Convergence in uncertain linear systems

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Abstract
State convergence is essential in several scientific areas, e.g. multi-agent consensus/disagreement, distributed optimization, monotone game theory, multi-agent learning over time-varying networks. In this paper, we study the state convergence in both continuous- and discrete-time linear systems affected by polytopic uncertainty. First, we characterize state convergence in linear time invariant systems via equivalent necessary and sufficient conditions. In the presence of uncertainty, we complement the canonical definition of (weak) convergence with a stronger notion of convergence, which requires the existence of a common kernel among the generator matrices of the difference/differential inclusion (strong convergence). We investigate under which the canonical definition of (weak) convergence with a stronger notion of convergence, which requires the existence of a common Lyapunov arguments, linear matrix inequalities and separability of the eigenvalues of the generator matrices. We also show conditions the two definitions are equivalent. Then, we characterize weak and strong convergence by means of LaSalle and linear time invariant systems via equivalent necessary and sufficient conditions. In the presence of uncertainty, we complement MOS (802348).

1 Introduction

Hand in hand with stability, state convergence of dynamical, possibly uncertain, systems represents a fundamental problem in system theory. However, while stability and asymptotic stability have been intensively studied in the system-and-control community, state convergence has received little attention, especially for uncertain systems. Unlike asymptotic convergence, where the state of a certain system is supposed to converge to a known (desired) state, with state convergence, we mean convergence of a system to some state, which in general is unknown a-priori. For this reason, we address the state convergence problem of continuous- and discrete-time linear systems, both time invariant and affected by polytopic uncertainty, by means of spectral and geometrical analysis, Lyapunov and LaSalle theories.

The problem of convergence to a constant, a-priori unknown equilibrium state is ubiquitous and spans, for instance, from distributed consensus problems to multi-agent optimization and games over networks, positive system dynamics and tuning of plants with unknown input-output map. The first two applications are used in the sequel as motivating examples, while the remaining are illustrated in the example section (§8).

Motivating applications: We consider polar opinion dynamics in social networks [13,1,19]. Given a group of \( N \) agents indexed by the set \( I := \{1, \ldots, N\} \) and connected in a directed social network, we refer to the model proposed in [1], where the collective opinion vector \( [x_1; \ldots; x_N] = x \in [-1,1]^N \), evolves according to

\[
\dot{x}(t) = -A(x(t))Lx(t).
\] (1)

Here, \( A(x(t)) \in \text{diag}([0,1]^N) \), for all \( t \geq 0 \), is the diagonal matrix that characterizes the susceptibility to persuasion of each agent, while \( L \in \mathbb{R}^{N \times N} \) is the Laplacian matrix of the graph. The fundamental question in these models is whether or not the state \( x(t) \) converges to some a-priori unknown state \( \bar{x} \). In that case, we have consensus, since \( A(\bar{x})L\bar{x} = 0 \), hence \( \bar{x} \) is a zero of the mapping \( x \mapsto A(x)L \). Since \( A(x(t)) \in \text{diag}([0,1]^N) \),
where \( \Omega \) and \( F(x) := \sum_{i=1}^{N} \nabla_{x_i} J_i(x_i, x_{-i}) \) is the so-called pseudo-gradient mapping. Clearly, convergence of the affine difference inclusion in (4) implies convergence of the original nonlinear dynamics in (3). In this paper, we study uncertain linear systems, since the affine case can be cast into the linear one via additional auxiliary states, see §8.1.

Essentially, the Nash equilibrium problem is the problem to ensure that the state \( x(t) \) of the system in (4) converges to some a-priori unknown state \( x^* \), which happens to be a Nash equilibrium due to the structure of the dynamics. Similarly, primal-dual projected pseudo-gradient dynamics can be designed for computing a generalized Nash equilibrium in games with coupling constraints [5,21,6]. While multi-agent optimization and game equilibrium dynamics are typically nonlinear, the analysis of the linear case provides necessary conditions for convergence [4], hence potential certificates of non-convergence, see [14] for an example of non-convergent linear time-varying primal-dual dynamics.

**Contribution:**

- We give necessary and sufficient conditions for the state convergence of linear time invariant systems. Specifically, we link state convergence with the existence of a (weak) Lyapunov function, the separability of the eigenvalues, the stability of an auxiliary system and linear matrix inequalities (§3 - for ease of readability, the proofs of this section are reported in Appendix A);
- We introduce the notions of weak and strong convergence for uncertain systems. We show that a sufficient condition, i.e., the kernel sharing property among the generator matrices of the difference/differential inclusion, implies that the two definitions are equivalent (§4);
- Strong convergence is possible if and only if the eigenvalues of the generator matrices can be separated from the critical ones. We show that marginal stability is necessary for strong convergence, while the existence of a quadratic or a polyhedral Lyapunov function is not sufficient (§5);
- We define the concept of weak kernel and, by exploiting LaSalle arguments, we investigate weak convergence, providing sufficient conditions and paving the way for novel research directions (§6);
- We associate sufficient linear matrix inequalities to weak and strong convergence (§5-6);
- We show the lack of duality of state convergence in uncertain systems. However, we show that the existence of a quadratic Lyapunov function guarantees duality of strong convergence (§7).

**Notation:** \( \mathbb{R} \) and \( \mathbb{R}_{\geq 0} \) denote the set of real and non-negative real numbers, respectively. \( \text{bdry}(S) \) denotes the boundary of a set \( S \). Given a matrix \( A \in \mathbb{R}^{n \times n} \), \( \ker(A) := \{ x \in \mathbb{R}^n \mid 0 = Ax \} \) denotes
its kernel; \( \mu(A) \) denotes its Lozinski measure, i.e.,
\[
\mu(A) = \lim_{h \to 0} \frac{\|A + hI\|_1}{h},
\]
\( S^n_{\geq 0} \) is the set of positive (semi-)definite symmetric matrices in \( \mathbb{R}^{n \times n} \). \( \mathbf{0}(1) \) denotes vectors with elements all equal to 0 (1).

2 Mathematical background

We start with some basic system-theoretic definitions to characterize equivalence results on the convergence of discrete-time (DT) and continuous-time (CT), both linear time-invariant (LTI) and uncertain systems.

First, we consider a DT LTI system
\[
x(k + 1) = A x(k),
\]
where \( x \in \mathbb{R}^n \) and \( A \in \mathbb{R}^{n \times n} \). For LTI systems, we are interested in global convergence of the state \( x \) to some vector \( \bar{x} \), which may depend on the initial conditions \( x(0) \), accordingly with the following definition.

Definition 1 (Convergence)
The system in (5) is convergent if, for all \( x(0) \in \mathbb{R}^n \),
\[
\lim_{k \to \infty} \|x(k) - \bar{x}\| = 0.
\]

The convergence of LTI systems is closely related with the location of the eigenvalues of the matrix \( A \) and their algebraic/geometric multiplicity. Thus, we recall the notion of semi-simple eigenvalue as follows [12].

Definition 2 ((Semi-) Simple eigenvalue)
An eigenvalue is semi-simple if it has algebraic and geometric multiplicity. An eigenvalue is simple if it has algebraic and geometric multiplicities both equal to 1.

Finally, we recall the fundamental definition of system (marginal) stability and (weak) Lyapunov function.

Definition 3 (Stability)
The system in (5) is stable if, for all \( x(0) \in \mathbb{R}^n \),
\[
\lim_{k \to \infty} \|x(k)\| = 0.
\]

Definition 4 (Weak Lyapunov function)
A positive definite function \( V : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) is a weak Lyapunov function (wLF) for the system in (5) if \( V(Ax) \leq V(x) \), for all \( x \in \mathbb{R}^n \).

In particular, to link convergence results and stability of dynamical systems, we will consider several classes of wLFs, as weak Quadratic Lyapunov functions (wQLFs), weak Polyhedral Lyapunov functions (wPLFs) and, more generally, weak Convex Lyapunov functions (wCVXLFs).

The same definitions of convergence and stability apply to the CT LTI system
\[
\dot{x}(t) = Ax(t),
\]
by substituting \( k \in \mathbb{N} \) with \( t \geq 0 \). The definition of wLF in Definition 4, instead, reads as \( V(x) \leq 0 \) for all \( x \in \mathbb{R}^n \).

3 Convergence in LTI systems

3.1 Discrete-time systems

In this section, we give some equivalence results that link the convergence of (5) with the stability of an auxiliary system. Linear Matrix Inequality (LMI) conditions and existence of a wLF. The proofs are reported in the appendix.

Lemma 1 The system in (5) is convergent if and only if there exists \( \eta \in (0, 1) \) such that the system
\[
x(k + 1) = \left( \frac{1}{\eta} A - \frac{1 - \eta}{\eta} I \right) x(k) = A^{\eta}_{\text{st}} x(k),
\]
is stable.

\[\square\]

Theorem 1 (Convergence in DT LTI systems)
The following statements are equivalent:
(a) The system in (5) is convergent;
(b) \( \exists \eta \in (0, 1) \) such that the system in (7) is stable;
(c) there exists a wPLF for the system in (7);
(d) there exists an invertible matrix \( T \in \mathbb{R}^{n \times n} \) such that
\[
T^{-1} A T = \begin{bmatrix}
A^{\text{ss}} & 0 \\
A^* & I
\end{bmatrix},
\]
for some Schur matrix \( A^{\text{ss}} \);
(e) \( \exists \eta \in [0, 1) \) and \( P \in S^n_{\geq 0} \) such that the following LMI holds:
\[
A^T P A \preceq (1 - \eta)(A^T P + PA) + (2\eta - 1)P;
\]
(f) \( \exists \eta \in [0, 1), Q \in S^n_{\geq 0} \) and \( P > 0 \), with \( \text{rank}(P) = n - \dim(\ker(A - I)) \), such that the following LMI holds:
\[
\eta(A^T P A - P) + (1 - \eta)(A^T - I)Q(A - I) \preceq 0.
\]

\[\square\]

Remark 1 The system in (5) is convergent if and only if the map \( x \mapsto Ax \) is averaged, i.e., if there exist \( P > 0 \) and \( \eta \in [0, 1) \) such that, for all \( x, y \in \mathbb{R}^n \),
\[
A^T P A \preceq (2\eta - 1)P + (1 - \eta)(A^T P + PA),
\]
which is equivalent to that in Theorem 1 (e), [4].

\[\square\]
Theorem 2 The LMI in item (e) in Theorem 1 can be equivalently rewritten as
\[ \eta(A^T PA - P) + (1 - \eta)(A^T - I)P(A - I) \preceq 0. \quad (8) \]

We conclude the subsection with an example of non-convergence, despite the presence of a wPLF.

Proposition 1 The existence of a wPLF for the DT LTI system in (5) does not imply convergence.

PROOF. Consider the rotation matrix \( A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \). The state does not converge, yet \( \|x(k)\|_\infty \) is constant. \( \blacksquare \)

3.2 Continuous-time systems

To characterize the convergence of the CT LTI in (6), let us introduce the following DT auxiliary system.

Definition 5 (Euler Auxiliary System)
Given \( \tau > 0 \), the DT system
\[ x(k + 1) = (I + \tau A) x(k) =: A_\tau x(k) \quad (9) \]
is the Euler Auxiliary System (EAS) of the CT system in (6).

Then, let us consider the following results.

Lemma 2 The system in (6) is convergent if and only if there exists \( \epsilon > 0 \) such that the system
\[ \dot{x}(t) = A(I + \epsilon A)^{-1} x(t) =: A_\epsilon^x x(t), \quad (10) \]
is stable. \( \blacksquare \)

Theorem 2 (Convergence in CT LTI systems)
The following statements are equivalent:

(a) The system in (6) is convergent;
(b) \( \exists \epsilon > 0 \) such that the system in (10) is stable;
(c) there exists a wPLF for the system in (6);
(d) there exists an invertible matrix \( T \in \mathbb{R}^{n \times n} \) such that
\[ T^{-1} A T = \begin{bmatrix} A^{as} & 0 \\ A^r & 0 \end{bmatrix}, \]
for some Hurwitz matrix \( A^{as} \);
(e) \( \exists \tau > 0 \) such that the EAS in (9) converges;
(f) \( \exists \epsilon > 0 \) and \( P \in S^n_{>0} \) such that the following LMI holds:
\[ A^T P + PA + \epsilon A^T PA \preceq 0; \]
(g) \( \exists \epsilon > 0, Q \in S^n_{>0} \) and \( P \succeq 0, \) with \( \text{rank}(P) = n - \dim(\ker(A)) \), such that the following LMI holds
\[ A^T P + PA + \epsilon A^T QA \preceq 0. \] \( \blacksquare \)

4 Uncertain linear systems

For difference or differential linear inclusions [2] the definition of convergence requires some care. Specifically, we shall distinguish between weak and strong convergence.

4.1 Weak and strong convergence

We consider uncertain DT linear systems of the form
\[ x(k + 1) = A(w(k)) x(k) \quad (11) \]
where \( A(w) \) satisfies the following assumption.

Standing Assumption 1 (Polytopic uncertainty)
\[ A(w(k)) := \sum_{i \in \mathcal{M}} A_i w_i(k), \]
with \( \mathcal{M} := \{1, 2, \ldots, m\} \) and \( w(k) \in \mathcal{W} \), defined as
\[ \mathcal{W} := \left\{ w \in \mathbb{R}^m \left| \sum_{i \in \mathcal{M}} w_i = 1, w_i \geq 0 \forall i \in \mathcal{M} \right. \right\}. \] \( \blacksquare \)

In CT, we consider the differential inclusion of the form
\[ \dot{x}(t) = A(w(t)) x(t), \quad (12) \]
with the same polytopic uncertainty structure. As for LTI systems, we investigate whether \( x(k) \) converges to some \( \bar{x} \), which in general depends on \( w(k) \). For instance, if \( A(w(k)) = \begin{bmatrix} a_{1,1}(k) & 0 \\ 0 & 0 \end{bmatrix} \), with \( a_{1,1}(k) \in \{1/2, 3/4\} \) for all \( k \in \mathbb{N} \), then \( x_1(k) \to 0 \) and \( x_2(k) \to 0 \), for \( x_1(0) = 1 \) and \( \sum_{k \geq 0} x_1(k) \), which is finite but depends on \( (a_{1,1}(k))_{k \in \mathbb{N}} \).

In view of this example, we give two different definitions of convergence.

Definition 6 (Weak convergence)
The difference inclusion in (11) (respectively, differential inclusion in (12)) is weakly convergent if, for all sequences \( w(k) \in \mathcal{W} \) (resp., for all \( w(t) \in \mathcal{W} \)) and initial conditions \( x(0) \in \mathbb{R}^n \), there exists a vector \( \bar{x} \in \mathbb{R}^n \) such that \( \lim_{k \to \infty} \|x(k) - \bar{x}\| = 0 \) (\( \lim_{t \to \infty} \|x(t) - \bar{x}\| = 0 \)). \( \blacksquare \)

Next, we introduce a stronger notion of convergence, i.e., convergence to the common kernel of the matrices \( \{A_i - I\}_{i \in \mathcal{M}} \) in DT (11), or \( \{A_i\}_{i \in \mathcal{M}} \) in CT (12).
Definition 7 (Strong convergence)
The difference inclusion in (11) (resp., differential inclusion in (12)) is strongly convergent if, for all sequences \( w(k) \in \mathcal{W} \) (resp., for all \( w(t) \in \mathcal{W} \)) and initial conditions \( x(0) \in \mathbb{R}^n \), there exists a vector \( \bar{x} \in \mathcal{X} \subseteq \mathbb{R}^n \), where
\[
\mathcal{X} := \bigcap_{i \in \mathcal{M}} \ker (A_i - I), \quad \text{resp., } \mathcal{X} := \bigcap_{i \in \mathcal{M}} \ker (A_i)
\]
such that \( \lim_{k \to \infty} \| x(k) - \bar{x} \| = 0 \left( \lim_{t \to \infty} \| x(t) - \bar{x} \| = 0 \right) \). □

Therefore, to have strong convergence, the limit vector must be a steady state. In other words, if we initialize the state \( x(0) \in \mathcal{X} \) then \( x(k) = x(0) \) for all \( k \geq 0 \) and for all possible sequences \( w(k) \in \mathcal{W} \). On the other hand, this is not ensured with a limit vector in the case of weak convergence. Moreover, as stressed in the following example, while strong convergence implies weak convergence, the converse does not necessarily hold.

Example 1 The scalar difference inclusion \( x(k + 1) \in \{1/2, 1\} x(k) \) is weakly convergent (because the sequence \( \{x(k)\}_{k \in \mathbb{N}} \) is non-increasing over \( k \geq 0 \)) but not strongly convergent. □

4.2 Kernel sharing property, when weak convergence implies strong convergence

We now investigate under which conditions the weak convergence of an uncertain system of the form (11) (or (12)) implies strong convergence.

Proposition 2 If the difference inclusion in (11) (resp., differential inclusion in (12)) is strongly convergent, then the matrices \( \{A_i - I\}_{i \in \mathcal{M}} \) (resp., \( \{A_i\}_{i \in \mathcal{M}} \)) have exactly the same kernel:
\[
\ker (A_i - I) = \bigcap_{i \in \mathcal{M}} \ker (A_i - I), \quad \forall i \in \mathcal{M} \quad \text{resp., } \ker (A_i) = \bigcap_{i \in \mathcal{M}} \ker (A_i), \quad \forall i \in \mathcal{M}
\]

The family of matrices \( \{A_i\}_{i \in \mathcal{M}} \) has the Kernel Sharing Property if (13) holds true.

Note that the KSP is equivalent to claiming that \( \ker (A(w)) = \mathcal{K} \) (resp., \( \ker (A(w) - I) = \mathcal{K} \)) is invariant with respect to \( w \in \mathcal{W} \). We now show that, under the KSP assumption, if the system is weakly convergent, then it is also strongly convergent. Some preliminary results are reported first.

Lemma 3 Let \( x(k) \) be a solution to (11). If \( x(k) \to \bar{x} \), then \( A(w(k)) - I\bar{x} \to 0 \).

Proof. Let \( z(k) := x(k) - \bar{x} \). In view of (11), we have
\[
z(k + 1) = A(w(k))z(k) + (A(w(k)) - I)\bar{x}.
\]
Therefore, \( z(k) \to 0 \) requires \( (A(w(k)) - I)\bar{x} \to 0 \). □

The CT counterpart is not immediate, due to some technical problems, as stressed by the following example.

Example 2 The scalar differential inclusion \( \dot{x}(t) \in \{0, -1\} x(t) \) is weakly convergent (because \( x(t) \) is non-increasing over \( t \geq 0 \)). Now, let us consider the case that \( A \) switches to \(-1\) with spikes of decreasing length \( \Delta_n < 1 \), i.e., let \( A(w(t)) = -1 \) when \( t \in [n, h + \Delta_n] \), and \( A(w(t)) = 0 \) otherwise. By considering an interval size such that \( \sum_{n \geq h} \Delta_n = \ln(2) \), we observe that \( x(t) \) tends asymptotically to \( x(0)/2 \), which is not in the common kernel of the \( A_i \) matrices. □

Thus, the natural extension of Lemma 3 to the CT case does not hold, i.e., \( x(t) \to \bar{x} \) does not imply that \( A(w(t))\bar{x} \to 0 \). However, convergence does happen on average, as formalized next.

Lemma 4 Let \( x(t) \) be a solution to (12). If \( x(t) \to \bar{x} \), then
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T A(w(t))\bar{x} \, dt = 0.
\]

Proof. Let \( \bar{z}(t) := x(t) - \bar{x} \to 0 \). Then, by (12),
\[
\dot{z}(t) = A(w(t))z(t) + A(w(t))\bar{x}.
\]
By integrating and dividing by \( T \) we obtain
\[
\frac{1}{T} \int_0^T A(w(t))\bar{x} \, dt = \frac{1}{T} \int_0^T (\dot{z}(t) - A(w(t))z(t)) \, dt = \frac{1}{T} \int_0^T [z(T) - z(0)] - \frac{1}{T} \int_0^T A(w(t))z(t) \, dt \to 0.
\]
For \( T \to \infty \), the first term converges to 0 because \( x(t) \) is bounded and, since \( A(w(t))z(t) \to 0 \), the average of
a function converging to 0 converges to 0 as well. Thus,
\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T A(w(t)) \dd x dt = 0. \]

Lemma 5 Let \( x(k) \) be a solution to (11) (resp., \( x(t) \) solution to (12)). If \( x(k) \to \bar{x} \) (resp., \( x(t) \to \bar{x} \)), then there exists some \( \bar{w} \in \mathcal{W} \) such that \( (A(\bar{w}) - I) \bar{x} = 0 \) (resp., \( A(\bar{w}) \bar{x} = 0 \)).

PROOF. In the DT case, by Lemma 3 we have:
\[
(u(k) := \sum_{i \in M} (A_i w_i(k) - I) \bar{x} = \sum_{i \in M} w_i(k) (A_i - I) \bar{x} \to 0.
\]

For all \( k \in \mathbb{N} \), since \( w(k) \in \mathcal{W} \), the vector \( u(k) \) belongs to the convex hull of the vectors \( \{ (A_i - I) \bar{x} \}_{i \in M} \), which is closed and convex. Thus, in the limit for \( k \to \infty \), \( u(k) \) shall belong to the convex hull as well, i.e., \( 0 = \sum_{i \in M} \bar{w}_i (A_i - I) \bar{x} \), namely, \( (A(\bar{w}) - I) \bar{x} = 0 \).

In the CT case, by Lemma 4, we have:
\[
\frac{1}{T} \int_0^T \sum_{i \in M} w_i(t) A_i \dd x dt = \left( \sum_{i \in M} \frac{1}{T} \int_0^T w_i(t) dt \right) A_i \bar{x} \to 0,
\]
where \( \omega_i(T) := \frac{1}{T} \int_0^T w_i(t) dt \) denotes the nonnegative average values of \( \{ w_i(t) \}_{i \in M} \) for every \( T > 0 \). Then, to derive \( A(\bar{w}) \bar{x} = 0 \) for some \( \bar{w} \), we use the same argument on the limit used in the DT case.

We are now ready to exploit the previous lemmas to link weak and strong convergence under the KSP.

Proposition 3 If the family of matrices \( \{ A_i \}_{i \in M} \) has the Kernel Sharing Property, then weak convergence implies strong convergence.

PROOF. In view of Lemma 5, the convergence (if it holds) shall be in the kernel of some \( (A(\bar{w}) - I) \) (resp., \( A(\bar{w}) \)), since the kernel is common.

Remark 3 In view of the properties of the graph Laplacian matrix, the family of matrices \( \{ A_i, L \}_{i \in M} \) in (2) has the KSP, since \( 1 \in \ker(A_i L) \), for all \( i \in M \).

The KSP, which can be efficiently verified by means of linear algebra arguments, is crucial to investigate strong convergence. In fact, it follows from Proposition 2 and 3 that if the matrices do not have a common kernel, there can not be strong convergence.

5 Strong convergence characterization

We first characterize the strong convergence property of difference (differential) inclusions via separability of the eigenvalues of the generator matrices. Essentially, if the KSP holds, then the convergence analysis reduces to investigate the asymptotic stability of a subsystem.

Theorem 3 The following statements are equivalent:

(a) The difference inclusion in (11) (resp., differential inclusion in (12)) is strongly convergent;
(b) there exists an invertible matrix \( T \in \mathbb{R}^{n \times n} \) such that, for all \( i \in M \),
\[
T^{-1} A_i T = \begin{bmatrix} A_{i1} & 0 \\ A_{i2} & I_m \end{bmatrix} \quad \text{(resp., } \begin{bmatrix} A_{i1} & 0 \\ A_{i2} & I_m \end{bmatrix} \text{)},
\]
where \( m := \dim(\bar{X}) \) and the matrices \( \{ A_{i1} \}_{i \in M} \) generate an asymptotically stable difference inclusion (resp., differential inclusion).

PROOF. We give the proof for the DT case, since the one in CT is analogous. Let \( T_2 \in \mathbb{R}^{n \times n} \) be the matrix generated by a basis of \( \bar{X} \), i.e., \( (A_i - I) T_2 = 0 \) for all \( i \in M \) and \( T_1 \in \mathbb{R}^{n \times (n-m)} \) be a complement of \( T_2 \) such that \( T := [T_1, T_2] \in \mathbb{R}^{n \times n} \) is invertible. Then, \( T \) determines a similarity transformation that separate the eigenvalues of the matrices \( \{ A_i \}_{i \in M} \) as follows:
\[
T^{-1} A_i T = \begin{bmatrix} A_{i1} & 0 \\ A_{i2} & I_m \end{bmatrix}, \quad \text{for all } i \in M.
\]

Since the difference inclusion (11) converges to \( \bar{X} \), this implies that the matrices \( \{ A_{i1} \}_{i \in M} \) shall generate an asymptotically stable difference inclusion.

5.1 Lyapunov-like results on strong convergence

Next, we characterize the strong convergence property for both difference and differential inclusions by means of the Lyapunov analysis. First, we show that strong convergence implies the existence of a polyhedral Lyapunov function.

Theorem 4 If the difference inclusion in (11) (resp., differential inclusion in (12)) is strongly convergent, then it admits a wPLF.

PROOF. We give the proof for the CT case (the one in DT is analogous). The strong convergence of (12) allows to partition each matrix \( A_i \) as in (14) with \( \{ A_{i1} \}_{i \in M} \) as follows:

\[
T^{-1} A_i T = \begin{bmatrix} A_{i1} & 0 \\ A_{i2} & I_m \end{bmatrix}, \quad \text{for all } i \in M.
\]

Since the difference inclusion (11) converges to \( \bar{X} \), this implies that the matrices \( \{ A_{i1} \}_{i \in M} \) shall generate an asymptotically stable difference inclusion.
that generate an asymptotically stable differential inclusion. By [10, Prop. 7.39], it admits a PLF whose vertices are in \( X \in \mathbb{R}^{(n-m) \times r} \) such that, for all \( i \in \mathcal{M} \),

\[
A_i^{\text{ass}} X = X P_i,
\]

with \( P_i \in \mathbb{R}^{r \times r} \) strictly column diagonally dominant, i.e., \( \mu(P_i) < 0 \), for all \( i \in \mathcal{M} \). Then, let us consider the augmented system with \( \beta > 0 \):

\[
\begin{bmatrix}
A_i^{\text{ass}} & 0 \\
A_i' & 0_m
\end{bmatrix}
\begin{bmatrix}
X \\
0 & \beta I_m
\end{bmatrix}
= 
\begin{bmatrix}
X & 0 \\
0 & \beta I_m
\end{bmatrix}
\begin{bmatrix}
P_i \\
1_\beta A_i X \\
0_m
\end{bmatrix}.
\]

For \( \beta \) sufficiently large, each matrix \( P_i^{\text{aug}} \) is such that \( \mu(P_i^{\text{aug}}) \leq 0 \), so that \( X^{\text{aug}} \) represents the matrix of vertices of a wPLF.

While strong convergence implies the existence of a wPLF, it does not imply the existence of a wQLF. This follows by seeing asymptotic stability as a special case of strong convergence to the kernel \( \{0\} \) (i.e., robust asymptotic stability). In literature, there are examples of systems that are asymptotically stable but do not admit a quadratic Lyapunov functions, see [10]. However, the existence of either a wPLF or a wQLF in general does not imply convergence, both strong and weak, as stated next.

**Proposition 4** The existence of either a wPLF or a wQLF for the difference inclusion in (11) (resp. differential inclusion in (12)) does not imply convergence. \( \square \)

**PROOF.** For the DT case, let us consider the scalar difference inclusion \( x(k+1) \in \{-1, 1\} x(k) \), with \( x(0) = 1 \). \( V(x) = x^2 \) is a wQLF, and \( \sqrt{V(x)} = |x| \) is a wPLF. By taking \( A(w(k)) = 1 \) for \( k \) even and \( A(w(k)) = -1 \) for \( k \) odd, the system does not converge. The proof in CT goes by means of two examples.

\( \exists \) wPLF \( \neg \) weak convergence of (12)\( ^{T} \): Let us consider the case \( A(w(t)) \in \left\{ \begin{bmatrix} -\alpha & \beta \\ -\beta & \alpha \end{bmatrix}, \begin{bmatrix} -\gamma & \delta \\ -\delta & \gamma \end{bmatrix} \right\} \) for some \( \alpha, \beta, \gamma, \delta > 0 \). This system is column weakly diagonally dominant, hence \( \|x\|_1 \) is a wPLF [10]. If we take \( x(t) \) such that \( x_1(0) + x_2(0) = 1 \), then \( x_1(t) + x_2(t) = 1 \). The motion on the plane \( x_1 + x_2 = 1 \) has no rest. Indeed, \( x_2 \) is \( -(\alpha + \beta)x_2 + \alpha x_1 = -(\alpha + \beta)x_2 + \alpha \) in the first mode, and \( x_2 = -(\gamma + \delta)x_2 + \gamma \) in the second mode. So \( x_2(t) \) is not constant under persistent switching.

\( \exists \) wQLF \( \neg \) weak convergence of (12)\( ^{T} \): The quadratic \( V(x) = \|x\|^2 \) is a wQLF for the differential inclusion \( A(w) = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix} \), with \( w > 0 \). In fact, \( \sqrt{V(x)} = 0 \), but the system exhibits persistent oscillations since the eigenvalues are placed on the imaginary axis, for all \( t \geq 0 \). \( \square \)

**Theorem 5** The difference inclusion in (11) (resp., differential inclusion in (12)) is strongly convergent if and only if there exist a full row rank matrix \( X \) and matrices

\[
P_i = \begin{bmatrix} P_i^{\text{ass}} & 0 \\
P_i' & I_m
\end{bmatrix} \quad \text{(resp., } P_i = \begin{bmatrix} P_i^{\text{ass}} & 0 \\
P_i' & 0_m
\end{bmatrix}\text{)},
\]

where \( \{P_i^{\text{ass}}\}_{i \in \mathcal{M}} \) are such that \( \|P_i^{\text{ass}}\|_1 < 1 \) (resp., \( \mu(P_i^{\text{ass}}) < 0 \)) and, for all \( i \in \mathcal{M} \),

\[
A_i X = X P_i.
\]

**PROOF.** We give the proof for the CT case, since the one in DT is analogous. In particular, the necessity part has been proved in Theorem 4. For the sufficiency, let \( T \) be the transformation such that, for all \( i \in \mathcal{M} \),

\[
T^{-1} A_i T = \begin{bmatrix} A_{i1} & 0 \\
A_{i2} & 0_m
\end{bmatrix}.
\]

Then, by letting

\[
\hat{X} = T^{-1} X = \begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\
\hat{X}_3 & \hat{X}_4
\end{bmatrix},
\]

we obtain:

\[
\begin{align*}
A_{i1} \hat{X}_1 &= \hat{X}_1 P_i^{\text{ass}} + \hat{X}_2 P_i', \\
A_{i2} \hat{X}_1 &= \hat{X}_3 P_i^{\text{ass}} + \hat{X}_4 P_i', \\
|A_{i1} A_{i2}| &\preceq 0.
\end{align*}
\]

The last equation suggests that \( \hat{X}_2 = 0 \) (otherwise \( \dim(\hat{X}) > m \)) so that \( A_{i1} \hat{X}_1 = \hat{X}_1 P_i^{\text{ass}} \). Since \( \{P_i^{\text{ass}}\}_{i \in \mathcal{M}} \) are strictly column diagonally dominant, we obtain the asymptotic stability of the uncertain polytopic subsystem with matrices \( \{A_{i1}\}_{i \in \mathcal{M}} \) and therefore, from Theorem 3, the strong convergence of (12). \( \square \)

As for deterministic systems, we give (in this case, sufficient) LMI conditions to characterize strong convergence of difference and differential inclusions.

**Theorem 6** The difference inclusion in (11) (resp., differential inclusion in (12)) is strongly convergent if there exist \( P \succ 0 \), with rank(\( P \)) = \( n - \dim(\hat{X}) \), \( Q \succ 0 \) and \( \epsilon > 0 \) such that, for all \( i \in \mathcal{M} \),

\[
A_i^T P A_i + \epsilon (A_i^T I - I) Q (A_i - I) \preceq P,
\]

\[
(\text{resp., } A_i^T P + P A_i + \epsilon A_i^T Q A_i \preceq 0).
\]

**PROOF.** We first prove the statement for the DT case. Let \( U \) be an orthogonal matrix such that

\[
\hat{P} = U^T Q^{1/2} P Q^{-1/2} U = \text{diag}(P_1, 0),
\]

with \( P \succ 0 \), and let \( \hat{A}_i = U^T Q^{1/2} A_i Q^{-1/2} U \). Thus,

\[
\hat{A}_i^T \hat{P} \hat{A}_i + \epsilon (\hat{A}_i^T I - I)^T (\hat{A}_i - I) \preceq \hat{P},
\]
which leads to \( \hat{A}_i = \begin{bmatrix} A_{i1} & 0 \\ A_{i2} & I \end{bmatrix} \), and

\[
A_{i1}^T P_i A_{i1} - P_i \preceq \epsilon (A_{i1} - I)^T (A_{i1} - I) + \epsilon A_{i2}^T A_{i2}.
\]

By the assumption on the rank of \( P \), the matrix on the right-hand side is positive definite, for all \( i \in \mathcal{M} \). Thus, the subsystem with dynamics induced by \( \{A_{i1}\}_{i \in \mathcal{M}} \) is asymptotically stable under arbitrary switching, hence the differential inclusion is strongly convergent.

The proof in CT mimics the one for the implication (g) \( \Rightarrow \) (d) of Theorem 2, by replacing \( A \) with \( \hat{A}_i \) and \( \hat{A}_i \) with \( A_{i1} \), for all \( i \in \mathcal{M} \). The positive definite matrix \( P_i \) generates a common wQLF for the subsystem with matrices \( \{\hat{A}_i\}_{i \in \mathcal{M}} \) and hence strong convergence. \( \square \)

**Remark 4** The first LMI in (15) can be equivalently rewritten as

\[
\eta(A_i^T P A_i - P) + (1 - \eta)(A_i^T - I) Q (A_i - I) \preceq 0,
\]

with \( \eta = 1/(1+\epsilon) \). This matches the inequality at item (f) in Theorem 1, with \( A \) replaced by \( A_i \).

Finally, we characterize the relation between the strong convergence of the differential inclusion in (12) and the following Euler auxiliary difference inclusion

\[
x(k + 1) = (I + \tau A(w(k))) x(k).
\]  

(16)

**Proposition 5** The differential inclusion in (12) is strongly convergent if and only if there exists \( \tau > 0 \) such that the difference inclusion in (16) is strongly convergent. \( \square \)

**PROOF.** For all \( i \in \mathcal{M} \), let \( A_{r_i} := I + \tau A_i \). The proof follows from Proposition 5 by first noticing that \( \ker(A_i) = \ker(A_{r_i} - I) \), for all \( i \in \mathcal{M} \). Moreover, the fact that \( \mu(P_i^{\text{ss}}) < 0 \) implies \( \|I + \tau P_i^{\text{ss}}\|_1 < 1 \) for all \( \tau < \min_k \{ -2/d_{kk} \} \), where \( d_{kk} \) are the diagonal entries of \( P_i^{\text{ss}} \). Finally, if \( \tau > 0 \) is such that \( \|P_i^{\text{ss}} - I\|_1 < 1 \), then \( \mu((P_i^{\text{ss}} - I)/\tau) > 0 \). \( \square \)

6 Weak convergence

In general, weak convergence is much harder to analyze than strong convergence. A necessary condition for convergence requires that the difference (differential) inclusion is at least marginally stable. To have convergence to a nonzero vector \( \hat{x} \), at least one matrix inside the convex hull of the matrices \( \{A_i - I\}_{i \in \mathcal{M}} \) (resp., \( \{A_i\}_{i \in \mathcal{M}} \)) shall be singular. In this case, the convergence does not need a common kernel and it can be associated with a specific singular element. As an example in CT, let us consider

\[
A(w(t)) \in \{A_1, A_2\} = \left\{ \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \right\},
\]

where \( \hat{A} = \frac{1}{2}(A_1 + A_2) \) is singular. Then, if \( A(w) \) “converges” to \( \hat{A} \), the trajectory converges to a nonzero vector of the form \( [\hat{x}1; \hat{x}0] \), otherwise \( x \rightarrow 0 \). This type of LaSalle arguments are considered next.

6.1 LaSalle-like criteria and the weak kernel

**Theorem 7** Let \( V \) be a positively homogeneous wLF for the differential inclusion in (12) with derivative

\[
D^+ V(x, A(w)x) := \limsup_{h \rightarrow 0^+} \frac{V(x + hA(w)x) - V(x)}{h},
\]

and assume that

\[
D^+ V(x, A(w)x) \leq -\phi(x, A(w)x),
\]  

(17)

for some positive semi-definite, locally Lipschitz function \( \phi \). Then \( x(t) \), solution to (12), converges to the set

\[
\mathcal{N} := \left\{ x \in \mathbb{R}^n \mid \min_{w \in \mathcal{W}} \{ \phi(x, A(w)x) \} = 0 \right\}.
\]  

(18)

**PROOF.** Let \( x(t) \) be any trajectory of (12). We have:

\[
V(x(0)) \geq \int_0^t -D^+ V d\sigma \geq \int_0^t \phi(x(\sigma), A(w(\sigma))x(\sigma)) d\sigma \geq \int_0^t \min_{w \in \mathcal{W}} \{ \phi(x(\sigma), A(w(\sigma))x(\sigma)) \} d\sigma.
\]  

(18)

The function \( \phi \) is Lipschitz continuous inside the set \( \{ x \in \mathbb{R}^n \mid V(x) \leq V(x(0)) \} \). Since \( \mathcal{W} \) is a compact and convex set, the min function is also nonnegative and Lipschitz [2, Th. 7, p. 93]. Therefore, in view of the Barbalat’s lemma, \( \lim_{t \rightarrow \infty} \min_{w \in \mathcal{W}} \{ \phi(x, A(w)x) \} = 0 \). \( \square \)

The next two corollaries follow as a direct consequence of Theorem 7.

**Corollary 1** Let \( V \) be a smooth wLF for the differential inclusion in (12). Then \( x(t), \) solution to (12), converges to the set

\[
\mathcal{N} := \left\{ x \in \mathbb{R}^n \mid \min_{i \in \mathcal{M}} -\nabla V(x) A_i x = 0 \right\}.
\]  

(19)

**PROOF.** The proof follows by applying Theorem 7 with \( \phi(x, A(w)x) = -\nabla V(x) A(w)x \geq 0 \) and by noticing that the minimum in (18) is achieved on the vertices, i.e., \( \min_{w \in \mathcal{W}} \{ -\nabla V(x) A(w)x \} = \min_{i \in \mathcal{M}} \{ -\nabla V(x) A_i x \} \). \( \square \)
Remark 5 Barbalat’s lemma requires Lipschitz continuity of $D^+V(x, A(w)x)$. Consequently, if the differential inclusion in (12) admits a non-smooth wLF, $V$, then we cannot guarantee the convergence to the set where $\min\{-D^+V(x, A(w)x)\} = 0$. However, we shall rely on some locally Lipschitz function $\phi$, as in Theorem 7. □

Corollary 2 Let $V$ be a positively homogeneous $wCVXLF$ for the Euler auxiliary difference inclusion in (16) and define

$$\Delta_\tau V(x, w) := \frac{V(x + \tau A(w)x) - V(x)}{\tau},$$

which is non-positive. Then $x(t)$, solution to (12), converges to the set

$$D := \left\{ x \in \mathbb{R}^n \middle| \min_{w \in \mathcal{W}} \{-\Delta_\tau V(x, w)\} = 0 \right\}.$$

PROOF. A property of convex functions is that, for all $x$ and $w$, $D^+V(x, A(w)x) \leq \Delta_\tau V(x, w)$ [18]. The proof then follows by noticing that $\Delta_\tau V(x, w)$ is Lipschitz continuous and by taking $-\phi = \Delta_\tau V$. ■

Corollary 3 Let $V$ be a positively homogeneous $wPLF$ for the differential inclusion in (12). Then, the function $\phi$ in (17) can be chosen as

$$\phi(x, A(w)x) = \Delta_\tau V(x, w)$$

for some $\tau > 0$ small enough. □

PROOF. A polyhedral function is a wPLF for the differential inclusion in (12) if and only if, for some $\tau > 0$, it is a wPLF for the associated Euler auxiliary difference inclusion. Moreover, by [18], $D^+V(x, A(w)x) \leq \Delta_\tau V(x, w)$. ■

To characterize weak convergence of differential inclusions, we now introduce the following definition.

Definition 9 (Weak Kernel)
The weak kernel of the differential inclusion in (12) is denoted by the set

$$\mathcal{K} := \{ x \in \mathbb{R}^n \mid \exists w \in \mathcal{W} \text{ s.t. } A(w)x = 0 \}. \quad \Box$$

By referring to the set $\mathcal{N}$ in (19), in general $\mathcal{K} \subseteq \mathcal{N}$. The next example shows a case in which $\mathcal{K}$ and $\mathcal{N}$ coincide, and the solution to (12) converges to the weak kernel.

Example 3 Let us consider the differential inclusion $A(w(t)) \in \{ A_1, A_2 \} = \{ \text{diag}(-1, 0), \text{diag}(0, -1) \}$, for which $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ is a wQLF. Then, the set $\mathcal{N}$ is characterized by the equation

$$\min_{i \in \mathcal{M}} \{ x^\top A_i x \} = \min \{ x_1^2, x_2^2 \} = 0,$$

which represents the (non-convex) union of the two axes. Note that, in general, if there exists a wQLF for the differential inclusion in (12), since $A_i P + PA_i = -Q_i$, with $Q_i \succ 0$ for all $i \in \mathcal{M}$, the set $\mathcal{N} = \{ x \in \mathbb{R}^n \middle| \min_{i \in \mathcal{M}} \{ x^\top Q_i x \} = 0 \}$ can be computed as the union of the kernels. □

If there exists a wPLF for the differential inclusion (12), we note an equivalence between asymptotic stability and the fact that the weak kernel is trivial.

Theorem 8 Let $V$ be a wPLF for the differential inclusion in (12). Then $A(w)$ is robustly non-singular, i.e., $\mathcal{K} = \{ 0 \}$, if and only if the solution to (12), $x(t)$, tends to 0 for all $x(0)$ and $w(t) \in \mathcal{W}_0$.

PROOF. If $A(w)$ is singular for some $w \in \mathcal{W}$, then we cannot have robust stability. Thus, we only need to prove the converse statement. Let $A(w)$ be robustly non-singular and let $V$ be defined by $\mathcal{K}$-planes, indexed in $\mathcal{S} := \{ 1, \ldots, s \}$. Given some $\rho > 0$, introduce the set $\mathcal{L}_{(V/\rho)} := \{ x \in \mathbb{R}^n \mid V(x) \leq \rho \} = \{ x \in \mathbb{R}^n \mid f_i^\top x \leq \rho, \forall i \in \mathcal{S} \}$. Moreover, let $A(x)$ be the set indexing the active planes, i.e., $A(x) := \{ i \in \mathcal{S} \mid f_i^\top x = V(x) \}$ and let $\mathcal{F}_A$ be any arbitrary $\ell$-dimensional face

$$\mathcal{F}_A := \{ x \in \mathcal{L}_{(V/\rho)} \mid f_i^\top x = \rho, \forall i \in \mathcal{E} \},$$

where $\mathcal{E}$ indexes the “active” planes at bdry ($\mathcal{L}_{(V/\rho)}$). The remainder of the proof exploits [9, Lemma 3]. Thus, for some $\ell$-dimensional face, and hence for some indices $i \in A(x)$, we shall have, for all $x \in \mathcal{F}_A$ and $w \in \mathcal{W}$,

$$D^+V(x, A(w)x) = f_i^\top A(w)x < 0.$$ 

Otherwise, the existence of some $w \in \mathcal{W}$ such that $f_i^\top A(w)x = 0$ for all $i \in A(x)$ implies the positive invariance of the LTI system $\dot{x} = A(w)x$, and hence the singularity of $A(w)$.

Thus, let $x(t)$ be the solution to (12) with initial condition $x(t_0)$. If $x(t_0) \in \mathcal{F}_A$, by continuity there exists a right neighborhood $[t_0, \tau_1)$ of $t_0$, with $\tau_1 > 0$, such that the active set can not grow, i.e., $A(x(t)) \subset A(x(t_0))$ for all $t \in [t_0, \tau_1]$. By repeating the same argument, $x(t)$ shall reach a face of dimension 1 where there is only a single active constraint, i.e., $A(x(t)) = \{ i \}$. Here, in view of the considerations above, $f_i^\top A(w)x < 0$ for all $x \in \mathcal{F}_A$ and $w \in \mathcal{W}$. Therefore, for any initial condition $x(t_0)$ such that $V(x(t_0)) = \rho$, in an arbitrary small neighborhood $[t_0, \tau_1]$ of $t_0$, for some $\tau > 0$, we have $V(x(t)) < \rho$. Then, $V(x(t)) \geq 0$ is monotonically non-increasing along any $x(t)$, and hence it has a limit $V(x(t)) \to \rho$ from above. From now on, the proof replicates the one in [9, Proof of Th. 2], and hence is here omitted. ■

Proposition 6 Nonsingularity, i.e., trivial weak kernel, along with the existence of a wLF (which is not polyhedral), does not imply convergence □

PROOF. It is sufficient to consider the system $\dot{x} = Ax$ with $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $V = \|x\|$. ■
Theorem 9 Let $V$ be a positively homogeneous $wCVXLF$ for the difference inclusion (11). Then $x(k)$, solution to (11), converges to the set

$$\mathcal{D} := \left\{ x \in \mathbb{R}^n \mid \min_{w \in \mathcal{W}} \{ V(A(w)x) - V(x) \} = 0 \right\}.$$

PROOF. We have $V(x(0)) = \sum_{k=0}^{\infty} \{ V(x(k)) - V(A(w(k))x(k)) \} \geq \sum_{k=0}^{\infty} \min_{w \in \mathcal{W}} \{ -\Delta V(x(k), w) \}$, where $-\Delta V(x(k), w(k)) = V(x(k)) - V(A(w(k))x(k))$.

The terms of the series are positive, hence boundedness implies that $\min_{w \in \mathcal{W}} \{ -\Delta V(x(k), w) \} \to 0$. \hfill □

We can apply the previous result to the CT case via the differential inclusion (12). Theorem 10

Corollary 4 Assume that the Euler auxiliary difference inclusion in (16) is weakly convergent. Then $x(t)$, solution to (12), converges to the set $\mathcal{N}$ in (19).

PROOF. Weak convergence of (16) implies the existence of a homogeneous $wCVXLF$. Then, the proof follows by Corollary 2. \hfill □

6.2 Lyapunov-like results on weak convergence

As for strong convergence, we now characterize the weak convergence property of uncertain systems by means of Lyapunov arguments.

Theorem 10 If the difference inclusion in (11) (resp., the differential inclusion (12)) is weakly convergent, then it admits a $wCVXLF$.

PROOF. Let us consider any polytopic, 0-symmetric, set $\mathcal{X} \subseteq \mathbb{R}^n$ including the origin in its interior, and denote $\mathcal{R}_k$ as the set of reachable states for the uncertain system (11) in at most $k$ steps from $x(0) \in \mathcal{X}_0$. Let $\mathcal{R}_\infty$ be the union of all these sets, i.e., $\mathcal{R}_\infty := \bigcup_{k \in \mathbb{N}} \mathcal{R}_k$.

By [3], $\mathcal{R}_\infty$ is bounded, namely there exists some $b \geq 0$ such that $\|x(k)\| \leq b$, for all $x \in \mathcal{R}_\infty$. Indeed, for all $k \in \mathbb{N}$, there exists some $x(k)$ that belongs to a trajectory originating from $x(0) \in \mathcal{X}_0$ (actually on a vertex [3]) which is on the boundary. Therefore, the boundedness of $\mathcal{R}_\infty$ is necessary for weak convergence. Thus, by defining $\bar{\mathcal{C}} := \text{conv}(\mathcal{R}_\infty)$, we note that its closure is a convex, compact, 0-symmetric set including the origin in its interior, and by [7, Th. 5.3] the norm induced by this set is a $wCVXLF$. The proof is similar (although a bit more involved) in the CT case. \hfill □

Sufficient LMI conditions for weak convergence are provided next, for both CT and DT cases.

Theorem 11 If there exist $\mathcal{P}_i > \mathcal{M}$ and $\mathcal{N}_i > \mathcal{M}$ such that, for all $i \in \mathcal{M}$, the LMI

$$A_i^\top \mathcal{P}_i + \epsilon (A_i^\top - I)P_i \preceq \mathcal{P}_i \quad (\text{resp.,} \; A_i^\top \mathcal{P}_i + P_i + \epsilon A_i^\top P_i \preceq \mathcal{P}_i)$$

holds, then the difference inclusion in (11) (resp., differential inclusion in (12)) is weakly convergent. \hfill □

PROOF. By considering the wQLF $V(x) = x^\top P x$, the Schur complement applied to the first LMI in (20) gives, for any $w \in \mathcal{W}$,

$$V(x(k+1)) - V(x(k)) \preceq \begin{bmatrix} x(k+1) - x(k) \\ P(x(k + 1) - x(k)) - \frac{1}{\epsilon} P \end{bmatrix} \preceq 0.$$

Then, $V(x(k+1)) - V(x(k)) \leq 0$ and $x(k+1) = x(k)$ for some $x(k) = \bar{x}$. By noticing that the first LMI in (20) can be equivalently rewritten as

$$\begin{bmatrix} P & A(w(k))^\top P & (A(w(k))^\top - I)P \\ P & 0 & 0 \\ P(A(w(k)) - I) & 0 & \frac{1}{\epsilon} P \end{bmatrix} \preceq 0,$$

for all $k \geq 0$, and by averaging in the interval $[0, K-1]$, we finally conclude that $\bar{x}$ belongs to ker$(A(\bar{w}))$, where $\bar{w} = \lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} w(k)$. Hence, the difference inclusion in (11) converges and $V(x)$ is a common wQLF.

The CT version is analogous: we consider the wQLF $V(x) = x^\top P x$ and we apply the Schur complement to the second LMI in (20), hence obtaining, for any $w \in \mathcal{W}$,

$$\begin{bmatrix} \dot{V}(x) \\ \dot{x}^\top P \\ P \dot{x} \\ -\frac{1}{\epsilon} P \end{bmatrix} \preceq 0.$$

Then, $\dot{V}(x) \leq 0$ and $\dot{V}(x) = 0$ for $\dot{x}(t) = 0$, i.e., for some $x = \bar{x}$. By noticing that the second LMI in (20) can be equivalently rewritten as

$$\begin{bmatrix} A(w(t))^\top P + PA(w(t)) & A(w(t))^\top P \\ P & -\frac{1}{\epsilon} P \end{bmatrix} \preceq 0,$$

for all $t \geq 0$, and by averaging in the interval $[0, T]$, we finally conclude that $\bar{x}$ is in the null space of $A(\bar{w})$, where $\bar{w} = \lim_{T \to \infty} \frac{1}{T} \int_0^T w(t) \, dt$. Thus, the differential inclusion converges and $V(x)$ is a common wQLF. \hfill □

Remark 6 The first LMI in (20) (resp., the second LMI in (20)) is equivalent to

$$(A_{\mathcal{H}_i}^{\text{ct}})^\top P A_{\mathcal{H}_i}^{\text{ct}} \preceq P \quad (\text{resp.,} \; (A_{\mathcal{H}_i}^{\text{ct}})^\top P + P A_{\mathcal{H}_i}^{\text{ct}} \preceq 0)$$
for all \( i \in \mathcal{M} \), where \( A_{\eta,i}^{dt} = \eta A_i - \frac{1}{\alpha} \eta I \) (resp., \( A_{\tau,i}^{rt} = A_i(I + \epsilon)^{-1} \)), implying weak stability of the auxiliary difference inclusion (resp., differential inclusion)

\[
x(k+1) = A_{\eta,i}^{dt}(w(k))x(k), \quad (\text{resp.,} \quad \dot{x}(t) = A_{\tau,i}^{rt}(w(t))x(t)) \leq \square
\]

Essentially, a comprehensive characterization of weak convergence is still an open research area.

### 7 Convergence lacks of duality

An interesting fact in studying the convergence of an uncertain system is the lack of duality. Indeed, it is known that, as long as we consider stability (asymptotic or marginal) of \( x(k+1) = A(w(k))x(k) \) (resp., \( \dot{x}(t) = A(w(t))x(t) \)), implies the stability of the dual system \( x(k+1) = A^\top(w(k))x(k) \) (resp., \( \dot{x}(t) = A^\top(w(t))x(t) \)).

**Proposition 7** Duality does not necessarily hold for convergence.

**PROOF.** The proof goes by means of two examples. We consider the DT case first. With \( a \in (0, 1) \), the system

\[
A(w(k)) \in \{A_1, A_2\} = \left\{ \begin{bmatrix} a & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a & 2 \\ 0 & 1 \end{bmatrix} \right\}
\]

is neither strong nor weak convergent. In fact, while \( x_{2k}(k) = x_2(0) \), for all \( k > 0 \), by taking alternatively \( A(w(k)) = A_1 \) for \( k \) even and \( A(w(k)) = A_2 \) for \( k \) odd, \( x_1(k) \) exhibits persistent oscillations. However, the dual system is strongly convergent, since it is in the block-triangular form in Theorem 3 (b).

In CT, the differential inclusion characterized by

\[
A(w(t)) \in \{A_1, A_2\} = \left\{ \begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix}, \begin{bmatrix} -\gamma & \delta \\ \gamma & -\delta \end{bmatrix} \right\},
\]

for some \( \alpha, \beta, \gamma, \delta > 0 \), does not converge (see Proposition 4) while the dual system with \( A_i^\top(w(t)) \) is strongly convergent to \( \kappa \mathbf{I} \), \( \kappa \in \mathbb{R} \). It follows by considering \( V(x) = |x_1 - x_2| \) as a common Lyapunov function.

However, an intriguing fact is that weak quadratic Lyapunov functions provide some kind of duality.

**Proposition 8** Assume that the difference inclusion in (11) (differential inclusion in (12)) admits a wQLF. Then, it is strongly convergent if and only if the dual system is strongly convergent.

**PROOF.** We prove the CT version first. By applying the state decomposition of Theorem 3, we consider a wQLF for (12) such that, for all \( i \in \mathcal{M} \),

\[
\begin{bmatrix} P & R \\ R^\top & Q \end{bmatrix} \begin{bmatrix} A_{is}^w & 0 \\ 0 & 0_m \end{bmatrix} + \begin{bmatrix} (A_{is}^w)^\top & (A_i^r)^\top \\ 0 & 0_m \end{bmatrix} \begin{bmatrix} P & R \\ R^\top & Q \end{bmatrix} = \begin{bmatrix} PA_{is}^w + (A_{is}^w)^\top P + RA_i^r + (RA_i^r)^\top (\ast) \\ R^\top A_{is}^w + QA_i^r \end{bmatrix} \leq 0,
\]

where \( (\ast) = (A_{is}^w)^\top R + (A_i^r)^\top Q \). The relation above implies \( R^\top A_{is}^w + QA_i^r = 0 \), for all \( i \in \mathcal{M} \). Then, we use the matrices \( R^\top \) and \( Q \) to introduce an additional state transformation as follows

\[
\begin{bmatrix} I & 0 \\ R^\top & Q \end{bmatrix} \begin{bmatrix} A_{is}^w & 0 \\ 0 & 0_m \end{bmatrix} \begin{bmatrix} I & 0 \\ R^\top & Q \end{bmatrix}^{-1} = \begin{bmatrix} A_i^r & 0 \\ 0 & 0_m \end{bmatrix},
\]

Since \( \{A_{is}^w\}_{i \in \mathcal{M}} \) generate asymptotically stable differential inclusions, the same holds for \( \{(A_{is}^w)^\top\}_{i \in \mathcal{M}} \). Hence the strong convergence of \( \text{diag}(A_{is}^w, 0_m) \) implies the strong convergence of the linear differential inclusion defined by \( \{A_i^r\}_{i \in \mathcal{M}} \), where

\[
A_i^r = \begin{bmatrix} (A_{is}^w)^\top & (A_i^r)^\top \\ 0 & 0_m \end{bmatrix}.
\]

The proof is similar for the differential inclusion in (11). Indeed, by considering a wQLF, we have

\[
\begin{bmatrix} (A_{is}^w)^\top & (A_i^r)^\top \\ 0 & I_m \end{bmatrix} \begin{bmatrix} P & R \\ R^\top & Q \end{bmatrix} \begin{bmatrix} A_{is}^w & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} P & R \\ R^\top & Q \end{bmatrix} = \begin{bmatrix} PA_{is}^w + (A_{is}^w)^\top P + RA_i^r + (RA_i^r)^\top (\ast) \\ R^\top A_{is}^w + QA_i^r \end{bmatrix} \leq 0,
\]

which implies, for all \( i \in \mathcal{M} \), \( R^\top A_{is}^w + QA_i^r = 0 \). Moreover, by adopting the same additional state transformation introduced above, we have

\[
\begin{bmatrix} I & 0 \\ R^\top & Q \end{bmatrix} \begin{bmatrix} A_{is}^w & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I & 0 \\ R^\top & Q \end{bmatrix}^{-1} = \begin{bmatrix} A_i^r & 0 \\ 0 & I_m \end{bmatrix},
\]

where \( \{A_{is}^w\}_{i \in \mathcal{M}} \) generate an asymptotically stable differential inclusion. So is \( \{(A_{is}^w)^\top\}_{i \in \mathcal{M}} \), and by following the same reasoning for the CT case, we obtain the strong convergence of the dual system.

### 8 Examples

#### 8.1 Uncertain linear systems with persistent input

Let us consider the system

\[
\dot{x}(t) = A(w(t))x(t) + Bu
\]
where $\bar{u} \in \mathbb{R}$ is a constant scalar, and assume that the unforced system is asymptotically stable for any signal $w(t) \in W$. We observe that the state of such system converges if and only if the following extended (marginally stable) system is strongly convergent:

$$
\begin{bmatrix}
\dot{x} \\
\dot{u}
\end{bmatrix} =
A(w)
\begin{bmatrix}
B \\
0_n^\top
\end{bmatrix}
\begin{bmatrix}
x \\
u
\end{bmatrix}.
$$

(21)

In fact, strong convergence implies that the extended state converges to the common kernel $\mathcal{K} = \{[\bar{x}; \bar{u}] \in \mathbb{R}^{n+1} | A(w)\bar{x} + B\bar{u} = 0\}$. This subspace has dimension 1, since $A(w)$ is non-singular. Let $(\bar{x}, \bar{u}) \in \mathcal{K}$:

with the transformation $T =
\begin{bmatrix}
I_n & \bar{x} \\
0_n^\top & \bar{u}
\end{bmatrix}$, we have that

$$
T^{-1}
\begin{bmatrix}
A(w) & \bar{B} \\
0_n^\top & 0
\end{bmatrix}
T = \text{diag}(A(w), 0),
$$

hence, by Theorem 3, the extended system in (21) is strongly convergent.

The DT case is analogous, since the uncertain affine system $x(k+1) = A(w_k)x(k) + B\bar{u}$, where $\bar{u} \in \mathbb{R}$ is constant, reads as the following uncertain linear system, where $u(0) = \bar{u}$:

$$
\begin{bmatrix}
x^+ \\
u^+
\end{bmatrix} =
\begin{bmatrix}
A(w) & \bar{B} \\
0_n^\top & 1
\end{bmatrix}
\begin{bmatrix}
x \\
u
\end{bmatrix}.
$$

8.2 Uncertain opinion dynamics in social networks

An example of uncertain system with persistent input is the following model of the probability distribution of a linear evaluative network of stochastic agents. Let $x^{(r)}(t) \in \mathbb{R}^M$ denote the unit-sum opinion probability vector for the $r$-th agent, $r \in \mathcal{N} := \{1, 2, \ldots, N\}$, in linear evaluative model with an arbitrary network topology. It is shown in [11] that the probability distribution of the $r$-th agent can be written as, for $j \in \mathcal{M} := \{1, 2, \ldots, M\}$,

$$
\dot{x}^{[r]}_j = \sum_{i \in \mathcal{M}} q_{i,j}^{[r]}x_i^{[r]} + \lambda \left( \sum_{k \in \mathcal{N}_r} x_k^{[r]} - x_j^{[r]} \right)
$$

where $\lambda > 0$ represents the influence intensity, equal for different opinions, $\mathcal{N}_r$ the set of neighbors of agent $r$, and $q_{i,j}^{[r]}$ the probability transition rate from opinion $i$ to opinion $j$ for agent $r$ in absence of influence. From now on, we assume that all the transition rate matrices $Q^{[r]}$, $r \in \mathcal{N}$, of the isolated agents are irreducible and $M = 2$. Since $1^\top r x^{[r]} = 1$, we can discard the second component of each probability vector, so that the reduced $N$-dimensional model reads as

$$
\dot{z}(t) = (-\Delta + \lambda G)z(t) + q
$$

where $G$ is the graph matrix, a Metzler matrix such that $1^\top G = 0_N^\top$, $\Delta = \text{diag}(\delta_1, \ldots, \delta_N)$ is a positive diagonal matrix with elements $\delta_r = q_{1,2}^{[r]} + q_{2,1}^{[r]}$ and $q$ is a column vector with $q_r = q_{2,1}^{[r]}$. Let us consider that the graph matrix is uncertain and given by

$$
G(w) = \sum_{i=1}^g G[i]w_i
$$

with $G[i]$, $i = 1, 2, \ldots, g$, suitable graph matrices of connected graphs. The augmented system is then

$$
\begin{bmatrix}
\dot{z} \\
\dot{u}
\end{bmatrix} =
\sum_{i=1}^g (\Delta + \lambda G[i])w_i
\begin{bmatrix}
q \\
u
\end{bmatrix}.
$$

(22)

We note that no common kernel exists unless $q = \bar{\beta}\Delta \mathbf{1}_N$, and in this case, the common (1-dimensional) kernel is given by span($[\mathbf{1}_N^\top, (\Delta \mathbf{1}_N^\top)]^\top$), since $(-\Delta + \lambda G[i])^{-1}\mathbf{1}_N = \mathbf{1}_N$, $\forall i \in \mathcal{N}$. This is the well known unbiased network case, i.e., the bias parameters $\beta_r = q_{2,1}^{[r]}/(q_{2,1}^{[r]} + q_{1,2}^{[r]})$ are all equal to $\bar{\beta} < 1$. It follows from §8.1 that the augmented uncertain system in (22) is strongly convergent, in particular, for all $z(0)$ and $u(0)$, $\lim_{t \to \infty} z(t) = \mathbf{1}_N^\top \beta_u(0)$.

8.3 Kolmogorov-like equations

Let us consider the system

$$
\dot{x}(t) = \sum_{i \in \mathcal{M}} A_i w_i(t) x(t),
$$

(23)

where each $A_i$ is an irreducible, Metzler matrix with strictly positive off-diagonal elements. If $A_i$ are column-stochastic matrices, i.e., if $1^\top A_i = 1^\top$ for all $i \in \mathcal{M}$, then $V(x) = \|x\|_1$ is a wPLF for the system in (23), while if $A_i$ are row-stochastic matrices, i.e., if $A_i 1 = 1$ for all $i \in \mathcal{M}$, then $V(x) = \|x\|_\infty$ is a wPLF.

In general, as shown in the special case of Proposition 4, column-stochastic matrices do not share a common kernel. Conversely, in the row-stochastic case, for $\tau > 0$ small enough, we note that the associated Euler difference inclusion $x(k+1) = F(w(x))x(k) = \sum_{i \in \mathcal{M}} w_i(k) F_i(x(k))$, where $F_i := I + \tau A_i$, $\forall i \in \mathcal{M}$, is a positive, row-stochastic matrix. Then, we have $\|F_i x\|_\infty \leq \|x\|_\infty$ where, unless $x_1 = x_2 = \cdots = x_n$, the relation holds as strict inequality since, for all $i \in \mathcal{M}$, $\sum_j (F_i b_{j1} x_j) = \sum_j (F_i b_{j1} x_j) < \max_j (x_j)$. Therefore, in view of Corollary 3, the state of the system converges to the set $\mathcal{K} = \{x \in \mathbb{R}^n | x_1 = x_2 = \cdots = x_n\}$. Under the irreducibility assumption, $\mathcal{K}$ corresponds to the common kernel of the generator matrices $\{A_i\}_{i \in \mathcal{M}}$.
Given a static plant governed by an unknown mapping
\[ y(t) = f(u(t)) \]
such that the Jacobian matrix \( J \) of \( f \) belongs to a polytope \( \mathcal{M} \) in which all the elements are non-singular (robust non-singularity) and such that \( f(\bar{u}) = 0 \) for some unique \( \bar{u} \), then there exists a dynamic control law \( \dot{u}(t) = \phi(y(t)) \) that steers \( u(t) \) to \( \bar{u} \) [8]. Conversely, let us consider the case in which there are isolated singular elements in \( \mathcal{M} \), hence we cannot guarantee convergence to 0, but we still need to ensure state convergence.

As an example, let us consider the flow system governed by the equations
\[
\begin{align*}
y_1 &= -\phi(u_1 - u_2) + r_1 \\
y_2 &= \phi(u_1 - u_2) + \psi(u_2) - r_1 + r_2,
\end{align*}
\]
where \( r_1 \) and \( r_2 \) are references, \( a_{\min} \leq a := \phi'(u) \leq a_{\max} \), \( b_{\min} \leq b := \psi'(u) \leq b_{\max} \). Next, we consider an integral control law \( \dot{u} = -k y \), with \( k > 0 \) and derive the dynamics \( \dot{y} = J f(u) \dot{u} \), i.e.,
\[
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} = -k
\begin{bmatrix}
a & -a \\
-a & a + b
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix},
\]
which is a linear differential inclusion since \( a \in [a_{\min}, a_{\max}] \) and \( b \in [b_{\min}, b_{\max}] \). For this system, \( V(y) = ||y|| \) is a Lyapunov function and, in view of Corollary 1, the state weakly converges to the set \( \mathcal{N} = \{ y \in \mathbb{R}^2 \mid y_1 - y_2 = 0 \} \).

9 Conclusion

The state convergence problem is highly relevant within the system-and-control community, since it occurs in several areas, from multi-agent learning, consensus and opinion dynamics to plant tuning. For LTI systems, state convergence can be characterized via necessary and sufficient matrix inequalities and Lyapunov-like conditions. In the presence of uncertainty, two different definitions of state convergence, i.e., strong and weak convergence, shall be considered. These two are equivalent under the kernel sharing property. In general, while strong convergence is structurally guaranteed by the separability of the eigenvalues of the generator matrices, weak convergence is not. By defining the notion of weak kernel, we derived sufficient, Lyapunov-like conditions for weak convergence. Deriving necessary and sufficient conditions for weak convergence is currently an open problem.

A Proofs of statements in Section 3

A.1 Proof of Lemma 1

Let \( \mathbb{D}_r := \{ z \in \mathbb{C} \mid |z - (1 - r)| \leq r \} \) denotes the disk of radius \( r > 0 \) centered in \((1 - r, 0)\). The system in (7) is stable if and only if the eigenvalues of \( A_t^u \) are contained in the unit disk \( \mathbb{D}_1 \) and those at \( \partial \mathbb{D}_1 \) are semi-simple. In view of [4, Lemma 4], this reflects on the eigenvalues of \( A \), that shall be located inside the disk \( \mathbb{D}_1 \) and those at \( \partial \mathbb{D}_1 \) shall be semi-simple. Since all the eigenvalues of \( A_t^u \) on the unit circle are semi-simple, then \( A \) has eigenvalues strictly inside \( \mathbb{D}_1 \), and those at 1 are semi-simple. Therefore, the system in (5) is stable and it converges.

A.2 Proof of Theorem 1

(a) \( \Leftrightarrow \) (b) follows directly from Lemma 1, while (b) \( \Leftrightarrow \) (c) follows by [10, Th. 4.50]. (a) \( \Leftrightarrow \) (d) comes from the fact that \( A \) has all the eigenvalues strictly inside the unit disk and the eigenvalues in 1 are semi-simple. Then, there always exists a state transformation that allows to separate stable and critical eigenvalues.

(c) \( \Leftrightarrow \) (e) : By the Lyapunov theorem, the system in (7) is stable if and only if \( (A_t^u)\top PA_t^u \preceq P \), which leads to the following inequalities:
\[
A\top PA - (1 - \eta)(A\top P + PA) + (1 - \eta)^2 P \preceq \eta^2 P,
\]
which corresponds to the LMI in (e).

(d) \( \Rightarrow \) (f) : Assume that \( A \) has the structure in (d). Then a solution is found by assuming \( T\top QT = I \) and \( T\top PT = \text{diag}(P_1, 0) \) with \( P_1 > 0 \) and \( \eta > 0 \) such that
\[
\eta ((A_{\text{as}})\top P_1 A_{\text{as}} - P_1) + (1 - \eta)W < 0,
\]
where \( W := (A_{\text{as}} - I)\top (A_{\text{as}} - I) + (A)\top A > 0 \).

(f) \( \Rightarrow \) (d) : Let \( \hat{A} = U\top Q^{1/2}AQ^{-1/2}U \), \( \hat{P} = U\top Q^{1/2}PQ^{-1/2}U \), where \( U \) is an orthogonal matrix such that \( \hat{P} = \text{diag}(P_1, 0) \), with \( P_1 > 0 \). From the LMI, that can be rewritten as
\[
\eta(\hat{A}\top \hat{P}\hat{A} - \hat{P}) + (1 - \eta)(\hat{A}\top I)Q(\hat{A} - I) \preceq 0,
\]
it follows that \( \hat{A} = \begin{bmatrix} \hat{A}_1 & 0 \\ \hat{A}_2 & I \end{bmatrix} \), and
\[
\eta(\hat{A}_1\top P_1 \hat{A}_1 - P_1) + (1 - \eta)\hat{W} \preceq 0,
\]
where \( \hat{W} := (\hat{A}_1 - I)(\hat{A}_1 - I) + \hat{A}_2\top \hat{A}_2 \). Due to the assumption on the rank of \( P \), \( \hat{W} \) is positive definite, hence \( \hat{A}_1 \) is Schur and the system is convergent. ■
A.3 Proof of Lemma 2

The proof follows by noticing that there exists a relation among the eigenvalues $\lambda \in \Lambda(A)$ and $\nu \in \Lambda(A_{\nu})$ that does not alter both geometric and algebraic multiplicities, i.e., $\nu = (\lambda)/(1 + \epsilon \lambda)$. The system in (6) converges if and only if all its eigenvalues have strictly negative real part and those on the imaginary axis are semi-simple and all equal to 0. Hence, there always exists $\epsilon > 0$ small enough that maps each $\lambda \in \Lambda(A)$ with strictly negative real part to some $\nu \in \Lambda(A_{\nu})$ belonging to the open left-half plane. Therefore, by noticing that the eigenvalues of $A$ at 0 are mapped in 0, the system in (10) is stable.

A.4 Proof of Theorem 2

(a) $\Rightarrow$ (b) follows directly from Lemma 2. (a) $\Rightarrow$ (c) $\Rightarrow$ (d) comes from [10, Th. 4.49].

(a) $\Rightarrow$ (e) : As in Lemma 2, there exists a relation among the eigenvalues $\lambda \in \Lambda(A)$ and $\nu \in \Lambda(A_{\nu})$ that does not alter both geometric and algebraic multiplicities, i.e., $\nu = 1 + \tau \lambda$. Moreover, the eigenvectors of $A_{\nu}$ are respectively the same of those of $A$. Therefore, the eigenvalues at 0 of $A$ are semi-simple if and only if the eigenvalues at 1 of $A_{\nu}$ are semi-simple.

(b) $\Rightarrow$ (f) : By the Lyapunov theorem, the system in (10) is stable if and only if $(A_{\nu}^T)^T P + P A_{\nu}^T \preceq 0$. Hence, by pre and post-multiplying by $(I + \epsilon A)^T$ and $(I + \epsilon A)$, respectively, we obtain the LMI in (f).

(d) $\Rightarrow$ (g) : Without restrictions, let $Q = I$. Then, $T = PT = \text{diag}(P_1, 0)$, $P_1 > 0$, and let $\epsilon > 0$ be such that

$$(A_{\nu}^T)^T P_1 + P_1 A_{\nu} + \epsilon \left((A_{\nu}^T)^T A_{\nu} + (A^T)^T A^T\right) \preceq 0.$$ 

(g) $\Rightarrow$ (d) : Let $\hat{A} = U^T Q^{1/2} A^T Q^{-1/2} U$, $\hat{P} = U^T Q^{1/2} P Q^{-1/2} U$, where $U$ is an orthogonal matrix such that $P = \text{diag}(P_1, 0)$, with $P_1 > 0$. From the LMI, $\hat{A}^T \hat{P} + \hat{P} \hat{A} + \epsilon \hat{A}^T \hat{A} \preceq 0$, it follows that $\hat{A} = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix}$.

and $A_1^T P_1 + P_1 A_1 \preceq -\epsilon (A_1^T A_1 + A_2^T A_2)$. The matrix $A_1^T A_1 + A_2^T A_2$, is positive definite in view of the assumption on the rank of $P$. Thus, the matrix $A_1$ is Hurwitz and the system is convergent.

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