ON THE EXACT CONTROLLABILITY AND THE STABILIZATION FOR THE BENNEY-LUKE EQUATION

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Abstract. In this work we consider the exact controllability and the stabilization for the generalized Benney-Luke equation
\[ u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} + pu_t u_x^{p-1}u_{xx} + 2u_x^p u_{xt} = f, \] (1)
on a periodic domain \( S \) (the unit circle on the plane) with internal control \( f \) supported on an arbitrary sub-domain of \( S \). We establish that the model is exactly controllable in a Sobolev type space when the whole \( S \) is the support of \( f \), without any assumption on the size of the initial and final states, and that the model is local exactly controllable when the support of \( f \) is a proper subdomain of \( S \), assuming that initial and terminal states are small. Moreover, in the case that the initial data is small and \( f \) is a special internal linear feedback, the solution of the model must have uniform exponential decay to a constant state.

1. Introduction. In this paper, we consider the controllability and the stabilization problems associated with the forced Benney-Luke equation
\[ u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} + pu_t u_x^{p-1}u_{xx} + 2u_x^p u_{xt} = f, \] (2)
in which \( u = u(x,t) \) with \( x,t \in \mathbb{R} \) and the subscripts denote the corresponding partial derivatives. For \( p = 1 \) and \( f = 0 \), this equation, known as the Benney-Luke equation, is a formally valid approximation for describing small-amplitude of the isotropic type for two-dimensional water waves (see [12, 2]), where the parameters \( a, b > 0 \) are such that \( a - b = \sigma - \frac{1}{3} \), with \( \sigma \) being associated with the surface tension (the Bond number). It is worth to mention that the Benney-Luke equation is an approximation formally valid for describing two-way water wave propagation, in contrast to one-way equations such as the KdV, or BBM equations.

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During the last 40 years, there have been many contributions to the exact controllability, stabilizability, and the boundary control (linear and nonlinear) for different models like the KdV, the improved Boussinesq equation, the Boussinesq equation with variable coefficients, the generalized Boussinesq equation, the Schrödinger equation (see for example, D. Russell in [15], B. Zhang in [19, 20], D. Russell and B. Zhang in [16, 17], D. Rosier in [13], M. Chapouly in [6], E. Cerpa and E. Crépeau in [4], E. Crépeau in [7], E. Cerpa and I. Rivas in [5], J. Amara and H. Bouzidi in [1], R. Capistrano and M. Cavalcante in [3], among others).

For instance, D. Russell in [15], L. Rosier in [13], and D. Russel and B. Zhang in [17] showed the exact boundary controllability of linear and nonlinear Korteweg-de Vries equation on bounded domains with various boundary conditions. In particular, L. Rosier in [13] derived the exact boundary controllability for the nonlinear KdV equation on bounded domains, when the initial and terminal states are sufficiently small (see also the work by C. Laurent and L. Rosier [8]).

B. Zhang in [20] considered a distributed control for the generalized Boussinesq equation on the periodic domain $S$, showing that the system is exactly controllable for either the control $f$ acting on the whole domain $S$, or the control $f$ acting only on a sub-domain of $S$, imposing a smallness condition on the initial and terminal states in the later case (see also the work [10]).

M. Chapouly in [6] considered both the global exact controllability to the trajectories (one boundary left control) and the global exact controllability of a nonlinear Korteweg-de Vries equation in a bounded interval. In this work, there were introduced, in addition to the boundary value at the left endpoint, a control in the boundary value at the right endpoint and an internal control in the right member of the equation, assumed both to be $x$-independent, proving the global exact controllability to the trajectories, for any positive time $T$. It was also obtained the global exact controllability, for any positive time, by introducing a fourth control on the first derivative at the right endpoint.

E. Cerpa and E. Crépeau in [4] established the controllability of the improved Boussinesq equation posed either on a bounded or periodic domain. For the bounded domain $[0, l]$ with boundary control, they showed that the linearized equation is not spectrally controllable, and so not null controllable, either. However, they proved an approximate controllability result.

J. Amara and H. Bouzidi in [1] considered the exact boundary controllability for a Boussinesq equation with variable physical parameters, when the control is acting at the end $x = l$ of the interval $[0, l]$. They established that the linearized problem is exactly controllable in any time $T > 0$, using a spectral analysis together with the moment method. They also established the local exact controllability for the nonlinear problem by fixed point argument.

C. Laurent, F. Linares and L. Rosier in [11] and F. Linares and L. Rosier in [9] considered the control and stabilization problem for the Benjamin-Ono equation in $L^2(T)$, where $T = \mathbb{R}/(2\pi \mathbb{Z})$. In the latter work, authors proved a controllability result in $L^2(T)$ that allows to get the global controllability in large time.

Our goal in this work is to study the generalized Benney-Luke equation

$$u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} + pu_x u_x^{-1} u_{xx} + 2 u_x^p u_{xt} = 0,$$

from a control point of view with a forcing term $f = f(x, t)$ added to the equation as a control input, which is assumed to be supported in a given open set $\omega \subseteq S$. In the controllability theory the following problems are essential:
Exact Control Problem: Given \( T > 0 \), an initial state \((u_0, v_0)^t\) and a terminal state \((u_T, v_T)^t\) in an appropriate space, can one determine an appropriate control input \( f \) so that the model (2) has a solution \( u \) satisfying that \( u_x(0) = u_0, u_x(T) = u_T, u_t(0) = v_0 \) and \( u_t(T) = v_T \)?

Stabilization Problem: Can one find a feedback law \( f = \Gamma(u_x, u_t) \) so that the resulting closed-loop system
\[
 u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} + pu_t u_x^{p-1} u_{xx} + 2u_x^p u_{xt} + \Gamma(u_x, u_t) = 0,
\]
is asymptotically stable as \( t \to \infty \)?

We point out that we follow the classical approach used by D. Russell and B. Zhang in the case of the KdV equation (see [17]), and by B. Zhang in the case of the good Boussinesq equation (see [20]).

Hereafter, for \( s \in \mathbb{R} \), we define the space \( V^{s+1} \) given by
\[
 V^{s+1} = \{ \psi \in C_{\text{per}}^\infty (\mathbb{R}) : \psi_x \in H^s(S) \}.
\]
We also defined the mass of \( u \) on \( S \) by \( [u] = \int_S u(x) \, dx \) (average of \( u \) in \( S \)).

In the case that the support of \( f \) is acting on the whole \( S \), we get an exact controllability result for the forced Benney-Luke equation (2) with a distributed control \( f \) having the general form
\[
 f(x, t) = g(x) \left( h(x, t) - \int_S g(y) h(y, t) \, dy \right) := Lh(x, t)
\]
with \( g \) being a smooth function defined on \( S \) having \( \int_S g(x) \, dx = 1 \), to guarantee that \( f(\cdot, t) \) has mean zero on \( S \).

**Theorem 1.1.** (Exact controllability) Let \( T > 0 \) and \( s \geq 2 \) be given. Suppose that the function \( g \) smooth on \( S \) is such that
\[
 |g(x)| \geq \beta > 0, \quad x \in S,
\]
then for any \((u_0, v_0)^t, (u_T, v_T)^t \in V^{s+1} \times H^s(S)\) such that
\[
 [u_0] = [u_T] = 0,
\]
there exists a control function \( f \in L^2((0,T), H^{s-2}(S)) \) such that the initial value problem associated with the nonlinear Benney-Luke equation (2) has a solution
\[
 u \in C((0,T), V^{s+1}) \bigcap C^1((0,T), H^s(S))
\]
satisfying that
\[
 u_x(0) = u_0, \quad u_x(T) = u_T, \quad u_t(0) = v_0, \quad u_t(T) = v_T.
\]

In the case that the support of \( f \) is acting on a proper subdomain \( \omega \subset S \), we get an exact controllability result for the forced Benney-Luke equation (2) with a distributed control \( f \) having the general form (3), assuming that the initial and terminal states are small.

**Theorem 1.2.** (Local exact controllability) Let \( T > 0 \) and \( s \geq 2 \) be given. Then there exists a \( \delta > 0 \) such that for \((u_0, v_0)^t, (u_T, v_T)^t \in V^{s+1} \times H^s(S)\) satisfying conditions
\[
 [u_0] = [u_T] = 0
\]
and
\[
 \|u_0\|_{V^{s+1}} + \|u_T\|_{V^{s+1}} + \|v_0\|_{H^s(S)} + \|v_T\|_{H^s(S)} < \delta,
\]
there exists a control function \( f = Lh \) with \( h \in L^2((0,T), H^{s-2}(S)) \) such that the initial value problem associated with the Benney-Luke equation (2) has a solution \( u \in C((0,T), V^{s+1}) \cap C^1((0,T), H^s(S)) \) satisfying that

\[
u_s(0) = u_0, \quad u_s(T) = u_T, \quad u_t(0) = v_0, \quad u_t(T) = v_T.
\]

We obtain previous result due to the exact controllability result for the system associated to the forced Benney-Luke equation with a distributed control \( f \) of the form (3) acting on the whole domain \( S \) or a subdomain of \( S \). Hereafter, we set

\[
h_1(z,w) = z + B^{-1}\left(\frac{1}{p+1}w^{p+1}\right),
\]

where \( B^{-1} \) is the inverse for the operator \( B = I - b\partial_x^2 \) with \( b > 0 \).

Regarding the stabilization issue, we get the following result.

**Theorem 1.3. (Stabilization)** Let \( K > 0 \) and \( s \geq 1 \) be given. For \( (u_0, v_0) \in V^{s+1} \times H^s(S) \) with

\[
[u_0] = 0, \quad [h_1(u_0, v_0)] = 0,
\]

the forced generalized Benney-Luke equation

\[
\begin{cases}
u_{tt} - u_{xx} + au_{xxxx} - bu_{uxt} + pu_t u_x^{p-1} u_{xx} + 2u_x^p u_{xt} + KBL(h_1(u, u_x)),
\end{cases}
\]

\[
u_x(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x),
\]

has a unique solution \( u \in C(\mathbb{R}, V^{s+1}) \cap C^1(\mathbb{R}, H^s) \). Moreover, there exist constants \( M > 0 \) and \( \gamma > 0 \) such that

\[
||u_s(\cdot, t)||_{H^s} + ||u_t(\cdot, t)||_{H^s} \leq M e^{-\gamma t} C_1(||u_0||_{H^s}, ||v_0||_{H^s}),
\]

where \( C_1(\cdot, \cdot) \) is a positive function.

The paper is organized as follows. In section 2, we consider the well-posedness of the initial value problem for the forced generalized Benney-Luke model on the periodic domain \( S \),

\[
\begin{cases}
u_{tt} - u_{xx} + au_{xxxx} - bu_{uxt} + pu_t u_x^{p-1} u_{xx} + 2u_x^p u_{xt} = f, \quad x \in S, \quad t \in \mathbb{R},
\end{cases}
\]

\[
u_x(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x),
\]

where the forcing term \( f \equiv f(x, t) \) is known as the control input. In section 3, we perform the spectral analysis for the operator

\[
\mathcal{M} = \begin{pmatrix} 0 & \partial_x \\ (B^{-1}A)\partial_x & 0 \end{pmatrix},
\]

defined in the space \( \mathcal{H}_s = H^s(S) \times H^s(S) \), where the operators \( A = I - a\partial_x^2 \) and \( B = I - b\partial_x^2 \) for \( a, b > 0 \) are defined using Fourier series. For the linear operator \( \mathcal{M} \), we prove the existence of a discrete spectral decomposition since the eigenvectors form a Riesz basis of the space \( \mathcal{H}_s \), for \( s \geq 0 \). In Section 4, we establish linear exact controllability by showing the existence of a bounded linear operator \( \Psi \) from the initial/end state pairs \((q_0, r_0), (q_T, r_T)\) in the space \( \mathcal{H}_s \), to the corresponding control \( h \) in the space \( L^2((0,T), H^{s-2}) \). In section 5, we prove the nonlinear controllability, which will be global in the case \( |g(x)| \geq \beta > 0 \) on \( S \) and local for general \( g \), by imposing smallness of the initial and terminal states in the later case. In section 6, we establish the stabilization result for the Benney-Luke with a forcing feedback depending on \((u_x, u_t)\).
2. Preliminary. Before we go further, we consider the Benney-Luke equation on the periodic domain $S$, the unit circle in the plane, consisting with the bounded domain $(-\pi, \pi)$ with periodic boundary conditions. For $s \in \mathbb{R}$, the periodic Sobolev space $H^s(S)$ is defined by

$$H^s(S) = \{ \psi \in C^\infty_{\text{per}}(\mathbb{R}) : (1 + n^2)^{s/2} \psi_n \in L^2(\mathbb{Z}) \}, \quad ||\psi||_s = \left( \sum_{n \in \mathbb{Z}} (1 + n^2)^s |\psi_n|^2 \right)^{1/2},$$

where $C^\infty_{\text{per}}(\mathbb{R})$ denotes the space of periodic functions on $\mathbb{R}$ with period $2\pi$ and $\psi_n$ denotes the $n$-Fourier coefficient with respect to the spatial variable $x$. We also define the Sobolev space $\dot{H}^s(S) = \{ \psi \in H^s(S) : \psi_0 = 0 \}$, which means that $\psi$ has average zero on $S$,

$$\int_S \psi(x) \, dx = 0.$$ 

On the other hand, for any $d > 0$, we define the operators $D = I - d\partial_x^2$ and $D^{-1}$, respectively by

$$D(\psi)(x) = \sum_{n \in \mathbb{Z}} (1 + dn^2) \psi_n e^{inx}, \quad D^{-1}(\psi)(x) = \sum_{n \in \mathbb{Z}} \frac{\psi_n}{1 + dn^2} e^{inx}.$$

We clearly have that $D : H^s(S) \to H^{s-2}(S)$ and $D^{-1} : H^s(S) \to H^{s+2}(S)$ with $s \in \mathbb{R}$. Moreover, we also have the $n$-Fourier symbols for the operators $\partial_x$ and $B^{-1}A$ are given respectively by

$$(\partial_x)_n = in, \quad (B^{-1}A)_n = \frac{1 + an^2}{1 + bn^2} := \Lambda^2(n).$$

In particular, we also have that $n$-Fourier symbol for the operator $M$ is given by

$$M_n = \begin{pmatrix} 0 & in \\ in\Lambda^2(n) & 0 \end{pmatrix}.$$

We begin this section by rewriting the Benney-Luke equation (2) ($f = 0$) as a first order equation. To do this, we consider the following variables

$$q = u_x, \quad r = u_t + B^{-1} \left( \frac{1}{p+1} q^{p+1} \right).$$

In this case, we see formally that

$$q_t = u_{xt} = r_x - \partial_x B^{-1} \left( \frac{1}{p+1} q^{p+1} \right),$$

and that the Benney-Luke equation (2) can be expressed as

$$\partial_t \left( Bu_t + \frac{1}{p+1} q^{p+1} \right) = Aq_x - \partial_x (q^p u_t).$$

So, we have that

$$r_t = \partial_x B^{-1} Aq - \partial_x B^{-1} \left( q^p \left( r - B^{-1} \left( \frac{1}{p+1} q^{p+1} \right) \right) \right).$$
Thus, the Benney-Luke equation (2) can be written as the first order system
\[
\begin{align*}
q_t &= r_x - \partial_x B^{-1} \left( \frac{1}{p+1} q^{p+1} \right), \\
r_t &= B^{-1} A q_x - \partial_x B^{-1} \left( q^p \left( r - B^{-1} \left( \frac{1}{p+1} q^{p+1} \right) \right) \right),
\end{align*}
\]
or equivalent to the first order system
\[
\partial_t U = MU + G(q,r),
\]
where \( U = (q,r)^t \), \( M \) and \( G \) are given by
\[
M = \begin{pmatrix} 0 & \partial_x \\ (B^{-1} A) \partial_x & 0 \end{pmatrix}, \quad G(q,r) = \begin{pmatrix} G_1(q,r) \\ G_2(q,r) \end{pmatrix},
\]
with \( G_i \) for \( i = 1, 2 \) being
\[
\begin{align*}
G_1(q,r) &= -\partial_x B^{-1} \left( \frac{1}{p+1} q^{p+1} \right), \\
G_2(q,r) &= -\partial_x B^{-1} \left( q^p \left( r - B^{-1} \left( \frac{1}{p+1} q^{p+1} \right) \right) \right).
\end{align*}
\]

**Remark 1.** Before we go further, if \( F = (0,F_2)^t \) is such that \( F_2 \) has average zero on \( S \), then we want to discuss the relationship between the Boussinesq type system
\[
\partial_t U(x,t) = MU(x,t) + G(q,r)(x,t) + F(x,t),
\]
and the forced generalized Benney-Luke equation (2). If we assume that \( q \in H^s(S) \) has zero average in \( S \) \( (q_0 = 0) \), then we have that the function \( u = \partial_x^{-1} q \in \mathcal{V}^{s+1} \) is such that \( q = u_x \), where the operator \( \partial_x^{-1} : H^s(S) \to H^{s+1}(S) \) is defined as
\[
\partial_x^{-1}(\psi)(x) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\psi_k}{ik} e^{ikx}.
\]

We first note that the quantity
\[
\mathcal{I}(q)(t) = \int_S q(t,x) \, dx,
\]
is conserved in time for classical solutions and even for mild solutions, as long as the solution exists. So, if we consider the Cauchy problem associated with the system in the variable \( (q,r) \) with the initial data \( q_0 \in H^s(S) \) with mean zero property
\[
\int_S q_0(x) \, dx = 0.
\]
Then, as long as the solution exists for \( t \), we have that
\[
\int_S q(x,t) \, dx = 0,
\]
meaning that \( q(\cdot,t) \in H^s(S) \) and has the mean zero property, as the solution exists for \( t \). In this case, the function \( u(x,t) = \partial_x^{-1} q(x,t) \in \mathcal{V}^{s+1} \) is such that
\[
q(x,t) = u_x(x,t), \quad u_t(x,t) = \partial_x^{-1} q_t(x,t) = r(x,t) - B^{-1} \left( \frac{1}{p+1} (u_x(x,t))^{p+1} \right).
\]

From this we see that \( u \) solves the forced Benney-Luke equation (2) for a special \( f \). In fact,
\[
Br_t = \partial_x A u_x - \partial_x (u_x^p u_t) + BF_2 = Au_{xx} - pu_x^{p-1} u_t - u_x^p u_{tx} + BF_2.
\]
On the other hand, we also have that

\[ Bu_t = Br_t - \partial_t \left( \frac{1}{p + 1} u_x^{p+1} \right) \]

\[ = Au_x - pu_x^{p-1} u_t - u_x^{p+1} u_{tx} + BF_2 - \partial_t \left( \frac{1}{p + 1} u_x^{p+1} \right) \]

\[ = Au_x - pu_x^{p-1} u_t - 2u_x^{p-1} u_{tx} + BF_2. \]

In other words, we have that \( u \) satisfies formally the forced generalized Benney-Luke equation

\[ u_t - u_{xxx} + au_{xxx} - bu_{xxx} + pu_t u_x^{p-1} u_{xx} + 2u_x^{p-1} u_{xt} = BF_2. \]  \hspace{1cm} (11)

Due to this fact, we focus our attention in the exact controllability and the stabilization associated with the system in the variable \((q,r)\).

2.1. Semigroup. We first note that \( M \in L_b(H_{s}, H_{s-1}) \) is the infinitesimal generator of a bounded \( C_0 \)-group \( S \) on \( H_s \). In fact, if we set

\[ S(t) = \sum_{n \in \mathbb{Z}} \hat{S}_n(t)e^{inx}, \]

where \( \hat{S}_n \) is defined using Fourier series by

\[ \hat{S}_n(t) = \begin{pmatrix} \cos(\sqrt{n^2} \Lambda^2(n)t) & i \sin(\text{sign}(n) \sqrt{n^2} \Lambda^2(n)t) \\ i \Lambda(n) \sin(\text{sign}(n) \sqrt{n^2} \Lambda^2(n)t) & \cos(\sqrt{n^2} \Lambda^2(n)t) \end{pmatrix}, \]  \hspace{1cm} \text{(12)}

\[ = \left( \begin{array}{cc} \frac{\Lambda(n) + i\text{sign}(n) \sqrt{n^2} \Lambda^2(n)t}{2} & i \frac{\Lambda(n) - i\text{sign}(n) \sqrt{n^2} \Lambda^2(n)t}{2} \\ i \Lambda(n) \frac{\sqrt{n^2} \Lambda^2(n)t}{2} & i \frac{\Lambda(n) + i\text{sign}(n) \sqrt{n^2} \Lambda^2(n)t}{2} \end{array} \right), \]  \hspace{1cm} \text{(13)}

where \( \Lambda_n = i \text{sign}(n) \sqrt{n^2} \Lambda^2(n) \) and \( \Lambda^2(n) = \frac{1 + qn^2}{1 + bn^2} \). In other words, \( \hat{S}_n \) is the exponential matrix of \( e^{tM_n} \), with \( M_n \) being the \( n \)-Fourier coefficient of \( M \). Moreover, we have the following estimate,

**Theorem 2.1.** Let \( T > 0 \) and \( s \geq 0 \) be given. Then there is a constant \( C > 0 \) such that for any given \( U \in H_s \),

\[ \sup_{t \in (0,T)} \| S(t)U \|_{H_s} \leq C(a,b) \| U \|_{H_s}, \]  \hspace{1cm} (14)

and for any given \( F = (0,F_2)^t \) with \( F_2 \in L^1((0,T), H^s(S)) \),

\[ \sup_{t \in (0,T)} \left\| \int_0^t S(t - \tau) F(\tau) \, d\tau \right\|_{H_s} \leq C(a,b) \| F_2 \|_{L^1((0,T), H^s(S))}. \]  \hspace{1cm} (15)

**Proof.** We see directly that for any \((q,r) \in \mathbb{R}^2\),

\[ \left\| \hat{S}_n(t)(q,r) \right\|^2 \leq C(a,b)(|q|^2 + |r|^2), \]

showing the estimate (14). From this estimate, we see that

\[ \sup_{t \in (0,T)} \left\| \int_0^t S(t - \tau) F(\tau) \, d\tau \right\|_{H_s} \leq \sup_{t \in (0,T)} \int_0^t \| S(t - \tau) F(\tau) \|_{H_s} \, d\tau \]

\[ \leq C(a,b) \| F_2 \|_{L^1((0,T), H^s(S))}. \]
In order to perform the nonlinear estimates in the case of the forced system (10), we combine the Sobolev Multiplication Law (see [18]) and a result by D. Roumégoux ([14]).

**Lemma 2.2. (Sobolev Multiplication Law)** Let \( s_1, s_2 \geq t \) be such that \( s_1 + s_2 > t + \frac{1}{2} \), there is a continuous multiplication map \( H^{s_1}(S) \times H^{s_2}(S) \to H^s(S) \),

\[
(\psi, \varphi) \mapsto \psi \varphi
\]

satisfying the estimate

\[
\|\psi \varphi\|_t \leq \|\psi\|_{s_1} \|\varphi\|_{s_2}.
\]

**Lemma 2.3. (D. Roumégoux [14])** Let \( D = I - \partial_x^2 \) with \( d > 0 \) and \( s \geq 0 \). Then, there exists a constant \( C(d) > 0 \) such that for any \( \psi, \varphi \in H^s \),

\[
\|D^{-1} \partial_x (\psi \varphi)\|_s \leq C(d) \|\psi\|_s \|\varphi\|_s.
\]

Now, we are able to get the nonlinear estimates. We set for \( y > 0 \) the following notation:

\[
\eta_p(y) = y^{2p} + y^p.
\]

**Lemma 2.4.** Let \( s > \frac{1}{2} \) and \( p \geq 1 \). There exists \( C(p, b) > 0 \) such that if \( U_i \in \mathcal{H}_s \) for \( i = 1, 2 \), then

\[
\|G(U_1) - G(U_2)\|_{\mathcal{H}_s} \leq C(p, b) \|U_1 - U_2\|_{\mathcal{H}_s} (\eta_p(\|U_1\|_{\mathcal{H}_s}) + \eta_p(\|U_2\|_{\mathcal{H}_s})).
\]

Moreover, for \( U \in \mathcal{H}_s \), we have that

\[
\|G(U)\|_{\mathcal{H}_s} \leq C(p, b) \eta_p(\|U\|_{\mathcal{H}_s}) \|U\|_{\mathcal{H}_s}.
\]

**Proof.** We must recall that \( H^s(S) \) is an algebra for \( s > \frac{1}{2} \). Let \( U_i = (q_i, r_i) \in \mathcal{H}_s \) with \( i = 1, 2 \). Then, we see using the estimate (17) in Lemma 2.3 that there is a positive constant \( C_1(p, b) \) such that

\[
\|G_1(U_1) - G_1(U_2)\|_{\mathcal{H}_s} = \frac{1}{p+1} \|\partial_x B^{-1} (q_1^{p+1} - q_2^{p+1})\|_s
\]

\[
\leq C_0(p, b) \|q_1^{p+1} - q_2^{p+1}\|_s
\]

\[
\leq C_1(p, b) \|q_1 - q_2\|_s (\|q_1\|_H^s + \|q_2\|_H^s).
\]

On the other hand, we set

\[
G_3(q, r) = -\partial_x B^{-1} (q^p r), \quad G_4(q, r) = -\partial_x B^{-1} (q^p B^{-1} (q^{p+1})).
\]

We see from the same approach that

\[
\|G_3(U_1) - G_3(U_2)\|_{\mathcal{H}_s} \leq C_2(p, b) (\|q_1 - q_2\|_{H^s} + \|r_1 - r_2\|_s) (\|q_1\|_H^s + \|q_2\|_H^s).
\]

Now, we perform the estimate for \( G_4 \) for \( p > 1 \), since \( p = 1 \) is similar.

\[
\|G_4(U_1) - G_4(U_2)\|_{\mathcal{H}_s} = \|\partial_x B^{-1} (q_1^p B^{-1} (q_1^{p+1}) - q_2^p B^{-1} (q_2^{p+1}))\|_{H^s}
\]

\[
\leq C_3 \left( \|q_1^p - q_2^p\|_s B^{-1} (q_1^{p+1}) + \|B^{-1} (q_1^{p+1}) - B^{-1} (q_2^{p+1})\|_s \right)
\]

\[
\leq C_4 \left( \|q_1 - q_2\|_s (\|q_1\|_H^{s-1} + \|q_2\|_H^{s-1}) \|B^{-1} (q_1^{p+1})\|_{s+2} + \|B^{-1} (q_2^{p+1})\|_{s+2} \right)
\]

\[
\leq C_4 \left( \|q_1 - q_2\|_s (\|q_1\|_H^{s-1} + \|q_2\|_H^{s-1}) \|q_1\|_H^{s+1} + \|q_2\|_H^{s+1} \right),
\]
where $C_l = C_l(p, b)$ for $l = 3, 4$ and we are using the Sobolev Multiplication Law with $t = s, s_1 = s$ and $s_2 = s + 2$, and that 
\[
\|B^{-1}(\psi^{p+1})\|_{s+2} \leq C(b)\|\psi^{p+1}\|_{s} \leq C(b)\|\psi\|^p_s,
\]
\[
\|q_{s+2}^{p+1} - q_{s+2}^{p+1}\|_{s} \leq C(p)\|q_1 - q_2\|_{s} (\|q_1\|_{s} + \|q_2\|_{s}).
\]

Now, using Hölder inequality with $\alpha = \frac{p+1}{p-1}$ for the product $\|q_2\|_{s}^{p-1}\|q_1\|_{s}^{p+1}$ and $\alpha = 2$ for the product $\|q_2\|_{s}\|q_1\|_{s}^p$, we conclude that 
\[
\|G_4(U_1) - G_4(U_2)\|_{H_s} \leq C_1(p, b)\|q_1 - q_2\|_{s} (\|q_1\|_{s}^{2p} + \|q_2\|_{s}^{2p}).
\]

In other words, we have shown that 
\[
\|G(U_1) - G(U_2)\|_{H_s} \leq C(p, b)\|U_1 - U_2\|_{H_s} (\|U_1\|_{H_s}^{2p} + \|U_1\|_{H_s}^{p} + \|U_2\|_{H_s}^{2p} + \|U_2\|_{H_s}^{p}).
\]

The second inequality follows by taking $U_1 = U$ and $U_2 = 0$. 

From the classical semigroup theory, we are able to establish the local well-posedness associated with the system (10) in the space $H_s$. For this, using the Banach fixed point Theorem, we will show the existence of a solution for the integral equation 
\[
U(t) = S(t)U_0 + \int_0^t S(t - \tau)(G(U)(\cdot, \tau) + F(\tau))\,d\tau. \tag{19}
\]

Before we go further, for $T > 0$ and $s \geq 0$, we define the space $Z_T^s = C([0, T], H_s)$.

**Theorem 2.5.** Let $s, p$ be as in Lemma 2.4 and $T > 0$ be given. For any given $U_0 \in H_s$ and for any given $F = (0, F_2)^t$ with $F_2 \in L^1((0, T), H^*(S))$, there exists $T^* > 0$, depending only on $\|U_0\|_{H_s}$ and $\|F_2\|_{L^1((0, T), H^*(S))}$, such that initial value problem

\[
\partial_t U(x, t) = \mathcal{A}U(x, t) + G(U)(x, t) + F(x, t), \quad U(0) = U_0 = (q_0, r_0)^t, \tag{20}
\]

has a unique solution $U \in Z_T^s$, and the corresponding solution map 
\[
\Phi : H_s \times L^1((0, T), H^*(S)) \to Z_T^s.
\]

given by $\Phi(U_0, f) = U$ is continuous.

**Proof.** For $R > 0$, let $\mathcal{X}_T^R = \{U \in Z_T^s : \sup_{t \in (0, T)} \|U\|_{H_s} \leq R\}$ be a bounded set of the space $Z_T^s$. Using the semigroup $S(t)$, we have that the integral form associated with (20) can be written as

\[
U(t) = S(t)U_0 + \int_0^t S(t - \tau)(G(U)(\cdot, \tau) + F(\tau))\,d\tau.
\]

So, we consider the mapping $\Phi$ defined as 
\[
\Phi(U) = S(t)U_0 + \int_0^t S(t - \tau)(G(U)(\cdot, \tau) + F(\tau))\,d\tau, \quad U \in H_s.
\]

From previous results, we conclude that 
\[
\sup_{t \in (0, T)} \int_0^t \|G(U)(\cdot, \tau)\|_{H_s}\,d\tau \leq \sup_{t \in (0, T)} \eta_p (\|U(\cdot, \tau)\|_{H_s}) \|U(\cdot, \tau)\|_{H_s},
\]

where $C_l = C_l(p, b)$ for $l = 3, 4$ and we are using the Sobolev Multiplication Law.
and also that
\[
\sup_{t \in (0, T)} \| \Phi(U) \|_{\mathcal{H}_s} \leq C(a, b) \| U_0 \|_{\mathcal{H}_s} + C(a, b) \int_0^t (\| G(U)(\cdot, \tau) \|_{\mathcal{H}_s} + \| F(\tau) \|_{\mathcal{H}_s}) \, d\tau
\]
\[
\leq C \left( \| U_0 \|_{\mathcal{H}_s} + \| F_2 \|_{L^1((0, T), \mathcal{H}_s(S))} \right) + C \int_0^t \| G(U)(\cdot, \tau) \|_{\mathcal{H}_s} \, d\tau.
\]
If we choose \( R > 0 \), and \( T^* > 0 \) such that
\[
R = 2C_1 \left( \| U_0 \|_{\mathcal{H}_s} + \| F_2 \|_{L^1((0, T), \mathcal{H}_s)} \right), \quad 2C_1(p, b)\eta_p(R)T^* = \frac{1}{N},
\]
with
\[
N > 2^{2p}(p + 1).
\]
(21)
From these choices, we conclude for any \( U \in \mathcal{X}_f^R \) that
\[
\sup_{t \in (0, T^*)} \| \Phi(U) \|_{\mathcal{H}_s} < \frac{R}{2} + T^*C_1\eta_p(R)R < R.
\]
From the same type of computations, we have that
\[
\sup_{t \in (0, T^*)} \| \Phi(U) - \Phi(V) \|_{\mathcal{H}_s} \leq T^*C_1 \sup_{0 \leq \tau \leq T^*} \| U(\tau) - V(\tau) \|_{\mathcal{H}_s} \left( \eta_p(\| U(\tau) \|_{\mathcal{H}_s}) + \eta_p(\| V(\tau) \|_{\mathcal{H}_s}) \right)
\]
\[
\leq 2T^*C_1\eta_p(R) \sup_{0 \leq \tau \leq T^*} \| U(\tau) - V(\tau) \|_{\mathcal{H}_s}
\]
\[
\leq C \sup_{0 \leq \tau \leq T^*} \| U(\tau) - V(\tau) \|_{\mathcal{H}_s}, \quad (C < 1)
\]
meaning that \( \Phi \) is a contraction on \( \mathcal{X}_f^R \), and so, the Contraction Mapping Theorem guaranties the existence and uniqueness of a local solution to the initial value problem (20) in the space \( Z^r \). \( \square \)

3. **Spectral analysis.** Let us consider the orthogonal basis for \( \mathcal{H}_s \) given by
\[
e_{1,k} = \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}, \quad e_{2,k} = \begin{pmatrix} 0 \\ e^{ikx} \end{pmatrix}
\]
for \( k \in \mathbb{Z} \). We see directly that
\[
\mathcal{M}(e_{1,k}, e_{2,k}) = (e_{1,k}, e_{2,k})\mathcal{M}_k.
\]
Moreover, we have that the eigenvalues for \( \mathcal{M}_k \) are
\[
\lambda_{1,k} = i\sqrt{k^2 + \Lambda(k)}, \quad \lambda_{2,k} = -i\sqrt{k^2 + \Lambda(k)}
\]
with corresponding eigenvector,
\[
\tilde{e}_{1,0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{e}_{2,0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tilde{e}_{1,k} = \begin{pmatrix} 1 \\ \lambda_{1,k} \end{pmatrix}, \quad \tilde{e}_{2,k} = \begin{pmatrix} 1 \\ \lambda_{2,k} \end{pmatrix}
\]
for \( k \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\} \). We also have that
\[
\mathcal{M}(e_{1,k}, e_{2,k})(\tilde{e}_{1,k}, \tilde{e}_{2,k}) = (e_{1,k}, e_{2,k})\Sigma_k(\tilde{e}_{1,k}, \tilde{e}_{2,k}) = (\lambda_{1,k}(e_{1,k}, e_{2,k})\tilde{e}_{1,k}, \lambda_{2,k}(e_{1,k}, e_{2,k})\tilde{e}_{2,k}),
\]
(22)
meaning that \( \lambda_{1,k} \) and \( \lambda_{2,k} \) are the eigenvalues for the operator \( \mathcal{M} \) with corresponding eigenvectors
\[
\eta_{j,k} = (e_{1,k}, e_{2,k})\tilde{e}_{j,k}, \quad j = 1, 2, \quad k \in \mathbb{Z}.
\]
On the other hand, we also have that
\[
\lim_{k \to \pm \infty} \frac{\lambda_{1,k}}{k} = \pm i, \quad \lim_{k \to \pm \infty} \frac{\lambda_{2,k}}{k} = \mp i,
\]
which implies that
\[
\lim_{k \to \pm \infty} (e_{1,k}, e_{2,k}) = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}
\]
and also that
\[
\lim_{k \to \pm \infty} \det(e_{1,k}, e_{2,k}) = \mp 2i \neq 0.
\]
In other words, \(\{\eta_{0,1}, \eta_{0,2}, \eta_{1,k}, \eta_{2,k} : k \in \mathbb{Z}^*\}\) forms a Riesz bases for \(H_s\), and also that \(\{\nu_{0,1}, \nu_{0,2}, \nu_{1,k}, \nu_{2,k} : k \in \mathbb{Z}^*\}\) forms an orthonormal bases for \(H_s\) with
\[
\nu_{j,k} = \frac{\eta_{j,k}}{\|\eta_{j,k}\|_{H_s}},
\]
so we have that
\[
\langle \nu_{j,k}, \nu_{l,m} \rangle_{H_s} = \begin{cases} 1, & j = l, \, k = m \\ 0, & \text{otherwise.} \end{cases}
\]
Moreover, we also have for \(j = 1, 2\) that \(\nu_{j,k}^{(2)} = b_{j,k} e^{ikx}\) with
\[
0 < C_1(a, b) \leq |b_{1,k}| \leq C_2(a, b), \quad k \in \mathbb{Z}.
\]
So, we have the following result,

**Theorem 3.1.** Let \(\lambda_k\) and \(\phi_{j,k}\) for \(j = 1, 2\) be given by
\[
\lambda_k = \text{isign}(k) \sqrt{k^2 \Lambda(k)}, \quad k \in \mathbb{Z},
\]

\[
\phi_{1,k} = \begin{cases} \nu_{1,k}, & k = 0, 1, 2, 3, \ldots \\ \nu_{2,k}, & k = -1, -2, -3, \ldots \end{cases}, \quad \phi_{2,k} = \begin{cases} \nu_{1,-k}, & k = 1, 2, 3, \ldots \\ \nu_{2,-k}, & k = 0, -1, -2, -3, \ldots \end{cases},
\]

a) \(\sigma(M) = \{\lambda_k : k \in \mathbb{Z}\}\), in which \(\lambda_k\) is a double eigenvalue with eigenvectors \(\phi_{1,k}\) and \(\phi_{2,k}\) for \(k \in \mathbb{Z}\).

b) The set \(\{\phi_{j,k} : k \in \mathbb{Z}\}\) forms an orthogonal bases for \(H_s\) such that for any \(U \in H_s\), we have the Fourier expansion
\[
U = \sum_{k \in \mathbb{Z}} (\alpha_{1,k} \phi_{1,k} + \alpha_{2,k} \phi_{2,k})
\]
with
\[
\alpha_{j,k} = \langle \phi_{j,k}, U \rangle_{H_s}, \quad k \in \mathbb{Z}.
\]

4. **Linear exact controllability.** In this section, we establish the linear exact controllability associated with the initial value problem (20) in the case \(G(U) \equiv 0\) and \(F = (0, B^{-1} f)\). In this case for \(s \geq 2\) and \(f \in L^1((0, T), H^{s-2}(S))\), the solution \(U\) of the first order linear system is given by
\[
U(t) = S(t)U_0 + \int_0^t S(t - \tau) F(\cdot, \tau) d\tau.
\]
Now using the spectral analysis on the operator \(\mathcal{M}\), we have that
\[
U(t) = \sum_{n \in \mathbb{Z}} e^{\lambda_n t} (\alpha_{1,n} \phi_{1,n} + \alpha_{2,n} \phi_{2,n})
\]
\[
+ \sum_{n \in \mathbb{Z}} \int_0^t e^{\lambda_n (t-\tau)} (\beta_{1,n}(\tau) \phi_{1,n} + \beta_{1,n}(\tau) \phi_{2,n}) d\tau, \quad (25)
\]
where \( \alpha_{j,n} \) and \( \beta_{j,n}(\tau) \) for \( j = 1, 2 \) are given by

\[
\alpha_{j,n} = \langle U_0, \phi_{j,n} \rangle_{\mathcal{H}_s},
\]

\[
\beta_{j,n}(\tau) = \langle F(\cdot, \tau), \phi_{j,n} \rangle_{\mathcal{H}_s} = \left\langle B^{-1} f(\cdot, \tau), \phi_{j,n}^{(2)} \right\rangle_s = \left\langle f(\cdot, \tau), B^{-1} \phi_{j,n}^{(2)} \right\rangle_{s-2,s+2},
\]

with \( \phi^{(l)} \) denoting the \( l \) component of \( \phi \), \( \langle \cdot, \cdot \rangle_s \) denoting the inner product in \( H^s(S) \) and \( \langle \cdot, \cdot \rangle_{s-2,s+2} \) denoting the pairing between \( H^{s-2}(S) \) and \( H^{s+2}(S) \). In the case \( f(x,t) = (Lh)(x,t) \) defined in (3), we have that

\[
\beta_{j,n}(\tau) = \left\langle Lh(\cdot, \tau), B^{-1} \phi_{j,n}^{(2)} \right\rangle_{s-2,s+2} = \left\langle h(\cdot, \tau), LB^{-1} \phi_{j,n}^{(2)} \right\rangle_{s-2,s+2}.
\]

Before we go further, we observe that \( P = \{e^{\lambda_k t} : k \in \mathbb{Z}\} \) is a Riesz basis for its closed span \( P_T \) generated in \( L^2(0,T) \), with a unique dual Riesz basis given by \( Q = \{q_k : k \in \mathbb{Z}\} \) satisfying that

\[
\int_0^T q_j(t)e^{\lambda_k t} dt = \delta_{jk}, \quad j,k \in \mathbb{Z}.
\]

We first assume that \( f \) has the form (3) with \( h \) given by the expansion

\[
h(x,t) = c_{1,0}q_0(t)L(B^{-1} \phi_{1,0}^{(2)}) + \sum_{l \in \mathbb{Z}^+} q_l(t) \left( c_{1,l}L(B^{-1} \phi_{1,l}^{(2)}) + c_{2,l}L(B^{-1} \phi_{2,l}^{(2)}) \right)
\]

\[
= \sum_{l \in \mathbb{Z}} q_l(t) \left( c_{1,l}L(B^{-1} \phi_{1,l}^{(2)}) + c_{2,l}L(B^{-1} \phi_{2,l}^{(2)}) \right), \quad (c_{2,0} = 0).
\]

From this representation, using the notation \( \rho_{j,n,k} = \left\langle L(B^{-1} \phi_{j,n}^{(2)}) \right\rangle_k \), we have that

\[
\int_0^T \beta_{j,0}(\tau) d\tau = \int_0^T \left\langle h(\cdot, \tau), L(B^{-1} \phi_{j,0}^{(2)}) \right\rangle_{s-2,s+2} d\tau
\]

\[
= \sum_k (1+k^2)s^{-2} \int_0^T \left( \int_S h(y,\tau) e^{-iky} \ dy \right)(\rho_{j,n,0}) d\tau
\]

\[
= \sum_k (1+k^2)s^{-2} \int_S \left( \int_0^T h(y,\tau) d\tau \right) e^{-iky} \ dy(\rho_{j,n,0})
\]

\[
= \sum_k (1+k^2)s^{-2} \int_S \left( c_{0,l}L(B^{-1} \phi_{2,0}^{(2)}) e^{-iky} \ dy(\rho_{j,n,0})
\]

\[
= c_{1,0} \left\langle L(B^{-1} \phi_{2,0}^{(2)}), L(B^{-1} \phi_{2,0}^{(2)}) \right\rangle_{s-2},
\]

where \( \langle \cdot, \cdot \rangle_l \) denotes the inner product in \( H^l(S) \) for \( \ell \geq 0 \). Now, for \( n \in \mathbb{Z}^+ \), we have that

\[
\int_0^T e^{-\lambda_n \tau} \beta_{j,n}(\tau) d\tau = \int_0^T e^{-\lambda_n \tau} \left\langle h(\cdot, \tau), L(B^{-1} \phi_{j,n}^{(2)}) \right\rangle_{s-2,s+2} d\tau
\]

\[
= \sum_k (1+k^2)s^{-2} \int_0^T e^{-\lambda_n \tau} \left( \int_S h(y,\tau) e^{-iky} \ dy \right)(\rho_{j,n,k}) d\tau
\]

\[
= \sum_k (1+k^2)s^{-2} \int_S \left( \int_0^T e^{-\lambda_n \tau} h(y,\tau) d\tau \right) e^{-iky} \ dy(\rho_{j,n,k}).
\]
On the other hand, we have that
\[ \int_S \int_0^T e^{-\lambda_n \tau} h(y, \tau) e^{-iky} d\tau dy = \int_S \left( c_{1,n} L(B^{-1} \phi_{1,n}^{(2)}) + c_{2,n} L(B^{-1} \phi_{2,n}^{(2)}) \right) e^{-iky} dy. \]
So, we also have that
\[
\sum_k (1 + k^2)^{s-2} \int_S \left( c_{1,n} L(B^{-1} \phi_{1,n}^{(2)}) + c_{2,n} L(B^{-1} \phi_{2,n}^{(2)}) \right) e^{-iky} dy \rho_{j,n,k} = c_{1,n} \left( L(B^{-1} \phi_{1,n}^{(2)}), L(B^{-1} \phi_{j,n}^{(2)}) \right)_S + c_{2,n} \left( L(B^{-1} \phi_{2,n}^{(2)}), L(B^{-1} \phi_{j,n}^{(2)}) \right)_S.
\]
Putting these estimates, we get that
\[
\int_0^T e^{-\lambda_n \beta_j n} (\tau) d\tau = c_{1,n} \left( L(B^{-1} \phi_{1,n}^{(2)}), L(B^{-1} \phi_{j,n}^{(2)}) \right)_S + c_{2,n} \left( L(B^{-1} \phi_{2,n}^{(2)}), L(B^{-1} \phi_{j,n}^{(2)}) \right)_S. \tag{28}
\]
We note for \( l \in \mathbb{Z} \) that
\[ L(B^{-1} e^{il\tau})(x) = (1 + b l^2)^{-1} L(e^{i\tau})(x) = (1 + b l^2)^{-1} g(x)(e^{i\tau}x - g_{-l}), \]
where we are using that
\[ g(x) = \sum_{m \in \mathbb{Z}} g_m e^{imx}. \]
We also have that \( k \)-Fourier coefficient for \( L(B^{-1} e^{il\tau}) \) is
\[ \sigma_{lk} = \int_S L(B^{-1} e^{il\tau})(x) e^{-ikx} dx = (1 + b l^2)^{-1} (g_{k-l} - g_{-l} g_k). \tag{29} \]
On the other hand, we have that
\begin{align*}
L(B^{-1} \phi_{1,l}^{(2)}) &= \frac{1}{(1 + b l^2)} L(\phi_{1,l}^{(2)}) = \frac{b_{1,l}}{(1 + b l^2)} L(e^{il\tau}) = \frac{b_{1,l} g(\cdot)(e^{il\tau} - g_{-l})}{(1 + b l^2)}, \tag{30} \\
L(B^{-1} \phi_{2,l}^{(2)}) &= \frac{1}{(1 + b l^2)} L(\phi_{2,l}^{(2)}) = \frac{b_{2,l}}{(1 + b l^2)} L(e^{-il\tau}) = \frac{b_{2,l} g(\cdot)(e^{-il\tau} - g_{l})}{(1 + b l^2)}. \tag{31}
\end{align*}
Now, we expand \( L(B^{-1} \phi_{r,l}^{(2)}) \) with \( r = 1, 2 \) as
\[ L(B^{-1} \phi_{r,l}^{(2)})(x) = \sum_{k \in \mathbb{Z}} \alpha_{r,l,k} e^{ikx}, \tag{32} \]
with the \( k \)-coefficient given by
\begin{align*}
\alpha_{1,l,k} &= \left< L(B^{-1} \phi_{1,l}^{(2)}), e^{ik} \right>_{L^2(S)} = b_{1,l} (1 + b l^2)^{-1} (g_{k-l} - g_{-l} g_k), \\
\alpha_{2,l,k} &= \left< L(B^{-1} \phi_{2,l}^{(2)}), e^{ik} \right>_{L^2(S)} = b_{2,l} (1 + b l^2)^{-1} (g_{k+l} - g_{l} g_k).
\end{align*}
From previous calculations, we see that
\begin{align*}
|\alpha_{1,l,k}|^2 &\leq C(b) (1 + b l^2)^{-2} (|g_{k-l}|^2 + |g_{-l} g_k|^2), \\
|\alpha_{2,l,k}|^2 &\leq C(b) (1 + b l^2)^{-2} (|g_{k+l}|^2 + |g_l|^2 |g_k|^2).
\end{align*}
Before we go further, we see the following estimate for \( s \geq 2 \) and \( l, k \in \mathbb{R} \)
\[ (1 + |k + l|)^{2(s-2)} \leq (1 + |l|)^{2(s-2)} (1 + |l|)^{2(s-2)}. \]
From this estimate, we conclude for \( s \geq 2 \) that
\[
\sum_{k \in \mathbb{Z}} (1 + |k|)^{2(s-2)} |g_k|^2 \leq \sum_{k \in \mathbb{Z}} (1 + |k + l|)^{2(s-2)} |g_k|^2 \leq (1 + |l|)^{2(s-2)} \sum_{k \in \mathbb{Z}} (1 + |k|)^{2(s-2)} |g_k|^2 \leq (1 + |l|)^{2(s-2)} \|g\|_{s-2}^2.
\]
So, we finally get that
\[
\sum_{k \in \mathbb{Z}} (1 + |k|)^{2(s-2)} |\alpha_{1,l,k}|^2 \leq C(b)(1 + |l|)^{-4} \big( (1 + |l|)^{2(s-2)} \|g\|_{s-2}^2 + |g_{-l}|^2 |g_k|^2 \big) + |g_{-l}|^2 \sum_{k \in \mathbb{Z}} (1 + |k|)^{2(s-2)} |g_k|^2 \leq C(b)(1 + |l|)^{-4} \big( (1 + |l|)^{2(s-2)} + |g_{-l}|^2 \big) \|g\|_{s-2}^2 \leq C_1(b)(1 + |l|)^{2(s-4)} \|g\|_{s-2}^2.
\]
In a similar fashion, we have that
\[
\sum_{k \in \mathbb{Z}} (1 + |k|)^{2(s-2)} |\alpha_{2,l,k}|^2 \leq C_1(b)(1 + |l|)^{2(s-4)} \|g\|_{s-2}^2.
\]
From (26) and using the expansion (32), we see that
\[
h(x, t) = \sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} q_l(t) (c_{1,l} \alpha_{1,l,k} + c_{2,l} \alpha_{2,l,k}) e^{ikx}.
\]
If we assume that
\[
\sum_{l \in \mathbb{Z}} (1 + |l|)^{2(s-4)} (|c_{1,l}|^2 + |c_{2,l}|^2)
\]
is finite, then we have that \( h \in L^2([0, T]; H^{s-2}(S)) \), according with previous computations. In fact,
\[
\|h\|_{L^2([0, T]; H^{s-2}(S))}^2 = \int_0^T \sum_{k \in \mathbb{Z}} (1 + |k|)^{2(s-2)} \left| \sum_{l \in \mathbb{Z}} q_l(t) (c_{1,l} \alpha_{1,l,k} + c_{2,l} \alpha_{2,l,k}) \right|^2 dt \leq C \sum_{k \in \mathbb{Z}} (1 + |k|)^{2(s-2)} \left( \sum_{l \in \mathbb{Z}} (|c_{1,l}|^2 |\alpha_{1,l,k}|^2 + |c_{2,l}|^2 |\alpha_{2,l,k}|^2) \right) \leq C \sum_{l \in \mathbb{Z}} (|c_{1,l}|^2 + |c_{2,l}|^2) \sum_{k \in \mathbb{Z}} (1 + |k|)^{2(s-2)} \left( |\alpha_{1,l,k}|^2 + |\alpha_{2,l,k}|^2 \right) \leq C_2(b) \|g\|_{s-2}^2 \sum_{l \in \mathbb{Z}} (1 + |l|)^{2(s-4)} (|c_{1,l}|^2 + |c_{2,l}|^2).
\]
Now, we define the space \( \mathcal{Y}_s \subset \mathcal{X}_s = \mathcal{H}_s \times \mathcal{H}_s \) by
\[
\mathcal{Y}_s = \left\{ (U_0, U_T) \in \mathcal{X}_s \times \mathcal{X}_s : \int_S U_0^{(1)} dx = \int_S U_T^{(1)} dx = 0 \right\},
\]
where \( U^{(1)} \) denotes the first component of \( U \).

**Theorem 4.1.** Let \( s \geq 2 \). For a given \( T > 0 \), there exist a bounded linear operator

\[
\Psi_T : \mathcal{Y}_s \to L^2((0, T), H^{s-2}(S))
\]

such that for any \((U_0, U_T) \in \mathcal{Y}_s\), the solution \( U(t) \) of the initial value problem associated with the linear system

\[
\partial_t U(x, t) = MU(x, t) + F(x, t),
\]

for initial data \( U_0 \) and \( F = (0, B^{-1}L\Psi_T(U_0, U_T))^t \), satisfies that

\[
U(T) = U_T.
\]

Moreover, we also have that

\[
\|\Psi_T(U_0, U_T)\|_{L^2((0, T), H^{s-2}(S))} \leq C(T)(\|U_0\|_{H^s} + \|U_T\|_{H^s}).
\]

**Proof.** Let \( U_0 \) and \( U_T \) be having the following decompositions

\[
U_0 = \sum_{k \in \mathbb{Z}} (\alpha_{1,k} \phi_{1,k} + \alpha_{2,k} \phi_{2,k}), \quad U_T = \sum_{k \in \mathbb{Z}} (\gamma_{1,k} \phi_{1,k} + \gamma_{2,k} \phi_{2,k}).
\]

We know that

\[
U(T) = \sum_{n \in \mathbb{Z}} e^{\lambda_n T} (\alpha_{1,n} \phi_{1,n} + \alpha_{2,n} \phi_{2,n}) + \sum_{n \in \mathbb{Z}} \int_0^T e^{\lambda_n (T-\tau)} (\beta_{1,n}(\tau) \phi_{1,n} + \beta_{2,n}(\tau) \phi_{2,n}) \, d\tau.
\]

So, in each node, we have for \( j = 1, 2 \) and \( n \in \mathbb{Z}^* \) that

\[
\alpha_{j,0} + \int_0^T \beta_{j,0}(\tau) \, d\tau = \gamma_{j,0}
\]

\[
\alpha_{j,n} + \int_0^T e^{-\lambda_n \tau} \beta_{j,n}(\tau) \, d\tau = \gamma_{j,n} e^{-\lambda_n T}.
\]

Now, we suppose the \( \Psi_T(U_0, U_T) \) has the form (26). In other words,

\[
\Psi_T(U_0, U_T)(\cdot, t) = c_{1,0} q_0(t) L(B^{-1} \phi_{2,0}^{(2)}) + \sum_{l \in \mathbb{Z}^*} q_l(t) \left( c_{1,l} L(B^{-1} \phi_{1,l}^{(2)}) + c_{2,l} L(B^{-1} \phi_{2,l}^{(2)}) \right). \tag{36}
\]

From the computations above, we are able to characterize the coefficients \( c_{0,1,l} \) and \( c_{2,l} \) later on in such a way that the series (26) converges appropriately. In fact, for \( n = 0 \), we have from (27) that

\[
c_{1,0} \left( L(B^{-1} \phi_{2,0}^{(2)}), L(B^{-1} \phi_{2,0}^{(2)}) \right)_{s-2} = \alpha_{1,0} - \gamma_{1,0},
\]

and for \( l \in \mathbb{Z}^* \), we have from (28) that \( c_{1,l} \) and \( c_{2,l} \) must satisfy the linear system

\[
\begin{pmatrix}
  a_{11} & a_{21} \\
  a_{12} & a_{22}
\end{pmatrix} \begin{pmatrix}
  c_{1,l} \\
  c_{2,l}
\end{pmatrix} = \begin{pmatrix}
  -\alpha_{1,l} + \gamma_{1,l} e^{-\lambda_l T} \\
  -\alpha_{2,l} + \gamma_{2,l} e^{-\lambda_l T}
\end{pmatrix},
\]

where \( a_{jr} = \left( L(B^{-1} \phi_{r,j}^{(2)}), L(B^{-1} \phi_{r,j}^{(2)}) \right)_{H^s(S)} \). Using the fact that \( L(B^{-1} e^{ikr}) \) and \( L(B^{-1} e^{ik\cdot}) \) are linear independent, we conclude that \( L(B^{-1} \phi_{1,l}^{(2)}) \) and \( L(B^{-1} \phi_{r,l}^{(2)}) \)
are linear independent for any \( l \in \mathbb{Z}^* \). On the other hand, we have that
\[
\begin{align*}
a_{11} &= \left\langle L(B^{-1}\phi_{1,l}^{(2)}), L(B^{-1}\phi_{1,l}^{(2)}) \right\rangle_{s-2} = b_{l1}^2 (1 + b l^2)^{-2} \left\langle L(e^{i l t}), L(e^{i l t}) \right\rangle_{s-2} \\
a_{12} &= \left\langle L(B^{-1}\phi_{1,l}^{(2)}), L(B^{-1}\phi_{2,l}^{(2)}) \right\rangle_{s-2} = b_{l1} b_{l2} (1 + b l^2)^{-2} \left\langle L(e^{i l t}), L(e^{-i l t}) \right\rangle_{s-2} \\
a_{21} &= \left\langle L(B^{-1}\phi_{2,l}^{(2)}), L(B^{-1}\phi_{1,l}^{(2)}) \right\rangle_{s-2} = b_{l1} b_{l2} (1 + b l^2)^{-2} \left\langle L(e^{-i l t}), L(e^{i l t}) \right\rangle_{s-2} \\
a_{22} &= \left\langle L(B^{-1}\phi_{2,l}^{(2)}), L(B^{-1}\phi_{2,l}^{(2)}) \right\rangle_{s-2} = b_{l2}^2 (1 + b l^2)^{-2} \left\langle L(e^{-i l t}), L(e^{-i l t}) \right\rangle_{s-2},
\end{align*}
\]
which implies that
\[
\begin{align*}
\|L(B^{-1}\phi_{1,l}^{(2)})\|_{s-2}^2 &= (1 + b l^2)^{-2} \|L(\phi_{1,l}^{(2)})\|_{s-2}^2, \quad (37) \\
\left\langle L(B^{-1}\phi_{1,l}^{(2)}), L(B^{-1}\phi_{2,l}^{(2)}) \right\rangle_{s-2} &= (1 + b l^2)^{-2} \left\langle L(\phi_{1,l}^{(2)}), L(\phi_{2,l}^{(2)}) \right\rangle_{s-2} \quad (38) \\
&= (1 + b l^2)^{-2} \left\langle L(\phi_{1,l}^{(2)}), L(\phi_{2,l}^{(2)}) \right\rangle_{s-2}. \quad (39)
\end{align*}
\]
Then we have that
\[
\begin{align*}
\Delta_l &= \begin{vmatrix}
    a_{11} & a_{21} \\
    a_{12} & a_{22}
\end{vmatrix} \\
&= \|L(B^{-1}\phi_{1,l}^{(2)})\|^2 \|L(B^{-1}\phi_{2,l}^{(2)})\|^2 - \left( \left\langle L(B^{-1}\phi_{1,l}^{(2)}), L(B^{-1}\phi_{2,l}^{(2)}) \right\rangle_{s-2} \right)^2 \\
&= (1 + b l^2)^{-4} \left( \|L(\phi_{1,l}^{(2)})\|_{H^{s-2}} \|L(\phi_{2,l}^{(2)})\|_{s-2} - \left| \left\langle L(\phi_{1,l}^{(2)}), L(\phi_{2,l}^{(2)}) \right\rangle_{s-2} \right| \right)^2 \\
&= (1 + b l^2)^{-4} \begin{vmatrix}
    \hat{a}_{11} & \hat{a}_{21} \\
    \hat{a}_{12} & \hat{a}_{22}
\end{vmatrix} \\
&= (1 + b l^2)^{-4} \hat{\Delta}_l \neq 0.
\end{align*}
\]
Moreover, using that \( \nu_{r,l}(x) = b_{r,l} e^{i l x} \) and the estimate (24), we have that
\[
\|L(\phi_{2,l}^{(2)})\|_{s-2} \geq |b_{r,l}| \geq C > 0, \quad \left\langle L(\phi_{1,l}^{(2)}), L(\phi_{2,l}^{(2)}) \right\rangle_{s-2} \to 0, \quad n \to \infty.
\]
So, we conclude that \( c_{1,l} \) and \( c_{2,l} \) are uniquely determine by
\[
\begin{align*}
c_{1,l} &= \begin{vmatrix}
    \alpha_{1,l} - \gamma_{1,l} e^{-\lambda_T} & a_{21} \\
    \alpha_{2,l} - \gamma_{2,l} e^{-\lambda_T} & a_{22}
\end{vmatrix} / \Delta_l, \quad c_{2,l} = \begin{vmatrix}
    \alpha_{1,l} - \gamma_{1,l} e^{-\lambda_T} & a_{21} \\
    \alpha_{2,l} - \gamma_{2,l} e^{-\lambda_T} & a_{22}
\end{vmatrix} / \Delta_l.
\end{align*}
\]
Using the estimates of E. Cerpa and I. Rivas in [5] for the Boussinesq equation and B. Zhang in [20] for the generalized Boussinesq equation, we also have for any \( l \in \mathbb{Z} \) that \( \hat{\Delta}_l \geq C_0 > 0 \), and so we get that
\[
\begin{align*}
|c_{r,l}|^2 &\leq \frac{C(b)(1 + b l^2)^{-4} \left( \sum_{i,j=1}^2 |\hat{a}_{ij}|^2 \right) \left( |\alpha_{1,l}|^2 + |\alpha_{2,l}|^2 + |\gamma_{1,l}|^2 + |\gamma_{1,l}|^2 \right)}{(1 + b l^2)^{-8} \hat{\Delta}_l^2} \\
&\leq C_1 (1 + |l|)^8 \left( |\alpha_{1,l}|^2 + |\alpha_{2,l}|^2 + |\gamma_{1,l}|^2 + |\gamma_{1,l}|^2 \right).
\end{align*}
\]
From this and the estimate (34), we conclude that
\[
\|\Psi_T(U_0,U_1)\|^2_{L^2((0,T);H^{-2}(S))} \leq C_2\|g\|^2_{s-2} \sum_{l \in \mathbb{Z}} (1 + |l|)^{2(s-4)}(|c_{1,l}|^2 + |c_{2,l}|^2)
\leq C_3\|g\|^2_{s-2} \sum_{l \in \mathbb{Z}} (1 + |l|)^{2s} (|\alpha_{1,l}|^2 + |\alpha_{2,l}|^2 + |\gamma_{1,l}|^2 + |\gamma_{1,l}|^2)
\leq C_3(\|U_0\|^2_{H^s(S)} + \|U_T\|^2_{H^s(S)})\|g\|_{H^{-2}(S)}.
\]

Moreover, we also have the following modification of previous result for \(s \geq 0\).

**Corollary 1.** Let \(s \geq 0\) and let \(T > 0\) be given. There exists a bounded linear operator
\[
\Upsilon_T : \mathcal{Y}_s \to L^2((0,T),H^s(S))
\]
such that for any \((U_0,U_T) \in \mathcal{Y}_s\), the solution \(U(t)\) of the initial value problem associated with the linear system
\[
\partial_t U(x,t) = \mathcal{M}U(x,t) + F(x,t),
\]
for initial data \(U_0\) and \(F = (0,LY_T(U_0,U_T))^t\), satisfies that
\[
U(T) = U_T.
\]
Moreover, we also have that
\[
\|
\Upsilon_T(U_0,U_T)\|_{L^2((0,T),H^s(S))} \leq C(T)(\|U_0\|_{\mathcal{H}_s} + \|U_T\|_{\mathcal{H}_s}).
\]

The proof follows by the same type of arguments as in previous theorem. In this case, we have that \(\Upsilon_T(U_0,U_T)\) has the form
\[
\Upsilon_T(U_0,U_T)(x,t) = c_{1,0}q_0(t)L(\phi_1^{(2)}) + \sum_{l \in \mathbb{Z}} q_l(t) \left( c_{1,l}L(\phi_{1,l}^{(2)}) + c_{2,l}L(\phi_{2,l}^{(2)}) \right).
\]

5. **Nonlinear control.**

**Global case.** We first discuss the exact controllability issue for the system (10) in the case that the support is the whole \(S\).

**Theorem 5.1.** (Exact Controllability) Let \(T > 0\) and \(s \geq 2\) be given and suppose that \(g\) is a smooth function such that
\[
|g(x)| \geq \beta > 0, \quad x \in S.
\]
Then for any \((q_0,r_0)^t,(q_T,r_T)^t \in \mathcal{H}_s\) such that
\[
[q_0] = [q_T] = 0,
\]
there exists a control function \(H \in L^2((0,T),\mathcal{H}_{s-2}(S))\) such that the initial value problem associated with the nonlinear system
\[
\partial_t U(x,t) = \mathcal{M}U(x,t) + G(U)(x,t) + B^{-1}LH(x,t),
\]
has a solution \(U \in C((0,T),\mathcal{H}_s) \cap C^1((0,T),\mathcal{H}_{s-1})\) that satisfies for \(t \geq 0\),
\[
U(0,x) = (q_0(x),r_0(x))^t, \quad U(T,x) = (q_T(x),r_T(x))^t, \quad [U^{(1)}(t,\cdot)] = 0.
\]
Proof. From Theorem 4.1, for $s \geq 2$ and a given $T > 0$, there exist a bounded linear operator
\[ \Psi_T : \mathcal{Y}_s \to L^2((0, T); H^{s-2}(S)) \]
such that for any $(U_0, U_T) \in \mathcal{Y}_s$, there exists $U \in C((0, T), \mathcal{H}_s) \cap C^1((0, T), \mathcal{H}_{s-1})$ satisfying
\[ U_t = \mathcal{M}U + B^{-1}L\Psi_T(U_0, U_T), \quad U(0) = U_0, \quad U(T) = U_T. \] (45)

So, we know that $U = (q, r)^T \in H^s(S) \times H^s(S)$ and $G(q, r) \in H^s(S) \times H^s(S)$. Now, as done by D. Russell and B. Zhang in [17] for the case the KdV equation, we define the linear operator $\tilde{L} : H^s_g(S) \to H^s_g(S)$ as
\[ \tilde{L}(v) = v - \int_S g(y)v(y) \, dy, \]
where the weighted space $H^s_g(S)$ is the space $H^s(S)$ endowed with the norm
\[ ||v||_{H^s_g(S)} = \langle gv, v \rangle_{H^s(S)}, \]
derived from the assumption that $g > 0$ on $S$ (otherwise, we change $g$ for $-g$). For this operator, we have the following properties: $\tilde{L}$ is a Fredholm operator, $\tilde{L}^* = \tilde{L}$, $N(\tilde{L}) = \{1\}$, and $\tilde{L}$ has a bounded inverse from $H^s_g(S)/\{1\} \to H^s_g(S)$. Now, we see that
\[ \langle BG_1(q, r), 1 \rangle_{H^s_g} = -\langle \partial_x \left( \frac{q^{p+1}}{1+q^{p+1}} \right), 1 \rangle_{H^s_g} = -\langle \partial_x \left( \frac{1}{q^{p+1}} \right), 1 \rangle_{H^s_g} = 0, \]
\[ \langle BG_2(q, r), 1 \rangle_{H^s_g} = -\langle \partial_x \left( \frac{q^p \left( r - B^{-1} \left( \frac{1}{1+q^{p+1}} \right) \right)}{g} \right), 1 \rangle_{H^s_g} = -\langle \partial_x \left( q^p \left( r - B^{-1} \left( \frac{1}{1+q^{p+1}} \right) \right) \right), 1 \rangle_{H^s_g} = 0. \]
From these facts, we conclude that there are functions $h_2(\cdot, t), h_3(\cdot, t) \in H^{s-2}(S)$ such that for each fixed $t$,
\[ Lh_2 = \partial_x \left( \frac{1}{q^{p+1}} \right), \]
\[ Lh_3 = \partial_x \left( q^p \left( r - B^{-1} \left( \frac{1}{1+q^{p+1}} \right) \right) \right). \]

So, adding and subtracting $G(U)$ in the right side of equation (45), we conclude that
\[ U_t = \mathcal{M}U + G(U) + \begin{pmatrix} 0 \\ B^{-1}Lh_2 \\ B^{-1}Lh_3 \end{pmatrix}, \quad U(0) = U_0, \quad U(T) = U_T. \] (46)

So, we set $H \in L^2([0, T], \mathcal{H}_{s-2})$ as follows
\[ H = \left( \Psi_T(U_0, U_T) + h_3 \right). \]
Local case. Now, we consider the local exact controllability problem. From the semigroup theory, we know that the solution of the initial value problem associated with the system

$$\partial_t U(x, t) = \mathcal{M}U(x, t) + G(U)(x, t) + F(x, t), \quad U(0) = U_0,$$

with $F = (0, B^{-1}Lh)^t$ has the general form

$$U(t) = \mathcal{S}(t)U_0 + \int_0^t \mathcal{S}(t - \tau)(G(U)(\cdot, \tau) + F(\cdot, \tau)) d\tau.$$

Note that, if we define $\Psi_U = (0, B^{-1}Lh)^t$ using Theorem 4.1 with $G(U) = 0$ as

$$w_1(U, T) = \int_0^T \mathcal{S}(T - \tau)G(U)(\cdot, \tau) d\tau$$

and choose $h$ using Theorem 4.1 with $G(U) = 0$ as

$$h = \Psi_T(U_0, U_T - w_1(U, T)),$$

then, we see that

$$U(t) = \mathcal{S}(t)U_0 + \int_0^t \mathcal{S}(t - \tau)\left[G(U)(\cdot, \tau) + \left(B^{-1}L\Psi_T(U_0, U_T - w_1(U, T))\right)(\cdot, \tau)\right] d\tau$$

satisfies that $U(0) = U_0$. On the other hand, if

$$U(t) = V(t) + \int_0^t \mathcal{S}(t - \tau)G(U)(\cdot, \tau) d\tau,$$

due to the definition of $\Psi_T$, we have that

$$U(T) = V(T) + w_1(U, T) = U_T - w_1(U, T) + w_1(U, T) = U_T.$$

**Theorem 5.2.** (Exact Controllability) For $T > 0$ and $s \geq 2$ be given. Then there exists $\delta > 0$ such that for $(U_0, U_T) \in \mathcal{Y}_s$ and $\|U_0, U_T\|_{\mathcal{Y}_s} < \delta$, there exists a control function $h \in L^2((0, T), H^s(S))$ such that the initial value problem associated with nonlinear system

$$\partial_t U(x, t) = \mathcal{M}U(x, t) + G(U)(x, t) + F(x, t),$$

with $F = (0, B^{-1}Lh)^t$ has a solution $U \in C([0, T), \mathcal{H}_s) \cap C^2([0, T), \mathcal{H}_{s-1})$ that satisfies

$$U(0) = U_0, \quad U(T) = U_T.$$

**Proof.** According to previous discussion, in order to obtain the exact controllability in $\mathcal{H}_s$, we need to prove that the operator

$$\Phi(U) = \mathcal{S}(T)U_0 + \int_0^T \mathcal{S}(T - \tau)\left[G(U)(\cdot, \tau) + \left(B^{-1}L\Psi_T(U_0, U_T - w_1(U, T))\right)(\cdot, \tau)\right] d\tau$$

from $\mathcal{H}_s$ to $\mathcal{H}_s$ defined by the right side of formula (49) has a fixed point. Recall that for $T > 0$ and $s \geq 0$, we define the space $Z_T^s = C([0, T], \mathcal{H}_s)$. Then, if $R > 0$, we consider the bounded set of the space $Z_T^s$ defined by

$$X_T^R = \{U \in Z_T^s : \sup_{t \in (0, T)} \|U\|_{\mathcal{H}_s} \leq R\}.$$

If $\tilde{F}$ is given by

$$\tilde{F} = \left(B^{-1}L\Psi_T(U_0, U_T - w_1(U, T))\right),$$
using the estimate in Theorem 4.1, we have that
\[
\|\tilde{F}\|_{L^1(0,T,H^s)} \leq \|L\Psi_T(U_0, U_T - w_1(U, T))\|_{L^1(0,T,H^{s-2}(S))}
\leq C(T) \left( \|U_0\|_{H^s(S)} + \|U_T\|_{H^s(S)} + \|w_1(U, T)\|_{H^s(S)} \right).
\]
Moreover, from the nonlinear estimates (18) in Lemma 2.4, we have that
\[
\sup_{t \in (0,T)} \|w_1(U, T)\|_{H^s} \leq C_2(T)\eta_p(R)R.
\]
Then we see that
\[
\sup_{t \in (0,T)} \|\Phi_1(U)\|_{H^s} \leq C_2(T) \left( \|U_0\|_{H^s(S)} + \|U_T\|_{H^s(S)} \right) + 2TC_2(T)\eta_p(R)R
\]
(see proof Theorem 2.5). If we choose \( \delta > 0 \) and \( R > 0 \) such that
\[
4C_2\delta(T) < R, \quad 4C_2\eta_p(R)T < 1,
\]
we conclude for any \( U \in X_T^R \) that
\[
\sup_{t \in (0,T)} \|\Phi_1(U)\|_{H^s} \leq R,
\]
provided that \( \|U_0\|_{H^s(S)} < \delta, \|U_T\|_{H^s(S)} < \delta \). Now, using the same type of computations in Theorem 2.5 and using the nonlinear estimates (18) in Lemma 2.4, we have that \( \Phi_1 \) is a contraction on \( X_T^B \). From the Contraction Mapping Theorem, there is a unique local solution (nonlinear control) for the integral equation (49) in the space \( Z_T^s \).

**On the exact controllability for the generalized Benney-Luke model.** As we discussed in the well-posedness section, it is possible to build solutions for the forced generalized Benney-Luke model (2) from solutions for the Boussinesq type system (10). So, we derive the global and the local exact controllability for the Benney-Luke model from the analysis for the Boussinesq type system (10).

**Proof of Theorem 1.1.** From the hypothesis, we have that
\[
[u_0] = [u_T] = 0.
\]
So, we set
\[
q_0 = u_0, \quad q_T = u_T, \quad r_0 = u_0, \quad r_T = v_T.
\]
Then, we have that \( [q_0] = [q_T] = 0 \). From the global exact controllability result (Theorem 5.1), we have that
\[
\begin{align*}
U_t = MU + G(U) + \begin{pmatrix} 0 \\ B^{-1}L\Psi_T(u_0, u_T, v_0, v_T) \end{pmatrix} + \begin{pmatrix} B^{-1}Lh_2 \\ B^{-1}Lh_3 \end{pmatrix},
\end{align*}
\]
has a solution \( U = (q, r)^T \in H_s(S) \) that satisfies \( U(0) = U_0 \) and \( U(T) = U_T \). If we set \( u(x, t) = \partial_x^{-1}q(x, t) \), which makes sense since \( q(\cdot, t) \) has average zero for any \( t \), then from the first component we see that
\[
u_t(x, t) = \partial_x^{-1}q_t(x, t) = r - B^{-1}\left( \frac{1}{\rho + 1}(u_x(x, t))^\rho \right) + \partial_x^{-1}B^{-1}Lh_2.
\]
We point out that \( B^{-1}Lh_2 \) and \( B^{-1}Lh_3 \) have mean zero on \( S \), so \( \partial_x^{-1}(B^{-1}Lh_2) \) and \( \partial_x^{-1}(B^{-1}Lh_3) \) make complete sense. Using this in the second equation, we get that
\[
\begin{align*}
u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} + pu_t u_{x}^{\rho - 1}u_x + 2u_x^p u_{xt} + H(x, t) = 0,
\end{align*}
\]
where the control $H$ is given by

$$H = -L \Psi_T(u_0, u_T, v_0, v_T) - \partial_x^{-1} L \partial_t h_2 - L h_3 - L h_4,$$

with $h_4 \in H^s(S)$ being taking in such a way that

$$L h_4(x, t) = \partial_x \left( (u_x(x, t))^p B^{-1} \partial_x^{-1} (L h_2(x, t)) \right),$$

using the same argument in the proof of Theorem 5.1, due to the fact that the term $\partial_x \left( (u_x(x, t))^p B^{-1} \partial_x^{-1} (L h_2(x, t)) \right) \in H^s(S)$ has mean zero on $S$ for any $t \in \mathbb{R}$. □

**Proof of Theorem 1.2.** From the hypothesis, we have that

$$[u_0] = [u_T] = 0.$$

So, we set

$$q_0 = u_0 \quad q_T = u_T, \quad r_0 = h_1(u_0, v_0), \quad r_T = h_1(u_T, v_T),$$

Then, we have that $[u_0] = [u_T] = 0$. From the local exact controllability result (Theorem 5.2), there exists a function $h \in L^2((0, T), H^{s-2}(S))$ such that the initial value problem associated with the system

$$U_t = MU + G(U) + F,$$

with $F = (0, B^{-1} L h)^t$, has a solution $U = (q, r)^t \in H_q$ that satisfies $U(0) = U_0$ and $U(T) = U_T$. As we discussed in Remark (1) in Section 2, we know that $u(x, t) = \partial_x^{-1} q(x, t)$ satisfies the forced generalized Benney-Luke equation

$$u_{ttt} - u_{xx} + a u_{xxxx} - b u_{xxtt} + p u_t u_x^{p-1} u_{xx} + 2 u_x^p u_{xt} - L h = 0.$$

Moreover, we also have that $u_x(x, 0) = u_0$ and $u_x(x, T) = u_T$. On the other hand, we have that $r(\cdot, 0) = v_0 + B^{-1} \left( \frac{1}{p+1} u_0^{p+1} \right) = r_0$, which implies that

$$r(x, 0) = u_t(x, 0) + B^{-1} \left( \frac{1}{p+1} (u_x(x, 0))^{p+1} \right)$$

$$= u_x(x, 0) + B^{-1} \left( \frac{1}{p+1} (u_0(x))^{p+1} \right)$$

$$= v_0(x) + B^{-1} \left( \frac{1}{p+1} (u_0(x))^{p+1} \right) = r_0(x),$$

meaning that $u_t(x, 0) = v_0(x)$. A similar argument shows that $u_t(x, T) = v_T(x)$. □

6. **Stabilization problem.** In this section, we study the stabilization problem associated with the nonlinear system

\begin{align*}
q_t &= r_x - \partial_x B^{-1} \left( \frac{1}{p+1} q^{p+1} \right), \\
\frac{r_t}{t} &= B^{-1} A q_x - \partial_x B^{-1} \left( q^p \left( r - B^{-1} \left( \frac{1}{p+1} q^{p+1} \right) \right) \right) - KL(r),
\end{align*}

with the conditions

$$q(x, 0) = q_0(x), \quad r(x, 0) = r_0(x),$$

having the property $[q_0] = [r_0] = 0$ and $K > 0$. 

\[51\]

\[52\]
6.1. Linear problem. We start to show the exponential decay result for the linear equation
\[
\begin{align*}
q_t &= r_x, \\
r_t &= B^{-1} Aq_x - KL(r),
\end{align*}
\] (53)
with $K$ being a positive constant and the conditions
\[
q(x,0) = q_0(x), \quad r(x,0) = r_0(x),
\] (54)
having the property $[q_0] = 0$.

**Theorem 6.1.** Given $s \geq 1$, for any $(q_0,r_0)^t \in \mathcal{H}_s$ with $[q_0] = 0$, the problem (53)-(54) has a unique solution $(q,r)^t \in C(\mathbb{R}, \mathcal{H}_s)$. Moreover, there exist constants $C, \gamma > 0$ such that
\[
\| (q(\cdot), t), (r(\cdot), t)) - (0, [r_0]) \|_{\mathcal{H}_s} \leq C e^{-\gamma t} \| (q_0, r_0) - (0, [r_0]) \|_{\mathcal{H}_s}.
\] (55)

**Proof.** Without loss of generality, we perform the proof assuming that $[q_0] = 0$, since the couple $(q,r - [r_0])^t$ is also a solution of the system (53)-(54).

We must note that the local existence result is a direct consequence of the semigroup properties in Theorem 2.1. In other words, given $(q_0,r_0) \in \mathcal{H}_s$, there is $T^* > 0$ (depending on $\| (q_0, r_0) \|_{\mathcal{H}_s}$) and a unique solution
\[
(q,r)^t \in C([0, T^*), \mathcal{H}_s).
\]
We start to show that the estimate (55) holds for $s = 1$. A direct computation shows that the energy of the solution $(q,r)^t$ is given by
\[
E(t) = \int_S (q(x,t)A(q)(x,t) + r(x,t)B(r)(x,t)) dx \simeq \| (q(\cdot), t), (r(\cdot), t)) \|^2_{\mathcal{H}_s}.
\]
We note for $t \in [0, T^*)$ that there is $\alpha \in (0, 1)$ such that
\[
E(t) \leq \alpha E(0).
\] (56)
To see this, we note that for given $t \in [0, T^*)$, using integration by parts yields,
\[
\frac{d}{dt} \int_S (q^2 + aq_x^2 + r^2 + Br^2) dx = \int_S (qq_x + aq_x q_{x\tau} + rr_\tau + Br_x r_{x\tau}) dx
\]
\[
= \int_S (q_\tau(q - aq_x) + r_\tau(r - Br_x)) dx d\tau
\]
\[
= \int_S (q_\tau Aq + r_\tau Br) dx
\]
\[
= -K \int_S rLrdx.
\]
On the other hand, using this estimate
\[
E(t) - E(0) = \frac{1}{2} \int_0^t \left( \frac{d}{d\tau} \int_S (q^2(x,\tau) + aq_x^2(x,\tau) + r^2(x,\tau) + Br_x^2(x,\tau)) dx \right) d\tau
\]
\[
= -K \int_0^t \int_S r(x,\tau)Lr(x,\tau) dx d\tau
\]
\[
= -K \int_0^t \int_S r(x,\tau)g(x) \left( r(x,\tau) - \int_S g(s)r(s,\tau) ds \right) dx d\tau
\]
\[
= -K \int_0^t \int_S g(x) \left( r(x,\tau) - \int_S g(s)r(s,\tau) ds \right)^2 dx d\tau,
\]
where the last equality holds since that

\[
\int_S g(x) \left( r(x, \tau) - \int_S g(s)r(s, \tau) ds \right) dx = 0.
\]

On the other hand, for \( t \in [0, T^*) \) we have that \((q(\cdot, t), r(\cdot, t)) \in \mathcal{H}_1(S)\) and \([q(\cdot, t)] = [r(\cdot, t)] = 0\). So, from Corollary 1 there is \( h \in L^2((0, t), H^1(S)) \) such that the system

\[
\tilde{q}_t = \tilde{r}_x, \\
\tilde{r}_t = B^{-1} A\tilde{q}_x + L(h),
\]

with the conditions \( \tilde{q}(x, 0) = \tilde{r}(x, 0) = 0 \), has a solution satisfying

\[
\tilde{q}(x, t) = q(x, t), \quad \tilde{r}(x, t) = r(x, t),
\]

and

\[
\|h\|_{L^2((0, t), H^1(S))}, \|Lh\|_{L^2((0, t), H^1(S))} \leq C \|(q(\cdot, t), r(\cdot, t))\|_{\mathcal{H}_i}. \tag{57}
\]

Now, if we set

\[
R(t) = \frac{1}{2} \int_S \left( q(x, t)\tilde{q}(x, t) + aq_x(x, t)\tilde{q}_x(x, t) + r(x, t)\tilde{r}(x, t) + br_x(x, t)\tilde{r}_x(x, t) \right) dx,
\]

then, we see directly that

\[
R'(t) = \frac{1}{2} \int_S \left( q\tilde{q} + \tilde{q}_x q_x + \tilde{r}_x q_x + \tilde{r} r + \tilde{r}_t r + br_x \tilde{r}_x + br_x \tilde{r}_x \right) dx
\]

\[
= \frac{1}{2} \int \left( Lr(x, t) (h(x, t) - K\tilde{r}(x, t)) \right) dx.
\]

Now, using that \( R(0) = 0 \), and the final conditions for \( \tilde{q} \) and \( \tilde{r} \), we see that

\[
E(t) = \frac{1}{2} \int_S \left( q^2(x, t) + aq_x^2(x, t) + r^2(x, t) + br_x^2(x, t) \right) dx
\]

\[
= \frac{1}{2} \int_S \left( (q\tilde{q})(x, t) + a(q_x\tilde{q}_x)(x, t) + (r\tilde{r})(x, t) + b(r_x\tilde{r}_x)(x, t) \right) dx
\]

\[
= R(t) - R(0)
\]

\[
= \int_0^t \frac{d}{d\tau} R(\tau) d\tau
\]

\[
\leq \|Lr\|_{L^2((0, t), L^2(S))} \|h - K\tilde{r}\|_{L^2((0, t), L^2(S))}.
\]

On the other hand, we also have that

\[
\|h - K\tilde{r}\|_{L^2((0, t), L^2(S))} \leq C \|(q, r)\|_{\mathcal{H}_i} \leq C (E(\tau))^{\frac{1}{2}}.
\]

Moreover,

\[
\|Lr\|_{L^2((0, t), L^2(S))} \leq g^* \int_0^t \int_S g(x) \left( r(x, \tau) - \int_S r(x, s)g(s) ds \right)^2 dx d\tau
\]

\[
\leq g^* \int_0^t \int_S g(x) \left( r(x, \tau) - \int_S r(x, s)g(s) ds \right)^2 dx dt,
\]

where we denote \( g^* > 0 \) as the least upper bound of \( g \) in \( S \). Thus,

\[
E(t) \leq C \left[ g^* \int_0^t \int_S g(x) \left( r(x, \tau) - \int_S r(x, s)g(s) ds \right)^2 dx d\tau \right]^\frac{1}{2} (E(t))^{\frac{1}{2}}.
\]
Then we see that
\[
\int_0^t \int_S g(x) \left( r(x, \tau) - \int_S r(x, s) g(s) ds \right)^2 dx d\tau \geq (C^2 g^*)^{-1} E(t).
\]
Then we obtain that
\[
E(t) - E(0) \leq -\mathcal{K} \left( C^2 g^* \right)^{-1} E(t),
\]
so, we have for some \( 0 < \alpha < 1 \) that
\[
E(t) \leq \frac{C^2 g^*}{\mathcal{K} + C^2 g^*} E(0) := \alpha E(0),
\]
which implies that
\[
\lim_{t \to T^*} \| (q(\cdot, t), r(\cdot, t))^t \|^2_{\mathcal{H}_1} < \infty,
\]
meaning that the solution can be extended to \( \mathbb{R} \) in the case \( s = 1 \). Now, we claim that
\[
E(t) \leq C e^{-\gamma t} E(0). \tag{58}
\]
In fact, for a given \( T > 0 \), we have that
\[
E(T) \leq \alpha E(0).
\]
Repeating this estimate on successive intervals \([ (k-1)T, kT ] \), for \( k = 2, 3, \ldots \), with
\[
(q(x, 0), r(x, 0))^t, (q(x, T), r(x, T))^t
\]
replaced by
\[
(q(x, (k-1)T), r(x, (k-1)T))^t, \ (q(x, kT), r(x, kT))^t,
\]
then it follows that,
\[
E(kT) \leq \alpha^k E(0) = e^{kT(\frac{1}{\mathcal{K}} - \ln \alpha)} E(0), \tag{59}
\]
Then, we complete the proof for \( s = 1 \). Now, we consider the case \( s = 2 \). Set
\[
q_1 = q_t, \ r_1 = r_t.
\]
Then \((q_1, r_1)\) solves the system
\[
\begin{align*}
\partial_t q_1 &= \partial_x r_1, \\
\partial_t r_1 &= B^{-1} A \partial_x q_1 - \mathcal{K} L(r_1),
\end{align*}
\]
with the conditions
\[
q_1(x, 0) = r_0'(x), \ r_1(x, 0) = B^{-1} A q_0'(x) - \mathcal{K} L(r_0).
\]
Note that we have \([q_1(\cdot, 0)] = [r_1(\cdot, 0)] = 0\). By taking \( q_0, r_0, \in H^2(S) \), then we see that
\[
r_0', \ [B^{-1} A q_0' - \mathcal{K} L(r_0)] \in H^1(S).
\]
Thus, using (55) in the case \( s = 1 \) we have that
\[
\| (q_1(\cdot, t), r_1(\cdot, t)) \|_{\mathcal{H}_1(S)} \leq C e^{-\gamma t} \| (q_1(\cdot, 0), r_1(\cdot, 0)) \|_{\mathcal{H}_1(S)}. \tag{60}
\]
Now, since
\[
q_1 = q_t = r_x, \ r_1 = r_t = B^{-1} A q_x - \mathcal{K} L(r),
\]
and using that \( q_1(\cdot, t), r_1(\cdot, t) \) and \( L(r) \) has the mean zero property we see that
\[
r = \partial_x^{-1} q_1, \ B^{-1} A q = \partial_x^{-1} r_1 + \mathcal{K} \partial_x^{-1} L(r),
\]
which implies that
\[
\|r(\cdot, t)\|_{H^2(S)} \leq \|\partial_x^{-1} q_1(\cdot, t)\|_{H^2(S)} \leq C_1 \|q_1(\cdot, t)\|_{H^1(S)},
\]
\[
\|q(\cdot, t)\|_{H^2(S)} \simeq \|B^{-1} A q(\cdot, t)\|_{H^2(S)} \leq \|r_1(\cdot, t)\|_{H^1(S)} + K \|L(r)(\cdot, t)\|_{H^1(S)} \leq C_1 \left( \|r_1(\cdot, t)\|_{H^1(S)} + \|q_1(\cdot, t)\|_{H^1(S)} \right).
\]
In addition,
\[
\|q_1(\cdot, 0)\|_{H^1(S)} = \|r'_0\|_{H^1(S)} \leq C_1 \|r_0\|_{H^2(S)},
\]
\[
\|r_1(\cdot, 0)\|_{H^1(S)} \leq \|B^{-1} A q_0\|_{H^1(S)} + K \|L(r_0)\|_{H^1(S)} \leq C_1 \left( \|q_0\|_{H^2(S)} + \|r_0\|_{H^1(S)} \right).
\]

Then, using (60) and previous estimates we are able to conclude that
\[
\| (q(\cdot, t), r(\cdot, t)) \|_{H^2(S)} \leq C e^{-\gamma t} \| (q_0, r_0) \|_{H^2(S)}.
\]
So, we have that the estimate (55) holds for \( s = 2 \). Moreover, for \( 1 < s < 2 \), the estimate (55) can be gotten by interpolation, and for \( s > 2 \), it can be obtained by an inductive argument.

We note that the system (53) can be written as the first order system
\[
\partial_t U = \mathcal{M} U - K F(U),
\]
with the condition
\[
U(x, 0) = (q_0(x), r_0(x)) = U_0(x),
\]
where \( U = (q, r)^t \), \( \mathcal{M} \) and \( F \) are given by
\[
\mathcal{M} = \begin{pmatrix} 0 & \partial_x \\ (B^{-1} A) \partial_x & 0 \end{pmatrix}, \quad F(U) = \begin{pmatrix} 0 \\ L(r) \end{pmatrix}.
\]
We see that the solution \( U \) can be written as
\[
U(x, t) = \mathcal{S}_K(t) U_0,
\]
where \( \mathcal{S}_K(t) \) is the semigroup on \( H_s \) associated to (61)-(62). Then we have the following corollary.

**Corollary 2.** Given \( s \geq 1 \), for any \( U_0 = (q_0, r_0)^t \in H_s(S) \) with \( [q_0] = 0 \), the problem (61)-(62) admits a unique solution \( U \in C(\mathbb{R}, H_s(S)) \). Moreover, there exists \( C, \gamma > 0 \) such that
\[
\|\mathcal{S}_K(t) U_0\|_{H_s(S)} \leq C e^{-\gamma t} \|U_0\|_{H_s(S)}.
\]

### 6.2. Stability of the nonlinear system

In this section we consider the nonlinear system
\[
\begin{align*}
q_t &= r_x - \partial_x B^{-1} \left( \frac{1}{p+1} r^{p+1} \right), \\
\partial_t &= B^{-1} A q_x - \partial_x B^{-1} \left( q^p \left( r - B^{-1} \left( \frac{1}{p+1} r^{p+1} \right) \right) \right) - K L(r),
\end{align*}
\]
or the equivalent first order system in the variable \( U = (q, r)^t \)
\[
\partial_t U = \mathcal{M} U + G(q, r) - K F(U),
\]
with the condition
\[
U(x, 0) = U_0(x) = (q_0(x), r_0(x))^t,
\]
where \( \mathcal{M} \) and \( F \) are defined as in previous section and the components \( G_1 \) and \( G_2 \) of \( G \) are given by (8) and (9), respectively.
A direct observation shows that there is an explicit relation between the semigroups $S(t)$ and $S_K(t)$, as we state in the following result without including the proof.

**Lemma 6.2.** For $s \geq 1$, the semigroup $S_K(t)$ defined in (63) satisfies the following relation

$$S_K(t)U_0 = S(t)U_0 - \mathcal{K} \int_0^t S(t-\tau)F(S_K(\tau)U_0) \, d\tau,$$

for any $U_0 \in \mathcal{H}_s$, where $S(t)$ is defined by (12). In addition, for $V \in L^1(\mathbb{R}, \mathcal{H}_s(S))$,

$$\int_0^t S_K(t-\tau)V \, d\tau = \int_0^t S(t-\tau)V \, d\tau - \mathcal{K} \int_0^t S(t-\tau)F\left(\int_0^\tau S_K(t-\xi)V \, d\xi\right) \, d\tau.$$  

(68)

Using the definition of $S_K(t)$, the system (65) can be written as the integral equation

$$U(t) = S_K(t)U_0 - \int_0^t S_K(t-\tau)G(U) \, d\tau$$

$$= S_K(t)U_0 - \int_0^t S(t-\tau)G(U) \, d\tau$$

$$+ \mathcal{K} \int_0^t S(t-\tau)F\left(\int_0^\tau S_K(t-\xi)G(U) \, d\xi\right) \, d\tau.$$  

(69)

**Theorem 6.3.** Given $s \geq 1$, there exist some constants $\delta > 0$, $M > 0$ and $\gamma > 0$ such that every solution of the problem (65)-(66) with $U_0 = (q_0, r_0)^t \in \mathcal{H}_s(S)$ such that $[q_0] = [r_0] = 0$ and $\|U_0\|_{\mathcal{H}_s(S)} \leq \delta$ satisfies

$$\|U(\cdot, t)\|_{\mathcal{H}_s(S)} \leq Me^{-\gamma t}\|U_0\|_{\mathcal{H}_s(S)}.$$  

(70)

**Proof.** From Corollary 2, for $U_0 \in \mathcal{H}_s$, we have that

$$\|S_K(t)U_0\|_{\mathcal{H}_s(S)} \leq \frac{1}{4}\|U_0\|_{\mathcal{H}_s(S)},$$

by choosing $T > 0$ satisfying that

$$Ce^{-\gamma T} \leq \frac{1}{4}.$$  

We want to have a solution $U$ to the integral equation

$$\Phi(U) = S_K(t)U_0 - \int_0^t S(t-\tau)G(U) \, d\tau$$

$$+ \mathcal{K} \int_0^t S(t-\tau)F\left(\int_0^\tau S_K(t-\xi)G(U) \, d\xi\right) \, d\tau$$

in some ball $\mathcal{X}^R$, for any $t \in [0, T]$, where

$$\mathcal{X}^R := \{U \in \mathcal{H}_s(S) : \|U\|_{\mathcal{H}_s(S)} \leq R\}.$$  

In order to get this result, we need $\|U_0\|_{\mathcal{H}_s(S)} \leq \delta$, where $\delta$ is determined later. Moreover, to reach the exponential stability, we need to have $\delta$ and $R$ be chosen such that

$$\|U\|_{\mathcal{H}_s} \leq \frac{1}{2}\|U_0\|_{\mathcal{H}_s}.$$
From the properties of $S_K(t)$, for given $T > 0$ we have that
\[
\sup_{t \in [0,T]} \|S_K(t)U\|_{H_s} \leq \|U\|_{H_s}
\]
and also for $V \in L^1((0, T), H_s(S))$ that,
\[
\sup_{t \in [0,T]} \left\| \int_0^t S_K(t - \tau)V d\tau \right\|_{H_s} \leq C \|V\|_{L^1((0, T), H_s)}.
\]
From similar estimates done for $S(t)$, we are able to establish that
\[
\left\| \int_0^t S(t - \tau)G(U)(\tau) d\tau \right\|_{H_s} \leq C \sup_{t \in [0,T]} \eta_p (\|U\|_{H_s}) \|U\|_{H_s},
\]
where $\eta_p(y) = y^{2p} + y^p$ (see Lemma 2.4 and proof of local existence Theorem 2.5).

In addition, from the boundedness of $L$ we obtain that
\[
\left\| F \left( \int_0^\tau S_K(\tau - \xi)G(U)(\xi) d\xi \right) \right\|_{H_s} \leq C \int_0^\tau S_K(\tau - \xi)\tilde{G}(U)(\xi) d\xi \leq C \left\| \int_0^\tau S_K(\tau - \xi)G(U)(\xi) d\xi \right\|_{H_s},
\]
where $\tilde{G}(U) = (0, G_2(U))^t$. Thus,
\[
\left\| \int_0^t S(t - \tau)F \left( \int_0^\tau S_K(\tau - \xi)G(U)(\xi) d\xi \right) d\tau \right\|_{H_s} \\
\leq \left\| F \left( \int_0^t S_K(\tau - \xi)G(U)(\xi) d\xi \right) \right\|_{L^1((0, T), H_s)} \\
\leq C \sup_{t \in [0,T]} \eta_p (\|U\|_{H_s}) \|U\|_{H_s},
\]
From this facts we have that
\[
\|\Phi(U)\|_{H_s} \leq C\|U_0\|_{H_s} + C\eta_p(R)R,
\]
and also that
\[
\|\Phi(U) - \Phi(V)\|_{H_s} \leq CR\|U - V\|_{H_s},
\]
for some $C$ independent of $t, \delta$ and $R$ (see proof Theorem 2.5). On the other hand, using (70) and formula for $\Phi$ we see that
\[
\|\Phi(U)(T)\|_{H_s} \leq \frac{1}{4}\|U_0\|_{H_s} + C\eta_p(R)R.
\]
We define \( \delta = 4C\eta_p(R)R \) where $R > 0$ is chosen so that
\[
(4C^2 + C)\eta_p(R) \leq 1, \quad CR \leq \frac{1}{2}.
\]
Thus, we have that $\Phi$ is a contraction mapping in $X^R$. Moreover, using (71), the unique fixed point $U \in X^R$ satisfies
\[
\|U(T)\|_{H_s} = \|\Phi(U)(T)\|_{H_s} \leq \frac{\delta}{2}.
\]
Now, for \(0 < \|U_0\|_{\mathcal{H}_s} \leq \delta\), we define \(\delta' = \|U_0\|_{\mathcal{H}_s}\). By changing \(\delta'\) into \(\delta\) and \(R\) into \(R' = \left(\frac{\delta'}{\delta}\right)^{\frac{1}{2}} R\), we obtain

\[
\|U(T)\|_{\mathcal{H}_s} \leq \frac{\delta'}{\delta} = \frac{1}{2} \|U_0\|_{\mathcal{H}_s}.
\]

So, arguing by induction, we conclude that

\[
\|U(kT)\|_{\mathcal{H}_s} \leq \frac{\delta'}{\delta} = 2^{-k} \|U_0\|_{\mathcal{H}_s}.
\]

As in the estimate (59), we are able to establish that

\[
\|U(\cdot, t)\|_{\mathcal{H}_s} \leq Me^{-\gamma t} \|U_0\|_{\mathcal{H}_s}
\]

for some constant \(M > 0\) and \(\gamma > 0\), as desired. \(\square\)

Finally, we are able to establish the corresponding stabilization result related with the Benney-Luke model.

**Remark 2.** (Theorem 6.3). The condition \([r_0] = 0\) imposed in Theorem 6.3 is not required. In fact, if we assume that \([r_0] = c \neq 0\), then we need to observe that \((q(\cdot, t), r(\cdot, t))^t\) is conserved for any \(t\), due to the structure of the system (65)-(66).

We note that \(L(c) = 0\). So, if we set \(\tilde{r}(x, t) = r(x, t) - c\), then we have \((q(\cdot, t), \tilde{r}(\cdot, t))^t\) solves the new system

\[
\begin{align*}
q_t &= \tilde{r}_x - \partial_x B^{-1} \left( \frac{1}{p+1} q^{p+1} \right), \\
\tilde{r}_t &= B^{-1} A q_x + c \partial_x B^{-1} (q^p) - \partial_x B^{-1} \left( q^p \left( \tilde{r} - B^{-1} \left( \frac{1}{p+1} q^{p+1} \right) \right) \right) - KL(\tilde{r}),
\end{align*}
\]

with the conditions \(q(x, 0) = q_0(x)\), \(\tilde{r}_0(x, 0) = \tilde{r}_0(x) = r_0(x) - c\), and having the property \([q_0] = [\tilde{r}_0] = 0\). We point out that the proof for the new system (72) is quite similar to the case \([r_0] = 0\), and the conclusion of the Theorem 6.3 for small data should be

\[
\|U(\cdot, t) - (0, [r_0])\|_{\mathcal{H}_s(S)} \leq Me^{-\gamma t} \|U_0 - (0, [r_0])\|_{\mathcal{H}_s(S)}.
\]  

(73)

**Proof of Theorem 1.3.** From the hypothesis, we have that

\([u_0] = [h_1(u_0, v_0)] = 0\).

If we set \(q_0 = u_0\) and \(r_0 = h_1(u_0, v_0)\), then we have that \([q_0] = [r_0] = 0\). From the stabilization result Theorem 6.3, there exist constants \(\delta > 0\), \(M > 0\) and \(\gamma > 0\) such that every solution \(U = (q, r)^t \in H^s(S)\), satisfying \(U(0) = U_0\), of the system

\[
U_t = MU + G(U) + \begin{pmatrix} 0 \\ -KL(r) \end{pmatrix},
\]

with \(\|U_0\|_{\mathcal{H}_s(S)} \leq \delta\) satisfies

\[
\|q(\cdot, t)\|^2_{H^s(S)} + \|r(\cdot, t)\|^2_{H^s(S)} = \|U(\cdot, t)\|^2_{\mathcal{H}_s(S)} \leq Me^{-2\gamma t} \|U_0\|^2_{\mathcal{H}_s(S)}.
\]

As we discussed in section 2, we know that \(u(x, t) = \partial_x^{-1}q(x, t)\) is such that \(r = h_1(u_t, u_x)\) and that satisfies the forced Benney-Luke equation

\[
u_{tt} - u_{xx} + au_{xxx} + bu_{xxtt} + pu_{tt}u_x^{-1}u_{xx} + 2u_x^2u_{xt} + KBL(h_1(u_t, u_x)) = 0.
\]
Moreover, we also have that \( u_x(x,0) = u_0(x) = q_0(x) \) and \( u_t(x,0) = v_0(x) \). Finally, we note that \( ||q_0||_{H^r(S)} = ||u_0||_{H^r(S)} \) and that
\[
||r_0||_{H^r(S)} = \left\| v_0 + B^{-1} \left( \frac{1}{p+1} q_0^{p+1} \right) \right\|_{H^r(S)} \leq ||v_0||_{H^r(S)} + C(p) ||u_0||_{H^r(S)}^{p+1}.
\]
Moreover, we also have that
\[
||u_x(\cdot, t)||_{H^r(S)} = ||q(\cdot, t)||_{H^r(S)} \leq C e^{-\gamma t} ||U_0||_{H^r(S)},
\]
\[
||u_t(\cdot, t)||_{H^r(S)} = \left\| r(\cdot, t) - B^{-1} \left( \frac{1}{p+1} (u_t(\cdot, t))^{p+1} \right) \right\|_{H^r(S)} \leq ||r(\cdot, t)||_{H^r(S)} + C(p) ||u_x(\cdot, t)||_{H^r(S)}^{p+1} \leq M_1 e^{-\gamma t} ||U_0||_{H^r(S)} + M_2(p) e^{-\gamma(p+1)t} ||U_0||_{H^r(S)}^{p+1},
\]
which imply that
\[
||u_x(\cdot, t)||_{H^r(S)} + ||u_t(\cdot, t)||_{H^r(S)} \leq M(p) e^{-\gamma t} C_1 \left( ||v_0||_{H^r(S)}, ||u_0||_{H^r(S)} \right),
\]
where \( C_1(\cdot, \cdot) \) is a positive function.

7. Conclusions. In this work we investigated the problems of controllability and stabilization for the Benney-Luke equation on the unit circle on the plane \( S \), with internal control \( f \) supported on an arbitrary sub-domain of \( S \). We showed that the model is exactly controllable when \( S \) is the support of \( f \), without any assumption on the size of the initial and terminal states, and that is locally exactly controllable when the support of \( f \) is a proper sub-domain of \( S \), in the case of small initial and terminal states. Moreover, assuming that the initial data is small and \( f \) is a special internal linear feedback, the solution of the model must have uniform exponential decay to a constant state. The results are obtained by re-writing the Benney-Luke equation as a first order system in special variables. This allowed us to perform the spectral analysis and the existence of the linear and nonlinear control in a simpler form. The results obtained in the paper are in concordance with the results of controllability and stabilization for model like the KdV, the Benjamin-Ono, the best Boussinesq equation, among others.

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