Relations among divisors on the moduli space of curves with marked points

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1. Introduction

Let $\mathcal{M}_g$ be the coarse moduli space of stable curves of genus $g$. Eisenbud, Harris and Mumford proved a relation between certain divisors on $\mathcal{M}_g$ (the Brill-Noether divisors, to be described below). Calculating their classes in $\text{Pic}(\mathcal{M}_g) \otimes \mathbb{Q}$, they succeeded in proving that $\mathcal{M}_g$ is of general type for $g > 23$, $g + 1$ composite. In subsequent work, the restriction that $g + 1$ be composite was removed.

In this paper, I will generalize their relation to $\mathcal{M}_{g,n}$, the moduli space of stable curves of genus $g$ with $n$ marked points. This does not yield new results on the Kodaira dimension of $\mathcal{M}_{g,n}$, as the “divisors of Brill-Noether type”, which I introduce below, are less effective for this purpose than certain other divisors which I studied in my dissertation [L]. (I expect to publish the other results of [L] shortly.)

The remainder of this introduction will be devoted to stating the results. For basic facts on $\mathcal{M}_g$ and $\mathcal{M}_{g,n}$, the reader is referred to [HM] and [K] respectively.

**Definition.** Fix a nonnegative integer $g$, and let $r > 1, d > 1$ be integers such that $g - (r + 1)(g - d + r) = -1$. (Here, the left side is the expected dimension of the space of $g^r_d$'s on a curve of genus $g$.) A divisor of Brill-Noether type on $\mathcal{M}_g$ is a codimension-1 component of the locus of curves which have an admissible $g^r_d$.

(All $g^r_d$'s on nonsingular irreducible curves are admissible; in general, an admissible $g^r_d$ is a $g^r_d$ on each component with certain ramification conditions at the singular points of the curve. Again, see [HM] for details.)

The Brill-Noether Ray Theorem of Eisenbud, Harris, and Mumford then asserts that, for fixed $g$, all of these divisors on $\mathcal{M}_g$ are linearly dependent, and calculates their class. We generalize their definition and theorem as follows:

**Definition.** Fix $g$ and $n$, and let integers $r, d$ and sequences $\{Z_1\}, \ldots, \{Z_n\}$ of length $r + 1$ be such that

$$g - (r + 1)(g - d + r) - \sum Z_{i,j} = -1.$$  

(The left-hand side is the expected dimension of the set of $g^r_d$'s on a curve of genus $g$ with ramification sequences $\{Z_1\}, \ldots, \{Z_n\}$ at points $p_1, \ldots, p_n$.) A divisor of Brill-Noether type on $\mathcal{M}_{g,n}$ is a divisorial component of the locus of curves and sets of points such that there is a $g^r_d$ on the curve whose ramification sequences at the $p_i$ are at least $\{Z_i\}$.

**Theorem 1.1** For any $n > 2$, every divisor of Brill-Noether type on $\mathcal{M}_{g,n}$ is a linear combination of pullbacks of divisors of Brill-Noether type from $\mathcal{M}_{g,2}$. Also, the space of divisors of Brill-Noether type on $\mathcal{M}_{g,1}$ has dimension 2, for any $g > 2$. □

**Corollary 1.2** The dimension of the subspace of $\text{Pic}(\mathcal{M}_{g,n}) \otimes \mathbb{Q}$ spanned by the divisors of Brill-Noether type is $1 + n + \binom{n}{2}$. □

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2. The First Case

Throughout the paper, we work over an algebraically closed field of characteristic 0 (though this is surely unnecessary), and we deal only with genus $\geq 3$. We start by introducing some notation. Arbarello and Cornalba have given a basis for $\text{Pic}(\mathcal{M}_{g,n}) \otimes \mathbb{Q}$ [AC, Thm. 2]:

**Theorem 2.1** $\text{Pic}(\mathcal{M}_{g,n})$, the Picard group of the moduli stack $\mathcal{M}_{g,n}$, is free on the following generators: $\lambda, \delta_0, \psi_i (1 \leq i \leq n)$, and $\delta_{i,S} (0 \leq i \leq \lfloor g/2 \rfloor), S \subseteq \{1, 2, \ldots, n\}, \text{card } S > 1$ if $i = 0$. □

Here $\lambda$ is the pullback of $\lambda$ on $\mathcal{M}_g$, $\delta_0$ the divisor corresponding to the locus of curves with a nondisconnecting node, $\delta_{i,S}$ the divisor corresponding to the locus of curves with a node whose removal leaves
one component of genus \(i\) with precisely the marked points indexed by \(S_i\), and \(\psi_i\) is the divisor class on the moduli stack which takes the value \(-\pi_i(\sigma_i^2)\) on the family \(\mathcal{X} \to B\) with the \(\sigma_i\) as sections.

Usually we will prefer to work with \(\omega_1\), the relative dualizing sheaf of the \(i\)th projection map from \(\overline{\mathcal{M}}_{g,n}\) to \(\overline{\mathcal{M}}_{g,n-1}\). If we replace \(\psi_i\) by \(\omega_i\) in the above, the theorem remains true, because \(\omega_i = \psi_i - \sum_{i \in S} \delta_0;S_i\), as will be seen below.

We will be pulling back divisors, so it also seems appropriate to state the results concerning this. Again, the answer is given in [AC, p. 161]:

**Theorem 2.2** The pullback map is given as follows:

\[
\begin{align*}
\pi_n^* \lambda &= \lambda, \\
\pi_n^* \delta_0 &= \delta_0, \\
\pi_n^* \omega_i &= \omega_i, \\
\pi_n^* \psi_i &= \psi_i - \delta_0;\{i\}
\end{align*}
\]

except that

\[
\pi_1^* \delta_{g/2;\emptyset} = \delta_{g/2;\emptyset}
\]

for \(n = 1\).

For notational simplicity, we have only stated this for \(\pi_n\), but the action of the symmetric group \(S_n\) on \(\overline{\mathcal{M}}_{g,n}\) makes it easy to describe the effects of the other \(\pi_i\). Also observe that, by an easy induction starting from \(\psi = \omega\) on \(\overline{\mathcal{M}}_{g,1}\), we get

\[
\psi_i = \omega_i + \sum_{i \in S \subseteq \{1, 2, \ldots, n\}, S \neq \{i\}} \delta_0;S
\]

on \(\overline{\mathcal{M}}_{g,n}\), as asserted above.

We now start by dealing with the case \(n = 1\).

**Definition.** Let \(BN\) be the divisor class

\[
(g + 3)\lambda - \frac{(g + 1)}{6} \delta_0 - \sum_{i=1}^{\lfloor g/2 \rfloor} i(g - i) \delta_i
\]

on \(\overline{\mathcal{M}}_g\).

If \(g + 1\) is not prime, then \(BN\) is a positive multiple of the class of an effective divisor, namely that of the union of the codimension-1 components of the locus of curves admitting a \(g_d\), where \((r+1)(g-d+r) = g + 1\) [EH, Thm. 1] (the condition on \(r, d, \) and \(g\) ensures that there are such components).

This will be one of the generators of the Brill-Noether space on \(\overline{\mathcal{M}}_{g,1}\) when \(g + 1\) is not prime; the other will be the Weierstrass divisor, that is, the locus of Weierstrass points. Its class was computed by Cukierman [Cuk, Thm. 2.0.12 and following remark]. His result is as follows:

**Theorem 2.3** Let \(g \geq 2\). The class on \(\overline{\mathcal{M}}_{g,1}\) of the Weierstrass divisor \(W\), which is the closure of the divisor on \(\overline{\mathcal{M}}_{g,1}\) given by Weierstrass points of smooth curves, is

\[
\frac{g(g + 1)}{2} \omega - \lambda - \sum_{i=1}^{g-1} \frac{(g - i)(g - i + 1)}{2} \delta_i
\]

**Proof.** See [Cuk]. Alternatively, this can be done by the method of test curves.

We recall the Plücker formula, which counts the ramification points of a \(g_d\) on a smooth curve. It asserts that if \(C\) is a smooth curve of genus \(g\) and \(V\) a \(g_d\), then \(\sum_{p \in C} \beta(V, p) = (r + 1)(d + r - 1)\). Here \(\beta(V, p) = \sum a_i(V, p) - i\), where the \(a_i\) give the sequence of orders of vanishing of \(V\) at \(p\).

**Proposition 2.4** The classical Plücker formula remains valid for reducible curves with no nondisconnecting nodes. (Of course, \(\overline{g_d}\) is understood to mean “limit linear series”. Also, the ramification conditions at the node imposed by the definition of “limit linear series” are not considered as contributing to ramification.)
Proof. By induction, it suffices to consider curves with exactly two components. Applying the Plücker formula to each component separately, we get a total of \((r+1)(2d+r(g-2))\) there. However, the definition of limit linear series imposes exactly \((r+1)(d-r)\) conditions at the node, leaving \((r+1)(d+r(g-1))\) in total, as claimed.

**Definition.** Let \(r\) and \(d\) be positive integers such that \(a = g - (r+1)(d-r) \geq -1\), and \(Z\) a possible ramification sequence for a \(g_a^d\) which sums to \(a\). We define the divisor \(D_{g,r,d,Z}\) on \(\overline{\mathcal{M}}_{g,1}\) to be the divisor of curves and points \((C,p)\) such that \(C\) admits a \(g_a^d\), \(V\), whose ramification sequence at the marked point equals or exceeds \(Z\).

The \(D_{g,r,d}\) are the divisors of Brill-Noether type on \(\overline{\mathcal{M}}_{g,1}\). In order to study the \(D_{g,r,d}\), we consider the following two maps to \(\overline{\mathcal{M}}_{g,1}\):

1. Map \(\overline{\mathcal{M}}_{0,g+1}\) to \(\overline{\mathcal{M}}_{g,1}\) by attaching a fixed elliptic curve at each of the first \(g\) points.
2. Map \(\overline{\mathcal{M}}_{2,1} - \mathcal{W}\) to \(\overline{\mathcal{M}}_{g,1}\) by attaching a fixed general curve of genus \(g-2\) with two marked points.

We claim that the images of these maps are disjoint from \(D\), and that this implies the linear dependence claimed. For the first of these maps, it is enough to count ramification points; there are not enough to spare on the component of genus 0. In particular, at each point of attachment we must have \(\beta \geq r\) on this component, making at least \(gr\). Even if the remaining \((r+1)(d-r) - gr = g - (r+1)(d-r) = a\) units of ramification are all concentrated at one point, that isn’t quite enough. For the second map, this follows from the extended Brill-Noether theorem, \([EH, \text{Thm. 1.1}]\).

In order to deduce the dependence from this, it is necessary to determine the pullback map on divisors induced by the two maps given above. In both cases, most of the work takes care of itself, since these maps fit into commutative diagrams in which one arrow is the map in question, one arrow is a map for which the pullback is computed in \([EH, \text{proof of Thms. 2.1, 3.1}]\), and the other arrow(s) are easily understood. Specifically, to study the first map, insert it into a commutative diagram as follows, in which the vertical projections are simply the forgetful maps as shown:

\[
\begin{array}{ccc}
\overline{\mathcal{M}}_{0,g+1} & \xrightarrow{f^\ast} & \overline{\mathcal{M}}_{g,1} \\
\downarrow \pi_{g+1} & & \downarrow \pi_1 \\
\overline{\mathcal{M}}_{0,g} & \xrightarrow{f} & \overline{\mathcal{M}}_g
\end{array}
\]

This diagram being commutative, the two pullback maps on \(\text{Pic} \otimes \mathbb{Q}\) must be equal. Since the pullback maps on the vertical arrows are as described in Theorem 2.2, and the map on the bottom arrow is given by Eisenbud and Harris, we can easily determine \(f^\ast\) of any class on \(\overline{\mathcal{M}}_{g,1}\) pulled back from \(\overline{\mathcal{M}}_g\). For example, since \(\pi_{g+1}^\ast f^\ast \lambda = 0\) (indeed, \(f^\ast \lambda = 0\), as in \([EH]\), and since \(\pi_1^\ast \lambda = \lambda\), it follows that \(f^\ast \lambda = 0\), and likewise for \(\delta_0\) in place of \(\lambda\).

**Definition.** Let \(\theta_i \in \text{Pic} \overline{\mathcal{M}}_{0,g+1} = \sum_{S \in T} \delta_{S_i} \delta_0 S_i\), where \(T\) runs over subsets of \(\{1, \ldots, g+1\}\) of cardinality \(i + 1\) that contain \(g + 1\). Also, let \(\epsilon_i = \sum_{\text{card } S = i} \delta_{S_0} S_i\).

It is easy to see that \(\theta_i + \theta_{g-i} = \pi_{g+1}^\ast \delta_i\). (As usual, the case \(2i = g\) is an exception: then the pullback is just \(\theta_i\).) Since \(\epsilon_i = f^\ast \delta_i\) for \(i > 1\), and since for \(i < g - 1\) we have that \(f^\ast \delta_i\) is supported on \(\theta_i\), it follows immediately that \(f^\ast \delta_i = \theta_i\) in this range.

On the other hand,

\[
\pi_{g+1}^\ast f^\ast \delta_i = -\sum_{i=2}^{g-2} \frac{i(g-i)}{(g-1)} \theta_i,
\]

as follows from the calculation of \(f^\ast \delta_1\) in \([EH, \text{Thm. 3.1}]\). This, therefore, is \(f^\ast (\delta_1 + \delta_{g-1})\), and so

\[
f^\ast (\delta_{g-1}) = -\sum_{i=1}^{g-2} \frac{i(g-i)}{(g-1)} \theta_i.
\]

Finally it is necessary to compute \(f^\ast \omega\). The easiest way to do this is simply to use the fact that \(f^\ast \mathcal{W} = 0\). Using Cukierman’s formula (Theorem 2.3), one translates this into the statement that

\[
f^\ast \omega = \sum_{i=1}^{g-2} \frac{(g-i)(g-i-1)}{g(g-1)} \theta_i.
\]
Next we consider the second map, to which a similar procedure applies, complete with a similar commutative diagram:

\[
\begin{array}{ccc}
\mathcal{M}_{2,1} & \xrightarrow{g'} & \mathcal{M}_{g,1} \\
\downarrow = & & \downarrow \pi_1 \\
\mathcal{M}_{2,1} & \xrightarrow{g} & \mathcal{M}_g
\end{array}
\]

This time, according to [EH, Sect. 2], we have

\[
\begin{align*}
g^*\delta_0 &= \delta_0, \\
g^*\delta_1 &= \delta_1, \\
g^*\delta_2 &= -\omega, \\
g^*\lambda &= \lambda = \frac{\delta_0}{10} + \frac{\delta_1}{5}, \\
g^*\delta_i &= 0 \text{ for } i > 2.
\end{align*}
\]

Therefore we will get

\[
\begin{align*}
g^*\delta_0 &= \delta_0, \\
g^*\lambda &= \lambda = \frac{\delta_0}{10} + \frac{\delta_1}{5}, \\
g^*\delta_{g-1} &= \delta_1, \\
g^*\delta_{g-2} &= \omega,
\end{align*}
\]

with all \(\delta\) not yet mentioned going to 0. After all, \(\pi_1, \delta_i = \delta_i + \delta_{g-i}\), and as \(g^*\delta_i\) reflects the nodes of the curve of genus 2, the marked points are not on the genus-\(i\) side. Note also that the relation \(\lambda = \frac{\delta_0}{10} + \frac{\delta_1}{5}\) on \(\mathcal{M}_2\) pulls back to \(\mathcal{M}_{2,1}\) without any change in its appearance.

To complete the description, we show that \(g^*\omega = 0\). Using the push-pull formula, we see that it is enough to show that \(\omega \cdot g'_*C = 0\) for any curve \(C\) contained in \(\mathcal{M}_{2,1}\). But this is easy, as \(\omega\) is the self-intersection of a constant section on the component of genus \(g - 2\). The variation of the other component, which is attached at some other point, has no effect on this.

To prove the theorem, we show that the intersection of the kernels has dimension 2. This is most easily seen as follows: if we know the coefficients of \(\delta_0\) and \(\lambda\) in a divisor contained in \(\ker g^*\), that determines its coefficients of \(\delta_{g-1}\) and \(\delta_{g-2}\). However, a knowledge of these two coefficients determines the coefficient of \(\omega\) (in order that the coefficient of \(\theta_{g-2}\) in the pullback by \(f'\) be 0), and that forces the coefficients of all of the rest. Assuming that the \(\theta_i\) are independent, we conclude that the dimension is at most 2; it is at least 2 because the Weierstrass and Brill-Noether classes are contained in the kernel. (Of course, we have only shown that the Brill-Noether class is in the kernel when it is the class of an effective divisor, but one can check with no difficulty that it never survives the map \(f'^*\).

Now we prove that the \(\theta_i\) are actually independent in \(\text{Pic}(\mathcal{M}_{0,q+1}) \otimes \mathbb{Q}\). This is not quite trivial: there are relations between the different \(\delta_{0,S}\) on \(\mathcal{M}_{0,n}\). Since \(\mathcal{M}_{0,4} \cong \mathbf{P}^1\), for example, and the Picard group of \(\mathbf{P}^1\) has rank 1, we must have \(\delta_{0,1,2} = \delta_{0,1,3} = \delta_{0,1,4}\). But it is not difficult either.

To start with, only \(\theta_1\) has nonzero degree on a fiber of \(\pi_{g+1}\), so its coefficient must be 0 in any relation. Then \(\theta_{g-1}\) is the only other \(\theta\) with nonzero degree on a fiber of \(\pi_1\), so its coefficient must be 0 as well. For the rest, put a fixed number \(i\), from 1 to \(g - 3\), of the first \(g\) points on a \(\mathbf{P}^1\) together with \(p_{g+1}\), attach another \(\mathbf{P}^1\) at another point with the remaining marked points, and consider a curve in which one of these points moves along its component. This curve will intersect \(\theta_{g-1}\) (where the moving point meets another marked point on its component), \(\theta_i\) (generically), and \(\delta_{i+1}\) (where the moving point reaches the point of attachment), and the coefficient of \(\theta_{i+1}\) will be 1. Therefore, by easy induction on \(i\), all the coefficients of \(\theta_i\) from 2 to \(g - 2\) in our putative relation must be 0, and we are done.

Moreover, it is easy to see that the space spanned by these divisors is actually of dimension 2 (not 1). If \(g + 1\) is not prime, this is clear, because \(W\) has a nonzero coefficient of \(\omega\) while the pullback of the Brill-Noether class from \(\mathcal{M}_g\) does not. If \(g + 1\) is prime, it is not much harder. (Details will appear in a paper presenting the results of [L].)
3. The General Case

Recall the definition of the divisor class $BN$ on $\overline{M}_g$ as

$$(g + 3)\lambda - \frac{g + 1}{6} \delta_0 - \sum i(g - i)\delta_i.$$ 

If $g + 1$ is composite, $BN$ is effective, but otherwise it may not be. We may define divisors on $\overline{M}_{g,n}$ for any $g, n$ by taking any $r$ and $d$ for which the expected dimension of the space of $g_d^1$'s on a curve of genus $g$ is $k(\geq -1)$, and imposing exactly enough ramification conditions separately at the $n$ distinct points to reduce the expected dimension to $-1$. This produces a locus in $\overline{M}_{g,n}$ which may in general have components of various dimensions. We will refer to any codimension-1 component of any of these loci as a Brill-Noether type divisor, and to the subspace of $\text{Pic}(\overline{M}_{g,n}) \otimes \mathbb{Q}$ generated by all Brill-Noether type divisors as the Brill-Noether subspace. Observe that if we pull back a divisor of Brill-Noether type from $\overline{M}_{g,n-1}$ to $\overline{M}_{g,n}$, we get another divisor of Brill-Noether type: the same conditions are imposed at $p_1, \ldots, p_{n-1}$, and none at $p_n$. The main purpose of this paper is to study the classes of these divisors: this is what we will do in this section.

The proofs of the earlier results on linear dependence suggest a way to proceed, and we will follow it. As before, we consider maps

$$\overline{M}_{0,g+n} \to \overline{M}_{g,n}$$

given by attaching a fixed elliptic curve at each of the first $g$ points, and

$$\overline{M}_{2,1} \to \overline{M}_{g,n}$$

given by attaching a fixed $n + 1$-pointed curve of genus $g$ at a marked point. And as before, the Plücker formula shows that the first map misses all of these divisors, while the extended Brill-Noether theorem proves that the image of $\overline{M}_{2,1} - W$ by second map has trivial intersection with them. Therefore, their classes must lie in the intersections of the kernels of these two maps. These facts are not quite enough to characterize the Brill-Noether subspace, though. We will prove the following theorem instead:

**Theorem 3.1** For any $g > 2$, the Brill-Noether subspace of $\text{Pic}(\overline{M}_{g,n}) \otimes \mathbb{Q}$ has dimension $1 + n + \binom{n}{2}$, unless $g + 1$ is prime and $n = 0$. In particular, its projection with respect to the standard basis onto the subspace spanned by $\lambda$, the $\omega_i$, and the $\delta_{0;\{i,j\}}$ is an isomorphism.

**Proof.** To start the argument, observe that $BN$ has a nonzero coefficient of $\lambda$, and the pullback of $W$ by the map which forgets all but the $i$th point has nonzero coefficient of $\omega_i$ and $\lambda$, while all of their other coefficients in the space we have projected to are 0. This, together with the results of the previous section, takes care of the cases $n = 0, 1$.

We start by proving that the projection is surjective. The case $n = 2$ is disposed of as soon as we find a divisor of Brill-Noether type with nonzero coefficient of $\delta_{0;\{1,2\}}$. To do this, we need a bit of notation.

**Definition.** Let $A(g, m, n) = A'(g, d, n)$ be the number of $g_d^1$'s on a general curve of genus $g$ ramified to order $m - 1$ at a specified point and to order $n - 1$ at an unspecified point, where $2d = g + m + n - 1$.

**Theorem 3.2** ([L], thm. 3.2)

$$A'(g, d, n) = g! (n^2 - 1) \sum_{j = \max(0, m + n - d - 1)}^{\min(m - 1, n - 1, d)} \frac{(m + n - 2j - 1)}{((d - m - n + j + 1)!(d - j)!)}.$$ 

(It seems almost certain that this was known long before [L], but I know of no reference.)

In odd genus $g$, we consider the divisor $D$ on $\overline{M}_{g,2}$ of curves and points such that there is a $g_d^{-1}$ ramified at both of the points. On the one hand, there are $c_{(g+1)/2} g_d^{(g+3)/2}$'s ramified at a given point, and each of them is ramified at exactly $3g$ other points, so the degree on a fiber is $3g c_{(g+1)/2}$. (Here $c_n$ is the $n$th Catalan number $(2n)!/(n!(n+1)!).$) On the other hand, consider $\pi_1* D \cdot \delta_{0;\{1,2\}}$. An easy argument with linear series shows this to consist of the sum of the pullback of a Brill-Noether divisor from $\overline{M}_g$ and a divisor of Brill-Noether type on $\overline{M}_{g,1}$, to wit, that of curves and points where the curve has a $g_d^{(g+3)/2}$ doubly ramified


at the marked point. The degree of this on a fiber is \(A(g, 1, 3) = 24\left(\binom{g}{g+3}/2\right)\), and the coefficient of \(\delta_{0;\{1,2\}}\) will be 0 iff this is equal to \(6g\binom{g+1}{2}\). Expanding out both sides and multiplying through by 
\[
\frac{((g+1)/2)!((g+3)/2)!}{6g!},
\]
we get \(g(g+1) = (g-1)(g+1)\), which holds for no positive integer, so the coefficient is nonzero as desired.

In even genus, consider the divisor \(D\) on \(\overline{M}_{g,2}\) of curves and points such that there is a \(g\binom{g+4}{2}\) ramified doubly at the first point and singly at the second. When we cut and push forward, we get the divisor \(D_{g,2} + D_{g,4}\) on \(\overline{M}_{g,1}\). Again, to show that \(D\) has nonzero coefficient of \(\delta_{0;\{1,2\}}\), we must prove that the sum of the degrees on the fibers of \(D\) is not equal to the degree on the fibers of its pushforward; in other words, that \(A(g, 2, 3) + A(g, 3, 2) \neq A(g, 1, 2) + A(g, 1, 4)\). Writing these out in terms of the formula for \(A\) given in Theorem 3.2 and dividing through by \(g!\), we get
\[
11 \sum_{j=0}^{1} \frac{(4-2j)}{(g/2+j-2)!(g/2+j+2)!} \neq \frac{6}{(g/2-1)!(g/2+1)!} + \frac{60}{(g/2-2)!(g/2+2)!},
\]
which on multiplying through by \((g/2-1)!(g/2+2)!\) becomes
\[
44(g/2-1) + 22(g/2+2) \neq 6(g/2+2) + 60(g/2-1),
\]
a statement that is always true.

Then, on \(\overline{M}_{g,n}\), we can prescribe the coefficients of the \(\delta_{0;\{i,j\}}\) by pulling back these divisors from \(\overline{M}_{g,2}\) in appropriate ways. The \(\omega\)'s are dealt with by pulling back \(W\) from \(\overline{M}_{g,1}\), and \(\lambda\) by pulling back \(BN\) from \(\overline{M}_g\). We must show, now, that a divisor in the Brill-Noether subspace whose coefficients of \(\lambda\), the \(\omega\), and the \(\delta_{0;\{i,j\}}\) are all 0 is 0.

**Lemma 3.3** Let \(S = \{i, j\} \subset \{1, \ldots, n+1\}\). The map
\[
\text{Pic}(\overline{M}_{g,n+1}) \otimes \mathbb{Q} \to \text{Pic}(\overline{M}_{g,n}) \otimes \mathbb{Q}
\]
given by \(D \to \pi_{i*}(D \cdot \delta_{0,S})\) maps the Brill-Noether subspace to the Brill-Noether subspace. (Note that this is the pullback map on Picard groups induced by the map \(\overline{M}_{g,n} \to \overline{M}_{g,n+1}\) which takes the point representing \((C, p_1, \ldots, p_i, \ldots, p_n)\) to \((C', p_1, \ldots, P_1, \ldots, P_2, \ldots, p_n)\), where \(C' = C\) with a copy of \(\mathbb{P}^1\) attached at \(p_i\) and the \(P_i\) are points on this copy of \(\mathbb{P}^1\).

**Proof.** It suffices to prove this on a set of generators for the Brill-Noether subspace. So let \(D\) be a divisor of Brill-Noether type. Then for a point of \(\delta_{0,S}\) to be contained in \(D\) means that there is a limit linear series on the reducible curve the point corresponds to that satisfies the necessary ramification conditions. On the \(\mathbb{P}^1\) containing \(p_i\) and \(p_j\), there is no choice in the matter: we know what the total order of vanishing must be at the point of attachment. This forces the total order of vanishing on the genus-\(g\) component, and it is easily checked that this results in a codimension-1 condition for each way to distribute the vanishing between base points and ramification. Thus \(D\) maps to a sum of divisors of Brill-Noether type, and the lemma is proved.

**Lemma 3.4** Let \(D \in BNS\) be a divisor whose coefficients of \(\lambda\), the \(\omega\), and the \(\delta_{0;\{i,j\}}\) are all 0, with respect to the standard basis. Then for any \(S\), the coefficient of \(\delta_{0,S}\) in \(D\) is 0 as well. (Note that we do not yet assert that its coefficient of \(\delta_0\) must be 0).

**Proof.** Let \(S \subseteq \{1, \ldots, n\}\) (card \(S > 2\)), and fix a card \(S\)-pointed \(\mathbb{P}^1\) and a general \((n+1-\text{card } S)\)-pointed curve of genus \(g\). Consider a family of curves whose base is isomorphic to \(\mathbb{P}^1\), and whose fiber at a point \(P\) has the card \(S\)-pointed \(\mathbb{P}^1\) attached at \(P\) to the curve of genus \(g\) at the first point. For each element \(x \in S\), this family meets \(\delta_{0,S-(x)}\) once, and it meets \(\delta_{0,S}\) with multiplicity \(2-\text{card } S\)—the section on the curve of genus \(g\) is constant, so has self-intersection 0, while the section on the \(\mathbb{P}^1\) is a diagonal on \(\mathbb{P}^1 \times \mathbb{P}^1\), blown up at card \(S\) points. On the other hand, it is plain that this curve does not meet any of the other boundary components or \(\lambda\). In addition, I claim that the intersection with each \(\omega_i\) is 0.
For $i \notin S$, this is obvious. For $i \in S$, the self-intersection of the $i$th section is $-1$, so this contributes 1. For every element $x$ of $S$ other than $i$, we get a contribution to $\omega_i$ of 1 from $\delta_{0,S\setminus \{x\}}$, thus $\operatorname{card} S - 1$. Finally, we add the intersection with $\delta_{0,S}$, which is $2 - \operatorname{card} S$; total 0.

By the extended Brill-Noether theorem [EH, Thm. 1.1], this curve misses all divisors of Brill-Noether type entirely, and so for every $S$ of cardinality greater than 2 we get a relation

$$(2 - \operatorname{card} S)d_{0,S} + \sum_{x \in S} d_{0,S\setminus \{x\}} = 0.$$ 

(Roman letters are the coefficients of divisors named by similar Greek letters.) Therefore, if $D$ is a divisor in the Brill-Noether subspace with coefficients of $\delta_{0,\{i,j\}}$ equal to 0, an induction on $\operatorname{card} S$ proves the lemma.

Now we are ready to settle the case $n = 2$ in Theorem 3.1. Suppose that $D \in BNS$ has all coefficients of $\lambda, \omega$, and $\delta_{0,\{i,j\}}$ equal to 0, but suppose that the coefficient of $\delta_{i,0}$ is nonzero. If $S$ is empty, then consider $\pi_* \delta_{0,\{1,2\}} \cdot D$—it is in the Brill-Noether subspace and has a nonzero coefficient of $\delta_{i,0}$, contradiction. We may thus assume that all of the $\delta_{i,0}$ are 0.

We now know that $D$ pulls back to 0 on $\overline{\mathcal{M}}_{0,g+2}$. Define $\theta_{i,S}$ on $\overline{\mathcal{M}}_{0,g+2}$ to be the sum

$$
\sum_{T \in \{1,2,\ldots,g\}, \operatorname{card} T = i} \delta_{0,T \cup (S+n)},
$$

where $S + n$ is the set obtained by adding $n$ to each element of $S$, so that $\theta_{i,S}$ is the pullback of $\delta_{i,S}$. Let

$$
D = \sum a_i \delta_{i,\{1\}} + d \delta_0;
$$

then $\sum a_i \theta_{i,\{1\}} = 0$. We show that the $\theta_{i,\{1\}}$ are linearly independent.

To start, only $\theta_{1,\{1\}}$ has nonzero degree on a fiber of $\pi_{g+1}$, so it cannot appear in a relation. Next, only $\theta_{g-1,\{1\}}$ of the remaining $\theta$ has nonzero degree on a fiber of any of the other projection maps. For the rest, fix $1 < i < g - 2$, put $i$ of the first $g$ points and $p_{g+2}$ on a $\mathbf{P}^1$ and attach this $\mathbf{P}^1$ to another $\mathbf{P}^1$ which has the other $g - i$ of the first $g$ marked points and $p_{g+1}$. Then let one of the points vary on the first $\mathbf{P}^1$. This meets only $\theta_{g-1,\{1\}}$ and $\theta_{g-i+1,\{1\}}$ of the $\theta$, the latter with multiplicity 1. Again it is immediate that the $\theta$ are in fact linearly independent.

This proves that all of the $a_i$ are 0, so $D$ is just a multiple of $\delta_0$. That means that $D = 0$, though, because the map $\overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_{g,2}$ does not pull any nonzero multiple of $\delta_0$ back to a multiple of $\mathcal{W}$. This completes the proof in the case $n = 2$.

Finally, for general $n$, given $D$ we start by removing its coefficients of $\lambda$, the $\omega$, and the $\delta_{0,\{i,j\}}$. Its coefficients of $\delta_{0,S}$ are automatically 0; suppose that its coefficient of $\delta_{i,S}$ is nonzero. Choose $j, k$ so that either $j, k \in S$ or $j, k \notin S$, cut with $\delta_{0,\{j,k\}}$, and forget the $k$th point. This produces a smaller $n$ and $D$ with such a nonzero coefficient, and proceeding in this way we get down to the case $n = 2$, contradiction. So all the $\delta_{i,S}$ have a coefficient of 0. We finish the proof of the theorem by concluding as before that $\delta_0$ cannot appear.

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