The Integration Algorithm of Lax equation for both Generic Lax matrices and Generic Initial Conditions

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Abstract

Several physical applications of Lax equation require its general solution for generic Lax matrices and generic not necessarily diagonalizable initial conditions. In the present paper we complete the analysis started in [arXiv:0903.3771] on the integration of Lax equations with both generic Lax operators and generic initial conditions. We present a complete general integration formula holding true for any (diagonalizable or non diagonalizable) initial Lax matrix and give an original rigorous mathematical proof of its validity relying on no previously published results.

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1 Introduction

Lax equation appears in a variety of physical-mathematical problems which turn out to constitute integrable dynamical systems.

An integration algorithm for generic matrix Lax equation was originally derived in the mathematical literature in [1], [2] and it was applied in the context of supergravity to cosmic billiards in [3], [4], [5] and to black-holes in [6], [7], [8] on the basis of their 3D description pioneered in [9].

The latter application to the case of black-holes revealed that the integration algorithm of [1], [2] did not cover the case of non-diagonalizable initial conditions. Such a situation occurs in particular when the initial Lax operator at time $t = 0$ is nilpotent and this case is extremely relevant for Physics since it corresponds to extremal black-holes [9]. An extension of the integration algorithm to the case of nilpotent operators was presented in [8]. In the appendix of that paper it was also conjectured a general formula, which was verified for some nontrivial cases, that provides the integration of Lax equation for completely generic initial data.

In the present paper we recall the general formula of [8] and we provide the rigorous mathematical proof that indeed it solves Lax equation with arbitrary initial conditions (diagonalizable or non diagonalizable). Our proof is completely original and independent from the algorithm discussed in [1], [2].

2 The integration algorithm for generic Lax matrices and generic initial conditions

Let us consider Lax equation

\[ \frac{d}{dt} L(t) + [L_>(t) - L_<(t), L(t)] = 0 \]

(2.1)

where $L(t)$ denotes a time-dependent generic $N \times N$ matrix and remove any hypothesis on the nature of the initial conditions. We are interested in writing an integration algorithm for eq. (2.1) which should hold true for generic initial matrices $L(0) \equiv L_0$ independently from the fact that $L_0$ be diagonalizable or non diagonalizable, nilpotent or not, symmetric or non symmetric with respect to any definite or indefinite metric $\eta$.

It turns out that such an integration algorithm exists, is very simple and equally simple and elegant is the formal proof of its validity.

Following [8] we first present the integration formula and then provide the mathematical proof that it satisfies Lax equation.

\footnote{Hereafter, we denote by $L_>$ ($L_<$) the upper (lower) triangular part including the diagonal of the matrix $L$, and $L^T$ is the transposed matrix.}
2.1 Integration formula

The matrix elements $L_{pq}(t)$ of the time-evolving Lax operator which coincides with the given $L_0$ at time $t = 0$ are constructed as follows [8]:

$$L_{pq}(t) = \frac{1}{\sqrt{D_p(t)D_{p-1}(t)D_q(t)D_{q-1}(t)}} \sum_{k=1}^{p} \sum_{\ell=1}^{q} M_{pk}(t) (C(t) L_0)_{k\ell} \tilde{M}_{q\ell}(t). \quad (2.2)$$

The building blocks appearing in eq.(2.2) are defined as follows. We have

$$M_{ik}(t) := (-1)^{i+k} \text{Det} \begin{pmatrix} C_{1,1}(t) & \ldots & C_{1,i-1}(t) \\ \vdots & \ddots & \vdots \\ \widehat{C_{k,1}(t)} & \ldots & \widehat{C_{k,i-1}(t)} \\ \vdots & \ddots & \vdots \\ C_{i,1}(t) & \ldots & C_{i,i-1}(t) \end{pmatrix}, \quad 1 \leq k \leq i; 2 \leq i \leq N,$$

$$M_{11}(t) := 1 \quad (2.3)$$

where the hats on the entries corresponding to the $k$-th row mean that such a row has been suppressed giving rise to a squared $(i-1) \times (i-1)$ matrix of which one can calculate the determinant. Similarly:

$$\tilde{M}_{ik}(t) := (-1)^{i+k} \text{Det} \begin{pmatrix} C_{1,1}(t) & \ldots & \widehat{C_{1,k}(t)} & \ldots & C_{1,i}(t) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \widehat{C_{i-1,1}(t)} & \ldots & \widehat{C_{i-1,k}(t)} & \ldots & C_{i-1,i}(t) \\ C_{i,1}(t) & \ldots & C_{i,i-1}(t) \end{pmatrix}, \quad 1 \leq k \leq i; 2 \leq i \leq N; \quad \tilde{M}_{11}(t) := 1, \quad (2.4)$$

where the hatted $k$-th column is deleted just as in eq.(2.3) it was deleted the $k$-th row.

In the above formulae we have used the following definitions:

$$C(t) := e^{-2tL_0} \quad (2.5)$$

and

$$D_i(t) := \text{Det} \begin{pmatrix} C_{1,1}(t) & \ldots & C_{1,i}(t) \\ \vdots & \ddots & \vdots \\ C_{i,1}(t) & \ldots & C_{i,i}(t) \end{pmatrix}, \quad D_0(t) := 1. \quad (2.6)$$

2.2 The theorem and its proof

In order to prove the integration formula (2.2) we first recast it into another equivalent form which is not only the most convenient for the formal proof but also the simplest and best suited for computer implementation.
Theorem 2.1  The solution of Lax equation (2.1) with generic initial condition \( L(0) = L_0 \) is given by
\[
L(t) = Q(C) L_0 (Q(C))^{-1}
\]
where the \( N \times N \) matrix \( Q(C(t)) \) is
\[
Q_{ij}(C) := \frac{1}{\sqrt{D_i(t) D_{i-1}(t)}} \sum_{k=1}^{i} M_{ik}(t) (C^\frac{1}{2}(t))_{k,j}
\]
\[
\equiv \frac{1}{\sqrt{D_i(t) D_{i-1}(t)}} \text{Det} \left( \begin{array}{ccc}
C_{1,1}(t) & \cdots & C_{1,i-1}(t) & (C^\frac{1}{2}(t))_{i,j} \\
\vdots & \ddots & \vdots & \vdots \\
C_{i,1}(t) & \cdots & C_{i,i-1}(t) & (C^\frac{1}{2}(t))_{i,j}
\end{array} \right) \quad (2.8)
\]
and \( C^\frac{1}{2}(t) \equiv e^{-t L_0} \) (see, eq. (2.5)).

Proof 2.1  The matrix \( Q(C) \) satisfies three properties
\[
X_>(t) := Q(C) C^\frac{1}{2}(t) \equiv \text{upper triangular}, \quad (2.9)
\]
\[
(X_<(t))^{-1} := Q(C) (C^\frac{1}{2}(t))^{-1} \equiv \text{lower triangular}, \quad (2.10)
\]
\[
(X_>(t))_{ii} = (X_<(t))_{ii} \quad (2.11)
\]
which are crucial in what follows. Therefore, the matrix \( e^{-t L_0} \equiv C^\frac{1}{2}(t) \) admits the following two different representations:
\[
e^{-t L_0} = (Q(C))^{-1} X_>(t), \quad (2.12)
e^{-t L_0} = X_<(t) Q(C) \quad (2.13)
\]
resulting from eqs. (2.9) and (2.10). Differentiating with respect to time \( t \), one after the other equations (2.12), (2.13) and (2.7), then using them one can straightforwardly derive the following relations:
\[
Q(C) \frac{d}{dt} (Q(C))^{-1} = -L(t) - \left( \frac{d}{dt} X_>(t) \right) (X_>(t))^{-1}, \quad (2.14)
\]
\[
Q(C) \frac{d}{dt} (Q(C))^{-1} = +L(t) + (X_<(t))^{-1} \left( \frac{d}{dt} X_<(t) \right), \quad (2.15)
\]
\[
\frac{d}{dt} L(t) + \left[ Q(C) \frac{d}{dt} (Q(C))^{-1}, L(t) \right] = 0. \quad (2.16)
\]
\footnote{The matrix \((X_>(t))_{ij}\) \(2.9\) is upper triangular, since at \( j < i \) there is a coincidence of two columns in the determinant originating from \(2.8\) in eq. \(2.9\). The matrix \((X_<(t))_{ij}^{-1}\) \(2.10\) is lower triangular, since at \( j > i \) the all entries of the last column in the determinant originating from \(2.8\) in eq. \(2.10\) become equal to zero.}
A simple inspection of eqs. (2.14–2.15) with the use of relation (2.11) leads to the conclusion that the quantity 

\[ Q(C) \frac{d}{dt} (Q(C))^{-1} = L_>(t) - L_<(t), \]

then with this matrix \( Q(C) \frac{d}{dt} (Q(C))^{-1} \) equation (2.16) reproduces Lax equation (2.1) and formulae (2.7–2.8) give indeed its general solution for generic initial conditions \( L_0 \). This ends the proof of our proposition.

Now, we present the useful expression of the inverse matrix \( (Q(C))^{-1} \):

\[
(Q(C))^{-1}_{ji} := \frac{1}{\sqrt{D_i(t)D_{i-1}(t)}} \sum_{t=1}^{i} (C^\frac{1}{2}(t))_{j,t,\tilde{M}_{t\ell}(t)}
\]

\[
\equiv \frac{1}{\sqrt{D_i(t)D_{i-1}(t)}} \text{Det} \begin{pmatrix}
C_{1,1}(t) & \ldots & C_{1,i}(t) \\
\vdots & \ddots & \vdots \\
C_{i-1,1}(t) & \ldots & C_{i-1,i}(t) \\
(C^\frac{1}{2}(t))_{j,1} & \ldots & (C^\frac{1}{2}(t))_{j,i}
\end{pmatrix}
\]

which follows from eq. (2.2). It is a very simple exercise to verify that indeed

\[
(Q(C) (Q(C))^{-1})_{pq} = \frac{1}{\sqrt{D_p(t)D_{p-1}(t)D_q(t)D_{q-1}(t)}} \sum_{k=1}^{p} \sum_{\ell=1}^{q} \tilde{M}_{pk}(t) C_{k\ell}(t) \tilde{M}_{q\ell}(t) = \delta_{p,q}
\]

since it is equal to zero if \( p \neq q \), due to a coincidence of two rows or columns in the determinants originating from (2.8–2.18) in eq. (2.19).

It is interesting to note one more property of the matrix \( Q(C) \) (2.8):

\[
Q^T(C^T) = (Q(C))^{-1}
\]

which is obvious if one compares equations (2.8) and (2.18). When \( L_0 \) is symmetric the same is true of \( C^T \) and this implies that \( Q(t) \) is orthogonal, namely \( Q(t) \in \text{SO}(N) \). Similarly when \( L_0 \) is pseudo-orthogonal \( \eta \)-metric symmetric, \( Q(t) \) is pseudo-orthogonal, \( Q(t) \in \text{SO}(p, N-P) \). Yet \( Q(t) \) exists in general also for non \( \eta \)-symmetric initial data. In this case \( Q(t) \in \mathfrak{gl}(N) \).

### 2.3 The Generalized Linear System

Finally, based on the knowledge of the constructed general solution (2.7–2.8) with generic initial data \( L_0 \), we comment briefly on a way of a generalization (suggested by this solution...
with the aim to reproduce it) of Kodama–Ye algorithm, which was based on the inverse scattering method applied to the case of the diagonalizable $L_0$ [2].

A link towards the inverse scattering method is given by the following observation. Relations (2.7) and (2.17) corresponding to solution $Q(C)$ (2.8), being identically rewritten as follows:

$$L(t) Q(C) = Q(C) L_0, \quad (2.21)$$
$$\frac{d}{dt} Q(C) = -(L_>(t) - L_<(t)) Q(C), \quad (2.22)$$

represent a generalized linear system for the matrix $Q(C)$, whose consistency is provided by Lax equation (2.1). This generalized linear system could be a starting point to apply the inverse scattering method to construct the solution for $Q(C)$. But, there is a subtlety: up to now the inverse scattering method was always applied only when $L_0$ is a diagonal or a diagonalizable matrix; it was never applied before to the case of non-diagonalizable $L_0$. Just the classical case of diagonal (diagonalizable) $L_0$ was considered in [2] where the corresponding solution to the linear system was constructed. The very existence of our solution $Q(C)$ (2.8) for generic $L_0$ gives an evidence that a generalization of the algorithm of [2] to the case of the generalized linear systems (2.21–2.22) with generic $L_0$ should exist as well. It is indeed the case, and the generalization is rather straightforward. So it turns out that one can repeat all the steps of the algorithm, starting from the generalized linear system (2.21–2.22), in such a way that all the intermediate operations are formulated in purely matrix terms; thus they are insensitive to whether $L_0$ is a diagonalizable or a non-diagonalizable matrix: this process ends with the solution presented in (2.8).

3 Conclusions

In the present paper we completed the analysis started in [8] on the integration of Lax equations with both generic Lax operators and generic initial conditions by presenting a simple, original rigorous proof of the validity of the integration algorithm proposed in [8]. The presented proof is constructive, and moreover it is interesting for its own sake since it opens a new deep insight into the structure of the corresponding integrable Toda-like systems which we plan to discuss elsewhere in connection with relevant physical problems like the classification and construction of Supergravity Black Hole solutions. We also clarified the relation between our new integration algorithm and the inverse scattering framework adopted by Kodama et al for the integration of Lax equation in the diagonalizable case.

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