Complexity and Near Extremal Charged Black Branes

Mohsen Alishahiha*, Komeil Babaei Velni† and Mohammad Reza Tanhayi‡,*

* School of Physics, Institute for Research in Fundamental Sciences (IPM)
P.O. Box 19395-5531, Tehran, Iran
† Department of Physics, University of Guilan, P.O. Box 41335-1914, Rasht, Iran
‡ Department of physics, Islamic Azad University Central Tehran Branch, Tehran, Iran

E-mails: alishah@ipm.ir, babaeivelni@guilan.ac.ir, mtanhayi@ipm.ir

Abstract

We compute holographic complexity of charged black brane solutions in arbitrary dimensions for the near horizon limit of near extremal case using two different methods. The corresponding complexity may be obtained either by taking the limit from the complexity of the charged black brane, or by computing the complexity for near horizon limit of near extremal solution. One observes that these results coincide if one assumes to have a cutoff behind horizon whose value is fixed by UV cutoff and also taking into account a proper counterterm evaluated on this cutoff. We also consider the situation for Vaidya charged black branes too.
1 Introduction

In the context of black hole physics, horizon, black hole entropy, information paradox and physics behind the horizon are in the center of the most studies in last several decades. Quantum information theory might be capable to shed light on these subjects. Indeed recent progress on black hole physics has opened up a possibility to make a connection between quantum information theory and back hole physics (see [1] and its citations). To explore and understand this possible connection the AdS/CFT correspondence [2] has played rather an important role. In this context holographic entanglement entropy [3] and computational complexity [4,5] may be thought of as examples which could make this connection more concrete.

In particular holographic complexity, by definition, might be able to give us some information about the physics behind the horizon. To elaborate this point better it is worth noting that the holographic complexity may be obtained by the on shell action evaluated on a certain subregion of spacetime known as the Wheeler-DeWitt (WDW) patch [6,7], that includes some portion of the spacetime located behind the horizon.

More precisely the late time behavior of complexity growth is entirely given by the on shell action evaluated on the intersection of the WDW patch with the future interior [8], leading to an observation that the late time behavior is insensitive to the UV cutoff [9]. This, in turn, results to a conclusion that there could be a relation between UV cutoff and a cutoff defined behind the horizon [9,10]. Indeed this is an interesting feature of complexity that could probe physics behind the horizon.

It is important to mention that the result of [9] leading to the behind the horizon cutoff relies on two facts. The first one is what we have already mentioned that is according to explicit computations using CA proposal the late time behavior is UV blind. The second one is that according to Lloyd’s bound [11] the late time behavior is given by twice of energy of the system. In fact we should emphasise that in order to reach our conclusion the actual value of bound is not crucial. The important fact is that the late time behavior is governed by the physical charges defined at the boundary; such as mass or energy and so forth.

Indeed, it is known that in the context of holographic complexity the Lloyd’s bound may be violated [12–17]. Nonetheless the violation of Lloyd’s bound just modifies the relation between the cutoff behind that horizon the UV cutoff and does not affect the conclusion. Of course, one again, if the bound is given by charges at the boundary.

This is the aim of the present paper to further explore the significance of the cutoff behind the horizon. To do so, we will study holographic complexity for AdS-Reissner-Nordström (AdS-RN) black branes in arbitrary dimensions (see also [12,18–20]). More precisely, we will consider near horizon limit of near extremal charged black branes and compute complexity in two different ways. In the first approach we will obtain the corresponding rate of complexity growth by taking the near extremal limit of the complexity growth of the charged black brane. In the second approach
we compute complexity for a metric obtained by taking near horizon limit of the near extremal AdS-RN black brane that has the form of $\text{AdS}_2 \times \mathbb{R}^{d-1}$.

We note that if one naively computes complexity for geometries containing an AdS$_2$ factor the late time growth vanishes [9, 21] that is not consistent with what we may get from taking near extremal limit of the complexity growth. On the other hand if one assumes a cutoff behind the horizon that is fixed by the UV cutoff one finds the standard late time linear growth that is exactly the one obtained by the first approach. It is, however, important to note that to get a consistent result one needs also to consider certain counterterms to be evaluated on the cutoff surface behind the horizon [10].

Indeed we believe that the result of the present paper should be considered as an evidence supporting the proposed idea that the UV cutoff will automatically fix a cutoff behind the horizon. Moreover it is also crucial to take all counterterms into account to gets a physically consistent result.

The organization of the paper is as follows. In the next section in order to fix our notations we will review the computation of complexity for charged black branes. In section three we will study near horizon limit of near extremal cases where we compute the corresponding complexity from two different approaches. In section four we will redo the same computations for charged Vaidya black brane. The last section is devoted to summery and conclusions.

## 2 Complexity of charged black branes: a brief review

In this section in order to fix our notation we will review holographic complexity for charged black branes using “complexity equals to action” (CA) proposal. Actually the full time dependence of the complexity growth for charged black holes has been first studied in [12]. Of course the aim of this section is while reviewing the CV proposal for charged black branes, presenting a closed inspiring form for the corresponding complexity. To do so, we will use an appropriate coordinate system that simplifies the computations rather significantly. Although the result is known the presentation is new and by itself is interesting.

To start with let us consider an Einstein-Maxwell theory in $d+1$ dimensions whose action may be given as follows

\[
I_0 = \frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{-g} \left( R + \frac{d(d-1)}{L^2} - \frac{1}{2} F_{\mu\nu}F^{\mu\nu} \right),
\]  

where $G_N$ is the Newton’s constant and $L$ is a length scale in the theory. The corresponding equations of motion are

\[
R_{\mu\nu} - F_{\alpha\mu}F^{\alpha\nu} - \frac{1}{2} \left( R + \frac{d(d-1)}{L^2} - \frac{1}{2} F^2 \right) g_{\mu\nu} = 0, \quad \partial_{\alpha} (\sqrt{-g} F^{\alpha\beta} g_{\beta\gamma}) = 0.
\]
These equations admit AdS-RN black brane solutions that for \( d \geq 3 \) are given by

\[
ds^2 = \frac{L^2}{r^2} \left( -f(r) dt^2 + \frac{dr^2}{f(r)} + \sum_{i=1}^{d-1} dx_i^2 \right), \quad f(r) = 1 - mr^d + Q^2 r^{2(d-1)},
\]

where \( m \) and \( Q \) are related to the mass and charge of the black brane solution, respectively. Moreover, \( r_+ \) is the radius of horizon that is the smallest solution of \( f(r) = 0 \) (note that in our notation the boundary is at \( r = 0 \)). For \( d = 2 \) the solution exhibits a logarithmic behavior.

The aim is now to compute holographic complexity for the above charged black brane using CA proposal. To do so, one needs to compute on shell action on the WDW patch associated with a boundary state given at \( \tau = t_L + t_R \). Here \( t_L(t_R) \), is time coordinate of left (right) boundary on the charged black brane (see left panel of the figure 1). To proceed we note that the action consists of several parts including bulk, boundary and joint points as follows [22–24]

\[
I = I_0 + \frac{1}{8\pi G_N} \int_{\Sigma^d_t} K_t \, d\Sigma_t \pm \frac{1}{8\pi G_N} \int_{\Sigma^d_s} K_s \, d\Sigma_s \pm \frac{1}{8\pi G_N} \int_{\Sigma^d_n} K_n \, dS d\lambda \\
\pm \frac{1}{8\pi G_N} \int_{J^{d-1}} a \, dS \pm \frac{1}{8\pi G_N} \int_{\Sigma^d_\lambda} d\lambda dS \log \frac{|L\Theta|}{d-1}.
\]

(2.4)

Here the timelike, spacelike, and null boundaries and also joint points are denoted by \( \Sigma^d_t, \Sigma^d_s, \Sigma^d_n \) and \( J^{d-1} \), respectively. The extrinsic curvature of the corresponding boundaries are given by \( K_t, K_s \) and \( K_n \). The function \( a \) at the intersection of the boundaries is given by the logarithm of the inner product of the corresponding normal vectors and \( \lambda \) is the null coordinate defined on the null segments. The sign of different terms depends on the relative position of the boundaries and the bulk region of interest (see [24] for more details). Moreover in the last term the quantity \( \Theta \) is defined as follows

\[
\Theta = \frac{1}{\sqrt{\gamma}} \frac{\partial \sqrt{\gamma}}{\partial \lambda},
\]

(2.5)

where \( \gamma \) is determinant of induced metric on the joint points. This term is needed in order to remove the ambiguity associated with the normalization of null vectors [24, 25]. As we will see this term together with other counterterms play crucial role in order to get desired results.

It is worth noting that in general to write the last term one could use an arbitrary length scale, though for simplicity we have fixed the corresponding scale to be the radius of curvature \( L \). This choice just removes the most divergent term from the complexity and does not affect the physics that we are interested in.

It is also important to note that besides the above terms, there are also other boundary terms which could contribute to the complexity. These terms are those counterterms needed to make the on shell action finite when evaluated over whole spacetime [26]. The importance of these terms
Figure 1: *Left:* Penrose diagram of a charged black brane and the WDW patch associated with the state defined at $\tau = t_R + t_L$. The green region shows the Penrose diagram of near extremal limit of the charged black brane. The red dashed lines is a timelike cutoff behind the horizon. *Right:* Penrose diagram of an AdS$_2$ geometry which can be thought of as near horizon limit of the near extremal solution. The corresponding WDW patch is cut by the behind the horizon cutoff $\hat{r}_0$ that is fixed by the UV cutoff $\hat{r}_{\text{Max}}$.

have been also known from holographic renormalization point of view (see e.g. [27]). In the present case since the solution has flat boundary the only term remains to be considered is (for asymptotically AdS$_{d+1}$ metric)

$$\frac{1}{8\pi G_N} \int d\Sigma \frac{d-1}{L}.$$  

As we will see this counterterm plays an important role when we compute on shell action on the WDW patch.

Let us now compute on shell action for the WDW patch depicted in the left panel of the figure 1. To do so, it is found useful to consider the following change of coordinate

$$dt = dv + \frac{dr}{f(r)},$$

by which the solution (2.3) may be recast into the following form

$$ds^2 = \frac{L^2}{r^2} (-f(r)dv^2 - 2drdv + d\vec{x}^2), \quad A_v = \sqrt{\frac{d-1}{d-2}QL (r^{d-2} - r'^{d-2})}$$

In this notation the null boundaries of the corresponding WDW patch are given by $v =$constant and $u =$constant. Here the coordinate $u$ defined by $du = dt + \frac{dr}{f(r)}$. Of course since we are using $(r, v)$ coordinate system one needs to write $u$ coordinate in terms of $r$ and $v$

$$du = dv + 2\frac{dr}{f(r)}.$$
Therefore the null boundaries are given by \( v = -t_L \) and \( v = t_R \) for constant \( v \) and, for constant \( u \) one ends up with following equations for boundaries

\[
\begin{align*}
\int_{-t_L}^{v} dv &= -2 \int_{\epsilon}^{r_q(v, \tau)} \frac{dr}{f(r)} \rightarrow v + t_L = -2 \int_{\epsilon}^{r_q(v, \tau)} \frac{dr}{f(r)}, \\
\int_{t_R}^{v} dv &= -2 \int_{\epsilon}^{r_p(v, \tau)} \frac{dr}{f(r)} \rightarrow t_R - v = 2 \int_{\epsilon}^{r_p(v, \tau)} \frac{dr}{f(r)}.
\end{align*}
\] (2.10)

Before proceeding to compute on shell action it is useful to fix our notation for the null vectors of the null boundaries of the corresponding WDW patch. Indeed as we have already mentioned the boundaries are given at \( u \) and \( v \) constant. Therefore the corresponding null vectors are given by

\[
k_1 = \alpha \partial_v, \quad k_2 = \beta \partial_u = \beta \left( \partial_v + \frac{2}{f(r)} \frac{\partial}{\partial r} \right),
\] (2.11)

where \( \alpha \) and \( \beta \) are two free parameters appearing due to the ambiguity of the normalization of null vectors.

For the charged black brane which we are considering one has

\[
\sqrt{-g} \left( R + \frac{d(d-1)}{L^2} - \frac{1}{2} F^2 \right) = -2d \frac{L^{d-1}}{r^{d+1}} + 2(d-2)L^{d-1}Q^2 r^{d-3} ,
\] (2.12)

by which the bulk part of the on shell action reads

\[
I^{\text{bulk}} = \frac{L^{d-1}}{8\pi G_N} \int d^{d-1}x \int_{-t_L}^{t_R} dv \int_{r_p(v, \tau)}^{r_q(v, \tau)} dr \left( \frac{-d}{r^{d+1}} + Q^2 (d-2)r^{d-3} \right)
= \frac{V_{d-1}L^{d-1}}{8\pi G_N} \int_{-t_L}^{t_R} dv \left( f(r_q(v, \tau)) - f(r_p(v, \tau)) \right).
\] (2.13)

On the other hand using the fact that \( \frac{dr_p(v, \tau)}{dr} = \frac{1}{2} f(r_p(v, \tau)) \) and \( \frac{dr_q(v, \tau)}{dr} = -\frac{1}{2} f(r_q(v, \tau)) \) one can perform the above integration resulting to

\[
I^{\text{bulk}} = \frac{V_{d-1}L^{d-1}}{8\pi G_N} \left( \frac{2}{(d-1)r_p^{d-1}(\tau)} + \frac{2}{(d-1)r_q^{d-1}(\tau)} - \frac{4}{(d-1)e^{d-1}} \right).
\] (2.14)

Here we have used the fact that in our notation one has \( r_p(\tau) = r_p(-t_L, \tau), r_q(\tau) = r_q(t_R, \tau) \) and \( r_p(t_R, \tau) = r_q(-t_L, \tau) = \epsilon \).

Since all boundaries are null their contributions vanish using Affine parameterization for null direction, while for the joint points using \( (2.11) \) one gets

\[
I^{\text{joint}} = \frac{V_{d-1}L^{d-1}}{8\pi G_N} \left( \frac{\log \alpha \beta r_p^2(\tau)}{L^2 f(r_p)} + \frac{\log \alpha \beta r_q^2(\tau)}{L^2 f(r_q)} - 2 \log \frac{\alpha \beta \epsilon^2}{L^2 f(\epsilon)} \right).
\]
\[ I_{\text{amb}} = \frac{V_{d-1} L^{d-1}}{8\pi G_N} \left( -\log \frac{\alpha \beta \gamma_2^2(\tau)}{r_p^d} + \frac{2}{r_p^d} + \frac{2 (d-1)}{r_q^d} \right) \]

Now putting all terms together one finds the following expression for the total on shell action

\[ I_{\text{total}} = \frac{V_{d-1} L^{d-1}}{8\pi G_N} \left( \log \frac{|f(r_p(\tau))|}{r_p^d} + \log \frac{|f(r_q(\tau))|}{r_q^d} \right). \]

Interestingly enough the final result is very simple and indeed consists of the contributions of two joint points \( p \) and \( q \). It is also easy to compute its time derivative to find

\[ \frac{dI_{\text{total}}}{d\tau} = \frac{V_{d-1} L^{d-1}}{16\pi G_N} \left( \frac{f'(r_q(\tau))}{r_q^d} - \frac{f'(r_p(\tau))}{r_p^d} \right) \]

\[ + \left( \frac{(d-1)}{r_p^d} f(r_p(\tau)) \log |f(r_p(\tau))| - \frac{(d-1)}{r_q^d} f(r_q(\tau)) \log |f(r_q(\tau))| \right), \]

which at the late time reads

\[ \frac{dI_{\text{total}}}{d\tau} = \frac{V_{d-1} L^{d-1}}{16\pi G_N} \left( \frac{f'(r_-)}{r_-^d} - \frac{f'(r_+)}{r_+^d} \right). \]

Using the explicit form of the function \( f(r) \) one gets the expression know in the literature \([12,18,20]\).

Clearly in the extremal case where \( r_- = r_+ \) the complexity growth at the late time vanishes. Of course this is not the case for the near extremal black brane as we will study in the following section.

\(^1\)To find this expression we use the fact that \( \Theta = \pm \alpha \frac{(d-1)r}{L^d} \), or \( \pm \beta \frac{(d-1)r}{L^d} \), and \( \frac{\partial \Theta}{\partial \lambda} = \pm \alpha^2 \frac{2}{L^d}, \) or \( = \pm \beta^2 \frac{2}{L^d} \) depending on which null boundary is taken.
3 Complexity of near horizon limit of near extremal solutions

In this section we would like to study complexity of near extremal charged black branes. When one is considering near extremal case, it usually comes with taking near horizon limit too. Therefore to be more precise in what follows we will be considering complexity for near horizon limit of the near extremal black brane. The Penrose diagram of the near extremal solution is shown by green region in the left panel of figure 1, while that of the near horizon limit is depicted in the right panel. Note that taking the near horizon limit of the near extremal black brane one gets a geometry with an AdS\(^2\) factor parameterized by the Rindler coordinates.

In this section we shall compute the corresponding complexity for near horizon of the near extremal black brane case in two different ways. To proceed it is useful to adapt a notation that is more appropriate for studying the near extremal case. Indeed, since the blacking factor \(f(r)\) has two roots at \(r = r_-\) and \(r = r_+\), one may write

\[
f(r) = \left(1 - \frac{r}{r_-}\right)\left(1 - \frac{r}{r_+}\right) h(r),
\]

where \(h(r)\) is a function of \(r\) defined via the above equation with the assumption \(h(r_\pm) \neq 0\). In this notation the late time behavior of the complexity of the charged black brane may be recast into the following form

\[
\frac{dI_{\text{total}}}{d\tau} = \frac{V_{d-1} L^{d-1}}{4G_N} \frac{r_- - r_+}{r_- r_+} \left( \frac{h(r_-)}{r_-^{d-1}} + \frac{h(r_+)}{r_+^{d-1}} \right) = S_- T_- + S_+ T_+, \tag{3.2}
\]

with

\[
T_\pm = \frac{r_- - r_+}{4\pi r_- r_+} h(r_\pm), \quad S_\pm = \frac{V_{d-1} L^{d-1}}{4G_N} \frac{1}{r_\pm^{d-1}}, \tag{3.3}
\]

being temperature and entropy one may associate to inner and outer horizons.

To study near horizon of the near extremal solution it is useful to define new coordinates \((\hat{r}, \hat{t})\) as follows (see for example \[28\])

\[
r = \frac{r_- - r_+}{2} \frac{\hat{r}}{L} + \frac{r_- + r_+}{2}, \quad t = \frac{2r_- r_+}{r_- - r_+} \frac{\hat{t}}{L}. \tag{3.4}
\]

The limit is then defined by \(r_- \to r_+\) keeping the new coordinates fixed. Note that in this limit setting \(r_- \approx r_+ = r_e\) one has \(f(r_e) = f'(r_e) = 0\) and thus

\[
r_e^{d-1} = \sqrt{\frac{d}{d-2}} \frac{1}{Q}, \quad h(r_e) = \frac{r_e^2}{2} f''(r_e) = d(d-1). \tag{3.5}
\]
It is then easy to see that in this limit the solution (2.3) reduces to
\[ ds^2 = -\left( \frac{\hat{r}^2}{L^2} - 1 \right) h(r_e) d\hat{t}^2 + \frac{d\hat{r}^2}{(\frac{\hat{r}^2}{L^2} - 1) h(r_e)} + \frac{L^2}{r_e^2} d\vec{x}^2, \quad F_{\hat{r}\hat{t}} = \frac{\sqrt{d(d-1)}}{L}, \]  
(3.6)
that is an AdS$_2 \times \mathbb{R}^{d-1}$ geometry whose Ricci scalar is \( R = -\frac{2h(r_e)}{L^4} \). It is easy to check that this is also a solution of equations of motion (2.2).

Now let us study the late time behavior of the complexity for near horizon limit of the near extremal case. This may be done by taking the above limit from the late time behavior of complexity of the charged black brane given in the equation (2.19). To take the limit it is important to note that the boundary time, \( \tau \), should be also rescaled properly that is the same as that for coordinate \( t \) given in the equation (3.4). Therefore for \( d > 2 \) one gets
\[ \frac{dI_{\text{total}}}{d\hat{r}} = \frac{V_{d-1} L^{d-2}}{4\pi G_N} \frac{h(r_e)}{r_e^{d-1}} = 2S_{\text{NE}} T_{\text{NE}}, \]  
(3.7)
that should be compared with (3.2). Note that in this expression
\[ T_{\text{NE}} = \frac{h(r_e)}{2\pi L}, \quad S_{\text{NE}} = \frac{V_{d-1} L^{d-1}}{4G_N r_e^{d-1}}, \]  
(3.8)
are the Hawking temperature and entropy of the near horizon solution (3.6).

Alternatively one could compute the corresponding complexity growth directly form the near horizon solution (3.6). Since the solution we are interested in has an AdS$_2$ factor, one may reduce the theory along the extra \( (d-1) \) dimensions into two dimensions where we typically obtain a two dimensional gravity that might be thought of as generalized Jackiw-Teitelboim gravity with background charge. Of course we could still work within the higher dimensional gravity and compute the complexity for a solution in the form of AdS$_2 \times \mathbb{R}^{d-1}$. Indeed this is what we will do in what follows.

To proceed we note that the integrand of the action (2.1) vanishes for the solution (3.6) and thus there is no contribution to the on shell action from the bulk part. Therefore we are left with joint and boundary terms. If one considers the corresponding WDW patch as that shown in the right panel of the figure 1 containing two joint points at \( \hat{r}_q \) and \( \hat{r}_p \), one only has to compute contributions of four joint points which results to the same form as that given in the equation (2.15). It is then easy to see that the resultant complexity approaches a constant at the late time contradicting the result (3.7).

Of course this is not what we really should do. Actually as it was shown in [9,10] as soon as one sets a UV cutoff, it will automatically enforce us to have a cutoff behind the horizon preventing us to have an access to the joint point \( \hat{r}_q \). In other words the correct WDW patch we should consider is the one with a joint point at \( \hat{r}_p \) and a spacelike boundary behind the horizon at \( \hat{r}_0 \). Therefore
we will have to compute contributions of five joint points together with certain boundary terms defined on the cutoff surface $r_0$.

Denoting the normal vectors to the null and spacelike boundaries by

$$k_1 = \alpha \left( \partial_t + \frac{1}{f(\hat{r})} \partial_r \right), \quad k_2 = \beta \left( \partial_t - \frac{1}{f(\hat{r})} \partial_r \right), \quad n_r = \frac{1}{\sqrt{f(\hat{r}_0)}} \partial_r,$$

(3.9)

with $f(\hat{r}) = \left( \frac{\hat{r}^2}{\hat{L}^2} - 1 \right) h(r_e)$, the joint contributions are

$$I_{\text{joint}} = \frac{V_{d-1} L^{d-1}}{8 \pi G_N r_e^{d-1}} \left( \log \frac{\alpha \beta}{|f(\hat{r}_p)|} + \log \frac{\alpha}{\sqrt{|f(\hat{r}_0)|}} + \log \frac{\beta}{\sqrt{|f(\hat{r}_0)|}} - 2 \log \frac{\alpha \beta}{|f(\hat{r}_\text{Max})|} \right)$$

$$= \frac{V_{d-1} L^{d-1}}{8 \pi G_N r_e^{d-1}} \left( 2 \log |f(\hat{r}_\text{Max})| - \log |f(\hat{r}_p)| - \log |f(\hat{r}_0)| \right).$$

(3.10)

Here $\hat{r}_\text{Max}$ is a UV cutoff by which the cutoff behind the horizon is given by $\hat{r}_0 \sim \frac{\hat{L}^2}{\hat{r}_\text{Max}}$. Note also that the parameters $\alpha$ and $\beta$ drop from the final expression of the action of joint points and therefore we do not need extra boundary terms to remove the ambiguity associated with the normalization of null vectors.

As for the boundary part, using the Affine parameterization for the null boundaries their contributions vanish and we only need to consider boundary terms defined on the spacelike boundary behind the horizon$^2$

$$I_{\text{surf}} = -\frac{1}{8 \pi G_N} \int d^{d-1}x dt \sqrt{|h|} \left( K - \frac{\sqrt{h(r_e)}}{L} \right)|_{\hat{r}_0},$$

(3.11)

that results to

$$I_{\text{surf}} = \frac{V_{d-1} L^{d-1} h(r_e) \left( \frac{\hat{r}_0}{\hat{L}^2} + \frac{1}{\hat{L}} \right) (\tau + 2 \hat{r}^* (\hat{r}_\text{Max}) - 2 \hat{r}^* (\hat{r}_0))}{8 \pi G_N r_e^{d-1}}.$$  

(3.12)

Therefore altogether one arrives at

$$I_{\text{total}} = \frac{V_{d-1} L^{d-1}}{8 \pi G_N r_e^{d-1}} \left( 4 \log \frac{\hat{r}_\text{Max}}{L} - \log \left( 1 - \frac{\hat{r}_p^2}{\hat{L}^2} \right) \right) + \frac{V_{d-1} L^{d-1}}{8 \pi G_N r_e^{d-1}} \frac{h(r_e)}{L} \left( \tau + 2 \hat{r}^* (\hat{r}_\text{Max}) - 2 \hat{r}^* (\hat{r}_0) \right),$$

(3.13)

whose time derivative is

$$\frac{d I_{\text{total}}}{d \hat{r}} = \frac{V_{d-1} L^{d-2}}{8 \pi G_N} \frac{h(r_e)}{r_e^{d-1}} \left( 1 + \hat{r}_p \frac{1}{\hat{L}} \right),$$

(3.14)

that reduces to $\{3.7\}$ at the late time when $\hat{r}_p \to L$.

To conclude we note that in order to get a consistent result for the rate of complexity growth

$^2$ The counterterm is fixed in such a way to make the free energy for the near horizon solution $\{3.6\}$ finite. More specifically if one uses the equations of motion for gauge field in the near extremal limit one gets an action whose integrand is $\sqrt{-g} \left( R + \frac{2}{\hat{L}^2} \right)$ with $\hat{L} = \frac{L}{\sqrt{h(r_e)}}$. Then the corresponding counterterm must be proportional to $\frac{1}{\hat{L}}$. 

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for near horizon solution (3.6), one needs to assume that the UV cutoff induces a cutoff behind the horizon and moreover it is crucial to take into account the contribution of a proper counterterm evaluated on the cutoff surface behind the horizon.

4 Complexity of Charged Vaidya solution

In this section we would like to compute complexity for a charged Vaidya solution\(^3\). This solution can be thought of as collapsing charged null shell that produces a charged black brane. The solution can be found from the action (2.1) by adding a proper extra charged matter field. Indeed the Vaidya geometry we are looking for is sourced by an energy momentum tensor and a current density of a massless null charged matter. Therefore the equations of motion (2.2) should be modified as follows

\[
R_{\mu
u} - F_{\alpha\mu}F_{\nu}^{\alpha} - \frac{1}{2} \left( R + \frac{d(d-1)}{L^2} - \frac{1}{2} F^2 \right) g_{\mu\nu} = T_{\mu\nu}^{\text{ext}}, \quad \partial_\alpha \left( \sqrt{-g} F^{\alpha\beta} \right) = (J^{\text{ext}})_\beta.
\] (4.1)

In the present case the non-zero components of the energy momentum and charge current are \(T^{\text{ext}}_{vv}\) and \(J^{\text{ext}}_v\), that assuming to have time dependent mass and charge, are given by \(^4\)

\[
T^{\text{ext}}_{vv} = \frac{d-1}{2r} \frac{\partial f(r,v)}{\partial v}, \quad J^{\text{ext}}_v = \sqrt{(d-1)(d-2)} r^{d-1} \frac{\partial Q(v)}{\partial v}.
\] (4.2)

The charged Vaidya metric we are going to consider is\(^5\)

\[
ds^2 = \frac{L^2}{r^2} \left( -f(r)dv^2 - 2drdv + d\vec{x}^2 \right), \quad A_v = \theta(v) \sqrt{\frac{d-1}{d-2}} QL \left( r^{d-2} - r^{d-2} \right),
\] (4.3)

with

\[
f(r) = 1 + \theta(v) \left( -mr^d + Q^2 r^{2(d-1)} \right),
\] (4.4)

where \(\theta(v)\) is the step function. Therefore for \(v < 0\) one has an AdS solution while for \(v > 0\) it is the AdS-RN black brane we considered in the previous section. The corresponding Penrose diagram is shown in the figure 2 (see [36]).

To compute the complexity for this model one needs to evaluate on shell action on the WDW patch depicted in the figure 2. Following our notation the null boundaries of the corresponding WDW patch are given by constant \(v\) and \(u\). To proceed, it is useful to decompose the WDW patch into three parts: \(v < 0\) and \(v > 0\) and \(v = 0\) parts. It is known that the action on null shell gives zero contribution (see e.g. [30]) and therefore we will consider two parts given by \(v > 0\) and \(v < 0\).

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\(^3\) Using CA proposal the complexity for neutral Vaidya metric has been already studied in several papers including [17, 29, 33].

\(^4\) Complexity for charged Vaidya metric has also been studied in [35]. Of course our study is different than this paper where the author have considered a charged black brane in the present of a null neutral collapsing matter that results to another charged black brane with different mass.
The corresponding null boundaries of these two parts are

\begin{align*}
  v > 0 : \quad & v = 0, \quad v = \tau, \quad \tau - v = 2 \int_{\epsilon}^{r(v, \tau)} \frac{dr}{f(r)} \\
  v < 0 : \quad & v = 0, \quad v = -\infty, \quad r(v, \tau) = r_p'(\tau) - \frac{1}{2}v,
\end{align*}

where \( \epsilon \) is the UV cutoff.

Actually for \( v < 0 \) the solution is a pure AdS\(_{d+1} \) solution and we will have to compute three different contributions to the on shell action that come from bulk, joint and ambiguity terms of the action. Note that since all boundaries are null their contributions are zero using the Affine parametrization to parametrize the null directions. It is then straightforward to compute the non-zero contributions. In particular for the bulk term one finds

\[
I_{v<0}^{\text{bulk}} = -\frac{dL^{d-1}}{8\pi G_N} \int d^{d-1}x \int_{-\infty}^{0} dv \int_{r_p'(\tau)-\frac{1}{2}v}^{\infty} \frac{dr}{r^{d+1}} = -\frac{V_{d-1}L^{d-1}}{4\pi G_N} \frac{1}{d-1} \frac{1}{r_{p'}^{d-1}(\tau)}. \tag{4.6}
\]

While using the null vectors \( \mathbf{2.11} \) the contribution of joint points is

\[
I_{v<0}^{\text{joint}} = -\frac{V_{d-1}}{8\pi G_N} \frac{L^{d-1}}{r_{p'}^{d-1}(\tau)} \log \frac{\alpha \beta r_p^2(\tau)}{L^2}. \tag{4.7}
\]
Finally one has a contribution from the ambiguity term that is

\[ I_{\text{amb}} = \frac{V_{d-1}L^{d-1}}{8\pi G_N} \left( \log \frac{\alpha\eta^2(\tau)}{L^2} + \frac{2}{r_p^{d-1}(\tau)} - \frac{2}{(d-1)r_{p'}^{d-1}(\tau)} \right). \tag{4.8} \]

Putting all terms together we find that on shell action vanishes for the \( v < 0 \) part of WDW patch. Therefore all nonzero contributions to the on shell action come from \( v > 0 \) part.

Indeed in this part for the bulk term one has

\[
I_{v>0}^{\text{bulk}} = \frac{L^{d-1}}{8\pi G_N} \int d^{d-1}x \int_0^\tau dv \int_{r(v,\tau)}^{r^-} dr \left( -\frac{d}{r^{d+1}} + Q^2(d-2)r^{d-3} \right) = -\frac{V_{d-1}L^{d-1}}{8\pi G_N} \int_0^\tau dv \frac{f(r(v,\tau))}{r^{d}(v,\tau)}. \tag{4.9} \]

On the other hand using the fact that \( \frac{dr(v,\tau)}{dr} = \frac{1}{2} f(r(v,\tau)) \) one can perform the above integration resulting to

\[
I_{v>0}^{\text{bulk}} = \frac{V_{d-1}L^{d-1}}{8\pi G_N} \left( \frac{2}{(d-1)r_p^{d-1}(\tau)} - \frac{2}{(d-1)e^{d-1}} \right). \tag{4.10} \]

Here we have used that \( r(\tau,\tau) = \epsilon, r(0,\tau) = r_p(\tau) \). It is important to note that due to the null shock wave at \( v = 0 \) there would be a displacement in the null boundary and the joint point \( p \) is not necessary at the same point \( p' \) we have considered in the AdS part for \( v < 0 \) \cite{30}. Of course it does not affect our results since we compute the contribution of the two parts \( v < 0 \) and \( v > 0 \) separately.

The \( v > 0 \) part of the WDW has four null boundaries whose contributions to the on shell action vanish using Affine parameterization for the null directions. Therefore the remaining part needs to be computed is the contribution of joint points.

In this part of the WDW patch there are four joint points two of which are located at \( r = r_- \), one at \( r_p \) and one at \( r = \epsilon \). Although the contributions of two later points can be easily computed using the coordinate systems we have been using so far, for those at \( r_- \) it finds useful to utilize the following coordinate system

\[
U = -e^{-\frac{1}{2}f'(r_-)u}, \quad V = e^{\frac{1}{2}f'(r_-)v}, \tag{4.11} \]

by which the corresponding two joint points are located at \((U,V) = (\zeta, V_0)\) and \((U,V) = (\zeta, V_\tau)\), for \( \zeta \to 0 \). Denoting the \( r \) coordinate associated with these points by \( r_0 \) and \( r_\tau \), respectively, and using the normal vectors \( (2.11) \) one gets the following expression for the contribution of the joint points

\[
I_{v>0}^{\text{joint}} = \frac{V_{d-1}L^{d-1}}{8\pi G_N} \left( \log \frac{\alpha\eta^2(\tau)}{L^2f(r_0)} - \log \frac{\alpha\eta^2}{L^2f(r_\tau)} \right) - \frac{1}{(d-1)r_\tau^{d-1}} - \frac{1}{(d-1)r_0^{d-1}}. \tag{4.12} \]
On the other hand using the fact that 
\[ \log f(r) = \log |UV| + c_0 + \mathcal{O}(UV), \quad \text{for} \ UV \to 0, \quad (4.13) \]
the above expression reads
\[ I_{v>0}^{\text{joint}} = \frac{V_{d-1} L^{d-1}}{8 \pi G_{N}} \left( \frac{\log \frac{\alpha \beta_{0}^{2}(\tau)}{L^{2}}}{r_{p}^{d-1}(\tau)} - \frac{\log \frac{\alpha \beta_{0}^{2}}{L^{2}}}{r_{p}^{d-1}(\tau)} - \frac{\log f(r_{p})}{r_{p}^{d-1}(\tau)} + \frac{f'(r_{-})}{2 r_{-}^{d-1}} \right). \quad (4.14) \]

Here we have also used the fact that \( \{r_{0}, r_{+}\} \approx r_{-} \). Finally the contribution of ambiguity term is
\[ I_{v>0}^{\text{amb}} = \frac{V_{d-1} L^{d-1}}{8 \pi G_{N}} \left( - \frac{\log \frac{\alpha r_{-}}{r_{-}^{d-1}}}{(d-1)r_{-}^{d-1}} - \frac{1}{(d-1)e^{d-1}} + \frac{1}{(d-1)e^{d-1}} \right) \]
\[ + \frac{V_{d-1} L^{d-1}}{8 \pi G_{N}} \left( - \frac{\log \frac{\beta_{0}(\tau)}{L}}{r_{p}^{d-1}(\tau)} - \frac{1}{(d-1)r_{p}^{d-1}(\tau)} + \frac{1}{(d-1)e^{d-1}} \right) \]
\[ + \frac{V_{d-1} L^{d-1}}{8 \pi G_{N}} \left( - \frac{\log \frac{\alpha r_{-}}{L}}{r_{-}^{d-1}(\tau)} - \frac{1}{(d-1)r_{-}^{d-1}(\tau)} + \frac{1}{(d-1)e^{d-1}} \right) \]
\[ = \frac{V_{d-1} L^{d-1}}{8 \pi G_{N}} \left( - \frac{\log \frac{\alpha r_{-}^{2}(\tau)}{L^{2}}}{r_{-}^{d-1}(\tau)} - \frac{2}{(d-1)r_{-}^{d-1}(\tau)} + \frac{2}{(d-1)e^{d-1}} \right). \quad (4.15) \]

As we already mentioned the contribution of \( \nu < 0 \) part vanishes and therefore the total on shell is
\[ I_{v>0}^{\text{total}} = I_{v>0}^{\text{bulk}} + I_{v>0}^{\text{joint}} + I_{v>0}^{\text{amb}} = \frac{V_{d-1} L^{d-1}}{8 \pi G_{N}} \left( \frac{f'(r_{-})}{2 r_{-}^{d-1}} - \frac{\log f(r_{p}(\tau))}{r_{p}^{d-1}(\tau)} \right). \quad (4.16) \]

It is then easy to compute the time derivative of on shell action
\[ \frac{dI_{v>0}^{\text{total}}}{d\tau} = \frac{V_{d-1} L^{d-1}}{16 \pi G_{N}} \left( \frac{f'(r_{-})}{r_{-}^{d-1}} - \frac{f'(r_{p}(\tau))}{r_{p}^{d-1}(\tau)} + \frac{d-1}{r_{p}^{d-1}(\tau)} f(r_{p}(\tau)) \log |f(r_{p}(\tau))| \right). \quad (4.17) \]

Note that at late time where \( r_{p}(\infty) = r_{+} \) one gets
\[ \frac{dI_{v>0}^{\text{total}}}{d\tau} = \frac{V_{d-1} L^{d-1}}{16 \pi G_{N}} \left( \frac{f'(r_{-})}{r_{-}^{d-1}} - \frac{f'(r_{+})}{r_{+}^{d-1}} \right), \quad (4.18) \]
that is the same as that of eternal black brane given in the equation \( (2.19) \).

Since all non-zero contributions to the on shell action come from the \( v > 0 \) part where the geometry is essentially an AdS-RN black brane, one may also study a case where the corresponding black brane is in the stage of near extremal. Of course in order to get such a geometry the charge and the energy of infalling null matter should be fine tuned. Since in the present case the late time

\[ ^{5}\text{Here} \ c_{0} = \psi^{(0)}(1) - \psi^{(0)}(\frac{1}{\pi + \tau}) \text{is a positive number and} \ \psi^{(0)}(x) = \frac{\Gamma'(x)}{\Gamma(x)} \text{is the digamma function.} \]
behavior of complexity is the same as that of two-sided black brane considered in the previous section, taking the corresponding near horizon limit would lead to the same expression given in the equation (3.7) too.

One could also try to directly compute the complexity from the near horizon solution. We note, however, that in general it is not straightforward to write a solution that is a $d + 1$ dimensional AdS for $v < 0$ and an $\text{AdS}_2 \times \mathbb{R}^{d-1}$ for $v > 0$. Nonetheless as far as the computation of complexity is concerned it is possible to proceed to find the complexity for near horizon solution. Actually as we have already seen the contribution to the complexity from the $v < 0$ part where we have an AdS$_{d-1}$ is zero and therefore we are left with just $v > 0$ part. On the other hand in this part the geometry is the near horizon limit of near extremal case which we have considered in the previous section. Therefore we will also get the same expression for the late time behavior of complexity growth as (3.7).

Another interesting case related to what we have considered here is to study complexity for a process of creating a near extremal RN black hole from the extremal one by adding an in falling shock wave. Such a geometry has been studied in [38].

5 Conclusions

In this paper we have studied complexity for near horizon limit of near extremal charged black brane solutions. The aim was to compute the corresponding complexity from two different approaches. In the first approach we have taken the limit from the complexity of a charged black brane while in the second approach we have computed complexity for a near horizon solution.

We have observed that both approaches give the same results if one assumes that the UV cutoff enforces us to have a cutoff behind the horizon whose value is also fixed by the UV cutoff. Moreover it is crucial to add the contribution of a certain counterterm evaluated on the cutoff surface behind the horizon.

It is important to note that in order to find meaningful results one has to carefully take the near horizon limit of the near extremal black brane. Doing so, the metric one gets has an AdS$_2$ factor given in the Rindler coordinates. Although we have computed the complexity from $d + 1$ dimensional theory point of view, one could have reduced the near horizon solution into two dimensions where we would have gotten the generalized Jackiw-Teitelboim gravity (see for example [39]). The corresponding two dimensional gravity may be obtained by a dimensional reduction using the following general ansatz

$$ds^2 = ds_2^2 + \psi^2 \, d\vec{x}^2.$$ (5.1)
Plugging this metric in to the action (2.1) one gets (see e.g. [39] for more details)

\[ I = \frac{V_{d-1}}{16\pi G_N} \int d^2x \sqrt{-\hat{g}_2} \psi^{d-1} \left( \hat{R}_2 + (d - 1)(d - 2)|\hat{\nabla}\psi|^2 \psi^{-2} + \frac{d(d - 1)}{L^2} - \frac{1}{2} \hat{F}_2^2 \right). \]  

(5.2)

Then the complexity could have been computed from two dimensional point of view. We note that complexity for the (generalized) Jackiw-Teitelboim gravity has been studied in [10, 21]. In particular the authors of [21] have studied four dimensional charged black hole reduced into two dimensions. In order to get the desired late times linear growth the author have pointed out that the following boundary term is needed (here we have written the corresponding term for arbitrary dimensions)

\[ I = \frac{1}{8\pi G_N} \int d^d x \sqrt{|h|} n_\mu F^{\mu \nu} A_\nu, \]  

(5.3)

where \( n_\mu \) is normal vector to the boundary. This term is needed if one wants to impose different boundary conditions on the gauge field rather than fixing the potential at boundary. On the other hand in the present paper we have considered the counterterm given in the equation (3.11). It is, however, straightforward to see that as far as their contributions to the complexity is concerned they lead to the same result. More precisely using the expressions of metric, gauge field and the normal vector the above term at the boundary may be recast into the following

\[ I = -\frac{1}{8\pi G_N} \int d^d x \sqrt{|h|} \frac{\sqrt{h(r_c)}}{L} \]  

(5.4)

that is the counterterm we have considered. It is important to note that in the present paper we have evaluated this term at the cutoff surface behind the horizon and not at the boundary. Therefore in our case adding this term would not change the physics (boundary condition) we have originally considered.

As the final remark we would like to make a comment on the behind the horizon cutoff for the charged black holes. Although in this case we have not obtained the relation between UV cutoff and the behind horizon cutoff, since for small UV cutoff we would expect that the corresponding cutoff tends to infinity where the singularity is located the behind horizon cutoff should be behind the inner horizon, as shown by red dashed lines in the left panel of figure 1 and figure 2. Therefore it does not have any direct effect in the WDW patch and thus complexity.

Indeed it is consistent with the result of [40] where a charged black hole at a finite cutoff has been studied. In this case one could see that unlike the neutral black holes the complexity is not affected by the cutoff.

On the other hand going into the near horizon the region in which the behind horizon cutoff is located will be excluded from the spacetime and we will have to reconsider the cutoff in the new coordinates denoted by (\( \hat{t}, \hat{r} \)) in present paper. More precisely in this case the obtained spacetime has an AdS\(_2\) factor and the corresponding cutoff is depicted by dashed red line in the right panel.
of figure 1.

**Note added:** While we were to submit our paper to arXiv the article [41] appeared where the complexity for charged (dyonic) black holes and Jackiw-Teitelboim gravity have been extensively studied. The way the authors resolved the puzzle of getting undesired results was to take into account the contribution of certain boundary terms similar to that of [21].

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