 Ramsey numbers in Sierpinski triangle, International Journal of Pure and Applied Mathematics, 116(4) pp 967-975
[4] Michael Voskoglou Gr 2015 An application of triangular fuzzy numbers to learning assessment Journal of Physical Science 20 pp 63-79
[5] Senthilkumar L S 2017 Triangular approximation of fuzzy numbers-a new approach International Journal of Pure and Applied Mathematics 113(13) pp 115-121
[6] Jayalalitha G and Sudha T 2019 Analysis of diabetes based on fuzzy fractals International Journal of Recent Technology and Engineering 7 pp 653-659
[7] Yasin Ali Md, Abeda Sultana and Khodadad Khan A F M 2016 Comparison of fuzzy multiplication operation on triangular fuzzy number IOSR Journal of Mathematics 12 pp 35-41
[8] Karpagam A and Sumathi P 2014 New approach to solve fuzzy linear programming problems by the ranking function Bonfring International Journal of Data Mining 4(4) pp 22-25
[9] Uthayakumar P and Jayalalitha G 2016 Spectral decimation functions and forbidden eigen values in the graph of level Sierpinski triangles South Asian Journal of Engineering and Technology 2(14) pp 47-57