Analytical approach to quasiperiodic beam Coulomb field modeling

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Abstract. The paper is devoted to modeling of space charge field of quasiperiodic axial-symmetric beam. Particle beam is simulated by charged disks. Two analytical Coulomb field expressions are presented, namely, Fourier-Bessel series and trigonometric polynomial. Both expressions permit the integral representation. It provides the possibility of integro-differential beam dynamics description. Consequently, when beam dynamics optimization problem is considered, it is possible to derive the analytical formula for quality functional gradient and to apply directed optimization methods. In addition, the paper presents the method of testing of space charge simulation code.

1. Introduction

The problems of modeling of self-consistent charged particle distributions are urgent and widely discussed by the researchers [1-6]. This paper deals with analytical representation of space charge field. Such an approach is successfully used by many authors e.g. [1-4,7-13], when treating different problems of beam dynamics investigation and optimization.

The paper is devoted to modeling of Coulomb field of quasiperiodic axial-symmetric beam. Particle beam is represented by the set of charged disks. Two Coulomb field models considered are presented in terms of Fourier-Bessel series and trigonometric polynomial. The analytical expressions make it possible to analyze particle interaction and to test space charge field simulation program.

Both models allow integral representation of Coulomb force and provide the possibility of integro-differential beam dynamics description. Consequently, beam dynamics optimization problems may be treated on the basis of the approach suggested by D A Ovsyannikov [1]. Analytical expression for quality criterion gradient may be obtained; one can use directed optimization methods. This approach is successfully used in the papers [7,8,14-19].

2. Beam dynamics equations

For beam dynamics investigation particle-in-cell method is used. The beam is considered to have constant radius \( R \) and to move inside the conducting channel of radius \( a \). Every bunch is represented by \( N \) disks-clouds (cylinders) with radius \( R \) and thickness \( 2\Delta \). May independent variable be \( \tau = ct \), where \( c \) is the velocity of light, \( t \) is the time. Particle phase state is characterized by the vector \((z, p)\), where \( z \) is longitudinal coordinate, \( p \) is reduced impulse. Particle dynamics equations are as follows:

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\[
\frac{dz}{d\tau} = -\frac{p}{\sqrt{1 + p^2}}, \quad \frac{dp}{d\tau} = \frac{e}{m_0c^2} \left( E^{(RF)}(\tau, z) + E(\tau, z) \right).
\]

Here \( e = Q/N, \ m_0 = M_0/N, \ Q \) and \( M_0 \) are the charge and rest-mass of a bunch correspondingly; \( E^{(RF)}, \ E \) are the longitudinal intensity components characterizing respectively the effect of RF field and space charge field on the disk as a whole.

3. Model 1 of particle interaction account: Fourier-Bessel series

Let us assume particle beam to be periodic sequence of bunches. May \( \tau \) be the set of particle phase states of a bunch for \( \tau \) value of independent variable. Let us suppose this set to be continuous medium volume. Potential field intensity characterizing the action of the beam on model particle with \( z \) longitudinal coordinate is described by the following expression [4,9]:

\[
E(\tau, z) = \int G(z - \tilde{z}, \tilde{\rho}) \rho(\tau, \tilde{z}, \tilde{\rho}) d\tilde{z} d\tilde{\rho},
\]

\[
G_j(x, y) = \frac{Qa^2}{2\pi\varepsilon_0 R^2N} \sum_{k=1}^{\infty} \frac{J_1(\mu_k R/a)}{\mu_k^2 J_1^2(\mu_k)} \text{sh}(\mu_k \sqrt{1 + y^2 H/\alpha}) V_z \left(\sqrt{1 + y^2 x, \sqrt{1 + y^2 H}}\right),
\]

\[
V_z(\eta_x, \eta_y) = 2g_x(\eta_x, \eta_y) - g_x(\eta_x + 2\Delta, \eta_y) - g_x(\eta_x - 2\Delta, \eta_y),
\]

\[
g_x(\xi, \eta) = \text{sign}(\xi) \left[ \text{sh}(\mu_x \xi/a) - \text{sh}(\mu_x \xi/a) \right].
\]

Here \( \rho(\tau, z, p) \) is phase density corresponding to dynamic system (1); \( \varepsilon_0 \) is electric constant; \( J_1(\eta) \) is Bessel function of the first order; \( \mu_k, k = 1, 2, \ldots \) are the roots of Bessel function \( J_1(\eta) \) of zero order; \( 2H \) is the length of moving beam spatial quasiperiod. When using the formulae (3)-(5), it is supposed \( |x| \leq H \) in view of the assumption of beam periodicity. Phase density satisfies the following transfer equation on the trajectories of the system (1) [1]:

\[
\frac{\partial \rho}{\partial \tau} + \frac{\partial \rho}{\partial z} \frac{p}{\sqrt{1 + p^2}} + \frac{\partial \rho}{\partial \rho} \frac{e}{m_0c^2} \left( E^{(RF)} + E \right) = 0.
\]

For numerical Coulomb field simulation one can calculate the integral (2) using Monte Carlo method. The approximate formula is as follows:

\[
E(\tau, z) = \frac{1}{N} \sum_{n=1}^{\infty} G_j(z - z_n, p_n),
\]

where \( z_n, p_n, n = 1, N \) are respectively longitudinal coordinates and reduced impulses of model particles.

4. Model 2 of particle interaction account: trigonometric polynomial

Consider the bunch with center coordinate \( z_c \) and average reduced impulse \( p_c \). One can approximate charge density of a bunch by grid function

\[
S = \{ S_j, j = 0, 2M - 1 \},
\]

where \( 2M \) is the number of grid cells. This function may be obtained on the basis of model particles positions and the bunch center state \((z_c, p_c)\) using cloud-in-cell method.
Let us introduce analytical model of charge density in the form of trigonometric polynomial assuming the values (7) at grid points:

$$
\eta(r, z, z_e) = \sum_{m=0}^{n} \left[ A_m \cos \left( \frac{m\pi}{R} (z - z_e) \right) + B_m \sin \left( \frac{m\pi}{R} (z - z_e) \right) \right], \quad \begin{cases} 
1, & r \leq R \\
0, & R < r \leq a.
\end{cases}
$$

(8)

$$
A_0 = \frac{Q}{2\pi R^2 \hat{H}}, \quad A_m = \frac{(-1)^m}{M[1 + m/M]} \sum_{j=0}^{2M-1} S_j \cos(m j \pi / M), \quad m = \overline{1, M}. 
$$

(9)

$$
B_m = \frac{(-1)^m}{M} \sum_{j=0}^{2M-1} S_j \sin(m j \pi / M), \quad m = 0, \overline{1, M}.
$$

(10)

In formula (9) \(\lfloor 1 + m/M \rfloor\) is integer part of the value \(1 + m/M\).

When modeling Coulomb field, let us suppose every particle to have reduced impulse \(p = p_e\). Potential field intensity characterizing the action of periodic sequence of bunches with charge density (8) on model particle with \(z\) longitudinal coordinate is determined by the following expression [4,9]:

$$
E(z, z_e, p_e) = \frac{2a}{R^2 K_{\Delta}} \sum_{m=1}^{n_{\Delta}} \alpha_m \sin \left( \frac{m\pi \Delta}{H \sqrt{1 + p_e^2}} \right) \left[ A_m \sin \left( \frac{m\pi}{H} (z - z_e) \right) - B_m \cos \left( \frac{m\pi}{H} (z - z_e) \right) \right], 
$$

(11)

$$
\alpha_m = \frac{2a RH^2 \sqrt{1 + p_e^2}}{\varepsilon_0} \sum_{i=1}^{\infty} \frac{J_i^2(\mu_i R/a)}{\mu_i^2 J_i^2(\mu_i)} \left( \frac{m\pi}{H^2} a^2 + \mu_i^2 H^2 (1 + p_e^2) \right), \quad m = \overline{1, M}.
$$

(12)

It should be remarked that beam evolution mathematical model in this case includes bunch center dynamics equations.

To obtain Coulomb field integral representation, the dependence of coefficients (9), (10) on particle positions and bunch center state \((z_e, p_e)\) should be expressed analytically. Using the approach suggested in [20] one can derive the formula

$$
E(\tau, z, \tilde{z}, p_e) = \int G_2(z, \tilde{z}, z_e, p_e) \rho(\tau, \tilde{\tau}, \hat{p}) d\tilde{z} d\hat{\tau},
$$

where the function \(G_2(z, \tilde{z}, z_e, p_e)\) is determined by the mode of grid function (7) and polynomial (8) constructing. To obtain smooth integrand (that is necessary for mathematical optimization methods developing), the proper form-factor and interpolation function should be chosen; in this case model particles may be charged nonuniformly.

5. Method of testing of space charge simulation code

We have two models of particle interaction account, but it is not sufficient for code testing. Indeed, the models are too different because model 2 includes the constructing of grid function (7) and trigonometric polynomial (8).

To overcome this difficulty the following idea is suggested in [4,9]. We can introduce the special beam charge density given by trigonometric polynomial (8), where \(A_0 = Q/(2\pi R^2 H)\), and the remaining coefficients \(A_m, B_m, m = \overline{1, M}\) are arbitrary. This density may be simulated with the use of special particle distribution. Coulomb field generated may be calculated using two ways: model 1 and formulae (11)-(12). Due to special charge distribution we avoid constructing of grid function representing charge density as well as trigonometric polynomial (8). To test the code realizing model 1, one can compare the results obtained.
To implement this idea let us prescribe the number $M$ and coefficients $A_m, B_m, m = \overline{1, M}$ and introduce the uniform grid at spatial quasiperiod. May $J >> M$ be the number of grid cells. The number $J$ is to be chosen rather large to provide the proper accuracy of special charge density simulation. To simulate the density (8) we locate $N = J$ model particles in the centers $z_j, j = \overline{1, J}$ of grid cells. Let particle charge values be

$$q_j = \frac{2\pi R^2 H}{J} \sum_{m=0}^{M} A_m \cos\left(\frac{m\pi}{H} (z_j - z_0)\right) + B_m \sin\left(\frac{m\pi}{H} (z_j - z_0)\right), \quad j = \overline{1, J}$$

and particle reduced impulses be $p_c$. Assume the cloud size not to exceed the grid cell size.

It is easy to modify the formulae (3)-(6) for model particles with reduced impulse $p_c$ and different charges (13). For particle distribution presented we have two ways to calculate Coulomb field: the analogue of formulae (3)-(6) and the formulae (11)-(12). For the testing of the code realizing model 1, one can compare the space charge field values in the points $z_j, j = \overline{1, J}$ calculated by two ways mentioned. If the code is correct we can provide (by increasing $J$) the required matching accuracy $\varepsilon$.

6. Numerical experiments

Beam dynamics was investigated for klystron-type buncher with following main characteristics: initial energy of electrons $W_0 = 0.5$ MeV, average beam current $I = 15$ A. Beam dynamics simulation code was developed in cooperation with B.S. Zhuravlev. Two interaction account modes (discussed in this paper) are realized. The numerical results obtained support the conclusion that model 2 is preferred to use when electron bunches are mostly formed.

The testing of code realizing model 1 was successive. The coefficients of trigonometric polynomial (8) were chosen as follows: $A_0 = Q/(2\pi R^2 H)$, $A_m = 0.000003 m$, $B_m = 0.000007 \cos(m + 4)$, $m = \overline{1, M}$. The testing was carried out repeatedly. In particular, when $M = 3$, the accuracy $\varepsilon = 0.05$ was achieved for $J \geq 150$.

7. Conclusion

We discussed two models of particle interaction account in axial-symmetric quasiperiodic beam. Coulomb field representations in the form of Fourier-Bessel series and trigonometric polynomial may be used both for simulation and analysis. The method of testing of space charge simulation code is worked out on the basis of analytical approach.

In addition, both analytical expressions allow the integral representation of space charge field. It provides the possibility of beam dynamics description by integro-differential equations. Consequently, beam dynamics optimization problems may be treated on the basis of the approach suggested by D A Ovsyannikov [1]: analytical expression of quality functional variation may be obtained to enable the use of directed optimization methods.

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