A Reduced-order Extrapolation Model Based on Proper Orthogonal Decomposition Technique for Rayleigh–Bénard convection

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Abstract. In order to quickly calculate the flow characteristics in the simulated two-dimensional Rayleigh–Bénard convection model, a reduced-order extrapolation difference model for solving two-dimensional Rayleigh–Bénard is established based on the Proper Orthogonal Decomposition (POD) method. Firstly, the classical difference format is used to calculate the flow field on the initial time span, the sample data is to construct the POD basis, and the truncation error generated in the process of constructing the POD base is analysed. Then, the two-dimensional reduced-order extrapolation Rayleigh–Bénard model is constructed by combining the SVD and POD methods. By numerical examples, the classical difference model and the reduced-order extrapolation model are numerically simulated under the given initial boundary value. The reliability and effective of the reduced-order method are verified by comparing the flow field temperature cloud maps in the two calculation stages.

1. Introduction
Thermal convection is a natural phenomenon that is ubiquitous in nature and is closely related to our lives, such as air conditioning and refrigeration, atmospheric circulation, hot water boiling, industrial production, and so on. Rayleigh–Bénard convection is a classic hydrodynamic model in many thermal convections. Originally discovered by French physicist Henri Bénard during the experiment, Lord Rayleigh discovered this phenomenon in the convection test. The combination of the two opened up the research of Rayleigh–Bénard convection by researchers [1,2]. Malkus theoretically derived the correspondence between the Nu number and the Ra number through rigorous
mathematical analysis, and then the Rayleigh–Bénard thermal convection model was rapidly developed [3]. The famous physicist Kadanoff and the University of Chicago collaborated to propose the "mixed zone theory", and Cioni obtained the dependence between the Nu number and the Pr number through this theory [4]. There are later SS models [5] proposed by Shraiman and Siggia and based on the Prandet-Blasius boundary layer hypothesis, Grossmann and Lohse proposed the GL model [6-9]; Bao Yun et al have long been committed to Rayleigh–Bénard research, and the latest research results use efficient parallel computing methods for 3D Rayleigh–Bénard calculations. The validity of the calculation method is verified, which may be a major breakthrough in the computation of complex Rayleigh–Bénard problems [10].

Rayleigh–Bénard convection is an important research direction in current turbulence research, but it hinders the development of many scientific researches due to the huge amount of computation. This paper combines the POD extrapolation algorithm of Luo Zhendong to establish a finite difference reduction order based on POD method [11-15]. The model provides a reference for further study of the characteristics of the Rayleigh–Bénard convection model, and analyzes the validity of the method by numerical examples.

2. Two-dimensional Rayleigh–Bénard convection model

The two-dimensional Rayleigh–Bénard thermal convection system model used in this paper refers to ideally, in a closed rectangular cavity filled with water medium, as shown in Figure 1-1.

![Fig. 2-1 two-dimensional Rayleigh–Bénard convection calculation model](image)

The two-dimensional rectangular cavity Rayleigh–Bénard convection model is described by using partial differential equations. In order to reduce the effect of density difference of the upper and lower fluids, the partial differential equation is usually dimensionless. Under the assumption of Bussinesq equation, the two-dimensional Rayleigh–Bénard convection partial differential equations can be expressed as:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]  

(2.1)

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]

(2.2)
Ra (Rayleigh number) and Pr (Prandtl number) are the parameters of Rayleigh-Bénard convection model. The dimensionless mentioned above requires the introduction of feature length L, feature time t, feature temperature, and feature speed u, which satisfies:

\[
\nu = \left( \frac{P_r}{Ra} \right)^{1/2}
\]

(2-5)

\[
\kappa = \left( \frac{1}{RaPr} \right)^{1/2}
\]

(2-6)

3. Establishment of Rayleigh-Bénard convective reduced order extrapolation model

\[
\begin{align*}
&u^n_m = \left( u^n_1, u^n_2, \ldots, u^n_m \right), \\
v^n_m = \left( v^n_1, v^n_2, \ldots, v^n_m \right), \\
p^n_m = \left( p^n_1, p^n_2, \ldots, p^n_m \right), \\
&\theta^n_m = \left( \theta^n_1, \theta^n_2, \ldots, \theta^n_m \right) (n = 1, 2, \ldots, N)
\end{align*}
\]

\[
\begin{align*}
&\left( u^n_{m+1}, v^n_{m+1}, \theta^n_{m+1} \right)^T = \left( u^n_m, v^n_m, \theta^n_m \right)^T + F \left( u^n_m, v^n_m, \theta^n_m \right), \\
&\quad \quad \quad \quad \quad \quad 1 \leq n \leq N - 1
\end{align*}
\]

(3-2)

Where F and G can be solved by classical difference form.

Define:

\[
\begin{align*}
&\left( u^n_m, v^n_m, p^n_m, \theta^n_m \right)^T = \left( \Phi_u^{nM_u}, \Phi_v^{nM_v}, \Phi_p^{nM_p}, \Phi_{\theta}^{nM_{\theta}} \right)
\end{align*}
\]

(3-3)

Where \( u^n_m, v^n_m, p^n_m, \theta^n_m \) are the column vectors corresponding to \( u, v, p, \) and then satisfy

\[
\begin{align*}
&\begin{bmatrix}
u^n_m \\
v^n_m \\
p^n_m \\
\theta^n_m
\end{bmatrix} = \begin{bmatrix}
u^n_m \\
v^n_m \\
p^n_m \\
\theta^n_m
\end{bmatrix} = \begin{bmatrix}
u^n_m \\
v^n_m \\
p^n_m \\
\theta^n_m
\end{bmatrix} = \begin{bmatrix}
u^n_m \\
v^n_m \\
p^n_m \\
\theta^n_m
\end{bmatrix}
\end{align*}
\]

(3-4)
In the upper form, $u_m^n, v_m^n, p_m^n, \theta_m^n$ is the approximate representation of $u_m^n, v_m^n, p_m^n, \theta_m^n$, respectively. $\Phi_u, \Phi_v, \Phi_p, \Phi_\theta$ are known POD bases, from which we can obtain a two-dimensional Rayleigh-Bénard thermal convection differential reduction model based on POD method.

$$\begin{align*}
\alpha_u^n = \Phi_u u_m^n, & \quad \beta_v^n = \Phi_v v_m^n, & \quad \gamma_p^n = \Phi_p p_m^n, & \quad \delta_\theta^n = \Phi_\theta \theta_m^n, & \quad n = 1, 2, \ldots, L, \\
\gamma_{M_p}^n = \Phi_p G (\Phi_u \alpha_{M_u}^{n-1}, & \quad \Phi_v \beta_{M_v}^{n-1}, & \quad \Phi_p \gamma_{M_p}^{n-1}), & \quad n = L, L + 1, \ldots, N, \\
(\alpha_{M_u}^{n+1}, & \quad \beta_{M_v}^{n+1}, & \quad \gamma_{M_p}^{n+1})^T = (\alpha_{M_u}^n, & \quad \beta_{M_v}^n, & \quad \gamma_{M_p}^n)^T + F (\alpha_u^n, & \quad \beta_v^n, & \quad \gamma_p^n, & \quad \delta_\theta^n), \\
n = L, L + 1, \ldots, N - 1
\end{align*}$$

(3-5)

4. Calculation example and calculation result analysis

In this paper, the POD method is used to reduce the order of the two-dimensional Rayleigh-Bénard convection model. In the rectangular cavity, set the length of the rectangle to 2, high 1, on the surface temperature of $25^\circ$, the surface temperature $25^\circ$, the rest of the fluid related parameters for Re = 1 e2, Pr = 7, Ra = 1 e2. The two models were numerically simulated with a grid of 100 x 70. The results of the simulation diagram of three periods of thermal plume were compared:

1. Initial flow stage

![Figure 4-1(a) classical difference calculation results in the initial stage](image)

**Figure 4-1(a) classical difference calculation results in the initial stage**

![Figure 4-1(b) calculation results of the extrapolated model with reduced order in the initial stage](image)

**Figure 4-1(b) calculation results of the extrapolated model with reduced order in the initial stage**

2. Mobile development stage
Using the POD extrapolation method, the first 20 steps of numerical solution should be solved by using the classical difference method, and the instantaneous image matrix is formed to construct the POD basis. For a single variable, five POD bases are adopted here, and then the reduced order model is solved to calculate the solutions at different times. The system calculation time is as follows:

|                  | Classical difference scheme (s) | reduced order model (s) |
|------------------|---------------------------------|-------------------------|
| The initial stage \((t=10)\) | 10.93                           | 7.16                    |
| Developing Stage \((t=15)\)   | 16.95                           | 9.76                    |

5. Conclusion

This chapter mainly introduces the method of decreasing the order of the classical two-dimensional Rayleigh-Bénard convection model using the eigen orthogonal decomposition theory and singular value decomposition theory. As the sample transient image set, the data is represented in matrix form, and the corresponding POD basis is solved using POD method. The first five pods are selected, and the reduced order extrapolation model can be constructed according to the steps of constructing the reduced order extrapolation model, and then the finite difference reduced order extrapolation model of two-dimensional Rayleigh - Bénard can be constructed by combining the initial values.
Reduced order extrapolation model by an example analysis and comparison with classic difference in calculating the numerical simulation of the same time, the results show that the reduced order extrapolation has a low degree of freedom model and the classic difference calculation result is close to, or even better than the classical finite difference calculation, secondly, a reduced order extrapolation model USES the POD with a low degree of freedom, and reduces the computing space truncation errors accumulated, proved based on the superiority of POD order reduction method.

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