MIRROR SYMMETRY FOR QUIVER ALGEBROID STACKS

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Abstract. In this paper, we construct noncommutative algebroid stacks and the associated mirror functors for a symplectic manifold. First, we formulate a version of stack that is well adapted for gluing quiver algebras with different number of vertices.

Second, we develop a representation theory of $A_{\infty}$ categories by quiver stacks. A key step is constructing an extension of the $A_{\infty}$ category over a quiver stack of a collection of nc-deformed objects. The extension involves non-trivial gerbe terms, which play an important role for quiver algebroid stacks.

Third, we apply the theory to construct mirror quiver stacks of local Calabi-Yau manifolds. In this paper, we focus on nc local projective plane. This example has a compact divisor which gives rise to interesting monodromy and homotopy terms which can be found from mirror symmetry. Geometrically, we find a new method of mirror construction by gluing with a middle agent using Floer theory. The method makes a crucial use of the extension of Fukaya category over quiver stacks.

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1. Introduction

Stack is an important notion in the study of moduli spaces. Roughly speaking, a stack is a fibered category, whose objects and morphisms can be glued from local objects. Besides, a stack can also be understood as a generalization of a sheaf that takes values in categories rather than sets.

An algebroid stack is a natural generalization of a sheaf of algebras. It allows gluing of sheaves of algebras by a twist by a two-cocycle. Such gerbe terms arise from deformation quantizations of complex manifolds with a holomorphic symplectic structure, which are controlled by DGLA of cochains with coefficients in the Hochschild complex. By [BGN10], an obstruction for an algebroid stack to be equivalent to a sheaf of algebras is the first Rozansky-Witten invariant.

In this paper, we define and study a version of algebroid stacks that are glued from quiver algebras. We will see that gerbe terms appear naturally and play a crucial role, when the quivers that are glued have different numbers of vertices. See Figure 1. We will call these to be quiver algebroid stacks (or simply quiver stacks). The readers are invited to jump to Example 1.3 and 1.4 to quickly get a sense of a quiver stack.

There are several motivations for us to consider quiver stacks. First, from a purely algebraic point of view, it is natural to ask for a local-to-global understanding of a quiver algebra (which can also be identified as a category). Quiver stack is a perfect fit for this purpose. It allows us to define affine charts of a quiver algebra. See Definition 1.2.

Second, according to the SYZ program [SYZ96], mirror manifolds should be constructed as the quantum-corrected moduli space of possibly singular fibers of a Lagrangian fibration. In general, the singular fibers may have several components in their normalizations, and their deformations and obstructions are naturally formulated using quiver algebras (where the vertices of the quiver correspond to the components). In such a situation, we need to glue different quivers corresponding to the singular and smooth fibers, which have different numbers of vertices. Quiver stacks come up naturally as the quantum corrected moduli of Lagrangian fibers in such situations. They appear ‘in the middle’ of two large complex structure limits in the complex moduli.

In [CHL21], Cho, Hong and the first author constructed quiver algebras as noncommutative deformation spaces of Lagrangian immersions in a symplectic manifold. In another work [CHL], the authors globalized the mirror functor construction in the usual
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commutative setting \cite{CHL17}, by gluing local deformation spaces of Lagrangian immersions using isomorphisms in the (extended) Fukaya category. In this paper, we extend the Fukaya category over a quiver stack, and construct a mirror functor to the dg category of twisted complexes over the stack.

Third, we find a method of using 'middle agent' in mirror construction. Namely, fibers of a Lagrangian fibration do not intersect each other, and gluing their deformation spaces require a choice of a consistent system of homotopies. This is crucial in family Floer theory \cite{Fuk02,Tu14,Abo17}. Choosing these homotopies is a highly complicated task, especially when the base space has a non-trivial first fundamental group.

In this paper, we introduce an alternative approach. Instead of deforming the reference Lagrangians by diffeomorphisms (which is the main step in order to find homotopies of their Floer cohomologies), we find an immersed Lagrangian $L$ that serves as a middle agent. All the reference Lagrangians $L_i$ intersect with $L$ and are isomorphic to $L$ in some local deformation spaces. Then the Floer theoretical gluing between the deformation spaces of $L_i$ can be found by composing the gluing maps between that of $L_i$ and $L$.

In this method, the middle agent $L$ typically have more than one components in its normalization. Its deformation space will be a quiver algebra. Quiver algebroid stacks are crucial in order to glue the deformation spaces of $L$ and $L_i$ together.

Fourth, we are very much motivated from quiver crepant resolutions of singularities found by Van den Burgh \cite{VdB04}. Quiver algebras were used as crepant resolutions. The most well-known example is the resolved conifold, which corresponds to the quiver shown in Figure 2. Van den Bergh showed that these quiver algebras and the usual geometric crepant resolutions have equivalent derived categories. This proves a version of the Bondal-Orlov conjecture that two crepant resolutions of the same Gorenstein singularity have equivalent derived categories.

![Figure 2. Quiver for resolution of the conifold.](image)

It is a natural question to ask what such a derived equivalence correspond to in the mirror symplectic side. We propose that the equivalence can be constructed from isomorphisms of two different classes of immersed Lagrangians in the mirror side.

In \cite{CHL21}, algebras which are known as quiver crepant resolutions of toric Gorenstein singularities, together with Landau-Ginzburg superpotentials which are central elements of the algebras, were constructed as mirrors of certain Lagrangian immersions $L$ in punctured Riemann surfaces.

On the other hand, usual commutative crepant resolutions (together with superpotentials) were constructed as mirrors by gluing deformation spaces of Seidel’s immersed Lagrangians $L_i$ \cite{Sei11,Sei12} in pair-of-pants decompositions of the surfaces. Such mirror pairs are Landau-Ginzburg counterparts of the toric Calabi-Yau mirror pairs constructed in \cite{CLL12,AAK16} using wall-crossing. Homological mirror symmetry for these mirror pairs were proved by \cite{Lee15,Boc16}.
In this paper, we find an isomorphism between the immersed Lagrangian $L$ that produces quiver crepant resolutions, and the Seidel Lagrangians $L_i$ in a pair-of-pants decomposition, in mirrors of crepant resolutions of $\mathbb{C}^3/\mathbb{Z}_3$. The advantage of the mirror approach is that, the equivalence that it produces naturally extends to deformation quantizations of the crepant resolutions, which correspond to non-exact deformations in the symplectic side. The method is general, and we will study other toric Calabi-Yau manifolds in a future paper.

1.1. Quiver algebroid stacks. Noncommutative geometry arises naturally from quantum mechanics and field theory, in which particles are modeled by operators that do not commute with each other. Connes [Con94] has made a very deep foundation of the subject in terms of operator algebras and spectral theory. Moreover, the groundbreaking work of Kontsevich [Kon03] has constructed deformation quantizations from Poisson structures on function algebras. Deformation theory [KS, KR00] plays a central role. The subject is super rich and wide, contributed by many great mathematicians and we do not attempt to make a full list here.

In this paper, we focus on noncommutative algebras that come from quiver gauge theory. They are given by quiver algebras with relations

$$A = \mathbb{C}Q / R$$

where $Q$ is a quiver, $\mathbb{C}Q$ is the path algebra and $R$ is a two-sided ideal of relations. Such nc geometries have important physical meaning. Each vertex of the quiver represents a brane contained in the Calabi-Yau singularity, and arrows represent stringy interactions between the branes. The quiver algebra relations are derived from a specific element called the spacetime superpotential, which encodes the couplings between the branes. Deformations of this spacetime superpotential produce interesting noncommutative geometries. Such nc geometries have important physical meaning, namely, they provide the worldvolume theory for D-branes in a local Calabi-Yau twisted by non-zero B-fields [SW99, FO11]. Below shows an important example, which will be the main focus in the application part of this paper.

**Example 1.1** (NC local projective plane as an algebra). Consider the quiver $Q$ given on the right of Figure 1. We have the quiver algebra $A = \mathbb{C}Q / R$, where the ideal $R$ are generated by $a_2b_1 - b_2a_1$ and other similar relations, which are the cyclic derivatives of the spacetime superpotential

$$(a_3b_2 - b_3a_2)c_1 + (a_1b_3 - b_1a_3)c_2 + (a_2b_1 - b_2a_1)c_3.$$ 

$A$ is derived equivalent to the total space of the canonical line bundle $X = K_{\mathbb{P}^2}$ [VdB04], which is the crepant resolution of the orbifold $\mathbb{C}^3/\mathbb{Z}_3$.

$A$ admits interesting noncommutative deformations. The simplest one is given by the following deformation of the spacetime superpotential:

$$a_3b_2 - e^h b_3a_2)c_1 + (a_1b_3 - e^h b_1a_3)c_2 + (a_2b_1 - e^h b_2a_1)c_3.$$ 

For instance, this gives the commuting relation $a_2b_1 = e^h b_2a_1$. Let's denote the resulting algebra by $A^h$.

Indeed, Sklyanin algebras [AS87, ATVdB91, Smi15] provide an even more interesting class of deformations of $A$. Such deformations were constructed in [CHL21] using mirror symmetry. One of the relations take the form $p(h)a_2b_1 + q(h)b_2a_1 + r(h)c_2c_1$, where $(p(h), q(h), r(h))$ is given by theta functions and produces an embedding of an elliptic curve in $\mathbb{P}^2$. 
Van den Bergh [VdB04] showed that the quiver algebra $A$ is derived equivalent to the usual geometric crepant resolution $X$. The former provides a very efficient way to extract information, via its modules and Hochschild cohomology.

On the other hand, a ‘manifold’ description of a space is equally important, in order to extract and describe local structures of the space. In this paper, we would like to give such a local-to-global description also for a quiver algebra.

We understand a quiver algebra $A = CQ/R$ as the homogeneous coordinate ring of a $Q_0$-graded noncommutative variety, where $Q_0$ denotes the vertex set. It is natural to ask for affine charts of such a variety.

**Definition 1.2.** An affine chart of a quiver algebra $A$ is

$$\{A' = CQ'/R', G_{01}, G_{10}\}$$

where $Q'$ is a quiver with a single vertex and $R'$ is a two-sided ideal of relations;

$$G_{01}: A' \rightarrow A_{\text{loc}} \quad \text{and} \quad G_{10}: A_{\text{loc}} \rightarrow A'$$

are representations that satisfy

$$G_{10} \circ G_{01} = \text{Id};$$

$$G_{01} \circ G_{10}(a) = c(h_a) a c(t_a)^{-1}$$

for some function $c: Q_0 \rightarrow (A_{\text{loc}})^\times$ that satisfies $c(v) \in e_v R e_v : A_{\text{loc}} e_v$, where $v_0$ denotes the image vertex of $G_{01}$. Here, $A_{\text{loc}}$ is a localization of $A$ at certain arrows (meaning to add corresponding reverse arrows $a^{-1}$ and imposing $aa^{-1} = e_h$, $a^{-1} a = e_i$). $e_v$ denotes the trivial path at the vertex $v$.

**Example 1.3** (Free projective space). Consider the quiver $Q$ with two vertices $0,1$ and several arrows $a_k, k = 0, \ldots, n$ from vertex $0$ to $1$. An affine chart of the path algebra $CQ$ can be constructed by localizing $CQ$ at one arrow $a_l$ for some $l = 0, \ldots, n$. We take $A' = CQ'$ where $Q'$ is the quiver with a single vertex and $n$ loops $X_k, k \in \{0, \ldots, n\} \setminus \{l\}$. We fix the image vertex of $G_{01}$ to be $0$. Then define

$$G_{01}(X_k) = a_l^{-1} a_k$$

$$G_{10}(a_k) = X_k$$

$$c(0) = 0; c(1) = a_l^{-1}.$$  

One can easily check that the required equations are satisfied. This is a free algebra analog of the projective space, where $a_k, X_k$ are the homogeneous and inhomogeneous coordinates.

Gluing the quiver algebra $A$ together with its affine charts, we get a quiver stack. The following will be our main example.

**Example 1.4** (NC local projective plane as a quiver stack). Consider three copies of noncommutative $C^3 [1,4]$, denoted by $\mathcal{A}_i^h$ for $i = 1, 2, 3$, which correspond to the three corners of the polytope as shown in Figure 3. We use $(x_1,y_1,w_1), (y_2,z_2,w_2)$ and $(z_3,x_3,w_3)$ to denote their generating variables.

We glue these three copies of $nc C^3$ with localizations of the quiver algebra

$$\mathcal{A}_0^h := \mathcal{A}^h = CQ/R^h$$

given in Example [1,4] where the left-right ideal $R^h$ is generated by the cyclic derivatives of $(a_3b_2 - e^h b_3 a_2)c_1 + (a_1 b_2 - e^h b_1 a_3)c_2 + (a_2 b_1 - e^h b_2 a_1)c_3$. (For instance, $b_1 c_3 = e^h c_1 b_3$, by taking cyclic derivative in $a_2$.)
We take the localizations
\[ \mathcal{A}_0^h(U_{01}) := \mathbb{A}^h(a_1^{-1}, a_2^{-1}), \mathcal{A}_0^h(U_{02}) := \mathbb{A}^h(c_1^{-1}, c_2^{-1}), \mathcal{A}_0^h(U_{03}) := \mathbb{A}^h(b_1^{-1}, b_2^{-1}). \]
Here, \( U_{0i} \) denote the neighborhoods of the corners of the base polytope, so that the union of \( U_{0i} \) for \( i = 1, 2, 3 \) equals to the polytope.

For the gluing direction \( \mathcal{A}_1^h \to \mathcal{A}_0^h(U_{01}) \), we take the homomorphisms defined by:

\[
G_0:\begin{cases} x_1 \to c_1 a_1^{-1} \\ y_1 \to b_1 c_1^{-1} \\ w_1 \to a_1 a_2 a_3 \\ z_3 \to a_1 b_1^{-1} \\ y_2 \to b_1 c_1^{-1} \\ w_2 \to c_1 c_2 \\ z_3 \to b_1 b_2 b_3. 
\end{cases}
\]

It can be checked explicitly that the above is a homomorphism, once we set
\[ \tilde{h} = -3h. \]

For instance, \( x_1 y_1 - e^{-3h} y_1 x_1 = 0 \) is sent to \( c_1 a_1^{-1} b_1 a_1^{-1} - e^{-3h} b_1 a_1^{-1} c_1 a_1^{-1} = 0 \).

However, for the reverse direction, there is no algebra homomorphism \( \mathcal{A}_0^h(U_{01}) \to \mathcal{A}_1^h \). Thus the gluing cannot make sense using algebra homomorphisms. Rather, we need to use representations of \( \mathcal{A}_0^h(U_{01}) \) over \( \mathcal{A}_1^h \). A representation of a quiver algebra \( A \) over another quiver algebra \( A' \) means the following. Each vertex of the quiver for \( A \) is represented by a vertex of \( A' \), and each arrow of \( A \) is represented by an element of \( A' \), in a way such that the head and tail relations are respected, and the relations for algebras mod idempotents at vertices are respected (that is, any expression in arrows of \( A \) that equals to zero in \( A \) is still zero in \( A' \) upon substitution according to the representation). In other words, it is a functor if we regard the quiver algebras as categories.

We take the following representation of \( \mathcal{A}_0^h(U_{03}) \) by \( \mathcal{A}_3^h \):

\[
G_{30} : \begin{cases} (a_1, b_1, c_1) \to (z_3, 1, x_3) \\ (a_2, b_2, c_2) \to (e^h w_3 z_3, w_2, e^{-h} w_3 x_3) \\ (a_3, b_3, c_3) \to (e^{-h} z_3, 1, e^h x_3). \end{cases}
\]

The representations \( G_{1i} \) of \( \mathcal{A}_0^h(U_{01}) \) by \( \mathcal{A}_1^h \) for \( i = 2, 1 \) are obtained by cyclic permutation \( (a, b, c) \to (b, c, a) \to (c, a, b) \) and \((z_3, x_3, w_3) \to (y_2, z_2, w_2) \to (x_1, y_1, w_1) \) respectively.

It is easy to check that \( G_{0i} \circ G_{1i} \neq \text{Id}_{\mathcal{A}_i^h}. \) However,
\[ G_{0i} \circ G_{10} \neq \text{Id}_{\mathcal{A}_0^h(U_{0i})}. \]

In general, when \( \mathcal{A}_i \) has more vertices than \( \mathcal{A}_j \), such equality cannot hold simply because the representation of vertices is not a bijection. For instance,
\[ G_{03} \circ G_{30}(a_2) = e^h (b_1 b_3 b_2 a_1 b_1^{-1}) = b_1 b_3 \cdot a_2 \neq a_2. \]

Rather, we have
\[ G_{0i} \circ G_{1i}(a) = c_{0i0}(h_a) G_{0i}(a) e_{0i0}^{-1}(t_a) \]
for all arrows \( a \), if we set
\[
c_{030}(v_3) = b_1 b_3, c_{030}(v_1) = b_1, c_{030}(v_2) = e_2; 
c_{020}(v_3) = c_1 c_3, c_{020}(v_1) = c_1, c_{020}(v_2) = e_2; 
c_{010}(v_3) = a_1 a_3, c_{010}(v_1) = a_1, c_{010}(v_2) = e_2.
\]

For instance,
\[ G_{03} \circ G_{30}(a_3) = e^{-h} a_1 b_1^{-1} = b_1 \cdot a_3 \cdot (b_1 b_3)^{-1}. \]
Thus, gerbe terms $c_{0i0}$ are necessary for gluing quivers with different numbers of vertices. 

Now for any $i, j \in \{1, 2, 3\}$, we define

$$G_{ij} := G_{i0} \circ G_{0j} : \mathcal{A}_i(U_{ij}) \to \mathcal{A}_i(U_{ij}).$$

The localizations $\mathcal{A}_i(U_{ij})$ are the standard toric ones and can be read from the polytope picture (Figure 3). Explicitly, $\mathcal{A}_1(U_{12}) = \mathcal{A}_1(x_1^{-1})$ and $\mathcal{A}_1(U_{13}) = \mathcal{A}_1(y_1^{-1})$. The others $\mathcal{A}_2(U_{2j})$ and $\mathcal{A}_3(U_{3j})$ are obtained by the substitution $(x_1, y_1) \mapsto (y_2, z_2) \mapsto (z_3, x_3)$.

Then we have

$$G_{ij} \circ G_{jk}(x) = G_{i0} \circ (G_{0j} \circ G_{j0}) \circ G_{0k}(x) = G_{i0} \left( c_{0j0}(h_{G_{0k}(x)}) \cdot G_{0k}(x) \cdot c_{0j0}^{-1}(t_{G_{0k}(x)}) \right).$$

Note that in our definition (1.2) for $G_{0k}$, $G_{0k}(x)$ are loops at vertex 2 for all $x$. Moreover, $c_{0j0}(v_2) = e_2$. Hence $c_{0j0}(h_{G_{0k}(x)}) \cdot G_{0k}(x) \cdot c_{0j0}^{-1}(t_{G_{0k}(x)}) = G_{0k}(x)$, and we obtain the cocycle condition

$$G_{ij} \circ G_{jk} = G_{ik}$$

for any $i, j, k \in \{1, 2, 3\}$. Explicitly, one can check that the gluing maps $G_{ij}$ are the one given in Figure 3, producing the noncommutative local $\mathbb{P}^2$. This is an example of a noncommutative toric variety. Deformation quantizations of toric varieties were studied in [CLS13, CLS11].

In summary, we obtain a quiver algebroid stack consisting of four charts, $\mathcal{A}_i$ for $i = 0, 1, 2, 3$. If we forget the chart $\mathcal{A}_0$, then the remaining three charts glue up to an algebroid stack $\tilde{X}$ that has trivial gerbe term, that is, a sheaf of algebras.

Interesting phenomena arise as we turn on $\hbar$, due to the existence of a compact divisor. First, the deformation parameters of the algebra $\mathbb{A}_\hbar$ and the algebroid stack $X^\hbar$ are related in the non-trivial way

$$\tilde{\hbar} = -3\hbar.$$ 

Second, the toric gluing also needs to be deformed (by the factor $e^{-18\hbar}$ in this example) in order to satisfy the cocycle condition.

Figure 3. An algebroid stack which is a noncommutative deformation of $K_{\mathbb{P}^2}$.

These non-trivial factors only manifest when we turn on the deformation $\hbar \neq 0$. 

The quiver algebra \( A \) in the above example (quiver resolution of the orbifold \( \mathbb{C}^3/\mathbb{Z}_3 \) and its nc deformations) has a mirror construction from a three-punctured elliptic curve \( [\text{HL21}] \). It is natural to ask for the mirror correspondence of taking affine charts of \( A \). We will see that it is mirror to a pair-of-pants decomposition of the three-punctured elliptic curve.

We will construct algebroid stacks and the universal complexes via mirror symmetry. While our method of construction is general, this paper will focus on the case of \( K_{\mathbb{P}^2}^\circ \). We will work out the construction for the resolved conifold and \( A_n \) resolutions in a subsequent paper.

1.2. Mirror construction. Mirror symmetry is a fascinating subject that has attracted a lot of attentions in recent decades. It has made surprising and far-reaching predictions and breakthroughs in geometry, topology, and number theory. Homological mirror symmetry \([\text{Kon95}]\) asserted a deep duality between Lagrangian submanifolds in a symplectic manifold and coherent sheaves over the mirror algebraic variety. The program of Strominger-Yau-Zaslow \([\text{SYZ96}]\) has proposed a grand unified geometric approach to understand mirror symmetry via duality of Lagrangian torus fibrations, which triggers a lot of groundbreaking developments in geometry. The family Floer theory \([\text{Fuk02, Tu14, Abo17}]\) applies homotopy techniques of Floer theory to Lagrangian torus fibers to construct a family Floer functor for mirror symmetry.

In \([\text{Sei08, Sei11, Sei}]\), Seidel has made groundbreaking contributions to homological mirror symmetry. The Lagrangian immersion that he has invented plays a central role in the mirror symmetry part of this paper. We combine ideas and methods from HMS, SYZ, and powerful techniques from Lagrangian Floer theory developed by Fukaya-Oh-Ohta-Ono \([\text{FOOO09b}]\), to construct algebroid stacks \( \mathcal{X} \) by finding noncommutative boundary deformations of Lagrangian immersions and isomorphisms between them. We are going to develop a gluing scheme of local noncommutative mirrors. This combines the methods of \([\text{CHL21}]\) and \([\text{CHL}]\).

Homological mirror symmetry between noncommutative deformations of an algebra and non-exact deformations of a symplectic manifold was found by Aldi-Zaslow \([\text{AZ06}]\) for Abelian surfaces and Auroux-Katzarkov-Orlov \([\text{AKO06, AKO08}]\) for weighted projective spaces and del Pezzo surfaces. \([\text{CHL21}]\) systematically constructed quiver algebras mirror to a symplectic manifold, by extending the Maurer-Cartan deformations of \([\text{FOOO09b, FOOO10, FOOO11, FOOO16}]\). This paper glues local nc mirrors to an algebroid stack, by extending the gluing technique of \([\text{CHL}]\) over quiver algebras.

In this paper, we shall construct mirror algebroid stacks and the corresponding mirror functors. We will explicitly compute the mirror functor in object and morphism levels and apply it to construct universal sheaves for the cases of nc \( K_{\mathbb{P}^2}^\circ \). To illustrate, let’s start with the basic building block, namely nc \( \mathbb{C}^3 \).

**Example 1.5.** The immersed Lagrangian constructed by Seidel \([\text{Sei11}]\) is the most important source of motivation. See Figure 4a. It is descended from a union of three circles in a three-punctured elliptic curve, as shown in Figure 4b. The configuration in the elliptic curve is also interesting from physics perspective \([\text{BHLW06, JL07, GLW07}]\).

The Seidel Lagrangian has three degree-one immersed generators. It gives the free algebra \( \mathbb{C}\langle x, y, z \rangle \). In the obstruction term \( m_0^b \) of Floer theory, where \( b = xX + yY + zZ \) is a formal linear combination of the degree-one generators, the front and back triangles bounded by \( L \) contribute \( e^A xy - e^B yx \) at the generator \( Z \) (and similar for the other generators \( X \) and \( Y \)), where \( A \) and \( B \) are the areas of the front and back triangles respectively.
We quotient out these relations coming from obstructions and obtain the nc \(\mathbb{C}^3\)

\[
\mathbb{C}(x, y, z)/(e^A xy - e^B yx, e^A yz - e^B zy, e^A zx - e^B xz).
\]

Note that when \(A \neq B\), the equation \(e^A xy - e^B yx\) has no commutative solution. We are forced to consider deformations over a noncommutative algebra.

In a similar reasoning, for the \(3:1\) lifting in punctured elliptic curve in Figure 4b, \(L\) produces the quiver algebra in Example 1.1. More interestingly, [CHL21] constructed a family of Sklyanin algebras over an elliptic curve by taking symplectic compactification of the punctured elliptic curve.

**Remark 1.6.** In the above example, we take the Seidel Lagrangian together with a specific \(\mathbb{Z}\)-grading. Namely, the point class and fundamental class are assigned to be in degree 0 and 3, and the generators at the self-intersection points are assigned to be in degree 1 and 2, depending on the parity. Such a grading indeed comes from the fact that the Seidel Lagrangian corresponds to an immersed three-sphere in the threefold \({(u, v, x, y) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2 : uv = 1 + x + y} \) via the coamoeba picture [FHKV]. This is mirror to the toric Calabi-Yau threefold \(\mathbb{C}^3 - \{xyz = 1\}\) [CLL12, AAK16]. The pair-of-pants is identified as the mirror curve \(\{1 + x + y = 0\} \subset (\mathbb{C}^*)^2\).

For the local-to-global approach, we shall take a *pair-of-pants decomposition* of the Riemann surface, and consider a Seidel Lagrangian in each copy of pair-of-pants. See the left of Figure 5a for the three-punctured elliptic curve that appears in Example 1.5.

![Figure 4](image_url)

**Figure 4.** The left hand side shows the Seidel Lagrangian in a pair-of-pants. The right hand side shows a lifting to 3-to-1 cover by a three-punctured elliptic curve.

![Figure 5](image_url)

**Figure 5.** The left shows a pair-of-pants decomposition of the three-punctured elliptic curve and Seidel Lagrangians. The right shows a way to put Seidel Lagrangians so that they can be isomorphic to the ‘middle Lagrangian’ \(L\).

We want to glue up the noncommutative deformation spaces of the local Seidel Lagrangians in the pair-of-pants decomposition. However, these Lagrangians do not intersect each other, implying that their deformations spaces over the Novikov ring \(\Lambda\) do
not intersect with each other. We recall the Novikov ring

$$\Lambda_+ = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid \lambda_i \in \mathbb{R}_{\geq 0}, a_i \in \mathbb{C}, \lambda_i \text{ increases to } \infty \right\},$$

and the maximal ideal

$$\Lambda_0 = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid \lambda_i \in \mathbb{R}_{\geq 0}, a_i \in \mathbb{C}, \lambda_i \text{ increases to } \infty \right\}$$

of the Novikov field

$$\Lambda = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid \lambda_i \in \mathbb{R}, a_i \in \mathbb{C}, \lambda_i \text{ increases to } \infty \right\}.$$  

They play crucial role in the Lagrangian Floer theory of Fukaya-Oh-Ohta-Ono [FOOO09b].

We find a new method to get around this problem that the Seidel Lagrangians $$S_j$$ do not talk to each other. Instead of moving $$S_j$$ around to make them communicate, we take the Lagrangian $$L$$ shown in Figure 4b as a 'middle agent' that all $$S_j$$ can talk to. Then the gluing between different $$S_j$$'s can be found by composing that between $$S_j$$ and $$L$$. We call this $$L$$ to be a ‘middle Lagrangian’.

More precisely, we shall find noncommutative isomorphisms between $$(S_j, b_j)$$ and $$(L, b)$$, where the boundary deformations $$b_j$$ and $$b$$ are over different quiver algebras $$\mathcal{A}_j$$ and $$\mathcal{A}$$ respectively. We will solve for algebra embeddings $$\mathcal{A}_j \to \mathcal{A}_{\text{loc}}$$ (where $$\mathcal{A}_{\text{loc}}$$ is a certain localization of $$\mathcal{A}$$) such that the isomorphism equations hold for certain $$a_j \in \text{Gr}^0(L, S_j)_{\mathcal{A}_{\text{loc}}}, \beta_j \in \text{Gr}^0(S_j, L)_{\mathcal{A}_{\text{loc}}}$$,

\[
m_1^{b_j, b}(a_j) = 0, m_1^{b_j, b}(\beta_j) = 0;
\]

\[
m_2^{b_j, b}(a_j, \beta_j) = 1_L, m_2^{b_j, b}(\beta_j, a_j) = 1_{S_j}.
\]

This motivates us to develop a mirror construction of algebroid stacks in Section 3.3. We construct quiver stack $$\mathcal{X}$$ from a collection of Lagrangian immersions $$\mathcal{L} = \{L_0, \ldots, L_N\}$$, and show that:

**Theorem 1.7 (Theorem 3.24).** There exists an $$A_\infty$$ functor

$$\mathcal{F}^\mathcal{L} : \text{Fuk}(M) \to \text{Tw}(\mathcal{X}).$$

Now we have two mirrors, namely $$\mathcal{X}$$ constructed from $$\mathcal{L}_i := S_i$$ and $$\mathcal{A}$$ constructed from $$L$$. In order to compare these two mirror functors, we construct a twisted complex of $$(\mathcal{A}_i, \mathcal{A})$$-bimodules $$\mathcal{U}$$ over $$\mathcal{X}$$ by taking the mirror transform of $$(L, b)$$. In some interesting cases, $$\mathcal{U}$$ is the universal bundle over the moduli space of stable $$\mathcal{A}$$-module. Besides, this twisted complex induces a functor $$\mathcal{F}^\mathcal{U} := \text{Hom}(\mathcal{U}, -) : \text{Tw}(\mathcal{X}) \to d\text{g}(\mathcal{A} \text{- mod}).$$

\[
\text{Fuk}(M) \quad \text{Tw}(\mathcal{X}) \quad \mathcal{F}^\mathcal{U} \quad d\text{g}(\mathcal{A} \text{- mod})
\]

We show that:

**Theorem 1.8 (Theorem 3.27).** There exists a natural $$A_\infty$$-transformation $$\mathcal{I} : \mathcal{F}^{(\mathcal{A}, b)} \to \mathcal{A} \otimes (\mathcal{F}^\mathcal{U} \circ \mathcal{F}^\mathcal{L}).$$

Using the isomorphisms between $$(L, b)$$ and $$(L_j, b_j)$$, we deduce the injectivity of the natural transformation $$\mathcal{I}:$$
Theorem 1.9 (Theorem 3.28). Suppose there exist \( \alpha_i \in \mathcal{F}(L) \), \( \beta_i \in \mathcal{F}(L) \) that satisfies the above equation for some \( i \). Then the natural transformation \( \mathcal{T} : \mathcal{F}(L, b) \to \mathbb{A} \otimes (\mathcal{F}_U \circ \mathcal{F}_L) \) has a left inverse.

Remark 1.10 (Remark 3.29). To attain the natural equivalence of \( \mathcal{T} : \mathcal{F}(L, b) \to \mathbb{A} \otimes (\mathcal{F}_U \circ \mathcal{F}_L) \), we assume the objects in \( d\mathfrak{g}(\mathbb{A} - \text{mod}) \) behave like a sheaf, see equation 3.17. If the cohomology of \( U_i \) concentrates in the highest degree for all \( i \), \( \mathcal{T} \) is a natural equivalence. Namely, the diagram 1.5 commutes.

We carry out such a construction for mirror symmetry of nc local projective plane. We find rather non-trivial isomorphisms between \( L \) and \( S_i \), see Figure 12. It is rather interesting that we need to localize at the noncommutative quiver variables (for instance \( b_1, b_3 \)) for the existence of isomorphisms.

Theorem 1.11 (Theorem 4.5). The mirror construction for the Seidel Lagrangians \( S_i \) together with the middle Lagrangian \( L \) in the three-punctured elliptic curve produces the nc deformed local projective plane shown in Example 1.1.

Below is the plan of this paper. In Section 2, we define a version of algebroid stacks and twisted complexes that well adapts to quiver algebras. A main ingredient is concerning representation of a quiver algebra over another quiver algebra, in place of usual algebra homomorphisms, and isomorphisms between them.

Section 3 is the main part for our theory. We further develop the gluing techniques in [CHL] to the noncommutative setting of [CHL21]. The key step is to extend the \( A_\infty \) operations in Fukaya category over algebroid stacks. In gluing quiver algebras with different number of vertices, gerbe terms \( c_{ijk} \) in an algebroid stack will be unavoidable, and we need to carefully deal with them in extending the \( m_k \) operations. Another main construction is to compare functors constructed from two different reference Lagrangians. We need to extend the \( m_k \) operations for bimodules in a delicate way, so that we have desired morphisms of modules and natural transformations.

In Section 4, we construct \( \hbar \)-deformed \( K_{P^2} \) and twisted complexes over it using mirror symmetry. The key difficulty is to find a (noncommutative) isomorphism between local Seidel Lagrangians and an immersed Lagrangian coming from a dimer model. Another difficulty arises from the fact that the local Seidel Lagrangians do not intersect with each other. We employ the method of ‘middle agent’ to solve this problem. This will be particularly important in the construction of the universal bundle.

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2. QUIVER ALGEBROID STACKS

2.1. Review on algebroid stacks and twisted cochains. In this section, we will recall the definition of algebroid stacks and twisted cochains. For more detail, see [BGN108].

Definition 2.1. Let $B$ be a topological space. An algebroid stack $\mathcal{A}$ over $B$ consists of the following data.

1. An open cover $\{U_i : i \in I\}$ of $B$.
2. A sheaf of algebras $\mathcal{A}_i$ over each $U_i$.
3. An isomorphism of sheaves of algebras $G_{ij} : \mathcal{A}_i|_{U_{ij}} \cong \mathcal{A}_j|_{U_{ij}}$ for every $i, j$.
4. An invertible element $c_{ijk} \in \mathcal{A}_i|_{U_{ijk}}$ for every $i, j, k$ satisfying

\[
G_{ij}G_{jk} = Ad(c_{ijk})G_{ik},
\]

such that for any $i, j, k, l$,

\[
c_{ijk}\cdot c_{ikl} = G_{ij}(c_{ijk})c_{ikl}.
\]

We call $\mathcal{A}_i(U_i)$ nc charts. When there is no confusion, we call them charts directly since they are the main object we study. In this paper, we define the sheaf of algebras $\mathcal{A}_i$ over each $U_i$ by using the localization of $\mathcal{A}_i$ defined in the following:

Definition 2.2. Let $S \subseteq \mathcal{A} = CQ/R$ be a finite subset. For each $\gamma \in S$, we add one arrow, denoted by $\gamma^{-1}$, with $s(\gamma^{-1}) = t(\gamma)$ and $t(\gamma^{-1}) = s(\gamma)$, to the quiver $Q$. Moreover, we take the ideal $\hat{R}$ generated by $R$ and $\gamma^{-1} - e_{t(\gamma)}, \gamma^{-1} - e_{s(\gamma)}$ to be the new ideal of relations. The new quiver with relations $\hat{C}Q/\hat{R}$ is called the localized algebra at $S$, and is denoted as $\mathcal{A}(S^{-1})$.

We assign each $U_{l_0, \ldots, l_p}$ a subset $S_{l_0, \ldots, l_p} \subseteq \mathcal{A}_{l_0}$ such that $S_I \subseteq S_J$ whenever $J \subseteq I$. We define $\mathcal{A}_{l_0}(U_{l_0, \ldots, l_p}) := \mathcal{A}_{l_0}(S_{l_0, \ldots, l_p}^{-1})$. Then, the restriction maps in the sheaves of algebras are just the inclusion maps.

Let $E^*$ be a collection of graded sheaves $E^*_i$ over $U_i$, where $E^*_i(U_i)$ is a direct summand of a free graded $\mathcal{A}_i(U_i)$-module of finite rank, and $E^*_i(V)$ is the image of $E^*_i(U_i)$ under the restriction map $\mathcal{A}_i(U_i) \to \mathcal{A}_i(V)$ for any open $V \subseteq U_i$. (And the restriction map $E^*_i(V_1) \to E^*_i(V_2)$ is induced from the restriction $\mathcal{A}_i(V_1) \to \mathcal{A}_i(V_2)$ for any open $V_2 \subseteq V_1 \subseteq U_i$.) Let

\[
C^*(\mathcal{A}, E^*) = \prod_{p \geq 0} C^p(\mathcal{A}, E^q)
\]

where an element $a^{p,q}$ consists of sections $a^{p,q}_{l_0, \ldots, l_p}$ of $E^q_{l_0}(U_{l_0, \ldots, l_p})$ for all $l_0, \ldots, l_p$.

Consider another collection of graded sheaves $F = \{F^*_i\}$ as above. Let

\[
C^*(\mathcal{A}, \text{Hom}^*(E, F)) = \prod_{p \geq 0} C^p(\mathcal{A}, \text{Hom}^q(E, F)).
\]

An element $u^{p,q} \in C^p(\mathcal{A}, \text{Hom}^q(E, F))$ consists of sections

\[
u^{p,q}_{l_0, \ldots, l_p} \in \text{Hom}^q_{\mathcal{A}_{l_0}}(G_{l_0, l_p}(E^*_i), F^*_i)
\]

over $U_{l_0, \ldots, l_p}$ for all $l_0, \ldots, l_p$, where $G_{l_0, l_p}(E^*_i)$ (restricted on $U_{l_0, \ldots, l_p}$) is the $\mathcal{A}_{l_0}$-module which is the same as $E^*_i$ as a set, and the module structure is defined by

\[
a_{l_0} \cdot m = G^{-1}_{l_0, l_p}(a_{l_0})m.
\]
Then for $G_{ij} : A_{ij}(U_{ij}) \to A_j(U_{ij})$, we have the induced module map
\[ G_{ij}(u_{lp}) : G_{ij}(U_{ij}(E_{ij})) \to G_{ij}(U_{ij}(F_{ij})) \]
over $U_{ij}(U_{ij}, \ldots, l_p)$.

For an $\mathcal{A}_k$-module $M$, the multiplication by $G_{ik}^{-1}(c_{ijk})$ on $M$ defines a morphism
over $\mathcal{A}_l$, $G_{ij}G_{jk}(M) \to G_{lk}(M)$, which is denoted by $\hat{c}_{ijk}$, or simply again by $c_{ijk}$ if there
is no confusion. (Note that $G_{ik}^{-1}(c_{ijk}) = G_{ik}^{-1}G_{ij}^{-1}(c_{ijk})$ by using the equation $G_{ij}G_{ik} =
Ad(c_{ijk})G_{ik}$ on $\mathcal{A}_k$. Hence this can also be understood as multiplication of $c_{ijk}$ on the
$\mathcal{A}_l$-module $G_{ij}G_{jk}(M)$.) This is a morphism of $\mathcal{A}_l$-modules because for any element
$e \in G_{ij}(G_{jk}(M))$,
\[
\hat{c}_{ijk}(a_i \cdot e) = G_{ik}^{-1}(c_{ijk})G_{ij}^{-1}(a_i) e = G_{ik}^{-1}G_{ij}^{-1}(a_i)e = \hat{c}_{ijk}(a_i \cdot e).
\]

For $u^{p,r} \in CP(\mathcal{A}, \operatorname{Hom}^r(F^r, F^s))$, $v^{q,s} \in C^q(\mathcal{A}', \operatorname{Hom}^s(F^s, F^t))$, define the product
\begin{equation}
(u \cdot v)^{p+r,q+s}_{l_{0},l_{p},l_{q},l_{s}} = (-1)^{qr} u^{p,r}_{l_{0},l_{p},l_{q},l_{s}} \cup_c v^{q,s}_{l_{0},l_{p},l_{q},l_{s}}.
\end{equation}
and
\begin{equation}
u^{p,r}_{l_{0},l_{p},l_{q},l_{s}} \cup_c v^{q,s}_{l_{0},l_{p},l_{q},l_{s}} = u^{p,r}_{l_{0},l_{p},l_{q},l_{s}} \circ_c v^{q,s}_{l_{0},l_{p},l_{q},l_{s}}.\end{equation}

Below, we use $\cdot$ to denote the multiplication between two elements in an algebra; we use $\circ$ for the composition of module maps.

**Lemma 2.3.** The composition $\hat{c}_{ikl} \circ \hat{c}_{ijk} : G_{ij}G_{jk}G_{kl}(X_l) \to G_{kl}(X_l)$ is given by the multiplication
by $G_{ik}^{-1}(c_{ikl} \cdot c_{ikl}) \circ \mathcal{A}_l$ on $X_l$. (Note that as vector spaces, $G_{ij}G_{jk}(X_l)$, $G_{kl}(X_l)$
and $X_l$ are all the same.)

**Proof.** $\hat{c}_{ikl} \circ \hat{c}_{ijk}(e) = G_{ij}^{-1}(c_{ikl})G_{ik}^{-1}G_{ij}^{-1}(c_{ijk})e = G_{ik}^{-1}(c_{ikl})G_{ij}^{-1}G_{kl}^{-1}(c_{ikl} \cdot c_{ikl})e = G_{kl}^{-1}(c_{ikl} \cdot c_{ikl}) \cdot e.$

**Lemma 2.4.** $G_{ij}(\hat{c}_{ijk}) : G_{ij}G_{jk}(X_k) \to G_{kl}(G_{kl}(X_k))$ equals to the multiplication by $G_{ij}(c_{ijk})$
on the $\mathcal{A}_l$-module $G_{ij}G_{jk}(X_k)$.

**Proof.** $G_{ij}(\hat{c}_{ijk})(e) = \hat{c}_{ijk}(e) = G_{ij}^{-1}G_{jk}^{-1}(c_{ijk})e = G_{ij}^{-1}G_{jk}^{-1}G_{ij}^{-1}G_{kl}(c_{ijk})e$ which equals to
acting $G_{ij}(c_{ijk})$ on $e \in G_{ij}G_{jk}(X_k)$ as $\mathcal{A}_l$-module.

Applying the above two lemmas,
\[
\hat{c}_{ikl} \circ \hat{c}_{ijk}(e) = G_{ij}^{-1}(c_{ikl} \cdot c_{ikl})e = G_{ij}^{-1}(G_{ij}(c_{ikl}) \cdot c_{ijk})e = \hat{c}_{ijk} \circ G_{ij}(\hat{c}_{ikl})(e).
\]

For our purpose later, we take the inverse of this equation:

**Corollary 2.5.** $G_{ij}(\hat{c}_{ijk}^{-1}) = \hat{c}_{ijk}^{-1} \circ \hat{c}_{ikl}^{-1} \circ \hat{c}_{ijl}^{-1}$.

**Lemma 2.6.** Given any $s, p, q, r$ and $\mathcal{A}_q$-morphism $w : G_{qr}(X_r) \to X_q$,
\[
\hat{c}_{sq} \circ G_{sq}G_{pq}(w) \circ \hat{c}_{spq}^{-1} = G_{sq}(w) : G_{sq}G_{qr}(X_r) \to G_{sq}(X_q).
\]
Furthermore,
\begin{equation}
\hat{c}_{spq} \circ (G_{sp}G_{pq}(w)) \circ \hat{c}_{spq}^{-1} = G_{sq}(w) \circ \hat{c}_{sqr}^{-1}
\end{equation}
as $\mathcal{A}_r$-morphisms $G_{sr}(X_r) \to G_{sq}(X_q)$. 

Proof. Given any \( e \in G_{qr}(X_r) = G_{sq}G_{qr}(X_r) \),
\[
\hat{c}_{spq} \circ (G_{sp}G_{pq}(w)) \circ \hat{c}_{spq}(e)
\]
\[
= G_{sq}^{-1}(c_{spq})w(G_{pq}^{-1}(c_{spq}))e
\]
\[
= G_{pq}^{-1}(G_{sp}^{-1}(c_{spq})) \cdot w(G_{sq}^{-1}(c_{spq}))e
\]
\[
= w \left( G_{pq}^{-1}(G_{sp}^{-1}(c_{spq})) \cdot G_{sq}^{-1}(c_{spq}) \right)
\]
\[
= w \left( G_{pq}^{-1}(c_{spq}c_{spq}c_{spq}G_{sq}^{-1}(c_{spq})) \right)
\]
\[
= w \left( G_{pq}^{-1}(c_{spq}c_{spq}c_{spq})G_{sq}^{-1}(c_{spq}) \right)
\]
\[
= w \left( G_{pq}^{-1}(c_{spq}c_{spq})G_{sq}^{-1}(c_{spq}) \right)
\]
\[
= w \left( G_{pq}^{-1}(c_{spq}c_{spq})G_{sq}^{-1}(c_{spq}) \right)
\]
\[
= w(e).
\]
Thus we get \( \hat{c}_{spq} \circ G_{sp}G_{pq}(w) \circ \hat{c}_{spq}^{-1} = G_{sq}G_{pq}(w) \). By composing the equality with \( \hat{c}_{spq}^{-1} \) on the right and applying Corollary 2.5, we get the required equation.

\[ \square \]

From now on, we will take the abuse of notation of writing the morphism \( \hat{c}_{ijk} \) as \( c_{ijk} \).

**Proposition 2.7.** The product defined by Equation 2.3 is associative.

**Proof.** We can ignore signs for the moment, since we know the cup product is associative without \( G \) and \( c \); including \( G, c \) does not affect signs.

\[
(u \cdot (v \cdot w))_{l_0 \ldots l_r} = \sum_p u_{l_0 \ldots l_p}G_{l_0l_p}(v \cdot w)_{l_p \ldots l_r}c_{l_p l_r}^{-1}
\]
\[
= \sum_{p \leq q} u_{l_0 \ldots l_p}G_{l_0l_p}(v_{l_p \ldots l_q}G_{l_p l_q}(w_{l_q \ldots l_r}))c_{l_p l_q}^{-1}c_{l_q l_r}^{-1}
\]
\[
= \sum_{q \leq p} u_{l_0 \ldots l_q}G_{l_0l_q}(v_{l_q \ldots l_p}G_{l_q l_p}(w_{l_p \ldots l_r}))c_{l_q l_p}^{-1}c_{l_p l_r}^{-1}
\]
\[
= \sum q \left( (u \cdot v)_{l_0 \ldots l_q}G_{l_0l_q}(w_{l_q \ldots l_r})c_{l_q l_r}^{-1} \right) \text{ by Equation 2.5}
\]
\[
= ((u \cdot v) \cdot w)_{l_0 \ldots l_r}.
\]

\[ \square \]

The Čech differential is defined as
\[
(\hat{\delta}u)_{l_0 \ldots l_{p+1}} = \sum_{k=1}^p (-1)^k u_{l_0 \ldots l_k \ldots l_{p+1}}
\]
for \( u \in C^*(\mathcal{A}, \text{Hom}^*(E, F)) \).

A twisting complex is a collection of graded sheaves \( E^* \) as above, together with an element \( a \in C^*(\mathcal{A}, \text{Hom}^*(E, E)) \) with total degree being 1 that satisfies the Maurer-Cartan equation
\[ (2.6) \quad \hat{\delta}a + a \cdot a = 0. \]

Explicitly, the first few equations are:
\[ (2.7) \quad a_{i}^{0,1}G_{i j}(a_{j}^{0,1}) = 0, \]
\[ (2.8) \quad a_{i}^{0,1}G_{i j}(a_{j}^{1,0})c_{ij}^{-1} + a_{i j}^{1,0}G_{i j}(a_{j}^{0,1})c_{ij}^{-1} = 0, \]
\[ (2.9) \quad -a_{i k}^{1,0} + a_{i j}^{1,0}G_{ij}(a_{j k}^{1,0})c_{i j k}^{-1} + a_{i k}^{0,1}G_{i k}(a_{j k}^{2,1})c_{i j k}^{-1} + a_{i j k}^{2,1}G_{i k}(a_{j k}^{0,1})c_{i j k}^{-1} = 0. \]
The last equation is the cocycle condition, which is stating that $a^0_{i k} \cdot d a_{i j} \cdot c^1_{j k}$ are equal up to homotopy.

For morphisms, $\text{Hom}((E, a), (F, b)) := C^*(\mathcal{A}, \text{Hom}^*(E, F))$, which is a bi-graded complex using the Čech differential and the differential induced by $a^0_{i} \cdot b^0_{j}$. Moreover, the differential, denoted by $d_{\mathcal{A}}$, of a morphism $\phi$ is defined as:

$$d_{\mathcal{A}} \phi = \partial \phi + b \cdot \phi - (-1)^{|\phi|} \phi \cdot a.$$  

This form a dg-category of twisted complex, denoted by $TW(\mathcal{A})$. For convenience, we also denote $\text{Mor}_{TW(\mathcal{A})}((E, a), (F, b)) = C^*(\mathcal{A}, \text{Hom}^*(E, F)) = C^*_a(E, F)$, which may also be abbreviated as $C^*_a$ where $(E, a)$ and $(F, b)$ are fixed.

Proof. This is a direct application of associativity of the product. $d(\mu \cdot v)$ equals to

$$d(\mu \cdot v) = d\mu \cdot v + (-1)^{|\mu|} \mu \cdot d(v).$$

2.2. Algebroid Stacks for quiver algebras. In this section, we generalize the definition of algebroid stacks in the context of quiver algebras. The reason for such a generalization is that, in gluing two (localized) quiver algebras over a common intersection, it rarely happens that the two algebras are isomorphic if the two quivers have different number of vertices and arrows, and so the definition of algebroid stack in the last section is not enough for our mirror construction in Section 4 for nc local $\mathbb{P}^2$. Conceptually, our construction here can be understood as gluing via birational transformations. Interestingly, in our generalization, gerbe terms naturally come up and are usually unavoidable when the quivers have more than one vertices.
Let $B$ be a topological space as before, and $\{U_i : i \in I\}$ an open cover. Each $U_i$ is associated with a sheaf of algebras $\mathcal{A}_i$, where $\mathcal{A}_i(U_i)$ is a quiver algebra with relations, and $\mathcal{A}(V)$ are certain localizations at arrows of $Q^{(i)}$ for $V \subseteq U_i$. Correspondingly, we have quivers $Q^{(i)}_V$ corresponding to these localizations, which are obtained by adding the corresponding reverse arrows to $Q^{(i)}$.

In Definition 2.1, we require $G_{ij}(U_{ij}) : \mathcal{A}_j(U_{ij}) \cong \mathcal{A}_i(U_{ij})$ are isomorphisms. Here, we relax the condition and define $G_{ij}(U_{ij})$ as follows. First, associate each vertex $v \in Q^{(j)}_0$ with a vertex $G_{ij}(v) \in Q^{(i)}_0$. Next, represent each arrow from $v$ to $w$ in $Q^{(j)}_{ij}$ by elements $e_{G_{ij}(v)} \cdot \mathcal{A}_j(U_{ij}) \cdot e_{G_{ij}(w)}$ such that the relations for the paths are respected upon substitution. Later on, we will simply refer this as a (thin) representation of $Q^{(i)}_{ij}$ over $\mathcal{A}_i(U_{ij})$.

For our purpose, we fix a base vertex $v^{(i)}$ of $Q^{(i)}$ for each $j$, and require that $G_{ij}(v^{(j)}) = v^{(i)}$ for all $i, j$. We denote the corresponding trivial paths by $e^{(i)} := e_{v^{(i)}}$.

**Example 2.10.** In this paper, we always assume $G_{ii} = \text{Id}$, which is a representation of $Q^{(i)}$ over $\mathcal{A}_i$.

Given a sheaf of representations $G_{ij}$ of $Q^{(j)}_{ij}$ by elements in $\mathcal{A}_i(U_{ij})$, and a sheaf of representations $G_{jk}$ of $Q^{(k)}_{jk}$ by elements in $\mathcal{A}_j(U_{jk})$, we can restrict to the common intersection $U_{ijk}$ and compose them to get the representation $G_{ij}(U_{ijk}) \circ G_{jk}(U_{ijk})$ of $Q^{(k)}_{ijk}$ over $\mathcal{A}_j(U_{ijk})$. We will simply denote it by $G_{ij} \circ G_{jk}$ for simplicity.

The cocycle condition is that $G_{ij} \circ G_{jk}$ and $G_{ik}(U_{ijk})$ are isomorphic as representations. Namely, there exists an assignment of $c_{ijk}(v) \in \left( e_{G_{ij}(G_{jk}(v))} \cdot \mathcal{A}_j(U_{ijk}) \cdot e_{G_{ik}(v)} \right)^x$ to each vertex $v$ of $Q^{(k)}$, such that

\[(2.12) \quad G_{ij} \circ G_{jk} (a) = c_{ijk} (h_a) \cdot G_{ik} (a) \cdot c_{ijk}^{-1} (t_a).\]

This is a change of basis for representations. Gerbe terms $c_{ijk}$ arise in this way naturally, and unavoidably, since $Q^{(i)}, Q^{(j)}, Q^{(k)}$ are quivers of different sizes in general and the localized quiver algebras cannot be isomorphic.

In particular, at the base point $v^{(k)}$, $c_{ijk}(v^{(k)})$ is a cycle in $e^{(i)} \cdot \mathcal{A}_j(U_{ijk}) \cdot e^{(l)}$.

As in the last section, we assume that

\[c_{ijk}(G_{kl}(v))c_{kl}(v) = G_{ij}(c_{kl}(v))c_{ij}(v).\]

$(G_{ij}(c_{kl}(v))$ above is taken as $e_{G_{ij}(w)}$ if $c_{kl}(v)$ is a trivial path at $w.$)
Lemma 2.11. Under the above condition on $c_{ijk}$, $(G_{ij} \circ G_{jk}) \circ G_{kl}(a) = G_{ij} \circ (G_{jk} \circ G_{kl})(a)$ for all $a$.

Take $i = k$ in Equation (2.12). In this paper, we always take $G_{ii} = \text{Id}$. Then,

$$G_{ij} \circ G_{ji}(a) = c_{jj}(h_a) \cdot a \cdot c_{ij}^{-1}(t_a).$$

This replaces the condition of invertibility for $G_{ij}$. Note that

$$c_{iji}(v) \in \left( e_{G_{ij}(G_{ji}(v))} \cdot A_{iijk} \cdot e_v \right)^\times$$

for each vertex $v$ of $Q^{(i)}$.

Take $i = j$ in Equation (2.12). Since we take $G_{ii} = \text{Id}$, we simply get

$$G_{jj}(a) = c_{jj}(h_a) \cdot G_{ij}(a) \cdot c_{ij}^{-1}(t_a).$$

Then $c_{jjk}(v) = 1$ for all $v$ satisfies this equation. We will always take $c_{jjk} \equiv 1$ in this paper. Similarly, we take $c_{kk} \equiv 1$.

We summarize as follows.

Definition 2.12. Let $B$ be a topological space. A quiver algebroid stack consists of the following data.

1. An open cover $\{U_i : i \in I\}$ of $B$.
2. A sheaf of algebras $\mathcal{A}_i$ over each $U_i$, coming from localizations of a quiver algebra $\mathcal{A}_i(U_i) = \mathbb{C}Q^{(i)}/R^{(i)}$.
3. A sheaf of representations $G_{ij}$ of $Q^{(j)}_V$ over $\mathcal{A}_j(V)$ for every $i$, $j$, and $V$, open in $U_j$.
4. An invertible element $c_{ijk}(v) \in \left( e_{G_{ij}(G_{ji}(v))} \cdot A_{ijk} \cdot e_v \right)^\times$ for every $i$, $j$, $k$ and $v \in Q^{(k)}_0$, that satisfies

$$(2.13) \quad G_{ij} \circ G_{jk}(a) = c_{ijk}(h_a) \cdot G_{ik}(a) \cdot c_{ij}^{-1}(t_a)$$

such that for any $i$, $j$, $k$, $l$ and $v$,

$$(2.14) \quad c_{ijk}(G_{kl}(v))c_{kl}(v) = G_{ij}(c_{kl}(v))c_{ij}(v).$$

In this paper, we always set $G_{ii} = \text{Id}$, $c_{jj} \equiv 1 \equiv c_{kk}$.

Now we consider twisted complexes. In the previous section, $C_i(U_{ij})$, a $\mathcal{A}_i(U_{ij})$-module, can be treated as $\mathcal{A}_j(U_{ij})$-module via $G_{ij}$, and the transition map

$$\phi_{ij} : C_i(U_{ij}) \rightarrow C_j(U_{ij})$$

is required to be $\mathcal{A}_j(U_{ij})$-module map. However, in the current generalized setup, $C_i(U_{ij})$ can no longer be treated as $\mathcal{A}_j(U_{ij})$-module since $G_{ij}$ is no longer an algebra map. We consider the following instead.

Definition 2.13. Let $C_1$ and $C_2$ be modules of $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. A $\mathbb{C}$-linear map is said to be intertwining if

$$\phi_{ij}(h \cdot x) = G_{ij}(h) \cdot \phi_{ij}(x)$$

for all $h \in \mathcal{A}_i(U_{ij})$.

The space of intertwining chain maps $C_i(U_{ij}) \rightarrow C_j(U_{ij})$ forms a vector space. This is defined to be the morphism space.

To connect with module maps we use in the last section, we can enlarge $C_i(U_{ij})$ to make an $\mathcal{A}_j(U_{ij})$-module $\tilde{G}_{ij}(C_i(U_{ij}))$ as follows. Define

$$\tilde{G}_{ij}(C_i(U_{ij})) := (C_i(U_{ij}))^{\tilde{G}_{ij}(Q_{ij}^k)},$$
which is endowed with a structure of $\mathcal{A}_j(U_{ij})$-module:

$$ a \cdot \left( \sum_{x \in Q_0^j} x \right) := \left( G_{ij}(a) x_{t(a)} \right)_{h(a)}. $$

**Lemma 2.14.** The above defines a $\mathcal{A}_j(U_{ij})$-module $\hat{G}_{ij}(C_i(U_{ij}))$.

**Proof.**

$$ b \cdot a \cdot \left( \sum_{x \in Q_0^j} x \right) = \left( G_{ij}(b) G_{ij}(a) x_{t(a)} \right)_{h(b)} = (ba) \cdot \left( \sum_{x \in Q_0^j} x \right) $$

if $t(b) = h(a)$, and both sides are zero otherwise. \(\square\)

Then $\phi_{ji}: C_i(U_{ij}) \to C_j(U_{ij})$ induces a map $\hat{\phi}_{ji}: \hat{G}_{ij}(C_i(U_{ij})) \to C_j$ by

$$\hat{\phi}_{ji}(x_{v}) := \sum_{v \in Q_0^j} c_{ji}^{-1}(v) \cdot \phi_{ji}(x_{v}). \quad (2.15)$$

**Proposition 2.15.** The induced linear map $\hat{\phi}_{ji}$ is an $\mathcal{A}_j(U_{ij})$-module map if $\phi_{ji}$ is intertwining.

**Proof.** Suppose $\phi_{ji}$ is intertwining.

$$\hat{\phi}_{ji}(a \cdot (x_v)) = \hat{\phi}_{ji}\left( \left( G_{ij}(a) x_{t(a)} \right)_{h(a)} \right) = c_{ji}^{-1}(h(a)) \phi_{ji}(G_{ij}(a) x_{t(a)}) $$

$$= c_{ji}^{-1}(h(a)) G_{ji}(G_{ij}(a)) \cdot \phi_{ji}(x_{t(a)}) = ac_{ji}^{-1}(t(a)) \cdot \phi_{ji}(x_{t(a)})$$

which equals to

$$a \cdot \hat{\phi}_{ji}(x_v) = ac_{ji}^{-1}(t(a)) \cdot \phi_{ji}(x_{t(a)}).$$

The converse is based on the same calculation. \(\square\)

We make the following useful observation.

**Lemma 2.16.** If $C_i = \bigoplus_{p} \mathcal{A}_i \cdot e_{x_p}$ and $C_j = \bigoplus_{q} \mathcal{A}_j \cdot e_{x_q}$, and the components of $\phi_{ji}(x) \in C_j$ are given as a sum of terms in the form

$$G_{ji}(x_k \cdot y) \cdot a$$

for some $y \in \mathcal{A}_i(U_{ij})$ and $a \in \mathcal{A}_j(U_{ij})$ (and $x_p$ are the components of $x \in C_i$), then $\phi_{ji}(x)$ is intertwining.

The relation between intertwining maps and module maps is delicate. An intertwining map $\phi_{ji}$ lifts as a module map $\hat{\phi}_{ji}$. In the reverse way, given a map

$$\psi_{ji}: \hat{G}_{ij}(C_i(U_{ij})) \to C_j,$$

we can always restrict to define

$$(\psi_{ji})_{p} := c_{ji}(v_{ij}) \cdot \psi_{ji}(C_i(U_{ij})) \cdot \phi_{ji}: C_i(U_{ij}) \to C_j(U_{ij}).$$

However, $\psi_{ji}$ being an $\mathcal{A}_j(U_{ij})$-module map does not imply that $(\psi_{ji})_{p}$ is intertwining. It is obvious that $(\phi_{ji})_{p} = \phi_{ji}$. But it is not necessarily true that $(\psi_{ji})_{p} = \psi_{ji}$.

To have a better relation, consider the situation that

$$Q_0^{(j)} = \{ v \in Q_0^j : G_{ij}(v) = v^{(j)} \}.$$  

(This is always the case when $Q^{(j)}$ consists of a single vertex $v^{(j)}$.)
Proposition 2.17. Assume that \( Q_0^{(j)} = \{ v \in Q_0^{(j)} : G_{ij}(v) = v^{(j)} \} \). If
\[
\psi_{ji} : \hat{G}_{ij}(C_i(U_{ij})) \rightarrow C_j(U_{ij})
\]
is an \( \mathcal{A}_{\hat{i}}(U_{ij}) \)-module map and \((\psi_{ji})_e\) is intertwining, then \( \psi_{ji} = (\psi_{ji})_e \). In other words, the space of intertwining maps \( C_i(U_{ij}) \rightarrow C_j(U_{ij}) \) equals to the space of those module maps \( \psi_{ji} : \hat{G}_{ij}(C_i(U_{ij})) \rightarrow C_j(U_{ij}) \) with \((\psi_{ji})_e\) being intertwining.

Proof. Since for any \( v \in Q_0^{(j)} \), \( G_{ij}(v) = v^{(j)} \), we have \( c_{ij}^{-1}(v) \in \left( v^{(j)} \cdot \mathcal{A}_{\hat{i}}(i,j,k) \cdot v \right)^* \) and
\[
G_{ij} \circ G_{ij}(a) = c_{ij}(a) \cdot a \cdot c_{ij}^{-1}(ta) \in v^{(j)} \mathcal{A}_{\hat{i}}(i,j,k) v^{(j)}.
\]
In particular, \( G_{ij} \circ G_{ij}(c_{ij}(v)) = c_{ij}(v) \).

Let \( \phi_{ij}(x) := \psi_{ij}(x \cdot v^{(j)}) = c_{ij}^{-1}(v) \cdot (\psi_{ij}(x))_e \). It is intertwining by assumption. Since \( \psi_{ij} \) is a module map,
\[
c_{ij}^{-1}(v) \phi_{ij}(x) = c_{ij}^{-1}(v) \psi_{ij}(x) = \psi_{ij}(c_{ij}^{-1}(v) \cdot (x) \cdot v^{(j)}) = \psi_{ij}(G_{ij}(c_{ij}^{-1}(v)) \cdot x).
\]
Replacing \( x \) by \( G_{ij}(c_{ij}^{-1}(v)) \cdot x \), we get
\[
c_{ij}^{-1}(v) \phi_{ij}(G_{ij}(c_{ij}^{-1}(v)) \cdot x) = \psi_{ij}(x) \cdot v.
\]
On the other hand,
\[
c_{ij}^{-1}(v) \phi_{ij}(G_{ij}(c_{ij}^{-1}(v)) \cdot x) = c_{ij}^{-1}(v) \psi_{ij}(G_{ij}(c_{ij}^{-1}(v)) \cdot \phi_{ij}(x)) = c_{ij}^{-1}(v) c_{ij} \cdot \phi_{ij}(x).
\]
Thus, \( \psi_{ij}(x) = c_{ij}^{-1}(v) c_{ij} \cdot \phi_{ij}(x) \). That is, \( \psi_{ij} = (\psi_{ij})_e \).

Now we get back to the general situation (that \( Q_0^{(j)} \) may not equal to
\[
\{ v \in Q_0^{(j)} : G_{ij}(v) = v^{(j)} \},
\]
The higher terms \( \phi_I : C_{\hat{i}}(U_I) \rightarrow C_{\hat{0}}(U_I) \) (which are graded \( \mathbb{C} \)-linear maps) in defining a twisted complex are also required to be intertwining. Then it induces the \( \mathcal{A}_{\hat{0}}(U_I) \)-module map
\[
\hat{\phi}_I : \hat{G}_{\hat{i}} C_{\hat{0}}(U_I) \rightarrow C_{\hat{0}}(U_I)
\]
(where \( \hat{\phi}_I \) is defined from \( \phi_I \) by (2.15)).

Let \( I = (i_0, \ldots, i_k) \) and \( I' = (i_k, \ldots, i_0) \). Given intertwining maps \( \phi_I : C_{i_0}(U_I) \rightarrow C_{i_0}(U_I) \) and \( \psi_{I'} : C_{i_0}(U_{I'}) \rightarrow C_{i_0}(U_{I'}) \), we can take their composition
\[
\phi_I \circ \psi_{I'} : C_{i_0}(U_{I \cup I'}) \rightarrow C_{i_0}(U_{I \cup I'}).
\]
Unfortunately, \( \phi_I \circ \psi_{I'} \) is not intertwining. Rather,
\[
\phi_I \circ \psi_{I'}(ax) = G_{i_0,i_0}(G_{i_0,i_1}(a)) \phi_I \circ \psi_{I'}(x) = c_{i_0,i_0}^{-1}(h_a) G_{i_0,i_1}(a) c_{i_0,i_0}(t_a) \phi_I \circ \psi_{I'}(x) \neq G_{i_0,i_1}(a) \phi_I \circ \psi_{I'}(x).
\]
The above calculation tells us how to modify to make it intertwining. Namely, let \( G_{i} = \bigoplus_{p=1}^N \mathcal{A}_{\hat{i}} \cdot \nu_p \) for some vertices \( \nu_p \in Q_0^{(i)} \), and let \((X_1, \ldots, X_N)\) be the standard basis. Write \( x = \sum_p x_p X_p \). Then take
\[
(2.16) \quad \phi_I \cup \psi_{I'}(x) := \sum_p c_{i_0,i_0}^{-1}(h_{x_p}) \phi_I \circ \psi_{I'}(x_p X_p).
\]
Proposition 2.18. The above defined $\phi_I \cup \psi_I$ is intertwining.

Proof. 

$$
\phi_I \cup \psi_I(x) = \sum \frac{1}{G_{I_i}(G_{I_i}(x))} \phi_I \circ \psi_I(x)
$$

Thus,

$$
\phi_I \cup \psi_I(ax) = \sum \frac{1}{G_{I_i}(ax)} \phi_I \circ \psi_I(x).
$$

To simplify, we may write the short form $\phi_I \cup \psi_I(x) = \frac{1}{G_{I_i}(x)} \phi_I \circ \psi_I(x)$. However, note that $x$ is a module element rather than an element in $\mathcal{A}_{\mathcal{I}_i}^{(i)}$, and we need to write in basis like above in order to talk about $h_x$.

This can also be deduced in a systematic way like in last section, by considering the composition $\phi_I \circ G_{I_i}^I(\psi_I) \circ \xi_{I_i I_i}$ as explained below.

Given the module maps $\hat{\phi}_I : \hat{G}_{I_i}(C_{I_i}(U)) \to C_{I_i}(U)$ and $\hat{\psi}_I : \hat{G}_{I_i}(C_{I_i}(U)) \to C_{I_i}(U)$ where $U = U_{I_i I_i}$, we have the $\mathcal{A}_{\mathcal{I}_i}$-module map

$$
\hat{G}_{I_i}^I(\xi_{I_i I_i}^I) : \hat{G}_{I_i}(G_{I_i}(C_{I_i}(U))) \to \hat{G}_{I_i}(C_{I_i}(U)),
$$

where $\hat{G}_{I_i}(G_{I_i}(C_{I_i}(U))) = (C_{I_i}(U))_{G_{I_i}(C_{I_i}(U))}$, and $\hat{G}_{I_i}(\xi_{I_i I_i}^I)$ is simply taking $\hat{\psi}_I$ on each component labeled by an element in $Q_{I_i}^{(i)}$. By composition, we get an $\mathcal{A}_{\mathcal{I}_i}$-module map $\hat{\phi}_I \circ \hat{G}_{I_i}^I(\xi_{I_i I_i}^I) : \hat{G}_{I_i}^I(\hat{G}_{I_i}(C_{I_i}(U))) \to C_{I_i}(U)$. Next, we need to change the domain to $\hat{G}_{I_i}^I(\hat{G}_{I_i}(C_{I_i}(U)))$.

Proposition 2.19. There exist $\mathcal{A}_{\mathcal{I}_i}$-module maps

$$
\zeta_{I_i I_i}^I : \hat{G}_{I_i}(C_{I_i}(U)) \to \hat{G}_{I_i}^I(\hat{G}_{I_i}(C_{I_i}(U))),
$$

given by $\zeta_{I_i I_i}^I : x_{v,w} : v \in Q_{I_i}^{(i)}, w \in Q_{I_i}^{(i)} : c_{k j l}^I(w) \cdot x_{G_{I_i}(w,v)} : w \in Q_{I_i}^{(i)}$, and

$$
\zeta_{I_i I_i} : \hat{G}_{I_i}(C_{I_i}(U)) \to \hat{G}_{I_i}^I(\hat{G}_{I_i}(C_{I_i}(U))).
$$

Moreover, $\zeta_{I_i I_i} \circ \zeta_{I_i I_i} = \text{Id}$.

Then we take the composition

$$
\hat{\phi}_I \circ \hat{G}_{I_i}^I(\xi_{I_i I_i}^I) : \hat{G}_{I_i}(C_{I_i}(U)) \to C_{I_i}(U).
$$

This is the desired $\mathcal{A}_{\mathcal{I}_i}$-module map.

Proposition 2.20. $\hat{\phi}_I \circ \hat{G}_{I_i}^I(\xi_{I_i I_i}^I)$ equals to the lifting $\hat{\phi}_I \cup \psi_I$.

Proof. As in [2.16], we take a basis to write $x_w = \sum_{x,p} x_{w,p} X_p$. By definition,

$$
\hat{\phi}_I \circ \hat{G}_{I_i}^I(\xi_{I_i I_i}^I(x_w)) = \sum_{w,p} \frac{1}{G_{I_i}(G_{I_i}(w))} \phi_I \left( c_{I_i I_i}^{-1}(G_{I_i}(w)) \psi_I(c_{I_i I_i}(w)) \phi_I \left( x_{w,p} X_p \right) \right).
$$
First, we note that $c_{ij}^{-1}(w)$ can be expressed in terms of $c_{kli}^{-1}(w)$:

$$c_{ij}^{-1}(w) = c_{kli}^{-1}(w) c_{jl}^{-1}(G_{kl}(w)) G_{ij}(c_{kli}(w))$$

by taking $i = l$ in (2.14). Next, we use the intertwining property of $\phi_I$ and $\psi_I$. Also, note that $c_{li}^{-1}(w) x_w = 0$ if $G_{li}(w) \neq h(x_w,p)$. Then the right hand side equals to

$$\sum_{i,p} c_{li}^{-1}(w) c_{li}^{-1}(h(x_w,p)) G_{li}(c_{li}(w)) G_{li}(c_{li}(w)) = \phi_I \circ \psi_I(x_w,p).$$

Now we simplify $G_{li}(c_{li}(w)) G_{li}(c_{li}(w))$. Note that

$$c_{li}^{-1}(w) G_{li}(c_{li}(w)) = c_{li}(w) c_{li}(w) = c_{li}(w)$$

by taking $k = i$ in (2.14). Thus

$$G_{li}(c_{li}(w)) G_{li}(c_{li}(w)) = 1.$$

Thus,

$$\phi_I \circ \psi_I(x_w,p) = \sum_{i,p} c_{li}^{-1}(w) \cdot c_{li}^{-1}(h(x_w,p)) \phi_I \circ \psi_I(x_w,p)$$

and the right hand side is exactly $\phi_I \cup \psi_I$.

Once we have $\phi_I \cup \psi_I$, we define $\phi_I \cdot \psi_I$ as in Equation (2.5). Then a twisted complex is a collection of graded torsion-free sheaves (direct summands of free modules locally) over $U_i$, together with a collection of intertwining maps $a^{p,q}_i$ that satisfy the Maurer-Cartan equation (2.6). Similarly morphisms of twisted complexes are defined as in the last section. The essential changes are replacing module maps by intertwining maps, and defining their product by (2.16).

In concrete applications, the product is given as follows.

**Proposition 2.21.** Let $C_m = \bigoplus_{p=1}^{N_m} \mathcal{A}_m \cdot e_{ij}^{(p)}$ for $m = i, j, k$, and write every element in terms of the standard basis. Let

$$\phi_{ij}(x_s) = \left( \sum_{r=1}^{N_i} G_{ij}(x_r) \cdot a^{(j)}_{rs} \right)^{N_i},$$

$$\psi_{jk}(y_t) = \left( \sum_{s=1}^{N_k} G_{jk}(y_s) \cdot b^{(k)}_{st} \right)^{N_k},$$

for some $a^{(i)}_{rs} \in \mathcal{A}_i(U_{ijk}), a^{(j)}_{rs} \in \mathcal{A}_j(U_{ijk}), b^{(k)}_{st} \in \mathcal{A}_k(U_{ijk})$. Then

$$\phi_{ij} \cup \psi_{jk}(y_t) = \left( \sum_{s,t=1}^{N_sN_t} G_{ik}(y_s b^{(k)}_{st}) c_{lj}^{-1}(t_{b^{(k)}_{st}} G_{lj}(b^{(j)}_{st}) a^{(j)}_{rt}) \right)^{N_i}.$$
Proof. By (2.16),
\[
\phi_{ij} \cup \psi_{jk}(y) = \sum_{t} c_{ijk}^{-1}(h_{ij}) \phi_{ij} \circ \psi_{jk}(y_t) = \left( \sum_{s,t=1}^{N_i,N_k} c_{ij}^{-1}(h_{ij}) G_{ij} \circ G_{jk}(y_t b_{st}^{(k)}) G_{ij}(b_{st}^{(j)}) a_{s}^{(j)} \right)_{r=1}^{N_i} = \left( \sum_{s,t=1}^{N_i,N_k} G_{ik}(y_t b_{st}^{(k)}) c_{ij}^{-1}(t_{st}^{(k)}) G_{ij}(b_{st}^{(j)}) a_{s}^{(j)} \right)_{r=1}^{N_i}.
\]
\]

Remark 2.22. In applications, we take \( a_{s}^{(j)} \in \mathcal{A}(U_{i,j,k}) \), \( a_{s}^{(j)} \in \mathcal{A}(U_{i,j,k}) \), \( b_{st}^{(j)} \in \mathcal{A}(U_{i,j,k}) \), \( b_{st}^{(j)} \in \mathcal{A}(U_{i,j,k}) \). In particular, \( t_{st}^{(k)} = e^{(k)} \). If the gerbe term at base vertex \( c_{ijk}^{-1}(e^{(k)}) \) is taken to be 1, the above product formula becomes \( G_{ik}(y_t b_{st}^{(k)}) G_{ij}(b_{st}^{(j)}) a_{s}^{(j)} \).

In general, for \( \mathcal{A}_0, \ldots, \mathcal{A}_k \), let \( U = U_0, \ldots, k \), and define \( \mathcal{M}_{k,...,0} : \mathcal{A}(U) \otimes \ldots \mathcal{A}(U) \rightarrow \mathcal{A}_0(U) \),
\[
\mathcal{M}_{k,...,0}(z^{(k)} \otimes \ldots \otimes z^{(0)}) := G_{0k}(z^{(k)}) c_{0,k-1,k}^{-1}(t_{z^{(k)}}) G_{0,k-1}(z^{(k-1)}) \ldots c_{0,12}^{-1}(t_{z^{(2)}}) G_{01}(z^{(1)}) z^{(0)}.
\]
Proposition 2.23. Taken any \( 0 \leq p < q \leq k \). Let \( y^{(i)}, z^{(i)} \in \mathcal{A}(U) \) with \( t_{y^{(i)}} = h_{z^{(i)}} \) for \( i = 0, \ldots, k \). Then the product \( \mathcal{M}_{k,...,0}(y^{(k)} z^{(k)} \otimes \ldots \otimes y^{(0)} z^{(0)}) \) equals to the decomposition
\[
\mathcal{M}_{k,...,0}(y^{(k)} z^{(k)} \otimes \ldots \otimes y^{(0)} z^{(0)}) \text{ equals to to}
\]
\[
G_{0k}(y^{(k)} z^{(k)}) c_{0,k-1,k}^{-1}(t_{z^{(k)}}) G_{0,k-1}(y^{(k-1)}) \ldots c_{0,12}^{-1}(t_{z^{(2)}}) G_{01}(y^{(1)}) z^{(0)}
\]
where
\[
\phi' = G_{0,q}(h^{(q)}) c_{0,q-1,q}^{-1}(t_{z^{(q)}}) G_{0,q-1}(y^{(q-1)}) z^{(q-1)}) \ldots c_{0,0}^{-1}(t_{y^{(0)}}) G_{0,0}(y^{(0)}).
\]
We have \( G_{0,q}(h^{(q)}) c_{0,q-1,q}^{-1}(t_{z^{(q)}}) G_{0,q-1}(y^{(q-1)}) z^{(q-1)}) = c_{0,q-1,q}^{-1}(h_{z^{(q)}}) G_{0,q-1}(y^{(q-1)}) z^{(q-1)}) \]. Thus
\[
\phi' = c_{0,q-1,q}^{-1}(h_{z^{(q)}}) G_{0,q-1}(y^{(q-1)}) z^{(q-1)}) c_{0,q-2,q-1}^{-1}(t_{z^{(q-2)}}) \ldots c_{0,0}^{-1}(t_{y^{(0)}}) G_{0,0}(y^{(0)}).
\]
Then using
\[
c_{0,q-1,q}^{-1}(h_{z^{(q)}}) c_{0,q-2,q-1}^{-1}(h_{G_{0,q-1}(z^{(q)})}) = c_{0,q-2,q-1}^{-1}(h_{G_{0,q-1}(z^{(q)})}) G_{0,q-2}(y^{(q-2)}) z^{(q-2)} \ldots c_{0,0}^{-1}(t_{y^{(0)}}) G_{0,0}(y^{(0)}),
\]
we get
\[
\phi' = c_{0,q-2,q-1}^{-1}(h_{G_{0,q-1}(z^{(q)})}) G_{0,q-2}(y^{(q-2)}) z^{(q-2)} \ldots c_{0,0}^{-1}(t_{y^{(0)}}) G_{0,0}(y^{(0)}).
\]
Keep on doing this, we obtain
\[ \phi' = c_{0,p,q}^{-1} \left( h_{y(q)} G_{0,p} \left( G_{p,q} \left( z^{(q)} \right) \right) \right) \left( f_{z(q)} \right) \cdots \left( f_{z(0)} \right) G_{p,p+1} \left( y^{(p+1)} z^{(p+1)} \right) y^{(p)} \].

Note that \( h_{z(q)} = t_{y(q)} \). Thus \( \mathcal{M}_{k+0} \left( y^{(k)} z^{(k)} \otimes \cdots \otimes y^{(0)} z^{(0)} \right) \) equals to
\[ G_{0,k} \left( y^{(k)} z^{(k)} \right) c_{0,p,k-1}^{-1} \left( f_{z(0)} \right) G_{0,k-1} \left( y^{(k)} z^{(k)} \right) \cdots \left( G_{0,0,q} \right) \left( f_{y(q)} \right) \cdots \left( f_{y(0)} \right) \]
\[ \cdot \left( G_{0,p} \phi \cdot z^{(p)} \right) \left( f_{z(q)} \right) \cdots \left( f_{z(0)} \right) G_{0,12} \left( y^{(1)} z^{(1)} \right) y^{(0)} z^{(0)} \]
where \( \phi = G_{p,q} \left( z^{(q)} \right) c_{0,p,q}^{-1} \left( f_{z(q)} \right) \cdots \left( f_{z(0)} \right) G_{p,q} \left( y^{(p)} z^{(p)} \right) y^{(p)} \). This gives the desired expression.

**Remark 2.24.** In particular,
\[ \mathcal{M}_{k+0} \left( y^{(k)} z^{(k)} \otimes \cdots \otimes y^{(0)} z^{(0)} \right) \]
\[ = \mathcal{M}_{k+0} \left( y^{(k)} z^{(k)} \otimes \cdots \otimes y^{(0)} z^{(0)} \right) \]
\[ \cdot \left( 1 \otimes \mathcal{M}_{k+0} \right) \left( y^{(k)} z^{(k)} \otimes \cdots \otimes y^{(0)} z^{(0)} \right) \]
\[ \cdot \left( y^{(p)} \right) \cdots \left( y^{(0)} \right) \]
\[\text{RHS reads as} \]
\[ c_{0,p,k}^{-1} \left( h_{y(k)} \right) G_{0,p} \left( G_{p,q} \left( y^{(k)} z^{(k)} \right) \right) c_{0,p,k-1}^{-1} \left( f_{z(0)} \right) G_{0,k-1} \left( y^{(k)} z^{(k)} \right) \cdots \left( f_{z(0)} \right) G_{0,q} \left( y^{(0)} \right) z^{(0)} \]
\[ \cdot \left( G_{0,p} \phi \cdot z^{(p)} \right) \left( f_{z(q)} \right) \cdots \left( f_{z(0)} \right) G_{0,12} \left( y^{(1)} z^{(1)} \right) y^{(0)} z^{(0)} \].

In application, \( y^{(k)} \) is taken as a coefficient of an input module element. A linear combination of the product \( \mathcal{M}_{k+0} \left( y^{(k)} z^{(k)} \otimes \cdots \otimes y^{(0)} z^{(0)} \right) \) for various coefficients gives an intertwining map from a \( \mathcal{A}_k \)-module to an \( \mathcal{A}_0 \)-module. The above equation tells us that it can be written as the cup product of intertwining maps from the \( \mathcal{A}_k \)-module and from the \( \mathcal{A}_p \)-module to the \( \mathcal{A}_0 \)-module, where the maps are defined by
\[ G_{0,k} \left( y^{(k)} z^{(k)} \right) c_{0,p,k-1}^{-1} \left( f_{z(0)} \right) \cdots \left( f_{z(0)} \right) G_{0,p} \left( y^{(p)} \right) z^{(p)} \]
\[ \cdot \left( G_{0,p} \phi \cdot z^{(p)} \right) \left( f_{z(q)} \right) \cdots \left( f_{z(0)} \right) G_{0,12} \left( y^{(1)} z^{(1)} \right) y^{(0)} z^{(0)} \]
respectively. This will be important to establish \( A_{\infty} \)-equations over an algebroid stack.

Similarly, we can define
\[ \mathcal{M}_{k+0}^{op} \left( z^{(k)} \otimes \cdots \otimes z^{(0)} \right) := z^{(0)} G_{0,l} \left( z^{(1)} \right) c_{012} \left( h_{z(k)} \right) \cdots \left( G_{0,k-1} \left( z^{(k)} \right) \right) c_{0,k-1} \left( h_{z(q)} \right) G_{0,k} \left( z^{(k)} \right) \]
Similar to Proposition 2.23, it satisfies the following composition formula. The proof will not be repeated.

**Proposition 2.25.** \( \mathcal{M}_{k+0}^{op} \left( y^{(k)} z^{(k)} \otimes \cdots \otimes y^{(0)} z^{(0)} \right) \) equals to
\[ \mathcal{M}_{k+0}^{op} \left( y^{(k)} z^{(k)} \otimes \cdots \otimes z^{(q)} \otimes y^{(p)} \right) \]
\[ \cdot \mathcal{M}_{k+0}^{op} \left( y^{(q)} \otimes \cdots \otimes y^{(p+1)} \otimes y^{(p)} \otimes \cdots \otimes z^{(0)} \right) \]
Consider the case \( k = 1 \). Then
\[ \mathcal{M}_{1+0} \left( z^{(1)} \otimes z^{(0)} \right) = G_{01} \left( z^{(1)} \right) z^{(0)} \] and \( \mathcal{M}_{1+0}^{op} \left( z^{(1)} \otimes z^{(0)} \right) = z^{(0)} G_{01} \left( z^{(1)} \right) \).
\[ \mathcal{M}_{1+0} \left( y^{(1)} \otimes z^{(0)} \right) = G_{01} \left( y^{(1)} \right) z^{(0)} \] can be used to define an intertwining map from \( \mathcal{A}_0 \)-modules to \( \mathcal{A}_0 \)-modules, but \( \mathcal{M}_{1+0}^{op} \left( y^{(1)} \otimes z^{(0)} \right) \) cannot. On the other hand, \( \mathcal{M}_{1+0}^{op} \) preserves the left module structure of \( \mathcal{A}_0 \) on \( \mathcal{A}_0 \otimes \mathcal{A}_0 \) (where the module structure is defined by inserting \( \alpha \in \mathcal{A}_0 \) in the middle of \( z^{(1)} \otimes z^{(0)} \)). But \( \mathcal{M}_{1+0} \) destroys this module structure.
$\mathcal{M}_{k_0, \ldots, 0}^{\text{op}} (z^{(k)} \otimes \cdots \otimes z^{(0)})$ will be used in Section 3.2 for comparing two quiver algebras, while $\mathcal{M}_{k, \ldots, 0} (z^{(k)} \otimes \cdots \otimes z^{(0)})$ will be used in Section 3.3 for gluing mirror algebroid stacks.
3. Representation theory of $A_\infty$ category by algebroid stacks

In [CHL21], they introduced a non-commutative mirror functor from the Fukaya category to the category of matrix factorizations of the corresponding Landau-Ginzburg model. Later, in [CHL1], they developed a method of gluing the local mirrors and functors.

In this chapter, we will combine these two techniques. Namely, we will develop a gluing method for local nc mirror charts. We will use this to construct mirror algebroid stacks in later chapters. Moreover, we define the mirror transform of an nc family of Lagrangians. It is important for relating two different families of reference Lagrangians via a natural transformation.

3.1. Review on NC mirror functor. In this section, we firstly review some concepts about filtered $A_\infty$-algebra and bounding cochains in [FOOO09b]. Then we review the nc mirror functor construction in [CHL21].

The Novikov ring is defined as

$$\Lambda_0 = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} | \lambda_i \in \mathbb{R}_{\geq 0}, a_i \in \mathbb{C}, \lim_{i \to \infty} \lambda_i = \infty \right\}$$

with maximal ideal

$$\Lambda_+ = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} | \lambda_i \in \mathbb{R}_{> 0}, a_i \in \mathbb{C}, \lim_{i \to \infty} \lambda_i = \infty \right\}$$

and the universal Novikov field is defined as its field of fraction $\Lambda = \Lambda^0[T^{-1}]$. The filtration on $\Lambda$ is given by

$$F^A \Lambda = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \in \Lambda | \lambda_i \geq \lambda \right\}.$$ 

**Definition 3.1.** A filtered $A_\infty$-category $\mathcal{C}$ consists of a collection of objects $\text{Ob}(\mathcal{C})$, and torsion-free filtered graded $A_\infty$-module $\mathcal{C}(A_1, A_2)$ for each pair of objects $A_1, A_2 \in \text{Ob}(\mathcal{C})$, equipped with a family of degree one operations $m_k : \mathcal{C}(1)[A_0, A_1] \otimes \cdots \otimes \mathcal{C}(1)[A_{k-1}, A_k] \to \mathcal{C}(1)[A_0, A_k]$ for $A_i \in \text{Ob}(\mathcal{C})$ for $i = 0, 1, \cdots, k$, where $m_k$ is assumed to respect the filtration and satisfies the $A_\infty$-equations for $v_i \in \mathcal{C}(1)[(A_1, A_{i+1})]$:

$$\sum_{k_1 + k_2 = n + 1} \sum_{i=1}^{k_1} (-1)^{\epsilon_i} m_{k_1}(v_1, \cdots, m_{k_2}(v_i, v_{i+k_2-1}, v_{i+k_2, \cdots}, v_n)) = 0$$

where $\epsilon_i = \sum_{j=1}^{i-1} |v_j|$, and $|v'| = |v| - 1$, the shifted degree of $v$.

**Remark 3.2.** In this paper, a Novikov term $T^A$ show up to represent area of a polygon counted in $m_k$.

When a filtered $A_\infty$-category consists of only a single object, it is called a filtered $A_\infty$-algebra. Let $A$ be an $A_\infty$ algebra. When $m_{\geq 3} = m_0 = 0$, $A$ becomes a differential graded algebra, where $m_1$ and $m_2$ stand for differential and composition operation respectively according to $A_\infty$-equation.

With this understanding, we can also define unit in $\mathcal{C}^0(A, A)$, denoted by $1_A$, which satisfies

$$\begin{align*}
m_2(1_A, v) &= v & v & \in \mathcal{C}(A, A') \\
(-1)^{|w|}m_2(w, 1_A) &= w & w & \in \mathcal{C}(A', A) \\
m_k(\cdots, 1_A, \cdots) &= 0 & \text{otherwise.}
\end{align*}$$
Definition 3.3 ([FOOO09b]). An element in \( b \in F^+\mathcal{E}^1(A, A) \) is a weak Maurer-Cartan element if \( m^0_b := m(e^b) := \sum_{k=0}^{\infty} m_k(b, \ldots, b) = W(A, b) \cdot 1_A \) for some \( W(A, b) \in \Lambda \).

Given \( b \in F^+\mathcal{E}^1(A, A) \), we can define
\[
m^b_k(v_1, \ldots, v_k) = m(e^b, v_1, e^b, v_2, \ldots, e^b, v_k, e^b).
\]
In a similar fashion, one can also define \( m_k \) for several \((L_i, b_i)\), and we shall not repeat. The introduction of weak Maurer-Cartan elements gives a way to deform such that Floer cohomology is well-defined, even in the case that \( m_0 \) may not be zero.

In this paper, we will use the Fukaya category that also includes compact oriented spin immersed Lagrangians as objects. Their Floer theory was defined in [AJ10], generalizing the construction of [FOOO09b] for smooth Lagrangians.

Let \( X \) be a symplectic manifold, \( \mathbb{L} \to X \) a compact spin oriented unobstructed Lagrangian immersion with transverse doubly self-intersection points. The space of Floer cochains is
\[
\text{CF}^* (\mathbb{L}) := \text{CF}^*(\mathbb{L}) := C^*(\mathbb{L}) \oplus \bigoplus_p \text{Span}\{(p_-, p_+), (p_+, p_-)\}
\]
where \( p \) are doubly self-intersection points and \( p_-, p_+ \) are its preimage. \((p_-, p_+), (p_+, p_-)\) are treated as Floer generators that jump from one branch to the other at the angles of a holomorphic polygon. For \( C^* (\mathbb{L}) \), we shall use Morse model. Namely, we take a Morse function on each component of \((\text{the domain of}) \mathbb{L}\), and \( C^* (\mathbb{L}) \) is defined as the formal \( \mathbb{C} \)-span of the critical points. The Floer theory is defined by counting holomorphic pearl trajectories [OZ11][BC12][FOOO09a][She15].

By using homotopy method [FOOO09b][CW15], the algebra can be made to be unital. See [KLZ] Section 2.2 and 2.3 for detail in the case of Morse model. The unit is denoted by \( 1_\mathbb{L} \). It is homotopic to the formal sum of the maximum points of the Morse functions on all components (representing the fundamental class), denoted by \( 1_\mathbb{L}^1 \). Namely, \( 1_\mathbb{L} = 1_\mathbb{L}^1 \) (assuming \( \mathbb{L} \) bounds no non-constant disc of Maslov index zero).

The space of Floer cochains \( \text{CF}^* (\mathbb{L}_1, \mathbb{L}_2) \) for two Lagrangians (assuming they intersect cleanly) is similar and we shall not repeat. In general, \( \text{CF}^* (\mathbb{L}_1, \mathbb{L}_2) \) is only \( \mathbb{Z}_2 \)-graded. On the other hand, in Calabi-Yau situations where graded Lagrangians are taken, \( \text{CF}^* (\mathbb{L}_1, \mathbb{L}_2) \) is \( \mathbb{Z} \)-graded, meaning that each Floer generator is assigned an integer degree, compatible with the \( \mathbb{Z}_2 \)-grading, in a way such that the \( A_{\infty} \)-operations have the correct grading and satisfy \( A_{\infty} \) equations. Generators of degree one (which means odd degree when only \( \mathbb{Z}_2 \)-grading exists) play a particularly important role in deformation theory.

[CHL21] has made a construction of a noncommutative deformation space of a spin oriented Lagrangian immersion \( \mathbb{L} \subset M \). The construction is summarized as follows.

Construction 3.4.

1. Associate a quiver \( Q \) to \( \text{CF}^1 (\mathbb{L}) \). Namely, each component of \((\text{the domain of}) \mathbb{L}\) is associated with a vertex, and each generator in \( \text{CF}^1 (\mathbb{L}) \) is associated with an arrow.

2. Extend the Fukaya algebra \( A \) of \( \mathbb{L} \) over the path algebra \( \Lambda Q \) and obtain a noncommutative \( A_{\infty} \)-algebra
\[
\bar{A}^\Lambda = \Lambda Q \otimes_{\Lambda^e} \text{CF} (\mathbb{L}),
\]
whose unit is \( 1_\Lambda = \sum 1_{\Lambda_{e_i}} \). \( \Lambda^e \subset \Lambda Q \) denotes \( \bigoplus_i \Lambda \cdot e_i \) where \( e_i \) are the trivial paths at vertices of \( Q \). The fibered tensor product means that an element \( a \otimes X \) is nonzero only when tail of \( a \) corresponds to the source of \( X \). The \( A_{\infty} \) operations are defined by
\[
m_k(f_1 X_1, \ldots, f_k X_k) := f_k \ldots f_1 m_k(X_1, \ldots, X_k)
\](3.2)
where $X_i \in \text{CF}(L)$ and $f_i \in \Lambda Q$.

(3) Extend the formalism of bounding cochains of $\text{FOOO09b}$ over $\Lambda Q$. Namely, we take

\[ b = \sum_I b_I B_I \]

where $B_I$ are the generators of $\text{CF}^1(L)$, and $b_I$ are the corresponding arrows in $Q$. Then define the deformed $A_\infty$ structure $m_k^b$ as in $\text{FOOO09b}$ and via Equation (3.2).

(4) Quotient out the quiver algebra by the two-sided ideal $R$ generated by coefficients of the obstruction term $m_k^0$, so that $m_k^b = W \cdot 1_k$ over

\[ \mathbb{A} := \Lambda Q / R. \]

$\mathbb{A}$ is the space of noncommutative weakly unobstructed deformations of $L$. We call $(\mathbb{A},W)$ to be a noncommutative local mirror of $X$ probed by $L$.

(5) Extend the Fukaya category over $\mathbb{A}$, and enlarge the Fukaya category by including the noncommutative family of objects $(L,b)$ where $b$ in (3.3) is now defined over $\mathbb{A}$. This means for $L_1, L_2$ in the original Fukaya category, the morphism space is now extended as $\mathbb{A} \otimes \text{CF}(L_1, L_2)$. The morphism spaces between $(L,b)$ and $L$ are enlarged to be $\text{CF}((L,b), L) := \mathbb{A} \otimes_{\mathbb{A}^0} \text{CF}(\mathbb{L}, L)$ (and similarly for $\text{CF}(L, (\mathbb{L}, b))$). We already have $\text{CF}(\mathbb{L}, (\mathbb{L}, b))$ in Step 2 (except that $\Lambda Q$ is replaced by $\mathbb{A}$). The $m_k$ operations are extended in a similar way to (3.2).

**Remark 3.5.** Note that $m_k^b$ in Step 3 is no longer linear over $\Lambda Q$. For instance, suppose we have $m_k^b(X) = m_3(bB, X, bB) = b^2 \cdot \text{out}$ where out $= m_3(B, X, B)$. Then

\[ m_k^b(aX) = m_3(bB, aX, bB) = bab \cdot \text{out} \neq a \cdot m_k^b(X). \]

Boundary deformations are more non-trivial over noncommutative algebras in this sense.

On the other hand, if we consider $m_k^{b,0,...,0}$ on $\text{CF}((\mathbb{L}, b), L_1) \otimes \text{CF}(L_1, L_2) \otimes \text{CF}(L_2, L_3) \otimes \ldots \otimes \text{CF}(L_{k-1}, L_k)$ where none of $L_j$ is $(\mathbb{L}, b)$, then $m_k^{b,0,...,0}$ is still linear over $\mathbb{A}$. This is important in defining the mirror functor.

In Step 5, even though $L$ and $L'$ may not be $\mathbb{L}$, we can still take coefficients in $\mathbb{A}$ for the inputs. In other words, we define a family of Floer theory over $\mathbb{A}$. Using this, we obtain a canonical mirror transformation, which is analogous to the Yoneda functor, as follows.

**Definition 3.6.** For an object $L$ of $\text{Fuk}(X)$, its mirror matrix factorization of $(\mathbb{A},W)$ is defined as

\[ \mathcal{F}^L(L) := \left[ \mathbb{A} \otimes_{\mathbb{A}^0} \text{CF}^* (\mathbb{L}, L), d = (-1)^{|I|} m_1^{b,0} (\cdot) \right]. \]

The mirror of morphisms is given as follows: Given $L_1, L_2 \in \text{Fuk}(X)$ and an intersection point between them, $X \in \text{CF}(L_1, L_2)$, $\mathcal{F}^L(X) := (-1)^{|X|-1} (-1)^{|I|-1} m_2^{b,0,0} (\cdot, X) : \mathcal{F}^L(L_1) \to \mathcal{F}^L(L_2)$.

**Theorem 3.7 (CHL21).** The above definition of $\mathcal{F}^L$ extends to give a well-defined $A_\infty$ functor

\[ \text{Fuk}(X) \to \text{MF}(\mathbb{A}, W). \]

**Remark 3.8.** $m_0^b = W \cdot 1_k$ has degree 2. Thus in the $\mathbb{Z}$-graded situation, $W = 0$, and the above $\text{MF}(\mathbb{A}, W)$ reduces to the dg category of complexes of $\mathbb{A}$-modules.
Example 3.9. When $X$ is a symplectic surface, any compact oriented immersed curve (together with a weak bounding cochain) is an object inside $\text{Fuk}(X)$. The generators $(p_-, p_+)$ and $(p_+, p_-)$ can be visualized as angles at self-intersection points $p$, see Figure 6. The parity of degrees of generators are determined by orientation as shown in the figure.

![Figure 6](image)

**Figure 6.** Each transverse intersection point corresponds to two Floer generators.

For surfaces, we will use the following sign rule for a holomorphic polygon bounded by $\mathcal{L}$ constructed by Seidel [Sei08]. The spin structure is given by fixing spin points (marking where the non-triviality of the spin bundle occurs) in (the domain of) $\mathcal{L}$. Denote the input angles of the polygon $P$ by $X_1, \ldots, X_k$, and the output angle by $X_0$. If there is no spin point on the boundary of $P$ and the orientations of all edges of $P$ agree with that of $\mathcal{L}$, then the contribution of $P$ (via output evaluation) takes a positive sign. Otherwise, disagreement of the orientations on $\overline{X_iX_{i+1}}$, for $i = 2, \ldots, k - 1$, affects the sign by $(-1)^{|X_i|}$. Whether the orientation on $\overline{X_0X_1}$ agrees with $\mathcal{L}$ or not is irrelevant. If the orientations are opposite on $\overline{X_0X_1}$, then we multiply by $(-1)^{|X_1|+|X_0|}$. Finally, we multiply by $(-1)^l$ where $l$ is the number of times $\partial P$ passes through the spin points.

Remark 3.10. In many important situations, $A$ takes the form

$$\text{Jac}(Q, \Phi) = \frac{\Lambda Q}{(\partial_x, \Phi : e \in E)},$$

where $\Phi$ is called spacetime superpotential. The cases that we consider in this paper belong to this scenario.

3.2. **Fukaya category enlarged by two nc deformed Lagrangians.** In the last section, we have reviewed the weakly unobstructed nc deformation space of an immersed Lagrangian [CHL21]. In this section, we consider two immersed Lagrangians $\mathcal{L}_1, \mathcal{L}_2$ over their weakly unobstructed nc deformation spaces $A_1$ and $A_2$. The construction is important for relating different mirrors of the same symplectic manifold, for instance, the situation of twin Lagrangian fibrations [LY10] [LL19].

There are two closely related constructions in this situation. The first one is taking product. Namely, we take $(\mathcal{L}_1, b_1)$ as probes and transform $(\mathcal{L}_2, b_2)$ to a left $A_1$-module over $A_2$, or in other words, an $(A_1, A_2)$-bimodule. In commutative analog, this gives a universal sheaf over the product of local moduli of $\mathcal{L}_1$ and that of $\mathcal{L}_2$, whose fiber is the Floer cohomology $HF^*((\mathcal{L}_1, b_1), (\mathcal{L}_2, b_2))$. We concern about this in the current section.

The second construction is that we want to glue up the local nc mirrors of $\mathcal{L}_1$ and $\mathcal{L}_2$ by finding an nc family of isomorphisms between $(\mathcal{L}_1, b_1)$ and $(\mathcal{L}_2, b_2)$ over certain
localizations \((A_1)_{12} \equiv (A_2)_{12}\). \((L_1, b_1)\) are treated as objects in the same family. The construction is presented in the next section.

In Definition 3.6 we transform a single object \(L\) using \((L_1, b_1)\). Now we transform an nc family of objects \((L_2, b_2)\). Let’s define

\[
(3.4) \quad U := \mathcal{F}((L_1, b_1))((L_2, b_2)) := \left\{ A_1 \otimes (\Lambda^*)_{12} \ CF^*((L_1, L_2)) \otimes (\Lambda^*)_{12} \ A_2^{op}, d = (-1)^i m_1^{b_1, b_2}(\cdot) \right\}.
\]

For an algebra \(A\), recall that \(A^{op}\) is the opposite algebra which is the same as \(A\) as a set (and the corresponding elements are denoted as \(a^{op}\)), with multiplication \(a^{op} b^{op} := (ba)^{op}\). \(U\) is a (graded) \((A_1, A_2)\)-bimodule, where the right \(A_2\)-module structure on \(A_2^{op}\) is by taking \(a^{op}\). \(b := (ab)^{op} = b^{op} a^{op}\). The tensor product over \((\Lambda^*)_2\) and \((\Lambda^*)_1\) means that an element \(a_1 X a_2^{op}\) is non-zero only when the source of \(X\) matches with that of \(a_1\) and the target of \(X\) matches with that of \(a_2^{op}\).

Indeed, as a generalization of Step (5) to two algebras in Construction 3.4 we shall extend the whole Fukaya category over

\[
T(A_1, A_2) := \bigotimes_{k=0} \bigoplus_{|I|=k} A_{i_1} \otimes \ldots \otimes A_{i_k}
\]

where \(I = (i_1, \ldots, i_k)\) runs over multi-indices with entries in \(1, 2\) with no repeated adjacent entries. \(\bigoplus_{k=0} \bigotimes_{|I|=k}\) means that we take a completion over \(\Lambda\), meaning that we allow infinite series with valuation in \(\Lambda\) increasing to infinity. We think of this as the function algebra over the product. Moreover, we enlarge to include the objects \((L_1, b_1)\) and \((L_2, b_2)\).

**Definition 3.11.** The Fukaya category bi-extended over \(T(A_1, A_2)\) has the same objects as \(\text{Fuk}(M)\), and morphism spaces between any two objects \((L, L')\) are defined as \(T(A_1, A_2) \otimes \text{CF}(L, L') \otimes (T(A_1, A_2))^{op}\). The \(m_k\)-operations are defined by

\[
(3.5) \quad m_k(f_1 X_1 h_1^{op}, \ldots, f_k X_k h_k^{op}) := f_k \otimes \ldots \otimes f_1 m_k(X_1, \ldots, X_k) h_1^{op} \otimes \ldots \otimes h_k^{op},
\]

\[
= f_k \otimes \ldots \otimes f_1 m_k(X_1, \ldots, X_k) (h_1 \otimes \ldots \otimes h_k)^{op}.
\]

The enlarged Fukaya category has two more objects \((L_1, b_1)\) and \((L_2, b_2)\). The morphism spaces involving these objects are \(T(A_1, A_2) \otimes (\Lambda^*)_1 \otimes (\Lambda^*)_2 \ CF^*(L_i, L_j) \otimes (\Lambda^*)_1 \otimes (\Lambda^*)_2 \ (T(A_1, A_2) \otimes (\Lambda^*)_1) \ CF^*(L_1, L_2) \otimes (\Lambda^*)_1 \otimes (\Lambda^*)_2 \) for \(i, j = 1, 2\), and \(T(A_1, A_2) \otimes A_i \otimes (\Lambda^*)_1 \ CF^*(L, L) \ CF^*(L, L) \otimes (\Lambda^*)_1 \ (T(A_1, A_2) \otimes (\Lambda^*)_1)\). The \(m_k\) operations are extended like above. \(m_k^{b_1, b_2}\) is defined in the usual way, where \(b_1 \in A_{1} \otimes (\Lambda^*)_1 \ CF^*(L_1, L_1) \otimes (\Lambda^*)_1 \ A_1^{op}\) is in the form \(\Lambda^*\) (with non-trivial coefficients placed on the left; the coefficients on the right being simply 1).

It is easy that the extended \(m_k^{b_1, b_2}\) satisfy \(A_\infty\) equations. For notation simplicity, we will focus on the \(Z\)-graded situation where \(W^{(b_1, b_2)} = W^{(b_2, b_1)} = 0\). In particular, by the \(A_\infty\) equation for \(d_{b_1}^{i} := m_1^{b_1, b_2}\), \(d_{b_2}\) satisfies \(d_{b_2}^{2} = 0\).

Once we have extended and enlarged the Fukaya category, we can take further steps in (family) Yoneda embedding construction. We have two \(A_\infty\)-functors

\[
\mathcal{F}_{(1, b_1)} : \text{Fuk}(M) \to \text{dg}(A_1 \text{-mod})
\]

and

\[
\mathcal{F}_{(2, b_2)} : \text{Fuk}(M) \to \text{dg}(A_2 \text{-mod}).
\]

Moreover, we have the dg functor

\[
\mathcal{F}^U := \text{Hom}_{A_1}(U, -) : \text{dg}(A_1 \text{-mod}) \to \text{dg}(A_2 \text{-mod})
\]
where $\mathcal{U}$ is a complex of $(\mathcal{A}_1, \mathcal{A}_2)$-bimodules defined by (3.4). It takes $\text{Hom}_{\mathcal{A}_1}(\mathcal{U}, E)$ for each entry $E$ in a complex of $\mathcal{A}_1$-modules. We modify the signs as follows. The differential $(d_{\mathcal{F}^{\mathcal{U}}(E)}(\phi))$ is defined as $(-1)^{|\phi|}$ times the usual differential of $\phi$ as a homomorphism from $\mathcal{U}$ to $E$. Given $C, D \in \text{dg}(\mathcal{A}_1 - \text{mod})$, $f \in \text{Hom}_{\mathcal{A}_1}(C, D)$ and $\phi \in \text{Hom}_{\mathcal{A}_2}(\mathcal{U}, C)$,

$$
\mathcal{F}^{\mathcal{U}}(f)(\phi)(\cdot) = (-1)^{|f|}\cdot f \circ \phi(\cdot).
$$

We want to compare $\mathcal{F}^{[L_2, b_2]}$ and $\mathcal{F}^{[U, b_1]}$. For the computation in the following proof, we define the notation for simplicity:

$$
\sum_{i=1}^{r} : = \sum_{i=1}^{r} |\phi_i|'.
$$

**Theorem 3.12.** There exists a natural $\mathcal{A}_\infty$-transformation from $\mathcal{T}_1 = \mathcal{F}^{[L_2, b_2]}$ to $\mathcal{T}_2 = \mathcal{A}_2 \otimes (\mathcal{F}^{[U, b_1]}, \mathcal{A}_1)$.

**Proof.** First consider object level. Given an object $L$ of $\text{Fuk}(M)$, we have a morphism (of objects in $\text{dg}(\mathcal{A}_2 - \text{mod})$) from $\mathcal{F}^{[L_2, b_2]}(L) = \mathcal{A}_2 \otimes (\mathcal{A}_1^\circ \mathcal{A}_1^\circ \mathcal{A}_1, \text{CF}(L_2, L))$ to $\mathcal{A}_2 \otimes \mathcal{F}^{[U] \otimes [L_1, b_1]}(L) = \text{Hom}_{\mathcal{A}_2}(\mathcal{U}, \mathcal{A}_2 \otimes (\mathcal{A}_1^\circ \mathcal{A}_1, \text{CF}(L_1, L))$ (which is a left $\mathcal{A}_2$-module by the right multiplication of $\mathcal{A}_2$ on $\mathcal{U}$), given by

$$
\mathcal{T}_1(L) := (-1)^{|\phi|' - |L|'} R \left( m^{b_1, b_2, \circ}_{2, 0}(-, \phi) \right).
$$

On the RHS of the above expression, $m^{b_1, b_2, \circ}_{2, 0}(-, \phi) \in \mathcal{A}_2 \otimes (\mathcal{A}_1^\circ \mathcal{A}_1^\circ \mathcal{A}_1, \text{CF}(L_1, L) \otimes \mathcal{A}_2^\circ \mathcal{A}_1)$. The operator

$$
R : \mathcal{A}_2 \otimes (\mathcal{A}_1^\circ \mathcal{A}_1^\circ \mathcal{A}_1, \text{CF}(L_1, L) \otimes \mathcal{A}_2^\circ \mathcal{A}_1) \to \mathcal{A}_2 \otimes (\mathcal{A}_1^\circ \mathcal{A}_1^\circ \mathcal{A}_1, \text{CF}(L_1, L))
$$

moves an element $a_2^\circ \in \mathcal{A}_2^\circ$ on the right to $a_2$ multiplying on the left. More explicitly, let $pQq^\circ \in \mathcal{U}$ and $\phi = \phi_i X_i$ for $\phi_i \in \mathcal{A}_2$. Then $m^{b_1, b_2, \circ}_{2, 0}(pQq^\circ, \phi)$ takes the form

$$
m^{b_1, b_2, \circ}_{2, 0}(pQq^\circ, \phi) = \phi_i f_i (b_2) \otimes pg_i (b_1) \text{out}_i.
$$

where $f_i$ and $g_i$ are certain Novikov series. We get

$$
\left( m^{b_1, b_2, \circ}_{2, 0}(pQq^\circ, \phi) \right) = q\phi_i f_i (b_2) \otimes pg_i (b_1) \text{out}_i.
$$

Note that under scaling by $c \in \mathcal{A}_1$,

$$
R \left( m^{b_1, b_2, \circ}_{2, 0}(c pQq^\circ, \phi) \right) = c \cdot R \left( m^{b_1, b_2, \circ}_{2, 0}(pQq^\circ, \phi) \right)
$$

and so $\mathcal{T}_1(\phi)$ is an $\mathcal{A}_1$-module map. Under scaling by $c \in \mathcal{A}_2$,

$$
R \left( m^{b_1, b_2, \circ}_{2, 0}(pQq^\circ, c \phi) \right) = R \left( m^{b_1, b_2, \circ}_{2, 0}(pQq^\circ, \phi) \right)
$$

Thus $\mathcal{T}_1(c\phi) = c \cdot \mathcal{T}_1(\phi)$.

For morphisms and higher morphisms, let $L_0, \ldots, L_k$ be objects of $\text{Fuk}(M)$ and $\phi_1 \otimes \ldots \otimes \phi_k \in \text{CF}(L_0, L_1) \otimes \ldots \otimes \text{CF}(L_{k-1}, L_k)$. Then we have a corresponding morphism from $\mathcal{A}_2 \otimes (\mathcal{A}_1^\circ \mathcal{A}_1^\circ \mathcal{A}_1, \text{CF}(L_2, L_1))$ to $\text{Hom}_{\mathcal{A}_2}(\mathcal{U}, \mathcal{A}_2 \otimes (\mathcal{A}_1^\circ \mathcal{A}_1^\circ \mathcal{A}_1, \text{CF}(L_1, L_k)))$ given by

$$
\mathcal{T}(\phi_1; \ldots ; \phi_k)(\cdot) := (-1)^{|\phi_k|'} \sum_{i=1}^{r} R \left( m^{b_1, b_2, \circ}_{2, 0}(\cdot, \phi_1, \ldots, \phi_k) \right).
$$

(Recall that $\sum_{i=1}^{r} = \sum_{i=1}^{r} |\phi_i|'$ in (3.6).) For simplicity, let’s denote

$$
m^{b_1, b_2, \circ}_{2, 0} := R \circ m^{b_1, b_2, \circ}_{2, 0}.
$$
We want to check the equations for the $A_\infty$-natural transformation $\mathcal{T}$:

$$\delta \circ \mathcal{T}(\phi_1, \ldots, \phi_k)$$

$$= \sum_{r=0}^{k-1} (-1)^{r} \sum_{i=1}^{r} \mathcal{T}(\phi_{r+1}, \ldots, \phi_k) \circ \mathcal{T}(\phi_1, \ldots, \phi_r)$$

$$+ \sum_{r=0}^{k} \mathcal{T}(\phi_{r+1}, \ldots, \phi_k) \circ \mathcal{T}(\phi_1, \ldots, \phi_r)$$

$$- \sum_{r=0}^{k-1} \sum_{i=1}^{r} (-1)^{r-i} \mathcal{T}(\phi_1, \ldots, \phi_r, m_i(\phi_{r+1}, \ldots, \phi_{r+i}), \phi_{r+i+1}, \ldots, \phi_k) = 0.$$

For the first term, $\mathcal{T}(\phi_1, \ldots, \phi_k)(\phi) \in \text{Hom}_{A_}\text{L} \left(\cup, A_{2} \otimes A_1 \oplus (A_\text{A}), \text{CF}(L, L)\right)$, and $\delta$ is the differential on $\text{Hom}_{A_}\text{L} \left(\cup, A_{2} \otimes A_1 \oplus (A_\text{A}), \text{CF}(L, L)\right)$ defined by

$$(\delta \rho) := \rho \circ d^U + (-1)^{\rho'} d^L \circ \rho.$$ 

Thus the first term gives

$$\delta(\mathcal{T}(\phi_1, \ldots, \phi_k))(\phi)(\cdot) = (-1)^{\rho'} + \sum_{r=1}^{k} \left( m_{k+r} \cdot \phi \right)$$

$$= - \mathcal{T}(\phi_{r+1}, \ldots, \phi_k) \circ \mathcal{T}(\phi_1, \ldots, \phi_r)$$

$$= (-1)^{\rho'} + \sum_{r=1}^{k} \mathcal{T}(\phi_1, \ldots, \phi_r)$$

$$= - \sum_{r=0}^{k-1} \sum_{i=1}^{r} (-1)^{r-i} \mathcal{T}(\phi_1, \ldots, \phi_r, m_i(\phi_{r+1}, \ldots, \phi_{r+i}), \phi_{r+i+1}, \ldots, \phi_k) \rho(\cdot)$$

We compute the later terms as follows. First, $\mathcal{T}$ is in degree 0, and so $|\mathcal{T}'|= -1$.

Thus, it reduces to

$$(-1)^{\rho'} + \sum_{r=1}^{k} \left( m_{k+r} \cdot \phi \right)$$

$$+ \sum_{r=0}^{k} \sum_{i=1}^{r} (-1)^{r-i} \mathcal{T}(\phi_1, \ldots, \phi_r, m_i(\phi_{r+1}, \ldots, \phi_{r+i}), \phi_{r+i+1}, \ldots, \phi_k) \rho(\cdot)$$

which is the $A_\infty$ equation for $m_{k+r} = \sum_{r=0}^{k} \sum_{i=1}^{r} m_{k+r} \cdot \phi$, in the lemma below, with the common factor $(-1)^{\rho'} + \sum_{r=1}^{k} \left( m_{k+r} \cdot \phi \right)$. Thus, $\mathcal{T}$ is a natural transformation.
The operations of $m_k^{b_{ij},0,\ldots,0}$ and $R$ are carefully designed such that the following $A_\infty$ equation is satisfied.

**Lemma 3.13.** The operations $m_j^{b_{ij},0,\ldots,0} = R \circ m_j^{b_{ij},0,\ldots,0}$ (where $R$ is given in Equation (3.7)) satisfies the following $A_\infty$ equation for

$$\sum_{s=1}^{L} \sum_{r=1}^{l} (-1)^{s-r} |v_j|' m_{k-s}^{b_{ij},0,\ldots,0}, (v_1,\ldots,v_r-1, m_{s-r+1}^{b_{ij},0,\ldots,0}, (v_r,\ldots,v_s), v_{s+1},\ldots, v_k) = 0.$$ (3.9)

**Proof.** Let $v_j = y_j Q_j x_j^{op}$ for $j = 1,\ldots,l$ and $v_{l+1} = \phi X_{l+1}$, $v_j = X_j$ for $j = l+2,\ldots,k$, where $y_j \in A_{l-j+1}$, $x_j^{op} \in A_{l-j+1}^{op}$, $\phi \in \mathbb{A}_{l-j+1}$. For $s \leq l$, $m_{s-r+1}^{b_{ij},0,\ldots,0} (v_r,\ldots,v_s)$ takes the form

$$o(b_j) \otimes y_s o(b_{l-1}) \otimes \ldots \otimes y_1 o(b_0) \otimes m(\ldots,Q_r,\ldots,Q_s,\ldots) \otimes (x_r \otimes \ldots \otimes x_1)^{op}$$

where $o(b_j)$ are certain Novikov series in $b_j$. For $s > l$, $m_{s-r+1}^{b_{ij},0,\ldots,0} (v_r,\ldots,v_s)$ takes the form

$$(x_r \otimes \ldots \otimes x_1) \phi o(b_j) \otimes y_j o(b_{l-1}) \otimes \ldots \otimes y_1 o(b_0) \otimes m(\ldots,Q_r,\ldots,Q_s,\ldots).$$

We can check that all the terms in (3.9) have the general form

$$(x_1 \otimes \ldots \otimes x_1) \phi o(b_j) \otimes y_j o(b_{l-1}) \otimes \ldots \otimes y_1 o(b_0) \otimes m(\ldots,Q_r,\ldots,Q_s,\ldots) \otimes \ldots \otimes \ldots$$

Thus all terms have the same coefficient $(x_1 \otimes \ldots \otimes x_1) \phi o(b_j) \otimes y_j o(b_{l-1}) \otimes \ldots \otimes y_1 o(b_0)$ and the result follows from the usual $A_\infty$ equation without this common coefficient. 

Now we have an $A_\infty$-transformation from $\mathcal{F}^{[l_2,b_{21}]}$ to $A_{l_2} \otimes (\mathcal{F}^{l_1,b_{21}})$. If we fix a representation $G_{12}$ of $A_{l_2}$ over $A_{l_1}$, then the $A_\infty$-transformation can be made to $μ_{21}^{op} (x^{(2)} \otimes x^{(1)}) = x^{(1)} G_{12} (x^{(2)})$, and take the composition

$$\mathcal{M}_{21}^{op} \circ R \circ m_{k+2}^{b_{12},b_{21},0,\ldots,0}$$

in place of $R \circ m_{k+2}^{b_{12},b_{21},0,\ldots,0}$ in the definition of natural transformation (3.8). For instance, in the notation in the proof of Theorem 3.12

$$R \left( m_{k+2}^{b_{12},b_{21},0,\ldots,0} (pQ^{op}, \phi) \right) = q \phi f_1 (b_2) \otimes p g_1 (b_1) \text{ out}_i.$$ 

Then

$$\mathcal{M}_{21}^{op} \left( R \left( m_{k+2}^{b_{12},b_{21},0,\ldots,0} (pQ^{op}, \phi) \right) \right) = p g_1 (b_1) G_{12} (q \phi f_1 (b_2)) \text{ out}_i.$$ 

The scaling by $c \in A_{l_1}$ left on $p$ or $c \in A_{l_2}$ left on $\phi$ (or right on $Q^{op}$) enjoys the same nice properties as in the proof of Theorem 3.12. (If we used $\mathcal{M}_{21}$ instead, then it would be no longer $A_{l_1}$-linear on $p$.) The $A_\infty$ equation for $(L_1,\ldots,L_{l_2},L_{l_1},\ldots,L_k)$ continues to hold. In this way, we get an $A_\infty$ natural transformation from $\mathcal{F}^{[l_2,b_{21}]}$ to $\mathcal{F}^{l_1,b_{21}}$. 

Similarly, in the reverse direction, if we fix a representation $G_{21}$ of $\mathbb{A}_1$ over $\mathbb{A}_2$, then we have a natural $A_{\infty}$-transformation from $\mathcal{F}(L_1,b_1)$ to $\mathcal{F}^* \circ \mathcal{F}(L_2,b_2)$, where $\cup^* = \mathcal{F}(L_2,b_2)$ ($L_1, b_1$). Then we can compose the natural transformations

$$\mathcal{F}(L_2,b_2) \rightarrow \mathcal{F}^* \circ \mathcal{F}(L_1,b_1) \rightarrow \mathcal{F}^* \circ \mathcal{F}(L_2,b_2)$$

of functors from $\text{Fuk}(M)$ to $\text{dg}(\mathbb{A}_2 - \text{mod})$.

Given $\alpha \in \cup$ and $\beta \in \cup^*$, we have the evaluation natural transformation $\text{ev}_{(\alpha, \beta)} : \mathcal{F}^* \circ \mathcal{F}(L_2,b_2) \rightarrow \mathcal{F}(L_2,b_2)$. By composing all of these, we get a self natural transformation on $\mathcal{F}(L_2,b_2)$.

To go further, we consider a part of the setup in Section 2.2. Namely, suppose the representations $G_{12}$ and $G_{21}$ satisfy

$$(3.10) \quad G_{12} \circ G_{21}(a) = c_{121}(h_a) \cdot \alpha \cdot c_{212}^{-1}(t_a) \text{ and } G_{21} \circ G_{12}(a) = c_{212}(h_a) \cdot \alpha \cdot c_{121}^{-1}(t_a)$$

where $c_{121}(v) \in \{ e_{G_{12}(G_{21}(v))} \cdot \mathcal{A}_1 \cdot e_{v'} \}^*$ and $c_{212}(v') \in \{ e_{G_{21}(G_{12}(v))} \cdot \mathcal{A}_2 \cdot e_v \}^*$ for every $v \in Q_0^{(1)}$ and $v' \in Q_0^{(2)}$. Recall that we have defined the multiplication $\mathcal{M}_{l \to i} : \mathcal{A}_l \otimes \cdots \otimes \mathcal{A}_0 \rightarrow \mathcal{A}_0$ using $G_{12}$ and $G_{21}$ by (2.17). Then define

$$\mathcal{M}^{b_0 \ldots b_j}_{j} \circ \mathcal{M}^{b_0 \ldots b_i}_{i} = \mathcal{M} \circ \mathcal{M}^{b_0 \ldots b_i}_{i} \circ \mathcal{M}^{b_0 \ldots b_j}_{j} \circ \mathcal{M}^{b_0 \ldots b_1}_{1} \circ \mathcal{M}^{b_0 \ldots b_0}_{0}.$$

Explicitly, they take the form

$$\mathcal{M}^{b_0 \ldots b_j}_{j} \circ \mathcal{M}^{b_0 \ldots b_i}_{i} = \mathcal{M}^{b_0 \ldots b_j}_{j} \circ \mathcal{M}^{b_0 \ldots b_i}_{i} = \mathcal{M}^{b_0 \ldots b_i}_{i} \circ \mathcal{M}^{b_0 \ldots b_j}_{j} \circ \mathcal{M}^{b_0 \ldots b_1}_{1} \circ \mathcal{M}^{b_0 \ldots b_0}_{0}.$$

**Theorem 3.14.** The operations $\mathcal{M}^{b_0 \ldots b_j}_{j}$ and $\mathcal{M}^{b_0 \ldots b_i}_{i}$ satisfies the following $A_{\infty}$ equation for

$$(\mathcal{M}^{b_0 \ldots b_j}_{j} \circ \mathcal{M}^{b_0 \ldots b_i}_{i} \circ \mathcal{M}^{b_0 \ldots b_1}_{1} \circ \mathcal{M}^{b_0 \ldots b_0}_{0} :$$

$$= \mathcal{M}^{b_0 \ldots b_i}_{i} \circ \mathcal{M}^{b_0 \ldots b_j}_{j} \circ \mathcal{M}^{b_0 \ldots b_1}_{1} \circ \mathcal{M}^{b_0 \ldots b_0}_{0}.$$

**Proof.** As in the proof of Lemma 3.13 Let $v_j = y_j Q_j x_j^\text{op}$ for $j = 1, \ldots, l$ and $v_{l+1} = \phi X_l + 1$, $v_j = X_j$ for $j = l + 2, \ldots, k$, where $y_j \in \mathcal{A}_{i_{j-1}}$, $x_j^\text{op} \in \mathcal{A}_{i_j}^\text{op}$, $\phi \in \mathcal{A}_{i_l}$. The summands in the first term take the form

$$\mathcal{M} \circ \mathcal{M}^{b_0 \ldots b_j}_{j} \circ \mathcal{M}^{b_0 \ldots b_i}_{i} \circ \mathcal{M}^{b_0 \ldots b_1}_{1} \circ \mathcal{M}^{b_0 \ldots b_0}_{0} \circ m(\ldots, Q_1, \ldots, Q_{i-l}, \ldots, Q_{i+1}, \ldots, Q_{i+2}, \ldots, Q_l)$$

The summands in the second term take the form

$$\mathcal{M} \circ \mathcal{M}^{b_0 \ldots b_j}_{j} \circ \mathcal{M}^{b_0 \ldots b_i}_{i} \circ \mathcal{M}^{b_0 \ldots b_1}_{1} \circ \mathcal{M}^{b_0 \ldots b_0}_{0} \circ m(\ldots, Q_1, \ldots, Q_{i-l}, \ldots, Q_{i+1}, \ldots, Q_{i+2}, \ldots, Q_l).$$
By Proposition 2.25 in both cases, all the coefficients equal to
\[ A^{op} \{ x_1 \otimes \cdots \otimes x_t \phi_0(b) \otimes y_1 \otimes \cdots \otimes y_1 \otimes \phi_0(b_0) \}. \]
Then the result follows from the usual $A_\infty$ equation without this common coefficient.

Now we go back to the self natural transformation on $\mathcal{F}^{(l_2,b_2)}$ by composing the natural transformations
\[ \mathcal{F}^{(l_2,b_2)} \to \mathcal{F}^{(l_1,b_1)} \to \mathcal{F}^{(l_2,b_2)} \]
of functors from $\text{Fuk}(M)$ to $dg(\mathcal{A}_2 - \text{mod})$. The last one is by evaluation at $\alpha \in U$ and $\beta \in U^*$.

**Theorem 3.15.** Suppose $\alpha, \beta \in U$ and $\alpha, \beta \in U^*$ are of degree 0 satisfy $m_1^{b_1,b_2}(\alpha) = 0$, $m_2^{b_1,b_2}(\beta) = 0$, and $m_2^{b_2,b_1}(\beta, \alpha) = 1_{l_2}$. Then the natural transformation $\mathcal{F}^{(l_2,b_2)} \to \mathcal{F}^{(l_1,b_1)} \otimes \mathcal{F}^{(l_2,b_2)}$ has a left inverse, i.e.
\[ \mathcal{F}^{(l_2,b_2)} \to \mathcal{F}^{(l_2,b_2)} \]
is homotopic to the identity natural transformation.

**Proof.** Using the isomorphism $T(\mathcal{A}_1, \mathcal{A}_2) \cong \mathcal{A}_1$, we have a natural transformation $\mathcal{T}_{12} : \mathcal{F}^{(l_2,b_2)} \to \mathcal{F}^{(l_1,b_1)}$ and $\mathcal{T}_{21} : \mathcal{F}^{(l_1,b_1)} \to \mathcal{F}^{(l_2,b_2)}$. We want to show that the above composition
\[ \mathcal{F} := ev_{\alpha,\beta} \circ \mathcal{F}^{(l_2,b_2)} \circ \mathcal{T}_{12}, \]
is homotopic to the identity natural transformation $\mathcal{F}$ on $\mathcal{F}^{(l_2,b_2)}$.

First, in the object level, we need to show that $\mathcal{F}_L$ for a Lagrangian $L$, which is an endomorphism on $\mathcal{F}^{(l_2,b_2)}(L) = \mathcal{A}_2 \otimes (\mathcal{A}_2)^{op} \text{CF}(L_2, L)$, equals to the identity up to homotopy. For $\phi \in \mathcal{A}_2 \otimes (\mathcal{A}_2)^{op} \text{CF}(L_2, L)$,
\[ \mathcal{F}_L(\phi) = \bar{m}_2^{b_2,b_2,0}(\alpha, \phi) = \bar{m}_3^{b_2,b_2,0}(\alpha, \phi) + \bar{m}_3^{b_2,b_2,0}(\alpha, \phi) + \bar{m}_3^{b_2,b_2,0}(\alpha, \phi) = \bar{m}_3^{b_2,b_2,0}(\alpha, \phi).
\]
In the second line, we have used the $A_\infty$ equations by Theorem 3.14 with the terms $\bar{m}_1^{b_1,b_2}(\alpha)$ and $\bar{m}_1^{b_2,b_1}(\beta)$ vanish. We define
\[ \mathcal{H}_L := \bar{m}_3^{b_2,b_2,0}(\beta, \alpha, -) \]
as an endomorphism on $\mathcal{F}^{(l_2,b_2)}(L)$, and it is extended as a self pre-natural transformation on $\mathcal{F}^{(l_2,b_2)}$ by defining $\mathcal{H}(\phi_1, \ldots, \phi_k) : \mathcal{F}^{(l_2,b_2)}(L_0) \to \mathcal{F}^{(l_2,b_2)}(L_k)$ for $\phi_1 \otimes \cdots \otimes \phi_k \in \text{CF}(L_0, L_1) \otimes \cdots \otimes \text{CF}(L_{k-1}, L_k)$ to be
\[ \mathcal{H}(\phi_1, \ldots, \phi_k) := (-1)^{k+1} \bar{m}_3^{b_2,b_2,0}(\beta, \alpha, -, \phi_1, \ldots, \phi_k). \]
Then in the morphism level, for $\phi_1 \otimes \ldots \otimes \phi_k \in \text{CF}(L_0, L_1) \otimes \ldots \otimes \text{CF}(L_{k-1}, L_k)$ ($k \geq 1$),

$$\mathcal{F} (\phi_1, \ldots, \phi_k)(\phi) = \sum_{r=0}^{k} (-1)^k \sum_{i=1}^{k} \sum_{j=1}^{r} \left( \mathcal{F}_{r-j} \mathcal{H}_L (\phi_{r+1}, \ldots, \phi_k) \circ \mathcal{F}^{(r-r)} \mathcal{F}^{(l-1)(2, b_2)} (\phi_1, \ldots, \phi_r)(\phi) \right),$$

where $\mathcal{F}^{(l-1)(2, b_2)} (\phi_1, \ldots, \phi_r)(\phi)$ is the differential of the pre-natural transformation evaluated on $\phi_1 \otimes \ldots \otimes \phi_r$. This shows that $\mathcal{F} - \mathcal{F}''$ equals to the differential of $\mathcal{H}_L$. □

In some perfect cases, $\mathcal{F}^{(l-1)(2, b_2)}$ is naturally equivalent to $\mathcal{F}^{U} \circ \mathcal{F}^{(l-1)(2, b_2)}$.

**Theorem 3.16.** If $\mathcal{U}$ is a projective resolution, $\mathcal{F}^{(l-1)(2, b_2)}$ is naturally equivalent to $\mathcal{F}^{U} \circ \mathcal{F}^{(l-1)(2, b_2)}$.

**Proof.** The strategy is to show the natural transformation

$$\mathcal{F}^{U} \circ \mathcal{F}^{(l-1)(2, b_2)} \to \mathcal{F}^{U} \circ \mathcal{F}^{(l-1)(2, b_2)} \to \mathcal{F}^{(l-1)(2, b_2)} \to \mathcal{F}^{U} \circ \mathcal{F}^{(l-1)(2, b_2)}$$

is homotopic to the identity on $\mathcal{F}^{U} \circ \mathcal{F}^{(l-1)(2, b_2)}$. Combine with the previous theorem, we could get the desired equivalence.

Let $\mathcal{F}^{'} := \mathcal{F}_{12} \circ \mathcal{F}^{U}_{\alpha, b_2} \circ \mathcal{F}^{(l-1)(2, b_2)}$ (13). For each object $L$, we need to show that $\mathcal{F}^{'}_{L}$, which is an endomorphism on $\mathcal{F}^{U} \circ \mathcal{F}^{(l-1)(2, b_2)} (L) = \mathcal{U} \otimes \mathcal{A}_{\alpha, b_2} \otimes \mathcal{F}^{(l-1)(2, b_2)}$, is a quasi-isomorphism.

For $\phi \in \mathcal{A}_{\alpha, b_2} \otimes \mathcal{F}^{(l-1)(2, b_2)}$, denote the universal bundle by $U = (A^* \otimes \mathcal{A}_{\alpha, b_2} \otimes \mathcal{F}^{(l-1)(2, b_2)}), \mathcal{U}^*$ be its dual, i.e. $U^* := (A^* \otimes \mathcal{A}_{\alpha, b_2} \otimes \mathcal{F}^{(l-1)(2, b_2)}),$ then

$$\mathcal{F}^{'}_{L} (A^* \otimes \phi) = \mathcal{F}^{U}_{12} \circ \mathcal{F}^{U}_{\alpha, b_2} (A^* \otimes \mathcal{F}^{(l-1)(2, b_2)} (\phi) \otimes \mathcal{F}^{(l-1)(2, b_2)} (\phi)) = \mathcal{F}^{U}_{12} \circ \mathcal{F}^{U}_{\alpha, b_2} (\mathcal{F}^{U}_{\alpha, b_2} (\phi),) = A^* \otimes \mathcal{F}^{(l-1)(2, b_2)} (\phi),$$

where $\alpha = a_0 \cdot A^0$. Since $A^*$ is a resolution, the cohomology of $A^* \otimes \phi$ is generated by the class of $[A^0 \otimes \phi]$. Similarly for $A^* \otimes \mathcal{F}^{(l-1)(2, b_2)} (\phi)$, the total cohomology is generated by $[A^0 \otimes \mathcal{F}^{(l-1)(2, b_2)} (\phi)] = [A^0 \otimes \mathcal{F}^{(l-1)(2, b_2)} (\phi)].$
However, by the $A_\infty$ equations in Theorem 3.14
\[
[A^{0*} \otimes \tilde{m}^{b_1, b_2, 0}(\alpha, \tilde{m}^{b_1, b_2, 0}(\beta, \phi))]
\]
\[= [A^{0*} \otimes \tilde{m}^{b_1, b_2, 0}(\tilde{m}^{b_1, b_2, b_1}(\alpha, \beta, \phi) + \tilde{m}^{b_1, b_2, b_1}(\alpha, \beta, m_1^{b_1, 0}(\phi)) + m_1^{b_1, 0}(\tilde{m}^{b_1, b_2, b_1}(\alpha, \beta, \phi)))]
\]
\[= [A^{0*} \otimes \tilde{m}^{b_1, b_2, 0}(\hat{1}_{\pm}, \phi) + A^{0*} \otimes (\mathcal{H}_L^\prime \circ d_{\mathcal{F}}^{\prime, 1, b_1}(\hat{1}_{\pm}) \circ \phi) + (-1)^{|\phi|} d_{\mathcal{F}}^{\prime, 1, b_1}(\hat{1}_{\pm}) \circ \mathcal{H}_L^\prime(\phi)]
\]
\[= [A^{0*} \otimes \phi + A^{0*} \otimes (\mathcal{H}_L^\prime \circ d_{\mathcal{F}}^{\prime, 1, b_1}(\hat{1}_{\pm}) \circ \phi) + (-1)^{|\phi|} d_{\mathcal{F}}^{\prime, 1, b_1}(\hat{1}_{\pm}) \circ \mathcal{H}_L^\prime(\phi)]
\]
\[= [A^{0*} \otimes \phi].
\]
In the second line, we have used the $A_\infty$ equations by Theorem 3.14 with the terms $\tilde{m}^{b_1, b_2, b_1}(\alpha, \beta, \phi)$ vanish. And we define
\[
\mathcal{H}_L^\prime := \frac{m^{b_1, b_2, b_1, 0}}{m^{b_1, b_2, b_1}}(\alpha, \beta, -)
\]
as an endomorphism on $\mathcal{F}^{(\hat{1}_{\pm}, b_1)}(L)$. Hence, $\mathcal{F}_L^\prime : \mathcal{F}^\prime \circ \mathcal{F}^{(\hat{1}_{\pm}, b_1)}(L) \to \mathcal{F}^\prime \circ \mathcal{F}^{(\hat{1}_{\pm}, b_1)}(L)$ is a quasi-isomorphism.

With the previous theorem, we get the expected derived equivalence.\]

This also motivates the gluing construction via isomorphisms in the next section. We shall consider an algebroid stack in place of $\mathcal{A}_2$ in the rest of this paper.

3.3. Mirror algebroid stacks. In the last section, we have enlarged the Fukaya category by two families of nc deformed Lagrangians. It naturally generalizes to $n$ families. For the purpose of gluing in this section, we put all the coefficients on the left. Let $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be compact spin oriented immersed Lagrangians. We denote their algebras of unobstructed nc deformations by $\mathcal{A}_i$. We have
\[
T(\mathcal{A}_1, \ldots, \mathcal{A}_n) := \bigoplus_{m\geq 0} \bigoplus_{|i| = m} (\mathcal{A}_{i_0} \otimes \cdots \otimes \mathcal{A}_{i_m})
\]
that is understood as a product of the deformation spaces. The space of Floer chains and $A_\infty$ operations have been extended over $T(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. Namely, For two Lagrangians $L_0, L_1$ that are not any of these $\mathcal{L}_i$’s, the morphism space is $T(\mathcal{A}_1, \ldots, \mathcal{A}_n) \otimes \mathcal{CF}(L_0, L_1)$. The morphism spaces involving $(\mathcal{L}_i, b_j)$ are extended as $(\mathcal{A}_i \otimes T(\mathcal{A}_1, \ldots, \mathcal{A}_n) \otimes \mathcal{A}_j) \otimes (\mathcal{A}_1, b_j, \mathcal{A}_2, \ldots, \mathcal{A}_n) \otimes \mathcal{CF}^*(\mathcal{L}_1, L_1, L) \otimes \mathcal{CF}^*(\mathcal{L}_2, L_1, L) \otimes \mathcal{CF}^*(\mathcal{L}_3, L_1, L) \otimes \cdots \otimes \mathcal{CF}^*(\mathcal{L}_n, L_1, L)$. All coefficients are pulled to the left according to (3.2). This is analogous to Definition 3.11.

In this section, we would like to construct mirror algebroid stacks out of $(\mathcal{L}_j, b_j)$ for $i = 1, \ldots, n$. Naively, for every $k \neq j$, we want to find $a_{j, k} \in (\mathcal{A}_k \otimes \mathcal{A}_j) \otimes (\Lambda^*)_k \otimes (\Lambda^*)_j \otimes \mathcal{CF}^0(\mathcal{L}_j, \mathcal{L}_k)$ that satisfies
\[
m_1^{b_j, b_k}(\alpha_{j, k}) = 0,
\]
\[
m_2^{b_j, b_k, b_1}(\alpha_{j, k}, \alpha_{k, l}) = a_{j, l},
\]
\[
m_p^{b_j, \ldots, b_p}(\alpha_{i_1, \ldots, i_p}, \alpha_{i_{p-1}, l}) = 0 \text{ for } p \geq 3.
\]
We set $a_{j, j} = 1_{\mathcal{L}_j}$. Indeed, we can make a version that allows homotopy terms in the second equation, namely, the two sides are allowed to differ by $m_1^{b_j, b_k}(\gamma_{j, k})$ for some $\gamma_{j, k} \in (\mathcal{A}_j \otimes \mathcal{A}_j) \otimes (\Lambda^*)_j \otimes \mathcal{CF}^{-1}(\mathcal{L}_j, \mathcal{L}_k)$. (Similarly, we can also allow homotopy terms
in the third equation.) Such a system of equations of isomorphisms is a natural generalization of the equations $m_{i,j}^{b_i,b_j}(a_{j,k}) = 0$ and $m_{i,j}^{b_i,b_j}(a_{j,k}, a_{k,j}) = 1_{\mathcal{A}_j}$ raised and studied in [CHL17, HKL] in the two-chart case and before noncommutative extensions.

However, solving for $\alpha_{i,j}$ inside $(\mathcal{A}_j \oplus \mathcal{A}_i) \otimes (\mathcal{A}_j \oplus (\mathcal{A}_i)) \text{CF}^0(\mathcal{L}_j, \mathcal{L}_i)$ is not the right thing to do. $\mathcal{A}_j \oplus \mathcal{A}_i$ plays the role of a product. On the other hand, we want to find gluing between the charts so that the isomorphism equations hold over the resulting manifold, rather than over the product of the charts. To do so, we need to extend Fukaya category over an algebroid stack (in a modified version defined in Section 2.2).

To begin with, let’s motivate by the case of two charts. Given a representation $G_{ij}$ of $\mathcal{A}_i^{\text{loc}}$ over $\mathcal{G}_j$ and representation $G_{ij}$ of $\mathcal{A}_j^{\text{loc}}$ over $\mathcal{G}_j$ that satisfy (3.10), where $\mathcal{A}_i^{\text{loc}}$, $\mathcal{A}_j^{\text{loc}}$ are certain localizations of $\mathcal{A}_i$, $\mathcal{A}_j$ respectively, we can define $m_{i,j}^{b_i,b_j}$ with target in $\mathcal{A}_j^{\text{loc}} \otimes (\mathcal{A}_i \oplus (\mathcal{A}_i)) \text{CF}^0(\mathcal{L}_j, \mathcal{L}_i)$ by using

$$m_{i,j}^{b_i,b_j} = a_j \otimes a_i = a_j \cdot G_{ij}(a_i).$$

This is how we make sense of Equation (3.12). For higher $m_k$ operations, we need to use the multiplication defined by (2.17).

Let’s first state simple and helpful lemmas that follow directly from the definition of extended $m_k$-operations.

**Lemma 3.17.** Suppose $\phi \in (\mathcal{A}_k \cdot e_{i_1}^Q \otimes e_{i_0}^Q) \otimes (A_{i_k} \otimes (A_{i_k})) \text{CF}^*(\mathcal{L}_j, \mathcal{L}_k)$, where $e_{i_1}^Q$ and $e_{i_0}^Q$ are the trivial paths at the $i_1$-vertex in $Q_k$ and $i_0$-vertex in $Q_j$ respectively. Then the coefficient of each output $P \in \text{CF}^*(\mathcal{L}_j, \mathcal{L}_k)$ in $m_{i_1}^{b_i,b_j}(\phi)$ belongs to $e_{h_{i_{1}}} \cdot \mathcal{A}_k \cdot e_{i_1}^Q \otimes e_{i_0}^Q \cdot \mathcal{A}_j \cdot e_{i_{1}}^Q$.

Similarly, let in addition that $\psi \in (\mathcal{A}_j \cdot e_{i_{2}}^Q \otimes e_{i_{2}}^Q) \otimes (A_{i_k} \otimes (A_{i_k})) \text{CF}^*(\mathcal{L}_j, \mathcal{L}_i)$, Then the coefficient of each output $P \in \text{CF}^*(\mathcal{L}_j, \mathcal{L}_k)$ in $m_{i_2}^{b_i,b_j}(\phi, \psi)$ belongs to $e_{h_{i_{2}}} \cdot \mathcal{A}_i \cdot e_{i_{2}}^Q \otimes e_{i_{2}}^Q \cdot \mathcal{A}_j \cdot e_{i_{2}}^Q$.

**Lemma 3.18.** The map (3.15) restricted to $\mathcal{A}_j^{\text{loc}} \cdot e_{i_{2}}^Q \otimes e_{h_{i_{2}}}^{Q(i)}$ in $\mathcal{A}_j^{\text{loc}}$ is non-zero only if $e_{i_{2}}^{Q(i)} = G_{ij}(e_{i_{2}}^{Q(i)})$, where $t$ and $h$ are certain fixed vertices in $Q^{(i)}$ and $Q^{(i)}$ respectively. In particular, if $Q^{(i)}$ consists of only one vertex, then $G_{ij}$ takes image in the loop algebra of $\mathcal{A}_j^{\text{loc}}$ at the vertex $t$.

Now consider the general case. Suppose an algebroid stack $\mathcal{X}$ (in the version of Section 2.2) is given, where the charts $\mathcal{A}_j$ over $U_j$ are given by the quiver algebras of $\mathcal{L}_j$ and their localizations. We can simplify by fixing a base vertex $v_j$ for each $Q^{(i)}$ (although this is not a necessary procedure). Then we take

$$\alpha_{j,k} \in \left(\mathcal{A}_k^{\text{loc}} \cdot e_{i_{2k}}^Q \otimes e_{v_j}^{Q(i)} \cdot \mathcal{A}_j^{\text{loc}}\right) \otimes (\mathcal{A} \oplus (\mathcal{A})) \text{CF}^0(\mathcal{L}_j, \mathcal{L}_k)$$

and its corresponding image in $\mathcal{A}_k^{\text{loc}} \otimes (\mathcal{A} \oplus (\mathcal{A})) \text{CF}^0(\mathcal{L}_j, \mathcal{L}_k)$ (which is also denoted by $\alpha_{j,k}$ by abuse of notation). $(\mathcal{A})$ acts on $\mathcal{A}^{\text{loc}}$ via $G_{kj}$. By Lemma 3.18, we should only consider algebroid stacks whose transition maps satisfy $e_{i_{2k}}^{Q(i)} = G_{kj}(e_{v_j}^{Q(i)})$. $\alpha_{j,k} \in \left(\mathcal{A}_k^{\text{loc}} \cdot e_{i_{2k}}^Q \otimes e_{v_j}^{Q(i)} \cdot \mathcal{A}_j^{\text{loc}}\right) \otimes (\mathcal{A} \oplus (\mathcal{A})) \text{CF}^0(\mathcal{L}_j, \mathcal{L}_k)$ induces an element in $\mathcal{A}_j^{\text{loc}} \otimes (\mathcal{A} \oplus (\mathcal{A})) \text{CF}^0(\mathcal{L}_j, \mathcal{L}_k)$ which is again denoted by $\alpha_{j,k}$. 


Define

\[ \text{CF}(\ell_0^{(p)}, L_1^{(q)}) := \mathcal{A}_p(U_{pq}) \otimes \text{CF}(L_0, L_1), \]
\[ \text{CF}((\mathcal{L}, b_j), L_i^{(p)}) := \mathcal{A}_j(U_{jp}) \otimes (\Lambda^*)_j \text{CF}(\mathcal{L}, L_i), \]
\[ \text{CF}(L_0^{(p)}, (\mathcal{L}, b_j)) := \mathcal{A}_p(U_{jp}) \otimes (\Lambda^*)_j \text{CF}(L_0, \mathcal{L}), \]
\[ \text{CF}((\mathcal{L}, b_j), (\mathcal{L}_k, b_{jk})) := \mathcal{A}_j(U_{jk}) \otimes (\Lambda^*)_j \otimes (\Lambda^*)_j \text{CF}(\mathcal{L}, \mathcal{L}_k). \]

In above, \( L_0, L_1 \) denote Lagrangians that are not \((\mathcal{L}, b_j)\) for any \( j \). They are decorated with an index \( p \), meaning that they are treated over \( \mathcal{A}_p \). In the last line, \((\Lambda^*)_j\) left multiplies on \( \mathcal{A}_j|_{U_{jk}} \) via the representation \( G_{jk} \) of \( \mathcal{A}_j|_{U_{jk}} \) by \( \mathcal{A}_j|_{U_{jk}} \). (And similarly for the third line.) By restricting the sheaf of algebras over an open subset \( U \), we have the notion of \( \text{CF}_U \) (where \( U \) is a subset in the original domain, for instance \( U_{pq} \) in the first line).

By pulling all the coefficients to the left according to \( \ref{3.2} \) and multiplying using \( \ref{2.17} \), we have the operations

\[ m_{b_0,\ldots,b_k}^{b_0,\ldots,b_k} : \text{CF}(U_1(K_0, K_1) \otimes \cdots \otimes \text{CF}(U_k(K_{k-1}, K_k) \rightarrow \text{CF}(U_j(K_0, K_k)) \]

where \( K_i \) can be one of \((\mathcal{L}_{jk}, b_{jk})\) or other Lagrangians (in which case \( b_1 = 0 \) and \( K_i \) is decorated with an index of a chart which is denoted as \( \mathcal{A}_l \)).

**Theorem 3.20.** \( \{ m_{b_0,\ldots,b_k}^{b_0,\ldots,b_k} : k > 0 \} \) satisfies the \( A_{\infty} \) equations.

**Proof.** Recall the \( A_{\infty} \) equations for the original Fukaya category:

\[ \sum_{k_1+k_2=n+1}^{k_i} (-1)^{\epsilon_i} m_{k_1}(X_1,\ldots,m_{k_2}(X_{i+2-k_1-1},X_{i+2-k_2},\ldots,X_n) = 0 \]

where \( \epsilon_i = \sum_{j=1}^{i-1} |X_j|' \). Over \( T(\mathcal{A}_1(U_1,\ldots,n),\mathcal{A}_n(U_1,\ldots,n)) \), we have

\[ \sum_{k_1+k_2=n+1}^{k_i} (-1)^{\epsilon_i} m_{k_1}(b_0,\ldots,b_{i-1},b_{i+k_1-1},\ldots,b_{i+k_2-1},y_{i-1} \otimes x_{i-2} X_{i-1}, \]
\[ m_{b_0,\ldots,b_n}^{b_0,\ldots,b_n} (y_{i} \otimes x_{i-1} X_{i},\ldots,y_{i+k_2-1} \otimes x_{i+k_2-2} X_{i+k_2-1}) , y_{i+k_2} \otimes x_{i+k_2-1} X_{i+k_2-1}, \ldots,y_n \otimes x_{n-1} X_n \]
\[ \sum_{j=0}^{n+1-k_2} \sum_{l=1}^{n+1-l} (-1)^{\epsilon_i} m(B_0,\ldots,B_l,X_1,B_{l+1},\ldots,B_{l+k_2-1},B_{l+k_2},\ldots,B_{i+1},B_{i+2},\ldots,B_{i+k_2},\ldots,B_n), \]

which vanishes since the last two lines equal to zero. Here, we write \( b = \beta \cdot B \) in basis (understood as a linear combination) where \( |B|' = 0 \). The last summation above is over all the ways to split \( p_{i-1} \) copies of \( B_{i-1} \) into two sets, and \( p_{i+k_2-1} \) copies of \( B_{i+k_2-1} \) into two sets.
Then for the last expression, we multiply the coefficient for each \((p_0, \ldots, p_n)\) using \(2.17\), and we still have

\[
0 = \sum_{p_0, \ldots, p_n} \mathcal{M}_{b_0, \ldots, b_n} \left( \rho_n^{p_n} y_n \otimes \cdots \otimes x_1 \rho_1^{p_1} y_1 \otimes x_0 \rho_0^{p_0} \right)
\]

\[
\sum_{k=0}^{n+1} \sum_{l=1}^{n+1-k} (-1)^{i_l} \sum \left( m \left( B_0, \ldots, B_k, X_1, B_1, \ldots, X_l, B_l, \ldots, B_{n+1} \right), B_{l+2}, \ldots, B_{l+k-1}, B_{l+k}, \ldots, B_{n+1} \right).
\]

By Proposition \(2.23\) the coefficients equal to

\[
\mathcal{M}_{n, \ldots, l_i, l_j, \ldots, l_k} \left( \rho_n^{p_n} y_n \otimes \cdots \otimes x_l \rho_k^{p_k} y_k \otimes \cdots \otimes x_1 \rho_1^{p_1} y_1 \otimes \cdots \otimes x_0 \rho_0^{p_0} \right)
\]

\[
\mathcal{M}_{l_i, l_j, \ldots, l_k} \left( \rho_{l_i}^{p_{l_i}} y_{l_i} \otimes \cdots \otimes x_{l_j} \rho_{l_j}^{p_{l_j}} y_{l_j} \otimes \cdots \otimes x_{l_k} \rho_{l_k}^{p_{l_k}} y_{l_k} \otimes \cdots \otimes x_1 \rho_1^{p_1} y_1 \otimes \cdots \otimes x_0 \rho_0^{p_0} \right)
\]

where \(r_1 + r_2 = p_l + p_k\) and \(s_1 + s_2 = p_i, p_j, p_k\). By putting back the coefficients into the \(m_k\) operations, we obtain the \(A_\infty\) equations for \(m_{k, \chi}\).

**Remark 3.31.** We need to index the Lagrangians \(L_i\) by charts, since the multiplication \(2.17\) needs this information. \(b_i = 0\) for \(L_i\) not being any of \(\mathcal{L}_k\), but we still insert \(e^h = 1_{L_i}\) in the coefficient.

The following situation is particularly important for later use. Consider the sequence of Lagrangians \((\mathcal{L}_{q_0}, b_{q_0}), \ldots, (\mathcal{L}_{q_p}, b_{q_p}), L_{p+1}^{(q_1)} \cdots, L_{p}^{(q_1)}\), for \(i = p\). One of the terms in the corresponding \(A_\infty\) equation is

\[
b_{q_0, \ldots, q_p} \left( a_{q_1}, \ldots, a_{q_p}, L_{q_1}^{(q_1)} \cdots, L_{q_p}^{(q_1)} \right) = \psi(\chi) \cdot \text{out}
\]

for \(\psi(\chi) \in \mathcal{A}_{q_1}\) with \(h_{\psi(\chi)} = G_{q_1, q_1}(h_{q_1})\), and

\[
m_{q_0, \ldots, q_p} \left( a_{q_1}, \ldots, a_{q_p}, \text{out}, Q_{q_1}, \ldots, Q_p \right) = a_{q_0} \cdot \text{out}
\]

for \(a_{q_0} \in \mathcal{A}_{q_0}\). Then the above takes the form

\[
\mathcal{M}_{q_1, q_2} \left( e_{h_{q_1}}(\chi) \otimes e_{q_0}(\chi) \otimes a_{q_0} \right) \cdot \text{out} = G_{q_1, q_1}(h_{q_1})G_{q_0, q_1}(\psi(\chi))a_{q_0} = e_{q_0}^{-1}(h_{q_1})G_{q_0, q_1}(\psi(\chi))a_{q_0} = \phi(\psi(\chi) \otimes a_{q_0})
\]

where \(\phi(\cdot) := G_{q_1, q_1}(\cdot) a_{q_0} = m_{q_0, q_1, q_2} a_{q_0, \cdot, \cdot} (a_{q_1}, \ldots, a_{q_1}, \text{out}, Q_{q_1}, \ldots, Q_p), \) and \(\omega\) is defined by \(2.16\). This is the key ingredient in the proof of Theorem \(3.20\) later. (Note that we cannot get this if we take \(\mathcal{M}_{q_1, q_2} \cdot (\psi(\chi) \otimes a_{q_0})\) instead of \(\mathcal{M}_{q_1, q_2} \cdots (e_{h_{q_1}}(\chi) \otimes \psi(\chi) \otimes a_{q_0})\).)

Then Equation \(3.12\) and \(3.13\) are defined using \(m_{q_1, q_2} b_{q_1} b_{q_2}\) and \(m_{q_1, q_2} b_{q_1} b_{q_2}\). We can also use \(m_{q_1, q_2} b_{q_1} b_{q_2}\) to define an \(A_\infty\) functor from the Fukaya category to the dg category of twisted complexes over the algebroid stack.

We summarize our noncommutative gluing construction as follows.

**Construction 3.22.**

1. Fix a collection of spin oriented Lagrangian immersions \(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_N\).
2. Take their corresponding quivers \(Q^{(j)}\) of degree one endomorphisms, and algebras of weakly unobstructed deformations \(\mathcal{A}_j = \Lambda_\ast Q^{(j)} / R^{(j)}\).
3.4. Gluing noncommutative mirror functors. In this section, we construct the \( A_\infty \) functor

\[
\mathcal{F} L : \text{Fuk}(M) \rightarrow \text{Tw} L
\]

in object and morphism level, using the \( A_\infty \)-operations \( m^b_{k,\mathcal{X}} \) defined in the last section. The algebroid stack \( \mathcal{X} \) is constructed from a collection of Lagrangian immersions \( \mathcal{L} = \{ \mathcal{L}_1, \ldots, \mathcal{L}_N \} \) in last section. The construction is along the line of [CHI].

First, let's consider the object level. Given an object \( L \) in \( \text{Fuk}(M) \), we define the corresponding twisted complex \( \phi = \mathcal{F} L(L) \) on \( \mathcal{X} \) as follows. Over each chart \( U_i \), we take the complex

\[
\text{CF}(\mathcal{L}_1, b_1, \mathcal{L}_2, b_2, \ldots, \mathcal{L}_N, b_N), \phi_i = (-1)^{|i|} m^b_{i,\mathcal{X}}(-)
\]

Then the transition maps are defined by \( \phi_{ij}(\cdot) := m^b_{i,j,0} \). For \( \mathcal{F} L \), we define the corresponding\( \phi = \mathcal{F} L(L) \) on \( \mathcal{X} \) as follows. Over each chart \( U_i \), we take the complex

\[
\text{CF}(\mathcal{L}_1, b_1, \mathcal{L}_2, b_2, \ldots, \mathcal{L}_N, b_N), \phi_i = (-1)^{|i|} m^b_{i,\mathcal{X}}(-)
\]

Similarly, the higher maps \( \phi_{i_0 \ldots i_k} : \text{CF}(\mathcal{L}_{i_1}, \mathcal{L}_{i_2}, \ldots, \mathcal{L}_{i_k}) \rightarrow \text{CF}(\mathcal{L}_{i_1}, \mathcal{L}_{i_2}, \ldots, \mathcal{L}_{i_k}) \) for the twisted complex are defined by

\[
\phi_{i_0 \ldots i_k}(\cdot) := (-1)^{|i_0| - |i_k|} m^b_{i_0 \ldots i_k,0} \alpha_{i_0, i_1, \ldots, i_k}.
\]

Lemma 3.23. \( \phi \) above defines a twisted complex over \( \mathcal{X} \), namely, \( \phi \) is intertwining and it satisfies the Maurer-Cartan equation [2.5].

Proof. Since the coefficient of the input for \( \phi_{i_0 \ldots i_k} \) will be pulled out to the leftmost, and by the definition of \( \mathcal{M}_{i_0 \ldots i_k} \) [2.17], \( \phi_{i_0 \ldots i_k} \) is intertwining. The Maurer-Cartan quation for \( \phi \) follows from \( A_\infty \)-equations (Theorem 3.20 for \( (\alpha_{i_{01}}, \ldots, \alpha_{i_{k-1}k}, X) \), Namely,

\[
-(-1)^{|X|} (-1)^{p-1} m^b_{i_0 \ldots i_k,0} (\alpha_{i_0 i_1}, \ldots, \alpha_{i_{k-1}k}, X) = (-1)^{|X|} (-1)^p m^b_{i_0 \ldots i_k,0} (\alpha_{i_0 i_1}, \ldots, \alpha_{i_{k-1}k}, X) = (-1)^p \phi_{i_0 \ldots i_k}
\]

and

\[
-(-1)^{|X|} (-1)^{p} m^b_{i_0 \ldots i_k,0} (\alpha_{i_0 i_1}, \ldots, \alpha_{i_{k-1}k}, X) \alpha_{i_0 i_1}, \ldots, \alpha_{i_{k-1}k}, X) = (-1)^{|X|} (-1)^{p} m^b_{i_0 \ldots i_k,0} (\alpha_{i_0 i_1}, \ldots, \alpha_{i_{k-1}k}, X)
\]

Moreover, \( m^b_{i_0 \ldots i_k}(\alpha_{i_1, \ldots, i_{k-1}}) = 0 \) for \( k \neq 2 \) by [3.12] and [3.14]. The right hand sides of the above equations add up to the Maurer-Cartan equation for \( \phi \), while the LHS add up to zero by the \( A_\infty \) equation.

Next, let's consider the morphism level. For \( L, L' \) in \( \text{Fuk}(M) \) and \( Q \in \text{CF}(L, L') \), we want to define a morphism \( u = \mathcal{F} L(Q) : \mathcal{F} L(L) \rightarrow \mathcal{F} L(L') \). Over the charts \( U_i \), we define \( u_i(\cdot) := m^b_{i,0} \). Over \( U_{i_0 \ldots i_k} \), \( u_{i_0 \ldots i_k}(\cdot) := m^b_{i_0 \ldots i_k,0} \).

Similarly, given \( L_0, \ldots, L_p \) and morphisms \( Q_j \in \text{CF}(L_{j-1}, L_j) \), we define the higher morphism \( u = \mathcal{F}_L(Q_1, \ldots, Q_p) \) by

\[
u_{l_0, \ldots, l_k}(-) := (-1)^{k|\{l\}+S_p}m_{k+p+1,\mathcal{X}}^{b_{l_0}, \ldots, b_{l_k}, 0,0}(a_{l_0,i_1}, \ldots, a_{l_k,i_k}, -)Q_1, \ldots, Q_p),
\]

where \( S_p = \sum_{j=1}^{p} |Q_j|' \).

In the following computation, we denote \(|X|'\) by \( x \).

**Theorem 3.24.** The above defines an \( \infty \)-functor \( \mathcal{F}_L : \text{Fuk}(M) \to \text{Tw}(\mathcal{X}) \).

**Proof.** Consider the \( \infty \)-equation for \((a_{l_0,i_1}, \ldots, a_{l_k,i_k}, X, Q_1, \ldots, Q_p)\). It consists of terms

\[
(-1)^{k+x+S_{r-1}}b_{l_0, \ldots, b_{l_k}, 0,0}m_{k+1+(x-S_{r-1})}^{b_{l_0}, \ldots, b_{l_k}, 0,0}(a_{l_0,i_1}, \ldots, a_{l_k,i_k}, X, Q_1, \ldots, Q_t, Q_1, Q_1, \ldots, Q_1, Q_1, \ldots, Q_p).
\]

\[= (-1)\chi (k+x+S_{r-1}+1)(-1)^{k+x+S_{r-1}+x}m_{k+1+(x-S_{r-1})}^{b_{l_0}, \ldots, b_{l_k}, 0,0}(Q_1, l_0, \ldots, l_k, Q_1, \ldots, Q_1, Q_1, \ldots, Q_1, Q_1, \ldots, Q_p).\]

In the following computation, we denote \(|X|'\) by \( x \).

\[
\sum_{j=1}^{p} (-1)^{l}m_{k+p+1,\mathcal{X}}^{b_{l_0}, \ldots, b_{l_k}, 0,0}(a_{l_0,i_1}, \ldots, a_{l_k,i_k}, X, Q_1, \ldots, Q_p).
\]

Moreover, \( m_{k+1,\mathcal{X}}^{b_{l_0}, \ldots, b_{l_k}, 0,0}(a_{l_0,i_1}, \ldots, a_{l_k,i_k}, X, Q_1, \ldots, Q_p) = 0 \) for \( k \neq 2 \) by \( \text{[3.12]} \) and \( \text{[3.14]} \). With the common factor \(-1)\chi k+S_{r+1},1\), the right hand sides of the above equations add up to the equation for being an \( \infty \)-functor (keeping in mind that \( \text{Tw}(\mathcal{X}) \) is a dg category with no higher multiplication), while the LHS add up to zero by the \( \infty \)-equation.

\[
\text{3.5. Fourier-Mukai transform from an algebroid stack to an algebra.} \quad \text{Given a Lagrangian immersion } L, \text{ [CHL21] constructed an } \infty \text{-functor}
\]

\[
\text{Fuk}(M) \to \text{dg-mod}(A)
\]

where \( A \) is the quiver algebra associated to \( L \). (As in the last section, we have assumed that \( W = 0 \) for simplicity). On the other hand, for a collection of Lagrangian immersions
The transition maps of $A_\mathcal{X}$ and $a_{ij} \in \text{CF}((\mathcal{L}_i, b_i), (\mathcal{L}_j, b_j))$ that satisfy (3.12), (3.13) and (3.14). In this setting, we have constructed an $A_\infty$-functor

$$\text{Fuk}(M) \rightarrow \text{Tw}(\mathcal{X})$$

in the last section. We would like to compare these two functors. This is a natural extension of Section 3.2 for a transformation between two algebras.

We shall consider bimodules as in Section 3.2. Below is a combination of Definition 3.25 and Definition 3.11.

**Definition 3.25.** The enlarged Fukaya category bi-extended over $T := T(\mathcal{X})$ has objects in $\text{Fuk}(M)$ or $(\mathcal{L}_1, b_1), \ldots, (\mathcal{L}_N, b_N)$, and morphism spaces between any two objects $L, L'$ are defined as follows.

- $CL_i(L_0, L_1) := \{(\mathcal{X}_1, \mathcal{A}) \otimes \text{CF}(L_0, L_1) \otimes (\mathcal{X}_1, \mathcal{A}))^{op};$
- $CL_i(L, L_0, L_1) := \{(\mathcal{X}_1, \mathcal{A}) \otimes \mathcal{A} \otimes \Lambda^{\mathcal{X}_1} \text{CF}(L, L_0, L_1) \otimes (\mathcal{X}_1, \mathcal{A}))^{op};$
- $CL_i(L, b, L_0, L_1) := \{(\mathcal{X}_1, \mathcal{A}) \otimes \mathcal{A} \otimes \Lambda^{\mathcal{X}_1} \text{CF}(L, L_0, L_1) \otimes \Lambda^{\mathcal{X}_1})^{op};$
- $CL_i((\mathcal{L}_j, b), (\mathcal{L}_k, b)) := \{(\mathcal{X}_1, \mathcal{A}) \otimes \mathcal{A} \otimes \Lambda^{\mathcal{X}_1} \text{CF}(\mathcal{L}_j, \mathcal{L}_k) \otimes (\mathcal{X}_1, \mathcal{A}))^{op};$
- $CL_i((\mathcal{L}_j, b), (\mathcal{L}_k, b)) := \{(\mathcal{X}_1, \mathcal{A}) \otimes \mathcal{A} \otimes \Lambda^{\mathcal{X}_1} \text{CF}(\mathcal{L}_j, \mathcal{L}_k) \otimes (\mathcal{X}_1, \mathcal{A}))^{op};$

By pulling the coefficients to the left and right according to (3.5) and multiplying among $\mathcal{A}$ using $\mathcal{M}_{b_i...b_k}$ (2.17), we have the operations

$$m_{b_i...b_k} : \text{CF}(\mathcal{U}_i, K_0, K_1) \otimes \ldots \otimes \text{CF}(\mathcal{U}_i, K_{k-1}, K_k) \rightarrow \text{CF}((\mathcal{U}_i, K_0) \cap \ldots \cap (\mathcal{U}_i, K_{k-1}, K_k))(K_0, K_k)$$

where $K_i$ can be one of $(\mathcal{L}_j, b_j)$, $(\mathcal{L}_k, b_j)$, $(\mathcal{L}_j, b_k)$, $(\mathcal{L}_k, b_k)$ (in which case we set $j_{K_i} = 0$) or other Lagrangian (in which case $b_i = 0$ and $j_{K_i} = \emptyset$).

Similar to Theorem 3.20, $m_{b_i...b_k}$ satisfy $A_\infty$ equations.

**Definition 3.26.** The universal sheaf $\mathcal{U}$ is defined as $\mathcal{X}(\mathcal{L}, b)$, which is a twisted complex of right $\mathcal{A}$-modules over $\mathcal{X}$. Namely, over each chart $U_i$,

$$\mathcal{U}_i = \mathcal{A} \otimes \mathcal{X}_i \otimes (\Lambda^{\mathcal{X}_i} \text{CF}(\mathcal{L}_i, L) \otimes \Lambda^{\mathcal{X}_i})^{op}, \phi_i^{\mathcal{U}} := (-1)^{|i|} m_i^{b_i, b_i}.$$ 

The transition maps of $\mathcal{U}$ are defined by $\phi_{jij}^{\mathcal{U}}(-) := m_{2, \mathcal{X}, \mathcal{A}}^{b_i, b_j}(\mathcal{X}_i, \mathcal{A}(-)) \rightarrow \mathcal{U}_j((U_i)).$

Similarly, we have the higher maps $\phi_{j_{k-1}...b_{k-1}b_k}^{\mathcal{U}} : \mathcal{U}_{k+1} \rightarrow \mathcal{U}_k$ given by

$$\phi_{j_{k-1}...b_{k-1}b_k}^{\mathcal{U}}(-) := (-1)^{|k-1|-|i|} m_{k+1, \mathcal{X}}^{b_i...b_k}(\mathcal{X}_{i_{k-1}}, \ldots, \mathcal{X}_i, \mathcal{A}(-)).$$

Then we have the dg functor

$$\mathcal{F}^{\mathcal{U}} := \text{Hom}_{\mathcal{X}}(U, -) : \text{Tw}(\mathcal{X}) \rightarrow \text{dg}(\mathcal{A} \text{ mod}).$$
We modify the signs as follows. Given $\phi \in \text{Hom}_\mathcal{X}(U, E)$, its differential is given by

$$(d_{\mathcal{X}(U,E)} \phi) = (-1)^{\| \phi \|} d_{\mathcal{X}}(\phi)$$

where $d_{\mathcal{X}}$ is defined by 2.10. Given $C, D \in \text{dg}(\mathcal{X} - \text{mod})$, $f \in \text{Hom}_\mathcal{X}(C, D)$ and $\phi \in \text{Hom}_\mathcal{X}(U, C)$,

$$\mathcal{F}^U(f)(\phi)(-) = f \cdot \phi(-).$$

**Theorem 3.27.** There exists a natural $A_\infty$-transformation $\mathcal{T}$ from $\mathcal{F}_1 = \mathcal{F}^{(L,b)}_1$ to $\mathcal{F}_2 = A \otimes (\mathcal{F} \circ \mathcal{F}^L)$.

**Proof.** First consider object level. Given an object $L$ of $\text{Fuk}(M)$, we define the following morphism (of objects in $\text{dg}(\mathcal{A} - \text{mod})$)

$$\mathcal{F}^{(L,b)}(L) = A \otimes A L \text{ CF}(L, L) \rightarrow A \otimes \mathcal{F}^U \left( \mathcal{F}^L(L) \right) = \text{Hom}_\mathcal{X}(U, A \otimes \mathcal{F}^L(L)).$$

Over each chart $U_i$, for $\phi \in \mathcal{F}^{(L,b)}(L)$,

$$\mathcal{T}^L_{U_i}(\phi) := (-1)^{1(|\phi| + |\phi'|)} R \left( \sum_{m_{k+2,\mathcal{X},A}^0} m_{k+1,\mathcal{X},A}^0 (-, \phi) \right)$$

where $R$ is the operator that moves $A^0$ to the rightmost to $A$ to the leftmost, see 3.7. Over an intersection $U_{i_0 \cdots i_k}$,

$$\mathcal{T}^L_{U_{i_0 \cdots i_k}}(\phi) := (-1)^{k(|\phi| + |\phi'|)} R \left( \sum_{m_{k+2,\mathcal{X},A}^0} m_{k+1,\mathcal{X},A}^0 (-, \phi) \right)$$

In the above expression, all coefficients of $\alpha_{i_{j-1}i_j}$ and $\phi$ appear on the left (with coefficient on the right being 1); the only entry that can have non-trivial right-coefficients is the input $(-)$. As in the proof of Theorem 3.12, we denote

$$m_{k+2,\mathcal{X},A}^0 := R \circ m_{k+2,\mathcal{X},A}^1.$$

It satisfies an analogous $A_\infty$ equation as 3.9. Thus $\mathcal{T}^L_{U_{i_0 \cdots i_k}}$ is a chain map:

$$\sum_{j=1}^k (-1)^{j} m_{k+2,\mathcal{X},A}^0 (\alpha_{i_0i_1}, \ldots, m_{k+1,\mathcal{X},A}^0 (\alpha_{i_{j-1}i_j}, \alpha_{i_{j}i_{j+1}}, \ldots), \phi_1) = (-1)^{1+|\phi|} \mathcal{F}^L_{U_{i_0 \cdots i_k}}(\phi) = (-1)^{1+|\phi|} \mathcal{F}^L_{U_{i_0 \cdots i_k}}(\phi)$$

For morphisms and higher morphisms, let $L_0, \ldots, L_p$ be objects of $\text{Fuk}(M)$ and $\phi_1 \otimes \ldots \otimes \phi_p \in \text{CF}(L_0, L_1) \otimes \ldots \otimes \text{CF}(L_{p-1}, L_p)$. Then we define a corresponding morphism

$$\mathcal{T}(\phi_1, \ldots, \phi_p) : \mathcal{F}^{(L,b)}_p(L_0) \rightarrow \text{Hom}_\mathcal{X}(U, A \otimes \mathcal{F}^L(L_p)),$$

$$(\mathcal{T}(\phi_1, \ldots, \phi_p))(\phi)(-1) := (-1)^{k(|\phi| + |\phi'|)} R \left( \sum_{m_{k+2,\mathcal{X},A}^0} m_{k+1,\mathcal{X},A}^0 (-, \phi) \right)$$

(Recall that $\sum_{j=1}^p \phi_j$).
Now we show that it satisfies the equations for the $A_\infty$-natural transformation $\mathcal{F}$:

$$
(-1)^{1+\sum_1^p} d_{\text{Hom}, X(\cup_{i \in \mathcal{A}} \mathcal{F}, X)} \circ \mathcal{F}(\phi_1, \ldots, \phi_p)
$$

$$
+ \sum_{i=0}^{p-1} (-1)^{1+\sum_1^p} \mathcal{F}(\phi_{r+1}, \ldots, \phi_p) \circ \mathcal{F}(\phi_1, \ldots, \phi_r)
$$

$$
+ \sum_{i=1}^p \mathcal{F}(\phi_1, \ldots, \phi_p) \circ \mathcal{F}_1(\phi_1, \ldots, \phi_r)
$$

$$
- \sum_{r=0}^{p-1} \sum_{i=1}^p (-1)^{1+\sum_1^p} \mathcal{F}(\phi_1, \ldots, \phi_r, m_j(\phi_{r+1}, \ldots, \phi_{r+j}), \phi_{r+j+1}, \ldots, \phi_p) = 0.
$$

The first term gives

$$
(-1)^{1+\sum_1^p} d_{\text{Hom}, X(\cup_{i \in \mathcal{A}} \mathcal{F}, X)} \circ \mathcal{F}(\phi_1, \ldots, \phi_p)(\phi))|_{i_0 \ldots i_k}
$$

$$
= (-1)^{1+\sum_1^p} \mathcal{F}(\phi_1, \ldots, \phi_p)(\phi))|_{i_0 \ldots i_k}
$$

$$
+ (\mathcal{F}_1 \circ \mathcal{F}(\phi_1, \ldots, \phi_p)(\phi)))|_{i_0 \ldots i_k}
$$

$$
= (-1)^A \sum_{k=1}^p \sum_{j=0}^{1-1} (-1)^{1+\sum_1^p} (\mathcal{F}_1 \circ \mathcal{F}(\phi_1, \ldots, \phi_p)(\phi)))|_{i_0 \ldots i_k}
$$

where $A = p(|-|' + \sum_1^p)$.

We compute the later terms as follows.

$$
(-1)^{1+\sum_1^p} d_{\text{Hom}, X(\cup_{i \in \mathcal{A}} \mathcal{F}, X)} \circ \mathcal{F}(\phi_1, \ldots, \phi_p)(\phi))|_{i_0 \ldots i_k}
$$

$$
= \sum_{i=0}^{p-1} (-1)^A (-1)^{1+\sum_1^p} (m_{k-l+1} \mathcal{A})|_{i_0 \ldots i_k}
$$

$$
= (-1)^A (-1)^{1+\sum_1^p} (m_{k-l+1} \mathcal{A})|_{i_0 \ldots i_k}
$$

Result follows from $A_\infty$ equations for $m_{k \mathcal{A}, \mathcal{A}}$.

Similar to theorem 1.15 locally over a single chart, the $A_\infty$-transformation $\mathcal{F}_1 \rightarrow \mathcal{F}_2$ has a left inverse up to homotopy. □
Theorem 3.28. Assume there exist $a_{0i} \in \mathcal{F}^L(L), a_{i0} \in \mathcal{F}^L(L_i)$ for some $i$. Then the natural transformation $\mathcal{T} : \mathcal{F}^{L,b} \to \mathcal{A} \otimes (\mathcal{F}^U \circ \mathcal{F}^L)$ has a left inverse. Namely,

\[
\mathcal{F}^{[L,b]} \to \mathcal{A} \otimes (\mathcal{F}^L \circ \mathcal{F}^U) \to \mathcal{F}^{[L,b]} \to \mathcal{F}^{L,b}
\]

is homotopic to the identity natural transformation.

Proof. By previous theorem, we have natural transformations $\mathcal{T} : \mathcal{F}^{[L,b]} \to \mathcal{A} \otimes (\mathcal{F}^U \circ \mathcal{F}^L)$ and $\mathcal{F}^U(\mathcal{T}) : \mathcal{A} \otimes (\mathcal{F}^U \circ \mathcal{F}^L) \to \mathcal{A} \otimes (\mathcal{F}^U \circ \mathcal{F}^U \circ \mathcal{F}^{[L,b]}).$ Compose with $e v_{a_{i0}, a_{0i}},$ we get

\[
\mathcal{T} := e v_{a_{i0}, a_{0i}} \circ \mathcal{F}^U(\mathcal{T}) \circ \mathcal{T} : \mathcal{F}^{[L,b]} \to \mathcal{F}^{[L,b]}
\]

We want to show it is homotopic to the identity natural transformation on $\mathcal{F}^{[L,b]}$.

For a lagrangian $L$, we need to show $\mathcal{T}_L$, which is an endomorphism on $\mathcal{F}^{[L,b]}(L)$, equals to the identity up to homotopy.

Over an intersection $U_{i_0, i_k}$, for $\phi \in \mathcal{F}^{[L,b]}(L)$,

\[
\mathcal{F}^L_{i_0, i_k}(\phi) := (-1)^{k(|\phi|+|\phi'|)+|\phi|+|\phi'|} m_{i_0 i_1 \ldots i_k, 0} (\alpha_{i_0 i_1}, \ldots, \alpha_{i_{k-i}, i_k}, \ldots, \phi)
\]

as in the theorem 3.27.

Note that $\mathcal{F}^U(\mathcal{T}^L) \circ \mathcal{F}^L$ is a morphism of twisting complexes. Over an intersection $U_{i_0, i_k} \cap U_{j_0, j_l}$, with $j_1 = i_0$, up to sign we have

\[
\mathcal{F}^U(\mathcal{T}^L)_{i_0, j_0} \circ \mathcal{F}^L_{i_0, i_k}(\phi) := m_{i_0 i_1 \ldots i_k, 0} (\alpha_{i_0 j_1}, \ldots, \alpha_{i_{k-j} j_{k-j}}, \alpha_{i_{k-j}, i_k}) (\alpha_{i_0 j_1}, \ldots, \alpha_{i_{k-j}, i_k}, \ldots, \phi)
\]

If we further evaluate at $a_{0i}, a_{i0}$, by definition only $\mathcal{F}^U(\mathcal{T}^L)_{i_0, i_k}(\phi) = m_{i_0 i_1 \ldots i_k, 0} (\alpha_{i_0 j_1}, \ldots, \alpha_{i_{k-j}, i_k})$ contributes. Namely,

\[
\mathcal{F}^L(\phi) = m_{i_0 i_1 \ldots i_k, 0} (\alpha_{i_0 j_1}, \ldots, \alpha_{i_{k-j}, i_k}, \alpha_{i_0, i_k})
\]

\[
= m_{i_0 i_1 \ldots i_k, 0} (\alpha_{i_0 j_1}, \ldots, \alpha_{i_{k-j}, i_k}) (\alpha_{i_0 j_1}, \ldots, \alpha_{i_{k-j}, i_k}, \alpha_{i_0, i_k}) + m_{i_0 i_1 \ldots i_k, 0} (\alpha_{i_0 j_1}, \ldots, \alpha_{i_{k-j}, i_k}, \alpha_{i_0, i_k})
\]

\[
= \mathcal{F}^L(\phi) + \mathcal{H}_{L \otimes L} \circ d_{\mathcal{F}^U(\mathcal{T})_{i_0, i_k}}(\phi) + (-1)^{|\phi|} d_{\mathcal{F}^U(\mathcal{T})_{i_0, i_k}}(\phi) + \mathcal{H}_{L \otimes L} \circ d_{\mathcal{F}^U(\mathcal{T})_{i_0, i_k}}(\phi).
\]

In the second line, we have used the $A_{\infty}$-equations, with the terms $m_{i_0 i_1 \ldots i_k, 0} (\alpha_{i_0 j_1}, \ldots, \alpha_{i_{k-j}, i_k}, \alpha_{i_0, i_k})$ vanish. Define $\mathcal{H}_L := m_{i_0 i_1 \ldots i_k, 0} (\alpha_{i_0 j_1}, \ldots, \alpha_{i_{k-j}, i_k})$ as an endomorphism on $\mathcal{F}^{[L,b]}(L)$ as in theorem 3.15. Hence, over a single chart, the $A_{\infty}$-transformation $\mathcal{T}_1 \to \mathcal{T}_2$ has a left inverse up to homotopy.

In practical situations, we have $a_{0i}$ and $a_{i0}$ defined over certain localization $A_{\text{loc}, i}$. Then theorem 3.26 implies $\mathcal{F}^{[L,b]}|_{U_i} := A_{\text{loc}, i} \otimes \mathcal{A} \otimes (\mathcal{F}^U \circ \mathcal{F}^L)$ injective.

Assuming that there are enough charts of $\mathcal{A}$ such that $a_{0i}, a_{i0}$ are defined over certain localizations for all $i$, and any object $M$ in $\text{dg}(\mathcal{A}-\text{mod})$ satisfies $M \to \prod_i A_{\text{loc}, i} \otimes \mathcal{A} M$ is injective in the derived category of $\text{dg}(\mathcal{A}-\text{mod})$. We attain the injectivity of $\mathcal{F}^{[L,b]} \to \mathcal{A} \otimes (\mathcal{F}^U \circ \mathcal{F}^L)$.

Remark 3.29. If $U_i$ is a projective resolution for all $i$, with theorem 3.16 we know $\mathcal{F}^{[L,b]}|_{U_i} \to A_{\text{loc}, i} \otimes (\mathcal{F}^U \circ \mathcal{F}^L)$ is a quasi-isomorphism. Besides, these quasi-isomorphisms agree on the overlap. Assuming that any object $M$ in $\text{dg}(\mathcal{A}-\text{mod})$ satisfies that

\[
M' \to \prod_i A_{\text{loc}, i} \otimes \mathcal{A} M' \to \prod_i A_{\text{loc}, i} \otimes \mathcal{A} M'
\]

is an equalizer in the derived category of $\text{dg}(\mathcal{A}-\text{mod})$. We know $\mathcal{F}^{[L,b]}$ is natural equivalent to $\mathcal{A} \otimes (\mathcal{F}^U \circ \mathcal{F}^L)$.
4. NC Local Projective Plane

4.1. Construction of the Algebroid Stack. In this section, we will consider reference Lagrangians, \( L = L_1 \cup L_2 \cup L_3 \), in 3-punctured elliptic curve \( M \). We can decompose \( M \) into three pair-of-pants and assign three Seidel Lagrangian, denoted by \( S_1, S_2, S_3 \) in each of them. They are lifted to 3-to-1 cover and are shown in Figure 7. In the picture, \( S_1, S_2, S_3 \) are red, blue and green curves respectively while \( L_1, L_2, L_3 \) are light blue, light green and purple. Stars and circles are, respectively, spin structures and fundamental classes assigned to Lagrangians. Finally, \( Q^{i,j}'s \) are even degree generators while \( P^{i,j}'s \) are odd-degree generators in \( CF(L_i, S_j) \). In the picture, we only label generators for \( j = 3 \). For other generators, labels are just shifted downward with \( j \) changed to other numbers. We don't indicate which angles those generators represent since it can be determined by the given orientation of Lagrangians. For the assignment of area for each polygons in the picture, readers can refer to Appendix A.1.

Figure 7. Lagrangians in 3-to-1 cover of \( M \)

**Proposition 4.1.** \([CHL21]\) Consider the reference Lagrangians \( L = \{L_1, L_2, L_3\} \) shown in Figure 7. With the space of odd-degree weakly unobstructed formal deformations \( b = \sum a_i A_i + b_1 B_1 + c_1 C_1 \) of \( L \), the nc mirror of \( L \) is \( A = \Lambda Q / \partial \Phi \), where \( Q \) is the quiver in Figure 7 and \( \Phi = - T^h (b_1 c_3 a_2 + a_1 b_3 c_2 + c_1 a_3 b_2) + (c_1 b_3 a_2 + b_1 a_3 c_2 + a_1 c_3 b_2) \).

Now we will use three Seidel Lagrangians, denoted by \( S_i \), in the figure to construct the local charts of this deformation space \( \mathcal{A} \). We denote the local deformation space of each Seidel Lagrangian by \( \mathcal{A}_i \). By similar computation above, we have

**Proposition 4.2.** Consider the Seidel Lagrangian \( S_1 \) with the given orientation, fundamental class and spin structure in Figure 7. With the space of odd-degree weakly unobstructed formal deformations \( b = w_1 W_1 + y_1 Y_1 + x_1 X_1 \) of \( S_1 \), the noncommutative local mirror of \( S_1 \) is \( \mathcal{A}_1 = \Lambda < w_1, y_1, x_1 > i \partial \Phi_1 \), where \( \Phi_1 = y_1 x_1 w_1 - T^h x_1 y_1 w_1 \).
Similarly, the NC local mirror of $S_2$ is $\mathcal{A}_2 = \Lambda < w_2, z_2, y_2 > / \partial \Phi_2$, where $\Phi_2 = z_2 y_2 w_2 - T^3 y_2 z_2 w_2$, and the NC mirror of $S_3$ is $\mathcal{A}_3 = \Lambda < w_3, x_3, z_3 > / \partial \Phi_3$, where $\Phi_3 = x_3 z_3 w_3 - T^3 x_3 z_3 w_3$.

Next, we need to construct algebroid stacks from $\mathcal{A}_i$'s by finding their localizations and isomorphisms among localizations.

Being different from the conifold case, our Seidels in this case do not intersect. So, instead, we deform $L$ to $\tilde{L}$ and denote its local mirror by $\mathcal{A}_0$, which is the same as the mirror of $L$. Then, each Seidel Lagrangian $S_i$ intersects with $\tilde{L}$, which means we can use the method of counting polygons to find the relation between $\mathcal{A}_0, \mathcal{A}_i$ and even compute universal bundle for the algebroid stack over two charts $\mathcal{A}_0, \mathcal{A}_i$ once we find the appropriate isomorphisms. Notice that $\mathcal{A}_0$ is a quiver algebra of more than one vertex. Hence, we need to use the definition of modified algebroid stack in Section 2.2 instead. This gives us algebroid stacks $\mathcal{A}_{0i}$ for $i \in \{1, 2, 3\}$ defined by the following data:

1. $S_{0i} = \emptyset$ and $S_{00} = \{a_1, a_3\}, S_{02} = \{c_1, c_3\}, S_{03} = \{b_1, b_3\}$.
2. For $i \in \{1, 2, 3\}$, we define representations $\mathcal{G}_{0i} : \mathcal{A}_{0i} \to \mathcal{A}_{0i}^h$ according to the following relations:

\[
\begin{align*}
\begin{cases}
x_1 = T^{B_1} a_1 c_1^{-1} \\
y_1 = T^{B_1} b_1 c_1^{-1} \\
w_1 = T^{B_1} a_1 a_3 a_2 \\
z_2 = T^{B_1} b_1 c_1^{-1} \\
z_3 = T^{B_2} c_1 b_3 \cdot b_2 \\
w_2 = T^{B_3} c_3 c_2
\end{cases}
\end{align*}
\]

where $B_1 = A_{112345} + A_{1}'$, $B_2 = -A_{1345} + A_{4}'$, $B_3 = -A_{1345}' + A_4$.

3. $G_{30}$ is defined by the following assignment:

\[
\begin{align*}
\begin{cases}
e_1 & \mapsto 1 \\
a_1 & \mapsto T^{-B_1} z_3 \\
b_1^{-1} & \mapsto 1 \\
c_1 & \mapsto T^{-B_2} x_3 \\
e_2 & \mapsto 1 \\
a_2 & \mapsto T^{B_2} u_3 \cdot z_3 \\
b_3^{-1} & \mapsto 1 \\
d_3 & \mapsto T^{-B_2 + h} x_3
\end{cases}
\end{align*}
\]

Other $G_{i10}$ can be defined similarly according to the transformation rule 4.4.

4. $c_{010}(e_2) = e_2, c_{010}(e_3) = b_1 b_3 e_2, c_{010}(e_1) = b_1 e_2, c_{020}(e_3) = c_1 c_2 e_2, c_{010}(e_3) = a_1 a_3 e_2, c_{010}(e_1) = a_1 e_2$ and other terms are trivial.

With the algebroid stack defined above, we claim that

**Proposition 4.3.** Let

\[
\alpha_3 = -Q^{2,3} \in CF_{03}(\mathbb{L}, \mathbb{L}, (S_3, b_3)), \beta_3 = T^{-W} 1 \otimes b_3^{-1} b_1^{-1} p^{3,3} \in CF_{03}(S_3, b_3, \mathbb{L}, b).
\]

Then, $m_1^{b_1, b_2}(\alpha_3) = 0, m_1^{b_1, b_2}(\beta_3) = 0, m_2^{b_1, b_2}(\alpha_3, \beta_3) = 1_1$ and $m_2^{b_1, b_2}(\beta_3, \alpha_3) = 1_{S_3}$.

Similarly, we have pairs of isomorphisms

\[
\begin{align*}
\{ (a_2, \beta_2) = (-Q^{2,2}, T^{-W} 1 \otimes c_3^{-1} c_1^{-1} p^{3,2}) \\
(a_1, \beta_1) = (-Q^{2,1}, T^{-W} 1 \otimes a_3^{-1} a_1^{-1} p^{3,1})
\end{align*}
\]

The details of computations can be found in Appendix A.2.

We can reveal the hidden relations between $\mathcal{A}_i, \mathcal{A}_j$ because of the following proposition:
Proposition 4.4. Let \( S_{0,0,j} = S_{0,0,j} \cup S_{j,1,0} \cup S_{0,1,0} = \{ x_1 \}, S_{2,0,12} = \{ z_2 \}, S_{1,0,13} = \{ y_1 \}, S_{3,0,13} = \{ z_3 \}, S_{2,0,23} = \{ y_2 \}, S_{0,2,03} = \{ x_1 \} \) and \( S_{p,0,i,0} = \cup_{k_0 \neq \lambda, k_0 \neq \lambda k} \) for \( i_0, \ldots, i_p \in \{ 1,2,3 \} \). There exists an embedding \( G_{0,j,0,i,j} : \mathcal{A}_j(S_i,0,j)^{-1} \rightarrow \mathcal{A}_0(S_0,0,j)^{-1} \) such that \( \text{Im} G_{0,j,0,i,j} = \text{Im} G_{0,i,0,j} \) for \( i, j = 1,2,3 \).

\[
G_{0,j} : \mathcal{A}_j(S_i,0,j)^{-1} \rightarrow \mathcal{A}_0(S_0,0,j)^{-1}
\]

have the same image in \( \mathcal{A}_0(S_0,0,j)^{-1} \).

Proof. We show that \( G_{01} \) and \( G_{02} \) have the same image. Clearly, \( G_{01}(x_1) = T^{B_1+B_2} G_{02}(z_2^{-1}) \) and \( G_{01}(y_1) = T^{B_1} T^{'b_1} \) \( c_1 a_1^{-1} = T^{-B_1} G_{02}(y_2 z_2^{-1}) \). Finally,

\[
G_{01}(w_1 x_1^3) = T^{B_3+B_1} a_1 a_3 a_2(c_1 a_1^{-1})^3 = T^{B_3+B_1} a_1 a_3 a_2(c_1 a_1^{-1})^2
\]

For \( c_1 \), \( 0 \neq \delta \), we define \( \mathcal{A}_f \) for \( i \neq 0 \) glue as an algebroid stack, and we drop the chart \( \mathcal{A}_f \).

Theorem 4.5. The following data defines an algebraic stack as glued mirror of the graded immersed Lagrangians \( S, S_1, S_2, S_3 \) in the three-punctured elliptic curve.

1. The collection of NC local mirrors of Seidel Lagrangians, \( \mathcal{A}_f, \mathcal{A}_2, \mathcal{A}_3 \) with complex coefficients instead of Novikov field.
2. \( S_{i,j,k} = \{ x_1 \}, S_{2,1,2} = \{ z_2 \}, S_{1,1,3} = \{ y_1 \}, S_{3,1,3} = \{ z_3 \}, S_{2,2,3} = \{ y_2 \}, S_{2,3,2} = \{ x_2 \}, S_{3,2,3} = \{ x_3 \} \) and \( S_{i,j,k} = \cup_{k_0 \neq \lambda, k_0 \neq \lambda k} \) for \( i_0, \ldots, i_p \in \{ 1,2,3 \} \).
3. For \( i, j \in \{ 1,2,3 \} \), we define \( G_{i,j} : \mathcal{A}_{i,j}^{h} \rightarrow \mathcal{A}_{i,j}^{h} \) according to the relations (4.3). For instance, \( G_{1,2}(x_1) = z_2^{-1}, G_{1,2}(z_2) = x_1^{-1} \) and so on.
4. \( c_{i,j,k} = 1 \).

4.2. Construction of the Universal Bundle. We deform \( L \) as shown in Figure 8 and denote it by \( \mathcal{L} = L_1 \cup L_2 \cup L_3 \), where \( \beta^1, \beta^2, \beta^3 \) are the intersection points of \( L_1, L_2, L_3 \). Furthermore, we define \( \alpha, \beta \)'s obtained previously by intersection points between \( (\mathcal{L}, \beta), (S_i, b_i) \) instead of \( (L, \beta), (S_j, b_j) \).

We can obtain a local complex

\[
(E_i, a_i) := \mathcal{F}_i((\mathcal{L}, b_i)) = (\mathcal{A} \otimes \mathcal{A}_i \otimes CF((\mathcal{L}, b_i), (S_i, b_i)), (-1)^{\deg(c)} m_i^{b_i}(c_i))
\]

with appropriate Z-grading. For instance,
Figure 8. Deformed Lagrangian $L$

**Proposition 4.6.** With $Z$-grading, we obtain the following chain complex, denoted by $(E_3, a_3)$, is

$$
0 \longrightarrow Q^{2,3} \xrightarrow{a_3} P^{2,3} \oplus P_1^{1,3} \oplus P_2^{1,3} \xrightarrow{a_1} Q^{3,3} \oplus Q_1^{1,3} \oplus Q_2^{1,3} \xrightarrow{a_2} P^{3,3} \longrightarrow 0,
$$

where the horizontal arrows are defined in Appendix A.5.1.

**Proof.** The polygons involved can be found in Appendix A.3. The computation for each polygon is similar to previous computations. Once we have $a_i$, we can check that it defines the differential by brute force. \hfill $\square$

We only show the chain complex $(E_3, a_3)$ explicitly above. Other two complexes can be obtained by replacing variables according to the following transformation rule:

$$
\begin{align*}
x_3 & \leftrightarrow z_2 \leftrightarrow y_1 \\
z_3 & \leftrightarrow y_2 \leftrightarrow x_1 \\
w_3 & \leftrightarrow w_2 \leftrightarrow w_1
\end{align*}
\quad
\begin{align*}
a & \leftrightarrow b \leftrightarrow c \\
b & \leftrightarrow c \leftrightarrow a \\
c & \leftrightarrow a \leftrightarrow b
\end{align*}
\tag{4.4}
$$

**Remark 4.7.** Indeed, we are applying the mirror construction to a $Z$-graded $A_\infty$ category of Lagrangians, rather than the $Z_2$-graded Fukaya category of Lagrangians in Riemann surfaces. We give a certain $Z$-grading to the collection of immersed Lagrangians under consideration. In this paper, we check by hand that the resulting objects obtained from mirror transform are well-defined. In a future work, we will prove that our grading give an $A_\infty$ category, and construct a well-defined mirror functor which ensures the mirror objects and morphisms are automatically well-defined.
Warning: This transformation rule is only used to generate other complexes from the known complex. It has nothing to do with the relation of variables when we glue charts.

However, because Seidel Lagrangians do not intersect, we cannot construct vertical arrows among local complexes in the universal bundle as what we do for NC resolved conifold. Just like the construction of glued mirror, we make use of \( \tilde{\alpha} \) arrows among local complexes in the universal bundle as what we do for NC resolved known complex. It has nothing to do with the relation of variables when we glue charts.

With those isomorphisms defined in \( \mathcal{X} \), we can glue \( \mathcal{F}_{\mathcal{L}}^L \) and \( \mathcal{F}_{\mathcal{L}}^S \). This gives us the following commutative diagrams for each \( L_j \in \mathcal{L} \) where the vertical arrows are defined over \( \mathring{R}_{0,0,1} \):

\[
\begin{array}{c}
0 \longrightarrow Q^{2,i} \longrightarrow P^{2,i} \oplus P^{1,i} \oplus P^{1,i} \longrightarrow Q^{3,i} \oplus Q^{1,i} \oplus Q^{1,i} \longrightarrow P^{3,i} \longrightarrow 0 \\
0 \longrightarrow Q^{1,j} \longrightarrow \bigoplus_{k=1,2,3} P^{1,j} \longrightarrow \\
0 \longrightarrow P^{1,1} \longrightarrow \bigoplus_{k=1,2,3} P^{1,1} \longrightarrow \\
0 \longrightarrow P^{1,2} \longrightarrow \bigoplus_{k=1,2,3} P^{1,2} \longrightarrow \\
0 \longrightarrow P^{1,3} \longrightarrow \bigoplus_{k=1,2,3} P^{1,3} \longrightarrow \\
0 \longrightarrow P^{3,1} \longrightarrow \bigoplus_{k=1,2,3} P^{3,1} \longrightarrow \\
0 \longrightarrow P^{3,2} \longrightarrow \bigoplus_{k=1,2,3} P^{3,2} \longrightarrow \\
0 \longrightarrow P^{3,3} \longrightarrow \bigoplus_{k=1,2,3} P^{3,3} \longrightarrow 0,
\end{array}
\]

where the lower complex is \( \mathcal{F}_{\mathcal{L}}^L(L_j) \) and \( j \) is treated as an index in the cycle \( \{1, 2, 3\} \).

The vertical arrows are defined by \( m_2(a_{i,j}, \cdot) \) and \( m_2( \cdot, \beta_i) \). For convenience, we abuse the notation and denote them by \( a_{i,j}, \beta_i \). We define \( a_{ij} = \beta_i \circ a_{i,j} : E_{i,j} \rightarrow E_{i,ij} \) when \( i \neq j \) and \( a_{ij} = id_{i} : E_{i} \rightarrow E_{i} \). In the following proposition, we compute the vertical isomorphisms \( a_{23} : E_{3,23} \rightarrow E_{2,23} \). Other \( a_{ij} \) can be obtained via the transformation rule \( 4.4 \).

\textbf{Proposition 4.8.} Overall, with \( a_{23} : E_{3,23} \rightarrow E_{2,23} \) and \( a_{32} : E_{2,23} \rightarrow E_{3,23} \), the following diagram is commutative:

\[
\begin{array}{c}
0 \longrightarrow Q^{2,3} \longrightarrow P^{2,3} \oplus P^{1,2} \oplus P^{1,3} \longrightarrow Q^{3,3} \oplus Q^{1,2} \oplus Q^{1,3} \longrightarrow P^{3,3} \longrightarrow 0 \\
0 \longrightarrow Q^{1,2} \longrightarrow \bigoplus_{k=1,2,3} P^{1,2} \longrightarrow \\
0 \longrightarrow Q^{1,3} \longrightarrow \bigoplus_{k=1,2,3} P^{1,3} \longrightarrow \\
0 \longrightarrow Q^{3,2} \longrightarrow \bigoplus_{k=1,2,3} P^{3,2} \longrightarrow \\
0 \longrightarrow Q^{3,3} \longrightarrow \bigoplus_{k=1,2,3} P^{3,3} \longrightarrow 0,
\end{array}
\]

where \( Q^{i,1}, P^{i,1}, p^{i,1} \) are generators in \( CF(L_i, S_j) \) and \( a_{23}, a_{32} \) are defined in Appendix \( A_{5.5} \).

\textbf{Proof.} We firstly compute \( a_{i0} \) and \( a_{0j} \) for \( i \neq 0 \). The polygons in \( a_{i0} \) and \( a_{0j} \) can be found in Appendix \( A_{5.5} \). After we compose them, we obtain \( a_{ij} \). The commutativity of diagram can be checked by brute force.

\textbf{Definition 4.9.} \( a_{ijk}(\cdot) = \beta_i \circ m_3^{b,b'}(a_{j}, \beta_{j,k}, \cdot) \circ a_k \)

The following lemma and proposition can be checked by direct calculation. We show the polygons for \( m_3^{b,b'}(a_{j}, \beta_{j,k}, \cdot) \) in Appendix \( A_{4.4} \).

\textbf{Lemma 4.10.} \( a_{ijk} \) is only non-zero whenever there are no repeated indices. In particular, \( a_{321} \) is defined in Appendix \( A_{4.5} \). Other \( a_{ijk} \) can be obtained via transformation rule \( 4.4 \).

\textbf{Proposition 4.11.} \( -a_{1,0}^{1,0} + a_{1,0}^{1,0} G_{i,j}(a_{jk}^{1,0})c_{ik}^{-1} + a_{0,1}^{1,0} G_{i,j}(a_{jk}^{2,1})c_{ik}^{-1} + a_{1,0}^{2,1} G_{i,j}(a_{jk}^{1,0})c_{ik}^{-1} + a_{1,0}^{2,1} G_{i,j}(a_{jk}^{1,0})c_{ik}^{-1} = 0. \)

This produces the twisted complex \( U = \mathcal{F}_{\mathcal{L}}(L) \) that we need for the functor \( \mathcal{F}^U : T w(\mathcal{X}) \rightarrow dg - \text{mod}(\mathfrak{A}) \).

\textbf{Proposition 4.12.} The collections of \( (E_i, a_i) \) together with \( a_{ij}, a_{ijk} \) form a twisted complex \( U \) over \( \mathcal{X} \).
A. Computations and Figures for Mirror Symmetry for NC Local Projective Plane

A.1. Notation of Area Terms. The assignment of area is labeled in Figure 9, where the green triangle is labeled by $A'_1$, the pink triangle is labeled by $A_1$, and the red one is labeled as $A_2$. Then, $\{1,2\}$ are $A'_1 - A'_4, A_1 - A_2 - A_4$. For any other tiny polygons, we set area as zero. The area of any other non-labeled polygons can be obtained by shifting given labels.

To shorten the expression of entries, we use the following abbreviation of area terms:

- $A'_j = A'_j$
- $A_I = \sum_j A_{ij}$
- $A'_I = \sum_j A'_{ij}$
- $A_{I,j'} = \sum_k A_{ik} + \sum_k A'_{jk}$

In particular, to avoid counting, we prefer to denote $A_{i...i}$ by $k A_i$ for $k$ repeated indices.

Furthermore, we can simplify the expression by using the following variables:

- $W = A_{112345(5)}$
- $S_1 = A_{1345} - A'_4$
- $S_2 = A'_{1345} - A_4$
- $\Delta_i = A_i - A'_i$

Finally, we set $A_{112345(5)}' = A_{5(112345)}$.

A.2. Computation of Isomorphisms. In the following proof, we will show the proposition holds for $\alpha_3, \beta_3$. For other Seidel Lagrangians, $S_1$ and $S_2$, the computation is similar.

Proof of 4.3 According to Figure 10

$$m_2^{b_2,b_3}(\beta, \alpha) = m_4(T^{-W} b_1^{-1} b_3^{-1} P_{a,3}, b_2 b_3 b_1, b_1 Q_{2,3}) = T^{W} (T^{-W} b_1 b_3 b_1^{-1} b_1^{-1}) 1 S_3 = 1 S_3,$$

where the reversed orientation along $b_1 b_1^{-1} b_1 Q_{2,3}$ contributes $(-1)^2$ and spin structures along the boundary contribute $(-1)^3$ in the pink polygon.
In the orange polygon, the only clockwise edge is from $P_{3,3}$ to $Q_{2,3}$, whose degrees are even. So, the only $(-1)$ comes from the spin structure on this edge. Together with the negative sign in $\beta$, we have

$$m_2^{b_3 b_1}(\alpha, \beta) = (b_3 b_1^{-1})_1 L_1 + (b_1 b_3 b_1^{-1})_1 L_2 + (b_3^{-1} b_1 b_3)_1 L_3$$

$$= \sum_{i=1}^{3} e_i L_i = (e_1 + e_2 + e_3) L = L_1.$$

Now, we need to check $m_1(\alpha) = 0$. In Figure 11 there are three pairs of polygons from $Q^{2,3}$ to $P^{2,3}, P^{1,3}, P^{1,3}$. The leftmost one contributes to $m_2(c_1 C_1, 1 \otimes e_2 Q^{2,3}) = -TA^1 \otimes e_2 \otimes c_1 P^{1,3}$ and $m_4(b_1 B_1, 1 \otimes e_2 Q^{2,3}, x_3 X_3) = T^{A'_{1345}, X_3} \otimes 1 \otimes e_2 \otimes b_1 P^{1,3}_2$. Similarly, we can compute other pairs of polygons. Their coefficients in $m_1(\alpha)$ are

$$\begin{align*}
(w_3 \otimes 1 \otimes e_2 - T^W 1 \otimes e_2 \otimes b_1 \otimes b_3 \otimes b_2) \\
(-1 \otimes e_2 \otimes a_1 + T^{A'_{1345}, X_3} \otimes 1 \otimes e_2 \otimes b_1) \\
(-T^A 1 \otimes e_2 \otimes c_1 + T^{A'_{1345}, X_3} \otimes 1 \otimes e_2 \otimes b_1)
\end{align*}$$

With the relation 4.1, they all vanish after we apply $\mathcal{M}$ defined by Equation 2.17. For instance, the first sum above corresponds to

$$G_{03}(w_3) c_{033}^{-1}(1) e_2 - T^W G_{30}(1) c_{033}^{-1}(1) b_1 b_3 b_2 = T^W b_1 b_3 b_2 - T^W b_1 b_3 b_2 = 0.$$
The computation of $m_1(b) = 0$ is similar. So, we only show all polygons involved in Figure 12 instead of providing more details.

\[ \square \]

**Proposition A.1.** Consider the following reference Lagrangian $S$ with the given orientation, fundamental class and spin structure in Figure 13. With the space of odd-degree weakly unobstructed formal deformations $b = w_1 W_1 + y_1 Y_1 + x_1 X_1$ of $S$, noncommutative mirror $\mathcal{A}_1 = \Lambda < w_1, y_1, x_1 > / \partial \Phi$, where $\Phi = y_1 x_1 w_1 - T^3 x_1 y_1 w_1$

**Proof.** Let $b = w_1 W_1 + y_1 Y_1 + x_1 X_1$. There are only two polygons bounded by $S_1$, the shaded and unshaded polygons. (Notice that any unshaded region outside $S_1$ is not a polygon because there are other punctures outside this picture.) Hence, all non-zero terms in $m(e^b)$ comes from those two polygons. $m_2(x_1 X_1, y_1 Y_1)$, $m_2(y_1 Y_1, w_1 W_1)$, and $m_2(w_1 W_1, x_1 X_1)$ correspond to the orange triangle, and $m_2(x_1 X_1, w_1 W_1)$, $m_2(y_1 Y_1, x_1 X_1)$, and $m_2(w_1 W_1, y_1 Y_1)$ correspond to the pink one.

Then, the coefficient of $W_1$ in $m_2(x_1 X_1, y_1 Y_1)$ is $-T^A y_1 x_1$ and the coefficient of $W_1$ in $m_2(y_1 Y_1, x_1 X_1)$ is $T^B x_1 y_1$. Overall, the coefficient of $W_1$ in $m_2(b, b)$ is

$$-T^A y_1 x_1 + T^B x_1 y_1 = -T^A(y_1 x_1 - T^B-A x_1 y_1) = -T^A(y_1 x_1 - T^3 x_1 y_1)$$
, since $B - A = A_3' + A_2' + A_4' - (A_3 + A_2 + A_4) + A_1 - A_1' = 2A_1 - 2A_1' + A_1 - A_1' = 3A_1 - 3A_1' = 3h$

and $2A_1 + A_{234} = 2A_1' + A_{(234)'}$.

Similarly, we can obtain the coefficients of $\tilde{Y}_1$ and $\tilde{X}_1$. Then, $\Phi' = \sum \frac{1}{2\pi i} < m_2(b, b), b >= T^A(y_1x_1 - T^{3h}x_1y_1)w_1 \in \mathcal{A}_1 / [\mathcal{A}_1, \mathcal{A}_1]$. After rescaling the spacetime superpotential, we have $\Phi = (y_1x_1 - T^{3h}x_1y_1)w_1$.

\[ \square \]

A.3. Polygons in $a_i$, $a_{0i}$, $a_{00}$. In this section, we show the polygons involved for $a_i$, $a_{0i}$ and $a_{00}$. We first show the polygons involved for $a_3$. Other $a_i$'s are similar. In addition, $a_0$ is not computed here since it's not important for our final result.
Figure 14. Polygons in $\mathcal{A}^0_3$

Figure 15. Polygons in $\mathcal{A}^1_3$

Figure 16. Polygons in $\mathcal{A}^2_3$
Now we show polygons in $a_{30}$ and $a_{03}$. The polygons in other $a_{i0}, a_{0j}$ are similar. We firstly show polygons in $a_{30}$ and $a_{03}$ where $\tilde{b}$ is not involved.

In the following pictures, pink polygons are the polygons in $a_{03}$ and orange polygons are the ones in $a_{30}$:
Then, we show polygons where $\tilde{b}$ is involved.
A.4. Polygons in $m_3$. Like previous sections, we only show the polygons in $m_3(\alpha_2, \beta_2, \cdot)$. Other cases are similar.
A.5. Definition of Arrows in Universal Bundles.

A.5.1. Horizontal Arrows.

\[ a_0^3 = \begin{pmatrix} w_3 \cdot - T^{A_1 A_2 A_3 A_4 A_5}_3 & b_1 b_2 & T^{A_1}_1 \cdot a_1 + T^{A_1 A_2 A_3}_3 \cdot b_1 & - T^{A_1}_1 \cdot c_1 + T^{A_1 A_2 A_3}_3 \cdot b_1 \\ \end{pmatrix} \]

\[ a_1^3 = \begin{pmatrix} 0 \\ - T^{A_1}_1 \cdot c_3 + T^{A_1 A_2 A_3 A_4 A_5}_3 \cdot b_3 \\ - T^{A_1}_1 \cdot c_3 + T^{A_1 A_2 A_3 A_4 A_5}_3 \cdot b_3 \\ - T^{A_1}_1 \cdot c_3 + T^{A_1 A_2 A_3 A_4 A_5}_3 \cdot b_3 \\ 0 \\ \end{pmatrix} \]

\[ a_2^3 = \begin{pmatrix} w_3 \cdot - T^{A_1 A_2 A_3 A_4 A_5}_3 \cdot b_2 b_1 b_3 \\ - T^{A_1}_1 \cdot a_3 + T^{A_1 A_2 A_3 A_4 A_5}_3 \cdot b_3 \\ - T^{A_1}_1 \cdot c_3 + T^{A_1 A_2 A_3 A_4 A_5}_3 \cdot b_3 \\ \end{pmatrix} \]

A.5.2. Vertical Arrows.

\[ a_0^1 = 1 \]

\[ a_1^1 = \begin{pmatrix} - T^{A_1 A_2 A_3 A_4 A_5}_1 \cdot b_2 b_3 c_1 c_2 c_3 \cdot b_2 \\ T^{A_1 A_2 A_3 A_4 A_5}_1 \cdot b_2 b_3 c_1 c_2 c_3 \cdot b_2 \\ T^{A_1 A_2 A_3 A_4 A_5}_1 \cdot b_2 b_3 c_1 c_2 c_3 \cdot b_2 \\ 0 \\ 0 \\ \end{pmatrix} \]

\[ a_2^1 = \begin{pmatrix} - T^{A_1 A_2 A_3 A_4 A_5}_3 \cdot c_3 \cdot c_2 \cdot c_1 \cdot c_2 \cdot c_3 \cdot b_2 \\ T^{A_1 A_2 A_3 A_4 A_5}_3 \cdot c_3 \cdot c_2 \cdot c_1 \cdot c_2 \cdot c_3 \cdot b_2 \\ T^{A_1 A_2 A_3 A_4 A_5}_3 \cdot c_3 \cdot c_2 \cdot c_1 \cdot c_2 \cdot c_3 \cdot b_2 \\ 0 \\ 0 \\ \end{pmatrix} \]
A.5.3. Higher Homotopies. For $k = 0, 2, 3,$

$$a^k_{321} = 0$$

$$a^1_{321} = \begin{pmatrix}
A^{3245(5)} - A_4' & 0 & A^{3245(5)} - A_4' \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}$$

$$a^2_{32} = \begin{pmatrix}
A^{3245(5)} - A_4' & 0 & A^{3245(5)} - A_4' \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}$$

$$a^3_{32} = (b_1 b_3 c_3^{-1} c_1^{-1})$$

$$a^0_{23} = 1$$

$$a^1_{23} = \begin{pmatrix}
A^{3245(5)} - A_4' & 0 & A^{3245(5)} - A_4' \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}$$

$$a^2_{23} = \begin{pmatrix}
A^{3245(5)} - A_4' & 0 & A^{3245(5)} - A_4' \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}$$

A.5.3. Higher Homotopies. For $k = 0, 2, 3,$
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