Research Article

Malik Bataineh* and Rashid Abu-Dawwas

Graded $I$-second submodules

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Abstract: Let $G$ be a group with identity $e$, $R$ be a $G$-graded commutative ring with a nonzero unity $1$, $I$ be a graded ideal of $R$, and $M$ be a $G$-graded $R$-module. In this article, we introduce the concept of graded $I$-second submodules of $M$ as a generalization of graded second submodules of $M$ and achieve some relevant outcomes.

Keywords: graded second submodules, graded prime submodules, graded weakly prime submodules

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1 Introduction

A proper graded ideal $P$ of $R$ is said to be graded prime if whenever $x, y \in h(R)$ such that $xy \in P$, then either $x \in P$ or $y \in P$. Graded prime ideals have been admirably introduced and studied in [1]. Graded prime submodules have been introduced by Atani in [2]. A proper graded $R$-submodule $N$ of $M$ is said to be graded prime if whenever $r \in h(R)$ and $m \in h(M)$ such that $rm \in N$, then either $m \in N$ or $r \in (N :_RM)$. Graded prime submodules have been widely studied by several authors, for more details one can look in [3–6], Atani introduced in [7] the concept of graded weakly prime ideals. A proper graded ideal $P$ of $R$ is said to be a graded weakly prime ideal of $R$ if whenever $x, y \in h(R)$ such that $0 \neq xy \in P$, then $x \in P$ or $y \in P$. Also, Atani extended the concept of graded weakly prime ideals into graded weakly prime submodules in [8]. A proper graded submodule $N$ of $M$ is called graded weakly prime if for $r \in h(R)$ and $m \in h(M)$ with $0 \neq rm \in N$, either $m \in N$ or $r \in (N :_RM)$.

Let $M$ and $S$ be two $G$-graded $R$-modules. An $R$-homomorphism $f : M \to S$ is said to be graded $R$-homomorphism if $f(M_g) \subseteq S_g$ for all $g \in G$ (see [9]). Graded second submodules have been introduced by Ansari-Toroghy and Farshadifar in [10]. A nonzero graded $R$-submodule $N$ of $M$ is said to be graded second if for each $a \in h(R)$, the graded $R$-homomorphism $f : N \to N$ defined by $f(x) = ax$ is either surjective or zero. In this case, $\text{Ann}_R(N)$ is a graded prime ideal of $R$. Graded second submodules have been wonderfully studied by Çeken and Alkan in [11]. On the other hand, graded secondary modules have been introduced by Atani and Farzalipour in [12]. A nonzero graded $R$-module $M$ is said to be graded secondary if for each $a \in h(R)$, the graded $R$-homomorphism $f : M \to M$ defined by $f(x) = ax$ is either surjective or nilpotent.

The main purpose of this article is to follow [13] in order to introduce and study the concept of graded $I$-second submodules of a graded $R$-module $M$ as a generalization of graded second submodules of $M$ and achieve some relevant outcomes. Among several results, we show that a graded second submodule is a graded $I$-second submodule for every graded ideal $I$ of $R$, but we prove that the converse is not true in general (Examples 2.5, 2.6, and 2.7). We follow [14] to introduce the concept of graded $I$-prime ideals of a graded ring $R$, we show that a graded prime ideal is a graded $I$-prime ideal for every graded ideal $I$ of $R$, but
we prove that the converse is not true in general (Example 2.16). We prove that if $N$ is a graded $I$-second
$R$-submodule of $M$ such that $\text{Ann}_R((N :_R I)) \subseteq I\text{Ann}_R(N)$, then $\text{Ann}_R(N)$ is a graded $I$-prime ideal of $R$
(Proposition 2.21). We show that if $M$ is a graded comultiplication $R$-module and $N$ is a graded $R$-submodule
of $M$ such that $\text{Ann}_R(N)$ is an $I$-prime ideal of $R$, then $N$ is a graded $I$-second $R$-submodule of $M$
(Proposition 2.23). We prove that if $M$ is primary, then every proper graded $(0)$-second $R$-submodule of $M$
is a graded primary $R$-submodule of $M$ (Proposition 2.27). In Proposition 2.28, we study graded $I$-second submodules
under graded homomorphism. Finally, in Proposition 2.29, we study the relation between graded $I$-second
submodules of $M$ and $I_L$-second submodules of $M_e$ when $|G| = 2$.

1.1 Preliminaries

Throughout this article, $G$ will be a group with identity $e$ and $R$ will be a commutative ring with a nonzero
unity $1$. $R$ is said to be $G$-graded if $R = \bigoplus_{g \in G} R_g$ with $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$, where $R_g$
is an additive subgroup of $R$ for all $g \in G$. The elements of $R_g$ are called homogeneous of degree $g$. Consider $\text{supp}(R, G) = \{g \in G : R_g \neq 0\}$. If $x \in R$, then $x$ can be written uniquely as $\sum_{g \in G} x_g$, where $x_g$
is the component of $x$ in $R_g$.

Also, $h(R) = \bigcup_{g \in G} R_g$. Moreover, it has been proved in [9] that $R_g$ is a subring of $R$ and $1 \in R_g$.

Let $I$ be an ideal of a graded ring $R$. Then $I$ is said to be a graded ideal if $I = \bigoplus_{g \in G} (I \cap R_g)$, i.e., for $x \in I$, $x = \sum_{g \in G} x_g$, where $x_g \in I$ for all $g \in G$. Let $R$ be a $G$-graded ring and $I$ be a graded ideal of $R$. Then $R/I$ is
$G$-graded by $(R/I)_g = (R_g + I)/I$ for all $g \in G$.

Assume that $M$ is a left $R$-module. Then $M$ is said to be $G$-graded if $M = \bigoplus_{g \in G} M_g$ with $R_g M_h \subseteq M_{gh}$ for all
$g, h \in G$, where $M_g$ is an additive subgroup of $M$ for all $g \in G$. The elements of $M_g$ are called homogeneous
of degree $g$. Also, we consider $\text{supp}(M, G) = \{g \in G : M_g \neq 0\}$. It is clear that $M_g$ is an $R_e$-submodule of $M$
for all $g \in G$. Moreover, $h(M) = \bigcup_{g \in G} M_g$. Let $N$ be an $R$-submodule of a graded $R$-module $M$. Then $N$ is said
to be graded $R$-submodule if $N = \bigoplus_{g \in G} (N \cap M_g)$, i.e., for $x \in N$, $x = \sum_{g \in G} x_g$, where $x_g \in N$ for all $g \in G$. Let $M$
be a $G$-graded $R$-module and $N$ be a graded $R$-submodule of $M$. Then $M/N$ is a graded $R$-module by $(M/N)_g =
(M_g + N)/N$ for all $g \in G$.

Lemma 1.1. [15] Let $R$ be a $G$-graded ring and $M$ be a $G$-graded $R$-module.
1. If $I$ and $J$ are graded ideals of $R$, then $I + J$ and $I \cap J$ are graded ideals of $R$.
2. If $N$ and $K$ are graded $R$-submodules of $M$, then $N + K$ and $N \cap K$ are graded $R$-submodules of $M$.
3. If $N$ is a graded $R$-submodule of $M$, $r \in h(R)$, $x \in h(M)$, and $I$ is a graded ideal of $R$, then $Rx$, $IN$, and $rN$
are graded $R$-submodules of $M$. Moreover, $(N :_R M) = \{r \in R : rN \subseteq N\}$ is a graded ideal of $R$.

Similarly, if $M$ is a graded $R$-module, $N$ a graded $R$-submodule of $M$, and $m \in h(M)$, then $(N :_R m)$
is a graded ideal of $R$. Also, it has been proved in [16] that if $N$ is a graded $R$-submodule of $M$, then $\text{Ann}_R(N)$
$= \{r \in R : rN = \{0\}\}$ is a graded ideal of $R$.

In [17], a proper $Z$-graded $R$-submodule $N$ of $M$ is said to be graded completely irreducible if whenever
$N = \bigcap_{k \in \Delta} N_k$, where $\{N_k\}_{k \in \Delta}$ is a family of $Z$-graded $R$-submodules of $M$, then $N = N_k$ for some $k \in \Delta$. In [16],
the concept of graded completely irreducible submodules has been extended into $G$-graded case, for any
group $G$. It has been proved that every graded $R$-submodule of $M$ is an intersection of graded completely
irreducible $R$-submodules of $M$. In many instances, we use the following basic fact without further
discussion.

Remark 1.2. Let $N$ and $L$ be two graded $R$-submodules of $M$. To prove that $N \subseteq L$, it is enough to prove that if
$K$ is a graded completely irreducible $R$-submodule of $M$ such that $L \subseteq K$, then $N \subseteq K$. 

2 Graded $I$-second submodules

In this section, we introduce and study the concept of graded $I$-second submodules.

Let $\Omega(M)$ be the set of all graded completely irreducible $R$-submodules of $M$. Assume that $P$ is a graded prime ideal of $R$ and $N$ is a graded $R$-submodule of $M$. Then we define $I^M_P(N) = \bigcap_{K \in \Omega(M)} K : rN \subseteq K$ for some $r \in h(R) - P$. The following lemma gives some characterizations for graded second $R$-submodules.

**Lemma 2.1.** Let $N$ be a graded $R$-submodule of a graded $R$-module $M$. Then the following are equivalent.
1. If $N \neq \{0\}$, $K$ is a graded completely irreducible $R$-submodule of $M$ and $r \in h(R)$ such that $rN \subseteq K$, then either $rN = \{0\}$ or $N \subseteq K$.
2. $N$ is a graded second $R$-submodule of $M$.
3. $P = \text{Ann}_R(N)$ is a graded prime ideal of $R$ and $I^M_P(N) = N$.

**Proof.** (1) $\Rightarrow$ (2): Suppose that $r \in h(R)$ and $N \neq \{0\}$. If $rN \subseteq K$ for some graded completely irreducible $R$-submodule $K$ of $M$, then by assumption, $N \subseteq K$. Hence, $N \subseteq rN$.

(2) $\Rightarrow$ (3): By [10], $P = \text{Ann}_R(N)$ is a graded prime ideal of $R$. Now, let $K$ be a graded completely irreducible $R$-submodule of $M$ and $r \in h(R) - P$ such that $rN \subseteq K$. Then $N \subseteq K$ by assumption. Therefore, $N \subseteq I^M_P(N)$. The reverse inclusion is clear.

(3) $\Rightarrow$ (1): Since $\text{Ann}_R(N)$ is a graded prime ideal of $R$, $N \neq \{0\}$. Let $K$ be a graded completely irreducible $R$-submodule of $M$ and $r \in h(R)$ such that $rN \subseteq K$. Suppose that $rN \neq \{0\}$. Then $r \in h(R) - P$. Therefore, $I^M_P(N) \subseteq K$. But $I^M_P(N) = N$ by assumption. Hence, $N \subseteq K$, as desired. $\square$

**Lemma 2.2.** Let $M$ be a $G$-graded $R$-module and $N$ a graded $R$-submodule of $M$. If $r \in h(R)$, then $(N :_M r) = \{m \in M : rm \in N\}$ is a graded $R$-submodule of $M$.

**Proof.** Clearly, $(N :_M r)$ is an $R$-submodule of $M$. Let $m \in (N :_M r)$. Then $rm \in N$. Now, $m = \sum_{g \in G} m_g$, where $m_g \in M_g$ for all $g \in G$. Since $r \in h(R)$, $r \in h_g$ for some $h \in G$ and then $rm_g \in M_{h_g} \subseteq h(M)$ for all $g \in G$ such that $\sum_{g \in G} m_g = r \left( \sum_{g \in G} m_g \right) = rm \in N$. Since $N$ is graded, $rm_g \in N$ for all $g \in G$, which implies that $m_g \in (N :_M r)$ for all $g \in G$. Hence, $(N :_M r)$ is a graded $R$-submodule of $M$. $\square$

Similarly, if $N$ is a graded $R$-submodule of $M$ and $I$ is a graded ideal of $R$, then $(N :_M I)$ is a graded $R$-submodule of $M$.

**Proposition 2.3.** Let $M$ be a graded $R$-module, $I$ be a graded ideal of $R$, and $N$ be a nonzero graded $R$-submodule of $M$. Then the following statements are equivalent:
1. For each $r \in h(R)$, a graded $R$-submodule $K$ of $M$, $r \in (K :_R N) - (K :_R (N :_M I))$ implies that $N \subseteq K$ or $r \in \text{Ann}_R(N)$;
2. For each $r \notin (rN :_R (N :_M I)) \cap h(R)$, we have $rN = N$ or $rN = \{0\}$;
3. $(K :_R N) = \text{Ann}_R(N) \cup (K :_R (N :_M I))$, for any graded $R$-submodule $K$ of $M$ with $N \notin K$;
4. $(K :_R N) = \text{Ann}_R(N) \cup (K :_R (N :_M I))$, for any graded $R$-submodule $K$ of $M$ with $N \notin K$.

**Proof.** (1) $\Rightarrow$ (2): Let $r \notin (rN :_R (N :_M I)) \cap h(R)$. Then as $rN \subseteq rN$, we have $N \subseteq rN$ or $rN = \{0\}$ by part (1). Thus, $rN = N$ or $rN = \{0\}$.

(2) $\Rightarrow$ (1): Let $r \in h(R)$ and $K$ be a graded $R$-submodule of $M$ such that $r \in (K :_R N) - (K :_R (N :_M I))$. Then if $r \in (rN :_R (N :_M I))$, then $r \in (K :_R (N :_M I))$, which is a contradiction. Thus, $r \notin (rN :_R (N :_M I))$. Now, by part (2), $rN = N$ or $rN = \{0\}$. So, $N \subseteq K$ or $rN = \{0\}$, as desired.

(1) $\Rightarrow$ (3): Let $r \in (K :_R N)$ and $N \notin K$. If $r \notin (K :_R (N :_M I))$, then $r \in \text{Ann}_R(N)$ by part (1). Hence, $(K :_R N) \subseteq \text{Ann}_R(N)$, $rN \in (K :_R (N :_M I))$, then $K :_R N \subseteq (K :_R (N :_M I))$. Therefore, $(K :_R N) \subseteq \text{Ann}_R(N) \cup (K :_R (N :_M I))$.

The other inclusion is clear.

(3) $\Rightarrow$ (4): If a graded ideal is a union of two graded ideals, then it is equal to one of them.

(4) $\Rightarrow$ (1): Obvious. $\square$
**Definition 2.4.** Let $M$ be a graded $R$-module, $I$ be a graded ideal of $R$, and $N$ be a nonzero graded $R$-submodule of $M$. Then $N$ is said to be a graded $I$-second $R$-submodule of $M$ if $N$ satisfies the equivalent conditions of Proposition 2.3.

Clearly, every graded second submodule is a graded $I$-second submodule for every graded ideal $I$ of $R$. However, the following examples prove that the converse is not true in general.

**Example 2.5.** Every graded $R$-module $M$ is a graded $I = \{0\}$-second $R$-submodule of itself, but not every graded $R$-module is a graded second $R$-submodule of itself.

**Example 2.6.** Consider $R = \mathbb{Z}$, $M = \mathbb{Z}[i]$, and $G = \mathbb{Z}_2$. Then $R$ is trivially $G$-graded by $R_0 = R$ and $R_1 = \{0\}$. Also, $M$ is $G$-graded by $M_0 = \mathbb{Z}$ and $M_1 = i\mathbb{Z}$. Now, $N = \mathbb{Z}$ is a graded $R$-submodule of $M$. If $I = R$, then $N$ is a graded $I$-second $R$-submodule of $M$ that is not a graded second $R$-submodule of $M$.

**Example 2.7.** Consider $R = \mathbb{Z}$, $M = \mathbb{Z}_{12}[i]$, and $G = \mathbb{Z}_4$. Then $R$ is trivially $G$-graded by $R_0 = R$ and $R_1 = R_2 = R_3 = \{0\}$. Also, $M$ is $G$-graded by $M_0 = \mathbb{Z}_{12}$, $M_2 = i\mathbb{Z}_{12}$, and $M_4 = M_3 = \{0\}$. Now, $N = 3\mathbb{Z}_{12}$ is a graded $R$-submodule of $M$. If $I = 4\mathbb{Z}$, then $N$ is a graded $I$-second $R$-submodule of $M$ that is not a graded second $R$-submodule of $M$.

**Remark 2.8.**
1. If $I = R$, then every graded $R$-submodule of $M$ is a graded $I$-second $R$-submodule of $M$. So in the rest of our article, we can assume that $I \neq R$.
2. If Condition (1) in Proposition 2.3 holds for graded completely irreducible submodules, that is, if for each $r \in h(R)$, and a graded completely irreducible $R$-submodule $L$ of $M$, $r \in (L :_R N) - (L :_R (N :_M I))$ implies that $N \subseteq L$ or $r \in \text{Ann}_R(N)$, we cannot achieve that $N$ is a graded $I$-second $R$-submodule of $M$ (as in Lemma 2.1 for graded second submodules), see the following example:

**Example 2.9.** Consider $R = \mathbb{Z}$, $M = \mathbb{Z}[i]$, and $G = \mathbb{Z}_2$. Then $R$ is trivially $G$-graded by $R_0 = R$ and $R_1 = \{0\}$. Also, $M$ is $G$-graded by $M_0 = \mathbb{Z}$ and $M_1 = i\mathbb{Z}$. Now, $N = 2\mathbb{Z}$ is a graded $R$-submodule of $M$. If $I = 4\mathbb{Z}$, then $N$ is not a graded $I$-second $R$-submodule of $M$, but Condition (1) in Proposition 2.3 holds for graded completely irreducible $R$-submodules of $M$.

**Proposition 2.10.** Let $M$ be a graded $R$-module and $I_1$, $I_2$ be graded ideals of $R$ such that $I_1 \subseteq I_2$. If $N$ is a graded $I_1$-second $R$-submodule of $M$, then $N$ is a graded $I_2$-second $R$-submodule of $M$.

**Proof.** Since $I_1 \subseteq I_2$, we conclude that $(rN :_R N) - (rN :_R (N :_M I_2)) \subseteq (rN :_R N) - (rN :_R (N :_M I_1))$ for each $r \in h(R)$. So, the result holds. $\square$

**Corollary 2.11.** Let $M$ be a graded $R$-module. Then every graded $\{0\}$-second $R$-submodule of $M$ is a graded $I$-second $R$-submodule of $M$ for each graded ideal $I$ of $R$.

**Definition 2.12.** Let $M$ be a $G$-graded $R$-module, $I$ be a graded ideal of $R$, $N$ be a nonzero graded $R$-submodule of $M$, and $g \in G$. Then $N$ is said to be a $g$-$I$-second $R$-submodule of $M$ if for each $r \in R_g$, and a graded $R$-submodule $K$ of $M$, $r \in (K :_{R_g} N) - (K :_{R_g} (N :_{M_g} I))$ implies that $N \subseteq K$ or $r \in \text{Ann}_{R_g}(N)$.

**Definition 2.13.** Let $M$ be a $G$-graded $R$-module and $g \in G$. A nonzero graded $R$-submodule $N$ of $M$ is said to be a $g$-second $R$-submodule of $M$ if $K$ is a graded $R$-submodule of $M$ and $r \in R_g$ such that $rN \subseteq K$, then either $rN = \{0\}$ or $N \subseteq K$.

**Proposition 2.14.** Let $M$ be a $G$-graded $R$-module and $g \in G$. If $N$ is a $g$-$I$-second $R$-submodule of $M$ which is not graded $g$-second, then $\text{Ann}_{R_g}(N) (N :_{M_g} I) \subseteq N$. 
Suppose that \( \text{Ann}_{R_g}(N : M_g I) \not\subseteq N \). We show that \( N \) is a \( g \)-second \( R \)-submodule of \( M \). Let \( rN \subseteq K \) for some \( r \in R_g \) and a graded \( R \)-submodule \( K \) of \( M \). If \( r \not\in (K : R_g \langle N : M_g I \rangle) \), then \( N \) is a graded \( g \)-second \( R \)-submodule implies that \( N \subseteq K \) or \( r \in \text{Ann}_{R_g}(N) \) as required. Assume that \( r \in (K : R_g \langle N : M_g I \rangle) \). Suppose that \( r(N : M_g I) \not\subseteq N \). Then there exists a graded \( R \)-submodule \( L \) of \( M \) such that \( N \subseteq L \) with \( r(N : M_g I) \not\subseteq L \), and then \( r \in (K \cap L : R_g \langle N : M_g I \rangle) = (K \cap L : R_g \langle N : M_g I \rangle) \). So, \( N \subseteq K \cap L \) or \( r \in \text{Ann}_{R_g}(N) \) and hence \( N \subseteq K \) or \( r \in \text{Ann}_{R_g}(N) \). Assume that \( r(N : M_g I) \subseteq N \). If \( \text{Ann}_{R_g}(N)(N : M_g I) \not\subseteq K \), then there exists \( t \in \text{Ann}_{R_g}(N) \) such that \( t \not\in (K : R_g \langle N : M_g I \rangle) \). Then \( t + r \in (K : R_g \langle N : M_g I \rangle) \). Thus, \( N \subseteq K \) or \( t + r \in \text{Ann}_{R_g}(N) \) and hence, \( N \subseteq K \) or \( r \in \text{Ann}_{R_g}(N) \). Assume that \( \text{Ann}_{R_g}(N)(N : M_g I) \subseteq N \). Since \( \text{Ann}_{R_g}(N)(N : M_g I) \not\subseteq N \), there exist \( t \in \text{Ann}_{R_g}(N) \), and a graded \( R \)-submodule \( T \) of \( M \) such that \( N \subseteq T \) and \( t(N : M_g I) \not\subseteq T \). Now we have \( r + t \in (K \cap T : R_g \langle N \rangle) = (K \cap T : R_g \langle N : M_g I \rangle) \). So, \( N \) is a \( g \)-second \( R \)-submodule of \( M \) gives \( N \subseteq K \cap T \) or \( r + t \in \text{Ann}_{R_g}(N) \). Hence, \( N \subseteq K \) or \( r \in \text{Ann}_{R_g}(N) \), as needed.

In the following definition, we follow [14] to introduce the concept of graded \( I \)-prime ideals of a graded ring \( R \).

**Definition 2.15.** Let \( R \) be a graded ring and \( I \) be a graded ideal of \( R \). Then a proper graded ideal \( P \) of \( R \) is said to be graded \( I \)-prime if for \( x, y \in h(R) \) such that \( xy \in P - IP \), then either \( x \in P \) or \( y \in P \).

Clearly, every graded prime ideal is a graded \( I \)-prime ideal for every graded ideal \( I \) of \( R \). However, the following example shows that the converse is not true in general.

**Example 2.16.** Consider \( R = \mathbb{Z}_{12}[I] \) and \( G = \mathbb{Z}_4 \). Then \( R \) is \( G \)-graded by \( R_0 = \mathbb{Z}_{12}, R_2 = i\mathbb{Z}_{12}, \) and \( R_1 = R_3 = \{0\} \). If we take \( P = I = \langle h \rangle \), then \( P \) is a graded \( I \)-prime ideal of \( R \) which is neither graded prime nor graded weakly prime.

**Lemma 2.17.** Let \( R \) be a \( G \)-graded ring, \( I \) be an ideal of \( R \), and \( J \) be a graded ideal of \( R \) such that \( J \subseteq I \). Then \( I \) is a graded ideal of \( R \) if and only if \( I/J \) is a graded ideal of \( R/J \).

**Proof.** Suppose that \( I \) is a graded ideal of \( R \). Clearly, \( I/J \) is an ideal of \( R/J \). Let \( x + J \in I/J \). Then \( x \in I \) and since \( I \) is graded, \( x = \sum_{g \in G} x_g \), where \( x_g \in I \) for all \( g \in G \) and then \( (x + J)_g = x_g + J \in I/J \) for all \( g \in G \). Hence, \( I/J \) is a graded ideal of \( R/J \). Conversely, let \( x \in I \). Then \( x = \sum_{g \in G} x_g \), where \( x_g \in R_g \) for all \( g \in G \) and then \( (x + J)_g \in (R_g + J)/J = (R/J)_g \) for all \( g \in G \) such that

\[
\sum_{g \in G} (x + J)_g = \sum_{g \in G} (x_g + J) = \left( \sum_{g \in G} x_g \right) + J = x + J \in I/J.
\]

Since \( I/J \) is graded, \( x_g + J \in I/J \) for all \( g \in G \), which implies that \( x_g \in I \) for all \( g \in G \). Hence, \( I \) is a graded ideal of \( R \). \qed

**Proposition 2.18.** Let \( P \) be a proper graded ideal of \( R \). Then \( P \) is a graded \( I \)-prime ideal of \( R \) if and only if \( P/IP \) is a graded weakly prime ideal of \( R/IP \).

**Proof.** Suppose that \( P \) is a graded \( I \)-prime ideal of \( R \). By Lemma 2.17, \( P/IP \) is a graded ideal of \( R/IP \). Let \( x + IP, y + IP \in h(R/IP) \) such that \( 0 + IP \neq (x + IP)(y + IP) = P/IP \). Then \( x, y \in h(R) \) such that \( xy \in P - IP \), and then \( x \in P \) or \( y \in P \). So, \( x + IP \in P/IP \) or \( y + IP \in P/IP \). Hence, \( P/IP \) is a graded weakly prime ideal of \( R/IP \). Conversely, let \( x, y \in h(R) \) such that \( xy \in P - IP \). Then \( x + IP, y + IP \in h(R/IP) \) such that \( 0 + IP \neq (x + IP)(y + IP) = P/IP \), and then \( x + IP \in P/IP \) or \( y + IP \in P/IP \). So, \( x \in P \) or \( y \in P \). Hence, \( P \) is a graded \( I \)-prime ideal of \( R \). \qed
Proposition 2.19. Let $I$ and $J$ be two graded ideals of $R$ such that $I \subseteq J$. Then every graded $I$-prime ideal of $R$ is graded $J$-prime.

Proof. Let $P$ be a graded $I$-prime ideal of $R$. Then the result follows from the fact that $P - JP \subseteq P - IP$. □

The following example shows that if $I$ and $J$ are two graded ideals of $R$ such that $I \subseteq J$ and $P$ is a graded $J$-prime ideal of $R$, then $P$ does not need to be graded $I$-prime.

Example 2.20. Consider $R = \mathbb{Z}[x]$ and $G = \mathbb{Z}$. Then $R$ is $G$-graded by $R_j = \mathbb{Z}_{12\cdot j}^1$ for $j \geq 0$ and $R_0 = \{0\}$ otherwise. Choose $I = \langle 0 \rangle$, $J = \langle 4 \rangle$, and $P = \langle 4x \rangle$, then $I$, $J$, and $P$ are graded ideals of $R$ such that $I \subseteq J$, $P = \langle 4x \rangle - \{0\}$, and $P - JP = \emptyset$. Clearly, $P$ is a graded $J$-prime ideal of $R$ but not graded $I$-prime.

Proposition 2.21. Let $M$ be a graded $R$-module and $N$ be a graded $R$-submodule of $M$. If $N$ is a graded $I$-second $R$-submodule of $M$ such that $\text{Ann}_R(N :_M I) \subseteq I\text{Ann}_R(N)$, then $\text{Ann}_R(N)$ is a graded $I$-prime ideal of $R$.

Proof. By [16], $\text{Ann}_R(N)$ is a graded ideal of $R$. Let $xy \in \text{Ann}_R(N) - I\text{Ann}_R(N)$ for some $x, y \in h(R)$. Then $xN \subseteq (0 :_M y)$. As $xy \notin I\text{Ann}_R(N)$ and $\text{Ann}_R(N :_M I) \subseteq I\text{Ann}_R(N)$, we have $xy \notin \text{Ann}_R((N :_M I))$. This implies that $x \notin (0 :_M y) :_R(N :_M I)$, and $x \in \text{Ann}_R(N)$ or $N \subseteq (0 :_M y)$. Hence, $x \in \text{Ann}_R(N)$ or $y \in \text{Ann}_R(N)$, as required. □

Corollary 2.22. If $M$ is a graded faithful $R$-module and $N$ is a graded $\langle 0 \rangle$-second $R$-submodule of $M$, then $\text{Ann}_R(N)$ is a graded weakly prime ideal of $R$.

Proof. Apply Proposition 2.21 with $I = \langle 0 \rangle$. □

Graded comultiplication modules have been introduced by H. A. Toroghy and F. Farshadifar in [18]; a graded $R$-module $M$ is said to be graded comultiplication if for every graded $R$-submodule $N$ of $M$, $N = (0 :_M I)$ for some graded ideal $I$ of $R$, or equivalently, $N = (0 :_M \text{Ann}_R(N))$. The concept of graded comultiplication modules has been studied by several authors, for example, see [19,20].

Proposition 2.23. Let $M$ be a graded comultiplication $R$-module and $N$ be a graded $R$-submodule of $M$. If $\text{Ann}_R(N)$ is an $I$-prime ideal of $R$, then $N$ is a graded $I$-second $R$-submodule of $M$.

Proof. Let $r \in (K :_R N) - (K :_R (N :_M I))$ for some $r \in h(R)$ and a graded $R$-submodule $K$ of $M$. As $M$ is a graded comultiplication $R$-module, there exists a graded ideal $J$ of $R$ such that $K = (0 :_M J)$. Thus, $rJ \subseteq \text{Ann}_R(N)$. Since $r \notin (K :_R (N :_M I))$, we have $Jr(N :_M I) \neq \emptyset$. This implies that $Jr \notin \text{Ann}_R((N :_M I))$. Since clearly, $I\text{Ann}_R(N) \subseteq \text{Ann}_R((N :_M I))$, we have $rJ \notin I\text{Ann}_R(N)$. Thus, $r \in \text{Ann}_R(N)$ or $J \subseteq \text{Ann}_R(N)$ by ([14], Theorem 2.12), and so $N \subseteq (0 :_M J) = K$. □

Corollary 2.24. Let $M$ be a graded comultiplication $R$-module and $N$ be a graded $R$-submodule of $M$. If $\text{Ann}_R(N)$ is a weakly prime ideal of $R$, then $N$ is a graded $\langle 0 \rangle$-second $R$-submodule of $M$.

Proof. Apply Proposition 2.23 with $I = \langle 0 \rangle$. □

The next example shows that the condition “$M$ is a graded comultiplication $R$-module” in Corollary 2.24 is necessary.

Example 2.25. Let $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}$. Consider the trivial graduation of $R$ and $M$ by any group $G$. Then $M$ is not a graded comultiplication $R$-module. Now, $N = 2\mathbb{Z} \oplus \{0\}$ is a graded $R$-submodule of $M$ such that $\text{Ann}_R(N) = \{0\}$ is a weakly prime ideal of $R$, but $N$ is not a graded $\langle 0 \rangle$-second $R$-submodule of $M$. 
Proposition 2.26. Let $I$ be a graded ideal of a graded ring $R$ and $M$ be a graded $R$-module. Let $N$ be a graded $I$-second $R$-submodule of $M$. If $L$ is a graded $R$-submodule of $M$ with $L \subseteq N$, then $N/L$ is a graded $I$-second $R$-submodule of $M/L$.

Proof. Similar to the proof of Lemma 2.17, one can prove that $N/L$ is a graded $R$-submodule of $M$. The result follows by $r \notin (r(N/L) \cap (N/L :_{M/L} I))$ implies that $r \notin (rN :_R (N :_M I))$. □

Graded primary ideals have been introduced and studied in [21]. A proper graded ideal $P$ of $R$ is said to be graded primary if for $x, y \in h(R)$ such that $xy \in P$, then either $x \in P$ or $y \in \text{Grad}(P)$, where $\text{Grad}(P)$ is the graded radical of $P$, and is defined to be the set of all $r \in R$ such that for each $g \in G$, there exists a positive integer $n_g$ that satisfies $r_g^n \in P$. One can see that if $r \in h(R)$, then $r \in \text{Grad}(P)$ if and only if $r^n \in P$ for some positive integer $n$. In [22], a proper graded $R$-submodule $N$ of $M$ is said to be graded primary if whenever $r \in h(R)$ and $m \in h(M)$ such that $rm \in N$, then either $m \in N$ or $r \in \text{Grad}(N :_R M)$. An $R$-module $M$ is said to be a primary $R$-module if $\{0\}$ is a primary $R$-submodule of $M$.

Proposition 2.27. Let $M$ be a graded $R$-module. If $M$ is primary, then every proper graded $\{0\}$-second $R$-submodule of $M$ is a graded $R$-primary submodule of $M$.

Proof. Let $N$ be a proper graded $\{0\}$-second $R$-submodule of $M$ and $rm \in N$ for some $r \in h(R)$ and $m \in h(M)$. If $r \notin (rN :_R M)$, then $rN = \{0\}$ or $rN = N$ since $N$ is a graded $\{0\}$-second $R$-submodule of $M$. If $rN = \{0\}$, then $r^m \in rN = \{0\}$. Now as $M$ is primary, $m = 0$ or $r \in \text{Grad}(0 :_R M)$. This implies that $m \in N$ or $r \in \text{Grad}(N :_R M)$, as required. If $rN = N$, then $rm = ra$ for some $a \in N$. This implies that $m = a \in N$ or $r \in \text{Grad}(0 :_R M) \subseteq \text{Grad}(N :_R M)$ since $M$ is primary. Suppose that $r \in (rN :_R M)$. Then $rm \in rM \subseteq rN$. Therefore, similar to the previous case we are done. □

Let $M$ and $S$ be two $G$-graded $R$-modules. An $R$-homomorphism $f : M \rightarrow S$ is said to be graded $R$-homomorphism if $f(M_g) \subseteq S_g$ for all $g \in G$ (see [9]).

Proposition 2.28. Let $I$ be a graded ideal of a graded ring $R$, $M$ and $S$ be graded $R$-modules, and let $f : M \rightarrow S$ be an injective graded $R$-homomorphism. If $K$ is a graded $I$-second $R$-submodule of $S$ such that $K \subseteq \text{Im}(f)$, then $f^{-1}(K)$ is a graded $I$-second $R$-submodule of $M$.

Proof. Since $K \neq \{0\}$ and $K \subseteq \text{Im}(f)$, we conclude that $f^{-1}(K) \neq \{0\}$. Let $r \notin (rf^{-1}(K) :_R (f^{-1}(K) :_M I))$ for some $r \in h(R)$. Then $r \notin (rK :_R (K :_S I))$. Thus, $rK = \{0\}$ or $rK = K$. This implies that $rf^{-1}(K) = \{0\}$ or $rf^{-1}(K) = f^{-1}(K)$, as needed. □

Proposition 2.29. Let $G = \{e, g\}$, where $g \neq e$. Suppose that $R$ is a nontrivially $G$-graded ring with $R = R_e \oplus R_g$, $I$ is a graded ideal of $R$, and $M$ is a nontrivially $G$-graded $R$-module by $M = M_e \oplus M_g$. Assume that $N$ is an $R_e$-submodule of $M_e$. Then $N \oplus \{0\}$ is a graded $I$-second $R$-submodule of $M$ if and only if $N$ is an $I_e$-second $R_e$-submodule of $M_e$ and for $r \in (rN :_{R_e} (N :_{M_e} I_e))$ with $rN \neq \{0\}$ and $rN \neq N$, we have $r \in \text{Ann}_{R_e}(0 :_{M_e} I_e)$.

Proof. Suppose that $N \oplus \{0\}$ is a graded $I$-second $R$-submodule of $M$. Let $r \notin (rN :_{R_e} (N :_{M_e} I_e))$. Then $r \notin (r(N \oplus \{0\}) :_R (N \oplus \{0\} :_M I))$. Since $N \oplus \{0\}$ is graded $I$-second, either $r(N \oplus \{0\}) = N \oplus \{0\}$ or $r(N \oplus \{0\}) = \{0\} \oplus \{0\}$. Thus, either $rN = N$ or $rN = \{0\}$, so $N$ is $I_e$-second. Assume that $r \in (rN :_{R_e} (N :_{M_e} I_e))$ with $rN \neq \{0\}$ and $rN \neq N$. Suppose that $r \notin \text{Ann}_{R_e}(0 :_{M_e} I_e)$. Then there exists $x \in M_g$ such that $Ix = \{0\}$ and $rx \neq 0$. This implies that $r(0, x) \in r(N \oplus \{0\} :_M I) = r(N \oplus \{0\})$. So, since $N \oplus \{0\}$ is graded $I$-second, either $r(N \oplus \{0\}) = N \oplus \{0\}$ or $r(N \oplus \{0\}) = \{0\} \oplus \{0\}$. Thus, either $rN = N$ or $rN = \{0\}$, which is a contradiction. So, $r \in \text{Ann}_{R_e}(0 :_{M_e} I_e)$. Conversely, let $r \notin (r(N \oplus \{0\}) :_M I)$. Then if $rN = N$ or $rN = \{0\}$, the result is clear. So, suppose that $rN \neq N$ and $rN \neq \{0\}$. We show that $r \notin (rN :_{R_e} (N :_{M_e} I_e))$, and this contradiction proves the result because $N$ is an $I_e$-second $R_e$-submodule of $M_e$. Assume on the contrary that $r \in (rN :_{R_e} (N :_{M_e} I_e))$. Then by assumption,
\( r \in \text{Ann}_{R_0}(0 :_{M_0} I_0) \). This implies that if \((x, y) \in N \oplus (0 :_{M} I)\), then \(r(x, y) \in r(N \oplus \{0\})\). Therefore, \(r \in (r(N \oplus \{0\}) :_{M} I)\), which is a desired contradiction.

\[ \square \]

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References

[1] M. Refai, M. Hailat and S. Obiedat, *Graded radicals and graded prime spectra*, Far East J. Math. Sci. 1 (2000), 59–73.
[2] S. E. Atani, *On graded prime submodules*, Chiang Mai J. Sci. 33 (2006), no. 1, 3–7.
[3] R. Abu-Dawwas and K. Al-Zoubi, *On graded weakly classical prime submodules*, Iran. J. Math. Sci. Inform. 12 (2017), no. 1, 153–161.
[4] R. Abu-Dawwas, K. Al-Zoubi and M. Bataineh, *Prime submodules of graded modules*, Proyecciones 31 (2012), no. 4, 355–361.
[5] K. Al-Zoubi and R. Abu-Dawwas, *On graded quasi-prime submodules*, Kyungpook Math. J. 55 (2015), 259–266.
[6] K. Al-Zoubi, M. Jaradat and R. Abu-Dawwas, *On graded classical prime and graded prime submodules*, Bull. Iranian Math. Soc. 41 (2015), no. 1, 205–2013.
[7] S. E. Atani, *On graded weakly prime ideals*, Turkish J. Math. 30 (2006), 351–358.
[8] S. E. Atani, *On graded weakly prime submodules*, Int. Math. Forum 1 (2006), no. 2, 61–66.
[9] C. Nastasescu and F. van Oystaeyen, *Methods of Graded Rings*, Lecture Notes in Mathematics, 1836, Springer-Verlag, Berlin, 2004.
[10] H. Ansari-Toroghy and F. Farshadifar, *On graded second modules*, Tamkang J. Math. 43 (2012), no. 4, 499–505.
[11] S. Çeken and M. Alkan, *On graded second and coprimary modules and graded second representations*, Bull. Malaysian Math. Sci. Soc. Ser. 2 38 (2015), no. 4, 1317–1330.
[12] S. E. Atani and F. Farzalipour, *On graded secondary modules*, Turkish J. Math. 31 (2007), 371–378.
[13] F. Farshadifar and H. Ansari-Toroghy, *I-second submodules of a module*, Matematicki Vesnik 72 (2020), no. 1, 58–65.
[14] I. Akray, *I-prime ideals*, J. Algebra Relat. Topics 4 (2016), no. 2, 41–47.
[15] F. Farzalipour and P. Ghasvand, *On the union of graded prime submodules*, Thai J. Math. 9 (2011), no. 1, 49–55.
[16] D. Northcott, *Lessons on Rings, Modules, and Multiplicities*, Cambridge University Press, Cambridge, 1968.
[17] J. Chen and Y. Kim, *Graded irreducible modules are irreducible*, Comm. Algebra 45 (2017), no. 5, 1907–1913.
[18] H. Ansari-Toroghy and F. Farshadifar, *Graded comultiplication modules*, Chiang Mai J. Sci. 38 (2011), no. 3, 311–320.
[19] R. Abu-Dawwas and M. Ali, *Comultiplication modules over strongly graded rings*, Int. J. Pure Appl. Math. 81 (2012), no. 5, 693–699.
[20] R. Abu-Dawwas, M. Bataineh and A. Dakeek, *Graded weak comultiplication modules*, Hokkaido Math. J. 48 (2019), 253–261.
[21] M. Refai and K. Al-Zoubi, *On graded primary ideals*, Turkish J. Math. 28 (2004), no. 3, 217–229.
[22] K. H. Oral, Ü. Tekir and A. G. Argargün, *On graded prime and primary submodules*, Turkish J. Math. 35 (2011), 159–167.