The Futaki invariant on the blowup of Kähler surfaces∗

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Abstract

We prove the expansion formula for the classical Futaki invariants on the blowup of Kähler surfaces, which explains the balancing condition of Arezzo-Pacard in [3]. The relation with Stoppa’s result [18] is also discussed.

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1 Introduction

In [8], E. Calabi introduced the extremal Kähler metric on a compact Kähler manifold, which is a critical point of the Calabi functional. A special case of extremal Kähler metrics is the constant scalar curvature Kähler (cscK for brevity) metrics. The uniqueness of extremal Kähler metric

∗The first-named author is supported in part by NSFC No. 11001080 and No. 11131007 and the second-named author by NSFC No. 11101206.
was proved by Chen-Tian in [9]. However, the existence of extremal Kähler metrics or constant scalar curvature metrics is a long standing difficult problem, which is closely related to some stabilities conditions in algebraic geometry. In some special cases, the extremal metrics can be constructed explicitly and they have many interesting properties (cf. [8] [19][1]). In a series of papers, Arezzo-Pacard [2] [3] and Arezzo-Pacard-Singer [4] get a general existence result on the blowup of a Kähler manifold with extremal Kähler metrics or constant scalar curvature metrics at finite many points with some conditions by using a gluing method.

To state Arezzo-Pacard’s theorem, we introduce some notations. Let \((M, \omega)\) be a compact Kähler manifold with a Kähler metric \(\omega\), and \(K\) the group of automorphisms of \(M\) which are also exact symplectomorphisms of \((M, \omega)\). There is a normalized moment map \(\xi: M \rightarrow k^*\), where \(k\) is the Lie algebra of \(K\). Moreover, for any \(X \in k\), \(\langle \xi, X \rangle\) is a Hamiltonian generating function on \(X\) and satisfying the normalization condition
\[
\int_M \langle \xi, X \rangle \omega^n = 0.
\] (1.1)

**Theorem 1.1.** (Arezzo-Pacard [3]) Let \((M, \omega)\) be a compact Kähler manifold with constant scalar curvature metric \(\omega\), and \(\pi: \tilde{M} \rightarrow M\) the blow up at distinct points \(\{p_1, \cdots, p_k\}\) satisfying the following conditions

1. \(\xi\) \((p_1), \cdots, \xi\) \((p_n)\) span \(k\);

2. there exists \(a_1, \cdots, a_k > 0\) such that
\[
\sum_{j=1}^k a_j \xi\) \((p_j)\) = 0 \in k.
\] (1.2)

Then, there exist \(c > 0, \epsilon_0 > 0\) and for all \(\epsilon \in (0, \epsilon_0)\), there exists on \(\tilde{M}\) a constant scalar curvature metric \(\omega_\epsilon\) in the Kähler class
\[
\omega_\epsilon \in \pi^*[\omega] - \epsilon^2(a_{1, \epsilon}^{1/2} c_1([E_1]) + \cdots + a_{k, \epsilon}^{1/2} c_1([E_k]))
\]
where \(a_{j, \epsilon}\) satisfies \(|a_{j, \epsilon} - a_j| \leq c\epsilon^{2/3}\).

Condition (2) in Theorem 1.1 is called the balancing condition and it should be related with the stability property of the blowing up manifold. In [18] J. Stoppa gives the expansion of the Donaldson-Futaki invariant on the blown up manifold and he shows that the conditions (2) is naturally related to the Chow stability of 0-dimensional cycles. Using this formula, he proved that if we blow up a cscK manifold with integral Kähler class \([\omega]\) at a Chow unstable 0-cycle \(\sum a_{i, \epsilon}^{r_i - 1} p_i\), then for any rational \(0 < \epsilon \ll 1\), the class \(\pi^*[\omega] - \epsilon (\sum a_{i, \epsilon} E_i)\) does not contain a cscK metric, since this new polarized manifold is K-unstable.

A natural question is whether we can remove the rationality assumption in Stoppa’s theorem. Recall that when the Kähler class is polarized by an ample line bundle \(L\) (hence \(M\) is a projective
algebraic manifold) and the holomorphic vector field $X$ generates a $\mathbb{C}^\ast$ action, the Donaldson-Futaki invariant for the induced product test configuration coincides with the classical Futaki invariant of $X$ up to a universal constant [11]. Since the vanishing of Futaki invariant is an obstruction to the existence of cscK metric, we can prove the non-existence of cscK metrics by a corresponding expansion formula for the classical Futaki invariant on the blown up manifold. In this paper, we will prove such a formula. For technical reasons, we restrict our attention to complex dimension 2.

**Theorem 1.2.** Let $\pi : \tilde{M} \to M$ be the blowing up map of a compact Kähler surface $M$ at the distinct points $\{p_1, p_2, \ldots, p_n\}$. If the holomorphic vector field $X$ on $M$ vanishes and is non-degenerate at $p_i$ ($1 \leq i \leq n$), then we have

$$f_{\tilde{M}}(\tilde{\Omega}_\epsilon, \tilde{X}) = f_M(\Omega, X) + \sum_{i=1}^{n} \nu_{p_i}(\Omega, X) \cdot \epsilon_i + O(\epsilon^2),$$

where $\tilde{X}$ is the natural holomorphic extension of $X$ over $\tilde{M}$, and the Kähler class $\tilde{\Omega}_\epsilon$ is

$$\tilde{\Omega}_\epsilon = \pi^* \Omega - \sum_{i=1}^{n} \epsilon_i c_1([E_i]).$$

Here $\epsilon_i > 0$ are small numbers and $\nu_{p_i}(\Omega, X)$ are given by

$$\nu_{p_i}(\Omega, X) = -2\text{tr} \Omega(X)(p_i) + \frac{2}{3\Omega^2}f_M(\Omega, X) = -2(\theta_X - \theta_X^0)(p_i),$$

where $\theta_X$ is the holomorphy potential of $X$ with respect to $\Omega$, and $\theta_X^0$ is the average of $\theta_X$.

The notations in Theorem 1.2 will be introduced in Section 2. Note that $(\theta_X - \theta_X^0)(p_i)$ is independent of the choices of $\omega_k$ and $\theta_X$, see Lemma 2.1. When the manifold is a projective algebraic surface, the polarization is asymptotically Chow stable and the holomorphic vector field generates a $\mathbb{C}^\ast$ action, then $(\theta_X - \theta_X^0)(p_i)$ equals the Chow weight of $p_i$ up to a universal constant factor and hence our result coincides with Stoppa’s. For details, see section 6.

The proof of Theorem 1.2 is based on Futaki [14] and Tian’s localization formula in [20] for the Futaki invariant, which essentially uses Bott’s residue formula for characteristic numbers in [6]. However, Bott’s residue formula needs the non-degeneracy condition on the holomorphic vector fields, and it will be difficult to remove this condition when calculating the Futaki invariant. When we consider the blown up manifold as in Arezzo-Pacard’s result, under the non-degeneracy assumption the induced holomorphic vector field on the blown up manifold may still be degenerate somewhere and we need to calculate the residue carefully in this case.

The key step in the proof of Theorem 1.2 is to calculate the residue formula in the degenerate case, and we use only elementary calculus. It should be generalized to higher dimensions. We note that there is a vast amount of literatures discussing various residue formulas on $\mathbb{C}^n$ (cf. [17] [22] and references therein), but few of them focus on the case in the Kähler manifolds, which usually involves Kähler metrics. The calculation in this paper might be the first step toward this direction.
A direct corollary of Theorem 1.2 is the following result, which gives a partial converse of Theorem 1.1 in the special case of Kähler surfaces:

**Corollary 1.3.** Let $\pi : \tilde{M}(p_1, \cdots, p_n) \to M$ be the blowing up map of a compact Kähler surface $M$ at the points $\{p_i\} \subset \text{Zero}(X)$, where $X \in h_0(M)$ is non-degenerate at $\{p_i\}$. If

$$
\sum_{i=1}^n \langle \xi, X \rangle (p_i) \cdot \varepsilon_i \neq 0, \tag{1.3}
$$

where $\xi$ satisfies the normalization condition (1.1) and $\varepsilon_i > 0$ are small, then $\tilde{M}$ has no constant scalar curvature metrics in the Kähler class

$$
\tilde{\Omega}_\varepsilon = \pi^* \Omega - \sum_{i=1}^n \varepsilon_i c_1([E_i]).
$$

In fact, it is well-known that the moment map $\xi$ under the normalization condition (1.1) can be characterized by $\langle \xi, X \rangle = \theta_X - \theta_X$. Therefore, Corollary 1.3 follows directly from Theorem 1.2. Corollary 1.3 gives a criterion on the non-existence of constant scalar curvature metrics on the blown up manifold. Moreover, the condition (1.3) may be related to the $\bar{K}$-stability, which is introduced by Donaldson in [12],[13].

The structure of the paper is as follows: In section 2, we include basic facts concerning the Futaki invariant and also outline the proof of the localization formula of Futaki and Tian. To state our result in a clear way, we also define some local invariants on the zero locus of a holomorphic vector field $X$. In section 3, we prove Theorem 1.2 in the case when the blow up center is an isolated zero point of $X$, and in section 4 we consider the case when the blow up center lies on a 1-dimensional component of the zero locus of $X$. Note that the extension of $X$ on the blow up manifold is degenerate if and only if the blow up center $p$ is an isolated zero point of $X$ and the linearization of $X$ at $p$ is not semisimple. This is proved in section 3.1. The proof of the degenerate case is the most technical part of our paper, and occupies section 3.3 and 3.4. Then in section 5, we apply our result to the blowup of $\mathbb{CP}^1 \times \mathbb{CP}^1$ at 2 or 3 points. The Futaki invariants in the former case has already been calculated by LeBrun and Simanca in [16]. Our method can also obtain a full expression for the Futaki invariant. For simplicity, we only write down the first order terms, which suffices to prove the non-existence of cscK metrics in some Kähler classes. Finally in section 6, we compare our result with that of Stoppa.

In a forthcoming paper, we will use a different method to get the expansion of the Futaki invariant on compact Kähler manifolds of higher dimensions and general holomorphic vector fields. As an application, we will show Corollary 1.3 for more general cases.

## 2 Preliminaries

In this section, we will recall briefly the localization formula of Bott [6], Futaki [14] and Tian [20] for the calculation of the Futaki invariant.
Let \((M, \omega_\theta)\) be a compact Kähler manifold, where \(\omega_\theta = g_{ij} dz^i \wedge d\bar{z}^j\). Here we adopt the convention that \(\omega\) and \(Ric(\omega)\) are defined without the usual \(\sqrt{-1}\) factor. Let \(h_0(M)\) be the space of holomorphic vector fields with nonempty zero locus. For any \(X \in h_0(M)\), we denote by \(Zero(X)\) the zero set of \(X\), which consists of complex subvarieties \(Z_\lambda (\lambda \in \Lambda)\). We say \(X\) is non-degenerate on \(Z_\lambda\), if \(Z_\lambda\) is smooth and \(\det (DX|_{TM/Z_\lambda})\) is nowhere zero along \(Z_\lambda\). \(X\) is called non-degenerate on \(M\) if \(X\) is non-degenerate on all the \(Z_\lambda (\lambda \in \Lambda)\).

The holomorphy potential of \(X \in h_0(M)\) with respect to \(\omega_\theta\), denoted by \(\theta_X\), is given by the equation
\[
i_X \omega_\theta = -\bar{\partial} \theta_X.
\]

Such a \(\theta_X\) always exists and is unique up to a constant. Note that the function \(\theta_X\) restricted on any \(Z_\lambda\) is a constant, and we define
\[
tr_\Omega (X|_{Z_\lambda}) = \theta_X|_{Z_\lambda},
\]
where \(\Omega = \{ \omega_\theta \} \) is the Kähler class of \(\omega_\theta\). Note that \(tr_\Omega (X|_{Z_\lambda})\) depends on the choice of \(\omega_\theta\) and \(\theta_X\). However, we have

**Lemma 2.1.** Let \(\theta_X\) be the average of \(\theta_{X}\), then the value of \(\theta_X - \theta_X\) on a zero point of \(X\) is independent of the choices of \(\omega_\theta\) in \(\Omega\) and \(\theta_X\).

**Proof.** First we fix the Kähler form \(\omega_\theta\), then \(\theta_X\) is unique up to adding a constant. Then obviously \(\theta_X - \theta_X\) is independent of the choice of \(\theta_X\). Now let’s fix a \(\omega_\theta\) and \(\theta_X\) with \(\theta_X = 0\). We change \(\omega_\theta\) by \(\omega_\theta = \omega_\theta + \bar{\partial} \partial \phi\). Then we can choose the holomorphy potential with respect to \(\omega_\theta\) to be \(\theta_X - X(\phi)\). Since \(X(\phi)\) vanishes on any zero point of \(X\), we need only to prove \(\int_M (\theta_X - X(\phi)) \omega_\theta^n = 0\). Let \(f(t) := \int_M (\theta_X - tX(\phi)) \omega_\theta^n\). Then \(f(0) = 0\) and
\[
f'(t) = n \int_M (\theta_X - X(\phi)) \bar{\partial} \partial \phi \wedge \omega_\theta^{n-1} - \int_M X(\phi) \omega_\theta^n.
\]
Observe that \(0 = i_X (\bar{\partial} \phi \wedge \omega_\theta^n) = X(\phi) \omega_\theta^n - n \bar{\partial} \phi \wedge i_X \omega_\theta \wedge \omega_\theta^{n-1}\). Integrate this and using integration by parts, we get directly \(f'(t) = 0\), hence \(f(1) = 0\).

The Futaki invariant of \(\Omega\) and the holomorphic vector field \(X\) is defined by
\[
f_M(\Omega, X) = \left( \frac{\sqrt{-1}}{2\pi} \right)^n \int_M X(h_g) \omega_g^n,
\]
where \(h\) is a function satisfying
\[
s(g) - \bar{\xi} = \Delta_X h_g.
\]

Now we start with some general discussions. Let \(\phi\) be any symmetric \(GL\)-invariant polynomial of degree \(n + 1\), and \(E\) a vector bundle over \(M\). Assume that \(h\) is a hermitian metric on \(E\) and \(\theta_X(h)\) be an \(End(E)\)-valued function satisfying
\[
\bar{\partial} \theta_X(h) = -i_X R(h)
\]
(2.2)
where $R(h)$ denotes the curvature of $h$. Then we can check that
\[
    i_X \phi(R(h) + \theta_X(h)) = -\bar{\partial}\phi(R(h) + \theta_X(h)).
\]
We define a $(1,0)$ form $\eta$ on $M \setminus \text{Zero}(X)$ by $\eta(Y) = g(Y,X)/g(X,X)$ for any $Y \in h_0(M)$ and we define a formal series of forms by
\[
    \alpha = \phi(\theta_X(h) + R(h)) \wedge \frac{\eta}{1 + \bar{\partial}\eta}.
\]
Direct calculation shows that
\[
    \phi(\theta_X(h) + R(h)) - \bar{\partial}\alpha - i_X \alpha = 0. \tag{2.3}
\]
Let $[\beta]_k$ denote the degree $k$ term in $\beta$. By (2.3) we have
\[
    [\phi(\theta_X(h) + R(h))]_{2n} = \bar{\partial}[\alpha]_{2n-1}. \tag{2.4}
\]
Let $B_\varepsilon(Z_\lambda)$ be an $\varepsilon$-neighborhood of $Z_\lambda$. Using (2.4) and the Stokes formula, we have
\[
    \int_M \phi(\theta_X(h) + R(h)) = \lim_{\varepsilon \to 0^+} \int_{M \cup \bigcup_{\lambda \in \Lambda} B_\varepsilon(Z_\lambda)} \phi(\theta_X(h) + R(h)) \nonumber
\]
\[
    = \lim_{\varepsilon \to 0^+} \int_{M \cup \bigcup_{\lambda \in \Lambda} B_\varepsilon(Z_\lambda)} \bar{\partial}[\alpha]_{2n-1} \nonumber
\]
\[
    = -\sum_{\lambda \in \Lambda} \lim_{\varepsilon \to 0^+} \int_{\partial B_\varepsilon(Z_\lambda)} [\alpha]_{2n-1}, \tag{2.5}
\]
where $\partial B_\varepsilon(Z_\lambda)$ has the induced orientation such that the last equality holds. The following result was essentially proved by Bott in [6], and the readers are referred to Theorem 5.2.8 of [14] for the details.

**Lemma 2.2.** (6/14) If $X$ is non-degenerate on $M$, then
\[
    \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(Z_\lambda)} \phi(\theta_X(h) + R(h)) \cdot \frac{\eta}{1 + \bar{\partial}\eta} = -(-2\pi \sqrt{-1})^\nu \int_{Z_\lambda} \phi(\theta_X(h) + R(h)) \left| \det(L_\lambda(X) + K_\lambda) \right|^n \nonumber
\]
where $\nu$ is the complex codimension of $Z_\lambda$ in $M$, $L_\lambda(X)$ is the operator $L_\lambda(X)(Y) = (\nabla_Y X)^\perp$ for $Y \in N_M Z_\lambda$, and $K_\lambda$ is the curvature form of the induced metric on $N_M Z_\lambda$ by $g$.

Now we would like to apply Lemma 2.2 to the calculation of Futaki invariants. Direct calculation shows that the Futaki invariant can be expressed by
\[
    (n+1)2^{n+1}f_M(\Omega, X) \nonumber
\]
\[
    = \sum_{j=0}^n (-1)^j \frac{1}{j!(n-j)!} \left( \frac{\sqrt{-1}}{2\pi} \right)^n \int_M \left( -\Delta_\theta \theta_X + \text{Ric}(g) + (n-2j)(\theta_X + \omega_h) \right)^{n+1} \nonumber
\]
\[
    - (\Delta_\theta \theta_X - \text{Ric}(g) + (n-2j)(\theta_X + \omega_h))^{n+1} - n2^{n+1}\mu \cdot \left( \frac{\sqrt{-1}}{2\pi} \right)^n \int_M (\theta_X + \omega_h)^{n+1}. \tag{2.6}
\]
where $\mu = c_1(M)\Omega^{-1}$. In the following, we want to choose the polynomial $\phi$ and the vector bundle $E \to M$ such that (2.6) can be simplified by Lemma 2.2. We assume without loss of generality that $\Omega = c_1(L)$ for a holomorphic line bundle $L \to M$. If we choose

$$\phi = \frac{1}{(n+1)!}\text{tr}(x_1 x_2 \cdots x_{n+1}), \quad E_j = K_M^{-1} \otimes L^{n-2j}.$$  \hfill (2.7)

then we have

$$R(h) = \text{Ric}(g) + (n-2j)\omega, \quad \theta_X(h) = -\Delta \theta_X + (n-2j)\theta_X,$$

where $R(h)$ is the curvature of the Hermitian metric on $E_j$ and $\theta_X(h)$ is determined by (2.2). Therefore, we have

$$\int_M \left( -\Delta \theta_X + \text{Ric}(g) + (n-2j)(\theta_X + \omega) \right)^{n+1}$$

$$= - \sum_{\lambda \in \Lambda} \lim_{\varepsilon \to 0^+} \int_{\partial B_{\varepsilon}(Z_\lambda)} \left( -\Delta \theta_X + \text{Ric}(g) + (n-2j)(\theta_X + \omega) \right)^{n+1} \wedge \sum_{k=0}^{n-1} (-1)^k \eta \wedge (\bar{\partial} \eta)^k.$$  \hfill (2.8)

Similarly, if we choose $E_j = K_M \otimes L^{n-2j}$ in (2.7), we have

$$R(h) = -\text{Ric}(g) + (n-2j)\omega, \quad \theta_X(h) = \Delta \theta_X + (n-2j)\theta_X.$$

Combining this with (2.5), we have

$$\int_M \left( -\Delta \theta_X - \text{Ric}(g) + (n-2j)(\theta_X + \omega) \right)^{n+1}$$

$$= - \sum_{\lambda \in \Lambda} \lim_{\varepsilon \to 0^+} \int_{\partial B_{\varepsilon}(Z_\lambda)} \left( -\Delta \theta_X - \text{Ric}(g) + (n-2j)(\theta_X + \omega) \right)^{n+1} \wedge \sum_{k=0}^{n-1} (-1)^k \eta \wedge (\bar{\partial} \eta)^k.$$  \hfill (2.9)

For the last term of (2.6), we choose $E = L^{n+1-2k}$ and do the same calculation as above,

$$\int_M (\theta_X + \omega)^{n+1} = - \sum_{\lambda \in \Lambda} \lim_{\varepsilon \to 0^+} \int_{\partial B_{\varepsilon}(Z_\lambda)} (\theta_X + \omega)^{n+1} \wedge \sum_{k=0}^{n-1} (-1)^k \eta \wedge (\bar{\partial} \eta)^k.$$  \hfill (2.10)

Combining the equalities (2.6)-(2.10), we have

$$f_M(\Omega, X) = \sum_{\lambda \in \Lambda} \left( I_{Z_\lambda}(\Omega, X) - \frac{n}{n+1} \mu I_{Z_\lambda}(\Omega, X) \right).$$
where $I_{Z_h}(\Omega, X)$ and $J_{Z_h}(\Omega, X)$ are defined by

$$I_{Z_h}(\Omega, X) = \frac{1}{(n+1)2^{n+1}} \sum_{j=0}^{n} (-1)^j \frac{1}{j!(n-j)!} \left( \frac{\sqrt{-1}}{2\pi} \right)^n \left( - \lim_{\epsilon \to 0^+} \int_{\partial B_\epsilon(Z_h)} (-\Delta_\epsilon \theta_X + Ric(g) + (n-2j)(\theta_X + \omega_\epsilon)) \right)^{n+1} \wedge - \lim_{\epsilon \to 0^+} \int_{\partial B_\epsilon(Z_h)} (-\Delta_\epsilon \theta_X + Ric(g) + (n-2j)(\theta_X + \omega_\epsilon)) \right)^{n+1} \wedge \sum_{k=0}^{n-1} (-1)^k \eta \wedge (\bar{\partial} \eta)^k.$$  

(2.11)

and

$$J_{Z_h}(\Omega, X) = - \lim_{\epsilon \to 0^+} \left( \frac{\sqrt{-1}}{2\pi} \right)^n \int_{\partial B_\epsilon(Z_h)} (-\Delta_\epsilon \theta_X + Ric(g) + (n-2j)(\theta_X + \omega_\epsilon)) \right)^{n+1} \wedge \sum_{k=0}^{n-1} (-1)^k \eta \wedge (\bar{\partial} \eta)^k.$$  

(2.12)

Note that using the identities

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-2j)^k = \begin{cases} 0, & \text{if } k < n \text{ or } k = n+1; \\ 2^n n!, & \text{if } k = n, \end{cases}$$

the equality (2.11) can be simplified to

$$I_{Z_h}(\Omega, X) = - \lim_{\epsilon \to 0^+} \left( \frac{\sqrt{-1}}{2\pi} \right)^n \int_{\partial B_\epsilon(Z_h)} (-\Delta_\epsilon \theta_X + Ric(g) + (n-2j)(\theta_X + \omega_\epsilon) \right)^{n+1} \wedge \sum_{k=0}^{n-1} (-1)^k \eta \wedge (\bar{\partial} \eta)^k.$$  

(2.13)

Simple calculation shows that $-\Delta_\epsilon \theta_X(Z_h) = \text{tr}(L_\lambda(X))$. Applying Lemma 2.2 to (2.12)-(2.13), we have the following result:

**Theorem 2.3.** ([17], [20]) For a Kähler class $\Omega$ and non-degenerate $X \in h_0(M)$, the Futaki invariant is given by

$$f_M(\Omega, X) = \sum_{\lambda \in \Lambda} \left( I_{Z_h}(\Omega, X) - \frac{n}{n+1} \mu J_{Z_h}(\Omega, X) \right),$$

where $\mu = \frac{c_1(M, \Omega) n-1}{4\pi}$ and

$$I_{Z_h}(\Omega, X) = \int_{Z_h} \frac{\text{tr}(L_\lambda(X)) + c_1(M)) \text{tr}(\Omega(X))_{Z_h} + \Omega)^n}{\det(L_\lambda(X)) + \sqrt{-1} K_\lambda},$$  

(2.14)

and

$$J_{Z_h}(\Omega, X) = \int_{Z_h} \frac{\text{tr}(\Omega(X)) + \Omega)^n+1}{\det(L_\lambda(X)) + \sqrt{-1} K_\lambda}.$$  

(2.15)
Theorem 2.3 was first proved by Futaki in \cite{14} for the first Chern class and by Tian in \cite{20} for a general Kähler class. For simplicity, we introduce the following notations:

**Definition 2.4.** For a Kähler class $\Omega$ and $X \in h_0(M)$ with $\text{Zero}(X) = \cup_\lambda Z_\lambda$, the local Futaki invariant of $(\Omega, X)$ on $Z_\lambda$ is defined by
\[
 f_{Z_\lambda}(\Omega, X) = I_{Z_\lambda}(\Omega, X) - \frac{n}{n+1} \mu J_{Z_\lambda}(\Omega, X),
\]
where $I_{Z_\lambda}(\Omega, X)$ and $J_{Z_\lambda}(\Omega, X)$ are given by (2.14) and (2.15) respectively when $X$ is nondegenerate on $Z_\lambda$, and by (2.13) and (2.12) in the general case. Moreover, we define
\[
 J_M(\Omega, X) = \sum_{\lambda \in \Lambda} J_{Z_\lambda}(\Omega, X) = \left( \frac{\sqrt{-1}}{2\pi} \right)^n \int_M (\theta_X + \omega_\lambda)^{n+1} = (n+1) \int_M \theta_X \left( \frac{\sqrt{-1}}{2\pi} \omega_\lambda \right)^n. 
\] (2.16)

When $M$ has complex dimension 2, we can simplify the formula in Theorem 2.3 as follows. Write the set of indices $\Lambda = \Lambda_0 \cup \Lambda_1$ where $\Lambda_i$ consists of all $\lambda$ with $\dim_\mathbb{C} Z_\lambda = i(i = 0, 1)$ and we set
\[
 A_\lambda := \text{tr}(L_\lambda(X)), \quad B_\lambda := \text{tr}_\Omega(X)_{Z_\lambda}, \quad C_\lambda = \det(L_\lambda(X)).
\]

The following result is a direct corollary of Lemma 2.3.

**Corollary 2.5.** (\cite{20}, \cite{21}) Suppose that $\dim_\mathbb{C} M = 2$. If $X \in h_0(M)$ is non-degenerate on the zero set $Z_\lambda$, we have
\[
 f_{Z_\lambda}(\Omega, X) = \frac{A_\lambda^2 B_\lambda^2 - \frac{2\mu}{3} B_\lambda^3}{C_\lambda} 
\] (2.17)
for the case $\lambda \in \Lambda_0$, and
\[
 f_{Z_\lambda}(\Omega, X) = (2B_\lambda - 2\mu B_\lambda^2 A_\lambda^{-1}) \Omega([Z_\lambda]) + \left( \frac{2\mu}{3} A_\lambda^2 B_\lambda^3 \right) c_1(M)([Z_\lambda]) + \left( A_\lambda^{-1} B_\lambda^2 - \frac{2\mu}{3} A_\lambda^{-2} B_\lambda^3 \right)(2 - 2g(Z_\lambda)) 
\] (2.18)
for $\lambda \in \Lambda_1$, where $g(Z_\lambda)$ denotes the genus of $Z_\lambda$.

Corollary 2.5 is given by \cite{20} and the details of the proof are given by \cite{21}.

**Remark 2.6.** We will also use the expression of $I_{Z_\lambda}(\Omega, X)$ and $J_{Z_\lambda}(\Omega, X)$ when $\dim_\mathbb{C} Z_\lambda = 0, 1$ in later sections. So we write them down here:

- **When** $\dim_\mathbb{C} Z_\lambda = 0$, **we have**
  \[
  I_{Z_\lambda}(\Omega, X) = \frac{A_\lambda B_\lambda^2}{C_\lambda}, \quad J_{Z_\lambda}(\Omega, X) = \frac{B_\lambda^3}{C_\lambda}.
  \]

- **When** $\dim_\mathbb{C} Z_\lambda = 1$, **we have**
  \[
  I_{Z_\lambda}(\Omega, X) = 2B_\lambda \Omega([Z_\lambda]) + A_\lambda^{-1} B_\lambda^2 (2 - 2g(Z_\lambda)),
  
  J_{Z_\lambda}(\Omega, X) = 3A_\lambda^{-1} B_\lambda^2 \Omega([Z_\lambda]) - A_\lambda^{-2} B_\lambda^3 \left( c_1(M)([Z_\lambda]) + 2g(Z_\lambda) - 2 \right).
  \]
3 Blowing up at isolated zeros

In this section we will calculate the Futaki invariant of the blow up $\pi: \tilde{M} \to M$ of a Kähler surface $M$ at an isolated zero point $p \in M$ of $X$ with the exceptional divisor $\pi^{-1}(p) = E$. In this case, $X$ can be naturally extended to a holomorphic vector field $\tilde{X}$ on $\tilde{M}$. We would like to calculate the Futaki invariant of $(\tilde{\Omega}_\varepsilon, \tilde{X})$ on $\tilde{M}$ where $\tilde{\Omega}_\varepsilon = \pi^* \Omega - \varepsilon c_1([E])$.

Now we study the zero set of $\tilde{X}$ on the blown up $\tilde{M}$. The zero set $\text{Zero}(\tilde{X}) = \bigcup_{\lambda \in \tilde{\Lambda}} \tilde{Z}_\lambda$ of $\tilde{X}$ on $\tilde{M}$ can be divided into two types: one coincides with the zero set of $X$ on $M$ and we denote the set of the indices by $\Lambda$. The other belongs to the exceptional divisor $E$ and we denote the set of the indices by $\Upsilon$. Thus, the indices of the zeros sets $\tilde{Z}_\lambda$ has the decomposition $\tilde{\Lambda} = \Lambda \cup \Upsilon$. Set

$$\tilde{\mu} = c_1(\tilde{M}) \cdot \tilde{\Omega}_\varepsilon \Omega^2 \varepsilon,$$

$$\delta := \tilde{\mu} - \mu = -\frac{1}{\Omega^2 \varepsilon} + O(\varepsilon^2).$$

With these notations, we have

**Lemma 3.1.**

$$f_{\tilde{M}}(\tilde{\Omega}_\varepsilon, \tilde{X}) = f_M(\Omega, X) - \frac{n}{n+1} \delta J_M(\Omega, X) + \sum_{\lambda \in \Lambda} f_{\tilde{Z}_\lambda}(\tilde{\Omega}_\varepsilon, \tilde{X}) + \frac{n}{n+1} \delta J_p(\Omega, X) - f_p(\Omega, X),$$

where $J_M(\Omega, X)$ is defined by (2.16).

**Proof.** The Futaki invariant of $(\tilde{\Omega}_\varepsilon, \tilde{X})$ on $\tilde{M}$ is given by

$$f_{\tilde{M}}(\tilde{\Omega}_\varepsilon, \tilde{X}) = \sum_{\lambda \in \Lambda} f_{\tilde{Z}_\lambda}(\tilde{\Omega}_\varepsilon, \tilde{X}) + \sum_{\lambda \in \Upsilon} f_{\tilde{Z}_\lambda}(\tilde{\Omega}_\varepsilon, \tilde{X}).$$

Note that for any $\lambda \in \Lambda$ we have

$$f_{\tilde{Z}_\lambda}(\tilde{\Omega}_\varepsilon, \tilde{X}) = I_{\tilde{Z}_\lambda}(\tilde{\Omega}_\varepsilon, \tilde{X}) - \frac{n}{n+1} \tilde{\mu} J_{\tilde{Z}_\lambda}(\tilde{\Omega}_\varepsilon, \tilde{X})$$

$$= I_{\tilde{Z}_\lambda}(\Omega, X) - \frac{n}{n+1} (\mu + \delta) J_{\tilde{Z}_\lambda}(\Omega, X)$$

$$= f_{Z_\lambda}(\Omega, X) - \frac{n}{n+1} \delta J_\lambda(\Omega, X),$$

where we used the fact that $I_{\tilde{Z}_\lambda}(\tilde{\Omega}_\varepsilon, \tilde{X}) = I_{Z_\lambda}(\Omega, X)$ and $J_{\tilde{Z}_\lambda}(\tilde{\Omega}_\varepsilon, \tilde{X}) = J_{Z_\lambda}(\Omega, X)$ since $\tilde{Z}_\lambda$ and $E$ are disjoint for $\lambda \in \Lambda$. Note that

$$f_M(\Omega, X) = \sum_{\lambda \in \Lambda} f_{Z_\lambda}(\Omega, X) + f_p(\Omega, X).$$

The lemma follows from the above equalities. \qed
3.1 The zero set of the holomorphic vector field $\tilde{X}$

In this subsection, we will calculate the zero locus of the holomorphic vector field $\tilde{X}$ on $\tilde{M}$.

Let $p \in M$ be an isolated zero point of $X$ and $U$ be a neighborhood of $p$ with coordinates $(z,w)$. Near the point $p$ the vector field $X$ can be written as

$$X = X^1(z,w) \frac{\partial}{\partial z} + X^2(z,w) \frac{\partial}{\partial w}, \quad (3.1)$$

where $X^1(z,w)$ and $X^2(z,w)$ are holomorphic functions on $U$. We assume that the functions $X^1$ and $X^2$ can be expanded on $U$ near $p \in M$ as

$$X^1(z,w) = a_1z + b_1w + \sum_{i+j \geq 2} c_{ij}z^i w^j, \quad (3.2)$$

$$X^2(z,w) = a_2z + b_2w + \sum_{i+j \geq 2} d_{ij}z^i w^j, \quad (3.3)$$

where $a_i, b_i, c_{ij}$ and $d_{ij}$ are constants. By our non-degenerate assumption, the matrix

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

is non-singular.

Consider the blowing up map $\pi : \tilde{M} \rightarrow M$ at the point $p$.

**Lemma 3.2.** Let $p$ be a non-degenerate isolated zero point of $X$, where $X$ is locally given by (3.2)-(3.3). Then $\tilde{X}$ is non-degenerate if and only if the matrix

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

is semisimple (i.e. diagonalizable).

**Proof.** By a linear transform of coordinates, we may assume that the matrix

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

is a Jordan canonical form. In particular, $a_2 = 0$. We choose the coordinates on $\tilde{U} := \pi^{-1}(U)$ as

$$\tilde{U} := \left\{ ((z,w),[\xi,\eta]) \mid (z,w) \in U, z\eta = w\xi \right\} \subset U \times \mathbb{C}P^1,$$

which can be covered by two open sets $\tilde{U}_1 = \{ \xi \neq 0 \}$ and $\tilde{U}_2 = \{ \eta \neq 0 \}$. We choose the coordinate functions $(u_1, v_1)$ on $\tilde{U}_1$ where $u_1 = z, v_1 = \frac{z}{\bar{z}}$ and the coordinate functions $(u_2, v_2)$ on $\tilde{U}_2$ where $u_2 = \frac{\bar{\xi}}{\eta}, v_2 = w$. We want to compute the zero set of $\tilde{X}$ on $\tilde{U}_1$ and $\tilde{U}_2$. 

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On $\tilde{U}_1$ the holomorphic vector field $\tilde{X}$ can be written as

$$\tilde{X} = X^1(u_1, u_1v_1) \frac{\partial}{\partial u_1} + \frac{1}{u_1} \left( X^2(u_1, u_1v_1) - X^1(u_1, u_1v_1)v_1 \right) \frac{\partial}{\partial v_1}. $$

Thus, using the coordinates on $\tilde{U}_1$ the vector field $\tilde{X}$ can be expressed by

$$\tilde{X} = \tilde{X}^1(u_1, v_1) \frac{\partial}{\partial u_1} + \tilde{X}^2(u_1, v_1) \frac{\partial}{\partial v_1},$$

where

$$\tilde{X}^1 = u_1 \left( a_1 + b_1 v_1 + \sum_{i+j \geq 2} c_{ij} u_1^{i+j-1} v_1^j \right),$$

$$\tilde{X}^2 = (b_2 - a_1) v_1 - b_1 v_1^2 + \sum_{i+j \geq 2} (d_{ij} v_1^i - c_{ij} v_1^{i+1}) u_1^{i+j-1}. $$

(3.4) (3.5)

Since $p$ is an isolated zero of $X$, the zero set of $\tilde{X}$ on $\tilde{U}_1$ lies in the exceptional divisor and it is given by

$$Z_1 = \left\{ (u_1, v_1) \in \tilde{U}_1 \mid u_1 = 0, (b_2 - a_1) v_1 - b_1 v_1^2 = 0 \right\} \subset E \cap \tilde{U}_1. $$

which consists of the following cases:

- If

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

where $a \neq 0$, then $Z_1 = E \cap \tilde{U}_1$;

- If

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

where $a \neq b$, then $Z_1 = \{ p_1 \}$ where $p_1$ has the coordinates

$$p_1 : (u_1, v_1) = (0, 0);$$

- If

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix},$$

where $a \neq 0$, then $Z_1 = \{ p_1 \}$ where $p_1$ has the coordinates

$$p_1 : (u_1, v_1) = (0, 0),$$

and $\tilde{X}$ is degenerate at this point.
Now we calculate the zero set of $\tilde{X}$ on $\tilde{U}_2$. Using the coordinates on $\tilde{U}_2$ the holomorphic vector field $\tilde{X}$ can be written as

$$\tilde{X} = \tilde{X}^1(u_2,v_2) \frac{\partial}{\partial u_2} + \tilde{X}^2(u_2,v_2) \frac{\partial}{\partial v_2},$$

where $\tilde{X}^1(u_2,v_2)$ and $\tilde{X}^2(u_2,v_2)$ are given by

$$\tilde{X}^1(u_2,v_2) = b_1 + (a_1 - b_2)u_2 + \sum_{i+j \geq 2} (c_{ij}u_2^i - d_{ij}u_2^{i+1})v_2^{i+j-1}, \quad (3.6)$$

$$\tilde{X}^2(u_2,v_2) = v_2 \left( b_2 + \sum_{i+j \geq 2} d_{ij}u_2^i v_2^{i+j-1} \right). \quad (3.7)$$

Thus, the zero set $Z_2$ of $\tilde{X}$ on $E \cap \tilde{U}_2$ is given by

$$Z_2 = \left\{ (u_2,v_2) \mid v_2 = 0, \ b_1 + (a_1 - b_2)u_2 = 0 \right\}.$$

So we have:

- If $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, where $a \neq 0$, then $Z_2 = E \cap \tilde{U}_2$;

- If $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a \neq b$, then $Z_2 = \{ q_1 \}$ where $q_1$ has the coordinates $q_1 : (u_2,v_2) = (0,0)$.

- If $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$, where $a \neq 0$, then $Z_2 = \emptyset$.

Hence the lemma is proved. \(\square\)

### 3.2 The non-degenerate cases

In this subsection, we will calculate the Futaki invariant of $(\tilde{\Omega}_\epsilon, \tilde{X})$ for the non-degenerate cases in Lemma 3.2. It follows from Lemma 3.1 that we need to calculate the local Futaki invariant on the zero set of $\tilde{X}$ which lies in the exceptional divisor $E$. The calculation is not difficult since we have the nice formula in Corollary 2.5 when $\tilde{X}$ is non-degenerate.
Theorem 3.3. If \( \tilde{X} \) is non-degenerate on \( \tilde{M} \), then

\[
\tilde{f}_M(\tilde{\Omega}_e, \tilde{X}) = f_M(\Omega, X) + v_p(\Omega, X) \cdot \varepsilon + O(\varepsilon^2),
\]

where \( v_p(\Omega, X) \) is given by

\[
v_p(\Omega, X) = -2\text{tr}_\Omega(X)(p) + \frac{2}{3} \Omega^2 J_M(\Omega, X).
\]

Proof. By Lemma 3.1 it suffices to compute \( \tilde{f}_{\tilde{\Omega}_e}(\tilde{\Omega}_e, \tilde{X})(\lambda \in \mathcal{Y}) \) for the non-degenerate cases in Lemma 3.2. We divide the proof into two cases:

Case 1: For \( a_1 = b_2, a_2 = b_1 = 0 \) the zero set of \( \tilde{X} \) on \( \tilde{U} \) is given by \( Z = E \). Note that by (3.4)-(3.5) we have

\[
\tilde{A}_E := \text{tr}(L_E(\tilde{X})) = a_1 = \tilde{C}_E, \quad \tilde{\Omega}_e([E]) = \pi^* \Omega([E]) - \varepsilon E \cdot E = \varepsilon.
\]

To calculate \( B_E := \text{tr}(\tilde{\Omega}_e(\tilde{X})) \), we need to choose a suitable Kähler metric on \( \tilde{M} \) in the class \( \tilde{\Omega}_e \).

We shall choose such a metric as Griffiths and Harris did in their book [15]. The construction is as follows:

Following the notations in Section 3.1, the set \( \tilde{U}_1 \) has local coordinates \((u_1, v_1)\) with \( u_1 = z, v_1 = \frac{2}{3} \) and the exceptional divisor is given by \( u_1 = 0 \). The line bundle \([E]\) over \( \tilde{U} \) has transition function \( z/w \) on \( \tilde{U}_1 \cap \tilde{U}_2 \) and we can choose a global section \( \sigma \) of \([E]\) over \( \tilde{M} \) by \( \sigma = u_1 \) on \( \tilde{U}_1 \) and \( \sigma = 1 \) on \( \tilde{M} \backslash \tilde{B}_2 \) where \( \tilde{B}_r := \pi^{-1}(B_{2r}(p)) \). Here \( B_r(p) \) denotes the ball on \( M \) centered at \( p \) with \( |z|^2 + |w|^2 < r \) and we assume that \( \tilde{B}_1 \subset \tilde{U} \). Define the Hermitian metric \( h_1 \) of \([E]\) over \( \tilde{U} \) given in \( \tilde{U}_1 \) by

\[
h_1 = \frac{|\xi|^2 + |\eta|^2}{|\xi|^2},
\]

and \( h_2 \) the Hermitian metric of \([E]\) over \( \tilde{M} \backslash \tilde{B}_2 \) with \( |\sigma|^2_{h_2} = 1 \). Let \( \rho_1, \rho_2 \) be a partition of unity for the cover \((\tilde{B}_1, \tilde{M} \backslash \tilde{B}_2)\) of \( \tilde{M} \) and let \( h \) be the global Hermitian metric defined by

\[
h = \rho_1 h_1 + \rho_2 h_2.
\]

Then the function \( |\sigma|^2_h \) on \( \tilde{B}_2 \) is given by

\[
|\sigma|^2_h = \frac{|\xi|^2 + |\eta|^2}{|\xi|^2} \cdot |u_1|^2 = |u_1|^2 + |u_1 v_1|^2.
\]

Given a Kähler metric \( \omega_X \) with the Kähler class \( \Omega = \frac{1}{2\pi} [\omega_X] \) on \( M \), the induced metric \( \tilde{\omega}_e \) in the Kähler class \( \tilde{\Omega}_e = \pi^* \Omega - \varepsilon c_1(M) \) is given by

\[
\tilde{\omega}_e = \pi^* \omega_X + \varepsilon \partial \bar{\partial} \log h.
\]

Thus, the holomorphy potential \( \tilde{\theta}_X \) of \( \tilde{X} \) with respect to \( \tilde{\omega}_e \) is given by

\[
\tilde{\theta}_X = \pi^* \theta_X - \varepsilon \cdot \tilde{X} \left( \log |\sigma|^2_h \right).
\]
Using the expression (3.4)-(3.5) and (3.9), we have $\tilde{X} (\log |\sigma|^2_E) |_E = a_1$. In conclusion, we have

$$ B_E = \theta_p - a_1 \epsilon, \quad (3.10) $$

where $\theta_p = \theta_X(p)$.

Note that the genus of the exceptional divisor is zero and $\tilde{\mu} = \mu + \delta$, by Corollary 2.5 we have

$$ f_E (\tilde{\Omega}_E, \tilde{X}) = (2B_E - 2\tilde{\mu}B_E^2 \tilde{A}^{-1}_E) \tilde{\Omega}_E (\tilde{X}) ) $$

$$ + (\frac{2\tilde{\mu}}{3} \tilde{A}_E^{-2} \tilde{B}_E^3 c_1(\bar{M})([E]) + (\tilde{A}_E^{-1} \tilde{B}_E^2 - \frac{2\tilde{\mu}}{3} \tilde{A}_E^{-2} \tilde{B}_E^3)(2 - 2g(E)) $$

$$ = \frac{2\theta_p^2}{a_1} - \frac{2\theta_p^2}{3a_1^2} \mu - 2\theta_p \epsilon - \frac{2\theta_p}{3a_1^2} \delta + O(\epsilon^2), $$

where we used (3.3). On the other hand, using Corollary 2.5 again we can compute $f_p(\Omega, X)$ and $J_p(\Omega, X)$ as follows:

$$ f_p(\Omega, X) = \frac{2\theta_p^2}{a_1} - \frac{2\theta_p^2}{3a_1^2} \mu, \quad J_p(\Omega, X) = \frac{\theta_p^3}{a_1}, $$

where we used the fact that $A_p = 2a_1, B_p = \theta_p$ and $C_p = a_1^2$. Combining these with Lemma 3.1 we have

$$ f_M(\tilde{\Omega}_E, \tilde{X}) = f_M(\Omega, X) - \frac{2}{3} \delta J_M(\Omega, X) - 2\theta_p \epsilon + O(\epsilon^2). $$

**Case 2:** For $a_1 \neq b_2$ and $a_2 = b_1 = 0$, the zero set $Z = \{ p_1, q_1 \}$ where $p \in \bar{U}_1$ and $q \in \bar{U}_2$ and the coordinates are given by

$$ p_1 : (u_1, v_1) = (0, 0), \quad q_1 : (u_2, v_2) = (0, 0). $$

By the expression (3.4)-(3.5) of $\tilde{X}$ near $p_1$ we have

$$ \tilde{A}_{p_1} = b_2, \quad \tilde{B}_{p_1} = \theta_p - a_1 \epsilon, \quad \tilde{C}_{p_1} = a_1 (b_2 - a_1), $$

where $\tilde{B}_{p_1}$ can be calculated as Case 1. Thus, the local Futaki invariant of $p_1$ is given by

$$ f_{p_1}(\tilde{\Omega}_E, \tilde{X}) = \frac{b_2(\theta_p - a_1 \epsilon)^2}{a_1(b_2 - a_1)} - \frac{2(\theta_p - a_1 \epsilon)^3(\mu + \delta)}{3a_1(b_2 - a_1)}, \quad (3.11) $$

Similarly, by the expression (3.6)-(3.7) of $\tilde{X}$ near $q_1$ we have

$$ \tilde{A}_{q_1} = a_1, \quad \tilde{B}_{q_1} = \theta_p - b_2 \epsilon, \quad \tilde{C}_{q_1} = b_2 (a_1 - b_2). $$

The local Futaki invariant of $q_1$ is given by

$$ f_{q_1}(\tilde{\Omega}_E, \tilde{X}) = \frac{a_1(\theta_p - b_2 \epsilon)^2}{b_2(a_1 - b_2)} - \frac{2(\theta_p - b_2 \epsilon)^3(\mu + \delta)}{3b_2(a_1 - b_2)}, \quad (3.12) $$

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Next, we calculate the local Futaki invariant of $p$. Clearly, on the point $p \in M$,

$$A_p = a_1 + b_2, \quad B_p = \theta_p, \quad C_p = a_1 b_2$$

and we have

$$f_p(\Omega, X) = \frac{(a_1 + b_2)\theta_p^2}{a_1 b_2} - \frac{2\theta_p^3}{3a_1 b_2}, \quad J_p(\Omega, X) = \frac{\theta_p^3}{a_1 b_2}.$$ 

Collecting the above results, we have

$$f_{\tilde{M}}(\tilde{\Omega}_\varepsilon, \tilde{X}) = f_M(\Omega, X) - \frac{2}{3} \delta J_M(\Omega, X) + f_p(\Omega, X)$$

The theorem is proved.

### 3.3 The degenerate case

In this subsection, we will calculate the Futaki invariant when $\tilde{X}$ is degenerate on the exceptional divisor $E$. In this case, the calculation of Bott, Futaki and Tian fails and it should be related to the general theory of Residue currents (cf. [22] and reference therein). However, when $M$ has complex dimension 2, we can do the direct calculation using only the elementary calculus:

**Theorem 3.4.** Let $p$ be an isolated zero of $X$. If $\tilde{X}$ is degenerate at a zero point $\tilde{p} \in E$, then the Futaki invariant of $(\tilde{\Omega}_\varepsilon, \tilde{X})$ is given by

$$f_{\tilde{M}}(\tilde{\Omega}_\varepsilon, \tilde{X}) = f_M(\Omega, X) + v_p(\Omega, X) \cdot \varepsilon + O(\varepsilon^2),$$

where

$$v_p(\Omega, X) = -2 \text{tr}_\Omega(X)(p) + \frac{2}{3}\Omega^2 J_M(\Omega, X).$$

**Proof.** First, we claim that we can find a holomorphic coordinate transform around $p$ such that in the new coordinates, our holomorphic vector field contains only linear terms. The reason is the following:

We call a vector $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ to be “resonant”, if there is an integral relation of the form

$$\lambda_k = \sum_{i=1}^n m_i \lambda_i,$$

where $m_i$ are non-negative integers with $\sum m_i \geq 2$. And we say $\lambda$ belongs to the Poincaré domain if the convex hull of $\lambda_1, \ldots, \lambda_n$ in $\mathbb{C}$ does not contain the origin.

**Theorem 3.5 (Poincaré, [5], P190).** If the eigenvalues of the linear part of a holomorphic vector field at a singular point (i.e. zero point) belong to the Poincaré domain and are non-resonant, then the vector field is biholomorphically equivalent to its linear part in a neighborhood of the singular point.

The idea of this theorem is that if the linear part of the vector field satisfies the “non-resonant condition”, then we can construct a family of holomorphic coordinate transforms that eliminate the $k$-th order terms recursively for any $k \geq 2$. And if the eigenvalues are in the Poincaré domain,
then the compositions of the coordinate transforms also converge to a holomorphic coordinate transform. The interested reader can find a detail discussion in [5].

In our case, the linear part of $X$ clearly satisfies the conditions in Poincaré’s theorem, so in the following discussion, we can assume without loss of generality that

$$X^1(z,w) = az + w, \quad X^2(z,w) = aw.$$  

Then in the coordinates $(u_1,v_1)$ of previous subsections, $\tilde{X}$ can be written as the following on $\tilde{U}_1$:

$$\tilde{X} = u_1(a + v_1)\frac{\partial}{\partial u_1} - v_1^2\frac{\partial}{\partial v_1}.$$  

It is clear from the discussion in Section 2 that in defining $I_p$ and $J_p$, we can use any family of domains shrinking to $p$. So in this section, we choose special domains to simplify the computation. Let $B_r$ be a sufficiently small “distorted” ball around $p_1$, defined by $|\tilde{X}|^2(u_1,v_1) \leq r^4$. We have the following

**Lemma 3.6.** Let $\phi$ be any smooth function on $\tilde{U}_1$. Then we have

$$\lim_{r \to 0^+} \int_{\partial B_r} \phi \eta \wedge \bar{\eta} = \frac{4\pi^2}{a} \frac{\partial \phi}{\partial v_1}(0) - \frac{4\pi^2}{a^2} \phi(0).$$  

We use this lemma to calculate $J_{p_1}(\tilde{\Omega}_\varepsilon, \tilde{X})$. First, note that for any smooth 2-form $\chi$, we have

$$\lim_{r \to 0^+} \int_{\partial B_r} \chi \wedge \eta = 0.$$  

This can be seen from the expression of $\eta$ in the next subsection. By (2.12), we have

$$J_{p_1}(\tilde{\Omega}_\varepsilon, \tilde{X}) = -\lim_{r \to 0^+} \frac{1}{4\pi^2} \int_{\partial B_r} (\tilde{\theta}_X + \tilde{\omega}_X)^3 \wedge \eta \wedge \bar{\eta}$$

$$= -\lim_{r \to 0^+} \frac{1}{4\pi^2} \int_{\partial B_r} \tilde{\theta}_X^3 \cdot \eta \wedge \bar{\eta}.$$  

To calculate the last term, we need to expand the function $\tilde{\theta}_X$. In fact, near $p_1$ we have

$$\tilde{X}(\log |\sigma^2_n|) = u_1(a + v_1)\frac{\partial}{\partial u_1}(\log(|u_1|^2 + |u_1v_1|^2)) - v_1^2\frac{\partial}{\partial v_1}(\log(|u_1|^2 + |u_1v_1|^2))$$

$$= a + \frac{v_1}{1 + |v_1|^2}.$$  

It follows that

$$\tilde{\theta}_X(0) = \theta_p - a\varepsilon, \quad \frac{\partial \tilde{\theta}_X}{\partial v_1}(0) = -\varepsilon.$$  

By Lemma 3.6 and (3.13), we have

$$J_{p_1}(\tilde{\Omega}_\varepsilon, \tilde{X}) = \frac{3(\theta_p - a\varepsilon)^2\varepsilon}{a} + \frac{(\theta_p - a\varepsilon)^3}{a^2} = \frac{\theta_3^3}{a^2} + O(\varepsilon^2).$$  

(3.14)
Next, we calculate $I_p(\tilde{\Omega}_\varepsilon, \tilde{X})$. When $n = 2$ we have

$$I_p(\tilde{\Omega}_\varepsilon, \tilde{X}) = \lim_{r \to 0} \left( \frac{\sqrt{-1}}{2\pi} \right)^n \int_{\partial B_r} (-\Delta \tilde{\theta}_X + Ric(\tilde{g})(\tilde{\theta}_X + \omega_\varepsilon)^n) \wedge \sum_{k=0}^{n-1} (-1)^k \eta \wedge (\bar{\partial} \eta)^k$$

$$= -\frac{1}{4\pi^2} \lim_{r \to 0} \int_{\partial B_r} -\Delta \tilde{\theta}_X \tilde{\theta}_X^2 \cdot \eta \wedge \bar{\partial} \eta.$$

Direct computation shows that

$$-\Delta \tilde{\theta}_X(0) = a, \quad \frac{\partial}{\partial v_1}(-\Delta \tilde{\theta}_X)(0) = -1. \quad (3.15)$$

Combining this with (3.13) and Lemma 3.6, we have

$$I_p(\tilde{\Omega}_\varepsilon, \tilde{X}) = \frac{2\theta^2}{a} - 2\theta \varepsilon + O(\varepsilon^2).$$

On the other hand, we have

$$f_p(\Omega, X) = \frac{2\theta^2}{a} - \frac{2\theta^3}{3a^2} \mu, \quad J_p(\Omega, X) = \frac{\theta^3}{a^2}. \quad (3.16)$$

Combining the above results, we have

$$f_M(\tilde{\Omega}_\varepsilon, \tilde{X}) = f_M(\Omega, X) - \frac{2}{3} \delta J_M(\Omega, X) + f_p(\tilde{\Omega}_\varepsilon, \tilde{X}) - f_p(\Omega, X) + \frac{2}{3} \delta J_p(\Omega, X)$$

$$= f_M(\Omega, X) - 2\theta \varepsilon - \frac{2}{3} \delta J_M(\Omega, X) + O(\varepsilon^2).$$

The theorem is proved.

3.4 Proof of Lemma 3.6

We first write $\eta$ as (for simplicity, we sometimes use $(z^1, z^2)$ to denote $(u_1, v_1)$)

$$\eta = \eta_i dz^i,$$

where $\eta_i = \frac{\alpha_i}{|X|^4}$, and $\alpha_i = g_{ij} \bar{X}^j$. Direct computation shows that

$$|X|^2 = g_{11}|u_1(a + v_1)|^2 - 2Re(g_{12}u_1(a + v_1)\bar{v}_1^2) + g_{22}|v_1|^4,$$

and

$$\eta \wedge \bar{\partial} \eta = \frac{(\alpha_1 dz^1) \wedge \bar{\partial} \alpha_1 \wedge dz^1}{|X|^4} = \frac{\alpha_2 dz^2 \wedge \alpha_3 dz^3 \wedge dz^1}{|X|^4},$$

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where $\alpha_{\ldots, k}$ means derivative in the direction of $\bar{z}^k$. In our 2-dimensional case, we have

$$
\eta \wedge \partial \eta = \frac{\alpha_1 \alpha_{2,1} du_1 \wedge d\bar{u}_1 \wedge dv_1 + \alpha_1 \alpha_{2,2} du_1 \wedge d\bar{v}_1 \wedge dv_1 + \alpha_2 \alpha_{1,1} dv_1 \wedge d\bar{u}_1 + \alpha_2 \alpha_{1,2} dv_1 \wedge d\bar{v}_1 \wedge du_1}{|X_g|^4} + \frac{(\alpha_1 \alpha_{2,1} - \alpha_2 \alpha_{1,1}) du_1 \wedge d\bar{u}_1 \wedge dv_1}{|X_g|^4} + \frac{(\alpha_2 \alpha_{1,2} - \alpha_1 \alpha_{2,2}) du_1 \wedge dv_1 \wedge d\bar{v}_1}{|X_g|^4}.
$$

Now we have the following:

$$
\begin{align*}
\alpha_1 &= g_{11} u_1 (a + v_1) - g_{12} \bar{v}_1^2, \\
\alpha_2 &= g_{21} u_1 (a + v_1) - g_{22} \bar{v}_1^2, \\
\alpha_{1,1} &= g_{11,1} u_1 (a + v_1) - g_{12,1} \bar{v}_1^2 + g_{11} (a + v_1), \\
\alpha_{1,2} &= g_{11,2} u_1 (a + v_1) - g_{12,2} \bar{v}_1^2 + g_{11} (a + v_1), \\
\alpha_{2,1} &= g_{21,1} u_1 (a + v_1) - g_{22,1} \bar{v}_1^2 + g_{21} (a + v_1), \\
\alpha_{2,2} &= g_{21,2} u_1 (a + v_1) - g_{22,2} \bar{v}_1^2 + g_{21} (a + v_1).
\end{align*}
$$

**Proof of Lemma 3.6.** To compute the limit

$$
\lim_{r \to 0} \int_{\partial B_r} \phi \eta \wedge \overline{\partial \eta},
$$

We use scaling: Set $(u_1, v_1) = (r^2 u, rv)$, and for a function $f(u_1, v_1, \bar{u}_1, \bar{v}_1)$, the function $f^{(r)}$ is defined to be

$$
f^{(r)}(u, v, \bar{u}, \bar{v}) = f(r^2 u, rv, r^2 \bar{u}, r \bar{v}).
$$

Now in the coordinate $(u, v)$, the boundary $\partial B_r$ becomes

$$
S_r := \{(u, v)|g_{11}^{(r)}|u(a + rv)|^2 - 2Re(g_{12}^{(r)} u(a + rv)\bar{v}^2) + g_{22}^{(r)}|v|^4 = 1\}.
$$

Recall that on $\partial B_r$, we have $|X_g|^2 = r^4$. Then we have:

$$
\int_{\partial B_r} \phi \eta \wedge \overline{\partial \eta} = \int_{S_r} \phi^{(r)} r^5 (\alpha_1^{(r)} \alpha_{2,1}^{(r)} - \alpha_2^{(r)} \alpha_{1,1}^{(r)}) du \wedge d\bar{u} \wedge dv + r^4 (\alpha_2^{(r)} \alpha_{1,2}^{(r)} - \alpha_1^{(r)} \alpha_{2,2}^{(r)}) du \wedge dv \wedge d\bar{v}.
$$

Now when $r \to 0$, for any function $f$ we have $f^{(r)} \to f(p_1)$. Moreover, we have that

$$
\left( g_{11}^{(r)} |u(a + rv)|^2 - 2Re(g_{12}^{(r)} u(a + rv)\bar{v}^2) + g_{22}^{(r)} |v|^4 \right)^2 \to Q_0(a, -v^2),
$$

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where $Q_0$ is the hermitian quadratic form defined by $g_{ij}(p_1)$. Note that $Q_0(au, -v^2)$ is invariant under the symmetries $(u, v) \mapsto (u, -v)$ and $(u, v) \mapsto (-u, \sqrt{-1}v)$. Direct computation shows that

$$\frac{\alpha_1^{(r)}(u_1) - \alpha_2^{(r)}(u_1)}{r^3} = \frac{av^2 \det g^{(r)}}{r} + \bar{v}^3 \det g^{(r)} + r(\ldots)$$

and

$$\frac{\alpha_2^{(r)}(u_2) - \alpha_1^{(r)}(u_2)}{r^4} = \frac{2auv \det g^{(r)}(u_2)}{r} + u v^2(\ldots) + u^2(a + rv)^2(\ldots) + uv^2(a + rv)(\ldots) + \bar{v}^4(\ldots) + r(\ldots).$$

We claim that when taking limit, we need only to consider the terms with the factor $\frac{1}{r}$. First, for terms with a factor $r$, the limit vanishes automatically. For other terms without the factor $\frac{1}{r}$, the integration operation commutes with taking limit, and we can use the special symmetries of $Q_0(au, -v^2)$ to prove that the limit integral also vanishes. To sum up, we have

**Lemma 3.7.** We have

$$\lim_{r \to 0} \int_{\partial B_r} \phi \eta \wedge \bar{\eta} = \lim_{r \to 0} \frac{\Phi(r)}{r} = \lim_{r \to 0} \frac{\tilde{\phi} \det g^{(r)}(0)}{r} \left( v^2 du \wedge d\bar{u} \wedge dv + 2\bar{u} du \wedge dv \wedge d\bar{v} \right),$$

where $\Phi(r)$ is the integral over $S_r$.

Now we use the Taylor expansion of the function $\phi^{(r)} \det g^{(r)}$, and using the symmetry of $S_0$, we have

$$\lim_{r \to 0} \frac{\Phi(r)}{r} = (\phi \det g)(0) \lim_{r \to 0} \frac{1}{r} \int_{B_r} -4\bar{v} du \wedge d\bar{u} \wedge dv \wedge d\bar{v} + \frac{\partial}{\partial v_1} (\phi \det g)(0) \int_{S_0} |v|^2 (\bar{v} du \wedge d\bar{u} \wedge dv + 2\bar{u} du \wedge dv \wedge d\bar{v}),$$

where $B_r$ is the image of $B_r$ under the coordinate change. Next we evaluate the second integral. Since under the degree 2 map $(u, v) \mapsto (au, -v^2)$, the surface $S_0$ becomes

$$\tilde{S} = \{(s, t) | Q_0(s, t) = 1\}.$$ 

So we have

$$\int_{S_0} |v|^2 (\bar{v} du \wedge d\bar{u} \wedge dv + 2\bar{u} du \wedge dv \wedge d\bar{v})$$

$$= 2 \frac{1}{2|a|^2} \int_{\tilde{S}} \tilde{s} ds \wedge d\tilde{s} \wedge dt + \tilde{\bar{s}} ds \wedge dt \wedge d\bar{t}$$

$$= -2 \frac{1}{|a|^2} \int_{Q_0(s, t) \leq 1} ds \wedge d\tilde{s} \wedge dt \wedge d\bar{t}$$

$$= \frac{2}{|a|^2 \det g(0)} \int_{|\gamma|^2 + |\bar{\gamma}|^2 \leq 1} (\sqrt{-1})^2 ds \wedge d\bar{s} \wedge dt \wedge d\bar{t}$$

$$= \frac{4\pi^2}{|a|^2 \det g(0)}.$$
For the first limit we have the following lemma:

**Lemma 3.8.** We have

\[
\lim_{r \to 0} \frac{1}{r} \int_{B_r} \bar{u} \, du \wedge d\bar{u} \wedge dv \wedge d\bar{v} = \pi^2 \left( \frac{\partial}{\partial v}(\det g(0)) + \frac{\bar{a}}{|a|^4 \det g(0)} \right)
\]

Now combining the above results, we get Lemma 3.6.

**Proof of Lemma 3.8** To compute the integral

\[
\int_{\tilde{B}_r} \bar{u} \, du \wedge d\bar{u} \wedge dv \wedge d\bar{v},
\]

where \( \tilde{B}_r \) is given by

\[
\tilde{B}_r = \left\{ (u, v) \mid \tilde{g}_{11}^{(r)} |u(a + rv)|^2 - \tilde{g}_{12}^{(r)} u(a + rv) \bar{v}^2 - \tilde{g}_{21}^{(r)} u(a + rv) \bar{v}^2 + \tilde{g}_{22}^{(r)} |v|^4 \leq 1 \right\}.
\]

Consider the following differentiable coordinate transformation:

\[
s = \sqrt{\tilde{g}_{11}^{(r)}} u(a + rv) - \frac{\tilde{g}_{21}^{(r)}}{\sqrt{\tilde{g}_{11}^{(r)}}} \bar{v}^2, \quad t = \left( \frac{\det \tilde{g}^{(r)}}{\tilde{g}_{11}^{(r)}} \right)^{\frac{1}{2}} v.
\]

Then the domain \( \tilde{B}_r \) is transformed to \( \Omega_0 = \{(s, t) ||s|^2 + |t|^4 \leq 1 \} \). Direct computation shows that

\[
\frac{\partial t}{\partial u} = O(r^2), \quad \frac{\partial t}{\partial \bar{u}} = O(r^2).
\]

So we have

\[
dt \wedge d\bar{t} = \left( \left| \frac{\partial t}{\partial v} \right|^2 - \left| \frac{\partial t}{\partial \bar{v}} \right|^2 \right) dv \wedge d\bar{v} + O(r^2).
\]

Note that \( \frac{\partial t}{\partial v} = O(r) \), we have

\[
dt \wedge d\bar{t} = \left| \frac{\partial t}{\partial v} \right|^2 dv \wedge d\bar{v} + O(r^2).
\]

It follows that

\[
 ds \wedge d\bar{s} \wedge dt \wedge d\bar{t} = \left( \left| \frac{\partial t}{\partial v} \right|^2 \left( \left| \frac{\partial s}{\partial u} \right|^2 - \left| \frac{\partial s}{\partial \bar{u}} \right|^2 \right) + O(r^2) \right) du \wedge d\bar{u} \wedge dv \wedge d\bar{v}
\]

\[
 = \left( \left| \frac{\partial t}{\partial v} \right|^2 \left| \frac{\partial s}{\partial u} \right|^2 + O(r^2) \right) du \wedge d\bar{u} \wedge dv \wedge d\bar{v},
\]

where we used \( \frac{\partial s}{\partial \bar{u}} = O(r^2) \) in the last inequality.
Now we compute $\left| \frac{\partial u}{\partial v} \right|^2 \left| \frac{\partial v}{\partial u} \right|^2$. We have

$$\frac{\partial s}{\partial u} = (a + rv) \sqrt{g_{11}(r)} + O(r^2),$$

So

$$\left| \frac{\partial s}{\partial u} \right|^2 = (|a|^2 + rv\bar{a} + rv\bar{a}) \left( g_{11}(0) + rv\frac{\partial g_{11}}{\partial v_1}(0) + r\bar{v}\frac{\partial g_{11}}{\partial v_1}(0) \right) + O(r^2)$$

$$= |a|^2 g_{11}(0) + rv\left( |a|^2 \frac{\partial g_{11}}{\partial v_1}(0) + \bar{a}g_{11}(0) \right) + r\bar{v}\left( |a|^2 \frac{\partial g_{11}}{\partial v_1}(0) + a g_{11}(0) \right) + O(r^2).$$

Similarly, we have

$$\left| \frac{\partial t}{\partial v} \right|^2 = \left( \frac{\det g(0)}{g_{11}(0)} \right)^{-\frac{1}{2}} \left( \left( \frac{\det g(0)}{g_{11}(0)} \right)^2 + \frac{3}{4} \frac{\partial}{\partial v_1} \left( \frac{\det g}{g_{11}} \right)(0) \right) + \frac{3}{4} \frac{\partial}{\partial v_1} \left( \frac{\det g}{g_{11}} \right)(0) + O(r^2).$$

So we get

$$\left| \frac{\partial t}{\partial v} \right|^2 \left| \frac{\partial s}{\partial u} \right|^2 = \left( \frac{\det g(0)}{g_{11}(0)} \right)^{-\frac{1}{2}} (C_0 + rvC_1 + rvC_2) + O(r^2),$$

(3.17)

where

$$C_0 = |a|^2 \frac{(\det g(0))^2}{g_{11}(0)}$$

(3.18)

and

$$C_1 = \frac{\frac{3}{2} |a|^2 g_{11}(0) \det g(0) \frac{\partial g}{g_{11}^2}(0) + (\det g(0))^2 (\frac{1}{4} |a|^2 \frac{\partial g_{11}}{\partial v_1}(0) + \bar{a} g_{11}(0))}{(g_{11}(0))^2}.$$  

(3.19)

Using (3.17), we have

$$\lim_{r \to 0} \frac{1}{r} \int_{\Omega} \bar{v} du \wedge d\bar{u} \wedge dv \wedge d\bar{v}$$

$$= \lim_{r \to 0} \frac{1}{r} \left( \frac{\det g(0)}{g_{11}(0)} \right)^{-\frac{1}{2}} \int_{\Omega} \left( \frac{\det g}{g_{11}} \right)^{\frac{1}{2}} \frac{\bar{v} ds \wedge d\bar{s} \wedge dt \wedge d\bar{t}}{C_0 + rvC_1 + rvC_2 + O(r^2)}$$

$$= \left( \frac{\det g(0)}{g_{11}(0)} \right)^{\frac{1}{2}} \lim_{r \to 0} \frac{1}{r} \int_{\Omega} \left( \frac{\det g}{g_{11}} \right)^{\frac{1}{2}} - \frac{rt}{4} \left( \frac{\det g}{g_{11}} \right) \frac{\partial}{\partial v_1} \left( \frac{\det g}{g_{11}} \right)(0) \right) \frac{1}{C_0} \left( 1 - C_1 \left( \frac{\det g}{g_{11}} \right) \right)$$

$$\int_{\Omega} (\frac{\partial v}{\partial u} \left( \frac{\det g}{g_{11}} \right)^{\frac{1}{2}} \frac{\partial}{\partial v_1} \left( \frac{\det g}{g_{11}} \right)(0) + \frac{C_1}{C_0} \left( \frac{\det g}{g_{11}} \right)^{\frac{1}{2}} \right) \frac{1}{(\frac{\det g}{g_{11}})^{\frac{1}{2}}} ds \wedge d\bar{s} \wedge dt \wedge d\bar{t}.$$

Now using the 2-1 mapping $(s,t) \mapsto (s',t')$, we have

$$\int_{\Omega} (\frac{1}{(\frac{\det g}{g_{11}})^{\frac{1}{2}}} ds \wedge d\bar{s} \wedge dt \wedge d\bar{t} = \frac{1}{2} \int_{|s| + |t| \leq 1} (\frac{\det g}{g_{11}})^{\frac{1}{2}} ds \wedge d\bar{s} \wedge dt \wedge d\bar{t} = \pi^2.$$

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Finally using (3.18) and (3.19), we get
\[
\lim_{r \to 0} \frac{1}{r} \int_{B_r} \tilde{u} \, d\tilde{u} \wedge d\tilde{v} = \pi^2 \left( \frac{\partial}{\partial \tilde{v}} (\det g)(0) \right)^2 + \frac{\tilde{a}}{|\tilde{a}|^2 \det g(0)}.
\]

**Remark 3.9.** We can also prove Lemma 3.6 by a direct method without using Poincaré’s theorem, but the calculation is much more complicated.

### 4 Blowing up at non-isolated zeroes

#### 4.1 The zero set of holomorphic vector fields

In this section, we consider the non-isolated case. Let \( Z_\lambda \) be a one dimensional component of the zero set of \( X \) on a Kähler surface \( M \). We choose a coordinate \((z, w)\) on a neighborhood \( U \) of \( p \) such that \( Z_\lambda \cap U = \{ z = 0 \} \). Therefore, \( X \) can be locally written as
\[
X = z \cdot h(z, w) \frac{\partial}{\partial z} + z \cdot k(z, w) \frac{\partial}{\partial w},
\]
where \( h(z, w) \) and \( k(z, w) \) are holomorphic functions on \( U \). Since \( X \) is non-degenerate at \( Z_\lambda \), we have
\[
h(0, w) \neq 0, \quad (0, w) \in U
\]
and we can assume that
\[
h(z, w) = a_0 + a_1 z + a_2 w + \sum_{i+j \geq 2} a_{ij} z^i w^j, \quad a_0 \neq 0,
\]
\[
k(z, w) = b_0 + b_1 z + b_2 w + \sum_{i+j \geq 2} b_{ij} z^i w^j.
\]

Let \( \pi : \tilde{M} \to M \) be the blowing up of \( M \) at the point \( p \). We denote by \( L \) the strict transform of \( Z_\lambda \) under \( \pi \), and by \( \tilde{X} \) the zero locus of \( \tilde{X} \) over \( \tilde{M} \). Then obviously \( L \subset Z \). Now we study the zeroes of \( \tilde{X} \) on the exceptional divisor \( E \). Choose coordinate charts \( \tilde{U}_1 \) and \( \tilde{U}_2 \) of \( \tilde{U} = \pi^{-1}(U) \) as in Section 3:
\[
\tilde{U}_1 = \{ ((z, w), [\zeta, \eta]) \, | \, z\eta = w\zeta, \, \zeta \neq 0 \} \subset U \times \mathbb{CP}^1,
\]
\[
\tilde{U}_2 = \{ ((z, w), [\zeta, \eta]) \, | \, z\eta = w\zeta, \, \eta \neq 0 \} \subset U \times \mathbb{CP}^1.
\]

We choose coordinates \( u_1 = z, v_1 = \frac{w}{\zeta} \) on \( \tilde{U}_1 \) and we have \( E \cap \tilde{U}_1 = \{ u_1 = 0 \} \) and \( L \cap \tilde{U}_1 = \emptyset \). On \( \tilde{U}_1 \), \( \tilde{X} := \pi^* X \) can be written as
\[
\tilde{X} = \tilde{X}^1(u_1, v_1) \frac{\partial}{\partial u_1} + \tilde{X}^2(u_1, v_1) \frac{\partial}{\partial v_1}.
\]
Therefore, the zero set of $\tilde{X}$ is a non-degenerate zero of $\tilde{X}$ on $M$. We would like to compute the Futaki invariant of $\tilde{X}$.

In this section we will calculate the Futaki invariant of the blow-up $\pi: \tilde{M} \to M$ of Kähler surface $M$ at a point $p \in l$ where $l$ is a one-dimensional component of the set of $X$. We assume that $X$ is non-degenerate on $l$. As before $X$ can be naturally extended to a holomorphic vector field $\tilde{X}$ on $\tilde{M}$. We would like to compute the Futaki invariant of $(\tilde{\Omega}_e, \tilde{X})$ on $\tilde{M}$ where $\tilde{\Omega}_e = \pi^*\Omega - \varepsilon c_1([E])$, where $E = \pi^{-1}(p)$ is the exceptional divisor.

The zero set $Z$ of $\tilde{X}$ on $\tilde{U}_1$ is given by

$$Z \cap \tilde{U}_1 = \{ p_1 \}, \quad p_1 : (u_1, v_1) = (0, \frac{b_0}{a_0})$$

which is a non-degenerate zero of $\tilde{X}$.

On the other hand, we choose coordinates $u_2 = \frac{\xi}{\eta}, v_2 = w$ on $\tilde{U}_2$ and we have $E \cap \tilde{U}_2 = \{ v_2 = 0 \}$ and $L \cap \tilde{U}_2 = \{ u_2 = 0 \}$. Note that $\tilde{X}$ can be written as

$$\tilde{X} = \tilde{X}_1(u_2, v_2) \frac{\partial}{\partial u_2} + \tilde{X}_2(u_2, v_2) \frac{\partial}{\partial v_2},$$

where

$$\tilde{X}_1(u_2, v_2) = a_0 u_2 - b_0 u_2^2 + a_2 u_2 v_2 + (a_1 - b_2) u_2^3 v_2 - b_1 u_2^3 v_2 + \sum_{i+j \geq 2} (a_{ij} - b_{ij} u_2^i v_2^j),$$
$$\tilde{X}_2(u_2, v_2) = u_2 v_2 \left( b_0 + b_1 u_2 v_2 + b_2 v_2 + \sum_{i+j \geq 2} u_2^i v_2^j \right).$$

Therefore, the zero set of $\tilde{X}$ on $\tilde{U}_2$ consists of the following cases:

- If $b_0 = 0$, then $Z \cap \tilde{U}_2 = L \cap \tilde{U}_2$;
- If $b_0 \neq 0$, then $Z \cap \tilde{U}_2 = (L \cap \tilde{U}_2) \cup \{ q_1 \}$, where $q_1 : (u_2, v_2) = (\frac{a_0}{b_0}, 0)$. One can check easily that $q_1 = p_1 \in \tilde{U}_1$.

Combining the above results, we have

**Lemma 4.1.** The zero set $Z$ of $\tilde{X}$ on $\tilde{U}$ is given by $Z \cap \tilde{U} = (L \cap \tilde{U}) \cup \{ p_1 \}$, where $p_1 \in E$ is the point $((0, 0), [a_0, b_0]) \in U \times \mathbb{CP}^1$.

### 4.2 The local Futaki invariant

In this section we will calculate the Futaki invariant of the blow-up $\pi: \tilde{M} \to M$ of Kähler surface $M$ at a point $p \in l$ where $l$ is a one-dimensional component of the set of $X$. We assume that $X$ is non-degenerate on $l$. As before $X$ can be naturally extended to a holomorphic vector field $\tilde{X}$ on $\tilde{M}$. We would like to compute the Futaki invariant of $(\tilde{\Omega}_e, \tilde{X})$ on $\tilde{M}$ where $\tilde{\Omega}_e = \pi^*\Omega - \varepsilon c_1([E])$, where $E = \pi^{-1}(p)$ is the exceptional divisor.
Thus, using Corollary 2.5 we have

\[ \text{Lemma 4.2.} \]

Note that for any \( \lambda \in \Lambda, \lambda \neq 0 \), we have

\[
f_{2\lambda}(\tilde{\Omega}_e, \tilde{X}) = I_{2\lambda}(\tilde{\Omega}_e, \tilde{X}) - \frac{n}{n+1} \beta J_{2\lambda}(\tilde{\Omega}_e, \tilde{X})
\]

\[
= I_{2\lambda}(\Omega, X) - \frac{n}{n+1} (\mu + \delta) J_{2\lambda}(\Omega, X)
\]

\[
= f_{2\lambda}(\Omega, X) - \frac{n}{n+1} \delta J_{2\lambda}(\Omega, X),
\]

where we used the fact that \( I_{2\lambda}(\tilde{\Omega}_e, \tilde{X}) = I_{2\lambda}(\Omega, X) \) and \( J_{2\lambda}(\tilde{\Omega}_e, \tilde{X}) = J_{2\lambda}(\Omega, X) \) since \( Z_\lambda \) and \( E \) are disjoint for \( \lambda \neq 0 \). Note that

\[
\Omega_e([L]) = \Omega([l]) - \epsilon, \quad c_1(M)([L]) = c_1(M)([l]) - 1, \quad g(L) = g(l).
\]

Thus, using Corollary 2.3 we have

\[
f_L(\tilde{\Omega}_e, \tilde{X}) = (2B_0 - 2\beta B_0 A_0^{-1}) \tilde{\Omega}_e([L]) + \left( \frac{2\mu}{3} A_0^{-2} B_0 \right) c_1(M)([L]) + A_0^{-1} B_0^2 - \frac{2\mu}{3} A_0^{-2} B_0^3) (2 - 2g(L))
\]

\[
= f_L(\Omega, X) - \frac{2}{3} \delta J_L(\Omega, X) - \left( 2B_0 - 2\beta B_0 A_0^{-1} (\mu + \delta) \right) \epsilon - \frac{2}{3} A_0^{-2} B_0^3 (\mu + \delta).
\]  

(4.1)

Combine these formulas, we proved the lemma.

\[ \square \]

**Theorem 4.3.** Let \( \pi : \tilde{M} \rightarrow M \) be the blow-up of \( M \) at \( p \in l \) where \( l \) is a one-dimensional component of the zero set of \( X \). If \( X \) is non-degenerate on \( M \), then we have

\[
f_{\tilde{M}}(\tilde{\Omega}_e, \tilde{X}) = f_M(\Omega, X) + v(\Omega, X) \cdot \epsilon + O(\epsilon^2),
\]

where \( v(\Omega, X) \) is given by

\[
v(\Omega, X) = -2 \text{tr}_\Omega(X)(p) + \frac{2}{3\Omega^2} J_M(\Omega, X).
\]

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Proof. By Lemma 4.1, the zero set of $\tilde{X}$ on $\tilde{U}$ is given by $(L \cap \tilde{U}) \cup \{p_1\}$ where $p_1 : (u_1, v_1) = (0, 0)$. Note that using the local expression of $\tilde{X}$ at $p_1$ we have

$$\tilde{A}_{p_1} = 0, \quad \tilde{B}_{p_1} = \theta_X(p) - a_0 \epsilon, \quad \tilde{C}_{p_1} = -a_0^2.$$

Thus, we have

$$f_{p_1}(\tilde{\Omega}_\epsilon, \tilde{X}) = \frac{2(\mu + \delta)(\theta_X(p) - a_0 \epsilon)^3}{3a_0^3}.$$

Combining this with Lemma 4.2, we have

$$f_M(\tilde{\Omega}_\epsilon, \tilde{X}) = f_M(\Omega, X) - \frac{2}{3} \delta J_M(\Omega, X) + \frac{2(\mu + \delta)(\theta_X(p) - a_0 \epsilon)^3}{3a_0^3} - \frac{2\theta_X(p)\mu}{3a_0^2} - \frac{2\theta_X(p)}{3a_0^2} + O(\epsilon^2).$$

The theorem is proved.

5 Examples

In this section, we apply our theorem to some examples. Actually, we can get very explicit formulas as we do in the proof our theorems. For simplicity, we only write down the first order expansion, which in many cases suffices to prove the non-existence of cscK metrics.

5.1 $\mathbb{CP}^1 \times \mathbb{CP}^1$ blowing up two points

Let $M = \mathbb{CP}^1 \times \mathbb{CP}^1$, and $p_1 = (0, 0), p_2 = (\infty, \infty)$ are two points of $M$. We blow up $p_1, p_2$ to get $\tilde{M}_1$ and denote the blowing up map by $\pi$, with exceptional divisors $E_1$ and $E_2$. $\tilde{M}_1$ can also be realized as $\mathbb{CP}^2$ blowing up three generic points, and is a toric Fano manifold. Write the homogeneous coordinates of the two factors of $\mathbb{CP}^1 \times \mathbb{CP}^1$ by $[z_0, z_1]$ and $[w_0, w_1]$. Let $z = \frac{z_0}{z_1} = \frac{w_0}{w_1}$ and $w = \frac{w_0}{w_1} = \frac{1}{w}$. Then one can check easily that the vector fields $\frac{\partial}{\partial z}$ and $w \frac{\partial}{\partial w}$ extend naturally to holomorphic vector fields on $\tilde{M}_1$ and they form a basis of $\mathfrak{h}_0(\tilde{M}_1)$. We denote these two vector fields by $Z$ and $W$ respectively.

Any Kähler class on $M$ has the form $\Omega_{a,b} = ac_1([H_1]) + bc_1([H_2]), a, b > 0$, where $H_1, H_2$ are divisors in $M$ defined by $z = \text{const}$ and $w = \text{const}$, respectively. Since any Kähler class on $M$ admits a cscK metric, we know that the Futaki invariant vanishes identically on $\mathfrak{h}(M)$. Now we consider the Kähler class on $\tilde{M}_1$:

$$\tilde{\Omega}_{a,b,\epsilon_1,\epsilon_2} = \pi^*\Omega_{a,b} - \epsilon_1 c_1([E_1]) - \epsilon_2 c_1([E_2]).$$

We want to compute $f_{\tilde{M}_1}(\tilde{\Omega}_{a,b,\epsilon_1,\epsilon_2}, Z) := f_Z(a, b, \epsilon_1, \epsilon_2)$. (The computation of $f_W$ is the same.)
We choose the following Kähler form in the class \( \Omega_{a,b} \) on \( M \):

\[
\omega = a \left( \partial \bar{\partial} \log(|z_0|^2 + |z_1|^2) \right) + b \left( \partial \bar{\partial} \log(|w_0|^2 + |w_1|^2) \right),
\]

then we can choose the holomorphy potential of \( Z \) to be:

\[
\theta_Z = -a \frac{|z_1|^2}{|z_0|^2 + |z_1|^2}.
\]

By symmetry, we have \( \theta_Z = -\frac{\alpha}{2} \). We have the following:

\[
\nu_{p_1} = -2(\theta_Z - \theta_Z)(p_1) = -2(0 - (-\frac{a}{2})) = -a, \quad \nu_{p_2} = -2(\theta_Z - \theta_Z)(p_2) = -2(-a - (-\frac{a}{2})) = a.
\]

By Theorem [3.3] we have \( f_Z(a, b, \epsilon_1, \epsilon_2) = a(\epsilon_2 - \epsilon_1) + o(|\epsilon|) \).

**Corollary 5.1.** Let \( \tilde{M} \) and \( \tilde{\Omega}_{a,b,\epsilon_1,\epsilon_2} \) be as above. Then for small \( \epsilon_i > 0 \) with \( \epsilon_1 \neq \epsilon_2 \), there are no cscK metrics in the class \( \tilde{\Omega}_{a,b,\epsilon_1,\epsilon_2} \).

**Remark 5.2.** By our method of proving the main theorems, one can actually have the following exact formula:

\[
f_Z(a, b, \epsilon_1, \epsilon_2) = -2a(a + b - \epsilon_2) - \frac{2(2a + 2b - \epsilon_1 - \epsilon_2)}{3(2ab - \epsilon_1^2 - \epsilon_2^2)}(\epsilon_1^3 - \epsilon_2^3 + 3ae_1^2 - 3a^2b).
\]

### 5.2 \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) blowing up three points

Let \( M = \mathbb{C}P^1 \times \mathbb{C}P^1 \) and \( \Omega_{a,b} \) be as above, and \( p_1 = (0, 0), p_2 = (\infty, \infty), p_3 = (0, \infty) \) are three points of \( M \). We blow up \( p_1, p_2, p_3 \) to get \( \tilde{M} \) and denote the blowing up map by \( \pi \), with exceptional divisors \( E_1, E_2 \) and \( E_3 \). \( \tilde{M} \) is also a toric manifold. Write the homogeneous coordinates of the two factors of \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) by \([z_0, z_1]\) and \([w_0, w_1]\). Let \( z = \frac{z_0}{z_1} = \frac{1}{t} \) and \( w = \frac{w_0}{w_1} = \frac{1}{w} \). Then the natural extensions of the vector fields \( \frac{\partial}{\partial z} \) and \( \frac{\partial}{\partial w} \) form a basis of \( \mathfrak{h}_0(\tilde{M}_2) \). We denote these two vector fields by \( Z \) and \( W \) respectively.

The Kähler class we choose on \( \tilde{M}_2 \) is:

\[
\tilde{\Omega}_{a,b,\epsilon_1,\epsilon_2,\epsilon_3} = \pi^* \Omega_{a,b} - \epsilon_1 c_1([E_1]) - \epsilon_2 c_1([E_2]) - \epsilon_3 c_1([E_3]).
\]

We want to compute \( f_{\tilde{M}_2}(\tilde{\Omega}_{a,b,\epsilon_1,\epsilon_2,\epsilon_3}, Z) := f_Z(a, b, \epsilon_1, \epsilon_2, \epsilon_3) \) and similarly \( f_W(a, b, \epsilon_1, \epsilon_2, \epsilon_3) \).

The holomorphy potential of \( Z \) is still

\[
\theta_Z = -a \frac{|z_1|^2}{|z_0|^2 + |z_1|^2}.
\]

And we still have \( \theta_Z = -\frac{\alpha}{2} \). We have the following:

\[
\nu_{p_1} = -a, \quad \nu_{p_2} = a, \quad \nu_{p_3} = -a.
\]
By Theorem 4.3, we have
\[ f_{Z}(a, b, \epsilon_1, \epsilon_2, \epsilon_3) = a(\epsilon_2 - \epsilon_1 - \epsilon_3) + o(1). \]
Similarly, we have
\[ f_{W}(a, b, \epsilon_1, \epsilon_2, \epsilon_3) = b(\epsilon_2 - \epsilon_1 + \epsilon_3) + o(1). \]
Since \(\epsilon_2 - \epsilon_1 - \epsilon_3\) and \(\epsilon_2 - \epsilon_1 + \epsilon_3\) can not be both zero when all the \(\epsilon_i\)'s are positive, we have:

**Corollary 5.3.** Let \(\tilde{M}_2\) and \(\tilde{\Omega}_{a, b, \epsilon_1, \epsilon_2, \epsilon_3}\) be as above. Then for \(\epsilon_i > 0\) small enough, there are no cscK metrics in the class \(\tilde{\Omega}_{a, b, \epsilon_1, \epsilon_2, \epsilon_3}\).

## 6 Relation with Stoppa’s result

In this section, we point out the relation of Stoppa’s theorem with ours when the Kähler manifold is a polarized algebraic surface \((M, L)\), with Kähler class \(\Omega = c_1(L)\). Let’s first recall Stoppa’s result.

Let \(Z = \sum_i a_i p_i\) be a 0-dimensional cycle on a \(n\)-dimensional polarized algebraic manifold \((M, L)\), where \(p_i \in M\) are different points and \(a_i \in \mathbb{Z}_+\). We write \(\tilde{M} := Bl_{Z}M\) and denote by \(p: \tilde{M} \to M\) the blowing up map, with exceptional divisor \(E\). Assume \(X\) is a holomorphic vector field on \(M\) that generates a holomorphic \(\mathbb{C}^*\) action \(\alpha(t)\).

Define \(Z_i := \alpha(t)Z\) and taking the flat closure of \(\bigcup_{t \in \mathbb{C}^*} Z_t \times \{t\} \subset M \times \mathbb{C}\), we get a subscheme \(Y\) of \(M \times \mathbb{C}\). Then blowing up \(Y\), we get \(\tilde{X} = Bl_Y(M \times \mathbb{C})\), which is a test configuration for \((\tilde{M}, \tilde{L})\), where \(\tilde{L} = \gamma p^*L - E\) for some positive large integer \(\gamma\), here \(\frac{\gamma}{2}\) plays the same role as \(\epsilon_i\) in our setting. Stoppa got the following formula for the algebraic Donaldson-Futaki invariant of this test configuration:

\[
F(\tilde{X}) = F(M, L, X)\gamma^n - \mathcal{C}(\sum_i a_i^{n-1} p_i, \alpha) \frac{\gamma}{2(n-2)!} + O(1),
\]
where \(F(M, L, X)\) is the algebraic Donaldson-Futaki invariant of the product test configuration of \((M, L)\) with \(\mathbb{C}^*\) action induced by \(\alpha\), and \(\mathcal{C}(\sum_i a_i^{n-1} p_i, \alpha)\) is the Chow weight of \(\alpha(t)\) acting on 0-dimensional cycles.

In our case, the blowing up centers \(p_i\) are non-degenerate zero points of \(X\), so they are fixed by \(\alpha(t)\), and hence \(Y = Z \times \mathbb{C}\). So \(\tilde{X} = Bl_{Z}M \times \mathbb{C}\) is a product test configuration. In this case, both \(F(\tilde{X})\) and \(F(M, L, X)\) are the classical Calabi-Futaki invariants, up to a universal constant factor (see \([11]\)). Observe that in this case, the holomorphic vector field \(X\) also have non-degenerate zero locus. This is because \(X\) generates a \(\mathbb{C}^*\) action, so the linearization of \(X\) at any of its zero point is semisimple, and our Lemma 4.2 guarantees the non-degeneracy.

Now we give a formula for the Chow weight in this case, using the potential of \(X\). Assume \(L^n\) is very ample. For simplicity, we also assume that the induced action of \(\alpha\) on \(H^0(X, \gamma L)\) gives a 1-ps of \(SL(N + 1)\). We also assume that \((M, L)\) is asymptotically Chow polystable. First by Stoppa’s work ([18] 14-15), we know that \(\mathcal{C}(\sum_i a_i p_i, \alpha) = -\sum_i a_i^{n-1}\lambda(p_i)\). The definition

\footnote{As a manifold, \(\tilde{M}\) is the same as \(M\) blowing up all the points \(p_i\). If we denote the exceptional divisors by \(E_i\), then \(E = \sum_i a_i E_i\).}
of $\lambda(p_i)$ is as follows. Since the $\mathbb{C}^*$ action $\alpha(t)$ preserves the fiber of $L$ over $p_i$, we have a well-defined notion of weight for this action. This is $\lambda(p_i)$. We also write the induced linear $\mathbb{C}^*$ action on $\mathbb{P}^N$ as $\alpha(t)$. Suppose the image of $p_i$ is the point $[1,0,\ldots,0]$, and the action of $\alpha(t)$ is in a diagonal form $\text{diag}(t^{\lambda_0},\ldots,t^{\lambda_N})$. Then $\lambda_0 = -\gamma \lambda(p_i)$.

The holomorphy potential of $X$ is defined by the equation $-\bar{\partial} \theta_X = i X \omega$. Applying the $d$ operator, we get $-\bar{\partial} \partial \theta_X = L_X \omega$. We can choose a special metric to compute $\theta_X$. So let’s assume that $\omega$ is the pull-back metric

$$
\frac{1}{\gamma} \partial \bar{\partial} \log(\sqrt{|Z_0|^2 + \cdots + |Z_N|^2}).
$$

Since we assume that $(M,L)$ is asymptotically Chow polystable, we can choose $\omega$ to be a balanced metric. The real 1-parameter group associated with $X$ (and $\alpha(t)$) is $\beta(s) = \text{diag}(e^{\lambda_0 s},\ldots,e^{\lambda_N s})$. Then by the definition of Lie derivatives, we have

$$
L_X \omega = L_{ReX} \omega = \frac{d}{ds}|_{s=0} \beta(s)^* \omega = \frac{1}{\gamma} \partial \bar{\partial} \frac{2\lambda_0 |Z_0|^2 + \cdots + 2\lambda_N |Z_N|^2}{|Z_0|^2 + \cdots + |Z_N|^2}.
$$

The first equality is because the action is Hamiltonian. So we can take

$$
\theta_X = -\frac{1}{\gamma} \frac{2\lambda_0 |Z_0|^2 + \cdots + 2\lambda_N |Z_N|^2}{|Z_0|^2 + \cdots + |Z_N|^2}
$$

where the right handside means restriction to the image of $M$ under Kodaira’s embedding map. Evaluate at $p_i$, we get $\theta_X(p_i) = -\frac{2}{\gamma} \lambda_0 = 2 \lambda(p_i)$. Since $\omega$ is balanced, we have $\theta_X = 0$. So $\nu_{p_i}(\Omega,X) = -4 \lambda(p_i)$. So in this case, our result coincides with (6.1), up to a universal constant factor.

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