ONE BROWNIAN STOCHASTIC FLOW

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Abstract. The weak limits of the measure-valued processes organized as a mass carried by the interacting Brownian particles are described.

Introduction

Let $W$ be an $\mathbb{R}^d$-valued Wiener sheet on $\mathbb{R}^d \times [0; 1]$. Assume that $\varphi \in C_0^\infty(\mathbb{R}^d)$ is the spherically symmetric nonnegative function with the property

$$\int_{\mathbb{R}^d} \varphi(u) du = 1.$$

Define for $\varepsilon > 0$

$$\varphi_\varepsilon(u) = \varepsilon^{-\frac{d}{2}} \varphi(\varepsilon^{-1} u)^\frac{1}{2}, u \in \mathbb{R}^d.$$

Note that $\varphi_\varepsilon \in C_0^\infty(\mathbb{R}^d)$ under every $\varepsilon > 0$. Let us consider now the equation

$$\begin{cases}
  dx_\varepsilon(u, t) = \int_{\mathbb{R}^d} \varphi_\varepsilon(x_\varepsilon(u, t) - q) W(dq, dt), \\
  x_\varepsilon(u, 0) = u.
\end{cases}$$

The solution $x_\varepsilon$ has two important properties. The first one is that $x_\varepsilon$ is the flow of the homeomorphisms [1] and the second one is that for every $u \in \mathbb{R}^d \{x_\varepsilon(u, t); t \geq 0\}$ is the Wiener process. Really, denote by $\{\mathcal{F}_t; t \geq 0\}$ the flow of $\sigma$-field generated by $W$ in a usual way. Then $\{x_\varepsilon(u, t); t \geq 0\}$ is a continuous $\mathcal{F}_t$-martingale with the matrix characteristics

$$\langle x_\varepsilon(u, \cdot) \rangle_t = \int_0^t \int_{\mathbb{R}^d} \varphi_\varepsilon^2(x_\varepsilon(u, s) - q) dq ds I = tI,$$

$I$ is the identity matrix. Hence $\{x_\varepsilon(u, t); t \geq 0\}$ is the Wiener process. Note that for different $u_1, u_2 \in \mathbb{R}^d x_\varepsilon(u_1, \cdot)$ and $x_\varepsilon(u_2, \cdot)$ are not independent. Their joint characteristic equals to

$$\langle x_\varepsilon(u_1, \cdot), x_\varepsilon(u_2, \cdot) \rangle_t = \int_0^t \int_{\mathbb{R}^d} \varphi_\varepsilon(x_\varepsilon(u_1, s) - q) \varphi_\varepsilon(x_\varepsilon(u_2, s) - q) dq ds I.$$

So the flow $x_\varepsilon$ now consists of the Wiener processes which do not stick together. Let us mention that the support $\varphi_\varepsilon$ tends to the origin when $\varepsilon \to 0^+$. So, one can expect

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that in the limit $x_\varepsilon$ turns into the family of independent Wiener processes. But from
another side if $d = 1$ and $u_1 < u_2$, then, as it was mentioned above, $x_\varepsilon(u_1, t) < x_\varepsilon(u_2, t)$
for every $t$ with probability one. Consequently, on the non-normal level $x_\varepsilon$ in the limit
turns into the family of the Wiener particles, which start from every point of the space
and move independently up to the meeting after which continue the motion together.
The formal realization of this idea on the level of the description of the particle motion
meets some technical troubles (see [2]). So we propose here to speak not about the
particles but about the mass which they carry. In another words we will consider the
measure-valued processes related to the flow $x_\varepsilon$ and their weak limit under $\varepsilon \to 0 +$. Let
us fix the probability measure $\mu_0$. Define the measure-valued process $\{\mu_t^\varepsilon : t \in [0, 1]\}$ in
the following way
\begin{equation}
\mu_t^\varepsilon = \mu_0 \circ x_\varepsilon(\cdot, t)^{-1}.
\end{equation}
We will consider this process in the different spaces of measures. In order to introduce
this spaces define for $u \in \mathbb{R}^d$
\begin{align*}
\varphi_0(u) &= \frac{\|u\|}{1 + \|u\|}, \\
\varphi_n(u) &= \|u\|^n, \ n \geq 1.
\end{align*}
Denote by $\mathcal{M}_n$ the space of all probability measures $\mu$ on $\mathbb{R}^d$ which have the property
\[\int_{\mathbb{R}^d} \varphi_n(u) \mu(du) < +\infty.\]
Note that $\mathcal{M}_0$ contains all probability measures. Define the Wassershtain distance on $\mathcal{M}_n$ by the rule
\[\forall \mu, \nu \in \mathcal{M}_n : \]
\[\gamma_n(\mu, \nu)^{\|x\|_1} = \inf_{x \in C(\mu, \nu)} \int_{\mathbb{R}^d} \varphi_n(u - v) x(du, dv),\]
where the infimum is taken over all probability measures $x$ on $\mathbb{R}^d \times \mathbb{R}^d$ which have $\mu$ and $\nu$
as their marginal distributions. It is known, that $(\mathcal{M}_n, \gamma_n)$ is the complete
separable metric space [3] for every $n \geq 0$. The convergence in $\gamma_0$ is equivalent to the
weak convergence. Since $x_\varepsilon$ is continuous with respect to both variables then $\mu^\varepsilon$ is
continuous process in $\mathcal{M}_0$ with probability one. Here we will consider the weak limits of
$\mu^\varepsilon$ under $\varepsilon \to 0 +$.

**Main result**

In what follows we will need in the conditions for the weak compactness of the measure-valued
processes from [4]. Here we present the necessary notations and statements. Let
$C_b(\mathbb{R}^d)$ and $C_0(\mathbb{R}^d)$ be the spaces of bounded and finite continuous functions on $\mathbb{R}^d$.
Consider the sequence $\{f_n : n \geq 1\}$ from $C_0(\mathbb{R}^d)$ such that:
1) $\forall n \geq 1 : \max_{\mathbb{R}^d} |f_n| \leq 1$,
2) for every $\varphi_n \in C_0(\mathbb{R}^d)$ which is bounded by 1 there is the subsequence $\{f_{n_k} : k \geq 1\}$
such that
\[\max_{\mathbb{R}^d} |\varphi - f_{n_k}| \to 0, \ k \to \infty.\]
Now let $\{g_n : n \geq 1\}$ be the sequence from $C_b(\mathbb{R}^d)$ such, that:
1) $\forall x \in \mathbb{R}^d, n \geq 1 : 0 \leq g_n(x) \leq 1$,
2) $\forall x \in \mathbb{R}^d, \|x\| \leq n : g_n(x) = 0$,
3) $\forall x \in \mathbb{R}^d, \|x\| \geq n \geq 1 : g_n(x) = 1$.
The following statement was proved in [4].
Theorem 1. The family \( \{ \xi_\alpha; \alpha \in U \} \) of random elements in \( C([0; 1], \mathcal{M}_0) \) is weakly relatively compact if and only if the following conditions hold:

1) for every \( k \geq 1 \) the set of the random processes \( \{ \langle \xi_\alpha, f_k \rangle; \alpha \in U \} \) is weakly relatively compact in \( C([0; 1]) \),

2) the set \( \{ \langle \xi_\alpha, g_k \rangle; \alpha \in U, k \geq 1 \} \) is weakly relatively compact in \( C([0; 1]) \),

3) \( \forall t \in [0; 1] \forall \varepsilon > 0 : \sup_{\alpha \in U} P\{ \langle \xi_\alpha(t), g_k \rangle > \varepsilon \} \rightarrow 0, k \rightarrow \infty \).

Applying this theorem to the our processes \( \{ \mu^\varepsilon \} \) we can get the following statement.

Theorem 2. Let the initial measure \( \mu_0 \in \mathcal{M}_n, n > 2 \). Then the family \( \{ \mu^\varepsilon; \varepsilon > 0 \} \) is weakly compact in \( C([0; 1], \mathcal{M}) \).

Proof. Let us use the criterion of the weak compactness for the measure-valued processes from theorem 1. Take the functions \( \{ f_k; k \geq 1 \} \) and \( \{ g_k; k \geq 1 \} \) in such a way, that

1) for every \( k \geq 1 \), \( f_k \) and \( g_k \) satisfy the Lipshitz condition,

2) the Lipshitz constant for \( g_k \) equals 1 for every \( k \geq 1 \).

Now check the weak compactness of the families \( \{ \langle f_k, \mu^\varepsilon \rangle; \varepsilon > 0 \} \) and \( \{ \langle g_k, \mu^\varepsilon \rangle; \varepsilon > 0, k \geq 1 \} \) in \( C([0; 1]) \). Consider the function \( h \) on \( \mathbb{R}^d \) which satisfies the Lipshitz condition with the constant \( C \). For the such function

\[
E|\langle h, \mu^\varepsilon_1 \rangle - \langle h, \mu^\varepsilon_2 \rangle|^n \leq C^n \int_{\mathbb{R}^d} E\|x_\varepsilon(u, t_1) - x_\varepsilon(u, t_2)\|^n \mu_0(du) \leq C^n \cdot K_n \cdot |t_2 - t_1|^{\frac{n}{d}},
\]

where the constant \( K_n \) depends on only from \( n \) and the dimension \( d \). This estimation together with relation

\[
\lim_{k \rightarrow \infty} \langle g_k, \mu_0 \rangle = 0
\]
gives us that the conditions 1) and 2) of the theorem 1 hold. In order to check the condition 3) consider

\[
E \int_{\mathbb{R}^d} u^n \mu^\varepsilon_1(du) = E \int_{\mathbb{R}^d} u + (x_\varepsilon(u, t) - u)\|u\|^n \mu^\varepsilon_1(du) \leq D \left( \int_{\mathbb{R}^d} \|u\|^n \mu_0(du) + 1 \right),
\]

where the constant \( D \) depends only on the dimension \( d \) and \( n \). Then

\[
\lim_{k \rightarrow \infty} \sup_{\varepsilon > 0} P\{ \langle g_k, \mu^\varepsilon_1 \rangle > \delta \} \leq \lim_{k \rightarrow \infty} \sup_{\varepsilon > 0} \frac{1}{\delta} E\langle g_k, \mu^\varepsilon_1 \rangle \leq
\]
\[
\leq \lim_{k \to \infty} \sup_{\varepsilon > 0} \frac{1}{k^n} E \int_{R^d} \|u\|^n \mu^*_k(du) \leq \lim_{k \to \infty} \frac{1}{k^n} D \left( \int_{R^d} \|u\|^n \mu_0(du) + 1 \right) = 0
\]
for every \( t \in [0; 1] \) and \( \delta > 0 \). So the condition 3) holds and the theorem is proved.

In order to understand how many limit points the family \( \{\mu^\varepsilon\} \) admits under \( \varepsilon \to 0+ \) let us study the behaviour of the finite-time processes \( \{\bar{x}_\varepsilon(t) = (x_\varepsilon(u_1, t), \ldots, x_\varepsilon(u_n, t)); t \in [0; 1]\} \) under \( \varepsilon \to 0+ \). From now we will consider the case \( d = 1 \). We begin with the construction of the Markov process in \( R^n \) which can serve as a weak limit of \( \bar{x}_\varepsilon \). Nonformally this process can be described as follows. In the space \( R^n \) we consider the usual Wiener process up to the first time when the some of its coordinates became to be equal. After this time process turns into the Wiener process on the hyperplane, where this coordinates remains equal. This procedure goes on until we get the one-dimensional Wiener process. From that moment our process coincide with this Wiener process. In order to construct such a random process rigorously and check that it is the unique weak limit of \( \bar{x}_\varepsilon \) let us prove the next theorem.

**Theorem 3.** The family \( \{\bar{x}_\varepsilon; \varepsilon > 0\} \) weakly converges under \( \varepsilon \to 0+ \) in the space \( C([0; 1], R^n) \).

Let the weak compactness of \( \{\bar{x}_\varepsilon; \varepsilon > 0\} \) follows from the arguments which were mentioned in the proof of the theorem 2. So we only have to prove that under \( \varepsilon \to 0+ \) there is only one limit point. Fix \( \varepsilon > 0 \) and consider for \( x \in R \) the function
\[
g_\varepsilon(x) = \int_R \varphi_\varepsilon(x + q) \varphi_\varepsilon(q) dq.
\]
The process \( \bar{x}_\varepsilon \) is a diffusion process in \( R^n \) with zero drift and the diffusion matrix
\[
A(\bar{x}) = (g_\varepsilon(x_i - x_j))_{i,j=1}^n,
\]
where \( \bar{x} = (x_1, \ldots, x_n) \). It results from the condition on \( \varphi_\varepsilon \) that \( A \) coincide with the identity matrix on the set
\[
G_\varepsilon = \{\bar{x}: |x_i - x_j| > 2\varepsilon, i \neq j\}.
\]
Let us define the random moment \( \tau_\varepsilon \) as the first exit time from \( G_\varepsilon \). In our case \( \bar{w}(0) \in G_\varepsilon \) because we take the different initial values \( u_1, \ldots, u_n \). It results from theorem 1.13.2 [5] that the distribution of the process \( \{\bar{w}(\tau_\varepsilon t); t \in [0; 1]\} \) coincide with the distribution of the process \( \{\bar{w}(\tau_\varepsilon t); t \in [0; 1]\} \), where \( \bar{w} \) is the Wiener process in \( R^n \) starting from \( (u_1, \ldots, u_n) \) and \( \tau_\varepsilon \) has the same meaning for \( \bar{w} \) as for \( \bar{x}_\varepsilon \). Now suppose that \( u_1 < \ldots < u_n \) with out loss of generality. Consider in \( C([0; 1], R^n) \) the set
\[
G_\delta = \{\bar{f}: f_i(0) = u_i, i = 1, \ldots, n, \bar{f}(t) \in G_\delta, t \in [0; 1]\}.
\]
Note that \( G_\delta \) is an open set in \( C([0; 1], R^n) \) for sufficiently small \( \delta > 0 \) (we use the usual uniform norm in the space of continuous functions). Let \( \varkappa \) be the limit point of the distributions \( \bar{x}_\varepsilon \) under \( \varepsilon \to 0+ \). It results from the previous considerations that for all sufficiently small \( \varepsilon > 0 \) the restriction on \( G_\delta \) of the distribution of \( \bar{x}_\varepsilon \) coincide with the
restriction of the Wiener measure related to the initial value \((u_1, \ldots, u_n)\). Consequently, \(\kappa\) coincide with the Wiener measure on the set \(G_\delta\) for arbitrary \(\delta > 0\). Denote by \(G\) the closure of the union

\[
\bigcup_{\delta > 0} G_\delta.
\]

Note that for all \(\varepsilon > 0\)

\[
P\{\vec{x}_\varepsilon \in G\} = 1.
\]

So, by characterization of the weak convergence [6],

\[
\kappa(G) = 1.
\]

It remains to describe \(\kappa\) on the boundary of \(G\). In order to do this let us recall that the random process, which is obtained from \(\vec{x}_\varepsilon\) by choosing some of its coordinates has the same properties as \(\vec{x}_\varepsilon\) under \(\varepsilon \to 0^+\). Hence the measure \(\kappa\) has the following properties. The every coordinate has the Wiener distribution. Any two coordinates move as an independent Wiener processes up to their meeting and move together after this moment.

Now the uniqueness of \(\kappa\) can be obtained by induction. Theorem is proved.

This theorem has the following consequence.

**Corollary.** The measure-valued processes \(\{\mu^\varepsilon\}\) constructed in theorem 2 converge weakly under \(\varepsilon \to 0^+\).

**Proof.** In view of the theorem 3 we have only to check the uniqueness of the limit point for \(\{\mu^\varepsilon\}\) under \(\varepsilon \to 0^+\). Let \(\{\nu_t; t \in [0; 1]\}\) be the measure-valued process representing the limit point of \(\{\mu^\varepsilon\}\) under \(\varepsilon \to 0^+\). Take \(t_1, \ldots, t_d \in [0; 1]\) and for bounded continuous functions \(\varphi_1, \ldots, \varphi_d\) consider the value

\[
E \prod_{k=1}^d \int_R \varphi_k(u) \nu_{t_k}(du).
\]

Note, that the set of all such values uniquely define the distribution of \(\{\nu_t; t \in [0; 1]\}\).

Due to the previous theorem and the Lebesgue dominated convergence theorem

\[
E \prod_{k=1}^d \int_R \varphi_k(u) \nu_{t_k}(du) = \lim_{n \to 0^+} E \prod_{k=1}^d \int_R \varphi_k(u) \mu_{t_k}^\varepsilon(du) =
\]

\[
= \lim_{n \to 0^+} E \prod_{k=1}^d \int_R \varphi_k(x_{\varepsilon_n}(u, t_k)) \mu_0(du) =
\]

\[
= \lim_{n \to 0^+} \int_{R^d} \prod_{k=1}^d \varphi_k(x_{\varepsilon_n}(u_k, t_k)) \mu_0(du_1) \ldots \mu_0(du_k) =
\]

\[
= \int_{R^d} \lim_{n \to 0^+} \prod_{k=1}^d \varphi_k(x_{\varepsilon_n}(u_k, t_k)) \mu_0(du_1) \ldots \mu_0(du_k).
\]

Since the limit in the last integral does not depend on the choice of the sequence \(\varepsilon_n \to 0^+, n \to \infty\), then the value of (4) is uniquely defined. Hence we have only one limit point for \(\{\mu^\varepsilon\}\) under \(\varepsilon \to 0^+\) and the our statement is proved.
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