Renormalization fixed point of the KPZ universality class

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The one dimensional Kardar-Parisi-Zhang universality class is believed to describe many types of evolving interfaces which also have the same characteristic scaling exponents. These exponents lead to a natural renormalization group action on the evolution of such interfaces. In this Letter we introduce and describe the renormalization group fixed point of the Kardar-Parisi-Zhang universality class in terms of a random nonlinear semigroup with stationary independent increments, and via a variational formula. Furthermore, we compute the exact transition probabilities using replica Bethe ansatz. The semigroup is constructed from the Airy sheet, a four parameter space-time field which is the Airy$_2$ process in each of its two spatial coordinates. Minimizing paths through this field describe the renormalization group fixed point of directed polymers in a random potential.

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Interfaces evolving according to local stochastic growth rules have been extensively studied in physics, biology, material science, and engineering. There has been significant theoretical success over the last twenty-five years in describing such rough self-affine interfaces which evolve due to local processes. In short times such interfaces display properties particular to their local processes. However, in the long time limit, theoreticians predicted that certain universal scaling exponents and exact statistical distributions would accurately describe the fixed long time properties of a wide variety of rough interfaces. These predictions have been repeatedly confirmed through Monte-Carlo simulation as well as experiments. What is lacking, however, is a theoretical understanding and prediction of the temporal evolution of these interfaces, in the large-time scaling limit. In this Letter we describe, both phenomenologically and statistically, the universal temporal evolution of such interfaces. We also describe both experimental and numerical means to verify these predictions.

The 1 + 1 dimensional KPZ universality class includes a wide variety of forms of stochastic interface growth on a one dimensional substrate, randomly stirred one dimensional fluids (the stochastic Burgers equation), polymer chains directed in one dimension and fluctuating transversally in the other due to a random potential, and various lattice models such as the driven lattice gas model of ASEP and ground state polymer model of last passage percolation. All models can be transformed to a kinetically roughening, growing interface reflecting the competition between growth in a direction normal to the surface, a surface tension smoothing force, and a stochastic term which tends to roughen the interface. Numerical simulations along with some theoretical results have confirmed that in the long time $t$ scaling limit, fluctuations in the height of such evolving interfaces scale like $t^{1/3}$ and display non-trivially spatially correlated in the scale $t^{2/3}$. These scales were confirmed experimentally in studied involving paper wetting, burning fronts, bacterial colonies and liquid crystals.

Beyond the KPZ scalings, the universality class is characterized in terms of the long-time limits of the probability distribution of fluctuations. These depend on the initial data or geometry. Starting from (i) narrow wedge, or droplet, one sees the $F_{\text{GUE}}$ distribution of the Gaussian Unitary Ensemble of random matrix theory while starting from (ii) flat substrate, the $F_{\text{GOE}}$ distribution of the Gaussian Orthogonal Ensemble. A recent series of spectacular experiments involving turbulent liquid crystals have been able to not only confirm the predicted scaling laws but also the statistics (skewness and kurtosis) for the distribution of these fluctuations. The experiment involves studying the interface between two topologically different turbulent states in a nematic liquid crystal after an external voltage (in the form of a laser) is applied so as to trigger the Carr-Helfrich instability.

The multi-point joint distributions of scaled fluctuations are likewise given by the (i) Airy$_2$ and (ii) Airy$_1$ processes. showed that the two-point correlation function agrees closely with that of the Airy$_2$ process. A further natural initial geometry is two sided Brownian motion for which one sees, at later time, a new (though correlated) Brownian motion, with a global height shift given by the $F_0$ distribution. Note that all these spatial processes have $n$-point distributions given by Fredholm determinants.

In this work we answer two questions: 1. Can one compute the exact statistics and multi-point joint distributions for growth off of more general initial geometries; and 2. Can one predict multi-time statistics and distri-
The answers are provided through a detailed study of the **KPZ renormalization fixed point** which we denote as $h$. We describe the fixed point in two complementary ways: 1. Through a variational formulation similar to that of a stochastically forced Burgers equation, but with a new, nontrivial driving noise called the **Airy sheet**; and 2. Through an exact formula for the transition probabilities of this Markov process, derived by employing the methods of replica Bethe ansatz [13, 14] (see also [15]). The transition probabilities enable us to compute statistics for general initial geometries, and using the Markov property, this enables us to also compute multi-time statistics, thus answering both questions. Furthermore, we describe specific experimental and numerical means to confirm the theoretical results.

The **renormalization operator**. Kardar-Parisi-Zhang [8] proposed the model equation,

$$\partial_t h = \frac{1}{2}(\partial_x h)^2 + \frac{1}{4}\partial_x^4 h + \xi$$

(1)

from which the universality class takes its name. The noise $\xi$ is Gaussian space-time white noise $(\xi(t, x) \xi(s, y)) = \delta(t - s) \delta(y - x)$. Many of the discrete models have a tunable asymmetry and (1) appears as a continuum limit in the diffusive time scale as this parameter is critically tuned close to zero [16–18]. Recent progress has been recorded in solving this equation [5, 8, 10]. (1) is linearized through the Hopf-Cole transformation $Z = e^h$, becoming a stochastic heat equation with multiplicative noise $\partial_t Z = \frac{1}{2}\partial_x^2 Z + \xi Z$. The narrow wedge initial conditions correspond to starting $Z$ with a delta function, and the flat initial conditions correspond to starting with $Z \equiv 1$. Correspondingly, in the liquid crystal experiments the laser excites a single point, or a line.

The fluctuations are observed under the rescaling,

$$h_\epsilon(t, x) = (R_\epsilon h)(t, x) = \epsilon^{1/2}h(\epsilon^{-3/2}t, \epsilon^{-1}x).$$

(2)

which satisfies (1) with renormalized coefficients,

$$\partial_t h_\epsilon = \frac{1}{2}(\partial_x h_\epsilon)^2 + \epsilon^{1/2}\partial_x^2 h_\epsilon + \epsilon^{1/4}\dot{W}.$$

(3)

The **KPZ fixed point** $h$ is the $\epsilon \to 0$ limit of the properly centered process $h_\epsilon$. We now construct it.

The **Airy sheet**. Let $h(u, y; t, x)$ be the solution of (1) for times $t > u$ started at time $u$ with a delta function at $y$, all using the same noise. To center, set $\tilde{h}(u, y; t, x) = h(u, y; t, x) - \frac{u}{2\sqrt{\pi}} - \log \sqrt{2\pi(t - u)}$ and define $A_1$ by

$$\tilde{h}(u, y; t, x) = -\frac{(x - y)^2}{2(t - u)} + A_1(u, y; t, x).$$

After the rescaling $\tilde{x}$,

$$R_\epsilon \tilde{h}(u, y; t, x) = -\frac{(x - y)^2}{2(t - u)} + A_\epsilon(u, y; t, x)$$

where $A_\epsilon = R_\epsilon A_1$. As $\epsilon \to 0$, $A_\epsilon(u, y; t, x)$ converges to the **Airy sheet** $A_{\epsilon}(u, y; t, x)$. In each spatial variable it is an Airy$_2$ process [10]. It has several nice properties:

1. **Independent increments.** $A(u, y; t, x)$ is independent of $A(u', y'; t', x)$ if $(u, t) \cap (u', t') = \emptyset$;

2. **Space and time stationarity.** $A(u, y; t, x) \overset{\text{dist}}{=} A(u + h, y; t + h, x) = A(u, y + z; t, x + z)$;

3. **Scaling.** $A(0, y; t, x) \overset{\text{dist}}{=} t^{1/3}A(0, t^{-2/3}y; 1, t^{-2/3}x)$;

4. **Semigroup property.** For $u < s < t$,

$$A(u, y; t, x) = \sup_{z \in \mathbb{R}} \{-(x - y)^2/(2(t - u)) - (z - y)^2/(2(t - z)) + A(u, y; s, z) + A(s, z; t, x)\}.$$  

(4)

**KPZ fixed point.** Using $A(u, y; t, x)$ we construct the KPZ fixed point $h(t, x)$. By the Hopf-Cole transformation and the linearity of the stochastic heat equation, the rescaled solution of (1) with initial data $h_0$ is

$$R_\epsilon h(t, x) = \epsilon^{1/2}\log \int e^{-\epsilon^{-1/2}\left(-\frac{(x - y)^2}{2(t - u)} + A_0(0, y; t, x + R_\epsilon h_0(y))\right)}dy.$$  

If we choose initial data $h_0^\epsilon$ so that $R_\epsilon h_0^\epsilon$ converges to a fixed function $f$ in the limit, we can use Laplace’s method to evaluate $h(t, x) = \lim_{\epsilon \to 0} R_\epsilon h(t, x) = T_{0, t} f(x)$ where

$$T_{u,t} f(x) := \sup_y \left\{ -\frac{(x - y)^2}{2(t - u)} + A(u, y; t, x) + f(y) \right\}.$$  

(5)

The operators $T_{u,t}$, $0 < u < t$ form a semi-group, i.e.

$$T_{u,t} = T_{u,s} T_{s,t}$$

which is stationary with independent increments and $T_{0,t} \overset{\text{dist}}{=} \frac{1}{\epsilon^{2/3}} T_{0,t'}/\epsilon^{2/3}$.

By the Markov property, the joint distribution of the marginal spatial process of $h$ (for initial data $f$) at a set of times $t_1 < t_2 < \cdots < t_n$ is given by

$$(h(t_1), \ldots, h(t_n)) \overset{\text{dist}}{=} (T_{0,t_1} f, \ldots, T_{t_{n-1},t_n} \cdots T_{0,t_1} f).$$

The process of randomly evolving functions can be thought of as a high dimensional analogue of Brownian motion (with state space Brownian motions!), and the $T_{t_1,t_{i+1}}$ as analogous to the independent increments.

**Polymer fixed point.** The solution of (1) corresponds to the free energy of a directed random polymer $x(s), u < s < t$ starting at $y$ and ending at $x$, with quenched random energy

$$H(x(\cdot)) = \int_0^t \left\{ -\frac{(x(s)^2 - \xi(x(s))}{2}\right\} ds.$$  

(6)

Directed polymers model domain walls in disordered media [2] and arise as competition interfaces in multi-species growth [20]. Under the rescaling (2) this probability measure on paths converges to the polymer fixed point; a continuous path $\pi_{u,y}(t, x(s), u \leq s \leq t$) which at discrete times $u = s_0 < \cdots < s_{m-1} < t$ is given by the argmax over $x_0, \ldots, x_{m-1}$ of

$$(T_{u,x_0} \delta_y)(x_1) + (T_{s_1,x_1} \delta_z)(x_2) + \cdots + (T_{s_{m-1},x_{m-1}} \delta_x)(x).$$  

(7)

This is the analogue in the present context of the minimization of the action and the polymer fixed point paths
are analogous to characteristics in the randomly forced Burger’s equation. One might hope to take the analogy farther and find a limit of the renormalizations of \( \pi \), and minimize it to find that path \( \pi_{u,y,t,x} \). However, the limit does not appear to exist, so one has to be satisfied with the limiting paths themselves. The path \( \pi_{0,y,t,x} \) turns out to be H"older continuous with exponent 1/3−, as compared to Brownian motion where the H"older exponent is 1/2−. As the mesh of times is made finer, a limit does not appear to exist, so one has to be satisfied with the time evolution of the KPZ fixed point in terms of the polymer fixed point through the analogue of the Lax-Oleinik variational formula,

\[
\mathcal{h}(t,x) = \sup_{y \in \mathbb{R}} \{ \mathcal{E}(\pi_{0,y,t,x}) + f(\pi_{0,y;x,t}(0)) \}. \tag{8}
\]

**Universality.** The KPZ fixed point, Airy sheet, and polymer fixed point are universal and will arise in random polymers, last passage percolation and growth models — anything in the KPZ universality class. Just as for [14], at the microscopic scale, approximate versions of the variational problem [15] hold, becoming exact as \( \epsilon \to 0 \).

For example, consider the PNG model [10] with a finite collection of nucleations spaced order \( \epsilon^{-1} \) apart (Figure 1). At time \( \epsilon^{-3/2} t \) we look at \( \epsilon^{-1/2} \) scaled fluctuations in spatial locations \( \epsilon^{-1} x \). As \( \epsilon \) goes to zero, these fluctuations converge to \( \mathcal{h} \) where the initial data \( f \) is \( -\infty \) except at the nucleation points, where it is zero. By introducing additional nucleations at time on the order of \( \epsilon^{-1/2} \) and spatial locations order \( \epsilon^{-1} \) apart, it is possible to modulate the value of \( f \) at these non-\( \infty \) points. Taking the number of nucleation points large allows one to recover any \( f \). This scheme provides an easy to implement approach to run numerical simulations of the fixed point for general initial data.

The experiment of [8] is well described by the KPZ fixed point with a single nucleation. Future experiments could probe the effect of additional nucleations. Using statistics to differentiate between types of initial data given finite time observations is a driving force for the development of the following exact formulas which provide theoretical predictions.

**Transition probabilities.** We start with a simple version. Given \( c_i, y_i \in \mathbb{R} \), \( i = 1, \ldots, \nu \), and \( s_j, x_j \in \mathbb{R} \), \( j = 1, \ldots, \mu \), let \( \mathcal{h}(t,x) = (T_t f)(x) \) with \( f(y_i) = -c_i + (y_i - x_i)^2/2t \) and \( f = -\infty \) otherwise. Then,

\[
P(\mathcal{h}(t,x_j) \leq s_j - \frac{(x_j - x_i)^2}{2t}, \ j = 1, \ldots, \mu) = \det(I - \hat{L})
\]

where \( \hat{L} \) is the operator with kernel

\[
\hat{L}(z, z') = \int_{\mathcal{A}(s,c)} dv_1 \cdots dv_{\nu+1} du_1 \cdots du_{\mu} e^{(y_i - x_i)H} K[v_1] \langle v_1 | e^{(y_2 - y_1)H} | v_2 \rangle \cdots \langle u_{\mu-1} | e^{(x_{i-1} - x_i)H} | u_{\mu} \rangle \langle u_1 | e^{(x_{i-2} - x_{i-1})H} | u_{\mu+1} \rangle \delta(u_1 - v_{\nu+1}) \delta(u_{\mu} - v_{\nu+1})
\]

acting on \( L^2(\mathbb{R}) \), where \( H = -\frac{\partial^2}{\partial^2} + x \) is the Airy operator, \( K \) is the Airy operator which is the spectral projection of \( H \) onto its negative eigenvalues, and

\[
\mathcal{A}(s,c) = \{ (u, v) : \max_{i=1}^\mu \{ y_i - s_i \} + \max_{j=1}^\nu \{ y_j - c_j \} \geq v_{\nu+1} \}.
\]

We assume \( x_1 < x_2 < \cdots < x_\nu \) and \( y_1 > \cdots > y_\mu \). The formulas give a consistent family of finite dimensional distributions. Specializing to \( \mu = 1 \) or \( \nu = 1 \) one checks that the resulting process (in \( x_1 \) and \( y_1 \)) is the Airy\(_2\) process in each variable.

From the above the continuous transition probabilities can be obtained by approximation. Let \( Y_{[a,b]} \) be the operator defined by \( Y_{[a,b]} f(y) = u(b,y) \) where \( u \) is the solution of the boundary value problem, \( \partial_t u = -Hu \) for \( a < t < b \), with Dirichlet condition \( u(t, x) = 0 \) on \( x \geq g(t) \) and initial data \( u(a, x) = f(x) \).

Let \( c(x) \) and \( s(x) \) be finite in \( [c, r_c] \) and \( [\ell_s, r_s] \) and \( +\infty \) outside. Let \( f(x) = -(c(x) - (x - \ell_s)^2/2t) \), \( g(x) = s(x) - (x - \ell_s)^2/2t \) and

\[
p(0, f; t, g) = P(T_0 f(x) \leq g(x), \ -\infty < x < \infty)
\]

Then \( p(0, f; t, g) \) is given by a Fredholm determinant \( \det(I - K + L) \) where \( L(z, z') \) is given by

\[
\int dm du Y_{[\ell_s-r_s, \ell_c-r_c]} e^{(x-r_s)H} K(u, z) \frac{d}{dm} Y_{[\ell_s-r_s, \ell_c-r_c]} e^{(x-r_s)H} \langle z, u \rangle
\]

where \( \hat{c}(y) = -f(\ell_c + r_c - y) \), and \( K, H \) are as before.

Although the connection is not yet understood, our formula should be a scaling limit of a less explicit determinantal formula derived earlier for the microscopic model TASEP [11, 21].

**Replica Bethe ansatz.** Given \( s_j, x_j, j = 1, \ldots, \mu \) and \( c_i, y_i \in \mathbb{R} \), \( i = 1, \ldots, \nu \) consider \( \sum_{i=1}^\nu e^{-c_i \delta_{y_i}} \). From the linearity this can be written as \( \sum_{k=1}^\nu e^{-s_k \delta_{y_i}} Z(y_i, \ell_k ; t) \). Following [12, 15] we compute the generating function

\[
G(s, x; c) = \left\{ \exp(-e^{2t} \left[ \sum_{k,l=1}^\nu e^{-s_k \delta_{y_i}} Z(y_i, \ell_k; t) \right]) \right\}
\]

By studying asymptotics as \( t \) goes to infinity we obtain the formula for the transition probabilities.
Expanding the generating function exponential we write this using replicas as

\[ G(s, x; c, y) = 1 + \sum_{N=1}^{\infty} \frac{\langle (-1)^N e^{N/2} | I^N \rangle}{N!} \langle e^{-H_N t} | F^N \rangle \]

where the \( N \) particle wave functions of the \( \delta \)-Bose gas \( H_N = -\frac{1}{2} \sum_{x=1}^{N} \partial_x^2 - \frac{1}{2} \sum_{x \neq y} \delta(x - y) \) are \( \langle I^N \rangle = \sum_{\{y_1, \ldots, y_N\}} e^{-(c_1 + \cdots + c_N)} | y_1 \cdots y_N \rangle \) and \( \langle F^N \rangle = \sum_{\{s_1, \ldots, s_N\} = 1} e^{-\sum_{x=1}^{N} s_x} | x_1 \cdots x_N \rangle \). The wave functions are symmetric, so the propagator is only needed on the symmetric subspace. Thus we may employ the eigenfunction expansion of \( H_N \) [13]. Expanding the tensor products given above,

\[ G(s, x; c, y) = 1 + \sum_{N=1}^{\infty} \frac{\langle (-1)^N e^{N/2} | I^N \rangle}{N!} \sum_{\{y_1, \ldots, y_N\}} e^{-\sum_{x=1}^{N} s_x} \left[ \frac{\psi_r(y_1, \ldots, y_N)}{\psi_r(0)} \right]^N \]

where \( s_x = x/a \) and \( \sum_{a=1}^{M} s_x = \sum_{a=1}^{M} x/a \) are the parabolic shifts. After a series of integrations by parts [14] we arrive at (9).

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\[ \sum_{\{s_1, \ldots, s_N\} = 1} e^{-\sum_{x=1}^{N} s_x} \left[ \frac{\psi_r(y_1, \ldots, y_N)}{\psi_r(0)} \right] \]

holds asymptotically in \( t \) (as confirmed by the recovery of the Airy \( 2 \) process in the limit when \( \nu = 1 \) [14]). From this we obtain an asymptotically correct generating function by a Fredholm determinant \( \det(1 - H) \) where \( N(z, z') \) is

\[ \tau_{2x^2/2t, y^2/2t} 1_{z, z' \geq 0} e^{tx} e^{ty} \int du \Lambda_1(u + z) \Lambda_1(u + z') \Phi(au - s - c) \]

where for vectors \( s = (s_1, \ldots, s_M) \) and \( c = (c_1, \ldots, c_M) \),

\[ \Phi(s; c) = \frac{1}{(e^{s_1} + \cdots + e^{s_M}) (e^{c_1} + \cdots + e^{c_M})} \]

and \( J_\nu = -\sum_{b>0} x_b \partial_a a_b - \frac{1}{2} \sum_{a, \alpha} x_a \partial_a a_{\alpha} + \sum_{a, \alpha}^{\nu} y^2 \partial_a \partial_a a_{\alpha} \) is the parabolic shift. After a series of integrations by parts [14] we arrive at (9).

Conclusions. We have described new random objects: A Markov process, the KPZ fixed point described through a variational formula on a network of minimizing paths, the polymer fixed point. Both are described in terms of the Airy sheet. This description implies variational relations between the Airy processes, which are known to appear as spatial interface fluctuation processes for this class, and extend the universal limit processes of random matrices.

Furthermore, we have computed exact transition probabilities for the KPZ fixed point in terms of Fredholm determinants and prescribed both experimental and numerical methods to verify our prediction. For example, exciting several nucleations in turbulent liquid crystals and studying the correlations of interface height fluctuations at different points. These fixed points are expected to describe the asymptotic fluctuations over large space and time of a very wide class of one-dimensional physical systems with nonlinear transport of fluctuations, such as those modeled by directed random polymers, random growth models, stochastic Hamilton-Jacobi and Hamilton-Jacobi-Bellman equations, as well as the Kardar-Parisi-Zhang equation.