Thermodynamics of charged rotating dilaton black branes with power-law Maxwell field

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In this paper, we construct a new class of charged rotating dilaton black brane solutions, with complete set of rotation parameters, which is coupled to a nonlinear Maxwell field. The Lagrangian of the matter field has the form of the power-law Maxwell field. We study the causal structure of the spacetime and its physical properties in ample details. We also compute thermodynamic and conserved quantities of the spacetime such as the temperature, entropy, mass, charge, and angular momentum. We find a Smarr-formula for the mass and verify the validity of the first law of thermodynamics on the black brane horizon. Finally, we investigate the thermal stability of solutions in both canonical and grand-canonical ensembles and disclose the effects of dilaton field and nonlinearity of Maxwell field on the thermal stability of the solutions. We find that for $\alpha \leq 1$, charged rotating black brane solutions are thermally stable independent of the values of the other parameters. For $\alpha > 1$, the solutions can encounter an unstable phase depending on the metric parameters.

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I. INTRODUCTION

Historically, the presence of a scalar field in the context of general relativity dates back to Kaluza-Klein theory [1]. Kaluza-Klein theory was presented in order to give a unified theory describing gravity and electromagnetic. It may be considered as a pioneering theory to the string theory. Afterwards, in Brans-Dicke (BD) theory, the notion was seriously taken into account [2]. The root of this theory can be traced back to Mach’s principle encoded by a varying gravitational constant in BD theory. The variety of gravitational constant is indicated in this theory by considering a scalar field non-minimally coupled to gravity. BD theory is known as a non-quantum scalar-tensor theory [3]. From quantum viewpoint, the scalar field called dilaton field is emerged in the low energy limit of string theory [4].

On the other side, there are strong evidences on the observational sides that our Universe is currently experiencing a phase of accelerating expansion [5]. This acceleration phase cannot be explained through standard model of cosmology which is based on the Einstein theory of general relativity. One way for explanation of such an acceleration is to add a new unknown component of energy, usually called “dark energy”. Another way for challenging with this problem is to modify the Einstein theory of gravity. In this regards, some physicists have convinced that Einstein theory of gravitation cannot give a complete description of what really occurs in our Universe and one needs an alternative theory. As we mentioned above, string theory, in its low energy limit, suggests a scalar field which is nonminimally coupled to gravity and other fields called dilaton field [4]. Consequently, dilaton gravity as one of the alternatives of Einstein theory have encountered high interest in recent years. String theory also suggests dimensions of spacetime to be higher than four dimensions. It seemed for a while that dimensions higher than four should be of Planck scale, but recent theories expresses that our three-dimensional brane can be embedded in a relatively large higher dimensional bulk which is still unobservable [4, 6]. In such a scenario, all gravitational objects including black objects are higher dimensional. The action of dilaton gravity usually includes one or more Liouville-type potentials which can be justified as trace of supersymmetry breaking of spacetime in ten dimensions.

Many authors have explored exact charged black solutions in the context of Einstein-dilaton gravity. For instance, asymptotically flat black hole solutions in the absence of the dilaton potential are studied in Refs. [8–11]. In recent years, AdS/CFT correspondence have made asymptotically non-flat solutions more valuable. On the other hand, study of asymptotically non-flat and non-AdS solutions extends the validity range of tools already applied and tested in the cases of asymptotically flat or AdS spacetimes. Many efforts have been made to find and study static and rotating nonflat exact solutions in the literature. Static asymptotically nonflat and non AdS solutions and their
thermodynamics were explored in [10–17]. Different aspects of exact rotating nonflat solutions have also been studied in [18–27].

In the present paper we extend the study on the dilaton gravity to include the power-law Maxwell field term in the action. The motivation for this study comes from the fact that, as in the case of scalar field which has been shown that particular power of the massless Klein-Gordon Lagrangian shows conformal invariance in arbitrary dimensions [28], one can have a conformally electrodynamic Lagrangian in higher dimensions. It is worth mentioning that Maxwell Lagrangian, $F_{\mu\nu}F^{\mu\nu}$ is conformally invariant only in four dimensions, while it was shown that the Lagrangian $(F_{\mu\nu}F^{\mu\nu})^{(n+1)/4}$ is conformally invariant in $(n+1)$-dimensions [28]. In other words, this Lagrangian is invariant under the conformal transformation $g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}$ and $A_\mu \rightarrow A_\mu$. The studies on the black object solutions coupled to a conformally invariant Maxwell field have got a lot of attentions in the past decades [29–33]. Thermodynamics of higher dimensional Ricci flat rotating black branes with a conformally invariant power-Maxwell source in the absence of a dilaton field were studied in [32]. Recently, we explored exact topological black hole solutions in the presence of nonlinear power-Maxwell source as well as dilaton field [33]. Since these solutions [33] are static, it is worthwhile to construct the rotating version of the solutions. Charged rotating dilaton black branes in dilaton gravity and their thermodynamics in the presence of linear Maxwell field and nonlinear Born-Infeld electrodynamics have been studied in [27] and [34] respectively. Till now, charged rotating dilaton black objects coupled to a power-law Maxwell field has not been constructed. In this paper, we intend to construct a new class of $(n+1)$-dimensional rotating dilaton black branes in the presence of power-Maxwell field, where we relax the conformally invariant issue for generality. Of course, the solution do exist for the case of conformally invariant source. We find that the solutions exist provided one takes Liouville-type potentials with two terms. In the limiting case of Maxwell field, one of the Liouville potentials vanishes. We shall investigate thermodynamics as well as thermal stability of our rotating black brane solutions and explore the effects of the dilaton and power-Maxwell fields on thermodynamics and thermal stability of these black branes.

The outline of this paper is as follows. In the next section, we present the basic field equations of Einstein-dilaton gravity with power-law Maxwell field and find a new class of charged rotating black brane solutions of this theory and investigate their properties. In section III we obtain the conserved and thermodynamic quantities of the solutions and verify the validity of the first law of black hole thermodynamics. In Sec. IV we study thermal stability of the solutions in both canonical and grand-canonical ensembles. We finish our paper with closing remarks in the last section.

II. FIELD EQUATIONS AND ROTATING SOLUTIONS

We consider an $(n+1)$-dimensional $(n \geq 3)$ action of Einstein gravity which is coupled to dilaton and power-law Maxwell field,

$$I = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} \left\{ \mathcal{R} - \frac{4}{n-1} (\nabla \Phi)^2 - V(\Phi) + \left( -e^{-4\alpha \Phi/(n-1)} \mathcal{F} \right)^p \right\} - \frac{1}{8\pi} \int_{\partial \mathcal{M}} d^n x \sqrt{-\gamma} \Theta(\gamma), \quad (1)$$

where $\mathcal{R}$ is the Ricci scalar, $\Phi$ is the dilaton field and $V(\Phi)$ is the dilaton potential. Here $\mathcal{F} = F^{\mu\nu}F_{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor and $A_\mu$ is the electromagnetic potential, while $p$ and $\alpha$ are two constants that determine nonlinearity of electromagnetic field and coupling strength of the dilaton and electromagnetic fields, respectively. The well-known Einstein-Maxwell dilaton theory corresponds to the case $p = 1$. The last term in (1) is Gibbons-Hawking boundary term which is added to the action in order to make variational principle well-defined. We show the metric of manifold $\mathcal{M}$ with $g_{\mu\nu}$, with covariant derivative $\nabla_\mu$. The metric of the boundary $\partial \mathcal{M}$ is $g_{\mu\nu}$ and $\Theta = g_{\mu\nu} \Theta^{\mu\nu}$ is the trace of the extrinsic curvature of the boundary $\Theta^{\mu\nu}$. By varying action (1) with respect to the gravitational field $g_{\mu\nu}$, the dilaton field $\Phi$ and the gauge field $A_\mu$, one can obtain equations of motion as

$$\mathcal{R}_{\mu\nu} = g_{\mu\nu} \left\{ \frac{V(\Phi)}{n-1} + \frac{(2p-1)}{n-1} \left( -\mathcal{F} e^{-4\alpha \Phi/(n-1)} \right)^p \right\} + \frac{4 (\partial_\mu \Phi \partial_\nu \Phi)}{n-1} + 2pe^{-4\alpha \Phi/(n-1)} (-\mathcal{F})^{p-1} F_{\mu\lambda} F_{\nu}^{\lambda}, \quad (2)$$

$$\nabla^2 \Phi = \frac{n-1}{8} \frac{\partial V}{\partial \Phi} + \frac{p\alpha}{2} e^{-4\alpha \Phi/(n-1)} (-\mathcal{F})^p, \quad (3)$$

$$\nabla_\mu \left( e^{-4\alpha \Phi/(n-1)} (-\mathcal{F})^{p-1} F^{\mu\nu} \right) = 0. \quad (4)$$

Here we seek for higher dimensional rotating solutions of field equations (2)-(4). We know that rotation group in $(n+1)$-dimensions is $SO(n)$. Thus the number of independent rotation parameters for a localized object is $k \equiv \lfloor n/2 \rfloor$ where $\lfloor x \rfloor$ is the integer part of $x$. Note that $k$ is equal to number of Casimir operators. Therefore, the metric of
(n + 1)-dimensional rotating solution with cylindrical or toroidal horizons and complete set of rotation parameters can be written as \[35\]

\[
ds^2 = -f(r) \left( \Xi dt - \sum_{i=1}^{k} a_i d\phi_i \right)^2 + \frac{r^2}{l^2} R^2(r) \sum_{i=1}^{k} \left( a_i dt - \Xi^2 d\phi_i \right)^2
\]

\[-\frac{r^2}{l^2} R^2(r) \sum_{i<j} \left( a_i d\phi_j - a_j d\phi_i \right)^2 + \frac{d\phi^2}{f(r)} + \frac{r^2}{l^2} R^2(r) dX^2,
\]

\[
\Xi^2 = 1 + \sum_{i=1}^{k} \frac{a_i^2}{l^2},
\]

where \(a_i\)s are \(k\) rotation parameters and \(l\) has the dimension of length which is related to the cosmological constant \(\Lambda\) for the case of Liouville-type potential with constant \(\Phi\). The angular coordinates are in the range \(0 \leq \phi_i \leq 2\pi\) and \(dX^2\) is the Euclidean metric on the \((n - k - 1)\)-dimensional submanifold with volume \(\Sigma_{n-k-1}\). Integrating the Maxwell equation \(4\) leads to

\[
F_{tr} = q e^{\frac{4\alpha \rho \Phi(r)}{(n-1)(2+1-p)}} (rR)^{\frac{n+1}{p-1}}, \quad F_{\phi r} = -\frac{a_i}{\Xi} F_{tr}.
\]

where \(q\) is an integration constant related to the electric charge of the brane. Substituting \(5\) and \(6\) in the field equations \(2\) and \(3\), we find the following differential equations

\[
-\frac{\Xi^2 f''}{2} - \frac{(n-1)}{2} \left( \frac{f'R'}{R} + \frac{f'}{r} \right) + \left( \Xi^2 - 1 \right) \left[ 2\frac{(n-1)f R'}{rr} + (n-2) \left( \frac{f}{r^2} + \frac{f R^2}{R^2} \right) \right] - \frac{(n-3)}{2} \left( \frac{f'R'}{R} + \frac{f'}{r} \right) + \left( \Xi^2 - 1 \right) \left[ 2\frac{(n-1)f R'}{rr} + (n-2) \left( \frac{f}{r^2} + \frac{f R^2}{R^2} \right) \right]
\]

\[-\frac{f''}{2} - \frac{(n-1)}{2} \frac{f'R'}{R} + \frac{(n-1)}{2} \frac{f'}{r} - \frac{4f \Phi'^2}{n-1} - \frac{2(n-1)f R'}{r R} - \frac{(n-1)}{2} \frac{f R''}{R} - V(\Phi) \frac{n}{n-1} - \frac{2p^2 q^2 e^{\frac{4\alpha \rho \Phi}{(n-1)(2+1-p)}}}{(rR)^{\frac{2(n-1)}{2-p-1}}} \left( \frac{2p-1}{n-1} - p \Xi^2 \right) = 0,
\]

\[
\frac{f''}{2} - \frac{2(n-1)}{r R} \frac{f R'}{R} - \frac{f R''}{R} + \frac{n-3}{2} \left( \frac{f'}{r} + \frac{R'R'}{R} \right) - (n-2) \left( \frac{f}{r^2} + \frac{f R^2}{R^2} \right) - \frac{2p^2 q^2 e^{\frac{4\alpha \rho \Phi}{(n-1)(2+1-p)}}}{(rR)^{\frac{2(n-1)}{2-p-1}}} = 0,
\]

\[
\frac{a^2 f''}{2l^2} + \left( 1 + \frac{a^2}{l^2} \right) \left[ -2 \frac{f R'}{r R} - \frac{f R''}{R} - (n-2) \left( \frac{f}{r^2} + \frac{f R^2}{R^2} \right) \right]
\]

\[+ \left( \frac{(n-3)}{2l^2} - 1 \right) \left( \frac{f'}{r} + \frac{R'R'}{R} \right) - \frac{V(\Phi)}{n-1} - \frac{2p^2 q^2 e^{\frac{4\alpha \rho \Phi}{(n-1)(2+1-p)}}}{(rR)^{\frac{2(n-1)}{2-p-1}}} \left( \frac{2p-1}{n-1} + p a^2 \right) = 0,
\]

\[
-(n-2) f \left( \frac{1}{r^2} + \frac{R^2}{R^2} \right) - 2 \frac{n-1}{r R} \frac{f R'}{R} - \frac{f'}{r} - \frac{R'R'}{R} - \frac{f R''}{R} - V(\Phi) \frac{n}{n-1} - \frac{2p^2 q^2 e^{\frac{4\alpha \rho \Phi}{(n-1)(2+1-p)}}}{(rR)^{\frac{2(n-1)}{2-p-1}}} \left( \frac{2p-1}{n-1} - 1 \right) = 0,
\]

\[
f \Phi'' + (n-1) f \Phi' \left( \frac{1}{r} + \frac{R'}{R} \right) + f' \Phi' - \frac{(n-1)}{8} \frac{dV(\Phi)}{d\Phi} - \frac{2p^2 q^2 e^{\frac{4\alpha \rho \Phi}{(n-1)(2+1-p)}}}{(rR)^{\frac{2(n-1)}{2-p-1}}} = 0,
\]
\[ f'' - (n-2) f \left( \frac{1}{r^2} + \frac{R'^2}{R^2} \right) + \frac{(n-3)}{2} \left( \frac{f'}{r} + \frac{R' f'}{R} \right) - \frac{f R''}{R} - \frac{2(f - 1) f'}{r R} - \frac{2p q^{2p} e^{\frac{4p \Phi}{2 + p - 1}}}{(r R) \sqrt{2p - 1}} = 0, \tag{13} \]

where the prime denotes derivative with respect to \( r \). In order to construct exact rotating solutions of the theory given by action \( \text{(1)} \), the functions \( f(r) \), \( R(r) \) and \( \Phi(r) \) should be determined so that the system of equations \( \text{(7)-(13)} \) are satisfied. To do that, we make the ansatz \[ R(r) = e^{2\alpha \Phi(r)/(n-1)}, \tag{14} \]

and take the potential of the Liouville-type with two terms, namely

\[ V(\Phi) = 2 \Lambda_1 e^{2\zeta_1 \Phi} + 2 \Lambda e^{2\zeta_2 \Phi}, \tag{15} \]

where \( \Lambda_1 \), \( \Lambda \), \( \zeta_1 \) and \( \zeta_2 \) are constants. Using \( \text{(14)} \) and \( \text{(15)} \), one can easily show that equations \( \text{(7)-(13)} \) have solutions of the form

\[ f(r) = - \frac{2\Lambda b^2 (1 + \alpha^2) r^{2(1-\gamma)}}{(n-1)(n-\alpha^2)} - \frac{m}{r^{(n-1)(1-\gamma)-1}} + \frac{2p(1+\alpha^2)(2p-1)q^{2p}}{\Pi \Gamma b2(n-2)\gamma/(2p-1)p \gamma + (n-1)(1-\gamma)-1}, \tag{16} \]

\[ \Phi(r) = \frac{(n-1)\alpha}{2(1+\alpha^2)} \ln \left( \frac{b}{r} \right), \tag{17} \]

where \( b \) and \( m \) are arbitrary constants, \( \gamma = \alpha^2/(\alpha^2+1) \), \( \Psi = (n-2p+\alpha^2)/(2p-1)(1+\alpha^2) \) and \( \Pi = \alpha^2 + (n - 1 - \alpha^2) p \).

The above solutions will fully satisfy the system of equations provided,

\[ \zeta_1 = \frac{2p (n - 1 + \alpha^2)}{(n - 1) (2p - 1) \alpha}, \quad \zeta_2 = \frac{2\alpha}{n - 1}, \quad \Lambda_1 = \frac{2p^{-1} (2p - 1) (p - 1) \alpha^2 q^{2p}}{\Pi b^{2(n-1)p/(2p-1)}}. \]

It is worthwhile to note that the necessity of existence of the term \( 2\Lambda_1 e^{2\zeta_1 \Phi} \) in scalar field potential is due to nonlinearity of Maxwell field and this term vanishes for the case of linear Maxwell field where \( p = 1 \) \cite{27}. Note that in our solutions \( \Lambda \) remains as a free parameter which plays the role of the cosmological constant. Another thing to notice is that although these rotating solutions are locally the same as those found in \cite{33} with flat horizon \( (k = 0) \), they are not the same globally. One may also note that in the particular case \( p = 1 \) these solutions reduce to the \((n+1)\)-dimensional charged rotating dilaton black branes presented in \cite{27}. The parameter \( m \) in Eq. \( \text{(15)} \) is the integration constant which is known as the geometrical mass and can be written in term of the horizon radius as

\[ m(r_+) = \frac{2p (2p - 1) q^{2p}}{(1 - \gamma) \Pi b^{2(n-2)\gamma/(2p-1)r_+^2}} + \frac{b^{2\gamma n r_+^{1-\gamma}(n+1)-1}}{\Pi b^{2(1-\gamma)^2} (n - \alpha^2)}, \tag{18} \]

where \( r_+ \) is the positive real root of \( f(r_+) = 0 \). One can easily show that the vector potential \( A_\mu \) corresponding to the electromagnetic tensor \( \mathbf{E} \) can be written as

\[ A_\mu = \frac{q b^{p(2p+1-\alpha^2)}}{\Pi b^{p+1}} \left( \Xi \delta^i_\mu - \alpha_i \delta^i_\mu \right) \quad \text{(no sum on } i \text{).} \tag{19} \]

Here we pause to give some remarks about the value of \( A_\mu \) at infinity that apply some restrictions on the value of \( p \) and \( \alpha \). This discussion is given here because it is necessary to know these restrictions for next studies. In section \( \text{(III)} \), we will show that the mass of solutions is dependent on the value of \( A_\mu \) at infinity. Thus, the value of electromagnetic vector potential should be finite at infinity so that we have finite mass. In order to guarantee this behavior \( \Psi \) should be positive i.e.

\[ \frac{n - 2p + \alpha^2}{(2p - 1)(1 + \alpha^2)} > 0. \tag{20} \]

The above equation leads to the following restriction on the range of \( p \),

\[ \frac{1}{2} < p < \frac{n + \alpha^2}{2}. \tag{21} \]
On the other hand, the effect of $m$ in metric function should vanish in spacial infinity. This fact leads to a restriction on $\alpha$

$$\alpha^2 < n - 2.$$  \hfill (22)

Therefore, one can sum up the constraints (21) and (22) as follows:

$$\begin{align*}
&\text{for } \frac{1}{2} < p < \frac{n}{2}, \quad 0 \leq \alpha^2 < n - 2, \quad \text{(23)} \\
&\text{for } \frac{n}{2} < p < n - 1, \quad 2p - n < \alpha^2 < n - 2. \quad \text{(24)}
\end{align*}$$

It is worth mentioning that the solution is always well-defined in the above allowed ranges of $\alpha$ and $p$. In continue of this section we discuss properties and asymptotic behaviors of the solutions in allowed ranges of $p$ and $\alpha$.

### A. Asymptotic behaviors and properties of the solutions

Now, by considering the restrictions (23) and (24) on $p$ and $\alpha$, we are ready to discuss the behavior of our solutions both in the vicinity of $r = 0$ and infinity. First, one should note that in allowed ranges of $p$ and $\alpha$, the charge term in metric function disappears at infinity as the mass term does. This behavior is also seen in the special cases of $p = 1$ or $\alpha = 0$. Therefore, the first term in $f(r)$ (10) determines the behavior of it at infinity. Since we are interested in the solutions that go to infinity as $r \to \infty$, we assume $\Lambda < 0$ and take it in the standard form $\Lambda = -n(n-1)/2l^2$. It is also notable to mention that the spacetime is neither asymptotically flat nor AdS due to existence of dilaton field. About $r = 0$, the term including $q$ in metric function is dominant. This term is always positive in permitted ranges of $p$ and $\alpha$ because $\Pi, \Upsilon > 0$ in these ranges. Therefore as in the case of Reissner-Nordstrom black holes, we have timelike singularity and there are no Schwarzschild-type black hole solutions.

We continue our discussions about properties of the solutions by looking for curvature singularities. Since essential singularities are located at divergencies of Kretschmann scalar, we seek for these for our solutions. It is easy to show that the Kretschmann scalar $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$ diverges at $r = 0$ while it is finite for $r \neq 0$ and goes to zero as $r \to \infty$ and therefore there is an essential singularity located at $r = 0$. Although the location of event horizon cannot be determined analytically by using $f(r)$, fortunately we can get more insight about the solutions by calculating temperature corresponding to event horizon. The temperature and angular velocity of the horizon can be obtained by analytic continuation of the metric. The analytical continuation of the Lorentzian metric by $t \to i\tau$ and $a \to ia$ yields the Euclidean section. In order that Euclidean metric is regular at $r = r_+$, one should identify $\tau \sim \tau + \beta_+ k$ and $\phi_i \sim \phi_i + \beta_+ \Omega_i$, where $\beta_+$ and $\Omega_i$s are the inverse Hawking temperature and the $i$th component of angular velocity of the horizon. Then, temperature and $i$th component of angular velocity can be computed as

$$T_+ = \frac{f'(r_+)}{4\pi \Xi} = \frac{(1 + \alpha^2)}{4\pi \Xi} \left\{ \frac{nb^{2\gamma}r_+^{1-2\gamma}}{l^2} - \frac{2p p(2p - 1)q^{2p}}{\Pi b^{2(n-2)\gamma p/(2p-1)}r_+^{\gamma (1-\gamma)(n-1)}} \right\},$$  \hfill (25)

$$\Omega_i = \frac{a_i}{2l^2}. \hfill (26)$$

Numerical calculations show that temperature vanishes at event horizon for extreme black brane solutions. Therefore, one can see from Eq. (25) that we have extreme black brane if

$$m_{\text{ext}} = \frac{(\alpha^2 + 1)nb^{2\gamma}}{\Upsilon l^2} \left[ 1 + \frac{(1 + \alpha^2)\Upsilon}{(n - \alpha^2)} \right] r_{\text{ext}}^{(1-\gamma)(n+1)-1}, \hfill (27)$$

or

$$q_{\text{ext}}^{2p} = \frac{\Pi nb^{2\gamma(np-1)/(2p-1)}}{2p p(2p - 1)l^2} r_{\text{ext}}^{(1-\gamma)(n+1)-1}. \hfill (28)$$

The solutions have two inner and outer horizons located at $r_-$ and $r_+$, provided the charge parameter $q$ is lower than $q_{\text{ext}}$ or $m$ is greater than $m_{\text{ext}}$ and a naked singularity if $q > q_{\text{ext}}$ or $m < m_{\text{ext}}$ (see Fig. 1). Note that there is a relation between $m_{\text{ext}}$ and $q_{\text{ext}}$ as...
\[ m_{\text{ext}} = \left( \frac{(\alpha^2 + 1)n b^2 \gamma \left( (n - \alpha^2) + (1 + \alpha^2) \Upsilon \right)}{\Upsilon^2 (n - \alpha^2)} \right) \left( \frac{2^p p (2p - 1) l^2}{\Pi n b^2 \gamma (np - 1)/(2p - 1)} q_{\text{ext}}^{2p} \right)^{\frac{1 - (1 - \gamma)(n + 1) - 1}{1 - \gamma}(n + 1) - 1}, \]  

which reduces to extremal mass obtained in [27] for linear Maxwell field \((p = 1)\) and to one obtained in [42] in the absence of dilaton field \(\alpha = \gamma = 0\) and with linear Maxwell field. Finally, it is noticeable to mention that there is a Killing horizon in addition to event horizon for rotating solutions in our Einstein-dilaton gravity as in the case of rotating black solutions of the Einstein gravity. It is easy to show that the Killing vector field

\[ \tilde{\chi} = C \chi, \]

\[ \chi = \partial_t + \sum_{i=1}^{k} \Omega_i \partial_{\phi_i}, \]

is the null generator of the event horizon, where \(k\) denote the number of rotation parameters \([36]\) and \(C\) is a constant that we will fix it in next section. The Killing horizon is a null surface whose null generators are tangent to a Killing field.

![Graph](image.png)

**FIG. 1:** The behavior of \(f(r)\) versus \(r\) with \(l = b = 1, q = 0.5, \alpha = \sqrt{2}, n = 5, p = 2, r_{\text{ext}} = 1\). In this case \(m_{\text{ext}} = 60.00\) and \(q_{\text{ext}} = 1.06\).

### III. THERMODYNAMICS OF BLACK BRANES

In this section, we discuss thermodynamics of rotating black brane solutions in the presence of power-law Maxwell field. Since the discussion of thermodynamics of these solutions depends on the calculation of the mass and other conserved charges of the spacetime, we first compute these conserved quantities. The way we use for calculating conserved quantities is counterterm method. This method is a well-known method for Asymptotically AdS solutions to avoid divergencies in calculation of conserved quantities which is inspired by AdS/CFT correspondence \([37]\). This method can also be used for dilaton gravity and in the presence of Liouville-type potential where the spacetime is not asymptotically AdS \([22, 27, 38]\). Since boundary curvature of our spacetime is zero \((R_{abcd}(\gamma) = 0)\), the counterterm for the stress energy tensor should be proportional to \(\gamma^{ab}\). We find the finite stress-energy tensor in \((n + 1)\)-dimensional Einstein-dilaton gravity with Liouville-type potential as

\[ T^{ab} = \frac{1}{8\pi} \left[ \Theta^{ab} - \Theta \gamma^{ab} + \frac{n - 1}{l_{\text{eff}}} \gamma^{ab} \right], \]

where \(l_{\text{eff}}\) is given by

\[ l_{\text{eff}}^2 = \frac{(n - 1)(\alpha^2 - n)}{V(\Phi)}. \]
The first two terms in Eq. (31) are the variation of the action with respect to $\gamma_{ab}$, and the last term is counterterm which removes the divergences. Note that in the absence of the dilaton field ($\alpha = 0$), we have $V(\Phi) = 2\Lambda$, and the effective $I_{\text{eff}}^2$ of Eq. (32) reduces to $I^2 = -n(n - 1)/2\Lambda$ of the AdS spacetimes. In order to compute the conserved charges of the spacetime, one should first choose a spacelike surface $\mathcal{B}$ in $\partial M$ with metric $\sigma_{ij}$, and write the boundary metric in ADM (Arnowitt-Deser-Misner) form

$$\gamma_{ab} dx^a dx^b = -N^2 dt^2 + \sigma_{ij} \left( d\varphi^i + V^i dt \right) \left( d\varphi^j + V^j dt \right),$$

where the coordinates $\varphi^i$ are the angular variables parameterizing the hypersurface of constant $r$ around the origin, and $N$ and $V^i$ are the lapse and shift functions respectively. Then, the quasilocal conserved quantities associated with the stress tensors of Eq. (31) can be written as

$$Q(\xi) = \int_{\mathcal{B}} d^{n-1} \varphi \sqrt{\sigma} T_{ab} n^a \xi^b,$$

(33)

where $\sigma$ is the determinant of the metric $\sigma_{ij}$, $\xi$ and $n^a$ are the Killing vector field and the unit normal vector on the boundary $\mathcal{B}$. In order to calculate quasilocal mass and angular momentum, one should choose boundaries with timelike ($\xi = \partial / \partial t$) and rotational ($\xi = \partial / \partial \varphi$) Killing vector fields i.e.

$$M = \int_{\mathcal{B}} d^{n-1} \varphi \sqrt{\sigma} T_{ab} n^a \xi^b,$$

(34)

and

$$J = \int_{\mathcal{B}} d^{n-1} \varphi \sqrt{\sigma} T_{ab} n^a \xi^b,$$

(35)

provided the surface $\mathcal{B}$ contains the orbits of $\xi$. Eqs. (34) and (35) are conserved mass and angular momenta of the black hole surrounded by the boundary $\mathcal{B}$. It is remarkable to mention that although mass and angular momenta do not depend on specific choice of foliation $\mathcal{B}$ within the hypersurface $\partial M$, they are dependent on location of boundary $\mathcal{B}$ in the spacetime. Taking into account the cylindrical symmetry of the rotating black brane with $k$ rotation parameters along the angular coordinates $0 \leq \phi_i \leq 2\pi$, we denote the volume of the hypersurface boundary at constant $t$ and $r$ by $V_{n-1}$. Then, the mass and angular momentum per unit volume $V_{n-1}$ of the black branes can be calculated through the use of Eqs. (34) and (35). We find

$$M = \frac{b^{(n-1)/2} \left( \frac{n - \alpha^2}{1 + \alpha^2} \right) m}{16\pi l^{n-2}},$$

(36)

$$J_i = \frac{b^{(n-1)/2} \left( \frac{n - \alpha^2}{1 + \alpha^2} \right) \Xi m a_i},$$

(37)

As one can see from (37), the angular momentum per unit volume is proportional to $a_i$s and therefore it vanishes if $a_i = 0$ ($\Xi = 1$). Thus, it is physically reasonable to consider $a_i$s as rotational parameters of the spacetime. The last conserved quantity of our solutions is electric charge. Electric charge can be obtained by calculating the flux of the electric field at infinity. By projecting the electromagnetic field tensor on specific hypersurfaces, we can find the electric field as $E^\rho = g^{\mu\nu} e^{-4K\Phi/(n-1)} (-F)^{\rho - 1} F_{\mu\nu} u^\nu$ where $u^\nu$ is normal to such hypersurfaces. The components of $u^\nu$ are

$$u^0 = \frac{1}{N}, \quad u^r = 0, \quad u^i = -\frac{V^i}{N},$$

(38)

where $N$ and $V^i$ are the lapse function and shift vector. Eventually, the electric charge per unit volume $V_{n-1}$ can be calculated as

$$Q = \frac{\Xi \tilde{q}}{4\pi l^{n-2}},$$

$$\tilde{q} = 2^{p-1} q^{2p-1}.$$

(39)

One may note that $\tilde{q} = q$ for $p = 1$ [23].

Now, we calculate thermodynamic quantities entropy $S$ and electric potential $U$. Entropy of almost all black solutions including ones in Einstein gravity typically obeys the so-called area law [39, 40]. Dilaton black solutions are
not exceptions (see for instance [27, 33]). Thus, we can calculate the entropy per unit volume $V_{n-1}$ of our rotating black brane as

$$S = \frac{\Xi h(n-1) \gamma r^+(n-1)(1-\gamma)}{4n^2},$$

(40)

The electric potential $U$, can be calculated through the definition [41]

$$U = A_\mu \chi^\mu |_{r \to \infty} - A_\mu \chi^\mu |_{r = r_+},$$

(41)

where $\chi$ is the null generators of the event horizon given by Eq. (30). Therefore, the electric potential may be obtained as

$$U = \frac{Cq b^{(2p+1-\gamma)/(p-1)}}{\Xi r^4}.$$

(42)

Here, we are ready to seek for satisfaction of thermodynamics first law. First, we should obtain the mass $M$ in terms of extensive quantities $S$, $Q$ and $J$. Using Eqs. (36)-(39) and the fact that $f(r_+) = 0$, we receive

$$M(S, Q, J) = \frac{(n-\alpha^2)Z + \alpha^2 - 1}{(n-\alpha^2)l} \sqrt{Z(Z-1)}.$$

(43)

where $J = \sum_i J_i^2$ and $Z = \Xi^2$ which is the positive real root of the following equation:

$$(\alpha^2 + 1) \sqrt{Z - 1} (n - \alpha^2) p (2p - 1) \left[ \frac{(32\pi^2 Q)^2 p l^{n+4(p-1)} b^{n^2}}{\sqrt{Z^{(\alpha^2+2(n-2p-1))/n-1}}} \left( \frac{4l^{n-2} S}{Z} \right)^{(n-2p+\alpha^2)/(1-n)} \right]^{1/(2p-1)} + (n - 2p + \alpha^2) \Pi \left[ - \frac{(\alpha^2 + 1) \sqrt{Z - 1} b^{n^2} n}{l^{n-2} \sqrt{Z^{(\alpha^2-2(n+1))/(n-1)}}} \left( \frac{4l^{n-2} S}{Z} \right)^{(n-\alpha^2)/(n-1)} + 16 \pi l J \right] = 0.$$

(44)

Considering $S$, $Q$ and $J$ as a complete set of extensive quantities for the mass $M(S, Q, J)$, we should define conjugate intensive quantities to them. These quantities are temperature, angular velocities and electric potential

$$T = \left( \frac{\partial M}{\partial S} \right)_{J, Q}, \Omega_i = \left( \frac{\partial M}{\partial J_i} \right)_{S, Q}, U = \left( \frac{\partial M}{\partial Q} \right)_{S, J}.$$

(45)

One can check numerically that quantities defined by Eq. (45) coincide with Eqs. (25), (26) and (42) provided $C$ is chosen as $C = (n-1)p^2/\Pi$. Note that in the case of linear Maxwell field ($p = 1$), $C$ reduces to 1 as we expect [27]. Therefore one can conclude that thermodynamics first law

$$dM =TdS + \sum_{i=1}^{k} \Omega_i dJ_i + UdQ,$$

(46)

is satisfied. As one can see from (41), $U$ is proportional to value of $A_\mu$ at infinity and therefore $U$ diverges if $A_\mu$ diverges at infinity. On the other hand, it is obvious from thermodynamics first law that $M$ is dependent on the value of $U$. Therefore, as we mentioned in section (11), $A_\mu$ should be finite at infinity in order to have a finite mass. We took this fact into account for finding constraints on $p$ and $\alpha$ (see Eq. (24)).

IV. STABILITY IN CANONICAL AND GRAND-CANONICAL ENSEMBLES

In this section, we are going to study thermal stability of rotating black branes in the presence of a power-law Maxwell field in canonical and grand-canonical ensembles. It is necessary for a thermodynamic system to discuss its thermal stability. Thermal stability is investigated to ensure that the entropy of system is at local maximum or equivalently the internal energy of system is at local minimum. Therefore, the stability of a rotating black brane as a thermodynamic system can be studied in terms of entropy $S(M, Q, J)$ or its Legendre transformation $M(S, Q, J)$. 
In terms of mass $M(S, Q, J)$, the local stability in any ensemble implies that $M(S, Q, J)$ be a convex function of its extensive variables. Typically, this behavior is studied by calculating the determinant of the Hessian matrix of $M(S, Q, J)$ with respect to its extensive variables $X_i, \mathbf{H}^M_{X_i X_j} = \left[\frac{\partial^2 M}{\partial X_i \partial X_j}\right]_{[41, 42]}$. $\mathbf{H}^M_{X_i X_j} \geq 0$ guarantees that the system is thermally stable. The number of thermodynamic variables is ensemble-dependent. In canonical ensemble, the charge and angular momenta are fixed parameters and consequently the determinant of Hessian matrix $\mathbf{H}^M_{X_i X_j}$ reduces to $(\frac{\partial^2 M}{\partial S^2})_{Q, J}$. Therefore, in order to find the ranges where the system is at thermal stability, it is sufficient to find the ranges of positivity of $(\frac{\partial^2 M}{\partial S^2})_{Q, J}$ where the temperature $T$ is positive as well. In grand-canonical ensemble $Q$ and $J$ are not fixed parameters.

We discuss thermal stability for uncharged and charged cases, separately. First we discuss the uncharged case. It is notable to mention that in the case of uncharged rotating black branes $(\frac{\partial^2 M}{\partial S^2})_{J}$ is exactly one which is obtained in [27].

\[
\left(\frac{\partial^2 M}{\partial S^2}\right)_J = \frac{n (\alpha^2 + 1) \left[\Xi^2 - 1\right] \left(n + 1 - 2\alpha^2\right) + \Xi^2 \left(1 - \alpha^2\right)}{\pi^{2/3} (2 - n - \alpha^2)^{1/3} (\alpha^2 + n - 2) \Xi^2 + \alpha^2} \right)^{\left(2 - n - \alpha^2\right)/(\alpha^2 + 1)}. \quad (47)
\]

Therefore, in canonical ensemble we just briefly review the results of them. Since $\Xi^2 \geq 1$, (47) is positive provided $\alpha \leq 1$. Therefore uncharged rotating black branes are stable in canonical ensemble for $\alpha \leq 1$. Note that for uncharged
case temperature is always positive (see Eq. (25)). In grand-canonical ensemble $H_{S, J}^M$ can be calculated as:

$$H_{S, J}^M = \frac{16}{b^{2(n-1)}\Xi^4} \left[ (\alpha^2 + n - 2) \Xi^2 + 1 - \alpha^2 \right].$$

(48)

From Eq. (48) it is obvious that $H_{S, J}^M \geq 0$ provided $\alpha \leq 1$, which is similar to the case of the canonical ensemble. From the above arguments we conclude that the uncharged rotating black branes are thermally stable provided $\alpha \leq 1$ in both canonical and grand-canonical ensembles.

For charged case, we discuss the stability for $\alpha \leq 1$ and $\alpha > 1$, separately. Since charge does not change the stable solutions to unstable ones $[43]$, for $\alpha \leq 1$ we have thermally stable charged rotating black branes. This fact is shown in Figs. 2, 3, 5, 6, 8 and 9. Figs. 2 and 3 show that for $\alpha \leq 1$, the obtained solutions are always stable in canonical and grand-canonical ensembles for any value of $r_+$, as we expect. Since $T > 0$ guarantees that we have black branes for our choices, one should choose $q < q_{\text{ext}}$. The behavior of temperature is depicted in Fig. 4 with same constants as Figs. 2 and 3. Behaviors of $(\partial^2 M/\partial S^2)_{Q, J}$ and $H_{S, J}^M$ with respect to $\alpha \leq 1$ for different choices of charge $q$ are plotted in Figs. 5 and 6. These figures once again show that charge cannot change stable solutions to unstable ones and therefore we have stable charged rotating solutions for $\alpha \leq 1$ as we had in uncharged case. Fig. 7 shows positivity of temperature $T$ for solutions that their thermal stabilities have been depicted in Figs. 5 and 6. Figs. 8 and 9 depict stability of solutions for $\alpha \leq 1$ for different values of $p$ in canonical and grand-canonical ensembles respectively. The behavior of temperature for latter case is illustrated in Fig. 10. For $\alpha > 1$, one can understand from Fig. 11 that event horizon radius of stable black branes encounter an upper limit $r_{+\text{max}}$ in both canonical and grand-canonical ensembles. The value of this upper limit is greater in canonical ensemble than grand-canonical ensemble. The effect
of $\alpha$ on stability of solutions in both canonical and grand-canonical ensembles are depicted in Fig. 12. There is again an ensemble dependent upper limit this time on $\alpha$ that for values greater than it solutions are no longer stable. The value of $\alpha_{\text{max}}$ is smaller in the grand-canonical ensemble. In terms of $\frac{1}{2} < p < \frac{n}{2}$, stability is shown in Fig. 13. In contrast with $r_+$ and $\alpha$, there is a lower limit for $p$ i.e. for $p > p_{\text{min}}$, we have stable solutions. $p_{\text{min}}$ is again ensemble dependent as well as $r_{\text{+max}}$ and $\alpha_{\text{max}}$. For $\frac{n}{2} < p < n - 1$ where $\alpha$ has a $p$-dependent lower limit, numerical analysis confirm the result of investigation for $\frac{1}{2} < p < \frac{n}{2}$, i.e. there is again a lower limit $p_{\text{min}}$ that for values lower than it solutions are unstable. Stability in terms of charge is depicted in Fig. 14. In this case there is an ensemble dependent $q_{\text{min}}$ in each of ensembles that for $q$ greater than it black branes are stable. The value of $q_{\text{min}}$ is greater in grand-canonical ensemble.

V. CLOSING REMARKS

In this paper, we constructed a new class of higher dimensional nonlinear charged rotating black brane solutions in Einstein-dilaton gravity with complete set of rotation parameters. The nonlinear electromagnetic source was considered in the form of the power-law Maxwell field which guarantees conformal invariance of the electromagnetic Lagrangian in arbitrary dimensions for specific choices of power. Due to the presence of the dilaton field, our solutions are neither asymptotically flat nor (A)dS. We showed that in case of power-law Maxwell field, one needs two Liouville type potentials in order to have a rotating black brane solutions while in the case of linear Maxwell source just one
The term is needed [27]. The extra dilaton potential term disappears for $p = 1$. All our results reproduce the results of [27] in the case of linear charged rotating solutions where $p = 1$.

Demanding from one side that the value of total mass should be finite and from the other side that the effect of mass term in metric function should disappear at infinity, we found some restrictions on $p$ and $\alpha$. The allowed ranges of these two parameters is as follows. For $1/2 < p < n/2$, we have $0 \leq \alpha^2 < n - 2$ while for $n/2 < p < n - 1$, we have $2p - n < \alpha^2 < n - 2$. For these permitted ranges, our solutions are always well-defined. Also, in these ranges of $p$ and $\alpha$, the charge term in metric function $f(r)$ is always positive and dominant in the vicinity of $r = 0$. Therefore, Schwarzschild-like solutions are ruled out. However, solutions with two inner and outer horizons, extreme solutions and naked singularities are allowed.

In order to study thermodynamics of charged rotating black branes, we calculated mass, charge, temperature, entropy, electric potential energy and angular momentum. Using these quantities we obtained Smarr-type formula for the mass $M(S, Q, J)$ and showed that the first law of thermodynamics is satisfied. Next, we analysed thermal stability of the solutions in both canonical and grand-canonical ensembles. These investigations showed that for $\alpha \leq 1$ charged rotating black brane solutions are always stable with any value of the charge parameter. For $\alpha > 1$, there is a $r_{+\text{max}}$ in each of ensembles that we have stable solutions provided their radius are smaller than $r_{+\text{max}}$. In terms of $\alpha(> 1)$, the solutions is changed from stable ones to unstable ones when they meet an $\alpha_{\text{max}}$. Versus $p$ and $q$, we showed that the solutions encounter $p_{\text{min}}$ and $q_{\text{min}}$ respectively so that solutions with $q$ and $p$ parameters lower than them are unstable for $\alpha > 1$. All $r_{+\text{max}}$, $\alpha_{\text{max}}$, $q_{\text{min}}$ and $p_{\text{min}}$ values depend on the ensemble.
FIG. 10: The behavior of $T$ versus $\alpha \leq 1$ with $l = b = 1$, $r_+ = 1.5$, $\Xi = 1.25$, $n = 5$ and $q = 0.8$. Note that curve corresponding to $p = 0.75$ has been rescaled by the factor 1.5.

FIG. 11: The behavior of $T$ (solid curve), $10(\partial^2 M/\partial S^2)_{Q,J}$ (dashed curve) and $10^{-3}\mathbf{H}_{SQJ}^{M}$ (dashdot curve) versus $r_+$ with $l = b = 1$, $q = 0.5$, $\alpha = 1.35$, $\Xi = 1.25$, $n = 5$ and $p = 2$.

It is worth noting that the higher dimensional charged rotating solutions obtained here have flat horizon. One may interested in studying the rotating solutions with curved horizon. Specially the case of spherical horizon will be a good extension of Kerr-Newmann solution. It seems that the study of the general case is a difficult problem. However, it is possible to seek for slowly rotating nonlinear charged solutions with curved horizon. The latter is in progress.

Acknowledgments

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FIG. 12: The behavior of $T$ (solid curve), $(\partial^2 M/\partial S^2)_Q$, $J$ (dashed curve) and $H_{SQJ}^M$ (dashdot curve) versus $\alpha$ with $l = b = 1$, $q = 0.8$, $r_+ = 1.5$, $\Xi = 1.25$, $n = 5$ and $p = 2$.

FIG. 13: The behavior of $T$ (solid curve), $(\partial^2 M/\partial S^2)_Q$, $J$ (dashed curve) and $10^{-1}H_{SQJ}^M$ (dashdot curve) versus $p$ with $l = b = 1$, $q = 0.9$, $r_+ = 1.1$, $\Xi = 1.25$, $\alpha = 1.45$ and $n = 5$.

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FIG. 14: The behavior of $T$ (solid curve), $(\partial^2 M/\partial S^2)_{q,J}$ (dashed curve) and $10^{-2}H_{SQJ}^M$ (dashdot curve) versus $q$ with $l = b = 1$, $p = 4$, $r_+ = 0.9$, $\Xi = 1.25$, $\alpha = 1.5$ and $n = 6$.