A New Stability Criterion for Systems with Distributed Time-Varying Delays via Mixed Inequalities Method

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This paper is concerned with the delay-dependent stability of systems with distributed time-varying delays. The novelty relies on the use of some new inequalities which are less conservative than some existing inequalities. A less conservative stability criterion is obtained by constructing some new augmented Lyapunov–Krasovskii functionals, which are given in terms of linear matrix inequalities. The effectiveness of the presented criterion is demonstrated by two numerical examples.

1. Introduction

Consider the systems with distributed time-varying delays:

\[
\dot{x}(t) = Ax(t) + Bx(t - h(t)) + C \int_{t-h(t)}^{t} x(s)ds, \\
x(t) = \phi(t),
\]

where \( x(t) \in \mathbb{R}^n \) is the system state, \( A, B, C \in \mathbb{R}^{n \times n} \) are constant matrices, and \( h(t) \) is the time-varying delay satisfying

\[
0 \leq h(t) \leq h, \quad -u \leq \dot{h}(t) \leq u < 1.
\]

Since time delays occur in many dynamic systems, stability analysis of the delay system [1–5] has become a hot topic in the past few decades. Due to the representation of linear systems with time-varying delays, the delay-dependent stability analysis via the LKF method has attracted much attention. The conservatism of the LKF method comes from two aspects: the construction of the LKF and the bound on its derivative.

Selecting the LKF is crucial to derive less conservative criteria. An augmented LKF [6] is proposed to reduce the conservatism in the early literature. Recently, a new augmented LKF [7] is introduced by employing the information of a second-order Bessel–Legendre inequality. It is necessary to take the derivative of the LKF to derive a stability criterion. The difficulty lies in the bounds of the integrals that arise in the derivative of the LKF. There are two main methods for dealing with such integrals: the free-weighting matrix method [8] and the integral inequality method. The integral inequality method includes various integral inequalities, such as Jensen inequality [9–11], Wirtinger-based inequality [12–15], free matrix-based inequality [16, 17], auxiliary function-based inequality [18], relaxed integral inequality [19], and Bessel–Legendre inequality [20]. Very recently, the improved inequality-based functions approach [21] is proposed to derive less conservative results for systems with time-varying delays. However, when estimating \( \hat{V}(x, t) = \int_{a}^{b} x^T(s)R \dot{x}(s)ds \) is only estimated as \( \int_{a}^{b} x^T(s)R \dot{x}(s)ds \geq (1/b-a)\Omega_1 R \Omega_1 + (3/b-a)\Omega_2 R \Omega_2 + (5/b-a)\Omega_3 R \Omega_3 + \Omega_3 \), and \( \Omega_1, \Omega_2, \Omega_3 \) are the same as in Lemma 2. Then, a new integral inequality was proposed in [22] to...
further reduce the conservatism. But the integral inequality can only deal with the constant time delay. On the other hand, stability analysis for systems with distributed delays is of both practical and theoretical importance. Then, it is desirable to extend the system model to include distributed delays. In recent years, the stability analysis of systems with distributed delays has been received considerable attention [23–27]. But only the authors in [25, 26] consider the systems with distributed time-varying delays.

This paper is concerned with the delay-dependent stability of systems with distributed time-varying delays. Based on some new inequalities and some new augmented LKFs, a less conservative stability criterion is obtained in terms of LMIs. Our paper has two characteristics: (1) the integral inequality is estimated as \( \int_{a}^{b} x^T(s) R \dot{x}(s) ds \geq \frac{1}{b-a} \Omega_1^T R \Omega_1 + \frac{3}{b-a} \Omega_2^T R \Omega_2 + \frac{5}{b-a} \Omega_3^T R \Omega_3 + \frac{7}{b-a} \Omega_4^T R \Omega_4 \), which includes those in [9, 13, 20] as special cases. \( \Omega_i, i = 1, 2, 3, 4 \), is the same as in Lemma 2. (2) An augmented LKF which contains more information about \( h(t) \) is proposed to reduce the conservatism. The effectiveness of the presented criterion is demonstrated by two numerical examples.

Throughout this paper, the set \( S^n \) denotes the set of symmetric matrices and the set \( S^n_{0} \) denotes the set of symmetric positive definite matrices. For any square matrix \( P \), we define \( \text{Sym}(P) = P + P^T \).

### 2. Main Results

In this section, the following lemmas are introduced to derive the main results.

**Lemma 1** (see [20]). For any matrices \( \Theta \in S^n_{0} \), \( M_1, M_2 \in R^{m \times n} \), \( Y \in R^{2n \times m} \), and \( \forall \alpha \in (0, 1) \), the inequality

\[
-Y^T \begin{bmatrix}
\frac{1}{\alpha} \\
\frac{1}{1-\alpha}
\end{bmatrix}
Y \leq -Y^T \Sigma(\alpha) Y - \text{Sym} \left( Y^T \begin{bmatrix}
(1-\alpha)M_1^T \\
\alpha M_2^T
\end{bmatrix} \right)
+ \alpha M_1 \Theta^{-1} M_1^T + (1-\alpha)M_2 \Theta^{-1} M_2^T,
\]

holds, where

\[
\Sigma(\alpha) = \begin{bmatrix}
(2-\alpha)\Theta & 0 \\
0 & (1+\alpha)\Theta
\end{bmatrix}.
\]

**Lemma 2** (see [22]). For a matrix \( R \in S^n_{0} \) and any continuously differentiable function \( x: [a, b] \to R^n \), the inequality

\[
\int_{a}^{b} x^T(s) R \dot{x}(s) ds \geq \frac{1}{b-a} \Omega_1^T R \Omega_1 + \frac{3}{b-a} \Omega_2^T R \Omega_2 + \frac{5}{b-a} \Omega_3^T R \Omega_3 + \frac{7}{b-a} \Omega_4^T R \Omega_4,
\]

then, system (1) is asymptotically stable, where

\[
\begin{align*}
\Omega_1 &= x(b) - x(a), \\
\Omega_2 &= x(b) + x(a) - \frac{2}{b-a} \int_{a}^{b} x(s) ds, \\
\Omega_3 &= x(b) - x(a) + \frac{6}{b-a} \int_{a}^{b} x(s) ds - \frac{12}{(b-a)^2} \int_{a}^{b} x(s) ds du, \\
\Omega_4 &= x(b) + x(a) - \frac{12}{(b-a)^2} \int_{a}^{b} x(s) ds + \frac{60}{(b-a)^3} \int_{a}^{b} x(s) ds du - \frac{120}{(b-a)^4} \int_{a}^{b} x(s) ds dvdu.
\end{align*}
\]
\( \phi(\alpha, \beta) = \text{Sym}(\Pi_i^T P \Pi_i) + \Pi_i^T Q_i \Pi_i \)
\( - (1 - \beta) \Pi_i^T Q_i \Pi_i + \text{Sym}(\Pi_i^T Q_2 \Pi_0) + h^2 \varepsilon_0 \Pi_i \varepsilon_0, \)
\( \Pi_1 = \begin{bmatrix} e_1^T & \alpha e_i^T + (1 - \alpha) h \varepsilon_6^T \end{bmatrix}^T, \)
\( \Pi_2 = \begin{bmatrix} A e_i^T + B e_i^T + C a e_i^T & \alpha^2 e_i^T + \varepsilon_5^T \end{bmatrix}^T, \)
\( \Pi_3 = \begin{bmatrix} e_1^T & \lambda e_i^T + B e_i^T + C a e_i^T \end{bmatrix}^T, \)
\( \Pi_4 = \begin{bmatrix} e_i^T & e_i^T \end{bmatrix}^T, \)
\( \Pi_5 = \begin{bmatrix} e_i^T \end{bmatrix}^T, \)
\( \Pi_6 = \begin{bmatrix} A e_i^T + B e_i^T + C a e_i^T & (1 - \beta) e_i^T \end{bmatrix}^T, \)
\( \Pi_7 = \varepsilon_1 - \varepsilon_2, \)
\( \Pi_8 = \varepsilon_1 + \varepsilon_2 - 2 \varepsilon_5, \)
\( \Pi_9 = \varepsilon_1 - \varepsilon_2 + 6 \varepsilon_5 - 12 \varepsilon_7, \)
\( \Pi_{10} = \varepsilon_1 + \varepsilon_2 - 12 \varepsilon_5 + 60 \varepsilon_7 - 120 \varepsilon_9, \)
\( \Pi_{11} = \varepsilon_2 - \varepsilon_3, \)
\( \Pi_{12} = \varepsilon_2 + \varepsilon_2 - 2 \varepsilon_6, \)
\( \Pi_{13} = \varepsilon_2 - \varepsilon_3 + 6 \varepsilon_6 - 12 \varepsilon_8, \)
\( \varepsilon_0 = A e_i + B e_i + C a e_i, \)
\( \gamma = \begin{bmatrix} \Pi_i^T \Pi_i^T \Pi_i^T \varepsilon_0^T \Pi_i^T \Pi_i^T \Pi_i^T \end{bmatrix}^T, \)
\( \Theta = \text{diag}(Q_3, 3Q_5, 5Q_7, 7Q_9), \)
\( \text{and } \varepsilon_i \in \mathbb{R}^{nx10n} \text{ is defined as } \varepsilon_i = [0_{nx(i-1)n} I_n 0_{nx(10-i)n}] \)
\( \text{for } i = 1, 2, \ldots, 10. \)

**Proof.** Introduce an LKF candidate as

\[
V(x_i) = V_1(x_i) + V_2(x_i) + V_3(x_i) + V_4(x_i), \tag{16}
\]

where

\[
V_1(x_i) = \int_{t-h}^{t} [x(t) - x(t-h)]^T P [x(t) - x(t-h)],
\]

\[
V_2(x_i) = \int_{t-h}^{t} [x(t) - x(t-h)]^T Q_1 [x(t) - x(t-h)],
\]

\[
V_3(x_i) = \int_{t-h}^{t} [x(t) - x(t-h)]^T Q_2 [x(t) - x(t-h)],
\]

\[
V_4(x_i) = \int_{t-h}^{t} \int_{t-h}^{t} [x(t) - x(t-h)]^T Q_3 [x(t) - x(t-h)].
\]

Calculate the derivative of \( V(x_i) \) along the solution of system (1) as follows:

\[
\dot{V}_1(x_i) = 2 \int_{t-h}^{t} [x(t) - x(t-h)]^T P [x(t) - x(t-h)],
\]

\[
\dot{V}_2(x_i) = \int_{t-h}^{t} [x(t) - x(t-h)]^T Q_1 [x(t) - x(t-h)],
\]

\[
\dot{V}_3(x_i) = \int_{t-h}^{t} [x(t) - x(t-h)]^T Q_2 [x(t) - x(t-h)],
\]

\[
\dot{V}_4(x_i) = \int_{t-h}^{t} \int_{t-h}^{t} [x(t) - x(t-h)]^T Q_3 [x(t) - x(t-h)].
\]
Lemma 2, and we have

\[ V_4(x_t) = \mathcal{H}_T(\mathbf{x}_T(t)) - h \int_{t-h}^{t} \mathbf{x}_T(s) \mathbf{x}_s(s) \, ds. \]  

(22)

According to (18)–(22), we can obtain

where

\[ V(x_t) = \zeta(t) \left[ \text{Sym}(\Pi^T_1 \Pi^T_5) + \Pi^T_2 Q_2 \Pi_1 - (1 - \beta) \Pi^T_4 Q_3 \Pi_4 \right. 
+ \text{Sym}(\Pi^T_3 Q_4 \Pi_6) + h^2 \epsilon^T_2 Q_5 \epsilon_3 \zeta(t) 
- h \int_{t-h}^{t} \mathbf{x}_T(s) \mathbf{x}_s(s) \, ds, \]  

(23)

\begin{align*}
\zeta(t) & = \begin{bmatrix}
(t) x^T(t) & x^T(t-h) & \dot{x}^T(t-h) & \Pi^T_1(t) & \Pi^T_2(t) & \Pi^T_3(t)
\end{bmatrix}^T, \\
\pi_1(t) & = \frac{1}{h} \int_{t-h}^{t} x^T(s) \, ds - \frac{1}{h} \int_{t-h}^{t} x^T(s) \, ds \\
\pi_2(t) & = \frac{1}{h^2} \int_{t-h}^{t} \int_{u}^{t} x^T(s) \, ds \, du - \frac{1}{(h-h)^2} \int_{t-h}^{t} \int_{u}^{t-h} x^T(s) \, ds \, du \\
\pi_3(t) & = \frac{1}{h^3} \int_{t-h}^{t} \int_{u}^{t} \int_{v}^{t} x^T(s) \, ds \, dv \, du - \frac{1}{(h-h)^3} \int_{t-h}^{t} \int_{u}^{t-h} \int_{v}^{t-h} x^T(s) \, ds \, dv \, du. 
\end{align*}

(24)

Let \( \alpha = (h(t)/h) \), then \( 1 - \alpha = (h - h(t)/h) \), applying Lemma 2, and we have

\begin{align*}
- h \int_{t-h}^{t} \mathbf{x}_T(s) \mathbf{x}_s(s) \\
= -h \int_{t-h}^{t} \mathbf{x}_T(s) \mathbf{x}_s(s) \, ds - h \int_{t-h}^{t} \mathbf{x}_T(s) \mathbf{x}_s(s) \, ds \\
\leq - h \int_{t-h}^{t} \mathbf{x}_T(s) \mathbf{x}_s(s) \, ds \\
= - h \frac{1}{\alpha} \zeta^T(t) \left( \Pi^T_1 Q_3 \Pi_1 + 3 \Pi^T_2 Q_8 \Pi_8 + 5 \Pi^T_4 Q_4 \Pi_4 + 7 \Pi^T_6 Q_6 \Pi_6 \right) \zeta(t) \\
- h \frac{1}{1 - \alpha} \zeta^T(t) \left( \Pi^T_1 Q_3 \Pi_1 + 3 \Pi^T_2 Q_8 \Pi_8 + 5 \Pi^T_4 Q_4 \Pi_4 + 7 \Pi^T_6 Q_6 \Pi_6 \right) \zeta(t) \\
= - \zeta^T(t) \left[ \frac{1}{\alpha} \mathbf{Y} \mathbf{Y}^T \mathbf{Y} \right] \zeta(t),
\end{align*}

(25)
Table 1 shows that our method produces the larger upper bound \(\beta\) estimated as state vector, which may yield less conservative criteria. Remark 2. So our method can yield less conservative results.

Remark 1. The integral \(\int_a^b x^T(s)R\dot{x}(s)ds\) in [9, 13, 20] is estimated as \(\int_a^b x^T(s)R\dot{x}(s)ds \geq (1/b-a)\Omega_1^TR\Omega_1 + (3/b-a)\Omega_2^TR\Omega_2\), \(\int_a^b x^T(s)\dot{\dot{x}}(s)R\dot{x}(s)ds \geq (1/b-a)\Omega_1^TR\Omega_1 + (3/b-a)\Omega_2^TR\Omega_2 + (5/b-a)\Omega_3^TR\Omega_3\), respectively. In this paper, the integral \(\int_a^b x^T(s)R\dot{x}(s)ds\) is estimated as \(\int_a^b x^T(s)R\dot{x}(s)ds \geq (1/b-a)\Omega_1^TR\Omega_1 + (3/b-a)\Omega_2^TR\Omega_2 + (5/b-a)\Omega_3^TR\Omega_3 + (7/b-a)\Omega_4^TR\Omega_4\), which includes those in [9, 13, 20] as special cases. So our method can yield less conservative results.

Remark 2. An augmented LKF which contains more information about time-varying delay \(h(t)\) which was proposed to reduce the conservatism. \(\dot{x}(t-h(t))\) is added as a state vector, which may yield less conservative criteria.

3. Numerical Examples

Two numerical examples are given to demonstrate advantages of the proposed criterion.

Example 1. Consider system (1) with

\[
A = \begin{bmatrix} 0.0 & 1.0 \\ -1.0 & -2.0 \end{bmatrix},
B = \begin{bmatrix} 0.0 & 0.0 \\ -1.0 & 1.0 \end{bmatrix},
C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]  

For different \(u\), Table 1 presents the allowable upper bound of \(h(t)\), which guarantees the stability of system (1). Table 1 shows that our method produces the larger upper bound \(h\) than those in [7, 12, 13, 16, 17, 21]. In this sense, our stability criterion is less conservative than those in [7, 12, 13, 16, 17, 21].

Example 2. Consider system (1) with

\[
A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix},
B = \begin{bmatrix} -1.0 & 0.0 \\ -1.0 & -1.0 \end{bmatrix},
C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

For different \(u\), Table 2 presents the allowable upper bound of \(h(t)\), which guarantees the stability of system (1). Table 2 shows that our method produces the larger upper bound \(h\) than those in [7, 11, 13, 15, 16, 21]. In this sense, our stability criterion is less conservative than those in [7, 11, 13, 15, 16, 21].

4. Conclusions

This paper focuses on delay-dependent stability analysis for systems with distributed time-varying delays. The novelty relies on the use of some new inequalities which are less conservative than some existing inequalities. A less conservative stability criterion is obtained by constructing some new augmented LKFs. The effectiveness of the presented criterion is demonstrated by two numerical examples. In addition, the proposed method can be applied to stability analysis of other dynamic systems such as fuzzy systems with time-varying delay and neutral systems with time-varying delay.

Data Availability

No additional data are available for this paper.
Conflicts of Interest
The authors declare that there are no conflicts of interest.

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