On an affirmative solution to Michael’s acclaimed problem in the theory of Fréchet algebras, with applications to automatic continuity theory

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Abstract. In 1952, Michael posed a question about the functional continuity of commutative Fréchet algebras in his memoir, known as Michael’s problem in the literature. We settle this in the affirmative along with its various equivalent forms, even for the non-commutative case. Indeed, we continue our recent works, and develop two approaches to directly attack these problems.

The first approach is to show that the test case for this problem – the Fréchet algebra \( U \) of all entire functions on \( \ell^\infty \) – is, in fact, a Fréchet algebra \( C[[X]] \), if there exists a discontinuous character on \( U \).

In the second approach, the existence of discontinuous character on \( U \) allows us to generate another Fréchet algebra topology on \( U \), inequivalent to the usual Fréchet algebra topology.

In both the approaches, an important tool is a topological version of the (symmetric) tensor algebra over a Banach space. The elementary, but crucial, idea is to express the test algebra as a weighted Fréchet symmetric algebra over the Banach space \( \ell^1 \).

We summarize effects of our affirmative solutions on these attempts in addition to giving various important applications in automatic continuity theory, answering some long-standing problems from 1978.
1 Introduction

History of Michael’s problem. An important subject in the theory of Fréchet algebras is certain questions from automatic continuity theory, which may have applications to commutative rings and algebras, theory of several complex variables (briefly: SCV) and complex dynamical systems; e.g., the (non-)uniqueness of the Fréchet algebra topology (briefly: the Fréchet topology in the following) on certain Fréchet algebras, the (dis)continuity property of certain homomorphisms between certain Fréchet algebras as well as that of (higher point) derivations on Fréchet algebras.

In 1952, Michael posed a question about the functional continuity of commutative Fréchet algebras in his memoir [M], known as Michael’s problem in the literature. As we shall see, this problem has a very strong historical background since then. In fact, it is likely that the question was already discussed by Mazur in Warsaw around 1937 [DPR]. Several significant analysts worked on this problem since 1952, giving affirmative solutions for special classes of Fréchet algebras, or discussing various test cases, or discussing various approaches, or discussing various other equivalent forms, or deriving other important automatic continuity results such as the (non-)uniqueness of the Fréchet topology for certain commutative Fréchet algebras; especially,
Despite a lot of efforts by various mathematicians to solve Michael’s problem, it seems that only six significant ideas appeared in the literature since 1952 [Ar, Sh, Cl, DE, E3, DPR]. The strong partial result was obtained by Arens in 1958 [Ar]; he showed that each finitely (resp., rationally) generated, commutative Fréchet algebra is functionally continuous. As far as we know, the most latest effort was made by Dales, Read and the author in 2010 [DPR]; we showed that a well known test case for this problem—the Fréchet algebra \( \mathcal{U} \) of all entire functions on \( \ell^\infty \) [Cl, DE, E3]—is, in fact, a Fréchet algebra of power series (briefly: FrAPS) [DPR], and so, the natural question of whether every character (i.e., non-zero complex homomorphism) is automatically continuous on a Fréchet algebra which is a FrAPS, is Michael’s acclaimed problem itself in disguise. Incidentally, the author asked Želazko in 2004 whether Michael’s problem has an affirmative solution for all FrAPS [Z4, Personal communication]. Obviously, an affirmative answer to this question would
extend the Arens result from the singly generated case to the non-singly generated case within the class of FrAPS, and would, of course, solve Michael’s problem in view of Thm. 10.1 of [DPR].

**Our past work as a background.** So far, we have established the uniqueness of the Fréchet topology for FrAPS in one indeterminate [P1], and in several indeterminates [P3]. Dales, Read and the author have also obtained an affirmative solution (in a stronger form) to Dales-McClure’s problem (1977), and have also obtained other interesting automatic continuity results, including a reduction of Michael’s problem (discussed below) [DPR]. It is worthwhile mentioning that in the forthcoming papers, we have used the discontinuity of derivations to give other inequivalent Fréchet topology to certain Fréchet algebras, while attempting to solve Loy’s problem (1974) [P4], and by using the Read’s method [R], we also have constructed two maiden examples of Fréchet algebras admitting countably many mutually inequivalent Fréchet topologies to discuss the famous Singer-Wermer conjecture (1955) in Fréchet algebras [SW, T2, R, P5]; such examples are not known even in the Banach algebra case. All this work has some connection with Michael’s acclaimed problem; in fact, all this work has turned out to be stepping stones for the complete solution to the problem as we shall see below.
Since 1958, it was not clear to (functional) analysts how to extend the Arens result from the finitely generated case to the countably generated case, in order to obtain functional continuity of the test algebra $\mathcal{U}$; we remark that the Arens result, which is the only consistent general partial result about this problem, was based itself upon the (abstract) Mittag-Leffler theorem (an essential ingredient in the proof). Also, one strongly believes that there must be some ways to apply appropriate methods within automatic continuity theory as the problem falls in this theory. So, we represent two approaches in this paper. In both the approaches, an important tool is a topological version of the (symmetric) tensor algebra over a Banach space; the elementary, but crucial, idea is to express the test algebra $\mathcal{U}$ as a weighted Fréchet symmetric algebra over the Banach space $\ell^1$ [P4, DPR]. We also use the notion of “tensor product by rows”, introduced by Read [R], in the second approach.

We take this opportunity to give honor to all the past work, especially Dixon and Esterle’s approach [DE] and Esterle’s approaches [E2, E3]. We hope that the present work will encourage some people to invest some time and energy in order to make progress on the associated problem in the theory of SCV (by showing that these two problems are, indeed, equivalent); in particular, Dixon and Esterle may like to revisit their approaches, and Fornaess
and Stensones (and their team) may like to study the associated problem in
the theory of SCV in the light of the present work [BH, FS, Fo1, Fo2, Gl,
Ki, St]. Later, an introductory text for the reader’s sake could be “Banach
Algebras and Several Complex Variables” [W].

Preliminaries. Throughout the paper, “algebra” will mean a non-zero,
complex, (non-)commutative algebra with identity unless otherwise speci-
fied. A character on an algebra $A$ is a non-zero homomorphism from $A$
on onto $\mathbb{C}$; the collection of all characters on $A$ is the character space of $A$. A
topological algebra is an algebra which is a topological vector space in which
the ring multiplication is separately continuous (if an algebra is complete,
metrizable, then the multiplication is necessarily jointly continuous). We
shall write $S(A)$ (resp., $M(A)$) for the space of all (continuous) characters
on a topological algebra $A$ w.r.t. the relative $\sigma(A^\times, A)$-topology. We say
that $A$ is functionally continuous if every character on $A$ is continuous, and
$A$ is functionally bounded if every character on $A$ is bounded on bounded
sets.

We recall that a Fréchet algebra is a complete, metrizable locally con-
 vex algebra $A$ whose topology $\tau$ may be defined by an increasing sequence
$(p_m)_{m \in \mathbb{N}}$ of submultiplicative seminorms. The basic theory of Fréchet alge-

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bras was introduced in [Go2, M]. The principal tool for studying Fréchet algebras is the Arens-Michael representation, in which $\mathcal{A}$ is given by an inverse limit of Banach algebras $\mathcal{A}_m$. Obviously, if $\mathcal{A}$ is a Fréchet algebra, then it is functionally continuous if and only if it is functionally bounded [DF, Thm. 1]. However, there are commutative, complete LMC-algebras which are functionally bounded, but not functionally continuous; e.g., see [M, pp. 52-53].

A Fréchet algebra $\mathcal{A}$ is called a *uniform Fréchet algebra* if for each $m \in \mathbb{N}$ and for each $x \in \mathcal{A}$, $p_m(x^2) = p_m(x)^2$. Let $k \in \mathbb{N}$ be fixed. We write $\mathcal{F}_k$ for the algebra $\mathbb{C}[[X_1, X_2, \ldots, X_k]]$ of all formal power series in $k$ commuting indeterminates $X_1, X_2, \ldots, X_k$, with complex coefficients. A fuller description of this algebra is given in [D2, §1.6], and for the algebraic theory of $\mathcal{F}_k$, see [ZS, Ch. VII]; we briefly recall some notation, which will be used throughout the paper. Let $k \in \mathbb{N}$, and let $J = (j_1, j_2, \ldots, j_k) \in \mathbb{Z}^{+k}$. Set $|J| = j_1 + j_2 + \cdots + j_k$; ordering and addition in $\mathbb{Z}^{+k}$ will always be component-wise. A generic element of $\mathcal{F}_k$ is denoted by

$$\sum_{J \in \mathbb{Z}^{+k}} \lambda_J X^J = \sum \{\lambda_{(j_1, j_2, \ldots, j_k)} X_1^{j_1} X_2^{j_2} \cdots X_k^{j_k} : (j_1, j_2, \ldots, j_k) \in \mathbb{Z}^{+k}\}.$$  

The algebra $\mathcal{F}_k$ is a Fréchet algebra when endowed with the weak topology $\tau_c$ defined by the coordinate projections $\pi_I : \mathcal{F}_k \to \mathbb{C}, I \in \mathbb{Z}^{+k}$, where
\[ \pi_I(\sum_{J \in \mathbb{Z}^k} \lambda_J X^J) = \lambda_I. \] A defining sequence of seminorms for \( \mathcal{F}_k \) is \( (p'_m) \), where \( p'_m(\sum_{J \in \mathbb{Z}^k} \lambda_J X^J) = \sum_{|J| \leq m} |\lambda_J| \) (\( m \in \mathbb{N} \)). A Fréchet algebra of power series in \( k \) commuting indeterminates (briefly: FrAPS in \( \mathcal{F}_k \)) is a sub-algebra \( A \) of \( \mathcal{F}_k \) such that \( A \) is a Fréchet algebra w.r.t. a Fréchet topology \( \tau \), containing the indeterminates \( X_1, X_2, \ldots, X_k \), and such that the inclusion map \( A \hookrightarrow \mathcal{F}_k \) is continuous (equivalently, the projections \( \pi_I, I \in \mathbb{Z}^k \), are continuous linear functionals on \( A \)) [P3]. We defined a power series generated Fréchet algebra and discussed several examples of power series generated FrAPS in [BP]. One extends this notion in finitely many indeterminates case appropriately, and discusses analogous examples of power series generated FrAPS in \( \mathcal{F}_k \); e.g., \( \mathcal{F}_k \), the Beurling-Banach (Fréchet) algebras in \( \mathcal{F}_k \) (called the Beurling-Fréchet algebras of (semi)weight types in [P3]), \( \text{Hol}(U) \), \( U \) the open unit disc in \( \mathbb{C}^k \), \( \text{Hol}(\mathbb{C}^k), A^\infty(\Gamma^k) \), \( \Gamma \) the unit circle in \( \mathbb{C} \). We shall also require the non-commutative version of FrAPS in \( \mathcal{F}_k \), as defined in the case of countably many non-commuting indeterminates below.

We write \( \mathcal{F}_\infty \) for the algebra \( \mathbb{C}[[X_1, X_2, \ldots]] \) of all formal power series in countably many commuting indeterminates \( X_1, X_2, \ldots \), with complex coefficients. We shall also require the non-commutative version of \( \mathcal{F}_\infty \), denoted by \( \mathcal{B} = \mathbb{C}_{nc}[[X_1, X_2, \ldots]] \). A fuller description of these algebras is given in
[DPR, §9]; see also [E3] for $\mathcal{F}_\infty$. Both are Fréchet algebras under the usual topology $\tau_c$ of coordinatewise convergence defined by the coordinate projections $\pi_I$, $I \in (\mathbb{Z}^+)^{<\omega}$ (resp., $\pi_I$, $I \in S_{nc}$ the free semigroup in countably many (non-commuting) indeterminates $X_1, X_2, \ldots$, for $\mathcal{B}$), with defining sequence $(p'_m)$ of seminorms, where $p'_m(\sum_{J \in (\mathbb{Z}^+)^{<\omega}} \lambda_J X^J) = \sum \{|\lambda_J| : J \in (\mathbb{Z}^+)^m, |J| \leq m\}$ (m $\in$ $\mathbb{N}$) (resp., $p'_m(\sum_{J \in S_{nc}} \lambda_J X^{\otimes J}) = \sum \{|\lambda_J| : J \in \mathbb{N}^m, \text{rank} J \leq m\}$ (m $\in$ $\mathbb{N}$) for $\mathcal{B}$). Clearly, each $p'_m$ is a proper seminorm on $\mathcal{F}_\infty$ (resp., on $\mathcal{B}$). A Fréchet algebra of power series in countably many commuting indeterminates (briefly: FrAPS in $\mathcal{F}_\infty$) is a subalgebra $A$ of $\mathcal{F}_\infty$ such that $A$ is a Fréchet algebra w.r.t. a Fréchet topology $\tau$, containing the indeterminates $X_1, X_2, \ldots$, and such that the inclusion map $A \hookrightarrow \mathcal{F}_\infty$ is continuous (equivalently, the projections $\pi_I$, $I \in (\mathbb{Z}^+)^{<\omega}$, are continuous linear functionals on $A$); also a Fréchet algebra of power series in countably many non-commuting indeterminates (briefly: FrAPS in $\mathcal{B}$) is a subalgebra $A$ of $\mathcal{B}$ such that $A$ is a Fréchet algebra w.r.t. a Fréchet topology $\tau$, containing the indeterminates $X_1, X_2, \ldots$, and such that the inclusion map $A \hookrightarrow \mathcal{B}$ is continuous (equivalently, the projections $\pi_I$, $I \in S_{nc}$, are continuous linear functionals on $A$). We extend the notion of Fréchet algebra with a power series generator in countably many indeterminates case appropriately, and
discuss analogous examples of power series generated FrAPS in $\mathcal{F}_\infty$ (resp., FrAPS in $\mathcal{B}$); for example, $\mathcal{F}_\infty$ (resp., $\mathcal{B}$), the Beurling-Banach (Fréchet) algebras in $\mathcal{F}_\infty$ (resp., in $\mathcal{B}$); e.g., the Banach algebras $\mathcal{U}_m$ (resp., $\mathcal{U}_{nc_m}$) for each $m \in \mathbb{N}$, and the test case $\mathcal{U}$ (resp., $\mathcal{U}_{nc}$) as we shall see below.

We remark that we shall consider these algebras in the indeterminates $X_0, X_1, \ldots$, depending on our requirements, but this change will make no difference on their algebraic/topological structures, except the notational freedom that we want to avail (e.g., in §4). Some remarks on $\mathcal{F}_\infty$ are in order. The algebra $\mathcal{F}_\infty$ is a graded algebra which is not noetherian, as the ideal generated by $X_1, X_2, \ldots$, is not closed [DPR] (see also [E3, Prop. 2.2] or [Z3, Thm. 5]; the algebra $\mathcal{B}$ is also a graded algebra which is not noetherian for the same reason (Żelazko posed a question whether Thm. 5 holds true in the non-commutative case [Z3]; we conjecture that the answer of Żelazko’s question is in the affirmative). In [E3], Esterle remarks that $\mathcal{F}_\infty$ is an integral domain, all principal ideals in $\mathcal{F}_\infty$ are closed, but that he does not know whether or not all finitely generated ideals in $\mathcal{F}_\infty$ are closed. Here, we remark that the algebra $\mathcal{F}_\infty$ is a Fréchet algebra of finite type (introduced by Kopp in [K]) and so, all finitely generated ideals are closed in $\mathcal{F}_\infty$ [K, Rem., p. 222]. It was shown that there is a continuous,
injective homomorphism from $\mathcal{F}_\infty$ into $\mathcal{F}_2$ as an extension of Thm. 2.2 of [DPR] (see [DPR, Thm. 9.1]). It was also shown that the algebra $\mathcal{F}_k$ cannot be embedded algebraically in $\mathcal{F}_1 = \mathcal{F}$ for each $k \geq 2$ [DPR, Thm. 2.6]; using some topological arguments, a second proof of this theorem was given in Thm. 11.8 of [DPR]. The similar arguments for the algebra $\mathcal{F}_\infty$ allows us to establish the following

**Theorem 1.1.** There is neither algebraic nor topological embedding of $\mathcal{F}_\infty$ into $\mathcal{F}$.  

We remark that for each $k \in \mathbb{N}$, $\mathcal{F}_k$ is noetherian [ZS, Z3] whereas $\mathcal{F}_\infty$ is not [E3, Z3, DPR]. We shall also require in a future proof the ’averaging map’ and symmetrizing map $\tilde{\sigma}$ on $\mathcal{B}$, and symmetric elements of $\mathcal{B}$ [DPR]. There is a product in $\mathcal{B}_{\text{sym}}$, and with this product $\lor$, it is a commutative, unital algebra and $\mathcal{B}_{\text{sym}} = \epsilon(\mathcal{F}_\infty)$, where $\epsilon : \mathcal{F}_\infty \to \mathcal{B}$ is a continuous linear embedding [DPR]; in fact, it is a continuous, injective homomorphism as we shall see in Section 4.

**Structure of the article, with comments on the strategies.** In this paper, we shall be concerned with the affirmative solutions to Michael’s two problems in both the cases (commutative as well as non-commutative). We shall briefly discuss some general theory of topological (symmetric) tensor
algebras over a Banach space in the next section, in order to establish notation that will be used throughout the paper. We shall also discuss the Arens-Michael representations of such algebras. In particular, we are mainly interested in certain semigroup (Banach/Fréchet) algebras over two specific semigroups, which are graded subalgebras of $F_\infty$ (respectively, of $B$ in the non-commutative case). As we shall see, these algebras over the Banach space $\ell^1$ is the technical main-spring of the paper.

The first approach is discussed in Section 3. The essential ingredient is to describe the test algebra $U$ in terms of weighted Fréchet symmetric algebra $\hat{\bigvee}_WE$ over the Banach space $E = \ell^1$ (this was the example, discussed in [P4], that motivated the ideas of the present paper), and then, to show that the existence of a discontinuous character on $U$ leads us to produce a non-degenerate, totally discontinuous higher point derivation $(d_n)_{n \in \mathbb{Z}^+}$ on $U$ at this discontinuous character. This would imply that $U = \mathcal{F}$ by Thms. 10.1 and 11.2 of [DPR], a contradiction. (Discussions along these lines always skirt Dales-McClure’s problem (1977), solved affirmatively by Dales, Read and the author in 2010 [DPR], as well as Loy’s problem (1974) [L5], solved affirmatively in the Fréchet case by the author recently [P4].) As consequences, we have affirmative solutions to Michael’s two problems as follows.
Statements. 1. Every commutative Fréchet algebra is functionally continuous (see Cor. 3.2 below; for the non-commutative analogue, see Cor. 4.5 below). 2. Every character on a commutative, complete LMC-algebra is bounded (see Cor. 3.3 below).

We extend certain results about functional continuity of Stein algebras, established by Forster [For], Markoe [Ma] and Ephraim [Ep]. We also discuss implications of our solution in view of [E3, Thm. 2.7], and give further interesting remarks in automatic continuity theory.

The second approach is discussed in Section 4. Although we shall show that both the non-commutative case and the commutative case are dependent on each other ([DE] and Thm. 4.1 below), we shall work along the Read’s method to give another inequivalent Fréchet topology to the (non-)commutative test case $U$ (resp., $U_{nc}$), a contradiction to the fact that $U$ (resp., $U_{nc}$), being a (non-)commutative, semisimple FrAPS (even in $\mathcal{F}_\infty$) (resp., even in $\mathcal{B}$), has a unique Fréchet topology [C1, P1, P3, DPR] (resp., non-commutative analogue of Cor. 5.2, or Cor. 5.5 below). Our argument for this section is kept short because it uses the key ideas involved in [R].

In Section 5, we summarize the previous developments on Michael’s two problems in view of our affirmative solutions; e.g., we shall discuss Cor. 6 of
Carpenter’s result [C1] and shall also discuss Michael’s problem in view of (dis)continuity of the derivation $\partial/\partial X_0$ on $\mathcal{U}$ (resp., on $\mathcal{U}_{nc}$) [C2] (resp., Cor. 5.5 below). It is a known fact that another equivalent form of Michael’s problem is the long-standing problem of continuity of a homomorphism from a Fréchet algebra $B$ into a semisimple Fréchet algebra $A$ (an affirmative answer would give us the shortest proof of the uniqueness of the Fréchet topology for (non-)commutative, semisimple Fréchet algebras). We shall establish this result in both the cases (commutative as well as non-commutative). In fact, it is a surprising consequence that an attempt to solve the non-commutative analogue of this problem (as well as the problem of continuity of derivations on non-commutative, semisimple Fréchet algebras) leads us to certain important automatic continuity results as well as another approach to affirmatively answer Michael’s problem for more general complete, metrizable topological algebras (briefly: $(F)$-algebras) in both the cases (commutative as well as non-commutative) by extending Esterle’s result [E1], many thanks to Thomas stability lemma [T1]. We also extend some automatic (dis)continuity results of [DPR, §12], in order to complete the circle of ideas.
2 Topological tensor algebras over a Banach space

In the next sections, our important tool is a topological version of the (symmetric) tensor algebra over a vector space [Gre, Ch. III]. The topological version of the tensor algebra over a Banach (Fréchet) space appear in [Co, Le] ([V2]). For a general information about topological tensor products, we refer to [Gro, Tr]. However, a fuller description of what we require is given in [DM2, D2]; we briefly recall some notation, which will be used throughout the paper. Let \( E = \hat{\otimes}^1 E \) be a Banach space, and take \( \hat{\otimes}^0 E = \mathbb{C} \). For each \( p \geq 2 \), write \( \hat{\otimes}^p E \) (resp., \( \check{\otimes}^p E \)) for the completion of \( \otimes^p E \) w.r.t. the projective tensor product norm \( \| \cdot \|_\pi \) (resp., the equicontinuous tensor product norm \( \| \cdot \|_e \)). When it is unnecessary to distinguish, we may write \( \hat{\otimes}^p E \) for either completion, and \( \| \cdot \| \) for the specified norm. Then \( \hat{\otimes}E \) is a non-commutative, unital Fréchet tensor algebra over \( E \) (if \( \dim E > 1 \)) w.r.t. the product

\[
(\sum_p u_p) \otimes (\sum_p v_p) = \sum_p (\sum_{i+j=p} u_i \otimes v_j)
\]

and the coefficientwise convergence topology defined by an increasing sequence \( (\| \cdot \|_m) \) of seminorms, where \( \| \sum_p u_p \|_m = \sum_{p=0}^{m} \| u_p \| \). We refer to
the subspaces $\hat{\bigotimes}^p E$ as the homogeneous subspaces of $\hat{\bigotimes} E$.

We shall also need in a future proof a commutative analogue $\hat{\bigvee} E$ of $\hat{\bigotimes} E$, and a closed, linear subspace $\hat{\bigvee}^p E$ of $\hat{\bigotimes}^p E$. Then $\hat{\bigvee} E$ is a commutative, unital Fréchet symmetric algebra over $E$ w.r.t. the product

\[(2.2) \quad (\sum_p u_p) \vee (\sum_p v_p) = \sum_p (\sum_{i+j=p} u_i \vee v_j)\]

and the same coefficientwise convergence topology defined by $(\| \cdot \|_m)$. We remark that if $\tilde{\sigma} = \bigoplus \tilde{\sigma}_p$ is the continuous, symmetrizing epimorphism $\tilde{\sigma} : \hat{\bigotimes} E \to \hat{\bigvee} E$, then $\ker \tilde{\sigma}$ is the closed, two-sided ideal of $\hat{\bigotimes} E$ generated by $\{u \otimes v - v \otimes u : u, v \in \hat{\bigotimes}^1 E = E\}$, so that the algebras in [Co] and [Le] and the algebras $\hat{\bigvee} E$ are topologically isomorphic.

There are some important Banach subalgebras of the above examples [D2, Ex. 2.2.46 (ii), p. 186]. Let $\omega$ be a weight on $\mathbb{Z}^+$ and let $A$ be one of the algebras $\hat{\bigotimes} E$ and $\hat{\bigvee} E$. (When it can cause no confusion, the symbol $(\mathbb{Z}^+)^{<\omega}$ was used to denote the subsemigroup consisting of all $\mathbb{Z}^+$-valued sequences that are eventually 0 in Section 1.) Define $\hat{\bigotimes}_\omega E$ and $\hat{\bigvee}_\omega E$, respectively, as

\[(2.3) \quad \{u = \sum_{p=0}^{\infty} u_p \in A : \|u\|_\omega := \sum_{p=0}^{\infty} \|u_p\|\omega(p) < \infty\}.\]

We obtain two unital Banach subalgebras, with $\hat{\bigvee}_\omega E$ being commutative. These algebras are called weighted Banach tensor algebras and weighted
Banach symmetric algebras, respectively. The algebra $\hat{\otimes}_\omega E$, where $\omega \equiv 1$ on $\mathbb{Z}^+$, is the Banach tensor algebra which is the starting point for the constructions in [Co] and [Le]. We recall the following theorem from [DM2] (also, cf. [Le, Satz 2 and Satz 3]), describing the maximal ideal space and semisimplicity of $\hat{\vee}_\omega E$.

**Theorem 2.1.** (i) The space of characters on $\hat{\vee}_\omega E$ is homeomorphic with the ball $\{\lambda \in E' : \|\lambda\| \leq \inf_p \omega(p)^{\frac{1}{p}} \equiv \omega_\infty\}$ ($w^* - \text{topology}$).

(ii) $\hat{\vee}_\omega E$ (w.r.t. the equicontinuous tensor norm $\|\cdot\|_\varepsilon$) is semisimple if and only if $\omega_\infty > 0$.

(iii) $\hat{\vee}_\omega E$ (w.r.t. the projective tensor norm $\|\cdot\|_\pi$) is semisimple if and only if $\omega_\infty > 0$ and $E$ has the approximation property.

Along the same lines, for an increasing sequence $W = (\omega_m)$ of weights on $\mathbb{Z}^+$, we define $\hat{\otimes}_W E$ and $\hat{\vee}_W E$, respectively, as

\[
(2.4) \quad \{u = \sum_{p=0}^{\infty} u_p \in A : p_m(u) := \sum_{p=0}^{\infty} \|u_p\|\omega_m(p) < \infty \text{ for all } m \in \mathbb{N}\}.
\]

We obtain two unital Fréchet subalgebras, with $\hat{\vee}_W E$ being commutative. These algebras are called weighted Fréchet tensor algebras and weighted Fréchet symmetric algebras, respectively. Clearly, from (2.4), their Arens-
Michael representations are given by \( \hat{\bigvee}_W E = \bigcap_{m \in \mathbb{N}} (\hat{\bigvee}_{\omega_m} E, p_m) \) and \( \hat{\bigotimes}_W E = \bigcap_{m \in \mathbb{N}} (\hat{\bigotimes}_{\omega_m} E, p_m) \).

Next, we discuss the Arens-Michael representation of \( \hat{\bigvee} E \). For each \( m \), \( \hat{\bigvee} E / \ker \| \cdot \|_m \) is a Banach algebra. It is isomorphic with

\[
[\hat{\bigvee} E]_m = \left\{ \sum_{p=0}^m u_p : u_p \in \hat{\bigvee} E \right\},
\]

the norm being \( \| \cdot \|_m \), and the product being

\[
\left( \sum_{p=0}^m u_p \right) \lor \left( \sum_{p=0}^m v_p \right) = \sum_{p=0}^m \left( \sum_{i+j=p} u_i \lor v_j \right).
\]

Similarly, we also have an Arens-Michael representation of \( \hat{\bigotimes} E \).

We shall also require to study certain semigroup (Banach or Fréchet) algebras from [DPR, §9, 10], which are graded subalgebras of \( \mathcal{F}_\infty \) (resp., of \( \mathcal{B} \) in the non-commutative case). Mainly, we consider these algebras on the semigroup \( S = (\mathbb{Z}^+)^{<\omega} \) in the commutative case, and on the free semigroup \( S_{nc} \) in the non-commutative case; we shall discuss these examples, required for our approaches to solve Michael’s problem affirmatively, in Section 3. Most importantly, these algebras may be viewed as weighted Banach tensor algebras, weighted Banach symmetric algebras, and the test cases for Michael’s problem as their Fréchet analogues.

Set \( E = l^1(\mathbb{Z}^+) \), the Banach space for the remaining part of this section.
Then, as in [DPR, §10], for each $p \in \mathbb{N}$, $\hat{\otimes}^p E$ can be identified with $\ell^1((\mathbb{Z}^+)^p)$ as a Banach space. This Banach space can also be viewed as the space of absolutely summable functions on $(\mathbb{Z}^+)^p$ [Tr, Ch. 45] (equivalently, the space $A^+(D^p)$ of functions in $A(D^p)$ ($D^p$ the product of $p$-copies of $D$ in $\mathbb{C}$), with absolutely convergent Taylor series on $\Gamma^p$ (the product of $p$-copies of the unit circle $\Gamma$ in $\mathbb{C}$), w.r.t. the $\| \cdot \|_{\ell^1}$-norm on the space).

Here, we remark that the Banach space $\hat{\otimes}^p E$ is also a Banach algebra as follows. By [D2, 1.3.11], $\otimes^p E$ is a commutative, unital algebra w.r.t. the product

\[(2.7) \quad (f_1 \otimes \cdots \otimes f_p) \cdot (g_1 \otimes \cdots \otimes g_p) = (f_1 g_1 \otimes \cdots \otimes f_p g_p) \quad (f_i, g_i \in E, \ i \in \mathbb{N}_p).\]

Then, it is easy to see that it is a normed algebra w.r.t. the projective tensor norm $\| \cdot \|_\pi$ (note that $\| \cdot \|_\pi$ is submultiplicative on $\otimes^p E$). Now $\hat{\otimes}^p E$ is, indeed, a Banach algebra by noticing that the product can be extended to $\hat{\otimes}^p E$ (cf. [DM2, p. 315]). Thus, for each $p \in \mathbb{N}$, $\otimes^p E$ can be identified with $\ell^1((\mathbb{Z}^+)^p)$ as a Banach algebra via the homomorphism $\theta : (f_1 \otimes \cdots \otimes f_p)(r) \mapsto f_1(r_1) \cdots f_p(r_p)$, where $r = (r_1, \ldots, r_p) \in (\mathbb{Z}^+)^p$, and so, the character space of $\otimes^p E$ is homeomorphic with the polydisc $D^p$ for each $p \in \mathbb{N}$. The main point of this identification should be emphasized. The completion of the tensor power of finite copies of the Banach space (which is also a singly
generated Banach algebra) w.r.t. $\| \cdot \|_*$ is identified with a finitely generated Banach algebra, but, unfortunately, we cannot extend this to $\mathbb{N}$-fold tensor product within the framework of Banach algebras; however, we can extend this to $\mathbb{N}$-fold tensor product within the framework of Fréchet algebras (e.g., the algebra $\widehat{\bigotimes} E$ and its commutative analogue $\widehat{\bigvee} E$ below).

As above, $\widehat{\bigvee}^p E$ is a closed, linear subspace of $\widehat{\bigotimes}^p E$, consisting of the symmetric elements. Recall that $\widehat{\bigvee}^p E$ is the range of the symmetrizing map $\sigma_p$ (equivalently, the projection of norm 1) on $\widehat{\bigotimes}^p E$. In fact, since every element of $\widehat{\bigotimes}^p E$ is symmetric due to the fact that $\ell^1((\mathbb{Z}^*)^p)$ is identified with $A^+(D^p)$ and so, $\partial_k f(z) (z \in D^p)$ will be invariant under permutations of 1, 2, \ldots, $k$ (cf. [DM2, p. 320]). So, $\widehat{\bigvee}^p E = \widehat{\bigotimes}^p E$ by [Tr, Ch. 45] and [DM2, p. 321]. Thus, as above, for each $p \in \mathbb{N}$, $\widehat{\bigvee}^p E$ can be identified with $\ell^1((\mathbb{Z}^*)^p)$ as a Banach algebra.

Now, $\widehat{\bigotimes} E$ is a non-commutative, unital Fréchet tensor algebra over $E$ w.r.t. the product given in (2.1), and $\widehat{\bigvee} E$ its commutative analogue (w.r.t. the product given in (2.2)), consisting of the symmetric elements. Recall that $\widehat{\bigvee} E$ is the range of the symmetrizing map $\bar{\sigma}$ on $\bigotimes E$. The algebra $\bigotimes E$ is naturally identified with a graded subalgebra of $B$, and $\bigvee E$ is naturally identified with a graded subalgebra of $(B_{\text{sym}}, \lor)$, called the unital Fréchet
symmetric algebra over $E$. The latter algebra can also be viewed as the algebra of $p$-absolutely summable symmetric functions on $(\mathbb{Z}^+)_{<\omega} = \bigcup_{p \in \mathbb{N}} (\mathbb{Z}^+)^p$ for all $p \in \mathbb{N}$; i.e., if $f = \sum_p f_p$ is a symmetric function on $(\mathbb{Z}^+)_{<\omega}$, then $f_p$ is an absolutely summable function on $(\mathbb{Z}^+)^p$ for each $p \in \mathbb{N}$, and the former algebra $\hat{\otimes} E$ can also be viewed as the algebra of $p$-absolutely summable functions on $(\mathbb{Z}^+)_{<\omega}$.

Let $m \in \mathbb{N}$ be fixed. As commutative Banach algebras,

\[(2.8) \quad [\hat{\bigvee} E]_m \cong ([\hat{\bigvee} E]_m)_\mu = \{ \sum_{p=0}^m u_p \in [\hat{\bigvee} E]_m : \sum_{p=0}^m \| u_p \| = \sum_{p=0}^m \| u_p \| \},\]

where $\mu \equiv 1$ is a weight on the finite semigroup $(\mathbb{Z}^+)^{m+1} = \{0, 1, \ldots, m\}$ with semigroup operation addition modulo $m+1$ (cf. [BP, p. 144]). Similarly, there is a non-commutative analogue of this identification.

**Remarks A.** 1. We shall often write the Banach algebra $\hat{\bigvee}_\omega E$, where $\omega \equiv 1$ on $\mathbb{Z}^+$ as $\mathcal{U}_1 = \ell^1((\mathbb{Z}^+)_{<\omega})$ throughout the paper (see §3 below), since it is isometrically isomorphic with $\ell^1((\mathbb{Z})_{<\omega})$ [DPR]. So, by Thm. 2.1, it may be viewed as the algebra of functions in $A(\ell^\infty_{[1]})$ with absolutely convergent Taylor series (equivalently, the algebra of absolutely summable symmetric functions on $(\mathbb{Z}^+)_{<\omega}$). It is interesting to note that the proof of showing this algebra a BAPS, could drastically be shortened by noticing that the homomorphism, discussed in the proof of (ii) of Thm. 1.2 of [DM2], from this algebra into
\( \ell^1(\mathbb{Z}^+) \), is, indeed, an injection, and so, the range algebra is a BAPS, w.r.t. the norm transferred from this algebra (see a very long proof of (i) of Thm. 10.1 of [DPR] to claim the same fact).

2. Richard Aron and his team have worked a lot on the algebras of analytic functions on a Banach space. We, here, represent a “tensor approach” for the study of such algebras. We hope that the present work will encourage some people to invest some time and energy in order to make progress on the study of (locally) Stein algebras on (reduced Stein-)Banach spaces in the theory of SCV, possibly through this approach. In particular, they may find some interest (especially, from “tensor approach” point of view) in the Banach algebras \( U_1 = \ell^1((\mathbb{Z}^+)^\omega) \) and \( U_m \), for \( m \geq 2 \), and the Fréchet algebras \( \hat{\bigvee} E \) and \( U \) (see §3) in view of their study of these kinds of algebras of analytic functions on infinite-dimensional Banach spaces ([ACGLM, ACLM, ACG, AGGM, CGJ, Mu, Ry]).

3 **First Approach**

**Background.** Our first approach is based on generating a totally discontinuous higher point derivation \( (d_n)_{n \in \mathbb{Z}^+} \) of infinite order on the test algebra
$\mathcal{U}$ at a discontinuous character $\phi$, and then, applying the Dales-Patel-Read’s method from [DPR] to arrive at a contradiction. For general information about higher point derivation $(d_n)$ of infinite order on a topological algebra $A$ at a character, we refer to [DM1, D1]. We call $(d_n)$ “natural” on a topological algebra $A$ at a character $d_0$ (continuous or not), if $d_1$ is induced by $d_0$. In other words, $d_1$ is continuous (resp., discontinuous), if so is $d_0$. We remark that this notion extends the notion of totally discontinuous $(d_n)$, stated in [DM2, DPR]; in particular, for us, if $(d_n)$ is natural and totally discontinuous, then all $d_n$ $(n \in \mathbb{Z}^+)$ are discontinuous (of course, all $d_n$ $(n \in \mathbb{Z}^+)$ are continuous, if $(d_n)$ is natural and continuous). For examples of natural $(d_n)$, we refer to [D1, Ex. 6.4] and [DM1, p. 177] (in the Fréchet case, one can take $d_n(f) = \frac{f^{(n)}(z)}{n!}$, where $f \in \text{Hol}(U)$, $U$ the open unit disc in $\mathbb{C}$, and $d_n(f) = \frac{f^{(n)}(x)}{n!}$, where $f \in C^\infty([0,1])$ [DM1, p. 188]21), whereas for examples of non-natural $(d_n)$, we refer to [DM2, §2] (in the Fréchet case, one can take $d_n = \lambda_n \circ P_n$ on the test algebra $\mathcal{U}$ at a continuous character $d_0 = P_0$, where $\lambda_1$ is any discontinuous linear functional on the Banach space $E = \ell^1$). In fact, it is well-known that every higher point derivation $(d_n)$ of infinite order on the disc algebra $A(D)$ (resp., $H^\infty(U)$) at a character is always natural and continuous; the same holds for $\text{Hol}(U)$ and $\mathcal{F}$ in the
Fréchet case [D1, §8] (also, see Thm. 3.7 of [DM1] for the Fréchet algebra $C^\infty([0, 1])$). One can claim the same result for the analogous examples in the finitely many variables case, but certainly not in the infinitely many variables case, as discussed above, which is crucial in this approach. Thus, we will consider only “natural” higher point derivation $(d_n)$ of infinite order on $U$ at a character $d_0$ (continuous or not). For such a natural $(d_n)$ on $U$, it is easy to see that there is a (continuous or not) linear functional $\lambda = \lambda_1$ on $E$ such that $d_1 = \lambda_1 \circ P_1$ (equivalently, the diagram is commutative for a suitable linear functional $\lambda$ on $E$), since $U$ is a weighted Fréchet symmetric algebra over $E$ and $P_1$ is a projection onto $E$. Further, we remark that if $(d_n)$ is discontinuous on $U$ (i.e., if at least one of the $d_n$ is discontinuous), then $(d_n)$ must be totally discontinuous higher point derivation of infinite order (this is clear by looking at the identities (2.1) of [DM2]). As a consequence, if $(d_n)$ is natural and discontinuous on $U$, then $(d_n)$ is natural and totally discontinuous on $U$ in our sense; equivalently, the character $d_0$ must be discontinuous (the converse is trivial, since in that case, $d_1$ is discontinuous). So, if a character is continuous on $U$, then either $(d_n)$ is not natural (this situation arises in the Banach case as well; see [DM2, §2]), or $(d_n)$ is natural and continuous (i.e., $U$ is a FrAPS [DPR, Thm. 10.1 (ii)]).
Strategy. We now discuss our first approach, in order to solve Michael’s problem affirmatively. We see that the test case \(U\) is a Fréchet algebra of the kind \(\hat{\bigvee}_{\omega} E\), where \(E = \ell^1(\mathbb{Z}^+)\) and \(W = (\omega_m)\) is an increasing sequence of weights on \(\mathbb{Z}^+\) [P4], as follows. The Banach algebras \(\ell^1(S, \omega) = U_1\), where \(\omega \equiv 1\) on \(S = (\mathbb{Z}^+)^{<\omega}\), and for each \(m \geq 2, \ell^1(S, \omega_m) = U_m \cong \hat{\bigvee}_{\omega_m} E\), where \(\omega_m\) is a weight on \(\mathbb{Z}^+\) defined by \(\omega_m(|r|) = m^{|r|}, r \in S = (\mathbb{Z}^+)^{<\omega}\) (or, one may take \(\omega_m\) as a weight on \(S\) defined by \(\omega_m(r) = m^{|r|}\)), and a Fréchet algebra \(U = \bigcap_{m \in \mathbb{N}} U_m = \ell^1(S, W) \cong \bigcap_{m \in \mathbb{N}} \hat{\bigvee}_{\omega_m} E = \hat{\bigvee}_W E\), all graded subalgebras of \(F_\infty\) in the commutative case [DPR, Def. 9.2]. The non-commutative analogues are the Banach algebras \(\ell^1_{nc}(S_{nc}, \omega) = U_{1nc}\), where \(\omega \equiv 1\) on \(S_{nc}\), and for each \(m \geq 2, \ell^1_{nc}(S_{nc}, \omega_m) = U_{mnc} \cong \hat{\bigotimes}_{\omega_m} E\), where \(\omega_m\) is a weight on \(\mathbb{Z}^+\) defined by \(\omega_m(|r|) = m^{|r|}, r \in S_{nc}, |r| = \text{rank } r\) (or, one may take \(\omega_m\) as a weight on \(S_{nc}\) defined by \(\omega_m(r) = m^{|r|}\)), and a Fréchet algebra \(U_{nc} = \bigcap_{m \in \mathbb{N}} U_{mnc} \cong \bigcap_{m \in \mathbb{N}} \hat{\bigotimes}_{\omega_m} E = \hat{\bigotimes}_W E\) [DE, §2], all graded subalgebras of \(B\). We remark that the map \(\epsilon\), restricted to \(U_m\), is an isometric isomorphism of \(U_m\) onto \(\hat{\bigvee}_{\omega_m} E\) and the same map \(\epsilon\), restricted to \(U\), is an isometric isomorphism of \(U\) onto \(\hat{\bigvee}_W E\) [DPR, p. 142].

It is important to note the following chains of (dense) continuous inclu-
sions of certain algebras and their corresponding character spaces. We have

\[(3.1) \quad \mathcal{U} \hookrightarrow \mathcal{U}_m \hookrightarrow \mathcal{U}_1 = \ell^1(S, \{1\}) \cong \bigvee_{\{1\}} E \hookrightarrow \hat{\bigvee} E \rightarrow [\hat{\bigvee} E]_m,\]

where the last map is an epimorphism, being quotient Banach algebra, and the density follows from the fact that all algebras are countably generated by the monomials \(X_1, X_2, \ldots\). By [Go2, Lem. 3.2.5], we have

\[(3.2) \quad M\left(\bigvee_{\{1\}} E\right) \cong \ell^1_{[1]} \cong M(\ell^1(S, \{1\})) \hookrightarrow M(\mathcal{U}_m) \cong \ell^\infty_{[m]} \hookrightarrow M(\mathcal{U}) \cong \ell^\infty\]

\((w^*-\text{topology}),\) where the homeomorphisms are due to Thm. 2.1 and [DE], respectively. We also have non-commutative analogues of (3.1) and (3.2), but with the same character spaces in (3.2) (e.g., the character spaces of \(\mathcal{U}\) and \(\mathcal{U}_{nc}\) are same; see [DE, Rem. 2.2]).

Next, we recall that the test case \(\mathcal{U}\) is, indeed, a FrAPS in \(\mathcal{F}\) by [DPR, Thm. 10.1 (ii)]. We claim that there exists a natural, discontinuous higher point derivation \((d_n)\) on \(\mathcal{U}\), induced by a discontinuous character \(d_0 = \phi\); such a \((d_n)\) obviously turns out to be non-degenerate and totally discontinuous in our sense, as discussed above. Note that if we prove our claim we have our desire result; for since such a \((d_n)\) gives us an epimorphism \(\theta\) from \(\mathcal{U}\) onto \(\mathcal{F}\) by [DPR, Thm. 11.2], and then, we may apply the Dales-Patel-
Read’s method, used in the proof of Thm. 10.1, in our case to show that \( \theta \) is indeed an isomorphism (i.e., \( \mathcal{U} = \mathcal{F} \)), a contradiction to our assumption of the existence of a discontinuous character on \( \mathcal{U} \) (cf. Remarks D. 5 below).

The main point of the following theorem should be emphasized. It is a surprising consequence of the fact that one is able to show that the test case \( \mathcal{U} \) is, indeed, a weighted Fréchet symmetric algebra \( \hat{\bigvee}_w E \) as above, and so, if one starts with a discontinuous character on \( \mathcal{U} \), then one can construct \( (d_n) \) by applying the Dales-McClure method \([DM2, D2]\) as follows.

**Proof of Theorem 3.1 below.** Assume that there is a discontinuous character \( \phi \) on \( \mathcal{U} = \bigcap_{m \in \mathbb{N}} \mathcal{U}_m \). Then, for each \( m \in \mathbb{N} \), \( \phi \) is a discontinuous character on the normed algebra \( (\mathcal{U}, q_m) \), because if it is continuous for some \( m \), then there is \( c > 0 \) such that \( |\phi(f)| \leq cq_m(f), f \in (\mathcal{U}, (q_m)) \), a contradiction to the assumption that \( \phi \) is discontinuous on \( \mathcal{U} \). Since the norm \( q_1 \) is equivalent to the norm \( \| \cdot \|_{\ell^1} \) on \( \ell^1 \subset \mathcal{U}, \phi\|_{\ell^1} \) is a discontinuous linear functional on \( \ell^1 \) (because if it is a continuous linear functional on \( \ell^1 \), then it can be extended to a continuous character on \( (\mathcal{U}_1, q_1) \) (as shown at the end of this paper, but in the Banach case), a contradiction to the fact that it is discontinuous on the normed algebra \( (\mathcal{U}, q_1) \) as above). There is also a bit longer route to obtain a discontinuous linear functional on \( \ell^1 \), induced
by \( \phi \), as follows. \( \phi \) can be extended to a discontinuous linear functional \( \lambda_1 \) on \((U_1, q_1)\) using a Hamel basis of a complement of \( U \) in \( U_1 \) (remark that there are infinitely many ways of extending \( \phi \); also, we cannot extend \( \phi \) as a discontinuous character on \( U_1 \)). Hence \( \lambda_1|_{\ell^1} \) is also a discontinuous linear functional on \( \ell^1 = \bigvee_{\omega_1}^1 E \) [DM2, §2]. (This argument uses the ZFC alone; see Remarks D. 5 below.) We write \( \phi|_{\ell^1} \) as \( \lambda_1 = \lambda \).

Now, by [DM2, Thm. 2.1 (ii)], there are linear functionals \( \lambda_n \) on \( \text{SE}_n(n) \), for \( n = 2, 3, \ldots \), such that (2.1) of [DM2] holds for any positive integers \( n \) and \( m \), any \( u \in \text{SE}_m(n) \), and any \( v \in \text{SE}_n(m) \). As discussed in the beginning of §2 of [DM2], define \( d_n = \lambda_n \circ P_n \), for \( n \in \mathbb{N} \). Then \( (d_n) \) will be a “natural”, discontinuous higher point derivation of infinite order on \( U \) at a discontinuous character \( d_0 = \phi \) (note that since \( U \) is a subalgebra of \( \bigvee E \), we are free to apply certain arguments from §2 of [DM2]). Moreover, \( (d_n) \) is indeed non-degenerate and totally discontinuous in our sense. This is clear since \( d_1 \) is discontinuous on \( U \) as \( d_0 \) is discontinuous on \( U \) (see proof of Thm. 11.2 of [DPR]), and the discontinuity of the remaining \( d_n \) on \( U \) also follows from the fact that each \( \lambda_n \) \((n \geq 2)\) is discontinuous since \( \lambda_1 \) is discontinuous and the fact that (2.1) of [DM2] holds.

Thus we have the required existence of a natural, discontinuous higher
point derivation \((d_n)\) of infinite order on \(\mathcal{U}\) at a discontinuous character \(\phi\).

By [DPR, Thm. 11.2], this \((d_n)\) induces an epimorphism \(\theta\) from \(\mathcal{U}\) onto \(\mathcal{F}\).

Now, we show that \(\theta\) is, indeed, an injection. Our argument here is kept short because it uses the key ideas involved in the proof of Thm. 10.1 of [DPR], but the map \(\theta\) that we are working with, is entirely different (i.e., induced by a natural, totally discontinuous \((d_n)\) in our sense).

Our first remark is the following. There is a sequence \((g_i)_{i \in \mathbb{N}}\) of singular elements in \(\mathcal{F}\) such that \(\theta(X_i) = g_i\). We claim that at least one member of this sequence has the order 1. Indeed, assume towards a contradiction that \(o(g_i) > 1\) for all \(i \in \mathbb{N}\). Since \((X_i)_{i \in \mathbb{N}}\) generates the Fréchet algebra \(\mathcal{U}\), for any singular element \(f\), \(o(\theta(f)) > 1\); equivalently, \(X\) is not in the range of \(\theta\), a contradiction of the fact that \(\theta\) is an epimorphism. In fact, w.l.o.g., we may assume that \(\theta(X_1) = g_1 = X\), since if \(g_1\) has the initial form \(X\) in \(\mathcal{F}\), then there is an automorphism \(\alpha\) of \(\mathcal{F}\) such that \(\alpha(g_1) = X\), and in this case, \(\alpha \circ \theta = \Theta\) is a discontinuous epimorphism such that \(\Theta(X_1) = X\).

Our main claim is that we can choose the sequence \((g_i)_{i \in \mathbb{N}}\) of singular elements in \(\mathcal{F}\) with \(g_1 = X\) such that \(o(g_i) \geq i\) \((i \in \mathbb{N})\) so that the map \(\theta\) is an injection. Incidentally, it is a somewhat surprising fact that we can follow proof of Thm. 10.1 of [DPR] in our case since the arguments there
are purely set-theoretic and algebraic while dealing with selection of \( (g_i) \) in \( \mathcal{F} \) with the stated conditions and Lem. 10.2 of [DPR] save perhaps for the arguments to show that \( \theta \) is an injection before proof of Lem. 10.2. Even that part of arguments can be managed, if we suppose that \( q_1(f) = 1 \) for a non-zero element \( f \in \mathcal{U} \subset \mathcal{U}_1 \) (recall that \( q_1 = \| \cdot \|_1 \) and \( \mathcal{U}_1 = A = \ell^1(S) \) in our notations). Thus, \( \mathcal{U} = \mathcal{F} \), a contradiction.

It is essential to emphasize two important remarks of our proof as follows. First, we can choose such a sequence \( (g_i) \) in \( \mathcal{F} \) since the range of \( \theta \) is the whole \( \mathcal{F} \), which allows necessary freedom in selection of \( (g_i) \). Secondly, having had the sequence \( (g_i) \) in \( \mathcal{F} \) with the stated conditions, it is not necessary that the map \( \theta \) on \( \mathcal{U} \) would always be a continuous injection; one might have a “unique” discontinuous isomorphism (in other words, the uniqueness of the injective map \( \theta \) in proof of Thm. 10.1 of [DPR] is up to the continuity, and the uniqueness of the discontinuous isomorphism \( \theta \) in our proof is up to the injectivity, which means that if we drop the injectivity, then we may have a discontinuous epimorphism, say, induced by a “non-natural”, discontinuous \( (d_n) \) on \( \mathcal{U} \); see Remarks B. 1 below).

**Theorem 3.1.** All characters on the commutative Fréchet algebra \( (\mathcal{U}, (q_m)) \) are continuous. In particular, these characters are, indeed, the point evalu-
ation mappings, that is, if \( \phi \) is a character on \( U \), then there exists \( z \in \ell^\infty \) such that \( \phi = \phi_z \), where \( \phi_z(f) = f(z) \) for all \( f \in U \).

\[ \Box \]

**Remarks B.** 1. The author asked whether every (surjective) homomorphism \( \theta : B \to \mathcal{F} \) from a non-Banach Fréchet algebra \( B \) is continuous [P1, p. 135]. Then, Dales, Read and the author further took up this question, in order to affirmatively answer the Dales-McClure’s problem from 1977 [DPR, Thm. 11.2]. Now, we apply the Dales-McClure’s method and the Dales-Patel-Read’s method to obtain Thm. 3.1 above. This clearly establishes the fact that how all this work has turned out to be the stepping stones for the complete solution to Michael’s problem since 2004, as discussed in beginning of Introduction.

In view of our approach here, some crucial remarks on the Dales-McClure’s method and the Dales-Patel-Read’s method are in order. As discussed above (and in Section 2 of [DM2] in the Banach case), one may start with any discontinuous linear functional \( \lambda \) on \( \ell^1 \), in order to produce a “non-natural”, totally discontinuous \( (d_n) \) of infinite order on \( U \) (resp., \( U_m \) for each \( m \) in the Banach case) at a continuous character \( d_0 = P_0 \). This would then lead us to a discontinuous epimorphism \( \theta \) from \( U \) (resp., \( U_m \) for each \( m \)) onto \( \mathcal{F} \) by Thm. 11.2 of [DPR] (resp., Thm. 2.3 of [DM2]). Suppose that this \( \theta \) is an
injection, then it is a unique, discontinuous isomorphism, and so, \( U = F \)
(resp., \( U_m = F \) for each \( m \)), a contradiction of the fact that \( U \) (resp., \( U_m \)
for each \( m \)) is a semisimple algebra and \( F \) is a local algebra. Thus, this \( \theta \)
cannot be an injection. (Alternatively, it is easy to see that this \( \theta \) cannot
be an injection, because \( \ker \theta \) is obviously a prime ideal of \( U \) (resp., \( U_m \)
for each \( m \)), which is dense (and so, non-null as well) in the kernel of a continu-
ous character \( d_0 \) of \( U \) (resp., \( U_m \) for each \( m \)), such that the quotient algebra
\( U/\ker \theta \) (resp., \( U_m/\ker \theta \) for each \( m \)) is isomorphic to \( F \) (see [E3, Rem. 3-17
(2)] for the Banach case).) On the contrary, if one starts with a discontinuous
linear functional \( \phi|_{\ell^1} = \lambda_1 \), induced by a discontinuous character \( d_0 = \phi \)
on \( U \), then one produces a natural, totally discontinuous \( (d_n) \) on \( U \) in our
sense, and this \( (d_n) \) would then produce a unique, discontinuous isomorphism
\( \theta \) by applying the key ideas from the Dales-Patel-Read’s method, a con tra-
diction. (Alternatively, as discussed above, \( \ker \theta \) is obviously a prime ideal
of \( U \), which is dense in the kernel of a discontinuous character \( d_0 = \phi \) of
\( U \) (and so, it is also dense in \( U \)), such that the quotient algebra \( U/\ker \theta \) is
isomorphic to \( F \); the density of \( \ker \theta \) in \( U \) shows that either \( \ker \theta = \ker d_0 \)
or, \( \ker d_0 = U \), but since \( \ker d_0 \) is a maximal ideal, \( \ker d_0 \neq U \) and since
\( U/\ker \theta = F \), \( \ker \theta \neq \ker d_0 \), which implies that \( \ker \theta \) is null, leading to a
contradiction.)

We remark that Dales and McClure cooked their method up to show that the statement (C) of [DM1] is false, leading to an important contribution in the theory of automatic continuity (cf. Thm. 2.2 and a question in [DM2]). Thus, this method provides an ingenious construction of linear functionals $\lambda_n$ on $\text{SE}_\pi(n)$ (for $n \geq 2$), and was used on purpose (so is in our case as well). Finally, the Dales-Patel-Read’s method has been used in two extreme cases: (1) to show that the test case $U$ for Michael’s problem is indeed a FrAPS, where the corresponding $(d_n)$ is natural and continuous [DPR, Thm. 10.1]; and (2) to solve Michael’s problem in the affirmative, and here, the corresponding $(d_n)$ is natural and totally discontinuous in our sense to arrive at a contradiction.

2. There are no characters $\phi$ on $U = \text{Hol}(\ell^\infty)$, not equal to evaluation at $z \in \ell^\infty$, such that $\phi(f) = f(z)$ whenever $f \in \text{Hol}(\ell^\infty)$ is finitely determined (cf. [Cl, Thm. 9]). More generally, by [Cr, Thm. 4.4], there are no characters $\phi$ on $\Upsilon(\ell^\infty) = O(\ell^\infty)$, not equal to evaluation at $z \in \ell^\infty$, such that $\phi(\tilde{f}) = f(z)$ whenever $\tilde{f} \in \Upsilon(\ell^\infty) = O(\ell^\infty)$ is a finitely determined function germ.

3. Esterle showed that the continuous character on $U$ is a point evaluation mapping [E3, Prop. 2.5]. Dales, Read and the author showed that the
character $\pi_0$ on a FrAPS $\mathcal{U}$ is continuous [DPR, Cor. 11.5]. These two results jointly was another starting point for the author to affirmatively answer Michael’s problem, because it is easy to see that, for some $z \in \ell^\infty$, $\phi_z = \pi_0$.

Recall that the author asked Želazko in 2004 [Z4] whether every FrAPS is functionally continuous. The reason to ask this question was that the author had a strong belief that it should be very difficult to find a FrAPS with a discontinuous character (because in that case, the maximal ideal of this character would be dense in the closed maximal ideal $\ker \pi_0$, and the author has been consistently using the fact somewhat in the opposite direction for the case of principal maximal ideal [P2, p. 472] (also, [P6, pp. 21-22])).

As a corollary, we have an affirmative solution to the Michael’s problem.

**Corollary 3.2.** Every commutative Fréchet algebra is functionally continuous. 

**Remark C.** The author extends the notion of Stein (resp., Riemann) algebras to locally Stein (resp., Riemann) algebras in [P6] (resp., in [P2]). As an application of this corollary, we see that every character on a (locally) Stein (resp., Riemann) algebra is necessarily continuous, extending the results of [For], [Ma] and [Ep] (Markoe extended the Forster’s result by taking the dimension of $S(X)$, the singularity set of $X$, is finite in place of the dimension
of $X$ is finite, and Ephraim further extended the Markoe’s result (see [Ep] for details) by exposing the elementary nature of the Forster’s theorem. Recall that any character on a Stein algebra $A$ is a point evaluation map if $M(A)$ is finite dimensional [For, Thm. 4], and Ephraim extended this result [Ep, Thm. 2.3]. We deduce from Thm. 3.1 that the Forster’s theorem holds true for any Stein algebra $A$, that is, all characters on the Stein algebra $A$ are the point evaluation mappings. Not only this, but we have $X = M(A)$ in view of our result and [For, Thm. 1], that is, $X$ is a domain of holomorphy (cf. [Go2, Chs. 2, 11, 12]).

We have a few more interesting consequences as follows. First, recall that Michael also asked whether every character on a commutative, complete LMC-algebra is bounded [M, §12, Que. 2]. Dixon and Fremlin showed that the two problems are, in fact, equivalent [DF]. Akkar gave in a nice interpretation of this fact: the two problems are equivalent because every complete LMC algebra is, as a “bornological algebra”, isomorphic to inductive limit of a family of Fréchet algebras [Ak]. The equivalence of the two questions follows also from the fact that there is a complete LMC-algebra, produced by Craw [Cr] (see below), such that the existence of a discontinuous character on some commutative, unital Fréchet algebra would imply the
existence of an unbounded character on the complete LMC-algebra. We have the following

**Corollary 3.3.** Every character on a commutative, complete LMC-algebra is bounded.

We remark that it is easy to find a non-metrizable, commutative, complete LMC-algebra that is not functionally continuous [M, Prop. 12.2 and Rem.] (that is, there is a bounded, discontinuous character on this algebra, however, every continuous character on $C(T)$ is a point evaluation mapping for some $t \in T$ [Cr, §4]); surprisingly, below, we give an example of a commutative, non-metrizable, non-complete LMC-algebra that is functionally continuous and functionally bounded. We take this opportunity to correct an obvious typo in (b) of Thm. 9.3 of [DPR] as follows (cf. [Cl] and [E2]).

**Corollary 3.4.** Let $\lambda = (\lambda_n) \in \ell^1 \setminus c_{00}$, and let $g = \sum_{n=1}^{\infty} \lambda_n X_n$. The following statements are equivalent.

(i) There are no non-zero characters on the quotient algebra $\mathcal{M}/\mathcal{I}$, where $\mathcal{M}$ is the closed maximal ideal $\{f \in \mathcal{U} : f(0, 0, \ldots) = 0\}$ and $\mathcal{I} = \bigcup_{n \in \mathbb{N}} \mathcal{I}_n$, where $\mathcal{I}_n = X_1 \mathcal{U} + \cdots + X_n \mathcal{U}$, is a prime ideal in $\mathcal{U}$, which is dense in $\mathcal{M}$ and distinct from $\mathcal{M}$.
(ii) There are no characters on the quotient algebra $\mathcal{U}/\mathcal{I} + (g - 1)\mathcal{U}$. □

**Remarks D.** 1. Esterle introduced notions of Picard-Borel ideal and Picard-Borel algebra in [E3]. From above corollary, it is clear that there does not exist a maximal Picard-Borel ideal of $\mathcal{V} = \mathcal{U}/\mathcal{I}$ distinct from $\mathcal{M}/\mathcal{I}$ of codimension 1. So, all maximal Picard-Borel ideals are of infinite codimension.

2. In view of [E3, §6], the quotient algebra $\mathcal{V}$ certainly does not possess the finite extension property, if we show that the converse of Thm. 3.3 (or, Cor. 3.4) of [DE] holds true. We remark that since there are maximal Picard-Borel ideals of infinite codimension, one has a finite family $(v_1, \ldots, v_p)$ of elements of $\mathcal{V}$ such that $\sigma(v_1, \ldots, v_p) \neq (\chi_0(v_1), \ldots, \chi_0(v_p))$, where $\chi_0$ is the unique character of $\mathcal{V}$. It would be interesting to investigate whether one can show that $\mathcal{V}$ does not possess the finite extension property by using this fact.

Esterle pointed out in abstract of [E3] new algebraic obstructions to the construction of discontinuous characters on $\mathcal{U}$ related to the Picard theorem, and relate to extension properties of joint spectra of finite families of a quotient of $\mathcal{U}$ a question about iteration of Bieberbach mappings raised in [DE]. So, he found some difficulties while looking for a negative answer to Michael’s problem. This was also one of the starting points for the author to work on positive direction(s).
3. In view of [E3, Cor. 2.20], in presence of the continuum hypothesis (CH), the quotient algebras \( V \) and \( V(g) = U/I + (g - 1)U \) are not normable. It would be interesting to check whether there exist models of set theory, including the axiom of choice (AC), for which the algebra \( V \) is not normable (cf. the commutative analogue of Thm. 5.3 below). However, one believes that there may exist models of set theory, including AC, in which \( V \) is normable. For this, one may work along the line of the result of Allan in [A1, A2], obtained by using the ZFC alone (see below remark), and so, \( U \) is normable, since it is a FrAPS, by [DPR, Thm. 10.1 (ii)]. Then one requires to use certain facts about the algebras \( U, U/I_m \) (resp., \( U/I_m + (g - 1)U \)) for \( m \in \mathbb{N} \) and \( V \) (resp., \( V(g) \)) from [E3] in order to show that the quotient algebra \( V \) (resp., \( V(g) \)) can (algebraically) be embedded in \( \mathcal{F} = \mathbb{C}[[X]] \). Thus, in presence of ZFC alone, the quotient algebras \( V \) and \( V(g) \) may be normable.

4. By [Cl], \( (\ell^\infty, w^*) \) is not a first countable (and hence, not metrizable) topological space; in fact, it is not even a \( k \)-space, but it is a Lindelöf and completely regular space. Hence the algebra \( C(\ell^\infty) \) of continuous functions on \( \ell^\infty \) is not a Fréchet algebra by [Go2, Thm.]; it is just a commutative, non-complete LMC algebra [D2, Prop. 4.10.20] (cf. comments succeeding to Cor. 3.2 of [E4]), however, \( \text{Hol}(\ell^\infty) \) is a Fréchet algebra [Cl, Prop. 3]. Thus one
sees the existence of a functionally continuous, functionally bounded, non-complete, non-metrizable LMC algebra (remark that all the characters on $C(\ell^\infty)$ are bounded, since $(\ell^\infty, w^*)$ is replete [D2, Cor. 4.10.23]; not only this, but $C(\ell^\infty)$ is functionally continuous by [D2, Thm. 4.10.24]), whose closed subalgebra $\text{Hol}(\ell^\infty)$ is also a functionally continuous, functionally bounded Fréchet algebra. The author does not know any such examples in the literature. The commutative, complete, unital LMC-algebra $\Upsilon(\ell^\infty) = O(\ell^\infty)$ of germs of analytic functions in a neighbourhood of $\ell^\infty$, is another interesting algebra which is functionally continuous (and hence, functionally bounded) [Cr, §4].

5. Allan exhibited a discontinuous homormorphism between two commutative, unital Fréchet algebras having certain properties [A2, Thm. 8]; but, in the construction of discontinuous homomorphism, he used a continuous homomorphism from $A$ into $\mathcal{F}$, induced by a natural, continuous higher point derivation $(d_n)$ of infinite order on $\text{Hol}(\mathbb{C})$ at a continuous character $d_0$. He used only the ZFC axioms (and not the CH). Similarly, Read used the ZFC axioms alone to give an inequivalent Fréchet algebra topology on the algebra $\mathcal{F}_\infty$, in order to show that the Singer-Wermer conjecture fails in the Fréchet case. However, we use the ZFC alone to establish the Michael problem in the
affirmative (see §4 below). Thus we, here, establish $N\mu H_{F, \mathbb{C}}$, where $F$ is any Fréchet algebra (see [D1, §9] for more details). We recall that Dales, Read and the author also used only the ZFC axioms to establish several important results in automatic continuity theory, including an affirmative answer to the Dales-McClure problem from 1977 and showing that the test algebra $U$ for the Michael problem is, indeed, a FrAPS (we remark that Michael’s problem arises in the ZFC).

6. Other test algebras were discussed by Dixon and Esterle [DE] (non-commutative case), Esterle [E3], Schottenloher [Sc], Mujica [Mj], Muro [Mu] and Vogt [V2].

## 4 Second approach

**Background.** Having solved Michael’s problem for commutative Fréchet algebras, we now discuss the second approach to solve this problem for non-commutative Fréchet algebras. It is clear that the method used in the first approach works only for commutative Fréchet algebras, so it is the need of the hour to develop another approach, which would work for the non-commutative case. However, we show that solving problem for the commu-
tative case would suffice to solve the problem for the non-commutative case in the following proposition. We remark that if \( \phi \) is a discontinuous character on some non-commutative, unital Fréchet algebra \( A \), then there is a discontinuous character on the test case \( U_{nc} \) [DE, Pro. 2.1]; the same statement holds for the case \( U \) as well [E3, Thm. 2.7] and [DPR, Thm. 9.3].

**Proposition 4.1.** There is a discontinuous character on a commutative, unital Fréchet algebra if and only if there is a discontinuous character on a non-commutative, unital Fréchet algebra.

**Proof.** First, we remark that it is sufficient to prove this proposition for the test cases \((U, \tau_0)\) and \((U_{nc}, \tau_0)\). Let \( \phi \) be a discontinuous character on \( U_{nc} \). Then \( \phi \) does not belong to \( M(U_{nc}) \cong \ell^\infty [Cl, E2] \). Since \( M(U_{nc}) = M(U) \cong \ell^\infty \) by [DE, Rem. 2.2], there is a discontinuous character, say \( \phi \), on \( U \) as well. The same holds in the reverse direction also. In fact, we can associate these discontinuous characters as follows. Recall that \( U_{nc} \cong \hat{\bigotimes}_WE \), a weighted Fréchet tensor subalgebra of \( \hat{\bigotimes}E \), is a graded subalgebra of \( \mathcal{B} \). Hence, we have a continuous symmetrizing epimorphism \( \hat{\sigma} = \bigoplus_{p} \sigma_p : U_{nc} \to U_{nc} \), where \( \sigma_p : U^{(p)}_{nc} \to U^{(p)}_{nc} \) is the averaging map on \( U^{(p)}_{nc} \). Clearly, if \( \phi \) is a discontinuous character on \( U \), then \( \phi \circ \hat{\sigma} = \psi \) is a discontinuous character on \( U_{nc} \). For the reverse direction, since \( \epsilon_{|U} : U \to U_{nc} \) is a natural inverse of \( \pi_{|U=\hat{\sigma}(U_{nc})} \),
(defined below), if $\psi$ is a discontinuous character on $U_{nc}$, then $\phi = \psi \circ \epsilon|_U$ is a discontinuous character on $U$. □

**Strategy.** Next, we show that if $\phi$ is a discontinuous character on $U$ (resp., on $U_{nc}$), then $\phi|_{\ell^1}$ is a discontinuous linear functional on $U^{(1)} = U^{(1)}_{nc}$. We recall that $U$ and $U_m$ ($m \in \mathbb{N}$) are graded subalgebras of $F_\infty$; i.e., $U = \sum_{p=0}^{\infty} U^{(p)}$ and $U_m = \sum_{p=0}^{\infty} U^{(p)}_m$, with $U^{(1)}_1 = \ell^1(\mathbb{Z}^+)$ = $\bigvee^1 E$ [DPR]. So, $U^{(p)} = \bigcap_{m \in \mathbb{N}} U^{(p)}_m$, the subspace of $p$-homogeneous formal power series $\sum_{r \in (\mathbb{Z}^+)^p} \alpha_r X^r$, and, for each $m$, there is a continuous, dense embedding from $U^{(p)}_m$ into $U^{(p)}_1$, with $U^{(p)}_m$ is a closed linear subspace of $U_m$, spanned by the monomials $X^r$, $|r| = p$. So, clearly, for each $m$, $U^{(1)}_m$ is also isomorphic with $U^{(1)}_1 = \ell^1(\mathbb{Z}^+)$ ($\| \cdot \| \sim \| \cdot \|_m$ on $U^{(1)}_m$, where $U^{(1)}_m = \{ \sum_{i=1}^{\infty} \alpha_i X^i : \sum_{m=1}^{\infty} |\alpha_i|m < \infty \}$). In particular,

\[(4.1) \quad U^{(p)} = \bigcap_{m \in \mathbb{N}} U^{(p)}_m = \{ \sum_{r \in S} \alpha_r X^r \in U_1 : \sum_{r \in S, |r|=p} |\alpha_r|m^p < \infty \text{ for all } m \in \mathbb{N} \}.\]

Therefore,

\[(4.2) \quad U = \sum_{p=0}^{\infty} \left( \bigcap_{m \in \mathbb{N}} U^{(p)}_m \right) = \bigcap_{m \in \mathbb{N}} \left( \sum_{p=0}^{\infty} U^{(p)}_m \right) = \bigcap_{m \in \mathbb{N}} U_m,\]

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since $U_m = \sum_{p=0}^{\infty} U_m^{(p)}$. In particular,

(4.3) $$U^{(1)} = \bigcap_{m \in \mathbb{N}} U_m^{(1)} = \{ \sum_{i=1}^{\infty} \alpha_i X_i : \sum_{i=1}^{\infty} |\alpha_i| m < \infty, \text{ for all } m \in \mathbb{N} \}, U_m^{(1)} \cong \ell^1(\mathbb{Z}^+)$$

as above. Here,

(4.4) $$U_1 = \sum_{p=0}^{\infty} U_1^{(p)} = \sum_{p=0}^{\infty} \ell^1(S^{(p)}); U_1^{(1)} = \ell^1(S^{(1)}) \cong \ell^1(\mathbb{Z}^+) = \bigvee^1 E.$$

Also, $U_m^{(p)} \subseteq U_m^{(p)}$ for all $m \in \mathbb{N}$ and $p \in \mathbb{Z}^+$. We remark that $U^{(1)}$ is a closed linear subspace of $U$, spanned by $X_i$, whereas $U_m^{(1)}$ is a closed linear subspace of $U_m$ for each $m$, spanned by $X_i$ (the latter is a Banach space and the former is a Fréchet space). Obviously, all $X_i \in U^{(1)}$, but $\sum_{i=1}^{\infty} \frac{X_i}{i}$ does not belong to $U^{(1)}$ (resp., does not belong to even $U_1^{(1)}$) whereas $\sum_{i=1}^{\infty} \frac{X_i}{i^2} \in U^{(1)}$ (resp., belongs to $U_1^{(1)}$).

Recall that an extension of $\phi$ (again denoted by $\phi$) is a discontinuous linear functional on $U_m$ for each $m \in \mathbb{N}$, and that $\phi|_{\ell^1}$ (again denoted by $\phi$ below) is a discontinuous linear functional on $\ell^1(\mathbb{Z}^+) \cong U_1^{(1)}$ by §3. So, $\phi|_{\ell^1}$ is also a discontinuous linear functional on $U_m^{(1)}$ for each $m \in \mathbb{N}$ (the continuous embedding from $U_m^{(1)}$ into $U_1^{(1)}$ actually turns out to be a topological isomorphism), and so, $\phi|_{\ell^1}$ is also a discontinuous linear functional on $U^{(1)}$ (because if it is a continuous linear functional on $U^{(1)}$, then it can be
extended to a continuous character on $U$ (as shown at the end of this paper),
a contradiction to our assumption above).

Now, we take $\phi(X_n) = 1$ for all $n \in \mathbb{Z}^+$, and using the axiom of choice,
we extend $\phi$ to a Hamel basis of $U^{(1)}$, so that $\phi$ becomes a discontinuous
linear functional on $U^{(1)}$. (We remark that this idea was used by ˙Zelazko
in [Z1] while defining a (discontinuous) linear functional on $\text{Hol}(\mathbb{C})$, much
before Read [R, Def. 1.7] (Read used this idea to define a discontinuous
linear functional on a closed linear subspace $\mathcal{A}^{(1)}$ of $\mathcal{B}$), and, in our case,
$U^{(1)} \cong \text{Hol}(\mathbb{C})$, $\sum_n \alpha_n X_n \mapsto \sum_n \alpha_n z^n$, as shown in the final remark at the
end of this section; so, $H_0 = \{ \psi \in \text{Hol}(\mathbb{C}) : \psi(z) = e^{z^n}, n \in \mathbb{N} \}$, is a
countable linearly independent subset of $H$ [Z1, Lem. 1], and $f$ could be
defined on $\text{Hol}(\mathbb{C})$ using this $H_0$.) We shall generate another Fréchet topology
$\tau$ (inequivalent to the usual one, generated by the norms $q_m$) on $U_{nc}$ (and
thus, on $U$ as well), using this discontinuous linear functional. Our argument
here is kept short because it uses the key ideas involved in the Read’s method
(and we follow notations in align with the Read’s notations), but the Fréchet
topologies on $U_{nc}$ (resp., on $U$) that we are working with, are entirely different.

We start with a remark that the derivations on a (non-)commutative,
semisimple Fréchet algebra $(U, (q_m))$ (resp., $(U_{nc}, (q_m))$) are continuous [C2]
(Cor. 5.5 below). However, we shall show that the derivation $\partial/\partial X_0$ is discontinuous w.r.t. $\tau$, a contradiction to the Carpenter’s result (resp., Cor. 5.5 below) in the commutative case (resp., in the non-commutative case). We shall also require in a future proof the following “locally finite” linear map

\begin{equation}
T : \mathcal{U}_{nc} \to \mathcal{U}_{nc}, \quad T(\sum_{r \in S_{nc}} \alpha_r X^r) = \sum_{r \in S_{nc}} \alpha_r X_r
\end{equation}

(actually, this $T$ is the composition of the inclusion map with the $T$, discussed by Read in [R]; also, one requires to restrict the range as well). Similarly we may define “locally finite” linear maps $\mathcal{U}_{nc} \to \mathcal{U}, \mathcal{U} \to \mathcal{U}_{nc}$ and $\mathcal{U} \to \mathcal{U}$; they are precisely the linear maps between these two spaces that are continuous w.r.t. their natural Fréchet topologies (two such mappings were considered in the proof of Prop. 4.1 above).

We shall require the concept of “tensor products by rows”, taken from [R]. First, for each $n \in \mathbb{Z}^+$, let $P_n : \mathcal{U}_{nc} \to \mathcal{U}_{nc}$ be the linear map such that $P_n(1) = 0$, and

\begin{equation}
P_n(X_{i_1} \otimes X_{i_2} \otimes \cdots \otimes X_{i_m}) = \begin{cases}
0, & \text{when } i_1 \neq n, \\
X_{i_2} \otimes \cdots \otimes X_{i_m}, & \text{when } i_1 = n.
\end{cases}
\end{equation}

Thus $P_n$ takes the quotient on division from the left by $X_n$, and discards the remainder. Let $\pi : \mathcal{U}_{nc} \to \mathcal{U}$ be the natural map that the $X_i$ commute,
i.e., the locally finite map such that \( \pi(X^i) = X^1 \) for all \( i \). Then \( \pi|_{\bar{\sigma}(U_{nc})} \) is bijective and \( \epsilon|_U : U \to \bar{\sigma}(U_{nc}) \) be its inverse (recall that \( \epsilon : U \to U_{nc} \) is a continuous, injective homomorphism such that \( \epsilon = \oplus \epsilon_p \), where \( \epsilon_p : U^{(p)} \to U_{nc}^{(p)} \) is a continuous linear embedding for each \( p \)). Thus \( \epsilon \) is the natural right inverse to \( \pi \).

Next, we have a linear functional \( \phi : U^{(1)} \to \mathbb{C} \) such that \( \phi(X_n) = 1 \) for all \( n \in \mathbb{Z}^+ \), as discussed earlier. Recall that \( U_{nc} \) is a graded subalgebra of \( B \). Write \( U_{nc}^{(p)} \) for the subspace of \( p \)-homogeneous formal power series \( \sum_{i \in S_{nc}^{(p)}} b_i X^i \) and write \( U^{(p)} = \pi(U_{nc}^{(p)}) \), \( U_{nc}^{(1)} = U^{(1)} \). If \( b \in U_{nc} = \oplus_{p=0}^\infty U_{nc}^{(p)} \) we write \( (b^{(p)})_{p \in \mathbb{N}} \) such that \( b = \sum_{p=1}^\infty b^{(p)} \). If \( \phi_1, \phi_2 : U^{(1)} \to \mathbb{C} \) are linear functionals, we define the “tensor product by rows” \( \phi_1 \otimes \phi_2 : U^{(2)} \to \mathbb{C} \) by

\[
\phi_1 \otimes \phi_2 (b) = \phi_1(\sum_{j=0}^\infty X_j \cdot \phi_2(P_j(b))).
\]

Tensor product by rows of \( n \) linear functionals are then defined inductively by

\[
\otimes_{i=1}^n \phi_i(a) = \phi_1(\sum_{j=0}^\infty X_j \cdot \otimes_{i=2}^n \phi_i(P_j(a))),(4.8)
\]

and we see that Lem. 1.10 of [R] holds for the elements \( a \in U^{(r)} \), \( b \in U^{(p-r)} \) for \( 0 \leq r \leq p \).
Corollary 4.2. If \((\phi_n)\) is a sequence of linear functionals on \(U^{(1)}\), then, for each \(m \in \mathbb{N}\), the seminorm \(\| \cdot \|_m\) on \(U_{nc}\) given by

\[
\|a\|_m = |a^{(0)}| + \sum_{r=1}^{\infty} \sum_{1 \in (\mathbb{Z}^+)^r} |\otimes_{j=1}^{r} \phi_{ij}(a^{(r)})|
\]

is a submultiplicative seminorm.

As discussed in [R], since the order of appearance of the \(\phi_{ij}\) can be permuted arbitrarily in (4.9), one has \(\|\tilde{\sigma}\|_m = 1\) for all \(m \in \mathbb{N}\), where \(\tilde{\sigma}\) is as in Prop. 4.1 above. Hence, these seminorms are also submultiplicative seminorms on \(U\), when \(U\) is identified with the linear subspace \(\tilde{\sigma}(U_{nc}) \subset U_{nc}\) (the multiplication of \(U\) is then implemented by \((a, b) \rightarrow \tilde{\sigma}(a \otimes b)\)). However, instead of the ‘usual’ coordinate linear functionals, discussed in [R], we, here, need the ‘weighted’ coordinate linear functionals, defined as follows, in order to give Fréchet topologies \(\tau\) and \(\tau_0\) on \(U_{nc}\) below. Let

\[
\phi^m_n : U^{(1)} \to \mathbb{C}, \phi^m_n(\sum_{n=0}^{\infty} \alpha_n X_n) = \alpha_n \cdot m
\]

for each \(m \in \mathbb{N}\), then \((\phi^m_n)_{n \in \mathbb{Z}^+, m \in \mathbb{N}}\) is a sequence of weighted coordinate linear functionals on \(U^{(1)}\) for each \(m \in \mathbb{N}\). Let \((\phi_n)_{n \in \mathbb{Z}^+}\) be a sequence of linear functionals on \(U^{(1)}\) as follows: (a) \(\phi_0 = \phi\), the discontinuous linear functionals defined above; and (b) for \(n, m \in \mathbb{N}\), \(\phi^m_n\) be the weighted coordinate functionals on \(U^{(1)}\). Apply the above corollary to \((\phi_n)\) to define a
locally multiplicatively convex topology $\tau$ on $\mathcal{U}_{nc}$.

We claim that $(\mathcal{U}_{nc}, \tau)$ is a Fréchet algebra. Since $\tilde{\sigma}$ is a $\tau$–continuous projection, the subspace $\mathcal{U} = \tilde{\sigma}(\mathcal{U}_{nc}) = \ker(I - \tilde{\sigma})$ is closed, so $(\mathcal{U}, \tau)$ is a commutative Fréchet algebra. Note that if we prove that $\mathcal{U}_{nc}$ is, in fact, complete under the topology $\tau$, then we arrive at a contradiction to the fact that $(\mathcal{U}_{nc}, \tau_0)$ (resp., $(\mathcal{U}, \tau_0)$) is a non-(commutative), semisimple Fréchet algebra with a unique Fréchet topology by Cor. 5.5 below (resp., by [C1, P1, P3, DPR]). Since $\phi_0(X_N - X_0) = 0$ for $N > 0$ one sees that $\|X_N - X_0\|_{\tau_0}^n = 0$ for all $n, m$ with $n < N$; hence $X_N \to X_0$ in $\tau$.

Let $\tau_0$ be the “usual” topology that makes $\mathcal{U}_{nc}$ a Fréchet algebra (cf. [DE]). One sees that $\tau_0$ could be obtained by applying Cor. 4.2 to the sequence $(\phi_n)$, where $\phi_0$ is the usual continuous coordinate functional, defined by $\phi_0(\sum_{n=0}^{\infty} \alpha_n X_n) = \alpha_0$, inducing submultiplicative norms $|\cdot|_m^n$, where

$$|a|_m^n = |a^{(0)}| + \sum_{r=1}^{\infty} \sum_{i \in (\mathbb{Z}^+)^r} |a^{(r)}| m!.$$ (4.11)

One may say that $\tau_0$ is the topology of “weighted” convergence w.r.t. all these norms $|\cdot|_m^n$; that is, $\tau_0$ is the topology of “weighted” convergence of all the coefficients $a_i$. 

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Next, we define a linear map $\Psi : \mathcal{U}_{nc} \to \mathcal{U}_{nc}$ by

\begin{equation}
\Psi(a) = a^{(0)} + \sum_{r=1}^{\infty} \sum_{i \in (\mathbb{Z}^+)^r} \otimes_{j=1}^{r} \phi_{ij}(a^{(r)}).
\end{equation}

We note that $\Psi : (\mathcal{U}_{nc}, \tau) \to (\mathcal{U}_{nc}, \tau_0)$ is continuous because convergence under $\tau$ is precisely convergence of all the weighted linear functionals $\otimes_{j=1}^{r} \phi_{ij}^{m}(a^{(r)})$, corresponding to the usual linear functional $\otimes_{j=1}^{r} \phi_{ij}(a^{(r)})$ that are involved in $\Psi(a)$.

**Theorem 4.3.** $\Psi$ is bijective.

**Proof.** The proof is the same as that of [R, Thm. 2.3], with a remark that one requires to replace $\mathcal{B}$ by $\mathcal{U}_{nc}$ throughout that proof (the reader would like to notice that the Read’s proof was purely algebraic, and so one survives under the replacement, because $\mathcal{U}_{nc}$ is a graded subalgebra of $\mathcal{B}$, so it inherits all the graded algebraic structure that $\mathcal{B}$ has).

**Theorem 4.4.** $(\mathcal{U}_{nc}, \tau)$ is complete with respect to $(\| \cdot \|_{m})$. The derivation $\partial/\partial X_0 : (\mathcal{U}_{nc}, \tau) \to (\mathcal{U}_{nc}, \tau)$ is discontinuous, and its separating subspace is all of $\mathcal{U}_{nc}$. The derivation $\partial/\partial X_0 : (\mathcal{U}, \tau) \to (\mathcal{U}, \tau)$ is also discontinuous, and its separating subspace is all of $\mathcal{U}$.

**Proof.** The proof is the same as that of [R, Thm. 2.5], with a remark that the notations are same, but the two topologies $\tau$, generated by
(\| \cdot \|^n_m), and \( \tau_0 \), generated by (\| \cdot \|^n_m), are not the same that were discussed by Read. Obviously, the derivation \( \partial/\partial X_0 \) is discontinuous on \((U_{nc}, \tau)\), since \( \partial/\partial X_0(X_0 - X_N) = 1 \) whereas \( X_N \to X_0 \) in \( \tau \). Clearly, the separating subspace of the derivation is a two-sided ideal, and so, it is all of \( U_{nc} \). Similarly, the derivation \( \partial/\partial X_0 : (U, \tau) \to (U, \tau) \) is also discontinuous, and its separating subspace is all of \( U \). \( \square \)

**Corollary 4.5.** All characters of \( U \) are continuous and so is for \( U_{nc} \). In particular, every Fréchet algebra (commutative or not) is functionally continuous.

*Proof.* From Thm. 4.4, \( U \) (resp., \( U_{nc} \)) admits two inequivalent Fréchet topologies, a contradiction to the fact that it has a unique Fréchet topology, being semisimple Fréchet algebra by [C1, P1, P3, DPR] (resp., by Cor. 5.5 below). Since all characters of \( U \) are continuous, all characters of \( U_{nc} \) are also continuous by Prop. 4.1. \( \square \)

**Remarks E.** 1. Until now, we know that \( M(A) \) is dense in \( S(A) \), where \( S(A) \) is the space of all characters on \( A \) w.r.t. the Gel’fand topology [Go2, Lem. 10.1.1]. By Cor. 4.5, we have \( M(A) = S(A) \).

2. Finally, we remark that Vogt studies the tensor algebras over a Fréchet space in [V1, V2], mostly linear topological properties (and, in particular,
linear isomorphisms) of these algebras. A particular topic of interest, close to
the algebra \( U \) of all entire functions on \( \ell^\infty \), should be the study of the space of
entire functions on a nuclear sequence spaces, which is the symmetric tensor
algebra of a suitable nuclear reflexive Köthe space (see [BMV]).

**Fréchet analogues of Sections 2 and 3, an alternative of the first
approach.** It is clear that \( U^{(1)} \) is isometrically isomorphic to the Beurling-
Fréchet algebra \( \ell^1(\mathbb{Z}^+, W) \), where \( W = (\omega_m) \) is an increasing sequence of
weights on \( \mathbb{Z}^+ \) defined by \( \omega_m(n) = m \) for all \( n, m \in \mathbb{N} \) and \( \omega_m(0) = 1 \) for
all \( m \in \mathbb{N} \) [BP, Ex. 1.2], which is, in turn, isomorphic to the Fréchet algebra
\( \text{Hol}(\mathbb{C}) \) of entire functions [BP, Ex. 1.4], the nuclear power series space
of infinite type [V2, §2] or [Gro] (the linear isomorphism \( \sum_{n=1}^\infty \alpha_n X^n \mapsto \sum_{n=1}^\infty \alpha_n X^n \) between \( U^{(1)} \) and \( \ell^1(\mathbb{Z}^+, W) \) can be used to define a multipli-
cation on the Fréchet space \( U^{(1)} \) which makes it a Fréchet algebra). Since
\( E = U^{(1)} \) is a nuclear Fréchet space (and so, it has approximation prop-
erty), if we apply fairly straightforward arguments of Vogt [V2, §1], then it
is easy to see that the tensor algebra \( T(E) \) (resp., the symmetric algebra
\( S(E) \) in the commutative case) is, indeed, the algebra \( U_{nc} \) (resp., \( U \)). In fact,
one develops the theory of topological tensor algebra over the Fréchet space
\( E = U^{(1)} \) as discussed in §2. For example, write \( \bigotimes^p E \) for the completion

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of $\otimes E$ w.r.t. the projective tensor product metric induced by the metric $d$ on $E$, then we have $\hat{\otimes}^p E = U^{(p)}_{nc}$, the subspace of $p$-homogeneous formal power series, and $\hat{\otimes} E = U_{nc}$, the Fréchet tensor algebra. Similarly, we have $\hat{\vee}^p E = U^{(p)}$, the closed subspace of $\hat{\otimes}^p E = U^{(p)}_{nc}$ and $\hat{\vee} E = U$, the Fréchet symmetric algebra.

We then apply the method of §3 to construct a totally discontinuous higher point derivation $(d_n)$ on $U$ at a discontinuous character $\phi$ (remark that we start with a discontinuous linear functional $\phi$ on $U^{(1)}$, and it is easy to see that we can apply the Dales-McClure’s method in the Fréchet case). The main point of this attempt should be emphasized. There is no need to possibly construct $(d^m_n)$ on $U_m$ for each $m$ and $n$, and so, one does not require to properly restrict all $d^m_n$ to $d_n$ on $U$.

The tensor algebra $T(E) = U_{nc}$ is a nuclear power series space of infinite type by [V2, Lem. 2.1 and Thm. 3.1], which is further linearly isomorphic to $s$ by [V2, Thm. 4.1]. Then, the isomorphism of Thm. 4.1 of [V2] equips the space $s$ of all rapidly decreasing sequences with a multiplication which turns it into a Fréchet algebra, called $s_{\cdot}$. By Cor. 4.5 above, the algebra $T(s) = s_{\cdot} = T(E) = U_{nc}$ is now functionally continuous; thus, the algebra $s_{\cdot}$ was, indeed, one of the test cases for Michael’s problem (cf. [V2, p.
190]). However, our approaches to the algebras $U_{nc}$ and $U$ seem to be more convenient for our purposes in this section and §3, respectively.

3. Read used his method of constructing another Fréchet topology on $F_{\infty}$, inequivalent to the usual Fréchet topology on $F_{\infty}$, to show that the famous Singer-Wermer conjecture fails in the Fréchet case whereas we use the same method to affirmatively answer Michael’s problem. Similarly, Dales and McClure used the topological version of the tensor algebra over a Banach space to obtain a somewhat negative result in automatic continuity theory whereas we use the same technique to again answer Michael’s problem affirmatively (also, along the same lines, recall our comments, given in Remarks B. 1, on the use of the Dales-Patel-Read’s method in Thm. 10.1 of [DPR] as well as in Thm. 3.1 above). Thus these two approaches confirm the impression that one may not expect the phenomena to hold in the Fréchet algebra theory that hold in the Banach algebra theory. However, we are lucky enough to answer Michael’s problem in our favor [R, Conclusion]. Thus, the situation on Fréchet algebras is markedly different from that on Banach algebras, and that a structure theory for Frechet algebras behaves in a very distinctive manner from Banach algebras.
5 Applications to automatic continuity theory

History of automatic continuity theory. As promised, we have answered affirmatively Michael’s problem within automatic continuity theory. In the past, there were some significant works in this theory, which relate to our present work in some or the other way and giving applications within this theory or in the theory of commutative rings and algebras as follows. Feldman showed that the Wedderburn principal theorem does not hold for Banach algebras by giving two inequivalent complete norms to a specific Banach algebra $\ell^2 \oplus \mathbb{C}$ [F]. Johnson established the uniqueness of the complete norm for semisimple Banach algebras [Jo1], and for BAPS [Jo2]; this result was extended by Loy to FrAPS satisfying the equicontinuity condition (E) [L3] among other papers [L1, L2, L4], and the author settled this for all FrAPS (in $\mathcal{F}_k$) [P1] (P3) and for certain FrAPS in $\mathcal{F}_\infty$ [P3]. Thomas showed that the image of a derivation on a Banach algebra is contained in the radical, solving the Singer-Wermer conjecture affirmatively [SW, T2]. However, Read showed that this conjecture fails in the Fréchet case [R]; the author extended this work, giving countably many inequivalent Fréchet topologies to two specific
and maiden) Fréchet algebras [P5] (we remark that Vogt gave uncountably many inequivalent Fréchet space topologies to spaces of holomorphic functions [V3]; however, all these spaces are semisimple Fréchet algebras with a unique Fréchet topology [C1, P1, DPR]). Loy gave method to construct commutative Banach algebras with inequivalent complete norms by using the discontinuity of derivations [L5]; we extend his work to the Fréchet case, while attempting to answer Loy’s problem from 1974 [P4]. The germ of the ideas for the first approach, discussed in §3, lies in [P4]; the germ of the ideas for the second approach, discussed in §4, lies in [P5]. Among other works, we quote works of Grabiner [Gr] on automatic continuity of derivations and homomorphisms on BAPS and some results of Laursen on automatic continuity of linear operators [La].

Remarks F. 1. Our first remark is on the original two problems of Michael [M, §12], posed exclusively for commutative algebras. However, their non-commutative analogues also exist, and Dixon and Esterle discussed the test case \( U_{nc} \) (the non-commutative analogue of \( U \)) in [DE]. Dales posed a question about the continuity of characters on a commutative, locally convex \((F)\)-algebras [D1, Q. 3 (ii)], and Dixon discussed the non-equivalence of Michael’s two problems for more general commutative, locally convex \((F)\)-algebras and
commutative, complete, locally convex algebras (the reason is that he could not extend Thm. 1 of [DF] in this case); also, he showed that there is a commutative, complete, locally convex algebra, with jointly continuous multiplication and with an unbounded character [Di, §4, Ques. (3) and (4), and Thm. 4.2]. We shall show that every character on a commutative (F)-algebra is automatically continuous by Cor. 5.4 below. Thus, every commutative, complete LMC-algebra is functionally bounded by Cor. 3.2 (or Cor. 4.5), but this is not true for every commutative, complete, locally convex algebra.

2. We have throughout assumed that our algebras have an identity. It is a triviality to show that if a (non-)commutative Fréchet algebra has a discontinuous character, then so does the algebra obtained by adjoining an identity (see [DE, Prop. 2.1] and [Cr, Remarks 4.5 (2)]). Thus, as far as Michael's problem is concerned, no loss of generality is involved in our assumption.

3. It is obvious that a discontinuous character on \((\mathcal{U}, \tau_0)\) (resp., on \((\mathcal{U}_{nc}, \tau_0)\)) is also discontinuous on \((\mathcal{U}, \tau)\) (resp., \((\mathcal{U}_{nc}, \tau)\)) by [E3, Thm. 2.7] (resp., by Prop. 4.1 above). So, Cor. 6 of [C1], which was stated for commutative, semisimple Fréchet algebras, is, here, established for all Fréchet algebras in a more general form by Cor. 4.5 and Cor. 5.5 below.
4. Another topology $\tau$ on $\mathcal{U}$ (resp., on $\mathcal{U}_{mc}$) also contradicts to the fact that every derivation is automatically continuous on a (non-)commutative, semisimple Fréchet algebra [C2] (Cor. 5.5 below), because the derivation $\partial/\partial X_0$ is discontinuous w.r.t. $\tau$ by Thm. 4.4.

5. Carpenter [C1] showed that every commutative semisimple Fréchet algebra $A$ admits a unique Fréchet topology. The proof was divided into four parts, but if characters were continuous on $A$, then the proof could be derived from the third part only.

Another equivalent form of Michael’s problem. Below, we shall show that every homomorphism $\theta : B \rightarrow A$ from a Fréchet algebra $B$ into a semisimple Fréchet algebra $A$ is automatically continuous; this is another long-standing open problem in the theory of Fréchet algebra, equivalent to Michael’s problem [D1, p. 143] and [P1, p. 134]. Once we show this, we have a very short proof for the uniqueness of the Fréchet topology for a (non-)commutative semisimple Fréchet algebra using the open mapping theorem for Fréchet spaces.

**Theorem 5.1.** Let $A$ be a commutative semisimple Fréchet algebra and $B$ be any commutative Fréchet algebra. Let $\theta : B \rightarrow A$ be a homomorphism. Then $\theta$ is automatically continuous.
Proof. First, we assume w.l.o.g. that \( B \) is unital, because if \( B \) is not unital, then we can adjoin the identity \( e \) and we can extend \( \theta \) on \( B_e \) by taking \( \theta(e) = e' \) (if \( A \) is not unital, then we can also adjoin the identity \( e' \) to \( A \)). Obviously, \( \theta \) is continuous if and only if an extension of \( \theta \) is continuous.

Suppose that \( \theta \) is a discontinuous, unital homomorphism. Let \( \phi \) be a character on \( A \). Then \( \phi \) is continuous by Cor. 3.2 or Cor. 4.5. Set \( \psi = \phi \circ \theta \). Then \( \psi \) is a character on \( B \). Since \( \theta \) is discontinuous, \( \psi \) is also discontinuous, a contradiction to the fact that all characters on \( B \) are continuous. Thus \( \theta \) is automatically continuous. \( \square \)

As an application to the above theorem, we can drastically shorten the proof of Carpenter [C1], establishing the uniqueness of the Fréchet topology of a commutative, semisimple Fréchet algebra in the following

**Corollary 5.2.** Every commutative, semisimple Fréchet algebra admits a unique Fréchet algebra topology.

**Proof.** Let \( A \) be a commutative, semisimple Fréchet algebra with respect to another Fréchet topology \( \sigma \), distinct from the Fréchet topology \( \tau \). Consider the identity mapping from \( (A, \tau) \) into \( (A, \sigma) \). Evidently it is a continuous homomorphism by Thm. 5.1. By the open mapping theorem for Fréchet spaces, it is a linear, homeomorphism, and so \( \sigma = \tau \). \( \square \)
6. What about the non-commutative analogue of 5. above? That is, whether a non-commutative, semisimple Fréchet algebra has a unique Fréchet topology. More generally, whether Thm. 5.1 holds for a non-commutative Fréchet algebra \( B \) and a non-commutative, semisimple Fréchet algebra \( A \). There is another parallel question about the continuity of derivations on a non-commutative, semisimple Fréchet algebra. As far as we know, the second problem was discussed by Johnson in the Banach case [Jo1] (a generalization of this result was given by Jewell and Sinclair [JS], which turned out to be a starting point for Esterle and Thomas to generalize the Jewell and Sinclair stability lemma to a more general \((F)\)-space case [E1, T1] as well as for the author to discuss the three questions here [D1, p. 141-144]). Also, we do not know any progress on the third question.

To answer the first two questions, we remark that it is easy to follow the proof of Thm. 5.1 in the non-commutative case, since we have already shown that every non-commutative Fréchet algebra is functionally continuous by Cor. 4.5. It is also interesting to note that we can affirmatively answer Michael’s acclaimed problems as an application of Thm. 5.1 (and its non-commutative analogue) by taking \( A = \mathbb{C} \). In fact, if \( \theta \) is a homomorphism from a Fréchet algebra \( B \) (commutative or not) into a commutative,
semisimple Fréchet algebra $A$, then it is automatically continuous if and only if every character on a Fréchet algebra $B$ is continuous. However, we, here, take an alternate approach which also answers the third question. For this, we first remark that a proper generalization of Thm. 2 of [JS] holds true in the Fréchet case as follows (also, we can take a (non-)commutative complete, metrizable algebra in the domain of an epimorphism, in order to state the below theorem in a fuller generality).

**Theorem 5.3.** Let $B$ be a Fréchet algebra such that

(i) for each infinite-dimensional closed two-sided ideal $J$ in $B$ there is a sequence $(b_n)$ in $B$ such that the closed ideal $\overline{(Jb_{n(1)} \ldots b_{1})^k}$ is a proper subset of the closed ideal $\overline{(Jb_{n(1)})^k}$ for each $n(1) \in \mathbb{N}$ and for some $k \in \mathbb{N}$;

(ii) $B$ contains no non-zero finite-dimensional nilpotent two-sided ideals.

Then $B$ has a unique Fréchet algebra topology, and every epimorphism from a Fréchet algebra onto $B$ is automatically continuous. Moreover, every derivation on $B$ is automatically continuous.

**Proof.** The proof is the same as that of [JS, Thm. 2], with a remark that one requires to obtain a contradiction by applying Lem. 1.1a of [T1]. We
also emphasize that no improvement in the condition (ii) such as “$B$ contains no non-zero finite-dimensional locally nilpotent two-sided ideals”, is possible here because since the separating space is a closed, finite-dimensional ideal, every locally nilpotent element of the separating space is, indeed, nilpotent, and so, for finite-dimensional two-sided ideals, the notions of locally nilpotent ideals (defined appropriately in the Fréchet case) and nilpotent ideals coincide (cf. [A2, p. 277]).

We remark that if $B$ is commutative in the above theorem, then the condition (i) above can be expressed in a more neater form: for each infinite-dimensional closed ideal $J$ in $B$ there exists $b \in B$ such that the infinite-dimensional closed ideal $(Jb)^k$ is a proper subset of $J_k$ for some $k \in \mathbb{N}$. Moreover, a similar hypothesis such as the condition (i) above, but in the opposite direction, was considered by Allan in Thm. 8 of [A2], in order to obtain a discontinuous homomorphism between certain commutative Fréchet algebras.

**Third approach to Michael’s problem.** It is a surprising consequence of the above theorem (with $B$ a Banach algebra) that we have another approach to affirmatively answer Michael’s problems in the following corollary (see a comment on p. 144 of [D1], or [E1]). Esterle remarked that Cor. 3 (or
Thm. 1) of [E1] cannot be applied in the case $B = \mathbb{C}$ for obvious reason. However, $B = \mathbb{C}$ surely satisfies both the conditions of Thm. 2 of [JS] (the first condition is vacuously satisfied, and since $\mathbb{C}$ is a field, the second condition is also satisfied). In particular, $B$ does satisfy Cor. 3 of [E1], if the second condition is replaced by “if $B$ has no non-zero nilpotent ideal of finite codimension” (equivalently, $B$ has no non-zero finite dimensional nilpotent ideal, which is the condition (ii) of Thm. 2 of [JS]). We can take $A$ a (non-)commutative complete, metrizable algebra in the below corollary, as mentioned in Remarks F. 1 above; also, Esterle’s stability lemma [E1, Lem. 1] would suffice to obtain a contradiction in the proof since $B$ is a Banach algebra.

**Corollary 5.4.** Let $A$ be a non-commutative Fréchet algebra, and $B$ be a Banach algebra as in Thm. 2 of [JS]. Then every epimorphism from a non-commutative Fréchet algebra onto $B$ is automatically continuous. In particular, every character on $A$ is automatically continuous. Further, every Fréchet algebra (commutative or not) is functionally continuous. □

**Non-commutative analogues of some important results in automatic continuity theory.** Next, if $B$ is a non-commutative, semisimple Fréchet algebra, then it has no nilpotent two-sided ideals. To see whether
the condition (i) of Thm. 5.3 holds, we follow the proof of Cor. 9 of [JS] in the Fréchet case by working with an infinite dimensional irreducible left $B$–module $X$, which is a Fréchet module under the Fréchet space topology. Then we have the following

**Corollary 5.5.** Let $B$ be a non-commutative, semisimple Fréchet algebra. Then $B$ has a unique Fréchet algebra topology, and every epimorphism from a Fréchet algebra onto $B$ is automatically continuous. Moreover, every derivation on $B$ is automatically continuous. \[\square\]

We remark that Cor 5.5 affirmatively answers Que. 9 of [D1]. Esterle deduced in [E1] that, if $B$ is a Banach algebra satisfying the condition (i) of Thm. 5.3 (but in the Banach case; see [JS, Thm. 2]), and if $B$ contains no non-zero finite-dimensional two-sided ideal, then every epimorphism from an $(F)$–algebra onto $B$ is automatically continuous. We see that a generalization of Esterle’s result does not hold in the Fréchet case; for example, the algebra $\mathcal{F}$ clearly satisfies the conditions of Esterle’s theorem, but every epimorphism from an $(F)$–algebra onto $\mathcal{F}$ is discontinuous by [DPR, Thm. 11.2], answering affirmatively a question of Dales-McClure’s problem (1977) for a more general case (see [DM2, Thm. 2.3] for the case the domain algebra a Banach algebra).
7. In [P1, P3, Thms. 4.1], we can now drop the condition “the range of $\theta$ is not one-dimensional”, because there are no discontinuous characters on the domain algebra $B$ which would have given the discontinuous homomorphism $b \mapsto \phi(b)1, B \to A$.

Completing the circle of ideas. In [DPR, Thm. 12.3], it was shown that a Banach algebra $\ell^1(S) \cong \mathbb{V}_{\{1\}}E$, is such that $\mathbb{C}[X_1, X_2] \subset \ell^1(S) \subset F_2$, but the embedding $(\ell^1(S), \| \cdot \|) \to (F_2, \tau_c)$ is not continuous. We did not know whether there is a non-Banach Fréchet algebra with these properties. We show that the test case $U$ is such an example in the following

**Theorem 5.6.** The test case $(U, \tau_0)$ for Michael’s problem is such that $\mathbb{C}[X_1, X_2] \subset U \subset F_2$, but the embedding $(U, \tau_0) \to (F_2, \tau_c)$ is not continuous.

**Proof.** The proof is the same as that of [DPR, Thm. 12.3]. Recall that $U^{(1)}$ is the closed linear subspace of $U$ spanned by the elements $X_i$, and so this Fréchet space is not isometrically isomorphic to $\ell^1$, but we can still choose a non-zero, discontinuous linear functional $\lambda$ on $U^{(1)}$ (as the element $\sum_{i=2}^{\infty} \frac{X_i}{x_i} \in U^{(1)}$), and then define a linear map

$$\psi : U^{(1)} \to F_2, \ u \mapsto \theta(u) + \lambda(u)Y,$$
where $\theta$ is taken from Thm. 10.1 (ii) of [DPR]. Our main claim is that $\psi$ can be extended to a homomorphism $\Psi' : \mathcal{U} \to \mathcal{F}_2$ such that $\pi \circ \Psi' = \theta$. To establish this claim, we shall require the following slightly more general theorem, whose proof we omit.

Theorem 5.7. Let $\beta : \mathcal{U}^{(1)} \to \mathcal{M}_2$ be a linear map such that $\pi \circ \beta : \mathcal{U}^{(1)} \to \mathcal{F}$ is continuous. Then there is a unital homomorphism $\overline{\beta} : \mathcal{U} \to \mathcal{F}_2$, extending $\beta$, such that $\pi \circ \overline{\beta} : \mathcal{U} \to \mathcal{F}$ is continuous.

In fact, a slight digression of the proof of Thm. 5.7 (which is analogous to the proof of Thm. 12.4 of [DPR], and a similar proof is discussed below) enables us to state the following theorem in its fuller generality. Both Thms. 5.7 and 5.8 are of independent interest in view of the first approach to Michael’s problem.

Theorem 5.8. Let $\beta : \mathcal{U}^{(1)} \to \mathcal{F}$ be a continuous linear map. Then there is a continuous, unital homomorphism $\overline{\beta} : \mathcal{U} \to \mathcal{F}_2$, extending $\beta$.

As opposite to Thm. 5.6 above, we provide a much shorter and elegant way to embed $\mathcal{U}$ into $\mathcal{F}$ in the following corollary (cf. [DPR, Thm. 10.1]).

Corollary 5.9. The Fréchet algebra $\mathcal{U}$ is (isometrically isomorphic to) a Fréchet algebra of power series.
Proof. First, we remark that we again require to take further digression in the proof of Thm. 5.8 as follows. For each \( i \in \mathbb{Z}^+ \), we take \( \beta(i) \) to be the usual co-ordinate linear functional on \( \mathcal{U}^{(1)} \). We then define a mapping \( \beta : \mathcal{U}^{(1)} \to \mathcal{F} \) such that \( \beta(f) = \sum_{i=0}^{\infty} \beta(i)(f)X_i \). Clearly, such a \( \beta \) is continuous and injective, and \( \beta(X_1) = X \). We extend each \( \beta(i) \) to a linear functional \( \beta(i) \) on \( \mathcal{F}_{\infty}^{(1)} \). Next we define a linear functional \( \beta_{(i)}^{(n)} \) on \( \mathcal{F}_{\infty}^{(n)} \) for each \( n \in \mathbb{N} \) by the following formula:

\[
\beta_{(i)}^{(n)}(f) = \sum \{(\beta_{(i)}^{(1)} \otimes \cdots \otimes \beta_{(i)}^{(n)})(\epsilon_n(f))\} \quad (f \in \mathcal{F}_{\infty}^{(n)}),
\]

where the sum is taken over all \( n \)-tuples \( (i) = (i^{(1)}, \ldots, i^{(n)}) \in (\mathbb{Z}^+)^n \) such that \( i^{(1)} + 2i^{(2)} + \cdots + ni^{(n)} = \omega((i)) \), a weighted order of \( (i) \) [DPR, pp. 137-138].

We now claim that the map \( \bar{\beta} : \mathcal{U} \to \mathcal{F} \), defined for \( f \in \mathcal{F}_{\infty} \) by the formula

\[
\bar{\beta}(f) = \sum_{k=0}^{\infty} \{(\sum \{\beta_{(i)}^{(n)}(f^{(n)} : n \in \mathbb{N}_i)\}X_i : \omega((i)) = k\},
\]

where we set \( \bar{\beta}^{(0)}(f) = f(0)1 \), is a unital homomorphism \( \bar{\beta} : \mathcal{U} \to \mathcal{F} \) satisfying the stated conditions. Follow the proof of Thm. 12.4 of [DPR] to show that \( \bar{\beta} \) is a unital homomorphism that extends \( \beta \). Again follow the proof of Thm. 12.4 of [DPR] to show that \( \bar{\beta} \) is continuous; we need to use the continuity of co-ordinate linear functionals \( \beta(i) \) and the fact that the 'tensor product by rows' agrees with the usual tensor product when the linear
functionals are continuous. Finally, we need to show that $\bar{\beta}$ is injective. Now, follow the proof of Thm. 9.1 of [DPR] for this to derive. Thus the algebra $\mathcal{U}$ is continuously embedded in $\mathcal{F}$. In this case, $\bar{\beta}(\mathcal{U})$ is a FrAPS, w.r.t. the metric transferred from $\mathcal{U}$, and so $\mathcal{U}$ is isometrically isomorphic to a FrAPS. $\square$

A similar argument also enables us to extend a continuous linear functional $\beta$ on $\mathcal{U}^{(1)}$ (resp., on $\ell^1$) to a continuous character $\phi$ on $\mathcal{U}$ (resp., on the Banach algebra $\mathcal{U}_1$; this fact was used in §3); in the process, one can either consider ‘tensor product by rows’ or usual tensor product of a continuous linear functional, it makes no difference. The main point should be emphasized here: if one starts with a discontinuous character on $\mathcal{U}$, then one obtains a discontinuous linear functional on $\mathcal{U}^{(1)}$ (this fact was used in §4).

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