On Berinde’s paper

“Comments on some fixed point theorems in metric spaces”

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Abstract

In Theorem 1 of the paper [V. Pata, A fixed point theorem in metric spaces, J. Fixed Point Theory Appl., 10 (2011), 299–305] it is proved that Picard’s iterates for a function converge to a fixed point if a certain condition (C) is verified for all parameters in the interval [0,1]. In the recent paper [V. Berinde, Comments on some fixed point theorems in metric spaces, Creat. Math. Inform. 27 (2018), 15–20], the author claims that Pata’s result does not hold at least in the two extremal cases for the parameter involved in (C). In this note we point out that Berinde’s Theorem 1.1 has only a visual similarity with Pata’s Theorem 1, and the conclusion of Theorem 1.1 is verified only by constant functions.

The paper mentioned in the title, that is [1], begins with the following text:

“Let (X,d) be a metric space. By selecting an arbitrary point x₀ ∈ X, which we call the zero of the metric space X, we denote, according to the terminology and notations in [19], ∥x∥ := d(x,x₀) ∀x ∈ X. We also consider an increasing function Ψ : [0,1] → [0,∞) which is vanishing with continuity at zero and the (vanishing) sequence ωn(α) = (α) \sum_{k=1}^{\infty} Ψ(k), (1.1) where α ≥ 1.

The following theorem is the main result in [19].

**Theorem 1.1.** Let (X,d) be a complete metric space and f : X → X a self mapping of X. Let Λ ≥ 0, α ≥ 1 and β ∈ [0,α] be fixed constants. If the inequality
\[ d(f(x),f(y)) ≤ (1−\epsilon)d(x,y) + \Lambda \epsilon^\alpha Ψ(\epsilon)(1 + \|x\| + \|y\|) \] (1.2)
is satisfied for every ε ∈ [0,1] and every x,y ∈ X, then f possesses a unique fixed point x* = f(x*). Furthermore, by denoting the nth iterate of f by f^n, we have the estimate
\[ d(x*,f^n(x₀)) ≤ Cωn(α), (1.3) \] for some positive constant C ≤ Λ(1 + 4\|x*\|)β.

Our aim in this note is to show that Theorem 1.1 does not hold at least for two extremal cases of the parameter ε involved in the contraction condition (1.2). We also provide a correct version (but not fully in the spirit) of Theorem 1.1 and discuss some other related results.”

The reference “[19]” mentioned in the quoted text is Pata’s paper [3]. The result envisaged by Berinde in [1] is the following theorem quoted from [3 page 299]:

**Theorem 1.** Let Λ ≥ 0, α ≥ 1 and β ∈ [0,α] be fixed constants. If the inequality
\[ d(f(x),f(y)) ≤ (1−\epsilon)d(x,y) + \Lambda \epsilon^\alpha Ψ(\epsilon)(1 + \|x\| + \|y\|) \] (1.1)
is satisfied for every ε ∈ [0,1] and every x,y ∈ X, then f possesses a unique fixed point x* = f(x*). Furthermore, calling f^n = f ◦ ··· ◦ f (n times),

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\[
d(x_*, f^n(x_0)) \leq C\omega_n(\alpha) \tag{1.2}
\]
for some positive constant \(C \leq \Lambda(1 + 4\|x_*\|)^\beta\).”

In \([3\text{ Theorem 1}]\) \((X, d)\), \(f\) and \(\psi\) are as in \([1\text{ Theorem 1.1}]\), while \(\omega_n(\alpha)\) is \((\frac{\alpha}{\alpha})^n \sum_{k=1}^n \psi(\frac{\alpha}{k})\).

Comparing the statements of \([3\text{ Theorem 1}]\) and \([1\text{ Theorem 1.1}]\) we emphasize the following differences:

(a) \([1 + \|x\| + \|y\|]^\beta\) in \([3]\) versus \([1 + \|x\| + \|y\|]\) in \([1]\);
(b) \(C \leq \Lambda(1 + 4\|x_*\|)^\beta\) in \([3]\) versus \(C \leq \Lambda(1 + 4\|x\|)^\beta\) in \([1]\);
(c) \(\omega_n(\alpha) = (\frac{\alpha}{\alpha})^n \sum_{k=1}^n \psi(\frac{\alpha}{k})\) in \([3]\) versus \(\omega_n(\alpha) = (\frac{\alpha}{\alpha})^n \sum_{k=1}^n \psi(\frac{\alpha}{k})\) in \([1]\).

Of course, one could (reasonably) assume that the differences (a), (b) and (c) mentioned above are misprints.

The fact that the missing \(\beta\) in \([1\text{ Eq. (1.2) }]\) is not a misprint is proved by the following text from \([1\text{ page 17}]\):

“If in (1.2) we have \(\varepsilon = 1\) (or we let \(\varepsilon \to 1\)), then this condition becomes
\[
d(f(x), f(y)) \leq L \cdot [1 + d(x, x_0) + d(y, x_0)], \quad \forall x, y \in X, \quad (3.10)
\]
for a fixed element \(x_0 \in X\) and a constant \(L = \Lambda \psi(1)^\gamma\).

In the sequel we denote \(\Psi\) from \([1]\) with \(\psi\). Having in view the above differences, the natural question is if \([3\text{ Theorem 1}]\) and \([1\text{ Theorem 1.1}]\) are equivalent.

A first remark is that in \([3\text{ Theorem 1}]\) and \([1\text{ Theorem 1.1}]\) \(\omega_n(\alpha)\) does not make sense if \(\alpha > 1\) because \(\psi(\frac{\alpha}{k})\) is not well defined at least for \(k = 1\). The solution is simple: set \(\psi(t) := \psi(1)\) for \(t > 1\), or to take from the beginning \(\psi : \mathbb{R}_+ := [0, \infty[ \to \mathbb{R}_+\) with the mentioned properties; this is practically done in \([3\text{ Corollary 2}]\) and its proof by taking \(\psi(\varepsilon) = \varepsilon^\gamma\) (with \(\gamma > 0\) and, without mentioning it explicitly, \(\varepsilon \geq 0\).

We did not detect other drawbacks in the proof of \([3\text{ Theorem 1}]\).

For further discussions, we rewrite the condition that \((X, f, \alpha, \psi, \Lambda, \beta)\) (with \((X, d)\) metric space, \(f : X \to X, \alpha \geq 1, \psi : \mathbb{R}_+ \to \mathbb{R}_+\) increasing and continuous at 0 with \(\psi(0) = 0, \Lambda \geq 0, \beta \in [0, \alpha]\)) from \([1\text{ Theorem 1.1}]\) must satisfy in the form

\[
\forall \varepsilon \in [0, 1], \quad \forall x, y \in X : d(f(x), f(y)) \leq (1 - \varepsilon)d(x, y) + \Lambda \varepsilon^\alpha \psi(\varepsilon)[1 + \|x\| + \|y\|]. \tag{1}
\]

So, \((X, f, \alpha, \psi, \Lambda, \beta)\) satisfies \([1]\) if and only if \((X, f, \alpha, \psi, \Lambda, \beta = 0)\) satisfies \([1]\). Having in view this remark we get the following statement.

**Fact B.** Assume that \([1\text{ Theorem 1.1}]\) is true. If \((X, f, \alpha, \psi, \Lambda, \beta)\) verifies the hypothesis of \([1\text{ Theorem 1.1}]\), then \(f\) is constant.

Proof. As observed above, \((X, f, \alpha, \psi, \Lambda, \beta = 0)\) satisfies \([1]\). By \([1\text{ Theorem 1.1}]\) there exists a unique \(\overline{x} \in X\) such that \(f(\overline{x}) = \overline{x}\) and \(d(\overline{x}, f^n(x_0)) \leq C\omega_n(\alpha)\) for every \(n \geq 1\), where \(0 \leq C \leq \Lambda(1 + 4\|x_*\|)^\beta = 0\). Therefore, \(f(x_0) = \overline{x}\). Fix now \(x'_0 \in X\) and set \(\|x\|' := d(x, x'_0)\). Then for \(x, y \in X\) one has

\[
1 + \|x\| + \|y\| = 1 + d(x, x_0) + d(y, x_0) \leq 1 + 2d(x, x'_0) + d(x, x'_0) + d(y, x'_0) \\
\leq [1 + 2d(x, x'_0)] \cdot [1 + \|x\|' + \|y\|'].
\]

It follows that \([1\text{ Eq. (1.2) }]\) holds for \(x_0, \Lambda\), and \(\beta = 0\) replaced by \(x'_0, \Lambda' := \Lambda \cdot [1 + 2d(x, x'_0)], \beta' := 0\) (and the same \(\alpha\) and \(\psi\)). Applying again \([1\text{ Theorem 1.1}]\), we get a (unique) \(\overline{x} \in X\).
such that \( f(x') = x' = f(x_0') \). By the uniqueness of the fixed point \( x' \) we obtain that \( f(x_0') = x' \). Since \( x_0' \) is arbitrary, it follows that \( f \) is constant.

Note that any \( \lambda \)-contraction \( f \) verifies condition (1) for \( \alpha := 1 \), \( \psi(\varepsilon) := \varepsilon \) for \( \varepsilon \geq 0 \), \( \Lambda := [4(1 - \lambda)]^{-1} \) and \( \beta := 0 \); indeed \( 1 - \varepsilon + \Lambda \varepsilon^2 \geq 1 - \frac{1}{\lambda} = \lambda \) for \( \varepsilon \in \mathbb{R} \), and so
\[
\begin{align*}
d(f(x), f(y)) &\leq \lambda d(x, y) \\
&\leq [1 - \varepsilon + \Lambda \varepsilon^2] d(x, y) \leq (1 - \varepsilon) d(x, y) + \Lambda \varepsilon \psi(\varepsilon)[1 + \|x\| + \|y\|]
\end{align*}
\]

for all \( \varepsilon \in [0, 1] \) and all \( x, y \in X \). So, we conclude that [1, Theorem 1.1] is false.

Therefore, [3, Theorem 1] and [1, Theorem 1.1] are not equivalent.

Recall that Berinde’s aim in his “note is to show that Theorem 1.1 does not hold at least for two extremal cases of the parameter \( \varepsilon \) involved in the contraction condition (1.2),” aim which was achieved as mentioned in [1, page 18]: “By summarizing the comments above, we conclude that Theorem 1.1 does not hold if we have \( \varepsilon = 0 \) or \( \varepsilon = 1 \) in (1.2)”.

Indeed, Berinde proved that the following statement is false for \( \varepsilon \in \{0, 1\} : \)

**Theorem 1.1b.** Let \((X, d)\) be a complete metric space and \( f : X \to X \) a self mapping of \( X \). Let \( \Lambda \geq 0 \), \( \alpha \geq 1 \) and \( \beta \in [0, \alpha] \) be fixed constants. If there exists \( \varepsilon \in [0, 1] \) such that
\[
\forall x, y \in X : d(f(x), f(y)) \leq (1 - \varepsilon) d(x, y) + \Lambda \varepsilon^\alpha \psi(\varepsilon)[1 + \|x\| + \|y\|],
\]
then \( f \) possesses a (unique) fixed point \( x^* = f(x^*) \).

In fact Theorem 1.1b is false not only for \( \varepsilon \in \{0, 1\} \); it is false for any fixed \( \varepsilon \in [0, 1] \). Indeed, take, for example, \( X := \{-1, 1\} \subset (\mathbb{R}, |.|) \) with \( f(x) := -x \). Fix some \( \alpha \geq 1 \), \( \psi : [0, 1] \to \mathbb{R}_+ \) (increasing, continuous at 0 with \( \psi(0) = 0 \)), and \( \varepsilon \in [0, 1] \). Clearly \( |f(x) - f(y)| = |x - y| \leq (1 - \varepsilon) d(x, y) + \varepsilon (1 + \|x\| + \|y\|) \) for \( x, y \in X \). Setting \( \Lambda := \varepsilon^{\alpha - 1} \psi(\varepsilon) \) for \( \varepsilon \in (0, 1] \) and \( \Lambda = 0 \) for \( \varepsilon = 0 \), (2) is verified. Clearly, \( f \) has not fixed points.

Recall also that besides the main aim of [1], that is to prove that “Theorem 1.1 does not hold if we have \( \varepsilon = 0 \) or \( \varepsilon = 1 \) in (1.2)”, another aim was to “provide a correct version (but not fully in the spirit) of Theorem 1.1”.

That “correct version (but not fully in the spirit) of Theorem 1.1” is “the following interesting existence result.

**Theorem 3.5.** Let \((X, d)\) be a complete metric space and \( f : X \to X \) a self mapping of \( X \). Let \( \Lambda \geq 0 \), \( \alpha \geq 1 \) and \( \beta \in [0, \alpha] \) be fixed constants. If the inequality
\[
d(f(x), f(y)) \leq (1 - \varepsilon) d(x, y) + \Lambda \varepsilon^\alpha \Psi(\varepsilon) d(y, f(x))
\]
is satisfied for some \( \varepsilon \in (0, 1) \) and every \( x, y \in X \), then
1) \( \text{Fix}(T) = \{x \in X : TX = x\} \neq \emptyset \);
2) For any \( x_0 \in X \), Picard iteration \( \{x_n\}_{n=0}^{\infty} \), \( x_n = T_{n}x_0 \), converges to some \( x^* \in \text{Fix}T \);
3) The following estimate holds
\[
d(x_{n+i-1}, x^*) \leq \frac{(1-\varepsilon)^i}{\varepsilon}d(x_n, x_{n-1}) \quad n = 0, 1, 2,...; \quad i = 1, 2,...
\]

Related to [1, Theorem 3.5] let us observe the following: 1) Surely, one must have \( f \) instead of \( T \). 2) The introduction of \( \beta \) in its statement is superfluous because \( \beta \) is neither involved in Eq. (3.16), nor in its conclusion; even more, the introduction of \( \alpha \) is superfluous because for \( \varepsilon \in (0, 1) \) fixed, setting \( \delta := 1 - \varepsilon \) and \( L := \Lambda \varepsilon^\alpha \psi(\varepsilon) \), the hypothesis of [1, Theorem 3.5] is saying that \( f \) is “an almost contraction” (see [1, Definition 3.1]), and so [1, Theorem 3.5] is nothing but [1, Theorem 3.4].

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As a conclusion, one can say that [1, Theorem 1.1] has only a visual similarity with [3, Theorem 1]. It seems that the aim of [1] is to advertise its author’s results on fixed point theorems because we may not admit he does not know the meaning of the quantifiers $\forall$ and $\exists$.

We end this note with the following remark extracted from [2]:

“Remark 1 There is a scope of misunderstanding with the inequality (1) in the work of Pata and also in similar other inequalities like (2)–(4) in works incorporating the ideas of Pata. Berinde noted in [7] that if the condition (1) is satisfied, not for all $\varepsilon \in [0,1]$, but just for some specific values, the conclusion of Theorem 1 might not hold. For example, if (1) holds just for $\varepsilon = 0$, then one has just the non-expansive condition
\[ d(fx, fy) \leq d(x, y) \quad \forall x, y \in X, \]
which obviously does not imply the existence of fixed point. Similarly, if $\varepsilon = 1$, (1) reduces to
\[ d(fx, fy) \leq L[1 + \|x\| + \|y\|]^{\beta}, \]
with some constant $L$, which is also known to be insufficient for the existence of a fixed point of $f$.

Similar conclusions hold for conditions (2)–(4).

From this observation Berinde concludes that the Pata-type result is incorrect. But this is not so. There is no contradiction between the above observations and conclusions of the Pata-type theorems for the following reasons. As we have already noted in the introduction, the Pata type results are obtained for functions satisfying a family of inequalities and any single inequality from the above mentioned family will not provide us with a sufficient condition for the existence of a fixed point. Had it been so, then there is no need of considering a family of inequalities. Thus the argument of Berinde is not tenable.”

References

[1] V. Berinde, Comments on some fixed point theorems in metric spaces, Creat. Math. Inform. 27(1) (2018), 15–20.

[2] B.S. Choudhury, Z. Kadelburg, N. Metiya, et al., A survey of fixed point theorems under Pata-type conditions, Bull. Malays. Math. Sci. Soc. 43 (2020), 1289–1309.

[3] V. Pata, A fixed point theorem in metric spaces, J. Fixed Point Theory Appl. 10 (2011), 299–305.