The graph of a Weyl algebra endomorphism

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Abstract

Endomorphisms of Weyl algebras are studied using bimodules. Initially, for a Weyl algebra over a field of characteristic zero, Bernstein’s inequality implies that holonomic bimodules finitely generated from the right (respectively, left) form a monoidal category.

The most important bimodule in this paper is the graph of an endomorphism. We prove that the graph of an endomorphism of a Weyl algebra over a field of characteristic zero is a simple bimodule. The simplicity of the tensor product of the dual graph and the graph is equivalent to the Dixmier conjecture.

It is also shown how the graph construction leads to a non-commutative Gröbner basis algorithm for detecting invertibility of an endomorphism for Weyl algebras and computing the inverse over arbitrary fields.

Introduction

Let \(X = \text{Spec}_k(R)\) and \(Y = \text{Spec}_k(S)\) denote affine \(k\)-schemes, where \(k\) is a field and \(R\) and \(S\) commutative \(k\)-algebras. The graph of a \(k\)-morphism \(\varphi : X \to Y\) is the closed subscheme in \(X \times_k Y \cong \text{Spec}(S \otimes_k R)\) given by the ideal \(J\) generated by \(a \otimes 1 - 1 \otimes \varphi^*(a)\) for \(a \in S\), where the \(k\)-algebra homomorphism \(\varphi^* : S \to R\) defines \(\varphi\). The \(S \otimes_k R\)-module \(S \otimes_k R/J\) identifies with the \(R\)-\(S\) bimodule \(R\) with the natural left multiplication, but with right multiplication induced by \(\varphi^*\).

This observation in commutative affine geometry is the basis of an interpretation of the graph in the non-commutative case: the graph of a homomorphism \(f : S \to T\) of general \(k\)-algebras \(S\) and \(T\) is the \(T\)-\(S\) bimodule \(T^f\) defined by \(txs = txf(s)\), where \(t, x \in T\) and \(s \in S\). The graph has several appealing properties, for example, it is finitely generated from the left and a homomorphism is invertible if and only if its graph is invertible as a bimodule.

We first give some general properties of bimodules over Weyl algebras. Let \(A := A_n(k)\) be a Weyl algebra over a field \(k\) of characteristic zero and \(M\) a bimodule over \(A\). Then we prove that \(M\) is holonomic as a bimodule, that is, as a left module over the enveloping algebra \(A^e \cong A_{2n}(k)\) if it is finitely generated from the left or right. If \(N\) is a holonomic bimodule and \(M\) is finitely generated from the right, we show that \(M \otimes_A N\) is a holonomic bimodule. Introducing the \emph{left} and \emph{right rank} of a holonomic bimodule \(M\) over \(A\), we show that the dual bimodule \(\text{Hom}_A(M, A)\) is holonomic. These introductory results are basically all proved using Bernstein’s inequality.

For an endomorphism \(f : A \to A\), we show by yet another invocation of Bernstein’s inequality that the graph \(A^f\) is a simple bimodule. This prompts a question for general algebras \(B\): if \(S \subset B\) is a subalgebra and \(B\) is a simple \(B\)-\(S\) bimodule, is \(S = B\)? If \(B\) is the matrix algebra over an algebraically closed field, this question has an affirmative answer as a consequence of Burnside’s classical theorem. A generalization to infinite-dimensional algebras is not immediate. For the Weyl algebra \(A = A_1(\mathbb{C})\), there are several examples of (intricate) proper subalgebras \(S\), such that \(A\) is a simple \(A\)-\(S\) bimodule.

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The tensor product $\Hom(A^f, A) \otimes_A A^f$ of the dual graph and the graph, is isomorphic to the bimodule $A^f$ introduced by Bavula in [2]. Bavula’s result on holonomicity of $A^f$ is put in the framework of tensor products of bimodules. The simplicity of $A^f$ as a bimodule is equivalent to the Dixmier conjecture that every endomorphism of a Weyl algebra (over a field of characteristic zero) is an automorphism [9, 11.1]. This conjecture is open even for the first Weyl algebra. The Dixmier conjecture for the $n$th Weyl algebra implies the Jacobian conjecture for affine $n$-space [1, p. 297]. Kanel-Belov, Kontsevich [14] and Tsuchimoto [19] have proved, reducing the Weyl algebra modulo prime numbers, that the Jacobian conjecture for affine $2n$-space implies the Dixmier conjecture for the $n$th Weyl algebra.

One may prove that the Dixmier conjecture is true if the graph $A^f$ is finitely generated as a right module [15].

Finally, using the graph construction, a Gröbner basis algorithm emerges for detecting whether an endomorphism of $A_n(k)$ is an automorphism, where $k$ is a general field.

In the non-commutative case, $S$-polynomials of differential operators with relatively prime initial terms do not necessarily reduce to zero. From the bimodule setting, we encounter a sequence of commuting differential operators in $A_{2n}(k)$ with pairwise relatively prime initial terms. In this situation, reduction to zero from the commutative setting generalizes rather easily. To complete the proof of the validity of the algorithm, we need that a surjective endomorphism of a Weyl algebra is an automorphism. This is well known for fields of characteristic zero. For fields of positive characteristic, this invokes reduction to the center.

Even though it is not completely obvious from the current version, this paper was inspired (see also Example 3.7) by computer calculations with Gröbner bases in the Weyl algebra. Experiments with Macaulay2 [13] originally indicated to us some time ago that the left ideal $(x_2 - Q, \partial_2 + P)$ in the second Weyl algebra $A_2(Q)$ in the variables $x_1, x_2, \partial_1, \partial_2$ with $P, Q$ in the subalgebra generated by $x_1$ and $\partial_1$, is proper if and only if $[P, Q] = 1$ (an analogous statement in positive characteristic is false). It only gradually became clear to us that the proper framework for explaining this phenomenon could be found in bimodules and Bernstein’s inequality.

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1. Preliminaries

Let $k$ be a field and $A$ a $k$-algebra. Our focus in this paper is endomorphisms of $k$-algebras and for the sake of simplicity we will only consider bimodules over $A$ in this paper. We will denote by $A^o$ the opposite of $A$, which is $A$ as an abelian group but with multiplication given by $a \cdot b = ba$, where $a, b \in A^o$. Note that right modules over $A$ correspond to left modules over $A^o$.

1.1. Bimodules over a ring

A bimodule over $A$ is a $k$-vector space $M$ with (compatible) left and right $A$-module structures, such that

$$(a_1 m) a_2 = a_1 (ma_2),$$

where $m \in M$ and $a_1, a_2 \in A$.

A homomorphism $f : M \to N$ of bimodules $M$ and $N$ over $A$ is a vector space homomorphism, which is a homomorphism of left and right modules.

Bimodules over $A$ and their homomorphisms form an abelian category equivalent to the category of left modules over the enveloping algebra $A^e := A \otimes_k A^o$. Here a bimodule $M$ over...
A becomes a left $A^e$-module through $(a \otimes b)m = amb$ for $m \in M$ and $a, b \in A$. Let $M$ and $N$ be bimodules over $A$. Then

$$M \otimes_A N, \quad \text{Hom}_A(M, N) \quad \text{and} \quad \text{Hom}_{A^e}(M, N)$$

are bimodules over $A$. Notice that $\text{Hom}_A(M, N)$ (left module homomorphisms) is a bimodule through $(afs)(x) = f(xa)s$ and that $\text{Hom}_{A^e}(M, N)$ (right module homomorphisms) is a bimodule through $(afs)(x) = af(sx)$.

A bimodule $P$ over $A$ is called invertible if $- \otimes_A P$ is an auto-equivalence of the category of right $A$-modules (or equivalently if $P \otimes_A -$ is an auto-equivalence of left $A$-modules). The invertibility of $P$ is equivalent to the existence of a bimodule $Q$ with isomorphisms

$$P \otimes_A Q \cong A \quad \text{and} \quad Q \otimes_A P \cong A,$$

where the natural diagrams commute (see [7, (55.2)]). In this case $Q \cong \text{Hom}_A(P, A)$.

For a $k$-algebra endomorphism $f : A \to A$ and a bimodule $M$ over $A$, we let $M^f$ denote the $A$-bimodule $M$ with its right module structure twisted by $f$, that is, $ams = amf(s)$ for $a, s \in A$ and $m \in M$. Similarly $^fM$ denotes the bimodule $M$ with the left module structure twisted by $f$, that is, $ams = f(a)ms$ for $a, s \in A$ and $m \in M$. The graph of $f$ is defined as $A^f$.

**Proposition 1.1.** Let $f : A \to A$ be an endomorphism and $M$ a bimodule over $A$. Then

(a) $M \otimes_A A^f \cong M^f$

(b) $\text{Hom}_A(A^f, A) \cong ^fA$

(c) $f$ is invertible if and only if $A^f$ is invertible. In this case

$$A^{f^{-1}} \cong \ ^fA$$

(d) If $f$ is invertible, then

$$\text{Ann}_{A^e} 1_{f^{-1}A} = \text{Ann}_{A^e} 1_{A^f}.$$

**Proof.** For (a), the natural map $M \otimes_A A^f \cong M^f$ given by $m \otimes a \mapsto ma$ is an isomorphism of bimodules. In (b), define $\text{Hom}_A(A^f, A) \to ^fA$ by $\varphi \mapsto \varphi(1)$. This is an isomorphism of bimodules, since $(a\varphi s)(1) = \varphi(1a)s = \varphi(f(a))s = f(a)\varphi(1)s$.

In (c), suppose first that $f$ is an invertible. Then $f$ is an isomorphism of bimodules $A^{f^{-1}} \to ^fA \cong \text{Hom}_A(A^f, A)$. The natural isomorphisms $A^{f^{-1}} \otimes_A A^f \to A$ and $A^f \otimes_A A^{f^{-1}} \to A$ coming from (a) show that $A^f$ is invertible. Now suppose that $A^f$ is an invertible bimodule. Then we have a bimodule isomorphism $\varphi : A \to ^fA \cong A^f \otimes_A \text{Hom}_A(A^f, A)$. Since $\varphi(a) = f(a)\varphi(1) = \varphi(1)f(a)$, it follows that $f$ must be injective and surjective and therefore an isomorphism.

The identity in (d) follows from

$$a_1f(b_1) + \cdots + a_mf(b_m) = 0 \iff f^{-1}(a_1)b_1 + \cdots + f^{-1}(a_m)b_m = 0,$$

where $a_1, b_1, \ldots, a_m, b_m \in A$. \hfill \Box

### 1.2. The Weyl algebra

We briefly summarize relevant properties of the Weyl algebra. For proofs and further details, we refer to the monograph [4, Chapter 1] and the textbook [5].
1.2.1. **Arbitrary characteristic.** Let \( k \) be a field of arbitrary characteristic. The Weyl algebra \( A_n(k) \) of order \( n \) over \( k \) is the free algebra on \( \partial_1, \ldots, \partial_n, x_1, \ldots, x_n \) with relations

\[
\begin{align*}
[x_i, x_j] &= 0 \\
[\partial_i, \partial_j] &= 0 \\
[\partial_i, x_j] &= \delta_{ij}
\end{align*}
\] (1.2.1)

for \( i, j = 1, \ldots, n \).

**Proposition 1.2.** The set

\[
M = \{x^\alpha \partial^\beta \mid \alpha, \beta \in \mathbb{N}^n\}
\]

is a vector space basis of \( A_n(k) \) over \( k \), where \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) and \( \partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n} \) for \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \beta = (\beta_1, \ldots, \beta_n) \) in \( \mathbb{N}^n \).

**Proof.** See [12, §2, Lemma 3]\( \square \)

The degree of a monomial \( x^\alpha \partial^\beta \in M \) is \(|\alpha| + |\beta|\), where \(|\alpha| = \alpha_1 + \cdots + \alpha_n \) and \(|\beta| = \beta_1 + \cdots + \beta_n \) with \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \beta = (\beta_1, \ldots, \beta_n) \) both in \( \mathbb{N}^n \). The degree, \( \deg(f) \), of \( f \in A_n(k) \setminus \{0\} \) is the maximum degree of the monomials occurring with non-zero coefficient in the expansion of \( f \) in the basis (1.2.2). For \( f, g \in A_n(k) \setminus \{0\} \), \( \deg(fg) = \deg(f) + \deg(g) \).

The map \( \alpha : A_n(k) \otimes_k A_n(k) \to A_{2n}(k) \) given by

\[
\begin{align*}
\alpha(x_i \otimes 1) &= x_i, & \alpha(1 \otimes x_i) &= x_{n+i} \\
\alpha(\partial_i \otimes 1) &= \partial_i, & \alpha(1 \otimes \partial_i) &= \partial_{n+i}
\end{align*}
\] (1.2.3)

for \( i = 1, \ldots, n \), is an isomorphism of \( k \)-algebras.

The map \( \tau : A_n(k) \to A_n(k)^\alpha \) given by

\[
\tau(x_i) = x_i, \quad \tau(\partial_i) = -\partial_i,
\]

for \( i = 1, \ldots, n \) defines an isomorphism of \( k \)-algebras. Combined, (1.2.3) and (1.2.4) give an equivalence between \( A_n(k) \)-bimodules and left \( A_n(k)^\tau = A_n(k) \otimes_k A_n(k)^\alpha \cong A_{2n}(k) \)-modules.

1.2.2. **Characteristic zero.** Let \( k \) be a field of characteristic zero. The increasing sequence \( k = B_0^\alpha \subset B_1^\alpha \subset \cdots \) of finite-dimensional subspaces given by

\[
B_m^\alpha = \text{Span}_k \{x^\alpha \partial^\beta \mid \deg(x^\alpha \partial^\beta) \leq m\} \subset A_n(k)
\]

is called the Bernstein filtration of \( A_n(k) \). Clearly \( \cup_i B_i^\alpha = A_n(k) \) and \( B_i^\alpha B_j^\alpha \subset B_{i+j}^\alpha \) for \( i, j \in \mathbb{N} \).

Furthermore, \( \text{Gr}_{B}(A_n(k)) = B_0^\alpha \oplus B_1^\alpha / B_0^\alpha \oplus \cdots \) is the commutative polynomial ring over \( k \) in the \( 2n \) variables \( [x_1], \ldots, [x_n], [\partial_1], \ldots, [\partial_n] \) in \( B_i^\alpha / B_0^\alpha \).

A filtration \( \Gamma \) of a left module \( M \) over \( A_n(k) \) is an increasing sequence \( \Gamma_0 \subset \Gamma_1 \subset \cdots \) of finite-dimensional subspaces of \( M \), such that \( \cup_i \Gamma_i = M \) and \( B_i^\alpha \Gamma_j \subset \Gamma_{i+j} \) for \( i, j \in \mathbb{N} \).

Such a filtration is called good if \( \text{Gr}_{\Gamma}(M) = \Gamma_0 \oplus \Gamma_1 / \Gamma_0 \oplus \cdots \) is finitely generated as a \( \text{Gr}_{B}(A_n(k)) \)-module. A left module with a good filtration is finitely generated. If \( M \) is finitely generated by \( m_1, \ldots, m_r \in M \), then \( \Gamma_i = B_i^\alpha m_1 + \cdots + B_i^\alpha m_r \) is a good filtration of \( M \).

For a good filtration \( \Gamma \) of a left module \( M \), there exists a polynomial \( p = \frac{1}{d} x^d + \cdots \in \mathbb{Q}[x] \) with \( e \in \mathbb{N} \), such that

\[
\dim_k \Gamma_i = p(i) \quad \text{for } i \gg 0.
\]

The degree \( d \) and the leading coefficient of this polynomial are independent of the good filtration of \( M \). For a module \( M \) with a good filtration, we denote \( \dim(M) := d \) the dimension of \( M \) and \( e(M) := e \) the multiplicity of \( M \). The following important result is called Bernstein’s inequality.
Theorem 1.3 (Bernstein). If $M$ is a finitely generated non-zero left module over $A_n(k)$, then $\dim(M) \geq n$.

A finitely generated left module $M$ over $A_n(k)$ is called holonomic if $M = 0$ or $M \neq 0$ and $\dim(M) = n$. Submodules and quotient modules of holonomic modules are holonomic. If $0 \to N \to M \to M/N \to 0$ is a short exact sequence of holonomic modules, then

$$e(M) = e(N) + e(M/N).$$

We have the following analogue [2, Theorem 2.5] of a classical result due to Bernstein [3, Corollary 1.4].

Theorem 1.4. Let $M$ be a left module over $A_n(k)$. If $M$ has a filtration $M_v$ with

$$\dim_k M_v \leq av^n + \text{lower order terms}$$

with $a > 0$, then $M$ is holonomic.

Proof. Let $N$ be a non-zero finitely generated submodule of $M$ and let $N_0$ be a finite-dimensional generating subspace. If $N_0 \subset M_j$, then

$$N_i := B^n_i N_0 \subset B^n_i M_j \subset M_{i+j}$$

and therefore

$$\dim_k N_i \leq \dim_k M_{i+j} = a(i+j)^n + \text{lower degree terms}.$$

Theorem 1.3 implies that $N$ is holonomic with multiplicity bounded by $n!a$. By (1.2.5), this implies that $M$ has a maximal finitely generated submodule, which has to be $M$ itself. □

2. Bimodules over the Weyl algebra

In this chapter, $k$ denotes a field of characteristic zero. A bimodule over the Weyl algebra $A_n(k)$ is called holonomic if it is holonomic as a left module over the Weyl algebra $A_{2n}(k)$ through the isomorphism $A_{2n}(k) \cong A_n(k) \otimes_k A_n(k) \circ$ given by (1.2.3) and (1.2.4).

2.1. Finite generation from the left or right

The following theorem underlies the theoretical explanation of several results in this paper.

Theorem 2.1. Let $M$ be a bimodule over $A_n(k)$.

(i) If $M$ is finitely generated as a left or right module over $A_n(k)$, then $M$ is holonomic.

(ii) If $M$ is cyclic as a left (respectively, right) $A_n(k)$-module, then $M$ is free as a left (respectively, right) module and simple as a bimodule.

Proof. We will give the proof of (i) and (ii) with respect to the left module structure. The proof for the right module structure is similar.

Let $m_1, \ldots, m_r$ denote a set of generators of $M$ as a left module over $A_n(k)$. For $a \in A_n(k)$,

$$m_ia = \sum_{j=1}^r a_{ij}m_j$$

(2.1.1)
with $a_{ji} \in A_n(k)$ and $i = 1, \ldots, r$. With $a$ in (2.1.1), we put $d(a) = \max\{\deg(a_{ji}) \mid i, j = 1, \ldots, r\}$. Let $D = \max\{d(a) \mid a = x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\}$. Then

$$M_i = \sum_{j=1}^{r} B_{ij}^{2n} m_j \subseteq \sum_{j=1}^{r} B_{iD}^{n} m_j,$$

(2.1.2)

and $M_i$ is a good filtration of $M$. Since

$$\dim_k(M_j) \leq r \cdot \dim_k(B_{jD}^{n}) = rD^{2n} \left(\frac{2n!}{2n} \right) + \text{lower degree terms in } j,$$

(2.1.3)

it follows by Theorem 1.3 that $M$ is holonomic.

For the proof of (ii), let $m$ be a generator of $M$ as a left module over $A$. By (i) we know that $M$ is a holonomic bimodule. Consider the surjective map

$$f : A_n \to M$$

of left modules given by $f(a) = am$. Let $J$ denote the kernel $\text{Ker}(f)$ of $f$. With $M_j$ denoting $B_{jD}^{n} m$, we obtain, using similar arguments as in the proof of (i), that $M_j$ is contained in $B_{jD}^{n} m$, for some integer $D$. Thus

$$\dim_k(M_j) \leq \dim_k(B_{jD}^{n} [1]),$$

where $[1] = 1 + J \in A/J$. If $J \neq 0$, then

$$\dim_k(B_{jD}^{n} [1]) = cj^d + \text{lower degree terms in } j,$$

with $c > 0$ and $d \leq 2n - 1$ (cf. [5, Chapter 9, 3.5 COROLLARY]). This contradicts Theorem 1.3 for the bimodule $M$.

If $0 \subsetneq N \subsetneq M$ is a proper sub-bimodule, then $M/N$ is a non-zero cyclic left module with torsion contradicting what we just proved. Therefore $M$ is irreducible as a bimodule if it is cyclic as a left module. □

2.2. The tensor product

THEOREM 2.2. Let $M$ and $N$ be bimodules over $A_n(k)$. If $M$ is finitely generated as a right module and $N$ is holonomic, then the bimodule

$$M \otimes_{A_n(k)} N$$

is holonomic.

Proof. Choose generators $m_1, m_2, \ldots, m_r$ of $M$ as a right $A_n(k)$-module. Similar to the proof of Theorem 2.1, we may find an integer $D$, such that

$$M_i = \sum_{j=1}^{r} B_{ij}^{2n} m_j = \sum_{j=1}^{r} m_j B_{iD}^{n}.$$

(2.2.1)

Let $\{N_j\}_{j \geq 0}$ denote a good filtration of $N$, and consider the filtration $\{T_s\}_{s \geq 0}$ of $T = M \otimes_{A_n(k)} N$, defined by letting $T_s$ denote the $k$-span of elements of the form $m \otimes n$, with $m \in M_i$ and $n \in N_j$, with $i + j \leq s$. By (2.2.1), $T_s$ is contained in the subspace spanned by

$$m_j \otimes \hat{n},$$

where $\hat{n} \in N_{sD}$. In particular,

$$\dim_k(T_s) \leq r \dim_k(N_{sD}).$$

The right-hand side of the inequality is polynomial of degree $2n$ in $s$, and $T$ is holonomic by Theorem 1.4. □
2.3. The left and right rank of a bimodule

**Proposition 2.3.** Let $M$ be a holonomic bimodule over $A_n(k)$. If $m_1, \ldots, m_r \in M$ are left or right linearly independent over $A_n(k)$, then
\[ r \leq e(M). \]

**Proof.** We give the proof for left linearly independent elements. The right linear independence is similar. Let $m_1, m_2, \ldots, m_r$ denote elements in $M$ which are linearly independent under the left action of $A_n(k)$. As holonomic modules are cyclic, we may choose an element $m \in M$ generating $M$ as a bimodule over $A_n(k)$.

The elements $m, m_1, \ldots, m_r \in M$ then generate $M$ as a bimodule over $A_n(k)$. Therefore
\[ M_i = B_{2n}^i m + B_{2n}^i m_1 + \cdots + B_{2n}^i m_r \]
defines a good filtration of $M$. Using the left action, we define the subspace
\[ N_i = B_{2n}^i m_1 + \cdots + B_{2n}^i m_r \subset M_i. \]

Here
\[ \dim_k(N_i) = r \dim_k(B_{2n}^i) = r \binom{2n + i}{2n} = r \frac{i^{2n}}{2n!} + \text{lower degree terms in } i, \]
since $m_1, \ldots, m_r$ are left linearly independent. Since $M$ is holonomic and $\dim_k N_i \leq \dim_k M_i$, this implies by comparing leading terms that $r \leq e(M)$. □

This leads to the following definition.

**Definition 2.4.** Let $M$ be a holonomic bimodule over the Weyl algebra $A_n(k)$. Then the left (right) rank $\text{lrk}(M)$ ($\text{rrk}(M)$) of $M$ is defined as the rank of the largest free left (right) submodule in $M$.

**Theorem 2.5.** Let $A = A_n(k)$ and let $M$ be a holonomic bimodule over $A$. Then $\text{Hom}_A(M, A)$ (respectively, $\text{Hom}_{A^o}(M, A)$) is finitely generated as a right (respectively, left) module and a holonomic bimodule over $A$.

**Proof.** We will give the proof for $\text{Hom}_A(M, A)$. The proof for $\text{Hom}_{A^o}(M, A)$ is similar.

Let $m_1, \ldots, m_r$ denote a basis of a maximal free left submodule $F$ of $M$, where $r = \text{lrk}(M)$. The map
\[ \alpha : \text{Hom}_A(M, A) \to A^r \quad (2.3.1) \]
given by
\[ \alpha(f) = (f(m_1), \ldots, f(m_r)) \]
is a homomorphism of right modules over $A$. We claim that $\alpha$ is injective. If $\alpha(f) = 0$, then $f$ defines a homomorphism $\overline{f} : M/F \to A$. Since $F$ is a maximal free submodule, $M/F$ is a torsion module. This implies that $\overline{f} = 0$, since $A$ has no zero divisors. Therefore $f = 0$ and $\alpha$ is injective.

Finally, $A$ being right Noetherian implies by the injectivity of $\alpha$ in (2.3.1) that the dual bimodule is finitely generated as a right $A$-module. Now holonomicity follows from Theorem 2.1. □

2.4. The graph of a Weyl algebra endomorphism

An immediate application of Theorem 2.1 is the following.
Corollary 2.6. Let $k$ be a field of characteristic zero and let $A = A_n(k)$ denote the $n$th Weyl algebra over $k$. If $f$ is an endomorphism of $A$, then the graph $A^f$ and the dual graph $f^!A$ are holonomic and simple bimodules over $A$.

Proof. This follows from part (ii) of Theorem 2.1, since $A^f$ is cyclic as a left module and $f^!A$ is cyclic as a right module. □

We also obtain the following result due to Bavula [2, Corollary 1.4].

Corollary 2.7. Let $f : A \to A$ be an endomorphism. Then the bimodule

$$f^!A^f = f^!A \otimes_A A^f$$

is holonomic.

Proof. This is a consequence of Corollary 2.6 and Theorem 2.2. □

A famous question (cf. [9, 11. Problèmes 11.1]) due to Dixmier asks if an endomorphism $f$ of the first Weyl algebra $A_1(k)$ over a field $k$ of characteristic zero is an automorphism. This is equivalent to $f(A_1(k)) = A_1(k)$, since an endomorphism of the Weyl algebra is injective.

One may ask the same question for the $n$th Weyl algebra. This is equivalent to the simplicity of the bimodule tensor product of the two simple modules in (2.4.1), since $f(A)$ is a sub-bimodule of $f^!A^f$ (cf. [2, p. 686]).

2.5. Burnside’s theorem on matrix algebras

Based on the simplicity of the graph (Corollary 2.6), the following question emerges.

If $S$ is a subalgebra of an algebra $A$ and $A$ is a simple $A$–$S$ bimodule, is $S = A$?

Phrased in this generality, the answer is no. However, for a matrix algebra $A$ over an algebraically closed field, the question has a positive answer. This is related to Burnside’s classical theorem [8, (27.4)] that every proper subalgebra of $A$ has an invariant subspace.

Theorem 2.8. Let $k$ be an algebraically closed field and $S$ a subalgebra of $A = \text{Mat}_n(k)$. If $A$ is simple as an $A$–$S$ bimodule, then $S = A$.

Proof. Let $v$ be a non-zero row vector in $k^n$ and $f \in A$ the matrix with rows $v$. Then by assumption $AfS = A$. Let $I(vS)$ be the left ideal of matrices in $A$ whose row space is contained in the subspace $vS \subset k^n$. Clearly, $fS \subset I(vS)$. Therefore $I(vS) = A = I(k^n)$ and $vS = k^n$. The classical Burnside theorem on matrix algebras now implies $S = A$. □

We give three examples showing that this result does not, however, generalize easily to the infinite-dimensional case. In each of the examples, $k$ denotes an algebraically closed field of characteristic zero. Example 2.10 was communicated to us by Søren Jøndrup.

Example 2.9. Let $S$ be the subalgebra of $A = A_1(k)$ given by $S = k + xA$. It follows from Proposition 1.2 that $\partial \notin S$ and therefore $S \neq A$. For $f \in A \setminus \{0\}$, $AfS \supset AfxA = A$, since $A$ is a simple ring and $x$ is not a zero divisor. Therefore $S$ is a proper subalgebra of $A$, but $A$ is a simple $A$–$S$ bimodule.
**Example 2.10.** Consider the quantum plane $A_q = k[x, y]/(xy - qyx)$, where $q \in k \setminus \{0\}$ is not a root of unity. Localizing $A_q$ in powers of $x$ and $y$, we get the quantum torus

$$T_q = k[x^{\pm 1}, y^{\pm 1}]/(xy - qyx) = \left\{ \sum_{i,j} \lambda_{ij} x^i y^j \mid \lambda_{ij} \in k, i, j \in \mathbb{Z} \right\},$$

which is a simple Noetherian ring. The subalgebra $S$ generated by $x^{-2}, y^{-2}, x^2, y^2$ is the subset of Laurent polynomials in even powers of $x$ and $y$. This subalgebra is proper and simple. Furthermore, $T_q$ is a simple $T_q$-bimodule. To see this, consider first

$$f = a_0 + a_1 y + \cdots + a_n y^n \in T_q,$$

with $n > 0$ and $a_i \in k$ for $i = 0, \ldots, n$, with $a_0 \neq 0$. Then

$$q^{2n} f - x^2 f x^{-2}$$

is of the form (2.5.1) but of degree $n - 1$ and non-zero constant term $(q^{2n} - 1)a_0$. By induction, we have $T_q f S = T_q$. To reduce to the case (2.5.1), it suffices to consider

$$g = b_0 + b_1 x + \cdots + b_n x^n \in T_q,$$

with $n > 0$, $b_i \in k[y^{\pm 1}]$ for $i = 0, \ldots, n$ and $b_0 \neq 0$. Then $q^{2n} g - y^{-2} gy^2$ is of the form (2.5.2) of degree $n - 1$ and non-zero constant term $(q^{2n} - 1)b_0$. By induction and multiplication by a power of $y$ from the left, we have reduced to (2.5.1) and therefore $T_q g S = T_q$.

The subalgebra $S$ in Example 2.9 is not simple, since $xA \subset S$ is a non-trivial two-sided ideal. Like the quantum torus in Example 2.10, the Weyl algebra $A$ in Example 2.9 also affords proper simple subalgebras $S$, such that $A$ is a simple $A$-$S$ bimodule.

**Example 2.11.** This example can be traced back to [10]. Let $U$ denote the enveloping algebra for the semisimple Lie algebra $\mathfrak{sl}_2(k)$. Recall that $U$ is generated by $E$, $F$ and $H$ with the relations

$$[E, F] = H, \quad [H, E] = 2E \quad \text{and} \quad [H, F] = -2F.$$

The center of $U$ is generated by the Casimir element $Q = 4FE + H^2 + 2H$. It is known that the primitive ideals in $U$ all have the form $I_c = U(Q - c)$ for $c \in k$ and that $U/I_c$ is a simple ring for $c \neq n^2 + 2n$ with $n = 0, 1, 2, \ldots$. There is an embedding (cf. [10, Remarques 7.2]) $U/I_c \to A_1(k)$ given by

$$E \mapsto -\mu x - \partial x^2, \quad F \mapsto \partial \quad \text{and} \quad H \mapsto \mu + 2\partial x$$

for $c = \mu^2 + 2\mu$. We let $S$ denote the image of this embedding for $c = \mu = -1$. Notice that $S$ is a simple subalgebra containing $\partial$.

In general, if $R$ is a simple subalgebra of $T$ and $TfR \cap R \neq \{0\}$ for $f \in T \setminus \{0\}$, then $TfR = T$. The fact that $\partial \in S$ implies that $A\{S \cap S \neq \{0\}$ for $f \in A \setminus \{0\}$, since $\text{ad}(\partial)^n(f) \in S \setminus \{0\}$ for some $n \geq 0$. This shows that $A_1(k)$ is a simple $A_1(k)$-$S$ bimodule. Furthermore, $M = x^{-1}k[x^{-1}]$ is an $S$-stable subset for the natural action of $A_1(k)$ on $k[x, x^{-1}]$ corresponding to the Verma module with highest weight $-1$ for $\mathfrak{sl}_2(k)$. Since $xM \not\subset M$, we get $x \notin S$ and $S$ is a proper subalgebra. One may prove that $A$ is not finitely generated as a right or left $S$-module. Furthermore $A$ is not flat as an $S$-module [18, 7, Remark, 2]. This should be compared with the flatness of endomorphisms of Weyl algebras proved in [16].
3. A Gröbner basis algorithm for the inverse

Let $A = A_n(k)$ be the $n$th Weyl algebra over an arbitrary field $k$. Gröbner basis theory for the Weyl algebra was initiated by Galligo [11] and later extended to a wider class of non-commutative algebras with monomial bases by several authors. Here we will focus on the Weyl algebra and briefly recall the theory, which is rather close to the classical theory of Gröbner bases for the commutative polynomial ring in $n$ variables.

3.1. Gröbner basics for the Weyl algebra

Let $\prec$ be a term order on the monomials $M = \{x^\alpha \partial^\beta \mid \alpha, \beta \in \mathbb{N}^n\} \subset A$, that is,

\begin{equation}
\prec \text{ is a total order},
\end{equation}

\begin{equation}
1 \prec x^\alpha \partial^\beta,
\end{equation}

\begin{equation}
x^\alpha \partial^\beta \prec x^{\alpha'} \partial^{\beta'} \Rightarrow x^{\alpha+\gamma} \partial^{\beta+\delta} \prec x^{\alpha'+\gamma} \partial^{\beta'+\delta},
\end{equation}

for every $\alpha, \beta, \alpha', \beta', \gamma, \delta \in \mathbb{N}^n$. The support, $\text{supp}(f)$, of a differential operator

\begin{equation}
f = \sum_{(\alpha, \beta) \in \mathbb{N}^{2n}} \lambda_{\alpha, \beta} x^\alpha \partial^\beta \in A \setminus \{0\}
\end{equation}

is the set of monomials $x^\alpha \partial^\beta$ in (3.1.4) with $\lambda_{\alpha, \beta} \neq 0$. The leading monomial $\text{lm}(f)$ of $f$ is the maximal monomial in $\text{supp}(f)$. If $\text{lm}(f) = x^u \partial^v$, then $\text{lt}(f) := \lambda_{u,v} x^u \partial^v$ is called the leading term and $\text{lc}(f) := \lambda_{u,v}$, the leading coefficient of $f$.

Let $\leq$ denote the partial order on $\mathbb{N}^{2n}$ given by $(u_1, \ldots, u_{2n}) \leq (v_1, \ldots, v_{2n})$ if and only if $u_1 \leq v_1 \land \cdots \land u_{2n} \leq v_{2n}$. This partial order is transferred to $M$ and reads $m_1 \leq m_2$ if and only if $m_1$ divides $m_2$ as commutative monomials.

If $\text{lm}(d) = x^\alpha \partial^\beta \leq \text{lm}(f) = x^{\alpha'} \partial^{\beta'}$ for $f, d \in A$, then

\begin{align*}
f - qd &= 0 \quad \text{or} \\
\text{lm}(f - qd) &\preceq \text{lm}(f),
\end{align*}

where $q = \text{lc}(d)^{-1} \text{lc}(f) x^{\alpha' - \alpha} \partial^{\beta' - \beta}$. This is basically a consequence of (3.1.3) and the identity

\begin{equation}
(x^\alpha \partial^\beta)(x^{\alpha'} \partial^{\beta'}) = x^{\alpha + \alpha'} \partial^{\beta + \beta'} + \sum_{i \geq 1} c_i x^{\alpha + \alpha' - i} \partial^{\beta + \beta' - i}, \quad c_i \in k
\end{equation}

for multiplication of monomials in the first Weyl algebra. An element $f \in A \setminus \{0\}$ is called reducible by a finite subset $S \subset A$ if there exists $m \in \text{supp}(f)$ with $\text{lm}(g) \leq m$ for some $g \in S \setminus \{0\}$.

These observations allow for the commutative theory to be carried over almost verbatim, that is, the division algorithm, the $S$-polynomial, Buchberger’s $S$-criterion, and uniqueness of reduced Gröbner bases etc. The computations, however, are severely complicated by the ‘non-commutative’ multiplication (3.1.5).

Let $AS$ denote the left ideal generated by $S$, where $S \subset A$.

**Definition 3.1.** (i) A finite subset $G \subset A$ is called a Gröbner basis (with respect to $\prec$) if for every $f \in AG \setminus \{0\}$, there exists $d \in G$ with

\begin{equation}
\text{lm}(d) \leq \text{lm}(f).
\end{equation}

(ii) A Gröbner basis $G = \{f_1, \ldots, f_r\}$ is called reduced if $f_i \neq 0, \text{lc}(f_i) = 1$ and $f_i$ is not reducible by $G \setminus \{f_i\}$ for $i = 1, \ldots, n$.

(iii) A Gröbner basis $G$ is called a Gröbner basis for a left ideal $I \subset A$ if $I = AG$. 
Theorem 3.2. Let $I \subset A$ be a non-zero left ideal and $\prec$ a term order. There exists a unique reduced Gröbner basis for $I$ with respect to $\prec$.

We briefly recall [6, §2.9].

Definition 3.3. For a finite subset $G = \{g_1, \ldots, g_m\} \subset A$, we say that $f \in A$ reduces to zero modulo $G$ (denoted $f \rightarrow_G 0$) if

$$f = a_1 g_1 + \cdots + a_m g_m$$

for $a_1, \ldots, a_m \in A$ and $\text{lm}(a_i g_i) \prec \text{lm}(f)$ for every $i = 1, \ldots, m$ with $a_i g_i \neq 0$.

The $S$-polynomial of $f, g \in A \setminus \{0\}$ is

$$S(f, g) := \text{lc}(g)m_f(g)f - \text{lc}(f)m_g(f)g,$$

where $m_p(q)$ is the monomial given by the least common multiple of $\text{lm}(p)$ and $\text{lm}(q)$ divided by $\text{lm}(p)$ for $p, q \in A \setminus \{0\}$. In particular, $\text{lt}(\text{lc}(g)m_f(g)f) = \text{lt}(\text{lc}(f)m_p(f)g)$.

Theorem 3.4. A finite subset $G \subset A$ is a Gröbner basis if and only if $S(f, g) \rightarrow_G 0$ for every $f, g \in G$.

The following simple example illustrates a significant difference from commutative Gröbner bases.

Example 3.5. The subset $S = \{\partial, x\} \subset A_1(k)$ is not a Gröbner basis, since $\partial x - x \partial = 1 \in A_1(k)S$ and $x \not\equiv 1, \partial \not\equiv 1$.

The monomials $x$ and $\partial$ in Example 3.5 are relatively prime, but they do not commute. A remedy for this situation is given by the following result.

Theorem 3.6. A set $G = \{f_1, \ldots, f_r\} \subset A$ of commuting differential operators, that is, $f_i f_j = f_j f_i$ for $i, j = 1, \ldots, r$ is a Gröbner basis if their leading monomials, $\text{lm}(f_i)$ and $\text{lm}(f_j)$ are relatively prime for $i \neq j$.

Proof. The proof of [6, §2.9, Proposition 4] carries over verbatim: to show that $S(f_i, f_j) \rightarrow_G 0$, one only needs $f_i f_j = f_j f_i$ along with $\text{lm}(f_i)$ and $\text{lm}(f_j)$ being relatively prime. □

Example 3.7. The following example illustrates the previous definitions and results in the context that lead to this article. Let $k$ be any field and consider the second Weyl algebra $A_2(k)$ in the variables $x_1, x_2, \partial_1, \partial_2$ with relations

$$[\partial_1, x_1] = 1$$
$$[\partial_2, x_2] = 1$$
$$[x_1, x_2] = 0$$
$$[\partial_1, \partial_2] = 0.$$

The first Weyl algebra $A_1(k)$ in the variables $x_1, \partial_1$ with the relation $[\partial_1, x_1] = 1$ embeds naturally into $A_2(k)$. Suppose that $p, q \in A_1(k)$ with $[p, q] = 1$ and consider the left ideal

$$I = (x_2 - q, \partial_2 + p) \subset A_2(k).$$
If we fix a term order $\prec$ on $A_2(k)$ with the property that monomials containing $x_2$ or $\partial_2$ are greater than monomials in $x_1$ and $\partial_1$ (this could be the lexicographic order determined by $x_1 \prec \partial_1 \prec x_2 \prec \partial_2$), then
\[ G = \{ x_2 - q, \partial_2 + p \} \]
is a Gröbner basis for $I$ by Theorem 3.6, since $\lfloor x_2 - q, \partial_2 + p \rfloor = 0$, $\text{lcm}(x_2 - q) = x_2$ and $\text{lcm}(\partial_2 + q) = \partial_2$. However if $\prec$ is replaced by $\prec'$, where monomials containing $x_1$ or $\partial_1$ are greater than monomials in $x_2$ and $\partial_2$ (this could be the lexicographic term order given by $x_2 \prec' \partial_2 \prec' x_1 \prec' \partial_1$), then $G$ is not necessarily a Gröbner basis for $I$ (with respect to $\prec'$).

If the endomorphism $\varphi$ of $A_1(k)$ given by $\varphi(x_1) = q$ and $\varphi(\partial_1) = p$ is invertible, then $\varphi^{-1}$ can be read off the reduced Gröbner basis for $I$ with respect to the term order $\prec'$. This is explained in the following two sections.

3.2. From the graph to the left ideal

**Lemma 3.8.** Let $k$ be an arbitrary field and $A = A_n(k)$. A surjective endomorphism $f : A \to A$ is an automorphism.

*Proof.* If $k$ is a field of characteristic zero, $A$ is a simple ring. Therefore $f$ is injective in this case. Let $k$ be a field of positive characteristic $p > 0$. Then the center of $A$ is the commutative polynomial ring
\[ C = k[x_1^p, \ldots, x_n^p, \partial_1^p, \ldots, \partial_n^p] \]
in $2n$ variables and $A$ is a free module over $C$ with the two bases (cf. §3.2 and [20, §3.3] and the Nousiainen lemma in [17])
\begin{align*}
\{ x_1^{i_1} \cdots x_n^{i_n} \partial_1^{j_1} \cdots \partial_n^{j_n} | 0 \leq i_1, \ldots, i_n, j_1, \ldots, j_n \leq p - 1 \} \\
\{ f(x_1)^{i_1} \cdots f(x_n)^{i_n} f(\partial_1)^{j_1} \cdots f(\partial_n)^{j_n} | 0 \leq i_1, \ldots, i_n, j_1, \ldots, j_n \leq p - 1 \}.
\end{align*}
(3.2.1)
The endomorphism $f$ restricts to an endomorphism $\overline{f} : C \to C$, since $f(C) \subset C$. Applying $f$ in terms of the two bases in (3.2.1), it follows that $\overline{f}$ is surjective, since $f$ is surjective. Therefore $\overline{f}$ and hence $f$ is an automorphism. \qed

**Remark 3.9.** Let $k$ be a field of positive characteristic. An endomorphism of the first Weyl algebra $A_1(k)$ is always injective. Surprisingly, endomorphisms of $A_n(k)$ fail to be injective for $n \geq 2$ (cf. [17, p. 793]).

**Theorem 3.10.** Let $k$ be an arbitrary field and $A = A_n(k)$. For an endomorphism,
\[ f : A \to A, \]
we define the left ideal
\[ J = (1 \otimes x_1 - f(x_1) \otimes 1, \ldots, 1 \otimes x_n - f(x_n) \otimes 1, 1 \otimes \partial_1 - f(\partial_1) \otimes 1, \ldots, 1 \otimes \partial_n - f(\partial_n) \otimes 1) \subset A^e. \]
This left ideal equals the annihilator $\text{Ann}_{A^e} 1_{A^f}$.

(i) If $f$ is invertible, then
\[ J = (x_1 \otimes 1 - 1 \otimes f^{-1}(x_1), \ldots, x_n \otimes 1 - 1 \otimes f^{-1}(x_n), \partial_1 \otimes 1 - 1 \otimes f^{-1}(\partial_1), \ldots, \partial_n \otimes 1 - 1 \otimes f^{-1}(\partial_n)) \subset A^e. \]
(ii) If

\[ J = (x_1 \otimes 1 - 1 \otimes q_1, \ldots, x_n \otimes 1 - 1 \otimes q_n), \]
\[ \partial_1 \otimes 1 - 1 \otimes p_1, \ldots, \partial_n \otimes 1 - 1 \otimes p_n) \subset A^e, \]

then \( f \) is invertible with

\[ f^{-1}(x_1) = q_1 \]
\[ \vdots \]
\[ f^{-1}(x_n) = q_n \]
\[ f^{-1}(\partial_1) = p_1 \]
\[ \vdots \]
\[ f^{-1}(\partial_n) = p_n. \]

Proof. Let us prove that

\[ \text{Ann}_{A^e} 1_{A^e} = (f(x_1) \otimes 1 - 1 \otimes x_1, \ldots, f(x_n) \otimes 1 - 1 \otimes x_n, \]
\[ f(\partial_1) \otimes 1 - 1 \otimes \partial_1, \ldots, f(\partial_n) \otimes 1 - 1 \otimes \partial_n). \] (3.2.2)

If \( a_1 f(b_1) + \cdots + a_m f(b_m) = 0 \), for \( a_1, b_1, \ldots, a_m, b_m \in A \), then

\[ a_1 \otimes b_1 + \cdots + a_m \otimes b_m = - (a_1 \otimes 1)(f(b_1) \otimes 1 - 1 \otimes b_1) - \cdots - (a_m \otimes 1)(f(b_m) \otimes 1 - 1 \otimes b_m). \]

This shows that \( \text{Ann}_{A^e} 1_{A^e} \) is generated by \( \{f(a) \otimes 1 - 1 \otimes a \mid a \in A\} \). The identity

\[ f(ab) \otimes 1 - 1 \otimes (ab) = f(a)f(b) \otimes 1 - 1 \otimes (ab) \]
\[ = (f(a) \otimes 1)(f(b) \otimes 1 - 1 \otimes b) + (1 \otimes b)(f(a) \otimes 1 - 1 \otimes a) \]

is verified using \( A^e = A \otimes_k A^o \) and shows (3.2.2) inductively as \( x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \) generate \( A \). An analogous argument gives

\[ \text{Ann}_{A^e} 1_{A^e} \gamma_A = (x_1 \otimes 1 - 1 \otimes f^{-1}(x_1), \ldots, x_n \otimes 1 - 1 \otimes f^{-1}(x_n), \]
\[ \partial_1 \otimes 1 - 1 \otimes f^{-1}(\partial_1), \ldots, \partial_n \otimes 1 - 1 \otimes f^{-1}(\partial_n)). \]

Therefore (i) is a consequence of (d) in Proposition 1.1.

From

\[ J = \text{Ann}_{A^e} 1_{A^e} = (x_1 \otimes 1 - 1 \otimes q_1, \ldots, x_n \otimes 1 - 1 \otimes q_n, \]
\[ \partial_1 \otimes 1 - 1 \otimes p_1, \ldots, \partial_n \otimes 1 - 1 \otimes p_n) \subset A^e, \]

it follows that

\[ x_1 = f(q_1) \]
\[ \vdots \]
\[ x_n = f(q_n) \]
\[ \partial_1 = f(p_1) \]
\[ \vdots \]
\[ \partial_n = f(p_n). \]

Now (ii) is consequence of Lemma 3.8. □
3.3. The Gröbner basis in $A_{2n}(k)$

In Theorem 3.11, Theorem 3.10 is translated into statements about Gröbner basis in $A_{2n}(k) \cong A^e$ using the isomorphisms in (1.2.3) and (1.2.4). This involves viewing $f$ as an endomorphism of the Weyl algebra in the variables $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ and the potential inverse as an endomorphism of the Weyl algebra in the variables $x_{n+1}, \ldots, x_{2n}, \partial_{n+1}, \ldots, \partial_{2n}$.

**Theorem 3.11.** Let $k$ be an arbitrary field and $A = A_n(k)$. For an endomorphism $f : A \rightarrow A$, we define the left ideal

$$J = (x_{n+1} - f(x_1), \ldots, x_{2n} - f(x_n), \partial_{n+1} + f(\partial_1), \ldots, \partial_{2n} + f(\partial_n))$$

in $A_{2n}(k)$.

(i) If $f$ is invertible, then

$$J = (x_1 - \tau f^{-1}(x_{n+1}), \ldots, x_n - \tau f^{-1}(x_{2n}), \partial_1 - \tau f^{-1}(\partial_{n+1}), \ldots, \partial_n - \tau f^{-1}(\partial_{2n})),$$

where $\tau$ is the antihomomorphism given by $\tau(x_{n+i}) = x_{n+i}$ and $\tau(\partial_{n+i}) = -\partial_{n+i}$.

The generators of $J$ given above form a commuting set of differential operators.

(ii) If

$$J = (x_1 - q_1, \ldots, x_n - q_n, \partial_1 + p_1, \ldots, \partial_n + p_n)$$

with $p_1, \ldots, p_n, q_1, \ldots, q_n$ in the subalgebra generated by $x_{n+1}, \ldots, x_{2n}, \partial_{n+1}, \ldots, \partial_{2n}$, then $f$ is invertible with

$$f^{-1}(x_{n+1}) = \tau q_1$$

$$\vdots$$

$$f^{-1}(x_{2n}) = \tau q_n$$

$$f^{-1}(\partial_{n+1}) = -\tau p_1$$

$$\vdots$$

$$f^{-1}(\partial_{2n}) = -\tau p_n.$$

(iii) Let $\prec$ denote the lexicographic term order on monomials in $A_{2n}(k)$ given by

$$x_1 \succ \cdots \succ x_n \succ \partial_1 \succ \cdots \succ \partial_n \succ x_{n+1} \succ \cdots \succ x_{2n} \succ \partial_{n+1} \succ \cdots \succ \partial_{2n}.$$  

Then $f$ is invertible if and only if the reduced Gröbner basis of $J$ is

$$G = \{x_1 - q_1, \ldots, x_n - q_n, \partial_1 + p_1, \ldots, \partial_n + p_n\}$$

with $p_1, \ldots, p_n, q_1, \ldots, q_n$ in the subalgebra generated by $x_{n+1}, \ldots, x_{2n}, \partial_{n+1}, \ldots, \partial_{2n}$.

Proof. The items (i) and (ii) are straightforward translations of the corresponding statements in Theorem 3.10. The generators given in (ii) are seen to commute using that $\tau$ is an anti-homomorphism of $A$.

For the claim (iii), note by (i) that if $f$ is invertible, then the reduced Gröbner basis of $J$ must be of the form $G$, since the generators in (i) already form the reduced Gröbner basis with respect to the given lexicographic term order by Theorem 3.6. If the reduced Gröbner basis is $G$, then it follows from (ii) that $f$ must be invertible.

A perhaps surprising consequence of (ii) in Theorem 2.1 is the following result.
Theorem 3.12. Let $k$ be a field of characteristic zero and $A_{2n}(k)$ the Weyl algebra in the variables $x_1, \ldots, x_{2n}, \partial_1, \ldots, \partial_{2n}$. If $p_1, \ldots, p_n, q_1, \ldots, q_n$ are in the subalgebra generated by $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$, then the left ideal

$$J = (x_{n+1} - q_1, \ldots, x_{2n} - q_n, \partial_{n+1} + p_1, \ldots, \partial_{2n} + p_n)$$

(3.3.4)
is proper if and only if

$$[q_i, q_j] = 0$$
$$[p_i, p_j] = 0$$
$$[p_i, q_j] = \delta_{ij}. \tag{3.3.5}$$

Proof. If $J$ is a proper ideal, then $A_{2n}/J$ can be viewed as a non-zero bimodule over $A_n(k)$, which is cyclic generated by $1 \in A_{2n}(k)$ as a left module over $A_n(k)$ identified with the subalgebra generated by $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$. The commutators in (3.3.5) are all elements of the left ideal $J$ except for $[p_i, q_i]$. Here

$$[\partial_{n+i} + p_i, x_{n+i} - q_i] = 1 - [p_i, q_i] \in J.$$ 

Part (ii) of Theorem 2.1 implies that $J$ cannot contain non-zero elements from the subalgebra generated by $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$. Therefore the commutator identities in (3.3.5) must hold if $J$ is a proper ideal.

If the commutator identities in (3.3.5) hold, then the generators of $J$ in (3.3.4) form the unique reduced Gröbner basis with respect to the term order in Theorem (3.3.3), since they are commuting with relatively prime leading monomials (cf. Theorem 3.6). This shows that $J$ is proper. \hfill \Box

Remark 3.13. Theorem 3.12 fails for a field $k$ of positive characteristic $p > 0$. Consider the left $A_2(k)$-module given by

$$M = k[x_1, x_2]/(x_1^p, x_2^p)$$
as a quotient of the natural left $A_2(k)$-module $k[x_1, x_2]$. The annihilator of the element $[(x_2 - x_1)^{p-1}] \in M$ contains the left ideal

$$J = (x_2 - x_1, \partial_2 + \partial_1 + \partial_1^{p+1}) \subseteq A_2(k),$$

but $[\partial_1 + \partial_1^{p+1}, x_1] = 1 + \partial_1^p$.

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