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Sign Reversal of the Hall Response in a Crystalline Superconductor

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We consider the Hall conductivity due to the motion of a vortex in a lattice-model of a clean superconductor, using a combination of general arguments, unrestricted Hartree-Fock calculations, and exact diagonalization. In the weak coupling limit, \( k_F \xi \gg 1 \), the sign of the Hall response of the superconducting state is the same as that of the normal (non-superconducting) state. For intermediate and strong coupling, however, \( (k_F \xi \sim 1) \) we find that the sign of the Hall response in the superconducting state can be opposite to that of the normal state. In addition, we find that the sign reversal of the Hall response is correlated with a discontinuous change in the density profile at the vortex core. Implications for experiments in the cuprate superconductors are discussed.

I. INTRODUCTION

The sign of the Hall response in a metal or semiconductor reveals the charge of the underlying charge carriers (electrons or holes). In the absence of a crystalline lattice or disorder, the Hall conductivity is fixed by Galilean invariance, and given by

\[
\sigma_{xy} = \frac{nec}{B}, \tag{1}
\]

where \( n \) is the electron density, \( c \) is the speed of light, \( B \) is the applied magnetic field, and \( e \) is the electron charge. When a lattice is introduced, the Hall conductivity can deviate from (1), both in magnitude and in sign. The explanation of the Hall conductivity was one of the early successes of the Bloch theory of metals.

When a metal undergoes a superconducting transition, its Hall conductivity can change dramatically. Deep in the superconducting phase, if the superconducting vortices become pinned by disorder, the Hall resistance vanishes (as does the diagonal resistance). If the pinning is sufficiently weak (in clean samples, at high temperatures, or high magnetic fields), vortex motion leads to a finite Hall resistance. If we neglect the crystalline lattice, \( \sigma_{xy} \) is again fixed by Galilean invariance and given by (1); in the presence of a lattice or impurities, however, the magnitude and sign of the Hall response can then differ from that of the normal (non-superconducting) state. A change in the sign of the Hall response upon approaching the superconducting phase has been detected in the cuprate superconductors in the underdoped1–3 and optimally doped4 regime. A similar phenomenon has been observed in conventional superconductor films4.

The Hall response of a superconductor in the presence of disorder has been investigated thoroughly in the literature5–7. The effects of the crystalline lattice, however, have not been fully clarified. Traditionally, the problem has been analyzed phenomenologically in a time-dependent Ginzburg-Landau framework8. The key is to analyze the motion of a single superconducting vortex in the presence of a lattice and a background superflow5,9,10; this is a collective phenomena in terms of the electrons (or Cooper pairs), and is thus not easy to describe.

In this paper, we study the Hall response of a microscopic model of a lattice superconductor. We follow the strategy of Refs. 11–13, which studied the Hall conductivity of interacting lattice bosons: the Hall conductivity of a system containing a single vortex is formulated as a property of the many-body wave-function, which is calculated non-perturbatively in the inter-particle interactions. We find a rich behavior of the Hall conductivity as a function of the number of electrons per unit cell, \( n \), and the strength of the attractive interaction that leads to superconductivity, \( U \) (parametrized by the dimensionless number \( k_F \xi \), where \( k_F \) is the Fermi momentum and \( \xi \) is the coherence length).

A representative phase diagram for the Hall conductivity is seen in Fig. 1. In the Bardeen-Cooper-Schrieffer (BCS) limit, \( k_F \xi \gg 1 \), we find that the Hall conductivity changes sign as a function of density at the point where
the normal state Fermi surface changes its topology from particle-like to hole-like. In this regime, therefore, the sign of the Hall response is the same as in the underlying normal state. When \( k_F \xi \) is of the order of unity, however, the density at which \( \sigma_{xy} \) changes sign can be different from that of the normal state. A non-monotonic behavior of this critical density as a function of \( k_F \xi \) is found. In a certain range of densities, the Hall conductivity changes its sign as a function of temperature upon cooling from the normal to the superconducting state. We discuss the origin of this behavior, and possible implication for short-coherence length superconductors, such as the cuprates.

II. MODEL

We consider a model of interacting electrons hopping on a square lattice. The Hamiltonian is given by

\[
H = - \sum_{\vec{r}, \vec{r}', \sigma = \uparrow, \downarrow} t_{\vec{r}, \vec{r}'} c_{\vec{r}'}^\dagger \sigma c_{\vec{r}\sigma} + \text{H.c.} + \sum_{\vec{r}} \left[ -U \left( n_{\vec{r}\uparrow} - \frac{1}{2} \right) \left( n_{\vec{r}\downarrow} - \frac{1}{2} \right) - \mu n_{\vec{r}} \right].
\] (2)

Here, \( c_{\vec{r}\sigma} \) annihilates an electron on site \( \vec{r} \) with spin \( \sigma \), \( n_{\vec{r}\sigma} = c_{\vec{r}\sigma}^\dagger c_{\vec{r}\sigma} \), and \( n_{\vec{r}} = n_{\vec{r}\uparrow} + n_{\vec{r}\downarrow} \). The hopping parameters are chosen to be \( t_{\vec{r}, \vec{r}'} = te^{i \phi_{\vec{r}, \vec{r}'}} \) for nearest-neighbor sites, \( t_{\vec{r}, \vec{r}'} = t' e^{i \phi_{\vec{r}, \vec{r}'}} \) for next-nearest neighbors, and 0 otherwise. (We choose in units such that \( \hbar = c = 1 \).) The spin-dependent gauge field \( A_\sigma^{\vec{r}} \) is introduced in order to induce vortices in the system, and will be defined below. \( U \) is an on-site attractive interaction. When \( U > 0 \) (which we assume in the following) and \( \mu \neq 0 \), the ground state of \( H \) in Eq. (2) is superconducting for all values of \( U \), crossing over smoothly from the BCS limit \( U \ll \max\{|t|, |t'|\} \) to a Bose-Einstein condensate of tightly bound pairs in the large \( U \) limit. To suppress a competing charge ordering instability, it will sometimes be useful to add an extended interaction term \( H_V = V \sum_{\langle i, j \rangle} (n_i - 1)(n_j - 1) \), where \( \langle i, j \rangle \) are nearest-neighbor sites.

III. COMPUTATION OF \( \sigma_{xy} \)

In order to calculate \( \sigma_{xy} \) due to the motion of a single vortex, we define the lattice model on a torus of size \( L_x \times L_y \). Next, we need to choose the gauge field \( A_{\vec{r}}^\sigma \). The flux of the gauge field \( A_{\vec{r}}^\sigma \) through the system is quantized to \( 2\pi N_\sigma^\phi \), where \( N_\sigma^\phi \) is an integer. The ground state is a condensate of Cooper pairs, composed of one electron of each spin species; therefore, the total flux seen by the condensate is \( 2\pi (N_\uparrow^\phi + N_\downarrow^\phi) \). We see that if \( N_\uparrow^\phi = N_\downarrow^\phi \), the total number of superconducting flux quanta is an even integer. Then, there are at least two vortices on the torus, and the analysis of the contribution of a single vortex to \( \sigma_{xy} \) is complicated by inter-vortex interactions.

Alternatively, we may choose \( \{N_\uparrow^\phi, N_\downarrow^\phi\} = \{1, 0\} \), for which there is a single vortex on the torus. This choice may look odd at first glance, since it does not correspond to a physical magnetic field. Nevertheless, we argue that it captures correctly the contribution of a single vortex to \( \sigma_{xy} \) in the limit \( L_x, L_y \gg \xi \). To understand why, we note that the magnetic field in the vortex core region is small. As we will argue in the following, the Hall conductivity is determined by the structure of the core. Therefore, in the limit of large system size, the only role of the external magnetic field is to guarantee that the ground state has a single vortex; the dynamics of the vortex is independent of the precise way it was induced.

In the following, we set \( A_{\vec{r}}^\sigma = 0 \), and choose \( A_{\vec{r}}^\sigma \) such that electrons with spin up are subject to a uniform flux of \( 2\pi / L_x L_y \) per unit cell. An explicit gauge choice for \( A_{\vec{r}}^\sigma \) is shown in Appendix A. We imposed twisted boundary conditions such that the electron operators satisfy and \( c_{\vec{r} + L_x \vec{e}_x, \sigma} = e^{i \Theta_{\vec{r}}} c_{\vec{r}, \sigma} \) for \( \sigma = x, y \).

The Hall conductivity of the system at \( T = 0 \) may then be expressed as

\[
\sigma_{xy} = \frac{e^2}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} d\Theta_x d\Theta_y \text{Im} \langle \partial_{\Theta_x} \Psi \partial_{\Theta_y} \Phi \rangle,
\] (3)

where \( |\Psi(\Theta_x, \Theta_y)\rangle \) is the many-body ground state wavefunction, which depends on the boundary conditions. Eq. (3) requires that \( |\Psi(\Theta_x, \Theta_y)\rangle \) is unique for all \( (\Theta_x, \Theta_y) \in [0, 2\pi] \), as is generically the case for our finite-size system. Then, \( \sigma_{xy} \) is quantized in units of \( e^2 / 2\pi \).

Eq. (3) relates \( \sigma_{xy} \) to the Berry phase accumulated when \( \Theta_x, \Theta_y \) are changed adiabatically. Changing \( \Theta_x \), \( \Theta_y \) moves the position of the center of mass of the vortex on the torus, so the Hall conductivity per site can be viewed as the Berry’s phase \( \Phi_B \) associated with the adiabatic motion of the vortex core around a single unit cell. This Berry’s phase is related by a “generalized Luttinger theorem” to the density of charged particles in the system \( k \) times.
of the vortex and $\vec{u}_v$ is its velocity. In a lattice system, however, the force on a vortex is, strictly speaking, an ill defined concept. Nevertheless, the connection of Eq. (4) to the Hall conductivity of a single vortex, through Eq. (3), is still valid.

Upon varying parameters of the Hamiltonian (2), the integer $p$ can only change discontinuously via a level crossing in the many-body spectrum, at which the integrand of Eq. (3) is ill-defined. Since far away from the vortex core the system is gapped, any level crossing must occur within the vortex core. In the following, we will focus our attention to the vortex core and map out the location of the jumps of the integer $p$ in Eq. (4).

IV. GENERAL ARGUMENTS

We are interested in $\sigma_{xy}$ as a function of the electron density $n$ [tuned by the chemical potential in Eq. (2)], the interaction strength $U/t$, and the next-nearest neighbor hopping $t'/t$. Below, we discuss general arguments that can be used to constrained the form of $\sigma_{xy}$ as a function of model parameters.

Let us first discuss the case $t' = 0$. Then, under a particle-hole transformation $C$ defined through $C_Q c_{\vec{r},\sigma} C_Q^{-1} = e^{i\vec{Q} \cdot \vec{r}} c_{\vec{r},\sigma}$ where $\vec{Q} = (\pi, \pi)$, the Hamiltonian satisfies $H(\mu) = H^*(-\mu)$. Because of the complex conjugation operation, $\sigma_{xy}$ is odd under $C_Q$: $\sigma_{xy}(\mu, U/t) = -\sigma_{xy}(-\mu, U/t)$. Under $C_Q$, $n \rightarrow 2 - n$; therefore, $\sigma_{xy}$ must change its sign at $n = 1$ for any value of $U/t$. This change of sign occurs through a jump in the integer $p$ in Eq. (4), which is associated with a degeneracy at the vortex core.

For $t' \neq 0$, particle-hole symmetry is broken, and the critical density $n_c$ at which $p$ jumps (and $\sigma_{xy}$ changes sign) can depend on $U/t$. Nevertheless, in the extreme limits of weak and strong coupling, the position of the jump can be deduced from the following arguments. For $U/t = 0$, the system is non-interacting with a single particle dispersion given by $\varepsilon_{\vec{r}} = -2t (\cos k_x + \cos k_y) - 4t' \cos k_x \cos k_y - \mu$. The ground state is a filled Fermi sea. The Fermi surface undergoes a van Hove singularity at $\mu = 4t'$ (corresponding to a density $n_{vH}$), changing its character from a particle-like to a hole-like Fermi surface (see Fig. 2). By standard semiclassical reasoning,$^{20}$ $\sigma_{xy}$ is expected to change sign at $n = n_{vH}$. Continuity implies that in the limit $U/t \rightarrow 0$, $n_c(U/t) \rightarrow n_{vH}$. In Appendix C, we show that $n_c$ indeed changes continuously in the limit $U/t \rightarrow 0$ and approaches $n_{vH}$, by analyzing the spectrum of a vortex core in the weak-coupling limit.

In the opposite limit, $U/t \gg 1$ (keeping $t'/t$ fixed), $n_c$ can also be easily determined. To zeroth order in $t/U$, there are infinitely many degenerate ground states, corresponding to an occupation of either two or zero electrons in every site. Expanding in powers of $t/U$, one obtains the following effective hard-core boson Hamiltonian:

$$H_b = -\sum_{\vec{r},\vec{r}'} \hat{t}_{\vec{r},\vec{r}'} b_{\vec{r}}^d \hat{b}_{\vec{r}'} + \text{H.c.} - 2\mu \sum_{\vec{r}} n_{b,\vec{r}} + \sum_{\vec{r},\vec{r}'} \hat{V}_{\vec{r},\vec{r}'} (n_{b,\vec{r}} - \frac{1}{2})^2 + O\left(\frac{t^3}{U^2}\right), \quad (5)$$

where $b_{\vec{r}}^d$ creates a pair of electrons with opposite spins on site $\vec{r}$, $n_{b,\vec{r}} = b_{\vec{r}}^d b_{\vec{r}}$, and $\hat{t}_{\vec{r},\vec{r}'}$, $\hat{V}_{\vec{r},\vec{r}'}$ are effective boson hopping and interaction, which scale as $t'^2/U$. Explicit expressions for $\hat{t}_{\vec{r},\vec{r}'}$, $\hat{V}_{\vec{r},\vec{r}'}$, as well as higher order terms in $t/U$, are given in Appendix E.

To order $t^2/U$, the system is particle-hole symmetric: $C_B H_b(\mu) C_B^{-1} = H_b(-\mu)$, independently of $t'/t$, where $C_B b_{\vec{r}} C_B^{-1} = b_{\vec{r}}$. This fixes $n_c(U/t \rightarrow \infty) = 1$, up to corrections of higher order in $t/U$: Particle-hole symmetry breaking terms appear in Eq. (5) at order $t^3/U^2$, and their sign depends on the sign of $t'$. In Appendix E, we analyze the correction to $n_c(U/t)$ due to these terms, and find that for positive (negative) $t'$, $n_c(U/t)$ approaches 1 from below (above) in the $U/t \rightarrow \infty$ limit. This is supported by our numerical calculations which we discuss next.

V. NUMERICAL CALCULATIONS

In the intermediate interaction regime, $U/t \sim 1$, we lack a small expansion parameter and therefore resort to a numerical calculation of $\Phi_B$. We apply an unrestricted Hartree-Fock approximation by minimizing the energy $\langle \Psi(\{\Delta \vec{r}, \mu_{\vec{r}}\}) | H | \Psi(\{\Delta \vec{r}, \mu_{\vec{r}}\}) \rangle$ where $| \Psi(\{\Delta \vec{r}, \mu_{\vec{r}}\}) \rangle$ is
Below (above) \( n_c \), corresponding to negative (positive) \( \sigma_{xy} \), respectively. As expected, for \( U/t \ll 1 \), \( n_c \to n_{\text{th}} \), while for \( U/t \gg 1 \), \( n_c \to 1 \). The critical density has a different asymptotic behavior for \( U \ll t \) and \( U \gg t \): \( n_c(U/t \to 0) < 1 \) while \( n_c(U/t \to \infty) > 1 \). Hence, we find that \( n_c(U/t) \) has a non-monotonous dependence on \( U/t \).

Fig. 3 shows the density and pairing potential profiles for two solutions on either side of the critical density \( n_c(U/t) \). While the pairing potential profiles look similar, the density profiles show a clear distinction: Below the critical density \( [n < n_c, \text{panel (a)}] \), the vortex core is depleted, while the situation is reversed above the critical density \( [n > n_c, \text{panel (b)}] \), where the core carries an excess density. These two solutions cross in energy at \( n_c(U/t) \). Such a level crossing indicates a possible change in the Chern number (3) and hence the Berry phase \( \Phi_B \).

VI. FINITE TEMPERATURE

The zero-temperature results, summarized in Fig. 3, show that there is a range of densities near half-filling in which the sign of \( \sigma_{xy} \) in the superconductor is different from the normal state. Within this range, one expects also a sign change in \( \sigma_{xy} \) when we destroy superconductivity by raising temperature at a fixed value of \( U/t \). We now generalize our approach to address the temperatures dependence of \( \sigma_{xy} \).

We use a thermal state of Hamiltonian (6) to determine self-consistently the parameters \( \{\Delta_{\vec{r}}, \mu_{\vec{r}}\} \). Then, one can calculate the thermally averaged Chern number\(^\text{(12)}\)

\[
\sigma_{xy}(T) = \frac{e^2}{(2\pi)^2} \int_0^{2\pi} d\Theta_x d\Theta_y e^{-\frac{E_0}{k_B T}} \text{Im}\langle \partial_{\Theta_\alpha} \Psi_{\alpha} | \partial_{\Theta_y} \Psi_{\alpha} \rangle,
\]

where \( \alpha \) runs over all excited states. Fig. 4 summarizes our results for one density \( n \approx 1 \) at \( t'/t = -0.3 \) and \( V/t = -0.05 \). In panel (b) we show one temperature trace of \( \Phi_B \) evaluated in a thermal ensemble for \( U/t = 2.6 \). We see that \( \Phi_B \) changes its sign before the order-parameter vanishes, i.e., within the superconductor. Moreover, we see that the sharp jump at \( T = 0 \) is washed out by thermally excited states. We can now define the temperature \( T_{\sigma} \) where \( \Phi_B \) changes sign and \( T_c \) where superconductivity is lost. Panel (a) shows a phase diagram that summarizes our finite temperature results.
VII. DISCUSSION

We have analyzed the Hall response of a lattice-superconductor, by examining the motion of a single vortex in a background super-flow. By combining general arguments with numerical simulations, we have shown that in the vicinity of half filling, it is possible to find situations where the normal state Hall response is opposite in sign compared to that of the superconducting ground state, leading to a sign change in $\sigma_{xy}$ as a function of temperature. Unlike previously proposed mechanisms, the present mechanism is an effect of the crystalline lattice, and does not disappear in the clean limit.

It would be interesting to generalize our results to models applicable to the cuprate superconductors. In particular, one needs to include the d-wave symmetry of the order parameter and the strong correlations due to on-site repulsive interactions. Interestingly, that the density of carriers in the hole-doped cuprates satisfies $n_{vH} < n < 1$. Therefore, within the simple model used here, it lies in the range in which the sign of $\sigma_{xy}$ in the superconductor at intermediate coupling is different from that of the normal state.

Appendix A: Choice of gauge

In Fig. 5 we show the Landau gauge we used for the numerical calculations. The arrow indicate the phases picked up by the $\uparrow$-electrons when hopping over the respective link. The arrows denote the following values

$$\lambda = \Phi_{2L_x L_y} \quad \text{and} \quad \lambda = 2L_x \times \lambda,$$

(A1)

where $L_x$ is the extent of the lattice in horizontal direction in Fig. 5 and $\Phi = 2\pi$ for one vortex.

Note that at the right edge of the lattice there is a winding of $2\pi$ for the hopping in the horizontal direction (blue arrows). When comparing to the phase pattern $\phi(\vec{r}) = \arg[\Delta(\vec{r})]$ of the superconducting order parameter in Fig. 3 of the paper, we see that the gauge-invariant current $j \sim \nabla \phi(\vec{r}) - eA(\vec{r})$ is indeed continuous.\(^{12}\)

FIG. 5: Gauge choice.
Appendix B: Vortex Berry’s phase and the Hall conductivity

In the following we clarify the origin of the “generalized Luttinger theorem” for the Berry’s phase accumulated when taking a vortex around a single unit cell. First, consider twisting only the boundary condition for the spin up electrons by Θ^↑_{x,y}, by defining the gauge field as
\begin{equation}
A^\uparrow_{\mathbf{p},\mathbf{r}} = (A_L)^\uparrow_{\mathbf{p},\mathbf{r}} + \left( \frac{\Theta^x_L}{L_x} \frac{\Theta^y_L}{L_y} \right) \cdot (\mathbf{r} - \mathbf{p}) / |\mathbf{r} - \mathbf{p}|, \tag{B1}
\end{equation}
where \((A_L)^\uparrow_{\mathbf{p},\mathbf{r}}\) is the gauge field configuration appearing in Fig. 5. In this case, changing \(\Theta^\uparrow_{x,y}\) by \(2\pi/L_{x,y}\) moves the vortex by one lattice site in the \(x\) or \(y\) direction. Moreover, such a change in \(\Theta^\uparrow_{x,y}\) and gives a Hamiltonian which is unitarily equivalent to the original one. Therefore, a similar construction to the one appearing in Ref.\(^{12,13}\) yields a “generalized Luttinger theorem” which gives the Berry’s phase acquired when taking the vortex around a single unit cell as
\begin{equation}
\Phi_B = 2\pi(n^\uparrow + p). \tag{B2}
\end{equation}

In the model analyzed in this manuscript, \(n^\uparrow = n^\downarrow = n/2\).

Now consider twisting the boundary conditions for both spin values \(\Theta^\uparrow_{x,y} = \Theta^\downarrow_{x,y} = \Theta_{x,y}\). The vortex moves by one lattice site upon changing \(\Theta_{x,y}\) by \(\pi/L_{x,y}\) (as a cooper pair is twisted by \(2\pi\Theta_{x,y}\)). When \(L_x, L_y \gg \xi\), we expect that the Berry phase accumulated when moving the vortex around a single unit cell using a twist for both spin flavours to also give \(\Phi_B\) as in Eq. (B2). This is indeed verified by our numerical calculations. Therefore, for \(L_x, L_y \gg \xi\), we obtain the relation between the Hall conductivity and \(\Phi_B\) as
\begin{equation}
\sigma_{xy} = \frac{4e^2}{(2\pi)^2} L_x L_y \Phi_B. \tag{B3}
\end{equation}

Appendix C: Weak coupling limit

For non-interacting electrons, semiclassical reasoning\(^\text{20}\) shows that the sign of \(\sigma_{xy}\) is determined by the topology of the Fermi surface. For an electron-like (a hole-like) Fermi surface, \(\sigma_{xy}\) is negative (positive), respectively. In the limit \(U/t \ll 1\), one may expect \(\sigma_{xy}\) to approach its non-interacting value, and hence \(n_c(U/t) \rightarrow n_cH\) as \(U/t \rightarrow 0\). This is not entirely obvious, however, since the ground state changes its nature singularly in the limit \(U \rightarrow 0\). In this section, we show explicitly that \(\sigma_{xy}\) is indeed smooth for \(U \rightarrow 0\).

By the arguments in the main text, the critical density \(n_c\) in which \(\sigma_{xy}\) changes sign is associated with a level crossing at the vortex core. Let us analyze the spectrum of Marticon-Caroli-de Gennes (McdG) states in the core. The Bogoliubov-de Gennes (BdG) Hamiltonian is of the form
\begin{equation}
H_{\text{BdG}} = \varepsilon(\mathbf{k}) \tau^z + \Delta(\mathbf{r}) \tau^\uparrow + \text{H.c.}, \tag{C1}
\end{equation}
where \(\varepsilon(\mathbf{k})\) is a general energy dispersion as a function of the crystal momentum \(\mathbf{k}\), \(\Delta(\mathbf{r}) = i\nabla \cdot \mathbf{r}\) is a pairing potential that includes a vortex at \(\mathbf{r} = \mathbf{r}_V\), and \(\tau^\uparrow\) are Pauli matrices acting in Nambu space on the spinor \(\psi^\uparrow = (c_{k,1}^\uparrow, c_{-k,1}^\uparrow)\). We assume that the system is defined on a square lattice, such that \(\varepsilon(\mathbf{k})\) is symmetric under \(C_4\). The Fermi surface is at \(\varepsilon(\mathbf{k}) = 0\). For simplicity, let us consider a linear gap function which describes a vortex at \(\mathbf{r}_V = 0\):
\begin{equation}
\Delta(\mathbf{r}) = \Delta_0 / \xi (x - iy), \tag{C2}
\end{equation}
where \(\Delta_0\) and \(\xi\) are parameters. The precise gap profile is not expected to change our results qualitatively.

We now transform the coordinates to a new frame \((\mathbf{k}_\parallel, \mathbf{k}_\perp)\) such that \(\mathbf{k}_\parallel \parallel (\text{perpendicular})\) to the Fermi surface, respectively. The transformation is depicted in Fig. 6a. The new coordinates are related to the old ones by
\begin{equation}
dk_x = \cos \theta dk_\parallel - J \sin \theta dk_\perp, \tag{C3}
dk_y = \sin \theta dk_\parallel + J \cos \theta dk_\perp.
\end{equation}
Here, \(\theta(\mathbf{k}_\parallel/k_p)\) is the angle between the normal to the Fermi surface and the horizontal axis \((k_p\parallel\text{ the perimeter of the Fermi surface})\), and \(J\) is the Jacobian of the transformation. We fix the orientation of \(\mathbf{k}_\parallel\) by requiring
\begin{equation}
\int_0^{k_p} dk_\parallel d\theta / dk_\parallel = 2\pi.
\end{equation}
Since Eqs. (C3) must be total differentials, we find that \(\partial k_\parallel, J = \partial k_\parallel, \theta\). The following choice of \(J\) is consistent with this constraint:
\begin{equation}
J(k_\perp, k_\parallel) = 1 + \frac{k_\perp}{k_p} \theta' \left( \frac{k_\parallel}{k_p} \right). \tag{C5}
\end{equation}
Performing the coordinate transformation (C3), linearizing the dispersion near the Fermi surface, and finally performing a similarity transformation \(H_{\text{BdG}} \rightarrow \tilde{H}_{\text{BdG}} = J^{1/2} U H_{\text{BdG}} U^{-1} J^{-1/2}\) where \(U = \exp[i\theta \tau^z/2]\), the BdG Hamiltonian takes the form
\begin{equation}
\tilde{H}_{\text{BdG}} = H_\parallel + H_\perp, \tag{C6}
\end{equation}
where
\begin{align}
H_\parallel &= \frac{\Delta_0}{\xi} \left( -i \mathbf{J} \right) \tau^y \left( \partial_{k_\parallel} + \frac{1}{2J} \partial J \right), \nonumber \\
H_\perp &= \frac{\Delta_0}{\xi} \tau^z i \partial_{k_\perp} + v_F(k_\parallel) k_\perp \tau^z. \tag{C7}
\end{align}
Here, \(v_F(k_\parallel) = \nabla_\mathbf{k} \varepsilon(k_\parallel = 0)\cdot \hat{n}_\perp\) is the Fermi velocity, where \(\hat{n}_\perp = (\cos \theta, \sin \theta)\) is unit vector normal to the Fermi surface. Note that, due to the unitary transformation \(U\), the eigenstates of (C6) satisfy antiperiodic boundary conditions as a function of \(k_\parallel\).
Anticipating $H_\parallel / \Delta_0 \sim \xi^{-1} \partial_{k_\parallel} \sim (k_p \xi)^{-1} \ll 1$, we diagonalize $\tilde{H}_{\text{BdG}}$ perturbatively in $H_\parallel$. To zeroth order in $H_\parallel$, $\tilde{H}_{\text{BdG}}$ has a family of zero modes parametrized by $k_\parallel$, of the form

$$\varphi(k_\perp, k_\parallel) = \varphi_0(k_\perp a(k_\parallel)) \left( \frac{1}{-\text{sgn}(v_F)} \right). \quad (C8)$$

Here, $\varphi_0(x) = \sqrt{2/\pi e^{-x^2/2}}$ is the harmonic oscillator ground state wavefunction, and we have defined $a(k_\parallel) = \sqrt{v_F(k_\parallel)\xi/2\Delta_0}$. Note that away from the van Hove point, $v_F \neq 0$ for all $k_\parallel$, and therefore $\text{sgn}(v_F)$ is independent of $k_\parallel$.

$H_\parallel$ lifts the degeneracy within the zero-energy subspace of $F_\perp$. Inserting $\psi(k_\parallel, k_\perp) = \chi(k_\parallel)\varphi(k_\perp, k_\parallel)$ into the eigenvalue equation for $\tilde{H}_{\text{BdG}}$ and projecting both sides onto the subspace of zero modes defined by Eq. (C8), we get the following eigenvalue equation for $\chi$ [to leading order in $(k_p \xi)^{-1}$]:

$$\Delta_0 \text{sgn}(v_F) \left( \partial_{k_\parallel} - \frac{1}{a} \frac{\partial a}{\partial k_\parallel} \right) \chi(k_\parallel) = E \chi(k_\parallel). \quad (C9)$$

The eigenstates are of the form $\chi_n = a(k_\parallel) \exp[ik_\parallel/q_n]$, where $1/q_n = \pi(2n+1)/k_p$, $n \in \mathbb{Z}$, and the corresponding eigenenergies are $E_n = -\Delta_0 \text{sgn}(v_F)$. These are nothing but the well-known MCdG states, whose minimum energy is $\Delta_0 / \xi \sim \Delta_0^2 / E_F$ (where we have used the estimates $\xi \sim v_F / \Delta_0$ and $E_F \sim v_F k_p$).

Now, let us consider the low-energy spectrum on either side of the van Hove point, in which the Fermi surface undergoes a change of topology. According to our definition of the orientation of the Fermi surface, Eq. (C4), $\text{sgn}(v_F) > 0$ for a particle-like Fermi surface ($n < n_{vH}$), while $\text{sgn}(v_F) < 0$ for a hole-like Fermi surface ($n > n_{vH}$). Therefore, we see that across the van Hove point, the two states $\chi_{-1}$ and $\chi_0$ interchange their energy. These two states transform differently under $C_4$ (e.g., under rotation by $\pi/2$, $\chi_0$ and $\chi_{-1}$ pick up a phases of $e^{\pm \pi/4}$, respectively). Hence, $\chi_{0,-1}$ cannot hybridize with each other. We conclude that, as the density crosses $n_{vH}$, there must be a level crossing between $\chi_{-1}$ and $\chi_0$.

More generally, under rotation by $\pi/2$, $\chi_n$ acquires a phase of $\exp[i\theta_n]$, where $\theta_n = (2n+1)\pi/4$. Near the van Hove point, $\chi_{2n-1}$ and $\chi_{2n}$ cross in energy. The distance of the level crossing point to the van Hove point goes to zero in the weak-coupling limit, $k_p \xi \to \infty$. The spectrum in either side of the van Hove point is shown in Fig. 6b.

In terms of the many-body spectrum, a zero energy state in the BdG spectrum corresponds to a level crossing between the ground state and the first excited state. Since the Chern number can only change via a level crossing, this implies that in the weak-coupling limit, the jump in the Hall conductivity occurs arbitrarily close to the van Hove point. This conclusion is consistent with our numerical simulations (Fig. 1 in the main text).

Note that our argument relies on the presence of a $C_4$ symmetry. Indeed, if the $C_4$ symmetry is broken, there is generically no single van Hove density in which the Fermi surface changes its character from particle-like to hole-like. The regions of electron and hole-like Fermi surfaces are generically separated by a density range with an open Fermi surface, in which the normal-state Hall conductivity is ill-defined in the clean limit.

**Appendix D: Overlaps of Bogoliubov-de Gennes wavefunctions**

In order to calculate Chern numbers [Eq. (3) of the main text], we need to compute overlaps of many-body wavefunctions. In the Hartree-Fock approximation, these wavefunctions are ground states of a variational quadratic Hamiltonian [Eq. (6) in the main text]. In order to derive a formula for the overlap between two such wavefunctions, it is convenient to perform a particle-hole transformation on one of the spin species:
In terms of the new operators $d_{1,\vec{r}}$, $H_{\text{HF}}$ has the form

$$H_{\text{HF}} = -\sum_{\vec{r},\vec{r}^\prime,\sigma=\uparrow,\downarrow} \left( t_{\vec{r}\vec{r}^\prime}^\dagger d_{1,\vec{r}}^\dagger c_{1,\vec{r}^\prime,\sigma} - t_{\vec{r}\vec{r}^\prime}^\uparrow d_{1,\vec{r}}^\dagger c_{1,\vec{r}^\prime,\uparrow} + \text{H.c.} \right)$$

$$- \Delta_{\vec{r}} d_{2,\vec{r}}^\dagger d_{1,\vec{r}}^\dagger c_{1,\vec{r},\uparrow} + \text{H.c.}$$

$$- \sum_{\vec{r},\sigma=\uparrow,\downarrow} \left( \mu_{\sigma} d_{1,\vec{r}}^\dagger c_{1,\vec{r},\sigma} - \mu_{\sigma} d_{1,\vec{r}}^\dagger c_{1,\vec{r},\sigma} \right).$$

Note that $H_{\text{HF}}$ contains no anomalous terms. The conservation of the number of $d$ particles, $\mathcal{N}_d = \sum_{\vec{r}} \left( d_{1,\vec{r}}^\dagger c_{1,\vec{r}} + d_{1,\vec{r}}^\dagger c_{2,\vec{r}} \right)$, corresponds to the conservation of the total spin in the $z$ direction in the original problem.

One can diagonalize (D2) by performing a unitary (Bogoliubov) transformation

$$\gamma_1 \begin{pmatrix} \vdots \\ \vdots \\ \gamma_{N+1} \\ \gamma_{N+1} \end{pmatrix} = U \begin{pmatrix} d_{1,\vec{r}_1}^\dagger \\ \vdots \\ d_{2,\vec{r}_N}^\dagger \end{pmatrix},$$

Here, $U$ is a $2N \times 2N$ unitary matrix, and $N$ is the number of lattice points. After this transformation, $H_{\text{HF}}$ can be written as

$$H_{\text{HF}} = \sum_{j=1}^{2N} E_j^\dagger \gamma_j^\dagger \gamma_j,$$

where $|\tilde{0}\rangle$ is defined as the vacuum state of the $d_{1,\vec{r}}$ operators, and $|0\rangle$ is the original vacuum of the $c_{\uparrow,\downarrow}$ operators. Let us define $\tilde{W}$ as an $N_0 \times N$ matrix containing the $N_0$ first rows of the matrix $U$. Then, one can verify that the overlap between two ground states is

$$\langle \Psi\{\Delta_{\vec{r}},\mu_{\vec{r}}\} | \Psi\{\Delta_{\vec{r}},\mu_{\vec{r}}\} \rangle = \det[\tilde{W}\tilde{W}^\dagger].$$

Appendix E: Strong coupling expansion

In the strong coupling limit, $U/t \gg 1$, the physics of the negative-$U$ Hubbard model is dominated by strongly bound pairs of electrons. We can use a Schrieffer-Wolff transformation to obtain an effective hard-core boson...
The different couplings can be expressed as

\[
H_{\text{eff}} = \sum_{\langle i,j \rangle} \left[ -J(S_i^z S_j^z + S_i^y S_j^y) + (J + J_a)S_i^\pi S_j^\pi \right] + \sum_{\langle\langle i,j \rangle\rangle} \left[ -J_2(S_i^z S_j^z + S_i^y S_j^y) + J_2 S_i^\pi S_j^\pi \right] + 12J_{\text{ring}} \sum_i S_i^z S_j^+ S_k^- - 2J_{\text{ring}} \sum_{i,j,k} [S_i^z S_j^+ S_k^- + \text{H.c}].
\]

(E1)

The different couplings can be expressed as

\[
J = \frac{2t^2}{U}, \quad J_2 = \frac{2t'^2}{U}, \quad J_a = 4V, \quad J_{\text{ring}} = \frac{t'^2}{U^2}.
\]

(E2)

The first line in (E1) contains processes up to second order in \( t' \). Note that this Hamiltonian is manifestly particle-hole symmetric at half filling independently of \( t' \). The easiest way to see this is that the sub-lattice gauge-transformation \( e^{i\tilde{Q} \cdot \pi} \) with \( \tilde{Q} = (\pi, \pi) \) appearing in the particle-hole transformation for the fermions is absent in the bosonic case as no fermionic signs have to be corrected for.

The first particle-hole symmetry breaking terms appear in third order in the hopping. In (E1) we only show those third order terms that lead to such a symmetry breaking. They all contain \( J_{\text{ring}} \) where exactly one hopping takes place over a next-to-nearest neighbor bond.

Note that we suppress the gauge field in (E1) for simplicity.

To study the effect of the particle-hole symmetry breaking terms \( \propto t'^2 \) we use exact diagonalization on clusters up to \( L_x \times L_y = 4 \times 4 \) using the hard-core boson model. We investigate the half-filled lattice and observe a degeneracy between two different vortex states at \( t' = 0 \) as expected, cf. Fig. 7. When turning on a \( t'/t < 0 \) as in the main text we find that the vortex with a density depletion is lower in energy. From this we conclude that the line where the integer \( p \) jumps from zero to \(-1\) moves towards densities \( n > 1 \), in accordance with the results in the main text. These findings are also supported by the calculation of the Chern number\(^{13}\). For the actual calculations we used values for \( J_2 \), \( J_a \), and \( J_{\text{ring}} \) that don’t obey the rules (E2) for the following reason: Our aim is to study the breaking of particle-hole (PH) symmetry by the application of \( J_{\text{ring}} \) terms. For that we need a \( t' \) in the fermionic Hamiltonian. However, this induces also \( J_2 \), which does not break PH symmetry but which has to be counter-acted by a larger \( J_a \) in order to fight the competing CDW. To keep the vortex smaller than the system size, we use \( J = 1 \), \( J_2 = 0.2 \), \( J_a = 0.7 \) and we change \( J_{\text{ring}} \) from 0 to 0.2. There is no reason to expect that by changing \( J_2 \) and \( J_a \) the main conclusion would change.

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19. In this statement, we have ignored the Goldstone modes in the superfluid phase, whose gap goes to zero as the system size increases. However, this gap is not expected to vary significantly as a function of \( \Theta_x, \Theta_y \). Moreover, in a charged superconductor, there are no gapless Goldstone modes. One can imagine adding to the Hamiltonian (2) a repulsive long-range interaction which decays logarithmically with distance. This interaction gaps out the Goldstone modes. However, if the coefficient of the long-range interaction is sufficiently small, it does not affect the spectrum of the core much, and therefore it does not shift the position of the level crossing that occurs at the core.
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