Gravitational waves from inspiraling compact binaries: The quadrupole-moment term

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(Submitted to Physical Review D, September 12, 1997)

A rotating star’s oblateness creates a deformation in the gravitational field outside the star, which is measured by the quadrupole-moment tensor. We consider the effect of the quadrupole moment on the orbital motion and rate of inspiral of a compact binary system, composed of neutron stars and/or black holes. We find that in the case of circular orbits, the quadrupole-monopole interaction affects the relation between orbital radius and angular velocity, and also the rate of inspiral, by a quantity of order \((v/c)^4\), where \(v\) is the orbital velocity and \(c\) the speed of light.

Pacs numbers: 04.25.Nx, 04.30.Db, 97.60.Jd, 97.60.Lf

I. INTRODUCTION AND SUMMARY

Inspiring compact binaries, composed of neutron stars and/or black holes, are the most promising source of gravitational waves for kilometer-scale interferometric detectors such as the American LIGO [1] and the French-Italian VIRGO [2]. The current construction of these detectors has motivated a lot of recent work on strategies to detect and measure such waves [3-11]. In the course of this work, the need for extremely accurate predictions of the expected signal has repeatedly been demonstrated, as reliable model signals will be required for analyzing the data with the well-known technique of matched filtering [12].

A large effort is currently underway to calculate inspiraling binary waveforms to high order in a post-Newtonian approximation to the exact laws of general relativity [13-15]. This method is based on the assumption that the orbital motion is sufficiently slow that the waves can be expressed as a power series in \(v\), the orbital velocity. Thus far, the waveforms have been calculated to order \(v^5\) beyond the leading-order expressions, under the assumption that the spinning motion of the binary companions can be ignored [14]. Rotational corrections to the waveforms have been computed only up to order \(v^4\) [16].

Of particular importance for matched filtering is the phasing of the waves [17], which is determined by the rate of increase of the gravitational-wave frequency \(f\) (equal to twice the orbital frequency) in response to the system’s loss of energy and angular momentum to gravitational radiation. Assuming, as is usual, that the orbital motion is circular, this is given by [17]

\[
\frac{df}{dt} = \frac{96}{5\pi \mathcal{M}^2} (\pi \mathcal{M} f)^{11/3} \left[ 1 - \left( \frac{743}{336} + \frac{11}{4} \eta \right) v^2 \right.
\]

\[
+ (4\pi - \beta) v^3
\]

\[
+ \left( \frac{13661}{18144} + \frac{34103}{2016} \eta + \frac{59}{18} \eta^2 + \sigma \right) v^4
\]

\[
+ O(v^5) \right],
\]

where we use units such that \(c = G = 1\). We have introduced a number of symbols. Let \(m_1\) and \(m_2\) denote the masses of the two companions, and let \(M = m_1 + m_2\) be the total mass and \(\mu = m_1m_2/M\) the reduced mass. Then we define the chirp mass \(\mathcal{M}\) and the mass ratio \(\eta\) as

\[
\mathcal{M} = \eta^{3/5} M, \quad \eta = \mu/M.
\]

The orbital velocity \(v\) is defined in terms of the gravitational-wave frequency \(f\) by

\[
v = (\pi M f)^{1/3}.
\]

Finally, the quantities \(\beta\) and \(\sigma\) have to do with the spin angular momenta of the companions, denoted \(s_1\) and \(s_2\). The “spin-orbit” parameter is given by [16]

\[
\beta = \frac{1}{12} \sum_A \left[ 113(m_A/M)^2 + 75\eta \right] \hat{L} \cdot \chi_A,
\]

where \(\hat{L}\) is the direction of the orbital angular momentum, and \(\chi_A = s_A/m_A^2\) is the dimensionless spin of companion \(A\). (The index \(A\) runs over the values 1 and 2.) Here and throughout the paper, vectors and tensors are defined in three-dimensional flat space, and a hat indicates that the vector has a unit norm.) On the other hand, the “spin-spin” contribution to \(\sigma\) is [16]

\[
\sigma_{ss} = \frac{\eta}{48} \left[ -247(\chi_1 \cdot \chi_2) + 721(\hat{L} \cdot \chi_1)(\hat{L} \cdot \chi_2) \right].
\]

Our purpose in this paper is to derive an additional contribution to \(\sigma\), namely \(\sigma_{qm}\), which is due to the quadrupole moments of the companions. The physical picture is the following. The spinning motion of companion \(A\) creates a distortion in its mass distribution which, in turn, creates a distortion in the gravitational field outside the star, measured by \(Q_A^{\alpha\beta}\), the (treefree)
quadrupole-moment tensor. The quadrupole term in the gravitational potential affects the orbital motion of the companions, and it affects also the emission of gravitational waves. This is the effect that we consider in this paper. It is important to understand that we are not considering the gravitational waves emitted by the time variations of the quadrupole moments $Q^{ab}_{2}$. Although such an effect exists (because of spin precession), it is much weaker (by an estimated factor of order $v^{30}$) than the one considered here.

Assuming that the spinning body $A$ is axially symmetric about the direction of $\hat{s}_A$, the quadrupole-moment tensor can be expressed as

$$Q^{ab}_{A} = Q_A (\hat{s}^a \hat{s}^b - \frac{1}{3} \delta^{ab}),$$

where $Q_A$ is the quadrupole-moment scalar. In Newtonian theory, this is given in terms of the mass density $\rho$ by $Q_A = \int_A \rho(x) |x|^2 P_2(\hat{s} \cdot \hat{x}) \, d^3x$, where $P_2(x) = \frac{1}{2} (3x^2 - 1)$. In general relativity, $Q_A$ is defined in a coordinate-invariant manner in terms of the falloff behavior of the metric outside the star (see Ref. [19] and references therein). The general relativistic definition reduces to the Newtonian one when the gravitational field is weak everywhere inside the star.

In the following sections of this paper we calculate that the quadrupole-moment contribution to $\sigma$ is

$$\sigma_{qm} = -\frac{5}{2} \sum_A \frac{Q_A}{m_AM^2} \left[3(\hat{L} \cdot \hat{s}_A)^2 - 1\right],$$

where $Q_A$ is the general-relativistic quadrupole-moment scalar. This is the main result of this paper. To the best of the author’s knowledge, this contribution was never presented in the literature before, except in the specific reference [20]. We note, however, that similar calculations were carried out independently, but not published, by Kidder [21].

The quadrupole-monopole interaction responsible for $\sigma_{qm}$ is a Newtonian effect which formally takes the appearance of a second post-Newtonian correction in $df/dt$. Indeed, apart from the facts that $Q_A$ is defined in a general-relativistic manner and the Einstein quadrupole formula is used to compute the radiation, the calculation leading to Eq. (6) involves only Newtonian theory. In other words, although the gravitational field is not assumed to be weak inside the compact objects, the relative separation of the companions is assumed to be sufficiently large that the mutual gravitational potential will be small. Under such conditions, Newtonian theory can be used to describe the orbital motion, provided that the quadrupole moments are defined in a suitable, strong-field manner.

It is well known that the quadrupole moment of a rotating black hole is given by $Q_A = -\chi_A^2 m_A^3$, where $\chi_A \equiv |\chi_A|$ is the dimensionless spin [22]. (The minus sign reflects the oblateness of the black hole.) Quadrupole moments of realistic rotating neutron stars have been computed in Ref. [19], where it is shown that for a neutron star of $1.4 M_\odot$,

$$Q_A \simeq -a \chi_A^2 m_A^3.$$  

Here, the parameter $a$ ranges from approximately 4 to 8 depending on the equation of state for neutron-star matter. Stiffer equations of state give larger values of $a$, and $a = 1$ for a rotating black hole. Inserting Eq. (8) into Eq. (6) shows that whatever the nature of the compact object (black hole or neutron star), $\sigma_{ss}$ and $\sigma_{qm}$ are of the same order of magnitude. For example, for two $1.4 M_\odot$ neutron stars with $a = 5.0$, we find $\sigma_{ss} \in (-1.04, 1.04)$, while $\sigma_{qm} \in (-2.64, 5.28)$. We have used the fact that for such neutron stars, $\chi < 0.65$ [19].

The rest of the paper is organized as follows. In Sec. II we integrate the Newtonian equations of motion for the centers of mass of the two companions, focusing on the motion over a time scale comparable with the orbital period. In Sec. III we consider the precession motions of the orbital and spin angular momenta, which occur over a time scale much longer than the orbital period. Finally, in Sec. IV we calculate the contribution to the rate of change of the gravitational-wave frequency due to the quadrupole-monopole interaction.

II. ORBITAL MOTION

The Newtonian equations of motion for the centers of mass of two bodies with masses $m_1$ and $m_2$ and quadrupole moments $Q_1$ and $Q_2$ can be derived from an effective one-body Lagrangian, $\hat{L} = \frac{1}{2} \mu \hat{x}^2 - V(x)$, where $\mu$ is the reduced mass, $x = x_2 - x_1$ the relative separation of the centers, and $\hat{x}$ the relative velocity. The gravitational potential is [23]

$$V(x) = -\frac{\mu M}{r} - \frac{3}{2} \left(m_1 Q^{ab}_{2} + m_2 Q^{ab}_{1}\right) \frac{n_a n_b}{r^3},$$

where $M$ is the total mass, $r \equiv |x|$, $n \equiv \hat{x} = x/r$, and $Q^{ab}_{2}$ is constructed from $Q_A$ as in Eq. (6). For what follows it is useful to define the dimensionless quantities

$$p_A = \frac{Q_A}{m_A M^2},$$

and to introduce the angles $\alpha_A$ and $\beta_A$ such that

$$\hat{s}_A = (\sin \alpha_A \cos \beta_A, \sin \alpha_A \sin \beta_A, \cos \alpha_A).$$

Although these angles vary because of spin precession (see Sec. III), this variation occurs on a time scale much longer than the orbital period. We shall therefore take them to be constant for the purpose of calculating the motion over a time scale comparable with the orbital
period. We also introduce the spherical coordinates \( \{r, \theta, \phi\} \), such that \( x = r \mathbf{n} \) and

\[
\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).
\]

Substituting these relations, together with Eq. (6), into the equations of motion derived from Eq. (3) yields

\[
V(x) = -\frac{\mu M}{r} - \frac{\mu}{2} \left( \frac{M}{r} \right)^3 \sum_A p_A (3 \cos^2 \gamma_A - 1),
\]

where \( \cos \gamma_A \equiv \mathbf{\hat{s}}_A \cdot \mathbf{n} = \sin \alpha_A \sin \theta \cos(\phi - \beta_A) + \cos \alpha_A \cos \theta. \)

The second term in Eq. (13) is smaller than the first by a factor of order \((M/r)^2 = v^2\), and it can be treated as a perturbation when integrating the equations of motion. Taking the background motion to be a circular orbit in the equatorial plane, we write \( r(t) = r_0 [1 + \epsilon R(t)] \), \( \theta(t) = \pi/2 + \epsilon \Theta(t) \), and \( \phi(t) = \Omega t + \epsilon \Phi(t) \), where \( r_0 \) and \( \Omega = M^{1/2} r_0^{-3/2} \) are the background radius and angular velocity, respectively, and \( \epsilon \equiv (M/r_0)^2 \). Substituting these relations into the equations of motion derived from \( \mathcal{L} \) and linearizing with respect to \( \epsilon \) leads to differential equations for the unknowns \( R, \Theta, \) and \( \Phi \). Integration of these equations is straightforward, and we find

\[
r(t) = r_0 \left[ 1 + \frac{3 \epsilon}{4} \sum_A p_A (3 \cos^2 \alpha_A - 1) \right. \\
\left. + \frac{\epsilon}{4} \sum_A p_A \sin^2 \alpha_A \cos 2(\Omega t - \beta_A) \right],
\]

\[
\theta(t) = \frac{\pi}{2} - \frac{3 \epsilon}{4} \Omega t - \sum_A p_A \sin 2\alpha_A \sin(\Omega t - \beta_A),
\]

\[
\phi(t) = \Omega t \left[ 1 - \frac{3 \epsilon}{2} \sum_A p_A (3 \cos^2 \alpha_A - 1) \right. \\
\left. + \frac{\epsilon}{4} \sum_A p_A \sin^2 \alpha_A \cos 2(\Omega t - \beta_A) \right].
\]

These equations show that the orbits cannot be circular in the strict sense, although \( r \) and \( \dot{\phi} \) are constant after averaging over an orbital period. As we shall see in Sec. III, the linear growth of \( \theta \) signals the occurrence of orbital precession. Because we are concerned only with the motion over a time scale comparable with the orbital period, and because the precessional motion occurs over a much longer time scale, we shall ignore this issue here.

From Eqs. (14) and (16) we find that the averaged orbital radius is given by

\[
\langle r \rangle = r_0 \left[ 1 + \frac{3 \epsilon}{4} \sum_A p_A (3 \cos^2 \alpha_A - 1) \right],
\]

while the averaged angular velocity is

\[
\langle \dot{\phi} \rangle = \Omega t \left[ 1 - \frac{3 \epsilon}{2} \sum_A p_A (3 \cos^2 \alpha_A - 1) \right].
\]

Eliminating \( r_0 \) from these equations, we arrive at (we discard all terms of order \( \epsilon^2 \) and higher)

\[
\langle \dot{\phi} \rangle = \left( \frac{M}{\langle r \rangle^3} \right)^{1/2} \left[ 1 - \frac{3}{8} \left( \frac{M}{\langle r \rangle} \right)^2 \sum_A p_A (3 \cos^2 \alpha_A - 1) \right].
\]

Inverting this gives

\[
\frac{M}{\langle r \rangle} = v^2 \left[ 1 + \frac{1}{4} v^4 \sum_A p_A (3 \cos^2 \alpha_A - 1) \right],
\]

where \( v \equiv (\langle \dot{\phi} \rangle)^{1/3}. \)

We shall need also an expression for the orbital energy, given by \( E = \frac{\mu}{2} \dot{\mathbf{x}} + V(x) \). Substituting the relations (13)–(20) and discarding all terms of order \( \epsilon^2 \) and higher, we find

\[
E = -\frac{1}{2} \mu v^2 \left[ 1 + \frac{1}{4} v^4 \sum_A p_A (3 \cos^2 \alpha_A - 1) \right].
\]

Equations (13)–(21) describe the (averaged) orbital motion over time scales comparable to the orbital period.

### III. PRECESSIONS

It is well known that the Newtonian quadrupole-monopole interaction produces a precession of the orbital angular momentum, as well as a precession of the spin axis of each of the binary companions [23]. We examine this effect here, as well as the precessional motions caused by the general relativistic spin-orbit and spin-spin interactions.

Calculating the torque associated with the potential (13) and averaging over an orbital period leads to this equation describing the orbital precession:

\[
\langle \mathbf{\dot{L}} \rangle = \Omega_{\text{op}} \times \mathbf{L},
\]

where the angular velocity of orbital precession is given by

\[
\Omega_{\text{op}} = \frac{3}{2} \left( \frac{M}{r} \right)^2 \Omega \sum_A p_A \cos \alpha_A \mathbf{\hat{s}}_A.
\]

Here, \( r \equiv \langle r \rangle \) is the orbital radius, and \( \Omega \equiv \langle \dot{\phi} \rangle \) is the orbital angular velocity. Equation (23) shows that the time scale for orbital precession is longer than the orbital period by a factor of order \( \langle r/M \rangle^2 \). Equations (22) and (23) also imply that \( \cos \alpha_A \equiv \mathbf{\hat{L}} \cdot \mathbf{\hat{s}}_A \) does not stay constant during the motion. We note that Eq. (15) may be recovered by integrating Eq. (23) with \( \mathbf{\hat{L}} \) set equal to \( \mathbf{\hat{z}} \) on the right-hand side.

On the other hand, treating companion \( A \) as a rigid body under the influence of the potential (13) leads to this equation describing the spin precession:
\[ (\dot{s}_A) = \Omega_{sp,A} \times s_A, \]  
where the angular velocity of spin precession is given by
\[ \Omega_{sp,A} = \frac{3}{2} \frac{\mu M}{m_A^2} \left( \frac{M}{r} \right)^{3/2} \frac{p_A \cos \alpha_A}{\chi_A} \hat{L}. \]  
(25)

This shows that the time scale for spin precession is longer than the orbital period by a factor of order \((r/M)^{3/2}\). It should be noted that for simplicity, we have not accounted for the time variation of \(\cos \alpha_A\) in this calculation.

The quadrupole-monopole interaction is not alone in inducing a precession of \(\mathbf{L}\) and \(s_A\). The general relativistic spin-orbit and spin-spin interactions also contribute to these precessions, and this was studied in detail in Ref. [24]. The spin-orbit contributions to the precessional angular velocities are given by
\[ \Omega_{sp} = \frac{4m_1 + 3m_2}{2m_1 r^3} s_1 + \frac{4m_2 + 3m_1}{2m_2 r^3} s_2 \]  
and
\[ \Omega_{sp,1} = \frac{4m_1 + 3m_2}{2m_1 r^3} L, \]  
(27)

with a similar equation holding for \(\Omega_{sp,2}\). On the other hand, the spin-spin contributions are
\[ \Omega_{sp} = -\frac{3}{2r^3} \left[ (s_2 \cdot \hat{L}) s_1 + (s_1 \cdot \hat{L}) s_2 \right], \]  
(28)

where \(L \equiv |\mathbf{L}| = \mu r^2 \Omega\), and
\[ \Omega_{sp,1} = \frac{1}{2r^3} \left[ s_2 - 3(s_1 \cdot \hat{L}) \hat{L} \right], \]  
(29)

with a similar equation holding for \(\Omega_{sp,2}\).

From these equations it is easy to see that the spin-orbit contribution to the precession dominates over both the spin-spin and quadrupole-monopole contributions. In fact,
\[ \frac{\text{spin-orbit}}{\text{spin-spin}} \sim \frac{\text{spin-orbit}}{\text{quadrupole-monopole}} \sim \left( \frac{r}{M} \right)^{1/2}. \]  
(30)

The main lesson here is that while the quadrupole-monopole precession is entirely a Newtonian effect, it is a small one compared with the general relativistic spin-orbit precession, and it is of the same order of magnitude as the spin-spin precession.

These precessional motions are important, because they produce modulations in the amplitude and phase of the gravitational waves emitted by the binary system. The modulations produced by the spin-orbit and spin-spin interactions were studied in detail in Ref. [24]. While it may be worthwhile to repeat this analysis so as to also incorporate the modulations due to the quadrupole-monopole interaction, it is doubtful that any of the conclusions would be affected.

\[ IV. \text{GRAVITATIONAL WAVE} \]

We now use the Einstein quadrupole formula [25],
\[ \dot{E} = -\frac{1}{5} (Q^{ab} Q^{ab}), \]  
(31)

to calculate the rate of loss of orbital energy to gravitational radiation, for a binary system moving on a circular orbit of (averaged) radius \(r \equiv \langle r \rangle\) and (averaged) angular velocity \(\Omega \equiv \langle \dot{\phi} \rangle\), where \(\langle \cdot \rangle\) and \(\langle \dot{\cdot} \rangle\) are related by Eqs. (19) and (20). The quadrupole moment of the centers of mass is given by \(Q^{ab} = \sum_A m_A (x_A^a x_A^b - \frac{1}{3} r^2 A^2 \delta^{ab})\), where, in terms of the relative separation \(\mathbf{x}\), \(\mathbf{x}_1 = -m_2 \mathbf{x}/M\) and \(\mathbf{x}_2 = m_1 \mathbf{x}/M\). Substituting \(\mathbf{x} = r(\cos \Omega t, \sin \Omega t, 0)\) into \(Q^{ab}\), and the result into Eq. (31), we find
\[ \dot{E} = -\frac{32}{5} \mu^2 r^4 \Omega^6, \]  
(32)

irrespective of the relation between \(r\) and \(\Omega\). Using Eq. (20) yields
\[ \dot{E} = \frac{32}{5} \left( \frac{\mu}{M} \right)^2 v^{10} \left[ 1 - v^4 \sum_A p_A (3 \cos^2 \alpha_A - 1) \right], \]  
(33)

where \(v = (M \Omega)^{1/3}\).

The gravitational-wave frequency \(f\) is equal to twice the orbital frequency, so
\[ \pi M f = M \Omega = v^3. \]  
(34)

The loss of orbital energy is accompanied by an increase in orbital frequency, and therefore, an increase in \(f\). The rate of change of the gravitational-wave frequency is calculated as \(\dot{f} = \dot{E}/(dE/df) = (3v^2/M) \dot{E}/(dE/dv)\), where \(\dot{E}(v)\) is given by Eq. (21). This gives
\[ \dot{f} = \frac{96 \mu}{5 \pi M^5} v^{11} \left[ 1 - \frac{5}{2} v^4 \sum_A p_A (3 \cos^2 \alpha_A - 1) \right]. \]  
(35)

The coefficient of the \(O(v^4)\) term within the square brackets is the quadrupole-moment contribution to the parameter \(\sigma\) appearing in Eq. (1). Taking into account the definitions (10) and (11), this is the same as what was given in Eq. (7).

\[ \text{ACKNOWLEDGMENTS} \]

This work was supported by the Natural Sciences and Engineering Research Council of Canada. It is a pleasure to thank Bernie Nickel for useful discussions on orbital precession.
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