ON ALGEBRAIC VARIETIES WITH
FINITE POLYHEDRAL MORI CONE

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Abstract. The fundamental property of Fano varieties with mild singularities is
that they have a finite polyhedral Mori cone. Thus, it is very interesting to ask:
What we can say about algebraic varieties with a finite polyhedral Mori cone? I give
a review of known results.

All of them were obtained applying methods which were originated in the theory
of discrete groups generated by reflections in hyperbolic spaces with a fundamental
chamber of finite volume.

1. Introduction

Let $X$ be a projective algebraic variety with $\mathbb{Q}$-factorial singularities over an
algebraically closed field. Let $N_1(X)$ be the $\mathbb{R}$-linear space generated by all algebraic
curves on $X$ by the numerical equivalence, and the $N^1(X)$ be the $\mathbb{R}$-linear space
generated by all Cartier (or Weil) divisors on $X$ by numerical equivalence. The
$N_1(X)$ and $N^1(X)$ are dual to each other.

The convex cone $NE(X) \subset N_1(X)$ generated by all effective curves on $X$ is
called Mori cone. Its dual $NEF(X) = \{ x \in N^1(X) \mid x \cdot NE(X) \geq 0 \}$ is called nef
cone.

One of basic properties of Fano varieties with log-terminal singularities which
follows from Mori Theory [Mo1], [Mo2] and its development by Kawamata [Ka1],
and Shokurov [Sho] is that the Mori cone $NE(X)$ is finite polyhedral. It is generated
by a finite set of rays which are called extremal rays.

Thus, it is interesting to ask: What can we say about algebraic varieties with
finite polyhedral Mori cone? Another question could be: What can we get from
this property for Fano varieties?

Nowadays, we understand that well for surfaces, but there are some results also
for 3-folds, for Fano 3-folds and Calabi–Yau 3-folds. All these results were obtained
during last 15 years, and I want to make a review of these results and also to speak
about open problems.

All methods, I know, were originated in the Theory of discrete groups generated
by reflection in hyperbolic spaces with a fundamental chamber of finite volume.

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They were developed by the author and Vinberg more than 20 years ago. We exploit
the idea that, in some cases, the polyhedron $M(X) = (\text{NEF}(X) - \{0\})/\mathbb{R}^+$ has very
similar properties to properties of a fundamental chamber $M$ of finite volume of a
discrete reflection group in a hyperbolic space. In some cases, Algebraic Geometry
is very similar to Hyperbolic Geometry.

2. METHODS AND RESULTS FROM THE THEORY OF
DISCRETE REFLECTION GROUPS IN HYPERBOLIC SPACES

There are two methods which were developed in the Theory of discrete reflection
groups $W$ generated by reflections in hyperbolic spaces $\mathcal{L}^n$. That is in simply-
connected Riemannian manifolds $\mathcal{L}^n$ of a constant negative curvature, where $n \geq 2$
is dimension. They are based on studying of geometric and combinatorial properties
of the fundamental chamber $M$ of $W$.

The first method is the Method of narrow parts of polyhedra. See [N3]. It is
a metric property of a finite closed convex polyhedron in a hyperbolic space. We
formulate it in Sect. 3 considering application of this method to algebraic surfaces.
It is valid for any finite convex closed polyhedron in a hyperbolic space. Applying
this method to fundamental chambers $M$ of reflection groups, it was shown in
[N3] (see also [N4]) that the number of maximal arithmetic reflection groups $W$
equivalently, the number of polyhedra $M$) is finite for the fixed dimension $n \geq 2$
of the hyperbolic space and the fixed degree $N = [K : \mathbb{Q}]$ of the ground field $K$
(the field of definition) of $W$. Here $K$ is some purely real algebraic number field.
The number of fields $K$ of the fixed degree $N$ for all $n \geq 2$ is also finite. Arithmetic
groups always have a finite fundamental domain of finite volume.

Applying the Method of narrow parts of polyhedra to algebraic varieties, we
expect to get boundedness, existence of a very ample divisor of bounded degree.
We consider these results in Sect. 3. The method of narrow parts of polyhedra is
now developed and applied only to surfaces. It would be very interesting to develop
it for 3-folds.

The second method is the Diagram Method. It was started for arithmetic reflection
groups in [N4] and was developed for arbitrary reflection groups with bounded
fundamental chambers $M$ by Vinberg in [V]. It uses the combinatorial property of
a bounded fundamental chamber $M$ which is that the number of faces of highest
dimension of $M$ containing a vertex is equal to dimension. Equivalently, the dual
polyhedron to $M$ is simplicial. The important development of this method was
obtained by Khovansky [Kh] and Prokhorov [P]. They generalized the method for
polyhedra $M$ which are simplicial in 1-dimensional faces. This is important for
reflection groups with fundamental chambers $M$ of finite volume but with some
vertices at infinity. We formulate the method in Sect. 4 considering algebraic sur-
faces. Using this method, it was proved in [N4] for arithmetic reflection groups
that the dimension $n \leq 9$ if the degree $N$ is large; in [V] that $n \leq 29$ for arbitrary
discrete reflection groups with bounded fundamental chamber; in [Kh], [P] that
$n \leq 995$ for arbitrary discrete reflection groups with fundamental chamber of finite
Applying the Diagram Method to algebraic varieties, we expect to get an estimate for the Picard number $\rho = \dim N_1(X)$ which is absolute for the considered class of algebraic varieties. This method is very effective for Del Pezzo surfaces with log-terminal singularities. See Sect. 4. It is very interesting that there are some generalizations of this method also to 3-fold: Fano 3-folds and Calabi–Yau 3-folds. See Sect. 5.

3. The method of narrow parts of polyhedra and algebraic surfaces.

We consider non-singular projective algebraic surfaces $X$ over an algebraically closed field, and we assume that $\rho = \dim N_1(X) \geq 3$. If the Mori cone $NE(X)$ is finite polyhedral, by Hodge Index Theorem,

$$NE(X) = \sum_{E \in \text{Exc}(X)} \mathbb{R}^+ E$$

is generated by the finite set $\text{Exc}(X)$ of all exceptional curves $E$ on $X$. We remind that a curve $E \subset X$ is exceptional, if $E$ is irreducible and $E^2 < 0$. We have natural invariants

$$\rho = \dim N_1(X) \geq 3, \delta = \max_{E \in \text{Exc}(X)} (-E^2), p = \max_{E \in \text{Exc}(X)} (p_a(E))$$

where $p_a(E) = \frac{E^2 + E \cdot K}{2} + 1$ is the arithmetic genus of $E$ and $K = K_X$ the canonical class.

We have

Theorem 1 ([N15]). For any algebraic surface $X$ with finite polyhedral Mori cone and the invariants $\rho \geq 3$, $\delta$, $p$, there exists a very ample divisor $H$ such that

$$H^2 \leq N(\rho, \delta, p)$$

where $N(\rho, \delta, p)$ is a constant depending only on $\rho$, $\delta$ and $p$.

There are examples in [N15] that Theorem 1 is not true if at least one of constants $\rho$, $\delta$, $p$ is not fixed. The proof of Theorem 1 follows from standard results on symmetric matrices with non-negative coefficients (the Perron–Frobenius Theorem) and the following result.

Lemma 1 (Method of narrow parts of polyhedra [N3, Appendix]). There exist $E_1, ..., E_\rho \in \text{Exc}(X)$ such that

1) $E_1, ..., E_\rho$ generate $N_1(X)$;
2) $\frac{M(E_i \cdot E_j)^2}{E_i^2 E_j^2} < 62^2$ (an absolute constant) for any $1 \leq i, j \leq n$ ;
3) The Dynkin diagram of $E_1, \ldots, E_\rho$ is connected (the set cannot be divided into two non-empty orthogonal to each other subsets).

Using Lemma 1, the very ample divisor $H$ of Theorem 1 can be obtained as $H = a_1E_1 + \cdots + a_\rho E_\rho$ where $a_i \in \mathbb{N}$ depend only on the intersection matrix $(E_i \cdot E_j) = 1 \leq i, j \leq \rho$, the number of these matrices is finite.

Idea of Proof of Lemma 1. It is very geometrical. I can explain how one can find the first element $E_1$.

Let us fix a very ample $D$. We find $E_1, F \in \text{Exc}(X)$ such $[E_1, F]$ is negative definite, $(F \cdot D)/\sqrt{-F^2} + (E_1 \cdot D)/\sqrt{-E_1^2}$ is maximal, and $(F \cdot D)/\sqrt{-F^2} \geq (E_1 \cdot D)/\sqrt{-E_1^2}$. Roughly speaking, we are looking for an exceptional curve $F$ of maximal degree with respect to $D$.

By the Hodge Index Theorem, the space $N_1(X)$ is hyperbolic for the intersection pairing. It has the signature $(1, \rho - 1)$. Then one can relate with $N_1(X)$ the hyperbolic space $\mathcal{L}(X) = V^+(X)/\mathbb{R}^+$ where $V^+(X)$ is the half (containing $D$) of the light cone $V(X)$ of all elements $x \in N_1(X)$ with $x^2 > 0$. The $\mathcal{L}(X)$ is equipped with the hyperbolic distance $|\mathbb{R}^+ x \mathbb{R}^+ y| = (x, y)/\sqrt{x^2 y^2}$. The polyhedron

$$\mathcal{M}(X) = \text{NEF}(X)/\mathbb{R}^+ = \{ \mathbb{R}^+ x \in \mathcal{L}(X) \mid x \cdot \text{Exc}(X) \geq 0 \} \subset \mathcal{L}(X)$$

is a finite closed convex polyhedron in $\mathcal{L}(X)$. The set $\text{Exc}(X) \subset N_1(X)$ gives all orthogonal vectors to faces of $\mathcal{M}(X)$ of codimension one. An element $E \in \text{Exc}(X)$ gives the face of codimension one of $\mathcal{M}(X)$ containing in the hyperplanes $\mathcal{H}_E = \{ \mathbb{R}^+ x \mid x \cdot E = 0 \} \subset \mathcal{L}(X)$ which is orthogonal to $E$.

The same Lemma 1 is valid for any finite closed convex polyhedron $\mathcal{M}$ in $\mathcal{L}(X)$ if one replaces $\text{Exc}(X)$ by the set $P(\mathcal{M}) \subset N_1(X)$ of orthogonal vectors to faces of $\mathcal{M}$ of codimension one. Lemma 1 does not change if one replaces $E_i$ by $\lambda_i E_i$, $\lambda_i > 0$. The statement 2) means that the hyperplanes of the corresponding faces are on absolutely bounded distance to each other. The statements 1) and 3) mean that they are in general position.

Because of Theorem 1, it is interesting to classify surfaces $X$ with finite polyhedral Mori cone for small invariants $\rho, \delta$ and $p$. It is done for $\delta = 2$ and $p = 0$, see [N15] for details. Equivalently, $E^2 = -1$ or $-2$, and $E$ is non-singular rational for any exceptional curve $E \in \text{Exc}(X)$. Then any $E \in \text{Exc}(X)$ defines a reflection $x \rightarrow x - 2(x \cdot E)/E^2$, $x \in NS(X)$. Here $NS(X) \subset N_1(X)$ is the Neron-Severi lattice of $X$ generated by all algebraic curves on $X$. Thus, $\mathcal{M}(X)$ is a fundamental chamber of the corresponding arithmetic reflection group, and then $\rho = \dim N_1(X)$ is absolutely bounded. The surfaces $X$ are divided in several types (a) — (d) described below.

(a) Minimal resolutions of singularities of Del Pezzo surfaces with Du Val (or index 1 log-terminal) singularities. This is the case $K^2 > 0$ for the canonical class $K$ of the surface. All these surfaces are rational, $\rho \leq 9$. Classification of Del Pezzo surfaces with Du Val singularities was obtained by Nagata in [Na]. It is also a part
of classification of Del Pezzo surfaces with log-terminal singularities of index \( \leq 2 \) which was obtained in [AN1] and [AN2]. We shall discuss it in Sect. 4.

(b) **Rational surfaces with** \( K^2 = 0 \) **and finite polyhedral Mori cone.** This is the case \( K^2 = 0, \) but \( K \not\equiv 0. \) Then \( \rho = 10. \) Classification of these surfaces was obtained in [N15]. There are two different cases of that surfaces. First case: \( \dim |-nK| = 0 \) for any \( n \geq 1. \) These surfaces \( X \) are classified by the graphs of exceptional curves \( \text{Exc}(X), \) there are three possible graphs, given below (the black vertex denotes an exceptional curve \( E \) with \( E^2 = -2. \)).

**Graph** \( H\widetilde{E}_8 \)

\[
E_{10} \quad E_9 \quad E_8 \quad E_7 \quad E_6 \quad E_5 \quad E_4 \quad E_3 \quad E_2
\]

**Graph** \( H\widetilde{D}_8 \)

\[
E_{11} \quad E_{10} \quad E_7 \quad E_5 \quad E_3 \quad E_2 \quad E_6 \quad E_8 \quad E_9
\]

**Graph** \( H\widetilde{A}_8 \)

\[
E_{10} \quad E_4 \quad E_7 \quad E_2 \quad E_5
\]

Second case: \( \dim |-nK| = 1 \) for some \( n \geq 1. \) The linear system \( |-nK| \) defines
then an elliptic fibration on $X$ with a singular fibre of multiplicity $n$. The invariant $n \in \mathbb{N}$ can take any value, and the surfaces $X$ have infinite number of connected components of moduli. Construction and classification of these surfaces uses general theory of rational elliptic surfaces [H], [CD], [D1], and Ogg-Shafarevich Theory of elliptic surfaces [O], [Sha]. To have a finite polyhedral Mori cone, the sum of ranks of degenerate fibres of the elliptic fibration $| - nK |$ must be maximal, equals to 8. Equivalently, the automorphism group of $X$ (it is mainly the Mordell–Weil group) must be finite.

(c) $K3$ surfaces with $\rho \geq 3$ and finite automorphism group. This is the case when $K = 0$. All exceptional curves $E$ have $E^2 = -2$. By the description of automorphism groups of $K3$ surfaces in [P-ŠS], for $\rho \geq 3$ the Mori cone $NE(X)$ is finite polyhedral, if and only if $Aut(X)$ is finite (over $\mathbb{C}$). All $K3$ surfaces $X$ with $\rho \geq 3$ and finite automorphism group were classified in [N1], [N2], [N5] and [N7]. For $\rho = 4$, this classification is due to Vinberg, see the result in [N7]. These $K3$ surfaces are classified by their Picard lattices $S = NS(X)$ which have the property that the group $W^{(-2)}(S)$ generated by reflections in all roots $\alpha$ of $S$ with $\alpha^2 = -2$ has finite index in the automorphism group $O(S)$ of the lattice. There is a finite number of such hyperbolic lattices $S$ of $\rho = \text{rk } (S) \geq 3$. All of them were found in [N1], [N2], [N5] and [N7]. Here is their number for each $\rho \geq 3$:

| $\rho$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | $\geq 20$ |
|-------|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|--------|
| Number| 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | $\geq 20$ |
|       | 27| 17| 10| 10| 9 | 12| 10| 9  | 4  | 4  | 3  | 3  | 1  | 1  | 1  | 1  | 1  | 1  | 0      |

A $K3$ surface $X$ may have very arbitrary Picard lattice $S$. For example, any even hyperbolic lattice $S$ of rank $\leq 11$ is a Picard lattice of some $K3$ surface. But only finite number of $S$ with $\rho = \text{rk } S \geq 3$ corresponds to $K3$ surfaces with finite automorphism group, and equivalently with finite polyhedral Mori cone. $K3$ surfaces with $\rho \geq 3$ having a finite polyhedral Mori cone are extremely rare and exceptional.

(d) Enriques surfaces with finite automorphism group. This is the case when $K \equiv 0$, but $K \neq 0$. All exceptional curves $E$ have $E^2 = -2$, the $\rho = 10$. There are seven types of Enriques surfaces with finite automorphism group. Two of them depend on 1 moduli, and five give isolated Enriques surfaces. They were classified in [N6] (see also [N7]), and by Kondō in [Ko] where one missing in [N6] type was also found.

The first example of Enriques surfaces with finite automorphism group was found by Gino Fano [F] in 1910. It is hard to follow Fano, but his example really gives an Enriques surface with the finite automorphism group! Since Enriques surfaces depend on 10 parameters, Fano’s ideas were genius and correct. This paper by Fano was rediscovered by Dolgachev in 1985 during his visit of the University of Torino where he looked Fano’s archive. One year before, Dolgachev published the paper [D2] where he discovered the first contemporary example of Enriques surfaces depending on one parameter with finite automorphism group. It is completely non-obvious that Enriques surfaces with finite automorphism group do exist! The papers [N6], [N7] and [Ko] followed this paper.
Unfortunately, no 3-dimensional generalization of the Method of narrow parts of polyhedra is known, in spite of its very clear geometric ideas.

4. The Diagram Method for Surfaces

Let \( X \) be a projective algebraic surface with isolated singularities and \( \pi : \tilde{X} \to X \) the minimal resolution of singularities. By results of Mori [Mo1] applied to nonsingular surfaces, \( \tilde{X} \) has a finite polyhedral Mori cone, if \( \rho(\tilde{X}) \geq 3 \). The Mori cone \( NE(\tilde{X}) \) is generated by exceptional curves which are in pre-images of singular points, and by exceptional curves of the first kind on \( \tilde{X} \).

The diagram method can be applied to all these surfaces \( \tilde{X} \), and all surfaces with finite polyhedral Mori cone as well, but we get the most astonishing results if singularities are log-terminal (or \( X \) is a log Del Pezzo surface). Over \( \mathbb{C} \) log-terminal singularities are the same as quotient singularities.

Applying Diagram Method, we get

**Theorem 2 ([N8]—[N10]).** There are functions \( A(n) \), \( B(n) \) of \( n \in \mathbb{N} \) such that for any Del Pezzo surface \( X \) with log-terminal singularities one has:

\[
\rho(\tilde{X}) < A(\text{maximal index of singularities});
\]

\[
\rho(\tilde{X}) < B(\text{maximal multiplicity of singularities}).
\]

It follows boundedness for moduli of \( X \) if maximal index or multiplicity of singularities is bounded (e.g. apply Theorem 1).

**Theorem 3 (Alexeev, [A]).** There is an absolute constant \( C > 0 \) such that for any Del Pezzo surface \( X \) with log-terminal singularities

\[
\rho(\tilde{X}) < \frac{C}{\epsilon(X)} \quad \text{where}
\]

\[
1 \geq \epsilon(X) = \min_{E \text{ of } 1\text{st kind}} (-E \cdot \pi^* K_X) > 0.
\]

It follows (Shokurov’s Conjecture) that the set of fractional indices of all log Del Pezzo surfaces is contained in \((0, 1]\), all its accumulation points are \( 1/n \), \( n \in \mathbb{N} \), and all of them are accumulation points from above, but not from below.

Remind that a positive rational number \( r > 0 \) is the fractional index of a log Del Pezzo surface \( X \) if \( K_X = rH \) where \( H \in PicX \) is a primitive element of the Picard lattice.

Proofs of Theorems 2, 3 are based on the Lemma 2 below which constitutes the Diagram Method. A subset \( \mathcal{L} \subset \text{Exc}(X) \) is called *Lanner* if it is minimal hyperbolic. A subset \( \mathcal{E} \subset \text{Exc}(X) \) is *elliptic*, if it is negative definite. The set \( \text{Exc}(X) \) defines a graph in a usual way. The set of its vertices is \( \text{Exc}(X) \). Two vertices \( E_1, E_2 \) are connect by an edge if \( E_1 \cdot E_2 > 0 \). Thus, we can consider the distance, the diameter and so on.
Lemma 2 (Diagram Method, [N8], [N14]). Let \( \tilde{X} \) be an algebraic surface with a finite polyhedral Mori cone. Assume that

(a) \( \text{diam} (L) \leq d \) for any Lanner subset \( L \subset \text{Exc}(\tilde{X}) \);

(b) 
\[ \#\{\{E_1, E_2\} \subset \mathcal{E} \mid 1 \leq \rho(E_1, E_2) \leq d\} \leq C_1 \#\mathcal{E} \]

and

\[ \#\{\{E_1, E_2\} \subset \mathcal{E} \mid d + 1 \leq \rho(E_1, E_2) \leq 2d + 1\} \leq C_2 \#\mathcal{E} \]

for any elliptic subset \( \mathcal{E} \subset \text{Exc}(\tilde{X}) \). Then

\[ \rho(\tilde{X}) < 96(C_1 + C_2/3) + 68 \]

To prove Theorems 2 and 3, one should just show that the constants \( d, C_1 \) and \( C_2 \) of Lemma 2 can be estimated by functions depending on the maximal index, maximal multiplicity of singularities of \( X \), or the invariant \( 1/\epsilon(X) \). This is not easy, but it can be done. See [N8]—[N11] and [A].

Idea of Proof of Lemma 2. Like for the proof of Lemma 1, we have a finite closed hyperbolic polyhedron

\[ \mathcal{M} = \text{NEF}(\tilde{X})/\mathbb{R}^+ = \{\mathbb{R}^+ x \in \mathcal{L}(\tilde{X}) \mid x \cdot \text{Exc}(\tilde{X}) \geq 0\} \subset \mathcal{L}(\tilde{X}) \]

of dimension \( n = \rho - 1 \) with the set \( \text{Exc}(\tilde{X}) \subset N_1(X) \) of orthogonal vectors to faces of \( \mathcal{M} \) of codimension one. Since \( E \cdot E' \geq 0 \) for \( E \neq E' \in \text{Exc}(\tilde{X}) \), the polyhedron \( \mathcal{M} \) has acute angles. By Perron-Frobenius Theorem, \( \mathcal{M} \) is simplicial in its finite vertices and in its edges (1-dimensional faces).

For simplicity, let us assume that all vertices of \( \mathcal{M} \) are finite. Then \( \mathcal{M} \) is dual to a simplicial polyhedron. It follows that the combinatorial polynomial of \( \mathcal{M} \) (where \( \alpha_i \) is the number of \( i \)-dimensional faces of \( \mathcal{M} \)),

\[ R(s) = \alpha_0 + (s - 1)\alpha_1 + \cdots + \alpha_{n-1}(s - 1)^{n-1} + (s - 1)^n, \]

is reversible and has positive coefficients (e.g. see [St]). It follows [N4] that the average number \( A^{0.2} \) of vertices of plane faces of \( \mathcal{M} \) is bounded as \( A^{0.2} \leq 4 + 4/(n - 2) \). Thus, almost all 2-dimensional faces of \( \mathcal{M} \) are quadrangles or triangles if \( n \) is large. In particular, since \( \mathcal{M} \) is a hyperbolic polyhedron, \( \mathcal{M} \) has a lot of non-right plane angles.

A plane angle \( A \) of \( \mathcal{M} \) is defined by an elliptic subset \( \mathcal{E} \subset \text{Exc}(\tilde{X}) \) with \( n \) vertices and by two its distinguished elements \( E_1 \neq E_2 \in \mathcal{E} \). The vertex of \( A \) is defined as intersection of all hyperplane orthogonal to elements of \( \mathcal{E} \). Two sides of the angle \( A \) are intersections of all hyperplanes which are orthogonal to elements of \( \mathcal{E} - \{E_1\} \) and \( \mathcal{E} - \{E_2\} \). The angle \( A \) is called combinatorially right, if the distance between \( E_1 \) and \( E_2 \) in \( \mathcal{E} \) is greater than \( 2d + 1 \). Otherwise, \( A \) is not combinatorially right.
It is easy to see that a Lanner subset $L$ is connected. Two Lanner subsets cannot be orthogonal. It follows that any triangle of $M$ has at least two combinatorially non-right angles. Any quadrangle of $M$ has at least one combinatorially non-right angle. Since almost all 2-dimensional faces of $M$ are triangles or quadrangles, $M$ has a lot of combinatorially non-right angles. On the other hand, by the condition (b) of of the lemma, $M$ has not too many of combinatorially non-right angles. If $n$ is big, we get a contradiction, which gives some estimate on dimension of $M$ depending on the constants $d$, $C_1$ and $C_2$.

In [N4], for arithmetic reflection groups, usual right angles in hyperbolic geometry were used. Introducing of the combinatorial notion of right angles is due to Vinberg [V]. The generalization of these considerations for polyhedra $M$ with some vertices at infinity is due to Khovanskii [Kh] and Prokhorov [P]. They had to use 3-dimensional angles of $M$ instead of 2-dimensional. Already in [N4], 3-dimensional angles were used with the estimate $A^{2,3} \leq 6 + 12/(n - 2)$ for the average number of 2-dimensional faces in 3-dimensional faces of $M$. In [N4], a general estimate for the average number or $k$-dimensional faces in $m$-dimensional faces of $M$ was given. Khovanskii [Kh], without using of combinatorial polynomials, had shown that the same estimates are valid for polyhedra $M$ which are simplicial only in edges.

See details in [N14] and the cited papers.

Theorem 2 shows that for a fixed $k$, it is possible in principle to classify Del Pezzo surfaces with log-terminal singularities of index $\leq k$. It is now known only for $k = 2$. In [AN1] and [AN2], classification of Del Pezzo surfaces $X$ with log-terminal singularities of index $\leq 2$ is given. It is shown that the linear system $|-2K_X|$ has a non-singular curve $C$ which does not contain singular points of $X$. There exists an appropriate right resolution of singularities $p : X_1 \to X$ such that $|-2K_{X_1}|$ has a non-singular curve $\widetilde{C} = p^{-1}(C) + E_1 + \cdots + E_k$ where $E_1, \ldots, E_k$ are non-singular rational curves with $E_i^2 = -4$. The double covering of $X_1$ ramified in $\widetilde{C}$ gives a K3 surface $Y$ with a non-symplectic involution $\sigma$ of the double covering. It reduces classification of $X$ to classification of K3 surfaces $Y$ with non-symplectic involutions $\sigma$ such that the fixed point set $Y^\sigma$ has a connected component of genus $\geq 2$. The surface $X_1$ has a finite polyhedral Mori cone with the polyhedron $M(X_1) = \text{NEF}(X_1)/\mathbb{R}^+$ which is a fundamental chamber for an arithmetic reflection group. The classification of the Del Pezzo surfaces $X$ enumerates possible graphs of the finite sets $\text{Exc}(X_1)$ of exceptional curves on the surfaces $X_1$. There are plenty of cases.

In [K-M] a different geometric approach to log Del Pezzo surfaces is developed. It is based on studying of rational curves on the log Del Pezzo surfaces.

Boundedness of log Del Pezzo surfaces of a fixed index (Theorem 2 for index) is now generalized by Alexandr Borisov [B] to Fano 3-folds with log-terminal singularities. Similar result for $n$-dimensional Fano varieties was recently announced by McKernan [McK]. All other results of Theorems 2 and 3 are now known only for surfaces, and their proofs use the diagram method.
We can say that the property to have a finite polyhedral Mori cone for $\tilde{X}$ is now crucial for description of Del Pezzo surfaces $X$ with log-terminal singularities.

5. The Diagram Method for 3-folds

Generalizing the diagram method, two results about 3-folds were obtained. One of them is valid for Fano 3-folds with terminal $\mathbb{Q}$-factorial singularities, another one is valid for 3-dimensional Calabi–Yau manifolds.

We remind that an extremal ray $R$ of the Mori cone $NE(X)$ is called divisorial if effective curves of $R$ fill out a divisor $D(R)$ of $X$; in theorems below it is always irreducible. An extremal ray $R$ is called small if effective curves of $R$ fill out a curve of $X$. A proper face $\gamma$ of $NE(X)$ has Kodaira dimension $d$, $d \leq 3$, if the contraction of $\gamma$ (in theorems below, it always exists) is a morphism such that its image has dimension $d$. An irreducible curve $C$ belongs to $\gamma$ if and only if its image is a point for the contraction of $\gamma$.

**Theorem 4 ([N12]).** Let $X$ be a Fano 3-fold with terminal $\mathbb{Q}$-factorial singularities (by [Mo1], [Mo2], it has a finite polyhedral Mori cone). Then

$$\rho(X) \leq 7$$

except the following two cases:

1. The Mori cone $NE(X)$ has a face of Kodaira dimension $\leq 2$ (i.e. its contraction gives a fibration $\pi : X \to Y$ where $\dim Y \leq 2$).
2. There exists a small extremal ray (it does not exist if $X$ is non-singular).

Of course, we know a lot about non-singular Fano 3-folds and about Fano 3-folds with terminal singularities, e.g. see [M-M] and [Ka2], and this result is not something extraordinary. But Theorem 4 is obtained by a very elementary, purely combinatorial method.

We have the following statement about 3-dimensional Calabi–Yau manifolds.

**Theorem 5 (Nikulin and Shokurov, [N13]).** Let $X$ be a 3-dimensional Calabi–Yau manifold (here the Mori cone is not necessarily finite polyhedral, and we have to add the case (3) below). Then

$$\rho(X) \leq 40$$

except the following three cases:

1. There exists a rational nef element $D$ with $D^3 = 0$ (it is expected that $|nD|$ gives a fibration of $X$ for big $n$).
2. There exists a small extremal ray (the corresponding flop gives then another birational model).
3. The Mori cone is not finite polyhedral (it is expected that $\text{Aut}(X)$ is then infinite, [Mor]).
Sketch of Proofs. If we exclude exceptions (1) — (3), Mori cone of $X$ is finite polyhedral, all extremal rays are divisorial, there are no nef rational elements $D$ with $D^3 = 0$. Further proof is divided in several steps.

**Step A:** We prove that two different extremal rays $R_1$ and $R_2$ cannot have the same divisor $D(R_1) = D(R_2)$. The picture below is impossible. This is one of the most difficult steps which reflects problems of 3-dimensional geometry. The proof for Calabi–Yau manifolds is especially complicated.

**Step B:** We can introduce diagrams like below:

Here arrow $R_1 \rightarrow R_2$ means $R_1 \cdot D(R_2) > 0$. Thus, the set $\text{Exc}(X)$ of all extremal rays (all of them are divisorial) defines an oriented graph. Using this graph, we can consider distances in subsets of $\text{Exc}(X)$, their diameters. Since the graph is oriented, the distance function might be not symmetric.

**Step C:** A subset $E \subset \text{Exc}(X)$ is called **elliptic** if it is contained in a proper face of $NE(X)$. A subset $L \subset \text{Exc}(X)$ is called a **E-set (extremal)** if it is minimal non-elliptic. (E-sets are similar to Lanner subsets for surfaces.) We have the following statement which constitutes the diagram method for 3-folds.

**Lemma 3 (Diagram Method, [N12], [N13]).** Assume that there are constants $d, C_1, C_2$ such that the conditions (a) and (b) below hold:

(a) $$\text{diam}(L) \leq d$$

for any $E$-subset $L \subset \text{Exc}(X)$.

(b) $$\sharp\{(R_1, R_2) \in E \times E \mid 1 \leq \rho(R_1, R_2) \leq d\} \leq C_1 \sharp E;$$

and

$$\sharp\{(R_1, R_2) \in E \times E \mid d + 1 \leq \rho(R_1, R_2) \leq 2d + 1\} \leq C_2 \sharp E.$$
for any extremal subset $\mathcal{E} \subset \text{Exc}(X)$.

Then

$$\rho(X) \leq (16/3)C_1 + 4C_2 + 6.$$  

About proof of Lemma 3. The proof of Lemma 3 is very similar to the proof of Lemma 2 for surfaces. Again one should work with the polyhedron $\mathcal{M}(X) = \text{NEF}(X)/\mathbb{R}^+$. Its faces are also orthogonal to elements of $\text{Exc}(X)$. The crucial step is to prove that $\mathcal{M}(X)$ is dual to a simplicial polyhedron. Then one can use the same estimates for the average number of vertices of 2-dimensional faces of $\mathcal{M}(X)$. Further, considering oriented plane angles of $\mathcal{M}(X)$ instead of non-oriented, one should go through all steps of the proof of Lemma 2 with some small changes.

**Step D:** To apply Lemma 3, one needs to describe (classify) elliptic subsets and $\mathcal{E}$-subsets. For 3-dimensional Calabi-Yau manifolds, one can see some of them below:

**Table 1.** Calabi–Yau elliptic diagrams without single arrows (classical Dynkin diagrams).
For classification of elliptic and E-subsets, one needs the statement: any divisorial extremal ray $R$ of a 3-dimensional Calabi-Yau manifold has a curve $C \in R$ such that $C \cdot D(R) = -k$ where $1 \leq k \leq 3$. This was proved by Shokurov (see Appendix to [N13]).

As a result, using Lemma 3, we get some estimate on $\rho(X)$. The estimate $\rho(X) \leq 40$ of Theorem 5 requires additional very delicate considerations with elliptic and E-subsets, and their combinatorics.

It seems, Theorem 5 is one of the strongest known results about the structure of Mori (or Kähler) cone of 3-dimensional Calabi–Yau manifolds. See other related results in papers of Miyaoka [Mi], Oguiso [Og] and Wilson [W1]—[W3], and their discussion in [N13].

It is expected that 3-dimensional Calabi–Yau manifolds very often have finite
polyhedral Mori cone. Morrison [Mor] conjectured that the Kähler cone of a Calabi–Yau manifold $X$ is rational finite polyhedral up to action of the automorphism group $\text{Aut } X$. In particular, the Kähler (and Mori) cone is finite polyhedral if $\text{Aut } X$ is finite. It follows that a 3-dimensional Calabi–Yau manifold has a finite polyhedral Mori cone if its cubic intersection hypersurface is non-singular. We can expect that very often. K3 surfaces with $\rho \geq 3$ very rare have a finite polyhedral Mori cone (see Sect. 3). We expect that for Calabi–Yau 3-folds situation is opposite.

Algebraic surfaces with finite polyhedral Mori cone are now understood well. Our knowledge about 3-folds with finite polyhedral Mori cone is very poor in spite of it could be very useful for understanding of some very important algebraic varieties, for example Calabi–Yau 3-folds.

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