CONVOLUTIONS OF SINGULAR MEASURES AND APPLICATIONS TO THE ZAKHAROV SYSTEM

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Abstract. Uniform $L^2$-estimates for the convolution of singular measures with respect to transversal submanifolds are proved in arbitrary space dimension. The results of Bennett-Bez are used to extend previous work of Bejenaru-Herr-Tataru. As an application, it is shown that the 3D Zakharov system is locally well-posed in the full subcritical regime.

1. Introduction and main results

In this paper we complete the development of a geometric multilinear $L^2$-estimate which streamlines the analysis of a general class of bilinear forms which appear in various types of nonlinear PDE. In [2] Tataru and the authors proved uniform estimates for the convolution of $L^2$ measures supported on transversal surfaces in three dimensions. This complemented the result of Bennett-Carbery-Wright in [4]. In the present paper we generalize our previous result to higher dimensions by using the recent work of Bennett-Bez in [3].

As an application, we establish a sharp result for the Zakharov system in 3D. Our result, when combined with the results in [1] [8], closes the full subcritical regime (in the sense of [8, p. 387]) for the Zakharov system in all dimensions. As a consequence, the remaining part of the paper is organized in two sections, each containing results of independent interest.

1.1. Convolutions of singular measures. The first part of the paper is dedicated to a generalization to higher dimensions of the results in [2]. We consider three subsets $\Sigma_1, \Sigma_2, \Sigma_3$ of submanifolds of $\mathbb{R}^n$ whose codimensions add up to $n$ and which are transversal in the sense that the normal spaces at each point span $\mathbb{R}^n$ and which satisfy certain regularity assumptions. In this set-up we study the restriction to $\Sigma_3$ of the convolution of two measures supported on $\Sigma_1, \Sigma_2$. Our main results are global $L^2$ estimates.

We build on the result on nonlinear Brascamp-Lieb inequalities proved in [3], see also [4]. More precisely, we utilize the $m = 3$ case of [3, Theorem 1.3] in order to extend the trilinear case of [3, Theorem 7.1] to submanifolds of general codimensions, formulated under global, quantitative assumptions in the spirit of [2].

Before we formulate the precise assumptions on the submanifolds, let us introduce some notation: For numbers $m_1, m_2, m_3 \in \mathbb{N}$ we define the sets of indices $M_1 = \{1, \ldots, m_1\}$, $M_2 = \{m_1 + 1, \ldots, m_1 + m_2\}$, and $M_3 = \{m_1 + m_2 + \ldots\}$.

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Assumption 1.1. There exist $0 < \beta \leq 1$, $b > 0$, $\theta > 0$, $R > 0$, and $m_i \in \mathbb{N}$ for $i = 1, 2, 3$, with $n = m_1 + m_2 + m_3$, such that

\begin{enumerate}[-]
  \item for every $i = 1, 2, 3$ there exists an open set $U_i \subset \mathbb{R}^{m_i}$, $n_i = n - m_i$, and $\phi_i \in C^{1, \beta}(U_i; \mathbb{R}^{m_i})$ with the property
    \begin{equation}
    R^{-\beta} \sup_{x \in U_i} |D\phi_i(x)| + \sup_{x, x' \in U_i} \frac{|D\phi_i(x) - D\phi_i(x')|}{|x - x'|^\beta} \leq b,
    \end{equation}
    \end{enumerate}

such that $\Sigma_i$ is relatively open, and compactly contained in $G_i \text{graph}(\phi_i)$ for some orthogonal transformation $G_i \in O(n)$;

\begin{enumerate}[-] \setcounter{enumi}{2}
  \item for every $i = 1, 2, 3$ and $\sigma_i \in \Sigma_i$ and any orthonormal basis $\{n_k\}_{k \in M_i}$ of the normal space $N_{\sigma_i}(\Sigma_i)$ the determinant
    \[ d(\sigma_1, \sigma_2, \sigma_3) = \det(n_1(\sigma_1), \ldots, n_{m_1}(\sigma_1), \ldots, n_{m_1+m_2+1}(\sigma_3), \ldots, n_n(\sigma_3)) \]
    satisfies the uniform transversality condition
    \begin{equation}
    \inf_{\sigma_1, \sigma_2, \sigma_3} |d(\sigma_1, \sigma_2, \sigma_3)| = \theta; \tag{1.2}
    \end{equation}

\end{enumerate}

\begin{enumerate}[-] \setcounter{enumi}{3}
  \item for every $i = 1, 2, 3$ it holds
    \[ \text{diam}(\Sigma_i) \leq R. \tag{1.3} \]
\end{enumerate}

Remark 1. The quantity $d(\sigma_1, \sigma_2, \sigma_3)$ is invariant under changes of the orthonormal bases within each normal space.

We identify $f \in L^2(\Sigma_1) = L^2(\Sigma_1, \mu_1) - \mu_1$ being the $n_1$-dimensional Hausdorff-measure — with the distribution

\[ \langle f, \psi \rangle = \int_{\Sigma_1} f(y)\psi(y)d\mu_1(y), \quad \psi \in C_0^\infty(\mathbb{R}^n). \]

For $f \in L^2(\Sigma_1)$, $g \in L^2(\Sigma_2)$ with compact support the convolution $f \ast g$ is defined as the distribution

\[ \langle f \ast g, \psi \rangle = \int_{\Sigma_1} \int_{\Sigma_2} f(x)g(y)\psi(x + y)d\mu_1(x)d\mu_2(y), \quad \psi \in C_0^\infty(\mathbb{R}^n). \]

Since a-priori the restriction of $f \ast g$ to sets of measure zero is not well-defined, we begin with $f \in C_0(\Sigma_1)$ and $g \in C_0(\Sigma_2)$. Then $f \ast g \in C_0(\mathbb{R}^n)$ and has a well-defined trace on $\Sigma_3$. Once we have proved an appropriate $L^2$-bound, the trace of $f \ast g$ on $\Sigma_3$ can be defined by density for arbitrary $f \in L^2(\Sigma_1)$ and $g \in L^2(\Sigma_2)$.

Following the ideas of [2] we first note the behavior under linear transformations.

Proposition 1.2. Let $\Sigma_1$, $\Sigma_2$, $\Sigma_3$ satisfy the Assumption 1.1 with

\[ \theta \leq |d(\sigma_1, \sigma_2, \sigma_3)| \leq 2\theta, \]

and suppose that the estimate

\[ \|f \ast g\|_{L^2(\Sigma_3)} \leq C\theta^{-\frac{b}{2}}\|f\|_{L^2(\Sigma_1)}\|g\|_{L^2(\Sigma_2)}, \tag{1.4} \]

holds true for all functions $f \in L^2(\Sigma_1)$, $g \in L^2(\Sigma_2)$. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible, linear map and $\Sigma'_i = T\Sigma_i$, then the estimate

\[ \|f' \ast g'\|_{L^2(\Sigma'_3)} \leq 2C\theta^{-\frac{b}{2}}\|f'\|_{L^2(\Sigma'_1)}\|g'\|_{L^2(\Sigma'_2)} \tag{1.5} \]
holds true for all functions \( f' \in L^2(\Sigma'_1), g' \in L^2(\Sigma'_2) \), where
\[
\theta' = \inf_{\sigma'_1, \sigma'_2, \sigma'_3} |d'(\sigma'_1, \sigma'_2, \sigma'_3)|
\]
is defined in analogy to Assumption 1.1 ii).

In summary, the size of the constant is determined only by the transversality properties of the submanifolds.

Next, we look at the fully transversal case. The dual formulation of a local version of the following result for codimension 1 submanifolds is contained in [3, Theorem 7.1].

**Theorem 1.3.** Let \( \Sigma_1, \Sigma_2, \Sigma_3 \) be submanifolds in \( \mathbb{R}^n \) which satisfy Assumption [7] with parameters \( 0 < \beta \leq 1, b = 1 \) and \( \theta = \frac{1}{2}, \) and \( R = 1. \) Then for each \( f \in L^2(\Sigma_1) \) and \( g \in L^2(\Sigma_2) \) the restriction of the convolution \( f \ast g \) to \( \Sigma_3 \) is a well-defined \( L^2(\Sigma_3) \)-function which satisfies
\[
\|f \ast g\|_{L^2(\Sigma_3)} \leq C\|f\|_{L^2(\Sigma_1)}\|g\|_{L^2(\Sigma_2)},
\]
where the constant \( C \) depends only on \( \beta \) and \( n. \)

Finally, in view of future applications let us note how the estimate depends on the more general hypothesis of Assumption 1.1.

**Corollary 1.4.** Let \( \Sigma_1, \Sigma_2, \Sigma_3 \) be submanifolds in \( \mathbb{R}^n \) which satisfy Assumption [7] with parameters \( 0 < \beta \leq 1, b > 0, 0 < \theta \leq 1/2. \) Then for each \( f \in L^2(\Sigma_1) \) and \( g \in L^2(\Sigma_2) \) the restriction of the convolution \( f \ast g \) to \( \Sigma_3 \) is a well-defined \( L^2(\Sigma_3) \)-function which satisfies
\[
\|f \ast g\|_{L^2(\Sigma_3)} \leq C\theta^{-\frac{b}{2}}\|f\|_{L^2(\Sigma_1)}\|g\|_{L^2(\Sigma_2)},
\]
where \( C \) depends only on \( \beta, n, \) and the size of the quantity \( R^{\beta}b\theta^{-1}. \)

1.2. **The 3D Zakharov system.** In this section we consider the initial value problem associated with the Zakharov system

\[
\begin{align*}
\partial_t u + \Delta u &= nu \quad \text{in} \ (0, T) \times \mathbb{R}^3, \\
\frac{\partial^2_n}{2} n - \Delta n &= \Delta |u|^2 \quad \text{in} \ (0, T) \times \mathbb{R}^3, \\
(u, n, \partial_t n)|_{t=0} &\in H^\sigma(\mathbb{R}^3) \times H^\sigma(\mathbb{R}^3) \times H^{\sigma-1}(\mathbb{R}^3).
\end{align*}
\]

The Zakharov system is a model for Langmuir oscillations in a plasma, cf. [12] and [11, Chapter 13] for more information.

Local weak solutions for \((1.8)\) with smooth data were constructed by Sulem-Sulem in [10], and local well-posedness for data in \( H^2 \times H^1 \times L^2 \) was established by Ozawa-Tsutsumi in [9]. Provided that the Schrödinger part is small in \( H^1, \) global well-posedness for data in the energy space, see [6] for details, was established by Bourgain-Colliander in [6].

We are interested in the low regularity well-posedness theory of \((1.8)\). Our notion of well-posedness includes existence of generalized solutions, uniqueness in a suitable subspace, local Lipschitz continuity and persistence of initial regularity. It has been shown by Ginibre-Tsutsumi-Velo in [8] that \((1.8)\) is locally well-posed for \( \sigma \geq 0, 2s \geq \sigma + 1, \sigma \leq s \leq \sigma + 1. \) We extend this result to the full subcritical range in the sense of [8, p. 387].

**Theorem 1.5.** The Cauchy problem \((1.8)\) is locally well-posed in \( H^\sigma(\mathbb{R}^3) \times H^\sigma(\mathbb{R}^3) \times H^{\sigma-1}(\mathbb{R}^3) \) for \( \sigma > -\frac{1}{2}, \sigma \leq s \leq \sigma + 1, 2s > \sigma + \frac{1}{2}. \)
For a more detailed statement we refer the reader to [1] Theorem 1.1.

The almost admissible endpoint \((s, \sigma) = (0, -\frac{1}{2})\), i.e. bottom left corner of the convex region of admissible \((s, \sigma)\), matches the 2D result obtained in [1] Theorem 1.1] and extends the result of [8, formula (1.10)] for dimensions \(d \geq 4\) to \(d = 3\).

2. Convolution estimates

Proof of Proposition 1.2. By density and duality, the claimed estimate is equivalent to

\[
I(f, g, h) := \int f(\sigma') g(\sigma'_2) h(\sigma'_3) \delta(\sigma'_1 + \sigma'_2 - \sigma'_3) d\mu'_1(\sigma'_1) d\mu'_2(\sigma'_2) d\mu'_3(\sigma'_3) \\
\leq 2C\theta^{-\frac{1}{2}} \|f\|_{L^2(\Sigma'_1)} \|g\|_{L^2(\Sigma'_2)} \|h\|_{L^2(\Sigma'_3)},
\]

for all nonnegative, continuous \(f, g, h\). We assume that \(\varphi_i : \mathbb{R}^n \supset \Omega_i \to \mathbb{R}^n\) is a global parametrization for \(\Sigma_i, i = 1, 2, 3\). In that case \(\varphi'_i := T\varphi_i\) is a parametrization for \(\Sigma'_i, i = 1, 2, 3\).

Using the above parameterizations (2.1) is given as follows

\[
I(f, g, h) = \int f(\varphi'_1(x)) g(\varphi'_2(y)) h(\varphi'_3(z)) (g'_1 g'_2 g'_3)^{\frac{1}{2}} d\nu'(x, y, z)
\]

where \(g'_i = \det(D\varphi'_i) D\varphi'_i\) and with respect to the measure

\[
d\nu'(x, y, z) = \delta(\varphi'_1(x) + \varphi'_2(y) - \varphi'_3(z)) dx dy dz.
\]

With \(g_i = \det(D\varphi_i) D\varphi_i\) and the measure

\[
d\nu(x, y, z) = \delta(\varphi_1(x) + \varphi_2(y) - \varphi_3(z)) (g_1 g_2 g_3)^{\frac{1}{2}} dx dy dz
\]

an upper bound on \(I(f, g, h)\) is given by

\[
\sup M(x, y, z) \frac{1}{|\det T|} \int f(\varphi_1(x)) g(\varphi_2(y)) h(\varphi_3(z)) (g_1 g_2 g_3)^{\frac{1}{2}} d\nu(x, y, z) \\
\leq 2\theta_2^{-\frac{1}{2}} \theta'^{-\frac{1}{2}} \int f(\sigma_1) g(\sigma_2) h(\sigma_3) \delta(\sigma_1 + \sigma_2 - \sigma_3) d\mu_1(\sigma_1) d\mu_2(\sigma_2) d\mu_3(\sigma_3) \\
\leq 2C\theta^{-\frac{1}{2}} \|f\|_{L^2(\Sigma'_1)} \|g\|_{L^2(\Sigma'_2)} \|h\|_{L^2(\Sigma'_3)},
\]

where we have used the definitions

\[
M = \prod_{i=1}^{3} g'_i g_i^{-\frac{1}{2}}, \quad \tilde{f} = g'_1 g_1^{-\frac{1}{2}} f(T),
\]

similarly for \(\tilde{g}, \tilde{h}\). We have also used that Dirac’s \(\delta\) obeys the simple rule

\[
\delta(T\varphi_1(x) + T\varphi_2(y) - T\varphi_3(z)) = (\det T)^{-1} \delta(\varphi_1(x) + \varphi_2(y) - \varphi_3(z))
\]

and the following identity

\[
M(x, y, z) \frac{1}{\det T} = \left( \frac{d(\varphi_1(x), \varphi_2(y), \varphi_3(z))}{d'(\varphi'_1(x), \varphi'_2(y), \varphi'_3(z))} \right)^{\frac{1}{2}},
\]

so that (2.2) is the only claim which remains to be proved.

For brevity, let \(\sigma_1 = \varphi_1(x)\), \(\sigma_2 = \varphi_2(y)\), \(\sigma_3 = \varphi_3(z)\), \(\sigma' = T\sigma_1\), be arbitrary points on \(\Sigma_i\), which will be fixed for the subsequent calculation.

For \(i = 1, 2, 3\) we fix orthonormal bases \(\{n_k(\sigma_i)\}_{k \in M_i}\) of the normal spaces and define the invertible matrix

\[
S = S(\sigma_1, \sigma_2, \sigma_3) = (n_1(\sigma_1), \ldots, n_{m_1}(\sigma_1), \ldots, n_{m_3+1}(\sigma_3), \ldots, n_n(\sigma_3))^t
\]
as well as $R = R(\sigma_1, \sigma_2, \sigma_3) = TS^{-1}$. Then, $T = RS$ and $S$ has the property that if $\Sigma_i'' = S\Sigma_i$ then $\{\xi_k\}_{k \in M_i}$ is an orthonormal basis of the normal space of $\Sigma_i''$ at $S\sigma_i$, $i = 1, 2, 3$. We observe that

$$\frac{\det((TD\phi_i)^tTD\phi_i)}{\det((D\phi_i)^tD\phi_i)} = \frac{\det((SD\phi_i)^tSD\phi_i)}{\det((D\phi_i)^tD\phi_i)} \cdot \frac{\det((RSD\phi_i)^tRSD\phi_i)}{\det((SD\phi_i)^tSD\phi_i)}. \quad (2.3)$$

Thus, without restricting the generality of the problem, we can assume that an orthonormal basis of the normal space of $\Sigma_i'$ at $\sigma_i' = T\sigma_i$ is given as $\{\xi_k\}_{k \in M_i}$, since this takes care of the first factor and it also provides the computation for the reverse situation which takes care of the second factor.

Under this assumption the rows $a_k^i$ of $T$, i.e. $n_k := T^t\xi_k$, $k \in M_i$ form a basis of the normal space of $\Sigma_i$ at $\sigma_i$, but not necessarily an orthonormal basis. We rely on two basic geometric facts: The first is that $(\det(A'))^\frac{1}{2}$ is the $p$-dimensional volume of the parallelepiped spanned by the columns of $A \in \mathbb{R}^{n \times p}$. The second is that if

$$A = (A_1|A_2), \ A_k \in \mathbb{R}^{n \times p_k}, \ p_1 + p_2 = n, \ \text{and} \ R(A_1) \perp R(A_2),$$

then the volume of the parallelepiped spanned by the columns of $A$ is the product of the volumes of the parallelepipeds spanned by the columns of $A_1, A_2$, respectively, i.e.

$$\det(A) = (\det(A_1^tA_1))^\frac{1}{2}(\det(A_2^tA_2))^\frac{1}{2}.$$

We define the submatrices

$$N_i' = (\xi_{k_i+1}, \ldots, \xi_{k_i+m_i}),$$

with $k_i$ such that $M_i = \{k_i + 1, \ldots, k_i + m_i\}$, and $N_i = T^tN_i'$, where the columns $n_k$ are normal to $\Sigma_i$, but do not necessarily form an orthonormal set. We compute for $i = 1, 2, 3$ based on the considerations above that

$$\frac{\det((TD\phi_i)^tTD\phi_i)}{\det((D\phi_i)^tD\phi_i)} = \frac{\det((TD\phi_i)^t(TD\phi_i) \det(N_i'|N_i))}{\det^2(T) \det((TD\phi_i)^tTN_i) \det(N_i'|N_i]}$$

where here in in the sequel we suppress the evaluation of $\phi_i$ at $x, y, z$, respectively. Next, we use

$$\det(TD\phi_i|TN_i) = \det(TD\phi_i|P_iTN_i),$$

where $P_i$ is the orthogonal projection onto $N_i'$, and conclude

$$\det(TD\phi_i|TN_i) = (\det((P_iTN_i)^tP_iTN_i))^\frac{1}{2}(\det((TD\phi_i)^tTD\phi_i))^\frac{1}{2},$$

such that in summary

$$\frac{\det((TD\phi_i)^tTD\phi_i)}{\det((D\phi_i)^tD\phi_i)} = \frac{\det^2(T) \det(N_i'|N_i)}{\det((P_iTN_i)^tP_iTN_i) \det(N_i'|N_i)} = \frac{\det^2(T)}{\det(N_i'|N_i)}.$$
The above computation holds for all $i \in \{1, 2, 3\}$, therefore
\[
\frac{M(x, y, z)}{\det T} = (\det T)^{-1} \prod_{i=1}^{3} \left( \frac{\det^2 T}{\det(N_{i1}N_{i1})} \right)^{\frac{1}{2}} = \left( \prod_{i=1}^{3} (\det(N_{i1}N_{i1}))^{\frac{1}{2}} \right)^{-1}
\]
This expression is invariant with respect to the choice of normal vectors in $N_i$, hence we can use an orthonormal set to obtain
\[
\frac{M(x, y, z)}{\det T} = (d(\varphi_1(x), \varphi_2(y), \varphi_3(z)))^{\frac{1}{2}}
\]
and in view of our previous reduction in (2.3) the claim (2.2) follows. This ends the proof of Proposition 1.2. \hfill \Box

**Proof of Theorem 1.3** In what follows we use Landau’s notation $o(1)$ for scalars, vectors or matrices to denote a quantity which can be made arbitrarily small as $R = \max(\text{diam}(\Sigma_1), \text{diam}(\Sigma_2), \text{diam}(\Sigma_3)) \to 0$. For brevity we introduce the shorthand notation
\[
(x_i, \ldots, x_j)^t = x_{i,j}, \quad i < j.
\]
We subdivide the proof into two steps:

**Step 1.** By a finite partition (depending only on the dimension), linear changes of coordinates as in the proof of Corollary 1.3 below we can reduce the problem to the following set-up: There exists a triplet $(\sigma_1^0, \sigma_2^0, \sigma_3^0) \in \Sigma_1 \times \Sigma_2 \times \Sigma_3$ where $\{e_k\}_{k \in M_i}$ is a basis for $N_{\sigma_i^0}(\Sigma_i)$, $i = 1, 2, 3$, such that by the implicit function theorem we have $C^{1,\beta}$-parametrizations $\varphi_i : \Omega_i \to \mathbb{R}^n$, for open subsets $\Omega_i$ of the unit ball in $\mathbb{R}^n_i$, centered at $a_i^0$, given as
\[
\varphi_1(x_{m_1+1,n}) = (\phi_1(x_{m_1+1,n}), \ldots, \phi_{m_1}(x_{m_1+1,n}), x_{m_1+1,n})^t,
\]
\[
\varphi_2(x_{m_1+1,n}, x_{m_1+m_2+1,n}) = (x_{m_1+1,n}, \phi_{m_1+1}(x_{m_1+1,n}), \ldots, \phi_{m_1+m_2+1,n})^t,
\]
\[
\varphi_3(x_{m_1+m_2+1,n}) = (x_{m_1+m_2+1,n}, \phi_{m_1+m_2+1,n})^t,
\]
where $m_i = n - n_i$, such that $\Sigma_i = \varphi_i(\Omega_i)$, $\text{diam}(\Sigma_i)$ is small enough, $\varphi_i(a_i^0) = \sigma_i^0$, where the submanifolds intersect $\varphi_1(a_i^0) + \varphi_2(a_i^0) = \varphi_3(a_i^0)$, and
\[
\partial_i \phi_j(a_i^0) = 0, \quad \text{for all } j \in M_i, \text{ and all } 1 \leq l \leq n_i. \tag{2.4}
\]

**Step 2.** We have that for each $i \in \{1, 2, 3\}$
\[
\det[D\varphi_1 D\varphi_i](a_i^0) = 1, \quad \det[D\varphi_2 D\varphi_i] = 1 + o(1) \tag{2.5}
\]
and the determinant of the normals satisfies
\[
d(\sigma_1^0, \sigma_2^0, \sigma_3^0) = 1, \quad d(\sigma_1, \sigma_2, \sigma_3) = 1 + o(1).
\]
In this set-up, we need to estimate
\[
\int (f \circ \varphi_1)(x_{m_1+1,n})(g \circ \varphi_2)(x_{m_1+1,n}, y_{m_1+m_2+1,n})(h \circ \varphi_3)(y_{m_1+m_2+1,n})
\]
\[
\delta(\varphi_1(x_{m_1+1,n}) + \varphi_2(x_{m_1+1,n}, y_{m_1+m_2+1,n}) - \varphi_3(y_{m_1+m_2+1,n}))
\]
\[
(det[D\varphi_1 D\varphi_1] \det[D\varphi_2 D\varphi_2] \det[D\varphi_3 D\varphi_3])^\frac{1}{2} dx_{m_1+1,n} dy_{m_1+m_2+1,n}
\]
For the function
\[
F(x_{m_1+1,n}, y_{m_1+m_2+1,n}) = \varphi_1(x_{m_1+1,n}) + \varphi_2(x_{m_1+1,n}, y_{m_1+m_2+1,n}) - \varphi_3(y_{m_1+m_2+1,n})
\]
it follows from the implicit function theorem that there exists a $C^1$ function $G$ such that $F(x_1, y_1, n) = 0$ if and only if $y_1, n = G(x_1, n)$, since

$$|\det \partial_{y_1, n} F| = 1 + o(1),$$

(2.6)

because (2.4) yields that the matrix is close to the diagonal matrix with $-1$ as the first $n_3$ diagonal entries and $+1$ as the remaining $m_3$ diagonal entries. Since the following is true

$$\delta(F(x_1, y_1, n)) = |\det \partial_{y_1, n} F|^{-1} \delta(y_1, n - G(x_1, n)),$$

the above integral the above integral can be rewritten as

$$\int (f \circ \varphi_1)(x_{m_1+1, n}) (g \circ \varphi_2)(x_{1, m_1}, G_{m_1+m_2+1, n}(x_1, n))$$

$$\quad (h \circ \varphi_3)(G_{n_3}(x_1, n)) m(x_1, n) dx_1, n$$

where $m(x_1, n) = 1 + o(1)$ in the domain of integration, which follows from (2.5) and (2.6). Then, following the ideas in [3, 4], we define the maps $B_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$B_1 x_1, n = x_{m_1+1, n}, \quad B_2 x_1, n = (x_{1, m_1}, G_{m_1+m_2+1, n}(x_1, n)),$$

$$B_3 x_1, n = G_{n_3}(x_1, n).$$

From the properties of $\varphi_i$ and (2.5) it follows that $B_1, B_2, B_3$ are $C^1, \beta$ functions. With these notations the above integral becomes

$$\int (f \circ \varphi_1)(B_1 x_1, n) (g \circ \varphi_2)(B_2 x_1, n) (h \circ \varphi_3)(B_3 x_1, n) m(x_1, n) dx_1, n$$

Next, we will verify the assumptions of [3 Theorem 1.3] on the kernels of $DB_i(x_0)$, where $x_0 = ([a_0^{0,1}, a_1^{0,1}] \in \mathbb{R}^n$. We start with $i = 1$:

$$DB_1(x_0) = \begin{pmatrix} 0 & I_{m_1} \end{pmatrix}$$

hence an orthonormal basis of $\ker DB_1(x_0)$ is of the form $\{e_k \}_{k \in M_1}$. For $i = 2$ we compute

$$DG(x_0) = -[(\partial_{y_1, n} F(x_0, G(x_0)))^{-1} \partial_{x_1, n} F(x_0, G(x_0))] = \begin{pmatrix} I_{n_3} & 0 \\ 0 & -I_{m_3} \end{pmatrix}$$

which implies

$$DB_2(x_0) = \begin{pmatrix} I_{m_1} \\ 0 \end{pmatrix}$$

and an orthonormal basis of $\ker DB_2(x_0)$ is of the form $\{e_k \}_{k \in M_2}$. Concerning $i = 3$, the computation of $DG(x_0)$ above immediately yields

$$DB_3(x_0) = \begin{pmatrix} I_{n_3} \\ 0 \end{pmatrix}$$

and an orthonormal basis of $\ker DB_3(x_0)$ is given as $\{e_k \}_{k \in M_3}$.

From the above characterizations of the kernels of $dB_j$, it follows from [3 formula (25)] that

$$\left| \bigstar_{j=1}^3 * X_j(DB_j(x_0)) \right| = 1,$$

where we use the notation of [3]. This allows us to invoke the result of [3 Theorem 1.3] in a small neighborhood of $x_0$, whose size depends only on $\beta$ and $n$. \hfill \Box

For the remaining proof we will follow closely the argument in [2 Proof of Corollary 1.6].
Proof of Corollary 1.4 Step 1. We first carry out the proof under the additional hypothesis

\[ R^3b\theta^{-1} \ll 1. \tag{2.7} \]

We have \( \|D\phi_i\| \leq bR^3 \ll 1 \) throughout \( U_i \subset \mathbb{R}^{n_i} \).

Let \( i = 1, 2, 3 \) and \( \sigma_i^0 \in \Sigma_i \) be fixed. Define the normal vectors \( \{n_k(\sigma_i)\}_{k \in M_i} \) at \( \sigma_i = G_i \cdot (x, \phi_i(x))^t \) to be the columns of the matrix

\[ G_i \left( -D\phi_i^t(x) \right) I_{m_i} \in \mathbb{R}^{n \times m_i}. \]

These vectors satisfy

\[ |n_k(\sigma_i) - n_k(\sigma_i^0)| \lesssim bR^3 \ll \theta, \tag{2.8} \]

for all \( k \in M_i, \ i = 1, 2, 3 \). By the Gram-Schmidt orthonormalization procedure, we can also construct from \( \{n_k(\sigma_i)\}_{k \in M_i} \) an orthonormal basis \( \{n_k(\sigma_i)\}_{k \in M_i} \) of the normal space at \( \sigma_i \in \Sigma_i \) satisfying (2.8), which shows that

\[ |d(\sigma_i^0, \sigma_0^0, \sigma_3^0) - d(\sigma_1, \sigma_2, \sigma_3)| \ll \theta. \tag{2.9} \]

Moreover, we observe that

\[ |(\sigma_i - \sigma_i^0) \cdot n_k(\sigma_i)| \lesssim bR^{1+\beta} \ll R\theta, \ k \in M_i, \]

which shows that \( \Sigma_i \) is contained in a plain layer of thickness \( \ll R \theta \) with respect to the \( n_k(\sigma_i) \) direction, for all \( k \in M_i \). By \( L^2(\mathbb{R}^n) \)-orthogonality with respect to such layers it suffices to prove the desired bound (1.7) in the case when the other two submanifolds are contained in similar regions, i.e.

\[ |(\sigma_i - \sigma_i^0) \cdot n_k(\sigma_i^0)| \ll R\theta, \ k \in M_j, \ i, j = 1, 2, 3. \tag{2.10} \]

We will apply Proposition 1.2 with the matrix

\[ T = R\theta (A')^{-1}, \quad A = (n_1(\sigma_1^0), \ldots, n_{m_1}(\sigma_1^0), \ldots, n_{m_3}(\sigma_3^0), \ldots, n_{m_3}(\sigma_3^0)). \]

It remains to show that the submanifolds \( \Sigma_i := T^{-1}\Sigma_i \) satisfy the assumptions of Theorem 1.3 i.e.

1) the size condition \( \text{diam}(\Sigma_i) \leq 1 \),
2) the transversality condition (1.2) with \( \theta = \frac{1}{2} \),
3) the regularity condition (1.1) with \( R = b = 1 \).

Concerning item 1) we observe that

\[ T^{-1}(\sigma_i - \sigma_i^0) = \frac{1}{R\theta} (n_1(\sigma_1^0) \cdot (\sigma_i - \sigma_i^0), \ldots, n_3(\sigma_3^0) \cdot (\sigma_i - \sigma_i^0))^t, \]

such that (2.10) shows \( \text{diam}(\Sigma_i) \leq 1 \).

In order to obtain the transversality condition in 2) we estimate

\[ \|A^{-1}\| \lesssim \|\det A\|^{-1} \sim \theta^{-1}, \quad \|T\| \lesssim R. \tag{2.11} \]

Let \( k \in M_i \). We define at \( \tilde{\sigma}_i \in \tilde{\Sigma}_i \) a normal vector \( \tilde{n}_k(\tilde{\sigma}_i) \) to \( \tilde{\Sigma}_i \) by

\[ \tilde{n}_k(\tilde{\sigma}_i) = A^{-1}n_k(T\tilde{\sigma}_i). \tag{2.12} \]

By construction for \( \tilde{\sigma}_i^0 = T^{-1}\sigma_i^0 \) we have \( \tilde{n}_k(\tilde{\sigma}_i^0) = \epsilon_k \). By (2.8) and (2.11) it follows that

\[ |	ilde{n}_k(\tilde{\sigma}_i) - \epsilon_k| \lesssim R^3b \ll \theta. \tag{2.13} \]
Thus, we have found a basis \( \{ \tilde{n}_k(\tilde{\sigma}_i) \}_{k \in M_1} \) of \( N_{\tilde{\delta}}(\tilde{\Sigma}_i) \). By the Gram-Schmidt process, we can recursively construct an orthonormal basis \( \{ \check{n}_k(\tilde{\sigma}_i) \}_{k \in M_1} \) with the property \( (\ref{p14}) \). This in turn yields the desired transversality condition

\[
d(\tilde{\sigma}_1, \check{\sigma}_2, \check{\sigma}_3) \geq 1/2.
\]

Concerning the regularity condition in \( \text{(iii)} \) we define

\[
\check{\Phi}_i(p) = (R\theta)^{-1}[G_i^{-1}Tp_{n_i+1,n} - \phi_i((G_i^{-1}Tp)_{1,n_i})],
\]

such that with \( Q_i = T^{-1}G_i(U_i \times \mathbb{R}^m) \) it is

\[
\tilde{\Sigma}_i = \{ p \in Q_i \subset \mathbb{R}^n : \check{\Phi}_i(p) = 0 \}.
\]

We would like to resolve this equation for \( p_k, k \in M_i \). Now, for \( k \leq l \) let \( I_{k,l} \) be the \( l - k + 1 \times n \) matrix, such that \( I_{k,l}p = p_{k,l} \). It is

\[
D\check{\Phi}_i(p) = I_{n_i+1,n}G_i^t(A^t)^{-1} - D\phi_i((G_i^tTp)_{1,n_i})I_{1,n_i}G_i^t(A^t)^{-1}.
\]

To keep the exposition clear we discuss the case \( i = 1 \) only.

\[
D_{1,m_1}\check{\Phi}_i(p) = (I_{n_i+1,n}G_i^t - D\phi_1((G_1^tTp)_{1,n_i})I_{1,n_i}G_i^t) (A^t)^{-1}I_{1,m_1}.
\]

Since \( \|D\phi_i\| \ll 1 \) in \( U_i \) and by construction of \( A \) it holds

\[
\|I_{n_i+1,n}G_i^t - D\phi_1((G_1^tTp)_{1,n_i})I_{1,n_i}G_i^t - I_{1,m_1}A^t\| \ll 1,
\]

which shows that

\[
\|D_{1,m_1}\check{\Phi}_i(p) - I_{m_1}\| \ll 1.
\]

It also implies that for

\[
D_{m_1+1,n}\check{\Phi}_1(p) = (I_{n_i+1,n}G_i^t - D\phi_1((G_1^tTp)_{1,n_i})I_{1,n_i}G_i^t) (A^t)^{-1}I_{m_1+1,n}
\]

we have

\[
\|D_{m_1+1,n}\check{\Phi}_1(p)\| \ll 1.
\]

At \( p = \tilde{\sigma}_1^0 \) we evaluate

\[
D_{1,m_1}\check{\Phi}_1(\tilde{\sigma}_1^0) = I_{m_1} \text{ and } D_{m_1+1,n}\check{\Phi}_1(\tilde{\sigma}_1^0) = 0.
\]

The implicit function theorem yields a global resolution \( \tilde{\phi}_1 \in C^{1,\beta}(\tilde{U}_1) \) with domain \( \tilde{U}_1 = I_{1,n_1}(Q_1) \) such that \( \tilde{\Phi}_1(\tilde{\phi}_1(\tilde{x}), \tilde{x}) = 0 \) with \( D\tilde{\phi}_1(\tilde{x}^0) = 0 \) and the analog of \( (\ref{p14}) \) is satisfied with \( R = b = 1 \).

**Step 2.** Finally, we remove the additional assumption \( (2.7) \). In general we have \( R^\theta b \theta^{-1} \geq 1 \). We partition each submanifold \( \Sigma_i \) into about \( R\delta^{-1} \) pieces of diameter \( \delta \) for \( \delta^\beta b \ll \theta \). It remains to prove that for each such piece we can find a graph representation satisfying Assumption \( (\ref{p14}) \) with \( R \) replaced with \( \delta \). In order to do so, in each piece we select a point \( G_i(a_i^0, \phi_i(a_i^0))^t \) and define a rotation \( O_i \in \mathbb{R}^{n \times n} \) with the property

\[
O_i \text{ range } \left( \begin{array}{c} 0 \\ I_{m_i} \end{array} \right) = \text{ range } \left( \begin{array}{c} -D\phi_i^t(a_i^0) \\ I_{m_i} \end{array} \right),
\]

and the implicit function theorem yields a representation of the piece as \( G_iO_i \text{ graph}(\tilde{\phi}_i) \) with vanishing differential at a point. This implies \( (\ref{p14}) \) with \( R \) replaced by \( \delta \). \( \square \)
3. The Zakharov system

3.1. Notation and function spaces. We adopt the notation from [1]: We write \( A \lesssim B \) if there exists a harmless constant \( c > 0 \) such that \( A \leq cB \). Moreover, we write \( A \gtrsim B \) if \( B \lesssim A \) and \( A \sim B \) if \( A \lesssim B \) and \( A \gtrsim B \). Throughout this paper we will denote dyadic numbers \( 2^n \) for \( n \in \mathbb{N} \) by the corresponding upper-case letters, e.g. \( N = 2^n, L = 2^j, \ldots \).

Let \( \psi \in C_0^\infty((-2, 2)) \) be an even, non-negative function with the property \( \psi(r) = 1 \) for \( |r| \leq 1 \). We use it to define a partition of unity in \( \mathbb{R} \),

\[
1 = \sum_{N \geq 1} \psi_N, \psi_1 = \psi, \psi_N(r) = \psi\left(\frac{r}{N}\right) - \psi\left(\frac{2r}{N}\right), \quad N = 2^n \geq 2.
\]

Thus \( \operatorname{supp} \psi_1 \subset [-2, 2] \) and \( \operatorname{supp} \psi_N \subset [-2N, -N/2] \cup [N/2, 2N] \) for \( N \geq 2 \). For \( f : \mathbb{R}^3 \to \mathbb{C} \) we define the dyadic frequency localization operators \( P_N \) by

\[
F_x(P_N f)(\xi) = \psi_N(|\xi|)F_x f(\xi).
\]

For \( u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C} \) we define \( (P_N u)(x, t) = (P_N u(\cdot, t))(x) \). We will often write \( u_N = P_N u \) for brevity. We denote the space-time Fourier support of \( P_N \) by the corresponding Gothic letter

\[
\mathcal{P}_1 = \{ (\xi, \tau) \in \mathbb{R}^3 \times \mathbb{R} \mid |\xi| \leq 2 \}, \quad \mathcal{P}_N = \{ (\xi, \tau) \in \mathbb{R}^3 \times \mathbb{R} \mid |\xi| \leq 2N \}.
\]

Moreover, for dyadic \( L \geq 1 \) we define the modulation localization operators

\[
F(S_L u)(\tau, \xi) = \psi_L(\tau + |\xi|^2)F u(\tau, \xi) \quad \text{(Schrödinger case),} \quad (3.1)
\]

\[
F(W^+_L u)(\tau, \xi) = \psi_L(\tau \pm |\xi|)F u(\tau, \xi) \quad \text{(Wave case),} \quad (3.2)
\]

and the corresponding space-time Fourier supports

\[
\mathcal{S}_1 = \{ (\xi, \tau) \in \mathbb{R}^3 \times \mathbb{R} \mid |\tau + |\xi|^2| \leq 2 \},
\]

\[
\mathcal{S}_L = \{ (\xi, \tau) \in \mathbb{R}^3 \times \mathbb{R} \mid |\tau + |\xi|^2| \leq 2L \},
\]

respectively

\[
\mathcal{W}_1^+ = \{ (\xi, \tau) \in \mathbb{R}^3 \times \mathbb{R} \mid |\tau + |\xi|| \leq 2 \},
\]

\[
\mathcal{W}_L^+ = \{ (\xi, \tau) \in \mathbb{R}^3 \times \mathbb{R} \mid |\tau + |\xi|| \leq 2L \}.
\]

Next we introduce the decompositions with respect to angular variables. For each \( A \in \mathbb{N} \) we choose a decomposition \( \{ \omega^j_A \}_{j \in \Omega_A} \) of \( \mathbb{S}^2 \) with the following properties:

i) Each \( \omega^j_A \) is a spherical cap with angular opening \( A^{-1} \), i.e. the angle \( \angle(x, y) \) between any two vectors in \( x, y \in \omega^j_A \) satisfies

\[
|\angle(x, y)| \leq A^{-1}.
\]

ii) \( \mathbb{S}^2 \) is the almost disjoint union of \( \{ \omega^j_A \}_{j \in \Omega_A} \), i.e. if \( \chi_{\omega^j_A} \) denotes the characteristic function of the cap \( \omega^j_A \), then have the following

\[
1 \leq \chi(x) := \sum_{j \in \Omega_A} \chi_{\omega^j_A}(x) \leq 3, \quad \forall x \in \mathbb{S}^2,
\]

and we require that any two centers of caps in our collection are separated by a distance \( \sim A^{-1} \) such that \( \#\Omega_A \lesssim A^2 \).
Related to this we define the function

\[ \alpha(j_1, j_2) = \inf \{ |\angle(\pm x, y)| : x \in \omega_{j_1}^A, y \in \omega_{j_2}^A \} \]

which measures the minimal angle between any two straight lines through the caps \( \omega_{j_1}^A \) and \( \omega_{j_2}^A \), respectively.

Based on the above construction, for each \( j \in \Omega_A \) we define

\[ \Omega_A^j = \left\{ (\xi, \tau) \in \mathbb{R}^3 \setminus \{0\} \times \mathbb{R} : \frac{\xi}{|\xi|} \in \omega_A^j \right\} . \]

and the corresponding localization operator

\[ \mathcal{F}(Q_A^j u)(\xi, \tau) = \frac{X_{\omega_A^j}(\xi)}{\lambda(\xi)^j} \mathcal{F}u(\xi, \tau). \]

For \( k, \ell \in \mathbb{R} \) and \( T > 0 \) we define the space \( \mathcal{Z}_{k, \ell}^T \) as the Banach space of all pairs of space-time distributions \((u, n)\) which satisfy

\[ u \in C([0, T]; H^k(\mathbb{R}^3; \mathbb{C})), \]
\[ n \in C([0, T]; H^\ell(\mathbb{R}^3; \mathbb{R}) \cap C^1([0, T]; H^{\ell - 1}(\mathbb{R}^3; \mathbb{R})), \]

endowed with the standard norm \( || \cdot || \}_{\mathcal{Z}_{k, \ell}^T} \) defined as

\[ ||(u, n)||_{\mathcal{Z}_{k, \ell}^T}^2 = \sup_{t \in [0, T]} \left\{ ||u(t)||^2_{H^k_x} + ||n(t)||^2_{H^\ell_x} + ||\partial_t n(t)||^2_{H^{\ell - 1}_x} \right\}. \] (3.4)

Let \( \sigma, b \in \mathbb{R}, 1 \leq p < \infty \). In connection to the operator \( i\partial_t + \Delta \) we define the Bourgain space \( X^{S}_{\sigma, b, p} \) of all \( u \in \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R}) \) for which the norm

\[ ||u||_{X^{S}_{\sigma, b, p}} = \left( \sum_{N \geq 1} N^{2\sigma} \left( \sum_{L \geq 1} L^p \|S_L P_N u\|_{L^2}^p \right)^\frac{1}{p} \right)^{\frac{1}{2}} \]

is finite. Similarly, to the half-wave operators \( i\partial_t \pm (\nabla) \) we associate the Bourgain spaces \( X^{W}_{\sigma, b, p, \pm} \) of all \( v \in \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R}) \) for which the norm

\[ ||v||_{X^{W}_{\sigma, b, p, \pm}} = \left( \sum_{N \geq 1} N^{2\sigma} \left( \sum_{L \geq 1} L^p \|W^\pm_L P_N u\|_{L^2}^p \right)^\frac{1}{p} \right)^{\frac{1}{2}} \]

is finite. For \( p = \infty \) we modify the definition as usual. In cases where the Schwartz space \( \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}) \) is not dense in \( X^{W}_{\sigma, b, p, \pm} \) or \( X^{S}_{\sigma, b, p} \), respectively, we redefine the spaces and take the closure of \( \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}) \) instead.

For a normed space \( B \subset \mathcal{S}'(\mathbb{R}^n \times [0, T]) \) of space-time distributions we denote by \( \overline{B} \) the space of complex conjugates with the induced norm.

For \( T > 0 \) we define the space \( B(T) \) of restrictions of distributions in \( B \) to the set \( \mathbb{R}^n \times (0, T) \) with the induced norm

\[ ||u||_{B(T)} = \inf \{ ||\tilde{u}||_B : \tilde{u} \in B \text{ is an extension of } u \text{ to } \mathbb{R}^n \times \mathbb{R} \}. \]
3.2. Multilinear estimates. This section is devoted to the proof of the crucial multilinear estimates which imply the well-posedness result for the Zakharov system in Theorem 3.1. The detailed reduction to multilinear estimates as explained in [1] Section 3 remains true verbatim, cf. also [8], so we do not reproduce it here. Given these multilinear estimates, Theorem 1.5 can be deduced by the standard Picard iteration argument as described in [1] Section 5 for the 2d case. Therefore, in the sequel we will focus on the proof of the following:

**Theorem 3.1.** Assume that $s > 0, \sigma > -\frac{1}{2}, \sigma \leq s \leq \sigma + 1, \sigma - 2s < -\frac{1}{2}$.

i) For all $0 < T \leq 1$ and for all functions $u, u_1, u_2 \in X^{s,1}_{\sigma,1}(T)$ and $v \in X^{s,1}_{\sigma,1}(T)$ the following estimates hold true:

\[
\|uv\|_{X^{s,\frac{1}{2}}_{\sigma,\frac{1}{2}}(T)} \lesssim \|u\|_{X^{s,\frac{1}{2}}_{\sigma,\frac{1}{2}}(T)}\|v\|_{X^{s,\frac{1}{2}}_{\sigma,\frac{1}{2}}(T)},
\]

(3.5)

\[
\|u\bar{v}\|_{X^{s,1}_{\sigma,1}(T)} \lesssim \|u\|_{X^{s,1}_{\sigma,1}(T)}\|v\|_{X^{s,1}_{\sigma,1}(T)},
\]

(3.6)

\[
\left\| \frac{\Delta}{\langle v \rangle}(u_2 \bar{u}) \right\|_{X^{s,1}_{\sigma,1}(T)} \lesssim \|u_1\|_{X^{s,1}_{\sigma,1}(T)}\|u_2\|_{X^{s,1}_{\sigma,1}(T)}.
\]

(3.7)

ii) There exists $\theta = \theta(s, \sigma) > 0$ in the above regime for $s, \sigma$ such that all the inequalities can be improved with a factor of $T^\theta$ on the right hand side.

We have split the above result in two parts for the following reason. Part i) contains the "clean" estimates without keeping track of the gains of powers of $T$ which may distract the reader from the main ideas. However, from part ii) we would be able to claim only a small data result for the Zakharov system. It is part ii) that allows us to claim the local well-posedness result for large data.

We introduce the notation

\[
I(f, g_1, g_2) = \int f(\zeta_1 - \zeta_2)g_1(\zeta_1)g_2(\zeta_2)d\zeta_1d\zeta_2,
\]

where $\zeta_i = (\xi_i, \tau_i), i = 1, 2$. Using duality and the fact that $\mathcal{F}u = \mathcal{F}\mathcal{F}(u)$, we can reduce Theorem 3.1 to the following trilinear estimates:

**Proposition 3.2.** Assume that $s > 0, \sigma > -\frac{1}{2}, \sigma \leq s \leq \sigma + 1, \sigma - 2s < -\frac{1}{2}$.

i) For all $v, u_1, u_2 \in S(\mathbb{R}^3 \times \mathbb{R})$ it holds

\[
|I(\mathcal{F}v, \mathcal{F}u_1, \mathcal{F}u_2)| \lesssim \|u_1\|_{X^s_{-\frac{1}{2},\infty}}\|u_2\|_{X^s_{-\frac{1}{2},\infty}}\|v\|_{X^{s,\frac{1}{2}}_{\sigma,\frac{1}{2}}},
\]

(3.8)

\[
|I(\mathcal{F}v, \mathcal{F}u_1, \mathcal{F}u_2)| \lesssim \|u_1\|_{X^s_{\frac{1}{2},\infty}}\|u_2\|_{X^s_{\frac{1}{2},\infty}}\|v\|_{X^{s,\frac{1}{2}}_{\sigma,\frac{1}{2}}},
\]

(3.9)

ii) There exists $b = b(s, \sigma) < \frac{1}{2}$ in the above regime for $s, \sigma$ such that the above inequalities hold true with $\|u_2\|_{X^s_{1,\infty}}$ instead.

Obviously part i) in Theorem 3.1 follows from part i) in the Proposition. Part ii) in Theorem 3.1 follows from part ii) of the Proposition and the following estimate

\[
\|f\|_{X^{s,b,1}_{\sigma,1}(T)} \lesssim T^{\frac{1}{2} - b}\|f\|_{X^{s,\frac{1}{2},1}(T)}
\]

(3.10)

whenever $0 \leq b < \frac{1}{2}$. Here $X^{s,b,1}_{\sigma,1}(T)$ stands both for $X^{s}_{\sigma,b,1}(T)$ and $X^{s,1}_{\sigma,b,1}(T)$. A proof of (3.10) can be found in [1] Section 5.

The proof of Proposition 3.2 is given at the end of this section. As building blocks we provide a number of preliminary estimates first. These are concerned
with functions which are dyadically localized in frequency and modulation. In some cases we additionally differentiate frequencies by their angular separation.

We start this analysis by recalling the well-known bilinear generalization of the linear $L^4$ Strichartz estimate for the Schrödinger equation in dimension 2 which is essentially due to Bourgain [5, Lemma 111]. We observe that a similar estimate is true for a Wave-Schrödinger interaction.

**Proposition 3.3** (Bilinear Strichartz estimates).

i) Let $u_1, u_2 \in L^2(\mathbb{R}^4)$ be dyadically Fourier-localized such that

\[ \text{supp } \mathcal{F} u_i \subset \mathcal{P}_{N_i} \cap \mathcal{S}_{L_i} \]

for $L_1, L_2 \geq 1$, $N_1, N_2 \geq 1$. Then the following estimate holds:

\[ \|u_1 u_2\|_{L^4(\mathbb{R}^4)} \lesssim N_1^{-\frac{3}{4}} N_2^{-\frac{1}{4}} L_1^{\frac{3}{4}} L_2^{\frac{1}{4}} \|u_1\|_{L^2} \|u_2\|_{L^2}. \] (3.11)

ii) Let $u, v \in L^2(\mathbb{R}^4)$ be such that

\[ \text{supp } \mathcal{F} v \subset C \times \mathbb{R} \cap \mathcal{M}_C^+, \text{ supp } \mathcal{F} u \subset \mathcal{P}_{N_1} \cap \mathcal{S}_{L_1} \]

for $L, L_1 \geq 1$, $N \geq 1$ and a cube $C \subset \mathbb{R}^3$ of sidelength $d \geq 1$. Then the following estimate holds:

\[ \|uv\|_{L^2(\mathbb{R}^4)} \lesssim \min\{d, N\} N_1^{-\frac{3}{4}} L_1^{\frac{1}{4}} \|u\|_{L^2} \|v\|_{L^2}. \] (3.12)

In particular, if

\[ \text{supp } \mathcal{F} v \subset \mathcal{P}_{N} \cap \mathcal{M}_1^+, \text{ supp } \mathcal{F} u \subset \mathcal{P}_{N_1} \cap \mathcal{S}_{L_1} \]

for $L, L_1 \geq 1$, $N, N_1 \geq 1$, it follows

\[ \|uv\|_{L^2(\mathbb{R}^4)} \lesssim \min\{N, N_1\} N_1^{-\frac{3}{4}} L_1^{\frac{1}{4}} \|u\|_{L^2} \|v\|_{L^2}. \] (3.13)

On the left hand side of (3.11), (3.13) and (3.11) we may replace each function with its complex conjugate.

**Proof.** As remarked above the estimate (3.11) is due to Bourgain [5, Lemma 111] for two dimensions and has been generalized in [7, Lemma 3.4] to higher dimensions. It remains to show (3.12) and (3.13). With $f = \mathcal{F} v$ and $g = \mathcal{F} u$ it follows

\[ \left\| \int f(\xi_1, \tau_1) g(\xi - \xi_1, \tau - \tau_1) d\xi_1 d\tau_1 \right\|_{L^2_{\xi, \tau}} \lesssim \sup_{\xi, \tau} |E(\xi, \tau)|^{\frac{1}{2}} \|f\|_{L^2} \|g\|_{L^2} \]

by the Cauchy-Schwarz inequality, where

\[ E(\xi, \tau) = \{(\xi_1, \tau_1) \in \text{supp } f \mid (\xi - \xi_1, \tau - \tau_1) \in \text{supp } g\} \subset \mathbb{R}^4. \]

With $L = \min\{L, L_1\}$ and $\overline{L} = \max\{L, L_1\}$ the volume of this set can be estimated as

\[ |E(\xi, \tau)| \leq L \cdot \{ |\xi_1| + |\tau_1| + |\xi - \xi_1|^2 \} \lesssim \overline{L} \xi_1 \in C, \ |\xi - \xi_1| \sim N_1 \}, \]

by Fubini’s theorem. The latter subset of $\mathbb{R}^3$ is contained in a cube of sidelength $m$, where $m \sim \min\{d, N_1\}$, so if $N_1 = 1$ the estimate follows. If $N_1 \geq 2$ and one component $\xi_{1,i}, i \in \{1, 2, 3\}$ is fixed, then the other two components $\xi_{1,j}, j \neq i$ are confined to an interval of length $m$. For each $i \in \{1, 2, 3\}$, we notice that in the [note that the proof of [7, Lemma 3.4] applies with $\delta = 0$ for functions which are dyadically Fourier-localized]
subset where $|(ξ − ξ_1)i| \gtrsim N_1$ we have that $|\partial_{ξ_1},(τ ± |ξ_i| + |ξ − ξ_1|^2)| \gtrsim N_1$. This shows that

$$|\{ξ \mid |τ ± |ξ| + |ξ − ξ_1|^2| \lesssim \frac{1}{ξ}, ξ_1 \in C, |ξ − ξ_1| \sim N_1\}| \lesssim N_1^{-17} m^2,$$

and the claim (3.12) follows. This also implies the claim (3.13) because the dyadic annulus of radius $N$ is contained in a cube of sidelength $d ∼ N$.

\[ \Box \]

Proposition 3.4 (Transverse high-high interactions, low modulation). Let $f, g_1, g_2 \in L^2$ with $\|f\|_{L^2} = \|g_1\|_{L^2} = \|g_2\|_{L^2} = 1$ and

$$\text{supp}(f) \subset Ψ_N \cap Ψ^0_N, \quad \text{supp}(g_k) \subset Ω^j_k \cap Ψ_N \cap Ω_{L_k} \quad (k = 1, 2),$$

where the frequencies $N, N_1, N_2$ and modulations $L, L_1, L_2$ satisfy

$$1 \ll N \lesssim N_1 \sim N_2, \quad L_1, L_2, L \lesssim N_2^2$$

while the angular localization parameters $A$ and $j_1, j_2 \in Ω_A$ satisfy

$$1 \leq A \ll N_1, \quad \alpha(j_1, j_2) \sim A^{-1}.$$

Then the following estimate holds

$$|I(f, g_1, g_2)| \lesssim N_1^{-1/2}(L_1L_2L)^{1/2}.$$  \hspace{1cm} (3.14)

Proof. We abuse notation and replace $g_2$ by $g_2(−·)$ and change variables $ξ_2 \mapsto −ξ_2$ to obtain the usual convolution structure. From now on it holds $|τ_2 − |ξ_2|^2| \sim L_2$ within the support of $g_2$. We consider only the case $\text{supp}(f) \subset Ψ^0_N$ since in the case $\text{supp}(f) \subset Ψ_N$ the same arguments apply.

For fixed $ξ_1, ξ_2$ we change variables $c_1 = τ_1 + |ξ_1|^2$, $c_2 = τ_2 − |ξ_2|^2$. By decomposing $f$ into $L$ pieces and applying the Cauchy-Schwarz inequality, it suffices to prove

$$\left| \int g_1(Φ_{c_1}(ξ_1))g_2(Φ_{c_2}^+(ξ_2))f(Φ_{c_1}^-(ξ_1) + Φ_{c_2}^+(ξ_2))dξ_1dξ_2 \right| \lesssim N_1^{-1/2} \|g_1 \circ Φ_{c_1}^-\|_{L^2_1} \|g_2 \circ Φ_{c_2}^+\|_{L^2_2} \|f\|_{L^2}$$

where $f$ is now supported in $c \leq τ − |ξ| \leq c + 1$ and $Φ_{c_k}^±(ξ) = (ξ, ±|ξ|^2 + c_k), k = 1, 2$, and the implicit constant is independent of $c, c_1, c_2$.

We refine the localization of the $ξ$ and $τ$ components by orthogonality methods. Since the support of $f$ in the $τ$ direction is confined to an interval of length $\lesssim N_1$, $|ξ|^2 − |ξ|^2$ is localized in an interval of length $\sim N_1$ which in turn localizes $|ξ|^2 − |ξ|^2$ in an interval of size $\sim 1$. By decomposing the plane into annuli of size $\sim 1$ and using the Cauchy-Schwarz inequality, we reduce (3.15) further to the additional assumption that $|ξ_1|$ and $|ξ_2|$ are localized in two intervals of length $\sim 1 \lesssim N_1A^{-1}$.

Recalling the additional angular localization, we can assume that $g_1, g_2$ and $f$ are each localized in cubes of size $N_1A^{-1}$ with respect to the $ξ$ variables.

We use the parabolic scaling $(ξ, τ) \mapsto (N_1ξ, N_1^2τ)$ to define

$$\tilde{f}(ξ, τ) = f(N_1ξ, N_1^2τ), \quad \tilde{g}_k(ξ_k, τ_k) = g_k(N_1ξ_k, N_1^2τ_k), \quad k = 1, 2.$$

If we set $c_k = c_kN_1^{-2}$, the estimate (3.15) reduces to

$$\left| \int \tilde{g}_1(Φ_{c_1}^-(ξ_1))\tilde{g}_2(Φ_{c_2}^+(ξ_2))\tilde{f}(Φ_{c_1}^-(ξ_1) + Φ_{c_2}^+(ξ_2))dξ_1dξ_2 \right| \lesssim N_1^{-1} \|\tilde{g}_1 \circ Φ_{c_1}^-\|_{L^2_1} \|\tilde{g}_2 \circ Φ_{c_2}^+\|_{L^2_2} \|\tilde{f}\|_{L^2_2},$$

\hspace{1cm} (3.16)
where now $\tilde{g}_k$ is supported in a cube of size $\sim A^{-1}$ with $|\xi_1|, |\xi_2| \sim 1$ and the supports are separated by $\sim A^{-1}$. $\tilde{f}$ is supported in a neighborhood of size $N^{-2}_1$ of the submanifold $S_3$ parametrized by $(\xi, \psi_{N_1}(\xi))$ for $\psi_{N_1}(\xi) = \frac{|\xi|}{N_1} + \frac{\xi}{N_1}$. Let us put $\varepsilon = N^{-2}_1$ and denote this neighborhood by $S_3(\varepsilon)$. The separation of $\xi_1$ and $\xi_2$ above implies also that in the support of $\tilde{f}$ we have $|\xi| \gtrsim A^{-1} \geq N^{-1}_1$.

By density and duality it is enough to consider continuous $\tilde{g}_1, \tilde{g}_2$ and we can further rewrite the above estimate as

$$
\|\tilde{g}_1|_{S_1} \ast \tilde{g}_2|_{S_2}\|_{L^2(S_3(\varepsilon))} \lesssim \varepsilon^\beta \|\tilde{g}_1\|_{L^2(S_1)} \|\tilde{g}_2\|_{L^2(S_2)}
$$

where $S_i$, $i = 1, 2$ are parametrized by $\phi^i_{\varepsilon}$. The above localization properties of the support of $\tilde{g}_i$ are inherited by $S_i$, which implies that the maximal diameter of the $S_1, S_2$ and $S_3$ is at most $R \sim A^{-1}$.

It is a straightforward to check that $S_1, S_2$ and $S_3$ verify Part ii) of Assumption 1.1 with $\beta = 1$ and $b \sim 1$.

We now turn our attention to the transversality condition, i.e. Part ii) of Assumption 1.1 Since $\alpha(j_1, j_2) \sim A^{-1}$, there exists a unit vector $v$ which is almost orthogonal to any $\xi_1 \in \Omega^1_{2\varepsilon}$ and any $\xi_2 \in \Omega^{2\varepsilon}_{2\varepsilon}$ in the following sense

$$
|\det \left( \frac{\xi_1}{|\xi_1|}, \frac{\xi_2}{|\xi_2|}, v \right) | = \text{vol} \left( \frac{\xi_1}{|\xi_1|}, \frac{\xi_2}{|\xi_2|}, v \right) \sim |\sin \angle \left( \frac{\xi_1}{|\xi_1|}, \frac{\xi_2}{|\xi_2|}, v \right) | \sim A^{-1}
$$

The codimensions of $S_1, S_2, S_3$ add up to 3 instead of 4. In order to be able to apply the results in the first part of the paper, we foliate one of the surfaces to increase its codimension by one. We do this for $S_3$ as follows:

$$
S_3 = \bigcup_{c \in I} S_3^c
$$

where $S_3^c = S_3 \cap \{(\xi, \tau) : \xi \cdot v = c\}$ and $c$ varies in an interval $I$ of length $|I| \sim A^{-1}$. Each $S_3^c$ retains its $C^{1,1}$ structure. In addition,

$$
\|f\|_{L^2(S_3)}^2 = \int_I \|f\|_{L^2(S_3^c)}^2 \, dc \lesssim A^{-1} \sup_c \|f\|_{L^2(S_3^c)}^2
$$

For fixed $c \in I$, let us identify a basis of unit normals to $S_3^c$. For the following calculations, we set $\xi = \xi_0$ and denote the components as

$$
\xi_k = (\xi_{k,1}, \xi_{k,2}, \xi_{k,3}), k = 0, 1, 2.
$$

At each point we keep the normal to the cone

$$
n_{S_3} = \left( \frac{\xi_{0,1}}{|\xi| (N_1)}, \frac{\xi_{0,2}}{|\xi| (N_1)}, \frac{\xi_{0,3}}{|\xi| (N_1)} - \frac{N_1}{\langle N_1 \rangle} \right).
$$

Another convenient normal is $n_{S_3} = (v, 0)$. This choice is simple, but it has the disadvantage that $\{n_{S_3}, n_{S_3}^c\}$ is not an orthonormal basis. On the other hand,

$$
|n_{S_3} \cdot n_{S_3}^c| = |v \cdot \left( \frac{\xi_{0,1}}{|\xi| (N_1)}, \frac{\xi_{0,2}}{|\xi| (N_1)}, \frac{\xi_{0,3}}{|\xi| (N_1)} - \frac{N_1}{\langle N_1 \rangle} \right)| \leq \frac{1}{\langle N_1 \rangle} \ll 1.
$$

Therefore, a correct orthonormal set of normals to $S_3^c$ is $\{n_{S_3}, n_{S_3}^c\}$, with

$$
n_{S_3}^c = \frac{n_{S_3} - (n_{S_3} \cdot n_{S_3}) n_{S_3}}{|n_{S_3} - (n_{S_3} \cdot n_{S_3}) n_{S_3}|}.
$$
Now, we can analyze the transversality properties of our submanifolds $S_1, S_2, S_3$ in the sense of (1.2). Let $n_1, n_2$ be the unit normals at $S_1$, respectively $S_2$. Then we need to determine the absolute value of the determinant

$$d = \det(n_1, n_2, n_{S_3}) = \frac{1}{|n_{S_3} - (n_{S_3} \cdot n_{S_1})n_{S_1}|} \det(n_1, n_2, n_{S_1}, n_{S_3}).$$

In view of (3.20) we obtain

$$d \sim \det(n_1, n_2, n_{S_1}, n_{S_3}) = \begin{vmatrix} 2\xi_1 + 2\xi_2 + 2\xi_3 & 2\xi_1 + 2\xi_2 + 2\xi_3 & 2\xi_2 - 2\xi_3 \xi_0, v_1 \xi_0, v_2 \xi_0, v_3 \end{vmatrix},$$

Expansion along the third column shows that

$$|\det(n_1, n_2, n_{S_1}, n_{S_3}) - d| \leq N^{-1},$$

i.e. the main contribution comes from the (4,3)-minor

$$\tilde{d} = \frac{N}{(N_1)} \begin{vmatrix} 2\xi_1 + 2\xi_2 + 2\xi_3 & 2\xi_1 + 2\xi_2 + 2\xi_3 & 2\xi_2 - 2\xi_3 \xi_0, v_1 \xi_0, v_2 \xi_0, v_3 \end{vmatrix},$$

which can be rewritten as

$$\tilde{d} = \frac{N}{(N_1)} \begin{vmatrix} 2\xi_1 + 2\xi_2 + 2\xi_3 & 2\xi_1 - 2\xi_3 \xi_0, v_1 \xi_0, v_2 \xi_0, v_3 \end{vmatrix} \det \left( \frac{\xi_1}{\xi_0}, \frac{\xi_2}{\xi_0}, v \right)$$

which, by (3.18), implies that $|\tilde{d}| \sim A^{-1} \gg N^{-1}$. Therefore we have established that $|d| \sim A^{-1}$. Recalling that the diameters of $S_1, S_2, S_3$ are $\sim A^{-1}$, we can now apply Corollary 1.4 which implies

$$\|\tilde{g}_1|_{S_1} \ast \tilde{g}_2|_{S_2}\|_{L^2(S_3)} \lesssim A^{-\frac{3}{2}}\|\tilde{g}_1\|_{L^2(S_1)}\|\tilde{g}_2\|_{L^2(S_2)}.$$
This shows that $|\xi_1 + \xi_2| = |\xi_0| \sim N$, $|\xi_1|, |\xi_2| \sim N_1$.

In the following, we use almost orthogonality methods to further localize all functions to smaller pieces, for which the claim is trivial.

By decomposing $f, g_1, g_2$ into $L, L_1, L_2$ pieces, respectively, and applying the Cauchy-Schwarz inequality, it suffices to prove

$$\left| \int g_1(\xi, \tau_1)g_2(\xi, \tau_2) f(\xi + \xi_2, \tau_1 + \tau_2) d\xi_1 d\xi_2 d\tau_1 d\tau_2 \right| \leq N_1^{-\frac{2}{3}} \|g_1\|_{L^2} \|g_2\|_{L^2} \|f\|_{L^2},$$

where $f$ is now supported in $c \leq \tau - |\xi| \leq c + 1$ and $g_k$ is supported in $c_k \leq \tau_k - |\xi_k| \leq c_k + 1$. Therefore, with respect to the $\tau$ variable, $f$ is supported in an interval of length $\sim N$. Using orthogonality, we can further localize $g_k$ with respect to the second variable $\tau_k$ to intervals of length $\sim N, k = 1, 2$. In turn this implies that the spatial frequencies $\xi_k$ can be localized further to annuli of width $\sim NN_1^{-1} \leq 1$. In light of (3.22) we can strengthen the localization of $g_k$ with respect to $\xi_k$ to cubes of side-length $\sim 1$. As a consequence, we also improve the localization of the $\xi$-support of $f$ to cubes of size $\sim 1$, which then also allows to localize $f$ with respect to $\tau$ to intervals of length $\sim 1$. Now, we repeat the above procedure: We can further localize $g_k$ with respect to $\tau_k$ to intervals of length $\sim 1$, which also implies a better localization for $g_k$ with respect to $\xi_k$ to annuli of width $\sim N_1^{-1}$.

In summary, we have reduced the problem to the case when the volume of the supports of $g_1$ and $g_2$ is $\sim N_1^{-1}$ which then trivially gives (3.23) by virtue of the Cauchy-Schwarz inequality. \qed

Next, we summarize the previous two results in the following Corollary, which settles the high-high to low interactions with low modulation.

**Corollary 3.6** (high-high to low interactions, low modulation). Assume that $f, g_1, g_2 \in L^2$ with $\|f\|_{L^2} = \|g_1\|_{L^2} = \|g_2\|_{L^2} = 1$ and

$$\text{supp}(f) \subset \Psi_N \cap \mathcal{W}_L^+,$$
$$\text{supp}(g_k) \subset \Psi_{N_k} \cap \mathcal{S}_{L_k} \ (k = 1, 2),$$

where $N, N_1, N_2$ and $L, L_1, L_2$ satisfy

$$1 \ll N \lesssim N_1 \sim N_2, \quad L_1, L_2, L \lesssim N_2^2.$$  

Then, the following estimate holds

$$|I(f, g_1, g_2)| \lesssim (L_1 L_2 L)^{\frac{1}{4}} N_1^{-\frac{1}{4}} \log N_1. \quad (3.24)$$

**Proof.** It suffices to consider non-negative $f, g_1, g_2$. We choose a threshold $M = CN_1$ such that for $A < M$ Proposition 3.4 respectively for $A = M$ Proposition 3.5 is applicable, and decompose

$$|I(f, g_1, g_2)| \leq \sum_{A=1}^{M-1} \sum_{(j_1, j_2) \sim A^{-1}} I(f, Q_{i_1}^{j_1} g_1, Q_{i_2}^{j_2} g_2) + \sum_{(j_1, j_2) \sim M^{-1}} I(f, Q_{i_1}^{j_1} g_1, Q_{i_2}^{j_2} g_2).$$
Concerning the first sum, we use (3.14) for fixed $A$ and obtain
\[
\sum_{\alpha(j_1,j_2) \approx A^{-1}} I(f, Q_A^{j_1} g_{1}, Q_A^{j_2} g_{2}) \lesssim \frac{(L_1 L_2 L)^{\frac{1}{2}}}{N_1^{\frac{1}{2}}} \sum_{\alpha(j_1,j_2) \approx A^{-1}} \| Q_A^{j_1} g_{2} \|_2 \| Q_A^{j_2} g_{2} \|_2 \lesssim N_1^{-\frac{1}{2}} (L_1 L_2 L)^{\frac{1}{2}},
\]
where we use Cauchy-Schwarz in the last step. Concerning the second sum, we use (3.21) for fixed $A$ and obtain the same bound as above. Dyadic summation with respect to $A$ introduces the additional factor $\log N_1$, which leads to (3.24).

The case of high-high to low interactions with high modulation is covered by the following Proposition.

**Proposition 3.7** (high-high to low interactions, high modulation). Assume that $f, g_1, g_2 \in L^2$ with $\|f\|_2 = \|g_1\|_2 = \|g_2\|_2 = 1$ and
\[
supp(f) \subset \mathcal{P}_N \cap \mathcal{W}_L^\perp, \quad supp(g_k) \subset \mathcal{P}_{N_k} \cap \mathcal{H}_{L_k} \quad (k = 1, 2),
\]
where $N, N_1, N_2$ and $L, L_1, L_2$ satisfy
\[
1 \leq N \lesssim N_1 \sim N_2, \quad N_2^2 \lesssim \max\{L, L_1, L_2\}.
\]

Then, the following estimate holds
\[
|I(f, g_1, g_2)| \lesssim N_1^{-\frac{1}{2}} (L_1 L_2 L)^{\frac{1}{2}} (\max\{L, L_1, L_2\}^{-2} N_1^{-2})^{-\frac{1}{2}}. \tag{3.25}
\]

**Proof.** Case a) $L = \max\{L, L_1, L_2\}$: Use Cauchy-Schwarz and (3.11).

Case b) $L_1 = \max\{L, L_1, L_2\}$ or $L_2 = \max\{L, L_1, L_2\}$: Use Cauchy-Schwarz and (3.13). □

The next proposition covers the case of low-high interactions.

**Proposition 3.8** (low-high interactions). Let $f, g_1, g_2 \in L^2$ be functions with $\|f\|_2 = \|g_1\|_2 = \|g_2\|_2 = 1$ such that
\[
supp(f) \subset \mathcal{P}_N \cap \mathcal{W}_L^\perp, \quad supp(g_k) \subset \mathcal{P}_{N_k} \cap \mathcal{H}_{L_k} \quad (k = 1, 2),
\]
with $1 \leq N_1 \ll N_2$.

i) If $L_2 \ll N_2$, then we have
\[
|I(f, g_1, g_2)| \lesssim N_1^{-\frac{1}{2}} (L_1 L_2 L)^{\frac{1}{2}} \max\{L, L_1, L_2\}^{-\frac{1}{2}}. \tag{3.26}
\]

ii) If $L_2 \gtrsim N_2^2$, then we have
\[
|I(f, g_1, g_2)| \lesssim N_1^{\frac{1}{2}} \min\{L, L_1\}^{\frac{1}{2}} \min\{N_1^2, \max\{L, L_1\}\}^{\frac{1}{2}}. \tag{3.27}
\]

**Proof.** The integral vanishes unless $N_2 \sim N$ and
\[
\max\{L, L_1, L_2\} \gtrsim \|\xi_1\|^2 - |\xi_2|^2 \pm |\xi_1 - \xi_2| \gtrsim N_2^2. \tag{3.28}
\]

We split the proof into two cases:

Case a) $L_2 \ll N_2^2$.

Subcase i) $L = \max\{L, L_1, L_2\}$. The bilinear $L^2$ estimate (3.11) yields
\[
|I(f, g_1, g_2)| \lesssim \|f\|_2 \|F^{-1} g_1 \overline{F^{-1} g_2}\|_2 \lesssim L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} N_1 N_2^{-\frac{1}{2}}.
\]

Subcase ii) $L_1 = \max\{L, L_1, L_2\}$. Since $g_1$ is localized to a cube of sidelength $N_1$ with respect to the $\xi_1$ variable, by almost orthogonality the estimate reduces to
the case when $f$ and $g_2$ are similarly localized to cubes of sidelength $N_1$. Then we use the bilinear $L^2$ estimate (3.12) with $d = N_1$ to obtain

$$|I(f, g_1, g_2)| \lesssim \|g_1\|_{L^2} \|\mathcal{F}^{-1} f\|_{L^2} \|g_2\|_{L^2} \lesssim L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} N_1 N_2^{-\frac{1}{2}}.$$  

This finishes the proof of (3.20).

**Case b)** $L_2 \gtrsim N_2^2$. Again, since $g_1$ is localized to a cube of sidelength $N_1$ with respect to the $\xi$ variable, the estimate reduces to the case when $f$ and $g_2$ are localized to cubes of sidelength $N_1$ with respect to the $\xi$ variables.

**Subcase i)** $L \leq L_1$ and $N_1^2 \leq \max\{L, L_1\}$. The volume of the support of $f$ is $\lesssim N_1^3 L$, and we estimate

$$|I(f, g_1, g_2)| \lesssim \|f\|_{L^1} \|g_1\|_{L^2} \|g_2\|_{L^2} \lesssim N_1^2 L^{\frac{3}{2}}.$$

**Subcase ii)** $L_1 < L$ and $N_1^2 \leq \max\{L, L_1\}$. The volume of the support of $g_1$ is $\lesssim N_1^3 L_1$, and we estimate

$$|I(f, g_1, g_2)| \lesssim \|f\|_{L^1} \|g_1\|_{L^1} \|g_2\|_{L^2} \lesssim N_1^2 L_1^{\frac{3}{2}}.$$

**Subcase iii)** $N_1^2 > \max\{L, L_1\}$. Cauchy-Schwarz and the bilinear $L^2$ estimate (3.12) yields

$$|I(f, g_1, g_2)| \lesssim \|\mathcal{F}^{-1} f\|_{L^1} \|g_1\|_{L^1} \|g_2\|_{L^2} \lesssim N_1^2 L_1^{\frac{3}{2}} L_1^{\frac{3}{2}},$$

which finishes the proof of (3.27). \qed

Finally, we deal with the case where the wave frequency is very small.

**Proposition 3.9** (very small wave frequency). Assume that $f, g_1, g_2 \in L^2$ with $\|f\|_{L^2} = \|g_1\|_{L^2} = \|g_2\|_{L^2} = 1$ such that

$$\text{supp}(f) \subset \mathcal{P}_N \cap \mathcal{W}_L^k, \quad \text{supp}(g_k) \subset \mathcal{P}_{N_k} \cap \mathcal{S}_L \quad (k = 1, 2),$$

and assume that $N \lesssim 1$. Then,

$$|I(f, g_1, g_2)| \lesssim \min\{L, L_1, L_2\}^{\frac{1}{2}}. \quad (3.29)$$

**Proof.** Using orthogonality we reduce the problem to the case when both $g_1$ and $g_2$ are supported in cubes of size $\sim 1$ with respect to the $\xi_k$ variables. Then, the volume of the support of $f$ is $L$, while the volume of the support of $g_k$ is $L_k$. If $L = \min\{L, L_1, L_2\}$, then by using the trivial estimate

$$|I(f, g_1, g_2)| \lesssim \|f\|_{L^1} \|g_1\|_{L^2} \|g_2\|_{L^2} \lesssim L^{\frac{1}{2}}$$

the claim follows. If $L_1 = \min\{L, L_1, L_2\}$ then we obtain

$$|I(f, g_1, g_2)| \lesssim \|f\|_{L^2} \|g_1\|_{L^1} \|g_2\|_{L^2} \lesssim L_1^{\frac{1}{2}}.$$

The case $L_2 = \min\{L, L_1, L_2\}$ follows in a similar manner. \qed

3.3. **Proof of Proposition 3.2** We prove parts [i] and [ii] at the same time. We focus on establishing (3.8) and (3.9) as stated in part [i]. Then, at any step we show that we can improve the corresponding estimate by using the $X_{s, b, \infty}$ norm instead of $X_{s, b, \infty}$ norm on of the terms involved in the estimate, where $b$ is a parameter which depends on $s$ and $\sigma$. The conditions on $b$ will accumulate in several steps but one has to keep in mind that $b < \frac{1}{2}$ is the starting condition and it will not be repeated.
Proof of Proposition 3.2. By definition of the norms it is enough to consider functions with non-negative Fourier transform. We dyadically decompose

\[ u_k = \sum_{N_k, L_k \geq 1} S_{L_k} P_{N_k} u_k, \quad v = \sum_{N, L \geq 1} W_{L} P_{N} v. \]

Setting \( g_k^{L_k, N_k} = FS_{L_k} P_{N_k} u_k \) and \( f^{L, N} = FW_{L} P_{N} v \), we observe

\[ I(\mathcal{F}v, \mathcal{F}u_1, \mathcal{F}u_2) = \sum_{N, N_1, N_2 \geq 1} \sum_{L, L_1, L_2 \geq 1} I(f^{L, N}, g_1^{L_1, N_1}, g_2^{L_2, N_2}). \]

Case a) high-high-low interactions, i.e. \( N_1 \sim N_2 \gtrsim N \gg 1 \). Using (3.24) and (3.25) it follows that

\[
\sum_{L, L_1, L_2 \geq 1} |I(f^{L, N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| \\
\lesssim N_1^{-\frac{3}{4}} \log N_1 \sum_{L_1, L_2 \leq N_1^2} L_1^2 \| f^{L, N} \|_{L^2} L_2 \| g_1^{L_1, N_1} \|_{L^2} L_2 \| g_2^{L_2, N_2} \|_{L^2} \\
+ N_1^{-\frac{3}{4}} \sum_{\max\{L, L_1, L_2\} > N_1^2} N_1 \frac{1}{\max\{L, L_1, L_2\}} L_1^2 \| f^{L, N} \|_{L^2} L_2 \| g_1^{L_1, N_1} \|_{L^2} L_2 \| g_2^{L_2, N_2} \|_{L^2} \\
\lesssim N_1^{-\frac{3}{4}} (\log N_1) \| P_{N_1} v \|_{X^{w_{\frac{1}{2}}, \frac{1}{2}}_{s, \frac{1}{4}}} \| P_{N_1} u_1 \|_{X^s_{\frac{1}{2}, \frac{1}{4}}} \| P_{N_2} u_2 \|_{X^s_{\frac{1}{2}, \frac{1}{4}}}.
\]

A straightforward modification also shows the bound

\[ \lesssim N_1^{\frac{1}{4} - 2b} (\log N_1)^4 \| P_{N_1} v \|_{X^{w_{\frac{1}{2}}, \frac{1}{2}}_{s, \frac{1}{4}}} \| P_{N_1} u_1 \|_{X^s_{\frac{1}{2}, \frac{1}{4}}} \| P_{N_2} u_2 \|_{X^s_{\frac{1}{2}, \frac{1}{4}}} \]

In order to prove (3.8) we perform the summation with respect to \( 1 \ll N \leq N_1 \sim N_2 \) and obtain

\[
\sum_{1 \ll N \leq N_1 \sim N_2} N^{-\sigma} N_1^{-\frac{3}{4}} (\log N_1)^4 \| P_{N_1} v \|_{X^{w_{\frac{1}{2}}, \frac{1}{2}}_{s, \frac{1}{4}}} \| P_{N_1} u_1 \|_{X^s_{\frac{1}{2}, \frac{1}{4}}} \| P_{N_2} u_2 \|_{X^s_{\frac{1}{2}, \frac{1}{4}}} \\
\lesssim \| v \|_{X^{w_{\frac{1}{2}}, \frac{1}{2}}_{s, \frac{1}{4}}} \sum_{1 \ll N \leq N_1 \sim N_2} \| P_{N_1} u_1 \|_{X^s_{\frac{1}{2}, \frac{1}{4}}} \| P_{N_2} u_2 \|_{X^s_{\frac{1}{2}, \frac{1}{4}}} \\
\lesssim \| v \|_{X^{w_{\frac{1}{2}}, \frac{1}{2}}_{s, \frac{1}{4}}} \| u_1 \|_{X^s_{\frac{1}{2}, \frac{1}{4}}} \| u_2 \|_{X^s_{\frac{1}{2}, \frac{1}{4}}},
\]

where we have used that \( \sigma > -\frac{1}{2} \). If we choose \( b \) such that \( \frac{1}{2} - 2b - \sigma < 0 \), then we also obtain

\[
\sum_{1 \ll N \leq N_1 \sim N_2} |I(f^{N_1}, g_1^{N_1}, g_2^{N_2})| \lesssim \| v \|_{X^{w_{\frac{1}{2}}, \frac{1}{2}}_{s, \frac{1}{4}}} \| u_1 \|_{X^s_{\frac{1}{2}, \frac{1}{4}}} \| u_2 \|_{X^s_{\frac{1}{2}, \frac{1}{4}}}.
\]

For proving (3.5) in this case, we perform the summation as follows:

\[
\sum_{1 \ll N \leq N_1 \sim N_2} \left( \frac{N}{N_1} \right)^{1 + \sigma} N_1^{-\frac{3}{4} + \sigma - 2s} (\log N_1)^4 \| P_{N_1} v \|_{X^{w_{\frac{1}{2}}, \frac{1}{2}}_{s, \frac{1}{4}}} \| P_{N_1} u_1 \|_{X^s_{\frac{1}{2}, \frac{1}{4}}} \| P_{N_2} u_2 \|_{X^s_{\frac{1}{2}, \frac{1}{4}}} \\
\lesssim \| v \|_{X^{w_{\frac{1}{2}}, \frac{1}{2}}_{s, \frac{1}{4}}} \sum_{1 \ll N \leq N_1 \sim N_2} \| P_{N_1} u_1 \|_{X^s_{\frac{1}{2}, \frac{1}{4}}} \| P_{N_2} u_2 \|_{X^s_{\frac{1}{2}, \frac{1}{4}}} \\
\lesssim \| v \|_{X^{w_{\frac{1}{2}}, \frac{1}{2}}_{s, \frac{1}{4}}} \| P_{N_1} u_1 \|_{X^s_{\frac{1}{2}, \frac{1}{4}}} \| P_{N_2} u_2 \|_{X^s_{\frac{1}{2}, \frac{1}{4}}},
\]
where we have used that \( \sigma - 2s < -\frac{1}{2} \). By picking \( b \) such that \( \frac{3}{2} - 2b + \sigma - 2s < 0 \), we also obtain

\[
\sum_{1 < N \leq N_1 \sim N_2} |I(f^{N}, g_1^{N_1}, g_2^{N_2})| \lesssim \|v\|_{X^0_{-1 + \frac{3}{2}, \infty}} \|u_1\|_{X^s_{0, \frac{1}{2}, \infty}} \|u_2\|_{X^s_{0, \frac{1}{2}, \infty}}.
\]

Case b) very small wave frequency, i.e. \( N \ll 1 \). In this case, either \( N_1 \sim N_2 \) or \( N, N_1, N_2 \ll 1 \). We use \((3.29)\) and obtain

\[
\sum_{L, L_1, L_2 \geq 1} |I(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2})| \lesssim \sum_{1 \leq L, L_1, L_2} \left( \frac{\min \{L, L_1, L_2\}}{LL_1L_2} \right)^{\frac{1}{2}} \|P_N v\|_{X^0_{-1 + \frac{3}{2}, \infty}} \|P_{N_1} u_1\|_{X^s_{0, \frac{1}{2}, \infty}} \|P_{N_2} u_2\|_{X^s_{0, \frac{1}{2}, \infty}} \\
\lesssim \|P_N v\|_{X^0_{-1 + \frac{3}{2}, \infty}} \|P_{N_1} u_1\|_{X^s_{0, \frac{1}{2}, \infty}} \|P_{N_2} u_2\|_{X^s_{0, \frac{1}{2}, \infty}}.
\]

A similar argument shows

\[
\sum_{L, L_1, L_2 \geq 1} |I(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2})| \lesssim \|P_N v\|_{X^0_{-1 + \frac{3}{2}, \infty}} \|P_{N_1} u_1\|_{X^s_{0, \frac{1}{2}, \infty}} \|P_{N_2} u_2\|_{X^s_{0, \frac{1}{2}, \infty}}.
\]

provided that \( b > 0 \). \((3.28)\) and \((3.30)\) and their counterpart in ii) follow from these estimates since \( N_1 \sim N_2 \) or \( N_1, N_2 \ll 1 \).

Case c) high-low interactions, i.e. \( N_1 \ll N_2 \) or \( N_1 \gg N_2 \). We focus on the case \( N_1 \ll N_2 \), the other one being similar. Since we apply Proposition \((3.8)\) we need to differentiate between the cases \( L_2 \ll N_2^2 \) and \( L_2 \gg N_2^2 \). In the first case, by \((3.29)\) and the observation \((3.28)\) which implies that \( \max (L, L_1) \gg N_2^2 \) for non-vanishing interactions, we have

\[
\sum_{L_2 \ll N_2^2} \sum_{L, L_1 \geq 1} |I(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2})| \\
\lesssim N_1 N_2^{-\frac{3}{2}} \sum_{L_2 \ll N_2^2} \sum_{L, L_1 \geq 1} (\max (L, L_1))^{-\frac{1}{2}} \|P_N v\|_{X^0_{-1 + \frac{3}{2}, \infty}} \|P_{N_1} u_1\|_{X^s_{0, \frac{1}{2}, \infty}} \|P_{N_2} u_2\|_{X^s_{0, \frac{1}{2}, \infty}} \\
\lesssim N_1 N_2^{-\frac{3}{2}} (\ln N_2)^2 \|P_N v\|_{X^0_{-1 + \frac{3}{2}, \infty}} \|P_{N_1} u_1\|_{X^s_{0, \frac{1}{2}, \infty}} \|P_{N_2} u_2\|_{X^s_{0, \frac{1}{2}, \infty}}.
\]

By the same reasoning we also have

\[
\sum_{L_2 \ll N_2^2} \sum_{L, L_1 \geq 1} |I(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2})| \\
\lesssim N_1 N_2^{-\frac{3}{2} - 2b} (\ln N_2)^2 \|P_N v\|_{X^0_{-1 + \frac{3}{2}, \infty}} \|P_{N_1} u_1\|_{X^s_{0, \frac{1}{2}, \infty}} \|P_{N_2} u_2\|_{X^s_{0, \frac{1}{2}, \infty}}.
\]
In the case \( L_2 \geq N_2^2 \) we use (3.27) and obtain
\[
\sum_{L_2 \geq N_2^2} \sum_{L_1 \geq 1} |I(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2})| \lesssim \sum_{L_2 \geq N_2^2} \sum_{L_1 \leq N_1} L_2^{-\frac{1}{2}} \|P_N v\|_{X^{1,\infty}} \|P_{N_1} u_1\|_{X^{\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^{\frac{1}{2},\infty}} + N_1^2 \sum_{L_2 \geq N_2^2} \sum_{L_1 \leq \max\{L, L_1\}} (\max\{L, L_1\} L_2)^{-\frac{1}{2}} \|P_N v\|_{X^{1,\infty}} \|P_{N_1} u_1\|_{X^{\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^{\frac{1}{2},\infty}} \lesssim N_1^2 (\ln N_1)^2 N_2^{-1} \|P_N v\|_{X^{1,\infty}} \|P_{N_1} u_1\|_{X^{\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^{\frac{1}{2},\infty}}
\]

In a similar manner we obtain
\[
\sum_{L_2 \geq N_2^2} \sum_{L_1 \geq 1} |I(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2})| \lesssim N_1^{1-s+\frac{1}{2}} N_{-s+\frac{1}{2}} (\ln N_1)^2 N_2^{-1} \|P_N v\|_{X^{1,\infty}} \|P_{N_1} u_1\|_{X^{\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^{\frac{1}{2},\infty}}
\]

For proving (3.3) we estimate the above term in the worst case in which we place the low Schrödinger frequency in the space with positive Sobolev regularity and the high Schrödinger frequency in the space with negative Sobolev regularity. It is obvious that the other case gives better estimates. From the above inequalities we deduce
\[
\sum_{N_1 \ll N_2} |I(f^{N}, g_1^{N_1}, g_2^{N_2})| \lesssim \sum_{N_1 \ll N_2} N_1^{-s+\frac{1}{2}} N_{-s+\frac{1}{2}} (\ln N_1)^2 \|P_N v\|_{X^{1,\infty}} \|P_{N_1} u_1\|_{X^{\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^{\frac{1}{2},\infty}}
\]

If \( s \leq \frac{1}{2} \), then we can bound the above sum by
\[
\|P_{N_1} u_1\|_{X^{\frac{1}{2},\infty}} \sum_{N \sim N_2} N^{-s-\frac{1}{4}} (\ln N_1)^3 \|P_N v\|_{X^{1,\infty}} \|P_{N_2} u_2\|_{X^{s+\frac{1}{2},\infty}} \lesssim \|P_N v\|_{X^{1,\infty}} \|P_{N_1} u_1\|_{X^{s+\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^{s+\frac{1}{2},\infty}}
\]

where we have used that \( \sigma > -\frac{1}{4} \).

If \( s > \frac{1}{2} \), then we can bound the above sum by
\[
\|P_{N_1} u_1\|_{X^{s+\frac{1}{2},\infty}} \sum_{N \sim N_2} N^{s-\frac{1}{4}} \|P_N v\|_{X^{1,\infty}} \|P_{N_2} u_2\|_{X^{s-\frac{1}{2},\infty}} \lesssim \|P_N v\|_{X^{1,\infty}} \|P_{N_1} u_1\|_{X^{s+\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^{s+\frac{1}{2},\infty}}
\]

where we have used that \( s \leq 1 + \sigma \). As noted earlier, it is obvious that in the case \( N_2 \ll N_1 \sim N \) the above estimate is easier. With similar arguments we can verify the counterpart in ii) of these estimates, but we omit the details.

Concerning (3.9) we proceed as follows:
\[
\sum_{N_1 \ll N_2} |I(f^{N}, g_1^{N_1}, g_2^{N_2})| \lesssim \sum_{N_1 \ll N_2} N_1^{-s+\frac{1}{2}} N_{-s+\frac{1}{2}} (\ln N_1)^2 \|P_N v\|_{X^{1,\infty}} \|P_{N_1} u_1\|_{X^{s+\frac{1}{2},\infty}} \|P_{N_2} u_2\|_{X^{s+\frac{1}{2},\infty}}
\]
If $s \leq \frac{1}{2}$, then we can bound the above sum by

$$
\|P_{N_1}u_1\|_{X^{\frac{1}{2}, \infty}} \leq \sum_{N \approx N_2} N^{-2s+\frac{1}{2}+\sigma} (\ln N)^2 \|P_{N_2}v\|_{X^{w_{-1-s, \frac{1}{2}}, \infty}} \|P_{N_2}u_2\|_{X^{\frac{1}{2}, \infty}}
$$

where we have used that $\sigma + \frac{1}{2} < 2s$.

If $s > \frac{1}{2}$, then we can bound the above sum by

$$
\|P_{N_1}u_1\|_{X^{\frac{1}{2}, \infty}} \leq \sum_{N \approx N_2} N^{-s+\sigma} \|P_{N_2}v\|_{X^{w_{-1-s, \frac{1}{2}}, \infty}} \|P_{N_2}u_2\|_{X^{\frac{1}{2}, \infty}}
$$

where we have used that $\sigma \leq s$.

It is an easy exercise to verify the counterpart in ii) of these estimates. This concludes the proof of Proposition 3.

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