Stable and unstable cosmological models in bimetric massive gravity

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Nonlinear, ghost-free massive gravity has two tensor fields; when both are dynamical, the mass of the graviton can lead to cosmic acceleration that agrees with background data, even in the absence of a cosmological constant. Here the question of the stability of linear perturbations in this theory is examined. Instabilities are presented for several classes of models, and simple criteria for the cosmological stability of massive bigravity are derived. In this way, we identify a particular self-accelerating bigravity model, infinite-branch bigravity (IBB), which exhibits both viable background evolution and stable linear perturbations. We discuss the modified gravity parameters for IBB, which do not reduce to the standard ΛCDM result at early times, and compute the combined likelihood from measured growth data and type Ia supernovae. IBB predicts a present matter density \( \Omega_{m0} = 0.18 \) and an equation of state \( w(z) = -0.79 + 0.21z/(1 + z) \). The implications of the linear instability for other bigravity models are discussed: the instability does not necessarily rule these models out, but rather presents interesting questions about how to extract observables from them when linear perturbation theory does not hold.

I. INTRODUCTION

Testing gravity beyond the limits of the solar system is an important task of present and future cosmology. The detection of any modification of Einstein’s gravity at large scales or in past epochs would be an extraordinary revolution and change our view of the evolution of the Universe.

General relativity is the unique theory of a massless spin-2 field \([1–6]\). Consequently, most modifications of gravity proposed so far introduce one or more new dynamical fields, in addition to the massless metric tensor of standard gravity. This new field is usually a scalar field, typically through the so-called Horndeski Lagrangian \([7, 8]\), or a vector field, such as in Einstein-aether models (see Refs. \([9, 10]\) and references therein). A complementary approach which has gained significant attention in recent years is, rather than adding a new dynamical field, to promote the massless spin-2 graviton of general relativity to a massive one.

The history of massive gravity is an old one, dating back to 1939, when the linear theory of Fierz and Pauli was published \([11]\). We refer the reader to the reviews \([12, 13]\) for a reconstruction of the steps leading to the modern approach, which has resulted in a ghost-free, fully nonlinear theory of massive gravity \([14]\) (see also Refs. \([15–19]\)). A key element of these new forms of massive gravity is the introduction of a second tensor field, or “reference metric,” in addition to the standard metric describing the curvature of spacetime. When this reference metric is fixed (e.g., Minkowski), this theory propagates the five degrees of freedom of a ghost-free massive graviton.

However, the reference metric can also be made dynamical, as proposed in Refs. \([20, 21]\). This promotes massive gravity to a theory of bimetric gravity. This theory is still ghost-free and has the advantage of allowing cosmologically-viable solutions. The cosmology of bimetric gravity has been studied in several papers, e.g., in Refs. \([22–26]\). The main conclusion is that bimetric gravity allows for a cosmological evolution that can approximate the \( \Lambda \)CDM universe and can therefore be a candidate for dark energy without invoking a cosmological constant. Crucially, the parameters and the potential structure leading to the accelerated expansion are thought to be stable under quantum corrections \([29]\), in stark contrast to a cosmological constant, which would need to be fine-tuned against the energy of the vacuum \([30, 31]\).

Bimetric gravity has been successfully compared to background data (cosmic microwave background, baryon acoustic oscillations, and type Ia supernovae) in Refs. \([22, 24]\) and to linear perturbation data in Refs. \([32, 33]\). The comparison

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with linear perturbations has been undertaken on subhorizon scales assuming a quasi-static approximation, in which
the potentials are assumed to be slowly varying. This assumption makes it feasible to derive the modification to the
Poisson equation and the anisotropic stress, two functions of scale and time which completely determine observational
effects at the linear level.

The quasi-static equations are, however, a valid subhorizon approximation only if the full system is stable for large
wavenumbers. Previous work \cite{25,34,35} has identified a region of instability in the past. \footnote{This should not be
confused with the Higuchi ghost instability, which affects most massive gravity cosmologies and some in bigravity,
but is, however, absent from the simplest bimetric models which produce ΛCDM-like backgrounds \cite{34}.} Here we investigate
this problem in detail. We reduce the linearized Einstein equations to two equations for the scalar modes, and analytically
determine the epochs of stability and instability for all the models with up to two free parameters which have been
shown to produce viable cosmological background evolution. The behavior of more complicated models can be reduced
to these simpler ones at early and late times.

We find that several models which yield sensible background cosmologies in close agreement with the data are in
fact plagued by an instability that only turns off at recent times. This does not necessarily rule these regions of the
bimetric parameter space out, but rather presents a question of how to interpret and test these models, as linear
perturbation theory is quickly invalidated. Remarkably, we find that only a particular bimetric model — the one in
which only the $\beta_1$ and $\beta_4$ parameters are nonzero (that is, the linear interaction and the cosmological constant for
the reference metric are turned on) — is stable at all times when the evolution is within a particular branch. This
shows that a cosmologically viable bimetric model without an explicit cosmological constant does indeed exist, and
raises the question of how to nonlinearly probe the viability of other bimetric models.

This paper is part of a series dedicated to the cosmological perturbations of massive bigravity and their properties,
following Ref. \cite{33}.

## II. BACKGROUND EQUATIONS

We start with the action of the form \cite{21}

\[
S = -\frac{M_g^2}{2} \int d^4x \sqrt{-\det g} R(g) - \frac{M_f^2}{2} \int d^4x \sqrt{-\det f} R(f) + m^2 M_g^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n \varepsilon_n \left( \sqrt{g^{\alpha\beta} f^{\beta\gamma}} \right) + \int d^4x \sqrt{-\det g} \mathcal{L}_m(g, \Phi),
\]

where $\varepsilon_n$ are elementary symmetric polynomials and $\beta_n$ are free parameters. Here $g_{\mu\nu}$ is the standard metric coupled
to the matter fields $\Phi$ in the matter Lagrangian, $\mathcal{L}_m$, while $f_{\mu\nu}$ is a new dynamical tensor field with metric properties.
In the following we express masses in units of $M_g$ and absorb the mass parameter $m^2$ into the parameters $\beta_n$. The
graviton mass is generally of order $m^2 \beta_n$. The action then becomes

\[
S = -\frac{1}{2} \int d^4x \sqrt{-\det g} R(g) - \frac{M_f^2}{2} \int d^4x \sqrt{-\det f} R(f) + \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n \varepsilon_n \left( \sqrt{g^{\alpha\beta} f^{\beta\gamma}} \right) + \int d^4x \sqrt{-\det g} \mathcal{L}_m(g, \Phi).
\]

There has been some discussion in the literature over how to correctly take square roots. We will find solutions in
which $\det \sqrt{g^{-1} f}$ becomes zero at a finite point in time (and only at that time), and so it is important to determine
whether to choose square roots to always be positive, or to change sign on either side of the $\det = 0$ point. This was
discussed in some detail in Ref. \cite{37} (see also Ref. \cite{38}), where continuity of the vielbein corresponding to $\sqrt{g^{-1} f}$
demanded that the square root not be positive definite. We will take a similar stance here, and make the only choice
that renders the action differentiable at all times, i.e., such that the derivative of $\sqrt{g^{-1} f}$ with respect to $g_{\mu\nu}$ and $f_{\mu\nu}$
is continuous everywhere. In particular, using a cosmological background with $f_{\mu\nu} \equiv \text{diag}(-X^2, b^2, b^2, b^2)$, this
choice implies that we assume $\sqrt{-\det f} = X b^3$, where $X = \dot{b}/H$ with $H$ the $g$-metric Hubble rate. This is important
because, as we will see later on, it turns out that in the cosmologically-stable model, the $f$ metric bounces, so $X$
changes sign during cosmic evolution. Consequently the square roots will change sign as well, rather than develop
cusps. Note that sufficiently small perturbations around the background will not lead to a different sign of this square root.

\[\footnote{This should not be confused with the Higuchi ghost instability, which affects most massive gravity cosmologies and some in bigravity, but is, however, absent from the simplest bimetric models which produce ΛCDM-like backgrounds \cite{34}.}\]
Varying the action with respect to $g_{\mu\nu}$, one obtains the following equations of motion:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \sum_{n=0}^{3} (-1)^n \beta_n g_{\mu\lambda} Y_{(n)\nu}^{\lambda} \left( \sqrt{g^{\alpha\beta} f_{\beta\gamma}} \right) = T_{\mu\nu}. \quad (5)$$

Here the matrices $Y_{(n)\nu}^{\lambda} \left( \sqrt{g^{\alpha\beta} f_{\beta\gamma}} \right)$ are defined as, setting $X = \left( \sqrt{g^{-1} f} \right)$,

$$Y_{(0)}(X) = \mathbb{I},$$

$$Y_{(1)}(X) = X - \mathbb{I}[X],$$

$$Y_{(2)}(X) = X^2 - X[X] + \frac{1}{2} \mathbb{I} \left( [X]^2 - [X^2] \right),$$

$$Y_{(3)}(X) = X^3 - X^2[X] + \frac{1}{2} X \left( [X]^2 - [X^2] \right) - \frac{1}{6} \mathbb{I} \left( [X]^3 - 3[X][X^2] + 2[X^3] \right), \quad (9)$$

where $\mathbb{I}$ is the identity matrix and $[...]$ is the trace operator. Varying the action with respect to $f_{\mu\nu}$ we find

$$\bar{R}_{\mu\nu} - \frac{1}{2} f_{\mu\nu} \bar{R} + \frac{1}{M_f^2} \sum_{n=0}^{3} (-1)^n \beta_n \lambda_{\alpha\beta\gamma} Y_{(n)\nu}^{\lambda} \left( \sqrt{g^{\alpha\beta} f_{\beta\gamma}} \right) = 0, \quad (10)$$

where the overbar indicates the curvature of the $f_{\mu\nu}$ metric.

The $f$-metric Planck mass, $M_f$, is a redundant parameter and can be freely set to unity $^{[39]}$. To see this, consider the rescaling $f_{\mu\nu} \rightarrow M_f^{-2} f_{\mu\nu}$. The Ricci scalar transforms as $\bar{R}(f) \rightarrow M_f^2 \bar{R}(f)$, so the full Einstein-Hilbert term in the action becomes

$$\frac{M_f^2}{2} \sqrt{-\det f} \bar{R}(f) \rightarrow \frac{1}{2} \sqrt{-\det f} \bar{R}(f). \quad (11)$$

The other term in the action that depends on $f_{\mu\nu}$ is the mass term, which transforms as

$$\sum_{n=0}^{4} \beta_n c_n \left( \sqrt{g^{-1} f} \right) \rightarrow \sum_{n=0}^{4} \beta_n c_n \left( M_f^{-1} \sqrt{g^{-1} f} \right) = \sum_{n=0}^{4} \beta_n M_f^{-n} c_n \left( \sqrt{g^{-1} f} \right), \quad (12)$$

where in the last equality we used the fact that the elementary symmetric polynomials $c_n(X)$ are of order $X^n$. Therefore, by additionally redefining the interaction couplings as $\beta_n \rightarrow M_f^2 \beta_n$, we end up with the original bigravity action but with $M_f = 1$. Consequently we set $M_f = 1$ in the following.

Let us now consider the background cosmology of massive bigravity. We assume a spatially-flat FLRW metric, $ds^2 = a^2(\tau) (-d\tau^2 + dx_i dx^i)$, where $\tau$ is conformal time and an overdot represents the derivative with respect to it. The second metric is chosen as

$$ds^2 = - \left[ b(\tau)^2 / H^2(\tau) \right] d\tau^2 + b(\tau)^2 dx_i dx^i, \quad (14)$$

where $H \equiv \dot{a}/a$ is the conformal-time Hubble parameter associated with the physical metric, $g_{\mu\nu}$. The particular choice for the $f$-metric lapse, $f_{00}$, ensures that the Bianchi identity is satisfied (see, e.g., Ref. $^{[20]}$).

Inserting the FLRW ansatz for $g_{\mu\nu}$ into Eq. (5) we get

$$3H^2 = a^2 (\rho_{\text{tot}} + \rho_{\text{mg}}), \quad (15)$$

where we define an effective massive-gravity energy density as

$$\rho_{\text{mg}} = B_0 \equiv \beta_0 + 3\beta_1 r + 3\beta_2 r^2 + \beta_3 r^3 \quad (16)$$

$^2$ Recall that we are expressing masses in units of the Planck mass, $M_p$. In more general units, the redundant parameter is $M_f/M_g$. 
with
\[ r \equiv \frac{b}{a}, \]  
(17)
while \( \rho_{\text{tot}} \) is the density of all other matter components (e.g., dust and radiation). The total energy density follows the usual conservation law,
\[ \dot{\rho}_{\text{tot}} + 3H\rho_{\text{tot}} = 0. \]  
(18)
It is useful to define the density parameter for the mass term (which will be the effective dark energy density):
\[ \Omega_{mg} \equiv \frac{\rho_{mg}}{\rho_{\text{tot}} + \rho_{mg}} = 1 - \Omega_m - \Omega_r, \]  
(19)
where \( \Omega_i = \rho_i / (\rho_{\text{tot}} + \rho_{mg}) \) for matter and radiation.

The background dynamics depend entirely on the the g-metric Hubble rate, \( \mathcal{H} \), and the ratio of the two scale factors, \( r = b/a \) [23]. Moreover, by using \( N = \log a \) as time variable, with \( ^{'} \) denoting derivatives with respect to \( N \), the background equations can be conveniently reformulated as a first-order autonomous system [40]:
\[ 2\mathcal{H}'\mathcal{H} + \mathcal{H}^2 = a^2(B_0 + B_2r' - \omega_{\text{tot}}\rho_{\text{tot}}), \]  
(20)
\[ r' = \frac{3(1 + \omega_{\text{tot}})B_1\Omega_{\text{tot}}r}{\beta_1 - 2\beta_3r^2 - 2\beta_4r^3 + 3B_2r^2}, \]  
(21)
\[ \Omega_{\text{tot}} = 1 - B_0/B_1r, \]  
(22)
where
\[ B_1 \equiv \beta_1 + 3\beta_2r + 3\beta_3r^2 + \beta_4r^3, \]  
(23)
\[ B_2 \equiv \beta_1 + 2\beta_2r + \beta_3r^2, \]  
(24)
and \( \omega_{\text{tot}} \) denotes the equation of state corresponding to the sum of matter and radiation density parameter \( \Omega_{\text{tot}} \). We can define the effective equation of state
\[ w_{\text{eff}} \equiv \Omega_{mg}w_{mg} + \Omega_{\text{tot}}\omega_{\text{tot}} = -\frac{1}{3}(1 + 2\frac{\mathcal{H}'}{\mathcal{H}}) = -\frac{r(B_0 + B_2r')}{B_1} \]  
(25)
\[ = -1 + \Omega_{\text{tot}} - \frac{B_2rr'}{B_1}, \]  
(26)
from which we obtain
\[ w_{mg} = -1 + \frac{B_2rr'}{\Omega_{mg}B_1} = -1 - \frac{B_2r'}{B_0}. \]  
(27)
Another useful relation gives the Hubble rate in terms of \( r \) without an explicit \( \rho \) dependence,
\[ \mathcal{H}^2 = \frac{a^2B_1}{3r}. \]  
(28)

The background evolution of \( r \) will follow Eq. (21) from an initial value of \( r \) until \( r' = 0 \), unless \( r \) hits a singularity. In Ref. [40] it was shown that cosmologically viable evolution can take place in two distinct ways, depending on initial conditions: when \( r \) evolves from 0 to a finite value (we call this a finite branch) and when \( r \) evolves from infinity to a finite value (infinite branch). In all viable cases, the past asymptotic value of \( r \) corresponds to \( \Omega_m = 1 \) while the final point corresponds to a de Sitter stage with \( \Omega_m = 0 \) (see Fig. 1 for an illustrative example).

In the following, we consider only pressureless matter, or dust, with \( \omega_{\text{tot}} = 0 \). The reason is that we are interested only in the late time behavior of massive gravity when the Universe is dominated by dust. We also assume \( r \geq 0 \), although in principle nothing prevents a negative value of \( b \).

We will find it convenient to express all the \( \beta_i \) parameters in units of \( H_0^2 \) and \( \mathcal{H} \) in units of \( H_0 \). In this way all the quantities that enter the equations are dimensionless.

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3 With this convention, our \( \beta_i \) parameters are equivalent to the \( B_i = m^2\beta_i/H_0^2 \) used in Refs. [23] [24] [33].
III. PERTURBATION EQUATIONS

In this section we study linear cosmological perturbations. We define our perturbed metrics in Fourier space by

\begin{align}
g_{\alpha\beta} &= g_{0,\alpha\beta} + h_{\alpha\beta}, \\
f_{\alpha\beta} &= f_{0,\alpha\beta} + h_{f,\alpha\beta},
\end{align}

where $g_{0,\alpha\beta}$ and $f_{0,\alpha\beta}$ are the background metrics with line elements

\begin{align}
ds^2_g &= a^2(t)(-dt^2 + dx_i dx^i), \\
ds^2_f &= -[b(t)^2/H^2(t)] dt^2 + b(t)^2 dx_i dx^i,
\end{align}

while $h_{\alpha\beta}$ and $h_{f,\alpha\beta}$ are perturbations around the backgrounds $g_{0,\alpha\beta}$ and $f_{0,\alpha\beta}$, respectively, whose line elements are

\begin{align}
ds^2_g &= 2a^2 \left[-\Psi dt^2 + (\Phi \delta_{ij} + k_i k_j E_j) dx^i dx^j\right] \exp(i k \cdot r), \\
ds^2_f &= 2b^2 \left[-\frac{\dot{b}^2 \Psi_f}{b^2 H^2} dt^2 + (\Phi_f \delta_{ij} + k_i k_j E_f) dx^i dx^j\right] \exp(i k \cdot r).
\end{align}

After transforming to gauge-invariant variables \cite{23},

\begin{align}
\Phi &\rightarrow \Phi - \mathcal{H}^2 E', \\
\Psi &\rightarrow \Psi - \mathcal{H}(\mathcal{H}'E' + \mathcal{H}(E'' + E')), \\
\Phi_f &\rightarrow \Phi_f - \frac{\mathcal{H}^2 E'}{r' + r}, \\
\Psi_f &\rightarrow \Psi_f - \frac{\mathcal{H}r^2 H' (r' + r) E_f' + \mathcal{H} r^2 \left( r (r' + r) E_f'' + E_f' (2r'^2 + r (2r' - r'') + r^2) \right)}{(r' + r)^3},
\end{align}

and using $N = \log a$ as the time variable, the perturbation equations for the $g_{\mu\nu}$ metric read:

\begin{align}
[00] &\left( \frac{2k^2}{3B_2 a^2} + 1 \right) \Phi - \Phi_f + \frac{1}{3} k^2 \Delta E + \frac{2\mathcal{H} r (\mathcal{H} - \mathcal{H}')}{A_2} E' - \frac{\mathcal{H}^2 A_1}{A_2} \Delta E' \\
&- \frac{2\mathcal{H}^2 (A_1 + \sigma^2 r^2 B_2) (\mathcal{H} - \mathcal{H}')}{a^2 B_2} \theta - \frac{\delta \rho}{3 B_2} = 0, \\
[0i] &\Phi' - \Psi + \frac{\sigma^2}{2 \mathcal{H} B_2} \theta + (\mathcal{H}^2 - \mathcal{H} \mathcal{H}') E' = 0, \\
[i j] &\Phi + \Psi + \frac{1}{2} \mathcal{H}^2 A_1 \Delta E = 0, \\
[i i] &\left( \frac{2k^2}{3B_2 a^2} + \frac{A_1}{B_2} \right) \Phi + \left( \frac{2k^2}{3B_2 a^2} + 1 \right) \Psi - \frac{A_4}{B_2} \Phi_f - \frac{A_4}{A_2} \Psi_f + \frac{k^2 A_3}{B_2} \Delta E - \frac{2\mathcal{H}^3 r (\mathcal{H} - \mathcal{H}')}{A_2} E'' \\
&- \frac{\mathcal{H}^2 A_1}{A_2} \Delta E'' + A_4 E' + A_5 \Delta E' = 0,
\end{align}

while the corresponding equations for $f_{\mu\nu}$ are

\begin{align}
[00] &\Phi - \left( 1 + \frac{2k^2}{3a^2 B_2} \right) \Phi_f + \frac{k^2}{3} \Delta E - \frac{A_1}{A_2} \mathcal{H}^2 \Delta E - \frac{2\mathcal{H}^3 r (\mathcal{H} - \mathcal{H}')}{A_2} E' = 0, \\
[0i] &\Phi_f - \frac{A_2}{A_1} \Psi + \frac{\sigma^2 \mathcal{H} B_2 (\mathcal{H}' - \mathcal{H})}{A_2} \Delta E' - \frac{\sigma^2 \mathcal{H} B_2 (\mathcal{H}' - \mathcal{H})}{A_2} E' = 0, \\
[i j] &\Phi + \Psi - \frac{\sigma^2 A_1 A_2}{2 r A_2} \Delta E = 0, \\
[i i] &\left( \frac{2k^2 A_3}{3a^2 B_2 A_1} + \frac{A_1}{B_2} \right) \Phi_f + \left( \frac{2k^2 r A_2}{3a^2 B_2 A_1} + \frac{A_2}{A_1} \right) \Psi_f - \frac{A_2}{B_2} \Phi - \Psi - \frac{k^2 A_3}{B_2} \Delta E + \frac{2\mathcal{H}^3 r (\mathcal{H}' - \mathcal{H})}{A_2} E'' \\
&+ \frac{\mathcal{H}^2 A_1}{A_2} \Delta E'' - A_4 E' - A_5 \Delta E' = 0,
\end{align}

where $\Delta E \equiv E - E_f$ and the $A_i$ coefficients are defined as

\begin{align}
A_1 &= \sigma^2 B_2 - 2 \mathcal{H}^2 r, \\
A_2 &= \sigma^2 B_2 - 2 \mathcal{H} r \mathcal{H}',
\end{align}
\[ A_3 = 2B_2 + B'_2, \]
\[ A_4 = -\frac{(A_1 - A_2)^2 (a^4 (1 + 2r^2) B_2^2 + A_1 (A_1 + A_2) + a^2 r^2 B_2 (2A_1 + A_2))}{2r (a^2 r^2 B_2 + A_1) A_2^2} + \frac{(-a^2 B_2 + A_1) (A_1 - A_2) (A_1 A_2 - a^2 B_2 ((1 + r^2) A_1 - r^2 A_2)) B'_2}{2r B_2 (a^2 r^2 B_2 + A_1) A_2^2}, \]
\[ A_5 = \frac{A_2^2 (A_1 - A_1 A_2 - 4A_2^2)}{2r (a^2 r^2 B_2 + A_1) A_2^2} + \frac{a^2 B_2 (A_1 (A_1 A_2 + a^2 B_2 ((1 + r^2) A_1 - (1 + 2r^2) A_2))}{2r B_2 (a^2 r^2 B_2 + A_1) A_2^2}. \]

These equations are in agreement with those presented in Refs. [25, 33, 39] (for a more detailed derivation see, e.g., Ref. [11]).

The matter equations are
\[ \delta' + \theta \mathcal{H}^{-1} + 3 \Phi' - 3 \mathcal{H}^2 E'' - 6 \mathcal{H} \mathcal{H}' E' + k^2 E' = 0, \]
\[ \theta' + \theta + k^2 E' \mathcal{H}' - k^2 \Psi \mathcal{H}^{-1} + k^2 \mathcal{H} (E'' + E') = 0, \]
where \( \delta \) and \( \theta \) are the matter density contrast and peculiar velocity divergence, respectively. Differentiating and combining Eqs. (52) and (53) we obtain
\[ \delta'' + \left( 1 + \frac{\mathcal{H}'}{\mathcal{H}} \right) \delta' + \frac{k^2 \Psi}{\mathcal{H}^2} - 6 \mathcal{E}' (2 \mathcal{H}^2 + \mathcal{H} (\mathcal{H}'' + \mathcal{H}')) - 3 \mathcal{H} E'' (5 \mathcal{H}' + \mathcal{H}) - 3 \mathcal{E} (3) \mathcal{H}^2 + 3 \left( 1 + \frac{\mathcal{H}'}{\mathcal{H}} \right) \Phi' + 3 \Psi'' = 0. \]

Note that \( E \) enters the equations only with derivatives; one could then define a new variable \( Z = E' \) to lower the degree of the equations.\(^4\) One could also adopt the gauge-invariant variables
\[ \delta \to \delta + 3 \mathcal{H}' E', \]
\[ \theta \to \theta - k^2 \mathcal{H} E' \]

to bring the matter conservation equations into the standard form of a longitudinal gauge but since this renders the other equations somewhat more complicated we will not employ them.

**IV. QUASI-STATIC LIMIT**

Large-scale structure experiments predominantly probe modes within the horizon. Conveniently, in the subhorizon and quasi-static limit, the cosmological perturbation equations simplify dramatically. In this section we consider this quasi-static (QS) limit of subhorizon structures in massive bigravity.

The subhorizon limit is defined by assuming \( k \gg \mathcal{H} \), while the QS limit assumes that modes oscillate on a Hubble timescale: \( \Xi' \sim \Xi \) for any variable \( \Xi \).\(^5\) Concretely, this means that we consider the regime where \( (k^2 / \mathcal{H}^2) \Xi_i \gg \Xi_i \sim \Xi'_i \sim \Xi''_i \) for each field \( \Xi_i = \{ \Psi, \Phi, \Psi_f, \Phi_f, \Delta E, E \} \). We additionally take \( \delta (k / \mathcal{H})^2, \delta' (k / \mathcal{H})^2 \gg \theta / \mathcal{H} \). In this limit we obtain the system of equations
\[ 3k^2 \Delta E + \left( 9 + \frac{6k^2}{B_2 a^2 \tau} \right) \Phi - 9 \Phi_f - \frac{3 \delta \rho}{B_2 \tau} = 0, \]
\[ \frac{1}{2} a^2 r A_3 \Delta E + \Phi + \Psi = 0, \]
\[ 3 \frac{k^2 A_3}{B_2} \Delta E + \left( 9 \frac{A_1}{B_2} + \frac{6k^2}{B_2 a^2 \tau} \right) \Phi + \left( 9 + \frac{6k^2}{B_2 a^2 \tau} \right) \Psi - 9 \frac{A_3}{B_2} \Phi_f - 9 \frac{A_4}{A_1} \Psi_f = 0, \]

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\(^4\) \( E \) only appears without derivatives in the mass terms, specifically in differences with \( E_f \), and so all appearances of \( E \) are accounted for by the separate gauge-invariant variable \( \Delta E \).

\(^5\) Recall that we are using the dimensionless \( N = \log a \) as our time variable.
3k^2 \Delta E - \left(9 + \frac{6k^2}{\delta B_2^2}\right) \Phi_f + 9\Phi = 0, \quad (60)
\frac{-\sigma^2 A_1 A_2}{2r A_2^2} \Delta E + \Phi_f + \Psi_f = 0, \quad (61)
3k^2 \frac{A_3}{B_2} \Delta E + \frac{9A_3}{B_2} \Phi - \left(6r^2 A_2 - \frac{A_3}{B_2} \right) \Phi_f - \left(\frac{6k^2 r A_2}{a^2 B_2 A_1^3} + \frac{9A_2}{A_1^3}\right) \Psi_f = 0, \quad (62)

where we have used the momentum contraints, Eqs. (40) and (44), to replace time derivatives of \Phi and \Phi_f. The above set of equations can be solved for \Psi, \Phi, \Psi_f, \Phi_f, and \Delta E in terms of \delta (see also Ref. 33):

\Psi = \frac{3}{k^4} \left(3a^2 A_1 A_3 B_2^3 + 3a^2 A_2 A_3 B_2^3 r^2 + k^2 (2A_1 A_2^3 r^3 - 2B_2 r (A_2 B_2 - 2 A_1 A_3)) \right) \frac{\Omega_m H^2}{a^2} \delta, \quad (63)
\Phi = -\frac{3}{k^4} \left(3a^2 A_1 A_3 B_2 + 3a^2 A_2 A_3 B_2 r^2 + k^2 (r (4A_1 A_3 - 2 A_2 B_2) + 2 A_1 A_3 r^3) \right) \frac{\Omega_m H^2}{a^2} \delta, \quad (64)
\Psi_f = -\frac{3}{k^4} \left(-3a^2 A_1 A_2 A_3 B_2^2 - 3a^2 A_2 A_3 A_3 B_2^2 r^2 + 2A_1 k^2 r (a^2 A_1 A_3^2 - a^2 (A_1 + A_2) A_3 B_2 + A_2 B_2) \right) \frac{\Omega_m H^2}{a^2} \delta, \quad (65)
\Phi_f = -\frac{3}{k^4} \left(3a^2 A_1 A_3 B_2 + 3a^2 A_1 A_3 B_2 r^2 + 2A_1 k^2 r (A_3 - B_2) \right) \frac{\Omega_m H^2}{a^2} \delta, \quad (66)
\Delta E = \frac{3r (3a^2 (A_1 - A_3) A_2^2 + 2A_1 k^2 r (B_2 - A_3) \right) \frac{\Omega_m H^2}{a^2} \delta. \quad (67)

The QS limit is, however, only a good approximation if the full set of equations produces a stable solution for large k. In fact, if the solutions are not stable, the derivative terms we have neglected are no longer small (as their mean values vary on a faster timescale than Hubble), and the QS limit is never reached. We therefore need to analyze the stability of the full theory.

V. INSTABILITIES

Let us go back to the full linear equations, presented in section III. While we have ten equations for ten variables, there are only two independent degrees of freedom, corresponding to the scalar modes of the two gravitons. The degrees-of-freedom counting goes as follows (see Ref. 12 for an in-depth discussion of most of these points): four of the metric perturbations (\delta g_{00}, \delta g_{0i}, \delta f_{00}, and \delta f_{0i}) and \theta are nondynamical, as their derivatives do not appear in the second-order action. These can be integrated out in terms of the dynamical variables and their derivatives. We can further gauge fix two of the dynamical variables. Finally, after the auxiliary variables are integrated out, one of the initially-dynamical variables becomes auxiliary (its derivatives drop out of the action) and can itself be integrated out.

This leaves us with two independent dynamical degrees of freedom. The aim of this section is to reduce the ten linearized Einstein equations to two coupled second-order equations, and then ask whether the solutions to that system are stable. We will choose to work with \Phi and \Psi as our independent variables, eliminating all of the other perturbations in their favor.

We can begin by eliminating \Psi_f, \Phi_f, \Delta E, and their derivatives using the \(0 - 0\), \(i - i\), and \(i - j\) components of the g-metric perturbation equations. We will herein refer to these equations as \(g_{00}, g_{ii}, \text{ and } g_{ij}\) and so for the sake of conciseness. Doing this we see also that the \(g_{ij}\) and \(f_{ij}\) equations are linearly related. Then we can replace \delta and \theta with the help of the \(g_{00}\) and \(f_{00}\) equations. Finally, one can find a linear combination of the \(f_{00}\) and \(g_{ii}\) equations which allows one to express \(E^0\) as a function of \(\Phi, \Psi, \text{ and their derivatives. In this way, we can write our original ten equations as just two second order equations for } X_i \equiv \{\Phi, \Psi\} \text{ with the following structure:}

\[X''_i + F_{ij} X'_j + S_{ij} X_j = 0, \quad (68)\]

where \(F_{ij}\) and \(S_{ij}\) are complicated expressions that depend only on background quantities and on \(k\). The eigenfrequencies of these equations can easily be found by substituting \(X = X_0 e^{i \omega N}\), assuming that the dependence of \(\omega\) on

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6 We thank Macarena Lagos and Pedro Ferreira for discussions on this point.
time is negligibly small. For instance, assuming that only $\beta_1$ is nonzero, in the limit of large $k$ we find

$$\omega_{\beta_1} = \pm \frac{k}{H} \frac{\sqrt{-1 + 12r^2 + 9r^4}}{1 + 3r^2},$$

(69)

plus two other solutions that are independent of $k$ and are therefore subdominant. One can see then that real solutions (needed to obtain oscillating, rather than growing and decaying, solutions for $X$) are found only for $r > 0.28$, which occurs for $N = -0.4$, i.e., $z \approx 0.5$. At any epoch before this, the perturbation equations are unstable for large $k$. In other words, we find an imaginary sound speed. This behavior invalidates linear perturbation theory on subhorizon scales and may rule out the model, if the instability is not cured at higher orders, for instance by a phenomenology related to the Vainshtein mechanism.

Now let us move on to more general models. Although the other one-parameter models are not viable in the background (i.e., none of them have a matter dominated epoch in the asymptotic past and produce a positive Hubble rate), it is worthwhile to study the eigenfrequencies in these cases too, particularly because they will tell us the early-time behavior of the viable multiple-parameter models. For simplicity, from now on we refer to a model in which, e.g., only $\beta_1$ and $\beta_2$ are nonzero as the $\beta_1\beta_2$ model, and so on.

At early times, every viable, finite-branch, multiple-parameter model reduces to the single-parameter model with the lowest-order interaction. For instance, the $\beta_1\beta_2$, $\beta_1\beta_3$, and $\beta_1\beta_2\beta_3$ models all reduce to $\beta_1$, the $\beta_2\beta_3$ model reduces to $\beta_2$, and so on. Similarly, in the early Universe, the viable, infinite-branch models reduce to single-parameter models with the highest-order interaction. Therefore, in order to determine the early-time stability, we need to only look at the eigenfrequencies of single-parameter models, for which we find

$$\omega_{\beta_2} = \pm \frac{k}{H r},$$

(70)

$$\omega_{\beta_3} = \pm \frac{ik\sqrt{r^4 - 8r^2 + 3}}{\sqrt{3} H (r^2 - 1)},$$

(71)

$$\omega_{\beta_4} = \pm \frac{k}{\sqrt{2H}}.$$  

(72)

Therefore, the only single-parameter models without instabilities at early times are the $\beta_2$ and $\beta_4$ models. Using the rules discussed above, we can now extend these results to the rest of the bigravity parameter space.

Since much of the power of massive bigravity lies in its potential to address the dark energy problem in a technically natural way, let us first consider models without an explicit $g$-metric cosmological constant, i.e., $\beta_0 = 0$. On the finite branch, all such models with $\beta_1 \neq 0$ reduce, at early times, to the $\beta_1$ model, which has an imaginary eigenfrequency for large $k$ (69) and is therefore unstable in the early Universe. Hence the finite-branch $\beta_1\beta_2\beta_3\beta_4$ model and its subsets with $\beta_1 \neq 0$ are all plagued by instabilities. All of these models have viable background evolution. This leaves the $\beta_2\beta_3\beta_4$ model; this is stable on the finite branch as long as $\beta_2 \neq 0$, but its background is not viable. We conclude that there are no models with $\beta_0 = 0$ which live on a finite branch, have a viable background evolution, and predict stable linear perturbations at all times.

This conclusion has two obvious loopholes: either including a cosmological constant, $\beta_0$, or turning to an infinite-branch model. We first consider including a nonzero cosmological constant, although this may not be as interesting theoretically as the models which self-accelerate. Adding a cosmological constant can change the stability properties, although it turns out not to do so in the finite-branch models with viable backgrounds. In the $\beta_0\beta_1$ model, the eigenfrequencies

$$\omega_{\beta_0\beta_1} = \pm \frac{k\sqrt{9r^4 + 2(\beta_0/\beta_1) r + 12r^2 - 1}}{H (3r^2 + 1)},$$

(73)

are unaffected by $\beta_0$ at early times and therefore still imply unstable modes in the asymptotic past. This result extends (at early times) to the rest of the bigravity parameter space with $\beta_0, \beta_1 \neq 0$. No other finite-branch models yield viable backgrounds. Therefore, all of the solutions on a finite branch, for any combination of parameters, are either unviable (in the background) or linearly unstable in the past.

Let us now turn to the infinite-branch models. In this case, it turns out that there exists a small class of viable models which have stable cosmological evolution: models where the only nonvanishing parameters are $\beta_0, \beta_1, \text{and} \beta_4$.
Therefore, the condition for the stability of this model in the infinite branch, where one might wonder whether this expression for \( \omega_{\beta_0 \beta_1 \beta_4} \) evolves from infinity in the past and asymptotes to a finite de Sitter value in the future. For these \( \beta_0 \beta_1 \beta_4 \) models we perform a similar eigenfrequency analysis and obtain

\[
\omega_{\beta_0 \beta_1 \beta_4} = \pm \frac{k}{\sqrt{3}} \sqrt{9 + 2 \beta_0 \beta_4 / \beta_1^2} r^4 + 2 (\beta_0 / \beta_1) r + 12 r^2 - 1 + (\beta_4 / \beta_1) [2(\beta_4 / \beta_1) r^6 - 6r^5 - 8r^3].
\] (74)

Restricting ourselves to the self-accelerating models (i.e., \( \beta_0 = 0 \)), we obtain

\[
\omega_{\beta_1 \beta_4} = \pm \frac{k}{\sqrt{3}} \sqrt{9r^4 + 12r^2 - 1 + (\beta_4 / \beta_1) [2(\beta_4 / \beta_1) r^6 - 6r^5 - 8r^3]].
\] (75)

Notice that, for large \( r \), the eigenvalues (74) and (75) reduce to the expression (72) for \( \omega_{\beta_4} \). This frequency is real, and therefore the \( \beta_1 \beta_4 \) model, as well as its generalization to include a cosmological constant, is stable on the infinite branch at early times.

It is interesting to note that the eigenfrequencies can also be written as

\[
\omega_{\beta_0 \beta_1 \beta_4} = \pm \frac{ik}{\sqrt{3}} \sqrt{r''/r'}. 
\] (76)

Therefore, the condition for the stability of this model in the infinite branch, where \( r' < 0 \), is simply \( r'' > 0 \). One might wonder whether this expression for \( \omega \) is general or model specific. While it does not hold for the \( \beta_2 \) and \( \beta_3 \) models, Eqs. (69) and (71), it is valid for all of the submodels of \( \beta_0 \beta_1 \beta_4 \), including Eqs. (69) and (72). We can see from this, for example, that the finite-branch (\( r' > 0 \)) \( \beta_1 \) model is unstable at early times because initially \( r'' \) is positive. In Fig. 1 we show schematically the evolution of the \( \beta_1 \beta_4 \) model on the finite and infinite branches. The stability condition on either branch is \( r''/r' = dr'/dr < 0 \). For the parameters plotted, \( \beta_1 = 0.5 \) and \( \beta_4 = 1 \), one can see graphically that this condition is met, and hence the model is stable, only at late times on the finite branch but for all times on the infinite branch. Our remaining task is to extend this to other parameters.

Let us now prove that the infinite-branch \( \beta_1 \beta_4 \) model is stable at all times for all viable choices of the parameters. We have already seen that at early times, \( r \to \infty \), and the eigenfrequencies match those in the \( \beta_4 \) model (72) which are purely real. What about later times? The discriminant is positive and hence the model is stable whenever \( r > 1 \). Furthermore, in previous work we showed that background viability and the condition that we live on the infinite branch restrict us to the parameter range \( 0 < \beta_4 < 2\beta_1 \). The question then is: do the infinite-branch models in this region of the parameter space always have \( r > 1 \)?

The answer is yes. To see this, consider the algebraic equation for \( r \), which can be determined by combining the \( g \) and \( f \)-metric Friedmann equations (see Eq. (2.17) of Ref. [33]), and focus on the asymptotic future by taking \( \rho \to 0 \). This gives

\[
\beta_4 r_c^3 - 3\beta_1 r_c^2 + \beta_1 = 0, \tag{77}
\]
where \( r_c \) is the far-future value of \( r \). When \( \beta_4 = 2\beta_1 \) exactly, this is solved by \( r_c = 1 \). We must then ask whether for \( 0 < \beta_4 < 2\beta_1 \), \( r_c \) remains greater than 1. Writing \( p \equiv r_c - 1 \), using Descartes’ rule of signs, and restricting ourselves to \( 0 < \beta_4 < 2\beta_1 \), we can see that \( p \) has one positive root, i.e., there is always exactly one solution with \( r_c > 1 \) in that parameter range. Therefore, in all infinite-branch solutions with \( 0 < \beta_4 < 2\beta_1 \), \( r \) evolves to some \( r_c > 1 \) in the asymptotic future. We conclude that all of the infinite-branch \( \beta_1\beta_4 \) cosmologies which are viable at the background level are also linearly stable at all times, providing a clear example of a bimetric cosmology which is a viable competitor to \( \Lambda\text{CDM} \).

### VI. QUASI-STATIC LIMIT OF INFINITE-BRANCH BIGRAVITY

In the previous section we found that most bigravity models which are viable at the background level suffer from a linear instability at early times. A prominent exception was the model with the \( \beta_1 \) and \( \beta_4 \) interactions turned on (i.e., the first-order interaction between the two metrics and the f-metric cosmological constant) in the case of solutions on the “infinite branch”, where \( r \) evolves from infinity at early times to a finite value in the far future. This means that we can safely use the QS approximation for the subhorizon modes in the infinite-branch \( \beta_1\beta_4 \) model, hereafter referred to (interchangeably) as infinite-branch bigravity (IBB); in this section, we compare the QS limit of this model to observations.

The background cosmology of IBB was studied in Refs. [33, 40]. Ref. [33] further studied the linear perturbations and quasi-static limit, finding results in agreement with those presented in the following two sections. Using the Friedmann equations, it has been shown that the background cosmology only selects a curve in the parameter space, given by

\[
\beta_4 = \frac{3\Omega_{mg,0} \beta_1^2 - \beta_1^4}{\Omega_{mg,0}},
\]

where we recall that \( \Omega_{mg,0} \equiv \beta_1 r_0 \) is the present-day effective density of dark energy that appears in the Friedmann equation (15). This does not need to coincide with the value of \( \Omega_A \) derived in the context of \( \Lambda\text{CDM} \) models; indeed, the best-fit value to the background data is \( \Omega_{mg,0} = 0.84^{+0.03}_{-0.02} \) [40]. Furthermore, as discussed in the previous subsection, to ensure that we are on the infinite branch we impose the condition \( 0 < \beta_4 < 2\beta_1 \).

The QS-limit equations in terms of \( \delta \) now read (recall \( B_1 = \beta_1 + \beta_4 r^3 \), see Eq. (23):

\[
k^2 \Psi = \frac{(\frac{3}{2}a^2 \beta_1 (9\beta_1 (r^2 - 1) r^2 + (r^2 - 2) B) - \frac{1}{2}k^2 r (9\beta_1 (r^2 - 1) + (8r^2 + 9) B)) \Omega_m H^2}{3a^2 \beta_1 (r^2 + 1)^2 B + k^2 (2r^3 B + 3\beta_1 (r^2 - 1) r + 3rB)}\delta,
\]

\[
k^2 \Phi = \frac{(3a^2 \beta_1 (r^2 + 1) B + \frac{1}{2}k^2 r (9\beta_1 (r^2 - 1) + (4r^2 + 9) B)) \Omega_m H^2}{2a^2 \beta_1 (r^2 + 1)^2 B + k^2 (2r^3 B + 3\beta_1 (r^2 - 1) r + 3rB)}\delta,
\]

\[
k^2 \Phi_f = \frac{(3a^2 \beta_1 (r^2 - 1) (2r^2 - 1) B + k^2 r (3\beta_1 (r^2 - 1) + (2r^2 + 3) B)) \Omega_m H^2}{2a^2 \beta_1 (r^2 + 1)^2 B + k^2 r (3\beta_1 (r^2 - 1) + (2r^2 + 3) B)}\delta,
\]

\[
k^2 \Psi_f = \frac{(3a^2 \beta_1 (r^2 + 1)^2 B + k^2 r (3\beta_1 (r^2 - 1) + (2r^2 + 3) B)) \Omega_m H^2}{2a^2 \beta_1 (r^2 + 1)^2 B + k^2 r (3\beta_1 (r^2 - 1) + (2r^2 + 3) B)}\delta,
\]

\[
k^2 \delta = \frac{(2k^2 r^2 B - 2a^2 \beta_1 r (3\beta_1 (r^2 - 1) + B)) \Omega_m H^2}{2a^2 \beta_1 (r^2 + 1)^2 B + \beta_1 a^2 k^2 r (3\beta_1 (r^2 - 1) + (2r^2 + 3) B)}\delta
\]

where we have used the combination \( B \equiv 3\beta_1 (r^2 + 1) - 2B_1 \) to further simplify the expressions.

In order to compare with observations, we calculate two common modified gravity parameters: the anisotropic stress, \( \eta \equiv -\Phi/\Psi \), and the effective gravitational coupling for the growth of structures, \( Y \equiv -2k^2 \Psi/(3H^2 \Omega_m \delta_m) \). In general relativity with \( \Lambda\text{CDM} \), \( \eta = Y = 1 \), while in \( \beta_1\beta_4 \) IBB they possess the following structure,

\[
\eta = H_2 \frac{1 + H_4(k/H)^2}{1 + H_3(k/H)^2},
\]

\[
Y = H_1 \frac{1 + H_3(k/H)^2}{1 + H_5(k/H)^2},
\]

with coefficients

\[
H_1 = -\frac{9\beta_1 (r^2 - 1) r^2 + (r^2 - 2) B}{2 (r^2 + 1)^2 B},
\]

(86)
FIG. 2. The modified-gravity parameters, $Y$ and $\eta$, for the $\beta_1\beta_4$ infinite-branch (IBB) model, from $z = 5$ until the asymptotic (de Sitter) future. Notice that the parameters approach a constant late-time value until a late era of horizon exit, when the $k = 0.1/h/\text{Mpc}$ mode becomes superhorizon and the QS limit breaks down. The horizontal line corresponds to the $\Lambda$CDM prediction for $Y$ and $\eta$, and the vertical line is the present day. These curves are very weakly dependent on $k$. For concreteness, we use the best-fit values $\beta_1 = 0.48$ and $\beta_4 = 0.94$, calculated in Sec. [VII].

\begin{align}
H_2 &= -\frac{2(r^2 + 1)B}{9\beta_1(r^2 - 1)r^2 + (r^2 - 2)B}, \\
H_3 &= -\frac{\mathcal{H}^2r(9\beta_1(r^2 - 1) + (8r^2 + 9)B)}{3a^2\beta_1(9\beta_1(r^2 - 1)r^2 + (r^2 - 2)B)}, \\
H_4 &= \frac{\mathcal{H}^2r(9\beta_1(r^2 - 1) + (4r^2 + 9)B)}{6a^2\beta_1(r^2 + 1)B}, \\
H_5 &= \frac{\mathcal{H}^2r(6r^2B + 9\beta_1(r^2 - 1) + 9B)}{6a^2\beta_1(r^2 + 1)^2B}.
\end{align}

As a side remark, we note that in this model the asymptotic past corresponds to the limit $r \to \infty$ and $r' \to -\frac{3}{2}r$, i.e., $r \to a^{-3/2}$. This implies that $b \sim a^{-1/2}$, i.e., the second metric initially collapses while “our” metric expands. On the approach to the final de Sitter stage, $r$ approaches a constant $r_c$, so the scale factors $a$ and $b$ both expand exponentially. The $f$-metric scale factor, $b$, therefore undergoes a bounce in this model.

This bounce has an unusual consequence. Recall from Eq. [14] that, after imposing the Bianchi identity, we have $f_{00} = -\dot{b}/\mathcal{H}^2$. Therefore, when $b$ bounces, $f_{00}$ becomes zero: at that one point, the lapse function of the $f$ metric vanishes.\footnote{Moreover, the square root of this, $\dot{b}/\mathcal{H}$, appears in the mass terms. This quantity starts off negative at early times and then becomes positive.} We believe, however, that this does not render the solution unphysical, for the following reasons. First, the $f$ metric does not couple to matter and so, unlike the $g$ metric, it does not have a geometric interpretation. A singularity in the $f$-metric therefore does not necessarily imply a singularity in observable quantities. In fact, we find no singularity in any of our background or perturbed variables. Second, although the Riemann tensor for the $f$ metric is singular when $f_{00} = 0$, the Lagrangian density $\sqrt{-\det fR_f}$ remains finite and non-zero at all times, so the equations of motion can be derived at any points in time.

In the asymptotic past, every infinite-branch $\beta_1\beta_4$ model satisfies

$$\lim_{N \to -\infty} \eta = \frac{1}{2} \quad \text{and} \quad \lim_{N \to -\infty} Y = \frac{4}{3}$$

and therefore does not reduce to the standard $\Lambda$CDM. In the future one finds $\eta \to 1$ if $k$ is kept finite, but this is somewhat fictitious: for any finite $k$ there will be an epoch of horizon exit in the future after which the subhorizon...
QS approximation breaks down. We can see both this asymptotic past and future behavior in Fig. 2 although the late-time approach of η to unity is not easily visible.

VII. COMPARISON TO MEASURED GROWTH DATA

In this section we compare the predictions in the quasi-static approximation to the measured growth rate. In Ref. [33], we discussed the numerical results of the modified-gravity parameters, Eq. (54) [55], for β1β4 infinite-branch bigravity and their early-time limits [10] and compared to the data. Although we found strong deviations from the ΛCDM values, the model is at present still in agreement with the observed growth data. However, as we mentioned, future experiments will be able to distinguish between the predictions of the ΛCDM and bimetric gravity for η and Y.

We use the dataset compiled by Ref. [45] containing the current measurements of the quantity $f(z)σ_S(z) = f(z)G(z)σ_S$, (92)

where $f(z) ≡ δ'/δ$ and $G(z)$ is the growth factor normalized to the present. The data come from the 6dFGS [16], LRG200, LRG 3.1, BOSS [38], WiggleZ [49], and VEPERS [50] surveys. These measurements can be compared to the theoretical growth rate which follows from integrating Eq. (54) in the QS limit:

$$δ''_m + δ'_m \left(1 + \frac{H'}{H}\right) - \frac{3}{2}Y(k)Ω_m δ_m = 0.$$ (93)

The theoretically expected and observed data, $t_i$ and $d_i$, respectively, can be compared to compute

$$χ^2_{fσ_S} = \sum_{ij} (d_i - σ_st_i) C_{ij}^{-1} (d_j - σ_st_j),$$ (94)

where $C_{ij}$ denotes the covariance matrix. Since no model-free constraints on $σ_S$ exist, one can remove this dependency with a marginalization over positive values which can be performed analytically:

$$χ^2_{fσ_S} = S_{20} - \frac{S_{11}}{S_{02}} + \log S_{02} - 2 \log \left(1 + \text{Erf} \left(\frac{S_{11}}{\sqrt{2S_{02}}}\right)\right).$$ (95)

Here, $S_{11} = d_i C_{ij}^{-1} t_j$, $S_{20} = d_i C_{ij}^{-1} t_j$, and $S_{02} = t_i C_{ij}^{-1} t_j$. Note that $Y$ is (weakly) scale-dependent but the current observational data are averaged over a range of scales. For the computation of the likelihood, we assume an average scale $k = 0.1h/\text{Mpc}$.

As shown in Fig. 3, the confidence region obtained from the growth data is in agreement with type Ia supernovae (SNe) data (see Ref. [40] for the likelihood from the SCP Union 2.1 Compilation of SNe Ia data [51]). The growth data alone provides $β_1 = 0.40^{+0.14}_{-0.15}$ and $β_2 = 0.67^{+0.33}_{-0.31}$ with a $χ^2_{min} = 9.72$ (with 9 degrees of freedom) for the best-fit value and is in agreement with the SNe Ia likelihood. The likelihood from growth data is, however, a much weaker constraint than the likelihood from background observations. Thus, the combination of both likelihoods, providing $β_1 = 0.48^{+0.05}_{-0.16}$ and $β_2 = 0.94^{+0.11}_{-0.05}$, is similar to the SNe Ia result alone.

Note that those favored parameter regions were obtained by integrating the 2-dimensional likelihood and are not Gaussian distributed due to the degeneracy in the parameters $β_1$ and $β_2$ (see Eq. (78)). This degeneracy curve is unaffected by additional growth data and is still parameterized by the SNe Ia result $Ω_{m0} = 1 - Ω_{mg0} = 0.16^{+0.02}_{-0.03}$ (note that the combination of the most likely parameters predicts, however, $Ω_{m0} = 0.18$). According to Eq. (27), the equation of state (EOS) of modified gravity, $w_{mg}$, is best fit by $w_0 = -0.79$ and $w_a = 0.21$, where we use the CPL parametrization [52] [53].

$$w(z) = w_0 + w_az/(1 + z).$$ (96)

However, since we approximated the EOS near the present time, we can not expect Eq. (96) to fit the real EOS well at early times or in the future. As shown in Fig. 4 the fit is in fact valid in the past only up to $z ≈ 0.5$, while in the future the limit $w_{mg} → -1$ is lost.

10 Note that Ref. [33] uses a slightly different effective gravitational constant, $Q ≡ ηY$. 
FIG. 3. Likelihood from measured growth rates, where the red, orange, and light orange filled regions correspond to 68%, 95% and 99.7% confidence levels. Both black (68%) and gray (99.7%) regions illustrate the combination of the likelihoods from measured growth data and type Ia supernovae. The blue line indicates the degeneracy curve corresponding to the background best-fit points. Note that the viability condition enforces the likelihood to vanish when β_4 > 2β_2.

FIG. 4. The equation of state (EOS, solid blue) in the IBB model with β_1 = 0.48, β_4 = 0.94, along with the CPL approximation w(a) ≈ w_0 + w_a z/(1 + z) (dotted green). In the asymptotic future, w_{mg} tends to −1, i.e., the EOS of a cosmological constant (dashed red).

For one specific choice of parameters, corresponding to the best-fit values, we compared the quantity f(z)G(z) with the measured growth data and fits from ΛCDM in Fig. 5. Although the modified-gravity parameters differ significantly from the ΛCDM result Y = η = 1, the prediction for f(z)G(z) is in good agreement with measurements and is close to the ΛCDM result.

The difference between the growth rate f(z) in the best-fit model and ΛCDM is, however, quite large. Therefore, the common approximation f ≈ Ω_m fits the growth rate very badly, even if the range in the redshift is small (where
$f(z)$ is still smaller than unity \[33\]. We have found a two-parameter scheme,
\[
f(z) \approx \Omega_m^{\gamma_0}(1 + \alpha z/(1 + z)),
\]
which is able to provide a much better fit (see Fig. 5). Using this approximation, we obtain $\gamma_0 = 0.47$ and $\alpha = 0.21$ as best-fit values.

**VIII. CONCLUSIONS AND OUTLOOK**

We have investigated the stability of linear cosmological perturbations in bimetric massive gravity. Many models with viable background cosmologies exhibit an instability on small scales until fairly recently in cosmic history. However, we also found a class of viable models which are stable at all times: infinite-branch bigravity (IBB) with the interaction parameters $\beta_1$ and $\beta_4$ turned on. In these models, the ratio $r = b/a$ of the two scale factors decreases from infinity to a finite late-time value. IBB is able to fit observations at the level of both the background (type Ia supernovae) and linear, subhorizon perturbations (growth histories) without requiring an explicit cosmological constant for the physical metric, although the region of likely parameters is small. The combination of both likelihoods yields the parameter constraints $\beta_1 = 0.48^{+0.05}_{-0.06}$ and $\beta_4 = 0.94^{+0.11}_{-0.05}$. IBB with these best-fit parameters predicts $\Omega_{m0} = 0.18$ and an equation of state $w(z) \approx -0.79 + 0.21 z/(1 + z)$. The growth rate, $f \approx d \ln \delta/d \ln a$, is approximated very well by the two-parameter fit $f(z) \approx \Omega_m^{\gamma_0}[1 + 0.21 z/(1 + z)]$. Additionally, the two main modified-gravity parameters, the anisotropic stress $\eta$ and modification to Newton’s constant $Y$, tend to $\eta = 1/3$ and $Y = 4/3$ for early times and therefore do not reduce to the standard $\Lambda$CDM result. The predictions of this two-parameter model will be testable by near-future experiments \[54\].

On the surface, our results would seem to place in jeopardy a large swath of bigravity’s parameter space, such as the “minimal” $\beta_1$-only model which is the only single-parameter model that is viable at the background level \[40\]. It is important to emphasize that the existence of such an instability does not automatically rule these models out.
merely impedes our ability to use linear theory on deep subhorizon scales (recall that the instability is problematic specifically for large \( k \)). Models that are not linearly stable can still be realistic if only the gravitational potentials become nonlinear, or even if the matter fluctuations also become nonlinear but in such a way that their properties do not contradict observations. The theory can be saved if, for instance, the instability is softened or vanishes entirely when nonlinear effects are taken into account. We might even expect such behavior: bigravity models exhibit a Vainshtein mechanism \[43, 44\] which restores general relativity in environments where the new degrees of freedom are highly nonlinear.

Consequently there are two very important questions for future work: can these unstable models still accurately describe the real Universe, and if so, how can we perform calculations for structure formation?

Until these questions are answered, the \( \beta_1 \beta_4 \) infinite-branch model seems to be the most promising target at the moment for studying massive bigravity. Because this instability appears to be absent in the superhorizon limit, it may also be feasible to test the unstable models using large-scale modes.

What other escape routes are there? Throughout this analysis we have assumed that only one of the metrics couples to matter. A possible way to cure bimetric gravity from instabilities while only allowing one nonvanishing \( \beta \) parameter could be to allow matter to couple to both metrics \[24, 55\]. In such a theory, the finite-branch solutions asymptote to a nonzero value for \( r \) in the far past, so these theories may avoid the instability. This would introduce a new coupling parameter, so if only one \( \beta \) parameter is turned on, there are two free parameters and such a model is arguably as predictive as the \( \beta_1 \beta_4 \) model. The cosmological solutions in this type of theory and its consequences for linear perturbations will be discussed in a future work (in prep.).

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