On the Treves Criterion for the Boussinesq
and other GD Hierarchies.

L. A. Dickey
University of Oklahoma

Abstract
This is an addition to our paper (Lett.Math.Phys 65, 2003,187-197)
where an analogue of the Treves criterion for the first integrals of the
KdV hierarchy was suggested for the Boussinesq hierarchy and its neces-
sity was proven. In the present paper, it is proven that there exists the
second, “conjugated”, group of tests. Besides, the relationship between
our method and the Bäcklund transformation method recently suggested
by Morosi and Pizzocchero is discussed.

1. This is an addition to our paper [1]. The Boussinesq hier-
archy was discussed there: the set of equations
\[
\frac{\partial L}{\partial t_i} = [L^{i/3}, L]
\]
where \( L = \partial^3 + u_1 \partial + u_0 \), and \( \partial = \partial/\partial x \). As it is well-known, the
first integrals of these equations are \( \text{res}_9 L^{m/3} \) (\( \text{res}_9 \) symbolizes the
coefficient in \( \partial^{-1} \)) modulo derivatives of any differential polynomi-
als of \( u_1 \) and \( u_0 \). The following proposition was proven in [1] (\( \text{res}_x \)
symbolizes the coefficient in the term with \( x^{-1} \) in a Laurent series):1:

**Proposition.** The differential polynomials \( P[u_1, u_0] = \text{res}_9 L^{m/3} \)
satisfy the following two criteria:

(i) \[ \text{res}_x P[\tilde{u}_1(x), \tilde{u}_0(x)] = 0 \] (1)

when the series
\[
\tilde{u}_1(x) = -3x^{-2} + \sum_{i=1}^{\infty} u_{1i} (x^i/i!), \quad \tilde{u}_0(x) = 3x^{-3} + \sum_{i=0}^{\infty} u_{0i} (x^i/i!) \] (2)

with \( u_{01} = u_{12}/2 \) are substituted for variables \( u_1 \) and \( u_0 \).

(ii) The same equality (1) is true when the series are
\[
\tilde{u}_1(x) = -6x^{-2} + \sum_{i=1}^{\infty} u_{1i} (x^i/i!), \quad \tilde{u}_0(x) = 12x^{-3} + \sum_{i=0}^{\infty} u_{0i} (x^i/i!), \] (3)

1 There are misprints in [1]: the limits of summation in the first sums in both (0.4) and
(0.5) are 1 and \( \infty \).
with 

\[ u_{01} = u_{12} \text{ and } u_{02} = 2u_{13}/3. \]

This proposition is an analogue of the necessity part of the Treves theorem for the KdV hierarchy where \( L \) is an operator of the second order, see [2].

We are going to show that besides this group of two tests there is the second, “conjugated”, group of two tests (i∗) and (ii∗) of which the first one coincides with (i) but the second is different:

(ii∗) Eq. (1) is true if \( \tilde{u}_1(x) \) and \( \tilde{u}_2(x) \) are replaced by

\[
\tilde{u}_1^*(x) = -6x^{-2} + \sum_{i=1}^{\infty} u_{1i}^*(x^i/i!), \quad \tilde{u}_0^*(x) = \sum_{i=0}^{\infty} u_{0i}^*(x^i/i!),
\]

with \( u_{01}^* = 0 \) and \( u_{02}^* = u_{13}^*/3. \)

Let us prove this. It is shown in [1] that if the series (2) or (3) are substituted for \( u_1 \) and \( u_0 \) in the operator \( \tilde{L} = \partial^3 + \tilde{u}_1(x)\partial + \tilde{u}_0(x) \) can be represented in a “dressing” form \( \tilde{L} = w\partial^3w^{-1} \) where \( w = \sum_{i=0}^{\infty} w_i\partial^i, w_0 = 1 \) and all \( w_i \) are Laurent series in \( x \). The equation (1) follows from this fact. Let us go to the conjugated operators: \( \tilde{L}^* = -(w^*)^{-1}\partial^3w^* \). The operator \( -\tilde{L}^* \) is

\[
-\tilde{L}^* = \partial^3 + u_1^*\partial + u_0^* = (w^*)^{-1}\partial^3w^*, \quad \text{where } u_1^* = u_1, \quad u_0^* = -u_0 + u_1'.
\]

It is represented in the same dressing form where the dressing operator has the Laurent series coefficients. Therefore, the equality (1) is true when \( \tilde{u}_1^* = \tilde{u}_1 \) and \( \tilde{u}_0^* = -\tilde{u}_0 + \tilde{u}_1' \) are substituted for \( \tilde{u}_1 \) and \( \tilde{u}_0 \): \( \text{res}_xP[\tilde{u}_1^*(x), \tilde{u}_0^*(x)] = 0 \) or

\[
\text{res}_xP[\tilde{u}_1(x), -\tilde{u}_0(x) + \tilde{u}_1'(x)] = 0.
\]

This gives us the second, “conjugated”, group of tests.

We find for the test (i):

\[
\tilde{u}_1^* = \tilde{u}_1 = -3x^{-2} + \sum_{i=1}^{\infty} u_{1i}(x^i/i!)
\]

and

\[
\tilde{u}_0^* = -\tilde{u}_0 + \tilde{u}_1' = -3x^{-3} - \sum_{i=0}^{\infty} u_{0i}x^i/i! + 6x^{-3} + \sum_{i=1}^{\infty} u_{1,i+1}x^i/i!.
\]
\[ = 3x^{-3} + \sum_{0}^{\infty} u_{0i}^* x^i / i!, \quad u_{0i}^* = -u_{0i} + u_{1,i+1}. \]

In particular,
\[ u_{01}^* = -u_{01} + u_{12} = -u_{12}/2 + u_{12} = u_{12}^*/2. \]

We see that this test is the same as (i).

Now, for the test (ii):
\[ \tilde{u}_1^* = \tilde{u}_1 = -6x^{-2} + \sum_{1}^{\infty} u_{1i}(x^i / i!) \]

and
\[ \tilde{u}_0^* = -\tilde{u}_0 + \tilde{u}_1' = -12x^{-3} - \sum_{0}^{\infty} u_{0i}(x^i / i!) + 12x^{-3} + \sum_{0}^{\infty} u_{1,i+1}x^i / i! \]
\[ = \sum_{0}^{\infty} u_{0i}^* x^i / i!, \quad u_{0i}^* = -u_{0i} + u_{1,i+1}. \]

In particular,
\[ u_{01}^* = -u_{01} + u_{12} = 0, \]
\[ u_{0,2}^* = -u_{0,2} + u_{13} = -2u_{13}/3 + u_{13} = u_{13}/3 = u_{13}^*/3. \]

We obtained the test (4).

There was a conjecture in [1] that if a differential polynomial \( P[u_1, u_0] \) satisfies conditions (i) and (ii) then it is a linear combination of \( \text{res}_\partial L^{m/3} \). It is not clear: maybe all three conditions, (i), (ii) and (ii*), should be checked for this, maybe not, and it suffices to check one group of tests, either the first or the conjugated. We know nothing about the independence of these tests.

2. There was an explanation in [1] how the singular parts of the series (2) and (3) were guessed. More than that, a recipe was given how to find these singular parts if the operator \( L = \partial^n + u_{n-2}\partial^{n-2} + ... + u_0 \) is of an arbitrary order \( n \). Namely, the singular part of the operator \( \tilde{L} \) (i.e., only the singular terms of the Laurent series \( \tilde{u}_i \) are preserved) can be found from the formula
\[ \tilde{L}_N = \left( \partial - \frac{1}{x} \right)^N \partial^n \left( \partial - \frac{1}{x} \right)^{-N} \]

3
When $N = 0$, we obtain $\tilde{L}_0 = \partial^n$, an operator without singularities. This operator does not correspond to any test.

**Examples.** Let $n = 2$ and $N = 1$. Then $L_1 = \partial^2 - 2x^{-2}$, this is the singular part of the Treves test for the KdV hierarchy, see [1].

Let $n = 3$ and $N = 1$. Then $L_1 = \partial^3 - 6x^{-2}\partial + 12x^{-3}$ which gives the singular parts of $u_1$ and $u_0$ for the test (i) (Eq. (2)).

Let $n = 3$ and $N = 2$. Then $L_1 = \partial^3 - 6x^{-2}\partial + 12x^{-3}$ which corresponds to the test (ii) (Eq. (3)).

What about the second, conjugated, group of tests? As we know, it is generated by the operators $(-1)^n\tilde{L}_N^*$. The singular parts are

\[
(-1)^n\tilde{L}_N^* = \left(\partial + \frac{1}{x}\right)^{-N}\partial^n\left(\partial + \frac{1}{x}\right)^N
\]

Notice that both operators, (6) and (7), coincide when $N = 1$:

\[
(\partial - x^{-1})\partial^n(\partial - x^{-1})^{-1} = (\partial + x^{-1})^{-1}\partial^n(\partial + x^{-1})
\]

since $(\partial + x^{-1})(\partial - x^{-1}) = \partial^2$. Therefore, one of tests of the first group ($N = 1$) coincides with the corresponding conjugated test.

For the KdV ($n=2$) one has only one test, for the Boussinesq ($n = 3$) one has 3 different tests, and so on.

3. In a recent work by Morosi and Pizzocchero [3], another method of the proof of the necessity of the Treves criterion is suggested (they considered the case of KdV, $n = 2$). It is based on the multiplicative representation of the operator $L$, as a product of first-order differential operators, rather than in the dressing form we exploited. They notice that if the term $u$ in the operator is replaced by the Laurent series $\tilde{u}(x)$ satisfying the Treves condition, then the operator is Bäcklund equivalent to an operator without singularities. Then, they use the fact that for two Bäcklund equivalent operators $L_1$ and $L_2$ the first integrals res$_\partial L_1^{m/n}$ and res$_\partial L_2^{m/n}$
differ by a derivative of a differential polynomial (see below) and, therefore,

\[ \text{res}_x \text{res}_\theta \tilde{L}_1^{m/n} = \text{res}_x \text{res}_\theta \tilde{L}_2^{m/n} \]

and the fact that if an operator has no singularities then this quantity is zero. They conclude that for the given operator this is zero, too. The relation between two method, at least for singular parts of operators is clearly seen on the same formula (6). There are two representations of the operators \( L_N \) there: the first is the dressing formula, the second is the product of first-order operators. Two such operators with distinct \( N \) differ by a cyclic permutation of the first-order operators, i.e., they are Bäcklund equivalent. One of them (when \( N = 0 \)) has no singularities. For the conjugated operators, we use the formula (7).

Example. Let \( n = 3 \). We have

\[ -\tilde{L}_2^* = (\partial - 2x^{-1})(\partial + x^{-1})(\partial + x^{-1}) = \partial^3 - 6x^{-1}\partial \]

which is the singular part of the test (\( \text{ii}^* \)). How can we find the whole series (4), following the logic of [4], after we got the singular part? Firstly, we can write

\[ -L_0^* = (\partial + x^{-1} + v_1(x))(\partial + x^{-1} + v_2(x))(\partial - 2x^{-1} - v_1(x) - v_2(x)) \]

where \( v_1(x) = \sum_0^\infty v_1(x^i)/i! \) and \( v_2(x) = \sum_0^\infty v_2(x^i)/i! \). Then we must require that this operator be regular. After some rather lengthy but straightforward computations, one can show that this requirement is equivalent to \( v_1(x) \) and \( v_2(x) \) being \( O(x^2) \), i.e., the series should start with terms with \( x^2 \). After that we calculate

\[ -L_2^* = (\partial - 2x^{-1} - v_1(x) - v_2(x))(\partial + x^{-1} + v_1(x))(\partial + x^{-1} + v_2(x)) \]

\[ = \partial^3 + \tilde{u}_1^*(x)\partial + \tilde{u}_0^*(x). \]

We can find that the only restriction which should be imposed on the series for \( \tilde{u}_1^*(x) \) and \( \tilde{u}_2^*(x) \) in order they can be expressed in terms of \( v_1(x) \) and \( v_2(x) \) are \( u_{01}^* = 0 \) and \( u_{02}^* = u_{13}^*/3 \).

As it was said earlier, this guarantees that

\[ \text{res}_x \text{res}_\theta (L_2^*)^{m/3} = \text{res}_x \text{res}_\theta (L_0^*)^{m/3} = 0 \]
if the lemma is proven:

**Lemma.** If two \( n \)th order operators are Bäcklund equivalent,

\[
L_1 = (\partial + w_1) \cdots (\partial + w_n), \quad L_2 = (\partial + w_2) \cdots (\partial + w_n)(\partial + w_1)
\]

where \( w_i \) belong to a differential algebra, then \( \text{res}_\partial L_{1/m/n} \) and \( \text{res}_\partial L_{2/m/n} \) differ by a derivative of an element of the same differential algebra.

**Proof.** Evidently, \( L_1(p + w_1) = (\partial + w_1)L_2, \) or \( L_2 = (\partial + w_1)^{-1}L_1(\partial + w_1). \) This easily implies

\[
L_{2/m/n} = (\partial + w_1)^{-1}L_{1/m/n}(\partial + w_1) = L_{1/m/n}(\partial + w_1)(\partial + w_1)^{-1}
\]

\[+[(\partial + w_1)^{-1}, L_{1/m/n}(\partial + w_1)] = L_{1/m/n} + [(\partial + w_1)^{-1}, L_{1/m/n}(\partial + w_1)].\]

It remains to take the \( \text{res}_\partial \) of this equality and to notice that the residue of the commutator of two operators with coefficients in some differential algebra is a derivative of an element of the same algebra.

\( \square \)

In our case, the differential algebra is the algebra of the Laurent series, and we obtain the test (ii*), Eq. (4).

**References.**

1. Dickey, L. A.: On a generalization of the Treves criterion for KdV hierarchy to higher hierarchies, *Lett. in Math. Phys.* 65 (2003), 187-197.
2. Treves, F. An algebraic characterization of the Korteweg- de Vries hierarchy, *Duke Math. J.* 108(2) (2001), 251-294.
3. Morosi, P. and Pizzocchero, L.: On a theorem by Trèves, arXiv: nlin.SI/0405007 (2004), 1-7.