Coherent acoustic response of a screen containing a random distribution of scatterers: comparison between different approaches

J. Dubois, C. Aristégui, O. Poncelet, A. L. Shuvalov
Université de Bordeaux, CNRS, UMR 5469, Laboratoire de Mécanique Physique, 351, Cours de la Libération, Talence, F-33405, France
E-mail: j.dubois@lmp.u-bordeaux1.fr

Abstract. Theoretical models underlying the ultrasonic study of suspensions and bubble swarms in liquids often rely on the concept of a coherent wave response, within which the given medium is viewed as an effective homogeneous medium. More specifically, the coherent response can be formulated in a straightforward manner via the effective wave number and the effective impedance. These in turn can be expressed through the actual material properties of the given scattering medium. The derivation of the effective wave number and impedance is the issue of a number of different approaches existing in the multiple-scattering theory.

The present work deals with a coherent response of acoustic wave impinging on a screen of cylindrical inclusions randomly distributed in a fluid. The reflection and transmission coefficients have been expressed via the effective wave number and impedance that are provided by three different models due to Foldy, Waterman & Truell, and Linton & Martin. The obtained expressions have been analyzed with a view to illuminate what is the quantitative difference between those approaches as revealed in the coherent response, and how this difference depends on the basic parameters of the problem such as the frequency, the concentration of scatterers and their contrast relatively to the fluid matrix. Another aspect of this work is to compare the above analytical results with the numerical data. It has been obtained by means of a deterministic computational code which delivers the coherent wave field through averaging the outputs for various samplings of the positions of scatterers. Knowing this numerical benchmark allows us to specify the validity domains for each of the three analytical methods under study.

1. Introduction

Propagation of waves in many types of inhomogeneous media, e.g. of seismic waves in the crust, ultrasonic waves in composites, waves in bubbly fluids or in suspended particles, is accompanied by multiple scattering. If the exact location of each scatterer is known, it is theoretically possible to solve the deterministic problem, but the solution becomes not reachable when the scatterers are too numerous. Moreover, in most of the cases, only a statistical description of the medium is accessible.

In his seminal paper [1], Foldy studied multiple scattering of scalar waves by a random distribution of isotropic point-like scatterers in a fluid matrix. By averaging the field over all possible configurations of the positions of scatterers, he determined the effective wave number $k_{\text{eff}}$ of the coherent wave in the form $k_{\text{eff}}^2 = k_0^2 - 4\pi gn_0$, where $k_0$ is the wave number in the matrix, $g$ is the scattering coefficient for an individual scatterer and $n_0$ is the number of scatterers.

Published under licence by IOP Publishing Ltd
per unit volume. Extending the model to anisotropic point-like scatterers, Waterman & Truell [2] obtained the effective wave number with the quadratic term in $n_0$: $k_{\text{eff}}^2 = k_0^2 + \delta_1 n_0 + \delta_2 n_0^2$, where $\delta_{1,2}$ are computable quantities (see § 3.1). Taking into account the real size of anisotropic scatterers, Lloyd & Berry [3] derived a formula, which differs from that of [2] by the form of the coefficient $\delta_2$. Using the assumption of [3], Linton & Martin [4] provided a similar formula for $k_{\text{eff}}$ in the two-dimensional case. Other expressions for the effective wave number may be found e.g. in [5-7].

Experiments show that a coherent wave, which is created by a plane wave impinging on a random medium, is also plane. Therefore a random medium can be replaced by an equivalent homogeneous medium from the viewpoint of the coherent wave. Kuster & Toksöz [8] expressed the effective properties of such media for the long-wavelength limit by considering only simple scattering. Aristégui & Angel [9] developed a procedure for evaluating the mechanical properties of an effective medium in the framework of Waterman & Truell, while Luppé & Conoir [10] proceeded in a similar way but in the framework of Linton & Martin. The effective properties obtained by either of these methods are complex-valued and frequency-dependent.

The aim of this work is to compare the theoretical predictions of the homogenization techniques [9, 10], which are based on the Foldy (F), Waterman & Truell (WT) and Linton & Martin (LM) models of multiple scattering theory (MST), with our numerical simulations performed by means of a finite difference time domain (FDTD) code. For this purpose, a fluid matrix containing fluid cylindrical scatterers will be considered. The paper is organized as follows. First, the statement of the problem and the numerical scheme for simulating the coherent reflected and transmitted waves are presented. Second, we recapitulate the explicit results of [9, 10], which provide the theoretical predictions for the reflection and transmission coefficients on an infinite screen of scatterers. Then the numerical results are presented and compared with the above theoretical estimations at different concentrations and at different contrasts between the properties of the matrix and of the scatterers.

2. Statement of the problem

We consider a screen of fluid cylindrical scatterers, immersed in a nonviscous fluid matrix, see Fig. 1. The scatterers of radius $a$, which are parallel and identical to each other, are uniformly and randomly distributed with an area fraction $\phi$. The infinite screen of thickness $e$ is insonified by a normally incident plane wave.

![Screen of scatterers immersed in a matrix.](image)

Multiple scattering gives rise to a reflected field and distorts the transmitted one. These signals consist of a coherent and incoherent part. The incoherent part vanishes due to averaging over a sufficiently large number of configurations of positions of the scatterers. In practice it is not easy to access the coherent-wave properties with only one sample of the random medium, for this requires to perform plenty of local measurements along the sample to restore the averaging procedure. Therefore, numerical simulations prove to be a good alternative.
Fig. 2 shows the geometrical parameters that were used in the FDTD scheme. The horizontal extent of the screen is restricted to a finite length with periodical boundary conditions imposed at each lateral side in order to ensure a plane front of the incident wave. This length is taken to be large enough with respect to the scatterer radius (200a) for making the effect of the periodical boundary conditions negligible. For each simulation, a new random configuration of the positions of scatterers is generated. The only condition imposed for any distribution is that at least one scatterer is tangent to each of the virtual interfaces of the screen. The reflection and transmission coefficients are defined by dividing the coherent reflected and transmitted spectra by the incident one. We have made between 90 and 210 simulations (depending on the parameters of the system) in order to achieve the convergence of the averaged field.

![Figure 2. Geometry of the FDTD scheme (PML are perfectly matched layer).](image)

The thickness \( e \) of the screen may admit two different values. According to the standard ansatz [1] of the MST, the thickness \( e_1 \) is defined by the centers of the extreme scatterers, see Fig. 3. The corresponding concentration \( \phi_1 \) is given by \( \phi_1 = \frac{N\pi a^2}{A_{e_1}} \), where \( N \) is the number of scatterers inside the screen and \( A_{e_1} \) is the area of the screen determined by the thickness \( e_1 \). On the other hand, actually the screen is limited by the tangents to the extreme scatterers, which define the thickness \( e_2 \) (Fig. 3). From this perspective, the concentration \( \phi_2 \) of scatterers inside the screen is \( \phi_2 = \frac{N\pi a^2}{A_{e_2}} \), where \( A_{e_2} \) is the area of the screen determined by the thickness \( e_2 \). For the simulations \( e_2 (e_2 = e_1 + 2a) \) is taken to be 18a.

3. Theoretical background (the effective-medium models)

3.1. Multiple scattering theory: expressions of the effective wavenumber

Under the assumption \( \frac{Q_s}{k_0} \approx (k_0 a)^3 \ll 1 \), where \( Q_s \) is the scattering cross-section and \( n_0 \) is the number of scatterers per unit area, the coherent wave number provided by Waterman & Truell [2] is

\[
\frac{k_{eff}^2}{k_0^2} = 1 - \frac{4i n_0}{k_0^2} f(0) - \frac{4m_0^2}{k_0^4} \left[ f^2(0) - f^2(\pi) \right],
\]

where

\[
f(\theta) = \sum_{n=0}^{\infty} \epsilon_n C_n \cos(n\theta)
\]

is the angular shape function of a cylinder (in the three-dimensional case, the angular shape function of a sphere is used), \( C_n \) is the scattering amplitude factors of the \( n \)th mode, and \( \epsilon_n \) is the Neumann symbol equal to 1 if \( n = 0 \) and to 2 for \( n > 0 \).
For isotropic scatterers, \( f \equiv f_0 \) is constant, hence \( f(\pi) = f(0) \), and so Eq. (1) simplifies to the form:

\[
\frac{k_{\text{eff}}^2}{k_0^2} = 1 - \frac{4i\eta_0}{k_0^2} f_0.
\]  

(3)

This expression is similar to the result of Foldy, which, however, takes into account only the first term of the modal series (2).

Fikioris & Waterman [7] introduced the “hole correction” to avoid overlapping of the scatterers by means of applying a pair correlation function in the calculation of the mean field. Using this, Linton & Martin [4] determined the low-frequency limit of the coherent wave number for the case of cylindrical scatterers as:

\[
\frac{k_{\text{eff}}^2}{k_0^2} = 1 - \frac{4i\eta_0}{k_0^2} f(0) - \frac{8\eta_0^2}{k_0^4} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |m-n| C_n C_m.
\]  

(4)

Note that Eqs. (1), (3) and (4) are identical to the first order in \( \eta_0 \).

### 3.2. Homogenization techniques: expression of the effective impedance

For the purpose of subsequent comparison with the numerical results, the reflection and transmission coefficients \( R \) and \( T \) shall be evaluated by way of the classical formulas for a homogeneous isotropic slab but with the effective mechanical properties, i.e.:

\[
R = -\frac{Qe^{-ik_0e}}{1 - Q^2e^{2ik_{\text{eff}}e}} \left(1 - e^{2ik_{\text{eff}}e}\right),
\]  

(5)

\[
T = \frac{1 - Q^2}{1 - Q^2e^{2ik_{\text{eff}}e}} e^{i(k_{\text{eff}} - k_0)e},
\]  

(6)

with

\[
Q = \frac{Z_0 - Z_{\text{eff}}}{Z_0 + Z_{\text{eff}}},
\]

where \( Z \) is the impedance and the subscripts “0” and “eff” indicate the matrix and the effective medium, respectively.

By analyzing the reflection and transmission for a screen of scatterers immersed in a nonviscous fluid matrix (Fig. 1), Aristégui & Angel [9] identified the effective impedance as

\[
\frac{Z_{\text{eff}}}{Z_0} = \frac{k_0}{k_{\text{eff}}} \left(1 - \frac{2i\eta_0}{k_0^2} [f(0) - f(\pi)]\right).
\]  

(7)

For isotropic scatterers, Eq. (7) becomes

\[
\frac{Z_{\text{eff}}}{Z_0} = \frac{k_0}{k_{\text{eff}}},
\]  

(8)

which in view of the definition \( Z = \rho \omega/k \) means that the effective mass density \( \rho_{\text{eff}} \) is equal to the mass density of the matrix.

Luppé & Conoir [10] identified the effective impedance in the LM framework as:

\[
\frac{Z_{\text{eff}}}{Z_0} = \frac{k_0}{k_{\text{eff}}} \left(1 - \frac{2i\eta_0}{k_0^2} [f(0) - f(\pi)] + C \frac{4\eta_0^2}{k_0^4}\right),
\]  

(9)
where

$$C = \begin{bmatrix} \frac{1}{2} (f(0)^2 - f(\pi)^2) + \frac{f(\pi) - C_0 - f(0)}{f(0)} J(0) + \frac{C_0}{f(0)} J(\pi) \\ + C_0 \frac{f(0) - f(\pi)}{f(0)} I(\pi) - C_0 \frac{2f(\pi) - f(0) - C_0 I(0)}{f(0)} \end{bmatrix},$$

$$J(\alpha) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |m-n| C_n C_m e^{in\alpha},$$

$$I(\alpha) = \sum_{n=-\infty}^{\infty} |n| C_n e^{in\alpha}.$$ 

4. Numerical results and their comparison with the theory

The following simulations utilize water scatterers and one of the two other fluids as the matrix (Table 1). The ratios between the impedance of these two fluids and of the water are relatively close to each other, while the density and velocity contrasts are weak for the oil/water case, and strong for the FC-72 (perfluorohexane)/water case.

| Fluid     | Density $\rho$ [kg.m$^{-3}$] | Velocity $v$ [m.s$^{-1}$] |
|-----------|-----------------------------|---------------------------|
| water     | $1000$                      | $1500$                    |
| oil       | $800$                       | $1200$                    |
| FC-72     | $1680$                      | $512$                     |

4.1. Reflection and transmission coefficients

On calculating the reflection and transmission coefficients, we have taken into account a possibility of two different definitions of the screen thickness $e_1$ and $e_2$ (Fig. 3) and hence of the concentration of scatterers. It was observed that choosing one or another definition has little effect on the transmission coefficient but affects noticeably the reflection, especially near its resonances. This is highlighted on the low-frequency zooms (Figs. 5, 7, 11), while all other diagrams are referred to the thickness $e_1$.

4.1.1. Oil matrix and water scatterers

Figs. 4-7 present modulus of reflection coefficient $|R|$ versus adimensional frequency $k_0a$ that has been numerically calculated for two different concentrations $\phi_1 = 4.42\%$ and $34.4\%$. These data are compared to the theoretical predictions obtained by [9] in the F and WT frameworks and by [10] in the LM framework. For the low contrast case in hand, the results of [9] in the WT framework and of [10] are too close to be discerned at the scale of the figures.

It is seen that the theoretical predictions of the reflection coefficient obtained by [9] in the F framework do not fit with the numerical data for both concentrations. At low concentration ($\phi_1 = 4.42\%$), the other predictions are in a qualitatively good agreement with the numerical reflection coefficient up to $k_0a = 3$, although they underestimate it. Fig. 5 shows the impact of the optional definitions $e_1$ or $e_2$ of the screen thickness. It is seen that the numerical data lie in between the predictions assuming the two different thickness values. For the concentration $\phi_1 = 13.2\%$ (not shown here), the results are similar, except that it is only the first lobe that
Figure 4. Reflection coefficient for oil matrix and water scatterers, $\phi_1 = 4.42\%$.

Figure 5. Low frequency zoom of Fig. 4.

Figure 6. Reflection coefficient for oil matrix and water scatterers, $\phi_1 = 34.4\%$.

Figure 7. Low-frequency zoom of Fig. 6.

The numerical results for the transmission-coefficient modulus $|T|$ and their comparison with the theoretical predictions of [9, 10] are shown in Figs. 8,9. It is seen that the results of [9, 10] provide a similar evaluation of the transmission at low contrast. The relative error is less than
1% for $\phi_1 = 4.42\%$ and increases to 7% for $\phi_1 = 13.2\%$. For the high concentration $\phi_1 = 34.4\%$, the error reaches 45% at $k_0a = 3$ but does not exceed 5% for $ka \leq 0.5$.

We have also performed simulations for the case of oil scatterers in a water matrix which yields similar results except that the relative errors are slightly higher.

4.1.2. FC-72 matrix and water scatterers

This case is characterized by stronger density and velocity contrasts between the two phases than in the previous setting. For the reflection coefficient, this leads to a quantitatively greater difference between [9] in the WT framework and [10], which is most noticeable at low frequencies and high concentration, see Figs. 10, 11. This difference increases as concentration grows: it is equal to 1.8% and 8% for $\phi_1 = 4.42\%$, and 31.4% respectively. It is seen that the numerical data are closer on the first lobe to the theoretical prediction given by [10] than to the other predictions. The theoretical prediction of [9] in the F framework is found to be no longer valid for the high-contrast case: even for the low concentration $\phi_1 = 4.42\%$, it is five times greater than the numerical data.

Contrary to the low-contrast case of § 4.1.1, the three theories predict slightly different values for the transmission coefficient, which though are still inferior to the numerical data (Fig. 12). The difference between the three theories becomes noticeable only for the high concentration. The relative error is greater than for the low-contrast case: it is 2.3%, 18% and 76% (12% if $k_0a < 0.5$) for $\phi_1 = 4.42\%, 13.2\%$ and 31.4% respectively.
4.2. The effective thickness

As remarked in § 2, the definition of the screen thickness is ambiguous. The thickness-resonances calculated for the two different thickness values $e_1$ and $e_2$ do not coincide with the numerical ones, see Figs. 5, 7. Let us introduce in Eq. (5) the effective thickness $e_{\text{eff}}$, for which the first theoretical and numerical resonance fit together. The concentration is adapted to this thickness similarly to the case of $e_1$ and $e_2$ explained in § 2.

The so-defined effective thicknesses $e_{\text{eff}}$ is equal to 16.82$a$, 16.64$a$ and 17.54$a$ for the concentrations $\phi_1 = 4.42\%$, 13.2%, 31.4%, respectively. Even though the resonances are actually not periodic in $k_0a$ due to a dispersive $k_{\text{eff}}$, it is observed that the effective thickness enables the coincidence of theoretical and numerical resonances up to $k_0a = 1.5$ and 0.5 for $\phi_1 = 4.42\%$ and 31.4%, respectively, see Figs. 13, 14. This observation confirms that, within these frequency ranges, the effective velocity $v_{\text{eff}} = \omega/\Re(k_{\text{eff}})$ is well defined (here $\Re$ means real part).

Figure 10. Reflection coefficient for FC72 matrix and water scatterers, $\phi_1 = 31.4\%$.

Figure 11. Low-frequency zoom of Fig. 10.

Figure 12. Transmission coefficient for FC72 matrix and water scatterers, $\phi_1 = 31.4\%$. 
5. Conclusion

The coherent wave transmitted through a random medium is numerically reachable with not so many simulations (less than thirty). On the other hand, access to the coherent reflected wave is more effortful and time-consuming, because one needs between one and two hundreds of simulations to achieve the convergence to the average.

The performed numerical simulations and their comparison with the theoretical results of [9, 10], obtained within three different theoretical frameworks, yield the following observations. The transmission coefficient predicted by the three theories differs only for high contrast and high concentration. Thus, if only the transmission is dealt with, the results of [9] in the Foldy framework, which is the simplest model, are well-suited for the purpose. At the same time, this model is too “simple” in that it leads to the effective density being equal to the mass density of the matrix and, as a result, does not agree with the numerical data for the reflection coefficient.

It is seen that for low contrast between the matrix and the scatterers, the predictions obtained by [9] in the framework of Waterman & Truell and by [10] are similar. For high contrast, a noticeable difference appears for high concentrations of scatterers. In this case, the model of [10] is closer to the numerical data for the reflection coefficient than the model of [9].

The overall conclusion is that the homogenization techniques in the context of multiple scattering theory remain valid at low frequencies even for high concentrations.

Acknowledgment

We are grateful to F. Luppé and J.-M. Conoir for making the result of their paper [10] available to us. This work has been supported by the grant ANR-08-BLAN-0101-01 from the ANR (Agence Nationale de la Recherche) and also the project SAMM (Self-Assembled MetaMaterials) supported by the cluster AMA (Advanced Materials in Aquitaine).

References

[1] Foldy L L 1945 Phys. Rev. 67 107-19
[2] Waterman P C and Truell R 1961 J. Math. Phys. 2 512-37
[3] Lloyd P and Berry M V 1967 Proc. Phys. Soc. 91 678-88
[4] Linton C M and Martin P A 2005 *J. Acoust. Soc. Am.* **117** 3413–23
[5] Lax M 1951 *Rev. Mod. Phys.* **23** 287–310
[6] Twersky V 1952 *J. Acoust. Soc. Am.* **24** 42–6
[7] Fikioris J G and Waterman P C 1964 *J. Math. Phys.* **10** 1413–20
[8] Kuster G T and Tolsöz M N 1974 *Geophysics* **39** 587–606
[9] Aristégui C and Angel Y C 2007 *Wave Motion* **44** 153–64
[10] Luppé F and Conoir J M 2010 *J. Phys.: Conf. Ser.* (this issue)