Invariant differential operators for non-compact Lie algebras parabolically related to conformal Lie algebras

V.K. Dobrev

Theory Division, Department of Physics, CERN, CH-1211 Geneva 23, Switzerland
Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Tsarigradsko Chaussee 72, BG-1784 Sofia, Bulgaria
E-mail: Vladimir.Dobrev@cern.ch

ABSTRACT: In the present paper we continue the project of systematic construction of invariant differential operators for non-compact semisimple Lie groups. Our starting points is the class of algebras, which we call ‘conformal Lie algebras’ (CLA), which have very similar properties to the conformal algebras of Minkowski space-time, though our aim is to go beyond this class in a natural way. For this we introduce the new notion of parabolic relation between two non-compact semisimple Lie algebras $G$ and $G'$ that have the same complexification and possess maximal parabolic subalgebras with the same complexification. Thus, we consider the exceptional algebra $E_{7(7)}$ which is parabolically related to the CLA $E_{7(-25)}$, the parabolic subalgebras including $E_{6(6)}$ and $E_{6(-26)}$. Other interesting examples are the orthogonal algebras $so(p,q)$ all of which are parabolically related to the conformal algebra $so(n,2)$ with $p + q = n + 2$, the parabolic subalgebras including the Lorentz subalgebra $so(n - 1, 1)$ and its analogs $so(p - 1, q - 1)$. We consider also $E_{6(6)}$ and $E_{6(2)}$ which are parabolically related to the hermitian symmetric case $E_{6(-14)}$, the parabolic subalgebras including real forms of $sl(6)$.

We also give a formula for the number of representations in the main multiplets valid for CLAs and all algebras that are parabolically related to them. In all considered cases we give the main multiplets of indecomposable elementary representations including the necessary data for all relevant invariant differential operators. In the case of $so(p,q)$ we give also the reduced multiplets. We should stress that the multiplets are given in the most economic way in pairs of shadow fields. Furthermore we should stress that the classification of all invariant differential operators includes as special cases all possible conservation laws and conserved currents, unitary or not.

KEYWORDS: Conformal and W Symmetry, Space-Time Symmetries

ArXiv ePrint: 1208.0409
1 Introduction

Invariant differential operators play a very important role in the description of physical symmetries — starting from the early occurrences in the Maxwell, d’Alembert, Dirac, equations (for more examples cf., e.g., [1]), to the latest applications of (super-)differential operators in conformal field theory, supergravity and string theory (cf. e.g., [2, 3]).

For example, applications of invariant differential operators in supersymmetry involved the study of multiplets, superfields and supercurrents [4–8], of harmonic superspaces [9–19], of auxiliary fields of supergravity [20, 21], on the coupling of supersymmetric Yang-Mills theories to supergravity [22–24], twistor formulation of superstrings [25–27], Landau-Ginzburg description of $N = 2$ minimal models [28, 29], in various other applications to superstrings and supergravity [30–34].
Invariant differential operators played important role in the group-theoretical approach to conformal field theory [35–38], e.g., in the derivation of operator product expansion of two scalar fields.

Invariant super-differential operators were crucial in the derivation of the classification of positive energy unitary irreducible representations of extended conformal supersymmetry in 4D [39–41], later in 3D & 5D [42], in 6D [42, 43], (see also [44, 45]), then for the derivation of the character formulae in 2D [46]. Later applications include [47–71].

Special mentioning requires the applications of exceptional groups, cf. [72–94], since they play important role in the present paper. Exceptional groups recently appeared also as symmetries of Freudenthal dual Lagrangians, as investigated, e.g., in [95].

Finally, among our motivations are the mathematical developments — see the relevant mathematical references: [96–120], and others throughout the text.

Thus, it is important for the applications in physics to study systematically such operators. In a recent paper [121] we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the parabolic subgroups and subalgebras from which the necessary representations are induced. Thus we have set the stage for study of different non-compact groups.

Since the study and description of detailed classification should be done group by group we had to decide which groups to study first. A natural choice would be non-compact groups that have discrete series of representations. By the Harish-Chandra criterion [122, 123] these are groups where holds:

\[ \text{rank } G = \text{rank } K, \]

where \( K \) is the maximal compact subgroup of the non-compact group \( G \). Another formulation is to say that the Lie algebra \( \mathcal{G} \) of \( G \) has a compact Cartan subalgebra.

Example: the groups \( SO(p, q) \) have discrete series, except when both \( p, q \) are odd numbers.

This class is still rather big, thus, we decided to consider a subclass, namely, the class of Hermitian symmetric spaces. The practical criterion is that in these cases, the maximal compact subalgebra \( \mathcal{K} \) is of the form:

\[ \mathcal{K} = so(2) \oplus \mathcal{K}'. \]  

The Lie algebras from this class are:

\[ so(n, 2), \ sp(n, R), \ su(m, n), \ so^*(2n), \ E_6(-14), \ E_7(-25) \]

These groups/algebras have highest/lowest weight representations, and relatedly holomorphic discrete series representations.

The most widely used of these algebras are the conformal algebras \( so(n, 2) \) in \( n \)-dimensional Minkowski space-time. In that case, there is a maximal Bruhat decomposition [124]:

\[ so(n, 2) = \mathcal{P} \oplus \mathcal{N} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N} \oplus \mathcal{N'}, \]

\[ \mathcal{M} = so(n - 1, 1), \ \ \dim \mathcal{A} = 1, \ \ \dim \mathcal{N} = \dim \mathcal{N'} = n \]
that has direct physical meaning, namely, $so(n-1,1)$ is the Lorentz algebra of $n$-dimensional Minkowski space-time, the subalgebra $A = so(1,1)$ represents the dilatations, the conjugated subalgebras $\mathcal{N'}, \tilde{\mathcal{N}}$ are the algebras of translations, and special conformal transformations, both being isomorphic to $n$-dimensional Minkowski space-time. The subalgebra $\mathcal{P} = \mathcal{M} \oplus A \oplus \mathcal{N}' (\cong \mathcal{M} \oplus A \oplus \tilde{\mathcal{N}})$ is a maximal parabolic subalgebra.\footnote{The precise general definition is given in section 2.}

There are other special features which are important. In particular, the complexification of the maximal compact subgroup is isomorphic to the complexification of the first two factors of the Bruhat decomposition:

$$K^C = so(n, \mathbb{C}) \oplus so(2, \mathbb{C}) \cong so(n-1,1)^C \oplus so(1,1)^C = \mathcal{M}^C \oplus A^C.$$ \hspace{1cm} (1.4)

In particular, the coincidence of the complexification of the semi-simple subalgebras:

$$K'^C = \mathcal{M}^C$$ \hspace{1cm} (1.5)

means that the sets of finite-dimensional (nonunitary) representations of $\mathcal{M}$ are in 1-to-1 correspondence with the finite-dimensional (unitary) representations of $so(n)$. The latter leads to the fact that the corresponding induced representations are representations of finite $K$-type \cite{122,123}.

It turns out that some of the hermitian-symmetric algebras share the above-mentioned special properties of $so(n,2)$. That is why, in view of applications to physics, these algebras should be called 'conformal Lie algebras' (CLA), (or groups).

This subclass consists of:

$$so(n,2), \ sp(n,\mathbb{R}), \ su(n,n), \ so^*(4n), \ E_{7(-25)}$$ \hspace{1cm} (1.6)

the corresponding analogs of Minkowski space-time $V$ being:

$$\mathbb{R}^{n-1,1}, \ Sym(n,\mathbb{R}), \ Herm(n,\mathbb{C}), \ Herm(n,\mathbb{Q}), \ Herm(3,\mathbb{O}).$$ \hspace{1cm} (1.7)

The corresponding groups are also called 'Hermitian symmetric spaces of tube type' \cite{125}. The same class was identified from different considerations in \cite{126} called there 'conformal groups of simple Jordan algebras'. In fact, the relation between Jordan algebras and division algebras was known long time ago. Our class was identified from still different considerations also in \cite{127} where they were called 'simple space-time symmetries generalizing conformal symmetry'. For more references on Jordan algebras relevant in our approach cf., e.g., \cite{128–144,148}.

We have started the study of the above class in the framework of the present approach in the cases: $so(n,2), \ su(n,n), \ sp(n,\mathbb{R}), \ E_{7(-25)}$, in \cite{149–152}, resp., and we have considered also the algebra $E_{6(-14)}$, \cite{153,154}.

In the present paper we are mainly interested in non-compact Lie algebras (and groups) that are ‘parabolically’ related to the conformally Lie algebras.

• Definition: Let $\mathcal{G}, \mathcal{G}'$ be two non-compact semi-simple Lie algebras with the same complexification $\mathcal{G}^C \cong \mathcal{G}'^C$. We call them \textit{parabolically related} if they have parabolic subalgebras $\mathcal{P} = \mathcal{M} \oplus A \oplus \mathcal{N}, \mathcal{P}' = \mathcal{M}' \oplus A' \oplus \mathcal{N}'$, such that: $\mathcal{M}^C \cong \mathcal{M}'^C$ ($\Rightarrow \mathcal{P}^C \cong \mathcal{P}'^C$).\hfill\Diamond
Certainly, there are many such parabolic relationships for any given algebra $\mathcal{G}$. Furthermore, two algebras $\mathcal{G}, \mathcal{G}'$ may be parabolically related with different parabolic subalgebras. For example, the exceptional Lie algebras $E_6(6)$ and $E_6(2)$ are considered in section 7 (as related also to $E_6(-14)$) with maximal parabolics such that $\mathcal{M}^C \cong \mathcal{M}'^C \cong \mathfrak{sl}(6, \mathbb{C})$. But these two algebras are related also by another pair of maximal parabolics $\mathcal{P}^C, \mathcal{P}'^C$ such that $\mathcal{M}^C \cong \mathcal{M}'^C \cong \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$, cf. [121], (11.4), (11.7).

Another interesting example are the algebras $\mathfrak{so}^*(2m)$ and $\mathfrak{so}(p, q)$ which have a series of maximal parabolics with $\mathcal{M}$-factors [121], :

\[
\mathcal{M}_j = \mathfrak{su}^*(2j) \oplus \mathfrak{so}^*(2m - 4j), \quad j \leq \left\lfloor \frac{m}{2} \right\rfloor, \\
\mathcal{M}'_j = \mathfrak{sl}(2j, \mathbb{R}) \oplus \mathfrak{so}(p - 2j, q - 2j), \quad j \leq \left\lfloor \frac{q}{2} \right\rfloor \leq \left\lfloor \frac{m}{2} \right\rfloor.
\]

whose complexifications coincide for $p + q = 2m$

\[
(\mathcal{M}_j)^C = (\mathcal{M}'_j)^C = \mathfrak{sl}(2j, \mathbb{C}) \oplus \mathfrak{so}(2m - 4j, \mathbb{C}), \quad j \leq \left\lfloor \frac{q}{2} \right\rfloor \leq \left\lfloor \frac{m}{2} \right\rfloor = \left\lfloor \frac{p + q}{4} \right\rfloor .
\]

As we know only for $m = 2n$, i.e., for $\mathfrak{so}^*(4n)$ is fulfilled relation (1.5), with $\mathcal{M} = \mathcal{M}_n = \mathfrak{su}^*(2m)$ from (1.8), (recalling that $\mathcal{K}' \cong \mathfrak{su}(2m)$). Obviously, $\mathfrak{so}(p, q)$ is parabolically related to $\mathfrak{so}^*(4n)$ with this $\mathcal{M}$-factor only when $p = q = 2n$, i.e., $\mathcal{G}' = \mathfrak{so}(2n, 2n)$ with $\mathcal{M}'_n = \mathfrak{sl}(2n, \mathbb{R})$ (which is outside the range of (1.9)).

We leave the classification of the parabolic relations between the non-compact semisimple Lie algebras for a subsequent publication. In the present paper we consider mainly algebras parabolically related to conformal Lie algebras with maximal parabolics fulfilling (1.5). We summarize the relevant cases in table 1, where we have included also the algebra $E_6(-14)$; we display only the semisimple part $\mathcal{K}'$ of $\mathcal{K}$; $\mathfrak{sl}(n, \mathbb{C})_\mathbb{R}$ denotes $\mathfrak{sl}(n, \mathbb{C})$ as a real Lie algebra, (thus, $(\mathfrak{sl}(n, \mathbb{C})_\mathbb{R})^C = \mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{sl}(n, \mathbb{C})$); $\mathfrak{e}_6$ denotes the compact real form of $\mathfrak{e}_6$; and we have imposed restrictions to avoid coincidences or degeneracies due to well known isomorphisms: $\mathfrak{so}(1, 2) \cong \mathfrak{sp}(1, \mathbb{R}) \cong \mathfrak{su}(1, 1)$, $\mathfrak{so}(2, 2) \cong \mathfrak{so}(1, 2) \oplus \mathfrak{so}(1, 2)$, $\mathfrak{su}(2, 2) \cong \mathfrak{so}(4, 2)$, $\mathfrak{sp}(2, \mathbb{R}) \cong \mathfrak{so}(3, 2)$, $\mathfrak{so}^*(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(2, 1)$, $\mathfrak{so}^*(8) \cong \mathfrak{so}(6, 2)$.

After this extended introduction we give the outline of the paper. In section 2 we give the preliminaries, actually recalling and adapting facts from [121]. We add a remark on conservation laws and conserved currents which are an integral part of our approach. In section 3 we consider the case of the pseudo-orthogonal algebras $\mathfrak{so}(p, q)$ which are parabolically related to the conformal algebra $\mathfrak{so}(n, 2)$ for $p + q = n + 2$. We add historical remarks and a remark on shadow representations. In section 4 we consider the algebras $\mathfrak{su}^*(4k)$ and $\mathfrak{sl}(4k, \mathbb{R})$ as parabolically related to the CLA $\mathfrak{su}(2k, 2k)$. In section 5 we consider the algebra $\mathfrak{sp}(r, r)$ as parabolically related to the CLA $\mathfrak{sp}(2r)$ (of rank $2r$). In section 6 we consider the algebra $E_7(7)$ as parabolically related to the CLA $E_7(-25)$. In section 7 we consider the algebras $E_6(6)$ and $E_6(2)$ as parabolically related to the hermitian symmetric case $E_6(-14)$. In section 8 we give Summary and Outlook.

2 Preliminaries

Let $G$ be a semisimple non-compact Lie group, and $K$ a maximal compact subgroup of $G$. Then we have an Iwasawa decomposition $G = KA_0N_0$, where $A_0$ is Abelian simply con-
Table 1. Table of conformal Lie algebras (CLA) $\mathcal{G}$ with $\mathcal{M}$-factor fulfilling (1.5) and the corresponding parabolically related algebras $\mathcal{G}'$.

connected vector subgroup of $G$, $N_0$ is a nilpotent simply connected subgroup of $G$ preserved by the action of $A_0$. Further, let $M_0$ be the centralizer of $A_0$ in $K$. Then the subgroup $P_0 = M_0A_0N_0$ is a minimal parabolic subgroup of $G$. A parabolic subgroup $P = M'A'N'$ is any subgroup of $G$ which contains a minimal parabolic subgroup.

Further, let $\mathcal{G}, \mathcal{K}, \mathcal{P}, \mathcal{M}, \mathcal{A}, \mathcal{N}$ denote the Lie algebras of $G, K, P, M, A, N$, resp.

For our purposes we need to restrict to maximal parabolic subgroups $P = MAN$, i.e. rank $A = 1$, resp. to maximal parabolic subalgebras $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ with dim $\mathcal{A} = 1$.

Let $\nu$ be a (non-unitary) character of $A$, $\nu \in \mathcal{A}^*$, parameterized by a real number $d$, called the conformal weight or energy.

Further, let $\mu$ fix a discrete series representation $D^\mu$ of $M$ on the Hilbert space $V_\mu$, or the finite-dimensional (non-unitary) representation of $M$ with the same Casimirs.

We call the induced representation $\chi = \text{Ind}\mathcal{G}_P(\mu \otimes \nu \otimes 1)$ an elementary representation of $G$ [37]. (These are called generalized principal series representations (or limits thereof)
in [155]). Their spaces of functions are:

\[ \mathcal{C}_\chi = \{ \mathcal{F} \in C^\infty(G, V_\mu) \mid \mathcal{F}(g \text{man}) = \} \]

\[ = e^{-\nu(H)} \cdot D^\mu(m^{-1}) \mathcal{F}(g) \}

where \( a = \exp(H) \in A', \ H \in A', \ m \in M', \ n \in N'. \) The representation action is the left regular action:

\[ (T^\chi(g) \mathcal{F})(g') = \mathcal{F}(g^{-1}g'), \ g, g' \in G. \] (2.2)

- An important ingredient in our considerations are the highest/lowest weight representations of \( G^C. \) These can be realized as (factor-modules of) Verma modules \( V^\Lambda \) over \( G^C, \) where \( \Lambda \in (H^C)^+, \) \( H^C \) is a Cartan subalgebra of \( G^C, \) weight \( \Lambda = \Lambda(\chi) \) is determined uniquely from \( \chi \) [156].

Actually, since our ERs may be induced from finite-dimensional representations of \( M \) the Verma modules are always reducible. Thus, it is more convenient to use generalized Verma modules \( \hat{V}^\Lambda \) such that the role of the highest/lowest weight vector \( v_0 \) is taken by the (finite-dimensional) space \( V_\mu v_0. \) For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight \( d. \) Relatedly, for the intertwining differential operators only the reducibility w.r.t. non-compact roots is essential.

- One main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called multiplets. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible ERs and the lines (arrows) between the vertices correspond to intertwining operators. The explicit parametrization of the multiplets and of their ERs is important for understanding of the situation. The notion of multiplets was introduced in [157, 158] and applied to representations of \( SO_o(p, q) \) and SU(2, 2), resp., induced from their minimal parabolic subalgebras. Then it was applied to the conformal superalgebra [159], to infinite-dimensional (super-)algebras [160–163], to quantum groups [164].

**Remark:** Note that the multiplets of Verma modules include in general more members, since there enter Verma modules which are induced from infinite-dimensional representations of \( M \) but nevertheless have the same Casimirs. The main multiplets in this case contain as many members as the Weyl group \( W(G^C) \) of \( G^C. \) For example, for \( su(2, 2) \) the maximal multiplets contain 24 members (\(|W(sl(\ell, \mathbb{C}))| = \ell!\), which were considered in [158] and the \( su(2, 2) \) sextets of ERs induced from the maximal parabolic with \( M = sl(2, \mathbb{C}) \) are submerged in the 24-member multiplets.

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consists of the pair \( (\beta, m), \) where \( \beta \) is a (non-compact) positive root of \( G^C, \) \( m \in \mathbb{N}, \) such that the BGG Verma module reducibility condition (for highest weight modules) is fulfilled:

\[ (\Lambda + \rho, \beta^\vee) = m, \ \ \beta^\vee \equiv 2\beta/\langle \beta, \beta \rangle \] (2.3)

where \( \rho \) is half the sum of the positive roots of \( G^C. \) When the above holds then the Verma module with shifted weight \( V^{\Lambda - m\beta} \) (or \( \hat{V}^{\Lambda - m\beta} \) for GVM and \( \beta \) non-compact) is embedded

\[ 2 \text{For other applications we refer to [165–168].} \]
in the Verma module $V^\Lambda$ (or $\tilde{V}^\Lambda$). This embedding is realized by a singular vector $v_s$ expressed by a polynomial $P_{m,\beta}(G^-)$ in the universal enveloping algebra $(U(G^-))v_0$, $G^-$ is the subalgebra of $G_C$ generated by the negative root generators [169]. More explicitly, [156], $v^s_{m,\beta} = P_{m,\beta} v_0$ (or $v^s_{m,\beta} = P_{m,\beta} V_\mu v_0$ for GVMs).

Then there exists [156] an intertwining differential operator of order $m = m_\beta$:

$$D_{m,\beta} : C_{\chi(\Lambda)} \rightarrow C_{\chi(\Lambda - m\beta)}$$

(2.4)

given explicitly by:

$$D_{m,\beta} = P_{m,\beta}(\hat{G}^-)$$

(2.5)

where $\hat{G}^-$ denotes the right action on the functions $F$.

Thus, in each such situation we have an invariant differential equation of order $m = m_\beta$:

$$D_{m,\beta} f = f', \quad f \in C_{\chi(\Lambda)}, \quad f' \in C_{\chi(\Lambda - m\beta)}.$$  

(2.6)

In most of these situations the invariance operator $D_{m,\beta}$ has a non-trivial invariant kernel in which a subrepresentation of $G$ is realized. Thus, studying the equations with trivial r.h.s.:

$$D_{m,\beta} f = 0, \quad f \in C_{\chi(\Lambda)},$$

(2.7)

is also very important. For example, in many physical applications in the case of first order differential operators, i.e., for $m = m_\beta = 1$, equations (2.7) are called conservation laws, and the elements $f \in \ker D_{m,\beta}$ are called conserved currents.

The above construction works also for the subsingular vectors $v_{ssv}$ of Verma modules. Such a vector is also expressed by a polynomial $P_{ssv}(G^-)$ in the universal enveloping algebra: $v^s_{ssv} = P_{ssv}(G^-) v_0$, cf. [172, 173]. Thus, there exists a conditionally invariant differential operator given explicitly by: $D_{ssv} = P_{ssv}(\tilde{G}^-)$, and a conditionally invariant differential equation, for many more details, see [172, 173]. (Note that these operators/equations are not of first order.)

Below in our exposition we shall use the so-called Dynkin labels:

$$m_i \equiv (\Lambda + \rho, \alpha_i^\vee), \quad i = 1, \ldots, n,$$

(2.8)

where $\Lambda = \Lambda(\chi)$, $\rho$ is half the sum of the positive roots of $G_C$.

We shall use also the so-called Harish-Chandra parameters:

$$m_\beta \equiv (\Lambda + \rho, \beta),$$

(2.9)

where $\beta$ is any positive root of $G_C$. These parameters are redundant, since they are expressed in terms of the Dynkin labels, however, some statements are best formulated in their terms.\(^4\)

\(^3\)For explicit expressions for singular vectors we refer to [170, 171].

\(^4\)Clearly, both the Dynkin labels and Harish-Chandra parameters have their origin in the BGG reducibility condition (2.3).
3 The pseudo-orthogonal algebras $so(p, q)$

3.1 Choice of parabolic subalgebra

Let $G = so(p, q)$, $p \geq q$, $p + q > 4$.\(^5\) Most of the results here are known for $q = 1, 2$, cf. [149, 174–176], and the purpose of the consideration is to extend those for arbitrary $q$.

For fixed $p, q$ this algebra has at least $q$ maximal parabolic subalgebras [121]. For example, when $p > q$ there are the following possibilities for $M$-factor (cf. (7.11) of [121]):

\[
M_j^{\text{max}} = sl(j, R) \oplus so(p - j, q - j), \quad j = 1, 2, \ldots, q .
\]

(3.1)

(There are more choices when $p = q$.)

We would like to consider a case, which would relate parabolically all $G = so(p, q)$ for $p + q$-fixed. Thus, in order in order to include the case $q = 1$ (where there is only one parabolic which is both minimal and maximal), we choose the case $j = 1$:

\[
M = M_1^{\text{max}} = so(p - 1, q - 1) .
\]

(3.2)

Then we have:

\[
dim N = dim \tilde{N} = p + q - 2 .
\]

(3.3)

With this choice we get for the conformal algebra exactly the Bruhat decomposition in (1.3).

We label the signature of the ERs of $G$ as follows:

\[
\chi = \{ n_1, \ldots, n_h; c \}, \quad
\]

\[
n_j \in \mathbb{Z}/2, \quad c = d - \frac{p+q-2}{2}, \quad h \equiv \left\lfloor \frac{p+q-2}{2} \right\rfloor ,
\]

\[
|n_1| < n_2 < \cdots < n_h, \quad p + q \text{ even,}
\]

\[
0 < n_1 < n_2 < \cdots < n_h, \quad p + q \text{ odd,}
\]

where the last entry of $\chi$ labels the characters of $A$, and the first $h$ entries are labels of the finite-dimensional nonunitary irreps of $M \cong so(p - 1, q - 1)$.

The reason to use the parameter $c$ instead of $d$ will become clear below.

3.2 Main multiplets

Following results of [149, 174–176] we present the main multiplets (which contain the maximal number of ERs with this parabolic) with the explicit parametrization of the ERs

\(^5\)We shall explain the last restriction at the end of this section.
in the multiplets in a simple way (helped by the use of the signature entry \(c\)):

\[
\chi^\pm_i = \left\{ \epsilon n_1, \ldots, n_h; \pm n_{h+1} \right\},
\]

\(n_h < n_{h+1},\)

\[\chi^\pm_2 = \left\{ \epsilon n_1, \ldots, n_{h-1}, n_{h+1}; \pm n_h \right\}\]

\[\chi^\pm_3 = \left\{ \epsilon n_1, \ldots, n_{h-2}, n_h, n_{h+1}; \pm n_{h-1} \right\}\]

\[\cdots\]

\[\chi^\pm_{h-1} = \left\{ \epsilon n_1, n_2, n_4, \ldots, n_h, n_{h+1}; \pm n_3 \right\}\]

\[\chi^\pm_h = \left\{ \epsilon n_1, n_3, \ldots, n_h, n_{h+1}; \pm n_2 \right\}\]

\[\chi^\pm_{h+1} = \left\{ \epsilon n_2, n_3, \ldots, n_h, n_{h+1}; \pm n_1 \right\}\]

\[\epsilon = \begin{cases} 
\pm, & p + q \text{ even} \\
1, & p + q \text{ odd} 
\end{cases}\]

(\(\epsilon = \pm\) is correlated with \(\chi^\pm\)). Clearly, the multiplets correspond 1-to-1 to the finite-dimensional irreps of \(so(p+q, \mathbb{C})\) with signature \(\{n_1, \ldots, n_h, n_{h+1}\}\) and we are able to use previous results due to the parabolic relation between the \(so(p, q)\) algebras for \(p + q\)-fixed.

Note that the two representations in each pair \(\chi^\pm\) were called shadow fields in the 1970s, see more on this towards the end of this section.

Further, the number of ERs in the corresponding multiplets is equal to \(2 \left\lfloor \frac{p+q}{2} \right\rfloor = 2(h+1)\). This maximal number is equal to the following ratio of numbers of elements of Weyl groups:

\[
\frac{|W(\mathcal{G}^\mathbb{C}, \mathcal{H}_C)|}{|W(\mathcal{M}^\mathbb{C}, \mathcal{H}_m^\mathbb{C})|},
\]

(3.6)

where \(\mathcal{H}_C^\mathbb{C}, \mathcal{H}_m^\mathbb{C}\) are Cartan subalgebras of \(\mathcal{G}^\mathbb{C}, \mathcal{M}^\mathbb{C}\), resp.

The above formula actually holds for all conformal Lie algebras and those parabolically related to them. More precisely, we have:

- The number of elements of the main multiplets of a conformal Lie algebra \(\mathcal{G}\) with \(\mathcal{M}\)-factor fulfilling (1.5) is given by (3.6). The same number holds for any algebra \(\mathcal{G}'\) parabolically related to \(\mathcal{G}\) w.r.t. \(\mathcal{M}\).

Further, we denote by \(C^\pm_i\) the representation space with signature \(\chi^\pm_i\).

We first give the multiplets pictorially in figures 1 and 2 for \(p+q\) even and odd, resp., and then explain notations and results:

The ERs in the multiplet are related by intertwining integral and differential operators.

The integral operators were introduced by Knapp and Stein [177, 178]. They correspond to elements of the restricted Weyl group of \(\mathcal{G}\). In fact, these operators are defined for any ER, not only for the reducible ones, the general action being in the context of (3.4), (3.5):

\[
G : C_\chi \rightarrow C_{\chi'},
\]

\[\chi = \left\{ n_1, \ldots, n_h; c \right\},\]

\[\chi' = \left\{ (-1)^{p+q+1}n_1, \ldots, n_h; -c \right\} .\]
Figure 1. Main multiplet for SO($p,q$) for $p+q = 2h + 2 \geq 6$, with maximal parabolic subalgebra $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ where $\mathcal{M}^\mathbb{C} = \text{so}(2h,\mathbb{C})$ (arrows are differential operators $d_i$, $d'_i$, dashed arrows are integral operators) $\varepsilon_1 \pm \varepsilon_k$, are the non-compact roots.

These operators intertwine the pairs $C_i^\pm$ (cf. (3.5)):
\[
G_i^\pm : C_i^\mp \rightarrow C_i^\pm, \quad i = 1, \ldots, 1 + h .
\] (3.8)

In the conformal setting (both Euclidean $q = 1$ and Minkowskian $q = 2$) the integral kernel of the Knapp-Stein operator is given by the conformal two-point function [37].
Figure 2. Main multiplet for SO($p, q$) for $p + q = 2h + 3 \geq 5$, with maximal parabolic subalgebra \( P = M \oplus A \oplus N \) where \( M^C = so(2h + 1, \mathbb{C}) \) (arrows are differential operators \( d_i \), \( d'_i \), dashed arrows are integral operators) \( \varepsilon_1 \pm \varepsilon_k \), \( \varepsilon_1 \) are the non-compact roots.

The intertwining differential operators correspond to non-compact positive roots of the root system of \( so(p + q, \mathbb{C}) \), cf. [156]. In the current context, compact roots of \( so(p + q, \mathbb{C}) \) are those that are roots also of the subalgebra \( so(p + q - 2, \mathbb{C}) \), the rest of the roots are non-compact. In more detail, we briefly recall the root systems:
For \( p + q = 2h + 2 \) even, the positive root system of \( \text{so}(2h + 2, \mathbb{C}) \) may be given by vectors \( \epsilon_i \pm \epsilon_j, \ 1 \leq i < j \leq h + 1 \), where \( \epsilon_i \) form an orthonormal basis in \( \mathbb{R}^{h+1} \), i.e., 
\[
(\epsilon_i, \epsilon_j) = \delta_{ij}.
\]
The non-compact roots may be taken as \( \epsilon_1 \pm \epsilon_1, \ 2 \leq i \leq h + 1 \). The root \( \epsilon_1 - \epsilon_i \) corresponds to the operator \( d_i - 1 \), the root \( \epsilon_1 + \epsilon_i \) corresponds to the operator \( d_i^+ - 1 \).

For \( p + q = 2h + 3 \) odd, the positive root system of \( \text{so}(2h + 3, \mathbb{C}) \) may be given by vectors \( \epsilon_i \pm \epsilon_j, \ 1 \leq i < j \leq h + 1 \), \( \epsilon_1 \pm \epsilon_1, \ 1 \leq k \leq h + 1 \). The non-compact roots may be taken as \( \epsilon_1 \pm \epsilon_i, \ \epsilon_1 \). The root \( \epsilon_1 - \epsilon_i \) corresponds to the operator \( d_i - 1 \), the root \( \epsilon_1 + \epsilon_i \) corresponds to the operator \( d_i^+ - 1 \). The root \( \epsilon_1 \) has a special position since it intertwines the same ERs that are intertwined by the Knapp-Stein integral operator \( G_{h+1}^+ \). The latter means that \( G_{h+1}^+ \) degenerates to the differential operator \( d_{h+1} \), and this degenerations determines that \( d_{h+1} \sim \Box^n \), (for \( n_1 \in \mathbb{N} \)), where \( \Box \) is the d’Alembert operator, as explained explicitly for the case \( \text{so}(3, 2) \) in [179]. (The latter phenomenon happens for the Knapp-Stein integral operators at critical points, but usually there is no non-compact root involved, cf., e.g., [37].)

The degrees of these intertwining differential operators are given just by the differences of the \( c \) entries [176]:
\[
\begin{align*}
\deg d_i &= \deg d_i' = n_{h+2-i} - n_{h+1-i}, & i = 1, \ldots, h, \\
\deg d_h &= n_2 + n_1, & p + q \text{ even}, \\
\deg d_{h+1} &= 2n_1, & p + q \text{ odd}.
\end{align*}
\]
where \( d_h' \) is omitted from the first line for \( (p + q) \) even. By our construction they are equal to the Harish-Chandra parameters for the non-compact roots:
\[
\begin{align*}
\deg d_i &= m_{\epsilon_1 - \epsilon_i+1}, & \quad (3.10) \\
\deg d_i' &= m_{\epsilon_1 + \epsilon_i+1}, & i = 1, \ldots, h, \\
\deg d_{h+1} &= m_{\epsilon_1}.
\end{align*}
\]

Matters are arranged so that in every multiplet only the ER with signature \( \chi_1^+ \) contains a finite-dimensional nonunitary subrepresentation in a subspace \( \mathcal{E} \). The latter corresponds to the finite-dimensional unitary irrep of \( \text{so}(p, q) \) with signature \( \{n_1, \ldots, n_h, n_{h+1}\} \). The subspace \( \mathcal{E} \) is annihilated by the operator \( G_1^+ \), and is the image of the operator \( G_1^- \).

Although the diagrams are valid for arbitrary \( \text{so}(p, q) \) \( (p + q \geq 5) \) the contents is very different. We comment only on the ER with signature \( \chi_1^+ \). In all cases it contains an UIR of \( \text{so}(p, q) \) realized on an invariant subspace \( \mathcal{D} \) of the ER \( \chi_1^+ \). That subspace is annihilated by the operator \( G_1^- \), and is the image of the operator \( G_1^+ \). (Other ERs contain more UIRs.)

If \( pq \in 2\mathbb{N} \) the mentioned UIR is a discrete series representation. Other ERs contain more discrete series UIRs. The number of discrete series is given by the formula [155]:
\[
|W(G^C, \mathcal{H}^C)| / |W(K^C, \mathcal{H}^C)|,
\]
where \( \mathcal{H}^C \) is a Cartan subalgebra of both \( G^C \) and \( K^C \).

And if \( q = 2 \) the invariant subspace \( \mathcal{D} \) is the direct sum of two subspaces \( \mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^- \), in which are realized a holomorphic discrete series representation and its conjugate anti-holomorphic discrete series representation, resp. These are contained only in \( \chi_1^+ \) and count.
for two series in the formula (3.12). Furthermore, any holomorphic discrete series representation is infinitesimally equivalent to a lowest weight GVM of the conformal algebra so(p, 2), while an anti-holomorphic discrete series representation is infinitesimally equivalent to a highest weight GVM.

Highest/lowest weight GVMs are related to other pairs besides $\chi_1^+$. A detailed analysis of these occurrences is done for the conformal algebra so(3, 2) in [149] and for so(4, 2) in [149, 175].

3.3 Reduced multiplets

Besides the main multiplets which are 1-to-1 with the finite-dimensional irreps of so(p + q, C), there are other multiplets which we describe here.

- We start with the case $p + q$ even. In this case there are $h + 1 (= (p + q)/2)$ multiplets — doublets — each consisting of a pair with signatures $\tilde{\chi}_i^\pm$ given explicitly as follows:

$$\tilde{\chi}_1^\pm = \{ \pm n_1, \ldots, n_h; \pm n_h \}$$

$$\tilde{\chi}_2^\pm = \{ \pm n_1, \ldots, n_{h-1}, n_{h+1}; \pm n_{h-1} \}$$

$$\tilde{\chi}_3^\pm = \{ \pm n_1, \ldots, n_{h-2}, n_{h}, n_{h+1}; \pm n_{h-2} \}$$

$$\vdots$$

$$\tilde{\chi}_{h-1}^\pm = \{ \pm n_1, n_2, n_4, \ldots, n_h, n_{h+1}; \pm n_2 \}$$

$$\tilde{\chi}_h^\pm = \{ \pm n_1, n_3, \ldots, n_h, n_{h+1}; \pm n_1 \}, \quad n_1 \neq 0$$

$$\tilde{\chi}_{h+1}^\pm = \{ \mp n_1, n_3, \ldots, n_h, n_{h+1}; \pm n_1 \}, \quad n_1 \neq 0$$

Clearly, the signature $\tilde{\chi}_i^\pm$ may be obtained from $\chi_i^\pm$ by setting the corresponding Harish-Chandra parameter equal to zero:

$$m_{\varepsilon_1 \pm \varepsilon_{i+1}} = \deg d_i = \deg d'_i = n_{h+2-i} - n_{h+1-i} = 0, \quad i = 1, \ldots, h-1,$$

$$m_{\varepsilon_1 - \varepsilon_{h+1}} = \deg d_h = n_2 - n_1 = 0, \quad \text{for } \tilde{\chi}_h^\pm,$$  

$$m_{\varepsilon_1 + \varepsilon_{h+1}} = \deg d'_h = n_2 + n_1 = 0, \quad \text{for } \tilde{\chi}_{h+1}^\pm.$$  

Although written compactly as (3.5) no pair is related to any other pair. This may be seen easily as follows. Consider (3.5) and set formally $n_{h+1} = n_h$. The signatures $\chi_1^\pm$ and $\chi_2^\pm$ coincide are become equal to $\tilde{\chi}_1^\pm$, but the rest of the signatures $\chi_i^\pm, i \geq 3$ are not allowed in our class, e.g.,

$$\chi_3^\pm \rightarrow \{ \epsilon n_1, \ldots, n_{h-2}, n_{h}, n_{h+1}; \pm n_{h-1} \}$$

is not allowed since it violates (3.4) due to equality of two $M$-signature entries ($n_h$). Thus, from the whole multiplet only the pair $\tilde{\chi}_1^\pm$ remains in our class. Similarly for the rest of the pairs.

Inside a fixed pair $\tilde{\chi}_i^\pm, i = 1, \ldots, h+1$, act two operators: a Knapp-Stein integral operator from $\tilde{\chi}_i^+$ to $\tilde{\chi}_i^-$, and a differential operator from $\tilde{\chi}_i^-$ to $\tilde{\chi}_i^+$. In more detail:

- Let first $i = 1, \ldots, h-1$. Inside a fixed pair $\tilde{\chi}_i^\pm$, acts the Knapp-Stein integral operator $G_i^-$ (3.8) (coinciding with $G_{i+1}^-$ for this signature), and a differential operator $\tilde{d}_i$ of degree
which is a degeneration of the Knapp-Stein integral operator $G^+_{i+1}$ (coinciding with $G^+_{i+1}$ for this signature). For this differential operator for $n_1 = 0$ we have: $\tilde{d}_i \sim \Box^{n_{h+1-i}}$, 
$(n_{h+1-i} \in \mathbb{N})$.\(^6\)

- Inside the fixed pair $\hat{\chi}^\pm_h$ acts the Knapp-Stein integral operator $G^-_h$ \((3.8)\) (coinciding with $G^-_{h+1}$ for this signature), and the differential operator $d'_h$ of degree $2n_1$ (cf. the previous subsection) which in addition is a degeneration of the Knapp-Stein integral operator $G^+_h$ (coinciding with $G^+_{h+1}$ for this signature).

- Inside the fixed pair $\hat{\chi}^\pm_{h+1}$ acts the Knapp-Stein integral operator $G^-_{h+1}$ \((3.8)\) (coinciding with $G^+_{h+1}$ for this signature), and the differential operator $d_h$ of degree $2n_2$ which in addition is a degeneration of the Knapp-Stein integral operator $G^+_h$ (coinciding with $G^-_{h}$ for this signature).

- We continue with the case $p + q$ odd. In this case there are $h$ doublets\(^7\) with signatures $\hat{\chi}^\pm$ given similarly to the even case as follows:

\[
\begin{align*}
\hat{\chi}^\pm_1 &= \{n_1, \ldots, n_h; \pm n_h\} \\
\hat{\chi}^\pm_2 &= \{n_1, \ldots, n_{h-1}, n_{h+1}; \pm n_{h-1}\} \\
\hat{\chi}^\pm_3 &= \{n_1, \ldots, n_{h-2}, n_h, n_{h+1}; \pm n_{h-2}\} \\
&\vdots \\
\hat{\chi}^\pm_h &= \{n_1, n_3, \ldots, n_h, n_{h+1}; \pm n_1\}
\end{align*}
\]

The signature $\hat{\chi}^\pm_i$ may be obtained from $\chi^\pm_i$ by setting the corresponding Harish-Chandra parameter equal to zero:

\[
m_{\varepsilon_1 \pm \varepsilon_{i+1}} = \deg d_i = \deg d'_i = n_{h+2-i} - n_{h+1-i} = 0, \quad i = 1, \ldots, h.
\]

Inside a fixed pair $\hat{\chi}^\pm_i$, $i = 1, \ldots, h$, acts the Knapp-Stein integral operator $G^-_i$ \((3.8)\) (coinciding with $G^-_{i+1}$ for this signature), and a differential operator $\tilde{d}_i$ of degree $2n_{h+1-i}$ which is a degeneration of the Knapp-Stein integral operator $G^+_i$ (coinciding with $G^+_{i+1}$ for this signature). For the differential operators we have $\tilde{d}_i \sim \Box^{n_{h+1-i}}$, \((\text{when } n_{h+1-i} \in \mathbb{N})\).

The difference with the even situation is only for $i = h$, where the degeneration of $G^+_h$ was present already in the main multiplet.

If $pq \in 2\mathbb{N}$ the representations $\hat{\chi}^\pm_1$, $\hat{\chi}^\pm_1$, contain an UIR called limits of the discrete series representations. And if $q = 2$ that UIR is the direct sum of two subspaces in which are realized limits of holomorphic discrete series representation and its conjugate limits of anti-holomorphic discrete series representation, resp. The latter do not happen in any other doublet, while limits of discrete series representations happen in other doublets. (For more on this see \cite{149} for $\text{so}(3, 2)$ and \cite{149, 175} for $\text{so}(4, 2)$.)

\(^6\)For $\text{so}(4, 2)$, $(h = 2, i = 1)$, when $n_1 = 0, n_2 = 1$ the latter d’Alembert operator arises also as a conditionally invariant differential operator due to the presence of a subsingular vector in the corresponding Verma module \cite{172, 173}.

\(^7\)In the case $\text{so}(3, 2)$ there are two additional doublets \cite{149} involving the two singleton representations, which are special for $\text{so}(3, 2)$. \footnote{In the case $\text{so}(3, 2)$ there are two additional doublets \cite{149} involving the two singleton representations, which are special for $\text{so}(3, 2)$.}
3.4 Remarks on shadow fields and history

• We labelled the signature of the ERs in (3.4) as

\[ \chi = \{ n_1, \ldots, n_h ; c \} \]

using the parameter \( c \) instead of the conformal weight \( d = c + \frac{p+q-2}{2} \). This was used already in [37] since the multiplets were given more economically in terms of pairs of ERs in which the parameter \( c \) just changes sign. (Also mathematicians use the parameter \( c \) due to the fact that in its terms the representation parameter space looks simple: the principal unitary series representation induced from a maximal parabolic is given by \( c = i\rho, \rho \in \mathbb{R} \); the supplementary series of unitary representations is given by \(-s < c < s, s \in \mathbb{R} \), etc.)

Otherwise in the current context we should use for each Knapp-Stein operators conjugated doublet of shadow fields:

\[ \chi^+ = [ n_1, \ldots, n_h ; d ], \quad n_j \in \mathbb{Z}/2, \]
\[ \chi^- = [ (-1)^{p+q+1} n_1, \ldots, n_h ; d_{\text{shadow}} = p + q - 2 - d ]. \] (3.18)

The reason the representations \( \chi^\pm \) in the 1970s were called ”shadow fields” in the context of the conformal algebra \( \mathfrak{so}(n,2) \) is that the sum of their conformal weights equals the dimension \( n \) of Minkowski space-time - isomorphic to \( \mathcal{N} \) or \( \tilde{\mathcal{N}} \), cf. (3.3). This continues to be true for general \( \mathfrak{so}(p,q) \):

\[ d + d_{\text{shadow}} = p + q - 2 = n, \] (3.19)

and also for all conformal Lie algebras considered in the next sections.

Shadow fields appear all the time in conformal field theory. For example, in [180] we showed that in the generic case each field on the AdS bulk has two boundary fields which are shadow fields being related by a integral Knapp-Stein operator. Later Klebanov-Witten [181] showed that these two boundary fields are related by a Legendre transform.

For a current discussion on shadow fields we refer to [182].

• The diagram for \( p + q \) even appeared first for the Euclidean conformal group in four-dimensional space-time \( \text{SU}^*(4) \cong \text{Spin}(5,1) \) in [174]. Later it was generalised to the Minkowskian conformal group in four-dimensional space-time \( \text{SO}(4,2) \) in [175]. In both cases, the three \((p + q)/2\) doublets (from the previous subsection) were also given together the corresponding degeneration of the Knapp-Stein integral operators.

The exposition above including figures 1 & 2 follows the exposition for Euclidean case \( \mathfrak{so}(n+1,1) \) in [176]. Later the results were generalised to the Minkowskian case \( \mathfrak{so}(n,2) \) [149].

• Actually, the case of Euclidean conformal group in arbitrary dimensions \( \mathfrak{so}(p,1) \) was studied in [37] for representations of \( \mathcal{M} = \mathfrak{so}(p-1) \) which are symmetric traceless tensors. This means in (3.4) we should set \( n_1 = n_2 = \cdots = n_{h-1} = 0 \), and then only the first two pairs \( \chi_1^\pm, \chi_2^\pm \) in (3.5) are possible. Thus from the two figures only the upper quadrants are relevant, and were given in [37], cf. figure 1 there.

• Above we restricted to \( p + q \geq 5 \). The excluded cases are: \( \mathfrak{so}(3,1), \mathfrak{so}(2,2) \cong \mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1), \mathfrak{so}(2,1), \mathfrak{so}(1,1) \) is abelian).
In the case $so(3, 1) \cong sl(2, \mathbb{C})$ the multiplet in general contains only four ERs, and is in fact representable by the diagram in the case of symmetric traceless tensors of $so(p, 1)$, $p > 3$, cf. [37], appendix B.

The case $so(2, 1) \cong sl(2, \mathbb{R})$ is special and must be treated separately. But in fact, it is contained in what we presented already. In that case the multiplets contain only two ERs which may be depicted by the top pair $\chi_{1 \pm}$ in both figures. (Formally, set $h = 0$ in both figures.) They have the properties that we described, including the (anti)holomorphic discrete series which are present in this case. That case was the first given already in 1946-7 independently by Gel’fand et al [183] and Bargmann [184].

4 The Lie algebras $su^*(2n)$ and $sl(n, \mathbb{R})$

4.1 Case $su^*(2n)$

Let $\mathcal{G} = su^*(2n)$. It has maximal compact subalgebra $\mathcal{K} = sp(n)$, and thus $\mathcal{G}$ does not have discrete series representations (as $\text{rank} \mathcal{K} = n < \text{rank} su^*(2n) = 2n - 1$).

The algebra $\mathcal{G} = su^*(2n)$ has $n - 1$ maximal parabolic subalgebras with $\mathcal{M}$-factors (cf. (5.8) from [121]):

\[
\mathcal{M}_{k}^{\text{max}} = su^*(2k) \oplus su^*(2(n - k)), \quad 1 \leq k \leq n - 1,
\]

with complexification:

\[
(\mathcal{M}_{k}^{\text{max}})^{\mathbb{C}} = sl(2k, \mathbb{C}) \oplus sl(2(n - k), \mathbb{C}) .
\]

We would like to relate parabolically this algebra with the appropriate conformal Lie algebra, namely, with $su(n, n)$. It was considered in [150] with $\mathcal{M}$-factor: $\mathcal{M}' = sl(n, \mathbb{C})_{\mathbb{R}}$ which has complexification:

\[
\mathcal{M}^{\mathbb{C}} = sl(n, \mathbb{C}) \oplus sl(n, \mathbb{C}) .
\]

Clearly, the latter expression can match (4.2) only if $n = 2k$, i.e., $n$ must be even.

Thus, we set $n = 2k$ and consider:

\[
\mathcal{G} = su^*(4k),
\]
\[
\mathcal{M} = su^*(2k) \oplus su^*(2k),
\]
\[
\mathcal{M}^{\mathbb{C}} = sl(2k, \mathbb{C}) \oplus sl(2k, \mathbb{C}) .
\]

4.2 Case $sl(n, \mathbb{R})$

Let $sl(n, \mathbb{R})$. Its maximal compact subalgebra is $\mathcal{K} = so(n)$, and thus it does not have discrete series representations. It has $\left[ \frac{n}{2} \right]$ maximal parabolic subalgebras obtained by deleting a node from its standard Dynkin diagram and taking into account the symmetry (cf. [121]):

\[
\mathcal{M}_j = sl(j, \mathbb{R}) \oplus sl(n - j, \mathbb{R}) , \quad 1 \leq j \leq \left[ \frac{n}{2} \right] .
\]
We would like to match this with (4.3). Obviously this can happen only for $n = 4k$ and $j = n/2 = 2k$, so we consider:

$$\mathcal{G} = sl(4k, \mathbb{R}),$$  \hspace{1cm} (4.6)

$$\mathcal{M} = sl(2k, \mathbb{R}) \oplus sl(2k, \mathbb{R}),$$

$$\mathcal{M}^C = sl(2k, \mathbb{C}) \oplus sl(2k, \mathbb{C}).$$

### 4.3 Representations and multiplets

Above we have chosen the $\mathcal{M}$-factors of the Lie algebras $su^*(4k)$ and $sl(4k, \mathbb{R})$ so that they are parabolically related to the conformal Lie algebra $su(2k, 2k)$ with $\mathcal{M}$-factor $\mathcal{M}^C = sl(2k, \mathbb{C}) \oplus sl(2k, \mathbb{C})$, cf. (4.4), (4.6), thus, we shall discuss them together.

The signature of the ERs of both $\mathcal{G}$ may be denoted as:

$$\chi = \{n_1, \ldots, n_{2k-1}, n_{2k+1}, \ldots, n_{4k-1}; c\},$$ \hspace{1cm} (4.7)

$$n_j \in \mathbb{N}, \quad c = d - 2k,$$

same as for $su(2k, 2k)$.

The Knapp-Stein restricted Weyl reflection mapping $\chi$ to its shadow is given by:

$$G : C_\chi \rightarrow C_{\chi'},$$ \hspace{1cm} (4.8)

$$\chi' = \{(n_1, \ldots, n_{2k-1}, n_{2k+1}, \ldots, n_{4k-1})^*; -c\},

(n_1, \ldots, n_{2k-1}, n_{2k+1}, \ldots, n_{4k-1})^* \equiv

(n_{2k+1}, \ldots, n_{4k-1}, n_1, \ldots, n_{2k-1})$$

Further, we use the root system of the complex algebra $sl(4k, \mathbb{C})$. The positive roots $\alpha_{ij}$ in terms of the simple roots $\alpha_i$ are:

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \quad 1 \leq i < j \leq 4k - 1,$$

$$\alpha_{ii} \equiv \alpha_i, \quad 1 \leq i \leq 4k - 1$$

from which the non-compact are:

$$\alpha_{ij}, \quad 1 \leq i \leq 2k, \quad 2k \leq j \leq 4k - 1$$

The correspondence between the signatures $\chi$ and the highest weight $\Lambda$ is through the Dynkin labels:

$$n_i = m_i \equiv (\Lambda + \rho, \alpha_i^\vee) = (\Lambda + \rho, \alpha_i), \quad i = 1, \ldots, 4k - 1,$$

$$c = -\frac{1}{2} (m_\tilde{\alpha} + m_{2k}) = -\frac{1}{2} (m_1 + \cdots + m_{2k-1} + 2m_{2k} + m_{2k+1} + \cdots + m_{4k-1})$$

$$\Lambda = \Lambda(\chi), \quad \tilde{\alpha} = \alpha_1 + \cdots + \alpha_{4k-1}$$

is the highest root.

In our diagrams we need also the Harish-Chandra parameters for the non-compact roots using the following notation:

$$m_{ij} \equiv m_{\alpha_{ij}} = m_i + \cdots + m_j, \quad i < j$$

- 17 -
The number of ERs in the corresponding multiplets is according to (3.6):

\[
\frac{|W(\mathcal{G}^\mathbb{C}, \mathcal{H}^\mathbb{C})|}{|W(\mathcal{M}^\mathbb{C}, \mathcal{H}^\mathbb{C})|} = \frac{|W(sl(4k, \mathbb{C}))|}{|W(sl(2k, \mathbb{C}))|^2} = \frac{(4k)!}{((2k)!)^2} = \frac{(4k)}{2k}
\]  

(4.11)

(which was given for \(su(n, n)\) in [150]).

Below we give the diagrams for the cases \(k = 1, 2\). Of course, the case \(k = 1\) is known long time ago, first as \(su^*(4) \cong so(5, 1)\), cf. [174], then as \(su(2, 2) \cong so(4, 2)\), cf. [175], and also as \(sl(4, \mathbb{R}) \cong so(3, 3)\), as we recalled already in the previous section on \(so(p, q)\) algebras. We present it here using a new diagram look which can handle the more complicated cases that follow further. In this new look only the invariant differential operators are presented explicitly. The integral Knapp-Stein operators, more precisely the restricted Weyl reflection action is understood by a symmetry of the picture, either w.r.t. a central point, or w.r.t. middle line.

Thus, in figure 3 we give the case \(k = 1\), where the Knapp-Stein symmetry is w.r.t. to the bullet in the middle of the figure. Then in figure 4 we give the diagram figure 1 for the special case \(h = 2\), as given originally for \(so(5, 1)\) in [174], and \(so(4, 2)\) in [175], stressing that both figures 3 and 4 have the same content.

Next we give the case \(k = 2\), in figure 5, which applies to \(su^*(8), sl(8, \mathbb{R})\) and \(su(4, 4)\). (For reduced multiplets we refer to [150].) The diagram is very complicated and just to be able to depict all the relevant information we must use the following condensing conventions. Each intertwining differential operator is represented by an arrow accompanied by a symbol \(i_j\ldots\ell\) encoding the root \(\beta_{j\ldots\ell}\) and the number \(m_{\beta_{j\ldots\ell}}\) which is involved in the BGG criterion. This notation is used to save space, but it can be used due to the fact that only intertwining differential operators which are non-composite are displayed, and that the data \(\beta, m_{\beta, \Lambda}\) which is involved in the embedding \(V^\Lambda \rightarrow V^{\Lambda - m_{\beta, \beta}}\) turns out to involve only the \(m_i\) corresponding to simple roots, i.e., for each \(\beta, m_{\beta}\) there exists \(i = i(\beta, m_{\beta, \Lambda}) \in \{1, \ldots, r\}\), \(r = \text{rank} \mathcal{G}\), such that \(m_{\beta} = m_i\). Hence the data \(\beta_{j\ldots\ell}, m_{\beta_{j\ldots\ell}}\) is represented by \(i_j\ldots\ell\) on the arrows.

5 The Lie algebras \(sp(p, r)\)

Let \(\mathcal{G} = sp(p, r), p \geq r\). It has maximal compact subalgebra \(K = sp(p) \oplus sp(r)\) and has discrete series representations (as \(\text{rank} K = p + r = \text{rank} \mathcal{G}\)). It has \(r\) maximal parabolic subalgebras with \(M\)-factors (cf. (9.8) from [121]):

\[
\mathcal{M}_{j}^{\text{max}} = su^*(2j) \oplus sp(p - j, r - j), \quad 1 \leq j \leq r
\]  

(5.1)

with complexification:

\[
(\mathcal{M}_{j}^{\text{max}})^\mathbb{C} = sl(2j, \mathbb{C}) \oplus sp(p + r - 2j, \mathbb{C}).
\]  

(5.2)

We would like to match this algebra with the appropriate conformal Lie algebra, namely, with \(sp(n, \mathbb{R})\). It was considered in [151] with \(M\)-factor: \(M' = sl(n, \mathbb{R})\) with
Figure 3. Main multiplets for $su^*(4) \cong so(5,1)$ and $su(2,2) \cong so(4,2)$ with parabolic factor $\mathcal{M} = sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})$. The pairs of shadow fields are symmetric w.r.t. the bullet.

Figure 4. Sextet of partially equivalent ERs and intertwining operators for $so(5,1) \cong su^*(4)$ and $so(4,2) \cong su(2,2)$ cf. [140, 141], resp. (arrows are differential operators, dashed arrows are integral operators).
Figure 5. Main multiplets for $su(4, 4)$ and $su^*(8)$ with parabolic factor $\mathcal{M}^C = sl(4, \mathbb{C}) \oplus sl(4, \mathbb{C})$.

complexification $\mathcal{M}^C = sl(n, \mathbb{C})$. Obviously, the latter can match (5.2) only if $n$ is even and $p = r = j = n/2$. Thus, we shall consider

$$G = sp(r, r),$$

$$\mathcal{M} = su^*(2r),$$

$$\mathcal{M}^C = sl(2r, \mathbb{C}).$$

The signature of the ERs of $G$ is:

$$\chi = \{ n_1, \ldots, n_{2r-1}; c \}, \quad n_j \in \mathbb{N}, \quad c = d - r - \frac{1}{2}.$$

(5.4)
The Knapp-Stein restricted Weyl reflection acts as follows:

\[ G : C\chi \rightarrow C\chi', \]

\[ \chi' = \{ (n_1, \ldots, n_{2r-1})^* : c \} , \quad (n_1, \ldots, n_{2r-1})^* \equiv (n_{2r-1}, \ldots, n_1) \]

In terms of an orthonormal basis \( \varepsilon_i \), \( i = 1, \ldots, n \), the positive roots of \( sp(2r, \mathbb{C}) \) are:

\[ \Delta^+ = \{ \varepsilon_i \pm \varepsilon_j, \quad 1 \leq i < j \leq 2r; \quad 2\varepsilon_i, \quad 1 \leq i \leq 2r \} , \]

the simple roots are:

\[ \pi = \{ \alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad 1 \leq i \leq 2r - 1; \quad \alpha_{2r} = 2\varepsilon_{2r} \} , \]

the positive non-compact roots are:

\[ \beta_{ij} \equiv \varepsilon_i + \varepsilon_j, \quad 1 \leq i \leq j \leq 2r , \]

the Harish-Chandra parameters: \( m_\beta \equiv (\Lambda + \rho, \beta) \) for the noncompact roots are:

\[ m_{\beta_{ij}} = \left( \sum_{s=i}^{2r} + \sum_{s=j}^{2r} \right) m_s , \quad i < j , \]

\[ m_{\beta_{ii}} = \sum_{s=i}^{2r} m_s \]

The correspondence between the signatures \( \chi \) and the highest weight \( \Lambda \) is:

\[ n_i = m_i , \quad c = -\frac{1}{2}(m_\alpha + m_{2r}) = -\frac{1}{2}(m_1 + \cdots + m_{2r-1} + 2m_{2r}) \]

where \( \alpha = \beta_{11} \) is the highest root.

The number of ERs in the corresponding multiplets is according to (3.6):

\[ \frac{|W(G^C, \mathcal{H}^C)|}{|W(M^C, \mathcal{H}_m^C)|} = \frac{|W(sp(2r, \mathbb{C}))|}{|W(sl(2r, \mathbb{C}))|} = \frac{2^{2r}(2r)!}{((2r)!)^2} = 2^{2r} \]

(which was given for \( sp(n, \mathbb{R}) \) in [151]).

Below we give pictorially the multiplets for \( sp(r, r) \) for \( r = 1, 2 \), valid also for \( sp(2r, \mathbb{R}) \). (The case \( r = 3 \), together with the reduced multiplets and \( sp(5, \mathbb{R}) \) are given in [151].)

In fact, the case \( r = 1 \) is known long time as \( sp(1, 1) \cong so(4, 1) \), cf. [37], then later as \( sp(2, \mathbb{R}) \cong so(3, 2) \), cf. [179], as we recalled already in the previous section on \( so(p, q) \) algebras. We present it here using the new diagram look which we already used in the previous section. Thus, in figure 6 we give the case \( r = 1 \), where the Knapp-Stein symmetry is w.r.t. to the bullet in the middle of the figure. Thus, it is seen that the action of the differential operator indexed by \( 1_{12} \) is the same as the Knapp-Stein operator from \( \Lambda^- \) to \( \Lambda^+ \), so that the latter operator degenerates as discussed in section 1. Then in figure 7 we give the diagram figure 2 for the special case \( h = 1 \), stressing that both figures 6 and 7 have the same content.

Finally, in figure 8 we give the case \( r = 2 \).
Figure 6. Main multiplets for $sp(1, 1) \cong so(4, 1)$ and $sp(2, \mathbb{R}) \cong so(3, 2)$ with parabolic factor $M^C = sl(2, \mathbb{C})$.

Figure 7. Quartet of partially equivalent ERs and intertwining operators for $so(4, 1) = sp(1, 1)$ and $so(3, 2) \cong sp(2, \mathbb{R})$ cf. [125, 148] resp. (arrows are differential operators, dashed arrows are integral operators).

6 The non-compact Lie algebra $E_{7(7)}$

Let $\mathcal{G} = E_{7(7)}$. This is the split real form of $E_7$ which is denoted also as $E_7^r$ or $EV$. The maximal compact subgroup is $K \cong su(8)$. This algebra has discrete series representations (as rank $\mathcal{G} = rank K$).

It has the following Dynkin-Satake diagram (same as for $E_7$) [185]:

\[
\begin{array}{cccccccc}
& & & & & & & \\
\circ \alpha_1 & \circ \alpha_3 & \circ \alpha_4 & \circ \alpha_5 & \circ \alpha_6 & \circ \alpha_7 & \\
& & & & & & & \\
\end{array}
\]

(6.1)

The real algebra $E_{7(7)}$ has seven maximal parabolics which are obtained by deleting one node as explained in [121]. We choose the one which is most suitable w.r.t. the maximal compact subgroup $K = su(8)$, as will become clear below. This parabolic is obtained by deleting the root $\alpha_7$ from the Dynkin-Satake diagram (6.1), i.e., we shall use as $M$-factor $E_{6(6)}$ (the split real form of $E_6$).
Figure 8. Main multiplets for $sp(2,2)$ and $sp(4,\mathbb{R})$ with parabolic factor $M^C = sl(4,\mathbb{C})$.

Thus, our maximal parabolic is

$$\mathcal{P} = M \oplus A \oplus N, \quad A \cong so(1,1), \quad M \cong E_{6(6)}, \quad \dim_{\mathbb{R}} N = 27,$$

cf. (11.17) of [121].

We label the signature of the ERs of $G$ as follows:

$$\chi = \{ n_1, \ldots, n_6; c \}, \quad n_j \in \mathbb{N}, \quad c = d - 9$$

where the last entry of $\chi$ labels the characters of $A$, and the first 6 entries are labels of the finite-dimensional nonunitary irreps of $M$, (or of the finite-dimensional unitary irreps of the compact $e_6$).

Further, we need the root system of the complex algebra $E_7$. With Dynkin diagram enumerating the simple roots $\alpha_i$ as in (6.1), the positive roots are:

first there are 21 roots forming the positive root system of $sl(7)$ (with simple roots $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$), then 21 positive roots which are positive roots of the $E_6$ subalgebra including the non-$sl(7)$ root $\alpha_2$, and finally the following 21 roots including the
non-$E_6$ root $\alpha_7$:

\begin{align}
\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, & \quad \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \\
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, & \\
\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, & \quad \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \\
\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, & \quad \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \\
\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, & \\
\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, & \\
\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, & \\
\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, & \\
\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + \alpha_6 + \alpha_7, & \\
\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + \alpha_6 + \alpha_7, & \\
\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + \alpha_6 + \alpha_7, & \\
\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + \alpha_6 + \alpha_7, & \\
2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 = \tilde{\alpha}, &
\end{align}

where $\tilde{\alpha}$ is the highest root of the $E_7$ root system.

The differential intertwining operators that give the multiplets correspond to the non-compact roots, and since we shall use the latter extensively, we introduce more compact notation for them. Namely, the non-simple roots will be denoted in a self-explanatory way as follows:

\begin{align}
\alpha_{ij} &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, & \alpha_{i,j} &= \alpha_i + \alpha_j, & i < j, \\
\alpha_{ij,k} &= \alpha_{k,i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j + \alpha_k, & i < j, \\
\alpha_{ij,k,m} &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_j + \alpha_k + \alpha_{k+1} + \cdots + \alpha_m, & i < j, & k < m, \\
\alpha_{ij,k,m,4} &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_j + \alpha_k + \alpha_{k+1} + \cdots + \alpha_m + \alpha_4, & i < j, & k < m,
\end{align}

i.e., the non-compact roots will be written as:

\begin{align}
\alpha_7, & \quad \alpha_{67}, \quad \alpha_{57}, \quad \alpha_{47}, \quad \alpha_{37}, \quad \alpha_{137}, \\
\alpha_{2,37}, & \quad \alpha_{27}, \quad \alpha_{17}, \quad \alpha_{27,4}, \quad \alpha_{17,4}, \quad \alpha_{27,45}, \\
\alpha_{17,34}, & \quad \alpha_{17,45}, \quad \alpha_{27,46}, \quad \alpha_{17,35}, \quad \alpha_{17,46}, \quad \alpha_{17,36}, \\
\alpha_{17,35,4}, & \quad \alpha_{17,25,4}, \quad \alpha_{17,36,4}, \quad \alpha_{17,26,4}, \\
\alpha_{17,36,45}, & \quad \alpha_{17,26,45}, \quad \alpha_{17,26,45,4}, \quad \alpha_{17,26,35,4}, \quad \alpha_{17,16,35,4} = \tilde{\alpha},
\end{align}
where the first six roots in (6.6a) are from the \( sl(7) \) subalgebra, and the 21 in (6.6b) are those from (6.4).

Further, we give the correspondence between the signatures \( \chi \) and the highest weight \( \Lambda \). The connection is through the Dynkin labels (2.8) \( m_i, \ i = 1, \ldots, 7 \), and is given explicitly by:

\[
m_i = m_i, \quad i = 1, \ldots, 6, \\
c = -\frac{1}{2}(m_\tilde{\alpha} + m_7) = -\frac{1}{2}(2m_1 + 2m_2 + 3m_3 + 4m_4 + 3m_5 + 2m_6 + 2m_7) \tag{6.7}
\]

Here we note that the simple root system of the \( su(8) \) compact subalgebra of \( E_7(7) \), or equivalently, of the \( sl(8) \) subalgebra of \( E_7 \), is given by the \( sl(7) \) simple roots plus the highest root \( \hat{\alpha} \) of the \( E_6 \) subalgebra:

\[
\alpha_1, \alpha_3, \alpha_4, \alpha_6, \alpha_7, \hat{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \tag{6.8}
\]

Indeed, it is easy to check that:

\[
(\alpha_i, \hat{\alpha}) = 0, \quad i = 1, 3, 4, 5, 6, \quad (\alpha_7, \hat{\alpha}) = -1.
\]

Now we should connect our considerations with the case of another real form of \( E_7 \), namely, the Lie algebra \( E_{7(-25)} \), cf. [152]. In that paper we chose as maximal parabolic \( P' = M' \oplus A' \oplus N' \), where \( M' \cong E_{6(-26)} \), \( \dim R N' = 27 \), cf. (11.24) of [121].

Since the algebras \( E_{7(7)} \) and \( E_{7(-25)} \) are parabolically related they have the same signatures, and thus the same main multiplets.

The number of ERs in the corresponding main multiplets is according to (3.6):

\[
\frac{|W(G^C, H^C)|}{|W(M^C, H^m)|} = \frac{|W(E_7)|}{|W(E_6)|} = \frac{2^{10} 3^4 5.7}{2^7 3^4 5} = 56 \tag{6.9}
\]

(which was given for \( E_{7(-25)} \) in [152]).

Below we give the main multiplets valid for both algebras in figure 9. For reduced multiplets cf. [152].

### 7 Two real forms of \( E_6 \)

#### 7.1 The Lie algebra \( E_{6(6)} \)

Let \( G = E_{6(6)} \). This is the split real form of \( E_6 \) denoted also as \( E_6' \) or \( E_1 \). The maximal compact subgroup is \( K \cong sp(4) \). This real form does not have discrete series representations (as \( \text{rank} \ G \neq \text{rank} \ K \)).

We use the following Dynkin-Satake diagram (same as for \( E_6 \)):

\[
\begin{array}{cccccc}
\circ & \circ \alpha_2 \\
\alpha_1 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6
\end{array}
\]

\tag{7.1}

The real algebra \( E_{6(6)} \) has four maximal parabolics which are obtained by deleting one node as explained in [121]. (Note that deleting node 1 or node 6 produces the same
Figure 9. Main type of multiplets for $E_7(\gamma)$ and $E_7(-25)$ with parabolic factor $M^C = E_6$. 
parabolic, same for deleting node 3 or node 5.) We choose the parabolic obtained by deleting node 2.

Thus, the maximal parabolic is

$$P = M \oplus A \oplus N, \quad A \cong so(1,1), \quad M \cong sl(6,\mathbb{R}), \quad \text{dim}_\mathbb{R} N = 21,$$

(7.2) cf. (11.4) of [121].

7.2 The Lie algebra $E_6(2)$

Let $G = E_6(2)$. This is another real form of $E_6$ sometimes denoted as $E_6''$, or $E_{II}$. The maximal compact subalgebra is $K \cong su(6) \oplus su(2)$. This real form has discrete series representations.

The Satake diagram is:

```
  o-----o-----o-----o-----o-----o
  α₁   α₃   α₄   α₅   α₆
```

(7.3)

The real algebra $E_6(2)$ has four maximal parabolics which are obtained by deleting one node as explained in [121] (taking into account $E_6$ symmetry as in the previous case). We choose the parabolic obtained by deleting node 2.

Thus, the maximal parabolic is

$$P = M \oplus A \oplus N, \quad A \cong so(1,1), \quad M \cong su(3,3), \quad \text{dim}_\mathbb{R} N = 21,$$

(7.4) cf. (11.7) of [121].

7.3 Representations and multiplets

We note that the $M$-factors of the two real forms of $E_6$ discussed in the previous subsections have the same complexification:

$$sl(6,\mathbb{R})^\mathbb{C} = su(3,3)^\mathbb{C} = sl(6,\mathbb{C})$$

i.e., they are parabolically related and we can discuss them together.

The signature of the ERs of $G$ is:

$$\chi = \{ n_1, n_3, n_4, n_5, n_6 ; c \}, \quad c = d - \frac{11}{2},$$

expressed through the Dynkin labels as:

$$n_i = m_i, \quad -c = \frac{1}{2}m_\delta = \frac{1}{2}(m_1 + 2m_2 + 2m_3 + 3m_4 + 2m_5 + m_6)$$

Further, we need the root system of the complex algebra $E_6$. With Dynkin diagram enumerating the simple roots $\alpha_i$ as in (7.1), the positive roots are:
first there are 15 roots forming the positive root system of $sl(6)$ (with simple roots $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6$), then the following 21 roots including the non-$sl(6)$ root $\alpha_2$:

\begin{align}
\alpha_2, & \quad \alpha_2 + \alpha_4, \quad \alpha_2 + \alpha_3 + \alpha_4, \quad \alpha_2 + \alpha_4 + \alpha_5, \\
\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, & \quad \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \quad \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \\
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, & \quad \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6, \quad \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \\
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6, & \quad \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \\
\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, & \quad \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \\
\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, & \quad \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \\
\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, & \quad \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \\
\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 & \equiv \hat{\alpha},
\end{align}

where $\hat{\alpha}$ is the highest root of the $E_6$ root system.

Relative to our parabolic subalgebra, the roots in (7.5) are non-compact, while the rest are compact. As before we introduce more condensed notation for the noncompact roots:

$$
\begin{align*}
\alpha_2, & \quad \alpha_{14}, \quad \alpha_{15}, \quad \alpha_{16}, \quad \alpha_{24}, \quad \alpha_{25}, \quad \alpha_{26} \\
\alpha_{24}, & \quad \alpha_{2,45}, \quad \alpha_{2,45'}, \quad \alpha_{25,4}, \quad \alpha_{15,4}, \quad \alpha_{26,4} \\
\alpha_{15,4}, & \quad \alpha_{15,34}, \quad \alpha_{26,45}, \quad \alpha_{16,34}, \quad \alpha_{16,45} \\
\alpha_{16,35}, & \quad \alpha_{16,35,4}, \quad \alpha_{16,25,4} = \hat{\alpha}
\end{align*}
$$

Now we should connect our considerations with the case of another real form of $E_6$, namely, the Lie algebra $E_6(-14)$, cf. [153, 154]. In that paper we chose as maximal parabolic $\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$, where $\mathcal{M}' \cong su(5,1)$, $\dim \mathcal{N} = 21$, cf. (11.21) of [121].

Since both the algebras and the maximal parabolics have the same complexification, this means that they are parabolically related, thus, we have the same non-compact roots, the same signatures, and the same multiplets. We show only the main multiplet in figure 10, referring to [153, 154] for the diagrams of reduced multiplets. The main multiplet has 70 members and the figure has the standard $E_6$ symmetry, namely, conjugation exchanging indices $1 \leftrightarrow 6$, $3 \leftrightarrow 5$. The Knapp-Stein operators act pictorially as reflection w.r.t. the dotted line separating the $\mathcal{H}^-$... members from the $\mathcal{H}^+$... members. Note that there are five cases when the embeddings correspond to the highest root $\hat{\alpha}$: $V^{\Lambda_-} \rightarrow V^{\Lambda^+}$, $\Lambda^+ = \Lambda^- - m_4 \hat{\alpha}$. In these five cases the weights are denoted as: $\Lambda_{k,\nu}^\pm$, $\Lambda_{k,\nu'}^\pm$, $\Lambda_{k}^\pm$, $\Lambda_{k,4}^\pm$, $\Lambda_{k,5}^\pm$, then: $m_\hat{\alpha} = m_1, m_3, m_4, m_5, m_6$, resp. We recall that Knapp-Stein operators $G^+$ intertwine the corresponding ERs $\mathcal{T}_{\chi^-}$ and $\mathcal{T}_{\chi^+}$. In the above five cases the Knapp-Stein operators $G^+$ degenerate to differential operators as we discussed earlier.

8 Summary and outlook

In the present paper we continued the project of systematic construction of invariant differential operators for non-compact semisimple Lie groups. Our aim in this paper was
Figure 10. Main type of multiplets for $E_6(6)$, $E_6(2)$ and $E_6(-14)$ with parabolic factor $\mathcal{M}^C = sl(6, \mathbb{C})$. 
to extend our considerations beyond the class of algebras, which we call 'conformal Lie algebras' (CLA). For this we introduce the new notion of parabolic relation between two non-compact semisimple Lie algebras $\mathcal{G}$ and $\mathcal{G}'$ that have the same complexification and possess maximal parabolic subalgebras with the same complexification. Thus, we considered the algebras $so(p, q)$ all of which are parabolically related to the conformal algebra $so(n, 2)$ with $p + q = n + 2$, then the algebras $su^*(4k)$ and $sl(4k, \mathbb{R})$ parabolically related to the CLA $su(2k, 2k)$, then $sp(r, r)$ as parabolically related to the CLA $sp(2r)$ (of rank $2r$), then the exceptional Lie algebra $E_7(7)$ which is parabolically related to the CLA $E_7(-25)$, finally the exceptional Lie algebras $E_6(6)$ and $E_6(2)$ parabolically related to the hermitian symmetric case $E_6(-14)$.

We have given a formula for the number of representations in the main multiplets valid for CLAs and all algebras that are parabolically related to them. In all considered cases we have given the main multiplets of indecomposable elementary representations including the necessary data for all relevant invariant differential operators. In the case of $so(p, q)$ we have given also the reduced multiplets. We note that the multiplets are given in the most economic way in pairs of shadow fields related by the Knapp-Stein restricted Weyl symmetry (and the corresponding integral operators).

Finally, we should stress that the classification of all invariant differential operators includes as special cases all possible conservation laws and conserved currents, unitary or not.

We plan also to extend these considerations to the supersymmetric cases and also to the quantum group setting. Such considerations are expected to be very useful for applications to string theory and integrable models. It is interesting to note that almost all of the algebras that appear in table 1 of [80] are treated in the present paper, though our motivations and approach are different (see also [186–188]).

Acknowledgments

The author thanks S. Ferrara for stimulating discussions. The author thanks the Theory Division of CERN for hospitality during the course of this work. This work was supported in part by the Bulgarian National Science Fund, grant DO 02-257.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

[1] J.F. Cornwell, Group Theory in Physics. Vol. III, Academic Press, London U.K. (1989).
[2] J. Maldacena, Large $N$ Field Theories, String Theory and Gravity, in Lectures on Quantum Gravity, Series of the Centro De Estudios Científicos, A. Gomberoff and D. Marolf eds., Springer, New York U.S.A. (2005), pg. 91.
[3] J. Terning, International Series of Monographs on Physics. Vol. 132: Modern Supersymmetry: Dynamics and Duality, Oxford University Press, Oxford U.K. (2005).
[4] S. Ferrara, J. Wess and B. Zumino, *Supergauge Multiplets and Superfields*, Phys. Lett. **B 51** (1974) 239 [SPIRE].

[5] S. Ferrara and B. Zumino, *Supergauge Invariant Yang-Mills Theories*, Nucl. Phys. **B 79** (1974) 413 [SPIRE].

[6] P. Fayet and S. Ferrara, *Supersymmetry*, Phys. Rept. **32** (1977) 249 [SPIRE].

[7] V. Ogievetsky and E. Sokatchev, *On Vector Superfield Generated by Supercurrent*, Nucl. Phys. **B 124** (1977) 309 [SPIRE].

[8] V. Ogievetsky and E. Sokatchev, *Structure of supergravity group*, Phys. Lett. **B 79** (1978) 222 [SPIRE].

[9] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, *Unconstrained N = 2 Matter, Yang-Mills and Supergravity Theories in Harmonic Superspace*, Class. Quant. Grav. **1** (1984) 469 [SPIRE].

[10] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, *Unconstrained Off-Shell N = 3 Supersymmetric Yang-Mills Theory*, Class. Quant. Grav. **2** (1985) 155 [SPIRE].

[11] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, *Harmonic superspace: key to N = 2 supersymmetry theories*, JETP Lett. **40** (1984) 912 [SPIRE].

[12] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, *Harmonic Supergraphs. Green Functions*, Class. Quant. Grav. **2** (1985) 601 [SPIRE].

[13] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, *HyperK"ahler metrics and harmonic superspace*, Commun. Math. Phys. **103** (1986) 515 [SPIRE].

[14] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, *N = 2 supergravity in superspace: different versions and matter couplings*, Class. Quant. Grav. **4** (1987) 1255 [SPIRE].

[15] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, *Gauge field geometry from complex and harmonic analyticities. K"ahler and selfdual Yang-Mills cases*, Annals Phys. **185** (1988) 1 [SPIRE].

[16] E. Sokatchev, *Projection Operators and Supplementary Conditions for Superfields with an Arbitrary Spin*, Nucl. Phys. **B 99** (1975) 96 [SPIRE].

[17] E. Sokatchev, *Light cone harmonic superspace and its applications*, Phys. Lett. **B 169** (1986) 209 [SPIRE].

[18] E. Sokatchev, *Harmonic superparticle*, Class. Quant. Grav. **4** (1987) 237 [SPIRE].

[19] F. Delduc, A. Galperin and E. Sokatchev, *Lorentz harmonic (super)fields and (super)particles*, Nucl. Phys. **B 368** (1992) 143 [SPIRE].

[20] D.Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, *Progress Toward a Theory of Supergravity*, Phys. Rev. **D 13** (1976) 3214 [SPIRE].

[21] S. Ferrara and P. van Nieuwenhuizen, *The Auxiliary Fields of Supergravity*, Phys. Lett. **B 74** (1978) 333 [SPIRE].

[22] E. Cremmer, J. Scherk and S. Ferrara, *SU(4) Invariant Supergravity Theory*, Phys. Lett. **B 74** (1978) 61 [SPIRE].

[23] E. Cremmer, S. Ferrara, L. Girardello and A. Van Proeyen, *Coupling Supersymmetric Yang-Mills Theories to Supergravity*, Phys. Lett. **B 116** (1982) 231 [SPIRE].
[24] E. Cremmer, S. Ferrara, L. Girardello and A. Van Proeyen, Yang-Mills Theories with Local Supersymmetry: Lagrangian, Transformation Laws and SuperHiggs Effect, Nucl. Phys. B 212 (1983) 413 [SPIRE].

[25] F. Delduc and E. Sokatchev, Superparticle with extended worldline supersymmetry, Class. Quant. Grav. 9 (1992) 361 [SPIRE].

[26] F. Delduc, A. Galperin, P.S. Howe and E. Sokatchev, A Twistor formulation of the heterotic $D = 10$ superstring with manifest $(8,0)$ world sheet supersymmetry, Phys. Rev. D 47 (1993) 578 [hep-th/9207050] [SPIRE].

[27] A. Galperin and E. Sokatchev, A Twistor like $D = 10$ superparticle action with manifest $N = 8$ worldline supersymmetry, Phys. Rev. D 46 (1992) 714 [hep-th/9203051] [SPIRE].

[28] E. Witten, On the Landau-Ginzburg description of $N = 2$ minimal models, Int. J. Mod. Phys. A 9 (1994) 4783 [hep-th/9304026] [SPIRE].

[29] E. Witten, SL(2, $\mathbb{Z}$) Action On Three-Dimensional Conformal Field Theories With Abelian Symmetry, in From Fields to Stings: Circumnavigating Theoretical Physics. Vol. 2, M. Shifman et al. eds., World Scientific, Singapore (2004), pg. 1173 [hep-th/0307041].

[30] A. Ceresole, R. D’Auria and S. Ferrara, The symplectic structure of $N = 2$ supergravity and its central extension, Nucl. Phys. Proc. Suppl. 46 (1996) 67 [hep-th/9509160] [SPIRE].

[31] A. Ceresole, G. Dall’Agata, R. D’Auria and S. Ferrara, Spectrum of type IIB supergravity on $AdS_5 \times T^1$: Predictions on $N = 1$ SCFT’s, Phys. Rev. D 61 (2000) 066001 [hep-th/9905226] [SPIRE].

[32] I. Antoniadis, S. Ferrara and T. Taylor, $N = 2$ heterotic superstring and its dual theory in five-dimensions, Nucl. Phys. B 460 (1996) 489 [hep-th/9511108] [SPIRE].

[33] I. Antoniadis, S. Ferrara, R. Minasian and K. Narain, $R^4$ couplings in $M$ and type-II theories on Calabi-Yau spaces, Nucl. Phys. B 507 (1997) 571 [hep-th/9707013] [SPIRE].

[34] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, Large $N$ field theories, string theory and gravity, Phys. Rept. 323 (1999) 183 [hep-th/9905111].

[35] V.K. Dobrev, G. Mack, I.T. Todorov, V.B. Petkova and S.G. Petrova, On the Clebsch-Gordan Expansion for the Lorentz Group in $n$ Dimensions, Rept. Math. Phys. 9 (1976) 219 [SPIRE].

[36] V.K. Dobrev, V.B. Petkova, S.G. Petrova and I.T. Todorov, Dynamical Derivation of Vacuum Operator Product Expansion in Euclidean Conformal Quantum Field Theory, Phys. Rev. D 13 (1976) 887 [SPIRE].

[37] V.K. Dobrev, G. Mack, V.B. Petkova, S.G. Petrova and I.T. Todorov, Harmonic Analysis on the $n$-Dimensional Lorentz Group and Its Applications to Conformal Quantum Field Theory, Lect. Notes Phys. 63 (1977) 1.

[38] I.T. Todorov, M.C. Mintchev and V.B. Petkova, Conformal invariance in quantum field theory, in Publications of the Scuola Normale Superiore, Edizioni della Normale, Pisa Italy (1978).

[39] V.K. Dobrev and V.B. Petkova, All Positive Energy Unitary Irreducible Representations of Extended Conformal Supersymmetry, Phys. Lett. B 162 (1985) 127 [SPIRE].

[40] V.K. Dobrev and V.B. Petkova, On the group theoretical approach to extended conformal supersymmetry: classification of multiplets, Lett. Math. Phys. 9 (1985) 287 [SPIRE].
[41] V.K. Dobrev and V.B. Petkova, Group theoretical approach to extended conformal supersymmetry: function space realizations and invariant differential operators, Fortsch. Phys. 35 (1987) 537 [inSPIRE].

[42] S. Minwalla, Restrictions imposed by superconformal invariance on quantum field theories, Adv. Theor. Math. Phys. 2 (1998) 781 [hep-th/9712074] [inSPIRE].

[43] V.K. Dobrev, Positive energy unitary irreducible representations of $D = 6$ conformal supersymmetry, J. Phys. A 35 (2002) 7079 [hep-th/0201076] [inSPIRE].

[44] C. Carmeli, G. Cassinelli, A. Toigo and V. Varadarajan, Unitary representations of super Lie groups and applications to the classification and multiplet structure of super particles, Commun. Math. Phys. 263 (2006) 217 [Erratum ibid. 307 (2011) 565-566] [hep-th/0501061] [inSPIRE].

[45] V.S. Varadarajan, Unitary representations of super Lie groups, lectures given at the University of Oporto, Oporto Portugal, 20–23 July 2006.

[46] V.K. Dobrev, Characters of the unitarizable highest weight modules over the $N = 2$ superconformal algebras, Phys. Lett. B 186 (1987) 43 [inSPIRE].

[47] P.C. Argyres, M.R. Plesser, N. Seiberg and E. Witten, New $N = 2$ superconformal field theories in four-dimensions, Nucl. Phys. B 461 (1996) 71 [hep-th/9511154] [inSPIRE].

[48] E. Witten, Conformal Field Theory In Four And Six Dimensions, arXiv:0712.0157 [inSPIRE].

[49] L. Andrianopoli et al., $N = 2$ supergravity and $N = 2$ super Yang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map, J. Geom. Phys. 23 (1997) 111 [hep-th/9605032] [inSPIRE].

[50] L. Andrianopoli, R. D’Auria and S. Ferrara, U duality and central charges in various dimensions revisited, Int. J. Mod. Phys. A 13 (1998) 431 [hep-th/9612105] [inSPIRE].

[51] L. Andrianopoli, S. Ferrara, E. Sokatchev and B. Zupnik, Shortening of primary operators in $N$ extended SCFT(4) and harmonic superspace analyticity, Adv. Theor. Math. Phys. 4 (2000) 1149 [hep-th/9912007] [inSPIRE].

[52] S. Ferrara and J.M. Maldacena, Branes, central charges and U duality invariant BPS conditions, Class. Quant. Grav. 15 (1998) 749 [hep-th/9706097] [inSPIRE].

[53] S. Ferrara and C. Fronsdal, Conformal Maxwell theory as a singleton field theory on AdS$_5$, IIB three-branes and duality, Class. Quant. Grav. 15 (1998) 2153 [hep-th/9712239] [inSPIRE].

[54] S. Ferrara, C. Fronsdal and A. Zaffaroni, On $N = 8$ supergravity on AdS$_5$ and $N = 4$ superconformal Yang-Mills theory, Nucl. Phys. B 532 (1998) 153 [hep-th/9802203] [inSPIRE].

[55] B. Eden, P.S. Howe, C. Schubert, E. Sokatchev and P.C. West, Extremal correlators in four-dimensional SCFT, Phys. Lett. B 472 (2000) 323 [hep-th/9910150] [inSPIRE].

[56] B. Eden, A.C. Petkou, C. Schubert and E. Sokatchev, Partial nonrenormalization of the stress tensor four point function in $N = 4$ SYM and AdS/CFT, Nucl. Phys. B 607 (2001) 191 [hep-th/0009106] [inSPIRE].
[57] G. Arutyunov, B. Eden, A. Petkou and E. Sokatchev, Exceptional nonrenormalization properties and OPE analysis of chiral four point functions in $N = 4$ SYM(4), Nucl. Phys. B 620 (2002) 380 [hep-th/0103230] [SPIRE].

[58] L. Dolan, C.R. Nappi and E. Witten, Conformal operators for partially massless states, JHEP 10 (2001) 016 [hep-th/0109096] [SPIRE].

[59] S. Ferrara and E. Sokatchev, Short representations of SU(2,2/N) and harmonic superspace analyticity, Lett. Math. Phys. 52 (2000) 247, [hep-th/9912168].

[60] S. Ferrara and E. Sokatchev, Conformal superfields and BPS states in $AdS_4$ geometries, Int. J. Theor. Phys. 40 (2001) 935 [hep-th/0005151] [SPIRE].

[61] S. Ferrara and E. Sokatchev, Superconformal interpretation of BPS states in $AdS_4$ geometries, Int. J. Mod. Phys. B 14 (2000) 2315 [hep-th/0007058] [SPIRE].

[62] S. Ferrara and E. Sokatchev, Representations of superconformal algebras in the $AdS_7/CFT_3$ correspondence, J. Math. Phys. 42 (2001) 3015 [hep-th/0010117] [SPIRE].

[63] S. Ferrara and E. Sokatchev, Conformal primaries of $OSp(8/4,R)$ and BPS states in $AdS_4$, JHEP 05 (2000) 038 [hep-th/0003051] [SPIRE].

[64] S. Ferrara and E. Sokatchev, Superconformal interpretation of BPS states in $AdS$ geometries, Int. J. Theor. Phys. 40 (2001) 935 [hep-th/0005151] [SPIRE].

[65] S. Ferrara and E. Sokatchev, Universal properties of superconformal OPEs for $1/2$ BPS operators in $3 \leq D \leq 6$, New J. Phys. 4 (2002) 2 [hep-th/0110174].

[66] S. Ferrara and E. Sokatchev, Non-anticommutative $N = 2$ super Yang-Mills theory with singlet deformation, Phys. Lett. B 579 (2004) 226 [hep-th/0308021] [SPIRE].

[67] A.E. Faraggi, C. Kounnas and J. Rizos, Spinor-Vector Duality in fermionic $Z(2) \times Z(2)$ heterotic orbifold models, Nucl. Phys. B 774 (2007) 208 [hep-th/0611251] [SPIRE].

[68] J. Kinney, J.M. Maldacena, S. Minwalla and S. Raju, An Index for 4 dimensional super conformal theories, Commun. Math. Phys. 275 (2007) 209 [hep-th/0510251] [SPIRE].

[69] S. Gurrieri, A. Lukas and A. Micu, Heterotic String Compactifications on Half-flat Manifolds. II., JHEP 12 (2007) 081 [arXiv:0709.1932] [SPIRE].

[70] D.M. Hofman and J. Maldacena, Conformal collider physics: energy and charge correlations, JHEP 05 (2008) 012 [arXiv:0803.1467] [SPIRE].

[71] Sh. Mizoguchi, Localized Modes in Type II and Heterotic Singular Calabi-Yau Conformal Field Theories, JHEP 11 (2008) 022 [arXiv:0808.2857].

[72] L.C. Biedenharn and P. Truini, Exceptional Groups And Elementary Particle Structures, Physica A 114 (1982) 257 [SPIRE].

[73] P. Truini and L. Biedenharn, An $E_6 \times U(1)$ invariant quantum mechanics for a Jordan pair, J. Math. Phys. 23 (1982) 1327 [SPIRE].

[74] P. Truini, G. Olivieri and L. Biedenharn, The Jordan pair content of the magic square and the geometry of the scalars in $N = 2$ supergravity, Lett. Math. Phys. 9 (1985) 255 [SPIRE].

[75] P. Truini, Scalar manifolds and Jordan pairs in supergravity, Int. J. Theor. Phys. 25 (1986) 509 [SPIRE].
[76] R. Dundarer, F. Gursey and H.C. Tze, Generalized vector products, duality and octonionic identities in $D = 8$ geometry, *J. Math. Phys.* 25 (1984) 1496 [inSPIRE].

[77] S. Ferrara and M. Güneydīn, Orbits of exceptional groups, duality and BPS states in string theory, *Int. J. Mod. Phys.* A 13 (1998) 2075 [hep-th/9708025] [inSPIRE].

[78] S. Ferrara, BPS black holes, supersymmetry and orbits of exceptional groups, *Fortsch. Phys.* 47 (1999) 159 [hep-th/9801095] [inSPIRE].

[79] M. Güneydīn, K. Koepsell and H. Nicolai, Conformal and quasiconformal realizations of exceptional Lie groups, *Commun. Math. Phys.* 221 (2001) 57 [hep-th/9908063] [inSPIRE].

[80] S. Ferrara, R. Kallosh and A. Marrani, Degeneration of Groups of Type $E_7$ and Minimal Coupling in Supergravity, *JHEP* 06 (2012) 074 [arXiv:1202.1290] [inSPIRE].

[81] M. Duff and S. Ferrara, $E_6$ and the bipartite entanglement of three qutrits, *Phys. Rev.* D 76 (2007) 124023 [arXiv:0704.0507] [inSPIRE].

[82] F. Bernardoni, S.L. Cacciatori, Bianca L. Cerchiai and A. Scotti, Mapping the geometry of the $E_6$ group, *J. Math. Phys.* 49 (2008) 012107 [arXiv:0710.0356].

[83] S.L. Cacciatori, F.D. Piazza and A. Scotti, $E_7$ groups from octonionic magic square, *Adv. Theor. Math. Phys.* 15 (2011) 1605 [arXiv:1007.4758] [inSPIRE].

[84] R. Kallosh and M. Soroush, Explicit Action of $E_7(7)$ on $N = 8$ Supergravity Fields, *Nucl. Phys.* B 801 (2008) 25 [arXiv:0802.4106] [inSPIRE].

[85] R. Kallosh and T. Kugo, The footprint of $E_7$ in amplitudes of $N = 8$ supergravity, *JHEP* 01 (2009) 072 [arXiv:0811.3414].

[86] M. Bianchi and S. Ferrara, Enriques and Octonionic Magic Supergravity Models, *JHEP* 02 (2008) 054 [arXiv:0712.2976] [inSPIRE].

[87] M. Cederwall and J. Palmkvist, The octic $E_8$ invariant, *J. Math. Phys.* 48 (2007) 073505 [hep-th/0702024] [inSPIRE].

[88] L. Brink, Maximal supersymmetry and exceptional groups, *Mod. Phys. Lett.* A 25 (2010) 2715 [arXiv:1006.1558] [inSPIRE].

[89] M. Güneydīn and O. Pavlyk, Quasiconformal Realizations of $E_{6(6)}, E_{7(7)}, E_{8(8)}$ and $SO(n+3,m+3), N \geq 4$ Supergravity and Spherical Vectors, *Adv. Theor. Math. Phys.* 13 (2009) 1 [arXiv:0904.0784].

[90] S.L. Cacciatori, B.L. Cerchiai and A. Marrani, Iwasawa $N = 8$ Attractors, *J. Math. Phys.* 51 (2010) 102502 [arXiv:1005.2231] [inSPIRE].

[91] S.L. Cacciatori, B.L. Cerchiai and A. Marrani, Magic Coset Decompositions, arXiv:1201.6314 [inSPIRE].

[92] L. Borsten, M. Duff, A. Marrani and W. Rubens, On the Black-Hole/Qubit Correspondence, *Eur. Phys. J. Plus* 126 (2011) 37 [arXiv:1101.3559] [inSPIRE].

[93] A. Marrani, E. Orazi and F. Riccioni, Exceptional Reductions, *J. Phys.* A 44 (2011) 155207 [arXiv:1012.5797] [inSPIRE].

[94] M. Güneydīn, H. Samtleben and E. Sezgin, On the Magical Supergravities in Six Dimensions, *Nucl. Phys.* B 848 (2011) 62 [arXiv:1012.1818] [inSPIRE].

[95] L. Borsten, M. Duff, S. Ferrara and A. Marrani, Freudenthal Dual Lagrangians, arXiv:1212.3254 [inSPIRE].

– 35 –
[96] Harish-Chandra, *Discrete series for semisimple Lie groups: II*, Ann. Math. 116 (1966) 1.

[97] I.N. Bernstein, I.M. Gel’fand and S.I. Gel’fand, *Structure of representations generated by highest weight vectors*, Funkts. Anal. Prilozh. 5 (1971) 1 [Funct. Anal. Appl. 5 (1971) 1].

[98] G. Warner, *Harmonic Analysis on Semi-Simple Lie Groups I*, Springer, Berlin Germany (1972).

[99] R.P. Langlands, *On the classification of irreducible representations of real algebraic groups*, Math. Surveys and Monographs. Vol. 31, AMS Publications, Providence U.S.A. (1988).

[100] D.P. Zhelobenko, *Harmonic Analysis on Semisimple Complex Lie Groups* in Russian, Nauka, Moscow USSR (1974).

[101] B. Kostant, *Verma modules and the existence of quasi-invariant differential operators*, in *Lecture Notes in Math.* Vol. 466, A. Dold and B. Eckmann eds., Springer-Verlag, Berlin Germany (1975), pg. 101.

[102] J. Wolf, *Unitary Representations of Maximal Parabolic Subgroups of the Classical Groups*, Memoirs Amer. Math. Soc. Vol. 180, AMS Publications, Providence U.S.A. (1976).

[103] J. Wolf, *Classification and Fourier inversion for parabolic subgroups with square integrable nilradical*, Memoirs Amer. Math. Soc. Vol. 225, AMS Publications, Providence U.S.A. (1979).

[104] A.W. Knapp and G.J. Zuckerman, *Classification theorems for representations of semisimple groups*, in *Lect. Notes Math.* Vol. 587, Springer, Berlin Germany (1977), pg. 138.

[105] A.W. Knapp and G.J. Zuckerman, *Classification of irreducible tempered representations of semisimple groups*, Ann. Math. 116 (1982) 389.

[106] B. Speh and D.A. Vogan Jr., *Reducibility of generalized principal series representations*, Acta Math. 145 (1980) 227.

[107] D. Vogan, *Progress in Mathematics. Vol. 15: Representations of Real Reductive Lie Groups*, Birkhäuser, Boston U.S.A. (1981).

[108] T. Enright, R. Howe and W. Wallach, *A classification of unitary highest weight modules*, in: *Representations of Reductive Groups*, P. Trombi eds., Birkhäuser, Boston U.S.A. (1983), pg. 97.

[109] T.P. Branson, G. Olafsson and B. Orsted, *Spectrum generating operators, and intertwining operators for representations induced from a maximal parabolic subgroup*, J. Funct. Anal. 135 (1996) 163.

[110] M. Eastwood, *Notes on conformal geometry*, Suppl. Rend. Circ. Mat. Palermo (2), 43 (1996) 57.

[111] P. Truini and V. Varadarajan, *Universal deformations of reductive Lie algebras*, Lett. Math. Phys. 26 (1992) 53 [inSPIRE].

[112] P. Truini and V. Varadarajan, *CGTMP Salamanca 92 Proc. Anales de Fisica. Monografias. Vol. I*, M.A. del Olmo et al. eds., CIEMAT/RSEF, Madrid Spain (1993) pg. 208.

[113] P. Truini and V. Varadarajan, *Quantization of reductive Lie algebras: Construction and univerasality*, Rev. Math. Phys. 5 (1993) 363 [inSPIRE].

[114] P. Truini and V. Varadarajan, in *Symmetries in Science VI*, B. Gruber eds., Plenum Press, New York U.S.A. (1993), pg. 731.
[115] V.G. Kac and M. Wakimoto, *Integrable highest weight modules over affine superalgebras and number theory*, in: Progress in Mathematics. Vol. 123: Lie Theory and Geometry, Birkhäuser, Boston U.S.A. (1994), pg. 415.

[116] T. Kobayashi, *Discrete decomposability of the restriction of $A_q(\lambda)$ with respect to reductive subgroups and its applications*, Invent. Math. **117** (1994) 181.

[117] A.W. Knapp, *Progress in Mathematics. Vol. 140: Lie Groups Beyond an Introduction*, second edition, Birkhäuser, Boston U.S.A. (2002)

[118] V. Kac, S.S. Roan and M. Wakimoto, *Quantum reduction for affine superalgebras*, Commun. Math. Phys. **241** (2003) 307.

[119] B. Kostant, *Powers of the Euler product and commutative subalgebras of a complex simple Lie algebra*, Invent. Math. **158** (2004) 181.

[120] K. Baur and N. Wallach, *Nice parabolic subalgebras of reductive Lie algebras*, Represent. Theor. **9** (2005) 1.

[121] V.K. Dobrev, *Invariant differential operators for non-compact Lie groups: Parabolic subalgebras*, Rev. Math. Phys. **20** (2008) 407 [hep-th/0702152] [INSPIRE].

[122] Harish-Chandra, *Representations of semisimple Lie groups: IV*, Am. J. Math. **77** (1955) 743.

[123] Harish-Chandra, *Representations of semisimple Lie groups: V*, Am. J. Math. **78** (1956) 1.

[124] F. Bruhat, *Sur les representations induites des groupes de Lie*, Bull. Soc. Math. France **84** (1956) 97.

[125] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Oxford Mathematical Monographs, Clarendon Press, Oxford U.K. (1994).

[126] M. Günyaydin, *Generalized conformal and superconformal group actions and Jordan algebras*, Mod. Phys. Lett. A **8** (1993) 1407 [hep-th/9301050] [INSPIRE].

[127] G. Mack and M. de Riese, *Simple space-time symmetries: Generalizing conformal field theory*, J. Math. Phys. **48** (2007) 052304.

[128] S. Okubo, *Pseudoquarternion And Pseudooctonion Algebras*, Hadronic J. **1** (1978) 1250.

[129] M. Günyaydin and C. Saclioglu, *Oscillator like unitary representations of noncompact groups with a Jordan structure and the noncompact groups of supergravity*, Commun. Math. Phys. **87** (1982) 159 [INSPIRE].

[130] P. Truini and L. Biedenharn, *An $E_6 \times U(1)$ invariant quantum mechanics for a Jordan pair*, J. Math. Phys. **23** (1982) 1327 [INSPIRE].

[131] M. Günyaydin, G. Sierra and P. Townsend, *Exceptional Supergravity Theories and the MAGIC Square*, Phys. Lett. B **133** (1983) 72 [INSPIRE].

[132] M. Günyaydin, G. Sierra and P. Townsend, *VAningishing potentials in gauged $n = 2$ supergravity: an application of Jordan algebras*, Phys. Lett. B **144** (1984) 41 [INSPIRE].

[133] M. Günyaydin, G. Sierra and P. Townsend, *The Geometry of $N = 2$ Maxwell-Einstein Supergravity and Jordan Algebras*, Nucl. Phys. B **242** (1984) 244 [INSPIRE].

[134] M. Günyaydin, G. Sierra and P. Townsend, *Gauging the $D = 5$ Maxwell-Einstein Supergravity Theories: More on Jordan Algebras*, Nucl. Phys. B **253** (1985) 573 [INSPIRE].

[135] G. Sierra, *An application to the theories of Jordan algebras and fredenthal triple systems to particles and strings*, Class. Quant. Grav. **4** (1987) 227 [INSPIRE].
[136] S. Cecotti, S. Ferrara and L. Girardello, *A topological formula for the Kähler potential of $4-d$ $N = 1$, $N = 2$ strings and its implications for the moduli problem*, Phys. Lett. B 213 (1988) 443 [inSPIRE].

[137] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of Type II Superstrings and the Moduli of Superconformal Field Theories*, Int. J. Mod. Phys. A 4 (1989) 2475 [inSPIRE].

[138] M. Cederwall, *Jordan algebra dynamics*, Phys. Lett. B 210 (1988) 169 [inSPIRE].

[139] C. Hull, *Higher spin extended conformal algebras and W gravities*, Nucl. Phys. B 353 (1991) 707 [inSPIRE].

[140] R. Iordanescu and P. Truini, *Quantum groups and Jordan structures*, Bull. Univ. Politecnica Appl. Math. Sect. - Bucharest (1994) [hep-th/9406099] [inSPIRE].

[141] P. Ramond, *Algebraic dreams*, hep-th/0112261 [inSPIRE].

[142] S. Catto, *Exceptional projective geometries and internal symmetries*, hep-th/0302079 [inSPIRE].

[143] S. Ferrara and A. Marrani, *Symmetric Spaces in Supergravity*, arXiv:0808.3567 [inSPIRE].

[144] L. Borsten, D. Dahanayake, M.J. Duff, S. Ferrara, A. Marrani and W. Rubens, *Observations on Integral and Continuous U-duality Orbits in $N = 8$ Supergravity*, Class. Quant. Grav. 27 (2010) 185003 [arXiv:1002.4223] [inSPIRE].

[145] L. Borsten, M. Duff, S. Ferrara, A. Marrani and W. Rubens, *Small Orbits*, Phys. Rev. D 85 (2012) 086002 [arXiv:1108.0424] [inSPIRE].

[146] B.L. Cerchiai, S. Ferrara, A. Marrani and B. Zumino, *Charge Orbits of Extremal Black Holes in Five Dimensional Supergravity*, Phys. Rev. D 82 (2010) 085010 [arXiv:1006.3101] [inSPIRE].

[147] S. Ferrara, R. Kallosh, A. Linde, A. Marrani and A. Van Proeyen, *Superconformal Symmetry, NMSSM and Inflation*, Phys. Rev. D 83 (2011) 025008 [arXiv:1008.2942] [inSPIRE].

[148] G. Allemandi, M. Capone, S. Capozziello and M. Francaviglia, *Conformal aspects of Palatini approach in extended theories of gravity*, Gen. Rel. Grav. 38 (2006) 33 [hep-th/0409198] [inSPIRE].

[149] V.K. Dobrev, *Positive Energy Representations, Holomorphic Discrete Series and Finite-Dimensional Irreps*, J. Phys. A 41 (2008) 425206 [arXiv:0712.4375] [inSPIRE].

[150] V.K. Dobrev, *Invariant differential operators for non-compact Lie groups: the main $su(n,n)$ cases*, plenary talk at SYMPHYS XV, Dubna Russia, 12–16 July 2011.

[151] V.K. Dobrev, *Invariant Differential Operators for Non-Compact Lie Groups: the $sp(n,R)$ Case*, arXiv:1205.5521 [inSPIRE].

[152] V.K. Dobrev, *Exceptional Lie Algebra $E_{7(-25)}$: Multiplets and Invariant Differential Operators*, J. Phys. A 42 (2009) 285203 [arXiv:0812.2690] [inSPIRE].

[153] V.K. Dobrev, *Invariant Differential Operators for Non-Compact Lie Groups: the $E_{6(-14)}$ case*, arXiv:0812.2655 [inSPIRE].

[154] V.K. Dobrev, *Invited Lectures at 5th Mathematical Physics Meeting: Summer School and Conference on Modern Mathematical Physics*, Belgrade Serbia (2008).
[155] A.W. Knapp, *Representation Theory of Semisimple Groups. An Overview Based on Examples*, Princeton University Press, Princeton U.S.A. (1986).

[156] V.K. Dobrev, Canonical construction of intertwining differential operators associated with representations of real semisimple Lie groups, *Rept. Math. Phys.* 25 (1988) 159.

[157] V.K. Dobrev, Multiplet classification of the reducible elementary representations of real semisimple Lie groups: the $SO_c(p, q)$ example, *Lett. Math. Phys.* 9 (1985) 205 [INSPIRE].

[158] V.K. Dobrev, Elementary representations and intertwining operators for $SU(2, 2)$. I., *J. Math. Phys.* 26 (1985) 235 [INSPIRE].

[159] V.K. Dobrev and V.B. Petkova, On the group theoretical approach to extended conformal supersymmetry: classification of multiplets, *Lett. Math. Phys.* 9 (1985) 287 [INSPIRE].

[160] V.K. Dobrev, Multiplet classification of the indecomposable highest weight modules over the Neveu-Schwarz and Ramond superalgebras, *Lett. Math. Phys.* 11 (1986) 225 [INSPIRE].

[161] V.K. Dobrev, Multiplet classification of the reducible Verma modules over affine Lie algebras and invariant differential operators, talk at Conference on Algebraic Geometry and Integrable Systems, Oberwolfach Germany (1984).

[162] V.K. Dobrev, Multiplets of indecomposable highest weight modules over infinite-dimensional Lie algebras: the Virasoro-$A^{(1)}_1$ correspondence, in *Proceedings of XIII Int. Conf. Diff.-Geom. Meth. Theor. Phys.*, Shumen Bulgaria (1984), H.-D. Doebner and T.D. Palev eds., World Scientific, Singapore (1986), pg. 348.

[163] V.K. Dobrev, Multiplets of Verma modules over the $osp(2, 2)^{(1)}$ super Kac-Moody algebra, in *Proceedings of Int. Symp. Topol. Geom. Methods Field Theory*, Espoo Finland (1986), J. Hietarinta and J. Westerholm eds., World Scientific, Singapore (1986), pg. 93.

[164] V.K. Dobrev, Multiplet classification of highest weight modules over quantum universal enveloping algebras: the $U_q(sl(3, C))$ example, talk at Int. Group Theory Conference, St. Andrews U.K. (1989), [London Math. Soc. Lecture Note Ser. 159 (1991) 87].

[165] V.K. Dobrev, Positive energy unitary irreducible representations of $D = 6$ conformal supersymmetry, *J. Phys. A* 35 (2002) 7079 [hep-th/0201076] [INSPIRE].

[166] V.K. Dobrev, Characters of the positive energy UIRs of $D = 4$ conformal supersymmetry, *Phys. Part. Nucl.* 38 (2007) 564 [hep-th/0406154] [INSPIRE].

[167] V.K. Dobrev and A.Ch. Ganchev, Modular invariance for the $N = 2$ twisted superconformal algebra, *Mod. Phys. Lett. A* 3 (1988) 127 [INSPIRE].

[168] V.K. Dobrev and P.J. Moylan, Finite dimensional singletons of the quantum anti-de Sitter algebra, *Phys. Lett. B* 315 (1993) 292 [INSPIRE].

[169] J. Dixmier, *Enveloping Algebras*, North Holland, New York U.S.A. (1977).

[170] V.K. Dobrev, Singular vectors of quantum groups representations for straight Lie algebra roots, *Lett. Math. Phys.* 22 (1991) 251 [INSPIRE].

[171] N. Chair, V.K. Dobrev and H. Kanno, $SO(2, C)$ invariant ring structure of BRST cohomology and singular vectors in $2 – D$ gravity with $c < 1$ matter, *Phys. Lett. B* 283 (1992) 194 [hep-th/9201071] [INSPIRE].

[172] V.K. Dobrev, Subsingular vectors and conditionally invariant ($q$-deformed) equations, *J. Phys. A* 28 (1995) 7135.
[173] V.K. Dobrev, "Kazhdan-Lusztig polynomials, subsingular vectors, and conditionally invariant (q-deformed) equations," in *Proceedings of Symposium "Symmetries in Science IX*, Bregenz Austria (1996), B. Gruber and M. Ramek eds., Plenum Press, New York U.S.A. (1997), pg. 47.

[174] V.K. Dobrev and V.B. Petkova, *Elementary Representations and Intertwining Operators for the Group SU*+(4),* Rept. Math. Phys. 13 (1978) 233 [inSPIRE].

[175] V.B. Petkova and G.M. Sotkov, *The six point families of exceptional representations of the conformal group,* Lett. Math. Phys. 8 (1984) 217 [Erratum ibid. 9 (1985) 83] [inSPIRE].

[176] V.K. Dobrev, *q*-difference conformal invariant operators and equations, Suppl. Rend. Circ. Mat. Palermo (2) 43 (1996) 15.

[177] A.W. Knapp and E.M. Stein, *Intertwining operators for semisimple groups,* Ann. Math. 93 (1971) 489.

[178] A.W. Knapp and E.M. Stein, *Intertwining operators for semisimple groups. II,* Invent. Math. 60 (1980) 9.

[179] V.K. Dobrev, *Invariant differential operators and characters of the AdS4 algebra,* J. Phys. A 39 (2006) 5995 [hep-th/0512354] [inSPIRE].

[180] V.K. Dobrev, *Intertwining operator realization of the AdS/CFT correspondence,* Nucl. Phys. B 553 (1999) 559 [hep-th/9812194] [inSPIRE].

[181] I.R. Klebanov and E. Witten, *AdS/CFT correspondence and symmetry breaking,* Nucl. Phys. B 556 (1999) 89 [hep-th/9905104] [inSPIRE].

[182] R.R. Metsaev, *Anomalous conformal currents, shadow fields and massive AdS fields,* Phys. Rev. D 85 (2012) 126011 [arXiv:1110.3749] [inSPIRE].

[183] I.M. Gelfand and M.A. Naimark, *Unitary Representations of the Lorentz Group,* Acad. Sci. USSR. J. Phys. 10 (1946) 93.

[184] V. Bargmann, *Irreducible unitary representations of the Lorentz group,* Ann. Math. 48 (1947) 568.

[185] I. Satake, *On representations and compactifications of symmetric Riemannian spaces,* Ann. Math. 71 (1960) 77.

[186] S. Ferrara, A. Marrani, E. Orazi, R. Stora and A. Yeranyan, *Two-Center Black Holes Duality-Invariants for STU Model and its lower-rank Descendants,* J. Math. Phys. 52 (2011) 062302 [arXiv:1011.5864] [inSPIRE].

[187] S. Ferrara, A. Marrani, M. Trigiante, A. Marrani and M. Trigiante, *Super-Ehlers in Any Dimension,* JHEP 11 (2012) 068 [arXiv:1206.1255] [inSPIRE].

[188] S. Ferrara, A. Marrani and B. Zumino, *Jordan Pairs, E6 and U-duality in Five Dimensions,* arXiv:1208.0347 [inSPIRE].