Some eigenstates for a model associated with solutions of tetrahedron equation.

III. Tetrahedral Zamolodchikov algebras and perturbed strings

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March 1997

Abstract

This paper continues the series begun with works solv-int/9701016 and solv-int/9702004. Here we show how to construct eigenstates for a model based on tetrahedron equation using the tetrahedral Zamolodchikov algebras. This yields, in particular, new eigenstates for the model on infinite lattice—‘perturbed’, or ‘broken’, strings.

Introduction

In this work we continue the study, begun in [1, 2], of eigenstates of the model based on solutions to the tetrahedron equation. The solutions can be found in paper [3]. The method we will be using is based upon the trigonometrical tetrahedral Zamolodchikov algebras described in [4].

The idea is that sometimes we can control the evolution under the action of transfer matrix powers for the states arising from a (kagome) lattice of five-legged ‘$R$-operators’, with given boundary conditions. Here we continue to use the notations of [1, 2, 3] where a four-legged object is called ‘$L$’, a five-legged one is called ‘$R$’, and a six-legged one is called ‘$S$’. The object ‘$L$’ is a usual 1 + 1-dimensional $L$-operator obeying the free-fermion condition. $L$’s are not used directly in this paper. The object ‘$R$’ is, roughly speaking, some special pair of $L$’s. The fifth leg of $R$ serves
to bear an additional superscript taking values 0 and 1 and intended to mark two elements within the pair.

Some special pair of $R$’s enters in the defining relation of a tetrahedral Zamolodchikov algebra, namely

$$R_{12}^a \tilde{R}_{13}^b \tilde{R}_{23}^c = \sum_{d,e,f} S_{def}^{abc} \tilde{R}_{23}^d \tilde{R}_{13}^e \tilde{R}_{12}^d.$$  \hspace{1cm} (1)

This is illustrated by Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

Suppose we have fixed some boundary conditions in the tensor product of spaces denoted 1, 2 and 3 (which means, most generally, that we have taken the trace of a product of each side of (1) and some linear operator acting in the mentioned tensor product). This yields, in the l.h.s. of (1), some vector in the tensor product of spaces corresponding to indices $a$, $b$ and $c$, and in the r.h.s. of (1)—the result of $S$-operator action upon a similar vector. For different boundary conditions, this provides enough (consistent) relations for $S$-operator to be determined uniquely.

We will look at this, however, from another point of view, using boundary conditions for $R$’s as a means to define vectors in the space where $S$’s act. Of course, we will take, instead of just three $R$-operators, a large lattice made up of them, whose fragment is depicted in Figure 2, and apply a layer of $S$-operators—a hedgehog transfer matrix—to it. We will see that sometimes, for simple boundary conditions, this can be a reasonable way of describing the vectors on which the transfer matrix acts, as well as the results of such action.
The $R$-operators we will be dealing with in this paper are the simplest possible—trigonometrical—ones. In this connection, let us refer to Theorem 2.3 of the work [5] wherefrom it follows that to an $S$-matrix corresponds a two-parameter family of triples of $R$-operators. If we restrict ourselves to only trigonometrical $R$-matrices from [4], then there remains a one-parameter family of those. So, below it is implied that we are constructing one-parameter families of states for a given transfer matrix.

1 Two kinds of strings on a finite lattice

1.1 Eigenstates with eigenvalue 1 yielded by the lattice with a given “polarization”

For a finite lattice on a torus, the “periodic” boundary conditions seem, at first sight, to be already fixed. However, trigonometrical $R$-operators of work [1] conserve the “number of particles”, sometimes called also “polarization” (because they are very much like the usual 6-vertex model $L$-operators), and this provides more possibilities. Namely, the reader can easily verify that the following construction yields some states that are transformed into themselves by the hedgehog transfer matrix.

Let us declare some edges of the kagome lattice (Figure 2) ‘black’ and the others ‘white’ in such a way that the number of black lines is conserved at each vertex (the incoming edges being situated below and to the left of the vertex, and the outgoing edges—above and to the right). It can be said that such a configuration of black edges—we will call it permitted configuration—forms a cycle belonging
to some homology class of the torus. Let us say that vectors \((0 \ 1)\) correspond to white edges, and vectors \((1 \ 0)\)—to black edges. This selects some matrix element for each \(R\)-operator, but as there are really two operators numbered by the upper index, this selects a pair of numbers forming a vector in the two-dimensional space corresponding to a vertical edge in Figure 2. The tensor product of such vectors lies in the space where the transfer matrix acts. Now let us take a sum of those vectors over all black edges configurations belonging to the same homological class. It is quite straightforward to see that the transfer matrix transforms this sum into exactly the same sum.

1.2 How the moving strings arise from the lattice of \(R\)-operators

A slight modification of the above construction yields the moving strings from work \([1]\). Namely, fix arbitrarily some straight lines of the kagome lattice of Figure 2 and paint black all the edges belonging to them. Then those lines will move under the action of transfer matrix as described in \([1]\).

The only nontrivial point here is with oblique lines. To make this clear, let us draw some more pictures. First, let us interchange l.h.s. and r.h.s. in Figure 1 and redraw it like the following formula:

\[
S \sum \text{configurations of } \frac{\partial}{\partial} = \sum \text{configurations of } \frac{\partial}{\partial},
\]

where “configurations” means “vectors corresponding to permitted configurations of three black edges within a triangle with given ‘boundary condition’ for six external edges”. In these terms, Figure 1 itself says only that

\[
S \left( \frac{\partial}{\partial} + \frac{\partial}{\partial} \right) = \frac{\partial}{\partial} + \frac{\partial}{\partial}
\]

(here only the black edges are depicted), and not that

\[
S \frac{\partial}{\partial} = \frac{\partial}{\partial}.
\] (2)

However, (2) is proved by direct calculation involving the explicit expressions for matrix elements of \(R\)-operators.
2 ‘Broken’ infinite strings

Consider now the infinite kagome lattice. Let the edges painted black be placed in such manner that they form two horizontal rays, one going to the right and one going to (or rather coming from) the left, as in Figure 3, and those rays be connected by some path going along the lattice edges and permitted in the sense of Subsection 1.1. Consider the vector—the formal infinite tensor product—corresponding to such black edge configuration, and take a sum over all the permitted paths linking the two rays (in particular, the ends A and B of the rays can move anywhere to the left and/or to the right).

This gives some generalization of the straight strings of paper [1] or, at all events, of one such string—we leave the careful consideration of the case of several strings for a separate work. As in [1], formal eigenvectors can be built out of translations of such string.

3 Discussion

3.1 Tetrahedral Zamolodchikov algebras and vacuum vectors

Tetrahedral Zamolodchikov algebras are not, of course, exactly the same thing that the $S$-operator’s vacuum vectors, but they are intimately connected. To be exact, in general case one can obtain a modified Zamolodchikov algebra from vacuum vectors as follows. Let it be known that for some $S$-operator, taken e.g. from the work [7],

$$S(X \otimes Y \otimes Z) = U \otimes V \otimes W,$$

(3)
and let the tetrahedron equation hold (where the dots stand for the parameters on which the $S$’s depend, each of four $S$’s having its own parameters):

$$S_{123}(\ldots) S_{145}(\ldots) S_{246}(\ldots) S_{356}(\ldots) = S_{356}(\ldots) S_{246}(\ldots) S_{145}(\ldots) S_{123}(\ldots) \quad (4)$$

In contrast with the paper [2], here each of the six spaces in whose tensor product the $S$-operators act is marked by a single number. In formula (3) those numbers—superscripts at $S = S_{123}(\ldots)$—are dropped. We will, if necessary, attach those superscripts at vectors as well, to mark the number of space where a vector belong, so that (3) will become

$$S_{123}(\ldots) (X_1 \otimes Y_2 \otimes Z_3) = U_1 \otimes V_2 \otimes W_3.$$  

Now let us apply both sides of (3) to the product $(X_1 \otimes Y_2 \otimes Z_3)$. We will get an equation like (1), but with six, generally speaking, different $R$’s:

$$SR_{45}(\ldots) R_{46}(\ldots) R_{56}(\ldots) = R'_{56}(\ldots) R'_{46}(\ldots) R'_{45}(\ldots),$$

where e.g.

$$R_{45}(\ldots) = S_{145}(\ldots) X_1, \quad R'_{45}(\ldots) = S_{145}(\ldots) U_1$$

e tc. It is clear that a “usual” Zamolodchikov algebra will appear if

$$X = U, \quad Y = V, \quad Z = W.$$  

Probably, the obvious connections between the tetrahedral Zamolodchikov algebras and vacuum vectors, described here, are worth further investigation.

### 3.2 General conclusions

The papers [1, 2] and the present one show that

- even the model corresponding to the simplest solutions of tetrahedron equation possesses a large variety of eigenstates which are probably not easy to classify,

- eigenvalues seem sometimes rather trivial—maybe, it is due to relation $S^2 = 1$, see [3]—but probably they will be more interesting for the models from [3, 4],

- some states can be introduced that are countable sums of formal tensor products throughout the infinite lattice, but it is unclear what to do for a finite lattice,

- there probably does not exists—at least, it was not discovered in papers [1, 4]—a complete analog of the 6-vertex model in 1 + 1 dimensions with its “conservation of particle number”, but something can be built even upon the fact that that number is conserved “sometimes”,
• there exists a huge amount of symmetries multiplying eigenvalues by constants (roots of unity for a finite lattice) and unknown for the 1 + 1-dimensional models, and

• eigenstates can be constructed with making no use of invertibility of tetrahedron equation solutions—so probably it makes sense to search for non-invertible ones.

References

[1] I.G. Korepanov, *Some eigenstates for a model associated with solutions of tetrahedron equation*, solv-int/9701016, 7p.

[2] I.G. Korepanov, *Some eigenstates for a model associated with solutions of tetrahedron equation. II. A bit of algebraization*, solv-int/9702004, 8p.

[3] I.G. Korepanov, *Tetrahedron equation and the algebraic geometry*, Zapiski Nauchnyh Seminarov POMI (S-Petersburg) 209, 137–149 (1994), or hep-th/9401076, 12p.

[4] I.G. Korepanov, *Tetrahedral Zamolodchikov algebra and the two-layer flat model in statistical mechanics*, Mod. Phys. Lett. B 3:3, 201–206 (1989).

[5] I.G. Korepanov, *Tetrahedral Zamolodchikov Algebras Corresponding to Baxter’s L-operators*, Commun. Math. Phys. 154, 85–97 (1993).

[6] J. Hietarinta, *Labeling schemes for tetrahedron equations and dualities between them*, J. Phys. A27, 5727–5748 (1994), also hep-th/9402139.

[7] S.M. Sergeev, V.V. Mangazeev, Yu.G. Stroganov, *The vertex formulation of the Bazhanov-Baxter Model*, hep-th/9504033, 20 p.