Singular Centre in Quantum Mechanics as a Black Hole

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Abstract. We consider the radial Schrödinger equation with an attractive potential
singular in the origin. The additional continuum of states caused by the singularity,
that usually remain nontreatable, are shown to correspond to particles, asymptotically
free near the singularity (in the inner channel). Depending on kinematics, they are
either confined by the centre or may escape to infinity (to the outer channel).

The orthonormality within the continuum of confined states is established and the
scattering phase of the particle emitted by the centre and then reflected back to it is
found.

For the deconfinement case a unitary 2×2 S-matrix is found in terms of the Jost
functions, and describes transitions within and between the two channels. The volume
elements in the two channels are different.

The two-channel situation is analogous to the known behaviour of radiation in the
black hole metrics. We discuss the black hole essence of singularly attracting centre
for classical motion and the relativity of time inherent to this problem.

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1. Introduction

There are many reasons to be interested in studying singular potentials in quantum mechanics. The most important ones lie in that these are inherent to the theory and cannot be merely brushed aside referring to its physical non-treatability. For instance one cannot help considering the case of supercritical nuclei with the charge \( Z > 137 \) that creates effective singularity in the Dirac equation (see Berestetsky et al [1]). When studying a charged massive vector boson in a magnetic field it is not consistent to confine oneself to considering only low fields below the threshold \( B = m^2 c^3 / e \hbar \). Studying the Schrödinger equation in the region of purely imaginary angular momentum is important in the framework of the Regge theory, especially in proving the Mandelstam dispersion relations with respect to the scattering angle cosine (DeAlfaro and Regge [2]). The attractive singular centrifugal term that appears for imaginary angular momentum causes the fall down onto the centre already at the free (potentialless) level. The order-to-order growing divergencies that accompany the perturbation expansion in the Lippmann-Schwinger equations in singular attractive case are believed to model the nonrenormalizability in quantum field theory [3], especially the growing divergencies in the ladder Bethe-Salpeter equation. There are experimentally probed problems [4] of a charged straight wire with a magnetic field applied along it or with the Aharonov-Bohm potential, the both problems possessing [5] the \( -\beta/r^2 \) singularity in the Schrödinger equation (see [6, 7]). We have listed these cases to illustrate the simple idea that consideration of singular potential is inevitable.

The singular potential problem in quantum mechanics attracted essential efforts during, perhaps, half a century. In spite of a mathematical advancement achieved by applying the theory of a self-ajoint extension of the Hamiltonian [8, 9] (see Audretsch et al [6, 7] for the most recent results in this direction), we cannot state that a satisfactory physical description and understanding of the situation is developed of the fall down onto the center phenomenon, peculiar to the singular potential problem. Nevertheless, at least one lesson should be taught from these studies: there appears an extra continuum of states, whose wave functions decrease at infinity. (These are sometimes inadequately called a continuum of bound states).

In the present paper we study the radial Schrödinger equation and deal with the special case, mentioned above, when the singularity is caused by the centrifugal term \( \lambda^2/r^2 \) with the value \( \lambda^2 = l(l+1) + 1/4 \) (where \( l \) is the angular momentum) taken negative. The external potential is sufficiently well behaving and real. We are going to fully exploit the appearance of extra degrees of freedom by giving a direct physical interpretation to the continuum of states, presented by the asymptotic behaviour of the wave function near the origin \( r = 0 \)

\[
\psi(r) \propto r^{\pm i \lambda} \left( \frac{\ln(r/r_0)}{r_0^\lambda} \right) r^{1/2} e^{-i \lambda}
\]

(1)

where \( r_* = r_0 \ln(r/r_0) \), and \( r_0 \) is a dimensional parameter. In our approach this continuum is in the end responsible for the waves that behave like (asymptotically)
free particles near the origin, with \( \text{Im} \lambda/r_0 \) playing the role of their momentum. These waves are absorbed by the singular centre or emitted by it. For negative energy \( k^2 < 0 \) they form a standing wave in the vicinity of the origin, no overall probability being absorbed/emitted by the centre nor escaping to infinity. This is a regime, that can be thought of as confinement and is described as purely elastic process in the inner channel, when particles emitted from the centre are fully scattered back onto it. When \( k^2 > 0 \) a two-channel regime occurs, i.e. the particles, emitted by the center, not only are reflected back to the centre, but also partially escape to infinity. Correspondingly, particles, incoming from infinity, are partially reflected back into infinity and partially penetrate into the inner region to become asymptotically free in the vicinity of the centre (and be absorbed by it).

To the best of our knowledge the existing papers, including that of Alliluev [10], where a set of waves, diverging from and converging to the centre were considered, deal with the universal measure \( dr \) that serves the scalar product both for \( r \to \infty \) and \( r \to 0 \). The key difference of our approach is that we show that another measure should be used, which is singular near the origin. Its use makes the norm of the oscillating in the origin wave function linearly divergent. This fact is definitely necessary to make it possible to interpret such wave functions as corresponding to particles that are free in the origin and have at their disposal a sufficiently ample volume element in the centre to be such. This is analogous to the linear divergency of the norm of the wave function of an ordinary free particle, infinitely remote from the centre.

The above picture is similar to the behaviour of scalar, electromagnetic, spinor and gravitational waves in the gravitational field of a spinning and nonspinning black hole, with particles disappearing behind the event horizon (see the book by Chandrasekhar [11] and the references to original works by himself, Zel’dovich, Starobinsky, Teukolsky and others therein).

The analogy of the singular centre and a black hole is, perhaps, more profound than just the above two-channel similarity. In section 2 we discuss the classical motion in an attractive \( -\beta/r^2 \) potential and argue that the character of the unbounded motion towards the singular centre proposes that the latter may be associated with a sort of pre-Einstein black hole, i.e. the one, not depending on the General Relativity Theory (GRT), but showing a relativity of time and two different time scales, one of which is singular.

In the rest of the paper we concentrate in the quantum mechanical case.

In section 3 the basic solutions of the radial Schrödinger equation and the corresponding Jost functions are listed.

In section 4 various two-channel and one-channel regimes are described in four domains (called sectors) of the momentum and angular momentum values. This description is summarized in Table 1.

In subsections 4.1, 4.2 the ordinary bound states and elastic scattering sectors are briefly reviewed. In subsection 4.3 the confining sector is studied. The behaviour of the confined particle is described by a one-dimensional Schrödinger equation obtained by
mapping the origin \( r = 0 \) to the negative infinity \( r_* = -\infty \). The effective potential in the transformed Schrödinger equation disappears at \( r_* = -\infty \) and exponentially grows for \( r_* \to \infty \), thus providing the locking of the particle in the inner space (half-bounded motion to and from the centre). The exponential growth of the locking potential is the consequence of the singularity in the initial Schrödinger equation.

The orthonormality condition for confined states is formulated referring to a scalar product with a new measure, singular in the origin: \( dr/r^2 \). This measure appears naturally in the transformed Schrödinger equation and provides the necessary divergency of the norm of the wave function near \( r = 0 \).

In subsection 4.4 the two-channel regime is studied. To describe it a \( 2 \times 2 \) \( S \)-matrix (cf [11] for the black hole case) is obtained, whose unitarity \( SS^\dagger = 1 \) follows from the properties of solutions of the Schrödinger equation, and reflects the overall probability conservation, as well as transformation laws of the Jost functions under the complex conjugation - the time reflection. The \( S \)-matrix contains three angles: two scattering phases and one extra angle, responsible for the mixing of the two channels - the inelasticity angle. The three angles are expressed in terms of three independent Jost functions.

It is instructive, perhaps, for axiomatic quantum field theory, that the \( S \)-matrix unitarity cannot be formulated in terms of states, free at infinity, alone. The states, free near the origin, should be included.

The orthogonality for the two-channel scattering states is established with a measure that tends to \( dr \) for large distances from the centre and to \( r_0^2 dr/r^2 \) for small. For properly arranged orthogonal enelastic scattering states the relation \( k = -\text{Im} \lambda/\tau_0 \) holds (\( \text{Im} k = \text{Re} \lambda = 0 \)), which looks like a quantum number conservation condition, if one understands \( r_0 \) as a conventional border between two channels. Finally, some prospects of future work are mentioned.

In concluding section 5 the general insight into results is sketched.

2. Fall down onto a centre as an unbounded motion in classical mechanics.
PreEinstein black hole

In this section we confine ourselves to classical consideration of the central-symmetric motion in a potential that behaves in an attractive singular way near the origin \( r = 0 \)

\[
U(r) = -\frac{\beta}{r^2}.
\]  

(2)

It dominates over the centrifugal barrier when

\[
\beta > \frac{L^2}{2m},
\]  

(3)

where \( m \) is the mass and \( L \) the angular momentum of a probe particle. When (2) is fulfilled, the origin becomes a classically admissible point, since the square root

\[
R = \left[ (E - U(r)) \frac{2}{m} - \frac{L^2}{m^2 r^2} \right]^{1/2}
\]  

(4)
where \( E > 0 \) is the energy, is real at \( r = 0 \) (the perihelion point corresponds to a complex value of \( r \)). Then the motion is "unbounded" near the origin in the sense that the particle turns around the centre infinite number of times before it reaches it following the logarithmic spiral. This is seen from the expression for the angle differential

\[
d\phi = R^{-1} \frac{dr}{r^2 m}, \tag{5}
\]

which leads to a logarithmic divergency in the origin \( \phi = -(\ln r)L(2\beta m - L^2)^{-1/2} \). On the other hand, the time \( t \), necessary for the particle to reach the centre, and the distance \( s \), which it passes before it does so, are both finite, since their differentials

\[
dt = drR^{-1}, \quad ds = rdr\phi \tag{6}
\]

are integrable in the point \( r = 0 \). (Note that in the bordering case \( \beta = L^2/2m \) the angle \( \phi \) diverges like \( L/r\sqrt{2Em} \), and \( s \) also diverges like \( -\ln r(L/\sqrt{2Em}) \), while the time remains finite). If a potential \( U_1(r) \) behaves near the origin like \( 2 \) and is positive in some finite region, there is an apohelium \( r = r_{\text{max}} \) to be found from the equation \( U_1(r_{\text{max}}) = E \), and the trajectory is half-bounded: the particle may fall down onto the centre but cannot escape to infinity. On the other hand, there is also a half-bounded motion outside, provided that \( U_1(r) \to 0 \) when \( r \to \infty \). The situation when the motion is unbounded both at zero and at infinity is also possible (for instance, if \( U_1(r) < 0 \) everywhere). The same character of motion is peculiar to the quantum case, as it will be demonstrated in the next section. The main difference is that in quantum mechanics the particle splits into reflected and transmitted parts.

Now we are going to discuss what the spiral motion with infinite wending around the centre may have to do with the black hole. The black hole in GRT is characterized by the fact that there exist different ideas of time held by different observers. The stationary (external) observer finds that the time necessary for a particle attracted by a black hole to disappear behind the horizon is infinite, whereas a comoving (internal) observer would spend only a finite (proper) time to reach the black hole. The possibility that different times may exist is the manifestation of the principle of relativity of time, that reads that the time appears no sooner than means are pointed of how to make counts of it. The relativity of time is best pronounced within Einstein’s theory of relativity, especially after the limitation on the maximum speed of a signal is imposed. It is certain, however, that the relativity of time, understood as stated above, is less special and exists beyond Einstein’s relativity theory.

We advocate the theory that the famous Achilles-and-Tortoise paradox, invented by Zeno of Elea as early as the fifth century BC, may be understood as introducing the relativity and singularity of time. Let us imagine, that an observer number 1 does not have at his disposal another physical process for defining time except the Achilles-Tortoise pursuit. So he makes time count each time Achilles covers the distance that separated him from Tortoise at the moment of the previous count. Observer number 1 will make infinite number of counts before Achilles catches up with Tortoise. The clock of this observer will indicate an infinite time. On the other hand, an observer number
who may use other physical processes, like planet (bounded) motion, pendulums, atomic transitions etc. to define segments of time taken as equal, would state that Achilles catches up with Tortoise after only finite time passes. The time of observer number 1 turns into infinity, i.e. it possesses a singularity as a function of the time of observer number 2 in a finite point of the latter. In this respect we say that Zeno’s paradox introduces the situation of a sort of preEinstein black hole.

Let us return to the singular centre (2), (3). Define observer number 1 as the one who does not have at his disposal any bounded motion to be used to produce time intervals, taken as equal, only the (unbounded) process of the fall down onto the centre. By definition, the only way for him to measure time is by counting revolutions around the centre. These are infinite in number. Therefore, this observer would state that infinite time passes before the centre is reached, whereas the time obtained by integrating $dt$ in (2) is finite.

The contents of the next sections do not depend directly on the consideration of the present section. What is remarkable is that in our classical consideration time and angle change their roles in the falling down onto the centre process: time is finite, whereas angle is infinite. On the contrary, for the motion, unbounded at infinity, the (scattering) angle is finite and the time, necessary to reach infinity, is infinite. In quantum case we shall see in the next sections that the quantities, canonically conjugated to time and angle, i.e. the energy and the angular momentum (squared) also change their roles near the origin, leading to the change of the volume elements (the metrics).

3. Basic definitions

In the rest of the paper we study the radial Schrödinger equation corresponding to spherically symmetric case

$$H\psi(r) = k^2\psi(r)$$  \hspace{1cm} (7)

$$H = -\frac{d^2}{dr^2} + \frac{\lambda^2 - \frac{1}{4}}{r^2} + V(r), \quad 0 \leq r < \infty,$$  \hspace{1cm} (8)

where $k^2 = E$ is the energy, and $\lambda$ is related to the angular momentum quantum number as $\lambda = l + \frac{1}{2}$. The radial part of the full 3D Schrödinger equation is connected with the solution of (7) as $R(r) = \psi/r$. The potential $V(r)$ is real and good enough as not to essentially affect the character of the solutions, as compared with the free case of $V(r) \equiv 0$. Namely:

1) $V(r)$ decreases at $r \to \infty$ sufficiently fast so that the integral

$$\int_{c}^{\infty} |V(r)|dr < \infty$$  \hspace{1cm} (9)

might converge at the upper limit

2) $V(r)$ is less singular in zero $r = 0$ than the centrifugal term in (8), so that the integral

$$\int_{0}^{c'} r|V(r)|dr < \infty$$  \hspace{1cm} (10)
converge at the lower limit, $c, c' > 0$.

We shall give the parameters $\lambda$ and $k$ in (7), (8) real or purely imaginary values. Together with the reality of the potential $V(r)$ this keeps coefficient functions in (8) real, so that if $\psi(r)$ is a solution to equation (7) the complex conjugate $\psi^*(r)$ is a solution of the same equation, too.

The singular character of the problem under consideration is only due to the imaginarity of $\lambda$ and not to the potential, that does not cause singularity according to (10). For this reason in treating our case many formal results may be borrowed from the so-called Regge pole theory, dealing with analytical continuation in the complex angular momentum plane, that was popular in the sixtieth-seventieth, although no special emphasis was made at that time on the fact that there is a singular dynamics, underlying the formalism. The case of $V(r) \equiv 0$ will be referred to as free, although this seemingly kinematic case is already far from being trivial and contains all the features we are interested in.

In introducing basic definitions below we follow the book by DeAlfaro and Regge [2]. Consider two fundamental pairs of solutions of the Schrödinger equation.

The functions $f_{\lambda,k}(r)$ and $f_{\lambda,-k}(r)$ satisfy equation (7) and have each a single-term asymptotic behaviour at infinity

$$f_{\lambda,\pm k}(r) \propto \exp(\mp ikr), \quad r \to \infty. \tag{11}$$

The form (11) is the leading term of a solution to equation (7) in the asymptotic region $r \to \infty$. The functions $\phi_{\pm \lambda,k}(r)$ satisfy (7) and have each a single-term asymptotic behaviour near the origin

$$\phi_{\pm \lambda,k}(r) \propto r^{\pm \lambda + \frac{1}{2}}, \quad r \to 0. \tag{12}$$

The solution $f_{\lambda,k}(r)$ is even with respect to the change of sign of $\lambda$, since neither the differential operator (8), nor the boundary condition (11) depends on this sign: $f_{\lambda,k}(r) = f_{-\lambda,k}(r)$. Analogously, $\phi_{\lambda,k}(r) = \phi_{-\lambda,k}(r)$.

There are linear relations between the solutions:

$$f_{\lambda,k}(r) = A\phi_{\lambda,k}(r) + B\phi_{-\lambda,k}(r), \tag{13a}$$

$$f_{\lambda,-k}(r) = C\phi_{\lambda,k}(r) + D\phi_{-\lambda,k}(r), \tag{13b}$$

$$\phi_{\lambda,k}(r) = E f_{\lambda,k}(r) + G f_{\lambda,-k}(r), \tag{13c}$$

$$\phi_{-\lambda,k}(r) = H f_{\lambda,k}(r) + K f_{\lambda,-k}(r). \tag{13d}$$

Here the coefficients $A, B, C, D, E, G, H, K$ depend on the parameters $\lambda$ and $k$, and are expressed in terms of the four Jost functions $f(\pm \lambda, \pm k)$, defined as

$$f(\lambda, k) = W(f_{\lambda,k}(r), \phi_{\lambda,k}(r)) = f_{\lambda,k}(r) \frac{d\phi_{\lambda,k}(r)}{dr} - \frac{df_{\lambda,k}(r)}{dr} \phi_{\lambda,k}(r), \tag{14}$$
where $W(f, \phi)$ is the Wronsky determinant of the two independent solutions, as follows

\[
A = -f(-\lambda, k)/2\lambda, \quad K = i(\lambda/k)A,
E = i f(\lambda, -k)/2k, \quad D = -i(k/\lambda)E,
G = -i f(\lambda, k)/2k, \quad B = i(k/\lambda)G,
C = -f(-\lambda, -k)/2\lambda, \quad H = -i(\lambda/k)C.
\]

In obtaining (15) the use was made of (14) and the other two, trivial Wronskians

\[
W(\phi_{\lambda,k}(r), \phi_{-\lambda,k}(r)) = -2\lambda, \quad W(f_{\lambda,k}(r), f_{\lambda, -k}(r)) = 2ik,
\]

as well as of the evenness of $f_{\lambda,k}(r)$ with respect to $\lambda$, and of $\phi_{\lambda,k}(r)$ with respect to $k$. From the Volterra integral equations for $f_{\lambda,k}(r), \phi_{\lambda,k}(r)$, equivalent to the differential equation (7) with the boundary conditions (11), (12), resp., one can get the complex conjugation properties for the solutions [2]

\[
(f_{\lambda,k}(r))^* = f_{\lambda^*, -k^*}(k),
(\phi_{\lambda,k}(r))^* = \phi_{\lambda^*, k^*}(r).
\]

The Jost function satisfies the relation [2]

\[
f^*(\lambda, k) = f(\lambda^*, k^* \exp(-i\pi)).
\]

In the free case, $V(r) \equiv 0$, the above solutions and the Jost function are [2]

\[
\phi_{\lambda,k}^{(0)} = \Gamma(\lambda + 1) \left( \frac{k}{2} \right)^{-\lambda} r^{1/2} J_{\lambda}(kr),
\]

\[
f_{\lambda,k}^{(0)} = \left( \frac{\pi kr}{2} \right)^{1/2} \exp(-i\pi/2(\lambda + 1/2)) H_{\lambda}^{(2)}(kr),
\]

\[
f^{(0)}(k, \lambda) = \left( \frac{ik}{2} \right)^{-\lambda+1/2} (2/\sqrt{\pi}) \Gamma(\lambda + 1).
\]

Here $J_{\lambda}(kr)$ and $H_{\lambda}^{(2)}(kr)$ are cylindric functions in the standard notations, $\Gamma$ is the Euler gamma-function. The free solutions $f_{\lambda,k}^{(0)}(r), \phi_{\lambda,k}^{(0)}(r)$ posses the same asymptotic forms (11), (12), resp., as $f_{\lambda,k}(r), \phi_{\lambda,k}(r)$ owing to the properties (9), (10) of the potential.

4. Four kinematic sectors

We are now in a position to describe four different sectors of the problem: sector I (imaginary $k$, real $\lambda$), sector II (real $k$, real $\lambda$), sector III (imaginary $\lambda$, imaginary $k$), and sector IV (imaginary $\lambda$, real $k$). This consideration is summarized in Table 1.

Sectors I and II are quite ordinary, whereas sectors III and IV reveal the fall down onto the centre features. For the sake of completeness and comparability we describe sectors I and II in the words that will be appropriate for describing sectors III and IV as well (see subsections 4.3 and 4.4).
Table 1. Four sectors I, II, III, IV of the singular Schrödinger problem for various domains of the momentum \( k \) and the angular-momentum-related parameter \( \lambda \)

| \( \lambda \) | \( i \) | \( m \) | \( a \) | \( l \) | \( r \) | \( e \) | \( Sector \) I | \( Sector \) II |
|---------------|-----|-----|-----|-----|-----|-----|----------|----------|
| \( \lambda \) | \( i \) | \( n \) | \( a \) | \( r \) | \( g \) | \( y \) | \( \text{Continuum of} \) | \( \text{BOUND STATES} \) |
| \( \lambda \) | \( i \) | \( n \) | \( a \) | \( r \) | \( g \) | \( y \) | \( \text{Continuum of} \) | \( \text{ELASTIC SCATTERING states} \) |
| \( \lambda \) | \( i \) | \( n \) | \( a \) | \( r \) | \( g \) | \( y \) | \( \text{in the inner channel} \) | \( \text{TWO CHANNELS} \) |

4.1. Bound states

In sector I a discrete manifold of states may appear as usual.

For \( \text{Re} \lambda \geq \frac{1}{2}, \text{Im} \lambda = 0 \) we face repulsion in the vicinity of the origin, the wave function \( \phi_{\lambda,k}(r) \) according to (12) decreases fast enough to keep the radial part of the wave function of the 3D Schrödinger equation \( R(r) = \psi(r)/r \) finite (see Landau’s and Lifshits’ Quantum Mechanics [13]). Simultaneously, for \( \text{Re}k = 0, \text{Im}k > 0 \) the wave function \( f_{\lambda,-k}(r) \) is decreasing and square-integrable near infinity \( \| f_{\lambda,-k}(r) \|_\infty < \infty \), where

\[
\| \psi \|_\infty = \int_0^\infty |\psi(r)|^2 \, dr, \tag{20}
\]

while the function \( f_{\lambda,k}(r) \) is not. Correspondingly, the physically acceptable states are given by zeros of the coefficient \( E \) in (13), i.e. satisfy the equation \( f(\lambda, -k) = 0 \) according to (15). Its solutions, if any, make a discrete manifold of trajectories \( \text{Im}k_{n_r}(\text{Re}\lambda) \), labelled by an integer, \( n_r = 0, 1, \ldots \) (For half-integral \( \lambda_l = l + \frac{1}{2}, l = 0, 1, \ldots \) these are physical bound states, labelled by a couple of discrete quantum numbers, \( n_r \) and \( l \).) Finally, the wave function of the acceptable states is \( \phi_{\lambda,k}(r) = f_{\lambda,-k}(r) \). As for the function \( \phi_{-\lambda,k}(r) \), it cannot provide the finiteness of \( R(r) \) for any \( \text{Im} \lambda = 0, \text{Re} \lambda \geq \frac{1}{2} \), and is not to be used.

We are left with the interval \( 0 \leq \text{Re} \lambda < \frac{1}{2}, \text{Im} \lambda = 0 \), not considered yet. In this interval the interaction in (4, 5) is already attractive, but not strong enough to give rise to the fall down onto the centre (13), since (12) does not oscillate. The criterion of the finiteness of \( R(r) \), used above, is no longer applicable to select solutions: this function is infinite in \( r = 0 \) for the both functions \( \phi_{\pm\lambda,k}(r) \). Generally the requirement that \( R(r) \)
should be finite in the origin lies beyond the context of the equation (4.8) under study, is not natural from the formal point of view and should not be used at all. As for the norm (21), it is not indicative within the interval $0 \leq \text{Re}\lambda < \frac{1}{2}$, since it converges for the both functions $\phi_{\pm\lambda,k}(r)$ there.

As a matter of fact (see the next subsections 4.3, 4.4), a special measure, which behaves like $dr/r^2$ near the origin, appears naturally in the problem, to be used when calculating norms and scalar products. Appealing to this measure we state that the norm

$$\| \psi \|_0 = \int_0^\infty |\psi(r)|^2 \frac{dr}{r^2}$$

converges at the lower limit in sector I, for $\psi = \phi_{\lambda,k}(r)$, $\text{Re}\lambda \geq 0$, while the same integral (21) for $\psi = \phi_{-\lambda,k}(r)$ diverges for $\text{Re}\lambda \geq 0$.

We conclude that physically accepted solutions $\phi_{\lambda,k}(r) = f_{\lambda,-k}(r)$ in sector I, that make - provided they exist - a discrete set specified by trajectories $\text{Im}k_{\lambda,n}(\text{Re}\lambda)$, are selected by the requirement that the norm (21) for them converges at the upper limit, and the norm (21) at the lower. (The convergences of (20) at the lower, and of (21) at the upper limit are then automatically guaranteed). We shall add more concerning the orthonormality relations among the acceptable solutions of sector I in subsection 4.4.2.

4.2. Elastic scattering states

In sector II (real $k$, real $\lambda$) usual continuum of states that are free at infinity lies. Again, out of the two solutions $\phi_{\lambda,k}(r), \phi_{-\lambda,k}(r)$ according to (12) only $\phi_{\lambda,k}(r)$ has the norm (21) converging at the lower limit for $\text{Re}\lambda \geq 0$. According to (13c), (11) the wave function $\phi_{\lambda,k}(r)$ has two nondecreasing oscillating asymptotic forms at infinity, an incoming and outgoing waves, of which none is better than the other. For the both of them the norm (20) $\| f_{\lambda,\pm k}(r) \|_\infty = \infty$ diverges linearly with respect to the upper cutoff $R$, if the latter is introduced. The normalization of scalar products by dividing over $R$ is used and ensures that the particles stay mostly at an infinite distance from the centre.

The Wronsky determinant of two mutually complex conjugate solutions is (up to a factor of $i$) the probability flux density to or from the centre.

$$P(\psi) = i \left( \psi \frac{d\psi^*}{dr} - \psi^* \frac{d\psi}{dr} \right).$$

The normalization of the probability flux density remains at this stage nonfixed and depends upon normalization of the appropriate wave function. Since there is no singularity between the points $r = 0, \infty$, the probability flux density is independent of $r$. In sector II the probability flux carried by the wave function $\phi_{\lambda,k}(r)$ disappears: $P(\phi_{\lambda,k}(r)) = 0$ already because this function is real near the origin in accord with (12). Thus, this function is responsible for elastic scattering of the incoming wave by turning it into outgoing wave, with no probability being transferred to the centre. The incoming and outgoing waves form a standing wave at infinity with equal amplitudes and a phase difference, which is the scattering phase. The elastic scattering reduces to
pure reflection with no transmission of the wave towards the centre. The absence of penetration is due to the singular repulsive (locking out) barrier (of kinematical nature) provided by the centrifugal term \((\lambda^2 - 1/4)/r^2\) in (8). The probability flux densities (22) associated with the two functions \(f_{\lambda,k}(r)\) and \(f_{\lambda,-k}(r)\) forming \(\phi_{\lambda,k}(r)\) in view of (13) are equal in absolute value but differ in sign, the crossing terms being zero. The overall probability, transferred to or from the infinity is also zero in virtue of the constancy of the Wronsky determinant. “The infinity absorbs all it emits”.

All these well known facts will find their close analogy in the situation characteristic of sector III, to be described in the next subsection 4.3.

4.3. Continuum of confined states

In sector III of Table 1, wherein \(\text{Re}\lambda = 0\), \(\text{Re}\kappa = 0\) consider the wave function \(f_{\lambda,-k}(r)\) in (13b). For \(\text{Im}\kappa > 0\) owing to (11) this function is exponentially decreasing, as \(r \to \infty\), and real. It has the norm (24) convergent at the upper limit \(\|f_{\lambda,-k}(r)\|_{\infty} < \infty\), and carries to/from the infinity a zero probability flux (22) \(P(f_{\lambda,-k}(r)) = 0\). As for its two asymptotic forms near the origin (12), they both oscillate, none of them being any worse or better than the other. These states make a continuum, since there is no reason to require that any of the coefficients \(C\) or \(D\) in (13b) should turn to zero. We insist that the equality in rights of the two asymptotically oscillating solutions (12) in sector III leads to analogous interpretation as the same circumstance in sector II, i.e. they are responsible for free particles, this time incoming from or outgoing to the centre.

4.3.1. Transformed Schrödinger equation

To make this interpretation explicit let us perform the transformation of the variable in the Schrödinger equation (7), (8)

\[
r_* = r_0 \ln \frac{r}{r_0},
\]

where \(r_0\) is an arbitrary dimensional parameter, accompanied with the transformation of the wave function

\[
\left(\frac{r_0}{r}\right)^{\frac{1}{2}} \psi(r) = \tilde{\psi}(r_*)
\]

or

\[
\tilde{\psi}(r_*) = \exp\left(-\frac{r_*}{2r_0}\right) \psi(r_0 \exp \frac{r_*}{r_0})
\]

The transformation (23) maps the origin \(r = 0\) to the minus infinity \(r_* = -\infty\). The transformation (24) is intended to annihilate the first-order derivative term in the resulting differential equation. With (23), (24) the Schrödinger equation (7), (8) turns into

\[
H_{\text{cent}} \tilde{\psi}(r_*) = \frac{(\text{Im}\lambda)^2}{r_0^2} \tilde{\psi}(r_*), \quad -\infty < r_* < \infty,
\]

\[
H_{\text{cent}} = -\frac{d^2}{dr_*^2} - \exp\left(\frac{2r_*}{r_0}\right) \left(k^2 - V(r_0 \exp \frac{r_*}{r_0})\right).
\]
In what follows we use the tilde sign to denote the transform (24), (25) of any solution of equation (7), (8).

The probability flux density (22) - like any other Wronsky determinant - remains form-invariant under the transformation (24), (23)

\[ i \left( \psi(r) \frac{d\psi^*(r)}{dr} - \psi^*(r) \frac{d\psi(r)}{dr} \right) = i \left( \tilde{\psi}(r) \frac{d\tilde{\psi}^*(r)}{dr} - \tilde{\psi}^*(r) \frac{d\tilde{\psi}(r)}{dr} \right). \]

This invariance would be violated if we have chosen different scales in the \( r \)– and \( r_* \)– coordinates by using the transformation \( r_* = r_0^* \ln(r/r_0) \) with \( r_0^* \neq r_0 \) in place of (23).

Bearing in mind the property (10) of the potential one sees that in the limit \( r_* \to -\infty \) the potential-containing term in (27) may be omitted (as well as the kinematic term \( k^2 \)). This is the property of asymptotic freedom near the singular centre. The asymptotic form of (26) is the free equation, with \( \text{Im} \lambda/r_0 \) playing the role of asymptotic momentum in the inner channel,

\[ -\frac{d^2}{dr_*^2} \tilde{\psi}_{-\infty}(r_*) = \frac{(\text{Im} \lambda)^2}{r_0^2} \tilde{\psi}_{-\infty}(r_*). \]

Its solutions are

\[ \tilde{\psi}_{-\infty}(r_*) = c_\pm \exp \left( \pm i \frac{\text{Im} \lambda}{r_0} r_* \right), \]

where \( c_\pm \) are arbitrary constants, and \( \text{Im} \lambda \equiv \sqrt{(\text{Im} \lambda)^2} > 0 \). This is in agreement with (12).

Consider

\[ \tilde{f}_{\lambda,-k}(r_*) = \exp(-\frac{r_*}{2r_0}) f_{\lambda,-k} \left( r_0 \exp \frac{r_*}{r_0} \right). \]

This function is a combination of two waves: the one incoming from the negative infinity of \( r_* \), and the other outgoing to the negative infinity of \( r_* \) (the waves, emitted and absorbed by the center \( r = 0 \)). This is seen from (13) when written as follows

\[ \tilde{f}_{\lambda,-k}(r_*) = \exp(-\frac{r_*}{2r_0}) \left[ C\phi_{\lambda,k} \left( r_0 \exp \frac{r_*}{r_0} \right) + D\phi_{-\lambda,k} \left( r_0 \exp \frac{r_*}{r_0} \right) \right] \]

\[ = C\tilde{\phi}_{\lambda,k}(r_*) + D\tilde{\phi}_{-\lambda,k}(r_*) \]

\[ \asymp C \exp \left( \frac{\text{Im} \lambda}{r_0} r_* \right) r_0^{i\text{Im} \lambda + \frac{1}{2}} + D \exp \left( -i \frac{\text{Im} \lambda}{r_0} r_* \right) r_0^{-i\text{Im} \lambda + \frac{1}{2}}, \]

which agrees with (23).

The norm \( \| \tilde{\psi} \|_{-\infty} = \int_{-\infty}^{\infty} |\tilde{\psi}(r_*)|^2 dr_* \), associated with (26) (and not (24)). Taken for the transform of \( f_{\lambda,-k}(r) \), it diverges linearly with the lower cutoff \( L \)

\[ \lim_{L \to \infty} \int_{-L}^{\infty} |\tilde{f}_{\lambda,-k}(r_*)|^2 dr_* = L(|C|^2 + |D|^2), \]

as it should for the continuum of asymptotically free particles (cf sector II). This implies that the particle is located mostly at negative infinity of the variable \( r_* \). Such is the meaning to be given to the phrase: fall down onto the centre.
At the positive infinity of \( r \) also \( r_* \to \infty \) and \( V(r_0 \exp(r_*/r_0)) \) may be as usual neglected as compared to \( k^2 \) in (27) due to the property (3). Equation (26), (27) becomes
\[
- \frac{d^2}{dr_*^2} \tilde{\psi}_\infty(r_*) - k^2 \exp \left( \frac{2r_*}{r_0} \right) \tilde{\psi}_\infty(r_*) = \frac{(\text{Im} \lambda)^2}{r_0^2} \tilde{\psi}_\infty(r_*)
\] (32)
In agreement with (11), (23), (24) the asymptotic behaviour of (31) for \( r_* \to \infty \) is
\[
\tilde{f}_{\lambda, -k}(r_*) \propto \exp \left( -r_0 \text{Im} k \frac{r_*}{r_0} \right) \exp \left( -\frac{r_*}{2r_0} \right).
\] (33)
This is the leading asymptotic term in the solution of (32). Remind that in sector III \( k^2 < 0 \), and we are considering \( \text{Im} k > 0 \). Hence the second term in the l.-h. side of (32) is an infinitely growing positive (repulsive) potential, locking the particle away from the outer world. It is just \( k^2 r^2 \) and originates from the \( 1/r^2 \) singularity in the initial equation (7,8). It prevents the particle from escaping to the positive infinity \( r \to \infty \) in the same way, as the infinite repulsive centrifugal term in sector II prevented it from approaching the origin (cf subsection 4.2). Again, the locking potential in sector III is a purely kinematic (kinetic energy) term that originates from the r.-h. side of (7), whereas in sector II it was the centrifugal potential, forming the r.-h. side of (26). The function (33) for \( \text{Im} k > 0 \) decreases, as it should.

Thus, \( \tilde{f}_{\lambda, -k}(r_*) \) represents the process of elastic scattering of particles incoming from the negative infinity of \( r_* \) (i.e. from the origin of \( r \)) with all the probability reflected inwards (towards \( r = 0 \)), nothing escaping to positive infinity. In other words, one has purely elastic scattering process in sector III, similar to the one in sector II, with the only difference that the asymptotically free particles income and outgo from/to the origin \( r = 0, r_* \to -\infty \).

In the asymptotic domain \( r_* \to -\infty \) the solution (31) may be presented as
\[
\tilde{f}_{\lambda, -k}(r_*) \propto \left( S_{\text{cent}} \exp \left[ \frac{\text{Im} \lambda}{r_0} r_* \right] + \exp \left( -\frac{\text{Im} \lambda}{r_0} r_* \right) \right) D r_0^{-\text{Im} \lambda + \frac{i}{2}},
\] (34)
where the scattering 1×1 matrix (just a complex number) in sector III for the elastic scattering from the centre back to the centre is
\[
S_{\text{cent}} = \frac{C}{D} r_0^{2\text{Im} \lambda} = -\frac{f(-\lambda, -k)}{f(\lambda, -k)} r_0^{2\text{Im} \lambda}.
\] (35)
We have used (13) in deriving the last equality in (33).

Since in sector III the equality \( S_{\text{cent}} S_{\text{cent}}^* = 1 \) holds true due to (15), (18), the scattering matrix (33) can be presented in the form, analogous to the well-known elastic scattering matrix in sector II
\[
S_{\text{cent}} = \exp(i\delta).
\]
In the free case, \( V \equiv 0 \), using (19) one obtains for the scattering phase \( \delta = \delta_0 \), where
\[
\delta_0(\lambda, k) = \pi + 2\text{Im} \lambda \ln \left( \frac{r_0 \text{Im} k}{2} \right) + \arg \frac{\Gamma(1 - \text{Im} \lambda)}{\Gamma(1 + \text{Im} \lambda)}
\]
\[
= \pi + 2\text{Im} \lambda \left[ \ln \left( \frac{r_0 \text{Im} k}{2} \right) + C_E \right]
\]
\[ + 2 \arctan(\text{Im}\lambda) + 2 \sum_{s=1}^{\infty} \left[ \arctan \left( \frac{\text{Im}\lambda}{s+1} \right) - \frac{\text{Im}\lambda}{s} \right], \]  

(36)

where \( C_E = 0.577... \) is the Euler constant, and the series in the r.-h.side converges.

4.3.2. Orthonormality of confined states. Now we follow the common procedure, applicable irrespective of whether sector III or II is concerned, to demonstrate that the confined wave functions \( \tilde{\eta}_{\lambda, -k}(r_*) \), normalized as

\[ \tilde{\eta}_{\lambda, -k}(r_*) = \frac{\text{Im}\lambda \sqrt{2}}{\sqrt{r_0} |f(\lambda, -k)|} \tilde{f}_{\lambda, -k}(r_*), \]  

(37)

satisfy the orthnormality condition, taken for coinciding \( k \)’s and different or equal \( \lambda \)’s. The orthogonality for different \( \lambda \)’s is independently and straightforwardly demonstrated in subsection 4.4.2.

Consider the function

\[ \Delta_L(\lambda, \lambda', k) \equiv \int_{-L}^{\infty} \tilde{\eta}_{\lambda, -k}(r_*) \tilde{\eta}^*_{\lambda', -k}(r_*) dr_* \]  

(38)

and its integral with a test function \( T(\text{Im}\lambda'/r_0) \)

\[ \tau \left( \frac{\text{Im}\lambda}{r_0} \right) = \int_0^{\infty} \Delta_L(\lambda, \lambda', k) T \left( \frac{\text{Im}\lambda'}{r_0} \right) d\frac{\text{Im}\lambda'}{r_0} \]  

(39)

as a limit \( L \to \infty \), where \( -L \) is the size of the box \( (-L, \infty) \), into which the system is placed. The solution (37) in the box is subject to two homogeneous boundary conditions. The first one

\[ \tilde{\eta}_{\lambda, -k}(\infty) = 0 \]  

(40)

is fulfilled due to (11) or (33). The second boundary condition imposed at the other end of the interval

\[ \tilde{\eta}_{\lambda, -k}(-L) = 0 \]  

(41)

is satisfied for discrete values of \( \text{Im}\lambda \). For large \( L \) these values are found from the equation that follows from (34), (37)

\[ - \frac{\text{Im}\lambda_n}{r_0} L + \frac{1}{2} \delta(\lambda_n, k) = -n\pi + \frac{\pi}{2} \]  

(42)

to be

\[ \frac{\text{Im}\lambda_n}{r_0} = \frac{n\pi}{L} \]  

(43)

with \( n = 0, 1, 2, \ldots \infty \) unless \( \delta \) grows too fast when \( \lambda \to 0 \). With (43) in mind the integral (39) is presented as the limit at \( L \to \infty \) of the Darboux sum (we take \( d(\text{Im}\lambda'/r_0) = \pi/L \))

\[ \tau \left( \frac{\pi n}{r_0} \right) = \lim_{L \to \infty} \sum_{n'=0}^{\infty} \Delta_L(\lambda_n, \lambda_{n'}, k) \frac{\pi}{L} T \left( \frac{\pi n'}{L} \right) \]  

(44)

In calculating the limit \( \Delta_L/L \) we take into account that only the diverging part of \( \Delta_L \) may give a finite contribution into it, and hence the integrand in (38) should be replaced.
by its asymptotic form at $r_* \to -\infty$ according to (31) (integral (38) converges at the upper limit $r_* \to \infty$ in sector III due to (11)). With the use of (13), (31) one obtains

$$
\lim_{L \to \infty} \frac{\Delta L}{L} = \lim_{L \to \infty} \frac{1}{L} \int_{-L}^{L} \tilde{\eta}_{\lambda, -k}(r_*) \tilde{\eta}_{\lambda', -k}(r_*) dr_* = \frac{1}{2} \left| \frac{f(\lambda, -k) f(\lambda', -k)}{\int f(-\lambda, -k)^2} \right|
$$

$$
\times \lim_{L \to \infty} \frac{1}{L} \int_{-L}^{L} \left[ f(-\lambda, -k) f(\lambda', -k) \rho_{\lambda, -k}(r_*) \rho_{\lambda', -k}(r_*) \right] dr_* + f(\lambda, -k) f(\lambda', -k) \rho_{\lambda, -k}(r_*) \rho_{\lambda', -k}(r_*) \left( \frac{r_0}{r_*} \right) \left( \frac{1 + \frac{r_0}{r_*}}{\rho_{\lambda, -k}(r_*)} \right),
$$

(45)

where $c$ is any finite number. The difference between the integrals in the first and second lines in (43) is finite and gives vanishing contribution into the limiting value of $\Delta_L/L$. Also for this reason we omitted the crossing terms $\exp(\pm \text{Im}(\lambda + \lambda') r_*/r_0)$, since $\lambda$ and $\lambda'$ are of the same sign and the sum $\lambda + \lambda'$ is nonzero. As long as $\lambda \neq \lambda'$ the integral in (43) is finite and the limit (43) disappears as $L \to \infty$. For $\lambda = \lambda'$ expression (43) is unity in virtue of (18). We conclude, that

$$
\lim_{L \to \infty} \frac{\Delta L}{L} = \delta_{n, n'}.
$$

(46)

Hence, (14) becomes $\tau(\text{Im}\lambda/r_0) = \pi T(\text{Im}\lambda/r_0)$, which implies that $\Delta_\infty$ in (39) is $\pi$ times the Dirac $\delta$-function. Referring to (38) we finally get the orthonormality relation in sector III (Re$\lambda$, Re$\lambda'$, 0, Im$\lambda$, Im$\lambda'$, 0, Im$k$ > 0, Re$k$ = 0)

$$
\Delta_\infty(\lambda, \lambda', k) = \int_{-\infty}^{\infty} \tilde{\eta}_{\lambda, -k}(r_*) \tilde{\eta}_{\lambda', -k}(r_*) dr_* = \pi \delta \left( \frac{\text{Im}\lambda}{r_0} - \frac{\text{Im}\lambda'}{r_0} \right)
$$

(47)

Note, that as long as the parameter $\text{Im}\lambda$ can be viewed upon as a strength of the singular attraction, the orthonormality relation (47) expresses spectral properties with respect to a ”coupling constant”. This may be interesting when singular potentials other than $1/r^2$ are concerned.

To conclude this subsection let us transform the integral (47) to the primary variable. Using (23), (24), (37) we obtain from (47)

$$
\int_{0}^{\infty} \rho_{\lambda, -k}(r) \rho_{\lambda', -k}(r) \frac{dr}{r^2} = \pi \delta(\text{Im}\lambda - \text{Im}\lambda'),
$$

(48)

where

$$
\rho_{\lambda, -k}(r) = \frac{\text{Im}\lambda \sqrt{2}}{|f(\lambda, -k)|} f_{\lambda, -k}(r)
$$

(49)

Relation (48) does not contain $r_0$. Note the singular measure $r^{-2} dr$ that appears in (48).

For the free case, $V(r) \equiv 0$, it follows from the first line of (36) that (13) is indeed the solution to equation (12). After substituting (19) into (19) we find the normalized wave function of a confined state

$$
\rho_{\lambda, -k}^{(0)}(r) = r^{\frac{1}{2}} K_{\text{Im}\lambda}(r \text{Im}k)[\text{Im}\lambda \sinh(\pi \text{Im}\lambda)]^{\frac{1}{2}} \left( \frac{2\lambda}{\pi} \right)^{\frac{1}{2}}
$$

(50)
and (48) takes the form of a relation for McDonald functions with imaginary indices $K_{i\text{Im} \lambda}(r)$:

$$\frac{2}{\pi} \text{Im} \lambda \sinh(\pi \text{Im} \lambda) \int_0^\infty K_{i\text{Im} \lambda}(r)K_{i\text{Im} \lambda'}(r) \frac{dr}{r} = \pi \delta(\text{Im} \lambda - \text{Im} \lambda')$$  \hspace{1cm} (51)

Here, the orthogonality for $\text{Im} \lambda \neq \text{Im} \lambda'$ may be directly seen from the formula 6.576.4 of Gradshteyn and Ryzhik [14] after one makes the power $\lambda$ tend to 1 in it and puts $a = b$. The $\delta$-function is given rise to by the divergency of (51) near $r = 0$. The normalization may be checked using the asymptotic behaviour of McDonald function near $r = 0$ following the example [13] of one-dimensional Schrödinger equation with the barrier reflection.

### 4.4. Two-channel sector

In sector II the (elastic) scattering states were described by one function $\phi_{\lambda,k}(r)$ out of the set of two fundamental solutions $\phi_{\pm \lambda,k}$, while the other, $\phi_{-\lambda,k}(r)$ was physically useless. In sector III, again, only one function, $f_{\lambda,-k}(r)$ was responsible for elastic scattering, while its partner from the fundamental couple, $f_{\lambda,k}(r)$ was meaningless. Correspondingly, the scattering matrices in these sectors were unitary $1 \times 1$ matrices, i.e. just complex numbers with their absolute values equal to unity. One had a single scattering phase in each sector.

In sector IV, where $\lambda$ is imaginary and $k$ real, both solutions from each fundamental couple are meaningful and represent waves incoming from and outgoing to infinity, as (11), and incoming from and outgoing to the origin, as (12) (or from/to the negative infinity of $\mathbf{r}$, as (23), as (24)). Any linear combination (13) of two meaningful fundamental solutions is again a meaningful solution. Two $2 \times 2$ mutually inverse matrices transform two fundamental sets of two solutions according to (13).

#### 4.4.1. $2 \times 2$ scattering matrix

To calculate the probability flux density $P(\phi_{\lambda,k}(r))$ brought from the centre $r = 0$ in the physical state described by the wave function $\phi_{\lambda,k}(r)$ it is sufficient to substitute (14) into (22). This gives $P(\phi_{\lambda,k}(r)) = \text{Im} \lambda$. The same quantity (mind the probability conservation, i.e. the Wronskian (22) independence of $r$) is to be obtained, provided we use the r.-h.side of (13) in the asymptotic form near $r \to \infty$

$$\phi_{\lambda,k}(r) = E \exp(-ikr) + G \exp(ikr)$$  \hspace{1cm} (52)

and substitute this into (22). The crossing term $\sim EG$ vanishes, while the first term gives the contribution $-k|E|^2$ responsible for the influx of probability from infinity, and the second term contributes as $k|G|^2$, which is the probability carried away to infinity. Therefore

$$\frac{|G|^2}{|E|^2} = \frac{\text{Im} \lambda}{k|E|^2} = 1.$$  \hspace{1cm} (53)

For $\text{Im} \lambda < 0, k > 0$ the flux density $P(\phi_{\lambda,k}(r))$ is negative and the wave $\phi_{\lambda,k}(r)$ is absorbed by the centre. Then relation (53) states that the sum of reflection and
transmission coefficients is unity. For $\text{Im}\lambda > 0, k > 0$ the centre emits, and the reflection coefficient $|G|^2/|E|^2$ in (53) is greater than unity (cf the superradiation of the Kerr black hole [11]).

Analogously, in the state represented by the wave function $f_{\lambda, -k}(r), k > 0$ the probability flux density (22) emitted outwards is $P(f_{\lambda, -k}(r)) = k$. By equating this to the same quantity calculated using (13b) and (12) we obtain the probability conservation relation

$$k = \text{Im}\lambda |C|^2 - \text{Im}\lambda |D|^2,$$

where the first term in the r.-h.side describes the probability flux emitted by the centre, when $\text{Im}\lambda > 0$, or absorbed by it, when $\text{Im}\lambda < 0$, while the second term vice versa. Thus, for $\text{Im}\lambda$ negative, after normalizing to the wave incoming from the centre, we obtain from (54)

$$1 = \frac{|C|^2}{|D|^2} - \frac{k}{\text{Im}\lambda |D|^2}.$$

This reads: the coefficient of reflection of the wave emitted from the origin back into the origin plus the coefficient of transmission of the emitted wave to infinity is unity. For positive $\text{Im}\lambda$ the relation that expresses the same physical statement is obtained from (54) by dividing it over $\text{Im}\lambda |C|^2$, which is again the normalization to the emitted wave. This relation is

$$1 = \frac{|D|^2}{|C|^2} + \frac{k}{\text{Im}\lambda |C|^2}.$$

If we assume $k < 0$ in (55), (56) we obtain that the reflection coefficient in the inner channel is greater than unity, which means the internal superradiation due to the fact that there is the overall incoming probability flux from the infinity.

In what follows we consider only negative $\text{Im}\lambda$ and positive $k$ for definiteness.

Represent (13c), (13d) in the form

$$f_{\lambda, k}(r) = -\frac{G}{E} f_{\lambda, -k}(r) + \frac{1}{\sigma E} \sigma \phi_{\lambda, k}(r),$$

$$\sigma \phi_{-\lambda, k}(r) = \frac{\sigma}{D} f_{\lambda, -k}(r) - \frac{C}{D} \sigma \phi_{\lambda, k}(r),$$

where the parameter $\sigma$ is introduced to appropriately normalize the wave function $\phi_{\lambda, k}(r)$

$$\sigma = \left( \frac{k}{\text{Im}\lambda} \right)^{\frac{1}{2}}.$$

Define a $2 \times 2$ scattering matrix $S_{ij}, i, j = 1, 2$. Let $S_{11} = -G/E$ be the coefficient that relates the wave, reflected to infinity, contained in the asymptotic behaviour of $f_{\lambda, -k}(r)$ in (57), to the wave incoming from infinity, contained in $f_{\lambda, k}(r)$. Let, further, $S_{12} = (1/\sigma E)$ be the coefficient that relates the wave transmitted to the origin, contained in $\sigma \phi_{\lambda, k}(r)$ in (57), to the wave incoming from infinity, contained in $f_{\lambda, k}(r)$. Also $S_{22} = -C/D$ relates the wave reflected to the origin, contained in $\sigma \phi_{\lambda, k}(r)$ in (58), to the wave incoming from
the origin, contained in $\sigma \phi_{-\lambda,k}(r)$, while $S_{21} = (\sigma/D)$ relates the wave transmitted to infinity, contained in $f_{-\lambda,k}(r)$ in (58), to the wave incoming from the origin $\sigma \phi_{-\lambda,k}(r)$. Using (15) we can write (57), (58) with the help of the $S$-matrix defined above:

$$
\begin{pmatrix}
  f_{\lambda,k}(r) \\
  \sigma \phi_{-\lambda,k}(r)
\end{pmatrix} =
\begin{pmatrix}
  f(\lambda,k) \\
  2i\sqrt{k}\text{Im} \lambda
\end{pmatrix}^{-1}
\begin{pmatrix}
  f(-\lambda,-k) \\
  f(\lambda,-k)
\end{pmatrix}
\begin{pmatrix}
  f_{\lambda,-k}(r) \\
  \sigma \phi_{\lambda,k}(r)
\end{pmatrix}
$$

(59)

The nondiagonal part of the $S$-matrix in (59) contains only three out of the four Jost functions $f(\pm \lambda, \pm k)$. On the other hand, the Hermitian conjugate matrix $S^\dagger$ contains via (18) another triple of the Jost functions, so that the unitarity property (60) implies a relation between the four Jost functions in sector IV. Such a relation is known (2)

$$
\begin{vmatrix}
  f(\lambda,-k) & f(-\lambda,-k) \\
  f(\lambda,k) & f(-\lambda,k)
\end{vmatrix} = -4k\text{Im}\lambda
$$

(61)

and equivalent to (53), (55).

The most general form of a unitary matrix with antisymmetric nondiagonal part is

$$
S = e^{i\delta_2}
\begin{pmatrix}
  \exp(i\delta_1) \cos \alpha & \sin \alpha \\
  -\sin \alpha & \exp(-i\delta_1) \cos \alpha
\end{pmatrix}
$$

(62)

By comparing this with (53) we find

$$
\delta_1 = \arg f(\lambda,k), \quad \delta_2 = \arg f(\lambda,-k) + \frac{\pi}{2},
$$

$$
\tan \alpha = \frac{2\sqrt{-k}\text{Im}\lambda}{|f(\lambda,k)|}.
$$

(63)

The two scattering phases $\delta_1, \delta_2$ and the channel-mixing nonelasticity angle $\alpha$ are real in sector IV, $\text{Im}\lambda < 0, k > 0$. The element $S_{11}$ in (59) is the usual form of the $S$-matrix in sector II, where the scattering of the particles incoming from infinity is elastic: $|S_{11}| = 1$ due to the relation $f(\lambda,k) = f^*(-\lambda,-k)$, valid for real $\lambda$, real $k$ (see (18)). The element $S_{22}$ in (59) should be compared with the $S_{\text{cent}}$-matrix in sector III (53), where the scattering of particles emitted by the centre back to the centre is elastic: $|S_{22}| = 1$ owing to the relation $f(\lambda,-k) = f^*(-\lambda,-k)$, valid for imaginary $\lambda$, imaginary $k$, (see (18)).

The above definition of the $S$-matrix is subject to an arbitrariness. Without affecting the unitarity and the meaning of the $S$-matrix elements we can change the normalization in (57), (58) by multiplying $\sigma$ by a unit length complex number, say
exp(i\(\delta_3\)). Correspondingly, the nondiagonal elements \(S_{12}\) and \(S_{21}\) in (62), (59) are multiplied by \(\exp(-i\delta_3)\) and \(\exp(i\delta_3)\), resp. with the phase \(\delta_3\) arbitrary. This tells us that the antisymmetricity of the \(S\)–matrix we used for a particular choice of \(\delta_3\) is not an invariant property.

The partial transmission of the wave incoming from infinity to the centre is considered as a transition from the outer channel, formed by the environment of the infinitely remote point, to the inner channel, formed by the environment of the origin. Vice versa, the partial escaping of the wave, emitted by the centre to infinity is considered as the reciprocal transition between the two channels. Each interchannel transition means absorption in the sense that the probability partially leaves the initial channel and is lost for it. For this reason in sector IV the \(S\)-matrix elements \(S_{11}\) and \(S_{22}\) are no longer unit length complex numbers, and the unitarity can only be formulated with the inclusion of the elements \(S_{12},\ S_{21}\) responsible for transitions between the two channels as it was done above.

It would not be appropriate to try to take into account the nonelasticity of the scattering process in sector IV by analytic continuation with respect to \(k\) or \(\lambda\) from any of the elastic sectors II or III. The analytic continuation makes the corresponding scattering phase complex but is unable to create the lacking phase and the nonelasticity angle: a description of the system with a greater number of degrees of freedom cannot be achieved by mere analytic continuation.

For the free case, \(V(r) \equiv 0\), one readily gets from (53), (19)

\[
\delta_{1}^{(0)} = \frac{\pi}{4} - \Im \lambda \ln \frac{k}{2} + \frac{1}{2} \arg \frac{\Gamma(1 + i\Im \lambda)}{\Gamma(1 - i\Im \lambda)}, \quad \delta_{2}^{(0)} = \delta_{1}^{(0)} - \frac{\pi}{2},
\]

(64)

\[
\tan \alpha^{(0)} = \left( e^{-2\pi \Im \lambda} - 1 \right)^{\frac{1}{2}}.
\]

(65)

The transmission coefficient in (53) is

\[
T^{(0)} = \frac{-\Im \lambda}{|F^{(0)}|^2} = \frac{-4\Im \lambda}{|f^{(0)}(\lambda, -k)|^2} = 1 - e^{2\pi \Im \lambda}.
\]

(66)

The transmission vanishes, \(\alpha^{(0)} = 0, \ T^{(0)} = 0\) for \(\Im \lambda = 0\), and is a maximum, \(\alpha^{(0)} = \pi/2, \ T^{(0)} = 1\) for \(\Im \lambda = -\infty\). Expression (66) is in agreement with the absorption cross section of [10].

4.4.2. Orthogonality and measure. By combining any two solutions \(\psi_{1,2}\) of different Schrödinger equations (7), corresponding to different values of \(k^2\) and \(\lambda^2\), and integrating by parts we obtain

\[
\int_{0}^{\infty} \left[ \frac{\lambda_1^2 - \lambda_2^2}{r^2} - (k_1^2 - k_2^2) \right] \psi_1^{*}(r)\psi_2(r)dr
\]

\[
= \left( \psi_1^{*}(r) \frac{d\psi_2}{dr} - \psi_2(r) \frac{d\psi_1^{*}}{dr} \right)_{r=0}^{r=\infty}.
\]

(67)

This holds true both with and without the complex conjugation sign over \(\psi_1(r)\). To avoid a possible misunderstanding, stress that the Wronskian-like form in the r.-h.side
of (67) is not $r$-independent, since $\psi_{1,2}(r)$ are solutions of differential equations with different coefficients in them.

We were/are interested in the functions $\psi_{1,2}(r)$ that:

- decrease at the both ends of the interval $0 \leq r < \infty$. These make the discrete set of functions $\psi_1(r) = \phi_{\lambda_1,k_1}(r) = f_{\lambda_1,-k_1}(r)$ and $\psi_2(r) = \phi_{\lambda_2,k_2}(r) = f_{\lambda_2,-k_2}(r)$ of sector I (see subsection 4.1).

- decrease at the upper end, $r = \infty$, and turn to zero at the lower end when the latter is understood as the limit of the left box end coordinate $r_* = -L$, when $L \to \infty$ ($r = r_0 \exp(-L/r_0) \to 0$). These are the functions $\psi_{1,2}(r) = f_{\lambda_1,-k}(r)$, $k_1 = k_2 = k$, corresponding to the continuum of confined states of sector III (see subsection 4.3).

- turn to zero at the lower end, $r = 0$, and at the upper end, provided that the latter is understood as the limit of the right box wall coordinate $r = R$, when $R \to \infty$. These are the wave functions $\psi_{1,2}(r) = \phi_{\lambda,k_1}(r)$, $\lambda_1 = \lambda_2 = \lambda$ of the continuum of states of sector II (see subsection 4.2).

- turn to zero at the both ends of the interval when these are both understood as limits of the left and right box wall coordinates $L, R$ tending to infinity ($r_* = -L \to -\infty$, $r = R \to \infty$). These are the functions $\psi_{1,2}(r)$ formed by linear combinations of the functions $\phi_{\lambda_1,k_1}(r), \phi_{\lambda_2,k_2}(r)$ and $f_{\lambda_1,k_1}(r), f_{\lambda_2,k_2}(r)$ that belong to the continuum of the two-channel scattering states of sector IV, we are discussing in the present subsection.

In all these cases the r.-h.side of (67) vanishes, provided that the integration limits in the l.-h.side are understood as stated above. Now, the orthogonality relations that follow from (67) are different in different cases and sectors. In sector I for bound states and in sector II for elastic scattering states we are interested in orthogonality of states with different energies $k_1^2 \neq k_2^2$ and coinciding angular momenta. So we put $\lambda_1^2 = \lambda_2^2$ in (67) and see that such states are orthogonal with the measure $dr$ taken in the scalar product. The states with equal momenta squared $k_1^2 = k_2^2$ but different angular momenta $\lambda_1^2 \neq \lambda_2^2$ belonging to the discrete spectrum of sector I or to the continuum of confined states of sector III are orthogonal with the measure $dr/r^2$ like in (68).

In sector IV, let us rearrange the set of solutions of the Schrödinger equation labelled by two parameters $k$ and $\Im \lambda$ into another two-parametric set labelled by $r_0$ and $k$, so that

\[ \Im \lambda = -r_0 k. \]  

(68)

Then (67) takes the form (we assume the zero boundary conditions imposed as discussed above)

\[ (k_1^2 - k_2^2) \int_0^\infty \psi_1^*(r)\psi_2(r) \left( \frac{r_0^2}{r^2} + 1 \right) dr = 0. \]

(69)

Here $\psi_1(r)$ refers to the set of quantum numbers $k = k_1, \lambda = -ir_0k_1$, whereas $\psi_2(r)$ to the set $k = k_2, \lambda = -ir_0k_2$. Hence (69) implies the orthogonality of solutions with
different $k^2$'s (simultaneously, with different $\lambda^2$'s) with the measure
\[
\left(\frac{r_0^2}{r^2} + 1\right) dr
\]
that tends to $dr$ for $r \to \infty$ to provide the possibility of free life of particles at infinity, like in sector II, and tends to $(r_0^2/r^2) dr$ for $r \to 0$ to provide the freedom in the vicinity of the centre, like in sector III.

Relation (68) should be substituted into (59), (63) and other equations of subsection 4.4.

From (70) the meaning of the dimensional parameter $r_0$ becomes clear. It characterizes the separation between the outer and inner worlds in the sense that: the greater $r_0$, the greater the weight of the first term in the measure (70), in other words, the farther the inner world spreads. So $r_0$ may be thought of as the size of the singular centre. The parameter $r_0$ represents the barrier that reflects particles inwards and outwards. Relation (68) is a sort of quantum number conservation law at the "border": the angular momentum, labelling the states in the inner world, is equal to the momentum, labelling the states in the outer world, multiplied by the radius of the border.

Remarkably, with the parametrization (68) the eigenvalues in the r.-h.sides of equations (7) and (26) coincide.

The final remark is in order. Let us represent the radial Schrödinger equation (7, 8) in the form of a special eigenvalue problem
\[
\left(-\frac{d^2}{dr^2} + V(r)\right) \psi(r) = \kappa(r) \psi(r)
\]
where $\kappa(r)$ is not a constant eigenvalue, but a function:
\[
\kappa(r) = k^2 - \frac{\lambda^2 - \frac{1}{4}}{r^2}.
\]
According to the standard reference book [15] if $\kappa(r)$ is continuous and preserves its sign inside an interval $(r_1, r_2)$, and the potential $V(r)$ is smooth enough, equation (71) creates a self-adjoint eigenvalue problem, provided that the boundary conditions at $r = r_1, r_2$ belong to the so-called regular class. Then a complete orthonormal set of functions is associated to the problem (71), the function $\kappa(r) dr$ acting as a measure in the orthonormality relations and the generalized Fourier expansions.

The zero boundary conditions at the box walls ($r_1 = r_0 \exp(-L/r_0)$, $r_2 = R$) proposed above in this subsection, are regular, besides the measure (72) has in sector IV a definite sign. So, the conditions for the complete set to exist are fulfilled. We are going to explicitely accomplish their construction, as well as build the matrix Green functions and the set of Lippmann-Schwinger equations in the forthcoming paper.

5. Concluding remarks

We presented a new physical approach to the singular potential problem, using the $1/r^2$ attractive potential as an example. A great deal of symmetricity in handling the two
singular points $r = 0$ and $r = \infty$ of the radial Schrödinger equation is peculiar for this approach. The singular centre is analogous to the remote infinity in the sense that it also produces incoming and outgoing waves to form the scattering process. The waves, incoming from and outgoing to the centre contribute into the unitarity in the same way as the ones incoming from and outgoing to infinity. The interaction at short distances disappears, the same as it does at large distances. This was seen after a transformation of the Schrödinger equation (24), (25), that maps the origin to minus infinity, had been performed. This transformation turns the initial singularity into a growing potential, that locks the particle away from the outer world, but leaves it free in the origin and supplies it with a sufficiently ample volume element - a singular measure $\frac{dr}{r^2}$ - to be living in. The strength of the singular potential $\text{Im}\lambda$ becomes a momentum-like quantum number in the inner world. As long as the above confinement regime is maintained, the centre absorbs all what it emits and hence remains stable. On the contrary, there is a two-channel regime, where exchanges between the inner and outer worlds take place. There a deconfinement - we called it escaping to infinity in the body of the paper - may occur, as well as its inverse - the absorption. We believe that this unstable regime may be responsible for transitional processes in the course of life of a system, described by the singular potential. During the deconfinement/absorption the particle changes its quantum number following the conservation law (68): that what was the momentum in the outer world becomes the strength-of-the-singular-potential quantum number in the inner. The overall measure (70) serving the two-channel regime corresponds to two different volume elements (metrics) - its limits into the inner and outer worlds. An arbitrary dimensional parameter $r_0$, introduced by the transformation (24), (25) or, alternatively, by (68) viewed upon as reparametrization of states, is intrinsic in the system. It characterizes the size of the system, or the spread border between the outer and inner worlds.

The above pattern contains many features, customary in quantum chromodynamics, such as asymptotic freedom and confinement. It is notable that these phenomena are reproduced basing on a very simple and also very fundamental model, proposed by nonrelativistic quantum mechanics, which makes the primary basis of elementary particle theory. The singular case considered also reveals many features of a black hole beyond GRT. To use a black hole as an elementary particle prototype is sometimes considered a tempting possibility. We believe that the features under discussion should be laid into the foundation of QFT at the level of first and second quantization and might also modify its basic axioms, especially in what concerns the completeness of asymptotic states.

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