FACTORIZATION OF PLATONIC POLYTOPES
INTO CANONICAL SPHERES

Richard H. Hammack
Department of Mathematics and Applied Mathematics
Virginia Commonwealth University
Richmond, VA 23284-2014 USA
rhammack@vcu.edu

Paul C. Kainen
Department of Mathematics and Statistics
Georgetown University
Washington, D.C. 20057 USA
kainen@georgetown.edu

Abstract

Factorization into spheres is achieved for skeleta of the simplex, cube, and cross-polytope, both explicitly and using Keevash’s proof of existence of designs.

Key Phrases: Decomposition of skeleta, Steiner triples, Hanani quadruples

Decomposing a polytope’s 2-skeleton (i.e., partitioning its 2-cells) into closed manifolds was studied in [1], [3], [5], [7], [9], while in [6] we found factorizations of hypercube 2-skeleta into boundaries of (pairwise isomorphic) 3-cubes. Here we obtain such sphere factorizations for k-skeleta of all (non-exceptional) Platonic polytopes both explicitly for low values of k and, as a consequence of Keevash’s result [11], also existentially with k ≥ 1 arbitrary. Note we only use the case of multiplicity λ = 1 of [11].

Let ∆n denote the n-dimensional simplex, whose 1-skeleton is the graph K_{n+1}. Let O_n denote the n-dimensional cross-polytope, whose 1-skeleton is the graph K_{2n} − F, where F is a 1-factor. Let Q_n denote the n-dimensional hypercube, whose 1-skeleton is the graph Q_n, the n-fold graph Cartesian product of K_2. These are the three families of Platonic polytopes.

For any n-dimensional polytope A and nonnegative integer k, let A^k denote the k-skeleton of A; see, e.g., [2]. If L and K are polytopal ℓ-complexes,
let $L|K$ mean that $L$ factors $K$: that is, $K$ and $L$ are of equal dimension $\ell$ and $K$ is the union of subcomplexes $L_1, \ldots, L_r$, each isomorphic to $L$, such that every $\ell$-face of $K$ is contained in exactly one of the $L_i$. An $\ell$-complex is even if each $(\ell-1)$-face is in a positive even number of $\ell$-faces.

For $X \in \{\Delta, \Omega, \mathcal{Q}\}$ we study the factorization of $K = X_\ell^n$ by $L = X_{\ell+1}^\ell$ for $\ell \leq n-1$. We call $L$ the canonical $\ell$-sphere as it is the boundary of a type $X$ polytope of dimension $\ell+1$. For $\ell \geq 1$, such factorizations always exist for $\Omega$ as the proof of the first theorem below shows explicitly.

Observe, however, that these canonical spheres might not be composed of the minimum number of $\ell$-faces. Indeed, by Theorem 1 below, the 1-skeleton of $\Omega_3$ is factored into three 4-cycles, rather than four 3-cycles.

Clearly, $X_\ell^n$ even is a necessary condition for a factorization into spheres. The $\ell$-skeleton of an $n$-simplex (or an $n$-cube) is an even complex if and only if $n-\ell+1$ (the number of $\ell$-faces containing an $(\ell-1)$-face) is positive and even; for the cross-polytope $\Omega_n$, the $\ell$-skeleton is even for all $1 \leq \ell < n$. Evenness is automatic for $\Omega_\ell^n$ as each $(\ell-1)$-face is a simplex and is part of two $\ell$-faces (also simplexes) one for each vertex in a copy of $K_2$.

The following extends [5] for $\ell = 2$; cf. Speer [13], which partitions $\Omega_2^n$ into face-disjoint surfaces that contain the 1-skeleton $\Omega_1^n$; such surfaces are called 1-Hamiltonian. Observe that our result does not depend on [11].

**Theorem 1.** If $1 \leq \ell < n$, then $\Omega_\ell^{\ell+1} | \Omega_\ell^n$.

**Proof.** For each $k \geq 1$, $\partial \Omega_k$ is the iterated topological join of $k$ copies of the 2-point complex $K_2$, which is an $(k-1)$-sphere. The $\ell$-skeleton of $\Omega_n$ is the $\ell$-cell-disjoint union of the $n \choose \ell+1$ copies of $\Omega_\ell^{\ell+1}$ formed by the iterated join of each $(\ell+1)$-element subset of the set $n K_2$.

For the simplex, we use the theory of Steiner-type configurations $(v, k, \ell)$, which consist of a $v$-element set $X$ endowed with a family of $k$-element subsets of $X$ (called blocks) such that each $\ell$-subset of $X$ is in exactly one block. The case $k = \ell+1$ will be what we apply to factor $\Delta_{v-1}^{\ell-1}$ by $\Delta_{\ell-1}^{\ell-1}$.

The translation from design to combinatorial factorization works for special cases, such as the Steiner triple systems as we noted in [6]. But by Keevash [11] (see also Gowers [4] and Kalai [10]), this sort of factorization must occur almost always for the $\ell$-skeleta of simplexes and, hence, of cubes, as we shall explain. Write $a | b$ to mean $b$ is an integer multiple of $a$.

Define the **divisibility set** (e.g., [8]) to be the set of all feasible values

\[
\mathcal{D}(k, \ell) := \left\{ v : \binom{k-h}{\ell-h}, \binom{v-h}{\ell-h} \mid 0 \leq h \leq \ell-1 < k \leq v \right\},
\]  

(1)
Four rotations of this configuration by angles of $k\frac{\pi}{2}$ for $k = 0, 1, 2, 3$.

Four rotations of this configuration by angles of $k\frac{\pi}{4}$ for $k = 0, 1, 2, 3$.

Two rotations of this configuration by angles of 0 and $\frac{\pi}{4}$.

Figure 1: Factoring $\Delta^2_7$ into 14 2-face-disjoint tetrahedral boundaries $\Delta^2_3$

By [11], for $1 \leq \ell \leq k$, \exists finite set $X(k, \ell) \subset D(k, \ell)$ such that
\[ \exists (v, k, \ell)\text{-configuration} \iff v \in D(k, \ell) \setminus X(k, \ell). \quad (2) \]

We interpret the combinatorics geometrically as applying to faces of the simplex. Since the $(j+1)$-sets from the $n+1$ vertices of $\Delta_n$ correspond to the $j$-faces of $\Delta_n$, we obtain a factorization into canonical spheres.

Theorem 2. For $1 \leq \ell$ and, assuming Keevash’s result [11],
\[ \Delta^\ell_{\ell+1} \mid \Delta^\ell_n \iff n + 1 \in D(\ell+2, \ell+1) \setminus X(\ell+2, \ell+1). \quad (3) \]

Any $(v, 3, 2)$-configuration is a Steiner triple system (as observed in [11]), and corresponds to a factorization of the 1-skeleton of $\Delta_{v-1}$ (i.e., $K_v$) into edge-disjoint 3-cycles, the canonical (and minimum) spheres in 1-dimension.

For the case $(v, 4, 3)$, corresponding to the $(v-1)$-simplex, Hanani constructed a family of 4-sets (quadruples) that cover each 3-set exactly once, and he proved in [8] that they exist for $v \geq 4$ if and only if $v \equiv 2$ or 4 (mod 6), so again independent of [11], we can characterize when the 2-skeleton of an $n$-simplex is factored by the boundary of a tetrahedron:
\[ \Delta^2_4 \mid \Delta^2_n \iff n \equiv 1 \text{ or } 3 \text{ (mod 6)}, n \geq 3. \quad (4) \]

See Fig. 1 for an illustration of the first non-trivial case of [11].
In order to factor cube-skeleta, we use an “exponentiation” method that goes back to Danzer [3], [12]. If \( \sigma = (i_1, \ldots, i_{k+1}) \) is any \( k \)-simplex in the \( n \)-simplex with vertex set \([n+1] := \{1, \ldots, n+1\}\), let
\[
2^\sigma := A_1 \times \cdots \times A_{n+1},
\]
where \( A_i = [0, 1] \) if \( i \in \sigma \), \( A_i = \{0, 1\} \) if \( i \notin \sigma \), and \( \times \) is topological product. Thus, \( 2^\sigma \) is a family of \( 2^{n-k} \) pairwise-vertex-disjoint \((k+1)\)-dimensional cubes in the \((n+1)\)-cube. Let \( 2^K \) be the union of the \( 2^\sigma \) over all simplexes \( \sigma \in K \) for \( K \) a simplicial complex. Then we have (see Kühnel & Schulz [12])
\[
(a) \ 2^{\Delta_{n-1}^\ell} = Q_n^\ell; \quad (b) \ 2^{\partial K} = \partial (2^K); \quad \text{and}
\]
\[
2^K \text{ is a manifold } \iff \text{K is a combinatorial sphere.}
\]

By (6), (a) and (b), a factorization of the \((\ell-1)\)-skeleton of the \((v-1)\)-simplex by the boundary of \( \Delta_\ell \) maps under the exponential correspondence to a factorization of the \( \ell \)-skeleton of the \( n \)-cube into subcomplexes isomorphic to \( \partial(Q_{\ell+1}^v) \). We used this in [9] to factor the 2-skeleton of a cube by the boundary of a 3-cube exactly when Steiner triples exist. In [7] we used (7) to show that, for \( n \) even, the Hamiltonian factorization of the graph \( K_{n+1} = \Delta_n^1 \) into spanning 1-spheres yields a factorization of the 2-skeleton of \( Q_{n+1}^1 \) into orientable minimum-genus surfaces which are 1-Hamiltonian.

Similarly, Hanani’s factorization (4) of the 2-skeleton of the simplex gives
\[
Q_n^3 | Q_n^4 \iff n \geq 4, \ n \equiv 2 \text{ or } 4 \pmod{6}.
\]

Again, (8) doesn’t depend on [11], while by Theorem [2] the exponential correspondence (5), and using [11], we have

**Theorem 3.** For \( 1 \leq \ell \), if \( n \in D(\ell+1, \ell) \setminus X(\ell+1, \ell) \), then \( Q_{\ell+1}^\ell | Q_n^\ell \).

We conjecture that the condition is also necessary. Note that the result for \( \ell = 1 \) follows from the true though not stated case of Theorem [2] for \( \ell = 0 \) since \( D(2, 1) \) is the set of positive even integers and \( X(2, 1) = \emptyset \).

By the results of [11], factorability into canonical spheres is ubiquitous for skeleta of Platonic polytopes. A geometric explanation for this phenomenon (in special cases) would imply the existence theorem (in those cases).
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