Explicit bounds of unknown function of some new weakly singular retarded integral inequalities for discontinuous functions and their applications

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Abstract
The purpose of the present paper is to establish some new retarded weakly singular integral inequalities of Gronwall-Bellman type for discontinuous functions, which generalize some known weakly singular and impulsive integral inequalities. The inequalities given here can be used in the analysis of the qualitative properties of certain classes of singular differential equations and singular impulsive equations.

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1 Introduction
Being an important tool in the study of qualitative properties of solutions of differential equations and integral equations, various generalizations of Gronwall-Bellman integral inequality and their applications have attracted great interest of many mathematicians (such as [1–11] and the references therein). Gronwall [11] and Bellman [5] established the integral inequality

\[ u(t) \leq c + \int_a^t f(s)u(s) \, ds, \quad t \in [a,b], \]

for some constant \( c \geq 0 \), obtained the estimation of an unknown function,

\[ u(t) \leq c \exp\left( \int_a^t f(s) \, ds \right), \quad t \in [a,b]. \]

Abdeldaim [12] discussed the following nonlinear integral inequality:

\[ u(t) \leq u_0 + \int_0^{u(t)} f(s) \left[ u^{2-p}(s) + \int_0^s g(\tau)u(\tau) \, d\tau \right]^{\theta} \, ds, \quad p \in [0,1), \]

\[ u(t) \leq n(t) + \int_0^{u(t)} f(s) \left[ u(s) + \int_0^s g(\tau)u(\tau) \, d\tau \right]^{\theta} \, ds, \quad p \in [0,1). \]
Usually, this type integral inequalities have regular or continuous integral kernels, but some problems of theory and practicality require us to solve integral inequalities with singular kernels. For example, to prove a global existence and an exponential decay result for a parabolic Cauchy problem. Henry [13] investigated the following linear singular integral inequality:

\[ u(t) \leq a + b \int_0^t (t-s)^{\beta-1} u(s) \, ds. \]

Sano and Kunimatsu [14] generalized Henry’s type inequality to

\[ 0 \leq u(t) \leq c_1 + c_2 t^{\alpha-1} + c_3 \int_0^t u(s) \, ds + c_4 \int_0^t (t-s)^{\beta-1} u(s) \, ds, \]

and gave a sufficient condition for stabilization of semilinear parabolic distributed systems. Ye et al. [15] discussed the linear singular integral inequality

\[ u(t) \leq a(t) + b(t) \int_0^t (t-s)^{\beta-1} u(s) \, ds, \]

and they used it to study the dependence of the solution and the initial condition to a certain fractional differential equation with Riemann-Liouville fractional derivatives. All inequalities of this type are proved by an iteration argument and the estimation formulas are expressed by a complicated power series which is sometimes not very convenient for applications. To avoid the weakness, Medveď [16] presented a new method to solve integral inequalities of Henry-Gronwall type, then he got the explicit bounds with a quite simple formula, similar to the classic Gronwall-Bellman inequalities. Furthermore, he also obtained global solutions of the semilinear evolutions in [17]. In 2008, Ma and Pečarić [18] used the modification of Medveď’s method to study a new weakly singular integral inequality,

\[ u'(t) \leq a(t) + b(t) \int_0^t (t^\beta - s^\beta)^{\gamma-1} s^{\xi-1} f(s) u^\rho(s) \, ds, \quad t \in [0, +\infty). \]

Besides the results mentioned above, various investigators have discovered many useful and new weakly singular integral inequalities, mainly inspired by their applications in various branches of fractional differential equations (see [14, 16–27] and the references therein).

In analyzing the impulsive phenomenon of a physical system governed by certain differential and integral equations, by estimating the unknown function in the integral inequality of the discontinuous functions, Some properties of the solution of some impulsive differential equations can be studied. These inequalities and their various linear and nonlinear generalizations are crucial in the discussion of the existence, uniqueness, boundedness, stability, and other qualitative properties of solutions of differential and integral equations (see [10, 25, 28–34] and the references therein). Tatar [25] discussed the following class of integral inequalities:

\[
 u(t) \leq a(t) + b(t) \int_0^t k_1(t,s) u^m(s) \, ds + c(t) \int_0^t k_2(t,s) u^n(s - \tau) \, ds \\
 + d(t) \sum_{0 < t_k < t} \eta_k u(t_k), \quad t \geq 0,
\]
where \( k_i(t, s) = (t - s)\beta_i s^n F_i(s), \) \( i = 1, 2. \) Iovane [28] studied the following discontinuous function integral inequality:

\[
u(t) \leq a(t) + \int_{t_0}^{t} f(s)\nu(\tau(s)) \, ds + \sum_{t_0 < t_i < t} \beta_i \nu'(t_i - 0), \quad \forall t \geq t_0,
\]

where \( a(t) > 0, f(t) \geq 0, g(t) \geq 0, \beta_i \geq 0, m > 0. \) Gllo et al. [10] studied the impulsive integral inequality

\[
u(t) \leq a(t) + g(t) \int_{t_0}^{t} q(s)\nu'(\tau(s)) \, ds + p(t) \sum_{t_0 < t_i < t} \beta_i \nu'(t_i - 0), \quad \forall t \geq t_0,
\]

where \( a(t) \) is a nondecreasing function as \( t \geq t_0, g(t) \geq 1, p(t) \geq 1, q(s) \in C(\mathbb{R}_+, \mathbb{R}_+), \tau : \mathbb{R} \to \mathbb{R}, \) \( \tau(s) \leq s, \lim_{|s| \to \infty} \tau(s) = \infty, \beta_i \geq 0, m > 0. \) Yan [32] discussed the impulsive integral inequality with delay

\[
u(t) \leq a(t) + \int_{t_0}^{t} f(t, s)\nu(\tau(s)) \, ds + \int_{t_0}^{t} f(t, s) \left( \int_{t_0}^{\tau(s)} g(s, \theta)\nu(\tau(\theta)) \, d\theta \right) ds

+ q(t) \sum_{t_0 < t_i < t} \beta_i \nu'(t_i - 0), \quad \forall t \geq t_0,
\]

where \( a(t) \in C(\mathbb{R}_+, \mathbb{R}_+), f, g \in C(\mathbb{R}_+, \mathbb{R}_+), \tau : \mathbb{R} \to \mathbb{R}, \) \( \tau(s) \leq s, \lim_{|s| \to \infty} \tau(s) = \infty, \beta_i \geq 0, m > 0. \) Mi et al. [30] studied the integral inequality of complex functions with unknown function

\[
u(t) \leq a(t) + \int_{t_0}^{t} f(t, s) \int_{t_0}^{s} g(s, \tau)w(\nu(\tau)) \, d\tau ds

+ q(t) \sum_{t_0 < t_i < t} \beta_i \nu'(t_i - 0), \quad \forall t \geq t_0,
\]

where \( w(u) \) is monotone decreasing continuous function defined on \([0, \infty), \) and \( w(u) > 0 \) when \( u > 0. \) Liu et al. [29] investigated the impulsive integral inequality with delay

\[
u^p(t) \leq a(t) + b(t) \int_{t_0}^{t} \left[ f(s)\nu^p(s) + h(s)\nu'(\sigma(s)) \right] \, ds + \sum_{t_0 < t_i < t} \beta_i \nu'(t_i - 0), \quad \forall t \geq t_0,
\]

where \( a(t), b(t) \geq 1 \) are both nondecreasing functions at \( t \geq t_0, f(s), h(s) \in C(\mathbb{R}_+, \mathbb{R}_+), \) \( \sigma(s) \leq s, \lim_{|s| \to \infty} \sigma(s) = \infty, \beta_i \geq 0, m > 0, p \geq q \geq 0, p \geq r \geq 0. \) Zheng et al. [34] studied the following integral inequality for discontinuous function:

\[
u^p(t) \leq a_0(t) + \frac{p}{p - 1} \sum_{i=1}^{N} \int_{t_0}^{t} g_i(s)\nu^p(\phi_i(s)) \, ds + \sum_{j=1}^{l} \int_{t_0}^{t} b_j(s) \int_{t_0}^{s} c_j(\theta)\nu^p(w_j(\theta)) \, d\theta ds

+ \sum_{t_0 < t_i < t} \beta_i \nu'(t_i - 0),
\]
Let $p > 1$, $\frac{1}{q} + \frac{1}{p} = 1$. Then
\begin{equation}
\int_c^d f(s)g(s) \, ds \leq \left( \int_c^d f^p(s) \, ds \right)^{1/p} \left( \int_c^d g^q(s) \, ds \right)^{1/q}.
\end{equation}

Let $\alpha(t)$ be a continuous, differentiable and increasing function on $[t_0, +\infty)$ with $\alpha(t) \leq t, \alpha(t_0) = t_0$, then
\begin{equation}
\int_{\alpha(t_0)}^{\alpha(t)} f(s)g(s) \, ds \leq \left( \int_{\alpha(t_0)}^{\alpha(t)} f^p(s) \, ds \right)^{1/p} \left( \int_{\alpha(t_0)}^{\alpha(t)} g^q(s) \, ds \right)^{1/q}.
\end{equation}
Proof. We prove the inequality (5). Using the inequality (4), we obtain

\[ \int_{\alpha(t_0)}^{\alpha(t)} f(s)g(s)\,ds = \int_{t_0}^{t} f(\alpha(s))g(\alpha(s))\alpha'(s)\,ds = \int_{t_0}^{t} f(\alpha(s))(\alpha'(s))^{1/p}g(\alpha(s))(\alpha'(s))^{1/q}\,ds \]

\[ \leq \left( \int_{t_0}^{t} f^p(\alpha(s))\,ds \right)^{1/p} \left( \int_{t_0}^{t} g^q(\alpha(s))\,ds \right)^{1/q} \]

\[ = \left( \int_{\alpha(t_0)}^{\alpha(t)} f^p(s)\,ds \right)^{1/p} \left( \int_{\alpha(t_0)}^{\alpha(t)} g^q(s)\,ds \right)^{1/q}. \]

Lemma 2 ([35]) Let \(a_1, a_2, \ldots, a_n\) be nonnegative real numbers, \(m > 1\) is a real number, and \(n\) is a natural number. Then

\[ (a_1 + a_2 + \cdots + a_n)^m \leq n^{m-1}(a_1^m + a_2^m + \cdots + a_n^m). \] (6)

Lemma 3 ([18, 21]) Let \(\beta, \gamma, \xi\) and \(p\) be positive constants. Then

\[ \int_{0}^{t} (t^\beta - s^\beta)^p(\gamma - s^{\xi - 1})^p\,ds = \frac{t^\beta}{\beta} B \left[ \frac{p(\xi - 1) + 1}{p(\gamma - 1) + 1} \right], \quad t \in [0, +\infty). \]

Let \(\alpha(t)\) be a continuous, differentiable and increasing function on \([t_0, +\infty)\) with \(\alpha(t) \leq t\), \(\alpha(t_0) = t_0\), then

\[ \int_{\alpha(t_0)}^{\alpha(t)} (t^\beta - s^\beta)^p(\gamma - s^{\xi - 1})^p\,ds \leq \frac{\alpha^\beta(t)}{\beta} B \left[ \frac{p(\xi - 1) + 1}{p(\gamma - 1) + 1} \right], \quad t \in [0, +\infty), \]

where \(B[x, y] = \int_{0}^{1} s^{x-1}(1 - s)^{y-1}\,ds (x > 0, y > 0)\) is the well-known beta-function and \(\theta = p[\beta(\gamma - 1) + \xi - 1] + 1\). Suppose that the positive constants \(\beta, \gamma, \xi, p_1\) and \(p_2\) satisfy conditions:

1. If \(\beta \in (0, 1), \gamma \in (1/2, 1)\) and \(\xi \geq 3/2 - \gamma, p_1 = 1/\gamma\);
2. If \(\beta \in (0, 1), \gamma \in (0, 1/2)\) and \(\xi > (1 - 2\gamma^2)/(1 - \gamma^2), p_2 = (1 + 4\gamma)/(1 + 3\gamma), \) then

\[ B \left[ \frac{p_1(\xi - 1) + 1}{p_1(\gamma - 1) + 1} \right] \in [0, +\infty), \]

and \(\theta_i = p_i[\beta(\gamma - 1) + \xi - 1] + 1 \geq 0\) are valid for \(i = 1, 2\).

Lemma 4 Let \(u(t), a(t), b(t), h(t) \in C(\mathbb{R}, \mathbb{R})\), \(\alpha(t)\) be a continuous, differentiable and increasing function on \(\mathbb{R}\), with \(\alpha(t) \leq t, \alpha(0) = 0\). If \(u(t)\) satisfies the following inequality:

\[ u(t) \leq a(t) + b(t) \int_{0}^{\alpha(t)} h(s)u(s)\,ds. \] (7)

Then

\[ u(t) \leq a(t) + \frac{b(t)}{e(\alpha(t))} \int_{0}^{\alpha(t)} h(s)\alpha(s)e(s)\,ds, \] (8)

where

\[ e(t) = \exp \left( -\int_{0}^{t} h(s)b(s)\,ds \right). \] (9)
Theorem 1 Let $a(t), f_1(t), f_2(t), f_3(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$, and $a(t)$ is a nondecreasing function, and let $a(t)$ be a continuous, differentiable and increasing function on $\mathbb{R}_+$, with $a(t) \leq t, a(0) = 0$. Let $\beta, \gamma, \xi$ be positive constants. Suppose that $u(t)$ satisfies the inequality (1).

(1) If $\beta \in (0, 1], \gamma \in (1/2, 1)$ and $\xi \geq 3/2 - \gamma$, we have

$$u(t) \leq \left( \tilde{a}_1(t) + \tilde{b}_1(t) \int_0^{a(t)} \tilde{h}_1(s) \tilde{a}_1(s) \tilde{c}_1(s) \, ds \right)^{1-\gamma}, \quad t \in \mathbb{R}_+,$$

Proof Define a function $v(t)$ on $\mathbb{R}$, by

$$v(t) = e(\alpha(t)) \int_0^{a(t)} h(s)u(s) \, ds,$$

we have $v(0) = 0$. Differentiating $v(t)$ with respect to $t$ and using (7) and (9), we have

$$v'(t) = \alpha'(t)h(\alpha(t))u(\alpha(t))e(\alpha(t)) - \alpha'(t)h(\alpha(t))b(\alpha(t))e(\alpha(t)) \int_0^{a(t)} h(s)u(s) \, ds$$

$$\leq \alpha'(t)h(\alpha(t))a(\alpha(t))e(\alpha(t)) + \alpha'(t)h(\alpha(t))e(\alpha(t)) \int_0^{a(t)} h(s)u(s) \, ds$$

$$- \alpha'(t)h(\alpha(t))b(\alpha(t))e(\alpha(t)) \int_0^{a(t)} h(s)u(s) \, ds$$

$$\leq \alpha'(t)h(\alpha(t))a(\alpha(t))e(\alpha(t)).$$

Integrating both sides of the inequality (11) from 0 to $t$, since $v(0) = 0$ we get

$$v(t) \leq \int_0^t \alpha'(s)h(\alpha(s))a(\alpha(s))e(\alpha(s)) \, ds = \int_0^{a(t)} h(s)a(s)e(s) \, ds.$$

From (10) and (12), we obtain

$$\int_0^{a(t)} h(s)u(s) \, ds \leq \frac{1}{e(\alpha(t))} \int_0^{a(t)} h(s)a(s)e(s) \, ds.$$

Substituting the inequality (13) into (7) we get the required estimation (8). The proof is completed.

Lemma 5 Let $a \geq 0, p \geq q \geq 0$ and $p \neq 0$, then

$$a^\frac{q}{p} \leq \frac{q}{p} - a + \frac{p - q}{p}.$$

Proof If $q = 0$, the inequality above is obviously valid. On the other hand, if $q > 0$, let $\delta = q/p$, then $\delta \leq 1$, by [36], [18] (Lemma 2.1), we obtain

$$a^\delta \leq \frac{q}{p} K^{(q-p)/p} \alpha + \frac{p - q}{p} K^{q/p},$$

for any $K > 0$. Let $K = 1$, we get (14).
where
\[
\tilde{a}_1(t) = 3^{\frac{1}{\gamma}} a^{\frac{1}{\gamma}}(t),
\]
\[
\tilde{b}_1(t) = (3M_1 a^{\alpha_1}(t))^{\frac{1}{\gamma}},
\]
\[
\tilde{h}_1(t) = f_1^{\frac{1}{\gamma}}(s) + \left( f_2(s) \int_0^t f_3(\tau) d\tau \right)^{\frac{1}{\gamma}},
\]
\[
\tilde{e}_1(t) = \exp \left( - \int_0^t \tilde{h}_1(s) \tilde{b}_1(s) ds \right),
\]
\[
M_1 = \frac{1}{\beta} B \left[ \frac{\gamma + \xi - 1}{\beta \gamma}, \frac{2\gamma - 1}{\gamma} \right],
\]
\[
\theta_1 = 1 + \left( \beta(\gamma - 1) + \xi - 1 \right) + 1.
\]

(2) If \( \beta \in (0, 1], \gamma \in (0, 1/2] \) and \( \xi > (1 - 2\gamma^2)/(1 - \gamma^2) \), we have
\[
w(t) \leq \left( \tilde{a}_2(t) + \frac{\tilde{b}_2(t)}{\tilde{e}_2(\alpha(t))} \right) \int_0^{\alpha(t)} \tilde{h}_2(s) \tilde{a}_2(s) \tilde{e}_2(s) ds \right)^{\frac{1}{\gamma}}, \quad t \in \mathbb{R},
\]
where
\[
\tilde{a}_2(t) = 3^{1/\gamma} \alpha^{1/\gamma}(t),
\]
\[
\tilde{b}_2(t) = (3M_2 a^{\alpha_2}(t))^{1/\gamma},
\]
\[
\tilde{h}_2(t) = f_1^{1/\gamma}(s) + \left( f_2(s) \int_0^s f_3(\tau) d\tau \right)^{1/\gamma},
\]
\[
\tilde{e}_2(t) = \exp \left( - \int_0^t \tilde{h}_2(s) \tilde{b}_2(s) ds \right),
\]
\[
M_2 = \frac{1}{\beta} B \left[ \frac{\xi(1 + 4\gamma) - \gamma}{\beta(1 + 3\gamma)}, \frac{4\gamma^2}{1 + 3\gamma} \right],
\]
\[
\theta_2 = 1 + \frac{4\gamma}{1 + 3\gamma} \left( \beta(\gamma - 1) + \xi - 1 \right) + 1.
\]

**Proof** If \( \beta \in (0, 1], \gamma \in (1/2, 1) \) and \( \xi \geq 3/2 - \gamma \), let
\[
p_1 = \frac{1}{\gamma}, \quad q_1 = \frac{1}{(1 - \gamma)},
\]
if \( \beta \in (0, 1], \gamma \in (0, 1/2] \) and \( \xi > (1 - 2\gamma^2)/(1 - \gamma^2) \), let
\[
p_2 = \frac{(1 + 4\gamma)}{(1 + 3\gamma)}, \quad q_2 = \frac{(1 + 4\gamma)}{\gamma},
\]
then
\[
\frac{1}{p_i} + \frac{1}{q_i} = 1, \quad i = 1, 2.
\]
Using Hölder’s inequality in Lemma 1 applied to (1), we have

\[
\begin{align*}
    u(t) & \leq a(t) + \left[ \int_0^{a(t)} (\alpha^\beta(t) - s^\beta)^{p(t)/p(\xi)} s^{q(\xi)/p(\xi)} \, ds \right]^{1/p(\xi)} \left[ \int_0^{a(t)} f_1^{\theta_i}(s) u^{\theta_i}(s) \, ds \right]^{1/q_i} \\
    & \quad + \left[ \int_0^{a(t)} (\alpha^\beta(t) - s^\beta)^{p(t)/p(\xi)} s^{q(\xi)/p(\xi)} \, ds \right]^{1/p(\xi)} \\
    & \quad \times \left[ \int_0^{a(t)} \left( f_2(s) \int_0^s f_3(\tau) \, d\tau \right)^{q_i} \, ds \right]^{1/q_i}.
\end{align*}
\]

Set

\[
\begin{align*}
    z(t) & = a(t) + \left[ \int_0^{a(t)} (\alpha^\beta(t) - s^\beta)^{p(t)/p(\xi)} s^{q(\xi)/p(\xi)} \, ds \right]^{1/p(\xi)} \left[ \int_0^{a(t)} f_1^{\theta_i}(s) z^{\theta_i}(s) \, ds \right]^{1/q_i} \\
    & \quad + \left[ \int_0^{a(t)} (\alpha^\beta(t) - s^\beta)^{p(t)/p(\xi)} s^{q(\xi)/p(\xi)} \, ds \right]^{1/p(\xi)} \\
    & \quad \times \left[ \int_0^{a(t)} \left( f_2(s) \int_0^s f_3(\tau) \, d\tau \right)^{q_i} z^{\theta_i}(s) \, ds \right]^{1/q_i}.
\end{align*}
\]

Then \( z(t) \) is a nondecreasing function, and \( u(t) \leq z(t) \), from (17), we have

\[
\begin{align*}
    z(t) & \leq a(t) + \left[ \int_0^{a(t)} (\alpha^\beta(t) - s^\beta)^{p(t)/p(\xi)} s^{q(\xi)/p(\xi)} \, ds \right]^{1/p(\xi)} \left[ \int_0^{a(t)} f_1^{\theta_i}(s) z^{\theta_i}(s) \, ds \right]^{1/q_i} \\
    & \quad + \left[ \int_0^{a(t)} (\alpha^\beta(t) - s^\beta)^{p(t)/p(\xi)} s^{q(\xi)/p(\xi)} \, ds \right]^{1/p(\xi)} \\
    & \quad \times \left[ \int_0^{a(t)} \left( f_2(s) \int_0^s f_3(\tau) \, d\tau \right)^{q_i} z^{\theta_i}(s) \, ds \right]^{1/q_i}.
\end{align*}
\]

Using the discrete Jensen inequality (6) in Lemma 2 with \( n = 3, m = q_i \), we obtain

\[
\begin{align*}
    z^{\theta_i}(t) & \leq 3^{q_i-1} a^{\theta_i}(t) + 3^{q_i-1} \left[ \int_0^{a(t)} (\alpha^\beta(t) - s^\beta)^{p(t)/p(\xi)} s^{q(\xi)/p(\xi)} \, ds \right]^{q_i/p(\xi)} \left[ \int_0^{a(t)} f_1^{\theta_i}(s) z^{\theta_i}(s) \, ds \right]^{q_i/p(\xi)} \\
    & \quad + 3^{q_i-1} \left[ \int_0^{a(t)} (\alpha^\beta(t) - s^\beta)^{p(t)/p(\xi)} s^{q(\xi)/p(\xi)} \, ds \right]^{q_i/p(\xi)} \\
    & \quad \times \left[ \int_0^{a(t)} \left( f_2(s) \int_0^s f_3(\tau) \, d\tau \right)^{q_i} z^{\theta_i}(s) \, ds \right]^{q_i/p(\xi)}.
\end{align*}
\]

Using Lemma 3, the inequality (18) can be restated as

\[
\begin{align*}
    z^{\theta_i}(t) & \leq 3^{q_i-1} a^{\theta_i}(t) + 3^{q_i-1} \left( M \alpha^\beta(t) \right)^{q_i/p(\xi)} \\
    & \quad \times \left[ \int_0^{a(t)} \left( f_1^{\theta_i}(s) + f_2(s) \int_0^s f_3(\tau) \, d\tau \right)^{q_i} z^{\theta_i}(s) \, ds \right]^{q_i/p(\xi)}.
\end{align*}
\]
for \( t \in \mathbb{R}_+ \), where

\[
M_i = \frac{1}{\beta} \left[ \frac{\beta^i - 1}{\beta}, \beta^i \right], \quad \theta_i = \frac{\beta^i}{\beta^i + \xi - 1} + 1 \geq 0,
\]

for \( i = 1, 2 \). Applying Lemma 4 to (19), we obtain

\[
u(t) = z(t) \leq \tilde{a}_i(t) + \frac{\tilde{b}_i(t)}{\tilde{e}_i(\alpha(t))} \int_0^t \tilde{h}_i(s) \tilde{a}_i(s) \tilde{e}_i(s) \, ds, \quad i = 1, 2, t \in \mathbb{R}_+,
\]

where

\[
\tilde{a}_i(t) = 3^{i-1} a_i(t), \quad \tilde{b}_i(t) = 3^{i-1} (M_i a_i(t))^{\theta_i}, \quad \tilde{h}_i(t) = f_i(t) + \left( \frac{f_i(t)}{f_i(t)} \right)^{\theta_i}, \quad \tilde{e}_i(t) = \exp \left( - \int_0^t \tilde{h}_i(s) \tilde{b}_i(s) \, ds \right),
\]

for \( i = 1, 2 \). Substituting \( p_1 = 1/\gamma, q_1 = 1/(1 - \gamma) \) and \( p_2 = (1 + 4\gamma)/(1 + 3\gamma), q_2 = (1 + 4\gamma)/\gamma \) to (20), respectively, we can get the desired estimations (15) and (16). This completes the proof. \( \square \)

**Theorem 2** Let \( u(t) \) is a nonnegative piecewise continuous function with discontinuous of the first kind in the points \( t_i \) \( (t_0 < t_1 < t_2 < \cdots, \lim_{i \to \infty} t_i = \infty) \), \( a(t), f(t) \in C(\mathbb{R}_+, \mathbb{R}_+) \), \( a(t) \geq 1 \), and let \( \alpha(t) \) be a continuous, differentiable and increasing function on \([t_0, +\infty)\) with \( \alpha(t) \leq t \), \( \alpha(t) = t \), \( i = 0, 1, 2, \ldots \). Let \( p, \beta, \gamma \) be positive constants, \( \beta_i \in [0, \infty) \). If \( u(t) \) satisfies the inequality (2), then we have

\[
u(t) \leq \left( \tilde{a}_i(t) + \frac{1}{\tilde{e}_i(\alpha(t))} \int_{t_i}^t \tilde{h}(s) \tilde{a}_i(s) \tilde{e}_i(s) \, ds \right)^{1-\gamma}, \quad t \in [t_i, t_{i+1}), i = 0, 1, 2, \ldots,
\]

where

\[
\tilde{a}_i(t) = A_i(t), \quad t \in [t_i, t_{i+1}), i = 0, 1, 2, \ldots,
\]

\[
A_i(t) = a(t) + \sum_{j=1}^i \int_{t_{j-1}}^{t_j} (t^\beta - s^\beta)^{\gamma-1} f(s) u(s) \left[ u^2(s) + \int_{t_0}^s g(t) u(t) \, dt \right]^\beta \, ds
\]

\[
+ \sum_{j=1}^i \beta_j u(t_j - 0), \quad i = 0, 1, 2, \ldots,
\]

\[
\tilde{h}(t) = (t^\beta - s^\beta)^{\gamma-1} f(t) \Omega \left( \frac{\alpha^{-1}(s)}{\alpha'(s)} \right),
\]

\[
\tilde{e}_i(t) = \exp \left( - \int_{t_i}^t \tilde{h}(s) \, ds \right).
\]
Proof. Firstly, we consider the case \( t \in [t_0, t_1] \), denoting

\[
v(t) = a(t) + \int_{t_0}^{a(t)} (t^\beta - s^\beta)^{-1} f(s) u(s) \left[ u^2(s) + \int_{t_0}^{a(t)} g(s) u(s) \, ds \right]^p \, ds,
\]

then \( v(t) \) is a nonnegative and nondecreasing continuous function, and

\[
u(t) \leq v(t), \quad v(t_0) = a(t_0).
\]

Differentiating (22) with respect to \( t \), we have

\[
v'(t) = a'(t) + \alpha'(t)(t^\beta - \alpha^\beta(t))^{-1} f(\alpha(t)) u(\alpha(t)) \left[ u^2(\alpha(t)) + \int_{t_0}^{a(t)} g(s) u(s) \, ds \right]^p \]

\[
\leq a'(t) + \alpha'(t)(t^\beta - \alpha^\beta(t))^{-1} f(\alpha(t)) v(\alpha(t)) \left[ v^2(\alpha(t)) + \int_{t_0}^{a(t)} g(s) v(s) \, ds \right]^p.
\]

Let

\[
\Gamma(t) = v^2(\alpha(t)) + \int_{t_0}^{a(t)} g(s) v(s) \, ds,
\]

then \( \Gamma(t) \) is a nonnegative and nondecreasing function, and \( \Gamma(t_0) = a^2(t_0) \), since \( a(t) \geq 1 \), we can conclude that \( v(t) \leq \Gamma(t) \), differentiating (25), from (24), we obtain

\[
\Gamma'(t) = 2v(\alpha(t)) v'(\alpha(t)) \alpha'(t) + a'(t) g(\alpha(t)) v(\alpha(t))
\]

\[
\leq 2 \Gamma(\alpha(t)) \alpha'(t) (a'(t) + \alpha'(t)(t^\beta - \alpha^\beta(t))^{-1} f(\alpha(t)) \Gamma(\alpha(t)) \Gamma' p(\alpha(t))
\]

\[+ a'(t) g(\alpha(t)) \Gamma(\alpha(t))
\]

\[
\leq 2 \Gamma(t \alpha'(t) (a'(t) + \alpha'(t)(t^\beta - \alpha^\beta(t))^{-1} f(\alpha(t)) \Gamma(t) \Gamma' p(\alpha(t))
\]

\[+ a'(t) g(\alpha(t)) \Gamma(t).
\]

From (26), we have

\[
\Gamma^{-(p+1)} \Gamma'(t) \leq \Gamma^{-(p+1)}(t) (2a'(t)a'(t) + a'(t) g(\alpha(t)))
\]

\[+ 2(a'(t))^2 (t^\beta - \alpha^\beta(t))^{-1} f(\alpha(t)).
\]

Let \( \eta(t) = \Gamma^{-(p+1)}(t) \), then \( \eta'(t) = -(p+1) \Gamma^{-(p+1)} \Gamma'(t) \), (27) can be restated as

\[
\eta'(t) + (p+1) \eta(t) (2a'(t)a'(t) + a'(t) g(\alpha(t)))
\]

\[\geq -2(p+1)(a'(t))^2 (t^\beta - \alpha^\beta(t))^{-1} f(\alpha(t)).
\]
Multiplying by \( \exp((p + 1) \int_{t_0}^{a(t)} (2a'(\alpha^{-1}(s)) + g(s)) \, ds) \) on both sides of (28), we have
\[
\left[ \eta(t) \exp \left( (p + 1) \int_{t_0}^{a(t)} (2a'(\alpha^{-1}(s)) + g(s)) \, ds \right) \right]'
\geq -2(p + 1)(p'(t)) \left( (t^\beta - a^\beta(t))^{\gamma-1} f(a(t)) \times \exp \left( (p + 1) \int_{t_0}^{a(t)} (2a'(\alpha^{-1}(s)) + g(s)) \, ds \right) \right)
\]
integrating both sides of (29) from \( t_0 \) to \( t \), we obtain
\[
\eta(t) \exp \left( (p + 1) \int_{t_0}^{a(t)} (2a'(\alpha^{-1}(s)) + g(s)) \, ds \right) - \eta(t_0)
\geq -2(p + 1)(p'(t)) \left( (t^\beta - s^\beta(t))^{\gamma-1} f(s) \times \exp \left( (p + 1) \int_{t_0}^{a(s)} (2a'(\alpha^{-1}(\tau)) + g(\tau)) \, d\tau \right) \right) ds,
\] (30)

since \( \eta(t_0) = \Gamma^{-(p+1)}(t_0) = a^{2(p+1)}(t_0) \), denoting \( \Delta(t) = \exp((p + 1) \int_{t_0}^{a(t)} (2a'(\alpha^{-1}(\tau)) + g(\tau)) \, d\tau) \), from (30), we have
\[
\eta(t) \geq \frac{1 - 2a^{2(p+1)}(t_0) (p + 1) \int_{t_0}^{a(t)} (t^\beta - s^\beta(t))^{\gamma-1} f(s) \Delta(s) ds}{a^{2(p+1)}(t_0) \Delta(t)},
\] (31)

by \( \eta(t) = \Gamma^{-(p+1)}(t) \), from (31), we have
\[
\Gamma^p(t) \leq \left[ \frac{a^{2(p+1)}(t_0) \Delta(t)}{1 - 2a^{2(p+1)}(t_0) (p + 1) \int_{t_0}^{a(t)} (t^\beta - s^\beta(t))^{\gamma-1} f(s) \Delta(s) ds} \right]^{\frac{p}{p+1}},
\] (32)

where \( 1 - 2a^{2(p+1)}(t_0) (p + 1) \int_{t_0}^{a(t)} (t^\beta - s^\beta(t))^{\gamma-1} f(s) \Delta(s) ds > 0 \), setting
\[
\Omega(t) = \left[ \frac{a^{2(p+1)}(t_0) \Delta(t)}{1 - 2a^{2(p+1)}(t_0) (p + 1) \int_{t_0}^{a(t)} (t^\beta - s^\beta(t))^{\gamma-1} f(s) \Delta(s) ds} \right]^{\frac{p}{p+1}},
\] (33)

from (24), (25), (32) and (33), we have
\[
v'(t) \leq a'(t) + \alpha'(t)(t^\beta - a^\beta(t))^{\gamma-1} f(\alpha(t)) v(\alpha(t)) \Omega(t).
\] (34)

Integrating both side of (34) from \( t_0 \) to \( t \), we get
\[
v(t) \leq a(t) + \int_{t_0}^{t} \alpha'(s)(t^\beta - a^\beta(s))^{\gamma-1} f(\alpha(s)) v(\alpha(s)) \Omega(s) ds
\]
\[
= a(t) + \int_{t_0}^{a(t)} (t^\beta - s^\beta(t))^{\gamma-1} f(s) v(s) \Omega \left( \frac{\alpha^{-1}(s)}{\alpha'(s)} \right) ds.
\] (35)
Equation (35) has the same form as Lemma 4, and the functions of (35) satisfy the conditions of Theorem 1. Consequently, by using a similar procedure to Lemma 4 and Theorem 1, we can get the desired estimations (21) for $t \in [t_0, t_1)$.

Next, let us consider the interval $[t_1, t_2)$, when $t \in [t_1, t_2)$, (2) can be restated as

$$u(t) \leq a(t) + \int_{t_0}^{a(t)} (t^\beta - s^\beta) \gamma^{-1} f(s) u(s) \left[ u^2(s) + \int_{t_0}^{s} g(\tau) u(\tau) d\tau \right]^p ds$$

$$+ \int_{t_1}^{a(t)} (t^\beta - s^\beta) \gamma^{-1} f(s) u(s) \left[ u^2(s) + \int_{t_0}^{s} g(\tau) u(\tau) d\tau \right]^p ds + \beta u(t_1 - 0), \quad (36)$$

setting

$$A_1(t) = a(t) + \int_{t_0}^{a(t)} (t^\beta - s^\beta) \gamma^{-1} f(s) u(s) \left[ u^2(s) + \int_{t_0}^{s} g(\tau) u(\tau) d\tau \right]^p ds + \beta u(t_1 - 0),$$

$$\Psi(t) = a(t) + \int_{t_0}^{a(t)} (t^\beta - s^\beta) \gamma^{-1} f(s) u(s) \left[ u^2(s) + \int_{t_0}^{s} g(\tau) u(\tau) d\tau \right]^p ds$$

$$+ \int_{t_1}^{a(t)} (t^\beta - s^\beta) \gamma^{-1} f(s) u(s) \left[ u^2(s) + \int_{t_0}^{s} g(\tau) u(\tau) d\tau \right]^p ds + \beta u(t_1 - 0), \quad (37)$$

then $\Psi(t)$ is a nonnegative and nondecreasing function, and

$$u(t) \leq \Psi(t), \quad u(t_1) \leq \Psi(t_1) = A_1(t_1).$$

Differentiating with respect to $t$ both sides of (37), we obtain

$$\Psi'(t) = A_1'(t) + \alpha'(t) (t^\beta - \alpha(t)^\beta) \gamma^{-1} f(\alpha(t)) u(\alpha(t)) \left[ u^2(\alpha(t)) + \int_{t_0}^{\alpha(t)} g(s) u(s) ds \right]^p$$

$$\leq A_1'(t) + \alpha'(t) (t^\beta - \alpha(t)^\beta) \gamma^{-1} f(\alpha(t)) \Psi(\alpha(t))$$

$$\times \left[ \Psi^2(\alpha(t)) + \int_{t_0}^{\alpha(t)} g(s) \Psi(s) ds \right]^p, \quad (38)$$

(38) has the same form of (24), and using a similar procedure for $t \in [t_1, t_2)$, we can get the desired estimations (21) for $t \in [t_1, t_2)$.

Consequently, by using a similar procedure for $t \in [t_i, t_{i+1})$, we can get the desired estimations (21) for $t \in [t_i, t_{i+1})$. Thus we complete the proof of Theorem 2.

**Theorem 3** Let $u(t)$ is a nonnegative piecewise continuous function with discontinuous of the first kind in the points $t_i (t_0 < t_1 < t_2 < \cdots, \lim_{t \to \infty} t_i = \infty$, $a(t), f(t) \in C(R_+, R_+)$, $a(t) \geq 1$, and let $\alpha(t)$ be a continuous, differentiable and increasing function on $[t_0, +\infty)$ with $\alpha(t) \leq t$, $\alpha(t_i) = t_i$, $i = 0, 1, 2, \ldots, \lim_{t \to \infty} t_i = \infty$, $a(t), f(t) \in C(R_+, R_+)$, $a(t) \geq 1$, and let $\alpha(t)$ be a continuous, differentiable and increasing function on $[t_0, +\infty)$ with $\alpha(t) \leq t$, $\alpha(t_i) = t_i$, $i = 0, 1, 2, \ldots, \lim_{t \to \infty} t_i = \infty$. Let $p, q, m, n, \xi, \beta, \gamma$ be positive constants with $p \geq m$, $p \geq n$, $q \in [0, 1], \beta \in [0, \infty)$. If $u(t)$ satisfies the inequality (3).

(1) If $\beta \in (0, 1], \gamma \in (1/2, 1)$ and $\xi \geq 3/2 - \gamma$, we have

$$u(t) \leq \left[ E_i(t) + \left( \tilde{a}_i(t) + \frac{\tilde{h}_i(t)}{\varepsilon_i(\alpha(t))} \int_{t_i}^{\alpha(t)} \tilde{h}_i(s) \tilde{a}_i(s) \varepsilon_i(s) ds ds \right) \left[ 1 - \gamma \right] \right]^{1/p},$$

$$t \in [t_i, t_{i+1}), i = 0, 1, 2, \ldots, \quad (39)$$
where $M_1, \theta_1$ are the same as in Theorem 1, and

$$E_{0}(t) = a(t), \quad t \in [t_0, t_1),$$

$$E_i(t) = a(t) + b(t) \sum_{j=0}^{i} \int_{t_j}^{a(t)} \left( (\alpha^\beta(t) - s^\beta)^{\gamma-1} s^{\beta-1}f(s) \right) u^m(s) + \int_{t_j}^{s} g(\tau) u^n(\tau) d\tau \right]^{q} ds$$

$$+ \sum_{j=1}^{i} \beta_j \mu^\alpha(t_j - 0), \quad t \in [t_i, t_{i+1}), i = 1, 2, \ldots,$$

$$\tilde{a}_i(t) = 3^{\frac{1}{\gamma - r}} A_i^{\frac{1}{\gamma - r}} (t), \quad i = 0, 1, 2, \ldots, $$

$$A_i(t) = b(t) \int_{t_i}^{a(t)} \left( t^\beta - s^\beta \right)^{\gamma-1} s^{\beta-1} B_i(s) ds, \quad i = 0, 1, 2, \ldots,$$

$$B_i(t) = f(t) \left[ (1 - q) + q \left( \frac{m}{p} E_i(t) + \frac{p - m}{p} \right) \right]$$

$$+ q f(t) \int_{t_i}^{t} g(\tau) \left[ \frac{n}{p} E_i(\tau) + \frac{p - n}{p} \right] d\tau, \quad i = 0, 1, 2, \ldots,$$

$$\tilde{b}_1(t) = (3 M_1 a^\beta(t))^{\frac{1}{\gamma - r}} b^{\frac{1}{\gamma - r}} (t),$$

$$\tilde{\varepsilon}_i(t) = \exp \left( - \int_{t_i}^{t} \tilde{h}_i(s) \tilde{b}_1(s) ds \right), \quad i = 0, 1, 2, \ldots,$$

$$\tilde{h}_i(t) = g_i^{\frac{1}{\gamma - r}} (t) + \left( g_2(t) \int_{t_i}^{t} g_3(\tau) d\tau \right)^{\frac{1}{\gamma - r}},$$

$$g_1(t) = \frac{m q}{p} f(t), \quad g_2(t) = q f(t), \quad g_3(t) = \frac{n}{p} g(t).$$

(2) If $\beta \in (0, 1], \gamma \in (0, 1/2]$ and $\xi > (1 - 2 \gamma^2)/(1 - \gamma^2)$, we have

$$u(t) \leq \left[ E_i(t) + \left( \tilde{a}_i(t) + \frac{\tilde{b}_2(t)}{\tilde{\varepsilon}_i(a(t))} \int_{t_i}^{a(t)} \tilde{h}_i(s) \tilde{a}_i(s) \tilde{\varepsilon}_i(s) ds \right)^{\frac{1}{\gamma - r}} \right]^{1/p},$$

$$t \in [t_i, t_{i+1}), i = 0, 1, 2, \ldots,$$  \hspace{1cm} (40)

where $M_2, \theta_2$ are the same as in Theorem 1 and $E_i, A_i, B_i, h_i, i = 0, 1, 2, \ldots$, are the same in (1) of Theorem 3,

$$\tilde{a}_i(t) = 3^{\frac{1 + \beta \gamma}{\gamma - r}} A_i^{\frac{1}{\gamma - r}} (t), \quad i = 0, 1, 2, \ldots,$$

$$\tilde{b}_2(t) = (3 M_2 a^\beta(t))^{\frac{1}{\gamma - r}} b^{\frac{1}{\gamma - r}} (t),$$

$$\tilde{\varepsilon}_i(t) = \exp \left( - \int_{t_i}^{t} \tilde{h}_i(s) \tilde{b}_2(s) ds \right), \quad i = 0, 1, 2, \ldots,$$

Proof: When $t \in [t_0, t_1)$, (3) can be restated as

$$u^p(t) \leq a(t) + b(t) \int_{t_0}^{a(t)} \left( (\alpha^\beta(t) - s^\beta)^{\gamma-1} s^{\beta-1}f(s) \right) u^m(s) + \int_{t_0}^{s} g(\tau) u^n(\tau) d\tau \right]^{q} ds,$$  \hspace{1cm} (41)
by Lemma 5, we obtain

\[
\left[u^m(s) + \int_{t_0}^{s} g(\tau)u^n(\tau) \, d\tau \right]^q \leq q \left[u^m(s) + \int_{t_0}^{s} g(\tau)u^n(\tau) \, d\tau \right] + (1-q). \tag{42}
\]

Substituting (42) into (41), we have

\[
u^n(t) \leq a(t) + b(t) \int_{t_0}^{a(t)} (a^\beta(t) - s^\beta)^{1/s} - 1 f(s) \, ds \\
\times \left[q \left(u^m(s) + \int_{t_0}^{s} g(\tau)u^n(\tau) \, d\tau \right) + (1-q) \right] \, ds. \tag{43}
\]

Define a function \(w(t)\) by

\[
w(t) = b(t) \int_{t_0}^{a(t)} (a^\beta(t) - s^\beta)^{1/s} - 1 (1-q)f(s) \, ds \\
+ b(t) \int_{t_0}^{a(t)} (a^\beta(t) - s^\beta)^{1/s} - 1 qf(s)u^m(s) \, ds \\
+ b(t) \int_{t_0}^{a(t)} (a^\beta(t) - s^\beta)^{1/s} - 1 qf(s) \int_{t_0}^{s} g(\tau)u^n(\tau) \, d\tau \, ds,
\]

from (43) and (44), we have

\[
u^n(t) \leq a(t) + w(t) \quad \text{or} \quad u(t) \leq (a(t) + w(t))^{1/p}. \tag{45}
\]

By Lemma 5 and (45), we obtain

\[
u^m(t) \leq (a(t) + w(t))^{m/p} \leq \frac{m}{p} (a(t) + w(t)) + \frac{p-m}{p}, \tag{46}
\]

\[
u^n(t) \leq (a(t) + w(t))^{n/p} \leq \frac{n}{p} (a(t) + w(t)) + \frac{p-n}{p}. \tag{47}
\]

Substituting the inequality (46) and (47) into (44) we have

\[
w(t) \leq b(t) \int_{t_0}^{a(t)} (a^\beta(t) - s^\beta)^{1/s} - 1 (1-q)f(s) \, ds \\
+ b(t) \int_{t_0}^{a(t)} (a^\beta(t) - s^\beta)^{1/s} - 1 qf(s) \left[m/p (a(s) + w(s)) + p-m/p \right] \, ds \\
+ b(t) \int_{t_0}^{a(t)} (a^\beta(t) - s^\beta)^{1/s} - 1 qf(s) \int_{t_0}^{s} g(\tau) \left[n/p (a(\tau) + w(\tau)) + p-n/p \right] \, d\tau \, ds \\
\leq b(t) \int_{t_0}^{a(t)} (a^\beta(t) - s^\beta)^{1/s} - 1 f(s) \left[(1-q) + q \left(m/p (a(s) + w(s)) + p-m/p \right) \right] \, ds \\
+ b(t) \int_{t_0}^{a(t)} (a^\beta(t) - s^\beta)^{1/s} - 1 qf(s) \int_{t_0}^{s} g(\tau) \left[n/p (a(\tau) + w(\tau)) + p-n/p \right] \, d\tau \, ds \\
+ b(t) \int_{t_0}^{a(t)} (a^\beta(t) - s^\beta)^{1/s} - 1 \frac{mq}{p} f(s) w(s) \, ds
\]
Theorem 1, applying Theorem 1 to (48), considering equation (45), we can get the desired estimations (39) and (40) for 

where 

\[
A_0(t) = b(t) \int_{t_0}^{\alpha(t)} (t^\alpha - s^\alpha)^{\gamma - 1} s^{\delta - 1} B_0(s) \, ds, \\
B_0(t) = f(t) \left[ (1-q) + q \left( \frac{m}{p} a(t) + \frac{p-m}{p} \right) \right] \\
+ qf(t) \int_{t_0}^{t} g(\tau) \left[ \frac{n}{p} a(\tau) + \frac{p-n}{p} \right] \, d\tau, \\
g_1(t) = \frac{mq}{p} f(t), \quad g_2(t) = qf(t), \quad g_3(t) = \frac{n}{p} g(t).
\]

Since (48) have the same form as (1) and the functions of (48) satisfy the conditions of Theorem 1, applying Theorem 1 to (48), considering equation (45), we can get the desired estimations (39) and (40) for \( t \in [t_0, t_1] \).

Then, when \( t \in [t_1, t_2] \), (3) can be restated as 

\[
u(t) \leq a(t) + b(t) \int_{t_0}^{\alpha(t)} (\alpha^\beta(t) - s^\beta)^{\gamma - 1} s^{\delta - 1} f(s) \left[ u^m(s) + \int_{t_0}^{s} g(\tau) u^p(\tau) \, d\tau \right]^q \, ds \\
+ \beta_1 u^p(t_1 - 0) + b(t) \int_{t_1}^{\alpha(t)} (\alpha^\beta(t) - s^\beta)^{\gamma - 1} s^{\delta - 1} f(s) \\
\times \left[ u^m(s) + \int_{t_1}^{s} g(\tau) u^p(\tau) \, d\tau \right]^q \, ds.
\]

Let 

\[
E_1(t) = a(t) + b(t) \int_{t_0}^{\alpha(t)} (\alpha^\beta(t) - s^\beta)^{\gamma - 1} s^{\delta - 1} f(s) \left[ u^m(s) + \int_{t_0}^{s} g(\tau) u^p(\tau) \, d\tau \right]^q \, ds \\
+ \beta_1 u^p(t_1 - 0),
\]

then we have 

\[
u(t) \leq E(t) + b(t) \int_{t_1}^{\alpha(t)} (\alpha^\beta(t) - s^\beta)^{\gamma - 1} s^{\delta - 1} f(s) \left[ u^m(s) + \int_{t_1}^{s} g(\tau) u^p(\tau) \, d\tau \right]^q \, ds. \tag{49}
\]

From (49), we can conclude that the estimates (39) and (40) are valid for \( t \in [t_1, t_2] \). Consequently, by using a similar procedure for \( t \in [t_1, t_{i+1}] \), we complete the proof of theorem. \qed
3 Some applications

Example 1 Consider the following Volterra type retarded weakly singular integral equations:

\[ y'(t) - \int_{a(t)}^{t} \left( \alpha^{\beta}(t) - s^{\beta} \right)^{\gamma - 1} s^{\beta(1+\delta)-1} \left[ y(s) + \int_{s}^{t} g(\tau) y(\tau) \, d\tau \right]^{q} \, ds = h(t), \tag{50} \]

which arises very often in various problems, especially describing physical processes with aftereffects. Ma and Pečarić [18] discussed the case \( \alpha(t) = t, g(t) \equiv 0 \) in (50).

Theorem 4 Let \( y(t), g(t) \) and \( h(t) \) be continuous functions on \([0, +\infty)\), and let \( \alpha(t) \) be continuous, differentiable and increasing functions on \([0, +\infty)\) with \( \alpha(t) \leq t, \alpha(t_0) = t_0 \). Let \( p, q, \beta, \gamma, \delta \) be positive constants with \( p \geq q \). Assume \( y(t) \) satisfies equation (50).

(1) If \( \beta \in (0, 1], \gamma \in (1/2, 1) \) and \( \beta(1 + \delta) \geq 3/2 - \gamma \), we have

\[ |y(t)| \leq \left[ |h(t)| + \left( \tilde{a}_1(t) + \frac{\tilde{b}_1(t)}{\tilde{e}_1(\alpha(t))} \int_{0}^{\alpha(t)} \tilde{h}_1(s) \tilde{a}_1(s) \tilde{e}_1(s) \, ds \right)^{1-\gamma} \right]^{1/p}, \quad t \in \mathbb{R}_+, \tag{51} \]

where

\[ \tilde{a}_1(t) = 3^{\frac{\gamma}{\gamma - 1}} \int_{0}^{\alpha(t)} A^{1-\gamma}_{\gamma}(s) \, ds, \]

\[ \tilde{b}_1(t) = (3M_1 \alpha^{\beta_0}(t))^{\frac{1}{\gamma - 1}}, \]

\[ \tilde{h}_1(t) = A^{1-\gamma}_{\gamma}(t) \beta \left[ \frac{\gamma + \xi - 1}{\gamma}, \frac{2\gamma - 1}{\gamma} \right], \]

\[ \tilde{e}_1(t) = \exp \left( -\int_{0}^{t} \tilde{h}_1(s) \tilde{h}_1(s) \, ds \right), \]

\[ M_1 = \frac{1}{\gamma} \left[ \beta(\gamma - 1) + \xi - 1 \right] + 1, \]

\[ A_1(t) = (1-q) + q \left( \frac{1}{p} |h(t)| + \frac{p-1}{p} \right) \]

\[ + qK^{q-1} \int_{0}^{t} |g(\tau)| \left[ \frac{1}{p} |h(t)| + \frac{p-1}{p} \right] \, d\tau, \]

\[ A_2(t) = \frac{q}{p}, \quad A_3(t) = qK^{q-1}, \quad A_4(t) = \frac{1}{p} |g(t)|. \]

(2) If \( \beta \in (0, 1], \gamma \in (0, 1/2) \) and \( \xi > (1 - 2\gamma^2)/(1 - \gamma^2) \), we have

\[ |y(t)| \leq \left[ |h(t)| + \left( \tilde{a}_2(t) + \frac{\tilde{b}_2(t)}{\tilde{e}_2(\alpha(t))} \int_{0}^{\alpha(t)} \tilde{h}_2(s) \tilde{a}_2(s) \tilde{e}_2(s) \, ds \right)^{1-\gamma} \right]^{1/p}, \quad t \in \mathbb{R}_+, \tag{52} \]

where

\[ \tilde{a}_2(t) = (3M_2 \alpha^{\beta_0}(t))^{\frac{1}{\gamma - 1}} \int_{0}^{\alpha(t)} A_{1\gamma}^{1-\gamma}(s) \, ds, \]
\[ \tilde{b}_2(t) = (3M_2 \alpha^{2+}(t))^{\frac{1+4\gamma}{\gamma}}, \]
\[ \tilde{h}_2(t) = A_2^{\frac{1+4\gamma}{\gamma}}(s) + \left( A_3(t) \int_{t_0}^t A_4(\tau) \, d\tau \right)^{\frac{1+4\gamma}{\gamma}}, \]
\[ \tilde{e}_2(t) = \exp\left( -\int_{t_0}^t \tilde{h}_2(s) \tilde{b}_2(s) \, ds \right), \]
\[ M_2 = \frac{1}{\beta} \left[ \frac{\xi(1+4\gamma) - \gamma}{\beta(1+3\gamma)} - 4\gamma^2 \right], \]
\[ \theta_2 = \frac{1+4\gamma}{1+3\gamma} \left[ \beta(\gamma - 1) + \xi - 1 \right] + 1. \]

**Proof** From (50), we have
\[ |y(t)|^p \leq |h(t)| + \int_{t_0}^t \left| (a^{\beta}(t) - s^{\beta})^{\gamma-1} s^{\beta(1+\gamma)-1} \right| \gamma \, |y(s)| + \int_{t_0}^t |g(t)| |y(t)| \, d\tau \, ds. \]  
Applying Theorem 3 for \( t \in [t_0, t_i) \) (with \( m = n = 1 \), \( a(t) = |h(t)| \), \( b(t) = |\lambda(t)|^{t^{\beta}}/\Gamma(\gamma) \), \( \xi = \beta(1+\delta) \)) to (53), we obtain the desired estimations (51) and (52).

**Example 2** Consider the following impulsive differential system:
\[ \frac{d(x(t))}{dt} = F(t,x), \quad t \neq t_i, t \in [t_0, \infty), \]
\[ \Delta(x)|_{t=t_i} = \beta_i x(t_i - 0), \]
\[ x(t_0) = x_0, \]
where \( 0 \leq t_0 < t_1 < t_2 < \cdots, \lim_{i \to \infty} t_i = \infty, x_0 > 0 \) is a constant, \( F(t,x) \) is continuous with respect to \( t \) and \( x \) on \([t_0, \infty) \times (0, +\infty)\). Suppose \( F(s,x) \) satisfies
\[ F(s,x) \leq (t^\beta - s^\beta)^{\gamma-1} f(s) \sqrt{x(s)}, \]
where \( f(t) \in C(\mathbb{R}_+, \mathbb{R}_+), \beta \in (0,1], \gamma \in (1/2,1). \)

Then the impulsive differential system (54) and (55) are equivalent to the integral equation
\[ x(t) = x_0 + \int_{t_0}^t F(s,x(s)) \, ds + \sum_{t_0<t_i<ct} \beta_i x(t_i - 0). \]  
By using the condition (56), from (57), we have
\[ |x(t)| \leq x_0 + \int_{t_0}^t (t^\beta - s^\beta)^{\gamma-1} f(s) \sqrt{x(s)} \, ds + \sum_{t_0<t_i<ct} \beta_i |x(t_i - 0)|. \]  
Let \( u(t) = |x(t)| \), from (58), we get
\[ u(t) \leq x_0 + \int_{t_0}^t (t^\beta - s^\beta)^{\gamma-1} f(s) \sqrt{u(s)} \, ds + \sum_{t_0<t_i<ct} \beta_i u(t_i - 0). \]
By Lemma 5, we have

$$u^1(t) \leq \frac{1}{2} u(t) + \frac{1}{2}. \quad (60)$$

Substituting (60) to (59), we have

$$u(t) \leq x_0 + \int_{t_0}^t (t^\beta - s^\beta)^{\gamma-1} f(s) \left( \frac{1}{2} u(s) + \frac{1}{2} \right) ds + \sum_{t_0 < t_i < t} \beta_i u(t_i - 0)$$

$$\leq x_0 + \int_{t_0}^t (t^\beta - s^\beta)^{\gamma-1} f(s) u(s) ds + \int_{t_0}^t (t^\beta - s^\beta)^{\gamma-1} f(s) \left( \frac{1}{2} u(s) + \frac{1}{2} \right) ds + \sum_{t_0 < t_i < t} \beta_i u(t_i - 0)$$

$$\leq a(t) + \int_{t_0}^t (t^\beta - s^\beta)^{\gamma-1} f(s) u(s) ds + \sum_{t_0 < t_i < t} \beta_i u(t_i - 0), \quad (61)$$

where $$a(t) = x_0 + \int_{t_0}^t (t^\beta - s^\beta)^{\gamma-1} f(s) \frac{1}{2} ds.$$

We see that (61) is the particular form of (3), and the functions of (54) satisfy the conditions of Theorem 3, using the result of Theorem 3, we can conclude that we have the estimated solutions for the impulsive system

$$u(t) \leq E_i(t) + \left( \tilde{a}_i(t) + \tilde{b}_i(t) \int_{t_i}^t \tilde{h}(s) \tilde{e}_i(s) ds \right)^{1-\gamma},$$

$$t \in [t_i, t_{i+1}), i = 0, 1, 2, \ldots,$$

where $$M_i, \theta_i$$ are the same as in Theorem 3, and

$$E_0(t) = a(t), \quad t \in [t_0, t_1),$$

$$E_i(t) = a(t) + \sum_{j=0}^i \int_{t_j}^{t_i} (\alpha^\beta(t) - s^\beta)^{\gamma-1} f(s) \sqrt{u(s)} ds$$

$$+ \sum_{j=1}^i \beta_j u(t_j - 0), \quad t \in [t_i, t_{i+1}), i = 1, 2, \ldots,$$

$$\tilde{a}_i(t) = 2^{1-\gamma} A_i \frac{1}{\alpha^\beta(t)}, \quad i = 0, 1, 2, \ldots,$$

$$A_i(t) = \int_{t_i}^t (t^\beta - s^\beta)^{\gamma-1} B_i(s) ds, \quad i = 0, 1, 2, \ldots,$$

$$B_i(t) = f(t) \left( \frac{1}{2} + \frac{1}{2} E_i(t) \right), \quad i = 0, 1, 2, \ldots,$$

$$\tilde{b}_i(t) = (2M_i \alpha^\beta(t))^{1-\gamma},$$

$$\tilde{e}_i(t) = \exp \left( - \int_{t_i}^t \tilde{h}(s) \tilde{b}_i(s) ds \right), \quad i = 0, 1, 2, \ldots,$$

$$\tilde{h}(t) = g_i^{1-\gamma}(t), \quad g_i(t) = \frac{1}{2} f(t).$$
4 Conclusion
In this paper, we generalized the weakly singular integral inequality. The first inequality was a generally weak singular type, the second inequality was a like-weakly singular type with discontinuous functions, the third inequality was a type of weakly singular integral inequality with impulsive. We used analytical methods, reducing the inequality with the known results in the lemma, and the estimations of the upper bound of the unknown functions were given. The results were applied to the weakly singular integral equation and the impulsive differential system.

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Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
LZZ organized and wrote this paper. WWS examined all the steps of the proofs in this research and gave some advice. All authors read and approved the final manuscript.

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