Tail estimations for functions belonging to Grand Lebesgue Spaces builded on the set with infinite measure.

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Abstract.

We establish the bilateral exact reciprocal interrelations between a tail behavior of a measurable functions and its norm in the suitable Grand Lebesgue Space (GLS) as well as Orlicz one, builded over the set with infinite measure.

We bring also some examples in order to illustrate the exactness of offered estimates.

Key words and phrases.

Measurability, tail of function, ordinary Lebesgue - Riesz and Grand Lebesgue Spaces (GLS) and norms, subgaussian and anti - subgaussian functions, upper and lower estimates, Young - Fenchel transform, Young - Orlicz function, Orlicz norm and space, examples, generating and natural generating function.

1 Statement of problem.

Let \((X = \{x\}, M, \mu)\) be measurable space with non-trivial sigma-finite measure \(\mu\).
We will consider in this report only the case when \( \mu (X) = \infty \).

Denote as usually for arbitrary measurable numerical valued function \( f : X \to R \) its Lebesgue - Riesz \( L(p) = L(p; X) \) norm
\[
||f||_p := \left[ \int_X |f(x)|^p \mu(dx) \right]^{1/p}, \quad 1 \leq p < \infty.
\]

We recall yet another important for us definition of the tail function \( T[f](t), \ t > 0 \) for arbitrary (measurable) function \( f : X \to R \):
\[
T[f](t) \overset{def}{=} \mu \{ x, \ |f(x)| > t \}.
\]

Let also \( \psi = \psi(p), \ p \in (a, b), \ 1 \leq a < b \leq \infty \) be certain finite strictly positive: \( \inf_{p \in (a, b)} \psi(p) > 0 \) numerical valued function.

Recall the definition of the so - called Grand Lebesgue Space (GLS) norm for the function \( f(\cdot) : \)
\[
||f||_{G\psi} = ||f||_{G\psi[a, b]} \overset{def}{=} \sup_{p \in (a, b)} \left\{ \frac{||f||_p}{\psi(p)} \right\},
\]
and correspondent Banach (complete) functional rearrangement invariant space \( G\psi = G\psi[a, b] := \{ f : ||f||_{G\psi} < \infty \} \).

The set of all such a functions will be denoted by \( \Psi[a, b] = \{ \psi(\cdot) \} \); and we take the notations \( G\psi_0 \overset{def}{=} G\psi[1, \infty) \) as well as
\[
G\Psi := \cup_{(a,b): 1 \leq a < b < \infty} G\psi(a, b)
\]

and
\[
(a, b) \overset{def}{=} \text{supp}(\psi).
\]

The function \( \psi = \psi(p), \ p \in (a, b) \) is named as ordinary generating function for this space.

Define formally \( \psi(p) = \infty \) for the values \( p \notin (a, b) \).

The theory of these spaces is represented in many works, see e.g. [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13],[15], [17], [18], [19], [20], [21], [18], [23] etc. As a rule it was considered the case of finite measure: \( \mu(X) = 1 \). The case \( \mu(X) = \infty \) and the operators acting in these spaces was covered in particular in [14], [24] - [25], [27] - [28].

We intent in this short report to establish the bilateral exact reciprocal interrelations between tail behavior of a measurable functions and its norm in the suitable Grand Lebesgue Space (GLS), builded over the set with infinite measure.
We establish also the connections between these and Orlicz spaces.

**Remark 1.1.** Notice that in the considered here unbounded case $\mu(X) = \infty$, it is very essential to investigate the behavior of the tail function $T[f](t)$ not only as $t \to \infty$, but also as $t \to 0^+$, as long as in the general case, e.g. when the function $f = f(t)$ is strictly positive,

$$t \to 0^+ \Rightarrow T[f](t) \to \text{mes}\{ t : |f(t)| > 0 \} = \text{mes}(X) = \infty.$$

**Example 1.0.** The bounded case $\mu(X) = 1$ (probabilistic case) was investigated in [16], [18], [20], chapters 1.2. Let for instance $X = \mathbb{R}_+$ and consider the measurable functions of the form

$$h(x) := h_0(x) I(x \in [0, 1]),$$

where $I(x \in A) = I(A)$ denotes an indicator function for the measurable set (or predicator) $A$. We return to the probabilistic case.

In particular, suppose

$$\exists m = \text{const} \in (0, \infty) \Rightarrow \forall p \in [1, \infty) \||h||_p \leq C_1 p^{1/m}, \ C_1 \in (0, \infty). \quad (3)$$

The last relation is quite equivalent to the following tail estimate

$$\exists c(m) \in (0, \infty] \Rightarrow T[h](t) \leq \exp\left(-c(m) t^m \right), \ t \geq 0. \quad (4)$$

The case $m = 2$ correspondent to the famous subgaussian function (random variable.)

**Example 1.1.** Let $g : X \to \mathbb{R}$ be measurable function for which $\exists(a, b) : 1 \leq a < b \leq \infty \forall p \in (a, b) \Rightarrow ||g||_p < \infty$. The function

$$\psi_g(p) \overset{\text{def}}{=} ||g||_p, \ p \in (a, b) \quad (5)$$

is named as a **natural function** for the function $g = g(x)$. Evidently, $||g|| \psi_g = 1$. Of course, it may be chosen as a generating function for suitable GLS; named: natural generating function.

**Sub-example 1.2.** Let $X = \mathbb{R}_+$ equipped with ordinary Lebesgue measure $d\mu = dx$. Choose the (positive) function

$$g[c, \theta](x) := \exp\left(-cx^\theta \right), \ c, \theta = \text{const} \in (0, \infty).$$

We find

$$|| g[c, \theta] ||_p^p = \int_0^\infty g_p[c, \theta](x) \ dx = \int_0^\infty \exp\left(-cpx^\theta \right) \ dx = \theta^{-1} c^{-1/\theta} p^{-1/\theta} \Gamma(1/\theta),$$
where as usually $\Gamma(\cdot)$ denotes the Euler’s Gamma function. So, the natural function for $g[c, \theta]$ has a form

$$
||g[c, \theta]||_p = \theta^{-1/p} (cp)^{-1/(\theta p)} \Gamma^{1/p}(1/\theta), \ 1 \leq p < \infty.
$$

A particular case: $c = \theta = 1 : g[1, 1](x) = \exp(-x), \ x \geq 0$;

$$
||g[1, 1]||_p = p^{-1/p}, \ a = 1, \ b = \infty.
$$

Another particular case: $\theta = 2, \ c = 1$:

$$
g[1, 2](x) := \exp \left(-x^2\right), \ x \geq 0, \ -
$$

the so - called anti - subgaussian case. We find

$$
||g[1, 2]||_p = 0.5 \pi^{1/2} p^{-1/2}, \ p \in (0, \infty).
$$

Recall that the classical subgaussian case for the function $\tilde{g}$ implies that

$$
||\tilde{g}||_p \asymp \sqrt{p}, \ p \geq 1.
$$

Note that in the considered examples one can suppose $p \in (0, \infty]$.

**Remark 1.2.** It is interest in our opinion to note that in the all last examples

$$
\lim_{p \to \infty} ||g[c, \theta]||_p = 0,
$$

but

$$
\inf_{p \in [1, \infty)} ||g[c, \theta]||_p > 0.
$$

**Remark 1.3.** One can apply for our purpose, in the finite measure case, $\mu(X) = 1$ the so - called moment generating function

$$
\exp(\phi(\lambda)) \overset{def}{=} \int_X \exp(\lambda f(x)) \mu(dx),
$$

if this function there exists for certain non - trivial neighborhood of origin $\lambda : |\lambda| < \lambda_0$, where $\exists \lambda_0 = \text{const} \in (0, \infty]$.

This possibility absent in general case under considered here convention $\mu(X) = \infty$. In particular,

$$
\exp(\phi(0)) = \int_X \mu(dx) = \mu(X) = \infty.
$$
2 Main result. Upper estimate.

Assume that under formulated before conditions the non-zero function $f : X \to \mathbb{R}$ belongs to the certain Grand Lebesgue Space $G\psi; \exists \psi \in G\Psi$. Introduce the following notations:

$$\nu(p) = \nu[\psi](p) := p \ln \psi(p),\ p \in \text{supp}(\psi);\ \gamma := ||f||_{G\psi} \in (0, \infty);$$

$$\nu^*(t) = \nu^*[\psi](t) \overset{def}{=} \sup_{p \in \text{supp}(\psi)} (pt - \nu[\psi](p)) -$$

the (regional) Young - Fenchel transform of the function $\nu$.

**Theorem 2.1.**

$$T[f](t) \leq \exp\left( -\nu^*(\ln(t/\gamma)) \right),\ t > e \gamma. \quad (8)$$

**Proof** is quite alike to one in the finite - measure case [15] - [16]. Indeed, one can suppose without loss of generality

$$\gamma = ||f||_{G\psi} = 1.$$

It follows immediately from the direct definition of the Grand Lebesgue Norm that

$$\forall p \in \text{supp}(\psi) \Rightarrow ||f||_p \leq \psi(p),$$

or equally

$$\forall p \in \text{supp}(\psi) \Rightarrow \int_X |f(x)|^p \mu(dx) \leq \psi^p(p) = \exp(p \ln \psi(p)) = \exp(\nu(p)).$$

We apply the Markov - Tchebychev’s inequality:

$$T[f](t) \leq \frac{\exp(\nu(p))}{t^p} = \exp\left( -(p \ln t - \nu(p)) \right),\ t \geq e.$$

It remains to take the minimum over all the admissible values of the parameter $p \in \text{supp}(\psi)$.

**Example 2.1.** Consider the function $g = g[1, \theta](x),\ x \in \mathbb{R}_+ :$

$$g(x) = \exp\left( -x^\theta \right),\ \theta = \text{const} > 0.$$

The tail function for $g(\cdot)$ has a form

$$T[g](t) = 0,\ t > 1;\ T[g](t) = \ln|t|^{1/\theta},\ t \in (0, 1).$$
The upper estimate for this function based on the theorem 2.1 and example 1.1 give at the same up to constants result.

3 Main result: lower estimates.

Given: the tail function \( T[f](t), \ t > 0, \) (or its upper estimate.) We intent to establish under our conditions the exact up to multiplicative constant estimate for the GLS norm for this function \( ||f||G\psi \) for suitable generating function \( \psi(\cdot) \).

Note first of all that

\[
||f||_p^p = \int_X |f(x)|^p \mu(dx) = p \int_0^\infty t^{p-1} T[f](t) \ dt, \ p \in (0, \infty). \tag{9}
\]

Therefore, the natural generating function for the function \( f(\cdot) \) namely \( \psi = \psi(p) = \psi[f](p) \) has a form

**Proposition 3.1.**

\[
\psi[f](p) = \left[ p \int_0^\infty t^{p-1} T[f](t) \ dt \right]^{1/p}, \tag{10}
\]

if of course there exists for some non-trivial segment \( p \in (a, b) = \text{supp}(\psi(p)), \ 0 < a < b \leq \infty, \) so that

\[
f \in G\psi[f], \ ||f||G\psi[f] = 1.
\]

**Example 3.1.** Let as above \( X = (-1, 0), \ g(x) := | \ln |x| |, \ x \in X; \)

\[
Y := (0, \infty); \ h(y) = e^{-y}.
\]

Define also

\[
Z = \{z\} = \{(x, y)\} = X \otimes Y, \ X \cap Y = \emptyset; \ f(z) = f(x, y) := g(x) + h(y).
\]

There holds

\[
||g||_p^p = \Gamma(p + 1), \ ||h||_p^p = \frac{1}{p}, \ p > 0.
\]

As long as this functions \( f, g \) are disjoint: \( f(x) \cdot g(y) = 0, \)

\[
||f||_p^p = ||g||_p^p + ||h||_p^p
\]

and
\[
\|f\|_p^p \lesssim \Gamma(p + 1), \ p \to \infty; \quad \|f\|_p^p \gtrsim \frac{1}{p}, \ p \in (0, 1).
\]

The last relations stands in complete accordance with the tail behavior of the function \( f \):

\[
T[f](t) \asymp \frac{1}{|\ln t|}, \ t \in (0, 1); \ T[f](t) \asymp e^{-t}, \ t \in (1, \infty).
\]

4 Orlicz space characterization of tail behavior.

Statement of problem: given a tail function \( T = T(t), \ t > 0 \); find an Young - Orlicz function \( N = N(u) = N[T](u), \ u \geq e \), such that for arbitrary measurable function \( f : X \to R \) for which \( T[f](t) \leq T(t) \) this function belongs also the Orlicz space \( L(N) \) builded over \((X, \mathcal{M}, \mu)\).

We understand as the Young - Orlicz function \( N = N(u), \ u \in [0, \infty) \) the continuous non - negative strictly increasing function for which

\[
\lim_{u \to 0^+} \frac{N(u)}{u} = 0; \ \lim_{u \to \infty} \frac{N(u)}{u} = \infty;
\]

not necessarily to be convex.

The finite - measure case \( \mu(X) = 1 \) is investigated in particular in [16].

It is convenient for us to represent the tail function \( T = T(t) \) as an exponential form

\[
T(t) = \exp(-w(t)), \ t \geq 1. \quad (11)
\]

Note first of all that if for some increasing positive function \( G = G(t) \)

\[
I := \int_X G(|f(x)|) \mu(dx) < \infty,
\]

then by virtue of Tchebychev - Markov’s inequality

\[
T[f](t) \leq I/G(t), \ t > 0, \quad (12)
\]

or equally

\[
G(t) \leq I/T[f](t), \ t < 0. \quad (13)
\]

Therefore, it is reasonable to choose as the Young - Orlicz function the tail one \( T = T(t) : \)

\[
N[T](t) := G_0(t) = G_0[T](t) \overset{def}{=} \frac{1}{T(t)}; \ t \geq 1. \quad (14)
\]
Theorem 4.1. Suppose
\[ \exists k = \text{const} > 0 \Rightarrow \int_0^\infty \frac{|dT(t)|}{T(t/k)} < \infty. \quad (15) \]

Then the arbitrary measurable function \( f : X \to R \) for which \( T[f](t) \leq T(t), \ t > 0 \) belongs to the Orlicz space \( L(N[T]) \) built over source measurable space \( (X, \mathcal{M}, \mu) \).

Proof. We will use the following fact
\[ \int_X H(|f|(x)) \mu(dx) = \int_0^\infty H(t) |dT[f](t)|, \]
see e.g. [26], chapters 1,2; we conclude following that for all the sufficiently greatest positive values of the constant \( K > 0 \)
\[ \int_X N[T](|f(x)|/K) \mu(dx) \leq \int_0^\infty N[T](t/K)|dT(t)| = \int_0^\infty \frac{|dT(t)|}{T(t/K)} < \infty, \ K > k; \]
we used a comparison inequality, see e.g. [16].

Examples 4.1 - 4.2. The condition (15) is satisfied for the tail functions of the form
\[ \exists C, m \in (0, \infty) \Rightarrow T_{m,C}(t) = \exp(-Ct^m), \ t > 0, \]
and is not satisfies for the next tail functions
\[ T^{(\theta)}(t) = |\ln t|^{1/\theta} \cdot I(t \in (0, 1)), \ \theta \in (0, \infty). \]

The finding of the correspondent Orlicz function for this tail function is an open problem.

5 Concluding remarks.

Possible generalization. It is interest, in our opinion, to generalize the obtained results on the multidimensional case, i.e. when the functions \( \{f\} \) are vector valued, in the spirit of the preprint [22].

Acknowledgement. The first author has been partially supported by the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and by Università degli Studi di Napoli Parthenope through the project “sostegno alla Ricerca individuale”.
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