Complexities of 3-manifolds from triangulations, Heegaard splittings, and surgery presentations

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Abstract. We study complexities of 3-manifolds defined from triangulations, Heegaard splittings, and surgery presentations. We show that these complexities are related by linear inequalities, by presenting explicit geometric constructions. We also show that our linear inequalities are asymptotically optimal. Our results are used in [Chab] to estimate Cheeger-Gromov $L^2$ $\rho$-invariants in terms of geometric group theoretic and knot theoretic data.

1. Introduction and main results

In this paper we study the relationship between various notions of complexities of 3-manifolds. In what follows, we always assume that 3-manifolds are compact.

Simplicial complexity. The first notion of complexity we consider is defined from triangulations. In this paper a triangulation designates a simplicial complex structure.

Definition 1.1. For a 3-manifold $M$, the simplicial complexity $c^{\text{simp}}(M)$ is defined to be the minimal number of 3-simplices in a triangulation of $M$.

A similar notion of complexity defined from more flexible triangulations is often considered in the literature (e.g., see [MPV09, JRT09, JRT11, JRT13]): a pseudo-simplicial triangulation of a 3-manifold $M$ is defined to be a collection of 3-simplices together with affine identifications of faces from which $M$ is obtained as the quotient space. The pseudo-simplicial complexity, or the complexity $c(M)$ of $M$ is defined to be the minimal number of 3-simplices in a pseudo-simplicial triangulation. For closed irreducible 3-manifolds, $c(M)$ agrees with Matveev’s complexity [Mat90] defined in terms of spines, unless $M = S^3, \mathbb{R}P^3, \text{or } L(3,1)$. Since the second barycentric subdivision of a pseudo-simplicial triangulation is a triangulation and a 3-simplex is decomposed to $(4!)^2 = 576$ 3-simplices in the second barycentric subdivision, we have

$$\frac{1}{576} c^{\text{simp}}(M) \leq c(M) \leq c^{\text{simp}}(M).$$

Heegaard-Lickorish complexity. Recall that a Heegaard splitting of a closed 3-manifold is represented by a mapping class in the mapping class group $\text{Mod}(\Sigma_g)$ of a surface $\Sigma_g$ of genus $g$. (Our precise convention is described in the beginning of Section 3.) The identity mapping class gives the standard Heegaard splitting of $S^3$ shown in Figure 11. It is well known that $\text{Mod}(\Sigma_g)$ is finitely generated; Lickorish showed that $\text{Mod}(\Sigma_g)$ is generated by the $\pm 1$ Dehn twists about the $3g - 1$ curves $\alpha_i, \beta_i$, and $\gamma_i$ shown in Figure 1 [Lic62, Lic64].

From this, a geometric group theoretic notion of complexity is defined for 3-manifolds as follows.

Definition 1.2. The Heegaard-Lickorish complexity $c^{\text{HL}}(M)$ of a closed 3-manifold $M$ is defined to be the minimal word length, with respect to the Lickorish generators, of a mapping class $h \in \text{Mod}(\Sigma_g)$ which gives a Heegaard splitting of $M$. 

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By definition, \( c^{HL}(S^3) = 0 \).

We remark that the Heegaard-Lickorish complexity tells us more delicate information than the Heegaard genus. It turns out that the difference of the Heegaard-Lickorish complexities of two 3-manifolds with the same Heegaard genus can be arbitrarily large, whereas the Heegaard genus of a 3-manifold is bounded by twice its Heegaard-Lickorish complexity. See Lemma 3.1 and related discussions in Section 3.

Our first result is the following relationship of the two complexities defined above.

**Theorem A.** For any closed 3-manifold \( M \neq S^3 \), \( c^{simp}(M) \leq 692 \cdot c^{HL}(M) \).

**Surgery complexity.** To define another notion of complexity of 3-manifolds from knot theoretic information, we consider Dehn surgery with integral coefficients. For a framed link \( L \) in \( S^3 \), let \( f(L) = \sum_i |f_i(L)| \) where \( f_i(L) \in \mathbb{Z} \) is the framing on the \( i \)th component of \( L \). Let \( n(L) \) be the number of split unknotted zero framed components of \( L \). We denote by \( c(L) \) the crossing number of a link \( L \) in \( S^3 \), that is, \( c(L) \) is the minimal number of crossings in a planar diagram of \( L \). As a convention, if \( L \) is empty, then \( c(L) = f(L) = n(L) = 0 \).

**Definition 1.3.** The surgery complexity of a closed 3-manifold \( M \) is defined by

\[
c^{surg}(M) = \min_L \{ 2c(L) + f(L) + n(L) \}
\]

where \( L \) varies over framed links in \( S^3 \) from which \( M \) is obtained by surgery.

We remark that we bring in \( n(L) \) to detect \( S^1 \times S^2 \) summands, which can be added to any 3-manifold by connected sum without altering \( c(L) \) and \( f(L) \) of a framed link \( L \) giving the 3-manifold. Note that \( n(L) = 0 \) for any \( L \) that gives \( M \) if \( M \) has no \( S^1 \times S^2 \) summand. In particular it is the case if \( M \) is irreducible. Note that \( c^{surg}(S^3) = 0 \) by our convention.

Our second result is the following relationship of the simplicial complexity and the surgery complexity.

**Theorem B.** For any closed 3-manifold \( M \neq S^3 \), \( c^{simp}(M) \leq 96 \cdot c^{surg}(M) \).

The proofs of Theorems A and B consist of geometric arguments which explicitly construct efficient triangulations from Heegaard splittings and from surgery presentations. Details are given in Sections 2 and 3.

**Optimality of Theorems A and B** It is natural to ask how sharp the inequalities in Theorems A and B are. This seems to be a nontrivial problem, since it appears to be hard to determine the complexities we consider, or even to find an efficient lower bound for them. We remark that the determination and lower bound problems for the pseudo-simplicial complexity \( c(M) \) have been studied extensively in the literature and regarded as difficult problems [Mat03, JRT13].
We show that the linear inequalities in Theorems A and B are asymptotically optimal. As explicit examples, the lens spaces $L(n,1)$ satisfy the following:

**Theorem C.** For any $n > 3$,

$$\frac{1}{4357080} c^{HL}(L(n,1)) \leq c^{simp}(L(n,1)),$$

$$\frac{1}{4357080} c^{surg}(L(n,1)) \leq c^{simp}(L(n,1)).$$

We also prove a similar inequality for a larger class of 3-manifolds. See Theorem 4.4 and related discussions in Section 4.

The optimality of our linear inequality can also be understood in terms of standard notations for asymptotic growth, as follows. Recall that we write $f(n) \in O(g(n))$ if $f$ is bounded above by $g$ asymptotically, that is, $\limsup_{n \to \infty} |f(n)/g(n)|$ is finite. Also, $f(n) \in o(g(n))$ if $f(n)$ is dominated by $g(n)$ asymptotically, that is, $\limsup_{n \to \infty} |f(n)/g(n)| = 0$.

We write $f(n) \in \Omega(g(n))$ if $f(n)$ is not dominated by $g(n)$.

Define two functions $s_{HL}(\ell)$ and $s_{surg}(k)$ by

$$s_{HL}(\ell) = \sup \{ c^{simp}(M) | c^{HL}(M) \leq \ell \},$$

$$s_{surg}(k) = \sup \{ c^{simp}(M) | c^{surg}(M) \leq k \},$$

where the supremums exist by Theorems A. In other words, $s_{HL}(\ell)$ is the “largest possible value” of the simplicial complexity for 3-manifolds with Heegaard-Lickorish complexity $\ell$ or less. We can interpret $s_{surg}(k)$ similarly. The following is a consequence of Theorem C:

**Corollary D.** $s_{HL}(\ell) \in O(\ell) \cap \Omega(\ell)$ and $s_{surg}(k) \in O(k) \cap \Omega(k)$.

The proofs of Theorem C and Corollary D are given in Section 4.

**Applications to universal bounds for Cheeger-Gromov invariants.** Results in this paper are closely related to the recent development of a topological approach to the universal bounds of Cheeger-Gromov $L^2$ $\rho$-invariants in [Chab]. In fact, Theorems A and B of this paper are used as essential ingredients in [Chab] to give explicit linear estimates of Cheeger-Gromov $\rho$-invariants of 3-manifolds in terms of geometric group theoretical and knot theoretical data. See Theorems 1.8 and 1.9 of [Chab]. This application is a major motivation of the present paper. Our inequalities in Theorems A and B are sharp enough to give results that the linear estimates in [Chab] are asymptotically optimal. See Theorem 7.8 of [Chab].

On the other hand, the lower bounds in Theorem C are proven by employing results of [Chab] which relate triangulations and the Cheeger-Gromov $\rho$-invariants. See Section 4 for more details.

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2. Linear complexity triangulations from surgery presentations

In this section we present a construction of an efficient triangulation from a surgery presentation. First we consider the special case of a link diagram with the blackboard framing.

**Lemma 2.1.** Suppose $D$ is a planar diagram of a nonempty link $L$ with $c$ crossings, in which each component is involved in a crossing. Let $M$ be the 3-manifold obtained by
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surgery on \( L \) along the blackboard framing of \( D \). Then \( M \) has simplicial complexity at most 96c.

Before we prove Lemma 2.1 we discuss its application.

Example 2.2. Consider the stevedore knot, which is \( 6_1 \) in the table in Rolfsen [Rol76], or KnotInfo [CL]. It has a planar diagram with 6 crossings, where 2 of them have the same sign but the other 4 have the opposite sign. Applying Reidemeister move I twice, we obtain a planar diagram with 8 crossings and writhe zero. Since the blackboard framing is the zero framing for this diagram, it follows that the zero surgery manifold \( M \) of \( 6_1 \) satisfies \( c_{\text{simp}}(M) \leq 96 \cdot 8 = 768 \).

The argument of Example 2.2 generalizes to the following observation, which tells us how to reduce a general integral coefficient surgery to the special case of Lemma 2.1. We say that a component of a link in \( S^3 \) is split if there is an embedded 3-ball in \( S^3 \) which contains the component and is disjoint from the other components.

Lemma 2.3. Suppose \( L \) is a framed link in \( S^3 \). Then there is a planar diagram for \( L \) with \( 2c(L) + f(L) \) or less crossings such that the blackboard framing agrees with the given framing of \( L \). Furthermore, a component of \( L \) is involved in a crossing unless it is a split unknotted zero framed component.

Proof. Choose a minimal planar diagram for \( L \), which has \( c(L) \) crossings. Let \( K_i \) be the \( i \)th component. Let \( w_i \) be the writhe of \( K_i \) (forgetting other components), that is, the blackboard framing on \( K_i \) is \( w_i \in \mathbb{Z} \). Since a crossing in the diagram contributes 1, 0, or \(-1\) to \( w_i \) for some \( i \), it follows that \( |w_1| + \cdots + |w_r| \leq c(L) \). Observe that if we add a local kink by Reidemeister move I, then the blackboard framing changes by \( \pm 1 \). Let \( n_i \in \mathbb{Z} \) be the given framing on \( K_i \). By adding \( n_i - w_i \) local kinks to \( K_i \), we obtain a new diagram, say \( D \), for which the blackboard framing agrees with the framing \( n_i \) on each component. The number of crossings of \( D \) is at most

\[
c(L) + |w_1| + \cdots + |w_r| + |n_1| + \cdots + |n_r| \leq 2c(L) + f(L).
\]

Since we have added \( n_i - w_i \) local kinks to \( K_i \), it follows that \( K_i \) is involved in no crossings only if \( K_i \) is an embedded circle in the planar diagram which is disjoint from other components and \( n_i = w_i = 0 \). Such a component is split, unknotted, and zero framed.

Proof of Theorem B. Recall that Theorem B says

\[
c_{\text{simp}}(M) \leq 96 \cdot c_{\text{sur}}(M)
\]

for \( M \neq S^3 \).

We need the following two observations: first, we have

\[
c_{\text{simp}}(M_1 \# M_2) \leq c_{\text{simp}}(M_1) + c_{\text{simp}}(M_2) - 2,
\]

since the connected sum of two triangulated 3-manifolds can be performed by deleting a 3-simplex from each and then gluing faces. Second, we have

\[
c_{\text{simp}}(S^1 \times S^2) \leq 96.
\]

For instance, by taking the product of a triangle triangulation of \( S^1 \) and its suspension which is a triangulation of \( S^2 \) and then by applying the standard prism decomposition to each product \( \Delta^1 \times \Delta^2 \) (see Figure 2), we obtain a triangulation of \( S^1 \times S^2 \) with \( 3 \cdot 6 \cdot 3 = 54 \) tetrahedra.
Choose a framed link \( L \) such that \( M \) is obtained by surgery on \( L \) and \( 2c(L) + f(L) + n(L) = c_{\text{surf}}(M) \). Choose a planar diagram \( D \) for \( L \) by invoking Lemma 2.3 and let \( D_0 \) be the subdiagram of the components of \( D \) not involved in any crossing. Since each component of \( D_0 \) is split, unknotted, and zero framed by Lemma 2.3, the number of them is at most \( n(L) \). Let \( M' \) be the 3-manifold obtained by surgery along the blackboard framing of \( D - D_0 \). Since each component of \( D_0 \) contributes an \( S^1 \times S^2 \) summand, \( M = M' \# k(S^1 \times S^2) \) with \( k \leq n(L) \).

If \( D - D_0 \) is empty, then \( c(L) = f(L) = 0 \) and \( M' = S^3 \); also, \( k \geq 1 \) since \( M \neq S^3 \). It follows that
\[
c_{\text{simp}}(M) \leq k \cdot c_{\text{simp}}(S^1 \times S^2) \leq 96 \cdot n(L)
\]
by using (2.1) and (2.2). This is the desired conclusion for this case.

If \( D - D_0 \) is nonempty, we have
\[
c_{\text{simp}}(M) \leq c_{\text{simp}}(M') + n(L) \cdot c_{\text{simp}}(S^1 \times S^2) \leq c_{\text{simp}}(M') + 96 \cdot n(L)
\]
by using (2.1) and (2.2). Since each component of \( D - D_0 \) is involved in a crossing and since the number of crossings of \( D - D_0 \) is equal to that of \( D \), which is not greater than \( 2c(L) + f(L) \) by our choice of \( D \) (see Lemma 2.3), we have
\[
c_{\text{simp}}(M') \leq 96 \cdot (2c(L) + f(L))
\]
by Lemma 2.1. From (2.3) and (2.4), the desired conclusion follows. \( \square \)

Proof of Lemma 2.1. We will construct a triangulation of the exterior of \( L \) which is motivated from J. Weeks’ SnapPea, and then will triangulate the Dehn filling tori in a compatible way.

First, by the subadditivity (2.1), we may assume that the diagram \( D \) is nonsplit. Also, \( D \) is nonempty by the hypothesis.

In what follows we view \( D \) as a planar diagram lying on \( S^2 \). Consider the dual graph \( G \) of \( D \), whose regions are quadrangles corresponding to crossings, as illustrated in the left of Figure 3. View the link \( L \) as a submanifold of \( S^2 \times [-1,1] \), and remove from \( S^2 \times [-1,1] \) an open tubular neighborhood \( \nu(L) \) of \( L \) which is tangential to \( S^2 \times \{-1,1\} \) at (each crossing) \( \times \{-1,1\} \); cutting along \( G \times [-1,1] \), we obtain pieces corresponding to the crossings of \( D \), as shown in the middle of Figure 3.

Cutting each piece along \( D \times [-1,1] \), we obtain 4 equivalent subpieces; a subpiece is shown in the right of Figure 3. The hatched regions represent \( \partial \nu(L) \). Each subpiece can be viewed as a cube shown in Figure 4. Let \( p \) be the vertex shown in the left of Figure 4 and triangulate the three faces not adjacent to \( p \) as in the right of Figure 4. By taking a cone from \( p \), we obtain a triangulation of the cubic subpiece. Since the triangulation of the faces away from \( p \) has 14 triangles, the cone triangulation of the
subpiece has 14 tetrahedra. By applying this to each subpiece, we obtain a triangulation of \(S^2 \times [-1, 1] \setminus \nu(L)\), which has \(14 \cdot 4c = 56c\) tetrahedra.

For \(t = -1, 1\), the triangulation restricts to a triangulation of \(S^2 \times \{t\}\) with \(8c\) triangles. Attaching two 3-balls triangulated as the cone of these triangulations, we obtain a triangulation of \(S^3 \setminus \nu(L)\) which has \(56c + 2 \cdot 8c = 72c\) tetrahedra.

In our triangulation, the \(8c\) hatched quadrangular regions are paired up to form \(4c\) annuli, and a boundary component of \(\nu(L)\) is a union of \(4k\) such annuli, where \(k\) is the number of times the corresponding component of \(L\) passes through a crossing. (Since a component may pass through the same crossing twice, \(k\) may not be equal to the number of crossings that the component passes through.) See the left of Figure 5, the hatched meridional annulus is one of these \(4k\) annuli. Also, the circle \(\alpha\) in the left of Figure 5 is the union of the top edges of the hatched quadrangles in Figure 4. Obviously \(\alpha\) is a longitude of \(L\) taken along the blackboard framing, along which we perform surgery. Similarly, the bottom edges of the hatched quadrangles form a parallel of \(\alpha\), which we denote by \(\alpha'\).

Let \(r\) be the number of the components of \(L\). Take \(r\) copies of the solid torus \(D^2 \times S^1\). Attach them to the exterior \(S^3 \setminus \nu(L)\) along orientation reversing homeomorphisms of boundary tori which takes the curves \(\alpha\) and \(\alpha'\) to meridians bounding disks and takes our hatched annulus to a longitudinal annulus, as shown in Figure 5. Pulling back the triangulation of \(\partial(S^3 \setminus \nu(L))\), we obtain a triangulation of \(\partial(D^2 \times S^1)\). It extends to a triangulation of \(D^2 \times S^1\) as follows. By cutting the \(D^2 \times S^1\) along the meridional disks

![Figure 3](image-url)  
**Figure 3.** A decomposition of a link diagram.

![Figure 4](image-url)  
**Figure 4.** A decomposition of a subpiece.
Figure 5. A boundary component and a Dehn filling torus.

bounded by $\alpha$ and $\alpha'$, we obtain two solid cylinders $D^2 \times [0, 1]$. Note that we already have $4k$ vertices on $\partial D^2 \times 0$. We triangulate $D^2 \times 0$ into $4k$ triangles, by drawing lines joining the vertices to the center of $D^2 \times 0$. See the bottom picture in Figure 5. Taking the product with $[0, 1]$, we decompose $D^2 \times [0, 1]$ into $4k$ triangular prisms. Note that each hatched quadrangle gives exactly one prism. Finally we apply the standard prism decomposition (Figure 2) to each prism. Since each prism gives 3 tetrahedra and there are $8c$ hatched quadrangles, the union of all the Dehn filling solid tori is decomposed into $24c$ tetrahedra.

The triangulation of our surgery manifold $M$ is obtained by adjoining the Dehn filling tori triangulations to that of the exterior. By the above tetrahedra counting, it follows that the number of tetrahedra in $M$ is at most $72c + 24c = 96c$. □

3. Linear complexity triangulations from Heegaard splittings

In this section we present an explicit construction of an efficient triangulation from a Heegaard splitting given by a mapping class.

Recall from Definition 1.2 that the Heegaard-Lickorish complexity of a closed 3-manifold $M$ is the minimal word length, in the Lickorish generators, of a mapping class which gives a Heegaard splitting of $M$. Here the Lickorish generators of the mapping class group $\text{Mod}(\Sigma_g)$ of an oriented surface $\Sigma_g$ of genus $g$ are defined to be the $\pm 1$ Dehn twists along the curves $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_{g-1}$ shown in Figure 1.

To make it precise, we use the following convention. We fix a standard embedding of a surface $\Sigma_g$ of genus $g$ in $S^3$ as in Figure 1. Then $\Sigma_g$ bounds the inner handlebody $H_1$ and the outer handlebody $H_2$ in $S^3$. Let $i_j: \Sigma_g \to H_j$ ($j = 1, 2$) be the inclusion. The mapping class $h \in \text{Mod}(\Sigma_g)$ of a homeomorphism $f: \Sigma_g \to \Sigma_g$ gives a Heegaard splitting $(\Sigma_g, \{\beta_i\}, \{f(\alpha_i)\})$ of the 3-manifold

$$M = (H_1 \cup H_2)/i_1(f(x)) \sim i_2(x), \ x \in \Sigma_g.$$ 

In other words, $M$ is obtained by attaching $g$ 2-handles to the inner handlebody $H_1$ with boundary $\Sigma_g$ along the curves $f(\alpha_i)$ and then attaching a 3-handle. Under our convention, the identity mapping class gives us $S^3$.

The Heegaard-Lickorish complexity can be compared with the Heegaard genus by the following lemma.
Lemma 3.1. Suppose $M$ is a closed 3-manifold with a Heegaard splitting given by a mapping class $h \in \text{Mod}(\Sigma_g)$ which is a product of $\ell$ Lickorish generators. Then for some $g' \leq 2\ell$, $M$ admits a Heegaard splitting given by a mapping class $h' \in \text{Mod}(\Sigma_{g'})$ which is a product of $\ell$ Lickorish generators.

From Lemma 3.1, it follows immediately that the Heegaard genus is not greater than twice the Heegaard-Lickorish complexity. On the other hand, it is easily seen that a 3-manifold may be drastically more complicated than another with the same Heegaard genus. For example, all the lens spaces $L(n,1)$ have Heegaard genus one, but $L(n,1)$ is represented by a genus one mapping class of Heegaard-Lickorish word length $n$. In fact, by results of [Chab] (see also Lemma 4.2 and related discussions in the present paper), $c_{\text{simp}}(L(n,1)) \to \infty$ as $n \to \infty$, and consequently $c_{\text{HL}}(L(n,1)) \to \infty$ and $c_{\text{surg}}(L(n,1)) \to \infty$ by Theorems A and B.

Proof of Lemma 3.1. For a Lickorish generator $t \in \text{Mod}(\Sigma_g)$, we say that $t$ passes through the $i$th hole of $\Sigma_g$ if $t$ is a Dehn twist along either one of the curves $\alpha_i, \beta_i, \gamma_i$ or $\gamma_i-1$ (see Figure 1). It is easily seen from Figure 1 that a Lickorish generator can pass through at most two holes of $\Sigma_g$. Therefore, the Lickorish generators which appear in the given word expression of $h$ of length $\ell$ can pass through at most $2\ell$ holes. If $g > 2\ell$, then for some $i$, no Lickorish generator used in $h$ passes through the $i$th hole. By a destabilization which removes the $i$th hole from $\Sigma_g$, we obtain a Heegaard splitting of $M$ of genus $g-1$ given by a mapping class which is a product of $\ell$ Lickorish generators. By an induction, the proof is completed. □

Lickorish’s work [Lic62, Lic64] presents a construction of a surgery presentation from a Heegaard splitting. From his proof, we obtain the following:

Theorem 3.2. For any closed 3-manifold $M$, $c_{\text{surg}}(M) \leq 2 \cdot c_{\text{HL}}(M)^2 + 3 \cdot c_{\text{HL}}(M)$.

Proof. Suppose $M$ has a Heegaard splitting represented by a mapping class of Lickorish word length $\ell$. By the arguments in Lickorish [Lic62, Lic64] (see also Rolfsen’s book [Rol76, Chapter 9, Section I]), $M$ is obtained by surgery on a link $L$ with $\ell$ ($\pm 1$)-framed components, which admits a planar diagram in which no component has a self-intersection and any two distinct components can intersect at most twice. See Figure 6 for an example. It follows that $n(L) = 0$, $f(L) = \ell$, and $c(L) \leq 2 \cdot \left(\frac{\ell}{2}\right) = \ell(\ell + 1)$. By definition, we have $c_{\text{surg}}(M) \leq 2c(L) + f(L) + n(L) = 2\ell^2 + 3\ell$. □

Figure 6. An example of Lickorish’s surgery link.

Remark 3.3. Conversely, a surgery presentation can be converted to a Heegaard splitting. For instance, Lu’s method in [Lu92] tells us how to obtain a Heegaard splitting from a surgery link, as a product of explicit Dehn twists on an explicit surface. By rewriting those Dehn twists in terms of the Lickorish twists, for instance by following the arguments of existing proofs that Lickorish twists generate the mapping class group...
(e.g., see [Lic62, Lic64] or [FM12]), one would obtain a word in the Lickorish twists which represents the mapping class, and in turn an upper bound for the Heegaard-Lickorish complexity of the 3-manifold. We do not address details here.

**Remark 3.4.** Theorem [3.2] and (the proof of) Theorem [B] immediately give a triangulation from a Heegaard splitting, together with the following complexity estimate:

\[ c_{\text{simp}}(M) \leq 96 \cdot (2 \cdot c_{HL}(M))^2 + 3 \cdot c_{HL}(M). \]

It tells us that the simplicial complexity is bounded by a quadratic function in the Heegaard-Lickorish complexity. A quadratic bound seems to be the best possible result from this method (unless one finds a clever simplification of the resulting surgery link).

For instance, by generalizing the rightmost 5 components in Figure 6 and considering the corresponding mapping class, one sees that there is actually a genus one mapping class of Lickorish word length \( \geq \ell \leq \text{class of Lickorish word length} \).

The rest of this section is devoted to the proof of Theorem [A]. The key idea used in our proof below, which enables us to produce a more efficient triangulation (cf. Remark [3.3]), is that we view Lickorish’s surgery link (Figure 6) as a link in the thickened Heegaard surface.

**Proof of Theorem [A]** Here we will prove the following statement, which is slightly sharper than Theorem [A] if a closed 3-manifold \( M \neq S^3 \) has Heegaard-Lickorish complexity \( \ell \), then the simplicial complexity of \( M \) is not greater than \( 692\ell - 128 \).

Suppose \( h \in \text{Mod}(\Sigma_g) \) gives a Heegaard splitting of a given 3-manifold \( M \), and suppose \( h \) is a product of \( \ell \) Lickorish generators. Both \( g \) and \( \ell \) are nonzero, since \( M \neq S^3 \).

Lickorish showed that \( M \) is obtained by surgery on an \( \ell \)-component link \( L_0 \) in \( S^3 \), where each component has either \((+1)\) or \((-1)\)-framing [Lic62]. His proof tells us more about \( L_0 \) (another useful reference for this is [Rol76, Chapter 9, Section I]). In fact, \( L_0 \) lies in a bicollar \( \Sigma_g \times [-1, 1] \) of \( \Sigma_g \) in \( S^3 \), and each component is of the form \( \alpha_i \times \{t\}, \beta_i \times \{t\}, \) or \( \gamma_i \times \{t\} \) for some \( i \) and \( t \in [-1, 1] \). An example is shown in Figure 6. In particular, \( L_0 \) lies on \((\alpha_1 \cup \cdots \cup \alpha_g \cup \beta_1 \cup \cdots \cup \beta_g \cup \gamma_1 \cup \cdots \cup \gamma_{g-1}) \times [-1, 1] \subset \Sigma_g \times [-1, 1] \subset S^3 \).

By adding a local kink to each \( \alpha_i, \beta_i, \gamma_i \) on \( \Sigma_g \) and by taking their union, we obtain a graph \( D \) embedded in \( \Sigma_g \), which is shown in Figure 7. We regard the embedded curves \( \alpha_i, \beta_i, \gamma_i \) as subsets of \( D \).

**Figure 7.** The graph \( D \) obtained by adding kinks to the Dehn twist curves.

Note that for a link in the bicollar \( \Sigma_g \times [-1, 1] \), if each component is regular with respect to the projection of \( \Sigma_g \times [-1, 1] \to \Sigma_g \), then the blackboard framing with respect to \( \Sigma_g \) is well-defined; the preferred parallel with respect to the blackboard framing is defined to be the push-off along the \([-1, 1]\) direction. In particular, for our surgery link \( L_0 \), the blackboard framing with respect to \( \Sigma_g \) is the zero framing. We apply Reidemeister move \( I \) to each component of \( L_0 \) to obtain a new link \( L \) which lies in \( D \times [-1, 1] \subset \Sigma \times [-1, 1] \subset S^4 \) and whose blackboard framing is the desired \((\pm 1)\)-framing for surgery; see Figure 8 for an example.

Now, in order to construct a triangulation of \( \Sigma_g \times [-1, 1] \setminus \nu(L) \), we proceed similarly to the proof of Lemma [2.1] the difference is that we now use a “link diagram” on \( \Sigma_g \),
instead of a planar diagram. Let $G$ be the dual graph of $D$ on $\Sigma_g$. Each face of $G$ is a quadrangle. Cutting $\Sigma_g \times [-1,1] \setminus \nu(L)$ along $G \times [-1,1]$, we obtain cubic pieces with tubes removed, as shown in the left of Figure 9. Cutting along $D \times [-1,1]$, each piece is divided into 4 subpieces; see the middle of Figure 9.

We triangulate the three front faces of each subpiece as in the right of Figure 9 and then triangulate the subpiece by taking a cone at the opposite vertex, as we did in the proof of Lemma 2.1. We claim that there are $6k + 6$ tetrahedra in this triangulation, where $k$ is the number of hatched quadrangles in the right of Figure 9. The number of tetrahedra in the subpiece is equal to the number of triangles in the three front faces. There are two triangles in the top face. To count triangles in the remaining two faces, observe that the front middle vertical edge is divided into $2k + 1$ 1-simplices. There are $4k + 2$ triangles that have one of these 1-simplices as an edge, and there are $2k + 2$ remaining triangles. Therefore there are total $6k + 6$ triangles, as we claimed.

Collecting the triangulations of the subpieces, we obtain a triangulation of $\Sigma_g \times [-1,1] \setminus \nu(L)$. To estimate the number of tetrahedra, first observe that the graph $D$ has $6g - 3$ vertices, where $g$ is the genus of the Heegaard surface $\Sigma_g$. Therefore its dual graph $G$ has $6g - 3$ faces, and since each face of $G$ gives us 4 subpieces of $\Sigma_g \times [-1,1] \setminus \nu(L)$, we have total $24g - 12$ subpieces. Also, observe that a component of $L$ is cut into at most 5 pieces by $G$, and so can contribute at most 20 hatched quadrangles. It follows...
that there are at most
\[ 6 \cdot 20 \ell + 6 \cdot (24g - 12) = 120 \ell + 144g - 72 \]
tetrahedra in our triangulation of \( \Sigma_g \times [-1, 1] \setminus \nu(L) \).

Now we triangulate the inner and outer handlebodies which are the components of \( S^3 \setminus (\Sigma_g \times (0, 1)) \). Let \( D_1, \ldots, D_g \) be disjoint disks in the outer handlebody bounded by the curves \( \alpha_i \) in Figure 1, and \( D'_1, \ldots, D'_g \) be disjoint disks in the inner handlebody bounded by the curves \( \beta_i \). Our triangulation on \( \Sigma \times \{ \pm 1 \} \) divides each of \( \partial D_1 \) and \( \partial D_g \) into six 1-simplices, \( D_i \) for \( i = 2, \ldots, g - 1 \) into eight 1-simplices, and \( \partial D'_i \) into four 1-simplices. Extending this, we triangulate each of \( D_1, D_g \) into 4 triangles, \( D_i \) for \( i = 2, \ldots, g - 1 \) into 6 triangles, and each \( D'_i \) into 2 triangles by drawing arcs joining vertices. Cutting the handlebodies along the disks \( D_i \) and \( D'_i \), we obtain two 3-balls.

The triangulations of \( \Sigma_g \times 1 \) and \( D_i \) give us a triangulation of the boundary of the outer 3-ball. Recall that the top face of each subpiece we considered above consists of two triangles, and there are \( 24g - 12 \) subpieces. Therefore the boundary of the outer 3-ball has at most \( 2(24g - 12) + 2(6g - 4) = 60g - 32 \) triangles. Taking a cone at the center, the outer 3-ball is triangulated into at most \( 60g - 32 \) tetrahedra. Similarly the inner 3-ball is triangulated into \( 2(24g - 12) + 2 \cdot 2g = 52g - 24 \) tetrahedra.

We triangulate the Dehn filling tori as in Lemma 2.1. Since there are at most \( 20 \ell \) hatched quadrangles and each hatched quadrangle contributes at most 3 tetrahedra (= one triangular prism) in the Dehn filling tori, there are at most \( 60 \ell \) tetrahedra in the Dehn filling tori.

It follows that our triangulation of the surgery manifold \( M \) has at most
\[ (120 \ell + 144g - 72) + (60g - 32) + (52g - 24) = 180 \ell + 256g - 128 \]
tetrahedra. By Lemma 3.1 we may assume that \( g \leq 2 \ell \). It follows that the simplicial complexity of \( M \) is at most \( 692 \ell - 128 \). \( \square \)

4. Theorems A and B are asymptotically optimal

In this section we prove Theorem C and related results. For this purpose we use some results in [Chab]. First, we need the following lower bound of the simplicial complexity. In [CG85], Cheeger and Gromov introduced the von Neumann \( L^2 \) \( \rho \)-invariant \( \rho^{(2)}(M, \phi) \in \mathbb{R} \) which is defined for a smooth closed \( (4k-1) \)-manifold \( M \) and a homomorphism \( \phi: \pi_1(M) \to G \). By deep analytic arguments, they showed that for each \( M \) there is a universal bound for the values of \( \rho(M, \phi) \) [CG85]: that is, there is \( C_M > 0 \) satisfying that \( |\rho^{(2)}(M, \phi)| \leq C_M \) for any \( \phi \). In [Chab], a topological approach to the universal bound for \( \rho^{(2)}(M, \phi) \) is presented, and in particular, an explicit linear universal bound is given in terms of the simplicial complexity of 3-manifolds:

**Theorem 4.1 ([Chab] Theorem 1.5).** Suppose \( M \) is a closed 3-manifold. Then
\[ |\rho^{(2)}(M, \phi)| \leq 363090 \cdot c^{\text{simp}}(M) \]
for any homomorphism \( \phi \).

In this paper, we will use the Cheeger-Gromov \( \rho \)-invariant as a lower bound of the simplicial complexity.
For the lens space \( L(n, 1) \) and the identity map \( \text{id} : \pi_1(L(n, 1)) \to \mathbb{Z}_n \), Lemma 7.1 of [Chab] gives the following value of the Cheeger-Gromov \( \rho \)-invariant, using the computation of Atiyah-Patodi-Singer [APS75, p. 412]:
\[
\rho^{(2)}(L(n, 1)), \text{id}) = \frac{n}{3} + \frac{2}{3n} - 1.
\]
From this and Theorem 4.1, a lower bound of the simplicial complexity of \( L(n, 1) \) is obtained. We state it as a lemma:

**Lemma 4.2.** \( c^{\text{simp}}(L(n, 1)) \geq \frac{|n|}{1089270} - 3 \).

We remark that a pseudo-simplicial complexity analog is given in [Chab, Corollary 1.15].

Now we are ready to proof Theorem C. In fact, the following stronger inequalities hold, and Theorem C follows immediately from them.

**Theorem 4.3.** For any \( n \neq 0 \),
\[
\frac{1}{1089720} \cdot \left( 1 - \frac{3}{|n|} \right) \cdot c^{\text{HL}}(L(n, 1)) \leq c^{\text{simp}}(L(n, 1)),
\]
\[
\frac{1}{1089720} \cdot \left( 1 - \frac{3}{|n|} \right) \cdot c^{\text{surg}}(L(n, 1)) \leq c^{\text{simp}}(L(n, 1)).
\]

**Proof.** Since \( L(n, 1) \) is obtained by the \( n \)-framed surgery on the unknot, it is easily seen that \( c^{\text{HL}}(M), c^{\text{surg}}(M) \leq n \). The desired inequalities follow from this and Lemma 4.2. \( \square \)

In what follows we discuss a generalization and a specialization of the lens space case we considered in Theorem 4.3.

First, the second inequality in Theorem 4.3 generalizes for a larger class of 3-manifolds. For a knot \( K \) in \( S^3 \), let \( M(K, n) \) be the 3-manifold obtained by \( n \)-framed surgery on \( K \). Let \( g_4(K) \) be the (topological) slice genus of \( K \).

**Theorem 4.4.** For any \( n \neq 0 \),
\[
\frac{1}{1089720} \cdot \left( 1 - \frac{3 + 6g_4(K)}{|n|} \right) \cdot c^{\text{surg}}(M(K, n)) - 2c(K) \leq c^{\text{simp}}(M(K, n)).
\]

**Proof.** Let \( \phi : \pi_1(M(K, n)) \to \mathbb{Z}_{|n|} \) be the abelianization. Due to [Chaa, Equation (2.8)],
\[
|\rho^{(2)}(M(K, n), \phi)| \geq \frac{1}{3} \cdot (|n| - 3 - 6g_4(K)).
\]
By Theorem 4.1, it follows that
\[
c^{\text{simp}}(M(K, n)) \geq \frac{1}{1089270} \cdot (|n| - 3 - 6g_4(K)).
\]
By definition, \( c^{\text{surg}}(M(K, n)) \leq 2c(K) + |n| \). From this and 4.1, the desired inequality follows. \( \square \)

On the other hand, if we consider the special case of lens spaces \( L(2k, 1) \), then the inequalities in Theorem 4.3 (and hence those in Theorem C) can be improved significantly as follows.

**Theorem 4.5.** For \( k > 1 \), the following hold:
\[
\left( 1 - \frac{3}{2k} \right) \cdot c^{\text{HL}}(L(2k, 1)) \leq c^{\text{simp}}(L(2k, 1)),
\]
\[
\left( 1 - \frac{3}{2k} \right) \cdot c^{\text{surg}}(L(2k, 1)) \leq c^{\text{simp}}(L(2k, 1)).
\]
Proof. Due to Jaco, Rubinstein, and Tillman [JRT09], the pseudo-simplicial complexity of $L(2k, 1)$ is equal to $2k - 3$ for $k > 1$, and consequently $c_{\text{simp}}(L(2k, 1)) \geq 2k - 3$. Using this in place of Lemma 4.2 in the proof of Theorem 4.3 we obtain the inequalities. □

We finish this section with a proof of Corollary 4.4.

Proof of Corollary 4.4. Recall the definition of the “largest possible value” of the simplicial complexity for Heegaard-Lickorish complexity $\leq \ell$:

$$s_{\text{HL}}(\ell) := \sup \{c_{\text{simp}}(M) \mid c_{\text{HL}}(M) \leq \ell\}.$$  

The first assertion of Corollary 4.4, which says $s_{\text{HL}}(\ell) \in O(\ell \cap \Omega(\ell))$, follows immediately from the estimate

$$\frac{1}{1089270} \leq \limsup_{\ell \to \infty} \frac{s_{\text{HL}}(\ell)}{\ell} \leq 692.$$  

which we prove in what follows.

Fix $\ell$. For any $M$ with $c_{\text{HL}}(M) \leq \ell$, we have

$$\frac{c_{\text{simp}}(M)}{\ell} \leq \frac{c_{\text{simp}}(M)}{c_{\text{HL}}(M)} \leq 692$$

by Theorem A. Taking the supremum over all such $M$, we obtain $s_{\text{HL}}(\ell)/\ell \leq 692$. From this we obtain the upper bound in (4.2).

By the definition of $s_{\text{HL}}(\ell)$, we have

$$\frac{c_{\text{simp}}(M)}{c_{\text{HL}}(M)} \leq s_{\text{HL}}(c_{\text{HL}}(M))$$

for any $M$. By Theorem 4.3 the limit supremum of the left hand side as $c_{\text{HL}}(M) \to \infty$ is bounded from below by $1/1089270$. From this the lower bound in (4.2) follows.

The analog for the function $s_{\text{surg}}(k)$ is proved by the same argument. □

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