Virtual mutations of weighted surface algebras

by

Thorsten Holma, Andrzej Skowrońskib, and Adam Skowyrskib

Abstract. The finite-dimensional symmetric algebras over an algebraically closed fields, based on surface triangulations, motivated by the theory of cluster algebras, have been extensively investigated and applied. In particular, the weighted surface algebras and their deformations were introduced and studied in [16]-[20], and it was shown that all these algebras, except few singular cases, are symmetric tame periodic algebras of period 4. In this article, using the general form of a weighted surface algebra from [19], we introduce and study so called virtual mutations of weighted surface algebras, which constitute a new large class of symmetric tame periodic algebras of period 4. We prove that all these algebras are derived equivalent but not isomorphic to weighted surface algebras. We associate such algebras to any triangulated surface, first taking blow-ups of a family of edges to 2-triangle discs, and then virtual mutations of their weighted surface algebras. The results of this paper form an essential step towards a classification of all tame symmetric periodic algebras.

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E-mail addresses: holm@math.uni-hannover.de (T. Holm), skowron@mat.umk.pl (A. Skowroński), skowyr@mat.umk.pl (A. Skowyrski)

1 Introduction and the main result

Throughout this paper, K will denote a fixed algebraically closed field. By an algebra we mean an associative finite-dimensional K-algebra with an identity. For an algebra A, we denote by mod A the category of finite-dimensional
right $A$-modules and by $D$ the standard duality $\text{Hom}_K(\cdot, K)$ on $\text{mod} \ A$. An algebra $A$ is called self-injective if $A_A$ is injective in $\text{mod} \ A$, or equivalently, the projective modules in $\text{mod} \ A$ are injective. A prominent class of self-injective algebras is formed by the symmetric algebras for which there exists an associative, non-degenerate symmetric $K$-bilinear form $(\cdot, \cdot) : A \times A \rightarrow K$. Classical examples of symmetric algebras are provided by blocks of group algebras of finite groups and the Hecke algebras of finite Coxeter groups. In fact, any algebra $A$ is the quotient algebra of its trivial extension algebra $T(A) = A \ltimes D(A)$, which is a symmetric algebra.

Let $A$ be an algebra. Given a module $M$ in $\text{mod} \ A$, its syzygy is defined to be the kernel of a minimal projective cover of $M$ in $\text{mod} \ A$. The syzygy operator $\Omega_A$ is a very important tool to construct modules in $\text{mod} \ A$ and relate them. For a self-injective algebra $A$, it induces an equivalence of the stable module category $\text{mod} \ A$, and its inverse is the shift of a triangulated structure on $\text{mod} \ A$ [29]. A module $M$ in $\text{mod} \ A$ is said to be periodic if $\Omega^n_A(M) \cong M$, for some $n \geq 1$, and minimal such $n$ is called the period of $M$. The action of $\Omega_A$ on $\text{mod} \ A$ can effect the algebra structure of $A$. For example, if all simple modules in $\text{mod} \ A$ are periodic, then $A$ is a self-injective algebra. An algebra $A$ is defined to be periodic if it is periodic viewed as a module over the enveloping algebra $A^e = A^{op} \otimes_K A$, or equivalently, as an $A$-$A$-bimodule. It is known that if $A$ is a periodic algebra of period $n$, then for any indecomposable non-projective module $M$ in $\text{mod} \ A$ the syzygy $\Omega^n_A(M)$ is isomorphic to $M$. Periodic algebras are self-injective and have periodic Hochschild cohomology. Periodicity of an algebra, as well as its period, are invariant under derived equivalences. Therefore, to study periodic algebras we may assume that the algebras are basic and indecomposable.

Finding or possibly classifying periodic algebras is an important problem as it has connections with group theory, topology, singularity theory, cluster algebras and algebraic combinatorics. For details, we refer to the survey article [15] and the introductions of [2, 16, 18].

We are concerned with the classification of all periodic tame symmetric algebras. Dugas proved in [11] that every representation-finite self-injective algebra, without simple blocks, is a periodic algebra. The representation-infinite, indecomposable, periodic algebras of polynomial growth were classified in [2]. It is conjectured in [16, Problem] that every indecomposable symmetric periodic tame algebra of non-polynomial growth is of period 4. The large class of tame symmetric algebras of period 4 is provided by the weighted surface algebras associated to compact real surfaces (these and their deformations are investigated in [16]-[20]). Surface triangulations have been also used to study cluster algebraic structures in Teichmüller theory [25, 28], cluster algebras of topological origin [26], and the classification of all cluster
algebras of finite mutation type with skew symmetric exchange matrices [24].

We also mention that there exist wild symmetric periodic algebras of period 4, with arbitrary large number (at least 4) of pairwise non-isomorphic simple modules. These wild periodic algebras arise as stable endomorphism rings of cluster-tilting Cohen-Macaulay modules over one-dimensional hypersurface singularities (see [5] and [18, Corollary 2]).

Periodic algebras based on surface triangulations lead also to interesting non-periodic symmetric tame algebras. Namely, we get new symmetric tame algebras by taking the idempotent algebras $e\Lambda e$ of periodic surface algebras $\Lambda$.

In particular, every Brauer graph algebra is of this form (see [21, Theorem 4]). We refer also to [22] for the related classification of all algebras of generalized dihedral type. Summing up, the classification of all symmetric tame periodic algebras of period 4 is currently an important problem.

In order to present our main result we recall briefly the nature of weighted surface algebras, introduced and investigated in [16, 19, 23]. By a surface we mean a connected, compact, 2-dimensional real manifold $S$, with or without boundary. Then $S$ admits a structure of a finite 2-dimensional triangular cell complex, and hence a triangulation. We say that $(S, \overrightarrow{T})$ is a directed triangulated surface if $S$ is a surface, $T$ a triangulation of $S$ with at least two edges, and $\overrightarrow{T}$ an arbitrary choice of orientations of triangles in $T$. To such $(S, \overrightarrow{T})$ one associates a triangulation quiver $(Q(S, \overrightarrow{T}), f)$, where $Q(S, \overrightarrow{T})$ is a 2-regular quiver, that is every vertex is a source and target of exactly two arrows. The vertices of this quiver are the edges of $T$, and $f$ is a permutation of the arrows of $Q(S, \overrightarrow{T})$ reflecting the orientation $\overrightarrow{T}$ of triangles in $T$. Since $Q(S, \overrightarrow{T})$ is 2-regular there is a second permutation, denoted by $g$, of the arrows of $Q(S, \overrightarrow{T})$. For the set $O(g)$ of $g$-orbits of arrows in $Q(S, \overrightarrow{T})$, two functions $m_\ast : O(g) \rightarrow \mathbb{N}^*$ and $c_\ast : O(g) \rightarrow K^*$, called weight and parameter functions, are considered. Then, under some restrictions on $m_\ast$ and $c_\ast$, the weighted surface algebra $\Lambda(S, \overrightarrow{T}, m_\ast, c_\ast)$ is defined as a quotient algebra $KQ(S, \overrightarrow{T})/I(S, \overrightarrow{T}, m_\ast, c_\ast)$ of the path algebra $KQ(S, \overrightarrow{T})$ of $Q(S, \overrightarrow{T})$ over $K$ by an ideal $I(S, \overrightarrow{T}, m_\ast, c_\ast)$ of $KQ(S, \overrightarrow{T})$. It has been proved in [19] Theorems 1.1-1.3 that, if $\Lambda = \Lambda(S, \overrightarrow{T}, m_\ast, c_\ast)$ is a weighted surface algebra other than a singular disc, triangle, tetrahedral or spherical algebra, then $\Lambda$ is a tame symmetric periodic algebra of period 4. We mention that the Gabriel quiver $Q_\Lambda$ of such an algebra is at most 2-regular, which means that every vertex is a source and target of at most two arrows. Moreover, $Q_\Lambda$ is 2-regular if and only if $Q_\Lambda = Q(S, \overrightarrow{T})$. This holds exactly when $m_\circ |O| \geq 3$ for any orbit $O$ in $O(g)$. In general, it is assumed only $m_\circ |O| \geq 2$ for any orbit $O$ in $O(g)$, and some other minor restrictions (see Section 3). Conversely, it has been
shown in [18, Main Theorem] that a basic indecomposable algebra $A$ with 2-
regular Gabriel quiver $Q_A$ having at least three vertices is a tame symmetric
periodic algebra of period 4 (more generally, algebra of generalized quaternion
type) if and only if $A$ is socle equivalent to a weighted surface algebra $\Lambda(S, \rightarrow T, m, c)$ from
[20]. Furthermore, new relevant exotic families of the trivial extensions algebras
occurred in the Erdmann's classification of algebras of quaternion type [13]
(see also [9]).

The main theorem of this paper is to show that there are numerous tame symmetric
algebras of period 4, with arbitrary large ranks of the Grothendieck
group, which are not isomorphic to weighted surface algebras. We also note
that some relevant exotic families of algebras with 2 and 3 simple modules
occurred in the Erdmann's classification of algebras of quaternion type [13].

The following theorem is the main result of this paper.

Theorem: Let $A$ be a tame symmetric periodic algebra of period 4 with
arbitrary large rank of the Grothendieck group, which is not isomorphic to a
weighted surface algebra $\Lambda(S, \rightarrow T, m, c)$. Then there exists a
sequence of virtual arrows $\xi$ in $Q(S, \rightarrow T)$, such that $A$ is
not equivalent to a weighted surface algebra $\Lambda(S, \rightarrow T, m, c)$ with
respect to the chosen sequence $\xi$.
Main Theorem. Let $\Lambda(\xi) = \Lambda(S, \overrightarrow{T}, m_*, c_*, \xi)$ be a virtual mutation of a weighted surface algebra $\Lambda = \Lambda(S, \overrightarrow{T}, m_*, c_*)$. Then the following statements hold.

(1) $\Lambda(\xi)$ is a finite-dimensional symmetric algebra.

(2) $\Lambda(\xi)$ is derived equivalent to $\Lambda$.

(3) $\Lambda(\xi)$ is not isomorphic to a weighted surface algebra.

(4) $\Lambda(\xi)$ is a representation-infinite tame algebra.

(5) $\Lambda(\xi)$ is a periodic algebra of period 4.

The quiver $Q(S, \overrightarrow{T}, \xi)$ describing $\Lambda(\xi)$ has the same vertices as $Q(S, \overrightarrow{T})$, which are the edges of $T$. All arrows of the quiver $Q(S, \overrightarrow{T})$ which are not connected to the sources of the chosen virtual arrows $\xi_1, \ldots, \xi_r$ are the arrows of $Q(S, \overrightarrow{T}, \xi)$. On the other hand, the quiver $Q(S, \overrightarrow{T}, \xi)$ contains (with few exceptions) vertices which are sources or targets of 3 or 4 arrows, so this explains the statement (3). We describe a canonical basis for the algebra $\Lambda(\xi)$ and then provide formula for its dimension over $K$ (see Section 4). For the proof of (2) we construct in Section 6 a tilting complex $T^\xi$ in the homotopy category $K^b(P_\Lambda)$ of bounded complexes of projective modules in mod $\Lambda$ and prove that the endomorphism algebra $\text{End}_{K^b(P_\Lambda)}(T^\xi)$ is isomorphic to $\Lambda(\xi)$. Then the remaining statements of the above theorem follow from known general results.

We note that in general the directed triangulated surface $(S, \overrightarrow{T})$ of a weighted surface algebra $\Lambda = \Lambda(S, \overrightarrow{T}, m_*, c_*)$ may not contain 2-triangle discs, and then $\Lambda$ does not admit a virtual mutation. In Section 7 we discuss the following construction of weighted surface algebras which admit virtual mutations.

Let $\Lambda = \Lambda(S, \overrightarrow{T}, m_*, c_*)$ be a weighted surface algebra, say with $S$ an orientable surface and $\overrightarrow{T}$ a coherent orientation of triangles in $T$ (due to [21, Theorem 3.1] it is not restriction of generality). We take a non-empty set $I$ of edges in $T$ (possibly all edges of $T$) and a function $\epsilon : I \to \{-1, 1\}$. Then we may associate to $(S, \overrightarrow{T})$, in a unique way, a directed triangulated surface $(S, \overrightarrow{T_I})$, where $T_I$ is the new triangulation of $S$ obtained from $T$ by blowing-up of every edge of $I$ to a 2-triangle disc, and in such a manner that $\overrightarrow{T}$ is extended to a coherent orientation $\overrightarrow{T_I}$ of triangles in $T_I$. This creates canonically a new weighted surface algebra $\Lambda_I = \Lambda(S, \overrightarrow{T_I}, m'_*, c'_*)$ and a sequence $\xi = (\epsilon_i)_{i \in I}$ of virtual arrows of $Q(S, \overrightarrow{T_I})$, defined by the function
Then the virtual mutation $\Lambda_\epsilon^I = \Lambda_I(\epsilon)$ of $\Lambda_I$ with respect to $\epsilon$ is defined as a deformation of $\Lambda$ at the set of edges $I$, with respect to function $\epsilon$.

We have the following consequence of the Main Theorem.

**Corollary 1.** Let $\Lambda = \Lambda(S, \overrightarrow{T}, m_\bullet, c_\bullet)$ be a weighted surface algebra, $I$ a set of edges of $T$, $\epsilon : I \to \{-1, 1\}$ a function, and $\Lambda_\epsilon^I$ the associated deformation of $\Lambda$. Then $\Lambda_\epsilon^I$ is a finite-dimensional, tame, symmetric and periodic algebra of period 4. Moreover, $\Lambda_\epsilon^I$ is not isomorphic to a weighted surface algebra.

The paper is organized as follows. In Sections 2 and 3 we present necessary background on derived equivalences of algebras and weighted triangulation (surface) algebras, needed for further parts of the article. In Section 4 we introduce and study the virtual mutations of weighted triangulation (surface) algebras. Section 5 presents several examples illustrating the concept of a virtual mutation of a weighted surface algebra. Section 6 is devoted to the proof of Main Theorem. In final Section 7 we introduce deformations of weighted surface algebras with respect to collections of edges of the underlying surface triangulation.

For general background on the relevant representation theory we refer to the books [1, 13, 30, 42, 45] and the survey article [44].

## 2 Derived equivalences of algebras

In this section, we recall basic facts on derived equivalences of algebras, needed in our article.

For an algebra $A$, we denote by $K^b(\text{mod } A)$ the homotopy category of bounded complexes of modules in $\text{mod } A$ and by $K^b(P_A)$ its subcategory formed by bounded complexes of projective modules. The derived category $D^b(\text{mod } A)$ of $A$ is the localization of $K^b(\text{mod } A)$ with respect to quasi-isomorphisms, and admits structure of a triangulated category, where the suspension functor is given by left shift $(-)[1]$ (see [29]). Two algebras $A$ and $B$ are called derived equivalent provided their derived categories $D^b(\text{mod } A)$ and $D^b(\text{mod } B)$ are equivalent as triangulated categories. Moreover, a complex $T$ in $K^b(P_A)$ is called a tilting complex [39], if the following conditions are satisfied:

1. $\Hom_{K^b(P_A)}(T, T[i]) = 0$, for all integers $i \neq 0$,
2. $\text{add}(T)$ generates $K^b(P_A)$ as triangulated category.

We have the following handy criterion for verifying derived equivalence of algebras, due to Rickard [39, Theorem 6.4].
**Theorem 2.1.** Two algebras $A$ and $B$ are derived equivalent if and only if there exists a tilting complex $T$ in $K^b(P_A)$ such that $\text{End}_{K^b(P_A)}(T) \cong B$.

We recall also the following two theorems (see [41, Corollary 5.3] and [15, Theorem 2.9]).

**Theorem 2.2.** Let $A$ and $B$ be derived equivalent algebras. Then $A$ is symmetric if and only if $B$ is symmetric.

**Theorem 2.3.** Let $A$ and $B$ be derived equivalent algebras. Then $A$ is periodic if and only if $B$ is periodic. Moreover, if this is the case, then $A$ and $B$ have the same period.

Because in the class of self-injective algebras, derived equivalence implies stable equivalence (see [40, Corollary 2.2] and [41, Corollary 5.3]), one may conclude from [7, Theorems 4.4 and 5.6] and [33, Corollary 2] that the following theorem holds.

**Theorem 2.4.** Let $A$ and $B$ be derived equivalent self-injective algebras. Then the following equivalences are valid.

1. $A$ is tame if and only if $B$ is tame.
2. $A$ is of polynomial growth if and only if $B$ is of polynomial growth.

We present now a simple construction of tilting complexes of length 2 over symmetric algebras, observed first by Okuyama [38] and Rickard [40]. These tilting complexes have been used extensively to verify various cases of Broué’s abelian defect group conjecture [5], as well as in realizing derived equivalences between symmetric algebras (see [3, 4, 31, 32, 37, 40]).

Let $A$ be a basic, indecomposable, symmetric algebra with the Grothendieck group $K_0(A)$ of rank at least 2 and $A = A_A = P \oplus Q$ a proper decomposition in $\mod A$. Consider a left $\add(Q)$-approximation $f : P \to Q'$ of $P$, that is $Q'$ is a module in $\add(Q)$ and $f$ induce surjective map

$$\text{Hom}_A(f, Q'') : \text{Hom}_A(Q', Q'') \to \text{Hom}(P, Q''),$$

for any module $Q''$ in $\add(Q)$.

In this case, we may consider two complexes

$$T_1 : \quad 0 \longrightarrow Q \longrightarrow 0 \quad \text{and} \quad T_2 : \quad 0 \longrightarrow P \overset{f}{\longrightarrow} Q' \longrightarrow 0$$

concentrated, respectively in degree 0 and in degrees 1 and 0. Then we have the following proposition (for a proof we refer to [12, Proposition 2.1]).

**Proposition 2.5.** $T := T_1 \oplus T_2$ is a tilting complex in $K^b(P_A)$. 7
3 Weighted triangulation algebras

A quiver is a quadruple $Q = (Q_0, Q_1, s, t)$ consisting of a finite set $Q_0$ of vertices, a finite set $Q_1$ of arrows and two maps $s, t : Q_1 \to Q_0$ assigning to each arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and target $t(\alpha) \in Q_0$. We denote by $KQ$ the path algebra of $Q$ over $K$, where underlying $K$-vector space has as its basis the set of all paths in $Q$ of length $\geq 0$, and by $R_Q$ the arrow ideal of $KQ$ generated by all paths in $Q$ of length $\geq 1$. For a vertex $i \in Q_0$, let $e_i$ be the path of length 0 at $i$, and then $e_i$ are pairwise orthogonal idempotents, which sum up to identity of $KQ$. We will consider algebras of the form $A = KQ/I$, where $I$ is an ideal of $KQ$ such that $R_Q^0 \subseteq I \subseteq R_Q$ for some $m \geq 2$, so that $A$ will be a basic finite-dimensional algebra. Then the Gabriel quiver $Q_A$ of $A$ is the full subquiver of $Q$ obtained by removing all arrows $\alpha$ with $\alpha + I \in R_Q^2 + I$.

A quiver $Q$ is 2-regular if for each vertex $i \in Q_0$ there are precisely two arrows with source $i$ and precisely two arrows with target $i$. Such a quiver $Q$ has involution $\overline{\cdot} : Q_1 \to Q_1$, which is a function assigning to each arrow $\alpha \in Q_1$ the unique arrow $\overline{\alpha} \neq \alpha$ with $s(\alpha) = s(\overline{\alpha})$.

Following [16, 35], a triangulation quiver is a pair $(Q, f)$, where $Q = (Q_0, Q_1, s, t)$ is a connected 2-regular quiver and $f : Q_1 \to Q_1$ is a permutation such that $s(f(\alpha)) = t(\alpha)$, for any arrow $\alpha \in Q_1$, and $f^3$ is the identity on $Q_1$. In particular, all cycles in $Q_1$ induced by $f$ (that is the orbits of $f$) have length 1 or 3. We also assume that $|Q_0| \geq 2$. It was shown in [16, Theorem 4.11] (see also [21, Theorem 3.1]) that every triangulation quiver $(Q, f)$ is the triangulation quiver $(Q(S, \overrightarrow{T}), f)$ coming from a triangulation $T$ of a compact connected real surface $S$, with or without boundary, and where $\overrightarrow{T}$ is an arbitrary choice of orientation of triangles in $T$. We may even assume that $S$ is an orientable surface and $\overrightarrow{T}$ is a coherent orientation of triangles in $T$. We refer to Section 7 for more details.

Let $(Q, f)$ be a triangulation quiver. Then we have the composed permutation $g : Q_1 \to Q_1$, where $g(\alpha) = \overrightarrow{f(\alpha)}$, if $\alpha \in Q_1$. For each arrow $\alpha \in Q_1$, we denote by $O(\alpha)$ the $g$-orbit of $\alpha$ in $Q_1$, and set $n_\alpha = n_{O(\alpha)} = |O(\alpha)|$. Hence the $g$-orbit $O(\alpha)$ is of the form $O(\alpha) = \langle \alpha, g(\alpha), \ldots, g^{n_\alpha - 1}(\alpha) \rangle$. We note that $n_\alpha$ may be arbitrary large natural number. We write $O(g)$ for the set of all $g$-orbits in $Q_1$. Following [16], we call a function

$$m_\bullet : O(g) \to \mathbb{N}^* := \mathbb{N} \setminus \{0\}$$

a weight function of $(Q, f)$, and a function

$$c_\bullet : O(g) \to K^* := K \setminus \{0\}$$
a parameter function of \((Q, f)\). We write briefly \(m_\alpha := m_\mathcal{O}(\alpha)\) and \(c_\alpha := c_\mathcal{O}(\alpha)\), for \(\alpha \in Q_1\).

**Definition 3.1.** We say that an arrow \(\alpha \in Q_1\) is virtual if \(m_\alpha n_\alpha = 2\). Note that this condition is preserved under permutation of \(g\), and the virtual arrows form \(g\)-orbits of size 1 or 2.

We have also the following general assumption [19, Assumption 2.7].

**Assumption 3.2.** We assume that a weight function \(m_\star\) of \((Q, f)\) satisfies the following conditions:

1. \(m_\alpha n_\alpha \geq 2\) for all arrows \(\alpha\),
2. \(m_\alpha n_\alpha \geq 3\) for all arrows \(\alpha\) such that \(\bar{\alpha}\) is virtual but not a loop,
3. \(m_\alpha n_\alpha \geq 4\) for all arrows \(\alpha\) such that \(\bar{\alpha}\) is a virtual loop.

For each arrow \(\alpha \in Q_1\), we consider the path

\[
A_\alpha := \alpha g(\alpha) \ldots g^{m_\alpha n_\alpha - 2}(\alpha)
\]

along the \(g\)-cycle of \(\alpha\) of length \(m_\alpha n_\alpha - 1\), and the \(g\)-cycle

\[
B_\alpha := \alpha g(\alpha) \ldots g^{m_\alpha n_\alpha - 1}(\alpha)
\]

of \(\alpha\) of length \(m_\alpha n_\alpha\). We observe that \(B_\alpha = A_\alpha g^{n_\alpha - 1}(\alpha)\). Moreover, if \(n_\alpha \geq 3\), we consider also the path

\[
A_\alpha' := \alpha g(\alpha) \ldots g^{m_\alpha n_\alpha - 3}(\alpha)
\]

along the \(g\)-cycle of \(\alpha\) of length \(m_\alpha n_\alpha - 2\). Let us only mention that \(f^2 = g^{n_\alpha - 1}(\bar{\alpha})\), for any arrow \(\alpha \in Q_1\) (see [16, Lemma 5.3]).

The definition of a weighted triangulation algebra looks as follows.

**Definition 3.3.** The algebra \(\Lambda = \Lambda(Q, f, m_\star, c_\star) = KQ/I\) is a said to be a weighted triangulation algebra if \((Q, f)\) is a triangulation quiver and \(I = I(Q, f, m_\star, c_\star)\) is the ideal of \(KQ\) generated by the following relations:

1. \(\alpha f(\alpha) - c_\bar{\alpha} A_\bar{\alpha}\), for any arrow \(\alpha \in Q_1\),
2. \(\alpha f(\alpha) g(f(\alpha))\), for all arrows \(\alpha \in Q_1\) unless \(f^2(\alpha)\) is virtual or unless \(f(\bar{\alpha})\) is virtual with \(m_{\bar{\alpha}} = 1\) and \(n_{\bar{\alpha}} = 3\),
3. \(ag(\alpha) f(g(\alpha))\), for all arrows \(\alpha \in Q_1\) unless \(f(\alpha)\) is virtual or unless \(f^2(\alpha)\) is virtual with \(m_{f(\alpha)} = 1\) and \(n_{f(\alpha)} = 3\).
We note that the relations (2) and (3) are corrections of the relations (2) and (3) in \[19\] Definition 2.8 (see [23] for a discussion).

The following theorem is a consequence of the main results proved in \[19\].

**Theorem 3.4.** Let $\Lambda = \Lambda(Q,f,m_\bullet,c_\bullet)$ be a weighted triangulation algebra other than a singular disc, triangle, tetrahedral or spherical algebra. Then the following statements hold:

1. $\Lambda$ is a finite-dimensional algebra with $\dim_K \Lambda = \sum_{O \in \mathcal{O}(g)} m_O n^2_O$.
2. $\Lambda$ is a symmetric algebra.
3. $\Lambda$ is a tame algebra.
4. $\Lambda$ is a periodic algebra of period 4.

**Definition 3.5.** Let $(S, \overrightarrow{T})$ be a directed triangulated surface, $(Q(S, \overrightarrow{T}), f)$ the associated triangulation quiver and $m_\bullet, c_\bullet$ weight and parameter functions of $Q(S, \overrightarrow{T})$. Then the weighted triangulation algebra

$$\Lambda(Q(S, \overrightarrow{T}), f, m_\bullet, c_\bullet)$$

is called a weighted surface algebra, and it is denoted by $\Lambda(S, \overrightarrow{T}, m_\bullet, c_\bullet)$.

We recall also the following description of bases of indecomposable projective modules over a weighted triangulation algebra, established in \[19\], Lemma 4.7.

**Proposition 3.6.** Let $\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)$ be a weighted triangulation algebra, $i$ a vertex of $Q$ and $\alpha, \overline{\alpha}$ the two arrows in $Q_1$ starting at $i$. Then the following statements hold.

1. Assume that $\alpha$ is virtual. Then the module $e_i \Lambda$ has basis $\mathcal{B}_i$ formed by all initial submonomials of $B_\overline{\alpha}$ together with $e_i$ and $\overline{\alpha}f(\overline{\alpha})$.
2. If $\alpha$ and $\overline{\alpha}$ are not virtual, then $e_i \Lambda$ has basis $\mathcal{B}_i$ formed by all proper initial submonomials of $B_\alpha$ and $B_\overline{\alpha}$ together with $e_i$ and $B_\alpha$.
3. We have the equalities

$$\alpha f(\alpha) f^2(\alpha) = c_\alpha B_\alpha = c_\overline{\alpha} B_\overline{\alpha} = \overline{\alpha} f(\overline{\alpha}) f^2(\overline{\alpha})$$

and this element generates the socle of $e_i \Lambda$. 

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Example 3.7. Following [22], by a 2-triangle disc we mean the unit disc $D = D^2$ in $\mathbb{R}^2$ with the triangulation $T$ given by two triangles

with 1 and 3 boundary edges, and the coherent orientation $\vec{T} : (1 \ 2 \ 4)$ and $(2 \ 3 \ 4)$ of these two triangles. Then the associated triangulation quiver $(Q(D, \vec{T}), f)$ is the following quiver

with $f$-orbits $(\alpha \xi \delta)$, $(\beta \nu \mu)$, $(\rho)$ and $(\gamma)$. We mention that the associated permutation $g = \tilde{f}$ has two orbits $\mathcal{O}(\alpha) = (\alpha \beta \gamma \nu \delta \rho)$ and $\mathcal{O}(\xi) = (\xi \mu)$. Let $m \in \mathbb{N}^*$ and $\lambda \in K^*$. Consider the weight function $m_\bullet : \mathcal{O}(g) \to \mathbb{N}^*$ and the parameter function $c_\bullet : \mathcal{O}(g) \to K^*$ such that $m_{\mathcal{O}(\alpha)} = m$, $m_{\mathcal{O}(\xi)} = 1$, $c_{\mathcal{O}(\alpha)} = \lambda$, $c_{\mathcal{O}(\xi)} = 1$, and the associated weighted disc (triangulation) algebra

which we call the disc algebra of degree $m$. We note that $\xi$ and $\mu$ are virtual arrows, and $\xi = \beta \nu$ and $\mu = \delta \alpha$ in $D(m, \lambda)$, so the disc algebra $D(m, \lambda)$ is given by its Gabriel quiver

and the relations

$$\alpha \beta \nu = \lambda(\rho \alpha \beta \gamma \nu \delta)^{m-1} \rho \alpha \beta \gamma \nu, \ \beta \nu \delta = \lambda(\beta \gamma \nu \delta \rho \alpha)^{m-1} \beta \gamma \nu \delta \rho,$$

$$\delta \alpha \beta = \lambda(\delta \rho \alpha \beta \gamma \nu)^{m-1} \delta \rho \alpha \beta \gamma, \ \nu \delta \alpha = \lambda(\gamma \nu \delta \rho \alpha \beta)^{m-1} \gamma \nu \delta \rho \alpha,$$
\[ \rho^2 = \lambda (\alpha \beta \gamma \nu \delta \rho)^{m-1} \alpha \beta \gamma \nu \delta, \quad \gamma^2 = \lambda (\nu \delta \rho \alpha \beta \gamma)^{m-1} \nu \delta \rho \alpha \beta, \]
\[ \alpha \beta \nu \delta \alpha = 0, \quad \beta \nu \delta \rho = 0, \quad \rho^2 \alpha = 0, \]
\[ \nu \delta \alpha \beta \nu = 0, \quad \delta \alpha \beta \gamma = 0, \quad \gamma^2 \nu = 0, \]
\[ \beta \nu \delta \alpha \beta = 0, \quad \rho \alpha \beta \nu = 0, \quad \delta \rho^2 = 0, \]
\[ \delta \alpha \beta \nu \delta = 0, \quad \gamma \nu \delta \alpha = 0, \quad \beta \gamma^2 = 0. \]

Observe also that we have no zero relations of the forms:
\[ \delta \alpha \beta = \delta f(\delta) g(f(\delta)), \quad \text{because} \ f^2(\delta) = \xi \text{ is virtual}, \]
\[ \beta \nu \delta = \beta f(\beta) g(f(\beta)), \quad \text{because} \ f^2(\beta) = \mu \text{ is virtual}, \]
\[ \alpha \beta \nu = \alpha g(\alpha) f(g(\alpha)), \quad \text{because} \ f(\alpha) = \xi \text{ is virtual, and} \]
\[ \nu \delta \alpha = \nu g(\nu) f(g(\nu)), \quad \text{because} \ f(\nu) = \mu \text{ is virtual}. \]

Moreover, \( \dim_K D(m, \lambda) = m \cdot 6^2 + 2^2 = 36m + 4 \) and, according to Proposition 3.6, \( D(m, \lambda) \) admits basis \( B = B_1 \cup B_2 \cup B_3 \cup B_4 \), where
\[
B_1 = \{(\alpha \beta \gamma \nu \delta \rho)^r \alpha, (\alpha \beta \gamma \nu \delta \rho)^r \alpha \beta, (\alpha \beta \gamma \nu \delta \rho)^r \alpha \beta \gamma \nu, \\
(\alpha \beta \gamma \nu \delta \rho)^r \alpha \beta \gamma \nu \delta; 0 \leq r \leq m - 1 \} \cup \{(\rho \alpha \beta \gamma \nu \delta \rho)^r \rho, (\rho \alpha \beta \gamma \nu \delta \rho)^r \rho \alpha \beta, \\
(\rho \alpha \beta \gamma \nu \delta \rho)^r \rho \alpha \beta \gamma \nu \delta \rho; 0 \leq r \leq m - 1 \} \cup \{e_1, (\alpha \beta \gamma \nu \delta \rho)^s, (\rho \alpha \beta \gamma \nu \delta \rho)^t; 1 \leq s \leq m, 1 \leq t \leq m - 1 \},
\]
\[
B_2 = \{(\beta \gamma \nu \delta \rho \alpha)^r \beta, (\beta \gamma \nu \delta \rho \alpha)^r \beta \gamma \nu, (\beta \gamma \nu \delta \rho \alpha)^r \beta \gamma \nu \delta, \\
(\beta \gamma \nu \delta \rho \alpha)^r \beta \gamma \nu \delta \rho; 0 \leq r \leq m - 1 \} \cup \{e_2, \beta \nu, (\beta \gamma \nu \delta \rho \alpha)^s; 1 \leq s \leq m \},
\]
\[
B_3 = \{(\nu \delta \rho \alpha \beta \gamma \nu)^r \nu, (\nu \delta \rho \alpha \beta \gamma \nu)^r \nu \delta, (\nu \delta \rho \alpha \beta \gamma \nu)^r \nu \delta \rho \alpha, \\
(\nu \delta \rho \alpha \beta \gamma \nu)^r \nu \delta \rho \alpha \beta \gamma \nu \rho; 0 \leq r \leq m - 1 \} \cup \{(\gamma \nu \delta \rho \alpha \beta \gamma \nu)^r \gamma, (\gamma \nu \delta \rho \alpha \beta \gamma \nu)^r \gamma \nu, \\
(\gamma \nu \delta \rho \alpha \beta \gamma \nu)^r \gamma \nu \delta, (\gamma \nu \delta \rho \alpha \beta \gamma \nu)^r \gamma \nu \delta \rho \alpha \beta \gamma \nu \rho; 0 \leq r \leq m - 1 \} \cup \{e_3, (\nu \delta \rho \alpha \beta \gamma \nu)^s, (\gamma \nu \delta \rho \alpha \beta \gamma \nu)^t; 1 \leq s \leq m, 1 \leq t \leq m - 1 \},
\]
\[
B_4 = \{(\delta \rho \alpha \beta \gamma \nu)^r \delta, (\delta \rho \alpha \beta \gamma \nu)^r \delta \rho, (\delta \rho \alpha \beta \gamma \nu)^r \delta \rho \alpha, \\
(\delta \rho \alpha \beta \gamma \nu)^r \delta \rho \alpha \beta \gamma \nu \rho; 0 \leq r \leq m - 1 \} \cup \{e_4, \delta \alpha, (\delta \rho \alpha \beta \gamma \nu)^s; 1 \leq s \leq m \}.
\]

In particular, the Cartan matrix \( C_{D(m, \lambda)} \) of \( D(m, \lambda) \) is of the form
\[
\begin{bmatrix}
4m & 2m & 4m & 2m \\
2m & m + 1 & 2m & m + 1 \\
4m & 2m & 4m & 2m \\
2m & m + 1 & 2m & m + 1
\end{bmatrix}
\]
4 Virtual mutations of weighted surface algebras

The aim of this section is to introduce the main objective of the paper, namely the virtual mutations of weighted triangulation (surface) algebras, and describe their linear bases. In particular, we will show that these are finite-dimensional algebras and determine their dimensions.

Let \((Q, f)\) be a triangulation quiver, \(Q = (Q_0, Q_1, s, t)\), \(g = \bar{f}\) the associated permutation of \(Q_1\), and \(\mathcal{O}(g)\) the set of all \(g\)-orbits in \(Q_1\). Moreover, let \(m_\bullet : \mathcal{O}(g) \to \mathbb{N}^+\) be a weight function, \(c_\bullet : \mathcal{O}(g) \to K^*\) a parameter function, and \(\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)\) the associated weighted triangulation algebra. We keep the notation introduced in the previous section. We also assume that \(|Q_0| \geq 4\) and that \(\Lambda\) is not a singular spherical algebra, introduced in [19, Example 3.6].

Assume that \(\mathcal{O}(g)\) contains a family \(\mathcal{O}_1, \ldots, \mathcal{O}_r\) of orbits with \(|\mathcal{O}_i| = 2\) and \(m_{\mathcal{O}_i} = 1\), for any \(i \in \{1, \ldots, r\}\) (we mention that \(\mathcal{O}(g)\) may contain other virtual orbits besides the chosen ones). For a given element

\[ \xi = (\xi_1, \ldots, \xi_r) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_r \]

we shall define a virtual mutation

\[ \Lambda(\xi) = \Lambda(Q, f, m_\bullet, c_\bullet, \xi) \]

of \(\Lambda\) with respect to the sequence \(\xi\) of virtual arrows.

Observe that, for each \(i \in \{1, \ldots, r\}\), the triangulation quiver \((Q, f)\) contains a subquiver of the form

\[
\begin{array}{c}
\sigma_i \quad \alpha_i \quad \beta_i \quad \gamma_i \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\xi_i \quad \mu_i \quad \nu_i \quad \omega_i \\
\rho_i \quad \delta_i \quad \gamma_i \quad \delta_i
\end{array}
\]

with \(f\)-orbits \((\alpha_i, \xi_i, \delta_i)\) and \((\beta_i, \nu_i, \mu_i)\) and \(f(\sigma_i) = \rho_i, f(\omega_i) = \gamma_i, g(\sigma_i) = \alpha_i, g(\alpha_i) = \beta_i, g(\beta_i) = \gamma_i, g(\omega_i) = \nu_i, g(\nu_i) = \delta_i\) and \(g(\delta_i) = \rho_i\). We also note that the \(g\)-orbits \(\mathcal{O}(\alpha_i) = (\alpha_i, \beta_i, \gamma_i, \ldots, \sigma_i)\) and \(\mathcal{O}(\nu_i) = (\nu_i, \delta_i, \rho_i, \ldots, \omega_i)\) are of length at least 3, and may coincide (as shown in Example 3.7). But we have always \(|\mathcal{O}(\alpha_i)| \geq 4\) or \(|\mathcal{O}(\nu_i)| \geq 4\), because otherwise, due to 2-regularity of \(Q\), there is an \(f\)-orbit of length 2, which is impossible. The special cases
\(|\mathcal{O}(\alpha_i)| = 3\) or \(|\mathcal{O}(\nu_i)| = 3\) are as follows:

\[
\begin{array}{ccc}
\alpha_i & \Rightarrow & b_i \\
| & | & | \\
\omega_i & \Rightarrow & \gamma_i = \sigma_i \\
| & | & | \\
a_i & \Rightarrow & c_i \\
\end{array}
\]

Further, since the arrows \(\xi_i\) and \(\mu_i\) are virtual, we have in \(\Lambda\) the equalities

\[
\delta_i \alpha_i = c_i \mu_i\ 	ext{and} \ eta_i \nu_i = c_i \xi_i, \ \text{with} \ c_i \xi_i = c_i \mu_i = c_i \Omega_i.
\]

Replacing \(\xi_i\) by \(c_i \xi_i\) and \(\mu_i\) by \(c_i \mu_i\), we may assume that \(c_i \Omega_i = 1\), for all \(i \in \{1, \ldots, r\}\). We also note that in the presentation of \(\Lambda\) by its Gabriel quiver \(Q_\Lambda\) and the induced relations, the virtual arrows \(\xi_i\) and \(\mu_i\), \(i \in \{1, \ldots, r\}\), are removed (as well as all other virtual arrows of \(Q\)).

We will define the algebra \(\Lambda(\xi) = \Lambda(Q, f, m, c, \xi)\) by a quiver \(Q(\xi)\) and a set of relations, keeping triangle nature of the most of the relations defining \(\Lambda\). There are also added some new types of the relations.

The quiver \(Q(\xi) = (Q(\xi)_0, Q(\xi)_1, s, t)\) is defined in the following way. We take \(Q(\xi)_0 = Q_0\) and the set \(Q(\xi)_1\) of arrows is obtained from the set of arrows \(Q_1\) by three types of operations:

- removing the virtual arrows \(\xi_i\) and \(\mu_i\), for \(i \in \{1, \ldots, r\}\),
- replacing the arrows \(a_i \overset{\alpha_i}{\Rightarrow} c_i\) and \(c_i \overset{\beta_i}{\Rightarrow} b_i\) by arrows \(c_i \overset{\alpha_i}{\Rightarrow} a_i\) and \(b_i \overset{\beta_i}{\Rightarrow} c_i\), for \(i \in \{1, \ldots, r\}\),
- adding the arrows \(a_i \overset{\tau_i}{\Rightarrow} b_i\), for \(i \in \{1, \ldots, r\}\).

Therefore, for any \(i \in \{1, \ldots, r\}\), the quiver \(Q(\xi)\) contains a subquiver of one of the forms:
if $|\mathcal{O}(\alpha_i)| \geq 4$ and $|\mathcal{O}(\nu_i)| \geq 4$,

(2) if $|\mathcal{O}(\alpha_i)| = 3$, and

(3) for $|\mathcal{O}(\nu_i)| = 3$. 

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We consider also the quiver $Q(\xi)^*$ obtained from $Q(\xi)$ by removing all vertices $c_i, i \in \{1, \ldots, r\}$, and arrows attached to them (that is the arrows $\alpha_i, \beta_i$). Note that every vertex of $Q(\xi)^*$ except $d_i, i \in \{1, \ldots, r\}$, is 2-regular. Consequently, each arrow $\eta$ of $Q(\xi)^*$ different from $\delta_i, i \in \{1, \ldots, r\}$, admits the second arrow $\tilde{\eta}$ with $s(\tilde{\eta}) = s(\eta)$. Setting $\tilde{\delta}_i := \delta_i$, for each $i \in \{1, \ldots, r\}$, we obtain an involution $\sim : Q(\xi)^* \to Q(\xi)^*$.

The quiver $Q(\xi)^*$ also admits a triangulation-like structure given by the permutation $f^* : Q(\xi)^* \to Q(\xi)^*$ such that $s(f^*(\eta)) = t(\eta)$, for each arrow $\eta \in Q(\xi)^*$, and $(f^*)^3$ is the identity on the set of arrows. Indeed, for any arrow $\eta \in Q(\xi)^*$ different from $\nu_i, \delta_i, \tau_i, i \in \{1, \ldots, r\}$, we set $f^*(\eta) = f(\eta)$, whereas for any $i \in \{1, \ldots, r\}$, we put $f^*(\nu_i) = \delta_i$, $f^*(\delta_i) = \tau_i$, and $f^*(\tau_i) = \nu_i$. Composing $f^*$ with the involution $\sim : Q(\xi)^* \to Q(\xi)^*$, we obtain also the permutation $g^* = \tilde{f}^* : Q(\xi)^* \to Q(\xi)^*$ on the set of arrows of $Q(\xi)^*$.

For each arrow in $\eta \in Q(\xi)_1^*$, we denote by $O^*(\eta)$ the $g^*$-orbit of $\eta$ in $Q(\xi)_1^*$, and put $n^*_{\eta} = |O^*(\eta)|$. By $O(g^*)$ we denote the set of all $g^*$-orbits in $Q(\xi)_1^*$. We note that for any arrow $\eta \in Q_1$ with $O(\eta)$ different from $O(\alpha_1), \ldots, O(\alpha_r)$, we have $O^*(\eta) = O(\eta)$.

The following lemma describes the orbits in $O(g^*)$ without arrows in $Q_1$.

**Lemma 4.1.** Let $O^*$ be an orbit in $O(g^*)$. The following statements are equivalent.

1. $O^*$ does not contain an arrow $\eta \in Q_1$.
2. $(Q, f)$ is of the form

$$
\begin{array}{ccccccccc}
& \alpha_1 & \beta_1 & & \alpha_2 & \beta_2 & & \cdots & & \alpha_r & \beta_r \\
\downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \cdots & & \downarrow & \downarrow \\
(\xi_1, \delta_1, \mu_1, \nu_1) & b_1 = a_2 & b_2 = a_3 & \cdots & b_{r-1} = a_r & (\xi_r, \delta_r, \mu_r, \nu_r) & b_r = a_1 \\
\end{array}
$$

with $r \geq 2$, the $f$-orbits $(\alpha_i, \xi_i, \delta_i)$ and $(\beta_i, \nu_i, \mu_i), i \in \{1, \ldots, r\}$, the $g$-orbits $O(\alpha_1) = (\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_r, \beta_r)$, $O(\nu_r) = (\nu_r, \delta_r, \nu_1, \delta_1, \nu_2, \delta_2)$ and $O(\xi_i) = (\xi_i, \mu_i) = O(\mu_i), i \in \{1, \ldots, r\}$, $m_{O(\xi_i)} = 1$, for any $i \in \{1, \ldots, r\}$, and $\xi = (\xi_1, \ldots, \xi_r)$.

3. $Q(\xi)$ is of the form

$$
\begin{array}{ccccccccc}
& \alpha_1 & \beta_1 & & \alpha_2 & \beta_2 & & \cdots & & \alpha_r & \beta_r \\
\downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \cdots & & \downarrow & \downarrow \\
(\tau_1, \delta_1, \nu_1, \mu_1) & b_1 = a_2 & b_2 = a_3 & \cdots & b_{r-1} = a_r & (\tau_r, \delta_r, \nu_r, \mu_r) & b_r = a_1 \\
\end{array}
$$

with $r \geq 2$, the $f$-orbits $(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_r, \beta_r)$, $O(\tau_r) = (\nu_r, \delta_r, \delta_1, \nu_1, \delta_2, \nu_2)$ and $O(\tau_i) = (\tau_i, \mu_i) = O(\mu_i), i \in \{1, \ldots, r\}, m_{O(\tau_i)} = 1$, for any $i \in \{1, \ldots, r\}$, and $\tau = (\tau_1, \ldots, \tau_r)$. [16]
where \( r \geq 2 \) and \( \mathcal{O}^* = \mathcal{O}^*(\tau_1) = (\tau_1 \tau_2 \ldots \tau_r) \).

Proof. If \( \mathcal{O}^* \) does not contain an arrow from \( Q_1 \), then by definition of \( Q(\xi)^* \) and \( g^* \), \( \mathcal{O}^* \) consists only of arrows of type \( \tau_j \), and this yields that \( (Q, f) \) and \( Q(\xi) \) have shapes required in (2) and (3), respectively. The remaining implications are obvious from the definition of \( g^* \). \( \square \)

Remark 4.2. The triangulation quiver \((Q, f)\) occurring in the above lemma is the triangulation quiver \((Q(S^2, T(r)), f)\) of the following triangulation \( T(r) \) of the unit sphere \( S^2 \) in \( \mathbb{R}^3 \):

![Triangulation Diagram]

with \( r \geq 2 \) and coherent orientation \( T(r) \) of triangles in \( T(r) \): \((a_i, c_i, d_i)\) and \((c_i, a_{i+1}, d_i)\), \( i \in \{1, \ldots, r\} \) (see [22, Example 7.5]).

We also have the following lemma.

**Lemma 4.3.** Let \( \eta \) be an arrow in \( Q(\xi)_1^* \). Then the following equivalences hold.

1. \(|\mathcal{O}^*(\eta)| = 1\) if and only if \( \eta \in Q_1 \) and \(|\mathcal{O}(\eta)| = 1\). In this case, we also have \( \mathcal{O}^*(\eta) = \mathcal{O}(\eta) \).

2. \(|\mathcal{O}^*(\eta)| = 2\) if and only if one of the cases holds:
   - (a) \( \eta \in Q_1 \) with \(|\mathcal{O}(\eta)| = 2\);
   - (b) \( \mathcal{O}^*(\eta) = \mathcal{O}^*(\tau_i) \) for some \( i \in \{1, \ldots, r\} \) with \(|\mathcal{O}(\alpha_i)| = 3\);
   - (c) \( \mathcal{O}^*(\eta) = (\tau_1 \tau_2) \) for the case \( r = 2 \) described in the previous lemma.

Proof. This is straightforward from the definition of \( Q(\xi)^* \). \( \square \)

We define now two functions

\[
m^*_\circ : \mathcal{O}(g^*) \to \mathbb{N}^* \quad \text{and} \quad c^*_\circ : \mathcal{O}(g^*) \to \mathbb{K}^*
\]
which assign to each orbit $\mathcal{O}$ in $\mathcal{O}(g^*)$ the elements

$$m_{\mathcal{O}}^* = \begin{cases} m_{\mathcal{O}(\eta)}, & \text{if } \mathcal{O}^* = \mathcal{O}^*(\eta) \text{ for some arrow } \eta \in Q_1, \\ m_{\mathcal{O}(\alpha_1)}, & \text{otherwise (see Lemma 4.1)} \end{cases}$$

$$c_{\mathcal{O}}^* = \begin{cases} c_{\mathcal{O}(\eta)}, & \text{if } \mathcal{O}^* = \mathcal{O}^*(\eta) \text{ for some arrow } \eta \in Q_1, \\ c_{\mathcal{O}(\alpha_1)}, & \text{otherwise (see Lemma 4.1)} \end{cases}$$

Note that the two conditions exclude in both definitions. We abbreviate $m_\eta^* = m_{\mathcal{O}^*}(\eta)$ and $c_\eta^* = c_{\mathcal{O}^*}(\eta)$, for any arrow $\eta \in Q(\xi)^*_1$. Using assumptions imposed on $m_\bullet$ one can deduce from Lemma 4.1 that $m_\eta^* m_\eta^* \geq 2$, for any arrow $\eta \in Q(\xi)^*_1$. The description of all arrows $\eta \in Q(\xi)^*_1$ with $m_\eta^* m_\eta^* = 2$ follows from Lemma 4.3, and we call these arrows $\eta$ virtual arrows.

For each $\eta \in Q(\xi)^*_1$, we define the path $A^*_\eta$ as follows:

- $A^*_\eta = (\eta g^*(\eta) \ldots (g^*)^{n_\eta^*_1 - 1}(\eta))^{m_\eta^*_1 - 1} \eta g^*(\eta) \ldots (g^*)^{n_\eta^*_2 - 2}(\eta)$, if $n_\eta^*_2 \geq 2$,
- $A^*_\eta = A_\eta = \eta^{m_\eta^*_1 - 1}$, if $n_\eta^*_2 = 1$ (equivalently, $\eta \in Q_1$ with $n_\eta = 1$).

Moreover, we set

$$B^*_\eta = A^*_\eta (g^*)^{n_\eta^*_1 - 1}(\eta) = (\eta g^*(\eta) \ldots (g^*)^{n_\eta^*_1 - 1}(\eta))^{m_\eta^*_1}.$$  

For each $i \in \{1, \ldots, r\}$, we denote by $C^*_\delta_i$ the subpath of $A^*_\delta_i$ such that $A^*_\delta_i = \delta_i C^*_\delta_i$, and let

- $A^*_\alpha_i = \alpha_i C^*_\delta_i$, and
- $B^*_\alpha_i = A^*_\alpha_i \beta_i = \alpha_i C^*_\delta_i \beta_i$.

We note that $C^*_\delta_i$ is of length $\geq 1$, since $|\mathcal{O}(\delta_i)| \geq 3$.

The definition of a virtual mutation of a weighted triangulation algebra is now as follows.

**Definition 4.4.** The algebra $\Lambda(\xi) = \Lambda(Q, f, m_\bullet, c_\bullet, \xi) := KQ(\xi)/I(\xi)$ is called a virtual mutation of a weighted triangulation algebra $\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)$ with respect to a sequence $\xi = (\xi_1, \ldots, \xi_r)$ of virtual arrows if $I(\xi)$ is the ideal of the path algebra $KQ(\xi)$ generated by the elements:

1. $\nu_i \delta_i - \beta_i \alpha_i - c_\delta^*_i A^*_\delta_i$, $\alpha_i \tau_i$, and $\tau_i \beta_i$, for all $i \in \{1, \ldots, r\}$,
2. $\eta f^*(\eta) - c_\eta A^*_\eta$, for all arrows $\eta \in Q(\xi)^*_1$ different that $\eta_i$, $i \in \{1, \ldots, r\}$,
3. $\eta f^*(\eta) g^*(f^*(\eta))$, for all arrows $\eta \in Q(\xi)^*_1$ different from $\tau_i$, $\nu_i$, for $i \in \{1, \ldots, r\}$, and $\eta \in Q_1$ such that $f^2(\eta)$ is virtual or $f(\eta)\eta$ is virtual with $m_\eta = 1$ and $n_\eta = 3$,
(4) \( \eta g^*(\eta) f^*(g^*(\eta)) \) for all arrows \( \eta \in Q(\xi)_1^1 \) different from \( (g^*)^{-1}(\nu_i) \), \( \nu_i \), \( \tau_i \)
with \( m_{\nu_i} n_{\nu_i} = 3 \), for \( i \in \{1, \ldots, r\} \), and \( \eta \in Q_1 \) such that \( f(\eta) \) is virtual
or \( f^2(\eta) \) is virtual with \( m_{f(\eta)} = 1 \) and \( n_{f(\eta)} = 3 \).

(5) \( A^*_\alpha \nu_i \) and \( A^*_\delta \beta_i \), for all \( i \in \{1, \ldots, r\} \).

We will identify an arrow \( \eta \) of \( Q(\xi) \) with the corresponding element of
\( \Lambda(\xi) = KQ(\xi)/I(\xi) \).

**Definition 4.5.** Let \( (S, \overrightarrow{T}) \) be a directed triangulated surface, \( (Q(S, \overrightarrow{T}), f) \)
the associated triangulation quiver, \( m_\bullet \) and \( c_\bullet \) be weight and parameter functions of \( (Q(S, \overrightarrow{T}), f) \), and \( \xi = (\xi_1, \ldots, \xi_r) \) a sequence of virtual arrows from
pairwise different orbits in \( \mathcal{O}(g) \) of length 2 (and trivial weights). Then the virtual mutation \( \Lambda(Q(S, \overrightarrow{T}), f, m_\bullet, c_\bullet, \xi) \) is said to be a virtual mutation of the
weighted surface algebra \( \Lambda(S, \overrightarrow{T}, m_\bullet, c_\bullet) \), and denoted by \( \Lambda(S, \overrightarrow{T}, m_\bullet, c_\bullet, \xi) \).

We shall present now some consequences of the relations defining a virtual mutation \( \Lambda(\xi) \) of a weighted triangulation algebra \( \Lambda = \Lambda(Q, f, m_\bullet, c_\bullet) \).

**Lemma 4.6.** Assume that \( i \in \{1, \ldots, r\} \) and \( m_{\alpha_i} = 1 \) and \( n_{\alpha_i} = |\mathcal{O}(\alpha_i)| = 3 \). Then the following relations hold in \( \Lambda(\xi) \).

(1) \( \nu_i \delta_i = \beta_i \alpha_i + c_{\gamma_i} \gamma_i \) and \( \nu_i \omega_i = c_{\gamma_i} \tau_i \).

(2) \( \omega_i \nu_i \delta_i = \omega_i \beta_i \alpha_i + c_{\gamma_i} c_{\rho_i} A^*_{\gamma_i} \).

(3) \( \nu_i \delta_i \rho_i = \beta_i \alpha_i \rho_i + c_{\gamma_i} c_{\nu_i} A^*_{\nu_i} \).

(4) \( \rho_i \omega_i \nu_i = c_{\gamma_i} c_{\rho_i} A^*_{\rho_i} \) and \( \rho_i \omega_i \nu_i \delta_i = c_{\rho_i} B^*_{\rho_i} \).

(5) \( \delta_i \rho_i \omega_i = c_{\gamma_i} c_{\delta_i} A^*_{\delta_i} \) and \( \delta_i \rho_i \omega_i \nu_i = c_{\delta_i} B^*_{\delta_i} \).

(6) \( \alpha_i \omega_i = 0 \) and \( \rho_i \omega_i \beta_i = 0 \).

In particular, the arrows \( \gamma_i \) and \( \tau_i \) do not occur in the Gabriel quiver of \( \Lambda(\xi) \).

**Proof.** For (1), we note that \( \tilde{\nu}_i = \gamma_i \), \( \tilde{\rho}_i = \tau_i \), and \( A^*_{\gamma_i} = \gamma_i \) and \( A^*_{\tau_i} = \tau_i \),
because \( \omega_i = f(\rho_i) = f^*(\rho_i) \) and \( \mathcal{O}^* = \mathcal{O}^*(\gamma_i) = \mathcal{O}^*(\tau_i) \) is an orbit of length 2
with weight \( m_{\alpha_i} = m_{\mathcal{O}(\alpha_i)} = 1 \). Then required relations follow from (1) and (2) in Definition 4.3.

For (2) and (3), observe that \( \omega_i \gamma_i = \omega_i f^*(\omega_i) = c_{\omega_i} A^*_{\omega_i} \) and \( \gamma_i \rho_i = \gamma_i f^*(\gamma_i) = c_{\gamma_i} A^*_{\gamma_i} \), by relations (2) defining \( \Lambda(\xi) \), so the required relations
follow, since \( \tilde{\gamma}_i = \nu_i \) and \( \tilde{\omega}_i = g(\rho_i) \).

The equalities in (4) and (5) follow from (1) and the relations \( \tau_i \nu_i = c_{\gamma_i} A^*_{\gamma_i} \)
and \( \delta_i \tau_i = c_{\delta_i} A^*_{\delta_i} \), because \( \tilde{\gamma}_i = \rho_i \) and \( \tilde{\delta}_i = \delta_i \).

Finally, (6) is a consequence of (1) and the relations \( \alpha_i \tau_i = 0 \) and \( \tau_i \beta_i = 0 \).
Proposition 4.7. Let η be an arrow in $Q(\xi)^*$. Then the following statements hold.

1. $B^*_\eta$ is a non-zero element of the right (respectively, left) socle of $\Lambda(\xi)$.
2. $c^*_\eta B^*_\eta = \eta f^*(\eta)(f^*)^2(\eta) = \tilde{\eta} f^*(\tilde{\eta})(f^*)^2(\tilde{\eta}) = c^*_\tilde{\eta} B^*_\tilde{\eta}$.

Proof. (1) For $\eta \in Q_1$, the cycle $B^*_\eta$ is obtained from the cycle $B^*_\eta$ of $Q$ by replacing all subpaths $\alpha_j \beta_j$, $j \in \{1, \ldots, r\}$, by arrows $\gamma$. Similarly, if $\eta = \tau_i$ for some $i \in \{1, \ldots, r\}$, then $B^*_\eta$ is obtained from the cycle $B_{\alpha_i}$ of $Q$ in the same way. We know from Proposition 3.6(3) that every cycle $B_{\alpha_i}$ in $Q$, $\alpha \in Q_1$, is a non-zero element of $\Lambda = \text{soc}(\Lambda)$, and hence is a non-zero element with $\gamma B_{\alpha} = 0$ and $B_{\alpha}\sigma = 0$ for all arrows $\gamma, \sigma \in Q_1$. Hence it follows from the relations defining $\Lambda(\xi)$ that $B^*_\eta$ is a non-zero element of $\Lambda(\xi)$ satisfying $\phi B^*_\eta = 0$ and $B^*_\eta \psi = 0$ for all arrows $\phi, \psi \in Q(\xi)^*$. Since the only arrows in $Q(\xi)_1$ which do not belong to $Q(\xi)^*_1$ are the arrows $\alpha_i, \beta_i$, for $i \in \{1, \ldots, r\}$, it is now sufficient to observe that the equalities

$$\alpha_i B^*_\nu = A^*_\alpha, \nu \delta_i = 0$$

are consequences of relations (5). Summing up, we conclude that $B^*_\eta$ is a non-zero element of the right (or left) socle of $\Lambda(\xi)$, for any arrow $\eta \in Q(\xi)^*$.

(2) It follows from Proposition 3.6 that for any arrow $\alpha \in Q_1$ we have the equalities in $\Lambda$

$$\alpha f(\alpha)f^2(\alpha) = c^*_\alpha B_{\alpha} = c^*_\alpha B_{\tilde{\alpha}} = \tilde{\alpha} f(\tilde{\alpha})f^2(\tilde{\alpha}).$$

Hence, if $\eta$ is an arrow of $Q_1$ different from $\nu_i, \delta_i$, $i \in \{1, \ldots, r\}$, then the following equalities hold in $\Lambda(\xi)$

$$\eta f^*(\eta)(f^*)^2(\eta) = c^*_\eta B^*_\eta = c^*_\tilde{\eta} B^*_\tilde{\eta}.$$
Lemma 4.8. If \( i \in \{1, \ldots, r\} \), then \( B_{\alpha_i}^* \) is a non-zero element of the right (respectively, left) socle of \( \Lambda(\xi) \).

**Proof.** The fact that \( B_{\alpha_i}^* \) is a non-zero element of \( \Lambda(\xi) \) follows from the defining relations. Moreover, \( \alpha_i \) is the unique arrow in \( Q(\xi) \) with source \( c_i \) and \( \beta_i \) is the unique arrow in \( Q(\xi) \) with target \( c_i \). It follows from Proposition 4.7(1) that

\[
B_{\alpha_i}^* \alpha_i = \alpha_i C_{\delta_i}^* \beta_i \alpha_i = \alpha_i B_{\phi_i}^* (\delta_i) = 0,
\]

because \( g^*(\delta_i) \in Q(\xi)_i^* \). Moreover, we have

\[
\beta_i B_{\alpha_i}^* = \beta_i \alpha_i C_{\delta_i}^* \beta_i = \nu_i \delta_i C_{\delta_i}^* \beta_i - c_{\bar{\nu}} \phi \alpha_i C_{\delta_i}^* \beta_i.
\]

Observe also that \( \nu_i \delta_i C_{\delta_i}^* \beta_i = \nu_i A_{\phi_i}^* \beta_i = B_{\phi_i}^* \beta_i = 0 \), again by (1) in Proposition 4.7, since \( \nu_i \in Q(\xi)_i^* \). Now we claim that also \( A_{\phi_i}^* \beta_i = 0 \), and consequently \( \beta_i B_{\alpha_i}^* = 0 \). By definition of \( g^* \) we have \( (g^*)^{-1}(\nu_i) = \tau_i \), and there are three cases to consider.

(a) Assume that \( (g^*)^{-1}(\tau_i) = \tau_j \) for some \( j \in \{1, \ldots, r\} \). Then \( A_{\delta_i}^* = \tilde{\nu}_i \ldots \tau_j \) and \( C_{\delta_i}^* = \nu_i \delta_j \ldots (g^*)^{-1}(\nu_i) \). Moreover, it follows from Proposition 4.7 that the element \( \tau_j \nu_j \delta_j = \tau_j f^*(\tau_j) (f^*)^2(\tau_j) = \tau_j f^*(\tau_j) \) belongs to the right socle of \( \Lambda(\xi) \), and so \( A_{\phi_i}^* \beta_i = 0 \).

(b) Assume that \( (g^*)^{-1}(\tau_i) = \sigma_i \) is an arrow in \( Q_1 \) with \( \bar{\sigma}_i \) virtual but not in one of the orbits \( O_1, \ldots, O_r \). Then \( Q(\xi) \) contains a subquiver of the form

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We shall now describe a basis of the virtually mutated weighted triangulation algebra $\Lambda(\xi)$.

Let $\eta$ be an arrow in $Q(\xi)_1 \setminus \{\beta_1, \ldots, \beta_r\}$. Then we denote by $B_\eta$ the set of all proper initial submonomials of $B^*_\eta$, and by $B^c_\eta$ the set of all paths of the form $u\beta_i$, for all paths $u$ such that $uv_i$ is a subpath of $B^*_\eta$ for some $i \in \{1, \ldots, r\}$.

**Proposition 4.9.** Let $x$ be a vertex in $Q(\xi)_0$. The following statements hold.

1. Assume $x$ is the starting vertex of two arrows $\eta$ and $\tilde{\eta}$ in $Q(\xi)_1$, which are not virtual. Then the module $e_x\Lambda(\xi)$ has basis of the form
   $$B_x = B_\eta \cup B^c_\eta \cup B^*_\eta \cup \{e_x, B^*_\eta\}.$$

2. Assume $x$ is the starting vertex of two arrows $\eta$ and $\tilde{\eta}$ in $Q(\xi)_1$, and $\tilde{\eta}$ is virtual. Then the module $e_x\Lambda(\xi)$ has basis
   $$B_x = B_\eta \cup B^c_\eta \cup \{e_x, B^*_\eta, \eta f(\eta)\}.$$

3. Assume that $x = c_i$ for some $i \in \{1, \ldots, r\}$. Then the module $e_x\Lambda(\xi)$ has basis
   $$B_x = B_{\alpha_i} \cup B^c_{\alpha_i} \cup \{e_x, B^*_{\alpha_i}\}.$$

4. Assume that $x = d_i$ for some $i \in \{1, \ldots, r\}$. Then the module $e_x\Lambda(\xi)$ has basis
   $$B_{\delta_i} \cup B^c_{\delta_i} \cup \{e_x, B^*_{\delta_i}\}.$$

**Proof.** It follows from the relations defining $\Lambda(\xi)$, Proposition 4.7 and Lemmas 4.6 and 4.8.

The next aim is to determine the dimension of $\Lambda(\xi)$.

For each arrow $\eta \in Q(\xi)_1$, we denote
$$n^\nu_\eta = |\{i \in \{1, \ldots, r\} : O^*(\nu_i) = O^*(\eta)\}|.$$

**Lemma 4.10.** For each $i \in \{1, \ldots, r\}$, we have $|B_{c_i}| = |B_{d_i}| = m^{*}_{\delta_{i}}(n^*_{\delta_{i}} + n^\nu_{\delta_{i}})$.

**Proof.** Clearly $|B_{c_i}| = |B_{d_i}|$. Moreover, $m^{*}_{\delta_{i}} = m_{\delta_{i}}$ and $|B_{\delta_{i}}| = m_{\delta_{i}}n^*_{\delta_{i}} - 1$, and $|B^c_{\delta_{i}}| = m_{\delta_{i}}n^\nu_{\delta_{i}} - 1$, hence the required equality follows.

**Lemma 4.11.** Let $\eta$ be an arrow in $Q(\xi)_1$ such that $\tilde{\eta}$ is virtual, and let $x = s(\eta)$. Then we have
$$|B_x| = m^*_{\eta}(n^*_n + n^\nu_n) + 2.$$
Proof. We note that \( m^*_\eta = m_\eta \), \( |B_\eta| = m_\eta n^*_\eta - 1 \) and \( |B'_\eta| = m_\eta n^\nu_\eta \), so the required equality holds by Proposition 4.9(2).

**Lemma 4.12.** Let \( x \) be the starting vertex of two arrows \( \eta \) and \( \tilde{\eta} \) in \( Q(\xi)_1^* \), which are not virtual. Then

\[
|B_x| = m^*_\eta (n^*_\eta + n^\nu_\eta) + m^{*\nu}_{\tilde{\eta}} (n^{*\nu}_{\tilde{\eta}} + n^\nu_{\tilde{\eta}}).
\]

Proof. We have \( |B_\eta| = m^*_\eta n^*_\eta - 1 \), \( |B'_\eta| = m^*_\eta n^\nu_\eta \), \( |B_{\tilde{\eta}}| = m^*_{\tilde{\eta}} n^\nu_{\tilde{\eta}} \), and the required equality follows from Proposition 4.9.

**Remark 4.13.** We note that if \( \varepsilon \) is a virtual arrow of \( Q(\xi)_1^* \), then \( m^*_\varepsilon (n^*_\varepsilon + n^\nu_\varepsilon) = 2 \).

Summing up, we obtain the following theorem.

**Theorem 4.14.** \( \Lambda(\xi) \) is a finite-dimensional algebra with

\[
\dim_K \Lambda(\xi) = \sum_{\eta \in Q(\xi)_1^*} m^*_\eta (n^*_\eta + n^\nu_\eta).
\]

Moreover, we have the following proposition.

**Proposition 4.15.** \( \Lambda(\xi) \) is not isomorphic to a weighted triangulation algebra.

Proof. This is a consequence of the shape of the quiver \( Q(\xi) \) and relations of type (1) in Definition 4.4 (see also Lemma 4.6).

5 **Examples**

In this section we present several examples of virtually mutated weighted triangulation (surface) algebras.

**Example 5.1.** Let \( D(m, \lambda) = \Lambda(Q, f, m_\bullet, c_\bullet) \) be the disc algebra of degree \( m \) described in Example 3.7. Hence \( (Q, f) = (Q(D, T^2), f) \) and we have two \( g \)-orbits \( \mathcal{O}(\alpha) = (\alpha \beta \gamma \nu \delta \rho) = \mathcal{O}(\nu) \) and \( \mathcal{O}(\xi) = (\xi \mu) \), where \( \xi \) and \( \mu \) are unique virtual arrows. We take \( \xi \) as the chosen element of the orbit \( \mathcal{O}(\xi) \).

Then the virtually mutated algebra

\[
D(m, \lambda, \xi) = \Lambda(\xi) = \Lambda(Q, f, m_\bullet, c_\bullet, \xi)
\]

is the algebra given by the quiver \( Q(\xi) \) of the form

\[
\begin{array}{c}
1 \\
\alpha & \beta & \gamma \\
\delta & \tau & \nu \\
3 \\
\end{array}
\]

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and the relations:

\[ \nu \delta = \beta \alpha + \lambda (\gamma \nu \delta \rho \tau)^{m-1} \gamma \nu \delta \rho, \alpha \tau = 0, \tau \beta = 0, \]
\[ \tau \nu = \lambda (\rho \tau \gamma \nu \delta)^{m-1} \rho \tau \gamma \nu, \delta \tau = \lambda (\delta \rho \tau \gamma \nu)^{m-1} \delta \rho \tau \gamma, \]
\[ \rho^2 = \lambda (\tau \gamma \nu \delta \rho)^{m-1} \tau \gamma \nu \delta, \gamma^2 = \lambda (\nu \delta \rho \tau \gamma)^{m-1} \nu \delta \rho \tau, \]
\[ \rho^2 \tau = 0, \gamma^2 \nu = 0, \delta \tau \gamma = 0, \]
\[ \delta \rho^2 = 0, \tau \gamma^2 = 0, \rho \tau \nu = 0. \]

We note the following consequences of those relations:

\[ \rho^3 = \lambda (\tau \gamma \nu \delta \rho)^m = \lambda (\rho \tau \gamma \nu \delta)^m = \tau \nu \delta, \]
\[ \gamma^3 = \lambda (\nu \delta \rho \tau \gamma)^m = \nu \delta \tau, \]
\[ \beta \alpha \rho = \nu \delta \rho, \gamma \beta \alpha = \gamma \nu \delta. \]

We observe that, according to Proposition 4.9, \( D(m, \lambda, \xi) \) has the basis \( B^\xi = B_1^\xi \cup B_2^\xi \cup B_3^\xi \cup B_4^\xi \) given by the sets

\[ B_1^\xi = \{(\rho \tau \gamma \nu \delta)^{t-1} \}\]  \( \cup \{(\gamma \nu \delta \rho \tau \gamma)^{t-1} \}\)  \( \cup \{e_1, (\rho \tau \gamma \nu \delta)^s, (\gamma \nu \delta \rho \tau \gamma)^t\} \),

\[ B_2^\xi = \{\alpha (\rho \tau \gamma \nu \delta)^{t-1} \}\]  \( \cup \{e_2, \alpha (\rho \tau \gamma \nu \delta)^s, \alpha (\gamma \nu \delta \rho \tau \gamma)^t\} \),

\[ B_3^\xi = \{(\gamma \nu \delta \rho \tau \gamma)^{t-1} \}\]  \( \cup \{(\gamma \nu \delta \rho \tau \gamma)^{t-1} \}\)  \( \cup \{e_3, (\gamma \nu \delta \rho \tau \gamma)^s, (\gamma \nu \delta \rho \tau \gamma)^t\} \),

\[ B_4^\xi = \{\delta (\rho \tau \gamma \nu \delta)^{t-1} \}\]  \( \cup \{e_4, \delta (\rho \tau \gamma \nu \delta)^s, \delta (\gamma \nu \delta \rho \tau \gamma)^t\} \).
In particular, the Cartan matrix $C_{D(m,\lambda,\xi)}$ of $D(m,\lambda,\xi)$ is of the form

$$
\begin{bmatrix}
4m & 2m & 4m & 2m \\
2m & m+1 & 2m & m-1 \\
4m & 2m & 4m & 2m \\
2m & m-1 & 2m & m+1 \\
\end{bmatrix}.
$$

and $\dim_K D(m,\lambda,\xi) = 36m$.

We also note that, for the choice $\mu \in \mathcal{O}(\xi)$, the algebra $D(m,\lambda,\mu)$ is isomorphic to $D(m,\lambda,\xi)$.

**Example 5.2.** Let $(Q,f)$ be the triangulation quiver

![Triangulation Quiver Diagram]

with $f$-orbits $(\alpha \xi \delta)$, $(\beta \nu \mu)$, $(\epsilon \gamma \sigma)$ and $(\rho)$. Then for the associated permutation $g = \bar{f}$ we have orbits

$$
\mathcal{O}(\alpha) = (\alpha \beta \gamma \rho \sigma), \mathcal{O}(\nu) = (\nu \delta \epsilon), \mathcal{O}(\xi) = (\xi \mu).
$$

Let $m \in \mathbb{N}^*$ and $\lambda \in K^*$. We consider the weight function $m_* : \mathcal{O}(g) \rightarrow \mathbb{N}^*$ and the parameter function $c_* : \mathcal{O}(g) \rightarrow K^*$ given by

$$
m_{\mathcal{O}(\alpha)} = m, \quad m_{\mathcal{O}(\nu)} = 1, \quad m_{\mathcal{O}(\xi)} = 1,
$$

$$
c_{\mathcal{O}(\alpha)} = \lambda, \quad c_{\mathcal{O}(\nu)} = 1, \quad c_{\mathcal{O}(\xi)} = 1.
$$

Then the associated weighted triangulation algebra $A = A(m,\lambda) = \Lambda(Q,f,m_*,c_*)$ is given by its Gabriel quiver $Q_A$ of the form

![Weighted Triangulation Algebra Quiver Diagram]
and the relations:

\[ \beta \nu \delta = \lambda (\beta \gamma \rho \sigma)^{m-1} \beta \gamma \rho \sigma, \quad \alpha \beta \nu = \epsilon \nu, \]
\[ \nu \delta \alpha = \lambda (\gamma \rho \sigma \alpha \beta)^{m-1} \gamma \rho \sigma \alpha, \quad \delta \alpha \beta = \delta \epsilon, \]
\[ \epsilon \gamma = \lambda (\alpha \beta \gamma \rho \sigma)^{m-1} \alpha \beta \gamma \rho, \quad \gamma \sigma = \nu \delta, \]
\[ \sigma \epsilon = \lambda (\rho \sigma \alpha \beta \gamma)^{m-1} \rho \sigma \alpha \beta, \quad \rho^2 \sigma = 0, \]
\[ \rho^2 = \lambda (\sigma \alpha \beta \gamma \rho)^{m-1} \sigma \alpha \beta \gamma, \quad \gamma \rho^2 = 0, \]
\[ \alpha \beta \nu \delta \alpha = 0, \beta \nu \delta \epsilon = 0, \nu \delta \alpha \beta \nu = 0, \delta \alpha \beta \gamma = 0, \]
\[ \epsilon \gamma \rho = 0, \sigma \epsilon \nu = 0, \epsilon \nu \delta \alpha = 0, \sigma \alpha \beta \nu = 0, \]
\[ \beta \nu \delta \alpha = 0, \delta \alpha \beta \nu \delta = 0. \]

Observe that we have no zero relations of the shape

\[ \delta \alpha \beta = \delta f(\delta) g(f(\delta)) = 0, \quad \text{because} \ f^2(\delta) = \xi \text{ is virtual}, \]
\[ \beta \nu \delta = \beta f(\beta) g(f(\beta)) = 0, \quad \text{because} \ f^2(\beta) = \mu \text{ is virtual}, \]
\[ \gamma \sigma \alpha = \gamma f(\gamma) g(f(\gamma)) = 0, \quad \text{because} \ f(\gamma) = \mu \text{ is virtual and} \ m_\gamma = 1, n_\gamma = 3, \]
\[ \alpha \beta \nu = \alpha g(\alpha) f(g(\alpha)) = 0, \quad \text{because} \ f(\alpha) = \xi \text{ is virtual}, \]
\[ \nu \delta \alpha = \nu g(\nu) f(g(\nu)) = 0, \quad \text{because} \ f(\nu) = \mu \text{ is virtual}, \]
\[ \beta \gamma \sigma = \beta g(\beta) f(g(\beta)) = 0, \quad \text{because} \ f^2(\beta) = \mu \text{ is virtual and} \ m_{f(\beta)} = 1, n_{f(\beta)} = 3. \]

We also observe that \( \gamma \sigma \alpha = \nu \delta \alpha \) and \( \beta \gamma \sigma = \beta \nu \delta \). Moreover, \( \dim_K A(m, \lambda) = m \cdot 5^2 + 3^2 + 2^2 = 25m + 13 \). In fact, the Cartan matrix \( C_{A(m, \lambda)} \) of \( A(m, \lambda) \) is of the form

\[
\begin{bmatrix}
  m + 1 & m & m + 1 & 2m & 1 \\
  m & m + 1 & m & 2m & 1 \\
  m + 1 & m & m + 1 & 2m & 1 \\
  2m & 2m & 2m & 4m & 0 \\
  1 & 1 & 1 & 0 & 2
\end{bmatrix}.
\]

We now take \( \mathcal{O} = \mathcal{O}(\xi) \) and consider two virtually mutated algebras with respect to two possible choices of elements from \( \mathcal{O} \).

(1) Let first \( A(\xi) = A(m, \lambda, \xi) = \Lambda(Q, f, m_\bullet, c_\bullet, \xi) \). Then the algebra \( A(\xi) \) is given by the quiver \( Q(\xi) \) of the form

![Quiver Diagram](attachment:quiver.png)
and the relations:

\[ \nu \delta = \beta \alpha + \lambda (\gamma \rho \sigma \tau)^{m-1} \gamma \rho \sigma, \tau \nu = \epsilon \nu, \delta \tau = \delta \epsilon, \]
\[ \alpha \tau = 0, \tau \beta = 0, \gamma \sigma = \nu \delta, \epsilon \gamma = \lambda (\tau \gamma \rho \sigma)^{m-1} \tau \gamma \rho, \]
\[ \sigma \epsilon = \lambda (\rho \sigma \tau \gamma)^{m-1} \rho \sigma \tau, \rho^2 = \lambda (\sigma \tau \gamma \rho)^{m-1} \sigma \tau \gamma, \]
\[ \delta \tau \gamma = 0, \epsilon \gamma \rho = 0, \sigma \epsilon \nu = 0, \rho^2 \sigma = 0, \]
\[ \delta \epsilon \gamma = 0, \tau \gamma \rho = 0, \gamma \rho^2 = 0, \rho \sigma \epsilon = 0. \]

The Cartan matrix \( C_{A(\xi)} \) of \( A(\xi) \) is of the form

\[
\begin{bmatrix}
  m + 1 & 1 & m + 1 & 2m & 1 \\
  1 & 2 & 1 & 0 & 0 \\
  m + 1 & 1 & m + 1 & 2m & 1 \\
  2m & 0 & 2m & 4m & 0 \\
  1 & 0 & 1 & 0 & 2
\end{bmatrix}
\]

Moreover, we have \( \dim_K A(\xi) = 16m + 16 \).

(2) Now, consider \( A(\mu) = A(m, \lambda, \mu) = A(Q, f, m_\bullet, c_\bullet, \mu) \). Then \( A(\mu) \) is given by the quiver \( Q(\mu) \) of the form

and the relations (we note that \( m_\nu n_\nu = 3 \)):

\[ \alpha \beta = \delta \nu + c_\alpha^* A_\alpha^* = \delta \nu + A^*_\alpha = \delta \nu + \epsilon, \]
\[ \tau \alpha = c_\gamma^* A_\gamma^* = c_\gamma A_\gamma^* = \lambda (\gamma \rho \sigma \alpha \beta)^{m-1} \gamma \rho \sigma \alpha, \]
\[ \beta \tau = c_\beta A_\beta^* = \lambda (\beta \gamma \rho \sigma \alpha)^{m-1} \beta \gamma \rho \sigma, \]
\[ \tau \delta = 0, \nu \tau = 0, \]
\[ \epsilon \gamma = \lambda (\alpha \beta \gamma \rho \sigma)^{m-1} \alpha \beta \gamma \rho, \sigma \epsilon = \lambda (\rho \sigma \alpha \beta \gamma)^{m-1} \rho \sigma \alpha \beta, \]
\[ \gamma \sigma = c_\gamma^* A_\gamma^* = c_\gamma A_\gamma^* = \tau, \]
\[ \rho^2 = \lambda (\sigma \alpha \beta \gamma \rho)^{m-1} \sigma \alpha \beta \gamma, \]
\[\sigma \epsilon \tau = 0, \epsilon \gamma \rho = 0, \rho^2 \sigma = 0,\]
\[\tau \epsilon \gamma = 0, \epsilon \tau \delta = 0, \gamma \rho^2 = 0, \rho \sigma \epsilon = 0.\]

In particular, we have \(\epsilon = \alpha \beta - \delta \nu\) and \(\tau = \gamma \sigma\). Hence \(A(\mu)\) is given by its Gabriel quiver

\[\tau \nu \sigma \rho\]

and the following relations:
\[
\gamma \sigma \alpha = \lambda (\gamma \rho \sigma \alpha \beta)^{m-1} \gamma \rho \sigma \alpha, \beta \gamma \sigma = \lambda (\beta \gamma \rho \sigma \alpha)^{m-1} \beta \gamma \rho \sigma,
\]
\[
\gamma \sigma \delta = 0, \nu \gamma \sigma = 0, \alpha \beta \gamma = \delta \nu \gamma = \lambda (\alpha \beta \gamma \rho \sigma)^{m-1} \alpha \beta \gamma \rho,
\]
\[
\sigma \alpha \beta = \sigma \delta \nu + \lambda (\rho \sigma \alpha \beta \gamma)^{m-1} \rho \sigma \alpha \beta, \rho^2 = \lambda (\sigma \alpha \beta \gamma \rho)^{m-1} \sigma \alpha \beta \gamma,
\]
\[
\sigma \alpha \beta \gamma \sigma = 0, \alpha \beta \gamma \rho = \delta \nu \gamma \rho, \rho^2 \sigma = 0,
\]
\[
\gamma \sigma \alpha \beta \gamma = 0, \alpha \beta \gamma \sigma \delta, \gamma \rho^2 = 0, \rho \sigma \alpha \beta = \rho \sigma \delta \nu.
\]

The Cartan matrix \(C_{A(\mu)}\) of \(A(\mu)\) is of the form

\[
\begin{bmatrix}
m + 1 & m & m + 1 & 2m & m \\
m & m + 1 & m & 2m & m - 1 \\
m + 1 & m & m + 1 & 2m & m \\
2m & 2m & 2m & 4m & 2m \\
m & m - 1 & m & 2m & m + 1
\end{bmatrix}.
\]

In particular, \(\dim_K A(\mu) = 36m + 4\).

We note that the algebras \(A(\xi)\) and \(A(\mu)\) are not isomorphic, since their Gabriel quivers are different, as well as their dimensions never coincide.

Example 5.3. (Virtually mutated generalized spherical algebras).

We consider the following triangulation \(T(2)\) of the sphere \(S^2\) in \(\mathbb{R}^3\).
and the coherent orientation $\overrightarrow{T(2)}$ of triangles in $T(2)$: $(1\ 2\ 5)$, $(2\ 3\ 5)$, $(3\ 4\ 6)$ and $(4\ 1\ 6)$. Then the associated triangulation quiver $(Q, f) = (Q(S^2, \overrightarrow{T(2)}), f)$ is of the form

$\xymatrix{ 1 \ar[dr]_{\sigma} & \ar[dl]^{\delta} \ar[rr]^\rho & & \ar[dl]^{\xi} \ar[rr]^\mu & & 2 \ar[dr]_{\nu} \ar[dl]^{\alpha} \ar[dl]^{\beta} & & \ar[dl]^{\gamma} \ar[rr]^\omega & & 3 \ar[dl]^{\zeta} \ar[dl]^{\eta} \ar[dl]^{\nu} & & 4 \ar[dr]_{\zeta} \ar[dl]^{\eta} \ar[dr]_{\omega} & & 5 \ar[dr]_{\alpha} \ar[dl]^{\delta} \ar[dr]_{\beta} & & 6 \ar[dl]^{\xi} \ar[dr]_{\mu}.}$

with $f$-orbits $(\alpha \xi \delta)$, $(\beta \nu \mu)$, $(\gamma \eta \omega)$, $(\sigma \rho \zeta)$. Then the the permutation $g = \overline{f}$ has four orbits

$O(\alpha) = (\alpha \beta \gamma \sigma)$, $O(\rho) = (\rho \omega \nu \delta)$, $O(\xi) = (\xi \mu)$, $O(\eta) = (\eta \zeta)$.

Let $m, n \in \mathbb{N}^*$ and $a, b \in K^*$. We define the weight function $m_\bullet : O(g) \to \mathbb{N}^*$ and the parameter function $c_\bullet : O(g) \to K^*$ such that

$m_{O(\alpha)} = m$, $m_{O(\rho)} = n$, $m_{O(\xi)} = 1$, $m_{O(\eta)} = 1$,

$c_{O(\alpha)} = a$, $c_{O(\rho)} = b$, $c_{O(\xi)} = 1$, $c_{O(\eta)} = 1$.

Hence the orbits $O(\xi)$ and $O(\eta)$ consist of virtual arrows. Moreover, if $m = n = 1$, then we assume that $ab \neq 1$ (see [19, Example 3.6]). We consider the associated triangulation algebra $S(m, n, a, b) = \Lambda(Q, f, m_\bullet, c_\bullet)$, and call it a generalized spherical algebra. Note that $S(1, 1, a, b)$ is isomorphic to the non-singular spherical algebra $S(ab)$ introduced in [19].

The algebra $S(m, n, a, b)$ is given by its Gabriel quiver $Q_S$ of the form

$\xymatrix{ 1 \ar[dr]_{\sigma} & \ar[dl]^{\delta} \ar[rr]^\rho & & \ar[dl]^{\xi} \ar[rr]^\mu & & 2 \ar[dr]_{\nu} \ar[dl]^{\alpha} \ar[dl]^{\beta} & & \ar[dl]^{\gamma} \ar[rr]^\omega & & 3 \ar[dl]^{\zeta} \ar[dr]_{\eta} \ar[dr]_{\omega} & & 4 \ar[dr]_{\zeta} \ar[dl]^{\eta} \ar[dr]_{\omega} & & 5 \ar[dr]_{\alpha} \ar[dl]^{\delta} \ar[dr]_{\beta} & & 6 \ar[dl]^{\xi} \ar[dr]_{\mu}.}$
and the relations

\[\alpha\beta\nu = b(\rho\omega\nu\delta)^{n-1}\rho\omega\nu, \quad \beta\nu\delta = a(\beta\gamma\sigma\alpha)^{m-1}\beta\gamma\sigma,\]
\[\delta\alpha\beta = b(\delta\rho\omega)^{m-1}\delta\rho\omega, \quad \nu\delta\alpha = a(\gamma\sigma\alpha\beta)^{m-1}\gamma\sigma\alpha,\]
\[\gamma\sigma\rho = b(\nu\delta\rho\omega)^{n-1}\nu\delta\rho, \quad \sigma\rho\omega = a(\sigma\alpha\beta\gamma)^{m-1}\sigma\alpha\beta,\]
\[\omega\gamma\sigma = b(\omega\nu\delta\rho)^{n-1}\omega\nu\delta, \quad \rho\omega\gamma = a(\alpha\beta\gamma\sigma)^{m-1}\alpha\beta\gamma,\]
\[\alpha\beta\nu\delta\alpha = 0, \beta\nu\delta\rho = 0, \nu\delta\alpha\beta\nu = 0, \delta\alpha\beta\gamma = 0,\]
\[\gamma\sigma\rho\omega\gamma = 0, \sigma\rho\omega\nu = 0, \rho\omega\gamma\sigma\rho = 0, \omega\gamma\sigma\alpha = 0,\]
\[\beta\gamma\sigma\rho = 0, \sigma\alpha\beta\nu = 0, \delta\rho\omega\gamma = 0, \omega\nu\delta\alpha = 0,\]
\[\beta\nu\delta\alpha\beta = 0, \delta\alpha\beta\nu\delta = 0, \sigma\rho\omega\gamma\sigma = 0, \omega\gamma\sigma\rho\omega = 0.\]

Moreover, a minimal set of relations defining \(S(m, n, a, b)\) is given by the above eight commutativity relations and the four zero relations:

\[\beta\nu\delta\rho = 0, \delta\alpha\beta\gamma = 0, \sigma\rho\omega\nu = 0, \omega\gamma\sigma\alpha = 0.\]

Further, we have

\[\dim_K S(m, n, a, b) = 16m + 16n + 4 + 4 = 16(m + n) + 8.\]

We shall now show some representative families of virtually mutated algebras of \(S(m, n, a, b)\).

(1) Let \(O = O(\xi)\) and \(\xi\) be the chosen element of \(O\). We use the following notation

\[S(m, n, a, b, \xi) := \Lambda(Q, f, m_\bullet, c_\bullet, \xi).\]

We note that \(s(\xi) = 2\). The algebra \(S(m, n, a, b, \xi)\) is given by the quiver \(Q(\xi)\) of the form

![Quiver Diagram]
and the relations:

\[
\begin{align*}
\nu \delta &= \beta \alpha + a(\gamma \sigma \tau)^{m-1} \gamma \sigma, \alpha \tau = 0, \tau \beta = 0, \\
\tau \nu &= b(\rho \omega \nu \delta)^{n-1} \rho \omega \nu, \delta \tau = b(\delta \rho \omega \nu)^{n-1} \delta \rho \omega, \\
\gamma \sigma \rho &= b(\nu \delta \rho \omega)^{n-1} \nu \delta \rho, \sigma \rho \omega = a(\sigma \tau \gamma)^{m-1} \sigma \tau, \\
\omega \gamma \sigma &= b(\omega \nu \delta \rho)^{n-1} \omega \nu \delta, \rho \omega \gamma = a(\tau \gamma \sigma)^{m-1} \tau \gamma, \\
\delta \tau \gamma &= 0, \rho \omega \gamma \sigma \rho = 0, \gamma \sigma \rho \omega \gamma = 0, \\
\delta \rho \omega \gamma &= 0, \tau \gamma \sigma \rho = 0, \sigma \tau \nu = 0.
\end{align*}
\]

It follows from Theorem 4,4 that

\[
\dim_K S(m, n, a, b, \xi) = 9m + 25n + 4.
\]

In particular, we have

\[
\begin{align*}
\dim_K S(1, n, a, b, \xi) &= 25n + 13, \\
\dim_K S(m, 1, a, b, \xi) &= 9m + 29.
\end{align*}
\]

(2) Let \( O_1 = O(\xi) \) and \( O_2 = O(\eta) \) and \( (\xi, \eta) \) be the chosen element of \( O_1 \times O_2 \). We set

\[
S(m, n, a, b, \xi, \eta) := \Lambda(Q_m, f, m, c, (\xi, \eta)).
\]

Then obviously \( s(\xi) = 2 = t(\alpha) \), \( s(\eta) = 4 = t(\gamma) \) and \( \alpha, \gamma \) belong to the same \( g \)-orbit. The algebra \( S(m, n, a, b, \xi, \eta) \) is given by the quiver \( Q(\xi, \eta) \) of the form

\[
\begin{align*}
&1 \\
&\downarrow \quad \downarrow \\
2 &\rightarrow \alpha \quad \delta \\
&\uparrow \quad \uparrow \\
5 &\rightarrow \nu \quad \beta \end{align*}
\]

and the relations:

\[
\nu \delta = \beta \alpha + a(\epsilon \tau)^{m-1} \epsilon, \alpha \tau = 0, \tau \beta = 0,
\]
\[ \tau \nu = b(\rho \omega \nu \delta)^{n-1} \rho \omega \nu, \quad \delta \tau = b(\delta \rho \omega \nu)^{n-1} \delta \rho \omega, \]
\[ \rho \omega = \sigma \gamma + a(\tau \epsilon)^{m-1}, \quad \gamma \epsilon = 0, \quad \epsilon \sigma = 0, \]
\[ \nu \delta \rho = b(\nu \delta \rho \omega)^{n-1} \nu \delta \rho, \quad \omega \nu \epsilon = b(\omega \nu \delta \rho)^{n-1} \omega \nu \delta, \]
\[ \tau \epsilon \rho = 0, \quad \epsilon \tau \nu = 0. \]
Moreover, it follows from Theorem 4.14 that
\[ \dim_K S(m, n, a, b, \xi, \eta) = 4m + 36n. \]
Note that if \( m \geq 2 \) then the above quiver is the Gabriel quiver of \( S(m, n, a, b, \xi, \eta) \).

Now, assume that \( m = 1 \). Then we have the equalities \( \nu \delta = \beta \alpha + a \epsilon \) and \( \rho \omega = \sigma \gamma + a \tau \), so \( \tau, \epsilon \) are not occurring in the Gabriel quiver, and hence \( S(1, n, a, b, \xi, \eta) \) is given by a quiver isomorphic to \( Q_S \) and the following relations:
\[ \rho \omega \nu = \sigma \gamma \nu + ab(\rho \omega \nu \delta)^{n-1} \rho \omega \nu, \quad \alpha \rho \omega = \alpha \sigma \gamma, \]
\[ \delta \rho \omega = \delta \sigma \gamma + ab(\delta \rho \omega \nu)^{n-1} \delta \rho \omega, \quad \rho \omega \beta = \gamma \sigma \beta, \]
\[ \nu \delta \rho = \beta \alpha \rho + ab(\nu \delta \rho \omega)^{n-1} \nu \delta \rho, \quad \gamma \nu \delta = \gamma \beta \alpha, \]
\[ \omega \nu \delta = \omega \beta \alpha + ab(\omega \nu \delta \rho)^{n-1} \omega \nu \delta, \quad \nu \delta \sigma = \beta \alpha \sigma, \]
\[ (\rho \omega \nu \delta)^{n} \rho = 0, \quad (\nu \delta \rho \omega)^{n} \nu = 0. \]
We observe that the last two zero relations are consequences of the zero relations of the form \( (\rho \omega - \sigma \gamma)(\nu \delta - \beta \alpha)\rho = 0, \quad (\nu \delta - \beta \alpha)(\rho \omega - \sigma \gamma)\nu = 0 \), and some of the above relations. Therefore, we conclude that, for \( n \geq 2 \) and \( \lambda = ab \), the algebra \( S(1, n, a, b, \xi, \eta) \) is isomorphic to the so called higher spherical algebra \( S(n, \lambda) \) investigated in [20].

(3) Let now \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) be as in previous case (2), but take \( (\xi, \zeta) \) as the chosen element of \( \mathcal{O}_1 \times \mathcal{O}_2 \). We consider the virtually mutated algebra
\[ S(m, n, a, b, \xi, \zeta) := \Lambda(Q, f, m_\bullet, c_\bullet, (\xi, \zeta)). \]
Note that \( s(\xi) = 2 = t(\alpha) \) and \( s(\zeta) = 6 = t(\omega) \) and \( \alpha, \omega \) belong to different \( g \)-orbits in \( (Q, f) \). The algebra \( S(m, n, a, b, \xi, \zeta) \) is given by the
quiver $Q(\xi, \zeta)$ of the form

and the relations:

$$
\nu \delta = \beta \alpha + a(\gamma \sigma \tau)^{m-1} \gamma \sigma, \alpha \tau = 0, \tau \beta = 0,
\gamma \sigma = \rho \omega + b(\nu \delta \epsilon)^{n-1} \nu \delta, \rho \epsilon = 0, \epsilon \omega = 0,
\tau \nu = b(\epsilon \nu \delta)^{n-1} \epsilon \nu, \delta \tau = b(\delta \epsilon \nu)^{n-1} \delta \epsilon,
\epsilon \gamma = a(\tau \gamma \sigma)^{m-1} \tau \gamma, \sigma \epsilon = a(\sigma \tau \gamma)^{m-1} \sigma \tau,
\delta \epsilon \gamma = 0, \sigma \tau \nu = 0.
$$

Applying Theorem 4.14, we conclude that

$$\dim_K S(m, n, a, b, \xi, \zeta) = 16(m+n).$$

6 Proof of Main Theorem

Let $\Lambda = \Lambda(Q, f, m, c)$ be a weighted triangulation algebra. For each vertex $i$ of the quiver $Q$, we denote by $P_i = e_i \Lambda$ the associated projective module in $\text{mod} \Lambda$. Moreover, for any arrow $\theta$ from $j$ to $k$ in $Q$, we identify $\theta$ with the homomorphism $\theta: P_k \to P_j$ in $\text{mod} \Lambda$ given by the left multiplication by $\theta$.

Assume that $O(g)$ contains a family $O_1, \ldots, O_r$ of orbits of $|O_i| = 2$ and $m_{O_i} = 1$, for any $i \in \{1, \ldots, r\}$, and $\xi = (\xi_1, \ldots, \xi_r) \in O_1 \times \cdots \times O_r$. In Section 4 we defined the virtual mutation $\Lambda(\xi) = \Lambda(Q, f, m, c, \xi)$ of $\Lambda$ with respect to the sequence $\xi$ of virtual arrows. We use the notation established in Section 4. In particular, for each $i \in \{1, \ldots, r\}$, the triangulation quiver $(Q, f)$ contains a subquiver of the form

$$
\begin{array}{c}
\bullet & \overset{\alpha_i}{\rightarrow} & \xi_i & \overset{\beta_i}{\rightarrow} & \bullet \\
\beta_i & \overset{\gamma_i}{\leftarrow} & \beta_i & \overset{\delta_i}{\leftarrow} & \beta_i
\end{array}
$$
with $f$-orbits $(\alpha_i, \xi_i, \delta_i)$ and $(\beta_i, \nu_i, \mu_i)$, and $O_i = (\xi_i, \mu_i)$. We also note that in the Gabriel quiver $Q_\Lambda$ of $\Lambda$ the virtual arrows do not occur, and hence $Q_\Lambda$ has subquivers

$$
\begin{array}{c}
\alpha_i \\
a_i \\
\beta_i \\
c_i \\
\delta_i \\
d_i \\
b_i
\end{array}
$$

where $\alpha_i$ is the unique arrow of $Q_\Lambda$ with target $c_i$ and $\beta_i$ is the unique arrow of $Q_\Lambda$ with source $c_i$, for $i \in \{1, \ldots, r\}$. We consider the decomposition $\Lambda = P' \oplus P''$ in $\text{mod } \Lambda$, where

$$
P' = \bigoplus_{i=1}^r P_{c_i} \oplus \bigoplus_{x \in Q_0 \setminus \{c_1, \ldots, c_r\}} P_x.
$$

Moreover, we consider the following complexes in the homotopy category $K^b(P_\Lambda)$ of projective modules modules in $\text{mod } \Lambda$:

$$
T^\xi_{c_i} : 0 \longrightarrow P_{c_i} \longrightarrow P_{a_i} \longrightarrow 0
$$

concentrated in degrees 1 and 0, for all $i \in \{1, \ldots, r\}$. We also set

$$
T^\xi = \bigoplus_{x \in Q_0} T^\xi_x.
$$

**Lemma 6.1.** $T^\xi$ is a tilting complex in $K^b(P_\Lambda)$.

**Proof.** Let $\alpha : \bigoplus_{i=1}^r P_{c_i} \to \bigoplus_{i=1}^r P_{a_i}$ be the diagonal homomorphism given by the homomorphisms $\alpha_i : P_{c_i} \to P_{a_i}$. Since $\alpha_i$ is the unique arrow in $Q_\Lambda$ ending at $c_i$, for any $i \in \{1, \ldots, r\}$, we conclude that $\alpha$ is a left $\text{add}(P'')$-approximation of $P'$, so $T^\xi$ is indeed a tilting complex in $K^b(P_\Lambda)$, by Proposition 2.3.

We define $\Gamma = \Gamma(\xi) := \text{End}_{K^b(P_\Lambda)}(T^\xi)$. The following theorem is a direct consequence of Theorems 2.2, 2.3, 2.4 and 3.4.

**Theorem 6.2.** The following statements hold.

1. $\Gamma(\xi)$ is a finite-dimensional symmetric algebra.
(2) \(\Gamma(\xi)\) is a representation-infinite tame algebra.

(3) \(\Gamma(\xi)\) is a periodic algebra of period 4.

We shall prove now that the algebras \(\Lambda(\xi)\) and \(\Gamma(\xi)\) are isomorphic.

Observe first that \(\tilde{\alpha}_i : T_{a_i}^\xi \to T_{c_i}^\xi\) given by \(\text{id} : P_{a_i} \to P_{a_i}\), \(\tilde{\beta}_i : T_{c_i}^\xi \to T_{b_i}^\xi\) given by \(\nu_i \delta_i - c_{\gamma_i}^A : P_{a_i} \to P_{b_i}\), \(\tilde{\tau}_i : T_{b_i}^\xi \to T_{a_i}^\xi\) given by \(\alpha_i \beta_i : P_{b_i} \to P_{a_i}\), for all \(i \in \{1, \ldots, r\}\), and \(\tilde{\eta} : T_{t(\eta)}^\xi \to T_{s(\eta)}^\xi\) given by \(\eta : P_{t(\eta)} \to P_{s(\eta)}\), for any arrow \(\eta \in Q_1\) different from the arrows \(\alpha_i, \beta_i, i \in \{1, \ldots, r\}\).

Then applying covariant functor \(\text{Hom}_{K^b(P_\Lambda)}(T^\xi, -)\) to above morphisms, we obtain the following homomorphisms between indecomposable projective modules in \(\text{mod} \Gamma\):

\[
\alpha_i = \text{Hom}_{K^b(P_\Lambda)}(T^\xi, \tilde{\alpha}_i) : \tilde{P}_{a_i} \to \tilde{P}_{c_i},
\beta_i = \text{Hom}_{K^b(P_\Lambda)}(T^\xi, \tilde{\beta}_i) : \tilde{P}_{c_i} \to \tilde{P}_{b_i},
\tau_i = \text{Hom}_{K^b(P_\Lambda)}(T^\xi, \tilde{\tau}_i) : \tilde{P}_{b_i} \to \tilde{P}_{a_i},
\]

for all \(i \in \{1, \ldots, r\}\), and

\[
\eta = \text{Hom}_{K^b(P_\Lambda)}(T^\xi, \tilde{\eta}) : \tilde{P}_{t(\eta)} \to \tilde{P}_{s(\eta)},
\]

for any arrow \(\eta \in Q_1\) different from \(\alpha_i, \beta_i, i \in \{1, \ldots, r\}\). We observe also that these homomorphisms correspond to the arrows of the quiver \(Q(\xi)\) defining the algebra \(\Lambda(\xi)\).

We will identify the homomorphisms \(\alpha_i, \beta_i, \tau_i\) and \(\eta\) with the elements of \(e_{c_i}(\text{rad} \Gamma)e_{a_i}, e_{b_i}(\text{rad} \Gamma)e_{c_i}, e_{a_i}(\text{rad} \Gamma)e_{b_i}\) and \(e_{s(\eta)}(\text{rad} \Gamma)e_{t(\eta)}\), respectively, corresponding to them. We note the following obvious fact.

**Lemma 6.3.** The elements \(\alpha_i, \beta_i, \tau_i, i \in \{1, \ldots, r\}\), and \(\eta \in Q_1\), different from \(\alpha_i, \beta_i, i \in \{1, \ldots, r\}\), generate \(\text{rad} \Gamma\).
Proposition 6.4. The relations (1)-(5) defining the algebra $\Lambda(\xi)$ hold also in $\Gamma(\xi)$.

Proof. (1) We fix $i \in \{1, \ldots, r\}$ and prove that relations from (1) hold in three steps. First, we prove that the equality $\nu_i \delta_i = \beta_i \alpha_i + c^\bullet_{\alpha_i} A^\bullet_{\alpha_i}$ holds. Indeed, we have $\hat{\nu}_i = \tau_i$, so $c^\bullet_{\alpha_i} = c_{\gamma_i}$, and moreover

$$\mathcal{O}(\alpha_i) = (\alpha_i, \beta_i, \gamma_i \ldots g^{n_{n-3}}(\gamma_i)),$$

hence $\mathcal{O}^*(\tau_i) = (\tau_i, \gamma_i \ldots (g^*)^{-1}(\gamma_i))$. It is easy to see that $A^\bullet_{\gamma_i}$ is obtained from $A^\bullet_{\alpha_i}$ by replacing all paths of the form $\alpha_j \beta_j$ by $\tau_j$, $j \in \{1, \ldots, r\}$. Further, $\beta_i \alpha_i$ in $\Gamma(\xi)$ is identified with the map $T^\xi_{\alpha_i} \to T^\xi_{\beta_i}$ given by $\nu_i \delta_i - c_{\gamma_i} A^\gamma_{\gamma_i} : P_{\alpha_i} \to P_{\beta_i}$. It is now clear that the equality $\beta_i \alpha_i = \nu_j \delta_j - c^\bullet_{\gamma_i} A^\gamma_{\gamma_i}$ holds in $\Gamma(\xi)$, so we are done (note that $\tau_j$ as an element of $\Gamma(\xi)$ is identified with the map $T^\xi_{\beta_i} \to T^\xi_{\alpha_i}$ given by $\alpha_i \beta_j : P_{\beta_i} \to P_{\alpha_i}$).

Since $\alpha \tau_i$ in $\Gamma$ is identified with the map $\tilde{\alpha_i} \tilde{\tau_i} : T^\xi_{\alpha_i} \to T^\xi_{\beta_i}$ given by $\alpha_i \beta_j : P_{\beta_i} \to P_{\alpha_i}$ (in degree 0), we deduce that $\alpha_i \tau_i = 0$ in $\Gamma$, because $\tilde{\alpha_i} \tilde{\tau_i}$ is homotopic to zero, as it is given by the homomorphism which factors through $\alpha_i$. The equality $\tau_i \beta_i = 0$ holds in $\Gamma$, because $\tilde{\tau_i} \tilde{\beta_i}$ is given by $\alpha_i \beta_i (\nu_i \delta_i - c_{\gamma_i} A^\gamma_{\gamma_i}) : P_{\alpha_i} \to P_{\beta_i}$ and we have also the following equalities in $\Lambda$

$$\alpha_i \beta_i \nu_i \delta_i = \alpha_i \xi_i \delta_i = c_{\beta_i} \alpha_i A_{\beta_i} = c_{\beta_i} B_{\alpha_i} = c_{\gamma_i} \alpha_i \beta_i A^\gamma_{\gamma_i}.$$

(2) Let $\eta$ be an arrow in $Q(\xi)_{\lambda}$ different from $\nu_i$. Then $\eta = \tau_i$ or $\eta = \delta_i$ (for $i \in \{1, \ldots, r\}$ or $\eta$ is an arrow of $Q_1$ different from $\alpha_i, \beta_i, \nu_i, \delta_i$, $i \in \{1, \ldots, r\}$.

If $\eta = \tau_i$, $i \in \{1, \ldots, r\}$, then $\eta f^*(\eta) = \tau_i \nu_i$ and we have the equalities in $\Lambda$

$$\alpha_i \beta_i \nu_i = \alpha_i \xi_i = \alpha_i f(\alpha_i) = c_{\alpha_i} A_{\alpha_i} = c_{\rho_i} A_{\rho_i}.$$

Moreover, $\rho_i = \tilde{\tau_i}$, $c_{\rho_i} = c^\bullet_{\rho_i}$, and $A^\bullet_{\rho_i}$ is obtained from $A^\bullet_{\rho_i}$ by replacing all paths $\alpha_i \beta_j$ by $\tau_j$, $j \in \{1, \ldots, r\}$. Since $\tau_i$ is identified with $\alpha_i \beta_i$ in $\Lambda$, we conclude that the required equality $\eta f^*(\eta) = c^\bullet_{\rho_i} A^\rho_{\rho_i}$ holds in $\Gamma$.

For $\eta = \delta_i$, $i \in \{1, \ldots, r\}$, we have $\eta f^*(\eta) = \delta_i \tau_i$ and the following equalities hold in $\Lambda$

$$\delta_i \alpha_i \beta_i = \mu_i \beta_i = \mu_i f(\mu_i) = c_{\mu_i} A_{\mu_i} = c_{\delta_i} \delta_i,$$

because $\tilde{\mu_i} = \delta_i$. Observe also that $\delta_i = \delta_i$, and $A^*_{\delta_i}$ is obtained from $A^\bullet_{\delta_i}$ by replacing all paths $\alpha_i \beta_j$ by $\tau_j$, for $j \in \{1, \ldots, r\}$. Hence the required equality $\eta f^*(\eta) = c^\bullet_{\delta_i} A^\rho_{\delta_i}$ holds in $\Gamma$, because $\tau_i$ is given by $\alpha_i \beta_i$. 36
Finally, assume that \( \eta \) is in \( Q_1 \) and different from \( \alpha_i, \beta_i, \nu_i, \delta_i \). Then also \( f(\eta) \in Q_1 \) is different from all these arrows, and the equality \( \eta f(\eta) = c_{\eta}A_{\eta} \) holds in \( \Lambda \). Moreover, if \( \eta = \rho_i \), \( i \in \{1, \ldots, r\} \), then \( \tilde{\eta} = \tau_i \neq \tilde{\eta} = \alpha_i \), and otherwise \( \tilde{\eta} = \tilde{\eta} \), and in both cases we have \( f^*(\eta) = f(\eta) \). Note that then \( A_{\eta}^* = A_{\eta} \) or \( A_{\eta}^* \) is obtained from \( A_{\eta} \) by similar replacement as above, so the required equality \( \eta f^*(\eta) = c_{\eta}A_{\eta}^* \) holds.

(3) Let \( \eta = \delta_i \) for some \( i \in \{1, \ldots, r\} \). Then we have \( f^*(\eta) = \tau_i \), and \( g^*(\tau_i) = \gamma_i \) or \( g^*(\tau_i) = \tau_j \) for some \( j \in \{1, \ldots, r\} \). If \( g^*(\tau_i) = \gamma_i \), then we have the equalities

\[
\eta f^*(\eta) g^*(f^*(\eta)) = \delta_i \tau_i \gamma_i = \delta_i \alpha_i \beta_i \gamma_i = \mu_i \beta_i \gamma_i = \mu_i f(\mu_i) g(f(\mu_i)) = 0.
\]

For \( g^*(\tau_i) = \tau_j \), the following equalities hold

\[
\eta f^*(\eta) g^*(f^*(\eta)) = \delta_i \tau_i \tau_j = \delta_i \alpha_i \beta_i \gamma_i = \mu_i \beta_i \gamma_j = \mu_i f(\mu_i) g(f(\mu_i)) \beta_j = 0.
\]

Therefore \( \eta f^*(\eta) g^*(f^*(\eta)) = 0 \) holds for \( \eta = \delta_i \).

Now assume \( \eta \) is an arrow in \( Q_1 \) such that \( f^2(\eta) \) is not virtual and \( f(\eta) \) is not virtual if \( m_{\eta} = 1 \) and \( n_{\eta} = 3 \). Moreover, let \( \eta \) be different from \( \nu_i, \tau_i, i \in \{1, \ldots, r\} \). Clearly, then we have \( f^*(\eta) = f(\eta) \). If also \( g^*(f(\eta)) = g(f(\eta)) \), then \( \eta f^*(\eta) g^*(f^*(\eta)) = \eta f(\eta) g(f(\eta)) = 0 \), because it is one of relations defining \( \Lambda \). Assume that \( g^*(f(\eta)) \neq g(f(\eta)) \). This is the case only when \( g^*(f(\eta)) = \tau_i \) for some \( i \in \{1, \ldots, r\} \), and hence \( f(\eta) = f^{-1}(\alpha_i) = \sigma_i \) and \( \eta = f^{-1}(\sigma_i) = f(\rho_i) \). But then we have

\[
\eta f^*(\eta) g^*(f^*(\eta)) = \eta \sigma_i \tau_i = \eta \sigma_i \alpha_i \beta_i = \eta f(\eta) g(f(\eta)) \beta_i = 0,
\]

because of the relations defining \( \Lambda \) and restrictions imposed on \( \eta \).

(4) Assume \( \eta \) is an arrow in \( Q(\xi)_1^* \) different from \( (g^*)^{-1}(\nu_i), \nu_i \), and \( \tau_i \) with \( m_{\nu_i} n_{\nu_i} = 3 \), for \( i \in \{1, \ldots, r\} \), and \( \eta \in Q_1 \) such that \( f(\eta) \) is virtual or \( f^2(\eta) \) is virtual with \( m_{f(\eta)} = 1 \) and \( n_{f(\eta)} = 3 \). In particular, if \( g^*(\eta) = g(\eta) \) and \( f^*(g^*(\eta)) = f(g(\eta)) \), then \( \eta g^*(\eta) f^*(g^*(\eta)) = \eta g(\eta) f(g(\eta)) = 0 \), by the restrictions imposed on \( \eta \) and the zero relations defining \( \Lambda \). If \( g^*(\eta) \neq g(\eta) \), that is \( g^*(\eta) = \tau_i \) for some \( i \in \{1, \ldots, r\} \), then

\[
\eta g^*(\eta) f^*(g^*(\eta)) = \eta \tau_i \nu_i = \eta \nu_i \beta_i \nu_i = \eta \nu_i \xi_i = \eta g(\eta) f(g(\eta)) = 0.
\]

Assume now that \( \eta = \tau_i \) for some \( i \in \{1, \ldots, r\} \). Then \( g^*(\eta) = \gamma_i \) or \( g^*(\eta) = \tau_j \) for some \( j \in \{1, \ldots, r\} \). Suppose \( g^*(\eta) = \gamma_i \). Then we have

\[
\eta g^*(\eta) f^*(g^*(\eta)) = \tau_i \gamma_i f(\gamma_i) = \alpha_i \beta_i \gamma_i f(\gamma_i) = 0,
\]

because \( \beta_i \gamma_i f(\gamma_i) = \beta_i g(\beta_i) f(g(\beta_i)) = 0 \), due to assumptions imposed on \( \tau_i \). If \( g^*(\tau_i) = \tau_j \), then

\[
\eta g^*(\eta) f^*(g^*(\eta)) = \tau_i \tau_j \nu_j = \alpha_i \beta_i \alpha_j \nu_j = \alpha_i \beta_i \alpha_j \xi_j = \epsilon_{\nu_i} \alpha_i \beta_i A_{\nu_i} = \epsilon_{\alpha_i} A_{\nu_i}.
\]
If \( m_{\alpha_i} \) belongs to \( \text{soc} \Lambda \).

(5) Fix \( i \in \{1, \ldots, r\} \). We first prove that \( A_{\alpha_i}^* \nu_i = 0 \) in \( \Gamma \). We recall that \( A_{\alpha_i}^* \nu_i = \nu_i C_{\alpha_i}^* \cdot \delta_i \), where \( A_{\alpha_i}^* \nu_i = \nu_i C_{\alpha_i}^* \cdot \alpha_i A_{\alpha_i}^* \). Hence \( A_{\alpha_i}^* \nu_i = \nu_i C_{\alpha_i}^* \cdot \alpha_i A_{\alpha_i}^* \).

Further, we have in \( \Lambda \) the equality \( A_{\alpha_i}^* \nu_i = \nu_i C_{\alpha_i}^* \cdot \alpha_i A_{\alpha_i}^* \). Moreover, we have in \( \Lambda \) the equality \( A_{\alpha_i}^* \nu_i = \nu_i C_{\alpha_i}^* \cdot \alpha_i A_{\alpha_i}^* \).

Moreover, we have \( \alpha_i : T_{\alpha_i}^\xi \to T_{\alpha_i}^\xi \) is given by identity on \( P_{\alpha_i} \) (in degree 0), so the morphism \( \tilde{\alpha}_i : \tilde{T}_{\alpha_i}^\xi : T_{\alpha_i}^\xi \to T_{\alpha_i}^\xi \) is zero in \( K^\theta(P_{\alpha_i}) \), since \( A_{\alpha_i}^* \) factors through \( \alpha_i \).

Therefore, the required equality \( A_{\alpha_i}^* \nu_i = 0 \) holds in \( \Gamma \).

In the next step we show that \( A_{\alpha_i}^* \beta_i = 0 \) in \( \Gamma \). Recall that \( \tilde{\beta}_i : T_{\beta_i}^\xi \to T_{\beta_i}^\xi \) is given by \( \nu_i \delta_i - c_{\gamma_i} A_{\gamma_i}^* \cdot P_{\beta_i} \rightarrow P_{\beta_i} \). We also have in \( \Lambda \) the following equalities \( A_{\gamma_i} \nu_i \delta_i = \beta_i \delta_i = 0 \), because element \( \beta_i \delta_i \) is in the right socle of \( \Lambda \). We claim that also \( A_{\gamma_i}^* \nu_i = 0 \) holds in \( \Lambda \). We will proceed in three cases.

(a) Assume that \( m_{\nu_i} n_{\nu_i} = 3 \). Then \( A_{\gamma_i} = \delta_i \gamma_i = \delta_i \gamma_i f(g(\delta_i)) = 0 \), since \( f(\delta_i) = \alpha_i \) is not virtual. Clearly then \( A_{\gamma_i}^* \nu_i = 0 \).

(b) Next, assume that \( m_{\nu_i} n_{\nu_i} \geq 4 \) with \( \omega_i \) not virtual. Hence we have \( \mathcal{O}(\delta_i) = (\delta_i \rho_i \ldots \omega_i \nu_i) \) and \( A_{\gamma_i} = \delta_i \rho_i \ldots \omega_i \). In this case \( g^{-1}(\omega_i) \omega_i \gamma_i = g^{-1}(\omega_i) \omega_i f(\omega_i) = 0 \) holds in \( \Lambda \), because \( f(g^{-1}(\omega_i)) = \omega_i \) is assumed not virtual, and consequently, \( A_{\gamma_i}^* \nu_i = 0 \).

(c) Finally, let \( m_{\nu_i} n_{\nu_i} \geq 4 \) and \( \omega_i \) is virtual. Then we have in \( (Q, f) \) a subquiver of the form

\[
\begin{array}{c}
\alpha_i \quad \beta_i \quad \gamma_i \\
\xi_i \quad \nu_i \quad \psi_i \\
\delta_i \quad \omega_i \quad \phi_i \\
\end{array}
\]

Moreover, we have \( A_{\gamma_i}^* = \gamma_i \phi_i \ldots \sigma_i \) and \( A_{\beta_i} = \delta_i \ldots \psi_i \omega_i \). Observe also that \( \psi_i \omega_i \gamma_i \phi_i = c_{\gamma_i} \psi_i \xi_i \phi_i = c_{\gamma_i} c_{\psi_i} f(\psi_i) f^2(\psi_i) = c_{\gamma_i} c_{\psi_i} B_{\psi_i} \), by Proposition 3.6. Hence, if \( m_{\gamma_i} n_{\gamma_i} \geq 5 \) or \( m_{\nu_i} n_{\nu_i} \geq 5 \), we obtain \( A_{\beta_i} A_{\gamma_i}^* = 0 \), since \( B_{\psi_i} \) is an element of socle. Therefore, it remains to consider case \( m_{\gamma_i} n_{\gamma_i} = 4 = m_{\nu_i} n_{\nu_i} \). Clearly, then \( m_{\gamma_i} = m_{\nu_i} = 1, n_{\gamma_i} = n_{\nu_i} = 4 \) and \( z_i = a_i \), because we have two pairs of virtual arrows. But this implies \( A_{\gamma_i} = \delta_i \gamma_i \phi_i \) and \( A_{\gamma_i}^* = \gamma_i \phi_i \), so we conclude that \( A_{\beta_i} A_{\gamma_i}^* = \delta_i \gamma_i \phi_i \gamma_i \phi_i = c_{\gamma_i} c_{\phi_i} A_{\gamma_i}^* B_{\psi_i} = 0 \).
Theorem 6.5. The algebras $\Lambda(\xi)$ and $\Gamma(\xi)$ are isomorphic.

Proof. It follows from Theorem 6.2 that $\Gamma(\xi)$ is a finite-dimensional symmetric algebra. Moreover, by Lemma 6.3 and Proposition 6.4 we conclude that the Gabriel quivers $Q_{\Lambda(\xi)}$ and $Q_{\Gamma(\xi)}$ coincide, and the relations defining $\Lambda(\xi)$ are satisfied in $\Gamma(\xi)$. Hence $\Gamma(\xi)$ is an epimorphic image of the algebra $\Lambda(\xi)$. Finally, because $\Gamma(\xi)$ is symmetric, we may then conclude, as in Proposition 4.9, that $\Gamma(\xi)$ has the same basis as $\Lambda(\xi)$, so $\dim_K \Lambda(\xi) = \dim_K \Gamma(\xi)$. Therefore, $\Lambda(\xi)$ and $\Gamma(\xi)$ are indeed isomorphic as $K$-algebras.

Concluding, the proof of Main Theorem is now complete.

For each vertex $x \in Q(\xi)_0 = Q_0$, we denote by $P_x \Lambda(\xi)$ the indecomposable projective module $e_x \Lambda(\xi)$ in mod $\Lambda(\xi)$ at $x$. Consider the following complexes in the homotopy category $K^b(P_{\Lambda(\xi)})$ of projective modules in mod $\Lambda(\xi)$:

$$
\hat{T}_x^\xi : 0 \longrightarrow P_x^\xi \longrightarrow 0
$$
concentrated in degree 0, for all vertices $x \neq c_1, \ldots, c_r$, and

$$
\hat{T}_{c_i}^\xi : 0 \longrightarrow P_{c_i}^\xi \longrightarrow P_{b_i}^\xi \longrightarrow 0
$$
concentrated in degrees 1 and 0, for all $i \in \{1, \ldots, r\}$. We set

$$
\hat{T}^\xi := \bigoplus_{x \in Q(\xi)_0} \hat{T}_x^\xi.
$$

Moreover, let

$$
\hat{P}' = \bigoplus_{i=1}^r P_{c_i}^\xi \quad \text{and} \quad \hat{P}'' = \bigoplus_{x \in Q(\xi)_0 \setminus \{c_1, \ldots, c_r\}} P_x^\xi.
$$

Let $\beta : \bigoplus_{i=1}^r P_{c_i}^\xi \rightarrow \bigoplus_{i=1}^r P_{b_i}^\xi$ be the diagonal homomorphism given by the homomorphisms $\beta_i : P_{c_i}^\xi \rightarrow P_{b_i}^\xi$. Then $\beta$ is a left add($\hat{P}'$)-approximation of $\hat{P}'$ in mod $\Lambda(\xi)$, since $\beta_i$ is the unique arrow in $Q(\xi)$ ending at $c_i$, for any $i \in \{1, \ldots, r\}$. Hence, applying Proposition 2.5, we obtain that $\hat{T}^\xi$ is a tilting complex in $K^b(P_{\Lambda(\xi)})$. We define

$$
\hat{\Lambda}(\xi) = \text{End}_{K^b(P_{\Lambda(\xi)})}(\hat{T}^\xi).
$$

Theorem 6.6. The algebras $\Lambda$ and $\hat{\Lambda}(\xi)$ are isomorphic.
Proof. We observe that \( \hat{P}_x = \text{Hom}_{K^b(P_{\Lambda(\xi)})}(\hat{T}_x, \hat{T}_y), \ x \in Q_0 \), form a complete family of pairwise non-isomorphic indecomposable projective modules in \( \text{mod} \Lambda(\xi) \). We define the following morphisms between indecomposable summands of \( \hat{T}_x \) in \( K^b(P_{\Lambda(\xi)}) \):

\[
\hat{\alpha}_i : \hat{T}_{a_i} \to \hat{T}_{b_i}, \quad \text{given by} \ \tau_i : P_{a_i} \to P_{b_i}, \\
\hat{\beta}_i : \hat{T}_{b_i} \to \hat{T}_{c_i}, \quad \text{given by} \ id : P_{b_i} \to P_{b_i}, \\
\hat{\epsilon}_i : \hat{T}_{a_i} \to \hat{T}_{b_i}, \quad \text{given by} \ \beta_i \alpha_i : P_{a_i} \to P_{b_i},
\]

for all \( i \in \{1, \ldots, r\} \), and

\[
\hat{\eta} : \hat{T}_{\hat{t}(\eta)} \to \hat{T}_{s(\eta)}, \quad \text{given by} \ \eta : P_{t(\eta)} \to P_{s(\eta)},
\]

for any arrow \( \eta \in Q(\xi)_1 \) different from the arrows \( \alpha_i, \beta_i, \ i \in \{1, \ldots, r\} \). We obtain then the homomorphisms between indecomposable projective modules in \( \text{mod} \hat{\Lambda}(\xi) \):

\[
\alpha_i = \text{Hom}_{K^b(P_{\Lambda(\xi)})}(\hat{T}_x, \hat{\alpha}_i) : \hat{P}_{a_i} \to \hat{P}_{b_i}, \\
\beta_i = \text{Hom}_{K^b(P_{\Lambda(\xi)})}(\hat{T}_x, \hat{\beta}_i) : \hat{P}_{b_i} \to \hat{P}_{c_i}, \\
\epsilon_i = \text{Hom}_{K^b(P_{\Lambda(\xi)})}(\hat{T}_x, \hat{\epsilon}_i) : \hat{P}_{a_i} \to \hat{P}_{b_i},
\]

for all \( i \in \{1, \ldots, r\} \), and

\[
\eta = \text{Hom}_{K^b(P_{\Lambda(\xi)})}(\hat{T}_x, \hat{\eta}) : \hat{P}_{t(\eta)} \to \hat{P}_{s(\eta)},
\]

for any arrow \( \eta \in Q(\xi)_1 \) different from \( \alpha_i, \beta_i, \ i \in \{1, \ldots, r\} \). Recall that we have in \( \Lambda(\xi) \) the relations:

\[
\nu_i \delta_i = \beta_i \alpha_i + c_{\beta_i} A^*_{\alpha_i}, \alpha_i \tau_i = 0, \tau_i \beta_i = 0, \\
\tau_i \nu_i = c_{\epsilon_i} A^*_{\delta_i}, \delta_i \tau_i = c_{\epsilon_i} A^*_{\delta_i},
\]

for all \( i \in \{1, \ldots, r\} \). Moreover, we have in \( K^b(P_{\Lambda(\xi)}) \) the equalities \( \hat{\tau}_i = \hat{\alpha}_i \hat{\beta}_i, \)

\( \hat{\epsilon}_i \hat{\alpha}_i = 0 \) and \( \hat{\beta}_i \hat{\epsilon}_i = 0 \), for any \( i \in \{1, \ldots, r\} \). Note that \( \epsilon_i \) is not irreducible, so there is no arrow in \( Q_{\Lambda(\xi)} \) corresponding to \( \epsilon_i \), for any \( i \in \{1, \ldots, r\} \).

Consider the quiver \( \hat{Q}(\xi) = (\hat{Q}(\xi)_0, \hat{Q}(\xi)_1, s, t) \) defined as follows. We take \( \hat{Q}(\xi)_0 = Q(\xi)_0 = Q_0 \), and the set \( \hat{Q}(\xi)_1 \) of arrows is obtained from the set \( Q(\xi)_1 \) of arrows of \( Q(\xi) \) by the following operations:

- replacing the arrows \( c_i \xleftarrow{\alpha_i} a_i \) and \( b_i \xleftarrow{\beta_i} c_i \) by the arrows \( a_i \xleftarrow{\alpha_i} c_i \)
  and \( c_i \xleftarrow{\beta_i} b_i \),
removing the arrows $a_i \xrightarrow{\tau_i} b_i$, for all $i \in \{1, \ldots, r\}$.

We observe that the quiver $\hat{Q}(\xi)$ is the quiver obtained from the triangulation quiver $(Q, f)$, defining the algebra $\Lambda = \Lambda(Q, f, m_\bullet, c_\bullet)$, by removing the virtual arrows $\xi_i$ and $\mu_i$, for $i \in \{1, \ldots, r\}$. Moreover, it follows from the above relations that the radical of $\hat{\Lambda}(\xi)$ is generated by the arrows of $\hat{Q}(\xi)$. We also mention that for any arrow $\eta \in \hat{Q}(\xi)$ different from $\alpha_i, \beta_i$, $i \in \{1, \ldots, r\}$, we have in $\hat{Q}(\xi)$ the path $\hat{A}_\eta$ obtained from the path $A^*_\eta$ in $Q(\xi^*)$ by replacing every arrow $\tau_i$ by the path $\alpha_i \beta_i$, and hence $\hat{A}_\eta$ coincides with the path $A_\eta$ in $Q$. Clearly, we have also in $\hat{Q}(\xi)$ the paths $\hat{A}_{\alpha_i} = A_{\alpha_i}$ and $\hat{A}_{\beta_i} = A_{\beta_i}$, for $i \in \{1, \ldots, r\}$.

Finally, note that the following relations in $\hat{\Lambda}(\xi)$ are consequences of the relations in $\Lambda(\xi)$ presented above:

$$
\nu_i \delta_i \alpha_i = \epsilon_i \alpha_i + c_{\tilde{\alpha}_i} A^*_\tilde{\alpha}_i \alpha_i = c_{\tilde{\alpha}_i} \hat{A}_{\tilde{\alpha}_i},
$$

$$
\beta_i \nu_i \delta_i = \beta_i \epsilon_i + c_{\tilde{\beta}_i} \beta_i A^*_\tilde{\beta}_i = c_{\tilde{\beta}_i} \hat{A}_{\tilde{\beta}_i},
$$

$$
\delta_i \alpha_i \beta_i = \delta_i \tau_i = c_{\tilde{\delta}_i} A^*_\tilde{\delta}_i = c_{\tilde{\delta}_i} \hat{A}_{\tilde{\delta}_i},
$$

$$
\alpha_i \beta_i \nu_i = \tau_i \nu_i = c_{\tilde{\alpha}_i} A^*_\tilde{\alpha}_i = c_{\tilde{\alpha}_i} \hat{A}_{\tilde{\alpha}_i}.
$$

Then we may conclude that the algebra $\hat{\Lambda}(\xi)$ is given by the quiver $\hat{Q}(\xi)$ and the same relations as the algebra $\Lambda$. Therefore, the algebras $\hat{\Lambda}(\xi)$ and $\Lambda$ are isomorphic.

7 Virtual edge deformations of weighted surface algebras

In this section, by a surface we mean a connected, compact, two-dimensional orientable real manifold, with or without boundary. It is well known that every surface $S$ admits an additional structure of a finite two-dimensional triangular cell complex and hence a triangulation, by the deep Triangulation Theorem (see for example [6, Section 2.3]).

For a positive natural number $n$, we denote by $D^n$ the unit disc in the $n$-dimensional Euclidean space $\mathbb{R}^n$, formed by all points of distance $\leq 1$ from the origin. Then the boundary $\partial D^n$ of $D^n$ is the unit sphere $S^{n-1}$ in $\mathbb{R}^n$ formed by all points of distance 1 from the origin. Further, by an $n$-cell we mean a topological space homeomorphic to the open disc $O^n = \text{int}(D^n) = D^n \setminus \partial D^n$. In particular, $S^0 = \partial D^1$ consists of two points and $\partial D^0 = D^0$ is a single
point. We refer to [30, Appendix] for some basic topological facts about cell complexes.

Let \( S \) be a surface. By a finite two-dimensional triangular cell complex structure on \( S \) we mean a finite family of maps \( \phi_i^n : D_i^n \to S \), with \( n \in \{0,1,2\} \) and \( D_i^n = D^n \), satisfying the following conditions:

1. Each \( \phi_i^n \) restricts to a homeomorphism \( O^n \to \phi_i^n(O^n) \) and the \( n \)-cells \( e_i^n := \phi_i^n(O^n) \) of \( S \) are pairwise disjoint and their union is \( S \).

2. For each two-dimensional cell \( e_2^i \), \( \phi_2^i(\partial D_2^i) \) is the union of \( k \) 1-cells and \( m \) 0-cells, where \( k \in \{2,3\} \) and \( m \in \{1,2,3\} \), and different from \( \bullet \square \bullet \).

Then the closures \( \phi_2^i(D_2^i) \) of all 2-cells \( e_2^i \) are called triangles of \( S \), and the closures \( \phi_1^i(D_1^i) \) of all 1-cells \( e_1^i \) are called edges of \( S \). The collection of all triangles is said to be a triangulation of \( S \). We assume that such a triangulation \( T \) of \( S \) has at least two different edges, so then \( T \) is a finite collection of triangles of the form

\[
\begin{array}{c}
\text{or}
\end{array}
\]

\[
\begin{array}{c}
\text{a, b, c pairwise different} \\
\text{a, b different (self-folded triangle)}
\end{array}
\]

such that every edge is either the edge of exactly two triangles, is the self-folded edge, or lies on the boundary of \( S \). We note that a given surface \( S \) admits many finite two-dimensional triangular cell complex structures, and hence triangulations.

In this section, by a directed triangulated surface we mean a pair \((S, \overrightarrow{T})\), where \( S \) is a surface, \( T \) a triangulation of \( S \), and \( \overrightarrow{T} \) a coherent orientation of triangles in \( T \). We may assume that all triangles of \( T \) have clockwise orientation in \( \overrightarrow{T} \). Then the associated triangulation quiver \((Q(S, \overrightarrow{T}), f)\) is defined as follows:

- The set \( Q(S, \overrightarrow{T})_0 \) of vertices consists of the edges of \( T \).
- The set \( Q(S, \overrightarrow{T})_1 \) of arrows and permutation \( f : Q(S, \overrightarrow{T})_1 \to Q(S, \overrightarrow{T})_1 \) are given by the orientation of triangles in \( T \), namely

\[
\begin{array}{c}
(a) \\
\text{a} \alpha \to \beta \gamma \sset{100}{100}{100} \text{b} \\
\beta \gamma \sset{100}{100}{100} \text{c}
\end{array}
\]

\[
f(\alpha) = \beta, f(\beta) = \gamma, f(\gamma) = \alpha
\]
for any oriented triangle \( \Delta = (abc) \) in \( \overrightarrow{T} \), and \( a, b, c \) pairwise different edges,

\[(b) \quad \alpha \xrightarrow{\beta} a \xrightarrow{\gamma} b \]  
\[f(\alpha) = \beta, f(\beta) = \gamma, f(\gamma) = \alpha \]

for a self-folded triangle \( \Delta = (aab) \) in \( \overrightarrow{T} \) (\( a, b \) different edges), and

\[(c) \quad \alpha \xrightarrow{a} \beta \xrightarrow{\gamma} a \]  
\[f(\alpha) = \alpha, \]

for any boundary edge \( a \) of \( T \).

**Definition 7.1.** Let \((S, \overrightarrow{T})\) be a directed triangulated surface and \( I \) a non-empty set of edges in \( T \) (possibly all the edges). The **blow-up** of \((S, \overrightarrow{T})\) at \( I \) is the directed triangulated surface \((S, \overrightarrow{T}_I)\) obtained from \((S, \overrightarrow{T})\) by replacing each edge \( i \in I \) by a 2-triangle disc

![Diagram](image)

with \((a_i, c_i, d_i)\) and \((c_i, b_i, d_i)\) in \( \overrightarrow{T}_I \). We note that since we require \( \overrightarrow{T}_I \) to be a coherent orientation of triangles in \( T_I \), this blow-up is uniquely determined by \( I \) (and the coherent orientation \( \overrightarrow{T} \) of triangles in \( T \)).

Let \((S, \overrightarrow{T}_I)\) be the blow-up of a directed triangulated surface \((S, \overrightarrow{T})\) at a set \( I \) of edges of \( T \). Let \((Q, f)\) denote the triangulation quiver \((Q(S, \overrightarrow{T}), f)\) and \((Q^I, f^I)\) the triangulation quiver associated to \((S, \overrightarrow{T}_I)\). We write \( g : Q_1 \to Q_1 \) for the permutation associated to \( f \) and \( g^I : Q^I_1 \to Q^I_1 \) for the permutation associated to \( f^I \). For an arrow \( \eta \in Q^I_1 \), we denote by \( \mathcal{O}(\eta) \) the \( g^I \)-orbit of \( \eta \) in \( Q^I_1 \). Moreover, let \( \mathcal{O}(g^I) \) be the set of all \( g^I \)-orbits in \( Q^I_1 \). For each edge \( i \in I \), we abbreviate \((S, \overrightarrow{T}_i) = (S, \overrightarrow{T}_{\{i\}}), f_i = f^{(i)} \) and \( g_i = g^{(i)} \). We mention that \((S, \overrightarrow{T}_I)\) (respectively, \((Q^I, f^I)\)) is obtained from \((S, \overrightarrow{T})\) (respectively, from \((Q, f)\)) by an iterated application of single blow-ups at all edges from \( I \) (in arbitrary fixed order).

We will illustrate the changes for single blow-up at an edge \( i \) of \( T \).
(1) Assume $i$ is a common edge of two triangles in $T$

with $(x \ i \ y)$ and $(i \ z \ t)$ in $\vec{T}$, where possibly $x = y$ or $z = t$. Then $(Q, f)$ has arrows

with $f(\sigma_i) = \rho_i$, $f(\omega_i) = \gamma_i$, $g(\sigma_i) = \gamma_i$ and $g(\omega_i) = \rho_i$. Then the blow-up $(S, \vec{T}_i)$ contains triangles

with $(x \ a_i \ y), (a_i \ c_i \ d_i), (c_i \ b_i \ d_i), (b_i \ z \ t) \in \vec{T}_i$, so $(Q(S, \vec{T}_i), f_i)$ contains a subquiver of the form

with $f$-orbits $(\alpha_i \ \xi_i \ \delta_i), (\beta_i \ \nu_i \ \mu_i)$, $f(\sigma_i) = \rho_i$, $f(\omega_i) = \gamma_i$, and $g(\sigma_i) = \alpha_i$, $g(\alpha_i) = \beta_i$, $g(\beta_i) = \gamma_i$, $g(\omega_i) = \nu_i$, $g(\nu_i) = \delta_i$, $g(\delta_i) = \rho_i$, $g(\xi_i) = \mu_i$ and $g(\mu_i) = \xi_i$. 

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(2) Assume that $i$ is a self-folded edge of the following self-folded triangle in $T$

\[
\begin{array}{c}
\includegraphics{triangle}
\end{array}
\]

with $(i \, i \, x) \in \overrightarrow{T}$, and hence $(Q, f)$ admits a subquiver

\[
\begin{array}{c}
\includegraphics{subquiver}
\end{array}
\]

with $f$-orbit $(\rho_i \, \gamma_i \, \sigma_i)$, and $g(\rho_i) = \rho_i$, $g(\sigma_i) = \gamma_i$.

Then the blow-up $(S, \overrightarrow{T_i})$ contains the triangles

\[
\begin{array}{c}
\includegraphics{blow-up}
\end{array}
\]

with $(x \, b_i \, a_i), (a_i \, c_i \, d_i), (c_i \, b_i \, d_i)$ in $\overrightarrow{T_i}$, and so $(Q(S, \overrightarrow{T_i}), f_i)$ has a subquiver of the form

\[
\begin{array}{c}
\includegraphics{subquiver_blow-up}
\end{array}
\]

where $f_i$-orbits are $(\alpha_i \, \xi_i \, \delta_i), (\beta_i \, \nu_i \, \mu_i), (\sigma_i \, \rho_i \, \gamma_i)$. Moreover, we have $g_i$-orbits $(\xi_i \, \mu_i), (\alpha_i \, \beta_i \, \gamma_i)$ and $(\delta_i \, \rho_i \ldots \omega_i \, \nu_i)$.
(3) Assume $i$ is a border edge of $T$, so we have in $T$ a triangle

with $(x \ i \ y)$ in $\overrightarrow{T}$ and possibly $x = y$. Then $(Q, f)$ contains a subquiver

![Diagram](https://via.placeholder.com/150)

with $f(\gamma_i) = \gamma_i$, $f(\sigma_i) = \rho_i$, $g(\sigma_i) = \gamma_i$, $g(\gamma_i) = \rho_i$.

The blow-up $(S, \overrightarrow{T_i})$ is then containing the triangles

![Diagram](https://via.placeholder.com/150)

with $(x \ a_i \ y)$, $(a_i \ c_i \ b_i)$, $(c_i \ b_i \ d_i)$ in $\overrightarrow{T_i}$, and $b_i$ the boundary edge. In this case $(Q(S, \overrightarrow{T_i}), f_i)$ contains a subquiver of the form

![Diagram](https://via.placeholder.com/150)

with $f_i$-orbits $(\alpha_i \ \xi_i \ \delta_i)$, $(\beta_i \ \nu_i \ \mu_i)$, $(\gamma_i)$, $f(\sigma_i) = \rho_i$, and the $g_i$-orbits $(\xi_i \ \mu_i)$ and $(\alpha_i \ \beta_i \ \gamma_i \ \nu_i \ \delta_i \ \rho_i \ldots \sigma_i)$.

Therefore, for the single blow-up $(S, \overrightarrow{T_i})$ of $(S, \overrightarrow{T})$ at an edge $i$, the set $\mathcal{O}(g_i)$ of the $g_i$-orbits in $Q(S, \overrightarrow{T_i})$ has the following structure: we have the new orbit $\mathcal{O}_i = (\xi_i \ \mu_i)$ of length 2, one or two orbits in $\mathcal{O}(g)$ are extended by the arrows $\alpha_i, \beta_i, \nu_i, \delta_i$, and the remaining orbits in $\mathcal{O}(g)$ become orbits in $\mathcal{O}(g_i)$.
In general, for an arbitrary non-empty set $I$ of edges of $T$, the set $O(g^I)$ of all $g^I$-orbits in $Q^I_1 = Q(S, \overrightarrow{T_I})_1$ consists of the new orbits $O_i = (\xi_i, \mu_i)$, $i \in I$, of length 2, and extensions $O^I(\eta)$ of the $g$-orbits $O(\eta)$, $\eta \in Q_1$. Hence, for any choice of a weight function $m_\bullet : O(g) \to \mathbb{N}^*$ and a parameter function $c_\bullet : O(g) \to K^*$, we may consider the weight function $m^I_\bullet : O(g^I) \to \mathbb{N}^*$ and the parameter function $c^I_\bullet : O(g^I) \to K^*$ by setting

\begin{align*}
\bullet \quad m^I_{O_i} &= 1 \quad \text{and} \quad c^I_{O_i} = 1, \text{ for any } i \in I, \text{ and} \\
\bullet \quad m^I_{O^I(\eta)} &= m_{O(\eta)} \quad \text{and} \quad c^I_{O^I(\eta)} = c_{O(\eta)}, \text{ for any arrow } \eta \in Q_1.
\end{align*}

**Definition 7.2.** The weighted surface algebra $\Lambda(S, \overrightarrow{T_I}, m^I_\bullet, c^I_\bullet)$ is said to be the blow-up of the weighted surface algebra $\Lambda(S, \overrightarrow{T}, m_\bullet, c_\bullet)$ at the set $I$ of edges of $T$.

Let $\Lambda_I = \Lambda(S, \overrightarrow{T_I}, m^I_\bullet, c^I_\bullet)$ be the blow-up of a weighted surface algebra $\Lambda = \Lambda(S, \overrightarrow{T}, m_\bullet, c_\bullet)$ at a non-empty set $I$ of edges of $T$. Then for each edge $i \in I$, we have the pair of virtual arrows

\[ d_i \xrightarrow{\xi_i} \xleftarrow{\mu_i} c_i \]

from the $g^I$-orbit $O_i = (\xi_i, \mu_i)$. Moreover, let $\epsilon : I \to \{-1, 1\}$ be an arbitrary function. We assign to $\epsilon$ a sequence of virtual arrows $\underline{\epsilon} = (\epsilon_i)_{i \in I}$, with $\epsilon_i \in O_i$, where for any $i \in I$, we set

\[ \epsilon_i := \begin{cases} 
\xi_i & \text{if } \epsilon(i) = 1 \\
\mu_i & \text{if } \epsilon(i) = -1.
\end{cases} \]

**Definition 7.3.** Let $\Lambda = \Lambda(S, \overrightarrow{T}, m_\bullet, c_\bullet)$ be a weighted surface algebra, $I$ a non-empty set of edges of $T$, and $\epsilon : I \to \{-1, 1\}$ a function. Then the virtual mutation $\Lambda^I_\epsilon := \Lambda(S, \overrightarrow{T_I}, m^I_\bullet, c^I_\bullet, \underline{\epsilon})$ of the weighted surface algebra $\Lambda_I = \Lambda(S, \overrightarrow{T_I}, m^I_\bullet, c^I_\bullet)$ is said to be a virtual edge deformation of $\Lambda = \Lambda(S, \overrightarrow{T}, m_\bullet, c_\bullet)$ at the set $I$ of edges in $T$.

We have the following direct consequence of the main result of this paper.

**Theorem 7.4.** Let $\Lambda^I_\epsilon = \Lambda(S, \overrightarrow{T_I}, m^I_\bullet, c^I_\bullet, \underline{\epsilon})$ be a virtual edge deformation of a weighted surface algebra $\Lambda = \Lambda(S, \overrightarrow{T}, m_\bullet, c_\bullet)$. Then $\Lambda^I_\epsilon$ is a finite-dimensional symmetric tame periodic algebra of period 4.
Example 7.5. Let \((S, \vec{T})\) be the directed surface given by the torus \(S = \mathbb{T}\) with the triangulation \(T\) and orientation \(\vec{T}\) of triangles as follows

Then \((Q(S, \vec{T}), f)\) is the Markov triangulation quiver (see [18, Section 5] and [36])

with the \(f\)-orbits \((\gamma \omega \rho)\) and \((\sigma \eta \phi)\), given by the upper and the lower oriented triangles of \(\vec{T}\), respectively. We note that \(O(g)\) consists of one \(g\)-orbit \(O(\gamma) = (\gamma \eta \rho \sigma \omega \phi)\). Let \(I = \{2, 3\}\) be the chosen set of edges of \(T\). Then the blow-up \((S, \vec{T}_I)\) of \((S, \vec{T})\) at \(I\) is of the form

with the following orientation of triangles of \(T_I\): \((1 \ b_2 \ a_3),\ (1 \ a_2 \ b_3),\ (a_2 \ c_2 \ d_2),\ (c_2 \ b_2 \ d_2),\ (a_3 \ c_3 \ d_3),\ (c_3 \ b_3 \ d_3)\). Hence the associated triangulation quiver
\( (Q^I, f^I) \) of \((S, \overrightarrow{T_I})\) is of the form

with \(f^I\)-orbits are \((\gamma \omega \rho), (\sigma \eta \phi), (\alpha_2 \xi_2 \delta_2), (\beta_2 \nu_2 \mu_2), (\alpha_3 \xi_3 \delta_3), (\beta_3 \nu_3 \mu_3)\).

Moreover, the set \(\mathcal{O}(g^I)\) of \(g^I\)-orbits in \(Q^I_1\) consists of the two orbits

\[
\mathcal{O}_2 = \mathcal{O}^I(\xi_2) = (\xi_2 \mu_2) = \mathcal{O}^I(\mu_2) \quad \text{and} \quad \mathcal{O}_3 = \mathcal{O}^I(\xi_3) = (\xi_3 \mu_3) = \mathcal{O}^I(\mu_3)
\]

of length 2, and the following orbit of length 14

\[
\mathcal{O}^I(\gamma) = (\gamma \nu_2 \delta_2 \eta \nu_3 \delta_3 \rho \sigma \alpha_2 \beta_2 \omega \alpha_3 \beta_3 \phi).
\]

Let \(m_\bullet : \mathcal{O}(g) \to \mathbb{N}^*\) and \(c_\bullet : \mathcal{O}(g) \to K^*\) be the weight and parameter functions with \(m_{\mathcal{O}(\gamma)} = 1 = c_{\mathcal{O}(\gamma)}\). Then the weight function \(m^I_\bullet : \mathcal{O}(g^I) \to \mathbb{N}^*\) and the parameter function \(c^I_\bullet : \mathcal{O}(g^I) \to K^*\) are both constantly equal 1. Consider also the associated weighted surface algebras

\[
\Lambda = \Lambda(S, \overrightarrow{T}, m_\bullet, c_\bullet) \quad \text{and} \quad \Lambda_I = \Lambda(S, \overrightarrow{T_I}, m^I_\bullet, c^I_\bullet).
\]

It follows from general theory (Theorem 3.4) that

\[
\dim_K \Lambda = 6^2 = 36 \quad \text{and} \quad \dim_K \Lambda_I = 2^2 + 2^2 + 14^2 = 204.
\]

Let \(\epsilon : I \to \{-1, 1\}\) be the function with \(\epsilon(2) = \epsilon(3) = 1\). Then the associated sequence of virtual arrows \(\epsilon \in \mathcal{O}_2 \times \mathcal{O}_3\) is equal to \((\xi_2, \xi_3)\). Take the virtual edge deformation algebra \(\Lambda^\epsilon_I\) of \(\Lambda\) at \(I\), with respect to \(\epsilon\). Directly from Definition 4.4 we obtain that the algebra \(\Lambda^\epsilon_I\) is given by the quiver \(Q^I(\epsilon)\)

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of the form

\[ \nu_2 \delta_2 = \beta_2 \alpha_2 + \omega \tau_3 \phi \gamma \nu_2 \delta_2 \eta \nu_3 \delta_3 \rho \sigma, \nu_3 \delta_3 = \beta_3 \alpha_3 + \phi \gamma \nu_2 \delta_2 \eta \nu_3 \delta_3 \rho \sigma \tau_2 \omega, \]
\[ \alpha_2 \tau_2 = 0, \alpha_3 \tau_3 = 0, \tau_2 \beta_2 = 0, \tau_3 \beta_3 = 0, \]

and the relations:

(1) \[ \nu_2 \delta_2 = \beta_2 \alpha_2 + \omega \tau_3 \phi \gamma \nu_2 \delta_2 \eta \nu_3 \delta_3 \rho \sigma, \nu_3 \delta_3 = \beta_3 \alpha_3 + \phi \gamma \nu_2 \delta_2 \eta \nu_3 \delta_3 \rho \sigma \tau_2 \omega, \]
\[ \alpha_2 \tau_2 = 0, \alpha_3 \tau_3 = 0, \tau_2 \beta_2 = 0, \tau_3 \beta_3 = 0, \]

(2) \[ \nu_2 \delta_2 = \eta \nu_2 \delta_2 \rho \sigma \tau_2 \omega \tau_3 \phi \gamma \nu_2 \delta_2 \eta \nu_3 \delta_3, \]
\[ \delta_2 \tau_2 = \delta_2 \eta \nu_2 \delta_2 \rho \sigma \tau_2 \omega \tau_3 \phi \gamma, \delta_3 \tau_3 = \delta_3 \rho \sigma \tau_2 \omega \tau_3 \phi \gamma \nu_2 \delta_2 \eta, \]
\[ \gamma \omega = \sigma \tau_2 \omega \tau_3 \phi \gamma \nu_2 \delta_2 \eta \nu_3 \delta_3, \omega \rho = \nu_2 \delta_2 \eta \nu_2 \delta_2 \rho \sigma \tau_2 \omega \tau_3 \phi, \]
\[ \rho \gamma = \tau_3 \phi \gamma \nu_2 \delta_2 \rho \sigma \tau_2 \omega \tau_3 \phi \gamma \nu_2 \delta_2, \]
\[ \phi \sigma = \nu_3 \delta_3 \rho \sigma \tau_2 \omega \tau_3 \phi \gamma \nu_2 \delta_2, \]

(3) \[ \gamma \omega \tau_3 = 0, \omega \rho \sigma = 0, \rho \gamma \nu_2 = 0, \sigma \eta \nu_3 = 0, \eta \phi \gamma = 0, \phi \sigma \tau_2 = 0, \]

(4) \[ \omega \tau_3 \nu_3 = 0, \rho \sigma \eta = 0, \sigma \tau_2 \nu_2 = 0, \phi \gamma \omega = 0, \]

(5) \[ \alpha_2 \nu_3 \delta_3 \rho \sigma \tau_2 \omega \tau_3 \phi \gamma \nu_2 \delta_2 \eta = 0, \]
\[ \delta_2 \eta \nu_2 \delta_2 \rho \sigma \tau_2 \omega \tau_3 \phi \gamma \beta_3 = 0, \delta_3 \rho \sigma \tau_2 \omega \tau_3 \phi \gamma \nu_2 \delta_2 \eta \beta_3 = 0. \]

We also observe that we have only one \( g^* \)-orbit of arrows in the quiver \( Q_1^I(\xi)^* \) consisting of all arrows of \( Q_1^I(\xi) \) different from \( \alpha_2, \beta_2, \alpha_3, \beta_3 \). Hence, for any arrow \( \theta \) in \( Q_1^I(\xi) \) we have \( n^*_0 = 10 \), while \( n^*_\theta = 2 \), and clearly \( m^*_\theta = 1 \). Then, applying Theorem 4.14, we conclude that

\[ \dim_K \Lambda_I^* = \sum_{\theta \in Q_1^I(\xi)^*} \left( m^*_\theta (n^*_\theta + n^*_\theta) \right) = 12(12 + 2) = 168. \]
Example 7.6. Let \((S, \overrightarrow{T})\) be the collection of three self-folded triangles with \((1 \ 1 \ 4), (2 \ 2 \ 5), (3 \ 3 \ 6)\) and \((4 \ 5 \ 6)\) in \(\overrightarrow{T}\). Then the associated triangulation quiver \((Q(S, \overrightarrow{T}), f)\) is of the form

\[ m_{\mathcal{O}(\rho_1)} = m_{\mathcal{O}(\rho_2)} = m_{\mathcal{O}(\rho_3)} = m \geq 3 \text{ and } m_{\mathcal{O}(\gamma_1)} = 1, \]
\[ c_{\mathcal{O}(\rho_1)} = c_{\mathcal{O}(\rho_2)} = c_{\mathcal{O}(\rho_3)} = 1 \text{ and } c_{\mathcal{O}(\gamma_1)} = \lambda \in K^*. \]

Let \(I = \{1, 2, 3\}\) be the chosen set of edges of \(T\). Then the blow-up \((S, \overrightarrow{T_I})\)
of $(S, \overrightarrow{T})$ at $I$ is of the form

with the following orientation of triangles: $(4\ 5\ 6), (b_1\ a_1\ 4), (b_2\ a_2\ 5), (b_3\ a_3\ 6)$, and $(a_i\ c_i\ d_i), (c_i\ b_i\ d_i)$, for $i \in I$. Hence the associated triangulation quiver $(Q^I, f^I) = (Q(S, \overrightarrow{T}_I), f^I)$ is of the form
with $f^I$-orbits $(\phi \omega \psi)$ and $(\alpha_i \xi_i \delta_i)$, $(\beta_i \nu_i \mu_i)$, $(\rho_i \gamma_i \sigma_i)$, for $i \in \{1, 2, 3\}$. Further, the set $\mathcal{O}(g^I)$ of all $g^I$-orbits in $(Q^I, f^I)$ consists of the orbits
\[ \mathcal{O}_i = (\xi_i \mu_i) \quad \mathcal{O}^I(\alpha_i) = (\alpha_i \beta_i \rho_i), \quad i \in \{1, 2, 3\}, \]
and one larger orbit of length 15
\[ \mathcal{O}^I(\gamma_1) = (\gamma_1 \phi \sigma_2 \nu_2 \delta_2 \gamma_2 \omega \sigma_3 \nu_3 \delta_3 \gamma_3 \psi \sigma_1 \nu_1 \delta_1). \]
Moreover, the weight function $m^I : \mathcal{O}(g^I) \to \mathbb{N}^*$ and the parameter function $c^I : \mathcal{O}(g^I) \to K^*$ are given by $m^I_{\mathcal{O}_i} = 1$, $m^I_{\mathcal{O}^I(\alpha_i)} = m$, $c^I_{\mathcal{O}_i} = 1$, $c^I_{\mathcal{O}^I(\alpha_i)} = 1$, for $i \in \{1, 2, 3\}$, and $m^I_{\mathcal{O}^I(\gamma_1)} = 1$, $c^I_{\mathcal{O}^I(\gamma_1)} = \lambda$. Consider the associated weighted surface algebras
\[ \Lambda = \Lambda(S, \text{T}, m_\bullet, c_\bullet) \quad \text{and} \quad \Lambda_I = \Lambda(S, \overrightarrow{T}, m^I_\bullet, c^I_\bullet) \]
It follows from Theorem 3.4 that
\[ \dim_K \Lambda = 3 + 3 + 3 \cdot 9^2 = 90, \quad \text{and} \]
\[ \dim_K \Lambda_I = 4 + 4 + 4 + m \cdot 3^2 + m \cdot 3^2 + m \cdot 3^2 + 15^2 = 27m + 237. \]
Let $\epsilon : I \to \{-1, 1\}$ be the function given by $\epsilon(1) = \epsilon(2) = 1$ and $\epsilon(3) = -1$. Then the associated sequence of virtual arrows $\xi \in \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3$ is equal to $(\xi_1, \xi_2, \mu_3)$. Then the virtual edge deformation $\Lambda^\epsilon_I = \Lambda(S, \text{T}, m^I_\bullet, c^I_\bullet, \xi)$ of $\Lambda$ at $I$ (with respect to $\epsilon$) is defined by the quiver $Q^I(\xi)$ of the form

![Diagram](image-url)
We recall also that the associated quiver \( Q^{f}(\xi)^{*} \) is obtained from \( Q^{f}(\xi) \) by removing the arrows \( \alpha_{1}, \beta_{1}, \alpha_{2}, \beta_{2}, \nu_{3}, \delta_{3} \). Further, the set \( O^{f}(g^{*}) \) of all \((g^{f})^{*}\)-orbits of the associated permutation \((g^{f})^{*:} Q^{f}(\xi)^{*} \rightarrow Q^{f}(\xi)^{*} \) consists of the orbits

\[
(\tau_{1} \rho_{1}), (\tau_{2} \rho_{2}), (\alpha_{3} \beta_{3} \rho_{3}), \text{ and }
(\tau_{3} \gamma_{3} \psi \sigma_{1} \nu_{1} \delta_{1} \gamma_{1} \phi \sigma_{2} \nu_{2} \delta_{2} \gamma_{2} \omega \sigma_{3}).
\]

Moreover, the associated permutation \((f^{f})^{*:} Q^{f}(\xi)^{*} \rightarrow Q^{f}(\xi)^{*} \) has the following orbits

\[
(\nu_{1} \delta_{1} \tau_{1}), (\gamma_{1} \sigma_{1} \rho_{1}), (\nu_{2} \delta_{2} \tau_{2}), (\gamma_{2} \sigma_{2} \rho_{2}), (\alpha_{3} \beta_{3} \tau_{3}), (\gamma_{3} \sigma_{3} \rho_{3}), (\phi \omega \psi).
\]

Then it follows from Definition 4.4 that the algebra \( \Lambda_{f} \) is given by the quiver \( Q^{f}(\xi) \) and the following relations (here we abbreviate \( f^{*} := (f^{f})^{*} \) and \( g^{*} := (g^{f})^{*} \):

\[
(1) \quad \nu_{1} \delta_{1} = \beta_{1} \alpha_{1} + (\rho_{1} \tau_{1})^{m-1} \rho_{1}, \alpha_{1} \tau_{1} = 0, \tau_{1} \beta_{1} = 0, \nu_{2} \delta_{2} = \beta_{2} \alpha_{2} + (\rho_{2} \tau_{2})^{m-1} \rho_{2}, \alpha_{2} \tau_{2} = 0, \tau_{2} \beta_{2} = 0, \alpha_{3} \beta_{3} = \delta_{3} \nu_{3} + \lambda \gamma_{3} \psi \sigma_{1} \nu_{1} \delta_{1} \gamma_{1} \phi \sigma_{2} \nu_{2} \delta_{2} \gamma_{2} \omega \sigma_{3}, \nu_{3} \tau_{3} = 0, \tau_{3} \delta_{3} = 0,
\]

\[
(2) \quad \delta_{1} \tau_{1} = \lambda \delta_{1} \gamma_{1} \phi \sigma_{2} \nu_{2} \delta_{2} \gamma_{2} \omega \sigma_{3} \tau_{3} \gamma_{3} \psi \sigma_{1}, \tau_{1} \nu_{1} = \lambda \gamma_{1} \phi \sigma_{2} \nu_{2} \delta_{2} \gamma_{2} \omega \sigma_{3} \tau_{3} \gamma_{3} \psi \sigma_{1} \nu_{1}, \rho_{1} \gamma_{1} = \lambda \nu_{1} \delta_{1} \gamma_{1} \phi \sigma_{2} \nu_{2} \delta_{2} \gamma_{2} \omega \sigma_{3} \tau_{3} \gamma_{3} \psi \sigma_{1}, \sigma_{1} \rho_{1} = \lambda \phi \sigma_{2} \nu_{2} \delta_{2} \gamma_{2} \omega \sigma_{3} \tau_{3} \gamma_{3} \psi \sigma_{1} \nu_{1} \delta_{1}, \delta_{2} \tau_{2} = \lambda \delta_{2} \gamma_{2} \omega \sigma_{3} \tau_{3} \gamma_{3} \psi \sigma_{1} \nu_{1} \delta_{1} \gamma_{1} \phi \sigma_{2} \nu_{2} \delta_{2} \gamma_{2} \omega \sigma_{3} \tau_{3} \gamma_{3} \psi \sigma_{1}, \tau_{2} \nu_{2} = \lambda \gamma_{2} \omega \sigma_{3} \tau_{3} \gamma_{3} \psi \sigma_{1} \nu_{1} \delta_{1} \gamma_{1} \phi \sigma_{2} \nu_{2} \delta_{2} \gamma_{2} \omega \sigma_{3} \tau_{3} \gamma_{3} \psi \sigma_{1}, \rho_{2} \gamma_{2} = \lambda \nu_{2} \delta_{2} \gamma_{2} \omega \sigma_{3} \tau_{3} \gamma_{3} \psi \sigma_{1} \nu_{1} \delta_{1} \gamma_{1} \phi \sigma_{2} \nu_{2} \delta_{2} \gamma_{2} \omega \sigma_{3} \tau_{3} \gamma_{3} \psi \sigma_{1}, \sigma_{2} \rho_{2} = \lambda \omega \sigma_{3} \tau_{3} \gamma_{3} \psi \sigma_{1} \nu_{1} \delta_{1} \gamma_{1} \phi \sigma_{2} \nu_{2} \delta_{2} \gamma_{2} \omega \sigma_{3} \tau_{3} \gamma_{3} \psi \sigma_{1}, \rho_{3} \gamma_{3} = \lambda \tau_{3} \gamma_{3} \gamma_{3} \psi \sigma_{1} \nu_{1} \delta_{1} \gamma_{1} \phi \sigma_{2} \nu_{2} \delta_{2} \gamma_{2} \omega \sigma_{3} \tau_{3} \gamma_{3} \psi \sigma_{1}, \sigma_{3} \rho_{3} = \lambda \psi \sigma_{1} \nu_{1} \delta_{1} \gamma_{1} \phi \sigma_{2} \nu_{2} \delta_{2} \gamma_{2} \omega \sigma_{3} \tau_{3} \gamma_{3} \psi \sigma_{1}, \gamma_{1} \tau_{1} = (\tau_{1} \rho_{1})^{m-1} \tau_{1}, \gamma_{2} \tau_{2} = (\tau_{2} \rho_{2})^{m-1} \tau_{2}, \gamma_{3} \tau_{3} = (\alpha_{3} \beta_{3} \rho_{3})^{m-1} \alpha_{3} \beta_{3}, \beta_{3} \tau_{3} = (\beta_{3} \rho_{3} \alpha_{3})^{m-1} \beta_{3} \rho_{3}, \tau_{3} \alpha_{3} = (\rho_{3} \alpha_{3} \beta_{3})^{m-1} \rho_{3} \alpha_{3},
\]

\[
(3) \quad \eta f^{*}(\eta) g^{*}(f^{*}(\eta)) = 0 \text{ for any arrow } \eta \text{ from the set }
\{\rho_{1}, \gamma_{1}, \sigma_{1}, \psi, \phi, \omega, \rho_{2}, \gamma_{2}, \sigma_{2}, \rho_{3}, \gamma_{3}, \sigma_{3}\},
\]

\[
(4) \quad \eta g^{*}(\eta) f^{*}(g^{*}(\eta)) = 0 \text{ for any arrow } \eta \text{ from the set }
\{\delta_{1}, \tau_{1}, \rho_{1}, \gamma_{1}, \psi, \phi, \omega, \delta_{2}, \tau_{2}, \rho_{2}, \gamma_{2}, \beta_{3}, \tau_{3}, \gamma_{3}, \sigma_{3}\},
\]

\[
(5) \quad \alpha_{1} \gamma_{1} \phi \sigma_{2} \nu_{2} \delta_{2} \gamma_{2} \omega \sigma_{3} \tau_{3} \gamma_{3} \psi \sigma_{1} \nu_{1} = 0, \delta_{1} \gamma_{1} \phi \sigma_{2} \nu_{2} \delta_{2} \gamma_{2} \omega \sigma_{3} \tau_{3} \gamma_{3} \psi \sigma_{1} \beta_{1} = 0, \alpha_{2} \gamma_{2} \omega \sigma_{3} \tau_{3} \gamma_{3} \psi \sigma_{1} \nu_{1} \delta_{1} \gamma_{1} \phi \sigma_{2} \nu_{2} = 0, \delta_{2} \gamma_{2} \omega \sigma_{3} \tau_{3} \gamma_{3} \psi \sigma_{1} \nu_{1} \delta_{1} \gamma_{1} \phi \sigma_{2} \beta_{2} = 0, \nu_{3} (\rho_{3} \alpha_{3} \beta_{3})^{m-1} \rho_{3} \alpha_{3} = 0, (\beta_{3} \rho_{3} \alpha_{3})^{m-1} \beta_{3} \rho_{3} \delta_{3} = 0.
\]

Moreover, applying Theorem 4.14 we have

\[
\dim_{K} \Lambda_{f} = 4m + 4m + 12 + 14 \cdot 16 = 20m + 224.
\]
Remark 7.7. We would like to mention that there are virtual edge deformations $\Lambda'_{\ell}$ of weighted surface algebras $\Lambda$ whose Gabriel quivers contain an arbitrary large number of subquivers of the forms

![Diagram](https://via.placeholder.com/150)

We may obtain such algebras in the following way.

Let $(S, \overrightarrow{T})$ be a directed triangulated surface with non-empty boundary and $X$ a fixed set of boundary edges of $T$. Then one may enlarge $(S, \overrightarrow{T})$ to the directed surface $(S(X), \overrightarrow{T}(X))$ by gluing each edge $x \in X$ with the self-folded triangle

![Diagram](https://via.placeholder.com/150)

Next, we take the set $I(X)$ of all created self-folded edges $i(x), x \in X$, and the associated blow-up $(S(X), \overrightarrow{T}(X)_{I(X)})$, so we have in $\overrightarrow{T}(X)_{I(X)}$ the triangles

![Diagram](https://via.placeholder.com/150)

with orientation $(x b_x a_x), (a_x c_x d_x)$ and $(c_x b_x d_x)$ in $\overrightarrow{T}(X)_{I(X)}$, for all $x \in X$. Hence the associated triangulation quiver $(Q(S(X), \overrightarrow{T}(X)_{I(X)}), f_{I(X)})$
contains subquivers

We also note that the triangulation quiver \((Q(S(X), T(X)), f^X)\) has the subquivers

with \(f^X\)-orbits \((\gamma_x, \sigma_x, \rho_x)\), and hence \(g^X(\rho_x) = \rho_x\), for all \(x \in X\), where \(g^X\) is the permutation induced by \(f^X\). Finally, we take a weight function \(m^X : O(g^X) \to \mathbb{N}^*\) with value \(m \geq 3\) on all the orbits of \(\rho_x, x \in X\), and an arbitrary parameter function \(\epsilon^X : O(g^X) \to K^*\). Then the suitable edge deformations \(\Lambda'_I(X)\) of \(\Lambda = \Lambda(S(X), T(X), m^X, \epsilon^X)\) with respect to \(I(X)\) and functions \(\epsilon : I(X) \to \{-1, 1\}\) provide the required algebras.

We also note that the directed triangulated surfaces and algebras considered in Example 7.6 are special cases of the above procedures, with the starting directed triangulated surface being the single triangle

and the set \(X = \{4, 5, 6\}\) of its boundary edges.

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