MULTIPlicative ZETA FUNCTION AND LOGARITHmic
POINT COUNTING OVER FINITE FIELDS

O. BRAUNLING

Abstract. The zeta function of a motive over a finite field is multiplicat ive with
respect to the direct sum of motives. It has beautiful analytic properties, as were
predicted by the Weil conjectures. There is also a multiplicative zeta function, which
instead respects the tensor product of motives. There is no analogue of the Weil
conjectures, and we give a sufficient criterion for an analytic continuation to exist.
This happens, for example, for cellular varieties, abelian varieties, or genus \( \geq 2 \)
curves with a supersingular Jacobian.

1. Introduction

Let \( X/\mathbb{F}_q \) be a variety over a finite field. The usual zeta function
\begin{equation}
Z(X, t) := \exp \left( \sum_{r \geq 1} |X(\mathbb{F}_{q^r})| \cdot \frac{t^r}{r} \right)
\end{equation}
behaves well under disjoint union, \( Z(X_1 \coprod X_2, t) = Z(X_1, t) \cdot Z(X_2, t) \). Generalized to
motives, it respects the symmetric monoidal structure coming from the direct sum
of motives. However, we might instead be interested in the question: Can our variety \( X \)
be written as a product \( X = X_1 \times X_2 \)? For this type of question the multiplicative zeta
function
\begin{equation}
Z_{\log}(X, t) := \exp \left( \sum_{r \geq 1} \log |X(\mathbb{F}_{q^r})| \cdot \frac{t^r}{r} \right),
\end{equation}
when defined, is better suited. It satisfies
\[ Z_{\log}(X_1 \times X_2, t) = Z_{\log}(X_1, t) \cdot Z_{\log}(X_2, t). \]
The definition can also be generalized to motives, and then respects the symmetric
monoidal structure coming from the tensor product of motives. In a way, \( Z \) resp. \( Z_{\log} \)
belong to the two natural symmetric monoidal structures on a Tannakian category, “\( \oplus \)”
resp. “\( \otimes \)”.

We know a lot about the ordinary zeta function thanks to the Weil conjectures, for
example:
(A) The function \( Z \) is rational; in particular it has an analytic continuation to the entire
complex plane.
(B) Poincaré Duality of a smooth projective variety \( X \) induces a functional equation
\[ Z(X, (q^d t)^{-1}) = \pm q^{d \chi(X)} \cdot \chi(X) \cdot Z(X, t). \]
(C) Zeros and poles can be described in terms of the cohomology of \( X \).

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And of course all properties of $Z$ follow from the existence of a Weil cohomology theory. There is no indication that the function $Z_{\log}$ can be obtained from something like a Grothendieck–Lefschetz trace formula, the logarithm term is just too disruptive, so its analytic properties are far less clear. Basically, following Murphy’s Law, one might suspect its properties are random at best.

But this is not so. Firstly, we shall show that $Z_{\log}$ behaves well for all abelian varieties:

**Theorem 1.1.** Let $A/\mathbb{F}_q$ be an abelian variety of dimension $g \geq 1$. Then $Z_{\log}$ has radius of convergence 1. If $\alpha_1, \ldots, \alpha_{2g}$ denote the Weil $q$-numbers of weight one, $Z_{\log}$ admits a multi-valued analytic continuation to

$$
\mathbb{C} \setminus \{1, \alpha_1^{Z_{\geq 2}}, \ldots, \alpha_{2g}^{Z_{\geq 2}}\}.
$$

See Theorem 5.3 for a precise statement. We will give a precise formulation for ‘multi-valued analytic continuation’ below. Secondly, we show that $Z_{\log}$ has an analytic continuation for all cellular varieties. Indeed, it suffices if all summands in the motivic decomposition of the variety are (Tate or) supersingular. The latter means that all its Frobenius eigenvalues are of the shape $\zeta \cdot q^{w/2}$ for $\zeta$ a root of unity and $w$ an integer. This encompasses all Artin and Tate motives. If one believes in the Tate conjecture, numerical pure motives over $\mathbb{F}_q$ are generated by all abelian varieties; and these supersingular motivic summands would be those coming from the supersingular abelian varieties.

**Theorem 1.2.** Suppose $X/\mathbb{F}_q$ is a smooth projective variety with an $\mathbb{F}_q$-rational point. Suppose its motive splits as a direct sum

$$
\mathcal{M}(X) = \bigoplus M_i
$$

such that each summand $M_i$ is supersingular, e.g. a Tate motive. Then $Z_{\log}$ has a multi-valued analytic continuation to

$$
\mathbb{C} \setminus \Delta,
$$

with $\Delta$ some discrete subset of $\mathbb{C}$.

See Theorem 5.7 for details. This theorem covers for example: projective space, Grassmannians, or more broadly all projective homogeneous varieties. It also covers smooth projective curves of arbitrary genus, as long as their Jacobian is supersingular.

We also extend the definition of $Z_{\log}$ to motives. It cannot always be defined then, but whenever it exists, it is multiplicative with respect to the tensor product of motives. There are plenty of motives not coming from a variety, for which we also get the existence of analytic continuations.

The above theorems also have an implication which no longer makes any reference to $Z_{\log}$:

**Corollary.** If $X/\mathbb{F}_q$ is a smooth projective variety of dimension $\geq 1$, meeting the hypotheses of Theorem 1.1 or Theorem 1.2, then the sequence

$$
 n \mapsto \log |X(\mathbb{F}_{q^n})|
$$

does not satisfy any linear recurrence equation.

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1Here ‘motive’ refers to pure Grothendieck motives. It can be taken to mean numerical motives, or $\ell$-adic homological motives for any $\ell$ different from the characteristic of $\mathbb{F}_q$. 

This result may not be particularly important; and probably admits a direct proof based on the rationality of $Z$. However, it falls out with no extra work from the previous results: If the sequence satisfies a linear recurrence, then its generating function, which is nothing but $Z_{\log}/Z_{\log}$, would be rational. Rational functions have a single-valued analytic continuation to $\mathbb{C} \setminus \{ \text{finite set} \}$, contradicting the analytic properties which our theory yields. This needs the more precise versions in the main body of the text, and not the shortened formulations above. See Theorem 5.18.

**Corollary.** Suppose $X/\mathbb{F}_q$ is a geometrically connected smooth projective curve with an $\mathbb{F}_q$-rational point. If

1. the genus is $g = 0, 1$ or
2. the genus is $g \geq 2$ and the Jacobian of $X$ is supersingular,

then $Z_{\log}(X, t)$ admits a multi-valued analytic continuation to $\mathbb{C} \setminus \Delta$, with $\Delta$ some discrete subset of $\mathbb{C}$.

See Theorem 5.15

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2. **Definitions**

2.1. **Conventions.** For us, a variety $X/k$ is a finite type separated $k$-scheme for $k$ some field. A morphism of varieties is tacitly understood to mean a finite type separated $k$-morphism. Suppose $k = \mathbb{F}_q$ is a finite field. By “Frobenius” we always refer to the geometric Frobenius, i.e. it acts as $x \mapsto x^{q^{-1}}$ on elements $x \in \mathbb{F}_q$. This agreement only really plays a rôle to ensure that $Z_{\ell}(1) := \lim_{\ell \to \infty} \mu_{\ell}$ has weight $-2$.

The term “logarithm” will usually refer to the standard branch, i.e.

$$\log z = \log |z| + i\arg z \quad \text{with} \quad \arg z \in (-\pi, \pi]$$

for all $z \in \mathbb{C}^\times$. When we work with a more general branch of the logarithm, we denote it by a capitalized ‘Log’.

We will freely use some aspects of the theory of pure (Grothendieck) motives. All we need is explained in [Mil94] or [And04]. Our conventions are as follows: Let $F$ be any field of characteristic zero, which will serve as our field of coefficients. We pick $F$ once and for all and henceforth drop it from the notation. Let $\text{Mot}_{\sim}(k)$ be the category of effective pure (Grothendieck) motives over $k$ with coefficients in $F$, and “$\sim$” denotes an adequate equivalence relation. Objects are of the shape $(X, p)$ with $X$ a smooth projective $k$-variety and $p$ an idempotent correspondence from $X$ to itself.

Concretely, we write $\text{Mot}_{\text{num}}(k)$ for numerical motives, and $\text{Mot}_{\text{hom}}(k)$ for homological motives ($\ell \neq \text{char} k$), using $\ell$-adic cohomology as the underlying Weil cohomology theory. Conjecturally, homological and numerical equivalence agree, and in particular the choice of a Weil cohomology theory should not matter. So, speculatively, $\text{Mot}_{\text{num}}(k) = \text{Mot}_{\text{hom}}(k)$. However, this remains open. Many aspects of the formalism in this text can be extended to motives. We discuss this in the Appendix A.
2.2. Definition. First of all, we give a definition of $Z_{\log}$ for varieties.

**Definition 2.1.** If $X/F_q$ is a variety with an $F_q$-rational point, we define the multiplicative zeta function as the power series

$$Z_{\log}(X, t) := \exp \left( \sum_{r \geq 1} \log |X(F_q^r)| \cdot \frac{t^r}{r} \right).$$

Thanks to the condition $X(F_q) \neq \emptyset$, we have $|X(F_q^r)| \geq 1$ for all $r \geq 1$, making this expression well-defined as a formal power series over the reals. If $X_1, X_2$ are varieties with $X_i(F_q) \neq \emptyset$ for $i = 1, 2$, we get the fundamental property:

$$Z_{\log}(X_1 \times X_2, t) = Z_{\log}(X_1, t) \cdot Z_{\log}(X_2, t).$$

For the ordinary zeta function, denoted by $Z$, we instead have

$$Z(X_1 \coprod X_2, t) = Z(X_1, t) \cdot Z(X_2, t).$$

**Remark 2.2.** There is also a formula for product varieties for the function $Z$, but it relies on the more complicated so-called Witt product `$\ast$' (it is not due to Witt, but named so as it is related to the ring of big Witt vectors). Then $Z(X_1 \times X_2, t) = Z(X_1, t) \ast Z(X_2, t)$.

We refer to [Nau07] and [Ram15] for more on this perspective.

**Definition 2.3.** We say that a series $f(t) = \sum b_r t^r$, $b_r \in \mathbb{C}$, with positive radius of convergence, has a (possibly multi-valued) analytic continuation, or in brief (AC), if there exists a discrete set $\Delta \subset \mathbb{C}$, $0 \notin \Delta$, with the property: For every simply connected domain $U$ with $0 \in U \subset \mathbb{C} \setminus \Delta$, there exists a holomorphic function $f_U : U \to \mathbb{C}$ such that $f_U$ agrees with $f$ in some neighbourhood of $t = 0$.

Equivalently, regard $(\mathbb{C} \setminus \Delta, 0)$ as a pointed space. If $V$ denotes a sufficiently small neighbourhood of 0, the above datum defines a unique lift of $f$ on $V$ to the universal covering space $(\tilde{X}, *)$:

$$\begin{align*}
\tilde{X} \times \ast & \to (\mathbb{C} \setminus \Delta, 0) \\
(V, 0) & \to (\mathbb{C} \setminus \Delta, 0),
\end{align*}$$

Giving this lift of $f$ is equivalent to providing the collection of all $(f_U)_U$ as above. This formulation is more elegant, but less practical for explicit computations.

We will take Definition 2.3 as the meaning for the term ‘analytic continuation’ in this text in order to avoid having to repeat more precise qualifiers again and again. We call it multi-valued because different choices of $U$ might yield different continuations.

**Remark 2.4 (Existence).** The existence of a (single- or multi-valued) analytic continuation is a non-trivial statement. Even for the ordinary zeta function, as in Equation 1.1, having an explicit power series expansion does not easily let us read off whether $Z$ is a rational function. For example, the power series

(A) $\sum_{r \geq 1} t^r$, \hspace{1cm} (B) $\sum_{r \geq 1} \frac{t^r}{r}$, \hspace{1cm} (C) $\sum_{r \geq 1} t^{2r}$,

all have radius of convergence precisely one. The first one is a rational function, namely $\frac{t}{1-t}$, and thus admits a meromorphic continuation to the entire plane, while the second...
is $-\log(1-t)$, so while it does admit a holomorphic extension to all of $\mathbb{C} \setminus \{0\}$, it requires multiple branches, and yet the last power series has the unit circle as its natural boundary. This means that it is impossible to find an analytic continuation anywhere outside the open unit disc – for a dense set inside the unit circle, its values tend to go off to infinity as one approaches the radius of convergence. Both (A) and (B) satisfy our definition of (AC), while (C) does not.

**Example 2.5.** For affine space we have $|A^n(F_q)| = q^{nr}$ and thus

$$Z_{\log}(A^n, t) = \exp \left( \sum_{r \geq 1} \log(q^{nr}) \cdot \frac{t^r}{r} \right) = \exp \left( n \log(q) \cdot \frac{t}{1-t} \right) = (q^{\frac{1}{1-t}})^n.$$  

This defines a single-valued holomorphic continuation to all of $\mathbb{C} \setminus \{0\}$. We have (AC) for $\Delta := \{1\}$. We also see the property of the multiplicativity; it would have sufficed to deal with $A^1$. (The usual zeta function is $Z(A^n, t) = \frac{1}{1-q^{nt}}$)

**Example 2.6.** Suppose we want to deal with the torus $(G_m)^n$ resp. $(P_1)^n$. It suffices to treat $n = 1$. However, we get

$$Z_{\log}(-, t) := \exp \left( \sum_{r \geq 1} \log(q^{r \mp 1}) \cdot \frac{t^r}{r} \right).$$

The radius of convergence of the inner series is $R = 1$, and the values of $\log(q^{r \mp 1})$ will always be very close to $r \log q$, yet not quite the same. So the question whether (AC) holds is a priori unclear. Later, we will be able to answer this affirmatively.

See the Appendix, [A] for the extension of the definition of $Z_{\log}$ to motives. Most of this text can be read without having to deal with motives.

### 2.3. Pseudo-divisors

We shall use the word ‘divisor’ in the sense of complex manifolds, i.e. instead of defining it to be a finite linear combination as customary in algebraic geometry, we just demand local finiteness in the complex topology:

**Definition 2.7.** A pseudo-divisor on $\mathbb{C}$ is a set-theoretic function

$$\mathcal{D} : \mathbb{C} \to \mathbb{Z} \cup \{\infty\}.$$

We may express this datum in the notation $\mathcal{D} = \sum_{P \in \mathbb{C}} n_P[P]$ with $n_P \in \mathbb{Z} \cup \{\infty\}$, reminiscent of divisors. Define the support of $\mathcal{D}$ by

$$\text{supp} \mathcal{D} := \{P \in \mathbb{C} \mid n_P \neq 0\},$$

where the closure is taken with respect to the complex topology (not Zariski!). We say that $\mathcal{D}$ is a locally finite divisor on $\mathbb{C}$ if the support of $\mathcal{D}$ is locally finite, i.e. for any point $z \in \mathbb{C}$ there exists an open neighbourhood of $z$ which contains only finitely many points in the support of $\mathcal{D}$.

**Definition 2.8.** Suppose $\mathcal{D}$ is a pseudo-divisor. We also define a $2\pi i$-periodic version, called $\mathcal{D}_{\text{per.}}$, of a pseudo-divisor by

$$\mathcal{D}_{\text{per.}} := \sum_{j=1}^{\infty} T_j^* \mathcal{D},$$

where $T_j$ is the translation $z \mapsto z + 2\pi i j$ (i.e. $T_j^*$ translates the divisor $\mathcal{P}$ by the multiple $2\pi i j$ in the plane). We write $\mathcal{D}_{\text{per}}$ if $j$ runs through all of $\mathbb{Z}$, so $\mathcal{D}_{\text{per}} := \sum_{j=1}^{\infty} T_j^* \mathcal{D}$. 


If for any point \( P \in \mathbb{C} \) these definitions would require us to evaluate a sum of infinitely many non-zero terms, we define the multiplicity of the sum at \( P \) to be \( n_P = \infty \).

3. Step I

In this section we will begin developing the technical tools necessary to establish the existence of an analytic continuation.

3.1. Idea. Our method is as follows: We want to apply Abel–Plana summation, which is a technique which succeeds excellently in producing analytic continuations for the polylogarithm \( \text{Li}_s(z) \) or the Hurwitz zeta function \( \zeta(s; q) \). It belongs to the family of results around Euler–MacLaurin summation. In brief: Firstly, we transform the power series in question into an integral, and then we evaluate the integral in a different fashion. If this is designed in a careful way, one may arrange to arrive at an expression which remains sensible in a larger domain of definition than the original power series. As we shall see, this strategy frequently succeeds for \( Z_{\log} \). Whether it does, will turn out to be controlled by a certain pseudo-divisor.

References are Olver’s book [Olv97, Ch. 8, §3] or Hardy’s classic treatise [Har92, §13.14]. The statement is as follows:

**Proposition 3.1** (Abel, Plana). Suppose \( h : \{ s \mid \text{Re } s \geq 0 \} \to \mathbb{C} \) is a function such that the following conditions are met:

1. For every integer \( n \geq 0 \) and the vertical strip \( S_n := \{ s \mid n \leq \text{Re } s \leq n + 1 \} \), the function \( h \) is continuous in \( S_n \) and is holomorphic in the interior of \( S_n \) and at \( s = 0 \).
2. For every \( a \geq 0 \), we have \( \lim_{b \to \pm \infty} |h(a \pm bi)| \cdot e^{-2\pi b} = 0 \) and this convergence is uniform in \( a \).
3. We have \( \lim_{a \to +\infty} \int_0^\infty |h(a \pm bi)| e^{-2\pi b} \, db = 0 \).

Then the identity

\[
\sum_{n=0}^\infty h(n) = \int_0^\infty h(s) \, ds + \frac{1}{2} h(0) + i \int_0^\infty \frac{h(ib) - h(-ib)}{e^{2\pi b} - 1} \, db
\]

holds and the integrals on the right-hand side exist.

We shall shortly see that we will need to modify this method a little bit in order to apply it to our problem.

3.2. Setup. In this section we shall work with some running assumptions and notation throughout: Pick \( N \geq 1 \). Suppose for \( i = 1, \ldots, N \) we are given complex numbers \( \varepsilon_i, \lambda_i \in \mathbb{C} \) such that \( 0 < |\lambda_i| < 1 \). Define a pseudo-divisor (in the sense of Definition 2.7) and depending on our data \( (\varepsilon_i, \lambda_i)_{i=1,\ldots,N} \) by

\[
\mathcal{P} := \sum_{l=1}^{\infty} \sum_{k_1 + \cdots + k_N = l} \left( \frac{l-1}{k_1, \ldots, k_N} \varepsilon_1^{k_1} \cdots \varepsilon_N^{k_N} \cdot [k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N] \right).
\]

Or, equivalently, one might prefer: To any point \( z \in \mathbb{C} \) attach a multiplicity by

\[
\sum_{l=1}^{\infty} \sum_{k_1 + \cdots + k_N = l} \left( \frac{l-1}{k_1, \ldots, k_N} \varepsilon_1^{k_1} \cdots \varepsilon_N^{k_N} \cdot \delta_z = \sum_{i} k_i \log \lambda_i \right).
\]
where \( \delta_A = 1 \) if \( A \) is a true statement, and \( \delta_A = 0 \) otherwise. If there are infinitely many such summands, we just declare the multiplicity to be \( \infty \).

We define \( M := \max_i |\varepsilon_i| \). If \( r_0 \geq 1 \) is any integer, we will intend to expand the logarithm in the expression

\[
\sum_{r \geq r_0} \log \left( 1 - \sum_{i=1}^N \varepsilon_i \lambda_i^r \right) \cdot e^{-wr}
\]

in terms of its usual power series \( \log(1 - z) = -\sum_{l \geq 1} \frac{1}{l} z^l \). Ideally, we would like to pick \( r_0 := 1 \). However, this is not necessarily possible because for small \( r \) the expression \( \sum_{i=1}^N \varepsilon_i \lambda_i^r \) need not lie within the radius of convergence of the logarithm series. We fix this as follows:

We compute

\[
\left| \sum_{i=1}^N \varepsilon_i \lambda_i^r \right| \leq M \sum_{i=1}^N |\varepsilon_i \lambda_i^r| \leq M \cdot \max_i |\lambda_i|^{\text{Re}(r)} \cdot \sum_{i=1}^N e^{-\text{Im}(r) \arg(\lambda_i)},
\]

where the second inequality stems from the computation \( |e^{r \log \lambda_i}| = |e^{x \log |\lambda_i| - y \arg \lambda_i}| = |\lambda_i|^{\text{Re}(r)} e^{-\text{Im}(r) \arg(\lambda_i)} \), where we have written \( r = x + iy \) with \( x, y \in \mathbb{R} \). Thus, if we pick some \( K > 0 \) and constrain

\[ |\text{Im}(r)| \leq K, \]

then there exists some sufficiently large integer \( r_0 \geq 1 \) such that the following two properties hold:

1. For all \( i = 1, \ldots, N \), and all \( r \) with \( \text{Re}(r) \geq r_0 \) and \( |\text{Im}(r)| \leq K \),

\[ |e^{r \log \lambda_i}| < \frac{1}{2}, \tag{3.2} \]

2. And moreover, for all \( r \) with \( \text{Re}(r) \geq r_0 \) and \( |\text{Im}(r)| \leq K \),

\[ \left| \sum_{i=1}^N \varepsilon_i \lambda_i^r \right| < \frac{1}{2}. \tag{3.3} \]

It is here where we have used the condition \( |\lambda_i| < 1 \). From now on, fix once and for all some \( K > 0 \) and pick \( r_0 \) (tacitly depending on \( \varepsilon_i, \lambda_i, K \)) so that both inequalities, 3.2, 3.3 hold.

**Remark 3.2.** Suppose we pick a different \( K' \) such that \( 0 < K' < K \). Then the above conditions still remain valid for the same choice of \( r_0 \).

### 3.3. Continuation of auxiliary functions.

With this data chosen, we shall show:
Theorem 3.3. Suppose we are given \((\epsilon_i, \lambda_i)_{i=1,\ldots,N}\) and have picked \(K, r_0\) as explained above. Then for all \(w \in \mathbb{C}\) in the open right half-plane we have the equality
\[
\sum_{r=r_0}^{\infty} \log \left( 1 - \sum_{i=1}^{N} \epsilon_i \lambda_i^r \right) e^{-wr} = \frac{1}{2} \log \left( 1 - \sum_{i=1}^{N} \epsilon_i \lambda_i^{r_0} \right) \cdot e^{-wr_0} + \sum_{l=1}^{\infty} \sum_{j \in \mathbb{Z}} \sum_{k_1, \ldots, k_N = l} \left( \frac{l-1}{k_1, \ldots, k_N} \right) \epsilon_1 \ldots \epsilon_N \epsilon_i \lambda_1^{k_1} \ldots \lambda_N^{k_N} e^{r_0 (k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N - w + 2\pi i j)} \right) \frac{k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N - w + 2\pi i j}{k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N - w + 2\pi i j}.
\]

1. The series on the left side is uniformly convergent in any compactum in the open right half-plane.
2. The series on the right side is uniformly convergent in any compactum \(A \subset \mathbb{C}\) outside the support of \(P^{\text{per}}\), the periodization of the pseudo-divisor in Equation 3.1. The support of \(P^{\text{per}}\) lies in the closed left half-plane.

Note that the support of \(P^{\text{per}}\) might be the entire closed left half-plane, so the statement of (2) can happen to be no stronger than (1).

The rest of the section will be devoted to the proof of this. Define
\[
(3.4) \quad h : [r_0, +\infty) \times i[-K, K] \to \mathbb{C}.
\]
\[
(3.5) \quad h(r) := \log \left( 1 - \sum_{i=1}^{N} \epsilon_i \lambda_i^r \right) \cdot e^{-wr},
\]
where \(\lambda^r\) means \(\lambda^r := \exp(r \cdot \log \lambda)\). Inside the half-infinite box \([r_0, +\infty) \times i[-K, K]\), we remain inside the radius of convergence of the logarithm series, and in particular cannot hit the branch cut of the outer logarithm. Thus, \(h\) is holomorphic (and as is easy to see, it is even holomorphic in an open neighbourhood of this box).

We begin with the following observation:

Remark 3.4. Clearly \(\log(1 - z)\) is bounded inside any closed disc inside the open unit disc, say \(\leq C_0\). Then since line 3.3 implies that \(\sum_{i=1}^{N} \epsilon_i \lambda_i^r\) lies inside this circle, we get
\[
\left| \log \left( 1 - \sum_{i=1}^{N} \epsilon_i \lambda_i^r \right) e^{-wr} \right| \leq C_0 \left| e^{-wr} \right| = C_0 \left| e^{-((\text{Re} w)(\text{Re} r) + (\text{Im} w)(\text{Im} r))} \right|.
\]

Hence, given any compactum \(A \subset \mathbb{C}\) in the open right half-plane, then for any \(w \in A\) and any sequence of values \(r_n\) with \(|\text{Im} r_n|\) bounded (e.g. \(\leq K\)) and \(\text{Re}(r_n) \to +\infty\), we have exponential decay of
\[
|h(r)| = \left| \log \left( 1 - \sum_{i=1}^{N} \epsilon_i \lambda_i^r \right) e^{-wr} \right|
\]
towards zero.

So the initial idea would be to apply Proposition 3.1 to the function in Equation 3.5. However, this does not quite work because we only have a function \(h : [r_0, +\infty) \times i[-K, K] \to \mathbb{C}\), i.e. defined on a much smaller domain as would be required. Of course the formula in Equation 3.5 can be extended to make sense on all of \(\mathbb{C}\), however not in
a holomorphic way. Indeed, the branch cut of the logarithm will generally (depending on \( \varepsilon, \lambda \)) make it impossible to meet the holomorphicity condition of Proposition 3.1.

**Example 3.5.** The following figure shows an example of the set of those \( r \in \mathbb{C} \), where \( \text{Re}(1 - \lambda_1 r + \lambda_2 r) \leq 0 \) for suitable \( \lambda_1, \lambda_2 \) on the left, and an example with \( N = 3 \) on the right. As one can see, the resulting geometry is complicated.

The terms \( \lambda_i^r = \exp(r \log \lambda_i) \) have a periodicity built in, caused by the \( 2\pi i \)-periodicity of the exponential function. This explains why we have so many distinct connected components. Correspondingly, we get many pairwise disjoint copies of the logarithm branch cut, which lie inside these copies of the negative half-plane.

**Example 3.6.** As a complex plot, the function \( r \mapsto \log(1 - \lambda_1 r + \lambda_2 r) \) can, depending on \( \lambda_1, \lambda_2 \), for example look as follows:

One can see the jumps at the many copies of the logarithmic branch cut. Any contour integration running through this territory is necessarily problematic. It is best to avoid such regions altogether.

We shall therefore work with a more complicated variant of the Abel–Plana method. Suppose \( a \leq b \) are integers with \( r_0 \leq a \):

**Proposition 3.7.** Suppose \( b : [a, +\infty) \times i[-K, K] \to \mathbb{C} \) is any function which admits a holomorphic continuation to an open neighbourhood of this box. Then for all integers
1 ≤ a < b, we have
\[
\sum_{r=a}^{b} h(r) = \frac{1}{2} h(a) + \frac{1}{2} h(b) + \int_{a}^{b} h(s) ds
\]
\[
+ i \int_{0}^{K} \frac{h(a + iy) - h(a - iy)}{e^{2\pi y} - 1} dy - \int_{0}^{K} \frac{h(b + iy) - h(b - iy)}{e^{2\pi y} - 1} dy
\]
\[
- \int_{a+iK}^{b+iK} \frac{h(s)}{1 - e^{-2\pi is}} ds + \int_{a-iK}^{b-iK} \frac{h(s)}{1 - e^{2\pi is}} ds.
\]

Proof. This is proven in a similar fashion as the original result, see for example [Olv97, Ch. 8, §3]. A detailed proof is given as [Bra17, Prop. 4.3].

Next, using Prop. 3.7 with the holomorphic function of Equation 3.4 we obtain
\[
\sum_{r=a}^{b} \log \left( 1 - \sum_{i=1}^{N} \varepsilon_i \lambda_i^r \right) e^{-wr} = \frac{1}{2} \log \left( 1 - \sum_{i=1}^{N} \varepsilon_i \lambda_i^a \right) \cdot e^{-wa}
\]
\[
+ \frac{1}{2} \log \left( 1 - \sum_{i=1}^{N} \varepsilon_i \lambda_i^b \right) \cdot e^{-wb} + \int_{a}^{b} h(s) ds
\]
\[
+ i \int_{0}^{K} \frac{h(a + iy) - h(a - iy)}{e^{2\pi y} - 1} dy - i \int_{0}^{K} \frac{h(b + iy) - h(b - iy)}{e^{2\pi y} - 1} dy
\]
\[
- \int_{a+iK}^{b+iK} \frac{h(s)}{1 - e^{-2\pi is}} ds + \int_{a-iK}^{b-iK} \frac{h(s)}{1 - e^{2\pi is}} ds.
\]

It remains to compute these integrals. We will first treat them for the summation from a to b on the left-hand side, and then afterwards let b → +∞. We begin with the integrals which appear in the last line of the above equation:

**Proposition 3.8.** Pick a sign “+/−”. Suppose the pseudo-divisor \( \mathcal{P}^{\text{per.} \pm} \) (see Definition 2.8) is locally finite. Then the integral
\[
V^{\pm}(w) := \int_{a+iK}^{b+iK} \frac{\log \left( 1 - \sum_{i=1}^{N} \varepsilon_i \lambda_i^r \right) \cdot e^{-wr}}{1 - e^{-2\pi ir}} dr,
\]
resp.
\[
V^{−}(w) := \int_{a-iK}^{b-iK} \frac{\log \left( 1 - \sum_{i=1}^{N} \varepsilon_i \lambda_i^r \right) \cdot e^{-wr}}{e^{2\pi r} - 1} dr,
\]
exists and defines a holomorphic function on the open right half-plane. A meromorphic analytic continuation to the entire complex plane is given by
\[
\tilde{V}^{\pm}(w) = \pm \sum_{l=1}^{\infty} \sum_{k_1 + \cdots + k_N = l} \left( \frac{l - 1}{k_1, \ldots, k_N} \right) e^{k_1 \cdots e_{k_N}^{k_N}}
\]
\[
\sum_{j=1}^{\infty} e^{r(k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N - w \pm 2\pi ij)} \bigg|_{r=b\pm iK}^{r=a\pm iK}
\]
All its poles have order 1 and lie precisely at the support of \( \mathcal{P}^{\text{per.} \pm} \). This series converges uniformly in any compactum in \( \mathbb{C} \) which avoids the support of \( \mathcal{P}^{\text{per.} \pm} \).
Proof. Note that for \( r = x \pm iy \) with \( x, y \in \mathbb{R} \) and \( y > 0 \), we get \( |e^{2\pi i(x \pm iy)}| = e^{\pm 2\pi y} \).

Rewrite \( V^+ \) as

\[
\int_{a + iK}^{b + iK} \frac{\log \left( 1 - \sum_{i=1}^{N} \varepsilon_i \lambda_i^r \right) \cdot e^{-wr}}{1 - e^{-2\pi i r}} \, dr = \int_{a + iK}^{b + iK} \frac{\log \left( 1 - \sum_{i=1}^{N} \varepsilon_i \lambda_i^r \right) \cdot e^{-wr}}{-e^{-2\pi i r} (1 - e^{2\pi i r})} \, dr
\]

and since all throughout the path of integration we have \( \text{Im} \, r > 0 \), we have \( |e^{2\pi i r}| < 1 \). Thus, we may unravel the term \( 1 - e^{2\pi i r} \) in terms of a geometric series, yielding

\[
= \sum_{j=1}^{\infty} \int_{a + iK}^{b + iK} \log \left( 1 - \sum_{i=1}^{N} \varepsilon_i \lambda_i^r \right) \cdot e^{(-w \mp 2\pi ij)r} \, dr.
\]

For \( V^- \) one proceeds analogously (in this case we will expand \( 1 - e^{-2\pi i r} \) as a geometric series. It converges since \( r > 0 \) for \( \lambda_i^r \)).

Thus, for our claim it suffices to handle the numerators. We have the general

\[
\sum_{l=1}^{\infty} \sum \left( 1 - \sum_{i=1}^{N} \varepsilon_i \lambda_i^r \right) \cdot e^{(-w \pm 2\pi ij)r} \, dr,
\]

but this integral is easy to compute: We find

\[
\int_{a \pm iK}^{b \pm iK} (\lambda_1^r)^{k_1} \cdots (\lambda_N^r)^{k_N} \cdot e^{(-w \pm 2\pi ij)r} \, dr = \frac{e^{r(k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N \mp w \pm 2\pi ij)}}{k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N \pm w \pm 2\pi ij} \bigg|_{r=a \pm iK}^{r=b \pm iK}.
\]

Now,

\[
\int_{a \pm iK}^{b \pm iK} \log \left( 1 - \sum_{i=1}^{N} \varepsilon_i \lambda_i^r \right) \cdot e^{(-w \mp 2\pi ij)r} \, dr
\]

We still need to settle the uniform convergence: Let \( A \subset \mathbb{C} \) be any compactum avoiding the support of \( \mathcal{D}^{\text{per,} \pm} \). This means that for all \( w \in A \) the denominator \( k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N - w \pm 2\pi ij \) is non-zero and thus we can give a uniform lower bound \( > 0 \) valid on all of \( A \). Thus, for our claim it suffices to handle the numerators. We have the general

\[
\sum_{j=1}^{\infty} \sum \left( 1 - \sum_{i=1}^{N} \varepsilon_i \lambda_i^r \right) \cdot e^{(-w \pm 2\pi ij)r} \, dr.
\]
formula \( |e^{\alpha \beta}| = e^{(\Re \alpha)(\Re \beta) - (\Im \alpha)(\Im \beta)} \) for all \( \alpha, \beta \in \mathbb{C} \). It yields

\[
\left| (e^{\pm i y})(k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N - w \pm 2\pi i j) \right| = (e^{-2\pi y})^j \cdot e^{-x \cdot \Re w} \cdot e^{\pm y \cdot \Im w} \cdot \prod_{n=1}^{N} e^{k_n (x \log |\lambda_n| + \pi y \arg \lambda_n)}
\]

and this can be rewritten as

\[
= (e^{-2\pi y})^j \cdot e^{-x \cdot \Re w} \cdot e^{\pm y \cdot \Im w} \cdot \prod_{n=1}^{N} (e^{x \log |\lambda_n| + \pi y \arg \lambda_n})^k_n.
\]

So far we had worked with \( r = x \pm iy \) and \( y > 0 \). We may instead write \( r = x + iy \) and allow all \( y \in \mathbb{R}, y \neq 0 \). Then this expression becomes

\[
(3.7) \quad = (e^{-2\pi |y|})^j \cdot e^{-x \cdot \Re w} \cdot e^{y \cdot \Im w} \cdot \prod_{n=1}^{N} (e^{x \log |\lambda_n| + y \arg \lambda_n})^k_n
\]

\[
= (e^{-2\pi |x|})^j \cdot e^{-x \cdot \Re w} \cdot e^{y \cdot \Im w} \cdot \prod_{n=1}^{N} e^{r \log |\lambda_n|}^k_n
\]

Now, by our choice of \( r_0 \), we have \( |e^{r \log \lambda_n}| < \frac{1}{2} \) for all \( r \in [r_0, +\infty) \times i[-K, K] \) (see Equation 3.72). We observe: We have a sum over \( k_1, \ldots, k_N \) and \( j \). Since \( |\Re r| > 0 \) (remember that we only need the cases \( r = a \pm iK \) and \( r = b \pm iK \), so we even have \( |\Re r| = K \), but this stronger statement is not truly needed here), the term \( (e^{-2\pi |x|})^j \) guarantees exponential decay in \( j \), and the product term on the right guarantees exponential decay in each of \( k_1, \ldots, k_N \). As a result, the entire sum over \( k_1, \ldots, k_N \) and \( j \) can be dominated by convergent geometric series in each of these variables. Thus, the numerators converge uniformly in \( A \).

Next, we will use the previous result and let \( b \to +\infty \):

**Corollary 3.9.** In every compactum \( A \subset \mathbb{C} \) inside the open right half-plane, we have

\[
\int_{a+iK}^{\infty+iK} \log \left( 1 - \sum_{i=1}^{N} \xi_i \lambda_i^r \right) \cdot e^{-w r} \frac{d r}{1 - e^{-2\pi i r}}
\]

\[
= -\sum_{l=1}^{N} \sum_{k_1 + \cdots + k_N = l} \left( \frac{l-1}{k_1, \ldots, k_N} \right) e^{k_1} \cdots e^{k_N} \sum_{j=1}^{\infty} \frac{e^{(a+iK)(k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N - w + 2\pi i j)}}{k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N - w + 2\pi i j}
\]

as a uniformly convergent series in \( w \in A \). The series itself converges uniformly in any compactum in \( \mathbb{C} \) which avoids the support of \( \mathcal{P}^{\text{per}, \pm} \). This corresponds to the \( V^+ \)-case of Prop. 3.8; the analogous statements in the \( V^- \)-case also holds.

**Proof.** We continue with the same notation as in the proof of Prop. 3.8. To show convergence: Let \( A \subset \mathbb{C} \) be a compactum avoiding the support of \( \mathcal{P}^{\text{per}, \pm} \). Now Equation 3.77 suffices to see uniform convergence. To show agreement with the integral: This time \( A \subset \mathbb{C} \) is a compactum in the right half-plane. Hence, there exists some \( \epsilon > 0 \) such that \( \Re w > \epsilon \) holds for all \( w \in A \). Now we take the limit \( b \to +\infty \). As \( \Re w > \epsilon \), the term \( e^{-x \cdot \Re w} \cdot e^{y \cdot \Im w} \) in Equation 3.77 therefore gives an exponential decay also in \( b \to +\infty \); irrespective of the imaginary part as \( |y| \leq K \) and \( |\Im w| \) is also bounded since \( w \in A \), which is compact. Thus, the other factors stay bounded.

**Remark 3.10.** If we drop the condition that \( A \) has to lie in the right half-plane, the last claim will indeed fail.
These proofs have handled two of the integrals which appear in Equation 3.6. For the remaining integrals, the computation can be carried out in virtually the same way. We leave the details to the reader and only list the results:

**Proposition 3.11.** In every compactum $A \subset \mathbb{C}$ inside the open right half-plane, we have

$$
\int_a^\infty \log \left(1 - \sum_{i=1}^N \varepsilon_i \lambda_i^r\right) \cdot e^{-w r} \, dr
$$

$$
= \sum_{l=1}^\infty \sum_{k_1, \ldots, k_N = l} \left(1 - \frac{l - 1}{k_1, \ldots, k_N}\right) \varepsilon_1^{k_1} \cdots \varepsilon_N^{k_N} \frac{e^{a(k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N - w)}}{k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N - w}.
$$

The series is uniformly convergent in any compactum $A \subset \mathbb{C}$ avoiding the support of $P$ (i.e. not necessarily contained in the open right half-plane).

**Proposition 3.12.** For $0 < \varepsilon < K$, the integrals

$$
H^+_{u}(w) := \pm i \int_{\varepsilon}^{K} \frac{h(u \pm iy)}{e^{2\pi y} - 1} \, dy
$$

exist and define holomorphic functions in the open right half-plane. Then $\hat{H}^+_{u}(w) :=$

$$
= e^{-w u} \sum_{l=1}^\infty \sum_{k_1, \ldots, k_N} \left(1 - \frac{l - 1}{k_1, \ldots, k_N}\right) \varepsilon_1^{k_1} \cdots \varepsilon_N^{k_N} \frac{1}{k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N - w - 2\pi j}.
$$

**Corollary 3.13.** Pick $u \geq r_0$ an integer. In every compactum $A \subset \mathbb{C}$ inside the open right half-plane, we have

$$
\pm i \int_{0}^{K} \frac{h(u \pm iy)}{e^{2\pi y} - 1} \, dy = e^{-w u} \sum_{l=1}^\infty \sum_{k_1, \ldots, k_N} \left(1 - \frac{l - 1}{k_1, \ldots, k_N}\right) \varepsilon_1^{k_1} \cdots \varepsilon_N^{k_N} \frac{1}{k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N - w - 2\pi j}.
$$

**Proof of Theorem 3.3.** We use Equation 3.6 and let $b \rightarrow +\infty$. After this limit has been carried out, we invoke Remark 3.2. We may run the same computation for any $K'$ such that $0 < K' < K$ without having to change $r_0$ and our assumptions will remain met. In particular, we can let $K' \rightarrow 0$. By inspection, the resulting series then converge to the statement of the theorem. This finishes the proof. □

**Example 3.14.** We make the continuation provided by Theorem 3.3 explicit: We pick $q := 2^2$. For $N := 3$, let $\lambda_1 := q^{-1/2}$, $\lambda_2 := -q^{-1/2}$, $\lambda_3 := q^{-1}$, $\varepsilon_1 := \varepsilon_2 := 1$ and $\varepsilon_3 := -1$. Below, on the left, we plot the original series $\sum_{r=1}^\infty \log(1 - \sum_{i=1}^N \varepsilon_i \lambda_i^r) e^{-w r}$,
and on the right we plot the analytic continuation:

The dots represent the support of the locally finite pseudo-divisor $P_{\text{per}}$. Secondly, pick $q := 11$. Consider the polynomial $x^2 - x + 11$. It is irreducible over the rationals. If $\alpha_1, \alpha_2$ are its two solutions, we have $|\alpha_j| = q^{1/2}$ for $j = 1, 2$, so these are Weil $q$-numbers of weight one. Pick $N := 3$, $\lambda_j := \alpha_j/q$ for $j = 1, 2$ and $\lambda_3 := 1/q$, $\varepsilon_1 := \varepsilon_2 := 1$ and $\varepsilon_3 := -1$.

The input data for this example was not picked at random. See Example 5.17.

Example 3.15. The following figure sketches a pseudo-divisor $P_{\text{per}}$ which fails to be locally finite:

In such a situation the above picture may represent the locus of poles for a truncated series for some choice for $b$, but when we take the limit $b \to +\infty$ as in the proof of Theorem 3.3 these poles accumulate and we cannot hope for $(\text{AC})$ to hold.
4. Step II

Suppose we are given \((\varepsilon_i, \lambda_i)_{i=1,...,N}\) and have picked \(r_0\) as explained in §3.3. We define a new pseudo-divisor
\[
\mathcal{D} := \Phi^*(P_{\text{per}}) \quad \text{for} \quad \Phi(w) := e^{-w},
\]
and consider the power series
\[
J_{\varepsilon, \lambda, r_0}(z) := \sum_{r=r_0}^{\infty} \log \left( 1 - \sum_{i=1}^{N} \varepsilon_i \lambda_i^r \right) z^r.
\]

(4.1)

The estimate in Remark 3.3 implies that this power series has radius of convergence \(\geq 1\).

**Proposition 4.1.** Suppose we are given \((\varepsilon_i, \lambda_i)_{i=1,...,N}\) and have picked \(r_0\) as explained in §3.3. Then for every compactum \(A \subset \mathbb{C}^x \setminus \text{supp} \mathcal{D}\) and choice of a branch of the logarithm \(\text{Log}_A : A \to \mathbb{C}\) which extends to a holomorphic function in some neighbourhood of \(A\), the series
\[
\tilde{J}_{\varepsilon, \lambda, r_0}(z) = \frac{1}{2} \log \left( 1 - \sum_{i=1}^{N} \varepsilon_i \lambda_i^r \right) z^r
\]
\[
+ \sum_{l=1}^{\infty} \sum_{j \in \mathbb{Z}} \sum_{k_1+\cdots+k_N=l} \left( \begin{array}{c} l-1 \varepsilon_1^{k_1} \cdots \varepsilon_N^{k_N} \\ k_1, \ldots, k_N \end{array} \right) \frac{e^{r_0(k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N + \text{Log}_A(z) + 2\pi i j)}}{k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N + \text{Log}_A(z) + 2\pi i j},
\]
is uniformly convergent in \(A\). It defines a holomorphic function and is independent of the choice of \(\text{Log}_A\). If \(\mathcal{D}\) is locally finite, \(\tilde{J}_{\varepsilon, \lambda, r_0}\) defines a meromorphic function on \(\mathbb{C}\) whose poles all have order one and the support of the divisor of poles agrees with \(\text{supp} \mathcal{D}\).

Inside the unit circle, \(\tilde{J}_{\varepsilon, \lambda, r_0}\) agrees with \(J_{\varepsilon, \lambda, r_0}\).

**Proof.** Let us write \(T(w)\) for the function described by Theorem 3.3. Let \(z \in \mathbb{C}^x\) be any point outside the support of \(\mathcal{D}\). Let \(U\) be an open neighbourhood of \(z\) and \(\text{Log}_U : U \to \mathbb{C}\) a branch of the logarithm which is holomorphic on all of \(U\) (always possible after shrinking \(U\)). We define a function \(g_U : U \to \mathbb{C}\) by the formula \(g_U(w) := T(-\text{Log}_U(w))\) so that, thanks to the definition of \(\mathcal{D}\), \(g_U\) is holomorphic on \(U\). If we do this for opens \(U_1, U_2\) (which overlap), the value of \(-\text{Log}_U\) is independent of \(j = 1, 2\) up to an integer multiple of \(2\pi i\), but since \(T\) is \(2\pi i\)-periodic in \(w\), we will have \(g_{U_1} |_{U_1 \cap U_2} = g_{U_2} |_{U_1 \cap U_2}\). Thus, \(g\) glued (without any monodromy!) and as a result we get a uniquely determined function \(g\), defined on all of \(\mathbb{C}^x \setminus \text{supp} \mathcal{D}\). However, by Theorem 3.3 this function is locally given by \(\tilde{J}_{\varepsilon, \lambda, r_0}\) as in the claim, and this also guarantees the uniform convergence. Since the exponential function is everywhere locally a homeomorphism, \(\mathcal{D}\) is locally finite if and only if \(P_{\text{per}}\) is locally finite. The meromorphy and the statement about the poles follow.

We have \(\tilde{J}_{\varepsilon, \lambda, r_0} = J_{\varepsilon, \lambda, r_0}\) inside the open unit disc, as under \(z = e^{-w}\) this corresponds to the open right half-plane.

**Definition 4.2.** Let \(U \subset \mathbb{C}\) be any simply connected domain containing \(0 \in \mathbb{C}\) and not intersecting the support of \(\mathcal{D}\). Define
\[
J_{\varepsilon, \lambda, r_0}^U(w) := \int_{\gamma} \frac{\tilde{J}_{\varepsilon, \lambda, r_0}(w)}{w} \, dw,
\]
where \( \gamma \) is any path inside of \( U \) from \( 0 \in \mathbb{C} \) to \( z \in U \).

As we demand that \( U \) is simply connected, the integral is independent of the choice of \( \gamma \). The integrand \( \tilde{J}_{z, \Delta, r_0}(w)/w \) can be regarded as holomorphic since \( \tilde{J}_{z, \Delta, r_0} \) has a zero of order \( \geq 1 \) at the origin (by Prop. 4.1 it agrees with \( J_{z, \Delta, r_0} \) inside the unit circle, and by Equation 4.1 and \( r_0 \geq 1 \), \( §3.2 \) the latter function has no constant coefficient). Thus, \( I_{z, \Delta, r_0} \) is a holomorphic function on \( U \).

**Lemma 4.3.** Let \( U \) be as in Definition 4.2. At \( z = 0 \), the function \( I_{z, \Delta, r_0} \) has the power series expansion

\[
I_{z, \Delta, r_0}(z) = \sum_{r=r_0}^{\infty} \log \left( 1 - \sum_{i=1}^{N} \varepsilon_i \lambda_i^r \right) \frac{z^r}{r},
\]

whose radius of convergence is \( \geq 1 \). In particular, near \( z = 0 \) it is independent of the choice of \( U \).

**Proof.** By Prop. 4.1 in a neighbourhood of the origin, \( \tilde{J}_{z, \Delta, r_0}(w)/w = J_{z, \Delta, r_0}(w)/w \), so by Equation 4.1 we have the power series expansion

\[
\tilde{J}_{z, \Delta, r_0}(w) = \sum_{r=r_0}^{\infty} \log \left( 1 - \sum_{i=1}^{N} \varepsilon_i \lambda_i^r \right) w^{r-1}
\]

in a neighbourhood of \( w = 0 \) and by termwise integration, we get the power series in the claim. Termwise integration leaves the radius of convergence invariant. \( \square \)

Unlike the procedure in the proof of Prop. 4.1, the analytic continuation \( I_{z, \Delta, r_0} \) will have non-trivial monodromy.

**Proposition 4.4 (Monodromy).** We keep the assumptions of the section. Moreover, suppose the pseudo-divisor \( D \) is locally finite. Let \( \gamma \) be any closed path inside the open set \( \mathbb{C} \setminus \text{supp} \, D \). Then

\[
\int_{\gamma} \frac{J_{z, \Delta, r_0}(w)}{w} dw \in 2\pi i \mathbb{Q}[\varepsilon_1, \ldots, \varepsilon_N],
\]

where the latter is the subring generated by \( \varepsilon_1, \ldots, \varepsilon_N \) inside \( \mathbb{C} \) over the rationals. If \( \varepsilon_1, \ldots, \varepsilon_N \) are algebraic, this is a number field.

**Proof.** By Proposition 4.1 we have

\[
\frac{J_{z, \Delta, r_0}(z)}{z} = \frac{1}{2} \log \left( 1 - \sum_{i=1}^{N} \varepsilon_i \lambda_i^{r_0} \right) z^{r_0-1} + \sum_{l=1}^{\infty} \frac{1}{l} \sum_{j \in \mathbb{Z}} \sum_{k_1+\cdots+k_N=l} \left( \frac{l}{k_1, \ldots, k_N} \right) \varepsilon_i^{k_1} \cdots \varepsilon_N^{k_N} e^{r_0(k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N + \text{Log}_A(z) + 2\pi i j)} \frac{1}{k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N + \text{Log}_A(z) + 2\pi i j} \cdot \frac{1}{z}.
\]

Note that \( r_0 \geq 1 \), so the initial term on the right is holomorphic on all of \( \mathbb{C} \). By using the additivity of the integral with respect to the path and \( D \) being locally finite, it suffices to
compute the integral around sufficiently small circles going around each of the isolated poles. Concretely, it suffices to compute
\[ \frac{1}{\pi i} \int \frac{e^{r_0(C + \log A(z))}}{C + \log A(z)} \frac{dz}{z} \]
for every choice \( C := k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N + 2\pi i j \) and \( \eta \) a sufficiently small circle around \( z \) such that \( \log A(z) = -C \). If we can show that the value of this integral lies in \( 2\pi i \mathbb{Q} \), our claim is proven. Write \( C = -\log A(W) \) for \( W \) suitably chosen (since we only need to work in some neighbourhood of \( z \), this is possible for the same branch). By the Residue Theorem, since this is a pole of order 1 at worst,
\[ \frac{2\pi i}{l} \lim_{z \to W} (z - W) \cdot \frac{e^{r_0(-\log A(W) + \log A(z))}}{-\log A(W) + \log A(z)} \]
Since the derivative of the logarithm, irrespective of the choice of a branch, is
\[ \lim_{z \to W} \log \frac{z - \log W}{z - W} = 1, \]
this is easy to compute and we get
\[ = \frac{2\pi i}{l} \cdot W \cdot \frac{1}{W} \cdot \lim_{z \to W} e^{r_0(-\log A(W) + \log A(z))} = \frac{2\pi i}{l}. \]
Our claim follows.

Remark 4.5. As one can see from the proof, as soon as there are poles at all, we will have monodromy in \( 2\pi i \mathbb{Q} \), and it will usually not happen that this can be reduced to \( 2\pi i \mathbb{Z} \) for some fixed \( M \geq 1 \), even if \( \varepsilon_1, \ldots, \varepsilon_N \in \mathbb{Z} \). To see this, note that among the coefficients of the series we have \( \frac{1}{\varepsilon_j} \) for all \( j = 1, \ldots, N \) and all \( l \geq 1 \). As \( \varepsilon_j \) has a fixed prime factorization, as \( l \to \infty \), \( l \) will infinitely often have prime factors which cannot be cancelled by \( \varepsilon_j \).

Proposition 4.6. Suppose we are given \( (\varepsilon_i, \lambda_i)_{i=1, \ldots, N} \) and have picked \( r_0 \) as explained in \( \S 3.2 \). Suppose the pseudo-divisor \( \mathcal{D} \) is locally finite, and on top of our running assumptions we demand \( \varepsilon_1, \ldots, \varepsilon_N \in \mathbb{Q} \).

1. Then the power series
\[ F_{\mathbb{Q}^{\mathcal{D} \cdot r_0}}(z) := \exp \left( \frac{1}{2} \sum_{r=r_0}^{\infty} \log \left( 1 - \sum_{i=1}^{N} \varepsilon_i \lambda_i^r \right) \frac{z^r}{r} \right) \]
has positive radius of convergence.

2. (AC) For every simply connected domain \( U \) with \( 0 \in U \subset \mathbb{C} \setminus \text{supp } \mathcal{D} \), there exists a unique holomorphic function \( F_{\mathbb{Q}^{\mathcal{D} \cdot r_0}}^U : U \to \mathbb{C} \) such that \( F_{\mathbb{Q}^{\mathcal{D} \cdot r_0}}^U \) agrees with \( F_{\mathbb{Q}^{\mathcal{D} \cdot r_0}} \) in some neighbourhood of \( t = 0 \). Moreover, \( F_{\mathbb{Q}^{\mathcal{D} \cdot r_0}}^U \) has no zeros in \( U \).

3. On the intersection on any two domains \( U_1, U_2 \) as in (2) we have,
\[ F_{\mathbb{Q}^{\mathcal{D} \cdot r_0}}^{U_1}/F_{\mathbb{Q}^{\mathcal{D} \cdot r_0}}^{U_2} \in \mu_\infty(U_1 \cap U_2), \]
i.e. two branches of the analytic continuation differ by a root of unity, and this fraction is locally constant.

4. The logarithmic derivative \( (F_{\mathbb{Q}^{\mathcal{D} \cdot r_0}}^U)'/F_{\mathbb{Q}^{\mathcal{D} \cdot r_0}}^U \) has a single-valued meromorphic continuation to \( \mathbb{C} \). Its locus of poles agrees with \( \text{supp } \mathcal{D} \) and all poles have order 1.
Note that the logarithmic derivative \((F^U_{\Delta r_0})'/F^U_{\Delta r_0}\) has a single-valued analytic continuation.

**Proof.** By Lemma 4.3 the integral

\[
I^U_{\Delta r_0}(z) = \int_\gamma \frac{\hat{J}_{\Delta r_0}(w)}{w} \, dw
\]

defines a holomorphic analytic continuation of \(\sum_{r=r_0}^\infty \log \left(1 - \sum_{i=1}^N \varepsilon_i \lambda_i r^\gamma \right) r^\varepsilon \) inside the domain \(U\). Define \(F^U_{\Delta r_0} := \exp\left(\frac{1}{2} I^U_{\Delta r_0}\right)\). It follows that \(F^U_{\Delta r_0} : U \to \mathbb{C}\) is holomorphic, cannot have zeros, and agrees with \(F^U_{\Delta r_0}\) in a neighbourhood of the origin. This proves (1) and (2). For (3) and \(z \in U_1 \cap U_2\) we compute

\[
\frac{F^U_{\Delta r_0}(z)}{F^U_{\Delta r_0}(z)} = \exp\left(\frac{1}{2} I^U_{\Delta r_0}(z) - \frac{1}{2} I^U_{\Delta r_0}(\zeta)\right) = \exp\left(\frac{1}{2} \left( \int_{\gamma_1} - \int_{\gamma_2} \frac{\hat{J}_{\Delta r_0}(w)}{w} \, dw \right)\right),
\]

where \(\gamma_i\) is a path inside \(U_i\) and thus inside \(\mathbb{C} \setminus \text{supp } D\), which goes from \(w = 0\) to \(w = z\). Hence, \(\gamma_1 \circ \gamma_2^{-1}\) is a closed path in \(\mathbb{C} \setminus \text{supp } D\) and by Monodromy (Prop. 4.4) we get \(\exp(\tau)\) for some \(\tau \in 2\pi i \mathbb{Q}\). Thus, \(\exp(\tau)\) is a root of unity, locally constant, \(F^U_{\Delta r_0}(z)/F^U_{\Delta r_0}(\zeta) \in \mu_\infty\). For (4), note that

\[
\left(\frac{F^U_{\Delta r_0}}{F^U_{\Delta r_0}}\right)' = \frac{1}{2} \frac{\partial I^U_{\Delta r_0}}{\partial z}(z) = \frac{1}{2} \frac{\hat{J}_{\Delta r_0}(z)}{z}
\]

and we get all the required properties from Proposition 4.1 and the fact that \(\hat{J}_{\Delta r_0}\) has a zero of order \(\geq 1\) at the origin. \(\square\)

**Theorem 4.7.** Suppose we are given \((\varepsilon_i, \lambda_i)_{i=1,\ldots,N}\) and have picked \(r_0\) as explained in \(\mathbb{W}\). Suppose the pseudo-divisor \(D\) is locally finite, and on top of our running assumptions we demand \(\varepsilon_1, \ldots, \varepsilon_N \in \mathbb{Q}\). Define a pseudo-divisor

\[
E := D + c^* D,
\]

where \(c\) denotes complex conjugation on \(\mathbb{C}\) and \(c^*\) the pullback. Then \(E\) is also locally finite.

(1) Then the power series

\[
f(z) := \exp\left(\sum_{r=r_0}^\infty \log \left|1 - \sum_{i=1}^N \varepsilon_i \lambda_i r^\gamma \right| \frac{z^r}{r}\right)
\]

has positive radius of convergence.

(2) (AC) For every simply connected domain \(U\) with \(0 \in U \subset \mathbb{C} \setminus \text{supp } E\), there exists a unique holomorphic function \(f_U : U \to \mathbb{C}\) such that \(f_U\) agrees with \(f\) in some neighbourhood of \(t = 0\). Moreover, \(f_U\) has no zeros in \(U\).

(3) On the intersection on any two domains \(U_1, U_2\) as in (2) we locally have

\[
f_{U_1}/f_{U_2} \in \mu_{\infty}(U_1 \cap U_2),
\]

i.e. two branches of the analytic continuation differ by a root of unity.

(4) The logarithmic derivative \(f_{U}'/f_U\) has a single-valued meromorphic continuation to \(\mathbb{C}\). Its locus of poles agrees with \(\text{supp } E\) and all poles have order 1.
Proof. It is easy to see that our assumptions on \((\varepsilon_i, \lambda_i)\) in Example 5.2 imply that the complex conjugates \((\overline{\varepsilon_i}, \overline{\lambda_i})\) also satisfy them; perhaps (for a given fixed \(K\)) for a different choice of \(r_0\). However, we can without loss of generality pick an \(r_0\) sufficiently large for both \((\varepsilon_i, \lambda_i)\) and \((\overline{\varepsilon_i}, \overline{\lambda_i})\) simultaneously. For \((\overline{\varepsilon_i}, \overline{\lambda_i})\) the pseudo-divisor \(D\) gets replaced by \(c^*D\). Next, note that for all integers \(r \geq r_0\) we have
\[
\log \left| 1 - \sum_{i=1}^{N} \varepsilon_i \lambda_i^r \right| = \frac{1}{2} \log \left( 1 - \sum_{i=1}^{N} \varepsilon_i \lambda_i^r \right) + \frac{1}{2} \log \left( 1 - \sum_{i=1}^{N} \overline{\varepsilon_i} \overline{\lambda_i^r} \right)
\]
and thus \(f = F_{(\varepsilon_i, \lambda_i), r_0} \cdot F_{(\overline{\varepsilon_i}, \overline{\lambda_i}), r_0}\). Now apply Prop. 4.6 to both factors. \(\square\)

5. Step III

So far, we have not looked into the matter of detecting whether the pseudo-divisor \(D\) (or equivalently \(P^\text{per}\)) is locally finite.

Lemma 5.1. Suppose we are given \((\varepsilon_i, \lambda_i)\) as explained in Example 5.2. If \(N \leq 2\), \(\mathcal{P}\) is a locally finite divisor. If \(N \leq 1\), \(P^{\text{per,+}}, P^{\text{per,-}}\) and \(P^{\text{per}}\) are locally finite divisors.

Proof. Regarding \(\mathcal{P}\), the cases \(N = 0, 1\) are obvious. Suppose \(N = 2\). Then
\[
\mathcal{P} = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \delta_{k_1+k_2 \geq 1} \frac{(k_1 + k_2 - 1)!}{k_1!k_2!} \varepsilon_1^{k_1} \varepsilon_2^{k_2} \cdot [k_1 \log \lambda_1 + k_2 \log \lambda_2]
\]
There are only two cases: (A) If \(\log \lambda_1, \log \lambda_2\) are \(\mathbb{R}\)-linearly independent, then this spans a two-dimensional cone (with its apex removed) in the complex plane. In particular, \(\mathcal{P}\) is a locally finite divisor. By our standing assumption \(|\lambda_1|, |\lambda_2| < 1\), this cone lies entirely in the open left half-plane. (B) If they are linearly dependent,
\[
\alpha \log \lambda_1 + \beta \log \lambda_2 = 0
\]
for some \(\alpha, \beta \in \mathbb{R}\), taking the real part shows that \(\alpha, \beta\) must both be non-zero and have opposite parity, say \(\alpha < 0 < \beta\) without loss of generality. Thus, \(\log \lambda_1 = \gamma \cdot \log \lambda_2\) for some positive real number \(\gamma\). In particular, for every constant \(C' > 0\) there are only finitely many \((k_1, k_2)\) such that \(-C' < \text{Re}(k_1 \log \lambda_1 + k_2 \log \lambda_2) < 0\). In summary, \(\mathcal{P}\) lies discretely in a ray in the negative open half plane. The situation with the periodic divisor \(P^{\text{per,+}}\) is analogous, just replace one spanning vector by \(\pm 2\pi i\). Since \(\text{Re} \log \lambda_1 < 0\), this always spans a rank 2 integral cone, which is discrete. \(\square\)

Example 5.2. Having \(N \geq 3\) does not hinder \(P\) or \(P^{\text{per}}\) to be a locally finite divisor. We shall construct an example with \(N = 2^j - 1\) for any given \(j \geq 1\) and \(P^{\text{per}}\) locally finite: Pick input data, \(\varepsilon_i\) and \(\lambda_i\) with \(i \in X := \{1, 2, \ldots, j\}\) as in Example 5.2. Denote by \(2^X\) the power set and define for all subsets \(I \in 2^X\) with \(|I| \geq 1\):
\[
\varepsilon_I := (-1)^{|I|-1} \cdot \prod_{i \in I} \varepsilon_i \quad \text{and} \quad \lambda_I := \prod_{i \in I} \lambda_i.
\]
Clearly we still have \(|\lambda_I| < 1\). Hence, \((\varepsilon_I, \lambda_I)\) also determines valid input data as in Example 5.2.

Write \(\mathcal{P}\) for the corresponding pseudo-divisor. On the other hand, we can handle for each \(i \in X\), the singleton system \((\varepsilon_i, \lambda_i)\), i.e. we remove all entries except for the one of index \(i\), so that seen individually it has \(N = 1\). Write \(\mathcal{P}_i\) for the corresponding pseudo-divisor.
Since $N = 1$ for these singleton systems, all $\mathcal{P}_i^{\text{per}}$ are locally finite divisors (Lemma 5.1). We compute
\[
g(r) = \log \left( \prod_{i=1}^{N} (1 - \varepsilon_i \lambda_i^t) \right) \cdot e^{-wr} = \log \left( 1 - \sum_{l \in \mathbb{Z}^{1\ldots N}, \|l\| \geq 1} \tilde{\varepsilon}_l \tilde{\lambda}_l^t \right) \cdot e^{-wr}.
\]
Since all $\mathcal{P}_i^{\text{per}}$ are locally finite, Theorem 5.3 implies that $g$ possesses an analytic continuation. Thus, the same is true for the function determined by the above equation. Thus, $\mathcal{P}$ is necessarily also locally finite. To give more context: If we number the slots of $(\tilde{\varepsilon}_l, \tilde{\lambda}_l)$ instead of indexing them by subsets of $2^N$, the pseudo-divisor
\[
\mathcal{P} := \sum_{l=1}^{\infty} \sum_{k_1 + \ldots + k_{2^{j-1}} = l} \left( \frac{l-1}{k_1, \ldots, k_{2^{j-1}}} \right) \varepsilon_1^{k_1} \cdots \varepsilon_{2^{j-1}}^{k_{2^{j-1}}} \cdot \left[ \sum_{l=1}^{2^{j-1}} k_l \log \lambda_l \right]
\]
has the property that the integral cone
\[
\mathbb{Z}_{\geq 0} \left( k_1 \log \lambda_1, \ldots, k_{2^{j-1}} \log \lambda_{2^{j-1}} \right)
\]
will (usually, once $j > 2$ and a generic choice of $\lambda_i$) define a non-discrete subset of the complex plane, and indeed very generically for large $j$ be dense. Nonetheless, supp $\mathcal{P}$ will always be a locally finite divisor thanks to the (not so obvious) heavy cancellation of terms, based on the combinatorics of the multinomial coefficients.

Finally, we can prove the existence of an analytic continuation for various varieties and motives.

**Theorem 5.3.** Let $A/\mathbb{F}_q$ be an abelian variety of dimension $g \geq 1$. Let $\alpha_1, \ldots, \alpha_{2g}$ be its Weil numbers. Then the following properties hold:

1. **(Radius of Convergence)** The power series $Z_{\log}$ has radius of convergence precisely one.
2. **(AC)** It admits a holomorphic analytic continuation $\tilde{Z}_{\log}$ to any simply connected domain $U \ni 0$ avoiding $\{1, \alpha_1^{Z_{\geq 2}}, \ldots, \alpha_{2g}^{Z_{\geq 2}}\}$.
3. **(Periods)** On the intersection of any two domains $U_1, U_2$ as in (2), we have $\tilde{Z}_{\log}^{U_1}/\tilde{Z}_{\log}^{U_2} \in \mu_{\infty}(U_1 \cap U_2)$, i.e. all branches of the analytic continuation differ by multiplication with a root of unity.
4. **(Logarithmic derivative)** The logarithmic derivative $\tilde{Z}_{\log}'/\tilde{Z}_{\log}$ has a single-valued meromorphic continuation to the entire complex plane with a pole of order 2 at $z = 1$ and poles of order 1 at all positive integer powers of the weight one Weil numbers of $A$, i.e. $\{\alpha_1^{Z_{\geq 1}}, \ldots, \alpha_{2g}^{Z_{\geq 1}}\}$.
5. **(Monodromy)** Around $z = 1$, the function $\tilde{Z}_{\log}$ has an essential singularity. Around any point $w := \alpha_i^l, l \geq 2$, has an open neighbourhood in which $\tilde{Z}_{\log}$ has the shape
\[
(z - w)^\frac{n}{m} \cdot g,
\]
where $g$ is non-zero holomorphic in a neighbourhood and $n_j$ is the multiplicity of $\alpha_j$ as a root of the Frobenius characteristic polynomial.
6. **(Zeros)** Suppose $A$ is simple. Then the zeros of $\tilde{Z}_{\log}^U : U \to \mathbb{C}$, for $U$ as in (2), are precisely
\[
\{\alpha_1, \ldots, \alpha_{2g}\} \cap U.
\]
Statement (6) can also be rephrased as follows: If we consider the analytic continuation as a lift to the respective covering space where it becomes single-valued, as in Figure 2.1 the zeros of this lift are precisely the fibers of \( \{ \alpha_1, \ldots, \alpha_{2g} \} \) under the covering map.

**Proof.** We give a proof without motives: The \( \ell \)-adic cohomology algebra of \( A \) is canonically isomorphic to an exterior algebra,

\[
H^*(A \times_{\mathbf{F}_q} \mathbf{F}_{q^{sep}}, \mathbf{Q}_\ell) = \wedge^* H^1(A \times_{\mathbf{F}_q} \mathbf{F}_{q^{sep}}, \mathbf{Q}_\ell).
\]

As a result, thanks to the Grothendieck–Lefschetz trace formula, we have the point count

\[
N_r = |A(\mathbf{F}_q)| = \prod_{j=1}^{2g} (1 - \alpha_j^r),
\]

where \( \alpha_1, \ldots, \alpha_{2g} \) are the Weil \( q \)-numbers (of weight 1) of the abelian variety, or equivalently the eigenvalues of the (geometric) Frobenius, acting on \( H^1(A \times_{\mathbf{F}_q} \mathbf{F}_{q^{sep}}, \mathbf{Q}_\ell) \) as a Galois module (This particular result is actually due to Weil and predates the work of the Grothendieck school). Thus,

\[
\log |A(\mathbf{F}_q)| = \sum_{j=1}^{2g} \log |\alpha_j^r - 1| = \sum_{j=1}^{2g} \log \left| (\alpha_j^r) \cdot (1 - (\alpha_j^{-1})^r) \right|
\]

\[
= \sum_{j=1}^{2g} \left( r \log q + \log \left| 1 - (\alpha_j^{-1})^r \right| \right)
\]

since \( |\alpha_i| = q^{1/2} \) for all \( j = 1, \ldots, 2g \). So by Definition 2.1 the function \( Z_{log}(A, t) \) literally equals

\[
(5.1) \quad Z_{log}(A, t) = \exp \left( \sum_{r \geq 1} \log |A(\mathbf{F}_q)| \cdot \frac{t^r}{r} \right)
\]

\[
= \exp \left( g \log q \cdot \frac{t}{1-t} \right) \cdot \prod_{j=1}^{2g} \exp \left( \sum_{r \geq 1} \log \left| 1 - (\alpha_j^{-1})^r \right| \cdot \frac{t^r}{r} \right).
\]

Thus, our claims (2)-(4) are proven if we can show that the desired properties hold for all factors \( \exp \left( \sum_{r \geq 1} \log \left| 1 - (\alpha_j^{-1})^r \right| \cdot \frac{t^r}{r} \right) \). To this end, we apply Theorem 4.7 for each \( i = 1, \ldots, 2g \) in the following situation: \( N := 1, \varepsilon_1 := +1, \lambda_1 := \alpha_i^{-1} \) (which has \( 0 < |\alpha_i^{-1}| < 1 \) as required), and \( r_0 = 1 \). Indeed, \( \varepsilon_1 \in \mathbf{Q}_\ell \), and the resulting pseudo-divisor \( \mathcal{P}^{\text{per}} \) is locally finite by Lemma 5.1 since \( N = 1 \). Thus, the theorem is applicable and we leave it to the reader to compute that the divisor of poles agrees with \( \mathcal{D} \). This settles (1)-(4), albeit only for the smaller set \( \mathbf{C} \setminus \{ 1, \alpha_1 \zeta_{2g}, \ldots, \alpha_{2g} \zeta_{2g} \} \). We address (5): Equation 5.1 settles the essential singularity at \( t = 1 \): The first factor has such a singularity at \( t = 1 \), while the remaining \( 2g \) factors can holomorphically be extended across \( t = 1 \) by Theorem 4.7. As the logarithmic derivative, by (4), has poles of order 1 at all points \( w \in \{ \alpha_1 \zeta_{2g}, \ldots, \alpha_{2g} \} \), we locally have

\[
(\log Z_{log})'(z) = \frac{\beta}{z-w} + h(z),
\]

where "Log" is a locally defined branch of the logarithm, \( \beta \in \mathbf{C} \), and \( h \) a holomorphic function defined in some neighbourhood of \( w \). A local computation shows that that
\[ \beta = \frac{n_j}{l} \text{ for } w = \alpha_j^l, \text{ where } 1 \leq j \leq 2g, l \geq 1 \text{ and } n_j \geq 1 \text{ is the multiplicity of } \alpha_j \text{ as a root of the Frobenius characteristic polynomial} \]

Integrating the above equation then leads to

\[ \left( \log \tilde{Z}_\log^U \right)(z) = \frac{n_j}{l} \log(z - w) + H(z) \]

for a new holomorphic function, defined in a neighbourhood. Then \( \tilde{Z}_\log^U(z) = (z - w)^{n_j/l} \cdot \exp(H(z)) \), proving (5). Moreover, it shows that for \( l = 1 \), we have \( \frac{n_j}{l} \in \mathbb{Z}_{\geq 1} \), and thus the isolated singularities at \( \alpha_1, \ldots, \alpha_{2g} \) are removable, thus extending (1)-(4) to \( \mathbb{C} \setminus \{ 1, \alpha_1^{z_2}, \ldots, \alpha_{2g}^{z_2} \} \). It remains to prove (6): As we had used Theorem 4.4, we know that \( \tilde{Z}_\log^U \) has no zeros in \( U \setminus \{ \alpha_1 z \geq 0, \ldots, \alpha_{2g} z \geq 0 \} \), irrespective of what \( U \) is. Thus, it remains to check what happens at the remaining points in \( U \): Suppose \( A \) is simple, so \( n_j = 1 \) for all \( j \). At \( w = 1 \), we know that \( \tilde{Z}_\log^U \) has an essential singularity, and at \( w = \alpha_j^l \) for \( i = 1, \ldots, 2g \) and \( l \geq 2 \), the local description shows that no holomorphic extension to these points is possible (indeed: The real part admits a continuous continuation by zero, but the imaginary part has a jump thanks to the \( l \)-th root function), so they cannot be contained in the domain of any analytic continuation. The points \( w = \alpha_j^l \) with \( l = 1 \) and \( w \in U \) remain. As we have seen above, we indeed have zeros at these points. \( \Box \)

Remark 5.4. A motivic proof would use Shermenev’s theorem, providing an isomorphism \( \mathcal{M}(A) = \bigwedge^* h^1(A) \). The original paper is [Sh17] (or as an alternative source [DM91, Kim94]). The proof then proceeds in exactly the same way.

By Honda–Tate theory [Hon68] the abelian varieties over \( \mathbb{F}_q \) (up to isogeny) are classified by Galois orbits of Weil \( q \)-numbers of weight one:

\[ \Phi : \{ \text{\( q \)-isogeny classes of abelian varieties over } \mathbb{F}_q \} \]

\[ \sim \rightarrow \{ \text{Galois orbits of Weil numbers with } |x|^2 = q \} \]

Thus, knowing \( Z_{\log} \) one can explicitly reconstruct the Weil numbers from the zeros of the analytic continuation, and get the isogeny class of \( A \) back.

Corollary 5.5. Given an abelian variety \( A/\mathbb{F}_q \), the order 1 poles of the logarithmic derivative \( Z_{\log} \) at points of absolute value \( |z| = \sqrt{q} \) are precisely the weight one Weil \( q \)-numbers of the abelian variety.

The next result covers a wide range of examples. We refer to the Appendix [A] for background and notation regarding motives.

Theorem 5.6. Suppose \( M \) is a motive which decomposes as a finite direct sum of supersingular motives (e.g. Tate or Artin motives). Suppose it has a unique top weight (Definition A.4). If \( Z_{\log}(M, t) \) is defined (i.e. \( N_l \geq 1 \) for all \( l \geq 1 \)), then it has (AC). More specifically: There is a discrete subset \( \Delta \subset \mathbb{C} \), \( 0 \notin \Delta \) such that:

1. (AC) For every simply connected domain \( U \) with \( 0 \in U \subset \mathbb{C} \setminus \Delta \), there exists a unique holomorphic function \( \tilde{Z}_{\log}^U : U \to \mathbb{C} \) such that \( \tilde{Z}_{\log}^U \) agrees with \( Z_{\log} \) in some neighbourhood of \( t = 0 \).

2. (Periods) On the intersection on any two domains \( U_1, U_2 \) as in (1) we have

\[ \frac{\tilde{Z}_{\log}^{U_1}}{\tilde{Z}_{\log}^{U_2}} \in \mu_\infty(U_1 \cap U_2), \]

i.e. two branches of the analytic continuation differ by a root of unity.
(3) (Logarithmic derivative) The logarithmic derivative \((\tilde{Z}_U^\log)'/\tilde{Z}_U^\log\) has a single-valued meromorphic continuation to all of \(\mathbb{C}\). Its locus of poles agrees with \(\Delta\). With at most finitely many exceptions, the poles have order 1.

(4) (Monodromy) In a neighbourhood around any point \(w \in \Delta \cap U\), the continuation \(\tilde{Z}_U^\log\) of (1) locally has the shape
\[
(z - w)^{\frac{v}{q}} \cdot g \quad \text{with} \quad \frac{v}{q} \in \mathbb{Q},
\]
and \(g\) some non-zero holomorphic function.

Proof. Suppose \(M = \bigoplus M_v\), where each \(M_v\) consists entirely of summands of weight \(v\). This is possible by assumption. Consider the motivic (virtual) point count numbers \(N_l\) as in \([\Lambda,\Pi]\). Then \(N_l = \sum_{v,j} (-1)^v \alpha_{v,j}\), where, \(\alpha_{v,j} = \zeta_{v,j} q^{v / 2}\) for some root of unity \(\zeta_{v,j}\) (depending on \(v, j\) as indices; these have nothing to do with its exponent as a torsion element in the multiplicative group). The summation over \(v\) runs through the individual weights, while \(j\) runs through the collection of eigenvalues appearing in each weight part. By our assumption of a unique top weight, say it is \(q^m\) for some \(m\), we may write \(N_r = \sum_{v < 2m, j} (-1)^v \alpha_{v,j}^r + q^{mr}\). Thus, the definition of \(Z_{\log}(M, t)\) unravels as
\[
\begin{align*}
= \exp \left( m \log q \frac{t}{1-t} \right) \exp \sum_{v=1}^{r_{\max}} \log \left( 1 + \sum_{v < 2m, j} (-1)^v \left( \frac{\alpha_{v,j}}{q^m} \right)^r \frac{\lambda_{v,j}}{r} \right) \\
\cdot \exp \left( \sum_{r=0}^{\infty} \log \left( 1 + \sum_{v < 2m, j} (-1)^v \left( \frac{\alpha_{v,j}}{q^m} \right)^r \frac{\lambda_{v,j}}{r} \right) \right),
\end{align*}
\]
where \(|\alpha_{v,j}/q^m| = q^{(v-2m)/2}\) and since \(v < 2m\), we have \(0 < |\alpha_{v,j}/q^m| < 1\). Next, we wish to apply Theorem \([\Lambda,\Pi]\) with the datum \((\xi_i, \lambda_i)\) given by \(\xi_i \in \{1, -1\}\) and \(\lambda_i\) running through the values \((\alpha_{v,j}/q^m)\) for all summands which appear; and pick \(r_0\) sufficient for the assumptions of the theorem to be applicable (this is possible: the above computation works for any \(r_0 \geq 1\), and by our remarks in \([\Lambda,\Pi]\) any sufficiently large choice of \(r_0\) meets the conditions). To this end, it only remains to check that the pseudo-divisor \(D\) is locally finite. This is equivalent to checking that \(P_{\text{per}}\) is locally finite (as \(D\) is just the pullback of the latter under a map which is everywhere a local homeomorphism). However,
\[
P_{\text{per}} = \sum_{j=-\infty}^{\infty} \sum_{l=1}^{\infty} \sum_{k_1, \ldots, k_N} \left( \frac{l-1}{k_1, \ldots, k_N} \right) \xi_{k_1} \cdots \xi_{k_N} \cdot \left[ k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N + 2\pi ij \right],
\]
where all \(\lambda_i\) are of the shape \((\zeta_{v(s), j(s)} q^{v(s)/2})/q^m\), so our \(P_{\text{per}}\) satisfies
\[
\text{supp} P_{\text{per}} \subseteq \bigcup_{k_1, \ldots, k_N \geq 0} \{k_1 \log \lambda_1 + \cdots + k_N \log \lambda_N + 2\pi ij\}
\]
where, splitting these points into their real and imaginary part,
\[
2\pi ij + \sum_{s=1}^{N} k_s \log \lambda_s = \sum_{s=1}^{N} k_s \left( \frac{v(s) - 2m}{2} \right) \log(q) + i \left( 2\pi j + \sum_{s=1}^{N} k_s \arg(\zeta_{v(s), j(s)}) \right).
\]
Since the \(\zeta_{v(s), j(s)}\) are all roots of unity and the sum is finite, there exists some fixed integer \(M \geq 1\) such that we have \(\arg(\zeta_{v(s), j(s)}) \in \frac{1}{M} 2\pi i\mathbb{Z}\). Hence, all these values are contained in the lattice spanned by \(\mathbb{Z} \langle \frac{1}{2} \log q, \frac{1}{M} 2\pi i \rangle\). Hence, \(\text{supp} P_{\text{per}}\) is contained in
a discrete rank 2 lattice in the complex plane, and thus necessarily locally finite. Hence, we can indeed invoke the Theorem and it guarantees that the last factor, line 5.2, has (AC) for \( C \setminus \text{supp } D \). The remaining two factors are

\[
\exp \left( m \log(q) \cdot \frac{t}{1-t} \right) \exp \left( \sum_{v \geq 2, m,j} (-1)^v \left( \frac{\alpha_{v,j}}{q^m} \right) \frac{\alpha_{v,j}^r}{r} \right).
\]

In either case, we face the exponential of a function which is rational in \( t \). Clearly, this is immediately defined on all of \( C \) except for the isolated pole locus of the rational function in question. This proves (AC) for \( Z_{\log}(M, t) \) for some set \( \Delta := \text{supp } D \cup \{ \text{finite set of points} \} \), so if \( D \) is locally finite, \( \Delta \) is necessarily a discrete subset of the complex plane. Note that any multi-valuedness, i.e. distinct branches, can only stem from the factor controlled by Theorem 4.7, i.e. line 5.2, so the description of the possible monodromy remains valid for \( Z_{\log} \). It remains to prove (4): As the logarithmic derivative is meromorphic on the entire complex plane, locally around \( w \in \Delta \) we have

\[
(\log \hat{Z}_W^U)'(z) = \beta \log(z - w) + H(z),
\]

where “Log” is a locally defined branch of the logarithm, \( m \in \mathbb{Z}_{\geq 1}, \beta \in C \) some constant, and \( H \) a meromorphic function defined in some neighbourhood of \( w \) of pole order at most \( m-1 \). Thus, if \( m \geq 2 \),

\[
(\log \hat{Z}_W^U)(z) = \frac{1}{1 - m (z - w)^{m-1}} + H(z),
\]

for suitable \( H \), meromorphic of pole order at most \( m-2 \), and so \( \hat{Z}_W^U \) has an essential singularity nearby \( w \). If \( m = 1 \),

\[
(\log \hat{Z}_W^U)(z) = \beta \log(z - w) + H(z)
\]

for suitable \( H \), and by Monodromy (Prop. 4.4) the monodromy of \( (\log \hat{Z}_W^U)' \) is rational, so \( \beta \in \mathbb{Q} \). Thus, \( \hat{Z}_W^U = (z - w)^\beta \exp(H(z)) \) in an open neighbourhood of \( w \).

\[\square\]

**Theorem 5.7.** Suppose \( X/\mathbb{F}_q \) is a geometrically connected smooth projective variety

1. whose motive decomposes as a finite direct sum of supersingular motives (e.g. Artin or Tate motives), and
2. which has a \( \mathbb{F}_q \)-rational point.

Then \( Z_{\log}(X, t) \) is defined, and both have (AC).

**Proof.** Since \( X \) is geometrically connected, Poincaré Duality implies that it has a unique top weight. The function \( Z_{\log} \) is defined since we have a \( \mathbb{F}_q \)-rational point, so \( N_l \geq 1 \) for all \( l \geq 1 \). Thus, we can apply Theorem 5.6 to the motive of the variety and are done. \( \square \)

5.1. **Cellularity and the functions \( \Lambda_n \).**

**Corollary 5.8.** Suppose \( X/\mathbb{F}_q \) is a geometrically connected smooth projective cellular variety. Then \( Z_{\log}(X, t) \) has (AC).

**Proof.** If the variety is cellular, even its Chow motive (i.e. \( \text{Mot}_{\text{ch}}(k) \) with rational equivalence) splits into a finite direct sum of Tate motives, see [CGM05, Theorem 7.2] or [Kar00, Corollary 6.11]. This induces the same statement for \( \text{Mot}_{\text{num}}(k) \) and \( \text{Mot}_{\text{num}}(k) \). Now use the previous theorem. \( \square \)
The following definition turns out to be convenient:

**Definition 5.9.** Define

$$\Lambda_n(t) := Z_{\log}(A^n - \{0\}, t),$$

i.e.

$$\Lambda_n(t) = \exp \left( \sum_{r \geq 1} \log(q^{nr} - 1) \cdot \frac{t^r}{r} \right).$$

Theorem 4.7 easily implies that (AC) holds (we have $N = 1$, so by Lemma 5.1 the finiteness of $\mathcal{P}^{per}$ is automatic).

**Example 5.10.** For projective space we have

$$|P^n(F_q^r)| = 1 + q^r + q^{2r} + \cdots + q^{nr}.$$

As $P^n$ is a cellular variety, we could directly invoke Cor. 5.8 However, we will handle this example manually. We compute

$$Z_{\log}(P^n, t) = \exp \left( \sum_{r \geq 1} \log \left( \sum_{m=0}^{n} (q^r)^m \right) \cdot \frac{t^r}{r} \right) = \exp \left( \sum_{r \geq 1} \log (q^r) \cdot \frac{t^r}{r} \right) = \Lambda_{n+1}(t)/\Lambda_1(t).$$

Hence, besides (AC) we have a suggestive result: At least over an algebraically closed base field we can also interpret $P^n$ as $(A^{n+1} - \{0\})/\sim$, where the equivalence relation identifies all points on a shared line, i.e. on a shared $A^1$.

**Example 5.11.** For the general linear group one finds

$$|GL_k(F_q)| = q^\frac{k(k-1)}{2}(q^k - 1)(q^{k-1} - 1)\cdots(q - 1).$$

We can directly plug this into the definition of $Z_{\log}$. Thanks to the multiplicative nature of this formula, this leads to a factorization of the function: Namely, we compute

$$Z_{\log}(GL_k, t) = \exp \left( \sum_{r \geq 1} \log \left( (q^r)^{\frac{k(k-1)}{2}}((q^r)^k - 1)((q^r)^{k-1} - 1)\cdots(q^r - 1) \right) \cdot \frac{t^r}{r} \right) = \exp \left( \frac{k(k-1)}{2} \log q \cdot \frac{t}{1-t} \right) \cdot \exp \left( \sum_{r \geq 1} \log \left( \prod_{l=1}^{k} (q^{rl} - 1) \right) \cdot \frac{t^r}{r} \right).$$

Pulling the product out of the logarithm, we get the factorization

$$= q^{\frac{k(k-1)}{2}} \cdot \prod_{l=1}^{k} \Lambda_l(t).$$

Although the linear group $GL_k$ is not a projective variety, it follows that (AC) holds.
Example 5.12. In a similar fashion, one can treat the Grassmannians,

$$|G(k, n)(F_q)| = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1) \cdots (q - 1)}$$

being cellular, (AC) itself immediately follows from Corollary 5.8. However, much as in the previous example, we also get a pleasant formula:

$$Z_{\log}(G(k, n), t) = \frac{\exp \left( \sum_{r \geq 1} \log \left( \prod_{l=r}^{n-k+1} (q^r - 1) \cdot \frac{t^r}{r} \right) \right)}{\exp \left( \sum_{r \geq 1} \log \left( \prod_{l=1}^{k} (q^r - 1) \cdot \frac{t^r}{r} \right) \right)} = \prod_{l=n-k+1}^{n} \Lambda_l(t) \prod_{l=1}^{k} \Lambda_l(t).$$

The properties of the analytic continuation follow from the properties of the $\Lambda_l(t)$ alone.

Example 5.13. Suppose $Q$ is a quadratic form and consider the inhomogeneous affine solution space $X := \text{Spec} F_q[t_1, \ldots, t_n]/(Q - \alpha)$ for some non-zero $\alpha \in F_q^\times$. While being cellular, $X$ is not projective. We follow [LS99, Theorem 2.7]: If $Q$ is of Type 1, we have (for suitable $m$)

$$|X(F_q)| = q^{r(n-m)} \left( q^{rm} - q^{r(m-1)/2} \right) = \left( q^{n-m-1} \right)^r \cdot \left( q^{\frac{m-1}{2}} \right)^r - 1.$$

Thus,

$$Z_{\log}(X, t) = \left( q^{(n-m-1)} \right)^r \cdot \Lambda_{\frac{m-1}{2}}(t).$$

Again, the (AC) falls out from the properties of these two factors although $X$ itself is not projective.

If $X$ is smooth projective cellular, the functions $\Lambda_n$ obviously are a convenient ingredient to decompose $Z_{\log}$ into factors. If one allows non-projective cellular varieties, an additional term of the shape $q^{(\cdots)} \cdot \Lambda_n$ plays a rôle. Some of the above computations have a deeper structural reason on the level of motives. This following all directly follows from the work of N. Karpenko [Kar00], who in turn attributes the basic idea (in the case of quadrics) to M. Rost.

Proposition 5.14. Let $V$ be a finite-dimensional $F_q$-vector space. Let $\text{Grass}(V)$ denote the full Grassmannian of all vector subspaces of $V$ (of any dimension). Then $\text{Grass}(V)$ is a smooth projective $F_q$-variety. For every short exact sequence

$$0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$$

of finite-dimensional vector spaces, one has

$$Z_{\log}(\text{Grass } V) = Z_{\log}(\text{Grass } V') \cdot Z_{\log}(\text{Grass } V'').$$

The analogous statement holds for the varieties of length $r$ flags, for any $r$. 
This follows immediately from [Kar00, Corollary 9.13] resp. [Kar00, Corollary 11.5]: The Grassmannian of $V$ is not the product of the Grassmannians of $V'$ and $V''$, but as it just differs by a fibration into affine spaces, the motive is the tensor product motive nonetheless.

5.2. Curves.

**Theorem 5.15.** Suppose $X/F_q$ is a geometrically connected smooth projective curve with an $F_q$-rational point. If

1. the genus is $g = 0, 1$ or
2. the genus is $g \geq 2$ and the Jacobian of $X$ is supersingular,

then $Z_{\log}(X, t)$ has (AC).

The supersingular condition can be checked if one understands global 1-forms:

**Theorem 5.16** (Nygaard [Nyg81, Theorem 4.1]). Suppose $X/F_q$ is a geometrically connected smooth projective curve with an $F_q$-rational point. Suppose the Cartier operator $C : H^0(X, \Omega^1) \to H^0(X, \Omega^1)$ induces the zero map. Then the Jacobian of $X$ is supersingular.

Most people appear to expect that there exist curves of arbitrarily high genus and supersingular Jacobian over any finite field, so that this theorem would give a rich supply of high genus curves with (AC). However, it is not easy to make this claim solid:

**Problem 1** (van der Geer [EMO01, Problem 19]). Do there exist smooth projective curves of arbitrary genus with supersingular Jacobian over all finite fields?

To the best of our knowledge this problem is only settled (and affirmatively so) in the case of characteristic two [vdGvdV95].

We have the feeling that the converse of (2) might have a chance to be true:

**Problem 2.** Is it true: A geometrically connected smooth projective curve with an $F_q$-rational curve and genus $g \geq 2$ has (AC) if and only if the Jacobian is supersingular?

**Proof of Theorem 5.15**. If $X$ has genus 0 and a rational point, it must be $\mathbb{P}^1$. Thus, it is cellular and the claim follows from Corollary 5.8. If $X$ has genus 1 and a rational point, it is an elliptic curve and Theorem 5.3 applies. Finally, suppose $X$ has arbitrary genus and $\text{Pic}^0(X)$ is supersingular. As $X$ has an $F_q$-rational point by assumption, $Z_{\log}$ is defined (i.e. we have $N_r \geq 1$ for all $r \geq 1$). Moreover, the $\ell$-adic homological motive of the curve splits as $\mathcal{M}(X) = \mathbb{Z} \oplus h^1(\text{Pic}^0(X)) \oplus \mathbb{Z}(1)$. As the Jacobian is supersingular, its motive, and in particular its weight one part $h^1(\text{Pic}^0(X))$ (the entire motive is the full exterior algebra over this weight one part) is supersingular. Hence, $\mathcal{M}(X)$ splits as a finite direct sum of Tate motives and supersingular motives, so Theorem 5.7 applies. □

**Example 5.17.** When we invoke Theorem 5.3 for a supersingular elliptic curve over $F_{22}$, or an ordinary elliptic curve over $F_{11}$ with Frobenius characteristic polynomial $x^2 - x + 11$, the input data for our constructions as in [3.12] corresponds to those in Example 3.14.

5.3. Linear recurrences.

**Theorem 5.18.** If $X/F_q$ is a smooth projective variety of dimension $\geq 1$, meeting the hypotheses of Theorem [1.1] or Theorem [1.2], then the sequence

$$n \mapsto \log |X(F_{q^n})|$$
does not satisfy any linear recurrence equation.

Proof. If the coefficients of a power series satisfy a linear recurrence, then the power series describes a rational function, and thus it has a meromorphic analytic continuation to the entire complex plane. Thus, in our situation, this continuation agrees with the ones given for the logarithmic derivative of $Z_{\log}$ by Theorem 5.3 resp. 5.6. In either case, the locus of poles is governed by a suitable pseudo-divisor. However, a rational function has only finitely many poles, so we reach a contradiction as soon as we can show that the relevant pseudo-divisors have support larger than a finite set of points. In either case this is easy to see.

\appendix

\section{Construction for motives}

In this appendix we collect a survey on motives and discuss how to extend the definition of $Z_{\log}$ to motives. In many ways this appears to be the more natural habitat for the theory.

Recall our conventions from \S 2.1.

\subsection{Numerical motives over a finite field}

The category $(\text{Mot}_{\text{num}}(F_q), \otimes_{\text{twisted}})$ is an abelian semi-simple $F$-linear Tannakian category. Let us briefly recall the ingredients for this: (1) Thanks to Jannsen’s Theorem \cite[Theorem 1]{Jan92} any category of numerical motives $\text{Mot}_{\text{num}}(k)$ over an arbitrary field $k$ is $F$-linear abelian semi-simple. (2) Numerical motives have a canonical finite weight decomposition, 

$$M = \bigoplus_i h^i(M),$$

where the sum runs over finitely many $i$, depending on $M$. We call $h^i(M)$ the \textit{weight $i$ part}. This is based on the algebraicity of Künneth projectors, following Katz–Messing \cite[\S III]{KM74}. (3) The naïve tensor product on $\text{Mot}_{\text{num}}(k)$ can impossibly yield a Tannakian category. However, using a twisted tensor product due to Deligne one can resolve this issue over finite fields \cite[Corollary 2]{Jan92} and Remark (2) following this Corollary, loc. cit.

If $M$ is a numerical motive (so, concretely $=(X, p)$ for $X$ a smooth projective variety and $p$ a correspondence, idempotent up to numerical equivalence), then the Frobenius of $X$ gives a well-defined endomorphism, usually denoted by $\pi_X$, of $X$. Its characteristic polynomial has coefficients in $\mathbb{Q}$. Define the ordinary zeta function by

$$Z_{\text{num}}(M, t) := \prod_r \det(1 - \pi_X \cdot t | h^r(M))^{(-1)^{r+1}},$$

where $h^r(X)$ is the weight $r$ part. See \cite[Prop. 2.1]{Mi94} for details. As the individual characteristic factors are polynomials, and there are only finitely many non-zero weight parts, $Z_{\text{num}}(M, t)$ is a rational function. We also observe $Z_{\text{num}}(M, 0) = 1$ for all $M$. For more on the Tannakian viewpoint, see \cite{Kah09}.

\subsection{Homological motives over a finite field}

On the other hand, $\text{Mot}_{\text{hom}}(\ell)$ denotes homological motives with respect to $\ell$-adic cohomology, $\ell \neq \text{char} k$. That is, define

$$X \mapsto H_\ell(X) := \bigoplus H^i(X \times_{F_q} F_q^{\text{sep}}, \mathbb{Q}_\ell)$$

\section{Appendix A. Construction for motives}

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Recall our conventions from \S 2.1.

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where $h^r(X)$ is the weight $r$ part. See \cite[Prop. 2.1]{Mi94} for details. As the individual characteristic factors are polynomials, and there are only finitely many non-zero weight parts, $Z_{\text{num}}(M, t)$ is a rational function. We also observe $Z_{\text{num}}(M, 0) = 1$ for all $M$. For more on the Tannakian viewpoint, see \cite{Kah09}.
for smooth projective varieties $X$. This is functorial in homological correspondences. If $(X,p)$ is a homological motive and $\pi_X$ again denotes the Frobenius, one may define the ordinary zeta function by

$$Z^{\text{hom}}((X,p),t) := \prod_r \det (1 - \pi_X^* \cdot t | H^r((X,p), \mathbb{Q}_\ell))(-1)^{r+1}$$

where $p_*H^r(X, \mathbb{Q}_\ell)$ denotes the direct summand of the $\ell$-adic cohomology which is cut out by the idempotent $p$, and $\pi_X^*$ denotes the action of the Frobenius, as induced to cohomology. Again, we observe $Z^{\text{hom}}(M,0) = 1$ for all $M$.

**Fact A.1.** Both constructions yield the same zeta function, $Z^{\text{hom}} = Z^{\text{num}}$, for all $\ell \neq \text{char } k$.

A.3. Tate motives, supersingular motives. A simple numerical motive has a characteristic polynomial via the Tannakian structure, as in [Mi94, Prop. 2.1]. A simple $\ell$-adic homological motive has a characteristic polynomial by taking the action of the Frobenius on its $\ell$-adic cohomology. Thus, either way, we have a notion of characteristic polynomial and we will call its roots the Frobenius eigenvalues. We shall use the following conventions:

**Definition A.2.** We call a simple (numerical or homological) motive Tate if its Frobenius eigenvalues are of the shape $q^w$ for some $w \in \mathbb{Z}$. We call it supersingular if its Frobenius eigenvalues are of the shape

$$\zeta \cdot q^{w/2}$$

for some $w \in \mathbb{Z}$ and $\zeta$ any root of unity.

If we believe in the Tate conjecture, the motives over $\mathbb{F}_q$ are (Tannakian) generated by abelian varieties and the supersingular abelian varieties then generate the same motives as when we take the supersingular ones with the above definition. This justifies the term ‘supersingular’.

**Example A.3.** We have $\mathcal{M}(\mathbb{P}^n) = \mathbb{Z} \oplus \mathbb{Z}(1) \oplus \cdots \oplus \mathbb{Z}(n)$, sitting in $h^0, h^2, \ldots, h^{2n}$ respectively. In particular, these summands are all supersingular.

A.4. Multiplicative zeta for motives. The following considerations make sense both in $\ell$-adic homological or numerical motives; and under sending a homological motive to its numerical counterpart, they are compatible.

So, let $Z$ denote either $Z^{\text{hom}}$ or $Z^{\text{num}}$, according to which viewpoint we may prefer. Expanding it as a power series around $t = 0$, we may define numbers $N_m$ for a motive $(X,p)$ by

$$\sum_{m \geq 1} N_m \frac{t^m}{m} := \log Z((X,p),t).$$

To make sense of this, we note that $Z((X,p),t)$ at $t = 0$ is $+1$. Hence,

$$\log Z((X,p),t) = \log \left(1 - \left(\sum_{r \geq 1} a_r t^r \right)\right) = -\sum_{l \geq 1} \frac{1}{l} \left(\sum_{r \geq 1} a_r t^r \right)^l$$
makes sense as a (formal or genuine) power series, and moreover its constant coefficient vanishes. Thus, we may write it in the form \( \log Z((X,p),t) = \sum_{r \geq 1} N_r \cdot \frac{t^r}{r} \) for uniquely determined values \( N_r \in \mathbb{Q} \). Now, define

\[ Z_{\log}(X,t) := \exp \left( \sum_{r \geq 1} \log |N_r| \cdot \frac{t^r}{r} \right). \]

This may not make sense for arbitrary motives since we could (and can!) have \( N_r = 0 \). This corresponds to the issue that we can only define \( Z_{\log} \) in the context of varieties when we demand the existence of a rational point. We will content ourselves with this definition, which will make sense for many motives. Otherwise, we will simply say that the multiplicative zeta function is not defined. Nonetheless, a convenient observation is the following:

**Definition A.4.** We say that a motive \( M \) has unique top weight if it decomposes as a finite direct sum

\[ M = M_1 \oplus \cdots \oplus M_r \oplus Z(m) \]

such that the Frobenius eigenvalues of all factors \( M_i, i = 1, \ldots, r \), are strictly \( < q^m \).

**Lemma A.5.** If \( M \) has unique top weight, \( N_r \) can only vanish for finitely many \( r \) (and this can be effectively bounded).

This condition is clearly met for geometrically connected smooth projective varieties \( X/\mathbb{F}_q \) because the Poincaré dual partner of the \( h^0(X) = Z(0) \) of the single connected component provides the single summand \( Z(d) \) with \( d := \dim X \).

**Proof.** The ordinary zeta function of a motive has the shape

\[ Z(M,t) = \prod_{i,j} (1 - \alpha_{i,j} \cdot t)^{-1 + i}, \]

where \( |\alpha_{i,j}| = q^{i/2} \) are Weil \( q \)-numbers of weight \( i \) (this follows from working with \( \ell \)-adic homological motives, using the Weil conjectures there, and then using equality of characteristic polynomials for numerical vs. \( \ell \)-adic homological motives). Thus,

\[ \sum_{l \geq 1} N_l \frac{t^l}{l} = \sum_{i,j} (-1)^{i+1} \log(1 - \alpha_{i,j} \cdot t) \]

\[ = \sum_{i,j} (-1)^i \sum_{l=1}^{\infty} \frac{1}{l} \alpha_{i,j}^l t^l = \sum_{i,j} \left( \sum_{i,j} (-1)^i \alpha_{i,j}^l \right) \frac{t^l}{l}. \]

Thus, \( N_l = \sum_{i,j} (-1)^i \alpha_{i,j}^l \). By our assumption precisely one \( \alpha_{i,j} \) has an absolute value strictly larger than any other of the \( \alpha_{i,j} \). Call the corresponding index \((i_{\text{top}}, j_{\text{top}})\). Then

\[ N_l = (-1)^{i_{\text{top}}} \alpha_{i_{\text{top}}, j_{\text{top}}}^l \left( 1 + \sum_{(i,j) \neq (i_{\text{top}}, j_{\text{top}})} (-1)^{-i_{\text{top}}} \alpha_{i,j} / \alpha_{i_{\text{top}}, j_{\text{top}}} \right) \]

with \( |\alpha_{i,j}/\alpha_{i_{\text{top}}, j_{\text{top}}}| < 1 \) for all \((i,j) \neq (i_{\text{top}}, j_{\text{top}})\). For sufficiently large \( t \), the term in the bracket is too close to 1 to ever vanish again. \( \square \)

We give some examples which show phenomena specific to \( Z_{\log} \) for general pure motives:
Example A.6. Suppose $A/F_p$ is a supersingular elliptic curve. Then for $h^1(A)$ we get $N_r = -(\sqrt[r]{p}) + (-\sqrt[r]{p}) = -(1 + (-1)^r)p^{r/2}$. Every $N_{2r+1}$ is zero. Hence, we cannot define $Z_{\log}(h^1(A), t)$.

This kind of problem is specific to motives. For smooth projective varieties the assumption of having an $F_q$-rational point settles $N_r \neq 0$ for all $r \geq 1$.

Example A.7. We continue Example A.6. Define $Y := h^0(A) \oplus h^1(A)$, i.e. we truncate the top degree summand from the motive. Although the simple summands of the motive $Y$ are all supersingular, Theorem 5.6 does not apply because $Y$ does not have a unique top weight (Definition A.4). However, we can still establish (AC) manually: We get

$$N_r = 1 - (1 + (-1)^r)p^{r/2}$$

and thus, after some series manipulations,

$$Z_{\log}(Y, t)^2 = (1 - t^2)^{\log(2)} \cdot p(\frac{r^2}{1 - t^2}) \cdot W(t^2).$$

The reference to the left accounts for $\frac{1}{2}$ in Equation A.3 while $t^2$ on the right accounts for the squared variable in loc. cit. Since the existence of an analytic continuation for $W$ implies the existence of a continuation for $t \mapsto \sqrt{W(t^2)}$, we get (AC) for $Z_{\log}(Y, t)$. This is our first example where we needed $\varepsilon_1 \neq \pm 1$.

Example A.8. We continue Example A.6 in a different way. Define $W := h^1(A) \oplus h^2(A)$, i.e. this time we truncate the degree zero part. Since $h^2(A) \cong \mathbb{Z}(2)$, $W$ has supersingular summands and unique top weight. Theorem 5.6 applies. We compute

$$N_r = -(1 + (-1)^r)p^{r/2} + p^r$$

and after some series manipulations, this leads to

$$Z_{\log}(W, t) = p(\frac{r^2}{t}) \exp(\frac{1}{2} \sum_{r \geq 1} \log |1 - 2(p^{-1})^r| \cdot \frac{(t^2)^r}{r}).$$

With $N := 1$, $\varepsilon_1 := 2$ and $\lambda_1 := p^{-1}$, the Theorem A.7 can be applied directly.

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