ORDER AND HYPER-ORDER OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

SANJAY KUMAR AND MANISHA SAINI

Abstract. We have discussed the problem of finding the condition on coefficients of \( f'' + A(z)f' + B(z)f = 0 \), \( B(z) \neq 0 \) so that all non-trivial solutions are of infinite order. The hyper-order of non-trivial solution of infinite order is found when \( \lambda(A) < \rho(B) \) and \( \rho(B) \neq \rho(A) \) or \( B(z) \) has Fabry gap.

1. Introduction

The study of growth of solutions of complex differential equation starts with Wittich’s work in [26]. For the fundamental results of complex differential equations one may consult to [14] and [19]. The Nevanlinna’s value distribution theory has been a useful tool in investigating the complex differential equations. For the notion of value distribution theory one may refer to [30].

For an entire function \( f(z) \) the order of growth is defined as:

\[
\rho(f) = \limsup_{r \to \infty} \frac{\log^+ M(r, f)}{\log r} = \limsup_{r \to \infty} \frac{\log^+ T(r, f)}{\log r},
\]

where \( M(r, f) = \max \{ |f(z)| : |z| = r \} \) is the maximum modulus of the function \( f(z) \) over the circle \( |z| = r \) and \( T(r, f) \) is the Nevanlinna characteristic function of the function \( f(z) \).

In this paper we investigate the growth of solutions \( f(\neq 0) \) of the second order linear differential equation

\[
f'' + A(z)f' + B(z)f = 0
\]

where the coefficients \( A(z) \) and \( B(z) \neq 0 \) are entire functions. We know that all solutions of the equation \([11]\) are entire function \([19]\). It is necessary and sufficient condition that all solutions of the equation \([11]\) are of finite order if the coefficients \( A(z) \) and \( B(z) \) are polynomials \([19]\). Therefore, if any of the coefficients is a transcendental entire

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function then almost all solutions are of infinite order. However, there is a necessary condition for equation (1) to have a solution of finite order:

**Theorem 1.** Suppose that \( f(z) \) be a finite order solution of the equation (1) then \( \rho(B) \leq \rho(A) \).

Therefore, if \( \rho(A) < \rho(B) \) then all non-trivial solutions \( f(z) \) of the equation (1) are of infinite order. It is well known that above condition is not sufficient, for example: \( f'' - e^z f' - e^z f = 0 \) has solution \( f(z) = e^{e^z} \). Therefore, it is interesting to find conditions on \( A(z) \) and \( B(z) \) so that all solutions \( f(\not\equiv 0) \) are of infinite order. Many results have been given in this context. Gundersen [9] and Hellerstein [13] proved

**Theorem 2.** Let \( f \not\equiv 0 \) be a solution of the equation (1) with the coefficients satisfying

1. \( \rho(B) < \rho(A) \leq \frac{1}{2} \) or
2. \( A(z) \) is a transcendental entire functions with \( \rho(A) = 0 \) and \( B(z) \) is a polynomial.

then \( \rho(f) = \infty \).

Frei [6], Ozawa [24], Amemiya and Ozawa [1], Gundersen [7] and Langley [20] proved that all non-trivial solutions are of infinite order for the differential equation

\[ f'' + Ce^{-z} f' + B(z)f = 0 \]

for any nonzero constant \( C \) and for any nonconstant polynomial \( B(z) \). J. Heittokangas, J. R. Long, L. Shi, X. Wu, P. C. Wu, X. B. Wu, and Zhang in [22, 23, 27, 28] gave conditions on the coefficients \( A(z) \) and \( B(z) \) so that all solutions \( f(\not\equiv 0) \) are of infinite order.

The concept of hyper-order were used to further investigate the growth of infinite order solutions of complex differential equations. In this context, K. H. Kwon [18] proved that:

**Theorem 3.** Suppose \( P(z) = a_n z^n + \ldots + a_0 \) and \( Q(z) = b_n z^n + \ldots + b_0 \) be non-constant polynomials of degree \( n \) such that either \( \arg a_n \neq \arg b_n \) or \( a_n = c b_n \) (0 < \( c < 1 \)), \( h_1(z) \) and \( h_0(z) \) be entire functions satisfying \( \rho(h_i) < n \), \( i = 1, 2 \). Then every non-trivial solutions \( f(z) \) of

\[ f'' + h_1 e^{P(z)} f' + h_0 e^{Q(z)} f = 0, \quad Q(z) \not\equiv 0 \] (2)

are of infinite order with \( \rho_2(f) \geq n \).

For an entire function \( f(z) \) the hyper-order is defined in the following manner:

\[ \rho_2(f) = \limsup_{r \to \infty} \frac{\log^+ \log^+ M(r, f)}{\log r} = \limsup_{r \to \infty} \frac{\log^+ \log^+ T(r, f)}{\log r} \]

C. Zongxuan [31], investigate the differential equation (2) for some special cases and proved the following theorem:
Theorem 4. Let $b \neq -1$ be any complex constant, $h(z)$ be a non-zero polynomial. Then every solution $f(\neq 0)$ of the equation
\[ f'' + e^{-z} f' + h(z)e^{bz}f = 0 \] (3)
has infinite order and $\rho_2(f) = 1$.

In [17], K. H. Kwon found the lower bound for the hyper-order of all solutions $f(\neq 0)$ in the following theorem:

Theorem 5. [17] Suppose that $A(z)$ and $B(z)$ be entire functions such that (i) $\rho(A) < \rho(B)$ or (ii) $\rho(B) < \rho(A) < \frac{1}{2}$ then
\[ \rho_2(f) \geq \max\{ \rho(A), \rho(B) \} \]
for all solutions $f \neq 0$ of the equation (1).

In [16], we have proved:

Theorem 6. Suppose $A(z)$ be an entire function with $\lambda(A) < \rho(A)$ and

1. $B(z)$ be a transcendental entire function with $\rho(B) \neq \rho(A)$ or
2. $B(z)$ be an entire function having Fabry gap

then all non-trivial solutions of the equation (1) are of infinite order.

An entire function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ has Fabry gap if the sequence $(\lambda_n)$ satisfies
\[ \frac{\lambda_n}{n} \to \infty \]
as $n \to \infty$. An entire function $f(z)$ with Fabry gap satisfies $\rho(f) > 0$ [11]. However, the growth of an entire function with infinite order can be measured by its hyper-order. Therefore, in this paper we are proving following theorem:

Theorem 7. Let $A(z)$ be a finite ordered entire function satisfying $\lambda(A) < \rho(A)$ and

1. $B(z)$ be a transcendental entire function with $\rho(B) \neq \rho(A)$ or
2. $B(z)$ be an entire function having Fabry gap

then all non-trivial solutions $f(z)$ of the equation (1) have $\rho_2(f) = \max\{ \rho(A), \rho(B) \}$.

In Theorem [6] (i), we have replaced the condition $\rho(B) \neq \rho(A)$ with $\mu(B) \neq \rho(A)$ and proved the next theorem:

Theorem 8. Suppose that $A(z)$ and $B(z)$ be transcendental entire functions satisfying $\lambda(A) < \rho(A)$ and $\mu(B) \neq \rho(A)$ then all non-trivial solutions $f(z)$ of the equations satisfies $\rho(f) = \infty$.

For an entire function $f(z)$ the lower order of growth is defined as follows:
\[ \mu(f) = \liminf_{r \to \infty} \frac{\log^+ M(r,f)}{\log r} = \liminf_{r \to \infty} \frac{\log^+ T(r,f)}{\log r} \]
In Theorem 8, the order of the coefficients \(A(z)\) and \(B(z)\) may be equal. Next theorem presents the hyper-order of solutions of the differential equation satisfying the conditions of the Theorem 8.

**Theorem 9.** Suppose that \(A(z)\) be an entire function with finite order and \(B(z)\) be a transcendental entire function satisfying \(\lambda(A) < \rho(A)\) and \(\mu(B) \neq \rho(A)\) then

\[
\rho_2(f) = \max\{\rho(A), \mu(B)\}
\]

for all non-constant solutions \(f(z)\) of the equation (4).

This paper is organised in following manner: section 2 presents preliminary results to be used for proving Theorem 7. Section 3 include proof of Theorem 7. In section 4 we conclude by giving proof of Theorem 8.

### 2. Preliminary Results

This section includes the results which we need in proving our theorems.

For a set \(F \subset [0, \infty)\), the Lebesgue linear measure of \(F\) is defined as \(m(F) = \int_F dt\) and for a set \(G \subset [1, \infty)\), the logarithmic measure of \(G\) is defined as \(m_1(G) = \int_G \frac{1}{t} dt\). For set \(G \subset [1, \infty)\), the upper and lower logarithmic densities are defined, respectively, as follows:

\[
\log dens(G) = \limsup_{r \to \infty} \frac{m_1(G \cap [1, r])}{\log r}
\]

\[
\log dens(G) = \liminf_{r \to \infty} \frac{m_1(G \cap [1, r])}{\log r}
\]

Next lemma is due to Gundersen which provide the estimates for transcendental meromorphic function.

**Lemma 1.** Let \(f\) be a transcendental meromorphic function and let \(\Gamma = \{(k_1, j_1), (k_2, j_2), \ldots, (k_m, j_m)\}\) denote finite set of distinct pairs of integers that satisfy \(k_i > j_i \geq 0\) for \(i = 1, 2, \ldots, m\). Let \(\alpha > 1\) and \(\epsilon > 0\) be given real constants. Then the following three statements holds:

(i) there exists a set \(E_1 \subset [0, 2\pi)\) that has linear measure zero and there exists a constant \(c > 0\) that depends only on \(\alpha\) and \(\Gamma\) such that if \(\psi_0 \in [0, 2\pi) \setminus E_1\), then there is a constant \(R_0 = R(\psi_0) > 0\) so that for all \(z\) satisfying \(\arg z = \psi_0\) and \(|z| \geq R_0\), and for all \((k, j) \in \Gamma\), we have

\[
\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq c \left( \frac{T(\alpha r, f)}{r} \log^\alpha r \log T(\alpha r, f) \right)^{(k-j)}
\]

(4)

If \(f\) is of finite order then \(f(z)\) satisfies:

\[
\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho-1+\epsilon)}
\]

(5)
for all $z$ satisfying $\arg z = \psi_0 \notin E_1$ and $|z| \geq R_0$ and for all $(k, j) \in \Gamma$

(ii) there exists a set $E_2 \subset (1, \infty)$ that has finite logarithmic measure and there exists a constant $c > 0$ that depends only on $\alpha$ and $\Gamma$ such that for all $z$ satisfying $|z| = r \notin E_2 \cup [0, 1]$ and for all $(k, j) \in \Gamma$, inequality (4) holds.

If $f(z)$ is of finite order then $f(z)$ satisfies inequality (5), for all $z$ satisfying $|z| = r \notin E_3$ and for all $(k, j) \in \Gamma$.

(iii) there exists a set $E_3 \subset [0, \infty)$ that has finite linear measure and there exists a constant $c > 0$ that depends only on $\alpha$ and $\Gamma$ such that for all $z$ satisfying $|z| = r \notin E_3$ and for all $(k, j) \in \Gamma$ we have

$$|f^{(k)}(z)| \leq c (T(\alpha r, f)r^c \log T(\alpha r, f))^{(k-j)}$$  \hfill (6)

If $f(z)$ is of finite order then

$$|f^{(k)}(z)| \leq |z|^{(k-j)(\rho + \epsilon)}$$ \hfill (7)

for all $z$ satisfying $|z| \notin E_3$ and for all $(k, j) \in \Gamma$.

Next lemma is due to [25] and is proved using Phragmén-Lindelöf theorem.

**Lemma 2.** Let $A(z)$ be an entire function such that $\rho(A) \in (0, \infty)$ then there exists sector $S(\alpha, \beta)$ where $\alpha < \beta$ and $\beta - \alpha \geq \frac{\pi}{\rho(A)}$ such that

$$\limsup_{r \to \infty} \frac{\log \log |A(re^{i\theta})|}{\log r} = \rho(A)$$

for all $\theta \in (\alpha, \beta)$.

For a non-constant polynomial $P(z) = a_nz^n + \ldots + a_0$ of degree $n$ we denote $\delta(P, \theta) = \Re(a_n e^{in\theta})$. The rays $z = \theta$ such that $\delta(P, \infty) = 0$ divides the complex plane into $2n$ sectors of equal width $\frac{\pi}{n}$. Also $\delta(P, \theta) > 0$ and $\delta(P, \theta) < 0$ in the alternative sectors. We state next lemma which is due to [2] and is useful for estimating an entire function $A(z)$ satisfying $\lambda(A) < \rho(A)$.

**Lemma 3.** Let $A(z) = v(z)e^{P(z)}$ be an entire function with $\lambda(A) < \rho(A)$, where $P(z)$ is a non-constant polynomial of degree $n$ and $v(z)$ is an entire function. Then for every $\epsilon > 0$ there exists $E \subset [0, 2\pi)$ of linear measure zero such that

(i) for $\theta \notin E$ such that $\delta(P, \theta) > 0$ and there exists $R > 1$ such that

$$|A(re^{i\theta})| \geq \exp ((1 - \epsilon)\delta(P, \theta) r^n)$$  \hfill (8)

for $r > R$. 
(ii) for $\theta \notin E$ such that $\delta(P, \theta) < 0$ there exists $R > 1$ such that

$$|A(re^{i\theta})| \leq \exp((1 - \epsilon)\delta(P, \theta)r^\sigma)$$  \hspace{1cm} (9)

for $r > R$.

**Lemma 4.** [17] Let $f(z)$ be a non-constant entire function. Then there exists a real number $R > 0$ such that for all $r \geq R$ we have

$$\left|\frac{f(z)}{f'(z)}\right| \leq r$$  \hspace{1cm} (10)

where $|z| = r$.

Next lemma give property of an entire function with Fabry gap and can be found in [21], [28].

**Lemma 5.** Let $g(z) = \sum_{n=0}^{\infty} a_\lambda z^{\lambda_n}$ be an entire function of finite order with Fabry gap, and $h(z)$ be an entire function with $\rho(h) = \sigma \in (0, \infty)$. Then for any given $\epsilon \in (0, \sigma)$, there exists a set $H \subset (1, +\infty)$ satisfying $\log \text{dense} H \geq \xi$, where $\xi \in (0, 1)$ is a constant such that for all $|z| = r \in H$, one has

$$\log M(r, h) > r^{\sigma - \epsilon}, \quad \log m(r, g) > (1 - \xi) \log M(r, g),$$

where $M(r, h) = \max\{|h(z)| : |z| = r\}$, $m(r, g) = \min\{|g(z)| : |z| = r\}$, and $M(r, g) = \max\{|g(z)| : |z| = r\}$.

The following remark follows from the above lemma.

**Remark 1.** Suppose that $g(z) = \sum_{n=0}^{\infty} a_\lambda z^{\lambda_n}$ be an entire function of order $\sigma \in (0, \infty)$ with Fabry gap then for any given $\epsilon > 0$, $(0 < 2\epsilon < \sigma)$, there exists a set $H \subset (1, +\infty)$ satisfying $\log \text{dense} H \geq \xi$, where $\xi \in (0, 1)$ is a constant such that for all $|z| = r \in H$, one has

$$|g(z)| > M(r, g)^{(1 - \xi)} > \exp((1 - \xi)r^{\sigma - \epsilon}) > \exp(r^{\sigma - 2\epsilon}).$$

**Lemma 6.** [5] Let $f(z)$ be an entire function of infinite order then

$$\rho_2(f) = \limsup_{r \to \infty} \frac{\log \log v(r, f)}{\log r}$$

where $v(r, f)$ is the central index of the function $f(z)$.

In [31], C. Zongxuan provides the upper bound for the hyper-order of solutions $f(z)$ of the equation (II).

**Theorem 10.** [31] Suppose that $A(z)$ and $B(z)$ are entire functions of finite order. Then

$$\rho_2(f) \leq \max\{\rho(A), \rho(B)\}$$

for all solutions $f$ of the equation (II).

The following result is from Wiman-Valiron theory and we use this result to prove our next lemma which is motivated from Theorem [10].
Theorem 11. [19] Let $g$ be a transcendental entire function, let $0 < \delta < \frac{1}{4}$ and $z$ be such that $|z| = r$ and
\[ |g(z)| > M(r, g) v(r, g)^{-\frac{1}{4} + \delta} \]
holds. Then there exists a set $F \subset \mathbb{R}_+$ of finite logarithmic measure such that
\[ g^{(m)}(z) = \left( \frac{v(r, g)}{z} \right)^m (1 + o(1)) g(z) \]
holds for all $m \geq 0$ and for all $r \notin F$, where $v(r, g)$ is the central index of the function $g(z)$.

Lemma 7. Let us suppose that $A(z)$ and $B(z)$ be entire functions such that $\rho(A)$ and $\mu(B)$ are finite then
\[ \rho_2(f) \leq \max \{ \rho(A), \mu(B) \} \]
for all solutions $f$ of the equation (7).

Proof. Suppose $\max \{ \rho(A), \mu(B) \} = \rho$. Thus for $\epsilon > 0$ we have
\[ |A(re^{io})| \leq \exp r^{\rho+\epsilon} \] (11)
and
\[ |B(re^{io})| \leq \exp r^{\rho+\epsilon} \] (12)
for sufficiently large $r$. From Theorem [11], we choose $z$ satisfying $|z| = r$ and $|f(z)| = M(r, f)$ then there exists a set $F \subset \mathbb{R}_+$ having finite logarithmic measure such that
\[ \frac{f^{(m)}(z)}{f(z)} = \left( \frac{v(r, f)}{z} \right)^m (1 + o(1)) \] (13)
for $m = 1, 2$ and for all $|z| = r \notin F$, where $v(r, f)$ is the central index of the function $f(z)$. Thus using equation (11), (11), (12) and (13) we get
\[ \left( \frac{v(r, f)}{z} \right)^2 |(1 + o(1))| \leq \exp (r^{\rho+\epsilon}) \left( \frac{v(r, f)}{z} \right) |(1 + o(1))| + \exp (r^{\rho+\epsilon}) \] (14)
for all $|z| = r \notin F$, from here we get
\[ \limsup_{r \to \infty} \frac{\log \log |v(r, f)|}{\log r} \leq \rho + \epsilon. \] (15)
Since $\epsilon > 0$ chosen is arbitrary we get $\rho_2(f) \leq \rho$. □

Lemma 8. [29] Suppose $B(z)$ be an entire function with $\mu(B) \in [0, 1)$. Then for every $\alpha \in (\mu(B), 1)$, there exists a set $E \subset [0, \infty)$ such that $\log \text{d} \mu E \geq 1 - \frac{\mu(B)}{\alpha}$ and $m(r) > M(r) \cos \frac{\pi}{\alpha}$, for all $r \in E$, where $m(r) = \inf_{|z|=r} \log |B(z)|$, and $M(r) = \sup_{|z|=r} \log |B(z)|$.

The above lemma is also true for an entire function $B(z)$ with $\rho(B) < \frac{1}{2}$. We can get next lemma easily using Lemma [8].
Lemma 9. If $B(z)$ be an entire function with $\mu(B) \in (0, \frac{1}{2})$. Then for any $\epsilon > 0$ there exists $(r_n) \to \infty$ such that

$$|B(r_ne^{i\theta})| > \exp r_n^{\mu(B)-\epsilon}$$

for all $\theta \in [0, 2\pi)$.

Proof. Since $\mu(B) \in (0, \frac{1}{2})$, using Lemma 8 we have, for $\alpha_0 = \frac{\mu(B)+\frac{1}{2}}{2}$, there exists $E \subset \mathbb{R}$ such that $\log \text{dens} E > 0$ and

$$m(r) > M(r) \cos \pi \alpha_0$$

where $m(r) = \inf_{|z|=r} \log |f(z)|$ and $M(r) = \sup_{|z|=r} \log |B(z)|$.

Thus, for $\epsilon > 0$ there exists $(r_n) \subset E$, $(r_n) \to \infty$ such that

$$|B(r_ne^{i\theta})| > \exp r_n^{\mu(B)-\epsilon}$$

for all $\theta \in [0, 2\pi)$.

Lemma 10. Let $B(z)$ be an entire function with $\mu(B) \in [\frac{1}{2}, \infty)$. Then there exists a sector $S(\alpha, \beta)$, $\beta - \alpha \geq \frac{\pi}{\mu(B)}$, such that

$$\limsup_{r \to \infty} \frac{\log \log |B(re^{i\theta})|}{\log r} \geq \mu(B)$$

for all $\theta \in S(\alpha, \beta)$, where $0 \leq \alpha < \beta \leq 2\pi$.

3. Proof of Theorem 7

Proof. (1) We know that all solutions $f \neq 0$ of the equation (11) are of infinite order, when $\rho(B) \neq \rho(A)$. Then from Lemma 11 for $\epsilon > 0$, there exists a set $E_3 \subset [0, \infty)$ that has finite linear measure such that for all $z$ satisfying $|z| = r \notin E_3$ we have

$$\left| \frac{f''(z)}{f'(z)} \right| \leq cr [T(2r, f)]^2 \quad (16)$$

where $c > 0$ is a constant.

If $\rho(A) < \rho(B)$ then from Theorem 5 and Theorem 10 we get that $\rho_2(f) = \max\{\rho(A), \rho(B)\}$.

If $\rho(B) < \rho(A)$ then we can choose $\beta$ such that $\rho(B) < \beta < \rho(A)$. Now choose $\theta \notin E$, $\delta(P, \theta) > 0$ and $(r_m) \notin E_3$ such that equations (8), (10) and (16) are satisfied for $z_m = r_me^{i\theta}$. Using equation (11), (8), (10) and (16) for $z_m = r_me^{i\theta}$ we have

$$\exp \left( (1 - \epsilon)\delta(P, \theta)r_m^n \right) \leq |A(r_me^{i\theta})|$$

$$\leq \left| \frac{f''(r_me^{i\theta})}{f'(r_me^{i\theta})} \right| + |B(r_me^{i\theta})| \left| \frac{f(r_me^{i\theta})}{f'(r_me^{i\theta})} \right|$$

$$\leq cr_m [T(2r_m, f)]^2 + \exp (r_m^\beta) r_m$$
which implies that
\[
\limsup_{r \to \infty} \frac{\log \log T(r,f)}{\log r} \geq \rho(A)
\]
(17)
then from Theorem [10] and equation (17) we have
\[
\rho_2(f) = \max\{\rho(A), \rho(B)\}
\]
(2) It has been proved that all non-trivial solutions \( f(z) \) of the equation (1), with \( A(z) \) and \( B(z) \) satisfying the hypothesis of the theorem, are of infinite order. Also if \( \rho(A) \neq \rho(B) \) then from part [1],
\[
\rho_2(f) = \{\rho(A), \rho(B)\}
\]
Now let \( \rho(A) = \rho(B) = \rho \). Using Lemma [11], for \( \epsilon > 0 \), there exists \( E_3 \subset [0, \infty) \) with finite linear measure such that for all \( z \) satisfying \( |z| = r \notin E_3 \) we have
\[
\left| \frac{f^{(k)}(z)}{f(z)} \right| \leq crT(2r,f)^{2k}
\]
(18)
where \( c > 0 \) is a constant and \( k \in \mathbb{N} \). Also from Lemma [5], for \( \epsilon > 0 \), there exist \( H \subset (1, \infty) \) satisfying \( \log \text{dens}H \geq 0 \) such that for all \( |z| = r \in H \) we have
\[
|B(z)| > \exp \left( r^{\rho-\epsilon} \right)
\]
(19)
Next choose \( \theta \notin E, \delta(P,\theta) < 0 \) and \( r_m \in H \setminus E_3 \), from equations (11), (12), (18) and (19) we have
\[
\exp \left( r_m^{\rho-\epsilon} \right) < |B(r_m e^{i\theta})| \leq \left| \frac{f''(r_m e^{i\theta})}{f'(r_m e^{i\theta})} \right| + \left| \frac{A(r_m e^{i\theta})}{f'(r_m e^{i\theta})} \right| \leq cr_m T(2r_m, f)^4 + \exp ((1-\epsilon)\delta(P,\theta)r^\rho)cr_m T(2r_m, f)^2 \leq cr_m T(2r_m, f)^4(1+o(1)).
\]
Thus we conclude that
\[
\limsup_{r \to \infty} \frac{\log \log T(r,f)}{\log r} \geq \rho.
\]
(20)
Using Theorem [10] and equation (20) we get
\[
\rho_2(f) = \{\rho(A), \rho(B)\}.
\]
4. Proof of Theorem [8]

Proof. If $\rho(A) = \infty$ then result follows from equation (11). Therefore suppose that $\rho(A) < \infty$.

If $\rho(A) < \mu(B)$ then result follows from Theorem [1]. Let us suppose that $\mu(B) < \rho(A)$ and $f$ be a non-trivial solution of the equation (11) with finite order. Then using Lemma [1], for each $\epsilon > 0$, there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) \setminus E_1$, then there is a constant $R_0 = R_0(\psi_0) > 0$ and

$$|f^{(k)}(z)/f(z)| \leq |z|^{2\rho(f)}, \quad k = 1, 2 \quad (21)$$

for all $z$ satisfying $\arg z = \psi_0$ and $|z| \geq R_0$. Since $\lambda(A) < \rho(A)$ therefore $A(z) = v(z)e^{P(z)}$, where $P(z)$ is a non-constant polynomial of degree $n$ and $v(z)$ is an entire function such that $\rho(v) = \lambda(A) < \rho(A)$. Then using Lemma [3], there exists $E \subset [0, 2\pi)$ with linear measure zero such that for $\theta \notin E \cup E_1$ and $\delta(P, \theta) < 0$ there exists $R_1 > 1$ such that

$$|A(re^{i\theta})| \leq \exp \left((1 - \epsilon)\delta(P, \theta)r^n\right) \quad (22)$$

for $r > R_1$.

We have following three cases on lower order of $B(z)$:

1. when $0 < \mu(B) < \frac{1}{2}$ then from Lemma [10], there exists $(r_n) \to \infty$ such that

$$|B(re^{i\theta})| > \exp \left(r^{\frac{\mu(B)}{2}}\right) \quad (23)$$

for all $\theta \in [0, 2\pi)$ and $r > R$, $r \in (r_n)$.

Using equation (11), (21), (22) and (23) we have

$$\exp \left(r^{\frac{\mu(B)}{2}}\right) < |B(z)| \leq \frac{|f''(z)|}{|f(z)|} + |A(z)| \frac{|f'(z)|}{|f(z)|} \leq r^{2\rho(f)} \left\{1 + \exp \left((1 - \epsilon)\delta(P, \theta)r^n\right)\right\} = r^{2\rho(f)} \left\{1 + o(1)\right\}$$

for all $\theta \in [0, 2\pi) \setminus (E \cup E_1)$ and $r > R$.. This will conduct a contradiction for sufficiently large $r$.

Thus all non-trivial solutions are of infinite order in this case.

2. Now if $\mu(B) \geq \frac{1}{2}$ then by Lemma [10] we have that there exists a sector $S(\alpha, \beta)$, $0 \leq \alpha < \beta \leq 2\pi$, $\beta - \alpha \geq \frac{2\pi}{\mu(B)}$ such that

$$\limsup_{r \to \infty} \frac{\log \log |B(re^{i\theta})|}{\log r} \geq \mu(B) \quad (24)$$

for all $\theta \in S(\alpha, \beta)$.

Since $\mu(B) < \rho(A)$ therefore there exists $S(\alpha', \beta') \subset S(\alpha, \beta)$ such that for all $\phi \in S(\alpha', \beta')$ we have

$$|A(re^{i\phi})| \leq \exp \left((1 - \epsilon)\delta(P, \theta)r^n\right) \quad (25)$$
for all \( r > R \). From equation (24) we get
\[
\exp \left( r^{\mu(B)} - \epsilon \right) \leq |B(re^{\phi})| \tag{26}
\]
for \( \phi \in S(\alpha', \beta') \) and \( r > R \). As done in above case, using equation (1), (21), (25) and (26) we get contradiction for sufficiently large \( r \).

(3) If \( \mu(B) = 0 \) then from Lemma [8] for \( \alpha \in (0, 1) \), there exists a set \( E_2 \subset [0, \infty) \) with finite linear measure such that
\[
|B(re^{\theta})| > \log M(r, B) \frac{1}{\sqrt{2}} \tag{27}
\]
for all \( \theta \in [0, 2\pi) \) and \( r \in E_2 \). Now using equation (1), (21), (25) and (27) we get
\[
M(r, B) \frac{1}{\sqrt{2}} < |B(re^{\theta})| \leq r^{2\rho(f)} \{ 1 + \exp (1 - \epsilon) \delta(P, \theta) r^n \}
\]
for \( \theta \notin E \cup E_1, \delta(P, \theta) < 0 \) and \( r > R, r \in E_2 \). This implies that
\[
\liminf_{r \to \infty} \frac{\log M(r, B)}{\log r} < \infty
\]
which is not so as \( B(z) \) is an transcendental entire function. Thus non-trivial solution \( f \) with finite order of the equation (1) can not exist in this case also.

Therefore all non-trivial solutions of the equation (1) are of infinite order. \( \square \)

5. PROOF OF THEOREM [9]

Proof. We know that under the hypothesis of the theorem, all non-trivial solutions \( f(z) \) of the equation (1) are of infinite order. Therefore from Lemma [1] for \( \epsilon > 0 \), there exists a set \( E_3 \subset [0, \infty) \) with finite linear measure
\[
\left| \frac{f''(z)}{f'(z)} \right| \leq cr[T(2r, f)]^2 \tag{28}
\]
for all \( z \) satisfying \( |z| = r \notin E_3 \) and \( c > 0 \) is a constant.

If \( \rho(A) < \mu(B) \) then from Theorem [5] and Lemma [7] we get that
\[
\rho_2(f) = \max\{ \rho(A), \mu(B) \}
\]
for all non-trivial solutions \( f(z) \) of the equation (1).

Now let \( \mu(B) < \rho(A) \). It is easy to choose \( \eta \) such that \( \mu(B) < \eta < \rho(A) \). From Lemma [3], we have
\[
\exp ((1 - \epsilon) \delta(P, \theta) r^n) \leq |A(re^{\theta})| \tag{29}
\]
for all $\theta \notin E$, $\delta(P, \theta) > 0$ and for sufficiently large $r$. Also

$$\left| B(re^{i\theta}) \right| \leq \exp r^n \quad (30)$$

for sufficiently large $r$ and for all $\theta \in [0, 2\pi)$.

Thus from equations (1), (10), (28), (29) and (30) we have

$$\exp ((1 - \epsilon)\delta(P, \theta)r^n) \leq |A(re^{i\theta})| \leq \frac{f''(re^{i\theta})}{f'(re^{i\theta})} + |B(re^{i\theta})| \frac{|f(re^{i\theta})|}{f'(re^{i\theta})} \leq cr[T(2r, f)]^2 + \exp (r^\eta)r$$

for all $\theta \notin E$, $\delta(P, \theta) > 0$ and for sufficiently large $r$. Since $\eta < n$ we have

$$\exp ((1 - o(1))r^n) \leq dr[T(2r, f)]^2 \quad (31)$$

for sufficiently large $r$. Thus

$$\limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r} \geq n.$$}

Now using $\rho_2(f) \geq \max\{\rho(A), \mu(B)\}$ and Lemma 7 we get the required result.

□

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Department of Mathematics, Deen Dayal Upadhyaya College, University of Delhi, New Delhi-110078, India.

E-mail address: sanjpant@gmail.com

Department of Mathematics, University of Delhi, New Delhi-110007, India.

E-mail address: sainimanisha210@gmail.com