ON PERIOD POLYNOMIALS OF DEGREE $2^m$ AND WEIGHT DISTRIBUTIONS OF CERTAIN IRREDUCIBLE CYCLIC CODES

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ABSTRACT. We explicitly determine the values of reduced cyclotomic periods of order $2^m$, $m \geq 4$, for finite fields of characteristic $p \equiv 3$ or $5$ (mod 8). These evaluations are applied to obtain explicit factorizations of the corresponding reduced period polynomials. As another application, the weight distributions of certain irreducible cyclic codes are described.

Keywords: Cyclotomic period; $f$-nomial Gaussian period; period polynomial; reduced period polynomial; factorization; irreducible cyclic code; weight distribution.

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1. Introduction

Let $\mathbb{F}_q$ be a finite field of characteristic $p$ with $q = p^s$ elements, $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$, and let $\gamma$ be a fixed generator of the cyclic group $\mathbb{F}_q^*$. By $\text{Tr} : \mathbb{F}_q \to \mathbb{F}_p$ we denote the trace mapping, that is, $\text{Tr}(x) = x + x^p + x^{p^2} + \cdots + x^{p^{s-1}}$ for $x \in \mathbb{F}_q$. Let $e$ and $f$ be positive integers such that $q = ef + 1$. Denote by $\mathcal{H}$ the subgroup of $e$-th powers in $\mathbb{F}_q^*$. For any positive integer $n$, write $\zeta_n = \exp(2\pi i/n)$.

The cyclotomic (or $f$-nomial Gaussian) periods of order $e$ for $\mathbb{F}_q$ with respect to $\gamma$ are defined by

$$\eta_j = \sum_{x \in \gamma^j \mathcal{H}} \zeta_p^{\text{Tr}(x)} = \sum_{h=0}^{f-1} \zeta_p^{\text{Tr}(\gamma^{eh+j})}, \quad j = 0, 1, \ldots, e - 1.$$ 

The reduced cyclotomic (or reduced $f$-nomial Gaussian) periods of order $e$ for $\mathbb{F}_q$ with respect to $\gamma$ are defined by

$$\eta_j^* = \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}(\gamma^j x^e)} = 1 + e\eta_j, \quad j = 0, 1, \ldots, e - 1.$$ 

The period polynomial of degree $e$ for $\mathbb{F}_q$ is the polynomial

$$P_e(X) = \prod_{j=0}^{e-1} (X - \eta_j),$$

and the reduced period polynomial of degree $e$ for $\mathbb{F}_q$ is

$$P_e^*(X) = \prod_{j=0}^{e-1} (X - \eta_j^*).$$
The polynomials $P_e(X)$ and $P_e^*(X)$ have integer coefficients and are independent of the choice of generator $\gamma$. They are irreducible over the rationals when $s = 1$, but not necessarily irreducible when $s > 1$. More precisely, $P_e(X)$ and $P_e^*(X)$ split over the rationals into $\delta = \gcd(e, (q-1)/(p-1))$ factors of degree $e/\delta$ (not necessarily distinct), and each of these factors is irreducible or a power of an irreducible polynomial. Furthermore, the polynomials $P_e(X)$ and $P_e^*(X)$ are irreducible over the rationals if and only if $\gcd(e, (q-1)/(p-1)) = 1$. For proofs of these facts, see [11].

In the case $s = 1$, the period polynomials were determined explicitly by Gauss for $e \in \{2, 3, 4\}$ and by many others for certain small values of $e$. In the general case, Myerson [11] derived the explicit formulas for $P_e(X)$ and $P_e^*(X)$ when $e \in \{2, 3, 4\}$, and also found their factorizations into irreducible polynomials over the rationals. Gurak [8] obtained similar results for $e \in \{6, 8, 12, 24\}$; see also [7] for the case $s = 2$, $e \in \{6, 8, 12\}$. Hoshi [9] considered the case $e = 5$. Note that if $-1$ is a power of $p$ modulo $e$, then the period polynomials can also be easily obtained. Indeed, if $e > 2$ and $e \mid (p^v + 1)$, with $v$ chosen minimal, then $2v \mid s$, and [11, Proposition 20] yields

$$P_e^*(X) = (X + (-1)^{s/2v}(e - 1)q^{1/2})(X - (-1)^{s/2v}q^{1/2})^{e-1}.$$ 

Baumert and Mykkeltveit [2] found the values of cyclotomic periods in the case when $e > 3$ is a prime, $e \equiv 3 \pmod{4}$ and $p$ generates the quadratic residues modulo $e$; see also [11, Proposition 21].

It is seen immediately from the definitions that $P_e^*(eX + 1) = e^eP_e(X)$, and so it suffices to factorize only $P_e^*(X)$.

The aim of this paper is to find the values of reduced cyclotomic periods of order $2^m$, $m \geq 4$, for finite fields of characteristic $p \equiv 3$ or 5 (mod 8) and obtain explicit factorizations of the corresponding reduced period polynomials. The traditional approach to cyclotomic periods is to express them in terms of Gauss sums and to apply known results about these sums. Instead, we observe that the values of reduced cyclotomic periods of order $2^m$ have already appeared implicitly in our recent paper [1], and so they can easily be deduced from our results on diagonal equations. The main result in Section 3 is Theorem 5, which gives the explicit factorization of $P_{2^m}^*(X)$ in the case $p \equiv 3 \pmod{8}$. Our main result of Section 4 is Theorem 8, in which we treat the case $p \equiv 5 \pmod{8}$. In Section 5, we apply the results of previous sections to describe the weight distributions of certain irreducible cyclic codes. All the evaluations in Sections 3–5 are effected in terms of parameters occurring in quadratic partitions of some powers of $p$.

2. Preliminary lemmas

We denote by $N[x_1^e + \cdots + x_n^e = 0]$ the number of solutions to the equation $x_1^e + \cdots + x_n^e = 0$ in $\mathbb{F}_q^n$.

Lemma 1. We have

$$N[x_1^e + \cdots + x_n^e = 0] = q^{n-1} + \frac{q-1}{eq} \sum_{j=0}^{e-1} (\eta_j^*)^n.$$
Proof. See [3, Theorem 10.10.7 and Problem 22 in Exercises 12] (this lemma is equivalent to a special case of Proposition 1 in [13]). \qed

Lemma 2. Assume that there exist complex numbers \( \omega_0, \omega_1, \ldots, \omega_{e-1} \) such that for \( n = 1, 2, \ldots, e \),

\[
N[x_1^n + \cdots + x_e^n = 0] = q^{n-1} + \frac{q-1}{eq} \sum_{j=0}^{e-1} \omega_j^n.
\]

Then

\[
P_e^*(X) = \prod_{j=0}^{e-1} (X - \omega_j),
\]

that is, the sequence \( \{\omega_0, \omega_1, \ldots, \omega_{e-1}\} \) is just a permutation of the sequence \( \{\eta_0^*, \eta_1^*, \ldots, \eta_{e-1}^*\} \).

Proof. Let \( \sigma_1, \ldots, \sigma_e \) be elementary symmetric polynomials in the variables \( y_1, \ldots, y_e \) and \( s_n = s_n(y_1, \ldots, y_e) = y_1^n + \cdots + y_e^n \) for \( n \geq 1 \). Lemma 1 yields

\[
s_n(\eta_0^*, \eta_1^*, \ldots, \eta_{e-1}^*) = s_n(\omega_0, \omega_1, \ldots, \omega_{e-1}) \quad \text{for} \quad n = 1, 2, \ldots, e.
\]

Using Newton’s formula

\[
n\sigma_n = s_1\sigma_{n-1} - s_2\sigma_{n-2} + \cdots + (-1)^{n-2}s_{n-1}\sigma_1 + (-1)^{n-1}s_n
\]

for \( n = 2, \ldots, e \), we infer that

\[
\sigma_n(\eta_0^*, \eta_1^*, \ldots, \eta_{e-1}^*) = \sigma_n(\omega_0, \omega_1, \ldots, \omega_{e-1}) \quad \text{for} \quad n = 1, 2, \ldots, e.
\]

Therefore, \( \eta_0^*, \eta_1^*, \ldots, \eta_{e-1}^* \) and \( \omega_0, \omega_1, \ldots, \omega_{e-1} \) are roots of the same monic polynomial of degree \( e \), as desired. \( \quad \Box \)

Lemma 2 shows that the reduced cyclotomic periods of order \( e \) and \( P_e^*(X) \) can be easily computed once we know a formula of the type (1). Let us present two examples.

Example 1. Assume that \( q = 2^s \), \( f \) is a prime, \( f \mid (q-1) \), 2 is a primitive root modulo \( f \) and \( e = (q-1)/f \). Then

\[
N[x_1^e + \cdots + x_n^e = 0] = 2^{n-f+1} \sum_{j=0}^{(f-1)/2} \binom{f}{2j} (2^{s-1} - 2je)^n
\]

(see [14, Theorem 5.1]), or, equivalently,

\[
N[x_1^e + \cdots + x_n^e = 0] = q^{n-1} + \frac{q-1}{eq} \left[ \frac{2^{s-f+1} - 1}{f} \cdot q^n + 2^{s-f+1} \sum_{j=1}^{(f-1)/2} \frac{(2j)^f}{(2j)^j} \cdot (q-4je)^n \right].
\]

Note that \( 2^{s-f+1} = 2^{(f-1)((s/(f-1))-1)} \equiv 1 \pmod{f} \) and \( \binom{f}{2j} \equiv 0 \pmod{f} \) for \( j = 1, 2, \ldots, (f-1)/2 \). Thus

\[
P_e^*(X) = (X - q)^{(2^{s-f+1}-1)/f} \prod_{j=1}^{(f-1)/2} (X - q + 4je)^{2^{s-f+1}(f)/(2j)}/f.
\]
Example 2. Assume that $\ell$ divides $s$ and $e = (q - 1)/(p^\ell - 1)$. Rewriting the result of Wolfmann [14, Theorem 5.2] as

$$N[x_1^e + \cdots + x_n^e = 0] = q^{n-1} + \frac{q - 1}{eq} \left[ \frac{e - 1}{p^\ell} \cdot q^n + p^{s-e}(1-e)^n \right]$$

and observing that $e - 1 = (p^s - p^\ell)/(p^\ell - 1)$ is divisible by $p^\ell$, we obtain

$$P_e^*(X) = (X - q)^{(e-1)/p^\ell}(X + e - 1)^{p^{s-e}}.$$  

3. Factorization of $P_{2m}^*(X)$ in the case $p \equiv 3 \pmod{8}$

In this section, $p \equiv 3 \pmod{8}$, $2^m \mid (q - 1)$, $m \geq 4$. Notice that $\text{ord}_2(q - 1) = \text{ord}_2(p^s - 1) = \text{ord}_2(p^2 - 1) + \text{ord}_2 s - 1 = \text{ord}_2 s + 2$ (for a proof, see [4, Proposition 1]). Hence,

$$\gcd(2^m, (q - 1)/(p - 1)) = \begin{cases} 2^{m-1} & \text{if } 2^m - 1 \mid s, \\ 2^{m-2} & \text{if } 2^m - 2 \mid s. \end{cases}$$

Appealing to [11, Theorem 4], we conclude that in the case when $2^m - 1 \mid s$, $P_{2m}^*(X)$ splits over the rationals into linear factors. If $2^m - 2 \mid s$, then $P_{2m}^*(X)$ splits into irreducible polynomials of degrees at most 2.

For $3 \leq r \leq m$, define the integers $A_r$ and $B_r$ by

$$p^{s/2^r - 2} = A_r^2 + 2B_r^2, \quad A_r \equiv -1 \pmod{4}, \quad p \nmid A_r. \tag{2}$$

It is well known [3, Lemma 3.0.1] that for each fixed $r$, the conditions (2) determine $A_r$ and $|B_r|$ uniquely.

Lemma 3. Let $p \equiv 3 \pmod{8}$ and $m \geq 4$. If $2^m - 1 \mid s$, then

$$N[x_1^{2m} + \cdots + x_n^{2m} = 0] = q^{n-1} + \frac{q - 1}{2mq} \left[ 2^{m-2} \cdot \left( (q^\frac{1}{2} + 4B_2q^\frac{1}{4})^n + (q^\frac{1}{2} - 4B_2q^\frac{1}{4})^n \right) \\
+ 2^{m-3} \cdot \left( (q^\frac{1}{2} + 8B_4q^\frac{3}{8})^n + (q^\frac{1}{2} - 8B_4q^\frac{3}{8})^n \right) \\
+ \sum_{t=2}^{m-3} 2^{m-t-2} \left( (-3q^\frac{1}{2} + \sum_{r=3}^{t} 2^{r-1} A_r q^{\frac{r^2-1}{2^{r-1}}} - 2^t A_{t+1} q^{\frac{t^2-1}{2^t}} + 2t^2 B_{t+3} q^{\frac{t^2+1}{2^t+1}} - 2^t A_{t+1} q^{\frac{t^2-1}{2^t}} - 2^t A_{t+1} q^{\frac{t^2+1}{2^t+1}} n) \\
+ (-3q^\frac{1}{2} + \sum_{r=3}^{m-2} 2^{r-1} A_r q^{\frac{r^2-1}{2^{r-1}}} - 2^m A_{m-1} q^{\frac{m^2-1}{2^{m-1}}} n) \\
+ 2 \cdot (-3q^\frac{1}{2} + \sum_{r=3}^{m-2} 2^{r-1} A_r q^{\frac{r^2-1}{2^{r-1}}} - 2^m A_{m-1} q^{\frac{m^2-1}{2^{m-1}}} n) \right) \\
+ (-3q^\frac{1}{2} + \sum_{r=3}^{m-1} 2^{r-1} A_r q^{\frac{r^2-1}{2^{r-1}}} - 2^{m-1} A_{m-1} q^{\frac{m^2-1}{2^{m-1}}} n) \right).$$
If $2^{m-2} \parallel s$ and $m \geq 5$, then
\[
N[x_1^{2m} + \cdots + x_n^{2m} = 0] = q^{-n+1} + \frac{q-1}{2^m q} \left[ 2^{m-2} \cdot \left( (q^{\frac{1}{2}} + 4B_3q^{\frac{3}{2}})^n + (q^{\frac{1}{2}} - 4B_3q^{\frac{3}{2}})^n \right) + 2^{m-3} \cdot (q^{\frac{1}{2}} + 8B_4q^{\frac{3}{2}})^n + (q^{\frac{1}{2}} - 8B_4q^{\frac{3}{2}})^n \right] + \sum_{t=2}^{m-4} 2^{m-t-2} \left( (-3q^{\frac{1}{2}} + \sum_{r=3}^{m-3} 2^{r-1} A_r q^{2^{r-2} 2^{r-1}} - 2^{m-3} A_{m-2} q^{2^{m-4} 2^{m-3}} + 2^{m-1} B_m q^{2^{m-2} 2^{m-1} i})^n \right) \]
+ \sum_{t=2}^{m-4} 2^{m-t-2} \left( (-3q^{\frac{1}{2}} + \sum_{r=3}^{m-3} 2^{r-1} A_r q^{2^{r-2} 2^{r-1}} - 2^{m-3} A_{m-2} q^{2^{m-4} 2^{m-3}} - 2^{m-1} B_m q^{2^{m-2} 2^{m-1} i})^n \right) + 2 \cdot \left( (-3q^{\frac{1}{2}} + \sum_{r=3}^{m-3} 2^{r-1} A_r q^{2^{r-2} 2^{r-1}} - 2^{m-2} A_{m-1} q^{2^{m-3} 2^{m-2}}) \right)^n + \left( (-3q^{\frac{1}{2}} + \sum_{r=3}^{m-3} 2^{r-1} A_r q^{2^{r-2} 2^{r-1}} + 2^{m-1} A_m q^{2^{m-2} 2^{m-1} i})^n \right) + \left( (-3q^{\frac{1}{2}} + \sum_{r=3}^{m-3} 2^{r-1} A_r q^{2^{r-2} 2^{r-1}} - 2^{m-1} A_m q^{2^{m-2} 2^{m-1} i})^n \right).
\]

If $4 \parallel s$, then
\[
N[x_1^{16} + \cdots + x_n^{16} = 0] = q^{-n+1} + \frac{q-1}{16 q} \left[ 4 \cdot \left( (q^{\frac{1}{2}} + 4B_3q^{\frac{3}{2}})^n + (q^{\frac{1}{2}} - 4B_3q^{\frac{3}{2}})^n \right) + 2 \cdot \left( (q^{\frac{1}{2}} + 8B_4q^{\frac{3}{2}} i)^n + (q^{\frac{1}{2}} - 8B_4q^{\frac{3}{2}} i)^n \right) + 2 \cdot \left( (-3q^{\frac{1}{2}} - 4A_3q^{\frac{1}{2}} i)^n \right) + \left( (-3q^{\frac{1}{2}} + 4A_3q^{\frac{1}{2}} + 8A_4q^{\frac{3}{2}} i)^n + (-3q^{\frac{1}{2}} - 4A_3q^{\frac{1}{2}} - 8A_4q^{\frac{3}{2}} i)^n \right) \right] .
\]

The integers $A_r$ and $|B_r|$ are uniquely determined by (2).

Proof. See [1, Theorems 18 and 19].

Combining Lemmas 2 and 3, we deduce the following corollary.

**Corollary 4.** Under the conditions of Lemma 3, the reduced cyclotomic periods of order $2^m$ are given by Tables 1–3.

We are now in a position to prove the main result of this section.
TABLE 1. The reduced cyclotomic periods of order $2^m$ in the case when $p \equiv 3 \pmod{8}$, $2^{m-1} \mid s$, $m \geq 4$ ($t$ runs from 2 to $m-3$).

| Value                        | Multiplicity |
|------------------------------|--------------|
| $q^2 \pm 4B_3q^3$           | $2^{m-2}$    |
| $q^2 \pm 8B_4q^3$           | $2^{m-3}$    |
| $-3q^2 + \sum_{t=3}^{m-3} 2^{t-1} A_t q^{2^{t-1} - 1} - 2^t A_{t+1} q^{2^{t-1} - 1} \pm 2^{t+2} B_{t+3} q^{2^{t+1} - 1}$ | $2^{m-t-2}$ |
| $-3q^2 + \sum_{r=3}^{m-3} 2^{r-1} A_r q^{2^{r-1} - 1} - 2^{m-2} A_{m-1} q^{2^{m-3} - 1}$ | 2 |
| $-3q^2 + \sum_{r=3}^{m-3} 2^{r-1} A_r q^{2^{r-1} - 1} \pm 2^{m-1} A_m q^{2^{m-3} - 1}$ | 1 |

TABLE 2. The reduced cyclotomic periods of order $2^m$ in the case when $p \equiv 3 \pmod{8}$, $2^{m-2} \parallel s$, $m \geq 5$ ($t$ runs from 2 to $m-4$).

| Value                        | Multiplicity |
|------------------------------|--------------|
| $q^2 \pm 4B_3q^3$           | $2^{m-2}$    |
| $q^2 \pm 8B_4q^3$           | $2^{m-3}$    |
| $-3q^2 + \sum_{r=3}^{m-3} 2^{r-1} A_r q^{2^{r-1} - 1} - 2^t A_{t+1} q^{2^{t-1} - 1} \pm 2^{t+2} B_{t+3} q^{2^{t+1} - 1}$ | $2^{m-t-2}$ |
| $-3q^2 + \sum_{r=3}^{m-3} 2^{r-1} A_r q^{2^{r-1} - 1} - 2^{m-3} A_{m-2} q^{2^{m-4} - 1} \pm 2^{m-1} B_m q^{2^{m-2} - 1} i$ | 2 |
| $-3q^2 + \sum_{r=3}^{m-3} 2^{r-1} A_r q^{2^{r-1} - 1} - 2^{m-2} A_{m-1} q^{2^{m-3} - 1}$ | 2 |
| $-3q^2 + \sum_{r=3}^{m-3} 2^{r-1} A_r q^{2^{r-1} - 1} \pm 2^{m-1} A_m q^{2^{m-3} - 1} i$ | 1 |

TABLE 3. The reduced cyclotomic periods of order 16 in the case when $p \equiv 3 \pmod{8}$, $4 \parallel s$.

| Value                        | Multiplicity |
|------------------------------|--------------|
| $q^2 \pm 4B_3q^3$           | 4            |
| $q^2 \pm 8B_4q^3i$          | 2            |
| $-3q^2 - 4A_3q^3$           | 2            |
| $-3q^2 + 4A_3q^3 \pm 8A_4q^3i$ | 1 |

Theorem 5. Let $p \equiv 3 \pmod{8}$ and $m \geq 4$. Then $P_{2^m}^*(X)$ has a unique decomposition into irreducible polynomials over the rationals as follows:

(a) if $2^{m-1} \mid s$, then

$$P_{2^m}^*(X) = (X - q^2 \pm 4B_3q^3)^{2^{m-2}} (X - q^2 \pm 4B_3q^3)^{2^{m-2}}$$

$$\times (X - q^2 \pm 8B_4q^3)^{2^{m-3}} (X - q^2 \pm 8B_4q^3)^{2^{m-3}}$$

$$\times \left( X + 3q^2 - \sum_{r=3}^{m-2} 2^{r-1} A_r q^{2^{r-1} - 1} + 2^{m-2} A_{m-1} q^{2^{m-2} - 1} \right)^2$$
\[ \times \left( X + 3q^{\frac{1}{2}} - \sum_{r=3}^{m-1} 2^{r-1} A_r q^{2^{r-2}-1} + 2^{m-1} A_m q^{2^{m-2}-1} \right) \]
\[ \times \left( X + 3q^{\frac{1}{2}} - \sum_{r=3}^{m} 2^{r-1} A_r q^{2^{r-2}-1} \right) \prod_{t=2}^{m-3} Q_t(X)^{2^{m-t-2}}; \]

(b) if \( 2^{m-2} \parallel s \) and \( m \geq 5 \), then
\[ P_{2m}^*(X) = (X - q^{\frac{1}{2}} + 4B_3q^{\frac{1}{2}})^{2^{m-2}} (X - q^{\frac{1}{2}} - 4B_3q^{\frac{1}{2}})^{2^{m-2}} \]
\[ \times (X - q^{\frac{1}{2}} + 8B_4q^{\frac{3}{2}})(X - q^{\frac{1}{2}} - 8B_4q^{\frac{3}{2}})^{2^{m-3}} \]
\[ \times \left( X + 3q^{\frac{1}{2}} - \sum_{r=3}^{m-1} 2^{r-1} A_r q^{2^{r-2}-1} + 2^{m-2} A_{m-1} q^{2^{m-3}-1} \right)^2 \]
\[ \times \left( X + 3q^{\frac{1}{2}} - \sum_{r=3}^{m-3} 2^{r-1} A_r q^{2^{r-2}-1} + 2^{m-3} A_{m-2} q^{2^{m-4}-1} \right)^2 \]
\[ + 2^{2(m-1)} B_{m}^2 q^{2^{m-2}-1} \prod_{t=2}^{m-4} Q_t(X)^{2^{m-t-2}}; \]

(c) if \( 4 \parallel s \), then
\[ P_{16}^*(X) = (X + 3q^{\frac{1}{2}} + 4A_3q^{\frac{3}{2}})^2 (X - q^{\frac{1}{2}} + 4B_3q^{\frac{1}{2}})^4 (X - q^{\frac{1}{2}} - 4B_3q^{\frac{1}{2}})^4 \]
\[ \times \left( X + 3q^{\frac{1}{2}} - 4A_3q^{\frac{1}{2}} \right)^2 + 64A_4^2 q^{\frac{1}{2}} \left( (X - q^{\frac{1}{2}})^2 + 64B_4^2 q^{\frac{3}{2}} \right)^2. \]

The integers \( A_r \) and \( |B_r| \) are uniquely determined by (2), and

\[ Q_t(X) = (X + 3q^{\frac{1}{2}} - \sum_{r=3}^{t} 2^{r-1} A_r q^{2^{r-2}-1} + 2^t A_{t+1} q^{2^t+1} + 2^{t+2} B_{t+3} q^{2^{t+1}+1}) \]
\[ \times \left( X + 3q^{\frac{1}{2}} - \sum_{r=3}^{t} 2^{r-1} A_r q^{2^{r-2}-1} + 2^t A_{t+1} q^{2^t-1} + 2^{t+2} B_{t+3} q^{2^{t+1}-1} \right). \]

Proof. First assume that \( 2^{m-1} \mid s \). In this case, all the cyclotomic periods are integers (see Table 1), and the result follows.

Next assume that \( 2^{m-2} \parallel s \) and \( m \geq 5 \). In this case, \( P_{2m}^*(X) \) has two pairs of complex conjugate roots, namely,
\[ -3q^{\frac{1}{2}} + \sum_{r=3}^{m-1} 2^{r-1} A_r q^{2^{r-2}-1} \pm 2^{m-1} A_m q^{2^{m-2}-1} i. \]
and
\[ -3q^2 + \sum_{r=3}^{m-3} 2^{r-1} A_r q^{2^{r-2} - 1} - 2^{m-3} A_{m-2} q^{2^{m-4} - 1} \pm 2^{m-1} B_m q^{2^{m-2} - 1} i, \]
and the remaining roots are integers (see Table 2). Hence \( P_{2m}^* (X) \) has the irreducible quadratic factors
\[ \left( X + 3q^2 - \sum_{r=3}^{m-3} 2^{r-1} A_r q^{2^{r-2} - 1} \right)^2 + 2^{2(m-1)} A_m^2 q^{2^{m-2} - 1} \]
and
\[ \left( X + 3q^2 - \sum_{r=3}^{m-3} 2^{r-1} A_r q^{2^{r-2} - 1} + 2^{m-3} A_{m-2} q^{2^{m-4} - 1} \right)^2 + 2^{2(m-1)} B_m^2 q^{2^{m-2} - 1} \]
(the last one occurs with multiplicity 2). The remaining factors are linear and occur with the multiplicities given in Table 2.

Finally, if \( 4 \parallel s \), then \( P_{16} (X) \) has complex conjugate roots \( q^{\frac{1}{2}} \pm 8B_4 q^{\frac{3}{2}} i, -3q^{\frac{1}{2}} + 4A_3 q^{\frac{1}{2}} \pm 8A_4 q^{\frac{3}{2}} i \), and the other roots are integers (see Table 3). Taking into account the multiplicities given in Table 3, we obtain the desired factorization. This completes the proof. \( \square \)

**Remark 1.** The result of Gurak [8, Proposition 3.3(iii)] can be reformulated in terms of \( A_3 \) and \( B_3 \). Namely, \( P_s^* (X) \) has the following factorization into irreducible polynomials over the rationals:
\[ P_s^* (X) = (X - q^{1/2})^2 (X - q^{1/2} + 4B_3 q^{1/4})^2 (X - q^{1/2} - 4B_3 q^{1/4}) \]
\[ \times (X + 3q^{1/2} + 4A_3 q^{1/4})(X + 3q^{1/2} - 4A_3 q^{1/4}) \quad \text{if} \ 4 \mid s, \]
\[ P^*_s (X) = (X - q^{1/2})^2 \]
\[ \times ((X + q^{1/2})^2 + 16A_3^2 q^{1/2}) ((X + q^{1/2})^2 + 16B_3^2 q^{1/2})^2 \quad \text{if} \ 2 \parallel s. \]

We see that Theorem 5 is not valid for \( m = 3 \).

4. Factorization of \( P_{2m}^* (X) \) in the case \( p \equiv 5 \pmod{8} \)

In this section, \( p \equiv 5 \pmod{8}, 2^m \mid (q - 1), m \geq 3 \). As in the previous section, we have \( \text{ord}_2 (q - 1) = \text{ord}_2 (p^s - 1) = \text{ord}_2 s + 2 \), and thus
\[ \gcd(2^m, (q - 1)/(p - 1)) = \begin{cases} 2^m & \text{if} \ 2^m \mid s, \\ 2^{m-1} & \text{if} \ 2^{m-1} \parallel s, \\ 2^{m-2} & \text{if} \ 2^{m-2} \parallel s. \end{cases} \]

Using [11, Theorem 4], we see that \( P_{2m}^* (X) \) splits over the rationals into linear factors if \( 2^m \mid s \), splits into linear and quadratic irreducible factors if \( 2^{m-1} \parallel s \), and splits into linear, quadratic and biquadratic irreducible factors if \( 2^{m-2} \parallel s \).

For \( 2 \leq r \leq m - 1 \), define the integers \( C_r \) and \( D_r \) by
\[ p^{s/2^{r-1}} = C_r^2 + D_r^2, \quad C_r \equiv -1 \pmod{4}, \quad p \nmid C_r. \quad (3) \]
If $2^{m-1} \mid s$, we extend this notation to $r = m$. It is well known [3, Lemma 3.0.1] that for each fixed $r$, the conditions (3) determine $C_r$ and $|D_r|$ uniquely.

**Lemma 6.** Let $p \equiv 5 \pmod{8}$ and $m \geq 3$. If $2^{m-1} \mid s$, then

$$N[x_1^{2^m} + \cdots + x_n^{2^m} = 0]$$

$$= q^{n-1} + \frac{q - 1}{2mq} \cdot \left[ 2^{m-2} \cdot (q^{\frac{1}{2}} + 2D_2q^{\frac{1}{2}})^n + (q^{\frac{1}{2}} - 2D_2q^{\frac{1}{2}})^n \right]$$

$$+ \sum_{t=1}^{m-2} 2^{m-t-2} \left( (-q^{\frac{1}{2}} + \sum_{r=2}^{t} 2^{r-1} C_r q^{\frac{r-1}{2^t}} - 2^t C_{t+1} q^{\frac{2t-1}{2^{t+1}}} + 2^{t+1} D_{t+2} q^{\frac{2^{t+1}-1}{2^{t+2}}})^n \right.$$

$$\left. + (-q^{\frac{1}{2}} + \sum_{r=2}^{m-1} 2^{r-1} C_r q^{\frac{r-1}{2^t}} - 2^{m-1} C_1 q^{\frac{2m-1}{2^m}} - 2^{t+1} D_{t+2} q^{\frac{2^{t+1}-1}{2^{t+2}}})^n \right].$$

If $2^{m-2} \mid s$, then

$$N[x_1^{2^m} + \cdots + x_n^{2^m} = 0]$$

$$= q^{n-1} + \frac{q - 1}{2mq} \cdot \left[ 2^{m-2} \cdot (q^{\frac{1}{2}} + 2D_2q^{\frac{1}{2}})^n + (q^{\frac{1}{2}} - 2D_2q^{\frac{1}{2}})^n \right]$$

$$+ \sum_{t=1}^{m-3} 2^{m-t-2} \left( (-q^{\frac{1}{2}} + \sum_{r=2}^{t} 2^{r-1} C_r q^{\frac{r-1}{2^t}} - 2^t C_{t+1} q^{\frac{2t-1}{2^{t+1}}} + 2^{t+1} D_{t+2} q^{\frac{2^{t+1}-1}{2^{t+2}}})^n \right.$$

$$\left. + (-q^{\frac{1}{2}} + \sum_{r=2}^{m-3} 2^{r-1} C_r q^{\frac{r-1}{2^t}} - 2^t C_{t+1} q^{\frac{2t-1}{2^{t+1}}} - 2^{t+1} D_{t+2} q^{\frac{2^{t+1}-1}{2^{t+2}}})^n \right)$$

$$+ \left( -q^{\frac{1}{2}} + \sum_{r=2}^{m-2} 2^{r-1} C_r q^{\frac{r-1}{2^t}} - 2^m C_{m-1} q^{\frac{2m-2}{2^m-1}} \right. + \left( 2^{m-2} q^{\frac{2m-1}{2^m-1}} i \sqrt{2(q^{\frac{1}{2}} - C_{m-1})} \right)^n$$

$$\left. + (-q^{\frac{1}{2}} + \sum_{r=2}^{m-2} 2^{r-1} C_r q^{\frac{r-1}{2^t}} + 2^m C_{m-1} q^{\frac{2m-2}{2^m-1}} - 2^{m-2} q^{\frac{2m-1}{2^m-1}} i \sqrt{2(q^{\frac{1}{2}} - C_{m-1})} \right)^n$$
TABLE 4. The reduced cyclotomic periods of order $2^m$ in the case when $p \equiv 5 \pmod{8}$, $2^{m-1} \mid s$, $m \geq 3$ ($t$ runs from 1 to $m-2$).

| Value | Multiplicity |
|-------|--------------|
| $q^4 \pm 2D_2 q^4$ | $2^{m-2}$ |
| $-q^4 + \sum_{r=2}^t 2^{r-1}C_r q^{(2^r - 1)/2} - 2^t C_{t+4} q^{2^t-1} + 2^{t+1}D_{t+4} q^{2^{t+1}-1}$ | $2^{m-t-2}$ |
| $-q^4 + \sum_{r=2}^{m-1} 2^{r-1}C_r q^{(2^r - 1)/2} \pm 2^{m-1}C_m q^{2^{m-1}-1}$ | 1 |

TABLE 5. The reduced cyclotomic periods of order $2^m$ in the case when $p \equiv 5 \pmod{8}$, $2^{m-2} \mid s$, $m \geq 3$ ($t$ runs from 1 to $m-3$).

| Value | Multiplicity |
|-------|--------------|
| $q^4 \pm 2D_2 q^4$ | $2^{m-2}$ |
| $-q^4 + \sum_{r=2}^t 2^{r-1}C_r q^{(2^r - 1)/2} - 2^t C_{t+4} q^{2^t-1} + 2^{t+1}D_{t+4} q^{2^{t+1}-1}$ | $2^{m-t-2}$ |
| $-q^4 + \sum_{r=2}^{m-2} 2^{r-1}C_r q^{(2^r - 1)/2} + 2^{m-2}C_{m-1} q^{2^{m-2}-1}$ | 1 |
| $= 2^{-2m-1} \sum_{i=1}^{2m} q^{(2^i - 1)/2} \pm 2^{m-2} q^{2^{m-1}-1} i \sqrt{2(q^{2^{m-1}} - C_{m-1})}$ | 1 |
| $-q^4 + \sum_{r=2}^{m-2} 2^{r-1}C_r q^{(2^r - 1)/2} - 2^{m-2}C_{m-1} q^{2^{m-2}-1}$ | 1 |
| $= 2^{-2m-1} \sum_{i=1}^{2m} q^{(2^i - 1)/2} \pm 2^{m-2} q^{2^{m-1}-1} i \sqrt{2(q^{2^{m-1}} + C_{m-1})}$ | 1 |

\[ + \left( -q^4 + \sum_{r=2}^{m-2} 2^{r-1}C_r q^{(2^r - 1)/2} - 2^{m-2}C_{m-1} q^{2^{m-2}-1} \right. \]
\[ + 2^{-2m-1} \sum_{i=1}^{2m} q^{(2^i - 1)/2} \pm 2^{m-2} q^{2^{m-1}-1} i \sqrt{2(q^{2^{m-1}} + C_{m-1})} \right)^n \]
\[ + \left( -q^4 + \sum_{r=2}^{m-2} 2^{r-1}C_r q^{(2^r - 1)/2} - 2^{m-2}C_{m-1} q^{2^{m-2}-1} \right. \]
\[ - 2^{m-2} q^{2^{m-1}-1} i \sqrt{2(q^{2^{m-1}} + C_{m-1})} \right)^n \].

The integers $C_r$ and $|D_r|$ are uniquely determined by (3).

Proof. See [1, Theorems 22 and 23].

Combining Lemmas 2 and 6, we obtain the following corollary.

**Corollary 7.** Under the conditions of Lemma 6, the reduced cyclotomic periods of order $2^m$ are given by Tables 4 and 5.

We are now ready to establish our second main result.

**Theorem 8.** Let $p \equiv 5 \pmod{8}$ and $m \geq 4$. Then $P_{2m}^+(X)$ has a unique decomposition into irreducible polynomials over the rationals as follows:
(a) if $2^m \mid s$, then

$$P_{2^m}(X) = (X - q^{\frac{1}{2}} + 2D_2q^\frac{1}{4})^{2m-2} (X - q^{\frac{1}{2}} - 2D_2q^\frac{1}{4})^{2m-2}$$

$$\times \left( X + q^{\frac{1}{4}} - \sum_{r=2}^{m-1} 2^{r-1}C_r q^{2^{r-1}-1} + 2^{m-1}C_m q^{2m-2}\right)$$

$$\times \left( X + q^{\frac{1}{4}} - \sum_{r=2}^{m} 2^{r-1}C_r q^{2^{r-1}-1} \prod_{t=1}^{m-2} R_t(X)^{2^{m-t-2}} \right);$$

(b) if $2^{m-1} \parallel s$, then

$$P_{2^m}(X) = (X - q^{\frac{1}{2}} + 2D_2q^\frac{1}{4})^{2m-2} (X - q^{\frac{1}{2}} - 2D_2q^\frac{1}{4})^{2m-2}$$

$$\times \left( X + q^{\frac{1}{4}} - \sum_{r=2}^{m-1} 2^{r-1}C_r q^{2^{r-1}-1} - 2^{2(m-1)}C_m q^{2m-2}\right)$$

$$\times \left( X + q^{\frac{1}{4}} - \sum_{r=2}^{m} 2^{r-1}C_r q^{2^{r-1}-1} + 2^{m-2}C_{m-1} q^{2m-2}\right)^2$$

$$- 2^{2(m-1)}D_m^2 q^{2m-2}\prod_{t=1}^{m-3} R_t(X)^{2^{m-t-2}};$$

(c) if $2^{m-2} \parallel s$, then

$$P_{2^m}(X) = (X - q^{\frac{1}{2}} + 2D_2q^\frac{1}{4})^{2m-2} (X - q^{\frac{1}{2}} - 2D_2q^\frac{1}{4})^{2m-2}$$

$$\times \left( X + q^{\frac{1}{4}} - \sum_{r=2}^{m-3} 2^{r-1}C_r q^{2^{r-1}-1} + 2^{m-3}C_{m-2} q^{2m-2}\right)^2$$

$$- 2^{2(m-2)}D_{m-1}^2 q^{2m-2}\prod_{t=1}^{m-4} R_t(X)^{2^{m-t-2}}.$$
The integers $C_r$ and $|D_r|$ are uniquely determined by (3), and

$$R_t(X) = \left( X + q^\frac{1}{2} - \sum_{r=2}^{t} 2^{r-1}C_r q^{\frac{r-1}{2}} + 2^t C_{t+1} q^{\frac{t}{2}} + 2^{t+1}D_{t+2} q^{\frac{t+1}{2}} \right) \times \left( X + q^\frac{1}{2} - \sum_{r=2}^{t} 2^{r-1}C_r q^{\frac{r-1}{2}} + 2^t C_{t+1} q^{\frac{t}{2}} - 2^{t+1}D_{t+2} q^{\frac{t+1}{2}} \right).$$

Proof. First suppose that $2^m \mid s$. It follows from Table 4 that all the cyclotomic periods are integers in this case. This yields the desired factorization.

Next suppose that $2^{m-1} \parallel s$. In this case, we have two pairs of algebraic conjugates of degree 2 among the cyclotomic periods, namely,

$$-q^\frac{1}{2} + \sum_{r=2}^{m-1} 2^{r-1}C_r q^{\frac{r-1}{2}} \pm 2^{m-1}C_m q^{\frac{m-1}{2m-1}}$$

and

$$-q^\frac{1}{2} + \sum_{r=2}^{m-2} 2^{r-1}C_r q^{\frac{r-1}{2}} - 2^{m-2}C_{m-1} q^{\frac{m-2}{2m-1}} \pm 2^{m-1}D_m q^{\frac{m-1}{2m-1}},$$

and the remaining roots of $P_{2m}^*(X)$ are integers (see Table 4). Therefore $P_{2m}^*(X)$ has the irreducible quadratic factors

$$\left( X + q^\frac{1}{2} - \sum_{r=2}^{m-1} 2^{r-1}C_r q^{\frac{r-1}{2}} \right)^2 - 2^{2(m-1)}C_m^2 q^{\frac{m-1}{2m-1}}$$

and

$$\left( X + q^\frac{1}{2} - \sum_{r=2}^{m-2} 2^{r-1}C_r q^{\frac{r-1}{2}} + 2^{m-2}C_{m-1} q^{\frac{m-2}{2m-1}} \right)^2 - 2^{2(m-1)}D_m^2 q^{\frac{m-1}{2m-1}},$$

and the remaining factors are linear. Taking into account the multiplicities given in Table 4, we obtain the asserted result.

Finally, suppose that $2^{m-2} \parallel s$. In this case, there is a pair of algebraic conjugates of degree 2 among the cyclotomic periods, namely,

$$-q^\frac{1}{2} + \sum_{r=2}^{m-3} 2^{r-1}C_r q^{\frac{r-1}{2}} - 2^{m-3}C_{m-2} q^{\frac{m-3}{2m-2}} \pm 2^{m-2}D_{m-1} q^{\frac{m-2}{2m-2}}.$$
occurring with multiplicity 2. Furthermore, the polynomials 

\[
\left( X + q^{\frac{1}{2}} - \sum_{r=2}^{m-2} 2^{r-1} C_r q^{\frac{2r-1}{2^r}} - 2^{m-2} C_{m-1} q^{m-1} \right) 
+ 2^{m-2} q^{\frac{m-1}{2^{m-1}}} i \sqrt{2(\frac{1}{2^{2^{m-1}}} - C_{m-1})}
\]

\[
\times \left( X + q^{\frac{1}{2}} - \sum_{r=2}^{m-2} 2^{r-1} C_r q^{\frac{2r-1}{2^r}} - 2^{m-2} C_{m-1} q^{m-1} \right) 
- 2^{m-2} q^{\frac{m-1}{2^{m-1}}} i \sqrt{2(\frac{1}{2^{2^{m-1}}} - C_{m-1})}
\]

\[
= \left( X + q^{\frac{1}{2}} - \sum_{r=2}^{m-2} 2^{r-1} C_r q^{\frac{2r-1}{2^r}} - 2^{m-2} C_{m-1} q^{m-1} \right)^2 
+ 2^{2m-3} q - 2^{m-3} q^{\frac{m-1}{2^{m-1}}} C_{m-1}
\]

and

\[
\left( X + q^{\frac{1}{2}} - \sum_{r=2}^{m-2} 2^{r-1} C_r q^{\frac{2r-1}{2^r}} + 2^{m-2} C_{m-1} q^{m-1} \right) 
+ 2^{m-2} q^{\frac{m-1}{2^{m-1}}} i \sqrt{2(\frac{1}{2^{2^{m-1}}} + C_{m-1})}
\]

\[
\times \left( X + q^{\frac{1}{2}} - \sum_{r=2}^{m-2} 2^{r-1} C_r q^{\frac{2r-1}{2^r}} + 2^{m-2} C_{m-1} q^{m-1} \right) 
- 2^{m-2} q^{\frac{m-1}{2^{m-1}}} i \sqrt{2(\frac{1}{2^{2^{m-1}}} + C_{m-1})}
\]

\[
= \left( X + q^{\frac{1}{2}} - \sum_{r=2}^{m-2} 2^{r-1} C_r q^{\frac{2r-1}{2^r}} + 2^{m-2} C_{m-1} q^{m-1} \right)^2 
+ 2^{2m-3} q + 2^{m-3} q^{\frac{m-1}{2^{m-1}}} C_{m-1}
\]

belong to \( \mathbb{R}[X] \setminus \mathbb{Q}[X] \) and are irreducible over the reals. Since \( 2^{m-2} \mid s \) and \( \mathbb{R}[X] \) is a unique factorization domain, it follows that the product of the above polynomials, namely,

\[
\left( \left( X + q^{\frac{1}{2}} - \sum_{r=2}^{m-2} 2^{r-1} C_r q^{\frac{2r-1}{2^r}} \right)^2 + 2^{2(m-2)} C_{m-1}^2 q^{m-2} + 2^{2m-3} q \right)^2 
- 2^{2(m-1)} C_{m-1}^2 q^{\frac{m-2}{2^{m-1}}} \left( X + (2^{m-2} + 1)q^{\frac{1}{2}} - \sum_{r=2}^{m-2} 2^{r-1} C_r q^{\frac{2r-1}{2^r}} \right)^2
\]
belongs to \( \mathbb{Q}[X] \) and is irreducible over the rationals. We see from Table 5 that it occurs with multiplicity 1. The remaining factors are linear and occur with the multiplicities given in Table 5. This concludes the proof. \( \square \)

Remark 2. Myerson has shown [11, Theorem 17] that \( P_s^*(X) \) is irreducible if \( 2 \nmid s \),

\[
P_s^*(X) = (X + q^{1/2} + 2C_2q^{1/4})(X + q^{1/2} - 2C_2q^{1/4})
\]

\[
\times (X - q^{1/2} + 2D_2q^{1/4})(X - q^{1/2} - 2D_2q^{1/4})
\]

if \( 4 \mid s \),

and, with a slight modification,

\[
P_s^*(X) = \left((X + q^{1/2})^2 - 4C_2^2q^{1/2}\right) \left((X - q^{1/2})^2 - 4D_2^2q^{1/2}\right)
\]

if \( 2 \parallel s \),

where in the latter case the quadratic polynomials are irreducible over the rationals. Furthermore, the result of Gurak [8, Proposition 3.3(ii)] can be reformulated in terms of \( C_2, D_2, C_3 \) and \( D_3 \). Namely, \( P_s^*(X) \) has the following factorization into irreducible polynomials over the rationals:

\[
P_s^*(X) = (X - q^{1/2} + 2D_2q^{1/4})^2(X - q^{1/2} - 2D_2q^{1/4})^2
\]

\[
\times \left((X + q^{1/2} - 2C_2q^{1/4} + 4C_3q^{3/8}\right)
\]

\[
\times \left((X + q^{1/2} - 2C_2q^{1/4} - 4C_3q^{3/8}\right)
\]

\[
\times \left((X + q^{1/2} + 2C_2q^{1/4} + 4D_3q^{3/8}\right)
\]

\[
\times \left((X + q^{1/2} + 2C_2q^{1/4} - 4D_3q^{3/8}\right)
\]

if \( 8 \mid s \),

\[
P_s^*(X) = (X - q^{1/2} + 2D_2q^{1/4})^2(X - q^{1/2} - 2D_2q^{1/4})^2
\]

\[
\times \left((X + q^{1/2} - 2C_2q^{1/4})^2 - 16C_3^2q^{3/4}\right)
\]

\[
\times \left((X + q^{1/2} + 2C_2q^{1/4})^2 - 16D_3^2q^{3/4}\right)
\]

if \( 4 \parallel s \),

\[
P_s^*(X) = \left((X - q^{1/2})^2 - 4D_2^2q^{1/2}\right)^2
\]

\[
\times \left((X + q^{1/2})^2 + 4C_2^2q^{1/2} + 8q\right)^2
\]

\[
\left(16C_2^2q^{3/2}(X + 3q^{1/2})^2\right)
\]

if \( 2 \parallel s \).

Thus part (a) of Theorem 8 remains valid for \( m = 2 \) and \( m = 3 \). Moreover, for \( m = 3 \), part (b) of Theorem 8 is still valid (cf. Remark 1).

5. Weight distributions of certain irreducible cyclic codes

Let \( \mathbb{F}_q^{\ell} \) be a subfield of \( \mathbb{F}_q \) (i.e., \( \ell \) divides \( s \)). A \( k \)-dimensional linear subspace \( \mathcal{C} \) of \( \mathbb{F}_q^{\ell} \) is called a linear \([n, k] \) code over \( \mathbb{F}_q^{\ell} \) and \( n \) is called the length of \( \mathcal{C} \). The elements of \( \mathcal{C} \) are called codewords, and the number \( w(c) \) of nonzero components in \( c \in \mathcal{C} \) is called the Hamming weight of \( c \). The polynomial \( 1 + a_1X + a_2X^2 + \cdots + a_nX^n \) is called the weight enumerator of \( \mathcal{C} \) and the vector \((1, a_1, \ldots, a_n)\) is called the weight distribution of \( \mathcal{C} \), where \( a_j \) denotes the number of codewords with Hamming weight \( j \) in \( \mathcal{C} \).

A linear \([n, k] \) code \( \mathcal{C} \) is called cyclic if \((c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C} \) implies \((c_{n-1}, c_0, c_1, \ldots, c_{n-2}) \in \mathcal{C} \). Assume that \( p \nmid n \). By identifying any vector
(c_0, c_1, \ldots, c_{n-1}) \in \mathbb{F}_{p^\ell}^n$ with $c_0 + c_1 X + \cdots + c_{n-1} X^{n-1} \in \mathbb{F}_{p^\ell}[X]/(X^n - 1)$, any linear cyclic code $C$ of length $n$ over $\mathbb{F}_{p^\ell}$ corresponds to an ideal of the principal ideal ring $\mathbb{F}_{p^\ell}[X]/(X^n - 1)$. If $f(X)$ is an irreducible divisor of $X^n - 1$ and $C$ corresponds to the ideal generated by $(X^n - 1)/f(X)$, then $C$ is called an irreducible cyclic code.

Now let $N > 1$ be a positive divisor of $q - 1$. As before, $\gamma$ denotes a generator of the cyclic group $\mathbb{F}_q^*$. Put $\theta = \gamma^N$. If $s/\ell$ is the multiplicative order of $p^\ell$ modulo $(q - 1)/N$ and $C$ is an irreducible cyclic $[(q - 1)/N, s/\ell]$ code over $\mathbb{F}_{p^\ell}$, then $C$ is isomorphic to $\mathbb{F}_q$ and can be represented as

$$C = \{(\text{Tr}_{\mathbb{F}_q/\mathbb{F}_{p^\ell}}(\beta), \text{Tr}_{\mathbb{F}_q/\mathbb{F}_{p^\ell}}(\beta \theta), \ldots, \text{Tr}_{\mathbb{F}_q/\mathbb{F}_{p^\ell}}(\beta \theta^{((q-1)/N)-1})) : \beta \in \mathbb{F}_q \},$$

(4)

where $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_{p^\ell}}$ denotes the trace mapping from $\mathbb{F}_q$ to $\mathbb{F}_{p^\ell}$. It has been observed in [5] that the determination of the weight distribution of $C$ is equivalent to that of the cyclotomic periods of order $e = \gcd(N, (q - 1)/(p^\ell - 1))$ for $\mathbb{F}_q$. More precisely, if $\beta \in \gamma^j \mathcal{H}$ for some $j \in \{0, 1, \ldots, e - 1\}$, then

$$w(c(\beta)) = \frac{(p^\ell - 1)(q - 1 - e \eta_j)}{p^\ell N} = \frac{(p^\ell - 1)(q - \eta_j^*)}{p^\ell N},$$

(5)

where

$$c(\beta) = (\text{Tr}_{\mathbb{F}_q/\mathbb{F}_{p^\ell}}(\beta), \text{Tr}_{\mathbb{F}_q/\mathbb{F}_{p^\ell}}(\beta \theta), \ldots, \text{Tr}_{\mathbb{F}_q/\mathbb{F}_{p^\ell}}(\beta \theta^{((q-1)/N)-1}))$$

(see [5, Equation (12)]). Using this formula, the authors of [5] computed weight enumerators of $C$ when $e = \gcd(N, (q - 1)/(p^\ell - 1)) = 1$ or 2 or 3 or 4; and when there exists an integer $v$ such that $p^v \equiv 1 \pmod{e}$. They also noticed that the case $e \in \{5, 6, 8, 12\}$ can be treated in a similar manner, however, the weight formulas will be very complicated; see [12] for some results in this direction.

Assume now that $p \equiv 3$ or 5 (mod 8) and $\gcd(N, (q - 1)/(p^\ell - 1)) = 2^m$ with $m \geq 3$. We claim that $s/\ell$ is the multiplicative order of $p^\ell$ modulo $(q - 1)/N$. Indeed, it follows from [4, Proposition 1] that

$$\text{ord}_2 \frac{q - 1}{p^\ell - 1} = \text{ord}_2 \frac{p^s - 1}{p^\ell - 1} = \begin{cases} \text{ord}_2 \frac{s}{\ell} & \text{if } \ell \text{ is even,} \\ \text{ord}_2 (p + 1) + \text{ord}_2 s - 1 & \text{if } \ell \text{ is odd.} \end{cases}$$

Therefore, 4 divides $s/\ell$. This implies that $(p^\ell - 1) \mid (p^{s/2} - 1)$ and $(p^{s/2} + 1)/2$ is odd. Since

$$\gcd\left(N, \frac{p^{s/2} + 1}{2} \cdot \frac{2(p^{s/2} - 1)}{p^\ell - 1}\right) = 2^m, \quad m \geq 3,$$

we have $\gcd(N, (p^{s/2} + 1)/2) = 1$. Thus $(p^{s/2} + 1)/2$ divides $(q - 1)/N$. Hence, if $\nu$ is the multiplicative order of $p$ modulo $(q - 1)/N$, then $2 \cdot \frac{p^{s/2} + 1}{2} \mid (p^\nu - 1)$, which yields $\nu > s/2$. Since $\nu$ must divide $s$, we conclude that $\nu = s$, and so $s/\ell$ is the multiplicative order of $p^\ell$ modulo $(q - 1)/N$. Combining the results given in Table 1 and Remark 1 with (5), we deduce the following theorem.
Theorem 9. Let \( p \equiv 3 \pmod{8} \), \( N \) be a positive divisor of \( q - 1 \) with \( \gcd(N, (q - 1)/(p^f - 1)) = 2^m, m \geq 3 \). Then \( C \) in (4) is an irreducible cyclic \( [(q - 1)/N, s/\ell] \) code over \( \mathbb{F}_{p^f} \) with the weight enumerator

\[
1 + \frac{q - 1}{4} X^{(p^f - 1)(q - q^2)}/p^f N + \frac{q - 1}{4} X^{(p^f - 1)(q - q^2 + 4B_3 q^{\frac{1}{2}})}/p^f N + \frac{q - 1}{4} X^{(p^f - 1)(q - q^2 - 4B_3 q^{\frac{1}{2}})}/p^f N
\]

\[
+ \frac{q - 1}{4} X^{(p^f - 1)(q + 3q^2 + 4A_3 q^{\frac{1}{2}})}/p^f N + \frac{q - 1}{4} X^{(p^f - 1)(q + 3q^2 - 4A_3 q^{\frac{1}{2}})}/p^f N
\]

if \( m = 3 \), and with the weight enumerator

\[
1 + \frac{q - 1}{4} X^{(p^f - 1)(q - q^2 + 4B_3 q^{\frac{1}{2}})}/p^f N + \frac{q - 1}{4} X^{(p^f - 1)(q - q^2 - 4B_3 q^{\frac{1}{2}})}/p^f N
\]

\[
+ \frac{q - 1}{8} X^{(p^f - 1)(q - q^2 + 8B_3 q^{\frac{1}{2}})}/p^f N + \frac{q - 1}{8} X^{(p^f - 1)(q - q^2 - 8B_3 q^{\frac{1}{2}})}/p^f N
\]

\[
+ \sum_{t=2}^{m-3} \frac{q - 1}{2^t+2} X^{(p^f - 1)(q + 3q^2 - \sum_{r=3}^{t-1} 2r^{-1} A_r q^\frac{r-2}{2}} + 2^{r-1} A_{t+1} q^\frac{r-1}{2} + 2^{-1} B_{t+3} q^\frac{r+1+1}{2} )}/p^f N
\]

\[
+ \sum_{t=2}^{m-3} \frac{q - 1}{2^t+2} X^{(p^f - 1)(q + 3q^2 - \sum_{r=3}^{t-1} 2r^{-1} A_r q^\frac{r-2}{2}} - 2^{r-1} A_{t+1} q^\frac{r-1}{2} - 2^{-1} B_{t+3} q^\frac{r+1+1}{2} )}/p^f N
\]

\[
+ \frac{q - 1}{2^{m-1}} X^{(p^f - 1)(q + 3q^2 - \sum_{r=3}^{m-2} 2r^{-1} A_r q^\frac{r-2}{2}} + 2^{-1} A_{m-1} q^\frac{m-3}{2m-1})}/p^f N
\]

\[
+ \frac{q - 1}{2^{m-1}} X^{(p^f - 1)(q + 3q^2 - \sum_{r=3}^{m-1} 2r^{-1} A_r q^\frac{r-2}{2}} - 2^{r-1} A_{m-1} q^\frac{m-3}{2m-1})}/p^f N
\]

if \( m \geq 4 \). The integers \( A_r \) and \( |B_r| \) are uniquely determined by (2).

In a similar manner, making use of Table 4 and equality (5), we obtain the following result.

Theorem 10. Let \( p \equiv 5 \pmod{8} \), \( N \) be a positive divisor of \( q - 1 \) with \( \gcd(N, (q - 1)/(p^f - 1)) = 2^m, m \geq 3 \). Then \( C \) in (4) is an irreducible cyclic \( [(q - 1)/N, s/\ell] \) code over \( \mathbb{F}_{p^f} \) with the weight enumerator

\[
1 + \frac{q - 1}{4} X^{(p^f - 1)(q - q^2 + 2D_2 q^{\frac{1}{2}})}/p^f N + \frac{q - 1}{4} X^{(p^f - 1)(q - q^2 - 2D_2 q^{\frac{1}{2}})}/p^f N
\]

\[
+ \sum_{t=1}^{m-2} \frac{q - 1}{2^t+2} X^{(p^f - 1)(q + q^2 - \sum_{r=2}^{t-1} 2r^{-1} C_r q^\frac{r-2}{2}} + 2^t C_{t+1} q^\frac{t-1}{2} + 2^{-1} D_{t+1} q^\frac{t+1+1}{2} )}/p^f N
\]
\[
\begin{align*}
+ \sum_{t=1}^{m-2} \frac{q-1}{2^t+1} X^{(p^t-1)(q^{\frac{1}{2}} - \sum_{r=2}^{m-1} 2^{r-1}C_r q^{\frac{2^{r-1}-1}{2^r}} + 2^{t+1}q^{\frac{2^t-1}{2^t}} - 2^{t+1}D_t + 2q^{\frac{2^t-1}{2^t}}})/p^t N \\
+ \frac{q-1}{2^m} X^{(p^m-1)(q^{\frac{1}{2}} - \sum_{r=2}^{m-1} 2^{r-1}C_r q^{\frac{2^{r-1}-1}{2^r}} + 2^{m-1}C_m q^{m^{m-1}})/p^m N} \\
+ \frac{q-1}{2^m} X^{(p^m-1)(q^{\frac{1}{2}} - \sum_{r=2}^{m-1} 2^{r-1}C_r q^{\frac{2^{r-1}-1}{2^r}} - 2^{m-1}C_m q^{m^{m-1}})/p^m N}.
\end{align*}
\]

The integers \(C_r\) and \(|D_r|\) are uniquely determined by (3).

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