Supersymmetry and discrete transformations of the Dirac operators in Taub-NUT geometry

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Abstract

It is shown that the $N = 4$ superalgebra of the Dirac theory in Taub-NUT space has different unitary representations related among themselves through unitary $U(2)$ transformations. In particular the $SU(2)$ transformations are generated by the spin-like operators constructed with the help of the same covariantly constant Killing-Yano tensors which generate Dirac-type operators. A parity operator is defined and some explicit transformations which connect the Dirac-type operators among themselves are given. These transformations form a discrete group which is a realization of the quaternion discrete group. The fifth Dirac operator constructed using the non-covariant Killing-Yano tensor of the Taub-NUT space is quite special. This non-standard Dirac operator is connected with the hidden symmetry and is not equivalent to the Dirac-type operators of the standard $N = 4$ supersymmetry.

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1 Introduction

The theory of the usual or hidden symmetries of the Lagrangian quantum field theory on curved spacetimes, give rise to interesting mathematical problems concerning the properties of the physical observables. It is known that one of the largest algebras of conserved operators is produced by the Euclidean Taub-NUT geometry since beside usual isometries this has a hidden symmetry of the Kepler type [1, 2]. When discussing the geodesic equations in the Taub-NUT metric the existence of extra conserved quantities was noticed. These reflect a symmetry of the phase space of the system and enable the Schrödinger [1, 3] and Dirac equations [4, 5, 6] to be separated in a special coordinate system. This is related to the existence of a Stäckel-Killing tensor of rank 2 in Taub-NUT space.

The theory of the Dirac equation in the Kaluza-Klein monopole field was studied in the mid eighties [4]. An attempt to take into account the Runge-Lenz vector of this problem was done in [5]. We have continued this study showing that the Dirac equation is analytically solvable [3] and determining the energy eigenspinors of the central modes. Moreover, we derived all the conserved observables of this theory, including those associated with the hidden symmetries of the Taub-NUT geometry. Thus we obtained the Runge-Lenz vector-operator of the Dirac theory, pointing out its specific properties [7]. The consequences of the existence of this operator were studied in [8] showing that the dynamical algebras of the Dirac theory corresponding to different spectral domains are the same as in the scalar case [2] but involving other irreducible representations.

The Taub-NUT space is also of mathematical interest, the main features of the Taub-NUT metric relevant here are the fact that it is a 4 dimensional hyper-Kähler metric and possesses special tensors - Stäckel-Killing and Killing-Yano tensors [2, 9]. A hyper-Kähler manifold is a Riemannian manifold modeled on a quaternion inner-product space. In fact a hyper-Kähler manifold is a manifold whose Riemannian metric is Kähler with respect to three different complex structures. In the 4 dimensional case the holonomy group $Sp(1) \subset SO(4)$ is the same as $SU(2) \subset SO(4)$.

In the Taub-NUT geometry four Killing-Yano tensors are known to exist. Three of these are special because they are covariantly constant and define the complex structures of the manifold. Using these covariantly constant Killing-Yano tensors it is possible to construct new Dirac-type operators [10] which anticommute with the standard Dirac operator. The aim of this paper
is to prove explicitly that these operators and the standard Dirac one are equivalent among themselves.

We show that the representation of the whole theory can be changed using the $U(2)$ transformations among them the $SU(2)$ ones are generated just by the spin-like operators constructed using the above mentioned three Killing-Yano tensors [6]. Based on these results, we define the parity transformation and a discrete group with eight elements formed by the transformations which relate to each other the four Dirac operators and their parity transformed as well. We show that this discrete group is a realization of the quaternion group which is isomorphic with the dicyclic group of order eight.

The Taub-NUT space also possesses a Killing-Yano tensor which is not covariantly constant. The corresponding non-standard operator, constructed with the general rule [10] anticommutates with the standard Dirac operator but is not equivalent to it. This non-standard Dirac operator is connected with the hidden symmetries of the space allowing the construction of a conserved vector operator analogous to the Runge-Lenz vector of the Kepler problem [4]. The final objective here is to discuss the behavior of this operator under discrete transformations pointing out that the hidden symmetries are in some sense decoupled from the discrete symmetries studied here. The explanation of this distinction is that the standard $N = 4$ supersymmetry are linked to the hyper-Kähler structure of the Taub-NUT space. The corresponding supercharges close on the Hamiltonian of the theory. The quantal anticommutator of the Dirac-type operators closes on the square of the Hamiltonian operator. On the other hand, the non-standard supercharge involving the non-covariant Killing-Yano tensor does not close on the Hamiltonian. The appearance of the non-covariant Killing-Yano tensor in this context is not surprising since it also plays an essential role in the existence of hidden symmetries. Its existence requires the Weyl tensor to be of Petrov type $D$. The quantal anticommutator of the non-standard Dirac operator does not close on the square of the Hamiltonian, as would Dirac-type operators, rather on a combination of different conserved operators of the theory.

The paper is organized as follows. In the next two sections we introduce the first four Dirac operator which constitute the $N = 4$ superalgebra and we define the transformations leading to equivalent representations of the whole theory. These allow us to extract in section 4 the discrete transformations which show that these Dirac operators are equivalent among themselves. The role of the fifth Dirac operator is briefly discussed in section 5. Section 6 contains some discussion.
2 Dirac operators of the Taub-NUT space

Let us consider the Taub-NUT space and the chart with Cartesian coordinates \( x^\mu (\mu, \nu, ..., = 1, 2, 3, 4) \) having the line element

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{1}{V} dl^2 + V(dx^4 + A_i dx^i)^2 ,
\]

where \( dl^2 = (d\vec{x})^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \) is the Euclidean three-dimensional line element and \( \vec{A} \) is the gauge field of a monopole. Another chart suitable for applications is that of spherical coordinates, \( (r, \theta, \phi, \chi) \), among them the first three are the spherical coordinates commonly associated with the Cartesian space ones, \( x^i (i,j,... = 1,2,3) \), while \( \chi + \phi = -x^4/\mu \). The real number \( \mu \) is the parameter of the theory which enters in the form of the function \( 1/V(r) = 1 + \mu/r \). The unique non-vanishing component of the vector potential in spherical coordinates is \( A_\phi = \mu (1 - \cos \theta) \). This space has the isometry group \( G_s = SO(3) \otimes U(1)_4 \) formed by the rotations of the Cartesian space coordinates and \( x^4 \) translations. The \( U(1)_4 \) symmetry is important since this eliminates the so called NUT singularity if \( x^4 \) has the period \( 4\pi \mu \).

For the theory of the Dirac operators in Cartesian charts of the Taub-NUT space, it is convenient to consider the local frames given by tetrad fields \( e^i(x) \) and \( \hat{e}^i(x) \) as defined in \([11]\) while the four Dirac matrices \( \gamma^\alpha \), that satisfy \( \{ \gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}} \} = 2 \delta^{\hat{\alpha}\hat{\beta}} \), have to be written in the following representation

\[
\gamma^i = -i \begin{pmatrix}
0 & \sigma_i \\
-\sigma_i & 0
\end{pmatrix}, \quad \gamma^4 = \begin{pmatrix}
0 & 1_2 \\
1_2 & 0
\end{pmatrix},
\]

where all of them are self-adjoint. In addition we consider the matrix

\[
\gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}
\]

which is denoted by \( \gamma^0 \) in Kaluza-Klein theory explicitly involving the time \( \ddot{\theta} \).

The \textit{standard} Dirac operator of the theory without explicit mass term is defined as \( D_s = \gamma^{\hat{\alpha}} \hat{\nabla}_{\hat{\alpha}} \) \([3,7]\) where the spin covariant derivatives with local indices, \( \hat{\nabla}_{\hat{\alpha}} \), depend on the momentum operators, \( P_i = -i(\partial_i - A_i \partial_4) \) and \( P_4 = -i \partial_4 \), and spin connection \([4]\), such that the Hamiltonian operator
can be expressed in terms of Pauli operators,
\[ \alpha = \sqrt{V} (\hat{\sigma} \cdot \hat{P} - \frac{iP_4}{V}) , \]  \hspace{1cm} (5)  
\[ \alpha^* = V (\hat{\sigma} \cdot \hat{P} + \frac{iP_4}{V}) \frac{1}{\sqrt{V}} , \]  \hspace{1cm} (6)

involving the Pauli matrices, \( \sigma_i \). These operators give the (scalar) Klein-Gordon operator of the Taub-NUT space \( \Delta = -\nabla_\mu g^{\mu\nu} \nabla_\nu = \alpha^* \alpha \). We specify that here the star superscript is a mere notation that does not represent the Hermitian conjugation because we are using a non-unitary representation of the algebra of Dirac operators. Of course, this is equivalent to the unitary representation where all of these operators are self-adjoint \( \Box \).

The first three Killing-Yano tensors of the Taub-NUT space \( \Box \),

\[ f^i = f^i_{\alpha\beta} \hat{e}^\alpha \wedge \hat{e}^\beta = 2 \hat{e}^4 \wedge \hat{e}^i + \varepsilon_{ijk} \hat{e}^j \wedge \hat{e}^k \]  \hspace{1cm} (7)

are rather special since they are covariantly constant. The \( f^i \) define three anticommuting complex structures of the Taub-NUT manifold, their components realizing the quaternion algebra

\[ f^i f^j + f^j f^i = -2\delta_{ij} , \quad f^i f^j - f^j f^i = -2\varepsilon_{ijk} f^k . \]  \hspace{1cm} (8)

The existence of these Killing-Yano tensors is linked to the hyper-Kähler geometry of the manifold and shows directly the relation between the geometry and the \( N = 4 \) supersymmetric extension of the theory \( \Box \). Moreover we can give a physical interpretation of these Killing-Yano tensors defining the spin-like operators,

\[ \Sigma_i = -\frac{i}{4} f^i_{\alpha\beta} \gamma^\alpha \gamma^\beta = \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix} , \]  \hspace{1cm} (9)

that have similar properties to those of the Pauli matrices. In the pseudo-classical description of a Dirac particle \( \Box \), the covariantly constant Killing-Yano tensors correspond to components of the spin which are separately conserved.
Here, since the Pauli matrices commute with the Klein-Gordon operator, the spin-like operators \((\mathbf{9})\) commute with \(H^2\). Remarkable the existence of the Killing-Yano tensors allows one to construct Dirac-type operators \([10]\)

\[ Q_i = -i f^{i}_{\alpha\beta} \gamma^\alpha \nabla^\beta = \{H, \Sigma_i\} = \begin{pmatrix} 0 & \sigma_i \alpha^* \\ \alpha \sigma_i & 0 \end{pmatrix} \] (10)

which anticommute with \(D_s\) and \(\gamma^5\) and commute with \(H\) \([7]\). Another Dirac operator can be defined using the fourth Killing-Yano tensor but this will be discussed separately in Sec. 5.

### 3 Equivalent representations

In \([6]\) we have shown that in the massless case the operators \(Q_i\) \((i = 1, 2, 3)\) and the new supercharge \(Q_0 = iD_s = i\gamma^5H\) form the basis of a \(N = 4\) superalgebra obeying the anticommutation relations

\[ \{Q_A, Q_B\} = 2\delta_{AB}H^2, \quad A, B, ... = 0, 1, 2, 3 \] (11)

linked to the hyper-Kähler geometric structure of the Taub-NUT space. In addition, we associate to each Dirac operator \(Q_A\) its own Hamiltonian operator \(\bar{Q}_A = -i\gamma^5Q_A\) obtaining thus another set of supercharges,

\[ \bar{Q}_0 = H, \quad \bar{Q}_i = i[H, \Sigma_i], \] (12)

which obey the same anticommutation relations as \((11)\). Thus we find that there are two similar superalgebras of operators with precise physical meaning. Obviously, since all of these operators must be self-adjoint we have to work only with unitary representations of these superalgebras, up to an equivalence.

The concrete form of these supercharges depends on the representation of the Dirac matrices which can be changed at any time with the help of a non singular operator \(T\) such that all of the \(4 \times 4\) matrix operators of the Dirac theory transform as \(X \rightarrow X' = TXT^{-1}\). In this way one obtains an equivalent representation which preserves the commutation and the anticommutation relations. In \([8]\) we have used such transformations for pointing out that the convenient representations where we work are equivalent to an unitary one. We note that some properties of the transformations changing
representations in theories with two Dirac operators and their possible new applications are discussed in [14].

In order to produce explicit operators $T$ which connect different Dirac operators in what follows we shall consider constant non-singular matrices $T$ commuting with $H^2 = \text{diag}(\Delta, \alpha \alpha^*)$ giving transformations $Q_A \rightarrow Q'_A = TQ_AT^{-1}$ that lead to equivalent representations of our superalgebra, $\{Q'_A, Q'_B\} = 2\delta_{AB}H^2$, with the same $H^2$. Since $\Delta$ is a scalar differential operator while $\alpha \alpha^*$ has complicated spin terms, it is suitable to choose matrices of the form $T = \text{diag}(\hat{T}, 1_2)$ where $\hat{T}$ can be any non singular $2 \times 2$ constant matrix. In general, these transformations lead to new supercharges $Q'_A$ which are linear combinations of the original ones with mixing coefficients that can be complex numbers. The basic principles of quantum mechanics require the Dirac-type operators to be self-adjoint (up to an equivalence) as the standard Dirac operator $[10]$. Therefore, if one starts with a suitable representation then it is recommendable to use only unitary transformations of the form

$$U(\beta, \vec{\xi}) = \begin{pmatrix} \hat{U}(\beta, \vec{\xi}) & 0 \\ 0 & 1_2 \end{pmatrix},$$

where $\hat{U}(\beta, \vec{\xi}) = e^{-i\beta\hat{U}(\vec{\xi})} \in U(2) = U(1) \otimes SU(2)$ with $\hat{U}(\vec{\xi}) \in SU(2)$. This is because among these transformations one could find those linking equivalent Dirac operators.

It is interesting to observe that the $SU(2)$ transformations are generated just by the above defined spin-like operators as

$$U(\vec{\xi}) = U(0, \vec{\xi}) = e^{-i\vec{\xi} \cdot \vec{\sigma}/2} = \begin{pmatrix} \hat{U}(\vec{\xi}) & 0 \\ 0 & 1_2 \end{pmatrix},$$

If we take now $\vec{\xi} = 2\varphi \vec{n}$ with $|\vec{n}| = 1$ and $\varphi \in [0, \pi]$, we find that

$$\hat{U}(\vec{\xi}) = e^{-i\vec{\xi} \cdot \vec{\sigma}/2} = 1_2 \cos \varphi - i\vec{n} \cdot \vec{\sigma} \sin \varphi$$

and after a little calculation we can write the concrete action of (14) as

$$Q'_0 = U(\vec{\xi})Q_0U^+(\vec{\xi}) = Q_0 \cos \varphi + n_iQ_i \sin \varphi$$
$$Q'_i = U(\vec{\xi})Q_iU^+(\vec{\xi}) = Q_i \cos \varphi - (n_iQ_0 + \varepsilon_{ijk}n_jQ_k) \sin \varphi.$$
these transformations correspond to an irreducible representation since the supercharges transform like the real components of a Pauli spinor. In other words, the usual $SU(2)$ transformations $\psi_Q \rightarrow \psi'_Q = \hat{U}^+(\xi)\psi_Q$ of the spinor-operator

$$\psi_Q = \begin{pmatrix} Q_0 - iQ_3 \\ Q_2 - iQ_1 \end{pmatrix}$$

(18)
give just the transformations (16) and (17).

4 Discrete transformations

Let us focus now only on the transformations which transform the supercharges $Q_A$ among themselves without to affect their form. From (16) and (17) we see that there exists particular transformations,

$$Q_k = U_k Q_0 U_k^+, \quad k = 1, 2, 3,$$

(19)

where the matrix $U_k = \text{diag}(-i\sigma_k, 1_2)$ is given by $-i\sigma_k \in SU(2)$. In addition, we consider the parity operator $P = P^{-1} = -\gamma^5$ which changes the sign of supercharges,

$$PQ_A P = -Q_A, \quad A = 0, 1, 2, 3.$$

(20)

Then it is not hard to verify that the identity $I = 1_4$, $P$ and the sets of matrices $U_k$ and $PU_k (k = 1, 2, 3)$ form a discrete group of order eight the multiplication table of which is determined by the following rules

$$P^2 = I, \quad PU_k = U_k P,$$

$$U_1^2 = U_2 = U_3^2 = P,$$

$$U_1 U_2 = U_3, \quad U_2 U_1 = PU_3,$$

(21)

etc.

We denote this group by $G_Q$ since it is a realization of the quaternion group $Q$ which is isomorphic with the dicyclic group $\langle 2, 2, 2 \rangle$ [15, 16] (see the Appendix). In the representation (2) of the $\gamma$-matrices, its operators are defined by proper unitary matrices (which satisfy $G^{-1} = G^+$ and $\det G = 1, \forall G \in G_Q$) constructed using the elements $\pm 1_2, \pm i\sigma_1, \pm i\sigma_2, \pm i\sigma_3$ of the natural realization of $Q$ as a discrete subgroup of $SU(2)$.

The group $G_Q$ is interesting because it brings together the parity that produces the transformations (20) and the operators $U_k$ giving sequences of the form

$$Q_1 = U_3^+ Q_2 U_3 = U_2 Q_3 U_2^+ = U_1 Q_0 U_1^+, \ldots \text{etc.}$$

(22)
which lead to the conclusion that the Dirac operators and their parity trans-
formed, \( \pm Q_A \) \((A = 0, 1, 2, 3)\), are equivalent among themselves. All these
operators constitute the orbit \( \Omega_Q = \{Q \mid Q = GQ_0G^+, \forall G \in G_Q\} \) of the
group \( G_Q \) in the algebra of the \( 4 \times 4 \) matrix operators. A similar orbit, \( \tilde{\Omega}_Q \),
can be constructed for the associated Hamiltonian operators, \( \pm \tilde{Q}_A \) defined by
\( (12) \), if we start with \( \tilde{Q}_0 \) instead of \( Q_0 \). It is remarkable that each of these two
orbits includes only operators representing (up to sign) supercharges obeying
superalgebras of the form \( (11) \).

In the Kaluza-Klein theory with the time trivially added \( \mathbb{R} \), the time
dependent term of the whole massless Dirac operator commutes with all
the operators of \( G_Q \) such that it remains unchanged when one replaces
the space parts using the discrete transformations of this group. In these condi-
tions all the Dirac operators of \( \Omega_Q \) lead to equivalent Dirac equations from
the physical point of view. These can be written in Hamiltonian form as
\[ i\partial_t \psi^{(\pm)}_A = \pm \tilde{Q}_A \psi^{(\pm)}_A \] \((A = 0, 1, 2, 3)\) and produce the same energy spectrum
which coincides to that of the Klein-Gordon equation as it results from the
superalgebra \( (11) \) \([5, 6]\).

The existence of this discrete symmetry among the four supercharges of
the superalgebra of the Dirac and Dirac-type operators (or the corresponding
Hamiltonian operators) must be understood as a consequence of the fact that
the Taub-NUT space has a hyper-Kähler structure modeled on a quaternion
inner-product space \( \mathbb{H} \). In other words, the Dirac theory in this space picks
up the basic quaternion character of the tangent space showing it off as the
discrete symmetry due to the group \( G_Q \sim Q \), naturally related to the specific
supersymmetries of this geometry.

5 Hidden symmetries and the fifth Dirac
operator

In the Taub-NUT space, in addition to the above discussed covariantly con-
stant Killing-Yano tensors, there exists a fourth Killing-Yano tensor,
\[ f^Y = -\frac{x}{r} f^i + \frac{2x}{\mu V} \varepsilon_{ijk} \hat{e}^j \wedge \hat{e}^k, \tag{23} \]
which is not covariantly constant. The presence of \( f^Y \) is due to the existence of
the hidden symmetries of the Taub-NUT geometry which are encapsulated
in three non-trivial Stäckel-Killing tensors. These are interpreted as the components of the so-called Runge-Lenz vector of the Taub-NUT problem and are expressed as symmetrized products of the Killing-Yano tensors $f^Y$ and $f^i$, $(i = 1, 2, 3)$.

As in the case of the Dirac operators (10), one can use $f^Y$ for defining the fifth Dirac operator

$$Q^Y_0 = -i\gamma^\alpha \left( f^Y_{\alpha\beta} \hat{\nabla}^\beta - \frac{1}{6} \gamma^\alpha \gamma^\beta f_{\alpha\beta\delta}^Y \right),$$

(24)
called here the non-standard or hidden Dirac operator to emphasize the connection with the hidden symmetry of the Taub-NUT problem. It is denoted by $Q^Y_0$ instead of $Q^Y$ as in [7] to point out its relation to the standard Dirac operator since it can be put in the form

$$Q^Y_0 = i \frac{r}{\mu} \left[ Q_0, \begin{pmatrix} \sigma_r & 0 \\ 0 & \sigma_r V^{-1} \end{pmatrix} \right],$$

(25)

where $\sigma_r = \vec{x} \cdot \vec{\sigma}/r$. We showed that $Q^Y_0$ commutes with $\tilde{Q}_0 = H$ and anticommutes with $Q_0$ and $\gamma^5$ [7]. This operator is important because it allowed us to derive the explicit form of the Runge-Lenz operator, $\tilde{K}$, of the Dirac field in Taub-NUT background establishing its properties [7]. We recall that the components of the conserved total angular momentum, $\vec{J}$, and the operators $R_i = F^{-1} K_i$ with $F^2 = P_4^2 - H^2$ are just the generators of the dynamical algebra of the Dirac theory in Taub-NUT background [8].

Starting with $Q^Y_0$ we can construct a new orbit, $\Omega^Y$, of $G_Q$ defining

$$Q^Y_k = U_k Q^Y_0 U_k^+ = i \frac{r}{\mu} \left[ Q_k, \begin{pmatrix} \sigma_k \sigma_r \sigma_k & 0 \\ 0 & \sigma_r V^{-1} \end{pmatrix} \right],$$

(26)

(for $k = 1, 2, 3$) and observing that

$$PQ^Y_A P = -Q^Y_A, \quad A = 0, 1, 2, 3.$$  

(27)

From the explicit form (26) we deduce that, in contrast with the operators of the orbits $\Omega_Q$ and $\tilde{\Omega}_Q$, those of the orbit $\Omega^Y$ have more involved algebraic properties. We can convince that calculating, for example, the identity

$$H^2(Q^Y_0)^2 = H^4 + \frac{4}{\mu^2} H^2 \left( \vec{J}^2 + \frac{1}{4} \right) + 4 F^2 P_4^2,$$

(28)
and it is worth comparing it with equation (11). The Dirac-type operators 
$Q_A$ are characterized by the fact that their quantal anticommutator close on
the square of the Hamiltonian of the theory. No such expectation applies to
the non-standard, hidden Dirac operators $Q^Y_A$ which close on a combination
of different conserved operators. Also from equation (28) it results that $(Q^Y_A)^2 \neq
(Q^Y_B)^2$ if $A \neq B$ (because $\vec{J}^2$ does not commute with $U_k$). Moreover, one can
show that the commutators $[Q^Y_A, Q^Y_B]$ have complicated forms which can not
be expressed in terms of operators $Q^Y_A$. Therefore, neither the commutator
nor the anticommutator of the pairs of operators of this orbit do not lead
to significant algebraic results as the anticommutation relations (11) of the
operators $Q_A, (A = 0, 1, 2, 3)$.

Thus we conclude that the operators of the orbit $\Omega^Y$ do not form a closed
algebraic structure. The unique virtue of the equivalent operators $\pm Q^Y_A$ is
that they commute with the corresponding Hamiltonian operators $\tilde{Q}_A, (A =
0, 1, 2, 3)$. In this way we see that the discrete symmetry given by $G_Q$ is
decoupled from the hidden symmetries which have a different geometric origin.
Its existence requires the Weyl tensor to be of Petrov type D. For this reason
it is pointless to use the whole orbit $\Omega^Y$, the operator $Q^Y_0$ being enough for
deriving the components of the Runge-Lenz operator.

6 Discussion

In this article we pointed out the existence the discrete symmetry group
$G_Q \sim Q$ of the Dirac theory in Taub-NUT space which plays here the same
role as the simpler discrete group $Z_2 \subset Q$ of the usual theory of the Dirac field
in Minkowski background (formed only by the identity and parity operators).
The operator $P \in G_Q$ is interpreted as the parity operator in the massless
case of the Kaluza-Klein theories with the time trivially added because it
is a proper transformation which changes the sign of the space part of the
Dirac operator (or of the Hamiltonian one). In other theoretical conjectures
the interpretation of $P$ can be different. One of the first examples given
of a gravitational instanton was the self-dual Taub-NUT solution [17]. The
gravitational instantons are complete non-singular Einstein metrics, usually
taken to have $(+ + +)$ signature. For this reason in these theories the
operator $P$ is interpreted as the $TP$ reversal, changing the sign of all the
coordinates $x^\mu$.

As it is expected the parity operator $P$ is involved in the relation between
the indices of the Dirac-type operators. Taking into account relations (11) and (20) we observe that any pair of operators \((Q_A, PQ_B)\) with \(A \neq B\) can always be diagonalized simultaneously. Hereby it results that the kernels of all four Dirac-type operators coincide. In even-dimensional spaces the index of a Dirac operator can be defined as the difference in the number of linearly independent zero modes with eigenvalues +1 and −1 under \(\gamma^5\). It is quite simple to get the remarkable result that the index of all Dirac operators is the same \([14, 18]\). An immediate consequence is that the operators \(Q_A\) have the same zero-modes. However the zero-modes of the fifth non-standard Dirac operator \(Q^Y\) coincide with those of the other Dirac-type operators only in some peculiar cases \([13]\) even though the index of the operator \(Q^Y\) is equal to the index of the Dirac-type operators \(Q^A\) \([18]\).

In conclusion we can say that the Taub-NUT space has a special geometry where the covariantly constant Killing-Yano tensors exist by virtue of the metric being self-dual and the Dirac-type operators generated by them are equivalent with the standard one. All of these operators which form the orbit \(\Omega_Q\) of \(\mathcal{G}_Q\) accomplish the anticommutation relation (11). The fourth Killing-Yano tensor \(f^Y\) which is not covariantly constant exists by virtue of the metric being of type \(D\). The corresponding non-standard or hidden Dirac operator does not close on \(H\) as it can be seen from equation (28) and is not equivalent to the Dirac-type operators. As it was mentioned, it is associated with the hidden symmetries of the space allowing the construction of the conserved vector-operator analogous to the Runge-Lenz vector of the Kepler problem. Here we have shown how the discrete symmetry given by \(\mathcal{G}_Q\) is naturally related only to the supersymmetries, being decoupled from the hidden symmetries which have another geometric source.

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Appendix: The quaternion group

The dicyclic group \langle 2, 2, m \rangle of order 4m is defined as the discrete group generated by two elements, \( x \) and \( y \), obeying

\[
x^4 = e, \quad x^2 = y^m, \quad yx = xy^{-1}
\]  

(A.1)

where \( e \) is the unit element \([16]\).

For \( m = 2 \) we denote \( u_1 = x \), \( u_2 = y \) and \( u_3 = xy \) finding that the new element \( p = u_1^2 = u_2^2 = u_3^2 \) satisfies \( p^2 = e \). If, in addition, we consider the elements \( pu_1, pu_2 \) and \( pu_3 \), we recover similar multiplication rules as \([21]\). On the other hand, taking \( e = 1, p = -1 \) and \( u_1, u_2, u_3 \) the quaternion complex constants one generates the quaternion group \( Q \sim \langle 2, 2, 2 \rangle \). We recall that the pair \((e, p) \sim (1, -1)\) forms the cyclic group \( \mathbb{Z}_2 \subset Q \).

References

[1] G. W. Gibbons and N. S. Manton, *Nucl. Phys.* **B274** (1986) 183.

[2] G. W. Gibbons and P. J. Ruback, *Phys. Lett.* **B188** (1987) 226; *Commun. Math. Phys.* **115** (1988) 267; L. Gy. Feher and P. A. Horváthy, *Phys. Lett.* **B183** (1987) 182; B. Cordani, Gy. Feher and P. A. Horváthy, *Phys. Lett.* **B201** (1988) 481.

[3] I. I. Cotăescu and M. Visinescu, *hep-th/9911014*, *Mod. Phys. Lett.* **A15** (2000) 145.

[4] Z. F. Ezawa and A. Iwazaki, *Phys. Lett.* **B138** (1984) 81; M. Kobayashi and A. Sugamoto, *Progr. Theor. Phys.* **72** (1984) 122; A. Bais and P. Batenberg, *Nucl. Phys.* **B245** (1984) 469.

[5] A. Comtet and P. A. Horváthy, *Phys. Lett.* **B349** (1995) 49.

[6] I. I. Cotăescu and M. Visinescu, *hep-th/0008181*, *Int. J. Mod. Phys.* **A16** (2001) 1743.

[7] I. I. Cotăescu and M. Visinescu, *hep-th/0101163*, *Phys. Lett.* **B502** (2001) 229.

[8] I. I. Cotăescu and M. Visinescu, *hep-th/0102083*, *Class. Quantum Grav.* **18** (2001) 3383.
[9] N. Hitchin, *Monopoles, Minimal Surfaces and Algebraic Curves* (Séminaire de Mathématiques Supérieures, vol. 105, Les Presses de l’Université de Montréal, 1987).

[10] B. Carter and R. G. McLenaghan, *Phys. Rev.* D19 (1979) 1093; R. G. McLenaghan and Ph. Spindel, *Phys. Rev.* D20 (1979) 409.

[11] H. Boutaleb - Joutei and A. Chakrabarti, *Phys. Rev.* D21 (1979) 2280.

[12] G. W. Gibbons, R. H. Rietdijk and J. W. van Holten, hep-th/9303112, *Nucl. Phys.* B404 (1993) 42.

[13] J. W. van Holten, hep-th/9409139, *Phys. Lett.* B342 (1995) 47.

[14] V. V. Kliśhevič, *Class. Quantum Grav.* 17 (2000) 305.

[15] A. O. Barut and R. Rączka, *Theory of Group Representations and Applications* (PWN, Warszawa 1977).

[16] H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups* (Springer-Verlag, Berlin, 1965).

[17] S. W. Hawking, *Phys. Lett.* 60A (1977) 81.

[18] J. W. van Holten, S. Waldron and K. Peeters, hep-th/9901163, *Class. Quantum Grav.* 16 (1999) 2537.

[19] D. Vaman and M. Visinescu, hep-th/9707175, *Phys. Rev.* D57 (1998) 3790; D. Vaman and M. Visinescu, hep-th/9805116, *Fortschr. Phys.* 47 (1999) 493.