AN ALMOST PERIODIC DENGUE TRANSMISSION MODEL WITH AGE STRUCTURE AND TIME-DELAYED INPUT OF VECTOR IN A PATCHY ENVIRONMENT

JING FENG AND BIN-GUO WANG∗
School of Mathematics and Statistics, Lanzhou University
Lanzhou, Gansu 730000, China

(Communicated by Shigui Ruan)

Abstract. In this paper, we propose an almost periodic multi-patch SIR-SEI model with age structure and time-delayed input of vector. The existence of the almost periodic disease-free solution and the definition of the basic reproduction ratio \( R_0 \) are given. It is shown that the disease is uniformly persistent if \( R_0 > 1 \), and it dies out if \( R_0 < 1 \) under the assumptions that there exists a small invasion and the same travel rate of susceptible, infective and recovered host population in different patches. Finally, we illustrate the above results by numerical simulations. In addition, a simple example shows that the basic reproduction ratio may be underestimated or overestimated if an almost periodic coefficient is approximated by a periodic one.

1. Introduction. Dengue diseases, as one of the most common vector borne infectious disease, is present for more than 100 countries. The prevalence of dengue disease has a great impact on human population and has become a major public health concern in recently years. According to World Health Organization [36], there are estimated about 50-100 million new dengue infectious. Almost fifty percent of the world’s population lives in the countries where dengue disease is endemic. Since Kermack and Mckendrick [15] considered an SIR model, more and more researchers have been taking into account in studying the transmission of epidemic diseases [1, 2, 6, 32, 34]. About dengue disease transmission, Esteva and Vargas [7, 8] proposed an SIR-SI model with constant and variable host populations, respectively, and proved the existence and global asymptotic stability of the endemic equilibrium. Phaijoo and Gurung [20] considered a multi-patch SIR-SI model of dengue disease transmission with constant coefficients, and obtained that the disease dies out if \( R_0 < 1 \) and the disease takes hold if \( R_0 \geq 1 \). More work on dengue disease transmission can be found in [10, 16, 21, 22, 28].

Dengue disease is regarded as an infectious disease and transmitted to humans by the bite of infected adult female Aedes mosquitoes. There are four serotypes of virus that play an important role in the transmission of dengue fever, denoted by DEN-1, DEN-2, DEN-3 and DEN-4. People infected with one of the four serotypes of dengue virus will not be infected by the same type of virus again, but will

2020 Mathematics Subject Classification. 34D08, 34C27, 34D45.

Key words and phrases. Almost periodic equations, basic reproduction ratio, disease-free solution.

∗ Corresponding author: Bin-Guo Wang.
temporarily lose immunity of other viruses [11]. On one hand, for some diseases, the main risk group is the adult population who has not been adequately immunized against dengue disease or whose immunity has decreased since childhood [17], and newborns are vaccinated and have a strong immune system because of the protection of the outside world, but the immunization will decline with the growth of age. So we divide the population into two stages, juvenile stage and adult stage, and it is supposed that disease transmission occurs only in adult individuals and that juvenile individuals are immune to the disease. Therefore, it is reasonable to assume that the adult individuals may be only infected and have an impact on human reproduction.

As noted in [31], Wang and Li studied the epidemic model with age structure, and obtained practical uniform persistence and a threshold result for the global attractivity of the disease free state. On the other hand, Aedes mosquitoes undergo four life stages: egg, larva, pupa and adult. As in [23], adult mosquitoes only work on the transmission of dengue disease. Naturally, there is a delay from birth to the stage of adult. Combined with the age structure of host population, it is realistic to consider the epidemic model with age structure and the time delayed input of vector.

As it is well known that seasonal factors play an important role in the many infectious diseases. For example, Watts et.al [35] showed that temperature and rainfall affect mosquitoes vector abundance, development and biting rate, and there is a different timing of outbreak for dengue disease during hot-dry and rainy season. Zhang and Zhao [37] proposed a periodic epidemic in a patchy environment in order to study the role of the seasonality. In this periodic model, all the involved periodic coefficients are assumed to share a common period. But from applied perspective, the birth rate, the death rate, the recovery rate and the other parameters are not necessary to share a common period. Therefore, it is more reasonable to know the dengue transmission model which all coefficient functions have complex oscillation. Especially, if all coefficient functions have no common period or the periods of these coefficient functions have no common integer multiple, then the model is not periodic system. Mathematically, we can treat such a model as an almost periodic system [33]. Since almost periodic functions are a generalization of periodic functions [4, 9], it is more reasonable and general to characterize the transmission of dengue disease. In this paper, we consider an almost periodic dengue transmission model with age structure and time delayed input of vector in a patchy environment.

We denote the number of juvenile individuals and adult individuals in patch \( i \) by \( J^h_i \) and \( H_i \), respectively. When the patches are isolated, \( J^h_i \) is assumed to satisfy

\[
\frac{dJ^h_i}{dt} = B^h_i(t, H_i)H_i - \mu^h_i(t)J^h_i,
\]

where \( B^h_i(t, H_i) \) and \( \mu^h_i(t) \) are the per capita birth rate of adult individuals and the per capita death rate of juveniles in patch \( i \) at time \( t \), respectively.

Refer to [17, 31], consider the following model

\[
\begin{align*}
\frac{dJ^h_i}{dt} &= B^h_i(t, H_i)H_i - \mu^h_i(t)J^h_i - A^h_i(t, H_i) + \sum_{j=1}^{n} m^j_i(t)J^h_j, \\
\frac{dS^h_i}{dt} &= A^h_i(t, H_i) - \frac{b(t)\beta^h(t)}{H_i} S^h_i I^v_i - \frac{b(t)\beta^h(t)}{H_i} S^h_i + \sum_{j=1}^{n} m^j_i(t)S^h_j, \\
\frac{dI^h_i}{dt} &= \frac{b(t)\beta^h(t)}{H_i} S^h_i I^v_i - (\gamma^h(t) + d^h(t))I^h_i + \sum_{j=1}^{n} m^j_i(t)I^h_j,
\end{align*}
\]
\[\frac{dR_i^h(t)}{dt} = \gamma_i(t)I_i^h(t) - d_i^h(t)R_i^h(t) + \sum_{j=1}^{n} m_{ij}^R(t)R_j^h(t),\]

where \(S_i^h, I_i^h\) and \(R_i^h\) are the number of susceptible, infectious and recovered host populations in patch \(i\), respectively. Then \(H_i(t) = S_i^h(t) + I_i^h(t) + R_i^h(t)\) denotes the total host population in patch \(i\), and \(A_i^h, d_i^h, \gamma_i^h\) are the recruitment rate of host population, the death rate in host population, recovery rate of infectious host population, and \(b, \beta_i^h\) represent the biting rate of vector and transmission probability from vector to host, and \(m_i^T, m_i^S, m_i^I, m_i^R\) represent the travel rate of juvenile population, susceptible host population, infectious host population and recovered host population from patch \(i\) to \(j\), respectively. Moreover, during the dispersal process, we assume the births and deaths are omitted because the time scale of an epidemic is generally much shorter than the demographic time scale \([37]\), then

\[\sum_{j=1}^{n} m_{ij}^T(t) = 0, \quad \sum_{j=1}^{n} m_{ij}^S(t) = 0, \quad \sum_{j=1}^{n} m_{ij}^I(t) = 0, \quad \sum_{j=1}^{n} m_{ij}^R(t) = 0, \quad \forall i = 1, \ldots, n.\]

Moreover, We have the following assumptions:

(A1) The matrices \((m_i^S(t))_{n \times n}, \ (m_i^T(t))_{n \times n}, \ (m_i^R(t))_{n \times n}\) are strongly irreducible;

(A2) \(B_i^h(t, H_i) > 0\) for all \(H_i > 0, t > 0\) and \(i = 1, \ldots, n;\)

(A3) The function \(B_i^h(t, H_i)\) is continuous differentiable and satisfies \(\frac{\partial B_i^h(t, H_i)}{\partial H_i} < 0\) for all

\(H_i > 0, t > 0\) and \(i = 1, \ldots, n.\)

Note that (A1) ensures that immigration always occurs between \(n\) separated patches; (A2) means the birth rate function is positive; (A3) implies that the per-capita birth rate is a decreasing function of \(H_i\).

In the following, we find \(J_i^h(t, a)\). Let \(J_i^h(t, a) = (J_i^h(t, a), \ldots, J_n^h(t, a))^T\), where \(J_i^h(t, a)\) is the number of the juveniles in the \(i\)th patch at time \(t\) with age \(a\). Hence,

\[J_i^h(t, 0) = (J_1^h(t, 0), \ldots, J_n^h(t, 0))^T = (B_1^h(t, H_1)H_1, \ldots, B_n^h(t, H_n)H_n)^T =: G(t, H(t)).\]

Assume \(\tau_h\) is the length of the juvenile period. As noted in \([31]\), we have

\[(\partial_t + \partial_a)J_i^h(t, a) = \sum_{j=1}^{n} m_{ij}^T(t)J_j^h(t, a) - (\sum_{j=1}^{n} m_{ij}^T(t) + \mu_i^T(t))J_i^h(t, a)\]

\[= \sum_{j=1}^{n} m_{ij}^T(t)J_j^h(t, a) - \mu_i^T(t)J_i^h(t, a).\]

Let \(J_i(t, a) = J_i(t, t - a)\) for all \(t \geq a \geq 0\). Then

\[\frac{\partial J_i(t, a)}{\partial t} = E_{j_i}(t)J_i(t, a),\]

\[(2)\]
where
\[
E_{J^h}(t) = \begin{pmatrix}
-\mu_1^h(t) + m_{11}^J(t) & m_{12}^J(t) & \cdots & m_{1p}^J(t) \\
m_{21}^J(t) & -\mu_2^h(t) + m_{22}^J(t) & \cdots & m_{2p}^J(t) \\
\vdots & \vdots & \ddots & \vdots \\
m_{n1}^J(t) & m_{n2}^J(t) & \cdots & -\mu_n^h(t) + m_{nn}^J(t)
\end{pmatrix}.
\]

Let \( \Phi(t, a), t \geq a \geq 0 \) be the fundamental solution matrix of (2), that is, \( \frac{\partial \Phi(t, a)}{\partial t} = E_{J^h}(t) \Phi(t, a) \) and \( \Phi(t, a) \) is an identity matrix. Hence,
\[
J^h(t, a) = \Phi(t, a) J^h(a, a), \quad \forall t \geq a.
\]

Since
\[
J^h(t, s) = J^h(t, t-s) = \Phi(t, t-s) J^h(t-s, t-s)
= \Phi(t, t-s) J^h(t-s, 0)
= \Phi(t, t-s) G(t-s, H(t-s)), \quad \forall t \geq s.
\]

It then follows that
\[
J^h(t, \tau_h) = \Phi(t, t-\tau_h) G(t-\tau_h, H(t-\tau_h)) := A^h(t, H(t-\tau_h)),
\]

where \( A^h(t, H(t-\tau_h)) = (A^h_1(t, H(t-\tau_h)), \ldots, A^h_n(t, H(t-\tau_h))) \). Let \( \Phi(t, t-\tau_h) := (p_{ij}(t))_{n \times n}. \) In view of \( E_{J^h}(t) \) is cooperative, it then follows from [3, Lemma2] that \( p_{ij}(t) > 0 \) for all \( i, j = 1, \ldots, n \) and \( t > 0 \). Obviously, \( A^h \) is independent of the variables of juveniles, and we rewrite system (1) as follows:
\[
\begin{aligned}
d\frac{dS_i^h}{dt} &= A^h_i(t, H(t-\tau_h)) - \frac{b_i^h(t) \beta_i^h(t)}{H_i} S_i^h R_i^v - d_i^h(t) S_i^h + \sum_{j=1}^{n} m_{ij}^S(t) S_j^h, \\
d\frac{dI_i^h}{dt} &= \frac{b_i^h(t) \beta_i^h(t)}{H_i} S_i^h R_i^v - (\gamma_i^h(t) + d_i^h(t)) I_i^h + \sum_{j=1}^{n} m_{ij}^I(t) I_j^h, \\
d\frac{dR_i^h}{dt} &= \gamma_i^h(t) I_i^h - d_i^h(t) R_i^h + \sum_{j=1}^{n} m_{ij}^R(t) R_j^h, \\
A^h_i(t, H(t-\tau_h)) &= \sum_{j=1}^{n} p_{ij}(t) G_j(t-\tau_h, H_j(t-\tau_h)).
\end{aligned}
\]

As noted in [23], let \( b_i^v(t, V_i(t)) \) be the egg reproduction rate which incorporates the death rate of immature mosquitoes at time \( t \), where \( V_i(t) \) is a function of the total number of mature mosquitoes in the \( i \)th patch at the time \( t \), and let \( A^v_i(t, V_i(t)) = b_i^v(t, V_i(t)) V_i(t), i = 1, \ldots, n. \) We suppose \( \tau_v \) is the average mature period of vector. From the mechanism of transmission of dengue disease between host and vector [20], we have
\[
\begin{aligned}
d\frac{dS_i^v}{dt} &= A^v_i(t, H(t-\tau_v)) - \frac{b_i^v(t) \beta_i^v(t)}{H_i} S_i^v R_i^v - d_i^v(t) S_i^v + \sum_{j=1}^{n} m_{ij}^S(t) S_j^v, \\
d\frac{dI_i^v}{dt} &= \frac{b_i^v(t) \beta_i^v(t)}{H_i} S_i^v R_i^v - (\eta_i^v(t) + d_i^v(t)) I_i^v + \sum_{j=1}^{n} m_{ij}^I(t) I_j^v, \\
d\frac{dR_i^v}{dt} &= \eta_i^v(t) I_i^v - d_i^v(t) R_i^v + \sum_{j=1}^{n} m_{ij}^R(t) R_j^v, \\
A^v_i(t, H(t-\tau_v)) &= \sum_{j=1}^{n} p_{ij}(t) G_j(t-\tau_v, H_j(t-\tau_v)),
\end{aligned}
\]

(3)
with the initial value \( S_i^h(\theta_1) = \phi_i^h(\theta_1), I_i^h(\theta_1) = \psi_i^h(\theta_1), R_i^h(\theta_1) = \phi_i^h(\theta_1), \theta_1 \in [-\tau_h, 0] \), \( S_i^0(\theta_2) = \phi_i^0(\theta_2), E_i^0(\theta_2) = \psi_i^0(\theta_2), I_i^0(\theta_2) = \phi_i^0(\theta_2), \theta_2 \in [-\tau_v, 0] \), \( \phi_i^h, \psi_i^h, \phi_i^0, \psi_i^0 \in C_{\mathbb{R}}^3 \), where \( C_{\mathbb{R}} = ([-\tau_h, 0], \mathbb{R}_+^{3}) \times ([-\tau_v, 0], \mathbb{R}_+^{3}) \). Here \( S_i^h, E_i^0, I_i^0 \) represent the number of susceptible, exposed, infectious vector populations, respectively. \( A_i^h, d_i^h, \beta_i^h, \eta_i^h \) are the recruitment rate of vector population, the death rate in vector population, transmission probability from host to vector, transfer rate of vector from the exposed class to the infectious class, respectively. In an almost periodic environment, we assume that \( B_i^h(t, H_r), A_i^h(t, H(t-\tau_h)), A_i^e(t-\tau_v, V_i(t-\tau_v)) \) are uniformly almost periodic in \( t \), and \( b(t), \beta_i^h(t), d_i^h(t), \gamma_i^h(t), m_{ij}^S(t), m_{ij}^I(t), m_{ij}^R(t), d_i^v(t), \beta_i^v(t), \eta_i^v(t) \) are almost periodic in \( t \). Thus system (3) is an almost periodic with age structure and time delayed differential system.

The remaining parts are organized as follows: In section 2, we introduce the basic reproduction rate \( R_0 \). In section 3, we prove the threshold dynamics behaviour on \( R_0 \) for the model. In Section 4, we illustrate the obtained theoretical results by numerical simulations of an example. A discussion completes the paper.

2. Basic reproduction ratios. A function \( f \in C(\mathbb{R}, \mathbb{R}^n) \) is said to be almost periodic if for any \( \varepsilon > 0 \), there exists \( l = l(\varepsilon) > 0 \) such that every interval of \( \mathbb{R} \) of length \( l \) contains at least one point of the set

\[
T(f, \varepsilon) = \{ s \in \mathbb{R} : |f(t+s) - f(t)| < \varepsilon, \forall t \in \mathbb{R} \},
\]

where \(|\cdot|\) is the Euclidean norm in \( \mathbb{R}^n \).

Let \( AP(\mathbb{R}, \mathbb{R}^n) := \{ f \in C(\mathbb{R}, \mathbb{R}^n) : f \text{ is an almost periodic function} \} \). Then \( AP(\mathbb{R}, \mathbb{R}^n) \) is a Banach space equipped with the supremum norm \( ||\cdot|| \) (33). Let \( \Omega \subset \mathbb{R}^n \). A function \( f \in C(\mathbb{R} \times \Omega, \mathbb{R}^n) \) is said to be uniformly almost periodic in \( t \) if \( f(\cdot, x) \) is almost periodic for each \( x \in \Omega \), and for any compact set \( \mathcal{D} \subset \Omega \), \( f \) is uniformly continuous on \( \mathbb{R} \times \mathcal{D} \) (4, 9). If \( f \) is an almost periodic function, then there exists a Fourier series \( \sum_{q=1}^{\infty} A_q e^{i\lambda_q t} \) associated with the function \( f \), that is, \( f(t) \sim \sum_{q=1}^{\infty} A_q e^{i\lambda_q t} \). We call \( \lambda_q, q = 1, 2, \ldots \), the Fourier exponent of \( f(t) \), and \( A_q \) is the Fourier coefficient of \( f(t) \). For an almost periodic function \( f \), the module of \( f \), denoted by \( mod(f) \), is defined as the smallest additive group of real numbers that contains the Fourier exponent of \( f(t) \).

A square matrix \( M \) is said to be cooperative if all off-diagonal entries of \( M \) are nonnegative. If all entries of \( M \) are nonnegative, then we call \( M \) nonnegative.

A nonnegative matrix \( M \) is called positive if \( M \) is not the zero matrix.

A square matrix \( M = (m_{ij}) \) is said to be quasi-positive if it is not the zero matrix and all off-diagonal entries \( M \) are nonnegative, that is, \( m_{ij} \geq 0 \) for \( i \neq j \).

A \( n \times n \) square matrix \( (m_{ij}) \) is said to be strongly irreducible if there exists a \( \theta > 0 \) such that if two nonempty subsets \( \mathcal{C}_1, \mathcal{C}_2 \) form a partition \( \mathcal{C} = \{1, 2, \ldots, n\} \), then there exist \( i \in \mathcal{C}_1 \) and \( j \in \mathcal{C}_2 = \mathcal{C} \setminus \mathcal{C}_1 \) with \( |m_{ij}| \geq \theta \).

From system (3), the total vector population \( V_i(t) \) satisfies the following delayed differential system:

\[
\frac{dV_i}{dt} = A_i^e(t-\tau_v, V_i(t-\tau_v)) - d_i^v(t)V_i. \tag{4}
\]

In order to guarantee that system (4) admits positive almost periodic solution, we make the following assumptions:
(C1) \( b_i^t(t, V_i) \) is a positive uniformly almost periodic function, and its partial derivative is also uniformly almost periodic in \( t \), which satisfies \( \frac{\partial b_i^t(t, V_i)}{\partial t} < 0 \);

(C2) \( \frac{\partial A_i^t(t, V_i)}{\partial V_i} \geq 0 \) for all \((t, V_i) \in \mathbb{R}^2\), and \( A_i^t(t, V_i) \) is strictly sublinear for \( t \in \mathbb{R}_+ \), that is for any \( \rho \in (0, 1) \), \( A_i^t(\rho V_i) > \rho A_i^t(t, V_i) \), \( i = 1, \ldots, n \).

Observe that (C1) means that the egg reproduction rate is positive and is strictly decreasing with the total number of vector; (C2) implies that input rate of mature vectors is increasing with the total number of vectors. Based on the above assumptions, it then follows from [38, Theorem 3.3.2] that each solution of (4) exists on \([0, \infty)\) and there exists a unique positive almost periodic solution \( S_i^t(t) \) satisfying

\[
\lim_{t \to \infty} |V_i(t, \varphi_i^t) - S_i^t(t)| = 0, \quad \forall \varphi_i^t(0) > 0, \ i = 1, \ldots, n,
\]

where \( \varphi_i^t = (\varphi_i^1(t), \ldots, \varphi_i^n(t)) \in C([-\tau_0, 0], \mathbb{R}_+^n) \).

Let

\[
c_{ij} = \max\{m_{ij}^S, m_{ij}^L, m_{ij}^R\}, \quad c_{ii} = \min\{m_{ii}^S, m_{ii}^L, m_{ii}^R\}.
\]

Further, we assume that:

(H1) \( \frac{\partial G_i(t, H_i)}{\partial H_i} > 0 \) for all \((t, H_i) \in \mathbb{R}^2_+\) and \( i = 1, \ldots, n \);

(H2) There exists \( l_0 > 0 \) such that \( \sum_{i=1}^n p_{ij}(t) G_j(t - \tau_h, l) \leq d_i^p(t) - \sum_{j=1}^n c_{ij}(t) \) for all \( t \geq \tau_h, l \geq l_0, i = 1, \ldots, n \);

(H3) When \( V_i(t) > \max\{K_1, \ldots, K_n\} \), where \( K_i \) is the maximum capacity of the vector in patch \( i \), we have \( \frac{d V_i(t)}{dt} < 0 \), \( i = 1, \ldots, n \).

Biologically, (H1) implies that the birth rate in each patch increases when the number of adult individuals in the same patch increases. (H2) means that the sizes of susceptible host populations will decrease as long as they are large enough. (H3) means that the number of susceptible host and vector will not grow indefinitely. Then we have the following.

**Lemma 2.1.** Assume that (H1)-(H3) hold. Then system (3) admits a unique nonnegative solution, and all solutions are ultimately bounded and uniformly bounded.

**Proof.** We use the skew-product semiflow approach to prove the boundedness of solutions. In the following, we embed system (3) into a skew-product semiflow.

Define the hull of \( A_i^t(t) \) by

\[
H(A_i^t) = \text{cls}\{(A_i^t)_s : s \in \mathbb{R}, (A_i^t)_s(t, \cdot) = A_i^t(s + t, \cdot)\},
\]

where the closure is taken under the compact open topology. We can define the hulls of \( \beta_i^t(t) \), \( d_i^p(t) \), \( \gamma_i^h(t) \), \( \Delta_i^t(t) \), \( \mu_i^L(t) \), \( \mu_i^R(t) \), \( p_{ij}(t) \), \( \rho_i^t(t) \), \( \delta_i^v(t) \), \( \nu_i^p(t) \), denoted by \( H(\beta_i^t), H(\Delta_i^t), H(d_i^p), H(\gamma_i^h), H(\mu_i^L), H(\mu_i^R), H(p_{ij}), H(\rho_i^t), H(\beta_i^v), H(\nu_i^p) \), respectively. Let \( \xi_i = (A_i^t(t), b(t), \beta_i^v(t), d_i^p(t), m_i^S(t), \ldots, m_i^S(t), \gamma_i^h(t), m_i^L(t), \ldots, \gamma_i^h(t), m_i^R(t), \ldots, \gamma_i^h(t), \mu_i^L(t), \ldots, \mu_i^L(t), \mu_i^R(t), \ldots, \mu_i^R(t), p_{i1}(t), \ldots, p_{im}(t), A_i^t(t), \beta_i^v(t), d_i^p(t), \nu_i^p(t)) \), \( i = 1, \ldots, n \).

Set \( H(\xi_i) = H(A_i^t) \times H(b) \times H(\beta_i^v) \times H(\gamma_i^h) \times H(\mu_i^L) \times \cdots \times H(\mu_i^R) \times H(p_{i1}) \times \cdots \times H(\mu_i^R) \times H(p_{im}) \times H(A_i^t) \times H(\beta_i^v) \times H(\nu_i^p) \). Let \( \eta = (\eta_1, \ldots, \eta_n) \in H(\xi_1) \times \cdots \times H(\xi_n) = H(\xi) \), where \( \eta_i = (A_i^t \beta_i^v \gamma_i^h \mu_i^L \mu_i^R \cdots \mu_i^L \mu_i^R p_{i1} \cdots p_{im} A_i^t \beta_i^v d_i^p \nu_i^p) \in H(\xi_i) \). The translation

\[
\zeta : \mathbb{R} \times H(\xi) \to H(\xi), \quad (t, \eta) \mapsto \eta,
\]

such that \( \zeta(t, \eta) = \eta \).
with $\eta_s(t) = \eta(s + t)$, defines a compact, almost periodic minimal and distal flow, denoted by $(H(\xi), \zeta, \mathbb{R})$.

Consider the family system

$$\frac{dS^h}{dt} = \bar{A}^h_i(t, H(t - \tau_h)) - \frac{\bar{b}^h_i(t)}{H_i} S^h_i I^h_i - \bar{d}^h_i(t) S^h_i + \sum_{j=1}^{n} \bar{m}^S_{ij}(t) S^h_j,$$

$$\frac{dI^h_i}{dt} = \frac{\bar{b}^h_i(t)}{H_i} S^h_i I^h_i - (\gamma^h_i(t) + \bar{d}^h_i(t)) I^h_i + \sum_{j=1}^{n} \bar{m}^I_{ij}(t) I^h_j,$$

$$\frac{dR^h_i}{dt} = \gamma^h_i(t) I^h_i - \bar{d}^h_i(t) R^h_i + \sum_{j=1}^{n} \bar{m}^R_{ij}(t) R^h_j,$$

$$\frac{d\bar{A}^h_i(t, H(t - \tau_h))}{dt} = \sum_{j=1}^{n} \bar{p}_{ij}(t) G_j(t - \tau_h, H_j(t - \tau_h)),$$

with the initial value $S^h_i(\theta_1, \eta) = \varphi^h_i(\theta_1), I^h_i(\theta_1, \eta) = \psi^h_i(\theta_1), R^h_i(\theta_1, \eta) = \phi^h_i(\theta_1), \theta_1 \in [-\tau_h, 0], S^h_i(\theta_2, \eta) = \varphi^h_i(\theta_2), E^h_i(\theta_2, \eta) = \psi^h_i(\theta_2), R^h_i(\theta_2, \eta) = \phi^h_i(\theta_2), \theta_2 \in [-\tau_h, 0], (\varphi^h_i, \psi^h_i, \phi^h_i) \in C^1_C$, where $C_+ := C([-\tau_h, 0], \mathbb{R}_+^n \times C([-\tau_h, 0], \mathbb{R}_+^n)$, and $(\bar{A}^h_i(t), \bar{I}^h_i(\cdot), \bar{R}^h_i(\cdot), \bar{S}^h_i(\cdot), \bar{E}^h_i(\cdot), \gamma^h_i(t), \bar{m}^S_{ij}(t), \ldots, \bar{m}^R_{ij}(t), \bar{p}_{ij}(t), \bar{q}_{ij}(t), \bar{r}_{ij}(t)) \in H(\xi_i), i = 1, \ldots, n$.

Let

$$\bar{c}_{ij} = \max\{\bar{m}^S_{ij}, \bar{m}^I_{ij}, \bar{m}^R_{ij}\}, i \neq j, \quad \bar{c}_{ii} = \min\{\bar{m}^S_{ii}, \bar{m}^I_{ii}, \bar{m}^R_{ii}\}.$$

Since $H_i(t) = S^h_i(t) + I^h_i(t) + R^h_i(t)$, we have

$$\frac{dH_i}{dt} \leq \bar{A}^h_i(t, H(t - \tau_h)) - \bar{d}^h_i(t) H_i + \sum_{j=1}^{n} \bar{c}_{ij}(t) H_j.$$

Consider the auxiliary system

$$\frac{dH_i}{dt} = \bar{A}^h_i(t, \bar{H}(t - \tau_h)) - \bar{d}^h_i(t) \bar{H}_i + \sum_{j=1}^{n} \bar{c}_{ij}(t) \bar{H}_j := \bar{Q}_i(t, \bar{H}(t), \bar{H}(t - \tau_h)).$$

It is easy to verify that

$$\frac{\partial \bar{Q}_i(t, u, v)}{\partial u_j} = \bar{c}_{ij}(t) \geq 0, \quad i \neq j,$$

$$\frac{\partial \bar{Q}_i(t, u, v)}{\partial v_k} = \bar{p}_{ik}(t) \frac{\partial G_k(t - \tau_h, \bar{v}_k)}{\partial v_k} > 0, \quad i,j,k = 1, \ldots, n.$$

Obviously, $\frac{\partial \bar{Q}}{\partial u}$ and $\frac{\partial \bar{Q}}{\partial v}$ is strongly irreducible. Furthermore, for each $j$, there exists $\alpha > 0$ and $i$ such that

$$\frac{\partial \bar{Q}_j(t, u, v)}{\partial v_j} > \alpha.$$

Thus, $\bar{Q}$ satisfies the [31, assumption (B1)]. For any $\phi \in C([-\tau_h, 0], \mathbb{R}_+^n)$ with $\phi_i(0) = 0$

$$\bar{Q}_i(t, \phi(0), \phi(-\tau_h)) = \sum_{j=1}^{n} \bar{p}_{ij}(t) G_j(t - \tau_h, \phi_j(-\tau_h)) + \sum_{j=1}^{n} \bar{c}_{ij}(t) \phi_j(0) \geq 0.$$

By (H2), there exists $l_0 > 0$ such that

$$\bar{Q}_i(t, C, C) \leq 0, \quad \forall t \geq \tau_h, C > l_0.$$
For any $\beta \in (0, 1)$, $u \gg 0$ and $v \gg 0$, we have
\[
\beta \sum_{j=1}^{n} \bar{p}_{ij}(t)B_j^h(t-\tau_h, \beta v_j)v_j - \beta \bar{d}_i^h(t)u_i + \sum_{j=1}^{n} \bar{c}_{ij}(t)\beta u_j >
\]
which implies that $\bar{Q}$ is strictly subinear. Hence, $\bar{Q}(t, \bar{H}(t), \bar{H}(t-\tau)) = (\bar{Q}_1(t, \bar{H}(t), \bar{H}(t-\tau)), \ldots, \bar{Q}_n(t, \bar{H}(t), \bar{H}(t-\tau)))$ satisfies (B1)-(B4) in [31]. By [31, Theorem 2.4], system (6) has a unique positive almost periodic solution
\[
\tilde{H}^*(t, \eta) = (\tilde{H}_1^*(t, \eta), \ldots, \tilde{H}_n^*(t, \eta)).
\]
By comparison principle,
\[
\limsup_{t \to \infty} \{S_i^h(t, \eta) + I_i^h(t, \eta) + R_i^h(t, \eta)\} \leq \max_{t \in (0, \infty)} \tilde{H}_i^*(t, \eta), \quad \forall \eta \in H(\xi), i = 1, \ldots, n,
\]
which implies that $S_i^h(t, \eta), I_i^h(t, \eta)$ and $R_i^h(t, \eta)$ are ultimately bounded. Moreover, (H2) insures that $\tilde{S}_i^h(t, \eta), \tilde{I}_i^h(t, \eta)$ and $\tilde{R}_i^h(t, \eta)$ are uniformly bounded.

Similarly, let $V_i(t) = S_i^v(t) + E_i^v(t) + I_i^v(t)$, then we have
\[
\frac{dV_i}{dt} = \check{A}_i^v(t-\tau_v, V_i(t-\tau_v)) - \check{d}_i^v(t)V_i.
\]
By the similar arguments to those in (4), system (7) admits a positive almost periodic solution $S_i^v(t, \eta)$, which satisfies $\lim_{t \to \infty} (V_i(t, \eta) - S_i^v(t, \eta)) = 0$. Thus, $S_i^v(t, \eta), E_i^v(t, \eta)$ and $I_i^v(t, \eta)$ are ultimately bounded. Further, (H3) implies that $S_i^v(t, \eta), E_i^v(t, \eta)$ and $I_i^v(t, \eta)$ are uniformly bounded.

In conclusion, we have
\[
\frac{dH_i}{dt} \leq \check{A}_i^h(t, H(t-\tau_h)) - \check{d}_i^h(t)H_i + \sum_{j=1}^{n} \check{c}_{ij}(t)H_j,
\]
\[
\frac{dV_i}{dt} = \check{A}_i^v(t-\tau_v, V_i(t-\tau_v)) - \check{d}_i^v(t)V_i.
\]
Thus, by the comparison principle, all solutions of system (5) are ultimately bounded and uniformly bounded.

Based on the above proof, we conclude that system (3) admits a unique nonnegative solution, and all solutions are ultimately bounded and uniformly bounded. \hfill \Box

In order to find the disease-free almost periodic solution of (3), we consider
\[
\begin{align*}
\frac{dS_i^h}{dt} &= \check{A}_i^h(t, S_i^h(t-\tau_h)) - \check{d}_i^h(t)S_i^h + \sum_{j=1}^{n} \check{m}_{ij}(t)S_j^h, \\
\frac{dS_i^v}{dt} &= \check{A}_i^v(t-\tau_v, S_i^v(t-\tau_v)) - \check{d}_i^v(t)S_i^v.
\end{align*}
\tag{8}
\]
Set
\[
\frac{dS_i^h}{dt} = \check{A}_i^h(t, S_i^h(t-\tau_h)) - \check{d}_i^h(t)S_i^h + \sum_{j=1}^{n} \check{m}_{ij}(t)S_j^h := \check{D}_i(t, S_i^h(t), S_i^h(t-\tau_h)).
\tag{9}
\]
Consider the following delay system
\[
\frac{dS_i^h}{dt} = \check{D}(t, S_i^h(t), S_i^h(t-\tau_h)), \quad \check{D} \in H(D),
\tag{10}
\]
where $H(D)$ is the hull of $D$. For each $(\bar{D}, x) \in H(D) \times C([−\tau_0, 0), \mathbb{R}_+^n)$, by Lemma 2.1, system (10) has a solution $S^h(t, \bar{D}, x)$ with $S^h(\theta, \bar{D}, x) = x(\theta), \theta \in [−\tau_0, 0]$. 

**Definition 2.2.** Let $K \subseteq H(D) \times C([−\tau_0, 0], \mathbb{R}_+^n)$ be a compact positively invariant. For $(\bar{D}, x) \in K$, we define the Lyapunov exponent $\lambda(\bar{D}, x)$ as

$$\lambda(\bar{D}, x) = \limsup_{t \to \infty} \frac{\ln \| S^h_x(t, \bar{D}, x) \|}{t}.$$

The number $\lambda_K = \sup_{(\bar{D}, x) \in K} \lambda(\bar{D}, x)$ is called the upper Lyapunov exponent over $K$. If $\lambda_K \leq 0$, then $K$ is said to be linearly stable.

In the following, we assume that the trivial solution is unstable as $D(t, 0, 0) = 0$.

Further, we have

(H4) If $\tilde{A}^h_i(t, \cdot) > 0$, then $\tilde{A}^h_i(t, \cdot) > 0$ for all $\tilde{A}^h_i(t, \cdot) \in H(A^h_i)$;

(H5) If $D(t, 0, 0) = 0$, then $\lambda_{O_n} > 0$, where $\lambda_{O_n}$ be the upper Lyapunov exponent associated with system (9) and $O_n = \{\eta \times 0, \eta \in H(\xi)\}$.

Observe that (H5) insures that susceptible host population must increase as long as it is small.

Similar the argument to the proof of Lemma 2.1, we know that $D(t, S^h(t), S^h(t − \tau_0)) = (D_1(t, S^h(t), S^h(t − \tau_0)), \ldots, D_n(t, S^h(t), S^h(t − \tau_0)))$ satisfies the conditions (B1) − (B4) [31]. It then follows from [31, Theorem 3.2] that system (9) has a unique positive almost periodic solution $S^h(t, \eta) = (S^h_1(t, \eta), \ldots, S^h_n(t, \eta))$, which is globally attractive in $C([−\tau_0, 0], \mathbb{R}^n_+) \times H(\xi)$. Then, consider

$$\frac{dS^v_i}{dt} = \tilde{A}^v_i(t − \tau_v, S^v_i(t − \tau_v)) − \tilde{d}^v_i(t)S^v_i. \quad (11)$$

In view of the assumption (C1)−(C2), system (11) admits a unique positive almost periodic solution $S^v_i(t, \eta) = (S^v_1(t, \eta), \ldots, S^v_n(t, \eta))$, which is uniformly asymptotically stable in $C([−\tau_v, 0], \mathbb{R}_+^n) \times H(\xi)$.

**Lemma 2.3.** If (H1) − (H5) and (C1) − (C2) hold, then system (3) has two disease-free almost periodic solutions, denoted by $(S^h_{1r}(t), 0, 0, S^v_{1r}(t), 0, 0)$ and $(S^h_{ir}(t), 0, 0, 0)$ for all $i = 1, \ldots, n$.

Linearizing (3) at the almost periodic solution disease-frees state $(S^h_{1r}(t), 0, 0, S^v_{1r}(t), 0, 0)$, then

$$\frac{dI^h_i}{dt} = b(t)\beta^h_i(t)I^v_i − (\gamma^h_i(t) + d^h_i(t))I^h_i + \sum_{j=1}^n m^h_{ij}(t)I^h_j,$$

$$\frac{dR^h_i}{dt} = \gamma^h_i(t)I^h_i − d^h_i(t)R^h_i + \sum_{j=1}^n m^h_{ij}(t)R^h_j,$$

$$\frac{dE^v_i}{dt} = \frac{b(t)\beta^v_i(t)}{S^v_{1r}}S^v_iI^h_i − \eta^v_i(t)E^v_i − d^v_i(t)E^v_i,$$

$$\frac{dI^v_i}{dt} = \eta^v_i(t)E^v_i − d^v_i(t)I^v_i.$$ 

Take

$$Y_1(t) = \text{diag}(b(t)\beta^h_i(t)), \quad Y_2(t) = \text{diag}(\frac{b(t)\beta^v_i(t)}{S^h_{1r}}S^v_i).$$
Let $\Psi$ be the evolution operator of (13) equals $\Psi$. Then

$$
\begin{align*}
\Psi(t) &= \left( \begin{array}{cccc}
\gamma_1^R(t) + d_1^R(t) - m_{11}^I(t) & -m_{12}^I(t) & \cdots & -m_{1n}^I(t) \\
-m_{21}^I(t) & \gamma_2^R(t) + d_2^R(t) - m_{22}^I(t) & \cdots & -m_{2n}^I(t) \\
\vdots & \vdots & \ddots & \vdots \\
-m_{n1}^I(t) & -m_{n2}^I(t) & \cdots & \gamma_n^R(t) + d_n^R(t) - m_{nn}^I(t)
\end{array} \right)
\end{align*}
$$

and

$$
Z_2(t) = \left( \begin{array}{cccc}
d_1^R(t) - m_{11}^R(t) & -m_{12}^R(t) & \cdots & -m_{1n}^R(t) \\
-m_{21}^R(t) & d_2^R(t) - m_{22}^R(t) & \cdots & -m_{2n}^R(t) \\
\vdots & \vdots & \ddots & \vdots \\
-m_{n1}^R(t) & -m_{n2}^R(t) & \cdots & d_n^R(t) - m_{nn}^R(t)
\end{array} \right).
$$

Then

$$
Y(t) = \left( \begin{array}{cccc}
0 & 0 & 0 & Y_1(t) \\
0 & 0 & 0 & 0 \\
Y_2(t) & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right),
$$

$$
Z(t) = \left( \begin{array}{cccc}
Z_1(t) & 0 & 0 & 0 \\
-diag(\gamma_1^h(t)) & Z_2(t) & 0 & 0 \\
0 & diag(\eta_1^v(t) + d_1^v(t)) & 0 & 0 \\
0 & 0 & -diag(\eta_1^v(t)) & diag(d_1^v(t))
\end{array} \right).
$$

Thus, system (12) can be written as

$$
\frac{du(t)}{dt} = (Y(t) - Z(t))u(t),
$$

where $u(t) = (I^h(t), R^h(t), E^v(t), I^v(t))^T$, and $I^h(t) = (I_1^h(t), \ldots, I_n^h(t)), R^h(t) = (R_1^h(t), \ldots, R_n^h(t)), E^v(t) = (E_1^v(t), \ldots, E_n^v(t)), I^v(t) = (I_1^v(t), \ldots, I_n^v(t))$.

Consider the linear almost periodic system

$$
\frac{dw}{dt} = -Z(t)w.
$$

Let $\Psi_Z(t, s)(t \geq s, s \in \mathbb{R})$ be the evolution operator of the system (13), that is, $\Psi_Z(t, s)$ satisfies

$$
\frac{d}{dt} \Psi_Z(t, s) = -Z(t)\Psi_Z(t, s), \quad \forall t \geq s, s \in \mathbb{R}
$$

and

$$
\Psi_Z(s, s) = I_d, \quad \forall s \in \mathbb{R},
$$

where $I_d$ is the $n \times n$ identity matrix. And the fundamental solution matrix $\Phi_Z(t)$ of (13) equals $\Psi_Z(t, 0), t \geq 0$. We define the exponential growth bound of the evolution operator $\Psi_Z(t, s)$ as

$$\omega(\Psi_Z) = \inf\{\omega \in \mathbb{R} : \exists K_0 \geq 1, \| \Psi_Z(t + s, s) \| \leq K_0 e^{\omega t}, \forall s \in \mathbb{R}, t \geq 0\}.$$

Note that $\omega(\Psi_Z)$ is finite because almost periodic function $Z(t)$ is bounded on $\mathbb{R}$. Define

$$AP_{(Y, Z)} := \{ \phi : \phi \in AP(\mathbb{R}, \mathbb{R}^n), \text{mod}(\phi) \subset \text{mod}(Y, Z) \}.$$ 

By [33, Lemma 2.1], $AP_{(Y, Z)}$ is a Banach space with the supremum norm $\| \cdot \|$, and the positive cone $AP^+_{(Y, Z)} = \{ \phi \in AP_{(Y, Z)} : \phi(t) \geq 0, \forall t \in \mathbb{R} \}$. Then $(AP_{(Y, Z)}, AP^+_{(Y, Z)})$ is an ordered Banach space.
We assume that $\phi(s) \in AP_{(Y,Z)}$ is the initial distribution of infectious individuals, then $Y(t)\phi(s)$ is the distribution of new infectious produced by the infected individuals who were introduced at the time $s$, then $\Psi(t,s)Y(t)\phi(s)$, $t \geq s$ denotes the distribution of those infected individuals who were newly infected at time $s$ and are still in the infected compartments at time $t$. It then follows that

$$\psi(t) := \int_{-\infty}^{t} \Psi(t,s)Y(s)\phi(s)\,ds = \int_{0}^{\infty} \Psi(t,t-a)Y(t-a)\phi(t-a)\,da$$

is the distribution of accumulative new infections at time $t$ produced by all those infected individuals $\phi(s)$ introduced at the previous time to $t$. Define the linear map $L : AP_{(Y,Z)} \rightarrow AP_{(Y,Z)}$ by

$$(L\phi)(t) = \int_{0}^{\infty} \Psi(t,t-a)Y(t-a)\phi(t-a)\,da, \quad \forall \phi \in AP_{(Y,Z)}, t \in \mathbb{R}.$$ 

Motivated by the concept of next generation matrices introduced in [5, 29], we call $L$ the next generation operator and define the spectral radius of $L$

$$R_0 := r(L)$$

as the basic reproduction ratio of model (3).

**Lemma 2.4.** [33, Theorem 3.2] Let (H1) – (H5) and (C1) – (C2) hold. Set $\omega(\Psi_{Y-Z}) = \inf \{ \omega \in \mathbb{R} : \exists K_0 \geq 1, \| \Psi_{Y-Z}(t+s,s) \| \leq K_0 e^{\omega t}, \forall s \in \mathbb{R}, t \geq 0 \}$. Then the following statements are valid:

(i): $R_0 < 1$ if and only if $\omega(\Psi_{Y-Z}) < 0$;

(ii): $R_0 = 1$ if and only if $\omega(\Psi_{Y-Z}) = 0$;

(iii): $R_0 > 1$ if and only if $\omega(\Psi_{Y-Z}) > 0$.

3. Threshold dynamics.

**Theorem 3.1.** Assume (C1) – (C2) and (H1) – (H5) hold. If $R_0 > 1$, then there exists $\varepsilon > 0$ such that any solution of system (3) with $I^h_i(0) > 0$, $R^h_i(0) > 0$, $E^v_i(0) > 0$, $I^v_i(0) > 0$ satisfies

$$\liminf_{t \rightarrow \infty} (I^h_i(t), R^h_i(t), E^v_i(t), I^v_i(t)) \geq (\varepsilon, \varepsilon, \varepsilon), \quad \forall t \geq 0, i = 1, \ldots, n.$$ 

**Proof.** Similar to the proof of Lemma 2.1, we use the skew-product semiflow approach to prove the desired uniform persistence with delay [33]. Define

$$X = C^3_+, \quad C_+ = C([-\tau_h, 0], \mathbb{R}^n_+) \times C([-\tau_v, 0], \mathbb{R}^n_+),$$

$$X_0 = \{ \nu = (\varphi, \psi, \varphi^h, \varphi^v, \psi^h, \psi^v) \in X : \varphi^h > 0, \psi^h > 0, \psi^v > 0, \varphi^v > 0, i = 1, \ldots, n \},$$

$$\partial X_0 = X \times H(\xi), \quad \Lambda = X \times H(\xi), \quad \partial \Lambda_0 = \Lambda \times H(\xi).$$

Then $\Lambda_0$ and $\partial \Lambda_0$ are relatively open and closed in $\Lambda$, respectively.

Let $X(t, \nu, \eta) := (S^h(t, \eta), I^h(t, \eta), R^h(t, \eta), S^v(t, \eta), E^v(t, \eta), I^v(t, \eta))$ be the unique solution of system (5) satisfying the initial value $X(0, \nu, \eta) = \nu$. Then we can define a skew-product semiflow $\Pi_t : \Lambda \rightarrow \Lambda$ by

$$\Pi_t(\nu, \eta) = (X_t(\nu, \eta), \eta), \quad \forall t \geq 0,$$

where $X_t(\nu, \eta)(s) = X(t + s, \nu, \eta), \forall s \in [-\tau_h, 0]$ or $s \in [-\tau_v, 0]$. By Lemma 2.1, $\Pi_t$ is point dissipative on $\Lambda$, thus, $\Pi_t : \Lambda \rightarrow \Lambda$ has a global compact attractor $\mathcal{A}$ [12, Theorem 3.4.8].
From the second, the third, the fifth and the sixth equations of system (5), for any \((\nu, \eta) \in \Lambda_0, (P^\nu(t, \eta), R^\nu(t, \eta), E^\nu(t, \eta), I^\nu(t, \eta)) \in \text{Int}(\mathbb{R}^{4n}_+)\) for all \(t \geq 0\). Thus, \(\Lambda_0\) is positively invariant for \(\Pi_t\), that is, \(\Pi_t(\Lambda_0) \subset \Lambda_0\) for all \(t \geq 0\).

Let
\[
M_\partial := \{(\nu, \eta) \in \partial \Lambda_0 : \Pi_t(\nu, \eta) \in \partial \Lambda_0, \forall t \geq 0\},
\]
\[
C := \{(\nu, \eta) \in \Lambda : \psi^\nu(0) = \phi^\nu(0) = \psi^v(0) = \phi^v(0) = 0\},
\]
\[
C' := \{(\nu, \eta) \in \Lambda : \phi^v = \psi^v = \phi^v = 0\}.
\]

Now, we show that \(M_\partial = C \cup C'\).

For any \((\nu, \eta) \in C\), we can define
\[
y^{\nu}_{2i}(t, \eta) := y^{\nu}_{2i}(t, \eta) = y^{\nu}_{4i}(t, \eta) = y^{\nu}_{6i}(t, \eta) = 0, \quad \forall t \geq 0, \eta \in H(\xi).
\]
with \(y^{\nu}_{2i}(0) = \psi^\nu(0), y^{\nu}_{3i}(0) = \phi^\nu(0), y^{\nu}_{4i}(0) = \psi^v(0), y^{\nu}_{6i}(0) = \phi^v(0)\).

Let \(y^{\nu}_{2i}(t, \eta)\) be the solution of
\[
\frac{dy^{\nu}_{2i}}{dt} = A^\nu(t, \varphi^\nu(t - \tau_h) + \psi^\nu(t - \tau_h) + \phi^\nu(t - \tau_h)) - \bar{d}_i^\nu(t)y^{\nu}_{2i} + \sum_{j=1}^{n} \bar{m}_{ij}^\nu(t)y^{\nu}_{4j},
\]
\[
\frac{dy^{\nu}_{4i}}{dt} = A^\nu(t, y^{\nu}_{4i}(t - \tau_h)) - \bar{d}_i^\nu(t)y^{\nu}_{4i} + \sum_{j=1}^{n} \bar{m}_{ij}^\nu(t)y^{\nu}_{4j}, \quad t \geq \tau_h.
\]

with \(y^{\nu}_{2i}(0) = \varphi^\nu(0), i = 1, \ldots, n, t \in [0, \tau_h].\)

Let \(y^{\nu}_{4i}(t, \eta)\) be the solution of
\[
\frac{dy^{\nu}_{4i}}{dt} = A^\nu(t, \varphi^\nu(t - \tau_\nu) + \psi^\nu(t - \tau_\nu) + \phi^\nu(t - \tau_\nu)) - \bar{d}_i^\nu(t)y^{\nu}_{4i}, \quad t \in [0, \tau_\nu],
\]
\[
\frac{dy^{\nu}_{4i}}{dt} = A^\nu(t, y^{\nu}_{4i}(t - \tau_\nu)) - \bar{d}_i^\nu(t)y^{\nu}_{4i}, \quad t \geq \tau_\nu.
\]
with \(y^{\nu}_{4i}(0) = \varphi^\nu(0), i = 1, \ldots, n. \) And
\[
y(t, \eta) = (y^{\nu}_{1i}(t, \eta), y^{\nu}_{2i}(t, \eta), y^{\nu}_{3i}(t, \eta), y^{\nu}_{4i}(t, \eta), y^{\nu}_{5i}(t, \eta), y^{\nu}_{6i}(t, \eta))
\]
\[
= (y^{\nu}_{2i}(t, \eta), 0, 0, y^{\nu}_{4i}(t, \eta), 0, 0)
\]
\[
= (y^{\nu}_{4i}(t, \eta), \ldots, y^{\nu}_{n4i}(t, \eta), 0, 0, y^{\nu}_{4i}(t, \eta), \ldots, y^{\nu}_{n4i}(t, \eta), 0, 0).
\]
Then \(y(t, \eta)\) is a solution of system (5) through \(\nu\) for \(\eta \in H(\xi)\). It can be obtained that \(\mathcal{X}(t, \nu, \eta) = y(t, \eta)\) for all \(t \geq 0\) by the uniqueness of the solution. Thus, \(C \subset M_\partial\).

For any \((\nu, \eta) \in C'\), it follows that \(S^\nu(t, \eta) = E^\nu(t, \eta) = I^\nu(t, \eta) = 0\) for all \(t \geq 0\), then \(C' \subset M_\partial\). Hence, \(C \cup C' \subset M_\partial\). For any \((\nu, \eta) \in \partial \Lambda_0 \setminus (C \cup C')\), we have \(\varphi^\nu(0) + \psi^\nu(0) = \phi^\nu(0) > 0\), then \(\lim_{t \to \infty} \|S^\nu_i(t, \eta) + E^\nu_i(t, \eta) + I^\nu_i(t, \eta) - S^\nu_i(t, \eta)\| = 0\).

From the forth equation of system (5), there exists \(t_0 > 0\) such that \(S^\nu_i(t, \eta) > 0\) for all \(t \geq t_0, \eta \in H(\xi), i = 1, \ldots, n\). It is not hard to see that \(S^\nu_i(t, \eta) > 0\) for all \(t > 0, \eta \in H(\xi), i = 1, \ldots, n\) from the first equation of system (5). If \(\psi^\nu(0) > 0\), we can get that \(I^\nu_i(t, \eta) > 0\) and \(R^\nu_i(t, \eta) > 0\) for all \(t > 0, \eta \in H(\xi), i = 1, \ldots, n\) from the second and the third equations of system (5). If \(\psi^\nu(0) > 0\), we see that \(R^\nu_i(t, \eta) > 0\) for all \(t > 0, \eta \in H(\xi), i = 1, \ldots, n\) from the third equation of system (5). If \(\psi^\nu(0) > 0\), we have \(E^\nu_i(t, \eta) > 0\) and \(I^\nu_i(t, \eta) > 0\) for all \(t > 0, \eta \in H(\xi), i = 1, \ldots, n\) from the fifth and the sixth equations of system (5). If \(\phi^\nu(0) > 0\), then from the second, the third and the sixth equations of system (5), we obtain
\[
I^\nu_i(t, \eta) > 0, \quad I^\nu_i(t, \eta) > 0, \quad R^\nu_i(t, \eta) > 0, \quad \forall t > 0, \eta \in H(\xi).
\]
Hence, $\Pi_1(\nu, \eta) \notin \partial \Lambda_0$. It follows that $M_0 \subset C \cup C'$. This proves that $M_0 = C \cup C'$.

Let
\[
O_1 = \{ (S^h_{t_0}(\varphi_*^h, \eta), 0, 0, S^v_{t_0}(\varphi_*^v, \eta), 0, 0, 0) : \eta \in H(\xi) \},
\]
\[
O_2 = \{ (S^h_{t_0}(\varphi_*^h, \eta), 0, 0, 0, 0, 0) : \eta \in H(\xi) \},
\]
where $\varphi_*^h = S^h_{t_0}(s) = S^h_{t_0}(s), s \in [-\tau_h, 0]$, $\varphi_*^v = S^v_{t_0}(s) = S^v_{t_0}(s), s \in [-\tau_v, 0]$ and $S^h_{t_0}(\varphi_*^h, \eta)(s) = S^h_{t_0}(s, \varphi_*^h, \eta), s \in [-\tau_h, 0]$, $S^v_{t_0}(\varphi_*^v, \eta)(s) = S^v_{t_0}(s, \varphi_*^v, \eta), s \in [-\tau_v, 0]$. By the uniqueness and continuity of solutions, we have $S^h_{t_0}(\varphi_*^h, \eta) = S^h_{t_0}(\varphi_*^h, \eta), S^v_{t_0}(\varphi_*^v, \eta) = S^v_{t_0}(\varphi_*^v, \eta)$. Hence, $O_1$ and $O_2$ are disjoint and compact invariant sets for $\Pi : \Lambda \to \Lambda$ in $M_0$.

For any $(\nu, \eta) \in M_0$, let $\omega(\nu, \eta)$ be the omega limit set of $(\nu, \eta)$ for $\Pi_1$. Hence, for any $(\tilde{\nu}, \tilde{\eta}) \in \omega(\nu, \eta)$, there exists a sequence $t_n \to \infty$ such that $\lim_{t_n \to \infty} \Pi_{t_n}(\nu, \eta) = (\tilde{\nu}, \tilde{\eta})$. Note that $\Pi_{t_n}(\nu, \eta) = (X_{t_n}(\nu, \eta), \eta_{t_n})$ and
\[
\lim_{n \to \infty} \| X_{t_n}(\nu, \eta) - (S^h_{t_{t_n}}(\varphi_*^h, \eta), 0, 0, S^v_{t_{t_n}}(\varphi_*^v, \eta), 0, 0) \| = 0
\]
or
\[
\lim_{n \to \infty} \| X_{t_n}(\nu, \eta) - (S^h_{t_{t_n}}(\varphi_*^h, \eta), 0, 0, 0, 0) \| = 0.
\]

By the continuous dependence of solutions on parameters, we have $S^h_{t_0}(\varphi_*^h, \eta) : H(\xi) \to C([-\tau_h, 0], \mathbb{R}^n)$ and $S^v_{t_0}(\varphi_*^v, \eta) : H(\xi) \to C([-\tau_v, 0], \mathbb{R}^n)$ is continuous in $\Lambda$. Since $H(\xi)$ is compact, $S^h_{t_0}(\varphi_*^h, \eta)$ and $S^v_{t_0}(\varphi_*^v, \eta)$ is uniformly continuous in $\eta \in H(\xi)$. It follows that
\[
S^h_{t_{t_n}}(\varphi_*^h, \eta) = S^h_{t_0}(\varphi_*^h, \eta_{t_n}) \to S^h_{t_0}(\varphi_*^h, \eta), \quad \text{as } n \to \infty,
\]
\[
S^v_{t_{t_n}}(\varphi_*^v, \eta) = S^v_{t_0}(\varphi_*^v, \eta_{t_n}) \to S^v_{t_0}(\varphi_*^v, \eta), \quad \text{as } n \to \infty.
\]
we get that $(\tilde{\nu}, \tilde{\eta}) = (S^h_{t_0}(\varphi_*^h, \eta), 0, 0, 0, 0, 0, 0) = (S^h_{t_0}(\varphi_*^h, \eta), 0, 0, 0, 0, 0, 0)$ or $(\tilde{\nu}, \tilde{\eta}) = (S^h_{t_0}(\varphi_*^h, \eta), 0, 0, 0, 0, 0, 0)$, that is, $(\tilde{\nu}, \tilde{\eta}) \in O_1 \cup O_2$ and $\omega(\nu, \eta) \subseteq O_1 \cup O_2$. Thus, $O_1$ and $O_2$ are disjoint, compact and isolated invariant sets for $\Pi : \Lambda \to \Lambda$ in $\Lambda_0$, $\bigcup_{(\nu, \eta) \in M_0} \omega(\nu, \eta) \subseteq O_1 \cup O_2$.

By the compact open topology. Further, we have the following claims:

**Claim 1.** For $(\nu, \eta) \in \Lambda_0$, we have
\[
\lim_{t \to \infty} \sup d(\Pi_t(\nu, \eta), O_1) \geq \delta_0.
\]

On the contrary, we assume that for some $(\tilde{\nu}, \tilde{\eta}) \in \Lambda_0$, $\limsup_{t \to \infty} d(\Pi_t(\nu, \eta), O_1) < \delta_0$. Then there exists $t_1 > 0$ such that $d(\Pi_t(\nu, \eta), O_1) < \delta_0$ for all $t \geq t_1$. Hence, $d(\Pi_t(\nu, \eta), O_1) < \delta_0$ for all $t \geq 0$, where $(\nu, \eta) = \Pi_{t_1}(\tilde{\nu}, \tilde{\eta}) \in \Lambda_0$. Thus, there exists $\tilde{\tilde{\eta}} \in H(\xi)$ such that $\| X_t(\nu, \eta) - (S^h_{t_0}(\varphi_*^h, \eta), 0, 0, S^v_{t_0}(\varphi_*^v, \eta), 0, 0) \| < \delta_0$ and
Consider the following comparison system

\[
\omega_R \text{ and Sect. 11.4 implies that system (15) admits a unique almost periodic solution}
\]

forward calculation shows that

\[
0. Then, we have
\]

which implies that \( \rho_i(t, \eta) < \varepsilon_0 < \varepsilon' \) and \( E_i^\eta(t, \eta) < \varepsilon_0 < \varepsilon' \) for all \( t \geq t_i, i = 1, \ldots, n \).

From the third equation of system (5),

\[
\frac{dR^h_i}{dt} \leq \gamma_i^h(t)\varepsilon_0 - d_i^h(t)R^h_i + \sum_{j=1}^n \tilde{m}_{ij}(t)R^h_j.
\]

Consider the following comparison system

\[
\frac{dR^h_i}{dt} = \gamma_i^h(t)\varepsilon_0 - d_i^h(t)R^h_i + \sum_{j=1}^n \tilde{m}_{ij}(t)R^h_j.
\]

Then system (15) can be written as

\[
\frac{dR^h}{dt} = -\bar{Z}_2(t)R^h + F_0,
\]

where \( \bar{Z}_2(t) \) belongs to the hull \( H(Z_2), R^h = (R^h_1, \ldots, R^h_n)^T, F_0 = (\gamma_1^h(t)\varepsilon_0, \ldots, \gamma_n^h(t)\varepsilon_0)^T \).

Let \( \Psi_{-\bar{Z}_2}(t, s), t \geq s \) be the evolution operator associated with \( -\bar{Z}_2(t) \). Then we can define the exponential growth bound of \( \Psi_{-\bar{Z}_2} \), denoted by \( \omega(\Psi_{-\bar{Z}_2}) \). A straightforward calculation shows that \( \omega(\Psi_{-\bar{Z}_2}) < 0 \). By [33, Lemma 2.4], \( \omega(\Psi_{-\bar{Z}_2}) = -d_i^h(t) \varepsilon_0 \) for all \( \bar{Z}_2 \in H(Z_2). \) Hence, for each \( \bar{Z}_2 \in H(Z_2), [9, Theorem 7.7 and Sect. 11.4] \) implies that system (15) admits a unique almost periodic solution \( R_{-\varepsilon_0}^h(t, \bar{Z}_2) = (R_{-\varepsilon_0}^h(t, Z_2), \ldots, R_{-\varepsilon_0}^h(t, Z_2)), \) which is uniformly asymptotically stable in \( \mathbb{R}_n^+ \). Further, in view of the continuity of \( \Psi_{-\bar{Z}_2} \) in \( -\bar{Z}_2 \), there exist \( K_1 > 0 \) and \( \sigma_1 > 0 \) such that for any \( \bar{Z}_2 \in H(Z_2), \)

\[
\| \Psi_{-\bar{Z}_2}(t, s) \| \leq K_1 e^{-\sigma_1(t-s)}, \quad \forall t \geq s, s \in \mathbb{R}.
\]

By the constant-variation formula,

\[
R_{-\varepsilon_0}^h(t, \bar{Z}_2) = \Psi_{-\bar{Z}_2}(t, s)R_{-\varepsilon_0}^h(s, \bar{Z}_2) + \int_s^t \Psi_{-\bar{Z}_2}(t, \tau)F_0 d\tau, \quad \forall t \geq s, s \in \mathbb{R}.
\]

Let \( s \to -\infty \) in (16), we obtain that

\[
\| R_{-\varepsilon_0}^h(t, \bar{Z}_2) \| \leq \int_{-\infty}^t \| \Psi_{-\bar{Z}_2}(t, \tau) \| F_0 \| d\tau \leq \frac{K_1}{\sigma_1} \| F_0 \| \leq \sup_{t \in [0, \infty)} \{ \gamma_1^h(t), \ldots, \gamma_n^h(t) \} \varepsilon_0 \frac{K_1}{\sigma_1} \to 0,
\]
as \( \epsilon_0 \to 0 \). Hence, \( \lim_{\epsilon_0 \to 0} R^h_{i\epsilon_0}(t, \tilde{Z}_2) = 0 \) uniformly for all \( t \in \mathbb{R} \) and \( \tilde{Z}_2 \in H(Z_2), i = 1, \ldots, n \). We can choose \( t_2 > 0 \) and restrict \( \epsilon_0 > 0 \) small enough such that for any given \( \epsilon' > 0 \), \( R^h_{i\epsilon_0}(t, \tilde{Z}_2) < \epsilon_0 < \epsilon' \) for all \( t \geq t_2, i = 1, \ldots, n \). Applying the comparison principle

\[
R^h_i(t, \tilde{Z}_2) < \epsilon', \quad \forall t \geq t_2, i = 1, \ldots, n.
\]

Similarly, we consider

\[
\frac{dI^v_i}{dt} = \bar{\eta}^v_i(t)\epsilon_0 - \bar{d}^v_i(t)I^v_i. \tag{17}
\]

Let \( \tilde{M}_0(t) = -\text{diag}(\bar{d}^v_i(t)), \quad M_0(t) = -\text{diag}(d^v_i(t)) \). Then system (17) can be written as

\[
\frac{dI^v_i}{dt} = \tilde{M}_0(t)I^v + F_1,
\]

where \( \tilde{M}_0(t) \) belongs to the hull \( H(M_0), I^v = (I^v_1, \ldots, I^v_n)^T, F_1 = (\bar{\eta}^v_i(t)\epsilon_0, \ldots, \bar{\eta}^v_i(t)\epsilon_0)^T \).

By the similar arguments to those in the previous system (15) and the comparison principle, it follows from \( \omega(\Psi_{\tilde{M}_0}) < 0 \) that we can choose \( t_3 > t_2 \) and restrict \( \epsilon_0 > 0 \) small enough such that

\[
I^v_i(t, \tilde{M}_0) < \epsilon', \quad \forall t \geq t_3, i = 1, \ldots, n.
\]

From the first and the fourth equations of system (5), we obtain

\[
\frac{dS^h_i}{dt} \leq \bar{A}^h_i(t, S(t-t_h) + 2\epsilon') - \bar{d}^h_i(t)S^h_i + \sum_{j=1}^{n} \bar{m}^S_{ij}(t)S^h_j,
\]

\[
\frac{dS^v_i}{dt} \geq \bar{A}^v_i(t-t_v, S^v_i(t-t_v)) - \frac{b(t)\bar{\gamma}^v_i(t)}{H^v_i}S^v_i\epsilon' - \bar{d}^v_i(t)S^v_i, \quad \forall t \geq t_3,
\]

where \( \epsilon' = (\epsilon', \ldots, \epsilon') \).

Consider the following system

\[
\frac{dS^h_i}{dt} = \bar{A}^h_i(t, S(t-t_h) + 2\epsilon') - \bar{d}^h_i(t)S^h_i + \sum_{j=1}^{n} \bar{m}^S_{ij}(t)S^h_j,
\]

\[
\frac{dS^v_i}{dt} = \bar{A}^v_i(t-t_v, S^v_i(t-t_v)) - \frac{b(t)\bar{\gamma}^v_i(t)}{H^v_i}S^v_i\epsilon' - \bar{d}^v_i(t)S^v_i, \quad \forall t \geq t_3.
\]

Note that system (18) is the perturbation (for sufficiently small \( \epsilon' \)) of system

\[
\frac{dS^h_i}{dt} = \bar{A}^h_i(t, S(t-t_h)) - \bar{d}^h_i(t)S^h_i + \sum_{j=1}^{n} \bar{m}^S_{ij}(t)S^h_j,
\]

\[
\frac{dS^v_i}{dt} = \bar{A}^v_i(t-t_v, S^v_i(t-t_v)) - \bar{d}^v_i(t)S^v_i, \quad \forall t \geq t_3.
\]

We can restrict \( \epsilon' \) such that system (18) admits a unique almost periodic solution \((S^h_{i\epsilon'}, (t, \eta), S^v_{i\epsilon'}(t, \eta)), i = 1, \ldots, n, \) which is globally attractive in \( \mathbb{R}^2_+ \). Thus, for \( \epsilon' > 0 \), there exists \( t_4 > t_3 \) such that

\[
S^h_{i\epsilon'}(t, \eta) < S^h_i(t, \eta) + \theta, \quad S^v_{i\epsilon'}(t, \eta) > S^v_i(t, \eta) - \theta, \quad \forall t \geq t_4, i = 1, \ldots, n.
\]

By the comparison principle, there exists \( t_5 > t_4 \) such that \( S^h_i(t, \eta) \leq S^h_{i\epsilon'}(t, \eta), \quad S^v_i(t, \eta) \geq S^v_{i\epsilon'}(t, \eta) \) for all \( t \geq t_5, i = 1, \ldots, n \). Then, \( S^h_i(t, \eta) < S^h_i(t, \eta) + \theta, \)
Choose $\varepsilon'$ small enough such that
\[
\begin{align*}
S^u_i(t, \eta) &> S^v_i(t, \eta) - \theta. \\
\frac{S^h_i(t, \eta)}{H_i(t, \eta)} &\geq \frac{S^h_i(t, \eta)}{S^v_i(t, \eta) + 2\varepsilon'} \geq 1 - \theta, \\
\frac{S^v_i(t, \eta)}{H_i(t, \eta)} &\geq \frac{S^v_i(t, \eta) - \theta}{S^v_i(t, \eta) + \theta + 2\varepsilon'}, \quad \forall t \geq t_5, i = 1, \ldots, n.
\end{align*}
\]

Then, we can get that
\[
\begin{align*}
\frac{dI^h_i}{dt} &\geq \bar{b}(t) \beta_i(t)(1 - \theta)I^v_i - (\bar{c}_i(t) + \bar{d}_i(t))I^h_i + \sum_{j=1}^n \tilde{m}_{ij}(t)I^h_j, \\
\frac{dR^h_i}{dt} &= \gamma_i(t)I^h_i - \phi_i(t)R^h_i + \sum_{j=1}^n \tilde{m}_{ij}(t)R^h_j, \\
\frac{dE^v_i}{dt} &\geq \bar{b}(t) \beta_i(t) \frac{S^v_i(t, \eta) - \theta}{S^v_i(t, \eta) + \theta + 2\varepsilon'} I^h_i - \eta_i(t)E^v_i - \bar{d}_i(t)E^v_i, \\
\frac{dt^v_i}{dt} &= \eta_i(t)E^v_i - \bar{d}_i(t)I^v_i, \quad \forall t \geq t_5.
\end{align*}
\]

Let
\[
\mathcal{M}_{(\theta, \varepsilon')} = \begin{pmatrix}
-Z_1(t) & 0 & 0 & \text{diag}(\bar{b}(t) \beta_i(t)(1 - \theta)) \\
diag(\gamma_i(t)) & -Z_2(t) & 0 & 0 \\
diag(\bar{b}(t) \beta_i(t) \frac{S^v_i(t, \eta) - \theta}{S^v_i(t, \eta) + \theta + 2\varepsilon'}) & 0 & -\text{diag}(\eta_i(t) + \bar{d}_i(t)) & 0 \\
0 & 0 & \text{diag}(\eta_i(t)) & -\text{diag}(\bar{d}_i(t))
\end{pmatrix}.
\]

Consider the linear system
\[
\frac{dv(t)}{dt} = \mathcal{M}_{(\theta, \varepsilon')} v(t).
\]

By Lemma 2.4, $R_0 > 1$ if and only if $\omega(\Psi_{Y-Z}) > 0$, where $\omega(\Psi_{Y-Z}) > 0$ is the exponential growth bound of $\Psi_{Y-Z}$ associated with the evolution operator $\Psi_{Y-Z}$ of system (5), that is,
\[
\omega(\Psi_{Y-Z}) = \inf\{\bar{\omega} \in \mathbb{R} : \exists K_2 \geq 1, \|\Psi_{Y-Z}(t + s, s)\| \leq K_2 e^{\bar{\omega}t}, \forall s \in \mathbb{R}, t \geq 0\}.
\]

Define the hull of $H(Y, Z)$ of $(Y(t), Z(t))$, it follows from [33, Lemma 2.4] that $\omega(\Psi_{Y-Z}) > 0$ for all $(Y, Z) \in H(Y, Z)$. Thus, we can choose $\theta > 0, \varepsilon' > 0$ small enough such that $\omega(\Psi_{\mathcal{M}_{(\theta, \varepsilon')}}) > 0$. Since $-Z(t)$ is strongly irreducible and cooperative, [33, Theorem 2.5] implies that there exist two almost periodic functions $p_{(\theta, \varepsilon')}(t, \eta)$ and $q_{(\theta, \varepsilon')}(t, \eta) \in \text{Int}(\mathbb{R}_{+}^{2n})$ such that
\[
v_{(\theta, \varepsilon')}(t, \eta) = e^{\int_0^t \mu_{(\theta, \varepsilon')}(s, \eta) ds} q_{(\theta, \varepsilon')}(t, \eta)
\]
is the solution of system (20) and
\[
\lim_{t \to \infty} \int_0^t \mu_{(\theta, \varepsilon')}(s, \eta) ds = \omega(\Psi_{\mathcal{M}_{(\theta, \varepsilon')}}) > 0.
\]

Since $(\nu, \eta) \in \Lambda_0$ implies that $\Gamma(t, \eta) := \mu(t, \eta), R^h(t, \eta), E^v(t, \eta), I^v(t, \eta)) \in \text{Int}(\mathbb{R}_{+}^{4n})$ for all $t \geq t_5$, we can choose $\lambda > 0$ small enough such that $\Gamma(t_5, \eta) \geq \lambda v_{(\theta, \varepsilon')}(t_5, \eta)$. Applying the comparison principle to system (19), we obtain that
\[
\Gamma(t, \eta) \geq \lambda v_{(\theta, \varepsilon')}(t, \eta) \geq \lambda e^{\int_0^t \mu_{(\theta, \varepsilon')}(s, \eta) ds} q_{(\theta, \varepsilon')}(t, \eta), \quad \forall t \geq t_5.
\]

Note that $q_{(\theta, \varepsilon')}(t, \eta)$ is almost periodic in $t$, and
\[
\lim_{t \to \infty} e^{\int_0^t \mu_{(\theta, \varepsilon')}(s, \eta) ds} = \lim_{t \to \infty} (e^{\frac{t}{\lambda}} \int_0^t \mu_{(\theta, \varepsilon')}(s, \eta) ds)^t = \infty.
\]
we have \( \lim_{t \to \infty} (I^h(t, \eta), R^h(t, \eta), E^v(t, \eta), I^v(t, \eta)) = \infty \) for all \( \eta \in H(\xi) \), a contradiction.

**Claim 2.** For \((\nu, \eta) \in \Lambda_0\), we have

\[
\limsup_{t \to \infty} d(\Pi_t(\nu, \eta), O_2) \geq \delta_1, \quad \delta_1 < \frac{\varepsilon_0}{3}.
\]

On the contrary, we assume that for some \((\bar{\nu}, \bar{\eta}) \in \Lambda_0\), \( \limsup_{t \to \infty} d(\Pi_t(\bar{\nu}, \bar{\eta}), O_2) < \delta_1 \). It follows that there exists \( t_0 > 0 \) such that \( d(\Pi_t(\bar{\nu}, \bar{\eta}), O_2) < \delta_1 \) for all \( t \geq t_0 \). Hence, \( d(\Pi_t(\nu, \eta), O_2) < \delta_1 \) for all \( t \geq 0 \), where \((\nu, \eta) = \Pi_{t_0}(\bar{\nu}, \bar{\eta}) \in \Lambda_0\). Thus, there exists \( \hat{\eta} \in H(\xi) \) such that \( \| \mathcal{X}_t(\nu, \eta) - (S^h_{t_0}(\varphi^h_0, \hat{\eta}), 0, 0, 0, 0) \| < \delta_1 \) and \( \bar{d}(\eta, \hat{\eta}) < \delta_1 \) for all \( t \geq 0 \). Then, we have

\[
\| \mathcal{X}_t(\nu, \eta) - (S^h_{t_0}(\varphi^h_0, \eta), 0, 0, 0, 0) \| = \| \mathcal{X}_t(\nu, \eta) - (S^h_{t_0}(\varphi^h_0, \eta), 0, 0, 0, 0) \| \leq \| \mathcal{X}_t(\nu, \eta) - (S^h_{t_0}(\varphi^h_0, \hat{\eta}), 0, 0, 0, 0) \| + \| (S^h_{t_0}(\varphi^h_0, \hat{\eta}), 0, 0, 0, 0) - (S^h_{t_0}(\varphi^h_0, \eta), 0, 0, 0, 0) \| < \delta_1 + \frac{\varepsilon_0}{3} < \varepsilon_0, \quad \forall t \geq 0,
\]

which implies that \( S^h_i(t, \eta) < \varepsilon_0, E^v_i(t, \eta) < \varepsilon_0, I^v_i(t, \eta) < \varepsilon_0 \) and \( V_i(t, \eta) < 3\varepsilon_0 \) for all \( t \geq 0, i = 1, \ldots, n \). Since \((\nu, \eta) \in \Lambda_0\), we have

\[
\lim_{t \to \infty} (V_i(t, \eta) - S^h_i(t, \eta)) = 0,
\]

a contradiction.

Since \( O_1 \) and \( O_2 \) are isolated invariant sets for \( \Pi_t \) in \( \partial \Lambda_0 \), two claims above implies that \( O_1 \) and \( O_2 \) are also isolated invariants sets for \( \Pi_t \) in \( \Lambda \) and show that

\[
W^s(O_i) \cap \Lambda_0 = \emptyset, \quad i = 1, 2,
\]

where \( W^s(O_i) = \{ (\nu, \eta) \in \Lambda : \omega(\nu, \eta) \neq \emptyset, \omega(\nu, \eta) \subset O_i \} \) is the stable of \( O_i \) for \( \Pi_t \).

By the continuous-time version of [38, Theorem 1.3.1 and Remark 1.3.1], the skew-product semiflow \( \Pi_t : \Lambda \to \Lambda \) is uniformly persistent with respect to \( \Lambda_0 \) in the sense that there exists \( \hat{\varepsilon} > 0 \) such that

\[
\inf_{(\nu, \eta) \in \omega(\nu, \eta)} d((\nu, \eta), \partial \Lambda_0) \geq \hat{\varepsilon}, \quad \forall (\nu, \eta) \in \Lambda_0.
\]

Define a continuous function \( g : \Lambda \to \mathbb{R}_+ \) by

\[
g(\nu, \eta) = \min_{1 \leq i \leq n} \{ \phi^h_i(0), \psi^h_i(0), \phi^v_i(0), \psi^v_i(0) \}, \quad \forall (\nu, \eta) \in \Lambda.
\]

Note that \( \partial \Lambda_0 = \{ (\nu, \eta) \in \Lambda : g(\nu, \eta) = 0 \} \). Let \( \Lambda_\omega = \bigcup_{(\nu, \eta) \in \Lambda_0} \omega(\nu, \eta) \). Since \( \Pi_t : \Lambda_0 \to \Lambda_0 \) has a globally compact attractor \( \mathcal{A} \), \( \lim_{t \to \infty} d(\Pi_{t_0}(\nu, \eta), \mathcal{A}) = 0 \) for all \((\nu, \eta) \in \Lambda_0\). It follows that \( \min_{(\nu, \eta) \in \Lambda_0} g(\nu, \eta) > 0 \). From [19, Proposition 3.3], we conclude that there exists \( \varepsilon > 0 \) such that \( \inf_{(\nu, \eta) \in \Lambda_\omega} g(\nu, \eta) \geq \varepsilon \), which implies that

\[
\liminf_{t \to \infty} (I^h_i(t, \eta), R^h_i(t, \eta), E^v_i(t, \eta), I^v_i(t, \eta)) \geq (\varepsilon, \varepsilon, \varepsilon, \varepsilon), \quad \forall (\nu, \eta) \in \Lambda_0.
\]

This completes the proof.

**Theorem 3.2.** Suppose that (C1) – (C2) and (H1) – (H5) hold. If \( R_0 < 1 \), let \( m^S_{ij}(t) = m^I_{ij}(t) = m^E_{ij}(t), i, j = 1, \ldots, n \), then the disease-free almost period state \( E_i(t) = (S^h_i(t), 0, 0, S^v_i(t), 0, 0) \) of (3) is globally stable.
Proof. We use the skew-product semiflow approach to prove the results. Since \( m_{ij}^S(t) = m_{ij}^S(t) = m_{ij}^R(t) \), we set \( m_{ij}^S(t) = m_{ij}^S(t) = m_{ij}^R(t) := m_{ij}(t) \). Hence, system (5) can be written as
\[
\frac{dH^t}{dt} = \tilde{A}^h_i(t, H(t - \tau_h)) - \tilde{d}^h_i(t)H_i + \sum_{j=1}^n \tilde{m}_{ij}(t)H_j,
\]
\[
\frac{dV^t}{dt} = \tilde{A}^v_i(t - \tau_v, V_i(t - \tau_v)) - \tilde{d}^v_i(t)V_i.
\]
(21)

By Lemma 2.3, system (21) has a unique positive almost periodic solution \((S^h_i(t, \eta), S^v_i(t, \eta))\) for all \( t \geq 0, i = 1, \ldots, n \), which is globally attractive in \( C_+ \times H(\xi) \), that is, \( \lim_{t \to \infty} (H_i(t, \eta) - S^h_i(t, \eta)) = 0 \) and \( \lim_{t \to \infty} (V_i(t, \eta) - S^v_i(t, \eta)) = 0 \). Hence, for any \( \varepsilon > 0 \), there exists \( t_0 > 0 \) such that
\[
H_i(t, \eta) > S^h_i(t, \eta) - \varepsilon, \quad V_i(t, \eta) < S^v_i(t, \eta) + \varepsilon, \quad \forall t \geq t_0, \eta \in H(\xi).
\]
Thus, we have
\[
\begin{align*}
\frac{dH^t}{dt} &\leq \tilde{b}(t)\tilde{\sigma}^h_i(t)I^v_i - (\tilde{\sigma}^h_i(t) + \tilde{d}^h_i(t))I^h_i + \sum_{j=1}^n \tilde{m}_{ij}(t)I^h_j, \\
\frac{dR^t}{dt} &= \tilde{\gamma}_i(t)I^h_i - \tilde{d}^h_i(t)R_i + \sum_{j=1}^n \tilde{m}_{ij}(t)R_j, \\
\frac{dE^t}{dt} &\leq \tilde{b}(t)\tilde{\sigma}^v_i(t)S^v_i(t, \eta) - \tilde{d}^v_i(t)E_i + \sum_{j=1}^n \tilde{m}_{ij}(t)E_j, \\
\frac{dI^t}{dt} &= \tilde{\gamma}_i(t)E_i - \tilde{d}^v_i(t)I^v_i + \sum_{j=1}^n \tilde{m}_{ij}(t)I^v_j.
\end{align*}
\]
(22)

Let
\[
\mathcal{M}_\varepsilon = \begin{pmatrix}
-\tilde{Z}_i(t) & 0 & 0 & \text{diag}(\tilde{b}(t)\tilde{\sigma}^h_i(t)) \\
\text{diag}(\tilde{\sigma}^h_i(t)) & -\tilde{Z}_i(t) & 0 & 0 \\
\text{diag}(\tilde{b}(t)\tilde{\sigma}^v_i(t)) & 0 & -\text{diag}(\tilde{n}_i(t) + \tilde{d}^v_i(t)) & 0 \\
0 & 0 & \text{diag}(\tilde{n}_i(t)) & -\text{diag}(\tilde{d}^v_i(t))
\end{pmatrix}.
\]

It follows from Lemma 2.4 that \( R_0 < 1 \) if and only if \( \omega(\Psi_{Y-Z}) < 0 \). By [33, Lemma 2.4], we have \( \omega(\Psi_{Y-Z}) < 0 \) for all \((Y, Z) \in H(Y, Z)\). Hence, we can restrict \( \varepsilon \) small enough such that \( \omega(\Psi_{\mathcal{M}_\varepsilon}) < 0 \). By [33, Theorem 2.5], there exist two almost periodic functions \( \tilde{x}_z(t, \eta) \) and \( \tilde{y}_z(t, \eta) \in \text{Int}(\mathbb{R}^n) \) such that \( \tilde{u}_z(t, \eta) = e^{\int_0^t \tilde{x}_z(s, \eta) ds} \tilde{y}_z(t, \eta) \) is a solution of \( \frac{dx}{dt} = \mathcal{M}_\varepsilon \tilde{u}_z \) and
\[
\omega(\Psi_{\mathcal{M}_\varepsilon}) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \tilde{x}_z(s, \eta) ds < 0.
\]

Since \( \tilde{y}_z(t, \eta) \) is almost periodic and is bounded on \( \mathbb{R}^n \) and
\[
\lim_{t \to \infty} e^{\int_0^t \tilde{x}_z(s, \eta) ds} = \lim_{t \to \infty} (e^{\frac{1}{2} \int_0^t \tilde{x}_z(s, \eta) ds})^t = 0,
\]
we conclude that \( \lim_{t \to \infty} \tilde{u}_z(t, \eta) = 0 \). Applying the comparison principle to system (22), we conclude that
\[
\lim_{t \to \infty} (I^h_i(t, \eta), R^h_i(t, \eta), E^v_i(t, \eta), I^v_i(t, \eta)) = (0, 0, 0, 0), \quad \forall \eta \in H(\xi).
\]
Since
\[
\lim_{t \to \infty} (S^h_i(t, \eta) + I^h_i(t, \eta) + R^h_i(t, \eta) - S^h_i(t, \eta) = 0,
\]
\[
\lim_{t \to \infty} (S^v_i(t, \eta) + E^v_i(t, \eta) + I^v_i(t, \eta) - S^v_i(t, \eta) = 0.
\]
It follows that
\[
\lim_{t \to \infty} S^h_i(t, \eta) - S^h_i(t, \eta) = 0, \quad \lim_{t \to \infty} S^v_i(t, \eta) - S^v_i(t, \eta) = 0.
\]
This completes the proof.

**Theorem 3.3.** Suppose that (C1) − (C2) and (H1) − (H5) hold. Let \( X(t, \nu) = (S^h(t), I^h(t), R^h(t), S^\nu(t), E^h(t), I^\nu(t)) \) be the solution of system (3) with initial value \( X(0, \nu) = \nu \). If \( R_0 < 1 \), then for \( L > \max\{K_1, \ldots, K_n\} \), there exists a \( \mu = \mu(L) > 0 \) such that for any \( \nu \in Y^3_L \) with \( (\psi^h_1(0), \phi^h_1(0), \psi^\nu_1(0), \phi^\nu_1(0)) \in [0, \mu]^4 \) and \( \varphi^h_i(0) + \psi^h_i(0) + \phi^\nu_i(0) > 0, i = 1, \ldots, n \), the solution \( X(t, \nu) \) of system (3) through \( \nu \) satisfies

\[
\lim_{t \to \infty} \| X(t, \nu) - (S^h(t), 0, 0, S^\nu(t), 0, 0) \| = 0,
\]

where \( Y_L = C([-\tau, 0], [0, L]) \times C([-\tau, 0], [0, L]) \).

**Proof.** Given \( L > \max\{K_1, \ldots, K_n\} \), it then follows from Lemma 2.1 that \( Y^3_L \) is positively invariant for the solution of system (5). Here, we have

\[
\Pi_0(\nu, \eta) \in Y^3_L \times H(\xi), \quad \forall t \geq 0, \nu \in Y^3_L.
\]

Consider the following system

\[
\begin{align*}
\frac{dS^h}{dt} &= A^h(t, S^h(t - \tau_h) + 2\varepsilon) - \bar{d}^h(t)S^h + \sum_{j=1}^{n} \bar{m}^S_{ij}(t)S^h_j, \\
\frac{dS^\nu}{dt} &= A^\nu(t, S^\nu(t - \tau_v) + 2\varepsilon) - \bar{d}^\nu(t)S^\nu,
\end{align*}
\]

where \( \varepsilon = (\varepsilon, \ldots, \varepsilon) \).

We can restrict \( \varepsilon \) small enough such that system (23) has an attractive and positive almost periodic solution \( (S^h(t, \varepsilon, \eta), S^\nu(t, \varepsilon, \eta)) \) for all \( t \geq 0 \). Obviously, \( \lim_{\varepsilon \to 0^+} S^h(t, \varepsilon, \eta) = S^h(t, \eta) \), \( \lim_{\varepsilon \to 0^+} S^\nu(t, \varepsilon, \eta) = S^\nu(t, \eta) \). Let \( (S^h(t, \eta, \varphi^h), S^\nu(t, \eta, \varphi^\nu)) \) be the solution of system (23) with the initial value \( (\varphi^h, \varphi^\nu) \). Then there exists \( t_0 > 0 \) such that

\[
S^h(t, \eta, \varphi^h) > S^\nu(t, \varepsilon, \eta) - \varepsilon, \quad S^\nu(t, \eta, \varphi^\nu) < S^\nu(t, \varepsilon, \eta) + \varepsilon, \quad \forall t \geq t_0.
\]

Let

\[
G_{\varepsilon} = \begin{pmatrix}
-\bar{d}^h(t) & 0 & 0 & \text{diag}(\bar{b}(t))
\end{pmatrix},
\]

Consider the following equation

\[
\frac{d \bar{v}}{dt} = G_{\varepsilon}(t)\bar{v}.
\]

Let \( \Psi_{G_{\varepsilon}}(t, s), t \geq s, s \in \mathbb{R} \) be the evolution operator of the linear almost periodic system (24). Similarly, we can define the exponential growth bound \( \omega(\Psi_{G_{\varepsilon}}) \). The continuous dependence of solutions on parameter implies that \( \lim_{\varepsilon \to 0^+} \Psi_{G_{\varepsilon}}(t, s) = \Psi_{\bar{Y} - \bar{Z}}(t, s) \), where \( \Psi_{\bar{Y} - \bar{Z}}(t, s) \) is the evolution operator when we replace \( (Y(t) - Z(t)) \) by \( (\bar{Y}(t) - \bar{Z}(t)) \). It follows from Lemma 2.4 that \( R_0 < 1 \) if and only if \( \omega(\Psi_{\bar{Y} - \bar{Z}}) < 0 \). Also by [33, Lemma 2.4], we have \( \omega(\Psi_{\bar{Y} - \bar{Z}}) = \omega(\Psi_{\bar{Y} - \bar{Z}}) < 0 \). Hence, we can restrict \( \varepsilon \) small enough such that \( \omega(\Psi_{G_{\varepsilon}}) < 0 \). By [33, Theorem 2.5], we obtain that there exist two almost periodic functions \( a(t, G_{\varepsilon}) \) and \( b(t, G_{\varepsilon}) \in \text{Int}(\mathbb{R}^{4n}) \) such that

\[
\bar{v}(t, G_{\varepsilon}) = e^{\int_0^t a(t, G_{\varepsilon}) \, dt} b(t, G_{\varepsilon}(
\]

\[
\end{pmatrix}
\]
is a solution of system (24) and
\[ \omega(\Psi_{G_e}) = \lim_{t \to \infty} \frac{1}{t} \int_0^t a(\tau, G_e) \, d\tau < 0. \]

Choose \( \lambda^* > 0 \) small enough such that \( \lambda^* b(t, G_e) < 4\bar{\varepsilon}, \forall t \in [0, t_0]. \) Consider the following system
\[
\begin{align*}
\frac{dI^h}{dt} &= \bar{b}(t)\bar{I}^{h}(t)\bar{I}^{\nu} - (\gamma^h_i(t) + \bar{d}^h_i(t))\bar{I}^{h}_i + \sum_{j=1}^{n} \bar{m}^{h}_ij(t)\bar{I}^{h}_{j}, \\
\frac{d\bar{R}^{h}_i}{dt} &= \gamma^h_i(t)\bar{I}^{h}_i - \bar{d}^h_i(t)\bar{R}^{h}_i + \sum_{j=1}^{n} \bar{m}^{h}_{ij}(t)\bar{R}^{h}_j, \\
\frac{d\bar{E}^{\nu}_i}{dt} &= \bar{b}(t)\bar{\gamma}^{\nu}_i(t)\bar{S}^{\nu}_i(t, \eta, \varepsilon) + \bar{e}^\nu_i(t)\bar{E}^{\nu}_i - \bar{\eta}^\nu_i(t)\bar{E}^{\nu}_i - \bar{d}^\nu_i(t)\bar{E}^{\nu}_i, \\
\frac{d\bar{v}^{\nu}_i}{dt} &= \bar{\eta}^\nu_i(t)\bar{E}^{\nu}_i - \bar{d}^\nu_i(t)\bar{v}^{\nu}_i.
\end{align*}
\]
Let \( (\bar{I}^{h}(t, \eta, \psi^{h}), \bar{R}^{h}(t, \eta, \phi^{h}), \bar{E}^{\nu}(t, \eta, \phi^{\nu}), \bar{v}^{\nu}(t, \eta, \phi^{\nu})) := \bar{\Gamma}(t, \eta, x^0) \) be the solution of system (25) with the initial value \( x^0 = (\psi^{h}_1, \phi^{h}_1, \psi^{\nu}_1, \phi^{\nu}_1) \in \mathbb{R}^{4n} \) for \( \eta \in H(\xi), i = 1, \ldots, n. \) We restrict \( \mu > 0 \) sufficiently small such that
\[
\bar{\Gamma}(t, \eta, x^0) < \lambda^* e^{\int_0^t a(\tau, G_e) \, d\tau} b(t, G_e) < 4\bar{\varepsilon}, \forall t \in [0, t_0].
\]
Let \( (S^{h}(t, \eta), I^{h}(t, \eta), R^{h}(t, \eta), S^{\nu}(t, \eta), E^{\nu}(t, \eta), I^{\nu}(t, \eta)) \) be a nonnegative solution of system (5) with the initial value \( (\psi^{h}, \phi^{h}, \psi^{\nu}, \phi^{\nu}, \eta) \in Y^3_{E^2} \times H(\xi), \phi^{h}, \phi^{\nu} \not= \bar{0} \) and \( \psi^{h}_i(0), \phi^{h}_i(0), \psi^{\nu}_i(0), \phi^{\nu}_i(0) \leq \mu, i = 1, \ldots, n. \)

In the following, we claim
\[
\Gamma(t, \eta) \leq \lambda^* e^{\int_0^t a(\tau, G_e) \, d\tau} b(t, G_e), \quad \forall t \geq t_0,
\]
where \( \Gamma(t, \eta) = (I^{h}(t, \eta, \psi^{h}), R^{h}(t, \eta, \phi^{h}), E^{\nu}(t, \eta, \psi^{\nu}), I^{\nu}(t, \eta, \phi^{\nu})). \)

On the contrary, by the comparison principle and (26), there exists \( k \in [1, 4n] \) and \( T \geq t_0 \) such that
\[
\begin{align*}
\Gamma(t, \eta) &\leq \lambda^* e^{\int_0^t a(\tau, G_e) \, d\tau} b(t, G_e), \quad t_0 \leq t \leq T, \\
\Gamma_k(T, \eta) &= \lambda^* e^{\int_0^t a(\tau, G_e) \, d\tau} b(T, G_e),
\end{align*}
\]
From the analysis above and (H2) and (H3), for any \( t_0 \leq t \leq T, \) we have
\[
\begin{align*}
\frac{dS^{h}}{dt} &= \bar{A}^{h}_i(t, S^{h}(t - \tau_e) + 2\bar{\varepsilon)) - \bar{d}^{h}_i(t)S^{h}_i + \sum_{j=1}^{n} \bar{m}^{h}_{ij}(t)S^{h}_j, \\
\frac{dS^{\nu}_i}{dt} &= \bar{A}^{\nu}_i(t - \tau_v, S^{\nu}_i(t - \tau_v) + 2\bar{\varepsilon)) - \bar{d}^{\nu}_i(t)S^{\nu}_i.
\end{align*}
\]
By the comparison principle, we obtain that \( S^{h}(T, \eta) < S^{h}_e(T, \varepsilon, \eta) + \bar{\varepsilon} \) and \( S^{\nu}(T, \eta) < S^{\nu}_e(T, \varepsilon, \eta) + \bar{\varepsilon}. \) Then, for \( 0 < t - T \ll 1, \) we get \( S^{h}(t, \eta) < S^{h}_e(t, \varepsilon, \eta) + \bar{\varepsilon} \) and \( S^{\nu}(t, \eta) < S^{\nu}_e(t, \varepsilon, \eta) + \bar{\varepsilon}. \) and hence,
\[
\begin{align*}
\frac{dI^h}{dt} &= \bar{b}(t)\bar{I}^{h}(t)\bar{I}^{\nu} - (\gamma^h_i(t) + \bar{d}^h_i(t))\bar{I}^{h}_i + \sum_{j=1}^{n} \bar{m}^{h}_ij(t)\bar{I}^{h}_{j}, \\
\frac{d\bar{R}^{h}_i}{dt} &= \gamma^h_i(t)\bar{I}^{h}_i - \bar{d}^h_i(t)\bar{R}^{h}_i + \sum_{j=1}^{n} \bar{m}^{h}_{ij}(t)\bar{R}^{h}_j, \\
\frac{d\bar{E}^{\nu}_i}{dt} &= \bar{b}(t)\bar{\gamma}^{\nu}_i(t)\bar{S}^{\nu}_i(t, \eta, \varepsilon) + \bar{e}^\nu_i(t)\bar{E}^{\nu}_i - \bar{\eta}^\nu_i(t)\bar{E}^{\nu}_i - \bar{d}^\nu_i(t)\bar{E}^{\nu}_i, \\
\frac{d\bar{v}^{\nu}_i}{dt} &= \bar{\eta}^\nu_i(t)\bar{E}^{\nu}_i - \bar{d}^\nu_i(t)\bar{v}^{\nu}_i.
\end{align*}
\]
Since \( \Gamma(T, \eta) \leq \lambda^* e^{\int_0^t a(\tau, G_e) \, d\tau} b(T, G_e), \) we can obtain that
\[
\Gamma(t, \eta) \leq \lambda^* e^{\int_0^t a(\tau, G_e) \, d\tau} b(t, G_e), \quad 0 < t - T \ll 1.
\]
Thus,
\[ \Gamma_k(t, \eta) \leq \lambda^* (\int_0^t a(\tau, \cdot, \cdot) \, d\tau b(t, \cdot)) k, \quad 0 < t - T \ll 1. \]
a contradiction. The claim is proved.

From the claim above, we can see that for any \( t \geq 0 \), system (27) also holds. By the comparison principle, \( S^h(t, \eta) < S^h(t, \varepsilon, \eta) + \varepsilon \) and \( S^v(t, \eta) < S^v(t, \varepsilon, \eta) + \varepsilon \) for all \( t \geq t_0 \). By similar argument to in the claim above, it follows that
\[ \Gamma(t, \eta) < \lambda^* (\int_0^t a(\tau, \cdot, \cdot) \, d\tau b(t, \cdot)) \]
Since \( b(t, \cdot, \cdot) \) is almost periodic,
\[ \lim_{t \to \infty} \Gamma(t, \eta) = \lim_{t \to \infty} (I^h(t, \eta), R^h(t, \eta), E^v(t, \eta), I^v(t, \eta)) = 0. \]

For any \( (\nu, \eta^*) := (\varphi^h, \psi^v, \phi^h, \psi^v, \phi^v, \eta^*) \in Y_*^h \times H(\xi), \varphi^h, \psi^v \neq 0 \) and \( \psi^v(0), \phi^h(0), \psi^v(0), \phi^v(0) < \mu, i = 1, \ldots, n \), let \( \omega(\nu, \eta^*) \) be the omega limit set of \( (\nu, \eta^*) \) for \( \Pi_t \) in system (5). The argument above implies that \( \lim_{t \to \infty} \Gamma(t, \eta^*, \xi^0) = 0 \).

Hence, \( \omega(\nu, \eta^*) = \{\bar{\omega}, \bar{\eta}\} \). We claim that \( \bar{\omega} \neq \{0, 0\} \). On the contrary, suppose that \( \bar{\omega} = \{0, 0\} \). Then \( \lim_{t \to \infty} S^h(t, \eta^*, \varphi^h) = 0 \) and \( \lim_{t \to \infty} S^v(t, \eta^*, \varphi^v) = 0 \), which is contradictory to the claim in the proof of Theorem 3.1. The disease-free space
\[ X_\ast = \{(S^h, I^h, R^h, S^v, E^v, I^v) \in \mathbb{R}_+^6 : I^h = R^h = E^v = I^v = 0, i = 1, \ldots, n\} \]
is variant. Let \((S^h(t, \eta, \varphi^h), S^v(t, \eta, \varphi^v))\) be the unique solution of system (8) satisfying \( S^h(\theta_1, \eta, \varphi^h) = \varphi^h(\theta_1), \theta_1 \in [-\tau_h, 0] \) and \( S^v(\theta_2, \eta, \varphi^v) = \varphi^v(\theta_2), \theta_2 \in [-\tau_v, 0] \). Then, we can define the skew-product semiflow
\[ \tilde{\Pi}_t : \tilde{C}_+ \times H(\xi) \rightarrow \tilde{C}_+ \times H(\xi) \]
\[ (\varphi^h, \varphi^v, \eta) \rightarrow (\tilde{S}^h_t(\eta, \varphi^h), \tilde{S}^v_t(\eta, \varphi^v), \eta_t), \]
Let
\[ \tilde{A} = \{(S^h_\ast(\eta), S^v_\ast(\eta), \eta_t) : t \in \mathbb{R}, \eta \in H(\xi)\} \]
and
\[ W^s(\tilde{A}) = \{(\varphi^h, \varphi^v, \eta) \in C_+ \times H(\xi) : \tilde{\omega}(\varphi^h, \varphi^v, \eta) \neq \emptyset, \tilde{\omega}(\varphi^h, \varphi^v, \eta) \subset \tilde{A}\}, \]
which is the stable set of \( \tilde{A} \) for \( \tilde{\Pi}_t \), where \( C_+ = C([-\tau_h, 0], \mathbb{R}_+^n) \times C([-\tau_v, 0], \mathbb{R}_+^n) \)
and \( \tilde{\omega}(\varphi^h, \varphi^v, \eta) \) be the omega limit set for \( \tilde{\Pi}_t \).

By the continuous-time version of \[38, \text{Lemma 1.2.1}\], \( \omega(\varphi^h_\ast, \varphi^v_\ast, \eta^*) \) is an internal chain transitive set for \( \Pi_t \), it follows that \( \tilde{\omega} \) is an internal chain transitive set for \( \tilde{\Pi}_t \). Since \( \tilde{A} \) is globally attractive for \( \tilde{\Pi}_t \) in \( C_+ \times H(\xi) \setminus \{0\} \). Hence, we conclude that \( \bar{\omega} \cap W^s(\tilde{A}) \neq \emptyset \). Similarly, by continuous-time version of the \[38, \text{Theorem 1.2.1}\], we obtain that \( \bar{\omega} = \tilde{A} \), and thus
\[ \omega(\varphi^h_\ast, \varphi^v_\ast, \eta^*) = \{(S^h_\ast(\eta), S^v_\ast(\eta), \eta_t) : t \in \mathbb{R}, \eta \in H(\xi)\}. \]
Hence, \( \lim_{t \to \infty} (S^h(t, \eta) - S^h_\ast(t, \eta)) = 0 \), \( \lim_{t \to \infty} (S^v(t, \eta) - S^v_\ast(t, \eta)) = 0 \) and
\[ \lim_{t \to \infty} (I^h(t, \eta), R^h(t, \eta), E^v(t, \eta), I^v(t, \eta)) = 0. \]
This completes the proof.
Choose in this section, we consider the case of two patches to support our analytical conclusions by numerical simulations. Let the biting rate \( b(t) = k(3 + \cos 4t) \) month\(^{-1}\) in our model. As in [18], we choose \( B_i(t, H_i) = \frac{a_i}{H_i} + b_i t \) humans \times \) month\(^{-1}\) with \( a_i > 0, i = 1, 2 \), take \( a_1 = 0.1, a_2 = 0.09, b_1 = 0.05, b_2 = 0.045 \) and \( b_i(t, V_i) = \frac{q_i}{p_i + V_i} \) mosquitoes \times \) month\(^{-1}\) with \( p_i, q_i > 0, i = 1, 2 \).

Let \( \gamma_1(t) = 500(306 + \cos 6\sqrt{2}t), p_1 = 10 \) and \( q_2(t) = 5000 + 550(306 + \cos 6\sqrt{2}t), p_2 = 12, d_1(t) = 5 + 2 \sin 6\sqrt{3}t \) month\(^{-1}\), \( d_2(t) = 4 + 2 \sin 6\sqrt{3}t \) month\(^{-1}\). Obviously, the case is almost periodic one. For simplicity, we assume that other parameters are independent of time \( t \). In view of assumptions (1.2), we have

\[
\begin{align*}
    m_{21}^J &= -m_{11}^J, \quad m_{12}^J = -m_{22}^J, \quad m_{21}^S = -m_{12}^S, \quad m_{12}^S = -m_{21}^S, \\
    m_{21}^R &= -m_{11}^R, \quad m_{12}^R = -m_{22}^R, \\
    d_{11}^J &= -d_{11}^J, \quad d_{22}^J = -d_{22}^J, \quad d_{11}^S &= -d_{11}^S, \quad d_{22}^S = -d_{22}^S, \quad d_{11}^R &= -d_{11}^R, \quad d_{22}^R = -d_{22}^R.
\end{align*}
\]

Then, system (3) becomes

\[
\begin{align}
    \frac{dS_i}{dt} &= A_1^S(t, H(t - \tau_h)) - \frac{b_i(t)S_i}{H(t - \tau_h)} - (d_i + m_{2i}^S)S_i, \\
    \frac{dH_i}{dt} &= A_2^H(t, H(t - \tau_h)) - \frac{b_i(h_i + m_{2i}^H)H_i}{H(t - \tau_h)} - (d_i + m_{2i}^H)H_i, \\
    \frac{dI_i}{dt} &= \frac{b_i(t)S_i}{H(t - \tau_h)} - (\gamma_i + d_i + m_{2i}^I)I_i, \\
    \frac{dS_1}{dt} &= A_1^{S_1}(t, V(t - \tau_v)) - \frac{b_i(t)S_1}{V(t - \tau_v)} - (d_i^{S_1} + m_{21}^{S_1}S_1) + (d_i^{S_1} + m_{21}^{S_1}S_1), \\
    \frac{dH_1}{dt} &= A_2^{H_1}(t, V(t - \tau_v)) - \frac{b_i(h_1 + m_{21}^{H_1}H_1)}{V(t - \tau_v)} - (d_i^{H_1} + m_{21}^{H_1}H_1) + (d_i^{H_1} + m_{21}^{H_1}H_1), \\
    \frac{dI_1}{dt} &= \frac{b_i(t)V_1(t - \tau_v)}{V(t - \tau_v)} - (\gamma_i + d_i + m_{21}^{I_1}I_1), \\
    \frac{dS_2}{dt} &= A_1^{S_2}(t, V(t - \tau_v)) - \frac{b_i(t)S_2}{V(t - \tau_v)} - (d_i^{S_2} + m_{22}^{S_2}S_2) + (d_i^{S_2} + m_{22}^{S_2}S_2), \\
    \frac{dH_2}{dt} &= A_2^{H_2}(t, V(t - \tau_v)) - \frac{b_i(h_2 + m_{22}^{H_2}H_2)}{V(t - \tau_v)} - (d_i^{H_2} + m_{22}^{H_2}H_2) + (d_i^{H_2} + m_{22}^{H_2}H_2), \\
    \frac{dI_2}{dt} &= \frac{b_i(t)V_2(t - \tau_v)}{V(t - \tau_v)} - (\gamma_i + d_i + m_{22}^{I_2}I_2).
\end{align}
\]

Now we show that system (29) admits a unique positive almost periodic equilibrium under assumptions (C1)-(C2), and it is uniformly asymptotically stable. Refer to [20], we take \( \beta_1 = 0.25, \beta_2 = 0.375, d_1^2 = 0.044 \) month\(^{-1}\), \( d_2^2 = 0.0442 \) month\(^{-1}\), \( m_{21}^S = 0.95, m_{22}^S = 1.05, \gamma_1 = 0.01 \) month\(^{-1}\), \( m_{21}^R = 1, m_{22}^R = 0.8, m_{21}^R = 1.1, \beta_1 = 0.4, \beta_2 = 0.5, \gamma_1 = 1 \) month\(^{-1}\), \( \gamma_2 = 1.5 \) month\(^{-1}\). Additionally, we assume \( \mu_1^{H_1} = 0.01 \) month\(^{-1}\), \( m_{21}^R = m_{22}^R = 1, \)

\[
E_{Jh}(t) = \begin{pmatrix}
    -\mu_1^R - m_{21}^R & m_{12}^J - \mu_2^J - m_{12}^J
\end{pmatrix} \equiv E_{Jh}.
\]

It then follows that

\[
\Phi(t, t - \tau_h) = \exp(E_{Jh} \cdot \tau_h) = \frac{1}{2} \begin{pmatrix}
    p_1 & p_2 \\
    p_2 & p_1
\end{pmatrix},
\]

where \( p_1 = e^{-\mu_1^{H_1} \tau_h} + e^{-\mu_1^{H_1}(2m_{21}^S) \tau_h}, p_2 = e^{-\mu_1^{H_1} \tau_h} - e^{-\mu_1^{H_1}(2m_{21}^S) \tau_h} \).

Choose \( \tau_h = 3 \) month, \( \tau_v = 0.4 \) month, and linearized system (29), we get
AN ALMOST PERIODIC DENGUE TRANSMISSION MODEL

Figure 1. The number of equilibrium.

\[
\begin{align*}
\frac{dS_i}{dt} &= A_i^h(t, H(t - \tau_h)) - (d_i^h + m_{i2}^h)S_i^h + m_{i1}^h S_i^h, \\
\frac{dS_i}{dt} &= A_i^v(t, V(t - \tau_v)) - (d_i^v + m_{i1}^v)S_i^v + m_{i2}^v S_i^v, \\
\frac{dS_i}{dt} &= A_i^v(t - \tau_v, V_1(t - \tau_v)) - d_i^v S_i^v, \\
\frac{dS_i}{dt} &= A_i^v(t - \tau_v, V_2(t - \tau_v)) - d_i^v S_i^v.
\end{align*}
\]

Obviously, system (30) is almost periodic, we can get the graphs of solution \( S_i^h \) and \( S_i^v, i = 1, 2 \), see Fig. 1.

In the following, we consider the system

\[
\frac{du(t)}{dt} = (Y(t) - Z(t))u(t)
\]

with the initial value \( u(0) = (1000, 1000, 1000, 1000) \) and \( 1000 = (1000, 1000) \). By Lemma 2.4, we get that \( R_0 - 1 \) and \( \omega(\Psi_{Y-Z}) \) have the same sign, which implies that \( R_0 > 1( < 1) \) if and only if \( \omega(\Psi_{Y-Z}) > 0( < 0) \). As in [23, Theorem A.2], we can judge the sigh of \( \omega(\Psi_{Y-Z}) \) by choosing \( t \) large enough. If we take \( k = 1 \), through numerical simulations we obtain that \( \|u(t)\| \to 0 \) as \( t \to \infty \), that is, \( \ln \|u(t)\| \to 0 \) and \( \ln \|\omega(t)\| \to 0 \) as \( t \to \infty \), see Fig. 2. By Lemma 2.4, we get that \( \omega(\Psi_{Y-Z}) < 0 \), that is \( R_0 < 1 \). By [33, Theorem 3.4], \( R_0 \) satisfies \( \omega(\Psi_{-Z+Y_R}) = 0 \). Then we can calculate \( R_0 \) numerically, that is, \( R_0 = 0.8 \).

Figure 2. The graph of \( \|u(t)\| \) and \( \ln \|u(t)\| \) of (4.3) when \( k = 1 \).

When \( R_0 < 1 \), we get the long-term behavior of host and vector populations at patch 1 and 2 by Fig. 3. The figure shows that the disease will die out, it is consistent with the theoretical results obtained in previous section.

By [23], \( R_0 \) is increasing in \( k \). Hence, we can get greater \( R_0 \) by choosing greater \( k \). Set \( k = 4 \), by numerical simulations we obtain that \( \|u(t)\| \to \infty \) as \( t \to \infty \), that
Figure 3. The long-term behavior host and vector populations at patch 1 and 2 when $R_0 < 1$.

Figure 4. The graph of $\|u(t)\|$ and $\ln \|u(t)\|$ of (4.3) when $k = 4$.

Figure 5. The long-term behavior host and vector populations at patch 1 and 2 when $R_0 > 1$.

Figure 6. Relationship between $k$ and $R_0$. 


Figure 7. The graph of $\|u(t)\|$ and $\ln\|u(t)\|$ of (4.3) when $m_{21}^I = 0$.

Figure 8. The graph of $\|u(t)\|$ and $\ln\|u(t)\|$ of (4.3) when $m_{21}^I = 1$.

Figure 9. Relationship between $m_{21}^I$ and $R_0$.

Figure 10. Relationship between $n$ and $R_0$. 
is, $\ln \|u(t)\| > 0$ and $\frac{\ln \|u(t)\|}{t} > 0$ as $t \to \infty$, see Fig. 4. By Lemma 2.4, we get that $\omega(\Psi_{Y-Z}) > 0$, that is $R_0 > 1$. We can numerically get that $R_0 = 3.2$. Similar to the case of $R_0 < 1$, we can get that long-term behavior of host and vector populations at patch 1 and 2 when $R_0 > 1$, see Fig. 5. The figure shows that the disease is uniformly persistent, which is consistent with the theoretical results obtained in previous section.

From the numerically simulations and the relationship between $b(t)$ and $R_0$, we can get the relationship between $k$ and $R_0$, see Fig. 6. The figure shows that we can control dengue disease by reducing biting rate, that is, when $k < 1.25$, $R_0 < 1$. For measures to reduce the biting rate, we can hang mosquito nets or apply drugs on mosquito nets and so on.

In the following, fix $k = 2$, we discuss the effect of immigration and emigration of infected host on the basic reproduction number $R_0$. If $m_{21}^f = 0$, other parameters remain unchanged, through numerical simulations we obtain that $\|u(t)\| \to 0$ as $t \to \infty$, that is, $\ln \|u(t)\| < 0$ and $\frac{\ln \|u(t)\|}{t} < 0$ as $t \to \infty$, see Fig. 7. By Lemma 2.4, we get that $\omega(\Psi_{Y-Z}) < 0$, that is $R_0 < 1$. We can numerically get that $R_0 = 0.905$. If $m_{21}^f = 1$, by numerical simulations we obtain that $\|u(t)\| \to \infty$ as $t \to \infty$, that is, $\|u(t)\| > 0$ and $\frac{\ln \|u(t)\|}{t} > 0$ as $t \to \infty$, see Fig. 8. By Lemma 2.4, we get that $\omega(\Psi_{Y-Z}) > 0$, that is $R_0 > 1$. We can numerically get that $R_0 = 1.6$.

By the numerically simulations, we can get the relationship between $m_{21}^f$ and $R_0$, see Fig. 9. The figure shows that the disease in patch 2 can be brought under control by restricting the immigration of infected hosts from patch 1 to patch 2. That is, if the infected host in patch 1 to the patch 2 is rare, then $R_0 < 1$ and the disease is under control; if the number of infected host in patch 1 to patch 2 increases gradually, then $R_0 > 1$ and the disease is uniformly persistent.

Finally, motivated by [24], we make a comparison between the case $b(t)$ is periodic and the case $b(t)$ is almost periodic. Let $b(t) = \frac{2}{5}(3 + \cos(4t)) + \sum_{i=1}^{\infty} \frac{1}{2^n}(1 + \sin(\frac{i}{2^n}))$ month$^{-1}$. Then $b(t)$ is almost periodic. Let $b^n(t) = \frac{2}{5}(3 + \cos(4t)) + \sum_{i=1}^{\infty} \frac{1}{2^n}(1 + \sin(\frac{i}{2^n}))$ month$^{-1}$. Clearly, $\lim_{t \to \infty} \|b^n(t) - b(t)\| = 0$. The relationship between $R_0$ and $n$ is shown in Fig.10 in the case where $b(t) = b^n(t)$, that is $R_0$ is increasing in $n$ in the case where $b(t) = b^n(t)$. In another way, it is easy to see that $b^n$ is increasing in $n$. Furthermore, [33, Theorem 3.4] implies that $R_0$ is also increasing in $n$. Thus, the value of $R_0$ will be underestimate if we use $b^n$ to replace $b$. Similarly, when $b(t) = \frac{2}{5}(3 + \cos(4t)) - \sum_{i=1}^{\infty} \frac{1}{2^n}(1 + \sin(\frac{i}{2^n}))$ month$^{-1}$, if we use $b^n(t) = \frac{2}{5}(3 + \cos(4t)) - \sum_{i=1}^{n} \frac{1}{2^n}(1 + \sin(\frac{i}{2^n}))$ month$^{-1}$ to replace $b(t)$, then the value of $R_0$ will be overestimated.

5. Discussion. In an infectious disease model, the basic reproduction number $R_0$ is one of the most important threshold parameter in the study of disease transmission, which is defined as the expected number of secondary cases produced by a typical infective individual in a completely susceptible population. In this paper, we have considered an almost periodic dengue transmission model with age structure and time-delayed input of vector in a patchy environment. By applying developed theory, we have given the definition of the basic reproduction number corresponding to the dengue disease model and show that the basic reproduction number is an important threshold parameter determining whether the disease is transmitted or extinct. In one case that the basic reproduction number $R_0 > 1$, the disease is uniformly persistent. In the other case the basic reproduction number $R_0 < 1$, if the travel rate of the susceptible host population is equal to the travel
rate of the infectious host population and is equal to the travel rate of the recovered host population in each patch, then we get that the disease-free almost periodic solution of dengue model is globally stable, that is, the disease will die out. In the section of numerical stimulations, we consider the case of two patches and obtain results, which is consistent with the theoretical results obtained in previous section. Additionally, we numerically simulated the effect of the biting rate on the basic reproduction number $R_0$ (Fig. 6). The result shows that if $k < 1.25$, the disease will be controlled. Similarly, fix $k = 2$, we discuss the effect of immigration and emigration infected host on the basic reproduction number $R_0$ (Fig. 9). The figure shows that the disease in patch 2 can be brought under control by restricting the immigration of infected hosts from patch 1 to patch 2.

In this paper, we assume that the length of the juvenile period and the average mature period of vector are constant $\tau_h$ and $\tau_v$, respectively. In reality, because of different life environment, the length of juvenile period will change with time and may be almost periodic. Furthermore, rainfall and availability of resources play an important role in average mature periodic of vector, which implies that $\tau_v$ may also be almost periodic. Therefore, it is more reasonable to consider that $\tau_h$ and $\tau_v$ are time-dependent. For this problem, this will occur more new challenges and increase the difficulty, for example, the definition of the basic reproduction number, the existence of positive disease-free almost periodic solution and so on. We leave those interesting problems for further investigation of theoretical analysis.

Acknowledgments. The authors are very grateful to the referees for careful reading and valuable comments which led to important improvements of the original manuscript. The second author was supported by NSF of China (11501269, 11801241) and the Fundamental Research Funds for the Central Universities Lzujbky-2020-13.

REFERENCES

[1] S. Altizer, A. Dobson, P. Hosseini, P. Hudson, M. Pascual and P. Rohani, Seasonality and the dynamics of infectious diseases, *Ecology Letters*, 9 (2006), 467–484.
[2] J. Arino and P. van den Driessche, A multicity epidemic model, *Math. Popul. Stud.*, 10 (2003), 175–193.
[3] G. Aronsson and R. B. Kellogg, On a differential equation arising from compartmental analysis, *Math. Biosci.*, 38 (1978), 113–122.
[4] C. Corduneanu, *Almost Periodic Functions*, Chelsea Publishing Company New York, N.Y., 1989.
[5] O. Diekmann, J. A. P. Heesterbeek and J. A. J. Metz, On the definition and the computation of the basic reproduction ratio $R_0$ in models for infectious diseases in heterogeneous populations, *J. Math. Biol.*, 28 (1990), 365–382.
[6] D. J. D. Earn, P. Rohani, B. M. Bolker and B. T. Grenfell, A simple model for complex dynamical transitions in epidemics, *Science*, 287 (2000), 667–670.
[7] L. Esteva and C. Vargas, Analysis of a dengue disease transmission model, *Math. Biosci.*, 150 (1998), 131–151.
[8] L. Esteva and C. Vargas, A model for dengue disease with variable human population, *J. Math. Biol.*, 38 (1999), 220–240.
[9] A. M. Fink, *Almost Periodic Differential Equations*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1974.
[10] S. Gakkhar and N. C. Chavda, Impact of awareness on the spread of Dengue infection in human population, *Appl. Math.*, 4 (2013), 142–147.
[11] D. Gubler, Dengue and Dengue hemorrhagic fever. *Clinical Microbiology Reviews*, 3 (1998), 480–496.
[12] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Math. Surveys and Monographs 25, Amer. Math. Soc., Providence, RI, 1988.

[13] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Appl. Math. Sci., Vol. 99, Springer-Verlag, New York, 1993.

[14] Y. Hino, S. Murakami and T. Naiko, *Functional Differential Equations with Infinite Delay*, Lecture Notes in Mathematics, Vol. 1473, Springer-Verlag, Berlin, 1991.

[15] W. O. Kermack and A. G. McKendrick, A contribution to the mathematical theory of epidemics, *Proc. R. Soc. Lond.*, 115 (1927), 700–721.

[16] S. Lee and C. Castillo-Chavez, The role of residence times in two-patch dengue transmission dynamics and optimal strategies, *J. Theoret. Biol.*, 374 (2015), 152–164.

[17] X. Liu and X.-Q. Zhao, A periodic epidemic model with age structure in a patchy environment, *SIAM J. Appl. Math.*, 71 (2011), 1896–1917.

[18] Y. Lou and X.-Q. Zhao, Threshold dynamics in a time-delayed periodic SIS epidemic model, *Discrete Contin. Dyn. Syst. Ser. B*, 12 (2009), 169–186.

[19] P. Magal and X.-Q. Zhao, Global attractors and steady states for uniformly persistent dynamical systems, *SIAM J. Math. Anal.*, 37 (2005), 251–275.

[20] G. R. Phaijoo and D. B. Gurung, Mathematical study of dengue disease transmission in multi-patch environment, *Appl. Math.*, 7 (2016), 1521–1533.

[21] G. R. Phaijoo and D. B. Gurung, Mathematical study of dengue disease with and without awareness in host population, *Int. J. Adv. Eng. Res. Appl.*, 1 (2015), 239–245.

[22] P. Pongsumpun, Mathematical model of dengue disease with the incubation period of virus, *World Academy of Science, Engineering and Technology*, 44 (2008), 328–332.

[23] L. Qiang and B.-G. Wang, An almost periodic malaria transmission model with time-delayed input of vector, *Discrete Contin. Dyn. Syst. Ser. B*, 22 (2017), 1525–1546.

[24] L. Qiang, B.-G. Wang and X.-Q. Zhao, Basic reproduction ratios for almost periodic compartmental epidemic with time delay, *J. Diff. Equ.*, 269 (2019), 4440–4476.

[25] G. R. Sell, *Topological Dynamics and Ordinary Differential Equations*, Van Nostrand Reinhold Co., London, 1971.

[26] W. Shen and Y. Yi, Almost automorphic and almost periodic dynamics in skew-product semiflows, *Mem. Amer. Math. Soc.*, 136 (1998), 93pp.

[27] H. L. Smith, *Monotone Dynamical Systems: An Introductionto the Theory of Competitive and Cooperative Systems*, Amer. Math. Soc., Providence, RI, 1995.

[28] E. Soewono and A. K. Supriatna, A two-dimensional model for the transmission of dengue fever disease, *Bull. Malays. Math. Sci. Soc.*, 24 (2001), 49–57.

[29] P. van den Driessche and J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, *Math. Biosci.*, 180 (2002), 29–48.

[30] B.-G. Wang, W.-T. Li and L. Qiang, An almost periodic epidemic model in a patchy environment, *Discrete Contin. Dyn. Syst. Ser. B*, 21 (2016), 271–289.

[31] B.-G. Wang, W.-T. Li and L. Zhang, An almost periodic epidemic model with age structure in a patchy environment, *Discrete Contin. Dyn. Syst. Ser. B*, 21 (2016), 291–311.

[32] W. Wang and G. Mulone, Threshold of disease transmission in a patch environment, *J. Math. Anal. Appl.*, 285 (2003), 321–335.

[33] B.-G. Wang and X.-Q. Zhao, Basic reproduction ratios for almost periodic compartmental epidemic models, *J. Dyn. Diff. Equ.*, 25 (2013), 535–562.

[34] W. Wang and X.-Q. Zhao, An Epidemic Model in a Patchy Environment, *Math. Biosci.*, 190 (2004), 97–112.

[35] D. M. Watts, D. S. Burke, B. A. Harrison, R. E. Whitmire and A. Nisalak, Effect of temperature on the vector efficiency of Aedes aegypti for dengue 2 virus, *Am. J. Trop. Hyg.*, 36 (1987), 143–152.

[36] World Health Organization (2012), *Global Strategy for Dengue Prevention and Control 2012–2020*, World Health Organization, Geneva.

[37] F. Zhang and X.-Q. Zhao, A periodic epidemic model in a patchy environment, *J. Math. Appl.*, 325 (2007), 496–516.

[38] X.-Q. Zhao, *Dynamical Systems in Population Biology*, Springer, Cham, 2017.

Received March 2019; 1st revision April 2020; 2nd revision May 2020.

E-mail address: fengj16@lzu.edu.cn (J. Feng)
E-mail address: wangbinguo@lzu.edu.cn (B.-G. Wang)