CONTINUOUS VS DISCRETE SPINS IN THE HYPERBOLIC PLANE

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Abstract. We study the $O(n)$ model on planar hyperbolic cocompact lattices, with free boundary conditions. We observe that the pair correlations decay exponentially with distance, for all temperatures, if and only if $n > 1$.

We wish to report here on a curious contrast between the behaviour of models with discrete and continuous spins on cocompact lattices in the hyperbolic plane, lattices such as the seven-regular planar triangulation. For concreteness we study the $O(n)$ model, where, we recall, every vertex $v$ of the graph is endowed with a spin $s_v$, which are unit vectors in the $n$-sphere $S^n$. The Hamiltonian is given by:

$$H = - \sum_{\{v,u\} \in E} \langle s_v, s_u \rangle.$$

We take an exhaustion of our lattice, say the balls in the graph distance $B_r$, and define the pair correlation as

$$\lim_{r \to \infty} \frac{\int \langle s_u, s_v \rangle \exp(-\beta H_r(s)) \, ds}{\int \exp(-\beta H_r(s)) \, ds}$$

where the integrals are taken over $(S^n)^{|B_r|}$ and $H_r$ is the Hamiltonian with the sum restricted to edges inside $B_r$. One may also wire all the vertices of the boundary of $B_r$ to get the Hamiltonian with wired boundary conditions, and the corresponding pair correlation.

Let us start with the case of discrete spins, specifically Ising (or $n = 1$), which is well-known. In [5] it was shown, using a Peierls-type argument that the threshold for uniqueness of the infinite cluster, $p_u$, is strictly smaller than 1 for Bernoulli percolation on any transitive planar nonamenable graphs with one end. In particular this applies to the Voronoi tessellation of a cocompact
lattice in the hyperbolic plane (the Voronoi cells are all hyperbolic polygons, and the graph we consider has the corners of the polygons as its vertices and the sides of the polygons as its edges). For any graph this implies that the Ising model has a phase transition for the pair correlation, see [1, §4] (the argument of [1] is sufficiently simple to sketch here: since the functions $1\{u \leftrightarrow v\}$ and $2\#\text{clusters} + \#\text{open edges}$ are both increasing, the FKG inequality allows to control $u \leftrightarrow v$ in the FK representation at $2p/(1+p)$ which is the same as the Ising model at $\beta = |\log(2p/(1+p))|$. We get that the Ising model has an ordered phase, where pair correlations do not decay. This does not depend on boundary conditions (free or wired, for example). See also [9] for the Ising model on hyperbolic graphs.

To study continuous spins, recall that one may view a graph as an electrical network, where each edge is a 1 Ohm conductor. For hyperbolic lattices the electric resistance is proportional to the distance [2]. This implies that for any two vertices $x$ and $y$ and for any $r$ sufficiently large one may find a function $a$ defined on $B(x, r)$, the ball of radius $r$ around $x$ in the graph distance, with the properties that

$$a(y) - a(x) > c_1 d(x, y) \quad \sum_{u \sim v} |a(u) - a(v)|^2 \leq \frac{1}{2} |a(y) - a(x)|$$

$$\forall u \sim v \quad |a(u) - a(v)| \leq \frac{1}{10}$$

where $c_1$ is some constant, $u \sim v$ means that $u$ and $v$ are neighbours in the graph (and are implicitly assumed to both be in $B(x, r)$), and $d(x, y)$ is the graph distance. See e.g. [6] for the various equivalent definitions of electrical resistance.

The existence of such an $a$ gives that the pair correlations of the $O(n)$ model, $n \geq 2$, decay exponentially, using the argument of McBryan and Spencer [7]. The proof in that paper carries through with no change. Let us sketch the argument of [7] as well (even though [7] is very short and to the point). It is a version of the Mermin-Wagner argument where quadratic control of the error is achieved using a complex change of variables. Specifically, the integral from the definition of the pair correlation is written as

$$\int \exp \left( i(\theta(x) - \theta(y)) - \beta H(\tilde{\theta}) \right) \theta(u) \in [0, 2\pi) \forall u$$
and then the change of variables $\theta(x) \mapsto \theta(x) + ia(x)$ is applied. The first term makes one “earn” $\exp(a(y) - a(x))$ while the term containing the Hamiltonian makes one lose approximately $\exp \left( \sum_{u \sim v} |a(u) - a(v)|^2 \right)$.

We thus arrive at the following observation:

**Theorem.** The $O(n)$ model, $n \geq 2$ on a cocompact lattice in the hyperbolic plane with free boundary conditions has exponentially decaying pair correlations.

A similar result can be obtained even without the assumption of transitivity. In [4] it was shown that every transient bounded-degree planar graph admits a non constant harmonic function, with gradient in $L^2$. This implies that the resistance between a pair of vertices grows to infinity with the distance [3, 8]. For recurrent graph this holds as well. We conclude that there is no magnetisation phase for $O(n), n > 1$ models on any bounded degree planar graph.

One is tempted to conjecture that for wired boundary conditions the behaviour is different and there is a phase transition, because the electric resistance is bounded. Similarly, the electric resistance (free or wired) between a pair of vertices in lattices in the hyperbolic space (i.e. in dimension 3 or higher) is uniformly bounded. Thus we expect a phase transition for all $n$.

Another interesting variation is to take the spins in various subsets of $S^1$. For example, when the spins take the value in two intervals, is it true that one has a phase transition for the choice of the interval, but exponential decay of correlations inside the intervals? What happens if the spins take value in a Cantor set, or in the $p$-adic numbers? The same questions can be asked in $Z^2$.

**References**

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