Run-and-tumble particle in inhomogeneous media in one dimension

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Abstract. We investigate the run-and-tumble particle (RTP), also known as persistent Brownian motion, in one dimension. A telegraphic noise $\sigma(t)$ drives the particle which changes between $\pm 1$ values at certain rates. Denoting the rate of flip from 1 to $-1$ as $R_1$ and the converse rate as $R_2$, we consider the position- and direction-dependent rates of the form $R_1(x) = \left(\frac{|x|}{l}\right)^\alpha [\gamma_1 \theta(x) + \gamma_2 \theta(-x)]$ and $R_2(x) = \left(\frac{|x|}{l}\right)^\alpha [\gamma_2 \theta(x) + \gamma_1 \theta(-x)]$ with $\alpha \geq 0$. For $\alpha = 0$ and 1, we solve the master equations exactly for arbitrary $\gamma_1$ and $\gamma_2$ at large $t$. From our analytical expression for the time-dependent probability distribution $P(x,t)$ we find that for $\gamma_1 > \gamma_2$ the distribution relaxes to a steady state exponentially, whereas for $\gamma_1 \leq \gamma_2$ the distribution does not reach a steady state and can be described by a non-trivial scaling form. We interestingly find that these features of the probability distribution $P(x,t)$ in the two regimes $\gamma_1 > \gamma_2$ and $\gamma_1 \leq \gamma_2$ also remain valid for general $\alpha > 0$. In particular, for general $\alpha$, we argue and numerically demonstrate that the approach to the steady state in $\gamma_1 > \gamma_2$ case is exponential. On the other hand, for $\gamma_1 \leq \gamma_2$, the distribution $P(x,t)$ remains time dependent and possesses certain scaling behavior. For $\gamma_1 = \gamma_2$ we derive the scaling behavior as well as the scaling function rigorously, whereas for $\gamma_1 < \gamma_2$ we provide heuristic arguments to obtain the scaling behavior and the corresponding scaling functions. We also study the dynamics on a semi-infinite line with an absorbing barrier at
Run-and-tumble particle in inhomogeneous media in one dimension

the origin. For \( \alpha = 0 \) and 1, we analytically compute the survival probabilities and the corresponding first-passage time distributions. For general \( \alpha > 0 \), we provide approximate calculations to compute the behavior of the survival probability for \( t \to \infty \) in which limit it approaches a finite value for \( \gamma_1 < \gamma_2 \) but goes to zero for \( \gamma \geq \gamma_2 \). We also study the approach to the large \( t \) value in both cases. Finally, we consider RTP in a finite interval \([0, M]\) and compute the associated exit probabilities from that interval for all \( \alpha \). All our analytic results are verified with a numerical simulation.

**Keywords:** stationary states, active matter, persistence, Brownian motion

Contents

1. Introduction .................................................. 3

2. The probability density function \( P(x, t) \) ............. 6
   2.1. Case I: \( \alpha = 0 \) ........................................ 8
   2.2. Case II: \( \alpha = 1 \) ....................................... 10
      2.2.1. \( \Delta = 0 \) .......................................... 10
      2.2.2. \( \Delta \neq 0 \) ......................................... 11
   2.3. Case III: general \( \alpha \) ................................. 13
      2.3.1. \( \Delta > 0 \) ........................................... 14
      2.3.2. \( \Delta \leq 0 \) .......................................... 15

3. Survival probability ........................................... 18
   3.1. Case I: \( \alpha = 0 \) ........................................ 20
   3.2. Case II: \( \alpha = 1 \) ....................................... 23
      3.2.1. \( \Delta = 0 \) .......................................... 23
      3.2.2. \( \Delta \neq 0 \) ......................................... 24
   3.3. General \( \alpha \) ........................................... 26
      3.3.1. \( \Delta > 0 \) ........................................... 26
      3.3.2. \( \Delta \leq 0 \) .......................................... 27
      3.3.3. \( \Delta = 0 \) .......................................... 28

4. Exit probability of RTP from a finite interval for general \( \alpha \) 28

5. Conclusions .................................................. 29

Acknowledgments ................................................ 32

Appendix A. Derivation of \( G(X, s) \) and \( \bar{P}(x, s) \) for \( \alpha = 0 \) 32

Appendix B. Derivation of \( P(x, s) \) for \( \alpha = 0 \) in equation (18) 33
   B.1. Derivation of equation (B.1) .......................... 33
   B.2. Derivation of \( P(x, t) \) in equation (16) ............. 34

https://doi.org/10.1088/1742-5468/aba7b1
1. Introduction

Active matter is a class of non-equilibrium systems that can transduce the supplied energy to a systematic movement through some internal mechanisms [1–6]. The dynamics of these systems does not respect time-reversal symmetry and thus breaks the detailed balance. A plethora of interesting phenomena like motility-induced phase transition [7–10], flocking [11, 12], clustering [13, 14], non-existence of the equation of state in terms of pressure [15], etc, have been observed and studied in these systems, and they arise due to the activity and interaction among the particles. Also, at the level of single particles, such systems exhibit interesting behaviors like accumulation at the boundaries inside confinement [16–18], non-Boltzmann stationary distribution [19–24], and anomalous behaviors [25, 26], which are remarkably different than their passive counterparts. The run-and-tumble particle (RTP) and active Brownian particle (ABP) are two paradigmatic models of the dynamics of active particles that have extensively been studied in the past few years. A single ABP, free or in harmonic trap, shows rich features like anomalous first passage distributions [26], re-entrant phase transition [27, 28], position distribution [29, 30] and many more. These particles have also been used as models of microscopic constituents in many theoretical studies of active matter and the collective behavior of many active agents [20, 31].

The RTP mechanism describes the stochastic dynamics of a particle that moves in a straight line for some time $t_{\text{run}}$ and undergoes tumble, a state of rest, which lasts for another time $t_{\text{tum}}$. The particle then chooses the direction randomly for the next run. For example, *E. coli* bacteria runs for some time along a straight line and then tumbles.
to randomly choose a new direction of run [32, 33]. For bacteria such timescales are of the order of \( t_{\text{run}} \sim 1 \text{s} \) and \( t_{\text{tum}} \sim 0.1 \text{s} \), respectively [33, 34]. In RTP model, such tumble events are often considered to occur instantaneously, and after each tumble the direction of motion is changed. The time for which the particle runs is taken from exponential distribution at a certain rate. In one dimension the particle tumbles between the positive and the negative direction and its equation of motion is given by

\[
\frac{dx}{dt} = v \sigma(t),
\]

where \( x(t) \) is the position of the particle at time \( t \), \( v(>0) \) is its speed and \( \sigma(t) \) represents its instantaneous direction of motion governed by the telegraphic or dichotomous noise. This noise \( \sigma(t) \) switches between \( \pm 1 \) with rate \( R \) and consequently its values at different times are correlated exponentially as \( \langle \sigma(t_1)\sigma(t_2) \rangle = 2Re^{-2R|t_1-t_2|} \), which makes the evolution of the position \( x(t) \) non-Markovian. The RTP model is in some sense an amalgamation of ballistic motion and Brownian motion as by tuning the parameters \( R \) and \( v \), one can go from a pure ballistic particle \( (R\to0) \) to a pure Brownian particle \( (R\to\infty \text{ and } v\to\infty \text{ keeping } v^2/R \text{ fixed}) \). In the physics literature the telegraphic noise and the RTP process have been studied in various settings starting from persistent Brownian motion, electromagnetic theory, optics, and Lorentz gas to polymers [35–43]. The model has gained renewed interest in recent years due to its applicability in mimicking the movement of \textit{E. coli} [32]. Also, the exact solvability of this model makes it a quintessential candidate for study of the rich and remarkably different behaviors of the active systems. The model has been extensively studied and a variety of its properties are known. Some examples are the joint distribution of the maximum and minimum of the position [36], distributions of first-passage times and exit times from an interval [18, 44–46], behavior under resetting [47], large deviation forms [48–50], convex hull [51], distributions in a harmonic trap and other confining potentials [52, 53]. Recently the authors have also investigated the ‘generalized’ arcsine laws for this model and found some interesting features in comparison to a pure Brownian particle [54]. There has also been a reasonable amount of study of the microscopic dynamics of multiple interacting RTPs on continuous and lattice spaces [55–60].

It is imperative to emphasize that the telegraphic noise considered in the above settings is characterized only by the constant rate \( R \). The flip from \( +1 \) to \( -1 (1 \rightarrow -1) \) occurs at the same rate as from \( -1 \) to \( +1 (-1 \rightarrow 1) \). This consideration, however, is a cliché, especially when the particle is exposed to some chemoattractants or chemorepellents. For example it is seen experimentally (and used theoretically) [33, 61–64] that in \textit{E. coli} the run-time depends strongly on the concentration of the nutrients and the nutrient gradient. In [33], the run duration for \textit{E. coli} is found to depend on whether the bacteria are moving towards or away from the chemo-attractants although the distribution for the times is still exponential. This observation is suggestive of generalizing the telegraphic noise in equation (1) whereby the flips from \( 1 \leftrightarrow -1 \) occur with position-dependent rates. In this paper we consider the dynamics of a single RTP particle in one dimension with generalized telegraphic noise that is characterized by position- and direction-dependent rates \( R_1(x) \) and \( R_2(x) \) given by

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Figure 1. Plot of rates defined in equation (2) for $\alpha = 0.5$ and various signatures of $\Delta$. In the left panel, we have plotted $R_1(x)$ vs $x$ for (i) $\gamma_1 = 2, \gamma_2 = 1$ (green), (ii) $\gamma_1 = \gamma_2 = 1.5$ (blue) and (iii) $\gamma_1 = 0.8, \gamma_2 = 1.8$ (red). Inset shows the same plot for $\alpha = 0$. In the right panel, we have plotted $R_2(x)$ vs $x$ for the same choice of parameters and color. For all plots $l = 1$.

$$R_1(x) = \left(\frac{|x|}{l}\right)^\alpha \left[\gamma_1 \theta(x) + \gamma_2 \theta(-x)\right],$$

$$R_2(x) = \left(\frac{|x|}{l}\right)^\alpha \left[\gamma_2 \theta(x) + \gamma_1 \theta(-x)\right],$$

where $\alpha \geq 0$, $\theta(x)$ is the Heaviside function and $\gamma_1$ and $\gamma_2$ (both positive) are position-independent rates (see figure 1). Here $l$ is a length scale over which the rate functions vary. Similar generalizations of RTP motions have been considered in various settings such as Markovian robots [65], active diffusion [66], response to stochastic input [67], chemotaxis [20, 68, 69], quorum sensing [70, 71], active gel and active fluids [72, 73] and motion with space-dependent speed $v(x)$ [74]. These studies mostly deal with either steady-state behaviors or hydrodynamic descriptions. In a different setting, the motion of a single RTP has been recently studied in a periodic force field in one dimension [75], where, by analytically computing the stationary measure, the velocity and the diffusion constant, it has been shown that the dynamics exhibits interesting phase transitions. In addition, it has been shown that on a random but periodic force field the particle exhibits anomalous diffusion under certain conditions.

In this paper, we study the probability density function (PDF) of $x$ on an infinite line, the survival problem on semi-infinite line and the exit problem from a box of length $M$, going beyond the steady-state properties of non-interacting RTPs with flipping rates depending on both position and orientation. We find that, in addition to being proximal to realistic situations, this model of the dynamics of RTP also exhibits interesting features like the existence of steady-state, non-trivial and richer large-time properties of the PDF as well as a survival probability compared to pure Brownian particles, which are otherwise absent in RTPs at a constant rate. We find that for this generalized RTP the survival probability $S(t)$ for large-time decays as $S(t) \sim t^{-\theta}$ with a persistent exponent $\theta$, which strongly depends on $\alpha$ and the rates $\gamma_1$ and $\gamma_2$. Note that for a pure Brownian particle and for an RTP with constant rate, $\theta = 1/2$.

The paper is organized as follows. We start with the computation of the PDF $P(x, t)$ in section 2 for different $\alpha$ and $\Delta = (\gamma_1 - \gamma_2)/2$. We perform computations for $\alpha = 0$ in section 2.1, for $\alpha = 1$ in section 2.2 and for general $\alpha$ in section 2.3. For each choice of $\alpha$, three different cases of $\Delta = 0$, $\Delta > 0$ and $\Delta < 0$ are discussed in subsequent sections.

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After studying $P(x, t)$ we study the survival probability of the RTP in inhomogeneous media with an absorbing site at the origin in section 3. In this case also we perform computations for different $\alpha$ separately in section 3.1 (for $\alpha = 0$), section 3.2 (for $\alpha = 1$) and in section 3.3 (for general $\alpha$). Finally we study the exit probability of the RTP from a finite box in section 4 which is followed by our conclusion in section 5.

2. The probability density function $P(x, t)$

Let $P_\sigma(x, t)$ denote the probability distribution for the RTP, starting at the origin with orientations $\pm 1$ (chosen with probabilities $a_+ + a_- = 1$), to be at position $x$ in time $t$ with velocity direction $\sigma \in \{+,-\}$. Starting from the Langevin equation (1), it is easy to show that the distributions $P_\pm(x, t)$ satisfy the following master equations [18]

$$\partial_t P_+(x, t) = -v \partial_x P_+(x, t) - R_1(x) P_+(x, t) + R_2(x) P_-(x, t),$$

$$\partial_t P_-(x, t) = v \partial_x P_-(x, t) + R_1(x) P_+(x, t) - R_2(x) P_-(x, t),$$

where $R_1(x)$ and $R_2(x)$ are the position- and direction-dependent rates defined in equation (2). To solve these equations we need to specify the initial and boundary conditions. The initial conditions of the problem are $P_\pm(x, 0) = a_\pm \delta(x)$. Note that for a given finite time $t$, the particle can at most travel a distance $\pm vt$ depending on the initial velocity direction which implies the boundary conditions $P_\pm(x \to \pm \infty, t) = 0$. Throughout the paper, we will work with the symmetric initial condition $a_+ = a_- = \frac{1}{2}$ for which the particle starts with $\pm v$ velocity with equal probability. It is interesting to note that the choice of rates $R_1(x)$ and $R_2(x)$ are such that the timescale over which the particle moves towards the origin is $\sim \frac{1}{\gamma_2}$. Similarly, the timescale going away from the origin is $\frac{1}{\gamma_1}$. For $\gamma_1 > \gamma_2$, the motion drifts on average towards the origin, which leads one to anticipate a stationary state distribution at large times even when the particle is moving on an infinite line. On the other hand, for $\gamma_1 \leq \gamma_2$, the probability distribution of the particle never reaches a steady state.

To solve the master equations (3), we first take the Laplace transformation of the distributions with respect to time $t$, defined as

$$\tilde{P}_\pm(x, s) = L_{t \to s}[P_\pm(x, t)] = \int_0^\infty dt e^{-st} P_\pm(x, t),$$

on both sides and get the following ordinary but coupled differential equations for $\tilde{P}_\pm(x, s)$ as

$$(v \partial_x + R_1(x) + s) \tilde{P}_+ = R_2(x) \tilde{P}_- + \frac{1}{2} \delta(x),$$

$$(-v \partial_x + R_2(x) + s) \tilde{P}_- = R_1(x) \tilde{P}_+ + \frac{1}{2} \delta(x).$$

Defining

$$\tilde{P}(x, s) = \tilde{P}_+(x, s) + \tilde{P}_-(x, s),$$
Run-and-tumble particle in inhomogeneous media in one dimension

\( \tilde{Q}(x, s) = \tilde{P}_+(x, s) - \tilde{P}_-(x, s), \)  

we rewrite the above equations as

\[
v \partial_x \tilde{P} + s \tilde{Q} + \frac{2 \text{sgn}(x) \Delta |x|^{\alpha}}{l^{\alpha}} \tilde{P} + \frac{2 \gamma |x|^{\alpha}}{l^{\alpha}} \tilde{Q} = 0,
\]

\[
s \tilde{P} + v \partial_x \tilde{Q} = \delta(x),
\]

with \( 2 \Delta = \gamma_1 - \gamma_2 \) and \( 2 \gamma = \gamma_1 + \gamma_2 \). The signum function \( \text{sgn}(x) \) takes values 1 for \( x > 0 \), 0 for \( x = 0 \) and \(-1 \) for \( x < 0 \). Substituting \( \tilde{P}(x, s) \) from equation (10) in equation (9), one can eliminate \( \tilde{P}(x, s) \) and get a second-order differential equation of \( \tilde{Q}(x, s) \) valid for \( x \neq 0 \) as

\[
\partial_x^2 \tilde{Q} + \frac{2 \text{sgn}(x) \Delta |x|^{\alpha}}{v l^{\alpha}} \partial_x \tilde{Q} - \left( \frac{2 \gamma s |x|^{\alpha}}{v^2 l^{\alpha}} + \frac{s^2}{v^2} \right) \tilde{Q} = 0.
\]

Similarly, eliminating \( \tilde{Q}(x, s) \), one can get the following equation for \( \tilde{P}(x, s) \):

\[
s \tilde{P}(x, s) - \delta(x) = \partial_x \left( \frac{v^2}{s + \frac{2 \text{sgn}(x) \Delta}{v^2 l^{\alpha}}} \left[ \partial_x \tilde{P}(x, s) + \frac{2 \Delta \text{sgn}(x) |x|^{\alpha}}{v l^{\alpha}} \tilde{P}(x, s) \right] \right).
\]

To obtain \( P(x, t) \) one can in principle directly solve this equation; however, as we will see it turns out to be convenient to first solve equation (11) and then obtain \( \tilde{P}(x, s) \) from equation (10). To get rid of the first-order derivative term (second) in the LHS of equation (11) we define

\[
\tilde{Q}(x, s) = e^{-\frac{\Delta |x|^{\alpha+1}}{\alpha l^{\alpha+1}}} G(x, s),
\]

substituting which in equation (11) and simplifying we get

\[
\partial_x^2 G - \left[ \frac{\Delta \alpha |x|^{\alpha-1}}{v l^{\alpha}} + \frac{2 \gamma s |x|^{\alpha}}{v^2 l^{\alpha}} + \frac{\Delta^2 |x|^{2\alpha}}{v^2 l^{2\alpha}} + \frac{s^2}{v^2} \right] G = 0.
\]

We solve this equation with boundary conditions that \( G(x, s) \) should not diverge at \( x \to \pm \infty \) for arbitrary \( \alpha \) and \( \Delta \). Solving this equation exactly for general \( \alpha \) turns out to be a difficult task except for \( \alpha = 0 \) and \( \alpha = 1 \). However, for general \( \alpha \), it is possible to derive few general results in some cases. For example, the probability distribution \( P_\pm(x, t) \) reaches a stationary state at large times for \( \Delta > 0 \) with arbitrary \( \alpha \geq 0 \). For this case \( (\gamma_1 > \gamma_2) \) the particle tumbles from +1 to −1 more frequently if it is on the positive side and from −1 to +1 more frequently if it is on the negative side. As a result there is an overall effective bias on the particle towards the origin, which makes the RTP reach a stationary state. The explicit form of the stationary distribution depends on the value of \( \alpha \). On the other hand, for \( \Delta \leq 0 \) the distribution never reaches a stationary state, where some properties of the time-dependent distribution \( P(x, t) = P_+(x, t) + P_-(x, t) \) can be obtained in the asymptotically large time limit. In this limit we demonstrate that the dynamics of the RTP can be described by an effective Langevin equation of

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a particle diffusing in an inhomogeneous medium with position-dependent drift and a diffusion constant.

In what follows, we first consider \( \alpha = 0 \) and \( \alpha = 1 \) cases separately and consider the general \( \alpha \) case in the subsequent section. For each value of \( \alpha \), we discuss the following cases (i) \( \Delta > 0 \) (ii) \( \Delta = 0 \) and (iii) \( \Delta < 0 \) separately.

### 2.1. Case I: \( \alpha = 0 \)

For this case the rates \( R_1(x) \) and \( R_2(x) \) are independent of the magnitude of \( x \) but depend on the sign of \( x \). In this case, equation (14) reduces to

\[
\partial_s^2 G - \lambda^2 G = 0, \tag{15}
\]

where \( \lambda(s) = \frac{1}{2} \sqrt{\Delta^2 + 2 \gamma s + s^2} \). We solve this equation with the boundary conditions \( G(x \to \pm \infty, s) = 0 \), and using the solution in equations (10) and (13) we finally get \( \bar{Q}(x, s) \) and \( \bar{P}(x, s) \). For clarity and compactness of presentation, we have relegated the details of calculation of \( G(x, s) \) to appendix A. We here instead present the final expression of \( \bar{P}(x, s) \), which reads as

\[
\bar{P}(x, s) = \frac{1}{2s} \left( \lambda(s) + \frac{\Delta}{v} \right) e^{-\left(\lambda(s) + \frac{\Delta}{v}\right)|x|}. \tag{16}
\]

Recall that for \( \Delta > 0 \) (i.e. \( \gamma_1 > \gamma_2 \)) one anticipates a stationary state distribution at late times given by

\[
P_{0}^{st}(x) = \lim_{s \to 0} \left[ s \bar{P}(x, s) \right] = \frac{\Delta}{v} e^{-\frac{\Delta}{v}|x|}. \tag{17}
\]

This is an exponential distribution decaying over the length scale \( l_d = \frac{v}{2\Delta} \). In figure 2(a), we have plotted our analytic result of \( P_{0}^{st}(x) \) in equation (17) with the numerical simulation of the same and find excellent agreement. Note that the decay length \( l_d \) diverges as \( \Delta \to 0 \), which indicates that there is no stationary state for \( \Delta = 0 \). For \( \Delta \leq 0 \), the \( \lim_{s \to 0} \left[ s \bar{P}(x, s) \right] = 0 \) again implies that there is no stationary state for this case either.

To get the full distribution \( P(x, t) \) for arbitrary time \( t \), one has to perform the inverse Laplace transform over \( s \) (which for a function \( f(t) \) is denoted by \( f(t) = L_{s=0}^{-1}[\tilde{f}(s)] \) where \( \tilde{f}(s) = L_{t \to s}[f(t)] \)). The details of the inversion of \( \bar{P}(x, s) \) are provided in appendix B, and we here provide only the final result:

\[
P(x, t) = \frac{1}{2} e^{-\gamma_1 t} \delta(|x| - vt) + \frac{\gamma_1}{2v} \left( 1 + \frac{\gamma_2 |x|}{2v} \right) e^{-\frac{\gamma_1 |x|}{2v}} \Theta(vt - |x|) \]
\[
- \frac{\sqrt{\gamma_1 \gamma_2}}{2v} \int_0^t d\tau \ e^{-\gamma \tau} \left( \frac{d \mathcal{I}(u, \tau)}{du} \right)_{u=|x|} \Theta(v \tau - |x|), \tag{18}
\]

where \( \mathcal{I}(u, t) = \frac{ue^{-u\gamma_1 t}}{v} I_1 \left( \sqrt{\gamma_1 \gamma_2 (t - \frac{u}{v})} \right) \) with \( I_1(z) \) being the modified Bessel function of the first kind. Note that the distribution \( P(x, t) \) contains \( \delta \) function terms at \( x = \pm vt \). They
from the fact that for $\gamma > \gamma_1$ but approximate expression of the distribution 
To gain more insights into this distribution, it is instructive to get a more explicit 

discontinuity at $\gamma = 1 = 1$.

\[
\frac{\partial}{\partial t} P(x, t) = -\alpha x P(x, t) + \sqrt{2 \alpha} \xi(t) \quad \text{for} \quad \gamma > 1 \quad \text{and} \quad \gamma < 1.
\]

For $\gamma > 1$, the derivative of $P(x, t)$ has a discontinuity at $x = 0$ for $\gamma_1 \neq \gamma_2$, while it is continuous for $\gamma_1 = \gamma_2$. For $\alpha = 0$ and $\gamma_1 \neq \gamma_2$, the rates $R_1(x)$ and $R_2(x)$ in equation (2) have discontinuity at $x = 0$, which amounts to the discontinuity in the derivative of $P(x, t)$. To gain more insights into this distribution, it is instructive to get a more explicit but approximate expression of the distribution $P(x, t)$ in the large $t$ limit for both the $\gamma_1 > \gamma_2$ and $\gamma_1 \leq \gamma_2$ cases. After some algebra (presented in appendices C.1 and C.2 with details) we find the following approximate expressions valid for large $t$:

\[
P(x, t) \approx \begin{cases} 
\frac{1}{\sqrt{4\pi D_0 t}} e^{-\frac{x^2}{4Dt}} & \text{if } \gamma_1 = \gamma_2 \\
\frac{\sqrt{\gamma^2 - \Delta^2}}{2\pi t} \frac{\Delta}{2v(1 - \Delta^2)} \sqrt{\gamma^2 - \Delta^2 - \gamma^2(1 - \Delta^2)} e^{-\frac{\Delta^2}{2v(1 - \Delta^2)}} & \text{if } \gamma_1 < \gamma_2 \\
\frac{1}{\sqrt{4\pi D_0 t}} e^{-\frac{x^2}{4Dt}} & \text{if } \gamma_1 = \gamma_2 \\
\frac{8\gamma t}{\pi} \left[ \sqrt{8\gamma t} \Delta \rho \right] - te^{i\rho} \text{Erfc}[\sqrt{\rho\Delta}] \\
\times \left( \Delta - \frac{\rho^2 - \rho^2}{2} \right) + te^{i\rho} \text{Erfc}[\sqrt{\rho\Delta}] & \text{if } \gamma_1 > \gamma_2 
\end{cases}
\]

(19)

Figure 2. (a) Comparison of the stationary distribution $P_{\text{st}}^0(x)$ in equation (17) for $\alpha = 0$ with the same obtained from the simulation of the microscopic dynamics (shown by filled circles). The parameters chosen are $v = 1$, $\gamma_1 = 2$ and $\gamma_2 = 1$. The histogram is constructed for $10^6$ realizations at time $t = 20$. (b) The time-dependent distribution $P(x, t)$ in equation (18) of the position of the RTP for the $\alpha = 0$ case (solid lines) has been shown in comparison with simulation data (filled circles) for $t = 2.5$. The blue, black and red correspond to $\Delta > 0$, $\Delta = 0$ and $\Delta < 0$. The explicit values for the parameters for the three curves are given as follows: (i) $\gamma_1 = 1.5$, $\gamma_2 = 1$ (blue) (ii) $\gamma_1 = \gamma_2 = 1$ (black) and (iii) $\gamma_1 = 1$, $\gamma_2 = 1.5$ (red). For all plots we have taken $v = 1$. 

arise from those trajectories in which the particle has not changed its velocity direction until time $t$ starting from $x = 0$ with equal probability for $\pm v$. In figure 2(b), we plot the above result for $P(x, t)$ for three cases and compare them against the direct simulation of the Langevin equation (1). It is interesting to note that $P(x, t)$ has a dip at $x = 0$ for $\gamma_1 < \gamma_2$ and a peak for $\gamma_1 > \gamma_2$. The appearance of this behavior can be understood from the fact that for $\gamma_1 < \gamma_2$, the particle effectively drifts away from the origin, while for $\gamma_1 > \gamma_2$ the drift is towards the origin. Another interesting point to note is that the derivative of $P(x, t)$ has discontinuity at $x = 0$ for $\gamma_1 \neq \gamma_2$, while it is continuous for $\gamma_1 = \gamma_2$. For $\alpha = 0$ and $\gamma_1 \neq \gamma_2$, the rates $R_1(x)$ and $R_2(x)$ in equation (2) have discontinuity at $x = 0$, which amounts to the discontinuity in the derivative of $P(x, t)$. To gain more insights into this distribution, it is instructive to get a more explicit but approximate expression of the distribution $P(x, t)$ in the large $t$ limit for both the $\gamma_1 > \gamma_2$ and $\gamma_1 \leq \gamma_2$ cases. After some algebra (presented in appendices C.1 and C.2 with details) we find the following approximate expressions valid for large $t$: 

\[
P(x, t) \approx \begin{cases} 
\frac{1}{\sqrt{4\pi D_0 t}} e^{-\frac{x^2}{4Dt}} & \text{if } \gamma_1 = \gamma_2 \\
\frac{\sqrt{\gamma^2 - \Delta^2}}{2\pi t} \frac{\Delta}{2v(1 - \Delta^2)} \sqrt{\gamma^2 - \Delta^2 - \gamma^2(1 - \Delta^2)} e^{-\frac{\Delta^2}{2v(1 - \Delta^2)}} & \text{if } \gamma_1 < \gamma_2 \\
\frac{8\gamma t}{\pi} \left[ \sqrt{8\gamma t} \Delta \rho \right] - te^{i\rho} \text{Erfc}[\sqrt{\rho\Delta}] \\
\times \left( \Delta - \frac{\rho^2 - \rho^2}{2} \right) + te^{i\rho} \text{Erfc}[\sqrt{\rho\Delta}] & \text{if } \gamma_1 > \gamma_2 
\end{cases}
\]

(19)

https://doi.org/10.1088/1742-5468/aba7b1
Run-and-tumble particle in inhomogeneous media in one dimension

Figure 3. Comparison of the approximate expressions (red) of $P(x,t)$ given in equation (19) with the exact result (black) in equation (18) for $\Delta \neq 0$. In figure (a), we have plotted $P(x,t) - P_{0}^{st}(x)$ vs $x$ for $\gamma_{1} = 1.2$, $\gamma_{2} = 1$ and $t = 50$, while in figure (b), we plot $P(x,t)$ vs $x$ for $\gamma_{1} = 1$, $\gamma_{2} = 2$ and $t = 200$. For both plots $v = 1$.

with $D_{0} = \frac{v^{2}}{2\gamma}$, $\bar{x} = \frac{|x|}{vt}$ and $\rho_{\pm} = \sqrt{\gamma - \sqrt{\gamma_{1}\gamma_{2}}} \pm \frac{\sqrt{\gamma_{1}\gamma_{2}}}{2}$. In figure 3 we have compared these approximate results with the exact result in equation (18). From the expressions and the plots, we observe that for $\gamma_{1} > \gamma_{2}$ the distribution approaches the stationary state with exponential convergence at large $t$. On the other hand, for $\gamma_{1} = \gamma_{2}$ the distribution is peaked at $x = 0$, and at large $t$ it is described by a Gaussian function. For $\gamma_{1} < \gamma_{2}$ the distribution has two symmetric peaks at $x = \pm \frac{\Delta}{\gamma}vt$ that move ballistically. It is possible to see from the above expression that the distribution near these peaks, i.e. $|\bar{x}| \approx \frac{|\Delta|}{\gamma}$, behave as Gaussian at the leading order; however, while comparing them with numerical results we find that the distribution near the peaks quickly starts deviating from the Gaussian form, requiring one to include higher-order terms.

2.2. Case II: $\alpha = 1$

We now focus on the second analytically solvable case $\alpha = 1$, in which equation (14) becomes

$$\partial_{x}^{2}G - \frac{\Delta}{vl} + \frac{2s|x|}{v^{2}l} + \frac{s^{2}}{v^{2}} + \frac{\Delta^{2}x^{2}}{v^{2}l^{2}} G = 0. \quad (20)$$

We first look at the $\Delta = 0$ case.

2.2.1. $\Delta = 0$. For this case, equation (20) becomes

$$\partial_{x}^{2}G(x,s) - \left[ \frac{2s|x|}{v^{2}l} + \frac{s^{2}}{v^{2}} \right] G(x,s) = 0. \quad (21)$$

We identify this equation as the Airy differential equation whose general solutions are Airy functions. Upon satisfying appropriate boundary conditions, one finally gets an expression of $G(x,s)$ (see appendix D.1 for details). Using this expression in equations (10) and (13) provides

$$\tilde{P}(x,s) = -\frac{1}{2} \frac{\text{Ai}'\left(\frac{4|\Delta|s}{D_{1}^{1/3}} + d_{0} s^{4/3}\right)}{\text{Ai}(d_{0} s^{4/3})}, \quad (22)$$

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where $\mathcal{D}_1 = \frac{3^2}{2^3}, \ D_0 = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}}, \ Ai(z)$ is the Airy function of the first kind and $Ai'(z) = \frac{d}{dz}Ai(z)$. We here are interested in the behavior of $P(x,t)$ for large $t$, which can be equivalently obtained from the small $s$ limit of $P(x,s)$. From the expression in (22), it is clear that neglecting the term $d_0s^{4/3}$ inside the argument for a small $s$, one can get an appropriate scaling limit: $s \to 0$ and $|x| \to \infty$ such that $|x|s^{1/3}$ remains fixed. This in the $t$ domain translates to $t \to \infty$, $|x| \to \infty$ keeping $|x|/t^\frac{1}{6}$ finite, which implies the following scaling form of the PDF $P(x,t)$:

$$
P(x,t) \simeq \frac{1}{t^\frac{3}{2}} f_0 \left( \frac{|x|}{t^\frac{1}{2}} \right),
$$

(23)

To find this scaling function $f_0(y)$, we first use the identity $Ai(z) = \frac{x^\frac{3}{2}}{\sqrt{3\pi}} K_\frac{3}{2}(\frac{2}{3} z^\frac{3}{2})$ to simplify and get

$$
P(x,s) \simeq \frac{|x|}{2\sqrt{3\pi} \ D_0^{2/3} Ai(0)s^{4/3}} K_\frac{3}{2} \left( \frac{2}{3} \left( \frac{|x|}{\mathcal{D}_1^{1/3}} \right)^{\frac{3}{2}} \sqrt{s} \right),
$$

(24)

where $K_\frac{3}{2}(y)$ is a modified Bessel function. Now using the following relation between $K_\mu(2\sqrt{gs})$ and the Whittaker function $W_{k,\mu}(z)$ [76],

$$
L_{s-1}^{-1}[2\sqrt{g} s^k K_\mu(2\sqrt{gs})] = t^{\frac{k-1}{2}} e^{-\frac{y^3}{3}} W_{k-\frac{3}{2}, \mu} \left( \frac{y^3}{3} \right),
$$

(25)

for $k$ and $\mu$ real, and $g > 0$, we perform the inverse Laplace transform of $P(x,s)$ in equation (24) to get

$$
P(x,t) \simeq \frac{1}{t^\frac{3}{2}} f_0 \left( \frac{|x|}{t^\frac{1}{2}} \right), \quad \text{with,}
$$

$$
f_0(y) = \frac{3^{2/3}}{2\Gamma[1/3]} \sqrt{\frac{1}{y D_1^{1/3}}} e^{\frac{y^3}{3}} W_{1, \frac{3}{2}} \left( \frac{y^3}{3 D_1^{1/3}} \right),
$$

(26)

where we have used $Ai(0) = \frac{\Gamma[1/3]}{2\pi^{1/3}}$ with $\Gamma[z]$ being the gamma function. To go from the second line to the third line we have used the identities given in appendix D.2.

In figure 4(a), we numerically verify this result for $P(x,t)$ where we note excellent agreement. The above scaling form in equation (26) implies that for $\alpha = 1$ with $\Delta = 0$, the position $x$ of the particle scales as $x \sim t^{1/3}$ for large $t$ in contrast to $x \sim t^{1/2}$ obtained in the $\alpha = 0$ case. To numerically verify this scaling behavior, we plot variance $\langle x^2 \rangle$ against $t$ in figure 4(b) and at large $t$ we indeed observe $\langle x^2 \rangle \sim t^{\frac{5}{2}}$. We will later see that one also obtains similar scaling behavior for general $\alpha > 0$ in this case.

2.2.2. $\Delta \neq 0$. Here we consider the $\Delta \neq 0$ case for $\alpha = 1$, for which we solve equation (20). Note that the general solutions of this equation can be expressed in
Figure 4. Plots for the case $\alpha = 1$ and $\Delta = 0$. In figure (a) we plot $P(x, t)$ obtained from equation (26) (solid line) and compare it with the numerical simulation of equation (1) (filled circles) for $t = 10$. In (b), we verify the scaling of $x \sim t^{1/3}$ by plotting $\langle x^2 \rangle$ against $t$. The green solid line is the analytic result obtained by using expression for $P(x, t)$ in equation (26), while blue filled circles are the results of numerical simulation. For both plots, the chosen parameters are $v = 1$, $\gamma_1 = \gamma_2 = 1$, and $l = 1$.

terms of parabolic cylinder functions $D_{\mu - 1/2}(z)$. Choosing the integration constants appropriately to satisfy the boundary conditions, one obtains $G(x, s)$, using which in equations (10) and (13) we get (see appendix D.3 for details)

$$
\tilde{P}(x, s) = -\frac{1}{2s} \left[ \frac{D_{\beta s^2 - \Theta(\Delta)} \left( \frac{2i|\Delta|}{v} \left( u + \frac{2i}{\Delta} \right) \right)}{D_{\beta s^2 - \Theta(\Delta)} \left( \frac{2i|\Delta|}{v} \gamma s l \Delta^2 \right)} \right]_{u = |x|},
$$

(27)

where $\beta = \frac{(\gamma_1^2 - \Delta^2)}{2v|\Delta|}$ and $\Theta(\Delta) = 0$ if $\Delta \leq 0$; otherwise 1. Just like the $\alpha = 0$ case, in this case also we anticipate a stationary state for $\Delta > 0$, which can be determined from the limit $\lim_{s \to 0} [s \tilde{P}(x, s)]$. Using $D_{-1}(z) = \sqrt{\frac{\pi}{2}} e^{\frac{-z^2}{2}} \text{Erfc}(\frac{z}{\sqrt{2}})$, we get

$$
P_{1}^{st}(x) = \sqrt{\frac{\gamma_1 - \gamma_2}{2\pi vl}} e^{\frac{-x^2}{vl}},
$$

(28)

for $\Delta > 0$ where the subscript 1 in $P_{1}^{st}(x)$ stands for $\alpha = 1$. In figure 5(a), we have plotted $P_{1}^{st}(x)$ and compared it with the direct numerical simulation of microscopic equation (1). We find excellent agreement between the two. To understand the relaxation to this stationary state one needs to take into account the contribution from the pole with the second largest real part (the largest pole is at $s = 0$, which gives the steady state) in the Laplace inversion procedure. The poles of $\tilde{P}(x, s)$ in equation (27), come from the zeros of $D_{\beta s^2 - 1} \left( \frac{2i|\Delta| \gamma s l}{v|\Delta|} \right)$, which lie on the negative real axis. The pole $\zeta$ with largest real part other than 0 will set the timescale $|\zeta|^{-1}$ for the exponential relaxation, which can be determined by solving

$$
D_{\beta \zeta^2 - 1} \left( \frac{2i|\Delta| \gamma \zeta l}{v|\Delta|} \right) = 0,
$$

(29)

numerically for $\Delta > 0$. To verify this result we compute $d(t) = \text{var}(\infty) - \text{var}(t)$, where $\text{var}(t) = \langle x^2 \rangle - \langle x \rangle^2$ is the variance obtained from the numerical simulation, which
should decay to zero as $\sim e^{-\zeta t}$. In figure 5(b) we plot $d(t)$ as a function of $t$ and indeed observe the exponential decay with timescale $|\zeta|^{-1}$.

For $\Delta < 0$, as we have argued earlier, there is no stationary state. In this case, the solution $\bar{P}(x,t)$ in the Laplace space, given in equation (27), becomes

$$
\bar{P}(x,s) = -\frac{1}{2s} \left[ \frac{d}{du} \left( e^{\frac{u}{2} \zeta^2} \frac{\beta \sigma^2}{s} \left( \frac{2|\zeta|}{vl} \left( u + \frac{\gamma_s l}{\Delta^2} \right) \right) \right) \right]_{u=|x|}, \tag{30}
$$

where $\beta = \frac{\zeta(\gamma^2 - \Delta^2)}{2v|\Delta^2|}$. Performing the inverse Laplace transform, we can obtain $P(x,t)$. For the parameters $\gamma_1 = 1, \gamma_2 = 3, v = 1, l = 1$ and $t = 5$ we perform the inverse Laplace transform numerically to get $P(x,t)$ at $t = 5$, which we compare with simulation results in figure 6 and observe excellent agreement. The convergence of the numerical inversion procedure becomes poor with increasing $t$. However, following a different approximate procedure, explained in the next section, we find that for large $t$, the distribution $P(x,t)$ has the following scaling form: $P(x,t) \sim \frac{1}{2\sigma_1(t)} G \left( \frac{|x| - \mu(t)}{\sigma_1(t)} \right)$ with $\mu(t) = \langle |x| \rangle = \frac{vl|\Delta|}{3}$ and $\sigma_1^2(t) = \langle x^2 \rangle - \langle |x| \rangle^2 = (vl/|\Delta|) \ln(t)$. In figure 7 we verify this scaling behavior numerically. In the next section we show that $G(u)$ is a mean zero and unit variance Gaussian.

2.3. Case III: general $\alpha$

We now look at the general $\alpha > 0$ case. For this case making concrete analytical progress from equation (20) for any $\Delta$ seems difficult. However, it is possible to obtain some asymptotic results for the probability distribution in the large time limit. To proceed, in this case, it seems convenient to start from the original master equations in (3) which, by defining $P(x,t) = P_+(x,t) + P_-(x,t)$ and $Q(x,t) = P_+(x,t) - P_-(x,t)$, can be rewritten, in terms of $R_\pm(x) = \frac{R_0(x) \pm R_1(x)}{2}$, as

https://doi.org/10.1088/1742-5468/aba7b1
Run-and-tumble particle in inhomogeneous media in one dimension

Figure 6. Comparison of the probability distribution $P(x, t)$ for $\alpha = 1$ and $\Delta < 0$ with the numerical simulation. The theoretical curve (solid line) is obtained by performing an inverse Laplace transform in equation (30). For this plot the parameters we have taken are $\gamma_1 = 1, \gamma_2 = 3, v = 1l = 1$ and $t = 5$.

Figure 7. Verification of the scaling behavior of the distribution $P(x, t) = \frac{1}{2\sigma_1(t)} G_1\left(\frac{|x| - \mu(t)}{\sigma_1(t)}\right)$ for $\alpha = 1$ where $\mu(t) = \langle |x| \rangle$ and $\sigma_1^2(t) = \langle x^2 \rangle - \langle |x| \rangle^2$. Here we have shown the scaling behavior only for positive $x$ as the distribution $P(x, t)$ is symmetric. Other parameters of the plot are $\gamma_1 = 1.5, \gamma_2 = 1.6, v = 1$ and $l = 1$.

2.3.1. $\Delta > 0$. We first present the $\Delta > 0$ case for reasons that will be self-evident later. In this case the particle reaches a stationary state; to obtain which we equate the time derivative on the left-hand side of equations (31) and (32) to zero and then solve for the $x$ dependence. We get the following expression for the stationary state distribution:

$$\partial_t P(x, t) = -v \partial_x Q(x, t),$$

$$\partial_t Q(x, t) = -R_+(x)Q(x, t) - R_-(x)P(x, t) - v \partial_x P(x, t).$$

(31) (32)

Note that for $\alpha = 0$ and $\alpha = 1$, this expression correctly reduces to the exponential and Gaussian distributions given in equations (17) and (28) respectively. In figure 8(a) we numerically verify the above form of the steady-state distribution $P_{\alpha}^{st}(x)$ for three choices of $\alpha$ different from $\alpha = 0$ and 1. The approach to this steady state can in principle be understood by looking at the time-dependent solutions of equations (31) and (32) at large times; however, finding such solutions is a difficult task for which one has to solve the eigenvalue equation (14). Note that this eigenvalue equation looks similar to
Figure 8. (a) Comparison of the stationary state distribution $P_{\alpha}^{st}(x)$ obtained in equation (33) (solid lines) with the numerical simulation data (filled circles) for three values of $\alpha$. The histogram has been constructed using $10^6$ realizations at $t = 15$. (b) Numerical simulation of $d(t)$ defined as $d(t) = \text{var}(\infty) - \text{var}(t)$, where $\text{var}(t) = \langle x(t)^2 \rangle - \langle x(t) \rangle^2$. We find $d(t) \sim e^{-\zeta t}$ (shown by the red line) where we numerically find $\zeta = 0.31$. For both plots, we have taken $\gamma_1 = 2$, $\gamma_2 = 1$, $\lambda = 1$ and $v = 1$.

2.3.2. $\Delta \leq 0$. As seen in the previous two exactly solvable cases $\alpha = 0$ and $\alpha = 1$, for general $\alpha \geq 0$ also we expect that the distribution $P(x, t)$ does not reach a stationary state for $\Delta \leq 0$. To proceed, we first note in equations (31) and (32) that the equation for $P(x, t)$ is in the form of a continuity equation. On the other hand, the equation for $Q(x, t)$ is not in this form, but has decay terms proportional to the rates $R_{\pm}(x)$ which are non-negative functions. As a result, at a large time the difference distribution $Q(x, t)$ would not depend on time explicitly. Only time dependence would come from $P(x, t)$. Neglecting $\partial_t Q(x, t)$ for large $t$, we get $R_+(x)Q(x, t) = -R_-(x)P(x, t) - v \partial_x P(x, t)$, inserting which in equation (31) we get

$$
\partial_t P(x, t) \simeq \frac{v^2 \alpha}{2 \gamma} \partial_x (|x|^{-\alpha} \partial_x P(x, t)) + \text{sgn}(x) \frac{v \Delta}{\gamma} \partial_x P(x, t). \quad (34)
$$

The approximate equality in the above equation indicates that this equation is valid only at a large $t$ limit. This equation effectively describes a diffusion process with position-dependent drift and diffusivity. We mention that a similar diffusion equation with $\Delta = 0$ appears in the context of diffusion in a turbulent medium [77]. This equation can also be derived from equation (12). Performing an inverse Laplace transform over the $s$ variable on both sides of this equation, we get

$$
\partial_t P(x, t) = \partial_x \left[ \int_0^t dt' e^{-2s|\text{sgn}(x)|^{\alpha} \partial_x P(x, t')} + \frac{2v \Delta}{\gamma} \delta(t-t') \right] \left[ v^2 \partial_x P(x, t') + \frac{2v \Delta}{\gamma} \frac{\text{sgn}(x)|x|^\alpha}{2|\text{sgn}(x)|^{\alpha}} P(x, t) \right]. \quad (35)
$$

For large $t$ (at which $|x|$ is also typically large as will be seen later), the exponential term in the above equation can be approximated by $\sim \frac{v}{2|\text{sgn}(x)|^{\alpha}} \delta(t - t')$ and as a result the above equation reduces to equation (34). Note that this approximation does not work for...
Figure 9. Comparison of the probability distribution obtained from simulation of the effective Langevin equation, (36) (red circles), with the same from the original RTP dynamics (1) (black squares) for \( \alpha = 0.5 \) and \( \alpha = 1.0 \). For both values of \( \alpha \) the histograms are obtained at \( t = 1000 \) with \( \gamma_1 = 1.5 \) and \( \gamma_2 = 1.6 \). Other common parameters are \( v = 1 \) and \( l = 1 \).

\( \alpha = 0. \) It works only for \( \alpha > 0 \). Equation (34) can be interpreted as the Fokker–Planck equation of a particle diffusing in an inhomogeneous environment of diffusion constant
\[
D(x) = \frac{v^2 \gamma}{|x|^{\alpha}}
\]
and drift
\[
V(x) = -\text{sgn}(x) \left[ \frac{\Delta}{\gamma} + \frac{\alpha v^2 \gamma}{2|\gamma|^{\alpha+1}} \right].
\]
The corresponding Ito–Langevin equation is given by
\[
\frac{dx}{dt} = V(x) + \sqrt{D(x)} \eta(t),
\]
where \( \eta(t) \) is the Gaussian white noise with \( \langle \eta(t) \rangle = 0 \) and \( \langle \eta(t) \eta(t') \rangle = \delta(t - t') \). Comparisons between the two dynamics are shown in figure 9, where we have plotted \( P(x, t) \) vs \( x \) obtained from the simulation of the effective Langevin equation, (36), and the same from the original RTP dynamics (1) at \( t = 1000 \) for two values of \( \alpha = 0.5 \) (figure 9(b)) and \( \alpha = 1.0 \) (figure 9(a)).

\( \Delta = 0 \): for this case the second term on the right-hand side of equation (34) is absent. For \( \alpha = 0 \) and \( \alpha = 1 \), we have seen that \( P(x, t) \) at large \( t \), satisfies scaling forms shown in equations (19) and (26) respectively. These results suggest to us to expect the following scaling solution for \( P(x, t) \) at large \( t \) for general \( \alpha > 0 \):
\[
P(x, t) \approx \frac{1}{t^{|\alpha|/2}} f_\alpha \left( \frac{|x|}{t^{1/2}} \right),
\]
\( \alpha = 0. \) It works only for \( \alpha > 0 \). Equation (34) can be interpreted as the Fokker–Planck equation of a particle diffusing in an inhomogeneous environment of diffusion constant
\[
D(x) = \frac{v^2 \gamma}{|x|^{\alpha}}
\]
and drift
\[
V(x) = -\text{sgn}(x) \left[ \frac{\Delta}{\gamma} + \frac{\alpha v^2 \gamma}{2|\gamma|^{\alpha+1}} \right].
\]
The corresponding Ito–Langevin equation is given by
\[
\frac{dx}{dt} = V(x) + \sqrt{D(x)} \eta(t),
\]
where \( \eta(t) \) is the Gaussian white noise with \( \langle \eta(t) \rangle = 0 \) and \( \langle \eta(t) \eta(t') \rangle = \delta(t - t') \). Comparisons between the two dynamics are shown in figure 9, where we have plotted \( P(x, t) \) vs \( x \) obtained from the simulation of the effective Langevin equation, (36), and the same from the original RTP dynamics (1) at \( t = 1000 \) for two values of \( \alpha = 0.5 \) (figure 9(b)) and \( \alpha = 1.0 \) (figure 9(a)).

\( \Delta = 0 \): for this case the second term on the right-hand side of equation (34) is absent. For \( \alpha = 0 \) and \( \alpha = 1 \), we have seen that \( P(x, t) \) at large \( t \), satisfies scaling forms shown in equations (19) and (26) respectively. These results suggest to us to expect the following scaling solution for \( P(x, t) \) at large \( t \) for general \( \alpha > 0 \):
\[
P(x, t) \approx \frac{1}{t^{|\alpha|/2}} f_\alpha \left( \frac{|x|}{t^{1/2}} \right),
\]
\( \alpha = 0. \) It works only for \( \alpha > 0 \). Equation (34) can be interpreted as the Fokker–Planck equation of a particle diffusing in an inhomogeneous environment of diffusion constant
\[
D(x) = \frac{v^2 \gamma}{|x|^{\alpha}}
\]
and drift
\[
V(x) = -\text{sgn}(x) \left[ \frac{\Delta}{\gamma} + \frac{\alpha v^2 \gamma}{2|\gamma|^{\alpha+1}} \right].
\]
The corresponding Ito–Langevin equation is given by
\[
\frac{dx}{dt} = V(x) + \sqrt{D(x)} \eta(t),
\]
where \( \eta(t) \) is the Gaussian white noise with \( \langle \eta(t) \rangle = 0 \) and \( \langle \eta(t) \eta(t') \rangle = \delta(t - t') \). Comparisons between the two dynamics are shown in figure 9, where we have plotted \( P(x, t) \) vs \( x \) obtained from the simulation of the effective Langevin equation, (36), and the same from the original RTP dynamics (1) at \( t = 1000 \) for two values of \( \alpha = 0.5 \) (figure 9(b)) and \( \alpha = 1.0 \) (figure 9(a)).

\( \Delta = 0 \): for this case the second term on the right-hand side of equation (34) is absent. For \( \alpha = 0 \) and \( \alpha = 1 \), we have seen that \( P(x, t) \) at large \( t \), satisfies scaling forms shown in equations (19) and (26) respectively. These results suggest to us to expect the following scaling solution for \( P(x, t) \) at large \( t \) for general \( \alpha > 0 \):
\[
P(x, t) \approx \frac{1}{t^{|\alpha|/2}} f_\alpha \left( \frac{|x|}{t^{1/2}} \right),
\]
\( \alpha = 0. \) It works only for \( \alpha > 0 \). Equation (34) can be interpreted as the Fokker–Planck equation of a particle diffusing in an inhomogeneous environment of diffusion constant
\[
D(x) = \frac{v^2 \gamma}{|x|^{\alpha}}
\]
and drift
\[
V(x) = -\text{sgn}(x) \left[ \frac{\Delta}{\gamma} + \frac{\alpha v^2 \gamma}{2|\gamma|^{\alpha+1}} \right].
\]
The corresponding Ito–Langevin equation is given by
\[
\frac{dx}{dt} = V(x) + \sqrt{D(x)} \eta(t),
\]
where \( \eta(t) \) is the Gaussian white noise with \( \langle \eta(t) \rangle = 0 \) and \( \langle \eta(t) \eta(t') \rangle = \delta(t - t') \). Comparisons between the two dynamics are shown in figure 9, where we have plotted \( P(x, t) \) vs \( x \) obtained from the simulation of the effective Langevin equation, (36), and the same from the original RTP dynamics (1) at \( t = 1000 \) for two values of \( \alpha = 0.5 \) (figure 9(b)) and \( \alpha = 1.0 \) (figure 9(a)).

\( \Delta = 0 \): for this case the second term on the right-hand side of equation (34) is absent. For \( \alpha = 0 \) and \( \alpha = 1 \), we have seen that \( P(x, t) \) at large \( t \), satisfies scaling forms shown in equations (19) and (26) respectively. These results suggest to us to expect the following scaling solution for \( P(x, t) \) at large \( t \) for general \( \alpha > 0 \):
\[
P(x, t) \approx \frac{1}{t^{|\alpha|/2}} f_\alpha \left( \frac{|x|}{t^{1/2}} \right),
\]
In figure 10, we have numerically verified this scaling form by plotting data for \( P(x, t) \) for three different values of times and for \( \alpha = 0.75 \) and \( \alpha = 0.5 \). The excellent data collapse observed in both cases establishes that the scaling form in equation (37) is indeed true. To obtain the explicit analytic form of \( f_\alpha \), we substitute the scaling form of \( P(x, t) \) from equation (37) in equation (34). This gives a differential equation for \( f_\alpha(y) \) as

\[
\mathcal{D}_\alpha \frac{d}{dy} \left[ y^{-\alpha} \frac{df_\alpha}{dy} \right] + \frac{1}{2 + \alpha} \left( y \frac{df_\alpha}{dy} + f_\alpha \right) \approx 0,
\]

where \( \mathcal{D}_\alpha = \frac{\alpha^2}{2} \). Changing variable \( u = \frac{1}{(2+\alpha)^2} y^{2+\alpha} \), the above equation simplifies to

\[
u \frac{d^2 f_\alpha}{du^2} + \left( \frac{1}{2 + \alpha} + u \right) \frac{df_\alpha}{du} + \frac{f_\alpha}{2 + \alpha} = 0.
\]

The two independent general solutions of this differential equation are given by \( e^{-u} \) and \( u^{\frac{1+\alpha}{2+\alpha}} F_1 \left( \frac{\alpha}{2+\alpha}; 2 - \frac{1}{2+\alpha}; u \right) \), where \( F_1 \) stands for a confluent hypergeometric function of first kind. The latter solution diverges as \( u \to \infty \). Hence, keeping only the physically relevant solution and using the normalization condition \( \int_{-\infty}^{\infty} dx \ P(x, t) = 1 \), which translates to \( \int_0^\infty dz \ f_\alpha(z) = 1/2 \), we finally get

\[
f_\alpha(y) = \frac{(2 + \alpha) \frac{\Gamma(\alpha)}{\Gamma(1 + \alpha)}}{2 \Gamma \left( \frac{1}{2+\alpha} \right) \mathcal{D}_\alpha^\alpha} \ e^{-y^{2+\alpha}}.
\]

Note that this expression for the scaling function \( f_\alpha(y) \) correctly reduces to the scaling function given in equation (26) for \( \alpha = 1 \) and to the middle expression in equation (19) for \( \alpha = 0 \). The function \( f_\alpha(y) \) is plotted in figure 10 for two different values of \( \alpha \) and compared with the numerical simulation for three different times. We observe excellent agreement between the theory and the simulation results.

\( \Delta < 0 \): For this case the drift \( V(x) \) on the particle, as given in equation (34), is away from the origin on both sides at large \( x \). As a result the particle never reaches a steady state (as has been argued earlier). We have already observed numerically in figure 9 that at large times the dynamics of the RTP can be well described by Langevin equation (36). From this comparison, we also observe that the distribution \( P(x, t) \) is symmetric with respect to the origin, which implies that the mean position of the particle is zero; however, the average value of the absolute position \( \mu(t) = \langle \vert x \vert \rangle \) is not zero. The two symmetric peaks on the opposite sides of the origin are situated at \( x = \pm \mu(t) \). For large \( t \), this quantity increases linearly as \( \langle \vert x \vert \rangle = \mu(t) \sim \frac{\Delta}{\gamma} t \), which can be easily verified numerically. In particular, it can be shown that using the following variable transformations, \( \vert x \vert = \mu(t) + z \) and \( \tau = t^{1-\alpha} \), equation (34) at the large \( t \) limit becomes the following diffusion equation:

\[
\partial_\tau P(z, \tau) = \mathcal{D}_\alpha \partial_z^2 P(z, \tau),
\]

where \( \mathcal{D}_\alpha = \frac{\alpha^2}{\gamma (1-\alpha)} \left( \frac{\Delta}{\gamma} \right)^\alpha \) for \( 0 < \alpha < 1 \). This immediately implies that for large \( t \), the distribution \( P(x, t) \) has the following scaling form:

https://doi.org/10.1088/1742-5468/aba7b1
Run-and-tumble particle in inhomogeneous media in one dimension

Figure 11. (a) Numerical verification of $\sigma^2(t) \sim D_\alpha t^{1-\alpha}$ with $D_\alpha = \frac{v^2 l^{\alpha}}{(1-\alpha)\gamma} (\frac{\gamma}{\sqrt{2\Delta}})^\alpha$ for large $t$. (b) Numerical verification of the scaling behavior in equation (42) with $G_\alpha(u)$ given by $G_\alpha(u) = e^{-u^2/2\sqrt{2\pi}}$ (shown by the solid line). For both plots, we have taken $\alpha = 0.5$, $\gamma_1 = 1.5$, $\gamma_2 = 1.6$, $l = 1$ and $v = 1$.

$$P(x, t) \approx \frac{1}{2\sigma_\alpha(t)} G \left( \frac{|x| - \mu(t)}{\sigma_\alpha(t)} \right),$$

(42)

$G(u)$ satisfies the differential equation $\partial_u^2 G(u) + u \partial_u G(u) + G(u) = 0$ and the variance $\sigma^2_\alpha(t) = \langle x^2 \rangle - \langle |x| \rangle^2$ is given by $\sigma^2_\alpha(t) \sim D_\alpha t^{1-\alpha}$. The solution of this equation is very simple and given by the zero mean and unit variance Gaussian, $G(u) = e^{-u^2/2\sqrt{2\pi}}$. The same procedure can also be followed for $\alpha = 1$ and one gets same scaling behavior (as in equation (42)) with the same scaling function $G(u)$, but now $\sigma^2_\alpha(t) \sim (vL/|\Delta|) \ln(t)$ for $\alpha = 1$. The time dependence of the variance $\sigma^2_\alpha(t)$ is verified numerically in figure 11(a) for $\alpha < 1$. In the numerical simulation of the equation of motion, (1), with the rates given in equation (2), we have chosen $\alpha = 0.5$. It turns out that this value is optimal for the numerical verification. For given $\gamma$ and $\Delta$, the description given by the FP equation (34) starts becoming valid at (large) times, which increase with decreasing $\alpha$. Performing numerical simulation over such huge time durations turns out to be highly expensive. On the other hand, for larger $\alpha$, even though the effective inhomogeneous diffusion equation (34) starts becoming valid at smaller times, the rates (being $\propto |x|^\alpha$) start increasing faster with time because for $\Delta < 0$ the particle effectively drifts away from the origin. As a result, one requires a very small $dt$ in the numerical simulation in order to get good convergence, which in turn again makes computation expensive. Hence we choose an intermediate value $\alpha = 0.5$ for numerical demonstrations.

The scaling behavior in equation (42) is demonstrated and verified in figure 11(b) numerically. Note that this scaling behavior is valid for $0 < \alpha < 1$. For $\alpha > 1$; we numerically observe that the variance decreases with time. As a result, at a very large time we expect the distribution $P(x, t)$ to shrink to a sum of two delta functions at $x = \pm \mu(t)$. This feature can be easily verified by following the same procedure for $\alpha > 1$ as done for $0 < \alpha < 1$.

3. Survival probability

In this section we study the motion of the RTP with space-dependent rates, given in equation (2) in the presence of an absorbing barrier. In many physical settings, the question of how long a particle survives at a given absorbing site is of primary interest.

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[78, 79]. In particular, we will consider the absorbing site to be at the origin and address the question of survival probability for the particle starting from some position $x_0 > 0$. Let $S_\pm(x_0, t)$ denote the survival probability for the RTP with initial position $x_0$ and initial velocity $\pm v$ in the presence of an absorbing wall at $x = 0$. We start by deriving the backward master equations satisfied by $S_\pm(x_0, t)$ and then solve them explicitly. Below we briefly discuss the derivation of the backward master equations.

The probability that RTP with initial velocity direction $+\,v$ survives from the absorbing wall at $x = 0$ until time $t + dt$ is $S_+(x_0, t + dt)$. One can break the total time duration $t + dt$ into two parts (i) $[0, dt]$ and (ii) $[dt, t + dt]$. In the first interval of duration $dt$, the RTP can do two things: (a) without flipping its direction move to position $x_0 + vdt$ with probability $1 - R_1(x_0)dt$ and (b) flip its direction of motion with probability $R_1(x_0)dt$. After time $dt$, the RTP survives the remaining interval with probability $S_+(x_0 + vdt, t)$ if event (a) occurs, and with probability $S_-(x_0, t)$ if event (b) occurs. Adding all these probabilities with appropriate weights one gets $S_+(x_0, t + dt) = (1 - R_1(x_0)dt)S_+(x_0 + vdt, t) + R_1(x_0)dtS_-(x_0, t)$. Similarly, if the initial velocity direction is negative, then one has $S_-(x_0, t + dt) = (1 - R_2(x_0)dt)S_-(x_0 - vdt, t) + R_2(x_0)dtS_+(x_0 + vdt, t)$. Performing a Taylor series expansion in $dt$ and taking the $dt \to 0$ limit, one gets the following backward master equations for $S_\pm(x_0, t)$:

$$\begin{align*}
\partial_t S_+(x_0, t) &= v\partial_{x_0}S_+(x_0, t) - R_1(x_0)S_+(x_0, t) + R_1(x_0)S_-(x_0, t), \\
\partial_t S_-(x_0, t) &= -v\partial_{x_0}S_-(x_0, t) + R_2(x_0)S_+(x_0, t) - R_2(x_0)S_-(x_0, t).
\end{align*}$$

To solve these equations, one needs to specify the initial and the boundary conditions. Note that if the particle starts from $x_0 \to \infty$ initially, then for all finite $t$ it survives regardless of its initial velocity direction. This gives rise $S_\pm(x_0 \to \infty, t) = 1$. To understand the other boundary condition $S_-(0, t) = 0$, the particle will be instantly absorbed if it starts at $x_0 = 0$ with $-\,v$ velocity. However, if the particle starts from $x_0 = 0$ with $+\,v$ velocity, it will not get absorbed instantly and accordingly one gets $S_+(0, t) \neq 0$. Hence for any $x_0 \neq 0$ we have $S_\pm(x_0, 0) = 1$.

To solve equation (43) we take the Laplace transformation $\tilde{S}_\pm(x_0, s) = \int_0^\infty dt e^{-st}S_\pm(x_0, t)$ on both sides to get

$$
[-v\partial_{x_0} + R_1(x_0) + s] \tilde{S}_+(x_0, s) = 1 + R_1(x_0)\tilde{S}_-(x_0, s),
$$

$$
[v\partial_{x_0} + R_2(x_0) + s] \tilde{S}_-(x_0, s) = 1 + R_2(x_0)\tilde{S}_+(x_0, s),
$$

(44)

where we have used the initial conditions $S_\pm(x_0, 0) = 1$. Under this transformation the boundary conditions become $\tilde{S}_\pm(x_0 \to \infty, s) = \frac{1}{s}$ and $\tilde{S}_-(0, s) = 0$. Note that the differential equations in equation (44) are inhomogeneous. To make them homogeneous we define $\tilde{U}_\pm(x_0, s)$ such that

$$\tilde{S}_\pm(x_0, s) = \frac{1}{s} + \tilde{U}_\pm(x_0, s),$$

(45)

which also simplifies the boundary conditions as $\tilde{U}_\pm(x_0 \to \infty, s) = 0$. Equation (44) now becomes
Run-and-tumble particle in inhomogeneous media in one dimension

\[-v \partial_{x_0} + R_1(x_0) + s] \dot{U}_+ = R_1(x_0) \dot{U}_-,
[ v \partial_{x_0} + R_2(x_0) + s] \dot{U}_- = R_2(x_0) \dot{U}_+.

(46)

Further, defining

\[\bar{U}(x_0, s) = \bar{U}_+(x_0, s) + \bar{U}_-(x_0, s),\]
\[\bar{H}(x_0, s) = \bar{U}_+(x_0, s) - \bar{U}_-(x_0, s),\]

we get

\[\partial_{x_0}^2 \bar{H} - \frac{2\Delta x_0^\alpha}{vl^\alpha} \partial_{x_0} \bar{H} - \left[ \frac{2\Delta x_0^{\alpha-1}}{vl^\alpha} + \frac{2\gamma s x_0^\alpha}{v^2 l^\alpha} + \frac{s^2}{v^2} \right] \bar{H} = 0,\]

(48)

and

\[\bar{U}(x_0, s) = \frac{v}{s} \partial_{x_0} \bar{H} - \frac{2\Delta x_0^\alpha}{sl^\alpha} \bar{H}.\]

(49)

One can get rid of the first-order derivative in equation (48) by making the transformation

\[\bar{H}(x_0, s) = e^{\Delta v \frac{\alpha}{l^\alpha} x_0^\alpha} F(x_0, s),\]

(50)

using which in equation (48), one gets

\[\partial_{x_0}^2 F(x_0, s) - \left[ \frac{\Delta x_0^{\alpha-1}}{v l^\alpha} + \frac{2\gamma s x_0^\alpha}{v^2 l^\alpha} + \frac{\Delta^2 x_0^{2\alpha}}{v^2 l^{2\alpha}} + \frac{s^2}{v^2} \right] F(x_0, s) = 0.\]

(51)

Note that this equation is identical to equation (14) for \(G(x, s)\) obtained in the previous section except for the boundary conditions, which are different for the two cases. In what follows, we will solve this equation for \(\alpha = 0\) and \(\alpha = 1\) separately and then address the case of general \(\alpha\).

3.1. Case I: \(\alpha = 0\)

We start with the simplest case of \(\alpha = 0\). For this case the rates \(R_{1,2}(x)\) are actually \(x\) independent. Recall that \(x_0\) is the initial position of the RTP, which is greater than 0. Hence noting that \(F(x_0, s)\) is finite as \(x_0 \to \infty\), one gets \(F(x_0, s) \sim e^{-\lambda(s)x_0}\), where \(\lambda(s) = \frac{1}{4} \sqrt{2\gamma s + s^2 + \Delta^2}\) (see appendix A for details). Inserting this in equations (49) and (50) and finally writing for \(\bar{S}_\pm(x_0, s)\), the expressions read as

\[\bar{S}_\pm(x_0, s) = \frac{1}{s} \frac{A}{2s} e^{(\Delta - \lambda(s))x_0} [-\Delta - v\lambda(s) \pm s],\]

(52)

where \(A\) is a constant independent of \(x_0\). To evaluate \(A\), we use the boundary condition \(\bar{S}_-(0, s) = 0\), which gives \(A(s) = \frac{2}{\Delta_+ s + v\lambda(s)}\). Finally inserting this in equation (52) we get the following expressions for \(\bar{S}_\pm(x_0, s)\):

\[\bar{S}_-(x_0, s) = \frac{1}{s} - \frac{1}{s} e^{(\Delta - \lambda(s))x_0},\]

(53)

https://doi.org/10.1088/1742-5468/aba7b1
Run-and-tumble particle in inhomogeneous media in one dimension

\[ S_+(x_0, s) = \frac{1}{s} - \frac{s + \gamma - v\lambda(s)}{s\gamma_2} e^{\left(\frac{s}{s\gamma_2}\right)x_0}. \] (54)

Using the following results for inverse Laplace transformations,

\[
L_{s\rightarrow t} \left[ e^{-\lambda(s)x_0} \right] = -ve^{-\frac{\Delta s}{v}} \frac{dx_0}{dx_0} \left[ e^{-vt} I_0 \left( \sqrt{\gamma_1\gamma_2} \left( t^2 - \frac{x_0^2}{v^2} \right) \right) \right],
\]

\[
L_{s\rightarrow t} \left[ (s + \gamma - v\lambda(s))e^{-\lambda(s)x_0} \right] = \sqrt{\gamma_1\gamma_2} \frac{e^{-vt - \Delta s}}{\frac{v}{\gamma_2}} \left[ \frac{x_0\sqrt{\gamma_1\gamma_2}}{v} I_0 \left( \sqrt{\gamma_1\gamma_2} \left( t^2 - \frac{x_0^2}{v^2} \right) \right) \right]
+ \sqrt{\frac{v^2 - x_0^2}{v^2 + x_0^2}} I_1 \left( \sqrt{\gamma_1\gamma_2} \left( t^2 - \frac{x_0^2}{v^2} \right) \right) \Theta(vt - x_0) \quad (55)
\]

we get

\[
S_-(x_0, t) = 1 + ve^{-\frac{\Delta s}{v}} \frac{dx_0}{dx_0} \int_0^t dt e^{-vt} I_0 \left( \sqrt{\gamma_1\gamma_2} \left( \tau^2 - \frac{x_0^2}{v^2} \right) \right) \Theta(v\tau - x_0),
\]

\[
S_+(x_0, t) = 1 - \sqrt{\frac{\gamma_1}{\gamma_2}} e^{-\frac{\Delta s}{v}} \int_0^t dt e^{-vt} I_0 \left( \sqrt{\gamma_1\gamma_2} \left( \tau^2 - \frac{x_0^2}{v^2} \right) \right) \Theta(v\tau - x_0)
+ \sqrt{\frac{v^2 - x_0^2}{v^2 + x_0^2}} I_1 \left( \sqrt{\gamma_1\gamma_2} \left( \tau^2 - \frac{x_0^2}{v^2} \right) \right) \Theta(vt - x_0) \quad (56)
\]

For \( \gamma_1 = \gamma_2 \) the above expressions match with the previously obtained results in [18, 54]. In figure 12 we have plotted our above theoretical results for \( S_\pm(x_0, t) \) and compared them with the numerical simulations. We observe excellent agreement between them. Note that for both \( S_\pm(x_0, t) \) remain at 1 until time \( t_b = \frac{2x_0}{v} \). This is because the RTP initially starting from \( x_0 \) will take at least time \( t_b \) to reach the absorbing wall at \( x = 0 \). Before \( t_b \), the RTPs do not feel the presence of the barrier. Once they reach the wall with velocity \(-v\) at time \( t_b \), a fraction of them will change velocity from \(-v\) to \( +v\) and survive the wall, while others will get absorbed at time \( t_b^* \). This results in a sudden drop in the population of RTPs as indicated by the sudden drop in \( S_-(x_0, t) \). However, no sudden drop occurs in \( S_+(x_0, t) \) because the particles do not reach the wall with \(+v\) velocity.

It is worth noting that the particle drifts away from the origin for \( \Delta < 0 \), which gives rise to non-zero \( S_\pm(x_0, t) \) as \( t \to \infty \). This can be, in principle, verified by putting \( t \to \infty \) in the expressions of \( S_\pm(x_0, t) \) given in equation (56). However, it turns out to be more convenient to obtain this from the Laplace transforms given in equations (53) and (54) by putting \( s = 0 \), which corresponds to the \( t \to \infty \) limit. Hence for \( \Delta < 0 \) we
The details of this calculation are given in appendix E.

One can similarly compute $S_\pm(x_0)$ for $\Delta > 0$ which turns out to be 0 as the particle will definitely reach the origin after a sufficient time interval.

We now discuss the behavior of $S_\pm(x_0, t)$ for large $t$ that would provide the relaxation to the stationary values for $\Delta < 0$ and to zero for $\Delta > 0$. For this, we take the large $t$ approximation in equation (56). The details of this calculation are given in appendix E and we present only the final results here. Defining $L_\pm(x_0, t) = S_\pm(x_0, t) - S_\pm(x_0)$, where $S_\pm(x_0)$ is 0 for $\Delta \geq 0$ and given by equation (57) for $\Delta < 0$, we obtain

\begin{align}
L_+(x_0, t) &\approx \begin{cases} 
\sqrt{\frac{1}{\gamma_2}} \left( x_0 \frac{1}{v} + \frac{1}{\sqrt{\gamma_1 \gamma_2}} \right) e_{\alpha_0}^{\frac{\Delta_0}{4\pi v^2}} e^{-t(\gamma - \sqrt{\gamma_1 \gamma_2})}, & \text{if } \gamma_1 \neq \gamma_2 \\
\frac{1}{\sqrt{\pi \gamma}} \left( x_0 + \frac{v}{\gamma} \right), & \text{if } \gamma_1 = \gamma_2.
\end{cases} \\
L_-(x_0, t) &\approx \begin{cases} 
\frac{v}{\sqrt{2\pi \gamma^3}} (\gamma_1 \gamma_2)^{\frac{1}{4}} e_{\alpha_0}^{\frac{\Delta_0}{4\pi v^2}} e^{-t(\gamma - \sqrt{\gamma_1 \gamma_2})}, & \text{if } \gamma_1 \neq \gamma_2 \\
\frac{1}{\sqrt{\pi \gamma}} x_0 \frac{v}{\sqrt{2\gamma}}, & \text{if } \gamma_1 = \gamma_2.
\end{cases}
\end{align}

Our results for $\Delta = 0$ match with those in [18, 59]. Note that for $\Delta > 0$, the survival probabilities decay exponentially, whereas for $\Delta = 0$ case it decays as a power law $\sim 1/\sqrt{t}$. In fact the timescale $\tau_\pm = \frac{1}{\gamma - \sqrt{\gamma_1 \gamma_2}}$ associated with this exponential decay diverges in the $\Delta \to 0 (\gamma_1 \to \gamma_2)$ limit which is consistent with the power law behavior for $\Delta = 0$. For the $\Delta < 0$ case, we observe that $S_\pm$ relaxes exponentially to their stationary values over the same timescale $\tau_\pm = \frac{1}{\gamma - \gamma_1 \gamma_2}$. The fact that $\tau_\pm$ diverges and the prefactor approaches...
Figure 13. Comparison of the asymptotic behavior of \(S_{\pm}(x_0, t)\) given by equations (58) and (59) with the exact expression in equation (56) for \(\alpha = 0\) and for both \(\Delta > 0\) and \(\Delta < 0\). We have plotted \(L_{\pm}(x_0, t) = S_{\pm}(x_0, t) - S_{\pm}(x_0, t \to \infty)\). In both (a) and (b) we have chosen (i) \(\gamma_1 = 3, \gamma_2 = 1\) for red and (ii) \(\gamma_1 = 1, \gamma_2 = 3\) for magenta. Other common parameters are \(v = 1\) and \(x_0 = 1\).

to zero as \(\Delta \to 0\) implies that the stationary survival probabilities do not exist for the \(\Delta = 0\) case. From the expressions, we see that for \(x_0 = 0\) while \(S_{-}\) is exactly 0, \(S_{+}\) still has a non-zero value. The particle can survive if it starts from the origin with positive velocity. In figure 13 we have compared these asymptotic behaviors with the exact results in equation (56).

3.2. Case II: \(\alpha = 1\)

We now turn to the \(\alpha = 1\) case. For this case the rates \(R_{1,2}(x)\) of the orientation flipping increases as \(\sim |x|\), which, as we will see, makes the large time behavior for the survival probability remarkably different from the \(\alpha = 0\) case. This difference is most prominent in the \(\Delta = 0\) case which we consider next. In the subsequent sections we discuss the \(\Delta \neq 0\) cases.

3.2.1. \(\Delta = 0\). We start with equation (51), which for \(\alpha = 1\) and \(\Delta = 0\) takes the form

\[
\partial_{x_0} F(x_0, s) - \left( \frac{2\gamma s x_0}{v^2 l} + \frac{s^2}{v^2} \right) F(x_0, s) = 0. \tag{60}
\]

Since the general solutions of this equation are the same as equation (21) we take the solutions from appendix D and write a general solution for \(F(x_0, s)\) in terms of Airy functions and the integration constants \(C_{\pm}\). Inserting the \(F(x_0, s)\) in equation (50) and then in equation (49), and fixing the integration constants through the boundary conditions, we finally get

\[
\bar{S}_{\pm}(x_0, s) = \frac{1}{s} - \frac{1}{s} \frac{v}{s} \frac{\text{Ai}' \left( \frac{x_0 s^4}{\mathcal{D}_1^{1/3}} + d_0 s^4 \right) \pm s^2 \mathcal{D}_1^{1/3} \text{Ai} \left( \frac{x_0 s^4}{\mathcal{D}_1^{1/3}} + d_0 s^4 \right)}{v \text{Ai}'(0) - s^2 \mathcal{D}_1^{1/3} \text{Ai}(0)}, \tag{61}
\]

with \(\mathcal{D}_1 = \frac{v^2 l}{\gamma v}\) and \(d_0 = \left( \frac{l}{2\gamma v} \right)^{2/3}\), where we have used the definitions in equation (47). To get \(S_{\pm}(x_0, t)\) one needs to perform the inverse Laplace transform. Since here also we are interested in the behavior at a large \(t\) limit, we first neglect the \(d_0 s^{4/3}\) term in the argument of the Airy function and get the following simpler equation:

https://doi.org/10.1088/1742-5468/aba7b1
Run-and-tumble particle in inhomogeneous media in one dimension

\[ S_{\pm}(x_0, s) \simeq \frac{1}{s} - \frac{1}{sv|A\Gamma'(0)|} \left[ vA\Gamma\left(\frac{x_0s^3}{D_1^{1/3}}\right) \pm s^{2/3} D_1^{1/3} A\Gamma\left(\frac{x_0s^3}{D_1^{1/3}}\right) \right. \\
\left. - \frac{s^{2/3} D_1^{1/3} A\Gamma(0)}{|A\Gamma'(0)|} A\Gamma\left(\frac{x_0s^3}{D_1^{1/3}}\right) \right]. \] (62)

Now once again using \( A\Gamma(z) = \frac{\pi}{\sqrt{\pi}z} \left( \frac{2z}{\sqrt{\pi}} \right) \) and using equation (25) we perform the inverse of the Laplace transform to get the following approximate expressions for \( S_{\pm}(x_0, t) \) for large \( t \):

\[ S_{\pm}(x_0, t) \simeq 1 + \frac{e^{-\frac{\theta t}{vD_1}}}{4\sqrt{3g_1 v}|A\Gamma'(0)|} \left[ x_0 A\Gamma(0) \left( \frac{x_0^3}{9D_1 t} \right)^{1/3} \pm \frac{v}{D_1^{1/3}} \left( \frac{x_0^3}{9D_1 t} \right)^{1/3} \right] \\
\pm \left( \frac{x_0}{t^{1/3}} D_1^{1/3} W_{\frac{1}{3}} \left( \frac{x_0}{9D_1 t} \right) \right), \] (63)

where \( g_1 = \frac{x_0^2}{9D_1} \). In figure 14, we have plotted our result of \( S_{\pm}(x_0, t) \) along with the results from numerical simulations, where we observe a perfect match at large \( t \). The mismatch at a smaller \( t \) is self-explanatory. Although the expression in equation (63) is efficient but still less illuminating. It is more instructive to find the true asymptotic of \( S_{\pm}(x_0, t) \) for \( t \to \infty \). For this, we use the following representation of the Whittaker function \( W_{k,m}(z) = e^{-\frac{z}{2}} z^{m+\frac{1}{2}} U\left(\frac{3}{2} + m - k, 1 + 2m, z\right) \), where \( U(a, b, z) \) is the Tricomi confluent hypergeometric function, which, for \( z \to 0 \), behaves as \( U(a, b, z) \approx \frac{\Gamma(a-1)}{\Gamma(a)} z^{-a-b} + \frac{\Gamma(1-b)}{\Gamma(a-b+1)} \).

Using this asymptotic form in equation (63) we get

\[ S_{+}(x_0, t) \approx \frac{1}{2\pi^{3/2} (D_1 t)^{3/4}} \left( -x_0^2 \Gamma\left( \frac{2}{3} \right) + \frac{6 D_1 \Gamma\left( \frac{3}{2} \right)}{v} \right), \] (64)

\[ S_{-}(x_0, t) \approx -\frac{1}{2\pi^{3/2} (D_1 t)^{3/4}} x_0^2 \Gamma\left( \frac{2}{3} \right), \]

suggesting a power law decay with persistent exponent \( \theta = 2/3 \). Note that this exponent is different from the exponent \( \theta = 1/2 \) in the \( \alpha = 0 \) case (see equations (58) and (59)). Another interesting feature to note is while \( S_{-}(x_0, t) \) vanishes for \( x_0 = 0 \), \( S_{+}(x_0, t) \) is non-zero (although it decays with \( t \)). This is in contrast with the passive Brownian particles where survival probability is 0 for all \( t \) when the particle starts from the origin (position of the absorbing barrier) \([79]\). The non-zero value of \( S_{+}(x_0, t) \) at \( x_0 = 0 \) is a signature of ‘activity’ as observed in previous studies with \( \alpha = \Delta = 0 \) \([18, 59]\).

3.2.2. \( \Delta \neq 0 \). We now consider the \( \Delta \neq 0 \) case for which equation (51) reduces to

\[ \partial_{x_0}^2 F - \left[ \frac{\Delta}{v l} + \frac{2 \gamma s x_0}{v^2 l} + \frac{s^2}{v^2} + \frac{\Delta^2 x_0^2}{v^2 l^2} \right] F = 0. \] (65)
Run-and-tumble particle in inhomogeneous media in one dimension

Figure 14. Comparison of the analytic results of $S_{\pm}(x_0, t)$ with the numerical simulation for $\alpha = 1$ and $\Delta = 0$ (a), $\Delta > 0$ (b) and $\Delta < 0$ (c). In figure (a), we have plotted equation (63) for $\gamma_1 = \gamma_2 = 1$ and $x_0 = 1$. For figures (b) and (c), we define $L_{\pm}(x_0, t) = S_{\pm}(x_0, t) - S_{\pm}(x_0)$ with $S_{\pm}(x_0)$ equal to 0 for $\Delta > 0$ and given by equation (68) for $\Delta < 0$. The analytic expressions for $L_{\pm}(x_0, t)$, given by equation (69) for both $\Delta < 0$ and $\Delta > 0$, are plotted (solid line) and compared with the simulation data (filled circles). For figure (b), we have taken $x_0 = 1.5$, $\gamma_1 = 2$ and $\gamma_2 = 1$, while for figure (c), we have taken $x_0 = 1.5$, $\gamma_1 = 1$ and $\gamma_2 = 1.5$. For all three figures, other common parameters are $v = 1$ and $l = 1$.

We note that this equation is identical to equation (20) although the boundary conditions of the two equations are different. However, the general solutions of the two equations are same and can be expressed in terms of the parabolic cylinder functions $D_{\mu}(y)$ as shown in appendix D. Inserting this general solution in equations (49) and (50) and using the boundary conditions $\bar{S}_-(x_0 \to 0, s) = 0$ we, after performing some simplifications, get

$$\bar{S}_{\pm}(x_0, s) = \frac{1}{s} - \frac{1}{s} e^{\frac{\Delta}{l} \frac{N_{\pm}(x_0, s)}{N_{\pm}(0, s)}} \left( \frac{2|\Delta|}{vl} \left( x_0 + \frac{\gamma s l}{\Delta^2} \right) \right)$$

where

$$N_{\pm}(x_0, s) = \sqrt{\frac{2v|\Delta|}{l}} \left\{ \beta s^2 \Theta(-\Delta) + \Theta(\Delta) \right\} D_{\beta s^2 - \Theta(\Delta)} \left( \sqrt{\frac{2|\Delta|}{vl}} \left( x_0 + \frac{\gamma s l}{\Delta^2} \right) \right)$$

$$- s \frac{\gamma \pm \Delta}{|\Delta|} D_{\beta s^2 - \Theta(\Delta)} \left( \sqrt{\frac{2|\Delta|}{vl}} \left( x_0 + \frac{\gamma s l}{\Delta^2} \right) \right),$$

with $\beta = \frac{v(\gamma^2 - \Delta^2)}{2l|\Delta|^3}$. To get the survival probabilities in the time domain one needs to perform the inverse Laplace transform. Once again, we are interested in the behavior at large $t$ and to get that we look at the behavior of $\bar{S}_{\pm}(x_0, s)$ for a small $s$. In particular, the $\lim_{s \to 0} [s \bar{S}_{\pm}]$ gives the survival probability as $t \to \infty$. For $\Delta < 0$ using $D_0(z) = e^{-z^2}$, we get

$$S_+(x_0) = 1 - \frac{\gamma - |\Delta|}{\gamma + |\Delta|} e^{\frac{|\Delta|}{\gamma + |\Delta|}},$$

$$S_-(x_0) = 1 - e^{\frac{-|\Delta|^2}{\gamma + |\Delta|}}.$$
sufficient time. To get the approach to the stationary value \( S_\pm(x_0) \) for \( \Delta < 0 \) and the decay to 0 for \( \Delta > 0 \), at large \( t \), we study the zeroes of \( \mathcal{N}_-(0, s) \). Note from equation (66) that there is a simple pole at \( s = 0 \), which provides \( S_\pm(x_0) \) for \( \Delta < 0 \) and 0 for \( \Delta > 0 \). Subtracting this part we define \( L_\pm(x_0, t) = S_\pm(x_0, t) - S_\pm(x_0, t \to \infty) \) which can be obtained from the poles of \( \check{S}_\pm(x_0, s) \) other than \( s = 0 \) on the negative \( s \) axis. These poles come from the solution \( \mathcal{N}_-(0, s) = 0 \). For large \( t \), the solution of \( \mathcal{N}_-(0, s) = 0 \) with the largest real part (say \( s^* \)) sets the timescale for the decay of \( L_\pm(x_0, t) \). We get

\[
L_\pm(x_0, t) \simeq -e^{\frac{\Delta}{2s^*}} U(x_0, s^*) \frac{\mathcal{N}_\pm(x_0, s^*)}{\mathcal{N}_\pm^2(0, s^*)},
\]

(69)

where \( s^* \) is the root of \( \mathcal{N}_-(0, s^*) = 0 \) with the largest real part. In figures 14(b) and (c), we have verified our analytic results with numerical simulation and observe an excellent match.

### 3.3. General \( \alpha \)

For this case it seems convenient to solve equation (46) directly. Making analytical progress for arbitrary \( t \) seems difficult. We instead look at the large time limit which necessarily requires \( x_0 \) to be large so that the particle survives for a long time. In this limit, the difference \( H(x_0, t) \) of survival probabilities \( U_\pm(x_0, t) \) of the particle starting with \pm velocities would decay fast (exponentially), which allows one to neglect the difference \( H(x_0, t) \). Making this approximation for large \( x_0 \) and large \( t \), as shown in appendix F, the equation for \( \check{U}(x_0, s) \) becomes

\[
s \check{U}(x_0, s) \simeq \frac{\alpha^2}{2\gamma} \partial_{x_0} \left( \frac{1}{x_0^\alpha} \partial_{x_0} \check{U} \right) - \frac{\gamma}{2} \partial_{x_0} \check{U}.
\]

(70)

where \( \check{U}(x_0, s) = \check{U}_+(x_0, s) + \check{U}_-(x_0, s) \). To solve this equation, we need to specify the boundary conditions in terms of \( \check{U}(x_0, s) \). The boundary condition at \( x_0 = 0 \) is discussed previously for \( S_\pm(x_0, t) \) which can be translated in terms of \( \check{U}_\pm(x_0, s) \) as

\[
\check{U}(0, s) = -\frac{2}{s}.
\]

(71)

Note that we have neglected the \( \check{H}(0, s) \) term in equation (71) as it decays faster than \( \check{U}(x_0, s) \). We now solve equation (70) separately for \( \Delta = 0 \) and \( \Delta \neq 0 \) cases.

#### 3.3.1. \( \Delta = 0 \)

For \( \Delta = 0 \), equation (70) reduces to

\[
s \check{U}(x_0, s) \simeq \mathcal{D}_\alpha \partial_{x_0} \left( \frac{1}{x_0^{\alpha+1}} \partial_{x_0} \check{U} \right), \quad \text{with} \quad \mathcal{D}_\alpha = \frac{\alpha^2}{2\gamma}.
\]

(72)

This equation is solved in appendix G and we here write the solution

\[
\check{U}(x_0, s) \simeq -\frac{4x_0^{1+\alpha}}{\Gamma \left( \frac{1+\alpha}{2} \right)} \frac{s^{\frac{1+\alpha}{2}}}{(2+\alpha)^{\frac{1+\alpha}{2}}} K_{\frac{1+\alpha}{2}}(2\sqrt{2}s),
\]

(73)

where \( K_\nu(z) \) is the modified Bessel function of the second kind and \( g_\alpha = \frac{x_0^{2+\alpha} \Gamma \left( \frac{1+\alpha}{2} \right)}{\mathcal{D}_\alpha (2+\alpha)^{\frac{1+\alpha}{2}}} \). To find the probability in the time domain, one has to perform the inversion of the Laplace
Laplace transform. Looking at the expression of $\bar{U}$, we use equation (25) to invert the Laplace transform.

$$S(x_0, t) = \frac{S_+(x_0, t) + S_-(x_0, t)}{2},$$

$$\approx 1 - \frac{1}{2} L^{-1}_{s-m}[\bar{U}(x_0, s)],$$

$$\approx 1 - e^{-x_0^2/(2\alpha)} \left\{ \sqrt{\Gamma(2 + \alpha)} \right\}_{\frac{1}{2}} \frac{x_0^2 t^{\frac{1}{2(2+\alpha)}}}{\Gamma(\frac{1}{2(2+\alpha)})} W_{\frac{1}{2(2+\alpha)}, \frac{1}{2(2+\alpha)}} \left( \frac{x_0^{2+\alpha}}{\alpha (2 + \alpha)^2 t} \right).$$  (74)

To find asymptotics, we use the following representation of the Whittaker function $W_{m,k}(z) = e^{-\frac{1}{2} z} z^{m+\frac{1}{2}} U(\frac{1}{2} + m - k, 1 + 2m, z)$ in terms of the Tricomi confluent hypergeometric function $U(a, b, z)$ of the second kind, whose asymptotic behavior as $z \to 0$ is $U(a, b, z) \approx \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} + \frac{\Gamma(b)}{\Gamma(a-s)} \frac{1}{(a-s+1)}$, which gives

$$S(x_0, t) \approx \frac{\left| \Gamma\left(-\frac{1+\alpha}{2+\alpha}\right) \right|}{\Gamma\left(\frac{1+\alpha}{2+\alpha}\right)} \frac{x_0^{1+\alpha}}{\Gamma\left(\frac{1+\alpha}{2+\alpha}\right)} \frac{1}{(2 + \alpha)^{\frac{1}{2(2+\alpha)}} (D_{a, b})^{\frac{1}{2(2+\alpha)}}}. $$  (75)

In figure 15(a), we have plotted our analytic result in equation (74) and compared it with the numerical simulation. We see an excellent match between them for large $x_0$. For a small $x_0$, our result in equation (74) does not match with the simulation results as this expression is not valid although the power law decay $t^{-\frac{1+\alpha}{2+\alpha}}$ is correctly predicted. This mismatch arises from the approximation $\bar{H}(x_0, s) \approx 0$ for large $t$, which is true only for large $x_0$.

### 3.3.2. $\Delta < 0$

When $\Delta < 0$, the particle drifts away from the origin with a higher probability, implying a non-zero survival probability for the particle even for an infinite $t$. To find this probability we solve the original equation (43) directly for the stationary value of the survival probability by putting $\partial_t S_\pm(x_0, t) = 0$. We present here the final
The approach to the stationary values of both $S_\pm(x_0, t \to \infty)$ and relegate the details of the derivation to appendix H. Defining $S_\pm(x_0) = S_\pm(x_0, t \to \infty)$, we get

$$S_+(x_0) = 1 - \frac{\gamma - |\Delta|}{\gamma + |\Delta|} e^{-\frac{2|\Delta|}{\gamma + |\Delta|} x_0},$$

$$S_-(x_0) = 1 - e^{-\frac{2|\Delta|}{\gamma + |\Delta|} x_0}.$$  \hfill (76) (77)

To study the approach to the steady state, we numerically find $S_\pm(x_0, t)$ and plot $L_\pm(x_0, t) = S_\pm(x_0, t) - S(x_0)$ as functions of $t$ in figure 15(b). From these plots we see that the approach to the stationary values of both $S_\pm(x_0, t)$ is exponential with the same relaxation time.

3.3.3. $\Delta > 0$. When $\Delta > 0$, the particle drifts towards the origin. Unlike in the previous case, the particle will now definitely hit the origin. In figure 15(c), we numerically find that the survival probability decays exponentially to zero.

4. Exit probability of RTP from a finite interval for general $\alpha$

In the previous sections, we have considered RTP in an infinite or semi-infinite line. This section deals with RTP in a finite interval $[0, M]$. The question that is addressed in this section is what the probability is that the RTP will exit from the side $x = 0$ (or equivalently $x = M$) for general $\alpha$. Let $E_\pm(x_0)$ denote the exit probability of the particle from side $x = 0$ starting from $x_0$ with velocity $\pm v$. Following [79], one can write a coupled backward equations for $E_\pm(x_0)$ and solve them explicitly. Below we discuss the derivation of these equations.

Consider that the RTP starts at $x_0$ with $+v$. In the small time $dt$, RTP can (i) flip its velocity with probability $R_1(x_0)\frac{dx}{v}$ and move to $x_0 - dx$ or (ii) continue to move with $+v$ velocity with probability $\left[1 - R_1(x_0)\frac{dx}{v}\right]$ and reach $x_0 + dx$. Starting from this new position, the particle then exits from $x = 0$ without touching $x = M$. One can then write for $E_\pm(x_0)$,

$$E_+(x_0) = \left[1 - R_1(x_0)\frac{dx}{v}\right] \frac{dx}{v} E_+(x_0 + dx) + R_1(x_0)\frac{dx}{v} E_-(x_0 - dx),$$

$$E_-(x_0) = \left[1 - R_2(x_0)\frac{dx}{v}\right] \frac{dx}{v} E_-(x_0 - dx) + R_2(x_0)\frac{dx}{v} E_+(x_0 + dx).$$

(78)

Performing the Taylor series expansion in $dx$ and then taking the $dx \to 0$ limit, one gets the backward equations for $E_\pm(x_0)$ which read as

$$v\partial_{x_0}E_+ - R_1(x_0)E_+ + R_1(x_0)E_- = 0,$$

$$-v\partial_{x_0}E_- + R_2(x_0)E_+ - R_2(x_0)E_- = 0.$$  \hfill (79)

Note that the rates $R_1(x_0)$ and $R_2(x_0)$ are defined in equation (2). Before solving these equations, we need to specify the boundary conditions. The boundary conditions are $E_+(x_0 = M) = 0$ and $E_-(x_0 = 0) = 1$. The first boundary condition comes from the fact that if the particle starts at $x_0 = M$ with positive velocity, it will exit from $x = M$.
Run-and-tumble particle in inhomogeneous media in one dimension

Figure 16. Comparison of the exit probability $E_{\pm}(x_0)$ given in equations (80) and (81) with the numerical simulation of the same (filled circles) for $\alpha = 0.5$. The three curves correspond to (i) green: $\gamma_1 = 2, \gamma_2 = 1$, (ii) brown: $\gamma_1 = 1, \gamma_2 = 1$ and (iii) orange: $\gamma_1 = 1, \gamma_2 = 2$. For both figures, we have chosen $v = 1$ and $l = 1$

wall in the next time-step. Likewise, the second boundary condition appears because if the particle starts from $x_0 = 0$ with $-v$, it will, in the next time-step, exit from $x = 0$. Given these boundary conditions, one can solve these coupled differential equations in equation (79) for general $\alpha$. After a straightforward but tedious calculation, we find the following final expressions for the exit probabilities:

$$E_+(x_0) = \frac{e^{\Delta M^{1+\alpha}} - e^{\Delta x_0^{1+\alpha}}}{e^{\Delta M^{1+\alpha}} - \frac{2\alpha}{\gamma_1}},$$

$$E_-(x_0) = \frac{e^{\Delta M^{1+\alpha}} - \frac{\gamma_2}{\gamma_1} e^{\Delta x_0^{1+\alpha}}}{e^{\Delta M^{1+\alpha}} - \frac{2\alpha}{\gamma_1}},$$

where $\Delta = \frac{2\Delta}{v(1+\alpha)l}$. One can, in principle, compute $E_{\pm}(x_0)$ also by integrating the current $j(0, t)$ through side $x = 0$ over all $t$. Although these two approaches yield the same result, the backward equation written in equation (79) is more illustrative and instructive, especially for general $\alpha$, where the computation of $j(x, t)$ with absorbing barriers at $x = 0$ and $x = M$ is still a theoretical challenge. We also remark that taking $\Delta \to 0$ limit in equations (80) and (81) correctly gives the results of [18] for $\alpha = 0$. In figure 16, we have plotted our results in equations (80) and (81) with the numerical simulation of the same. The match between the two is excellent. In figure 16, we note that for a given $x_0$, $E_{\pm}$ is smallest for $\Delta < 0$ and largest for $\Delta > 0$. For $\Delta > 0 (\gamma_1 > \gamma_2)$, the particle experiences an effective drift towards the origin, which enhances the chance for the particle to escape the origin. Similarly for $\Delta < 0 (\gamma_1 < \gamma_2)$ the particle drifts away from the origin.

Another interesting point to remark is that in the limit $M \to \infty$, $E_{\pm}(x_0)$ is equal to $1 - S_{\pm}(x_0)$. One can easily verify that equations (80) and (81) in this limit indeed reduce to 1 for $\Delta \geq 0$ and $1 - S_{\pm}(x_0)$ for $\Delta < 0$, where $S_{\pm}(x_0)$ are given by equations (76) and (77) respectively.

5. Conclusions

To summarize, we have studied the motion of a run-and-tumble particle in one-dimensional inhomogeneous media. The inhomogeneity was introduced by considering
the position- and direction-dependent rates of flipping given in equation (2). For $\gamma_1 > \gamma_2$, we have found that the particle reaches a stationary state in one dimension even in the absence of any external confining potential. We have obtained an exact expression of the probability distribution given in equation (33), which characterizes this non-equilibrium stationary state. The approach to this steady state is exponential for all $\alpha > 0$. While for $\alpha = 0$ and $\alpha = 1$ we have been able to compute the full distribution $P(x,t)$, which indeed shows exponential relaxation, for general $\alpha > 0$ performing an exact calculation turned out to be difficult. We have provided numerical evidence for the exponential relaxation for general $\alpha > 0$.

For $\gamma_1 \leq \gamma_2$ the RTP particle does not reach a stationary state. While for $\gamma_1 < \gamma_2$ the average absolute position $|x(t)|$ of the particle grows linearly with time, for $\gamma_1 = \gamma_2$, $\langle |x| \rangle = 0$. Note that the mean position $\langle x \rangle = 0$ in all cases. This suggests that the distribution $P(x,t)$ has two symmetric peaks moving with equal speed in the opposite direction for $\gamma_1 < \gamma_2$ and for $\gamma_1 = \gamma_2$; there is a single non-moving peak at $x = 0$. In this case for $\alpha = 0$ the distribution $P(x,t)$ was computed in [18], which was shown to be Gaussian at large $t$ with variance growing linearly with time. In this paper we have extended this result for general $\alpha > 0$ for which we have found that $\langle x^2(t) \rangle \sim t^{-\alpha}$. We also have proved that for large $t$, the distribution function $P(x,t)$ follows a scaling form $f_\alpha(y)$ with scaling variable $y = |x|/t^{1/(2+\alpha)}$ for $\gamma_1 = \gamma_2$. We have obtained an exact expression of this scaling function for $\alpha > 0$ in equation (40), which describes the distribution of the typical fluctuations of $x$ at large $t$. On the other hand, for

Table 1. Table summarizing $P(x,t)$ for various $\alpha$ and $\Delta$. Here $\sigma_\alpha^2(t) = \langle x^2(t) \rangle - \langle |x(t)| \rangle^2$ and ILT stands for inverse Laplace transform.

| $\Delta > 0$ | $\Delta = 0$ | $\Delta < 0$ |
|--------------|-------------|-------------|
| Stationary state $P'_\alpha(x)$ exists. | No stationary state, $P(x,t)$ with $\mu(t) = \langle |x(t)| \rangle = 0$. | No stationary state, $P(x,t)$ with $\mu(t) \sim t$. |
| $\alpha = 0$ | Relaxes as $e^{-\beta t}$ to $P'_{0}(x)$. | Exact expression for $P(x,t)$ given in equation (18). | Exact expression for $P(x,t)$ given in equation (18). |
| See equation (19). | $(x^2(t)) \sim t$ in equation (26). | $P(x,s)$ in equation (30). |
| $\alpha = 1$ | Relaxes as $e^{-ct}$ to $P'_{1}(x)$ with $c$ given by the solution of equation (29). | Large $t$ scaling form of $P(x,t)$ in equation (40). | $P(x,t)$ obtained from the ILT in equation (42). |
| See equation (19). | $\langle x^2(t) \rangle \sim t^{2/3}$. | $\sigma_1^2(t) \sim \log(t)$. |
| $\alpha > 0$ | Exponential relaxation verified numerically in figure 8(b). | Large $t$ scaling form of $P(x,t)$ in equation (40). | Large $t$ scaling form of $P(x,t)$ in equation (42). |

For $\alpha \leq 1$, with $\sigma_\alpha^2(t) \simeq D_\alpha t^{1-\alpha}$.
\[ S(x_0, t) \xrightarrow{t \to \infty} 0. \]

See equations (76) and (77).

| \( \Delta > 0 \) | \( \Delta = 0 \) | \( \Delta < 0 \) |
|-----------------|-----------------|-----------------|
| \( S(x_0, t) \xrightarrow{t \to \infty} 0. \) | \( S(x_0, t) \xrightarrow{t \to \infty} 0. \) | \( S(x_0, t) \xrightarrow{t \to \infty} S(x_0) \) |
| \( \alpha = 0 \) | Exact expression of \( S(x_0, t) \) given in equation (56). Decays to 0 exponentially. | Exact expression of \( S(x_0, t) \) given in equation (56). Decays to 0 as \( \sim \frac{1}{t^\gamma} \) for large \( t \). |
| \( \alpha = 1 \) | Decays to zero as \( e^{-|x'|t} \) at large \( t \); shown in equation (69). | Decays to zero as \( \frac{1}{t^\theta} \) at large \( t \); shown in equation (64). | Relaxes to \( S_{\pm}(x_0) \) exponentially. |
| \( \alpha > 0 \) | For large \( t \), decays to 0 exponentially. Verified numerically in figure 15(c). | For large \( t \), decays to 0 as \( t^{-\frac{1}{\theta \alpha}} \), see equation (75). Verified numerically in figure 15(a). | For large \( t \), decays to \( S_{\pm}(x_0) \) exponentially. Verified numerically in figure 15(b). |

\( \gamma_1 < \gamma_2 \), the distribution \( P(x, t) \) does not satisfy this scaling form. In this case we have found that the dynamics of the particle at a large time can effectively be described by an Ito–Langevin equation with a position-dependent drift and diffusion constant (note that such effective Ito–Langevin dynamics also holds for \( \gamma_1 = \gamma_2 \)). While for \( \alpha = 0 \) and \( \alpha = 1 \), it is possible to solve the master equation exactly to find \( P(x, t) \), performing the same task for general \( \alpha > 0 \) is difficult. In such cases the Ito–Langevin description is particularly useful in order to obtain the scaling behavior of distribution \( P(x, t) \) at a large time for \( \gamma_1 < \gamma_2 \) case (for which the particle drifts away from the origin). In particular, using this description we have shown that \( \langle |x(t)| \rangle \sim t \) and \( \sigma_x^2(t) = \langle x^2(t) \rangle - \langle |x(t)| \rangle^2 \sim t^{1-\alpha} \) at large \( t \) for general \( \alpha \). In addition we have shown that scaling form of the distribution \( P(x, t) \) is in fact Gaussian as also verified through direct numerical simulation of the actual RTP dynamics in equation (1).

We also have studied the survival probability of the inhomogeneous RTP dynamics on a semi-infinite line from an absorbing boundary at \( x = 0 \). For \( \gamma_1 = \gamma_2 \) the survival probability, at large \( t \), decays as a power law with a persistent exponent \( \theta \), i.e. \( S(t) \sim t^{-\theta} \). We have shown that the persistent exponent is given by \( \theta = \frac{1+\alpha}{2+\alpha} \), which generalizes the result \( \theta = 1/2 \) for \( \alpha = 0 \) derived in [18]. For \( \gamma_1 < \gamma_2 \) the particle has a non-zero probability to survive at \( t \to \infty \) as it effectively drifts away from the origin. We explicitly computed this non-zero survival probability for all \( \alpha > 0 \). On the other hand, for \( \gamma_1 > \gamma_2 \), the probability decays to zero at large \( t \). In both cases, we have found that the approach to the value at \( t \to \infty \) is exponential. We have also looked at the exit probabilities of the RTP from a finite interval. Finally, we provide a summary of the results presented in the paper in tables 1 and 2.

We note that in this paper we have focused on the range \( \alpha \geq 0 \). However, we find that some of our results remain valid for \( \alpha < 0 \). For example the steady-state distribution in...
case of $\gamma_1 > \gamma_2$, as given in equation (33), is also valid for $-1 < \alpha < 0$. In figure 8(a) we show a numerical verification of this fact for $\alpha = -0.3$. The scaling form $f_\alpha(y)$ of the probability distribution $P(x,t)$ for $\gamma_1 = \gamma_2$ is given in equation (40). Quite remarkably it turns out that this scaling distribution holds true for $-2 < \alpha < 0$ also, which we have verified numerically (not shown here) as well. While these results remain valid for $\alpha < 0$, many results, for example the scaling form of $P(x,t)$ given in equation (42) for $\gamma_1 < \gamma_2$, are not valid for $\alpha < 0$. Extending these results for $\alpha < 0$ remains an interesting future direction. All our results are valid in one dimension. Extending our results to a higher dimension would be interesting to explore. Recently it was shown that the survival probability for RTP in $d-$dimension has some universal features [46]. It would be interesting to see what happens to the universality when rates become position dependent. It would also be interesting to study the situation in which the rates $R_1, R_2$ become time dependent where the time dependence may come from the coupling of the RTP motion to the evolution of the inhomogeneous media. Another interesting future direction would be to see how the effective velocity and diffusion constant of an RTP moving in a force field as done in [75] changes in the presence of space-dependent rates. Finally, in order to study more realistic situations where individual active agents like bacteria, micro-robots or Janus particles interact among themselves, one needs to look at interacting particle dynamics, which is another important future direction.

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Appendix A. Derivation of $G(x,s)$ and $\bar{P}(x,s)$ for $\alpha = 0$

In this appendix, we solve equation (15) explicitly to get $G(x,s)$. We will then insert this solution in equations (10) and (13) to get $\bar{Q}(x,s)$ and $\bar{P}(x,s)$. Turning to equation (15), it is straightforward to solve, and using the boundary conditions $G(x \to \pm \infty, s) = 0$, one gets

$$G(x,s) = \begin{cases} A_+ e^{-\lambda(s)x}, & \text{if } x > 0 \\ A_- e^{\lambda(s)x}, & \text{if } x < 0 \end{cases}$$  \tag{A.1}$$

where $A_{\pm}$ are position-independent constants. Inserting this solution in equation (13), we get $\bar{Q}(x,s)$, which can again be substituted in equation (10) to get $\bar{P}(x,s)$.

$$\bar{Q}(x,s) = \begin{cases} A_+ e^{-\lambda(s)+\Delta v)x}, & \text{if } x > 0 \\ A_- e^{\lambda(s)+\Delta v)x}, & \text{if } x < 0 \end{cases}$$  \tag{A.2}$$

https://doi.org/10.1088/1742-5468/aba7b1
\[ \bar{P}(x, s) = \frac{v}{s} \left( \lambda(s) + \frac{\Delta}{v} \right) \begin{cases} A_+ e^{-(\lambda(s)+\frac{\Delta}{v})x}, & \text{if } x > 0 \\ -A_- e^{(\lambda(s)+\frac{\Delta}{v})x}, & \text{if } x < 0 \end{cases} \tag{A.3} \]

The task now is to evaluate the constants \( A_\pm \) which demands two conditions. One condition comes by integrating equation (10) from \(-\epsilon\) to \(+\epsilon\) and taking \( \epsilon \to 0 \). This will result in the following discontinuity equation:

\[ \bar{Q}(x \to 0^+, s) - \bar{Q}(x \to 0^-, s) = \frac{1}{v} \tag{A.4} \]

The other condition comes by noting that for symmetric initial condition, the probability distribution \( P(x, t) \) is symmetric about \( x = 0 \) which gives

\[ \bar{P}(x \to 0^+, s) = \bar{P}(x \to 0^-, s) \tag{A.5} \]

Inserting the solutions of \( Q(x, s) \) and \( P(x, s) \) in equations (A.4) and (A.5), we get two linear equations for \( A_+ \) and \( A_- \), solving which one finally gets complete expressions for \( P(x, s) \) as

\[ \bar{P}(x, s) = \frac{1}{2s} \left( \lambda(s) + \frac{\Delta}{v} \right) e^{-(\lambda(s)+\frac{\Delta}{v})|x|}. \tag{A.6} \]

This expression for \( \bar{P}(x, s) \) is also written in equation (16).

Appendix B. Derivation of \( P(x, t) \) for \( \alpha = 0 \) in equation (18)

In this appendix we will derive the expression for \( P(x, t) \) for \( \alpha = 0 \) as written in equation (18). As we will see later, in order to prove this result the following inverse Laplace transform will be useful:

\[ L_{s \to t} \left[ e^{-\lambda(s)y} \right] = -\frac{1}{v} \frac{d}{dy} \left[ e^{-\gamma t} I_0 \left( \sqrt{\gamma_1 \gamma_2 \left( t^2 - \frac{y^2}{v^2} \right)} \right) \Theta(vt - y) \right], \tag{B.1} \]

where

\[ \lambda(s) = \frac{1}{v} \sqrt{\Delta^2 + 2\gamma s + s^2}. \tag{B.2} \]

So we first provide a derivation of this equation and then provide the derivation of equation (18).

B.1. Derivation of equation (B.1)

We begin with the following inversion:

\[ L_{s \to t} \left[ \frac{e^{-\lambda(s)y}}{\lambda(s)} \right] = \frac{1}{2\pi i} \int_{-\infty}^{\infty} ds \ e^{st} \frac{e^{-\lambda(s)y}}{\lambda(s)}. \tag{B.3} \]
Proceeding further, we can rewrite $\lambda(s)$ as

$$
\lambda(s) = \frac{1}{v} \sqrt{(s + \gamma + \sqrt{\gamma_1 \gamma_2})(s + \gamma - \sqrt{\gamma_1 \gamma_2})}.
$$

(B.4)

Substituting this in equation (B.3), we get

$$
L_{s \rightarrow t} \left[ \frac{e^{-\lambda(s)y}}{\lambda(s)} \right] = \frac{v}{2\pi i} \int_{-\infty}^{\infty} ds \ e^{st} \frac{e^{-\frac{s}{2} \sqrt{(s + \gamma + \sqrt{\gamma_1 \gamma_2})(s + \gamma - \sqrt{\gamma_1 \gamma_2})}}}{\sqrt{(s + \gamma + \sqrt{\gamma_1 \gamma_2})(s + \gamma - \sqrt{\gamma_1 \gamma_2})}}.
$$

(B.5)

Changing the variable $s + \gamma - \sqrt{\gamma_1 \gamma_2} = \sqrt{\gamma_1 \gamma_2} z$ in equation (B.5), we have

$$
L_{s \rightarrow t} \left[ \frac{e^{-\lambda(s)y}}{\lambda(s)} \right] = \frac{v}{2\pi i} e^{-(\gamma - \sqrt{\gamma_1 \gamma_2})t} \int_{-\infty}^{\infty} dz \ e^{zt\sqrt{\gamma_1 \gamma_2}} \frac{e^{-\frac{v}{2\sqrt{\gamma_1 \gamma_2}} \sqrt{z(2+z)}}}{\sqrt{z(2+z)}},
$$

(B.6)

The inverse Laplace transform in the right-hand side has been obtained in [59] (see equation (C2) there). Using this, we obtain

$$
L_{s \rightarrow t} \left[ \frac{e^{-\lambda(s)y}}{\lambda(s)} \right] = ve^{-\gamma t}I_0 \left( \sqrt{\frac{\gamma_1 \gamma_2}{2}} \left( t^2 - \frac{y^2}{v^2} \right) \right) \theta(\gamma t - y),
$$

(B.7)

where $I_0$ is the modified Bessel function of first kind and $\theta(x)$ is the Heaviside step function. To prove (B.1), we note that

$$
L_{s \rightarrow t} \left[ \frac{e^{-\lambda(s)y}}{\lambda(s)} \right] = -\frac{d}{dy} \left[ L_{s \rightarrow t} \left( \frac{e^{-\lambda(s)y}}{\lambda(s)} \right) \right].
$$

(B.8)

Substituting equation (B.7) in (B.8), we establish the equality in (B.1).

**B.2. Derivation of $P(x, t)$ in equation (16)**

Let us begin by rewriting $\bar{P}(x, s)$ in equation (16) as

$$
\bar{P}(x, s) = -\frac{1}{2s} \frac{d}{du} \left[ e^{-\left(\lambda(s)+\frac{u}{s}\right)x} \right]_{u=|x|}.
$$

(B.9)

The inverse Laplace transform $P(x, t)$ read as

$$
P(x, t) = -\frac{1}{2} \frac{d}{du} \left[ e^{-\frac{u}{s}x} L_{s \rightarrow t} \left( \frac{e^{-\lambda(s)x}}{s} \right) \right]_{u=|x|},
$$

$$
= -\frac{1}{2} \frac{d}{du} \left[ e^{-\frac{u}{s}x} \int_0^t d\tau \ L_{s \rightarrow t} \left( \frac{1}{s} \right) L_{s \rightarrow t} \left( e^{-\lambda(s)x} \right) \right]_{u=|x|}
$$

$$
= -\frac{1}{2} \frac{d}{du} \left[ e^{-\frac{u}{s}x} \int_0^t d\tau \ L_{s \rightarrow t} \left( \frac{e^{-\lambda(s)x}}{s} \right) \right]_{u=|x|}.
$$

(B.10)
In going from the first line to the second line, we have used the convolution property of the Laplace transform. Inserting equation (B.1) in (B.10), one recovers the expression of \( P(x, t) \) written in equation (18).

**Appendix C. Derivation of the approximate expression of \( P(x, t) \) given in equation (19) for \( \alpha = 0 \) at large \( t \)**

**C.1. \( \Delta \geq 0 \)**

Here we will provide a derivation of the large \( t \) behavior of \( P(x, t) \) for \( \Delta > 0 \) as written in equation (19). We begin with the exact expression of \( P(x, t) \) in equation (18). We first rewrite equation (18) by replacing the time integral by \( \int_0^t = \int_0^\infty - \int_t^\infty \) as

\[
P(x, t) = \frac{1}{2} e^{-\gamma t} \delta(|x| - vt) + \frac{\gamma_1}{2v} \left( 1 + \frac{\gamma_2 |x|}{2v} \right) e^{-\Delta \pi z}\Theta(v t - |x|)
- \frac{\sqrt{\gamma_1 \gamma_2}}{2v} \int_0^\infty d\tau \ e^{-\gamma \tau} \frac{dI(u, \tau)}{du} \Theta(v \tau - |x|)
+ \frac{\sqrt{\gamma_1 \gamma_2}}{2v} \int_t^\infty d\tau \ e^{-\gamma \tau} \frac{dI(u, \tau)}{du} \Theta(v \tau - |x|).
\]  

(C.1)

We note that as \( t \to \infty \), \( P(x, t) \) goes to the stationary distribution \( P_0^t(x) \). Also at large \( t \), the coefficient of \( \delta \) functions becomes very small. Hence for large \( t \), we get

\[
P(x, t) - P_0^t(x) \simeq \frac{\sqrt{\gamma_1 \gamma_2}}{2v} \int_t^\infty d\tau \ e^{-\gamma \tau} \frac{dI(u, \tau)}{du} \theta_{|x|}.
\]  

(C.2)

Note that \( I(u, \tau) = \frac{u^{-\frac{1}{2}} I_1 \left( \sqrt{\gamma_1 \gamma_2 (\tau - \frac{\pi z}{v})} \right)}{\sqrt{\tau - \frac{\pi z}{v}}}, \) with \( I_1 \) being the modified Bessel function of first kind. One can easily perform the differentiation \( \frac{dI(u, \tau)}{du} \) in equation (C.2). Also for large \( t \), \( \tau \) is also very large, which means we can use the large \( \tau \) form of \( \frac{dI(u, \tau)}{du} \). It turns out that we need to make use of the asymptotic form of \( I_v(z) \simeq \frac{e^z}{\sqrt{2\pi z}} \) for large \( z \). For \( vt \gg |x| \) along with the aforementioned approximations, we change the variable \( \tau = tw \) to get

\[
P(x, t) - P_0^t(x) \simeq C_1 \int_1^\infty dw \frac{e^{-\frac{(C_2 w + C_3)}{w^{3/2}}}}{w^{3/2}} \left( 1 - \frac{\Delta |x|}{v} - \frac{x^2 \sqrt{\gamma_1 \gamma_2}}{v^t w} \right),
\]  

(C.3)

where \( C_1 = \frac{(\gamma_1 \gamma_2)^{1/4}}{2\pi v^{3/2}} e^{\frac{\Delta |x|}{v}} \), \( C_2 = \gamma - \sqrt{\gamma_1 \gamma_2} \) and \( C_3 = \frac{x^2 \sqrt{\gamma_1 \gamma_2}}{2v^{3/2}} \). Interestingly, this integration can be performed exactly. First we write

\[
\int_1^\infty dw \frac{e^{-\frac{(C_2 w + C_3)}{w^{3/2}}}}{w^{3/2}} = \int_0^\infty dw \frac{e^{-\frac{(C_2 w + C_3)}{w^{3/2}}}}{w^{3/2}} - \int_0^1 dw \frac{e^{-\frac{(C_2 w + C_3)}{w^{3/2}}}}{w^{3/2}},
\]  

(C.4)

\[
= \sqrt{\frac{\pi}{C_3 t}} e^{-2t \sqrt{C_2 C_3}} - Z_{3/2},
\]  

(C.5)
where \( Z_\mu \) is defined as
\[
Z_\mu = \int_0^1 dw \frac{e^{-t(C_2w + \frac{w}{2})}}{w^\mu}.
\] (C.6)

Similarly,
\[
\int_1^\infty dw \frac{e^{-t(C_2w + \frac{w}{2})}}{w^{5/2}} = \frac{1}{2} \sqrt{\frac{\pi}{(C_3t)^3}} \left( 1 + 2t \sqrt{C_2C_3} \right) e^{-2t\sqrt{C_2C_3}} - Z_{5/2}.
\] (C.7)

Let us start by evaluating \( Z_{1/2} \) which can be easily shown to be
\[
Z_{1/2} = \frac{1}{2} \sqrt{\frac{\pi}{C_2t}} e^{-2t\sqrt{C_2C_3}} \left[ 1 + \text{Erf} \left( \sqrt{tC_2} - \sqrt{tC_3} \right) - e^{4t\sqrt{C_2C_3}} \text{Erfc} \left( \sqrt{tC_2} + \sqrt{tC_3} \right) \right].
\] (C.8)

The integrals \( Z_{3/2} \) and \( Z_{5/2} \) can now be obtained from \( Z_{3/2} = -\frac{d}{dt} Z_{1/2} \) and \( Z_{5/2} = -\frac{d}{dt} Z_{3/2} \). We get
\[
Z_{3/2} = \frac{1}{2} \sqrt{\frac{\pi}{C_3t}} e^{-2t\sqrt{C_2C_3}} \left[ 2 - \text{Erfc} \left( \sqrt{tC_2} - \sqrt{tC_3} \right) + e^{4t\sqrt{C_2C_3}} \text{Erfc} \left( \sqrt{tC_2} + \sqrt{tC_3} \right) \right],
\]
\[
Z_{5/2} = \frac{1}{2} \sqrt{\frac{\pi}{(C_3t)^3}} e^{-2t\sqrt{C_2C_3}} \left[ 1 + 2t \sqrt{C_2C_3} - \frac{1}{2} \left( 1 + 2t \sqrt{C_2C_3} \right) \text{Erfc} \left( \sqrt{tC_2} - \sqrt{tC_3} \right) 
\]
\[+ \frac{1}{2} \left( 1 - 2t \sqrt{C_2C_3} \right) e^{4t\sqrt{C_2C_3}} \text{Erfc} \left( \sqrt{tC_2} + \sqrt{tC_3} \right) \right] + \frac{1}{C_3t} e^{-t(C_2 + C_3)}. \] (C.9)

Using these explicit expressions of \( Z_{3/2} \) and \( Z_{5/2} \), one gets an explicit expression of the integrals in equations (C.5) and (C.7). Finally, substituting these results in equation (C.3) we get the result written in equation (19) valid for large \( t \). This technique can also be used to evaluate the asymptotic form for \( \gamma_1 = \gamma_2 \).

**C.2. \( \Delta < 0 \)**

Here we will provide a derivation of the large \( t \) behavior of \( P(x, t) \) for the \( \Delta \leq 0 \) case using saddle point approximation. Using the expression of \( P(x, s) \) in equation (16), we write \( P(x, t) \) as the Bromwich integral as
\[
P(x, t) = \frac{e^{-t\bar{x}}}{{2\pi i} \int_{\gamma_0 - i\infty}^{\gamma_0 + i\infty} ds \left( \frac{\lambda(s) + \Delta}{2s} \right) e^{s\phi(s)},
\] (C.10)

where \( \bar{x} = \frac{\lambda}{v}, \lambda(s) = \frac{1}{v} \sqrt{\Delta^2 + 2\gamma s + s^2} \) and \( \phi(s) \) is given by
\[
\phi(s) = s - \bar{x}v\lambda(s).
\] (C.11)

From equation (C.10), we see that for \( t \to \infty \) the integral will be dominated by the saddle point of \( \phi(s) \). The saddle points are given by the solution of \( \frac{d\phi}{ds} = 0 \). It may
appear that this equation has two solutions, $s_\pm$

$$s_\pm = -\gamma \pm \sqrt{\frac{\gamma^2 - \Delta^2}{1 - \bar{x}^2}}, \quad (C.12)$$

however, only $s_+$ satisfies $\frac{d \phi}{ds} = 0$. Expanding $\phi(s)$ about $s_+$ and substituting in equation (C.10), we get

$$P(x, t) \simeq e^{t[\phi(s_+) - \Delta \bar{x}]} \frac{\lambda(s_+) + \Delta}{2s_+} \int_{\gamma_0 - i\infty}^{\gamma_0 + i\infty} ds \ e^{\phi'(s_+)(s-s_+)^2},$$

$$\simeq \frac{e^{i\phi(s_+)} - \Delta \bar{x}}{2\pi} \frac{\lambda(s_+) + \Delta}{2s_+} \int_{-\infty+i(s_+ + \gamma_0)}^{\infty+i(s_+ + \gamma_0)} dy \ e^{-\frac{\phi'(s_+)}{2}y^2}$$

$$\simeq \frac{e^{i\phi(s_+)} - \Delta \bar{x}}{2\pi} \frac{\lambda(s_+) + \Delta}{2s_+} \int_{-\infty}^{\infty} dz \ e^{-\frac{\phi'(s_+)}{2}z^2}, \quad (C.13)$$

In going from the first to the second line, we have changed the integration from the complex domain to the real line by substituting $s - s_+ = iy$, and from the second to third line we have used the fact that $\phi''(s_+) = \frac{d \phi}{ds}|_{s_+} = \frac{(1-\bar{x})^{3/2}}{\gamma\sqrt{\gamma^2 - \Delta^2}}$ is greater than zero. This also implies that the integral in equation (C.13) is always convergent. Performing the integration gives the asymptotic behavior of $P(x, t)$ as written in equation (19).

Appendix D. Derivation of $G(x, s)$ and $\bar{P}(x, s)$ for $\alpha = 1$

In this appendix, we will provide the solution of $G(x, s)$ for $\alpha = 1$ as given in equation (20). We consider the $\Delta = 0$ and $\Delta \neq 0$ cases separately.

D.1. Case I: $\Delta = 0$

We make the change of variable $x = \left(\frac{y^2}{2\gamma\bar{x}}\right)^{1/3} y$ and then write equation (21) in terms of $y$ as

$$\partial_y^2 G - \left(|y| + d_0 s^\pm\right) G = 0, \quad (D.1)$$

where $d_0 = \left(\frac{1}{2\gamma\bar{x}}\right)^{1/3}$. Solving this equation gives $G(y, s)$ in terms of Airy functions $\text{Ai} \left(|y| + d_0 s^\pm\right)$ and $\text{Bi} \left(|y| + d_0 s^\pm\right)$. However, $\text{Bi}(|y| \to \infty)$ diverges while $G(y, s)$ should remain finite. We thus get

$$G(y, s) = \begin{cases} C_+ \text{Ai} \left(y + d_0 s^\pm\right), & \text{if } y > 0 \\ C_- \text{Ai} \left(-y + d_0 s^\pm\right), & \text{if } y < 0 \end{cases} \quad (D.2)$$
Run-and-tumble particle in inhomogeneous media in one dimension

Using this expression in equation (13) one gets $\tilde{Q}(x, s)$, inserting which in equation (10), one finds

$$
P(y, s) = - \frac{v}{s} \frac{d}{dx} \begin{cases} 
\frac{C_+ \text{ Ai} \left( \frac{s^{1/3}x}{\mathcal{D}_1^{1/3}} + d_0 s^{4/3} \right)}{\mathcal{D}_1^{1/3}}, & \text{if } x > 0 \\
\frac{C_- \text{ Ai} \left( -\frac{s^{1/3}x}{\mathcal{D}_1^{1/3}} + d_0 s^{4/3} \right)}{\mathcal{D}_1^{1/3}}, & \text{if } x < 0
\end{cases}
$$

where $\mathcal{D}_1 = \frac{\sqrt{i}}{2s}$. Next we evaluate the constants $C_+$ and $C_-$. Integrating equation (10) from $-\epsilon$ to $\epsilon$ and taking the $\epsilon \to 0$ limit, one gets a discontinuity equation in $G(x, s)$. Also for the symmetric initial condition, the probability distribution will be symmetric about $x = 0$, which means $P(x, s)$ is continuous about $x = 0$. Therefore, one finally gets

$$
G(x \to 0^+, s) - G(x \to 0^-, s) = \frac{1}{v}, \\
P(x \to 0^+, s) = P(x \to 0^-, s),
$$

These equations give rise to two linear equations for $C_\pm$ which can be easily solved to get them to be a function of $s$. Next, inserting $C_\pm(s)$ in equation (D.2), one finally gets equation (22) for $\bar{P}(x, s)$.

### D.2. Identities

For $k$ and $\mu$ real, we use the following identity between the Whittaker function $W_{k,\mu}(z)$ and the Tricomi confluent hypergeometric function $U(a, b, z)$:

$$
W_{k,\mu}(y) = e^{-y/2} y^{m+1/2} U(1/2 + \mu - k ; 1 + 2\mu ; y).
$$

This implies

$$
W_{\frac{1}{2}+\alpha, \frac{1}{2}+\alpha}(y) = e^{-y/2} y^\frac{1+\alpha}{2} U \left( \frac{1+\alpha}{2+\alpha} ; 1 \right) U \left( \frac{1+\alpha}{2+\alpha} ; y \right)
$$

$$
= e^{-y/2} y^\frac{1}{2+\alpha}, \quad \text{because, } U(b ; 1 + b ; y) = y^{-b}, \quad \text{for real } b.
$$

### D.3. Case II: $\Delta \neq 0$

At first, we make the transformation $\tilde{z} = \frac{|\Delta|}{\beta} x + \text{sgn}(x) \frac{\sqrt{i}}{|\Delta|} \sqrt{\frac{l}{\beta |\Delta|}}$, which reduces equation (20) to

$$
\partial^2_\tilde{z} G - \left( \frac{z^2}{4} - \beta s^2 + \frac{\text{sgn}(\Delta)}{2} \right) G = 0
$$

where $\beta = \frac{l(\alpha^2 - \Delta^2)}{2|\Delta|}$. This equation is a standard one in the literature and whose solution is given by parabolic cylinder functions $D_{\beta s^2 - \frac{1+\alpha}{2} + \frac{\text{sgn}(\Delta)}}(\pm z)$. Recalling that $D_{\mu}(-|z|)$ diverges

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for some $\mu(s)$ as $z \to \pm \infty$, one gets the solution for $G(x, s)$ as

$$
G(x, s) = \begin{cases}
B_+ D_{\beta s^2 - \frac{1+\text{sgn}(\Delta)}{2}} \left( \sqrt{\frac{2|\Delta|}{vl}} \left( x + \frac{\gamma s l}{\Delta^2} \right) \right), & \text{if } x > 0 \\
B_- D_{\beta s^2 - \frac{1+\text{sgn}(\Delta)}{2}} \left( \sqrt{\frac{2|\Delta|}{vl}} \left( -x + \frac{\gamma s l}{\Delta^2} \right) \right), & \text{if } x < 0
\end{cases}
$$

where $B_\pm$ are the position-independent constants. Next we use equation (13) to write $\bar{Q}(x, s)$ and equation (10) to write $\bar{P}(x, s)$ as

$$
\bar{Q}(x, s) = e^{-\frac{\Delta x^2}{2}} \begin{cases}
B_+ D_{\beta s^2 - \frac{1+\text{sgn}(\Delta)}{2}} \left( \sqrt{\frac{2|\Delta|}{vl}} \left( x + \frac{\gamma s l}{\Delta^2} \right) \right), & \text{if } x > 0 \\
B_- D_{\beta s^2 - \frac{1+\text{sgn}(\Delta)}{2}} \left( \sqrt{\frac{2|\Delta|}{vl}} \left( -x + \frac{\gamma s l}{\Delta^2} \right) \right), & \text{if } x < 0
\end{cases}
$$

$$
\bar{P}(x, s) = -\frac{1}{2s} \begin{cases}
B_+ \frac{d}{dx} \left[ e^{-\frac{\Delta x^2}{2}} D_{\beta s^2 - \frac{1+\text{sgn}(\Delta)}{2}} \left( \sqrt{\frac{2|\Delta|}{vl}} \left( x + \frac{\gamma s l}{\Delta^2} \right) \right) \right], & \text{if } x > 0 \\
B_- \frac{d}{dx} \left[ e^{-\frac{\Delta x^2}{2}} D_{\beta s^2 - \frac{1+\text{sgn}(\Delta)}{2}} \left( \sqrt{\frac{2|\Delta|}{vl}} \left( -x + \frac{\gamma s l}{\Delta^2} \right) \right) \right], & \text{if } x < 0
\end{cases}
$$

To evaluate constants $B_\pm$ we need two conditions. The first one comes by integrating equation (10) from $-\epsilon$ to $+\epsilon$ and taking the $\epsilon \to 0$ limit, which gives the discontinuity equation for $\bar{Q}(x, s)$. The other condition comes by noting that for the symmetric initial condition, $\bar{P}(x, s)$ is symmetric about $x = 0$. These two conditions can be summarized as

$$
\bar{Q}(x \to 0^+, s) - \bar{Q}(x \to 0^-, s) = \frac{1}{v}, \quad \bar{P}(x \to 0^+, s) = \bar{P}(x \to 0^-, s), \quad \text{(D.12)}
$$

and they give rise to two linear equations for $B_+$ and $B_-$, solving which we get $B_\pm$ as a function of $s$. Inserting the solution for $B_\pm(s)$ in equation (D.10), we get the final expression for $\bar{P}(x, s)$ which is written in equation (27).

**Appendix E. Derivation of the asymptotic forms of $S_\pm(x_0, t)$ for $\alpha = 0$**

Here we provide the derivation of $S_\pm(x_0, t)$ for large $t$ for $\alpha = 0$ as given in equations (58) and (59). Let us begin with the exact expression of $S_-(x_0, t)$ in equation (56). In this expression, writing $\int_0^t = \int_0^\infty - \int_t^\infty$, one gets

https://doi.org/10.1088/1742-5468/aba7b1
from equation (56). It is worth remarking that the analysis so far has assumed that 

$$
\int_{t}^{\infty} d\tau e^{-\gamma \tau} I_0 \left( \sqrt{\gamma_1 \gamma_2} \left( \tau^2 - \frac{x_0^2}{v^2} \right) \right) \Theta(v\tau - x_0)
$$

This integration can be now easily performed as

$$
S_-(x_0, t) = 1 + ve^{\Delta_0} \frac{d}{dx_0} \left[ \int_{0}^{\infty} d\tau e^{-\gamma \tau} I_0 \left( \sqrt{\gamma_1 \gamma_2} \left( \tau^2 - \frac{x_0^2}{v^2} \right) \right) \Theta(v\tau - x_0) \right]
$$

Note that in the limit \( t \to \infty \), the third term becomes zero. Hence we can rewrite equation (E.1) as

$$
S_-(x_0, t) = S_-(x_0, t \to \infty) - ve^{\Delta_0} \frac{d}{dx_0} \left[ \int_{0}^{\infty} d\tau e^{-\gamma \tau} I_0 \left( \sqrt{\gamma_1 \gamma_2} \left( \tau^2 - \frac{x_0^2}{v^2} \right) \right) \Theta(v\tau - x_0) \right],
$$

where \( S_-(x_0, t \to \infty) \) is 0 for \( \Delta > 0 \) and \( S_-(x_0) \) in equation (57) for \( \Delta < 0 \). The Heaviside step function inside integration becomes redundant when \( vt > x_0 \). Performing the differentiation over \( x_0 \) and changing the variable \( \tau = ut \), we get

$$
L_-(x_0, t) = \frac{x_0}{v} \sqrt{\gamma_1 \gamma_2} e^{\Delta_0} \int_{0}^{\infty} du \frac{e^{-\gamma ut}}{\sqrt{u^2 - \frac{x_0^2}{v^2}}} I_1 \left( \sqrt{\gamma_1 \gamma_2} \left( u^2 t^2 - \frac{x_0^2}{v^2} \right) \right),
$$

where \( L_-(x_0, t) = S_-(x_0, t) - S_-(x_0, t \to \infty) \). Note that in the integration, \( u \) is greater than or equal to 1. Hence for large \( t \), we can use the asymptotic form of the modified Bessel functions \( I_\nu(z) \simeq \frac{e^z}{\sqrt{2\pi z}} \) for large \( z \). Therefore for large \( t \), we have

$$
L_-(x_0, t) \simeq e^{\Delta_0} \frac{x_0}{v} \left( \gamma_1 \gamma_2 \right)^{\frac{1}{2}} \sqrt{\pi \tau} \int_{1}^{\infty} \frac{du}{\sqrt{u^2 - \frac{x_0^2}{v^2}}} e^{-u(\gamma - \sqrt{\gamma_1 \gamma_2})}.
$$

This integration can be now easily performed as

$$
\int_{1}^{\infty} \frac{du}{u^{3/2}} e^{-uy} = 2e^{-yt} - 2\sqrt{\pi yt} \text{Erfc} \left[ \sqrt{yt} \right],
$$

$$
\simeq e^{-yt} \frac{1}{yt}.
$$

In going from the first line to the second line we have used that for large \( t \), \( \text{Erfc} \left[ \sqrt{yt} \right] \simeq e^{-yt} \left( \frac{1}{\sqrt{\pi yt}} - \frac{1}{2\sqrt{\pi yt}^3} \right) \). Substituting equation (E.5) in (E.4), one recovers the expressions in equation (59). Proceeding similarly for \( S_+(x_0, t) \) one gets equation (58) for large \( t \) from equation (56). It is worth remarking that the analysis so far has assumed that \( \gamma_1 \neq \gamma_2 \). However, for \( \gamma_1 = \gamma_2 \) also, the same technique gives the correct asymptotic form as obtained in [18, 59] and written in equations (58) and (59).
Run-and-tumble particle in inhomogeneous media in one dimension

Appendix F. Effective equation for survival probability for general $\alpha$ at large $t$ and large $x_0$

In this appendix, we will derive effective differential equations for $\bar{U}(x_0,s)$ which is related to $\bar{S}_\pm(x_0,s)$ using equations (45) and (47). We start with equation (46) for $\bar{U}_\pm(x_0,s)$:

\[
[-v\partial x_0 + R_1(x_0) + s] \bar{U}_+ = R_1(x_0) \bar{U}_-,
\]

\[
[v \partial x_0 + R_2(x_0) + s] \bar{U}_- = R_2(x_0) \bar{U}_+,
\]

where $R_1(x_0)$ and $R_2(x_0)$ are defined in equation (2). One may rewrite them as

\[
\frac{1}{x^{\alpha}} \mathcal{O}_+ \bar{U}_+ = \gamma_1 \bar{U}_-, \quad \text{with} \quad \mathcal{O}_+ = -v \partial_x + \gamma_1 \frac{x^{\alpha}}{l^{\alpha}} + s,
\]

\[
\frac{1}{x^{\alpha}} \mathcal{O}_- \bar{U}_- = \gamma_2 \bar{U}_+, \quad \text{with} \quad \mathcal{O}_- = v \partial_x + \gamma_2 \frac{x^{\alpha}}{l^{\alpha}} + s.
\]

Operating both sides of equation (F.2) by $\mathcal{O}_-$ and equation (F.3) by $\mathcal{O}_+$, we can decouple these two equations and simplify them further to get

\[
-v^2 t^{\alpha} \partial_x \left( x_0^{-\alpha} \partial_{x_0} \bar{U}_+ \right) + v(\gamma_1 - \gamma_2) \partial_{x_0} \bar{U}_+ - \left[ \frac{-v s t^{\alpha}}{x_0^{\alpha+1}} + \frac{s^2 t^{\alpha}}{x_0^{\alpha}} + (\gamma_1 + \gamma_2) s \right] \bar{U}_+ = 0, \quad (F.4)
\]

\[
-v^2 t^{\alpha} \partial_x \left( x_0^{-\alpha} \partial_{x_0} \bar{U}_- \right) + v(\gamma_1 - \gamma_2) \partial_{x_0} \bar{U}_- - \left[ \frac{-v s t^{\alpha}}{x_0^{\alpha+1}} + \frac{s^2 t^{\alpha}}{x_0^{\alpha}} + (\gamma_1 + \gamma_2) s \right] \bar{U}_- = 0. \quad (F.5)
\]

Adding these two equations and recalling the definition in equation (47), one gets

\[
-v^2 t^{\alpha} \partial_x \left( \frac{1}{x_0^{\alpha}} \partial_{x_0} \bar{U} \right) + 2v \Delta \partial_{x_0} \bar{U} + \left( 2\gamma s + \frac{s^2 t^{\alpha}}{x_0^{\alpha}} \right) \bar{U} - \frac{\alpha v s t^{\alpha}}{x_0^{\alpha+1}} \bar{H} = 0 \quad (F.6)
\]

For large $t$ (equivalently small $s$) behavior, we neglect the term having $O(s^2)$ coefficient. Also for large $x_0$, we neglect the term containing $\bar{H}$ because this is sub-leading with respect to $x_0^{-\alpha} \partial_{x_0} \bar{U}$, which itself is of order $\bar{H}$ (see equations (48) and (49)). The final equation thus reduces to equation (70) of the main text.

Appendix G. Solution of effective equation for survival probability for general $\alpha$ and $\Delta = 0$

Here we provide a solution of equation (72) for general $\alpha$. Changing variable $y = \frac{x_0}{2+s}$ and writing equation (72) in terms of $y$, we get

\[
y \partial_y^2 \bar{U} - \frac{\alpha}{2 + \alpha} \partial_y \bar{U} = \frac{4sy}{2^{\alpha}(2 + \alpha)^2} \bar{U}. \quad (G.1)
\]

The solutions of this equation are given in terms of the modified Bessel functions of the first kind and second kind: $y^{\frac{\alpha}{2(2+\alpha)}} I_{\frac{\alpha}{2(2+\alpha)}} \left( \frac{2}{2+\alpha} \sqrt{\frac{4}{2\alpha}} y \right)$ and $y^{\frac{\alpha}{2(2+\alpha)}} K_{\frac{\alpha}{2(2+\alpha)}} \left( \frac{2}{2+\alpha} \sqrt{\frac{4}{2\alpha}} y \right)$. However, the

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first solution diverges as $x_0 \to \infty$, which leaves us with only the second solution. Writing the solution in terms of $x_0$, we get

\[
\bar{U}(x_0, s) = B \frac{1^{1+\alpha}}{2^{1+\alpha}} \sqrt{s} \sqrt{2^{1+\alpha}}.
\]  

(G.2)

To get $\bar{H}(x_0, s)$ we add equation (F.1), which gives

\[
\bar{H}(x_0, s) = \frac{1}{s} \gamma_{x_0} \frac{d}{dB} \bar{U}.
\]  

(G.3)

To evaluate the constant $B$, we use the other boundary condition $S(0, s) = -\frac{1}{s}$, which gives $\bar{U}(0, s) = 0$ and non-zero $\bar{H}(0, s)$. Finally substituting this in equation (G.4) gives $B(s)$ and the expression for $\bar{U}(x_0, s)$, which is written in equation (73).

**Appendix H.** $S_\pm(x_0, t)$ as $t \to \infty$ for general $\alpha$ and $\Delta < 0$

In this appendix, we will solve equation (43) for general $\alpha$ and $\Delta < 0$ in the limit $t \to \infty$. For $\Delta < 0$, the particle effectively drifts away from the absorbing wall at the origin which gives rise to non-zero $S_\pm$ as $t \to \infty$. For this case equation (43) can be rewritten as

\[
\begin{align*}
\left(-\frac{v}{dx_0} + \frac{\gamma_1 x_0^\alpha}{l^\alpha} \right) S_+ &= \frac{\gamma_1 x_0^\alpha}{l^\alpha} S_-, \\
\left(\frac{v}{dx_0} + \frac{\gamma_2 x_0^\alpha}{l^\alpha} \right) S_- &= \frac{\gamma_2 x_0^\alpha}{l^\alpha} S_+.
\end{align*}
\]  

(H.1) (H.2)

These two equations can be recast as

\[
\begin{align*}
\frac{l^\alpha}{x_0^\alpha} \mathcal{L}_+ S_+ &= \gamma_1 S_-, & \text{with } \mathcal{L}_+ &= -\frac{v}{dx_0} + \frac{\gamma_1 x_0^\alpha}{l^\alpha}, \\
\frac{l^\alpha}{x_0^\alpha} \mathcal{L}_- S_- &= \gamma_2 S_+, & \text{with } \mathcal{L}_- &= \frac{v}{dx_0} + \frac{\gamma_2 x_0^\alpha}{l^\alpha}.
\end{align*}
\]  

(H.3) (H.4)

These coupled equations can be easily decoupled by multiplying both sides of equation (H.3) by $\mathcal{L}_-$, which gives ordinary differential equation for $S_\pm(x_0)$ as
Run-and-tumble particle in inhomogeneous media in one dimension

$$\frac{d}{dx_0} \left( \frac{v l''}{x_0'} \frac{d}{dx_0} - 2\Delta \right) S_+ = 0. \quad (H.5)$$

One can now solve equation (H.5) to get $S_+(x_0)$ which can be substituted in equation (H.1) to get $S_-(x_0)$. The solved expressions for $S_\pm(x_0)$ read as

$$S_+(x_0) = C_1 + C_2 e^{-\frac{2|\Delta|}{\gamma(\alpha+1)}x_0^{\alpha+1}}, \quad (H.6)$$

$$S_-(x_0) = C_1 + C_2 e^{\frac{2|\Delta|}{\gamma(\alpha+1)}x_0^{\alpha+1}}, \quad (H.7)$$

where $C_1$ and $C_2$ are constants that remain to be evaluated. To evaluate them, we use the boundary conditions $S_\pm(x_0 \to \infty) = 0$ and $S_-(x_0 \to 0) = 0$, which give $C_1 = 1$ and $C_2 = -\gamma - |\Delta|$. Inserting them in equations (H.6) and (H.7), one gets the final expression for $S_\pm(x_0)$ written in equations (76) and (77).

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