Brownian semistationary processes and related processes

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Abstract

In this paper we find a pathwise decomposition of a certain class of Brownian semistationary processes (BSS) in terms of fractional Brownian motions. To do this, we specialize in the case when the kernel of the BSS is given by \( \varphi_\alpha(x) = L(x)x^\alpha \) with \( \alpha \in (-1/2, 0) \cup (0, 1/2) \) and \( L \) a continuous function slowly varying at zero. We use this decomposition to study some path properties and derive Itô’s formula for this subclass of BSS processes.

Keywords: Brownian semistationary processes, fractional Brownian motion, stationary processes, Volterra processes, Itô’s formula.

1 Introduction

During recent years, Brownian semistationary processes (BSS) and their tempo-spatial analogues, Ambit fields, have been widely studied in the literature. BSS processes were originally introduced in Barndorff-Nielsen and Schmiegel (2009) as a model for the temporal component of the velocity field of a turbulent fluid. They form a class of processes analogous to that of Brownian semimartingales in the stationary context. Specifically, BSS constitute the family of stochastic processes which can be written as

\[
Y_t = \theta + \int_{-\infty}^t g(t-s) \sigma_s dB_s + \int_{-\infty}^t q(t-s) a_s ds, \quad t \in \mathbb{R},
\]

where \( \theta \in \mathbb{R}, B \) is a Brownian motion, \( g \) and \( q \) are deterministic functions such that \( g(x) = q(x) = 0 \) for \( x \leq 0 \), and \( \sigma \) and \( a \) are predictable processes. A very remarkable property is that BSS are not semimartingales in general.

A natural extension, the so-called Lévy semistationary processes (LSS), is obtained by replacing \( B \) by a Lévy process. Note that the class LSS has been used as models for energy spot prices in Barndorff-Nielsen et al. (2013), Benth et al. (2014), Veraart and Veraart (2014) and Bennedsen (2015).

We would like to emphasize that any LSS is a null-space Ambit field: for every \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^d \) an Ambit field follows the equation

\[
Y_t(x) := \theta + \int_{A_t(x)} g(t; x, y) \sigma_s(y) Z(dy, ds) + \int_{D_t(x)} q(t; x, y) a_s(y) dy ds,
\]

where \( Z \) is a Lévy basis on \( \mathbb{R}^{d+1} \), \( g \) and \( q \) are deterministic functions and \( \sigma \) and \( a \) are suitable stochastic fields for which the integral exists. Furthermore, the Ambit sets satisfy that \( A_t(x), D_t(x) \subset (-\infty, t] \times \mathbb{R}^d \). For surveys related to the theory and applications of Ambit fields, we refer to Podolskii (2015), Barndorff-Nielsen et al. (2015) and Barndorff-Nielsen et al. (2016). See also Barndorff-Nielsen et al. (2011), Barndorff-Nielsen et al. (2014) and Pakkanen (2014).

In this paper we consider the subclass of BSS given by

\[
Y_t := \int_{-\infty}^t \varphi_\alpha(t-s) \sigma_s dB_s, \quad t \in \mathbb{R},
\]

where

\[
\varphi_\alpha(x) := L(x)x^\alpha, \quad x > 0,
\]

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\( \alpha \in (-1/2, 0) \cup (0, 1/2) \) and \( \sigma \) is an adapted stationary càdlàg process. The function \( L \) is continuous and slowly varying at 0, where the latter means that for every \( t > 0 \), \( \lim_{x \to 0} L(tx)/L(x) = 1 \).

Kernels of the type \( (2) \) are building blocks for the standard class of BSS/LSS considered in the literature. For example, most of the existing limit theory for BSS and LSS processes is developed under this framework. See Corcuera et al. (2013) and Basse-O’Connor et al. (2017). We refer also to Barndorff-Nielsen et al. (2011), Barndorff-Nielsen et al. (2013), Gärtnert and Podolskii (2015) and Basse-O’Connor et al. (2017). BSS processes of the type \( (1) \) have also proved to be a potential model for turbulent time series as it was shown in Márquez and Schmiegel (2016), cf. Barndorff-Nielsen and Schmiegel (2009). Furthermore, Bennedsen et al. (2016a) have shown that processes of the kind of \( (1) \) can be used as a parsimonious way of decoupling the short and long term behavior of time series in terms of \( \alpha \) and \( L \), respectively.

It was noticed in Barndorff-Nielsen (2012), cf. Corcuera et al. (2013), that in the case when \( L(x) = e^{-ax} \), the second order structure of the increments of \( Y \) behaves as that of a fractional Brownian motion with Hurst parameter \( H = \alpha + 1/2 \). A generalization in this direction that includes certain subclass of functions of the form of \( (2) \) can be found in Bennedsen et al. (2016b). Such a behaviour means that, in the second order sense, the fractional Brownian motion (fBm for short) only differs from this kind of BSS by a process of bounded variation. The main goal of this paper is to show that, up to a modification, \( Y \) only differs from a fBm by an absolutely continuous process. To do this, we study a Wiener-type stochastic integral with respect to volatility modulated Volterra processes on the real line (VMVP). Let us remark that this class of integrals can be seen as a particular case of those introduced in Alòs et al. (2001), Barndorff-Nielsen et al. (2014) and Mocioalca and Viens (2004), for the case of non-random integrands. Finally, as a way to show the potential of our main result, we derive some path properties of \( Y \) and, based on the existing literature of stochastic calculus for the fBm, we establish Itô’s formulae for \( Y \).

The paper is organized as follows: Section 2 introduces the basic notation and definitions. Our basic examples are also introduced in that section and we finish by giving some basic properties of fractional Brownian motions. In Section 3, we introduce the class of VMVP on the real line and, by similar heuristic arguments as in Barndorff-Nielsen et al. (2014), cf. Mocioalca and Viens (2004), we define Wiener-type stochastic integrals. We show that in the case of bounded variation integrands, such integral operator is just of the Lebesgue-Stieltjes type. By using this, we study the class of solutions of the associated Langevin equation. Section 4 provides a pathwise decomposition of BSS processes admitting the representation of \( (1) \), as a sum of an absolutely continuous process and the so-called volatility modulated fractional Brownian motion. We end the section by establishing Itô’s formulae in the case when \( 1/2 > \alpha > 0 \). In Section 5 we resume our findings. We have also added an appendix which includes some technical results needed in the paper.

## 2 Preliminaries and basic results

Throughout this paper \( (\Omega, F, (F_t)_{t \in \mathbb{R}}, \mathbb{P}) \) denotes a filtered probability space satisfying the usual conditions of right-continuity and completeness. We will further assume that \( (\Omega, F, (F_t)_{t \in \mathbb{R}}, \mathbb{P}) \) is rich enough to support a two-sided Brownian motion \( (B_t)_{t \in \mathbb{R}} \), that is, \( B \) is a continuous centered Gaussian process satisfying that \( B_0 = 0 \) and for any \( t \geq 0 \) and \( s \in \mathbb{R} \), the random variable \( B_{t+s} - B_s \) has variance \( t \), is \( F_{t+s} \)-measurable and is independent of \( F_s \). A stochastic process \( (Z_t)_{t \in \mathbb{R}} \) is said to be an increment semimartingale on \( (\Omega, F, (F_t)_{t \in \mathbb{R}}, \mathbb{P}) \), if the process \( (Z_{t+s} - Z_s)_{s \geq 0} \) is a semimartingale on \( (F_{t+s})_{t \geq s} \) in the usual sense, for any \( s \in \mathbb{R} \). For a detailed study of increment semimartingales, included a stochastic integration theory, the reader can see Basse-O’Connor et al. (2013), cf. Basse-O’Connor et al. (2010).

### 2.1 Existence of BSS processes and its stationary structure

In this part we briefly discuss necessary and sufficient conditions for which the stochastic integral in \( (1) \) exists in the Itô’s sense.

Observe that, for any predictable process \( \phi \), the stochastic integral \( \int_\mathbb{R} \phi \, dB_s \) exists if and only if almost surely 
\[
\int_\mathbb{R} \phi_s^2 \, ds < \infty,
\]
see for instance the paper of Basse-O’Connor et al. (2013). Therefore, the process \( Y \) in \( (1) \) is well defined if and only if for every \( t \in \mathbb{R} \), a.s. 
\[
\int_0^\infty \phi_s^2(s) \, \sigma_{s+t}^2 \, ds < \infty.
\]
Since \( \sigma \) is assumed to be càdlàg, then its paths are bounded in compact sets almost surely. Thus, there exists a positive (random) constant \( M_t > 0 \) such that with probability 1 it holds that
\[
\int_0^1 \phi_s^2(s) \sigma_{s+t}^2 \, ds < \infty.
\]
Hence, \( Y \) is well defined if and only if \( \int_1^\infty \phi_s^2(s) \sigma_{s+t}^2 \, ds < \infty \) almost surely. Thus if \( \sigma \) is bounded in \( L^2(\Omega, F, \mathbb{P}) \),
that is, \( \sup_t \mathbb{E}(\sigma_t^2) < \infty \), then \( Y \) is well defined if \( \int_1^\infty \varphi^2_\alpha(s) \, ds < \infty \). For general conditions on the existence of \( Y \) in the non-Gaussian case, see Basse-O’Connor (2013) and Pedersen and Sauri (2015).

It is well known that if \( \sigma \) and \( B \) are independent and \( \sigma \) is weakly (strongly) stationary, then \( Y \) is weakly (strongly) stationary as well. Furthermore, its autocovariance function is given by

\[
\gamma(h) = \mathbb{E}(\sigma_0^2) \int_0^\infty \varphi_\alpha(s) \varphi_\alpha(s + h) \, ds.
\] (3)

In applications, \( \sigma \) might have a certain dependence structure on \( B \), so the imposed independence assumption does not seem to be general enough. Nevertheless, as the following result shows, we can drop the independence among \( \sigma \) and \( B \) but the price to pay is that we cannot have a closed formula for its autocovariance structure anymore. For a proof see the Appendix.

**Proposition 1.** Let \( \phi \) be a square integrable function. Assume that for all \( h \in \mathbb{R} \)

\[
(s + h, B^h_t) \overset{d}{=} (s, B_s) \in \mathbb{R},
\] (4)

where

\[
B^h_t := B_{t+h} - B_h, \quad t \in \mathbb{R},
\]

and \( \sigma \) is a stationary càglàd adapted process. Then the BSS given by

\[
Y_t := \int_{-\infty}^t \phi(t - s) \sigma_s dB_s, \quad t \in \mathbb{R},
\]

is stationary in the strong sense.

### 2.2 Basic examples

In this part we introduce two important subclasses of BSS of the type of (1).

**BSS with a gamma kernel**

Consider the case when \( \varphi_\alpha \) can be expressed as a gamma density, this is

\[
\varphi_\alpha(x) = \frac{\lambda^{\alpha+1}}{\Gamma(\alpha + 1)} e^{-\lambda x} x^\alpha, \quad \alpha > -1/2, x > 0.
\] (5)

The distinctive property of \( \varphi_\alpha \) is that it solves, though not uniquely, the so-called causal covariance equation

\[
\int_0^\infty \varphi_\alpha(x+h) \varphi_\alpha(x) \, dx = \frac{2^{-\alpha+1/2}}{\Gamma(\alpha + 1/2)} \mathcal{K}_{\alpha+1/2}(\lambda h), \quad h \geq 0,
\] (6)

where \( \mathcal{K}_\nu(u) = u^\nu K_\nu(u) \), with \( K_\nu \) the modified Bessel function of the second kind with parameter \( \nu \). The function on the right-hand side of (6) is known as the Whittle-Matérn covariance function. For a historical review on this class of autocovariance functions we refer to Guttorp and Gneiting (2005).

It is clear that \( \int_1^\infty \varphi^2_\alpha(s) \, ds < \infty \), meaning that the associated BSS process is well defined and its autocovariance function, in the case when \( \sigma \) is independent of \( B \), is proportional to that in (6). We will refer to \( Y \) as an BSS with a gamma kernel. This subclass of Brownian semistationary processes, as we will show later, are closely related with fractional Ornstein-Uhlenbeck processes. We would like to emphasize that \( Y \) has successfully studied and applied in the fields of econometrics, finance and turbulence. See for instance Barndorff-Nielsen and Schmiegel (2009), Barndorff-Nielsen et al. (2011), Pakkanen (2011), Barndorff-Nielsen et al. (2013), and Pedersen and Sauri (2015).

It was shown in Barndorff-Nielsen and Schmiegel (2009) that \( Y \) is not in general a semimartingale. More precisely, for any \( \alpha \geq 1/2 \), \( Y \) is a process of bounded variation. However, when \( \alpha \in (-1/2, 0) \cup (0, 1/2) \), \( Y \) is no longer a semimartingale. The case \( \alpha = 0 \) corresponds to the classic Ornstein-Uhlenbeck process.
Power-BSS processes

In Bennedsen et al. (2016b) the authors considered the BSS associated to the function

\[ \varphi_\alpha (x) = \frac{x^\alpha}{(1 + x)^{\alpha + 1}}, \quad \alpha \in (-1/2, 0) \cup (0, 1/2), \beta > 1/2, \]

and they referred to it as a power-BSS process. A remarkable property of this family of BSS processes is that, on

the one hand, the parameter \( \alpha \) controls the path regularities of \( Y \), while on the other, \( \beta \) characterizes the persistence

of \( Y \). More precisely, \( Y \) has \( p \)-Hölder-continuous paths almost surely for any \( p < \alpha + 1/2 \) and its autocovariance

function satisfies that as \( h \uparrow \infty \)

\[ \gamma(h) \sim \begin{cases} K_{\sigma,\beta,\alpha} h^{1-2\beta} & \text{if } \beta \in (1/2, 1); \\ K_{\sigma,\beta,\alpha} h^{-\beta} & \text{if } \beta > 1. \end{cases} \]

for some constant \( K_{\sigma,\beta,\alpha} > 0 \). For a proof of these properties see Bennedsen et al. (2016a). Therefore, \( Y \) has long

memory in the case when \( \beta \in (1/2, 1) \), while when \( \beta > 1 \), the process has short memory. The case \( \beta = 1 \) is a limit

case.

2.3 Fractional Brownian motion

A continuous centered Gaussian process \( (B_t^H)_{t \in \mathbb{R}} \) is called fractional Brownian motion (fBm for short) with Hurst

index \( H \in (0, 1) \), if its covariance structure is given by

\[ \mathbb{E} (B_t^H B_s^H) = \frac{c_H}{2} \left( |t|^{2H} + |s|^{2H} - |t-s|^{2H} \right), \quad \text{for any } s, t \in \mathbb{R}, \]

with \( c_H = \int_0^\infty (1 + x)^{H-1/2} - x^{H-1/2} \) \( dx + 1/2H \). It is well known that \( B^H \) has (up to modification) Hölder

continuous paths of order strictly smaller than \( H \) and it is a semimartingale if and only if \( H = 1/2 \), i.e. when \( B^H \)

is a Brownian motion. Furthermore, \( B^H \) admits the following spectral representation

\[ B_t^H = \int_{\mathbb{R}} K^H (t, s) dB_s, \quad t \in \mathbb{R}, \tag{7} \]

where

\[ K^H (t, s) := (t+s)^{H-1/2} - (-s)^{H-1/2}, \quad s, t \in \mathbb{R}, \]

with \( (x)_+ := \max \{0, x\} \). It follows from this that \( B^H \) is not adapted to the filtration generated by the increments

of \( B \) for \( t < 0 \) and it is a Volterra-type representation for \( t > 0 \). We claim that for \( t \geq 0 \)

\[ B_t^H = A_t + \tilde{B}_t^H, \tag{8} \]

where \( \tilde{B}_t^H = \int_0^t (t-s)^{H-1/2} dB_s \) and \( A_t := \int_{-\infty}^0 K^H (t, s) dB_s \) being a process with absolutely continuous paths.

Indeed, since for any \( t \geq 0 \)

\[ K^H (t, s) = (H-1/2) \int_0^t (u-s)^{H-3/2} du, \quad s < 0, \]

and

\[ \int_0^t \left( \int_{-\infty}^0 (u-s)^{2H-3} ds \right)^{1/2} du = \frac{1}{H(2-2H)^{1/2}} t^H, \]

the Stochastic Fubini Theorem can be applied (see Theorem 4 in the Appendix) to deduce that for any \( t \geq 0 \),

almost surely

\[ A_t := (H-1/2) \int_0^t \int_{-\infty}^0 (u-s)^{H-3/2} dB_s du, \]

meaning this that, up to a modification, the paths of \( A_t \) are absolutely continuous for \( t \geq 0 \).

More generally, any process admitting the integral representation

\[ B_t^{\sigma, H} = \int_{\mathbb{R}} K^H (t, s) \sigma_s dB_s, \quad t \in \mathbb{R}, \tag{9} \]

for an adapted càglàd process \( \sigma \), will be referred to as a volatility modulated fractional Brownian motion (vMBfBm

for short). Observe that by a localization argument on \( \sigma \), we can construct a modification of \( B^{\sigma, H} \) which has Hölder

continuous paths of index strictly smaller that \( H \). A similar decomposition to (8) can be obtained in the case when \( \sigma \) is bounded in \( L^2 (\Omega, \mathcal{F}, \mathbb{P}) \).
3 Wiener integrals for volatility modulated Volterra processes

In this section, by reasoning as in [Mocioalca and Viens (2004) as well as in Barndorf-Nielsen et al. (2014), we define Wiener-type stochastic integrals with respect to volatility modulated Volterra processes on the real line. Such an integral will play a key role in our main result.

3.1 Definition and basic properties

Let $K : \mathbb{R}^2 \to \mathbb{R}$ denotes a measurable function such that $K(t,s) = 0$ for $s > t$, $\int_{-\infty}^{t} K^2(t,s) ds < \infty$ for any $t \in \mathbb{R}$, and the mapping $t \mapsto K(t,s)$ is continuously differentiable on $(s, \infty)$ for almost all $s$. A stochastic process $(X_t)_{t \in \mathbb{R}}$ is called volatility modulated Volterra process ($\mathcal{VMVP}$ for short) on the real line if it admits the representation

$$X_t = \int_{-\infty}^{t} K(t,s) \sigma_s dB_s, \quad t \in \mathbb{R},$$

(10)

for some adapted càdlàg process that is bounded in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. Observe that $X$ is well defined because almost surely $\int_{-\infty}^{t} K(t,s)^2 \sigma^2_s ds < \infty$. Indeed, the $\mathcal{L}^2$-boundedness of $\sigma$ guarantees the existence of a constant $M > 0$ such that

$$\mathbb{E} \left[ \int_{-\infty}^{t} K(t,s)^2 \sigma^2_s ds \right] \leq M \int_{-\infty}^{t} K^2(t,s) ds < \infty, \quad \forall t \in \mathbb{R},$$

as claimed. The function $K$ will be referred as the kernel of $X$. For simplicity, for the rest of this paper we will assume that $K(0,s) = 0$ for almost all $s$. We will also assume that $K(t, \cdot) \rightarrow K(u, \cdot)$ in $\mathcal{L}^2(ds)$. This in particular implies that $X$ is continuous in probability (see [Basse-O’Connor et al. (2013)]) and therefore it admits a measurable modification.

Let us give some remarks:

Remark 1. We observe the following:

1. In general, $X$ is not adapted to the natural filtration of $B$ but rather to the filtration generated by the increments of $B$.

2. The fractional Brownian motion can be written as the sum of an $\mathcal{VMVP}$ and an increment semimartingale. Indeed, from (7), we have that

$$B_t^H = \int_{-\infty}^{t} K^H(t,s) dB_s - \int_{t \wedge 0}^{0} (-s)^{H-1/2} dB_s, \quad t \in \mathbb{R},$$

where, as usual, the symbol $t \wedge 0$ represents the minimum between $t$ and 0.

3. In general, $\mathcal{VMVP}$ are neither semimartingales nor increment semimartingales.

From the previous remark it follows that standard Itô’s stochastic integration cannot be applied to $\mathcal{VMVP}$. Nevertheless, proceeding as in [Barndorf-Nielsen et al. (2014), c.f. Mocioalca and Viens (2004), we introduce a Wiener-type stochastic integral with respect to (w.r.t. for short) $\mathcal{VMVP}$ as follows: Suppose for a moment that $K$ is smooth enough to make $X$ a semimartingale (see general conditions in Proposition 3 of [Barndorf-Nielsen et al. (2013)]), then by putting $dB_t^\sigma = \sigma_t dB_t$, we have

$$dX_t = K(t,t-) dB_t^\sigma + \int_{-\infty}^{t} \frac{\partial}{\partial t} K(t,s) dB_s^\sigma dt, \quad t \in \mathbb{R}.$$

(11)

In general, $K(t,t-) \underline{\text{might not be well defined, e.g.}}$ $K$ as in (7), but even when it is, one can have that $\frac{\partial}{\partial t} K(t, \cdot)$ is not integrable with respect to $B^\sigma$. However, we can formally infer from (11) that

$$\int_{-\infty}^{t} f(s) dX_s = \int_{-\infty}^{t} \int_{-\infty}^{s} f(s) \frac{\partial}{\partial s} K(s,r) dB_r^\sigma ds + \int_{-\infty}^{t} f(s) K(s,s-) dB_s^\sigma$$

(12)

$$= \int_{-\infty}^{t} \int_{-\infty}^{s} \left[ f(s) - f(r) \right] \frac{\partial}{\partial s} K(s,r) dB_r^\sigma ds + \int_{-\infty}^{t} f(s) K(s,s-) dB_s^\sigma$$

$$+ \int_{-\infty}^{t} \int_{-\infty}^{s} f(r) \frac{\partial}{\partial s} K(s,r) dB_r^\sigma ds.$$
but, by the stochastic Fubini theorem
\[
\int_{-\infty}^{t} \int_{-\infty}^{s} \left[ f(s) - f(r) \right] \frac{\partial}{\partial s} K(s,r) \, dB^\sigma_r \, ds = \int_{-\infty}^{t} \int_{s}^{t} \left[ f(u) - f(s) \right] \frac{\partial}{\partial u} K(u,r) \, du \, dB^\sigma_r,
\]
and
\[
\int_{-\infty}^{t} \int_{-\infty}^{s} f(r) \frac{\partial}{\partial s} K(s,r) \, dB^\sigma_r \, ds = \int_{-\infty}^{t} f(r) [K(t,r) - K(r,r-)] \, dB^\sigma_r.
\]
Thus, in the semimartingale case, we have that
\[
\int_{-\infty}^{t} f(s) \, dX_s = \int_{-\infty}^{t} \mathcal{K}_K f(t,s) \sigma_s \, dB_s,
\]
where
\[
\mathcal{K}_K f(t,s) := f(s) K(t,s) + \int_{s}^{t} \left[ f(u) - f(s) \right] \frac{\partial}{\partial u} K(u,s) \, du, \quad s < t.
\]
Note that the left-hand side of (13) is well defined (remember that \( \sigma \) is bounded in \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \)) if \( \mathcal{K}_K f(t,\cdot) \) is square integrable, which a priori does not require that \( K(t,t-) < \infty \). Thus, (13) gives a way to define stochastic integrals w.r.t. \( \mathcal{VMP} \) even in the non-semimartingale case. More precisely:

**Definition 1.** Let \( X \) be an \( \mathcal{VMP} \) as in (10) and put
\[
\mathcal{H}_K^t := \left\{ f \text{ measurable } : \mathcal{K}_K f \in L^2((-\infty,t]) \right\}, \quad t \in \mathbb{R}.
\]
For each \( f \in \mathcal{H}_K^t \), we define the stochastic integral of \( f \) w.r.t. \( X \) as
\[
\int_{-\infty}^{t} f(s) \, dX_s := \int_{-\infty}^{t} \mathcal{K}_K f(t,s) \sigma_s \, dB_s, \quad t \in \mathbb{R}.
\]

**Remark 2.** Observe that \( \int_{-\infty}^{t} f(s) \, dX_s \) is a particular case of the one introduced in [Alòs et al. (2001), Mocioalca and Viens (2004), and Barndorff-Nielsen et al. (2014)] for non-random integrands with support on \( \mathbb{R} \) rather than in an interval of the form \([0,T]\). Furthermore, \( \mathcal{H}_K^t \) is a Hilbert space with inner product
\[
(f,g)_{\mathcal{H}_K^t} = \langle \mathcal{K}_K f, \mathcal{K}_K g \rangle_{L^2((-\infty,t])}.
\]

**Remark 3.** We might be tempted to extend Definition 1 to any predictable process \( Y \) by putting
\[
\int_{-\infty}^{t} Y_s \, dX_s = \int_{-\infty}^{t} \mathcal{K}_K (Y)(t,s) \, dB_s.
\]
However, under this definition, the process \( (\mathcal{K}_K (Y)(t,s))_{s \leq t} \) is not adapted because the random variable \( \int_{-\infty}^{t} \left[ Y_u - Y_s \right] \frac{\partial}{\partial u} K(u,r) \, du \) is only \( \mathcal{F}_t \)-measurable. Hence, we cannot define such an integral in the Itô’s sense, but it is feasible in the Skorohod sense as it was done in [Barndorff-Nielsen et al. (2014)].

The following example shows that there is a natural connection between the stochastic integral defined in (15) and \( \mathcal{BSS} \) processes of the form of (1). It is also the starting point and motivation for our main result in the next section.

**Example 1 (BSS and \( \mathcal{VMP} \)).** For \( \alpha \in (-1/2,0) \cup (0,1/2) \) let \( K \) be the Mandelbrot-Van Ness kernel, i.e.
\[
K(t,s) = \{(t-s)^\alpha - (-s)^\alpha \} \mathbb{1}_{\{t>s\}},
\]
and consider
\[
f(s) = e^{\lambda s}, \quad s \in \mathbb{R}, \lambda > 0.
\]
Let us see that \( \int_{-\infty}^{t} e^{\lambda s} \, dX_s \) can be defined as in Definition 1 for every \( t \). First note that
\[
\mathcal{K}_K (f)(t,s) = e^{\lambda t} K(t,s) - \lambda \int_{s}^{t} e^{\lambda u} K(u,s) \, du, \quad s < t.
\]
Hence, BSS next section states that the process in the right-hand side of (18) has absolutely continuous paths almost surely.

Moreover, due to the Jensen’s inequality and Tonelli’s Theorem we have that

$$\int_{-\infty}^t \left( \int_s^t e^{\lambda u} K(u, s) \, du \right)^2 \, ds \leq \frac{e^{\lambda t}}{\lambda} \int_{-\infty}^t \int_s^t e^{\lambda u} K(u, s)^2 \, du \, ds = \frac{e_{\alpha} e^{\lambda t}}{\lambda} \int_{-\infty}^t e^{\lambda u} u^{2\alpha+1} \, du < \infty,$$

with \( c_\alpha \) a constant depending only on \( \alpha \). This, together with (17) shows that \( f \int_{-\infty}^t e^{\lambda s} \, dX_s \) is well defined and by linearity so \( f \int_{-\infty}^t e^{-\lambda(t-s)} \, dX_s \) is. Further, for \( t \geq 0 \) almost surely

$$\int_{-\infty}^t e^{-\lambda(t-s)} \, dX_s - Y_t = e^{\lambda t} \left( \int_{-\infty}^t \int_s^t \left[ e^{\lambda u} - e^{\lambda s} \right] \frac{\partial}{\partial u} K(u, s) \, du \, dB_s - \int_{-\infty}^0 e^{\lambda s} (-s)^\alpha \, \sigma_s \, dB_s \right), \quad (18)$$

where \( Y \) is a BSS with a gamma kernel. Observe that this decomposition is well defined. Our main result in the next section states that the process in the right-hand side of (18) has absolutely continuous paths almost surely. Hence, BSS processes with gamma kernel and \( \mathcal{VMVP} \) differs only by an absolutely continuous process.

The previous example motivates a further understanding of the integral introduced in Definition 1. Therefore, for the rest of this section we study some properties of the integral appearing in (15). Let us start with the connection between such an integral and the Lebesgue-Stieltjes integral:

**Proposition 2.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function of local bounded variation satisfying that for almost all \( s < t \)

$$\lim_{u \downarrow s} [f(u) - f(s)] K(u, s) = 0. \quad (19)$$

Then \( f \in \mathcal{H}_t^K \) if and only if the mapping \( s \mapsto f_s^t K(r, s) \) is square integrable. If in addition we have that

$$\int_{-\infty}^t \left( \int_s^t K(s, r)^2 \, dr \right)^{1/2} \, d|f| (s) < \infty, \quad (20)$$

where \( d|f| \) is the total variation of \( f \), then almost surely

$$\int_{-\infty}^t f(s) \, dX_s = f(t) X_t - \int_{-\infty}^t X_s \, df(s), \quad (21)$$

i.e. \( \int_{-\infty}^t f(s) \, dX_s \) coincides with the Lebesgue-Stieltjes integral of \( f \) with respect to \( X \).

**Proof.** Integration by part shows that for any continuous function of local bounded variation satisfying equation (19) it holds that

$$\int_s^t [f(u) - f(s)] \frac{\partial}{\partial u} K(u, s) \, du = [f(t) - f(s)] K(t, s) - \int_r^t K(u, s) \, df(u),$$

for \( s < t \). Thus

$$K_K f(t, s) = f(t) K(t, s) - \int_s^t K(r, s) \, df(r), \quad s < t. \quad (22)$$

The first conclusion of this proposition follows by noting that \( \int_{-\infty}^t K^2(s, t) \, ds < \infty \) for any \( t \in \mathbb{R} \) and using the fact that \( \mathcal{H}_t^K \) is a linear space. Moreover, due to (22), in the case when \( f \in \mathcal{H}_t^K \) satisfies (19) we have that

$$\int_{-\infty}^t f(s) \, dX_s = f(t) X_t - \int_{-\infty}^t \int_s^t K(r, s) \, df(r) \, dB_s^\sigma.$$

Suppose for a moment that the stochastic Fubini theorem can be applied to \( \int_{-\infty}^t \int_r^t K(s, r) \, \sigma df(r) \, dB_s^\sigma \), then almost surely

$$\int_{-\infty}^t f(s) \, dX_s = f(t) X_t - \int_{-\infty}^t \int_s^t K(r, s) \, df(r) \, dB_s^\sigma = f(t) X_t - \int_{-\infty}^t X_r \, df(r),$$
Indeed, if \( f \) is locally of bounded variation, it can be written as \( f = g_1 - g_2 \), where \( g_1 \) and \( g_2 \) are non-decreasing functions. By (20), we deduce that for \( i = 1, 2 \),
\[
\int_{-\infty}^{t} X_s dg_i (s) = \int_{-\infty}^{t} \int_{r}^{s} K (s, r) dg_i dB^r_r,
\]
which completes the proof.

Remark 4. We would like to remark that since the mapping \( u \mapsto \| K(t, \cdot) - K(u, \cdot) \|_{C^2 (d\alpha)} \) is continuous, then \( C^1_b \), the space of continuously differentiable functions with compact support, belongs to \( H^K_t \) just as in Proposition 14 in [Mocioalca and Viens (2004)].

Remark 5. Observe that a similar condition to (19) was imposed in [Barndorff-Nielsen et al. 2014] to determine the class of solutions associated to the Langevin equation driven by an \( \mathcal{VMP} \) indexed on the positive real line.

Remark 6. Equation (19) is trivial when \( f \) is continuous and \( K(t+ t) \) exists for every \( t \). Furthermore, when \( f \) satisfies (19), the condition
\[
\int_{-\infty}^{t} \int_{-\infty}^{s} |K(s, r)dg| < \infty
\]
together with (19), imply that all the conclusions of Proposition 2 hold for such \( f \). Indeed, by the Jensen’s inequality and Tonelli’s Theorem
\[
\left[ \int_{-\infty}^{t} \left( \int_{-\infty}^{s} |K(s, r)dg| \right)^{1/2} |f| (s) \right]^{2} \leq c_t \int_{-\infty}^{t} \int_{-\infty}^{s} |K(s, r)dg| |f| (s) < \infty,
\]
as well as
\[
\int_{-\infty}^{t} \int_{s}^{t} K(r, s)dr |df| \left[ ds \right]^{2} \leq c_t \int_{-\infty}^{t} \int_{s}^{t} K(r, s)dr |df| (s) = c_t \int_{-\infty}^{t} \int_{-\infty}^{s} K(r, s)dr |df| (s),
\]
where \( c_t = \int_{-\infty}^{t} |df| (s) \).

The next example concentrates in the case when the kernel \( K \) corresponds to the one of the fBm.

Example 2. Let \( K \) be as in Example 1. Observe that when \( \alpha > 0 \), (19) is satisfied for any continuous function. On the other hand, when \( \alpha < 0 \) we have that every function of bounded variation satisfies (19) for almost all \( s \in \mathbb{R} \). Indeed, if \( f \) is of bounded variation then its derivative exists almost everywhere, so pick \( s \in \mathbb{R} \setminus \{0\} \) for which the derivative of \( f \) exists. Then
\[
\lim_{u \downarrow s} \left[ f(u) - f(s) \right] K(u, s) = \lim_{u \downarrow s} \left( \frac{f(u) - f(s)}{u - s} \right) K(u, s) (u - s) = 0.
\]
where we have used that \( \frac{f(u) - f(s)}{u - s} \to f'(s) \) and \( K(u, s) (u - s) \to 0 \) as \( u \downarrow s \). Moreover, in view that
\[
\int_{-\infty}^{s} K(r, s)dr = c_\alpha s^{2\alpha + 1},
\]
for some \( c_\alpha > 0 \), then condition (20) becomes
\[
\int_{-\infty}^{t} s^{\alpha + 1/2} |df| (s) < \infty.
\]
In particular, if for \( s \leq t \) we let
\[
\begin{align*}
ft,\lambda(s) := & e^{-\lambda(t - s)}, \quad \lambda > 0; \\
ft,\beta,\alpha(s) := & (1 + t - s)^{-\alpha - \beta}, \quad \beta > 1/2,
\end{align*}
\]
then \( f_{t,\lambda} \) and \( f_{t,\beta,\alpha} \) satisfy (19) and (20). In Example 1 we have showed that \( f_{t,\lambda} \in H_{t}^{K} \), meaning this that almost surely

\[
\int_{-\infty}^{t} e^{-\lambda(t-s)}dX_s = X_t - \lambda e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s}X_s ds. \tag{24}
\]

Later we will see that also \( f_{t,\beta,\alpha} \) belongs to \( H_{t}^{K} \), which according to Proposition 2 lead us to the almost surely relation

\[
\int_{-\infty}^{t} (1 + t - s)^{-\alpha-\beta}dX_s = X_t - (\alpha + \beta) \int_{-\infty}^{t} (1 + t - s)^{-\alpha-\beta-1}X_s ds.
\]

Next, for the sake of completeness, we state the following result related to the continuity of \( \int_{-\infty}^{t} dX_s \) whose proof is straightforward and thus omitted.

**Proposition 3.** The mapping \( I_{t} : H_{t}^{K} \to L_{2}(\Omega, \mathcal{F}, P) \), defined by

\[
I_{t}(f) := \int_{-\infty}^{t} f(s) dX_s,
\]

is a continuous linear isometry.

**Remark 7.** Observe that from Basse-O’Connor et al. (2012), in general, we have that \( \{I_{t}(f) : f \in H_{t}^{K}\} \subseteq \text{span}\{X_u - X_v : v \leq u \leq t\} \), where \( \text{span} \) represents the closed linear span on \( L_{2}(\Omega, \mathcal{F}, P) \). See also Jolis (2010) and Pipiras and Taqqu (2000).

### 3.2 The Langevin equation and OU processes driven by an \( \mathcal{V}\mathcal{M}\mathcal{V}\mathcal{P} \)

Let \( X \) be an \( \mathcal{V}\mathcal{M}\mathcal{V}\mathcal{P} \) satisfying the assumptions of the previous subsection. Recall that a process \((Z_t)_{t \geq 0}\) solves the Langevin equation with parameter \( \lambda > 0 \) with respect to \( X \) if and only if for \( t \geq 0 \) almost surely

\[
Z_t = \xi + X_t - \lambda \int_{0}^{t} Z_s ds, \tag{25}
\]

for some random variable \( \xi \) which is \( \mathcal{F}_0 \)-measurable. Thus, by looking carefully into (24) in Example 2 one can actually see that the process \((\int_{-\infty}^{t} e^{-\lambda(t-s)}dX_s)_{t \geq 0}\) solves the Langevin equation associated to such an \( \mathcal{V}\mathcal{M}\mathcal{V}\mathcal{P} \). In this case, the initial condition is \( \xi = -\lambda \int_{-\infty}^{0} e^{\lambda u}X_u du \). In this part, we generalize such a result to the context of Proposition 2.

For \( \lambda > 0 \), set

\[
f_{t,\lambda}(s) := e^{-\lambda (t-s)}, \quad s \leq t.
\]

**Proposition 4.** Suppose that \( f_{t,\lambda} \) satisfies the assumptions of Proposition 2 for every \( t \geq 0 \). Then the process

\[
Z_t := \int_{-\infty}^{t} e^{-\lambda(t-s)}dX_s, \quad t \geq 0. \tag{26}
\]

is the unique solution (up to a modification) of the Langevin equation with parameter \( \lambda > 0 \) and initial condition \( \xi = -\lambda \int_{-\infty}^{0} e^{\lambda u}X_u du \).

**Proof.** Let us first show uniqueness. Let \( Z^1 \) and \( Z^2 \) be two solutions of the Langevin equation. Then the process

\[
H_t := Z^1_t - Z^2_t = -\lambda \int_{0}^{t} H_s ds, \quad t \geq 0,
\]

satisfies the ordinary differential equation

\[
\frac{d}{dt}H_t = -\lambda H_t, \quad t \geq 0,
\]

with initial condition \( H_0 = 0 \). This implies necessarily that \( H_t = 0 \) almost surely for every \( t \geq 0 \). The uniqueness of the solution follows from this.
Now, if for every \( t \geq 0 \), \( f_\lambda(t, \cdot) \) satisfies the assumptions of Proposition \( \ref{prop:2} \) we have that \( Z \) is well defined and almost surely
\[
Z_t = X_t - \lambda e^{-\lambda t} \int_{-\infty}^t e^{\lambda s} X_s ds.
\] (27)
This, together with Lemma \( \ref{lemma:2} \) show that \( Z \) has a modification that is locally Lebesgue integrable which also will be denoted by \( Z \). Furthermore, integration by parts gives that
\[
-\lambda \int_0^t e^{-\lambda u} \int_u^t e^{\lambda s} X_s ds du = e^{-\lambda t} \int_{-\infty}^t e^{\lambda s} X_s ds - \int_{-\infty}^0 e^{\lambda s} X_s ds - \int_0^t X_u du.
\]
Consequently, by (27), for every \( t \geq 0 \), almost surely
\[
-\lambda \int_0^t Z_u du = -\lambda [e^{-\lambda t} \int_{-\infty}^t e^{\lambda s} X_s ds - \int_{-\infty}^0 e^{\lambda s} X_s ds] = Z_t - \xi - X_t,
\]
as required. \( \blacksquare \)

**Remark 8.** Note that the previous result is in agreement with Barndorff-Nielsen and Basse-O’Connor (2011), Theorem 2.1, and Proposition 8 in Barndorff-Nielsen et al. (2014) in the context of VMVP. However, in general, the process \( Z \) is not stationary.

We conclude this section with an example that links the process appearing in (20) and the fractional Ornstein-Uhlenbeck process.

**Example 3 (Fractional OU processes).** In this example the processes \( B \) and \( \sigma \) are fixed, with \( \sigma \) bounded in \( \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}) \). Let \( X \) be as in Example \( \ref{example:1} \). As noticed in the beginning of this section, any volatility modulated fBm can be written as
\[
B_t^{H, \sigma} = X_t - \int_{t \wedge 0}^0 (-s)^{H-1/2} \sigma_s dB_s, \quad t \in \mathbb{R},
\] (28)
where \( H = \alpha + 1/2 \). Since the process \( A_t = \int_{t \wedge 0}^0 (-s)^{H-1/2} \sigma_s dB_s \) is an increment semimartingale, the integral \( \int_{-\infty}^t f(s) dA_s \) is well defined in the Itô’s sense if, in general not only if, \( \int_{-\infty}^t f^2(s) (-s)^{2H-1} ds < \infty \). We may thus define the integral of \( f \) w.r.t. to \( B_t^{H, \sigma} \) as
\[
\int_{-\infty}^t f(s) dB_t^{H, \sigma} = \int_{-\infty}^t f(s) dX_s + \int_{-\infty}^t f(s) dA_s.
\] (29)

We have seen in Example \( \ref{example:2} \) that \( f_{t, \lambda} \) satisfies the assumptions of Proposition \( \ref{prop:2} \). In view of this and the fact that \( \int_{-\infty}^t e^{-2\lambda(t-s)} (-s)^{2H-1} ds < \infty \), the process \( Z_{t}^{H} = \int_{-\infty}^t e^{-\lambda(t-s)} dB_t^{H, \sigma} \) is well defined in the sense of (29).

When \( \sigma \) is constant, \( Z \) coincides with the usual fractional Ornstein-Uhlenbeck process which was introduced in Cheridito et al. (2003). The most remarkable property of \( Z_t^H \) is that the process
\[
U_t := Z_t^H - \int_{-\infty}^t e^{-\lambda(t-s)} dB_s = e^{-\lambda t} \int_{-\infty}^t e^{\lambda s} (-s)^{H-1/2} \sigma_s dB_s,
\]
is of unbounded variation for \( t < 0 \), while for \( t \geq 0 \) it has absolutely continuous paths almost surely.

## 4 A pathwise decomposition of \( \mathcal{BSS} \) with respect to vmfBm and its application

Through several examples, in the previous section we showed that there is a strong connection between \( \mathcal{BSS} \) processes and a subclass of \( \mathcal{VMVP} \). In this section, such a connection is explicitly derived when we consider \( \mathcal{BSS} \) of the form of
\[
Y_t := \int_{-\infty}^t \varphi_\alpha(t-s) \sigma_s dB_s, \quad t \in \mathbb{R},
\] (30)
where
\[
\varphi_\alpha(x) = L(x) x^\alpha, \quad x > 0,
\] (31)
\[ X_t = \int_{-\infty}^{t} \left[ (t-s)^\alpha - (-s)^\alpha \right] \sigma_s dB_s \quad (32) \]

differ only by an absolutely continuous process.

### 4.1 Assumptions and statement of the result

Let us start by introducing our working assumption which is a refinement of that considered in Bennedsen et al. (2016). We recall to the reader that the notation \( f(x) = O(g(x)) \) as \( x \to c \) means that \( \limsup_{x \to c} |f(x)/g(x)| < \infty. \)

**Assumption 1.** The function \( L \) in (33) satisfies the following:

1. \( L \) is twice continuously differentiable on \([0, \infty)\) such that \( L(0) \neq 0 \) and \( \int_0^{\infty} |L(s)s^\alpha|^2 \, ds < \infty. \)
2. There is \( \alpha + 3/2 < \zeta_0 \) such that as \( s \uparrow \infty, L'(s) = O(s^{-\zeta_0}) \) and \( L''(s) = O(s^{-(\zeta_0+1)}). \)

Under this assumption we have:

**Theorem 1.** Let Assumption 1 holds and consider \( X \) as in (32). Then \( L(t - \cdot) \) is integrable w.r.t. \( X \) for every \( t \in \mathbb{R}. \) Moreover, if \( Y \) belongs to the class of BSS appearing in (30) and

\[ Y_t^X := \int_{-\infty}^{t} L(t-s) \, dX_s, \quad t \in \mathbb{R}, \]

then the process

\[ V_t := Y_t^X - Y_t, \quad t \geq 0 \]

has absolutely continuous paths almost surely.

**Proof.** Along the proof the constant \( c_\alpha > 0 \) is such that \( \int_{-\infty}^{t} K(t,s)^2 \, ds = c_\alpha t^{2\alpha+1}. \) Firstly we note that under Assumption 1 \( L, L' \) and \( L'' \) are totally bounded and satisfy (19). Thus, thanks to Proposition 2 and Lemma 1 below, \( L(t - \cdot) \in \mathcal{H}_t^K \) for any \( t \in \mathbb{R}. \) Furthermore, Assumption 1 also guarantees the existence of \( \alpha + 3/2 < \zeta \) such that for any \( x \geq 0 \)

\[
\begin{align*}
|L(x)| &\leq M_0 |x|^{-(\zeta-1)}; \\
|L'(x)| &\leq M_1 |x|^{-\zeta}; \\
|L''(x)| &\leq M_2 |x|^{-(\zeta+1)}. 
\end{align*}
\]  

for some constants \( M_0, M_1, M_2 > 0. \) In view of this and since \( L \) is twice continuously differentiable on \([0, \infty)\) we have that for any \( -\infty < s_0 < t \wedge 0 \)

\[
\int_{-\infty}^{t} s^{\alpha+1/2} |L'(t-s)| \, ds \leq \int_{s_0}^{t} s^{\alpha+1/2} |L'(t-s)| \, ds + M_1 \int_{-\infty}^{s_0} s^{\alpha+1/2} |t-s|^{-\zeta} \, ds < \infty. \quad (34)
\]

Therefore, from Example 2 and Proposition 2 almost surely

\[ L(0)X_t = Y_t^X - \int_{-\infty}^{t} L'(t-s)X_s \, ds. \]

On the other hand, reasoning as in (34) we obtain that \( \int_{-\infty}^{t} |L(t-s)s^\alpha|^2 \, ds < \infty. \) Moreover, since \( L \) is totally bounded we deduce that the mapping \( s \mapsto L(t-s)K(t,s) \) is square integrable. Hence, we get that almost surely

\[ L(0)X_t = Y_t + \int_{-\infty}^{t} [L(0) - L(t-s)] K(t,s) \sigma_s dB_s - \int_{-\infty}^{t} L(t-s)(-s)^\alpha \sigma_s dB_s. \]

All in all imply that for \( t \geq 0 \) almost surely

\[ V_t = U_t^1 + U_t^2 + U_t^3, \]
Hence, in order to finish the proof, it is enough to show that for \(i = 1, 2, 3\), \(U^i\) has almost surely absolutely continuous paths. Suppose for a moment that the following processes are well defined together with Proposition 2 and Lemma 1 below, show that

\[
\begin{align*}
M' &:= \frac{1}{2} \int_{0}^{t} s^2 dB_s,
M'' &:= \frac{1}{2} \int_{0}^{t} s^2 ds,
L &:= \int_{0}^{t} u_s dB_s,
\end{align*}
\]

with

\[
K_L(t, s) := [L(0) - L(t - s)] K(t, s), \quad t > s.
\]

Hence, in order to finish the proof, it is enough to show that for \(i = 1, 2, 3\), \(u^i\) is well defined and admits the representation

\[
\begin{align*}
U^1_t &:= \int_{-\infty}^{t} K_L(t, s) \sigma_s dB_s, \\
U^2_t &:= \int_{-\infty}^{t} [L(t - s) - L(0)] dX_s, \\
U^3_t &:= -\int_{-\infty}^{0} L(t - s)(-s)^{\alpha} \sigma_s dB_s,
\end{align*}
\]

and that the Fubini Theorem (stochastic and non-stochastic) can be applied. Then, we necessarily have that

\[
\int_{0}^{t} u^1_s ds = U^1_t - U^3_t, \quad t \geq 0,
\]

which would conclude the proof. Therefore, in what follows, we check that for \(i = 1, 2, 3\), \(u^i\) is well defined and that the Fubini theorem can be applied.

Fix \(t \geq 0 > s_0 > -\infty\). The estimates in (33) and the fact that \(L'\) and \(L''\) are totally bounded by, let’s say \(M', M'' > 0\), respectively, give us that

\[
\begin{align*}
\int_{-\infty}^{t} |L'(t - s)K(t, s)|^2 ds &\leq M' c_0 t^{2\alpha + 1} < \infty; \\
\int_{-\infty}^{t} \left| [L(0) - L(t - s)] \frac{\partial}{\partial t} K(t, s) \right|^2 ds &= \int_{0}^{\infty} |[L(0) - L(r)] r^{\alpha - 1}|^2 ds < \infty; \\
\int_{-\infty}^{0} |L'(t - s)(-s)^{\alpha}|^2 ds &\leq M' \int_{s_0}^{0} s^{2\alpha} ds + M_1 \int_{-\infty}^{s_0} s^{2(\alpha - \zeta)} ds < \infty;
\end{align*}
\]

which shows the well definiteness of \(u^1\) and \(u^3\), while

\[
\int_{-\infty}^{t} |L''(t - s)s^{\alpha + 1/2}| ds \leq M'' \int_{s_0}^{0} s^{\alpha + 1/2} ds + M_1 \int_{-\infty}^{s_0} s^{(\alpha - \zeta)-1/2} ds < \infty,
\]

together with Proposition 2 and Lemma 1 below, show that \(u^2\) is well defined and admits the representation

\[
\int_{-\infty}^{t} L'(t - s) dX_s.
\]

Therefore, it only rests to check that we can apply the stochastic Fubini Theorem. By using (35) and applying the triangle inequality one obtain that

\[
\int_{0}^{t} \left( \frac{\partial}{\partial u} K_L(u, s) \right) ds = \left( \frac{\partial}{\partial u} \right) \int_{-\infty}^{t} s^{\alpha + 1/2} ds + \left( \int_{0}^{\infty} \frac{\partial}{\partial d} \right) \int_{-\infty}^{t} s^{\alpha - 1/2} ds < \infty.
\]

which according to Theorem 4 justifies the interchange of the Itô’s integral with the Lebesgue integral for \(u^1\). Finally, thanks to (35), (36) and the Dominated Convergence Theorem, we deduce that the mappings \(u \mapsto \)
By the mean value theorem, we have that for any continuous function \( f \) on \((-\infty, t]\), let
\[
\ell_t(s) := \int_s^t \{(r - s)_+^\alpha - (-s)_+^\alpha\} f(r)dr, \quad s < t.
\]
If \( f(u) = O(|u|^{-\hat{\zeta}}) \) as \( u \downarrow -\infty \), for some \( \alpha + 3/2 < \hat{\zeta} \), then
\[
\int_{-\infty}^t \ell_t(s)^2 ds < \infty.
\]

\textbf{Proof.} Let \( c_\alpha > 0 \) be as in the proof of Theorem 1. Firstly, we claim that \( \ell_t \) is locally square integrable on \((-\infty, t]\) whenever \( f \) is continuous. Indeed, let \( -\infty < s_0 < t \). Then by the Jensen’s inequality and Tonelli’s Theorem, it holds that
\[
\left( \int_{s_0}^t |f(u)| du \right)^{-1} \int_{s_0}^t |\ell_t(s)|^2 ds \leq \int_{s_0}^t \int_{s_0}^r \{(r - s)_+^\alpha - (-s)_+^\alpha\}^2 ds |f(r)| dr \leq c_\alpha \int_{s_0}^t r^{2\alpha+1} |f(r)| dr < \infty,
\]
which proves our claim. Consequently, we only need to show that under our hypothesis on \( f, \ell_t \) is square integrable at infinity. Let \( s < (t \wedge 0) \), then
\[
\ell_t(s) = \begin{cases} \left|s\right|^{\alpha+1} \int_0^1 [(1-r)^\alpha - 1] f(sr) dr - \int_0^s \{(r-s)_+^\alpha - (-s)_+^\alpha\} f(r) dr & \text{if } t \leq 0; \\ \left|s\right|^{\alpha+1} \int_0^1 [(1-r)^\alpha - 1] f(sr) dr + \int_s^t \{(r-s)_+^\alpha - (-s)_+^\alpha\} f(r) dr & \text{if } t > 0. \end{cases}
\]
(38)

We show first that in general for any continuous \( f \), it holds that
\[
\ell_t(s) - \left|s\right|^{\alpha+1} \int_0^1 [(1-r)^\alpha - 1] f(sr) dr = O(|s|^{-\hat{\zeta} - 1}), \quad s \downarrow -\infty.
\]
(39)
By the mean value theorem, we have that for \( t > 0 \),
\[
\int_t^0 |\{(r-s)_+^\alpha - (-s)_+^\alpha\} f(r)| dr \leq |\alpha| |s|^{-\hat{\zeta} - 1} \int_0^t |rf(r)| dr,
\]
which implies trivially (39). In an analogous way, we get that for \( s < t \leq 0 \)
\[
\int_t^0 |\{(r-s)_+^\alpha - (-s)_+^\alpha\} f(r)| dr \leq |\alpha| |s|^{-\hat{\zeta} - 1} \int_0^t \left(1 - \frac{r}{s}\right)^{\hat{\zeta} - 1} |rf(r)| dr \leq |\alpha| \left(1 - \frac{1}{s}\right)^{\hat{\zeta} - 1} |s|^{-\hat{\zeta} - 1} \int_0^t |rf(r)| dr,
\]
from which (39) can be deduced. All above implies that if \( f \) is continuous, then \( \ell_t \) is square integrable if and only if the mapping \( s \mapsto |s|^{\alpha+1} \int_0^1 [(1-r)^\alpha - 1] f(sr) dr \) is square integrable at infinity. Let us show that under our hypothesis this holds. Since \( f \) is continuous on \((-\infty, t]\) and \( f(u) = O(u^{-\hat{\zeta}}) \) for some \( \alpha + 3/2 < \hat{\zeta} \) at \(-\infty\), then for any \( \alpha + 3/2 < \tilde{\zeta} \leq \hat{\zeta} \), we can choose a constant \( M > 0 \) such that for any \( x \leq 0 \)
\[
|f(x)| \leq M |x|^{-\tilde{\zeta}},
\]
where we use the convention that \( 0^{-1} = +\infty \). In particular, if we choose \( \tilde{\zeta} \) such that \( \alpha + 3/2 < \tilde{\zeta} < \min\{2, \hat{\zeta}\} \) then for any \( s < 0 \)
\[
|s|^{\alpha+1} \left|\int_0^1 [(1-r)^\alpha - 1] f(sr) dr\right| \leq M |s|^{\alpha+1-\tilde{\zeta}} \int_0^1 |(1-r)^\alpha - 1| r^{-\tilde{\zeta}} dr.
\]
Since \( \tilde{\zeta} < 2 \), we have that \( \int_0^1 |(1-r)^\alpha - 1| r^{-\tilde{\zeta}} dr < \infty \). The desired square integrability follows from the previous estimates and the fact that \( \alpha + 1 - \tilde{\zeta} < -1/2 \).
Corollary 1. Under the assumptions of Theorem 4.1, we have that for every \( t \in \mathbb{R} \), almost surely

\[ Y_t = L(0)B_t^{H,\sigma} + U_t + A_t, \]

where \( Y \) is a \( \text{BSS} \) process of the form of \((20)\), \( B_t^{H,\sigma} \) is a volatility modulated fractional Brownian motion with \( H = \alpha + 1/2 \), \( U \) has absolutely continuous paths, and \( A \) is an increment semimartingale for \( t < 0 \) and it vanishes for \( t \geq 0 \). Moreover, for \( t \geq 0 \), the following hold

1. \((Y_t)_{t \geq 0}\) is a semimartingale if and only if \( \alpha = 0 \) and has a version which is Hölder continuous with index \( \rho < \alpha + 1/2 \);

2. The \( p \)-variation of \( Y \) over \([0,T]\)

\[ V_p^Y ([0,T])^p := \sup_{\pi: \pi \text{ is a partition of } [0,T]} \left\{ \sum_{t_k \in \pi} \left| Y_{t_k} - Y_{t_{k-1}} \right|^p \right\}, \]

is finite almost surely for every \( p > \frac{1}{\alpha + 1/2} \). In particular, the quadratic variation of \( Y \) vanishes for \( \alpha > 0 \) and is not finite for \( \alpha < 0 \).

Proof. From the proof of Theorem 4.1, we have that almost surely

\[ L(0)X_t = Y_t - U_t, \]

where \( U \) is a process with absolutely continuous paths. Moreover, from \((28)\), we have that

\[ L(0)X_t = L(0)[B_t^{H,\sigma} + \int_{t>0} (-s)^{H-1/2} \sigma_s dB_s], \quad t \in \mathbb{R}, \]

The conclusion of this corollary follows by letting

\[ A_t := L(0) \int_{t>0} (-s)^{H-1/2} \sigma_s dB_s, \quad t \in \mathbb{R}. \]

Example 4. Let

\[ L_\lambda(x) = e^{-\lambda x}, \]
\[ L_{\beta,\alpha}(x) = (1 + x)^{-(\alpha + \beta)}, \]

for \( \lambda > 0 \), \( \alpha \in (-1/2,0) \cup (0,1/2) \) and \( \beta > 1/2 \). Since

\[ \lim_{x \downarrow \infty} L_\lambda'(x)x^p = \lim_{x \downarrow \infty} L_\lambda''(x)x^{p+1} = 0; \]
\[ \lim_{x \downarrow \infty} L_{\beta,\alpha}'(x)x^{\zeta_0} = \lim_{x \downarrow \infty} L_{\beta,\alpha}''(x)x^{(\zeta_0+1)} = 0, \]

for any \( p \geq 0 \) and \( \alpha + 3/2 < \zeta_0 < \alpha + \beta + 1 \) we have that \( L_\lambda \) and \( L_{\beta,\alpha} \) satisfy Assumption 1. This means that for \( t \geq 0 \), \( \text{BSS} \) processes with a gamma kernel and power-\( \text{BSS} \) differ from a vmfBm only by an absolutely continuous process. It is interesting to note that the former type of \( \text{BSS} \) has short memory, meaning this that \( L_\lambda \) kills all the persistence coming from the vmfBm. An analogous reasoning can be done to the power-\( \text{BSS} \) when \( \beta > 1 \), while for \( 1/2 < \beta < 1 \) the memory introduced by the vmfBm is not negligible anymore.

4.2 Itô’s formulae for \( \text{BSS} \) processes

In the previous subsection we have seen that any \( \text{BSS} \) of the form of \((11)\) are not in general semimartingales and they are in fact closely related to fractional Brownian motions. In this subsection, by using some stochastic integrals (with random integrands) for the fractional Brownian motion, we derive some Itô’s formulae. Observe that, as in the case of the fBm, stochastic integrals (with random integrands) can be defined in several ways. Let us note that the goal of this subsection is only to show an application of Theorem 4.1. We do not pretend to develop stochastic integrals with respect to \( Y \) for random integrands, but rather use the well-known results concerning to the fBm. For a survey in stochastic calculus for fBm we refer to [Conti, 2007].
Itô’s formula based on Young integrals  Let $T > 0$. Consider $Y$ as in \( \text{Eq. 1} \). Within the framework of Corollary \( \text{4.1} \) the $p$-variation of $Y$ is finite almost surely for every $p > \frac{1}{\alpha + 1/2}$. Therefore, if $(Z_t)_{t \geq 0}$ is a continuous process with finite $q$-variation, with $q < \frac{1}{\alpha + 1/2}$ and $\alpha > 0$, we have that the following Riemann sums

$$
\sum_{i=0}^{n-1} Z_{t_i} \left( Y_{t_{i+1}} - Y_{t_i} \right),
$$

converges almost surely. Here, $0 = t_0^n < t_1^n < \cdots < t_n^n = T$ and are such that $\sup |t_{i+1}^n - t_i^n| \to 0$ as $n \to \infty$. The limit does not depend on the partition $(t_i^n)$ . See Young (1936) for more details. Hence, we may define $\int_0^T Z_s dY_s$, the stochastic integral of $Z$ with respect to $Y$, as such limit. Using this result we obtain the following Itô’s formula based on Young (1936).

**Theorem 2.** Let $Y$ be as in \( \text{Eq. 30} \) with $0 < \alpha < 1/2$. For every $f : \mathbb{R} \to \mathbb{R}$ continuously differentiable function with Hölder continuous derivative of order $\beta > \frac{1}{\alpha + 1/2} - 1$ the following Itô’s formula holds

$$
f(Y_T) = f(Y_0) + \int_0^T f'(Y_s) dY_s,
$$

where $\int_0^T f'(Y_s) dY_s$ is understood pathwise.

Itô’s formula based on Malliavin calculus  In this part, for simplicity, we will assume that $L(0) = 1$ and $\sigma \equiv 1$. Due to Theorem \( \text{1.1} \) we can define stochastic integrals (with random integrands) with respect to $Y$ as follows: under the assumptions of Theorem \( \text{1.1} \) we have that for any $t \geq 0$

$$
Y_t = U_t + \tilde{B}_t^\alpha,
$$

with $U$ an absolutely continuous process and $\tilde{B}_t^\alpha := \int_0^t (t - s)^\alpha dB_s$. Thus, if $\mathcal{F} = \sigma \left( \tilde{B}_t^\alpha \right)$ and $(Z_t)_{0 \leq t \leq T}$ is a continuous $\mathcal{F}_T$-measurable process, we define the Skorohod integral of $Z$ with respect to $Y$ as

$$
\int_0^T Z_s dY_s := \int_0^T Z_s dU_s + \int_0^T Z_s \delta \tilde{B}_s^\alpha,
$$

(40)

where, in the nomenclature of Nualart (2006), $\int_0^T Z_s \delta \tilde{B}_s^\alpha$ represents the divergence operator evaluated at $Z$. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function. Then, by Taylor’s Theorem, for $0 = t_0^n < t_1^n < \cdots < t_n^n = T$ with $\sup |t_{i+1}^n - t_i^n| \to 0$ as $n \to \infty$, we get

$$
f(Y_T) - f(Y_0) = \sum_{i=0}^{n-1} f'(Y_{t_i}) \left( Y_{t_{i+1}} - Y_{t_i} \right) + \frac{1}{2} \sum_{i=0}^{n-1} f''(Y_{t_i}) \left( Y_{t_{i+1}} - Y_{t_i} \right)^2 + \sum_{i=0}^{n-1} R \left( Y_{t_{i+1}}, Y_{t_i} \right)
$$

where, from Theorem 1 and Proposition 5 in Alòs et al. (2001), as $n \to \infty$ the series $\sum_{i=0}^{n-1} f'(Y_{t_i}) \left( Y_{t_{i+1}} - Y_{t_i} \right)$ converges to

$$
\int_0^T f'(Y_s) dY_s + \alpha \int_0^T \int_s^T D_s f'(Y_u) (u - s)^{\alpha-1} duds,
$$

(41)

in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, provided that for some constants $c > 0$ and for every $\zeta > 0$ such that $\zeta < \left( 4 \int_0^\infty \varphi_\alpha^2(s) ds \right)^{-1}$, it holds that

$$
\max \{|f(x)|, |f'(x)|, |f''(x)|\} \leq ce^{\zeta |x|^2},
$$

(42)

plus the technical condition

$$
\lim_{n \to \infty} \int_0^T \sup_{u,v \in \{s,s+1/n\} \cap [0,T]} \mathbb{E} \left[ |D_s f'(Y_u) - D_s f'(Y_v)|^2 \right] ds = 0,
$$

(43)
with $D_s$ denoting the Malliavin derivative induced by $\tilde{B}^\alpha$. See for instance Nualart (2006). On the other hand, from (42) and Corollary 1 for $\alpha > 0$ we get that $n^{-1} \sum_{i=0}^{n-1} f''(Y_{t_i}) \left(Y_{t_{i+1}} - Y_{t_i}\right)^2 \to 0$ in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore, by standard arguments, it can be shown that the remaining term $n^{-1} \sum_{i=0}^{n-1} R(Y_{t_{i+1}}, Y_{t_i})$ tends to 0 in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. Hence, as a corollary to Theorem 4.1 and Alòs et al. (2001) we have that

**Theorem 3.** Let $Y$ be as in (30) with $0 < \alpha < 1/2$ and $\sigma \equiv 1$. For every $f$ of class $C^2$ satisfying (42) and (43), we have that the following Itô’s formula applies

$$f(Y_T) = f(Y_0) + \int_0^T f'(Y_s) \, dY_s + \alpha \int_0^T \int_s^T D_s f'(Y_u) \, (u-s)^{\alpha-1} \, duds,$$

(44)

where $\int_0^T f'(Y_s) \, dY_s$ is understood as in (40).

**Remark 9.** Note that the formula in (44) differs from the one in Alòs et al. (2001) by the term

$$\alpha \int_0^T \int_s^T D_s f'(Y_u) \, (u-s)^{\alpha-1} \, duds.$$

However, if we define

$$\int_0^T Z_s dY_s := \int_0^T Z_s \, dY_s + \int_0^T D_s \{K_K [Y(T, s)]\} \, ds,$$

(45)

where $K_K$ is the operator in (14) with $K(t, s) = (t-s)^\alpha$, then (44) can be written as

$$f(Y_T) = f(Y_0) + \int_0^T f'(Y_s) \, dY_s.$$

Thus, the integral in (45) coincides with that in Barndorff-Nielsen et al. (2014).

**Remark 10.** Observe that (44) still holds for $0 > \alpha > -1/4$. Nevertheless, at this point, it is not clear what is the limit behavior of $n^{-1} \sum_{i=0}^{n-1} R(Y_{t_{i+1}}, Y_{t_i})$ in this case.

## 5 Conclusions

In this paper, by using Wiener-type stochastic integral for volatility modulated Volterra processes on the real line, we have decomposed a subclass of $\mathcal{BSS}$ as a sum of a fractional Brownian motion and an absolutely continuous process. We exploit this decomposition in order to obtain the index of Hölder continuity of the $\mathcal{BSS}$ of interest. Furthermore, we derived Itô’s formula in the case when $1/2 > \alpha > 0$.

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## 6 Appendix

### Stochastic Fubini Theorem

Our main result is based mainly on the stochastic Fubini theorem for semimartingales. In the following, we present a review of some conditions for which this theorem holds. See Veraar (2012) for a detailed discussion and Barndorff-Nielsen and Basse-O’Connor (2011) for generalizations.

Let $T$ be an interval on $\mathbb{R}$ and $(\mathcal{X}, \mathcal{B}, \mu)$ be a $\sigma$-finite measure space. Consider a real-valued random field $\psi : \Omega \times T \times \mathcal{X} \to \mathbb{R}$. Assume that $\psi$ is jointly measurable and adapted on $T$. Let $(S_t)_{t \in T}$ be a continuous
semimartingale on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, P)\). The stochastic Fubini theorem gives necessary conditions for which almost surely
\[
\int_X \int_T \psi(t, x) dS_t \mu(dx) = \int_T \int_X \psi(t, x) \mu(dx) dS_t,
\] (46)
in which implicitly is also concluded that the Lebesgue and Itô integrals exist. Let \((M, A)\) be the canonical decomposition of \(S\), i.e. \(M\) is a continuous local martingale, \(A\) a continuous process of bounded variation and \(S = M + A\). We have that (46) holds under the assumptions of the following theorem whose proof can be found in [Veraar, 2012]:

**Theorem 4 (Stochastic Fubini Theorem).** Let \(\psi: \Omega \times T \times X \to \mathbb{R}\) be progressively measurable on \(T\) and assume that almost surely
\[
\int_X \left( \int_T \psi^2(s, x) d[M]_s \right)^{1/2} \mu(dx) < \infty; \tag{47}
\]
\[
\int_X \int_T \psi^2(s, x) |dA|_s \mu(dx) < \infty, \tag{48}
\]
where \([M]\) denotes the quadratic variation of \(M\) and \(|dA|\) the total variation of the Lebesgue-Stieltjes measure associated to \(A\). Then (46) holds almost surely.

**Remark 11.** Observe that when \(\mu\) is finite, (47) in the previous theorem can be replaced by
\[
\int_X \int_T \psi^2(s, x) d[M]_s \mu(dx) < \infty. \tag{49}
\]
For the particular case of the Brownian motion it is equivalent to
\[
\int_X \left( \int_T \psi^2(s, x) ds \right)^{1/2} \mu(dx) < \infty. \tag{50}
\]
Finally, when \(\psi\) is square integrable, (47) can be replaced by
\[
\int_X \left[ \mathbb{E} \left( \int_T \psi^2(s, x) d[M]_s \right) \right]^{1/2} \mu(dx) < \infty, \tag{51}
\]
or
\[
\int_X \mathbb{E} \left( \int_T \psi^2(s, x) d[M]_s \right) \mu(dx) < \infty, \tag{52}
\]
when \(\mu\) is finite.

The following result is a version of Theorem 4 into the context of \(\mathcal{VMP}\).

**Lemma 2.** Let \((X, \mathcal{G})\) be a measurable space and \(\mu\) a \(\sigma\)-finite measure on \(\mathcal{G} \otimes \mathcal{B}(\mathbb{R})\). Consider \((X_t)_{t \in \mathbb{R}}\) to be an \(\mathcal{VMP}\) satisfying the assumptions of Section 2. If \(f: X \times \mathbb{R} \to \mathbb{R}\) is a measurable function fulfilling the condition
\[
\int_X \int_{\mathbb{R}} |f(x, s)| \left( \int_{-\infty}^t K(s, u)^2 du \right)^{1/2} \mu(dxds) < \infty,
\]
then almost surely \(\int_X \int_{\mathbb{R}} |f(x, s)X_s| \mu(dxds) < \infty\) and
\[
\int_X \int_{\mathbb{R}} f(x, s)X_s \mu(dxds) = \int_{\mathbb{R}} \int_X K(s, r)f(x, s) \mu(dxds) \sigma_r dB_r.
\]

**Proof of Proposition 1**

In this part we give a proof of Proposition 1. We would like to emphasize that the proof, and consequently the Proposition, is valid for any Lévy process.
Proof. [Proof of Proposition 1] As it was mentioned in the beginning of this paper, for the BSS process to be well defined, we must assume that \( \int_0^\infty \phi(s)^2 \sigma_s^2 ds < \infty \) almost surely for any \( t \in \mathbb{R} \). Firstly, let us show that for all \( h \in \mathbb{R} \), almost surely

\[
\int_{-\infty}^{t+h} \phi(t+h-s) \sigma_s dB_s = \int_{-\infty}^t \phi(t-s) \sigma_{s+h} dB_s^h. \tag{53}
\]

Indeed, since \( \sigma \) is predictable and \( \phi \) is measurable, without loss of generality we may and do assume that \( \varphi_\alpha \) is non-negative and

\[
\sigma_t = \sum_{i=1}^N a_i \mathbf{1}_{(u_i, t_i]}(t) X_i, \quad t \in \mathbb{R},
\]

where \( a_i \in \mathbb{R}, -\infty < u_i < t_i < \infty \) and \( X_i \in \mathcal{F}_{u_i} \). Hence, we can choose a sequence of simple functions \( (\phi^n)_{n \in \mathbb{N}} \) with \( \phi^n \uparrow \phi \) and

\[
\phi^n(s) = \sum_{i=1}^{m_n} b^n_i \mathbf{1}_{(p_i, q_i]}(s), \quad s \in \mathbb{R},
\]

where \( b^n_i \in \mathbb{R}^+ \) and \( 0 \leq p_i < q_i < \infty \). We see that \( \int_{-\infty}^{t+h} \phi^n(t-s) \sigma_s dB_s \overset{p}{\to} Y_t \) for the reason that \( \phi^n(t-\cdot) \sigma \) is a sequence of simple predictable processes converging to \( \varphi_\alpha(t-\cdot) \sigma \). Observe that

\[
\int_{-\infty}^{t+h} \phi^n(t+h-s) \sigma_s dB_s = \sum_{i=1}^N \sum_{j=1}^{m_n} a_i b^n_j X_i \int_{-\infty}^{t+h} \mathbf{1}_{(u_i \vee t+h-q_i < s \leq t_i \wedge t+h-p_i]}(s) dB_s,
\]

and

\[
\int_{-\infty}^{t} \phi^n(t-s) \sigma_{s+h} dB_s^h = \sum_{i=1}^N \sum_{j=1}^{m_n} a_i b^n_j X_i \int_{-\infty}^{t} \mathbf{1}_{(u_i \wedge t-h < s \leq t_i \wedge t-h-p_i]}(s) dB_s^h.
\]

Since

\[
\int_{-\infty}^{t+h} \mathbf{1}_{(u_i \vee t+h-q_i < s \leq t_i \wedge t+h-p_i]}(s) dB_s = B_{t_i \wedge t+h-p_i} - B_{u_i \vee t+h-q_i},
\]

we get

\[
\int_{-\infty}^{t+h} \phi^n(t+h-s) \sigma_s dB_s = \int_{-\infty}^{t} \phi^n(t-s) \sigma_{s+h} dB_s^h.
\]

Taking limits in the last equation we obtain (53).

Thanks to (53), in order to finish the proof, we only need to prove that

\[
\left( \int_{-\infty}^{t} \phi(t-s) \sigma_{s+h} dB_s^h \right)_{t \in \mathbb{R}} \overset{d}{=} \left( \int_{-\infty}^{t} \phi(t-s) \sigma_s dB_s \right)_{t \in \mathbb{R}}.
\]

Since \( \sigma \) is càdlàg, we may and do assume that it is bounded, therefore for any \( h \in \mathbb{R} \)

\[
\phi(t-s) \sigma_{s+h} = \lim_{n \to \infty} \sum_{k \in \mathbb{Z}} \phi^n(t-s) \sigma_{s+h}^{k} \mathbf{1}_{\left\{ \frac{s+h}{h} < k \leq \frac{s+h+1}{h} \right\}}
\]

where the limit is pointwise. Note that \( \left| \psi^n_h \right| \leq \phi(t-\cdot) \), this jointly with (53) give \( \int_{-\infty}^{t} \psi^n_h(t-s) dB_s^h \overset{p}{\to} Y_{t+h} \). It only remains to show that

\[
\left( \int_{-\infty}^{t} \psi^n_h(t-s) dB_s^h \right)_{t \in \mathbb{R}} \overset{d}{=} \left( \int_{-\infty}^{t} \psi^n(t-s) dB_s \right)_{t \in \mathbb{R}}.
\]

Indeed, if it were true, we would have that \( \int_{-\infty}^{t} \psi^n_h(t-s) dB_s^h \Rightarrow Y \), where \( \Rightarrow \) denotes convergence in finite dimensional distributions, which gives the desired result. Now, observe that (1) implies that for any \( f : \mathbb{R}^2 \times M \to \mathbb{R} \) continuous function and \( (r_1, \ldots, r_M) \in \mathbb{R}^M \)

\[
f \left( \left\{ \sigma_{r_i+h}, B_{r_i}^h \right\}_{i=1}^{M} \right) \overset{d}{=} f \left( \left\{ (\sigma_{r_i}, B_{r_i}) \right\}_{i=1}^{M} \right). \tag{54}
\]
In view of
\[
\int_{-\infty}^{t} \psi_h^n(t-s) \, dB_s^h = \sum_{k \in \mathbb{Z}} \sigma_k \int_{-\infty}^{t} \phi^n(t-s) 1_{\{\frac{k}{h} \leq t < \frac{k+1}{h}\}} \, dB_s^h
\]
\[
= \sum_{k \in \mathbb{Z}} \sum_{j=1}^{m_n} b_j^n \sigma_k \int_{-\infty}^{t} 1_{\{\frac{k}{h} < t < \frac{k+1}{h}\}} \, dB_s^h
\]
\[
= \sum_{k \in \mathbb{Z}} \sum_{j=1}^{m_n} b_j^n \sigma_k \left( B_{\frac{k+1}{h} \wedge t - \pi_i} + h - B_{\frac{k}{h} \vee t - \pi_i} + h \right)
\]
and \[54\), we have that for any \((c_1, \ldots, c_M)\)
\[
\sum_{j=1}^{M} c_j \int_{-\infty}^{r_j} \psi_h^n(r_j-s) \, dB_s^h = \sum_{k \in \mathbb{Z}} \left( \sum_{j=1}^{M} \sum_{i=1}^{m_n} c_j b_i^n \right) \sigma_k \int_{-\infty}^{t} \phi^n(t-s) 1_{\{\frac{k}{h} \leq t < \frac{k+1}{h}\}} \, dB_s^h
\]
\[
\times \left( B_{\frac{k+1}{h} \wedge t - \pi_i} + h - B_{\frac{k}{h} \vee t - \pi_i} + h \right)
\]
\[
= \sum_{k \in \mathbb{Z}} \left( \sum_{j=1}^{M} \sum_{i=1}^{m_n} c_j b_i^n \right) \sigma_k \int_{-\infty}^{t} \phi^n(t-s) 1_{\{\frac{k}{h} \leq t < \frac{k+1}{h}\}} \, dB_s^h
\]
\[
\times \left( B_{\frac{k+1}{h} \wedge t - \pi_i} - B_{\frac{k}{h} \vee t - \pi_i} \right)
\]
\[
= \sum_{j=1}^{M} c_j \int_{-\infty}^{r_j} \psi_0^n(r_j-s) \, dB_s,
\]
the desired result.

\[\blacksquare\]

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