Dean-Kawasaki Equation with Singular Interactions and Applications to Dynamical Ising-Kac Model

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Abstract. Inspired by [Fehrman, Gess; Invent. Math., 2023] and [Fehrman, Gess; Arch. Ration. Mech. Anal., 2024], we consider the Dean-Kawasaki equation with singular interactions and correlated noise which can be viewed as fluctuating mean-field limits. By imposing the Ladyzhenskaya-Prodi-Serrin condition on the interaction kernel, the existence of probabilistic weak renormalized kinetic solutions is established. Further, under an additional integrability assumption on the divergence of the interaction kernel, a kinetic formulation approach is applied to derive pathwise uniqueness, leading to the strong well-posedness of the equation. As an application, we obtain the well-posedness of a conservative stochastic partial differential equations known as fluctuating Ising-Kac-Kawasaki dynamics, which paves a step on the conjecture concerning nonlinear fluctuations of Kawasaki dynamics proposed by [Giacomin, Lebowitz, Presutti; Math. Surveys Monogr., 1999].

1. Introduction

Fluctuating hydrodynamics provides a framework for simulating microscopic fluctuations by combining statistical mechanics and nonequilibrium thermodynamics. This framework leads to various conservative-type stochastic partial differential equations (SPDEs), which represent the fluctuation corrections of hydrodynamic limits of interacting particle systems, characterized by fluctuation-dissipation relations. In general, the fluctuation phenomena of these SPDEs is similar to that of the corresponding interacting particle systems. Taking into account the theory of fluctuating hydrodynamics, we consider a regularized version of Dean-Kawasaki equation, which can be viewed as fluctuating mean-field limits of the following mean-field systems:

\[ dX_i = -\frac{1}{N} \sum_{j=1}^{N} \nabla U(X_i - X_j)dt + \sqrt{2}dB_i, \quad i = 1, \ldots, N, \]  

(1.1)

where \( \{B_i\}_{i=1}^{N} \) represents independent Brownian motions, \( U \) denotes the interaction kernel. Let \( \pi_N \) be the empirical measure defined by \( \pi_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i} \). To approximate \( \pi_N \), the Dean-Kawasaki equation was proposed by Dean [Dea96] and Kawasaki [Kaw98] based on the covariance structure of the noise. The Dean-Kawasaki equation takes the following form:

\[ \partial_t \pi_N = \Delta \pi_N + \nabla \cdot \left( \pi_N \nabla U * \pi_N \right) - \sqrt{2}N^{-1/2} \nabla \cdot \left( \sqrt{\pi_N} \xi \right), \]  

(1.2)

where \( \xi \) is a space-time white noise. Note that (1.2) is a conservative supercritical singular stochastic PDE, which is ill-posed in the theories of Hairer’s regularity structure [Ha14] and Gubinelli, Imkeller, Perkowski’s paracontrolled distribution [GIP15]. The existence and uniqueness (in law) of trivial martingale solution to (1.2) was shown by Konarovskyi, Lehmann and von Renesse [KLvR19] under the condition that \( N \) is a non-negative integer. Specifically, the authors proved that the empirical measure of particle system solves the martingale problem, which gives rigorous mathematical meaning to the Dean-Kawasaki equation (1.2). However, the well-posedness of functional-valued solutions of (1.2), in particular, the pathwise uniqueness is challenging, due to the irregularity of the square root function and the nonlocal term even if the noise is sufficiently regular in space. In the case of...
local interaction, the obstacle was resolved by Fehrman and Gess [FG24]. Concretely, the authors considered the following general Dean-Kawasaki equation written by

$$\partial_t \rho = \Delta \Phi(\rho) - \nabla \cdot (\nu(\rho) - \nabla \cdot (\sigma(\rho) \circ W^F)),$$  \hspace{1cm} (1.3)

where $\circ$ represents the Stratonovich integral. The nonlinearity is in the form $\Phi(\xi) = \xi^m$ for all $m \geq 1$ corresponding to the hydrodynamic limit of zero-range process with a mean-local jump rate $\Phi$ (see Kipnis and Landim [KL99, Chapters 3 and 5]). The nonlinear flux function $\nu(\cdot)$ is of Burgers type. The noise coefficient $\sigma(\cdot)$ is locally 1/2-Hölder continuous and the noise $W^F$ is sufficiently regular in space and white in time.

To the best of our knowledge, the well-posedness of the correlated Dean-Kawasaki equation with nonlocal interactions, especially with singular interaction kernels, still remains open. In this paper, we devote to proving the well-posedness of the following equation:

$$\partial_t \rho = \Delta \rho - \nabla \cdot (\rho V(t) \ast \rho) - \nabla \cdot (\sqrt{\rho} \circ W^F).$$  \hspace{1cm} (1.4)

In the above equation, the symbol $\ast$ represents spatial convolution and the interaction kernel $V = -\nabla U$ is generalized to a singular nonlocal time-dependent function. More precisely, we consider the case of spatial dimension $d \geq 2$, and let $T > 0$ be a fixed time horizon in the whole context, the singular interaction kernel $V$ in (1.4) is assumed to satisfy that

**Assumption (A1):** $V \in L^{p^*}([0, T]; L^p(\mathbb{T}^d; \mathbb{R}^d))$ with $d/p + 2/p^* \leq 1$, $2 \leq p^* \leq \infty$ and $d < p \leq \infty$,

**Assumption (A2):** $\nabla \cdot V \in L^{q^*}([0, T]; L^q(\mathbb{T}^d))$ with $d/q + 2/q \leq 1$, $1 \leq q^* \leq \infty$ and $d \leq q \leq \infty$.

Let $\text{Ent}(\mathbb{T}^d)$ be the space of finite entropy functions defined by (2.6). Based on the above assumptions, the well-posedness of (1.4) reads as follows.

**Theorem 1.1.** (Uniqueness, cf. Theorem 4.2) Assume that $V$ satisfies Assumptions (A1) and (A2). Let $\hat{\rho}^1, \hat{\rho}^2 \in \text{Ent}(\mathbb{T}^d)$. Let $\rho^1, \rho^2$ be renormalized kinetic solutions of (1.4) in the sense of Definition 2.3 with initial data $\rho^1(\cdot, 0) = \hat{\rho}^1, \rho^2(\cdot, 0) = \hat{\rho}^2$. If $\hat{\rho}^1 = \hat{\rho}^2 \text{ a.e. in } \mathbb{T}^d$, then

$$\mathbb{P}\left( \sup_{t \in [0, T]} \| \rho^1(\cdot, t) - \rho^2(\cdot, t) \|_{L^1(\mathbb{T}^d)} = 0 \right) = 1.$$

**Theorem 1.2.** (Existence, cf. Theorem 6.2) Assume that $V$ satisfies Assumption (A1). Let $\hat{\rho} \in \text{Ent}(\mathbb{T}^d)$. Then there exist a stochastic basis $(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}(t)\}_{t \in [0, T]}, \hat{\mathbb{P}})$, a Brownian motion $\hat{W}^F$ and a stochastic process $\hat{\rho}$, which satisfy the definition of renormalized kinetic solution to (1.4) in the sense of Definition 2.3 with initial data $\hat{\rho}(\cdot, 0) = \hat{\rho}$. If $V$ is further assumed to satisfy Assumption (A2), then there exists a unique probabilistically strong renormalized kinetic solution of (1.4).

Regarding to the assumptions of the singular interaction kernels, we make some comments. Assumption (A1) is the Ladyzhenskaya-Prodi-Serrin (LPS) condition, which was firstly proposed by Prodi [Pro59], Serrin [Ser62], and Ladyzhenskaya [Lad67] as a regularity criterion when studying the uniqueness of the 3D Navier-Stokes equations. [CLI07] provides a representation of the solution of the Navier-Stokes equations by using stochastic flow. This leads to the study of stochastic differential equations (SDEs) with a singular drift that satisfies the LPS condition. Various literature in this direction includes [KR05, Zha05, Zha11, XXZZ20, RZ21b, RZ23]. Later, Röckner and Zhang [RZ21a] extended the results of [KR05] to distribution-dependent SDEs, which can be interpreted as large $N$ limits of interacting particle systems (1.1). As fluctuating mean-field limits of the same model (1.4), it is natural to assume the SPDE (1.4) to have an LPS-type interaction kernel, paralleling the theory of singular SDEs and distribution-dependent SDEs. Finally, we point out that the condition on $\nabla \cdot V$ is for technical reasons, which will be explained in detail at the end of the proof of uniqueness (see Section 4).

Employing a similar approach akin to that used for (1.4), we can establish the well-posedness of the following conservative SPDE given by

$$\partial_t \rho = \Delta \rho + \beta \nabla \cdot ((1 - \rho^2) \nabla J \ast \rho) - \gamma^{1/2} \nabla \cdot (\sqrt{1 - \rho^2} \circ \xi_d).$$  \hspace{1cm} (1.5)
Here, $J$ stands for the Kac potential, $\beta$ is a constant depends on temperature and the noise $\xi_\delta$ is white in time and correlated in space with a correlation length $\delta$ representing the simulation grid size. The equation (1.5) proposed by [GLP99] is related to the nonlinear fluctuations of Kawasaki dynamical Ising-Kac model (see subsection 1.1 (2) for details). The well-posedness of (1.5) is formulated as follows.

**Theorem 1.3.** *(Well-posedness, cf. Theorem 7.12)* For any spatial dimension $d \geq 1$, suppose that $\nabla J \in C^\infty(T^d, \mathbb{R}^d)$. Let the initial data $\hat{\rho}(\cdot, 0) \in \text{Ent}(T^d)$ that is defined by (7.7), then there exists a unique renormalized kinetic solution to (1.5) with initial data $\hat{\rho}$.

### 1.1. Applications

In this subsection, two important applications of Theorem 1.2 and Theorem 1.3 are presented.

1. **An application to fluctuations of mean-field systems.** We mention two landmark works by Wang, Zhao, Zhu [WZZ23] and Chen, Ge [CG22]. They studied the Gaussian fluctuations and large deviations for singular interacting mean-field systems

$$dX_i = \frac{1}{N} \sum_{j \neq i} V(X_i - X_j) dt + \sqrt{2} dB^i_t, \quad i = 1, \ldots, N, \quad (1.6)$$

respectively. Precisely, [WZZ23] proved the Gaussian fluctuation of the empirical measure $\pi^N$ around its mean-field limit

$$\partial_t \bar{\rho} = \Delta \bar{\rho} - \nabla \cdot (\bar{\rho} V \ast \bar{\rho}),$$

which reads as

$$\sqrt{N}(\pi^N - \bar{\rho}) \to \bar{\rho}^1 \quad \text{as} \quad N \to \infty.$$ 

Here, $\bar{\rho}^1$ satisfies the equation

$$\partial_t \bar{\rho}^1 = \Delta \bar{\rho}^1 - \nabla \cdot (\bar{\rho} V \ast \bar{\rho}^1) - \nabla \cdot (\bar{\rho}^1 V \ast \bar{\rho}) - \sqrt{2} \nabla \cdot (\sqrt{\bar{\rho}} \xi), \quad (1.7)$$

with $\xi$ being a space-time white noise. Let $\xi_{K(N)}$ be the ultra-violet noise, which converges to $\xi$ as $K(N) \to \infty$, $N \to \infty$. An informal computation shows that the Dean-Kawasaki equation with $\xi_{K(N)}$

$$\partial_t \rho^N + \nabla \cdot (\rho^N V \ast \rho^N) = \Delta \rho^N - \sqrt{\frac{2}{N}} \nabla \cdot (\sqrt{\rho^N} \xi_{K(N)}) \quad (1.8)$$

fulfills the same Gaussian fluctuation $\sqrt{N}(\rho^N - \bar{\rho}) \to \bar{\rho}^1$, where $\bar{\rho}^1$ fulfills (1.7).

Regarding to [CG22], the authors established the large deviations for the empirical measure of the two-dimensional interacting particle systems (1.6) with the rate function informally given by

$$I^0(\rho) = \sup_{\psi \in C^\infty(T^2 \times [0, T])} \left( \int_{T^2} \psi(x, T) \rho(x, T) dx - \int_{T^2} \psi(x, 0) \rho(x, 0) dx - \int_0^T \int_{T^2} \rho \partial_t \psi dx dt 
- \int_0^T \int_{T^2} \rho \Delta \psi dx dt - \int_0^T \int_{T^2} (\rho V \ast \rho) \nabla \psi dx dt - \frac{1}{2} \int_0^T \int_{T^2} \rho |\nabla \psi|^2 dx dt \right). \quad (1.9)$$

As discussed in [FG23, Theorem 39], Fehrman and Gess proved that the rate function for large deviations of the Dean-Kawasaki equation is governed by (1.9) as well when $V = 0$. However, upon closely following Fehrman and Gess's argument line by line, the result remains unchanged in the presence of an interaction kernel (at least when the kernel is “nice”). Consequently, the fluctuations of (1.6) can be predicted by (1.8).

In fact, aside from the result that the Dean-Kawasaki equation (1.8) preserves the fluctuations feature of the mean-field system (1.6), the regularity of the former is $L^2_t W^{1,1}_x$ (see the entropy estimate in Proposition 5.4) that is stronger than $L^2_t H^{\frac{d}{2}}_x$ for (1.6). Thus, it is a good alternative to study the Dean-Kawasaki equation (1.8) to capture the fluctuations of (1.6). Applying our result Theorem 1.2 to (1.8), we obtain its well-posedness which plays a fundamental role in studying fluctuations.
Finally, we mention that some quantitative error analysis has been built between discrete and continuous objects. For example, [DKP22] established a weak error estimate between the regularized Dean-Kawasaki equation and Brownian particles via a duality argument. [CF23b, CFIR23] provided error estimate results for the discretized Dean-Kawasaki equation, demonstrating that structure-preserving discretizations can closely approximate the density fluctuations of $N$ non-interacting diffusing particles to an arbitrary order in $N^{-1}$ within appropriate weak metrics. Thus, it is feasible to adopt an SPDE framework to mirror the fluctuations of the $d$-dimensional $N$-particle SDE systems (1.6), especially in the context of mean-field systems with $L^p$-type interactions (refer to [HRZ22, WZZ22, HHMT24]). This parallel allows the study of SPDE (1.8) to serve as a theoretical foundation for understanding the dynamics of the corresponding particle system.

**2. An application to nonlinear fluctuations of dynamical Ising-Kac model.** The Ising model, originally proposed by Ising in [EI25], serves as a fundamental model in statistical mechanics for exploring ferromagnetism. This model involves spins arranged on a lattice, where each spin interacts with its nearest neighbors. A variant of the Ising model called the Ising-Kac model was introduced to recover the van der Waals theory of phase transition, in which each spin interacts with all other spin variables in a large ball around its base point (see [HKU64]). Two main dynamics for the Ising-Kac model are Glauber dynamics and Kawasaki dynamics. The elementary events in the former are spin-flips (i.e., changes of sign of a single spin), while, for the latter, two spins exchange their positions (more details can be found in [Gla63, Spo12]). For both dynamics, it is an active field of research to derive the macroscopic behavior from microscopic models by hydrodynamical scaling limit. In [GLP99], it was conjectured that both the Glauber and the Kawasaki dynamics have anomalous fluctuating behaviors, i.e., nonlinear fluctuation, in a neighborhood of the critical temperature. Moreover, their nonlinear fluctuations are described by $\Phi^d_3$ and Cahn-Hilliard equation, respectively. Recently, the conjecture on the Glauber dynamics in 1, 2 and 3 dimensions is completely settled; see [BPRS94, FR95] for $d = 1$, [MW17] for $d = 2$, and [GMW23] for $d = 3$. However, the conjecture on the nonlinear fluctuation of Kawasaki dynamics remains widely open, and only some progress in 1 dimension are made by [Ibe18]. The present paper paves a step on this conjecture in a certain sense. In the following, we make some explanations.

In light of the theory of fluctuating hydrodynamics, a continuous analogue (a conservative SPDE) is conjectured to have the same scaling limits as microscopic particle system. A formal analysis from [GLP99] shows that the conservative SPDE related to the Ising-Kac model in one spatial dimension is the equation (1.5). Precisely, for any $\gamma > 0$ and $a \in \mathbb{R}$, let $\beta = 1 + a\gamma^{2/3}$ and define the rescaled density field

$$\rho_{\gamma}(x, t) := \gamma^{-1/3} \rho(\gamma^{-1/3} x, \gamma^{-4/3} t)$$

with $\rho$ satisfying (1.5). $\rho_{\gamma}$ is conjectured to converge to the conservative stochastic Cahn-Hilliard equation

$$\partial_t u = -\partial_{xx}^2 (\partial_{xx}^2 u - u^3 + au) - \partial_x \xi,$$

as $\gamma \to 0$. It implies that (1.5) has the same nonlinear fluctuations phenomenon as Ising-Kac model when the temperature is near criticality. Therefore, (1.5) can be regarded as a phenomenological model simulating the Kawasaki dynamics for the Ising-Kac model.

However, a rigorous proof of the conjecture still remains open, and we emphasize that any rigorous study of (1.5) is challenging due to the lack of well-posedness. Thanks to the structural resemblance with the Dean-Kawasaki equation, in this paper, we apply our main results to establish strong well-posedness for the Ising-Kac-Kawasaki equation (1.5).

**1.2. Key argument and technical comment.** As stated in [FG24] and [FG23], due to the singularity at zero of the Itô correction term $\frac{1}{4\gamma} \nabla \gamma \rho$, it is not clear how to define the concept of classical weak solutions to (1.3). To restrict the value of the solution away from zero, [FG24] employed the kinetic approach and introduced the concept of renormalized kinetic solutions. This concept was first proposed by Lions, Perthame, and Tadmor in [LPT94] when studying general multi-dimensional scalar conservation laws. The key insight is that the kinetic function has three variables $(t, x, \xi)$, which stand...
for time, spatial and velocity variable, respectively. Then, the test function for the renormalized kinetic solution can have compact support with respect to the velocity variable. As a result, it keeps the kinetic solution away from its singularities. This idea has been successfully applied to conservative stochastic PDEs, we refer readers to the works of Gess and Souganidis [GS17], Fehrman and Gess [FG19], and Dareiotis and Gess [DG20] for further details. In the present paper, we also adopt the concept of renormalized kinetic solutions.

In this paragraph, we outline the primary distinctions between our paper and [FG24] from the following three aspects. (i) The equation. Obviously, our equation has additional nonlocal interaction terms compared with [FG24]. Coming up with suitable conditions for the interaction terms to ensure the existence of a solution is the central challenge. The key observation is that the nonlocal interaction term is associated with the convolution type distributional dependence SDEs, hence it is natural to impose an LPS-type condition on the kernel. (ii) The technique. Compared with [FG24], all of the technical adjustments are to deal with the difficulties arising from nonlocal interaction terms. Due to the irregularity of the nonlocal interaction kernel, the $L^p$-theory ($p > 2$) of (1.3) established by [FG24] is no longer applicable to (1.4). As an alternative, we make an entropy dissipation estimate (see Proposition 5.4) for (1.4) assuming that the kernel $V$ satisfies Assumption (A1). Such an estimate not only provides the regularity of the square root of the solution, known as Fisher information, but also suggests a potential link between LPS-type conditions in fluid dynamics and the Boltzmann entropy in the fluctuating hydrodynamics. Moreover, it also plays a pivotal role in proving both the uniqueness and the existence of renormalized kinetic solutions to (1.4). For the uniqueness, the kernel terms cannot be expected to converge to zero or even be controlled since cut-off functions cannot bound the nonlocal term, which is different from [FG24] where all terms vanish. To solve this obstacle, we show that the interaction kernel terms in (1.4) can be controlled by the square root of the solution under Assumptions (A1) and (A2). Consequently, with the aid of entropy estimate, the uniqueness of (1.4) is achieved by applying a stochastic Gronwall’s lemma. A comprehensive proof can be found in Theorem 4.2. Regarding to the existence of renormalized kinetic solutions to (1.4), we also introduce a sequence of approximating equations with regular coefficients and smooth kernel similarly to [FG24]. However, it is different from [FG24] in finding a limiting kinetic measure. Due to the lack of $L^p$-theory, the kinetic measures of the approximating sequence are no longer uniformly bounded in the space of bounded Borel measures over $[0, T] \times \mathbb{T}^d \times \mathbb{R}$ (with norm defined by the total variation of measures). As a result, we cannot find a limiting kinetic measure by the method used in [FG24]. Instead, we make use of the entropy estimate to show that the kinetic measures are bounded in the space of nonnegative bounded Borel measures over $[0, T] \times \mathbb{T}^d \times [0, M]$ for any $M \geq 1$. Subsequently, the existence of a limiting kinetic measure can be proved by a diagonal argument. For more details, see Theorem 6.2 below. (iii) The applications. Aside from above, we have different applications in simulating microscopic particle systems compared with [FG24]. The well-posedness of (1.4) allows us to study the fluctuations of mean-field systems with singular interactions, especially the Kawasaki dynamics for the Ising-Kac model. The specific examples have been presented in subsection 1.1.

At the end of this subsection, we provide further commentary on technical details involved in studying the well-posedness of the fluctuating Ising-Kac-Kawasaki equation (1.5). As discussed in the above paragraph, the entropy estimate plays a central role. Due to the structural similarity between (1.5) and (1.4), we can anticipate it holds. Indeed, by choosing $\Psi(\zeta) = \int \log \left( \frac{1 + \zeta}{1 - \zeta} \right) d\zeta'$, the corresponding entropy dissipation estimate for (1.5) holds as well, see Proposition 7.6. Then, following the approach used in (1.4), an analogue of Theorems 4.2 and 5.7 can be established for (1.5). Finally, the results of Theorem 1.1 and 1.2 can be partially extended to (1.5). In addition, we emphasize some differences between (1.5) and (1.4). The first is that the kinetic solution of (1.4) is nonnegative, while, the solution of (1.5) takes values in the interval $[-1, 1]$, it causes the preservation of the $L^1(\mathbb{T}^d)$-norm to be invalid for the latter. The second is that the derivative of the diffusion coefficient $\sqrt{\zeta}$ for (1.4) only has one singularity at $\zeta = 0$, while, the derivative of the diffusion coefficient $\sqrt{1 - \zeta^2}$ for (1.5) has two singularities at $\zeta = +1$ and $\zeta = -1$. It leads to some technical differences in proving the tightness of the approximating equations. For (1.4), an $L^1([0, T]; L^1(\mathbb{T}^d))$-equivalent topology is constructed (see
1.3. Comments on the literature. The existence of solutions to corrected Dean-Kawasaki equations with smooth interacting kernel has been proved by von Renesse and Sturm [vRS09] by Dirichlet forms techniques, where the nonlocal interacting term is replaced by a nonlinear operator. Later, by correcting the drift term of the Dean-Kawasaki equation, the authors of [AvR10] and [KvR19] constructed a solution. Cornalba, Shardlow and Zimmer [CSZ19, CSZ20] derived a suitably regularized Dean-Kawasaki model of wave equation type in one dimension, which corresponds to second-order Langevin dynamics. In the case of local interaction, Fehrman and Gess [FG24] obtained the well-posedness of functional-valued solutions of the Dean-Kawasaki equation with correlated noise. Building on this framework in [FG24], the authors also addressed small noise large deviations in [FG23]. Furthermore, Clini and Fehrman [CF23a] expanded this research by developing a central limit theorem for the nonlinear Dean-Kawasaki equation with correlated noise.

In the framework of kinetic solution, the study of well-posedness of stochastic conservative law has attracted significant interests. Debussche and Vovelle [DV10] studied the Cauchy problem in any dimension and obtained the existence and uniqueness of the kinetic solutions. Later, Gess and Souganidis [GS17], Fehrman and Gess [FG19], and Dareiotis and Gess [DG20] extended the notion of kinetic solution to parabolic-hyperbolic stochastic PDE with conservative noise. Recently, the nonlocal conservative stochastic PDE was considered by Fehrman, Gess and Gvalani [FG24] and the mean-field stochastic PDE has been studied by Gess, Gvalani and Konarovskyi [GGK22]. For literatures on stochastic nonlinear diffusion equations, we refer to [BBDPR06, BDP08a, BDP08b, BDP16].

The literature contributed to Gaussian fluctuations of discrete interacting particle systems is notably comprehensive. Seminal studies such as [FPV87] investigated the zero-range process, while [Rav92] focused on the symmetric simple exclusion process, and [JM18] examined the weakly asymmetric exclusion process. The exploration of the mean field limit for singular interaction kernels has also yielded significant results. Key contributions in this area include the study of the vortex model by [Osa86, FHM14], and the analysis of more general singular kernels in works such as [JW18, BJW23, Ser20, RS93].

1.4. Structure of the paper. This paper is organized as follows. In Section 2, basic notations, assumptions on the noise and kernel, and the definition of renormalized kinetic solutions of (1.4) are given. Section 3 is dedicated to presenting various estimates associated with the nonlocal kernel. The uniqueness of the renormalized kinetic solutions of equation (1.4) is rigorously proved in Section 4. Section 5 shifts focus to a sequence of approximating equations for (1.4), where we derive an entropy estimate and establish $L^p(T^d \times [0, T])$-norm estimates in Section 5.1. The existence of renormalized kinetic solutions to these approximating equations is obtained in Section 5.2. Furthermore, Section 5.3 addresses the $L^1([0, T]; L^1(T^d))$-tightness of solutions for this sequence of approximating equations. In Section 6, we conclusively prove the existence of renormalized kinetic solutions of equation (1.4). Eventually, we adjust the proof of (1.4) to show the well-posedness of the fluctuating Ising-Kac-Kawasaki equation (1.5) in Section 7.

2. Preliminary

2.1. Notations. Throughout the paper, $T^d$ denotes a $d$-dimensional torus with volume 1. Let $\nabla$ represent the derivative operator and $\nabla \cdot$ be the divergence operator with respect to the space variable $x \in T^d$. In particular, for any $V : [0, T] \times T^d \to \mathbb{R}^d$, $\nabla \cdot V$ stands for the spatial divergence of $V$. Let $\| \cdot \|_{L^p(T^d)}$ denote the norm of Lebesgue space $L^p(T^d)$ (or $L^p(T^d; \mathbb{R}^d)$) for integer $p \in [1, \infty]$. The inner product in $L^2(T^d)$ will be denoted by $(\cdot, \cdot)$. Let $C^\infty(T^d \times (0, \infty))$ denote the space of infinitely differentiable functions on $T^d \times (0, \infty)$. $C^\infty_c(T^d \times (0, \infty))$ contains all infinitely continuous differentiable
functions with compact supports on $\mathbb{T}^d \times (0, \infty)$. For a non-negative integer $k$ and $p \in [1, \infty]$, denote by $W^{k,p}(\mathbb{T}^d)$ the usual Sobolev space on $\mathbb{T}^d$. Let $H^s(\mathbb{T}^d) = W^{s,2}(\mathbb{T}^d)$, and $H^{-s}(\mathbb{T}^d)$ stands for the topological dual of $H^s(\mathbb{T}^d)$. $C^k_{loc}(\mathbb{R})$ denotes $k$-th differential functions on any compact sets in $\mathbb{R}$. Let the bracket $\langle \cdot, \cdot \rangle$ stand for the duality between $C^\infty(\mathbb{T}^d)$ and the space of distributions over $\mathbb{T}^d$.

Let $X$ be a real Banach space with norm $\| \cdot \|_X$. The space $L^p([0, T]; X)$ denotes the standard Lebesgue space, and $W^{1,p}(0, T; X)$ denotes the standard Sobolev space. In the context, without confusion, for $1 \leq p \leq \infty$ we denote $L^p(\mathbb{T}^d; \mathbb{R})$ or $L^p(\mathbb{T}^d; \mathbb{R}^d)$ by $L^p(\mathbb{T}^d)$, $L^p([0, T] \times \mathbb{T}^d) := L^p([0, T]; L^p(\mathbb{T}^d; \mathbb{R}))$, $L^p([0, T] \times \mathbb{T}^d; \mathbb{R}^d) := L^p([0, T]; L^p(\mathbb{T}^d; \mathbb{R}^d))$ and $C([0, T]) := C([0, T]; \mathbb{R})$. Also, we will encounter integrals on a space $Z$ ($Z$ might be $[0, T] \times \mathbb{T}^d$, $[0, T] \times \mathbb{T}^d \times \mathbb{R}$, $[0, T] \times (\mathbb{T}^d)^2 \times \mathbb{R}^2$ and so on). For simplicity, we abbreviate all integrals $\int_Z f d\tau$ to $\int_Z f$. In addition, we always use $\nabla f$ to denote the weak derivative of $f$ with respect to the space variable.

In the sequel, the notation $a \lesssim b$ for $a, b \in \mathbb{R}$ means that $a \leq \mathcal{D} b$ for some constant $\mathcal{D} > 0$ independent of any parameters. We employ the letter $C$ to denote any constant that can be explicitly computed in terms of known quantities. The exact value denoted by $C$ may change from line to line.

2.2. Assumptions. Let $d \geq 2$. As stated in the introduction part, we impose LPS condition on $V$ and a technical condition on $\nabla \cdot V$. Specifically, we assume that

**Assumption (A1):** $V \in L^{p^*}([0, T]; L^p(\mathbb{T}^d; \mathbb{R}^d))$, with $\frac{d}{p} + \frac{2}{p} \leq 1$, $2 \leq p^* \leq \infty$ and $d < p \leq \infty$.

**Assumption (A2):** $\nabla \cdot V \in L^{q^*}([0, T]; L^q(\mathbb{T}^d))$, with $\frac{d}{2q} + \frac{1}{q^*} \leq 1$, $1 \leq q^* \leq \infty$ and $\frac{d}{2} < q \leq \infty$.

**Remark 2.1.** Assumption (A1) cannot be derived from Assumption (A2). We take two-dimensional Biot-Savart kernel as the counterexample. For every $x \in \mathbb{T}^2$, the Biot-Savart kernel is of the form $V(x) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} e_k(x)$, where $\{e_k\}_{k \in \mathbb{Z}^2}$ is a sequence of Fourier basis of $L^2(\mathbb{T}^2)$. Obviously, $\nabla \cdot V = 0$, however $V \notin L^p(\mathbb{T}^2; \mathbb{R}^2)$ for any $p < 2$, see [BFM16].

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in [0, T]}, \{\{B^k(t)\}_{t \in [0, T]}\}_{k \in \mathbb{N}})$ be a stochastic basis. Without loss of generality, here the filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$ is assumed to be complete and $\{B^k(t)\}_{t \in [0, T], k \in \mathbb{N}}$, are independent $(d$-dimensional) $\{\mathcal{F}_t\}_{t \in [0, T]}$-Wiener processes. We use $\mathbb{E}$ to denote the expectation with respect to $\mathbb{P}$. Let $F = \{f_k\}_{k \in \mathbb{N}} : \mathbb{T}^d \to \mathbb{R}$ be a sequence of continuously differentiable functions on $\mathbb{T}^d$. Define

$$W^F = \sum_{k=1}^{\infty} f_k(x) B^k_t.$$  

(2.1)

For the sequence $\{f_k\}_{k \in \mathbb{N}}$, define

$$F_1 = \sum_{k=1}^{\infty} |f_k|^2, \quad F_2 = \frac{1}{2} \sum_{k=1}^{\infty} \nabla f_k^2$$ and $$F_3 = \sum_{k=1}^{\infty} |\nabla f_k|^2.$$

In this paper, we always assume that $\{F_i\}_{i \in \{1, 2, 3\}}$ are continuous functions on $\mathbb{T}^d$ and $\nabla \cdot F_2 = \frac{1}{2} \Delta F_1 = 0$ on $\mathbb{T}^d$. A typical example satisfying the above conditions is the ultra-violet divergence noise, which lays the foundation when studying fluctuations in singular scaling limits, see [FG24, Remark 2.3] and [DFG20, Section 3] for details.

2.3. Renormalized kinetic solution to the Dean-Kawasaki equation. In this paper, we consider the Dean-Kawasaki equation

$$d\rho = \Delta \rho dt - \nabla \cdot (\rho V \ast \rho) dt - \nabla \cdot (\sqrt{\rho} \circ dW^F),$$

(2.2)

where $W^F$ is defined by (2.1). Then the Stratonovich equation (2.2) is formally equivalent to the Itô equation

$$d\rho = \Delta \rho dt - \nabla \cdot (\rho V \ast \rho) dt - \nabla \cdot (\sqrt{\rho} dW^F) + \frac{1}{8} \nabla \cdot (F_1 \rho^{-1} \nabla \rho + 2F_2) dt.$$  

(2.3)
Formally, the identities
\[ \chi(x, \xi, t) := 1_{\{0 < \xi < \rho(x, t)\}}. \]

hold. Then, the kinetic function \( \chi \) of \( \rho \) satisfies the equation
\[ \partial_t \chi = \nabla \cdot (\delta_0(\xi - \rho)\nabla \rho) + \frac{1}{8} \nabla \cdot (\delta_0(\xi - \rho) (F_1(\xi^{-1}\nabla \rho + 2F_2)) - \frac{1}{4} \partial_\xi (\delta_0(\xi - \rho) (\nabla \rho \cdot 2F_2 + 2\xi F_3)) + \partial_\xi \rho - \delta_0(\xi - \rho)\nabla \cdot (\rho \nabla \rho) - \delta_0(\xi - \rho)\nabla \cdot (\sqrt{W} F), \] (2.5)

where \( q = 4\delta_0(\xi - \rho)\xi|\nabla \sqrt{\rho}|^2 \) is the parabolic defect measure.

To define a renormalized kinetic solution to (2.2), we need a concept of kinetic measure.

**Definition 2.2.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with a filtration \((\mathcal{F}_t)_{t \in [0, \infty)}\). A kinetic measure is a map \( q \) from \( \Omega \) to the space of nonnegative, locally finite measures on \( \mathbb{T}^d \times (0, \infty) \times [0, T] \) that satisfies the property that the process
\[ (\omega, t) \in \Omega \times [0, T] \rightarrow \int_0^T \int_{\mathbb{T}^d} \psi(x, \xi) dq(x, \xi, t) \]
is \( \mathcal{F}_t \)-predictable, for every \( \psi \in C^\infty_c (\mathbb{T}^d \times (0, \infty)) \).

We will prove the well-posedness of (2.2) for initial data with finite entropy. Define
\[ \text{Ent}(\mathbb{T}^d) = \left\{ \rho \in L^1(\mathbb{T}^d) : \rho \geq 0 \text{ and } \int_{\mathbb{T}^d} \rho \log(\rho) < \infty \right\}. \] (2.6)

**Definition 2.3.** (Renormalized kinetic solution) Let \( \hat{\rho} \in \text{Ent}(\mathbb{T}^d) \). A renormalized kinetic solution of (2.2) with initial datum \( \rho(\cdot, 0) = \hat{\rho} \) is a nonnegative, almost surely continuous \( L^1(\mathbb{T}^d) \)-valued \( \mathcal{F}_t \)-predictable function \( \rho \in L^1(\Omega \times [0, T]; L^1(\mathbb{T}^d)) \) that satisfies the following properties.

1. **Conservation of mass:** almost surely for every \( t \in [0, T] \),
\[ \|\rho(\cdot, t)\|_{L^1(\mathbb{T}^d)} = \|\hat{\rho}\|_{L^1(\mathbb{T}^d)}. \] (2.7)

2. **Regularity of \( \sqrt{\rho} \):** there exists a constant \( c \in (0, \infty) \) depending on \( T, \|\hat{\rho}\|_{L^1(\mathbb{T}^d)}, \|V\|_{L^p([0,T];L^p(\mathbb{T}^d \times \mathbb{R}^d))} \) such that
\[ \mathbb{E} \int_0^T \int_{\mathbb{T}^d} |\nabla \sqrt{\rho}|^2 \leq c(T, d, \|\hat{\rho}\|_{L^1(\mathbb{T}^d)}, \|V\|_{L^p([0,T];L^p(\mathbb{T}^d \times \mathbb{R}^d))}). \] (2.8)

Furthermore, there exists a nonnegative kinetic measure \( q \) satisfying the following properties.

3. **Regularity:** almost surely
\[ 4\delta_0(\xi - \rho)\xi|\nabla \sqrt{\rho}|^2 \leq q \quad \text{on} \quad \mathbb{T}^d \times (0, \infty) \times [0, T]. \] (2.9)

4. **Vanishing at infinity:** we have
\[ \lim_{M \to \infty} \mathbb{E} \left[ q(\mathbb{T}^d \times [0, T] \times [M, M + 1]) \right] = 0. \] (2.10)

5. **The equation:** for every \( \varphi \in C^\infty_c (\mathbb{T}^d \times (0, \infty)) \), almost surely for every \( t \in [0, T] \),
\[ \int_{\mathbb{T}^d} \int_0^t \chi(x, \xi, t) \varphi(x, \xi) dx d\xi = \int_{\mathbb{T}^d} \int_0^t \hat{\rho}(\xi) \varphi(x, \xi) dx d\xi - \int_0^t \int_{\mathbb{T}^d} \nabla \varphi(\rho, \nabla \varphi)(x, \rho) dx d\xi - \frac{1}{8} \int_0^t \int_{\mathbb{T}^d} F_1(x) \rho^{-1} \nabla \cdot (\nabla \varphi)(x, \rho) dx d\xi - \frac{1}{4} \int_0^t \int_{\mathbb{T}^d} F_2(x) \cdot (\nabla \varphi)(x, \rho) dx d\xi - \int_0^t \int_{\mathbb{T}^d} \partial_t \varphi(x, \xi) dq + \frac{1}{4} \int_0^t \int_{\mathbb{T}^d} (\nabla \rho \cdot F_2(x) + 2F_3(x) \rho)(\partial_t \varphi)(x, \rho) \]
\[
- \int_0^t \int_{\mathbb{T}^d} \varphi(x, \rho) \nabla \cdot (\rho V(r) * \rho) - \int_0^t \int_{\mathbb{T}^d} \varphi(x, \rho) \nabla \cdot (\sqrt{\rho} dW^F(r)),
\]
where \( \chi(\hat{\rho})(x, \xi) := I_{\{0 < \xi < \hat{\rho}(x)\}} \).

**Remark 2.4.** The estimate (2.8) implies that for every \( K \in \mathbb{N} \),
\[
[(\rho \wedge K) \vee (1/K)] \in L^2(\Omega; L^2([0, T]; H^1(\mathbb{T}^d))).
\]
As a result, the term \(- \frac{1}{\rho} \int_0^t \int_{\mathbb{T}^d} F_1(x) \rho^{-1} \nabla \rho \cdot (\nabla \varphi)(x, \rho)\) on the righthand side of (2.11) is well-defined. In addition, the integrability of the kernel term \( \int_0^t \int_{\mathbb{T}^d} \varphi(x, \rho) \nabla \cdot (\rho V(r) * \rho) \) is guaranteed by Lemma 3.6 below.

Beyond the established vanishing property at infinity (2.10), the kinetic measure exhibits a decay at zero as well. In fact, this property can be derived by similar method as in the proof of [FG24, Proposition 4.5], thus we omit the proof.

**Lemma 2.5.** Suppose that Assumption (A1) is in force. Let \( \hat{\rho} \in \text{Ent}(\mathbb{T}^d) \). Let \( \rho \) be a renormalized kinetic solution of (1.4) in the sense of Definition 2.3 with initial data \( \rho(\cdot, 0) = \hat{\rho} \). Then it follows that, almost surely,
\[
\lim_{\beta \to 0} \left[ \beta^{-1/2} \| T^d \times [0, T] \times [\beta/2, \beta] \right] = 0.
\]

**3. Estimates for the kernel term**

In this section, we make a series of estimates for the nonlocal kernel term under Assumptions (A1) and (A2). These estimates will play a pivotal role in the substantiation of our main results.

Note that under Assumptions (A1) and (A2), it follows that \( \frac{2p}{p-d} = \frac{2}{1-\frac{d}{p}} \leq p^* \) and \( \frac{2q}{2q-d} = \frac{1}{1-\frac{d}{q}} \leq q^* \). A simple application of Hölder’s inequality implies the following results.

**Lemma 3.1.** Let \( f \in L^\infty([0, T]; L^1(\mathbb{T}^d)) \) be nonnegative with \( \sqrt{f} \in L^2([0, T]; H^1(\mathbb{T}^d)) \).

1. If \( V \) satisfies Assumption (A1), then there exists a constant \( C \) depending on \( T \) such that
\[
\int_0^T \| \nabla \sqrt{f}(t) \|_{L^2(\mathbb{T}^d)} \| V(t) \|_{L^p(\mathbb{T}^d)} \leq C(T) \| \nabla \sqrt{f} \|_{L^2([0, T]; L^2(\mathbb{T}^d))} \| V \|_{L^{p^*}([0, T]; L^p(\mathbb{T}^d; \mathbb{R}^d))}. \tag{3.1}
\]
2. If \( V \) satisfies Assumption (A2), then there exists a constant \( C \) depending on \( T \) such that
\[
\int_0^T \| \nabla \sqrt{f}(t) \|_{L^2(\mathbb{T}^d)} \| \nabla \cdot V(t) \|_{L^q(\mathbb{T}^d)} \leq C(T) \| \nabla \sqrt{f} \|_{L^2([0, T]; L^2(\mathbb{T}^d))} \| \nabla \cdot V \|_{L^{q^*}([0, T]; L^q(\mathbb{T}^d))}. \tag{3.2}
\]

We emphasize that the estimations presented in (3.1) and (3.2) will be employed when applying a stochastic Gronwall’s lemma to derive the pathwise uniqueness (see Theorem 4.2). In addition, the inequalities (3.1) and (3.2) imply the index relation imposed on \( V \) and \( \nabla \cdot V \), respectively.

We recall the following Gagliardo-Nirenberg interpolation inequality from [BM18].

**Lemma 3.2.** Suppose that \( j, m \) are nonnegative integers and \( 1 \leq p, q, r \leq +\infty \). Let \( \alpha \in [0, 1] \) be real numbers such that
\[
\frac{1}{p} = \frac{j}{d} + \left( \frac{1}{r} - \frac{m}{d} \right) \alpha + \frac{1 - \alpha}{q}
\]
and
\[
\frac{j}{m} \leq \alpha \leq 1.
\]
Let \( u : \mathbb{R}^d \to \mathbb{R} \) be a function in \( L^q(\mathbb{R}^d) \) with \( m \)th weak derivative in \( L^r(\mathbb{R}^d) \). Then the \( j \)th weak derivative of \( u \) lies in \( L^p(\mathbb{R}^d) \) and there exists a constant \( C \) that depends on \( m, d, j, q, r \) and \( \alpha \), but is independent of \( u \) such that
\[
\| \nabla^j u \|_{L^p(\mathbb{R}^d)} \leq C \| \nabla^m u \|_{L^r(\mathbb{R}^d)}^{\alpha} \| u \|_{L^q(\mathbb{R}^d)}^{1-\alpha}.
\]
The next lemma shows a product rule for the weak derivative.

**Lemma 3.3.** Let $f \in L^\infty([0,T];L^1(\mathbb{T}^d))$ be nonnegative with $\nabla \sqrt{T} \in L^2([0,T];L^2(\mathbb{T}^d;\mathbb{R}^d))$, then the chain rule for weak derivatives $\nabla f = 2\sqrt{T}\nabla \sqrt{T}$ holds for almost every $(x,t) \in \mathbb{T}^d \times [0,T]$. Moreover, we have $\nabla f \in L^2([0,T];L^1(\mathbb{T}^d;\mathbb{R}^d))$.

**Proof.** Let $F(\zeta) = \zeta^2$ for $\zeta \geq 0$, and let $\{\kappa_\delta\}_{\delta > 0}$ be a sequence of standard convolution kernels on $\mathbb{R}$. For any $M > 0$ and $\delta > 0$, define $F_M(\zeta) := (\zeta \wedge M)^2$ and $F_{M,\delta}(\zeta) := \kappa_\delta * F_M(\zeta)$. Then for every $\varphi \in C^\infty(\mathbb{T}^d)$, applying the chain rule (see Evans [Eva10, Chapter 5, Exercise 17]) to $(\nabla F_{M,\delta}(\sqrt{T}), \varphi)$, and then passing to the limits $\delta \to 0$, $M \to \infty$, this completes the proof.

**Lemma 3.4.** Let $g \in L^\infty([0,T];L^1(\mathbb{T}^d))$ and $f \in L^\infty([0,T];L^1(\mathbb{T}^d))$ be nonnegative functions. In addition, assume that $f$ satisfies $\sqrt{T} \in L^2([0,T];H^1(\mathbb{T}^d))$.

1. If $V$ satisfies Assumption (A1), then
   \[
   \int_0^T \|\nabla f \cdot V(t) \ast g\|_{L^1(\mathbb{T}^d)} \leq \int_0^T \|\sqrt{T} \ast f \cdot V(t)\|_{L^p(\mathbb{T}^d)} \|g\|_{L^q(\mathbb{T}^d)} \cdot \|V(t)\|_{L^p(\mathbb{T}^d)} \|f\|_{L^1(\mathbb{T}^d)}. \tag{3.3}
   \]

2. If $V$ satisfies Assumption (A2), then
   \[
   \int_0^T \|f \ast V(t) \cdot g\|_{L^1(\mathbb{T}^d)} \leq \int_0^T \|\sqrt{T} \ast f \cdot V(t)\|_{L^p(\mathbb{T}^d)} \|V(t)\|_{L^q(\mathbb{T}^d)} \|f\|_{L^1(\mathbb{T}^d)} \|g\|_{L^1(\mathbb{T}^d)}. \tag{3.4}
   \]

**Proof.** Based on Lemma 3.3, by using Hölder’s and convolution Young’s inequalities, we get
\[
\int_0^T \|\nabla f \cdot V(t) \ast g\|_{L^1(\mathbb{T}^d)} = \int_0^T \|2\sqrt{T} \nabla f \cdot V(t) \ast g\|_{L^1(\mathbb{T}^d)} \leq 2 \int_0^T \|\nabla \sqrt{T}\|_{L^2(\mathbb{T}^d)} \|g\|_{L^1(\mathbb{T}^d)} \|V(t)\|_{L^p(\mathbb{T}^d)} \|\sqrt{T}\|_{L^{p'}(\mathbb{T}^d)}, \tag{3.5}
\]
where $\frac{2}{p} + \frac{2}{p'} = 1$. Applying Lemma 3.2 to $\|\sqrt{T}\|_{L^{p'}(\mathbb{T}^d)}$, there exists a constant $c \in (0, \infty)$ depending on $d$ such that
\[
\|\sqrt{T}\|_{L^{p'}(\mathbb{T}^d)} \leq c(d) \|\nabla \sqrt{T}\|_{L^2(\mathbb{T}^d)} \|\sqrt{T}\|_{L^2(\mathbb{T}^d)} \tag{3.6}
\]
Substituting (3.6) into (3.5), we get (3.3).

Using Hölder’s and convolution Young’s inequalities again to see that
\[
\int_0^T \|f \ast V(t) \cdot g\|_{L^1(\mathbb{T}^d)} \leq \int_0^T \|f\|_{L^{p'}(\mathbb{T}^d)} \|\nabla \cdot V(t)\|_{L^q(\mathbb{T}^d)} \|g\|_{L^1(\mathbb{T}^d)}, \tag{3.7}
\]
where $\frac{1}{p} + \frac{1}{q} = 1$. Applying Lemma 3.2 to $\|f\|_{L^{p'}(\mathbb{T}^d)}$, there exists a constant $c \in (0, \infty)$ depending on $d$ such that
\[
\|f\|_{L^{p'}(\mathbb{T}^d)} \leq c(d) \|\nabla \sqrt{T}\|_{L^2(\mathbb{T}^d)} \|\sqrt{T}\|_{L^2(\mathbb{T}^d)} \tag{3.8}
\]
By substituting (3.8) into (3.7), we get the desired result (3.4).

Since the interaction kernel $V$ is irregular, a product rule for the weak derivatives is needed as well.

**Lemma 3.5.** The following two properties hold.

1. Let $f \in L^\infty([0,T];L^1(\mathbb{T}^d))$ and $g \in L^\infty([0,T];L^1(\mathbb{T}^d))$ be nonnegative functions with $\sqrt{T} \in L^2([0,T];H^1(\mathbb{T}^d))$ and $\sqrt{g} \in L^2([0,T];H^1(\mathbb{T}^d))$. Assume that $V$ satisfies Assumption (A1), then the product rule for weak derivatives $\nabla \cdot (fV \ast g) = \nabla f \cdot V \ast g + fV \ast (\nabla g)$ holds for almost every $(x,t) \in \mathbb{T}^d \times [0,T]$, where $V \ast (\nabla g) := \int_{\mathbb{T}^d} V(y) \cdot \nabla x g(x-y)dy$. 


(2) Let \( f \in L^\infty([0,T]; L^1(\mathbb{T}^d)) \) and \( g \in L^\infty([0,T]; L^1(\mathbb{T}^d)) \) be nonnegative functions with \( \nabla \sqrt{f} \in L^2([0,T]; L^2(\mathbb{T}^d; \mathbb{R}^d)) \). Assume that \( V \) satisfies Assumptions (A1) and (A2), then the product rule for weak derivatives \( \nabla \cdot (fV \ast g) = \nabla f \cdot V \ast g + f(\nabla \cdot V) \ast g \) holds for almost every \((x,t) \in \mathbb{T}^d \times [0,T]\), where \((\nabla \cdot V) \ast g := \int_\mathbb{T}^d (\nabla \cdot V(x-y))g(y)dy\).

The proof of Lemma 3.5 follows from considering a regularization of the kernel, and then passing to the limit, thus we omit the proof.

**Lemma 3.6.** Let \( f, g \in L^\infty([0,T]; L^1(\mathbb{T}^d)) \) be nonnegative functions with \( \sqrt{f} \in L^2([0,T]; H^1(\mathbb{T}^d)) \) and \( \sqrt{g} \in L^2([0,T]; H^1(\mathbb{T}^d)) \). Suppose that Assumption (A1) holds, then there exists a constant \( C < \infty \) such that

\[
\int_0^T \int_{\mathbb{T}^d} |\nabla \cdot (fV(t) \ast g)| \leq C.
\]

**Proof.** Referring to (1) in Lemma 3.5, we have

\[
\int_0^T \int_{\mathbb{T}^d} |\nabla \cdot (fV(t) \ast g)| \leq \int_0^T \|\nabla f \cdot V(t) \ast g\|_{L^1(\mathbb{T}^d)} + \int_0^T \|f(V(t) \ast \nabla g)\|_{L^1(\mathbb{T}^d)}.
\]

(3.3) and (3.1) together yield

\[
\int_0^T \|\nabla f \cdot V(t) \ast g\|_{L^1(\mathbb{T}^d)} \leq C(T) \|f\|_{L^\infty([0,T]; L^1(\mathbb{T}^d))} \|g\|_{L^\infty([0,T]; L^1(\mathbb{T}^d))}
\]

\[
\cdot \|\nabla \sqrt{f}\|_{L^2([0,T]; L^2(\mathbb{T}^d))} \|V\|_{L^{p^*}(\mathbb{T}^d; \mathbb{R}^d)}.
\]

For \( \frac{1}{p} + \frac{1}{p'} = 1 \), using Lemma 3.3 and Hölder’s inequality to see that

\[
\int_0^T \|f(V(t) \ast \nabla g)\|_{L^1(\mathbb{T}^d)} \leq 2 \int_0^T \|f\|_{L^{p'}(\mathbb{T}^d)} \|V(t) \ast (\sqrt{g} \nabla \sqrt{g})\|_{L^p(\mathbb{T}^d)}
\]

\[
= 2 \int_0^T \|f\|_{L^{p'}(\mathbb{T}^d)} \|V(t)\|_{L^p(\mathbb{T}^d)} \|\sqrt{g}\|_{L^2(\mathbb{T}^d)} \|\nabla \sqrt{g}\|_{L^2(\mathbb{T}^d)}
\]

\[
\lesssim \|f\|_{L^\infty([0,T]; L^1(\mathbb{T}^d))} \|g\|_{L^\infty([0,T]; L^1(\mathbb{T}^d))} \frac{1}{p}\|V\|_{L^{p^*}(\mathbb{T}^d; \mathbb{R}^d)} \cdot \|\nabla \sqrt{f}\|_{L^2(\mathbb{T}^d)} \|V(t)\|_{L^p(\mathbb{T}^d)} \|\nabla \sqrt{g}\|_{L^2(\mathbb{T}^d)},
\]

(3.10)

where (3.8) has been used for the last inequality. Similar to (3.1), we can derive that

\[
\int_0^T \|\nabla \sqrt{f}\|_{L^2(\mathbb{T}^d)} \|V(t)\|_{L^p(\mathbb{T}^d)} \|\nabla \sqrt{g}\|_{L^2(\mathbb{T}^d)}
\]

\[
\lesssim \left( \int_0^T \|\nabla \sqrt{f(t)}\|_{L^2(\mathbb{T}^d)}^2 \right)^{\frac{1}{2}} \left( \int_0^T \|\nabla \sqrt{g(t)}\|_{L^2(\mathbb{T}^d)}^2 \right)^{\frac{1}{2}} \|V\|_{L^{p^*}(\mathbb{T}^d; \mathbb{R}^d)}.
\]

(3.11)

Combining (3.9)-(3.11), we conclude the desired result. \(\square\)

### 4. Uniqueness of renormalized kinetic solutions

In this section, we aim to show the pathwise uniqueness of renormalized kinetic solutions of \( (2.3) \). The following stochastic Gronwall’s lemma will be employed, whose proof can be found in [GHZ09, Lemma 5.3].

**Lemma 4.1.** Let \( T > 0 \). Assume that \( X, Y, Z, R : [0,T] \times \Omega \to \mathbb{R} \) are real-valued, nonnegative stochastic processes. Let \( \tau < T \), \( \mathbb{P} - a.s. \) be a stopping time such that

\[
\mathbb{E} \int_0^T (RX + Z)ds < \infty.
\]

(4.1)
Assume that for some constant $M < \infty$,
\[
\int_0^\tau Rds < M, \quad P \text{- a.s.} \tag{4.2}
\]
Suppose that for all stopping times $0 \leq \tau_a \leq \tau_b \leq \tau$,
\[
\mathbb{E}\left( \sup_{t \in [\tau_a, \tau_b]} X + \int_{\tau_a}^{\tau_b} Yds \right) \leq C_0 \mathbb{E}\left( X(\tau_a) + \int_{\tau_a}^{\tau_b} (RX + Z)ds \right), \tag{4.3}
\]
where $C_0$ is a constant independent of $\tau_a$ and $\tau_b$. Then there exists a constant $C$ depending on $C_0, T$ and $M$ such that
\[
\mathbb{E}\left( \sup_{t \in [0, \tau]} X + \int_0^T Yds \right) \leq C(C_0, T, M) \mathbb{E}\left( X(0) + \int_0^T Zds \right).
\]

In order to restrict the values of kinetic solutions away from infinity and zero, we introduce cutoff functions by the same way as [FG24]. For every $\beta \in (0, 1)$, let $\varphi_\beta : \mathbb{R} \to [0, 1]$ be the unique nondecreasing piecewise linear function that satisfies
\[
\varphi_\beta(x) = 1 \text{ if } x \geq \beta, \quad \varphi_\beta(x) = 0 \text{ if } x \leq \frac{\beta}{2}, \quad \text{and } \varphi_\beta' = \frac{2}{\beta} \mathbb{1}_{\left(\frac{\beta}{2} < x < \beta\right)}.
\tag{4.4}
\]
For every $M \in \mathbb{N}$, let $\zeta_M : \mathbb{R} \to [0, 1]$ be the unique nonincreasing piecewise linear function satisfying
\[
\zeta_M(x) = 0 \text{ if } x \geq M + 1, \quad \zeta_M(x) = 1 \text{ if } x \leq M, \quad \text{and } \zeta_M' = -\mathbb{1}_{\{M < x < M+1\}}.
\tag{4.5}
\]
For every $\varepsilon, \delta \in (0, 1)$, let $\kappa_\varepsilon : \mathbb{T}^d \to [0, \infty)$ and $\kappa_\delta : \mathbb{R} \to [0, \infty)$ be standard convolution kernels of scales $\varepsilon$ and $\delta$ on $\mathbb{T}^d$ and $\mathbb{R}$, respectively. Let $\kappa^{\varepsilon, \delta}$ be defined by
\[
\kappa^{\varepsilon, \delta}(x, y, \xi, \eta) = \kappa_\varepsilon(x-y) \kappa_\delta(\xi-\eta) \text{ for every } (x, y, \xi, \eta) \in (\mathbb{T}^d)^2 \times \mathbb{R}^2.
\tag{4.6}
\]
Now, we are ready to prove the uniqueness of renormalized kinetic solutions of (2.3) by doubling variables method.

**Theorem 4.2.** Suppose that Assumptions (A1)-(A2) are in force. Let $\tilde{\rho}^1, \tilde{\rho}^2 \in \text{Ent} (\mathbb{T}^d)$. Let $\rho^1, \rho^2$ be renormalized kinetic solutions of (1.4) in the sense of Definition 2.3 with the corresponding initial data $\rho^1(\cdot, 0) = \tilde{\rho}^1, \rho^2(\cdot, 0) = \tilde{\rho}^2$. If $\tilde{\rho}^1(x) = \tilde{\rho}^2(x)$ for almost every $x \in \mathbb{T}^d$, then $P$-almost surely,
\[
\sup_{t \in [0, T]} \| \rho^1(\cdot, t) - \rho^2(\cdot, t) \|_{L^1(\mathbb{T}^d)} = 0.
\]

**Proof.** Let $\chi^1$ and $\chi^2$ be kinetic functions of $\rho^1$ and $\rho^2$, respectively. Recall that for every $\varepsilon, \delta \in (0, 1)$, $\kappa^{\varepsilon, \delta}$ is the convolution kernel given by (4.6). Then, for every $i \in \{1, 2\}$, we define
\[
\chi^{\varepsilon, \delta}_{i, \iota}(y, \eta) = (\chi^i(\cdot, \cdot) * \kappa^{\varepsilon, \delta})(y, \eta).
\]
According to Definition 2.3 and the Kolmogorov's continuity criterion, for every $\varepsilon, \delta \in (0, 1)$, there exists a subset of full probability such that, for every $i \in \{1, 2\}, (y, \eta) \in \mathbb{T}^d \times (\frac{\delta}{2}, \infty)$, and $t \in [0, T]$,
\[
\chi^{\varepsilon, \delta}_{i, \iota}(y, \eta) = \mathbb{E}_{y, \eta}\left( \int_0^t \int_{\mathbb{T}^d} \nabla \rho^i \kappa^{\varepsilon, \delta}(x, y, \rho^i, \eta) + \partial_{\rho^i} \left( \int_0^t \int_{\mathbb{T}^d} \kappa^{\varepsilon, \delta}(x, y, \xi, \eta) d\eta \right) \right)
\]
\[
+ \nabla_y \cdot \left( \int_0^t \int_{\mathbb{T}^d} \nabla \rho^i \kappa^{\varepsilon, \delta}(x, y, \rho^i, \eta) \right) + \partial_y \left( \int_0^t \int_{\mathbb{T}^d} \kappa^{\varepsilon, \delta}(x, y, \xi, \eta) d\eta \right)
\]
\[
- \partial_y \left( \int_0^t \int_{\mathbb{T}^d} (F_1(x)(\rho^i)^{-1}) \nabla \rho^i + 2F_2(x) \kappa^{\varepsilon, \delta}(x, y, \rho^i, \eta) \right)
\]
\[
- \partial_{\rho^i} \left( \int_0^t (F_3(x)(\rho^i)^{-1}) \nabla \rho^i + (\rho^i V(r) \rho^i) - \int_0^t \kappa^{\varepsilon, \delta}(x, y, \rho^i, \eta) \nabla \cdot (\sqrt{\rho^i} dW^F) \right).
\tag{4.7}
\]
Recall that the cutoff functions $\varphi_{\beta}$ and $\zeta_M$ are defined by (4.4) and (4.5). With the aid of properties of kinetic functions $\chi^1$ and $\chi^2$, we deduce that
\[
\begin{align*}
\frac{d}{dt} \int_{\mathbb{R}^d} & \left| \frac{\varepsilon}{\delta} \chi_{r,1}(y, \eta) - \frac{\varepsilon}{\delta} \chi_{r,2}(y, \eta) \right|^2 \varphi_{\beta}(\eta) \zeta_M(\eta) \\
= & d \int_{\mathbb{R}^d} \frac{\varepsilon}{\delta} \chi_{r,1}(y, \eta) \varphi_{\beta}(\eta) \zeta_M(\eta) + d \int_{\mathbb{R}^d} \frac{\varepsilon}{\delta} \chi_{r,2}(y, \eta) \varphi_{\beta}(\eta) \zeta_M(\eta) \\
& - 2d \int_{\mathbb{R}^d} \frac{\varepsilon}{\delta} \chi_{r,1}(y, \eta) \chi_{r,2}(y, \eta) \varphi_{\beta}(\eta) \zeta_M(\eta).
\end{align*}
\]
(4.8)

In view of (4.7) and [FG24, Lemma 4.3], by the approach similar to the derivation of (4.14) in [FG24], we have almost surely, for every $\varepsilon, \beta \in (0, 1)$, $M \in \mathbb{N}$, $\delta \in \left(0, \frac{1}{T}\right)$ and $t \in [0, T]$,
\[
\int_{\mathbb{R}} \int_{\mathbb{T}^d} \left| \frac{\varepsilon}{\delta} \chi_{r,1}(y, \eta) - \frac{\varepsilon}{\delta} \chi_{r,2}(y, \eta) \right|^2 \varphi_{\beta}(\eta) \zeta_M(\eta) = -2I_{t}^{\text{err}} + I_{t}^{\text{meas}} + I_{t}^{\text{cut}} + I_{t}^{\text{ker}}. \tag{4.9}
\]

Compared with [FG24, Theorem 4.6], the kernel term $I_{t}^{\text{err}}$ is the extra term that needs to be estimated. For the readers' convenience, we list the expressions for all the terms on the righthand side of (4.9).

In the following, the arguments of $\chi^1$ and its related quantities will be represented by $(x, \xi) \in \mathbb{T}^d \times \mathbb{R}$. For the arguments of $\chi^2$ and its related quantities, they will be referred to as $(x', \xi') \in \mathbb{T}^d \times \mathbb{R}$. Let
\[
\bar{\kappa}_{r,1}^{\varepsilon, \delta}(x, y, \eta) = \kappa_{r,1}^{\varepsilon, \delta}(x, y, \rho^1(x, r), \eta) \quad \text{and} \quad \bar{\kappa}_{r,2}^{\varepsilon, \delta}(x', y, \eta) = \kappa_{r,2}^{\varepsilon, \delta}(x', y, \rho^2(x', r), \eta).
\]

The error term. The error term $I_{t}^{\text{err}}$ can be written as
\[
I_{t}^{\text{err}} = -\frac{1}{8} \int_{0}^{t} \int_{\mathbb{R}} \int_{(\mathbb{T}^d)^3} \left( F_1(x) (\rho^1)^{-1} + F_1(x') (\rho^2)^{-1} - 2f_k(x)f_k(x') (\rho^1 \rho^2)^{-\frac{1}{2}} \right) \nabla \rho \cdot \nabla \varphi_{\beta} \zeta_M \left. \right|_{\epsilon=0} \tag{4.10}
\]

The measure term. The measure term is of the following form
\[
I_{t}^{\text{meas}} = \int_{0}^{t} \int_{\mathbb{R}^2} \int_{(\mathbb{T}^d)^3} \kappa_{r}^{\varepsilon, \delta}(x, y, \xi, \eta) \bar{\kappa}_{r,2}^{\varepsilon, \delta}(\eta) \zeta_M(\eta)dq^1(x, \xi, r)
\]
\[
+ \int_{0}^{t} \int_{\mathbb{R}^2} \int_{(\mathbb{T}^d)^3} \kappa_{r}^{\varepsilon, \delta}(x', y, \xi', \eta) \bar{\kappa}_{r,1}^{\varepsilon, \delta}(\eta) \zeta_M(\eta)dq^2(x', \xi', r)
\]
\[
- 2 \int_{0}^{t} \int_{\mathbb{R}} \int_{(\mathbb{T}^d)^3} \nabla \rho \cdot \nabla \varphi_{\beta} \zeta_M(\eta). \tag{4.11}
\]

The martingale term. The martingale term is given by
\[
I_{t}^{\text{mart}} = \int_{0}^{t} \int_{\mathbb{R}^2} \bar{\kappa}_{r,1}^{\varepsilon, \delta}(2\varphi_{\beta}(\eta) - 1) \zeta_M(\eta) \nabla \cdot (\sqrt{\rho^1}dW^F) \]
\[ + \int_0^t \int_{\mathbb{T}^d} R_{\kappa,2}^{\varepsilon,\delta}(2\chi_{r,1}^{\varepsilon,\delta} - 1) \varphi_{\beta}(\eta) \zeta_M(\eta) \nabla \cdot (\sqrt{\rho^2} dW^F) \]  

Referring to [FG24, (4.28)], it holds that almost surely for every \( t \in [0, T] \),

\[ \lim_{M \to \infty} \left( \lim_{\beta \to 0} \left( \lim_{\varepsilon \to 0} I_{t}^{\text{mart}} \left( \varphi_{\beta}(\eta) \right) \right) \right) = 0. \tag{4.12} \]

**The cutoff term.** The cutoff term has the following form

\[ I_{t}^{\text{cut}} = \int_0^t \int_{\mathbb{R}^2} \kappa_{\varepsilon,\delta}^r(x, y, \xi, \eta) \left( 2\chi_{r,2}^{\varepsilon,\delta} - 1 \right) \partial_\eta \left( \varphi_{\beta}(\eta) \zeta_M(\eta) \right) d\eta(x, \xi, r) \]

\[ - \frac{1}{4} \int_0^t \int_{\mathbb{R}^2} \left( 2F_3(x) \rho^1(x, r) + \nabla \rho^1 \cdot F_2(x) \right) \left( 2\chi_{r,2}^{\varepsilon,\delta} - 1 \right) \nabla \cdot (\varphi_{\beta}(\eta) \zeta_M(\eta)) \]

\[ + \int_0^t \int_{\mathbb{R}^2} \nabla \cdot (\rho^1 V(r) \ast \rho^1) \left( 2\chi_{r,2}^{\varepsilon,\delta} - 1 \right) \partial_\eta \left( \varphi_{\beta}(\eta) \zeta_M(\eta) \right) d\eta(x', \xi', r) \]

\[ - \frac{1}{4} \int_0^t \int_{\mathbb{R}^2} \left( 2F_3(x') \rho^2(x', r) + \nabla \rho^2 \cdot F_2(x') \right) \left( 2\chi_{r,1}^{\varepsilon,\delta} - 1 \right) \nabla \cdot (\varphi_{\beta}(\eta) \zeta_M(\eta)) \]

Thanks to the properties (2.10) and (2.13), by using the nonnegativity of kinetic measures and Fatou’s lemma, it yields that almost surely there exist subsequences \( \beta \to 0 \) and \( M \to \infty \) such that

\[ \lim_{M \to \infty} \left( \lim_{\beta \to 0} \left( \lim_{\varepsilon \to 0} I_{t}^{\text{cut}} \left( \varphi_{\beta}(\eta) \right) \right) \right) = 0. \]

Similarly, we can get the same result for the term related to \( q^2 \). For the other terms of \( I_{t}^{\text{cut}} \), they can be handled in the same way as [FG24, (4.30)]. Thus, it follows that there almost surely exist subsequences such that, for every \( t \in [0, T] \),

\[ \lim_{M \to \infty} \left( \lim_{\beta \to 0} \left( \lim_{\varepsilon \to 0} I_{t}^{\text{cut}} \left( \varphi_{\beta}(\eta) \right) \right) \right) = 0. \tag{4.13} \]

**The kernel term.** The kernel term is given by

\[ I_{t}^{\text{ker}} = \int_0^t \int_{\mathbb{T}^d} \kappa_{\varepsilon,\delta}^r \nabla \cdot (\rho^1 V(r) \ast \rho^1) \left( 2\chi_{r,2}^{\varepsilon,\delta} - 1 \right) \varphi_{\beta}(\eta) \zeta_M(\eta) \]

\[ + \int_0^t \int_{\mathbb{T}^d} \kappa_{\varepsilon,\delta}^r \nabla \cdot (\rho^1 V(r) \ast \rho^1) \left( 2\chi_{r,2}^{\varepsilon,\delta} - 1 \right) \varphi_{\beta}(\eta) \zeta_M(\eta) \]

By using Lemma 3.6, we deduce that \( \nabla \cdot (\rho^1 V(r) \ast \rho^1) \) is \( L^1(\Omega \times [0, T] \times \mathbb{T}^d) \)-integrable. It follows from the definition of \( \kappa_{\varepsilon,\delta}^r \), the boundedness of the kinetic functions and the dominated convergence theorem that, after passing to a subsequence \( \varepsilon \to 0 \), almost surely for every \( t \in [0, T] \),

\[ \lim_{\varepsilon \to 0} I_{t}^{\text{ker}} = \int_0^t \int_{\mathbb{T}^d} \kappa_{r,1}^{\delta} \left( 2\chi_{r,1}^{\delta} - 1 \right) \varphi_{\beta}(\eta) \zeta_M(\eta) \nabla \cdot (\rho^1 V(r) \ast \rho^1) \]

\[ + \int_0^t \int_{\mathbb{T}^d} \kappa_{r,1}^{\delta} \left( 2\chi_{r,1}^{\delta} - 1 \right) \varphi_{\beta}(\eta) \zeta_M(\eta) \nabla \cdot (\rho^1 V(r) \ast \rho^1), \tag{4.14} \]

where \( \chi_{r,i}^{\delta}(y, \eta) = (\chi_{r,i}^{\delta}(y, \eta) \ast \kappa_{i}^{\lambda}) (\eta) \) and \( \kappa_{r,i}^{\delta}(y, \eta) = \kappa_{r,i}^{\delta}(\rho^1(y, r) \ast \eta) \) for each \( i \in \{1, 2\} \). Let us focus on the first term on the righthand side of (4.14). In view of \( |2\chi_{r,2}^{\delta} - 1| \leq 1 \), by the definitions of \( \varphi_{\beta} \), \( \zeta_M \) and \( \kappa_{r,1}^{\delta} \), and the boundedness of the kinetic function, we deduce that there exists a constant \( c \in (0, \infty) \) depending on \( \beta \) such that for every \( t \in [0, T] \),

\[ \mathbb{E} \left| \int_0^t \int_{\mathbb{T}^d} \kappa_{r,1}^{\delta} \left( 2\chi_{r,2}^{\delta} - 1 \right) \left( \varphi_{\beta}(\eta) \zeta_M(\eta) - \varphi_{\beta}(\rho^1) \zeta_M(\rho^1) \right) \nabla \cdot (\rho^1 V(r) \ast \rho^1) \right| \]

\[ \leq \mathbb{E} \int_0^T \int_{\mathbb{T}^d} \left| \kappa_{r,1}^{\delta} \left( \varphi_{\beta}(\eta) \zeta_M(\eta) - \varphi_{\beta}(\rho^1) \zeta_M(\rho^1) \right) \nabla \cdot (\rho^1 V(r) \ast \rho^1) \right| \]
Indeed, from the definitions of \( \phi_{\beta} \) and Assumptions (A1) and (A2), we deduce that for every \( i \in \{ 1, 2 \} \),

\[
\lim_{\delta \to 0} \left| \int_0^t \int_{\mathbb{T}^d} \kappa_{r,1}^\delta (2 \chi_{r,1}^\delta - 1) \left( \varphi_{\beta}(\eta) \zeta_M(\eta) - \varphi_{\beta}(\rho) \zeta_M(\rho) \right) \nabla \cdot (\rho^1 V(r) \ast \rho^1) \right| = 0. \tag{4.15}
\]

Similarly, for the second term on the righthand side of (4.14), it holds that

\[
\lim_{\delta \to 0} \left| \int_0^t \int_{\mathbb{T}^d} \kappa_{r,2}^\delta (2 \chi_{r,2}^\delta - 1) \left( \varphi_{\beta}(\eta) \zeta_M(\eta) - \varphi_{\beta}(\rho^2) \zeta_M(\rho^2) \right) \nabla \cdot (\rho^2 V(r) \ast \rho^2) \right| = 0. \tag{4.16}
\]

Moreover, referring to [FG24, (4.22)] and in view of \( \varphi_{\beta}(0) = 0 \), we deduce that pointwise

\[
\lim_{\delta \to 0} \left( \int_0^t \int_{\mathbb{T}^d} \kappa_{r,2}^\delta (2 \chi_{r,2}^\delta - 1) \, d\eta \right) \varphi_{\beta}(\rho^1) = (1_{\{\rho^1 = \rho^2\}} + 21_{\{0 \leq \rho^1 < \rho^2\}} - 1) \varphi_{\beta}(\rho^1). \tag{4.17}
\]

Combining (4.14)-(4.17), passing to a subsequence \( \delta \to 0 \), almost surely,

\[
\lim_{\delta \to 0} \left( \lim_{\epsilon \to 0} I_{\text{der}}^R \right) = \int_0^T \int_{\mathbb{T}^d} \left( 1_{\{\rho^1 = \rho^2\}} + 21_{\{0 \leq \rho^1 < \rho^2\}} - 1 \right) \varphi_{\beta}(\rho^1) \zeta_M(\rho^1) \nabla \cdot (\rho^1 V(r) \ast \rho^1) \right| + \int_0^t \int_{\mathbb{T}^d} \left( 1_{\{\rho^1 = \rho^2\}} + 21_{\{0 \leq \rho^1 < \rho^2\}} - 1 \right) \varphi_{\beta}(\rho^2) \zeta_M(\rho^2) \nabla \cdot (\rho^2 V(r) \ast \rho^2). \tag{4.18}
\]

Now, we claim that along subsequences \( \beta \to 0 \) and \( M \to \infty \), almost surely, for every \( i \in \{ 1, 2 \} \),

\[
\lim_{M \to \infty} \left( \lim_{\beta \to 0} \varphi_{\beta}(\rho^i) \zeta_M(\rho^i) \nabla \cdot (\rho^i V(r) \ast \rho^i) \right) = \nabla \cdot (\rho^i V(r) \ast \rho^i) \quad \text{strongly in } L^1(\mathbb{T}^d \times [0,T]). \tag{4.19}
\]

Indeed, from the definitions of \( \varphi_{\beta} \) and \( \zeta_M \), it gives that for every \( i \in \{ 1, 2 \} \),

\[
\mathbb{E} \int_0^T \int_{\mathbb{T}^d} |\varphi_{\beta}(\rho^i) \zeta_M(\rho^i) \nabla \cdot (\rho^i V(r) \ast \rho^i) - \nabla \cdot (\rho^i V(r) \ast \rho^i)| \leq \mathbb{E} \int_0^T \int_{\mathbb{T}^d} 1_{\{0 \leq \rho^i < \beta\}} |\nabla \cdot (\rho^i V(r) \ast \rho^i)| + \mathbb{E} \int_0^T \int_{\mathbb{T}^d} 1_{\{\rho^i \geq M\}} |\nabla \cdot (\rho^i V(r) \ast \rho^i)|. \tag{4.20}
\]

Clearly, the second term on the righthand side of (4.20) converges to zero as \( M \to \infty \). For the first term on the righthand side of (4.20), by Lemma 3.3 and (2) in Lemma 3.5, it follows that for every \( i \in \{ 1, 2 \} \),

\[
\mathbb{E} \int_0^T \int_{\mathbb{T}^d} 1_{\{0 \leq \rho^i < \beta\}} |\nabla \cdot (\rho^i V(r) \ast \rho^i)| \leq \mathbb{E} \int_0^T \int_{\mathbb{T}^d} 1_{\{0 \leq \rho^i < \beta\}} |\rho^i \nabla \cdot (\rho^i V(r) \ast \rho^i)| + \beta \mathbb{E} \int_0^T \int_{\mathbb{T}^d} 1_{\{0 \leq \rho^i < \beta\}} |\nabla \cdot (\rho^i V(r) \ast \rho^i)|. \tag{4.21}
\]

By using Hölder’s inequality, convolution Young’s inequality and (2.7), we deduce that for every \( i \in \{ 1, 2 \} \),

\[
\mathbb{E} \int_0^T \|\nabla \sqrt{\rho^i} \cdot V(r) \ast \rho^i\|_{L^1(\mathbb{T}^d)} \leq \|\hat{\rho}^i\|_{L^1(\mathbb{T}^d)} \left( \int_0^T \|V(r)\|^2_{L^2(\mathbb{T}^d)} \right)^\frac{1}{2} \left( \mathbb{E} \int_0^T \|\nabla \sqrt{\rho^i}\|^2_{L^2(\mathbb{T}^d)} \right)^\frac{1}{2},
\]

\[
\mathbb{E} \int_0^T \int_{\mathbb{T}^d} 1_{\{0 \leq \rho^i < \beta\}} |\nabla \cdot (\rho^i V(r) \ast \rho^i)| \leq \mathbb{E} \int_0^T \|\nabla \cdot (\rho^i V(r) \ast \rho^i)\|_{L^1(\mathbb{T}^d)} \leq \|\hat{\rho}^i\|_{L^1(\mathbb{T}^d)} \int_0^T \|\nabla \cdot (V(r))\|_{L^1(\mathbb{T}^d)}.
\]

Owing to (4.21) and Assumptions (A1) and (A2), we get the desired result (4.19).
Combining (4.18) and (4.19), we get
\[
\lim_{M \to \infty} \left( \lim_{\beta \to 0} \left( \lim_{\varepsilon \to 0} \lim_{t \to T} H_t^{\text{ker}} \right) \right) = 
\int_0^t \int_{T^d} \left( 1_{\rho^1 = \rho^2} + 21_{\rho^1 < \rho^2} - 1 \right) \left( \nabla \cdot (\rho^1 V(r) \ast \rho^1) - \nabla \cdot (\rho^2 V(r) \ast \rho^2) \right)
+ \int_0^t \int_{T^d} \left( 1_{\rho^1 = \rho^2} + 21_{\rho^1 < \rho^2} - 1 + 1_{\rho^1 = \rho^2} + 21_{\rho^1 < \rho^2} - 1 \right) \nabla \cdot (\rho^2 V(r) \ast \rho^2)
= : \tilde{J}_1 + \tilde{J}_2.
\]
(4.22)

Since \(1_{\rho^1 < \rho^2} = 1 - 1_{\rho^1 = \rho^2} - 1_{\rho^1 < \rho^2}\), the \(L^1(\Omega \times [0, T] \times T^d)\)-integrability of \(\nabla \cdot (\rho^i V \ast \rho^j)\) for every \(i \in \{1, 2\}\) implies that
\[
\tilde{J}_2 = 0.
\]
(4.23)

For the term \(\tilde{J}_1\), by chain rule and the identity \(\text{sgn}(\rho^2 - \rho^1) = 1_{\rho^1 = \rho^2} + 21_{\rho^1 < \rho^2} - 1\), we have
\[
\tilde{J}_1 = : \tilde{J}_{11} + \tilde{J}_{12},
\]
where
\[
\tilde{J}_{11} = \int_0^t \int_{T^d} \text{sgn}(\rho^2 - \rho^1)\nabla \cdot ((\rho^1 - \rho^2)V(r) \ast \rho^1),
\]
\[
\tilde{J}_{12} = \int_0^t \int_{T^d} \text{sgn}(\rho^2 - \rho^1)\nabla \cdot (\rho^2 V(r) \ast (\rho^1 - \rho^2)).
\]

Define \(\text{sgn}^\delta := (\text{sgn} \ast \kappa^\delta)\) for every \(\delta \in (0, 1)\). By integration by parts formula, we get that almost surely for every \(t \in [0, T]\),
\[
\tilde{J}_{11} = \lim_{\delta \to 0} \int_0^t \int_{T^d} \text{sgn}^\delta(\rho^2 - \rho^1)\nabla \cdot ((\rho^1 - \rho^2)V(r) \ast \rho^1)
= - \lim_{\delta \to 0} \int_0^t \int_{T^d} (\text{sgn}^\delta)'(\rho^2 - \rho^1)(\rho^1 - \rho^2)\nabla(\rho^2 - \rho^1) \cdot V(r) \ast \rho^1.
\]

It follows from the uniform boundedness of \((\delta \kappa^\delta)\) in \(\delta \in (0, \beta/4)\) that there exists \(c \in (0, \infty)\) independent of \(\delta\) but depending on the convolution kernel such that for all \(\delta \in (0, \beta/4),\)
\[
|(\text{sgn}^\delta)'(\rho^2 - \rho^1)(\rho^1 - \rho^2)| = 2|\kappa^\delta(\rho^2 - \rho^1)(\rho^1 - \rho^2)| \leq c1_{(0 < |\rho^1 - \rho^2| < \delta)}. \]

Moreover, by (3.3), we get the \(L^1(\Omega \times [0, T] \times T^d)\)-integrability of \(\nabla(\rho^2 - \rho^1) \cdot V \ast \rho^1\). As a result, we deduce that, almost surely for every \(t \in [0, T]\),
\[
\tilde{J}_{11} = 0.
\]
(4.24)

Regarding to the term \(\tilde{J}_{12}\), by (2) in Lemma 3.5, (3.3) and (3.4), we deduce that almost surely for every \(t \in [0, T]\),
\[
\tilde{J}_{12} \leq \int_0^t \int_{T^d} |\nabla \rho^2 \cdot V(r) \ast (\rho^1 - \rho^2)| + \int_0^t \int_{T^d} |\rho^2(\nabla \cdot V(r)) \ast (\rho^1 - \rho^2)|
\leq C(\|\tilde{\rho}^2\|_{L^1(T^d)}, p, d) \int_0^t \|V(r)\|_{L^p(T^d)} \|\nabla \sqrt{\rho^2}\|_{L^2(T^d)}^{1+\frac{d}{2}} \|\rho^1 - \rho^2\|_{L^1(T^d)}
+ C(\|\tilde{\rho}^2\|_{L^1(T^d)}, q, d) \int_0^t \|\nabla \cdot V(r)\|_{L^q(T^d)} \|\nabla \sqrt{\rho^2}\|_{L^2(T^d)}^{1+\frac{d}{2}} \|\rho^1 - \rho^2\|_{L^1(T^d)}. \]
(4.25)

Combining (4.22)-(4.25), we conclude that almost surely for every \(t \in [0, T]\),
\[
\lim_{M \to \infty} \lim_{\beta \to 0} \lim_{\delta \to 0} \lim_{\varepsilon \to 0} H_t^{\text{ker}} \leq C(\|\tilde{\rho}^2\|_{L^1(T^d)}, p, d) \int_0^t \|V(r)\|_{L^p(T^d)} \|\nabla \sqrt{\rho^2}\|_{L^2(T^d)}^{1+\frac{d}{2}} \|\rho^1 - \rho^2\|_{L^1(T^d)}
\]
\[ + C(\|\rho^2\|_{L^1(T^2)}, q, d) \int_0^t \|\nabla \cdot V(r)\|_{L^q(T^2)} \|\nabla \sqrt{\rho^2}\|_{L^2(T^2)} \|\rho^1 - \rho^2\|_{L^1(T^2)}. \]  

(4.26)

**Conclusion.** Based on the properties of the kinetic function, (4.10)-(4.13), it follows that almost surely for every \( t \in [0, T] \),

\[
\int_{\mathbb{R}} \int_{T^d} \left| \chi_{r,1}^t - \chi_{r,2}^t \right|^2 d\mathbf{r} = \lim_{M \to \infty} \left( \lim_{\beta \to 0} \left( \lim_{\delta \to 0} \left( \lim_{\epsilon \to 0} \int_{\mathbb{R}} \int_{T^d} \left| \chi_{r,1}^\epsilon - \chi_{r,2}^\epsilon \right|^2 \varphi_{\beta} \xi M \right) \right) \right) = \lim_{M \to \infty} \left( \lim_{\beta \to 0} \left( \lim_{\delta \to 0} \left( \lim_{\epsilon \to 0} \left( -2 I_t^\text{err} - 2 I_t^\text{meas} + I_t^\text{mart} + I_t^\text{cut} + I_t^\text{ker} \right) \right) \right) \right) \leq \lim_{M \to \infty} \lim_{\beta \to 0} \lim_{\delta \to 0} \lim_{\epsilon \to 0} I_t^\text{ker}. \]  

(4.27)

For any \( N > 0 \), define a stopping time \( \tau_N := \inf \{ t \in [0, T]; \int_0^t \|\nabla \sqrt{\rho^2}\|_{L^2(T^2)} > N \} \). By Chebyshev’s inequality and (2.8), we have

\[
\mathbb{P}(\tau_N \leq T) \to 0, \quad \text{as } N \to \infty. \]  

(4.28)

Combining (4.27) with the definition of the kinetic function, we derive that for every stopping time \( 0 \leq \tau_a \leq \tau_b \leq \tau_N \wedge t, \)

\[
\mathbb{E} \sup_{r \in [\tau_a, \tau_b]} \left\| |\rho^1(r) - \rho^2(r)|_{L^1(T^2)} - \mathbb{E} \left( |\rho^1(\tau_a) - \rho^2(\tau_a)|_{L^1(T^2)} \right) \right\|_{L^1(T^2)} \leq C(\|\rho^2\|_{L^1(T^2)}, p, d) \int_{\mathbb{R}} \int_{T^d} \left\| V(r) \right\|_{L^q(T^2)} \left\| \nabla \sqrt{\rho^2}\|_{L^2(T^2)} \right\| \rho^1 - \rho^2 \right\|_{L^1(T^2)}^{-1} + C(\|\rho^2\|_{L^1(T^2)}, p, d) \int_{\mathbb{R}} \int_{T^d} \left\| V(r) \right\|_{L^q(T^2)} \left\| \nabla \sqrt{\rho^2}\|_{L^2(T^2)} \right\| \rho^1 - \rho^2 \right\|_{L^1(T^2)}. \]  

(4.29)

In the following, we aim to apply Lemma 4.1 to (4.29). Let \( \tau = \tau_N \wedge T, X = \|\rho^1 - \rho^2\|_{L^1(T^2)}, Y, Z = 0 \)

\[ R := C(\|\rho^2\|_{L^1(T^2)}, p, d) \left\| \nabla \cdot V(r) \right\|_{L^1(T^2)} \left\| \nabla \sqrt{\rho^2}\|_{L^2(T^2)} \right\| \rho^1 - \rho^2 \right\|_{L^1(T^2)} + C(\|\rho^2\|_{L^1(T^2)}, p, d) \left\| V(r) \right\|_{L^q(T^2)} \left\| \nabla \sqrt{\rho^2}\|_{L^2(T^2)} \right\| \rho^1 - \rho^2 \right\|_{L^1(T^2)}. \]  

Clearly, (4.29) implies (4.3). Moreover, with the aid of (3.1) and (3.2), it follows that

\[
\int_0^T R \leq C(T, \|\rho^2\|_{L^1(T^2)}, p, d) \left\| \nabla \cdot V(r) \right\|_{L^1(T^2)} \left\| \nabla \sqrt{\rho^2}\|_{L^2(T^2)} \right\| \left\| V \right\|_{L^q^*(0, T]; L^q(T^2)} \right\| \rho^1 - \rho^2 \right\|_{L^1(T^2)} + C(T, \|\rho^2\|_{L^1(T^2)}, p, d) \left\| \nabla \sqrt{\rho^2}\|_{L^2(T^2)} \right\| \left\| V \right\|_{L^q^*(0, T]; L^q(T^2)} \right\| \rho^1 - \rho^2 \right\|_{L^1(T^2)} \right\| \rho^1 - \rho^2 \right\|_{L^1(T^2)}^{-1}, \]  

(4.30)

For any \( \delta > 0 \), it follows from Chebyshev’s inequality and (4.30) that

\[
\mathbb{P} \left( \sup_{r \in [0, T]} \left\| |\rho^1(r) - \rho^2(r)|_{L^1(T^2)} > \delta \right\| \right) \leq \mathbb{P} \left( \sup_{r \in [\tau_N \wedge T]} \left\| |\rho^1(r) - \rho^2(r)|_{L^1(T^2)} > \delta, \tau_N > T \right\| + \mathbb{P}(\tau_N \leq T) \right) \leq \frac{1}{\delta} \mathbb{E} \sup_{r \in [0, T \wedge \tau_N]} \left\| |\rho^1(r) - \rho^2(r)|_{L^1(T^2)} \right\| + \mathbb{P}(\tau_N \leq T) \leq \frac{C(N, T, V, \|\rho^2\|_{L^1(T^2)}, p, q, d)}{\delta} \mathbb{E} \left\| \rho^1 - \rho^2 \right\|_{L^1(T^2)} + \mathbb{P}(\tau_N \leq T). \]  

(4.31)
When \( \hat{\rho}^1 = \hat{\rho}^2 \) a.e. in \( \mathbb{T}^d \), by (4.31) and (4.28), we get for every \( \delta > 0 \),
\[
P\left( \sup_{r \in [0, T]} \| \rho^1(\cdot, r) - \rho^2(\cdot, r) \|_{L^1(\mathbb{T}^d)} > \delta \right) \leq P(\tau_N \leq T) \to 0,
\]
as \( N \to \infty \). As a result, for every \( \delta > 0 \),
\[
P\left( \sup_{r \in [0, T]} \| \rho^1(\cdot, r) - \rho^2(\cdot, r) \|_{L^1(\mathbb{T}^d)} > \delta \right) = 0. \tag{4.32}
\]

Further, note that
\[
P\left( \omega : \sup_{t \in [0, T]} \| \rho^1(\cdot, t) - \rho^2(\cdot, t) \|_{L^1(\mathbb{T}^d)} = 0 \right) = P\left( \bigcap_{n \geq 1} \left\{ \omega : \sup_{t \in [0, T]} \| \rho^1(\cdot, t) - \rho^2(\cdot, t) \|_{L^1(\mathbb{T}^d)} < \frac{1}{n} \right\} \right),
\]
by (4.32), we get
\[
P\left( \sup_{r \in [0, T]} \| \rho^1(\cdot, r) - \rho^2(\cdot, r) \|_{L^1(\mathbb{T}^d)} = 0 \right) = 1.
\]

\[\square\]

**Remark 4.3.** We point out that the LPS condition (i.e., Assumption (A1) on \( V \)) is sufficient to guarantee the integrability of the kernel term \( \int_0^t \int_{\mathbb{T}^d} \varphi(x, \rho) \nabla \cdot (\rho V(\rho) * \rho) \) in (2.11) and the forthcoming entropy dissipation estimates (see Proposition 5.4 below). However, it is not strong enough to admit the uniqueness. In fact, if we want to avoid imposing any conditions on \( \nabla \cdot V \), it requires to handle \( \int_0^t \int_{\mathbb{T}^d} |\rho^2 V(\rho) * \nabla (\rho^1 - \rho^2)| \) instead of \( \int_0^t \int_{\mathbb{T}^d} |\rho^2 (\nabla \cdot V(\rho)) * (\rho^1 - \rho^2)| \) in (4.25). In this case, it is difficult to control \( \nabla (\rho^1 - \rho^2) \) by \( |\rho^1 - \rho^2|_{L^1(\mathbb{T}^d)} \), which results in the unapplicable of the stochastic Gronwall’s inequality. Thus, for technical reasons, we need an additional condition Assumption (A2).

As a consequence of (4.30) and Chebyshev’s inequality, we get the continuity of solutions with respect to the initial data.

**Lemma 4.4.** Let \( \{\rho^n, \hat{\rho}\}_{n \geq 1} \subset \text{Ent} (\mathbb{T}^d) \) satisfy \( \lim_{n \to \infty} \| \hat{\rho}^n - \hat{\rho} \|_{L^1(\mathbb{T}^d)} = 0 \). Let \( \rho_n, \rho \) be renormalized kinetic solutions of (2.3) in the sense of Definition 2.3 with initial values \( \rho_n(\cdot, 0) = \hat{\rho}^n, \rho(\cdot, 0) = \hat{\rho} \), respectively. Then for any \( \delta > 0 \), we have
\[
\lim_{n \to \infty} P\left( \sup_{t \in [0, T]} \| \rho_n(t) - \rho(t) \|_{L^1(\mathbb{T}^d)} > \delta \right) = 0.
\]

5. APPROXIMATION EQUATION

To demonstrate the existence of renormalized kinetic solutions to (2.3), we introduce an approximation equation with regularized coefficients. Specifically, we will introduce smooth sequences that approximate the square root function \( \sqrt{\cdot} \) and the kernel \( V \), respectively.

According to [FG24, Lemma 5.18], the choice of the approximation of the square root function is stated as follows.

**Lemma 5.1.** There exists a sequence \( \{\sigma_n\}_{n \in \mathbb{N}} \) that fulfills \( \sigma_n(\cdot) \to \sqrt{\cdot} ) \) in \( C^1_{\text{loc}}((0, \infty)) \) as \( n \to \infty \). Furthermore, \( \sigma_n \) has the following properties.

1. \( \sigma_n \in C([0, \infty)) \cap C^\infty((0, \infty)) \) with \( \sigma_n(0) = 0 \) and \( \sigma_n' \in C^\infty((0, \infty)) \) for every \( n \in \mathbb{N} \),
2. there exists \( c \in (0, \infty) \) such that for every \( \xi \in [0, \infty) \),
\[
|\sigma_n(\xi)| \leq c\sqrt{\xi} \text{ uniformly with respect to } n \in \mathbb{N}, \tag{5.1}
\]
3. for every \( \delta \in (0, 1) \), there exists \( c_\delta \in (0, \infty) \) such that
\[
|\sigma_n(\xi)|^2 \mathbf{1}_{(\xi \geq \delta)} + |\sigma_n(\xi)\sigma_n'(\xi)|^2 \mathbf{1}_{(\xi \leq \delta)} \leq c_\delta \text{ uniformly with respect to } n \in \mathbb{N}. \tag{5.2}
\]

For the kernel \( V \), let \( V_\gamma := ((V \land (1/\gamma)) \vee (1/\gamma)) * \eta_\gamma \), for every \( \gamma > 0 \) and \((t, x) \in \mathbb{R} \times \mathbb{T}^d \), where \( * \) denotes spatial convolution, and \( \eta_\gamma(x) = \frac{1}{\gamma} \eta\left(\frac{x}{\gamma}\right) \) is the standard convolution kernel on \( \mathbb{T}^d \). Suppose that \( V \) satisfies Assumption (A1), then \( V_\gamma \) satisfies the following properties.
Lemma 5.2. Let $V$ satisfy Assumption (A1). For $V_\gamma$ as defined above, we have

1. for every $\gamma > 0$, $V_\gamma$ satisfies Assumption (A1),
2. for every $\gamma > 0$, $V_\gamma \in L^\infty([0, T]; L^r(T^d; \mathbb{R}^d))$,
3. $V_\gamma \to V$ in $L^p$ ([0, T]; $L^p(T^d; \mathbb{R}^d)$) with $\frac{d}{p} + \frac{2}{p'} \leq 1$, $2 \leq p \leq \infty$ and $d < p \leq \infty$ as $\gamma \to 0$.

Proof. With the aid of convolution Young’s inequality and dominated convergence theorem, the proof can be easily achieved. \square

With the help of $\sigma_n$ and $V_\gamma$, we consider the following regularized stochastic PDE

\[
\frac{d\rho^{n,\gamma}}{dt} = \Delta \rho^{n,\gamma} dt - \nabla \cdot (\sigma_n(\rho^{n,\gamma}) dW_t^F) - \nabla \cdot (\rho^{n,\gamma} V_\gamma(t) \ast \rho^{n,\gamma}) dt
\]

\[
+ \frac{1}{2} \nabla \cdot (F_1(\sigma_n'(\rho^{n,\gamma})) \rho^{n,\gamma} + \sigma_n(\rho^{n,\gamma})\sigma_n'(\rho^{n,\gamma}) F_2) dt,
\]

(5.3)

with $\rho^{n,\gamma}(\cdot, 0) = \bar{\rho}$.

Now, we introduce the definition of weak solutions to (5.3).

Definition 5.3. Let $V$ satisfy Assumption (A1) and $\tilde{\rho} \in L^\infty(T^d)$ be nonnegative. For any $\gamma > 0$ and $n \in \mathbb{N}$, a weak solution of (5.3) with initial data $\rho^{n,\gamma}(\cdot, 0) = \tilde{\rho}$ is a nonnegative, $\mathcal{F}_t$-predictable, $L^m(T^d)$-continuous (for some $m \geq 2$) process $\rho^{n,\gamma}$ such that almost surely $\rho^{n,\gamma} \in L^2([0, T]; H^1(T^d))$ and for every $\psi \in C^\infty(T^d)$, almost surely for every $t \in [0, T]$,

\[
\int_{T^d} \rho^{n,\gamma}(x, t) \psi(x) \, dx = \int_{T^d} \tilde{\rho} \psi \, dx - \int_0^t \int_{T^d} \nabla \rho^{n,\gamma} \cdot \nabla \psi \, dx \, dt + \int_0^t \int_{T^d} (\rho^{n,\gamma} V_\gamma(r) \ast \rho^{n,\gamma}) \cdot \nabla \psi \, dx \, dr
\]

\[
+ \int_0^t \int_{T^d} \sigma_n(\rho^{n,\gamma}) \rho^{n,\gamma} \cdot \nabla \psi \, dW_t^F - \frac{1}{2} \int_0^t \int_{T^d} F_1(\sigma_n'(\rho^{n,\gamma})) \rho^{n,\gamma} \cdot \nabla \psi \, dx \, dt - \frac{1}{2} \int_0^t \int_{T^d} \sigma_n(\rho^{n,\gamma}) \sigma_n'(\rho^{n,\gamma}) F_2 \cdot \nabla \psi \, dx \, dr.
\]

(5.4)

For any nonnegative function $\tilde{\rho} \in L^\infty(T^d)$, let $\rho^{n,\gamma}$ be a weak solution of (5.3) in the sense of Definition 5.3 with initial data $\rho^{n,\gamma}(\cdot, 0) = \tilde{\rho}$ by $\psi = 1$ in (5.4) and using the nonnegativity of $\rho^{n,\gamma}$, we deduce that almost surely for every $t \in [0, T]$,

\[
\|\rho^{n,\gamma}(\cdot, t)\|_{L^1(T^d)} = \|\tilde{\rho}\|_{L^1(T^d)}.
\]

(5.5)

Moreover, we can show that the weak solution $\rho^{n,\gamma}$ of (5.3) satisfies the following entropy estimate and $L^m(T^d \times [0, T])$-norm estimate.

5.1. Entropy estimate and $L^m(T^d \times [0, T])$-norm estimate. Let $\Psi : [0, \infty) \to \mathbb{R}$ be the unique function satisfying $\Psi(0) = 0$ with $\Psi' (\xi) = \log(\xi)$. Recall that $\text{Ent}(T^d)$ is defined by (2.6).

Proposition 5.4. Suppose that $V$ satisfies Assumption (A1). Let $\hat{\rho} \in \text{Ent}(T^d)$. For any $\gamma > 0$ and $n \in \mathbb{N}$, let $\rho^{n,\gamma}$ be a weak solution of Definition 5.3 with initial data $\rho^{n,\gamma}(\cdot, 0) = \hat{\rho}$. Then there exists a constant $c \in (0, \infty)$ depending on $T, d, \|\hat{\rho}\|_{L^1(T^d)}$ and $\|V\|_{L^p([0, T]; L^p(T^d))}$ such that

\[
E\left[\sup_{t \in [0, T]} \int_{T^d} \Psi(\rho^{n,\gamma}(x, t)) \right] + E\left[\int_0^T \int_{T^d} |\nabla \sqrt{\rho^{n,\gamma}}|^2 \right] \leq \int_{T^d} \Psi(\rho^{\hat{\gamma}}) + c.
\]

(5.6)

Proof. For the above $\Psi$, we firstly introduce a sequence of smooth approximating functions denoted by $\Psi_\delta$ with $\delta \in (0, 1)$. Here, we require that $\Psi_\delta(0) = 0$ and $\Psi_\delta'(\xi) = \log(\xi + \delta)$. Applying Itô’s formula [Kry13] and by the nonnegativity of $\rho^{n,\gamma}$, we deduce that almost surely for every $t \in [0, T]$,

\[
\int_{T^d} \Psi_\delta(\rho^{n,\gamma}(x, r)) \bigg|_{r=0}^{r=t} = \int_0^t \int_{T^d} \frac{4\rho^{n,\gamma}}{\rho^{n,\gamma} + \delta} \nabla \sqrt{\rho^{n,\gamma}}^2 = K_t^{\text{ker}} + K_t^{\text{mart}} + K_t^{\text{err}},
\]

(5.7)

where

\[
K_t^{\text{ker}} = \int_0^t \int_{T^d} \frac{1}{\rho^{n,\gamma} + \delta} (\rho^{n,\gamma} V_\gamma(r) \ast \rho^{n,\gamma}) \cdot \nabla \rho^{n,\gamma}.
\]
\[ K_{t}^{\text{mart}} = \int_{0}^{t} \int_{T^{d}} \frac{2\sqrt{\rho^{n,\gamma}}\sigma_n(\rho^{n,\gamma})}{\rho^{n,\gamma} + \delta} \nabla \sqrt{\rho^{n,\gamma}} \cdot dW^{F}, \]

\[ K_{t}^{\text{err}} = \frac{1}{2} \int_{0}^{t} \int_{T^{d}} \frac{1}{\rho^{n,\gamma} + \delta} (\sigma_n(\rho^{n,\gamma})\sigma_n'(\rho^{n,\gamma})F_2 \cdot \nabla \rho^{n,\gamma} + \sigma_n^2(\rho^{n,\gamma})F_3). \]

We firstly proceed with the term \( K_{t}^{\text{err}} \). Thanks to the nonnegativity of \( \rho^{n,\gamma} \), (5.1) and the boundedness of \( F_3 \), by using \( \nabla \cdot F_2 = 0 \), combining with the assumption of \( \sigma_n \), it follows that there exists a constant \( c \in (0, \infty) \) independent of \( \delta \) such that almost surely

\[ \sup_{t \in [0, T]} K_{t}^{\text{err}} \leq c(T). \]  

(5.8)

Regarding to the term \( K_{t}^{\text{mart}} \), it follows from (5.1), the Burkholder-Davis-Gundy inequality and Hölder’s inequality that there exists \( c \in (0, \infty) \) depending on \( F_1 \) such that

\[ \mathbb{E} \sup_{t \in [0, T]} K_{t}^{\text{mart}} \leq \mathbb{E} \left[ \int_{0}^{T} \int_{T^{d}} \frac{\rho^{n,\gamma}}{\rho^{n,\gamma} + \delta} \left| \nabla \sqrt{\rho^{n,\gamma}} \right|^2 \right] + 4c^2, \]

(5.9)

where \( c \) is independent of \( \delta \in (0, 1) \). It remains to consider the term \( K_{t}^{\text{ker}} \). By Young’s inequality and (5.5), we have almost surely that

\[ \sup_{t \in [0, T]} K_{t}^{\text{ker}} \leq \int_{0}^{T} \int_{T^{d}} \frac{\rho^{n,\gamma}}{\rho^{n,\gamma} + \delta} \left| \nabla \sqrt{\rho^{n,\gamma}} \right|^2 + \int_{0}^{T} \int_{T^{d}} \rho^{n,\gamma} (V_\gamma \ast \rho^{n,\gamma})^2 \]

\[ \leq \int_{0}^{T} \int_{T^{d}} \frac{\rho^{n,\gamma}}{\rho^{n,\gamma} + \delta} \left| \nabla \sqrt{\rho^{n,\gamma}} \right|^2 + C(\|\hat{\rho}\|_{L^1(T^d)}) \int_{0}^{T} \|V_\gamma\|_{L^\infty(T^d)}^2. \]

(5.10)

Based on (5.7)-(5.10) and Lemma 5.2, there exists a constant \( c \in (0, \infty) \) independent of \( \delta \in (0, 1) \) such that

\[ \mathbb{E} \left[ \sup_{t \in [0, T]} \int_{T^{d}} \Psi_\delta(\rho^{n,\gamma}(x, t)) \right] + \mathbb{E} \left[ \int_{0}^{T} \int_{T^{d}} \frac{2\rho^{n,\gamma}}{\rho^{n,\gamma} + \delta} \left| \nabla \sqrt{\rho^{n,\gamma}} \right|^2 \right] \leq \int_{T^{d}} \Psi_\delta(\hat{\rho}) + c(T, \|\hat{\rho}\|_{L^1(T^d)}, \gamma). \]

(5.11)

By the definition of \( \Psi_\delta \) and (5.11), we deduce that there exists \( \delta_0 > 0 \) such that

\[ \sup_{0 < \delta \leq \delta_0} \mathbb{E} \int_{0}^{T} \left\| \frac{2\rho^{n,\gamma}}{\rho^{n,\gamma} + \delta} \nabla \sqrt{\rho^{n,\gamma}} \right\|_{L^2(T^d)}^2 \leq \int_{T^d} \Psi_\delta(\hat{\rho}) + c(T, \|\hat{\rho}\|_{L^1(T^d)}, \gamma). \]

(5.12)

Then, there exists a subsequence \( \{\delta_k\}_{k \geq 1} \subset \{\delta\}_{0 < \delta \leq \delta_0} \) and \( f^* \in L^2(\Omega \times [0, T]; L^2(T^d)) \) such that

\[ \frac{2\rho^{n,\gamma}}{\rho^{n,\gamma} + \delta} \nabla \sqrt{\rho^{n,\gamma}} \rightharpoonup f^* \text{ weakly in } L^2(\Omega \times [0, T]; L^2(T^d)) \] as \( k \to \infty \).

On the other hand, for almost every \( t \in [0, T] \), and \( \varphi \in C^\infty(T^d; \mathbb{R}^d) \), we have

\[ \left\langle \frac{2\rho^{n,\gamma}}{\rho^{n,\gamma} + \delta} \nabla \sqrt{\rho^{n,\gamma}} - \sqrt{2} \nabla \sqrt{\rho^{n,\gamma}}, \varphi \right\rangle = \sqrt{2} \left\langle \nabla \sqrt{\rho^{n,\gamma}}, \left( \frac{\xi}{\sqrt{\xi^2 + \delta}} - 1 \right) d\xi, \varphi \right\rangle \]

\[ = - \sqrt{2} \left\langle \int_{0}^{\frac{\rho^{n,\gamma}}{\rho^{n,\gamma} + \delta}} \left( \frac{\xi}{\sqrt{\xi^2 + \delta}} - 1 \right) d\xi, \nabla \cdot \varphi \right\rangle \]

\[ = - \sqrt{2} \left\langle \left[ \sqrt{\rho^{n,\gamma} + \delta} - \sqrt{\rho^{n,\gamma} - \sqrt{\delta}} \right], \nabla \cdot \varphi \right\rangle, \]

then, by (5.5) and the dominated convergence theorem, it gives

\[ \lim_{\delta \to 0} \mathbb{E} \int_{0}^{T} \left\langle \frac{2\rho^{n,\gamma}}{\rho^{n,\gamma} + \delta} \nabla \sqrt{\rho^{n,\gamma}} - \sqrt{2} \nabla \sqrt{\rho^{n,\gamma}}, \varphi \right\rangle dr = 0. \]

(5.13)

As a result of (5.13), for any \( \varphi \in C^\infty(T^d; \mathbb{R}^d) \), we deduce that

\[ \mathbb{E} \int_{0}^{T} (\sqrt{2} \nabla \sqrt{\rho^{n,\gamma}}, \varphi) dr = \lim_{k \to \infty} \mathbb{E} \int_{0}^{T} \left\langle \frac{2\rho^{n,\gamma}}{\rho^{n,\gamma} + \delta_k} \nabla \sqrt{\rho^{n,\gamma}}, \varphi \right\rangle dr = \mathbb{E} \int_{0}^{T} (f^*, \varphi) dr, \]

(5.14)
which implies $f^* = \sqrt{2} \nabla \sqrt{\rho^{n,\gamma}}$ almost surely for almost every $(t, x) \in [0, T] \times \mathbb{T}^d$. Thus, $\sqrt{2 \rho^{n,\gamma}} \nabla \sqrt{\rho^{n,\gamma}} \to \sqrt{2} \nabla \sqrt{\rho^{n,\gamma}}$ weakly in $L^2(\Omega \times [0, T]; L^2(\mathbb{T}^d))$, as $\delta \to 0$. By the lower semi-continuity of $L^2(\Omega \times [0, T]; L^2(\mathbb{T}^d))$-norm, we deduce from (5.12) that

$$E \int_0^T \| \nabla \sqrt{\rho^{n,\gamma}} \|^2_{L^2(\mathbb{T}^d)} \leq \int_{\mathbb{T}^d} \Psi_\delta(\hat{\rho}) + c(T, \| \hat{\rho} \|_{L^1(\mathbb{T}^d)}, \gamma).$$  

(5.15)

With the aid of (5.15), the kernel term can be reestimated as follows.

$$\sup_{t \in [0, T]} K^\kappa_{t, \mathbb{T}^d} \leq \int_0^T \| \nabla \rho^{n,\gamma} \cdot V_\gamma(r) * \rho^{n,\gamma} \|_{L^1(\mathbb{T}^d)}$$

$$\leq \int_0^T \| V_\gamma(r) \|_{L^p(\mathbb{T}^d; \mathbb{R}^d)} \| \hat{\rho} \|_{\frac{2}{\gamma} \frac{d}{d+1}}(\| \nabla \sqrt{\rho^{n,\gamma}} \|^2_{L^2(\mathbb{T}^d)})$$

$$\leq \int_0^T \int_{\mathbb{T}^d} \| \nabla \sqrt{\rho^{n,\gamma}} \|^2 + c(T, d, \| \hat{\rho} \|_{L^1(\mathbb{T}^d)}, \| V \|_{L^{p^*}(\Omega \cap [0, T]; L^{p^*}(\mathbb{T}^d))}, d, T),$$

(5.16)

where the nonnegativity of $\rho^{n,\gamma}$ is used.

Combining (5.7), (5.8), (5.9) and (5.16), we deduce that

$$E \left[ \sup_{t \in [0, T]} \int_{\mathbb{T}^d} \Psi_\delta(\rho^{n,\gamma}(t, x)) \right] + 3E \left[ \int_0^T \int_{\mathbb{T}^d} \frac{\rho^{n,\gamma}}{\rho^{n,\gamma} + \delta} \| \nabla \sqrt{\rho^{n,\gamma}} \|^2 \right]$$

$$\leq \int_{\mathbb{T}^d} \Psi_\delta(\hat{\rho}) + \int_0^T \int_{\mathbb{T}^d} \| \nabla \sqrt{\rho^{n,\gamma}} \|^2 + c(T, d, \| \hat{\rho} \|_{L^1(\mathbb{T}^d)}, \| V \|_{L^{p^*}(\Omega \cap [0, T]; L^{p^*}(\mathbb{T}^d))}).$$

(5.17)

Since $\{ \rho^{n,\gamma} = 0 \} = \{ \sqrt{\rho^{n,\gamma}} = 0 \}$, by (5.15) and Stampacchia’s lemma (see Evans [Eva10, Chapter 5, Exercise 18]), we deduce that almost surely

$$\int_0^T \int_{\mathbb{T}^d} 1\{ \rho^{n,\gamma} = 0 \} \| \nabla \sqrt{\rho^{n,\gamma}} \|^2 = 0.$$

Thus, with the aid of Fatou’s lemma, passing to the limit $\delta \to 0$ in (5.17), for some $c \in (0, \infty)$,

$$E \left[ \sup_{t \in [0, T]} \int_{\mathbb{T}^d} \Psi(\rho^{n,\gamma}(t, x)) \right] + E \left[ \int_0^T \int_{\mathbb{T}^d} \| \nabla \sqrt{\rho^{n,\gamma}} \|^2 \right] \leq \int_{\mathbb{T}^d} \Psi(\hat{\rho}) + c(T, d, \| \hat{\rho} \|_{L^1(\mathbb{T}^d)}, \| V \|_{L^{p^*}(\Omega \cap [0, T]; L^{p^*}(\mathbb{T}^d))}).$$

Regarding to the $L^m(\mathbb{T}^d \times [0, T])$-norm estimate, we have the following result.

**Proposition 5.5.** Suppose that $V$ satisfies Assumption (A1). Let $\hat{\rho} \in L^\infty(\mathbb{T}^d)$ be a nonnegative function. For any $\gamma > 0$ and $n \in \mathbb{N}$, let $\rho^{n,\gamma}$ be a solution of (5.3) in the sense of Definition 5.3 with initial data $\rho^{n,\gamma}(\cdot, 0) = \hat{\rho}$. Then there exists a constant $\lambda \in (0, \infty)$ depending on $m, T, \gamma, \| \hat{\rho} \|_{L^\infty(\mathbb{T}^d)}$, and $\| \hat{\rho} \|_{L^1(\mathbb{T}^d)}$ such that

$$\sup_{t \in [0, T]} \mathbb{E} \left[ \int_{\mathbb{T}^d} (\rho^{n,\gamma}(t, x))^m \right] + \mathbb{E} \left[ \int_0^T \int_{\mathbb{T}^d} \rho^{n,\gamma}(t, x)^{m-2} \| \nabla \rho^{n,\gamma} \|^2 \right] \leq \lambda.$$

(5.18)

**Proof.** Applying Itô’s formula [Kry13], we deduce that almost surely for every $t \in [0, T]$,

$$\frac{1}{m(m-1)} \int_{\mathbb{T}^d} (\rho^{n,\gamma}(r, x))^m \bigg|_{r=0}^{t=0} = - \int_0^t \int_{\mathbb{T}^d} |\rho^{n,\gamma}(t, x)^{m-2} \| \nabla \rho^{n,\gamma} \|^2 + \int_0^t \int_{\mathbb{T}^d} |\rho^{n,\gamma}(t, x)^{m-2} V_\gamma(r) \cdot \nabla \rho^{n,\gamma}$$

$$+ \int_0^t \int_{\mathbb{T}^d} \sigma_n(\rho^{n,\gamma}) |\rho^{n,\gamma}|^{m-2} \nabla \rho^{n,\gamma} \cdot dW^F + \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} F_3 |\rho^{n,\gamma}|^{m-2} \sigma_n(\rho^{n,\gamma})$$

$$+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} |\rho^{n,\gamma}|^{m-2} \sigma_n(\rho^{n,\gamma}) \sigma_n'(\rho^{n,\gamma}) F_2 \cdot \nabla \rho^{n,\gamma}. $$

(5.19)
The estimates of the items in \((5.19)\) closely follows the methodology outlined in [FG24, Proposition 5.7] except the nonlocal kernel term. Therefore, to maintain conciseness, we only focus on the kernel terms. By Hölder’s and convolution Young’s inequalities and \((5.5)\), it follows that
\[
\int_0^t \int_{\mathbb{T}^d} |\rho^{n,\gamma}|^{m-2}(\rho^{n,\gamma}V_\gamma(x) \ast \rho^{n,\gamma}) \cdot \nabla \rho^{n,\gamma} \leq \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} |\rho^{n,\gamma}|^{m-2} |\nabla \rho^{n,\gamma}|^2 + \frac{C}{\gamma^2} \|\hat{\rho}\|^2_{L^1(\mathbb{T}^d)} \int_0^t \int_{\mathbb{T}^d} |\rho^{n,\gamma}|^m.
\]
This, together with the estimations of other terms in [FG24, Proposition 5.7], implies that for every \(t \in [0, T]\),
\[
\mathbb{E} \left[ \int_{\mathbb{T}^d} (\rho^{n,\gamma})^m(x,t) \right] + \mathbb{E} \left[ \int_{\mathbb{T}^d} (\rho^{n,\gamma})^{m-2} |\nabla \rho^{n,\gamma}|^2 \right] \\
\leq \|\hat{\rho}\|^m_{L^\infty(\mathbb{T}^d)} + c(T) + c \left( 1 + \frac{\|\hat{\rho}\|^2_{L^1(\mathbb{T}^d)}}{\gamma^2} \right) \mathbb{E} \int_0^t |\rho^{n,\gamma}|^m_{L^\infty(\mathbb{T}^d)}. \tag{5.20}
\]
Applying Gronwall lemma to \((5.20)\), we derive that for every \(t \in [0, T]\),
\[
\mathbb{E} \left[ \int_{\mathbb{T}^d} (\rho^{n,\gamma})^m(x,t) \right] \leq \left( c(T) + \|\hat{\rho}\|^m_{L^\infty(\mathbb{T}^d)} \right) \exp \left\{ c(T) + c \frac{\|\hat{\rho}\|^2_{L^1(\mathbb{T}^d)}}{\gamma^2} (T) \right\}. \tag{5.21}
\]
By substituting \((5.21)\) into \((5.20)\), we get the desired result. \(\Box\)

**Remark 5.6.** Note that Lemma 5.2 implies that the entropy estimate holds uniformly with respect to the regularized parameters \(n\) and \(\gamma\). However, from Lemma 5.2, it follows that the \(L^m(\mathbb{T}^d \times [0, T])\)-norm estimate is not uniform with respect to \(\gamma\).

### 5.2. Existence of renormalized kientic solutions to the approxiamtion equation

We will firstly show the existence of weak solutions of approximation equation \((5.3)\), which are also weak in the probabilistic sense. Then, we introduce the definition of the renormalized kinetic solution to \((5.3)\).

**Theorem 5.7.** Suppose that \(V\) satisfies Assumption (A1). Let \(\hat{\rho} \in L^\infty(\mathbb{T}^d)\) be a nonnegative function. Then for any \(\gamma > 0\) and \(n \in \mathbb{N}\), there exists a stochastic basis \((\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}(t)\}_{t \in [0,T]}, \hat{\mathbb{F}})\), a Brownian motion \(\hat{W}^F\), and a process \(\hat{\rho}^{n,\gamma}\), which is a weak solution of \((5.3)\) in the sense of Definition 5.3 with initial data \(\hat{\rho}^{n,\gamma}(\cdot,0) = \hat{\rho}\). Furthermore, \(\hat{\rho}^{n,\gamma}\) satisfies the estimate \((5.18)\).

**Proof.** Let \(\{e_k\}_{k \in \mathbb{N}} \subset H^1(\mathbb{T}^d)\) be an orthonormal basis of \(L^2(\mathbb{T}^d)\). Let \(0 \leq \lambda_k < \infty\) be the corresponding eigenvalues such that \((\Delta - \lambda_k)e_k = \lambda_k e_k\). For every \(K \in \mathbb{N}\), define a finite dimensional noise \(W_{F,K}(x,t) := \sum_{k=1}^K f_k(x)B_k\). For every \(M \in \mathbb{N}\), let \(\Pi_M : L^2(\mathbb{T}^d \times [0,T]) \rightarrow L^2(\mathbb{T}^d \times [0,T])\) be the projection map defined by
\[
\Pi_M g(x, t) = \sum_{k=1}^M \langle g(t), e_k \rangle e_k(x) \quad \text{for every } g \in L^2(\mathbb{T}^d \times [0,T]).
\]
For any \(m > 0\), define a smooth function \(S^m : \mathbb{R} \rightarrow \mathbb{R}\) satisfying \(-m - 1 \leq S^m \leq m + 1\) and
\[
S^m(\xi) = \begin{cases} 
\xi, & \text{if } -m \leq \xi \leq m, \\
m + 1, & \text{if } \xi > m + 1, \\
-m - 1, & \text{if } \xi < -m - 1, \\
\text{smooth,} & \text{otherwise.}
\end{cases} \tag{5.22}
\]

We consider the following finite dimensional projected equation
\[
d\rho_{M}^{n,\gamma,m} = \Pi_M \left( \right. \Delta \rho_{M}^{n,\gamma,m} \right. dt - \nabla \cdot \left( \sigma_n(\rho_{M}^{n,\gamma,m}) \right) \left. \nabla \rho_{M}^{n,\gamma,m} \right) - \nabla \cdot \left( S^m(\rho_{M}^{n,\gamma,m}) V_\gamma(t) \ast \rho_{M}^{n,\gamma,m} \right) dt \\
+ \Pi_M \left( \int_0^t \frac{1}{2} \nabla \cdot \left( F^K_{1} \right. \left[ \nabla \rho_{M}^{n,\gamma,m} \right] \right. \left. \nabla \rho_{M}^{n,\gamma,m} + \sigma_n(\rho_{M}^{n,\gamma,m}) \sigma_n(\rho_{M}^{n,\gamma,m}) F^K_{2} \right) dt \right), \tag{5.23}
\]
where \(F^K_{1} = \sum_{k=1}^K f_k^2, F^K_{2} = \sum_{k=1}^K f_k \nabla f_k\).
Set $X^k(t) = (\rho^\gamma_{M}^{n,m}(t), e_k)$, (5.23) is equivalent to the following SDE-system:

$$
\frac{dX^k(t)}{dt} = -\lambda_k X^k(t) + \sum_{i=1}^{K} \left( \sigma_n \left( \sum_{j=1}^{M} X^j(t)e_j \right) f_i dB^i_t, \nabla e_k \right) + \sum_{i=1}^{M} S^m \left( \sum_{i=1}^{M} X^i(t)e_i \right) X^j(t) (V_j * e_j, \nabla e_k)
$$

$$
- \frac{1}{2} \sum_{j=1}^{M} X^j(t) \left( F^K_1 \left[ \sigma_n \left( \sum_{i=1}^{M} X^i(t)e_i \right) \right]^2 \nabla e_j, \nabla e_k \right)
$$

$$
- \frac{1}{2} \left( \sigma_n \left( \sum_{i=1}^{M} X^i(t)e_i \right) \right) \sigma_n \left( \sum_{i=1}^{M} X^i(t)e_i \right) \left( F^K_2, \nabla e_k \right), \ k = 1, 2, \ldots, M.
$$

Since $\sigma_n$ and $S^m$ are both smooth, bounded and the derivative of $\sigma_n$ is also bounded, equation (5.23) has a unique probabilistically strong solution $\rho^\gamma_{M}^{n,m}$ on the time interval $[0, T]$. Through a simple estimation, it can be inferred that the sequence $\{\rho^\gamma_{M}^{n,m}\}_{M \geq 1}$ is bounded in $L^\infty([0, T]; L^2(\mathbb{T}^d)) \cap L^2([0, T]; H^1(\mathbb{T}^d)).$

By a standard tightness argument, it follows from Prokhorov’s theorem and the Skorohod representation theorem that there exists a stochastic basis $(\Omega, \mathcal{F}, \{\tilde{F}(t)\}_{t \in [0, T]}, \mathbb{P}, \tilde{W}^{F,K})$ with expectation $\mathbb{E}$, random variables $\{\rho^\gamma_{M}^{n,m}\}_{M \geq 1}$ and $\tilde{\rho}^{\gamma,m} \in L^2(\Omega; L^2([0, T]; L^2(\mathbb{T}^d)))$ such that $\tilde{\rho}^{\gamma,m}$ has the same law as $\rho^\gamma_{M}^{n,m}$ and as $M \to +\infty$,

$$
\|\tilde{\rho}^{\gamma,m} - \tilde{\rho}^{\gamma,m}\|_{L^2([0, T]; L^2(\mathbb{T}^d))} \to 0, \quad \mathbb{P} - a.s.
$$

Next, we show that $\tilde{\rho}^{\gamma,m}$ is nonnegative. For the sake of simplicity, denote by $\rho = \tilde{\rho}^{\gamma,m}, \mathbb{P} = \mathbb{P}, \mathbb{E} = \mathbb{E}$. Let $\rho^\gamma$ be the negative part of $\rho$, i.e. $\rho^\gamma = -\min(\rho, 0)$. We aim to show that $\rho^\gamma = 0$, $\mathbb{P}$-almost surely. It follows from the definition of $\rho^\gamma$ that $\rho^\gamma = \mathbb{P}(\{\rho < 0\})$. Since for any $t \in [0, T]$, $\int_{\mathbb{T}^d} \rho^\gamma(t) = \int_{\mathbb{T}^d} \tilde{\rho} = \|\tilde{\rho}\|_{L^1(\mathbb{T}^d)}$, formally, we have $\partial_t \int_{\mathbb{T}^d} \rho^\gamma = \frac{1}{2} \partial_t \int_{\mathbb{T}^d} |\rho|$. Let $a(\xi) = |\xi|$ and $a^\delta = a * \kappa^\delta$, where $\kappa^\delta : \mathbb{R} \to [0, \infty)$ is a standard convolution kernel on $\mathbb{R}$. It follows from the Itô formula that for every $t \in [0, T]$,

$$
\mathbb{E} \int_{\mathbb{T}^d} a^\delta(\rho(t, x)) - \mathbb{E} \int_{\mathbb{T}^d} a^\delta(\tilde{\rho}) = \mathbb{E} \int_{0}^{t} \int_{\mathbb{T}^d} \kappa^\delta(\rho) \sigma_n(\rho) \sigma_n(\rho) \nabla \rho \cdot F^{K}_3 + \mathbb{E} \int_{0}^{t} \int_{\mathbb{T}^d} \kappa^\delta(\rho) \sigma_n(\rho)^2 F^{K}_3
$$

$$
- 2 \mathbb{E} \int_{0}^{t} \int_{\mathbb{T}^d} \kappa^\delta(\rho) |\nabla \rho|^2 + 2 \mathbb{E} \int_{0}^{t} \int_{\mathbb{T}^d} \kappa^\delta(\rho) S^m(\rho) \nabla \rho \cdot V_\gamma(r) * \rho
$$

where $F^{K}_3 = \sum_{k=1}^{K} |\nabla f_k|^2$. By integration by parts,

$$
\mathbb{E} \int_{0}^{t} \int_{\mathbb{T}^d} \kappa^\delta(\rho) \sigma_n(\rho) \sigma_n(\rho) \nabla \rho \cdot F^{K}_2 = 0.
$$

According to the definition of $\kappa^\delta$, there exists $c \in (0, \infty)$ independent of $\delta \in (0, 1)$ such that $|\kappa^\delta| \leq c/\delta$. As a result of the properties of $\sigma_n$ and $|S^m(\xi)| \leq |\xi|$, we have

$$
\left| \mathbb{E} \int_{0}^{t} \int_{\mathbb{T}^d} \kappa^\delta(\rho) \sigma_n(\rho)^2 F^{K}_3 \right| \leq c \mathbb{E} \int_{0}^{t} \int_{\mathbb{T}^d} I_{|\rho| < \delta} F_3,
$$

$$
\left| \mathbb{E} \int_{0}^{t} \int_{\mathbb{T}^d} \kappa^\delta(\rho) S^m(\rho) \nabla \rho \cdot V_\gamma(r) * \rho \right| \leq c \mathbb{E} \int_{0}^{t} \int_{\mathbb{T}^d} I_{|\rho| < \delta} |\nabla \rho| |V_\gamma(r) * \rho|.
$$

Since $\rho \in L^2([0, T]; H^1(\mathbb{T}^d))$ almost surely, by Assumption (A1) and the boundedness of $F_3$, thanks to the fact that $\hat{\rho}$ is nonnegative, we get when $\delta \to 0$,

$$
\mathbb{E} \int_{\mathbb{T}^d} \rho^\gamma(t, x) = \mathbb{E} \int_{\mathbb{T}^d} \rho^\gamma(t, x) - \mathbb{E} \int_{\mathbb{T}^d} (\hat{\rho})^\gamma(x) = \frac{1}{2} \mathbb{E} \int_{\mathbb{T}^d} |\rho(t, x)| - \frac{1}{2} \mathbb{E} \int_{\mathbb{T}^d} |\hat{\rho}| \leq 0.
$$

Since $\rho^\gamma \geq 0$, we obtain that almost surely $\rho^\gamma = 0$ for every $t \in [0, T]$ and almost every $x \in \mathbb{T}^d$. Thus, $\rho$ is almost surely nonnegative. It implies that $\|\tilde{\rho}^{\gamma,m}(\cdot, t)\|_{L^1(\mathbb{T}^d)} = \|\rho(\cdot, t)\|_{L^1(\mathbb{T}^d)} = \|\hat{\rho}\|_{L^1(\mathbb{T}^d)}$. Utilizing this $L^1$-norm conservation property and applying Itô formula, along with the
property of the smooth function $\mathcal{S}^m$, following a proof analogous to that of Proposition 5.5, we deduce that $\{\bar{\rho}^{n,\gamma,m}\}_{m \geq 0}$ is uniformly bounded in $L^\infty([0, T]; L^2(\mathbb{T}^d)) \cap L^2([0, T]; H^1(\mathbb{T}^d))$, with respect to the parameters $m$ and $K$. Repeating a process similar to the above, using a standard tightness argument along with Prokhorov’s theorem and the Skorohod representation theorem, there exists a stochastic basis $(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}(t)\}_{t \in [0, T]}, \hat{\mathbb{P}}, \hat{\mathcal{W}}^F)$ with expectation $\hat{\mathbb{E}}$, random variables $\{\tilde{\rho}^{n,\gamma,m}\}_{m \geq 0}$ and $\tilde{\rho}^{n,\gamma} \in L^2(\hat{\Omega}; L^2([0, T]; L^2(\mathbb{T}^d)))$ such that $\tilde{\rho}^{n,\gamma,m}$ has the same law as $\tilde{\rho}^{n,\gamma,m}$ and as $K \to +\infty$, $m \to +\infty,$

$$\|\tilde{\rho}^{n,\gamma,m} - \tilde{\rho}^{n,\gamma}\|_{L^2([0, T]; L^2(\mathbb{T}^d))} \to 0, \quad \hat{\mathbb{P}} - a.s..$$

Moreover, $\tilde{\rho}^{n,\gamma}$ satisfies (5.4) with respect to the new stochastic basis $(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}(t)\}_{t \in [0, T]}, \hat{\mathbb{P}}, \hat{\mathcal{W}}^F)$.

The method employed here is standard, thus we omit it. Finally, paralleling the approach used to demonstrate the nonnegativity of $\tilde{\rho}^{n,\gamma,m}$, we deduce that the solution $\tilde{\rho}^{n,\gamma}$ is almost surely nonnegative. With this conclusion, the proof is thereby completed.

**Remark 5.8.** The result of Theorem 5.7 is the starting point to prove the existence of the probabilistically strong solution to (1.4). Concretely, we will apply Lemma 6.1 below to the probabilistically weak solution constructed in Theorem 5.7 to find a solution living in the original probability space. Since the conditions of Lemma 6.1 are in the sense of distribution, it is not necessary to emphasize the difference between the original probability space and the new one. Thus, with a little abuse of notations, the probability space, the Brownian motion and the weak solution in Theorem 5.7 are still denoted by $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t \in [0, T]}, \mathbb{P}), \mathcal{W}^F$ and $\rho^{n,\gamma}$, respectively.

Now, we introduce the other definition of solutions to (5.3), which is called a renormalized kinetic solution.

**Definition 5.9.** Suppose that $V$ satisfies Assumption (A1). Let $\hat{\rho} \in L^\infty(\mathbb{T}^d)$ be a nonnegative function. For any $\gamma > 0$ and $n \in \mathbb{N}$, a renormalized kinetic solution of (5.3) with initial data $\rho^{n,\gamma}(\cdot, 0) = \hat{\rho}$ is a nonnegative, $L^m(\mathbb{T}^d)$-continuous (for some $m \geq 2$) $\mathcal{F}_t$-predictable process $\rho^{n,\gamma}$ such that almost surely $\rho^{n,\gamma} \in L^2([0, T]; H^1(\mathbb{T}^d))$ and almost surely for every $\psi \in C^\infty_c(\mathbb{T}^d \times \mathbb{R})$ and $t \in [0, T]$,

$$\int_{\mathbb{T}^d} \int_{\mathbb{R}} \chi^{n,\gamma}(x, \xi, t) \psi(x, \xi) = \int_{\mathbb{T}^d} \int_{\mathbb{R}} \chi^{n,\gamma}(\hat{\rho}) \psi(x, \xi) - \int_0^t \int_{\mathbb{T}^d} \nabla \rho^{n,\gamma} \cdot (\nabla \psi)(x, \rho^{n,\gamma})$$

$$- \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} F_1(x) \left[ \sigma_n(\rho^{n,\gamma}) \right] \nabla \rho^{n,\gamma} \cdot (\nabla \psi)(x, \rho^{n,\gamma}) - \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \sigma_n(\rho^{n,\gamma}) \sigma_n' (\rho^{n,\gamma}) F_2(x) \cdot (\nabla \psi)(x, \rho^{n,\gamma})$$

$$- \int_0^t \int_{\mathbb{T}^d} \left( \partial_t \psi \right)(x, \rho^{n,\gamma}) |\nabla \rho^{n,\gamma}|^2 + \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} \left( \sigma_n(\rho^{n,\gamma}) \sigma_n'(\rho^{n,\gamma}) \nabla \rho^{n,\gamma} \cdot F_2(x) + F_3(x) \sigma_n^2(\rho^{n,\gamma}) \right) (\partial_t \psi)(x, \rho^{n,\gamma})$$

$$- \int_0^t \int_{\mathbb{T}^d} \psi(x, \rho^{n,\gamma}) \nabla \rho^{n,\gamma} \cdot (\rho^{n,\gamma} V_r(r) \cdot \rho^{n,\gamma}) - \int_0^t \int_{\mathbb{T}^d} \psi(x, \rho^{n,\gamma}) \nabla \cdot (\sigma_n(\rho^{n,\gamma}) d\mathcal{W}^F),$$

where $\chi^{n,\gamma} : \mathbb{T}^d \times \mathbb{R} \times [0, T] \to \{0, 1\}$ is the kinetic function given by $\chi^{n,\gamma}(x, \xi, t) = 1_{\{0 < \xi < \rho^{n,\gamma}(x, t)\}}$, $\bar{\chi}^{n,\gamma}(\hat{\rho})(x, \xi) = 1_{\{0 < \xi < \hat{\rho}(x)\}}$ and the kinetic measure $q^{n,\gamma} = \delta_0(\xi - \rho^{n,\gamma}) |\nabla \rho^{n,\gamma}|^2$.

Similar to [FG21, Proposition 5.21], we can show that the weak solution is equivalent to the renormalized kinetic solution. It reads as follows.

**Proposition 5.10.** Suppose that $V$ satisfies Assumption (A1). Let $\hat{\rho} \in L^\infty(\mathbb{T}^d)$ be a nonnegative function. For any $\gamma > 0$ and $n \in \mathbb{N}$, let $\rho^{n,\gamma}$ be a weak solution of (5.3) in the sense of Definition 5.3 with initial data $\rho^{n,\gamma}(\cdot, 0) = \hat{\rho}$. Then $\rho^{n,\gamma}$ is a renormalized kinetic solution in the sense of Definition 5.9.

5.3. Tightness of approximating solutions. We aim to show the $L^1([0, T]; L^1(\mathbb{T}^d))$-tightness of the laws of $\{\rho^{n,\gamma}\}_{n \in \mathbb{N}, \gamma \in (0, 1)}$ constructed in Theorem 5.7. The method is based on the Aubin-Lions-Simon lemma ([Sim87, Corollary 5]). An important ingredient is to establish a stable $W^{\beta,1}(0, T; H^{-1}(\mathbb{T}^d))$-estimate. However, we cannot get such an estimation in the presence of the singular term $\nabla \cdot (F_1 \rho^{-1} \nabla \rho)$.
in (2.3) due to the singularity at $\rho = 0$. To solve this problem, we follow the idea used in [FG24] to introduce a smooth function $h_\delta$ defined by Definition 5.11 below to keep the solution away from zero. Hence, the $W^{3,1}((0,T]; H^{-1}(\mathbb{T}^d))$-estimate for $h_\delta(\rho)$ can be obtained in Proposition 5.13.

Let $B$ be a Banach space. Given $1 \leq p < \infty$, $0 < \sigma < 1$, let $W^{\sigma,p}((0,T]; B)$ be the fractional Sobolev space defined by [Mar87]. For every $\delta \in (0,1)$, let $\psi_\delta \in C^\infty([0,\infty))$ be a smooth nondecreasing function satisfying $0 \leq \psi_\delta \leq 1$ and

$$
\psi_\delta(\xi) = \begin{cases} 
1, & \text{if } \xi \geq \delta, \\
0, & \text{if } \xi \leq \delta/2, \\
\text{smooth, otherwise.}
\end{cases}
$$

Clearly, $|\psi_\delta'(\xi)| \leq c/\delta$ for some $c \in (0, \infty)$ independent of $\delta$.

**Definition 5.11.** For every $\delta \in (0,1)$, let $h_\delta \in C^\infty([0,\infty))$ be defined by

$$
h_\delta(\xi) = \psi_\delta(\xi) \xi \quad \text{for every } \xi \in [0,\infty).
$$

From the definition of $h_\delta$, it follows that $h_\delta'$ is supported on $[\frac{\delta}{2}, \infty)$ and

$$
h_\delta'(\xi) = \psi_\delta'(\xi) \xi + \psi_\delta(\xi) \leq c \mathbf{1}_{\{\xi \geq \delta/2\}},
$$

where $c$ is independent of $\delta$. Moreover, there exists a constant $c \in (0, \infty)$ depending on $\delta$ such that

$$
h_\delta''(\xi) = \psi_\delta''(\xi) \xi + 2 \psi_\delta'(\xi) \leq c(\delta) \mathbf{1}_{\{\frac{\delta}{4} \leq \xi \leq \delta\}}.
$$

To apply the Aubin-Lions-Simon lemma, we need the following result which can be easily proved by using (5.26), Lemma 3.3, Hölder’s inequality and (5.6).

**Lemma 5.12.** Suppose that $V$ satisfies Assumption (A1). Let $\hat{\rho} \in L^\infty(\mathbb{T}^d)$ be a nonnegative function. For any $\gamma > 0$ and $n \in \mathbb{N}$, let $\rho^{n,\gamma}$ be a weak solution of (5.3) in the sense of Definition 5.3 with initial data $\rho^{n,\gamma}(\cdot,0) = \hat{\rho}$. Then, there exists a constant $c \in (0, \infty)$ independent of $\delta$ such that

$$
\mathbb{E}\left[\|h_\delta(\rho^{n,\gamma})\|_{L^1(\{0,T]\cup[0,\infty); W^{1,1}(\mathbb{T}^d))}\right] \leq c(T, d, \|\hat{\rho}\|_{L^1(\mathbb{T}^d)}, \|V\|_{L^{p_1}(\{0,T]\cup[0,\infty); L^p(\mathbb{T}^d))}),
$$

Moreover, we need the following $W^{\beta,1}((0,T]; H^{-1}(\mathbb{T}^d))$-estimate for $h_\delta(\rho)$.

**Proposition 5.13.** Suppose that $V$ satisfies Assumption (A1). Let $\hat{\rho} \in L^\infty(\mathbb{T}^d)$ be a nonnegative function. For any $\gamma > 0$ and $n \in \mathbb{N}$, let $\rho^{n,\gamma}$ be a weak solution of (5.3) in the sense of Definition 5.3 with initial data $\rho^{n,\gamma}(\cdot,0) = \hat{\rho}$. Then, for every $\beta \in (0,1/2)$ and $l > \frac{d}{2} + 1$, there exists a constant $c \in (0, \infty)$ depending on $\delta, \beta, T, d, l, \|V\|_{L^{p_1}(\{0,T]\cup[0,\infty); L^p(\mathbb{T}^d))}$, and $\|\hat{\rho}\|_{L^1(\mathbb{T}^d)}$ such that

$$
\mathbb{E}\left[\|h_\delta(\rho^{n,\gamma})\|_{W^{\beta,1}(\{0,T]\cup[0,\infty); H^{-1}(\mathbb{T}^d))}\right] \leq c(\delta, \beta, T, d, l, \|V\|_{L^{p_1}(\{0,T]\cup[0,\infty); L^p(\mathbb{T}^d))}, \|\hat{\rho}\|_{L^1(\mathbb{T}^d)}).
$$

**Proof.** Applying Itô formula, for every $t \in (0,T)$, for every $\delta \in (0,1)$, as a distribution on $\mathbb{T}^d$, we have almost surely that $h_\delta(\rho^{n,\gamma}(x,t)) = h_\delta(\hat{\rho}) + J^{\text{v}}_t + J^{\text{ker}}_t + J^{\text{mart}}_t$. The finite variation part is defined by

$$
J^{\text{v}}_t = \int_0^t \nabla \cdot (h_\delta(\rho^{n,\gamma}) \nabla \rho^{n,\gamma}) - \int_0^t h_\delta'(\rho^{n,\gamma}) \nabla \rho^{n,\gamma} \cdot \nabla \rho^{n,\gamma} + \frac{1}{2} \int_0^t \left( (h_\delta'(\rho^{n,\gamma}) \sigma_n(\rho^{n,\gamma}) \sigma_n'(\rho^{n,\gamma}) F_1(x) F_2) + \frac{1}{2} \int_0^t h_\delta''(\rho^{n,\gamma}) \sigma_n(\rho^{n,\gamma}) \sigma_n'(\rho^{n,\gamma}) \nabla \rho^{n,\gamma} \cdot F_2 \right.
$$

$$
+ \frac{1}{2} \int_0^t h_\delta''(\rho^{n,\gamma}) F_3(x) \sigma_n^2(\rho^{n,\gamma}) \right) dx dt.
$$

The kernel part is defined by

$$
J^{\text{ker}}_t = - \int_0^t h_\delta'(\rho^{n,\gamma}) \nabla \cdot (\rho^{n,\gamma} V_n(x) + \rho^{n,\gamma}).
$$
The martingale part is defined by
\[ J^\text{mart}_t = -\int_0^t \nabla \cdot (h^{\beta}_0(\rho^{n,\gamma}) \sigma_n(\rho^{n,\gamma}) dW^F) + \int_0^t h^{\alpha}_0(\rho^{n,\gamma}) \sigma_n(\rho^{n,\gamma}) \nabla \rho^{n,\gamma} \cdot dW^F. \]

By a similar method to [FG21, Proposition 5.14], for every \( \beta \in (0, 1/2) \) and \( l > \frac{\beta}{\alpha} + 1 \), it gives
\[ E[\|J^F_v + J^\text{mart}\|_{W^{\beta,1}(0,T;L^{-1}(\mathbb{T}^d))}] \leq c \left( \delta, \beta, l, T, d, \|\hat{\rho}\|_{L^1(\mathbb{T}^d)}, \|V\|_{L^{p^*}(0,T;L^1(\mathbb{T}^d;\mathbb{R}))} \right). \quad (5.28) \]

In the sequel, we focus on the kernel term. By integration by parts formula, we get
\[ E\left[\|J^\text{ker}\|_{W^{1,1}(0,T;H^{-1}(\mathbb{T}^d))}\right] = E\int_0^T \left( \|J^\text{ker}_t\|_{H^{-1}(\mathbb{T}^d)} + \|\partial_t J^\text{ker}_t\|_{H^{-1}(\mathbb{T}^d)} \right) \]
\[ \leq (T + 1) E\int_0^T \sup_{\|\phi\|_{H^{1}(\mathbb{T}^d)} = 1} \left| \int_{\mathbb{T}^d} h''_0(\rho^{n,\gamma}) \phi(x) \nabla \rho^{n,\gamma} \cdot (\rho^{n,\gamma} V_\gamma(t) * \rho^{n,\gamma}) \right| \]
\[ + (T + 1) E\int_0^T \sup_{\|\phi\|_{H^{1}(\mathbb{T}^d)} = 1} \left| \int_{\mathbb{T}^d} h''_0(\rho^{n,\gamma}) \nabla \phi(x) \cdot (\rho^{n,\gamma} V_\gamma(t) * \rho^{n,\gamma}) \right| \]
\[ =: I_1 + I_2. \quad (5.29) \]

Note that when \( l > \frac{\beta}{\alpha} + 1 \), we have \( \|f\|_{L^\infty(\mathbb{T}^d)} \leq c \|f\|_{H^1(\mathbb{T}^d)} \) and \( \|
abla f\|_{L^\infty(\mathbb{T}^d)} \leq c \|f\|_{H^1(\mathbb{T}^d)} \). By Hölder’s and convolution Young’s inequalities, Lemma 3.3, 5.2 (5.27), and Proposition 5.4, we have
\[ I_1 \leq c(\delta, l) E\int_0^T \|\nabla \sqrt{\rho^{n,\gamma}} \cdot V_\gamma(t) * \rho^{n,\gamma}\|_{L^1(\mathbb{T}^d)} \]
\[ \leq c(\delta, l, \|\hat{\rho}\|_{L^1(\mathbb{T}^d)} \left( \int_0^T \|V_\gamma(t)\|_{L^2(\mathbb{T}^d)}^2 \right)^{\frac{1}{2}} \left( E\int_0^T \|
abla \sqrt{\rho^{n,\gamma}}\|_{L^2(\mathbb{T}^d)}^2 \right)^{\frac{1}{2}} \]
\[ \leq c \left( \delta, l, T, d, \|\hat{\rho}\|_{L^1(\mathbb{T}^d)}, \|V\|_{L^{p^*}(0,T;L^1(\mathbb{T}^d;\mathbb{R}))} \right). \quad (5.30) \]

Applying Hölder’s inequality to \( \frac{1}{2} + \frac{1}{2} = 1 \), by using convolution Young’s inequality, (5.26), Lemma 5.2, Proposition 5.4 and Lemma 3.2, there exists a constant \( c \in (0, \infty) \) depending on \( l \) such that
\[ I_2 \leq c(l) \|\hat{\rho}\|_{L^1(\mathbb{T}^d)} E\int_0^T \|ho^{n,\gamma}\|_{L^1(\mathbb{T}^d)} \|V_\gamma(t)\|_{L_p(\mathbb{T}^d)} \]
\[ \leq c(l, d) \|\hat{\rho}\|_{L^1(\mathbb{T}^d)}^2 E\int_0^T \|V_\gamma(t)\|_{L_p(\mathbb{T}^d)} \|
abla \sqrt{\rho^{n,\gamma}}\|_{L^2(\mathbb{T}^d)}^\frac{2}{p} \]
\[ \leq c \left( l, T, d, \|\hat{\rho}\|_{L^1(\mathbb{T}^d)}, \|V\|_{L^{p^*}(0,T;L^1(\mathbb{T}^d;\mathbb{R}))} \right). \quad (5.31) \]

Based on (5.29)-(5.31), we conclude that there exists a constant \( c \in (0, \infty) \) such that
\[ E\left[\|J^\text{ker}\|_{W^{1,1}(0,T;H^{-1}(\mathbb{T}^d))}\right] \leq c \left( \delta, l, T, d, \|\hat{\rho}\|_{L^1(\mathbb{T}^d)}, \|V\|_{L^{p^*}(0,T;L^1(\mathbb{T}^d;\mathbb{R}))} \right). \quad (5.32) \]

Combining (5.28) and (5.32), and by the embeddings \( W^{\beta,2}(\mathbb{T}^d), W^{1,1}(\mathbb{T}^d) \hookrightarrow W^{\beta,1}(\mathbb{T}^d) \) for every \( \beta \in (0, \frac{1}{2}) \), we complete the proof. \( \square \)

In the sequel, we follow the idea in [FG24] to obtain the tightness of the laws of \( \{\rho^{n,\gamma}\}_{n \in \mathbb{N}, \gamma \in (0,1)} \) on \( L^1([0,T];L^1(\mathbb{T}^d)) \).

**Proposition 5.14.** Suppose that \( V \) satisfies Assumption (A1). Let \( \hat{\rho} \in L^\infty(\mathbb{T}^d) \) be a nonnegative function. For any \( \gamma > 0 \) and \( n \in \mathbb{N} \), let \( \rho^{n,\gamma} \) be the renormalized kinetic solution of (5.3) with initial data \( \rho^{n,\gamma}(\cdot,0) = \hat{\rho} \), then the laws of \( \{\rho^{n,\gamma}\}_{n \in \mathbb{N}, \gamma \in (0,1)} \) are tight on \( L^1([0,T];L^1(\mathbb{T}^d)) \) in the strong norm topology.

**Proof.** According to the Aubin-Lions-Simon lemma ([Sim87, Corollary 5]), Lemma 5.12, and Proposition 5.13, it follows that the laws of \( \{h_\delta(\rho^{n,\gamma})\}_{n \in \mathbb{N}, \gamma \in (0,1)} \) are tight on \( L^1([0,T];L^1(\mathbb{T}^d)) \). The remaining task is to deduce the tightness of the laws of \( \{\rho^{n,\gamma}\}_{n \in \mathbb{N}, \gamma \in (0,1)} \) on \( L^1([0,T];L^1(\mathbb{T}^d)) \). It can
be achieved by closely following the approach used in [FG24, Proposition 5.22], where a new metric on $L^1([0,T]; L^1(\mathbb{T}^d))$ is introduced based on $h_k$ that is equivalent to the strong norm topology on $L^1([0,T]; L^1(\mathbb{T}^d))$. The explicit details of the proof are omitted for brevity.

Due to the lack of a stable $W^{\beta,1}([0,T]; H^{-1}(\mathbb{T}^d))$-estimate for $\rho^{n,\gamma}$, we are not able to show the laws of $\rho^{n,\gamma}$ are tight on the space $C([0,T]; H^{-1}(\mathbb{T}^d))$. For this reason, we prove the tightness of the martingale term directly. With the aid of Lemma 5.1, the following result can be easily proved by using the same method as [FG24, Proposition 5.23].

**Proposition 5.15.** Suppose that $V$ satisfies Assumption (A1). Let $\hat{\rho} \in L^\infty(\mathbb{T}^d)$ be a nonnegative function. For any $\gamma > 0$, $n \in \mathbb{N}$ and $\psi \in C_c^\infty(\mathbb{T}^d \times (0,\infty))$, let $\rho^{n,\gamma}$ be the renormalized kinetic solution of (5.3) with initial data $\rho^{n,\gamma}(\cdot,0) = \hat{\rho}$. Let

$$M^n_{t,\gamma,\psi} := \int_0^t \int_{\mathbb{T}^d} \psi(x,\rho^{n,\gamma}) \nabla \cdot \left( \sigma_n(\rho^{n,\gamma}) \, dW^F \right).$$

Then for every $\beta \in (0,1/2)$, the laws of the martingales $\{M^n_{t,\gamma,\psi}\}_{n \in \mathbb{N}, \gamma \in (0,1)}$ are tight on $C^\beta([0,T]; \mathbb{R})$.

6. Existence of renormalized kinetic solutions to Dean-Kawasaki equation

In this section, we aim to prove the existence of renormalized kinetic solutions (strong in the probabilistic sense) to the Dean-Kawasaki equation

$$d\rho = \Delta \rho \, dt - \nabla \cdot (\rho V(t) \ast \rho) \, dt - \nabla \cdot (\sqrt{\rho} dW^F(t)) + \frac{1}{8} \nabla \cdot (F_1 \rho^{-1} \nabla \rho + 2F_2) \, dt. \quad (6.1)$$

To achieve it, two steps are involved. Firstly, we show the existence of a probabilistically weak solution of (6.1) under Assumption (A1). Further, under Assumptions (A1) and (A2), by using the pathwise uniqueness (Theorem 4.2) and the following technical Lemma 6.1, we deduce that there exists a probabilistically strong solution to (6.1).

We need the following technical lemma, whose proof can be found in [FG23].

**Lemma 6.1.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{X_n : \Omega \rightarrow \bar{X}\}_{n \geq 1}$ be a sequence of random variables, where $\bar{X}$ is a complete separable metric space. Then, $X_n$ converges in probability as $n \rightarrow \infty$, if and only if for any sequences $\{(n_k, m_k)\}_{k=1}^\infty$ satisfying $n_k, m_k \rightarrow \infty$ as $k \rightarrow \infty$, there exists a further subsequence $\{(n_{k'}, m_{k'})\}_{k'=1}^\infty$ fulfilling $n_{k'}, m_{k'} \rightarrow \infty$ as $k' \rightarrow \infty$ such that the joint laws of $(X_{n_{k'}}, X_{m_{k'}})_{k' \in \mathbb{N}}$ converge weakly to a probability measure $\mu$ on $\bar{X} \times \bar{X}$ satisfying $\mu(\{ (x, y) \in \bar{X} \times \bar{X} : x = y \}) = 1$, as $k' \rightarrow \infty$.

**Theorem 6.2.** Let $\hat{\rho} \in \text{Ent}(\mathbb{T}^d)$. We have the following two results.

(i) Suppose that Assumption (A1) holds. Then there exists a stochastic basis $(\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{F}(t)\}_{t \in [0,T]}, \bar{\mathbb{P}})$, a Brownian motion $\bar{W}^F$, and a process $\bar{\rho}$, which is a renormalized kinetic solution of (6.1) in the sense of Definition 2.3 with initial data $\bar{\rho}(\cdot,0) = \hat{\rho}$.

(ii) Suppose that Assumptions (A1) and (A2) holds. Then there exists a probabilistically strong renormalized kinetic solution of (6.1) with initial data $\bar{\rho}(\cdot,0) = \hat{\rho}$.

**Proof.** We begin with a standard setting of a tightness statement and a Skorokhod’s representation argument. For every $n \in \mathbb{N}$, let $\rho^n := \hat{\rho} \wedge n$. It is obvious that for every $n \in \mathbb{N}$, $\rho^n \in L^\infty(\mathbb{T}^d)$. Since the entropy function is bounded, we have

$$\sup_n \int_{\mathbb{T}^d} \rho^n \log(\rho^n) < \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \|\rho^n - \hat{\rho}\|_{L^1(\mathbb{T}^d)} = 0.$$

For any $\gamma \in (0,1)$ and $n \in \mathbb{N}$, let $\rho^{n,\gamma}$ be the renormalized kinetic solution of (5.3) in the sense of Definition 5.9 with initial data $\rho^{n,\gamma}(\cdot,0) = \rho^n$. For every $\psi \in C_c^\infty(\mathbb{T}^d \times (0,\infty))$, the corresponding
martingales \( \{M^{n,\gamma};\psi\}_{n\in\mathbb{N},\gamma\in(0,1)} \) are defined by Proposition 5.15. The corresponding kinetic measures \( \{q^{n,\gamma}\}_{n\in\mathbb{N},\gamma\in(0,1)} \) are defined by

\[
q^{n,\gamma} := \delta_0 (\xi - \rho^{n,\gamma}) |\nabla \rho^{n,\gamma}|^2.
\]

According to Proposition 5.5, for any \( n \in \mathbb{N}, \gamma \in (0,1) \), \( q^{n,\gamma} \) is a finite kinetic measure in the sense of Definition 2.2. Moreover, it follows from (5.25) that, for every \( \psi \in C^\infty_c (\mathbb{T}^d \times (0,\infty)) \) and \( t \in [0,T] \), for the kinetic function \( \chi^{n,\gamma} \) of \( \rho^{n,\gamma} \), the martingale term defined by (5.33) can be written as

\[
M_t^{n,\psi,\gamma} = -\int_{\mathbb{R}} \int_{\mathbb{T}^d} \chi^{n,\gamma}(x,\xi,r)\psi(x,\xi) \bigg|_{r=0}^{r=t} - \int_{\mathbb{T}^d} \nabla \rho^{n,\gamma} \cdot (\nabla \psi) (x,\rho^{n,\gamma})
- \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} F_1 [\sigma_n(\rho^{n,\gamma})]^2 \nabla \rho^{n,\gamma} \cdot (\nabla \psi) (x,\rho^{n,\gamma}) \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} \sigma_n(\rho^{n,\gamma}) \sigma'_n(\rho^{n,\gamma}) F_2 \cdot (\nabla \psi) (x,\rho^{n,\gamma})
+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} (\partial_\xi \psi)(x,\rho^{n,\gamma}) \sigma_n(\rho^{n,\gamma}) \sigma'_n(\rho^{n,\gamma}) \nabla \rho^{n,\gamma} \cdot F_2 + \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} F_3 \sigma_n(\rho^{n,\gamma}) \sigma'_n(\rho^{n,\gamma}) (\partial_\xi \psi)(x,\rho^{n,\gamma})
\]

where \( \chi^{n,\gamma}(x,\xi) = 1_{\{0<\xi<\rho^{n,\gamma}(x)\}} \). Let \( l > \frac{4}{a} + 1 \) be some integer and fix a countable sequence \( \{\psi_j\}_{j\in\mathbb{N}} \) which is dense in \( C^\infty_c (\mathbb{T}^d \times (0,\infty)) \) in the strong \( H^l (\mathbb{T}^d \times (0,\infty)) \)-topology. For every \( n \in \mathbb{N} \) and \( \gamma \in (0,1) \), define the random variables

\[
X^{n,\gamma} = \left( \rho^{n,\gamma}, \nabla \sqrt{\rho^{n,\gamma}}, (M^{n,\gamma};\psi_j)_{j\in\mathbb{N}} \right),
\]

taking values in the space

\[
\bar{X} = L^1([0,T] \times \mathbb{T}^d) \times L^2([0,T] \times \mathbb{T}^d; \mathbb{R}^d) \times C([0,T])^N,
\]

where \( \bar{X} \) is equipped with the product metric topology induced by the strong topology on \( L^1([0,T] \times \mathbb{T}^d) \), the weak topology on \( L^2([0,T] \times \mathbb{T}^d; \mathbb{R}^d) \), and the topology on \( C([0,T]) \) induced by

\[
D \left((f_k)_{k\in\mathbb{N}}, (g_k)_{k\in\mathbb{N}}\right) = \sum_{k=1}^\infty 2^{-k} \frac{\|f_k - g_k\|_{C([0,T])}}{1 + \|f_k - g_k\|_{C([0,T])}}.
\]

In order to apply Lemma 6.1, we set \( \{n_k,\gamma_k\}_{k\in\mathbb{N}} \) and \( \{m_k,\eta_k\}_{k\in\mathbb{N}} \) as two subsequences satisfying \( n_k,m_k \to \infty \) and \( \gamma_k,\eta_k \to 0 \) as \( k \to \infty \). We consider the laws of

\[
(X^{n_k,\gamma_k}, X^{m_k,\eta_k}, B) \text{ on } \bar{Y} = \bar{X} \times \mathbb{R} \times C([0,T])^N,
\]

for \( B = \{B_j\}_{j\in\mathbb{N}} \). It follows from Propositions 5.4, 5.14 and 5.15 that the laws of \( (X^{n,\gamma})_{n\in\mathbb{N},\gamma\in(0,1)} \) are tight on \( \bar{X} \). Furthermore, according to Prokhorov's theorem, after passing to a subsequence still denoted by \( k \to \infty \), there exists a probability measure \( \mu \) on \( \bar{Y} \) such that, as \( k \to \infty \),

\[
(X^{n_k,\gamma_k}, X^{m_k,\eta_k}, B) \to \mu \text{ in law } \tag{6.3}
\]

Since the space \( \bar{X} \) is separable, \( \bar{Y} \) is separable as well. With the aid of the Skorokhod representation theorem, there exists a probability space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) \) and \( \tilde{Y} \)-valued random variables \( (\tilde{Y}^k, \tilde{Z}^k, \tilde{\beta}^k)_{k\in\mathbb{N}} \) and \( (\tilde{Y}, \tilde{Z}, \tilde{\beta}) \) on \( \tilde{\Omega} \) such that, for every \( k \in \mathbb{N} \),

\[
(\tilde{Y}^k, \tilde{Z}^k, \tilde{\beta}^k) = (X^{n_k,\gamma_k}, X^{m_k,\eta_k}, B) \text{ in law on } \tilde{Y}, \tag{6.4}
\]

such that

\[
(\tilde{Y}, \tilde{Z}, \tilde{\beta}) = \mu \text{ in law on } \tilde{Y}, \tag{6.5}
\]

and such that as \( k \to \infty \),

\[
\tilde{Y}^k \to \tilde{Y}, \tilde{Z}^k \to \tilde{Z}, \text{ and } \tilde{\beta}^k \to \tilde{\beta} \text{ } \tilde{\mathbb{P}}\text{-almost surely in } \tilde{X} \text{ and in } C([0,T])^N, \text{ respectively.}
\]
Our plan is as follows. Firstly, under Assumption (A1), we aim to prove the existence of a renormalized kinetic solution, which is weak in the probabilistic sense. This requires us to exhibit a specific component of $\tilde{Y}$. The corresponding result is formulated in (i). Secondly, by further assuming that Assumption (A2) holds, we employ the pathwise uniqueness to demonstrate that the limiting joint distribution $\mu$ is supported on the diagonal, i.e., $\mu(x,y) \in \tilde{X} \times \tilde{X} : x = y = 1$. Thereby we can establish the existence of a probabilistically strong solution by Lemma 6.1. The relevant results are shown in (ii). In the following, we start with the proof of (i).

**The proof of (i).**

**Step 1. Components of $\tilde{Y}^k$.** It follows from (6.4) that for every $k \in \mathbb{N}$, there exists $\tilde{\rho}^k \in L^1\left(\tilde{\Omega} ; L^1([0,T] \times \mathbb{T}^d)\right)$, $\tilde{G}^k \in L^2\left(\tilde{\Omega} ; L^2([0,T] \times \mathbb{T}^d ; \mathbb{R}^d)\right)$, and $(\tilde{M}^{k,\psi})_{j \in \mathbb{N}} \in L^2(\tilde{\Omega} ; C([0,T])^N)$ such that

$$
\tilde{Y}^k = (\tilde{\rho}^k, \tilde{G}^k, (\tilde{M}^{k,\psi})_{j \in \mathbb{N}}).
$$

Denote by $\mathcal{M}(\mathbb{T}^d \times \mathbb{R} \times [0,T])$ the space of all finite nonnegative Borel measures on $\mathbb{T}^d \times \mathbb{R} \times [0,T]$ equipped with weak topology. For any $k \in \mathbb{N}$, define

$$
\tilde{d}^k(\tilde{\omega}) := \delta_0(\tilde{\omega} - \tilde{\rho}^k(\tilde{\omega})) |\nabla \tilde{\rho}^k(\tilde{\omega})|^2
$$

for every $\tilde{\omega} \in \tilde{\Omega}$, (6.6)
on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Clearly, by Proposition 5.5, we have $\tilde{d}^k \in \mathcal{M}(\mathbb{T}^d \times \mathbb{R} \times [0,T])$ almost surely.

Regarding to the identification of the vector fields $\tilde{\xi}^k$, since the distributions of $(\sqrt{\tilde{\rho}^k}, \tilde{G}^k)$ and $(\sqrt{\tilde{\rho}^k \times \tilde{\rho}^k}, \nabla \sqrt{\tilde{\rho}^k \times \tilde{\rho}^k})$ are the same, by using the same approach as in [FG24, Theorem 5.25], it follows that $\tilde{G}^k = \nabla \sqrt{\tilde{\rho}^k}$ almost surely. Similarly, using the same joint distribution property in (6.4), one obtains

$$
\tilde{M}^{k,\psi} = -\int_{\mathbb{R}} \int_{\mathbb{T}^d} \chi^k(x,\xi,\rho) \psi_j(x,\xi) \, d\tilde{\omega} - \int_0^t \int_{\mathbb{T}^d} \nabla \tilde{\rho}^k \cdot (\nabla \psi_j) (x, \tilde{\rho}^k) \, dx
$$

$$
- \int_0^t \int_{\mathbb{T}^d} \rho \frac{1}{2} \left( \int_{\mathbb{T}^d} \sigma'_{nk}(\tilde{\rho}^k) \sigma_{nk}(\tilde{\rho}^k) \sigma'_{nk}(\tilde{\rho}^k) \cdot \nabla F_2 \right) \psi_j (x, \tilde{\rho}^k) \, dx
$$

$$
+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} \partial \xi \psi_j (x, \tilde{\rho}^k) \, d\tilde{\omega} + \int_0^t \int_{\mathbb{T}^d} \psi_j (x, \tilde{\rho}^k) \nabla \tilde{\rho}^k \cdot F_2
$$

$$
- \int_0^t \int_{\mathbb{T}^d} \partial \xi \psi_j (x, \xi) \, d\tilde{\omega} - \int_0^t \int_{\mathbb{T}^d} \psi_j (x, \tilde{\rho}^k) \nabla \tilde{\rho}^k \cdot \tilde{\rho}^k V_{nk} \, dx
$$

Now, let us focus on the random variable $\tilde{Y}$. There exist $\tilde{\rho} \in L^1(\tilde{\Omega} ; L^1([0,T] \times \mathbb{T}^d))$, $\tilde{G} \in L^2(\tilde{\Omega} ; L^2([0,T] \times \mathbb{T}^d ; \mathbb{R}^d))$ and $(\tilde{M}^{\psi})_{j \in \mathbb{N}} \in L^1(\tilde{\Omega} ; C([0,T])^N)$ such that

$$
\tilde{Y} = (\tilde{\rho}, \tilde{G}, (\tilde{M}^{\psi})_{j \in \mathbb{N}}).
$$

Using the dominated convergence theorem, it follows that as $k \to \infty$,

$$
\tilde{\rho}^k \to \tilde{\rho} \quad \text{strongly in } L^1([0,T] ; L^1(\mathbb{T}^d)), \quad \tilde{\mathbb{P}} - a.s.. \quad (6.8)
$$

Furthermore, the entropy dissipation estimate shows that

$$
\nabla \sqrt{\tilde{\rho}^k} \to \nabla \sqrt{\tilde{\rho}} \quad \text{weakly in } L^2(\tilde{\Omega} ; L^2([0,T] \times \mathbb{T}^d ; \mathbb{R}^d)),
$$

(6.9) Consequently, we have that $\tilde{G} = \nabla \sqrt{\tilde{\rho}}$, $\tilde{\mathbb{P}}$-a.s..

Moreover, with the aid of Proposition 5.15 and by using similar method as [FG24, Theorem 5.25], we deduce that $\tilde{\mathbb{P}}$-almost surely for every $j \in \mathbb{N}$ and $t \in [0,T]$,

$$
\tilde{M}^{\psi}_j = \int_0^t \int_{\mathbb{T}^d} \psi_j (x, \tilde{\rho}) \nabla \cdot (\sqrt{\tilde{\rho}} \tilde{W}^F)
$$

where $\tilde{W}^F$ is defined analogously to (2.1) by the Brownian motion $\tilde{\beta}$ on $\tilde{\Omega}$. Hence, for $\tilde{M}^{k,\psi}$ in (6.7), it follows that $\tilde{\mathbb{P}}$-almost surely for any $t \in [0,T]$, $j \in \mathbb{N}$,

$$
\tilde{M}^{k,\psi} \to \tilde{M}^{\psi}_j, \quad \text{as } k \to \infty. \quad (6.11)
$$
Step 2. Existence of a limiting kinetic measure. For any $M > 0$, set
\[
\theta_M(\xi) := I_{[0,M]}(\xi), \quad \Theta_M(\xi) := \int_0^\xi \int_0^r \theta_M(s) \, ds \, dr.
\]
With the aid of an approximation argument, we apply Itô formula to $\Theta_M(\rho^{n_k,\gamma_k}(x,T))$. According to Definition 5.9 and the definition of $q^{n_k,\gamma_k}$, it follows almost surely that for every $M > 0$ and $k \in \mathbb{N},$
\[
\begin{align*}
\int_0^T \int_{\mathbb{T}^d} \theta_M(\xi) dq^{n_k,\gamma_k} &= \int_0^T \Theta_M(\rho^{n_k}) - \int_0^T \Theta_M(\rho^{n_k,\gamma_k}(x,T)) \\
- \int_0^T \int_{\mathbb{T}^d} \Theta_M'(\rho^{n_k,\gamma_k}) \nabla \cdot (\rho^{n_k,\gamma_k} V_k \ast \rho^{n_k,\gamma_k}) + \int_0^T \int_{\mathbb{T}^d} \Theta_M(\rho^{n_k,\gamma_k}) \sigma_{n_k}(\rho^{n_k,\gamma_k}) \nabla \rho^{n_k,\gamma_k} \cdot dW^F \\
&+ \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \Theta_M(\rho^{n_k,\gamma_k}) \big( \sigma_{n_k}(\rho^{n_k,\gamma_k}) \sigma_{n_k}'(\rho^{n_k,\gamma_k}) \nabla \rho^{n_k,\gamma_k} \cdot F_2 + F_3 \sigma_{n_k}^2(\rho^{n_k,\gamma_k}) \big).
\end{align*}
\]
(6.12)
Since $\tilde{q}^k$ has the same law as $q^{n_k,\gamma_k}$ on $\mathcal{M}(\mathbb{T}^d \times \mathbb{R} \times [0,T])$ and $\rho^{n_k,\gamma_k}$ has the same law as $\tilde{\rho}^k$, we deduce that
\[
\begin{align*}
\mathbb{E} \left| \int_0^T \int_{\mathbb{T}^d} \theta_M(\xi) dq^{\tilde{q}^k} \right|^2 &\leq \mathbb{E} \left| \int_0^T \Theta_M(\rho^{n_k}) \right|^2 + \mathbb{E} \left| \int_0^T \Theta_M(\tilde{\rho}^k(x,T)) \right|^2 \\
&+ \mathbb{E} \left| \int_0^T \int_{\mathbb{T}^d} \Theta_M'(\tilde{\rho}^k) \sigma_{n_k}(\tilde{\rho}^k) \nabla \tilde{\rho}^k \cdot dW^F \right|^2 + \mathbb{E} \left| \int_0^T \int_{\mathbb{T}^d} \Theta_M'(\tilde{\rho}^k) \nabla \left( \tilde{\rho}^k V_k \ast \tilde{\rho}^k \right) \right|^2 \\
&+ \mathbb{E} \left| \int_0^T \int_{\mathbb{T}^d} \Theta_M(\tilde{\rho}^k) \left( \sigma_{n_k}(\tilde{\rho}^k) \sigma_{n_k}'(\tilde{\rho}^k) \nabla \tilde{\rho}^k \cdot F_2 + F_3 \sigma_{n_k}^2(\tilde{\rho}^k) \right) \right|^2.
\end{align*}
\]
(6.13)
According to the property $\Theta_M(\xi) \leq M(|\xi|)$, it follows that
\[
\mathbb{E} \left| \int_{\mathbb{T}^d} \Theta_M(\rho^{n_k}) \right|^2 + \mathbb{E} \left| \int_{\mathbb{T}^d} \Theta_M(\tilde{\rho}^k(x,T)) \right|^2 \leq C(M) \| \tilde{\rho} \|^2_{L^1(\mathbb{T}^d)}.
\]
For the martingale term on the righthand side of (6.13), by using Itô isometry, Assumption (A1) and Proposition 5.4, we deduce that there exists a constant $c \in (0,\infty)$ independent of $k$ such that
\[
\begin{align*}
\mathbb{E} \left| \int_0^T \int_{\mathbb{T}^d} \theta_M(\tilde{\rho}^k) \sigma_{n_k}(\tilde{\rho}^k) \nabla \tilde{\rho}^k \cdot dW^F \right|^2 &\leq \mathbb{E} \left| \int_0^T \int_{\mathbb{T}^d} F_1(x) I_{\{0 \leq \tilde{\rho}^k \leq M\}} |\sigma_{n_k}(\tilde{\rho}^k)|^2 |\nabla \tilde{\rho}^k|^2 \right|^2 \\
&\leq c \mathbb{E} \left| \int_0^T \int_{\mathbb{T}^d} F_1(x) I_{\{0 \leq \tilde{\rho}^k \leq M\}} (\tilde{\rho}^k)^2 |\nabla \sqrt{\tilde{\rho}^k}|^2 \right|^2 \\
&\leq c M^2 \mathbb{E} \left| \int_0^T \int_{\mathbb{T}^d} |\nabla \sqrt{\tilde{\rho}^k}|^2 \right|^2 \\
&\leq c(M,T,d,\|\tilde{\rho}\|_{L^1(\mathbb{T}^d)},\|V\|_{L^p([0,T];L^p(\mathbb{T}^d;\mathbb{R}))}).
\end{align*}
\]
Note that
\[
I_{\{0 \leq \tilde{\rho}^k \leq M\}} \sigma_{n_k}(\tilde{\rho}^k) \sigma_{n_k}'(\tilde{\rho}^k) \nabla \tilde{\rho}^k = \frac{1}{2} \nabla (\sigma_{n_k}^2(\tilde{\rho}^k \wedge M)),
\]
by integration by parts formula, we deduce that
\[
\int_0^T \int_{\mathbb{T}^d} \theta_M(\tilde{\rho}^k) \sigma_{n_k}(\tilde{\rho}^k) \sigma_{n_k}'(\tilde{\rho}^k) \nabla \tilde{\rho}^k \cdot F_2 = - \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \sigma_{n_k}^2(\tilde{\rho}^k \wedge M) \nabla \cdot F_2 = 0.
\]
Moreover, there exists a constant $c \in (0,\infty)$ independent of $k$ such that
\[
\int_0^T \int_{\mathbb{T}^d} \theta_M(\tilde{\rho}^k) F_3 \sigma_{n_k}^2(\tilde{\rho}^k) \leq c \int_0^T \int_{\mathbb{T}^d} I_{\{0 \leq \tilde{\rho}^k \leq M\}} F_3 \tilde{\rho}^k \leq c(M,T).
It remains to make estimate of the kernel term. By integration by parts formula, Hölder’s and convolution Young’s inequalities, and Proposition 5.4, we deduce that there exists a constant $c \in (0, \infty)$ independent of $k$ such that
\[
\mathbb{E} \left[ \int_0^T \int_{\mathbb{T}^d} \Omega_M(\tilde{\rho}^k) \nabla \cdot (\tilde{\rho}^k V_\gamma \ast \tilde{\rho}^k) \right]^2 \leq \mathbb{E} \left[ \int_0^T \int_{\mathbb{T}^d} I_{\{0 \leq \tilde{\rho}^k \leq M\}} \nabla \tilde{\rho}^k \cdot (\tilde{\rho}^k V_\gamma \ast \tilde{\rho}^k) \right]^2
\leq 4M^3 \mathbb{E} \left( \left\| \nabla \sqrt{\tilde{\rho}^k} \cdot V_\gamma \ast \tilde{\rho}^k \right\|_{L^1(T^d)} \right)^2
\leq 4M^3 \left\| \tilde{\rho}^k \right\|_{L^1(T^d)}^2 \left\| V \right\|_{L^2([0, T]; L^2(\mathbb{T}^d, \mathbb{R}^d))}^2 \int_0^T \left\| \nabla \sqrt{\tilde{\rho}^k} \right\|_{L^2(T^d)}^2
\leq c(M, T, d, \left\| \tilde{\rho} \right\|_{L^1(T^d)}, \left\| V \right\|_{L^p([0, T]; L^p(\mathbb{T}^d, \mathbb{R}^d))}).
\]
Combining all the above estimates, we conclude that there exists a constant $\Lambda$ such that
\[
\sup_{k \geq 1} \mathbb{E}(q^k([0, T] \times \mathbb{T}^d \times [0, M]))^2 \leq \Lambda(M, T, d, \left\| \tilde{\rho} \right\|_{L^1(T^d)}, \left\| V \right\|_{L^p([0, T]; L^p(\mathbb{T}^d, \mathbb{R}^d))}).
\]
(6.14)

For every $r \in \mathbb{N}$, define $K_r := \mathbb{T}^d \times [0, T] \times [0, r]$. Let $\mathcal{M}_r$ be the space of bounded Borel measures over $K_r$ (with norm given by the total variation of measures). Clearly, $\mathcal{M}_r$ is the topological dual of $C(K_r)$, which is the set of continuous functions on $K_r$. By (6.14), the sequence $\{\tilde{q}^k = \delta_0(-\tilde{\rho}^k) \nabla \tilde{\rho}^k \}_{k \geq 1}$ is uniformly bounded in $L^2(\Omega; \mathcal{M}_r)$. By the Banach-Alaoglu theorem, there exists $\tilde{q}_r \in L^2(\Omega; \mathcal{M}_r)$ and a subsequence still denoted by $\{\tilde{q}^k\}_{k \in \mathbb{N}}$ such that $\tilde{q}^k \rightharpoonup \tilde{q}_r$ in $L^2(\Omega; \mathcal{M}_r)$ weak star as $k \to \infty$. By a diagonal process, we extract a subsequence (not relabeled) and a Radon measure $\tilde{q}$ on $\mathbb{T}^d \times [0, T] \times [0, \infty)$ such that $\tilde{q}^k \rightarrow \tilde{q}$ weak star in $L^2(\Omega; \mathcal{M}_r)$ as $k \to \infty$ for every $r \in \mathbb{N}$.

The limiting measure $\tilde{q}$ is a kinetic measure in the sense of Definition 2.2, since the predictable property is stable with respect to weak limits. In addition, we claim that $\tilde{q}$ fulfills (2.9). That is,
\[
\tilde{q}(t, x, \xi) \geq 4\delta_0(\xi - \tilde{\rho}(t, x, \xi)) \tilde{\rho}(t, x, \xi) \sqrt{\tilde{\rho}(t, x, \xi)} \tilde{\Phi} - \text{a.s.,}
\]
(6.15)
Indeed, we can choose a subsequence (still denoted by $\tilde{\rho}^k$) such that for every $A \in \mathcal{F}$ and for all nonnegative $\phi \in C_c^\infty (\mathbb{T}^d \times [0, T] \times [0, \infty))$
\[
\nabla \sqrt{\tilde{\rho}^k} \sqrt{\phi(x, t, \tilde{\rho}^k)} I_A \rightharpoonup \nabla \sqrt{\tilde{\rho}} \sqrt{\phi(x, t, \tilde{\rho})} I_A,
\]
weakly in $L^2(\Omega \times [0, T]; L^2(\mathbb{T}^d))$, as $k \to \infty$. By the lower semi-continuity of the Sobolev norm, we have
\[
4\mathbb{E} \left( \int_0^T \int_{\mathbb{T}^d} |\nabla \sqrt{\tilde{\rho}^k}|^2 \Phi(x, t, \tilde{\rho}) I_A \right) \leq 4 \liminf_{k \to \infty} 4\mathbb{E} \left( \int_0^T \int_{\mathbb{T}^d} |\nabla \sqrt{\tilde{\rho}^k}|^2 \Phi(x, t, \tilde{\rho}) I_A \right) = \mathbb{E} \left( \tilde{q}(\phi) I_A \right)
\]
Step 3. The entropy estimate. Since $\tilde{\rho}^k$ satisfies the entropy estimates (5.6) uniformly on $k$, by (6.9) and the weak lower semi-continuity of the Sobolev norm, we deduce that
\[
\mathbb{E} \left[ \int_0^T \int_{\mathbb{T}^d} |\nabla \sqrt{\tilde{\rho}}|^2 \right] \leq \int_{\mathbb{T}^d} \Psi(\tilde{\rho}) + 1 + c(T, d, \left\| \tilde{\rho} \right\|_{L^1(T^d)}, \left\| V \right\|_{L^p([0, T]; L^p(\mathbb{T}^d, \mathbb{R}^d))}),
\]
where we have used $\min_x \Psi(\xi) = \Psi(1) = -1$. Thus, $\tilde{\rho}$ satisfies the condition (2.8) in Definition 2.3.

Step 4. Passing to the limits. From now on, we aim to show that $(\tilde{\rho}, \tilde{q}, \tilde{\beta})$ is a kinetic solution to (6.1) in the sense of Definition 2.3. To facilitate the proof of convergence, we denote $\sigma(\xi) = \sqrt{\xi}$ for every $\xi \in [0, \infty)$. Clearly, (1) and (2) in Definition 2.3 hold. To achieve the result, we need to verify that for every $j \in \mathbb{N}$ and $t \in [0, T]$, the kinetic function $\tilde{\chi}$ of $\tilde{\rho}$ satisfies
\[
\int_0^t \int_{\mathbb{T}^d} \psi_j(x, \tilde{\rho}) \nabla \cdot (\sigma(\tilde{\rho}) \bar{d}W^j) = -\int_0^t \int_{\mathbb{T}^d} \tilde{\chi}(x, \xi, r) \psi_j(x, \xi) \bigg|_{t=r} - \int_0^t \int_{\mathbb{T}^d} \nabla \tilde{\rho} \cdot (\nabla \psi_j) (x, \tilde{\rho})
- \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} F_1(\sigma'(\tilde{\rho})^2 \nabla \tilde{\rho} \cdot (\nabla \psi_j) (x, \tilde{\rho}) - \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} \sigma(\tilde{\rho}) \sigma'(\tilde{\rho}) F_2 \cdot (\nabla \psi_j) (x, \tilde{\rho})
\]
\[
\begin{align*}
&+ \frac{1}{2} \int_0^T \int_{T_t} \left( \partial_t \psi_j \right) (x, \hat{\rho}) \sigma (\hat{\rho}) \sigma' (\hat{\rho}) \nabla \hat{\rho} \cdot F_2 + \frac{1}{2} \int_0^T \int_{T_t} F_3 \sigma^2 (\hat{\rho}) \left( \partial_t \psi_j \right) (x, \hat{\rho}) \\
&- \int_0^T \int_{T_t} \partial_t \psi_j (x, \xi)d\xi - \int_0^T \int_{T_t} \psi_j (x, \hat{\rho}) \nabla \cdot \left( \hat{\rho} V (r) * \hat{\rho} \right).
\end{align*}
\]  

(6.16)

By the definition of $\tilde{\chi}^k$, (6.8) and the dominated convergence theorem, for every $j \in \mathbb{N}$ and $r \in [0, T]$, 
\[
\lim_{k \to \infty} \tilde{\mathbb{E}} \left[ \int_{T_t} \tilde{\chi}^k (x, \xi) \psi_j (x, \xi) - \int_{T_t} \tilde{\chi} (x, \xi, r) \psi_j (x, \xi) \right] = 0.
\]

Since $\psi_j \in C_c^\infty \left( \mathbb{T}^d \times (0, \infty) \right)$, we can choose $0 < \delta < M < \infty$ such that $[\delta, M]$ is the compact support of $\psi_j$ with respect to $\xi$. Then, by Lemma 3.3, (6.9), Proposition 5.4, the boundedness of $F_1$ and (5.2), it follows that for every $j \in \mathbb{N}$ and $t \in [0, T], 
\[
\lim_{k \to \infty} \tilde{\mathbb{E}} \int_0^t \int_{T_t} F_1 \left[ \sigma_n \left( \hat{\rho}^k \right) \right] \nabla \hat{\rho}^k \cdot \left( \nabla \psi_j \right) (x, \hat{\rho}^k) = \tilde{\mathbb{E}} \int_0^t \int_{T_t} F_1 \left[ \sigma' \left( \hat{\rho} \right) \right] \nabla \hat{\rho} \cdot \left( \nabla \psi_j \right) (x, \hat{\rho}).
\]

Since for every $r \in \mathbb{N}$, $\hat{q}^k \to \tilde{q}$ in $L^2 (\Omega; M_r)$–weak star as $k \to \infty$, it follows from the property $\{ \psi_j \}_{j \in \mathbb{N}} \subset C_c^\infty \left( \mathbb{T}^d \times (0, \infty) \right)$ that for every $j \in \mathbb{N}$ and $t \in [0, T], 
\[
\lim_{k \to \infty} \tilde{\mathbb{E}} \int_0^t \int_{T_t} \partial_t \psi_j (x, \xi)d\tilde{q}^k = \tilde{\mathbb{E}} \int_0^t \int_{T_t} \partial_t \psi_j (x, \xi)d\tilde{q}.
\]

Similar to the above, by the boundedness of $F_2$ and $F_3$, (5.2), Lemma 5.1, (6.8) and (6.9), we get the other terms of (6.16) except the last kernel term. For the kernel term, by integration by parts formula, we have 
\[
\tilde{\mathbb{E}} \left[ \int_0^t \int_{T_t} \psi_j (x, \hat{\rho}^k) \nabla \cdot \left( \hat{\rho}^k V_{\gamma_k} (r) * \hat{\rho}^k \right) - \psi_j (x, \hat{\rho}) \nabla \cdot \left( \hat{\rho} V (r) * \hat{\rho} \right) \right] \leq b_1^k + b_2^k,
\]

(6.17)

where 
\[
b_1^k = \tilde{\mathbb{E}} \int_0^t \int_{T_t} \left( \partial_t \psi_j \right) (x, \hat{\rho}^k) \nabla \hat{\rho}^k \cdot \hat{\rho}^k V_{\gamma_k} (r) * \hat{\rho}^k - \left( \partial_t \psi_j \right) (x, \hat{\rho}) \nabla \hat{\rho} \cdot \hat{\rho} V (r) * \hat{\rho},
\]

\[
b_2^k = \tilde{\mathbb{E}} \int_0^t \int_{T_t} \left( \nabla \psi_j \right) (x, \hat{\rho}^k) \cdot \hat{\rho}^k V_{\gamma_k} (r) * \hat{\rho}^k - \left( \nabla \psi_j \right) (x, \hat{\rho}) \cdot \hat{\rho} V (r) * \hat{\rho}.
\]

We firstly proceed with the term $b_1^k$. By Lemma 3.3, we have 
\[
b_1^k \leq b_{11}^k + b_{12}^k + b_{13}^k,
\]

(6.18)

where 
\[
b_{11}^k := 2 \tilde{\mathbb{E}} \left[ \int_0^t \int_{T_t} \left| \partial_t \psi_j \right| (x, \hat{\rho}^k) \left( \hat{\rho}^k \right)^{3/2} \left( V_{\gamma_k} (r) - V (r) \right) * \hat{\rho}^k \cdot \nabla \sqrt{\hat{\rho}^k} \right],
\]

\[
b_{12}^k := 2 \tilde{\mathbb{E}} \left[ \int_0^t \int_{T_t} \left( \partial_t \psi_j \right) (x, \hat{\rho}^k) \left( \hat{\rho}^k \right)^{3/2} V (r) * \hat{\rho}^k - \left( \partial_t \psi_j \right) (x, \hat{\rho}) \left( \hat{\rho} \right)^{3/2} V (r) * \hat{\rho} \right] \cdot \nabla \sqrt{\hat{\rho}^k},
\]

\[
b_{13}^k := 2 \tilde{\mathbb{E}} \left[ \int_0^t \int_{T_t} \left( \partial_t \psi_j \right) (x, \hat{\rho}) \left( \hat{\rho} \right)^{3/2} V (r) * \hat{\rho} \cdot \left( \nabla \sqrt{\hat{\rho}^k} - \nabla \sqrt{\hat{\rho}} \right) \right].
\]

Recall that $[\delta, M]$ is the compact support of $\psi_j$ with respect to $\xi$. For the term $b_{11}^k$, it follows from Hölder’s inequality, Proposition 5.4, the property of $\psi_j$ and convolution Young’s inequality that there exists a constant $c \in (0, \infty)$ independent of $k$ such that 
\[
b_{11}^k \leq 2 \left( \tilde{\mathbb{E}} \int_0^t \int_{T_t} \left| \partial_t \psi_j \right| (x, \hat{\rho}^k) \left( \hat{\rho}^k \right)^{3/2} \left( V_{\gamma_k} (r) - V (r) \right) * \hat{\rho}^k \right)^{\frac{1}{2}} \left( \tilde{\mathbb{E}} \int_0^t \int_{T_t} \left| \nabla \sqrt{\hat{\rho}^k} \right| ^2 \right)^{\frac{1}{2}} \leq c \left( \tilde{\mathbb{E}} \int_0^t \int_{T_t} \left| \partial_t \psi_j \right| (x, \hat{\rho}^k) \left( \hat{\rho}^k \right)^{3/2} (V_{\gamma_k} (r) - V (r)) * \hat{\rho}^k \right)^{\frac{1}{2}} \left( \tilde{\mathbb{E}} \int_0^t \int_{T_t} \left| \nabla \sqrt{\hat{\rho}^k} \right| ^2 \right)^{\frac{1}{2}}.
\]
\[c M^\frac{k}{2} \| \partial_t \psi_j \|_{L^\infty(\mathbb{R})} \| \hat{\rho} \|_{L^1(T^d)} \| V \|_{L^2(0,T;L^2(\mathbb{T}^d;\mathbb{R}^d))} \to 0, \quad \text{as } k \to \infty. \quad (6.19)\]

According to the properties of $V$ and $\{\psi_j\}_{j \in \mathbb{N}}$, by (6.9), it holds that for every $j \in \mathbb{N}$ and $t \in [0,T]$,

\[\lim_{k \to \infty} b_{13}^k = 0. \quad (6.20)\]

For the term $b_{12}^k$, it follows from Hölder’s inequality and Proposition 5.4 that there exists a constant $c \in (0,\infty)$ independent of $k$ such that

\[b_{12}^k \leq 2 \left( \mathbb{E} \int_0^T \int_{T^d} \left| (\partial_t \psi_j)(x,\hat{\rho}^k)(\hat{\rho}^k)^{3/2} V(r) * \hat{\rho}^k - (\partial_t \psi_j)(x,\tilde{\rho})(\tilde{\rho})^{3/2} V(r) * \tilde{\rho} \right|^2 \right)^{\frac{1}{2}} \left( \mathbb{E} \int_0^T \int_{T^d} |\nabla \sqrt{\hat{\rho}^k}|^2 \right)^{\frac{1}{2}} \]

\[\leq c \left( \mathbb{E} \int_0^T \int_{T^d} \left| (\partial_t \psi_j)(x,\hat{\rho}^k)(\hat{\rho}^k)^{3/2} V(r) * \hat{\rho}^k - (\partial_t \psi_j)(x,\tilde{\rho})(\tilde{\rho})^{3/2} V(r) * \tilde{\rho} \right|^2 \right)^{\frac{1}{2}} \]

\[\leq (b_{121}^k + b_{122}^k)^{\frac{1}{2}}, \quad (6.21)\]

\[\text{where} \quad b_{121}^k := 2 \mathbb{E} \int_0^T \int_{T^d} \left| (\partial_t \psi_j)(x,\hat{\rho}^k)(\hat{\rho}^k)^{3/2} V(r) * (\hat{\rho}^k - \tilde{\rho}) \right|^2, \]

\[b_{122}^k := 2 \mathbb{E} \int_0^T \int_{T^d} \left| \left( (\partial_t \psi_j)(x,\hat{\rho}^k)(\hat{\rho}^k)^{3/2} - (\partial_t \psi_j)(x,\tilde{\rho})(\tilde{\rho})^{3/2} \right) V(r) * \tilde{\rho} \right|^2. \]

For the term $b_{121}^k$, by using Hölder’s inequality and convolution Young’s inequality, we deduce that

\[b_{121}^k \leq 2M^3 \| \partial_t \psi_j \|_{L^\infty(\mathbb{R})}^2 \mathbb{E} \int_0^T \| V(r) * (\hat{\rho}^k - \tilde{\rho}) \|_{L^2(\mathbb{T}^d)}^2 \]

\[\leq 2M^3 \| \partial_t \psi_j \|_{L^\infty(\mathbb{R})}^2 \mathbb{E} \int_0^T \| V(r) \|_{L^2(\mathbb{T}^d;\mathbb{R}^d)}^2 \| \hat{\rho}^k - \tilde{\rho} \|_{L^1(\mathbb{T}^d)}^2 \]

\[\leq 2M^3 \| \partial_t \psi_j \|_{L^\infty(\mathbb{R})}^2 \left( \int_0^T \| V(r) \|_{L^2(\mathbb{T}^d;\mathbb{R}^d)}^2 \right)^{\frac{1}{2}} \left( \mathbb{E} \int_0^T \| \hat{\rho}^k - \tilde{\rho} \|_{L^1(\mathbb{T}^d)}^2 \right)^{\frac{p}{2p-2}}. \]

Since $\| \hat{\rho}^k - \tilde{\rho} \|_{L^1(\mathbb{T}^d)} \leq 2 \| \hat{\rho} \|_{L^1(\mathbb{T}^d)}$, by the dominated convergence theorem and (6.8), we have

\[\lim_{k \to \infty} b_{121}^k = 0. \quad (6.22)\]

Note that

\[\left| \left( (\partial_t \psi_j)(x,\hat{\rho}^k)(\hat{\rho}^k)^{3/2} - (\partial_t \psi_j)(x,\tilde{\rho})(\tilde{\rho})^{3/2} \right) V * \tilde{\rho} \right|^2 \leq 4M^3 \| \partial_t \psi_j \|_{L^\infty(\mathbb{R})}^2 \| V * \tilde{\rho} \|^2, \]

and

\[\mathbb{E} \int_0^T \int_{T^d} | V(r) * \tilde{\rho} |^2 \leq \| \tilde{\rho} \|^2 \| V \|^2 \int_0^T \| V(r) \|_{L^2(\mathbb{T}^d;\mathbb{R}^d)}^2 < \infty. \]

By (6.8) and applying the dominated convergence theorem, for every $j \in \mathbb{N}$ and $t \in [0,T]$, it yields that up to a subsequence

\[\lim_{k \to \infty} b_{122}^k = 0. \quad (6.23)\]

Based on (6.21)-(6.23), we deduce that for every $j \in \mathbb{N}$ and $t \in [0,T]$,

\[\lim_{k \to \infty} b_{12}^k = 0. \quad (6.24)\]

Combining (6.18), (6.19), (6.20) and (6.24), we get that for every $j \in \mathbb{N}$ and $t \in [0,T]$,

\[\lim_{k \to \infty} b_k^j \leq \lim_{k \to \infty} (b_{11}^k + b_{12}^k + b_{13}^k) = 0. \quad (6.25)\]
It remains to handle the term $b_2^k$. Clearly, we have

$$b_2^k \leq \mathbb{E} \left| \int_0^t \int_{\mathbb{T}^d} (\nabla \psi_j)(x, \hat{\rho}^k) \cdot (\hat{\rho}^k (V_{\gamma_k}(r) - V(r)) * \hat{\rho}^k) \right| + \mathbb{E} \left| \int_0^t \int_{\mathbb{T}^d} (\nabla \psi_j)(x, \hat{\rho}^k) \cdot \hat{\rho}^k V(r) * (\hat{\rho}^k - \hat{\rho}) \right|$$

$$+ \mathbb{E} \left| \int_0^t \int_{\mathbb{T}^d} ((\nabla \psi_j)(x, \hat{\rho}^k) - (\nabla \psi_j)(x, \hat{\rho})) \cdot V(r) * \hat{\rho} \right|$$

$$= b_{21}^k + b_{22}^k + b_{23}^k. \quad (6.26)$$

For the term $b_{21}^k$, by Hölder’s inequality, convolution Young’s inequality and $\lim_{k \to \infty} \|V_{\gamma_k} - V\|_{L^1([0, T]; L^p(\mathbb{T}^d; \mathbb{R}))} = 0$, for every $j \in \mathbb{N}$ and $t \in [0, T]$, as $k \to \infty$,

$$b_{21}^k \leq M \|\nabla \psi_j\|_{L^{\infty}(\mathbb{R})} \|\hat{\rho}\|_{L^1(\mathbb{T}^d)} \|V_{\gamma_k} - V\|_{L^1([0, T]; L^p(\mathbb{T}^d; \mathbb{R}))} \to 0. \quad (6.27)$$

For the term $b_{22}^k$, in view of the property of $\psi_j$, Hölder’s inequality and (6.8), as $k \to \infty$, we get

$$b_{22}^k \leq \|\nabla \psi_j\|_{L^{\infty}(\mathbb{R})} M \mathbb{E} \int_0^T \|V(r) * (\hat{\rho}^k - \hat{\rho})\|_{L^p(\mathbb{T}^d)}$$

$$\leq \|\nabla \psi_j\|_{L^{\infty}(\mathbb{R})} M \left( \int_0^T \|V(r)\|_{L^p(\mathbb{T}^d; \mathbb{R})} \right)^{\frac{p}{p-1}} \left( \mathbb{E} \int_0^T \|\hat{\rho}^k - \hat{\rho}\|_{L^1(\mathbb{T}^d)} \right)^{\frac{p-1}{p}} \to 0. \quad (6.28)$$

For the last term $b_{23}^k$, by Hölder’s and convolution Young’s inequalities, as $k \to \infty$, it holds that

$$\left| (\nabla \psi_j)(x, \hat{\rho}^k) \cdot (\nabla \psi_j)(x, \hat{\rho}^k) \cdot V(r) * \hat{\rho} \right| \leq 2 \|\nabla \psi_j\|_{L^{\infty}(\mathbb{R})} M |V * \hat{\rho}|.$$ 

Applying the dominated convergence theorem, for every $j \in \mathbb{N}$ and $t \in [0, T]$, we have that up to a subsequence

$$\lim_{k \to \infty} b_{23}^k = 0. \quad (6.29)$$

Based on (6.26)-(6.29), we deduce that for every $j \in \mathbb{N}$ and $t \in [0, T]$,

$$\lim_{k \to \infty} b_2^k = 0. \quad (6.30)$$

According to (6.17), (6.25) and (6.30), we have that for every $j \in \mathbb{N}$ and $t \in [0, T]$,

$$\lim_{k \to \infty} \left| \mathbb{E} \left[ \int_0^t \int_{\mathbb{T}^d} \psi_j(x, \hat{\rho}^k) \nabla \cdot (\hat{\rho}^k V(r) * \hat{\rho}^k) - \psi_j(x, \hat{\rho}) \nabla \cdot (\hat{\rho} V(r) * \hat{\rho}) \right] \right| = 0.$$ 

Taking $k \to \infty$ on both sides of (6.7), with the aid of the above estimates and (6.11), we conclude that for every $t \in [0, T]$ and $j \in \mathbb{N}$, the kinetic function $\tilde{\chi}$ of $\hat{\rho}$ satisfies (6.16).

**Step 5. Properties of $\tilde{q}$**. We will prove that $\tilde{q}$ satisfies (2.10) in Definition 2.3. For any $M > 0$, set

$$\tilde{\theta}_M(\xi) := I_{[M, M+1]}(\xi), \quad \tilde{\Theta}_M(\xi) := \int_0^\xi \int_0^r \tilde{\theta}_M(s)dsdr.$$

By a smooth approximation, we can apply Itô formula to $\tilde{\Theta}_M(\rho^{n_k, \gamma_k})$. Since $\tilde{q}^k$ has the same law as $q^{n_k, \gamma_k}$ on $\mathcal{M}(\mathbb{T}^d \times \mathbb{R} \times [0, T])$ and $\hat{\rho}^k$ has the same law as $\rho^{n_k, \gamma_k}$, it gives that for every $k \in \mathbb{N}$ and $M > 0$,

$$\mathbb{E} \left[ \int_0^T \int_{\mathbb{T}^d} \tilde{\theta}_M(\xi) d\tilde{q}^k \right] = \mathbb{E} \left[ \int_0^T \int_{\mathbb{T}^d} \tilde{\chi}(\hat{\rho}^{n_k}) \tilde{\Theta}_M(\xi) \right] - \mathbb{E} \int_0^T \int_{\mathbb{T}^d} \tilde{\chi}(\hat{\rho}^k(x, \xi, T) \tilde{\Theta}_M(\xi))$$

$$+ \mathbb{E} \int_0^T \int_{\mathbb{T}^d} \tilde{\Theta}_M(\hat{\rho}^k) (\hat{\rho}^k V_{\gamma_k} * \hat{\rho}^k) \cdot \nabla \hat{\rho}^k$$

$$+ \frac{1}{2} \mathbb{E} \int_0^T \int_{\mathbb{T}^d} \tilde{\Theta}_M(\hat{\rho}^k) (\sigma_{n_k}(\hat{\rho}^k) \sigma_{n_k}'(\hat{\rho}^k) \nabla \hat{\rho}^k \cdot F_2 + F_3 \sigma_{n_k}^2(\hat{\rho}^k)). \quad (6.31)$$
It follows from integration by parts formula that for every $k \in \mathbb{N}$ and $M > 0$,
\[ \mathbb{E} \int_0^T \int_{\mathbb{T}^d} \hat{\Theta}_M(\hat{\rho}^k) \sigma_{n_k}(\hat{\rho}^k) \sigma'_{n_k}(\hat{\rho}^k) \nabla \hat{\rho}^k \cdot F_2 = 0. \]

Taking $k \to \infty$ on both sides of (6.31), by a similar method to **Step 4**, we get that for every $M > 0$,
\[ \mathbb{E} \int_0^T \int_{\mathbb{R}^{\mathbb{T}^d}} \hat{\Theta}_M(\xi) d\hat{q} \]
\[ \leq \mathbb{E} \int_{\mathbb{T}^d} \hat{\chi}(\hat{\rho}) \hat{\Theta}_M(\xi) + \mathbb{E} \int_0^T \int_{\mathbb{T}^d} \hat{\Theta}_M(\hat{\rho})(\hat{\rho} V * \hat{\rho}) \cdot \nabla \hat{\rho} + \frac{1}{2} \mathbb{E} \int_0^T \int_{\mathbb{T}^d} \hat{\Theta}_M(\hat{\rho}) F_3 \hat{\rho} \]
\[ \leq \mathbb{E} \int_{\mathbb{T}^d} \mathbb{I}_{[M,M+1]}(\hat{\rho}) + \mathbb{E} \int_0^T \int_{\mathbb{T}^d} \mathbb{I}_{[M,M+1]}(\hat{\rho}(T)) + 2(M+1)^{3/2} \mathbb{E} \int_0^T \int_{\mathbb{T}^d} \mathbb{I}_{[M,M+1]}(\hat{\rho}) \nabla \sqrt{\hat{\rho}} \cdot V(r) * \hat{\rho} \]
\[ + \frac{M+1}{2} \mathbb{E} \int_0^T \int_{\mathbb{T}^d} F_3 \mathbb{I}_{[M,M+1]}(\hat{\rho}). \]

By convolution Young’s inequality and Proposition 5.4, it gives that $\nabla \sqrt{\hat{\rho}} \cdot V * \hat{\rho}$ is integrable in $L^1(\Omega \times [0,T] \times \mathbb{T}^d)$. Then, by [FG23, Lemma 7], it follows that
\[ \liminf_{M \to \infty} \mathbb{E} \hat{q}(0,T] \times \mathbb{T}^d \times [M,M+1] = 0. \] (6.32)

**Step 6. $L^1(\mathbb{T}^d)$ – continuity in time.** We will prove that the measure $\hat{q}$ has no atoms in time, and that the function $\hat{\rho}$ $\mathbb{P}$-almost surely admits a representative taking values in $C(\{0,T\}; L^1(\mathbb{T}^d))$. The proof process is quite similar to [FG24, Theorem 5.25], thus we omit it. In a conclusion, $\hat{\rho}$ has a representative taking values in $C(\{0,T\}; L^1(\mathbb{T}^d))$, which is a stochastic kinetic solution of (6.1) in the sense of Definition 2.3 with respect to $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t \in [0,T]}, \mathbb{P})$ and $\hat{W}^F$.

By further assuming Assumption (A2) holds, we are ready to establish the existence of the probabilistically strong solution.

**The proof of (ii).**

Revisiting equations (6.3), (6.4) and (6.5), thanks to the pathwise uniqueness obtained by Theorem 4.2, and adopting a methodology analogous to that in [FG24, Theorem 5.25, Conclusion], we find that the limiting joint distribution $\mu$ is supported on the diagonal, i.e., $\mu((x,y) \in \hat{X} \times \hat{X} : x = y) = 1$. Thus, the conditions of Lemma 6.1 are met. Thus, for the original solutions $\{\rho^{n,\gamma}\}_{n \in \mathbb{N}, \gamma \in (0,1)}$ on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$, after passing to a subsequence $\gamma_k \to 0, \gamma_k \to \infty$, there exists a random variable $\rho \in L^1(\Omega \times [0,T]; L^1(\mathbb{T}^d))$ such that $\{\rho^{n_k,\gamma_k}\}_{k \in \mathbb{N}}$ converges to $\rho$ in probability. A further subsequence still denoted by $\{\gamma_k, \gamma_k\}_{k \in \mathbb{N}}$ yields $\rho^{n_k,\gamma_k} \to \rho$ almost surely. A simplified variant of the previous argument confirms that $\rho$ satisfies the criteria of a renormalized kinetic solution of (6.1) in the sense of Definition 2.3 on $(\Omega, \mathcal{F}, \mathbb{P})$, thereby $\rho$ is a probabilistically strong solution to (6.1). The proof is completed with the assistance of Theorem 4.2. \( \square \)

7. **Well-posedness of the fluctuating Ising-Kac-Kawasaki equation**

In this section, we consider the fluctuating Ising-Kac-Kawasaki equation in any dimension $d \geq 1$, which is of the following form
\[ dp = \Delta \rho dt - \nabla \cdot [(1 - \rho^2) \nabla J \ast \rho] dt - \nabla \cdot (\sqrt{1 - \rho^2} \circ dW^F), \] (7.1)
where $W^F$ is defined by (2.1) and $J$ denotes the Kac potential satisfying $\nabla J \in C^\infty(\mathbb{T}^d; \mathbb{R}^d)$. As discussed in the Introduction part, (7.1) exhibits the same nonlinear fluctuations as Kawasaki dynamics for the Ising-Kac model. Thus, the well-posedness of (7.1) plays a fundamental role in studying nonlinear fluctuations of Kawasaki dynamics.

Due to the structural resemblance between (7.1) and the Dean-Kawasaki equation (2.2), we can employ a similar proof strategy as used in the previous sections to establish the well-posedness of the renormalized kinetic solution to (7.1). However, there are still some challenges that need to be
handled carefully, for instance, the lack of $L^1(\mathbb{T}^d)$-norm preservation and two singularities appear in the derivative of the diffusion coefficient $\sqrt{1 - \xi^2}$. Under this circumstance, to establish the tightness of the approximation equations for (7.1), the $L^1([0, T]; L^1(\mathbb{T}^d))$-equivalent topology introduced by [FG24, Definition 5.19] is inadequate. Instead, we employ the diagonal argument and utilize the boundness of the solutions to show the tightness on $L^2([0, T]; L^2(\mathbb{T}^d; [-1, 1]))$.

To maintain the clarity, we only present the proof of entropy estimate and the tightness (as seen in Theorem 6.2), which are two essential ingredients in proving the well-posedness of (7.1).

**Renormalized kinetic solution to (7.1).** The Stratonovich equation (7.1) is formally equivalent to the Itô equation

$$d\rho = \Delta \rho dt - \nabla \cdot (\sqrt{1 - \rho^2}) \nabla \rho dt - \nabla \cdot (\sqrt{1 - \rho^2} \, dW^F) + \frac{1}{2} \nabla \cdot (F_1 \frac{\rho^2}{1 - \rho^2} \nabla \rho - \rho F_2) dt. \quad (7.2)$$

For a given bounded solution $\rho$ of (7.2), we define the kinetic function $\chi : \mathbb{T}^d \times \mathbb{R} \times [0, T] \to \{-1, 1\}$ of $\rho$ as

$$\chi(x, \xi, t) := 1_{\{0 < \xi < \rho(x, t)\}} - 1_{\{\rho(x, t) < \xi < 0\}}.$$

Formally, the identities

$$\nabla \chi = 2\delta_0(\xi - \rho)\nabla \rho, \quad \partial_\xi \chi = 2\delta_0(\xi) - 2\delta_0(\xi - \rho) \quad \text{and} \quad \rho = \frac{1}{2} \int_{\mathbb{R}} \chi d\xi \quad (7.3)$$

hold. Then an informal application of Itô’s formula suggests that the kinetic function $\chi$ of $\rho$ satisfies the equation

$$\partial_t \chi = 2\nabla \cdot (\delta_0(\xi - \rho) \nabla \rho) + \nabla \cdot \left(\delta_0(\xi - \rho) \left(F_1 \frac{\xi^2}{1 - \xi^2} \nabla \rho - \xi F_2\right)\right) - 2\delta_0(\xi - \rho) \nabla \cdot (\sqrt{1 - \rho^2} \, dW^F)$$

$$+ 2\delta_0 q - \partial_\xi \left(\delta_0(\xi - \rho) \left(-\nabla \rho \cdot F_2 + (1 - \xi^2) F_3\right)\right) - 2\delta_0(\xi - \rho) \nabla \cdot (\sqrt{1 - \rho^2} \, dW^F), \quad (7.4)$$

where $q = \delta_0(\xi - \rho)|\nabla \rho|^2$ is the parabolic defect measure.

Similar to the Dean-Kawasaki equation, before defining a renormalized kinetic solution to (7.1), we have to specify the kinetic measure.

**Definition 7.1.** Let $(\Omega, \mathcal{F}^t, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)_{t \in [0, \infty)}$. A kinetic measure is a measurable map $q$ from $\Omega$ to the space of nonnegative, finite Radon measures on $\mathbb{T}^d \times [-1, 1] \times [0, T]$ that satisfies the property that the process

$$(\omega, t) \in \Omega \times [0, T] \rightarrow \int_0^t \int_{\mathbb{R}} \int_{\mathbb{T}^d} \psi(x, \xi) dq(x, \xi, r)$$

is $\mathcal{F}_t-$predictable, for every $\psi \in C^\infty_c(\mathbb{T}^d \times [-1, 1])$.

We will prove the well-posedness of (7.1) for initial data with finite mathematical entropy. Let

$$\psi(\xi) = \frac{1}{2} \log \frac{1 + \xi}{1 - \xi}, \quad (7.5)$$

and let $\Psi : [-1, 1] \to \mathbb{R}$ be a function defined by

$$\Psi(\xi) = \frac{1}{2} \left((\xi + 1) \log(\xi + 1) - (\xi + 1) + (1 - \xi) \log(1 - \xi) - (1 - \xi)\right). \quad (7.6)$$

A direct computation shows that $\Psi'(\xi) = \psi(\xi)$. Define

$$\text{Ent}(\mathbb{T}^d) = \left\{\rho : -1 \leq \rho \leq 1 \text{ a.e., and } \int_{\mathbb{T}^d} \Psi(\rho(x)) dx < \infty\right\}. \quad (7.7)$$

**Definition 7.2.** (Renormalized kinetic solution) Let $\rho_0 \in \text{Ent}(\mathbb{T}^d)$. A renormalized kinetic solution of (7.1) with initial datum $\rho(\cdot, 0) = \rho_0$ is an almost surely continuous $L^2(\mathbb{T}^d; [-1, 1])$-valued $\mathcal{F}_t$-predictable function $\rho \in L^2(\Omega \times [0, T]; L^2(\mathbb{T}^d; [-1, 1]))$ that satisfies the following properties.
(1) Essentially bounded: almost surely for every \( t \in [0, T] \),
\[
\rho(\cdot, t) \in [-1, 1], \text{ a.e.}
\] (7.8)

(2) Regularity of \( \sqrt{1 - \rho^2} \): there exists a constant \( c \in (0, \infty) \) depending on \( T, \rho_0, J \) and \( d \) such that
\[
E \int_0^T \int_{\mathbb{R}^d} \left[ |\nabla \sqrt{1 - \rho^2}|^2 + |\nabla \rho|^2 \right] dx ds \leq c(T, d, \rho_0, J).
\] (7.9)

Furthermore, there exists a finite nonnegative kinetic measure \( q \) satisfying the following properties.

(3) Regularity: almost surely
\[
\delta_0(\xi - \rho)|\nabla \rho|^2 \leq q \quad \text{on} \quad \mathbb{T}^d \times [-1, 1] \times [0, T].
\] (7.10)

(4) Optimal regularity: the measure \( \mu \) defined by
\[
d\mu = (1 - \xi^2)^{-1} dq \quad \text{is finite on} \quad \mathbb{T}^d \times (-1, 1) \times [0, T].
\] (7.11)

(5) The equation: for every \( \varphi \in C_c^\infty (\mathbb{T}^d \times (-1, 1)) \), almost surely for every \( t \in [0, T] \),
\[
\int_{\mathbb{R}} \int_{\mathbb{T}^d} \chi(x, \xi, t) \varphi(x, \xi) = \int_{\mathbb{R}} \int_{\mathbb{T}^d} \check{\chi}(\rho_0) \varphi(x, \xi) - 2 \int_0^t \int_{\mathbb{T}^d} \nabla \rho \cdot (\nabla \varphi)(x, \rho) \rho\int_0^t \int_{\mathbb{T}^d} F_1(x) \frac{\rho^2}{1 - \rho^2} \nabla \rho \cdot (\nabla \varphi)(x, \rho) + \int_0^t \int_{\mathbb{T}^d} \rho F_2(x) \cdot (\nabla \varphi)(x, \rho)
\] \[
- 2 \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}} \check{\rho} \partial_x \varphi(x, \xi) dq + \int_0^t \int_{\mathbb{T}^d} (\rho \nabla \rho \cdot F_2(x) + (1 - \rho^2) F_3(x)) (\partial_x \varphi)(x, \rho)
\] \[
- 2 \int_0^t \int_{\mathbb{T}^d} \varphi(x, \rho) \nabla \cdot ((1 - \rho^2) \nabla J * \rho) - 2 \int_0^t \int_{\mathbb{T}^d} \varphi(x, \rho) \nabla \cdot (\sqrt{1 - \rho^2} dW^F(r)),
\] (7.12)

where \( \check{\chi}(\rho_0)(x, \xi) := I_{(0 < \xi < \rho_0(x))} - I_{(\rho_0(x) < \xi < 0)} \).

**Approximation equation.** In order to prove the existence of renormalized kinetic solution to (7.2), we also need to introduce approximation equations with regularized coefficients. Similarly to [DFG20, Proposition 4.1] and [FG24, Lemma 5.18], we propose a smooth sequence to approximate \( \sqrt{1 - \xi^2} \).

**Lemma 7.3.** Let \( s^\frac{\xi}{\eta} : \mathbb{R} \to [0, 1] \) be defined by
\[
s^\frac{\xi}{\eta}(\xi) = \sqrt{1 - \xi^2} \quad \text{if} \quad \xi \in [-1, 1] \quad \text{and} \quad s^\frac{\xi}{\eta}(\xi) = 0 \quad \text{if} \quad \xi \notin [-1, 1].
\]

Then there exists a sequence of smooth, compactly supported approximations \( \{s^\frac{\xi}{\eta}\}_{\eta \in (0, 1)} \) satisfying
\[
\lim_{\eta \to 0} \|s^\frac{\xi}{\eta} - s^\frac{\xi}{\eta}\|_{L^\infty(\mathbb{R})} = 0.
\]

Furthermore, \( s^\frac{\xi}{\eta} \) has the following properties.

1. \( s^\frac{\xi}{\eta} \in C(\mathbb{R}) \cap C^\infty(\mathbb{R}) \) with \( s^\frac{\xi}{\eta}(1) = s^\frac{\xi}{\eta}(-1) = 0 \) and \( (s^\frac{\xi}{\eta})' \in C^\infty([-1, 1]) \),
2. there exists \( c \in (0, \infty) \) such that for every \( \xi \in [-1, 1] \),
\[
|s^\frac{\xi}{\eta}(\xi)| \leq c \sqrt{1 - \xi^2} \quad \text{uniformly with respect to} \quad \eta \in (0, 1),
\] (7.13)
3. for every \( \delta \in (0, 1) \), there exists \( c_\delta \in (0, \infty) \) such that uniformly with respect to \( \eta \in (0, 1) \),
\[
\left[ (s^\frac{\xi}{\eta})' (\xi) \right]^4 1_{\{1 - \delta \leq |\xi| \leq 1 - \delta \}} + \left| (s^\frac{\xi}{\eta})(\xi) (s^\frac{\xi}{\eta})' (\xi) \right|^2 1_{\{|1 - \delta \leq |\xi| \leq 1 - \delta\}} \leq c_\delta.
\] (7.14)
Now, we consider an approximating equation of (7.2),
\[
\partial_t \rho^\eta = \Delta \rho^\eta - \nabla \cdot \left[ (1 - (\rho^\eta)^2) \nabla J * \rho^\eta \right] - \nabla \cdot \left( s^{\frac{1}{2}}(\rho^\eta) \nabla F \right) + \frac{1}{2} \nabla \cdot \left( F_1 \left| s^{\frac{1}{2}}(\rho^\eta) \right| \nabla \rho^\eta - s^{\frac{1}{2}}(\rho^\eta) (s^{\frac{1}{2}}(\rho^\eta))' F_2 \right),
\]
with $\rho^\eta(0) = \rho_0$. The weak solution to this equation is defined as follows.

**Definition 7.4.** Let $\rho_0 \in \text{Ent}(T^d)$. A weak solution of (7.15) with initial data $\rho^\eta(\cdot, 0) = \rho_0$ is an $F_t$-predictable, $L^m(T^d; [-1, 1])$-continuous (for some $m \geq 2$) process $\rho^\eta$ such that almost surely $\rho^\eta \in L^2([0, T]; H^1(T^d))$ and for every $\psi \in C^\infty(T^d)$, almost surely for every $t \in [0, T]$,
\[
\int_{T^d} \rho^\eta(x, t) \psi(x) = \int_{T^d} \rho_0 \psi - \int_0^t \int_{T^d} \nabla \rho^\eta \cdot \nabla \psi + \int_0^t \int_{T^d} ((1 - (\rho^\eta)^2) \nabla J * \rho^\eta) \cdot \nabla \psi + \frac{1}{2} \int_0^t \int_{T^d} F_1 \left| s^{\frac{1}{2}}(\rho^\eta) \right|^2 \nabla \rho^\eta \cdot \nabla \psi + \frac{1}{2} \int_0^t \int_{T^d} s^{\frac{1}{2}}(\rho^\eta) \left( s^{\frac{1}{2}}(\rho^\eta) \right)' (\rho^\eta) F_2 \cdot \nabla \psi.
\]

**Remark 7.5.** Proceed similarly as in Theorem 5.7, we can show that if the initial value $\rho_0 \in [-1, 1]$, then $\rho^\eta \in [-1, 1]$ almost surely.

**Entropy estimate.** Recall that $\psi(\cdot)$ and $\text{Ent}(T^d)$ are defined by (7.5) and (7.7), respectively. We provide the following entropy dissipation estimate.

**Proposition 7.6.** Let $\rho_0 \in \text{Ent}(T^d)$. For any $\eta \in (0, 1)$, let $\rho^\eta$ be a weak solution of (7.15) in the sense of Definition 7.4 with initial data $\rho^\eta(\cdot, 0) = \rho_0$. Then there exists a constant $c \in (0, \infty)$ depending on $T$ and $\|\nabla J\|_{L^1(T^d; \mathbb{R}^d)}$ such that
\[
E \left[ \sup_{t \in [0, T]} \int_{T^d} \Psi(\rho^\eta(x, t)) \right] + E \int_0^T \int_{T^d} \frac{1}{2} \left( 1 - (\rho^\eta)^2 \right) |\nabla \rho^\eta|^2 \leq \int_{T^d} \Psi(\rho_0) + c.
\]

**Proof.** For the above $\Psi$, we introduce a sequence of smooth approximating functions denoted by $\Psi_\delta$ with $\delta \in (0, 1)$. Here, we require that $\Psi_\delta(0) = \Psi(0)$ and $\Psi_\delta'(\xi) = \frac{1}{2(1 + \delta)^2} \log \left( \frac{
abla \rho\eta}{1 + \delta} \right)$. Clearly, $\Psi_\delta'(\xi) = \frac{1}{1 + \delta - \xi^2}$. Applying Itô formula, we deduce that almost surely for every $t \in [0, T]$,
\[
\int_0^t \int_{T^d} \Psi_\delta(\rho^\eta(x, r)) \left| \frac{\rho^\eta(x, r)}{1 + \delta} \right|^2 \left| \nabla \rho^\eta \right|^2 = K_t^{\text{ker}} + K_t^{\text{mart}} + K_t^{\text{err}},
\]
where the term $K_t^{\text{ker}}$ is defined by
\[
K_t^{\text{ker}} = \int_0^t \int_{T^d} \frac{1}{2(1 + \delta)^2} \left( (1 - (\rho^\eta)^2) \nabla J * \rho^\eta \right) \cdot \nabla \rho^\eta,
\]
the term $K_t^{\text{mart}}$ is defined by
\[
K_t^{\text{mart}} = \int_0^t \int_{T^d} \frac{1}{2(1 + \delta)^2} \left( s^{\frac{1}{2}}(\rho^\eta) \right)^2 \nabla \rho^\eta \cdot \nabla \rho^\eta + \frac{1}{2} \int_0^t \int_{T^d} s^{\frac{1}{2}}(\rho^\eta) \left( s^{\frac{1}{2}}(\rho^\eta) \right)' (\rho^\eta) F_2 \cdot \nabla \rho^\eta + \frac{1}{2} \int_0^t \int_{T^d} s^{\frac{1}{2}}(\rho^\eta) \left( s^{\frac{1}{2}}(\rho^\eta) \right)' (\rho^\eta) F_2 + \frac{1}{2} \int_0^t \int_{T^d} s^{\frac{1}{2}}(\rho^\eta) \left( s^{\frac{1}{2}}(\rho^\eta) \right)' (\rho^\eta) F_2.
\]
For the error term, due to (7.13) and the boundness of $F_3$, we can derive that there exists a constant $c \in (0, \infty)$ such that
\[
E \int_0^t \int_{\mathbb{T}^d} \frac{1}{(1 + \delta)^2 - (\rho')^2} \left( s^{\pm, \eta}(\rho') \right)^2 F_3 \lesssim c(T). \tag{7.19}
\]
Using $\nabla \cdot F_2 = 0$ and integration by parts formula, we have that for every $\delta \in (0, 1)$,
\[
E \int_0^t \int_{\mathbb{T}^d} \frac{1}{(1 + \delta)^2 - (\rho')^2} s^{\pm, \eta}(\rho') \left( s^{\pm, \eta}(\rho') \right)' (\rho') F_2 \cdot \nabla \rho' = 0. \tag{7.20}
\]
For the martingale term, by Burkholder-Davis-Gundy inequality, Young inequality, the boundness of $F_1$, and (7.13), we deduce that there exists a constant $C \in (0, \infty)$ depending on $T$ such that
\[
E \sup_{t \in [0, T]} K^\text{mart}_t \leq E \sup_{t \in [0, T]} \left| \int_0^t \int_{\mathbb{T}^d} \frac{1}{(1 + \delta)^2 - (\rho')^2} s^{\pm, \eta}(\rho') \nabla \rho' \cdot dW^F \right| \lesssim E \left[ \int_0^T \int_{\mathbb{T}^d} \frac{1}{(1 + \delta)^2 - (\rho')^2} \left| \nabla \rho' \right|^2 \left( s^{\pm, \eta}(\rho') \right)^2 F_1(x) \right]^{\frac{1}{2}} \lesssim C (F_1) + \frac{1}{4} E \int_0^T \int_{\mathbb{T}^d} \frac{1}{(1 + \delta)^2 - (\rho')^2} |\nabla \rho'|^2. \tag{7.21}
\]
Finally, for the kernel term, since $|\rho'| \leq 1$ almost surely, it follows from the Young inequality that there exists a constant $c \in (0, \infty)$ such that
\[
E \sup_{t \in [0, T]} K^\text{ker}_t \leq E \int_0^T \int_{\mathbb{T}^d} \frac{1}{(1 + \delta)^2 - (\rho')^2} \nabla \rho' \cdot ((1 - (\rho')^2)) \nabla J * \rho') \right| \lesssim \frac{1}{2} E \int_0^T \int_{\mathbb{T}^d} \frac{1}{(1 + \delta)^2 - (\rho')^2} |\nabla \rho'|^2 + \frac{1}{2} E \int_0^T \int_{\mathbb{T}^d} \frac{1}{(1 + \delta)^2 - (\rho')^2} |1 - (\rho')^2|^2 |\nabla J * \rho'|^2 \lesssim E \int_0^T \int_{\mathbb{T}^d} \frac{1}{(1 + \delta)^2 - (\rho')^2} |\nabla \rho'|^2 + c(\|\nabla J\|_{L^1(\mathbb{T}^d; \mathbb{R})}). \tag{7.22}
\]
Based on (7.18)-(7.22), there exists a constant $c \in (0, \infty)$ independent of $\delta \in (0, 1)$ such that
\[
E \left[ \sup_{t \in [0, T]} \int_{\mathbb{T}^d} \Psi_\delta(\rho')(x, t) \right] + E \left[ \int_0^T \int_{\mathbb{T}^d} \frac{1}{(1 + \delta)^2 - (\rho')^2} |\nabla \rho'|^2 \right] \lesssim \int_{\mathbb{T}^d} \Psi_\delta (\rho_0) + c(T, \|\nabla J\|_{L^1(\mathbb{T}^d; \mathbb{R})}).
\]
The remaining proof can be done by a similar method as Proposition 5.4, thus we omit it.

\[\square\]

**Remark 7.7.** Similarly to Lemma 3.3, the weak derivative
\[
\frac{1}{1 - (\rho')^2} |\nabla \rho'|^2 = \left| \nabla \sqrt{1 - (\rho')^2} \right|^2 + |\nabla \rho'|^2
\]
holds for almost every $(x, t) \in \mathbb{T}^d \times [0, T]$. Therefore, the entropy estimate given in (7.17) not only implies the regularity of $\sqrt{1 - \rho^2}$, but also indicates that the kinetic measure is finite, which differs from the properties of the kinetic measure in the Dean-Kawasaki equation.

**Tightness of approximating solutions.** We aim to establish the $L^2 ([0, T]; L^2(\mathbb{T}^d; [-1, 1]))$-tightness of the laws of $\{\rho(t)\}_{t \in (0, 1)}$. The essential difficulty is also from singularities of the Stratonovich-to-Itô correction terms, which makes it intractable to obtain stable estimates on the time-derivative of the
solution. Further, different from the previous Dean-Kawasaki equation, two singularities at $\pm 1$ appear. In fact, we are in a similar situation as [DFG20], where 0 and 1 are two singularities. Thus, in the sequel, we modify the approach used in [DFG20] to make it applicable to our situation.

For every $\delta \in (0, 1/4)$, let $\psi_\delta \in C^\infty(\mathbb{R})$ be a smooth function satisfying $0 \leq \psi_\delta \leq 1$ and

$$
\psi_\delta(\xi) = \begin{cases} 
1, & \text{if } \xi \in [-1 + \delta, 1 - \delta], \\
0, & \text{if } \xi \in (-\infty, -1 + \frac{\delta}{2}] \cup [1 - \frac{\delta}{2}, \infty), \\
smooth, & \text{otherwise}.
\end{cases}
$$

Moreover, $|\psi'_\delta(\xi)| \leq c/\delta$ for some $c \in (0, \infty)$ independent of $\delta$. With the aid of $\psi_\delta$, we can define the function $h_\delta$.

**Definition 7.8.** For every $\delta \in (0, 1/4)$, let $h_\delta \in C^\infty(\mathbb{R})$ be defined by

$$
h_\delta(\xi) = \psi_\delta(\xi) \xi \quad \text{for every } \xi \in \mathbb{R}.
$$

Clearly, $|h_\delta(\xi)| \leq 1$ if $|\xi| \leq 1$. Moreover, by the definition of $h_\delta$, it follows that $h'_\delta$ is supported on $[-1 + \frac{\delta}{2}, 1 - \frac{\delta}{2}]$, $h''_\delta$ is supported on $[-1 + \frac{\delta}{2}, -1 + \delta] \cup [1 - \delta, 1 - \frac{\delta}{2}]$, and there exists a constant $c \in (0, \infty)$ depending on $\delta$ such that

$$
h'_\delta(\xi) = \psi'_\delta(\xi) + \psi_\delta(\xi) \leq c(\delta) \mathbf{1}_{\{-1 + \frac{\delta}{2} \leq \xi \leq 1 - \frac{\delta}{2}\}},
$$

$$
h''_\delta(\xi) = \psi''_\delta(\xi) + 2\psi'_\delta(\xi) \leq c(\delta) \mathbf{1}_{\{-1 + \frac{\delta}{2} \leq \xi \leq 1 - \frac{\delta}{2} + \delta \text{ or } 1 - \delta \leq \xi \leq 1 - \frac{\delta}{2}\}}.
$$

In order to apply the Aubin-Lions-Simon lemma to $h_\delta(\rho^n)$, we need the following results. Firstly, by using (7.23) and Proposition 7.6, we can easily derive the $L^1(\Omega, L^2([0, T]; H^1(\mathbb{T}^d)))$--norm.

**Lemma 7.9.** Let $\rho_0 \in \overline{\mathrm{Ent}}(\mathbb{T}^d)$. For any $\eta \in (0, 1)$, let $\rho^n$ be a weak solution of (7.15) in the sense of Definition 7.4 with initial data $\rho^n(\cdot, 0) = \rho_0$. Then there exists a constant $c \in (0, \infty)$ depending on $\delta, T, d, \|\nabla J\|_{L^1(\mathbb{T}^d; \mathbb{R}^d)}$, and $\rho_0$ such that

$$
\mathbb{E}\left[\|h_\delta(\rho^n)\|_{L^2([0, T]; H^1(\mathbb{T}^d))}\right] \leq \lambda(\delta, T, d, \|\nabla J\|_{L^1(\mathbb{T}^d; \mathbb{R}^d)}, \rho_0)
$$

Moreover, we need the following $W^{\beta, 1}([0, T]; H^{-1}(\mathbb{T}^d))$--estimate. The proof is very similar to that of Lemma 5.12, thus we omit it.

**Lemma 7.10.** Let $\rho_0 \in \overline{\mathrm{Ent}}(\mathbb{T}^d)$. For any $\eta \in (0, 1)$, let $\rho^n$ be a weak solution of (7.15) in the sense of Definition 7.4 with initial data $\rho^n(\cdot, 0) = \rho_0$. Then, for every $\beta \in (0, 1/2)$ and $l > \frac{d}{2} + 1$, there exists $\lambda \in (0, \infty)$ depending on $\delta, T, d, l, \|\nabla J\|_{L^1(\mathbb{T}^d; \mathbb{R}^d)}$, and $\rho_0$ such that

$$
\mathbb{E}\left[\|h_\delta(\rho^n)\|_{W^{\beta, 1}([0, T]; H^{-1}(\mathbb{T}^d))}\right] \leq \lambda(\delta, \beta, T, d, l, \|\nabla J\|_{L^1(\mathbb{T}^d; \mathbb{R}^d)}, \rho_0).
$$

Based on Lemmas 7.9 and 7.10, we can directly establish the tightness of the laws of $\rho^n$ through a diagonal argument.

**Proposition 7.11.** Let $\rho_0 \in \overline{\mathrm{Ent}}(\mathbb{T}^d)$. For any $\eta \in (0, 1)$, let $\rho^n$ be a weak solution of (7.15) in the sense of Definition 7.4 with initial data $\rho^n(\cdot, 0) = \rho_0$. Then the laws of $\{\rho^n\}_{n \in (0, 1)}$ are tight on $L^2([0, T]; L^2(\mathbb{T}^d; [-1, 1]))$.

**Proof.** In view of Lemmas 7.9 and 7.10, by the Aubin-Lions-Simon lemma, we get the tightness of the laws of $\{h_{1/5k}(\rho^n)\}_{n \in (0, 1)}$ on $L^2([0, T]; L^2(\mathbb{T}^d; [-1, 1]))$. Our aim is to derive the tightness of $\{\rho^n\}_{n \in (0, 1)}$ on $L^2([0, T]; L^2(\mathbb{T}^d; [-1, 1]))$. Let $n \in \mathbb{N}$ be an arbitrary integer. For each $k \in \mathbb{N}$, let $C_k$ be a compact subset of $L^2([0, T]; L^2(\mathbb{T}^d; [-1, 1]))$ such that for every $\eta \in (0, 1)$, $\mathbb{P}(h_{1/5k}(\rho^n) \notin C_k) \leq \frac{1}{n^2}$. Define the function $F_k : L^2([0, T]; L^2(\mathbb{T}^d; [-1, 1])) \to L^2([0, T]; L^2(\mathbb{T}^d; [-1, 1]))$ by $F_k(\rho) = h_{1/5k}(\rho)$. Due to the Lipschitz continuity of $h_{1/5k}$, $F_k$ is continuous. Denote by the preimage $D_k := F_k^{-1}(C_k)$,
then $D_k$ is a closed subset of $L^2([0,T];L^2(\mathbb{T}^d;[-1,1]))$ for every $k \in \mathbb{N}$. Define $D := \bigcap_{k=1}^{\infty} D_k$. Note that for every $\eta \in (0,1)$,
\[
\mathbb{P}(\rho^n \notin D) \leq \sum_{k=1}^{\infty} \mathbb{P}(h_{1/5k}(\rho^n) \notin C_k) \leq \sum_{k=1}^{\infty} \frac{1}{2^{2^k}} \eta \leq \frac{1}{n}.
\]
If $D$ is a compact subset of $L^2([0,T];L^2(\mathbb{T}^d;[-1,1]))$, by the arbitrary of $n$, it implies the tightness of $\{\rho^n\}_{n \in (0,1)}$ on $L^2([0,T];L^2(\mathbb{T}^d;[-1,1]))$.

It remains to demonstrate that $D$ is compact in $L^2([0,T];L^2(\mathbb{T}^d;[-1,1]))$. For any sequence $\{f_n\}_{n \in \mathbb{N}} \subseteq D$, we need to find a convergent subsequence $\{f_{n_l}\}_{l \in \mathbb{N}}$ whose limit is within $D$. In the following, the diagonal method is applied to find such a subsequence. Concretely, since $\{f_n\}_{n \in \mathbb{N}} \subseteq D_1$, by the definition of $D_1$, there exists a sequence $\{g_{n}^1\}_{n \in \mathbb{N}} \subseteq C_1$ such that $h_{1/5}(f_n) = g_{n}^1$ for each $n$. Since $C_1$ is compact, there exists a subsequence $\{n_1\}_{l \in \mathbb{N}} \subseteq \{n\}_{n \in \mathbb{N}}$ and an element $g^1$ such that $g_{n_1}^1 \to g^1$ in $L^2([0,T];L^2(\mathbb{T}^d;[-1,1]))$ and for almost every $(t,x) \in [0,T] \times \mathbb{T}^d$ as $l \to \infty$.

Moreover, $h_{1/5}(f_{n_1}) = g_{n_1}^1$. Since $\{f_{n_1}\}_{l \in \mathbb{N}} \subseteq D_2$, by the definition of $D_2$, there exists a sequence $\{g_{n_1}^2\}_{l \in \mathbb{N}} \subseteq C_2$ such that $h_{1/5}(f_{n_1}) = g_{n_1}^2$ for each $l \in \mathbb{N}$. Since $C_2$ is compact, there exists a subsequence $\{n_1^2\}_{l \in \mathbb{N}} \subseteq \{n_1\}_{l \in \mathbb{N}}$ and an element $g^2$ such that $g_{n_1}^2 \to g^2$ in $L^2([0,T];L^2(\mathbb{T}^d;[-1,1]))$ and for almost every $(t,x) \in [0,T] \times \mathbb{T}^d$ as $l \to \infty$. Moreover, $h_{1/5}(f_{n_1}) = g_{n_1}^2$. For the general $k \geq 3$, by the same method as above, there exists a subsequence $\{n_1^k\}_{l \in \mathbb{N}} \subseteq \{n_1\}_{l \in \mathbb{N}}$ and an element $g^k$ such that $g_{n_1}^k \to g^k$ in $L^2([0,T];L^2(\mathbb{T}^d;[-1,1]))$ and for almost every $(t,x) \in [0,T] \times \mathbb{T}^d$ as $l \to \infty$.

Moreover, $h_{1/5}(f_{n_1}) = g_{n_1}^k$. From $\{n_1^k\}_{l \in \mathbb{N},k \geq 1}$, we choose the diagonal elements denoted by $\{n_l\}_{l \in \mathbb{N}}$.

Then, for every $k \in \mathbb{N}$, it follows that $g_{n_l}^k \to g^k$ in $L^2([0,T];L^2(\mathbb{T}^d;[-1,1]))$ and for almost every $(t,x) \in [0,T] \times \mathbb{T}^d$ as $l \to \infty$. Moreover, $g_{n_l}^k = h_{1/5k}(f_{n_l})$. Then, by the continuity of $h_{1/5k}$, we deduce that for almost every $(t,x) \in [0,T] \times \mathbb{T}^d$,
\[
g^k(t,x) = \lim_{l \to \infty} h_{1/5k}(f_{n_l}(t,x)) = h_{1/5k}\left(\lim_{l \to \infty} f_{n_l}(t,x)\right),
\]
(7.25)

hence, $\lim_{l \to \infty} f_{n_l}(t,x)$ exists. Let $f(x,t) := \lim_{l \to \infty} f_{n_l}(t,x)$. Since (7.25) holds for each $k$, it follows that $f \in D$. Note that $\|f\|_{L^\infty([0,T] \times \mathbb{T}^d)} + \|f\|_{L^2([0,T] \times \mathbb{T}^d)} \leq 2$, by the dominated convergence theorem, we conclude that $f_{n_l} \to f$ in $L^2(\mathbb{T}^d \times [0,T])$ as $l \to \infty$. Thus, we complete the proof by choosing $\{n_l\}_{l \in \mathbb{N}} := \{n_l\}_{l \in \mathbb{N}}$.

**Well-posedness of (7.1).** Leveraging the previous entropy estimate and tightness of the approximation equation, we can prove the well-posedness of the renormalized kinetic solution for the fluctuating Ising-Kac-Kawasaki equation (7.1). The proof is achieved through an analogous approach to the well-posedness of the Dean-Kawasaki equation (2.2).

**Theorem 7.12.** For any spatial dimension $d \geq 1$, suppose that $\nabla J \in C^\infty(\mathbb{T}^d;\mathbb{R}^d)$. Let $\rho_0 \in \text{Ent}(\mathbb{T}^d)$.

Then there exists a unique renormalized kinetic solution to (7.1) in the sense of Definition 7.2 with initial data $\rho_0$.

**Proof.** For the uniqueness of renormalized kinetic solutions to (7.1), since (7.1) has two singularities at $\pm 1$, we need to introduce a new cutoff function in the velocity variable compared with the Dean-Kawasaki equation (2.2). Concretely, for each $\beta \in (0,\frac{1}{2})$, define a smooth function $\zeta_\beta : \mathbb{R} \to [0,1]$ which satisfies $\zeta_\beta = 0$ if $\xi \leq 1 - \beta$ or $\xi \geq 1 - \beta$, $\zeta_\beta = 1$ if $-1 + 2\beta \leq \xi \leq 1 - 2\beta$, and the condition $|\zeta_\beta'| \leq c/\beta$ for some constant $c \in (0,\infty)$ independent of $\beta$. Then, with the help of $\zeta_\beta$ and (7.11), we can apply a method akin to that used in [FG24, Theorem 4.6], [DFG20, Theorem 2.6] and Theorem 4.2 to establish the uniqueness.
Regarding to the existence of renormalized kinetic solutions to (7.1), since the entropy estimate from Proposition 7.6 provides the regularity properties of the solution, the proof can closely parallel that of [FG24, Theorem 5.25], [DFG20, Theorem 2.15] and Theorem 6.2.

At last, different from the Dean-Kawasaki equation (2.2), the well-posedness of (7.1) holds for all spatial dimensions \( d \geq 1 \). The reason is that we have assumed the condition \( \nabla f \in C^\infty(\mathbb{T}^d, \mathbb{R}^d) \), which can guarantee the integrability of the kernel term instead of applying the interpolation inequalities. \( \square \)

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**References**

[AvR10] Sebastian Andres and Max-K. von Renesse. Particle approximation of the Wasserstein diffusion. *J. Funct. Anal.*, 258(11):3879–3905, 2010.

[BBDPR06] Viorel Barbu, Vladimir I. Bogachev, Giuseppe Da Prato, and Michael Röckner. Weak solutions to the stochastic porous media equation via Kolmogorov equations: the degenerate case. *J. Funct. Anal.*, 237(1):54–75, 2006.

[BDPR08a] Viorel Barbu, Giuseppe Da Prato, and Michael Röckner. Existence and uniqueness of nonnegative solutions to the stochastic porous media equation. *Indiana Univ. Math. J.*, 57(1):187–211, 2008.

[BDPR08b] Viorel Barbu, Giuseppe Da Prato, and Michael Röckner. Some results on stochastic porous media equations. *Boll. Unione Mat. Ital.* (9), 1(1):1–15, 2008.

[BDPR09] Viorel Barbu, Giuseppe Da Prato, and Michael Röckner. Existence of strong solutions for stochastic porous media equation under general monotonicity conditions. *Ann. Probab.*, 37(2):428–452, 2009.

[BDPR16] Viorel Barbu, Giuseppe Da Prato, and Michael Röckner. *Stochastic porous media equations*, volume 2163 of *Lecture Notes in Mathematics*. Springer, [Cham], 2016.

[BFM16] Zdzislaw Brzeźniak, Franco Flandoli, and Mario Maurelli. Existence and uniqueness for stochastic 2D Euler flows with bounded vorticity. *Arch. Ration. Mech. Anal.*, 221(1):107–142, 2016.

[BJW23] Didier Bresch, Pierre-Emmanuel Jabin, and Zhenfu Wang. Mean field limit and quantitative estimates with singular attractive kernels. *Journal of Mathematical Physics*, 54(12):122701, 2013.

[BM18] Haim Brezis and Petru Mironescu. Gagliardo-Nirenberg inequalities and non-inequalities: the full story. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 35(5):1355–1376, 2018.

[BPRS94] Lorenzo Bertini, Errico Presutti, Barbara Rüdiger, and Ellen Saada. Dynamical fluctuations at the critical point: convergence to a nonlinear stochastic pde. *Theory of Probability & Its Applications*, 38(4):586–629, 1994.

[CF23a] Andrea Clini and Benjamin Fehrman. A central limit theorem for nonlinear conservative spdes. arxiv: 2310.19924, 2023.

[CF23b] Federico Cornalba and Julian Fischer. The dean–kawasaki equation and the structure of density fluctuations in systems of diffusing particles. *Archive for Rational Mechanics and Analysis*, 247, 08 2023.

[CFIR23] Federico Cornalba, Julian Fischer, Jonas Ingmann, and Claudia Raithel. Density fluctuations in weakly interacting particle systems via the dean-kawasaki equation. arXiv preprint arXiv:2303.00429, 2023.

[CG22] Beniamino Nemez, Michael Röckner, and Rongchan Zhu. Perturbation of the particle approximation of the Dean-Kawasaki equation. arXiv: 2205.11013, 2022.

[CI07] Lorenzo Bertini, Errico Presutti, Barbara Rüdiger, and Ellen Saada. Dynamical fluctuations at the critical point: convergence to a nonlinear stochastic pde. *Theory of Probability & Its Applications*, 38(4):586–629, 1994.

[CSZ19] Federico Cornalba, Tony Shardlow, and Johannes Zimmer. A regularized Dean-Kawasaki model: derivation and analysis. *SIAM J. Math. Anal.*, 51(2):1137–1187, 2019.

[CSZ20] Federico Cornalba, Tony Shardlow, and Johannes Zimmer. From weakly interacting particles to a regularised Dean-Kawasaki model. *Nonlinearity*, 33(2):864–881, 2020.

[Dea96] David S. Dean. Langevin equation for the density of a system of interacting Langevin processes. *J. Phys. A*, 29(24):L613–L617, 1996.

[DFG20] Nicolas Dirr, Benjamin Fehrman, and Benjamin Gess. Conservative stochastic pde and fluctuations of the symmetric simple exclusion process. arxiv: 2012.02126, 2020.
[DG20] Konstantinos Dareiotis and Benjamin Gess. Nonlinear diffusion equations with nonlinear gradient noise. *Electron. J. Probab.*, 25:Paper No. 35, 43, 2020.

[DKP22] Ana Djurdjevac, Helena Kremp, and Nicolas Perkowski. Weak error analysis for a nonlinear spde approximation of the dean-kawasaki equation. *arXiv preprint arXiv:2212.11714*, 2022.

[DV10] Arnaud Debussche and Julien Vovelle. Scalar conservation laws with stochastic forcing. *J. Funct. Anal.*, 259(4):1014–1042, 2010.

[EL25] Zeitschrift für Physik. E. Ising. Report on the theory of ferromagnetism. *31 (1925) 253–258*, 1925.

[Eva10] Lawrence Craig Evans. *Partial differential equations*, volume 19. American Mathematical Soc., 2010.

[FG19] Benjamin Fehrman and Benjamin Gess. Well-posedness of nonlinear diffusion equations with nonlinear, conservative noise. *Archive for Rational Mechanics and Analysis*, 233(1):249–322, 2019.

[FG21] Benjamin Fehrman and Benjamin Gess. Well-posedness of the dean–kawasaki and the nonlinear dawson–watanabe equation with correlated noise. *arXiv e-prints*, pages arXiv–2108, 2021.

[FG23] Benjamin Fehrman and Benjamin Gess. Non-equilibrium large deviations and parabolic-hyperbolic PDE with irregular drift. *Invent. Math.*, 234(2):573–636, 2023.

[FG24] Benjamin Fehrman and Benjamin Gess. Well-posedness of the dean–kawasaki and the nonlinear dawson–watanabe equation with correlated noise. *Archive for Rational Mechanics and Analysis*, 248(2):20, 2024.

[FGG22] Benjamin Fehrman, Benjamin Gess, and Rishabh S. Gvalani. Ergodicity and random dynamical systems for conservative spdes. *arXiv:2206.14789*, 2022.

[FHM14] Nicolas Fournier, Maxime Hauray, and Stéphane Mischler. Propagation of chaos for the 2D viscous vortex model. *J. Eur. Math. Soc.* (JEMS), 16(7):1423–1466, 2014.

[FV87] Pablo Augusto Ferrari, Enrico Presutti, and Maria Eulalia Vares. Local equilibrium for a one-dimensional zero range process. *Stochastic Process. Appl.*, 26(1):31–45, 1987.

[FR95] J Fritz and B Rüdiger. Time dependent critical fluctuations of a one dimensional local mean field model. *Probability theory and related fields*, 103:381–407, 1995.

[GGK22] Benjamin Gess, Rishabh S. Gvalani, and Vitalii Konarovskyi. Conservative spdes as fluctuating mean field equations. *Adv. Differential Equations*, 15(5-6):567–600, 2020.

[GHZ09] Nathan Glatt-Holtz and Mohammed Ziane. Strong pathwise solutions of the stochastic Navier-Stokes system. *Adv. Differential Equations*, 14(5-6):567–600, 2009.

[GIP15] Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski. Paracontrolled distributions and singular PDEs. *Forum Math.*, 3:e6, 75, 2015.

[Gla63] Roy J. Glauber. *Time-Dependent Statistics of the Ising Model*. Springer-Verlag, Berlin, 1963.

[GLP99] Giambattista Giacomin, Joel L Lebowitz, and Errico Presutti. Deterministic and stochastic hydrodynamic equations arising from simple microscopic model systems. *Mathematical Surveys and Monographs*, 64:107–152, 1999.

[GMW23] Paolo Grazieschi, Konstantin Matetski, and Hendrik Weber. The dynamical ising-kac model in 3d converges to $\phi^4_3$. *arxiv:2303.10242*, 2023.

[GS17] Benjamin Gess and Panagiotis E. Souganidis. Stochastic non-isotropic degenerate parabolic-hyperbolic equations. *Stochastic Process. Appl.*, 127(9):2961–3004, 2017.

[HA14] Martin Hairer. A theory of regularity structures. *Invent. Math.*, 198(2):269–504, 2014.

[HMT24] Jasper Hoeksema, Thomas Holding, Mario Maurelli, and Oliver Tse. Large deviations for singularly interacting diffusions. *60 I ANNALES DE L’INSTITUT HENRI POINCARÉ PROBABILITÉS ET STATISTIQUES* Vol. 60, No. 1 (February, 2024) 1–752, 60(1):492–548, 2024.

[HKU64] P. C. Hemmer, M. Kac, and G. E. Uhlenbeck. On the van der Waals theory of the vapor-liquid equilibrium. *J. Mathematical Phys.*, 5:60–74, 1964.

[HRZ22] Zimo Hao, Michael Röckner, and Xicheng Zhang. Strong convergence of propagation of chaos for mckean-vlasov sdes with singular interactions. *arxiv:2204.07952*, 2022.

[Ibe18] Massimo Iberti. *Ising-kac models near criticality*. 2018.

[JM18] Milton Jara and Otávio Menezes. Non-equilibrium fluctuations of interacting particle systems. *arxiv:1810.09526*, 2018.

[JW18] Pierre-Emmanuel Jabin and Zhenfu Wang. Quantitative estimates of propagation of chaos for stochastic systems with $W^{−1,\infty}$ kernels. *Invent. Math.*, 214(1):525–591, 2018.

[Kaw98] Kyozu Kawasaki. Microscopic analyses of the dynamical density functional equation of dense fluids. *J. Statist. Phys.*, 93(3-4):527–546, 1998.

[KL99] Claude Kipnis and Claudio Landim. *Scaling limits of interacting particle systems*, volume 320 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[KLVR19] Vitalii Konarovskyi, Tobias Lehmann, and Max-K. von Renesse. Dean-Kawasaki dynamics: ill-posedness vs. triviality. *Electron. Commun. Probab.*, 24:Paper No. 8, 9, 2019.

[KR05] Nicolai V. Krylov and Michael Röckner. Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields*, 131(2):154–196, 2005.
[Kry13] N. V. Krylov. A relatively short proof of Itô’s formula for SPDEs and its applications. *Stoch. Partial Differ. Equ. Anal. Comput.*, 1(1):152–174, 2013.

[KvR19] Vitalii Konarovskyi and Max-K. von Renesse. Modified massive Arratia flow and Wasserstein diffusion. *Comm. Pure Appl. Math.*, 72(4):764–800, 2019.

[Lad67] Olga Aleksandrovna Ladyzhenskaya. On the uniqueness and on the smoothness of weak solutions of the Navier–Stokes equations. *Zapiski Nauchnykh Seminarov POMI*, 5:169–185, 1967.

[LPT94] Pierre-Louis Lions, Benoît Perthame, and Eitan Tadmor. A kinetic formulation of multidimensional scalar conservation laws and related equations. *J. Amer. Math. Soc.*, 7(1):169–191, 1994.

[Mar87] Jürgen Marschall. The trace of Sobolev-Slobodeckij spaces on Lipschitz domains. *Manuscripta Math.*, 58(1-2):47–65, 1987.

[MW17] Jean-Christophe Mourrat and Hendrik Weber. Convergence of the two-dimensional dynamic Ising-Kac model to $\Phi^4_2$. *Comm. Pure Appl. Math.*, 70(4):717–812, 2017.

[Osa86] Hirofumi Osada. Propagation of chaos for the two-dimensional Navier-Stokes equation. *Proc. Japan Acad. Ser. A Math. Sci.*, 62(1):8–11, 1986.

[Pro59] Giovanni Prodi. Un teorema di unicità per le equazioni di Navier-Stokes. *Ann. Mat. Pura Appl.* (4), 48:173–182, 1959.

[Rav92] Krishnamurthi Ravishankar. Fluctuations from the hydrodynamical limit for the symmetric simple exclusion in $\mathbb{Z}^d$. *Stochastic Process. Appl.*, 48:173–182, 1992.

[RS93] L. C. G. Rogers and Z. Shi. Interacting Brownian particles and the Wigner law. *Probab. Theory Related Fields*, 95(4):555–570, 1993.

[RUZ21] Michael Röckner and Xicheng Zhang. Well-posedness of distribution dependent SDEs with singular drifts. *Bernoulli*, 27(2):1131 – 1158, 2021.

[RUZ21b] Michael Röckner and Guohuan Zhao. Sdes with critical time dependent drifts: strong solutions. *arXiv preprint arXiv:2103.05803*, 2021.

[RUZ23] Michael Röckner and Guohuan Zhao. Sdes with critical time dependent drifts: Weak solutions. *Bernoulli*, 29(1):757 – 784, 2023.

[Ser62] James Serrin. On the interior regularity of weak solutions of the Navier-Stokes equations. *Arch. Rational Mech. Anal.*, 9:187–195, 1962.

[Ser20] Sylvia Serfaty. Mean field limit for Coulomb-type flows. *Duke Math. J.*, 169(15):2887–2935, 2020. With an appendix by Mitia Duerinckx and Serfaty.

[Sim87] Jacques Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.

[Spo12] Herbert Spohn. *Large scale dynamics of interacting particles*. Springer Science & Business Media, 2012.

[vRS09] Max-K. von Renesse and Karl-Theodor Sturm. Entropic measure and Wasserstein diffusion. *Ann. Probab.*, 37(3):1114–1191, 2009.

[XXZZ20] Pengcheng Xia, Longjie Xie, Xicheng Zhang, and Guohuan Zhao. $L^p(L^p)$-theory of stochastic differential equations. *Stochastic Process. Appl.*, 139(8):5188–5211, 2020.

[Zha05] Xicheng Zhang. Strong solutions of SDES with singular drift and Sobolev diffusion coefficients. *Stochastic Process. Appl.*, 115(11):1805–1818, 2005.

[Zha11] Xicheng Zhang. Stochastic homeomorphism flows of SDEs with singular drifts and Sobolev diffusion coefficients. *Electron. J. Probab.*, 16:no. 38, 1096–1116, 2011.

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