2D continuous spectrum of shear Alfvén waves in the presence of a magnetic island

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Abstract
The radial structure of the continuous spectrum of shear Alfvén modes is calculated in the presence of a magnetic island in tokamak plasmas. Modes with the same helicity as the magnetic island are considered in a slab model approximation. In this framework, with an appropriate rotation of the coordinates the problem reduces to two dimensions. Geometrical effects due to the shape of the flux surface’s cross-section are retained to all orders. On the other hand, we neglect toroidal couplings but fully take into account curvature effects responsible for the beta-induced gap in the low-frequency part of the continuous spectrum. New continuum accumulation points are found at the O-point of the magnetic island. The beta-induced Alfvén eigenmodes (BAE) continuum accumulation point is found to be positioned at the separatrix flux surface. The most remarkable result is the modification of the BAE continuum accumulation point frequency, due to the presence of the magnetic island.

(Some figures in this article are in colour only in the electronic version)

1. Introduction
Plasma stability is one of the crucial issues for fusion devices. Shear Alfvén instabilities can resonate with energetic particles and are therefore particularly dangerous for a burning plasma [1–6]. Shear Alfvén waves (SAWs) are electromagnetic plasma waves propagating with the characteristic Alfvén velocity $v_A = B / \sqrt{4\pi \rho}$ ($B$ is the magnetic field and $\rho$ the mass...
density of the plasma). One of the main damping mechanisms of shear-Alfvén modes in non-uniform plasmas is continuum damping [7–10], due to singular structures that are formed at the SAW resonant surfaces. Due to non-uniformities along the field lines in toroidal geometry, gaps appear in the SAW continuous spectrum [11]. The mechanism is similar to that which creates forbidden energy bands for an electron traveling in a periodic lattice [12–14]. Two types of collective shear Alfvén instabilities exist in tokamak plasmas [15]: energetic particle continuum modes (EPMs) [16], with frequency determined by fast particle characteristic motions, and discrete Alfvén eigenmodes (AEs), with frequency inside SAW continuum gaps [12]. Discrete AEs are practically unaffected by continuum damping [5, 6]. For this reason, the importance of understanding the continuous spectrum structure is clear, if one faces the tokamak stability problem and its potential impact on reaching the ignition condition.

In a tokamak plasma, the SAW continuous spectrum is modified by the interaction with low-frequency MHD fluctuations, such as magnetic islands, which are formed when the original sheared equilibrium magnetic field lines break due to non-ideal effects (in particular finite resistivity) and reconnect with different magnetic topologies [17]. Since the typical island frequency and growth rate are much lower than the SAW oscillation frequency, we can model the equilibrium magnetic field as the sum of a tokamak axisymmetric part plus a quasi-static helical distortion due to the magnetic island.

Here, we derive the fluid theoretical description of the SAW continuum structure in the presence of a finite size magnetic island [18, 19] in finite-β plasmas, taking into account only toroidal effects due to geodesic curvature, which are responsible for the beta-induced Alfvén eigenmodes (BAEs) gap in the low-frequency part of the SAW continuous spectrum [20–22]. We adopt a linear ideal MHD model, and we calculate the continuous spectrum in the nonlinear equilibrium given by the axisymmetric tokamak field plus the magnetic island perturbation. We consider only shear Alfvén modes with the same helicity as the magnetic island. In this framework, with an appropriate rotation of the coordinates the problem reduces to two dimensions. Since we are interested in the structures of the SAW continuous spectrum, characterized by local 'radial' singular behavior, the problem can be further reduced to one dimension. Note that what we are calculating is essentially the distortion of the SAW continuous spectrum in a one-dimensional system (1D-slab or cylinder) due to the presence of a magnetic island. For this reason, for a given island helicity, we refer to the structures derived here as ‘2D continuous spectrum of SAW’ in the presence of a magnetic island. Note also that the problem investigated in this work is somewhat different with respect to that studied in [23], where modes with different helicities from that of the magnetic island are considered7, yielding 3D structures of the SAW continuous spectrum. In other words, this paper deals with SAW modes with the same symmetries of the considered nonlinear equilibrium (with island), whereas in [23] the modes inside the island have their helicity, which creates ellipticity induced gaps in the continuous spectrum and global modes inside the island.

A tokamak equilibrium modified by the presence of quasi-static magnetic island has analogies with a stellarator equilibrium, where Alfvén modes are present as in a tokamak [24]. The magnetic island O-point plays the role of the helical magnetic axis for stellarators and the plasma inside the separatrix topologically corresponds to the whole stellarator plasma. A simplification in our case is given by the fact that the island (plasma) cross-section has fixed shape while moving along the helical symmetry direction, while this is not the case in stellarator plasmas. In fact, magnetic islands in tokamaks are radially localized and the effect of the 1/R dependence of the equilibrium B-field on its shape is negligible at the lowest order;

7 Reference [23] analyzes this problem with a simple representation of the magnetic island as a straight flux-tube with non-circular cross-section.
meanwhile, there are no effects from external coils to be considered, which are dominant in the case of stellarators. With these considerations in mind, some analogies may be drawn between the problem analyzed here and those considered in stellarators (see, e.g., [24]). The more fundamental connection between SAW in nonlinear tokamak equilibria with helical structures, such as magnetic islands, and in stellarators emerges when considering resonant excitation of these fluctuations, which naturally bring in new invariants of particle motions and the corresponding intrinsic frequencies, which have obvious consequences not only on the mode drive, but on the nonlinear saturation and related transport processes as well. These problems are far beyond the scope of this paper and will be addressed elsewhere. As stated above, this work addresses radial (in magnetic flux coordinate) singular structures of the SAW continuous spectrum with the same symmetry as the nonlinear plasma equilibrium within a finite size magnetic island.

As suggested by the magnetic field line helicity behavior [25, 26], in an equilibrium with a magnetic island the separatrix flux surface plays an important role, hosting the BAE continuum accumulation point (BAE-CAP), which—without island—was positioned at the rational surface. Several branches of the nonlinear SAW continuum spectrum stem from the BAE-CAP. Inside the island, they reach continuum accumulation points at the magnetic island O-point (MiO-CAP), while outside the island and far from the separatrix, they asymptotically approach the typical spatial dependence of the SAW continuum in a sheared magnetic field in the absence of the island.

Understanding the modification of the SAW continuous spectrum due to the presence of a magnetic island has potential implications in explaining stability of Alfvén instabilities in tokamak plasmas. Modes in the BAE frequency range have been observed in the Frascati Tokamak Upgrade (FTU) [26, 27] in the presence of a magnetic island, where \( m_{\text{isl}}, n_{\text{isl}} = (-2, -1) \). A theoretical analysis showed that these modes can be interpreted as BAE modes, when thermal ion transit resonances and finite ion Larmor radius effects are accounted for, with good agreement of measured and calculated frequencies in the limit of vanishing magnetic island amplitudes [28]. In fact, measured frequencies were found to depend on the magnetic island amplitude as well [26], consistent with the dependence of the BAE-CAP frequency on the magnetic island size resulting from our theory. The modes were observed only when the magnetic island size was over a certain critical threshold [26]. Later on, similar observations have been reported in other tokamaks (see, for instance, the observations in HL-2A [29]).

The scheme of the paper is the following. In section 2 the equilibrium magnetic field and the model equations are given; meanwhile, a convenient coordinate system is defined inside and outside the magnetic island. In section 3 we solve the problem for the continuous spectrum both numerically, in the whole region inside the magnetic island and analytically, near the O-point, obtaining the value of the MiO-CAP. Similarly, in section 4 we solve the problem for the region outside the magnetic island and compare the numerical solution with the asymptotic analytical solution far from the separatrix. The frequency of the BAE-CAP is also calculated as a function of the island size and mode numbers and parity. Section 5 discusses a possible application of the results concerning the continuous spectrum modification in the presence of a small size magnetic island and provides the estimation of the island-induced frequency shift of BAE by means of a perturbative theory approach. Section 6 is devoted to a summary of the obtained novel results and their possible application to the study of BAE in the presence of a finite size magnetic island.

8 For the analysis of more general symmetries, the reader is referred to [23].
2. Equilibrium and model equations

2.1. Coordinate system

We consider a tokamak geometry for a torus with major radius \( R_0 \). The equilibrium is made of an axisymmetric tokamak magnetic field with a component \( B_{\text{pol}} \) in the toroidal direction \( \zeta_T \) and a component \( B_{\text{isl}} \) in the poloidal direction \( \theta_T \), plus an helical perturbation in the radial direction \( r_T \), generating a magnetic island. The magnetic island is located at a flux surface with minor radius \( r_0 \). Here, the subscripts \( T \) denote tokamak coordinates. We consider the region in the proximity of the rational surface of the magnetic island \( q_T = q_0 = m_{\text{isl}}/n_{\text{isl}} \), where \( m_{\text{isl}} \) and \( n_{\text{isl}} \) are, respectively, the poloidal and toroidal mode numbers of the magnetic island perturbation and \( q_T = r_T B_{\text{tor}}/(R_0 B_{\text{pol}}) \) is the safety factor. In this region, the toroidal magnetic field is assumed to be dominant and constant in space, \( B_{\text{tor}} = B_0/\gamma \), and the poloidal magnetic field is considered to vary linearly with \( q_T \). Here \( B_0 \) is the axisymmetric magnetic field amplitude,

\[
\gamma = \sqrt{1 + \varepsilon_0^2/q_0^2}, \quad \varepsilon_0 = r_0/R_0.
\]

We adopt a slab model with coordinates \((q_T, u, \zeta)\), where \( u \) is defined in equation (1), by applying a rotation of the coordinates in the \((\theta_T, \zeta_T)\) plane, as shown in Figure 1. In this model, the coordinate \( \zeta = (\zeta_T + \varepsilon_0^2 \theta_T/q_0)/(q_0 \gamma^2) \) is the coordinate of translational symmetry. The equilibrium magnetic field is \( \mathbf{B} = B_{\text{pol}, \hat{u}} \hat{u} + B_{\text{isl}, \hat{u}} + B_{\zeta, \hat{u}} \hat{\zeta} \), where the axisymmetric field components are \( B_{\text{pol}, \hat{u}} = B_0 \delta_0 (q_T - q_0)/(q_0^2 \gamma^2) \) and \( B_{\zeta, \hat{u}} = B_0 \) (the physical components are defined in detail in appendix A). The constant-\( \psi \) approximation is also adopted, assuming that the magnetic island field is \( B_{\text{pol}, \hat{u}} = B_{\text{pol}} \sin u \), with \( B_{\text{pol}} \) constant.

The flux surfaces of this equilibrium are labeled by \( \psi \), where the coordinates \( \psi \) and \( u \) are defined by

\[
\psi = (q_T - q_0)^2/2 + M(\cos u + 1), \quad u = n_{\text{isl}}(\zeta_T - q_0 \theta_T).
\]

The X-points of the magnetic island are at \((q_T - q_0, u) = (0, 0)\) and \((0, 2\pi)\) and the O-point at \((0, \pi)\). \( M \) is a constant with value \( M = (q_0 s/n_{\text{isl}})(B_{\text{pol}}/B_{\text{pol,0}}) \), determined by the condition \( \nabla \psi \cdot \mathbf{B} = 0 \), where \( s \) is the magnetic shear and \( B_{\text{pol,0}} \) is the poloidal magnetic field evaluated at the rational surface.

In the slab model approximation, the plasma inside the magnetic island is a straight flux tube with length \( Z_0 = \gamma q_0 R_0 \). The magnetic axis of the flux tube, directed along \( \hat{\zeta} \), and the O-point of the magnetic island are at \( \psi = 0 \) and the separatrix is labeled by \( \psi = \psi_{\text{isl}} = 2M \). An appropriate set of cylinder-like coordinates \((\rho, \theta, \zeta)\) is defined here to describe the region.
inside the magnetic island (see appendix A), with:

\[ \rho = \frac{r_0}{q_0} \sqrt{2 \psi}, \quad \theta = \arccos(\sqrt{M(\cos u + 1)/\psi}). \]

With these definitions, the magnetic axis is at \( \rho = 0 \) and the separatrix radius is \( \rho = \rho_{\text{ss}} \), which corresponds to the magnetic island half-width, \( W_{\text{isl}} \), given by the Rutherford formula [30]:

\[ W_{\text{isl}} \frac{r_0}{r_0} = 2 \frac{B_{\text{isl}}}{B_{\text{pol},0} q_0 s n_{\text{isl}}} = \frac{\rho_{\text{ss}}}{r_0} = \frac{2}{q_0} \sqrt{1 - \left(\frac{\rho}{\rho_{\text{ss}}} \right)^2} \]

with \( e \) defined below in this paragraph. The angle \( \theta \) is defined in the domain \( (0, \pi/2) \) and extended to \( (0, \pi) \) by reflection symmetry w.r.t. \( \theta = \pi/2 \). Further extension to \( (0, 2\pi) \) is obtained by reflection symmetry for \( \rho \leftrightarrow -\rho \). With this definition, \( \theta \) has values 0, \( \pi \) at the rational surface \( q = q_0 \). We also point out that the flux surface’s cross-section in the \( (\rho, \theta) \) plane and in the proximity of the O-point is an ellipse, with eccentricity \( e = 1 - M n_{\text{isl}}^2 / \gamma^2 \).

Typical magnetic islands in tokamak experiments have values of eccentricity close to \( e \approx 1 \). Outside the island, the flux surfaces have the same topology as the original axisymmetric equilibrium, and therefore the coordinate system \( (\rho, u, \zeta) \) is appropriate to describe the problem (see equation (1)).

2.2. Equilibrium magnetic field and model equations

The equilibrium magnetic field in the coordinates \( (\rho, u, \zeta) \) is described by the contra-variant physical components: \( B_{\text{ph}}^x = 0, B_{\text{ph}}^u = 2 \epsilon \delta B_0 \sqrt{M P / (q_0^2 \gamma^2)}, B_{\text{ph}}^\zeta = B_0 \). Here \( P = \sqrt{L^2 + (1 - e)(\sin^2 u)/4} \), where \( L = \sqrt{x^2 - (\cos u + 1)/2} \) and \( x = \rho / \rho_{\text{ss}} \). The contra-variant \( \text{physical} \) components of a vector \( V \), in a basis \( \{g, g_u, g_\zeta\} \) are defined here as the contra-variant components rescaled with the length of the correspondent basis vector, e.g. \( V^c = V^x |g_x| \) (see appendix A for further details). Similarly, in the coordinates defined inside the island we have \( B_{\text{ph}}^x = 0, B_{\text{ph}}^u = 2 \epsilon \delta B_0 \sqrt{M} \sqrt{\gamma} x / (q_0^2 \gamma^2), B_{\text{ph}}^\zeta = B_0 \). The function \( a \) is defined as \( a = \sin^2 \theta + \cos^2 \theta (1 - e) F^2 \), with \( F = \sqrt{1 - x^2 \cos^2 \theta} \.

The safety factor outside the island, \( q_{\text{out}} \), is defined as the flux-surface average of the ratio of the toroidal magnetic field and the sum of the poloidal and radial magnetic fields. The flux-surface average is performed separately on both sides of the magnetic island, with \( r > r_0 \) and with \( r < r_0 \), along the coordinate \( y_{\text{out}} = \int_{r_0}^{r_1} r_0 d\theta \sqrt{1 + B_{\text{q}, \text{ph}}^2 / B_{\text{q}, \text{ph}}^2} \). By definition, we have

\[ q_{\text{out}} = \int \frac{d y_{\text{out}}}{2 \pi r_0 B_{\text{ph}}} B_{\text{pol},0} = \frac{1}{2 \pi r_0} \int_{r_0}^{r_1} 2 \pi q_{\text{out}} d \theta \sqrt{1 + B_{\text{q}, \text{ph}}^2 / B_{\text{q}, \text{ph}}^2} \]

Far from the separatrix \( (x > 1) \), the safety factor reaches asymptotically the linear behavior typical of the slab model without island. When the magnetic island amplitude \( M \) vanishes, this behavior is recovered for all radial positions. In fact, in this case we have that \( 2 \sqrt{M} P \to (q_T - q_0) \) and therefore \( q_{\text{out}} \to q_T \). An important result is that the safety factor at the separatrix \( (x = 1) \) is not \( q_0 \), but there is a small difference, proportional to the magnetic island size. This difference is positive on one side of the island, where \( q_T > q_0 \), and negative on the side where \( q_T < q_0 \). The absolute value of this difference can be calculated at the separatrix from equation (3), in the limit \( M \ll 1 \). We obtain

\[ q_{\text{out,ss}} - q_0 = \frac{4 \sqrt{M}}{\pi} \approx 1.27 \sqrt{M}. \]
This result has important implications for the modification of the value of the BAE continuum accumulation point, which is located at the separatrix. This island-induced frequency shift of the BAE-CAP can be calculated in an approximated form as 

$$\Delta \omega \approx k_\parallel v_A \approx v_A m (q_{\text{out,}\parallel} - q_0) / (q_0 Z_0) \approx 1.27 \sqrt{M} (v_A / Z_0) (m / q_0),$$

where \(m\) is the poloidal mode number. In fact, we will see in section 4.2 that the BAE continuum accumulation point does not have the same frequency as without island, but a nonlinearly modified frequency which is proportional to \(\sqrt{M}\), consistent with the nonlinearly modified value of the safety factor given by equation (4).

Similarly, the safety factor inside the magnetic island can be defined as the average of \(Q = \rho B_\parallel / (Z_0 B^\theta)\) over \(\theta\), with \(B^\theta = B_{\text{pol}}^\theta \sqrt{1 - \varepsilon} / (\sqrt{\rho} \rho)\) (see also [23]):

$$q_{\text{in}} = \frac{2}{\pi} \frac{\gamma}{|s|} K(x) = \frac{2}{\pi} \frac{1}{\sqrt{M_{\text{isl}}}} K(x),$$

where we have introduced the complete elliptic integral of the first kind \(K(x) = \int_0^{\pi/2} d\theta / F\).

A similar definition for the safety factor was given in [25], but our result is a factor \(q_0\) smaller.

The basic difference of our derivation from that of [25] is that we take into account that the length of a magnetic island flux tube is \(Z_0 \simeq 2\pi q_0 R_0\), whereas in [25] a flux tube with length \(Z_0 \approx 2\pi R_0\) is considered, which is the major circumference of the tokamak. In other words, we point out that the magnetic island flux tube performs \(q_0\) loops before closing on itself, and therefore all periodicity boundary conditions have to be imposed in its whole length, and not in the length of the tokamak circumference. This difference enters in particular the definition of the coordinate along the axis of the magnetic island flux tube \(\zeta\), and approximated in [25] by the toroidal coordinate \(\xi_T\).

Before introducing the SAW model equations, we want to give an approximated evaluation of the continuous spectrum frequency, which although being strongly simplified, is useful to have a hint as to the dependence on the magnetic island size. To this aim, we can initially neglect compressibility and tokamak curvature, and use the formulation of the SAW continuous spectrum frequency valid in a geometry with cylindrical symmetry.

Inside the island, we decompose the SAW continuum modes in a Fourier expansion in the coordinates \(\theta\) and \(\zeta\), respectively, with quantum numbers \(j\) and \(\tilde{n}\). The parallel component of the wave-vector can be written, under these assumptions, as \(k_{\text{in,}\parallel} = (\tilde{n} q_{\text{in,}0} - j) / (q_{\text{in,}0} Z_0)\), with \(q_{\text{in,}0}\) given by equation (5). The SAW continuous spectrum frequency of modes with the same helicity as the magnetic island—i.e. \(\tilde{n} = 0\)—is written in terms of the safety factor in the approximated form

$$\omega^2_{\text{approx,}\text{in}} = v_A^2 k_{\text{in,}\parallel}^2 = \frac{v_A^2}{Z_0^2 q_{\text{in,}0}^2} j^2 K(x) = \frac{\pi^2 v_A^2}{4} \frac{M n_{\text{isl}}^2}{Z_0^2 K^2(x)},$$

Therefore, the squared frequency is proportional to \(M\)—i.e. to the squared magnetic island width—has a maximum at the O-point, and goes to zero at the separatrix. It will be shown in the next sections that this is indeed the main dependence of the continuous spectrum on the magnetic island size. Differences between this simplified formulation and the exact formula given in the next section are found especially near the separatrix, where the flux surfaces have different topologies from that of a cylinder.

Outside the island, the treatment is analogous. SAW continuum modes are decomposed in a Fourier expansion in the coordinates \(\theta_T\) and \(\xi_T\), respectively, with quantum numbers \(m\) and \(n\), then we give the simplified formulation of the SAW continuous spectrum as

$$\omega^2_{\text{approx,}\text{out}} = v_A^2 k_{\text{out,}\parallel}^2.$$ Here, the parallel component of the wave-vector is defined as \(k_{\text{out,}\parallel} = (n q_{\text{out,}0} - m) / (q_{\text{out,}0} R_0)\), with \(q_{\text{out,}0}\) given by equation (3) and \(m / n = q_0\).

Now, we introduce the SAW model equations. In this work, we want to study the SAW propagation near the resonant flux surfaces where the energy is absorbed by continuum
damping; therefore, we focus on the dynamics of modes that are characterized by radial singular structures. The linear equation for radially localized shear Alfvén modes in a compressible non-uniform tokamak plasma can be written in the form [20, 21]

\[
\frac{\omega_A^2}{\omega^2} \nabla^2 \phi + Z_0^2 \nabla_\perp \nabla_\parallel^2 \phi - \frac{\omega_{BAE-CAP}^2}{\omega_A^2} \nabla_\perp^2 \phi = 0,
\]

where \( \omega_A = v_A/Z_0 \). The frequency of the low-frequency SAW continuum accumulation point, delimiting the frequency gap of the beta-induced Alfvén eigenmode (BAE), is denoted as \( \omega_{BAE-CAP} \). We adopt, here, the value of the BAE-CAP given in [22] and valid in the fluid limit, i.e. for frequencies much larger than the ion transit frequency:

\[
\omega_{BAE-CAP} = \frac{1}{R_0} \sqrt{\frac{2T_i}{m_i}} \left( \frac{7}{4} + \frac{T_e}{T_i} \right)
\]

where \( T_i \) and \( T_e \) are the ion and electron temperatures and \( m_i \) is the ion mass. We focus on frequencies higher than \( \omega_{BAE-CAP} \) and consistently neglect kinetic effects associated with wave–particle resonances [22]. The operators \( \nabla_\parallel \) and \( \nabla_\perp \) are parallel and perpendicular gradients with respect to the equilibrium magnetic field.

In the following sections, the model equation for radially localized shear Alfvén modes, equation (7), is written in the coordinate systems inside and outside the magnetic island and the numerical solution is provided and compared with analytical expressions.

3. Solution inside the magnetic island

3.1. Eigenvalue problem

Here, we write the equation for SAW continuum modes, equation (7), in the coordinates inside the island, \((x, \theta, \zeta)\), introduced above. We consider continuum modes with the same helicity as the magnetic island: \( \partial / \partial \zeta = 0 \). In this framework, the problem reduces to two dimensions. The parallel and perpendicular differential operators have the following form

\[
Z_0 \nabla_\parallel = \sqrt{Mn_{\text{isl}}} F \frac{\partial}{\partial \theta},
\]

\[
\rho_{\text{isl}}^2 \nabla_\perp^2 = a \frac{\partial^2}{\partial x^2}
\]

and the boundary conditions of the problem are periodicity conditions in \( \theta \) on a \( 2\pi \) circle. By applying the differential operators in this explicit form, equation (7) can be written in the form of an eigenvalue problem:

\[
\left[ \frac{\Omega^2}{Mn_{\text{isl}}^2} + \frac{F}{a} \frac{\partial}{\partial \theta} a F \frac{\partial}{\partial \theta} \right] f = 0,
\]

where \( \Omega^2 = (\omega^2 - \omega_{BAE-CAP}^2)/\omega_A^2 \) is the eigenvalue, and \( f = \partial^2 \phi / \partial^2 x \) is the eigenfunction.

It is useful to write equation (9) in the form of a Schrödinger equation. The motivation is twofold. Firstly, we can use standard shooting method techniques to solve numerically the problem in the whole range inside the island: \( 0 < x < 1 \). Secondly, we can solve analytically the problem in the proximity of the island O-point, approximating the potential, and compare the numerical solution with the analytical one near the O-point. The eigenvalue problem takes the form

\[
\frac{\partial^2}{\partial \theta^2} h + \frac{\Omega^2}{Mn_{\text{isl}}^2} A_{\text{in}} h - V_{\text{in}} h = 0
\]
Figure 2. Continuous spectrum $\Omega^2(x)$ for the small eccentricity case, $e \ll 1$, corresponding to $M \approx 1$, plotted versus the radial position inside the island (such wide islands are not found in tokamak plasmas, but this limit represents a mathematical test case that makes a bridge with the well known results in cylinder geometry). Typical values of the equilibrium parameters have been chosen and $n_{isl} = 1$. The O-point is at $x = 0$ and the separatrix at $x = 1$. The branches of both even and odd eigenfunctions are shown. At the O-point, the continuum accumulation points (MiO-CAP) recover the value of the cylinder limit. At the separatrix, the first odd frequency branch tends to the original BAE-CAP ($\Omega^2 = 0$) and the others to the nonlinearly modified BAE-CAP ($\Omega^2 = Mn_{isl}^2$).

with $A_{in} = 1/F^2$, $V_{in} = (g''_{in} - g'_{in}/(2g_{in}))/2g_{in}$, $g_{in} = aF$, and $h = f \sqrt{g_{in}}$. Here the prime denotes the $\theta$-derivative, e.g. $g' = \partial g/\partial \theta$.

The solution of equation (10), can be found numerically with a shooting method code in the range $0 < x < 1$, for any value of the eccentricity $0 < e < 1$. The result is shown in figure 2 for $e \ll 1$, and in figure 4 for $e = 0.99$. We choose typical values for the equilibrium parameters, $q_0 = 2$, $s = 1$, $\epsilon_0 = 0.1$, $n_{isl} = 1$. The periodic boundary condition in $\theta$ is satisfied by an infinite set of solutions, labeled by $j = 1, 2, ...$. Three main features describe the continuous spectrum inside the island. (1) The continuous spectrum branches $\Omega^2_j$ have continuum accumulation points at the O-point, named here MiO-CAP. (2) Even and odd eigenfunctions have different eigenvalues, due to the non-uniformity of the magnetic field intensity along the field line. (3) At the separatrix, the continuum frequencies converge to two different BAE-CAP. The first odd eigenfunction has frequencies converging to the original BAE-CAP, whose value is the same as in an equilibrium without islands, corresponding to $\Omega^2 = 0$. All other branches converge to a nonlinearly modified BAE-CAP, whose value is $\Omega = M n_{isl}^2$.

The solution of equation (10) can also be found analytically near the O-point and near the separatrix, approximating the potential and obtaining the value of the continuum accumulation points (MiO-CAP and BAE-CAP). The analytical derivation of the MiO-CAP value is shown hereafter, and an estimation of the nonlinearly modified BAE-CAP is given in section 4.2.

3.2. Analytic solution near the O-point

The value of the continuum accumulation points at the O-point of the island can be calculated analytically from equation (10), by approximating the functions $A_{in}$ and $V_{in}$ near the O-point.
In this limit, equation (10) becomes
\[ \frac{\partial^2}{\partial \theta^2} h + \frac{\Omega_1^2}{Mn_{isl}^2} h - V_{in,0} h = 0, \] (11)
where the potential is defined by
\[ V_{in,0}(\theta) = e \left( \frac{1 - e}{1 - e \cos^2 \theta} \right) (2 - e \cos^2 \theta) - 1 \]. (12)

Two limiting cases are considered here: \( e \ll 1 \), corresponding to \( M \simeq 1 \), and \( e \approx 1 \), corresponding to \( M \ll 1 \). The former describes a magnetic island where the flux surfaces have circular cross-section near the O-point. Islands with these values of eccentricity are not found in tokamak plasmas, but we use this case as a mathematical test case, for the solution is well known from analytical theory. The latter case is representative of typical size magnetic islands in tokamak plasmas. In the former case, the function \( a \) can be approximated by \( a \approx 1 \) and the magnetic field intensity is independent of \( \theta \):
\[ B_{\theta} \approx \sqrt{\varepsilon s} B_0 \rho/(q_{in,0} \gamma^2) \]. This means that, in our model, the \( e \ll 1 \) case has cylindrical symmetry around the O-point. In this case, the potential \( V_{in,0} \) vanishes and equation (11) has the following eigenvalues:
\[ \Omega_{1,\text{O-CAP}}^2 = Mn_{isl}^2 j^2 = j^2/q_{in,0}^2 (\text{case } e \ll 1), \] (13)
where \( j \) is a natural number \((j = 1, 2, ... )\), and \( q_{in,0} \) is the safety factor given in equation (5), calculated at \( x = 0 \).

On the other hand, in the case of small island amplitude \( (e \simeq 1) \), the potential can be written as \( V_{in,0} \approx V_0 + \delta V \), where \( V_0 = -1 \) and \( \delta V = V_1 H(\theta_{\text{eff}}) \). Here \( V_1 = 1/(1 - e) = s^2/(Mn_{isl}^2 \gamma^2) \) and \( H(\theta_{\text{eff}}) \) is a function with unitary value inside the set \( (|\theta| < \theta_{\text{eff}} \cup (|\pi - \theta| < \theta_{\text{eff}}) \) and zero elsewhere, where \( \theta_{\text{eff}} = (1/2) (\sqrt{M/|s|})^{1/2} \). Since the potential \( V_{in,0} \) is not constant in \( \theta \), in this case the eigenvalue \( \Omega^2 \) has a different value for even and odd eigenfunctions. However, an approximated value of \( \Omega^2 \) can be calculated by making some considerations on the potential shape. In fact, having \( \delta V \) a support which is very localized in \( \theta \) for small amplitude islands, where odd modes are small, we can neglect its contribution and consider the problem determined by \( V_{in,0} = V_0 = -1 \). This gives the estimate for odd mode eigenvalues:
\[ \Omega_{1,\text{O-CAP}}^2 = Mn_{isl}^2 (j^2 - 1) \quad (\text{case } e \approx 1). \] (14)

4. Solution outside the magnetic island

4.1. Eigenvalue problem

In this section, we face the problem of SAW continuum modes, described by equation (7), outside the separatrix of a magnetic island, in the coordinate system, \((x, u, \zeta)\). The parallel and perpendicular differential operators have the following form:
\[ Z_0 \nabla_1 = 2\sqrt{Mn_{isl} L} \frac{\partial}{\partial u}, \]
\[ \rho_{\zeta}^2 \nabla_1^2 = \frac{p^2 z^2}{x^2} \frac{\partial^2}{\partial x^2}. \]
Only continuum modes with \( m = q_0 n \) are considered, where \( m \) and \( n \) are, respectively, the poloidal and toroidal mode numbers, in the tokamak coordinates \( \theta_T \) and \( \zeta_T \). Those are the modes with the same helicity as the magnetic island: \( \partial \theta_T/\partial \zeta = 0 \), and therefore, the problem reduces
to two dimensions. By applying the differential operators in this explicit form, equation (7) can be written as an eigenvalue problem:

$$\left[ \frac{\partial^2}{\partial u^2} + \frac{\Omega_0^2}{M n_{isl}^2} + \frac{4L}{P^2} \frac{\partial}{\partial u} \right] f = 0$$ (15)

and, as for equation (9), this can be put in the form of a Schrödinger equation. The Schrödinger equation can be solved numerically for $x > 1$ with standard techniques and the numerical solution can be compared with the asymptotic limit, obtained analytically for $x \gg 1$. Moreover, analytical estimations of the frequency can be found near the separatrix. The eigenvalue problem takes the form

$$\frac{\partial^2}{\partial u^2} h + \frac{\Omega_0^2}{M n_{isl}^2} A_{out} h - V_{out} h = 0$$ (16)

with $A_{out} = 1/(4L^2)$, $V_{out} = (g_{out}' - g_{out}'')/(2g_{out})/(2g_{out})$, $g_{out} = 2LP^2$, and $h = f \sqrt{g_{out}}$.

The periodic boundary condition in $u$ is satisfied by an infinite set of solutions, labeled with mode number $j$ (for $q_0$ integer, $j = n$, with $n$ the toroidal number in the tokamak $\zeta_T$ coordinate). The solution of equation (16) is shown in figures 3 and 4, for typical island size, $M = 10^{-2}$, and typical values of the equilibrium parameters, $q_0 = 2$, $s = 1$, $\varepsilon_0 = 0.1$, $n_{isl} = 1$.

Three main features describe the continuous spectrum outside the island:

1. The continuous spectrum for $x \gg 1$ reaches the asymptotical limit $\Omega_{0\infty}^2$. In this limit, namely for $M \ll 1$, we note that equation (15) takes the form of a Mathieu equation:

$$\frac{\Omega_0^2}{n_{isl}^2} f = M \left( 4x^2 - 2(\cos u + 1) \right) \frac{\partial^2}{\partial u^2} f = \left( 2\psi - 2M \right)f$$ (17)

with $M$ playing the role of the modulation coefficient, and the flux-surface coordinate $\psi$ approaching the value $\psi \simeq (q_T - q_0)/2$. The asymptotical limit $\Omega_{0\infty}^2$ can be calculated
Figure 4. Continuous spectrum $\Omega^2(x)$, for $M = 10^{-2}$, corresponding to $\varepsilon \simeq 0.99$. Typical equilibrium parameters have been chosen, and $n_{isl} = 1$. The region inside the island is at $0 < x < 1$, and the region outside the island at $x > 1$. The MiO-CAP are shown at the O-point ($x = 0$), and the linear and nonlinear BAE-CAP at the separatrix ($x = 1$).

from equation (17), and has a constant difference with the limit of the continuous spectrum in the limit of vanishing island, $\Omega^2_0$:

$$\Omega^2_\Lambda = n^2 \langle (q_T - q_0)^2 \rangle_0 = Mn_{isl}^2 j^2 (4x^2 - 2),$$
$$\Omega^2_0 = n^2 \lim_{M\to0} (q_T - q_0)^2 = Mn_{isl}^2 j^2 (4x^2).$$

This difference is due to the constant-$\psi$ approximation which is made modeling the island magnetic field. More realistic asymptotic behavior can be obtained by abandoning the constant-$\psi$ approximation and considering an improved model where the island field decays with growing $|q_T - q_0|$. The asymptotic behavior at $x \gg 1$ is depicted in figure 3 for the case $j = 1$.

(2) Even and odd eigenfunctions have different eigenvalues, due to the non-uniformity of the magnetic field intensity along the field line, but the difference is negligible with respect to the absolute value. In the case depicted in figure 3, the difference between odd and even solutions is $\Delta \Omega^2 \sim O(M^2)$.

(3) At the separatrix, all continuum frequencies converge to the non-linearly modified BAE-CAP, whose value is proportional to the magnetic island half-width:

$$\Omega_{nlBAE-CAP} = \sqrt{M} n_{isl} = \frac{q_0 sn_{isl}}{2} \frac{W_{isl}}{r_0}.$$

This is consistent with the qualitative asymptotic behavior of the continuous spectrum near the separatrix, which can be calculated by knowing the value of the safety factor outside the magnetic island (see equation (4)). Note that the nlBAE-CAP has the same frequency as the MiAE gap central frequency [23]. Note also that we refer here to the CAP at the separatrix as the non-linearly modified ‘BAE-CAP’ for we are interested in this paper to its application to the BAE dispersion relation. In fact, in the limit of vanishing beta this CAP still has non-zero value, which means that its existence is due to the topology of the magnetic island, rather than to the plasma pressure.

The non-linear modification of the continuous spectrum and in particular the upward shift in frequency of the BAE-CAP has important implications in the study of the dynamics of discrete
AE in tokamak equilibria in the presence of a magnetic island, as discussed in section 6. The dynamics of discrete AE is not treated here, and will be discussed in a dedicated paper. The analytical treatment of the continuous spectrum problem near the separatrix is presented in next section.

4.2. Analytical treatment near the separatrix

Here, we focus on the region outside the magnetic island, in the proximity of the separatrix. We also consider the limit of $e \simeq 1$, which is the case of a typical size magnetic island in tokamak plasmas. Under these assumptions, the functions $L$ and $P$ can be approximated as $L = P = \sqrt{1 - \cos u}/\sqrt{2}$. Consequently, the eigenvalue equation, equation (16), takes the form

$$\frac{d^2}{du^2} h - W_{\Omega^2} h = 0,$$

(19)

where the potential $W_{\Omega^2}$ is defined as

$$W_{\Omega^2} = - \frac{9 \cos^2 u - (12 + 8\tilde{\Omega}^2) \cos u + (3 + 8\tilde{\Omega}^2)}{16(1 - \cos u)^2}$$

and $\tilde{\Omega}^2 = \Omega^2 / (Mn_{ni}^2)$. This equation can be studied as a Schrödinger equation with zero energy. In this framework, we look for values of $\tilde{\Omega}^2$ such that the solution satisfies the boundary conditions of periodicity in $u$ on $(0, 2\pi)$. The potential $W_{\Omega^2}$ has positive second derivative for $\tilde{\Omega}^2 < 3/4$, and negative second derivative for $\tilde{\Omega}^2 > 3/4$. For $\tilde{\Omega}^2 = 3/4$, the potential has a constant value: $W_{\Omega^2} = -(3/4)^2$. For the given boundary conditions, this means that solutions of the problem exist for $\tilde{\Omega}^2 > 3/4$ only. On the other hand, when $\tilde{\Omega}^2 = 4$ the potential has maximum value $W_{\Omega^2} = -1$ at $u = \pi$, and therefore does not admit a solution with mode $j = 1$. With these considerations, we can estimate that $3/4 < \tilde{\Omega}_{nlBAE-CAP} < 4/3$ for the first mode number, $j = 1$. Numerically, we find that $\tilde{\Omega}^2 \simeq 1$ for $j = 1$ and that the difference in frequency for other mode numbers and different parities is negligible.

5. Estimation of BAE frequency in the presence of an island

Here, as an application of the results of last section, we estimate the BAE frequency shift due to the magnetic island, by adopting a perturbative theory. The BAE frequency has been given in [28] in the framework of a linear theory (namely in a tokamak equilibrium without magnetic islands), and the result is a value which lies below the BAE-CAP value (where the BAE-CAP value is given by equation (8)), as shown in figure 5. In our model, we apply the perturbative theory approach by considering BAE as Alfvénic instabilities with an eigenfunction peaked around the rational surface, and with a characteristic radial size which is supposed to be much larger than the magnetic island’s width. This hypothesis allows us to assume that the contribution of the continuous spectrum modification inside the magnetic island is negligible with respect to the contribution outside the magnetic island, in modifying the BAE dispersion relation. In other words, we assume that the main modification of the continuous spectrum is the upward shift in frequency of the BAE-CAP, resulting in the nonlinearly modified BAE-CAP described by equation (18). Therefore, we assume that the shift in frequency of the BAE due to the presence of the island follows linearly the modification of the BAE-CAP. We obtain

$$\omega_{\text{BAE}} = \omega_{\text{BAE,0}} \sqrt{1 + \frac{q_0^2 n_{i0}^2 W_{\text{is}}^2}{4 \omega_{\text{BAE-CAP}}^2}},$$

(20)
Figure 5. Schematic picture of the continuous spectrum squared frequency $\omega^2(r_T)$ in a tokamak plasma, with and without magnetic island. In an equilibrium without magnetic island the continuous spectrum is depicted by the dotted parabola, the frequency of the BAE-CAP (positioned at the rational surface $r_0$) is labeled by BAE-CAP$_0$ (given by equation (8)), and the BAE is depicted by a horizontal dotted line, labeled BAE$_0$. In an equilibrium with a magnetic island, the BAE-CAP splits into two nonlinear modified BAE-CAPS, and the BAE is expected to have a higher frequency, given by equation (20). The separatrix positions are labeled by $r_{sx1}$ and $r_{sx2}$, and the magnetic island half-width is given by $W_{isl} = |r_{sx} - r_0|$.

where all parameters are calculated at the rational surface, and $W_{isl}$ is the magnetic island half-width. The frequency $\omega_{BAE,0}$ is the frequency of BAE for a vanishing magnetic island, given in [28], and $\omega_{BAE-CAP}$ is given in equation (8). A schematic picture of the frequency and position of the nonlinearly modified BAE is given in figure 5. We expect the perturbative theory to provide consistent results only for low magnetic island amplitudes. In fact, when the magnetic island width is comparable with the BAE width, we expect the region inside the magnetic island to give an enhanced contribution in modifying the BAE dynamics. A general calculation of the nonlinear BAE dispersion relation in the presence of a finite size magnetic island (i.e. by dropping the perturbative theory approximation) will be given in a dedicated paper. The theoretical BAE frequency given in equation (20) has been compared with experimental data of BAE observed in FTU [31]. Agreement of theoretical and experimental data has been found at low magnetic island amplitudes, whereas a discrepancy has been found at higher magnetic island amplitudes.

6. Conclusions and discussion

Understanding the structure of the SAW continuous spectrum is one crucial step in studying the stability properties of a tokamak plasma. In fact, one of the main damping mechanisms of SAW instabilities in non-uniform plasmas is continuum damping, which occurs near the magnetic flux surfaces where the frequency of the instability matches the SAW continuum frequency. In toroidal devices, a frequency gap in the continuous spectrum represents a range of frequency where discrete Alfvén eigenmodes (AEs) can grow unstable, practically unaffected by continuum damping.
We have studied the radial structure of the SAW continuous spectrum in the presence of a magnetic island. Since magnetic islands are very common in fusion plasma experiments, for example in connection with tearing instabilities and sawtooth activity, the solution of this problem has important implications for understanding the stability regimes of AE in a fusion reactor. We have adopted a linear MHD model for radially localized SAW, where tokamak curvature is neglected except for the creation of the beta-induced Alfvén eigenmode (BAE) frequency gap, in the low-frequency range of the SAW continuous spectrum. We also assumed modes with the same helicity as the magnetic island. In this framework, the model reduces to two dimensions. The case of local singular structures with helicity different from that of the magnetic island is considered in a different paper [23]. Due to the time scale separation between island and the SAW dynamics, the island has been treated as a static helical distortion of the equilibrium.

We have shown that the SAW continuous spectrum of a finite-\(\beta\) tokamak plasma is modified by the presence of a magnetic island. In particular, the BAE continuum accumulation point (BAE-CAP) is shifted in space from the island rational surface, to the separatrix flux surface position. Moreover, we have shown that the non-uniformity of the magnetic field intensity along the field lines generates a frequency splitting between modes with even and odd parity. This splitting frequency has a finite value inside the island and is negligible outside for typical island sizes. Inside the magnetic island, we have found that the continuum frequency branches converge at the O-point to several magnetic island-induced CAP (MiO-CAP). At the separatrix, the frequency of the BAE-CAP is split into two values. The continuum branch corresponding to the first odd eigenfunction inside the island converges to the original BAE-CAP with the same frequency as without magnetic island; and the other branches converge to a nonlinearly modified BAE-CAP (namely the BAE-CAP in the nonlinear equilibrium given by the axisymmetric tokamak field plus the magnetic island perturbation), with a higher frequency given by equation (18). Outside the magnetic island, all continuum branches at the separatrix converge to the nonlinearly modified BAE-CAP. Far from the separatrix and outside the magnetic island, the continuum frequency converges asymptotically to the value calculated in an equilibrium with no island. We also note that, in the limit of vanishing beta, the nonlinearly modified CAP at the separatrix is still present, and therefore is not due to the plasma pressure but to the magnetic island topology itself.

Our results have two main implications in the study of the dynamics of Alfvén eigenmodes (AE). Inside the island, new magnetic island-induced AE (MiAE) could exist as bound states, essentially free of continuum damping, provided that plasma equilibrium effects and free-energy sources can drive and bind them locally (see also [23]). Outside the island, BAE are expected to have an eigenfunction radially peaked in the proximity of the magnetic island separatrix, namely at the BAE-CAP, other than at the rational surface. The BAE frequency is expected to have higher values when a magnetic island is present, consistent with the BAE-CAP frequency modification described by equation (18). The BAE frequency shift due to the magnetic island is calculated in section 5 in the limit of small island width, using a perturbation theory approach. Moreover, while inside the magnetic island the pressure and temperature profiles are flattened, outside the magnetic island, in the proximity of the separatrix, the equilibrium pressure gradients are highly increased. This may provide a free-energy source for BAE, which could become unstable in this configuration, even in the absence of fast particle drive. BAE are generally damped by thermal ion Landau damping; therefore, we may expect to detect BAE when the value of the pressure gradient is over a certain threshold, namely when the magnetic island exceeds a certain size. The general BAE dispersion relation in the presence of a finite size magnetic island and the corresponding drive and damping mechanisms will be treated in a different paper.
Experimental observations of Alfvénic modes in the presence of a magnetic island have been recently reported in several tokamak devices (see for instance the observations in FTU [26, 27] and in HL-2A [29]), even in purely Ohmic heated plasmas. The fluctuations frequency range confirms that they are BAE nonlinearly interacting with the magnetic island, characterized by a frequency shift dependence on the island size consistent with that of the nonlinearly modified BAE-CAP. Moreover, BAE were observed only above a certain threshold in the magnetic island amplitude.

The dependence on the magnetic island size of the continuous spectrum and in particular the MiO-CAP and the BAE-CAP suggests the possibility of using the MiAE and BAE frequency scalings as novel magnetic island diagnostics in a fashion similar to other commonly used Alfvén spectroscopy techniques [32–34]. The radial MiAE localization at the center of the island makes them more difficult to detect by external measurements than BAE and makes the use of internal fluctuation diagnostics, such as electron cyclotron emission (ECE) and soft x-rays, necessary. On the other hand, BAE are easier to detect with external diagnostic techniques, and therefore, the possibility of using BAE frequency scalings as a magnetic island diagnostic may be a more feasible one.

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Appendix A. Coordinate metric

Here, we provide the metric for the coordinates used in this work outside the magnetic island \((\rho, u, \zeta)\) and inside the magnetic island \((\rho, \theta, \zeta)\). This is necessary to calculate the differential operators in the equation for the dynamics of continuum modes.

The domains of the coordinates \((\rho, u, \zeta)\), describing the region outside the island, are \(\rho_{sx} = W_{isl} < \rho < \infty, 0 < u < 2\pi\) and \(0 < \zeta < 2\pi\), where \(\zeta\) is the coordinate of translational symmetry for both equilibrium and perturbations, and periodicity in \(u\) is assumed for the perturbations. The gradients of these coordinates are

\[
\nabla \rho = L/x \hat{q}_T - \sqrt{1 - e \sin u / (2x)} \hat{u} = (P/x) \hat{\rho},
\]

\[
\nabla u = \hat{u} / \rho_0,
\]

\[
\nabla \zeta = \hat{\zeta} / Z_0,
\]

where \(\rho_0 = r_0 / (q_0 n_{isl} \gamma)\), \(Z_0 = \gamma q_0 R_0\), and we use the notation \(\hat{V} = V / V\) for a versor (unit vector). Here \(P = \sqrt{L^2 + (1 - e) \sin^2 u / 4}\), where \(L = \sqrt{x^2 - (\cos u + 1)^2 / 2}\) and \(x = \rho / \rho_{sx}\).

A covariant basis \((g_\rho, g_u, g_\zeta)\) can also be defined as orthogonal to the contra-variant basis \((g^\rho = \nabla \rho, g^u = \nabla u, g^\zeta = \nabla \zeta)\), namely satisfying: \(g^j_i \cdot g_j = \delta_i^j\). With this notation, any vector can be written as \(V = \sum_i V^i g_i = \sum_i V^i_{ph} \hat{g}_i\), where we call \(V^i_{ph}\) the physical components.
of the vector, \( V_{ph}^i = V^i[\mathbf{g}_i] \). The Jacobian for the coordinates outside the island, defined as the determinant of the metric tensor \( g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j \), is \( g_{\text{out}} = \rho_{\text{st}}^2 \int_0^\infty \mathrm{d}x Z_0^2 x^2 / L^2 \). With these definitions, the Laplacian of a function \( f \) for 'radially' localized modes, can be written in the coordinates \((\rho, u, \zeta)\) as:

\[
\nabla^2_{\text{out}} f \simeq \frac{1}{g_{\text{out}}} \frac{\partial}{\partial \rho} \left( g_{\text{out}} \frac{\partial f}{\partial \rho} \right) \simeq \frac{1}{\rho_{\text{st}}^2} \frac{\partial^2 f}{\partial \rho^2},
\]

(A.1)

where we have used \( g^{\rho \rho} = g^\rho \cdot g^\rho = P^2 / x^2 \).

The domains of the coordinates \((\rho, \theta)\), for the region inside the island, are \(0 < \rho < \rho_{\text{st}}\), \(0 < \theta < 2\pi\) and \(0 < \zeta < 2\pi\), where \(\zeta\) is the coordinate of translational symmetry as above, and periodicity in \(\theta\) and \(\zeta\) is assumed for the perturbations. The gradients of the coordinates \(\rho\) and \(\theta\) are

\[
\nabla \rho = \sin \theta \hat{q}_T - \cos \theta \sqrt{1 - e F} \hat{u} = \sqrt{a} \hat{r},
\]

\[
\nabla \theta = (\cos \theta \hat{q}_T + \sin \theta \sqrt{1 - e F} \hat{u}) / \rho = \sqrt{a} \hat{\theta} / \rho.
\]

Moreover, we have the relation \( \nabla \rho \cdot \nabla \theta = c / \rho \). Here \( F = \sqrt{1 - x^2 \cos^2 \theta} \), \( a = \sin^2 \theta + \cos^2 \theta (1 - e) F^2 \), and \( c = \cos \theta \sin \theta (1 - (1 - e) F^2) \). The Jacobian for the coordinates inside the island is \( g_{\text{in}} = \rho_{\text{st}}^2 Z_0^2 x^2 / L^2 (1 - e) \). Using these definitions, and the fact that \( g^{\rho \rho} = g^\rho \cdot g^\rho = a \), we write the Laplacian for 'radially' localized modes in the coordinates \((\rho, \theta, \zeta)\) as

\[
\nabla^2_{\text{in}} f \simeq \frac{a}{\rho_{\text{st}}^2} \frac{\partial^2 f}{\partial x^2}.
\]

(A.2)

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