An application of the Gaussian correlation inequality to the small deviations for a Kolmogorov diffusion*

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Abstract

We consider an iterated Kolmogorov diffusion $X_t$ of step $n$. The small ball problem for $X_t$ is solved by means of the Gaussian correlation inequality. We also prove Chung’s laws of iterated logarithm for $X_t$ both at time zero and infinity.

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1 Introduction

Let $\{X_t\}_{0 \leq t \leq T}$ be an $\mathbb{R}^n$-valued stochastic process with continuous paths such that $X_0 = 0$ a.s. where $T > 0$ is fixed. Denote by $W_0(\mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued continuous functions on $[0,T]$ starting at zero. Given a norm $\| \cdot \|$ on $W_0(\mathbb{R}^n)$, the small ball problem for $X_t$ consists in finding the rate of explosion of

$$- \log \mathbb{P}(\|X\| < \varepsilon)$$

as $\varepsilon \to 0$. More precisely, a process $X_t$ is said to satisfy a small deviation principle with rates $\alpha$ and $\beta$ if there exist a constant $c > 0$ such that

$$\lim_{\varepsilon \to 0} -\varepsilon^\alpha \log \varepsilon^\beta \log \mathbb{P}(\|X\| < \varepsilon) = c. \quad (1.1)$$

The values of $\alpha$, $\beta$ and $c$ depend on the process $X_t$ and on the chosen norm on $W_0(\mathbb{R}^n)$. Small deviation principles have many applications including metric entropy estimates.

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and Chung’s law of the iterated logarithm. We refer to the survey paper [13] for more details.

We say that a process $X_t$ satisfies Chung’s law of the iterated logarithm (LIL) as $t \to \infty$ (resp. as $t \to 0$) with rate $a \in \mathbb{R}_+$ if there exists a constant $C$ such that

$$\lim \inf_{t \to \infty} \left( \frac{\log \log t}{t} \right)^a \max_{0 \leq s \leq t} |X_s| = C, \quad a.s. \tag{1.2}$$

(resp. $\lim \inf_{t \to 0} (\log \log t/t)^a \max_{0 \leq s \leq t} |X_s| = C$ a.s.). When $X_t$ is a Brownian motion, it was proven in a famous paper by K.-L. Chung in 1948 that (1.2) holds with $a = \frac{1}{2}$ and $C = \frac{\pi}{2}$. To find the rates $\alpha$ and $\beta$ such that the limit in (1.1) exists, and then find the constant $c$ is an extremely hard problem in general. Even the estimation of the rate of explosion of (1.1) is usually a difficult problem. Indeed, as can be surmised in [10, 15], the small ball problem for Gaussian processes is equivalent to metric entropy problems in functional analysis. In [11] and [20] a Brownian sheet in Hölder and uniform norm is considered, and the integrated Brownian motion in the uniform norm is the content of [8], and the m-fold integrated Brownian motion in both the uniform and $L^2$-norm is considered in [4]. In [17] and [3] a small deviation principle and Chung’s LIL are proved for a class of stochastic integrals and for a hypoelliptic Brownian motion on the Heisenberg group. When $X_t$ is a Gaussian process with stationary increments, upper and lower bounds on (1.1) can be found in [19, 16].

In this paper we consider the Kolmogorov diffusion of step $n$.

**Definition 1.1.** Let $T > 0$ and $b_t$ be a one-dimensional standard Brownian motion. The stochastic process $\{X_t\}_{0 \leq t \leq T}$ on $\mathbb{R}^n$ defined by

$$X_t := \left( b_t, \int_0^t b_{t_1} dt_1, \int_0^t \int_0^{t_2} b_{t_1} dt_2, \ldots, \int_0^t \int_0^{t_2} \cdots \int_0^{t_{n-1}} b_{t_n} dt_n \right)$$

is the Kolmogorov diffusion of step $n$.

$\{X_t\}_{0 \leq t \leq T}$ is a Markov process with generator given by $L = \frac{1}{2} \frac{\partial^2}{\partial x^2} + \sum_{d=2}^n x_{d-1} \frac{\partial}{\partial x_d}$. In particular, when $n = 2$ $X_t$ is the Markov process associated to the differential operator $L = \frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{x_{d-1}}{x_d} \frac{\partial}{\partial x_d}$, and it was first introduced by A. N. Kolmogorov in [9], where he obtained an explicit expression for its transition density. Later, L. Hörmander in [6] used $L$ as the simplest example of a hypoelliptic second order differential operator. More precisely, the operator $L$ satisfies the weak Hörmander condition. $\{X_t\}_{0 \leq t \leq T}$ is a Gaussian process and its law $\mu$ is a Gaussian measure on the Banach space $(W_0(\mathbb{R}^n), \| \cdot \|)$, where

$$\|f\| := \max_{0 \leq t \leq T} |f(t)|, \quad \forall f \in W_0(\mathbb{R}^n).$$

The main result of this paper is Theorem 2.6, where we prove the small deviation principle (1.1) for $X_t$ with rates $\alpha = 2$, $\beta = 0$, and constant $c = \frac{\pi}{2\sqrt{2}}$. Our proof relies on the Gaussian correlation inequality (GCI), see e.g. [18, 12], applied to the Gaussian measure $\mu$ on $W_0(\mathbb{R}^n)$. A different application of the GCI to estimate small balls probabilities is given in [14]. In Theorem 2.6 we also state Chung’s LIL at time zero and infinity for $X_t$ with rates given by $a = \frac{1}{2}$ and $a = \frac{2n-1}{2n}$ respectively.

The stochastic processes considered in [8, 17, 3] all satisfy a scaling property, that is, there exists a scaling constant $\delta \in (0, \infty)$ such that $X_{\delta t} \stackrel{(d)}{=} \delta X_t$. Properties of Gaussian measures on Banach spaces and scaling properties have been used to show the existence of a small deviation principle for some processes such as a Brownian motion with values in a finite dimensional Banach space in [5] and an integrated Brownian motion in [8]. Moreover, in [8, 17, 3] the scaling rate $\delta$ coincides with the rate of Chung’s LIL at infinity,
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and the small deviations’ rates are given by $\beta = 0$, $\alpha = \frac{1}{2}$. The Kolmogorov diffusion does not satisfy a scaling property with respect to the standard Euclidean norm, and the small deviations rate $\alpha$ is not related to the Chung’s LIL rate.

Lastly, large deviations and Chung’s LIL at time zero for the limsup of the Kolmogorov diffusion are discussed in Section [1, Section 4.2] and [2, Example 3.5] respectively.

The paper is organized as follows. In Section 2 we collect some examples and state the main result of this paper, namely, small deviation principle and Chung’s LIL at time zero and infinity for a step $n$ Kolmogorov diffusion. Section 3 contains the proof of the main result.

2 The setting and main results

Notation 2.1. Let $X_t$ be an $\mathbb{R}^n$-valued stochastic process with $X_0 = 0$ a.s. Then $X_t^*$ denotes the process defined by

$$X_t^* := \max_{0 \leq s \leq t} |X_s|,$$

where $|\cdot|$ denotes the Euclidean norm.

Notation 2.2. [Dirichlet eigenvalues in $\mathbb{R}^n$] We denote by $\lambda_1^{(n)}$ the lowest Dirichlet eigenvalue of $-\frac{1}{2} \Delta \mathbb{R}^n$ on the unit ball in $\mathbb{R}^n$.

Let us collect some examples of Chung’s LIL and small deviation principle.

Example 2.3. [Brownian motion] Let $X_t$ be a standard Brownian motion. Then $X_{\varepsilon t} \overset{(d)}{=} \varepsilon^{\frac{1}{2}} X_t$, and it satisfies the small deviation principle

$$\lim_{\varepsilon \to 0} -\varepsilon^2 \log \mathbb{P}(X_T^* < \varepsilon) = \lambda_1^{(1)} T,$$

(2.1)

where $\lambda_1^{(1)}$ is defined in Notation 2.2, see e.g. [7, Lemma 8.1]. Moreover, in a famous paper by K.-L. Chung in 1948 it was proven that

$$\liminf_{t \to \infty} \left( \frac{\log \log t}{t} \right)^{\frac{1}{2}} \max_{0 \leq s \leq t} |X_s| = \sqrt{\lambda_1^{(1)}} \ a.s. \quad (2.2)$$

Example 2.4. [Integrated Brownian motion]. Let $X_t := \int_0^t b_s ds$, where $b_s$ is a one-dimensional standard Brownian motion. It is easy to see that $X_{\varepsilon t} \overset{(d)}{=} \varepsilon^{\frac{1}{2}} X_t$. In [8] it is shown that there exists a finite constant $c_0 > 0$ such that

$$\lim_{t \to \infty} \left( \frac{\log \log t}{t} \right)^{\frac{1}{2}} \max_{0 \leq s \leq t} |X_s| = c_0 \ a.s. \quad (2.3)$$

and (2.3) was used to prove that

$$\lim_{\varepsilon \to 0} -\varepsilon^\frac{1}{2} \log \mathbb{P}(X_T^* < \varepsilon) = c_0^\frac{1}{2}.$$

Example 2.5. [Iterated integrated Brownian motion] Let $b_t$ be a one-dimensional Brownian motion starting at zero. Denote by $X_1(t) := b_t$ and

$$X_d(t) := \int_0^t X_{d-1}(s) ds, \ t \geq 0, d \geq 2,$$

the $d$-fold integrated Brownian motion for a positive integer $d$. Note that $X_d(\varepsilon t) \overset{(d)}{=} \varepsilon^{\frac{2d-1}{2}} X_d(t)$. In [4] it was shown that for any integer $d$ there exists a constant $\gamma_d > 0$ such
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that

\[ \lim_{\varepsilon \to 0} -\varepsilon^{2} \log P \left( \max_{0 \leq t \leq 1} |X_d(t)| < \varepsilon \right) = \gamma_d^{(2)} \]  
\[ \liminf_{t \to \infty} \left( \frac{\log \log t}{t} \right) \max_{0 \leq s \leq t} |X_d(s)| = \gamma_d \text{ a.s.} \]  

Our main object is the Kolmogorov diffusion on \( \mathbb{R}^n \) defined by

\[ X_t := (X_1(t), \ldots, X_n(t)) , \]

where

\[ X_d(t) := \int_{0}^{t_1} \int_{0}^{t_2} \cdots \int_{0}^{t_{d-1}} b_{d} dt_d \cdots dt_2 , \quad \text{for } d = 3, \ldots, n , \]

and \( X_2(t) := \int_{0}^{t} b_{1} ds , \) \( X_1(t) := b_{1} , \) where \( b_{i} \) is a one-dimensional standard Brownian motion. Note that \( X_d(\varepsilon t) \overset{(d)}{=} \varepsilon^{\frac{d-1}{2}} X_d(t) \) for all \( d = 1, \ldots, n , \) and hence the process \( X_t \) does not have a scaling property with respect to the Euclidean norm \( | \cdot | \) in \( \mathbb{R}^n . \)

**Theorem 2.6.** Let \( T > 0 \) and \( X_t \) be the Kolmogorov diffusion on \( \mathbb{R}^n . \) Then

\[ \lim_{\varepsilon \to 0} -\varepsilon^{2} \log P \left( X_T^* < \varepsilon \right) = \lambda_1^{(1)} T , \]

\[ \liminf_{t \to 0} \left( \frac{\log \log t}{t} \right) \max_{0 \leq s \leq t} |X_s| = \sqrt{\lambda_1^{(1)}} \text{ a.s.} \]

\[ \liminf_{t \to \infty} \left( \frac{\log \log t}{t} \right) \max_{0 \leq s \leq t} |X_s| = \gamma_n \text{ a.s.} \]

where \( \lambda_1^{(1)} \) is defined in Notation 2.2, and \( \gamma_n \) is given by (2.4)

**Remark 2.7.** By (2.2) and Brownian inversion, it follows that a standard Brownian motion satisfies Chung’s LIL at time zero and infinity with rate \( a = \frac{1}{2} , \) and it satisfies a small deviation principle with rate \( a = 2 . \) By (2.5), the \( n \)-step Kolmogorov diffusion \( X_t \) satisfies the same small deviation principle as a one-dimensional standard Brownian motion. As far as Chung’s LIL for \( X_t \) is concerned, the first component dominates when \( t \to 0 \) with rate \( a = \frac{1}{2} , \) and the \( n \)-th component dominates as \( t \to \infty \) with rate \( a = \frac{2n-1}{2} . \)

3 Proofs

**Proof of Theorem 2.6.** Let us first prove the small deviation principle (2.5). One has that

\[ P \left( X_T^* < \varepsilon \right) \leq P \left( b_T^* < \varepsilon \right) , \]

and hence by (2.1) it follows that

\[ \lambda_1^{(1)} T \leq \liminf_{\varepsilon \to 0} -\varepsilon^{2} P \left( X_T^* < \varepsilon \right) . \]

Let us now show that

\[ \limsup_{\varepsilon \to 0} -\varepsilon^{2} P \left( X_T^* < \varepsilon \right) \leq \lambda_1^{(1)} T . \]

For any \( x_1, \ldots, x_n \in (0, 1) \) such that \( x_1 + \cdots + x_n = 1 \) we have that

\[ P \left( X_T^* < \varepsilon \right) \geq P \left( \max_{0 \leq t \leq T} |X_1(t)| < x_1 \varepsilon , \ldots , \max_{0 \leq t \leq T} |X_n(t)| < x_n \varepsilon \right) \]

\[ \geq P \left( \max_{0 \leq t \leq T} |X_1(t)| < x_1 \varepsilon \right) \cdots P \left( \max_{0 \leq t \leq T} |X_n(t)| < x_n \varepsilon \right) , \]
where in the second line we used the Gaussian correlation inequality for the law of the process \( \{X_t\}_{0 \leq t \leq T} \) which is a Gaussian measure on \( W_0(\mathbb{R}^n) \). Thus,

\[
-\varepsilon^2 \log P (X_T^* < \varepsilon) \leq - \sum_{d=1}^{n} \varepsilon^2 \log P \left( \max_{0 \leq t \leq T} |X_d(t)| < x_d \varepsilon \right).
\]  

(3.1)

Note that, for any \( d = 2, \ldots, n \)

\[
\max_{0 \leq t \leq T} |X_d(t)| \leq \int_0^T \int_0^{t_2} \cdots \int_0^{t_{d-2}} \max_{0 \leq t \leq T} |X_2(t)| dt_2 \cdots dt_d = \frac{T^{d-2}}{(d-2)!} \max_{0 \leq t \leq T} |X_2(t)|,
\]

and hence

\[
P \left( \max_{0 \leq t \leq T} |X_2(s)| < \frac{(d-2)!}{T^{d-2}} x_d \varepsilon \right) \leq P \left( \max_{0 \leq t \leq T} |X_d(s)| < x_d \varepsilon \right),
\]  

(3.2)

and by [8, Theorem 1.1] we have that, for any \( d = 2, \ldots, n \)

\[
0 \leq \limsup_{\varepsilon \to 0} -\varepsilon^2 \log P \left( \max_{0 \leq t \leq T} |X_d(t)| < x_d \varepsilon \right)
\]

\[
\leq \lim_{\varepsilon \to 0} -\varepsilon^2 \log P \left( \max_{0 \leq t \leq T} |X_2(t)| < \frac{(d-2)!}{T^{d-2}} x_d \varepsilon \right)
\]

\[
= \lim_{\varepsilon \to 0} -\varepsilon^2 \log P \left( \max_{0 \leq t \leq T} \left| \int_0^t b_s \, ds \right| < \frac{(d-2)!}{T^{d-2}} x_d \varepsilon \right) = 0.
\]  

(3.3)

Thus, by (3.1) and (3.2)

\[
-\varepsilon^2 \log P (X_T^* < \varepsilon) \leq -\varepsilon^2 \log P (b_T^* < x_1 \varepsilon) - \sum_{d=2}^{n} \varepsilon^2 \log P \left( \max_{0 \leq t \leq T} |X_2(t)| < \frac{(d-2)!}{T^{d-2}} x_d \varepsilon \right),
\]

and by (3.3) and (2.1) it follows that

\[
\limsup_{\varepsilon \to 0} -\varepsilon^2 \log P (X_T^* < \varepsilon) \leq \frac{\lambda_1^{(1)}}{x_1 T}.
\]

The result follows by letting \( x_1 \) go to one.

Let us now prove (2.6). By (2.2) and Brownian time inversion one has that

\[
\liminf_{t \to 0} \left\{ \frac{\log |\log t|}{t} \max_{0 \leq s \leq t} |b_s| = \sqrt{\lambda_1} \ a.s. \right. \]

(3.4)

Note that

\[
|b_s|^2 \leq |X_s|^2 = |b_s|^2 + \sum_{d=2}^{n} |X_d(s)|^2
\]

\[
\leq |b_s|^2 + \max_{0 \leq u \leq s} |b_u|^2 \sum_{d=2}^{n} s^{2d-2} (d-1)!^2,
\]

and hence

\[
\frac{\log |\log t|}{t} \max_{0 \leq s \leq t} |b_s|^2 \leq \frac{\log |\log t|}{t} \max_{0 \leq s \leq t} |X_s|^2
\]

\[
\leq \frac{\log |\log t|}{t} \max_{0 \leq s \leq t} |b_s|^2 \left( 1 + \sum_{d=2}^{n} \frac{t^{2d-2}}{(d-1)!^2} \right).
\]
which completes the proof of (2.6). Let us now prove (2.7). Set

Thus, by (2.4) and (3.5) it follows that

and hence

\[
\liminf_{t \to 0} t^{2d-2} \log \left( \frac{\log t}{t} \right) \max_{0 \leq s \leq t} X_s^2 = \liminf_{t \to 0} t^{2d-2} \log \left( \frac{\log t}{t} \right) \max_{0 \leq s \leq t} b_s^2 = \lambda_1 \quad \text{a.s.}
\]

which completes the proof of (2.6). Let us now prove (2.7). Set

\[
\phi(t) := \frac{\log \log t}{t}.
\]

By (2.4) we have that, for any \( d, \ldots, n \)

\[
\liminf_{t \to \infty} \phi(t)^{n-1} \max_{0 \leq s \leq t} X_d(s) = 0 \quad \text{a.s.}
\]

since \( \phi(t) \to 0 \) as \( t \to \infty \). Note that

\[
|X_n(s)|^2 \leq |X_n(s)|^2 = \sum_{d=1}^{n-1} |X_d(s)|^2 + |X_n(s)|^2,
\]

and hence

\[
\phi(t)^{2n-1} \max_{0 \leq s \leq t} |X_n(s)|^2 \leq \phi(t)^{2n-1} \max_{0 \leq s \leq t} |X_n(s)|^2
\]

\[
\leq \sum_{d=1}^{n-1} \phi(t)^{2n-1} \max_{0 \leq s \leq t} |X_d(s)|^2 + \phi(t)^{2n-1} \max_{0 \leq s \leq t} |X_n(s)|^2.
\]

Thus, by (2.4) and (3.5) it follows that

\[
\liminf_{t \to \infty} \phi(t)^{2n-1} \max_{0 \leq s \leq t} |X_n(s)|^2 = \liminf_{t \to \infty} \phi(t)^{2n-1} \max_{0 \leq s \leq t} |X_n(s)|^2 = \gamma_n^2 \quad \text{a.s.}
\]

and (2.7) is proven. \( \square \)

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