Fat Hoffman graphs with smallest eigenvalue greater than $-3$

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Abstract
In this paper, we give a combinatorial characterization of the special graphs of fat Hoffman graphs containing $K_{1,2}$ with smallest eigenvalue greater than $-3$, where $K_{1,2}$ is the Hoffman graph having one slim vertex and two fat vertices.

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1 Introduction

In the field of Spectral Graph Theory, one of the important research problems is to characterize graphs with bounded smallest eigenvalue. In 1976, using root systems, P. J. Cameron, J. M. Goethals, J. J. Seidel, and E. E. Shult [2] characterized graphs whose adjacency matrices have smallest eigenvalue at least \(-2\). Their results revealed that graphs with smallest eigenvalue at least \(-2\) are generalized line graphs, except a finite number of graphs represented by the root system \(E_8\). We refer the reader to the monograph [3], for a complete account of this theory. In 1977, A. J. Hoffman [8] studied graphs whose adjacency matrices have smallest eigenvalue at least \(-1 - \sqrt{2}\) by using a technique of adding cliques to graphs. In 1995, R. Woo and A. Neumaier [14] formulated Hoffman’s idea by introducing the notion of Hoffman graphs and generalizations of line graphs. Hoffman graphs were subsequently studied in [9, 11, 12, 13, 15]. In particular, H. J. Jang, J. Koolen, A. Munemasa, and T. Taniguchi [9] proposed a scheme to classify fat indecomposable Hoffman graphs with smallest eigenvalue at least \(-3\). The present paper completes a partial case of this scheme.

Our result can also be regarded as a reformulation of a classical result of Hoffman [7] in terms of Hoffman graphs. Let \(\hat{A}(G, v^*)\) denote the adjacency matrix of a graph \(G\), modified by putting \(-1\) in the diagonal position corresponding to a vertex \(v^*\). Hoffman [7, Lemma 2.1] has shown that \(\hat{A}(L(T), e)\) has smallest eigenvalue greater than \(-2\) whenever \(e\) is an end edge of a tree \(T\), where \(L(T)\) denotes the line graph of \(T\). Moreover, under a conjecturally redundant assumption, Hoffman [7, Lemma 2.2] has shown that the smallest eigenvalue of \(\hat{A}(L(T), e)\) is a limit point of the set of smallest eigenvalues of graphs. Denoting by \(\lambda_{\text{min}}(A)\) the smallest eigenvalue of a real symmetric matrix \(A\), this implies that \(\lambda_{\text{min}}(\hat{A}(L(T), e)) - 1\) is also a limit point of the set of smallest eigenvalues of graphs. In fact, \(\lambda_{\text{min}}(\hat{A}(L(T), e)) - 1\) can be regarded as the smallest eigenvalue of a Hoffman graph, and by Hoffman’s limit theorem [8, Proposition 3.1] (see also [6, 9]), it is a limit point of the set of smallest eigenvalues of graphs. In this way, we obtain limit points of smallest eigenvalues of graphs with smallest eigenvalue greater than \(-3\).

The goal of this paper is to characterize the special graphs of fat indecom-
posable Hoffman graphs with smallest eigenvalue greater than $-3$ containing a slim vertex having two fat neighbors. As a consequence, we show in Theorem 5.2 that, if the smallest eigenvalue of $\hat{A}(G, v^*)$ is greater than $-2$, then $G$ is the line graph of a tree $T$ and $v^*$ corresponds to an end edge of $T$.

The organization of the paper is as follows. In Section 2, we give basic results on Hoffman graphs and block graphs which are needed in later sections. In Section 3, we show that various Hoffman graphs have smallest eigenvalue at most $-3$. These graphs will play a role of forbidden subgraphs for the family of fat Hoffman graphs with smallest eigenvalue greater than $-3$. In Section 4, we give our main theorem which characterizes the special graphs of fat indecomposable Hoffman graphs with smallest eigenvalue greater than $-3$ containing a slim vertex having two fat neighbors. Finally, in Section 5, we give an extension of a lemma of Hoffman [7] about the smallest eigenvalue of the modified adjacency matrix of a graph.

2 Preliminaries

2.1 Hoffman graphs

A Hoffman graph $\mathcal{H}$ is a pair of a (simple undirected) graph $(V(\mathcal{H}), E(\mathcal{H}))$ and a distinguished coclique $F \subseteq V(\mathcal{H})$. A vertex in $F$ is called a fat vertex and a vertex in $V(\mathcal{H}) \setminus F$ is called a slim vertex. We denote $F$ and $V(\mathcal{H}) \setminus F$ by $V_s(\mathcal{H})$ and $V_f(\mathcal{H})$, respectively. In this paper, we assume that no fat vertex is isolated.

For a vertex $x$ of a Hoffman graph $\mathcal{H}$, a slim neighbor (resp. a fat neighbor) of $x$ in $\mathcal{H}$ is a slim vertex (resp. a fat vertex) $y$ of $\mathcal{H}$ such that $\{x, y\}$ is an edge of $\mathcal{H}$. We denote by the set of slim neighbors (resp. fat neighbors) of $x$ in $\mathcal{H}$ by $N_s(\mathcal{H})(x)$ (resp. $N_f(\mathcal{H})(x)$). A Hoffman graph $\mathcal{H}$ is said to be fat if every slim vertex of $\mathcal{H}$ has a fat neighbor, and $\mathcal{H}$ is said to be slim if $\mathcal{H}$ has no fat vertex.

Two Hoffman graphs $\mathcal{H}$ and $\mathcal{H}'$ are said to be isomorphic if there exists a bijection $\phi : V(\mathcal{H}) \to V(\mathcal{H}')$ such that $\phi(V_s(\mathcal{H})) = V_s(\mathcal{H}')$, $\phi(V_f(\mathcal{H})) = V_f(\mathcal{H}')$, and $\{x, y\} \in E(\mathcal{H})$ if and only if $\{\phi(x), \phi(y)\} \in E(\mathcal{H}')$. A Hoffman graph $\mathcal{H}$ is called an induced Hoffman subgraph of a Hoffman graph $\mathcal{H}'$ if $V_s(\mathcal{H}) \subseteq V_s(\mathcal{H}')$, $V_f(\mathcal{H}) \subseteq V_f(\mathcal{H}')$, and $E(\mathcal{H}) = \{\{x, y\} \in E(\mathcal{H}) \mid x, y \in V(\mathcal{H})\}$.

Let $$A(\mathcal{H}) = \begin{pmatrix} A_s(\mathcal{H}) & C(\mathcal{H}) \\ C(\mathcal{H})^T & O \end{pmatrix}$$ be the adjacency matrix of a Hoffman graph $\mathcal{H}$, in a labeling in which the
slim vertices come first and the fat vertices come last. The eigenvalues of $\mathcal{F}$ are defined to be the eigenvalues of the real symmetric matrix

$$B(\mathcal{F}) = A'(\mathcal{F}) - C(\mathcal{F})C(\mathcal{F})^T.$$ 

We denote the smallest eigenvalue of $B(\mathcal{F})$ by $\lambda_{\text{min}}(\mathcal{F})$.

**Lemma 2.1 (\cite{Hoffman} Corollary 3.3).** If $\mathcal{F}'$ is an induced Hoffman subgraph of a Hoffman graph $\mathcal{F}$, then $\lambda_{\text{min}}(\mathcal{F}') \geq \lambda_{\text{min}}(\mathcal{F})$ holds.

A decomposition of a Hoffman graph $\mathcal{F}$ is a family $\{\mathcal{F}^i\}_{i=1}^n$ of non-empty induced Hoffman subgraphs of $\mathcal{F}$ satisfying the following conditions:

(i) $V(\mathcal{F}) = \bigcup_{i=1}^n V(\mathcal{F}^i)$;
(ii) $V^*(\mathcal{F}^i) \cap V^*(\mathcal{F}^j) = \emptyset$ if $i \neq j$;
(iii) For each $x \in V^*(\mathcal{F}^i)$, $N^f_{\mathcal{F}}(x) \subseteq V^*(\mathcal{F}^i)$
(iv) If $x \in V^*(\mathcal{F}^i)$, $y \in V^*(\mathcal{F}^j)$, and $i \neq j$, then $|N^f_{\mathcal{F}}(x) \cap N^f_{\mathcal{F}}(y)| \leq 1$, and $|N^f_{\mathcal{F}}(x) \cap N^f_{\mathcal{F}}(y)| = 1$ if and only if $\{x, y\} \in E(\mathcal{F})$.

A Hoffman graph $\mathcal{F}$ is said to be *decomposable* if $\mathcal{F}$ has a decomposition $\{\mathcal{F}^i\}_{i=1}^n$ with $n \geq 2$, and $\mathcal{F}$ is said to be *indecomposable* if $\mathcal{F}$ is not decomposable.

**Lemma 2.2 (\cite{Hoffman} Lemma 2.12).** If a Hoffman graph $\mathcal{F}$ has a decomposition $\{\mathcal{F}^i\}_{i=1}^n$, then $\lambda_{\text{min}}(\mathcal{F}) = \min\{\lambda_{\text{min}}(\mathcal{F}^i) \mid 1 \leq i \leq n\}$.

Let $\mathcal{F}$ be a Hoffman graph and let $m$ and $N$ be positive integers. A reduced representation of norm $m$ of $\mathcal{F}$ is a map $\psi : V^*(\mathcal{F}) \to \mathbb{R}^N$ such that

$$\langle \psi(x), \psi(y) \rangle = \begin{cases} m - |N^f_{\mathcal{F}}(x)| & \text{if } x = y, \\ 1 - |N^f_{\mathcal{F}}(x) \cap N^f_{\mathcal{F}}(y)| & \text{if } \{x, y\} \in E(\mathcal{F}), \\ -|N^f_{\mathcal{F}}(x) \cap N^f_{\mathcal{F}}(y)| & \text{otherwise}, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^N$. It follows immediately from the definitions that $B(\mathcal{F})_{x,y} = \langle \psi(x), \psi(y) \rangle$ holds for any distinct slim vertices $x, y$.

**Lemma 2.3 (\cite{Hoffman} Theorem 2.8).** Let $\mathcal{F}$ be a Hoffman graph and let $m$ be a positive integer. Then, $\lambda_{\text{min}}(\mathcal{F}) \geq -m$ if and only if $\mathcal{F}$ has a reduced representation of norm $m$.
Lemma 2.4 ([9, Lemma 3.6]). Let $\mathcal{S}$ be a fat Hoffman graph. If the matrix $B(\mathcal{S})$ has some off-diagonal entry less than $-1$, then $\lambda_{\min}(\mathcal{S}) < -3$.

Lemma 2.5. Let $\mathcal{S}$ be a Hoffman graph and let $m$ be a positive integer. Then, $\lambda_{\min}(\mathcal{S}) > -m$ if and only if $\mathcal{S}$ has a reduced representation $\psi$ of norm $m$ such that $\{\psi(v) \mid v \in V^*(\mathcal{S})\}$ is linearly independent.

Proof. See the proof of [9, Theorem 2.8].

An edge-signed graph $\mathcal{S}$ is a triple $(V(\mathcal{S}), E^+(\mathcal{S}), E^-(\mathcal{S}))$ of a set $V(\mathcal{S})$ of vertices, a set $E^+(\mathcal{S})$ of 2-subsets of $V(\mathcal{S})$ (called (+)-edges), and a set $E^-(\mathcal{S})$ of 2-subsets of $V(\mathcal{S})$ (called (-)-edges) such that $E^+(\mathcal{S}) \cap E^-(\mathcal{S}) = \emptyset$. The underlying graph $U(\mathcal{S})$ of an edge-signed graph $\mathcal{S}$ is the (unsigned) graph $(V(\mathcal{S}), E^+(\mathcal{S}) \cup E^-(\mathcal{S}))$.

An edge-signed graph $\mathcal{S}'$ is called an induced edge-signed subgraph of an edge-signed graph $\mathcal{S}$ if $V(\mathcal{S}') \subseteq V(\mathcal{S})$, $E^\pm(\mathcal{S}') = \{\{x, y\} \in E^\pm(\mathcal{S}) \mid x, y \in V(\mathcal{S}')\}$. Two edge-signed graphs $\mathcal{S}$ and $\mathcal{S}'$ are said to be isomorphic if there exists a bijection $\phi : V(\mathcal{S}) \rightarrow V(\mathcal{S}')$ such that $\{u, v\} \in E^\pm(\mathcal{S})$ if and only if $\{\phi(u), \phi(v)\} \in E^\pm(\mathcal{S}')$.

The special graph of a Hoffman graph $\mathcal{S}$ is the edge-signed graph $\mathcal{S}(\mathcal{S})$ defined by $V(\mathcal{S}(\mathcal{S})) = V^*(\mathcal{S})$ and

\[E^+(\mathcal{S}(\mathcal{S})) = \{\{u, v\} \mid u, v \in V^*(\mathcal{S}), u \neq v, \{u, v\} \in E(\mathcal{S}), N^f_S(u) \cap N^f_S(v) = \emptyset\},\]
\[E^-(\mathcal{S}(\mathcal{S})) = \{\{u, v\} \mid u, v \in V^*(\mathcal{S}), u \neq v, \{u, v\} \notin E(\mathcal{S}), N^f_S(u) \cap N^f_S(v) \neq \emptyset\}.

Lemma 2.6 ([9, Lemma 3.4]). A Hoffman graph $\mathcal{S}$ is indecomposable if and only if $U(\mathcal{S}(\mathcal{S}))$ is connected.

Lemma 2.7. Let $\mathcal{S}$ be a fat Hoffman graph with $\lambda_{\min}(\mathcal{S}) \geq -3$. Then for any two distinct vertices $x$ and $y$, $B(\mathcal{S})_{xy} = \pm 1$ if and only if $\{x, y\} \in E^\pm(\mathcal{S}(\mathcal{S}))$.

Proof. It follows from the definition of $B(\mathcal{S})$ and $E^\pm(\mathcal{S}(\mathcal{S}))$ that, for any two distinct slim vertices of $\mathcal{S}$,

\[\{x, y\} \in E^+(\mathcal{S}) \iff B(\mathcal{S})_{xy} = 1,\]
\[\{x, y\} \in E^-(\mathcal{S}) \iff B(\mathcal{S})_{xy} = -1.\]

Since $\lambda_{\min}(\mathcal{S}) \geq -3$, Lemma 2.4 applies, and $B(\mathcal{S})_{xy} \leq -1$ forces $B(\mathcal{S})_{xy} = -1$. □
2.2 Block graphs

A vertex \( v \) in a graph \( G \) is called a cut vertex of \( G \) if the number of connected components of \( G - v \) is greater than that of \( G \). A connected graph \( G \) is said to be 2-connected if \( G \) has no cut vertex. A block in a graph is a maximal 2-connected subgraph of the graph. Two distinct blocks have at most one vertex in common. An end block is a block having at most one cut vertex.

We define the block graph \( B(G) \) of a graph \( G \) to be the graph whose vertex set is the set of blocks of \( G \) and two distinct blocks are adjacent in \( B(G) \) if and only if they have a common vertex in \( G \). A block graph is a graph isomorphic to the block graph of some graph.

**Lemma 2.8** ([5, Theorems 1 and 2]). A graph \( G \) is a block graph if and only if every block of \( G \) is a clique.

**Lemma 2.9** ([1, Proposition 1]). If a graph \( G \) contains neither the diamond graph \( K_{1,1,2} \) or a cycle of length at least four as an induced subgraph, then \( G \) is a block graph.

A graph is said to be claw-free if it does not contain \( K_{1,3} \) as an induced subgraph.

**Lemma 2.10.** If a connected block graph \( G \) is claw-free, then \( B(G) \) is a tree. Let \( n(B) \) denote the number of non cut vertices of a block \( B \) of \( G \). Let \( T \) be the tree obtained from \( B(G) \) by attaching \( n(B) \) pendant edges to the vertex \( B \), for each vertex \( B \) of \( B(G) \). Then \( G \) is isomorphic to the line graph \( L(T) \) of \( T \).

**Proof.** Let \( B \) be a vertex of \( B(G) \) which is not a leaf. Then there are two neighbors \( B_1, B_2 \) of \( B \) in \( B(G) \). Since \( G \) is claw-free, there are distinct vertices \( v_1, v_2 \) of \( B \) such that \( B \cap B_i = \{v_i\} \) for \( i = 1, 2 \). Since \( v_i \) is a cut vertex, \( B - v_i \) and \( B_i - v_i \) belong to the different connected components of \( G - v_i \). If \( B(G) - B \) is connected, then there is a path in \( B(G) - B \) connecting \( B_1 \) and \( B_2 \). This implies that there is a path in \( G - v_2 \) connecting \( v_1 \) and a vertex of \( B_2 - v_2 \). But then \( B - v_2 \) and \( B_2 - v_2 \) belong to the same connected component of \( G - v_2 \), a contradiction. Therefore, every vertex of \( B(G) \) is either a leaf or a cut vertex. Since \( B(G) \) is connected, we conclude that \( B(G) \) is a tree.

Since there is a bijection between the set of edges of \( B(G) \) and the set of cut vertices of \( G \), the set of edges of \( T \) bijectively corresponds to the set of vertices \( G \). Then it is easy to see that this bijective correspondence between the vertices of \( L(T) \) and those of \( G \) preserves the adjacency. \( \square \)
A Hoffman graph is said to be $K$-with-$G$ greater than $-H$. Some Hoffman graphs then entries of $B$ containing two fat neighbors. Lemma 2.7 implies that unless all the off-diagonal entries of $t$ for any positive integer $v'$ are vertices in $B$ and $B'$, respectively, then

$$d_{B(G)}(B, B') = \begin{cases} 
  d_G(v, v') - 1 & \text{if } |V(P) \cap (V(B) \cup V(B'))| = 4, \\
  d_G(v, v') & \text{if } |V(P) \cap (V(B) \cup V(B'))| = 3, \\
  d_G(v, v') + 1 & \text{if } |V(P) \cap (V(B) \cup V(B'))| = 2,
\end{cases}$$

where $P$ is the shortest path between $v$ and $v'$ in $G$.

Proof. Let $P = (v = u_0, u_1, \ldots, u_k = v')$ be the shortest path between $v$ and $v'$ in $G$, where $k = d_G(v, v')$. Let $B_i$ be the block of $G$ containing $\{u_{i-1}, u_i\}$ for $i = 1, \ldots, k$. If $|V(P) \cap (V(B) \cup V(B'))| = 4$, then $B_1 = B$ and $B_k = B'$. Therefore $(B_1, \ldots, B_k)$ is the shortest path between $B$ and $B'$ in $B(G)$. Thus $d_{B(G)}(B, B') = k - 1$. If $|V(P) \cap (V(B) \cup V(B'))| = 3$, then either $B_1 \neq B$ and $B_k = B'$ or $B_1 = B$ and $B_k \neq B'$. Therefore $(B, B_1, \ldots, B_k)$ is the shortest path between $B$ and $B'$ in $B(G)$. Thus $d_{B(G)}(B, B') = k$. If $|V(P) \cap (V(B) \cup V(B'))| = 2$, then $B_1 \neq B$ and $B_k \neq B'$. Therefore $(B, B_1, \ldots, B_k, B')$ is the shortest path between $B$ and $B'$ in $B(G)$. Thus $d_{B(G)}(B, B') = k + 1$. Hence the lemma holds.

\section{Some Hoffman graphs $\mathcal{F}$ with $\lambda_{\min}(\mathcal{F}) \leq -3$}

For a positive integer $t$, let $\mathcal{R}_{1,t}$ be the connected Hoffman graph having exactly one slim vertex and $t$ fat vertices. Note that and $\lambda_{\min}(\mathcal{R}_{1,t}) = -t$. A Hoffman graph is said to be $\mathcal{R}_{1,t}$-free if it does not contain $\mathcal{R}_{1,t}$ as an induced Hoffman subgraph. If a Hoffman graph $\mathcal{F}$ has smallest eigenvalue greater than $-t$, then $\mathcal{F}$ is $\mathcal{R}_{1,t}$-free by Lemma 2.1. By a Hoffman graph containing $\mathcal{R}_{1,2}$, we mean a Hoffman graph in which some slim vertex has two fat neighbors.

In this section, we give some Hoffman graphs $\mathcal{F}$ with $\lambda_{\min}(\mathcal{F}) \leq -3$. Lemma 2.7 implies that unless all the off-diagonal entries of $B(\mathcal{F})$ are in $\{0, \pm 1\}$, we have $\lambda_{\min}(\mathcal{F}) < -3$. Thus, in the proofs of lemmas in this section, we may assume without loss of generality that all the off-diagonal entries of $B(\mathcal{F})$ are in $\{0, \pm 1\}$.

The graphs obtained in this section will form a set of forbidden subgraphs for Hoffman graphs with smallest eigenvalue greater than $-3$. This set of forbidden graphs will be used in the next section to determine the structure of the special graph $S(\mathcal{F})$ of a fat Hoffman graph $\mathcal{F}$ containing $\mathcal{R}_{1,2}$ with smallest eigenvalue greater than $-3$. 

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Lemma 3.1. Let $\mathcal{H}$ be a fat Hoffman graph. If $U(S(\mathcal{H}))$ is a path each of whose two end vertices has two fat neighbors in $\mathcal{H}$, then $\lambda_{\min}(\mathcal{H}) \leq -3$.

Proof. By the assumption, we have

$$B(\mathcal{H}) = \begin{pmatrix}
-2 & \epsilon_1 & 0 & \cdots & \cdots & 0 \\
\epsilon_1 & \delta_2 & \epsilon_2 & \cdots & \cdots & \vdots \\
0 & \epsilon_2 & \delta_3 & \cdots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \epsilon_{n-2} \\
\vdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \epsilon_{n-1} \\
0 & \cdots & \cdots & \cdots & \cdots & \epsilon_{n-1} - 2
\end{pmatrix},$$

where $n$ is the number of slim vertices of $\mathcal{H}$, and $\epsilon_1, \ldots, \epsilon_{n-1} \in \{\pm 1\}$, $\delta_2, \ldots, \delta_{n-1} \leq -1$. Let $B'$ be the matrix obtained from $B(\mathcal{H})$ by replacing $\delta_i$ by $-1$ for all $i = 2, \ldots, n - 1$. As $\lambda_{\min}(\mathcal{H}) \leq \lambda_{\min}(B')$, it suffices to show $\lambda_{\min}(B') = -3$. Multiplying $B'$ by the diagonal matrix whose diagonal entries are $1, \epsilon_1, \epsilon_1 \epsilon_2, \ldots, \prod_{i=1}^{n-1} \epsilon_i$ from both sides, we see that $B'$ is similar to the matrix

$$\begin{pmatrix}
-2 & 1 \\
1 & -1 & 1 \\
\vdots & \cdots & \cdots & \cdots & \cdots & 1 \\
1 & -1 & 1 \\
1 & -2
\end{pmatrix}.$$

The smallest eigenvalue of this matrix is $-3$ by [10, Theorem 3]. This implies $\lambda_{\min}(B') = -3$.

Lemma 3.2. Let $\mathcal{H}$ be a fat indecomposable Hoffman graph. If $\mathcal{H}$ contains at least two $K_{1,2}$, then $\lambda_{\min}(\mathcal{H}) \leq -3$.

Proof. Let $v_1$ and $v_2$ be the slim vertices of the two $K_{1,2}$. If $v_1 = v_2$, then $\mathcal{H}$ contains $K_{1,3}$ and thus $\lambda_{\min}(\mathcal{H}) \leq -3$. Now we assume that $v_1 \neq v_2$. Since $\mathcal{H}$ is indecomposable, $S(\mathcal{H})$ is connected by Lemma 2.6. Let $\mathcal{P}$ be a shortest path from $v_1$ to $v_2$ in $S(\mathcal{H})$. Let $\mathcal{H}_P$ be the Hoffman subgraph of $\mathcal{H}$ induced by the slim vertices which belong to $\mathcal{P}$ and their fat neighbors. Then $S(\mathcal{H}_P) = \mathcal{P}$. By Lemmas 2.1 and 3.1, $\lambda_{\min}(\mathcal{H}) \leq \lambda_{\min}(\mathcal{H}_P) \leq -3$.

Lemma 3.3. Let $\mathcal{H}$ be a fat Hoffman graph containing $K_{1,2}$. Suppose that the slim vertex $v^*$ in $K_{1,2}$ has two slim neighbors which are not adjacent in $\mathcal{H}$. Then $\lambda_{\min}(\mathcal{H}) \leq -3$. 

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Proof. Let \( u \) and \( w \) be slim neighbors of \( v \) which are not adjacent in \( S(H) \). Then the matrix \( B(H') \) of the Hoffman subgraph \( H' \) of \( H \) induced by \( u, v, w \) and their fat neighbors is one of the matrices

\[
\begin{pmatrix}
-1 & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & -1
\end{pmatrix}, \quad \begin{pmatrix}
-1 & -1 & 0 \\
1 & -2 & 1 \\
0 & 1 & -1
\end{pmatrix}, \quad \begin{pmatrix}
-1 & -1 & 0 \\
-1 & -2 & -1 \\
0 & -1 & -1
\end{pmatrix},
\]

which have smallest eigenvalue \(-3\) and thus \( \lambda_{\min}(H') \leq -3 \) by Lemma 2.1. □

**Lemma 3.4.** Let \( \mathcal{H} \) be a fat Hoffman graph containing \( K_{1,2} \). Let \( D_n \) \((n \geq 4)\) be the graph defined by \( V(D_n) = \{v_1, v_2, \ldots, v_n\} \) and \( E(D_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-2}, v_{n-1}\}, \{v_{n-2}, v_n\}\} \). If \( U(S(H)) = D_n \) \((n \geq 4)\) and \( v_1 \) has two fat neighbors in \( \mathcal{H} \), then \( \lambda_{\min}(H) \leq -3 \).

**Proof.** By assumption, we have

\[
B(\mathcal{H}) = \begin{pmatrix}
-2 & \epsilon_1 & 0 & \ldots & \ldots & 0 & 0 \\
\epsilon_1 & -1 & \epsilon_2 & \ddots & \ddots & \vdots & \vdots \\
0 & \epsilon_2 & -1 & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \epsilon_{n-3} & 0 & 0 \\
\vdots & \ddots & \ddots & -1 & \epsilon_{n-2} & \epsilon_{n-1} & 0 \\
0 & \ldots & \ldots & 0 & \epsilon_{n-2} & -1 & 0 \\
0 & \ldots & \ldots & 0 & \epsilon_{n-1} & 0 & -1
\end{pmatrix},
\]

where \( \epsilon_1, \ldots, \epsilon_{n-1} \in \{\pm 1\} \). Let \( x \in \mathbb{R}^n \) be a vector defined by

\[
(x)_i = \begin{cases}
2 \prod_{k=1}^{n-3} (-\epsilon_k) & \text{if } 1 \leq i \leq n-3, \\
2 & \text{if } i = n-2, \\
-\epsilon_{i-1} & \text{if } i = n-1, n.
\end{cases}
\]

Then \((B(\mathcal{H}) + 3I)x = 0\). Hence \( B(\mathcal{H}) \) has \(-3\) as an eigenvalue. This implies \( \lambda_{\min}(\mathcal{H}) \leq -3 \). □

**Lemma 3.5.** Let \( \mathcal{H} \) be a Hoffman graph. Let \( \mathcal{T} \) be a triangle in the special graph \( S(\mathcal{H}) \) such that every vertex in \( \mathcal{T} \) has exactly one fat neighbor in \( \mathcal{H} \). If \( \mathcal{T} \) has a \((-)\)-edge, then \( \lambda_{\min}(\mathcal{H}) \leq -3 \).

**Proof.** By [11, Lemma 3.11], the number of \((-)\)-edges in \( \mathcal{T} \) cannot be two. Therefore, the number of \((-)\)-edges in \( \mathcal{T} \) is one or three. Then, the matrix
\(B(\mathfrak{F}_T)\) of the Hoffman subgraph \(\mathfrak{F}_T\) induced by \(T\) and their fat neighbors is one of the matrices
\[
\begin{pmatrix}
-1 & 1 & 1 \\
1 & -1 & -1 \\
1 & -1 & -1
\end{pmatrix},
\begin{pmatrix}
-1 & -1 & -1 \\
-1 & -1 & -1 \\
-1 & -1 & -1
\end{pmatrix}
\]
which have smallest eigenvalue \(-3\). By Lemma 2.1 we obtain \(\lambda_{\text{min}}(\mathfrak{F}) \leq \lambda_{\text{min}}(\mathfrak{F}_T) = -3\).

**Lemma 3.6.** Let \(\mathfrak{F}\) be a Hoffman graph. Let \(T\) be a triangle in the special graph \(S(\mathfrak{F})\) such that two vertices \(v_1, v_2\) in \(T\) have exactly one fat neighbor in \(\mathfrak{F}\) and the other vertex \(v^*\) in \(T\) has exactly two fat neighbors in \(\mathfrak{F}\). If \(T\) has a \((-)\)-edge, then \(\lambda_{\text{min}}(\mathfrak{F}) \leq -3\) or \(E^+(T) = \{v_1, v_2\}\) and \(E^-(T) = \{v^*, v_1\}, \{v^*, v_2\}\).

**Proof.** Suppose that \(E^±(T)\) is different from the one described. It is enough to consider the following cases:

(a) \(E^+(T) = \{v^*, v_1\}, \{v^*, v_2\}\), \(E^-(T) = \{v_1, v_2\}\),

(b) \(E^+(T) = \{v^*, v_1\}, \{v_1, v_2\}\), \(E^-(T) = \{v^*, v_2\}\),

(c) \(E^+(T) = \{v^*, v_1\}\), \(E^-(T) = \{v^*, v_2\}, \{v_1, v_2\}\),

(d) \(E^+(T) = \emptyset, E^-(T) = \{v^*, v_1\}, \{v^*, v_2\}, \{v_1, v_2\}\),

First, consider the case (c). Since \(\{v^*, v_2\}\) and \(\{v_1, v_2\}\) are \((-)\)-edges in \(S(\mathfrak{F})\), \(v^*\) and \(v_2\) have a common fat neighbor, say \(f\), in \(\mathfrak{F}\), and \(v_1\) and \(v_2\) have a common fat neighbor, say \(f^*\), in \(\mathfrak{F}\). Since \(v_2\) has exactly one fat neighbor in \(\mathfrak{F}\), \(f = f^*\). Then \(f\) is a common fat neighbor of \(v^*\) and \(v_1\), which is a contradiction to the fact that \(\{v^*, v_1\}\) is a \((+)\)-edge in \(S(\mathfrak{F})\).

In the cases (a), (b), and (d), the matrices \(B(\mathfrak{F}_T)\), where \(\mathfrak{F}_T\) denotes the Hoffman subgraph of \(\mathfrak{F}\) induced by \(T\) and their fat neighbors, are the following matrices, respectively,
\[
\begin{pmatrix}
-2 & 1 & 1 \\
1 & -1 & -1 \\
1 & -1 & -1
\end{pmatrix},
\begin{pmatrix}
-2 & 1 & -1 \\
1 & -1 & 1 \\
-1 & 1 & -1
\end{pmatrix},
\begin{pmatrix}
-2 & -1 & -1 \\
-1 & -1 & -1 \\
-1 & -1 & -1
\end{pmatrix}
\]
which have smallest eigenvalue less than \(-3\). Then, by Lemma 2.1 we have \(\lambda_{\text{min}}(\mathfrak{F}) \leq \lambda_{\text{min}}(\mathfrak{F}_T) < -3\). Hence the lemma holds.

**Lemma 3.7.** Let \(\mathfrak{F}\) be a fat Hoffman graph containing \(\mathfrak{F}_{1,2}\). If \(U(S(\mathfrak{F})) = K_{1,1,2}\), then \(\lambda_{\text{min}}(\mathfrak{F}) \leq -3\).
Proof. Let $v^*$ be a slim vertex which has two fat neighbors in $\mathcal{S}$. If $v^*$ is a vertex of degree three in $U(\mathcal{S}) = K_{1,1,2}$, then it follows from Lemma 3.3 that $\lambda_{\text{min}}(\mathcal{S}) \leq -3$. Therefore, we assume that $v^*$ is a vertex of degree two in $U(\mathcal{S}) = K_{1,1,2}$. Then,

$$B(\mathcal{S}) = \begin{pmatrix}
-2 & \varepsilon_{12} & 0 & 0 & 0 & 0 \\
\varepsilon_{12} & -1 & \varepsilon_{13} & 0 & 0 & 0 \\
\varepsilon_{13} & -1 & -1 & \varepsilon_{23} & 0 & 0 \\
0 & \varepsilon_{24} & \varepsilon_{23} & 0 & 0 & 0 \\
0 & \varepsilon_{23} & 0 & 0 & 0 & 0 \\
0 & \varepsilon_{24} & 0 & 0 & 0 & 0 \\
\end{pmatrix},$$

where $\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}, \varepsilon_{24}, \varepsilon_{34} \in \{\pm 1\}$. By using computer, we can check that $B(\mathcal{S})$ has the smallest eigenvalue at most $-3$ for any case. Hence $\lambda_{\text{min}}(\mathcal{S}) \leq -3$.

Lemma 3.8. Let $\mathcal{S}$ be a fat Hoffman graph containing $\mathcal{K}_{1,2}$. Let $\mathcal{P}$ and $\mathcal{K}$ denote edge-signed graphs such that $U(\mathcal{P})$ is a path and $U(\mathcal{K}) \cong K_{1,1,2}$, respectively. If $\mathcal{S}(\mathcal{S})$ is the graph obtained from $\mathcal{P}$ and $\mathcal{K}$ by identifying an end vertex of $\mathcal{P}$ and a vertex in $\mathcal{K}$, and if another end vertex of $\mathcal{P}$ has two fat neighbors in $\mathcal{S}$, then $\lambda_{\text{min}}(\mathcal{S}) \leq -3$.

Proof. Since $\mathcal{S}(\mathcal{S})$ is connected, $\mathcal{S}$ is indecomposable by Lemma 2.6. If $\mathcal{S}$ contains at least two $K_{1,1,2}$, then it follows from Lemma 3.2 that $\lambda_{\text{min}}(\mathcal{S}) \leq -3$. Therefore, we assume that $\mathcal{S}$ contains exactly one $K_{1,1,2}$. Since the end vertex of $\mathcal{P}$ that is not in $\mathcal{K}$ has two fat neighbors in $\mathcal{S}$, every vertex in $\mathcal{K}$ has exactly one fat neighbor in $\mathcal{S}$. If $\mathcal{K}$ has a $(-)$-edge, it follows from Lemma 3.5 that $\lambda_{\text{min}}(\mathcal{S}) \leq -3$. Thus, now we assume that all the edges in $\mathcal{K}$ are $(+)$-edges. Let $n$ be the number of vertices in $\mathcal{P}$. Then

$$B(\mathcal{S}) = \begin{pmatrix}
-2 & \varepsilon_1 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \\
\varepsilon_1 & -1 & \varepsilon_2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \varepsilon_2 & -1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & \varepsilon_{n-1} & -1 & 1 & 1 & \varepsilon \\
0 & \cdots & \cdots & 0 & 1 & -1 & \delta & 1 & \varepsilon \\
0 & \cdots & \cdots & 0 & 1 & \delta & -1 & 1 & \varepsilon \\
0 & \cdots & \cdots & 0 & \varepsilon & 1 & 1 & -1 & \varepsilon \\
\end{pmatrix},$$

where $\varepsilon_1, \ldots, \varepsilon_{n-1} \in \{\pm 1\}$ and $\{\varepsilon, \delta\} = \{0, 1\}$. Let $x \in \mathbb{R}^{n+3}$ be a vector
defined by

$$(x)_i = \begin{cases} 2 \prod_{k=1}^{n-1} (-\epsilon_k) & \text{if } 1 \leq i \leq n - 1, \\ 2 & \text{if } i = n, \\ -1 & \text{if } i = n + 1, n + 2, \\ 2(\epsilon - 1)/(-\epsilon - 2) & \text{if } i = n + 3. \end{cases}$$

Since $\epsilon + \delta = 1$, $\epsilon \delta = 0$, $\epsilon^2 - \epsilon = 0$, we obtain $(B(\mathcal{H}) + 3I)x = 0$. Hence $B(\mathcal{H})$ has $-3$ as an eigenvalue. This implies $\lambda_{\min}(\mathcal{H}) \leq -3$. □

**Lemma 3.9.** Let $\mathcal{H}$ be a Hoffman graph. Let $C$ be a cycle of length at least four in the special graph $S(\mathcal{H})$ such that every vertex in $C$ has exactly one fat neighbor in $\mathcal{H}$. If the number of $(+)$-edges in $C$ is even, then $\lambda_{\min}(\mathcal{H}) \leq -3$.

**Proof.** Let $n$ be the length of the cycle $C$. Let $\mathcal{H}'$ be the Hoffman subgraph of $\mathcal{H}$ induced by $C$ and their fat neighbors. Then

$$B(\mathcal{H}') = \begin{pmatrix} -1 & \epsilon_1 & 0 & \cdots & 0 & \epsilon_n \\ \epsilon_1 & -1 & \epsilon_2 & \cdots & 0 \\ 0 & \epsilon_2 & -1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \epsilon_{n-2} & -1 & \epsilon_{n-1} \\ \epsilon_n & 0 & \cdots & 0 & \epsilon_{n-1} \end{pmatrix},$$

where $\epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$. Since the number of $(+)$-edges in $C$ is even, we have

$$\prod_{k=1}^{n} \epsilon_k = (-1)^n.$$

Let $x \in \mathbb{R}^n$ be the vector defined by

$$(x)_i = \begin{cases} 1 & i = 1, \\ \prod_{k=1}^{i-1} (-\epsilon_k) & 2 \leq i \leq n. \end{cases}$$

Then $(B(\mathcal{H}') + 3I)x = 0$. Thus $\mathcal{H}'$ has $-3$ as an eigenvalue. By Lemma 2.1, $\lambda_{\min}(\mathcal{H}) \leq \lambda_{\min}(\mathcal{H}') \leq -3$. □

**Remark 3.10.** In the proof of Lemma 3.9, multiplying $B(\mathcal{H}')$ by the diagonal matrix whose diagonal entries are $1, \epsilon_1, \epsilon_1 \epsilon_2, \ldots, \prod_{i=1}^{n-1} \epsilon_i$ from both sides, the resulting matrix is circulant with the first row $(-\epsilon_1, 1, 0, \ldots, 0, 1)$ when $n$ is even. Thus $B(\mathcal{H}')$ has smallest eigenvalue $-3$. If $n$ is odd, it contains the submatrix

$$\begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$
Let \( P_m \) denote the path of length \( m - 1 \) consisting of \( m \) vertices, and let \( C_n \) denote the cycle of length \( n \).

**Lemma 3.11.** Let \( \tilde{\mathcal{H}} \) be a fat Hoffman graph containing \( \mathcal{R}_{1,2} \). Let \( \mathcal{P} \) and \( \mathcal{C} \) denote edge-signed graphs such that \( U(\mathcal{P}) \cong P_m \) and \( U(\mathcal{C}) \cong C_n \), respectively. Suppose that \( \mathcal{S}(\tilde{\mathcal{H}}) \) is the graph obtained from \( \mathcal{P} \) and \( \mathcal{C} \) by adding two edges between an end vertex \( v_m \) of \( \mathcal{P} \) and two adjacent vertices \( v_{m+1} \) and \( v_{m+2} \) of \( \mathcal{C} \), and that another end vertex \( v_1 \) of \( \mathcal{P} \) has two fat neighbors in \( \tilde{\mathcal{H}} \). Then \( \lambda_{\text{min}}(\tilde{\mathcal{H}}) \leq -3 \).

**Proof.** If the number of \((+)-\)edges in \( \mathcal{C} \) is even, then we have \( \lambda_{\text{min}}(\tilde{\mathcal{H}}) \leq -3 \) by Lemma 3.9. Therefore, we assume that the number of \((+)-\)edges in \( \mathcal{C} \) is odd. Since \( \mathcal{S}(\tilde{\mathcal{H}}) \) is connected, \( \tilde{\mathcal{H}} \) is indecomposable by Lemma 2.6. If \( \tilde{\mathcal{H}} \) contains at least two \( \mathcal{R}_{1,2} \), then it follows from Lemma 3.2 that \( \lambda_{\text{min}}(\tilde{\mathcal{H}}) \leq -3 \). Therefore, we assume that \( \tilde{\mathcal{H}} \) contains exactly one \( \mathcal{R}_{1,2} \). Since the end vertex \( v_1 \) of \( \mathcal{P} \) has two fat neighbors in \( \tilde{\mathcal{H}} \), every vertex in \( \mathcal{S} - \{v_1\} \) has exactly one fat neighbor in \( \tilde{\mathcal{H}} \). If the triangle \( \{v_n, v_{n+1}, v_{n+2}\} \) has a \((-)-\)edge, it follows from Lemma 3.5 that \( \lambda_{\text{min}}(\tilde{\mathcal{H}}) \leq -3 \). Thus, now we assume that all the edges in the triangle \( \{v_n, v_{n+1}, v_{n+2}\} \) are \((+)-\)edges. Then \( B(\tilde{\mathcal{H}}) \) is the matrix

\[
\begin{pmatrix}
-2 & \delta_1 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\
\delta_1 & -1 & \delta_2 & \ddots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\
0 & \delta_2 & -1 & \ddots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\
\vdots & \ddots & \delta_{m-2} & -1 & \delta_{m-1} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & \delta_{m-1} & -1 & 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & 0 & 1 & -1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & 0 & 1 & 1 & -1 & \epsilon_2 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\
0 & \cdots & 0 & 0 & \epsilon_n & \cdots & \epsilon_{n-2} & -1 & \epsilon_{n-1} & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & 0 & \epsilon_n & 0 & \cdots & 0 & \epsilon_{n-1} & -1 & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix}
\]

where \( \delta_1, \ldots, \delta_{m-1}, \epsilon_2, \ldots, \epsilon_n \in \{\pm 1\} \). Since the number of \((+)-\)edges in \( \mathcal{C}_n \) is odd, we have

\[
\prod_{k=2}^{n} \epsilon_k = (-1)^{n-1}.
\]

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Let $\mathbf{x} \in \mathbb{R}^{m+n}$ be the vector defined by

$$(\mathbf{x})_i = \begin{cases} 
2 & \text{if } i = 1 \\
2 \cdot \prod_{k=1}^{i-1}(-\delta_k) & \text{if } 2 \leq i \leq m \\
-\prod_{k=1}^{m-1}(-\delta_k) & \text{if } m + 1 \leq i \leq m + 2 \\
-\prod_{k=1}^{m-1}(-\delta_k) \cdot \prod_{k=2}^{i-m-1}(-\epsilon_k) & \text{if } m + 3 \leq i \leq m + n 
\end{cases}$$

Then $(B(\mathcal{S}) + 3I)\mathbf{x} = 0$. Therefore, $\mathcal{S}$ has $-3$ as an eigenvalue and thus $\lambda_{\min}(\mathcal{S}) \leq -3$.

**Lemma 3.12.** Let $\mathcal{S}$ be a fat indecomposable Hoffman graph containing $\mathcal{R}_{1,2}$. If $U(\mathcal{S}(\mathcal{S}))$ contains $K_{1,1,2}$ as an induced subgraph, then $\lambda_{\min}(\mathcal{S}) \leq -3$.

**Proof.** If $\mathcal{S}$ contains at least two $\mathcal{R}_{1,2}$, then $\lambda_{\min}(\mathcal{S}) \leq -3$ by Lemma 3.2. So we assume that $\mathcal{S}$ contains exactly one $\mathcal{R}_{1,2}$. Let $v^*$ be the slim vertex of $\mathcal{R}_{1,2}$. Suppose that $U(\mathcal{S}(\mathcal{S}))$ contains $K_{1,1,2}$. Let $K$ be an induced subgraph of $U(\mathcal{S}(\mathcal{S}))$ such that $K \cong K_{1,1,2}$. If $v^*$ is in $K$, then it follows from Lemma 3.7 that $\lambda_{\min}(\mathcal{S}) \leq -3$. Consider the case where $v^*$ is not in $K$. Note that $S(\mathcal{S})$ is connected since $\mathcal{S}$ is indecomposable. Let $P$ be a shortest path in $U(\mathcal{S}(\mathcal{S}))$ from $v^*$ to a vertex in $K$. Let $\mathcal{S}'$ be the Hoffman subgraph of $\mathcal{S}$ induced by the slim vertices which belong to $P$ or $K$ and their fat neighbors. Then $U(\mathcal{S}(\mathcal{S}')) = P \cup K$, where the end vertex of $P$ other than $v^*$ is identified with a vertex in $K$. By Lemma 3.8, $\lambda_{\min}(\mathcal{S}') \leq -3$. Since $\mathcal{S}'$ is an induced Hoffman subgraph of $\mathcal{S}$, $\lambda_{\min}(\mathcal{S}) \leq \lambda_{\min}(\mathcal{S}')$ by Lemma 2.1. Therefore $\lambda_{\min}(\mathcal{S}) \leq -3$.

**Lemma 3.13.** Let $\mathcal{S}$ be a fat indecomposable Hoffman graph containing $\mathcal{R}_{1,2}$. If $U(\mathcal{S}(\mathcal{S}))$ contains $C_n$ ($n \geq 4$) as an induced subgraph, then $\lambda_{\min}(\mathcal{S}) \leq -3$.

**Proof.** Let $C$ be an induced subgraph of $U(\mathcal{S}(\mathcal{S}))$ such that $C \cong C_n$ for some $n \geq 4$. If $\mathcal{S}$ contains at least two $\mathcal{R}_{1,2}$, then $\lambda_{\min}(\mathcal{S}) \leq -3$ by Lemma 3.2. So we may assume that $\mathcal{S}$ contains exactly one $\mathcal{R}_{1,2}$. Let $v^*$ be the slim vertex of $\mathcal{R}_{1,2}$. If $v^*$ is in $C$, then the two slim neighbors of $v^*$ in $C$ are not adjacent, and the lemma follows from Lemma 3.3.

Consider the case where $v^*$ is not in $C$. Note that $S(\mathcal{S})$ is connected since $\mathcal{S}$ is indecomposable. Let $P$ be a shortest path in $U(\mathcal{S}(\mathcal{S}))$ from $v^*$ to a vertex in $C$. Let $m$ be the number of the vertices of the path $P$. Note that $m \geq 2$. Consider the subgraph $G$ of $U(\mathcal{S}(\mathcal{S}))$ induced by the vertices in $P \cup C$. Then it follows from the way of taking $P$ and $C$ that $G$ is the graph defined by $V(G) = \{v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+n}\}$ and $E(G) = \{(v_i, v_{i+1}) | 1 \leq i \leq n-1\} \cup \{\{v_{m+1}, v_{m+n}\}\} \cup F$, where $F \subseteq \{\{v_m, v_{m+j}\} | 2 \leq j \leq n\}$. If $F = \emptyset$, then the subgraph of $U(\mathcal{S}(\mathcal{S}))$ induced by $\{v_1, \ldots, v_m, v_{m+1}, v_{m+2}, v_{m+n}\}$ is isomorphic to $\mathcal{D}_{m+3}$. If $\{v_m, v_{m+j}\} \in F$ for some $3 \leq j \leq n-1$, then
the subgraph of $U(S(\mathcal{H}))$ induced by $\{v_1, \ldots, v_m, v_{m+1}, v_{m+j}\}$ is isomorphic to $D_{m+2}$. If $\{\{v_m, v_{m+2}\}, \{v_m, v_{m+n}\}\} \subseteq F$, then the subgraph of $U(S(\mathcal{H}))$ induced by $\{v_1, \ldots, v_m, v_{m+2}, v_{m+n}\}$ is isomorphic to $D_{m+2}$. In these three cases, it follows from Lemmas 2.1 and 3.4 that $\lambda_{\text{min}}(\mathcal{H}) \leq -3$. In the case where $F = \{\{v_m, v_{m+2}\}\}$ or $F = \{\{v_m, v_{m+n}\}\}$, it follows from Lemmas 2.1 and 3.11 that $\lambda_{\text{min}}(\mathcal{H}) \leq -3$. Hence $\lambda_{\text{min}}(\mathcal{H}) \leq -3$.

**Lemma 3.14.** Let $\mathcal{H}$ be a fat Hoffman graph containing $R_{1,2}$. If $U(S(\mathcal{H}))$ contains a claw $K_{1,3}$, then $\lambda_{\text{min}}(\mathcal{H}) \leq -3$.

**Proof.** Let $v^*$ be the vertex which has two fat neighbors in $\mathcal{H}$. Let $K$ be an induced subgraph of $U(S(\mathcal{H}))$ such that $K \cong K_{1,3}$. If $v^* \in V(K)$, then let $\mathcal{H}'$ be the Hoffman subgraph of $\mathcal{H}$ induced by $V(K)$ and their fat neighbors in $\mathcal{H}$. Then

$$B(\mathcal{H}') = \begin{pmatrix} -2 & \epsilon_1 & \epsilon_2 & \epsilon_3 \\ \epsilon_1 & -1 & 0 & 0 \\ \epsilon_2 & 0 & -1 & 0 \\ \epsilon_3 & 0 & 0 & -1 \end{pmatrix}, \quad B(\mathcal{H}) = \begin{pmatrix} -1 & \epsilon_1 & \epsilon_2 & \epsilon_3 \\ \epsilon_1 & -2 & 0 & 0 \\ \epsilon_2 & 0 & -1 & 0 \\ \epsilon_3 & 0 & 0 & -1 \end{pmatrix}$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}$. Therefore, we obtain $\lambda_{\text{min}}(\mathcal{H}') \leq -3$. If $v^* \notin V(K)$, then consider a shortest path $P$ in $U(S(\mathcal{H}))$ from $v^*$ to a vertex in $K$. Let $n$ be the number of vertices of $P$. Let $G$ be the subgraph of $U(S(\mathcal{H}))$ induced by $V(P) \cup V(K)$. Then we can easily verify that $G$ contains an induced subgraph $D$ isomorphic to $D_{n+2}$, $D_{n+3}$, or $D_{n+4}$. Let $\mathcal{H}'$ be the Hoffman subgraph of $\mathcal{H}$ induced by $V(D)$. By Lemmas 2.1 and 3.4 we obtain $\lambda_{\text{min}}(\mathcal{H}) \leq \lambda_{\text{min}}(\mathcal{H}') \leq -3$.

**Proposition 3.15.** Let $\mathcal{H}$ be a fat indecomposable Hoffman graph containing $R_{1,2}$. If $U(S(\mathcal{H}))$ contains $C_n$ ($n \geq 4$), $K_{1,1,2}$, or $K_{1,3}$ as an induced subgraph, then $\lambda_{\text{min}}(\mathcal{H}) \leq -3$.

**Proof.** This follows from Lemmas 3.13, 3.12, and 3.14.

### 4 Main result

**Lemma 4.1.** Let $\mathcal{H}$ be a fat indecomposable Hoffman graph containing $R_{1,2}$ with $\lambda_{\text{min}}(\mathcal{H}) > -3$. Then, the slim vertex $v^*$ in $R_{1,2}$ is not a cut vertex of $U(S(\mathcal{H}))$.

**Proof.** Suppose that the slim vertex $v^*$ in $R_{1,2}$ is a cut vertex of $U(S(\mathcal{H}))$. Let $v_1$ and $v_2$ be neighbors of $v^*$ in $S(\mathcal{H})$ such that $v_1$ and $v_2$ belong to different connected components in $U(S(\mathcal{H})) - v^*$. By Lemma 3.2 each of $v_1$ and $v_2$ has
only one fat neighbor in $\mathcal{S}$. Let $\mathcal{S}'$ be the Hoffman subgraph of $\mathcal{S}$ induced by $v^*, v_1, v_2$ and their fat neighbors. Then by Lemma 2.7,

$$B(\mathcal{S}') = \begin{pmatrix} -1 & \epsilon_1 & 0 \\ \epsilon_1 & -2 & \epsilon_2 \\ 0 & \epsilon_2 & -1 \end{pmatrix},$$

where $\epsilon_1, \epsilon_2 \in \{\pm 1\}$. Then, we obtain $\lambda_{\text{min}}(\mathcal{S}') = -3$ for any cases. Since $\mathcal{S}'$ is an induced Hoffman subgraph of $\mathcal{S}$, $\lambda_{\text{min}}(\mathcal{S}) \leq \lambda_{\text{min}}(\mathcal{S}')$. Therefore $\lambda_{\text{min}}(\mathcal{S}) \leq -3$, which is a contradiction. Hence the lemma holds.

We denote the $(+)$-complete graph on $n$ vertices by $K^+_n$ and the $(-)$-complete graph on 2 vertices by $K^-_2$. Let $T^*_1$ be the triangle defined by $V(T^*_1) = \{v^*, v_1, v_2\}$, $E^+(T^*_1) = \{\{v_1, v_2\}\}$, and $E^-(T^*_1) = \{\{v^*, v_1\}, \{v^*, v_2\}\}$.

Let $S$ be an edge-signed graph. By a block of $S$ we mean the subgraph of $S$ induced by a block of $U(S)$.

**Lemma 4.2.** Let $\mathcal{S}$ be a fat Hoffman graph containing $\mathcal{R}_{1,2}$ with $\lambda_{\text{min}}(\mathcal{S}) > -3$. Let $v^*$ be the slim vertex in the $\mathcal{R}_{1,2}$, and let $B^*$ be the block of $S(\mathcal{S})$ containing the vertex $v^*$. Then the block $B^*$ is $K^+_n$ ($n \geq 2$), $K^-_2$, or $T^*_1$.

**Proof.** If $E^-(B^*) = \emptyset$, then $B^* = K^+_n$ with $n \geq 2$ since each block has at least two vertices. We assume that $E^-(B^*) \neq \emptyset$. If $|V(B^*)| = 2$, then $B^* = K^-_2$. If $|V(B^*)| = 3$, then, by Lemma 3.6, $B^* = T^*_1$ since $\lambda_{\text{min}}(\mathcal{S}) > -3$.

We show that $|V(B^*)| \leq 3$ by contradiction. Suppose that $|V(B^*)| \geq 4$. Take any three vertices $v_1, v_2, v_3$ in $B^*$ other than $v^*$. Then by Lemma 3.2, each of the vertices $v_1, v_2$ and $v_3$ has exactly one fat neighbor. Since $\lambda_{\text{min}}(\mathcal{S}) > -3$, it follows from Lemma 3.5 that the edge-signed subgraph of $S(\mathcal{S})$ induced by $\{v_1, v_2, v_3\}$ is a $(+)$-triangle $K^+_3$. Since $E^-(B^*) \neq \emptyset$, without loss of generality, we may assume that $\{v^*, v_1\}$ is a $(+)$-edge in $S(\mathcal{S})$. Since $\lambda_{\text{min}}(\mathcal{S}) > -3$, Lemma 3.6 implies that both of the edges $\{v^*, v_2\}$ and $\{v^*, v_3\}$ are $(-)$-edges.

For $i = 1, 2, 3$, the vertices $v^*$ and $v_i$ have a common fat neighbor, say $f_i$, in $\mathcal{S}$ since $\{v^*, v_i\}$ is a $(+)$-edge in $S(\mathcal{S})$. Since the vertex $v^*$ has exactly two fat neighbors in $\mathcal{S}$, two of the three fat vertices $f_1, f_2, f_3$ are the same. Without loss of generality, we may assume that $f_1 = f_2$. Then $v_1$ and $v_2$ have a common fat neighbor, which is a contradiction to the fact that $\{v_1, v_2\}$ is a $(+)$-edge in $S(\mathcal{S})$. Thus $|V(B^*)| \leq 3$. Hence the lemma holds.

Now we are ready to give our main result.

**Theorem 4.3.** Let $\mathcal{S}$ be a fat indecomposable Hoffman graph containing a slim vertex $v^*$ having two fat neighbors. Then $\lambda_{\text{min}}(\mathcal{S}) > -3$ if and only if the following conditions hold:
(i) \(U(S(\mathcal{H}))\) is a claw-free block graph,

(ii) \(\mathcal{H}\) has exactly one induced Hoffman subgraph isomorphic to \(R_{1,2}\),

(iii) \(v^*\) is not a cut vertex of \(U(S(\mathcal{H}))\),

(iv) the block \(B^*\) of \(S(\mathcal{H})\) containing the vertex \(v^*\) is either \(K^+_n\) \((n \geq 2)\) or \(K^-_2\) or \(T^*_1\),

(v) each block of \(S(\mathcal{H})\) other than \(B^*\) is either \(K^+_n\) \((n \geq 2)\) or \(K^-_2\).

Proof. Suppose that \(\lambda_{\text{min}}(\mathcal{H}) > -3\). Then (i), (ii), (iii), (iv), and (v) follow by Proposition 3.15, Lemma 3.2, Lemma 4.1, Lemma 4.2, and Lemma 3.5 respectively.

Conversely, assume that (i)–(v) hold. Let \(\{B_0 = B^*, B_1, \ldots, B_p\}\) be the set of blocks of \(S(\mathcal{H})\). For each block \(B\) with \(B \cong K^-_2\), let \(V(B) = \{\sigma^+(B), \sigma^-(B)\}\). Let \(W = \{w_1, \ldots, w_q\}\) be the set of slim vertices of \(\mathcal{H} - v^*\) which are not cut vertices of \(U(S(\mathcal{H}))\). We define a map \(\psi : V^*(\mathcal{H}) \to \mathbb{R}^N\), where \(N = 1 + p + q\), by

\[
\psi(v)_i = \begin{cases}
1 & \text{if } i = 0, \quad v = v^*, \\
1 & \text{if } i = 0, \quad B_0 = K^+_n \text{ for some } n, \quad v \in V(B_0) - \{v^*\}, \\
-1 & \text{if } i = 0, \quad B_0 = K^-_2 \text{ or } T^*_1, \quad v \in V(B_0) - \{v^*\}, \\
1 & \text{if } 1 \leq i \leq p, \quad B_i \cong K^+_n \text{ for some } n, \quad v \in V(B_i), \\
1 & \text{if } 1 \leq i \leq p, \quad B_i \cong K^-_2, \quad v = \sigma^+(B_i), \\
-1 & \text{if } 1 \leq i \leq p, \quad B_i \cong K^-_2, \quad v = \sigma^-(B_i), \\
1 & \text{if } p + 1 \leq i \leq p + q, \quad v = w_{i-p} \in W, \\
0 & \text{otherwise.}
\end{cases}
\]

Then \(\psi\) is a reduced representation of \(\mathcal{H}\).

Next, we show that \(\{\psi(v) \mid v \in V^*(\mathcal{H})\}\) is linearly independent. Suppose that

\[
\sum_{v \in V^*(\mathcal{H})} a_v \psi(v) = 0
\]

where \(a_v \in \mathbb{R}\) for each \(v \in V^*(\mathcal{H})\). For each \(1 \leq i \leq q\), \(\psi(v)_{i+p} = \delta_{v,w_i}\), where \(\delta_{v,w}\) denotes the Kronecker’s delta. Since \(\sum_{v \in V^*(\mathcal{H})} a_v \psi(v)_{i+p} = 0\), we have \(a_{w_i} = 0\). Thus \(a_v = 0\) for any \(v \in W\).

Suppose that there exists \(u \in V^*(\mathcal{H}) - (W \cup \{v^*\})\) such that \(a_u \neq 0\). We take \(u\) as a vertex farthest from \(v^*\) among such vertices. Since \(u\) is a cut vertex, \(u\) is contained in two blocks, say \(B_i\) and \(B_j\). Let \(G = U(S(\mathcal{H}))\) for convenience, denote by \(B^* = B_0, B_1, \ldots, B_p\) the blocks in \(G\) corresponding to the blocks \(B_i = B_0, B_1, B_2, \ldots, B_p\), respectively. Note that \(\{B_i, B_j\}\) is an edge in \(B(G)\). By (i) and Lemma 2.10 \(B(G)\) is a tree. Therefore
\(d_{B(G)}(B^*,B_i) = d_{B(G)}(B^*,B_j) \pm 1\). Without loss of generality, we may assume that \(d_{B(G)}(B^*,B_i) = d_{B(G)}(B^*,B_j) + 1\). Let \(P\) be the shortest path from \(v^*\) to \(u\). If \(j \neq 0\), then \(|V(P) \cap (V(B^*) \cup V(B_i))| = 3\). Thus by Lemma 2.11

\[d_{B(G)}(B^*,B_i) = d_G(v^*,u)\]

If \(j = 0\), then this holds as well, since both sides equal 1.

Let \(u'\) be any cut vertex in \(B_i\) other than \(u\), i.e., \(u' \in V(B_i) - (W \cup \{u\})\). Since \(u\) is a cut vertex of \(G\), the shortest path from \(v^*\) to \(u'\) must pass the vertex \(u\). Therefore, \(d_G(v^*,u') = d_G(v^*,u) + d_G(u,u') = d_{B(G)}(B^*,B_i) + 1\). Then \(a_{u'} = 0\) by the choice of \(u\). Then we obtain

\[
0 = \sum_{v \in V^*(\mathcal{F})} a_v \psi(v)_i \\
= a_v \psi(u)_i + \sum_{u' \in V(B_i) - (W \cup \{u\})} a_{u'} \psi(u')_i + \sum_{v \in V(B_i) \cap W} a_v \psi(v)_i \\
= \pm a_u,
\]

which is a contradiction to \(a_u \neq 0\). Thus, we have \(a_v = 0\) for any \(v \in V^*(\mathcal{F}) - (W \cup \{v^*\})\).

Moreover, we obtain \(0 = \sum_{v \in V^*(\mathcal{F})} a_v \psi(v)_0 = a_{v^*}\). Thus we have \(a_v = 0\) for any \(v \in V^*(\mathcal{F})\). Hence \(\{\psi(v) | v \in V^*(\mathcal{F})\}\) is linearly independent. By Lemma 2.5 \(\lambda_{\min}(\mathcal{F}) > -3\). □

**Remark 4.4.** In the proof of Theorem 4.3 we constructed a reduced representation of norm 3 of the Hoffman graph \(\mathcal{F}\) satisfying (i)–(v) with integral entries. In general, a reduced representation may not be realizable in \(\mathbb{Z}^n\), but it is shown in [9, Theorem 2.8] that a graph satisfying the conditions of Theorem 4.3 admits such a reduced representation.

## 5 Concluding remarks

Our main theorem gives a characterization of fat indecomposable Hoffman graphs \(\mathcal{F}\) with \(\lambda_{\min}(\mathcal{F}) > -3\) containing a slim vertex \(v^*\) having two fat neighbors, in terms of their special graphs. This is natural, since the smallest eigenvalue of \(\mathcal{F}\) is determined only by its special graph \(\mathcal{S}(\mathcal{F})\). Given a connected edge-signed graph \(\mathcal{S}\) satisfying the following conditions:

(i) \(U(\mathcal{S})\) is a claw-free block graph,

(ii) No vertex other than \(v^*\) is incident with more than one \((-\))-edge,
(iii) $v^*$ is not a cut vertex of $U(S)$,  
(iv) the block $B^*$ of $S$ containing the vertex $v^*$ is either $K_n^+$ ($n \geq 2$) or $K_2^-$,  
(v) each block of $S$ other than $B^*$ is either $K_n^+$ ($n \geq 2$) or $K_2^-$, one can construct a fat Hoffman graph $H$ with $S(H) = S$, such that $v^*$ is the only slim vertex having two fat neighbors. Indeed, $H$ can be constructed from $S$ in the following manner:  
(a) for each $(-)$-edge, attach a common fat neighbor to its end vertices,  
(b) if $B^* \cong K_n^+$, then attach two pendant fat vertices to $v^*$; if $B^* \cong K_2^-$, then attach a pendant fat vertex to $v^*$,  
(c) for each vertex other than $v^*$ which is not incident to a $(-)$-edge, attach a pendant fat vertex,  
(d) replace every $(+)$-edge of $S$ by an edge, and remove all $(-)$-edges of $S$.

Then one can verify that $H$ is a fat Hoffman graph with $S(H) = S$, and $v^*$ has two fat neighbors.

It should be remarked, however, that a fat Hoffman graph $H$ with prescribed $S(H)$ is not unique. Indeed, let $S$ be the path of length 2 consisting of two $(+)$-edges, with vertex set \{$v^*, v_1, v_2$\}, where $v^*$ is an end vertex. Then the two Hoffman graphs $\tilde{S}^i (i = 1, 2)$ defined below satisfy $S(\tilde{S}^i) = S$.

\[ V(\tilde{S}^1) = \{f_+, f_-, f_1, f_2\}, \]
\[ E(\tilde{S}^1) = \{\{v^*, v_1\}, \{v_1, v_2\}, \{v^*, f_+\}, \{v^*, f_-\}, \{v_1, f_1\}, \{v_2, f_2\}\}, \]
\[ V(\tilde{S}^2) = \{f_0, f_1, f_2\}, \]
\[ E(\tilde{S}^2) = \{\{v^*, v_1\}, \{v^*, v_2\}, \{v_1, v_2\}, \{v^*, f_0\}, \{v^*, f_2\}, \{v_1, f_1\}, \{v_2, f_2\}\}. \]

Our main theorem also gives a generalization of a result of Hoffman [7]. Recall that $\tilde{A}(G, v^*)$ denotes the adjacency matrix of a graph $G$, modified by putting $-1$ in the diagonal position corresponding to a vertex $v^*$. As we mentioned in Section 1, Hoffman showed the following.

**Lemma 5.1 ([7] Lemma 2.1).** Let $L(T)$ be the line graph of a tree $T$ and let $e$ be an end edge of $T$. Then the smallest eigenvalue of $\tilde{A}(L(T), e)$ is greater than $-2$.

We can generalize Lemma 5.1 by using Theorem 4.3.
Theorem 5.2. Let $G$ be a graph and let $v^*$ be a vertex of $G$. Then the smallest eigenvalue of $\hat{A}(G,v^*)$ is greater than $-2$ if and only if $G$ is the line graph of a tree $T$ and $v^*$ corresponds to an end edge of $T$.

Proof. Suppose that the smallest eigenvalue of $\hat{A}(G,v^*)$ is greater than $-2$. Let $\mathcal{H}$ be the fat Hoffman graph obtained by attaching a pendant fat vertex to every vertex of $G$ except $v^*$, and attaching two pendant fat vertices to $v^*$. Then $B(\mathcal{H}) = \hat{A}(G,v^*) - I$, hence $\mathcal{H}$ has smallest eigenvalue greater than $-3$. By Theorem 4.3(i), $U(S(\mathcal{H}))$ is a claw-free block graph, hence it is a line graph of a tree by Lemma 2.10. Moreover, by Theorem 4.3(iii), $v^*$ is not a cut vertex of $U(S(\mathcal{H}))$. Thus $v^*$ corresponds to an end edge of $T$.

Conversely, suppose that $G$ is the line graph of a tree $T$ and $v^*$ corresponds to an end edge of $T$. It is easy to see that the Hoffman graph $\mathcal{H}$ constructed above satisfies all the conditions (i)–(v) of Theorem 4.3, hence $\lambda_{\text{min}}(\mathcal{H}) > -3$. Since $B(\mathcal{H}) = \hat{A}(G,v^*) - I$, we conclude that $\lambda_{\text{min}}(\hat{A}(G,v^*)) > -2$. □

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Appendix: Detailed proofs of Lemmas 3.4, 3.8, and 3.11

Proof of Lemma 3.4. By assumption, we have

\[
B(\delta) = \begin{pmatrix}
-2 & \epsilon_1 & & \\
\epsilon_1 & -1 & \epsilon_2 & \\
& \epsilon_2 & -1 & \ddots \\
& & \ddots & \ddots & \epsilon_{n-3} \\
& & & \epsilon_{n-3} & -1 & \epsilon_{n-2} & \epsilon_{n-1} \\
& & & & \epsilon_{n-2} & -1 & 0 \\
& & & & & \epsilon_{n-1} & 0 & -1
\end{pmatrix},
\]

Let \( x \in \mathbb{R}^n \) be a vector defined by

\[
(x)_i = \begin{cases}
2 \prod_{k=1}^{i-3} (-\epsilon_k) & \text{if } 1 \leq i \leq n-3, \\
2 & \text{if } i = n-2, \\
-\epsilon_{i-1} & \text{if } i = n-1, n.
\end{cases}
\]

Then \((B(\delta) + 3I)x = 0\). Indeed,

\[
(Bx)_1 = -2(x)_1 + \epsilon_1(x)_2 = -4 \prod_{k=1}^{n-3} (-\epsilon_k) + 2 \epsilon_1 \prod_{k=2}^{n-3} (-\epsilon_k) = -6 \prod_{k=1}^{n-3} (-\epsilon_k) = -3(x)_1.
\]

For \(2 \leq i \leq n-3\),

\[
(Bx)_i = \epsilon_{i-1}(x)_{i-1} - (x)_i + \epsilon_1(x)_{i+1} = 2 \epsilon_{i-1} \prod_{k=i-1}^{n-3} (-\epsilon_k) - 2 \prod_{k=i}^{n-3} (-\epsilon_k) + 2 \epsilon_{i+1} \prod_{k=i+1}^{n-3} (-\epsilon_k) = -2 \prod_{k=i}^{n-3} (-\epsilon_k) - 2 \prod_{k=i}^{n-3} (-\epsilon_k) = -6 \prod_{k=i}^{n-3} (-\epsilon_k) = -3(x)_i.
\]

\[
(Bx)_{n-2} = \epsilon_{n-3}(x)_{n-3} - (x)_{n-2} + \epsilon_{n-2}(x)_{n-1} + \epsilon_{n-1}(x)_n
\]

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where \( \epsilon \) defined by

\[
\epsilon = -6 = -3(x)_{n-2},
\]

\[
(Bx)_{n-1} = \epsilon_{n-2}(x)_{n-2} - (x)_{n-1} = 2\epsilon_{n-2} + \epsilon_{n-2} = 3\epsilon_{n-2} = -3(x)_{n-1},
\]

\[
(Bx)_n = \epsilon_{n-1}(x)_{n-2} - (x)_n = 2\epsilon_{n-1} + \epsilon_{n-1} = 3\epsilon_{n-1} = -3(x)_n.
\]

Hence \( B(\delta) \) has \(-3\) as an eigenvalue. This implies \( \lambda_{\min}(\delta) \leq -3. \)

**Proof of Lemma 3.8.** Consider the matrix

\[
B(\delta) = \begin{pmatrix}
-2 & \epsilon_1 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \\
\epsilon_1 & -1 & \epsilon_2 & \ddots & & & \vdots & \vdots & \vdots \\
0 & \epsilon_2 & -1 & \ddots & & \vdots & \vdots & \vdots & \vdots \\
& \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\
& & \epsilon_{n-2} & -1 & \epsilon_{n-1} & 0 & 0 & 0 & 0 \\
0 & \cdots & \cdots & 0 & \epsilon_{n-1} & -1 & 1 & 1 & \epsilon \\
0 & \cdots & \cdots & 0 & 1 & -1 & \delta & 1 & \\
0 & \cdots & \cdots & 0 & \epsilon & 1 & 1 & -1 & -1
\end{pmatrix},
\]

where \( \epsilon_1, \ldots, \epsilon_{n-1} \in \{\pm 1\} \) and \( \{\epsilon, \delta\} = \{0, 1\} \). Let \( x \in \mathbb{R}^{n+3} \) be a vector defined by

\[
(x)_i = \begin{cases}
2 \prod_{k=1}^{n-1} (-\epsilon_k) & \text{if } 1 \leq i \leq n-1, \\
2 & \text{if } i = n, \\
-1 & \text{if } i = n+1, n+2, \\
2(\epsilon - 1)/ (\epsilon - 2) & \text{if } i = n+3.
\end{cases}
\]

We claim \( B(\delta)x = -3x \). Indeed,

\[
(Bx)_1 = -2(x)_1 + \epsilon_1(x)_2
\]

\[
= -2(-1)^{n-1} \cdot 2 \prod_{k=1}^{n-1} \epsilon_k + \epsilon_1(-1)^{n-2} \cdot 2 \prod_{k=2}^{n-1} \epsilon_k
\]

\[
= -4(-1)^{n-1} \prod_{k=1}^{n-1} \epsilon_k - 2(-1)^{n-1} \prod_{k=1}^{n-1} \epsilon_k
\]

\[
= -3(-1)^{n-1} 2 \prod_{k=1}^{n-1} \epsilon_k = -3(x)_1.
\]

For \( 2 \leq j \leq n-1, \)

\[
(Bx)_j = \epsilon_{j-1}(-1)^{n-j+1} 2 \prod_{k=j}^{n-1} \epsilon_k - (-1)^{n-j} 2 \prod_{k=j}^{n-1} \epsilon_k + \epsilon_j(-1)^{n-j-1} 2 \prod_{k=j+1}^{n-1} \epsilon_k
\]

\[
= -3(-1)^{n-j} 2 \prod_{k=j}^{n-1} \epsilon_k = -3(x)_j.
\]
\[ \sum_{k=1}^{n-1} \epsilon_k ((-1)^{n-j+1} - (-1)^{n-j} + (-1)^{n-j-1}) \]
\[ = -3 \cdot (-1)^{n-j} \prod_{k=j}^{n-1} \epsilon_k = -3(x)_j. \]

\[ (Bx)_n = \epsilon_{n-1}(x)_{n-1} - (x)_n + (x)_{n+1} + (x)_{n+2} + \epsilon(x)_{n+3} \]
\[ = \epsilon_{n-1}(-2\epsilon_{n-1}) - 2 + (-1) + (-1) + \frac{2\epsilon(\epsilon - 1)}{\epsilon - 2} \]
\[ = -6 = -3(x)_n, \]

\[ (Bx)_{n+1} = (x)_n - (x)_{n+1} + (1 - \epsilon)(x)_{n+2} + (x)_{n+3} \]
\[ = 2 - (-1) + (1 - \epsilon)(-1) + \frac{2\epsilon(\epsilon - 1)}{\epsilon - 2} \]
\[ = 3 + \frac{(\epsilon - 1)((\epsilon - 2) + 2)}{\epsilon - 2} = -3(x)_{n+1}, \]

\[ (Bx)_{n+2} = (x)_n + (1 - \epsilon)(x)_{n+1} - (x)_{n+2} + (x)_{n+3} \]
\[ = 2 + (1 - \epsilon)(-1) - (-1) + \frac{2\epsilon(\epsilon - 1)}{\epsilon - 2} \]
\[ = 3 + \frac{(\epsilon - 1)((\epsilon - 2) + 2)}{\epsilon - 2} = 3 = -3(x)_{n+2}, \]

\[ (Bx)_{n+3} = \epsilon(x)_n + (x)_{n+1} + (x)_{n+2} - (x)_{n+3} \]
\[ = 2\epsilon - 1 - 1 - \frac{2(\epsilon - 1)}{\epsilon - 2} = 2(\epsilon - 1) - \frac{2(\epsilon - 1)}{\epsilon - 2} \]
\[ = \frac{(2(\epsilon - 2) - 2)(\epsilon - 1)}{\epsilon - 2} = \frac{-6(\epsilon - 1)}{\epsilon - 2} + \frac{2\epsilon(\epsilon - 1)}{\epsilon - 2} \]
\[ = \frac{-6(\epsilon - 1)}{\epsilon - 2} = -3(x)_{n+3}. \]

Thus \( B(\delta)x + 3x = 0. \) \[\square\]
Proof of Lemma 3.11. Consider the matrix

$$
\begin{pmatrix}
-2 & \delta_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\delta_1 & -1 & \delta_2 & \cdots & : & : & : & : \\
0 & \delta_2 & -1 & \cdots & \cdots & \delta_{m-2} & 0 & : \\
: & \cdots & \cdots & \delta_{m-2} & -1 & \delta_{m-1} & 0 & \cdots & 0 \\
0 & \cdots & 0 & \delta_{m-1} & -1 & 1 & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & -1 & 1 & 0 & \cdots & 0 & \epsilon_n \\
0 & \cdots & 0 & 1 & 1 & -1 & \epsilon_2 & \cdots & 0 \\
: & \cdots & 0 & 0 & \epsilon_2 & -1 & \cdots & \cdots & : \\
: & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & : \\
0 & \cdots & 0 & 0 & \cdots & \cdots & \epsilon_{n-2} \epsilon_{n-1} & -1 & \epsilon_{n-1} \\
0 & \cdots & 0 & 0 & \cdots & 0 & \epsilon_n & 0 & \cdots & 0 & \epsilon_n \epsilon_{n-1} -1 \\
\end{pmatrix},
$$

where $\delta_1, \ldots, \delta_{m-1}, \epsilon_2, \ldots, \epsilon_n \in \{\pm 1\}$. Since the number of $(+)$-edges in $C_n$ is odd, we have

$$
\prod_{k=2}^{n} \epsilon_k = (-1)^{n-1}.
$$

Let $x \in \mathbb{R}^{m+n}$ be the vector defined by

$$(x)_i = \begin{cases} 
2 & \text{if } i = 1, \\
2 \cdot \prod_{k=1}^{i-1} (-\delta_k) & \text{if } 2 \leq i \leq m, \\
- \prod_{k=1}^{m-1} (-\delta_k) & \text{if } m + 1 \leq i \leq m + 2, \\
- \prod_{k=1}^{m-1} (-\delta_k) \cdot \prod_{k=2}^{i-m-1} (-\epsilon_k) & \text{if } m + 3 \leq i \leq m + n.
\end{cases}
$$

Then $(B(\delta) + 3I)x = 0$. Indeed, set $\delta := \prod_{k=1}^{m-1} (-\delta_k)$. Then

$$(B(\delta)x)_1 = -2(x)_1 + \delta_1(x)_2 = -2 \cdot 2 + \delta_1 \cdot (2 \cdot (-\delta_1)) = -4 - 2\delta_1^2 = -6 = -3(x)_1.
$$

For $2 \leq j \leq m - 1$,

$$(B(\delta)x)_j = \delta_{j-1}(x)_{j-1} - (x)_j + \delta_j(x)_{j+1}
$$

$$
= \delta_{j-1} \cdot 2 \cdot \prod_{k=1}^{j-2} (-\delta_k) - 2 \cdot \prod_{k=1}^{j-1} (-\delta_k) + \delta_j \cdot 2 \cdot \prod_{k=1}^{j} (-\delta_k)
$$

$$
= (-2 - 2 - 2) \prod_{k=1}^{j-1} (-\delta_k) = -3(x)_j.
$$

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\[(B(\mathfrak{H}) \mathbf{x})_m = \delta_{m-1}(\mathbf{x})_{m-1} - (\mathbf{x})_m + (\mathbf{x})_{m+1} + (\mathbf{x})_{m+2}\]
\[
= \delta_{m-1} \cdot 2 \cdot \prod_{k=1}^{m-2} (-\delta_k) - 2\delta - \delta - \delta
\]
\[
= (-2 - 2 - 1 - 1)\delta = -3(\mathbf{x})_m,
\]
\[(B(\mathfrak{H}) \mathbf{x})_{m+1} = (\mathbf{x})_m - (\mathbf{x})_{m+1} + (\mathbf{x})_{m+2} + \epsilon_n(\mathbf{x})_{m+n}\]
\[
= 2\delta - (-\delta) + (-\delta) + \epsilon_n(-\delta) \prod_{k=2}^{n-1} (-\epsilon_k)
\]
\[
= 2\delta + \delta \prod_{k=2}^{n} (-\epsilon_k) = 2\delta + \delta = -3(-\delta) = -3(\mathbf{x})_{m+1},
\]
\[(B(\mathfrak{H}) \mathbf{x})_{m+2} = (\mathbf{x})_m + (\mathbf{x})_{m+1} - (\mathbf{x})_{m+2} + \epsilon_2(\mathbf{x})_{m+3}\]
\[
= 2\delta - \delta + \epsilon_2(-\delta(-\epsilon_2))
\]
\[
= 2\delta + \delta = -3(-\delta) = -3(\mathbf{x})_{m+2}.
\]

For \(m + 3 \leq j \leq m + n - 1,\)
\[(B(\mathfrak{H}) \mathbf{x})_j = \epsilon_{j-m-1}(\mathbf{x})_{j-1} - (\mathbf{x})_j + \epsilon_{j-m}(\mathbf{x})_{j+1}\]
\[
= \epsilon_{j-m-1}(-\delta \prod_{k=2}^{j-m-2} (-\epsilon_k)) - (-\delta \prod_{k=2}^{j-m-1} (-\epsilon_k)) + \epsilon_{j-m}(-\delta \prod_{k=2}^{j-m} (-\epsilon_k))
\]
\[
= (-1 - 1 - 1)(-\delta \prod_{k=2}^{j-m-1} (-\epsilon_k)) = -3(\mathbf{x})_j.
\]
\[(B(\mathfrak{H}) \mathbf{x})_{m+n} = \epsilon_n(\mathbf{x})_{m+1} + \epsilon_{n-1}(\mathbf{x})_{m+n-1} - (\mathbf{x})_{m+n}\]
\[
= \epsilon_n(-\delta) + \epsilon_{n-1}(-\delta \prod_{k=2}^{n-2} (-\epsilon_k)) - (-\delta \prod_{k=2}^{n-1} (-\epsilon_k))
\]
\[
= \epsilon_n \prod_{k=2}^{n} (-\epsilon_k) \cdot (-\delta) + \delta \prod_{k=2}^{n-1} (-\epsilon_k) + \delta \prod_{k=2}^{n-1} (-\epsilon_k)
\]
\[
= 3\delta \prod_{k=2}^{n} (-\epsilon_k) = -3(\mathbf{x})_{m+n}.
\]

Therefore, \(\mathfrak{H}\) has \(-3\) as an eigenvalue and thus \(\lambda_{\min}(\mathfrak{H}) \leq -3.\) □

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