Stochastic quasi-Gaussian models of atmospheric clouds

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Abstract. Random field simulation methods that can be used to simulate the stochastic structure of three-dimensional clouds are discussed. Examples of homogeneous random field realizations that reproduce one-dimensional distributions and correlation functions of experimental fields of stratus optical thicknesses are presented.

1. Introduction
Cloudiness is a major factor affecting the radiation balance of the Earth’s atmosphere. However, the stochastic structure of clouds brings considerable uncertainty into climatic models. An important problem is to construct numerical models that take into account the random optical properties and geometry of clouds. There are a lot of approaches to solving this problem (see, for example, [1] and the references therein). To simulate random fields of some of the characteristic properties (for instance, extinction coefficient or water content) of stratus clouds, we have analyzed observation data [2] and made some assumptions about the correlations along the vertical axis. We have used nonlinear transformations of Gaussian random fields [3] and simulated various models of Gaussian random fields based on a vector autoregressive scheme and a spectral decomposition. One of the objectives of this paper is to present some new simulation examples and to demonstrate, in addition to the results of some previous investigations, how the numerical methods work. These simulation methods seem to be universal (see [3, 4, 5], where similar approaches are used for various applications).

2. Quasi-Gaussian models
Assume that the stratus extinction coefficient \( \sigma(x, y, z) \) is a homogeneous random field and its one-dimensional distribution function \( F \) and its autocovariance function \( r(x, y, z) \) are defined. A well-known method to numerically simulate such random fields is based on nonlinear transformations of Gaussian homogeneous fields [6, 3]. The simulation formula is

\[
\tilde{\sigma}(x, y, z) = F^{-1}\left(\Phi(\omega(x, y, z))\right).
\]

(1)

Here \( \Phi(\omega) \) denotes a standard normal distribution function and \( \omega(x, y, z) \) is a homogeneous Gaussian random field with zero expectation, \( E\omega(x, y, z) = 0 \), unit variance, \( V\omega(x, y, z) = 1 \), and autocovariance function \( \rho(x, y, z) = E\omega(x, y, z)\omega(0, 0, 0) \). The autocovariance function \( r(x, y, z) \) of the field \( \sigma(x, y, z) \) has the form
\[ r(x, y, z) = \mathbf{E} \sigma(x, y, z) \sigma(0, 0, 0) = R_\mathbf{F}(\rho(x, y, z)) = \int_0^\infty \int_0^\infty F^{-1}(\Phi(\xi)) F^{-1}(\Phi(\eta)) \varphi_\rho(\xi, \eta) d\xi d\eta, \]

and \( \varphi_\rho(\xi, \eta) \) is the probability density of a two-dimensional Gaussian random vector with zero mean, unit variance and correlation coefficient \( \rho \) between the components

\[ \varphi_\rho(\xi, \eta) = \left[ 2\pi\sqrt{1-\rho^2} \right] \exp\left( -\frac{\xi^2 + \eta^2 - 2\rho\xi\eta}{2(1-\rho^2)} \right) \sim \]

Thus, the method implies the calculation of the function \( \rho(x, y) = R_\mathbf{F}^{-1}(r(x, y)) \), the subsequent simulation of the Gaussian field with this autocovariance function and, then, the transformation (1).

3. Statistical analysis of experimental fields

We have analyzed a random field of the Arctic stratus optical thickness \( \tau(x, y) \) retrieved from aircraft measurements obtained during the VERDI campaign 2012 [2, 7]. The estimations of the random field one-dimensional distribution densities \( f(r) \) have been made. Note that the one-dimensional distribution \( f(r) \) of \( \tau(x, y) \) can be approximated by a truncated Gaussian distribution in \([0, u_{max}]\) with the density

\[ f_{\mu, \sigma}(u) = \frac{\exp\left( -(u-\mu)^2/2\sigma^2 \right)}{|\Phi(\mu_{max}) - \Phi(\mu)| \sigma\sqrt{2\pi}}, \mu > 0, \]

or a Gamma distribution with the density

\[ g_{\lambda, \nu}(u) = \frac{\lambda^\nu u^{\nu-1}}{\Gamma(\nu)} e^{-\lambda u}, \lambda > 0, \nu > 0, \quad u > 0. \]

Examples of one-dimensional distributions \( f(r) \) of the fields \( \tau(x, y) \) in question and some approximations of \( f(r) \) are shown in Figure 1. When modeling cloud fields using the inverse function method, it is necessary, at first, to estimate the correlation functions distortion. For the one-dimensional distributions \( F(.) \) of the observed optical thickness fields \( \tau(x, y) \) St-01 - St-10, the normalized autocorrelation functions \( r(x, y) = R_\mathbf{F}(\rho(x, y)) \) of the corresponding quasi-Gaussian processes \( \tilde{\tau}(x, y) = F^{-1}(\Phi(\omega(x, y))) \) from (1) are computed according to the relation (2). In Table 1, the maximum values of the correlation distortion for the clouds St-01 -- St-10 are given for some intervals of \( \rho \). These results show that the distortion of the correlations in the nonlinear transformation for the above-considered one-dimensional distributions does not exceed 6 percent, and they can be neglected when constructing a numerical model, i.e. one can use the correlation function \( r(x, y) \) of a non-Gaussian field \( \tau(x, y) \) for constructing the Gaussian field \( \omega(x, y) \). In a statistical analysis of the available random fields \( \tau(x, y) \), unbiased \( \tilde{r}(k,l) \) and biased \( r(k,l) \) estimations of the correlation function are calculated using the following formulas:

\[ \tilde{r}(k,l) = \frac{1}{(M-k)(N-l)} \mathbf{V}_\tau \sum_{i=k+1}^{M} \sum_{j=l+1}^{N} \left( \tau(x_i + kh_x, y_j + lh_y) - \mathbf{E}\tau \right) \left( \tau(x_j, y_j) - \mathbf{E}\tau \right), \]

\[ r(k,l) = \frac{1}{MN} \mathbf{V}_\tau \sum_{i=k+1}^{M} \sum_{j=l+1}^{N} \left( \tau(x_i + kh_x, y_j + lh_y) - \mathbf{E}\tau \right) \left( \tau(x_j, y_j) - \mathbf{E}\tau \right). \]

\[ \mathbf{E}\tau = \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} \tau(x_i, y_j), \quad \mathbf{V}\tau = \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} (\tau(x_i, y_j) - \mathbf{E}\tau)^2 \]
According to the estimations obtained above, the one-dimensional correlation functions can be approximated with satisfactory accuracy by an exponential correlation function, at least, it can be made at distances not exceeding the correlation radius (the distance at which the correlation function decreases e times). Analyzing estimations of the correlation radii of arctic stratus fields [7], note that the mean value of the correlation radius is approximately equal 0.5 km, the minimal correlation radius is 0.09 km, and the largest correlation radius is 1.12 km. At the same time, the average correlation radius in the horizontal direction of the stratus water content of the temperate climatic zone is 1.4–1.5 km [8].

Table 1. Correlation distortion of quasi-Gaussian cloud models St-01 – St-10.

| Model  | max|ρ|-1,1| max|ρ|-0.5,1| max|ρ|0,1 |
|--------|----|-----|----|-----|------|----|-----|
| St-01  | 0.04417 | 0.01672 | 0.00618 |
| St-02  | 0.05408 | 0.02086 | 0.00782 |
| St-03  | 0.01385 | 0.00767 | 0.00675 |
| St-04  | 0.01301 | 0.00525 | 0.00248 |
| St-05  | 0.00664 | 0.00382 | 0.00269 |
| St-06  | 0.00386 | 0.00361 | 0.00361 |
| St-07  | 0.03071 | 0.01253 | 0.00597 |
| St-08  | 0.03735 | 0.01483 | 0.00638 |
| St-09  | 0.00885 | 0.00546 | 0.00422 |
| St-10  | 0.00846 | 0.00408 | 0.00321 |
Figure 2. Estimation of the autocorrelation function $r(x, y)$ of the optical thickness field St-01.

4. Numerical models of Gaussian stochastic fields

4.1. Multiplicative representation of the correlation function of a three-dimensional field

There are various ways of constructing three-dimensional random fields with a given correlation structure. Assuming that the correlation function of a homogeneous three-dimensional random field $u(x, y, z_i)$, $z_i = ih_z$, $i = 1, \ldots, N$ is represented by the product of two-dimensional and one-dimensional correlation functions $R(x, y, i) = E u(x, y, i) u(0,0) = R_{xy}(x,y) R_z(i)$, it is possible to build a combined algorithm for modeling such a random field, by constructing, at the first stage, a given number $N$ of two-dimensional fields $\omega(x, y, z_k) = \omega_k(x, y)$, $k = 1, \ldots, N$, for instance, using vector methods of conditional distributions and autoregressive models [3, 13]. This set of two-dimensional random fields can be considered as a set of dependent random vectors $(\omega_1(x, y), \ldots, \omega_N(x, y))$ with independent components. To obtain a three-dimensional random field with the correlation function $R(x, y, z)$ for each fixed point $(x, y)$, we use a method based on linear transformations, $\bar{u}_N(x, y) = A \bar{\omega}_N(x, y)$, where $A = (a_{ij})$ is a matrix such that $AA^* = B = (b_{ij})$, $b_{ij} = R_z(i-j)$. The factorization $AA^* = B$ can be obtained, for example, by using Cholesky decomposition [3, 12]. This algorithm is convenient if the two-dimensional fields are simulated on a discrete grid. A vector "conditional distributions" algorithm and a vector autoregression scheme will be briefly described in the next subsection.

To simulate an isotropic continuous Gaussian field with a correlation function $\rho(s) = \rho\left(\sqrt{x^2 + y^2 + z^2}\right)$ one can use a spectral model [10] of a random field $\phi(\vec{r}) = \phi(x, y, z)$. If the radial correlation function is exponentially decaying, $\rho(s) = \exp(-\lambda s)$, there is an algorithm [11] for simulating random harmonics of the spectral model $\phi(x, y, z)$. It will be described in subsection 3 of this section.

4.2. Vector methods of conditional distributions and autoregression [3, 13]

To simulate the homogeneous Gaussian random field $\omega(x, y)$ on a grid $(ih_x, jh_y)$, $i, j = 1, \ldots, N$ we used a two-dimensional autoregressive scheme [3, 13]. The main idea of the method is an iterative step by step simulation of the vectors $\xi$, taking into account the correlations both between the random elements of the simulated vector $\xi$ and between the elements of this vector and the $n$ previous vectors $\xi_j$, $j=1,n-1$ according to a given correlation function.
To simulate the stationary vector-valued autoregressive processes $\xi_t$ with a given covariance structure, the initial vectors $\xi_1, \ldots, \xi_n$ are calculated as

$$
\begin{align*}
\xi_1 &= C_0 \varphi_1 \\
\xi_2 &= B^T_1 [1] \xi_1 + C_1 \varphi_1 \\
\vdots & \quad \vdots \\
\xi_n &= B^T_{n-1} [n-1] \xi_{n-1} + \ldots + B^T_1 [n-1] \xi_1 + C_{n-1} \varphi_n 
\end{align*}
$$

and then the following autoregressive scheme is used:

$$
\xi_t = B^T_{t-1} \xi_{t-1} + \ldots + B^T_1 [n-1] \xi_1 + C_{n-1} \varphi_t, t \geq n. \tag{3}
$$

To calculate the matrices $B_i[j]$, it is necessary to solve systems of linear equations with a block Toeplitz matrix at each step. Details and various modifications of this method can be found in [3]. A general description of this problem solution is presented in [13]. Examples of simulating two-dimensional random fields of optical thicknesses are presented in [9].

4.3. Spectral model of Gaussian fields

We have simulated approximations of the homogeneous Gaussian fields $\tilde{\omega}(x, y, z)$ based on a spectral decomposition [10, 11]. We applied an isotropic stochastic field with a correlation function $\rho(s)$ depending on the distance between spatial points, $s = \sqrt{x^2 + y^2 + z^2}$. The correlation function of the field $\tilde{\omega}(x, y, z)$ can be represented in the form

$$
\rho(s) = \int_0^\infty \sin(\gamma s) \rho(\gamma s) d\gamma,
$$

where $h(\gamma)$ is the radial spectral distribution density on $[0, + \infty)$ of a homogeneous isotropic random field $\tilde{\omega}(x, y, z)$:

$$
h(\gamma) = \frac{2}{\pi} \int_0^\infty \gamma s \sin(\gamma s) \rho(\gamma s) ds.
$$

For the numerical simulation of the Gaussian field we use the following spectral model

$$
\tilde{\omega}_{nm}(\vec{r}) = \sum_{m=1}^{M} a_m \omega_m(\vec{r}) + \omega_{\tilde{\omega}}(\vec{r}) = \frac{1}{\sqrt{M}} \sum_{m=1}^{M} \left[ \xi_m \cos(\lambda_m \vec{r}) + \eta_m \sin(\lambda_m \vec{r}) \right],
$$

where $\xi_m$ and $\eta_m$ are independent standard normal variables, $\langle \lambda_m, \vec{r} \rangle$ is the scalar product of vectors in $\mathbb{R}^3$. For a model without splitting the spectrum, the vectors $\lambda_m$ have to be uniformly distributed on a hemisphere $S \subset \mathbb{R}^3$ of radius $\gamma$: $\lambda_m = \gamma e$, where $e = (e_1, e_2, e_3)$ is an isotropic vector, $\alpha_m^2 = 1/N$, $\gamma_m$ are independent random variables identically distributed on $[0, + \infty)$ with density $h(\gamma)$. It is convenient to model the isotropic direction in the hemisphere $S$ by the algorithm

$$
e_1 = \alpha_1, \quad e_2 = \sqrt{1-e_1^2} \cos(2\pi \alpha_2), \quad e_3 = \sqrt{1-e_1^2} \sin(2\pi \alpha_2)
$$

Here $\alpha_1, \alpha_2$ are independent uniformly distributed on $(0, 1)$ random values. For an exponential correlation function $\rho(s) = \exp(-\lambda s)$ of the random field $\tilde{\omega}(x, y, z)$, the radial spectral distribution density has a form

$$
h(\gamma) = \frac{4\lambda \gamma}{\pi \left( \lambda^2 + \gamma^2 \right)^{3/2}}.
$$
To simulate the random radius $\gamma$, one can use a special algorithm [11] which is based on simulating random variables by using the gamma distribution $\gamma = \lambda \sqrt{\zeta_1^2 - 2 \ln \alpha_1 / |\zeta_2|}$, where $\zeta_1, \zeta_2$ are independent standard normal variables and $\alpha_1$ is a uniformly distributed variable on (0,1).

5. Specific conversion of Gaussian fields

For obtaining a stochastic field with the marginal gamma distribution $f_{\nu, \lambda}(x) = \frac{\lambda^\nu x^{\nu-1} e^{-\lambda x}}{\Gamma(\nu)}$, $\lambda > 0, \nu > 0, x \geq 0$

where $\nu$ is an integer parameter, it is convenient to apply a nonlinear transformation of the Gaussian fields $\omega(x^k), k = 1, \ldots, 2\nu$:

$$\sigma_\nu(\bar{r}) = V \left( \omega^{(1)}(\bar{r}), \ldots, \omega^{(2\nu)}(\bar{r}) \right) = \delta^2 \left( \left( \omega^{(1)}(\bar{r}) \right)^2 + \ldots + \left( \omega^{(2\nu)}(\bar{r}) \right)^2 \right)$$

A similar transformation with $\nu = 1$ is described in [10, 11]. With this transformation, the autocorrelation function of the field $\sigma_\nu(\bar{r})$ has the form $r(s) = \rho^2(s)$, where $s = \sqrt{x^2 + y^2 + z^2}$, the mathematical expectation of the marginal distribution is $E\sigma_\nu = 2\nu \delta^2$, the variance is $V\sigma_\nu = 4\nu \delta^4$, and the scale parameter is $\lambda = \frac{1}{2\delta^2}$.

6. Numerical results

A three-dimensional random field of optical attenuation coefficients $\sigma(x, y, z)$ of visible radiation is constructed under the assumption that the one-dimensional distribution function of the two-dimensional optical thickness field is a gamma function with parameters $\lambda$ and $\nu$ (see Fig. 1), and the correlation function along the $z$ axis is exponential with correlation radius $\rho_z$. Let the geometric thickness of the cloud layer be $H = 0.5$ km, and the geometric length of the field in the horizontal direction $a = b = 1$ km. The number of field partitions along the $z$ axis is $N_z = 5$, along the $x$ and $y$ axes: $N_x = N_y = 100$. For each elementary three-dimensional cell $\tau_{ijk}$, the optical thickness is simulated, $i = 1 \ldots N_x, j = 1 \ldots N_y, k = 1 \ldots N_z$. The attenuation coefficient in the cell $ijk$ is constant, $\sigma_{ijk} = \tau_{ijk} \ast N_z / H$.

Due to the infinite divisibility of the gamma distribution [12], the one-dimensional distribution of the optical thickness field $\tau_{ijk}$ of unit cells is a gamma distribution with parameters $\lambda$ and $\nu/N_z$. Successive horizontal sections of the attenuation coefficient field $\sigma(x, y, z)$, simulated according to the algorithms in paragraphs 4.1 and 4.2, with initial data obtained from statistical analysis of the field St-01 [2] are presented in Fig. 3. The parameters of this model are as follows: $\rho_z = 0.2, \lambda = 0.26, \nu = 37, \nu / N_z = 7.2$. For such a one-dimensional distribution of the field, the distortion of the horizontal correlation function $R_{\nu, \lambda}(x, y)$ in the inverse function method does not exceed 6%, therefore, to simulate the two-dimensional Gaussian fields, a correlation function estimated from the experimental data (St-01) was used.
Figure 3. Successive horizontal sections of realizations of the attenuation coefficient field $\sigma (x, y, z)$ obtained by the algorithms of conditional distributions, inverse functions, and autoregression.

Figure 4. Successive horizontal sections of realizations of an isotropic spectral model of the attenuation coefficient field $\sigma (x, y, z)$ obtained with a one-dimensional gamma distribution and an exponential correlation function.

Figure 4 shows the attenuation coefficients $\sigma (x, y, z)$ for given fixed values of the vertical coordinate $z = (k-0.5) h$, $k = 1, ..., 5$, $h = 0.1$ km, simulated by using an isotropic spectral model (subsection 4.3) and a special nonlinear transformation (section 5). The correlation radius of the exponential function of the field $\sigma (x, y, z)$ is $\rho(s) = 0.25$ km, the number of partitions of the spectral space $N = 20$, $M = 100$, and the number of terms in the nonlinear transformation $K = 2^v = 20$, $\delta^2 = 1$.

7. Conclusions

The computation results show that the above-considered numerical methods for the stochastic field simulation (autoregressive schemes, spectral models, nonlinear transformations of Gaussian functions) can be effectively used in reproducing the geometrical and optical properties of atmospheric clouds. It is suggested to incorporate these methods into atmospheric models to study, in particular, the effects of clouds on gravity currents in the atmosphere [14, 15].
Acknowledgements
This work was supported by the Russian Foundation for Basic Research under grants 18-01-00149, 18-01-00609, and 17-01-00137, and ICMMG SB RAS under target program 0315-2019-0004.

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