On twin edge colorings of the direct product of paths

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Abstract. Let be a proper k-edge coloring of a simple connected graph of order at the latest 3, where the set of colors is. If a proper edge coloring of a graph that induces vertex coloring is a proper vertex coloring, then is called a twin edge k-coloring of a graph. the twin chromatic number of a graph is the least value k of colors of a graph, and denote it by. In this paper, we obtains the twin chromatic number of infinite paths and finite paths.

1. Introduction
Let be a simple connected graph. The maximum degree of is the maximum vertex degree in , denoted by Δ(G); the degree of a vertex of is the number of edges incident to , denoted by .

A k-edge coloring of a graph is a proper k-edge coloring if no two adjacent edges have the same color, where is an assignment of k colors to the edges of . The edge chromatic number, , of is the minimum k such that has a proper k-edges coloring.

A twin edge-coloring of a graph is a proper edge coloring of with the elements of the set such that the induced vertex coloring in which the color of a vertex in is the sum of the colors of the edges incident with is a proper vertex coloring. The twin chromatic number of is the least number of colors set of , denoted by .

Andrews [1] put forward the concept of twin edge colorings of a graph , and received the twin edge chromatic index of a path, a cycle, a complete graph and a complete bipartite graph.

Inspired by the concepts of the twin edge-coloring of described above, to determine the twin chromatic indexes of a simple connected graph , the following Lemmas are established clearly.

Lemma 1.1 If a simple connected graph has two adjacent vertices of maximum degree , then .

Lemma 1.2 For any connected graph , we have .

Lemma 1.3 If a k-regular graph exist an adjacent vertex distinguishing edge-coloring, we have .

The concept that is closely related to the twin edge-coloring of a graph is an adjacent vertex distinguishing edge-coloring of a graph . An adjacent vertex distinguishing edge-coloring [2] of a simple graph is a proper edge-coloring of such that no two adjacent vertices has the same set of colors in which . The minimum number of colors of is the adjacent vertex-distinguishing...
chromatic number, denoted by $\chi'_c(G)$ . Few is known about adjacent vertex distinguishing edge-coloring. In 2006, Baril [3] prove that the adjacent vertex-distinguishing chromatic number of the multidimensional mesh and the hypercube both are equal to the maximal degree of the both graphs plus 1. In 2007, Balister [2] prove for bipartite graphs with maximum degree $\Delta(G) = 3$, it have $\chi'_c(G) \leq \Delta(G)+2$ . In 2009, Dai [4] prove that a connected graph $G$ for a upper bound on the adjacent vertex-distinguishing chromatic index of graph is $3\Delta-1$. In 2010, Wang [5] investigate the chromatic number of a graph $G$ for the bounded in relation to the maximum average degree and the maximum degree. In 2011, Frigerio [6] research that the adjacent vertex-distinguishing edge-coloring of a regular graph, or a path, or a cycle, and obtained the adjacent vertex-distinguishing chromatic number of the regular graph, the path, or the cycle. In 2012, Yan [7] obtain that the adjacent vertex-distinguishing chromatic number of a planar graph $G$ is less than maximal degree plus 2 with girth at least 4 and maximum degree at least 6. In 2013, Hocquard [8] prove that the adjacent vertex-distinguishing chromatic number is equal to $\chi(G)+1$ for any graph $G$ with maximal degree is more than or equal 5 and the maximum average degree is less than $3\Delta-2\Delta$. In 2014, Zhang [9] prove that the adjacent vertex-distinguishing chromatic number is less than or equal $5(\Delta(G)+2)/2$ for any graph $G$ having maximum degree $\Delta(G)$ and isolated edges.

For more results about the adjacent vertex distinguishing edge coloring, the readers may refer to [10-25].

Here we address a natural extension of the very nice work by Jaradat [26] in which the chromatic number of products of graph were studied. The product we are taking is the usual direct product. The vertex set of $G \times H$ is the direct product $V(G \times H) = V(G) \times V(H)$ of the vertex sets of $G$ and $H$ and the edge set $E(G \times H) = \{(u,v) \mid u_1v_2 \in E(G) \text{ and } v_1u_2 \in E(H)\}$.

Let $P_m$, $P_n$ be a finite path, and $V(P_m) = \{0,1,\cdots,m-1\}$, $V(P_n) = \{0,1,\cdots,n-1\}$ are the vertex set of a finite path $P_m$, $P_n$, severally, where two vertices $x$ and $y$ are adjacent for every $x \in \{0,1,\cdots,n-1\}$, $y \in \{0,1,\cdots,m-1\}$ if and only if $|x-y| = 1$. And let $G = P_n \times P_m$ be the direct product of finite path $P_m$, $P_n$, clearly, the vertex set of $G$ may be denoted $\{(x,y) \mid x = 0,1,\cdots,n-1, y = 0,1,\cdots,m-1\}$.

Let $P_\infty$ be a infinite path, and $V(P_\infty) = \mathbb{Z}$ is vertex set of a infinite path $P_\infty$, where two vertices $x$ and $y$ are adjacent for every $x,y \in \mathbb{Z}$ if and only if $|x-y| = 1$. And let $Dp(d)$ be the direct product of $d$ infinite path, denoted by $Dp(d) = P_\infty \times P_\infty \times \cdots \times P_\infty$. Then the vertex set of $Dp(d)$ may be denoted $V(Dp(d)) = \{(x_1,x_2,\cdots,x_d) \mid x_i \in \mathbb{Z}\}$, where $d \geq 2$.

In this paper, we research that the twin edge-coloring of finite path and infinite path. Our thought here is to depict a somewhat general way to the twin edge-coloring, and acquires the chromatic number of finite path and infinite path. We refer to the books [27,28] for graph theory terminology and notation not defined in this paper.

### 2. The twin edge coloring of finite paths

For the twin edge coloring of the direct product of finite path $P_m$, $P_n$, we have the following result.

**Theorem 2.1** For any $m = 2$ and $n \geq 3$, if $n \equiv 2 \mod 3$, we have $\chi'_t(P_m \times P_n) = 3$.

**Proof** Let $G$ be a graph of the direct product of finite path $P_m$, $P_n$. By Lemma 1.1, we know that $\chi'_t(G) = \chi'_t(P_m \times P_n) \geq 3$. To prove that $\chi'_t(G) \leq 3$, we construct the twin edge 3-coloring of $G$.

Assumed that the color set is $\{0,1,2\}$, and for any $x = 0,1,\cdots,n-1$, $y = 0,1$, and $y \equiv 0 \mod 2$, command that

$$\sigma((x,y)(x+1,y+1)) = x \mod 3.$$

It can be easily seen that the chromatic number of $G$ is equal to 3.

In the first place, we proved that $\sigma$ is a proper edge coloring of $G$. For a vertex $(x,0)$ of $G$,
where $x = 0, 1, \ldots, n - 1$, by the definition of the coloring $\sigma$, we have

$$\sigma((x - 1, 1)(x, 0)) = (x - 1) \mod 3, \quad \sigma((0, 0)(x + 1, 1)) = x \mod 3.$$  

By definition of the coloring $\sigma$, $\sigma((x - 1, 1)(x, 0)) = \sigma((x, 0)(x + 1, 1))$ is equivalent to $2 \equiv 0 \mod 3$, which is done by contradiction. Therefore, the set of colors of the edges incident to $(x, 0)$ assigned the different color. Similarly, the set of colors of the edges incident to $(x, 1)$ assigned the different color.

From the above analysis, the coloring $\sigma$ is a proper edge coloring of $G$.

In the next place, we prove that a proper edge coloring $\sigma$ of a graph $G$ that induces vertex coloring $\sigma'$ is a proper vertex coloring. For any vertex $(x, 0), x = 1, 2, \ldots, n - 2$, by definition of the coloring $\sigma$ and $\sigma'$, we have

$$\sigma'(x, 0) = ((x - 1) \mod 3 + x \mod 3) \mod 3, \quad \sigma'(x - 1, 1) = ((x - 1) \mod 3 + (x - 2) \mod 3) \mod 3.$$  

Clearly, $\sigma'(x, 0) = \sigma'(x - 1, 1)$ and $\sigma'(x, 0) = \sigma'(x + 1, 1)$ is equivalent to $0 \equiv 1 \mod 3$ and $2 \equiv 1 \mod 3$, which is impossible. Otherwise, when $x = 0$ or $x = n - 1$, by definition of the coloring $\sigma$ and $\sigma'$, we have $\sigma'(0, 0) = 0, \sigma'(1, 1) = 0; \sigma'(n - 1, 0) = (n - 2) \mod 3, \sigma'(n - 2, 0) = (n - 2 - 2) \mod 3 \mod 3$.

If $\sigma'(0, 0) = \sigma'(1, 1)$, then $0 = 1$, therefore, $\sigma'(0, 0) \neq \sigma'(1, 1)$; If $\sigma'(n - 1, 0) = \sigma'(n - 2, 1)$, then $0 \equiv 2 \mod 3$ since $n \equiv 2 \mod 3$, therefore, $\sigma'(n - 1, 0) \neq \sigma'(n - 2, 1)$. In the same way, for any vertex $(x, 1), x = 1, 2, \ldots, n - 2$, we have the same results. Thus, It is clear that the coloring $\sigma'$ is a proper vertex colorings of $G$.

We can see from the above analysis, $\sigma$ is 3-twin edge colorings of $G$. Thus, $\chi'_pt(P_n \land P_m) = 3$.

If a graph $G$ exist two adjacent vertices of maximal degree $\Delta(G)$, we have $\chi'_pt(G) = \chi'_pt(G) \geq \Delta(G) + 1$. According to Theorem 2.1, for $n \geq m \geq 2$, we have $\chi'_pt(G) = \chi'_pt(G) = \Delta(G) + 1$.

3. The twin edge coloring of infinite paths

For the twin edge coloring of the direct product of infinite paths, we have the following result.

Theorem 3.1 $\chi'_pt (Dp(2)) = 5$.

Proof Clearly, $\chi'_pt (Dp(2)) = 4$. Therefore, $\chi'_pt (Dp(2)) \geq 5$. To prove that $\chi'_pt (Dp(2)) \leq 5$, we construct the twin edge 5-coloring of $Dp(2)$.

Assumed that the color set is $\{0, 1, 2, 3, 4\}$, and for any vertex $(x, y)$, command that

$$\sigma((x, y)(x + 1, y + \theta)) = (x - \theta + 1) \mod 5,$$

where $\theta = \pm 1$.

It can be easy to see that the chromatic number of $Dp(2)$ is equal to 5.

In the first place, we proved that the coloring $\sigma$ is a proper edge coloring of $Dp(2)$. For any vertex $(x, y)$ of $Dp(2)$, where $x = 0, 1, \ldots, m - 1, y = 0, 1, \ldots, n - 1$, by the definition of the coloring $\sigma$, we have

$$\sigma((x, y)(x + 1, y + \theta_1)) = (x - \theta_1 + 1) \mod 5, \quad \sigma((x, y)(x - 1, y + \theta_2)) = (x + \theta_2) \mod 5, \quad \theta_1 = \pm 1, \quad \theta_2 = \pm 1.$$

Obviously, when $\theta_1 = -1, \theta_1 = 1, \sigma((x, y)(x + 1, y - 1)) = \sigma((x, y)(x + 1, y + 1))$ is equivalent to $2 \equiv 0 \mod 5$, which is impossible. when $\theta_2 = -1, \theta_2 = 1, \sigma((x, y)(x - 1, y - 1)) = \sigma((x, y)(x + 1, y + 1))$ is equivalent to $4 \equiv 1 \mod 5$, which is contradictory. Analogously, when $\theta_1 = 1, \theta_1 = -1, \text{ or } \theta_2 = 1, \theta_2 = -1, \sigma((x, y)(x + 1, y + \theta_1)) = \sigma((x, y)(x - 1, y + \theta_2))$ is equivalent to $\theta_1 + \theta_2 = -1 = 3,4$. Thus, the coloring $\sigma$ is a proper edge coloring of $Dp(2)$.

In the second place, For any two vertices $u = (x, y), v = (x', y'),$ where $v = (x + 1, y \pm 1)$ or $v = (x - 1, y \pm 1)$. By the definition of the coloring $\sigma$ and $\sigma'$, we have

$$\sigma'(u) = (4x + 2) \mod 5, \quad \sigma'(v) = (4x + 1) \mod 5 \text{ or } \sigma'(v) = (4x + 3) \mod 5.$$
If $\sigma'(u) = \sigma'(v)$, then $2 \equiv 1 \mod 5$ or $2 \equiv 3 \mod 5$, consequently, $\sigma'(u) \neq \sigma'(v)$. It is clear that the coloring $\sigma'$ is a proper vertex colorings of $D_p(2)$.

We can see from the above analysis, the coloring $\sigma$ is 5-twin edge colorings of $D_p(2)$, hence $\chi'_s(D_p(2)) = 5$.

**Theorem 3.2** $\chi'_s(D_p(d)) = 2^d + 1$.

**Proof** We use mathematics induction to show them. When $d = 2$, by Theorem 3.1, the Theorem 3.2 to be true. Supposed that the Theorem 3.1 to be true when $2 \leq l \leq d$, that is to say $\chi'_s(D_p(l)) = 2^l + 1$. Let's prove $\chi'_s(D_p(l+1)) = 2^{l+1} + 1$. By the definition of the direct product, $\Delta(D_p(l+1)) \geq 2^{l+1} + 1$. Because $D_p(l)$ has two adjacent vertices of maximum degree vertex, then $\chi'_s(D_p(l+1)) \geq 2^{l+1} + 1$.

Now prove $\chi'_s(D_p(l+1)) \leq 2^{l+1} + 1$. Denote $H = D_p(l), G = P_2 \times H = D_p(l+1)$, and the edge disjoint subgraphs of $G$ is $H_{i,i+1} = P_2(l) \times H$, where $V(P_2(l)) = \{i,i+1\}$. Let the color set $C = A_0 \cup A_1$ and $A_0 \cup A_1 \neq \phi$, where $|A_0| = 2^l + 1$, $|A_1| = 2^l$. For any integer $i$, by induction, if $i = 0 \mod 2$, we determine a proper edge coloring of $A_0$ by using the $2^l + 1$ colors assigned by $C$ to the $H_{i,i+1}$. if $i = 1 \mod 2$, we determine a proper edge coloring of $A_1$ by using the $2^l$ colors assigned by $C$ to the $H_{i,i+1}$. The edges $H_{i,i+1}$ are the pairs $(i,i), (i,i+l)$ and $(i,i+1), (i,i)$, where $s,t$ are adjacent vertices of $H$. It is noteworthy that $H_{i,i+1}$ is bipartite, it is nonconnected and consists of two components isomorphic to $H$ if and only if $H$ is bipartite.

In any case, the maximal degree of $H_{i,i+1}$ coincides with the maximal degree of $H$, thus, as $H$ is regular of degree $d$, $H_{i,i+1}$ is regular of degree $d$ and $G$ has maximum degree $2^d$. there are adjacent vertices of degree $2^d$ and $\chi'_s(D_p(d)) \geq 2^d + 1$, and then $\chi'_s(D_p(d)) \geq 2^d + 1$ due to Lemma 1.2.

If a $k$-regular graph $G$ has an adjacent vertex distinguishing edge-coloring, we have $\chi'_s(G) \leq \chi'_s(G) = k + 1$. In accordance with the results of Theorem 3.1 and Theorem 3.2, we have $\chi'_s(D_p(d)) = \chi'_s(D_p(d)) = k + 1$.

**4. Conclusion**

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