A Unified Approach to Adaptive Regularization in Online and Stochastic Optimization

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Abstract

We describe a framework for deriving and analyzing online optimization algorithms that incorporate adaptive, data-dependent regularization, also termed preconditioning. Such algorithms have been proven useful in stochastic optimization by reshaping the gradients according to the geometry of the data. Our framework captures and unifies much of the existing literature on adaptive online methods, including the AdaGrad and Online Newton Step algorithms as well as their diagonal versions. As a result, we obtain new convergence proofs for these algorithms that are substantially simpler than previous analyses. Our framework also exposes the rationale for the different preconditioned updates used in common stochastic optimization methods.

1 Introduction

In Online Convex Optimization (Zinkevich, 2003; Shalev-Shwartz, 2012; Hazan, 2016) a learner makes predictions in the form of a vector belonging to a convex domain $\mathcal{X} \subseteq \mathbb{R}^d$ for $T$ rounds. After predicting $x_t \in \mathcal{X}$ on round $t$, a convex function $f_t : \mathcal{X} \mapsto \mathbb{R}$ is revealed to the learner, potentially in an adversarial or adaptive way, based on the learner’s past predictions. The learner then endures a loss $f_t(x_t)$ and also receives its gradient $\nabla f_t(x_t)$ as feedback.\footnote{Our analysis is applicable with minor changes to non-differentiable convex functions with subgradients as feedback.}

The goal of the learner is to achieve low cumulative loss, coined regret, with respect to any fixed vector in the $\mathcal{X}$. Formally, the learner attempts to cap the quantity

$$R_T = \sum_{t=1}^{T} f_t(x_t) - \min_{x \in \mathcal{X}} \sum_{t=1}^{T} f_t(x).$$

Online Convex Optimization has been proven useful in the context of stochastic convex optimization, and numerous algorithms in this domain can be seen and analyzed as online optimization methods; we again refer to (Hazan, 2016) for a thorough survey of many of these algorithms. Any online algorithm achieving a sublinear regret $R_T = o(T)$ can be readily converted to a stochastic convex optimization algorithm with convergence rate $O(R_T/T)$, using a standard technique called online-to-batch conversion (Cesa-Bianchi et al., 2004).

The online approach is particularly effective for the analysis of adaptive optimization methods, namely, algorithms that change the nature of their update rule on-the-fly so as to adapt to the geometry of the observed data (i.e., perceived gradients). The update rule of such algorithms often takes the form $x_{t+1} \leftarrow x_t - H_t g_t$ where $g_t$ is a (possibly stochastic) gradient of the

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function $f_t$ evaluated at $x_t$, and $H_t$ is a \textit{regularization matrix}, or a \textit{preconditioner}, used to skew the gradient step in a desirable way. Importantly, the matrix $H_t$ may be chosen in an adaptive way based on past gradients, and might even depend on the gradient $g_t$ of the same step. The online optimization apparatus, in which the objective functions $f_t$ may vary almost arbitrarily, is very effective in dealing with these intricate dependencies. For a recent survey on adaptive methods in online learning and their analysis techniques see (McMahan, 2014).

One of the well-known adaptive online algorithms is AdaGrad (Duchi et al., 2011) which is commonly used in machine learning for training sparse linear models. AdaGrad also became popular for training deep neural networks. Intuitively, AdaGrad employs an adaptive regularization for maintaining a step-size on a per-coordinate basis, and can thus perform aggressive updates on informative yet rarely seen features (a similar approach was also taken by McMahan and Streeter, 2010). Another adaptive algorithm known in the online learning literature is the Online Newton Step (ONS) algorithm (Hazan et al., 2007). ONS incorporates an adaptive regularization technique for exploiting directional (non-isotropic) curvature of the objective function. While these adaptive regularization algorithms appear similar to each other, their derivation and analysis are disparate and technically involved. Furthermore, it is often difficult to gain insights into the specific choices of the matrices used for regularization and what role do they play in the analysis of the resulting algorithms.

In this paper, we present a general framework from which adaptive algorithms such AdaGrad and ONS can be derived using a streamlined scheme. Our framework is parameterized by a \textit{potential function} $\Phi$. Different choices of $\Phi$ give rise to concrete adaptive algorithms. Morally, after choosing a potential $\Phi$, the algorithm computes its regularization matrix, a preconditioner, for iterate $t$ by solving a minimization of the form,

$$
\min_{H \succ 0} \left\{ \sum_{s=1}^{t} \|g_s\|_H^2 + \Phi(H) \right\}.
$$

Thus the algorithm strikes a balance between the potential of $H$, $\Phi(H)$, and the quality of $H$ as a regularizer for controlling the norms of the gradients with respect to the observations thus far. Not only does this balance give a natural interpretation of the regularization used by common adaptive algorithms, it also makes their analysis rather simple: an adaptive regularization algorithm can be viewed as a follow-the-leader (FTL) algorithm that operates over the class of positive definite matrices. We can thus analyze adaptive regularization methods using simple and well established FTL analyses.

Solving the minimization above over positive definite matrices is, in general, a non-trivial task. However, in certain cases we can obtain a closed form solution that gives rise to efficient algorithms. For instance, to obtain AdaGrad we pick the potential $\Phi(H) = \text{Tr}(H^{-1})$ and solve the minimization via elementary differentiation, which leads to regularizers of the form $H_t = \left( \sum_{s=1}^{t} g_s g_s^T \right)^{-1}/2$. To obtain ONS we pick $\Phi(H) = -\log |H|$ which yields $H_t = \left( \sum_{s=1}^{t} g_s g_s^T \right)^{-1}$, which constitutes the ONS update.

For both AdaGrad and ONS, we also derive diagonal versions of the algorithms by constraining the minimization to diagonal positive definite matrices. We also show that by further constraining the minimization to positive multiples of the identity matrix, one can recover familiar matrix-free (scalar) online algorithms that adaptively tune their step-size parameter according to observed gradients. As in the case of full matrices, the resulting minimization over matrices can be solved in closed form and the analyses follow seamlessly from the choice of the potential. Last we would like to note that the analysis applies to the mirror-descent family of algorithms; nevertheless, our approach can also be used to analyze dual-averaging-type algorithms, also referred to as follow-the-regularized-leader algorithms.

**Notation.** We denote by $\mathcal{S}_+$ the positive definite cone, i.e. the set of all $d \times d$ positive definite matrices. We use $\text{diag}(A)$ to denote the diagonal matrix whose diagonal coincides the diagonal
Algorithm 1: Adaptive regularization meta-algorithm.

Elements of $A$ and its off-diagonal elements are 0. The trace of the matrix $A$ is denoted as $\text{Tr}(A)$. The element-wise inner-product of matrices $A$ and $B$ is denoted as $A \bullet B = \text{Tr}(A^T B)$.

The spectral norm of a matrix $A$ is denoted $\|A\|_2 = \max \|Ax\|/\|x\|$ where $x \neq 0$. We denote by $\|x\|_H = \sqrt{x^T H x}$ the norm of $x \in \mathbb{R}^d$ with respect to a positive definite matrix $H \in \mathcal{S}_+$. The dual norm of $\| \cdot \|_H$ is denoted $\| \cdot \|_{H^*}$ and is equal to $\sqrt{x^T H^{-1} x}$. We denote by

$$\Pi^H_{\mathcal{X}}(x) = \arg\min_{x' \in \mathcal{X}} \|x' - x\|_H$$

the projection of $x$ onto a bounded convex set $\mathcal{X}$ with respect to the norm induced by $H \in \mathcal{S}_+$. When $H = I$, the identity matrix, we omit the superscript and simply use $\Pi_{\mathcal{X}}$ to denote the typical Euclidean projection operator.

Given a symmetric $d \times d$ matrix $A$ and a function $\phi : \mathbb{R} \mapsto \mathbb{R}$, we define $\bar{\phi}(A)$ as the $d \times d$ matrix obtained by applying $\phi$ to the eigenvalues of $A$. Formally, let us rewrite $A$ using its spectral decomposition, $\sum_{i=1}^d \lambda_i u_i u_i^T$ where $\lambda_i, u_i$ are $A$’s $i$’th eigenvalue and eigenvector respectively. Then, we define $\bar{\phi}(A) = \sum_{i=1}^d \phi(\lambda_i) u_i u_i^T$. The function $\bar{\phi}$ is said to be operator monotone if $A \succeq B \succeq 0$ implies that $\bar{\phi}(A) \succeq \bar{\phi}(B)$. A classic result in matrix theory used in our analysis is the Löwner-Heinz Theorem (see, for instance Theorem 2.6 in Carlen, 2010), which in particular asserts that the function $\phi(x) = x^\alpha$ is operator monotone for any $\alpha \in [0, 1]$. (Interestingly, it is not the case for $\alpha > 1$.) We also use an elementary identity from matrix calculus to compute derivatives of matrix traces: $\nabla_A \text{Tr}(\phi(A)) = \bar{\phi}'(A)$.

2 Unified Adaptive Regularization

In this section we describe and analyze the meta-algorithm for Adaptive Regularization (AdaReg). The pseudocode of the algorithm is given in Algorithm 1. AdaReg constructs a succession of matrices $H_t$, each multiplies its instantaneous gradient $g_t$. The matrices act as pre-conditioners which reshape the gradient-based directions. In order to construct the pre-conditioners AdaReg is provided with a potential function $\Phi : \mathcal{H} \mapsto \mathbb{R}$ over a subset $\mathcal{H}$ of the positive definite matrices. On each round, $\Phi$ casts a trade-off involving two terms. The first term promotes pre-conditioners which are inversely proportional to the accumulated outer products of gradients, namely,

$$G_t = G_0 + \sum_{s=1}^t g_s g_s^T.$$

(2)

The second term “pulls” back towards typically the zero matrix and is facilitated by $\Phi$. We define the initial regularizer $H_0 = \min_{H \in \mathcal{H}} \{G_0 \bullet H + \Phi(H)\}$.

We now state the main regret bound we prove for Algorithm 1, from which all the results in this paper are derived.
Theorem 1. For any $x^* \in \mathcal{X}$ it holds that
\[ \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) \leq \frac{1}{2} \min_{H \in \mathcal{H}} \{ G_T \cdot H + \Phi(H) - \Phi(H_0) \} + \frac{1}{2} \sum_{t=1}^{T} \Delta_t(x^*), \tag{3} \]
where $\Delta_t(x^*) = \|x_t - x^*\|^2_{H_t^*} - \|x_{t+1} - x^*\|^2_{H_t^*}$.

Note that
\[ G_T \cdot H = \sum_{t=1}^{T} \| g_t \|^2_{H} + G_0 \cdot H. \]

That is, the regret of the algorithm is controlled by the magnitude of the gradients measured by a norm $\| \cdot \|_H$ which is, in some sense, the best possible in hindsight: it is the one that minimizes the sum of the gradients’ norms plus a regularization term. The regularization term, that stems from the choice of the potential function $\Phi$, facilitates an explicit trade-off in the resulting regret bound between minimizing the gradients’ norms with respect to $\| \cdot \|_H$ and controlling the magnitude of $\Phi(H) - \Phi(H_0)$. The second summation term in the regret bound measures the stability of the algorithm in choosing its regularization matrices: an algorithm that changes the matrices $H_t$ frequently and abruptly is thus unlikely to perform well.

By definition, the minimization on the right-hand side of Eq. (3) is attained at $H_T$, thus the bound of Theorem 1 can be rewritten as
\[ \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) \leq \frac{1}{2} \{ G_T \cdot H_T + \Phi(H_T) - \Phi(H_0) \} + \frac{1}{2} \sum_{t=1}^{T} \Delta_t(x^*). \tag{4} \]

To prove Theorem 1, we rely on two standard tools in online optimization. The first is the Follow-the-Leader / Be-the-Leader (FTL-BTL) lemma.

Lemma 2 (FTL-BTL Lemma, Kalai and Vempala, 2005). Let $\psi_0, \ldots, \psi_T : \mathcal{X} \mapsto \mathbb{R}$ be an arbitrary sequence of functions defined over a domain $\mathcal{X}$. For $t \geq 0$, let $x_t \in \arg\min_{x \in \mathcal{X}} \sum_{s=0}^{t} \psi_s(x)$, then,
\[ \sum_{t=1}^{T} \psi_t(x_t) \leq \sum_{t=1}^{T} \psi_t(x_T) + (\psi_0(x_T) - \psi_0(x_0)). \]

(The term $\psi_0(\cdot)$ term is often used as regularization.)

The second tool is a standard bound for the Online Mirror Descent (OMD) algorithm, that allows for a different mirror map on each step (e.g., Duchi et al., 2011). The version of this algorithm relevant in the context of this paper starts from an arbitrary initialization $x_0 \in \mathcal{X}$ and makes updates of the form,
\[ x_{t+1} = \arg\min_{x \in \mathcal{X}} \left\{ g_t \cdot x + \frac{1}{2} \| x - x_t \|^2_{H_t^*} \right\}. \tag{5} \]

This update is equivalent to the one in step (4) of Algorithm 1, as shown in the appendix.

Lemma 3. For any $x^* \in \mathcal{X}$, $g_1, \ldots, g_T \in \mathbb{R}^d$ and $H_1, \ldots, H_T \in \mathcal{S}_+$, if $x_t$ are provided according to Eq. (5), the following bound holds,
\[ \sum_{t=1}^{T} g_t \cdot (x_t - x^*) \leq \frac{1}{2} \sum_{t=1}^{T} \Delta_t(x^*) + \frac{1}{2} \sum_{t=1}^{T} \| g_t \|^2_{H_t} . \]

For completeness, the proofs of both lemmas are given in Appendix A. We now proceed with a short proof of the theorem.
Proof of Theorem 1. From the convexity of $f_t$, it follows that $f_t(x_t) - f_t(x^*) \leq g_t \cdot (x_t - x^*)$. We thus get,
\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) \leq \sum_{t=1}^{T} g_t \cdot (x_t - x^*). 
\]
Hence, to obtain the claim from Lemma 3 we need to show that
\[
\sum_{t=1}^{T} \|g_t\|_{H_t}^2 \leq G_T \cdot H_T + \Phi(H_T) - \Phi(H_0). 
\]
To this end, define functions $\psi_0, \psi_1, \ldots, \psi_T$ by setting $\psi_0(H) = G_0 \cdot H + \Phi(H)$, and
\[
\psi_t(H) = g_t g_t^T \cdot H 
\]
for $t \geq 1$. Then, by definition, $H_t$ is a minimizer of $\sum_{s=0}^{t} \psi_s(H)$ over matrices $H \in \mathcal{H}$. Lemma 2 for the functions $\psi_0, \psi_1, \ldots, \psi_T$ now yields
\[
\sum_{t=1}^{T} \psi_t(H_t) \leq \sum_{t=1}^{T} \psi_t(H_T) + \psi_0(H_T) - \psi_0(H_0). 
\]
Expanding the expressions for the $\psi_t$, we get
\[
\sum_{t=1}^{T} \|g_t\|_{H_t}^2 \leq \sum_{t=1}^{T} \|g_t\|_{H_T}^2 + \Phi(H_T) + G_0 \cdot H_T - \Phi(H_0) - G_0 \cdot H_0 
\]
\[
= G_T \cdot H_T + \Phi(H_T) - \Phi(H_0) - G_0 \cdot H_0 
\]
\[
\leq G_T \cdot H_T + \Phi(H_T) - \Phi(H_0). 
\]

2.1 Spectral regularization

As we show in the sequel, the potential $\Phi$ will often have the form $\Phi(H) = -\text{Tr}(\phi(H))$, where $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a (scalar) monotonically increasing function with a positive first derivative. We call this a spectral potential. In this case $\nabla \Phi(H) = -\phi'(H)$, and further, if $\mathcal{H} = S_+$ then step (2) of the algorithm becomes
\[
H_t = (\phi')^{-1}(G_t). 
\]
Hence, the derivation of concrete algorithms from the general framework becomes extremely simple for spectral potentials and amounts to a simple transformation of the eigenvalues of the matrix $G_t$. Furthermore, as we discuss below, spectral potentials make the derivation of simplified diagonal (and scalar) versions of the algorithms a straightforward task.

2.2 Diagonal regularization

To obtain a diagonal version of Algorithm 1, i.e., a version in which the maintained matrices $H_t$ are restricted to be diagonal, the only modification required in the algorithm is to set $\mathcal{H}$ to be the set of all positive definite diagonal matrices, denoted $S_+^{\text{di}}$. Specifically, when $\Phi(H) = -\text{Tr}(\phi(H))$ is a spectral potential and $\mathcal{H} = S_+^{\text{di}}$, then step (2) of the algorithm becomes
\[
H_t = (\phi')^{-1}(\text{diag}(G_t)). 
\]
Indeed, for a diagonal $H$ we have $G_t \cdot H - \Phi(H) = \text{diag}(G_t) \cdot H - \Phi(H)$, and the minimizer of the latter over all positive definite matrices, according to Eq. (6), is the matrix $(\phi')^{-1}(\text{diag}(G_t))$. Since the latter is a diagonal matrix, it is also the minimizer of $G_t \cdot H - \Phi(H)$ over all diagonal positive definite matrices. Consequently, diagonal versions of adaptive algorithms are obtained by replacing the full matrix $G_t$ in Algorithm 1 with its diagonal counterpart $\text{diag}(G_t)$. $H_t = (\phi')^{-1}(\tilde{G}_t)$ instead of $H_t = (\phi')^{-1}(G_t)$. In Sections 3.2 and 4.2 below, we spell out how this is accomplished for the AdaGrad and ONS algorithms.
2.3 Isotropic regularization

To obtain the corresponding scalar versions of Algorithm 1 (namely, analogous algorithms that only adaptively maintain a single scalar step-size), we can modify the algorithm so that $H$ is optimized over the set $S_+^d = \{ sI : s > 0 \}$ of all positive multiples of the identity matrix. If we let $\Phi(H) = -\text{Tr}(\phi(H))$ be a spectral potential and let $\mathcal{H} = S_+^d$, then the update in step (2) of the algorithm is equivalent to

$$H_t = (\phi')^{-1}(\frac{1}{d} \text{Tr}(G_t) I). \quad (8)$$

To see this, note that for $H \in S_+^d$ we have

$$G_t \bullet H + \Phi(H) = \frac{1}{d} \text{Tr}(G_t) I \bullet H + \Phi(H).$$

Since the minimizer of the latter over all positive definite matrices is $(\phi')^{-1}(\frac{1}{d} \text{Tr}(G_t) I) \in S_+^d$, it is also the minimizer over $S_+^d$.

See Sections 3.3 and 4.3 on how scalar versions of the AdaGrad and Online Newton Step algorithms are obtained by means of this simple technique.

3 AdaReg ⇒ AdaGrad

We now derive AdaGrad (Duchi et al., 2011) from the AdaReg meta-algorithm. We first describe how to obtain the full-matrix version of the algorithm. In Section 3.2 we provide the derivation of AdaGrad’s diagonal version. Finally, in Section 3.3 we show that the well-studied adaptive version of online gradient descent can be viewed, and derived based on our framework, as a scalar version of AdaGrad. The three versions employ a potential parameterized by $\eta > 0$,

$$\Phi_{AG}(H) = \eta^2 \text{Tr}(H^{-1}), \quad (9)$$

and differ by the domain $\mathcal{H}$ of admissible matrices $H$. Since $\Phi_{AG}$ is a spectral potential, as we can rewrite, $\Phi_{AG}(H) = -\text{Tr}(\phi(H))$ for $\phi(x) = -\eta^2 x^{-1}$. Simple calculus yields that $(\phi')^{-1}(y) = \eta y^{-1/2}$, which in turn gives that

$$\arg\min_{H > 0} \{ H \bullet G + \Phi_{AG}(H) \} = \eta G^{-1/2}. \quad (10)$$

3.1 Full-matrix AdaGrad

AdaGrad employs the following update on each iteration,

$$x_{t+1} = \Pi_{\mathcal{X}}^{G_t^{1/2}} \left( x_t - \eta G_t^{-1/2} g_t \right), \quad \text{(AdaGrad)}$$

where $g_t = \nabla f_t(x_t)$, $G_t = \epsilon I + \sum_{s=1}^t g_s g_s^T$ for all $t \geq 0$, and $\eta > 0$ is the step-size parameter. In the analysis below, we only assume that the domain $\mathcal{X}$ is bounded and its Euclidean diameter is bounded by $b = \max_{x,x' \in \mathcal{X}} \|x - x'\|$.

To obtain AdaGrad from Algorithm 1, we choose the potential function $\Phi_{AG}$ over the domain $\mathcal{H} = S_+$ and set $G_0 = \epsilon I$. The values of the parameters $\eta$ and $\epsilon$ are determined in the sequel. According to Eq. (10), the norm-regularization matrices used by Algorithm 1 are indeed $H_t = \eta G_t^{-1/2}$, the same used by AdaGrad. Note that for projecting back to the domain $\mathcal{X}$, we can use a projection with respect to the norm $\| \cdot \|_{G_t^{1/2}}$ rather than $\| \cdot \|_{H_t^*}$. Since the two

\footnote{We note that we could have arrived at the same result by choosing the potential $\Phi(H) = \|H^{-1}\|_2$ and minimizing over $\mathcal{H} = S_+$. However, notice that this is not a spectral potential and its analysis is more technically involved.}
norms only differ by a scale, this difference has no effect on the projection step. We now invoke Theorem 1 and bound the second term of the bound in Eq. (3),
\[
\sum_{t=1}^{T} \Delta_t(x^*) = \frac{1}{\eta} (x_1 - x^*)^T G_1^{1/2} (x_1 - x^*) + \frac{1}{\eta} \sum_{t=2}^{T} (x_t - x^*)^T (G_t^{1/2} - G_{t-1}^{1/2}) (x_t - x^*) .
\]
We bound the left term using \( v^T M v \leq \| M \| \| v \|^2 \leq \text{Tr}(M)\| v \|^2 \) for a matrix \( M \succeq 0 \) and a vector \( v \). Setting \( v = x_1 - x^* \) and recalling that the diameter of \( \mathcal{X} \) is bounded by \( b \), we get
\[
\frac{1}{\eta} (x_1 - x^*)^T G_1^{1/2} (x_1 - x^*) \leq \frac{b^2}{\eta} \text{Tr}(G_1^{1/2}) .
\]
To bound the right term above we can use the same technique. We need to show though that \( G_t^{1/2} - G_{t-1}^{1/2} \succeq 0 \) for all \( t \). Indeed, the difference is PSD since \( G_t \succeq G_{t-1} \) and \( x \mapsto x^{1/2} \) is operator monotone. We thus get,
\[
(x_t - x^*)^T (G_t^{1/2} - G_{t-1}^{1/2}) (x_t - x^*) \leq \frac{b^2}{\eta} \text{Tr}(G_t^{1/2} - G_{t-1}^{1/2}) .
\]
Combining the two bounds we get,
\[
\sum_{t=1}^{T} \Delta_t(x^*) \leq \frac{b^2}{\eta} \text{Tr}(G_1^{1/2}) + \frac{b^2}{\eta} \sum_{t=2}^{T} \text{Tr}(G_t^{1/2} - G_{t-1}^{1/2}) = \frac{b^2}{\eta} \text{Tr}(G_T^{1/2}) .
\]
Since \( G_T \bullet H_T = \eta \text{Tr}(G_T G_T^{-1/2}) = \eta \text{Tr}(G_T^{1/2}) \), together with \( \Phi(H_T) = \eta \text{Tr}(G_T^{1/2}) \) and a choice of \( \eta = b/\sqrt{2} \), we have
\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) \leq \left( \eta + \frac{b^2}{2\eta} \right) \text{Tr}(G_T^{1/2}) = \sqrt{2} b \text{Tr}(G_T^{1/2}) ,
\]
for any \( x^* \in \mathcal{X} \). Note that \( \epsilon \) can be taken arbitrarily small.

### 3.2 Diagonal AdaGrad

Duchi et al. (2011) presented a diagonal version of AdaGrad that uses faster updates based on diagonal regularization matrices,
\[
x_{t+1} = \Pi_{\mathcal{X}} \tilde{G}_t^{1/2} (x_t - \eta \tilde{G}_t^{-1/2} g_t) ,
\]
where \( g_t = \nabla f_t(x_t) \) and \( \tilde{G}_t = \epsilon I + \text{diag}(\sum_{s=1}^{t} g_s g_s^T) \) for all \( t \). Following (Duchi et al., 2011), in the analysis of the diagonal algorithm we will assume a bound on the diameter of \( \mathcal{X} \) with respect to the \( \infty \)-norm, which we denote by \( b_\infty \).

In order to obtain the diagonal version of AdaGrad we choose the same potential \( \Phi_{AG} \), but optimize over a domain \( \mathcal{H} \) restricted to diagonal positive definite matrices. From Eq. (7) and Eq. (10), the induced regularization matrices are \( \tilde{H}_t = \eta \tilde{G}_t^{-1/2} \), which recovers the diagonal version of AdaGrad.

Invoking Theorem 1 and repeating the arguments for Full-matrix AdaGrad with \( \tilde{H}_t \) replacing \( H_t \), we obtain that
\[
\sum_{t=1}^{T} \Delta_t(x^*) = \frac{1}{\eta} (x_1 - x^*)^T \tilde{G}_1^{1/2} (x_1 - x^*) + \frac{1}{\eta} \sum_{t=2}^{T} (x_t - x^*)^T (\tilde{G}_t^{1/2} - \tilde{G}_{t-1}^{1/2}) (x_t - x^*) \leq \frac{b^2}{\eta} \text{Tr}(\tilde{G}_1^{1/2}) + \frac{b^2}{\eta} \sum_{t=2}^{T} \text{Tr}(\tilde{G}_t^{1/2} - \tilde{G}_{t-1}^{1/2}) = \frac{b^2}{\eta} \text{Tr}(\tilde{G}_T^{1/2}) .
\]
where the inequality uses the fact that for a diagonal positive semidefinite matrix $A$, it holds that $v^T A v \leq \|v\|_2^2 \text{Tr}(A)$ for any vector $v$. Furthermore, for the second sum in Eq. (3) we have

$$G_T \cdot H_T = \eta \text{Tr} \left( G_T \cdot \tilde{G}^{-1/2}_T \right) = \eta \text{Tr}(\tilde{G}^{-1/2}_T),$$

where we used the elementary yet constructive fact that the support of non-zeros of the product $G_T \cdot \tilde{G}^{-1/2}_T$ is the same as $\tilde{G}^{-1/2}_T \cdot \tilde{G}^{-1/2}_T$. Overall, with the choice of $\eta = b_\infty / \sqrt{2}$ we obtain the regret bound

$$\sum_{t=1}^T f_t(x_t) - \sum_{t=1}^T f_t(x^*) \leq 2b_\infty \sqrt{\text{Tr}(\tilde{G}^{-1/2}_T)}.$$

### 3.3 Isotropic AdaGrad: Adaptive Gradient Descent

A classic adaptive version of the Online Gradient Descent (OGD) algorithm uses standard projected gradient updates of the form

$$x_{t+1} = \Pi_{\mathcal{X}}(x_t - \eta_t g_t), \quad \text{(Adaptive OGD)}$$

with the decreasing step-size policy $\eta_t = c/\sqrt{\sum_{s=1}^t \|g_s\|^2}$ for an appropriate constant $c > 0$. For simplicity, we make the mild assumption that $\|g_1\| > 0$ to avoid degenerate cases.

We now show how the adaptive OGD algorithm is obtained from our framework as a scalar version of AdaGrad. To establish this, consider the potential $\Phi_{\text{AG}}$ and fix the domain $\mathcal{H} = \mathcal{S}_{++}^d = \{sI : s > 0\}$ to be the set of all positive multiples of the identity matrix. Let us also set $G_0 = 0$. Recalling Eq. (8), the resulting regularization matrices used by Algorithm 1 are $H_t = \eta_t \left( \frac{1}{2} \text{Tr}(G_t) \right)^{-1/2} I$. By setting $\eta = c/\sqrt{d}$ we get,

$$H_t g_t = \frac{c}{\sqrt{\sum_{s=1}^t \|g_s\|^2}} g_t = \eta_t g_t,$$

recovering the adaptive OGD algorithm. In order to obtain a regret bound for the Isotropic AdaGrad algorithm, we can apply Theorem 1 and repeat the arguments for Full-matrix AdaGrad. First, we have

$$\sum_{t=1}^T \Delta_t(x^*) = \frac{1}{c} \sqrt{\text{Tr}(G_1)} \|x_1 - x^*\|^2 + \frac{1}{c} \sum_{t=2}^T \left( \sqrt{\text{Tr}(G_t)} - \sqrt{\text{Tr}(G_{t-1})} \right) \|x_t - x^*\|^2 \leq \frac{b^2}{c} \sqrt{\text{Tr}(G_T)}.$$

We also use the following,

$$G_T \cdot H_T + \Phi_{\text{AG}}(H_T) - \Phi_{\text{AG}}(H_0)$$

$$= c \text{Tr}(G_T)^{-1/2} \text{Tr}(G_T) + c \text{Tr}(G_T)^{1/2} - c \text{Tr}(G_0)^{1/2}$$

$$\leq 2c \sqrt{\text{Tr}(G_T)}.$$

Overall, with the choice of $c = b/\sqrt{2}$ we obtain the regret bound

$$\sum_{t=1}^T f_t(x_t) - \sum_{t=1}^T f_t(x^*) \leq b \sqrt{2 \text{Tr}(G_T)} = \sqrt{2b} \left( \sum_{t=1}^T \|g_t\|^2 \right)^{1/2}.$$
3.4 A $p$-norm extension of AdaGrad

We conclude this section with a simple spectral extension of AdaGrad which regularizes according to the $p$-norm of the spectral coefficients. To do so, let us choose

$$\Phi_p(H) = \frac{p^{p+1}}{p} \text{Tr}(H^{-p}) .$$

We then have, $\Phi_p(H) = -\text{Tr}(\phi(H))$ where $\phi(x) = -(\eta^{p+1}/p)x^{-p}$, and therefore $\Phi_p(H)$ is a spectral potential. Elementary calculus yields $(\phi')^{-1}(y) = \eta y^{-1/(p+1)}$, which in turn gives

$$\arg\min_{H > 0} \{ H \cdot G + \Phi_p(H) \} = \eta G^{-1/(p+1)} . \tag{11}$$

We can use Theorem 1 as before to obtain the following regret bound,

$$\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) \leq \frac{1}{2\eta} b^2 \text{Tr}(G_T^{1/(p+1)}) + \frac{p+1}{2p} \text{Tr}(G_T^{p/(p+1)}) .$$

We now set

$$\eta = b \sqrt{\frac{p}{p+1} \text{Tr}(G_T^{1/(p+1)}) / \text{Tr}(G_T^{p/(p+1)})} ,$$

and obtain

$$\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) \leq b \sqrt{\frac{p+1}{p} \text{Tr}(G_T^{1/(p+1)}) \text{Tr}(G_T^{p/(p+1)})} . \tag{12}$$

Setting $p = 1$ yields the AdaGrad update with the same regret bound obtained above. Moreover, the choice of $p = 1$ provides the best regret bound among all choices for $p$. To see that, let us denote the eigenvalues of $G_T$ by $\lambda_1, \ldots, \lambda_d \geq 0$. The product of the two traces in Eq. (12) amounts to

$$\text{Tr}(G_T^{1/(p+1)}) \text{Tr}(G_T^{p/(p+1)}) = \left( \sum \lambda_i^{1/(p+1)} \right) \left( \sum \lambda_i^{p/(p+1)} \right) \geq \left( \sum \lambda_i^{1/2} \right)^2 = \text{Tr}(G_T^{1/2})^2 ,$$

by the Cauchy-Schwarz inequality.\footnote{We note, however, that the worst case analysis does not necessarily transfer to actual performance on real problems, and choosing a $p$-norm regularization with $p \neq 1$ may prove useful in practice.}

4 AdaReg ⇒ Online Newton Step

We now show how to derive the Online Newton Step (ONS) algorithm of (Hazan et al., 2007) from Algorithm 1. The ONS update takes the following form,

$$x_{t+1} = \Pi_{\mathcal{X}}^{G_t}(x_t - \eta G_t^{-1} g_t) , \quad \text{(ONS)}$$

where as before $g_t = \nabla f_t(x_t)$ and $G_t = \epsilon I + \sum_{s=1}^{t} g_s g_s^T$ for all $t \geq 0$. Here again $\eta$ is a fixed step-size. Throughout this section, we assume that the $f_t$ are $\gamma$-Lipschitz, namely, $|f_t(x) - f_t(y)| \leq \gamma \|x - y\|$ for $x, y \in \mathcal{X}$ and the domain’s diameter, $\max_{x,x' \in \mathcal{X}} \|x - x'\| \leq b$, both with respect to the Euclidean norm.
4.1 Full-matrix ONS

Let us first describe how the full-matrix version of ONS is derived through a specific choice for potential, $\Phi_{\text{ONS}}(H)$. We assume that the cost functions $f_1, \ldots, f_T$ are $\beta$-exp-concave over the domain $\mathcal{X} \subseteq \mathbb{R}^d$, with the following minor abuse of terminology. Concretely, we assume that for all $t \geq 1$ and $x, y \in \mathcal{X}$,

$$f_t(x) - f_t(y) \leq \nabla f_t(x) \cdot (x - y) - \frac{\beta}{2} (\nabla f_t(x) \cdot (x - y))^2 .$$

(13)

We refer the reader to (Hazan et al., 2007; Hazan, 2016) for further background on exp-concavity and its precise definition.

To obtain the ONS update, we use the potential,

$$\Phi_{\text{ONS}}(H) = -\frac{1}{\beta} \log |H| ,$$

over $\mathcal{H} = S_+$ and choose a fixed $G_0 = \epsilon I$ for some $\epsilon > 0$. Since $\Phi_{\text{ONS}}(H)$ is equal to $-\text{Tr}(\phi(H))$ where $\phi(x) = \beta^{-1} \log x$, $\Phi_{\text{ONS}}$ is a spectral potential. We get that

$$(\phi')^{-1}(z) = (\beta z)^{-1} ,$$

and thus

$$\text{argmin}_{H > 0} \{ H \bullet G + \Phi_{\text{ONS}}(H) \} = \frac{1}{\beta} G^{-1} .$$

(14)

Therefore, the pre-conditioning matrices induced by the potential $\Phi_{\text{ONS}}$ in Algorithm 1 are $H_t = (1/\beta) G_t^{-1}$, which gives rise to the update rule used by ONS.

We proceed to analyze the regret of ONS by means of Theorem 1. Bounding from above the terms of the right-hand side of Eq. (3), we have that

$$\sum_{t=1}^T \Delta_t(x^*) = \beta (x_1 - x^*)_T G_1 (x_1 - x^*) + \beta \sum_{t=2}^T (x_t - x^*)_T (G_t - G_{t-1}) (x_t - x^*)$$

$$= \beta (x_1 - x^*)_T G_0 (x_1 - x^*) + \beta \sum_{t=1}^T (x_t - x^*)_T (G_t - G_{t-1}) (x_t - x^*)$$

$$\leq \epsilon \beta b^2 + \sum_{t=1}^T \beta (g_t \cdot (x_t - x^*))^2 ,$$

since $G_t - G_{t-1} = g_t g_t^T$. In addition we have, $G_T \bullet H_T = \frac{1}{\beta} \text{Tr}(G_T^{-1} G_T) = \frac{d}{\beta}$. Let us denote the eigenvalues of $G_T - G_0 = \sum_{t=1}^T g_t g_t^T$ by $\lambda_1, \ldots, \lambda_d \geq 0$. Then, the eigenvalues of $G_T$ are $\lambda_1 + \epsilon, \ldots, \lambda_d + \epsilon$ and $\Phi_{\text{ONS}}(H_T)$ is bounded as follows,

$$\Phi_{\text{ONS}}(H_T) - \Phi_{\text{ONS}}(H_0) = \Phi_{\text{ONS}} \left( \frac{1}{\beta} G_T^{-1} \right) - \Phi_{\text{ONS}} \left( \frac{1}{\beta} G_0^{-1} \right)$$

$$= \frac{1}{\beta} \log \frac{|G_T|}{|G_0|} = \frac{1}{\beta} \sum_{i=1}^d \log \left( 1 + \frac{\lambda_i}{\epsilon} \right)$$

$$\leq \frac{d}{\beta} \log \left( 1 + \frac{\gamma^2 T}{\epsilon} \right) .$$

The inequality above stems from the fact that the eigenvalues $\lambda_i$ of $G$ are upper bounded by $\gamma^2 T$, since $\| g_t \|^2 \leq \gamma$ for all $t$ due to the $\gamma$-Lipschitz assumption. In order to put everything
together let us define \( \tilde{f}_t(x) = g_t \cdot x \) and apply Theorem Eq. (3) to \( \tilde{f}_t \), we obtain
\[
\sum_{t=1}^{T} (f_t(x_t) - f_t(x^*)) \leq \sum_{t=1}^{T} (g_t \cdot (x_t - x^*) - \frac{\beta}{2} (g_t \cdot (x_t - x^*))^2) \tag{From Eq. (13)}
\]
\[
= \sum_{t=1}^{T} (\tilde{f}_t(x_t) - \tilde{f}_t(x^*)) - \sum_{t=1}^{T} \frac{\beta}{2} (g_t \cdot (x_t - x^*))^2 \tag{Definition of \( \tilde{f}_t \)}
\]
\[
\leq \frac{1}{2} \sum_{t=1}^{T} (\Delta_t(x^*) - \beta (g_t \cdot (x_t - x^*))^2) + \frac{1}{2} (G_T \cdot H_T + \Phi_{\text{ONS}}(H_T) - \Phi_{\text{ONS}}(H_0)) \tag{Theorem 1}
\]
\[
\leq \frac{\epsilon \beta b^2}{2} + \frac{d}{2\beta} (1 + \log(1 + \gamma^2 T/\epsilon)) .
\]

Taking \( \epsilon = d/(\beta^2 b^2) \) gives the regret bound
\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) \leq \frac{d}{\beta} \left( 1 + \log \left( \frac{(\beta \gamma b^2 T)^d}{d} \right) \right) .
\]

### 4.2 Diagonal ONS

We now turn to develop a diagonal version of ONS. We will show that the diagonal version guarantees \( O(d \log T) \) regret under a coordinate-wise analogue of the exp-concavity property. Formally (and slightly abusing terminology again), we say that the function \( f_t \) is \( \beta \)-coordinate-wise exp-concave over a domain \( \mathcal{X} \) if the following holds for all points \( x, y \in \mathcal{X} \):
\[
f_t(x) - f_t(y) \leq \nabla f_t(x) \cdot (x - y) - \frac{\beta}{2} (\nabla f_t(x))^2 \cdot (x - y)^2 .
\] (15)

Here, we use the shorthand \( u^2 = (u_1^2, \ldots, u_d^2) \) for vectors \( u \in \mathbb{R}^d \). In general, coordinate-wise exp-concavity is not comparable to standard exp-concavity. Namely, if \( f_t \) satisfies Eq. (13) then it may not satisfy Eq. (15) and vice versa. Nonetheless, under either property one can obtain a \( O(d \log T) \) regret using a corresponding algorithm.

To obtain the diagonal ONS algorithm, we use the same potential \( \Phi_{\text{ONS}} \) and restrict \( \mathcal{H} \) to be the set of all diagonal positive definite matrices. Then, letting \( \tilde{G}_t = \text{diag}(G_t) \), Eq. (7) implies that the induced regularization matrices in Algorithm 1 are of the form \( \tilde{H}_t = (1/\beta) \tilde{G}_t^{-1} \), which gives rise to the following update rule,
\[
x_{t+1} = \Pi_{\mathcal{X}}^{\tilde{G}_t} \left( x_t - \frac{1}{\beta} \tilde{G}_t^{-1} g_t \right) \text{ where } \tilde{G}_t = \epsilon I + \text{diag} \left( \sum_{s=1}^{t} g_s g_s^T \right) . \tag{Diagonal ONS}
\]

We repeat the derivation for Full-matrix ONS with diagonal matrices
\[
H_t = (1/\beta) \tilde{G}_t^{-1} = (1/\beta) \text{diag}(G_t)^{-1} ,
\]
and get,
\[
\sum_{t=1}^{T} \Delta_t(x^*) \leq \epsilon \beta b^2 + \beta \sum_{t=1}^{T} \sum_{i=1}^{d} g_{t,i}^2 (x_{t,i} - x^*_{t,i})^2 .
\]
The bound on $G_T \cdot H_T$ amounts to,

$$G_T \cdot H_T = \frac{1}{\beta} \text{Tr}(\text{diag}(G_T)^{-1}G_T) = \frac{d}{\beta}.$$ 

Finally, we bound the difference in the potential of $H_0$ and $H_T$ as follows,

$$\Phi_{\text{ONS}}(H_T) - \Phi_{\text{ONS}}(H_0) = \frac{1}{\beta} \log \frac{|\text{diag}(G_T)|}{|\text{diag}(G_0)|} \leq \frac{d}{\beta} \log \left(1 + \frac{\gamma^2 T}{\epsilon}\right).$$

In summary, using Theorem 1 we obtain the bound,

$$\sum_{t=1}^{T} \sum_{i=1}^{d} \left( g_{t,i}(x_{t,i} - x_t^*) - \frac{\beta g_{t,i}^2}{2} (x_{t,i} - x_t^*)^2 \right) \leq \frac{\epsilon \beta b^2}{2} + \frac{d}{2\beta} \left(1 + \log \left(1 + \frac{\gamma^2 T}{\epsilon}\right)\right).$$

For $\beta$-coordinate-wise exp-concave functions, as defined above, the left-hand side upper-bounds the regret. Taking $\epsilon = d/(\beta^2 b^2)$ gives, for any $x^* \in \mathcal{X}$, the regret bound

$$\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) = \frac{d}{\beta} \left(1 + \log \left(\frac{\beta \gamma b^2 T}{d}\right)\right).$$

### 4.3 Isotropic ONS: Strongly-convex OGD

Finally, we derive a scalar version of the ONS algorithm and show that it yields an adaptive version of the Online Gradient Descent (OGD) algorithm in the strongly convex case (Hazan et al., 2007; Shalev-Shwartz and Kakade, 2009). This version performs the following update,

$$x_{t+1} = \Pi_{\mathcal{X}}(x_t - \eta g_t) \quad \text{(Strongly Convex OGD)}$$

with $\eta_t = \Theta(1/t)$. If the functions $f_t$ are $\alpha$-strongly convex, namely they satisfy

$$f_t(x) - f_t(y) \leq \nabla f_t(x) \cdot (x - y) - \frac{\alpha}{2} ||x - y||^2, \quad \forall x, y \in \mathcal{X},$$

then such an algorithm can achieve $O(\log T)$ regret (Hazan et al., 2007; Shalev-Shwartz and Kakade, 2009). To obtain and analyze this version, we yet again use the potential $\Phi_{\text{ONS}}$ and restrict the domain $\mathcal{H} = \mathcal{S}^d_+ = \{ sI : s > 0 \}$. In addition, we set $G_0 = (\epsilon/d)I$ where $\epsilon$ is determined below. From Eq. (8), we see that Algorithm 1 uses the regularization matrices $H_t = \frac{d}{\beta \text{Tr}(G_t)} I$. Since $\text{Tr}(G_t) = \epsilon + \sum_{s=1}^{t} ||g_s||^2$, the resulting update is of the same form as OGD with an adaptive learning rate of

$$\eta_t = \frac{d/\beta}{\epsilon + \sum_{s=1}^{t} ||g_s||^2}.$$ 

It is evident that $\eta_t$ roughly decays at a rate of $1/t$.

Applying Theorem 1 with $H_t = \frac{d}{\beta \text{Tr}(G_t)} I$ and repeating the same calculations above, we have

$$\sum_{t=1}^{T} \Delta_t(x^*) = \frac{\beta}{d} \text{Tr}(G_0) ||x_1 - x^*||^2 + \frac{\beta}{d} \sum_{t=1}^{T} ||g_t||^2 ||x_t - x^*||^2 \leq \frac{\epsilon \beta}{d} ||x_1 - x^*||^2 + \frac{\beta \gamma^2}{d} \sum_{t=1}^{T} ||x_t - x^*||^2;$$

$$G_T \cdot H_T + \Phi_{\text{ONS}}(H_T) - \Phi_{\text{ONS}}(H_0) = \frac{d}{\beta} + \frac{1}{\beta} \log \left(\frac{\text{Tr}(G_T)}{\text{Tr}(G_0)}\right)^d \leq \frac{d}{\beta} \left(1 + \log \left(1 + \frac{\gamma^2 T}{\epsilon}\right)\right).$$

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Overall, we obtain
\[
\sum_{t=1}^{T} \left( g_t \cdot (x_t - x^*) - \frac{\beta \gamma^2}{2d} \|x_t - x^*\|^2 \right) \leq \frac{\epsilon^2 \beta}{2d} \|x_1 - x^*\|^2 + \frac{d}{2\beta} \left( 1 + \log \left( 1 + \frac{\gamma^2 T}{\epsilon} \right) \right),
\]
Setting \( \beta = \alpha d / \gamma^2 \) and recalling Eq. (16), we see that the left-hand side bounds the regret for \( \alpha \)-strongly-convex functions, and we thus get
\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) \leq \sum_{t=1}^{T} \left( g_t \cdot (x_t - x^*) - \frac{\alpha}{2} \|x_t - x^*\|^2 \right) \leq \frac{\epsilon \alpha}{2\gamma^2} \|x_1 - x^*\|^2 + \frac{\gamma^2}{2\alpha} \left( 1 + \log \left( 1 + \frac{\gamma^2 T}{\epsilon} \right) \right).
\]
To bound \( \|x_1 - x^*\| \) let \( \tilde{x} \) be the minimizer of \( f(x) = \frac{1}{T} \sum_{t=1}^{T} f_t(x) \), which is a \( \gamma \)-Lipschitz and \( \alpha \)-strongly convex function. Thus,
\[
0 \leq f(x) - f(\tilde{x}) \leq \nabla f(x) \cdot (x - \tilde{x}) - \frac{\alpha}{2} \|x - \tilde{x}\|^2 \leq \gamma \|x - \tilde{x}\| - \frac{\alpha}{2} \|x - \tilde{x}\|^2,
\]
thus \( \|x - \tilde{x}\| \leq \frac{2\alpha}{\gamma} \) and \( \|x - x^*\| \leq \|x - \tilde{x}\| + \|\tilde{x} - x^*\| \leq \frac{4\alpha}{\gamma} \). A choice of \( \epsilon = \gamma^2 \) gives us the regret bound
\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) \leq \frac{\gamma^2}{\alpha} (8 + \log T).
\]

5 Brief Discussion

While our focus in this paper was on the derivation of AdaGrad and ONS as special cases of AdaReg, our apparatus can be used to derive new adaptive regularization algorithms. In addition to the \( p \)-norm regularization above, it is also seamless to derive block-diagonal variants of AdaReg, which structurally interpolate between the full and diagonal versions. A more challenging task is to generalize AdaReg to incorporate other, more complex structures of matrices that can efficiently capture intricate dependencies between different parameters. Another possible extension left to explore is allowing time-varying potentials in AdaReg. We plan to study additional potential functions and examine their empirical properties in future work.

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## A Online optimization mini-toolchest

**The FTL-BTL lemma.** We give for completeness a proof of the FTL-BTL Lemma. Note that \( \psi_0 \) can be viewed as a regularization term.

**Lemma 4 (FTL-BTL Lemma).** Let \( \psi_0, \ldots, \psi_T : \mathcal{X} \to \mathbb{R} \) be an arbitrary sequence of functions defined over a domain \( \mathcal{X} \). For each \( t \geq 0 \), let \( x_t \in \text{argmin}_{x \in \mathcal{X}} \sum_{s=0}^{t} \psi_s(x) \). Then, the following inequality holds for \( T \geq 1 \),

\[
\sum_{t=1}^{T} \psi_t(x_t) \leq \sum_{t=1}^{T} \psi_t(x_T) + (\psi_0(x_T) - \psi_0(x_0)) .
\]

**Proof.** We rewrite above equation as, \( \sum_{t=0}^{T} \psi_t(x_t) \leq \sum_{t=0}^{T} \psi_t(x_T) \). This inequality is true for \( T = 0 \). Assume it holds true for \( T - 1 \), then

\[
\sum_{t=0}^{T-1} \psi_t(x_t) \leq \sum_{t=0}^{T-1} \psi_t(x_{T-1}) \leq \sum_{t=0}^{T-1} \psi_t(x_T) ,
\]

where the second inequality follows since \( x_{T-1} \in \text{argmin}_{x \in \mathcal{X}} \sum_{s=0}^{T-1} \psi_s(x) \). Adding \( \psi_T(x_T) \) to both sides proves the result for \( T \), completing the induction. \( \square \)
Online Mirror descent with time-varying norms. The abstract version of online mirror descent (originated from Kivinen and Warmuth, 1997), which is an online version of the classic mirror descent method (Nemirovskii and Yudin, 1983), employs a Bregman divergence to construct its update. As this paper is focused solely on divergences of the form \( \| \cdot \|_H \) we present a simplified analysis of mirror descent for this specific setting. In this confined form, mirror descent sets the next iterate \( x_{t+1} \) as follows,

\[
x_{t+1} = \text{argmin}_{x \in \mathcal{X}} \{ g_t \cdot x + \frac{1}{2}\|x - x_t\|_H^2 \},
\]

where \( g_t \) is an arbitrary vector (usually, one takes \( g_t = \nabla f_t(x_t) \), but for analysis below this need not be the case). This update is equivalent to step (4) of Algorithm 1 as follows:

\[
\Pi^{H_t}_{\mathcal{X}} (x_t - H_t g_t) = \text{argmin}_{x \in \mathcal{X}} \{ \|x - (x_t - H_t g_t)\|_H^2 \}
\]

\[
= \text{argmin}_{x \in \mathcal{X}} \{ \|x - x_t\|_H^2 + 2(x - x_t)^T H_t^{-1} H_t g_t + \|H_t g_t\|_H^2 \}
\]

\[
= \text{argmin}_{x \in \mathcal{X}} \left\{ \frac{1}{2}\|x - x_t\|_H^2 + x \cdot g_t \right\},
\]

where we eliminated terms independent of \( x \).

The regret bound of mirror descent is provided by the following lemma.

**Lemma 5.** For any \( x \in \mathcal{X}, g_1, \ldots, g_T \in \mathbb{R}^d \) and \( H_1, \ldots, H_T \in \mathcal{S}_+ \), if \( x_t \) are given by Eq. (17), the following holds:

\[
\sum_{t=1}^{T} g_t \cdot (x_t - x) \leq \frac{1}{2} \sum_{t=1}^{T} \left( \|x_t - x\|_{H_t}^2 - \|x_{t+1} - x\|_{H_t}^2 \right) + \frac{1}{2} \sum_{t=1}^{T} \|g_t\|_{H_t}^2.
\]

**Proof.** First-order optimality conditions imply that the minimum \( y \) of a convex function \( \psi \) subject to domain constraints \( x \in \mathcal{X} \), then \( \nabla \psi(y) \cdot (y - x) \leq 0 \) for all \( x \in \mathcal{X} \). Taking as \( \psi \) the objective minimized by mirror descent on round \( t \), then \( \nabla \psi(x_{t+1}) = g_t + (x_{t+1} - x_t)^T H_t^{-1} \) and thus for \( x \in \mathcal{X} \),

\[
(g_t + (x_{t+1} - x_t)^T H_t^{-1}) \cdot (x_{t+1} - x) \leq 0.
\]

Re-arranging terms, we get:

\[
g_t \cdot (x_{t+1} - x) \leq (x_t - x_{t+1})^T H_t^{-1} (x_{t+1} - x)
\]

\[
= \frac{1}{2}\|x_t - x\|_{H_t}^2 - \frac{1}{2}\|x_{t+1} - x\|_{H_t}^2 - \frac{1}{2}\|x_t - x_{t+1}\|_{H_t}^2,
\]

where the equality can be verified by expanding both sides. On the other hand, using Hölder’s inequality and the fact that \( ab \leq \frac{1}{2}(a^2 + b^2) \) we obtain

\[
g_t \cdot (x_t - x_{t+1}) \leq \|g_t\|_{H_t} \cdot \|x_{t+1} - x_t\|_{H_t} \leq \frac{1}{2}\|g_t\|_{H_t}^2 + \frac{1}{2}\|x_{t+1} - x_t\|_{H_t}^2.
\]

Adding Eqs. (18) and (19), we have

\[
g_t \cdot (x_t - x) \leq \frac{1}{2}\|g_t\|_{H_t}^2 + \frac{1}{2}\|x_t - x\|_{H_t}^2 - \frac{1}{2}\|x_{t+1} - x\|_{H_t}^2.
\]

Summing over all \( t \), we have the required inequality. □