Scattering in Subluminal and Superluminal Space-Time Crystals

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In this paper, we develop the tools to calculate the scattering of electromagnetic waves from periodic structures having uniform velocities, which we name space-time (ST) crystals. We first extend the conventional transfer matrix method to ST layered structures. Using this method, we find the closed-form dispersion relation and plot the dispersion diagram of unbounded ST crystals and calculate the transmission and reflection of a bounded structure. Subluminal and superluminal modulation velocities are studied. Crystals with subluminal modulation velocity attenuate certain frequency bands, while those with superluminal modulation velocity amplify certain frequency bands.

I. INTRODUCTION

Space-time structures are structures with space and time varying parameters. They can be realized for instance by ionizing a plasma [1, 2], in electro-optic or acousto-optic structures [3], or using nonlinear elements [4]. These types of structures have been known to give rise to frequency shifts, associated to the Doppler effect [5, 6] and wave amplification, for instance in traveling-wave amplifiers [7, 8]. More recently, exotic effects such as an inverse Doppler shift [9, 10] and magnetless nonreciprocity [11–13] have been reported.

The aforementioned effects are typically related to subluminal modulation velocities. However modulation may be greater than velocity of light, for instance if the modulation is transverse to the propagation direction. This velocity regime is much less explored than the subluminal one, and it’s unusual scattering effects give rise to interesting phenomena, such as shifted refocusing [14]. Structures that are strictly time-varying may be seen as infinite-velocity modulations, and as such are a limiting case of superluminal modulation. Time-varying structures have been well explored [15–18] and their applications include, amongst others, frequency transitions [2], parametric amplification [19], inverse prism scattering [20] and refocusing [21].

Here, we develop the tools to calculate the scattering from subluminal and superluminal layered structures, and in particular periodic crystal structures. We first derive the transition and propagation matrices, then the complete transfer matrix of a unit cell. Using this tool, we find the closed-form dispersion diagram and the reflection and transmission amplitudes for a finite crystal. We show subluminal crystals filter certain frequency bands, but interestingly, the reflected wave is absorbed, leading to absorptive filters. Superluminal crystals on the other hand amplify waves in certain frequency bands.

II. LAYERED ST STRUCTURES

An aperiodic space-time layered structure is illustrated in the Minkowski diagram in Fig. 1. It consists of a stacking of N layers moving to the right at a uniform velocity. Different grey levels correspond to different refractive indices, and the slopes are inversely proportional to the velocity of the modulation.

The total transfer matrix relating the fields on the left to the fields on the right of the layered structure is found by cascading transition [T] and propagation [P] matrices

\[ [M_N] = \prod_{n=1}^{N} [P_{n+1}] [T_{n+1,n}] . \]  (1)

The transition matrix models the scattering at an interface between two layers, and the propagation matrix models the propagation of waves inside a given layer. In following sections, we derive the transition matrix of a single interface and the propagation matrix of a single layer for both subluminal and superluminal regimes.
A. Transition Matrices

We first model the scattering of a wave at a single interface separating two layers, as illustrated in Fig. 2(a). The transition between the layers is sharp, with the refractive index \( n(z, t) = n_1 + (n_2 - n_1)\Pi(z - vt) \), with II the step function and \( v \) the velocity of the modulation.

The scalar quantities \( \psi_{i,j}^{\pm} \) could be the amplitude of an acoustic, elastic, or electromagnetic wave, for instance. In this paper, they represent the amplitude of the electric field of an electromagnetic wave. In Fig. 2, the incident waves are found at earlier times and the scattered waves at later times.

The scattering from subluminal and superluminal interfaces is different: in the subluminal regime, Fig. 2(a), an incident and a scattered wave is found in each layer. In the superluminal regime, Fig. 2(b), both incident waves are in layer \( i \) and both scattered waves are in layer \( j \). This will have repercussions on the shape of the matrix as we will see next.

![Fig. 2. Scattering on space-time half-space. (a) Scalar fields, subluminal. (b) Scattering coefficients (c) Scalar fields, superluminal. (d) Scattering coefficients, superluminal.](image)

1. Subluminal Regime

To find the transition matrix, we read out Fig. 2(a). The following notation is used: the subscripts of \( \psi_{i,j}^{\pm} \) correspond to the layer of the wave and the superscript \( \pm \) to the propagation direction. For the coefficients \( \tau_{ji}, \gamma_{ji} \), the first subscript indicates the layer of the scattered field and the second the layer of the incident field. Reading out the figure, we find:

\[
\psi_j^+ = \tau_{ji}\psi_i^+ + \gamma_{ji}\psi_j^- \quad (2a)
\]
\[
\psi_j^- = \gamma_{ji}\psi_i^+ + \tau_{ij}\psi_j^- \quad (2b)
\]

Rearranging (2), we find the transition matrix, which relates the fields in layer \( j \) to fields in layer \( i \):

\[
\begin{bmatrix} \psi_j^+ \\ \psi_j^- \end{bmatrix} = \frac{1}{\tau_{ij}} \begin{bmatrix} \tau_{ji} & \gamma_{ji} \\ \gamma_{ji} & -\gamma_{ii} \end{bmatrix} \begin{bmatrix} \psi_i^+ \\ -\gamma_{ii}\psi_i^- \end{bmatrix} = \frac{1}{\tau_{ij}} \begin{bmatrix} \tau_{ii} + \tau_{ij} - \gamma_{ii} & \gamma_{ji} \\ \gamma_{ji} & -\gamma_{ii} \end{bmatrix} \begin{bmatrix} \psi_i^+ \\ \psi_i^- \end{bmatrix}.
\]

We note, in passing, that the scattering matrix is simply

\[
\begin{bmatrix} \psi_j^+ \\ \psi_j^- \end{bmatrix} = \begin{bmatrix} \tau_{ji} & \gamma_{ji} \\ \gamma_{ji} & -\gamma_{ii} \end{bmatrix} \begin{bmatrix} \psi_i^+ \\ -\gamma_{ii}\psi_i^- \end{bmatrix} = \frac{1}{\tau_{ij}} \begin{bmatrix} 1 & -t_{21} \\ -t_{12} & 1 \end{bmatrix} \begin{bmatrix} \psi_i^+ \\ \psi_i^- \end{bmatrix}.
\]

where \( t_{ab} \) represents the element of the \( a \) row and \( b \) column of the transition matrix. To express the transition matrix in terms of the refractive index and the velocity of the modulation, we substitute the scattering coefficients into (4). These are found in App. B. The transition matrix becomes:

\[
[T_{ji}^{ab}] = \frac{1}{2n_j} \begin{bmatrix} (n_i + n_j) & 1 - n_i v \\ 1 - n_j v & -(n_i - n_j) & 1 - n_i v \\ 1 - n_j v & (n_i + n_j) & 1 + n_i v \\ 1 - n_j v & 1 + n_j v \end{bmatrix}.
\]

Substituting \( v = 0 \) into (5), we find the conventional transition matrix for a stationary interface.

2. Superluminal Regime

We now derive the transition matrix for the superluminal regime, by reading out Fig. 2(b). The quantities \( \xi_{ji}, \xi_{ji} \) are the transmission and reflection coefficients for the wave traveling to the right, and the barred coefficients correspond to the reflection and transmission for the wave traveling to the left. We find:

\[
\psi_j^+ = \xi_{ji}\psi_i^+ + \bar{\xi}_{ji}\psi_i^- \quad (6a)
\]
\[
\psi_j^- = \bar{\xi}_{ji}\psi_i^+ + \xi_{ji}\psi_i^- \quad (6b)
\]

Equations (6) simply lead to the transition matrix:

\[
\begin{bmatrix} \psi_j^+ \\ \psi_j^- \end{bmatrix} = \begin{bmatrix} \xi_{ji} & \bar{\xi}_{ji} \\ \bar{\xi}_{ji} & \xi_{ji} \end{bmatrix} \begin{bmatrix} \psi_i^+ \\ \psi_i^- \end{bmatrix}.
\]

which has simpler form than (4), and is the same as the scattering matrix.

Although they have a very different form, when we express the transition matrix in terms of the refractive index and the modulation velocity by inserting the scattering coefficients (B7), we find the same transition matrix as the subluminal case. However the scattering matrices are different.
B. Propagation Matrix

As shown in Fig. 1, the propagation matrix relates the fields at two boundaries of a single layer

$$\begin{bmatrix} \psi_i^+ \\ \psi_i^- \end{bmatrix}_{n+1} = [P_i] \begin{bmatrix} \psi_i^- \\ \psi_i^+ \end{bmatrix}_{n}.$$  

(8)

There is no scattering inside the layer, only phase accumulation, such that the matrix simply reads:

$$[P_i] = \begin{bmatrix} e^{i\phi_i^+} & 0 \\ 0 & e^{-i\phi_i^-} \end{bmatrix} = e^{i\Delta\phi_i} \begin{bmatrix} e^{i\phi_i} & 0 \\ 0 & e^{-i\phi_i} \end{bmatrix},$$  

(9)

with the phase

$$\phi_i^+ = k_i^+\ell_i + \omega_i^+d_i = \phi_i + \Delta\phi_i.$$  

(10)

The moving interfaces induce frequency shifts, such that the frequencies in each layer and the frequencies travelling to the left and the right are related through (see App. B2)

$$\omega_i^- = \omega_i^+ \frac{1 - vn_i/c}{1 + vn_i/c}, \quad \omega_i^+ = \omega_i^- \frac{1 - vn_i/c}{1 + vn_i+1/c}.$$  

(11)

Quantities $\ell_i, d_i$ are the length and duration of the layer, related through $d_i = \ell_i v$ (see Fig. 1). $\phi_i$ is the average and $\Delta\phi_i$ the difference phase accumulation when going to the right or to the left:

$$\phi_i = \frac{\phi_i^+ + \phi_i^-}{2}, \quad \Delta\phi_i = \frac{\phi_i^+ - \phi_i^-}{2}.$$  

(12)

We must be careful when calculating the phase accumulation not to take the total length of the layer, $L_i$, but rather the projection of the propagating vector, $\ell_i$. The link between the total length and the projection length can be found geometrically in Fig. 1 as $L_i = \ell_i(1 + v^2/c^2)$.

III. ST CRYSTAL

A. Structure

We now study space-time crystal structures, illustrated in Fig. 3(a) and 22. The direction of the periodicity is given by the purple arrow, associated with quantities $\Lambda_B, T_B$, with $\Lambda_B = c^2/vT_B$ where B stands for Bloch. The periodicity is mathematically expressed as:

$$n(z, t) = n(z + m\Lambda_B, t - mT_B).$$  

(13)

The refractive index is expressed as $n(z, t) = n_1 + (n_2 - n_1)\text{sgn}(\cos(z/\Lambda_m - t/T_m))$, with $\Lambda_m, T_m$ the wavelength and the temporal period of the modulation, related through $\Lambda_m = vT_m$ with $v$ the velocity of modulation. The Bloch period and the modulation period are related through $\Lambda_m = \Lambda_B(1 + v^2/c^2)$.

B. Unit Cell Transfer Matrix

The transfer function of a period of a crystal, Fig. 3(b), is found by cascading interface and propagation matrices:

$$[M_B] = [P_1] [T_{12}] [P_2] [T_{21}]$$  

(14)

Inserting the transition matrix (77), and the propagation matrix (9) into (14), we obtain:

$$[M_B] = e^{i\Delta\phi} \begin{bmatrix} a & ib \\ ic & a^* \end{bmatrix} = e^{i\Delta\phi'[B^0]}$$  

(15)

where $\Delta\phi = \Delta\phi_1 + \Delta\phi_2$ is the difference in phase between forward and backward waves for a whole period of the crystal, and the parameters

$$a = \frac{e^{i\phi_1}}{2n_1n_2} \left( 2n_1n_2 \cos \phi_2 + i \left( n_1^2 + n_2^2 \right) \sin \phi_2 \right),$$  

(16a)

$$b = -\frac{e^{-i\phi_1}}{2n_1n_2} \frac{\left( n_1^2 - n_2^2 \right) \left( 1 + n_1v \right) \sin \phi_2}{1 - n_1v},$$  

(16b)

$$c = \frac{e^{i\phi_1}}{2n_1n_2} \frac{\left( n_1^2 - n_2^2 \right) \left( 1 - n_1v \right) \sin \phi_2}{1 + n_1v}.$$  

(16c)

Space-time crystals are in general nonreciprocal, and so $\det[M_B] \neq 1$, expect for the limiting cases of stationary crystals ($v = 0$) and temporal ($v = \infty$) crystals. For stationary crystals, $\det[M_B] = 1$ with $\phi_{1,2} = k_{z1,2}\ell_{1,2}$ and $\Delta\phi = 0$. For temporal crystals, $\det[M_B] = 1$ with $\phi_{1,2} = \omega_{1,2}d_{1,2}$ and $\Delta\phi = 0$. 

![FIG. 3. Space-time crystal structure (a) crystal, with spatial and temporal periods $\Lambda_m$, $T_m$ and $\Lambda_B, T_B$ the Bloch phase quantities. (b) Unit cell, left and right fields $\psi_L$, $\psi_R$ and transition and propagation matrices $T, P$.](image-url)
C. Closed-Form Dispersion Relation

The fields inside a space-time crystal can be represented in the space-time Bloch-Floquet form [23, 24]

\[ \psi(z, t) = e^{i\Phi_B} u(z, t) \]  \hspace{1cm} (17)

with Bloch-Floquet phase

\[ \Phi_B = K_B\Lambda_B - \Omega_B T_B, \]  \hspace{1cm} (18)

with \( K_B, \Omega_B \) the Bloch-Floquet frequencies and \( u(z, t) = u(z + \Lambda_B, t - T_B) \) a periodic function. The fields before and after a period of a crystal are related through \( \psi_R = e^{i\Phi_B}\psi_L \) where \( L \) and \( R \) stand for Left and Right. We thus have the following equation:

\[
\begin{bmatrix}
\psi_R^+ \\
\psi_R^-
\end{bmatrix} = [M_B] \begin{bmatrix}
\psi_L^+ \\
\psi_L^-
\end{bmatrix} = e^{i\Phi_B} \begin{bmatrix}
\psi_L^+ \\
\psi_L^-
\end{bmatrix}.
\]

Solving this system of equations, using property (15), yields (see App. C for derivations)

\[
\cos(\Phi_B - \Delta\phi) = \cos \phi_1 \cos \phi_2 - \frac{n_1^2 + n_2^2}{2n_1 n_2} \sin \phi_1 \sin \phi_2,
\]

which reduces to the conventional stationary case [24] when \( \Delta\phi = 0 \) and \( \phi_{1,2} = k_{1,2}^\ell_{1,2} \).

Equation (20) is underdetermined. In order to plot the dispersion diagram \( \Omega_B(K_B) \), we must eliminate \( \phi_{1,2}(\omega_{1,2}, k_{1,2}) \) and \( \Delta\phi_{1,2}(\omega_{1,2}, k_{1,2}) \) in equation (20). We enforce phase-matching of the wave \( \psi_1^+ \) with Bloch-Floquet solutions (17) at a moving interface \( I \), such that

\[
e^{i(k_1^+ z - \omega_1^+ t)} \left|_{z-\ell t=z_0} \right. = e^{i(K_B z - \Omega_B t)} \left|_{z-\ell t=z_0} \right.
\]

which yields

\[ k_1^+ v - \omega_1 = K_B v - \Omega_B. \]  \hspace{1cm} (22)

Equations (20) (22) form a determined system of equations, that we can solve to find the dispersion relation \( \Omega_B(K_B) \).

Fig. 4(a), (b) plots the dispersion relation \( \Omega_B(K_B) \) of a subluminal and a superluminal crystal, respectively. The gaps are aligned on a slope proportional to the velocity \( v \), as expected [23]. In the subluminal case, the gaps are vertical, whereas in the superluminal case, the gaps are horizontal [18, 24]. We will see that the former lead to attenuation, while the latter lead to amplification.

D. Transmission and Reflection Coefficients from Finite Multilayer

The transmission and reflection coefficients of a finite multilayer of \( N \) periods can be calculated by simply multiplying the transfer matrix of a single period [15] \( N \) times. An alternative approach uses the knowledge that

\[
[M_B] = e^{i\Delta\phi}[M_{B0}]^{15}, \text{ with } [M_{B0}] \text{ a unimodular matrix.}
\]

The Chebyshev identity [25] is applied to \( [M_{B0}] \), since this identity only applies to unimodular matrices

\[
[M_{B0}]^N = \begin{bmatrix}
m_{B0_1} U_{N-1}(a) - U_{N-2}(a) & m_{B0_2} U_{N-1}(a) - U_{N-2}(a) \\
m_{B0_2} U_{N-1}(a) & m_{B0_1} U_{N-1}(a) - U_{N-2}(a)
\end{bmatrix}
\]

where \( U_N \) are the Chebyshev polynomials of the second kind:

\[
U_N(a) = \frac{\sin((N + 1) \cos^{-1} a)}{\sqrt{1 - a^2}} \]  \hspace{1cm} (24)
and the argument:

$$a = \frac{1}{2} \left( m_{011} + m_{022} \right).$$  \hfill (25)

We then obtain the total result by applying (15)

$$[M_{\text{tot}}] = e^{iN\Delta\phi} [M_{B0}]^N.$$  \hfill (26)

The total transmission and reflection coefficients for a subluminal crystal are then inferred from (4)

$$\gamma_{11}^{\text{tot}} = -\frac{m_{21}^{\text{tot}}}{m_{22}^{\text{tot}}},$$  \hfill (27a)

$$\tau_{N1}^{\text{tot}} = m_{11}^{\text{tot}} - \frac{m_{12}^{\text{tot}} m_{21}^{\text{tot}}}{m_{22}^{\text{tot}}}. $$  \hfill (27b)

the results are plotted in Fig. 5. We notice that in the gap, energy is not conserved: $T + \Gamma < 1$. The reflected wave is partly absorbed by the structure, therefore leading to an absorptive filter. This could be useful in situations where the reflection from the filter is unwanted.

For the superluminal crystal, the coefficients are inferred from (7):

$$\zeta_{N1}^{\text{tot}} = m_{21}^{\text{tot}},$$  \hfill (28a)

$$\xi_{N1}^{\text{tot}} = m_{11}^{\text{tot}},$$  \hfill (28b)

and are plotted in Fig. 5. We see that in the gap, both the transmitted and the reflected wave are amplified. We thus have a structure that only amplifies certain frequency bands. This might be of interest for a new class of amplifiers.

### IV. CONCLUSION

This paper presents the tools to study space-time crystals, for both subluminal and superluminal modulation velocities. The transfer matrix for a unit cell is derived, and is used to calculate the dispersion diagram and the scattering coefficients of a crystal with a finite number of cells. Two interesting properties of space-time crystals are highlighted: an absorptive filter and a band-limited amplifier. The tools presented in this paper could be applied to other types of space-time structures, such as aperiodic structures, structures with more complex periodicities, and gradient structures.
Appendix A: Continuity Conditions

In this section, we derive the field continuity conditions for a moving perturbation half-space problem with permittivity \( \epsilon(z - vt) \) and permeability \( \mu(z - vt) \). Starting with one-dimensional Maxwell equations:

\[
\frac{\partial E_y}{\partial z} = \frac{\partial B_z}{\partial t} = \frac{\partial \mu(z - vt) H_x}{\partial t} \quad \text{(A1a)}
\]
\[
\frac{\partial H_x}{\partial z} = \frac{\partial D_y}{\partial t} = \frac{\partial \epsilon(z - vt) E_y}{\partial t} \quad \text{(A1b)}
\]
we apply the following transformation of coordinates:

\[
z' = z - vt, \quad t' = t. \quad \text{(A2)}
\]

The partial derivatives are found:

\[
\frac{\partial}{\partial z} = \frac{\partial}{\partial z'} \frac{\partial z'}{\partial z} = \frac{\partial}{\partial z'}, \quad \text{(A3a)}
\]
\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} \frac{\partial t'}{\partial t} + \frac{\partial}{\partial z'} \frac{\partial z'}{\partial t} = \frac{\partial}{\partial t'} - v \frac{\partial}{\partial z'}. \quad \text{(A3b)}
\]

We then substitute (A3) into (A1):

\[
\frac{\partial E_y}{\partial z'} = \left( \frac{\partial}{\partial t'} - v \frac{\partial}{\partial z'} \right) \mu(z') H_x, \quad \text{(A4a)}
\]
\[
\frac{\partial H_x}{\partial z'} = \left( \frac{\partial}{\partial t'} - v \frac{\partial}{\partial z'} \right) \epsilon(z') E_y, \quad \text{(A4b)}
\]
and rearrange (A4):

\[
\frac{\partial}{\partial z'} \left( E_y + v \mu(z') H_x \right) = \frac{\partial}{\partial t'} \mu(z') H_x, \quad \text{(A5a)}
\]
\[
\frac{\partial}{\partial z'} \left( H_x + v \epsilon(z') E_y \right) = \frac{\partial}{\partial t'} \epsilon(z') E_y. \quad \text{(A5b)}
\]

Observing these equations, we can deduce the continuity conditions by reasoning ad absurdum: if the arguments of the \( z' \) derivative on the left-hand side in (A5), \( E_y + v \mu(z') H_x \) or \( H_x + v \epsilon(z') E_y \), vary as a step function, then the right-hand side would be an impulse function, which is unphysical. Therefore, quantities \( E_y + v \mu(z') H_x \) and \( H_x + v \epsilon(z') E_y \) must be continuous at the interface. Integrating (A5) from \( z^- \) to \( z^+ \) and letting \( \Delta z \to 0 \), we find:

\[
(E_y + v \mu(z') H_x)_{z^+} = (E_y + v \mu(z') H_x)_{z^-} + \frac{\partial}{\partial t'} (E_y + v \mu(z') H_x), \quad \text{(A6a)}
\]
\[
(H_x + v \epsilon(z') E_y)_{z^+} = (H_x + v \epsilon(z') E_y)_{z^-} + \frac{\partial}{\partial t'} (H_x + v \epsilon(z') E_y), \quad \text{(A6b)}
\]

Therefore \( E_y + v B_z \) and \( H_x + v D_y \) are continuous at a moving interface. We can see that in the limiting case when \( v = 0 \), the quantities \( E_y, H_x \) are continuous, as expected. In the limiting case \( v = \infty \), the quantities \( D_y, B_z \) are continuous, as reported in [10].

Appendix B: Space-Time Half-Space Problems

1. Fresnel Coefficients

a. Subluminal Case

In this section, we derive the scattering coefficients of a wave scattering on a subluminal half-space, as in Fig. 2(a). We consider only one incident wave, \( \psi_1^+ \) scatters at a moving interface into two waves, \( \psi_2^+ \) and \( \psi_2^- \). The continuity of \( E_y + v B_z \) and \( H_x + v D_y \) was derived in App. A and so we write:

\[
[E_y^+ + v B_z^+]_z = \frac{E_y^+ + v B_z^-}{|v|} \delta(z), \quad \text{(B1a)}
\]
\[
[H_x^+ + v D_y^+]_z = \frac{H_x^+ + v D_y^-}{|v|} \delta(z), \quad \text{(B1b)}
\]
We consider a harmonic wave \( E_{yn} = A_0^+ e^{i(\omega t + k z)} \), and a magnetless medium, such that \( H_{xn} = \pm n/c E_{yn} \), \( D_{yn} = n^2 E_{yn} \), and \( B_{xn} = \pm n/\epsilon E_{yn} \). Therefore (B1) reduces to:

\[
A_1^+ (1 - v n/c) + A^-_1 (1 - v n/c) = A_1^+ (1 - v n/c), \quad \text{(B2a)}
\]
\[
n_1 (A_1^+ (1 - v n/c) - A^-_1 (1 - v n/c)) = n_2 A_2^+ (1 - v n/c). \quad \text{(B2b)}
\]

Solving this system of equations, we find:

\[
\frac{\tau_1}{A_1} = \frac{A_1^+}{A_1^-} = \frac{2 n_1}{n_1 + n_2} \left( \frac{1 - n_1 v/c}{1 - n_2 v/c} \right), \quad \text{(B3a)}
\]
\[
\frac{\gamma_1}{A_1} = \frac{A_1^-}{A_1^+} = \frac{n_1 - n_2}{n_1 + n_2} \left( \frac{1 - n_1 v/c}{1 + n_1 v/c} \right). \quad \text{(B3b)}
\]

Note that they reduce to the standard Fresnel coefficients when \( v = 0 \). Also note that this result is only valid for the interval \( c/n_1 < |v| < c/n_2 \) [27, 28].

To find the coefficients for the wave incident from the right, apply following substitutions:

\[
\tau_{12}(v, n_1, n_2) = \tau_{12}^+ (-v, n_2, n_1), \quad \text{(B4a)}
\]
\[
\gamma_{12}(v, n_1, n_2) = \gamma_{12}^+ (-v, n_2, n_1). \quad \text{(B4b)}
\]

b. Superluminal Case

For the superluminal case, the continuity conditions at the interface are the same as previously, but the continuity refers to the fields before and after the interface [13, 16], such that the two scattered waves are on the right-hand side of the equation (see Fig. 2(b)):

\[
[E_y^+ + v B_z^+]_z = \frac{E_y^+ + v B_z^-}{|v|} \delta(z), \quad \text{(B5a)}
\]
\[ H_{z1}^+ + vD_{y1}^+ = H_{z1}^- + vD_{y1}^- + H_{z2}^+ + vD_{y2}^+ \] 

(B5b)

As in the previous section, we consider a harmonic wave \( E_{yn}^\pm = A_n^\pm e^{i(\omega_n^\pm + k_L z, t)} \) and a magnetless medium, and substitute the relations \( H_{zn} = \mp n_n/mE_{yn}, \) \( D_{yn}^\pm = n_n^2 e_0 E_{yn}^\pm, \) and \( B_{zn} = \mp n_n/cE_{yn}^\pm, \) which yields

\[
A_1^+ (1-n_1 v/c) = A_2^+ (1-n_2 v/c) + A_2^- (1+n_1 v/c),
\]

(A6a)

\[
A_1^+ n_1 (1-n_1 v/c) = n_2 (A_2^+ (1-n_2 v/c) + A_2^- (1+n_1 v/c)),
\]

(A6b)

solving this system of equations yields

\[
\frac{\gamma_{12}^+}{A_1^+} = \frac{n_1 + n_2}{2n_2} \left( \frac{1-n_1 v/c}{1-n_2 v/c} \right), \quad \text{(B7a)}
\]

\[
\frac{\gamma_{12}^-}{A_1^-} = \frac{n_1 - n_2}{2n_2} \left( \frac{1-n_1 v/c}{1+n_2 v/c} \right). \quad \text{(B7b)}
\]

The coefficients associated to the scattering of the wave incident from the right \( \psi^-_2 \) are found by substituting:

\[
\gamma_{12}^+ (v) = \gamma_{12}^+ (-v), \quad \gamma_{12}^- (v) = \gamma_{12}^- (-v). \quad \text{(B8)}
\]

### 2. Frequencies

In this section, we calculate the frequencies of the waves scattered from a subluminal and a superluminal half-space. We rewrite the continuity condition \( \text{(B1)} \), with \( \psi' = (E_y + vB_z) \):

\[
\left[ \psi_{1}^{+} + \psi_{1}^{-} = \psi_{2}^{+} \right]_{z-\nu t = 0}. \quad \text{(B9)}
\]

Considering a plane harmonic wave

\[
\psi_{n}^\pm = A_n^\pm e^{i(k_{zn} z \pm \omega_n^\pm t)} \quad \text{(B10)}
\]

and inserting \( \text{(B10)} \) into \( \text{(B9)} \) yields

\[
A_1^+ e^{i(k_{1z} z - \omega_1^+ t)} + A_1^- e^{-i(k_{1z} z + \omega_1^- t)} \bigg|_{z-\nu t = 0} = A_2^+ e^{i(k_{2z} z - \omega_2^+ t)} \bigg|_{z-\nu t = 0}. \quad \text{(B11)}
\]

Enforcing the boundary condition \( z-\nu t = 0 \), we find:

\[
\omega_1^+ - \nu k_{1n}^+ = \omega_1^- + \nu k_{1n}^- = \omega_2^+ - \nu k_{2n}^+. \quad \text{(B12)}
\]

Inserting the Helmholtz relation for a 1D wave propagation \( k_{zn}^\pm = \pm \omega_n^\pm /c \) into \( \text{(B12)} \), we find

\[
\omega_1^+ (1-vn_1/c) = \omega_1^- (1+vn_1/c) = \omega_2^+ (1-vn_2/c). \quad \text{(B13)}
\]

and so

\[
\omega_1^+ = \omega_1^- \frac{1-vn_1/c}{1+vn_1/c}, \quad \omega_2^+ = \omega_2^- \frac{1-vn_1/c}{1-vn_2/c}. \quad \text{(B14)}
\]

### Appendix C: Floquet-Bloch Solution

In this section, we show how to obtain the closed-form dispersion relation from the matrix relation \( \text{(C1)} \)

\[
\begin{bmatrix}
\psi_R^+ \\
\psi_L^+
\end{bmatrix} = [M_B] \begin{bmatrix}
\psi_R^- \\
\psi_L^-
\end{bmatrix} = e^{i \Delta \phi} [M_B] \begin{bmatrix}
\psi_R^+ \\
\psi_L^+
\end{bmatrix}
\]

with \( \Delta \phi_n = \Delta \phi_1 + \Delta \phi_2 \) the difference in phase between forward and backward waves for a whole period of the crystal, and Bloch-Floquet phase \( \Phi_B \)

\[
\Phi_B = K_B A_B - \Omega_B T_B. \quad \text{(C2)}
\]

We start by setting determinant to zero:

\[
\begin{vmatrix}
m_{011} - e^{i(\Phi_n - \Delta \phi)} \\
m_{021} - e^{i(\Phi_n - \Delta \phi)}
\end{vmatrix} = 0 \quad \text{(C3)}
\]

which yields a quadratic equation, with solutions

\[
e^{i(\Phi_n - \Delta \phi)} = (m_{011} + m_{022}) \pm \sqrt{(m_{011} + m_{022})^2 - 4}.
\]

this is equivalent to

\[
\cos(\Phi_B - \Delta \phi) = \frac{m_{011} + m_{022}}{2} \quad \text{(C5)}
\]

(to check, substitute \( \text{(C5)} \) into \( \text{(C4)} \)).

Replacing the values \( m_{011}, m_{022} \) provided in \( \text{(C6)} \), we find:

\[
\cos(\Phi_B - \Delta \phi) = \cos \tilde{\phi}_1 \cos \tilde{\phi}_2 - \frac{n_1^2 + n_2^2}{2n_1 n_2} \sin \tilde{\phi}_1 \sin \tilde{\phi}_2. \quad \text{(C6)}
\]

### Appendix D: Quarter-Wave Stack

In this section, we derive the quarter-wave condition for the case of a subluminal crystal. The quarter-wave condition corresponds to a maximal reflection. The reflection from a unit cell was found to be \( \text{(C6)} \):

\[
b = -\frac{e^{-i\tilde{\phi}_1}}{2n_1 n_2} (n_1^2 - n_2^2)(1+n_1v) \sin \tilde{\phi}_2. \quad \text{(D1)}
\]

Differentiating this equation with respect to the average phase \( \partial / \partial \tilde{\phi}_2 \), we find

\[
\frac{\partial b}{\partial \tilde{\phi}_2} = -\frac{e^{-i\tilde{\phi}_1}}{2n_1 n_2} (n_1^2 - n_2^2)(1+n_1v) \cos \tilde{\phi}_2 = 0, \quad \text{(D2)}
\]

such that the average phase that maximises reflection is \( \tilde{\phi}_2 = \pi/2(2n + 1) \). Similarly for \( \tilde{\phi}_1 \). This is the quarter-stack condition.

We would like to express this condition in terms of the properties of the slab: its length, velocity and refractive
This reduces to the conventional quarter-wave condition.

\[
\bar{\phi}_1 = \frac{\phi^+_1 + \phi^-_1}{2} = \frac{k^+_1 \ell_1 - \omega^+_1 d_1 + k^-_1 \ell_1 + \omega^-_1 d_1}{2}
\]

or, substituting \(\omega^+_1 = \pm \omega_1^c n_i/c\) and \(d_1 = -v/c \ell_1\) yields

\[
\bar{\phi}_1 = \frac{k^+_1 \ell_1}{2} \left( \left( 1 + \frac{1 - n_i v/c}{1 + n_i v/c} \right) - \left( 1 - \frac{1 - n_i v/c}{1 + n_i v/c} \right) \frac{v}{cn_1} \right) \]

\[
= k^+_1 \ell_1 \left( \frac{1 - n_i v/c}{1 + n_i v/c} \right) \frac{v/c^2}{1 + n_i v/c} \]

\[
= k^+_1 \ell_1 \left( 1 + v^2/c^2 \right) \frac{1}{1 + n_i v/c} \]

The quarter-stack condition \(\bar{\phi}_1 = \pi/2\) is therefore written:

\[
k^+_1 \ell_1 \left( 1 + v^2/c^2 \right) = \pi/2, \quad (D3a)
\]

or, substituting \(\ell_1(1 + v^2/c^2) = L_1\),

\[
k^+_1 L_1 \frac{1}{1 + n_i v/c} = \pi/2. \quad (D3b)
\]

We finally express this in terms of the wavelength:

\[
L_1 = \frac{\lambda^+_1}{4} (1 + n_i v/c) \quad (D3c)
\]

and similarly,

\[
L_2 = \frac{\lambda^+_2}{4} (1 + n_2 v/c). \quad (D3d)
\]

This reduces to the conventional quarter-wave condition \(L_{1,2} = \lambda_{1,2}/4\) for the stationary case.

For the temporal counterpart, we have the same starting point:

\[
\bar{\phi}_1 = \frac{\phi^+_1 + \phi^-_1}{2} = \frac{k^+_1 \ell_1 - \omega^+_1 d_1 + k^-_1 \ell_1 + \omega^-_1 d_1}{2}
\]

or, in terms of the period:

\[
= \frac{\omega^+_1 d_1}{2} \left( 1 + \frac{1 - n_i v/c}{1 + n_i v/c} \right) \frac{1}{2} c/v \]

\[
= \omega^+_1 d_1 \left( 1 + \frac{1 - n_i v/c}{1 + n_i v/c} \right) \frac{1}{2} c/v \]

\[
= \omega^+_1 D_1 n_1 \frac{c/v}{1 + n_i v/c} \]

\[
= \omega^+_1 D_1 \frac{1}{n_1 + c/v} \quad (D3e)
\]

\[
= \omega^+_1 D_1 n_1 \frac{1 + v^2/c^2}{n_1 + c/v} \quad (D3f)
\]

\[
= \frac{\omega^+_1 D_1 n_1}{n_1 + c/v} \quad (D3g)
\]

\[
= \omega^+_1 D_1 \frac{1 + v^2/c^2}{n_1 + c/v} \quad (D3h)
\]

The quarter-stack condition \(\bar{\phi}_{1,2} = \pi/2\) is therefore written as:

\[
\frac{\omega^+_1 D_1 n_1}{n_1 + c/v} = \pi/2, \quad (D3i)
\]

or, in terms of the period:

\[
D_1 = \frac{T^+_1}{4n_1} (n_1 + c/v). \quad (D3j)
\]

for the purely temporal case, \(v = \infty\) and the quarter-wave condition becomes a quarter-period one: \(D_1 = T^+_1/4\).

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