AMENABILITY OF GROUPS IS CHARACTERIZED BY MYHILL’S THEOREM

LAURENT BARTHOLDI

with an appendix by DAWID KIELAK

Abstract. We prove a converse to Myhill’s “Garden-of-Eden” theorem and obtain in this manner a characterization of amenability in terms of cellular automata: A group $G$ is amenable if and only if every cellular automaton with carrier $G$ that has gardens of Eden also has mutually erasable patterns.

This answers a question by Schupp, and solves a conjecture by Ceccherini-Silberstein, Mackey and Scarabotti.

An appendix by Dawid Kielak proves that group rings without zero divisors are Ore domains precisely when the group is amenable, answering a conjecture attributed to Guba.

1. Introduction

Cellular automata were introduced in the late 1940’s by von Neumann as models of computation and of biological organisms [14]. We follow an algebraic treatment, as in [3]: let $G$ be a group. A cellular automaton carried by $G$ is a $G$-equivariant continuous map $\Theta: A^G \to A^G$ for some finite set $A$. Elements of $A^G$ are called configurations, and the action of $G$ on $A^G$ is given by $g \cdot \phi = \phi(-\cdot g)$ for all $\phi \in A^G, g \in G$.

One should think of $A$ as the stateset (e.g. “asleep” or “awake”) of a microscopic animal; then $A^G$ is the stateset of a homogeneous swarm of animals indexed by $G$, and $\Theta$ is an evolution rule for the swarm: it is identical for each animal by $G$-equivariance, and is only based on local interaction by continuity of $\Theta$. For example, fixing $f, \ell, r \in G$ the “front”, “left” and “right” neighbours, define $\Theta$ by “sleep if the guy in front of you sleeps, unless both your neighbours are awake”, or in formulæ, set for all $\phi \in A^G, g \in G$

$$\Theta(\phi)(g) = \begin{cases} \text{asleep} & \text{if } \phi(fg) = \text{asleep} \text{ and } \{\phi(\ell g), \phi(rg)\} \ni \text{asleep}, \\ \phi(g) & \text{else.} \end{cases}$$

Generally speaking, the memory set of a cellular automaton is the minimal $S \subseteq G$ such that $\Theta(\phi)(g)$ depends only on the restriction of $\phi$ to $Sg$, and is finite.

Two properties of cellular automata received particular attention. Let us call pattern the restriction of a configuration to a finite subset $Y \subseteq G$. On the one hand, there can exist patterns that never appear in the image of $\Theta$. These are called Gardens of Eden (GOE), the biblical metaphor expressing the notion of paradise the universe may start in but never return to.
On the other hand, $\Theta$ can be non-injective in a strong sense: there can exist patterns $\phi'_1 \neq \phi'_2 \in A^Y$ such that, however one extends $\phi'_1$ to a configuration $\phi_1$, if one extends $\phi'_2$ similarly (i.e. in such a way that $\phi_1$ and $\phi_2$ have the same restriction to $G \setminus Y$) then $\Theta(\phi_1) = \Theta(\phi_2)$. These patterns $\phi'_1, \phi'_2$ are called Mutually Erasable Patterns (MEP). Equivalently there are two configurations $\phi_1, \phi_2$ which differ on a non-empty finite set and satisfy $\Theta(\phi_1) = \Theta(\phi_2)$. The absence of MEP is sometimes called pre-injectivity [5, §8.G].

Amenability of groups was also introduced by von Neumann, in the late 1920's in [13]; there exist numerous formulations (see e.g. [17]), but we content ourselves with the following criterion due to Følner (see [4]) which we treat as a definition: a discrete group $G$ is amenable if for every $\epsilon > 0$ and every finite $S \subset G$ there exists a finite $F \subset G$ with $\#(SF) < (1 + \epsilon)\#F$. In words, there exist finite subsets of $G$ that are arbitrarily close to invariant under translation.

Cellular automata were initially considered on $G = \mathbb{Z}^n$. Celebrated theorems by Moore and Myhill [11, 12] prove that, in this context, a cellular automaton admits GOE if and only if it admits MEP; necessity is due to Myhill, and sufficiency to Moore. This result was generalized by Machi and Mignosi [8] to groups of subexponential growth, and by Ceccherini-Silberstein, Machi and Scarabotti [2] to amenable groups.

Our main result is a converse to Myhill's theorem:

**Theorem 1.1.** Let $G$ be a non-amenable group. Then there exists a cellular automaton carried by $G$ that admits Gardens of Eden but no mutually erasable patterns.

There is a natural measure, the Bernoulli measure, on the configuration space $A^G$: for every pattern $\phi \in A^Y$ it assigns measure $1/\#A^Y$ to the clopen set $\{ \psi \in A^G : \psi|_Y = \phi \}$. Note that the $G$-action on $A^G$ preserves this measure. Hedlund proved in [7, Theorem 5.4], for $G = \mathbb{Z}$, that a cellular automaton preserves Bernoulli measure if and only if it has no GOE. This result was generalized by Meyerovitch to amenable groups [10, Proposition 5.1].

Combining these with Theorem 1.1 and with the aforementioned results by Ceccherini-Silberstein et al. and the main result of [1], we deduce:

**Corollary 1.2.** Let $G$ be a group; then the following are equivalent:

1. the group $G$ is amenable;
2. all cellular automata on $G$ that admit MEP also admit GOE;
3. all cellular automata on $G$ that admit GOE also admit MEP;
4. all cellular automata on $G$ that do not preserve Bernoulli measure admit GOE.

**1.1. Origins.** Schupp had already asked in [15, Question 1] in which precise class of groups the theorems by Moore and Myhill hold. Ceccherini-Silberstein et al. conjecture in [2, Conjecture 6.2] that Corollary 1.2(1–3) are equivalent.

The implication (3$\Rightarrow$1) is the content of Theorem [14]. In case $G$ contains a non-abelian free subgroup, it was already shown by Muller in his University of Illinois 1976 class notes, see [8, page 55]; let us review the construction, in the special case $G = \langle x, y, z|x^2, y^2, z^2$. Fix a finite field $\mathbb{K}$, and set $A := \mathbb{K}^2$. View $A^G$ as $\mathbb{K}^G \times \mathbb{K}^G$, on which $2 \times 2$ matrices with coefficients in the group ring $\mathbb{K}G$ act from the left.
Define $\Theta: A^G \to \mathbb{C}$ by

$$\Theta(\phi) = \begin{pmatrix} x & y + z \\ 0 & 0 \end{pmatrix} \phi.$$ 

It obviously has gardens of Eden — any pattern with non-trivial second coordinate — and to show that it has no mutually erasable patterns it suffices, since $\Theta$ is linear, to show that $\Theta$ is injective on finitely-supported configurations; this is easily achieved by considering, in the support of a configuration $\phi$, a position $g \in G$ such that $xg$ and $yg$ don’t belong to the support of $\phi$.

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### 2. Proof of Theorem

We begin with a combinatorial

**Lemma 2.1.** Let $n$ be an integer. Then there exists a set $Y$ and a family of subsets $X_1, \ldots, X_n$ of $Y$ such that, for all $I \subseteq \{1, \ldots, n\}$ and all $i \in I$, we have

$$\#\left( X_i \setminus \bigcup_{j \in I \setminus \{i\}} X_j \right) \geq \frac{\#Y}{(1 + \log n)\#I}.$$ 

**Proof.** We denote by $\mathfrak{S}_n$ the symmetric group on $n$ letters. Define

$$Y := \{(i, \sigma) \sim (j, \sigma) \mid i \text{ and } j \text{ belong to the same cycle of } \sigma\};$$

in other words, $Y$ is the set of cycles of elements of $\mathfrak{S}_n$. Let $X_i$ be the natural image of $\mathfrak{S}_n$ in the quotient $Y$.

First, there are $(i - 1)!$ cycles of length $i$ in $\mathfrak{S}_i$, given by all cyclic orderings of $\{1, \ldots, i\}$; so there are $\binom{n}{i}(i - 1)!$ cycles of length $i$ in $\mathfrak{S}_n$, and they can be completed in $(n - i)!$ ways to a permutation of $\mathfrak{S}_n$; so

$$\#Y = \sum_{i=1}^{n} \binom{n}{i} (i - 1)! (n - i)! = \sum_{i=1}^{n} \frac{n!}{i!} \leq (1 + \log n)n!$$

since $1 + 1/2 + \cdots + 1/n \leq 1 + \log n$ for all $n$.

Next, consider $I \subseteq \{1, \ldots, n\}$ and $i \in I$, and set $X_{i,I} := X_i \setminus \bigcup_{j \in I \setminus \{i\}} X_j$. Then $X_{i,I} = \{(i, \sigma) : (i, \sigma) \sim (j, \sigma) \text{ for all } j \in I \setminus \{i\}\}$. Summing over all possibilities for the length-$(j + 1)$ cycle $(i, t_1, \ldots, t_j)$ of $\sigma$ intersecting $I$ in $\{i\}$, we get

$$\#X_{i,I} = \sum_{j=0}^{n-\#I} \binom{n-\#I}{j} j!(n - j - 1)!$$

$$= \sum_{k=n-j=\#I}^{n} (n-\#I)!(\#I-1)! \binom{k-1}{k-\#I}$$

$$= (n-\#I)!(\#I-1)! \binom{n}{n-\#I} = \frac{n!}{\#I}.$$
Combining (2.1) and (2.2), we get
\[
\#X_{i,t} = \frac{n!}{\#T} = \frac{(1 + \log n)n!}{(1 + \log n)\#T} \geq \frac{\#Y}{(1 + \log n)\#T}. \quad \square
\]

Let \( G \) be a non-amenable group. To prove Theorem 1.1, we construct a cellular automaton carried by \( G \), with GOE but without MEP. Since \( G \) is non-amenable, there exists \( \epsilon > 0 \) and \( S_0 \subset G \) finite with \( \#(S_0 F) \geq (1 + \epsilon)\#F \) for all finite \( F \subset G \). We then have \( \#(S_0^k F) \geq (1 + \epsilon)^k\#F \) for all \( k \in \mathbb{N} \). Let \( k \) be large enough so that \( (1 + \epsilon)^k > 1 + k \log \#S_0 \), and set \( S := S_0^k \) and \( n := \#S \). This set \( S \) will be the memory set of our automaton. We then have
\[
\#(S F) \geq (1 + \epsilon)^k\#F \geq (1 + k \log \#S_0)\#F \geq (1 + \log n)\#F \text{ for all finite } F \subset G.
\]

Apply Lemma 2.2 to this \( n \), and identify \( \{1, \ldots, n\} \) with \( S \) to obtain a set \( Y \) and subsets \( X_s \) for all \( s \in S \). We have
\[
\#(X_s \setminus \bigcup_{t \in T \setminus \{s\}} X_t) \geq \frac{\#Y}{(1 + \log n)\#T} \text{ for all } s \in T \subset S.
\]

Furthermore, since \( n \geq 2 \) these inequalities are sharp; so we may replace \( Y \) and \( X_s \) respectively by \( Y \times \{1, \ldots, k\} \) and \( X_s \times \{1, \ldots, k\} \) for some \( k \) large enough so that \( \#(X_s \setminus \bigcup_{t \in T \setminus \{s\}} X_t) \geq (\#Y + 1) / (1 + \log n)\#T \) holds; and then we replace \( Y \) by \( Y \cup \{\} \). If for \( T \subset S \) and \( s \in T \) we define
\[
X_{s,T} := X_s \setminus \bigcup_{t \in T \setminus \{s\}} X_t, \text{ then } \#X_{s,T} \geq \frac{\#Y}{(1 + \log n)\#T} \text{ for all } s \in T \subset S;
\]
and furthermore we have obtained \( \bigcup_{s \in S} X_s \subsetneq Y \).

Let \( \mathbb{K} \) be a large enough finite field (in a sense to be precised soon), and set \( A := \mathbb{K}Y \). For each \( s \in S \), choose a linear map \( \alpha_s : A \to \mathbb{K}X_s \subset A \), and for \( T \supset s \) denote by \( \alpha_{s,T} : A \to \mathbb{K}X_{s,T} \) the composition of \( \alpha_s \) with the coordinate projection \( \pi_{s,T} : A \to \mathbb{K}X_{s,T} \), in such a manner that, whenever \( \{T_s : s \in S\} \) is a family of subsets of \( S \) with \( \sum_{s \in S} \#X_{s,T_s} \geq \#Y \), we have
\[
(2.4) \quad \bigcap_{s \in S} \ker(\alpha_{s,T_s}) = 0.
\]

This is always possible if \( \mathbb{K} \) is large enough: indeed write each \( \alpha_s \) as a \( \#Y \times \#Y \) matrix and each \( \alpha_{s,T} \) as a submatrix. The condition is then that various vertical concatenations of submatrices have full rank, and the complement of these conditions is a proper algebraic subvariety of \( \mathbb{R}^{Y \times Y \times S} \) defined over \( \mathbb{Z} \), which is not full as soon as \( \mathbb{K} \) is large enough.

Define now a cellular automaton with stateset \( A \) and carrier \( G \) by
\[
\Theta(\phi)(g) = \sum_{s \in S} \alpha_s(\phi(sg)).
\]

Clearly \( \Theta \) admits gardens of Eden: for every \( \phi \in A^G \), we have \( \Theta(\phi)(1) \in \mathbb{K}(\bigcup_{s \in S} X_s) \subsetneq A \).
To show that \( \Theta \) admits mutually erasable patterns, it is enough to show, for \( \phi \in A^G \) non-trivial and finitely supported, that \( \Theta(\phi) \neq 0 \). Let thus \( F \neq \emptyset \) denote the support of \( \phi \). Define \( \rho : SF \to (0,1] \) by \( \rho(g) := 1/\# \{ s \in S : g \in sF \} \). Now

\[
\sum_{f \in F} \left( \sum_{s \in S} \rho(sf) \right) = \sum_{g \in SF} \sum_{s \in S} \rho(g) = \sum_{g \in SF} 1 = \#(SF),
\]

so there exists \( f \in F \) with \( \sum_{s \in S} \rho(sf) \geq \#(SF)/\#F \geq 1 + \log n \) by (2.3). For every \( s \in S \), set \( T_s := \{ t \in S : sf \in tF \} \), so \( \#T_s = 1/\rho(sf) \). We obtain

\[
\sum_{s \in S} \#X_{s,T_s} \geq \sum_{s \in S} \frac{\#Y}{(1 + \log n)\#T_s} \quad \text{by Lemma 2.1}
\]

\[
= \sum_{s \in S} \frac{\#Y\rho(sf)}{1 + \log n} \geq \#Y,
\]

so by (2.4) the map \( A \ni a \mapsto (\alpha_{s,T_s}(a))_{s \in S} \) is injective. Set \( \psi := \Theta(\phi) \). Since by assumption \( \phi(f) \neq 0 \), we get \( (\pi_{s,T_s}(\psi(sf)))_{s \in S} \neq 0 \), so \( \psi \neq 0 \) and we have proven that \( \Theta \) admits no mutually erasable patterns. The proof is complete.

**Appendix A. A characterization of amenability via Ore domains, by Dawid Kielak**

Let \( A \) be an associative ring without zero divisors, and let us write \( A^* = A \setminus \{0\} \). Recall that \( A \) is called an \emph{Ore domain} if it satisfies Ore’s condition: for every \( a \in A \), \( s \in A^* \) there exist \( b \in A \), \( t \in A^* \) with \( at = bs \). It then follows that \( A(A^*)^{-1} \), namely the set of expressions of the form \( as^{-1} \) with \( a \in A \), \( s \in A^* \) up to the obvious equivalence relation \( as^{-1} = at(st)^{-1} \), is a skew field called \( A \)'s \emph{classical field of fractions}.

A folklore conjecture, sometimes attributed to Victor Guba [6], asserts that group rings satisfy the Ore condition precisely when the group is amenable. We prove it in the following form:

**Theorem A.1.** Let \( G \) be a group, and let \( \mathbb{K} \) be a field such that \( \mathbb{K}G \) has no zero divisors. Then \( G \) is amenable if and only if \( \mathbb{K}G \) is an Ore domain.

**Proof.** (\( \Rightarrow \)) is due to Tamari [10]; we repeat it for convenience. Assume that \( G \) is amenable, and let \( a, s \in \mathbb{K}G \) be given. Let \( S \subseteq G \) be a finite set containing the supports of \( a \) and \( s \). By Følner’s criterion, there exists \( F \subseteq G \) finite such that \( \#(SF) < 2\#F \). Consider \( b, t \in \mathbb{K}F \) as variables; then the equation system \( as = bt \) is linear, has \( 2\#F \) unknowns, and at most \( \#(SF) \) equations, so has a non-trivial solution.

(\( \Leftarrow \)) Assume that \( G \) is non-amenable. The construction in the proof of Theorem 1.1 yields a finite extension \( L \) of \( \mathbb{K} \) and an \( n \times n \) matrix \( M \) over \( LG \) such that multiplication by \( M \) is an injective map \( (LG)^n \to \cup \) and \( M \)'s last row consists entirely of zeros. Forgetting that last row and restricting scalars, namely writing \( L = \mathbb{K}^d \) qua \( \mathbb{K} \)-vector space, we obtain an exact sequence of free \( \mathbb{K}G \)-modules

\[
0 \to (\mathbb{K}G)^{dn} \to (\mathbb{K}G)^{d(n-1)}.
\]

Suppose now that \( \mathbb{K}G \) is an Ore domain, with classical field of fractions \( \mathbb{F} \). Crucially, \( \mathbb{F} \) is a flat \( \mathbb{K}G \) module, that is the functor \( - \otimes_{\mathbb{K}G} \mathbb{F} \) preserves exactness.
of sequences (see e.g. [9] Proposition 2.1.16). Also, \( F \) is a skew field, and upon tensoring \((\mathbb{A}_1)\) with \( F \) we obtain an exact sequence 
\[
0 \longrightarrow F^dn \longrightarrow F^d(n-1)
\]
which is impossible for reasons of dimension. \( \square \)

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L.B.: DÉPARTEMENT DE MATHÉMATIQUES ET APPLICATIONS, ÉCOLE NORMALE SUPÉRIEURE, PARIS and MATHEMATISCHES INSTITUT, GEORG-AUGUST UNIVERSITÄT ZU GÖTTINGEN
E-mail address: laurent.bartholdi@gmail.com

D.K.: FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BIELEFELD
E-mail address: dkielak@math.uni-bielefeld.de