Unintegrated generalised parton distributions

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Abstract

We show how the generalised (or skewed) parton distributions of the proton, $\Pi(x, \xi; k_t^2, \mu^2)$, unintegrated over the partonic transverse momenta, can be calculated from the known conventional parton distributions, $q(x, \mu^2)$ and $g(x, \mu^2)$, for small values of the skewedness parameter $\xi$. We demonstrate the procedure by numerically evaluating the skewed unintegrated gluon. We also provide a simple approximate phenomenological form of the distribution, which may be used to make more rapid predictions of observables.
1 Introduction

Generalised (or skewed) unintegrated parton distributions are essential ingredients in the calculation of many high energy processes initiated by protons. Examples of such processes are virtual photon Compton scattering ($\gamma^* p \rightarrow \gamma p$); diffractive heavy photon dissociation processes, like diffractive vector particle electroproduction (e.g. $\gamma^* p \rightarrow \rho p, J/\psi p, \Upsilon p, Z p$) or dijet production ($\gamma^* p \rightarrow jj + p$) or heavy quark production ($\gamma^* p \rightarrow Q \bar{Q} + p$); double diffractive central particle production (e.g. Higgs or dijet production, $pp \rightarrow p + H + p$, $p +$ dijet $+ p$). In these examples, the $+$ sign is used to indicate the presence of a rapidity gap.

We use the 'symmetrized' distributions introduced by Ji [1]–[3]

$$H_a(x, \xi, \mu^2),$$

where $a = q, g$, with support $-1 \leq x \leq 1$. They depend on the momentum fractions $x_{1,2} = x \pm \frac{1}{2} \xi$ carried by the emitted and absorbed partons, see Fig. 1. Since, in general, $x_1 \neq x_2$, we speak of skewed or generalised distributions. We use the term unintegrated to indicate that the transverse momenta $k_t, k'_t$ of the partons have not been integrated over. The bar on $H$ is simply to denote that (1) is the unintegrated distribution. We reserve $H(x, \xi, \mu^2)$ for Ji’s distributions, which are integrated over the transverse momenta.

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![Figure 1: Schematic diagram of the unintegrated generalised distribution $H_a(x, \xi, k_t, k'_t; \mu^2)$ of (1), showing the last step of the evolution. We use the ‘symmetrized’ variables of Ji [2].](image)

The cross sections are calculated from these universal $H$ distributions using the $k_t$ factorization approach, and the explicit $k_t$ dependence allows the inclusion of the full kinematic behaviour. Assuming, for the moment, that $k_t$ has been appropriately integrated over, then, in the $\xi \rightarrow 0$ limit, the distributions reduce to the conventional diagonal quark and gluon distributions

$$H_q(x, 0) = \begin{cases} q(x) & \text{for } x > 0 \\ -\bar{q}(-x) & \text{for } x < 0 \end{cases}$$

$$H_g(x, 0) = xg(x).$$

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The equations for the $\mu^2$ evolution of the distributions of $H_a$ are well known, but as the $H_a$ possess several arguments it is impossible to determine the starting distributions by fitting to the available sparse data. However here we show that it is possible to determine the $x, \xi, k_t^2, \mu^2$ dependence of the distributions from the values of the conventional integrated distributions, $q(x, \mu^2)$ and $g(x, \mu^2)$, that have been determined in the global analyses of deep inelastic and related hard scattering data, assuming that $\xi^2 \ll 1$ and $k_t' \simeq k_t$. For most practical applications indeed $\xi \ll 1$, since it is specified by $\exp(-\Delta \eta)$ where $\Delta \eta$ is the size of the rapidity gap. We also assume that $|t| \ll \mu^2$, where $-t$ is the square of the momentum transfer. Indeed if $|t|$ were greater than the starting scale $\mu_0^2$, then the logarithmic structure of the evolution from $\mu_0^2$ to $|t|$ would be destroyed. Although we cannot predict the $t$ dependence, for small $t$ it is expected that the $t$ behaviour can be factored off as the proton form factor, see, for example, Ref. \[3\].

The procedure to determine generalised unintegrated distributions is to carefully combine two existing results. First, it has been shown that generalised (or skewed) distributions are completely determined from knowledge of the conventional distributions for small $\xi$ \[4\]. Second, it has been found that the unintegrated distributions can be determined from the conventional (integrated) distributions — the key observation is that angular ordering is only necessary in the last step of the evolution \[5\].

We briefly review these two results in Sections 2 and 3 below. In Section 4 we proceed to use this information to derive a formula for the unintegrated generalised distributions. Then in Section 5 we use the formula to compute the typical behaviour of the unintegrated skewed gluon distribution. To gain insight we also show other predictions for the gluon. Finally we give a simple phenomenological form that approximately reproduces the exact result. Section 6 contains a brief conclusion.

## 2 Generalised from conventional distributions

Here we describe how the skewed (or generalised) distributions $H_a(x, \xi)$, with $a = q$ or $g$, can be determined from the conventional parton distributions, $q(x)$ or $g(x)$, for $\xi^2 \ll 1$. For simplicity we omit the $\mu^2$ argument. The essential result, due to Shuvaev \[6\], is

$$H_a(x, \xi) = \int_{-1}^{1} dx' K_a(x, \xi; x') f_a(x')$$

(3)

with $a = q$ or $g$, where the kernels $K_a$ are known and where, for $\xi \ll 1$, $f_a(x')$ reduces to the conventional distributions $q(x')$ or $g(x')$. In this limit the Gegenbauer moments of the generalised distributions become equal to the Mellin moments of the usual parton distributions. For computational purposes, it is convenient to weaken the singularities in the integral expressions for $K_a$ by integration by parts. Then, for $t = 0$, (3) becomes\[^2\]

$$H_q(x, \xi) = \int_{-1}^{1} dx' \left[ \frac{2}{\pi} \operatorname{Im} \int_{0}^{1} ds \frac{y(s)}{\sqrt{1 - y(s)x'}} \right] \frac{d}{dx'} \left( q(x') \right),$$

(4)

\[^2\]Note that here we adopt Ji’s \[2\] definition $\xi = (x_1 - x_2)$, whereas in Ref. \[4\] $\xi \equiv (x_1 - x_2)/2$. 

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\[ H_a(x, \xi) = \int_{-1}^{1} dx' \left[ \frac{2}{\pi} \Im \int_0^1 ds \frac{x + \xi \left( \frac{1}{2} - s \right)}{y(s) \sqrt{1 - y(s)x'}} \right] \frac{d}{dx'} \left( \frac{g(x')}{|x'|} \right), \]  

\text{(5)}

where

\[ y(s) = \frac{4s(1-s)}{x + \xi \left( \frac{1}{2} - s \right)}. \]  

\text{(6)}

For small \( x \), where we may assume the conventional distributions behave as

\[ x f_a(x) \sim N_a x^{-\lambda_a}, \]  

\text{(7)}

\((\text{II})\) and \((\text{III})\) may be greatly simplified. Indeed we find that the ratios of the skewed to the diagonal distributions are

\[ \frac{H_a(x, \xi)}{H_a(x + \frac{1}{2} \xi, 0)} = R_a (x/\xi, \lambda_a), \]  

\text{(8)}

where \( R_a \) are simple known functions \([4]\).

### 3 Unintegrated from conventional distributions

We outline the procedure \([3]\) which enables the distributions \( f_a(x, k_t^2, \mu^2) \), unintegrated over \( k_t \), to be calculated from the conventional (integrated) distributions \( a(x, \mu^2) \equiv xq(x, \mu^2) \) or \( xg(x, \mu^2) \). We start from the DGLAP equation

\[ \frac{\partial a(x, \mu^2)}{\partial \ln \mu^2} = \frac{\alpha_s}{2\pi} \left[ \sum_{a'} \int_x^{1-\Delta} P_{aa'}(z) a'(\frac{x}{z}, \mu^2) \, dz - a(x, \mu^2) \, V^a(\Delta) \right], \]  

\text{(9)}

where in the virtual contribution

\[ V^a(\Delta) = \sum_{a'} \int_0^{1-\Delta} P_{a'a}(z) \, dz. \]  

\text{(10)}

In the case of the \( g \to gg \) splitting we have to insert a factor \( z \) in front of \( P_{gg} \) in \((\text{II})\), to account for the identity of the produced gluons. The real emission term in \((\text{III})\) gives the unintegrated parton density, except that we must allow for the modification due to virtual effects. The virtual contribution does not change the parton \( k_t \) and may be resummed to give the survival probability \( T^a \) that parton \( a \) remains untouched in the evolution up to \( \mu^2 \). This survival factor is

\[ T^a(k_t, \mu) = \exp \left( - \int_{k_t^2}^{\mu^2} \frac{dk_t^2}{k_t^2} \frac{\alpha_s(k_t^2)}{2\pi} V^a(\Delta) \right), \]  

\text{(11)}
and so the unintegrated distribution becomes
\[
f_a(x, k_t^2, \mu^2) = T^a(k_t, \mu) \left[ \frac{\partial a(x, k_t^2)}{\partial \ln k_t^2} \right]_{\text{real}}
\]
\[
= T^a(k_t, \mu) \left[ \frac{\alpha_s(k_t^2)}{2\pi} \sum_{a'} \int_x^{1-\Delta} P_{aa'}(z) a' \left( \frac{x}{z}, k_t^2 \right) dz \right].
\]

It is at the last step of the evolution, shown in (12), that the unintegrated gluon becomes dependent on \(\mu^2\). We have to make an appropriate choice of the cut-off \(\Delta\) in (11) and (12). We impose angular ordering in the last step of the evolution \cite{7,5}
\[
\Theta(\theta - \theta') \Rightarrow \mu > \frac{zk_t}{1-z}.
\]

Thus the maximum allowed value of the integration variable \(z\) is
\[
z_{\text{max}} = \frac{\mu}{\mu + k_t}, \quad \text{i.e. } \Delta \equiv 1 - z_{\text{max}} = \frac{k_t}{\mu + k_t},
\]
and similarly \(\Delta = k_t'/(\mu + k_t')\) in (11).

In summary, the \(\mu\) dependence of the unintegrated distributions \(f_a(x, k_t^2, \mu^2)\) enters at the last step, (12), of the evolution through the survival factor \(T^a\), which results from the resummation of the virtual contributions, and through the cut-off \(\Delta\).

### 4 Unintegrated generalised distributions

To calculate the generalised (or skewed) unintegrated distributions \(\overline{H}_a(x, \xi; k_t^2, \mu^2)\) from the conventional distributions \(a(x, \mu^2)\), we combine the procedures outlined in Sections 2 and 3. First, we extend (9) so as to describe the evolution of the skewed (integrated) distributions \(H_a(x, \xi, \mu^2)\)
\[
\frac{\partial H_a(x, \xi, \mu^2)}{\partial \ln \mu^2} = \frac{\alpha_s}{2\pi} \left[ \sum_{a'} \int_x^{1-\Delta} P_{aa'} \left( z, \frac{\xi}{x} \right) H_{a'} \left( \frac{x}{z}, \xi, \mu^2 \right) \frac{dz}{z} - H_a(x, \xi, \mu^2)(V_1 + V_2) \right].
\]

The splitting functions \(P_{aa'}\) are given by eq. (22) of Ref. [2], except that \(P_{gg}(x, \xi) = x P_{ji}^{gg}(x, \xi)\), which arises because \(H_g = xH_j\). For the antiquark contribution we should use \(-H_q(-x/z, \xi, \mu^2)\) in (15); see (2). \(V_1\) and \(V_2\) are the virtual loop contributions corresponding, in the axial gauge, to the self-energies of the emitted and absorbed partons respectively,
\[
V_i^a = \frac{1}{2} \sum_{a'} \int_0^{1-\Delta_i} P_{a'a}(z) dz,
\]
in analogy to (10). We will specify the cut-offs on the \(z\) integrations in a moment.
Resumming the virtual contributions, as before, introduces the survival factors $\sqrt{T_1}$ and $\sqrt{T_2}$, where now a square root occurs due to the factor $\frac{1}{2}$ in (16). In this way we obtain the $k_t^2$ dependence of the skewed distributions, namely

$$H_a(x, \xi; k_t^2, \mu^2) = \sqrt{T_1 T_2} \left[ \frac{\alpha_S(k_t^2)}{2\pi} \sum \int \frac{dz}{z} \frac{d\alpha}{\alpha} \frac{\alpha S(k_t^2)}{2\pi} P_{aa'} \left( \frac{\xi z}{x}, k_t^2, \mu^2 \right) \right]$$

with

$$T_i^a = \exp \left( - \int \frac{dk_i^2}{k_i^2} \frac{\alpha_S(k_i^2)}{2\pi} V_i^a (\Delta_i) \right).$$

Eq. (17) shows the modification necessary to extend (12) to skewed distributions. As in Section 3, the cut-offs in the $z$ integrations of (17) and (16) are chosen so as to respect angular ordering in the last step of the evolution. We obtain

$$\Delta = (1 - z_{\text{max}}) \left( 1 + \frac{\xi z_{\text{max}}}{2x + \xi - \xi z_{\text{max}}} \right)$$

$$\Delta_1 = 1 - z_{\text{max}}$$

$$\Delta_2 = (1 - z_{\text{max}}) \left( 1 + \frac{2\xi z_{\text{max}}}{2x + \xi - 2\xi z_{\text{max}}} \right),$$

where $z_{\text{max}} = \mu/\mu + k_t$ of (14). Although we have assumed that the emitted and absorbed partons of Fig. 1 have the same $k_t$, they have different momentum fractions, $x_1 \neq x_2$, and so the angular ordering gives different upper limits on the $z$ integration in (17), and in (16) for $V_1$ and $V_2$.

We are now in a position to use the results of Section 2. We substitute (4) and (5) for the integrated skewed distributions $H_{a'}$ into the right-hand-side of (17). In this way we determine the unintegrated skewed distribution $\Pi_a(x, \xi; k_t^2, \mu^2)$, for $\xi^2 \ll 1$, in terms of the conventional distributions, $q(x, \mu^2)$ or $g(x, \mu^2)$.

Note that there is no skewed effect coming from the $z \ll 1$ region of integration in (17). In this domain the second argument $\xi z/x$ of the splitting function $P_{aa'}$ is very small, while the first argument $(x/z \gg \xi)$ of the integrated distribution $H_{a'}$ becomes large. As a consequence the $\xi$ dependence is negligible. Therefore we may replace the singular $(1/z)$ part of $P_{aa'}$ by the (diagonal) BFKL kernel, and use this approach even for partons which incorporate the BFKL contribution in the small $x$ region, just as was done in the diagonal case [5]. Recall that in Ref. [5] it was seen that the unintegrated partons calculated from pure DGLAP partons (as in Section 3) are very similar to those calculated using partons [9] which incorporate BFKL effects. It was concluded that the imposition of the angular-ordering constraint in the last step of the evolution is more important than the BFKL effects.
5 Numerical results and discussion

Typical behaviour of the unintegrated generalised gluon, $\bar{H}_g$, is shown by the continuous curves in Fig. 2. The results are obtained by evaluating (17) for $\bar{H}_g$, using MRST99 conventional parton distributions [10] to first determine the integrated distributions, $H_{a'}$, from (4) and (5). We show results for the most relevant case, $x_2 \ll x_1$, that is for $x = \xi/2$. We take $\xi = 0.1, 0.01$ and $0.001$ with $\mu = 10$ GeV. To illustrate the $\mu$ dependence we also consider $\mu = 100$ GeV for the choice $\xi = 0.01$. In contrast to DGLAP $k_t$ ordering, the angular ordering, (13), imposed on the last step of evolution, enables the unintegrated distribution $\bar{H}_g$ to extend into the $k_t > \mu$ domain. However, $\bar{H}_g$ decreases for large $k_t$ when $z_{\text{max}}$ approaches $x$ — the lower limit of integral (17) (or (12)).

![Figure 2](image.png)

Figure 2: The unintegrated skewed gluon distribution, $\bar{H}_g(x, \xi = 2x; k_t^2, \mu^2)$, as a function of $k_t^2$ for various values of $x$ and $\mu^2$. The choice $\xi = 2x$ corresponds to the physically relevant fractions $x_1 = \xi$ and $x_2 = 0$ in Fig. 1. The continuous curves are the full result obtained from MRST99 partons [10] by inserting (4) and (5) into (17). The dashed, dot-dashed and dotted curves are shown only for information and correspond, respectively, to the values obtained from (22), (23) and (26).
To gain insight, we compare the results with different approximate formula for $\mathcal{H}_g$. The most naive possibility is to take

$$\mathcal{H}_g(x,\xi; k_t^2, \mu^2) = \left. \frac{\partial (yg(y, k_t^2))}{\partial \ln k_t^2} \right|_{y=x+\xi/2}.$$  \hfill (22)

This simplified distribution, which is widely used in the literature, is shown by the dot-dashed curves in Fig. 2. For not too small $x$ and large $k_t^2$, formula (22) gives a negative $\mathcal{H}_g$; the reason is that the virtual term in (9) starts to dominate as $x$ increases. This shortcoming may be avoided by developing a procedure originally implied in the work of DDT [11], see also [12]. For generalised partons, this leads to the formula

$$\mathcal{H}_a(x,\xi; k_t^2, \mu^2) = \left. \frac{\partial}{\partial \ln k_t^2} \left[ \sqrt{T_1 a T_2 a} a(y, k_t^2) \right] \right|_{y=x+\xi/2}.$$  \hfill (23)

In the diagonal case $\xi = 0$ and $T_1^a = T_2^a = T^a$. Then the derivative

$$\frac{\partial T^a}{\partial \ln k_t^2} = \frac{\alpha_s}{2\pi} (V_1^a + V_2^a)$$  \hfill (24)

cancels the virtual contribution in (9), and $\mathcal{H}_a$ is positive everywhere [3]. However for the skewed case the situation is worse. Our sample curves correspond to $x = \xi/2$ for which $\Delta_2 = 1$ and $T_2^a = 1$. Then only half of the negative term is cancelled by the derivative $\partial \sqrt{T_1^a} / \partial \ln k_t^2$, and even the DDT-improved formula (23) may give negative $\mathcal{H}_a$; see the dashed curves in Fig. 2 for $\xi = 0.1$ and $k_t^2 > 100 \text{ GeV}^2$. Note, however, the improvement of (23) as compared to (22) for the smaller $\xi$ values shown by Fig. 2. Due to the presence of the $T_1^a$ survival factors in (23), the gluon distribution $\mathcal{H}_g$ decreases for $k_t \ll \mu$ contrary to the behaviour of the simple derivative formula (22). This reflects the small chance for low $k_t$ partons to remain untouched in the long evolution up to $\mu$.

In order to reproduce the behaviour of the unintegrated generalised gluon $\mathcal{H}_g$, for $\xi^2 \ll 1$, in a simpler form than the full formula (17), it is useful to modify the DDT-like approximation. We retain the well-justified and important leading logarithm contribution coming from

$$\frac{\partial \sqrt{T_g}}{\partial \ln k_t^2} = \frac{N_c\alpha_s}{2\pi} \ln \left( \frac{1}{1 - z_{\text{max}}} \right),$$  \hfill (25)

but tune the non-leading contributions and an additional valence quark contribution (needed for $x \sim 0.1$) to approximately reproduce the full result shown by the continuous curves in Fig. 2. The result is the phenomenological form

$$\mathcal{H}_g \left( \frac{\xi}{2}, \xi; k_t^2, \mu^2 \right) = \sqrt{T_g} \left[ R_g \left. \frac{\partial yg(y, k_t^2)}{\partial \ln k_t^2} \right|_{y=x+\xi/2} + yg(y, k_t^2) \frac{N_c\alpha_s}{2\pi} \left( \ln \frac{\mu + \frac{1}{2}k_t}{k_t} + 1.2 \frac{\mu^2}{\mu^2 + k_t^2} \right) \right.$$

$$+ 5 \frac{\alpha_s}{2\pi} \left( yu_{\text{val}}(y, k_t^2) + yd_{\text{val}}(y, k_t^2) \right) \left. \right|_{y=\xi},$$  \hfill (26)
where \( R_g \) is defined in (8) and given in [4]. Form (26) gives the dotted curves in Fig. 2 and is a reasonable numerical approximation to the full result. It is a rough and ready way to estimate the values of \( \mathcal{P}_g \) for the physically relevant case \( x = \xi/2 \).

Of course, from a formal point of view, we can add to the generalised distribution, \( \mathcal{P} \) of (17), any function
\[
\Delta \mathcal{P}(x, \xi) = f(x, \xi) \Theta \left( \frac{\xi}{2} - |x| \right)
\]
with support only in the time-like ERBL region, \( |x| < \xi/2 \) [13]. Such a contribution is not obtained from the conventional partons determined from deep inelastic scattering (DIS) data. However for the processes mentioned in the Introduction we only need the unintegrated skewed distributions in the space-like (DIS) domain.

6 Conclusions

We have presented a procedure which allows the unintegrated skewed parton distributions, \( \mathcal{P}_a(x, \xi; k_t^2, \mu^2) \), to be calculated from the conventional distributions, \( q(x, \mu^2) \) and \( g(x, \mu^2) \), for \( \xi^2 \ll 1 \) and any \( x \). The key formula is (17), with (4) and (5) as input. A crucial advantage of using unintegrated distributions is that they enable the full kinematics of the hard subprocess to be retained, and in this way incorporate a major part of higher order QCD corrections. Moreover, the unintegrated skewed distributions are based on conventional partons fitted in global analyses of DIS and related data, and therefore have no uncertainties coming from the lack of knowledge of the (non-perturbative) initial conditions. They can therefore be used to make predictions for physical processes, including those for diffractive production listed in the Introduction.

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