SUMS OF TRIANGULAR NUMBERS
FROM THE FROBENIUS DETERMINANT

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Abstract. We show that the denominator formula for the strange series of affine superalgebras, conjectured by Kac and Wakimoto and proved by Zagier, follows from a classical determinant evaluation of Frobenius. As a limit case, we obtain exact formulas for the number of representations of an arbitrary number as a sum of \(4m^2/d\) triangles, whenever \(d \mid 2m\), and \(4m(m+1)/d\) triangles, when \(d \mid 2m \) or \( d \mid 2m + 2\). This extends recent results of Getz and Mahlburg, Milne, and Zagier.

1. Introduction

To count the number of representations of an integer \(n\) as a sum of \(k\) triangular numbers is a classical problem [D]. We will denote this number by \(\triangle_k(n)\). The most fundamental results are

\[
\triangle_2(n) = \sum_{d \mid 4n+1} (-1)^\frac{1}{2}(d-1),
\]

\[
\triangle_4(n) = \sum_{d \mid 2n+1} d,
\]

\[
\triangle_8(n) = \sum_{d \mid n+1, (n+1)/d \text{ odd}} d^3
\]

(where \(d\) is assumed to be positive), which may be compared with the sums of squares formulas

\[
□_2(n) = 4 \sum_{d \mid n, d \text{ odd}} (-1)^\frac{1}{2}(d-1),
\]

\[
□_4(n) = 8 \sum_{d \mid n, 4 \nmid d} d,
\]

\[
□_8(n) = 16 \sum_{d \mid n} (-1)^{n+d}d^3;
\]

see [2,5] below for the exact conventions used for defining \(\triangle_k\). The identities (1.1b) and (1.1c) were found by Legendre, while (1.2b) and (1.2c) are due to

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Jacobi. As for (1.2a), it was first published by Legendre in 1798. It implies (1.1a), in view of
\[ 4 \left( \frac{k(k+1)}{2} + \frac{l(l+1)}{2} \right) + 1 = (k+l+1)^2 + (k-l)^2. \]

There have been many attempts to find exact formulas for $\triangle_k$ and $\Box_k$ when $k \neq 2, 4, 8$. Some ten years ago, a breakthrough was made by Kac and Wakimoto [KW], who showed that many results of this type can be obtained from denominator formulas for affine superalgebras. In particular, they conjectured a denominator formula for the strange series $Q(k)$ of affine superalgebras (see [K]), and showed that it would imply new formulas for $\triangle_{4m^2}$ (when $k = 2m-1$) and $\triangle_{4m(m+1)}$ (when $k = 2m$). When $m = 1$ one recovers Legendre’s results for 4 and 8 triangles.

Roughly speaking, the Kac–Wakimoto formulas correspond to replacing the divisor sums in (1.1) by sums over solutions to an equation of the form $k_1l_1 + \cdots + k_ml_m = y$. We recall the exact statements below, see (3.8) and (3.12). It may be instructive to give here the case $m = 2$ explicitly, namely,

\begin{align*}
\triangle_{16}(n) &= \frac{1}{2^7 \cdot 3} \sum_{k_1l_1 + k_2l_2 = 2n+4} k_1k_2(k_1^2 - k_2^2)^2, \\
\triangle_{24}(n) &= \frac{1}{2^4 \cdot 3^2} \sum_{k_1l_1 + k_2l_2 = n+3} k_1^3k_2^3(k_1^2 - k_2^2)^2.
\end{align*}

By symmetry, one may impose the condition $k_1 < k_2$ if the right-hand sides are multiplied by 2; this is of course preferable for the purpose of computation.

The denominator formula for the strange series was proved by Zagier [Z], using elliptic function identities. He also gave a second proof of the corresponding triangular number identities, using modular forms. Previously, Milne [M1, M2] had obtained the triangular number identities by a third approach.

The present paper rests on two simple observations. First, we note that the elliptic function identities used by Zagier can be written as pfaffian evaluations, which follow from classical determinant evaluations due to Frobenius and Stickelberger [Fr, FS]. This leads to a new proof of the Kac–Wakimoto conjecture. Although it is closely related to Zagier’s proof, it has the advantage of showing how the result could have been discovered in the nineteenth century, without using affine superalgebras.

To explain our second observation, we recall that the denominator formulas for $Q(2m-1)$ and $Q(2m)$ contain $m$ free variables, $x_1, \ldots, x_m$. The Kac–Wakimoto triangular number identities are obtained as a limit case when $x_j \to 1$ for all $j$. Getz and Mahlburg [GM] showed that in the case of $Q(2m-1)$, letting $x_j \to \omega_{2m}^j$, where $\omega_{2m}$ denotes a primitive $2m$:th root of unity, one similarly obtains an identity...
for $\triangle_{2m}$, see (4.6) below. The case $m = 1$ gives back (1.1), while $m = 2$ gives

\begin{equation}
\triangle_4(n) = \sum_{k_1l_1+k_2l_2=8n+4} (-1)^{\frac{1}{2}(k_1-1)+\frac{1}{2}(k_2-3)}.
\end{equation}

In the case of $Q(2m)$, Getz and Mahlburg let $x_j \rightarrow \omega_j^{2m+1}$. This leads to an identity for $\triangle_{2m+1}$, but of a more complicated type that we will not consider here.

Our second observation is that, more generally, starting from the denominator formula for $Q(2m-1)$ and letting $x_j \rightarrow \omega_j^d$, where $d$ is any positive divisor of $2m$, gives an exact formula for $\triangle_{4m^2/d}$. Thus, for any $m$ we have results corresponding to $d = 1$ (Kac–Wakimoto, Milne, Zagier), $d = 2$ (new), $d = m$ (new) and $d = 2m$ (Getz–Mahlburg), and if $m$ is neither prime nor equal to 4 there are additional results related to the remaining divisors of $2m$. When $m = 2$, we may choose $d = 1, 2, 4$, which apart from (1.3) and (1.5) gives the, to our knowledge, new identity

$$\triangle_8(n) = \frac{1}{2^5} \left( \sum_{k_1l_1+k_2l_2=4n+4} (k_1+k_2)^2 - \sum_{k_1l_1+k_2l_2=4n+4} (k_1-k_2)^2 \right).$$

Moreover, in the case of $Q(2m)$, letting $x_j \rightarrow \omega_j^d$, where $d \mid 2m$ or $d \mid 2m+2$ gives an exact formula for $\triangle_{4m(m+1)/d}$, which is of a simpler type than for the choice $d = 2m+1$ in [GM]. When $m = 1$, we may choose $d = 1, 2, 4$, which gives back the three fundamental triangular number identities (1.1). Possibly, this unified proof of (1.1) is new. When $m = 2$, the admissible values of $d$ are $1, 2, 3, 4, 6$, which apart from (1.4) gives the new identities

$$\triangle_4(n) = \sum_{k_1l_1+k_2l_2=6n+3} (-1)^{\chi(k_1 \equiv -1 \mod 6) + \chi(k_2 \equiv -2 \mod 6)},$$

where $\chi(\text{true}) = 1$, $\chi(\text{false}) = 0$,

$$\triangle_6(n) = \frac{1}{2} \sum_{k_1l_1+k_2l_2=4n+3} (-1)^{\frac{1}{2}(k_1-1)k_2},$$

$k_1$ positive, $l_j$ odd positive
$k_1 \equiv \pm 1$, $k_2 \equiv \pm 2 \mod 6$

$k_1$ positive, $l_j$ odd positive
$k_1$ odd, $k_2 \equiv \pm 2 \mod 4$
\[ \triangle_8(n) = \frac{1}{2 \cdot 3^2} \left( \sum_{k_1 l_1 + k_2 l_2 = 3n+3} (k_1 + k_2)^2 \right. \\
\left. - \sum_{k_1 l_1 + k_2 l_2 = 3n+3} (k_1 - k_2)^2 \right), \]

\[ \triangle_{12}(n) = \frac{1}{2^3} \sum_{k_1 l_1 + k_2 l_2 = 2n+3} k_1 k_2^3. \]

The paper is organized as follows. Section 2 contains preliminaries on theta functions and pfaffians, in particular our new proof of the Kac–Wakimoto conjecture. In Section 3 we review the special case \( d = 1 \) considered by Kac and Wakimoto. Our main result is given in Theorem 4.1 and in a slightly different form in Corollary 4.3; it is proved in Section 5.

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2. Preliminaries

2.1. Notation. The letter \( q \) will denote a number such that \( 0 < q < 1 \), which will be suppressed from the notation whenever convenient. Thus, we write

\[ (x)_\infty = (x; q)_\infty = \prod_{j=0}^{\infty} (1 - xq^j). \]

Let \( \omega_d = e^{2\pi i/d} \). Then,

\[ \prod_{k=1}^{d} (x\omega_d^k; q)_\infty = (x^d; q^d)_\infty. \]

We introduce the theta function

\[ \theta(x) = \theta(x; q) = (x, q/x; q)_\infty, \]

which satisfies

\[ \theta(x^{-1}) = \theta(qx) = -x^{-1} \theta(x). \]
We will sometimes use the shorthand notation
\[(a_1, \ldots, a_n)_\infty = (a_1, \ldots, a_n;q)_\infty = (a_1;q)_\infty \cdots (a_n;q)_\infty,\]
\[\theta(a_1, \ldots, a_n) = \theta(a_1, \ldots, a_n;q) = \theta(a_1; q) \cdots \theta(a_n; q).\]

We need the classical Laurent expansions
\[(2.3) \quad \frac{1}{x} \frac{\theta(x)}{\theta(\sqrt[q]{x})} = -\sum_{k=\infty}^{\infty} \frac{(\sqrt[q]{x})^k}{1 - q^{k+\frac{1}{2}}}, \quad q^{\frac{1}{2}} < |x| < q^{-\frac{1}{2}},\]
\[(2.4) \quad x \frac{\theta'(x)}{\theta(x)} = -\sum_{k \neq 0} \frac{x^k}{1 - q^k}, \quad q < |x| < 1,\]
which can both be derived from Ramanujan’s $\psi_1$ summation [GR].

For the generating function for triangular numbers we use the notation
\[\triangle(q) = \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)} = \frac{1}{2} \sum_{n=\infty}^{\infty} q^{\frac{1}{2}n(n+1)}.\]

By Jacobi’s triple product identity, we have the product formula
\[\triangle(q) = (q, -q, -q; q)_\infty = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}.\]

We write $\triangle_k(n)$ for the coefficients in the Taylor expansion
\[(2.5) \quad \triangle(q)^k = \sum_{n=0}^{\infty} \triangle_k(n) q^n, \quad |q| < 1,\]
which count the representations of $n$ as a sum of $k$ triangular numbers. As is customary, representations obtained from each other by reordering the terms are considered as different.

2.2. Pfaffians. We recall some basic facts about pfaffians. The pfaffian of a skew-symmetric even-dimensional matrix $A = (a_{ij})_{i,j=1}^{2m}$ is given by
\[\text{pfaff}_{1 \leq i,j \leq 2m} (a_{ij}) = \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} \text{sgn}(\sigma) \prod_{i=1}^{m} a_{\sigma(2i-1), \sigma(2i)}.\]

Equivalently,
\[(2.6) \quad \text{pfaff}_{1 \leq i,j \leq 2m} (a_{ij}) = \sum_{\sigma \in S_{2m}/G} \text{sgn}(\sigma) \prod_{i=1}^{m} a_{\sigma(2i-1), \sigma(2i)},\]
with $G$ the subgroup of order $2^m m!$ consisting of permutations preserving the set of pairs $\{\{1, 2\}, \{3, 4\}, \ldots, \{2m-1, 2m\}\}$. Its main property is
\[\text{pfaff}(A)^2 = \det(A).\]
For an odd-dimensional skew-symmetric matrix $A = (a_{ij})_{i,j=1}^{2m+1}$, we similarly define

$$\text{pfaff}(A) = \frac{1}{2^m m!} \sum_{\sigma \in S_{2m+1}} \text{sgn}(\sigma) \prod_{i=1}^m a_{2(2i-1), 2(2i)}.$$

It is easy to check that, with this definition,

$$\text{pfaff}(A) = \text{pfaff}(B),$$

where $B = (b_{ij})_{i,j=1}^{2m+2}$ is the matrix

$$B = \begin{pmatrix} A & 1 \\ \vdots & \vdots \\ -1 & \cdots & -1 & 0 \end{pmatrix}.$$ 

2.3. The Frobenius determinant and elliptic pfaffians. Frobenius obtained the determinant evaluation

$$\det_{1 \leq i,j \leq n+1} \left( \frac{\theta(t x_i y_j)}{\theta(t, x_i y_j)} \right) = \frac{\theta(t x_1 \cdots x_n y_1 \cdots y_n) \prod_{1 \leq i < j \leq n} x_j y_j \theta(x_i/x_j, y_i/y_j)}{\theta(t) \prod_{i,j=1}^n \theta(x_i y_j)},$$

which will form the basis of our analysis. It gives an elliptic extension of the Cauchy determinant

$$\det_{1 \leq i,j \leq n} \left( \frac{1}{x_i + y_j} \right) = \frac{\prod_{1 \leq i < j \leq n} (x_i - x_j)(y_i - y_j)}{\prod_{i,j=1}^n (x_i + y_j)}.$$ 

For other recent applications of (2.7), see [CC, KN, R, Ru]. The reader interested in elliptic determinant evaluations should consult [Kr, RS] for more information and further references.

We also need a determinant evaluation due to Frobenius and Stickelberger, which may be obtained as a degenerate case of (2.7). Namely, rewriting (2.7) as

$$\det_{1 \leq i,j \leq n+1} \left( \begin{array}{ccc} \theta(t x_i y_j) & 0 \\ \theta(t, x_i y_j) & \vdots \\ -1 & \cdots & -1 & \theta(t) \end{array} \right) = \frac{\theta(t x_1 \cdots x_n y_1 \cdots y_n) \prod_{1 \leq i < j \leq n} x_j y_j \theta(x_i/x_j, y_i/y_j)}{\prod_{i,j=1}^n \theta(x_i y_j)}.$$
and then subtracting $\theta(t)^{-1}$ times the last row from the previous ones gives

$$
\det_{1 \leq i,j \leq n+1} \begin{pmatrix}
\theta(tx_iy_j) - \theta(x_iy_j) & 1 \\
\theta(t, x_iy_j) & \vdots \\
-1 & \cdots & -1 & 1 \\
\theta(t)
\end{pmatrix} = \frac{\theta(tx_1 \cdots x_n y_1 \cdots y_n) \prod_{1 \leq i < j \leq n} x_jy_j \theta(x_i/x_j, y_i/y_j)}{\prod_{i,j=1}^n \theta(x_iy_j)}.
$$

We now let $t \to 1$, obtaining in the limit the Frobenius–Stickelberger determinant

$$
(2.8) \quad \det_{1 \leq i,j \leq n+1} \begin{pmatrix}
x_iy_j\theta'(x_iy_j) & 1 \\
\theta'(1)\theta(x_iy_j) & \vdots \\
-1 & \cdots & -1 & 0 \\
1
\end{pmatrix} = \frac{\theta(x_1 \cdots x_n y_1 \cdots y_n) \prod_{1 \leq i < j \leq n} x_jy_j \theta(x_i/x_j, y_i/y_j)}{\prod_{i,j=1}^n \theta(x_iy_j)}.
$$

We are interested in pfaffian evaluations related to (2.7) and (2.8). In (2.7), we let $n = 2m$ and choose $t = \sqrt{q}$, $y_j = \sqrt{q}/x_j$. Using (2.2), the resulting identity can be written as

$$
\det_{1 \leq i,j \leq 2m} \left( \frac{\theta(x_j/x_i)}{x_j\theta(\sqrt{q}x_j/x_i)} \right) = q^{\frac{1}{2}m(m-1)} \prod_{i=1}^{2m} x_i^{2m-2i} \prod_{1 \leq i < j \leq 2m} \frac{\theta(x_j/x_i)^2}{\theta(\sqrt{q}x_j/x_i)^2}.
$$

The matrix on the left is skew-symmetric, so we can almost deduce that

$$
(2.9) \quad \text{pfaff}_{1 \leq i,j \leq 2m} \left( \frac{\theta(x_j/x_i)}{x_j\theta(\sqrt{q}x_j/x_i)} \right) = q^{\frac{1}{2}m(m-1)} \prod_{i=1}^{2m} x_i^{m-i} \prod_{1 \leq i < j \leq 2m} \frac{\theta(x_j/x_i)}{\theta(\sqrt{q}x_j/x_i)}.
$$

More precisely, we know that (2.9) holds up to a factor $\pm 1$ (possibly depending on $m$). It is not hard to show directly that this factor is always $+1$ (cf. the final paragraph of [Z]), but since that will anyway be clear from our computations below, see Remark 3.1, we will for the moment assume that (2.9) is valid.

Applying the same argument to (2.8), using also $\theta'(1) = -(q)_\infty^2$, $\theta(\sqrt{q}) = (\sqrt{q})_\infty^2$, we obtain, up to a factor $\pm 1$, the odd-dimensional pfaffian evaluation

$$
(2.10) \quad \text{pfaff}_{1 \leq i,j \leq 2m+1} \left( \frac{x_i\theta'(\sqrt{q}x_i/x_j)}{x_j\theta(\sqrt{q}x_i/x_j)} \right) = q^{\frac{1}{2}m(m-1)} \frac{(q)_\infty^{2m}}{(\sqrt{q})_\infty^{2m}} \prod_{i=1}^{2m+1} x_i^{m+1-i} \prod_{1 \leq i < j \leq 2m+1} \frac{\theta(x_j/x_i)}{\theta(\sqrt{q}x_j/x_i)}.
$$

We will see below that the sign chosen in (2.10) is correct.
The evaluations (2.9) and (2.10) appear as \[Z, \text{Eq. (7)}\] (where the pfaffians are written explicitly as alternating sums). As is demonstrated in \[Z\], they are equivalent to the denominator formula for the superalgebra \(Q(2m - 1)\) and \(Q(2m)\), respectively, which was conjectured by Kac and Wakimoto [KW]. Thus, our observation that these identities follow from (2.7) yields a new proof of the Kac–Wakimoto conjecture.

**Remark 2.1.** It is interesting to compare (2.9) with some other recent elliptic pfaffian evaluations. We first note that letting \(n = 2m\), \(t = -1\) and \(y_j = -1/x_j\) in (2.7) we obtain, up to a factor \(\pm 1\),

\[
\text{pfaff}_{1 \leq i,j \leq 2m} \left( \frac{x_j}{x_i} \right) \prod_{1 \leq i<j \leq 2m} \theta(x_j/x_i).
\]

For \(p = 0\), this is Schur’s identity [S]

\[
\text{pfaff}_{1 \leq i,j \leq 2m} \left( \frac{x_i - x_j}{x_i + x_j} \right) = \prod_{1 \leq i<j \leq 2m} \frac{x_i - x_j}{x_i + x_j};
\]

in particular, the sign in (2.11) is correct. Alternatively, one may obtain (2.11) as the modular dual of (2.9). Namely, the two results are related by the modular transformation for Jacobi theta functions, which in our notation takes the form

\[
\theta(e^{2\pi ix}; \tilde{q}) = -i\sqrt{\frac{q}{\tilde{q}^2}} e^{\pi x(1-x)q^{1/2}} \theta(q^{1/2}; q),
\]

where

\[
q = e^{-2\pi h}, \quad \tilde{q} = e^{-2\pi/\hbar}.
\]

A different elliptic extension of Schur’s identity was recently obtained by Okada [O], namely,

\[
\text{pfaff}_{1 \leq i,j \leq 2m} \left( \frac{x_i \theta(x_j/x_i, z x_i x_j, w x_i x_j)}{\theta(x_i x_j, z, w)} \right) = \frac{\theta(zx_1 \cdots x_{2m}, wx_1 \cdots x_{2m})}{\theta(z, w)} \prod_{1 \leq i<j \leq 2m} \frac{x_i \theta(x_j/x_i)}{\theta(x_i x_j)}.
\]

We also mention Rains’ pfaffian evaluation [R]

\[
\text{pfaff}_{1 \leq i,j \leq 2m} \left( \frac{\theta(x_i x_j, x_j/x_i)}{x_j \theta(\sqrt{q x_i x_j}, \sqrt{q x_j/x_i})} \right) = q^{\frac{1}{2}(m-1)} \prod_{1 \leq i<j \leq 2m} \frac{\theta(x_i x_j, x_j/x_i)}{x_j \theta(\sqrt{q x_i x_j}, \sqrt{q x_j/x_i})}.
\]

Its modular dual is

\[
\text{pfaff}_{1 \leq i,j \leq 2m} \left( \frac{\theta(x_i x_j, x_j/x_i)}{\theta(-x_i x_j, -x_j/x_i)} \right) = \prod_{1 \leq i<j \leq 2m} \frac{\theta(x_i x_j, x_j/x_i)}{\theta(-x_i x_j, -x_j/x_i)}.
\]

which gives a third elliptic extension of Schur’s pfaffian, different from both (2.11) and (2.12).
Remark 2.2. The Frobenius determinant (2.7) has been generalized to higher genus Riemann surfaces by Fay [F1, Corollary 2.19]. Similarly, it follows from the work of Fay [F2] that the pfaffian evaluations (2.9) and (2.11) can be extended to Prym varieties. We owe this piece of information to Eric Rains.

3. The special case \( d = 1 \)

Kac and Wakimoto utilized their (at that time conjectural) denominator formula to obtain new formulas for the number of representations of an integer as a sum of \( 4m^2 \) and \( 4m(m + 1) \) triangles. Before discussing generalizations, it will be convenient to review the details of this special case.

We first consider the pfaffian (2.9). The first step is to use (2.3) to expand the left-hand side as a multiple Laurent series. We must then assume

\[
|x_i/x_j| < q^{-1/2}, \quad i \neq j.
\]

After interchanging the finite and infinite summations, the left-hand side of (2.9) takes the form

\[
(3.1) \quad (-1)^m (\sqrt{q})^{2m} \sum_{k_1, \ldots, k_m = -\infty}^{\infty} \prod_{i=1}^{m} \frac{q^{\frac{1}{2}k_i}}{1 - q^{k_i + \frac{1}{2}}} \sum_{\sigma \in S_{2m}} \text{sgn}(\sigma) \prod_{i=1}^{m} x_{\sigma(2i)} x_{\sigma(2i-1)}.
\]

We next make the specialization \( x_j = t^j \). Then the inner sum in (3.1) becomes a special case of the Vandermonde determinant

\[
(3.2) \quad \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} y_i^{\sigma(i)-1} = \det (y_i^{j-1}) = \prod_{1 \leq i < j \leq n} (y_j - y_i).
\]

Namely, choosing \( n = 2m \), \((y_1, \ldots, y_n) = (t^{-k_1-1}, t^{k_1}, \ldots, t^{-k_m-1}, t^{k_m})\), we obtain after simplification

\[
\sum_{\sigma \in S_{2m}} \text{sgn}(\sigma) \prod_{i=1}^{m} t^{k_i\sigma(2i)-(k_i+1)\sigma(2i-1)}
\]

\[
= t^{-m} \prod_{i=1}^{m} (t^{k_i} - t^{-k_i-1})
\]

\[
\times \prod_{1 \leq i < j \leq m} (t^{k_j} - t^{k_i})(t^{k_j} - t^{-k_i-1})(t^{-k_j-1} - t^{k_i})(t^{-k_j-1} - t^{-k_i-1})
\]

\[
= (-1)^m t^{-\frac{m}{2}(m+1)} \prod_{i=1}^{m} t^{(1-2m)k_i} (1 - t^{2k_i+1}) \prod_{1 \leq i < j \leq m} (t^{k_j} - t^{k_i})^2 (1 - t^{k_i+k_j+1})^2.
\]
This gives

\[(3.4) \quad \frac{1}{2^m m!} \sum_{k_1, \ldots, k_m = -\infty}^{\infty} \prod_{i=1}^{m} (q^{1-2m} q^{\frac{1}{2}})^{k_i} \frac{1 - t^{2k_i + 1}}{1 - q^{k_i + \frac{1}{2}}} \prod_{1 \leq i < j \leq m} (t^{k_j} - t^{k_i})^2 (1 - t^{k_i + k_j + 1})^2 \]

\[= q^{\frac{1}{2} m(m-1)} t^{-\frac{1}{2} m(m-1)(4m+1)} \frac{(q)^{2m}}{(\sqrt{q})^{2m}} \prod_{1 \leq i < j \leq 2m} \frac{\theta(t^{j-i})}{\theta(\sqrt{q} t^{j-i})}, \]

which holds for \( q^{1/2} < |t^{2m-1}| < q^{-1/2} \). Here we also used

\[t^{\frac{1}{2} m(3m+1) - \frac{1}{2} m(m+1)(2m+1)} = t^{-\frac{1}{2} m(m-1)(4m+1)}. \]

It is clear that the left-hand side of \((3.2)\), and thus also of \((3.4)\), is invariant under the change of variables \( k_i \mapsto -k_i - 1 \), for any \( i \). Thus, if we multiply by \( 2^m \) we may assume that each \( k_i \) is positive. It will also be convenient to make the change of summation variables \( k_i \mapsto (k_i - 1)/2 \), giving

\[(3.5) \quad \frac{1}{m!} \sum_{k_1, \ldots, k_m = \text{odd positive}}^{\infty} \prod_{i=1}^{m} (q^{1-2m} q^{\frac{1}{2}})^{\frac{1}{2}(k_i-1)} \frac{1 - t^{k_i}}{1 - q^{\frac{1}{2} k_i}} \times \prod_{1 \leq i < j \leq m} (t^{\frac{1}{2}(k_j-1)} - t^{\frac{1}{2}(k_i-1)})^2 (1 - t^{\frac{1}{2}(k_i + k_j)})^2 \]

\[= q^{\frac{1}{2} m(m-1)} t^{-\frac{1}{2} m(m-1)(4m+1)} \frac{(q)^{2m}}{(\sqrt{q})^{2m}} \prod_{1 \leq i < j \leq 2m} \frac{\theta(t^{j-i})}{\theta(\sqrt{q} t^{j-i})}. \]

Following Kac and Wakimoto, we now divide both sides of \((3.5)\) by

\[\prod_{1 \leq i < j \leq 2m} (1 - t^{j-i})\]

and then let \( t \to 1 \). On the right-hand side, we have

\[(3.6) \quad \frac{(q)^{2m}}{(\sqrt{q})^{2m}} \prod_{1 \leq i < j \leq 2m} \frac{\theta(t^{j-i})}{(1 - t^{j-i}) \theta(\sqrt{q} t^{j-i})} = \prod_{i,j=1}^{2m} \frac{(q)^{t^{j-i}}}{(\sqrt{q} t^{j-i})} \to \Delta(\sqrt{q})^{4m^2}. \]

On the left-hand side, we consider the factor

\[\prod_{i=1}^{m} (1 - t^{k_i}) \prod_{1 \leq i < j \leq m} (t^{\frac{1}{2}(k_j-1)} - t^{\frac{1}{2}(k_i-1)})^2 (1 - t^{\frac{1}{2}(k_i + k_j)})^2 \prod_{1 \leq i < j \leq 2m} (1 - t^{j-i}). \]

Since \( m + 4 \binom{m}{2} = \binom{2m}{2} \), the denominator and numerator vanish of the same order at \( t = 1 \), so that the quotient tends to

\[\prod_{i=1}^{m} k_i \prod_{1 \leq i < j \leq m} \frac{(k_j - k_i)^2}{2} \frac{(k_i + k_j)^2}{2} \prod_{1 \leq i < j \leq 2m} (j - i) = \prod_{i=1}^{m} k_i \prod_{1 \leq i < j \leq m} \frac{(k_j^2 - k_i^2)^2}{4m(m-1) \prod_{j=1}^{2m-1} j^2}. \]
Thus, we obtain the identity

\[
\frac{q^{-\frac{1}{2}m(m-1)}}{4^{m(m-1)}m! \prod_{j=1}^{2m-1} j!} \sum_{k_1, \ldots, k_m \text{ odd positive}} \prod_{i=1}^{m} \frac{q^{\frac{1}{2}(k_i - 1)}}{1 - q^{\frac{1}{2}k_i}} \prod_{1 \leq i < j \leq m} (k_j^2 - k_i^2)^2 = \triangle(\sqrt{q})^{4m^2}.
\]

We now expand the left-hand side directly as a power series in $\sqrt{q}$, using

\[(3.7) \quad \frac{1}{1 - q^{\frac{1}{2}k_i}} = \sum_{l_{i} \text{ odd positive}} q^{\frac{1}{2}k_i(l_{i} - 1)},\]

which gives

\[
\frac{1}{4^{m(m-1)}m! \prod_{j=1}^{2m-1} j!} \sum_{k_1, \ldots, k_m, l_1, \ldots, l_m \text{ odd positive}} q^{\frac{1}{2}(k_1l_1 + \ldots + k_ml_m - m^2)} \prod_{i=1}^{m} k_i \prod_{1 \leq i < j \leq m} (k_j^2 - k_i^2)^2.
\]

In conclusion, this proves that

\[(3.8) \quad \triangle_{4m^2}(n) = \frac{1}{4^{m(m-1)}m! \prod_{j=1}^{2m-1} j!} \sum_{k_1l_1 + \ldots + k_ml_m = 2m + m^2} \prod_{i=1}^{m} k_i \prod_{1 \leq i < j \leq m} (k_j^2 - k_i^2)^2.
\]

**Remark 3.1.** Had we started from the identity similar to (2.9) but with the right-hand side multiplied by $-1$, we would have obtained a similarly modified version of (3.8), which would clearly be absurd. Thus, the sign chosen in (2.9) is correct.

We now turn to the case of (2.10), where we will be less detailed. However, we write down some intermediate steps for later reference. Still assuming (3.1), we apply (2.4) to rewrite the left-hand-side of (2.10) as

\[(3.9) \quad \frac{(-1)^m q^{-\frac{1}{2}m}}{2^mm!} \sum_{k_1, \ldots, k_m \neq 0} \prod_{i=1}^{m} \frac{q^{k_i/2}}{1 - q^{k_i}} \sum_{\sigma \in S_{2m+1}} \text{sgn}(\sigma) \prod_{i=1}^{m} x_{\sigma(2i-1)}^{k_i} x_{\sigma(2i)}^{k_i}.
\]

As before, we choose $x_i = t^i$. By the case $n = 2m + 1$, $(y_1, \ldots, y_n) = (t^{k_1}, t^{-k_1}, \ldots, t^{k_m}, t^{-k_m}, 1)$ of (3.3), the inner sum in (3.9) equals

\[
(-1)^m \prod_{i=1}^{m} t^{-2mk_i} (1 - t^{k_i}) (1 - t^{2k_i}) \prod_{1 \leq i < j \leq m} (t^{k_j} - t^{k_i})^2 (1 - t^{k_i + k_j})^2.
\]
Exploiting the symmetry $k_i \mapsto -k_i$, we reduce the summation to positive $k_i$, giving

\begin{equation}
\frac{1}{m!} \sum_{k_1, \ldots, k_m = 1}^{\infty} \prod_{i=1}^{m} \left( \frac{q^{1/2} t^{-2m} k_i}{1 - q^{k_i}} \right) (1 - t^{k_i})^2 (1 - t^{2k_i}) 
\times \prod_{1 \leq i < j \leq m} (t^{k_j} - t^{k_i})^2 (1 - t^{k_i + k_j})^2
= q^{1/2} m(m+1) t^{-1/2} m(m+1)(2m+1) \frac{(q)^{2m}}{(\sqrt{q})^{2m}} \prod_{1 \leq i < j \leq m+1} \frac{\theta(t^{j-i})}{\theta(\sqrt{q}t^{j-i})},
\end{equation}

which holds for $q^{1/2} < |t^{2m}| < q^{-1/2}$.

Dividing (3.10) by $\prod_{1 \leq i < j \leq 2m+1} (1 - t^{j-i})$ and letting $t$ tend to 1 gives

\begin{equation}
\frac{2^m q^{1/2} m(m+1)}{m! \prod_{j=1}^{2m} j!} \sum_{k_1, \ldots, k_m = 1}^{\infty} \prod_{i=1}^{m} \frac{q^{1/2} k_i^3}{1 - q^{k_i}} \prod_{1 \leq i < j \leq m} (k_j^2 - k_i^2)^2 = \Delta(\sqrt{q})^{4m(m+1)}.
\end{equation}

Expanding the denominator using

\begin{equation}
\frac{1}{1 - q^{k_i}} = \sum_{l_i \text{ odd positive}} q^{1/2} k_i (l_i - 1)
\end{equation}

and identifying the coefficient of $q^{n/2}$ we obtain

\begin{equation}
\Delta_{4m(m+1)}(n) = \frac{2^m}{m! \prod_{j=1}^{2m} j!} \sum_{k_1l_1 + \cdots + k_ml_m = n + 1/2 m(m+1)} \prod_{i=1}^{m} k_i^3 \prod_{1 \leq i < j \leq m} (k_j^2 - k_i^2)^2.
\end{equation}

In particular, we conclude that the choice of sign in (2.10) is correct.

4. The General Case

When $(k_1, \ldots, k_m)$ and $(l_1, \ldots, l_m)$ are multi-indices, let us write

$$(k_1, \ldots, k_m) \simeq (l_1, \ldots, l_m) \mod n$$

if they are equal modulo $n$ up to reordering and sign, that is, if there exists a permutation $\sigma \in S_m$ and numbers $\varepsilon_i \in \{\pm 1\}$ such that $k_{\sigma(i)} \equiv \varepsilon_i l_i \mod n$ for $i = 1, \ldots, m$. In this notation, we can state our main result as follows.
Theorem 4.1. Let \( d, m \) and \( x \) be non-negative integers. Then, if \( d \mid 2m \),

\[
(4.1) \quad \sum_{\substack{k_1 l_1 + \cdots + k_m l_m = m^2 + x \\text{and } l_i \text{ odd positive} \\text{for } (k_1, \ldots, k_m) \equiv (1,3,5,\ldots,2m-1) \ (2d)}} (-1)^{|\{i; d+1 \leq k_i \leq 2d-1 \ \text{mod } 2d\}|} 
\times \prod_{1 \leq i \leq m \atop k_i \equiv d \ (2d)} k_i \prod_{1 \leq i < j \leq m \atop k_i \equiv k_j \ (2d)} \left( \frac{k_j - k_i}{2} \right)^2 \prod_{1 \leq i < j \leq m \atop k_i \equiv -k_j \ (2d)} \left( \frac{k_j + k_i}{2} \right)^2 
\end{equation}

\[
= (-1)^{(d \prod_{1 \leq i \leq m \atop k_i \equiv 0 \ (d)} 2k_i^2 \prod_{1 \leq i < j \leq m \atop k_i \equiv d \ (d) / 2} 2k_i (k_j - k_i)^2 \prod_{1 \leq i < j \leq m \atop k_i \equiv -k_j \ (d)} (k_i + k_j)^2)}^{(2m-d)/d} m! \prod_{l=1}^{(2m-d)/d} l! d \Delta_{4m^2/d}(x/2d).
\]

Moreover, if \( d \mid 2m \) or \( d \mid 2m+2 \),

\[
(4.2) \quad \sum_{\substack{k_1 l_1 + \cdots + k_m l_m = \frac{1}{2} m(m+1) + x \\text{and } l_i \text{ odd positive} \\text{for } (k_1, \ldots, k_m) \equiv (1,2,\ldots,m) \ (d)}} (-1)^{|\{i; (d+1)/2 \leq k_i \leq d-1 \ \text{mod } d\}|} 
\times \prod_{1 \leq i \leq m \atop k_i \equiv 0 \ (d)} 2k_i^2 \prod_{1 \leq i < j \leq m \atop k_i \equiv d \ (d) / 2} 2k_i (k_j - k_i)^2 \prod_{1 \leq i < j \leq m \atop k_i \equiv -k_j \ (d)} (k_i + k_j)^2 
\end{equation}

\[
= C d^{(2m+2-d)/d} m! \Delta_{4m^2/(m+1)}(x/d),
\]

where

\[
(4.3) \quad C = \begin{cases} 
(-1)^{(d-1)/2} \left( \frac{2m^2}{2} \right) \frac{(2m)!}{(2m-d)/d} \prod_{l=1}^{(2m-d)/d} l! d, & d \mid 2m, \\
(-1)^{(d-1)/2} \left( \frac{2m^2}{2} \right) \frac{(2m+2)!}{(2m+2-d)/d} \prod_{l=1}^{(2m+2-d)/d} l! d, & d \mid 2m+2.
\end{cases}
\]

The right-hand side of (4.1) and (4.2) should be interpreted as zero if \( 2d \nmid x \) and \( d \nmid x \), respectively.

Remark 4.2. Since the sums in Theorem 4.1 are symmetric in \( k_i \) and vanish if \( k_i = k_j \) for \( i \neq j \), each term is repeated \( m! \) times. If one wants to compute the sums, one should first get rid of this redundancy. This can be done, for instance, by imposing the condition \( k_1 < k_2 < \cdots < k_m \) and deleting the factor \( m! \) from the right-hand side. However, it is more convenient to, in the case of (4.1), first impose the condition \( k_i \equiv \pm (2i-1) \ (2d) \) and then the condition \( k_i < k_j \) if \( i < j \) and \( k_i \equiv \pm k_j \ (2d) \), and similarly for (4.2).

In Theorem 4.1 we have tried to state the results in a unified form. However, this hides some structural differences between even and odd \( d \). In Corollary 4.3 we rewrite the identities in a way that emphasizes these differences, and is also better suited for computation as indicated in Remark 4.2. Moreover, we have removed a power of 2 from the left-hand sides, using (5.7) and (5.12) below.
Corollary 4.3. Let \( d, m \) and \( x \) be non-negative integers. Then, if \( d \) is odd and \( d \mid m \),

\[
\sum_{\substack{k_1 l_1 + \cdots + k_m l_m = m^2 + x \\ k_i \text{ and } l_i \text{ odd positive} \\ k_i \equiv \pm(2l-1) \mod 2d}} (-1)^{\{i; k_i \equiv d+2, d+4, \ldots, 2d-1 \mod (2d)\}} \prod_{1 \leq i \leq m} k_i \prod_{1 \leq i < j \leq m} (\mp \frac{k_j - k_i}{k_i - k_j}) \prod_{1 \leq i < j \leq m} (k_i + k_j) \prod_{1 \leq i < j \leq m} (k_j + k_i) = (-1)^{\frac{d+1}{2} m} (2^{m-d-1} d^{2m-d})^{m/d} \prod_{l=1}^{(2m-d)/d} l!^d \triangle_{4m^2/d}(x/2d),
\]

while if \( d \) is even and \( d \mid 2m \),

\[
\sum_{\substack{k_1 l_1 + \cdots + k_m l_m = m^2 + x \\ k_i \text{ and } l_i \text{ odd positive} \\ k_i \equiv \pm(2l-1) \mod 2d}} (-1)^{\{i; k_i \equiv d+1, d+3, \ldots, 2d-1 \mod (2d)\}} \prod_{1 \leq i < j \leq m} (k_j - k_i) \prod_{1 \leq i < j \leq m} (k_j + k_i) \prod_{1 \leq i < j \leq m} (k_i + k_j)^2 = (-1)^{\frac{d}{2}(2m/d)} (2d)^{(2m-d)/d} \prod_{l=1}^{(2m-d)/d} l!^d \triangle_{4m^2/d}(x/2d).
\]

Moreover, if \( d \) is odd and \( d \mid m \) or \( d \mid m+1 \),

\[
\sum_{\substack{k_1 l_1 + \cdots + k_m l_m = \frac{1}{2}m(m+1)+x \\ k_i \text{ positive, } l_i \text{ odd positive} \\ k_i \equiv \pm i \mod d}} (-1)^{\{i; k_i \equiv (d+1)/2, (d+3)/2, \ldots, d-1 \mod (d)\}} \prod_{1 \leq i \leq m} k_i^3 \prod_{1 \leq i < j \leq m} (k_j - k_i)^2 \prod_{1 \leq i < j \leq m} (k_i + k_j)^2 = A d^{(2m+2-d)/d} \triangle_{4m(m+1)/d}(x/d),
\]

where

\[
A = \begin{cases} 
(-1)^{\frac{d-1}{2} (\frac{2m}{d})^2} 2^{-m/d} (2m/d)! \prod_{l=1}^{(2m-d)/d} l!^d, & d \mid m, \\
(-1)^{\frac{d-1}{2} (\frac{2(m+1)}{d})^2} 2^{-(m+1-d)/d} \times ((2m+2-d)/d)!^{-1} \prod_{l=1}^{(2m+2-d)/d} l!^d, & d \mid m+1,
\end{cases}
\]
while if $d$ is even and $d \mid 2m$ or $d \mid 2m + 2$,

\[(4.5b) \sum_{k_1l_1 + \cdots + k_ml_m = \frac{1}{2}m(m+1)+x} (-1)^{|\{i; k_i \equiv (d+2)/2, (d+4)/2, \ldots, d-1 (d)\}|} \times \prod_{1 \leq i \leq m; k_i \equiv 0 (d)} k_i^3 \prod_{1 \leq i \leq m; k_i \equiv d/2 (d)} k_i \prod_{1 \leq i < j \leq m; k_i \equiv k_j (d)} (k_i - k_j)^2 \prod_{1 \leq i < j \leq m; k_i \equiv -k_j (d)} (k_i + k_j)^2,
\]

where

\[B = \begin{cases} (-1)^{(d-1)/2} (2m/d)! 2^{-2m/d} (2m/d)! \prod_{l=1}^{(2m-d)/d} l!^d, & d \mid 2m, \\ (-1)^{(d-1)/2} (2m+1)/2)! 2^{-(2m+2-d)/d} \times ((2m + 2 - d)/d)!^{-1} \prod_{l=1}^{(2m+2-d)/d} l!^d, & d \mid 2m + 2. \end{cases} \]

Apart from the case $d = 1$ discussed in Section 3, the other extremal case, $d = 2m$ of (4.1) (or (4.4b)) and $d = 2m + 2$ of (4.2) (or (4.5b)) is of special interest. Then, all products on the left are empty except for the power of $-1$, so that both sides of the identity have a clear combinatorial meaning. In the case of (4.6), we recover

\[(4.6) \sum_{k_1l_1 + \cdots + k_ml_m = m^2 + x} (-1)^{|\{i; k_i \equiv 1 \odot 2i \ (4m)\}|} = \Delta_{2m}(x/4m),
\]

which is equivalent to [GM, Corollary 1.3]. Similarly, (4.5b) gives

\[(4.7) \sum_{k_1l_1 + \cdots + k_ml_m = \frac{1}{2}m(m+1)+x} (-1)^{|\{i; k_i = -i \ (2m+2)\}|} = \Delta_{2m}(x/(2m + 2)).
\]

The fact that these sums vanish for $4m \uparrow x$ and $2m + 2 \uparrow x$, respectively, and are otherwise non-negative gives some non-trivial information, see [GM] for the case of (4.6).

5. Proof of Theorem 4.1

To prove Theorem 4.1 we will let $t \rightarrow \omega_d = e^{2\pi i/d}$ in (3.5) and (3.10), after dividing both sides by a suitable factor. We will assume $d \mid 2m$ in the case of (3.5) and $d \mid 2m$ or $d \mid 2m + 2$ in the case of (3.10), obtaining (4.1) and (4.2), respectively. As is already clear from [GM], other values of $d$ may also have arithmetic consequences, but we restrict here to the simplest situation.
Before working out the details, we collect some elementary but useful facts. Note that part (b) of the following Lemma gives a more explicit description of the range of summation in (4.1).

**Lemma 5.1.** Assume that \( d \mid 2m \) and consider the sequence \((1, 3, 5, \ldots, 2m - 1)\) reduced modulo \( 2d \). Then:

(a) The number of elements of the sequence that are congruent to \( i \) modulo \( 2d \) equals
\[
\frac{m}{d} - \frac{1}{2}, \quad i = d + 1, d + 3, \ldots, 2d - 1
\]
if \( 2m/d \) is odd (and thus \( d \) is even) and
\[
\frac{m}{d}, \quad i = 1, 3, \ldots, 2d - 1
\]
if \( 2m/d \) is even.

(b) The number of elements of the sequence congruent to \( \pm i \) modulo \( 2d \) equals
\[
2m/d, \quad i = 1, 3, \ldots, 2d - 1, \; i \neq d,
\]
\[
m/d, \quad i = d, \; d \text{ odd.}
\]

(c) The number of elements \( x \) of the sequence such that \( d + 1 \leq x \leq 2d - 1 \) modulo \( 2d \) has the same parity as \( \binom{d/2}{2} \binom{2m/d}{2} \).

**Proof.** The proof of (a) is trivial, and (b) follows immediately from (a). As for the final statement, it follows from (a) that the number of such \( x \) equals
\[
\frac{d}{2} \left( \frac{m}{d} - \frac{1}{2} \right), \quad \frac{d}{2} \cdot \frac{m}{d}, \quad \frac{d - 1}{2} \cdot \frac{m}{d}
\]
according to whether \( 2m/d \) is odd and \( d \) even, \( 2m/d \) and \( d \) are both even or \( 2m/d \) is even and \( d \) odd, respectively. The quotients of \( \binom{d/2}{2} \binom{2m/d}{2} \) by these numbers equal
\[
\frac{2m}{d} (d - 1), \quad \left( \frac{2m}{d} - 1 \right) (d - 1), \quad \left( \frac{2m}{d} - 1 \right) d,
\]
which in each case is odd. \( \square \)

In the case of \( (4.2) \), the corresponding facts are somewhat more tedious to state.

**Lemma 5.2.** Assume that \( d \mid 2m \) or \( d \mid 2m + 2 \) and consider the sequence \((1, 2, 3, \ldots, m)\) reduced modulo \( d \). Then:

(a) The number of elements of the sequence congruent to \( i \) modulo \( d \) equals
\[
\frac{m}{d} + \frac{1}{2}, \quad i = 1, 2, \ldots, \frac{d}{2},
\]
\[
\frac{m}{d} - \frac{1}{2}, \quad i = \frac{d}{2} + 1, \frac{d}{2} + 2, \ldots, d
\]
if $2m/d$ is odd:
\[
m/d, \quad i = 1, 2, \ldots, d
\]

if $2m/d$ is even:
\[
\begin{align*}
\frac{m+1}{d} + \frac{1}{2}, & \quad i = 1, 2, \ldots, \frac{d-1}{2}, \\
\frac{m+1}{d} - \frac{1}{2}, & \quad i = \frac{d}{2}, \frac{d+1}{2}, \ldots, d
\end{align*}
\]

if $(2m+2)/d$ is odd and
\[
\frac{m+1}{d} - 1, \quad i = d
\]

if $(2m+2)/d$ is even.

(b) The number of elements of the sequence congruent to $\pm i$ modulo $d$ equals
\[
\begin{align*}
2m/d, & \quad i = 1, 2, \ldots, d-1, i \neq d/2, \\
\frac{m}{d} + \frac{1}{2}, & \quad i = d/2, \\
\frac{m}{d} - \frac{1}{2}, & \quad i = d
\end{align*}
\]

if $2m/d$ is odd;
\[
\begin{align*}
2m/d, & \quad i = 1, 2, \ldots, d-1, i \neq d/2, \\
m/d, & \quad i = d/2 (d \text{ even}), i = d
\end{align*}
\]

if $2m/d$ is even;
\[
\begin{align*}
2(m+1)/d, & \quad i = 1, 2, \ldots, d-1, i \neq d/2, \\
\frac{m+1}{d} - \frac{1}{2}, & \quad i = d/2, i = d
\end{align*}
\]

if $(2m+2)/d$ is odd and
\[
\begin{align*}
2(m+1)/d, & \quad i = 1, 2, \ldots, d-1, i \neq d/2, \\
\frac{m+1}{d} - 1, & \quad i = d (d \text{ even}), \\
\frac{m+1}{d} - 1, & \quad i = d
\end{align*}
\]

if $(2m+2)/d$ is even.

(c) The number of elements $x$ of the sequence such that $(d+1)/2 \leq x \leq d-1$ modulo $d$ has the same parity as $(\frac{d-1}{2})(\frac{2m}{d})$ if $d \mid 2m$ and $(\frac{d-1}{2})(\frac{2(m+1)}{d})$ if $d \mid 2m+2.$
Proof of (4.1). We first assume \( d \mid 2m \) and consider the right-hand side of (3.5). After dividing by the factor

\[
P(t) = \prod_{1 \leq i < j \leq 2m} (1 - t^{j-i}),
\]

we have as in (3.6)

\[
(5.1) \quad \frac{(q)_{2m}}{(\sqrt{q})_{2m}} \prod_{1 \leq i < j \leq 2m} \left( \frac{\theta(t^{j-i})}{(1 - t^{j-i})} \right) = \prod_{i,j=1}^{2m} \left( \frac{q^{\omega_d^{-1}j-i}}{(\sqrt{q}^{\omega_d^{-1}j-i})} \right) = \prod_{k=1}^{d} \left( \frac{q^{\omega_d^{-1}k}}{(\sqrt{q}^{\omega_d^{-1}k})} \right)^{4m^2/d^2} = \Delta (q^{d/2})^{4m^2/d},
\]

where we used (2.1) in the last step.

We compute the multiplicity of \( t = \omega_d \) as a zero of the right-hand side of (3.5), or equivalently of \( P \). This is the number of pairs \( (i,j) \) such that \( 1 \leq i < j \leq 2m \) and \( i \equiv j \pmod{d} \). If we assume \( j = i + ld \), \( 1 \leq l \leq (2m - d)/d \), there are \( 2m - ld \) such pairs. Summing over \( l \) gives the total multiplicity

\[
(5.2) \quad (2m - d) + (2m - 2d) + \cdots + 2d + d = \frac{m(2m - d)}{d}.
\]

Next we turn to the left-hand side of (3.5).

**Lemma 5.3.** If \( d \mid 2m \) and \( k_1, \ldots, k_m \) are odd, then the multiplicity of \( t = \omega_d \) as a zero of

\[
\prod_{i=1}^{m} (1 - t^{k_i}) \prod_{1 \leq i < j \leq m} \left( \frac{t^{\frac{1}{2}(k_j-k_i)}}{t^{\frac{1}{2}(-k_i-k_j)}} \right)^2 (1 - t^{\frac{1}{2}(k_i+k_j)})^2
\]

is at least \( m(2m - d)/d \), with equality if and only if

\[
(5.3) \quad (k_1, \ldots, k_m) \simeq (1, 3, \ldots, 2m - 1) \pmod{2d}.
\]

**Proof.** Let \( a_i \) be the number of \( j, 1 \leq j \leq m \), such that \( k_j \equiv \pm i \pmod{2d} \). Here we take \( i = 1, 3, 5, \ldots, d \) or \( i = 1, 3, 5, \ldots, d-1 \) according to whether \( d \) is odd or even.

We note that there are single zeroes when \( k_i \equiv d \pmod{2d} \) and double zeroes when \( k_i \equiv \pm k_j \pmod{2d} \), \( i < j \). Thus, if \( d \) is odd, the total multiplicity is

\[
a_d + 2 \left( \frac{a_1}{2} \right) + 2 \left( \frac{a_3}{2} \right) + \cdots + 2 \left( \frac{a_{d-2}}{2} \right) + 4 \left( \frac{a_d}{2} \right),
\]

while if \( d \) is even, it is

\[
2 \left( \frac{a_1}{2} \right) + 2 \left( \frac{a_3}{2} \right) + \cdots + 2 \left( \frac{a_{d-1}}{2} \right).
\]
We want to minimize these expressions under the condition \( \sum_i a_i = m \). Using, for instance, Lagrange multipliers, one checks that in both cases the minimum is \( m(2m - d)/d \). Moreover, it is achieved precisely if

\[
a_1 = a_3 = \cdots = a_{d-2} = 2m/d, \quad a_d = m/d,
\]

for odd \( d \) and if

\[
a_1 = a_3 = \cdots = a_{d-1} = 2m/d
\]

for even \( d \). By Lemma 5.1.b, this is in both cases equivalent to (5.3). \( \square \)

We may now let \( t \to \omega_d \) termwise in (3.5), concluding that

\[
\frac{1}{m!} \sum_{k_1, \ldots, k_m} T_{k_1, \ldots, k_m}(t) \lim_{t \to \omega_d} \frac{P(t)}{P(t)} = q^{\frac{1}{2}m(m-1)} \omega_d^{-\frac{1}{2}m(m-1)(4m+1)} \Delta(q^{d/2})^{4m^2/d},
\]

where \( \sum' \) indicates that the summation variables are positive odd integers satisfying (5.3), and where

\[
T_{k_1, \ldots, k_m}(t) = \prod_{i=1}^{m} (t^{1-2m} q^{\frac{1}{2}(k_i-1)}) \frac{1 - t^{k_i}}{1 - q^{\frac{1}{2}k_i}} \times \prod_{1 \leq i < j \leq m} (t^{\frac{1}{2}(k_j-1)} - t^{\frac{1}{2}(k_i-1)})^2 (1 - t^{\frac{1}{2}(k_i+k_j)})^2.
\]

We factor \( T = T_{k_1, \ldots, k_m}(t) \) as \( T^1 T^2 \), where

\[
T^1 = \prod_{i=1}^{m} (t^{1-2m} q^{\frac{1}{2}(k_i-1)}) \prod_{1 \leq i < j \leq m} (1 - t^{k_i}) \prod_{1 \leq i \leq m} (1 - t^{k_i}) \prod_{k_i \equiv k_j \pmod{2d}} (1 - t^{k_i})(1 - t^{k_j})
\]

\[
\times \prod_{1 \leq i < j \leq m} (t^{\frac{1}{2}(k_i-1)} - t^{\frac{1}{2}(k_j-1)})^2 \prod_{1 \leq i < j \leq m} (1 - t^{\frac{1}{2}(k_i+k_j)})^2,
\]

\[
T^2 = \prod_{i=1}^{m} q^{\frac{1}{2}(k_i-1)} \prod_{1 \leq i \leq m} (1 - t^{k_i}) \prod_{1 \leq i \leq m} (1 - t^{k_i}) \prod_{k_i \equiv -k_j \pmod{2d}} (1 - t^{k_i})
\]

\[
\times \prod_{1 \leq i < j \leq m} (1 - t^{\frac{1}{2}(k_j-k_i)})^2 \prod_{1 \leq i < j \leq m} (1 - t^{\frac{1}{2}(k_i+k_j)})^2.
\]

Similarly, we write \( P = P^1 P^2 \), where

\[
P^1(t) = \prod_{1 \leq i < j \leq 2m} (1 - t^{j-i}),
\]

\[
P^2(t) = \prod_{1 \leq i < j \leq 2m} (1 - t^{j-i}).
\]
Lemma 5.4. In the notation above,

\begin{equation}
\left(5.4\right) \sum_{k_1,\ldots,k_m} T_{k_1,\ldots,k_m}(\omega_d) \frac{P_1^1(t)}{P_1(\omega_d)} = q^{\frac{1}{2}m(m-1)}\omega_d^{-\frac{1}{2}}q^{m(m-1)(4m+1)} \Delta(d^{2/2})^{4m^2/d},
\end{equation}

where \(T^1/P^1\) simplifies in view of the following lemma.

Lemma 5.4. In the notation above,

\begin{equation}
\left(5.5\right) \frac{T_{k_1,\ldots,k_m}(\omega_d)}{P_1(\omega_d)} = (-1)^{|\{i; d+1 \leq k_i \leq 2d-1 \text{ (2d)}\}| + \left(\frac{d}{2}\right)}(2m/2) \omega_d^{-\frac{1}{2}}m(m-1)(4m+1).
\end{equation}

Proof. Let \(\tau(k_1,\ldots,k_m) = T_{k_1,\ldots,k_m}(\omega_d)\). Then, \(\tau\) is visibly symmetric in the \(k_i\) and invariant under \(k_i \mapsto k_i + 2d\) for any \(i\). It is also easy to check that

\[\frac{\tau(-k_1,k_2,\ldots,k_m)}{\tau(k_1,k_2,\ldots,k_m)} = -1, \quad k_1 \neq 0 \quad (d).\]

Thus, \(\tau(k_1,\ldots,k_m)\) equals, up to a factor \(\pm 1, \tau(1,3,5,\ldots,2m-1)\). The sign may be computed using Lemma 5.1c, giving

\[\tau(k_1,\ldots,k_m) = (-1)^{|\{i; d+1 \leq k_i \leq 2d-1 \text{ (2d)}\}| + \left(\frac{d}{2}\right)}(2m/2) \tau(1,3,5,\ldots,2m-1).\]

Next, we write

\[P_1(t) = \prod_{1 \leq i < j \leq m} (1 - t^j-i) \prod_{m+1 \leq i < j \leq 2m} (1 - t^j-i) \prod_{1 \leq i \leq m, 1 \leq i < j \leq 2m} (1 - t^j-i).\]

In the second product, we replace \((i, j) \mapsto (m+i, m+j)\) and in the third product \((i, j) \mapsto (m+1-i, m+j)\), giving

\[P_1(t) = \prod_{1 \leq i < j \leq m} (1 - t^j-i)^2 \prod_{1 \leq i \leq m, 1 \leq j \leq m} (1 - t^i+j-1) \prod_{1 \leq i \leq j \leq m} (1 - t^j-i) \prod_{1 \leq i \leq j \leq m} (1 - t^i+j-1).\]

Comparing with the definition of \(T^1\) we find that, in general,

\[\frac{T_{1,3,\ldots,2m-1}(t)}{P_1(t)} = \prod_{i=1}^{m} t^{(1-2m)(i-1)} \prod_{1 \leq i < j \leq m} t^{2i-2} = t^{-\frac{1}{2}m(m-1)(4m+1)}.\]

This completes the proof. \(\square\)
Since $T_{k_1,\ldots,k_m}^2$ and $P^2$ vanish to the same order at $\omega_d$, we have

$$
(5.6) \quad \lim_{t \to \omega_d} \frac{T_{k_1,\ldots,k_m}^2}{P^2(t)} = \prod_{i=1}^{m} \frac{q^{\frac{1}{2}(k_i-1)}}{1 - q^{\frac{1}{2}k_i}} \prod_{1 \leq i \leq m \atop k_i \equiv d \pmod{2d}} k_i \prod_{1 \leq i < j \leq m \atop k_i \equiv k_j \pmod{2d}} \left( \frac{k_j - k_i}{2} \right)^2 \prod_{1 \leq i < j \leq m \atop k_i \equiv -k_j \pmod{2d}} \left( \frac{k_i + k_j}{2} \right)^2 \prod_{1 \leq i < j \leq m \atop i \equiv j \pmod{d}} \frac{1}{j - i},
$$

where, by the discussion leading to (5.2),

$$
\prod_{1 \leq i < j \leq 2m \atop i \equiv j \pmod{d}} (j - i) = d^{(2m-d)/d} \prod_{l=1}^{2m-ld} l^{(2m-d)/d} \prod_{l=1}^{d(2m-d)/d} l!^d.
$$

Plugging (5.5) and (5.6) into (5.4) gives

$$
\sum' (-1)^{|\{i; \ d+1 \leq k_i \leq 2d-1 \ (2d)\}|} \prod_{i=1}^{m} \frac{q^{\frac{1}{2}(k_i-1)}}{1 - q^{\frac{1}{2}k_i}} \prod_{1 \leq i \leq m \atop k_i \equiv d \pmod{2d}} k_i \prod_{1 \leq i < j \leq m \atop k_i \equiv k_j \pmod{2d}} \left( \frac{k_j - k_i}{2} \right)^2 \prod_{1 \leq i < j \leq m \atop k_i \equiv -k_j \pmod{2d}} \left( \frac{k_i + k_j}{2} \right)^2 \prod_{1 \leq i < j \leq m \atop i \equiv j \pmod{d}} \frac{1}{j - i} = (-1)^{\binom{d}{2}/2} \prod_{l=1}^{d(2m-d)/d} l!^d \Delta(q^{d/2})^{4m^2/d}.
$$

Using (3.7) to expand the left-hand side, we arrive at (4.1).

Finally we note that, by Lemma 5.1.b,

$$
(5.7) \quad \prod_{1 \leq i < j \leq m \atop k_i \equiv k_j \pmod{2d}} 4 \prod_{1 \leq i < j \leq m \atop k_i \equiv -k_j \pmod{2d}} 4 = \begin{cases} 4^{\frac{m}{d}+\frac{d}{2}(d-1)} \left( \frac{2m}{d} \right)^{\frac{d}{2}} = 2^m(2m-d-1)/d, & d \text{ odd,} \\ 4^{\frac{d}{2}(2m-d)/d} = 2^m(2m-d)/d, & d \text{ even,} \end{cases}
$$

which should be used when deriving (4.4) from (4.1).

Proof of (4.2). We repeat the analysis above, starting with (3.10) rather than (3.5). Consider first the right-hand side of (3.10). Dividing by

$$
P(t) = \prod_{1 \leq i < j \leq 2m+1} (1 - q^{j-i}),
$$

we have

$$
(5.8) \quad \frac{1}{P(t)} \frac{(q)_{\infty}}{(\sqrt{q})_{\infty}} \prod_{1 \leq i < j \leq 2m+1} \frac{\theta(t^{j-i})}{\theta(\sqrt{q}t^{j-i})} = \frac{(\sqrt{q})_{\infty}}{(q)_{\infty}} \prod_{i,j=1}^{2m+1} \frac{(q^{t^j-i})_{\infty}}{(\sqrt{q}t^{j-i})_{\infty}}.
$$
Lemma 5.5. If \( d \mid 2m \) or \( d \mid 2m+2 \), the expression (5.8) equals \( \Delta(q^{d/2})^{4m(m+1)/d} \) for \( t = \omega_d \).

**Proof.** If \( d \mid 2m \), (5.8) can be written
\[
\prod_{i,j=1}^{2m} \frac{(qt^j-i)^\infty_k}{(\sqrt{qt^j-i})^\infty_k} \prod_{i,j=1}^{2m} \frac{(qt^{2m+1-k}, qt^{k-2m-1})^\infty_k}{(\sqrt{qt^{2m+1-k}}, \sqrt{qt^{k-2m-1}})^\infty_k}.
\]
If \( t = \omega_d \), the double product is computed in (5.1) as \( \Delta(q^{d/2})^{4m^2/d} \). By (2.1), the single product can be written
\[
\prod_{k=1}^{d} \frac{(q\omega_d^{2m+1-k}, q\omega_d^{k-2m-1})^{2m/d}}{(\sqrt{q\omega_d^{2m+1-k}}, \sqrt{q\omega_d^{k-2m-1}})^{2m/d}} = \frac{(q^d, q^d, q^d)^{2m/d}}{(q^{d/2}, q^{d/2}, q^{d/2})^{2m/d}} = \Delta(q^{d/2})^{4m/d},
\]
which proves the result in this case.

If \( d \mid 2m+2 \) we may write (5.8) as
\[
\prod_{i,j=1}^{2m+2} \frac{(qt^j-i)^\infty_k}{(\sqrt{qt^j-i})^\infty_k} \prod_{i,j=1}^{2m+2} \frac{(qt^{2m+2-k}, qt^{k-2m-2})^\infty_k}{(\sqrt{qt^{2m+2-k}}, \sqrt{qt^{k-2m-2}})^\infty_k}.
\]
As above, the double product equals \( \Delta(q^{d/2})^{4(m+1)^2/d} \) and the single product \( \Delta(q^{d/2})^{-4(m+1)/d} \), which completes the proof. \( \square \)

Lemma 5.6. If \( d \mid 2m \) or \( d \mid 2m+2 \), the multiplicity of \( t = \omega_d \) as a zero of \( P \) is \( (2m + 2 - d)m/d \).

**Proof.** We must count the pairs \((i, j)\) such that \( 1 \leq i < j \leq 2m+1 \) and \( i \equiv j \pmod{d} \).
If \( j = i + ld \), there are \( 2m + 1 - ld \) such pairs, so the total number is
\[
\sum_{l=1}^{[2m/d]} (2m + 1 - ld).
\]
If \( d \mid 2m \), we have an arithmetic sum with \( 2m/d \) terms and average value \((2m + 2 - d)/2 \). If \( d \mid 2m + 2 \), \([2m/d] = (2m + 2 - d)/d \), unless \( d = 1 \) which is included in the previous case. This gives an arithmetic sum with \((2m + 2 - d)/d \) terms and average value \( m \). In both cases, the result is \((2m + 2 - d)m/d \). \( \square \)

We note in passing that the same argument gives, if \( d \mid 2m \),
\[
\prod_{1 \leq i < j \leq 2m+1 \atop i \equiv j \pmod{d}} (j-i) = d^{(2m+2-d)m/d} \prod_{l=1}^{2m/d} t^{2m+1-ld}
\]
\[
= d^{(2m+2-d)m/d}(2m/d)! \prod_{l=1}^{(2m-d)/d} l!^d,
\]
(5.9a)
while if \( d \mid 2m + 2 \),

\[
\prod_{1 \leq i \leq j \leq m+1 \atop i \equiv j (d)} (j - i) = d^{(2m+2-d)m/d} \prod_{l=1}^{(2m+2-d)/d} l^{2m+1-ld}
\]

\[
(5.9b)
\]

\[
= d^{(2m+2-d)m/d} \frac{1}{((2m + 2 - d)/d)!} \prod_{l=1}^{(2m+2-d)/d} l!d.
\]

Initially, we only obtain (5.9b) when \( d \neq 1 \), but when \( d = 1 \) it agrees with (5.9a) and thus remains valid.

**Lemma 5.7.** If \( d \mid 2m \) or \( d \mid 2m + 2 \), the multiplicity of \( t = \omega_d \) as a zero of

\[
\prod_{i=1}^{m} (1 - t^{k_i})^2 (1 - t^{2k_i}) \prod_{1 \leq i < j \leq m} (t^{k_j} - t^{k_i})^2 (1 - t^{k_i+k_j})^2
\]

is at least \((2m + 2 - d)m/d\), with equality if and only if

\[
(5.10) \quad (k_1, \ldots, k_m) \simeq (1, 2, \ldots, m) \mod d.
\]

**Proof.** Let \( a_i \) be the number of \( j, 1 \leq j \leq m \), such that \( k_j \equiv i (d) \). Here, \( i = 0, 1, 2, \ldots, (d-1)/2 \) or \( i = 0, 1, 2, \ldots, d/2 \) according to whether \( d \) is odd or even. If \( d \) is odd, the multiplicity is

\[
\mu_1 = 3a_0 + 4 \left( \begin{array}{c} a_0 \\ 2 \end{array} \right) + 2 \left( \begin{array}{c} a_1 \\ 2 \end{array} \right) + \cdots + 2 \left( \begin{array}{c} a_{(d-1)/2} \\ 2 \end{array} \right),
\]

while if \( d \) is even, it is

\[
\mu_2 = 3a_0 + 4 \left( \begin{array}{c} a_0 \\ 2 \end{array} \right) + 2 \left( \begin{array}{c} a_1 \\ 2 \end{array} \right) + \cdots + 2 \left( \begin{array}{c} a_{(d-2)/2} \\ 2 \end{array} \right) + a_{d/2} + 4 \left( \begin{array}{c} a_{d/2} \\ 2 \end{array} \right).
\]

In contrast to the case of Lemma 5.1, the minimum of these expressions subject to \( \sum_i a_i = m \) is achieved at non-integral values of \( a_i \). To prove that the minimum over the natural numbers equals \( \mu_0 = (2m + 2 - d)m/d \), we consider a number of different cases separately.

If \( d \) is odd and \( d \mid 2m \) (so that \( d \mid m \)), we write

\[
\mu_1 - \mu_0 = 2 \left( \frac{a_0}{d} - \frac{m}{d} \right) \left( \frac{a_0}{d} - \frac{m}{d} + 1 \right) + \sum_{k=1}^{(d-1)/2} \left( a_k - \frac{2m}{d} \right)^2.
\]

Since \( x(x + 1) \geq 0 \) for integer \( x \), we have that \( \mu_1 \geq \mu_0 \) with equality precisely if

\[
a_0 = m/d, \quad a_1 = \cdots = a_{(d-1)/2} = 2m/d;
\]

\( a_0 = (m - d)/d \) would contradict \( \sum a_i = m \).
If \( d \) is odd and \( d \mid 2m + 2 \), we similarly write
\[
\mu_1 - \mu_0 = 2 \left( a_0 - \frac{m + 1}{d} \right) \left( a_0 - \frac{m + 1}{d} + 1 \right) + \sum_{k=1}^{(d-1)/2} \left( a_k - \frac{2(m + 1)}{d} \right)^2,
\]
which shows that \( \mu_1 \geq \mu_0 \) with equality precisely if
\[
a_0 = \frac{m + 1}{d} - 1, \quad a_1 = \cdots = a_{(d-1)/2} = \frac{2(m + 1)}{d}.
\]

If \( d \) is even and \( d \mid 2m \), we write
\[
\mu_2 - \mu_0 = 2 \left( a_0 - \frac{m}{d} \right) \left( a_0 - \frac{m}{d} + 1 \right) + \sum_{k=1}^{(d-2)/2} \left( a_k - \frac{2m}{d} \right)^2 + 2 \left( a_{d/2} - \frac{m}{d} \right)^2.
\]

If \( d \mid m \), it follows as before that \( \mu_2 \geq \mu_0 \) with equality precisely if
\[
a_0 = a_{d/2} = m/d, \quad a_1 = \cdots = a_{(d-2)/2} = 2m/d.
\]

However, if \( 2m/d \) is odd, we observe that
\[
\left( a_0 - \frac{m}{d} \right) \left( a_0 - \frac{m}{d} + 1 \right) \geq -\frac{1}{4}, \quad \left( a_{d/2} - \frac{m}{d} \right)^2 \geq \frac{1}{4},
\]
so we still have \( \mu_2 \geq \mu_0 \), but with equality for
\[
a_0 = \frac{m}{d} - \frac{1}{2}, \quad a_1 = \cdots = a_{(d-2)/2} = \frac{2m}{d}, \quad a_{d/2} = \frac{m}{d} + \frac{1}{2}.
\]

Finally, if \( d \) is an even divisor of \( 2m + 2 \),
\[
\mu_2 - \mu_0 = 2 \left( a_0 - \frac{m + 1}{d} \right) \left( a_0 - \frac{m + 1}{d} + 1 \right) + \sum_{k=1}^{(d-2)/2} \left( a_k - \frac{2(m + 1)}{d} \right)^2 + 2 \left( a_{d/2} - \frac{m + 1}{d} \right)^2,
\]
and we conclude as above that \( \mu_2 \geq \mu_0 \) with equality precisely when
\[
a_0 = \frac{m + 1}{d} - 1, \quad a_1 = \cdots = a_{(d-2)/2} = \frac{2(m + 1)}{d}, \quad a_{d/2} = \frac{m + 1}{d}
\]
if \( (2m + 2)/d \) is even and when
\[
a_0 = a_{d/2} = \frac{m + 1}{d} - \frac{1}{2}, \quad a_1 = \cdots = a_{(d-2)/2} = \frac{2(m + 1)}{d}
\]
if \( (2m + 2)/d \) is odd.

In each case, the desired result now follows from Lemma 5.2.b. \( \square \)
Thus, using also Lemma 5.2.c, we have

\[
\frac{1}{m!} \sum_{k_1, \ldots, k_m} T^1_{k_1, \ldots, k_m}(\omega_d) \frac{P^1(\omega_d)}{P^1(\omega_d)} \lim_{t \to \omega_d} \frac{T^2_{k_1, \ldots, k_m}(t)}{P^2(t)},
\]

where the sum is over positive integers satisfying (5.10), and where

\[
\begin{align*}
T^1 &= \prod_{i=1}^{m} t^{-2m_i} \prod_{1 \leq i < j \leq m} t^{k_i} \prod_{1 \leq i \leq m} (1 - t^{k_i})^2 \prod_{1 \leq i \leq m} \prod_{2k_i \equiv 0 (d)} (1 - t^{2k_i}) \times \\
&\quad \times \prod_{1 \leq i < j \leq m} (t^{k_i} - t^{k_j})^2 \prod_{1 \leq i \leq m} \prod_{2k_i \equiv 0 (d)} (1 - t^{k_i + k_j})^2,
\end{align*}
\]

and

\[
\begin{align*}
T^2 &= \prod_{i=1}^{m} q^{\frac{1}{2}k_i} \prod_{1 \leq i \leq m} \prod_{k_i \equiv 0 (d)} (1 - t^{k_i})^2 \prod_{1 \leq i \leq m} \prod_{2k_i \equiv 0 (d)} (1 - t^{2k_i}) \\
&\quad \times \prod_{1 \leq i < j \leq m} (1 - t^{k_i - k_j})^2 \prod_{1 \leq i < j \leq m} \prod_{k_i \equiv k_j (d)} (1 - t^{k_i + k_j})^2,
\end{align*}
\]

and

\[
\begin{align*}
P^1(t) &= \prod_{1 \leq i < j \leq 2m+1} (1 - t^{-i}). \\
P^2(t) &= \prod_{1 \leq i < j \leq 2m+1} (1 - t^{-i}).
\end{align*}
\]

The expression \( T^1_{k_1, \ldots, k_m}(\omega_d) \) is visibly symmetric in the \( k_i \) and invariant under \( k_i \mapsto k_i + d \). It is straight-forward to check that it is odd in \( k_1 \) if \( 2k_1 \not\equiv 0 (d) \). Thus, using also Lemma 5.2.c, we have

\[
T^1_{k_1, \ldots, k_m}(\omega_d) = (-1)^{\left\lfloor (d-1)/2 \right\rfloor} t^{\left\lfloor (d-1)/2 \right\rfloor} t^{\left\lfloor (d-1)/2 \right\rfloor} \frac{1}{d} \frac{M}{2},
\]

where \( M = m \) if \( d \mid 2m \) and \( M = m + 1 \) if \( d \mid 2m + 2 \). Moreover, similarly as in Lemma 5.4 it is straight-forward to check that

\[
\frac{T^1_{1,2,\ldots,m}(\omega_d)}{P^1(t)} = \prod_{i=1}^{m} t^{-2m_i} \prod_{1 \leq i < j \leq m} t^{2i} = t^{-\frac{1}{2}m(m+1)(2m+1)}.
\]

Thus,

\[
\frac{T^1_{k_1,\ldots,k_m}(\omega_d)}{P^1(\omega_d)} = (-1)^{\left\lfloor (d+1)/2 \right\rfloor} t^{\left\lfloor (d+1)/2 \right\rfloor} t^{\left\lfloor (d+1)/2 \right\rfloor} \frac{1}{d} \frac{M}{2} \omega_d^{-\frac{1}{2}m(m+1)(2m+1)}.
\]
We also have
\[
\lim_{t \to \omega} \frac{T_{k_1, \ldots, k_m}^2(t)}{P^2(t)} = \prod_{i=1}^m \frac{q^{\frac{1}{2}k_i}}{1 - q^{k_i}} \prod_{1 \leq i \leq m} 2k_i^3 \prod_{1 \leq i \leq m} 2k_i \\
\times \prod_{1 \leq i < j \leq m} (k_j - k_i)^2 \prod_{1 \leq i < j \leq m} (k_i + k_j)^2 \prod_{1 \leq i < 2m + 1} \frac{1}{j - i},
\]
where the final factor is given by (5.9).

Plugging all this into (5.11), we obtain
\[
\sum' \frac{1}{\prod_{i=1}^m \frac{q^{\frac{1}{2}k_i}}{1 - q^{k_i}} \prod_{1 \leq i \leq m} 2k_i^3 \prod_{1 \leq i \leq m} 2k_i} \times \prod_{1 \leq i < j \leq m} (k_j - k_i)^2 \prod_{1 \leq i < j \leq m} (k_i + k_j)^2, \]
with \( C \) as in (4.3). Expanding the left-hand side using (3.11), we arrive at (4.2).

Finally, we note that, by Lemma 5.2.b,
\[
(5.12) \prod_{1 \leq i \leq m} 2 \prod_{1 \leq i \leq m} 2 = \begin{cases} \\
2^{2m/d}, & d \text{ even}, \ d \mid 2m \\
2^{m/d}, & d \text{ odd}, \ d \mid 2m \\
2^{(2m+2-d)/d}, & d \text{ even}, \ d \mid 2m + 2 \\
2^{(m+1-d)/d}, & d \text{ odd}, \ d \mid 2m + 2,
\end{cases}
\]
which should be used in deriving (4.15) from (4.2).

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