CHIO CONDENSATION
AND RANDOM SIGN MATRICES

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Abstract. This is to suggest a new approach to the old and open problem of counting the number \( f_n \) of \( \mathbb{Z} \)-singular \( n \times n \) matrices with entries from \( \{-1, +1\} \); comparison of two measures, none of them the uniform measure, one of them closely related to it, the other asymptotically under control by a recent theorem of Bourgain, Vu and Wood.

We will define a measure \( P_{\text{chio}} \) on the set \( \{-1,0,+1\}^{\lfloor (n-1)/2 \rfloor} \) of all \( (n-1) \times (n-1) \)-matrices with entries from \( \{-1,0,+1\} \) which (owing to a determinant identity published by M. F. Chio in 1853) is closely related to the uniform measures on \( \{-1,+1\}^{n^2} \) and \( \{0,1\}^{(n-1)^2} \) and at the same time it intriguingly mimics the so-called lazy coin flip distribution \( P_{\text{lcf}} \) on \( \{-1,0,+1\}^{\lfloor (n-1)/2 \rfloor} \), with the resemblance fading more and more as the events get smaller. This is relevant in view of a recent theorem of J. Bourgain, V. H. Vu and P. M. Wood (J. Funct. Anal. 258 (2010), 559–603) which proves that if the entries of an \( n \times n \) matrix whose \( \{-1,0,+1\} \)-entries are governed by \( P_{\text{lcf}} \) and fully independent (they are not when governed by \( P_{\text{chio}} \)), then an asymptotically optimal bound on the singularity probability over \( \mathbb{Z} \) can be proved. We will characterize \( P_{\text{chio}} \) graph-theoretically and use the characterization to prove that given a \( B \in \{-1,0,+1\}^{\lfloor (n-1)/2 \rfloor} \), deciding whether \( P_{\text{chio}}[B] = P_{\text{lcf}}[B] \) is equivalent to deciding an evasive graph property, hence the time complexity of this decision is \( \Omega(n^2) \). Moreover, we will prove \( k \)-wise independence properties of \( P_{\text{chio}} \). Many questions suggest themselves that call for further work. In particular, the present paper will close with more constrained equivalent formulations of the conjecture \( f_n/2n^2 \sim (\frac{1}{2} + o(1))^n \).

Keywords: Asymptotic enumeration, Betti number, Chio condensation, coboundary space, cut space, cycle space, determinant identities, discrete random matrices, lazy coin flip distribution, \( k \)-wise independence, signed graphs

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1. Introduction

For a commutative ring $R$ and a finite subset $U \subseteq R$ one may ask how many among the $|U|^2$ matrices $A \in U^{[n]^2}$ have $\det(A) = 0$. Much is known precisely when $R$ is the finite field $\mathbb{F}_q$ ($q$ power of a prime) and $U = R$. For instance, it follows from elementary linear algebra that the number of singular $n \times n$ matrices with entries from $\mathbb{F}_q$ is precisely $q^{n^2} - \prod_{0 \leq i < n-1} (q^n - q^i)$. As an advanced example, very precise statements can be proved for matrices over finite fields even if the entries are i.i.d. according to (quite) arbitrary distributions (cf. the work of Kahn–Komlós [22] and Maples [26].

In stark contrast, if $R = \mathbb{Z}$ and $U = \{-1, +1\}$, the correct order of decay of the density of singular matrices is still not known, but there is an old and plausible, yet still unproved conjecture of uncertain origin, on which the last two decades have brought several remarkable advances:

**Conjecture 1.** For $n \to \infty$, $\frac{|\{A \in \{-1, +1\}^{[n]^2} : \det(A) = 0 \in \mathbb{Z}\}|}{2^n} \sim (\frac{1}{2} + o(1))^n$.

Let us employ the abbreviations $- := -1$, $+ := +1$, $\{\pm\} := \{-1, +1\}$, $\{0, \pm\} := \{-1, 0, +1\}$, $\mathbb{P}[Ra_{<n}(\{\pm\}^{[n]^2})] := \{A \in \{\pm\}^{[n]^2} : \det(A) = 0\}$, and let $\mathbb{P}[\cdot]$ denote the uniform measure on $\{\pm\}^{[n]^2}$, (so that in particular Conjecture 1 now reads $\mathbb{P}[\{A \in \{\pm\}^{[n]^2} : \det(A) = 0\}] \sim (\frac{1}{2} + o(1))^n$). Moreover, let us introduce one of the two main protagonists of the present paper:

**Definition 2 (the measure $\mathbb{P}_lcf$).** For $(s, t) \in \mathbb{Z}^2_{\geq 0}$ and $\emptyset \subseteq I \subseteq [s - 1] \times [t - 1]$ let $\mathbb{P}_lcf$ denote the lazy coin flip distribution on $\{0, \pm\}^I$, i.e. the probability measure on $\{0, \pm\}^I$ defined by considering the values of a $B \in \{0, \pm\}^I$ as independent identically distributed random variables, each governed by the symmetric discrete distribution with values $-1$, $0$, $+1$ and probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$.

The name of $\mathbb{P}_lcf$ may stem from the fact that this is the distribution obtained when someone sets out to generate the entries of some $B \in \{0, \pm\}^I$ by performing $|I|$ independent fair coin flips, but there is a probability of $\frac{1}{2}$ at every single trial that out of a fleeting laziness the person decides to simply write 0 instead of flipping the coin.

The lazy coin flip distribution $\mathbb{P}_lcf$ plays a role in the recent article [6] of J. Bourgain, V. H. Vu and P. M. Wood in which the authors set the current record in a chain of successive improvements of upper bounds for $\mathbb{P}[Ra_{<n}(\{\pm\}^{[n]^2})]$ (see Komlós [24], Kahn–Komlós-Szemeredi [23] and Tao–Vu [29] [30]):

**Theorem 3 (Bourgain–Vu–Wood [6]).** For $n \to \infty$ it is true that

$$\mathbb{P}[\{A \in \{\pm\}^{[n]^2} : \det(A) = 0\}] \leq \left(\frac{1}{\sqrt{2}} + o(1)\right)^n ,$$

(1)

$$\mathbb{P}_lcf[\{B \in \{0, \pm\}^I : \det(B) = 0\}] \sim \left(\frac{1}{2} + o(1)\right)^n .$$

(2)

**Comments.** The upper bound within $\sim$ in (2) is the special case $\mu = \frac{1}{2}$ of [6, Corollary 3.1, p. 567]. The lower bound within the $\sim$ is obvious: consider the event that the first column has only zero entries (the lower bound is also explicity stated in [6, formula (7), p. 561]). The upper bound in (1) is the special case $S = \{\pm\}$ and $p = \frac{1}{2}$ in [6, Corollary 4.3, p. 576].

So Bourgain–Vu–Wood proved that the correct order of decay of $\mathbb{P}_lcf[\{B \in \{0, \pm\}^{[n]^2} : \det(B) = 0\}]$ is $(\frac{1}{2} + o(1))^n$—which is also the conjectured one for $\mathbb{P}[\{A \in \{\pm\}^{[n]^2} : \det(A) = 0\}]$. It is this latter achievement, combined with an observation made by the present author, which spurred the present paper. Note that using the uniform distribution on $\{\pm\}^{[n]^2}$ is equivalent to considering the $n^2$ entries as i.i.d. Bernoulli variables with probability $\frac{1}{2}$. The observation is this: When we apply one step of Chio condensation (see Definition 6) to this Bernoulli matrix, the result is a matrix
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whose entries are 3-wise (and ‘almost’ 6-wise, see Theorem 51 below) stochastically independent with \{-2, 0, +2\}-values which are distributed as if by the lazy coin flip distribution. Since Bourgain–Vu–Wood demonstrated that for \(P_{3t}\)-distributed entries an asymptotically correct order of decay can be proved, the observation feels like a hint at deeper connections and makes it seem imperative to investigate Chio condensation of sign matrices. A first step is taken in the present paper.

2. Definitions

Let \([n] := \{1, \ldots, n\}\). For any \(A = (a_{i,j})_{(i,j)\in [n]^2} \in \{\pm\}^{[n]^2}\), any \((i, j) \in [n-1]^2\) and any \(\emptyset \subseteq I \subseteq [n-1]^2\) let \(A[i, j] := a_{i,j}\) and \(A[I] := (a_{i,j})_{(i,j)\in I}\), hence in particular \(A[\emptyset]\) is the empty function and \(A[[n]^2] = A\). Let \(\mathcal{P}(X)\) denote the power set of a set \(X\).

For a cartesian product \(M \times N\) of two sets \(M\) and \(N\) let \(p_1 : M \times N \to M\) be the projection onto the first, and \(p_2 : M \times N \to N\) the projection onto the second factor. If \(M\) and \(N\) are finite and \(\emptyset \subseteq I \subseteq M \times N\) is some subset, then \(I\) is called rectangular if and only if \(|I| = |p_1(I)| \cdot |p_2(I)|\).

Let us view functions as sets and matrices as functions. If \(D\) is a set and \(f \colon D \to \mathbb{Z}\) is a function let us write \(D := \text{Dom}(f) \supseteq \text{Supp}(f) := \{d \in D : f(d) \neq 0\}\) for its domain and support, and let us employ the abbreviations \(|\text{Dom}(f)| := |\text{dom}(f)| \supseteq |\text{supp}(f)| := |\text{supp}(f)|\). We have \(\text{Dom}(\emptyset) = \text{Supp}(\emptyset) = \emptyset\) and therefore \(\text{dom}(\emptyset) = \text{supp}(\emptyset) = 0\). If \(U \subseteq \mathbb{Z}\), \(\emptyset \subseteq I \subseteq [s-1] \times [t-1]\) and \(B \subseteq U^I\), then we have \([s-1] \times [t-1] \supseteq I = \text{Dom}(B) \supseteq \text{Supp}(B) = \{(i, j) \in [s-1] \times [t-1] : B[i, j] \neq 0\}\). For a matrix \(M = (m_{i,j})_{(i,j)\in I} \in \mathbb{Q}^I\) and a \(q \in \mathbb{Q}\) we define, as usual, \(q \cdot M := (q \cdot m_{i,j})_{(i,j)\in I}\). The symbol \(\sqcup\) denotes a set union \(\cup\) and at the same time makes the claim that the union is disjoint. The term rank of matrix has its usual meaning (and we will only use it in the context of integral domains, so that row-rank, column-rank and determinantal rank are all the same). For a set \(S\), the group of all permutations of \(S\) is denoted by \(\text{Sym}(S)\).

The word ‘graph’ without any further qualifications means ‘finite simple graph’ (i.e. ‘finite 1-dimensional simplicial complex’). We will use \(V(X)\) (resp. \(E(X)\)) to denote the vertex set (resp. edge set) of a graph \(X\), and we follow [5] in reserving the more specific term circuit for what is called a cycle in [11] (i.e. closed walk without self-intersections). Moreover, we will use \(f_1(X)\) for the number of edges of a graph \(X\) and \(f_0(X)\) for the number of its vertices. The cycle space of \(X\) (i.e. 1-dimensional cycle group with \(\mathbb{Z}/2\)-coefficients in the sense of simplicial homology theory) will be denoted by \(Z_1(X; \mathbb{Z}/2)\) and the coboundary space of \(X\) by \(B^1(X; \mathbb{Z}/2)\) (this is the 1-dimensional coboundary group with \(\mathbb{Z}/2\)-coefficients in the sense of simplicial cohomology theory; a synonym is ‘cut space of \(X\)’). Let \(\beta_0(X)\) denote the number of connected components of a graph \(X\) and \(\beta_1(X) := \dim_{\mathbb{Z}/2} Z_1(X; \mathbb{Z}/2)\) the first Betti number (a synonym in the graph-theoretical literature is ‘cyclomatic number’ [27]). We will (without further notification) use the 1-dimensional case of the alternating sum relation between the ranks of the chain groups and the ranks of the homology groups of a free chain complex, i.e. \(\beta_1(X) - \beta_0(X) = f_1(X) - f_0(X)\) for every graph \(X\). For any two disjoint graphs \(X_1\) and \(X_2\), the graph obtained by identifying exactly one vertex of \(X_1\) with exactly one vertex of \(X_2\) is called the \((one-point)\) wedge of \(X_1\) and \(X_2\) and denoted by \(X_1 \vee X_2\). This is the standard wedge product of pointed topological spaces (but only vertices of a graph are allowed as basepoints); a synonym within the graph-theoretical literature is ‘coalescence’ [16, p. 140].

We will use the language of signed graphs (see [34] for a comprehensive overview). It is customary in signed graph theory to work with multigraphs (i.e. finite 1-dimensional CW-complexes) for reasons of higher flexibility in proofs and applications. However, in the present paper, all we will need are signed simple graphs, i.e. for us a signed graph \((X, \sigma)\) will simply consist of a graph \(X = (V, E)\) together with an arbitrary sign function \(\sigma : E \to \{\pm\}\). We call \((+)-edge\) (resp. \((-)-edge\) every \(e \in E(X)\) with \(\sigma(e) = +\) (resp. \(\sigma(e) = -\)). Define \((+)-paths\) (resp. \((-)-paths\) as paths all of whose edges are \((+)-edges\) (resp. \((-)-edges\)). For emphasizing the sign function we employ
the notation \((\sigma, +)-edge\). If \((X, \sigma)\) is a signed graph let \(f_1^{(\sigma)}(X, \sigma) := |\{e \in E(X) : \sigma(e) = -\}|\) denote the number of \((\sigma, -)-\)edges in it. A signed graph \((X, \sigma)\) is called balanced\(^1\) if and only if \(f_1^{(\sigma)}(C, \sigma)\) is even for every circuit \(C\) of \(X\). We will denote the set of all balanced signings of \(X\) by \(\mathcal{S}_{\text{bal}}(X) := \{\sigma \in (\pm)^{E(X)} : (X, \sigma)\text{ balanced}\}\).

**Definition 4** (Chio\(^2\) set). Let \((s, t) \in \mathbb{Z}^2\) and \(I \subseteq [s] \times [t]\). Then \(I\) is called a Chio set if and only if \((s, t) \in I\) and for every \((i, j) \in I\) we have \((i, t) \in I\) and \((s, j) \in I\).

**Definition 5** (Chio extension\(^3\)). For every \((s, t) \in \mathbb{Z}^2\) and every \(\emptyset \subseteq I \subseteq [s-1] \times [t-1]\),

\[
\tilde{I} := \{(s, t)\} \cup \bigcup_{i \in \pi_1(I)} \{(i, t)\} \cup \bigcup_{j \in \pi_2(I)} \{(s, j)\} \cup I.
\]

Note that \(\tilde{I} \subseteq [s] \times [t]\) for every \(\emptyset \subseteq I \subseteq [s-1] \times [t-1]\), in particular \(\emptyset = \{(s, t)\}\) and \(([s-1] \times [t-1]) = [s] \times [t]\). Moreover, a set \(I' \subseteq [s] \times [t]\) is a Chio set if and only if there exist an \(I \subseteq [s-1] \times [t-1]\) with \(I' = \tilde{I}\).

Now we come to Chio condensation. In the special (and very common) case of \(s = t\) (hence \([s] \times [t] = [n]^2\)) the following definition differs from the standard convention (as is to be found in [8] and [12]) in that the entry \(a_{n,n}\) instead of \(a_{1,1}\), is taken to be the pivot. This seems to be more convenient for handling the indices of a Chio condensate. There does not appear to be any logical necessity for multiplying by \(\frac{1}{2}\), but the author decided to keep the discussion within the realm of \(\{0, \pm\}\)-matrices (instead of \(-2, 0, +2\)-matrices).

**Definition 6** (Chio condensation, \(\frac{1}{2}C_{(s,t)}^I\)). For every \((s, t) \in \mathbb{Z}^2\) and every \(I \subseteq [s-1] \times [t-1]\) define the Chio map with pivot \(a_{s,t}\) as

\[
\frac{1}{2}C_{(s,t)}^I : \{\pm\}^I \rightarrow \{0, \pm\}^I, \quad A \mapsto \frac{1}{2} \cdot C_{(s,t)}(A),
\]

where \(C_{(s,t)}(A) := \{\det(a_{i,j} a_{i,t})\}_{(i,j) \in \tilde{I}} \in \{-2, 0, +2\}\). An image \(C_{(s,t)}(A)\) of some \(A \in \{\pm\}^I\) is referred to as the Chio condensate of \(A\).

**Definition 7** (the measure \(P_{\text{chio}}\)). For every \((s, t) \in \mathbb{Z}^2\) and every \(I \subseteq [s-1] \times [t-1]\) define

\[
P_{\text{chio}} : \mathfrak{P}(\{0, \pm\}^I) \rightarrow [0, 1], \quad B \mapsto \frac{1}{2^{|I|}} \sum_{B \in \mathcal{B}} |\left(\frac{1}{2}C_{(s,t)}^I\right)^{-1}(B)|.
\]

Note that in the special case of \(s := t := n\) and \(I := [n-1]^2\), the measure \(P_{\text{chio}}\) maps a single \(B \in \{0, \pm\}^{[n-1]^2}\) to \(P_{\text{chio}}[B] := P_{\text{chio}}[[B]] = 2^{-n^2} \cdot |\{A \in \{\pm\}^n : B = \frac{1}{2} \cdot C_{(n,n)}(A)\}|\).

We now define two additional measures. Later we will recognize both of them as familiar ones:

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\(^1\)The use of this term seems to have been initiated in [18]. The notion itself was already studied over seventy years ago by D. König [25, p. 149, Paragraph 3] under the name ‘p-Teilgraph’.

\(^2\)In earlier versions I wrote ‘Chii’ but this now seems wrong. All three spellings Chii, ‘Chii’ and ‘Chi’ are to be found in the literature. My sole reason for using ‘ô’ was that in [8] the authors consistently use the spelling ‘Chii’ and it is said [8, p. 790] that a copy of Chii’s original paper had been at the authors’ disposal. However, an original 1853 copy of [10] which I recently bought from an antiquarian bookstore in Italy gives strong circumstantial evidence in favour of the spelling ‘Chii’: on the title page and the inside-cover his given name ‘Félix’ is written with an accent whereas ‘Chii’ does not carry any accent. Moreover, the title page bears a hand-written dedication to a colleague, signed ‘L’auteur’. Therefore, to all appearances, Chii signed this title page himself, 158 years ago. Possibly extant autographs aside, putting this on record might come as close to a personal statement by Chii as one will ever get nowadays. Moreover, the spelling is further corroborated by the usage adopted by Cauchy in [9], Cauchy on several occasions consistently writes ‘M. Félix Chii’ [9, p. 110, pp. 112–113].

\(^3\)Due to the change of spelling explained in the previous footnote I now use a breve instead of a grave accent to denote Chii extension.
Whenever possible we will suppress the superscripts in such a situation and only write $X \in I$.

Define $\sigma$ by letting vertex-set and edge-set be defined as in the natural way (while ignoring the signs), and that $\sigma$ where $|I| = 3|B|^{-1}$ for arbitrary $\emptyset \subseteq J \subseteq [s-1] \times [t-1]$ and $B \subseteq \{0, \pm\}^I$.

In this paper we intend to use graph-theoretical language. For the sake of specificity and ease of reference, we will explicitly name the set of auxiliary labelled bipartite graphs that we will talk about (and give it a vertex set which blends well with the matrix setting).

Definition 11. For every $(s, t) \in \mathbb{Z}_+^2$, denote by $BG_{s,t}$ the $2^{(s-1)(t-1)}$-element set of all bipartite graphs $X = (V_1 \cup V_2, E)$ with $V_1 = \{(i, t) : 1 \leq i \leq s - 1\}$ and $V_2 = \{(s, j) : 1 \leq j \leq t - 1\}$.

There is an obvious bijection $BG_{s,t} \leftrightarrow \{0, 1\}^{(s-1)(t-1)}$. Associating with a (partially specified) $\{0, \pm\}$-matrix the following bipartite signed graph will be helpful in our study of $P_{\text{chio}}$. The definition can be summarized by saying that $X$ interprets a $B \in \{0, \pm\}^I$ as a bipartite adjacency matrix in the natural way (while ignoring the signs), and that $\sigma$ takes the signs in $B$ as a definition of a sign function.

Definition 12 (X and $\sigma$). For every $(s, t) \in \mathbb{Z}_+^2$ and every $0 \leq k \leq (s-1)(t-1)$ define

$$X^{k,s,t}_{BG} : \bigcup_{I \subseteq \{0, \pm\}^I \atop |I| = k} \{0, 1\}^I \rightarrow BG_{s,t}, \quad B \mapsto X^{k,s,t}_{B}$$

by letting vertex-set and edge-set be defined as

$$V(X^{k,s,t}_{BG}) := (\text{Dom}(B))^C \setminus \text{Dom}(B) \setminus \{(s, t)\},$$

$$E(X^{k,s,t}_{BG}) := \bigcup_{(i, j) \in \text{Supp}(B)} \{(i, t), (s, j)\}.$$

Define $\sigma_B : E(X_B) \rightarrow \{\pm\}$ by $\sigma_B((i, t), (s, j)) := b_{ij} \in \{\pm\}$ for every $\{(i, t), (s, j)\} \in E(X_B)$.

If $k = 0$, hence $I = \emptyset$, hence $B = \emptyset$ is the empty matrix, then $X^{k,s,t}_{BG}$ is the empty graph $(\emptyset, \emptyset)$ and $\sigma_B = \emptyset$ is the empty function. Note that while for a $B \in \{0, \pm\}^I$ the set $V(X_B)$ depends only on $I = \text{Dom}(B)$, the set $E(X_B)$ depends on Supp$(B)$ and $\sigma_B$ even depends on $B$ itself.

When we take the image of a matrix $B \in \{0, \pm\}^I$ under $X^{k,s,t}_{BG}$, then usually we will know what $I \in \{0, \pm\}^I$ we are talking about and then the superscripts $k, s, t$ give redundant information. Whenever possible we will suppress the superscripts in such a situation and only write $X_B$. When
we take preimages of a graph $X \in \mathcal{BG}_{s,t}$ under $X^{k,s,t}$, however, the full notation has to be used since in general it is not possible to tell $k$ from the labelled graph $X$ (let alone from its isomorphism type). As an example for this, consider the graph $X \in \mathcal{BG}_{4,4}$ with vertex set $\{(1,4), (2,4), (3,4)\} \cup \{(4,1), (4,2), (4,3)\}$ and edge set $\{(1,4), (4,1), (2,4), (4,1), (2,4), (4,2), (1,4), (4,2)\}$, which is isomorphic to a 4-circuit with two additional isolated vertices. Then we have $X^{5,4,4} = X^{6,4,4} = X$ for $B_1 := \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \neq B_2 := \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Here, $\text{dom}(B_1) = 5 \neq 6 = \text{dom}(B_2)$.

In the following, we deliberately do not define ‘isomorphism type of a graph’ more precisely. We would not have much use for any of the existing formalizations of an unlabelled graph.

Definition 13 (ul, $\beta^u_1$). Let $ul$ be the map which assigns a labelled graph to its isomorphism type. Let $\beta^u_1: \mathcal{BG}_{n,n} \to \mathbb{Z}_{\geq 0}$ be the map which assigns an unlabelled bipartite graph to its 1-dimensional Betti number.

Definition 14 ($ulX^{k,s,t}$). For $(s,t) \in \mathbb{Z}_{\geq 2}^2$ and $0 \leq k \leq (s-1)(t-1)$ let $ulX^{k,s,t} := ul \circ X^{k,s,t}$.

If $X$ is some (verbal, pictorial, ...) description of an isomorphism type of graphs, we can now take its preimage $(ulX^{k,s,t})^{-1}(X) \subseteq \bigsqcup_{\ell \in \{2s-1\}^{s-1}} \{0, \pm\}^t$. To analyse how $P_{\text{choio}}$ and $P_{\text{lef}}$ relate to one another, it is useful to have the following notations.

Definition 15 (failure sets). For every $k \geq 0$, $n \geq 2$, $\ell \in \mathbb{Q}_{\geq 0}$ and $p \in [0,1] \cap \mathbb{Q}$ let

1. $\mathcal{F}_M(k,n) := \{ B \in \{0, \pm\}^t : \ell \in \{n-k\}^2, P_{\text{choio}}[E_B^{n-1}] \neq P_{\text{lef}}[E_B^{n-1}] \}$,
2. $\mathcal{F}_\ell(k,n) := \{ B \in \mathcal{F}_M(k,n) : P_{\text{choio}}[E_B^{n-1}] = \ell \cdot P_{\text{lef}}[E_B^{n-1}] \} \subseteq \mathcal{F}_M(k,n)$,
3. $\mathcal{F}_p(k,n) := \{ B \in \mathcal{F}_M(k,n) : P_{\text{choio}}[E_B^{n-1}] = p \} \subseteq \mathcal{F}_M(k,n)$,
4. $\mathcal{F}_G(k,n) := ulX^{k,n,n}(\mathcal{F}_M(k,n))$, $\mathcal{F}_p(k,n) := ulX^{k,n,n}(\mathcal{F}_p(k,n))$.

We abbreviate $\mathcal{F}_M(k,n) := \mathcal{F}_M(k,n)$, and analogously for all the other sets just defined.

Obviously, $\mathcal{F}_M(k,n) = \emptyset$ and $\mathcal{F}_p(k,n) = \mathcal{F}_M(k,n)$ for all $k$ and $n$. Item (C3) in Theorem 30 will teach us that $\mathcal{F}_\ell(k,n) = \emptyset$ for every $\ell \notin \{0\} \cup \{2^i : i \in \mathbb{Z}_{\geq 0}\}$ (hence in particular $P_{\text{choio}}[E_B^{n-1}] \geq P_{\text{lef}}[E_B^{n-1}])$ for every $B \in \mathcal{F}_M(k,n)$ with $P_{\text{choio}}[E_B^{n-1}] > 0$.

Definition 16 (matrix-circuit, Cir(s,n)). For every $(s,t) \in \mathbb{Z}_{\geq 2}^2$ and every $\ell \subseteq \{s-1\} \times \{t-1\}$ with even $l := |\ell|$, the set $\ell$ is called a matrix-l-circuit if and only if $X^{\ell}_{(1,\ell)}$ is a graph-theoretical l-circuit. Moreover, Cir(l,s,t) := $\{ \ell \subseteq \{s-1\} \times \{t-1\} : |\ell| = l, L$ is a matrix-l-circuit $\}$ and Cir(l,n,n) := Cir(l,n,n).

Definition 17 ((−)-constant, (+)-proper vertex 2-colouring of a signed graph). For a graph $X = (V,E)$ and a $\sigma \in \{\pm\}^E$, a function $c \in \{\pm\}^V$ is called $(\sigma, -)-$constant, $(\sigma, +)$-proper if and only if $c(u) = c(v)$ for every $e = uv \in E(X)$ with $\sigma(e) = -$ and $c(u) \neq c(v)$ for every $e = uv \in E(X)$ with $\sigma(e) = +$.

Definition 18 (Col(X,σ)). For a graph $X = (V,E)$ and a $\sigma \in \{\pm\}^E$ let Col(X,σ) be the set of all $(\sigma, -,)$-constant, $(\sigma, +)$-proper vertex 2-colourings $c \in \{\pm\}^V$.

Definition 19 (rank-level-sets of matrices). For $(s,t) \in \mathbb{Z}_{\geq 2}^2$, $0 \leq r \leq \text{min}(s,t)$, $R$ an integral domain, $U \subseteq R$ and $\mathcal{R} \in \mathcal{P}(U \{0,1, \ldots, \text{min}(s,t)\})$ let $\text{Ra}_R(U^{[s] \times [t]}):= \{ A \in U^{[s] \times [t]} : \text{rk}(A) = r \}$, $\text{Ra}_R(\{\pm\}^{[s] \times [t]}):= \bigcup_{r \in \mathcal{R}} \text{Ra}_R(\{\pm\}^{[s] \times [t]})$ and $\text{Ra}_{<R}(U^{[s] \times [t]}):= \text{Ra}_{\{0,1, \ldots, r-1\}}(\{\pm\}^{[s] \times [t]})$. 

3. Lemmas

We will use the following elementary fact:

**Lemma 20.** \( f^{-1}(f(f^{-1}(U))) = f^{-1}(U) \) for any map \( f : A \rightarrow B \) and any subset \( U \subseteq B \). □

The following simple statement is essential for the approach developed in the present paper. More information on this identity can be found in [8, last paragraph of Section 9] and [1, Ch. 4, p. 282, Exerc. 43]. The formulation given here differs from those in [10, p. 11] and [8] in that \( a_{n,n} \) instead of \( a_{1,1} \) is taken to be the pivot. This seems more convenient for handling the indices of a Chio condensate.

**Lemma 21** (Chio’s identity). For \( n \geq 2 \), \( R \) an integral domain and \( (a_{i,j}) = A \in R^{[n]^2} \),

\[
\det(C(n,n)(A)) = a_{n,n}^{n-2} \cdot \det(A) .
\]

(12)

**Proof.** This is stated by Chio in [10, p. 11, Théorème 4, equation (20)] and he proves it on pp. 6–11 (the notation ‘\( \pm a_{0b_1} \)’ employed in [10, equation (13)’] is defined at the beginning of p. 6 of [10]). To contemporary eyes, this is an easy consequence of the behavior of determinants under linear transformations, cf. [12] for a direct proof of the version with pivot \( a_{1,1} \). Moreover, this is a special case of Sylvester’s determinant identity. To see this, set \( k = 1 \) in formula (8) of [8] to get a version of (12) with pivot \( a_{1,1} \). Obvious modifications in the proof in [8] yield the version with pivot \( a_{n,n} \). □

The following three assertions are obviously true:

**Corollary 22.** For every \( A \in \{\pm\}^{[n]^2} \), \( \det(A) = 0 \) if and only if \( \det(\frac{1}{2}C(n,n)(A)) = 0 \). □

**Lemma 23** (value of lazy coin flip distribution on single matrix). For every \( \emptyset \subseteq I \subseteq [n-1]^2 \) and every \( B \in \{0,\pm\}^I \), \( P_{\text{lef}}[\mathcal{E}_B] = (\frac{1}{2})^{|\text{dom}(B)+\text{supp}(B)} \). □

**Lemma 24.** For any two disjoint graphs \( X_1 \) and \( X_2 \) and any two sign functions \( \sigma_{X_1} \in \{\pm\}^{E(X_1)} \) and \( \sigma_{X_2} \in \{\pm\}^{E(X_2)} \), and for every graph \( X \) obtained by a one-point wedge of \( X_1 \) and \( X_2 \) at two arbitrary vertices, the sign function \( \sigma_X \in \{\pm\}^{E(X)} \) obtained by unifying the maps \( \sigma_{X_1} \) and \( \sigma_{X_2} \) is balanced if and only if both \( (X_1,\sigma_{X_1}) \) and \( (X_2,\sigma_{X_2}) \) are balanced. □

The following will be needed for counting failures of equality of \( P_{\text{chio}} \) and \( P_{\text{lef}} \).

**Lemma 25.** \( |\text{Cir}(2j,s,t)| = \left(\frac{s-1}{j}\right) \cdot \left(\frac{t-1}{j}\right) \cdot \frac{j!\cdot(2j-1)!}{2} \) for every \( n \geq 2 \) and every \( 1 \leq j \leq \min(|\frac{n}{2}|,|\frac{n}{2} - 1|) \).

**Proof.** For every \( N' \in \{\pm\}^{i,j-1} \) and \( N'' \in \{\pm\}^{i,j-1} \) let \( \text{Cir}(2j,s,t,N',N'') := \{S \in \text{Cir}(2j,s,t): p_1(S) = N', \ p_2(S) = N''\} \). Obviously, \( |\text{Cir}(2j,s,t)| = \sum_{N' \in \{\pm\}^{i,j-1}} \sum_{N'' \in \{\pm\}^{i,j-1}} |\text{Cir}(2j,s,t,N',N'')| \). To count \( \text{Cir}(2j,s,t,N',N'') \), define \( \text{Perm}(N',N'') : = \{P : P \text{ is a permutation matrix, Supp}(P) \subseteq N' \times N''\} \). Define a binary relation \( \mathcal{R} \subseteq \text{Cir}(2j,s,t,N',N'') \times \text{Perm}(N',N'') \) by letting \( (S,P) \in \mathcal{R} \) if and only if \( \text{Supp}(S) \supseteq \text{Supp}(P) \). For every \( S \in \text{Cir}(2j,s,t,N',N'') \) we have \( |\{P \in \text{Perm}(N',N'') : (S,P) \in \mathcal{R}\}| = 2 \). On the other hand, for every \( P \in \text{Perm}(N',N'') \) we have \( |\{S \in \text{Cir}(2j,s,t,N',N'') : (S,P) \in \mathcal{R}\}| = (j-1)! \cdot j! = (j-1)! \cdot |\text{Perm}(N',N'')| = \sum_{P \in \text{Perm}(N',N'')} |\{S \in \text{Cir}(2j,s,t,N',N'') : (S,P) \in \mathcal{R}\}| = \sum_{S \in \text{Cir}(2j,s,t,N',N'')} 2 \cdot |\text{Cir}(2j,s,t,N',N'')|. \)

The following is contained in König’s 1936 classic [25].

**Lemma 26** (D. König). Let \( \mathfrak{X} \) be a labelled or an unlabelled graph. Then:

(Kő1) \( S_{\text{bal}}(\mathfrak{X}) \neq \emptyset \) if and only if \( \text{Col}(\mathfrak{X},\sigma) \neq \emptyset \).

(Kő2) Let \( \sigma : \{\pm\}^E \) be arbitrary. Then \( \text{Col}(\mathfrak{X},\sigma) \neq \emptyset \) if and only if \( |\text{Col}(\mathfrak{X},\sigma)| = 2^{h_0(\mathfrak{X})} \).
(Kö3) \(|\{\sigma \in \{\pm\}^{E(X)} : (X, \sigma) \text{ balanced}\}| = 2^{f_0(X) - \beta_0(X)}\).

Proof. Modulo terminology a proof for (Kö1) can be found in [25, p. 152, Satz 11] (for the definition of ‘p-Teilgraph’ cf. [25, p. 149, Paragraph 3]). Statement (Kö1) is also proved in [18, Theorem 3]. Statement (Kö2) is implicit in the proof of [25, p. 152, Satz 14] and can easily be proved directly by induction on \(|E|\). For a proof of (Kö3) cf. e.g. [25, p. 152, Satz 14].

While for a given graph \(X = (V, E)\) and a given sign function \(\sigma : E \to \{\pm\}\), the decision problem of whether \((X, \sigma)\) is balanced is trivially in co-NP, the less obvious fact that it is also in NP follows from (Kö1): any \((\sigma, \{-\})\)-constant, \((\sigma, +)\)-proper vertex-2-colouring \(c : V \to \{\pm\}\) is a polynomially-sized certificate for \((X, \sigma)\) being balanced. However, the problem is not only in the intersection of these two classes but easily seen to be in P:

**Corollary 27.** For every graph \(X = (V, E)\) and every sign function \(\sigma : E \to \{\pm\}\), the decision problem whether \((X, \sigma)\) is balanced can be solved in time \(O(f_0(X) + f_1(X))\).

Proof. By (Kö1), the question is equivalent to whether there exists a \((\{-\})\)-constant, \((+, +)\)-proper vertex-2-colouring \(c : V \to \{\pm\}\). It is easy to see that an obvious greedy algorithm via a (e.g.) depth-first search on \(X\) succeeds in finding such a colouring if and only if such a colouring exists. Moreover, the algorithm requires time \(O(f_0(X) + f_1(X))\).

The following simple lemma encapsulates a basic mechanism linking Chio condensation with the auxiliary graph-theoretical viewpoint. For want of topologies on source or target, ‘\(k\)-fold cover’ is nothing but shorthand for ‘surjective map each of whose fibres has cardinality \(k\)’.

**Lemma 28.** For every \((s, t) \in \mathbb{Z}^2_{\geq 2}\), arbitrary \(\emptyset \subseteq I \subseteq J \subseteq [s-1] \times [t-1]\), every \(B \in \{0, \pm\}^I\), and with \(h(I, J) := |J| - |I| - |p_1(I)| - |p_2(I)| \in \mathbb{Z}_{\geq 1}\), there exists an \(2^{h(I, J)}\)-fold cover

\[
\Phi : \left(\frac{1}{2}C^J_{(s,t)}\right)^{-1}(E^J_B) \to \text{Col}(X_B, \sigma_B)
\]

(13)

Proof. Let \(s, t, I, J, B = (b_{i,j})_{(i,j) \in I}\) be given as stated. The claim \(h(I, J) \in \mathbb{Z}_{\geq 1}\) is true since Definition 5 implies \(|J| = 1 + |p_1(J)| + |p_2(J)| + |J|\) and because \(|J| \geq |J|\), \(|p_1(J)| \geq |p_1(J)|\) and \(|p_2(J)| \geq |p_1(J)|\).

If \(\text{Col}(X_B, \sigma_B) = \emptyset\), the statement of the lemma is vacuously true (there not being any point of the target for which the definition of a covering would have to hold). We therefore can assume that \(\text{Col}(X_B, \sigma_B) \neq \emptyset\). We now first show that this implies that \(\left(\frac{1}{2}C^J_{(s,t)}\right)^{-1}(E^J_B) \neq \emptyset\). Then we will construct a cover of the stated kind.

To prove \(\left(\frac{1}{2}C^J_{(s,t)}\right)^{-1}(E^J_B) \neq \emptyset\), choose an arbitrary \(c \in \text{Col}(X_B, \sigma_B)\) and define \(A = (a_{i,j}) \in \{\pm\}^J\) by \(a_{s,t} := +\), \(a_{s,t} := c((i, t))\) for every \(i \in p_1(J)\), \(a_{s,j} := c((s, j))\) for every \(j \in p_2(J)\). Moreover, for every \((i, j) \in J\) let \(a_{i,j} := b_{i,j}\) if \(b_{i,j} \neq 0\) and \(a_{i,j} := c((i, t)) \cdot c((s, j))\) if \(b_{i,j} = 0\). We now show that \(\frac{1}{2}C^J_{(s,t)}(A)|_{\text{Dom}(B)} = B\). Let \((i, j) \in I = \text{Dom}(B)\) be arbitrary. If \(b_{i,j} = 0\), then \(\frac{1}{2}C^J_{(s,t)}(A)[i, j] = \frac{1}{2}(a_{i,j} - a_{i, t}a_{s,t}) = \frac{1}{2}(a_{i,j} - c((i, t))c((s, j))) = \frac{1}{2}(c((i, t))c((s, j)) - c((i, t))c((s, j)))\) and because \(J\) implies \(|J| = |J|\), \(|p_1(J)| \geq |p_1(J)|\) and \(|p_2(J)| \geq |p_1(J)|\).

If \(b_{i,j} \neq 0\), then \(\frac{1}{2}C^J_{(s,t)}(A)[i, j] = \frac{1}{2}(a_{i,j}a_{s,t} - a_{i,t}a_{s,t}) = \frac{1}{2}(b_{i,j} - c((i, t))c((s, j)))\). Now if \(b_{i,j} = -\), then \(c \in \text{Col}(X_B, \sigma_B)\) implies \(c((i, t)) \neq c((s, j))\), hence \(\frac{1}{2}C^J_{(s,t)}(A)[i, j] = \frac{1}{2}((-) \cdot (-)) = = b_{i,j}\), and if \(b_{i,j} = +\), then \(c \in \text{Col}(X_B, \sigma_B)\) implies \(c((i, t)) \neq c((s, j))\), hence \(\frac{1}{2}C^J_{(s,t)}(A)[i, j] = \frac{1}{2}((+ \cdot -)) = = b_{i,j}\).

We have proved that \(A \in \left(\frac{1}{2}C^J_{(s,t)}\right)^{-1}(E^J_B)\), and therefore \(\left(\frac{1}{2}C^J_{(s,t)}\right)^{-1}(E^J_B) \neq \emptyset\). We can therefore define a nonempty map \(\Phi : \left(\frac{1}{2}C^J_{(s,t)}\right)^{-1}(E^J_B) \to \text{Col}(X_B, \sigma_B)\) as follows: for every \(A = (a_{i,j})_{(i,j) \in J} \in \left(\frac{1}{2}C^J_{(s,t)}\right)^{-1}(E^J_B) \subseteq \{\pm\}^J\) we let \(\Phi(A)\) be the function \(V(X_B) \to \{\pm\}\) defined by \(\Phi(A)((i, t)) := a_{i,t}\) and \(\Phi(A)((s, j)) := a_{s,j}\) for all \(i \in p_1(I)\) and \(j \in p_2(1)\).
Lemma 29. \( \Phi(A) \in \text{Col}(X_B, \sigma_B) \). Proof: Let \( \{(i, t), (s, j)\} \in E(X_B) \) be arbitrary. There are two cases. If \( \sigma_B \{(i, t), (s, j)\} = - \) then \( b_{i,j} = - \) by Definition 12. Moreover, since \( \left( \frac{1}{2} C_{(s,t)}^J(A) \right) |_{\text{Dom}(B)} = B \) by the choice of \( A \), it follows that for every \( (i, j) \in I \subseteq J \) we have \( - = b_{i,j} = \frac{1}{2} C_{(s,t)}^J(A)[i,j] = \frac{1}{2} (a_{i,j} a_{s,t} - a_{i,t} a_{s,j}) \). In view of \( a_{i,j}, a_{s,t}, a_{i,t}, a_{s,j} \in \{-\} \), this equation implies \( \Phi(A)((i,t)) = a_{i,t} = a_{s,j} = \Phi(A)((s,j)) \). This proves that \( \Phi(A) \) is \((\sigma_B, -)\)-constant. If \( \sigma_B \{(i, t), (s, j)\} = + \) then \( b_{i,j} = + \) by Definition 12. Again by the choice of \( A \), it is true that \( + = b_{i,j} = \frac{1}{2} C_{(s,t)}^J(A)[i,j] = \frac{1}{2} (a_{i,j} a_{s,t} - a_{i,t} a_{s,j}) \). In view of \( a_{i,j}, a_{s,t}, a_{i,t}, a_{s,j} \in \{-\} \), this equation implies \( \Phi(A)((i,t)) = a_{i,t} \neq a_{s,j} = \Phi(A)((s,j)) \). This proves that \( \Phi(A) \) is \((\sigma_B, +)\)-proper and hence Claim 1.

Claim 2. \( \Phi \) is surjective and every fibre under \( \Phi \) has cardinality \( 2^{h(I,J)} \). Proof: Let an arbitrary \( c \in \text{Col}(X_B, \sigma_B) \) be given. We are now looking for those \( A \in \left( \frac{1}{2} C_{(s,t)}^J \right)^{-1}(E_B^J) \) with \( \Phi(A) = c \). Since the definition of \( \Phi \) demands \( \Phi(A)((i, t)) = a_{i,t} \) and \( \Phi(A)((s, j)) = a_{s,j} \) for all \( (i, t) \) and \( (s, j) \) in \( V(X_B) \), it follows that with regard to the \( |p_1(I)| + |p_2(I)| \) different entries \( a_{i,j} \) with \( (i, j) \in V(X_B) \), we know from the outset that we have no choice but to define \( a_{i,t} := c((i, t)) \) and \( a_{s,j} := c((s, j)) \).

Furthermore, since \( A \) must be in \( \left( \frac{1}{2} C_{(s,t)}^J \right)^{-1}(E_B^J) \), there is, for every \( (i, j) \in I \subseteq J \), the condition that \( b_{i,j} = \frac{1}{2} C_{(s,t)}^J(A)[i,j] = \frac{1}{2} \cdot \left( C_{(s,t)}^J(A)[i,j] \right) = \frac{1}{2} \cdot \left( a_{i,j} a_{s,t} - a_{i,t} a_{s,j} \right) = \frac{1}{2} (a_{i,j} a_{s,t} - c((i, t)) \cdot c((s, j))) \), where in the last step we have used the information about \( A \) that we already have. Now there are three cases that can occur.

Case 1. \( b_{i,j} = - \). Then by Definition 12 we have \( \{(i, t), (s, j)\} \in E(X_B) \) and \( \sigma_B \{(i, t), (s, j)\} = - \). Therefore, due to the fact that \( c \in \text{Col}(X_B, \sigma_B) \) is \((\sigma_B, -)\)-constant, \( c((i, t)) = c((s, j)) \). Thus, in this case, \( - = \frac{1}{2} (a_{i,j} a_{s,t} - 1) \), equivalently, \( a_{i,j} = a_{s,t} \).

Case 2. \( b_{i,j} = 0 \). Then by Definition 12, \( \{(i, t), (s, j)\} \notin E(X_B) \), hence \( \sigma_B \{(i, t), (s, j)\} \) is not defined and therefore the product \( c((i, t)) \cdot c((s, j)) \) in the equation \( 0 = \frac{1}{2} (a_{i,j} a_{s,t} - c((i, t)) \cdot c((s, j))) \) cannot be simplified further, but the equation itself can: it is equivalent to \( a_{i,j} = c((i, t)) \cdot c((s, j)) \) \cdot \( a_{s,t} \in \{\pm\} \). (where we used that \( a_{i,j}^{-1} = a_{s,t} \)).

Case 3. \( b_{i,j} = + \). Then an entirely analogous argument as in Case 1, but this time using the \((\sigma_B, +)\)-properness of \( c \), shows that then there is the equation \( a_{i,j} = a_{s,t} \).

We now know what it means to require \( A \in \left( \frac{1}{2} C_{(s,t)}^J \right)^{-1}(E_B^J) \) in the present situation: among the \( |J| \) entries of \( A = (a_{i,j}) \in \{\pm\}^J \), there are the \( |p_1(I)| + |p_2(I)| \) ‘immediately determined’ entries \( a_{i,j} \) which have \((i \in p_1(I) \text{ and } j = t) \text{ or } (i = s \text{ and } j \in p_2(I)) \), and moreover the \( |I| \) different entries \( a_{i,j} \) with \((i, j) \in I \) which are determined by a system \( \{a_{i,j} = h_{i,j}; (i,j) \in I\} \) of \( |I| \) equations where the right-hand sides \( h_{i,j} \) are defined by the Cases 1-3 above. For the remaining \( h(I,J) \) different entries \( a_{i,j} \in \{\pm\} \) (note that the pivot \( a_{s,t} \) is among them since it is on the right-hand side in Case 2, hence not determined by the system), the choice of their value is free; any of the \( 2^{h(I,J)} \) possible choices gives an \( A \in \left( \frac{1}{2} C_{(s,t)}^J \right)^{-1}(E_B^J) \). This proves that the cardinality of the fibre \( \Phi^{-1}(c) \) is indeed \( 2^{h(I,J)} \), and in particular that \( \Phi \) is surjective. Now Claim 2 and Lemma 28 are proved.

Note that in the special case \( I = J \), i.e. \( \text{when all entries are specified, then } h(I,J) = 1 \) and the statement says that there is a double cover \( \Phi: \left( \frac{1}{2} C_{(s,t)}^J \right)^{-1}(\{B\}) \rightarrow \text{Col}(X_B, \sigma_B) \). This corresponds to the freedom of choosing the sign of the pivot. Now we can relate Chio condensation to balancedness:

**Lemma 29.** For every \((s, t) \in \mathbb{Z}_2^2 \), every \( \emptyset \subseteq I \subseteq J \subseteq [s-1] \times [t-1] \) and every \( B \in \{0, \pm\}^I \), the following statements are equivalent:

1. \( (X_B, \sigma_B) \) is balanced ,
2. \( \text{Col}(X_B, \sigma_B) \neq \emptyset \),
3. \( \left( \frac{1}{2} C_{(s,t)}^J \right)^{-1}(E_B^J) \neq \emptyset \),
4. \( \left| \left( \frac{1}{2} C_{(s,t)}^J \right)^{-1}(E_B^J) \right| = 2^{J - \text{dom}(B)} - f_0(X_B) + \beta_0(X_B) \).
Proof. Equivalence (1) \(\iff\) (2) is true by (Kö1) with \(X := X_B\). Equivalence (2) \(\iff\) (3) is an immediate consequence of Lemma 28 (non-emptiness of the target of a surjective map implies non-emptiness of its source; non-emptiness of the source of any map implies non-emptiness of its target). As to (3) \(\iff\) (4), note that by Lemma 28, there is the equation \(|(\frac{1}{2}C_{(s,t)})^{-1}(E_B^J)| = 2^{[J] - \dim(B) - f_0(X_B)} \cdot |\text{Col}(X_B, \sigma_B)|\), which may have the form \(0 = 0\). Now if (3), then \(\text{Col}(X_B, \sigma_B) \neq \emptyset\) by the already proved equivalence (2) \(\iff\) (3), therefore Lemma (Kö2) implies \(|\text{Col}(X_B, \sigma_B)| = 2^{\beta_0(X_B)}\) and hence (4) is true. Conversely, if (4) is true, then this formula alone implies (3). This completes the proof of (3) \(\iff\) (4) and also the proof of Lemma 29. \(\square\)

As an example, consider the special case \(s := t := n, \{1, 1\} =: I \subseteq J := [n - 1]^2, B([1, 1]) := 0\), i.e. \(E_B^J\) is the event that a \(\bar{B} = (\bar{b}_{i,j}) \in \{0, \pm\}^{[n-1]^2}\) has \(\bar{b}_{1,1} = 0\). For these data (4) in Lemma 29 yields \(2n^2 - 1\). And indeed, it is easy to convince oneself directly that there are \(2n^2 - 1\) possibilities to realize this event by Chio condensates of sign matrices \(A \in \{\pm\}^{[n]^2}\).

4. UNDERSTANDING THE CHIO MEASURE

**Theorem 30** (graph-theoretical characterization of the Chio measure of entry-specification events). For every \((s, t) \in \mathbb{Z}^2_\geq 2\), arbitrary \(\emptyset \subseteq I \subseteq J \subseteq [s - 1] \times [t - 1]\) and every \(B \in \{0, \pm\}^I\):

1. **(C1) positivity is determined by balancedness:**
   \[P_{\text{chio}}[E_B^I] > 0 \text{ if and only if } (X_B, \sigma_B) \text{ is balanced},\]

2. **(C2) absolute value is determined by the coboundary space:**
   \[P_{\text{chio}}[E_B^I] > 0 \text{ if and only if } \quad P_{\text{chio}}[E_B^I] = \left(\frac{1}{2}\right)^{\dim(B) + f_0(X_B) - \beta_0(X_B)} = \left(\frac{1}{2}\right)^{\dim(B)} \cdot |B^1(X_B; \mathbb{Z}/2)|;\]  \(14\)

3. **(C3) relative value is determined by the cycle space:**
   \[P_{\text{chio}}[E_B^I] > 0 \text{ if and only if } \quad P_{\text{chio}}[E_B^I] = 2^{\beta_1(X_B)} \cdot P_{\text{clf}}[E_B^I] = |Z_1(X_B; \mathbb{Z}/2)| \cdot P_{\text{clf}}[E_B^I].\]  \(15\)

**Proof.** As to (C1), Definition 7 implies that \(P_{\text{chio}}[E_B^I] > 0\) if and only if \(\left(\frac{1}{2}C_{(s,t)}\right)^{-1}(E_B^J) \neq \emptyset\), hence item (C1) follows from the equivalence (1) \(\iff\) (3) in Lemma 29.

As to (C2), by the just proved item (C1) we have \(P_{\text{chio}}[E_B^J] > 0\), if and only if \(\left(\frac{1}{2}C_{(s,t)}\right)^{-1}(E_B^J) = \emptyset\), and by equivalence (1) \(\iff\) (4) in Lemma 29 this is equivalent to \(\left|\left(\frac{1}{2}C_{(s,t)}\right)^{-1}(E_B^J)\right| = 2^{[J] - \dim(B) - f_0(X_B) + \beta_0(X_B)}\). Dividing by \(2^{[J]}\) in accordance with Definition 7 gives the first equality claimed in (C2). As to the second equality, this is a reformulation not necessary for the equivalence and is true by the known formula (e.g. [14, Theorem 14.1.1]) for the dimension of the coboundary space of a graph, together with the obvious formula for the number of elements of a finite-dimensional vector space over a finite field.

As to (C3), this follows by a simple calculation from (C2), Lemma 23 and \(\text{supp}(X_B) = f_1(X_B)\). The second equality in (C3) is true by definition of \(\beta_1(\cdot)\) (and therefore again a reformulation not necessary for the equivalence). The proof of Theorem 30 is now complete. \(\square\)

We will now derive several consequences of Theorem 30. Let us start with:

**Corollary 31.** Let \((s, t) \in \mathbb{Z}^2_\geq 2, B \in \{0, \pm\}^{[s-1] \times [t-1]}, \emptyset \subseteq I_1 \subseteq J_1 \subseteq [s - 1] \times [t - 1], \emptyset \subseteq I_2 \subseteq J_2 \subseteq [s - 1] \times [t - 1] \) with \([I_1] = [I_2], B_1 \in \{0, \pm\}^{I_1}\) and \(B_2 \in \{0, \pm\}^{I_2}\) be arbitrary. Then

1. \(\mathcal{F}^M(k, n) = (\beta_1 \circ X_{k,n,n}^{-1})^{-1}(\mathbb{Z}_{\geq 1}) = (\beta_1 \circ X_{k,n,n}^{-1})^{-1}(\mathbb{Z}_{\geq 1}),\)
2. \(B \in \text{im}(\frac{1}{2}C_{(s,t)}): \{\pm\}^{[s] \times [t]} \to \{0, \pm\}^{[s-1] \times [t-1]}\) if and only if \(P_{\text{chio}}[B] = \frac{2^{\beta_1(X_B)}}{2^{[s] \times [t]}}\),
3. \(P_{\text{chio}}[E_{B_1}^J] = P_{\text{chio}}[E_{B_2}^J]\) if \(X_{B_1}\) is a one-point wedge product of two components of \(X_{B_2}\).
Proof. As to (1), this is immediate from (C3) in Theorem 30. As to (2), this follows by setting $I := J := [s-1] \times [t-1]$ and combining the equivalence $(1) \Leftrightarrow (3)$ in Lemma 29 with (C1) $\Leftrightarrow (C2)$ in Theorem 30.

As to (3), let us first note that Lemma 24 implies that either $(X_{B_1}, \sigma_{B_1})$ and $(X_{B_2}, \sigma_{B_2})$ are both not balanced, or both are. If both are not balanced, then by item (C1) in Theorem 30, the claim is true in the form of $0 = 0$. If both are, then by item (C2) in Theorem 30, and using $|I_1| = |I_2|$, the equation $P_{\text{chio}}[\mathcal{E}_{B_1}^I] = P_{\text{chio}}[\mathcal{E}_{B_2}^I]$ is equivalent to $f_0(X_{B_1}) - \beta_0(X_{B_1}) = f_0(X_{B_2}) - \beta_0(X_{B_2})$. Since the one-point wedge product of two graphs keeps $f_0(\cdot) - \beta_0(\cdot)$ invariant, the equation is true also in this case and the proof is complete. \hfill \Box

Corollary 32 (the lazy coin flip measure is an averaged Chio measure). $P_{\text{ef}}[B] = \overline{P}_{\text{chio}}[B]$ for every $\emptyset \subseteq I \subseteq [n-1]^2$ and every $B \in \{0, \pm\}^I$.

Proof. It follows from Definition 12 that $\text{supp}(B) = f_1(X_B)$ and that $\{\tilde{B} \in \{0, \pm\}^I : \text{Supp}(\tilde{B}) = \text{Supp}(B)\} = \{\tilde{B} \in \{0, \pm\}^I : X_{\tilde{B}} = X_B\}$. Moreover, by (C1) in Theorem 30, every summand with the property that $(X_{\tilde{B}}, \sigma_{\tilde{B}})$ is not balanced vanishes. Thus, for every $B \in \{0, \pm\}^I$,

$$2^{f_I(X_B)} \cdot \overline{P}_{\text{chio}}[B] = \sum_{\tilde{B} \in \{0, \pm\}^I \text{ with } X_{\tilde{B}} = X_B \text{ and } (X_{\tilde{B}}, \sigma_{\tilde{B}}) \text{ balanced}} P_{\text{chio}}[\tilde{B}]$$

$(\text{C3})$ $= 2^{f_I(X_B)} \cdot P_{\text{ef}}[B] \cdot \big| \{\tilde{B} \in \{0, \pm\}^I : X_{\tilde{B}} = X_B \text{ and } (X_{\tilde{B}}, \sigma_{\tilde{B}}) \text{ balanced}\} \big|$\hspace{1cm}$(\text{K3})$

$= 2^{f_I(X_B)} \cdot P_{\text{ef}}[B] \cdot 2^{f_0(X_B) - \beta_0(X_B)} = 2^{f_I(X_B)} \cdot P_{\text{ef}}[B].$\hspace{1cm}$\Box$

Corollary 33 $(P_{\text{chio}}^{\pm}_I)$ is just the uniform distribution on $\{0, 1\}^I$. For every $(s, t) \in \mathbb{Z}_2^2$ and every $\emptyset \subseteq I \subseteq [s-1] \times [t-1]$ let $P_{0,1}^I$ denote the uniform distribution on $\{0, 1\}^I$. Then $P_{\text{chio}}^{\pm}_I = P_{0,1}^I$.

Proof. This is true since for $(s, t) \in \mathbb{Z}_2^2$, $\emptyset \subseteq I \subseteq [s-1] \times [t-1]$ and $B \in \{0, 1\}^I$ we have

$$2^{|I|} \cdot P_{\text{chio}}^{\pm}_I[B] = \big| \{A \in \{\pm\}^I : \frac{1}{2} \cdot C_{(s,t)}(A) = B\} \big|$$

$(\text{using (1)} \Leftrightarrow (3) \text{ in Lemma 29})$ $= \big| \{A \in \{\pm\}^I : \text{Supp}(\frac{1}{2}C_{(s,t)}(A)) = \text{Supp}(B)\} \big|$\hspace{1cm}$(\text{using (1)} \Leftrightarrow (3) \text{ in Lemma 29})$

$= \sum_{\tilde{B} \in \{0, \pm\}^I \text{ Supp}(\tilde{B}) = \text{Supp}(B), \text{ (X}_{\tilde{B}}, \sigma_{\tilde{B}}) \text{ balanced}} \big| (\frac{1}{2}C_{(s,t)}^I)^{-1}(\mathcal{E}_{B_1}^I) \big| \hspace{1cm}(\text{by (4) in Lemma 29})$

$= \big| \{\tilde{B} \in \{0, \pm\}^I : \text{Supp}(\tilde{B}) = \text{Supp}(B), \tilde{B} \text{ balanced}\} \cdot 2^{|I|-\text{dom}(B) - f_0(X_B) + \beta_0(X_B)} \big|$\hspace{1cm}$(\text{by (K3) in Lemma 26})$

$= 2^{f_0(X_B) - \beta_0(X_B)} \cdot 2^{|I| - \text{dom}(B) - f_0(X_B) + \beta_0(X_B)} = 2^{|I| - \text{dom}(B)} \hspace{1cm}\Box$

Let us state the special case $s := t := n$ and $I := [n-1]^2$ in graph-theoretical language:

Corollary 34. For random $A \in \{\pm\}^{[n]^2}$, the graph $X_{\frac{1}{2}C_{(n,n)}(A)}$ is a random bipartite graph with $n - 1$ vertices in each class and each edge chosen i.i.d. with probability $\frac{1}{2}$.

Theorem 30 also teaches us how fast $P_{\text{chio}}$ can be computed. In both Corollary 35 and 36 the asymptotic statements are referring to $n \to \infty$ and to sequences $I = I(n)$ of index sets with the property that $|I(n)| \to \infty$ (and therefore also $|p_1(I(n))| \cdot |p_2(I(n))| \to \infty$) as $n \to \infty$.

Corollary 35 (complexity of computing $P_{\text{chio}}$). For every $\emptyset \subseteq I \subseteq J \subseteq [n-1]^2$ and every $B \in \{0, \pm\}^I$, the value of $P_{\text{chio}}[\mathcal{E}_{B_1}^I] \in \mathbb{Q}$ can be computed exactly in time $O(|p_1(I)| + |p_2(I)| + |I|) \leq O(|p_1(I)| \cdot |p_2(I)|) \subseteq O(n^2)$. However, there does not exist a fixed algorithm computing $P_{\text{chio}}[\mathcal{E}_{B_1}^I] \in \mathbb{Q}$ exactly on arbitrary instances $B \subseteq \{0, \pm\}^{[n-1]^2}$ and $\emptyset \subseteq I \subseteq [n-1]^2$ and taking time $o(|p_1(I)| \cdot |p_2(I)|)$.
Proof. By items (C1) and (C2) in Theorem 30, to compute $P_{\text{chio}}[E_B']$ it suffices to first decide whether $\sigma_B$ (which in view of Definition 12 evidently can be read in time $O(|p_1(I)| \cdot |p_2(I)|) \subseteq O(n^2)$) is balanced, and, if so, to compute $f_0(X_B)$ and $\beta_0(X_B)$. By Corollary 27, and since the depth-first search mentioned there also computes the numbers $f_0(X_B)$ and $\beta_0(X_B)$, both tasks can be accomplished by one depth-first search in time $O(|p_1(I)||p_2(I)| + |I|) \subseteq O(n^2)$. If $(X_B, \sigma_B)$ is found to be not balanced, then $P_{\text{chio}}[E_B'] = 0$. Otherwise, the answer is $(\frac{1}{2})^{|I| + f_0(X_B) - \beta_0(X_B)}$. Since the bitlength of this dyadic fraction is $|I| + f_0(X_B) - \beta_0(X_B) = |I| + f_0(X_B) - \beta_1(X_B) \leq |I| + f_1(X_B) \leq 2|I| \in O(|p_1(I)| + |p_2(I)| + |I|)$ it is possible to write the output in the time claimed. This proves the first statement in Corollary 35.

As to the second statement, notice that any such fixed algorithm could in particular compute $P_{\text{chio}}[E_B'] \in \mathbb{Q}$ exactly on those instances $B \in \{0, \pm\}^{[n-1]^2}$ and $\emptyset \subseteq I \subseteq [n-1]^2$ for which $I$ is rectangular. But if $I$ is rectangular, then $|I| + f_0(X_B) - \beta_0(X_B) \geq |I| = |p_1(I)| \cdot |p_2(I)|$. Therefore, for these inputs, the bitlength of the dyadic fraction $(\frac{1}{2})^{|I| + f_0(X_B) - \beta_0(X_B)}$ is at least $|p_1(I)| \cdot |p_2(I)|$. Hence for such inputs the very task of writing the output takes time $\Omega(|p_1(I)| \cdot |p_2(I)|)$, which precludes a running time of $o(|p_1(I)| \cdot |p_2(I)|) \subseteq o(n^2)$. The proof of Corollary 35 is now complete.

A priori one might suspect that the task of merely deciding whether $P_{\text{chio}}[B] = P_{\text{lcf}}[B]$ could be accomplished much faster than the task of computing the value of $P_{\text{chio}}[B]$. Theorem 30 can also tell us that this is not the case.

Corollary 36 \text{(complexity of deciding whether $P_{\text{chio}}$ and $P_{\text{lcf}}$ agree)}. For every $\emptyset \subseteq I \subseteq J \subseteq [n-1]^2$ and every $B \in \{0, \pm\}^I$, the answer to the decision problem of whether $P_{\text{chio}}[E_B'] = P_{\text{lcf}}[E_B']$ can be computed in time $O(|p_1(I)| \cdot |p_2(I)|) \subseteq O(n^2)$. However, there does not exist a fixed algorithm (having only entry-wise access to $B$ and no further a priori information) which decides that question on arbitrary instances $B \in \{0, \pm\}^I$ with $\emptyset \subseteq I \subseteq [n-1]^2$ in time $o(|p_1(I)| \cdot |p_2(I)|)$.

Proof. Given $\emptyset \subseteq I \subseteq [n-1]^2$ and $B \in \{0, \pm\}^I$, it follows from item (C3) in Theorem 30 that the question of whether $P_{\text{chio}}[E_B'] = P_{\text{lcf}}[E_B']$ is equivalent to asking whether $X_B$ is a forest. The graph $X_B$ can obviously be computed from $B$ in time $O(|p_1(I)| \cdot |p_2(I)|) \subseteq O(n^2)$, and deciding whether $X_B$ is a forest, i.e. whether $X_B$ contains a circuit, can be done by a depth-first search in time $O(f_0(X_B) + f_1(X_B)) \subseteq O(|p_1(I)| + |p_2(I)| + |I|) \subseteq O(|p_1(I)| \cdot |p_2(I)|)$, so the first claim in Corollary 36 is proved.

As to the additional claim, suppose there were a fixed algorithm $A$ with the stated properties. Let $I$ be the set of all rectangular $\emptyset \subseteq I \subseteq [n-1]^2$. By assumption, the algorithm $A$ is in particular capable of deciding whether $P_{\text{chio}}[E_B'] = P_{\text{lcf}}[E_B']$ for each input $I \in I$ and for each of them taking time $o(|p_1(I)| \cdot |p_2(I)|)$. However, every bipartite graph with bipartition sizes of $|p_1(I)|$ and $|p_2(I)|$ can be realized as a $X_B$ with $I \in I$. By item (C3) in Theorem 30 the property $P_{\text{chio}}[E_B'] = P_{\text{lcf}}[E_B']$ is equivalent to $X_B$ being a forest. Therefore $A$ decides set membership for the set of all bipartite graphs which have the fixed (that is, fixed for every fixed value of $n$) bipartition classes $p_1(I)$ and $p_2(I)$ and do not contain a circuit. This set is a decreasing (i.e. closed w.r.t. deleting edges) graph property consisting of bipartite graphs only. Since all graphs in the property have the same bipartition classes $p_1(I)$ and $p_2(I)$ we may appeal to a theorem of A. C.-C. Yao [33, p. 518, Theorem 1] which says that every such property is evasive.

Hence there exists at least one $I \in I$ with the property that $A$ examines every entry of $B$. This takes time $\Omega(|I|) = \Omega(|p_1(I)| \cdot |p_2(I)|)$.

\[\text{Due to the fact that the bipartition classes are the same for all the graphs in the property, it is not necessary to appeal to the more general theorem of E. Triesch [31, p. 266, Theorem 4] in which the assumption of fixed bipartition classes is no longer made. An earlier version of the present paper stated that one would need this more general theorem. I was wrong regarding this particular point. Moreover I confused the adjectives ‘balanced’ and ‘fixed’. The theorem of Yao suffices.}\]
the equality being true because of \( |I| = |p_1(I)| \cdot |p_2(I)| \). This is a contradiction to the assumption about the running time of \( A \). The proof of Corollary 36 is now complete. □

We now take a more quantitative look at the relationship between \( P_{\text{chio}} \) and \( P_{\text{lcf}} \). We start with an enumeration of bipartite nonforests. The fact that we stop the enumeration at the \( f \)-vector \((f_0, f_1) = (8, 6)\), even though there are bipartite nonforests with \((f_0, f_1) = (8, 7)\), is explained by the application we have in mind; we will only be concerned with bipartite nonforests having up to six edges.

**Lemma 37** (bipartite nonforests ordered by their \( f \)-vectors). The isomorphism types of bipartite nonforests, ordered lexicographically by their \( f \)-vectors up to \((f_0, f_1) = (8, 6)\), are:

- \((t1) = C^4\)
- \((t2) = \text{disjoint union of } C^4 \text{ and one isolated vertex}\)
- \((t3) = C^4 \text{ intersecting one edge and one isolated vertex}\)
- \((t4) = K_{2,3}\)
- \((t5) = \text{disjoint union of } C^4 \text{ and two isolated vertices}\)
- \((t6) = C^4 \text{ intersecting one edge, and one extra isolated vertex}\)
- \((t7) = \text{disjoint union of } C^4 \text{ and one isolated vertex}\)
- \((t8) = C^4 \text{ intersecting two disjoint edges, the intersection set no edge of } C^4\)
- \((t9) = C^4 \text{ intersecting two disjoint edges, and an isolated edge}\)
- \((t10) = C^4 \text{ intersecting a 2-path in its inner vertex}\)
- \((t11) = C^4 \text{ intersecting two disjoint edges, and an isolated vertex}\)
- \((t12) = C^6\)
- \((t13) = \text{disjoint union of } C^4 \text{ and three isolated vertices}\)
- \((t14) = C^4 \text{ intersecting one edge, and two extra isolated vertices}\)
- \((t15) = \text{disjoint union of } C^4 \text{ and an edge, and one extra isolated vertex}\)
- \((t16) = C^4 \text{ intersecting one edge, and one extra isolated edge}\)
- \((t17) = \text{disjoint union of } C^4 \text{ and a 2-path}\)
- \((t18) = \text{disjoint union of } C^4 \text{ and four isolated vertices}\)
- \((t19) = \text{disjoint union of } C^4 \text{ and one edge and two extra isolated vertices}\)
- \((t20) = \text{disjoint union of } C^4 \text{ and two disjoint edges}\)

**Proof.** Easy to check since the graphs are required to be bipartite and have \( f_1 \leq 6 \). □

**Corollary 38** (isomorphism types for which equality of measures of entry specification events fails).

- (Fa3) \( F^G(k, n) = \emptyset \) for \( 0 \leq k \leq 3 \),
- (Fa5) \( F^G(5, n) = \{ (t2), (t3), (t5), (t7) \} \),
- (Fa4) \( F^G(4, n) = \{ (t1) \} \),
- (Fa6) \( F^G(6, n) = F^G(5, n) \cup \{ (t4), (t6), (t8), (t10), (t12), (t13), (t15) \} \).

**Proof.** By (C3) in Theorem 30 we have \( P_{\text{chio}}[C_{B}^{(n - 1)2}] \neq P_{\text{lcf}}[C_{B}^{(n - 1)2}] \) if and only if \( \beta_1(X_B) > 0 \). Moreover, directly from Definition 12 we have the bound \( f_1(X_B) \leq |I| \). Therefore, for every \( k \), we can get a set of candidates for membership in \( F^G(k, n) \) by collecting all isomorphism types in Corollary 38 having \( f_1 \leq k \). We then have to decide for each of these types whether it is possible to realize it as a \( X_B \) with \( B \in \{ 0, \pm \}^I \) and \( I \in \{ n - 1 \}^k \).

As to (Fa3), this is true since there do not exist bipartite nonforests with three edges or less.

As to (Fa4), i.e. \( k = 4 \), note that the only isomorphism types in Corollary 38 with \( f_1 \leq 4 \) are \( (t1) \) and \( (t2) \). Because of \( \beta_1(X_B) \geq 1 \) for every \( B \in F^M(4, n) \) the set \( I \) must be a matrix-4-circuit. This implies \( f_0(X_B) = 4 \). Since \( f_0(t2) = 5 \), it follows that \( (t2) \notin F^G(4, n) \). Since type \( (t1) \) obviously can be realized, (Fa4) is true.

As to (Fa5), i.e. \( k = 5 \), note that the only isomorphism types with \( f_1 \leq 5 \) in Corollary 38 are \( (t1), (t2), (t3), (t5), (t6) \) and \( (t7) \). \( C^4 \) is a subgraph of each of these types, it is necessary that there be a matrix-4-circuit \( S \subseteq I \). Since the sole non-matrix-circuit entry must create at least one addition vertex of \( X_B \), type \( (t1) \) cannot occur. The type \( (t6) \) cannot occur either since there is only one position \( u \in I \setminus S \subseteq \{ n - 1 \}^k \) left for us to choose freely and by the choice of \( u \) and \( B[u] \) we can either create an isolated vertex in \( X_B \) or an edge intersecting the \( C^4 \). The remaining types \( (t2), (t3), (t5) \) and \( (t7) \) can be indeed be realized, as is proved by the following examples. For all the examples let \( S := \{ (1, 1), (1, 2), (2, 1), (2, 2) \} \), \( B|S| := \{ -\}^S \) and \( \{ u \} := I \setminus S \).

For \( (t2) \) take e.g. \( n := 4, u := (2, 3) \) and \( B[u] := 0 \). For \( (t3) \) take e.g. \( n := 4, u := (2, 1) \) and \( B[u] := -1 \). For \( (t5) \) take e.g. \( n := 4, u := (3, 1) \) and \( B[u] := 0 \). For \( (t7) \) take e.g. \( n := 4, u := (3, 3) \) and \( B[u] := -1 \). This proves (Fa5).
As to (Fa6), i.e. \( k = 6 \), as far as only the necessary condition \( f_1(X_B) \leq |I| = k \) is concerned, all types in Lemma 37 are candidates. Type (t1) cannot be realized since \( f_0(t1) = 4 \) but \( f_0(X_B) \geq 5 \) for every \( I \in \binom{n-1}6 \). All others can, as will now be proved by giving one example for each of them. In all examples again let \( S := \{ (1, 1), (1, 2), (2, 1), (2, 2) \} \) and \( B, S := \{ - \} \). Here, \( \{ u, v \} := I \setminus S \). For (t2) take e.g. \( n := 4, u := (1, 3), v := (2, 3) \) and \( B[u] := B[v] := 0 \). For (t3) take e.g. \( n := 4, u := (1, 3), v := (2, 3) \) and \( B[u] := B[v] := 0 \). For (t4) take e.g. \( n := 4, u := (1, 3), v := (2, 3) \) and \( B[u] := B[v] := 0 \). For (t5) take e.g. \( n := 4, u := (1, 3), v := (3, 3) \) and \( B[u] := B[v] := 0 \). For (t6) take e.g. \( n := 4, u := (1, 3), v := (3, 3) \) and \( B[u] := B[v] := 0 \). For (t7) take e.g. \( n := 4, u := (1, 3), v := (3, 3) \) and \( B[u] := B[v] := 0 \). For (t8) take e.g. \( n := 5, u := (1, 3), v := (2, 4) \) and \( B[u] := B[v] := 0 \). For (t9) take e.g. \( n := 4, u := (1, 3), v := (3, 2) \) and \( B[u] := B[v] := 0 \). For (t10) take e.g. \( n := 4, u := (1, 3), v := (3, 3) \) and \( B[u] := B[v] := 0 \). For (t11) take e.g. \( n := 5, u := (1, 3), v := (1, 4) \) and \( B[u] := B[v] := 0 \). For (t12) we have to make an exception to our convention that \( \{ u, v \} = I \setminus S \) with \( S \) defined as above, and have to define the set \( I \) in its entirety. We can take e.g. \( n := 4, I := \{ (1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 3) \} \in \binom{n-1}5 \) and \( B = \{ - \} \). For (t13) take e.g. \( n := 5, u := (3, 3), v := (3, 4) \) and \( B[u] := B[v] := 0 \). For (t14) take e.g. \( n := 5, u := (1, 3), v := (3, 4) \) and \( B[u] := B[v] := 0 \). For (t15) take e.g. \( n := 5, u := (3, 3), v := (3, 4) \) and \( B[u] := B[v] := 0 \). For (t16) take e.g. \( n := 5, u := (1, 3), v := (3, 4) \) and \( B[u] := B[v] := 0 \). For (t17) take e.g. \( n := 5, u := (3, 3), v := (3, 4) \) and \( B[u] := B[v] := 0 \). For (t18) take e.g. \( n := 5, u := (3, 3), v := (4, 4) \) and \( B[u] := B[v] := 0 \). For (t19) take e.g. \( n := 5, u := (3, 3), v := (4, 4) \) and \( B[u] := B[v] := 0 \). For (t20) take e.g. \( n := 5, u := (3, 3), v := (4, 4) \) and \( B[u] := B[v] := 0 \). This proves (Fa6). The proof of Corollary 38 is now complete.

**Corollary 39** (ratios and absolute values of \( P_{\text{ch}n} \) for up to six entry specifications).

(1) \( F_{\text{G}}(k, n) = F_{\text{G}}^0(k, n) \cup \bigcup_{j \in \mathbb{Z} \setminus 1} F_{\text{G}}^j(k, n) \),
(2) \( F_{\text{M}}(k, n) = F_{\text{M}}^0(k, n) \cup \bigcup_{j \in \mathbb{Z} \setminus 1} F_{\text{M}}^j(k, n) \),
(3) \( F_{\text{G}}(4, n) = F_{\text{G}}^2(4, n), F_{\text{G}}(5, n) = F_{\text{G}}^2(5, n) \) and \( F_{\text{G}}(6, n) = F_{\text{G}}^2(6, n) \cup F_{\text{G}}(6, n) \),
(4) \( F_{\text{M}}(4, n) = F_{\text{M}}^0(4, n) \cup F_{\text{M}}^2(4, n), F_{\text{M}}(5, n) = F_{\text{M}}^0(5, n) \cup F_{\text{M}}^2(5, n) \)
and \( F_{\text{M}}(6, n) = F_{\text{M}}^0(6, n) \cup F_{\text{M}}^2(6, n) \cup F_{\text{M}}(6, n) \).

Moreover,

(4) \( F_{\text{G}}(4, n) = F_{\text{G}}(4, n), \quad \text{with} \quad F_{\text{G}}(4, n) = \{ (1) \} \),
(5) \( F_{\text{G}}(5, n) = F_{\text{G}}(5, n), \quad \text{with} \quad F_{\text{G}}(5, n) = \{ (2), (5), (t13), (t18) \} \),
(6) \( F_{\text{G}}(6, n) = F_{\text{G}}(6, n), \quad \text{with} \quad F_{\text{G}}(6, n) = \{ (1), (4), (6), (6), (6), (6), (11) \} \),

(1) \( F_{\text{G}}^1(6, n) = \{ (t2), (t5), (t13), (t18) \} \),
(2) \( F_{\text{G}}^1(6, n) = \{ (t3), (t4), (t6), (t7), (t14), (t15), (t19) \} \),
(3) \( F_{\text{G}}^1(6, n) = \{ (t8), (t9), (t10), (t11), (t12), (t16), (t17), (t20) \} \).

**Proof.** As to (R1), let us start with \( F_{\text{G}}(k, n) = F_{\text{G}}^0(k, n) \). The inclusion \( \supseteq \) is true directly by (4) in Definition 15. Conversely, let \( \mathcal{X} \in F_{\text{G}}(k, n) \). Then \( \beta_1(\mathcal{X}) \geq 1 \) by Corollary 31.1, hence \( \{ \pm \} \in S_{\text{bal}}(\mathcal{X}) = 2f_1(\mathcal{X}) - 2f_2(\mathcal{X}) > 0 \), hence there exists \( B \in F_{\text{M}}(k, n) \) with \( \mathcal{X} = \cup X_{\mathcal{X}, n}(B) \), hence \( \mathcal{X} \in \cup X_{\mathcal{X}, n}(F_{\text{G}}^0(k, n)) \) by Definition 15.4 \( F_{\text{G}}^0(k, n) \), proving \( \subseteq \).

As to the partition claimed in (R1), both claims follow immediately from (C3) in Theorem 30 (and the equality \( F_{\text{G}}(k, n) = F_{\text{G}}^0(k, n) \) is the reason why \( F_{\text{G}}^0(k, n) \) is missing in the disjoint union in (R1)). As to (R3) (respectively (R4)), this follows by combining (R1) (respectively (R2)) with (Fa4)–(F6) in Corollary 38. The claims (A4)–(A6) will be proved in reverse order.
As to (A6), this seems to require some calculations. However, Corollary 31.3 can be used to reduce the amount of work to be done: if (a) and (b) are isomorphism types of graphs, let us write (a) \rightarrow (b) if and only if (b) can be obtained from (a) by a single one-point wedge of two connected components of (a). Moreover, if (a) is any of the isomorphism types in \( \mathcal{F}_M(6, n) \), let us employ the abbreviation \( E_B := E_B^{\lfloor n-1 \rfloor^2} \) and let us write \( P_{\text{chio}}([a]) \) for the number \( P_{\text{chio}}[E_B^{\lfloor n-1 \rfloor^2}] \) with \( B \) an arbitrary \( B \in \{0, \pm 1\}^I \), \( I \in \binom{\lfloor n-1 \rfloor}{6} \), \( X_B \in \langle a \rangle \) and \( (X_B, \sigma_B) \) balanced. By (C2) in Theorem 30 we know that \( P_{\text{chio}}([a]) \) then does indeed only depend on (a), not on the choice of such a \( B \).

Since evidently \((t18) \rightarrow (t13) \rightarrow (t5) \rightarrow (t2)\), Corollary 31.3 implies \( P_{\text{chio}}([t2]) = P_{\text{chio}}([t5]) = P_{\text{chio}}([t13]) = P_{\text{chio}}([t18]) \). Since evidently \((t19) \rightarrow (t15) \rightarrow (t6) \rightarrow (t3)\), Corollary 31.3 implies \( P_{\text{chio}}([t19]) = P_{\text{chio}}([t15]) = P_{\text{chio}}([t6]) = P_{\text{chio}}([t3]) \). Since also \((t14) \rightarrow (t6)\), Corollary 31.3 implies \( P_{\text{chio}}([t14]) = P_{\text{chio}}([t6]) \). Since moreover \((t7) \rightarrow (t3)\), Corollary 31.3 implies \( P_{\text{chio}}([t7]) = P_{\text{chio}}([t3]) \). These equations together imply \( P_{\text{chio}}([t3]) = P_{\text{chio}}([t6]) = P_{\text{chio}}([t7]) = P_{\text{chio}}([t14]) = P_{\text{chio}}([t15]) = P_{\text{chio}}([t19]) \). Since evidently \((t20) \rightarrow (t17) \rightarrow (t10)\), Corollary 31.3 implies \( P_{\text{chio}}([t20]) = P_{\text{chio}}([t17]) = P_{\text{chio}}([t10]) \). Since evidently \((t20) \rightarrow (t16) \rightarrow (t10)\), Corollary 31.3 implies \( P_{\text{chio}}([t20]) = P_{\text{chio}}([t16]) = P_{\text{chio}}([t10]) \). Since evidently \((t17) \rightarrow (t11)\), Corollary 31.3 implies \( P_{\text{chio}}([t17]) = P_{\text{chio}}([t11]) \). Since evidently \((t16) \rightarrow (t19)\), Corollary 31.3 implies \( P_{\text{chio}}([t16]) = P_{\text{chio}}([t19]) \). Since evidently \((t16) \rightarrow (t8)\), Corollary 31.3 implies \( P_{\text{chio}}([t16]) = P_{\text{chio}}([t8]) \). These equations together imply \( P_{\text{chio}}([t8]) = P_{\text{chio}}([t9]) = P_{\text{chio}}([t10]) = P_{\text{chio}}([t11]) = P_{\text{chio}}([t16]) = P_{\text{chio}}([t17]) = P_{\text{chio}}([t20]) \).

This proves that it suffices (note that of the nineteen elements of \( \mathcal{F}_M(6, n) \) exactly \((t4)\) and \((t12)\) have not been part of one of the equality chains) to calculate only \( P_{\text{chio}}([t2]) \), \( P_{\text{chio}}([t3]) \), \( P_{\text{chio}}([t4]) \), \( P_{\text{chio}}([t8]) \) and \( P_{\text{chio}}([t12]) \). With the formulas in (C2) of Theorem 30 and in Lemma 23, this can be done as follows (keep in mind that, being within item (A6), \( \text{dom}(B) = |I| = 6 \) in each calculation):

If \( X = (t2) \), then \( f_0(X) = 5 \), \( \beta_0(X) = 2 \), hence \( P_{\text{chio}}([E_B]) = \left( \frac{1}{2} \right)^{|I|+5-2} = \left( \frac{1}{2} \right)^9 \).

If \( X \in \{(t3), (t4)\} \), then \( f_0(X) = 5 \), \( \beta_0(X) = 1 \), hence \( P_{\text{chio}}([E_B]) = \left( \frac{1}{2} \right)^{|I|+5-1} = \left( \frac{1}{2} \right)^{10} \).

If \( X \in \{(t8), (t12)\} \), then \( f_0(X) = 6 \), \( \beta_0(X) = 1 \), hence \( P_{\text{chio}}([E_B]) = \left( \frac{1}{2} \right)^{|I|+6-1} = \left( \frac{1}{2} \right)^{11} \).

As to (A5), it follows by an entirely analogous (but much shorter) argument as the one given for (A5) that it suffices to calculate only \( P_{\text{chio}}([t2]) \) and \( P_{\text{chio}}([t3]) \), and these calculations are identical to the ones made for \( P_{\text{chio}}([t2]) \) and \( P_{\text{chio}}([t3]) \) in the preceding paragraph, except that now \( |I| = 5 \).

As to (A4), in view of (F4) in Corollary 38, we only have to deal with the single type (t1) where \( f_0(t1) = 4 \), \( \beta_0(t1) = 1 \) and therefore \( P_{\text{chio}}([t1]) = \left( \frac{1}{2} \right)^{|I|+4-1} = \left( \frac{1}{2} \right)^{4+4-1} = \left( \frac{1}{2} \right)^7 \).

The results obtained so far can be turned into a set of instructions of how to tell the measure of \( P_{\text{chio}}[E_B^{\lfloor n-1 \rfloor^2}] \) from a given \( B \in \{0, \pm 1\}^I \) provided that \( |I| \leq 6 \). We formulate the instructions exclusively in terms of those data, avoiding any mention of the associated signed graph \( (X_B, \sigma_B) \).

**Corollary 40** (how to find the measure under \( P_{\text{chio}} \) of large entry-specification events). For every \( \emptyset \subseteq I \subseteq [n-1]^2 \) with \( |I| \leq 6 \) and every \( B \in \{0, \pm 1\}^I \), the following instructions lead to the correct Chio measure of \( E_B := E_B^{\lfloor n-1 \rfloor^2} : \)

\begin{align*}
\text{(H3)} & \quad \text{If } 0 \leq |I| \leq 3, \text{ then } P_{\text{chio}}[E_B] = P_{\text{ref}}[E_B] = \left( \frac{1}{2} \right)^{\text{dom}(B)+\text{supp}(B)}. \\
\text{(H4)} & \quad \text{If } |I| = 4, \text{ then check whether } I \text{ is a matrix-4-circuit such that } B \in \{-1\}^I. \text{ If not, then } P_{\text{chio}}[E_B] = P_{\text{ref}}[E_B] = \left( \frac{1}{2} \right)^{\text{dom}(B)+\text{supp}(B)}. \text{ If } B \text{ has this property, then check whether an odd number of the four nonzero values } B[i,j] \text{ with } (i,j) \in I \text{ are +. If so, } P_{\text{chio}}[E_B] = 0. \text{ If not, then } P_{\text{chio}}[E_B] = \left( \frac{1}{2} \right)^7 = 2 \cdot P_{\text{ref}}[E_B]. \\
\text{(H5)} & \quad \text{If } |I| = 5, \text{ then check whether there exists within } I \text{ a matrix-4-circuit } S \subseteq I \text{ such that } B[i] \in \{-1\}^S. \text{ If not, then } P_{\text{chio}}[E_B] = P_{\text{ref}}[E_B] = \left( \frac{1}{2} \right)^{\text{dom}(B)+\text{supp}(B)}. \text{ If so, then check whether an odd number of the four nonzero values } B[i,j] \text{ with } (i,j) \in S \text{ are +. If so,}
\end{align*}
Corollary 42.

In general we have \( \mathcal{F}^M(k,n) \) for every map \( \chi \in \mathcal{F}^G(k,n) \) (by Definition 15.44) as follows:

\[
(\mathcal{F}^M(k,n)) = \mathcal{F}^G(k,n) \quad \text{(for every map)}
\]

\[
\bigcup_{k \leq 6} (\mathcal{X}k_{k+6}^{k,n} - 1)(t_k)
\]

for \( \mathcal{X}k_{k+6}^{k,n} - 1(\mathcal{X}) \) and for the specific values \( 4 \leq k \leq 6 \) we can use Corollary 38 to obtain the claimed partitions.

While having the aim of explicitly determining the numbers \( ||(\mathcal{X}k_{k+6}^{k,n} - 1(\mathcal{X})) \) for certain \( k \) and \( \chi \) which interest us, we will start slowly by first formulating some linear relations among \( ||(\mathcal{X}k_{k+6}^{k,n} - 1(t_2)), \ldots, (\mathcal{X}k_{k+6}^{k,n} - 1(t_{20})) \) which will later serve as a check for the formulas given in Theorem 44.
Lemma 43 (linear relations among $|ulX^k,n,n,\rangle^{-1}(X)$ for $5 \leq k \leq 6$).

(11) $(3^1 - 1) \cdot |ulX^5,n,n,\rangle^{-1}(t2) = |ulX^5,n,n,\rangle^{-1}(t3)$,

(12) $(3^1 - 1) \cdot |ulX^5,n,n,\rangle^{-1}(t5) = |ulX^5,n,n,\rangle^{-1}(t7)$,

(13) $(3^2 - 1) \cdot |ulX^6,n,n,\rangle^{-1}(t5) = |ulX^6,n,n,\rangle^{-1}(t6) + \cdots + |ulX^6,n,n,\rangle^{-1}(t11)$,

(14) $(3^2 - 1) \cdot |ulX^6,n,n,\rangle^{-1}(t13) = |ulX^6,n,n,\rangle^{-1}(t14) + \cdots + |ulX^6,n,n,\rangle^{-1}(t17)$,

(15) $(3^2 - 1) \cdot |ulX^6,n,n,\rangle^{-1}(t18) = |ulX^6,n,n,\rangle^{-1}(t19) + |ulX^6,n,n,\rangle^{-1}(t20)$.

Proof. It follows from Definition 12 that $X_B \in (t3)$ if and only if equation (18) is true and $B[u] \in \{\pm\}$. This implies $|ulX^5,n,n,\rangle^{-1}(t3) = 2 \cdot |ulX^5,n,n,\rangle^{-1}(t2)$, proving (11). It also follows from Definition 12 that $X_B \in (t7)$ if and only if equation (19) is true and $B[u] \in \{\pm\}$. This implies $|ulX^6,n,n,\rangle^{-1}(t7) = 2 \cdot |ulX^6,n,n,\rangle^{-1}(t5)$, proving (12).

The isomorphism types $(t5)$–$(t11)$ are all the isomorphism types of bipartite nonforests with six vertices and exactly one copy of $C^4$. Therefore $|ulX^5,n,n,\rangle^{-1}(t5) + \cdots + |ulX^6,n,n,\rangle^{-1}(t11)$ is the number of all $B \in \{0, \pm\}^I$ with $I \in \binom{[n]}{6}$ such that $X_B$ contains exactly one $C^4$ and $f_0(X_B) = 0$. Imagine counting these $B$ by partitioning the set of all such $B$ according to the copy of $C^4$, and for each such copy, further partitioning the $B$ according to the mandatory values on the edges of the $C^4$, and then further partitioning according to the positions of the two elements of $I$ which are not responsible for the copy of $C^4$. When partitioning in that way, the number of blocks of the partition obtained so far equals $|ulX^5,n,n,\rangle^{-1}(t5)$. The reason for this is that to realize the type $(t5)$ there is no choice for the values indexed by the positions which are not responsible for the $C^4$, both must be zero. In the enumeration we are currently carrying out, however, there is still complete freedom left on how to choose any one of the $|ulX^6,n,n,\rangle^{-1}(t5) = 3^2$ values which can be indexed by these two positions, in other words, each of the blocks has size $3^2$. Therefore $|ulX^5,n,n,\rangle^{-1}(t5) + \cdots + |ulX^6,n,n,\rangle^{-1}(t11) = 3^2 \cdot |ulX^6,n,n,\rangle^{-1}(t5)$, which proves (13). Equations (14) and (15) are true for an entirely analogous reason. \(\square\)

We will now quantify the claims in Corollary 38 by determining $|ulX^k,n,n,\rangle^{-1}(X)$ for each $k$ and each isomorphism type $X$ mentioned there. A few preparatory comments seem in order. The behaviour of $|ulX^k,n,n,\rangle^{-1}(X)$ as a function of $k$ for a given isomorphism type $X$ is a little subtle. For example, note that Theorem 44 tells us that

$$|ulX^5,n,n,\rangle^{-1}(t2)| > |ulX^6,n,n,\rangle^{-1}(t2)|$$

(16)

in spite of the fact that in the case of $|ulX^6,n,n,\rangle^{-1}(t2)|$ we have one matrix entry more at our disposal to realize $(t2)$. The reason for this could be summarized thus: when wanting to keep the number of isolated vertices in $ulX_B$ at one, the additional matrix entry curtails our freedom more than it adds to it—after having chosen a position for one of the non-matrix-circuit-entries which ‘hides’ one of its two ‘shadows’ in one of the four shadows of the matrix-circuit-entries, we then have to position the second non-matrix-circuit-entry so as to hide both of its two shadows in already existing shadows, and this determines its position completely. Moreover, since $(t2)$ is an isomorphism type in which there do not exist edges outside the 4-circuit, the non-matrix-circuit positions must index the value 0. The net result of these rigid requirements are (since in effect for $|ulX^6,n,n,\rangle^{-1}(t2)|$ we are counting the possible 2-sets of non-circuit positions while for $|ulX^5,n,n,\rangle^{-1}(t2)|$ we counted the possible 1-sets of such positions) less possibilities. For other types it can happen that the mechanism just described is counterbalanced by the additional possibilities of indexing different values. This is the essential reason why $|ulX^5,n,n,\rangle^{-1}(t3)| = |ulX^6,n,n,\rangle^{-1}(t3)|$, despite (16) and despite the fact that the set of all domains in the preimages in question are the same as in (16), i.e.

$$\text{Dom}(|ulX^5,n,n,\rangle^{-1}(t2)|) = \text{Dom}(|ulX^5,n,n,\rangle^{-1}(t3)|), \quad \text{Dom}(|ulX^6,n,n,\rangle^{-1}(t2)|) = \text{Dom}(|ulX^6,n,n,\rangle^{-1}(t3)|).$$

(17)
Since biadjacency matrices are quite a fundamental topic, it would be of interest to treat these phenomena in more generality. It seems advisable to do this with a view towards the theory of \( \{0, 1\} \)-matrices with given row and column sums (for a start, cf. e.g. [21], [2] and [3]). However, so far the author could not harness the literature on this topic in any way which would lessen the burden of proving the following theorem:

**Theorem 44** (cardinality of preimages of \( a_1X^{k,n} \) on bipartite nonforests for \( 4 \leq k \leq 6 \)). The claims (Fa4) — (Fa6) can be quantified as follows (with \( \xi_n := 2^4 \cdot \left| \text{Cir}(4,n) \right| = 2^4 \cdot \left(\binom{n-1}{2}\right)^2 \))

**(QFa4)** For every \( n \geq 3 \), \( \left| \left( a_1X^{4,n} \right)^{-1}(t1) \right| = \xi_n \).

**(QFa5)** For every \( n \geq 3 \),

\[
\begin{align*}
(m5.12) \quad & \left| \left( a_1X^{5,n} \right)^{-1}(t12) \right| = 4 \cdot (n - 3) \cdot \xi_n, \\
(m5.13) \quad & \left| \left( a_1X^{5,n} \right)^{-1}(t13) \right| = 8 \cdot (n - 3) \cdot \xi_n,
\end{align*}
\]

**(QFa6)** For every \( n \geq 3 \),

\[
\begin{align*}
(m6.12) \quad & \left| \left( a_1X^{6,n} \right)^{-1}(t12) \right| = 2(n - 3)\xi_n, \\
(m6.13) \quad & \left| \left( a_1X^{6,n} \right)^{-1}(t13) \right| = 8(n - 3)\xi_n, \\
(m6.14) \quad & \left| \left( a_1X^{6,n} \right)^{-1}(t14) \right| = 2^7(\binom{n-1}{2})^2 \xi_n, \\
(m6.15) \quad & \left| \left( a_1X^{6,n} \right)^{-1}(t15) \right| = (8(n - 3)^2 + 8\binom{n-3}{2})\xi_n, \\
(m6.16) \quad & \left| \left( a_1X^{6,n} \right)^{-1}(t16) \right| = (24(n - 3)^2 + 32\binom{n-3}{2})\xi_n, \\
(m6.17) \quad & \left| \left( a_1X^{6,n} \right)^{-1}(t17) \right| = 8(n - 3)^2\xi_n, \\
(m6.18) \quad & \left| \left( a_1X^{6,n} \right)^{-1}(t18) \right| = 16\binom{n-2}{2}\xi_n, \\
(m6.19) \quad & \left| \left( a_1X^{6,n} \right)^{-1}(t19) \right| = 16(n - 3)^2\xi_n, \\
(m6.20) \quad & \left| \left( a_1X^{6,n} \right)^{-1}(t20) \right| = 8\binom{n-2}{2}\xi_n.
\end{align*}
\]

**Proof of (QFa4).** We have \( X_B \cong C^4 \) if and only if \( I \) is a matrix-4-circuit and \( \text{Supp}(B) = I \). By Lemma 25 there exist \( \binom{n-1}{2} \) possible matrix-4-circuits \( I \) and for each of them there are \( 2^4 \) possibilities for \( B \in \{0, \pm\}^I \) with \( \text{Supp}(B) = I \). □

Let us now prepare for the rest of the proof of Theorem 44 with some observations and definitions. Inspecting the isomorphism types in \( \mathcal{F}^G(6, n) \setminus \{(t4), (t12)\} \) (the types (t4) and (t12) are exceptions whose preimages are also exceptionally easy to count) we see that in each of them the graph contains exactly one \( C^4 \). We therefore know that for every \( X \in \mathcal{F}^G(6, n) \) (hence in particular for every \( X \in \mathcal{F}^G(5, n) \) since \( \mathcal{F}^G(5, n) \subseteq \mathcal{F}^G(6, n) \) by (Fa6) in Corollary 38), and for every \( I \in \binom{\{0, \pm\}^5}{6} \) it is necessary that there exist a matrix-4-circuit \( S \subseteq I \) with \( B \upharpoonright S \in \{\pm\}^S \). For this there are \( 2^4 |\text{Cir}(4,n)| \) possibilities. A priori it could be that the number of possibilities to realize an isomorphism type depends on the choice of this necessary \( S \subseteq I \). However, since we will take this \( S \) to be arbitrary in the proofs to follow, and since we will get results which do not depend on \( S \), it follows as a byproduct that they are not, more precisely that for each \( X \in \mathcal{F}^G(6, n) \setminus \{(t2), (t4)\} \) the values of \( \left| \left( a_1X^{k,n} \right)^{-1}(X) \right| \) are equal to the product of \( 2^4 \cdot |\text{Cir}(4,n)| \) and the number of possibilities to choose \( B \upharpoonright S \in \{\pm\}^S \) in such a way that \( X_B \in X \). By determining the latter number for each of the isomorphism types, we will prove all of the formulas (m5.12)—(m6.120), except, as already mentioned, (m6.14) and (m6.12), which do not fit into the overall plan of the proof (in the case of (m6.14) we would be overcounting the number of realizations since \( K_{2,3} \) contains three copies of \( C^4 \) but which are easy to count directly).

Let \( \prec \) denote the lexicographic ordering on \( \{n-1\}^2 \). Throughout the proof, we use the following conventions: we consider \( I \supseteq S \subseteq \binom{\{n-1\}^2}{4} \) and \( B \upharpoonright S \in \{\pm\}^S \) to be arbitrary. We set \( \{a, b, c, d\} := S, a_1 := p_1(a), a_2 := p_2(a) \) and analogously for \( b_1, b_2, c_1, c_2, d_1 \) and \( d_2 \). Since \( \prec \) is a total order, we may assume \( a \prec b \prec c \prec d \) which combined with the fact that \( S \) is a matrix-4-circuit implies \( a_1 = b_1, c_1 = d_1, a_2 = c_2 \) and \( b_2 = d_2 \). The cardinality of \( I \setminus S \) depends on whether we are proving formulas from (QFa5) or (QFa6). In the former case we set \( \{u\} := I \setminus S, \) in the latter \( \{u, v\} := I \setminus S \)
with the assumption that \( u < v \). Moreover, \( u_1 := p_1(u), u_2 := p_2(u), v_1 := p_1(v) \) and \( v_2 := p_2(v) \). Finally, let us use the abbreviation \( p(S) := p_1(S) \cup p_2(S) \).

**Proof of (QFa5).** As to (m5.t2), we start by noting that it follows directly from Definition 12 that \( X_B \in (t2) \) if and only if \( B[u] = 0 \) and

\[
|\{u_1\} \setminus p_1(S)| + |\{u_2\} \setminus p_2(S)| = 1.
\]

(18)

We distinguish cases according to how (18) is satisfied.

(C.(m5.t2).1) \( |\{u_1\} \setminus p_1(S)| = 0 \), i.e. \( u_1 \in p_1(S) \). Then (18) implies that \( u_2 \notin p_2(S) \). Since there are 2 different \( u_1 \) with \( u_1 \in p_1(S) \) for each of them there are \((n - 1) - 2 = (n - 3)\) different \( u_2 \) with \( u_2 \notin p_2(S) \) it follows that if (C.(m5.t2.1), then there are \( 2(n - 3) \) realizations of type (t2) by \( B[u] \).

(C.(m5.t2.2) \( |\{u_1\} \setminus p_1(S)| = 1 \). This case is easily seen to be symmetric to (C.(m5.t2.1) w.r.t. swapping the subscripts 1 and 2. Therefore, if (C.(m5.t2.2), there are also \( 2(n - 3) \) realizations of type (t2) by \( B[u] \).

It follows that there are \( 2(n - 3) + 2(n - 3) = 4(n - 3) \) realizations of type (t2) by \( B[u] \), proving (m5.t2). As to (m5.t3), this follows from (m5.t2) and Lemma 43.(11).

As to (m5.t5), it follows from Definition 12 that \( X_B \in (t5) \) if and only if \( B[u] = 0 \) and

\[
|\{u_1\} \setminus p_1(S)| + |\{v_2\} \setminus p_2(S)| = 2.
\]

(19)

Property (19) is equivalent to \( u_1 \notin p_1(S) \) and \( u_2 \notin p_2(S) \), and there are obviously \((n - 1) - 2 = (n - 3)^2 \) different \( u \in [n - 1]^2 \) satisfying this. Therefore, (m5.t5) is correct. As to (m5.t7), this follows from (m5.t5) and Lemma 43.(12). This completes the proof of (QFa5). □

We now take on the task of proving (QFa6), which will take some effort. We prepare by proving four lemmas characterizing the realizations of the types (t8)−(t11).

**Lemma 45.** For every \( B \in \{0, \pm\}^I \) with \( I \in \left[\binom{n-1}{2}\right] \), \( I = S \cup \{u, v\} \) and \( X_B |_S \cong C^4 \) we have \( X_B \cong (t8) \) if and only if

- (P.(t8).1) \( X_{(u \circ v),u} \cong (t5) \),
- (P.(t8).2) \( B[u] \in \{\pm\} \) and \( B[v] \in \{\pm\} \),
- (P.(t8).3) \( \{u_1, u_2\} \cap \{v_1, v_2\} = \emptyset \),
- (P.(t8).4) \( \{u_1 \in p_1(S) \) or \( v_1 \in p_1(S) \} or \( \{u_2 \in p_2(S) \) and \( v_2 \in p_2(S) \})

**Proof.** First suppose that \( X_B \cong (t8) \). Then Definition 12 implies that both (P.(t8).1) and (P.(t8.2) are true. To prove (P.(t8.3) and (P.(t8.4), let \( e \neq f \) be \( E(X_B[u]) \) denote the two edges in \( X_B[u] \), where \( e := E(X_B[u]) \) \( \{\{u_1, n\}, \{n, u_2\}\} \) and \( f := E(X_B[v]) \) \( \{\{v_1, n\}, \{n, v_2\}\} \). By hypothesis, \( e \cap f = \emptyset \) and this implies that (P.(t8.3) is true. Moreover, again by hypothesis, both \( e \) and \( f \) intersect \( X_B |_S \cong C^4 \) and the intersection set is not an edge of it.

If \( u_1 \in p_1(S) \), then there are still two possibilities for the intersection set \( f \cap V(X_B |_S) \), namely \( f \cap V(X_B |_S) = \{\{v_1, n\}\} \) (equivalently, \( v_1 \in p_1(S) \)) or \( f \cap V(X_B |_S) = \{\{v_2, n\}\} \) (equivalently, \( v_2 \in p_2(S) \)). It follows from Definition 12 that the vertex in the intersection set \( e \cap V(X_B |_S) = \{\{u_1, n\}\} \) is not adjacent to the vertex in \( f \cap V(X_B |_S) \) if and only if the first possibility is true, i.e. \( f \cap V(X_B |_S) = \{\{v_1, n\}\} \), i.e. \( v_1 \in p_1(S) \). This proves that the first clause of (P.(t8.4), and hence (P.(t8.4) itself, is true.

If \( u_2 \in p_2(S) \), then an entirely analogous argument as the one in the preceding paragraph shows that the second clause of (P.(t8.4), hence again (P.(t8.4) itself, is true. This proves that \( X_B \cong (t8) \) implies that (P.(t8.1)−(P.(t8.4) are true.

Conversely, assume (P.(t8.1)−(P.(t8.4). Then (P.(t8.2) implies that \( f_1(X_B) = 6 \) and (P.(t8.3) implies that the two edges in \( E(X_B[u]) \) do not intersect. Let \( e \) and \( f \) be defined as in the preceding proof of the other implication. It remains to show that \( \{e \cap V(X_B |_S)\} \cup \{f \cap V(X_B |_S)\} \notin E(X_B |_S) \).

By definition of \( e \), either \( e \cap V(X_B |_S) = \{\{u_1, n\}\} \) or \( e \cap V(X_B |_S) = \{\{n, u_2\}\} \).
In the former case we have \( u_1 \in p_1(S) \), hence the first clause of (P.t8.4) implies \( v_1 \in p_1(S) \), hence \( (v_1, n) \in f \cap V(X_B) \) by definition of \( f \), hence \( f \cap V(X_B) = \{(v_1, n)\} \) since \( f \cap V(X_B) \) is a singleton by construction. In view of Definition 12 this implies that indeed \( (e \cap V(X_{B|S})) \cup (f \cap V(X_{B|S})) = \{(u_1, n), (v_1, n)\} \notin E(X_{B|S}) \).

In the latter case we have \( u_2 \in p_2(S) \), hence the second clause of (P.t8.4) implies \( v_2 \in p_2(S) \), hence \( (n, v_2) \in f \cap V(X_B) \) by definition of \( f \), hence \( f \cap V(X_B) = \{(n, v_2)\} \) since \( f \cap V(X_B) \) is a singleton by construction. In view of Definition 12 this implies that indeed \( (e \cap V(X_{B|S})) \cup (f \cap V(X_{B|S})) = \{(n, u_2), (n, v_2)\} \notin E(X_{B|S}) \). This completes the proof that (P.t8.1)–(P.t8.4) imply \( X_B \cong (t8) \).

\[ \square \]

**Lemma 46.** For every \( B \in \{0, \pm\}^l \) with \( I \in \binom{[n-1]^2}{6} \), \( I = S \cup \{u, v\} \) and \( X_{B|S} \cong C^4 \) we have \( X_B \cong (t9) \) if and only if

\[ \begin{align*}
(P.t9.1) & \quad X_{\{0, \pm\} \cup B|S} \cong (t5), \\
(P.t9.2) & \quad B[u] \in \{\pm\} \text{ and } B[v] \in \{\pm\}, \\
(P.t9.3) & \quad \{u, v\} \cap \{v_1, v_2\} = \emptyset, \\
(P.t9.4) & \quad \{u_1 \in p_1(S) \text{ and } v_2 \in p_2(S)\} \text{ or } \{u_2 \in p_2(S) \text{ and } v_1 \in p_1(S)\}.
\end{align*} \]

**Proof.** First suppose that \( X_B \cong (t9) \). Then Definition 12 implies that both (P.t9.1) and (P.t9.2) are true. To prove (P.t9.3) and (P.t9.4), let \( e \neq f \in E(X_B) \) denote the two edges in \( X_{\{0, \pm\} \cup B|S} \), where \( e := E(X_{B[u]}) = \{(u_1, n), (n, u_2)\} \) and \( f := E(X_{B[v]}) = \{(v_1, n), (n, v_2)\} \). By hypothesis \( e \cap f = \emptyset \) and this implies that (P.t9.3) is true. Moreover, again by hypothesis, both \( e \) and \( f \) intersect \( X_{B|S} \cong C^4 \) and the intersection set is an edge of it.

If \( u_1 \in p_1(S) \), then there are still two possibilities for the intersection set \( f \cap V(X_{B|S}) \), namely \( f \cap V(X_{B|S}) = \{(v_1, n)\} \) (equivalently, \( v_1 \in p_1(S) \)) or \( f \cap V(X_{B|S}) = \{(n, v_2)\} \) (equivalently, \( v_2 \in p_2(S) \)). It is evident from Definition 12 that the vertex in the intersection set \( e \cap V(X_{B|S}) = \{(u_1, n)\} \) is adjacent to the vertex in \( f \cap V(X_{B|S}) \) if and only if the second possibility is true, i.e., \( f \cap V(X_{B|S}) = \{(n, v_2)\} \), i.e. \( v_2 \in p_2(S) \). This proves that the first clause of (P.t9.4), and hence (P.t9.4) itself, is true.

If \( u_2 \in p_2(S) \), an entirely analogous argument as the one in the preceding paragraph shows that then the second clause of (P.t9.4), hence again (P.t9.4) itself is true. This completes the proof that \( X_B \cong (t9) \).

Conversely, suppose that (P.t9.1)–(P.t9.4) are true. Then (P.t9.2) implies that \( f \mid X_B = 6 \) and (P.t9.3) implies that the two edges in \( E(X_{\{0, \pm\} \cup B|S}) \) do not intersect. Let \( e \) and \( f \) be defined as in the preceding proof of the other implication. It remains to show that \( (e \cap V(X_{B|S})) \cup (f \cap V(X_{B|S})) \in E(X_{B|S}) \). By definition of \( e \), either \( e \cap V(X_{B|S}) = \{(u_1, n)\} \) or \( e \cap V(X_{B|S}) = \{(n, u_2)\} \).

In the former case we have \( u_1 \in p_1(S) \), hence the first clause of (P.t9.4) implies that \( v_2 \in p_2(S) \), hence \( (n, v_2) \in f \cap V(X_{B|S}) \) by definition of \( f \), hence \( f \cap V(X_{B|S}) = \{(n, v_2)\} \) since \( f \cap V(X_{B|S}) \) is a singleton by construction. In view of Definition 12 this implies that indeed \( (e \cap V(X_{B|S})) \cup (f \cap V(X_{B|S})) = \{(u_1, n), (n, v_2)\} \in E(X_{B|S}) \).

In the latter case we have \( u_2 \in p_2(S) \), hence the second clause of (P.t9.4) implies that \( v_1 \in p_1(S) \), hence \( (v_1, n) \in f \cap V(X_{B|S}) \) by definition of \( f \), hence \( f \cap V(X_{B|S}) = \{(v_1, n)\} \) since \( f \cap V(X_{B|S}) \) is a singleton by construction. In view of Definition 12 this implies that indeed \( (e \cap V(X_{B|S})) \cup (f \cap V(X_{B|S})) = \{(n, u_2), (v_1, n)\} \in E(X_{B|S}) \). This completes the proof that (P.t9.1)–(P.t9.4) imply \( X_B \cong (t9) \).

\[ \square \]

**Lemma 47.** For every \( B \in \{0, \pm\}^l \) with \( I \in \binom{[n-1]^2}{6} \), \( I = S \cup \{u, v\} \) and \( X_{B|S} \cong C^4 \) we have \( X_B \cong (t10) \) if and only if

\[ \begin{align*}
(P.t10.1) & \quad X_{\{0, \pm\} \cup B|S} \cong (t5), \\
(P.t10.2) & \quad B[u] \in \{\pm\} \text{ and } B[v] \in \{\pm\}, \\
(P.t10.3) & \quad \{u, v\} \cap \{v_1, v_2\} \neq \emptyset, \\
(P.t10.4) & \quad \{u_1 \in p_1(S) \text{ and } v_2 \in p_2(S)\} = \emptyset \text{ or } \{v_1, v_2\} \cap p(S) = \emptyset.
\end{align*} \]

**Proof.** First suppose that \( X_B \cong (t10) \). Then Definition 12 implies that both (P.t10.1) and (P.t10.2) are true. To prove (P.t10.3) and (P.t10.4), let \( e \in E(X_B) \) denote the unique edge
which does not intersect $X_{B|\beta} \cong C^4$, and let $f \in E(B)$ denote the unique edge which intersects $X_{B|\beta} \cong C^4$. In view of Definition 12,
\[(e = ((u_1,n),(u,n_2)) \text{ and } f = ((v_1,n),(v,n_2))) \text{ or } (e = ((v_1,n),(v,n_2)) \text{ and } f = ((u_1,n),(u,n_2))]. \tag{20}
\]
By definition of $e$ and $f$ we have $e \cap f \neq \emptyset$, hence whatever of the two clauses of (20) is true, either $u_1 = v_1$ or $u_2 = v_2$. Therefore property (P.(t10).3) is true. By definition of $e$, if $e = ((u_1,n),(u,n_2))$, then $u_1 \notin p_1(S)$ and $u_2 \notin p_2(S)$, hence the first clause of (P.(t10).4) is true, and if $e = ((v_1,n),(v,n_2))$, then $v_1 \notin p_1(S)$ and $v_2 \notin p_2(S)$, hence then the second clause of (P.(t10).4) is true.

Conversely, suppose that properties (P.(t10).1)–(P.(t10).4) are true. Then (P.(t10).2) implies $f_1(X_B) = 6$ and (P.(t10).3) implies that the two edges corresponding to $u \neq v$ intersect. It remains to prove that the 2-path consisting of these edges intersects $X_{B|\beta} \cong C^4$ with one of its endvertices.

To see this, note first that (P.(t10).1) implies (23) which combined with $u \neq v$ implies that
\[\{u_1,u_2,v_1,v_2\} \cap p(S) \neq \emptyset. \tag{21}\]
Moreover, we know from (20) together with $e \cap f \neq \emptyset$ that $u_1 = v_1$ or $u_2 = v_2$.

If $u_1 = v_1$, then (21) cannot be true by virtue of $u_1 = v_1 \in p_1(S)$ since then (P.(t10).4) would become false. Therefore, if $u_1 = v_1$, then $u_1 = v_1 \notin p_1(S)$, and (21) implies $\{u_2,v_2\} \cap p_2(S) \neq \emptyset$.

It is impossible that $\{u_2,v_2\} \subseteq p_2(S)$ for this combined with $u_1 = v_1$ would imply $\{|u_1,v_1| \\setminus p_1(S)| + |\{u_2,v_2\} \setminus p_2(S)| = 1$, hence contradict (23). Therefore, $\{|u_2,v_2| \cap p_2(S)| = 1$, and since $u_1 = v_1$ implies $u_2 \neq v_2$, this is what we wanted to prove: exactly one of the two endvertices $(n,u_2),(n,v_2) \in V(X_B)$ intersects $X_{B|\beta} \cong C^4$.

If $u_2 = v_2$, then an entirely analogous argument as in the preceding paragraph shows that exactly one of the two endvertices $(u_1,n),(v_1,n) \in V(X_B)$ intersects the $X_{B|\beta} \cong C^4$.

The proof that properties (P.(t10).1)–(P.(t10).4) imply $X_B \cong (t10)$ is now complete. \hfill \Box

**Lemma 48.** For every $B \in \{0,\pm\}^I$ with $I \in \left(\mathbb{N}^2\right)$, $I = S \cup \{u,v\}$ and $X_{B|\beta} \cong C^4$ we have $X_B \cong (t11)$ if and only if

- (P.(t11).1) $X_{(0)}(u,v) \cap X_{B|\beta} \cong (t15)$,
- (P.(t11).2) $B[u] \in \{\pm\}$ and $B[v] \in \{\pm\}$,
- (P.(t11).3) $\{u_1,u_2\} \cap \{v_1,v_2\} \neq \emptyset$,
- (P.(t11).4) $\{u_1,u_2\} \cap p(S) \neq \emptyset$ and $\{v_1,v_2\} \cap p(S) \neq \emptyset$.

**Proof.** First suppose that $X_B \cong (t11)$. Then Definition 12 implies that both (P.(t11).1) and (P.(t11).2) are true. To prove (P.(t11).3) and (P.(t11).4), let $e \neq f \in E(B)$ denote the two edges in $E(X_B)$ forming the 2-path which intersects $X_{B|\beta} \cong C^4$ with its inner vertex. As in the proof of Lemma 47, we know that (20) is true. By definition of $e$ and $f$ we have $e \cap f \neq \emptyset$, hence whatever of the two clauses of (20) is true, either $u_1 = v_1$ or $u_2 = v_2$. Therefore property (P.(t11).3) is true. By definition of $e$ and $f$, both $e$ and $f$ intersect $X_{B|\beta} \cong C^4$. If the first clause in (20) is true then $e$ intersecting $X_{B|\beta} \cong C^4$ is equivalent to $(u_1 \in p_1(S) \text{ or } u_2 \in p_2(S))$ and $f$ intersecting $X_{B|\beta} \cong C^4$ is equivalent to $(v_1 \in p_1(S) \text{ or } v_2 \in p_2(S))$. Then (P.(t11).4) is indeed true. If the second clause in (20) is true then $e$ intersecting $X_{B|\beta} \cong C^4$ is true as well. This completes the proof that $X_B \cong (t11)$ implies (P.(t11).1)–(P.(t11).4).

Conversely, suppose that properties (P.(t11).1)–(P.(t11).4) are true. Then (P.(t11).2) implies $f_1(X_B) = 6$ and (P.(t11).3) implies that the two edges corresponding to $u \neq v$ intersect. It remains to prove that the 2-path consisting of these edges intersects $X_{B|\beta} \cong C^4$ with its inner vertex. Similar to the proof of Lemma 47 we know that (21) and that $u_1 = v_1$ or $u_2 = v_2$.

If $u_1 = v_1$, then $u_2 \in p_2(S)$ is impossible since this together with $u \neq v$ would imply $u_1 = v_1 \notin p_1(S)$ which due to the second clause of (P.(t11).4) would imply $v_2 \in p_2(S)$; but $u_2 \in p_2(S)$, $u_1 = v_1$ and $v_2 \in p_2(S)$ combined imply $\{|u_1,v_1| \setminus p_1(S)| + |\{u_2,v_2\} \setminus p_2(S)| = 1$, a contradiction to (23). For an entirely analogous reason $v_2 \in p_2(S)$ is impossible, too. Since both $u_2 \notin p_2(S)$ and $v_2 \notin p_2(S)$, it follows from (21) that $u_1 = v_1 \in p_1(S)$. This is what we wanted to prove: the
common vertex \((u_1, n) = (v_1, n)\) of \(e\) and \(f\) (i.e. the inner vertex of the 2-path formed by \(e\) and \(f\)) is also the unique vertex of intersection with \(X_{B|S} \cong C^4\).

If \(u_2 = v_2\), then an entirely analogous argumentation as in the preceding paragraph shows that the common vertex \((n, u_2) = (n, v_2)\) of \(e\) and \(f\) (i.e. the inner vertex of the 2-path which is formed by \(e\) and \(f\)) is also the unique vertex of intersection with \(X_{B|S} \cong C^4\).

The proof that properties \((P.(t11).1)-(P.(t11).4)\) imply \(X_B \cong (t11)\) is now complete. \(\Box\)

**Proof of \((QF\text{a}6)\).** As to \((m6.t2)\), it follows from Definition 12 that \(X_B \cong (t2)\) if and only if

\[ |\{u_1, v_1\} \setminus p_1(S)| + |\{u_2, v_2\} \setminus p_2(S)| = 1. \tag{22} \]

\[(C.(m6.t2).1) \ \{|\{u_1, v_1\} \setminus p_1(S)| = 0. \text{ Then (22) implies } |\{u_2, v_2\} \setminus p_2(S)| = 1 \text{ which is equivalent to (28). The property defining Case 1 is equivalent to } \{u_1, v_1\} \subseteq p_1(S). \text{ Property (28) implies two cases:} \]

(1) \(u_2 = v_2\) and \(\{u_2, v_2\} \setminus p_2(S) = \emptyset\). Then \(u_2 = v_2\) and \(u < v\) imply that \(u_1 < v_1\). This together with \(\{u_1, v_1\} \subseteq p_1(S)\) implies \(u_1 = a_1 = c_1\) and \(v_1 = b_1 = d_1\). Therefore, it is \(u_2 = v_2\) alone which determines the two pairs \(u, v\). The property \(u_2 = v_2\) and \(\{u_2, v_2\} \setminus p_2(S) = \emptyset\) is equivalent to \(u_2 = v_2 \notin p_2(S)\). It follows that if \((C.(m6.t2).1).1)\), then there are \((n - 1) - 2\) realizations of type \((t2)\) by \(u, v\).

(2) \(u_2 \neq v_2\) and \(\{u_2, v_2\} \setminus p_2(S) = 1\). Then either \(u_2 \in p_2(S)\) or \(v_2 \in p_2(S)\). If \(u_2 \in p_2(S)\), then because of \(\{u_1, v_1\} \subseteq p_1(S)\) it follows that \(u \in \{a, b, c, d\}\), a contradiction to \(I \setminus S = \{v, e\}\). Similarly, if \(v_2 \in p_2(S)\), then the same contradiction arises with regard to \(v\). Therefore, the case \((C.(m6.t2).1).2)\) cannot occur.

It follows that if \((C.(m6.t2).1)\), then there are exactly \((n - 1) - 2\) realizations of type \((t2)\) by \(B \setminus \{u, v\}\).

\[(C.(m6.t2).2) \ |\{u_1, v_1\} \setminus p_1(S)| = 1. \text{ Then (22) implies } |\{u_2, v_2\} \setminus p_2(S)| = 0 \text{ which is equivalent to } |\{u_2, v_2\} \subseteq p_2(S). \text{ The property defining (C.(m6.t2).2) is equivalent to (26). Now an argument entirely analogous to the one given for (C.(m6.t2).1) shows that if (C.(m6.t2).2), then there are exactly } (n - 1) - 2 \text{ realizations of type (t2) by } B \setminus \{u, v\}. \text{ It follows that there are exactly } 2 \cdot ((n - 1) - 2) \text{ different } I \setminus S = \{u, v\} \text{ with } X_B \cong (t2). \text{ This completes the proof of (m6.t2).} \]

As to \((m6.t3)\), notice that a necessary condition for type \((t3)\) is that \(|V(X_B) \setminus V(X_{B|S})| = 1\). Therefore the set of all suitable \(I \subseteq \{n-1\}^2\) is a subset (possibly nonproper) of those which are suitable for type \((t2)\). We may therefore reexamine the analysis carried out for \((m6.t2)\) and in each of the cases count the number of \(B \in \{0, \pm 1\}^I\) with \(X_B \cong (t3)\).

If \((C.(m6.t2).1).1)\), then properties \(u_1 = a_1 = c_1\) and \(v_1 = b_1 = d_1\) show that both \(u\) and \(v\) have the property that if one of them indexes a nonzero value of \(B\), then there is an edge intersecting \(X_{B|S} \cong C^4\). Since otherwise we would have \(K^{2,3}\), exactly one of them must be nonzero. This implies exactly 4 possibilities to realize type \((t3)\) for each of the \((n - 1) - 2\) realizations of type \((t2)\) which were offered in \((C.(m6.t2).1).1)\). Therefore, if \((C.(m6.t2).2).1)\), then there are exactly \(4 \cdot ((n - 1) - 2)\) realizations of type \((t3)\) by \(B \setminus \{u, v\}\). Since the case \((C.(m6.t2).1).2)\) is as impossible now as it was back then, it follows that this is also the number of realizations for the entire \((C.(m6.t2).1)\).

If \((C.(m6.t2).2)\), then by interchanging the subscripts 1 and 2 we may use the same analysis as for the case \((C.(m6.t2).1)\) to reach the conclusion that there are exactly \(4 \cdot ((n - 1) - 2)\) realizations of type \((t3)\) by \(B \setminus \{u, v\} = B \setminus \{l, S\}\).

It follows that there are exactly \(4 \cdot ((n - 1) - 2) + 4 \cdot ((n - 1) - 2) = 8 \cdot (n - 3)\) realizations of type \((t3)\) by \(B \setminus \{l, S\}\). This proves \((m6.t3)\).

As to \((m6.t4)\), it is evident that the number of possibilities to realize a \(K^{2,3}\) is \(2 \cdot 2^6 \cdot \binom{n-1}{2} \cdot \binom{n-1}{3}\), the first factor accounting for the two possibilities of either choosing two of the first indices and three of the last, or vice versa.
As to \((m6.t5)\), note that \(f_1(t5) = 4\), hence it is necessary that \(B[u] = B[v] = 0\). Therefore, the number of \(B \mid_{(u,v)} \in \{0, \pm\}^S \) with \(X_B \cong (t5)\) equals the number of \(u, v \in (\mathbb{Z}^S)^2\) such that
\[
|\{u_1, v_1\} \cap p_1(S)| + |\{u_2, v_2\} \setminus p_2(S)| = 2.
\] (23)

We now distinguish cases according to how (23) is satisfied.

\(\text{(C.}(m6.t5).1)\) \(|\{u_1, v_1\} \setminus p_1(S)| = 0\). Then (23) implies \(|\{u_2, v_2\} \setminus p_2(S)| = 2\), which is equivalent to (25). The property defining Case \(\text{(C.}(m6.t5).1)\) is equivalent to \(\{u_1, v_1\} \subseteq p_1(S)\). There are now two further cases:

1. \(u_1 = v_1\). Then there are exactly 2 possible set inclusions \(\{u_1, v_1\} = \{u_1\} \subseteq p_1(S) = \{a_1, c_1\}\). For each of them, there are exactly \((n-1)^2\) different sets \(\{u_2, v_2\}\) with property (25). Since \(u_1 = v_1\) and \(u \prec v\) imply \(u_2 < v_2\), each of these sets determines the two pairs \(u\) and \(v\). Therefore, if \(\text{(C.}(m6.t5).1).1\), then there are exactly \(2 \cdot (n-1)^2\) realizations of type \((t5)\) by \(B \mid_{(u,v)}\).

2. \(u_1 \neq v_1\). Then \(u \prec v\) implies \(u_1 < u_1\). Now there is exactly one possible set inclusion \(\{u_1, v_1\} \subseteq p_1(S) = \{a_1, c_1\}\). When this inclusion holds, there are exactly \((n-1)^2\) different sets \(\{u_2, v_2\}\) with property (25). Each of them can be realized in exactly 2 ways by \(u\) and \(v\), either by \(u_2 < v_2\) or by \(v_2 < u_2\).

Therefore, if \(\text{(C.}(m6.t5).1).2\), then there are again (with a qualitatively different reason for the factor 2) exactly \(2 \cdot (n-1)^2\) realizations of type \((t5)\) by \(B \mid_{(u,v)}\).

It follows that if \(\text{(C.}(m6.t5).1)\), then there are exactly \(4 \cdot (n-1)^2\) different realizations of type \((t5)\) by \(B \mid_{(u,v)}\).

\(\text{(C.}(m6.t5).2)\) \(|\{u_1, v_1\} \setminus p_1(S)| = 1\). Then (23) implies \(|\{u_2, v_2\} \setminus p_2(S)| = 1\). Hence, in the present situation, the equations (26) and (28) are simultaneously true. There are now two further cases depending on the manner in which (26) is true:

1. \(u_1 = v_1\) and \(\{u_1, v_1\} \cap p_1(S) = \emptyset\). There are \((n-1\cdot 2)\) possibilities for this. For each of them the clause \(u_2 = v_2\) and \(\{u_2, v_2\} \cap p_1(S) = \emptyset\) in (28) cannot be true because it would imply \(u = v\) and therefore for each of them \(u_2 \neq v_2\) and \(|\{u_2, v_2\} \cap p_2(S)| = 1\) must be true. In the following, keep in mind that \(u_1 = v_1\) and \(u \prec v\) implies \(u_2 < v_2\). If \(u_2 = a_2 = c_2\), then \(v_2 \neq b_2 = d_2\), hence there are exactly \(n-1-a_2-1\) different \(v_2\), and therefore as many different realizations of type \((t5)\) by \(u\) and \(v\). If \(u_2 = b_2 = d_2\), then there are exactly \(n-1-b_2\) different \(v_2\), and therefore as many different realizations of type \((t5)\) by \(u\) and \(v\). If \(v_2 = a_2 = c_2\), then there are exactly \(a_2-1\) different \(u_2\), and therefore as many different realizations of type \((t5)\) by \(u\) and \(v\). If \(v_2 = b_2 = d_2\), then \(u_2 \neq a_2 = c_2\), hence there are exactly \(b_2-1\) different \(u_2\) and therefore as many different realizations of type \((t5)\) by \(u\) and \(v\).

Therefore, if \(\text{(C.}(m6.t5).2).1\), then there are \((n-1-a_2-1)+(n-1-b_2)+(a_2-1)+(b_2-1)\cdot(n-1-2)\) realizations of type \((t5)\) by \(B \mid_{1\setminus S} = B \mid_{(u,v)}\).

2. \(u_1 \neq v_1\) and \(|\{u_1, v_1\} \cap p_1(S)| = 1\). Then \(u \prec v\) implies \(u_1 < v_1\) and there are two further cases:

1. \(|\{u_1, v_1\} \cap p_1(S)| = 1\) is true due to \(u_1 \in p_1(S) = \{a_1, c_1\}\). Then \(v_1 \notin \{a_1, c_1\} = p_1(S)\) and there are two further cases:

   1. \(u_1 = a_1 = b_1\). Since \(u_1 < v_1\) and \(v_1 \notin \{a_1, c_1\}\), there are then exactly \(n-1-a_1-1\) different \(v_1\). For each of them, there are two cases. If the first clause of (28) is true, then there are exactly \((n-1)\) different such \(u_2 = v_2\), and each such \(u_2 = v_2\) determines the two pairs \(u\) and \(v\). Therefore in this case there are exactly \((n-1-a_1-1)\cdot(n-1-2)\) realizations of type \((t5)\) by \(B \mid_{(u,v)}\). If the second clause of (28) is true, then since \(u_1 = a_1 = b_1\) combined with \(u \neq a\) and \(u \neq b\) implies \(u_2 \notin \{a_2 = c_2, b_2 = d_2\} = p_2(S)\), it
follows that $|\{u_2, v_2\} \cap p_2(S)| = 1$ is true as $v_2 \in p_2(S)$. Therefore in this case there are exactly $(n - 1) - 2$ different $u_2$ and exactly 2 different $v_2$ for each of them and hence exactly $(n - 1 - a_1 - 1) \cdot 2 \cdot ((n - 1) - 2)$ realizations of type (t5) by $B \mid \{u,v\}$.

Therefore, if (C.(m6.t5).2),(2),(1), then there are exactly $3 \cdot (n - 1 - a_1 - 1) \cdot (n - 1 - 2)$ different realizations of type (t5) by $B \mid \{u,v\}$.

(2) $u_1 = c_1 = d_1$. Since $u_1 < v_1$, there are then exactly $n - 1 - c_1$ different $v_1$. For each of them, there are two cases. If the first clause of (28) is true, then there are exactly $(n - 1 - c_1) \cdot (n - 1 - 2) \cdot ((n - 1) - 2) \cdot (n - 1 - 2)$ realizations of type (t5) by $B \mid \{u,v\}$. If the second clause of (28) is true, then since $u_1 = c_1 = d_1$ combined with $u \neq c$ and $u \neq d$ implies $u_2 \notin \{a_2 = c_2, b_2 = d_2\} = p_2(S)$, it follows that $|\{u_2, v_2\} \cap p_2(S)| = 1$ is true as $v_2 \in p_2(S)$. Therefore in this case there are exactly $(n - 1 - 2)$ different $u_2$ and for each of them exactly 2 different $v_2$, thus exactly $(n - 1 - c_1) \cdot 2 \cdot ((n - 1) - 2)$ realizations of type (t5) by $B \mid \{u,v\}$.

Therefore, if (C.(m6.t5).2),(2),(1), then there are exactly $3 \cdot (n - 1 - a_1 - 1) \cdot (n - 1 - 2) + 3 \cdot (n - 1 - c_1) \cdot (n - 1 - 2) = 3 \cdot (2n - a_1 - c_1 - 3) \cdot (n - 1 - 2)$ different realizations of type (t5) by $B \mid \{u,v\}$.

(2) $|\{u_1, v_1\} \cap p_1(S)| = 1$ is true due to $v_1 \in p_1(S) = \{a_1, c_1\}$. Then $u_1 \notin \{a_1, c_1\}$ and there are two further cases:

(1) $v_1 = a_1 = b_1$. Since $u_1 < v_1$, there are then exactly $a_1 - 1$ different $u_1$. For each of them, there are two cases. If the first clause of (28) is true, then there are exactly $(n - 1) - 2$ different $u_2 = v_2$ and each such $u_2$ determines the pair $\{u, v\}$. Hence in this case there are exactly $(a_1 - 1) \cdot (n - 1 - 2)$ realizations of type (t5) by $B \mid \{u,v\}$. If the second clause of (28) is true, then since $v_1 = a_1 = b_1$ combined with $v \neq a$ and $v \neq b$ implies $v_2 \notin \{a_2 = c_2, b_2 = d_2\} = p_2(S)$, we know that $|\{u_2, v_2\} \cap p_2(S)| = 1$ must be true as $u_2 \in p_2(S)$, hence there are 2 different $u_2$ and for each of them exactly $(n - 1) - 2$ different $v_2$, hence in this case there are exactly $(a_1 - 1) \cdot 2 \cdot (n - 1 - 2)$ realizations of type (t5) by $B \mid \{u,v\}$.

Therefore, if (C.(m6.t5).2),(2),(1), then there are exactly $3 \cdot (a_1 - 1) \cdot (n - 1 - 2)$ different realizations of type (t5) by $B \mid \{u,v\}$.

(2) $v_1 = c_1 = d_1$. Since $u_1 < v_1$ and $u_1 \notin \{a_1, c_1\}$, there are then exactly $c_1 - 1 - 1$ different $u_1$. For each of them, there are two cases. If the first clause of (28) is true, then there are exactly $(c_1 - 1 - 1) \cdot (n - 1 - 2)$ realizations of type (t5) by $B \mid \{u,v\}$. If the second clause of (28) is true, then since $v_1 = c_1 = d_1$ combined with $v \neq c$ and $v \neq d$ implies $v_2 \notin \{a_2 = c_2, b_2 = d_2\} = p_2(S)$, it follows that $|\{u_2, v_2\} \cap p_2(S)| = 1$ is true as $v_2 \in p_2(S)$. Therefore in this case there are exactly $(n - 1) - 2$ different $v_2$ and for each of them exactly 2 different $u_2$, hence exactly $(c_1 - 1 - 1) \cdot 2 \cdot (n - 1 - 2)$ realizations of type (t5) by $B \mid \{u,v\}$.

Therefore, if (C.(m6.t5).2),(2),(2), then there are exactly $3 \cdot (c_1 - 1 - 1) \cdot (n - 1 - 2)$ realizations of type (t5) by $B \mid \{u,v\}$.

It follows that if (C.(m6.t5).2),(2),(2), then there are exactly $3 \cdot (a_1 - 1) \cdot (n - 1 - 2) + 3 \cdot (c_1 - 1 - 1) \cdot (n - 1 - 2) = 3 \cdot (a_1 + c_1 - 3) \cdot (n - 1 - 2)$ realizations of type (t5) by $B \mid \{u,v\}$. 
It follows that if \((C.(m6.t5).2),(2)\), then there are exactly \(3 \cdot (2n - a_1 - c_1 - 3) \cdot ((n - 1) - 2) + 3 \cdot (a_1 + c_1 - 3) \cdot ((n - 1) - 2) = 6 \cdot ((n - 1) - 2)^2\) realizations of type \((t5)\) by \(B \upharpoonright \{u,v\}\).

It follows that if \((C.(m6.t5).2),\) then there are exactly \(2 \cdot ((n - 1) - 2)^2 + 6 \cdot ((n - 1) - 2)^2 = 8 \cdot ((n - 1) - 2)^2\) realizations of type \((t5)\) by \(B \upharpoonright \{u,v\}\).

\((C.(m6.t5).3)\) \(|\{u_1, v_1\}\backslash p_1(S)| = 2\). This is equivalent to \((27)\). Moreover \((23)\) implies \(|\{u_2, v_2\}\backslash p_2(S)| = 0\), which is equivalent to \(|\{u_2, v_2\}| \subseteq p_2(S)\). Swapping the subscripts 1 and 2 in the analysis of \((C.(m6.t5).1)\) shows that if \((C.(m6.t5).3)\), then there are exactly \(4 \cdot \left(\frac{(n-1)^2}{2}\right)\) realizations of type \((t5)\) by \(B \upharpoonright \{u,v\}\).

It follows that for the fixed \(S\), there are exactly \(4 \cdot \left(\frac{(n-1)^2}{2}\right) + 8 \cdot ((n - 1) - 2)^2 + 4 \cdot \left(\frac{(n-1)^2}{2}\right) = 8 \cdot (n - 3)^2 + 8 \cdot \left(\frac{(n-3)}{2}\right)\) different \(I \setminus S = \{u,v\}\) with the property \(X_B \cong (t5)\). This completes the proof of \((m6.t5)\).

As to \((m6.t6)\) and \((m6.t7)\), let us first note that both for \((t6)\) and for \((t7)\) a necessary condition is that \(X_B\) contain exactly two vertices not in \(X_{B \setminus S} \cong C^4\). Therefore, the set of all \(I \setminus S = \{u,v\}\) with \(X_B \cong (t6)\) is a subset of the set of those \(\{u,v\}\) with \(X_{\{u,v\}} \cong (t5)\), and likewise for \((t7)\). We may therefore determine both \((m6.t6)\) and \((m6.t7)\) by a single reexamination of the analysis given for type \((t5)\). In each of the cases which we distinguished there we now have to count the number of \(B \upharpoonright \{u,v\}\) with \(X_B \cong (t6)\) and also of those \(B \upharpoonright \{u,v\}\) with \(X_B \cong (t7)\).

We can prepare for this as follows. Consider the properties

\begin{enumerate}
  \item \((\{u_1, u_2\} \cap p(S) \neq \emptyset \text{ and } (v_1, v_2) \cap p(S) = \emptyset)\) or \((\{u_1, u_2\} \cap p(S) = \emptyset \text{ and } (v_1, v_2) \cap p(S) \neq \emptyset)\),
  \item \((\{u_1, u_2\} \cap p(S) \neq \emptyset \text{ and } (v_1, v_2) \cap p(S) \neq \emptyset)\).
\end{enumerate}

In each of the cases to be reexamined, these properties alone determine how many \(B \upharpoonright \{u,v\}\) realize \((t6)\) or \((t7)\).

Let us first focus on \((t6)\). In \((i1)\), each of the two clauses of that disjunction has the property that if it is true, then there are exactly 2 possibilities for a \(B \upharpoonright \{u,v\}\) with \(X_B \cong (t6)\). For the first clause these are \((B[u] \in \{\pm\} \text{ and } B[v] = 0)\), for the second clause \((B[u] = 0 \text{ and } B[v] \in \{\pm\})\) and likewise for \((i2)\).

Moreover, the disjunction is evidently exclusive. Therefore, if property \((i1)\) is true, then the number of \(B \upharpoonright \{u,v\}\) with \(X_B \cong (t6)\) is exactly 2-times as large as the number of \(\{u,v\} \in (\binom{n-1}{2}^2)\) with \(X_{\{u,v\}} \cong (t5)\) which was determined in the proof of \((m6.t5)\). If property \((i2)\) is true, then there are exactly 4 possibilities for a \(B \upharpoonright \{u,v\}\) with \(X_B \cong (t6)\): \((B[u] \in \{\pm\} \text{ and } B[v] = 0)\) or \((B[u] = 0 \text{ and } B[v] \in \{\pm\})\).

Therefore, if property \((i2)\) is true, then there are exactly 4-times as many realizations of isomorphism type \((t6)\) by \(B \upharpoonright \{u,v\}\) as there had been for type \((t5)\).

Let us now turn to \((t7)\). In case \((i1)\) there are \((\text{just as for type } (t6))\) exactly 2 possibilities for a \(B \upharpoonright \{u,v\}\) with \(X_B \cong (t7)\). This time, these are \((B[u] = 0 \text{ and } B[v] \in \{\pm\})\) for the first clause of \((i1)\), and \((B[u] \in \{\pm\} \text{ and } B[v] = 0)\) for the second clause. Again, due to the mutual exclusiveness of the clauses, it follows that whenever case \((i1)\) is true (no matter by way of which clause), there are exactly 2-times as many \(B \upharpoonright \{u,v\}\) with \(X_B \cong (t7)\) as there are \(\{u,v\} \in (\binom{n-1}{2}^2)\) with \(X_{\{u,v\}} \cong (t5)\). Concerning property \((i2)\), however, there is a genuine difference: when this property is true, there is no possibility to choose \(B \upharpoonright \{u,v\}\) so as to create exactly one edge disjoint from the \(X_{B \setminus S} \cong C^4\). We can now begin inspecting the cases.

If \((C.(m6.t5).1),\) then the inclusion \(\{u_1, v_1\} \subseteq p_1(S)\) alone, no matter whether \(u_1 = v_1\) or not, implies that property \((i2)\) is true and without going any deeper we know that there are exactly \(4 \cdot 4 \cdot \left(\frac{(n-1)^2}{2}\right) = 16 \cdot \left(\frac{(n-1)^2}{2}\right)\) realizations of type \((t6)\) and 0 realizations of type \((t7)\) by \(B \upharpoonright \{u,v\}\).

If \((C.(m6.t13).2),\) we have to descend one level deeper. If \((C.(m6.t13).2)\), then it is known that \(u_1 = v_1\), \(\{u_1, v_1\} \cap p_1(S) = \emptyset\), \(u_2 < v_2\) and \(|\{u_2, v_2\} \cap p_2(S)| = 1\) and obviously this implies that property \((i1)\) is true. Therefore without having to reexamine further subcases we then know...
that if \((C.(m6.t13.2).2),(1)\), then there are exactly \(2 \cdot 2 \cdot ((n - 1) - 2)^2 = 4 \cdot ((n - 1) - 2)^2\) realizations of type \((t6)\) and also exactly \(2 \cdot 2 \cdot ((n - 1) - 2)^2 = 4 \cdot ((n - 1) - 2)^2\) realizations of type \((t7)\) by \(B\mid_{\{u,v\}}\).

If \((C.(m6.t13.2).2),(2)\), however, then we have to go deeper still. Although we then already know that \(u_1 < v_1\) and \([\{u_1, v_1\} \cap p_1(S)] = 1\), and therefore know that
\[
\{u_1, u_2\} \cap p(S) \neq \emptyset \text{ or } \{v_1, v_2\} \cap p(S) \neq \emptyset,
\]
at the present stage of our knowledge this latter property is compatible with both \((i1)\) and \((i2)\) (i.e., we do not know yet whether the ‘or’ in (24) is true as an ‘and’). The reason is that we do not yet have any knowledge about \(u_2\) and \(v_2\). Therefore, neither descending down to \((C.(m6.t13.2).2),(2),(1)\) nor to \((C.(m6.t13.2).2),(2),(1),(1)\) is sufficient for us to know whether \((i1)\) or \((i2)\) is true. We therefore have to go all the way down to the two (anonymous) subcases of maximal depth within the case \((C.(m6.t13.2).2),(2),(1),(1)\). In the first of the two subcases we know that \([\{u_2, v_2\} \cap p_2(S) = \emptyset\) and combining this with our knowledge of \([\{u_1, v_1\} \cap p_1(S)] = 1\) we may conclude that exactly one of the two clauses in (24), and hence property \((i1)\) is true. Therefore, in the present subcase there are exactly \(2 \cdot (n - 1 - a_1 - 1) \cdot ((n - 1) - 2)\) realizations of type \((t6)\) and also \(2 \cdot (n - 1 - a_1 - 1) \cdot ((n - 1) - 2)\) realizations of type \((t7)\) by \(B\mid_{\{u,v\}}\).

In the second of the two subcases we know that \(u_2 \neq v_2\) and \([\{u_2, v_2\} \cap p_2(S) = 1\). Recall that at present we also know that \(u_1 < v_1\) and \([\{u_1, v_1\} \cap p_1(S)] = 1\). Keeping in mind the fact that because of \(u \notin S\) at most one projection of \(u\) can be contained in \(p(S)\) (and the analogous fact about \(v\)), we may argue that if \([\{u_1, v_1\} \cap p_1(S)] = 1\) (true as \(u_1 \in p_1(S)\) and \(v_1 \notin p_1(S)\)), then \([\{u_2, v_2\} \cap p_2(S)] = 1\) must be true as \((u_2 \notin p_2(S)\) and \(v_2 \in p_2(S)\), and if \([\{u_1, v_1\} \cap p_1(S)] = 1\) is true as \((u_1 \notin p_1(S)\) and \(v_1 \notin p_1(S)\), then \([\{u_2, v_2\} \cap p_2(S)] = 1\) must be true as \((u_2 \in p_2(S)\) and \(v_2 \in p_2(S)\)). Since in both cases both clauses of \((24)\) are true, it follows that \((i2)\) is true. Therefore, in the present subcase there are exactly \(4 \cdot (n - 1 - a_1 - 1) \cdot ((n - 1) - 2) = 8 \cdot (n - 1 - a_1 - 1) \cdot ((n - 1) - 2)\) realizations of type \((t6)\) and 0 realizations of type \((t7)\) by \(B\mid_{\{u,v\}}\). Adding up our findings, it follows that if \((C.(m6.t13.2).2),(2),(1),(1)\), then there are exactly \(2 \cdot (n - 1 - a_1 - 1) \cdot ((n - 1) - 2) + 8 \cdot (n - 1 - a_1 - 1) \cdot ((n - 1) - 2)\) realizations of type \((t6)\) and 0 realizations of type \((t7)\) by \(B\mid_{\{u,v\}}\).

The case \((C.(m6.t13.2).2),(2),(1),(2)\) is again not sufficient for us to know whether \((i1)\) or \((i2)\) is true and we again have to consider its anonymous subcases. In the first of them, an argument entirely analogous to the one given for the first subcase of \((C.(m6.t13.2).2),(2),(1),(1)\) proves that then property \((i1)\) is true and therefore we know that there are exactly \(2 \cdot (n - 1 - c_1) \cdot ((n - 1) - 2)\) realizations of type \((t6)\) and also exactly \(2 \cdot (n - 1 - c_1) \cdot ((n - 1) - 2)\) realizations of type \((t7)\) by \(B\mid_{\{u,v\}}\).

In the second of them, analogously to the second subcase of \((C.(m6.t13.2).2),(2),(1),(1)\) proves that then property \((i2)\) is true and therefore there are exactly \(4 \cdot (n - 1 - c_1) \cdot 2 \cdot ((n - 1) - 2) = 8 \cdot (n - 1 - c_1) \cdot ((n - 1) - 2)\) realizations of type \((t6)\) and 0 realizations of type \((t7)\) by \(B\mid_{\{u,v\}}\).

It follows that if \((C.(m6.t13.2).2),(2),(1),(2)\), then there are exactly \(2 \cdot (n - 1 - c_1) \cdot ((n - 1) - 2) + 8 \cdot (n - 1 - c_1) \cdot ((n - 1) - 2) = 10 \cdot (n - 1 - c_1) \cdot ((n - 1) - 2)\) realizations of type \((t6)\) but only \(2 \cdot (2n - a_1 - c_1 - 3) \cdot ((n - 1) - 2)\) of type \((t7)\) by \(B\mid_{\{u,v\}}\).

The case \((C.(m6.t13.2).2),(2),(2)\) will now be treated analogously to \((C.(m6.t13.2).2),(2),(1)\).

In the first subcase of \((C.(m6.t13.2).2),(2),(2),(1)\) we find that property \((i1)\) is true and therefore there are exactly \(2 \cdot (a_1 - 1) \cdot ((n - 1) - 2)\) realizations of type \((t6)\) and also \(2 \cdot (a_1 - 1) \cdot ((n - 1) - 2)\) realizations of type \((t7)\) by \(B\mid_{\{u,v\}}\). In the second subcase of \((C.(m6.t13.2).2),(2),(2),(1)\) we find that property \((i2)\) is true and therefore there are exactly \(4 \cdot (a_1 - 1) \cdot 2 \cdot ((n - 1) - 2) = 8 \cdot (a_1 - 1) \cdot ((n - 1) - 2)\) realizations of type \((t6)\) but 0 realizations of type \((t7)\) by \(B\mid_{\{u,v\}}\). Therefore, if
(C.(m6.t13),2), then there are exactly \(2 \cdot (a_1 - 1) \cdot ((n - 1) - 2)\) realizations of type (t6) but only \(2 \cdot (a_1 - 1) \cdot ((n - 1) - 2)\) realizations of type (t7) by \(B \mid \{u,v\}\).

In the first subcase of \((C.(m6.t13),2), \exists \exists (2)\), we conclude that property (i1) is true and therefore there exist exactly \(2 \cdot (c_1 - 1 - 1) \cdot ((n - 1) - 2)\) realizations of type (t6) and also \(2 \cdot (c_1 - 1 - 1) \cdot ((n - 1) - 2)\) realizations of type (t7) by \(B \mid \{u,v\}\). In the second subcase of \((C.(m6.t13),2), \exists \exists (2)\), we conclude that property (i2) is true and therefore there are exactly \(4 \cdot (c_1 - 1 - 1) \cdot ((n - 1) - 2)\) realizations of type (t6) and 0 realizations of type (t7) by \(B \mid \{u,v\}\). Therefore, if \((C.(m6.t13),2), \exists \exists (2)\), then there are exactly \(2 \cdot (c_1 - 1 - 1) \cdot ((n - 1) - 2) + 8 \cdot (c_1 - 1 - 1) \cdot ((n - 1) - 2)\) realizations of type (t6) but only \(2 \cdot (c_1 - 1 - 1) \cdot ((n - 1) - 2)\) realizations of type (t7) by \(B \mid \{u,v\}\).

It now follows that if \((C.(m6.t13),2), \exists \exists (2)\), there are exactly \(10 \cdot (a_1 - 1) \cdot ((n - 1) - 2) + 10 \cdot (c_1 - 1 - 1) \cdot ((n - 1) - 2)\) realizations of type (t6) but merely \(2 \cdot (c_1 - 1 - 1) \cdot ((n - 1) - 2)\) realizations of type (t7) by \(B \mid \{u,v\}\). Moreover we may now conclude that if \((C.(m6.t13),2), \exists \exists (2)\), then there are exactly \(10 \cdot (2n - a_1 - c_1 - 3) \cdot ((n - 1) - 2) + 10 \cdot (a_1 + c_1 - 3) \cdot ((n - 1) - 2)\) realizations of type (t6) but only \(2 \cdot (2n - a_1 - c_1 - 3) \cdot ((n - 1) - 2) + 2 \cdot (a_1 + c_1 - 3) \cdot ((n - 1) - 2)\) realizations of type (t7) by \(B \mid \{u,v\}\). Finally, we can conclude that if \((C.(m6.t13),2), \exists \exists (2)\), there are exactly \(4 \cdot ((n - 1) - 2)^2 + 20 \cdot (n - 1 - 2)^2\) realizations of type (t6) but only \(4 \cdot ((n - 1) - 2)^2\) realizations of type (t7) by \(B \mid \{u,v\}\).

If \((C.(m6.t5),3), \exists \exists \{w_2, v_2\} \subseteq p_2(S)\) alone implies that property (i2) is true and therefore there are exactly \(4 \cdot 4 \cdot ((n - 1)^{-2}) = 16 \cdot ((n - 1)^{-2})\) realizations of type (t6) but 0 realizations of type (t7) by \(B \mid \{u,v\}\).

Summing up, it follows that for each fixed \(S\) there are exactly \(2 \cdot 16 \cdot ((n - 1)^{-2}) + 24 \cdot (n - 1 - 2)^2\) realizations of type (t6) but only \(8 \cdot (n - 1 - 2)^2\) realizations of type (t7) by \(B \mid \{u,v\}\). This completes the proof of both (m6.t6) and (m6.t7).

We can now turn to counting the realizations of (t8)–(t11), i.e. to proving (m6.t8)–(m6.t11). By (P.(t8).1), (P.(t9).1), (P.(t10).1) and (P.(t11).1), for each of the four types (t8), (t9), (t10) and (t11) it is necessary that \(X_{(0)} x u \cup B \mid \exists (t5)\). We may therefore determine each of the four functions \((m6.t8), (m6.t9), (m6.t10), (m6.t11)\) in the course of one reexamination of the proof of (m6.t5). We consider each of the cases in turn, each time descending down just deep enough until we are able to decide which of the four isomorphism types (t8), (t9), (t10) can be realized in that case.

Since by (P.(t8).2), (P.(t9).2), (P.(t10).2) and (P.(t11).2) the property \(B[u] \in \{\pm\}\) and \(B[u] \in \{\pm\}\) is necessary for each of the four types, the positions \(u\) and \(v\) alone, not \(B \mid \{u,v\}\) itself, decide about which type can be realized. Therefore, if a decision is reached about which of the four types can be realized in a case, then we obtain the number of realizations by multiplying the number of realizations of type (t5) in that particular case by 4.

We now reexamine (C.(m6.t5).1). If \((C.(m6.t5).1), \exists \exists \{u_1, v_1\} \setminus p_1(S) = 0\) makes \((P.(t10).4)\) impossible, hence in the entire case (C.(m6.t5).1) the type (t10) is impossible.

If \((C.(m6.t5).1), \exists \exists \{u_1, v_1\} \mid \exists (P.(t11).4) \setminus p_1(S) = 0\) makes both \((P.(t10).4)\) and \((P.(t8).4)\) impossible. The only type remaining is (t11) and all the properties \((P.(t11).1)\)–\((P.(t11).4)\) are indeed satisfied. It follows that if \((C.(m6.t5).1), \exists \exists \{u_1, v_1\} \setminus p_1(S) = 0\) makes both \((P.(t10).4)\) and \((P.(t8).4)\) impossible. Moreover, since throughout \((C.(m6.t5).1)\) we also have \((25)\), in particular \(\{u_2, v_2\} \cap p_2(S) = \emptyset\), it follows that \((P.(t19).4)\) is impossible. The only type remaining is (t8) and all the properties \((P.(t8).1)\)–\((P.(t8).4)\) are indeed satisfied; notice in particular that in
(P.(t8).4) the first of the two mutually exclusive clauses is true). If follows that if (C.(m6.t5).1).(2), then there are exactly \(8 \cdot ((n-1)-2)^2\) realizations of type (t8) by \(B \mid \{u,v\}\). This completes our reexamination of (C.(m6.t5).1).

We now reexamine (C.(m6.t5).2). The information defining (C.(m6.t5).2) is by itself not yet sufficient to rule out any of the four types (t8)–(t11). The information defining (C.(m6.t5).2).(1), more specifically \(u_1 = v_1\), makes both (P.(t8).3) and (P.(t9).3) impossible, still leaving two types. We argued in (C.(m6.t5).2).(1) that we have \(u_2 \neq v_2\) and \(\{|u_2, v_2\} \cap p_2(S)\) = 1 in this case. If the latter is true as \(u_2 \in p_2(S)\), then \(v_2 \notin p_2(S)\), and combining this information with \(v_1 \notin p_1(S)\) (which we know since we are in (C.(m6.t5).2),(1)) makes the second clause of the conjunction (P.(t11).4) impossible. If on the other hand it is true as \(v_2 \in p_2(S)\), then \(u_2 \notin p_2(S)\), and combining this with \(u_1 \notin p_1(S)\) (which again we know since we are in (C.(m6.t5).2),(1)) makes (P.(t11).4) impossible (this time, the first clause). This rules out type (t11). The only type remaining is (t10) (and all the properties (P.(t10).1)–(P.(t10).4) are indeed satisfied; note that due to \(|\{u_2, v_2\} \cap p_2(S)\) = 1 the two clauses of (P.(t10).4) are mutually exclusive, the second being true if \(u_2 \in p_2(S)\) and the first if \(v_2 \in p_2(S)\). It follows that in case (1) of (C.(m6.t5).2) there are exactly \(8 \cdot ((n-1)-2)^2\) realizations of type (t10) by \(B \mid \{u,v\}\).

The information defining (C.(m6.t5).2).(2) is not enough to rule out any of the four types (t8)–(t11), and descending one level deeper to (C.(m6.t5).2).(2),(1) does not change this.

If (C.(m6.t5).2).(2),(1),(1), then the decision still cannot be made and depends on the (anonymous) subcases which we distinguished in that case, namely whether the first or the second clause of (28) is true:

If (C.(m6.t5).2).(2),(1),(1), and the first clause of (28) is true, then in particular we know that \(v_1 \notin p_1(S)\) and \(u_2 = v_2 \notin p_2(S)\). The latter contradicts (P.(t8).3) and (P.(t9).3). Moreover, \(v_1 \notin p_1(S)\) and \(v_2 \notin p_2(S)\) combined render the second clause of the conjunction (P.(t11).4) false. Note that for each of three discarded types we used in whole or in part the information \(u_2 = v_2 \notin p_2(S)\), which defines the present subcase. Hence deferring any decision about the types for so so long was necessary. We are now left with only the type (t10) (and indeed the properties (P.(t10).1)–(P.(t10).4) are all satisfied). Since within (C.(m6.t5).2),(2),(1),(1) we found that in this situation there are exactly \((n-1-a_1-1) \cdot ((n-1)-2)\) realizations of type (t5) by \(B \mid \{u,v\}\), it follows that here there are exactly \(4 \cdot ((n-1-a_1-1) \cdot ((n-1)-2)\) realizations of type (t10).

If (C.(m6.t5).2),(2),(1),(1) the second clause of (28) is true, then we know that \(u_1 \neq v_1\), \(u_1 \in p_1(S)\), \(v_1 \notin p_1(S)\), \(u_2 \neq v_2\), \(u_2 \notin p_2(S)\) and \(v_2 \in p_2(S)\). We can now rule out three types: properties \(v_1 \notin p_1(S)\) and \(u_2 \notin p_2(S)\) combined render both clauses of the disjunction (P.(t8).4) false. Properties \(u_1 \in p_1(S)\) and \(u_2 \in p_2(S)\) combined render both clauses of the disjunction (P.(t10).4) false. Properties \(u_1 \neq v_1\) and \(u_2 \neq v_2\) proves (P.(t11).3) to be false. Again note that in all three decisions we used the information defining the present subcase. The only type remaining now is (t9), and indeed all properties (P.(t9).1)–(P.(t9).4) are satisfied (in (P.(t9).4) only the first clause of the disjunction). Since in (C.(m6.t5).2),(2),(1),(1) we found that in this situation there are exactly \((n-1-a_1-1) \cdot 2 \cdot ((n-1)-2)\) realizations of type (t5) by \(B \mid \{u,v\}\), it follows that here there are exactly \(8 \cdot ((n-1-a_1-1) \cdot ((n-1)-2)\) realizations of type (t9).

If (C.(m6.t5).2),(2),(1),(2), then again we have to distinguish whether the first or the second clause of (28) is true to reach a conclusion:

If (C.(m6.t5).2),(2),(1),(2) and the first clause of (28) is true, then again we in particular know that \(v_1 \notin p_1(S)\) and \(u_2 = v_2 \notin p_2(S)\), and therefore an argument analogous to the one given for the first subcase of (C.(m6.t5).2),(2),(1),(1) shows that in the present situation there are exactly \(4 \cdot (n-1-c_1) \cdot (n-1-2)\) realizations of type (t10) by \(B \mid \{u,v\}\) and no other type possible.

If (C.(m6.t5).2),(2),(1),(2) and the second clause of (28) is true, then again we know that \(u_1 \neq v_1\), \(u_1 \in p_1(S)\), \(v_1 \notin p_1(S)\) and \(u_2 \neq v_2\), but this time we have \(u_2 \notin p_2(S)\) and \(v_2 \in p_2(S)\). We can now
rule out three types: properties \( v_1 \notin p_1(S) \) and \( u_2 \notin p_2(S) \) combined render both clauses of the disjunction \((P.(t8).4)\) false. Properties \( u_1 \in p_1(S) \) and \( v_2 \in p_2(S) \) combined render both clauses of the disjunction \((P.(t10).4)\) false. Properties \( u_1 \neq v_1 \) and \( u_2 \neq v_2 \) contradict \((P.(t11).3)\). What remains is type \((t9)\), and all properties \((P.(t9).1)\)–\((P.(t9).4)\) are satisfied (in \((P.(t9).4)\) only the first clause of the disjunction). Since in \((C.(m6.t5).2).2).1)\) we found that in this situation there are exactly \(2 \cdot (n - 1 - c_1) \cdot ((n - 1) - 2)\) realizations of type \((t5)\) by \(B_{\{u,v\}}\), it follows that now there are exactly \(8 \cdot (n - 1 - c_1) \cdot ((n - 1) - 2)\) realizations of type \((t9)\).

The next case to reexamine is \((C.(m6.t5).2).2).2)\) which again does not give enough information to decide about the types.

If \((C.(m6.t5).2).2).2).1)\), then the decision still cannot be made and once more depends on the nameless subcases that were distinguished in that case, namely whether the first or the second clause of \((28)\) is true:

If \((C.(m6.t5).2).2).2).1)\), and the first clause of \((28)\) is true, then we know that \( u_1 \neq v_1, u_1 \notin p_1(S), v_1 \in p_1(S) \) and \( u_2 = v_2 \notin p_2(S) \). The latter, more specifically \( u_2 = v_2 \), contradicts both \((P.(t8).3)\) and \((P.(t9).3)\). Combining \( u_1 \notin p_1(S) \) and \( u_2 \notin p_2(S) \) proves the first clause of the conjunction \((P.(t11).4)\) to be false. The only type remaining is \((t10)\), and all properties \((P.(t10).1)\)–\((P.(t10).4)\) are indeed satisfied (with only the first clause of the disjunction \((P.(t10).4)\) being true). Since in \((C.(m6.t5).2).2).2).1)\) we found that there are exactly \((a_1 - 1) \cdot ((n - 1) - 2)\) realizations of type \((t5)\) by \(B_{\{u,v\}}\), it follows that in the present situation there are exactly \(4 \cdot (a_1 - 1) \cdot ((n - 1) - 2)\) realizations of type \((t10)\).

If \((C.(m6.t5).2).2).2).1)\), and the second clause of \((28)\) is true, then we know that \( u_1 \neq v_1, u_1 \notin p_1(S), v_1 \in p_1(S) \) and \( u_2 \notin p_2(S) \) and \( v_2 \notin p_2(S) \). Since then \( u_2 \neq v_2 \) and \( u_1 \neq v_1 \), both \((P.(t10).3)\) and \((P.(t11).3)\) are impossible. Moreover, combining \( u_1 \notin p_1(S) \) and \( v_2 \notin p_2(S) \) shows that both clauses of the disjunction \((P.(t8).4)\) are false. The remaining type is \((t9)\) and all the properties \((P.(t9).1)\)–\((P.(t9).4)\) are indeed satisfied (as to the disjunction \((P.(t9).4)\), only its second clause is true). Since in \((C.(m6.t5).2).2).2).1)\) we found exactly \(2 \cdot (a_1 - 1) \cdot ((n - 1) - 2)\) realizations of type \((t5)\) by \(B_{\{u,v\}}\), it follows that right now there are exactly \(8 \cdot (a_1 - 1) \cdot ((n - 1) - 2)\) realizations of type \((t9)\).

If \((C.(m6.t5).2).2).2).2)\), then one more time we have to distinguish the anonymous subcases. If \((C.(m6.t5).2).2).2).2)\), and the first clause of \((28)\) is true, then we know \( u_1 \neq v_1, u_1 \notin p_1(S), v_1 \in p_1(S) \) and \( u_2 = v_2 \notin p_2(S) \), with \( u_2 = v_2 \) ruling out both \((t8)\) and \((t9)\). Moreover, combining \( u_1 \notin p_1(S) \) with \( u_2 \notin p_2(S) \) proves the first clause of the conjunction \((P.(t11).4)\) to be false. Again, for each decision the information in the first clause of \((28)\) was used. Now only \((t10)\) is left and indeed all properties \((P.(t10).1)\)–\((P.(t10).4)\) are true (with the disjunction \((P.(t10).4)\) satisfied only by way of its second clause). Since in \((C.(m6.t5).2).2).2).2)\) we found that in the first subcase there are exactly \((c_1 - 2) \cdot ((n - 1) - 2)\) realizations of type \((t5)\) by \(B_{\{u,v\}}\), it follows for our present situation that there are exactly \(4 \cdot (c_1 - 2) \cdot ((n - 1) - 2)\) realizations of \((t10)\) by \(B_{\{u,v\}}\).

If \((C.(m6.t5).2).2).2).2)\), and the second clause of \((28)\) is true, then we know \( u_1 \neq v_1, u_1 \notin p_1(S), v_1 \in p_1(S) \) and \( u_2 \notin p_2(S) \) and \( v_2 \notin p_2(S) \). Since then \( u_2 \neq v_2 \) it follows \( \{u_1,u_2\} \cap \{v_1,v_2\} = \emptyset \), contradicting both \((P.(t10).3)\) and \((P.(t11).3)\). Combining \( u_1 \notin p_1(S) \) and \( v_2 \notin p_2(S) \) we see that both clauses of the disjunction \((P.(t8).4)\) are false. We are left with type \((t9)\) and indeed, all properties \((P.(t9).1)\)–\((P.(t9).4)\) are satisfied (for the disjunction \((P.(t9).4)\) it is only the second clause, which is). Since in \((C.(m6.t5).2).2).2).2)\) we found that there exist exactly \(2 \cdot (c_1 - 2) \cdot ((n - 1) - 2)\) realizations of type \((t5)\) by \(B_{\{u,v\}}\), it follows that in the present situation there are exactly \(8 \cdot (c_1 - 2) \cdot ((n - 1) - 2)\) realizations of type \((t9)\) by \(B_{\{u,v\}}\).

The next case to reexamine is \((C.(m6.t5).3)\). This case is symmetric to \((C.(m6.t5).1)\) under swapping the subscripts 1 and 2. Therefore, we can analyse its subcases by reexamining the analysis
of (C.(m6.t5).1) with this swap in mind. First of all, if (C.(m6.t5).3), then \( \{u_2, v_2\} \subseteq p_2(S) \), and this renders both clauses of the disjunction \( (P.(t10).4) \) false.

Reading (C.(m6.t5).1),(1) this way implies that we know \( u_2 = v_2 \in p_2(S) \), and \( u_2 = v_2 \) contradicts both \( (P.(t8).3) \) and \( (P.(t9).3) \). The only type remaining is (t11) and indeed, all properties \( (P.(t11).1) \sim (P.(t11).4) \) are satisfied. Since in (C.(m6.t5).1),(1) we found that there are exactly \( 2 \cdot \binom{n-1}{2} \) realizations of type (t5) by \( B|_{\{u,v\}} \), it follows that there exist exactly \( 8 \cdot \binom{n-1}{2} \) realizations of type (t11) by \( B|_{\{u,v\}} \).

Reading (C.(m6.t5).1),(2) this way implies that we know \( u_2 \neq v_2 \), \( \{u_2, v_2\} \cap p_2(S) = \emptyset \), \( u_1 \neq v_1 \) and \( \{u_1, v_1\} \subseteq p_1(S) \). The properties \( u_1 \neq v_1 \) and \( u_2 \neq v_2 \) taken together contradict both \( (P.(t10).3) \) and \( (P.(t11).3) \). The property \( \{u_2, v_2\} \cap p_2(S) = \emptyset \) alone renders both clauses of the disjunction \( (P.(t9).4) \) false. What we are left with is type (t8) and indeed all properties \( (P.(t8).1) \sim (P.(t8).4) \) are satisfied. Since in (C.(m6.t5).1),(2) we found that there are exactly \( 2 \cdot \binom{n-1}{2} \) realizations of type (t5) by \( B|_{\{u,v\}} \), it follows by symmetry that here, too, there exist exactly \( 8 \cdot \binom{n-1}{2} \) realizations of type (t8) by \( B|_{\{u,v\}} \).

We have now completed reexamining the analysis of (m6.t5) and we may now add up (separately for each of the four types (t8)–(t11) the number of realizations we found during the reexamination.

For (t8) we found exactly \( 8 \cdot \binom{n-1}{2} + 8 \cdot \binom{n-1}{2} = 16 \cdot \binom{n-1}{2} \) realizations by \( B|_{\{u,v\}} \).

Now (m6.t8) is proved.

For (t9) we found exactly \( 8 \cdot (n-1-a_1-1) \cdot ((n-1) - 2) + 8 \cdot (n-1-c_1) \cdot ((n-1) - 2) + 8 \cdot (a_1-1) \cdot ((n-1) - 2) + 8 \cdot (c_1-2) \cdot ((n-1) - 2) = 16 \cdot ((n-1) - 2)^2 \) realizations by \( B|_{\{u,v\}} \).

Now (m6.t9) is proved.

For (t10) we found exactly \( 8 \cdot ((n-1) - 2)^2 + 4 \cdot (n-1-a_1-1) \cdot ((n-1) - 2) + 4 \cdot (n-1-c_1) \cdot ((n-1) - 2) + 4 \cdot (a_1-1) \cdot ((n-1) - 2) + 4 \cdot (c_1-2) \cdot ((n-1) - 2) = 8 \cdot ((n-1) - 2)^2 + 8 \cdot ((n-1) - 2)^2 \) realizations by \( B|_{\{u,v\}} \).

Now (m6.t10) is proved.

For (t11) we found exactly \( 8 \cdot \binom{n-1}{2} + 8 \cdot \binom{n-1}{2} = 16 \cdot \binom{n-1}{2} \) realizations by \( B|_{\{u,v\}} \).

Now (m6.t11) is proved.

As to (m6.t13), let us first note that Definition 12 implies:

**Lemma 49.** For every \( B \in \{0,\pm\}^I \) with \( I \subseteq \binom{\{u,v\}}{2} \), \( I = S \cup \{u,v\} \) and \( X_B|_{\{u,v\}} \cong C^4 \) we have \( X_B \cong (t13) \) if and only if

\[
(P.(t13).1) \quad B[u] = B[v] = 0, \quad (P.(t13).2) \quad |\{u_1, v_1\} \cap p_1(S)| + |\{u_2, v_2\} \cap p_2(S)| = 3.
\]

We now distinguish cases according to how \((P.(t13).2))\) is satisfied.

(C.(m6.t13).1) \( |\{u_1, v_1\} \cap p_1(S)| = 0 \). Then \( |\{u_2, v_2\} \cap p_2(S)| = 3 \) by \((P.(t13).2))\), which is impossible. Hence Case 1 does not occur.

(C.(m6.t13).2) \( |\{u_1, v_1\} \cap p_1(S)| = 1 \). Then \( |\{u_2, v_2\} \cap p_2(S)| = 2 \) by \((P.(t13).2))\), which is equivalent to \( u_2 \neq v_2 \) and \( \{u_2, v_2\} \subseteq p_2(S) = \emptyset \).

Since the condition defining \((C.(m6.t13).2))\) is equivalent to

\[
(u_1 = v_1 \text{ and } \{u_1, v_1\} \cap p_1(S) = \emptyset) \text{ or } (u_1 \neq v_1 \text{ and } |\{u_1, v_1\} \cap p_1(S)| = 1),
\]

there are two further cases.

1. \( u_1 = v_1 \text{ and } \{u_1, v_1\} \cap p_1(S) = \emptyset \). Then there are exactly \((n-1) - 2\) such \( u_1 = v_1 \).

Combining \((25))\) with \( u < v \) it follows that \( u_2 < v_2 \), therefore in the present case each of the \( \binom{n-1}{2} \) different sets \( \{u_2, v_2\} \) satisfying \((25))\) determines the two pairs \( u \) and \( v \). Therefore there are exactly \((n-1) - 2 \cdot \binom{n-1}{2} \) realizations of \((1))\).

2. \( u_1 \neq v_1 \text{ and } |\{u_1, v_1\} \cap p_1(S)| = 1 \). From \( u_1 \neq v_1 \) and the assumption \( u < v \) it follows that \( u_1 < v_1 \). By \((25))\), \( u_2 < v_2 \) or \( v_2 < u_2 \), but nothing more is known about \( u_2 \) and \( v_2 \). Therefore, both possibilities must be taken into account. Because of
\( p_1(S) = \{a_1, b_1, c_1, d_1\} = \{a_1, c_1\} \) and \( u_1 < v_1 \) there are exactly four possibilities for 
\(|\{u_1, v_1\} \cap p_1(S)| = 1 \) to be true:

1. \( u_1 = a_1 = b_1 \). Then because of \( u_1 < v_1 \) and \( v_1 \neq c_1 = d_1 \) it follows that there are exactly \( n - 1 - a_1 - 1 \) different \( v_1 \) with \( v_1 \notin p_1(S) \) in this case. For each of them, there exist exactly \( \binom{n - 1 - 2}{2} \) different \( \{u_2, v_2\} \) satisfying (25). Now the two pairs \( u \) and \( v \) are not determined by them: each of the sets can be realized in exactly two ways, both by \( u_2 < v_2 \) and by \( v_2 < u_2 \). Therefore, there are exactly 
\((n - 1 - a_1 - 1) \cdot 2 \cdot \binom{n - 1 - 2}{2}\) realizations of (1) by \( u \) and \( v \).

2. \( u_1 = c_1 = d_1 \). Then because of \( u_1 < v_1 \) it follows that there are exactly \( n - 1 - c_1 \) different \( v_1 \) with \( v_1 \notin p_1(S) \). As in the preceding case, for each of these \( v_1 \) there exist exactly \( \binom{n - 1 - 2}{2} \) different \( \{u_2, v_2\} \) satisfying (25), hence exactly
\( 2 \cdot \binom{n - 1 - 2}{2} \) different \( u \) and \( v \). Therefore there exist exactly \((n - 1 - c_1) \cdot 2 \cdot \binom{n - 1 - 2}{2}\) different realizations of type (t13) by \( B \mid \{u,v\} \).

3. \( v_1 = a_1 = b_1 \). Then because of \( u_1 < v_1 \) it follows that there are exactly \( a_1 - 1 \) different \( u_1 \) with \( u_1 \notin p_1(S) \) in this case. For the same reasons as in the preceding two cases we know that here there exist exactly \((a_1 - 1) \cdot 2 \cdot \binom{n - 1 - 2}{2}\) different realizations of type (t13) by \( B \mid \{u,v\} \).

4. \( v_1 = c_1 = d_1 \). Then because of \( u_1 < v_1 \) and \( u_1 \neq a_1 = b_1 \) it follows that there are \( c_1 - 1 - 1 \) different \( u_1 \) with \( u_1 \notin p_1(S) \) in this case. For the same reasons as in the preceding three cases we know that here there exist exactly 
\((c_1 - 1 - 1) \cdot 2 \cdot \binom{n - 1 - 2}{2}\) different realizations of type (t13) by \( B \mid \{u,v\} \).

It follows that if (C.(m6.t13),2), then there exist exactly \((n - 1 - a_1 - 1) + (a_1 - 1) + (c_1 - 1 - 1)) \cdot 2 \cdot \binom{n - 1 - 2}{2} = 4 \cdot (n - 1 - 2) \cdot \binom{n - 1 - 2}{2}\) different realizations of type (t13) by \( B \mid \{u,v\} \).

It follows that if (C.(m6.t13),2), then there exist exactly \((n - 1 - 2 + 4 \cdot (n - 1 - 2)) \cdot \binom{n - 1 - 2}{2}\) different realizations of type (t13) by \( B \mid \{u,v\} \).

\[(C.(m6.t13),3) \mid\{u_1, v_1\}\} \setminus p_1(S) = 2. \text{ This is equivalent to}
\begin{equation}
\{u_1, v_1\} \cap p_1(X) = \emptyset . \tag{27}
\end{equation}

Equation (P.(t13),2) implies \(|\{u_2, v_2\} \setminus p_2(S)| = 1\), which is equivalent to 
\[ (u_2 = v_2 \text{ and } \{u_2, v_2\} \cap p_2(S) = \emptyset) \text{ or } (u_2 \neq v_2 \text{ and } \{u_2, v_2\} \cap p_2(S) = 1) . \tag{28}
\]

By swapping the subscripts 1 and 2 in the argument given for Case 2 it now follows that if (C.(m6.t13),3), then there are exactly \(5 \cdot (n - 1 - 2) \cdot \binom{n - 1 - 2}{2}\) different realizations of type (t13) by \( B \mid \{u,v\} \).

\[(C.(m6.t13),4) \mid\{u_1, v_1\}\} \setminus p_1(S) = 3. \text{ This is impossible, hence } (C.(m6.t13),4) \text{ does not occur.}

It follows that for every fixed \( S \) there are exactly \(10 \cdot (n - 1 - 2) \cdot \binom{n - 1 - 2}{2}\) possibilities to position the two zeros indexed by \(I \setminus S\) such that \(X_B \cong (t13)\). This completes the proof of (m6.t13).

As to (m6.t14)–(m6.t17) we begin by noting that for each of the four isomorphism types (t14)–(t17), a necessary condition is that \(|V(X_B) \setminus V(X_{B|S})| = 3\). We can therefore prove (m6.t14)–(m6.t17) during one reexamination of the proof of (m6.t13).

Since (C.(m6.t13),1) and (C.(m6.t13),4) are impossible, we only have to consider (C.(m6.t13),2) and (C.(m6.t13),3). If (C.(m6.t13),2), we know (25) but this is not sufficient to rule out any of the types (t14)–(t17).

If (C.(m6.t13),2), we know that 
\[ u_1 = v_1, \{u_1, v_1\} \cap p_1(S) = \emptyset, u_2 \neq v_2, \{u_2, v_2\} \cap p_2(S) = \emptyset , \tag{29} \]

and will now consider the consequences of this for (m6.t14)–(m6.t17).
(1) Concerning contributions to (m6.t14), note that properties \( \{u_1, v_1\} \cap p_1(S) = \emptyset \) and \( \{u_2, v_2\} \cap p_2(S) = \emptyset \) make an edge intersecting \( X_{B|S} \cong C^4 \) impossible, hence the case \((C.(m6.t13).2),(1)\) does not contribute to \( |\mathcal{B}^{X^n,n,n}_{6}^{-1}(t14)| \).

(2) Concerning contributions to (m6.t15), note that (29) implies that \( X_B \cong (t15) \) if and only if either \( (B[u] \in \{\pm\} \) and \( B[v] = 0 \) or \( (B[u] = 0 \) and \( B[v] \in \{\pm\} \)). Each of these clauses corresponds to 2 different \( B \). It follows that if \((C.(m6.t13).2),(1)\), then there are 4-times as many realizations of type \((t15)\) as there are of type \((t13)\). Therefore, if \((C.(m6.t13).2),(1)\), there are exactly \( 4 \cdot ((n - 1) - 2) \cdot \binom{n-3}{2} \) realizations of type \((t15)\) by \( B \mid \{u,v\} \).

(3) Concerning contributions to (m6.t16), since properties \( \{u_1, v_1\} \cap p_1(S) = \emptyset \) and \( \{u_2, v_2\} \cap p_2(S) = \emptyset \) make an edge intersecting \( X_{B|S} \cong C^4 \) impossible, the case \((C.(m6.t13).2),(1)\) does not contribute to \( (m6.t16) \).

(4) Concerning contributions to (m6.t17), we see from (29) that \( X_B \cong (t17) \) if and only if \( (B[u] \in \{\pm\} \) and \( B[v] \in \{\pm\} \), and there are 4 different \( B \mid \{u,v\} \) satisfying this. Therefore, if \((C.(m6.t13).2),(1)\), there are exactly \( 4 \cdot ((n - 1) - 2) \cdot \binom{n-3}{2} \) realizations of type \((t17)\) by \( B \mid \{u,v\} \).

If \((C.(m6.t13).2),(2)\), then we know
\[
u_1 \neq v_1, \ |\{u_1, v_1\} \cap p_1(S)| = 1, \ u_2 \neq v_2, \ |\{u_2, v_2\} \cap p_2(S)| = 0. \tag{30}
\]
and will now consider the consequences of this for \((m6.t14)\)–\((m6.t17)\).

(1) Concerning contributions to (m6.t14), we can argue as follows: If \( |\{u_1, v_1\} \cap p_1(S)| = 1 \) is true as \( u_1 \in p_1(S) \), then there are exactly two \( B \mid \{u,v\} \) with \( X_B \cong (t14) \), namely those which satisfy \( (B[u] \in \{\pm\} \) and \( B[v] = 0 \). If it is true as \( v_1 \in p_1(S) \), then again there are exactly two such \( B \mid \{u,v\} \), namely those which satisfy \( (B[u] = 0 \) and \( B[v] \in \{\pm\} \)). It follows that without having to reexamine the subcases \((C.(m6.t13).2),(2),(1)-(C.(m6.t13).2),(2),(1)\) we know that there are twice as many realizations of \((t14)\) by \( B \mid \{u,v\} \) in the case \((C.(m6.t13).2),(2)\) than of \((m6.t13)\). Therefore, if \((C.(m6.t13).2),(2)\), then there are exactly \( 8 \cdot ((n - 1) - 2) \cdot \binom{n-3}{2} \) realizations of \((t14)\) by \( B \mid \{u,v\} \).

(2) Concerning contributions to (m6.t15), we have to distinguish in what way property \( |\{u_1, v_1\} \cap p_1(S)| = 1 \) in (30) is satisfied. If \( u_1 \in p_1(S) \) but \( v_1 \notin p_1(S) \), then \( X_B \cong (m6.t15) \) if and only if \( B[u] = 0 = B[v] \in \{\pm\} \), in this case there exist 2 different \( B \mid \{u,v\} \) with \( X_B \cong (m6.t15) \). If \( u_1 \notin p_1(S) \) but \( v_1 \in p_1(S) \), then \( X_B \cong (m6.t15) \) if and only if \( B[u] \in \{\pm\} \) and \( B[v] = 0 \) in this case there again exist 2 different \( B \mid \{u,v\} \) with \( X_B \cong (m6.t15) \). It follows that if \((C.(m6.t13).2),(2)\), then there are 2-times as many realizations of \((t15)\) than of \((t13)\) by \( B \mid \{u,v\} \). Therefore, if \((C.(m6.t13).2),(2)\), then there are exactly \( 2 \cdot 4 \cdot ((n - 1) - 2) \cdot \binom{n-3}{2} = 8 \cdot (n - 3) \cdot \binom{n-3}{2} \) realizations of \((t15)\) by \( B \mid \{u,v\} \).

(3) Concerning contributions to (m6.t16), note that no matter how (30) is satisfied, we have \( X_B \cong (m6.t16) \) if and only if \( B[u] \in \{\pm\} \) and \( B[v] \in \{\pm\} \). Hence, if \((C.(m6.t13).2),(2)\), then there are 4-times as many realizations of \((m6.t16)\) than there are of \((m6.t13)\), that is, if \((C.(m6.t13).2),(2)\), then there are exactly \( 4 \cdot 4 \cdot ((n - 1) - 2) \cdot \binom{n-3}{2} = 16 \cdot (n - 3) \cdot \binom{n-3}{2} \) realizations of type \((m6.t16)\) by \( B \mid \{u,v\} \).

(4) Concerning contributions to (m6.t17), note that (30) says that \( u_1 \neq v_1 \) and \( u_2 \neq v_2 \), and this makes it impossible to create a 2-path outside of \( X_{B|S} \cong C^4 \). Therefore, if \((C.(m6.t13).2),(2)\), there is no contribution to \((m6.t17)\).

---

5 The fact that neither \((C.(m6.t13).2),(1)\) nor the corresponding subcase of \((C.(m6.t13).3)\) (which due to symmetry was not spelled out in the proof of \((m6.t13)\) and therefore does not have a name) contribute to \(|\mathcal{B}^{X^n,n,n}_{6}^{-1}(t14)|\) is a reason why \( |\mathcal{B}^{X^n,n,n}_{6}^{-1}(t14)| \) is larger but not twice as large as \( |\mathcal{B}^{X^n,n,n}_{6}^{-1}(t13)\) even though in the cases where \((t14)\) can be realized the number of realizations is twice as large as for \((t13)\).
We now take stock of what we found in the subcases (C.(m6.t13).2),(1) and (C.(m6.t13).2),(2) in order to know what the entire case (C.(m6.t13).2) contributes to (m6.t14)–(m6.t17).

Since (C.(m6.t13).2),(1) did not contribute to \([u|X^{n,n,n}|^{-1}(14)]\) but (C.(m6.t13).2),(2) did contribute 8 \(\cdot (n-3)\cdot \binom{n-5}{3}\), it follows that if (C.(m6.t13).2), then there are exactly 8 \(\cdot (n-3)\cdot \binom{n-5}{3}\) realizations of type (t14) by \(B \{u,v\}\).

Since (C.(m6.t13).2),(1) contributed 4 \(\cdot (n-3)\cdot \binom{n-5}{3}\) to \([u|X^{n,n,n}|^{-1}(15)]\) while (C.(m6.t13).2),(2) contributed 8 \(\cdot (n-3)\cdot \binom{n-5}{3}\), it follows that if (C.(m6.t13).2), then there are exactly 12 \(\cdot (n-3)\cdot \binom{n-5}{3}\) realizations of type (t15) by \(B \{u,v\}\).

Since (C.(m6.t13).2),(1) did not contribute to \([u|X^{n,n,n}|^{-1}(16)]\) but (C.(m6.t13).2),(2) did contribute 16 \(\cdot (n-3)\cdot \binom{n-5}{3}\), it follows that if (C.(m6.t13).2), then there are exactly 16 \(\cdot (n-3)\cdot \binom{n-5}{3}\) realizations of type (t16) by \(B \{u,v\}\).

Since (C.(m6.t13).2),(1) contributed 4 \(\cdot (n-3)\cdot \binom{n-5}{3}\) to \([u|X^{n,n,n}|^{-1}(17)]\) while (C.(m6.t13).2),(2) did not contribute anything, it follows that if (C.(m6.t13).2), then there are exactly 4 \(\cdot (n-3)\cdot \binom{n-5}{3}\) realizations of type (t17) by \(B \{u,v\}\).

Since the case (C.(m6.t13).3) is symmetric to the case (C.(m6.t13).2) via interchanging the subscripts 1 and 2, we will get the same contributions to (m6.t14)–(m6.t17) as in the case (C.(m6.t13).2). We therefore have to double each of the four results found for (C.(m6.t13).2) to get the correct numbers of realizations of types (t14)–(t17). This proves (m6.t14)–(m6.t17).

As to (m6.t18), it suffices to note that the two values of \(B \{u,v\}\) are determined: since there does not exist an edge outside \(X_{B|s} \cong C^4\), they both must be zero. Therefore (m6.t18) is the number of \(\{u,v\} \in \binom{[n-1]}{2}\) such that \(X_{B|s\cup\{u,v\}} \cong (t18)\). By definition of \(S\) the latter is equivalent to saying that \(X_{B|s\cup\{u,v\}}\) has exactly eight vertices. It follows from Definition 12 that this is the case if and only if simultaneously

\[
|\{u_1,v_1\} \setminus p_1(S)| = 2 \quad \text{and} \quad |\{u_2,v_2\} \setminus p_2(S)| = 2 .
\]  

(31)

Due to \(u_1 < v_1\), the number of \(\{u_1,v_1\} \subseteq [n-1]\) with \(|\{u_1,v_1\} \setminus p_1(S)| = 2\) is \(\binom{n-3}{2}\). Since we only assume \(u < v\) and hence both \(u_2 < v_2\) and \(u_2 > v_2\) are possible, for each of these \(\binom{n-3}{2}\) different \(\{u_1,v_1\}\) there are 2 \(\cdot \binom{n-3}{2}\) different \(\{u_2,v_2\}\) with \(|\{u_2,v_2\} \setminus p_2(S)| = 2\). This proves (m6.t18).

As to (m6.t19) and (m6.t20), note that for both isomorphism types (t19) and (t20) it is necessary that \(X_{B|s\cup\{u,v\}} \cong (t18)\). We can therefore prove (m6.t19) and (m6.t20) by reexamining the proof of (m6.t18). In whatever way (31) is satisfied, there are exactly 4 different \(B \{u,v\}\) with \(X_B \cong (t19)\), namely those satisfying

\[
(B[u] \in \{\pm\} \text{ and } B[v] = 0) \quad \text{or} \quad (B[u] = 0 \text{ and } B[v] \in \{\pm\}) .
\]  

(32)

but there are also 4 different \(B \{u,v\}\) with \(X_B \cong (t20)\), namely those satisfying

\[
B[u] \in \{\pm\} \text{ and } B[v] \in \{\pm\} .
\]  

(33)

This proves both \(|u|X^{n,n,n}|^{-1}(19)| = |u|X^{n,n,n}|^{-1}(20)| = 4 \cdot |u|X^{n,n,n}|^{-1}(18)|\], and therefore both (m6.t19) and (m6.t20). The proof of (QFa6) is now complete. 

The relations (11)–(15) in Lemma 43 give us a plausibility check (i.e. necessary conditions) for the explicit formulas \([u|X^{n,n,n}|^{-1}(22)]\), \ldots , \([u|X^{n,n,n}|^{-1}(20)]\) that we found in (m5.t2)–(m6.t20). For brevity let \(x := n-3\) and \(y := \binom{n-3}{2}\). Then, indeed, the explicit formulas that we found in (QFa5) and (QFa6) pass the test: the formulas in (QFa5) evidently satisfy (11) and (12). Moreover, since \((3^2 - 1) \cdot (8x^2 + 8y) = 24x^2 + 32y + 8x^2 + 16y - 16x^2 + 16y^2 + 16y\), the formulas (m6.t5)–(m6.t11) satisfy (3). Since \((3^2 - 1) \cdot 10xy = 16xy + 24xy + 32xy + 8xy\), the formulas in (m6.t13)–(m6.t17) satisfy (14). Since \((3^2 - 1) \cdot 2y^2 = 8y^2 + 8y^2\), the formulas in (m6.t18)–(m6.t20) satisfy (15).
4.1. Counting failures of equality of $P_{\text{chio}}$ and $P_{\text{lcf}}$. While determining an absolute cardinality $|\{uX^{k,n}\}^{-1}(X)|$ seems to necessitate work specifically depending on the isomorphism type $X$, the ratio of all balanced matrix realizations to all realizations is easy to compute since it is determined by the Betti number of $X$ alone. This is the content of (E1) in the following lemma:

**Lemma 50.** For every $(s, t) \in \mathbb{Z}_{\geq 2}^2$, every $0 \leq k \leq (s - 1)(t - 1)$, every unlabelled bipartite graph $X$ and every $\beta \in \mathbb{Z}_{\geq 2}$.

(E1) $|\{B \in (uX^{k,s,t})^{-1}(X) : (X_B, \sigma_B) \text{ balanced} \}| = \left(\frac{1}{2}\right)^{\beta}(X) \cdot |(uX^{k,s,t})^{-1}(X)|$.

(E2) $|\mathcal{F}_0^M(k, s, t)| = \sum_{\beta \in \im(X^{k,s,t} : \beta(\beta) = \left(\frac{1}{2}\right)^{\beta}} |(uX^{k,s,t})^{-1}(X)|$.

(E3) $|\mathcal{F}_0^M(k, s, t)| = \sum_{\beta \in \im(X^{k,s,t})} |(uX^{k,s,t})^{-1}(X)|$.

**Proof.** If $M$ is a set of matrices, let us define $\text{Dom}(M) := \{\text{Dom}(B) : B \in M\}$ and $\text{Supp}(M) := \{\text{Supp}(B) : B \in M\}$. Moreover, if $S$ is a set, $S \subseteq \mathcal{P}(S)$ a set of subsets and $U \subseteq \mathcal{P}(S)$ a subset, then $U \cap S := \{U \cap S : S \in S\}$. Using these notations, we can prove (E1) by the following calculation: for every unlabelled $X$ we have $\{B \in (uX^{k,s,t})^{-1}(X) : (X_B, \sigma_B) \text{ balanced} \} = \sum_{\beta \in \im(X^{k,s,t})} |(uX^{k,s,t})^{-1}(X)|$.

(E2) this is true since $|\mathcal{F}_0^M(k, s, t)| = (C3) in Theorem 30 = \sum_{\beta \in \im(X^{k,s,t})} \beta(\beta) = \left(\frac{1}{2}\right)^{\beta}(X)$.

As to (E3), note that $|\mathcal{F}_0^M(k, s, t)| = (C1) in Theorem 30 = \sum_{\beta \in \im(X^{k,s,t})} \beta(\beta) = \left(\frac{1}{2}\right)^{\beta}(X)$.

The fewer the number $\text{dom}(B)$ of entries specified, the larger an entry-specification event $\mathcal{E}_B^{[n-1]}$ is (as a set). Any two probability measures by definition agree on the largest possible event, the entire sample space. The following theorem explores to what extent $P_{\text{chio}}$ and $P_{\text{lcf}}$ agree on successively smaller entry-specification events, descending down to as much as six specifications.

**Theorem 51.** (number of exceptions to equality of $P_{\text{chio}}$ and $P_{\text{lcf}}$ on large entry-specification events).

In the following statements let $\emptyset \subseteq I \subseteq [n-1]^2$, $B \in \{0, \pm 1\}$ and $\mathcal{E}_B := \mathcal{E}_B^{[n-1]}$.

(Ex3) For each of the $\sum_{0 \leq k < 3} 3^k \cdot \binom{n-1}{k} \sim 2^n \cdot n^6$ possible events $\mathcal{E}_B$ with $0 \leq \text{dom}(B) \leq 3$, it is true that $P_{\text{chio}}[\mathcal{E}_B] = P_{\text{lcf}}[\mathcal{E}_B] = \left(\frac{1}{2}\right)^{\text{dom}(B)+\text{supp}(B)}$.

(Ex4) Among the $3^4 \cdot \binom{n-1}{4} \sim 2^n \cdot n^6$ possible events $\mathcal{E}_B$ with $\text{dom}(B) = 4$, there are precisely $|\mathcal{F}_0^M(4, n)| = 2^4 \cdot \binom{n-1}{4} \sim 4 \cdot n^4$ events for which $P_{\text{chio}}[\mathcal{E}_B] = P_{\text{lcf}}[\mathcal{E}_B]$ does not hold.

Of these, we have $\frac{|\mathcal{F}_0^M(4, n)|}{|\mathcal{F}_0^M(4, n)|} = \left(\frac{1}{2}\right)$.

(Ex5) Among the $2^5 \cdot \binom{n-1}{5} \sim 2^n \cdot n^{10}$ different events $\mathcal{E}_B$ with $\text{dom}(B) = 5$, there are precisely $|\mathcal{F}_0^M(5, n)| = 2^5 \cdot \binom{n-1}{5} \sim 12 \cdot n^6$ events for which $P_{\text{chio}}[\mathcal{E}_B] = P_{\text{lcf}}[\mathcal{E}_B]$ does not hold.

Of these, we have $\frac{|\mathcal{F}_0^M(5, n)|}{|\mathcal{F}_0^M(5, n)|} = \left(\frac{1}{2}\right)$.

(Ex6) Among the $2^6 \cdot \binom{n-1}{6} \sim 2^n \cdot n^{12}$ different events $\mathcal{E}_B$ with $\text{dom}(B) = 6$, there are precisely $|\mathcal{F}_0^M(6, n)| = 2^6 \cdot \binom{n-1}{6} \sim 2^n \cdot n^{12}$ events for which $P_{\text{chio}}[\mathcal{E}_B] = P_{\text{lcf}}[\mathcal{E}_B]$ does not hold. Of these, we have $\frac{|\mathcal{F}_0^M(6, n)|}{|\mathcal{F}_0^M(6, n)|} = \left(\frac{1}{2}\right)$.
proves in Corollary 39 in Corollary 38 and the obvious Definition F.

As to (Ex3), this follows immediately from (Fa3) in Corollary 38. As to (Ex4), the claimed value of \(|F^M(4, n)|\) is true by (M4) in Corollary 42 combined with Lemma 25 and the obvious fact that \(|u^X4,n,n^{-1}(t1)| = \text{2}^4 \cdot |\text{Cir}(4, n)|\). The claimed ratios can be deduced as follows: note that \(\{X \in \im(u^X4,n,n): \beta_1(X) \geq 1\} = \{t1\}\) by Corollary 38, hence \(|F^M(4, n)| = \text{(by (E3))}\) = \(\frac{1}{2} |\text{F}^M(4, n)|\), which together with the equation \(F^M(4, n) = F^M_0(4, n) \cup F^M_2(4, n)\) from (R4) in Corollary 39 proves both \(|F^M_0(4, n)|/|F^M(4, n)| = \frac{1}{8}\) and \(|F^M_2(4, n)|/|F^M(4, n)| = \frac{1}{9}\).

As to (Ex5), the number stated first is obvious. The claimed value of \(|F^G(5, n)|\) can be deduced as follows: by (C3) in Theorem 30 we have \(P_{\text{chio}}[E_B] \neq P_{\text{ct}}[E_B]\) if and only if \(\beta_1(X_B) > 0\). Since due to Definition 12 we have \(0 \leq f_1(X_B) \leq \text{dom}(B) = |I| = 5\), it is easy to see that \(\beta_1(X_B) \leq 1\). Therefore \(P_{\text{chio}}[E_B] \neq P_{\text{ct}}[E_B]\) if and only if \(C^4 \leftrightarrow X_B\). The latter property is equivalent to the existence of a matrix-4-circuit \(S \subseteq I\) with \(S \subseteq \text{Supp}(B)\). Note that every \(I \in \binom{\binom{n-1}{5}}{}\) contains at most one matrix-4-circuit. Therefore, the number of all \(I \in \binom{\binom{n-1}{5}}{}\) with \(P_{\text{chio}}[E_B] \neq P_{\text{ct}}[E_B]\) is equal to the number of all matrix-4-circuits \(S \in \binom{\binom{n-1}{5}}{}\) with \(S \subseteq \text{Supp}(B)\), multiplied by the number of possibilities to choose an arbitrary position \(u \in \{n-1\}^2 \setminus S\) and an arbitrary \(B[u] \in \{0, \pm\}\), i.e. \(2^4 \cdot \binom{\binom{n-1}{2}}{4} \cdot (n-1)^2 - 4 = 48 \cdot (n-1)^2 - 4 = (n-1)^2 \cdot \left(\frac{n-1}{2}\right)^2\). This proves the second claim in (Ex5).

The claimed ratios can be deduced as follows: note that \(\{X \in \im(u^X5,n,n): \beta_1(X) \geq 1\} = \{t2, t3, t5, t7\}\) by Corollary 38, hence \(|F^M_0(5, n)| = \text{(by (E3))}\) = \(\left(1 - \frac{4}{5}\right)^{\beta_1(t2)} \cdot |u^X5,n,n^{-1}(t2)| + \left(1 - \frac{4}{5}\right)^{\beta_1(t3)} \cdot |u^X5,n,n^{-1}(t3)| + \left(1 - \frac{4}{5}\right)^{\beta_1(t5)} \cdot |u^X5,n,n^{-1}(t5)| + \left(1 - \frac{4}{5}\right)^{\beta_1(t7)} \cdot |u^X5,n,n^{-1}(t7)|\) by Corollary 42, which together with the equation \(F^M(5, n) = F^M_0(5, n) \cup F^M_2(5, n)\) from (R4) in Corollary 39 proves both \(|F^M_0(5, n)|/|F^M(5, n)| = \frac{1}{3}\) and \(|F^M_2(5, n)|/|F^M(5, n)| = \frac{1}{9}\).

As to (Ex6), the claimed value of \(|F^G(6, n)|\) can be deduced as follows: using (Fa6) in Corollary 38, and inspecting the list of isomorphism types in Lemma 37, we know that equality of the measures fails if and only \(C^4 \leftrightarrow X_B\) or \(C^6 \leftrightarrow X_B\). To count the events for which this is true let us define \(h_{c^n}(n) := |\{B \in \{0, \pm\}^7: I \in \binom{\binom{n-1}{6}}{6}\}|\), \(h_{K_{2,3}}(n) := |\{B \in \{0, \pm\}^7: I \in \binom{\binom{n-1}{6}}{6}\}|\), \(K^{2,3} \leftrightarrow X_B\) and \(h_{C^4 \leftrightarrow K_{2,3}}(n) := |\{B \in \{0, \pm\}^7: I \in \binom{\binom{n-1}{6}}{6}\}|\), \(C^4 \leftrightarrow X_B\), \(K^{2,3} \leftrightarrow X_B\)

Lemma 25 implies that \(h_{c^n}(n) = 2^6 \cdot \binom{\binom{n-1}{2}}{2}^2\), the factor of \(2^6\) accounting for the fact that the property \(C^6 \leftrightarrow X_B\) is indifferent to the choice of the six signs in \(B\). Moreover, evidently, \(h_{K^{2,3}}(n) = 2^6 \cdot \binom{\binom{n-1}{2}}{2}^2\), the first 2 accounts for the two possibilities \(|p_1(I)| = 2\) and \(|p_2(I)| = 3\) or \(|p_1(I)| = 3\) and \(|p_2(I)| = 2\).

In order to compute \(h_{c^4 \leftrightarrow K_{2,3}}(n)\), we will employ a simple inclusion-exclusion-argument: for any of the \(\binom{\binom{n-1}{2}}{2}\) choices \(1 \leq i_1 < i_2 \leq n-1\) and \(1 \leq j_1 < j_2 \leq n-1\) for the position of a matrix-4-circuit \(S = \{(i_1, j_1), (i_1, j_2), (i_2, j_1), (i_2, j_2)\}\), the number of distinct \(I \subseteq \{n-1\}^2\) with \(|I| = 6\) such that \(B\) has an matrix-4-circuit at least at the positions \((i_1, j_1), (i_1, j_2), (i_2, j_1), (i_2, j_2)\) is \(2^4 \cdot \binom{\binom{n-1}{2}}{2}^2\), the factor \(2^4\) accounts for the mandatory \((\pm)-values of the 4-circuit entries, the factor \(3^2\)
accounts for the arbitrary \( \{0, \pm\}\)-values of the two non-4-circuit-entries and the factor \( \frac{(n-1)^2}{2} \) accounts for the arbitrary positions of the two non-4-circuit-entries in \([n-1]^2 \setminus S\). Summing this expression over all \( \left(\frac{n-1}{2}\right)^2 \) possible choices of \( S \), we obtain \( h_{\geq}(n) := 2^4 \cdot \left(\frac{n-1}{2}\right)^2 \cdot 3^2 \cdot \left(\frac{(n-1)^2}{2}\right) \). But this is not \( h_{C^4, \sim K2,3}(n) \) yet: from the list in Lemma 37 we see that there is precisely one type which contains more than one copy of \( C^4 \), namely \( K2,3 \), which contains exactly three copies. Therefore, in \( h_{\geq}(n) \) every \( I \) with \( X_B \cong K2,3 \) has been counted exactly three times, and we did not overcount any of the realizations of the other isomorphism types. Thus, in order to arrive at \( h_{C^4, \sim K2,3}(n) \), we have to subtract three times \( h_{K2,3}(n) \). Hence, according to [19],

\[
h_{C^4, \sim K2,3}(n) = h_{\geq}(n) - 3 \cdot h_{K2,3}(n) = 18n^8 - 180n^7 + 612n^6 - 608n^5 - 774n^4 + 1348n^3 + 1200n^2 - 2864n + 1248. \tag{35}
\]

Substituting this into (34), one indeed arrives at the claimed value of \( |F^M(6, n)| \).

As to \( |F^M(6, n)| \), we can use (E2) to calculate \( |F^M(6, n)| = \sum_{\chi \in \text{im}(\mu_6, n^n)} \beta_1(\chi) = 1 \left(1 - \left(\frac{-1}{2}\right)^2\right) \cdot |X^6, n^n|^{-1}(\chi) + \sum_{\chi \in \text{im}(\mu_6, n^n)} \beta_1(\chi) = 2 \left(1 - \left(\frac{-1}{2}\right)^2\right) \cdot |X^6, n^n|^{-1}(\chi) = \left(1 - \left(\frac{-1}{2}\right)^2\right) \cdot |X^6, n^n|^{-1}(\chi)\) (from the list in Lemma 37) = \( \frac{1}{2} \cdot \sum_{\chi \in \{1, 2, \ldots, 20\}} |X^6, n^n|^{-1}(\chi) \) and it can be checked that this equals the claimed value of \( |F^M(6, n)| \).

Similarly, \( |F^M(6, n)| = \sum_{\chi \in \text{im}(\mu_6, n^n)} \beta_1(\chi) = 2 \left(\frac{-1}{2}\right)^2 \cdot |X^6, n^n|^{-1}(\chi) = (\text{from the list in Lemma 37}) = \frac{1}{2} \left( h_{C^4, \sim K2,3}(n) + h_{C^0}(n) \right) \) and it can be checked that this is equal to the value of \( |F^M(6, n)| \) which is claimed in (Ex6). Finally, \( |F^M(6, n)| = \sum_{\chi \in \text{im}(\mu_6, n^n)} \beta_1(\chi) = 2 \left(\frac{-1}{2}\right)^2 \cdot |X^6, n^n|^{-1}(\chi) = (\text{from the list in Lemma 37}) = \frac{1}{2} \cdot |X^6, n^n|^{-1}(t4) \), and this equals the value of \( |F^M(6, n)| \) claimed in (Ex6). \( \square \)

4.1.1. Alternative checks using Theorem 44. We did not need Theorem 44 in our proof of Theorem 51. It can, nevertheless, provide additional security since via Corollary 42 and Theorem 44 one may take an inclusion-exclusion-free (but, all told, much more laborious) alternative route to the claimed values of \( |F^M(5, n)| \) and \( |F^M(6, n)| \). As to the claimed value of \( |F^G(5, n)| \), by (M5) in Corollary 42 combined with Theorem 44 we have \( |F^G(5, n)| = (m5.t2) + (m5.t3) + (m5.t5) + (m5.t7) = 2^4 \cdot \left(n^{-1}\right)^2 \cdot (4 \cdot (n - 3) + 8 \cdot (n - 3) + 2 \cdot (n - 3)^2) = 48 \cdot ((n - 1)^2 - 2) \cdot (n^{-1}) \cdot (n^{-2}) \).

As to the claimed value of \( |F^G(6, n)| \), by (M6) in Corollary 42, the function \( |F^G(6, n)| \) equals the sum of the nineteen functions which were found in (m6.t2)–(m6.t20) of (QF6a) in Theorem 44, and one can check (e.g. with [19]) that indeed \( |F^G(6, n)| = \sum_{2 \leq k \leq 20} (m6.tk) = 18n^8 - 180n^7 + \frac{1856}{3}n^6 - \frac{2176}{3}n^5 - \frac{254}{3}n^4 + \frac{428}{3}n^3 - \frac{8144}{3}n^2 - \frac{11536}{3}n + 1504 \).

Incidentally, let us note that by summing all functions in (m6.t2)–(m6.t20) except (m6.t4) and (m6.t13) (i.e. by summing seventeen functions) one may also check the equation \( h_{C^4, \sim K2,3}(n) = h_{\geq}(n) - 3 \cdot h_{K2,3}(n) = 18n^8 - 180n^7 + 612n^6 - 608n^5 - 774n^4 + 1348n^3 + 1200n^2 - 2864n + 1248 \) claimed in the proof above.

4.1.2. Quantitatively dominant graph-theoretical reasons for \( \mathbf{P}_{\text{chio}} \neq \mathbf{P}_{\text{lef}} \). Let us note that Theorem 44 tells us that of the \( |F^M(6, n)| \in \Omega_{\sim \infty}(n^8) \) six-element-entry-specifications which cause non-agreement of \( \mathbf{P}_{\text{chio}} \) and \( \mathbf{P}_{\text{lef}} \), most of the failures are concentrated at only three out of the nineteen isomorphism types in (QF6a): only the types (t18), (t19) and (t20) have a preimage under \( \mu_6, n^n \) which is of size \( \Omega_{\sim \infty}(n^8) \). The quantitative domination of these isomorphism types is, however, a rather slow one in that

\[
\frac{\sum_{2 \leq k \leq 17} |(u^6, n^n)^{-1}(tk)|}{|u^6, n^n|^{-1}(t18)| + |u^6, n^n|^{-1}(t19)| + |u^6, n^n|^{-1}(t20)|} \in O(n^{-1}).
\]
4.1.3. Estimate of the number of failures of equality of $P_{\text{chiro}}$ and $P_{\text{cf}}$ for events $E_B$ with $B \in \{0, \pm\}^I$ and $I \in \left(\binom{n-1}{k}\right)$ and $k$ general.

**Proposition 52** (for fixed $k$ the measures $P_{\text{chiro}}$ and $P_{\text{cf}}$ agree for almost all entry-specifications). For every fixed $k \geq 1$ we have $|F^M(k, n)| / |\{B \in \{0, \pm\}^I, \ I \in \left(\binom{n-1}{k}\right)\}| \in \mathcal{O}_{n \to \infty}(n^{-2})$.

**Proof.** We will estimate numerator and denominator of this fraction separately. The denominator is equal to $3^k \cdot \left(\binom{n-1}{k}\right) \in \Omega_{n \to \infty}(n^{2k})$. Moreover, a very rough estimate suffices to obtain a bound on the numerator which nevertheless is sufficiently small to prove that the ratio vanishes: $|F^M(k, n)| = \sum_{0 \leq j \leq k} 1 \cup L \in \text{Cir}(2j, n) \{B \in \{0, \pm\}^I, \ I \in \left(\binom{n-1}{k}\right), \ L \in \text{Supp}(B)\} \leq \sum_{0 \leq j \leq k} 1 \cup L \in \text{Cir}(2j, n) |\{B \in \{0, \pm\}^I, \ I \in \left(\binom{n-1}{k}\right), \ L \in \text{Supp}(B)\}| \leq \sum_{0 \leq j \leq k} \sum_{0 \leq j \leq k} 2^{2j} \cdot 3^{k-2j} \cdot \left(\binom{n-1}{k-2j}\right) \cdot |\text{Cir}(2j, n)| = 25 \sum_{0 \leq j \leq k} |\text{Cir}(2j, n)| \in \mathcal{O}_{n \to \infty}(n^{2k-4j}) \in \mathcal{O}_{n \to \infty}(n^{2j}) \in \mathcal{O}_{n \to \infty}(1) \in \mathcal{O}_{n \to \infty}(n^{2k-2j}) \subseteq \mathcal{O}_{n \to \infty}(n^{2k-2j}). \quad \square

5. Connection to counting singular $\{\pm\}$-matrices

5.1. Basic connections. The Lemmas 53 and 54, which are consequences of Chio’s identity $P_{\text{chiro}}$ are the basic reason why the measure $P_{\text{chiro}}$ is relevant for the study of singular $\{\pm\}$-matrices.

**Lemma 53** (Chio condensation affects rank to the least possible degree). For every integral domain $R$, every $(s, t) \in \mathbb{Z}^2 \geq 2$ and every $A \in R^{[s] \times [t]}$ such that $\text{rk}(C_{(s, t)}(A)) \neq 0$ we have $\text{rk}(C_{(s, t)}(A)) = \text{rk}(A) - 1$.

**Proof.** If $\text{rk}(A) = 1$, then obviously $C_{(s, t)}(A) = \{0\}^{[s-1] \times [t-1]}$ and the claim is true. We may therefore assume that $r := \text{rk}(A) \geq 2$. By the equality of rank and determinantal rank over integral domains (cf. e.g. [1, Corollary 2.29(2)]) there exists $S \in \binom{[s]}{t}$ and $T \in \binom{[t]}{r}$ such that $\det(A |_{S \times T}) = 0$. If $s \notin S$, then by temporarily passing to the field of fractions of $R$ we may appeal to Steinitz’ exchange lemma for vector spaces to prove the existence of at least one $i \in S$ such that $\det(A |_{(S \setminus \{i\}) \cup \{s\} \times (T \setminus \{t\})}) = 0$. Analogously for $t \notin T$. Therefore we may assume that $s \in S$ and $t \in T$. Hence $S \times T = ((S \setminus \{s\}) \times (T \setminus \{t\}))$ and therefore $C_{(s, t)}(A |_{S \times T})$ is defined. By Lemma 21 we know that $\det(C_{(s, t)}(A |_{S \times T})) = a_{r, t}^{-1} \cdot \det(A |_{S \times T}) \neq 0$, the latter since $R$ is an integral domain and $a_{s, t} \neq 0$ by assumption. Since $C_{(s, t)}(A |_{S \times T}) = C_{(s, t)}(A) |_{(S \setminus \{s\}) \times (T \setminus \{t\})}$ is in $R^{(r-1) \times (r-1)}$, and by the equality of rank and determinantal rank, this implies $\text{rk}(C_{(s, t)}(A)) \geq r - 1$. On the other hand we also have $\text{rk}(C_{(s, t)}(A)) \leq r - 1$. To see this, it suffices to note that every $r \times r$ submatrix of $C_{(s, t)}(A)$ is the Chio condensate of an $(r+1) \times (r+1)$ submatrix of $A$, hence by Chio’s identity a nonvanishing $r \times r$ minor of $C_{(s, t)}(A)$ would imply a nonvanishing $(r+1) \times (r+1)$ minor of $A$, contrary to the assumption of $\text{rk}(A) = r$. \quad \square

**Lemma 54.** $P\left[\text{Ra}_{r}(\{\pm\}^{[s]} \times [t])\right] = P_{\text{chiro}}\left[\text{Ra}_{r-1}(\{0, \pm\}^{[s-1]} \times [t-1])\right]$ for every $(s, t) \in \mathbb{Z}^2 \geq 2$ and $1 \leq r \leq \min(s, t)$.

**Proof.** This follows from the calculation $P\left[\text{Ra}_r(\{\pm\}^{[s]} \times [t])\right] = \frac{1}{2^r} \left\{ A \in \{\pm\}^{[s]} \times [t] : \text{rk}(A) = r \right\} = \frac{1}{2^r} \sum_{B \in \{0, \pm\}^{[s-1]} \times [t-1] : \text{rk}(B) = r - 1} \left(\frac{1}{2} C_{(s, t)}\right)^{-1} (B) = \text{Definition} 3 P_{\text{chiro}}\left[\text{Ra}_{r-1}(\{0, \pm\}^{[s-1]} \times [t-1])\right]$. Note, incidentally, that with the third equality sign, many zero-summands are introduced. \quad \square

**Corollary 55.** $P\left[\text{Ra}_{r}(\{\pm\}^{[s]} \times [t])\right] = P_{\text{chiro}}\left[\text{Ra}_{r}(\{0, \pm\}^{[s-1]} \times [t-1])\right]$ for every $(s, t) \in \mathbb{Z}^2 \geq 2$ and every $\mathcal{R} \in \mathcal{P}(\min(s, t) \sqcup \{0\})$. 

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Proof. Immediate from Lemma 54 and Definition 19. □

Corollary 56. \( P_{\text{chio}}[R_{a<n-1}(\{0, \pm\}[n-1]^2)] \leq (1/\sqrt{2} + o(1))^n \) for \( n \to \infty \).

Proof. By combining Lemma 55 with (1) in Theorem 3. □

5.2. Sign functions which are both singular and balanced.

Definition 57 \((G_{s,t})\). For every \((s,t) \in \mathbb{Z}_{\geq 2}^2\) define \(G_{s,t} := (\bigoplus_{1 \leq i \leq s-1} \mathbb{Z}/2) \oplus (\bigoplus_{1 \leq j \leq t-1} \mathbb{Z}/2)\).

We will use the following group actions. Informally, \(\alpha_X\) is the action by switching signs of edges simultaneously in all ‘stars’ centered at those vertices for which \(g\) has nonzero components.

Definition 58. For every \((s,t) \in \mathbb{Z}_{\geq 2}^2\) define the group action \(\alpha_{s,t} : G_{s,t} \to \text{Sym}(\{0, \pm\}[s-1] \times [t-1]),\)

\[ ((g_i)_{i \in [s-1]}, (g_j)_{j \in [t-1]}) \mapsto (\left((0, \pm)[s-1] \times [t-1], (g_i)_{i \in [s-1]}\right) - (s-1)g_i, \left((0, \pm)[t-1], (g_j)_{j \in [t-1]}\right) - (t-1)g_j, \) \]

For every \(X \in \text{BG}_{s,t}\) define the group action \(\alpha_X : G_{s,t} \to \text{Sym}(\{0, \pm\}E(X))\) which is defined by \((\alpha_X(g)(\sigma))(e) := (-1)^{g_i} \cdot (-1)^{g_j} \cdot \sigma(e)\) for every \(e \in \{0, \pm\}E(X)\) and every \(e = ((i, t), (s, j)) \in E(X)\).

Note that neither \(\alpha_{s,t}\) nor \(\alpha_X\) are faithful group actions. More precisely, not only is both \(\ker(\alpha_{s,t})\) and \(\ker(\alpha_X)\) a 2-element set, but \(\alpha_{s,t}\) and \(\alpha_X\) are both double-covers onto their images. We could construct a faithful action by making an arbitrary choice of a \(i \in [s-1] \cup [t-1]\) and then refraining from switching at this index (analogously, by making an arbitrary choice of a star in \(X\) and then refraining from changing that particular star).

Balancedness is a very ‘rigid’ property of an edge-signing in that it is determined by the signing of an arbitrary spanning tree:

Lemma 59 \(\text{(rigidity of balanced edge signings)}\). For every connected graph \(X\) and every spanning tree \(T\) of \(X\), there is a bijection \(\{\pm\}E(T) \leftrightarrow \{\sigma \in \{\pm\}E(X) : (X, \sigma) \text{ balanced}\}\).

Sketch of proof. Since the balance-preserving sign of every edge \(e \in E(X) \setminus E(T)\) is determined by the unique circuit in \(E(T) \cup \{e\}\), for every given \(\sigma \in \{\pm\}E(T)\), there is at most one balanced extension of \(\sigma\), i.e. at most one balanced edge-signing \(\hat{\sigma} \in \{\pm\}E(X)\) with \(\sigma = \hat{\sigma} |_{E(T)}\) and \((X, \hat{\sigma})\) balanced. Moreover, this extension can be constructed in the obvious ‘greedy’ way by successively adding in the elements of \(E(X) \setminus E(T)\) in an arbitrary order while at each step of the construction choosing the sign of the added edge so as to avoid non-balanced circuits. That this is indeed possible can be proved by an induction on the number \(|E(X) \setminus E(T)|\) of edges to be added. A key observation (routine to prove and known since at least [18, Theorem 2]) is that at each step of the construction, for each pair of vertices either all paths within the partially constructed graphs which have these two vertices and all paths have sign (+), and therefore the greedy construction never stalls. □

Lemma 60. For every graph \(X\) the restriction \(\text{im}(\alpha_X) \upharpoonright_{S_{\text{bal}}(X)}\) is a transitive permutation group on \(S_{\text{bal}}(X)\).

Sketch of proof. One way to look at this is as ‘making use of the rigidity of balanced signings’: we can choose an arbitrary spanning tree \(T_1\) for each connected component \(X_i\) of \(X\), then show that \(\text{im}(\alpha_X) \upharpoonright_{S_{\text{bal}}(X)}\) is transitive on the set \(\{\pm\}E(T_i)\) of all edge-signings of \(T_i\), and then appeal to Lemma 59 which says that this transitivity already implies transitivity on the full set \(S_{\text{bal}}(X)\). □

Given a \([0, 1]\)-matrix, it can be possible to increase its \(\mathbb{Z}\)-rank by choosing signs for the entries. If we require the signed matrix to be balanced, however, the rank must stay the same. This follows quickly from the graph-theoretical considerations above:
Proposition 61 (all balanced signings of a \( \{0,1\} \)-matrix have the same rank). Let \( B \in \{0,1\}^{[s-1] \times [t-1]} \). Let \( \tilde{B} \) be an arbitrary `balanced signing of \( B^\ast ``, i.e. \( \tilde{B} \in \{0,\pm\}^{[s-1] \times [t-1]} \), \( \text{Supp}(\tilde{B}) = \text{Supp}(B) \) and \( (X_{\tilde{B}}, \sigma_{\tilde{B}}) = (X_B, \sigma_B) \) is a balanced signed graph. Then \( \text{rk}(\tilde{B}) = \text{rk}(B) \).

Proof. Since both \((X_B, \sigma_B)\) and \((X_{\tilde{B}}, \sigma_{\tilde{B}})\) are balanced, by Lemma 60 there exists \( g \in G_{s,t} \) such that \( \alpha_X(g)(\sigma_{\tilde{B}}) = \sigma_B \). In view of Definition 58 and Definition 12, this implies \( \alpha_{s,t}(g)(\tilde{B}) = B \).

Since \( \alpha_{s,t} \) obviously keeps the rank invariant, the claim is proved. \( \square \)

We will now use the knowledge established so far to analyse the tempting `absolute‘ route of using Corollary 55 and then partitioning according to isomorphism type of the associated bipartite graph.

The conclusion is that this will lead us onto a well-beaten path (counting singular \( \{0,1\} \)-matrices):

Proposition 62 (rank-level-sets, the Chio measure agrees with the uniform measure after forgetting the signs). Let \((s,t) \in \mathbb{Z}^2_{\geq 2} \) and \( R \in \mathcal{P}\{1, \ldots, \min(s,t)\} \). Then

\[
P_{\text{chio}}[\text{Ra}_R(\{0, \pm\}^{[s-1] \times [t-1]})] = P[\text{Ra}_R(\{0,1\}^{[s-1] \times [t-1]})]. \tag{36}
\]

Proof. This follows from the calculation

\[
P_{\text{chio}}[\text{Ra}_R(\{0, \pm\}^{[s-1] \times [t-1]})] = \sum_{X \in \text{ul}(BG_{s,t})} P_{\text{chio}} \left[ \left\{ B \in \{0, \pm\}^{[s-1] \times [t-1]} : \text{rk}(B) \in R, (X_B, \sigma_B) \right. \right. \left. \left. \text{balanced} \right\} \right] \tag{C1)}
\]

(by (2) in Corollary 31)

\[
= \sum_{X \in \text{ul}(BG_{s,t})} 2^{-s+2h_0(X)+1} \cdot \left\{ B \in \{0, \pm\}^{[s-1] \times [t-1]} : \text{rk}(B) \in R, (X_B, \sigma_B) \text{ balanced} \right\} \tag{by (KGS) in Lemma 26 and Proposition 61}
\]

\[
= \sum_{X \in \text{ul}(BG_{s,t})} 2^{f_0(X)-2t+1} \cdot \left\{ B \in \{0,1\}^{[s-1] \times [t-1]} : \text{rk}(B) \in R, X_B \cong X \right\} \tag{since \( f_0(X) \) is equal to \( (s-1)+(t-1) \) for every \( X \in \text{ul}(BG_{s,t}) \))
\]

\[
= \sum_{X \in \text{ul}(BG_{s,t})} 2^{-(s+2)(t-1)} \cdot \left\{ B \in \{0,1\}^{[s-1] \times [t-1]} : \text{rk}(B) \in R, X_B \cong X \right\}
\]

\[
= P[\text{Ra}_R(\{0,1\}^{[s-1] \times [t-1]})].
\]

The proof of Proposition 62 is now complete. \( \square \)

It should be noted that the special case \( P[\text{Ra}_{<n}(\{\pm\}^{[n]^2})] = P[\text{Ra}_{<n-1}(\{0,1\}^{[n-1]^2})] \) seems well-known (the author does not have an explicit reference corroborating this, but there are articles in which this is implicit (e.g. \cite{32})).

5.3. A relative point of view. What really appears to promise progress on Conjecture 1 is to use the theorem of Bourgain–Vu–Wood to reach a more relative vantage point:

Proposition 63 (relative formulations of Conjecture 1). The following statements are equivalent:

\[
(\text{Q1) } P[\text{Ra}_{<n}(\{\pm\}^{[n]^2})] \leq (\frac{1}{2} + o_{n \to \infty}(1))^n
\]

\[
(\text{Q2) } P_{\text{chio}}[\text{Ra}_{<n-1}(\{0, \pm\}^{[n-1]^2})] \leq (\frac{1}{2} + o_{n \to \infty}(1)) \cdot P[\text{Ra}_{<n-1}(\{0,\pm\}^{[n-1]^2})]
\]

\[
(\text{Q3) } \sum_{B' \in \text{Ra}_{<n-1}(\{0,\pm\}^{[n-1]^2})} P_{\text{chio}}[B'] \leq (\frac{1}{2} + o_{n \to \infty}(1)) \cdot \sum_{B'' \in \text{Ra}_{<n-1}(\{0,\pm\}^{[n-1]^2})} P[B'']
\]

\[
(\text{Q4) } |\{B' \in \{0,1\}^{[n-1]^2} : \text{rk}(B') < n-1\}| \leq (\frac{1}{2} + o_{n \to \infty}(1)) \cdot \sum_{B \in \{0,1\}^{[n-1]^2}} (\frac{1}{2})^{|\text{Supp}(B)|} \sum_{B' \in \{0,1\}^{[n-1]^2}} (\frac{1}{2})^{|\text{Supp}(B')|} \left\{ B'' \in \{0, \pm\}^{[n-1]^2} : \text{rk}(B'') < n-1 \right\}
\]

Proof. As to the equivalence (Q1) \( \Leftrightarrow \) (Q2), if (Q1), then \( P_{\text{chio}}[\text{Ra}_{<n-1}(\{0, \pm\}^{[n-1]^2})] \) \( \leq \) \( (\frac{1}{2} + o_{n \to \infty}(1))^n \) \( \leq \) \( (\frac{1}{2} + o_{n \to \infty}(1)) \cdot (\frac{1}{2} + o_{n \to \infty}(1))^{n-1} \) \( \text{Theorem 3} \) \( \leq \) \( (\frac{1}{2} + o_{n \to \infty}(1))^{n-1} \) \( \leq \) \( (\frac{1}{2} + o_{n \to \infty}(1))^{n-1} \) \( = \) \( (\frac{1}{2} + o_{n \to \infty}(1))^n \) \( \leq \) \( (\frac{1}{2} + o_{n \to \infty}(1))^n \)
Theorem 3 implies $P[\text{Ra}_{< n-1}((0, \pm)^{[n-1]^2})] \rightarrow \infty$, which is (Q2). As to the converse, (Q2) implies $P[\text{Ra}_{< n}((\pm)^{[n]^2})] \sim 40$. In Theorem 3, we have seen $\rightarrow \infty$ implies $P[\text{Ra}_{< n-1}((0, \pm)^{[n-1]^2})] \leq (\frac{1}{2} + o_{n\rightarrow \infty}(1)) \cdot P[\text{Ra}_{< n-1}((0, \pm)^{[n-1]^2})]$.

Note the ‘relativizing’ effect of having two sums over the same index set on either side of an (conjectured) inequality: thanks to commutativity of addition one may go about pitting (collections of) unequally indexed summands on both sides of (Q3) against one another, in the hope of finding a rearrangement that allows one to prove the inequality without any a priori knowledge about the size of the index set of the sums. Of course, if (Q1) is true, then the inequality is true for every permutation of the summands but the point is that this is not known and that it would suffice to prove the existence of only one suitable rearrangement of the summands to prove (or maybe disprove) Conjecture (Q1).

5.3.1. The inequality (Q2) must fail on entry-specification events. Let us remark that in view of the formula (C3) in Theorem 30 we find ourselves in a slightly ironic situation: while (Q2), which speaks about the $P_{\text{choio}}$-measure of the (rather mysterious) event $\text{Ra}_{< n-1}((0, \pm)^{[n-1]^2})$, may well be true, it cannot possibly be true in a non-trivial way (left-hand side nonzero) on any of the (rather simple) entry-specification events.

5.3.2. Worst possible failure of (Q2) for singleton events. Already in Corollary 40 we have seen examples that the inequality (Q2) can fail when the event $\text{Ra}_{< n-1}((0, \pm)^{[n-1]^2})$ is replaced by other events—the failure seeming more likely and more severe as the events get smaller. We will now see that (Q2) fails arbitrarily badly on every atom of the measure space we are dealing with (i.e. a singleton event $\{B\}$ with $B \in (0, \pm)^{[n-1]^2}$ and $P_{\text{choio}}[B] > 0$). For such events the ratio of $P_{\text{choio}}$ and $P_{\text{lef}}$ diverges as quickly as $2^{n^2}$ when $n \rightarrow \infty$, while (Q2) asserts a bounded ratio as $n \rightarrow \infty$.

By (C3) in Theorem 30, we know $P_{\text{choio}}[\mathcal{E}_B^f]/P_{\text{lef}}[\mathcal{E}_B^f] = 2^{\beta_1(X_B)}$. Therefore, to determine the maximum of $P_{\text{choio}}[\mathcal{E}_B^f]/P_{\text{lef}}[\mathcal{E}_B^f]$ over all entry specification events $\mathcal{E}_B^f$ it suffices to determine the maximum of $\beta_1(X_B)$ over all bipartite graphs $X_B \in \text{BG}_{n,n}$ with $B \in (0, \pm)^{[n-1]^2}$. The Betti number $\beta_1(X) = f_1(X) - f_0(X) + \beta_0(X)$ as a function of $X \in \text{BG}_{n,n}$ attains a unique maximum at $X = K^{n-1,n-1}$. The corresponding value is $(n-1)^2 - 2(n-1) + 1 = (n-2)^2$. Since $K^{n-1,n-1}$ can indeed occur as $X_B$ with $B \in \text{Ra}_{< n-1}((0, \pm)^{[n-1]^2}) \cap \text{im}(\frac{1}{2}C_{(n,n)})$: $\{\pm\}^{[n]^2} \rightarrow \{0, \pm\}^{[n-1]^2}$, it follows that for every fixed $n$, the maximum of $P_{\text{choio}}[\mathcal{E}_B^f]/P_{\text{lef}}[\mathcal{E}_B^f]$ over all $\mathcal{E}_B^f$ with $\emptyset \neq I \subseteq J \subseteq [n-1]^2$ and $B \in (0, \pm)^I \cap \text{im}(\frac{1}{2}C_{(n,n)})$: $\{\pm\}^{[n]^2} \rightarrow \{0, \pm\}^{[n-1]^2}$ is $2^{(n-2)^2} \sim 2^{n^2}$. Since Proposition 61 implies that every $B \in (0, \pm)^I \cap \text{im}(\frac{1}{2}C_{(n,n)})$: $\{\pm\}^{[n]^2} \rightarrow \{0, \pm\}^{[n-1]^2}$ which realizes the maximum, i.e. $X_B \cong K^{n-1,n-1}$, has rank 1, it follows that $2^{(n-2)^2}$ is also the maximum of $P_{\text{choio}}[\mathcal{E}_B^f]/P_{\text{lef}}[\mathcal{E}_B^f]$ over all $\mathcal{E}_B^f$ with $\emptyset \neq I \subseteq J \subseteq [n-1]^2$ and $B \in \text{Ra}_{< n-1}((0, \pm)^{[n-1]^2}) \cap \text{im}(\frac{1}{2}C_{(n,n)})$: $\{\pm\}^{[n]^2} \rightarrow \{0, \pm\}^{[n-1]^2}$. 


5.3.3. Extent of failure of (Q2) on those B which are Chio condensates of random $A \in \{\pm\}[n]^2$. Let us have a quick informal look at the typical value of the ratio $P_{\text{cho}}[B]$ and $P_{\text{lef}}[B]$ if $\{0, \pm\}[n-1]^2 \ni B := C_{Y,n}(A)$ with $A \in \{\pm\}[n]^2$ chosen uniformly at random. Of course, such a $B$ is (by Theorem 3 and Lemma 53) asymptotically almost surely not an element of $\{0, \pm\}[n-1]^2$.

By Corollary 34, for $A \in \{\pm\}[n]^2$ chosen uniformly at random, the graph $X_B$ is a random bipartite graph with partition classes of $n-1$ vertices on either side and having i.i.d. edges with probability $\frac{1}{2}$. Since this is very far above the threshold proved by Frieze [13] for hamiltonicity of a random bipartite graph, it follows that (for very strong reasons) a.a.s. $\beta_0(X_B) = 1$. As to $f_1(X_B)$, a standard argument using Chernoff’s bound shows that for every $\epsilon > 0$ a.a.s. (and approaching 1 exponentially fast) $(\frac{1}{2} - \epsilon) \cdot (n-1)^2 \leq f_1(X_B) \leq (\frac{1}{2} + \epsilon) \cdot (n-1)^2$. For simplicity let us pretend that $f_1(X_B) = \frac{1}{2} (n-1)^2$ exactly. Then $P_{\text{cho}}[B]/P_{\text{lef}}[B] = 2\beta_1(X_B) = 2f_1(X_B) - \ell_0(X_B) + \beta_0(X_B) = 2\frac{n^2}{2} - 3n + \frac{1}{4} \sim 2\frac{n^2}{2} \sim \sqrt{n^2} \rightarrow \infty$ as $n \rightarrow \infty$, and we have learned that the worst-case failure-ratio of (Q2) found in 5.3.2 arises roughly by squaring the failure-ratio for a $B = \frac{1}{2} C_{Y,n}(A)$ with $A \in \{\pm\}[n]^2$ random.

6. Concluding questions

Let us close with three questions:

6.1. What to make of the $k$-wise independence? Note that Theorem 51 in particular says that one application of $\phi C_{Y,n}$ to a $\{\pm\}$-valued $n \times n$-matrix yields a $\{0, \pm\}$-matrix whose entries are distributed as $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ but are merely $3$-wise stochastically independent (in the sense of [15, Definition 2.4, p.209]) (and ‘almost’ $k$-wise, the ‘almost’ quantified exactly for $k \in \{4, 5, 6\}$ in Theorem 51 and quantified roughly for general $k$ in Proposition 52). There appears to be ongoing research on how partial independence relates to full independence, a keyword being ‘$k$-wise independence’. To give only one very recent (the topic has been studied at least since the 1980s) example, which appears to summarize well the overall spirit, here is a quote from the abstract of [4]:

We pursue a systematic study of the following problem. Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be a (usually monotone) boolean function whose behaviour is well understood when the input bits are identically independently distributed. What can be said about the behaviour of the function when the input bits are not completely independent, but only $k$-wise independent $\ldots$? How high should $k$ be so that any $k$-wise independent distribution “fools” the function, i.e. causes it to behave nearly the same as when the bits are completely independent?

The author wonders whether the theory of $k$-wise stochastic independence has any bearing on the present problem. In particular, can the proof of Bourgain–Vu–Wood for $P_{\text{lef}}$ be deconstructed and somehow reassembled for $P_{\text{cho}}$, aided by knowledge about $k$-wise independence?

6.2. More accurate estimations? Note that Theorem 51, (Ex4)–(Ex6) teaches us that, asymptotically, the ratio $|F^M(k, n)|/|\{B \in \{0, \pm\}^k, I \in \{n-1\}^k\}|$ for $k \in \{4, 5, 6\}$ takes values $4n^4/12n^8, 32n^{-4}/11n^4, 12n^6/210n^{10} = 480/81 n^{-4}/5.9n^4, 6.0 n^{-4}$ and $18n^8/81 n^{12} = 1440/81 n^2 n^{-4}$ which are all—all although of course growing with $k$—vanishing with the same speed $O_{\rightarrow \infty}(n^{-4})$. This shows that the rough bound in Proposition 52 is not an asymptotically tight one. This of course raises two questions: Is it true that $|F^M(k, n)|/|\{B \in \{0, \pm\}^k, I \in \{n-1\}^k\}| \in O_{\rightarrow \infty}(n^{-4})$ for every fixed $k$? Moreover, note that while Proposition 52 shows that $|F^M(k, n)|/|\{B \in \{0, \pm\}^k, I \in \{n-1\}^k\}|$ vanishes as $n \rightarrow \infty$ for every fixed $k$, the discussion in 5.3.2 shows that for the extreme case of $k = (n - 1)^2$, i.e. if the entire matrix $B \in \{0, \pm\}[n-1]^2$ is specified, $P_{\text{cho}}[\epsilon B] \neq P_{\text{lef}}[\epsilon B]$ for the vast majority of $B \in \{0, \pm\}[n-1]^2$. In between these two extremes, i.e. almost sure agreement as opposed to almost sure non-agreement of $P_{\text{cho}}$ and $P_{\text{lef}}$, there should be a tipping point. This raises the question: For what order of growth
k = k(n) does |F^M(k,n)| / |{B ∈ {0, ±}^I, I ∈ \binom{[n-1]^2}{k}}| first become bounded away from zero? And for what order does it first tilt in favour of the non-agreement events?

6.3. Hidden connections to the Guralnick–Maróti-theorem? If \( \sigma: G \to \text{Aut}_K(V) \) is a representation of a group \( G \) on a \( K \)-vector space \( V \), then for every \( g \in G \) let \( \text{Fix}_V(g) \) denote the fixed-point space of \( g \), i.e. the \( K \)-linear subspace \( \{ v \in V : \sigma(g)(v) = v \} \).

In recent times there have been advances ([28], [20], [7]) concerning the problem of bounding averages of dimensions of fixed-point spaces by a fraction of the dimension of the representation, leading to a full proof (and in more general form) by R. M. Guralnick and A. Maróti [17] of a 1966 conjecture of P. M. Neumann:

**Theorem 64** (Guralnick–Maróti [17], Theorem 1.1). For every finite group \( G \) with smallest prime factor of \( |G| \) denoted by \( p \), every field \( K \), every finite-dimensional \( K \)-vector space \( V \), every homomorphism \( \sigma: G \to \text{Aut}_K(V) \), and every normal subgroup \( N \) of \( G \) which does not have a trivial composition factor on \( V \),

\[
\frac{1}{|N|} \sum_{g \in N\,g} \text{dim}_K(\text{Fix}_V(g)) \leq \frac{1}{p} \text{dim}_K(V) \ .
\]

(37)

Although the resemblance is likely to be merely superficial, the author cannot help being intrigued by the similarity of this inequality to (Q3), together with the fact that both in (Q3) and in (37) there can be zero-summands on the left-hand side. Moreover, when trying to combine Chio condensation of sign matrices with group actions, one gets the impression that groups of even order (i.e. \( p = 2 \)) play a natural role. One goal along these lines is to discover a vector space avatar of the lazy coin flip measure. Via Theorem 30 the author found the following formula (which is of course easy to check directly): for every \( B \in \{0, ±\}^{[n-1]^2} \) we have

\[
P_{\text{ref}}[B] = \left( \frac{1}{2} \right)^{(n-1)^2} \left( \frac{1}{2} \right)^{\text{dim}_{2/2}(B^1(X_N; \mathbb{Z}/2) \oplus Z_2(X_N; \mathbb{Z}/2))} \ .
\]

(38)

This suggests studying group actions on the direct sum \( B^1(X_N; C_{(n,n)}(A); \mathbb{Z}/2) \oplus Z_2(X_N; C_{(n,n)}(A); \mathbb{Z}/2) \). A sensible first choice are those actions which are induced by the ‘natural’ \( |\det(\cdot)| \)-preserving (hence intransitive) group actions on \( \{±\}^{[n-1]^2} \) (a merit of those ‘standard actions’ is that Chio condensation commutes with the corresponding actions on \( \{0, ±\}^{[n-1]^2} \)). So far the author did not detect much resonance with Theorem 64. Mirage or more?

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