Abstract
In this paper, we characterize NIP (Not the Independence Property) henselian valued fields modulo the theory of their residue field, both in an algebraic and in a model-theoretic way. Assuming the conjecture that every infinite NIP field is either separably closed, real closed, or admits a nontrivial henselian valuation, this allows us to obtain a characterization of all theories of NIP fields.

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1 INTRODUCTION

Since Macintyre showed in the early seventies that infinite $\omega$-stable fields are algebraically closed ([31]), the question of whether key model-theoretic tameness properties coming from Shelah’s classification theory (such as stability, simplicity, NIP) correspond to natural algebraic definitions when applied to fields has been studied extensively. The most prominent instance is the Stable Fields Conjecture, predicting that any infinite stable field is separably closed. In 1980, Cherlin and Shelah generalized Macintyre’s result to superstable fields ([6]), but despite much effort, no further progress was made for a long time. Very recently, the Stable Fields Conjecture was solved in the special case of large stable fields by Johnson, Tran, Walsberg, and Ye ([25]), using the newly introduced étale-open topology. Nevertheless, the Stable Fields Conjecture in full generality still seems to be far beyond our reach. Its generalization to NIP fields has received much attention.

Conjecture 1.1 (Conjecture on NIP fields). Let $K$ be an infinite NIP field. Then, $K$ is separably closed, real closed, or admits a nontrivial henselian valuation.
Conjecture 1.1 has many variants but no clear origin, and is usually attributed to Shelah (who stated a closely related conjecture on strongly NIP fields and asked for a similar description of NIP fields in [33]). The special case of fields of finite dp-rank was recently proven by Johnson in a series of spectacular preprints culminating in [24].

Apart from separably closed fields, real closed fields, and the p-adics plus their finite extensions, the only currently known method to construct NIP fields is by NIP transfer theorems in the spirit of Ax-Kochen/Ershov: under certain algebraic assumptions, if $(K, v)$ is a henselian valued field such that the residue field $Kv$ is NIP, then $(K, v)$ is NIP. The first such theorem was shown by Delon.

**Fact 1.2** (Delon [9]). Let $(K, v)$ be a henselian valued field of equicharacteristic 0. Then,

$$(K, v) \text{ is NIP in } \mathcal{L}_{\text{val}} \iff Kv \text{ is NIP in } \mathcal{L}_{\text{ring}}.$$ 

Here, $\mathcal{L}_{\text{ring}} = \{0, 1, +, \cdot\}$ denotes the language of rings and $\mathcal{L}_{\text{val}}$ is a three-sorted language with sorts for the field, the residue field and the value group (see the beginning of Section 2 for a precise definition of $\mathcal{L}_{\text{val}}$).

Note that Delon originally proved the theorem under the additional assumption that the value group $vK$ is NIP as an ordered abelian group. It was later shown by Gurevich and Schmidt that this holds for any ordered abelian group ([16, Theorem 3.1]). Several variants of Delon’s theorem were proven in mixed and positive characteristic, first by Bélair ([5]) and more recently by Jahnke and Simon ([22]). Bélair showed that an algebraically maximal Kaplansky field $(K, v)$ of positive characteristic is NIP in $\mathcal{L}_{\text{val}}$ if and only if its residue field $Kv$ is NIP in $\mathcal{L}_{\text{ring}}$, and that the same holds if $(K, v)$ is finitely ramified with perfect residue field. Jahnke and Simon generalized Bélair’s result to separably algebraically maximal Kaplansky fields of finite degree of imperfection and arbitrary characteristic. Conversely, they use the theorem by Kaplan, Scanlon, and Wagner, stating that NIP fields of positive characteristic admit no Artin–Schreier extensions ([26]) to show that NIP henselian valued fields of positive characteristic are separably algebraically maximal. The approach used by Jahnke and Simon builds on machinery developed by Chernikov and Hils in the NTP2 context ([7]). Also following this route, we prove what one might consider as the ultimate transfer theorem: our main result is that a henselian valued field $(K, v)$ is NIP (in $\mathcal{L}_{\text{val}}$) if and only if its residue field $Kv$ is NIP (in $\mathcal{L}_{\text{ring}}$) and the valued field satisfies a list of purely algebraic conditions (all of which are preserved under $\mathcal{L}_{\text{val}}$-elementary equivalence). More precisely, we show the following.

**Main Theorem** (Theorem 5.1). Let $(K, v)$ be a henselian valued field. Then $(K, v)$ is NIP in $\mathcal{L}_{\text{val}}$ if and only if both of the following hold.

1. $Kv$ is NIP.
2. Either
   \begin{align*}
   (a) & \quad \begin{cases} 
   (a.i) \quad (K, v) \text{ is of equal characteristic, and} \\
   (a.ii) \quad (K, v) \text{ is trivial or separably defectless Kaplansky;}
   \end{cases} \\
   \text{or} \\
   (b) & \quad \begin{cases} 
   (b.i) \quad (K, v) \text{ has mixed characteristic } (0, p), \text{ and} \\
   (b.ii) \quad (K, v_p) \text{ is finitely ramified, and} \\
   (b.iii) \quad (Kv_p, \bar{v}) \text{ is trivial or separably defectless Kaplansky;} 
   \end{cases}
   \end{align*}
or

\begin{enumerate}[\text{(c)}]
\item (c.i) \((K, v)\) has mixed characteristic \((0, p)\), and
\item (c.ii) \((Kv_0, \bar{v})\) is defectless Kaplansky.
\end{enumerate}

Here, for a valuation \(v\) of mixed characteristic \((0, p)\), we use \(v_0\) to denote the finest coarsening of \(v\) with residue characteristic 0 and \(v_p\) to denote the coarsest coarsening of \(v\) with residue characteristic \(p\) (see Remark 2.6).

We then apply our main theorem in two different ways. In Corollary 5.3, we show that the henselization \((K^h, v^h)\) of any NIP valued field \((K, v)\) is again NIP. Moreover, we give a classification of NIP fields, assuming that Conjecture 1.1 holds (see Theorem 7.1).

We now give an overview over the contents of this paper. In Section 2, we start by introducing the valuation-theoretic notions that appear in the main result, then we prove several lemmas about valued fields that are applied later in the paper. We also survey Ax-Kochen/Ershov principles, which are fundamental in the model theory of valued fields.

Next, in Section 3, we recall the definition of NIP and prove some facts about NIP valued fields (without assuming henselianity). These allow us to prove Theorem 3.5, which states that any NIP valued field (not assumed to be henselian) satisfies both of the properties (1) and (2) occurring in the Main Theorem. In particular, Theorem 3.5 entails the left-to-right implication of the Main Theorem.

Section 4 recalls known results about henselian NIP fields and contains two new NIP transfer results: In Proposition 4.1, we show that any separably algebraically maximal Kaplansky valued field of positive characteristic is NIP in \(L_{val}\) if and only if its residue field is NIP in \(L_{ring}\). This was previously known under the additional assumption of finite degree of imperfection ([22, Lemma 3.2]). As a consequence, we obtain a new proof of an unpublished result of Delon that the \(L_{val}\)-theory of any separably closed valued field is NIP (Corollary 4.2). The second NIP transfer result contained in this section is concerned with finitely ramified valued fields, that is, valued fields \((K, v)\) of mixed characteristic \((0, p)\) such that the interval \((0, v(p)]\) in the value group \(vK\) is finite.

We show that any henselian finitely ramified valued field with NIP residue field is NIP in \(L_{val}\) in Corollary 4.7. Moreover, this also holds if we compose a henselian finitely ramified valuation with a henselian NIP valuation on the residue field (Proposition 4.6). These results generalize a theorem by Bélaïr stating that a henselian unramified valued field with perfect NIP residue field is NIP in \(L_{val}\) ([5, Théorème 7.4(2)]). The key ingredient is a new understanding of the model theory of henselian unramified valued fields (where the residue field is allowed to be imperfect) by the authors, cf. [1].

In Section 5, we state and prove our Main Theorem (Theorem 5.1). We also give a number of examples showing that none of the clauses in the theorem is vacuous. As a consequence of Theorem 5.1, we show that the henselization of any NIP valued field is again NIP (Corollary 5.3). This gives an NIP analog to a result by Hasson and Halevi who showed that the henselization of every strongly NIP (also known as strongly dependent) valued field is again strongly NIP ([17]).

The last two sections contain variants of our Main Theorem, although not straightforward ones. The version presented in Section 6 has a distinctly more model-theoretic flavor: throughout the section, given a complete \(L_{ring}\)-theory \(T_k\) of NIP fields, we describe all the complete \(L_{val}\)-theories of NIP henselian valued fields \((K, v)\) such that the residue field is a model of \(T_k\). This is rather

\[\text{†}\]

Note that there are many examples of NIP valued fields that are not strongly NIP since all strongly NIP fields are perfect. For explicit examples, see Example 5.2.
easy and unsurprising in equal characteristic (and even well known in equicharacteristic 0), but noticeably harder in mixed characteristic. First, we introduce some theories of henselian valued fields of mixed characteristic and prove their completeness. After taking a closer look at finitely ramified fields (cf. Lemma 6.8), we give the desired characterization in Proposition 6.9.

In the final section, we apply this characterization to give a refinement of Conjecture 1.1: we give a list of complete \( \mathcal{L}_{\text{val}} \)-theories of henselian valued fields which are all NIP in \( \mathcal{L}_{\text{val}} \), and we show that Conjecture 1.1 implies that every NIP field \( K \) admits a henselian valuation \( v \) such that \((K, v)\) is a model of one of the theories listed (see Theorem 7.1). This is an NIP analog of a similar conjectural classification of strongly NIP fields which was obtained by Halevi, Hasson, and Jahnke ([18]).

2 BASIC NOTIONS FROM VALUATION THEORY

In this section, we introduce the valuation-theoretic notions that appear in our main result (Theorem 5.1). Furthermore, we prove a number of valuation-theoretic lemmas that will be applied in later sections.

Throughout this paper, we denote the valuation ring of a valued field \((K, v)\) by \( \mathcal{O}_v \), the valuation ideal by \( m_v \). When considering valued fields, we use the three sorted language \( \mathcal{L}_{\text{val}} \) with sorts \( K \) for the field, \( k \) for the residue field, and \( G \) for the value group together with an additional element for \( \infty \). On both \( K \) and \( k \), we have the language of rings \( \mathcal{L}_{\text{ring}} = \{0, 1, +, \cdot\} \), and on \( G \) the language \( \mathcal{L}_{\text{oag}} = \{0, +, <\} \) of ordered abelian groups together with a constant symbol for \( \infty \), all interpreted in the usual way. Furthermore, there are two function symbols connecting the sorts to one another, namely, \( v : K \to G \) and \( \text{res} : K \to k \). Whenever we consider a valued field \((K, v)\) as a first-order structure, we mean the corresponding \( \mathcal{L}_{\text{val}} \)-structure given by the field \( K \) in the sort \( K \), the residue field \( K_v \) in the sort \( k \), and the value group \( v_K \) with its additional element \( \infty \) in the sort \( G \). Naturally, \( v \) is interpreted by the valuation \( v \) and \( \text{res} \) is interpreted by the residue map \( \text{res} : \mathcal{O}_v \to K_v \) which we extend by setting \( \text{res}(x) = 0 \) for \( x \in K \) with \( v(x) < 0 \).

Note that we choose this language for convenience because it allows us to refer to the residue field and the value group as objects in our language. Other commonly used languages of valued fields include the one-sorted language \( \mathcal{L}_{\text{ring}} \cup \{\mathcal{O}\} \), the expansion of the language of rings by a predicate for the valuation ring, and the two-sorted language with sorts \( K \) and \( G \) together with a map \( v : K \to G \), where the \( K \)-sort is again endowed with \( \mathcal{L}_{\text{ring}} \) and the \( G \)-sort with \( \mathcal{L}_{\text{oag}} \cup \{\infty\} \). If we consider a valued field \((K, v)\) in any of these three languages, it is biinterpretable with each of the corresponding two structures in the other languages. Thus, as our focus is on the question of whether the valued field is NIP (and this is preserved under interpretability), the answer is independent from the language we choose.

**Definition 2.1.** A valued field \((K, v)\) is **Kaplansky** if either it is of equal characteristic zero, or if it is of residue characteristic \( p > 0 \) and

(i) \( vK \) is \( p \)-divisible,
(ii) \( K_v \) is perfect, and
(iii) \( K_v \) admits no proper separable algebraic extensions of degree divisible by \( p \).

**Remark 2.2.** Equivalently, a valued field \((K, v)\) of residue characteristic \( p > 0 \) is Kaplansky if and only if \( vK \) is \( p \)-divisible and \( K_v \) admits no proper finite extensions of degree divisible by \( p \).
We now introduce both the notions of defectlessness and separable defectlessness. Note that if a valued field \((K, v)\) is defectless, it is always separably defectless; the converse holds if we assume \(K\) to be perfect, but does not hold in general.

**Definition 2.3.** A valued field \((K, v)\) is (separably) defectless if, whenever \(L/K\) is a finite (separable) field extension, the fundamental equality holds:

\[
[L : K] = \sum_{\omega \supseteq v} e(\omega/v) f(\omega/v),
\]

where \(\omega\) ranges over all prolongations of \(v\) to \(L\), \(e(\omega/v) = (\omega L : vK)\) is the ramification degree, and \(f(\omega/v) = [L\omega : Kv]\) is the inertia degree of the extension \((L, \omega)/(K, v)\).

Note that defect can only occur in positive residue characteristic (cf. [13, Theorem 3.3.3]). The next lemma shows that defectlessness is an \(\mathcal{L}_{\text{val}}\)-elementary property.

**Lemma 2.4.** There is an \(\mathcal{L}_{\text{val}}\)-theory \(T_d\) that axiomatizes the class of defectless valued fields.

**Proof.** Note that a valued field is defectless if and only if the fundamental inequality holds for all finite normal extensions. For convenience, we fix a valued field \((K, v)\), and let \(n \in \mathbb{N}\). First, there is a standard method to uniformly interpret in \(K\) the family of normal extensions \(L/K\) of degree \(n\): one quantifies over tuples which form the coefficients of the minimal polynomial of a generator of such an extension. We fix one such tuple \((c_0, \ldots, c_{n-1})\), corresponding to a normal extension \(L/K\) of degree \(n\). We view the whole as an \(\mathcal{L}_{\text{val}}^1\)-structure \((K, L, v)\), where \(\mathcal{L}_{\text{val}}^1\) is the expansion of \(\mathcal{L}_{\text{val}}\) by an additional sort \(L\) equipped with \(\mathcal{L}_{\text{ring}}\) and interpreted by \(L\), as well as a distinguished embedding of \(K\) into \(L\). Next, we show that in \((K, L, v)\), the family of valuation rings on \(L\) corresponding to prolongations of \(v\) is definable using the parameters \((c_0, \ldots, c_{n-1})\). For this, we use an argument of Johnson, specifically the proof of [23, Lemma 9.4.8]. The only adjustment we need to make to Johnson’s argument is that in our case, \(\mathcal{O}_v\) is definable (not only \(\forall\)-definable), and so, the second condition in [23, Claim 9.4.9] is definable (not only type-definable). The rest of the argument goes through verbatim, and it follows that \(\mathcal{O}_w\) is definable. More precisely, there are parameters \(b_1, \ldots, b_m \in L\) and a formula \(\pi(x, y_1, \ldots, y_m, z_0, \ldots, z_{n-1})\) such that \(\pi(x, b_1, \ldots, b_m, c_1, \ldots, c_{n-1})\) defines in \((K, L, v)\) the valuation ring \(\mathcal{O}_w\). Therefore,

\[
\{\mathcal{O}_w \mid w \text{ prolongs } v\}
\]

is a definable family in \((K, L, v)\) using parameters \((c_0, \ldots, c_n)\). Finally, it is clear that there is an \(\mathcal{L}_{\text{val}}^1\)-theory that axiomatizes the class of those \((K, L, v)\) which satisfy the fundamental equality. Combining these steps, we are done. \(\square\)

Closely related to defectlessness is the following notion.

**Definition 2.5.** A valued field \((K, v)\) is (separably) algebraically maximal if it admits no proper (separable) algebraic immediate extensions.

We now explain how (separable) defectlessness is connected to (separable) algebraic maximality. If \((K, v)\) is a henselian valued (separably) defectless field, then \((K, v)\) is already (separably)
algebraically maximal. The converse implication fails in general, but holds in henselian NIP valued fields.†

In the cases (b) and (c) of Theorem 5.1, we decompose a mixed characteristic valuation into two equicharacteristic valuations and a rank-1 valuation of mixed characteristic. This is a standard trick for which we explain notation and give details below.

Remark 2.6. Let \((K, v)\) be a valued field of mixed characteristic \((0, p)\). First, let \(\Delta_p\) denote the maximal convex subgroup of \(vK\) that does not contain \(v(p)\), and let \(\Delta_0\) denote the minimal convex subgroup of \(vK\) that does contain \(v(p)\). So, we have a chain \(\Delta_p < \Delta_0 \leq vK\). Next, let \(v_p\) be the coarsening of \(v\) corresponding to \(\Delta_p\), and let \(v_0\) be the coarsening of \(v\) corresponding to \(\Delta_0\). We use \(\bar{v}\) to denote the valuation induced by \(v\) on each of the residue fields \(Kv_p\) and \(Kv_0\) of the coarsenings of \(v\), and \(\bar{v}_0\) to denote the valuation induced by \(v_p\) on the residue field of its coarsening \(Kv_0\). In particular, \((K, v_0)\) and \((Kv_p, \bar{v})\) are valued fields of equicharacteristic 0 and \(p\), respectively, and \((Kv_0, \bar{v}_0)\) is a rank-1 valued field of mixed characteristic with value group \(\Delta_0/\Delta_p\). We will make repeated use of this decomposition, which we call the standard decomposition.

It is illustrated by Figure 1, in which the arrows represent the places corresponding to the valuations rather than the valuations themselves.

Lemma 2.7. For valued fields of mixed characteristic \((0, p)\), the case distinction

(i) \(\Delta_0/\Delta_p\) is discrete
(ii) \(\Delta_0/\Delta_p\) is not discrete

is preserved under \(\mathcal{L}_{\text{val}}\)-elementary equivalence.

Proof. Let \((K, v)\) be a valued field of mixed characteristic \((0, p)\), viewed according to the standard decomposition. If \(\Delta_0/\Delta_p\) is not discrete, then for each \(n > 0\), there exists \(N_n > n\) and \(x \in K\) such that

\[0 < n v(x) \leq v_p \leq N_n v(x),\]

that is, \(v(x)\) is in the interval \([\frac{v(p)}{N_n}, \frac{v(p)}{n}] \subseteq vK\). The existence of such an element \(x\) is expressed by a sentence \(\varphi_{n,N_n}\) in the language of valued fields. Indeed, \(\Delta_0/\Delta_p\) is not discrete if and only if \((K, v) \models \bigwedge_{n > 0} \varphi_{n,N_n}\) for some function \(n \mapsto N_n\) such that \(N_n > n\), for all \(n\). Therefore, (i) (and hence also (ii)) is preserved under \(\mathcal{L}_{\text{ring}}\)-elementary equivalence. □

For lack of an appropriate reference, we state and prove the following lemmas, which ensure that several of the properties we are interested in behave well under compositions of valuations.

† By Theorem 3.5, NIP valued fields are compositions of finitely ramified and Kaplansky valued fields, and for such fields, algebraic maximality implies henselian defectlessness. For example, see [28, Theorem 3.2].
These will come in particularly handy when we use the standard decomposition to study mixed characteristic valued fields.

**Lemma 2.8.** Let $(K, v)$ be a valued field such that $v$ is equal to a composition $\delta \circ v^0$ of valuations, that is, the place corresponding to $v$ can be decomposed into two places as depicted in the following diagram:

\[ K \xrightarrow{v^0} Ku^0 \xrightarrow{\tilde{v}} Kv \]

(i) Assume that $L/K$ is an algebraic extension of fields, and let $w$ be a prolongation of $v$ to $L$. Then, there is a unique prolongation $w^0$ of $v^0$ to $L$ which is a coarsening of $w$.

(ii) Let $(K^h, v^h)$ be the henselization of $(K, v)$. Then, $(K^h w^0, v^h)$ is the henselization of $(K v^0, v)$, where $w^0$ denotes the unique coarsening of $v^h$ prolonging $v^0$.

*Proof.*

(i) The valuation $v^0$ corresponds to a convex subgroup $\Delta$ of $vK$, and likewise a coarsening $w^0$ of $w$ corresponds to a convex subgroup $\Delta' \leq wL$. Such a $w^0$ is a prolongation of $v^0$ if and only if $\Delta' \cap vK = \Delta$. Since $L/K$ is algebraic, $wL/vK$ is torsion, and the only possible choice for $\Delta'$ is the convex hull of $\Delta$ in $wL$.

(ii) Since $K^h/K$ is algebraic, there is a unique coarsening $w^0$ of $v^h$ that prolongs $v^0$. Since a composition of two valuations is henselian if and only if both components are henselian ([13, Corollary 4.1.4]), $(K^h w^0, v^h)$ is henselian. Let $(L, \tilde{u})$ be an extension of $(K v^0, v)$ that is henselian. There exists an extension $(M, \hat{w})/(K, v^0)$ that is henselian and has residue field $M\hat{w} = L$. Then, the composition $u := \tilde{u} \circ \hat{w}$ is a henselian valuation on $M$ that prolongs $v$. Thus, $(K^h, v^h)$ may identified with a valued subfield of $(M, u)$. The restriction of $\hat{w}$ to $K^h$ coincides with $w^0$, and induces an embedding $(K^h w^0, v^h) \subseteq (L, \tilde{u})$, which shows that $(K^h w^0, v^h)$ satisfies the universal property of the henselization of $(K v^0, v)$. \(\square\)

**Lemma 2.9.** Let $(K, v)$ be a valued field such that $v$ is equal to a composition $\delta \circ v^0$ of valuations. Then, $(K, v)$ is defectless if and only if both $(K, v^0)$ and $(K v^0, v)$ are defectless.

*Proof.* For a finite extension $L/K$ of fields, let $u^0_1, \ldots, u^0_s$ be the distinct prolongations of $v^0$ to $L$. For each $i \in \{1, \ldots, s\}$, let $\tilde{u}_{i,1}, \ldots, \tilde{u}_{i,r_i}$ be the distinct prolongations of $v$ to $Lu_i$, and write $u_{i,j} := \tilde{u}_{i,j} \circ u^0_i$. It follows from Lemma 2.8(i) that $(u_{i,j} : i \leq s, j \leq r_i)$ enumerates the set of prolongations of $v$ to $L$.

Applying the Fundamental Inequality ([13, Theorem 3.3.4]) several times, we have the following calculation:

\[ [L : K] \geq \sum_{i \leq s} e(u^0_i/v^0) f(u^0_i/v^0) \]
\[ \geq \sum_{i \leq s} e(u^0_i/v^0) \sum_{j \leq r_i} e(\tilde{u}_{i,j}/v) f(\tilde{u}_{i,j}/v) \]
\[ = \sum_{i \leq s} \sum_{j \leq r_i} e(u_{i,j}/v) f(u_{i,j}/v),\]
since \( e(u_{i,j}/v) = e(u_{i,j}^0/v^0) e(\bar{u}_{i,j}/\bar{v}) \) and \( f(u_{i,j}/v) = f(\bar{u}_{i,j}/\bar{v}) \). If \((Kv^0, \bar{v})\) is defectless, then
\[
f(u_{i,j}^0/v^0) = \sum_{j \leq r_i} e(\bar{u}_{i,j}/\bar{v}) f(\bar{u}_{i,j}/\bar{v}),
\]
for each \( i \). If both \((K, v^0)\) and \((Kv^0, \bar{v})\) are defectless, then the inequalities in (1) and (2) are equalities, which verifies that \((K, v)\) is defectless. On the other hand, if \((K, v)\) is defectless, then we have
\[
[L : K] = \sum_{i \leq s} \sum_{j \leq r_i} e(u_{i,j}/v) f(u_{i,j}/v).
\]
It follows that the inequalities in (1) and (2) are equalities again. The first of these equalities verify that \((K, v^0)\) is defectless, and the second equality implies Equations (3), for each \( i \).

Continuing to assume that \((K, v)\) is defectless, it remains to verify that \((Kv^0, \bar{v})\) is defectless, for which we consider an arbitrary finite extension \( E/Kv^0 \). For example, by [27, Theorem 2.14], we may take a finite extension \((F, w^0)/(K, v^0)\) such that \( Fw^0/Kv^0 \) is isomorphic to \( E/Kv^0 \) and
\[
[F : K] = e(w^0/v^0) f(w^0/v^0).
\]
By identifying \( L \) with \( F \), and \( w^0 \) with \( u_0^1 \), we are again in the situation considered above (where now \( s = 1 \)). We have already shown that there is equality in (3), which verifies that \((Kv^0, \bar{v})\) is defectless. □

The property of separable defectlessness does not behave under composition in the same way, nor does the property of being henselian (and) separably defectless. In order to give an example, we introduce the standard construction of a Cohen ring over an imperfect field. Cohen rings and their quotient fields occur at several points throughout this paper.

A Cohen ring (see, e.g., [8]) is a complete Noetherian local ring \( A \) with maximal ideal \( pA \), where \( p \) is the residue characteristic of \( A \). Such a ring is strict if it is an integral domain. For each field \( k \) of characteristic \( p > 0 \), there exists a strict Cohen ring \( C[k] \) with residue field \( k \), unique up to isomorphism. Its quotient field then admits a complete unramified henselian valuation \( v \) with valuation ring \( C[k] \), value group \( \mathbb{Z} \), and residue field \( k \). We denote it by \( C(k) \) and call it a Cohen field over \( k \). When \( k \) is perfect, the Witt ring \( \mathbb{W}[k] \) and the Cohen ring \( C[k] \) coincide. Note that \( C[k] \) is unique up to isomorphism, but — when \( k \) is imperfect — it is not unique up to unique isomorphism. For a recent treatment of the algebra and model theory of Cohen rings, see [1].

Remark 2.10. Let \((k, u)\) be a separably closed valued field of characteristic \( p > 0 \) of imperfection degree \( e > 0 \), so that \( k \) is imperfect. Let \((C(k), v)\) be a Cohen field over \( k \). Since \( k \) is separably closed, in particular, \((k, u)\) is henselian separably defectless. Moreover, \((C(k), v)\) is maximal, thus is even henselian defectless. Since \( k \) is imperfect, it admits a proper purely inseparable extension \( k'/k \), to which \( u \) extends uniquely to a valuation \( u' \). Then, \((k', u')/(k, u)\) is a proper immediate extension. Therefore, \((C(k'), u'\circ v')/(C(k), u\circ v)\) is a proper immediate extension of valued fields in characteristic 0, which, in particular, is separable. This shows that \((C(k), u\circ v)\) is not

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\(^1\) Here, we adopt the convention for a field \( K \) that \( p^e = [K : K^{(p)}] \) unless this degree is infinite, in which case we simply write \( e = \infty \).
separably defectless, despite $u \circ v$ being the composition of two henselian separably defectless valuations.

A generalization of algebraically maximal Kaplansky fields is given by tame\(^\ddagger\) valued fields. An algebraic extension $(L, w)/(K, v)$ of valued fields is tame if

(i) $(wL : vK)$ is coprime to the residue characteristic,
(ii) $Lw/Kv$ is separable, and
(iii) $(L, w)/(K, v)$ is defectless.

A valued field $(K, v)$ is tame if all algebraic extensions are tame. For more details and equivalents definitions, see [28]. We encounter tame valued fields in the final two sections of this paper.

**Ax-Kochen/Ershov principles**

A fundamental principle in the model theory of valued fields is that in “well-behaved” valued fields, the model theory of the valued field should be determined by the model theory of its residue field and value group. This goes back to the seminal work by Ax and Kochen and, independently, Ershov, on the model theory of the $p$-adic numbers. Our main theorem implies, in particular, that for all henselian NIP valued fields $(K, v)$, the valuation be decomposed into finitely many pieces, all of which fit into some Ax-Kochen/Ershov setting. There is one Ax-Kochen/Ershov principle for relative completeness and one for relative model completeness, both of which occur in this paper. Let $\mathcal{K}$ be an class of valued fields. We say that $\mathcal{K}$ satisfies $\text{AKE}^\equiv$ if for all $(K, v), (L, w) \in \mathcal{K}$, we have

\[
(K, v) \equiv (L, w) \iff vK \equiv wL \text{ in } \mathcal{L}_{\text{val}} \quad \text{and} \quad Kv \equiv Lw \text{ in } \mathcal{L}_{\text{oag}} .
\]

This principle holds in case $\mathcal{K}$ is the class of henselian fields of equicharacteristic 0 ([4, 14]). In equal positive characteristic $p$, it holds for the class of separably algebraically maximal Kaplansky valued fields of characteristic $p$ and fixed degree of imperfection $e$ ([10]), and for the class of tame valued fields of characteristic $p$ ([28]). In mixed characteristic, it holds for unramified henselian valued fields ([1, 5, 15]).

The second such principle gives criteria for an embedding to be elementary. We say that $\mathcal{K}$ satisfies $\text{AKE}^\leq$ if for all $(K, v), (L, w) \in \mathcal{K}$, with $(K, v) \subseteq (L, w)$, we have

\[
(K, v) \leq (L, w) \iff vK \leq wL \text{ in } \mathcal{L}_{\text{val}} \quad \text{and} \quad Kv \leq Lw \text{ in } \mathcal{L}_{\text{ring}} .
\]

This principle also holds for all the cases mentioned above, and moreover, in tame valued fields of mixed characteristic ([28]) and finitely ramified henselian valued fields ([15, 37]).

\(^\ddagger\) Tameness here is a purely algebraic notion of valued fields, not to be confused with model-theoretic tameness notions like NIP.
3 | NIP VALUED FIELDS

Let $T$ be a complete $\mathcal{L}$-theory. Recall that a formula $\varphi(\vec{x}, \vec{y})$ has the independence property (IP) if there is a model $M \models T$ and sequences $(\vec{a}_i)_{i \in \mathbb{N}}$ in $M^{\vec{x}}$ and $(\vec{b}_J)_{J \subseteq \mathbb{N}}$ in $M^{\vec{y}}$ such that we have

$$M \models \varphi(\vec{a}_i, \vec{b}_J) \iff i \in J.$$ 

If there is some formula with IP, we say that $T$ has IP. Otherwise, we say that $T$ has NIP. For an introduction to NIP theories, see [34].

Throughout this paper, we are interested in NIP fields and NIP valued fields. We call a field $K$ (respectively, a valued field $(K, v)$) NIP if its $\mathcal{L}_{\text{ring}}$-theory (respectively, the $\mathcal{L}_{\text{val}}$-theory corresponding to $(K, v)$) has NIP. If $K$ is an NIP field and $v$ is any valuation on $K$, then $(K, v)$ is not necessarily an NIP valued field. For example, the field $\mathbb{Q}_p$ is NIP in $\mathcal{L}_{\text{ring}}$ (in fact, $(\mathbb{Q}_p, v_p)$ is NIP, cf. [5, Corollaire 7.5]). However, if $v$ is any prolongation of the $l$-adic valuation on $\mathbb{Q}$ to $\mathbb{Q}_p$ (for $l \neq p$), then $(\mathbb{Q}_p, v)$ has IP (this is a special case of [19, Theorem 5.3] as $v_p$ is $\mathcal{L}_{\text{ring}}$-definable on $\mathbb{Q}_p$). For pure fields, the only known algebraic consequence of being NIP is a theorem of Kaplan, Scanlon, and Wagner ([26, Corollary 4.4]): an NIP field of characteristic $p > 0$ admits no Galois extensions of degree divisible by $p$. In this section, henselianity does not play a role. Background on henselian NIP valued fields can be found at the beginning of the next section.

We now prove the “left-to-right” implication of our main result (Theorem 5.1), namely, Theorem 3.5. For this direction, it is not necessary to assume henselianity. The main ingredient for this theorem, often implicit in our arguments, is the aforementioned theorem by Kaplan, Scanlon, and Wagner. From this, the equicharacteristic case of Theorem 3.5 is straightforward.

**Proposition 3.1.** If $(K, v)$ is NIP and of equal characteristic, then $(K, v)$ is trivial or separably defectless Kaplansky.

**Proof.** In the case of equal characteristic zero, there is nothing to show. So, we suppose that $v$ is nontrivial and that $\text{char}(K) = p > 0$. Then, $(K, v)$ is Kaplansky by [22, Proposition 4.1]. Let $(K^h, v^h)$ be the henselization of $(K, v)$. By [26, Corollary 4.4], $K$ has no separable algebraic extensions of degree divisible by $p$. Thus, $K^h$ also has no separable algebraic extensions of degree divisible by $p$, since $K^h/K$ is separably algebraic. By the fundamental equality ([13, Theorem 3.3.3]), $(K^h, v^h)$ is separably defectless. Since a valued field is separably defectless if and only if its henselization is, by [12, Theorem 18.2], $(K, v)$ is separably defectless. □

We now turn to the case of mixed characteristic. We quote the next two statements for the convenience of the reader.

**Lemma 3.2.** Let $(K, v)$ be an NIP valued field, possibly with additional structure, and let $v^0$ be a coarsening of $v$. Then, $(K, v^0, v)$ is NIP. Consequently, both $(K, v^0)$ and $(Kv^0, \bar{v})$ are NIP.

**Proof.** This is a straightforward application of Shelah’s expansion theorem [34, Corollary 3.14]. For more details, see, for example, [21, Example 2.2]. The final claim follows because both $(K, v^0)$ and $(Kv^0, \bar{v})$ are interpretable in $(K, v^0, v)$. □

In case the residue field of the coarsening is stably embedded as a pure field, the converse to Lemma 3.2 also holds.
**Proposition 3.3** [22, Proposition 2.4]. Let \((K, v)\) be a valued field and \(v^0\) a coarsening of \(v\). Assume that both \((K, v^0)\) and \((Kv^0, \bar{v})\) are NIP. If the residue field \(Kv^0\) is stably embedded as a pure field in \((K, v^0)\), then \((K, v)\) is NIP.

The last ingredient needed for Theorem 3.5 is the fact that an NIP valued field has at most one coarsening with imperfect residue field.

**Lemma 3.4.** Let \((K, v)\) be an NIP valued field. Then, \(v\) has at most one coarsening with imperfect residue field. If such a coarsening exists, then it is the coarsest coarsening \(w\) of \(v\) with \(\text{char}(Kw) > 0\).

**Proof.** Assume that \((K, v)\) is NIP and \(\text{char}(Kv) = p > 0\). Let \(v_p\) be the coarsest coarsening of \(v\) with \(\text{char}(Kv_p) = p\) (note that \(v_p\) might be the trivial valuation). Let \(u\) be any coarsening of \(v\). We claim that if \(u \neq v_p\), then \(Ku\) is perfect. Since the valuation rings of the coarsenings of \(v\) are linearly ordered by inclusion, we have either \(\mathcal{O}_u \subsetneq \mathcal{O}_{v_p}\) or vice versa. If \(\mathcal{O}_{v_p} \subsetneq \mathcal{O}_u\), \(\text{char}(Ku) = 0\), hence \(Ku\) is perfect. If \(\mathcal{O}_u \subsetneq \mathcal{O}_{v_p}\), Lemma 3.2 implies that \((K, v_p, u, v)\) is NIP, and hence, so is \((Kv_p, \bar{u})\). Since \(\bar{u}\) is nontrivial by assumption, \((Kv_p, \bar{u})\) is separably defectless Kaplansky, by Proposition 3.1. In particular, \(Ku\) is perfect. □

We are now in a position to prove that all NIP valued fields satisfy the properties \((1)\) and \((2)\) occurring the main theorem (Theorem 5.1).

**Theorem 3.5.** Let \((K, v)\) be an NIP valued field. Then, both of the following hold:

1. \(Kv\) is NIP.
2. Either
   
   \((a)\) \[\begin{cases} (a.i) & (K, v) \text{ is of equal characteristic, and} \\ (a.ii) & (K, v) \text{ is trivial or separably defectless Kaplansky;} \end{cases}\]
   
   or
   
   \((b)\) \[\begin{cases} (b.i) & (K, v) \text{ has mixed characteristic } (0, p), \text{ and} \\ (b.ii) & (K, v_p) \text{ is finitely ramified, and} \\ (b.iii) & (Kv_p, \bar{v}) \text{ is trivial or separably defectless Kaplansky;} \end{cases}\]
   
   or
   
   \((c)\) \[\begin{cases} (c.i) & (K, v) \text{ has mixed characteristic } (0, p), \text{ and} \\ (c.ii) & (Kv_0, \bar{v}) \text{ is defectless Kaplansky.} \end{cases}\]

**Proof.** Assume that \((K, v)\) is NIP. Then, \(Kv\) is NIP, because it is interpretable in \((K, v)\), so \((1)\) holds. We now show that one of the cases \((a)\), \((b)\), or \((c)\) holds. If \(\text{char}(K) = \text{char}(Kv)\), then \((a)\) is satisfied by Proposition 3.1. Assume that \(\text{char}(K) = 0\) and \(\text{char}(Kv) = p > 0\). We again use the standard decomposition for \((K, v)\):

\[
\begin{align*}
K & \xrightarrow{vK/\Delta_0} K_{v_0} \xrightarrow{\Delta_0/\Delta_p} K_{v_p} \xrightarrow{\Delta_p} Kv
\end{align*}
\]
By Lemma 3.2, \((K_v, \bar{v})\) is NIP. Since \((K_v, \bar{v})\) is of equal characteristic \(p\), it is either trivially valued or separably defectless Kaplansky, by Proposition 3.1. In particular, \(\Delta_p\) is \(p\)-divisible.

We work with the case distinction between whether or not \(\Delta_0/\Delta_p\) is discrete. If \(\Delta_0/\Delta_p \cong \mathbb{Z}\), then \((K_v, \bar{v})\) is finitely ramified, so (b) holds. Otherwise, \(\Delta_0/\Delta_p\) is not discrete. We let \((K^*, v^*)\) be an \(\aleph_1\)-saturated elementary extension of \((K, v)\), and consider the standard decomposition for \((K^*, v^*)\):

\[
\begin{align*}
K^* & \xrightarrow{v^*K^*/\Delta_0^*} K^*v_0^* \xrightarrow{\Delta_0^*/\Delta_p^*} K^*v_p^* \xrightarrow{\Delta_p^*} K^*v^*
\end{align*}
\]

By Lemma 2.7, \(\Delta_0^*/\Delta_p^*\) is also not discrete. By [3, §4], \(\Delta_0^*/\Delta_p^*\) is isomorphic to \(\mathbb{R}\). The argument above to show the \(p\)-divisibility of \(\Delta_p\) also applies to \(\Delta_p^*\). Combining these statements, \(\Delta^*_0\) is \(p\)-divisible, which means that \((K^*, v^*)\) is roughly \(p\)-divisible, that is, any element \(\gamma \in [0, v^*(p)] \subseteq v^*K^*\) is \(p\)-divisible. Since this is an elementary property, \((K, v)\) is also roughly \(p\)-divisible. To conclude that \((K, v)\) is in case (c), it remains to show that \((K\bar{v}_0, \bar{v})\) is defectless. We have already seen that \((K\bar{v}_p, \bar{v})\) is separably defectless, which also applies to \((K^*\bar{v}_p, \bar{v}^*)\). Next, we claim that \(K\bar{v}_p\) is perfect. To see this, we first pass to an \(\aleph_1\)-saturated elementary extension \((K', u') \supseteq (K, v_p)\). Since \((K, v_p)\) is not finitely ramified, by saturation \(u'\) admits a proper coarsening \(w'\) with char\(K'(w') > 0\). Once more applying Lemma 3.2, \((K, v_p)\) is NIP, and hence, so is \((K', u')\). By Lemma 3.4, \(K'\bar{u}'\) is perfect, and thus, \(K\bar{v}_p\) is perfect. Since this applies also to \(K^*\bar{v}_p\), it follows that \((K^*\bar{v}_p, \bar{v}^*)\) is defectless. By [3, §4], \((K^*\bar{v}_0, \bar{v}^*)\) is maximal (i.e., admits no immediate extensions), so, in particular, is henselian and defectless. Thus, by Lemma 2.9, \((K^*\bar{v}_0, \bar{v}^*)\) is defectless. Applying Lemma 2.9 once again, we conclude that \((K^*, v^*)\) is defectless. Therefore, by Lemma 2.4, \((K, v)\) is defectless, and so, \((K\bar{v}_0, \bar{v})\) is defectless by Lemma 2.9. This verifies that (c) holds.

\[\square\]

**Remark 3.6.** Let \((K, v)\) be a valued field and suppose that (2) from Theorem 3.5 holds. If \(Kv\) is finite, then \((K, v)\) is trivial or finitely ramified. To see this, we argue as follows. In case (a), if \(v\) is nontrivial, then \((K, v)\) is separably defectless Kaplansky. Residue fields of such valuations are, in particular, closed under Artin–Schreier extensions, which finite fields are not. So, \(v\) is trivial. In case (b), if \((K\bar{v}_p, \bar{v})\) is nontrivial, then again \(Kv\) is closed under Artin–Schreier extensions, which it is not, and therefore, \(\bar{v}\) is trivial. Thus, \(v_p = v\), and so, \((K, v)\) is finitely ramified. Finally, if \(Kv\) is finite, case (c) does not occur, as finite fields are never the residue fields of Kaplansky valuations.

### 4 NIP TRANSFER FROM RESIDUE FIELD TO VALUED FIELD

Ax-Kochen/Ershov principles, as discussed in Section 2, allow the transfer of properties like decidability from the theories of the residue field and value group to that of the valued field. A key observation is that this means that also model-theoretic tameness properties, like NIP, transfer from residue field and value group to the valued field. The first such theorem, proven by Delon ([9]), states that a henselian valued field of equicharacteristic 0 is NIP (in \(L_{\text{val}}\)) if and only if its residue field and its value group are NIP (in \(L_{\text{ring}}\) and \(L_{\text{oag}}\), respectively). By a result of Gurevich and Schmitt ([16]), the \(L_{\text{oag}}\)-theory of any ordered abelian group is NIP. Following Delon, several further such “NIP transfer theorems” have been proven for henselian valued fields: in particular, by Bélair for unramified henselian valued fields with perfect residue field ([5, Corollaire 7.5]) and by Jahnke and Simon for separably algebraically maximal Kaplansky valued fields of finite degree.
of imperfection ([22, Lemma 3.2]). In this section, we prove two more analogs of Delon’s theorem.

The first case that we consider is that of separably algebraically maximal Kaplansky valued fields of infinite degree of imperfection. The second NIP transfer theorem we prove is for henselian finitely ramified valued fields with imperfect residue field.

In both cases, we employ the proof method from [22] (which, in turn, builds on [7]), that is, we consider the following two properties of a theory \( T \) of valued fields from [22].

(Se) The residue field and the value group are stably embedded.

(Im) If \( K \models T \) and \( a \) is a singleton in an elementary extension \( (K^*, v^*) \succeq (K, v) \) such that \( K(a)/K \), together with the restriction of \( v^* \), is an immediate extension, then \( \text{tp}(a/K) \) is implied by instances of NIP formulas.

By [22, Theorem 2.3], if the theory of \( (K, v) \) satisfies both (Se) and (Im), and \( Kv \) is NIP, then \( (K, v) \) is NIP.

### 4.1 NIP transfer for separably algebraically maximal Kaplansky valued fields

**Proposition 4.1.** Let \( (K, v) \) be a valued field of equal characteristic \( p > 0 \) that is separably algebraically maximal and Kaplansky. Then, \( (K, v) \) is NIP if and only if \( Kv \) is NIP.

**Proof.** Clearly, if \( (K, v) \) is NIP, then \( Kv \) is NIP. We now show that the theory of any separably maximal Kaplansky field of positive characteristic satisfies both properties (Se) and (Im).

If \( K \) has finite degree of imperfection, \( (K, v) \) satisfies (Se) by [22, Lemma 3.1]. For the case of infinite degree of imperfection, note that by [10, Théorème 3.1], the theory of separably algebraically maximal Kaplansky valued fields of characteristic \( p \) and given imperfection degree (allowed to be infinite) with value group elementarily equivalent to \( vK \) and residue field elementarily equivalent to \( Kv \) is complete. Exactly as explained in the proof of [22, Lemma 3.1], the theory of \( (K, v) \) satisfies (Se) also in case the degree of imperfection is infinite; only citing Delon ([10, Théorème 3.1]) rather than Kuhlmann and Pal ([29, Theorem 5.1]).

In [22, Lemma 3.2], (Im) is proved in the case that the imperfection degree of \( K \) is finite. Accordingly, we suppose that \( K \) has infinite degree of imperfection. Let \( K(a)/K \) be an immediate extension, taken within an elementary extension \( (K^*, v^*) \succeq (K, v) \), and from now on we equip all subfields of \( K^* \) with the restriction of \( v^* \). Since \( K(a)/K \) is immediate, \( K(a) \) is also Kaplansky. We will show that the type \( \text{tp}(a/K) \) is implied by quantifier-free formulas. If \( a \) is algebraic over \( K \), then already \( a \) is a member of \( K \), and the rest of the argument is trivial. Otherwise, we suppose that \( a \) is transcendental over \( K \). Let \( M \) be a maximal algebraic purely wild extension of \( K(a) \), taken within \( K^* \). Since \( K(a)/K \) is immediate, \( K(a) \) is also Kaplansky. We will show that the subextension \( M/K(a) \) consisting of those elements of \( M \) that are separably algebraic over \( K(a) \). Then, \( L/K \) is again immediate, and \( L \) is a maximal separably algebraic purely wild extension of \( K(a) \), which determines \( L \) uniquely up to isomorphism over \( K(a) \). Moreover, \( L \) is separably algebraically maximal and Kaplansky. Since \( L/K \) is separable, the imperfection degree of \( L \) is also infinite. By [10, Théorème 3.1], we have \( K \preceq L \) as valued fields. This shows that the quantifier-free type of \( a \) over \( K \) determines the isomorphism type of an elementary submodel of \( K^* \) which contains \( a \); thus, the quantifier-free type of \( a \) over \( K \) implies the full type of \( a \) over \( K \). Recall that any quantifier-free \( L_{\text{val}} \)-formula has NIP because every nontrivially valued field embeds into an algebraically closed nontrivially valued field and the \( L_{\text{val}} \)-theory
ACVF has NIP, see [34, Theorem A.11]. Thus, \((K, \nu)\) has the property \((\text{Im})\). Therefore, if \(K\nu\) is NIP in the language of rings, \((K, \nu)\) is NIP in the language of valued fields, by [22, Theorem 2.3]. □

As a special case of the previous proposition, we get that the theory of any separably closed valued field is NIP. This was previously shown by Delon (although her proof remains unpublished), and — in the case of finite degree of imperfection — by Hong ([20, Corollary 5.2.13]).

**Corollary 4.2.** The complete theory of any separably closed valued field is NIP.

**Proof.** Let \((K, \nu)\) be any separably closed valued field. Then, \((K, \nu)\) is separably algebraically maximal and Kaplansky, because \(K\) has no nontrivial separable field extensions. As \(K\nu\) is separably closed, it is stable by [36]. In particular, \(K\nu\) is NIP and so \((K, \nu)\) is NIP by Proposition 4.1. □

### 4.2 NIP transfer for henselian finitely ramified valued fields

Recall that we call a valued field \((K, \nu)\) of mixed characteristic \((0, p)\) unramified if \(\nu(p)\) is the minimum positive element of the value group, and finitely ramified if the interval \((0, \nu(p)] \subseteq \nu(K)\) is finite. In particular, any henselian valued field of mixed characteristic with value group \(\mathbb{Z}\) is finitely ramified, and so, is any power series field over such a field (valued by the composition of the power series valuations and the mixed characteristic valuation with value group \(\mathbb{Z}\)).

For unramified henselian valued fields \((K, \nu)\) with perfect residue field \(K\nu\), Bélair proved that the valued field is NIP in case its residue field is NIP in [5, Théorème 7.4(2)]. Although he does not state the assumption explicitly that the residue field need be perfect, his proof relies crucially on properties of Witt rings that only hold over perfect fields. We use Cohen rings (see section 2) instead.

We first show an NIP transfer result for henselian unramified valued fields (allowing the residue field to be imperfect). In order to do this, we verify once more that the conditions \((\text{SE})\) and \((\text{Im})\) hold. The fact that the residue field and value group are stably embedded as a pure field or, respectively, a pure ordered abelian group, in a henselian unramified valued field was shown in [1, Theorem 1.4], so we only have to show \((\text{Im})\).

**Lemma 4.3.** Let \((K, \nu)\) be a henselian unramified valued field. Then, the \(L_{\text{val}}\)-theory of \((K, \nu)\) satisfies \((\text{Im})\).

**Proof.** Let \((K, \nu)\) be a henselian unramified valued field, and let \((L, \omega)\) be elementarily equivalent to \((K, \nu)\). Take any \(a \in (L^*, w^*) \subseteq (L, \omega)\) such that \((M, \omega) := (L(a), \omega^*|_{L(a)})\) is an immediate extension of \((L, \omega)\). We show that the type of \(a\) over \(L\) is determined by the quantifier-free type of \(a\) over \(L\). The isomorphism type of \(a\) over \(L\) determines the henselization \((M^h, \omega^h)\) of \((M, \omega)\) up to isomorphism over \((L, \omega)\). Since the extensions of the value group and residue field are trivial, they are elementary. Thus, by [15, Theorem 4.3.4], \((L, \omega) \preceq (M^h, \omega^h)\). This shows that the quantifier-free type of \(a\) over \(L\) completely determines a model containing \(L(a)\), and thus, determines the complete type of \(a\) over \(L\). □

---

† In fact, every henselian finitely ramified valued field is elementarily equivalent to such a composition, as we argue in the proof of Proposition 6.9.
Applying the two lemmas above, we now get an NIP transfer principle for henselian unramified valued fields.

**Proposition 4.4.** Let $k$ be an NIP field of characteristic $p > 0$. Then any henselian unramified valued field $(K, v)$, with residue field $k$, is NIP in the language $\mathcal{L}_{\text{val}}$ of valued fields.

**Proof.** Let $(K, v)$ be a henselian unramified valued field with NIP residue field. We verify that the conditions (SE) and (Im) hold for the theory of $(K, v)$. First, (SE) holds by [1, Theorem 1.4]. By Lemma 4.3, property (Im) also holds. Hence, applying [22, Theorem 2.3], $(K, v)$ is NIP. □

We deduce a version of NIP transfer for mixed characteristic valued fields $(K, v)$ in the case that $(K, v_p)$ is finitely ramified: Here, (SE) cannot be deduced by showing that any automorphism of the residue field (resp. value group) lifts to an automorphism of the valued field, as in finitely ramified fields, the existence of such a lift may fail (see [1, Example 11.5]).† Thus, we take a different route.

Before we start, we need some further details about finitely ramified fields. The key idea is that — up to elementary equivalence — every henselian finitely ramified valued field with value group elementarily equivalent to $\mathbb{Z}$ is, in fact, a finite extension of a henselian unramified valued field over the same residue field. This will be made precise in Lemma 6.8.

**Fact 4.5** [35, Theorem 22.7]. Let $(K, v)$ be a complete valued field of mixed characteristic with value group $vK \cong \mathbb{Z}$ and ramification $e > 0$, that is, the interval $(0, v(p)] \subseteq vK$ contains $e$ many elements. Then, $(K, v)$ is an extension of degree $e$ of a complete unramified valued field $(C(Kv), w)$ that has residue field $Kv$ and value group $\mathbb{Z}$. The latter is a called Cohen subfield of $(K, v)$.

We can now use Cohen subfields to tackle NIP transfer in finitely ramified henselian valued fields.

**Proposition 4.6.** Let $(K, v)$ be a henselian valued field of mixed characteristic $(0, p)$ such that $(K, v_p)$ is finitely ramified and $(Kv_p, \bar{v})$ is NIP. Then $(K, v)$ is NIP.

**Proof.** Let $(K^*, v^*) \geq (K, v)$ be an $\aleph_1$-saturated elementary extension. Since $v_p$ is $\mathcal{L}_{\text{ring}}$-definable in $K$, the corresponding valuation $v^*_p$ on $K^*$ is also finitely ramified, and $(K^*v_p^*, \bar{v}^*)$ is NIP. Also if $(K^*, v^*)$ is NIP, then $(K, v)$ is NIP. Therefore, without loss of generality, we may suppose from now on that $(K, v)$ is $\aleph_1$-saturated.

Consider the standard decomposition of $(K, v)$:

\[
K \xrightarrow{vK/\Delta_0} Kv_0 \xrightarrow{\Delta_0/\Delta_p} Kv_p \xrightarrow{\Delta_p} Kv
\]

The rank-1 valued field $(Kv_0, \bar{v}_p)$ is complete and finitely ramified. By Fact 4.5, there is a subfield $L$ of $Kv_0$ with $Kv_0/L$ finite, such that the restriction $w$ of $\bar{v}_p$ to $L$ is unramified and complete, and $Lw = Kv_p$. Hence, $(L, w)$ is NIP, by Proposition 4.4. Moreover, the residue field $Lw$ is stably embedded in $(L, w)$, by [1, Theorem 1.4]. Thus, by Proposition 3.3, $(L, w')$ is NIP, where the composition $w' := \bar{v}w$ is a valuation on $L$ with residue field $Lw' = Kv$. In fact, $(Kv_0, v)/(L, w')$

†That (SE) holds for finitely ramified fields has since been shown in [2, Theorem 6.2 and Remark 6.3].
is a finite extension of valued fields, and the valuation ring of \( \overline{v} \) is the integral closure in \( K\nu_0 \) of the valuation ring of \( \nu' \), since \( \nu' \) is henselian. Thus, \((K\nu_0, \overline{v})\) is interpretable in \((L, \nu')\), and it follows that \((K\nu_0, \nu)\) is NIP. Furthermore, \((K, \nu_0)\) is NIP, because it is an equicharacteristic zero henselian valued field with NIP residue field [34, Theorem A.15]. Moreover, the residue field \( K\nu_0 \) is stably embedded in \((K, \nu_0)\), by [11, Corollary 5.25]. Applying Proposition 3.3 again, we conclude that \((K, \nu)\) is NIP.

\[ \square \]

As a special case, we immediately get the following corollary.

**Corollary 4.7.** Let \((K, \nu)\) be a henselian finitely ramified valued field of mixed characteristic with \( K\nu \) NIP. Then, \((K, \nu)\) is NIP in the language \( L_{val} \) of valued fields.

## 5 \ THE MAIN THEOREM AND IMMEDIATE CONSEQUENCES

We are now in a position to prove our main theorem.

**Theorem 5.1.** Let \((K, \nu)\) be a henselian valued field. Then \((K, \nu)\) is NIP if and only if both of the following hold.

1. \( K\nu \) is NIP.
2. Either
   - \((a)\) \begin{align*}
     \begin{cases}
       (a.i) & \text{\((K, \nu)\) is of equal characteristic, and} \\
       (a.ii) & \text{\((K, \nu)\) is trivial or separably defectless Kaplansky;}
     \end{cases}
   \end{align*}
   or
   - \( (b) \) \begin{align*}
     \begin{cases}
       (b.i) & \text{\((K, \nu)\) has mixed characteristic \((0, p)\), and} \\
       (b.ii) & \text{\((K, \nu_p)\) is finitely ramified, and} \\
       (b.iii) & \text{\((K\nu_p, \overline{\nu})\) is trivial or separably defectless Kaplansky;}
     \end{cases}
   \end{align*}
   or
   - \((c)\) \begin{align*}
     \begin{cases}
       (c.i) & \text{\((K, \nu)\) has mixed characteristic \((0, p)\), and} \\
       (c.ii) & \text{\((K\nu_0, \overline{\nu})\) is defectless Kaplansky.}
     \end{cases}
   \end{align*}

**Proof.** The implication \( \Rightarrow \) is a special case of Theorem 3.5. It remains to prove the converse, and thus, from now on we assume that \( K\nu \) is NIP and \((K, \nu)\) satisfies one of \((a)\), \((b)\), or \((c)\).

In case \((a)\), if \( \nu \) is trivial, we automatically get that \((K, \nu)\) is NIP. Otherwise, \( \nu \) is separably defectless Kaplansky and henselian. Thus, it is separably algebraically maximal and Kaplansky. Now Proposition 4.1 implies that \((K, \nu)\) is NIP.

If \((K, \nu)\) satisfies \((b)\), then \((K\nu_p, \overline{\nu})\) is either trivial (in particular, \( K\nu_p = K\nu \)), and thus, NIP, or separably defectless Kaplansky. In the latter case, \((K\nu_p, \overline{\nu})\) is NIP by Proposition 4.1. Applying Proposition 4.6, we conclude that \((K, \nu)\) is NIP.

Finally, assume that \((K, \nu)\) is in case \((c)\). We show first that \((K\nu_0, \overline{\nu})\) is NIP. Either \( \overline{\nu} \) is trivial, or \((K\nu_0, \overline{\nu})\) is defectless Kaplansky and henselian, thus algebraically maximal. In either case, \((K\nu_0, \overline{\nu})\) is NIP ([22, Theorem 3.3]). In particular, \( K\nu_0 \) is NIP. Once more, \((K, \nu_0)\) is NIP by Delon’s theorem.
Furthermore, $K\nu_0$ is stably embedded as a pure field, see [11, Corollary 5.25]. Finally, applying Proposition 3.3 once again, we conclude that $(K, \nu)$ is NIP.

Note that none of the cases appearing in the theorem is vacuous.

Example 5.2.

(a) We give examples of NIP valued fields $(K, \nu)$ of equal characteristic where the valuation is trivial or separably defectless Kaplansky. Naturally, the fields $\mathbb{C}, \mathbb{R}$, and $\mathbb{F}_{\text{alg}}^p$, equipped with the trivial valuation, are suitable examples, as well as $\mathbb{C}((\Gamma))$ and $\mathbb{R}((\Gamma))$ with the power series valuation $\nu_G$, for any ordered abelian group $\Gamma$. Moreover, if $\Gamma$ is $p$-divisible, then $\mathbb{F}_{\text{alg}}^p((\Gamma))$ together with the power series valuation $\nu_G$ is NIP. In particular, ACVF$_{0,0}$, SCVF$_p$, and RCVF are NIP.

(b) We give examples of NIP valued fields $(K, \nu)$ of mixed characteristic $(0, p)$ such that $(K, \nu_p)$ is finitely ramified and $(K\nu_p, \bar{\nu})$ is trivial or separably defectless Kaplansky. The most basic examples are $\mathbb{Q}_p$ with the $p$-adic valuation $\nu_p$, and any finite extension thereof. In all of these examples, $(K\nu_p, \bar{\nu})$ is trivial. To illustrate the case where $(K\nu_p, \bar{\nu})$ is separably defectless Kaplansky and nontrivial, we start with any separably closed nontrivially valued field $(k, u)$ of characteristic $p$. In particular, $(k, u)$ is NIP by Proposition 4.1. Now, let $(K, w)$ be a Cohen field over $k$ and consider the valuation $v$ on $K$ defined to be the composition $u \circ w$. Since that $v_p = w$ since $w$ is the coarsest mixed characteristic coarsening of $v$. Since $(k, u) = (K\nu_p, \bar{\nu})$ is NIP, $(K, v)$ is NIP by Proposition 4.6. If $k$ is imperfect, then $(K\nu_p, \bar{\nu})$ is separably algebraically maximal but not algebraically maximal. Furthermore, any (generalized) power series field over any of the aforementioned examples, together with the composition of valuations, is again NIP.

(c) We give examples of NIP valued fields $(K, \nu)$ of mixed characteristic $(0, p)$ such that $(K\nu_0, \nu^0)$ is defectless Kaplansky. The most obvious examples are algebraically closed valued fields, that is, models of ACVF$_{0,p}$. More generally, given any perfect infinite NIP field $k$ of characteristic $p$, for example, $\mathbb{F}_{\text{alg}}^p$, we may construct suitable $(K, \nu)$ with residue field $k$, as follows.

Let $\Gamma$ be a nontrivial $p$-divisible ordered abelian group. By a standard construction, see, for example, [27, Theorem 2.14], there is a valued field $(L, \omega)$ of mixed characteristic $(0, p)$ with value group $\Gamma$ and residue field $k$. Now let $(K, \nu)$ be a maximal algebraic purely wild extension of $(L, \omega)$. Since $\Gamma$ is $p$-divisible and $k$ admits no finite extensions of degree divisible by $p$, $(K, \nu)$ is the unique such extension up to isomorphism over $L$, by [30, Theorem 5.1]. Moreover, $(K, \nu)/(L, \omega)$ is immediate, by [30, Lemma 5.2]. By construction, $(K, \nu)$ is defectless Kaplansky, and it is NIP by [22, Theorem 3.3]. As before, any (generalized) power series field over an example as just described, together with the composition of valuations, is NIP.

As a consequence of Theorem 5.1, we obtain an analog to a result by Halevi and Hasson, who proved that the henselization of every strongly dependent valued field is strongly dependent ([17, Theorem 2]).

Corollary 5.3. If $(K, \nu)$ is NIP, then its henselization $(K^h, \nu^h)$ is NIP.

Proof. Let $(\dagger)$ be one of the following properties of a valued field that all occur in the statement of Theorem 5.1:

(34, Theorem A.15).
(i) trivially valued,
(ii) of equal characteristic zero,
(iii) of equal characteristic $p$, for a given prime $p > 0$,
(iv) of mixed characteristic $(0, p)$, for a given prime $p > 0$,
(v) separably defectless,
(vi) defectless,
(vii) Kaplansky,
(viii) finitely ramified (for $(K, v)$ of mixed characteristic).

Claim 5.3.1. If $(K, v)$ satisfies $(†)$, then $(K^h, v^h)$ satisfies $(†)$.  

Proof of claim. If $v$ is trivial, then it is already henselian. The properties relating to characteristic, that is, (ii)–(iv), are preserved when taking any extension, so in particular when passing to the henselization. For “separably defectless,” we apply [12, Theorem 18.2]. For “defectless,” we argue as follows.

First, we consider a finite purely inseparable extension $L/K$, and denote by $w$ the unique prolongation of $v$ to $L$. The compositum $LK^h$ — equipped with the unique prolongation of $v^h$ — coincides with the henselization $(L^h, w^h)$ of $(L, w)$. Since $K^h/K$ is separably algebraic, $L/K$ is linearly disjoint from $K^h/K$, and so $[L^h : K^h] = [L : K]$. Moreover, any finite purely inseparable extension of $K^h$ arises in this way. Since both $(L^h, w^h)/(L, w)$ and $(K^h, v^h)/(K, v)$ are immediate, we have the equivalence

$$[L : K] = e(w/v) f(w/v) \iff [L^h : K^h] = e(w^h/v^h) f(w^h/v^h).$$

Combining this with the claim for the property of separable defectlessness, it follows that if $(K, v)$ is defectless, then $(K^h, v^h)$ is defectless.

Finally, the properties “Kaplansky” and “finitely ramified” each depend only on the value group and residue field. Since $(K^h, v^h)/(K, v)$ is an immediate extension, it follows that if $(K, v)$ is Kaplansky (respectively, finitely ramified), then $(K^h, v^h)$ satisfies the same property.

Let $(K, v)$ be NIP. By Theorem 3.5, $(K, v)$ is in one of the cases (a), (b), or (c). By repeated application of the claim, we will now show that $(K^h, v^h)$ is in the same case as $(K, v)$. First, we consider case (a). There are four properties of $(K, v)$ mentioned in (a), and each of those properties is shown in the claim to transfer from $(K, v)$ to $(K^h, v^h)$. Thus, if $(K, v)$ satisfies (a), then $(K^h, v^h)$ also satisfies (a).

Also, by the claim, if $(K, v)$ is of mixed characteristic $(0, p)$, then the same is true of $(K^h, v^h)$. We now consider the standard decomposition of $(K^h, v^h)$:

$$K^h \xrightarrow{v^h K^h/\Delta^h_0} K^h v^h_0 \xrightarrow{\Delta^h_0/\Delta^h_p} K^h v^h_p \xrightarrow{\Delta^h_p} K^h v^h.$$

Note that $vK = v^h K^h$ and $K v = K^h v^h$, from which we have $\Delta_0 = \Delta^h_0$ and $\Delta_p = \Delta^h_p$. Nonetheless, $v^h_0$ does not denote the henselization of $v_0$, but the coarsening of $v^h$ corresponding to the convex subgroup $\Delta^h_0$; and likewise for $v^h_p$. The coarsening $(K, v^h_p)$ is finitely ramified if and only if $\Delta_0/\Delta_p$ is discrete, which is purely a property of the value group $vK$. Since $\Delta_0/\Delta_p = \Delta^h_0/\Delta^h_p$, we have that $(K, v^h_p)$ is finitely ramified if and only if $(K^h, v^h_p)$ is also finitely ramified. By Lemma 2.8(ii), the
henselization of \((K_v, \bar{v})\) is \((K^h_v^h, \bar{v}^h)\). By the claim, if \((K_v, \bar{v})\) is trivial or separably defectless Kaplansky, so is \((K^h_v^h, \bar{v}^h)\). Thus, if \((K, \bar{v})\) satisfies (b), then \((K^h, \bar{v}^h)\) also satisfies (b).

On the other hand, by Lemma 2.8(ii), the henselization of \((K_v, \bar{v})\) is \((K^h_v^h, \bar{v}^h)\). By the claim, if \((K_v, \bar{v})\) is defectless Kaplansky, so is \((K^h_v^h, \bar{v}^h)\). Thus, if \((K, \bar{v})\) satisfies (c), then \((K^h, \bar{v}^h)\) also satisfies (c).

We have shown that each of the conditions (a), (b), and (c) is preserved by taking the henselization. Moreover, \(K_v\) is equal to \(K^h_v\), so one is NIP if and only if so is the other. Therefore, since \((K, \bar{v})\) satisfies the conjunction of (1) and (2), so does \((K^h, \bar{v}^h)\). Since \((K^h, \bar{v}^h)\) is henselian, it follows from Theorem 5.1 that \((K^h, \bar{v}^h)\) is NIP.

\[\square\]

6 | A MODEL-THEORETIC VERSION OF THEOREM 5.1

The aim for this section is to resolve the following.

Task 6.1. Given a complete \(\mathcal{L}_{\text{ring}}\)-theory \(T_k = \text{Th}(k)\) of NIP fields, describe all of the complete \(\mathcal{L}_{\text{val}}\)-theories of NIP henselian valued fields \((K, \bar{v})\) such that the residue field \(K_v\) is a model of \(T_k\).

6.1 | Equal characteristic

Fix a complete \(\mathcal{L}_{\text{ring}}\)-theory \(T_k\) of NIP fields. Of course, one complete theory of NIP henselian valued fields with residue field a model of \(T_k\) is the theory of \(k\) equipped with the trivial valuation. If \(\text{char}(k) = 0\), then — by the Ax–Kochen/Ershov theorem in equicharacteristic 0 ([11, Theorem 5.11]) — for each complete \(\mathcal{L}_{\text{oag}}\)-theory \(T_\Gamma = \text{Th}(\Gamma)\) of nontrivial ordered abelian groups, there is exactly one complete \(\mathcal{L}_{\text{val}}\)-theory of equicharacteristic zero henselian valued fields \((K, \bar{v})\) with residue field \(K_v \equiv T_k\) and value group \(vK \equiv T_\Gamma\), namely, the theory of \((k((\Gamma)), \bar{v}_\Gamma)\). We denote this theory by \(T(k, \Gamma)\). Vacuously, models of \(T(k, \Gamma)\) are separably defectless and Kaplansky.

If \(k\) is of positive characteristic \(p\), is perfect, and admits no Galois extensions of degree divisible by \(p\), then for each complete \(\mathcal{L}_{\text{oag}}\)-theory \(T_\Gamma = \text{Th}(\Gamma)\) of \(p\)-divisible nontrivial ordered abelian groups, and each \(e \in \mathbb{N} \cup \{\infty\}\), by [10, Théorème 3.1], there is exactly one complete \(\mathcal{L}_{\text{val}}\)-theory of equicharacteristic \(p\) henselian separably defectless valued fields \((K, v)\) of imperfection degree \(e\), and with residue field \(K_v \equiv T_k\) and value group \(vK \equiv T_\Gamma\), which we denote by \(T_{e, s}(k, \Gamma)\). Models of \(T_{e, s}(k, \Gamma)\) will be Kaplansky, by our assumptions on \(k\) and \(\Gamma\). Note that if \(k\) is imperfect, it is not the residue field of a Kaplansky valued field, and hence, we necessarily have that \(K = k\) and \(v\) is the trivial valuation.

By Theorem 5.1, these are all the complete \(\mathcal{L}_{\text{val}}\)-theories of NIP valued fields in case (a). Thus, we have determined all complete \(\mathcal{L}_{\text{val}}\)-theories of NIP henselian valued fields of equal characteristic with residue field of a model of \(T_k\).

6.2 | Mixed characteristic

A complete theory of henselian valued fields of mixed characteristic \((0, p)\) does not only depend on the complete theory of the value group and the residue field but also on the value of \(p\). Just like before, we use the standard decomposition to differentiate between the cases (b) and (c). As above, we fix a complete \(\mathcal{L}_{\text{ring}}\)-theory \(T_k = \text{Th}(k)\) of NIP fields of characteristic \(p > 0\).
Now, consider a triple \((k, \Gamma, \gamma)\) where \(\Gamma\) is an ordered abelian group, and \(\gamma \in \Gamma\) is such that \(\gamma > 0\). Our aim is to characterize the complete theories of NIP henselian valued fields \((K, v)\) such that \(K_v \equiv k\) and \((vK, v(p)) \equiv (\Gamma, \gamma)\). Mimicking the standard decomposition, but expressed purely for the ordered abelian group \(\Gamma\), rather than for a valued field, we let \(\Gamma_{\gamma^+}\) be the smallest convex subgroup of \(\Gamma\) containing \(\gamma\), and let \(\Gamma_{\gamma^-}\) be the greatest convex subgroup of \(\Gamma\) not containing \(\gamma\).

Complete theories at the heart of case (b)

Let \(e \in \mathbb{N} \cup \{\infty\}\) and suppose that the image of \(\gamma\) is minimum positive in \(\Gamma/\Gamma_{\gamma^-}\). Note that this is an elementary property of the ordered abelian group \(\Gamma\) with a constant symbol for \(\gamma\). Let \(T_e(k, \Gamma, \gamma)\) be the theory of valued fields \((K, v)\) of mixed characteristic \((0, p)\) such that

(i) \(K_v \equiv k\),
(ii) \((vK, v(p)) \equiv (\Gamma, \gamma)\),
(iii) \((Kv_p, \bar{v})\) is separably algebraically maximal of imperfection degree \(e\), and
(iv) \((K, v_p)\) is henselian.

Recall that the valuation \(v_p\) is \(\mathcal{L}_{\text{ring}}\)-definable in \(K\), without parameters, uniformly for all henselian unramified valued fields \((K, v_p)\) of mixed characteristic \((0, p)\). Thus, the above listed properties of \((K, v)\) are \(\mathcal{L}_{\text{val}}\)-axiomatizable: the axiomatizability of (i) and (ii) simply uses \(T_k\) and \(T_\Gamma\), and for (iii) and (iv), we use the uniform definability of \(v_p\) plus (for example) the axioms discussed in [28, Section 4].

Lemma 6.2. If \(k\) is infinite, perfect, and NIP, and if \(\Gamma_{\gamma^-}\) is \(p\)-divisible, then \(T_e(k, \Gamma, \gamma)\) is complete.

Proof. Let \((K, v)\) be a model of \(T_e(k, \Gamma, \gamma)\). Since \(k\) is infinite, perfect, and NIP of positive characteristic, it admits no finite extension of degree divisible by \(p\). Thus, \((Kv_p, \bar{v})\) is a separably algebraically maximal Kaplansky valued field of equal characteristic \(p\) and imperfection degree \(e\). By [10, Théorème 3.1], \(T_e(k, \Gamma, \gamma)\) determines the complete theory of \((Kv_p, \bar{v})\). Moreover, by [1, Théorème 1.2], the complete \(\mathcal{L}_{\text{val}}\)-theory of \((K, v_p)\) is determined by henselianity and the theories of the residue field and the value group, which all follow from \(T_e(k, \Gamma, \gamma)\). \[\square\]

Complete theories at the heart of case (c)

Now suppose that \(\Gamma/\Gamma_{\gamma^-}\) is not discrete. Again, this is an elementary property of the ordered abelian group \(\Gamma\) with a constant symbol for \(\gamma\), see the proof of Lemma 2.7. We denote by \([\Gamma]_\gamma\) the relative divisible hull in \(\Gamma\) of the subgroup generated by \(\gamma\). Note that \([\Gamma]_\gamma\) will always be a subgroup of \(\Gamma_{\gamma^+}\), although it will in general not be a convex subgroup.

We say that a valued field \((F, v_F)\) is compatible with \((\Gamma, \gamma)\) if it is algebraically maximal, an algebraic extension of \((Q, v_p)\), its residue field \(Fv_F\) is \(\overline{F}_p\), and its value group satisfies \((v_F F, v_F(p)) \cong ([\Gamma]_\gamma, \gamma)\). If \(\gamma\) is \(p\)-divisible, then \([\Gamma]_\gamma\) is \(p\)-divisible, and so, in this case, \((F, v_F)\) will be Kaplansky. Let \(T(k, \Gamma, \gamma)\) be the theory of valued fields \((K, v)\) of mixed characteristic \((0, p)\) such that

(i) \(Kv \equiv k\),
(ii) \((vK, v(p)) \equiv (\Gamma, \gamma)\), and
(iii) \((K, v)\) is algebraically maximal.
Given compatible \((F, v_F)\), we let \(T(k, \Gamma, \gamma, F, v_F)\) be the theory extending \(T(k, \Gamma, \gamma)\) by further requiring of \((K, v)\) that

(iv) the algebraic part of \((K, v)\) is isomorphic to \((F, v_F)\).

Again, these properties are obviously \(\mathcal{L}_{\text{val}}\)-axiomatizable (for (iii), see [28, Section 4]). The description of the complete theories of algebraically maximal Kaplansky valued fields of mixed characteristic is well known, and due independently to Ershov and Ziegler. However, for lack of a convenient reference, in the proof of the following lemma, we rely on more modern sources.

**Lemma 6.3.** If \(k\) is infinite, perfect, and NIP, and if \(\Gamma\) is \(p\)-divisible, then \(T(k, \Gamma, \gamma, F, v_F)\) is complete.

**Proof.** Let \((K, v)\) and \((L, w)\) be models of \(T(k, \Gamma, \gamma, F, v_F)\). Again, we argue that, since \(k\) is infinite, perfect, and NIP of positive characteristic, it admits no finite extension of degree divisible by \(p\). Thus, \((K, v)\) and \((L, w)\) are algebraically maximal Kaplansky valued fields of mixed characteristic \((0, p)\): in particular, they are tame. We may identify \((F, v_F)\) — which is also tame — with the algebraic part of both \((K, v)\) and \((L, w)\). Since the class of tame fields is relatively subcomplete, by [28, Theorem 7.3], we have

\[
(K, v) \equiv_{(F,v_F)} (L, w).
\]

In particular, \(T(k, \Gamma, \gamma, F, v_F)\) is complete. \(\square\)

Given a valued field \((K, v)\) of mixed characteristic \((0, p)\) with value group \(\Gamma\), we denote by \(\Gamma_{(p)}\) the maximal \(p\)-divisible convex subgroup of \(\Gamma\), and by \(v_{(p)}\) the corresponding coarsening of \(v\). Then, \(v\) induces a valuation \(\bar{v}\) on the residue field \(Kv_{(p)}\) which has value group \(\Gamma_{(p)}\), and the value group of \(v_{(p)}\) is isomorphic to \(\Gamma / \Gamma_{(p)}\).

**Lemma 6.4.** There is an \(\mathcal{L}_{\text{val}}\)-formula \(\pi_p(x)\) in the language of ordered abelian groups which defines the convex subgroup \(\Gamma_{(p)}\) in any ordered abelian group \(\Gamma\). Thus, the valuation ring corresponding to the valuation \(v_{(p)}\) is uniformly \(\mathcal{L}_{\text{val}}\)-definable, without parameters, in all valued fields.

**Proof.** Take \(\pi_p(x)\) to be the formula \(\forall y \ (0 \leq y \leq |x| \rightarrow \exists z \ p z = y)\). \(\square\)

**Lemma 6.5.** Let \(T\) be a theory of bivalued fields \((K, v', v)\) with \(v'\) an equal characteristic zero henselian coarsening of \(v\), and suppose that \(T\) entails complete theories of valued fields \((K, v')\) and \((Kv', \bar{v})\). Then, \(T\) is complete.

**Proof.** Let \((K, v', v)\) and \((L, w', w)\) be two saturated models of \(T\) of the same cardinality. By the saturation assumption, we may assume that there are isomorphisms \(\psi : (K, v') \rightarrow (L, w')\) and \(\phi : (Kv', \bar{v}) \rightarrow (Lw', \bar{w})\). By stable embeddedness of the residue field in equal characteristic zero as a pure field, there is an isomorphism \(\chi : (K, v') \rightarrow (L, w')\) inducing \(\phi\). By construction, \(\chi\) is also an isomorphism \((K, v) \rightarrow (L, w)\). Thus, \(T\) is complete. \(\square\)

The next lemma is a very simple modification of Lemma 6.3.

**Lemma 6.6.** If \(k\) is infinite, perfect and NIP, and if \([-\gamma, \gamma] \subseteq p\Gamma\), then \(T(k, \Gamma, \gamma, F, v_F)\) is complete.
Proof. Let \((K, v)\) be a model of \(T(k, \Gamma, \gamma, F, v_F)\), and we apply the standard decomposition to \((K, v)\) with the usual notation:

\[
\begin{align*}
K & \xrightarrow{v_K/\Delta_0} K v_0 \xrightarrow{\Delta_0/\Delta_p} K v_p \xrightarrow{\Delta_p} K v
\end{align*}
\]

The assumption \([-\gamma, \gamma] \subseteq p\Gamma\) entails that \(\Delta_0 \subseteq v_{K(p)}\), which, in turn, means that \(v_{(p)}\) is a coarsening of \(v_0\). Then, \((Kv_p, v)\) is of mixed characteristic \((0, p)\) and is algebraically maximal, since the composition \(v_{(p)} \circ \sigma = v\) is algebraically maximal. Moreover, its value group \(v(Kv_p)\) is \(p\)-divisible and elementarily equivalent to \(\Gamma(p)\) by Lemma 6.4; and its residue field \((Kv_p)\bar{v} = K\) admits no extension of degree divisible by \(p\) because it is elementarily equivalent to the infinite NIP field \(k\). Thus, \((Kv_p, \bar{v})\) is Kaplansky. By Lemma 6.3, the property “algebraically maximal,” together with the theories of \(k\) and \((\Gamma(p), \gamma)\), determines the complete \(\mathcal{L}_{\text{val}}\)-theory of \((Kv_p, \bar{v})\). The \(\mathcal{L}_{\text{val}}\)-theory of the equal characteristic zero valued field \((K, v_{(p)})\) is determined by its henselianity and by the theories of \(Kv_p\) and \(v_{(p)}K\) by the Ax–Kochen/Ershov principle ([11, Theorem 5.11]). From Lemma 6.5, it follows that \(T(k, \Gamma, \gamma, F, v_F)\) is complete. \(\square\)

6.2.1 NIP henselian valued fields of mixed characteristic

We now assemble the lemmas from the previous subsections into a list of the complete theories of NIP henselian valued fields of mixed characteristic with residue field elementarily equivalent to \(k\). We first recall that the class of finitely ramified henselian valued fields satisfies the AKE\(\leq\) principle.

**Fact 6.7.** Let \((K, v) \subseteq (L, u)\) be an extension of henselian finitely ramified fields.

\[
(K, v) \preceq (L, u) \iff v_K \preceq u_L \quad \text{in } \mathcal{L}_{\text{val}} \quad \text{and} \quad K v \preceq L u \quad \text{in } \mathcal{L}_{\text{ring}}.
\]

**Proof.** See [15, Theorem 4.3.4] or [37, Satz V.5 I) iii)]. \(\square\)

This fact now allows us to view henselian finitely ramified fields with value group a \(\mathbb{Z}\)-group essentially (i.e., up to elementary equivalence) as finite extensions of Cohen fields.

**Lemma 6.8.** Let \((K, v)\) be a henselian finitely ramified valued field of mixed characteristic with value group \(vK \equiv \mathbb{Z}\). Then \((K, v)\) is elementarily equivalent to a finite extension \((L, u)\) of a Cohen field \((C(Kv), w)\), where \(w\) denotes the unique nontrivial henselian valuation on \(C(Kv)\) of mixed characteristic, and \(u\) is its unique prolongation to \(L\).

**Proof.** Let \((K^*, v^*) > (K, v)\) be an \(\aleph_1\)-saturated elementary extension. Then \((K^*, v^*)\) is also finitely ramified with \(v^*K^* \equiv \mathbb{Z}\). Consider the standard decomposition

\[
K^* \xrightarrow{v^*K^*/\Delta_0^*} K^*v_0^* \xrightarrow{\Delta_0^*} K^*v^*
\]

of \((K^*, v^*)\). Since \(v_0^*\) is henselian, we may apply [11, Theorem 2.9], to choose a section \(\phi : K^*v_0^* \rightarrow K^*\) of the residue map \(\text{res}_{v_0^*}\) of \(v_0^*\); this is an embedding of fields such that \(\text{res}_{v_0^*} \circ \phi\) is the identity. In
fact, φ is an embedding of valued fields \((K^*v_0^*, \bar{v}^*) \rightarrow (K^*, v^*)\) such that the extension of residue fields is trivial and the extension of value groups (with the value of \(p\) distinguished) is elementary. Applying Fact 6.7, we have \((K^*v_0^*, \bar{v}^*) \preceq (K^*, v^*)\). By saturation, \((K^*v_0^*, \bar{v}^*)\) is complete with value group \(\Delta^*_0 \cong \mathbb{Z}\). By Fact 4.5, \((K^*v_0^*, \bar{v}^*)\) is a finite extension of a Cohen subfield \((C(K^*v^*), w^*)\). By [1, Theorem 1.2], and since \(Kv \equiv K^*v^*\), we have

\[(C(K^*v^*), w^*) \equiv (C(Kv), w),\]

where \((C(Kv), w)\) is a Cohen field over \(Kv\). Since \((C(K^*v^*), w^*)\) and \((C(Kv), w)\) are elementarily equivalent, the latter admits a finite extension \((L, u)\) to which \((K^*v_0^*, \bar{v}^*)\) is elementarily equivalent. Putting together this chain of elementary equivalences, this shows that \((K, v)\) is elementarily equivalent to a finite extension of \((C(Kv), w)\).

□

We are now in a position to prove the main result of this section.

**Proposition 6.9.** Let \(T_k = Th(k)\) be a complete \(\mathcal{L}_{\text{ring}}\)-theory of NIP fields of characteristic \(p\). Let \((K, v)\) be an NIP henselian valued field of mixed characteristic with residue field elementarily equivalent to \(k\). Then, each of the following holds.

(A) If \(k\) is finite, then \((K, v)\) is \(\mathcal{L}_{\text{val}}\)-elementarily equivalent to a finite extension of a model of \(T_0(F_p, \Gamma, \gamma)\), for some \((\Gamma, \gamma)\) such that \(\gamma\) is the minimum positive element of \(\Gamma\).

In other words: if \(k\) is finite, then \((K, v)\) is elementarily equivalent to a (generalized) power series field over a finite extension of the \(p\)-adics where \(v\) corresponds to the composition of the \(p\)-adic valuation and the power series valuation.

(B) If \(k\) has imperfection degree \(e \in \mathbb{N}_{>0} \cup \{\infty\}\), then \((K, v)\) is \(\mathcal{L}_{\text{val}}\)-elementarily equivalent to a finite extension of a model of \(T_e(k, \Gamma, \gamma)\), for some \((\Gamma, \gamma)\) such that \(\gamma\) is the minimum positive element of \(\Gamma\).

In other words: if \(k\) is imperfect, then \((K, v)\) is elementarily equivalent to a (generalized) power series field over a finite extension of the Cohen field \(C(k)\) where \(v\) corresponds to the composition of the Cohen valuation and the power series valuation.

(C) If \(k\) is perfect and infinite, then \((K, v)\) is either

(i) elementarily equivalent to a finite extension of a model of \(T_e(k, \Gamma, \gamma)\), such that \(e \in \mathbb{N} \cup \{\infty\}\), the image of \(\gamma\) in \(\Gamma/\Gamma_{\gamma^-}\) is minimum positive and \(\Gamma_{\gamma^-}\) is \(p\)-divisible,

or

(ii) a model of \(T(k, \Gamma, \gamma, F, v_F)\), such that \(\Gamma/\Gamma_{\gamma^-}\) is not discrete, \([-\gamma, \gamma] \subseteq p\Gamma\), and \((F, v_F)\) is compatible with \((\Gamma, \gamma)\); that is, \((K, v)\) is a model of any completion of \(T(k, \Gamma, \gamma)\).

Proof. Let \(k\) be an NIP field, and \((K, v)\) NIP henselian valued field with \(Kv \equiv k\). Once more, we consider the standard decomposition of \((K, v)\):

\[
K \xrightarrow{vK/\Delta_0} \frac{Kv_0}{\Delta_0/\Delta_p} \xrightarrow{\Delta_0/\Delta_p} Kp_v \xrightarrow{\Delta_p} Kv
\]

(A) If \(k\) is finite, then we have \(Kv = k\) and Remark 3.6 implies that \((K, v)\) is finitely ramified, so \(\Delta_p = \{0\}\) and \(vK\) has a minimum positive element. Thus, \((Kv_0, \bar{v})\) is a henselian valued field of mixed characteristic with value group \(\mathbb{Z}\) and finite residue field. Applying [32, Theorem 3.1], it is elementarily equivalent to a finite extension \(L\) of the \(p\)-adics \(\mathbb{Q}_p\), equipped
with the unique extension $w$ of the $p$-adic valuation. We now use $G$ to denote the ordered abelian group $\nu K/\Delta_0$. By the Ax-Kochen/Ershov theorem for equicharacteristic 0 ([11, Theorem 5.11]), $(K, \nu_0)$ is elementarily equivalent to the generalized power series field $(L((G)), \nu_G)$. Moreover, since $\nu$ is finitely ramified, it is $\emptyset$-definable in the language of rings. Thus, we have

$$(K, \nu) \equiv (L((G)), w \circ \nu_G),$$

The latter is a finite extension of $(Q_p((G)), w \circ \nu_G)$. The valued field $(Q_p((G)), w \circ \nu_G)$ a model of $T_0(F_p, \Gamma, \gamma)$, where $\Gamma$ is the value group of $w \circ \nu_G$ on $Q_p((G))$ with minimum positive element $\gamma$.

(B) If $k$ has imperfection degree $e \in \mathbb{N}_0 \cup \{\infty\}$, then it is not the residue field of any Kaplansky valued field. Thus, Theorem 5.1 implies that $(K\nu_p, \bar{\nu})$ is trivially valued and $(K, \nu_p)$ is finitely ramified. In particular, $\nu K$ has a minimum positive element $\gamma_0$ and $(K\nu_0, \bar{\nu})$ is a henselian valued field of mixed characteristic with value group $\mathbb{Z}$ and residue field $k$. By Lemma 6.8, $(K\nu_0, \bar{\nu})$ is elementarily equivalent to a finite extension $(L, w)$ of the Cohen field $(C(k), w)$. Again, writing $G = \nu K/\Delta_0$, we have $(K, \nu_0) \equiv (L((G)), \nu_G)$. Once more, the $\mathcal{L}_{\text{ring}}$-definability of $\nu$ implies that

$$(K, \nu) \equiv (L((G)), w \circ \nu_G)$$

holds. Like in the previous case, we note that the extension

$$(C(k)((G)), w \circ \nu_G) \subseteq (L((G)), w \circ \nu_G)$$

is finite and that $(C(k)((G)), w \circ \nu_G)$ is a model of $T_e(k, \Gamma, \gamma)$, where again $\Gamma$ is the value group of $w \circ \nu_G$ on $C(k)((G))$ with minimum positive element $\gamma$.

(C) Assume that $k$ is infinite and perfect. If $(K, \nu)$ is a henselian valued field of mixed characteristic with residue field $K\nu \equiv k$, then Theorem 5.1 implies that one of the following holds:

(i) $(K, \nu_p)$ is finitely ramified and $(K\nu_p, \bar{\nu})$ is trivial or separably defectless Kaplansky, that is, $(K, \nu)$ satisfies clause (b)

(ii) $(K\nu_0, \bar{\nu})$ is defectless Kaplansky, that is, $(K, \nu)$ satisfies clause (c).

We show that cases (i) and (ii) correspond exactly to (C.i) and (C.ii), respectively.

We first assume that $(K, \nu_p)$ is finitely ramified and $(K\nu_p, \bar{\nu})$ is trivial or separably defectless Kaplansky of imperfection degree $e$. If $(K\nu_p, \bar{\nu})$ is trivial, we have $K\nu_p \equiv k$ and hence $K\nu_p$ is perfect and admits no Galois extensions of degree divisible by $p$. In particular, the trivially valued field $(K\nu_p, \bar{\nu})$ is separably defectless Kaplansky (of imperfection degree 0). Thus, we may treat these two subcases simultaneously. Now, let $(K^*, \nu^*) > (K, \nu)$ be an $\aleph_1$-saturated elementary extension. Consider the standard decomposition of $(K^*, \nu^*)$:

$$K^* \xrightarrow{\nu^* K^*/\Delta^*_e} K^*\nu^*_0 \xrightarrow{\Delta^*_e/\Delta^*_p} K^*\nu^*_p \xrightarrow{\Delta^*_p} K^*\nu^*$$

As $\nu_p$ is finitely ramified, it is $\emptyset$-definable, and hence, we also have $(K^*, \nu^*_p) > (K, \nu_p)$ and $(K^*\nu^*_p, \bar{\nu}^*) > (K\nu_p, \bar{\nu})$. In particular, $(K^*\nu^*_p, \bar{\nu}^*)$ is separably algebraically maximal Kaplansky. By saturation, $(K^*\nu^*_0, \bar{\nu}^*_p)$ is a complete mixed characteristic valued field with value group isomorphic to $\mathbb{Z}$. By Fact 4.5, $(K^*\nu^*_0, \bar{\nu}^*_p)$ is a finite extension of a Cohen subfield $(C(K^*\nu^*_p), w)$.  


Hence, we get a finite extension
\[(C(K^* v^*_p), \bar{v}^* \circ \omega) \subseteq (K^* v^*_0, \bar{v}^* \circ \bar{v}^*_p)\]
of valued fields. This gives rise to a finite extension
\[(C(K^* v^*_p)((G)), \bar{v}^* \circ \omega \circ v_G) \subseteq (K^* v^*_0((G)), \bar{v}^* \circ \bar{v}^*_p \circ v_G)\]
of valued fields, where \(G = v^* K^*/\Delta^*_0\) and \(v_G\) denotes the corresponding power series valuation in both cases. Since \(\bar{v}^*\) is separably algebraically maximal Kaplansky of imperfection degree \(e\) on \(K^* v^*_p\), \((C(K^* v^*_p)((G)), u)\) is a model of \(T_e(k, \Gamma, \gamma)\), where \(u = \bar{v}^* \circ \omega \circ v_G\) and \(\Gamma\) denotes the value group of \(u\) on \(C(K^* v^*_p)((G))\) and \(\gamma = u(p)\). Since the Ax-Kochen/Ershov principle in equicharacteristic 0 still holds if one adds structure on the (purely stably embedded) residue field (i.e., \(K^* v^*_0\)), we have
\[(K^*, v^*) \equiv (K^* v^*_0((G)), \bar{v}^* \circ \bar{v}^*_p \circ v_G).\]

Thus, \((K, v)\) is indeed elementarily equivalent to a finite extension of a model of \(T_e(k, \Gamma, \gamma)\) as desired.

Finally, if \((K v_0, \bar{v})\) is defectless Kaplansky, then, in particular, \((K, v)\) is algebraically maximal. Therefore, \((K, v)\) is a model of \(T(k, vK, v(p))\). Moreover, \(\Delta_0\) is \(p\)-divisible, and so, the inclusion \([-v(p), v(p)]\) \(\subseteq p \cdot vK\) holds and \(vK/\Delta_p\) is not discrete. \(\square\)

## 7 A REFINEMENT OF CONJECTURE 1.1

The last result of this paper is a reformulation of Conjecture 1.1. Theorem 7.1 below gives a list of first-order theories of valued henselian fields such that, if Conjecture 1.1 holds, any NIP field \(K\) admits a henselian valuation \(v\) such that \((K, v)\) is a model of one of the theories on the list. Since all the theories appearing in Theorem 7.1 are either complete or their completions can easily be described (cf. Remark 7.2) and moreover (by Theorem 5.1) NIP, Theorem 7.1 gives a converse of sorts for Conjecture 1.1.

Recall from the previous section that for an ordered abelian group \(\Gamma\) and \(\gamma \in \Gamma\), we use \(\Gamma_{\gamma^-}\) to denote the maximal convex subgroup not containing \(\gamma\) and \(\Gamma_{\gamma^+}\) to denote the minimal convex subgroup containing \(\gamma\).

**Theorem 7.1.** Suppose that Conjecture 1.1 holds. If a field \(K\) is NIP, then it is finite or admits a henselian valuation \(v\), such that \((K, v)\) is

(I) a model of \(T(C, \Gamma)\), or equivalently, \((K, v) \equiv (C((\Gamma)), v_\Gamma)\),
(II) a model of \(T(\mathbb{R}, \Gamma)\), or equivalently, \((K, v) \equiv (\mathbb{R}((\Gamma)), v_\Gamma)\),
(III) a model of \(T_{sd}(\mathbb{F}_p, \Gamma)\), for \(e \in \mathbb{N} \cup \{\infty\}\), and where \(\Gamma\) is \(p\)-divisible. In particular, in case \(K\) is perfect, we have \((K, v) \equiv (\mathbb{F}_p((\Gamma)), v_\Gamma)\),
(IV) elementarily equivalent to a finite extension of a model of \(T_0(\mathbb{F}_p, \Gamma, \gamma)\), where \(\gamma\) is minimum positive in \(\Gamma\), or equivalently, \((K, v) \equiv (L((\Delta)), \omega \circ v_\Delta)\) where \(\Delta = \Gamma/\Gamma_{\gamma^+}\) and \((L, \omega)\) is a finite extension of \((\mathbb{Q}_p, \omega)\),
(V) elementarily equivalent to a finite extension of a model of \(T_e(\mathbb{F}_p, \Gamma, \gamma)\), where the image of \(\gamma\) is minimum positive in \(\Gamma/\Gamma_{\gamma^-}\), and \(\Gamma_{\gamma^-}\) is \(p\)-divisible, or equivalently, \((K, v) \equiv (L, v \circ \omega)\) where
(L, w) is a finitely ramified henselian valued field with value group Γ/Γγ− and with residue field k and such that (k, ν) ⊧ T^{sd}_e (𝔽̄_p, Γ, γ),
(VI) a model of T (𝔽̄_p, Γ, γ), where Γγ+ is p-divisible.

Before we prove Theorem 7.1, we comment on the extent to which the theories occurring in the statement are complete and, if not, how to complete them.

Remark 7.2. It follows from Theorem 5.1 that all of the valued fields in cases (i)–(vi) are NIP. In particular, assuming Shelah’s conjecture, the list (i)–(vi) in Theorem 7.1 gives a classification of the theories of NIP fields.

Each theory appearing in cases (i)–(v) is complete. Case (vi) of the previous theorem deals with the theories T(𝔽̄_p, Γ, γ), where (Γ, γ) is such that Γγ+ is p-divisible. By Lemma 6.3, the completions of such theories are exactly the theories T(𝔽̄_p, Γ, γ, F, v_F), where (F, v_F) is compatible with (Γ, γ). The class of valued fields (F, v_F) compatible with some such (Γ, γ) admits a simple algebraic description. Namely, it is the class of tame valued fields (F, v_F) that are algebraic extensions of the maximal unramified extension Q̄_{p,alg} of Q_p,alg, equipped with the unique extension of the p-adic valuation. In this way, Theorem 7.1 even provides a conjectural classification of the complete theories of NIP fields.

Proof of Theorem 7.1. Assume that K is an infinite NIP field that is neither separably closed nor real closed. By Conjecture 1.1, K admits a nontrivial henselian valuation ν. Without loss of generality, we may assume that ν is the canonical henselian valuation on K. By [21, Theorem B], (K, ν) is NIP, and, in particular, the residue field k := Kv is NIP. Since ν is the canonical henselian valuation, K is either separably closed or not henselian. Since K is also NIP, applying Conjecture 1.1 once more yields that K is either separably closed, real closed, or finite.

Applying Theorem 5.1, (K, ν) lies in case (a), (b), or (c) of that theorem. If char(k) = 0, then (K, ν) lies in case (a) and we must have that k is algebraically closed or real closed, that is, elementarily equivalent to either C or R. Note that in neither case is the value group Γ := νK divisible, since otherwise K is already algebraically closed or real closed, contrary to our assumption. If k is algebraically closed, (K, ν) is a model of T(C, Γ), so (i) holds. If k is real closed, (K, ν) is a model of T(R, Γ), so (ii) holds.

Next, if char(K) = p > 0, then (K, ν) again lies in case (a) from 5.1, and we must have that k is separably closed or finite. Moreover, since ν is nontrivial, (a) implies that (K, ν) is separably defectless Kaplansky. In particular, Γ is p-divisible and k is perfect and Artin–Schreier closed, and so, k is algebraically closed. It follows that (K, ν) is a model of T^{sd}_e (𝔽̄_p, Γ), where e denotes the imperfection degree of K, and hence (iii) holds.

We now turn to the case that (K, ν) is of mixed characteristic (0, p), with p > 0. In this case, (K, ν) lies in case (b) or case (c) from 5.1. Also, k is either separably closed or finite. If k is finite, then by Proposition 6.9(A), (K, ν) is L_{val}-elementarily equivalent to a finite extension of a model of T(𝔽̄_p, Γ, γ), for some pair (Γ, γ) such that γ is minimum positive in Γ. This verifies (iv). Suppose now that k is separably closed. Let w denote a finest valuation on k, that is, one which admits no proper refinement. Such valuations always exist, and in this case, w is henselian and the residue

† Recall that our definition of finitely ramified does not require the value group to have rank 1, cf. p. 2.
‡ Although in this proof, we reference [21] that, in turn, references the present paper, the argument is not circular since [21] only refers to results in Section 4 of the present paper.
field $k_w$ is $\overline{\mathbb{F}}_p$. By Proposition 6.9(C), there is a dichotomy between cases (C.i) and (C.ii). In case (C.i), $(K, v)$ is $L_{\text{val}}$-elementarily equivalent to a finite extension of a model of $T_e(\overline{\mathbb{F}}_p, \Gamma, \gamma)$ such that $e \in \mathbb{N} \cup \{\infty\}$ and the image of $\gamma$ in $\Gamma / \Gamma_{\gamma-}$ is minimum positive, and $\Gamma_{\gamma-}$ is $p$-divisible. This verifies (v). In case (C.ii), $(K, v)$ is a model of $T(\overline{\mathbb{F}}_p, \Gamma, \gamma)$ and $[-\gamma, \gamma] \subseteq p\Gamma$, which implies that $\Gamma_{\gamma+}$ is $p$-divisible. This verifies (vi).

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