Elliptic free-fermion model with OS boundary and elliptic Pfaffians

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Abstract

We introduce and study a class of partition functions of an elliptic free-fermionic face model. We study the partition functions under a triangular boundary using the off-diagonal $K$-matrix at the boundary (OS boundary), which was introduced by Kuperberg as a class of variants of the domain wall boundary partition functions. We find explicit forms of the partition functions under OS boundary using elliptic Pfaffians. We find two expressions based on two versions of Korepin’s trick, and we obtain an identity between two elliptic Pfaffians as a corollary.

1 Introduction

Special kinds of determinants and Pfaffians are not only interesting on their own but also because of their appearances in many fields of mathematics and mathematical physics. In mathematical physics, they often appear as partition functions of integrable lattice models [1, 2, 3, 4, 5, 6]. One of the most notable examples are the works by Korepin and Izergin. Korepin [7] introduced the domain wall boundary partition functions (DWBPF) of the $U_q(sl_2)$ six-vertex model, and also introduced a technique which enables one to reduce the problem of finding the explicit forms of the DWBPF to finding polynomials which satisfy several properties which uniquely define them. Later, Izergin [8] found the explicit determinant form which is now called as the Izergin-Korepin determinant. Several variants of the domain wall boundary partition functions were introduced and studied, sometimes with applications to the enumeration of the alternating sign matrices and connections with characters of classical groups (see [9, 10, 11, 12, 13, 14, 15, 16, 17] for examples). The seminal works are by Tsuchiya [9] and Kuperberg [11, 12], which they found determinant and Pfaffian representations for various variations of the DWBPF. There are also works on the free-fermionic model [18, 19].

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in which more simplified factorized representations of the partition functions were found, even for elliptic models.

Studying elliptic generalizations of the DWBPF are interesting, and it is interesting to find determinant and Pfaffian representations. In particular, finding representations using Pfaffians of a matrix whose matrix entries are elliptic functions are interesting, since there are only a few studies on elliptic Pfaffians. For example, Rosengren introduced a family of elliptic Pfaffians and showed that the partition functions of the Andrews-Baxter-Forrester (ABF) model at the supersymmetric point are expressed as a sum of two elliptic Pfaffians. We mention that expressions of the DWBPF of the ABF model which hold in generic parameters are derived in [23, 24, 25], a factorized expression at the free-fermion point is derived in [23, 24, 25], and a single determinant representation was recently derived in [32].

As for the properties of elliptic Pfaffians, Okada and Rosengren discovered several elliptic generalizations of the Pfaffian counterpart of the Cauchy determinant formulas. The properties of elliptic determinants are extensively studied, for example, several generalizations of the Cauchy determinant formula have been discovered [37]. On the other hand, there are only a few results on elliptic Pfaffians by Okada and Rosengren.

In this paper, we study partition functions of an elliptic free-fermionic face model under a triangular boundary, and show that can be explicitly expressed using elliptic Pfaffians. The face model we treat can be regarded as degenerations of the ABF model, Okado-Deguchi-Martin (elliptic Perk-Schultz) model, Foda-Wheeler-Zuparic (elliptic Felderhof) model, which are face-type counterparts of the elliptic vertex models, and are elliptic analogue of the trigonometric models of the $U_q(sl_2)$ six-vertex model, Perk-Schultz model and the Felderhof free-fermion model. In this paper, we treat a fundamental example of the variations of the DWBPF introduced by Kuperberg.

Kuperberg introduced a class of partition functions of the $U_q(sl_2)$ six-vertex model under a triangular boundary using an off-diagonal boundary $K$-matrix at the boundary, and showed that they have explicit expressions using Pfaffians. He called this boundary condition as the OS boundary. We introduce the partition functions of the elliptic free-fermionic face model with OS boundary, and study them using the elliptic version of the Izergin-Korepin analysis. We evaluate the explicit representations of the partition functions using elliptic Pfaffians.

As a corollary of the two elliptic Pfaffian representations of the same partition functions by the elliptic Izergin-Korepin analysis, we get an identity between the two elliptic Pfaffians. This paper is organized as follows. In the next section, we introduce and summarize formulas and properties of the Pfaffian and theta functions which will be used in later sections. In section 3, we introduce the elliptic free-fermionic face model using the dynamical $R$-matrix formalism, and introduce the partition functions under OS boundary. In section 4, we analyze and get the explicit expressions of the partition functions using elliptic Pfaffians. We also get an identity between the two elliptic Pfaffians as a corollary of the two representations of the partition functions. Section 5 is devoted to the conclusion of this paper.
2 Preliminaries

In this section, we introduce and present some formulas and properties of the Pfaffian and theta functions, which are going to be used in this paper for the analysis of the DWBPF under OS boundary.

The Pfaffian \( \text{Pf}X \) of a skew-symmetric matrix \( X = (x_{ij})_{1 \leq i,j \leq 2n} \) is defined as

\[
\text{Pf}X = \sum_{\sigma \in M_{2n}} \text{sgn}(\sigma) \prod_{j=1}^{n} x_{\sigma(2j-1)} x_{\sigma(2j)},
\]

(2.1)

where \( M_{2n} \) is a subset of the symmetric group \( S_{2n} \) satisfying

\[
M_{2n} = \left\{ \sigma \in S_{2n} \mid \sigma(1) < \sigma(3) < \cdots < \sigma(2n-1), \ \sigma(2j-1) < \sigma(2j), \ j = 1, \ldots, n \right\}.
\]

(2.2)

In this paper, besides the definition of the Pfaffian, we use the following expansion formula for the Pfaffian

\[
\text{Pf}X = \sum_{k=2}^{2n} (-1)^k x_{1k} \text{Pf}X_{1,k}.
\]

(2.3)

Here \( X_{1,k} \) is a \((2n-2) \times (2n-2)\) matrix which the first and \( k \)-th rows and columns are removed from the \( 2n \times 2n \) matrix \( X = (x_{ij})_{1 \leq i,j \leq 2n} \).

We introduce the notation \([u]\) as theta functions \([u] = H(\pi i u)\) where \( H(u) \) is

\[
H(u) = 2 \sinh u \prod_{j=1}^{\infty} (1 - 2q^{2j} \cosh 2u + q^{4j})(1 - q^{2j}).
\]

(2.4)

Here, \( q \) is the elliptic nome \((0 < q < 1)\).

The theta function \([u]\) is an odd function \([-u] = -[u]\) and hence \([0] = 0\). It also satisfies the quasi-periodicities

\[
[u + 1] = -[u],
\]

(2.5)

\[
[u - i \log(q)/\pi] = -q^{-1} \exp(-2\pi i u) [u].
\]

(2.6)

Using the above properties, we get

\[
[u - 1/2] = [-u - 1/2],
\]

(2.7)

for example. The addition formula for the theta functions

\[
[u + x][u - x][u + y][v - y] - [v + x][v - x][u + y][u - y] - [x + y][x - y][u + v][u - v] = 0,
\]

(2.8)

is one of the most important identities for the theta functions. For example, it is used to prove the Yang-Baxter relation for elliptic integrable models.

The following facts about the elliptic polynomials \([30, 64]\) turned out to be useful for the analysis of elliptic face-type integrable models \([28]\). They were used in developing the
method of quantum separation of variables for ABF model and the elliptic Gaudin model [61]. The facts justify the Izergin-Korepin analysis on elliptic integrable models and were used effectively on the computation of the DWBPF of elliptic integrable models. See Refs. [30], [29], and [31] for examples.

A character is a group homomorphism $\chi$ from multiplicative groups $\Gamma = \mathbb{Z} + \tau \mathbb{Z}$ to $\mathbb{C}^\times$. For each character $\chi$ and positive integer $n$, an $n$-dimensional space $\Theta_n(\chi)$ is defined that consists of holomorphic functions $\phi(y)$ on $\mathbb{C}$ satisfying the quasiperiodicities

$$
\phi(y + 1) = \chi(1)\phi(y),
$$

$$
\phi(y + \tau) = \chi(\tau)e^{-2\pi i y - \pi i \tau}\phi(y).
$$

The elements of the space $\Theta_n(\chi)$ are called elliptic polynomials. The space $\Theta_n(\chi)$ is $n$-dimensional [30, 64], and the following fact holds for the elliptic polynomials:

**Proposition 2.1.** [30, 64] Suppose there are two elliptic polynomials $P(y)$ and $Q(y)$ in $\Theta_n(\chi)$, where $\chi(1) = (-1)^n$ and $\chi(\tau) = (-1)^n e^\alpha$. If these two polynomials are equal at $n$ points $y_j$, $j = 1, \ldots, n$, satisfying $y_j - y_k \notin \Gamma$ and $\sum_{k=1}^N y_k - \alpha \notin \Gamma$, that is, $P(y_j) = Q(y_j)$, then the two polynomials are exactly the same: $P(y) = Q(y)$.

## 3 Elliptic free-fermionic face model

In this section, we introduce the free-fermionic face model using the dynamical $R$-matrix formalism [63, 66, 23, 24, 25], which enables one to describe the face model like a six-vertex model.

The dynamical $R$-matrix of the elliptic free-fermionic face model is given by

$$
R_{ab}(u, v|h) = \begin{pmatrix}
[u - v + 1/2] & 0 & 0 & 0 \\
0 & [h_{1/2}|u-v] & [h_{1/2}|u+v] & 0 \\
0 & [h_{1/2}|u+v] & [h_{1/2}|u-v] & 0 \\
0 & 0 & 0 & [u - v + 1/2]
\end{pmatrix},
$$

acting on the tensor product $W_a \otimes W_b$ of the complex two-dimensional space $W_a$. The free-fermionic dynamical $R$-matrix satisfies $R_{ab}(u, v|h + 1) = R_{ab}(u, v|h)$.

The dynamical $R$-matrix [30, 64] satisfies the dynamical Yang–Baxter relation (Fig. 3)

$$
R_{bc}(v, w|h)R_{ac}(u, w|h + 1/2)R_{ab}(u, v|h) = R_{ab}(u, v|h + 1/2)R_{ac}(u, w|h)R_{bc}(v, w|h + 1/2),
$$

acting on $W_a \otimes W_b \otimes W_c$.

We also introduce the following off-diagonal $K$-matrix acting on $W_a$ (see Fig. 3):

$$
K_a(u, h) = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
$$

One can easily check that the $K$-matrix [30, 64] together with the dynamical $R$-matrix (3.1) satisfy the relation

$$
R_{ba}(u - v, h)K_b(u, h)R_{ab}(v + u, h)K_a(v, h) = K_a(v, h)R_{ba}(u + v, h)K_b(u, h)R_{ab}(u - v, h),
$$

(3.5)
which is called the reflection equation or the boundary Yang–Baxter equation \([67]\) (Fig. 4). The reflection equation ensures integrability at the boundary. This off-diagonal \(K\)-matrix was used as local pieces of the partition functions for the case of the \(U_q(sl_2)\) six-vertex model by Kuperberg \([12]\). We also use this \(K\)-matrix for the elliptic integrable model in this paper.

It seems that it is hard or maybe impossible to extract the off-diagonal \(K\)-matrix \((3.4)\) from the general full \(K\)-matrices of elliptic integrable models \([68, 69, 70, 71]\). This \(K\)-matrix was used to impose the antiperiodic boundary condition on the ABF model in the paper by Felder-Schorr \([64, 72]\), in which they analyzed the antiperiodic boundary condition by the quantum separation of variables method.

### 4 Partition functions under OS boundary

In this section, we introduce and analyze the partition functions of the free-fermionic face model under OS boundary.

Let us denote the orthonormal basis of \(W_a\) and its dual by \(\{|0\rangle_a, |1\rangle_a\}\) and \(\{a\langle 0|_a, a\langle 1|_a\}\). Next, the Pauli spin operators \(\sigma^+\) and \(\sigma^-\) are defined as operators acting on the (dual) orthonormal basis as

\[
\sigma^+|1\rangle = |0\rangle, \quad \sigma^+|0\rangle = 0, \quad \langle 0|\sigma^+ = \langle 1|, \quad \langle 1|\sigma^+ = 0, \quad (4.1)
\]

\[
\sigma^-|0\rangle = |1\rangle, \quad \sigma^-|1\rangle = 0, \quad \langle 1|\sigma^- = \langle 0|, \quad \langle 0|\sigma^- = 0. \quad (4.2)
\]

To formulate the wavefunctions under a triangular boundary, we introduce the tensor product of the Fock spaces: \(W_1 \otimes \cdots \otimes W_{2n}\). Using the dynamical \(R\)-matrix \([3.1]\) and the \(K\)-matrix \([3.4]\), we next define a monodromy...
Figure 2: The free-fermionic dynamical Yang–Baxter relation (3.3). The left- and right-hand sides of the figure represent the left- and right-hand sides of the Yang-Baxter relation $R_{bc}(v, w|h)R_{ac}(u, w|h + 1/2)R_{ab}(u, v|h)$ and $R_{ab}(u, v|h + 1/2)R_{ac}(u, w|h)R_{bc}(v, w|h + 1/2)$, respectively.

matrix $T_j(u_j, \ldots, u_{2n}|h)$, $j = 1, \ldots, 2n$, as

$$T_j(u_j, \ldots, u_{2n}|h) = \prod_{k=j+1}^{2n} R_{jk}(u_j, -u_k|h + 1/4 + (-1)^{k-j}/4)K_j(u_j), \quad (4.3)$$

which acts on $W_j \otimes \cdots \otimes W_{2n}$. See Fig. 3 for a pictorial depiction of (4.3). Using this monodromy matrix, we introduce the partition functions $P_{2n}(u_1, \ldots, u_{2n}|h)$ as follows (Fig. 6):

$$P_{2n}(u_1, \ldots, u_{2n}, h) = 2^n \langle \Omega | T_{2n}(u_1|h) \cdots T_1(u_1, \ldots, u_{2n}|h) | \Omega \rangle_{2n}, \quad (4.4)$$

where the states $2^n \langle \Omega \rangle$ and $| \Omega \rangle_{2n}$ are defined as

$$2^n \langle \Omega \rangle = 1 \langle 0 | \otimes \cdots \otimes 2^n \langle 0 |, \quad (4.5)$$

$$| \Omega \rangle_{2n} = | 0 \rangle_1 \otimes \cdots \otimes | 0 \rangle_{2n}. \quad (4.6)$$

Now we perform the Izergin–Korepin analysis [7, 8] on the wavefunctions $P_{2n}(u_1, \ldots, u_{2n}|h)$. The Izergin-Korepin analysis is a technique introduced by Korepin, and the idea is to list the properties of the domain wall boundary partition functions which uniquely define them, and reduce the problem of explicitly computing the partition functions to that of finding the polynomials satisfying those properties. Izergin-Korepin needs the notion of degree of the polynomial for the uniqueness, and the notion of the degree and the property of the elliptic
polynomial stated in Proposition 2.1 in Section 2 ensures the elliptic version of the Izergin-Korepin analysis, which was effectively used for the computation of the ordinary domain wall boundary partition functions of the Andrews-Baxter-Forrester model [29, 30, 31].

**Proposition 4.1.** The partition functions under OS boundary $P_{2n}(u_1, \ldots, u_{2n}|h)$ satisfy the following properties:

1. The partition functions $P_{2n}(u_1, \ldots, u_{2n}|h)$ are elliptic polynomials of $u_1$ of degree $2n - 1$
   
   $\begin{align*}
   P_{2n}(u_1 + 1, \ldots, u_{2n}|h) &= (-1)^{2n-1} P_{2n}(u_1, \ldots, u_{2n}|h), \\
   P_{2n}(u_1 - i \log(q)/\pi, \ldots, u_{2n}|h) &= (-q)^{-1} (2n - 1) \exp \left( -2\pi i \left( (2n - 1) u_1 + h + \sum_{j=2}^{2n} u_j \right) \right) P_{2n}(u_1, \ldots, u_{2n}|h).
   \end{align*}$

2. The following relations among the partition functions hold (Fig. 7):

   $\begin{align*}
   P_{2n}(u_1, \ldots, u_{2n}|h)|_{u_1 = -u_\ell} &= \left[ \frac{1}{2} \right] \prod_{j=2}^{2n} \left[ u_j + u_\ell + 1/2 \right] \left[ u_j - u_\ell + 1/2 \right] P_{2n-2}(u_2, \ldots, \hat{u}_\ell, \ldots, u_{2n}|h),
   \end{align*}$

   for $\ell = 2, \ldots, 2n$, and $\hat{u}_\ell$ in $P_{2n-2}(u_2, \ldots, \hat{u}_\ell, \ldots, u_{2n}|h)$ means that $\hat{u}_\ell$ is removed.

3. The following evaluation holds:

   $\begin{align*}
   P_{2}(u_1, u_2|h) &= \left[ \frac{1}{2} \right] \left[ h + u_1 + u_2 \right].
   \end{align*}$

Figure 3: The off-diagonal $K$-matrix $K(u, h)$, (3.4). The horizontal lines carry a spectral parameter $u$, while the vertical lines carry $-u$. 

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Figure 4: The reflection equation (3.5). The left- and right-hand sides of the figure represent the left- and right-hand sides of the reflection equation $R_{ba}(u - v, h)K_b(u, h)R_{ab}(v + u, h)K_a(v, h)$ and $K_a(v, h)R_{ba}(u + v, h)K_b(u, h)R_{ab}(u - v, h)$, respectively.

Proof. Properties (1)–(3) can be proved in the standard way.

We first show property (1). We use the completeness relation in one up-spin sector,

$$\sum_{j=1}^{2n-1} |0^{j-1}10^{2n-j-1}\rangle \langle 0^{j-1}10^{2n-j-1}| = \text{Id}, \quad (4.11)$$

on the space $W_2 \otimes \cdots \otimes W_{2n}$ with

$$|0^{j-1}10^{2n-j-1}\rangle = |0\rangle_2 \otimes \cdots \otimes |0\rangle_j \otimes |1\rangle_{j+1} \otimes |0\rangle_{j+2} \otimes \cdots \otimes |0\rangle_{2n},$$

$$\langle 0^{j-1}10^{2n-j-1}| = 2|0\rangle \otimes \cdots \otimes j|0\rangle \otimes j+1|1\rangle \otimes j+2|0\rangle \otimes \cdots \otimes 2n|0\rangle,$$

and decompose $P_{2n}(u_1, \ldots, u_{2n}|h)$ as

$$P_{2n}(u_1, \ldots, u_{2n}|h) = \sum_{j=1}^{2n-1} 2^{n-1} \langle \Omega| T_{2n}(u_1|h) \cdots T_2(u_2, \ldots, u_{2n}|h) |0^{j-1}10^{2n-j-1}\rangle$$

$$\times \langle 0| \otimes \langle 0^{j-1}10^{2n-j-1}| T_1(u_1, \ldots, u_{2n}|h) |\Omega\rangle_{2n}, \quad (4.12)$$

where $2^{-1}\langle \Omega| = z|0\rangle \otimes \cdots \otimes 2|0\rangle$.

One can easily calculate the explicit forms of $1\langle 0| \otimes \langle 0^{j-1}10^{2n-j-1}| T_1(u_1, \ldots, u_{2n}|h) |\Omega\rangle_{2n} = f_j(u_1)$ in (4.12), which are

$$f_j(u_1) = \frac{1/2|h + (j - 1)/2 + u_1 + u_{j+1}]}{|h|}$$

$$\times \prod_{k=2}^{j} [u_1 + u_k] \prod_{k=j+2}^{2n} [u_1 + u_k + 1/2]. \quad (4.13)$$
\( T_j(u_j, \ldots, u_{2n}|h) \)

\[ T_j(u_j, \ldots, u_{2n}|h) \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{The monodromy matrix \( T_j(u_j, \ldots, u_{2n}|h) \), \( (4.3) \), constructed from one K-matrix \( (3.4) \) and \( 2n - j \) dynamical R-matrices \( (3.1) \).}
\end{figure}

It is easy to see from \( (4.13) \) and the quasi-periodicities of the theta functions \( (2.5) \) and \( (2.6) \) that the quasi-periodicities of \( f_j(u_1) \) are
\[ f_j(u_1 + 1) = (-1)^{2n-1} f_j(u_1), \]
\[ f_j(u_1 - i \log(q)/\pi) = (-q^{-1})^{2n-1} \exp \left( -2\pi i \left( (2n-1)u_1 + h + \sum_{\ell=2}^{2n} u_{\ell} \right) \right) f_j(u_1). \]

Since the quasi-periodicities for \( f_j(u_1) \)s \( (4.14) \), and \( (4.15) \), do not depend on \( j \), and noting that the dependence on \( u_1 \) for each summand in the right hand side of \( (4.12) \), comes only from \( f_j(u_1) \), one finds that the partition functions \( P_{2n}(u_1, \ldots, u_{2n}|h) \) are:
\[ P_{2n}(u_1 + 1, \ldots, u_{2n}|h) = (-1)^{2n-1} P_{2n}(u_1, \ldots, u_{2n}|h), \]
\[ P_{2n}(u_1 - i \log(q)/\pi, \ldots, u_{2n}|h) = (-q^{-1})^{2n-1} \exp \left( -2\pi i \left( (2n-1)u_1 + h + \sum_{j=2}^{2n} u_{j} \right) \right) P_{2n}(u_1, \ldots, u_{2n}|h). \]

From the above quasi-periodicities, one concludes that \( P_{2n}(u_1, \ldots, u_{2n}|h) \) are elliptic polynomials of \( u_1 \) of degree \( 2n - 1 \).

Next, we prove property (2). First, one shows \( (4.9) \) for the case \( \ell = 2 \) by using a graphical representation of the partition functions (Fig. \[ \] ), as is always the case when using the Korepin’s trick. First, one observes that the K-matrix at the bottom row is already frozen.
Figure 6: Partition functions $P_{2n}(u_1, \ldots, u_{2n}|h)$, (4.4), under OS boundary.

since the $K$-matrix we use for the partition functions $P_{2n}(u_1, \ldots, u_{2n}|h)$ is an off-diagonal one (3.4). If we set $u_1 = -u_2$, one finds that the $R$-matrix adjacent to the frozen $K$-matrix gets frozen since $1\langle 1|2\langle 0|R_{12}(-u_2, -u_2, h)|1\rangle_1|0\rangle_2 = 0$, and continue graphical observation using the ice-rule

$$a\langle \gamma|b\langle \delta|R_{ab}(u, v, h)|\alpha\rangle_a|\beta\rangle_b = 0, \quad \text{unless} \quad \alpha + \beta = \gamma + \delta,$$

(4.18)

one sees that the two bottom rows freeze (Fig. 7). The product of the matrix elements of the $R$-matrices of the frozen two rows is $[1/2] \prod_{j=3}^{2n} [u_j + u_2 + 1/2][u_j - u_2 + 1/2]$, and the remaining unfrozen part is $P_{2n-2}(u_3, \ldots, u_{2n}|h)$, and we get

$$P_{2n}(u_1, \ldots, u_{2n}|h)|_{u_1=-u_2}$$

$$= [1/2] \prod_{j=3}^{2n} [u_j + u_2 + 1/2][u_j - u_2 + 1/2]P_{2n-2}(u_3, \ldots, u_{2n}|h).$$

(4.19)

One can show that the partition functions $P_{2n}(u_1, \ldots, u_{2n}|h)$ are symmetric with respect to the spectral parameters $u_1, \ldots, u_{2n}$ by the standard railroad argument using the dynamical Yang-Baxter relation and the reflection equation (Kuperberg [12], see also [62]). From (4.19)
Figure 7: Partition functions $P_{2n}(u_1, \ldots, u_{2n} | h)$ evaluated at $u_1 = -u_2$ \textcolor{red}{(4.9)}. The bottom two rows are frozen due to the properties of the dynamical $R$-matrix \textcolor{red}{(3.1)} and the $K$-matrix \textcolor{red}{(3.4)}.

and the symmetry property, one finds the following relations hold

$$P_{2n}(u_1, \ldots, u_{2n} | h)_{u_1 = -u_\ell}$$

$$= \left[ \frac{1}{2} \right] \prod_{j=2, j \neq \ell}^{2n} [u_j + u_\ell + 1/2] [u_j - u_\ell + 1/2] P_{2n-2}(u_2, \ldots, \hat{u_\ell}, \ldots, u_{2n} | h), \quad (4.20)$$

for $\ell = 2, \ldots, 2n - 1$.

Finally, it is trivial to check Property (3) from the definition of the $R$-matrix \textcolor{red}{(3.1)}. \hfill \square

One can prove that there are explicit expressions for the partition functions under OS boundary $P_{2n}(u_1, \ldots, u_{2n} | h)$ in terms of elliptic Pfaffians by showing that the right hand side of \textcolor{red}{(4.21)} satisfy all the properties in Proposition \textcolor{red}{4.1}.

**Theorem 4.2.** The partition functions under OS boundary $P_{2n}(u_1, \ldots, u_{2n} | h)$ have the following expressions using elliptic Pfaffians:

$$P_{2n}(u_1, \ldots, u_{2n} | h)$$

$$= \prod_{1 \leq i < j \leq 2n} \frac{[u_j + u_i][u_j - u_i + 1/2]}{[u_j - u_i]} \text{Pf} \left( \begin{array}{c} [1/2] [u_j - u_i][u_i + u_j + h] \\ h [u_i + u_j][u_j - u_i + 1/2] \end{array} \right)_{1 \leq i, j \leq 2n}. \quad (4.21)$$

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Figure 8: Partition functions $P_{2n}(u_1, \ldots, u_{2n}|h)$ evaluated at $u_1 = -u_{2n} - 1/2$. The bottom row and the rightmost column are frozen due to the properties of the dynamical $R$-matrix (3.1) and the $K$-matrix (3.4).

Note that

$$
\begin{pmatrix}
\frac{1}{2} [u_j - u_i] [u_i + u_j + h] \\
h [u_i + u_j] [u_j - u_i + 1/2]
\end{pmatrix}_{1 \leq i,j \leq 2n}
$$

is a skew-symmetric matrix which can be checked using the facts that $[u]$ is an odd function and the property (2.7).

**Proof.** Let us denote the right hand side of (4.21) as $E_{2n}(u_1, \ldots, u_{2n}|h)$:

$$
E_{2n}(u_1, \ldots, u_{2n}|h) := \prod_{1 \leq i < j \leq 2n} \frac{[u_j + u_i][u_j - u_i + 1/2]}{[u_j - u_i]} \text{Pf}\left(\frac{[1/2][u_j - u_i][u_i + u_j + h]}{[h][u_i + u_j][u_j - u_i + 1/2]}\right)_{1 \leq i,j \leq 2n}.
$$

(4.22)

We show that $E_{2n}(u_1, \ldots, u_{2n}|h)$ satisfy all the properties in Proposition 4.1. Let us show Property (1). To check this, we view $E_{2n}(u_1, \ldots, u_{2n}|h)$ as a function of $u_1$ and split the function as $E_{2n}(u_1, \ldots, u_{2n}|h) = e_1(u_1) e_2(u_1)$, where $e_1(u_1)$ and $e_2(u_1)$ are the overall factor and the elliptic Pfaffians, respectively

$$
e_1(u_1) = \prod_{1 \leq i < j \leq 2n} \frac{[u_j + u_i][u_j - u_i + 1/2]}{[u_j - u_i]},
$$

(4.23)

$$
e_2(u_1) = \text{Pf}\left(\frac{[1/2][u_j - u_i][u_i + u_j + h]}{[h][u_i + u_j][u_j - u_i + 1/2]}\right)_{1 \leq i,j \leq 2n}.
$$

(4.24)
Using the quasi-periodicities of the theta functions (2.5) and (2.6), it is easy to calculate the quasi-periodicities of the overall factor $e_1(u_1)$:

$$e_1(u_1 + 1) = (-1)^{2n-1}e_1(u_1),$$
$$e_1(u_1 - i \log(q)/\pi) = (-q^{-1})^{2n-1} \exp \left(-2\pi i \left(2n - 1\right) u_1 + \sum_{j=2}^{2n} u_j - 1/2 \right) e_1(u_1).$$

Next, noting that the quasi-periodicities of the matrix elements of the first row of the matrix $X$, which are used to construct the Pfaffian $\text{Pf}X$, are cancelled by the denominator of the overall factor $e_1(u_1)$, one can calculate the quasi-periodicities of $e_2(u_1) = \text{Pf}X$ and get

$$e_2(u_1 + 1) = e_2(u_1),$$
$$e_2(u_1 - i \log(q)/\pi) = \exp(-2\pi i (h + 1/2))e_2(u_1).$$

Combining (4.26), (4.27), (4.29) and (4.30), we get the quasi-periodicities of $E_{2n}(u_1, \ldots, u_{2n}|h)$

$$E_{2n}(u_1 + 1, \ldots, u_{2n}|h) = (-1)^{2n-1}E_{2n}(u_1, \ldots, u_{2n}|h),$$
$$E_{2n}(u_1 - i \log(q)/\pi, \ldots, u_{2n}|h) = (-q^{-1})^{2n-1} \exp \left(-2\pi i \left(2n - 1\right) u_1 + \sum_{j=2}^{2n} u_j \right) E_{2n}(u_1, \ldots, u_{2n}|h).$$

We can also check that $E_{2n}(u_1 + 1, \ldots, u_{2n}|h)$ is holomorphic as a function of $u_1$. The factors $[u_i + u_j]$ and $[u_j - u_i + 1/2]$ in the denominators $X_{ij} = \left[1/2[u_j - u_i + u_j + h]\right]_{[u_j + u_i][u_j - u_i + 1/2]}$ of the matrix elements of $X$, which are used to construct the Pfaffian $X$, are cancelled by the overall factor $\prod_{1 \leq i < j \leq 2n} \left[\prod_{1 \leq i < j \leq 2n} \frac{[u_j + u_i][u_j - u_i + 1/2]}{[u_j - u_i]}\right]$. The factors $[u_k - u_i]$, $k = 2, \ldots, 2n$ in the denominator of the overall factor may lead to singularities at $u_1 = u_k$ ($k = 2, \ldots, 2n$), but does not. For example, let us see the case $k = 2$. If one expands the Pfaffian of the matrix $X$, all summands containing $X_{12}$ as a factor of the product have the factor $[u_2 - u_1]$ in the numerator. The sum of the summands which do not contain $X_{12}$ can be rearranged as a linear combination of the terms $(X_{1j}X_{2k} - X_{1k}X_{2j}) \prod_{i=3}^{n} X_{i(i-1)}(u_{i+2})$ ($j, k \neq 2$).
The definition of \( E \) functions the expression (4.34) from the expansion (4.33). (4.34) is the relation which the partition \( X \) following properties:

Here, we have used the fact that \( u = \) \( u_x \), \( k \)

Proposition 4.3. The version of the elliptic Izergin-Korepin analysis which is presented below.

which looks almost the same, but is slightly different from Proposition 4.1, i.e., a different property for the theta function \( \theta \) survives. Here, we use the basic property for the theta function \( \theta = 0 \) for this observation. After the substitution \( u_1 = -u_\ell \) in (4.33) and after simplifications, one finds

Here, we have used the fact that \( u \) is odd \( -u = -u \) and the property (2.7) to get the expression (4.37) from the expansion (4.33). (4.34) is the relation which the partition functions \( P_m(u_1, \ldots, u_{2n}|h) \) must satisfy, hence Property (2) is proved.

The only thing left to do is to check Property (3), which can be easily seen from the definition of \( E_m(u_1, \ldots, u_{2n}|h) \) in (4.21).

Theorem 4.2 is proved by showing that the right hand side of (4.21) satisfy the properties in Proposition 4.1.

We can make another Proposition (Korepin’s characterization of the partition functions) which looks almost the same, but is slightly different from Proposition 4.1 i.e., a different version of the elliptic Izergin-Korepin analysis which is presented below.

Proposition 4.3. The partition functions under OS boundary \( P_m(u_1, \ldots, u_{2n}|h) \) satisfy the following properties:

(1) The partition functions \( P_m(u_1, \ldots, u_{2n}|h) \) are elliptic polynomials of \( u_1 \) of degree \( 2n-1 \) with the following quasi-periodicities:

\[
P_m(u_1 + 1, \ldots, u_{2n}|h) = (-1)^{2n-1} P_m(u_1, \ldots, u_{2n}|h),
\]

\[
P_m(u_1 - i \log(q)/\pi, \ldots, u_{2n}|h)
\]

\[
= (-q^{-1})^{2n-1} \exp(-2\pi i (2n-1)u_1 + h + 2n \sum_{j=2}^{2n} u_j) \] \( P_m(u_1, \ldots, u_{2n}|h). \) (4.36)
The following relations among the partition functions hold (Fig. 8):

\[
P_{2n}(u_1, \ldots, u_{2n}|h)|_{u_1=-u_{\ell-1}} = \frac{[h - 1/2][1/2]}{[h]} \prod_{j=2}^{2n} [u_j + u_{\ell}] [u_j - u_{\ell} - 1/2] P_{2n-2}(u_2, \ldots, u_{2n-1}|h), \tag{4.37}
\]

for \( \ell = 2, \ldots, 2n \), and \( u_\ell \) in \( P_{2n-2}(u_2, \ldots, u_{2n-1}|h) \) means that \( u_\ell \) is removed.

The following evaluation holds:

\[
W_2(u_1, u_2|h) = \frac{[1/2][h + u_1 + u_2]}{[h]}, \tag{4.38}
\]

The difference between Proposition 4.4 and Proposition 4.3 is Property (2): The relation in Proposition 4.4 is the statement about the evaluations of the partition functions \( P_{2n}(u_1, \ldots, u_{2n}|h) \) at \( u_1 = -u_\ell, \ell = 2, \ldots, 2n \), while the relations in Proposition 4.3 are the evaluations at \( u_1 = -u_\ell - 1/2, \ell = 2, \ldots, 2n \).

**Proof.** The only additional thing to prove is (4.37). It is enough to prove the case \( \ell = 2n \) since the other cases \( \ell = 2, \ldots, 2n - 1 \) follow from the case \( \ell = 2n \) of (4.37) by using the symmetry of \( P_{2n}(u_1, \ldots, u_{2n}|h) \) with respect to the variables \( u_1, \ldots, u_{2n} \).

We again use the graphical representation of \( P_{2n}(u_1, \ldots, u_{2n}|h) \) (Fig. 8). We first realize that when we set \( u_1 \) to \( u_2 = -u_{2n} - 1/2 \), the \( R \)-matrix at the southeast corner starts to freeze since \( \langle 10 | 2n, 0 | R_{1,2n}(-u_{2n} - 1/2, -u_{2n}, h) | 01 \rangle_{2n} = 0 \), and continuing graphical observation using the ice-rule of the \( R \)-matrix (4.18), one finds that the \( R \)- and \( K \)-matrices at the bottom row and the rightmost column are frozen. The contribution of these frozen parts to the partition functions is the overall factor \( [h - 1/2][1/2][h]^{-1} \prod_{j=2}^{2n-1} [u_j + u_{2n}] [u_j - u_{2n} - 1/2] \), and the remaining unfrozen part is \( P_{2n-2}(u_2, \ldots, u_{2n-1}|h) \). Hence, we get

\[
P_{2n}(u_1, \ldots, u_{2n}|h)|_{u_1=-u_{2n}-1/2} = \frac{[h - 1/2][1/2]}{[h]} \prod_{j=2}^{2n-1} [u_j + u_{2n}] [u_j - u_{2n} - 1/2] P_{2n-2}(u_2, \ldots, u_{2n-1}|h). \tag{4.39}
\]

One can obtain a similar but different elliptic Pfaffian representation of the partition functions from (4.21) in Theorem 4.2 by finding a representation satisfying the properties in Proposition 4.3.

**Theorem 4.4.** The partition functions \( P_{2n}(u_1, \ldots, u_{2n}|h) \) under OS boundary have the following expressions using elliptic Pfaffians:

\[
P_{2n}(u_1, \ldots, u_{2n}|h) = \prod_{1 \leq i < j \leq 2n} \frac{[u_i + u_j + 1/2][u_j - u_i + 1/2]}{[u_j - u_i]} \text{Pf} \left( \frac{[1/2][u_i - u_j][u_i + u_j + h]}{[h][u_i + u_j + 1/2][u_j - u_i + 1/2]} \right)_{1 \leq i, j \leq 2n}. \tag{4.40}
\]
Proof. This can be proved in the same way as proving Theorem 4.2. For example, one can show that (4.40) satisfies (4.37) just as the same way with proving that (4.21) satisfies (4.9) using the expansion formula for the Pfaffian (2.3).

We derived two elliptic Pfaffian representations of the partition functions $P_{2n}(u_1, \ldots, u_{2n} | h)$ (4.21) in Theorem 4.2 and (4.40) in Theorem 4.4 based on two versions of the Korepin’s trick Proposition 4.1 and Proposition 4.3. By comparing (4.21) and (4.40), we get the following identity between two elliptic Pfaffians.

**Theorem 4.5.** The following identity between two elliptic Pfaffians holds:

$$
\prod_{1 \leq i < j \leq 2n} \frac{[u_j + u_i][u_j - u_i + 1/2]}{[u_j - u_i]} Pf\left(\frac{[1/2][u_j - u_i][u_i + u_j + h]}{[h][u_i + u_j][u_j - u_i + 1/2]}\right)_{1 \leq i, j \leq 2n} = \prod_{1 \leq i < j \leq 2n} \frac{[u_j + u_i + 1/2][u_j - u_i + 1/2]}{[u_j - u_i]} Pf\left(\frac{[1/2][u_j - u_i][u_i + u_j + h]}{[h][u_i + u_j + 1/2][u_j - u_i + 1/2]}\right)_{1 \leq i, j \leq 2n},
$$

(4.41)

or equivalently,

$$
\prod_{1 \leq i < j \leq 2n} [u_j + u_i] Pf\left(\frac{[u_j - u_i][u_i + u_j + h]}{[u_i + u_j][u_j - u_i + 1/2]}\right)_{1 \leq i, j \leq 2n} = \prod_{1 \leq i < j \leq 2n} [u_j + u_i + 1/2] Pf\left(\frac{[u_j - u_i][u_i + u_j + h]}{[u_i + u_j + 1/2][u_j - u_i + 1/2]}\right)_{1 \leq i, j \leq 2n},
$$

(4.42)

after cancelling common factors in both hand sides of (4.41).

We check (4.41), (4.42) for the case $n = 2$ in the Appendix by using addition formulas for the theta functions (2.8) repeatedly.

5 Conclusion

In this paper, we studied the partition functions of the elliptic free-fermionic face model under OS boundary, and analyzed them by using the elliptic Izergin-Korepin analysis. We obtained the representations of the partition functions using elliptic Pfaffians. Since use can use the Korepin’s trick in two ways, we can get two Pfaffian representations for the same partition functions. As a corollary of the two expressions, we get an identity between two elliptic Pfaffians.

It is interesting to extend the analysis performed on the OS boundary in this paper to other boundary conditions, i.e., consider various variations of the domain wall boundary partition functions introduced by Kuperberg [12] for the case of the elliptic face models. More complicated boundary conditions may lead to expressions as products of determinants and Pfaffians as is the case for the trigonometric models, and may also lead to various interesting identities between elliptic determinants and elliptic Pfaffians. It is also interesting to investigate if the elliptic Pfaffian identities by Okada [33] and Rosengren [34, 35] can be understood as different representations of the same partition functions of elliptic integrable models.
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A Appendix: An elementary proof of (4.41), (4.42) for the case \( n = 2 \)

In this Appendix, we check (4.41), (4.42) for the case \( n = 2 \) by elementary manipulations. In this case, one can see from the definition of Pfaffians (2.11) that proving (4.41), (4.42) is equivalent to showing the following identity

\[
\begin{align*}
&\frac{[u_2 - u_1][u_1 + u_2 + h][u_4 - u_3][u_3 + u_4 + h]}{[u_2 - u_1 + 1/2][u_4 - u_3 + 1/2]} \\
&\times [u_3 + u_1 + 1/2][u_4 + u_1 + 1/2][u_3 + u_2 + 1/2][u_4 + u_2 + 1/2] \\
&\quad - \frac{[u_3 - u_1][u_1 + u_3 + h][u_4 - u_2][u_2 + u_4 + h]}{[u_3 - u_1 + 1/2][u_4 - u_2 + 1/2]} \\
&\quad \times [u_2 + u_1 + 1/2][u_4 + u_1 + 1/2][u_3 + u_2 + 1/2][u_4 + u_3 + 1/2] \\
&\quad + \frac{[u_4 - u_1][u_1 + u_4 + h][u_3 - u_2][u_2 + u_3 + h]}{[u_4 - u_1 + 1/2][u_3 - u_2 + 1/2]} \\
&\quad \times [u_2 + u_1 + 1/2][u_3 + u_1 + 1/2][u_4 + u_2 + 1/2][u_4 + u_3 + 1/2] \\
&= \frac{[u_2 - u_1][u_1 + u_2 + h][u_4 - u_3][u_3 + u_4 + h]}{[u_4 - u_3 + 1/2][u_4 - u_2 + 1/2]} \\
&\quad - [u_3 - u_1][u_1 + u_3 + h][u_4 - u_2][u_2 + u_4 + h] \\
&\quad \times [u_2 + u_1 + 1/2][u_4 + u_1 + 1/2][u_3 + u_2 + 1/2][u_4 + u_3 + 1/2] \\
&\quad + [u_4 - u_1][u_1 + u_4 + h][u_3 - u_2][u_2 + u_3 + h] \\
&\quad \times [u_4 - u_1 + 1/2][u_3 - u_2 + 1/2]. \tag{A.1}
\end{align*}
\]

Let us show this using the addition formula for the theta functions (2.8) repeatedly. The difference of the left hand side and the right hand side of (A.1) can be expressed as

\[
\begin{align*}
&\frac{[u_2 - u_1][u_1 + u_2 + h][u_4 - u_3][u_3 + u_4 + h]}{[u_2 - u_1 + 1/2][u_4 - u_3 + 1/2]} \\
&\times ([u_3 + u_1 + 1/2][u_4 + u_1 + 1/2][u_3 + u_2 + 1/2][u_4 + u_2 + 1/2] \\
&\quad - [u_3 + u_1][u_4 + u_1][u_3 + u_2][u_4 + u_2]) \\
&\quad - [u_3 - u_1][u_1 + u_3 + h][u_4 - u_2][u_2 + u_4 + h] \\
&\quad \times ([u_2 + u_1 + 1/2][u_4 + u_1 + 1/2][u_3 + u_2 + 1/2][u_4 + u_3 + 1/2] \\
&\quad - [u_2 + u_1][u_4 + u_1][u_3 + u_2][u_4 + u_3]) \\
&\quad + [u_4 - u_1][u_1 + u_4 + h][u_3 - u_2][u_2 + u_3 + h] \\
&\quad \times ([u_4 - u_1 + 1/2][u_3 - u_2 + 1/2] \\
&\quad - [u_2 + u_1][u_3 + u_1][u_4 + u_2][u_4 + u_3]). \tag{A.2}
\end{align*}
\]
Using the addition formula for the theta functions (2.8) (and (2.7)), one finds

\[ u_3 + u_1 + 1/2][u_4 + u_1 + 1/2][u_3 + u_2 + 1/2][u_4 + u_2 + 1/2] \]
\[- [u_4 + u_1][u_4 + u_1][u_3 + u_2][u_4 + u_2] \]
\[= [1/2][u_1 + u_2 + u_3 + u_4 + 1/2][u_2 - u_1 + 1/2][u_4 - u_3 + 1/2], \] (A.3)
\[ [u_2 + u_1 + 1/2][u_4 + u_1 + 1/2][u_3 + u_2 + 1/2][u_4 + u_3 + 1/2] \]
\[- [u_2 + u_1][u_4 + u_1][u_3 + u_2][u_4 + u_3] \]
\[= [1/2][u_1 + u_2 + u_3 + u_4 + 1/2][u_3 - u_1 + 1/2][u_4 - u_2 + 1/2], \] (A.4)
\[ [u_2 + u_1 + 1/2][u_3 + u_1 + 1/2][u_4 + u_2 + 1/2][u_4 + u_3 + 1/2] \]
\[- [u_2 + u_1][u_3 + u_1][u_4 + u_2][u_4 + u_3] \]
\[= [1/2][u_1 + u_2 + u_3 + u_4 + 1/2][u_3 - u_2 + 1/2][u_4 - u_1 + 1/2]. \] (A.5)

Using the identities (A.3), (A.4) and (A.5), (A.2) reduces to

\[ [u_2 - u_1][u_1 + u_2 + h][u_4 - u_3][u_3 + u_4 + h] \]
\[= [u_2 - u_1 + 1/2][u_4 - u_3 + 1/2] \]
\[\times [1/2][u_1 + u_2 + u_3 + u_4 + 1/2][u_2 - u_1 + 1/2][u_4 - u_3 + 1/2] \]
\[+ [u_3 - u_1][u_1 + u_3 + h][u_4 - u_2][u_2 + u_4 + h] \]
\[= [u_3 - u_1 + 1/2][u_4 - u_2 + 1/2] \]
\[\times [1/2][u_1 + u_2 + u_3 + u_4 + 1/2][u_3 - u_1 + 1/2][u_4 - u_2 + 1/2] \]
\[+ [u_4 - u_1][u_1 + u_4 + h][u_3 - u_2][u_2 + u_3 + h] \]
\[= [u_4 - u_1 + 1/2][u_3 - u_2 + 1/2] \]
\[\times [1/2][u_1 + u_2 + u_3 + u_4 + 1/2][u_3 - u_2 + 1/2][u_4 - u_1 + 1/2] \]
\[= [1/2][u_1 + u_2 + u_3 + u_4 + 1/2][u_2 - u_1 + 1/2][u_4 - u_3 + 1/2] \]
\[- [u_3 - u_1][u_1 + u_3 + h][u_4 - u_2][u_2 + u_4 + h] + [u_4 - u_1][u_1 + u_4 + h][u_3 - u_2][u_2 + u_3 + h]. \] (A.6)

One can apply the addition formula (2.8) again to get

\[ [u_2 - u_1][u_1 + u_2 + h][u_4 - u_5][u_3 + u_4 + h] - [u_3 - u_1][u_1 + u_3 + h][u_4 - u_2][u_2 + u_4 + h] \]
\[+ [u_4 - u_1][u_1 + u_4 + h][u_3 - u_2][u_2 + u_3 + h] = 0. \] (A.7)

and we find the right hand side of (A.6) becomes zero, hence (4.41), (4.42) for the case \( n = 2 \) is proved.

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