On the number of limit cycles in quadratic
perturbations of quadratic codimension-four centres

Yulin Zhao
Department of Mathematics, Sun Yat-sen University, Guangzhou, 510275,
People’s Republic of China
E-mail: mcszyl@mail.sysu.edu.cn

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Abstract
This paper is concerned with the bifurcation of limit cycles in general quadratic
perturbations of quadratic codimension-four centres $Q_4$. Gavrilov and Iliev
set an upper bound of eight for the number of limit cycles produced from the
period annuli around the centre. Based on Gavrilov–Iliev’s proof, we prove in
this paper that the perturbed system has at most five limit cycles which emerge
from the period annuli around the centre. We also show that there exists a
perturbed system with three limit cycles produced by the period annuli of $Q_4$.

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1. Introduction and statement of the main results
In this paper we study the bifurcation of limit cycles in planar quadratic systems under small
quadratic perturbations. We assume that the unperturbed system has at least one centre. Taking
a complex coordinate $z = x + iy$ and using the terminology from [19], the list of quadratic
centres at $z = 0$ looks as follows:

$$
\dot{z} = -iz - z^2 + 2|z|^2 + (b + ic)\bar{z}^2, \quad \text{Hamiltonian (}Q^H_3\text{)},
\dot{z} = -iz + az^2 + 2|z|^2 + bz^2, \quad \text{reversible (}Q^R_3\text{)},
\dot{z} = -iz + 4z^2 + 2|z|^2 + (b + ic)\bar{z}^2, \quad |b + ic| = 2, \text{ codimension four (}Q_4\text{)},
\dot{z} = -iz + z^2 + (b + ic)\bar{z}^2, \quad \text{generalized Lotka–Volterra (}Q^{LV}_3\text{)},
\dot{z} = -iz + \bar{z}^2, \quad \text{Hamiltonian triangle},
$$

where $a$, $b$ and $c$ stand for arbitrary real constants. Let

$$
\dot{x} = \frac{H_x(x, y)}{M(x, y)}, \quad \dot{y} = -\frac{H_y(x, y)}{M(x, y)},
$$

(1)
be any of the above systems rewritten in \((x, y)\) coordinates. Here \(H(x, y)\) is a first integral of system \((1)\) with the integrating factor \(M(x, y)\). Consider a small quadratic perturbation of \((1)\):

\[
\begin{align*}
\dot{x} &= H_y(x, y) + \epsilon X_2(x, y, \epsilon), \\
\dot{y} &= -H_x(x, y) + \epsilon Y_2(x, y, \epsilon),
\end{align*}
\]

where \(X_2(x, y, \epsilon)\) and \(Y_2(x, y, \epsilon)\) are quadratic polynomials in \(x\) and \(y\) with coefficients depending analytically on the small parameter \(\epsilon\).

Each centre of system \((1)\) is surrounded by a continuous set of period annuli. Compactifying the phase plane \(\mathbb{R}^2\) of system \((1)\) to the Poincaré disc, the boundary of the period annulus of the centre has two connected components, the centre itself and a singular loop which consists of orbit(s) and at least one singularity. It is well known that the limit cycles of system \((2)\) can emerge from

(a) the centre (i.e. the inner boundary),
(b) the singular loop (i.e. the outer boundary),
(c) the period annulus.

Bautin [1] found that, at most, three limit cycles can appear near a focus or a centre of any quadratic system. This implies that the cyclicity of the centre of quadratic system is equal to three under quadratic perturbation. As usual, we use the notion of cyclicity for the total number of limit cycles which can emerge from a configuration of trajectories (centre, period annulus, a singular loop) under a perturbation.

The bifurcation of limit cycles from saddle loop in perturbations of quadratic Hamiltonian systems has been studied in [8]. Moreover, if a singular loop contains only one saddle under certain genericity conditions, then it was proved in [14] that the cyclicity of a singular loop can be transferred to the cyclicity of the period annuli. However, if a singular loop contains at least two saddles, this transfer in general is not true. For more details, we refer to [13] and references therein.

The cyclicity of the period annulus of system \((1)\) has been investigated by many authors. This problem is reduced to counting the number of zeros of the displacement function

\[
d(h, \epsilon) = \epsilon M_1(h) + \epsilon^2 M_2(h) + \cdots + \epsilon^k M_k(h) + \cdots,
\]

where \(d(h, \epsilon)\) is defined on a section to the flow, which is parametrized by the Hamiltonian value \(h\). The number of zeros of the first non-vanishing Melnikov function \(M_k(h)\) in \((3)\) determines the upper bound of the number of limit cycles in \((2)\) emerging from the periodic annulus of the unperturbed integrable system \((1)\). The corresponding Melnikov functions were determined in [10] for quadratic centres.

The cyclicity of the period annulus for quadratic Hamiltonian \(Q^H_3\) and Hamiltonian triangle was completely solved by several authors, see [3, 4, 9–11, 17, 18] and references therein. The generalized Lotka–Volterra \(Q^F_3\) was studied by Zoladek in [19]. Some results concerned with certain specific cases of \(Q^F_3\) can be found in [2, 12], etc. However, almost nothing is known about the generic reversible case \(Q^R_3\). Garu et al [7] have some new results in this area.

This paper deals with the cyclicity of the period annulus of quadratic codimension-four centres \(Q_4\). Using Picard–Fuchs equations and Petrov’s method (based on the argument principle) [15], Gavrilov and Iliev [5] proved that the cyclicity of the period annulus of \(Q_4\) is less or equal to eight, see theorem 2 in section 2. Based on their proof, we obtain the following theorem in this paper.

**Theorem 1.** Let system \((1)\) be a quadratic codimension-four system \(Q_4\) rewritten in \((x, y)\) coordinates. Then the perturbed quadratic system \((2)\) has at most five limit cycles which emerge from the period annuli around the centre. Moreover, the quadratic polynomials \(X_2(x, y, \epsilon)\)
and $Y_2(x, y, \epsilon)$ exist such that system (2) has at least three limit cycles produced by the period annuli of system (1).

The centres whose (generic complexified) periodic orbits are elliptic curves are called centres of genus one [6]. It can be seen that the codimension-four centre is a centre of genus one. Recently, the authors of [6] proposed a program for finding the cyclicity of period annuli of quadratic centres of genus one. The contents of [5] and this paper can be viewed as contributions of this program.

The rest of this paper is organized as follows. In section 2 we briefly sketch the proof of Gavrilov–Iliev’s theorem, which is crucial for our analysis. In section 3 the explicit forms of several related functions are given by revisiting Gavrilov–Iliev’s proof, and then we get the asymptotic expansions for these functions in section 4. Theorem 1 is proved in section 5. Finally we give some comments in section 6.

2. A sketch of the proof of Gavrilov–Iliev’s theorem

Gavrilov and Iliev proved the following theorem:

**Theorem 2 ([5]).** The cyclicity of the open period annulus surrounding the centre of any generic codimension-four plane quadratic system is less than or equal to eight.

We are going to sketch the proof of theorem 2.

It is well known that the cyclicity of the period annulus can be detected in a compact region by the number of zeros of the first non-vanishing $k$th order Melnikov function $M_k(h)$ in (3), which is sometimes called the generating function. The generating function for $Q_4$ is given in [10] by a complete elliptic integral. After a series of changes, the generating function becomes

$$I(h) = \mu_1 h I_{0,0} + \mu_2 I_{1,0} + \mu_3 I_{0,1} + \mu_4 (2I_{-1,0} + 3\kappa h I_{-1,1}).$$

where

$$I_{i,j}(h) = \int \int_{H(x,y)<h} x^i y^j \, dx \, dy, \quad h \in \left( -\frac{2}{3}, -\frac{2}{3\sqrt{\kappa}} \right),$$

with

$$H(x, y) \equiv \frac{2}{3}(\kappa - 1)x^3 - (\kappa - 1)x^2 y + \frac{\kappa}{3} y^3 - y = h, \quad \kappa > 1.$$

The integrals $I_{i,j}$ in (4) satisfy the following Picard-Fuchs system

$$I_{0,0} = \frac{3h}{2} I_{0,0} + I'_{0,1},$$
$$I_{1,0} = h I'_{0,0} + \frac{2}{3} I'_{1,1},$$
$$I_{0,1} = \frac{2}{3\kappa} I'_{0,0} + h I_{0,1} + \frac{2(\kappa - 1)}{3\kappa} I'_{1,1},$$
$$I_{1,1} = \frac{3h}{8} I_{0,0} + \frac{1}{2} I'_{1,0} + \frac{1}{4} I_{0,1} + \frac{3h}{4} I'_{1,1},$$
$$I_{-1,0} = 3h I_{-1,0} + 2I'_{-1,1},$$
$$I_{-1,1} = \frac{\kappa - 1}{\kappa} I_{1,0} + \frac{1}{\kappa} I_{-1,0} + \frac{3h}{2} I'_{-1,1}.$$
Using the above system, the authors obtain
\[
\frac{d}{dh} \left( \frac{I(h)}{h} \right) = \frac{h I'(h) - I(h)}{h^2} = \frac{\tilde{G}(h)}{h^2},
\]
where
\[
\tilde{G}(h) = (\mu_1 h^2 + \mu_3) I_{0,0}' + \mu_2 I_{1,1}' + \mu_4 [-4h I_{1,0}' + (3\kappa h^2 - 4) I_{1,1}'].
\] (9)

Therefore
\[
I(h) = h \int_{-2/3}^{b} \xi^{-2} \tilde{G}(\xi) \, d\xi
\]
and \(I(h)\) has at most as much zeros as \(\tilde{G}(h)\). It is proved in [5] that \(\tilde{G}(h)\) satisfies the following equation:
\[
L_2(h) \tilde{G}(h) = R(h) \triangleq \frac{h((\tilde{a}_0 + \tilde{a}_1 h^2 + \tilde{a}_3 h^4 + \tilde{a}_5 h^6) I_{0,0}' + (\tilde{b}_0 + \tilde{b}_1 h^2 + \tilde{b}_3 h^4) I_{1,1}')}{(9h^2 - 4)^2(9\kappa h^2 - 4)},
\]
where
\[
L_2(h) = 5\kappa h - (9\kappa h^2 - 8) \frac{d}{dh} + h(9\kappa h^2 - 4) \frac{d^2}{dh^2}.
\] (11)

Taking the changes
\[
h = -\frac{2}{3} \sqrt{\frac{s}{\kappa}}, \quad J_1(s) = I_{0,0}'(h(s)), \quad J_2(s) = I_{1,1}'(h(s)), \quad G(s) = \tilde{G}(h(s)),
\]
we obtain the equation
\[
L_2 G = \left( s(1 - s) \frac{d^2}{ds^2} - \frac{1}{2} \frac{d}{ds} - \frac{5}{36} \right) G(s) = \frac{P_3(s) J_1(s) + Q_2(s) J_2(s)}{(s - \kappa)^2(s - 1)},
\] (13)
where \(P_3(s)\) and \(Q_2(s)\) are real polynomials of degree, at most three and two, respectively. The dot denotes the differentiation with respect to \(s\). The integrals \(J_1(s)\) and \(J_2(s)\) satisfy the following Picard–Fuchs system
\[
6(s - 1)(s - \kappa) \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \begin{pmatrix} 1 - s & \kappa - 1 \\ 1 - s & s - 1 \end{pmatrix} \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}.
\] (14)

We say that \(V\) is a Chebyshev space, provided that each non-zero function in \(V\) has at most \(\dim(V) - 1\) zeros, counted with multiplicity.

**Proposition 3 ([5]).** The following statements hold:

(i) Suppose the solution space of the homogeneous equation \(x'' + a_1(t)x' + a_2(t)x = 0\) is a Chebyshev space and let \(R(t)\) be an analytic function on \((a, b)\) having \(k\) zeros (counted with multiplicity). Then every solution \(x(t)\) of the non-homogeneous equation
\[
x'' + a_1(t)x' + a_2(t)x = R(t)
\]
has at most \(k + 2\) zeros on \((a, b)\).

(ii) The solution space \(S\) associated with the differential operator \(L_2(h)\), defined in (11), is a Chebyshev space.

Let
\[
V_n = \{ P_n J_1(s) + Q_{n-1} J_2(s) : P_n, Q_{n-1} \in \mathbb{R}[s], \deg P_m, Q_m \leq m \}.
\]

**Proposition 4 ([5]).** The vector space \(V_n\) is Chebyshev on the interval \((1, \kappa)\): each element has at most \(\dim V_n - 1 = 2n\) zeros (counted with multiplicity).
Proof of theorem 2. It follows from proposition 4 that $P_n(s)J_1 + Q_n(s)J_2$ has 6 zeros in $(1, \kappa)$, and hence $R(h)$ has 6 zeros in $(-2/3, -2/(3\sqrt{\kappa}))$. Finally one gets theorem 2 from proposition 3 and (10).

Remark 5. Proposition 4 is proved using the argument principle in the complex domain $\mathbb{C} \setminus (-\infty, 1]$. The function $J_1(s)$ is a complete elliptic integral of the first kind and therefore does not vanish. Let

$$F(s) = \frac{P_n(s)J_1(s) + Q_{n-1}(s)J_2(s)}{J_1(s)}.$$ 

Along the interval $(-\infty, 1)$, the increase of the argument of $F$ is bounded by the number of zeros of $Q_{n-1}(s)$. Hence we have

$$\# F(s) \leq \deg P_n(s) + \# Q_{n-1}(s) + 1$$

in the complex domain $\mathbb{C} \setminus (-\infty, 1]$, where $\# F(s)$ denotes the number of zeros of $F(s)$.

Remark 6. In the rest of this paper we always suppose $\kappa > 1$ unless the opposite is claimed. For the convenience of the proof we also suppose that $H(x, y)$, defined in (6), is a first integral of the following system:

$$\dot{x} = \frac{\partial H}{\partial y} = -1 - (\kappa - 1)x^3 + \kappa y^2, \quad \dot{y} = -\frac{\partial H}{\partial x} = -2(\kappa - 1)x(x - y).$$

Hence the annulus $\Gamma_h = \{(x, y) | H(x, y) = h\}$ has the negative (clockwise) orientation for $h \in (-2/3, -2/(3\sqrt{\kappa}))$. The Hamiltonian value $h = -2/3$ and $h = -2/(3\sqrt{\kappa})$ correspond to the centre $(1, 1)$ and the homoclinic loop $\Gamma_{-2/(3\sqrt{\kappa})}$, respectively.

3. Some preliminary results

Proposition 3, obtained in [5], shows that the generating function $I(h)$ has at most $k + 2$ zeros in the interval $(-2/3, -2/(3\sqrt{\kappa}))$, where $k$ is the number of zeros of $R(h)$, defined in (10). In this paper we will prove $k \leq 3$, which yields the first part of theorem 1. To do it, we are going to give explicit forms of $I(h)$ and $R(h)$ (respectively $I(s)$ and $R(s)$) in this section.

As in [5], we introduce the variable $s \in (1, \kappa)$, defined in (12), and denote by a dot the differentiation with respect to $s$. Taking the changes (12) and

$$J_3(s) = I_{-1,0}^\prime(h(s)), \quad J_4(s) = I_{-1,1}^\prime(h(s)), \quad J_5(s) = I_{1,0}^\prime(h(s)),$$

$$J_6(s) = I_{0,1}^\prime(h(s)),$$

it follows from (4) and (7) that

$$I(s) = I(h(s)) = \frac{2s}{3\kappa} \mu_1 + \frac{2}{3\kappa} \mu_2 \right) J_1 + \left( \frac{2}{3} \mu_2 + \frac{2(\kappa - 1)}{3\kappa} \mu_3 \right) J_2$$

$$- 6\mu_4 \sqrt{s} \frac{\sqrt{3}}{\kappa} J_3 + (4 + 2s)\mu_4 J_4$$

$$- \frac{2}{3} \sqrt{s} \frac{\sqrt{3}}{\kappa} \mu_2 + 3(\kappa - 1)\mu_4 J_5 - \frac{2}{3} \sqrt{s} \left( \mu_1 + \mu_3 \right) J_6.$$ 

Suppose that $I(h)$ is defined as (4). By direct computation we have that, $\tilde{G}$, defined in (8), has the form

$$\tilde{G}(h) = \left( \mu_1 h^2 - \frac{2\mu_3}{3\kappa} \right) I_{0,0}^\prime + \left( \frac{2\mu_2}{3} - \frac{2(\kappa - 1)\mu_3}{3\kappa} \right) I_{0,1}^\prime - 4h I_{-1,0}^\prime$$

$$+ (3\kappa h^2 - 4) I_{-1,1}^\prime.$$ 

(20)
Here $\tilde{G}(h)$ is different from the one defined in (9). However, if we take
\[
\tilde{\mu}_3 = -\frac{2\mu_3}{3\kappa}, \quad \tilde{\mu}_2 = -\frac{2\mu_2}{3} - \frac{2(\kappa - 1)\mu_3}{3\kappa}
\]
and omit the tildes, then we obtain $\tilde{G}(h)$, defined in (9).

For convenience, in what follows we always suppose $\tilde{G}(h)$ is defined in (9) unless the opposite is claimed.

We note that $R(h)$, defined in (10), has no explicit form in [5]. Following the idea in [5], a direct calculation then yields
\[
R(h) = \frac{2h(p(h)I_{0,0} + q(h)I_{1,1})}{(9h^2 - 4)(9\kappa h^2 - 4)},
\]
where
\[
p(h) = (9\kappa h^2 - 4)(a_0 + a_1h^2 + a_2h^4), \quad q(h) = b_0 + b_1h^2 + b_2h^4,
\]
with
\[
a_0 = 64\mu_1 + 24\mu_2 + 8(5\kappa + 3)\mu_3 + 32(\kappa - 1)\mu_4,
\]
\[
a_1 = 40(\kappa - 9)\mu_1 - 162\mu_2 - 18(11\kappa + 9)\mu_3 + 24(\kappa - 1)(5\kappa - 9)\mu_4,
\]
\[
a_2 = 18(\kappa + 21)\mu_1 + 243\mu_2 + 486\kappa \mu_3 + 54\kappa(\kappa - 1)\mu_4,
\]
\[
b_0 = -32((5\kappa - 3)\mu_2 + 3(\kappa - 1)\mu_3 - 4(\kappa - 1)\mu_4),
\]
\[
b_1 = 72(4(\kappa - 1)\mu_1 + (5\kappa^2 + 8\kappa - 9)\mu_2 + 3(\kappa - 1)(2\kappa + 3)\mu_3 + 4(\kappa - 1)(\kappa - 3)\mu_4),
\]
\[
b_2 = -54(4(\kappa - 1)(2\kappa + 1)\mu_1 + 3(11\kappa - 9)\mu_2 + 36\kappa(\kappa - 1)\mu_3
\]
\[
+ 12\kappa(\kappa - 1)(2\kappa - 3)\mu_4).
\]

Taking the changes (12), equation (13) becomes
\[
L_2G = \left( s(1 - s) \frac{d^2}{ds^2} - \frac{1}{2} \frac{d}{ds} - \frac{5}{36} \right) G(s) = \frac{\kappa \mathcal{F}(s)}{1152(s - \kappa)^2(s - 1)},
\]
where
\[
\mathcal{F}(s) = P_3(s)J_1(s) + Q_2(s)J_2(s)
\]
with
\[
P_3(s) = p \left( \frac{2}{3} \sqrt{\frac{s}{\kappa}} \right), \quad Q_2(s) = q \left( -\frac{2}{3} \sqrt{\frac{s}{\kappa}} \right).
\]

Here $P_3(s)$ and $Q_2(s)$ are polynomials in $s$ with $\deg P_3(s) \leq 3$, $\deg Q_2(s) \leq 2$. It follows from (14) that
\[
\mathcal{F}(s) = P_2(s)J_1(s) + Q_1(s)J_2(s),
\]
where
\[
P_2(s) = P_3(s) - \frac{P_3(s) + Q_2(s)}{6(s - \kappa)} = \alpha_0 + \alpha_1s + \alpha_2s^2,
\]
\[
Q_1(s) = Q_2(s) + \frac{(\kappa - 1)P_3(s) + (s - 1)Q_2(s)}{6(s - \kappa)(s - 1)} = \beta_1s - \beta_0,
\]
with
\[
\alpha_0 = \frac{128(42 + 13\kappa)}{9\kappa} \mu_1 + \frac{16(54 + 13\kappa)}{3\kappa} \mu_2 + \frac{32(27 + 38\kappa + 15\kappa^2)}{3\kappa} \mu_3
\]
\[
= \frac{128(\kappa - 1)(2\kappa - 9)}{3\kappa} \mu_4.
\]
Perturbation of $Q_4$

$$
\begin{align*}
\alpha_1 &= \frac{128(-117 - 265\kappa + 30\kappa^2)}{27\kappa^2} \mu_1 + \frac{16(-174 + 5\kappa)}{3\kappa} \mu_2 = \frac{16(243 + 121\kappa)}{3\kappa} \mu_3 \\
+ & \frac{64(-1 + \kappa)(-119 + 60\kappa)}{9\kappa} \mu_4,
\end{align*}
\begin{align*}
\alpha_2 &= \frac{1088(21 + \kappa)}{27\kappa^2} \mu_1 + \frac{544}{\kappa} \mu_2 + \frac{1088}{\kappa} \mu_3 + \frac{1088(-1 + \kappa)}{9\kappa} \mu_4, \\
\beta_0 &= -\frac{256(-1 + \kappa)}{3\kappa} \mu_1 - \frac{32(-27 + 25\kappa + 15\kappa^2)}{3\kappa} \mu_2 = \frac{16(\kappa - 1)(54 + 31\kappa)}{3\kappa} \mu_3 \\
- & \frac{64(\kappa - 1)(-18 + 5\kappa)}{3\kappa} \mu_4, \\
\beta_1 &= -\frac{64(\kappa - 1)(18 + 77\kappa)}{27\kappa^2} \mu_1 - \frac{16(-111 + 137\kappa)}{3\kappa} \mu_2 - \frac{768(\kappa - 1)}{\kappa} \mu_3 \\
- & \frac{64(-1 + \kappa)(-116 + 77\kappa)}{9\kappa} \mu_4.
\end{align*}
$$

Solving $\mu_i$, $i = 1, 2, 3, 4$, from the above equations, we obtain

$$
\begin{align*}
\mu_1 &= \frac{9}{246400} \alpha_1 + \frac{9(162 - 213\kappa - 205\kappa^2)}{22400(-1 + \kappa)} \beta_0 = \frac{9(162 - 4236\kappa + 657\kappa^2 + 2845\kappa^3)}{3203200(-1 + \kappa)} \beta_1, \\
\mu_2 &= -\frac{3(54 - 77\kappa + 82\kappa^2)}{30800\kappa} \alpha_1 - \frac{3(3888 + 2502\kappa - 5184\kappa^2 + 11833\kappa^3)}{3403400\kappa} \alpha_2 \\
+ & \frac{3(9 - 7\kappa + 2\kappa^2)}{2800\kappa(\kappa - 1)} \beta_0 + \frac{3(27 - 709\kappa + 965\kappa^2 - 569\kappa^3)}{200200\kappa(\kappa - 1)} \beta_1, \\
\mu_3 &= -\frac{3(-54 + 71\kappa + 205\kappa^2)}{61600\kappa} \alpha_1 - \frac{3(7776 + 5868\kappa - 47598\kappa^2 - 59165\kappa^3)}{13613600\kappa} \alpha_2 \\
+ & \frac{3(9 - 6\kappa - 5\kappa^2)}{5600\kappa(\kappa - 1)} \beta_0 + \frac{3(54 - 1412\kappa - 1201\kappa^2 + 2845\kappa^3)}{800800\kappa(\kappa - 1)} \beta_1, \\
\mu_4 &= -\frac{3(486 - 1017\kappa + 90\kappa^2 + 205\kappa^3)}{246400\kappa(-1 + \kappa)} \alpha_1 \\
+ & \frac{3(69984 - 162\kappa - 276246\kappa^2 + 44405\kappa^3 + 59165\kappa^4)}{54454400\kappa(\kappa - 1)} \alpha_2 \\
- & \frac{3(\kappa - 3)(-27 + 30\kappa + 5\kappa^2)}{22400\kappa(\kappa - 1)^2} \beta_0 \\
- & \frac{3(486 - 13086\kappa + 16683\kappa^2 + 1050\kappa^3 - 2845\kappa^4)}{3203200\kappa(\kappa - 1)^2} \beta_1,
\end{align*}
$$

which yield that

$$
\alpha_0 = \beta_0 - \kappa \beta_1 - \kappa \alpha_1 - \kappa^2 \alpha_2. \tag{27}
$$

Remark 7. For the convenience of the proof in the rest of this paper we also take $\alpha_i$, $i = 1, 2$, and $\beta_i$, $i = 0, 1$, as the new parameters, instead of $\mu_i$, $i = 1, 2, 3, 4$.

Without loss of generality suppose $\beta_1 = 1$ if $\beta_1 \neq 0$. That is to say, $\beta_1 \in [0, 1]$.

Note that $J_1(s)$ is a complete elliptic integral of the first kind and therefore does not vanish [5]. Let

$$
\frac{g(s)}{J_1(s)} = \frac{P_2(s) + Q_1(s)w(s)}{J_1(s)} = \frac{P_2(s)}{J_1(s)} + \frac{Q_1(s)w(s)}{J_1(s)}, \tag{28}
$$
where

\[ w(s) = \frac{J_2(s)}{J_1(s)}, \quad P_2(s) = \alpha_2 s^2 + \alpha_1 s + \alpha_0, \quad Q_1(s) = \beta_1 s - \beta_0, \tag{29} \]

and \( \alpha_0 \) is defined in (27), \( \beta_1 \in [0, 1] \).

4. Asymptotic expansions for the related functions

In this section we give the asymptotic expansions of the related functions near the endpoints of their domain of definition.

Lemma 8. \( J_i(s) \), \( i = 1, 2, \ldots, 6 \), have the following asymptotic expansions near \( s = \kappa \):

\[
J_1(s) = J_1(\kappa) \left( 1 - \frac{5(s - \kappa)}{36(\kappa - 1)} + \frac{385(s - \kappa)^2}{5184(\kappa - 1)^2} - \frac{85085(s - \kappa)^3}{1679616(\kappa - 1)^3} \right.
\]
\[ + \frac{37182145(s - \kappa)^4}{967458816(\kappa - 1)^4} + \cdots \),

\[
J_2(s) = J_1(\kappa) \left( 1 + \frac{5(s - \kappa)}{36(\kappa - 1)} - \frac{35(s - \kappa)^2}{5184(\kappa - 1)^2} + \frac{5005(s - \kappa)^3}{1679616(\kappa - 1)^3} \right.
\]
\[ - \frac{1616615(s - \kappa)^4}{967458816(\kappa - 1)^4} + \cdots \),

\[
J_3(s) = J_1(\kappa) \left( 1 - \frac{17(s - \kappa)}{36(\kappa - 1)} + \frac{1837\kappa - 36(s - \kappa)^2}{5184\kappa(\kappa - 1)^2} \right.
\]
\[ - \frac{(5832 - 21276\kappa + 496709\kappa^2)(s - \kappa)^3}{1679616\kappa^2(\kappa - 1)^3} \]
\[ + \frac{5(-419904 + 1870128\kappa - 3388824\kappa^2 + 50126789\kappa^3)(s - \kappa)^4}{967458816\kappa^3(\kappa - 1)^4} + \cdots \). \]

\[
J_4(s) = J_1(\kappa) \left( 1 - \frac{(5\kappa + 12)(s - \kappa)}{36\kappa(\kappa - 1)} + \frac{(-432 + 1848\kappa + 385\kappa^2)(s - \kappa)^2}{5184\kappa^2(\kappa - 1)^2} \right.
\]
\[ - \frac{(69984 - 286416\kappa + 612612\kappa^2 + 85085\kappa^3)(s - \kappa)^3}{1679616\kappa^3(\kappa - 1)^3} \]
\[ + \frac{(37182145\kappa^4 + 356948592\kappa^3 - 250327584\kappa^2 + 122332032\kappa - 25194240)(s - \kappa)^4}{967458816\kappa^4(\kappa - 1)^4} \]
\[ - \frac{37182145\kappa^4 + 356948592\kappa^3 - 250327584\kappa^2 + 122332032\kappa - 25194240)(s - \kappa)^4}{967458816\kappa^4(\kappa - 1)^4} + \cdots \). \]
Perturbation of $Q$

$$J_0(s) = J_1(\kappa) \left(1 + \frac{(\kappa - 6)(s - \kappa)}{36\kappa(\kappa - 1)} - \frac{(216 - 672\kappa + 71\kappa^2)(s - \kappa)^2}{5184\kappa^3(\kappa - 1)^2}ight) + \frac{(-34992 + 121176\kappa - 185886\kappa^2 + 14617\kappa^3)(s - \kappa)^3}{1679616\kappa^3(\kappa - 1)^3} - \frac{5(2519424 - 10917504\kappa + 18833472\kappa^2 - 19076208\kappa^3 + 1204387\kappa^4)(s - \kappa)^4}{967458816\kappa^4(\kappa - 1)^4} + \ldots \right).$$

**Proof.** Differentiating both sides of system (7), we have

$$J_0''(s) = \frac{3h}{2} J_{0,0}'' + \frac{2}{3} J_{0,1}'',
$$

$$hJ_0'' + \frac{2}{3} J_1'' - \frac{4}{3} (\kappa - 1) J_0'' = 0,
$$

$$\frac{2}{3} J_0'' + \frac{2}{3} J_1'' = 0,
$$

$$\frac{3h}{8} J_0'' + \frac{1}{3} J_0'' + \frac{2}{3} J_1'' = 0,
$$

$$J_0'' - \frac{1}{\kappa} J_1'' + \frac{3h}{2} J_1'' = 0.
$$

Solving $I_{0,j}(h)$ from (30), one obtains

$$\begin{pmatrix} I_{0,0}'' \\ I_{0,1}'' \\ I_{-1,0}'' \\ I_{-1,1}'' \end{pmatrix} = M \begin{pmatrix} I_{0,0} \\ I_{1,0} \\ I_{1,0} \\ I_{1,1} \\ I_{-1,0} \\ I_{-1,1} \end{pmatrix},$$

where

$$M = \begin{pmatrix} -3h(9\kappa h^2 - 4) & 12(\kappa - 1)h & 0 & 0 \\ -3h(9\kappa h^2 - 4) & 3h(9\kappa h^2 - 4) & 0 & 0 \\ 8(\kappa - 1) & -8(\kappa - 1) & -6\kappa h(9h^2 - 4) & 2\kappa(9h^2 - 4) \\ -12(\kappa - 1)h & 12(\kappa - 1)h & 4(9h^2 - 4) & -3\kappa(9h^2 - 4) \\ 2(9h^2 - 4) & -2(9\kappa h^2 - 4) & 0 & 0 \\ 2(9\kappa h^2 - 4) & -18(\kappa - 1)h^2 & 0 & 0 \end{pmatrix}.$$
Therefore, we have

\[ I_i(s) = \sum_{j=0}^{\infty} c_{i,j}(s - \kappa)^j \] with

\[
M^* = \begin{pmatrix}
1 - s & \kappa - 1 & 0 & 0 \\
1 - s & s - 1 & 0 & 0 \\
\sqrt{\kappa}(\kappa - 1) / \sqrt{2} & \sqrt{\kappa}(\kappa - 1) / \sqrt{2} & -2(s - \kappa) / \sqrt{2(\kappa - s)} & \sqrt{\kappa}(s - \kappa) / \sqrt{2(\kappa - s)} \\
\sqrt{\kappa}(s - 1) / \sqrt{2} & \sqrt{\kappa}(s - 1) / \sqrt{2} & 2(s - \kappa) / \sqrt{2(\kappa - s)} & \kappa - s / \sqrt{2(\kappa - s)} \\
-\sqrt{\kappa}(s - 1) / (\kappa - 1) & -\sqrt{\kappa}(s - 1) / (\kappa - 1) & 0 & 0 \\
1 - \kappa & 1 - \kappa & 0 & 0
\end{pmatrix}.
\]

Since \( h = -2/3 \) corresponds to the centre \((1, 1)\) of Hamiltonian system (17), we have

\[
I_i(s)(-2/3) = 0, \quad i = 0, \pm 1, \quad j = 0, \pm 1.
\]

Substituting \( I_i(s)(-2/3) = 0 \) into (7) yields

\[
I_i(1,0)(-2/3) = I_i(1,0)(-2/3) = I_i(0,1)(-2/3) = I_i(0,0)(-2/3).
\]

Therefore \( J_i(\kappa) = J_2(\kappa) = \cdots = J_6(\kappa) \).

Since \( s = \kappa \) corresponds to the centre of Hamiltonian system (17), \( J_i(s) \) is analytic at \( s = \kappa \). Taking \( J_i(s) = \sum_{j=0}^{\infty} c_{i,j}(s - \kappa)^j \) with \( c_{i,0} = J_i(\kappa) \) into (32), we obtain the expansions. \( \square \)

**Corollary 9.** The following assertions hold:

(i) \( I(s) \equiv 0 \) if and only if \( \mu_1 = \mu_2 = \mu_3 = \mu_4 = 0 \).

(ii) There exists \( \mu_i, \quad i = 1, 2, 3, 4, \) such that \( I(s) \) has at most three zeros in \((1, \kappa)\).

**Proof.** It follows from (19) and lemma 8 that \( I(s) \) has the following asymptotic expansion at \( s = \kappa \):

\[
I(s) = v_1(s - \kappa) + v_2(s - \kappa)^2 + v_3(s - \kappa)^3 + v_4(s - \kappa)^4 + \cdots.
\]

where

\[
v_i = \frac{2\mu_1}{9\kappa} - \frac{\mu_2}{3\kappa} - \frac{\mu_3}{3\kappa} + \frac{2(\kappa - 1)\mu_4}{3\kappa},
\]

\[
v_2 = \frac{(13\kappa - 18)\mu_1 + (17\kappa - 18)\mu_2 + (17\kappa - 12)\mu_3 + (13\kappa + 18)\mu_4}{162\kappa^2(\kappa - 1)},
\]

\[
v_3 = \frac{(1944 - 4068\kappa + 1739\kappa^2)\mu_1}{11664\kappa^3(\kappa - 1)^2} - \frac{(1944 - 3780\kappa + 1801\kappa^2)\mu_2}{7776\kappa^3(\kappa - 1)^2} - \frac{(1296 - 2712\kappa + 1801\kappa^2)\mu_3}{3888\kappa^3(\kappa - 1)^2},
\]

\[
v_4 = \frac{5(-104976 + 324648\kappa - 338526\kappa^2 + 101837\kappa^3)\mu_1}{1259712\kappa^4(\kappa - 1)^3}
\]

\[
+ \frac{5(-104976 + 309096\kappa - 301374\kappa^2 + 96253\kappa^3)\mu_2}{839808\kappa^4(\kappa - 1)^3}
\]

\[
+ \frac{5(-69984 + 216432\kappa - 225684\kappa^2 + 96253\kappa^3)\mu_3}{839808\kappa^4(\kappa - 1)^3}
\]

\[
+ \frac{5(104976 - 180792\kappa + 7398\kappa^2 + 101837\kappa^3)\mu_4}{419904\kappa^4(\kappa - 1)^3}.
\]
System (34) is a linear system of equations in the variables \( \mu_i, \ i = 1, 2, 3, 4 \). The determinant of matrix of coefficients of (34) is equal to \( 125/(472392\kappa^4(\kappa-1)^4) > 0 \) for \( \kappa > 1 \). As shown by Cramer’s rule, system (34) has a unique solution. Therefore \( v_1 = v_2 = v_3 = v_4 = 0 \) if and only if \( \mu_1 = \mu_2 = \mu_3 = \mu_4 = 0 \), which yields that \( I(s) \equiv 0 \) if and only if \( \mu_i = 0, \ i = 1, 2, 3, 4 \). This proves the assertion (i).

Since system (34) has a unique solution, we can choose \( v_i \) as the independent parameters, instead of \( \mu_i, \ i = 1, 2, 3, 4 \). Denote by \( I(s, v_1, v_2, v_3, v_4) = I(s) \). Without loss of generality suppose \( v_4 > 0 \). To get more zeros of \( I(s) \), we choose \( v_i \) and \( s_i \in (1, \kappa) \), \( i = 4, 3, 2, 1 \), such that \( I(s_4, 0, 0, 0, v_4) > 0, I(s_3, 0, 0, v_3, v_4) < 0, I(s_2, 0, v_2, v_3, v_4) > 0, I(s_1, v_1, v_2, v_3, v_4) < 0 \) and \( 0 < |v_1| \ll |v_2| \ll |v_3| \ll |v_4| \), \( 1 < s_4 < s_3 < s_2 < s_1 < \kappa \). It is easy to show that \( I(s) \), defined as above, has at least three zeros which tend to \( \kappa \).

**Lemma 10.** \( J_i(s), \ i = 1, 2, \) have the following asymptotic expansions near \( s = 1 \):

\[
J_1(s) = -\frac{\ln(s-1)}{2\sqrt{\kappa-1}} + c_{12} - \frac{5(s-1)\ln(s-1)}{72(\kappa-1)^{3/2}} + \cdots,
\]

\[
J_2(s) = \frac{3}{\sqrt{\kappa-1}} - \frac{(s-1)\ln(s-1)}{12(\kappa-1)^{3/2}} + \cdots,
\]

where \( c_{12} \) is a real constant.

**Proof.** Since the period annuli of the vector field (17) begin at the centre \((1,1)\) and terminate at a homoclinic loop \( \Gamma_{-2/(3\sqrt{\kappa})} = \{(x, y) \mid H(x, y) = -2/(3\sqrt{\kappa})\} \), it follows from [16] that \( I_{i,j}(h), \ i \geq 0 \), have the asymptotic expansions of the form

\[
I_{i,j}(h) = \sum_{k=0}^{\infty} d_{i,j,k} \left(-\frac{2}{3\sqrt{\kappa}} - h\right)^k + \ln\left(-\frac{2}{3\sqrt{\kappa}} - h\right) \sum_{k=1}^{\infty} \tilde{d}_{i,j,k} \left(-\frac{2}{3\sqrt{\kappa}} - h\right)^k,
\]

as \( h \to -2/(3\sqrt{\kappa}) \), which implies that \( J_i(s), \ i = 1, 2 \), have the asymptotic expansions of the form

\[
J_i(s) = c_{i,1} \ln(s-1) + c_{i,2} + c_{i,3}(s-1)\ln(s-1) + \cdots
\]

(35) as \( s \to 1 \). On the other hand, it is well known (see for instance [16] or the appendix of [18]) that

\[
\tilde{d}_{i,j,1} = \frac{x^iy^j}{2\sqrt{\kappa-1}} \bigg|_{(x,y)=(0,1/\sqrt{\kappa})} = \begin{cases} 1 & \text{if } i = 0, \\ \frac{1}{2\sqrt{\kappa}}(\kappa-1) & \text{if } i = 1, \\ 0 & \text{if } i = 1. \end{cases}
\]

A simple calculation shows that \( c_{i,1} = -d_{i,j,1} \). Taking (35) with \( c_{i,1} = -d_{i,j,1} \) into (14), we obtain the asymptotic expansions near \( s = 1 \) for \( J_i(s), \ i = 1, 2 \).

By lemma 8 and lemma 10, a straight calculation shows that the following two lemmas hold.

**Lemma 11.** The following expressions hold:

\[
w(\kappa) = 1, \ w'(\kappa) = \frac{1}{6(\kappa-1)}, \ w''(\kappa) = -\frac{25}{216(\kappa-1)^2}, \ w'''(\kappa) = \frac{775}{3888(\kappa-1)^3},
\]

\[
g(\kappa) = 0, \ \mathcal{F}(\kappa) = \mathcal{F}'(\kappa) = 0, \ \mathcal{F}''(\kappa) = J_1(\kappa)g'(\kappa).
\]
Since $G(s)$ is analytic at $s = \kappa$, it follows from (24) that $F(\kappa) = F'(\kappa) = 0$. This is verified by lemma 11.

**Lemma 12.** The following expansions hold as $s \to 1$:

$$w(s) = -\frac{6}{\ln(s - 1)} + \cdots,$$

$$g(s) = \begin{cases} 
  P_2(1) - \frac{6Q_1(1)}{\ln(s - 1)} + \cdots, & \text{if } Q_1(1) \neq 0, \\
  P_2(s), & \text{if } \beta_0 = \beta_1 = 0, \\
  P_2(1) - \frac{6(s - 1)}{\ln(s - 1)} + \cdots, & \text{if } \beta_0 = \beta_1 = 1,
\end{cases}$$

where $\beta_1 \in \{0, 1\}$.

5. Proof of theorem 1

Using proposition 3 proved in [5], we get the following proposition.

**Proposition 13.** $\#F(s)$ denotes the number of zeros of $F(s)$. Then we have

$$\#F(s) \leq \#F'(s) = \#g(s), \quad \#I(s) \leq \#G(s) \leq \#F(s) + 2, \quad s \in (1, \kappa). \quad (36)$$

**Proof.** It follows from lemma 11 that $F(\kappa) = F'(\kappa) = 0$. This yields the first inequality of (36). The second inequality is obtained by proposition 3. \qed

Noting that the cyclicity of the period annulus is determined by $\#I(s)$, we will prove theorem 1 by estimating the number of zeros of $g(s)$. Since $g(\kappa) = 0$, we get that $g(s)$ has at most three zeros in $(1, \kappa)$ by argument principle, see remark 5. This implies that $\#I(s) \leq \#G(s) \leq 5$. However, to get more information about the number of zeros of $g(s)$ (hence $I(s)$), we prefer to prove theorem 1 by the following theorem (see the comments in the next section, and the note after the statement of theorem 14).

**Theorem 14.** Let $s \in (1, \kappa)$ and $\beta_1 \in \{0, 1\}$.

(a) If $\beta_1 = 1$, $\beta_0 \in (-\infty, (23\kappa - 54)/31] \cup [1, +\infty)$, then $g(s)$ has at most two zeros in $(1, \kappa)$.

(b) If $\beta_1 = 1$, $\beta_0 \in ((23\kappa - 54)/31, 1)$, then $g(s)$ has at most three zeros.

(c) If $\beta_1 = 0$, $\beta_0 \neq 0$, then $g(s)$ has at most two zeros.

(d) If $\beta_1 = 0$, $\beta_0 = 0$, then $g(s)$ has at most one zero.

If $\beta_1 = 1$, $\beta_0 \in [1, +\infty)$ (respectively $\beta_0 \in (-\infty, 1)$), then it can be proved that $g(s)$ has at most two (respectively three) zeros in $(1, \kappa)$ by argument principle, see the proof of proposition 4 [5], or remark 5. However, it seems that we cannot prove by argument principle that $g(s)$ has at most two zeros in $(1, \kappa)$ if $\beta_1 = 1$, $\beta_0 \in (-\infty, (23\kappa - 54)/31]$.

First we study the geometric properties of $w(s) = J_2(s)/J_1(s)$.

**Lemma 15.** The function $w(s)$ is monotonically increasing and concave in the interval $(1, \kappa)$, i.e. $w'(s) > 0$, $w''(s) < 0$ and $0 < w(s) < 1.$
Proof. It follows from (14) that \( w(s) \) satisfies
\[
6(s - 1)(s - \kappa)w' = 1 - s + 2(s - 1)w - (\kappa - 1)w^2 \leq U(s, w). \tag{37}
\]
Note that \( U(s, w) \), the right-hand side of (37), is a quadratic polynomial of \( w \). Since
\[
4(s - 1)^2 + 4(\kappa - 1)(1 - s) = 4(s - 1)(s - \kappa) < 0 \text{ for } s \in (1, \kappa) \text{ and } -(\kappa - 1) < 0,
\]
we have \( U(s, w) < 0 \). This yields \( w'(s) > 0 \) for \( s \in (1, \kappa) \). The inequality \( 0 < w(s) < 1 \)
follows from lemma 11 and lemma 12.

Differentiating both sides of (37), one obtains
\[
6(s - 1)(s - \kappa)w'' = \left( -6(2s - \kappa - 1) + \frac{\partial U(s, w)}{\partial w} \right) \frac{U(s, w)}{6(s - 1)(s - \kappa)} + \frac{\partial U(s, w)}{\partial s},
\]
which implies that
\[
\frac{d^2 w}{ds^2} = \frac{V_1(s, w)V_2(s, w)}{18(s - 1)^2(s - \kappa)^2} \tag{38}
\]
with
\[
V_1(s, w) = (\kappa - 1)w - (s - 1), \quad V_2(s, w) = (\kappa - 1)w^2 + (4s - 3\kappa - 1)w - 2(s - 1).
\]
By remark 5 (or the proof of proposition 4 [5]), we conclude that \((1 - s)J_1(s) + (\kappa - 1)J_2(s)\) has at most two zeros in \( C \cap (-\infty, 1] \). Since \((1 - s)J_1(s) + (\kappa - 1)J_2(s)\) has a zero at \( s = \kappa \), \((1 - s)J_1(s) + (\kappa - 1)J_2(s)\) has at most one zero in \((1, \kappa)\). Noting \( J_1(s) \neq 0 \), we know that
\[
\eta_1(s) = V_1(s, w(s)) = (1 - s) + (\kappa - 1)\frac{J_2(s)}{J_1(s)}
\]
has at most one zero in \((1, \kappa)\). By direct computation \( \eta_1(\kappa) = \eta_1(1) = 0, \lim_{s \to 1} \eta_1'(s) = +\infty, \eta_1'(k) = -5/6 \), which implies that the number of zeros of \( \eta_1(s) \) is even. Therefore \( \eta_1(s) \) has no zero in \((1, \kappa)\), which shows that \( V_1(s, w(s)) > 0 \) in \((1, \kappa)\).

Let
\[
D = \{(s, w)|1 \leq s \leq \kappa, \ 0 \leq w \leq 1\}. \tag{39}
\]
Now we study the two independent variables function \( V_2(s, w) \), defined in \( D \). The equations
\[
\partial V_2 / \partial s = \partial V_2 / \partial w = 0
\]
have a unique solution at \((s, w) = ((\kappa + 1)/2, 1/2) \in D \). Therefore, \( V_2(s, w) \) has a maximum point and a minimum point at either \((s, w) = ((\kappa + 1)/2, 1/2) \in D \), or the point in the boundary of \( D \). Since
\[
V_2(s, 0) = -2(s - 1) \leq 0, \quad V_2(s, 1) = 2(s - \kappa) \leq 0,
\]
\[
V_2(1, w) = (\kappa - 1)w(w - 3) \leq 0,
\]
we have \( V_2(s, w) \leq 0 \) for \((s, w) \in D \).

Let \( \eta_2(s) = V_2(s, w(s)) \). Then \( \eta_2(1) = \eta_2(\kappa) = 0, \lim_{s \to 1} \eta_2'(s) = -\infty, \eta_2'(\kappa) = 5/2 > 0 \), which implies that \( V_2(s, w(s)) < 0 \) in \((1, \kappa)\). It follows from (38) that \( w'(s) < 0 \) for \( s \in (1, \kappa) \).

Assume that \( \psi(x_1, x_2, \ldots, x_n) \) and \( \phi(x_1, x_2, \ldots, x_n) \) are two polynomials in \( x_1, x_2, \ldots, x_n \). Eliminating the variable \( x_i \) from the equations \( \psi(x_1, x_2, \ldots, x_n) = \phi(x_1, x_2, \ldots, x_n) = 0 \), we get the resultant of \( \psi(x_1, x_2, \ldots, x_n) \) and \( \phi(x_1, x_2, \ldots, x_n) \), denoted by Resultant(\( \psi, \phi, x_i \)).

Lemma 16. \( w''(s) > 0, \ s \in (1, \kappa) \).
**Proof.** It follows from (38) and (37) that
\[ w''(s) = \frac{\Phi(s, w)}{108(s - 1)^3(s - \kappa)^3}, \]
with
\[ \Phi(s, w) = -(s - 1)^2(20s + \kappa - 21) + 2(s - 1)(15 - \kappa + 6\kappa^2 - 29s - 11\kappa s + 20s^2)w \]
\[ - 2\kappa(1 + 18\kappa - 19s)(\kappa - s)w^2 + 6(\kappa - 1)^2(1 + 3\kappa - 4s)w^3 \]
\[ - 3(\kappa - 1)^3w^4. \]
Let \( \mathcal{D} \) be the closed rectangle, defined in (39). The maximum and minimum for \( \Phi(s, w) \) in \( \mathcal{D} \) necessarily occur either on the boundary of \( \mathcal{D} \), or the points inside \( \mathcal{D} \) whose coordinates satisfy equations
\[ \Phi_s = \frac{\partial \Phi(s, w)}{\partial s} = 0, \quad \Phi_w = \frac{\partial \Phi(s, w)}{\partial w} = 0. \]
(40)
\( \Phi_s \) and \( \Phi_w \), defined in (40), are two polynomials of \( s \) with the polynomial coefficients depending on \( w \) and \( \kappa \). Their resultant is
\[ \text{Resultant}(\Phi_s, \Phi_w, s) = -8000(\kappa - 1)^6(w - 1)^2w^2(2w - 1)\chi(w) \]
with
\[ \chi(w) = -1600425 + 316012w - 314956w^2 - 2112w^3 + 1056w^4. \]
It is nice for our study that \( \chi(w) \) does not depend on \( \kappa \). By Sturm’s theorem \( \chi(w) \) has no real zero in \((0, 1)\). Taking \( w = 1/2 \) into the first equation of system (40), we know that \((s, w) = ((\kappa + 1)/2, 1/2)\) is a solution of system (40) in \( \mathcal{D} \). Direct computation yields that if \((s, w) \in \mathcal{D}, \)
then
\[ \Phi \left( \frac{\kappa + 1}{2}, \frac{1}{2} \right) = -\frac{25}{16}(\kappa - 1)^3 < 0, \]
\[ \Phi(1, w) = -3(\kappa - 1)^3w^2(12 - 6w + w^2) \leq 0, \]
\[ \Phi(\kappa, w) = -3(\kappa - 1)^3(w - 1)^2(7 + 4w + w^2) \leq 0, \]
\[ \Phi(s, 0) = -(s - 1)^2(20s + \kappa - 21) \leq 0, \]
\[ \Phi(s, 1) = -(s - \kappa)^2(-1 + 21\kappa - 20s) \leq 0, \]
which imply that the maximum and minimum for \( \Phi(s, w) \) are non-positive. Therefore \( \Phi(s, w) \leq 0 \) for \((s, w) \in \mathcal{D} \).

Assume that there exists an interior point \((s^*, w^*)\) of \( \mathcal{D} \) such that \( \Phi(s^*, w^*) = 0 \). Since \( \Phi(s, w) \leq 0 \), \((s^*, w^*)\) must be a maximum point of \( \Phi(s, w) \) inside \( \mathcal{D} \). However, we have shown that the maximum and minimum for \( \Phi(s, w) \) inside \( \mathcal{D} \) necessarily occur at \((s, w) = ((\kappa + 1)/2, 1/2) \) and \( \Phi((\kappa + 1)/2, 1/2) < 0 \). This yields contradiction. Hence \( \Phi(s, w) < 0 \) for \((s, w) \in \{(s, w)|1 < s < \kappa, \ 0 < w < 1\}, \) which implies that \( w'''(s) > 0 \) for \( s \in (1, \kappa). \)

\[ \square \]

**Proposition 17.** Let \( \beta_1 = 1 \) and \( s \in (1, \kappa). \) The following statements hold.

(i) If \( \beta_0 \in (-\infty, (54 - 23\kappa)/31], \) then \( g'''(s) > 0. \)

(ii) If \( \beta_0 \in ((54 - 23\kappa)/31, 1), \) then \( g'''(s) \) has exactly one zero.

(iii) If \( \beta_0 \in [1, +\infty), \) then \( g'''(s) < 0. \)
Proof. By direct computation we have
\[ g'''(s) = 3w'''(s) + (s - \beta_0)w'''(s) = 3w'''(s)\Theta(s), \]
where
\[ \Theta(s) = \frac{w''(s)}{w'''(s)} + \frac{s - \beta_0}{3}. \]
Therefore,
\[ \frac{d\Theta(s)}{ds} = \frac{4(w'''(s))^2 - 3w'''(s)w^{(4)}(s)}{3(w'''(s))^2} = \frac{\Theta_1(s, w, \kappa)\Theta_2(s, w, \kappa)}{17496(s - 1)^6(s - \kappa)^5w''^2(s)} \quad (41) \]
with
\[ \Theta_1(s, w, \kappa) = 2(s - 1)^2(-9 + 4k + 5s) - (s - 1)(36 - 67\kappa + 51k^2 - 5s - 35ks) + 20s^2w + (k - 1)(18 - 41k + 18k^2 + 5s + 5ks - 5s^2)w^2 \]
\[ \triangleq \theta_0(s) + \theta_1(s)w + \theta_2(s)w^2, \]
\[ \Theta_2(s, w, \kappa) = (s - 1)^2(9 + 7k - 16s) - (s - 1)(-9 - 62k + 39k^2 + 80s + 16ks - 32s^2)w - (k - 1)(-9 + 55k + 18k^2 - 37s - 91ks + 64s^2)w^2 + 9(k - 1)^2(1 + k - 2s)w^3. \]
A calculation shows that \( \theta_2(1) = \theta_2(\kappa) = 18(k - 1)^3 > 0. \) Since \( \theta_2(s) \) is a quadratic polynomial in \( s \) with \( \lim_{s \to \pm \infty} \theta_2(s) = -\infty, \) we have \( \theta_2(s) > 0 \) for \( s \in [1, \kappa). \) On the other hand,
\[ (\theta_1(s))^2 - 4\theta_0(s)\theta_2(s) = 25(s - \kappa)^2(s - 1)^2(81 - 146\kappa + 81\kappa^2 - 16s - 16ks + 16s^2). \]
It follows from \( (16 + 16\kappa)^2 - 4(81 - 146\kappa + 81\kappa^2) \cdot 16 = -4928(k - 1)^2 < 0 \) that \( 81 - 146\kappa + 81\kappa^2 - 16s - 16ks + 16s^2 > 0, \) which implies \( (\theta_1(s))^2 - 4\theta_0(s)\theta_2(s) > 0. \)

Rewrite \( \Theta_1(s, w, \kappa) \) as
\[ \Theta_1(s, w, \kappa) = \theta_2(s)(w - W^+(s))(w - W^-(s)), \]
where
\[ W^\pm(s) = \frac{-\theta_1(s) \pm \sqrt{(\theta_1(s))^2 - 4\theta_0(s)\theta_2(s)}}{2\theta_2(s)}. \]
This gives that
\[ W^+(s) = \frac{8(s - 1)}{3(k - 1)} + \cdots, \quad W^-(s) = \frac{(s - 1)}{6(k - 1)} + \cdots, \]
as \( s \to 1^+, \) and
\[ W^+(s) = 1 + \frac{(s - \kappa)}{6(k - 1)} = \frac{115(s - \kappa)^2}{324(k - 1)^2} + \cdots, \]
\[ W^-(s) = 1 + \frac{8(s - \kappa)}{3(k - 1)} + \cdots, \]
as \( s \to \kappa^- \). Therefore it follows from lemma 11 and lemma 12 that \( W^-(s) < W^+(s) < w(s) = J_2(s)/J_1(s) \) as either \( s \to 1^+, \) or \( s \to \kappa^- \).

To prove \( \Theta_1(s, w, \kappa) \neq 0, \) we consider the system
\[ \dot{s} = 6(s - 1)(s - \kappa), \quad \dot{w} = U(s, w), \]
which has two saddle nodes at $(1, 0)$, $(\kappa, 1)$, and two invariant lines \( s = 1, s = \kappa \), respectively. It follows from (37) that the graph of the ratio \( w(s), 1 < s < \kappa \), is a trajectory starting from 
(1, 0) to \((\kappa, 1)\), see figure 1.

Since \( W^+(s) < w(s) \) as either \( s \to 1^+ \), or \( s \to \kappa^- \), \( w = w(s) \) and \( W^+(s) \) have at least two intersection points in the interval \((1, \kappa)\) if the intersection points exist. If \( w = w(s) \) and \( w = W^+(s) \) intersect at more than two points, counting their multiplicities, then we could find one point \((s^*, W^+(s^*))\) on \( w = W^+(s) \); at this point the tangent direction of \( w = W^+(s) \) coincides with the direction of the above vector field, i.e. \( w'(s^*) - (W^+)'(s^*) = 0 \), see figure 1.

Noting \((W^\pm)'(s) = -\Theta_{1s}/\Theta_{1w}\), it follows from (37) that \( s = s^* \) satisfies the following equations:
\[
\Theta_1(s, w, \kappa) = 0, \qquad \Theta_1(s, w, \kappa) = \Theta_{1w}U(s, w) + 6(s - 1)(s - \kappa)\Theta_{1s} = 0,
\]
where \( \Theta_{1s} = \frac{\partial\Theta_1(s, w, \kappa)}{\partial s} \). Since we have shown \( \Theta_2(s) > 0 \) for \( s \in (1, \kappa) \), one obtains
\[
\text{Resultant}(\Theta_1, \Theta_1, w) = -35083125(\kappa - 1)^2(\kappa - s)^2(s - 1)^2\Theta_2(s) \neq 0, \quad s \in (1, \kappa),
\]
which implies that two equations in (42) have no common zero. Therefore, \( w = w(s) \) does not intersect \( w = W^+(s) \) for \( s \in (1, \kappa) \). This yields \( W^-(s) < W^+(s) < w(s) \) for \( s \in (1, \kappa) \). Finally, we obtain that \( \Theta_1(s, w(s), \kappa) = 2\Theta_2(s)(w(s) - W^+(s))(w(s) - W^-(s)) > 0 \) for \( s \in (1, \kappa) \).

Now we consider \( \Theta_2(s, w, \kappa) \). Let \( D' = \{(s, \kappa)|1 \leq s \leq \kappa, 1 \leq \kappa \leq c\} \), \( c \) is a real large constant, \( |c| \gg 1 \), be a triangle in the \( \kappa s \)-plane and fix \( w \) as a real constant with \( 0 < w < 1 \). The maximum and minimum for \( \Theta_2(s, w, \kappa) \) in \( D' \) necessarily occur either on the boundary of \( D' \), or the points inside \( D' \) whose coordinates satisfy equations
\[
\frac{\partial\Theta_2(s, w, \kappa)}{\partial s} = 0, \qquad \frac{\partial\Theta_2(s, w, \kappa)}{\partial \kappa} = 0.
\]

Direct computation shows that
\[
\text{Resultant}(\Theta_2_1, \Theta_2_\kappa, w) = -6705(\kappa - 1)^2(s - \kappa)^2(s - 1)^2\gamma(s, \kappa)
\]
with
\[
\gamma(s, \kappa) = 362313 + 701586\kappa - 1012697\kappa^2 + 421884\kappa^3 - 1012697\kappa^4 + 701586\kappa^5
\]
\[
+ 362313\kappa^6 - 8(1 + \kappa)(359433 - 174116\kappa - 174026\kappa^2 - 174116\kappa^3
\]
\[
+ 359433\kappa^4)s + 8(991241 + 22492\kappa - 1044426\kappa^2 + 22492\kappa^3
\]
\[
+ 991241\kappa^4)s^2 - 16384(\kappa + 1)(649 - 978\kappa + 649\kappa^2)s^3
\]
\[
+ 8192(809 - 658\kappa + 809\kappa^2)s^4 - 1572864(1 + \kappa)s^5 + 524288s^6.
\]
Since \( \text{Resultant}(\gamma^4, \gamma', s) = c^6(k - 1)^2 \) with \( c^6 < 0 \), the maximum and the minimum for \( \gamma(s, \kappa) \) in \( \mathcal{D}' \) occur on the boundary of \( \mathcal{D}' \). \( \gamma(s, \kappa) \) is a polynomial of \( \kappa \) with degree 6 and the coefficient of \( \kappa^6 \) is 362313, which implies \( \gamma(s, c) > 0 \) as \( c \) is sufficiently large enough. Noting \( \gamma(1, \kappa) = \gamma(k, \kappa) = 362313(k - 1)^6 \), \( \gamma(s, \kappa) \) has its minimum value \( \gamma(1, 1) = 0 \) at \( (s, \kappa) = (1, 1) \). This yields \( \gamma(s, \kappa) > 0 \) for \( (s, \kappa) \in \mathcal{D}'(1, 1) \).

Therefore \( \text{Resultant}(\Theta_2, \Theta_2, w) \neq 0 \) for \( (s, \kappa) \in \mathcal{D}'(\{s = 1\} \cup \{s = \kappa\}) \), which implies that the maximum and minimum for \( \Theta_2(s, w, \kappa) \) in \( \mathcal{D}' \) necessarily occur on the boundary of \( \mathcal{D}' \). If \( 0 < w < 1 \) and \( (s, \kappa) \neq (1, 1) \), then

\[
\Theta_2(1, w, \kappa) = 9(k - 1)^2w^2(w - 2) < 0, \quad \Theta_2(k, w, \kappa) = -9(k - 1)^2(w - 1)^2(w + 1) < 0.
\]

Noting that \( \Theta_2(s, w, \kappa) \) is a polynomial in \( \kappa \) and the coefficient of the highest order term \( \kappa^3 \) is \( 9w^2(w - 2) < 0 \), we have \( \Theta_2(s, w, c) < 0 \), provided that \( c \) is sufficiently large enough and \( 0 < w < 1 \). Summing the above discussions and noting \( \Theta_2(1, w, 1) = 0 \), one obtains that \( \Theta_2(s, w, \kappa) \) has its maximum value zero at \( (s, \kappa) = (1, 1) \) in \( \mathcal{D}' \). Since we always suppose that \( \kappa > 1 \) in this paper, \( \Theta_2(s, w(s), \kappa) < 0 \) for \( s \in (1, \kappa) \).

It follows from (41) that \( \Theta(s) < 0 \). This yields that \( g''(s) \) has at most one zero in \((1, \kappa)\).

On the other hand, lemma 11 and lemma 12 give

\[
\lim_{s \to 1} g''(s) = \begin{cases} +\infty, & \text{if } \beta_0 < 1, \\ -\infty, & \text{if } \beta_0 \geq 1, \end{cases} \quad g''(k) = -\frac{25(23k + 31\beta_0 - 54)}{3888(k - 1)^3},
\]

which imply the assertions of this proposition.

**Corollary 18.** Let \( s \in (1, \kappa) \) and \( \beta_1 = 1 \).

(a) If \( \beta_0 \in (-\infty, (23k - 54)/31] \cup [1, +\infty) \), then \( g(s) \) has at most one inflection point.

(b) If \( \beta_0 \in ((23k - 54)/31, 1) \), then \( g(s) \) has at most two inflection points.

**Proof.** Note that the zero of \( g''(s) \) is the maximum or minimum point of \( g''(s) \) and \( g(k) = 0 \). The assertions of this proposition follow from proposition 17.

**Proof of theorem 14.** First we note that \( g(k) = 0 \).

If \( \beta_1 = 1 \), then the statements (a) and (b) follow from corollary 18.

If \( \beta_1 = 0 \), \( \beta_0 \neq 0 \), then lemma 16 shows that \( g''(s) = \beta_0 w''(s) \neq 0 \), which implies that \( g(s) \) has at most one inflection point. This yields the statement (c).

If \( \beta_0 = \beta_1 = 0 \), then \( g(s) = P_2(s) = (s - k)(\alpha_1 + k\alpha_2 + \alpha_2s) \). The assertion (d) follows.

At the end of this section, we prove theorem 1.

**Proof of theorem 1.** Proposition 13 and theorem 14 show that \( I(s) \) has at most five zeros in \((1, \kappa)\). This implies that the perturbed system (2) has at most five limit cycles which emerge from the period annulus around the centre. The second assertion of theorem 1 follows from corollary 9.

**6. Comments**

Zoladek conjectured that the exact upper bound of the cyclicity of the period annulus for \( Q_4 \) is three \([10, 19]\). Unfortunately, we cannot prove Zoladek’s conjecture in this paper.

As mentioned before, the argument principle gives a shorter proof of theorem 1. However, it seems clear to us that it does not allow us to go further in Zoladek’s conjecture. Our approach
is perhaps more involved from the computational point of view, but we think that it may provide a way to attack the problem in a future paper. For instance, we can get the following results from lemma 15, lemma 16 and corollary 18.

1. If \( \beta_1 = 1, \beta_0 < 1, \) \( P_2(\beta_0) \leq 0 \) (respectively \( \beta_0 > \kappa, P_2(\beta_0) \geq 0 \)), then \((P_2(s)/Q_1(s))'' + w''(s) = P_2(\beta_0)/(s - \beta_0)^3 + w''(s) < 0\). This implies that \( g(s) \) has at most one zero in \((1, \kappa)\). Therefore \( I(s) \) has three zeros in the same interval.

2. If \( \beta_1 = 1, \beta_0 < 1, 0 < P_2(\beta_0) < 25(1 - \beta_0)/(432(\kappa - 1)^2) \), then \((P_2(s)/Q_1(s))'' + w''(s) = P_2(\beta_0)/(s - \beta_0)^3 + w''(s) \leq 0\). This yields \( I(s) \) has three zeros in \((1, \kappa)\).

3. If \( \beta_1 = 1, \beta_0 \in (-\infty, (23\kappa - 54)/31) \cup [1, +\infty), P_2(1)g'(\kappa) > 0 \), then \( g(s) \) has at most one zero. Hence \( I(s) \) has at most three zeros.

Here we just list the partial results we have proved. We hope that the above results will be helpful for proving Zoladek’s conjecture in a future paper.

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