We investigate the long-time relaxation of the $q$-state kinetic Potts ferromagnet on the triangular lattice that is quenched to zero temperature from a random or an antiferromagnetic initial state. For $q = 3$, the final state is either the ground state (probability $\approx 0.75$), a two-stripe state (probability $\approx 0.09$), or a three-hexagon state (probability $\approx 0.16$). The relaxation to the hexagonal state is governed by a time that scales as $L^2 \ln L$. We provide a heuristic argument for this anomalous scaling and present additional new features of Potts coarsening on the triangular lattice for $q = 3$ and for $q > 3$.

When a ferromagnet with multiple degenerate ground states is quenched from above to below its critical point, a coarsening domain mosaic arises in which distinct phases compete with each other to survive the ordering dynamics [1, 2]. While continuum theories of coarsening predict that the ground state is ultimately reached, the long-time states that persist in discrete spin systems can be surprisingly complex when the quench is to zero temperature. Infinitely long-lived metastable states can appear, which may be static and geometrically simple, such as stripe states in the kinetic Ising model in spatial dimension $d = 2$ [3, 4], or non-stationary and topologically complex in the $d = 3$ kinetic Ising model [5, 6]. An unexpected feature of stripe configurations is that their probabilities of occurrence can be computed in terms of the spanning probabilities of continuum percolation [7–12].

The kinetic Potts model exhibits a richer domain geometry [13–19]. For quenches to zero temperature, it was found that the ground state may not always be reached [20, 21] and that “blinker” spins (spins that can flip forever without any energy cost) arise on the square lattice [22]. A domain mosaic may also get stuck for very long times in a nearly static state that is eventually disrupted by an avalanche that globally rearranges the domains. Interest in this model stems, in part, from its applications to a wide range of coarsening phenomena, such as soap froths [23–25], magnetic domains [26–31], cellular tissue and other natural tilings [32–34].

In this work, we investigate intriguing and apparently overlooked features of the coarsening of the 3-state ferromagnetic Potts model on the triangular lattice: (a) When quenched to zero temperature, roughly 75% of all trajectories end in the ground state, 9% end in a two-stripe state, and 16% end in a three-hexagon state (Fig. 1). (b) The approach to these final states is governed by three distinct times: (i) the conventional coarsening time $L^2$, with $L$ the linear size of the system, (ii) a time scale that appears to grow as $L^3$ that governs the relaxation of diagonal stripe states to the ground state, and (iii) a time that appears to grow as $L^2 \ln L$ that governs the approach to the three-hexagon state. It is worth noting that the zero-temperature coarsening of the 3-state Potts model on the square lattice is more complex still [22].

It is convenient in our simulations to represent the triangular lattice as a periodically bounded square array with additional diagonal interactions to the upper right and lower left next-nearest neighbors (on the square lattice). In each realization of the dynamics, the spins are initialized in either a random or an antiferromagnetic state, with equal numbers of spins in each type; both initial states give virtually identical results. The Hamiltonian of the system is

$$\mathcal{H} = -2J \sum_{i,j} \left[ \delta(s_i, s_j) - 1 \right], \quad (1)$$

where $J > 0$ is the coupling constant, $\delta(a,b)$ is the Kro-
necker delta function, and the sum runs over all nearest-neighbors spin pairs $i, j$. Each misaligned spin pair contributes $+J$ to the energy, while each aligned pair contributes zero. We use zero-temperature Glauber dynamics: spin-flip events that would decrease or conserve the systems energy are assigned rates of 1 or $\frac{1}{2}$, respectively, while spin-flip events that would increase the energy have rate zero. We use an event-driven algorithm to implement this dynamics in a rejection-free manner \[35\] and simulate systems with $L$ between 12 and 384, with $10^5$ realizations for each system size.

![Figure 2](image)

**FIG. 2.** Probabilities of freezing into the ground state, $P_G$, a two-stripe state, $P_S$, and a hexagonal state, $P_H$, as a function of the inverse of the logarithm of the system length. For $L \to \infty$, we estimate these probabilities as 0.75, 0.09, and 0.16, respectively (arrows).

Figure 2 shows the $L$ dependence of the probabilities for the system to reach one of three final states: ground state, two-stripe state, and three-hexagon state. An $ABC$ three-stripe state is reached with a probability of the order of $10^{-4}$ for the largest system simulated and plays a negligible role in the coarsening dynamics. Because of the non-monotonic and/or slow $L$ dependences of the final-state probabilities, we can give only crude estimates of their $L \to \infty$ values. Similar non-monotonicities occur for the kinetic Ising model, as well as the square-lattice Potts model \[3, 4, 22\]. Two-stripe states also arise in the coarsening of the kinetic Ising model on the square lattice, with an occurrence probability that is close to $\frac{1}{3}$ by the percolation correspondence \[3, 4\].

A curious feature of the three-hexagon state is that its energy equals $24L$, independent of the individual hexagon sizes. By examining Fig. 1, the total length of each of the vertical, horizontal, and tilted interfaces in the three-hexagon state must equal $L$. This gives $6L$ interfacial spins in all. Since an interfacial spin has four neighbors in the same spin state and two neighbors in a different state, each such spin contributes $4J$ to the total energy. Consequently, the final energy of any three-hexagon state is always $24L$. The distribution of hexagon areas $A$ is reasonably fit by $A \exp[(A - \langle A \rangle)^2 / \sigma^2]$, with $\langle A \rangle = L^2 / 3$.

The multiscale time dependence of the coarsening is illustrated by the probability $S(t)$ that the system has “survived” until time $t$, namely, the probability that flippable flips still exist at this time (Fig. 3). This probability decays to zero for all realizations in a finite time; for example, the longest-lived of $10^5$ realizations for $L = 384$ reaches its final state at $t/L^2 \approx 89$. All final states—ground state or otherwise—are static, with no flippable spins. This property contrasts with the kinetic Potts model on the square lattice where blinker spins persist \[22\], so that $S(t)$ never decays to zero.

![Figure 3](image)

**FIG. 3.** Time dependence of the survival probability $S(t)$ for (a) $t/L^2 < 1$ and (b) $t/L^2 > 2$. The data are based on a system of length $L = 384$. In (a), the three lines schematically indicate the different decay rates associated with coarsening, the relaxation to the three-hexagon state, and the relaxation to the diagonal stripe state.

![Figure 4](image)

**FIG. 4.** A diagonal stripe interface on the triangular lattice. The $A$ and $B$ spins on exterior corners can flip with no energy cost. To the right is shown the configuration after a spin flip.

Turning to the time dependence, at short times $(0.05 \leq t/L \leq 0.1)$, the decay of $S(t)$ corresponds to conventional coarsening, where $S(t)$ decays exponentially in time, with a characteristic decay time that scales as $L^2$. Conversely,
for \( t/L^2 \gtrsim 0.5 \), \( S(t) \) decays extremely slowly due to the relaxation from diagonal stripe states (Fig. 4) to the ground state. The probability for the Potts system to fall into diagonal stripe states is \( \approx 0.005 \) for the largest system simulated. When such a stripe forms, a large fraction of spins on the interface are freely flippable, which causes the interface to diffuse. When two diffusing diagonal stripe interfaces meet, the system then quickly falls into the ground state. For the analogous diagonal stripe states in the kinetic Ising model on the square lattice, we previously argued that this time to reach the ground state via the diagonal stripe state scales as \( L^4 \) [3, 4] (although numerical simulations gave an exponent value closer to 3.3). For the Potts model, we find that this diagonal relaxation time \( T_D \) scales as \( L^\mu \), with \( \mu \approx 3.13 \) (Fig. 5). A similar result arises by fitting the long-time tail of \( S(t) \) for different \( L \) to an exponential decay in time and then taking the inverse of this slope to estimate \( T_D \). This data is strongly compromised by finite-size effects, but from last few data points for the largest system sizes, we estimate \( \mu \approx 3.46 \).

In the intermediate time regime (0.2 \( \lesssim t/L \lesssim 0.4 \)), \( S(t) \) decays more slowly than in the coarsening regime, which is a manifestation of the relaxation to the three-hexagon state. We quantify this relaxation by measuring the average time \( T_H \) to reach the three-hexagon state for each \( L \). As a function of \( L \), a simple power-law fit suggests that \( T_H \sim L^\mu \), with \( \mu \approx 2.27 \). However, there is a small, but consistent downward curvature of \( T_H \) versus \( L \) on a double logarithmic scale (which becomes apparent by magnifying Fig. 5), and a power-law fit is not very satisfactory. Instead, we find that \( T_H \) is much better fit by the form \( T_H \sim L^2 \ln L \).

This unusual time dependence for \( T_H \) appears to have a simple geometrical explanation. For a realization to reach the three-hexagon state, it first has to condense to a state that consists of three clusters, none of which span the system. This three-cluster state will therefore have the same topology as the three-hexagon state, but with significant distortions. That is, the six T junction points where three interfaces meet will be out of registry with each other (compared to the registry of the T junctions in the three-hexagon state in the lower-right panel of Fig. 1). Consequently, each of the six interfaces between these T junctions will be tilted with respect to a triangular lattice direction. This means that a substantial fraction of the spins on each such interface are freely flippable.

These interfaces must gradually straighten out to reach the final three-hexagon state with the T junctions in registry. (See [37] for a visualization of a trajectory from the three-cluster to a static three-hexagon state.) The freely flippable spins on a single interface can each be viewed as independent random walks. Flipping one of these spins corresponds to the equivalent random walk hopping by one step (see Fig. 4). When a flippable spin reaches a T junction, the position of the latter moves by one lattice spacing. This displacement of the T junction corresponds to the random walk being absorbed. Thus we can view the interface straightening process as equivalent to the successive absorption of independent random walks on a finite interval whose length is of the order of \( L \). The initial number of walks \( n \) in this interval is also of the order of \( L \).

In this initial state, the typical separation between neighboring walkers is \( L/n \); this is also the distance \( x_1 \) between the end of the interval and the closest walk to the end. The time until this closest walk is absorbed at the end of the interval and the closest walker. At this stage, the time for the next absorption event is \( t_1 = x_1 (L-x_1) \) [36]. When there are \( k \) walkers remaining as a result of successive absorption events, the typical separation between walkers is now \( L/k \), which is also the typical distance between the end of the interval and the closest walker. At this stage, the time for the next absorption event is \( t_k = x_k (L-x_k) \), with \( x_k = L/k \). When all the walkers have been absorbed, the final three-hexagon state has been reached. Adding these individual absorption times gives, for the time to reach the three hexagon state,

\[
\tau = L(x_1 + x_2 + \cdots + x_L) - \left( x_1^2 + x_2^2 + \cdots + x_L^2 \right) = L^2 \sum_{k=1}^{L} \frac{1}{k} - L^2 \sum_{k=1}^{L} \frac{1}{k^2} \approx L^2 \ln L \tag{2}
\]

While our argument is crude, it appears to capture the basic mechanism that underlies the approach to the three hexagon state. As shown in Fig. 5, the prediction (2) agrees well with simulation results.

The dynamics and long-time state of the kinetic \( q \)-state Potts model with \( q > 3 \) states shares many features of the 3-state Potts models, but a number of oddities also occur. As the number of Potts states is increased, the coarsening mosaic becomes visually more picturesque and the possible final states are correspondingly richer. Additionally, final states that contain more than three hexagons now

\[ \text{FIG. 5. Dependence of the basic relaxational time scales of the kinetic triangular Potts model:} \]

\[ \text{(a) } T_D (\Delta), \text{ the diagonal stripe relaxation time. The green dashed line is a best-fit power law with exponent is 2.27. (b) } T_H (\diamondsuit), \text{ the three-hexagon relaxation time. The blue dashed line is the best-fit power law with exponent 2.02.} \]

\[ \text{The red dashed line is the best-fit power law with exponent 2.02.} \]
arise. For \( q = 5 \) and \( q = 6 \), five-hexagon states appear with probability of the order of \( 10^{-3} \). For \( q = 6 \) and \( L = 384 \), we also observed one realization out of \( 10^5 \) that reached an eight-hexagon state. Final states that contain blinker spins also exist, but these states are extremely rare. We observed blinker spins for \( q = 5 \) and \( q = 6 \) with a probability of the order of \( 10^{-4} \), but only for small system sizes. We did not observe blinker spins in any triangular Potts system with \( L > 40 \). Both of these exotic long-time states—multi-hexagon states and blinker spins—occur sufficiently rarely that they play a negligible role in characterizing the overall coarsening dynamics.

![FIG. 6. Time evolution of the densities of each spin type, \( \rho_n \), sorted by abundance order.](image)

Another intriguing aspect of the \( q \)-state Potts model for larger \( q \) is the nearly universal pattern for the long-time densities of the most-common spin type, the second most-common type, etc. (Fig. 6). Here, the system is always initialized in the antiferromagnetic state, with equal numbers of each spin type. We denote by \( \rho_1 \), the fraction of the most-common spin type in the final state, \( \rho_2 \), the second most-common spin fraction, etc. The time dependences of the fractions \( \rho_1, \rho_2, \rho_3 \) (red in Fig. 6) are almost the same, independent of the initial number of Potts states. The final fractions of the three most abundant spin types are \((\rho_1, \rho_2, \rho_3) \approx (0.871, 0.095, 0.034)\) for \( q = 3 \) and \((0.873, 0.081, 0.046)\), for \( q = 6 \). For all \( q \) values between 3 and 6, the fraction of spins types outside the top three abundances is less than \( 5 \times 10^{-3} \).

As the number of distinct initial spin types \( q \) is increased beyond \( q = 6 \), the final fractions \( \rho_n \) for the five most abundant spin types are nearly universal, while the final fractions \( \rho_n \) for \( n > 5 \) are negligibly small. Thus little additional information is gained about the nature of the final states of the kinetic \( q \)-state Potts model by simulating systems with more than 6 initial states.

To summarize, the coarsening of the kinetic \( q \)-state Potts model on the triangular lattice exhibits intriguing features that do not occur on the square lattice. For \( q = 3 \) spin states, the final configurations are either: the ground state, two-stripe states, or a three-hexagon state, with respective frequencies of 75\%, 9\%, and 16\%. The time dependence of the coarsening process is governed by three distinct time scales: a coarsening time that scales as \( L^2 \), a diagonal stripe relaxation time that roughly scales as \( L^3 \), and a hexagonal state relaxation time that appears to scale as \( L^2 \ln L \). We gave a heuristic argument based on mapping freely flippable spins on domain interfaces to random walks that are absorbed when they reach T junctions, to justify this latter time scale.

We also investigated the coarsening of the \( q \)-state Potts model on the triangular lattice with \( q > 3 \). Here, the main features of coarsening, such as multiple timescale relaxation and condensation into three-hexagon and stripe states, are qualitatively similar to that observed for the 3-state Potts model. One striking feature is that only the three most abundant spin types are present in measurable amounts in the final state when the initial state is symmetric. This fact suggests that little new dynamic information is gained by studying Potts models with a large number of states.

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