Explicit minimal resolution for certain monomial curves.

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Abstract

With a view to study problems of smoothability, we construct a minimal free resolution for the coordinate ring of an algebroid monomial curve associated to an AS numerical semigroup (i.e. generated by an arithmetic sequence), obtained independently of the result of [2] and equipped with the explicit description of all the involved maps.

Keywords: Numerical semigroup, Arithmetic sequence, Monomial curve, Free resolution, Betti number.
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0 Introduction

Let $k[x_0, ..., x_n]/I \cong k[t^s, s \in S]$ be the affine coordinate ring of a monomial algebraic curve $X \subseteq \mathbb{A}^{n+1}_k$ defined by a numerical semigroup $S$ and let $R = k[[x_0, ..., x_n]]/I$ ($k$ field). Several properties of the associated graded ring $G$ of $R$ have been studied in the recent years; on the other side, some questions related to the homological invariants of the local ring $R$ are still open. When $S$ is generated by an arithmetic sequence $m_0, ..., m_n$ (AS semigroup) the generating ideal $I$ of $X$ has a nice shape: in fact it is generated by the $2 \times 2$ minors of two matrices, as first noted in [1] and the Betti numbers of $G$ are calculated in [3].

In the recent paper [2] the authors give a minimal free resolution of the ring $k[x_0, ..., x_n]/I$, based on this “bide-terminantal” shape of the ideal $I$ and the mapping cone procedure. In particular they deduce the Betti numbers of $I$ [2, Theorem 4.1]. Essentially with an analogous technique, but independently, we have reached the same result: the difference is our more explicit definition of the maps involved in this construction. This punctual description is a basic tool to obtain a determinantal characterization of the first syzygies of the ideal $I$, which turns out to be useful in the study of the smoothability of these monomial curves (work in progress).

1 Setting.

Notation 1.1 (1) For a semigroup $S$ generated by an arithmetic sequence, $S = \sum_{0 \leq i \leq n} \mathbb{N}m_i$, where $m_i = m_0 + id_i$, $(1 \leq i \leq n)$ and $\text{GCD}(m_0, d) = 1$, let $a, b, \mu \in \mathbb{N}$ be such that

$$m_0 = an + b, \quad a \geq 1, \quad 1 \leq b \leq n, \quad \mu := a + d.$$ 

Let $P := k[x_0, ..., x_n]$ (k field), with weight($x_i$) := $m_i$, and let $k[S] = k[t^s, s \in S]$.

The defining ideal $I \subseteq P$ of the curve $X = \text{Spec}(k[S])$ (shortly AS monomial curve) is generated by the $2 \times 2$ minors of the following two matrices:

$$A := \begin{pmatrix} x_0 & x_1 & \ldots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & \ldots & x_{n-1} & x_n \end{pmatrix} \quad A' := \begin{pmatrix} x_0^n & x_0 & \ldots & x_{n-b} \#(n) \\ x_0 & x_0 & \ldots & x_0 \end{pmatrix}$$

and a minimal set of generators for $I$ can be obtained by the $(\begin{pmatrix} n \#(n) \end{pmatrix})$ maximal minors $\{f_1, ..., f_{\#(n)}\}$ (we choose lexicographic order) of the matrix $A$ and the $n - b + 1$ maximal minors $M_{1j}$ containing the first column of the matrix $A'$ (see [1, Theorem 1.1]).

(2) We call $C$ the codimension two ideal generated by the $2 \times 2$ minors of the matrix $A$ (which is the ideal of the cone over the rational normal curve of $\mathbb{P}^n$).

(3) For $h = 0, ..., n - b$, we shall denote by $g_h$ the minor $\text{det} \begin{pmatrix} x_0^n & x_0 & \ldots & x_{n-b} \#(n) \\ x_0 & x_0 & \ldots & x_0 \end{pmatrix}$ of $A'$ and by $\delta_h$ its weight:

$$g_0 := x_0^{n+1} - x_0^{n-b}, \quad g_h := x_0^n x_0^{n-h} - x_0^{n-b} x_0^{-h}, \quad g_{n-b} := x_0^n x_0^{n-b} - x_0^{n+1}, \quad \delta_h := am_n + m_{n-h}.$$ 

Lemma 1.2 With Notation [7]: $$(C + (g_0, ..., g_h)) : g_{h+1} = (x_1, ..., x_n), \quad \text{for each } h = 0, ..., n - b - 1.$$
Proof. First observe that \( x_{i+1}g_{n+1} - x_i g_h = x_{i+1}(x_n x_{n-1} - x_i) \) for each \( i = 0, \ldots, n - 1, \ h = 0, \ldots, n - b - 1. \) Hence the inclusion \( \mathfrak{g} \) is clear.

Now assume that \( \mathfrak{g}_0 g_{n+1} \in \mathfrak{g} + (g_0, \ldots, g_h). \) Then \( \mathfrak{g}_0 g_{n+1} = \beta + \alpha_0 g_0 + \cdots + \alpha_h g_h, \alpha_i \in P, \beta \in \mathfrak{g}. \) Hence \( \mathfrak{g}^2 g_{n+1} = x_\beta \alpha_0 x_0 g_0 + \cdots + \alpha_h g_h = \beta_1 + \alpha_0 g_1 + \cdots + \alpha_h g_{n+1} = \beta_2 + \alpha_0 g_2 + \cdots + (\alpha_{n-b} + \alpha g_{n+1}) = \mathfrak{g} + \beta + \alpha h g_{n+1}, \) with \( \beta \in \mathfrak{g}, \) \( \alpha \in \{x_1, \ldots, x_n\}. \) This would imply that \( \mathfrak{g}^2 g_{n+1} \not\in \mathfrak{g}, \) impossible since \( \mathfrak{g} \) is prime, \( g_{n+1} \not\in \mathfrak{g}, \mathfrak{g}_0 g_{n+1} \not\in \mathfrak{g} \) (because \( t^{m_0} \not\in (t^{m_1}, \ldots, t^{m_n}) \)).

2 Free resolution of the ideal \( I. \)

By means of the mapping cone technique, starting with the Eagon-Northcott resolution \( \mathbb{E} \) of the ideal \( \mathfrak{g} \) and the Koszul complex \( K \) of \( P/(x_1, \ldots, x_n) \), we can construct a free (non-minimal in general) resolution of the ideal \( I. \) This resolution is a lifting of the one found in \( \mathbb{E} \) for the associated graded ring \( \mathfrak{g} \) of the curve \( X \); in particular we’ll see that the Betti numbers of \( \mathfrak{g} \) and \( A \) are the same. We recall the main tools:

**Setting 2.1** With Notation \( \mathbb{I} \) let \( R = k[[x_0, \ldots, x_n]]/I. \)

1. The Eagon-Northcott free resolution for the \( R \)-ideal \( \mathfrak{g} = (f_1, \ldots, f_{n+2}), \) is the complex

\[
\mathbb{E} : \quad 0 \rightarrow E_{n-1} \rightarrow \cdots \rightarrow E_s \rightarrow E_{s-1} \quad \bigg\downarrow d_s \rightarrow E_s \rightarrow E_{s-1} \rightarrow \cdots \rightarrow E_0 \rightarrow P/\mathfrak{g} \rightarrow 0
\]

where \( E_0 \cong P, \ E_s = \wedge^{s+1} P^n \otimes (\text{Sym}_{s-1}(P^2))^* \cong P^{\beta_s}(-s-1), \) for \( 1 \leq s \leq n-1, \) \( \beta_s = s \binom{n}{s+1}, \) \( \text{Sym}_{s-1}(P^2) \) is a free \( P \)-module of rank \( s \) and basis \( \{x_0^{v_0} x_1^{v_1} \mid v_0 + v_1 = s - 1\} \) ( \( 1 \leq s \leq n - 1 \) ) (see \( \mathbb{I} \)).

Let \( \mathfrak{g} \) be the basis of \( E_s \) and let \( \mathfrak{g} \) be the basis of \( E_0. \) The maps in \( \mathbb{E} \) are:

\[
d_s : E_s \rightarrow E_{s-1}, \quad d_s \left((e_1 \wedge \cdots \wedge e_{s+1}) \otimes x_0^{v_0} x_1^{v_1}\right) = \Delta_0 (e_1 \wedge \cdots \wedge e_{s+1}) \otimes x_0^{v_0} x_1^{v_1-1}, \quad s \geq 2,
\]

where only the summands with non-negative powers of \( x_0, x_1 \) are considered, and, for \( q = 0, 1, s \geq 1, \) the maps \( \Delta_q : \wedge^s P^n \rightarrow \wedge^{s+1} P^n \) are defined as:

\[
\Delta_q (e_1 \wedge \cdots \wedge e_{s+1}) := \sum_{i=1}^s (-1)^{s+i+1} e_1 \wedge \cdots \wedge e_{i+1} \wedge \cdots \wedge e_{s+1}, \quad (1 \leq i_1 < \cdots < i_{s+1} \leq n), \ (2 \leq s \leq n),
\]

\[
\Delta_q (e_i) := x_{i-q} \epsilon_{i-q}, \quad (1 \leq i \leq n, \ s = 1).
\]

2. The Koszul complex \( K \) minimal free resolution for \( P/(x_1, \ldots, x_n) \), is:

\[
K : \quad 0 \rightarrow K_n \rightarrow K_{n-1} \rightarrow \cdots \rightarrow K_1 \rightarrow K_0 \rightarrow P/(x_1, \ldots, x_n) \rightarrow 0
\]

3. For \( 1 \leq h \leq n - b \) consider the ideal \( \mathfrak{g}_{h-1} := (\mathfrak{g}, g_0, g_1, \ldots, g_{h-1}) \subseteq P. \) By \( \mathbb{I} \), \( P/(\mathfrak{g}_{h-1} : g_h) = P/(x_1, \ldots, x_n); \) hence the Koszul complex gives a minimal free resolution \( \mathbb{K}(-\delta_h) \) of \( P/(x_1, \ldots, x_n): \)

\[
\mathbb{K}(-\delta_h) : \quad 0 \rightarrow K_n(-\delta_h) \rightarrow K_{n-1}(-\delta_h) \rightarrow \cdots \rightarrow K_0(-\delta_h) \rightarrow P(-\delta_h)/(\mathfrak{g}_{h-1} : g_h) \rightarrow 0
\]

where \( d_s \delta_h = \Delta_1, \ (1 \leq s \leq n). \)

4. For \( 0 \leq h \leq n - b \) denote respectively

\[
e_h, \quad \text{the (canonical) basis of} \ K_0(\delta_h),
\]

\[
e^{(h)}_{i_1} \wedge \cdots \wedge e^{(h)}_{i_s}, \quad \text{the (canonical) basis of} \ K_s(\delta_h) \ (1 \leq h \leq n - b),
\]

\[
e^{(0)}_{i_1} \wedge \cdots \wedge e^{(0)}_{i_s}, \quad \text{the (canonical) basis of} \ E_0(\delta_0),
\]

\[
e^{(0)}_{i_1} \wedge \cdots \wedge e^{(0)}_{i_s}, \quad \text{the (canonical) basis of} \ E_s(\delta_0) \ (1 \leq i_1 < \cdots < i_s \leq n)
\]

Further, for simplicity, when no confusion on indices occurs, we shall write

\[
W := e_1 \wedge \cdots \wedge e_s, \quad W^{(h)} = e^{(h)}_{i_1} \wedge \cdots \wedge e^{(h)}_{i_s}, \quad (0 \leq h \leq n - b).
\]
(5) Recalling that \( \text{weight}(x_i) = m_i = m_0 + i \cdot d \) (\( 0 \leq i \leq n \)), set \( \text{weight}(\lambda_0) := 0 \), \( \text{weight}(\lambda_1) := d \),
\[
\text{weight}(\varepsilon_h) = -\delta_h, \quad \text{weight}(e_i^{(1)} \wedge ... \wedge e_s^{(1)}) = m_{i_1} + ... + m_{i_s} - (s - 1)d.
\]
Therefore the modules \( K_s, E_s \) are graded as follows:
\[
K_s(-\delta_h) = \bigoplus_{1 \leq i_1 < ... < i_s \leq n} P(-\delta_h - m_{i_1} - ... - m_{i_s} + (s - 1)d), \quad \text{for} \quad 1 \leq s \leq n + 1, \quad h = 0, ..., n - b,
\]
\[
E_s = \bigoplus_{0 \leq e_i \leq s - 1} \left[ \bigoplus_{1 \leq i_1 < ... < i_s+1 \leq n} P(-m_{i_1} - ... - m_{i_{s+1}} + (s - v_1)d) \right], \quad \text{for} \quad 1 \leq s \leq n - 1, \quad E_0 \cong P
\]

(6) With the preceding assumptions we shall define by :
\[
F^{(0)}_0 := E_0, \quad \psi^{(0)}_s : E_s(-\delta_0) \rightarrow E_s, \quad \text{the multiplication by } g_0, \quad \text{for all } s \geq 0;
\]
\[
\psi^{(h)}_0 : K_0(-\delta_h) \rightarrow F^{(h-1)}_0 \quad \text{the multiplication by } g_h = x_n^a x_{n-h} - x_0^a x_{n-b-h}; \quad (h \geq 1)
\]
\[
\psi^{(h)}_s : K_s(-\delta_h) \rightarrow K_{s-1}(-\delta_{h-1}) \oplus E_{s-1}(-\delta_0) \oplus E_s \subseteq F^{(h-1)}_s \quad (s, h \geq 1):
\]
\[
\psi^{(1)}_s(e_i^{(1)}) = x_{i-1}e_0 + e_i \wedge (x_n^a e_{n-h} - x_0^a e_n) \in E_0(-\delta_0) \oplus E_1, \quad i = 1, ..., n, \quad (h = 1);
\]
\[
\psi^{(h)}_s(e_i^{(h)}) = -\Delta_0(e_i^{(h-1)}) + \phi^1_s(e_i^{(1)}), \quad (h \geq 2);
\]
\[
\psi^{(1)}_s(W^{(1)}) = -W^{(0)} \otimes \lambda_1^{s-2} \otimes \phi^{(1)}_s(W^{(1)}), \quad (s \geq 2);
\]
\[
\phi^{(h)}_s(W^{(h)}) = (-1)^{s-1} \Delta_0(W^{(h-1)}) + (-1)^h W^{(0)} \otimes \lambda_0^{h-1} \lambda_1^{s-h-1} + \phi^{(h)}_s(W^{(h)}), \quad (s, h \geq 2)
\]
where \( \phi^{(h)}_s : K_s(-\delta_h) \rightarrow E_s \) is
\[
\phi^{(h)}_s(W^{(h)}) := W \wedge \sum_{k=1}^h (-1)^k (x_n^a e_{n+k-h} - x_0^a e_{n-b+k-h}) \otimes \lambda_0^{k-1} \lambda_1^{s-k};
\]

(7) The maps \( d^{(h)}_s : F^{(h)}_s \rightarrow F^{(h)}_{s-1} \) are defined as follows:
\[
d^{(0)}_1(\alpha e_0 + \beta e_i \wedge e_j) = [\alpha g_0 + \beta (x_{i-1} x_j - x_i x_{j-1})] e_i;
\]
For \( s = 1, h \geq 1 \), with \( F^{(h)}_1 = K_0(-\delta_1) \oplus ... \oplus K_0(-\delta_1) \oplus E_0(-\delta_0) \oplus E_1, \quad F^{(h)}_0 = E_0(-\delta_h), \)
\[
d^{(h)}_1 : F^{(h)}_1 \rightarrow F^{(h)}_0 \quad \text{is the component-wise product}
\]
\[
(\alpha e_h \oplus ... \oplus \alpha e_0 e_0 + \beta e_i \wedge e_j) \cdot (g_0 + g_{h-1} \oplus ... \oplus g_0 + (x_{i-1} x_j - x_i x_{j-1}));
\]
\[
d^{(0)}_2 ((e_i \wedge e_j) \oplus (e_h \wedge e_k \wedge e_l \otimes \lambda_q)) = (x_{i-1} x_{j-1} x_{j-1}) e_0 \oplus (-g_0 e_i \wedge e_j + d_2 e_h e_k e_l \otimes \lambda_q), \quad (q = 1, 2);
\]
\[
d^{(0)}_s : E_{s-1}(-\delta_0) \oplus E_s \rightarrow E_{s-2}(-\delta_0) \oplus E_{s-1}, \quad s \geq 2, \quad \text{is}
\]
\[
\begin{pmatrix}
\frac{d_{s-1}}{(-1)^{s-1}(\cdot) g_0} \\
\frac{d_s}{(-1)^{s-1}(\cdot) g_0}
\end{pmatrix}
\]
\[
d^{(1)}_s : K_{s-1}(-\delta_1) \oplus E_{s-1}(-\delta_0) \oplus E_s \rightarrow K_{s-2}(-\delta_1) \oplus E_{s-2}(-\delta_0) \oplus E_{s-1}, \quad (s \geq 2) \text{ is given by}
\]
\[
\begin{pmatrix}
\frac{d_{s-1}}{(-1)^{s-1} \psi^{(1)}_{s-1}} & 0 \\
\frac{d_s}{(-1)^{s-1} \psi^{(1)}_{s-1}} & 0
\end{pmatrix}
\]
\[
\begin{pmatrix}
\Delta^{1,s-1}_{1,s-1} & 0 \\
-(&\otimes \lambda_1^{s-3}) & d_{s-1} & 0 \\
(... \otimes \lambda_0^{s-2}) & (-1)^{s-1}(\cdot) g_0 & d_s
\end{pmatrix}
\]
Now we construct a free resolution of the ideal $I$.

**2.1 Mapping cone construction**

We apply the mapping cone construction to the complex obtained by taking $I$ to be the cone of the graded ideal $I$. Thus, for $1 \leq k \leq h$,

- $d_s^{(h)}(W^{(k)}) = \Delta_1(W^{(k)}) + (-1)^s \psi_s(W^{(k)}) \in K_{s-1}(-\delta_k) \oplus K_{s-1}(-\delta_{k-1}) \oplus E_{s-1}(-\delta_0) \oplus E_s$;
- $d_s^{(h)}(W^{(0)} \otimes \lambda_0^{v_0} \lambda_1^{v_1}) = \Delta_1(W^{(0)} \otimes \lambda_0^{v_0-1} \lambda_1^{v_1}) + \Delta_1(W^{(0)} \otimes \lambda_0^{v_0} \lambda_1^{v_1-1} + (-1)^s \lambda_0^{v_0} \lambda_1^{v_1} g_0$;
- $d_s^{(h)}(W \otimes \lambda_0^{v_0} \lambda_1^{v_1}) = d_{s+1}(W \otimes \lambda_0^{v_0} \lambda_1^{v_1})$, $(v_0 + v_1 = s - 1)$.

The proof consists of several steps.

**STEP 0.** (Well-known). Let $g_0 := ax_{n-1}^a - x_0^a x_{n-b}$, $\delta_0 = \text{deg}(g_0)$. The following diagram is commutative:

\[
\begin{array}{cccccccc}
0 & \rightarrow & C_{n-1} & \rightarrow & \cdots & \rightarrow & C_s & \rightarrow & C_{s-1} & \rightarrow & C_2 & \rightarrow & C_1 & \rightarrow & P \\
& & & & & & & & & & & & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathbb{E}(-\delta_0) & \rightarrow & E_s(-\delta_0) & \rightarrow & E_2(-\delta_0) & \rightarrow & E_1(-\delta_0) & \rightarrow & E_0(-\delta_0) & \rightarrow & P(-\delta_0)/\mathcal{E} & \rightarrow & 0 \\
& & & & & & & & & & & & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
& & & & & & & & & & & & & g_0 \\
& & & & & & & & & & & & & P/(\mathcal{E}, g_0) \\
& & & & & & & & & & & & & 0 \\
\end{array}
\]

where the $\psi_i^{(0)}$ are the multiplication by $g_0$. By mapping cone one constructs the complex:

\[
0 \rightarrow E_{n-1}(-\delta_0) \rightarrow \cdots \rightarrow E_{s-1}(-\delta_0) \oplus E_s \rightarrow E_{s-2}(-\delta_0) \oplus E_{s-1} \rightarrow \cdots \rightarrow E_0(-\delta_0) \oplus E_1 \rightarrow E_0
\]

\[
d_1^{(0)}(\alpha \varepsilon_0 + \beta e_i \wedge e_j) = [\alpha g_0 + \beta (x_{i-1} x_j - x_i x_{j-1})]\varepsilon,
\]
The maps, as defined in (2.16), are:

1. The multiplication by \( g_1 = x_n^a x_{n-1} - x_0^b x_{n-b-1} \),

2. \( \psi^{(1)}_0 \), which is the multiplication by \( g_1 = x_n^a x_{n-1} - x_0^b x_{n-b-1} \),

3. \( \psi^{(1)}_1(e_i^{(1)}) = x_i - i \varepsilon_0 + e_i \wedge (x_0^b e_{n-b} - x_n^a e_n) \), for \( i = 1, \ldots, n \),

4. \( \psi^{(1)}_s(e_i^{(1)}) = -c^{(0)}_i \otimes 1^{s-2} + e_i \wedge (x_0^b e_{n-b} - x_n^a e_n) \otimes 1^{s-1} \), for \( s \geq 2 \).

One can check directly the commutativity of the diagrams for each \((s = 1, \ldots, n)\)

\[
\begin{array}{ccc}
K_s(-\delta_1) & \xrightarrow{d_s'} & K_{s-1}(-\delta_1) \\
\downarrow \psi^{(1)}_s & & \downarrow \psi^{(1)}_{s-1} \\
E_{s-1}(-\delta_0) \oplus E_s & \xrightarrow{d_s} & E_{s-2}(-\delta_0) \oplus E_{s-1}
\end{array}
\]

If \( s = 1 \), then

\[
d_1'(e_i^{(1)}) = x_i g_1 \varepsilon, \quad d_1'(\psi_1^{(1)}(e_i^{(1)})) = d_1'(x_i - i \varepsilon_0 + e_i \wedge (x_0^b e_{n-b} - x_n^a e_n)) = \\
x_i \varepsilon_0 + e_i \wedge (x_0^b e_{n-b} - x_n^a e_n) - x_i \varepsilon_0 = x_i g_1 \varepsilon.
\]

If \( s = 2 \), then

\[
d_2'(e_i^{(1)} \wedge e_j^{(1)}) = \Delta_1(e_i \wedge e_j) = x_i e_j^{(1)} - x_j e_i^{(1)} \quad \text{and for} \quad q = 1, 2
\]

\[
d_2'(e_i^{(0)} \wedge e_j^{(0)} \wedge e_k) = \Delta_1 e_k \wedge e_i \wedge e_j + d_2(e_k \wedge e_i \wedge e_j \wedge \lambda_q)
\]

\[
\psi_1^{(1)} d_2'(e_i^{(1)}) = \psi_1^{(1)}(x_i e_j^{(1)} - x_j e_i^{(1)}) = \\
x_i \varepsilon_0 + e_i \wedge (x_0^b e_{n-b} - x_n^a e_n) - x_j \varepsilon_0 + e_j \wedge (x_0^b e_{n-b} - x_n^a e_n)
\]

The conclusion and also the commutativity for \( s \geq 3 \), follow by the next Lemma 2.3 (see the proof given in the following "Step 3" which holds in the general case).
Now, again by the mapping cone construction, we get the exact complex:

\[
F_1: \quad 0 \to F_n^{(1)} \to \ldots \to F_s^{(1)} \to F_{s-1}^{(1)} \to \ldots \to F_1^{(1)} \to F_0^{(1)} \to P/\mathcal{C}_1 \to 0
\]

where

\[
F_s^{(1)} = K_{s-1}(-\delta_1) \oplus F_s^{(0)} = K_{s-1}(-\delta_1) \oplus E_{s-1}(-\delta_0) \oplus E_s, \quad \mathcal{C}_1 = \langle \mathcal{C}, g_0, g_1 \rangle.
\]

\[
d_1^{(1)}: K_0(-\delta_1) \oplus E_0(-\delta_0) \oplus E_1 \to E_0(-\delta_0), \quad d_1^{(1)}(a_1e_1 + a_0g_0 + x_{i-1}x_j - x_ix_{j-1}) = (a_1g_1 + a_0g_0 + x_{i-1}x_j - x_ix_{j-1})e_0
\]

Lemma 2.3 Let \( W := e_i \land \ldots \land e_i \), \( (i_1 < i_2 < \ldots < i_s) \); then for \( q \in \{ 0, 1 \} \), we have

(1) \( \Delta_q(W \land e_k) = \Delta_q(W) \land e_k + (-1)^s x_{k+q-1}W. \)

(2) \( d_s \circ \phi_s^{(h)}(W) = -\phi_s^{(h-1)}(\Delta_0(W)) + \phi_s^{(h)}(\Delta_1(W)) + (-1)^{s+h} g_0W \otimes \lambda_0^{-1} \lambda_1^{-h-1}. \)

Proof (1). Let \( p \) be the number of permutations to get \( i_1 < \ldots i_{s-p} < i_k < i_{s-p+1} < \ldots < i_s \). Then

\[
\Delta_q(W \land e_k) = (-1)^p[\Delta_q(e_{i_1} \land \ldots e_{i_{s-p}} \land e_k \land e_{i_{s-p+1}} \land \ldots \land e_{i_s})] = \Delta_q(e_{i_1} \land \ldots e_{i_{s-p}}) \land e_{i_{s-p+1}} \land \ldots \land e_{i_s} + (-1)^{s-p+2} x_{k+q-1}W + \Delta_q(e_{i_{s-p+1}} \land \ldots \land e_{i_s}) = \Delta_q(W) \land e_k + (-1)^{s+2} x_{k+q-1}W.
\]

(Note that this result holds also for \( k \in \{ i_1, \ldots, i_s \}. \))

(2). Recall that \( \phi^h(W) = \sum_{k=1}^{k=1} (-1)^k W \land (x_{i}^he_{n-k-h} - x_{i}^he_{n-b+k-h}) \otimes \lambda_0^{-1} \lambda_1^{-h} \), hence

\[
d_s \circ \phi_s^{(h)}(W) = \sum_{k=1}^{k=1} (-1)^k \Delta_0(W \land (x_{i}^he_{n-k-h} - x_{i}^he_{n-b+k-h})) \otimes \lambda_0^{-1} \lambda_1^{-h-1} + \sum_{k=1}^{k=1} (-1)^k \Delta_1(W \land (x_{i}^he_{n-k-h} - x_{i}^he_{n-b+k-h})) \otimes \lambda_0^{-1} \lambda_1^{-h-1}
\]

Let \( V_k := x_{i}^he_{n-k-h} - x_{i}^he_{n-b+k-h} \) and recall that \( g_i = x_{i}^he_{n-k-h} - x_{i}^he_{n-b+k-h} \); by (2.31) we get

\[
d_s \circ \phi_s(W) = \sum_{k=1}^{k=1} \left[ \Delta_0^k(W) \land V_k \otimes \lambda_0^{-1} \lambda_1^{-h-1} + \phi_s^{(h)}(\Delta_1(W)) \right] + \sum_{k=1}^{k=1} \left[ (-1)^k \Delta_1(W) \land V_k \otimes \lambda_0^{-1} \lambda_1^{-h-1} + (-1)^s g_i W \otimes \lambda_0^{-1} \lambda_1^{-h-1} \right]
\]

\[
= \sum_{k=1}^{k=1} (-1)^k \Delta_0^k(W) \land V_k \otimes \lambda_0^{-1} \lambda_1^{-h-1} + \sum_{k=1}^{k=1} \left[ (-1)^k \Delta_1(W) \land V_k \otimes \lambda_0^{-1} \lambda_1^{-h-1} + (-1)^s g_i W \otimes \lambda_0^{-1} \lambda_1^{-h-1} \right].
\]

\[
\text{STEP } h \quad (2 \leq h \leq n-b) \quad \text{By iterating this method, for all } h = 1, \ldots, n-b, \text{ let } \mathcal{C}_h = \mathcal{C} + (g_0, \ldots, g_h). \quad \text{By means of the following commutative diagram}
\]
To obtain a minimal free resolution we have to delete suitable subspaces of the modules $M_i$.

The complex $\mathbb{F}$:

\[
\begin{array}{ccccccc}
0 & \rightarrow & F_n \rightarrow & \cdots & F_s & \rightarrow & F_0 \rightarrow & P/\mathcal{E}_h \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \psi_s & & \downarrow \psi_s & & \downarrow g_n \\
F_{h-1} & \rightarrow & F_{h-1} & \rightarrow & \cdots & \rightarrow & F_0 & \rightarrow & P/\mathcal{E}_{h-1} \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & & & & & & & & \end{array}
\]

we construct the free complex $\mathbb{F}$:

\[
\begin{array}{ccccccc}
0 & \rightarrow & F_n \rightarrow & \cdots & F_s & \rightarrow & F_0 \rightarrow & P/\mathcal{E}_h \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \psi_s & & \downarrow \psi_s & & \downarrow g_n \\
F_{h-1} & \rightarrow & F_{h-1} & \rightarrow & \cdots & \rightarrow & F_0 & \rightarrow & P/\mathcal{E}_{h-1} \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & & & & & & & & \end{array}
\]

with $F_{h-1} = E_0(\delta_h)$, $F_s = K_{s-1}(\delta_h) \oplus \cdots \oplus K_s(\delta_h)$.

The commutativity for $s \geq 2$ of the above diagram follows by Lemma 2.4 since, with notation 2.16.

A. \[
\psi_s(W) = \psi_s(\Delta_1(W)) = \left[ \begin{array}{c}
(1)^s \Delta_0(\Delta_1(W)) \\
\oplus (1)^h \Delta_1(W) \otimes \lambda_0^{h-1} \lambda_1^{s-h-1} \\
\oplus \phi_s(\Delta_1(W))
\end{array} \right] \in K_{s-2}(\delta_{h-1})
\]

B. \[
d_s(\psi_s(W)) = d_s(\psi_s(W)) = (1)^s \Delta_0(\Delta_1(W)) \oplus d_s(\psi_s(W)) = \\
\left[ \begin{array}{c}
(1)^s \Delta_0(\Delta_1(W)) \\
\oplus (1)^h \Delta_1(W) \otimes \lambda_0^{h-1} \lambda_1^{s-h-1} + d_s(\psi_s(W)) \phi_s(\Delta_1(W))
\end{array} \right] \in K_{s-1}
\]

\[
d_s(\psi_s(W)) = d_s(\psi_s(W)) = (1)^s \Delta_0(\Delta_1(W)) \oplus d_s(\psi_s(W)) = \\
\left[ \begin{array}{c}
(1)^s \Delta_0(\Delta_1(W)) \\
\oplus (1)^h \Delta_1(W) \otimes \lambda_0^{h-1} \lambda_1^{s-h-1} + d_s(\psi_s(W)) \phi_s(\Delta_1(W))
\end{array} \right] \in K_{s-1}
\]

Remark 2.4 The complex $\mathbb{F}$ itself is a minimal free resolution for the ideal $I$ if $b = n$, but not in the other cases. In fact, if $b < n$ and $s \geq 2$ the maps $\psi_s$ give invertible entries in the matrix $M_{s+1}$ associated to the map $d_s(n-b)$ (note that in $M_2$ all the entries are non-invertible according to the definition of $\psi_s(1))$.

To obtain a minimal free resolution we have to delete suitable subspaces of the modules $C_s$ (see 2.2): the invertible entries in $M_{s+1}$ arise as $\psi_s(W(\psi_s(W)) \cap E_{s-1}(\delta_0)$, i.e. $(1)^h W(0) \otimes \lambda_0^{h-1} \lambda_1^{s-h-1}, h = 1, \ldots, n-b$. Since such elements are considered only with non-negative powers of $\lambda_0, \lambda_1$, to compute the dimension of the redundant subspace $D_s$ define

\[
\nu_s := \min \{ s-1, n-b \}.
\]

Then $\nu_s(W) = \dim \left( \text{im} \psi_s(W) \cap E_{s-1}(\delta_0) \right)$. Further $\psi_{s-1}$ gives invertible entries in $M_s$.

Therefore for each $s \geq 2$, the redundant subspace $D_s \subseteq C_s$ has dimension $\nu_s(W) + \nu_{s-1}(W) = \nu_s(W)$. (s $\geq 2$):

Theorem 2.5 A minimal free resolution of the ideal $I$ defining an AS curve with $b > 1$ is given by the complex

\[
\begin{array}{ccccccc}
R : & 0 & \rightarrow & R_n & \rightarrow & \cdots & \rightarrow & R_s & \rightarrow & R_2 & \rightarrow & R_1 & \rightarrow & P \\
\end{array}
\]
where \( R_s = C_s/D_s \) according to Remark 2.4. In particular

\[
R_n = \bigoplus_{0 \leq k \leq b-2} (e_1 \land \ldots \land e_n \lambda_0^{n-b+k} \lambda_1^{b-k-2}) P, \quad \text{if } b \geq 2
\]

\[
R_2 = K_1(-\delta_1) \oplus \ldots \oplus K_1(-\delta_1) \oplus E_2, \quad \text{with}
K_1(-\delta_j) = \bigoplus_{1 \leq i \leq n} P(-m_i - \delta_j),
E_2 = \bigoplus_{1 \leq i < j \leq n} P(-m_i - m_{i_2} - m_{i_3}) + (2 - q)d
\]

\[
R_1 = K_0(-\delta_0) \oplus \ldots \oplus K_0(-\delta_0) \oplus E_0(-\delta_0) \oplus E_1
\]

\[
\simeq \bigoplus_{0 \leq i \leq n-b} P(-\delta_i) \bigoplus_{1 \leq i < j \leq n} P(-m_i - m_{i_2} + d).
\]

**Corollary 2.6** (1) The Betti numbers of the ideal I are

\[
\beta_s = \text{dim}(R_s) = \begin{cases} 
(n - b + 2 - s)(n)_{s-1} + s(n)_{s+1}, & \text{if } 2 \leq s < n - b + 2 \\
(s - 1 - n + b)(n) + s(n)_{s+1}, & \text{if } n - b + 2 \leq s \leq n.
\end{cases}
\]

(2) The local ring \( R = k[[x_0, \ldots, x_n]]/I \) where I is the defining ideal of an AS monomial curve is of homogeneous type, i.e., \( \beta_s(R) = \beta_s(\mathcal{G}) \), where \( \mathcal{G} \) is the associated graded ring of \( R \).

**Proof.** (1). It suffices to recall that

\[
\text{dim}(R_s) = \text{dim}(C_s) - \text{dim}(D_s) = (n - b)(n)_{s-1} + (s - 1)(n) + s(n)_{s+1} - \nu_s(n) - \nu_{s-1}(n), \quad (s \geq 2).
\]

Since \( \nu_s := \min\{s - 1, n - b\} \), we get

\[
\beta_s = \text{dim}(R_s) = (n - b - \nu_{s-1})(n)_{s-1} + (s - 1 - \nu_s)(n) + s(n)_{s+1}.
\]

(2) follows immediately by [3] Theorem 4.1. \( \diamond \)

**References**

[1] P. Gimenez, I. Sengupta, H. Srinivasan “Minimal free resolutions for certain affine monomial curves” Commutative Algebra and Its connections to Geometry, *Contemporary Mathematics*, vol.555, AMS, (2011).

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