ON BOOSTED SPACE-TIMES WITH COSMOLOGICAL CONSTANT
AND THEIR ULTRARELATIVISTIC LIMIT

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Abstract

The problem of deriving a shock-wave geometry with cosmological constant by boosting a
Schwarzschild-de Sitter (or anti-de Sitter) black hole is re-examined. Unlike previous work in
the literature, we deal with the exact Schwarzschild-de Sitter (or anti-de Sitter) metric. In this
exact calculation, where the metric does not depend linearly on the mass parameter, we find a
singularity of distributional nature on a null hypersurface, which corresponds to a shock-wave ge-
ometry derived in a fully non-perturbative way. The result agrees with previous calculations, where
the metric had been linearized in the mass parameter.
I. INTRODUCTION

The subject of gravitational fields generated by sources which move at the speed of light has always received much attention in the literature. In the sixties it was already known that the fields produced by null sources are plane-fronted gravitational waves [1, 2]. In Ref. [3], Aichelburg and Sexl studied the field of a point particle with zero rest mass moving with the speed of light. They found that the gravitational field of such a particle is non-vanishing only on a plane containing the particle and orthogonal to the direction of motion. On this plane the Riemann tensor has a distributional (Dirac-delta-like) singularity and is exactly of Petrov type $N$ (i.e. all four principal null directions of the Weyl spinor, describing the Weyl conformal curvature, coincide). For this purpose, the authors of Ref. [3] used in part a set of Lorentz transformations in the ultrarelativistic limit.

Since then, other authors considered ‘boosting’ the Kerr or Kerr–Newman solutions [4, 5, 6, 7], while the work by Hotta and Tanaka in Ref. [8], motivated by the analysis of quantum effects of gravitons in de Sitter space-time, studied the problem of boosting the Schwarzschild–de Sitter metric to a similar limit (see also the work in Ref. [9]). For this purpose, the authors of Refs. [8, 9] approximated the Schwarzschild–de Sitter metric by a first-order perturbation of de Sitter, i.e.

$$ds^2 \approx -\left(1 - \frac{2m}{r} - \frac{r^2}{a^2}\right) dt^2 + \left(1 - \frac{r^2}{a^2}\right)^{-1} \left[1 + \left(1 - \frac{r^2}{a^2}\right)^{-1} \frac{2m}{r} \right] dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

They then found that a suitable change of coordinates, combined with the ultra-relativistic limit (cf. below), lead to a resulting space-time which differs from de Sitter space-time only by the inclusion of an impulsive wave located on a null hypersurface.

However, since the Einstein theory is ruled by non-linear effects in the first place, we remark that an exact analysis relying upon the full Schwarzschild–de Sitter metric, which depends non-linearly on the mass parameter, would be desirable if only possible (as will be shown below). Thus, in our problem, we start from the standard form of the metric for a Schwarzschild-de Sitter space-time, i.e.

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{r^2}{a^2}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r} - \frac{r^2}{a^2}\right)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

where $a^2$ is a parameter related to the cosmological constant $\Lambda$ by $\Lambda = \frac{3}{a^2}$. Moreover, we know that a de Sitter space-time in four dimensions can be viewed as a four-dimensional
A hyperboloid embedded in five-dimensional Minkowski space-time. Its metric reads
\[ ds^2 = -dZ_0^2 + \sum_{i=1}^{4} dZ_i^2, \tag{3} \]
with coordinates satisfying the hyperboloid constraint
\[ a^2 = -Z_0^2 + \sum_{i=1}^{4} Z_i^2, \tag{4} \]
so that the parameter \( a \) is just the ‘radius’ of this hyperboloid.

The plan of our paper is as follows. Section 2 studies the metric (2) in the \( Z_\mu \) coordinates above. Section 3 performs a boost in the \( Z_1 \)-direction, while section 4 is devoted to a similar analysis in Schwarzschild-anti-de Sitter space-time. Section 5 derives in detail a shock-wave geometry from the ultrarelativistic limit of such boosted space-times with cosmological constant, while concluding remarks are presented in section 6.

II. FIRST CHANGE OF COORDINATES

The work in \cite{8} exploits the relation between the \( Z_\mu \) coordinates in Eqs. (3) and (4) and the spherical static coordinates \((t, r, \theta, \phi)\):

\[ Z_0 \equiv \sqrt{a^2 - r^2} \sinh(t/a), \tag{5} \]
\[ Z_4 \equiv \pm \sqrt{a^2 - r^2} \cosh(t/a), \tag{6} \]
\[ Z_1 \equiv r \cos \theta, \tag{7} \]
\[ Z_2 \equiv r \sin \theta \cos \phi, \tag{8} \]
\[ Z_3 \equiv r \sin \theta \sin \phi. \tag{9} \]

The key point of our analysis is to rewrite the exact metric (2) in the \( Z_\mu \) coordinates, without resorting to a perturbative expansion up to terms linear in the black hole mass \( m \). Thus, on defining

\[ f^2 \equiv a^2 - r^2 = Z_4^2 - Z_0^2, \tag{10} \]
\[ F_m \equiv 1 - \frac{2a^2m}{f^2r} - \frac{a^2/r^2}{\left(1 - \frac{2a^2m}{f^2r}\right)}, \tag{11} \]
\[ Q \equiv 1 + \frac{2Z_0^2}{f^2}, \tag{12} \]
we re-express the Schwarzschild-de Sitter metric in the form (see details in Appendix A)

\[ ds^2 = h_{00}dZ_0^2 + h_{44}dZ_4^2 + 2h_{04}dZ_0dZ_4 + 2dZ_1^2 + 2dZ_2^2 + 2dZ_3^2, \]

(13)

where

\[ h_{00} \equiv -\frac{1}{2}(Q - 1)F_m - \left(1 - \frac{2a^2m}{f^2r}\right) - \frac{Z_0^2}{r^2}, \]

(14)

\[ h_{44} \equiv -\frac{1}{2}(Q + 1)F_m + \left(1 - \frac{2a^2m}{f^2r}\right) - \frac{Z_4^2}{r^2}, \]

(15)

\[ h_{04} \equiv \frac{Z_0Z_4}{f^2}F_m + \frac{Z_0Z_4}{r^2}, \]

(16)

with the ratio \( \frac{a^2m}{f^2r} \) given by

\[ \frac{a^2m}{f^2r} = \frac{a^2m}{(Z_4^2 - Z_0^2)\sqrt{a^2 + Z_0^2 - Z_4^2}}. \]

(17)

III. THE BOOST IN THE Z_1-DIRECTION

The next step, following [8], is to set

\[ m = p\sqrt{1 - v^2}, \]

(18)

where \( p > 0 \), and then introduce a boost in the Z_1-direction by defining yet new coordinates \( Y_\mu \), independent of \( v \), such that (hereafter \( \gamma \equiv (1 - v^2)^{-1/2} \))

\[ Z_0 = \gamma(Y_0 + vY_1), \]

(19)

\[ Z_1 = \gamma(vY_0 + Y_1), \]

(20)

\[ Z_2 = Y_2, \ Z_3 = Y_3, \ Z_4 = Y_4. \]

(21)

The metric (13) is therefore re-expressed, eventually, in the form (see Appendix A)

\[ ds^2 = \gamma^2(h_{00} + v^2)dY_0^2 + \gamma^2(1 + v^2h_{00})dY_1^2 + dY_2^2 + dY_3^2 + h_{44}dY_4^2 \\
+ 2v\gamma^2(1 + h_{00})dY_0dY_1 + 2\gamma h_{04}dY_0dY_4 + 2v\gamma h_{04}dY_1dY_4, \]

(22)

with the understanding that Eqs. (10)–(12) and (19)–(21) should be inserted into Eqs. (14)–(17) to express the metric components in Eq. (22) completely in \( Y_\mu \) coordinates (i.e. \( h_{ab} = h_{ab}(Y_\mu) \) in Eq. (22)).
IV. SCHWARZSCHILD-ANTI DE SITTER SPACE-TIME

To complete our starting set of formulae we now consider Schwarzschild-anti de Sitter space-time [9], whose metric takes originally the form (cf. Eq. (2))

\[
\text{ds}^2 = - \left(1 - \frac{2m}{r} + \frac{r^2}{a^2}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r} + \frac{r^2}{a^2}\right)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\] (23)

Such a geometry can be represented as the four-dimensional hyperboloid

\[
-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 - Z_4^2 = -a^2,
\] (24)

embedded in a five-dimensional space-time with metric

\[
\text{ds}^2 = -dZ_0^2 + dZ_1^2 + dZ_2^2 + dZ_3^2 - dZ_4^2,
\] (25)

corresponding to two timelike directions, i.e. $Z_0$ and $Z_4$. The natural parametrization of this is given by [9]

\[
Z_0 \equiv \sqrt{a^2 + r^2} \sin(t/a),
\] (26)

\[
Z_4 \equiv \sqrt{a^2 + r^2} \cos(t/a),
\] (27)

while $Z_1, Z_2$ and $Z_3$ remain defined as in Eqs. (7)–(9). Thus, on further defining (cf. Eqs. (10) and (11))

\[
\tilde{f}^2 \equiv a^2 + r^2 = Z_0^2 + Z_4^2,
\] (28)

\[
\tilde{F}_m \equiv 1 - \frac{2a^2m}{f^2 r} + \frac{a^2/r^2}{\left(1 - \frac{2a^2m}{f^2 r}\right)},
\] (29)

we first re-express the metric (23) in the form

\[
\text{ds}^2 = H_{00}dZ_0^2 + H_{44}dZ_4^2 + 2H_{04}dZ_0dZ_4 + dZ_1^2 + dZ_2^2 + dZ_3^2,
\] (30)

where

\[
H_{00} \equiv \frac{Z_0^2}{f^2} \tilde{F}_m - \left(1 - \frac{2a^2m}{f^2 r}\right) - \frac{Z_0^2}{r^2},
\] (31)

\[
H_{44} \equiv \left(1 - \frac{Z_0^2}{f^2}\right) \tilde{F}_m - \left(1 - \frac{2a^2m}{f^2 r}\right) - \frac{Z_4^2}{r^2},
\] (32)

\[
H_{04} \equiv \frac{Z_0Z_4}{f^2} \tilde{F}_m - \frac{Z_0Z_4}{r^2}.
\] (33)
Now the boost in the $Z_1$-direction is again given by Eqs. (18)–(21) [9], so that, in full analogy with Eq. (22), the boosted metric reads

\[
\tilde{H}_{00} = \gamma^2 (H_{00} + v^2),
\]
\[
\tilde{H}_{11} = \gamma^2 (1 + v^2 H_{00}) = 1 + v^2 + v^2 \tilde{H}_{00},
\]
\[
\tilde{H}_{22} = \tilde{H}_{33} = 1,
\]
\[
\tilde{H}_{44} = H_{44},
\]
\[
\tilde{H}_{01} = v \gamma^2 (1 + \tilde{H}_{00}) = v (1 + \tilde{H}_{00}),
\]
\[
\tilde{H}_{04} = \gamma H_{04},
\]
\[
\tilde{H}_{14} = v \gamma H_{04}.
\]

In these formulae, the ‘boosted’ $H_{00}, H_{44}$ and $H_{04}$ components can be written, upon defining (the subscript ‘minus’ refers here to the negative cosmological constant)

\[
W_- (a, v, Y) \equiv \frac{1}{(Y_0 + v Y_1)^2 + (1 - v^2)(Y_4^2 - a^2)},
\]
\[
\rho_- (a, v, Y) \equiv W_- (0, v, Y) - W_- (a, v, Y) - 2a^2 p (1 - v^2)^2 W_-^2 (0, v, Y) W_-^{1/2} (a, v, Y)
+ \frac{a^2 (1 - v^2) W_- (0, v, Y) W_- (a, v, Y)}{1 - 2a^2 p (1 - v^2)^2 W_- (0, v, Y) W_-^{1/2} (a, v, Y)},
\]

in the form

\[
H_{00} = (Y_0 + v Y_1)^2 \rho_- (a, v, Y) - \left(1 - 2a^2 p (1 - v^2)^2 W_- (0, v, Y) W_-^{1/2} (a, v, Y)\right),
\]
\[
H_{44} = (1 - v^2) Y_4^2 \rho_- (a, v, Y) - \left(1 - 2a^2 p (1 - v^2)^2 W_- (0, v, Y) W_-^{1/2} (a, v, Y)\right),
\]
\[
H_{04} = \frac{1}{\gamma} (Y_0 + v Y_1) Y_4 \rho_- (a, v, Y),
\]

where we have exploited Eqs. (28), (29) and (31)–(33). The advantage of Eqs. (34)–(45) is that they are non-perturbative, exact formulae for the boosted metric that relate the analysis of the singularity structure to one function only, i.e. $W_- (a, v, Y)$ defined in Eq. (41).
V. SHOCK-WAVE GEOMETRY FROM THE ULTARELATIVISTIC LIMIT

In the Schwarzschild-de Sitter geometry of Secs. II and III we find in analogous way, on defining (the subscript ‘plus’ refers here to the positive cosmological constant)

\[ W_+(a, v, Y) \equiv \frac{1}{(Y_0 + vY_1)^2 + (1 - v^2)(a^2 - Y_4^2)}, \] (46)

\[ \rho_+(a, v, Y) \equiv W_+(0, v, Y) - W_+(a, v, Y) + 2a^2p(1 - v^2)W^2_+(0, v, Y)W^{1/2}_+(a, v, Y) \]

\[ - \frac{a^2(1 - v^2)W_+(0, v, Y)W_+(a, v, Y)}{1 + 2a^2p(1 - v^2)W_+(0, v, Y)W^{1/2}_+(a, v, Y)}, \] (47)

the basic formulae

\[ h_{00} = (Y_0 + vY_1)^2\rho_+(a, v, Y) - \left(1 + 2a^2p(1 - v^2)W_+(0, v, Y)W^{1/2}_+(a, v, Y)\right), \] (48)

\[ h_{44} = (1 - v^2)Y_4^2\rho_+(a, v, Y) + \left(1 + 2a^2p(1 - v^2)W_+(0, v, Y)W^{1/2}_+(a, v, Y)\right), \] (49)

\[ h_{04} = -\frac{1}{\gamma}(Y_0 + vY_1)Y_4\rho_+(a, v, Y). \] (50)

At this stage, we exploit the fundamental identity \[8\] (here \(f\) is any summable function on the real line)

\[ \lim_{v \to 1} \gamma f (\gamma^2 (Y_0 + vY_1)^2) = \delta(Y_0 + Y_1) \int_{-\infty}^{\infty} f(x^2)dx, \] (51)

and add carefully the various terms in Eq. (47) to find that, on defining the even functions of \(x\) (hereafter \(a > Y_4\))

\[ f_1(x) \equiv (x^2 - Y_4^2)\sqrt{x^2 + a^2 - Y_4^2}\left[(x^2 - Y_4^2)(x^2 + a^2 - Y_4^2) + 2a^2p\sqrt{1 - v^2}\sqrt{x^2 + a^2 - Y_4^2}\right], \] (52)

\[ f_2(x) \equiv (x^2 - Y_4^2)\sqrt{x^2 + a^2 - Y_4^2}, \] (53)

\[ f_3(x) \equiv (x^2 - Y_4^2)\sqrt{x^2 + a^2 - Y_4^2}, \] (54)

one has (hereafter \(x_v \equiv \gamma(Y_0 + vY_1)\))

\[ \rho_+(a, v, Y) = 2a^2p\gamma \left[\frac{a^2}{f_1(x_v)} + \frac{1}{f_2(x_v)}\right], \] (55)

and hence, by virtue of the limit (51) (with our notation, \(\tilde{h}_{ab}\) is obtained from \(h_{ab}\) exactly as \(\tilde{H}_{ab}\) is obtained from \(H_{ab}\) in Eqs. (34)–(40), as is clear from Eq. (22)),

\[ \lim_{v \to 1} \tilde{h}_{00} = -1 + 2a^2p\delta(Y_0 + Y_1) \lim_{v \to 1} \int_{-\infty}^{\infty} \left[\frac{a^2x^2}{f_1(x)} + \frac{x^2}{f_2(x)} - \frac{1}{f_3(x)}\right] dx, \] (56)
which implies that

\[
\lim_{v \to 1} \frac{1}{h_{44}} = 1 + 2a^2 p \delta(Y_0 + Y_1) \lim_{v \to 1} (1 - v^2) \int_{-\infty}^{\infty} \left[ Y_4^2 \left( \frac{a^2}{f_1(x)} + \frac{1}{f_2(x)} \right) + \frac{1}{f_3(x)} \right] dx = 1, \quad (57)
\]

\[
\lim_{v \to 1} \frac{1}{h_{04}} = -2a^2 p Y_4 \lim_{v \to 1} \sqrt{1 - v^2} \int_{-\infty}^{\infty} x \left( \frac{a^2}{f_1(x)} + \frac{1}{f_2(x)} \right) dx = 0. \quad (58)
\]

In Eq. (56), the desired limit can be brought within the integral, and the \(v\)-dependent part of \(f_1(x)\) gives vanishing contribution to this limit. One thus finds

\[
\lambda_+ \equiv \lim_{v \to 1} \int_{-\infty}^{\infty} \left[ \frac{a^2 x^2}{f_1(x)} + \frac{x^2}{f_2(x)} - \frac{1}{f_3(x)} \right] dx
\]

\[
= 2 \left( (a^2 + Y_4^2) + (a^2 - Y_4^2) a \frac{\partial}{\partial a} \right) \int_{0}^{\infty} \frac{d\tau}{(x^2 - Y_4^2)^2 (x^2 + a^2 - Y_4^2)^{1/2}}. \quad (59)
\]

In this integral we now change integration variable so as to get rid of square roots in the integrand, by defining

\[
\tau \equiv x + \sqrt{x^2 + a^2 - Y_4^2}, \quad (60)
\]

which implies that

\[
\frac{dx}{\sqrt{x^2 + a^2 - Y_4^2}} = \frac{d\tau}{\tau}, \quad (61)
\]

\[
x^2 - Y_4^2 = \frac{g(\tau)}{4\tau^2}, \quad (62)
\]

where (hereafter both \((a - Y_4)\) and \((a + Y_4)\) are taken to be positive)

\[
g(\tau) \equiv (\tau^2 - (a^2 - Y_4^2)^2 - 2(\tau Y_4))^2 = (\tau - (Y_4 + a)) (\tau - (Y_4 - a)) (\tau + Y_4 + a) (\tau - (a - Y_4)). \quad (63)
\]

Hence we find

\[
\lambda_+ = 32 \left( (a^2 + Y_4^2) + (a^2 - Y_4^2) a \frac{\partial}{\partial a} \right) \int_{0}^{\infty} \frac{\tau^3}{\sqrt{a^2 - Y_1^2} g^2(\tau)} d\tau
\]

\[
= \left[ (a^2 + Y_4^2) + (a^2 - Y_4^2) a \frac{\partial}{\partial a} \right] \left[ -\frac{1}{(a Y_4)^2} + \frac{(a^2 + Y_4^2)}{2(a Y_4)^3} \log \left( \frac{a + Y_4}{a - Y_4} \right) \right]
\]

\[
= -\frac{4}{a^2} + \frac{2Y_4}{a^3} \log \left( \frac{a + Y_4}{a - Y_4} \right). \quad (64)
\]

Furthermore, in Eq. (57), the desired limit vanishes, since the integral therein is finite and is multiplied by a vanishing function of \(v\) (i.e. \((1 - v^2)\)) as \(v \to 1\), while the limit in Eq. (58) vanishes because \(\sqrt{1 - v^2}\) vanishes as well as the integral therein (being the integral of an odd function over the whole real line). In agreement with Ref. [8], we therefore obtain, from Eqs. (22), (56) and (64), the singular boosted metric (cf. Eq. (A17))

\[
ds^2 = -dY_0^2 + \sum_{i=1}^{4} dY_i^2
\]

\[
+ 4p \left[ -2 + \frac{Y_4}{a} \log \left( \frac{a + Y_4}{a - Y_4} \right) \right] \delta(Y_0 + Y_1) (dY_0 + dY_1)^2, \quad (65)
\]
i.e. de Sitter space plus a shock-wave singularity located on the null hypersurface described by the equations
\[ Y_0 + Y_1 = 0, \ Y_2^2 + Y_3^2 + (Y_4^2 - a^2) = 0. \] (66)

In an entirely analogous way, the ultrarelativistic limit of Schwarzschild-anti de Sitter space-time can be obtained, after adding carefully the terms in Eq. (42), by defining the even functions of \( x \) (hereafter \( Y_4 > a \))
\[ \varphi_1(x) \equiv (x^2 + Y_4^2) \sqrt{x^2 + Y_4^2 - a^2} \left[ (x^2 + Y_4^2)(x^2 + Y_4^2 - a^2) - 2a^2p\sqrt{1 - v^2}\sqrt{x^2 + Y_4^2 - a^2} \right], \] (67)
\[ \varphi_2(x) \equiv (x^2 + Y_4^2)^2 \sqrt{x^2 + Y_4^2 - a^2}, \] (68)
\[ \varphi_3(x) \equiv (x^2 + Y_4^2) \sqrt{x^2 + Y_4^2 - a^2}, \] (69)
which occur in the identity
\[ \rho_-(a, v, Y) = 2a^2p\gamma \left( \frac{a^2}{\varphi_1(x_v)} - \frac{1}{\varphi_2(x_v)} \right). \] (70)

By exploiting again the limit (51) we therefore obtain
\[ \lim_{v \to 1} \widetilde{H}_{00} = -1 + 2a^2p\delta(Y_0 + Y_1) \lim_{v \to 1} \int_{-\infty}^{\infty} \frac{a^2x^2}{\varphi_1(x)} - \frac{x^2}{\varphi_2(x)} + \frac{1}{\varphi_3(x)} \ dx, \] (71)
\[ \lim_{v \to 1} \widetilde{H}_{44} = -1 + 2a^2p\delta(Y_0 + Y_1) \lim_{v \to 1} (1 - v^2) \int_{-\infty}^{\infty} \frac{Y_4^2}{\varphi_1(x)} - \frac{1}{\varphi_2(x)} + \frac{1}{\varphi_3(x)} \ dx = -1, \] (72)
\[ \lim_{v \to 1} \widetilde{H}_{04} = 2a^2pY_4 \lim_{v \to 1} \int_{-\infty}^{\infty} x \left( \frac{a^2}{\varphi_1(x)} - \frac{1}{\varphi_2(x)} \right) \ dx = 0. \] (73)

In Eq. (71), the desired limit can be brought within the integral as in Eq. (56), and the \( v \)-dependent part of \( \varphi_1(x) \) gives vanishing contribution to this limit. One thus finds
\[ \lambda_- \equiv \lim_{v \to 1} \int_{-\infty}^{\infty} \frac{a^2x^2}{\varphi_1(x)} - \frac{x^2}{\varphi_2(x)} + \frac{1}{\varphi_3(x)} \ dx \]
\[ = 2 \left( a^2 + Y_4^2 \right) + 2a^2\sqrt{\frac{a^2}{(x^2 + Y_4^2)^2(x^2 + Y_4^2 - a^2)^2}}. \] (74)

In this integral, we now change integration variables according to (cf. Eq. (60))
\[ T \equiv x + \sqrt{x^2 + Y_4^2 - a^2}, \] (75)
which implies that
\[ \frac{dx}{\sqrt{x^2 + Y_4^2 - a^2}} = \frac{dT}{T}. \] (76)
\[ x^2 + Y_4^2 = \frac{G(T)}{4T^2}, \quad (77) \]

where (hereafter both \((Y_4 - a)\) and \((Y_4 + a)\) are taken to be positive)

\[ G(T) \equiv (T^2 - \frac{Y_4^2 + a^2}{2})^2 - (2iTY_4)^2 = (T - i(Y_4 + a))(T + i(Y_4 - a))(T - i(a - Y_4)). \quad (78) \]

Hence we find

\[
\begin{align*}
\lambda_- &= 32 \left[ (a^2 + Y_4^2) + (a^2 - Y_4^2) \alpha \frac{\partial}{\partial a} \right] \int_{-\infty}^{\infty} \frac{T^3}{\sqrt{Y_4^2 - a^2} G^2(T)} dT \\
&= -\frac{4}{a^2} + \frac{2Y_4}{a^3} \log \left( \frac{Y_4 + a}{Y_4 - a} \right). \quad (79)
\end{align*}
\]

The resulting singular boosted metric is, from Eqs. (34)-(40), (71) and (79),

\[
\begin{align*}
\begin{align*}
&dS^2 = -dY_0^2 + dY_1^2 + dY_2^2 + dY_3^2 - dY_4^2 \\
&\quad + 4p \left[ -2 + \frac{Y_4}{a} \log \left( \frac{Y_4 + a}{Y_4 - a} \right) \right] \delta(Y_0 + Y_1) (dY_0 + dY_1)^2,
\end{align*}
\end{align*}
\]

i.e. anti-de Sitter space plus a shock-wave singularity located on the null hypersurface described by the equations (cf. Eq. (66))

\[ Y_0 + Y_1 = 0, \quad Y_2^2 + Y_3^2 - (Y_4^2 - a^2) = 0. \quad (81) \]

By inspection of Eqs. (64) and (79) we notice that \(\lambda_-\) is not obtained from \(\lambda_+\) by the replacements \(a \to \pm ia, \ Y_4 \to \pm iY_4\), since the terms \(-\frac{4}{a^2}\) and \(\frac{2Y_4}{a^3}\) remain the same. Note also that, at \(Y_4 = a\), the limits (59) and (74) reduce to

\[
\lambda_\pm = 2a^2 \int_{-\infty}^{\infty} \frac{dx}{x(x^2 + a^2)^2}, \quad (82)
\]

which vanishes, being the integral of an odd function over the whole real line. Thus, only \(|a - Y_4| > 0\) concerns the shock-wave geometry.

VI. CONCLUDING REMARKS

Although our final result agrees with the findings in Ref. [8], our work is original and of interest for at least two reasons:

(i) We have performed an exact, fully non-perturbative analysis of boosted space-times with cosmological constant, dealing at all stages with the whole set of non-linearities of the metric.
This offers some advantages from the point of view of both physics and mathematics: for any fixed value of \( v \neq 1 \), the only reliable formulae for the metric are our Eqs. (13)–(17), (22), (30)–(50). Moreover, although the linearized metric is sufficient to derive the ultrarelativistic limit, it is instructive to understand how the final results come to agree in a background with cosmological constant.

(ii) We have provided detailed analytic formulae also for the Schwarzschild-anti-de Sitter space-time.

Our investigation is therefore part of the efforts aimed at a better understanding of the circumstances under which one can predict formation of shock-wave geometries \(^{10}\) in classical or quantum gravity \(^{11, 12}_\). In future work, we hope to be able to study the ‘boosted’ Riemann tensor, along the lines of Ref. \(^3\), but for Schwarzschild-de Sitter, Schwarzschild-anti de Sitter (see previous sections) and Kerr–Schild geometries \(^4\). Moreover, applications (if any) of our exact analysis to quantum gravity effects in space-times with cosmological constant should be investigated.

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**APPENDIX A: THE BOOSTED METRIC**

In the course of arriving at Eq. (13) from Eqs. (2) and (5)–(9), the relevant intermediate steps are the formulae (here \( \varepsilon \equiv \pm 1 \), \( \text{ch}_t \equiv \cosh(t/a) \), \( \text{sh}_t \equiv \sinh(t/a) \))

\[
\begin{align*}
    dZ_0 &= -\frac{r}{f} \text{sh}_t dr + \frac{f}{a} \text{ch}_t dt, \\
    dZ_4 &= -\frac{\varepsilon r}{f} \text{ch}_t dr + \frac{\varepsilon f}{a} \text{sh}_t dt, \\
    dZ_0^2 - dZ_4^2 &= \frac{f^2}{a^2} dt^2 - \frac{r^2}{f^2} dr^2.
\end{align*}
\]
\[ dZ_0^2 + dZ_4^2 = \frac{r^2}{f^2} \left( 1 + 2\text{sh}_t^2 \right) dr^2 + \frac{f^2}{a^2} \left( 1 + 2\text{sh}_t^2 \right) dt^2 - \frac{4r}{a} \text{sh}_t \text{ch}_t dr dt, \quad (A4) \]

\[ dZ_0 dZ_4 = \frac{\varepsilon r^2}{f^2} \text{sh}_t \text{ch}_t dr^2 + \frac{\varepsilon f^2}{a^2} \text{sh}_t \text{ch}_t dt^2 - \frac{\varepsilon r}{a} \left( 1 + 2\text{sh}_t^2 \right) dr dt. \quad (A5) \]

Thus, on defining (see Eqs. (5), (6), (10), (12))

\[ p_t \equiv \text{ch}_t \text{sh}_t, \quad (A6) \]

\[ q_t \equiv 1 + 2\text{sh}_t^2 - \frac{4p_t^2}{(1 + 2\text{sh}_t^2)} = (1 + 2\text{sh}_t^2)^{-1} = Q^{-1}, \quad (A7) \]

one arrives at

\[ dt^2 = \frac{a^2}{f^2} \left[ \frac{1}{2} \left( \frac{1}{q_t} + 1 \right) dZ_0^2 + \frac{1}{2} \left( \frac{1}{q_t} - 1 \right) dZ_4^2 - \frac{2}{\varepsilon} p_t dZ_0 dZ_4 \right], \quad (A8) \]

\[ dr^2 = \frac{f^2}{r^2} \left[ \frac{1}{2} \left( \frac{1}{q_t} - 1 \right) dZ_0^2 + \frac{1}{2} \left( \frac{1}{q_t} + 1 \right) dZ_4^2 - \frac{2}{\varepsilon} p_t dZ_0 dZ_4 \right], \quad (A9) \]

from which a little amount of calculations yields Eqs. (13)–(17). In the course of deriving Eq. (22), we have re-expressed Eq. (A9) in the form

\[ dr^2 = \frac{Z_0^2}{r^2} dZ_0^2 + \frac{Z_4^2}{r^2} dZ_4^2 - \frac{2Z_0 Z_4}{r^2} dZ_0 dZ_4, \quad (A10) \]

while bearing in mind that

\[ r^2(d\theta^2 + \sin^2 \theta d\phi^2) = dZ_1^2 + dZ_2^2 + dZ_3^2 - dr^2. \quad (A11) \]

Along similar lines, we exploit in Sec. IV the identities

\[ dt^2 = \frac{a^2}{f^2} \left[ \frac{Z_4^2}{f^2} dZ_0^2 + \frac{Z_0^2}{f^2} dZ_4^2 - \frac{2Z_0 Z_4}{f^2} \right], \quad (A12) \]

\[ dr^2 = \frac{Z_0^2}{r^2} dZ_0^2 + \frac{Z_4^2}{r^2} dZ_4^2 + \frac{2Z_0 Z_4}{r^2} dZ_0 dZ_4. \quad (A13) \]

The procedure for deriving a shock wave metric from a solution of the Einstein equations considers initially the metric for the latter in the form

\[ ds^2 = 2A(u, v') du dv' + g_{ij} dx^i dx^j. \quad (A14) \]

Let now \( w \) be a new coordinate defined by (\( \Theta \) being the step function)

\[ w \equiv v' + f(X^i) \Theta(u), \quad (A15) \]
so that
\[ dv' = dw - f(X^i)\delta(u)du, \]  
where we have exploited the distributional relation \( \Theta'(u) = \delta(u) \). The metric (A14) takes therefore the shock-wave form
\[ ds^2 = 2A(u,w)dudw - 2A(u,w)f(X^i)\delta(u)du^2 + g_{ij}dx^idx^j. \]  

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