OPTIMAL CONTROL PROBLEMS GOVERNED BY TWO DIMENSIONAL CONVECTIVE BRINKMAN-FORCHHEIMER EQUATIONS

MANIL T. MOHAN

Department of Mathematics
Indian Institute of Technology Roorkee-IIT Roorkee
Haridwar Highway, Roorkee, Uttarakhand 247667, INDIA

(Communicated by Andrei Fursikov)

Abstract. The convective Brinkman-Forchheimer (CBF) equations describe the motion of incompressible viscous fluids through a rigid, homogeneous, isotropic, porous medium and is given by

$$\begin{align*}
\partial_t u - \mu \Delta u + (u \cdot \nabla)u + \alpha u + \beta |u|^{r-1} u + \nabla p &= f, \quad \nabla \cdot u = 0.
\end{align*}$$

In this work, we consider some distributed optimal control problems like total energy minimization, minimization of enstrophy, etc governed by the two dimensional CBF equations with the absorption exponent $r = 1, 2$ and $3$. We show the existence of an optimal solution and the first order necessary conditions of optimality for such optimal control problems in terms of the Euler-Lagrange system. Furthermore, for the case $r = 3$, we show the second order necessary and sufficient conditions of optimality. We also investigate another control problem which is similar to that of the data assimilation problems in meteorology of obtaining unknown initial data, when the system under consideration is 2D CBF equations, using optimal control techniques.

1. Introduction. The motion of incompressible viscous fluids through a rigid, homogeneous, isotropic, porous medium can be characterized using the convective Brinkman-Forchheimer (CBF) equations. In this work, we consider the controlled CBF equations and investigate some optimal control problems. Let $O \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary $\partial O$. Let $u(t, x) \in \mathbb{R}^2$ denote the velocity field at time $t$ and position $x$, $p(t, x) \in \mathbb{R}$ represent the pressure field, and $f(\cdot, \cdot) \in \mathbb{R}^2$ stand for an external forcing. We consider the following controlled CBF equations (see [34] for physical interpretation of the model):

$$\begin{align*}
\frac{\partial u(t, x)}{\partial t} - \mu \Delta u(t, x) + (u(t, x) \cdot \nabla)u(t, x) + \alpha u(t, x) + \beta |u(t, x)|^{r-1} u(t, x) + \nabla p(t, x) &= f(t, x) + D\mathbf{U}(t, x), \quad \text{for } x \in O, \ t > 0,
\end{align*}$$

2020 Mathematics Subject Classification. Primary 49J20; Secondary 49K20, 35Q35, 76D03.

Key words and phrases. porous medium, convective Brinkman-Forchheimer equations, first order necessary conditions, Pontryagin’s maximum principle, optimal control.

*Corresponding author: Manil T. Mohan.
with the conditions
\[
\begin{align*}
(\nabla \cdot \mathbf{u})(t, x) &= 0, & & \text{for } x \in \mathcal{O}, \ t > 0, \\
\mathbf{u}(t, x) &= \mathbf{0}, & & \text{for } x \in \partial\mathcal{O}, \ t \geq 0, \\
\mathbf{u}(0, x) &= \mathbf{u}_0(x), & & \text{for } x \in \mathcal{O}, \\
\int_{\mathcal{O}} p(t, x) \, dx &= 0, & & \text{for } t \geq 0.
\end{align*}
\]

The system (1) can be considered as a modification (by an absorption term \(\alpha \mathbf{u} + \beta \|\mathbf{u}\|^{r-1} \mathbf{u}\)) of the classical controlled Navier-Stokes equations (NSE). That is, for \(\alpha = \beta = 0\), we obtain the classical controlled 2D NSE. The final condition in (2) is imposed for the uniqueness of the pressure \(p\). The constant \(\mu\) represents the positive Brinkman coefficient (effective viscosity), the positive constants \(\alpha\) and \(\beta\) represent the Darcy (permeability of porous medium) and Forchheimer (proportional to the porosity of the material) coefficient, respectively and the absorption exponent \(r \in [1, \infty)\). It has been observed in Proposition 1.1, [22] that the critical homogeneous CBF equations (for \(r = 3\)) have the same scaling as the Navier-Stokes equations (NSE) only when \(\alpha = 0\) (no scale invariance property for other values of \(\alpha\) and \(r\)).

The distributed control acting on the system is denoted by \(U(\cdot, \cdot)\) in (1) and \(D\) is a bounded linear operator. The model (2) is recognized to be more accurate when the flow velocity is too large for the Darcy’s law to be valid alone, and in addition, the porosity is not too small.

Let us first discuss the global solvability results for the uncontrolled system (1) (with \(U = 0\)) available in the literature. The existence of a global weak solution to the system (1) in two and three dimensional bounded domains is obtained in [1]. The authors have also established the uniqueness of weak solutions in two dimensions. The Brinkman-Forchheimer equations with fast growing nonlinearities is investigated in [23] and the authors established the existence of regular dissipative solutions and global attractors for the system (1) in three dimensions for \(r > 3\). This result assures the existence of global weak solutions to the 3D CBF equations (for \(r > 3\)) in the Leray-Hopf sense satisfying the energy equality. The authors in [16] showed that all weak solutions of the critical CBF equations \((r = 3\) with \(4\beta\mu \geq 1\)) posed on a bounded domain in \(\mathbb{R}^3\) satisfy the energy equality. The existence and uniqueness of a global weak solution in the Leray-Hopf sense satisfying the energy equality to the 3D CBF equations with \(r \geq 3\) \((2\beta\mu \geq 1, \text{ for } r = 3)\) is investigated in [30] by exploiting the monotonicity as well as the demicontinuity properties of the linear and nonlinear operators, and the Minty-Browder technique.

Control of fluid flow and turbulence inside a flow in a given physical domain, with known initial data and by means of body forces, boundary data, temperature are well-studied problems in fluid mechanics (cf. [2, 19, 21, 36] etc). After the mathematical advancements in the optimal control of infinite dimensional nonlinear system theory and partial differential equations, the optimal control problems governed by Navier-Stokes equations have been developed extensively for the past few decades (see for example [2, 3, 12, 19, 21, 26, 32, 36], etc). Various optimal control problems governed by Navier-Stokes equations have been considered in [4, 15, 41, 42], etc and the references therein. Pontryagin’s maximum principle for the time optimal control of 2D CBF equations is established in [31]. In this work, we consider the optimal control problems like total energy minimization, minimization of enstrophy, etc, governed by the two dimensional CBF equations (1). We show the existence of an optimal solution as well as the first order necessary conditions of optimality (Pontryagin’s maximum principle) for such optimal control problems in terms of the Euler-Lagrange system. For the cubic growth \((r = 3)\), we show the
second order necessary and sufficient conditions of optimality for such problems. Using optimal control techniques, we investigate an another control problem which is similar to that of data assimilation problems in meteorology of obtaining unknown initial data, when the system under consideration is 2D CBF equations (1). Due to technical difficulties (see Remark 3 below), in this work, we take the absorption exponent \( r = 1, 2 \) and \( 3 \) only (physically relevant cases). The case of \( r = 1 \) is not discussed in details as it is just a perturbation of the 2D NSE with \((\alpha + \beta)u\) and not much technical difficulties arise.

The rest of the paper is organized as follows. In the next section, we provide the necessary function spaces needed to obtain the existence of a unique weak solution satisfying the energy equality for the system (1). In the same section, we also discuss some important properties of the linear and nonlinear operators like local monotonicity, hemicontinuity, etc, which is used to obtain the global solvability results (Theorems 2.1, 2.2, 2.4, etc). We consider the linearized system corresponding to (1) and establish the global solvability results in the same section (Theorem 2.5). A distributed optimal control problem of minimizing the total energy as well as enstrophy subjected to the controlled 2D CBF equations (1) is formulated in section 3. We show the existence of an optimal solution for such problems in the same section (Theorem 3.3). In section 4, we prove the first order necessary conditions optimality for the optimal control problem via Pontryagin’s maximum principle (Theorem 4.1). We characterize the optimal control in terms of the adjoint variable. Furthermore, in the same section, we show the uniqueness of optimal control in small time interval for the optimal control problem (Theorem 4.4). For the critical case \((r = 3)\), the second order necessary and sufficient conditions of optimality for the optimal control problem is established in section 5 (Theorems 5.1 and 5.2). In the final section, we formulate a problem similar to that of the data assimilation problems of meteorology of optimizing the initial data, where we find the unknown optimal initial data by minimizing a suitable cost functional subject to 2D CBF equations. We find the first order necessary conditions as well as second order necessary and sufficient conditions of optimality (for the case \(r = 3\)) for such problems in section 6 (Theorems 6.2, 6.3, 6.4).

2. Mathematical Formulation. This section is devoted for the necessary function spaces needed to obtain the global solvability results of the system (1)-(2). We mainly follow the work [30] for the functional framework. We provide an abstract formulation of the system (1)-(2) and discuss some properties like local monotonicity, hemicontinuity, etc of the linear and nonlinear operators. The existence of a unique global weak solution to the linearized problem corresponding to the system (1)-(2) is also established.

2.1. Function spaces. We denote \( C_0^\infty(\mathcal{O}; \mathbb{R}^2) \) for the space of all infinitely differentiable functions (\( \mathbb{R}^2 \)-valued) with compact support in \( \mathcal{O} \subset \mathbb{R}^2 \). Let us define \( \mathcal{V} := \{u \in C_0^\infty(\mathcal{O}; \mathbb{R}^2) : \nabla \cdot u = 0\} \), \( \mathcal{H} \), \( \mathcal{V} \) and \( \mathcal{L}^p \) as the the closure of \( \mathcal{V} \) in the Lebesgue space \( L^2(\mathcal{O}) = L^2(\mathcal{O}; \mathbb{R}^2) \), in the Sobolev space \( H^1_0(\mathcal{O}) = H^1_0(\mathcal{O}; \mathbb{R}^2) \), in the Lebesgue space \( L^p(\mathcal{O}) = L^p(\mathcal{O}; \mathbb{R}^2) \), for \( p \in (2, \infty) \), respectively. Then under some smoothness assumptions on the boundary (one can take \( C^2 \)-boundary), we characterize the spaces \( \mathcal{H} \), \( \mathcal{V} \) and \( \mathcal{L}^p \) as

\[
\mathcal{H} = \{u \in L^2(\mathcal{O}) : \nabla \cdot u = 0, u \cdot n|_{\partial \mathcal{O}} = 0\},
\]

\[
\mathcal{V} = \{u \in H^1_0(\mathcal{O}) : \nabla \cdot u = 0\},
\]

for \( \mathcal{L}^p \).
\[ L^p = \{ u \in L^p(O) : \nabla \cdot u = 0, u \cdot n|_{\partial O} = 0 \}, \]

where \( n \) is the unit outward normal to \( \partial O \), and \( u \cdot n|_{\partial O} \) should be understood in the sense of trace in \( H^{-1/2}(\partial O) \) (cf. Theorem 1.2, Chapter 1, [38]). The norms in \( H, V, \overline{L}^p \) are denoted by \( \|u\|_H^2 := \int_O |u(x)|^2 \, dx, \|u\|_V^2 := \int_O |\nabla u(x)|^2 \, dx \) and \( \|u\|_{L^p}^p = \int_O |u(x)|^p \, dx \), respectively. Let \((\cdot, \cdot)\) denote the inner product in the Hilbert space \( H \) and \((\cdot, \cdot)_V\) denote the induced duality between the spaces \( V \) and its dual \( V' \) as well as \( L^p \) and its dual \( L^{p'} \), where \( \frac{1}{p} + \frac{1}{p'} = 1 \). Note that \( H \) can be identified with its own dual and we have the Gelfand triple \( V \subset H \subset V' \) and the embedding of \( V \subset H \) is compact. Let \( U \) be a separable Hilbert space identified with its own dual and we denote \( \| \cdot \|_V \) and \((\cdot, \cdot)_V\) as the norm and inner product defined on \( U \). We also denote \( \mathcal{L}(U, V') \) as the space of all bounded linear operators from \( U \) to \( V' \).

2.2. Linear operator. Let \( \mathcal{P}_p : L^p(O) \to \overline{L}^p, p \in (1, \infty) \), represent the Helmholtz-Hodge projection (cf. [18, 24]), and for simplicity we denote \( \mathcal{P}_p \) by \( \mathcal{P} \). For \( p = 2 \), it becomes an orthogonal projection. We define

\[
\begin{align*}
\text{D}(A) : & = V \cap H^2(O), \\
A : & \text{if } u \in \text{D}(A), \\
& = -\mathcal{P}\Delta u, \\
& \text{if } u \in \text{D}(A).
\end{align*}
\]

It can be easily seen that the operator \( A \) is a non-negative self-adjoint operator in \( H \) with \( V = D(A^{1/2}) \) and

\[
(Au, u) = \|u\|^2_V, \quad \text{for all } u \in V, \quad \text{so that } \|Au\|_{V'} \leq \|u\|_V. \tag{3}
\]

For a bounded domain \( O \), the operator \( A \) is invertible and its inverse \( A^{-1} \) is bounded, self-adjoint and compact in \( H \). Using spectral theorem, one can deduce that

\[
\|\nabla u\|_H^2 = (Au, u) = \sum_{k=1}^{\infty} \lambda_k |(u, e_k)|^2 \geq \lambda_1 \sum_{k=1}^{\infty} |(u, e_k)|^2 = \lambda_1 \|u\|_H^2, \tag{4}
\]

for all \( u \in V \), which is the well known Poincaré inequality.

2.3. Bilinear operator. Let us now define the trilinear form \( b(\cdot, \cdot, \cdot) : V \times V \times V \to \mathbb{R} \) by

\[
b(u, v, w) = \int_O (u(x) \cdot \nabla)v(x) \cdot w(x) \, dx = \sum_{i,j=1}^2 \int_O u_i(x) \frac{\partial v_j(x)}{\partial x_i} w_j(x) \, dx.
\]

If \( u, v \) are such that the linear map \( b(u, v, \cdot) \) is continuous on \( V \), the corresponding element of \( V' \) is denoted by \( B(u, v) \). We also denote \( B(u) = B(u, u) = \mathcal{P}[(u \cdot \nabla)u] \).

An integration by parts yields

\[
\begin{align*}
& \{ b(u, v, w) = 0, \text{ for all } u, v \in V, \\
& b(u, v, w) = -b(u, w, v), \text{ for all } u, v, w \in V. \tag{5}
\end{align*}
\]

The Sobolev inequality (Lemma 1, 2, Chapter 1, [25]) implies \( V \subset L^4 \subset H \subset \overline{L}^{4/3} \subset V' \). Using Hölder’s inequality, we deduce the following inequality:

\[
|b(u, v, w)| = |b(u, w, v)| \leq \|u\|_{L^4} \|\nabla w\|_H \|v\|_{L^4}, \quad \text{for all } u, v, w \in V,
\]

and hence applying Ladyzhenskaya’s inequality, we get

\[
\|B(u, v)\|_{V'} \leq \|u\|_{L^4} \|v\|_{L^4} \leq \sqrt{2} \|u\|_{L^2}^{1/2} \|v\|_{L^2}^{1/2}, \quad \text{for all } u, v \in V. \tag{6}
\]
For $u, v \in \mathcal{V}$, we further have
\[
\langle B(u) - B(v), u - v \rangle = \langle B(u - v), u - v \rangle \leq \|v\|\|u - v\|_{L^4},
\] (7)
using Hölder’s inequality.

2.4. Nonlinear operator. Let us now consider the operator $C(u) := P(|u|^{-1}u)$. It is immediate that $\langle C(u), u \rangle = \|u\|_{L^{r+1}}^{r+1}$, and $C : \mathbb{L}^{r+1} \to \mathbb{L}^{r+1}$. Furthermore, for all $u \in \mathbb{L}^{r+1}$, the map is Gateaux differentiable with Gateaux derivative
\[
C'(u)v = \begin{cases} P(|u|v), & \text{for } r = 1, \\ P(|u|v) + P\left(\frac{u}{|u|} (u \cdot v)\right), & \text{for } r = 2, \\ 0, & \text{for } u = 0, \\ P(|u|^r v) + (r - 1)P(u|u|^{r-3}(u \cdot v)), & \text{for } r \geq 3, \end{cases}
\]
for all $v \in \mathbb{L}^{r+1}$. For $u, v \in \mathbb{L}^{r+1}$, it can be easily seen that
\[
\langle C'(u)v, v \rangle = \int_{\mathcal{O}} |u(x)|^{r-1} |v(x)|^2 dx + (r - 1) \int_{\mathcal{O}} |u(x)|^{r-3} (u(x) \cdot v(x))^2 dx \geq 0,
\]
for $r \geq 1$ (note that for the case $r = 2$, (9) holds, since in that case the second integral becomes $\int_{x \in \mathcal{O}, u(x) \neq 0} \frac{1}{|u(x)|} |u(x) \cdot v(x)|^2 dx \geq 0$). For $r \geq 3$, we further have
\[
C''(u)(v \otimes w) = (r - 1)P\left(|u|^{r-3}[(u \cdot w)v + (u \cdot v)w + (w \cdot v)u]\right)
\]
\[+ (r - 1)(r - 3)P\left(|u|^{r-5}(u \cdot v)(u \cdot w)u\right),
\]
(10)
for all $u, v, w \in \mathbb{L}^{r+1}$ (for $r = 4$, the final term appearing in the right hand side of the equality (10) is defined for $u \neq 0$ and one has to take $0$ for $u = 0$). For $0 < \theta < 1$ and $u, v, w \in \mathbb{L}^{r+1}$, using Taylor’s formula (Theorem 6.5, [11], [9]), we find
\[
\langle P(|u|^{r-1}u) - P(|v|^{r-1}v), w \rangle \leq \|(|u|^{r-1}u) - (|v|^{r-1}v)\|_{L^{r+1}} \|w\|_{L^{r+1}}
\]
\[\leq \sup_{0 < \theta < 1} r\|\theta u + (1 - \theta)v\|_{L^{r+1}} \|u - v\|_{L^{r+1}} \|w\|_{L^{r+1}}
\]
\[\leq r\left(\|u\|_{L^{r+1}} + \|v\|_{L^{r+1}}\right)^{r-1} \|u - v\|_{L^{r+1}} \|w\|_{L^{r+1}},
\]
for all $u, v, w \in \mathbb{L}^{r+1}$. For any $r \in [1, \infty)$, from [30], we infer that
\[
\langle P(|u|^{r-1}u) - P(|v|^{r-1}v), u - v \rangle \geq \frac{1}{2}\|u\|_{L^{r+1}}^2 (u - v)\|_{L^{r+1}}^2 + \frac{1}{2}\|v\|_{L^{r+1}}^2 (u - v)\|_{L^{r+1}}^2 \geq 0.
\]
(11)
It is important to note that
\[
\|u - v\|_{L^{r+1}}^{r+1} \leq 2^{r-2}\|u\|_{L^2}^{r-1} (u - v)\|_{L^{r+2}}^2 + 2^{r-2}\|v\|_{L^2}^{r-1} (u - v)\|_{L^{r+2}}^2,
\]
(12)
for $r \geq 1$ (replace $2^{r-2}$ with $1$, for $1 \leq r \leq 2$).

Remark 1. Even though we have defined the the operator $C(\cdot)$ for $r \in [1, \infty)$, we consider the cases $r = 1, 2, 3$ only in the rest of the paper. For $r = 3$, from (10), we deduce that
\[
C''(u)(v \otimes w) = 2P\{[(u \cdot w)v + (u \cdot v)w + (w \cdot v)u],
\]
\[
C'''(u)(v \otimes w \otimes x) = 2P\{[(x \cdot w)v + (x \cdot v)w + (w \cdot v)x],
\]
for all $u, v, w, x \in \mathbb{L}^4$. 
2.5. Local monotonicity. Let us now discuss the local monotonicity property of the linear and nonlinear operators, which plays a crucial role in the global solvability of the system (1).

**Theorem 2.1** (Theorem 2.11, [30], Remark 2.4, [30]). Let \( u, v \in \mathcal{V} \). Then, for the operator \( G(u) = \mu Au + B(u) + \alpha u + \beta C(u) \), we have

\[
\langle (G(u) - G(v), u - v) + \frac{27}{32\mu^3} N^4 \| u - v \| \mathcal{H} \geq 0,
\]

for all \( v \in \mathcal{B}_N \), where \( \mathcal{B}_N \) is an \( \mathbb{L}^4 \)-ball of radius \( N \), that is, \( \mathcal{B}_N := \{ z \in \mathbb{L}^4 : \| z \|_{\mathbb{L}^4} \leq N \} \).

**Theorem 2.2** (Theorem 2.3, [30]). For the critical case \( r = 3 \) with \( 2\beta \mu \geq 1 \), the operator \( G(\cdot) : \mathcal{V} \to \mathcal{V}' \) is globally monotone, that is, for all \( u, v \in \mathcal{V} \), we have

\[
(G(u) - G(v), u - v) \geq 0.
\]

2.6. Abstract formulation and global solvability. On taking the Helmholtz-Hodge projection \( \mathcal{P} \) onto the system (1), we obtain

\[
\begin{aligned}
&\partial_t u(t) + \mu \partial_t \mathcal{A}(u(t)) + B(u(t)) + \alpha u(t) + \beta C(u(t)) = f(t) + DU(t), \quad \text{in } \mathcal{V}', \\
&u(0) = u_0,
\end{aligned}
\]

ea.e. \( t \in [0,T] \), where \( \partial_t u = \frac{du}{dt} \), \( D \in \mathcal{L}(U, \mathcal{V}') \) and for simplicity, we used the notation \( \mathcal{P}f = f \) and \( \mathcal{P}DU = DU \).

**Definition 2.3.** Let \( u_0 \in \mathcal{H} \), \( f \in L^2(0, T; \mathcal{V}') \) and \( U \in L^2(0, T; \mathcal{U}) \) be given. For \( r = 1, 2, 3 \), a function

\[
u \in L^\infty(0, T; \mathcal{H}) \cap L^2(0, T; \mathcal{V}'),
\]

with \( \partial_t u \in L^r(0, T; \mathcal{V}') \), where \( p = 2 \), for \( r = 1, 2 \) and \( p = 4/3 \), for \( r = 3 \), is called a weak solution to the system (15), if for \( u_0 \in \mathcal{H} \) and \( v \in \mathcal{V} \), \( u(\cdot) \) satisfies:

\[
\begin{aligned}
&\langle \partial_t u(t) + \mu \partial_t \mathcal{A}(u(t)) + B(u(t)) + \beta C(u(t)), v \rangle = \langle f(t), v \rangle + \langle DU(t), v \rangle, \\
&\lim_{t \downarrow 0} \int_\Omega u(t,x) v(x) dx = \int_\Omega u_0(x) v(x) dx,
\end{aligned}
\]

and the energy equality:

\[
\| u(t) \|_{\mathcal{H}}^2 + 2\mu \int_0^t \| u(s) \|_{\mathcal{V}}^2 ds + 2\alpha \int_0^t \| u(s) \|_{\mathcal{H}}^2 ds + 2\beta \int_0^t \| u(s) \|_{\mathcal{L}^{r+1}}^{r+1} ds
\]

\[
= \| u_0 \|_{\mathcal{H}}^2 + 2 \int_0^t \langle f(s), u(s) \rangle ds + 2 \int_0^t \langle DU(s), u(s) \rangle ds,
\]

for all \( t \in [0, T] \).

**Remark 2.** From the energy equality (17), one can easily deduce that

\[
\| u(t) \|_{\mathcal{H}}^2 + \mu \int_0^t \| u(s) \|_{\mathcal{V}}^2 ds + 2\alpha \int_0^t \| u(s) \|_{\mathcal{H}}^2 ds + 2\beta \int_0^t \| u(s) \|_{\mathcal{L}^{r+1}}^{r+1} ds
\]

\[
\leq \| u_0 \|_{\mathcal{H}}^2 + \frac{2}{\mu} \int_0^T \| f(t) \|_{\mathcal{V}}^2 dt + \frac{2}{\mu} \| D \|_{\mathcal{L}(U, \mathcal{V}')}^2 \int_0^T \| U(t) \|_{\mathcal{H}}^2 dt := K_T,
\]

for all \( t \in [0, T] \). For \( r = 2 \), since \( \partial_t u \in L^2(0, T; \mathcal{V}') \) and \( u \in L^2(0, T; \mathcal{V}) \), an application of Theorem 3, section 5.9.2, [13] implies that \( u \in C([0, T]; \mathcal{H}) \). For \( r = 3 \), we know that \( \partial_t u \in L^{4/3}(0, T; \mathcal{V}') \) and \( u \in L^\infty(0, T; \mathcal{H}) \), and an application of Theorem 2, section 5.9.2, [13] yields \( u \in C([0, T]; \mathcal{V}') \). In fact, one can show that
\( \mathbf{u} \in C([0, T]; \mathbb{H}) \) satisfying the energy equality (17) for \( r = 3 \), since Theorem 4.1, [20] (see Lemma 2.5, [20] for proper approximations) is applicable as \( \mathbf{u} \in L^4(0, T; L^4) \), for \( r = 3 \). Moreover, the existence of a strong solution (Theorem 2.10, [31]) to the system (15) also ensures that \( \mathbf{u} \in C([0, T]; \mathbb{H}) \).

The local monotonicity and hemicontinuity properties of the linear and nonlinear operators, and a localized version of the Minty-Browder technique imply the following global solvability results (similar methods were used in [17] for the global solvability of the 2D NSE and [28] for the existence and uniqueness of weak solutions of the 2D Oldroyd models of order one).

**Theorem 2.4** ([Theorem 3.4, [30]]). Let \( \mathbf{u}_0 \in \mathbb{H} \) be given. Then there exists a unique weak solution to the system (15) satisfying

\[
\mathbf{u} \in C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathbb{V}),
\]

and (18).

### 2.7. The linearized system

In this subsection, we consider the following linearized system and discuss its global solvability results under the assumption that \( \mathbf{u}(\cdot) \) is a weak solution to the system (15).

\[
\begin{aligned}
\partial_t \mathbf{w}(t) + \mu \mathbf{A}\mathbf{w}(t) + B'(\mathbf{u}(t))\mathbf{w}(t) + \alpha \mathbf{w}(t) + \beta C'(\mathbf{u}(t))\mathbf{w}(t) &= g(t), \quad \text{in } \mathbb{V}', \\
\mathbf{w}(0) &= \mathbf{w}_0,
\end{aligned}
\tag{19}
\]

for a.e. \( t \in [0, T] \), where \( B'(\mathbf{u})\mathbf{w} = B(\mathbf{u}, \mathbf{w}) + B(\mathbf{w}, \mathbf{u}), \ C'(\mathbf{u})\mathbf{w} \) is defined in (8), \( \mathbf{w}_0 \in \mathbb{H} \) and \( g \in L^2(0, T; \mathbb{V}') \). We first provide an a-priori energy estimate satisfied by the system (19) formally. A rigorous justification can be given using a standard Faedo-Galerkin’s approximation technique. Taking the inner product with \( \mathbf{w}(\cdot) \) to the first equation in (19), we obtain

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \| \mathbf{w}(t) \|_{\mathbb{H}}^2 + \mu \| \mathbf{w}(t) \|_{\mathbb{V}}^2 + \alpha \| \mathbf{w}(t) \|_{\mathbb{H}}^2 + \beta \| \mathbf{u}(t) \|_{\mathbb{H}}^{\frac{3}{2}} \| \mathbf{w}(t) \|_{\mathbb{H}}^\frac{1}{2} dt \\
+ (r - 1) \beta \| \mathbf{u}(t) \|_{\mathbb{H}}^{\frac{3}{2}} (\mathbf{u}(t) \cdot \mathbf{w}(t)) \|_{\mathbb{H}}^2 \\
= -\langle B(\mathbf{w}(t), \mathbf{u}(t)), \mathbf{w}(t) \rangle + \langle g(t), \mathbf{w}(t) \rangle \\
\leq \| \mathbf{u}(t) \|_{\mathbb{V}} \| \mathbf{w}(t) \|_{\mathbb{H}}^\frac{3}{2} + \| g(t) \|_{\mathbb{V}} \| \mathbf{w}(t) \|_{\mathbb{V}} \\
\leq \frac{\mu}{2} \| \mathbf{w}(t) \|_{\mathbb{V}}^2 + \frac{2}{\mu} \| \mathbf{u}(t) \|_{\mathbb{H}}^2 \| \mathbf{w}(t) \|_{\mathbb{H}} + \frac{1}{\mu} \| g(t) \|_{\mathbb{V}}^2, \tag{20}
\end{aligned}
\]

where we used Hölder’s, Ladyzhenskaya’s and Young’s inequalities. Note that for \( r = 2 \), the term \( \| \mathbf{u} \|_{\mathbb{H}}^{\frac{3}{2}} (\mathbf{u} \cdot \mathbf{w}) \|_{\mathbb{H}}^2 \) has to be interpreted as \( \begin{cases} \| \frac{\mathbf{u} \mathbf{w}}{\| \mathbf{w} \|_{\mathbb{V}}} \|_{\mathbb{H}}^2, & \text{if } \mathbf{u} \neq \mathbf{0}, \\ 0, & \text{if } \mathbf{u} = \mathbf{0}. \end{cases} \)

Integrating the above inequality from 0 to \( t \), we find

\[
\begin{aligned}
\| \mathbf{w}(t) \|_{\mathbb{H}}^2 + \mu \int_0^t \| \mathbf{w}(s) \|_{\mathbb{V}}^2 ds + 2 \alpha \int_0^t \| \mathbf{w}(s) \|_{\mathbb{H}}^2 ds + 2 \beta \int_0^t \| \mathbf{u}(s) \|_{\mathbb{H}}^{\frac{3}{2}} \| \mathbf{w}(s) \|_{\mathbb{H}}^\frac{1}{2} ds \\
+ 2(r - 1) \beta \int_0^t \| \mathbf{u}(s) \|_{\mathbb{H}}^{\frac{3}{2}} (\mathbf{u}(s) \cdot \mathbf{w}(s)) \|_{\mathbb{H}}^2 ds \\
\leq \| \mathbf{w}_0 \|_{\mathbb{H}}^2 + \frac{4}{\mu} \int_0^t \| \mathbf{u}(s) \|_{\mathbb{H}}^2 \| \mathbf{w}(s) \|_{\mathbb{V}}^2 ds + \frac{2}{\mu} \int_0^t \| g(s) \|_{\mathbb{V}}^2 ds, \tag{21}
\end{aligned}
\]
for all $t \in [0, T]$. An application of Gronwall’s inequality in (21) yields

$$\|w(t)\|_{H}^2 \leq \left\{ \|w_0\|_{H}^2 + \frac{2}{\mu} \int_0^T \|g(t)\|_{H}^2 \, dt \right\} \exp \left( \frac{4}{\mu} \int_0^T \|u(t)\|_{H}^2 \, dt \right), \quad (22)$$

for all $t \in [0, T]$. Using (22) in (21), we get

$$\sup_{t \in [0, T]} \|w(t)\|_{H}^2 + \mu \int_0^T \|w(t)\|_{H}^2 \, dt + 2\alpha \int_0^T \|w(t)\|_{H}^2 \, dt + 2\beta \int_0^T \|u(t)\|_{H}^2 \, dt + 2(r - 1)\beta \int_0^T \|w(t)\|_{H}^2 \, dt$$

$$\leq \left\{ \|w_0\|_{H}^2 + \frac{2}{\mu} \int_0^T \|g(t)\|_{H}^2 \, dt \right\} \exp \left( \frac{8}{\mu} \int_0^T \|u(t)\|_{H}^2 \, dt \right), \quad (23)$$

Let us now obtain the estimates for time derivatives. For $r = 2$ and $v \in L^2(0, T; \mathbb{V})$, from the first equation in (19), we find

$$\int_0^T \langle \partial_t w(t), v(t) \rangle \, dt$$

$$= \left| \int_0^T (-\mu \Lambda w(t) - B'(u(t))w(t) - \alpha w(t) - \beta C'(u(t))w(t) + g(t), v(t)) \, dt \right|$$

$$\leq \mu \int_0^T \|w(t)\|_V \|v(t)\|_V \, dt + \int_0^T \|u(t)\|_{L^4_4} \|v(t)\|_V \|w(t)\|_{L^4_4} \, dt$$

$$+ \int_0^T \|w(t)\|_{L^4_4} \|v(t)\|_V \|u(t)\|_{L^4_4} \, dt + \alpha \int_0^T \|w(t)\|_H \|v(t)\|_H \, dt$$

$$+ 2\beta \int_0^T \|u(t)\|_{L^4_4} \|w(t)\|_H \|v(t)\|_H + \int_0^T \|g(t)\|_V \|v(t)\|_V \, dt$$

$$\leq \left\{ \mu \left( \int_0^T \|w(t)\|_V^2 \, dt \right)^{1/2} + 2(1 + \beta) \left( \int_0^T \|u(t)\|_{L^4_4}^2 \|w(t)\|_{L^4_4}^2 \, dt \right)^{1/2} \right\}$$

$$+ \frac{\alpha}{\sqrt{\lambda_1}} \left( \int_0^T \|w(t)\|_H^2 \, dt \right)^{1/2} + \left( \int_0^T \|g(t)\|_V^2 \, dt \right)^{1/2} \right\} \left( \int_0^T \|v(t)\|_V^2 \, dt \right)^{1/2}, \quad (24)$$

so that we get $\|\partial_t w\|_{L^2(0, T; \mathbb{V}')} < \infty$. Thus, it is a consequence of Theorem 3, section 5.9.2, [13] that $w \in C([0, T]; \mathbb{H})$.

For $r = 3$ and $v \in L^2(0, T; \mathbb{V}) \cap L^4(0, T; \mathbb{L}^4)$, the estimate given in (24) holds true except for the estimate for the term involving $C'(u)w$. It can be easily seen that

$$\left| \int_0^T \langle C'(u(t))w(t), v(t) \rangle \, dt \right|$$

$$\leq r \int_0^T \|u(t)\|_{L^4_4}^2 \|w(t)\|_{L^4_4} \|v(t)\|_{L^4_4} \, dt$$

$$\leq r \left( \int_0^T \|u(t)\|_{L^4_4}^{8/3} \|w(t)\|_{L^4_4}^{4/3} \, dt \right)^{3/4} \left( \int_0^T \|v(t)\|_{L^4_4}^2 \, dt \right)^{1/4},$$
so that we find
\[
\left( \int_0^T \|C'(u(t))w(t)\|_{L^{4/3}}^3 \, dt \right)^{3/4} \leq \left( \int_0^T \|u(t)\|_4^4 \, dt \right)^{1/2} \left( \int_0^T \|w(t)\|_{L^4}^4 \, dt \right)^{1/4} < \infty,
\]
and hence \( \partial_t w \in L^2(0, T; \mathbb{V}') + L^{4/3}(0, T; \mathbb{L}^{4/3}) \). But we know that \( \mathbb{V} \subset \mathbb{L}^4 \subset \mathbb{H} \subset L^{4/3} \subset \mathbb{V}' \), and thus we get \( \partial_t w \in L^{4/3}(0, T; \mathbb{V}') \). Therefore an application of Theorem 2, section 5.9.2, [13] yields \( w \in C([0, T]; \mathbb{V}') \). The reflexivity of the space \( \mathbb{H} \) and Proposition 1.7.1 [10] gives \( w \in C_w([0, T]; \mathbb{H}) \) and the map \( t \mapsto \|w(t)\|_\mathbb{H} \) is bounded, where \( C_w([0, T]; \mathbb{H}) \) represents the space of functions \( w : [0, T] \to \mathbb{H} \) which are weakly continuous. Since \( w \in L^4(0, T; \mathbb{L}^4) \), one can show that \( w \in C([0, T]; \mathbb{H}) \) satisfying the energy equality by following similar arguments as in Theorem 4.1, [20]. Note that \( w \in L^2(0, T; \mathbb{V}) \) and \( \partial_t w \in L^{4/3}(0, T; \mathbb{V}') \) implies that the embedding of \( \{w \in L^2(0, T; \mathbb{V}) : \partial_t w \in L^{4/3}(0, T; \mathbb{V}')\} \) into \( L^2(0, T; \mathbb{H}) \) is compact due to an application of the Aubin-Lions compactness theorem (see [27], Theorem 1, [35]). The existence of weak solutions to the system (19) can be proved using a standard Faedo-Galerkin’s approximation technique and we have the following Theorem. Due to the linearity of the system (19), uniqueness of weak solutions follows immediately from the estimate (21).

**Theorem 2.5.** Let \( w_0 \in \mathbb{H} \) and \( f \in L^2(0, T; \mathbb{V}') \) be given. Then, there exists a unique weak solution to the system (19) satisfying
\[
w \in L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{V}),
\]
with \( \partial_t w \in L^p(0, T; \mathbb{V}) \), where \( p = 2 \), for \( r = 1, 2 \) and \( p = 4/3 \), for \( r = 3 \) (so that \( w \in C([0, T]; \mathbb{H}) \)), and the energy equality
\[
\begin{align*}
\|w(t)\|_\mathbb{H}^2 + 2\mu \int_0^t \|w(s)\|_\mathbb{V}^2 \, ds + 2\alpha \int_0^t \|w(s)\|_\mathbb{H}^2 \, ds + 2\beta \int_0^t \|u(s)\|_\mathbb{L}^4 \|w(s)\|_\mathbb{H}^2 \, ds + 2(2r - 1)\beta \int_0^t \|u(s)\|_\mathbb{L}^{4/3} |\tilde{w}(s)\|_\mathbb{H}^2 \, ds \\
+ 2\int_0^t \langle B(w(s), u(s)), w(s) \rangle \, ds + 2\int_0^t \langle g(s), w(s) \rangle \, ds,
\end{align*}
\]
for all \( t \in [0, T] \).

3. **Optimal Control Problem: Existence.** In this section, we formulate a distributed optimal control problem as the minimization of a suitable cost functional subject to the controlled 2D CBF system (15). The main objective of this section is to prove the existence of an optimal control that minimizes the cost functional
\[
J(u, U) := \frac{1}{2} \int_0^T \|u(t) - u_d\|_\mathbb{H}^2 \, dt + \frac{1}{2} \int_0^T \|\nabla \times u(t)\|_\mathbb{H}^2 \, dt + \int_0^T h(U(t)) \, dt + \frac{1}{2} \|u(T) - u'\|_\mathbb{H}^2,
\]
subject to the constraint (15). In (26), \( u_d \in L^2(0, T; \mathbb{H}) \) denotes the desired velocity field (or the reference velocity) and \( u' \in \mathbb{H} \) represents the desired velocity at time \( T \). On the physical background, our goal is to determine the control \( U \) in such a way that the velocity vector is as close as possible, in the sense of (P) (see below), to the desired velocity \( u_d \in L^2(0, T; \mathbb{H}) \) and the turbulence is minimal (enstrophy.
minimization). Since \( u(\cdot) \) is divergence free, we know that \( \| \text{curl} \ u \|_H^2 = \| \nabla \times u \|_H^2 = \| A^{1/2} u \|_H^2 \) and we replace \( \| \nabla \times u \|_H \) with \( \| \nabla u \|_H \) in (26) in the rest of the paper. In the sequel, we impose the following assumption on \( h(\cdot) \).

**Hypothesis 1.** The function \( h : \mathbb{U} \to (-\infty, \infty) \) is convex and lower semicontinuous. Moreover, it satisfies the coercivity condition:

\[
h(U) \geq \kappa_1 \| U \|_U^2 + \kappa_2, \quad \text{for all } U \in \mathbb{U},
\]

for some \( \kappa_1 > 0 \) and \( \kappa_2 \in \mathbb{R} \).

We take the set of all admissible control class \( \mathcal{U}_{ad} = L^2(0, T; \mathbb{U}) \). Next, we provide the definition of admissible class of solutions.

**Definition 3.1 (Admissible class).** The admissible class \( \mathcal{A}_{ad} \) of pairs

\[
(u, U) \in C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathcal{V'}) \times L^2(0, T; \mathbb{U})
\]

is defined as the set of states \( u(\cdot) \) solving the system (15) with the control \( U \in \mathcal{U}_{ad} \). That is,

\[
\mathcal{A}_{ad} := \{(u, U) : u \in C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathcal{V}) \text{ is the unique weak solution of (15)}
\]

with the control \( U \in L^2(0, T; \mathbb{U}) \}.

Clearly \( \mathcal{A}_{ad} \) is a nonempty set, as for any \( U \in \mathcal{U}_{ad} \), there exists a unique weak solution of the system (15). In view of the above definition, the optimal control problem we are considering can be formulated as:

\[
\min_{(u, U) \in \mathcal{A}_{ad}} \mathcal{J}(u, U).
\]

A solution to the problem (P) is called an optimal solution. The optimal pair is denoted by \((u^*, U^*)\) and the control \( U^* \) is called an optimal control.

### 3.1. The adjoint system

In order to establish the first order necessary conditions of optimality, we need to find the adjoint system corresponding to (15). Remember that optimal control is characterized via adjoint variable. Let the adjoint variable \( p(\cdot) \) satisfies the following adjoint system:

\[
\begin{cases}
-\partial_t p(t) + \mu p(t) + (B'(u(t)))^* p(t) + \alpha p(t) + \beta C'(u(t))p(t) = h(t), & \text{in } \mathcal{V}', \\
 p(T) = p_T,
\end{cases}
\]

for a.e. \( t \in [0, T] \), where \( \langle B'(u) p, q \rangle = \langle p, B'(u) q \rangle = \langle B(u, q), p \rangle + \langle B(q, u), p \rangle \) and \( h \in L^2(0, T; \mathcal{V}') \). Let us now obtain an a-priori energy estimate satisfied by \( p \). Taking the inner product with \( p(\cdot) \) to the first equation in (27), we find

\[
-\frac{1}{2} \frac{d}{dt} \| p(t) \|_{\mathcal{H}}^2 + \mu \| p(t) \|_{\mathcal{V}}^2 + \alpha \| p(t) \|_{\mathcal{H}}^2 + \beta \| u(t) \|_{\mathcal{H}}^2 \| p(t) \|_{\mathcal{H}}^2
\]

\[
+ (r - 1) \beta \| u(t) \|_{\mathcal{H}}^2 \| p(t) \|_{\mathcal{H}}^2
\]

\[
= \langle B(p(t), u(t)), p(t) \rangle + \langle h(t), p(t) \rangle
\]

\[
\leq \| u(t) \|_\mathcal{V} \| p(t) \|_{\mathcal{H}}^2 + \| h(t) \|_\mathcal{V} \| p(t) \|_{\mathcal{V}}
\]

\[
\leq \frac{\mu}{2} \| p(t) \|_{\mathcal{V}}^2 + \frac{2}{\mu} \| u(t) \|_{\mathcal{V}}^2 \| p(t) \|_{\mathcal{H}}^2 + \frac{1}{\mu} \| h(t) \|_{\mathcal{V}}^2,
\]

for a.e. \( t \in [0, T] \). Integrating the above inequality from \( t \) to \( T \), we obtain

\[
\| p(T) \|_{\mathcal{H}}^2 + \mu \int_t^T \| p(s) \|_{\mathcal{V}}^2 ds + 2\alpha \int_t^T \| p(s) \|_{\mathcal{H}}^2 ds + 2\beta \int_t^T \| u(s) \|_{\mathcal{H}}^2 \| p(s) \|_{\mathcal{H}}^2 ds
\]

\[
+ (r - 1) \beta \int_t^T \| u(s) \|_{\mathcal{H}}^2 \| p(s) \|_{\mathcal{H}}^2 ds
\]

\[
= \int_t^T \langle B(p(s), u(s)), p(s) \rangle + \langle h(s), p(s) \rangle ds
\]

\[
\leq \int_t^T \| u(s) \|_\mathcal{V} \| p(s) \|_{\mathcal{H}}^2 + \| h(s) \|_\mathcal{V} \| p(s) \|_{\mathcal{V}} ds + \frac{1}{\mu} \int_t^T \| h(s) \|_{\mathcal{V}}^2 ds.
\]
+ 2(r - 1)\beta \int_t^T \| u(s) \|^2 \| \nabla (u(s) \cdot p(s)) \|^2 ds
\leq \| p(T) \|^2_{\mathcal{H}} + \frac{4}{\mu} \int_t^T \| u(s) \|^2 \| p(s) \|^2 ds + \frac{2}{\mu} \int_t^T \| h(s) \|^2 \| \varphi \| ds.
\tag{29}

An application of Gronwall’s inequality in (29) yields
\| p(t) \|^2_{\mathcal{H}} \leq \left( \| p(T) \|^2_{\mathcal{H}} + \frac{2}{\mu} \int_0^T \| h(t) \|^2 \| \varphi \| dt \right) \exp \left( \frac{4}{\mu} \int_0^T \| u(t) \|^2 dt \right),
\tag{30}

for all \( t \in [0, T] \). Since \( u \in L^2(0, T; \mathcal{V}) \), the right hand side of the estimate in (30) is uniformly bounded. In order to obtain the time derivative estimate, for \( r = 2 \) and \( v \in L^2(0, T; \mathcal{V}) \cap L^4(0, T; \mathcal{L}^4) \), we consider
\[
\int_0^T \langle \partial_t p(t), v(t) \rangle dt 
\leq \mu \int_0^T \langle Ap(t), v(t) \rangle dt + \frac{4}{\mu} \int_0^T \| p(t) \|^2 \| v(t) \|^2 dt 
+ \int_0^T \| p(t) \| \langle B(v(t), u(t)) \rangle dt + \alpha \int_0^T \| p(t) \| \langle v(t) \rangle dt 
+ \beta \int_0^T \langle C'(u(t))p(t), v(t) \rangle dt + \int_0^T \| h(t) \| \langle v(t) \rangle dt
\]

so that \( \partial_t p \in L^2(0, T; \mathcal{V}') + L^2(0, T; \mathcal{L}^2) \). For \( r = 3 \), we just need to estimate the term \( \int_0^T \langle C'(u(t))p(t), v(t) \rangle dt \). An application of Hölder’s inequality yields
\[
\int_0^T \| C'(u(t))p(t), v(t) \| dt
\leq 3 \int_0^T \| u(t) \|^2 \| p(t) \|^2 \| v(t) \| dt
\leq 3 \left( \int_0^T \| u(t) \|^2 dt \right)^{1/2} \left( \int_0^T \| p(t) \|^2 dt \right)^{1/4} \left( \int_0^T \| v(t) \|^4 dt \right)^{1/4},
\]
so that \( C'(u)p \in L^2(0, T; \mathcal{V}') + L^2(0, T; \mathcal{L}^2) \). Thus, in both cases, we obtain \( \| \partial_t p \|_{L^2(T; \mathcal{V}') + L^2(T; \mathcal{L}^2)} \) is uniformly bounded. Once again using a Faedo-Galerkin approximation technique, one can obtain the global solvability results of the system (27). The following Theorem gives the global existence and uniqueness of weak solution to the system (27) satisfying the energy equality.
Theorem 3.2. For $h \in L^2(0,T;\mathbb{V}')$, there exists a unique weak solution to the system (27) satisfying

$$p \in L^\infty(0,T;\mathbb{H}) \cap L^2(0,T;\mathbb{V})$$

with $\partial_t p \in L^p(0,T;\mathbb{V}')$, where $p = 2$, for $r = 1$, and $p = \frac{4}{r}$, for $r = 2,3$ (so that $p \in C([0,T];\mathbb{H})$), and

$$\|p(t)\|_H^2 + 2\mu \int_t^T \|p(s)\|_H^2 \, ds + 2\alpha \int_t^T \|p(s)\|_H^2 \, ds + 2\beta \int_t^T \|u(s)\|_H^2 \, ds + 2(r-1)\beta \int_t^T \|u(s)\|_H^2 \|p(s)\|_H^2 \, ds$$

$$= \|p_0\|_H^2 - 2 \int_t^T \langle B(p(s),u(s)), p(s) \rangle \, ds - 2 \int_t^T \langle h(s), p(s) \rangle \, ds,$$

for all $t \in [0,T]$.

3.2. Existence of an optimal control. Our next aim is to show that an optimal pair $(u^*, U^*)$ exists for the problem (P).

Theorem 3.3 (Existence of an optimal pair). Let $u_0 \in \mathbb{H}$ and $f \in L^2(0,T;\mathbb{V}')$ be given. Then there exists at least one pair $(u^*, U^*) \in \mathcal{A}_{ad}$ such that the functional $J(u, U)$ attains its minimum at $(u^*, U^*)$, where $u^*$ is the unique weak solution of the system (15) with the control $U^* \in \mathcal{A}_{ad}$.

Proof. Let us first define $J := \inf_{U \in \mathcal{A}_{ad}} J(u, U)$. Since, $0 \leq J < +\infty$, there exists a minimizing sequence $\{U_n\} \in \mathcal{A}_{ad}$ such that $\lim_{n \to \infty} J(u_n, U_n) = J$, where $u_n$ is the unique weak solution of the system (15) with the control $U_n$ and also the initial data $u_n(0) = u_0 \in \mathbb{H}$. Therefore, we have

$$J \leq J(u_n, U_n) \leq J + \frac{1}{n},$$

where

$$\begin{cases}
\partial_t u_n(t) + \mu A u_n(t) + B(u_n(t)) + \alpha u_n(t) + \beta C(u_n(t)) = f(t) + D U_n(t), \quad \text{in} \quad \mathbb{V}', \\
u_n(0) = u_0,
\end{cases}$$

for a.e. $t \in [0,T]$. From the condition (31), we can find a large constant $\tilde{C} > 0$, such that

$$J \leq \int_0^T \|u_n(t)\|_H^2 \, dt \leq \tilde{C} < +\infty.$$
find
\[ \int_0^T |\partial_t u_n(t), v(t)| dt \]
\[ \leq \mu \int_0^T |(A u_n(t), v(t))| dt + \int_0^T |(B(u_n(t)), v(t))| dt + \alpha \int_0^T |(u_n(t), v(t))| dt \]
\[ + \beta \int_0^T |(C(u_n(t)), v(t))| dt + \int_0^T |(f(t), v(t))| dt + \int_0^T |(DU_n(t), v(t))| dt \]
\[ \leq \left\{ \left( \mu + \frac{\alpha}{\sqrt{\lambda_1}} \right) \left( \int_0^T \|u_n(t)\|_{L_x^2}^3 \right)^{1/2} + \left( 1 + \frac{\beta}{\sqrt{\lambda_1}} \right) \left( \int_0^T \|u_n(t)\|_{L_x^2}^4 \right)^{1/2} \right\} \]
\[ + \left( \int_0^T \|f(t)\|_{L_x^4} \right)^{1/2} + \|D\|_{L(U; V)} \left( \int_0^T \|U_n(t)\|_{L_x^4}^2 \right)^{1/2} \]
\[ \times \left( \int_0^T \|v(t)\|_{L_x^4} \right)^{1/2} . \]

For \( r = 3 \) and \( v \in L^4(0, T; \overline{L}^4) \), we estimate
\[ \int_0^T |(C(u_n(t)), v(t))| dt \leq \int_0^T \|u_n(t)\|_{L_x^4}^3 \|v(t)\|_{L_x^4} \, dt \]
\[ \leq \left( \int_0^T \|u_n(t)\|_{L_x^4}^4 \right)^{3/4} \left( \int_0^T \|v(t)\|_{L_x^4}^4 \right)^{1/4} , \]

so that \( C(u_n) \in L^4(0, T; \overline{L}^4) \). Following similar calculations as in the case of \( r = 2 \), we obtain \( \partial_t u_n \in L^4(0, T; \overline{V}') \) in the case of \( r = 3 \). Hence, by using the Banach-Alagou theorem, we can extract a subsequence \( \{u_{n_k}, U_{n_k}\} \) such that
\[
\begin{align*}
\{ u_{n_k} \to w^* \} & \text{ in } L^\infty(0, T; \mathbb{H}), u_{n_k} \to u^* \text{ in } L^2(0, T; \mathbb{V}), \\
\{ u_{n_k} \to u^* \} & \text{ in } L^{r+1}(0, T; \overline{L}^{r+1}), U_{n_k} \to U^* \text{ in } L^2(0, T; U), \\
\partial_t u_{n_k} & \to \partial_t u \text{ in } L^p(0, T; \mathbb{V}'),
\end{align*}
\]

as \( k \to \infty \), where \( p = 2 \) for \( r = 2 \) and \( p = 3 \) for \( r = 3 \). Using the Aubin-Lion compactness Theorem and the convergence in (33), we infer that
\[ u_{n_k} \to u^* \text{ in } L^2(0, T; \mathbb{H}), \]

(34)
as \( k \to \infty \) and along a subsequence, we obtain \( u_{n_k} \to u^* \), for a.e. \( (t, x) \in [0, T] \times \Omega \).
From the above convergences, one can pass limit in the equation corresponding to \( (u_{n_k}, U_{n_k}) \in \mathcal{A}_{ad} \) and conclude that \( u^*(\cdot) \) is the unique weak solution of (15) with the control \( U^* \in L^2(0, T; U) \). For \( r = 2 \), it is a consequence of the Theorem 3, section 5.9.2, [13] that \( u^* \in C([0, T]; \mathbb{H}) \). For the case of \( r = 3 \), one can follow the same arguments as in [20] to obtain \( u^* \in C([0, T]; \mathbb{H}) \). It should be noted that the initial condition \( u_n(0) = u_0 \in \mathbb{H} \), and the right continuity in time at 0 gives \( u^*(0) = u_0 \in \mathbb{H} \). Furthermore, \( u^*(\cdot) \) satisfies the following energy equality:
\[
\|u^*(t)\|_{\mathbb{H}}^2 + 2\mu \int_0^t \|u^*(s)\|_{\mathbb{V}}^2 ds + 2\alpha \int_0^t \|u^*(s)\|_{\mathbb{H}}^2 ds + 2\beta \int_0^t \|u^*(s)\|_{L_{r+1}}^{r+1} ds
\]
Finally, we show that \((u^*, U^*)\) is a minimizer, that is \(\mathcal{J} = \mathcal{J}(u^*, U^*)\). Since the cost functional \(\mathcal{J}(\cdot, \cdot)\) is continuous and convex (see Proposition III.1.6 and III.1.10, [14]) on \(L^2(0, T; \mathbb{H}) \times \mathcal{A}_{ad}\), it follows that \(\mathcal{J}(\cdot, \cdot)\) is weakly lower semicontinuous (Proposition II.4.5, [14]). That is, for a sequence 
\[(u_n, U_n) \rightharpoonup (u^*, U^*) \text{ in } L^2(0, T; \mathbb{H}) \times L^2(0, T; U),\]
we have 
\[\mathcal{J}(u^*, U^*) \leq \liminf_{n \to \infty} \mathcal{J}(u_n, U_n).\]
Therefore, we obtain 
\[\mathcal{J} \leq \mathcal{J}(u^*, U^*) \leq \liminf_{n \to \infty} \mathcal{J}(u_n, U_n) = \lim_{n \to \infty} \mathcal{J}(u_n, U_n) = \mathcal{J},\]
and hence \((u^*, U^*)\) is an optimizer of the problem \((P)\). \hfill \Box

4. First Order Necessary Conditions of Optimality. In this section, we prove the first order necessary condition of optimality for the optimal control problem \((P)\). We characterize optimal control in terms of the adjoint variable. In this section, we assume the following hypothesis on \(h(\cdot)\). The function \(h : U \to \mathbb{R}\) is convex, lower semicontinuous and there exist positive constants \(x_1\) and \(x_2\) such that:

\[h(U) \leq x_1\|U\|_U^2 + x_2, \quad \text{for all } U \in U.\]

We first provide Pontryagin’s maximum principle in terms of the Hamiltonian formulation. Here we are providing formal calculations only. Let us define the Lagrangian by 
\[\mathcal{L}(u, U) := \frac{1}{2}\|u - u_d\|_U^2 + \frac{1}{2}\|\nabla u\|_H^2 + h(U),\]
so that \(\mathcal{J}(u, U) = \int_0^T \mathcal{L}(u(t), U(t))dt + \frac{1}{2}\|u(T) - u^f\|_U^2\). We define the corresponding Hamiltonian by 
\[\mathcal{H}(u, U, p) := \mathcal{L}(u, U) + \langle p, N(u, U) \rangle,\]
where \(N(u, U) = -\mu A u - B(u) - \alpha u - \beta C(u) + f + U\) and the adjoint variable \(p(\cdot)\) satisfies system (38) given below. Since \((u^*, U^*)\) is an optimal pair for the problem \((P)\), Pontryagin’s minimum principle gives 
\[\mathcal{H}(u^*(t), U^*(t), p(t)) \leq \mathcal{H}(u^*(t), W, p(t)),\]
for all \(W \in U\) and a.e. \(t \in [0, T]\). That is, the following minimum principle is satisfied by an optimal pair \((u^*, U^*) \in \mathcal{A}_{ad}\) obtained in Theorem 3.3:

\[h(U^*(t)) + \langle p(t), DU^*(t) \rangle \leq h(W) + \langle p(t), DW \rangle,\]
for all \(W \in U\), and a.e. \(t \in [0, T]\). For \(\mathcal{A}_{ad} = L^2(0, T; U)\), from (37), we see that \(\partial h(\cdot) \in \mathcal{V}', \text{ a.e. } t \in [0, T]\), where \(\partial h(\cdot)\) denotes the subdifferential of \(h(\cdot)\). Since Pontryagin’s maximum principle provides the first order necessary conditions of optimality, we show the final conclusions in the next theorem. We follow similar techniques as in the works [2, 5, 29, 36], etc to get our main result. The main Theorem of this section is:
Theorem 4.1. Let \((u^*, U^*) \in \mathcal{A}_{ad}\) be an optimal solution for the problem \((P)\) obtained in Theorem 3.3. Then, there exists a unique weak solution \(p \in C([0, T]; \mathbb{V}) \cap L^2(0, T; \mathbb{V})\) of the adjoint system:

\[
\begin{aligned}
-\partial_t p(t) + \mu A p(t) + (B'(u^*(t)))^* p(t) + \alpha p(t) + \beta C'(u^*(t)) p(t) \\
= (u^*(t) - u_d(t)) - \Delta u^*(t), \quad \text{in } \mathbb{V}', \\
p(T) = u^*(T) - u^f,
\end{aligned}
\]

for a.e. \(t \in [0, T]\), such that

\[-D^* p(t) \in \partial h(U^*(t)), \quad \text{a.e. } t \in [0, T].\]

Before embarking to the proof, we prove two important Lemmas, which is useful in establishing the above Theorem. From (30), we infer that

\[
\sup_{t \in [0, T]} \|p(t)\|^2_{L^2} + \mu \int_0^T \|p(t)\|^2_{\mathbb{V}} dt \\
\leq \left(\|u^*(T) - u^f\|^2_{L^2} + \frac{4}{\mu} \int_0^T \|u^*(t) - u_d(t)\|^2_{\mathbb{V}} dt + \frac{4}{\mu} \int_0^T \|u^*(t)\|^2_{\mathbb{V}} dt\right) e^{\frac{\alpha \tau}{2}} K_T,
\]

and we denote the right hand side by \(M_T\), where

\[
K_T = \|u_0\|^2_{L^2} + \frac{2}{\mu} \int_0^T \|f(t)\|^2_{\mathbb{V}} dt + \frac{2}{\mu} \|D\|^2_{L^2(\mathbb{V}', \mathbb{V})} \int_0^T \|U^*(t)\|^2_{L^2} dt.
\]

Lemma 4.2. Let \((u^*, U^*) \in \mathcal{A}_{ad}\) be an optimal pair for the control problem \((P)\) obtained in Theorem 3.3. Let \(u_{U^*+\tau U}\) be the unique weak solution to the system (15) with the control \(U^* + \tau U\), for sufficiently small \(\tau\) and \(U \in \mathcal{U}_{ad}\). Then, we have

\[
\sup_{t \in [0, T]} \|u_{U^*+\tau U}(t) - u_{U^*}(t)\|^2_{L^2} + \mu \int_0^T \|u_{U^*+\tau U}(t) - u_{U^*}(t)\|^2_{\mathbb{V}} dt \\
+ 2\alpha \int_0^T \|u_{U^*+\tau U}(t) - u_{U^*}(t)\|^2_{\mathbb{V}} dt + \frac{\beta}{2\tau - \alpha} \int_0^T \|u_{U^*+\tau U}(t) - u_{U^*}(t)\|^2_{L^{r+1}} dt \\
\leq \left(\frac{2\tau^2}{\mu} \|D\|^2_{L^2(\mathbb{V}', \mathbb{V})} \int_0^T \|U(t)\|^2_{L^2} dt\right) e^{\frac{8K_T}{\mu^{r-1}}},
\]

where \(K_T\) is defined in (40).

Proof. Since \(u_{U^*+\tau U}\) and \(u_{U^*}\) are the unique weak solutions of the system (15) corresponding to the controls \(U^* + \tau U\) and \(U^*\), respectively, we know that \(\bar{u} = u_{U^*+\tau U} - u_{U^*}\) satisfies the following system:

\[
\begin{aligned}
\frac{\partial \bar{u}(t)}{\partial t} + \mu A \bar{u}(t) + B(u_{U^*+\tau U}(t)) - B(u_{U^*}(t)) + \alpha \bar{u}(t) \\
+ \beta (C(u_{U^*+\tau U}(t)) - C(u_{U^*}(t))) & = \tau DU(t), \quad \text{in } [0, T], \\
\bar{u}(0) & = 0.
\end{aligned}
\]

Taking the inner product with \(\bar{u}(\cdot)\) to the first equation in (42), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\bar{u}(t)\|^2_{L^2} + \mu \|\bar{u}(t)\|^2_{\mathbb{V}} + \alpha \|\bar{u}(t)\|^2_{L^2} \\
= -(B(u_{U^*+\tau U}(t)) - B(u_{U^*}(t)), \bar{u}(t)) - \beta (C(u_{U^*+\tau U}(t)) - C(u_{U^*}(t)), \bar{u}(t)) \\
+ \tau \langle DU(t), \bar{u}(t)\rangle
\]
where we have used (7) and (12). Integrating the above inequality from 0 to \( t \), we deduce that
\[
\begin{align*}
\| \bar{u}(t) \|^2_H + \mu \int_0^t \| \bar{u}(s) \|^2_H ds + 2\alpha \int_0^t \| \bar{u}(s) \|^2_H ds + \frac{\beta^2}{2} \int_0^t \| \bar{u}(s) \|^2_{L^2(U,V)} ds \\
= 4 \mu \int_0^t \| u_{U^*}(s) \|^2_H ds + \frac{2\alpha}{\mu} \int_0^t \| D(U(s)) \|^2_{L^2(U,V)} ds, \\
\end{align*}
\]
for all \( t \in [0,T] \). An application of Gronwall’s inequality in (44) yields
\[
\| \bar{u}(t) \|^2_H \leq \left\{ \frac{2\alpha}{\mu} \int_0^t \| u_{U^*}(s) \|^2_H ds + \frac{4}{\mu} \int_0^t \| u_{U^*}(t) \|^2_{L^2(U,V)} dt \right\} \exp \left( \frac{4}{\mu} \int_0^t \| u_{U^*}(t) \|^2_{L^2(U,V)} dt \right),
\]
for all \( t \in [0,T] \), which completes the proof. \( \square \)

The following lemma gives the differentiability of the mapping \( U \mapsto u_U \) from \( \mathcal{Y}_{ad} \) into \( C([0,T]; \mathbb{H}) \cap L^2(0,T; \mathbb{V}) \).

**Lemma 4.3.** Let \( u_0 \in \mathbb{H} \) and \( f \in L^2(0,T; \mathbb{V}') \) be given. Then the mapping \( U^* \mapsto u_{U^*} \) from \( \mathcal{Y}_{ad} \) into \( C([0,T]; \mathbb{H}) \cap L^2(0,T; \mathbb{V}) \) is Gateaux differentiable. Moreover, we have
\[
\lim_{\tau \downarrow 0} \frac{\| u_{U^*+\tau U}(t) - u_{U^*}(t) - \tau w(t) \|_\mathbb{H}}{\tau} = 0, \quad \text{for all} \quad t \in [0,T],
\]
\[
\lim_{\tau \downarrow 0} \frac{\| u_{U^*+\tau U} - u_{U^*} - \tau w \|_{L^2(0,T;\mathbb{V})}}{\tau} = 0,
\]
where \( w(\cdot) \) is the unique weak solution of the linearized system:
\[
\begin{cases}
\partial_t w(t) + \mu Aw(t) + B'(u_{U^*}(t))w(t) + \alpha w(t) + \beta C'(u_{U^*}(t))w(t) = DU(t), & \text{in} \ \mathbb{V}', \\
w(0) = 0,
\end{cases}
\]
a.e. \( t \in [0,T] \), and \( u_{U^*}(\cdot) \) and \( u_{U^*+\tau U}(\cdot) \) are the unique weak solutions of the controlled system (15) with the controls \( U^* \) and \( U^* + \tau U \), respectively.

**Proof.** Let us define \( y := u_{U^*+\tau U} - u_{U^*} - \tau w \). Then \( y(\cdot) \) satisfies the following system:
\[
\begin{cases}
\partial_t y(t) + \mu Ay(t) + B'(u_{U^*}(t))y(t) + \alpha y(t) + \beta C'(u_{U^*}(t))y(t) \\
= -[B(u_{U^*+\tau U}(t)) - B(u_{U^*}(t)) - B'(u_{U^*}(t))(u_{U^*+\tau U}(t) - u_{U^*}(t))] \\
- \beta [C(u_{U^*+\tau U}(t)) - C(u_{U^*}(t)) - C'(u_{U^*}(t))(u_{U^*+\tau U}(t) - u_{U^*}(t))] \\
y(0) = 0,
\end{cases}
\]
in \( \mathbb{V}' \), for a.e. \( t \in [0,T] \). Remember that the terms
\[
B(u_{U^*+\tau U} - u_{U^*}, u_{U^*+\tau U} - u_{U^*}) = B(u_{U^*+\tau U}) - B(u_{U^*}) - B'(u_{U^*})(u_{U^*+\tau U} - u_{U^*})
\]
and
\[
C(u_{U^*+\tau U}) - C(u_{U^*}) - C'(u_{U^*})(u_{U^*+\tau U} - u_{U^*})
\]
belong to $L^p(0,T; V')$, where $p = 2$, for $r = 2$ and $p = 4/3$, for $r = 3$, and hence the system (48) has a unique weak solution with $y \in C([0,T]; \mathbb{H}) \cap L^2(0,T; V)$. Taking the inner product with $y(\cdot)$ to the first equation in (48), we find
\[
\frac{1}{2} \frac{d}{dt} \|y(t)\|_H^2 + \mu \|y(t)\|_V^2 + \alpha \|y(t)\|_2^2 + \beta \|u_{U^*}(t)\|_{L^2}^2 \|y(t)\|_H^2 \leq -\langle B(y(t), u_{U^*}(t)), y(t) \rangle - \langle B(u_{U^*+t}, u_{U^*} + u_{U^*}(t) - u_{U^*}(t)), y(t) \rangle
\] 
\[ - \beta \langle (C(u_{U^*+t}) - C(u_{U^*}))(u_{U^*+t} - u_{U^*}(t)), y(t) \rangle. \quad (49)
\]
Making use of Hölder’s, Ladyzhenskaya’s and Young’s inequalities, we estimate
\[ -\langle B(y, u_{U^*}), y \rangle \]
\[ \leq \|u_{U^*}\|_V \|y\|_V^2 \leq \sqrt{2} \|u_{U^*}\|_V \|y\|_V \|\leq \frac{\mu}{4} \|y\|_V^2 + \frac{2}{\mu} \|u_{U^*}\|_V^2 \|y\|_H^2. \quad (50)
\]
A similar calculation yields
\[ -\langle B(u_{U^*+t} - u_{U^*}, u_{U^*+t} - u_{U^*}), y(t) \rangle \leq \frac{\mu}{4} \|y\|_V^2 + \frac{2}{\mu} \|u_{U^*+t} - u_{U^*}\|_{L^2}^2. \]
To estimate the term $-\beta \langle (C(u_{U^*+t}) - C(u_{U^*}))(u_{U^*+t} - u_{U^*})), y \rangle$, we consider the cases $r = 2$ and $r = 3$ separately. For $r = 2$, an application of Taylor’s formula (see Theorem 7.9.1, [9]), Hölder’s and Young’s inequalities yield
\[ -\beta \langle (C(u_{U^*+t}) - C(u_{U^*}))(u_{U^*+t} - u_{U^*}), y \rangle \]
\[ = -\beta \left( \int_0^1 C'(\theta u_{U^*+t} + (1 - \theta) u_{U^*})d\theta (u_{U^*+t} - u_{U^*}) \right) \]
\[ - C'(u_{U^*})(u_{U^*+t} - u_{U^*}), y \right) \]
\[ = -\beta \int_0^1 \left( \|\theta u_{U^*+t} + (1 - \theta) u_{U^*} - u_{U^*}\| (u_{U^*+t} - u_{U^*}), y \right) \]
\[ - \beta \int_0^1 \left( \left[ \frac{\theta u_{U^*+t} + (1 - \theta) u_{U^*}}{\|\theta u_{U^*+t} + (1 - \theta) u_{U^*}\|} \right] (u_{U^*+t} - u_{U^*}), y \right) \]
\[ \leq \beta \sup_{0 < \theta < 1} \langle \theta (u_{U^*+t} - u_{U^*})\| u_{U^*+t} - u_{U^*}, \|y \rangle \]
\[ + 3\beta \sup_{0 < \theta < 1} \langle \theta (u_{U^*+t} + (1 - \theta) u_{U^*} - u_{U^*}) u_{U^*+t} - u_{U^*}, \|y \rangle \]
\[ \leq 4\beta \|u_{U^*+t} - u_{U^*}\|_{L^2}^2 \|y\|_V \]
\[ \leq \frac{\mu}{4} \|y\|_V^2 + \frac{16}{\mu \lambda_1} \|u_{U^*+t} - u_{U^*}\|_{L^2}^2. \quad (51)
\]
Combining (50)-(51), substituting it in (49) and then integrating from 0 to t, we obtain
\[ \|y(t)\|_H^2 + \frac{\mu}{2} \int_0^t \|y(s)\|_V^2 ds + 2\alpha \int_0^t \|y(s)\|_2^2 ds + 2\beta \int_0^t \|u_{U^*}(s)\|_{L^2}^2 \|y(s)\|_H^2 ds \]
\[ \leq \frac{4}{\mu} \int_0^t \|u_{U^*}(s)\|_V^2 \|y(s)\|_H^2 ds + \frac{2}{\mu} \left( 2 + \frac{16}{\lambda_1} \right) \int_0^t \|u_{U^*+t} - u_{U^*}(s)\|_{L^2}^2 ds. \quad (52)
\]
An application of Gronwall’s inequality in (52) yields
\[
\|y(t)\|_{\mathbb{H}}^2 \\
\leq \left\{ \frac{2}{\mu} \left( 2 + \frac{16}{\lambda_1} \right) \int_0^T \|u_{U^*+\tau U}(t) - u_{U^*}(t)\|_{\mathbb{L}_4}^2 dt \right\} \exp \left( \frac{4}{\mu} \int_0^T \|u_{U^*}(t)\|_{\mathbb{H}}^2 dt \right) \\
\leq \frac{2}{\mu^2} \left( 2 + \frac{16}{\lambda_1} \right) \left\{ \frac{2\tau^2}{\mu} \|D\|^2_{\mathbb{L}(U,V')} \int_0^T \|U(t)\|_{\mathbb{H}}^2 dt \right\}^2 e^{\frac{12K_T}{\mu^2}},
\]
(53)
for all \( t \in [0, T] \), where we have used (41). Passing \( \tau \to 0 \) in (53) yields the required result (46) for the case \( r = 2 \).

For \( r = 3 \), let us estimate the term
\[-\beta \langle [C(u_{U^*+\tau U}) - C(u_{U^*}) - C'(u_{U^*})(u_{U^*+\tau U} - u_{U^*})], y \rangle\]
as follows. Once again an application of Taylor’s formula gives (see Theorem 6.5, [11])
\[-\beta \langle [C(u_{U^*+\tau U}) - C(u_{U^*}) - C'(u_{U^*})(u_{U^*+\tau U} - u_{U^*})], y \rangle \]
\[\leq \frac{2}{\mu} \sup_{0 < \theta < 1} \|C'(\theta u_{U^*} + (1 - \theta)u_{U^*})(u_{U^*+\tau U} - u_{U^*}) \otimes (u_{U^*+\tau U} - u_{U^*})\|_{\mathbb{L}_4/3} \]
\[\times \|y\|_{\mathbb{L}_4} \]
\[\leq 3\beta \sup_{0 < \theta < 1} \|\theta u_{U^*} \|_{\mathbb{L}_4} \|u_{U^*+\tau U} - u_{U^*}\|_{\mathbb{L}_4}^2 \|y\|_{\mathbb{L}_4} \]
\[\leq 2^{1/4} \beta \left( \|u_{U^*+\tau U}\|_{\mathbb{L}_4} + \|u_{U^*}\|_{\mathbb{L}_4} \right) \|u_{U^*+\tau U} - u_{U^*}\|^2_{\mathbb{L}_4} \|y\|_{\mathbb{H}}^{1/2} \|y\|_{\mathbb{Y}}^{1/2} \]
\[\leq \frac{\mu}{4} \|y\|_{\mathbb{Y}}^2 + \frac{6^{1/3} \beta^{4/3}}{4\mu^{1/3}} \left( \|u_{U^*+\tau U}\|_{\mathbb{L}_4} + \|u_{U^*}\|_{\mathbb{L}_4} \right)^{4/3} \|u_{U^*+\tau U} - u_{U^*}\|^{8/3}_{\mathbb{L}_4} \|y\|_{\mathbb{H}}^{2/3} \]
\[\leq \frac{\mu}{4} \|y\|_{\mathbb{Y}}^2 + \left( \|u_{U^*+\tau U}\|_{\mathbb{L}_4}^2 + \|u_{U^*}\|_{\mathbb{L}_4}^2 \right) \|y\|_{\mathbb{L}_4}^2 + \frac{9\beta^2}{\sqrt{2\mu}} \|u_{U^*+\tau U} - u_{U^*}\|^2_{\mathbb{L}_4},
\]
(54)
and a calculation similar to (53) yields the required result. \( \square \)

Let us now prove the main Theorem 4.1 by invoking Lemma 4.2 and Lemma 4.3.

**Proof of Theorem 4.1.** Let \((u^*, U^*) \in \mathcal{A}_{ad}\) be an optimal pair for the problem (P) established in Theorem 3.3. Let \(G(U) = \mathcal{J}(u, U), \) where \((u, U)\) is the unique weak solution to the controlled system (15) with the control \(U \in \mathcal{A}_{ad}\). Let \(\tau\) be sufficiently small such that \(U^{*} + \tau U \in \mathcal{A}_{ad}\) and \((u_{U^*+\tau U}, U^* + \tau U) \in \mathcal{A}_{ad}\). Then, we have (cf. [5, 29], etc)
\[G(U^*+\tau U) - G(U^*) = \mathcal{J}(u_{U^*+\tau U}, U^* + \tau U) - \mathcal{J}(u_{U^*}, U^*) \]
\[= \frac{1}{2} \int_0^T \|u_{U^*+\tau U}(t) - u_d(t)\|_{\mathbb{H}}^2 dt - \frac{1}{2} \int_0^T \|u_{U^*}(t) - u_d(t)\|_{\mathbb{H}}^2 dt \]
\[+ \frac{1}{2} \int_0^T \|\nabla u_{U^*+\tau U}(t)\|_{\mathbb{H}}^2 dt - \frac{1}{2} \int_0^T \|\nabla u_{U^*}(t)\|_{\mathbb{H}}^2 dt \]
\[+ \int_0^T h(U^*(t) + \tau U(t))dt - \int_0^T h(U^*(t))dt \]
\[+ \frac{1}{2} \|u_{U^*+\tau U}(T) - u(t)\|_{\mathbb{H}}^2 - \frac{1}{2} \|u_{U^*}(T) - u(t)\|_{\mathbb{H}}^2 \]
We also represent the directional derivative of $G$ have the convergence (46). Dividing by $\|u\|_H$, using the estimate (41) (see Lemma 4.2), we know that for all $t \in [0,T]$ and $\|u\|_H = 1$, we denote the directional derivative of $G$ in the direction $u$ as

$$h'(u^*, u) = \lim_{\lambda \to 0} \frac{h(u^* + \lambda u) - h(u^*)}{\lambda}.$$ 

We also represent the directional derivative of $G$ at $u^*$ in the direction of $U$ by $G'(u^*, U)$. Let $\nu(\cdot)$ satisfy the linearized system (47). From Lemma 4.3, we also have the convergence (46). Dividing by $\tau$ and then taking $\tau \to 0$ in (55), we obtain

$$0 \leq G'(u^*(t), U(t)) = \lim_{\tau \to 0} \frac{G(u^*(t) + \tau U(t)) - G(u^*(t))}{\tau}$$

$$= \int_0^T \langle \nu(t), u_{U^*}(t) - u_{\delta}(t) - \Delta u_{U^*}(t) \rangle dt$$

$$+ \int_0^T h'(u^*(t), U(t)) dt + (\nu(T), u_{U^*}(T) - u_f)$$

$$= \int_0^T \langle \nu(t), -\partial_t p(t) + \mu A p(t) + (B'(u_{U^*}(t)) \nu(t) + \alpha p(t) + \beta C'(u_{U^*}(t)) \nu(t) \rangle dt$$

$$+ \int_0^T h'(u^*(t), U(t)) dt$$

$$= \int_0^T \langle \partial_t \nu(t) + \mu A \nu(t) + B'(u_{U^*}(t)) \nu(t) + \alpha \nu(t) + \beta C'(u_{U^*}(t)) \nu(t), p(t) \rangle dt$$

$$+ \int_0^T h'(u^*(t), U(t)) dt$$

$$= \int_0^T (D'U(t), p(t)) dt + \int_0^T h'(u^*(t), U(t)) dt,$$ 

where we performed an integration by parts. Thus, from (56), we infer that

$$0 \leq G'(u^*(t), U(t)) = \int_0^T (U(t), D^*p(t)) dt + \int_0^T h'(u^*(t), U(t)) dt,$$

for all $U \in L^2(0, T; U)$. Hence, from the above relation, it is immediate that

$$-D^*p(t) \in \partial h(u^*(t)), \ a.e. \ t \in [0,T],$$

which completes the proof.  \qed
1. If we take \( h(U) = \frac{1}{2} \| U \|_2^2 \), then the function is Gateaux differentiable and the subdifferential contains a single point and from the relation (57), we find

\[
U^*(t) = -D^* p(t), \quad \text{a.e. } t \in [0, T].
\]

2. The difficulty for considering the case for \( r \in (3, \infty) \) is that the estimates similar to (31) and (54) are not controllable for \( r > 3 \). Thus, we are not able to obtain the convergence (46) in Lemma 4.3, for \( r > 3 \). One may think of modifying the admissible class \( \mathcal{A}_{ad} \) to the class of strong solutions \( u \in C([0, T]; V) \cap L^2(0, T; D(A)) \), with \( u_0 \in V \), \( f \in L^2(0, T; \mathbb{H}) \) and \( D \in L(U; \mathbb{H}) \). To the best of author’s knowledge, such strong solutions exist, for \( r \in [1, 3] \) only. As discussed in [23], we know that in bounded domains, \( P(|u|^{-1}u) \) need not be zero on the boundary, and \( P \) and \( -\Delta \) are not necessarily commuting (for a counterexample, see Example 2.19, [33]). Thus handling the term \( (\mathcal{C}(u), \Lambda u) = (P(|u|^{-1}u), \Lambda u) \), for \( r > 3 \) is difficult. Moreover, \( \Delta u \cdot n \big|_{\partial \Omega} \neq 0 \) in general and the term with pressure will not disappear (see [23]), while taking the inner product with \( -\Delta u \) to the first equation in (1). Therefore, the equality (22)

\[
\int_{\Omega} (-\Delta u(x)) \cdot |u(x)|^{-1} u(x) \, dx
= \int_{\Omega} |\nabla u(x)|^2 |u(x)|^{-1} \, dx + 4 \left[ \frac{r-1}{(r+1)^2} \right] \int_{\Omega} |\nabla |u(x)|^{-\frac{r}{2}}| \, dx
= \int_{\Omega} |\nabla u(x)|^2 |u(x)|^{-1} \, dx + \frac{r-1}{4} \int_{\Omega} |u(x)|^{-1} |\nabla |u(x)|^{-\frac{r}{2}}| \, dx,
\]

may not be useful in the context of bounded domains.

4.1. **Uniqueness of optimal control in small time interval.** In this subsection, we show the uniqueness of optimal control in small time interval for the optimal control problem (P) obtained in Theorem 3.3. Remember that the control to state mapping is nonlinear and getting global in time unique optimal control is difficult. Thus, we look for a time \( T > 0 \) such that this time ensures the uniqueness of optimal control. If we choose the final time \( T \) to be sufficiently small, then the state equation for \( u(\cdot) \) differ from the corresponding linearized problem \( w(\cdot) \) slightly only. In this case, the linearized state equation produces a strictly convex cost functional and the corresponding optimal control is unique. Here, we take \( h(U) = \frac{1}{2} \| U \|_2^2 \) in (26).

**Theorem 4.4.** Let \( (u^*, U^*) \in \mathcal{A}_{ad} \) be an optimal pair for the problem (P) with \( h(U) = \frac{1}{2} \| U \|_2^2 \). Then, if the final time \( T \) is sufficiently small and \( \mu \) is large, then an optimal pair \( (u^*, U^*) \in \mathcal{A}_{ad} \) obtained in Theorem 3.3 is unique.

**Proof.** We assume that there exists an another optimal pair \( (\tilde{u}, \tilde{U}) \in \mathcal{A}_{ad} \). From Theorem 4.1, we know that \( U^*(t) = -D^* p^*(t) \in U \), a.e. \( t \in [0, T] \), and \( \tilde{U}(t) = -D^* \tilde{p}(t) \in U \), a.e. \( t \in [0, T] \). Note also that \( p^*(\cdot) \) satisfies the adjoint system (38) and \( \tilde{p}(\cdot) \) satisfies the adjoint system (38) with \( u^*(\cdot) \) replaced by \( \tilde{u}(\cdot) \), and the forcing \( (u^*(\cdot) - u_d(\cdot)) - \Delta u^*(\cdot) \) replaced by \( (\tilde{u}(\cdot) - u_d(\cdot)) - \Delta \tilde{u}(\cdot) \). Thus, we have

\[
\int_0^T \| D(U^*(t) - \tilde{U}(t)) \|_{V'}^2 \, dt = \int_0^T \| p^*(t) - \tilde{p}(t) \|_{V'}^2 \, dt \leq \frac{1}{\gamma^2} \int_0^T \| p^*(t) - \tilde{p}(t) \|_{V'}^2 \, dt.
\]
The quantity on the right hand side of the inequality (59) can be estimated similarly as in (30), as the system satisfied by \( p(\cdot) = \bar{p}(\cdot) - p^*(\cdot) \) is given by

\[
\begin{cases}
-\partial_t p(t) + \mu A p(t) + (B'(\bar{u}(t)))^* \bar{p}(t) - (B'(u^*(t)))^* p^*(t) + \alpha p(t) \\
+ \beta(C'(\bar{u}(t))\bar{p}(t) - C'(u^*(t))p^*(t)) = u(t) - \Delta u(t),
\end{cases}
\]

\( p(T) = u(T), \)

in \( \mathbb{V} \), for a.e. \( t \in [0, T] \), where \( u = \bar{u} - u^* \). Taking the inner product with \( p(\cdot) \) to the first equation in (60) and then integrating from \( t \) to \( T \), we find

\[
\|p(t)\|_\mathbb{H}^2 + 2\mu \int_t^T \|p(s)\|_\mathbb{V}^2 ds + 2\alpha \int_t^T \|p(s)\|_\mathbb{H}^2 ds = \|u(T)\|_\mathbb{H}^2 - 2 \int_t^T \langle (B'(\bar{u}(s)))^* \bar{p}(s) - (B'(u^*(s)))^* p^*(s), p(s) \rangle ds \\
- 2\beta \int_t^T \langle C'(\bar{u}(s))\bar{p}(s) - C'(u^*(s))p^*(s), p(s) \rangle ds + 2 \int_t^T \langle u(s) - \Delta u(s), p(s) \rangle ds,
\]

for all \( t \in [0, T] \). Note that

\[
\langle (B'(\bar{u}))^* \bar{p} - (B'(u^*))^* p^*, p \rangle = \langle (B'(\bar{u}))^* p, p \rangle + \langle (B'(\bar{u}))^* - (B'(u^*))^* p^*, p \rangle = \langle B(p, \bar{u}), p \rangle + \langle B(u, p), p^* \rangle + \langle B(p, u), p^* \rangle.
\]

Using Hölder’s, Ladyzhenskaya’s and Young’s inequalities, we estimate the terms in the right hand side of the inequality (62) as

\[
\|B(p, \bar{u})\| \leq \|\bar{u}\|_\mathbb{V} \|p\|_\mathbb{H}^2 \leq \sqrt{2} \|\bar{u}\|_\mathbb{V} \|p\|_\mathbb{H} \|p\|_\mathbb{V} \leq \frac{\mu}{8} \|p\|_\mathbb{V}^2 + \frac{4}{\mu} \|\bar{u}\|_\mathbb{V} \|p\|_\mathbb{H}^2, \quad (63)
\]

\[
\|B(u, p)^*\| \leq \|u\|_\mathbb{H} \|p\|_\mathbb{H} \|p^*\|_\mathbb{H} \leq \frac{\mu}{8} \|p\|_\mathbb{V}^2 + \frac{2}{\mu} \|u\|_\mathbb{H} \|p^*\|_\mathbb{H}^2, \quad (64)
\]

\[
\|B(p, u)^*\| \leq \|p\|_\mathbb{H} \|u\|_\mathbb{H} \|p^*\|_\mathbb{H} \leq 2^{1/4} \|p\|_\mathbb{H}^{1/2} \|p^*\|_\mathbb{H}^{1/2} \|u\|_\mathbb{H} \|p^*\|_\mathbb{H} \leq \frac{\mu}{8} \|p\|_\mathbb{V}^2 + \frac{1}{2} \|p^*\|_\mathbb{H} \|p\|_\mathbb{H}^2 + \frac{1}{\sqrt{2} \mu} \|u\|_\mathbb{V}^2. \quad (65)
\]

For \( r = 2 \), the term \(-2\beta \langle C'(\bar{u})\bar{p} - C'(u^*)p^*, p \rangle\) can be estimated as

\[
-2\beta \langle C'(\bar{u})\bar{p} - C'(u^*)p^*, p \rangle = -2\beta \langle C'(\bar{u})p, p \rangle - 2\beta \langle (C'(\bar{u}) - C'(u^*))p^*, p \rangle \\
\leq -2\beta \langle (\|u\| - \|u^*\|)p^*, p \rangle - 2\beta \langle \frac{\bar{u}}{\|u\|}(\bar{u} \cdot p^*) - \frac{u^*}{\|u^*\|}(u^* \cdot p^*), p \rangle \\
\leq 8\beta \|u\|_\mathbb{H} \|p^*\|_\mathbb{H} \|p\|_\mathbb{H} \|p\|_\mathbb{H} \leq \frac{\mu}{2} \|p\|_\mathbb{V}^2 + \frac{32\beta^2}{\lambda_1 \mu} \|u\|_\mathbb{H} \|p^*\|_\mathbb{H} \|p\|_\mathbb{H}^2, \quad (66)
\]

since \( \langle C'(\bar{u})p, p \rangle \geq 0 \). Similarly, we obtain

\[
\|u - \Delta u, p\| \leq \|u\|_\mathbb{H} \|p\|_\mathbb{H} + \|u\|_\mathbb{V} \|p\|_\mathbb{V} \leq \frac{\mu}{8} \|p\|_\mathbb{V}^2 + \frac{4}{\mu} \left(1 + \frac{1}{\lambda_1}ight) \|u\|_\mathbb{V}^2. \quad (67)
\]
Combining (63)-(66) and substituting it in (61), we get
\[
\|p(t)\|_{H}^2 + \frac{\mu}{2} \int_{t}^{T} \|p(s)\|_{V}^2 ds + 2\alpha \int_{t}^{T} \|p(s)\|_{H}^2 ds \\
\leq \|u(T)\|_{H}^2 + \frac{8}{\mu} \int_{t}^{T} \|\tilde{u}(s)\|_{H}^2 \|p(t)\|_{V}^2 ds + \frac{4}{\mu} \left(1 + \frac{16\beta^2}{\lambda_1 \mu}\right) \int_{t}^{T} \|u(s)\|_{H}^2 \|p^*(s)\|_{V}^2 ds \\
+ \int_{t}^{T} \|p^*(s)\|_{V}^2 \|p(s)\|_{H}^2 ds + \left[\sqrt{\frac{2}{\mu}} + \frac{8}{\mu} \left(1 + \frac{1}{\lambda_1^2}\right)\right] \int_{t}^{T} \|u(s)\|_{H}^2 ds.
\] (68)

An application of Gronwall’s inequality in (68) yields
\[
\|p(t)\|_{H}^2 \leq \left\{ \|u(T)\|_{H}^2 + \frac{4}{\mu} \left(1 + \frac{16\beta^2}{\lambda_1}\right) \left(\int_{0}^{T} \|u(t)\|_{H}^2 dt\right) \right\}^{1/2} \left(\int_{0}^{T} \|p^*(t)\|_{V}^2 dt\right)^{1/2} \\
+ \left[\sqrt{\frac{2}{\mu}} + \frac{8}{\mu} \left(1 + \frac{1}{\lambda_1^2}\right)\right] \int_{0}^{T} \|u(t)\|_{H}^2 dt \\
\times \exp\left\{ \frac{8}{\mu} \int_{0}^{T} \|\tilde{u}(t)\|_{V}^2 dt + \int_{0}^{T} \|p^*(t)\|_{V}^2 dt \right\},
\] (69)
for all \(t \in [0, T]\). Using (69) in (68) and then taking \(t = 0\), we find
\[
\int_{0}^{T} \|p(t)\|_{V}^2 dt \\
\leq \frac{2}{\mu} \left\{ \|u(T)\|_{H}^2 + \frac{4}{\mu} \left(1 + \frac{16\beta^2}{\lambda_1}\right) \left(\int_{0}^{T} \|u(t)\|_{H}^2 dt\right) \right\}^{1/2} \left(\int_{0}^{T} \|p^*(t)\|_{V}^2 dt\right)^{1/2} \\
+ \left[\sqrt{\frac{2}{\mu}} + \frac{8}{\mu} \left(1 + \frac{1}{\lambda_1^2}\right)\right] \int_{0}^{T} \|u(t)\|_{H}^2 dt \\
\times \exp\left\{ \frac{16}{\mu} \int_{0}^{T} \|\tilde{u}(t)\|_{V}^2 dt + 2 \int_{0}^{T} \|p^*(t)\|_{V}^2 dt \right\} \\
\leq \frac{2}{\mu} \int_{0}^{T} \|D(U^*(t) - \tilde{U}(t))\|_{V}^2 dt \exp\left(\frac{2\gamma}{\nu} + M_T\right) \\
\times \frac{2}{\mu^2} \left(1 + \frac{4}{\mu^2} \left(1 + \frac{16\beta^2}{\lambda_1}\right) M_T + \left[\sqrt{\frac{2}{\mu}} + \frac{8}{\mu} \left(1 + \frac{1}{\lambda_1^2}\right)\right]\right) \\
\times \exp\left(\frac{4}{\mu^3} \int_{0}^{T} \|D(U^*(t) - \tilde{U}(t))\|_{V}^2 dt \right) \exp\left(\frac{8\gamma}{\nu}\right),
\] (70)
where we have used (41) and (30). Combining (59) and (70), it can be easily seen that
\[
\int_{0}^{T} \|D(U^*(t) - \tilde{U}(t))\|_{V}^2 dt \\
\leq \left\{ \frac{4}{\lambda_1^2 \mu^2} \int_{0}^{T} \|D(U^*(t) - \tilde{U}(t))\|_{V}^2 dt \right\} \exp\left(\frac{2\gamma}{\nu} + M_T\right) \\
\times \left(1 + \frac{4}{\mu^2} \left(1 + \frac{16\beta^2}{\lambda_1}\right) M_T + \left[\sqrt{\frac{2}{\mu}} + \frac{8}{\mu} \left(1 + \frac{1}{\lambda_1^2}\right)\right]\right) \exp\left(\frac{8\gamma}{\nu}\right)
\[ \times \exp \left\{ \frac{16}{\mu^3} \int_0^T \|D(U^*(t) - \hat{U}(t))\|_{L^2}^2 \, dt \right\} \exp \left( \frac{aK_T}{\mu} \right). \]

From the above relation, one can choose a time \( T \) sufficiently small such that
\[ \int_0^T \|D(U^*(t) - \hat{U}(t))\|_{L^2}^2 \, dt \leq 0. \]
Thus, we obtain \( U^*(t) = \hat{U}(t) \), a.e. \( t \in [0, T] \), for sufficiently small \( T \) and large \( \mu \).

For the case \( r = 3 \), we need to estimate the term \(-2\beta \langle C'(\bar{u})\bar{p} - C'(u^*)p^*, p \rangle\) only. An application of Taylor’s formula yields
\[
-2\beta \langle C'(\bar{u})\bar{p} - C'(u^*)p^*, p \rangle \\
\leq -2\beta \langle (C'(\bar{u}) - C'(u^*))p^*, p \rangle \\
\leq \beta \sup_{0 < \theta < 1} \|C'((\theta\bar{u} + (1-\theta)u^*)(u \otimes p^*))\|_{L^2} \|p\|_{L^2} \\
\leq 2^{1/4} \beta \|\bar{u}\|_{L^4} \|u\|_{L^4} \|p\|_{L^4} \|p\|_{L^2} \|p\|_{L^2}^{1/2} \\
\leq \frac{\mu}{2} \|p\|_{L^2}^2 + \frac{3}{4\mu^{1/3}} (6\beta)^{4/3} (\|u\|_{L^4} \|u\|_{L^4} \|p\|_{L^4} \|p\|_{L^2}^{1/2} + \|u\|_{L^4}^{4/3} \|p\|_{L^4}^{4/3} \|p\|_{L^2}^{2/3}) \\
\leq \frac{\mu}{2} \|p\|_{L^2}^2 + \frac{1}{2} \|p\|_{L^4}^4 \|p\|_{L^2}^2 + \frac{18\sqrt{2}\beta^2}{\sqrt{\mu}} \|u\|_{L^4}^4 \|u\|_{L^4}^4 + \frac{72\sqrt{2}\beta^2}{\sqrt{\mu}} \|u^*\|_{L^4}^4 \|u\|_{L^4}^4, \] (71)
and the proof can be completed using similar arguments as in the case of \( r = 2 \). \( \square \)

5. Second Order Necessary and Sufficient Conditions of Optimality. In this section, we obtain the second order necessary and sufficient conditions of optimality for the problem \((P)\). We prove it for the case \( r = 3 \) only, due to the lack of second order Gateaux derivative of \( C(u) \) for \( r = 2 \). Let \((\bar{u}, \bar{U}) \in A_{ad}\) be an arbitrary feasible pair for the optimal control problem \((P)\). Let us set
\[
M_{\bar{u}, \bar{U}} := \{(u, U) \in A_{ad}\} \setminus \{\bar{u}, \bar{U}\},
\] (72)
which denotes the differences of all feasible pairs of the problem \((P)\) with \((\bar{u}, \bar{U})\).

The next two Theorems provide the second order necessary and sufficient optimality conditions for the optimal control problem \((P)\) for the case \( r = 3 \). In this section, we take \( h(U) = \frac{1}{2} \|U\|_{L^2}^2 \) in (26). Motivated from [8], similar problems for Navier-Stokes equations have been considered in [39, 40]. Cahn-Hilliard-Navier-Stokes equations have been addressed in [7] and the primitive equations of the ocean have been given in [37].

**Theorem 5.1** (Necessary condition). Let \((u^*, U^*)\) be an optimal pair for the problem \((P)\). Then for any \((u, U) \in M_{u^*, U^*}\), we have
\[
\begin{align*}
&\frac{1}{2} \int_0^T \left( \|u(t)\|_{H^1}^2 + \|\nabla u(t)\|_{L^2}^2 + \|U(t)\|_{L^2}^2 \right) \, dt \\
&+ \frac{1}{2} \|u(T)\|_{H^1}^2 - \int_0^T \langle B(u(t), u(t)), p(t) \rangle \, dt \\
&- \beta \int_0^T \langle 2(u^*(t) \cdot u(t))u(t) + |u(t)|^2(u(t) + u^*(t)), p(t) \rangle \, dt \geq 0. \end{align*}
\] (73)
Proof. For any \((u, U) \in \mathcal{M}_{u^*, U^*}\), by the definition given in (72), there exists \((z, W) \in \mathcal{M}_{ad}\) such that \((u, U) = (z - u^*, W - U^*)\). Thus, from (15), one can derive that \((u, U)\) satisfies the following system:
\[
\begin{align*}
\partial_t u(t) + \mu A u(t) + B(u(t) + u^*(t)) - B(u^*(t)) + \alpha u(t) + \beta (C(u(t) + u^*(t)) - C(u^*(t))) &= DU(t), \quad \text{in } \mathbb{V}', \\
\partial_t u(0) &= 0.
\end{align*}
\]
for a.e. \(t \in [0, T]\). Taking the inner product with \(p(\cdot)\) to the first equation in (74) and then integrating over \(t \in [0, T]\), we obtain
\[
\int_0^T \langle \partial_t u(t) + \mu A u(t) + B'(u^*(t)) u(t) + \alpha u(t) + \beta C'(u^*(t)) u(t) - DU(t), p(t) \rangle dt
\]
\[
+ \int_0^T \langle B(u(t) + u^*(t)) - B(u^*(t)) - B'(u^*(t)) u(t), p(t) \rangle dt
\]
\[
+ \beta \int_0^T \langle C(u(t) + u^*(t)) - C(u^*(t)) - C'(u^*(t)) u(t), p(t) \rangle dt = 0.
\]
Using an integration by parts, we infer that
\[
(u(T), p(T))
\]
\[
+ \int_0^T \langle u(t), -p_t(t) + \mu A p(t) + (B'(u^*(t)))^* p(t) + \alpha p(t) + \beta C'(u^*(t)) p(t) \rangle dt
\]
\[
+ \int_0^T \langle B(u(t) + u^*(t)) - B(u^*(t)) - B'(u^*(t)) u(t) - DU(t), p(t) \rangle dt
\]
\[
+ \beta \int_0^T \langle C(u(t) + u^*(t)) - C(u^*(t)) - C'(u^*(t)) u(t), p(t) \rangle dt = 0. \quad (75)
\]
Since \((u^*, U^*)\) is an optimal pair, we know that \((u^*, U^*)\) satisfies the first order necessary condition
\[
U^*(t) = -D^* p(t), \quad \text{for a.e. } t \in [0, T]. \quad (76)
\]
Using (76) and (38) in (75), we find
\[
(u(T), u^*(T) - u^f) + \int_0^T (U^*(t), U(t)) dt + \int_0^T (u^*(t) - u_0(t) - \Delta u^*(t), u(t)) dt
\]
\[
+ \int_0^T \langle B(u(t) + u^*(t)) - B(u^*(t)) - B'(u^*(t)) u(t), p(t) \rangle dt
\]
\[
+ \beta \int_0^T \langle C(u(t) + u^*(t)) - C(u^*(t)) - C'(u^*(t)) u(t), p(t) \rangle dt = 0. \quad (77)
\]
Since \((u, U) \in \mathcal{M}_{u^*, U^*}\), by the definition (72), we know that \((u + u^*, U + U^*)\) is a feasible pair for the problem \((P)\). Remembering that \((u^*, U^*)\) is an optimal pair, from (26), we also obtain
\[
J(u + u^*, U + U^*) - J(u^*, U^*)
\]
\[
= \frac{1}{2} \int_0^T \left[ \|u(t) + u^*(t) - u_0(t)\|_H^2 + \|\nabla (u(t) + u^*(t))\|_H^2 \right] dt
\]
\[
+ \frac{1}{2} \int_0^T \|U(t) + U^*(t)\|_H^2 dt + \frac{1}{2} \|u(T) + u^*(T) - u^f\|_H^2.
\]
Thus, it is immediate that
\[
\int_0^T \left[ (u^*(t) - u_d(t), u(t)) + \frac{1}{2} \| u(t) \|^2_B + \langle -\Delta u^*(t), u(t) \rangle + \frac{1}{2} \| \nabla u(t) \|^2_{E^0} \right] dt
\]
\[
+ \int_0^T \left[ (U^*(t), U(t))_\mathcal{V} + \frac{1}{2} \| U(t) \|^2_{\mathcal{U}} \right] dt + (u^*(T) - u^f, u(T)) + \frac{1}{2} \| u(T) \|^2_{H^0} \geq 0.
\] (78)

From (77), it follows that
\[
\frac{1}{2} \left\{ \int_0^T (\| u(t) \|^2_B + \| \nabla u(t) \|^2_{E^0} + \| U(t) \|^2_{\mathcal{U}}) dt \right\} + \frac{1}{2} \| u(T) \|^2_{H^0}
\]
\[
- \int_0^T \langle B(u(t) + u^*(t)) - B(u^*(t)) - B'(u^*(t))u(t), p(t) \rangle dt
\]
\[
- \beta \int_0^T \langle C(u(t) + u^*(t)) - C(u^*(t)) - C'(u^*(t))u(t), p(t) \rangle dt \geq 0.
\] (79)

But, we know that
\[
\langle B(u + u^*) - B(u^*) - B'(u^*)u, p \rangle
\]
\[
= \langle B(u, u) + B(u, u^*) + B(u^*, u) + B(u^*, u^*), p \rangle
\]
\[
- \langle B(u^*, u^*), p \rangle - \langle B(u^*, u) - B(u, u^*), p \rangle
\]
\[
= \langle B(u, u), p \rangle.
\] (80)

Using Taylor’s series expansion (see Theorem 7.9.1, [9]), we have (see Remark 1)
\[
\langle C(u + u^*) - C(u^*) - C'(u^*)u, p \rangle
\]
\[
= \frac{1}{2} \langle C''(u^*)(u \otimes u), p \rangle + \frac{1}{2} \int_0^1 (1 - \theta)^2 \langle C'''(u^* + \theta u)(u \otimes u \otimes u) \rangle d\theta, p \rangle
\]
\[
= \langle 2u^* \cdot u, u \rangle p + \langle |u|^2 u^*, u \rangle + \langle |u|^2 u, p \rangle.
\] (81)

Using (80) and (81) in (79), we finally get (73).

**Remark 4.** For \( r = 1 \), the condition (73) becomes
\[
\frac{1}{2} \left\{ \int_0^T (\| u(t) \|^2_B + \| \nabla u(t) \|^2_{E^0} + \| U(t) \|^2_{\mathcal{U}}) dt \right\}
\]
\[
+ \frac{1}{2} \| u(T) \|^2_{H^0} - \int_0^T \langle B(u(t), u(t)), p(t) \rangle dt \geq 0,
\]
which is same as the case for the optimal control problems governed by 2D Navier-Stokes equations (cf. [40]).

**Theorem 5.2** (Sufficient condition). Let \((u^*, U^*)\) be a feasible pair for the problem (P) and assume that the first order necessary condition holds (see (76)). Let us also assume that for any \((u, U) \in \mathcal{M}_{u^*, U^*}\), the inequality (73) holds. Then \((u^*, U^*)\) is an optimal pair for the problem (P).
For any \((z, W) \in A_{ad}\), by the definition of (72), we know that \((z - u^*, W - U^*) \in M_{u^*, U^*}\) and it satisfies:

\[
\begin{cases}
\partial_t (z(t) - u^*(t)) + \mu A(z(t) - u^*(t)) + B(z(t)) - B(u^*(t)) + \alpha(z(t) - u^*(t)) \\
+ \beta(C(z(t)) - C(u^*(t))) = D(W(t) - U^*(t)), \quad z(0) - u^*(0) = 0.
\end{cases}
\]

(82)

in \(V', \) for a.e. \(t \in [0, T].\) Taking the inner product with \(p(\cdot)\) to the first two equations in (82), integrating over \([0, T]\) and then performing an integration by parts, we get

\[
\begin{align*}
&\int_0^T \langle z(t) - u^*(t), -\partial_t p(t) + \mu A p(t) + B'(u^*(t))p(t) + \alpha p(t) + \beta C'(u^*(t))p(t) \rangle \, dt \\
&+ \int_0^T \langle B(z(t)) - B(u^*) - B'(u^*)(z(t) - u^*)(t) - (W(t) - U^*(t)), D^* p(t) \rangle \, dt \\
&+ \beta \int_0^T \langle C(z(t)) - C(u^*) - C'(u^*)(z(t) - u^*), p(t) \rangle \, dt \\
&+ \langle (Z(T) - u^*(T), p(T) \rangle = 0.
\end{align*}
\]

(83)

A calculation similar to (80) yields

\[
\langle B(z) - B(u^*) - B'(u^*)(z - u^*), p \rangle = \langle B(z - u^*, z - u^*), p \rangle.
\]

(84)

Using Taylor’s formula (see (81)), we obtain

\[
\beta \langle C(z) - C(u^*) - C'(u^*)(z - u^*), p \rangle \\
= \frac{\beta}{2} \langle C''(u^*)((z - u^*) \odot (z - u^*)), p \rangle \\
+ \frac{\beta}{2} \int_0^1 (1 - \theta)^2 \langle C'''(u^* + \theta(z - u^*))((z - u^*) \odot (z - u^*) \odot (z - u^*)), d\theta, p \rangle \\
= 2\beta \langle (u^* \cdot (z - u^*) - (z - u^*)^2, p \rangle + \beta \langle |z - u^*|^2 u^*, p \rangle + \beta \langle |z - u^*|^2 (z - u^*), p \rangle.
\]

Note that \((u^*, U^*)\) satisfies the first order necessary condition

\[
\int_0^T (U(t), D^* p(t)) \, dt + \int_0^T (U(t), U^*(t)) \, dt = 0, \text{ for all } U \in L^2(0, T; U).
\]

It is true for any \(U \in \mathcal{H}_{ad}\), and in particular for \(W - U^* \in \mathcal{H}_{ad}\), that is, we have

\[
\int_0^T (W(t) - U^*(t), D^* p(t)) \, dt = - \int_0^T (W(t) - U^*(t), U^*(t)) \, dt.
\]

(85)

Using the adjoint system (38) and (84)-(85) in (83), we further find

\[
\begin{align*}
&\int_0^T \langle z(t) - u^*(t), (u^*(t) - u_d(t)) - \Delta u^*(t) \rangle \, dt + \int_0^T (W(t) - U^*(t), U^*(t)) \, dt \\
&+ \langle (Z(T) - u^*(T), u^*(T) - u^*) \rangle + \int_0^T \langle (B(z(t) - u^*(t), z(t) - u^*(t)), p(t) \rangle \, dt \\
&+ \beta \int_0^T 2\langle (u^*(t) \cdot (z(t) - u^*(t))) (z(t) - u^*(t)), p \rangle + \langle (z(t) - u^*(t))^2 z(t), p(t) \rangle \, dt \\
&= 0.
\end{align*}
\]

(86)
An easy calculation yields
\[
\frac{1}{2} \int_0^T \left( \| z(t) - u_d(t) \|_A^2 + \| \nabla z(t) \|_A^2 + \| W(t) \|_A^2 \right) dt + \frac{1}{2} \| z(T) - u^f \|_A^2 \\
- \frac{1}{2} \int_0^T \left[ \| u^*(t) - u_d(t) \|_A^2 + \| \nabla u^*(t) \|_A^2 + \| U^*(t) \|_A^2 \right] dt - \frac{1}{2} \| u^*(T) - u^f \|_A^2 \\
= \int_0^T \left[ (u^*(t) - u_d(t), z(t) - u^*(t)) + \frac{1}{2} \| z(t) - u^*(t) \|_A^2 \right] dt \\
+ \int_0^T \left[ \langle -\Delta u^*(t), z(t) - u^*(t) \rangle + \frac{1}{2} \| \nabla z(t) - u^*(t) \|_A \right] dt \\
+ \int_0^T \left[ (U^*(t), W(t) - U^*(t))_U + \frac{1}{2} \| W(t) - U^*(t) \|_A^2 \right] dt \\
+ (u^*(T) - u^f(T), z(T) - u^*(T)) + \frac{1}{2} \| z(T) - u^*(T) \|_A^2.
\] (87)

Using (86) in (87), we get

\[
\frac{1}{2} \int_0^T \left( \| z(t) - u_d(t) \|_A^2 + \| \nabla z(t) \|_A^2 + \| W(t) \|_A^2 \right) dt + \frac{1}{2} \| z(T) - u^f \|_A^2 \\
- \frac{1}{2} \int_0^T \left[ \| u^*(t) - u_d(t) \|_A^2 + \| \nabla u^*(t) \|_A^2 + \| U^*(t) \|_A^2 \right] dt - \frac{1}{2} \| u^*(T) - u^f \|_A^2 \\
= \frac{1}{2} \int_0^T \left[ \| z(t) - u^*(t) \|_A^2 + \| \nabla (z(t) - u^*(t)) \|_A^2 + \| W(t) - U^*(t) \|_A^2 \right] dt \\
+ \frac{1}{2} \| z(T) - u^*(T) \|_A^2 - \int_0^T \langle B(z(t) - u^*(t), z(t) - u^*(t)), p(t) \rangle dt \\
+ \beta \int_0^T 2\langle (u^*(t) \cdot (z(t) - u^*(t))), (z(t) - u^*(t)), p \rangle + \langle (z(t) - u^*(t))^2 z(t), p(t) \rangle dt \\
\geq 0,
\]

where we have used (73). Therefore, for any \((z, W) \in \mathcal{A}_{ad}\), we find that the following inequality holds:

\[
\frac{1}{2} \int_0^T \left( \| z(t) - u_d(t) \|_A^2 + \| \nabla z(t) \|_A^2 + \| W(t) \|_A^2 \right) dt + \frac{1}{2} \| z(T) - u^f \|_A^2 \\
\geq \frac{1}{2} \int_0^T \left[ \| u^*(t) - u_d(t) \|_A^2 + \| \nabla u^*(t) \|_A^2 + \| U^*(t) \|_A^2 \right] dt + \frac{1}{2} \| u^*(T) - u^f \|_A^2,
\]

which implies that the pair \((u^*, U^*)\) is an optimal pair for the problem (P).

\[ \Box \]

**Remark 5.** Note that the second order necessary and sufficient conditions of optimality (cf. (73)) is true for all \(\alpha, \beta \geq 0\) and 2D CBF equations is the damped 2D NSE. Thus as \(\alpha, \beta \to 0\), it coincides with the second order necessary and sufficient conditions of optimality for 2D NSE available in [40].

6. **Data Assimilation Problem.** In this section, we consider a problem similar to that of the data assimilation problems of meteorology. In the data assimilation problems arising from meteorology, determining the accurate initial data for the future predictions is a challenging problem. This motivates us to consider a similar
problem for 2D CBF equations. We formulate an optimal data initialization problem, where we find the unknown optimal initial data by minimizing suitable cost functional subject to 2D CBF equations (see [36] for the case of the incompressible Navier-Stokes equations, [6] for the Cahn-Hilliard-Navier-Stokes equations and [29] for the 2D Tidal dynamics equations).

We formulate the initial data optimization problem as finding an optimal initial velocity \( U \in \mathbb{H} \) such that \((u, U)\) satisfies the following system:

\[
\begin{aligned}
\partial_t u(t) + \mu A u(t) + B(u(t)) + \alpha u(t) + \beta \mathcal{C}(u(t)) &= f(t), \quad \text{in } \mathcal{V}', \\
\alpha u(0) = U,
\end{aligned}
\]

(88)

for a.e. \( t \in [0, T] \), and minimizes the cost functional

\[
J(u, U) := \frac{1}{2} \|U\|_{\mathcal{H}}^2 + \frac{1}{2} \int_0^T \|u(t) - u_M(t)\|_{\mathcal{H}}^2 dt + \frac{1}{2} \int_0^T \|\nabla \times u(t)\|_{\mathcal{H}}^2 dt \]

(89)

\[
+ \frac{1}{2} \|u(T) - u_M^f\|_{\mathcal{H}}^2,
\]

where \( u_M \) is the measured velocity of the fluid and \( u_M^f \) is the measured final velocity at time \( T \). We assume that

\[
u
u_M \in L^2(0, T; \mathcal{H}), \quad u_M^f \in \mathcal{H}.
\]

(90)

In this context, we take the set of admissible control class, \( \mathcal{U}_{ad} = \mathbb{H} \). Also, the admissible class \( \mathcal{A}_{ad} \) consists of all pairs \((u, U)\) such that the set of states \( u(\cdot) \) is a unique weak solution of the 2D CBF equations (88) with the control \( U \in \mathcal{U}_{ad} \). We formulate the optimal control problem as:

\[
(\text{P}_0) \quad \min_{(u, U) \in \mathcal{A}_{ad}} J(u, U).
\]

The next theorem provides the existence of an optimal pair \((u^*, U^*)\) for the problem \((\text{P}_0)\). A proof of the following Theorem can be established in a similar way as that of Theorem 3.3.

**Theorem 6.1 (Existence of an optimal pair).** Let \( f \in L^2(0, T; \mathcal{V}') \) be given. Then there exists at least one pair \((u^*, U^*) \in \mathcal{A}_{ad} \) such that the functional \( J(u, U) \) given in (89) attains its minimum at \((u^*, U^*)\), where \( u^*(\cdot) \) is the unique weak solution of the system (88) with the initial data control \( U^* \in \mathcal{U}_{ad} \).

The first order necessary conditions of optimality for this control problem can also be obtained by following similar arguments as in the proof of Theorem 4.1. Since \( \mathcal{U}_{ad} = \mathbb{H} \), the optimal control is given by \( U^* = -p(0) \), where \( p(\cdot) \) is the unique weak solution of the following adjoint system:

\[
\begin{aligned}
-\partial_t p(t) + \mu A p(t) + (B'(u^*(t)) \cdot p(t) + \alpha p(t) + \beta \mathcal{C}'(u^*(t)) p(t)
\end{aligned}
\]

(91)

\[
= (u^*(t) - u_M(t)) - \Delta u(t), \quad \text{in } \mathcal{V}',
\]

\[
p(T) = u^*(T) - u_M^f,
\]

for a.e. \( t \in [0, T] \). A similar calculation as in the proof of Theorem 3.2 provides the existence of a unique weak solution to the system (91) such that

\[
p \in C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathcal{V}).
\]

(92)

Using the continuity of \( p(\cdot) \) in time at \( t = 0 \in \mathbb{H} \), we infer that \( p(0) \in \mathbb{H} \) and hence we obtain \( U^* = -p(0) \in \mathcal{U}_{ad} = \mathbb{H} \). Thus, we have the following Theorem:
Theorem 6.2 (Optimal initial control). Let \((u^*, U^*) \in \mathcal{A}_{ad}\) be an optimal pair. Then there exists a unique weak solution \(p(\cdot)\) of the adjoint system (91) satisfying (92) such that the optimal control is obtained as

\[ U^* = -p(0) \in \mathcal{U}_{ad} = \mathcal{H}. \tag{93} \]

For \(r = 3\), the second order necessary and sufficient conditions of optimality can be obtained in a similar way as in the Theorems 5.1 and 5.2.

Theorem 6.3 (Necessary condition). Let \((u^*, U^*)\) be an optimal pair for the problem \((P_0)\). Then, for any \((u, U) \in \mathcal{M}_{u^*, U^*}\), where \(\mathcal{M}_{u^*, U^*} := \{(u, U) \in \mathcal{A}_{ad} \setminus \{(u^*, U^*)\}\}, we have

\[
\frac{1}{2} \left[ \int_0^T (\|u(t)\|_{\mathcal{H}}^2 + \|\nabla u(t)\|_{\mathcal{H}}^2) \, dt \right] + \frac{1}{2} \|u\|_{\mathcal{H}}^2 + \frac{1}{2} \|u(T)\|_{\mathcal{H}}^2 - \int_0^T \langle B(u(t), u(t)), p(t) \rangle \, dt \\
- \beta \int_0^T \langle 2(u^*(t) \cdot u(t))u(t) + |u(t)|^2(u(t) + u^*(t)), p(t) \rangle \, dt \geq 0. \tag{94} \]

Theorem 6.4 (Sufficient condition). Let \((u^*, U^*)\) be a feasible pair for the problem \((P_0)\) and assume that the first order necessary condition holds (see (93)). Let us also assume that for any \((u, U) \in \mathcal{M}_{u^*, U^*}\), the inequality (94) holds. Then \((u^*, U^*)\) is an optimal pair for the problem \((P_0)\).

Acknowledgments. M. T. Mohan would like to thank the Department of Science and Technology (DST), India for Innovation in Science Pursuit for Inspired Research (INSPIRE) Faculty Award (IFA17-MA110). The author sincerely would like to thank the reviewer for his/her valuable comments and suggestions.

REFERENCES

[1] S. N. Antontsev and H. B. de Oliveira, The Navier-Stokes problem modified by an absorption term, Applicable Analysis, 89 (2010), 1805–1825.
[2] F. Abergel and R. Temam, On some control problems in fluid mechanics, Theoretical and Computational Fluid Dynamics, 1 (1990), 303–325.
[3] V. Barbu, Optimal Control of Variational Inequalities, Pitman Res. Notes Math. Ser. 100, Pitman, Boston, 1984.
[4] V. Barbu, Optimal control of Navier-Stokes equations with periodic inputs, Nonlinear Analysis: Theory, Methods & Applications, 31 (1998), 15–31.
[5] T. Biswas, S. Dharmatti and M. T. Mohan, Pontryagin’s maximum principle for optimal control of the nonlocal Cahn-Hilliard-Navier-Stokes systems in two dimensions, Analysis (Berlin), 40 (2020), 127–150.
[6] T. Biswas, S. Dharmatti and M. T. Mohan, Maximum principle and data assimilation problem for the optimal control problem governed by 2D nonlocal Cahn-Hilliard-Navier-Stokes equations, Journal of Mathematical Fluid Mechanics, 22 (2020), Article No. 34, 1–42.
[7] T. Biswas, S. Dharmatti and M. T. Mohan, Second order optimality conditions for optimal control problems governed by 2D nonlocal Cahn-Hilliard-Navier-Stokes equations, Nonlinear Studies, 28 (2021), 29–43.
[8] E. Casas and F. Tröltzsch, Second-order necessary and sufficient optimality conditions for optimization problems and applications to control theory, SIAM J. Optim., 13 (2002), 406–431.
[9] P. G. Ciarlet, Linear and Nonlinear Functional Analysis with Applications, SIAM Philadelphia, 2013.
[10] P. Cherier and A. Milani, Linear and Quasi-linear Evolution Equations in Hilbert Spaces, American Mathematical Society Providence, Rhode Island, 2012.
[11] R. F. Curtain and A. J. Pritchard, *Functional Analysis in Modern Applied Mathematics*, Mathematics in Science and Engineering, Vol. 132, Academic Press, London-New York, 1977.

[12] S. Doboszcak, M. T. Mohan and S. S. Sritharan, Existence of optimal controls for compressible viscous flow, *Journal of Mathematical Fluid Mechanics, 20* (2018), 199–211.

[13] L. C. Evans, *Partial Differential Equations*, Grad. Stud. Math., vol. 19, Amer. Math. Soc., Providence, RI, 1998.

[14] I. Ekeland and T. Turnbull, *Infinite-dimensional Optimization and Convexity*, The University of Chicago press, Chicago and London, 1983.

[15] H. O. Fattorini and S. S. Sritharan, Necessary and sufficient conditions for optimal controls in viscous flow problems, *Proc. Roy. Soc. Edinburgh A, 124* (1994), 211–251.

[16] C. L. Fefferman, K. W. Hajduk and J. C. Robinson, Simultaneous approximation in Lebesgue and Sobolev norms via eigenspaces, arXiv:1904.03337.

[17] B. P. W. Fernando and S. S. Sritharan, Nonlinear filtering of stochastic Navier-Stokes equation with Lévy noise, *Stochastic Analysis and Applications, 31* (2013), 381–426.

[18] D. Fujiwara and H. Morimoto, An $L^r$-theorem of the Helmholtz decomposition of vector fields, *J. Fac. Sci. Univ. Tokyo Sect. IA Math., 24* (1977), 685–700.

[19] A. V. Fursikov, *Optimal Control of Distributed Systems: Theory and Applications*, American Mathematical Society, Rhode Island, 2000.

[20] G. P. Galdi, An introduction to the Navier–Stokes initial-boundary value problem, *Fundamental Directions in Mathematical Fluid Mechanics*, Adv. Math. Fluid Mech. Birkhäuser, Basel, 2000, 1–70.

[21] M. D. Gunzburger, *Perspectives in Flow Control and Optimization*, SIAM’s Advances in Design and Control series, Philadelphia, 2003.

[22] K. W. Hajduk and J. C. Robinson, Energy equality for the 3D critical convective Brinkman-Forchheimer equations, *Journal of Differential Equations, 263* (2017), 7141–7161.

[23] V. K. Kalantarov and S. Zelik, Smooth attractors for the Brinkman-Forchheimer equations with fast growing nonlinearities, *Commun. Pure Appl. Anal., 11* (2012), 2037–2054.

[24] H. Kozono and T. Yanagisawa, $L^r$-variational inequality for vector fields and the Helmholtz-Weyl decomposition in bounded domains, *Indiana Univ. Math. J., 58* (2009), 1853–1920.

[25] A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1969.

[26] J.-L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer, 1971.

[27] J.-L. Lions and E. Magenes, *Nonhomogeneous Boundary-Value Problems and Applications*, Springer Verlag, Berlin, Germany, 1972.

[28] M. T. Mohan, Deterministic and stochastic equations of motion arising in Oldroyd fluids of order one: existence, uniqueness, exponential stability and invariant measures, *Stochastic Analysis and Applications, 38* (2020), 1–61.

[29] M. T. Mohan, First order necessary conditions of optimality for the two dimensional tidal dynamics system, *Mathematical Control and Related Fields, 2020*.

[30] M. T. Mohan, On the convective Brinkman-Forchheimer equations, Submitted.

[31] M. T. Mohan, The time optimal control of two dimensional convective Brinkman-Forchheimer equations, *Applied Mathematics & Optimization, 2021*.

[32] J. P. Raymond, Optimal control of partial differential equations. Université Paul Sabatier, Lecture Notes, 2013.

[33] J. C. Robinson, J. L. Rodrigo and W. Sadowski, *The Three-dimensional Navier–Stokes Equations, Classical Theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, UK, 2016.

[34] A. K. Sharma, M. K. Khandelwal and P. Bera, Finite amplitude analysis of non-isothermal parallel flow in a vertical channel filled with high permeable porous medium, *Journal of Fluid Mechanics, 857* (2018), 469–507.

[35] J. Simon, Compact sets in the space $L^p(0,T; B)$, *Annali di Matematica Pura ed Applicata, 146* (1986), 65–96.

[36] S. S. Sritharan, *Optimal Control of Viscous Flow*, SIAM Frontiers in Applied Mathematics, Philadelphia. Society for Industrial and Applied Mathematics, 1998.

[37] T. Tachim Medjo, Second-order optimality conditions for optimal control of the primitive equations of the ocean with periodic inputs, *Applied Mathematics and Optimization, 63* (2011), 75–106.
[38] R. Temam, *Navier-Stokes Equations, Theory and Numerical Analysis*, North-Holland, Amsterdam, 1984.
[39] F. Tröltzsch and D. Wachsmuth, Second-order sufficient optimality conditions for the optimal control of Navier-Stokes equations, *ESAIM: Control, Optimisation and Calculus of Variations*, 12 (2006), 93–119.
[40] L. Wang and P. He, Second order optimality conditions for optimal control problems governed by 3-dimensional Navier-Stokes equations, *Acta Mathematica Scientia*, 26 (2006), 729–734.
[41] G. Wang, Optimal controls of 3 dimensional Navier-Stokes equations with state constraints, *SIAM Journal on Control and Optimization*, 41 (2002), 583–606.
[42] G. Wang and L. Wang, Maximum principle of state-constrained optimal control governed by fluid dynamic systems, *Nonlinear Analysis: Theory, Methods & Applications*, 52 (2003), 1911–1931.

Received May 2020; revised March 2021.

E-mail address: maniltmohan@gmail.com, maniltmohan@ma.iitr.ac.in