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Extensions of Positive Definite Functions: Applications and Their Harmonic Analysis

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Dedicated to the memory of

William B. Arveson
(22 November 1934 – 15 November 2011)

Edward Nelson
(May 4, 1932 – September 10, 2014)

1 William Arveson (1934 – 2011) worked in operator algebras and harmonic analysis, and his results have been influential in our thinking, and in our approach to the particular extension questions we consider here. In fact, Arveson’s deep and pioneering work on completely positive maps may be thought of as a non-commutative variant of our present extension questions. We have chosen to give our results in the commutative setting, but readers with interests in non-commutative analysis will be able to make the connections. While the non-commutative theory was initially motivated by the more classical commutative theory, the tools involved are different, and there are not always direct links between theorems in one area and the other. One of Arveson’s earlier results in operator algebras is an extension theorem for completely positive maps taking values in the algebra of all bounded operators on a Hilbert space. His theorem led naturally to the question of injectivity of von-Neumann algebras in general, which culminated in profound work by Alain Connes relating injectivity to hyperfiniteness. One feature of Arveson’s work dating back to a series of papers in the 60’s and 70’s, is the study of noncommutative analogues of notions and results from classical harmonic analysis, including the Shilov and Choquet boundaries. The commutative analogues are visible in our present presentation.

2 Edward Nelson (1932 – 2014) was known for his work on mathematical physics, stochastic processes, in representation theory, and in mathematical logic. Especially his work in the first three areas has influenced our thinking. In more detail: infinite-dimensional group representations, the mathematical treatment of quantum field theory, the use of stochastic processes in quantum mechanics, and his reformulation of probability theory. Readers looking for beautiful expositions of the foundations in these areas are referred to the following two set of very accessible lecture notes by Nelson, *Dynamical theory of Brownian motion; and Topics in Dynamics 1: Flows.*, both in Princeton University Press, the first 1967, and the second 1969.

Our formulation of the present extension problems in the form of Type I and Type II Extensions (see Chapter 5 below) was especially influenced by independent ideas and results of both Bill Arveson and Ed Nelson. We use the two Nelson papers [Nel59, NS59] in our analysis of extensions of locally defined positive definite functions on Lie groups.
We study two classes of extension problems, and their interconnections:

(i) Extension of positive definite (p.d.) continuous functions defined on subsets in locally compact groups $G$;
(ii) In case of Lie groups, representations of the associated Lie algebras $\mathfrak{L}(G)$ by unbounded skew-Hermitian operators acting in a reproducing kernel Hilbert space (RKHS) $\mathcal{H}_F$.

Our analysis is non-trivial even if $G = \mathbb{R}^n$, and even if $n = 1$. If $G = \mathbb{R}^n$, we are concerned in (ii) with finding systems of strongly commuting selfadjoint operators $\{T_i\}$ extending a system of commuting Hermitian operators with common dense domain in $\mathcal{H}_F$.

Why extensions? In science, experimentalists frequently gather spectral data in cases when the observed data is limited, for example limited by the precision of instruments; or on account of a variety of other limiting external factors. (For instance, the human eye can only see a small portion of the electromagnetic spectrum.) Given this fact of life, it is both an art and a science to still produce solid conclusions from restricted or limited data. In a general sense, our monograph deals with the mathematics of extending some such given partial data-sets obtained from experiments. More specifically, we are concerned with the problems of extending available partial information, obtained, for example, from sampling. In our case, the limited information is a restriction, and the extension in turn is the full positive definite function (in a dual variable); so an extension if available will be an everywhere defined generating function for the exact probability distribution which reflects the data; if it were fully available. Such extensions of local information (in the form of positive definite functions) will in turn furnish us with spectral information. In this form, the problem becomes an operator extension problem, referring to operators in a suitable reproducing kernel Hilbert spaces (RKHS). In our presentation we have stressed hands-on-examples. Extensions are almost never unique, and so we deal with both the question of existence, and if there are extensions, how they relate back to the initial completion problem.
By a theorem of S. Bochner, the continuous p.d. functions are precisely the Fourier transforms of finite positive measures. In the case of locally compact Abelian groups $G$, the two sides in the Fourier duality is that of the group $G$ itself vs the dual character group $\hat{G}$ to $G$. Of course if $G = \mathbb{R}^n$, we may identify the two.

But in practical applications a p.d. function will typically be given only locally, or on some open subset, typically bounded; say an interval if $G = \mathbb{R}^1$; or a square or a disk in case $G = \mathbb{R}^2$. Hence four questions naturally arise:

(a) Existence of extensions.
(b) If there are extensions, find procedures for constructing them.
(c) Moreover, what is the significance of choice of different extensions from available sets of p.d. extensions?
(d) Finally, what are the generalizations and applications of the results in (a)-(c) to the case of an infinite number of dimensions?

All four questions will be addressed, and the connections between (d) and probability theory will be stressed.

While the theory of p.d. functions is important in many areas of pure and applied mathematics, ranging from harmonic analysis, functional analysis, spectral theory, representations of Lie groups, and operator theory on the pure side, to such applications as mathematical statistics, to optimization (and more, see details below), and to quantum physics, it is difficult for students and for the novice to the field, to find accessible presentations in the literature which cover all these disparate points of view, as well as stressing common ideas and interconnections.

We have aimed at filling this gap, and with a minimum number of prerequisites. We do expect that readers have some familiarity with measures and their Fourier transform, as well as with operators in Hilbert space, especially the theory of unbounded symmetric operators with dense domain. When needed, we have included brief tutorials. Further, in our cited references we have included both research papers, and books. To help with a historical perspective, we have included discussions of some landmark presentations, especially papers of S. Bochner, M.G. Krein, J. von Neumann, W. Rudin, and I.J. Schöenberg.

The significance to stochastic processes of the two questions, (i) and (ii) above, is as follows. To simplify, consider first stochastic processes $X_t$ indexed by time $t$, but known only for “small” $t$. Then the corresponding covariance function $c_X(s,t)$ will also only be known for small values of $s$ and $t$; or in the case of stationary processes, there is then a locally defined p.d. function $F$, known only in a bounded interval $J = (-a,a)$, and such that $F(s-t) = c_X(s,t)$. Hence, it is natural to ask how much can be said about extensions to values of $t$ in the complement of $J$? For example; what are the possible global extensions $X_t$, i.e., extension to all $t \in \mathbb{R}$? If there are extensions, then how does information about the locally defined covariance function influence the extended global process? What is the structure of the set of all extensions? The analogues questions are equally interesting for processes indexed by groups more general than $\mathbb{R}$. 
Specifically, we consider partially defined p.d. continuous functions $F$ on a fixed group. From $F$ we then build a RKHS $H_F$, and the operator extension problem is concerned with operators acting in $H_F$, and with unitary representations of $G$ acting on $H_F$. Our emphasis is on the interplay between the two problems, and on the harmonic analysis of our RKHS $H_F$.

In the cases of $G = \mathbb{R}^n$, and $G = \mathbb{R}^n/\mathbb{Z}^n$, and generally for locally compact Abelian groups, we establish a new Fourier duality theory; including for $G = \mathbb{R}^n$ a time/frequency duality, where the extension questions (i) are in time domain, and extensions from (ii) in frequency domain. Specializing to $n = 1$, we arrive of a spectral theoretic characterization of all skew-Hermitian operators with dense domain in a separable Hilbert space, having deficiency-indices $(1,1)$.

Our general results include non-compact and non-Abelian Lie groups, where the study of unitary representations in $H_F$ is subtle.

While, in the most general case, the obstructions to extendibility (of locally defined positive definite (p.d.) functions) is subtle, we point out that it has several explicit features: algebraic, analytic, and geometric. In Section 4.6, we give a continuous p.d. function $F$ in a neighborhood of 0 in $\mathbb{R}^2$ for which $\text{Ext}(F)$ is empty. In this case, the obstruction for $F$, is geometric in nature; and involves properties of a certain Riemann surface. (See Figure 0.1, and Section 4.6, Figures 4.9-4.10, for details.)

Note on Presentation. In presenting our results, we have aimed for a reader-friendly account. We have found it helpful to illustrate the ideas with worked examples. Each of our theorems holds in various degrees of generality, but when appropriate, we have not chosen to present details in their highest level of generality. Rather, we typically give the result in a setting where the idea is more transparent, and easier to grasp. We then work the details systematically at this lower level of generality. But we also make comments about the more general versions; sketching these in rough outline. The more general versions of the respective theorems will typically be easy for readers to follow, and to appreciate, after the idea has already been fleshed out in a simpler context.

We have made a second choice in order to make it easier for students to grasp the ideas as well as the technical details: We have included a lot of worked examples.
And at the end of each of these examples, we then outline how the specific details (from the example in question) serve to illustrate one or more features in the general theorems elsewhere in the monograph. Finally, we have made generous use of both Tables and Figures. These are listed with page-references at the end of the book; see the last few items in the Table of Contents. And finally, we included a list of Symbols on page 18, after Table of Contents.
Preface

On the one hand, the subject of positive definite (p.d.) functions has played an important role in standard graduate courses, and in research papers, over decades; and yet when presenting the material for a particular purpose, the authors have found that there is not a single source which will help students and researchers quickly form an overview of the essential ideas involved. Over the decades new ideas have been incorporated into the study of p.d. functions, and their more general cousin, p.d. kernels, from a host of diverse areas. An influence of more recent vintage is the theory of operators in Hilbert space, and their spectral theory.

A novelty in our present approach is the use of diverse Hilbert spaces. In summary; starting with a locally defined p.d. $F$, there is a natural associated Hilbert space, arising as a reproducing kernel Hilbert space (RKHS), $H_F$. Then the question is: When is it possible to realize globally defined p.d. extensions of $F$ with the use of spectral theory for operators in the initial RKHS, $H_F$? And when will it be necessary to enlarge the Hilbert space, i.e., to pass to a dilation Hilbert space; – a second Hilbert space $K$ containing an isometric copy of $H_F$ itself?

The theory of p.d. functions has a large number of applications in a host of areas; for example, in harmonic analysis, in representation theory (of both algebras and groups), in physics, and in the study of probability models, such as stochastic processes. One reason for this is the theorem of Bochner which links continuous p.d. functions on locally compact Abelian groups $G$ to measures on the corresponding dual group. Analogous uses of p.d. functions exist for classes for non-Abelian groups. Even the seemingly modest case of $G = \mathbb{R}$ is of importance in the study of spectral theory for Schrödinger equations. And further, counting the study of Gaussian stochastic processes, there are even generalizations (Gelfand-Minlos) to the case of continuous p.d. functions on Fréchet spaces of test functions which make up part of a Gelfand triple.

These cases will be explored below, but with the following important change in the starting point of the analysis; – we focus on the case when the given p.d. function is only partially defined, i.e., is only known on a proper subset of the ambient group, or space. How much of duality theory carries over when only partial information is available?
Applications. In machine learning extension problems for p.d. functions and RHKSs seem to be used in abundance. Of course the functions are initially defined over finite sets which is different than present set-up, but the ideas from our continuous setting do carry over mutatis mutandis. Machine learning [PS03] is a field that has evolved from the study of pattern recognition and computational learning theory in artificial intelligence. It explores the construction and study of algorithms that can learn from and make predictions on limited data. Such algorithms operate by building models from “training data” inputs in order to make data-driven predictions or decisions. Contrast this with strictly static program instructions. Machine learning and pattern recognition can be viewed as two facets of the same field.

Another connection between extensions of p.d. functions and neighboring areas is number theory: The pair correlation of the zeros of the Riemann zeta function [GM87, HB10]. In the 1970’s Hugh Montgomery (assuming the Riemann hypothesis) determined the Fourier transform of the pair correlation function in number theory (– it is a p.d. function). But the pair correlation function is specified only in a bounded interval centered at zero; again consistent with pair correlations of eigenvalues of large random Hermitian matrices (via Freeman Dyson). It is still not known what is the Fourier transform outside of this interval. Montgomery has conjectured that, on all of \( \mathbb{R} \), it is equal to the Fourier transform of the pair correlation of the eigenvalues of large random Hermitian matrices; this is the “Pair correlation conjecture”. And it is an important unsolved problem.

Yet another application of the tools for extending locally defined p.d. functions is that of the pioneering work of M.G. Krein, now called the inverse spectral problem of the strings of Krein, see [Kot13, Kei99, KW82]. This in turn is directly related to a host of the symmetric moment problems [Chi82]. In both cases we arrive at the problem of extending a real p.d. function that is initially only known on an interval.

In summary: the purpose of the present monograph is to explore what can be said when a continuous p.d. function is only given on a subset of the ambient group (which is part of the application setting sketched above.) For this problem of partial information, even the case of p.d. functions defined only on bounded subsets of \( G = \mathbb{R} \) (say an interval), or on bounded subsets of \( G = \mathbb{R}^n \), is of substantial interest.
The co-authors thank the following for enlightening discussions: Professors Daniel Alpay, Sergii Bezuglyi, Dorin Dutkay, Paul Muhly, Rob Martin, Robert Niedzialomski, Gestur Olafsson, Judy Packer, Wayne Polyzou, Myung-Sin Song, and members in the Math Physics seminar at the University of Iowa.

We also are pleased to thank anonymous referees for careful reading, for lists of corrections, for constructive criticism, and for many extremely helpful suggestions; – for example, pointing out to us more ways that the question of extensions of fixed locally defined positive definite functions, impact yet more areas of mathematics, and are also part of important applications to neighboring areas. Remaining flaws are the responsibility of the co-authors.
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Symbols

$(\Omega, \mathcal{F}, \mathbb{P})$  probability space: sample space $\Omega$, sigma-algebra $\mathcal{F}$, probability measure $\mathbb{P}$ (pg. 24, 168)

$\delta_x$  Dirac measure, also called Dirac mass. (pg. 37, 112)

$\int_t f(t) \, dX_t$  Ito-integral, defined for $f \in L^2(J)$. (pg. 26, 168)

$\langle \cdot, \cdot \rangle$ or $\langle \cdot, \cdot \rangle_H$  Inner product; we add a subscript (when necessary) in order to indicate which Hilbert space is responsible for the inner product in question. Caution, because of a physics tradition, all of our inner products are linear in the second variable. (This convention further has the advantage of giving simpler formulas in case of reproducing kernel Hilbert spaces (RKHSs).) (pg. 36, 201)

$\langle \lambda, x \rangle$  duality pairing $\hat{G} \leftrightarrow G$ of locally compact Abelian groups. (pg. 63, 69)

$\mathbb{E}$  expectation, $\mathbb{E}(\cdots) = \int_\Omega \cdots \, d\mathbb{P}$. (pg. 26, 168)

$\mathbb{R}$  the real line

$\mathbb{R}^n$  the n-dimensional real Euclidean space

$\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$  tori (We identify $\mathbb{T}$ with the circle group.)

$\mathbb{Z}$  the integers

$\mathcal{M}_2(\Omega, F)$  Hilbert space of measures on $\Omega$ associated to a fixed kernel, or a p.d. function $F$. (pg. 39)

$\mathcal{B}(G)$  The sigma–algebra of all Borel subsets of $G$. (pg. 83, 212)

$\mathcal{H}_F$  The reproducing kernel Hilbert space (RKHS) of $F$. (pg. 29, 36)

$\mathcal{M}(G)$  All Borel measures on $G$. (pg. 55, 64)

$\perp$  Perpendicular w.r.t. a fixed Hilbert inner product. (pg. 84, 133)
\( \hat{G} \) the dual character group, where \( G \) is a fixed locally compact Abelian group, i.e., \( \lambda : G \to \mathbb{T} \), continuous, \( \lambda (x + y) = \lambda (x) \lambda (y) \), \( \forall x, y \in G \) \( \lambda (-x) = \overline{\lambda (x)} \). (pg. 63, 69)

\( \{ B_t \}_{t \in \mathbb{R}} \) Standard Brownian motion. (pg. 111, 166)

\( \{ X_g \}_{g \in G} \) Stochastic process indexed by \( G \). (pg. 24, 29)

\( D(\phi) \) The derivative operator \( \phi \mapsto \int dx \, F \phi \). (pg. 41, 48)

\( D^* \) Adjoint of a linear operator \( D \). (pg. 41, 60)

\( \text{DEF} \) The deficiency space of \( D(\phi) \) acting in the Hilbert space \( \mathcal{H}_F \). (pg. 41, 138)

\( \text{dom}(D) \) Domain of a linear operator \( D \). (pg. 41, 103)

\( \text{Ext}(F) \) Set of unitary representations of \( G \) on a Hilbert space \( \mathcal{H} \), i.e., the triples \( (U, \mathcal{H}, k_0) \), that extend \( F \). (pg. 55, 107)

\( \text{Ext}_1(F) \) The triples in \( \text{Ext}(F) \), where the representation space is \( \mathcal{H}_F \), and so the extension of \( F \) is realized on \( \mathcal{H}_F \). (pg. 55, 107)

\( \text{Ext}_2(F) \) \( \text{Ext}(F) \setminus \text{Ext}_1(F) \), i.e., the extension is realized on an enlargement Hilbert space. (pg. 55, 107)

\( F_\phi \) The convolution of \( F \) and \( \phi \), where \( \phi \in C_c(\Omega) \). (pg. 36, 64)

\( G \) group, with the group operation written \( x \cdot y \), or \( x + y \), depending on the context.

\( J \) conjugation operator, acting in the Hilbert space \( \mathcal{H}_F \) for a fixed local p.d. function \( F \). To say that \( J \) is a conjugation means that \( J \) is a conjugate linear operator which is also of period 2. (pg. 44, 45)

\( L\alpha(G) \) The Lie algebra of a given Lie group \( G \). (pg. 71, 76)

\( \text{Rep}(G, \mathcal{H}) \) Set of unitary representations of a group \( G \) acting on some Hilbert space \( \mathcal{H} \). (pg. 105)

\( S(= S_2) \) the isometry \( S : \mathcal{M}_2(\Omega, F) \to \mathcal{H}_F \) (onto) (pg. 39, 100)

\( T_F \) Mercer operator associated to \( F \). (pg. 128, 162)

\( X \) random variable, \( X : \Omega \to \mathbb{R} \). (pg. 26, 168)

\( \text{ONB} \) orthonormal basis (in a Hilbert space) (pg. 100, 127)

p.d. positive definite (pg. 19, 32)

\( \text{RKHS} \) reproducing kernel Hilbert space (pg. 36, 51)
Positive-definiteness arises naturally in the theory of the Fourier transform. There are two directions in transform theory. In the present setting, one is straightforward, and the other (Bochner) is deep. First, it is easy to see directly that the Fourier transform of a positive finite measure is a positive definite function; and that it is continuous. The converse result is Bochner’s theorem. It states that any continuous positive definite function on the real line is the Fourier transform of a unique positive and finite measure. However, if some given positive definite function is only partially defined, for example in an interval, or in the planar case, in a disk or a square, then Bochner’s theorem does not apply. One is faced with first seeking a positive definite extension; hence the theme of our monograph.

Definition 1.1. Fix $0 < a$, let $\Omega$ be an open interval of length $a$, then $\Omega - \Omega = (-a, a)$. Let a function

$$F : \Omega - \Omega \to \mathbb{C}$$

be continuous, and defined on $\Omega - \Omega$. $F$ is positive definite (p.d.) if

$$\sum_{i} \sum_{j} c_{i} c_{j} F(x_{i} - x_{j}) \geq 0,$$

for all finite sums with $c_{i} \in \mathbb{C}$, and all $x_{i} \in \Omega$. Hence, $F$ is p.d. iff the $N \times N$ matrix $(F(x_{i} - x_{j}))_{i,j=1}^{N}$ is p.d. for all $x_{1}, \ldots, x_{N}$ in $\Omega$, and all $N \in \mathbb{N}$.

Applications of positive definite functions include statistics, especially Bayesian statistics, and the setting is often the case of real valued functions; while complex valued and Hilbert space valued functions are important in mathematical physics.

In some statistical applications, one often takes $n$ scalar measurements (sampling of a random variable) of points in $\mathbb{R}^{n}$, and one requires that points that are closely separated have measurements that are highly correlated. But in practice, care must be exercised to ensure that the resulting covariance matrix (an $n$-by-$n$ matrix) is always positive definite. One proceeds to define such a correlation matrix $A$ which is then multiplied by a scalar to give us a covariance matrix: It will be positive definite, and Bochner’s theorem applies: If the correlation between a pair of points depends only on the distance between them (via a function $F$), then this function
$F$ must be positive definite since the covariance matrix $A$ is positive definite. In lingo from statistics, Fourier transform becomes instead characteristic function. It is computed from a distribution (“measure” in harmonic analysis).

In our monograph we shall consider a host of diverse settings and generalizations, e.g., to positive definite functions on groups. Indeed, in the more general setting of locally compact Abelian topological groups, Bochner’s theorem still applies. This is the setting naturally occurring in the study of representation of groups; representations typically acting on infinite-dimensional Hilbert spaces (so we study the theory of unitary representations). The case of locally compact Abelian groups was pioneered by W. Rudin, and by M. Stone, M.A. Naimark, W. Ambrose, and R. Godement\(^1\). Now, in the case of non-Abelian Lie groups, the motivation is from the study of symmetry in quantum theory, and there are numerous applications of non-commutative harmonic analysis to physics. In our presentation we will illustrate the theory in both Abelian and the non-Abelian cases. This discussion will be supplemented by citations to the References in the back of the book.

We study two classes of extension problems, and their interconnections. The first class of extension problems concerns (i) continuous positive definite (p.d.) functions on Lie groups $G$; and the second deals with (ii) Lie algebras of unbounded skew-Hermitian operators in a certain family of reproducing kernel Hilbert space (RKHS).

Our analysis is non-trivial even if $G = \mathbb{R}^n$, and even if $n = 1$.

If $G = \mathbb{R}^n$, we are concerned in (ii) with the study of systems of $n$ skew-Hermitian operators $\{S_i\}$ on a common dense domain in Hilbert space, and in deciding whether it is possible to find a corresponding system of strongly commuting selfadjoint operators $\{T_i\}$ such that, for each value of $i$, the operator $T_i$ extends $S_i$.

The version of this for non-commutative Lie groups $G$ will be stated in the language of unitary representations of $G$, and corresponding representations of the Lie algebra $\mathfrak{L}(G)$ by skew-Hermitian unbounded operators.

In summary, for (i) we are concerned with partially defined continuous p.d. functions $F$ on a Lie group; i.e., at the outset, such a function $F$ will only be defined on a connected proper subset in $G$. From this partially defined p.d. function $F$ we then build a RKHS $H_F$, and the operator extension problem (ii) is concerned with operators acting on $H_F$, as well as with unitary representations of $G$ acting on $H_F$. If the Lie group $G$ is not simply connected, this adds a complication, and we are then making use of the associated simply connected covering group. For an overview of high-points, see Sections 1.4 and 1.6 below.

Readers not familiar with some of the terms discussed above may find the following references helpful [Zie14, KL14, Gne13, MS12, Ber12, GZM11, Kol11, HV11, BT11, Hid80, App09, App08, Itô04, EF11, Lai08, Aro50, BCR84, DS88, Jor90, Jor91, Nel57, Rud70]. The list includes both basic papers and texts, as well as some recent research papers.

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\(^1\) M.H. Stone, Ann. Math. 33 (1932) 643-648
M.A. Naimark, Izv. Akad. Nauk SSSR. Ser. Mat. 7 (1943) 237-244
W. Ambrose, Duke Math. J. 11 (1944) 589-595
R. Godement, C.R. Acad. Sci. Paris 218 (1944) 901-903
1.1 Two Extension Problems

Our main theme is the interconnection between (i) the study of extensions of \textit{locally defined continuous and positive definite (p.d.) functions \( F \)} on groups on the one hand, and, on the other, (ii) the question of extensions for an associated \textit{system of unbounded Hermitian operators with dense domain} in a reproducing kernel Hilbert space (RKHS) \( \mathcal{H}_F \) associated to \( F \).

Because of the role of p.d. functions in harmonic analysis, in statistics, and in physics, the connections in both directions are of interest, i.e., from (i) to (ii), and vice versa. This means that the notion of “extension” for question (ii) must be inclusive enough in order to encompass all the extensions encountered in (i). For this reason enlargement of the initial Hilbert space \( \mathcal{H}_F \) is needed. In other words, it is necessary to consider also operator extensions which are realized in a dilation-Hilbert space; a new Hilbert space containing \( \mathcal{H}_F \) isometrically, and with the isometry intertwining the respective operators.

An overview of the extension correspondence is given in Section 2.2.1, Figure 2.3.

To appreciate these issues in concrete examples, readers may wish to consult Chapter 2, especially the last two sections, 4.4 and 4.5. To help visualization, we have included tables and figures in Sections 5.4, 6.4, and 7.1.

Where to find it.

\textbf{Caption to the table in item (i) below}: A number of cases of the extension problems (we treat) occur in increasing levels of generality, interval vs open subsets of \( \mathbb{R} \), \( T = \mathbb{R}/\mathbb{Z} \), \( \mathbb{R}^n \), locally compact Abelian group, and finally of a Lie group, both the specific theorems and the parameters differ from one to the other, and we have found it worthwhile to discuss the settings in separate sections. To help readers make comparisons, we have outlined a roadmap, with section numbers, where the cases can be found.

(i) Extension of \textit{locally defined p.d. functions}.

\begin{itemize}
  \item Type I vs Type II: Sections 2.4 (p.55), 6.1 (p.55)
  \item Level of generality:
\end{itemize}
Ambient Space | Section
---|---
interval $\subset \mathbb{R}$ | §2.4 (p.55), §6.4 (p.152), §7.1 (p.161)
$\Omega \subset \mathbb{R}^n$ | §4.1 (p.83), §7.2 (p.83)
$\Omega \subset \mathbb{T} = \mathbb{R}/\mathbb{Z}$ | §4.2 (p.88), §4.3 (p.90), §4.4 (p.92)
$\Omega \subset G$, locally cpt Abelian group | §3.1 (p.63)
$\Omega \subset G$, general Lie group | §3.2 (p.70)
$\Omega \subset \mathcal{S}$, in a Gelfand-triple | p.9, §1.3 (p.24), §7.1 (p.161)

(ii) Connections to extensions of system of unbounded Hermitian operators

Models for indices - (1, 1) operators: Section 10.2 (p.207)

Models for indices - $(d, d)$, $d > 1$, operators: Section 10.3 (p.212)

While each of the two extension problems has received a considerable amount of attention in the literature, the emphasis here will be the interplay between the two. The aim is a duality theory; and, in the case $G = \mathbb{R}^n$, and $G = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, the theorems will be stated in the language of Fourier duality of Abelian groups: With the time frequency duality formulation of Fourier duality for $G = \mathbb{R}^n$, both the time domain and the frequency domain constitute a copy of $\mathbb{R}^n$. We then arrive at a setup such that our extension questions (i) are in time domain, and extensions from (ii) are in frequency domain. Moreover we show that each of the extensions from (i) has a variant in (ii). Specializing to $n = 1$, we arrive of a spectral theoretic characterization of all skew-Hermitian operators with dense domain in a separable Hilbert space, having deficiency-indices $(1, 1)$.

A systematic study of densely defined Hermitian operators with deficiency indices $(1, 1)$, and later $(d, d)$ for $d > 1$, was initiated by M. Krein [Kre46], and is also part of de Branges’ model theory [dB68, dBR66]. The direct connection between this theme and the problem of extending continuous p.d. functions $F$ when they are only defined on a fixed open subset to $\mathbb{R}^n$ was one of our motivations. One desires continuous p.d. extensions to $\mathbb{R}^n$.

If $F$ is given, we denote the set of such extensions $\text{Ext} (F)$. If $n = 1$, $\text{Ext} (F)$ is always non-empty, but for $n = 2$, Rudin gave examples when $\text{Ext} (F)$ may be empty [Rud70, Rud63]. Here we extend these results, and we also cover a number of classes of p.d. functions on locally compact groups in general.

Our results in the framework of locally compact Abelian (l.c.a) groups are more complete than their counterparts for non-Abelian Lie groups, one reason is the availability of Bochner’s duality theorem for l.c.a groups [BC49, BC48, Boc47, Boc46]; – not available for non-Abelian Lie groups.
1.2 Quantum Physics

The axioms of quantum physics (see e.g., [BM13, OH13, KS02, CRKS79, ARR13, Fan10, Maa10, Par09] for relevant recent papers), are based on Hilbert space, and selfadjoint operators.

A brief sketch: A quantum mechanical observable is a Hermitian (selfadjoint) linear operator mapping a Hilbert space, the space of states, into itself. The values obtained in a physical measurement are in general described by a probability distribution; and the distribution represents a suitable “average” (or “expectation”) in a measurement of values of some quantum observable in a state of some prepared system. The states are (up to phase) unit vectors in the Hilbert space, and a measurement corresponds to a probability distribution (derived from a projection-valued spectral measure). The particular probability distribution used depends on both the state and the selfadjoint operator. The associated spectral type may be continuous (such as position and momentum; both unbounded) or discrete (such as spin); this depends on the physical quantity being measured.

Symmetries are ubiquitous in physics, and in dynamics. Because of the axioms of quantum theory, they take the form of unitary representations of groups $G$ acting on Hilbert space; the groups are locally Euclidian, (this means Lie groups). The tangent space at the neutral element $e$ in $G$ acquires a Lie bracket, making it into a Lie algebra. For describing dynamics from a Schrödinger wave equation, $G = \mathbb{R}$ (the real line, for time). In the general case, we consider strongly continuous unitary representations $U$ of $G$; and if $G = \mathbb{R}$, we say that $U$ is a unitary one-parameter group.

From unitary representation $U$ of $G$ to positive definite function: Let

$$U : G \to \left\{ \text{unitary operators in some Hilbert space } \mathcal{H} \right\},$$

(1.3)

let $v_0 \in \mathcal{H} \setminus \{0\}$, and let

$$F(x) := \langle v_0, U(x) v_0 \rangle_{\mathcal{H}}, \quad x \in G;$$

(1.4)

then $F$ is positive definite (p.d.) on $G$. The converse is true too, and is called the Gelfand-Naimark-Segal (GNS)-theorem, see Sections 3.2.1-3.2.2; i.e., from every p.d. function $F$ on some group $G$, there is a triple $(U, \mathcal{H}, v_0)$, as described above, such that (1.4) holds.

In the case where $G$ is a locally compact Abelian group with dual character group $\hat{G}$, and if $U$ is a unitary representation (see (1.3)) then there is a projection valued measure $P_U$ on the Borel subsets of $\hat{G}$ such that

$$U(x) = \int_{\hat{G}} \langle \lambda, x \rangle dP_U(\lambda), \quad x \in G,$$

(1.5)

where $\langle \lambda, x \rangle = \lambda(x)$, for all $\lambda \in \hat{G}$, and $x \in G$. The assertion in (1.5) is a theorem of Stone, Naimark, Ambrose, and Godement (SNAG), see Section 3.1. In (1.5), $P_U(\cdot)$
is defined on the Borel subsets $S$ in $\hat{G}$, and $P_U(S)$ is a projection, $P_U(\hat{G}) = I$; and $S \mapsto P_U(S)$ is countably additive.

Since the Spectral Theorem serves as the central tool in the study of measurements, one must be precise about the distinction between linear operators with dense domain which are only Hermitian as opposed to selfadjoint\textsuperscript{2}. This distinction is accounted for by von Neumann’s theory of deficiency indices (see e.g., [vN32a, Kre46, DS88, AG93, Nel69]).

(Starting with [vN32a, vN32b], J. von Neumann and M. Stone did pioneering work in the 1930s on spectral theory for unbounded operators in Hilbert space; much of it in private correspondence. The first named author has from conversations with M. Stone, that the notions “deficiency-index,” and “deficiency space” are due to them; suggested by MS to vN as means of translating more classical notions of “boundary values” into rigorous tools in abstract Hilbert space: closed subspaces, projections, and dimension count.)

1.3 Stochastic Processes

... from its shady beginnings devising gambling strategies and counting corpses in medieval London, probability theory and statistical inference now emerge as better foundations for scientific models, especially those of the process of thinking and as essential ingredients of theoretical mathematics, ...  

— David Mumford. From: “The Dawning of the Age of Stochasticity.” [Mum00]

Early Roots

The interest in positive definite (p.d.) functions has at least three roots:

1. Fourier analysis, and harmonic analysis more generally, including the non-commutative variant where we study unitary representations of groups. (See [Dev59, Nus75, Rud63, Rud70, Jor86, Jor89, Jor90, Jor91, KL07, KL14, Kle74, MS12, Ørs79, OS73, Sch86a, Sch86b, SF84] and the papers cited there.)

2. Optimization and approximation problems, involving for example spline approximations as envisioned by I. Schöenberg. (See [Pól49, Sch38a, Sch38b, Sch64, SZ07, SZ09, PS03] and the papers cited there.)

3. Stochastic processes. (See [Boc46, Ber46, Itô04, PS75, App09, Hid80, GSS83] and the papers cited there.)

Below, we sketch a few details regarding (3). A stochastic process is an indexed family of random variables based on a fixed probability space. In our present analy-

\textsuperscript{2} We refer to Section 2.1 for details. A Hermitian operator, also called formally selfadjoint, may well be non-selfadjoint.
1.3 Stochastic Processes

sis, the processes will be indexed by some group \( G \) or by a subset of \( G \). For example, 
\( G = \mathbb{R} \), or \( G = \mathbb{Z} \), correspond to processes indexed by real time, respectively discrete time. A main tool in the analysis of stochastic processes is an associated covariance function, see (1.19).

A process \( \{ X_g \mid g \in G \} \) is called Gaussian if each random variable \( X_g \) is Gaussian, i.e., its distribution is Gaussian. For Gaussian processes we only need two moments. So if we normalize, setting the mean equal to 0, then the process is determined by its covariance function. In general the covariance function is a function on \( G \times G \), or on a subset, but if the process is stationary, the covariance function will in fact be a p.d. function defined on \( G \), or a subset of \( G \).

We will be using three stochastic processes, Brownian motion, Brownian Bridge, and the Ornstein-Uhlenbeck process, all Gaussian, and Ito integrals.

A probability space is a triple \((\Omega, \mathcal{F}, P)\) where \( \Omega \) is a set (sample space), \( \mathcal{F} \) is a (fixed) sigma-algebra of subsets of \( \Omega \), and \( P \) is a (sigma-additive) probability measure defined on \( \mathcal{F} \). (Elements \( E \) in \( \mathcal{F} \) are "events", and \( P(E) \) represents the probability of the event \( E \).)

A real valued random variable is a function \( X : \Omega \to \mathbb{R} \) such that, for every Borel subset \( A \subset \mathbb{R} \), we have that \( X^{-1}(A) = \{ \omega \in \Omega \mid X(\omega) \in A \} \) is in \( \mathcal{F} \). Then

\[
\mu_X(A) = P(X^{-1}(A)) , \quad A \in \mathcal{B}
\]

defines a positive measure on \( \mathbb{R} \); here \( \mathcal{B} \) denotes the Borel sigma-algebra of subsets of \( \mathbb{R} \). This measure is called the distribution of \( X \). For examples of common distributions, see Table 5.2.

The following notation for the \( P \) integral of random variables \( X (\cdot) \) will be used:

\[
\mathbb{E}(X) := \int_{\Omega} X(\omega) dP(\omega),
\]

denoted expectation. If \( \mu_X \) is the distribution of \( X \), and \( \psi : \mathbb{R} \to \mathbb{R} \) is a Borel function, then

\[
\int_{\mathbb{R}} \psi d\mu_X = \mathbb{E}(\psi \circ X).
\]

An example of a probability space is as follows:

\[
\Omega = \prod_{i \in \mathbb{N}} \{ \pm 1 \} = \text{infinite Cartesian product}
\]

\[
= \{ \{ \omega_i \}_{i \in \mathbb{N}} \mid \omega_i \in \{ \pm 1 \}, \forall i \in \mathbb{N} \}, \quad \text{and}
\]

\( \mathcal{F} \): subsets of \( \Omega \) specified by a finite number of outcomes (called “cylinder sets”).

\( P \): the infinite-product measure corresponding to a fair coin \( \left( \frac{1}{2}, \frac{1}{2} \right) \) measure for each outcome \( \omega_i \).

The transform

\[
\widehat{\mu}_X(\lambda) = \int_{\mathbb{R}} e^{i\lambda x} d\mu_X(x)
\]
is called the Fourier transform, or the generating function.

Let $a$ be fixed, $0 < a < 1$. A random $a$-power series is the function

$$X_a(\omega) = \sum_{i=1}^{\infty} \omega_i a^i, \quad \omega = (\omega_i) \in \Omega.$$  \hfill (1.9)

One checks (see Chapter 9 below) that the generating function for $X_a$ is as follows:

$$\hat{\mu}_{X_a}(\lambda) = \prod_{k=1}^{\infty} \cos(a^k \lambda), \quad \lambda \in \mathbb{R}$$  \hfill (1.10)

where the r.h.s. in (1.10) is an infinite product. Note that it is easy to check independently that the r.h.s. in (1.10), $F_a(\lambda) = \prod_{k=1}^{\infty} \cos(a^k \lambda)$ is positive definite and continuous on $\mathbb{R}$, and so it determines a measure. See also Example 5.8.

An indexed family of random variables is called a stochastic process.

Example 1.1 (Brownian motion).

$\Omega$: all continuous real valued function on $\mathbb{R}$;
$\mathcal{F}$: subsets of $\Omega$ specified by a finite number of sample-points;
$P$: Wiener-measure on $(\Omega, \mathcal{F})$, see [Hid80].

For $\omega \in \Omega$, $t \in \mathbb{R}$, set $X_t(\omega) = \omega(t)$; then it is well known that $\{X_t\}_{t \in \mathbb{R}'}$ is a Gaussian-random variable with the property that:

$$d\mu_{X_t}(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx, \quad x \in \mathbb{R}, \quad t > 0,$$  \hfill (1.11)

$$X_0 = 0,$$ and

whenever $0 \leq t_1 < t_2 < \cdots < t_n$, then the random variables

$$X_{t_1}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$$  \hfill (1.12)

are independent; see [Hid80]. (The r.h.s. in (1.11) is Gaussian distribution with mean 0 and variance $t > 0$. See Figure 1.1.)

In more detail, $X_t$ satisfies:

1. $\mathbb{E}(X_t) = 0$, for all $t$; mean zero;
2. $\mathbb{E}(X_t^2) = t$, variance $= t$;
3. $\mathbb{E}(X_s X_t) = s \wedge t$, the covariance function; and
4. $\mathbb{E}((X_{b_1} - X_{a_1})(X_{b_2} - X_{a_2})) = |[a_1, b_1] \cap [a_2, b_2]|$, for any pair of intervals.

This stochastic process is called Brownian motion (see Figure 1.2).

Lemma 1.1 (The Ito integral [Hid80]). Let $\{X_t\}_{t \in \mathbb{R}'}$ be Brownian motion, and let $f \in L^2(\mathbb{R}^+)$. For partitions of $\mathbb{R}^+$, $\pi : \{t_i\}, t_i \leq t_{i+1}$, consider the sums

$$S(\pi) := \sum_i f(t_i) (X_{t_i} - X_{t_{i-1}}) \in L^2(\Omega, \mathbb{P})$$  \hfill (1.13)
Then the limit (in $L^2(\Omega, \mathbb{P})$) of the terms (1.13) exists, taking limit on the net of all partitions s.t. $\max_i (t_{i+1} - t_i) \to 0$. The limit is denoted
\[
\int_0^\infty f(t) \, dX_t \in L^2(\Omega, \mathbb{P}), \tag{1.14}
\]
and it is called the Ito-integral. The following isometric property holds:
\[
\mathbb{E} \left( \left\| \int_0^\infty f(t) \, dX_t \right\|^2 \right) = \int_0^\infty |f(t)|^2 \, dt. \tag{1.15}
\]
Eq (1.15) is called the Ito-isometry.

Proof. We refer to [Hid80] for an elegant presentation, but the key step in the proof involves the above mentioned properties of Brownian motion.

The first step is the verification of
\[
\mathbb{E} \left( |S(\pi)|^2 \right) = \sum_i |f(t_i)|^2 (t_i - t_{i-1}), \tag{1.16}
\]
which is based on (1.12).

An application of Lemma 1.1: A positive definite function on an infinite dimensional vector space.

Let $S$ denote the real valued Schwartz functions (see [Trè06]). For $\phi \in S$, set $X(\phi) = \int_0^\infty \phi(t) \, dX_t$, the Ito integral from (1.14). Then we get the following:
\[
\mathbb{E} \left( e^{X(\phi)} \right) = e^{-\frac{1}{2} \int_0^\infty |\phi(t)|^2 \, dt} \left( = e^{-\frac{1}{2} ||\phi||_{L^2}^2} \right), \tag{1.17}
\]
where $\mathbb{E}$ is the expectation w.r.t. Wiener-measure.

It is immediate that
\[
F(\phi) := e^{-\frac{1}{2} ||\phi||_{L^2}^2}, \tag{1.18}
\]
i.e., the r.h.s. in (1.16), is a positive definite function on $S$. To get from this an associated probability measure (the Wiener measure $\mathbb{P}$) is non-trivial, see e.g., [Hid80, AJ12, AJL11]: The dual of $S$, the tempered distributions $S'$, turns into a measure space, $(S', \mathcal{F}, \mathbb{P})$ with the sigma-algebra $\mathcal{F}$ generated by the cylinder sets in $S'$. With this we get an equivalent realization of Wiener measure (see the cited papers); now with the l.h.s. in (1.16) as $\mathbb{E}(\cdots) = \int_{S'} \cdots d\mathbb{P}(\cdot)$. But the p.d. function $F$ in (1.17) cannot be realized by a sigma-additive measure on $L^2$, one must pass to a “bigger” infinite-dimensional vector space, hence $S'$. The system
\[
S \hookrightarrow L^2_{\mathbb{R}}(\mathbb{R}) \hookrightarrow S'
\]
is called a Gelfand-triple. The second right hand side inclusion \( L^2 \hookrightarrow S' \) in (1.18) is obtained by dualizing \( S \hookrightarrow L^2 \), where \( S \) is given its Fréchet topology, see [Tré06].

Let \( G \) be a locally compact group, and let \( (\Omega,\mathcal{F},\mathbb{P}) \) be a probability space, \( \mathcal{F} \) a sigma-algebra, and \( \mathbb{P} \) a probability measure defined on \( \mathcal{F} \). A stochastic \( L^2 \)-process is a system of random variables \( \{X_g\}_{g \in G}, X_g \in L^2(\Omega,\mathcal{F},\mathbb{P}) \). The covariance function \( c_X : G \times G \to \mathbb{C} \) of the process is given by

\[
c_X(g_1,g_2) = \mathbb{E}(X_{g_1} X_{g_2}), \quad \forall (g_1,g_2) \in G \times G.
\]

To simplify, we will assume that the mean \( \mathbb{E}(X_g) = \int_{\Omega} X_g d\mathbb{P}(\omega) = 0 \) for all \( g \in G \).

We say that \( (X_g) \) is stationary iff
In this case $c_X$ is a function of $g_1^{-1}g_2$, i.e.,

$$E(X_{g_1}, X_{g_2}) = c_X(g_1^{-1}g_2), \quad \forall g_1, g_2 \in G; \quad (1.21)$$

(setting $h = g_1^{-1}$ in (1.20).)

The covariance function of Brownian motion $E(X_t, X_r)$ is computed in Example 1.2 below.

**1.4 Overview of Applications of RKHSs**

In a general setup, reproducing kernel Hilbert spaces (RKHSs) were pioneered by Aronszajn in the 1950s [Aro50]; and subsequently they have been used in a host of applications; e.g., [SZ09, SZ07].

The key idea of Aronszajn is that a RKHS is a Hilbert space $H_K$ of functions $f$ on a set such that the values $f(x)$ are “reproduced” from $f$ and a vector $K_x$ in $H_K$, in such a way that the inner product $\langle K_x, K_y \rangle = K(x, y)$ is a positive definite kernel.

Since this setting is too general for many applications, it is useful to restrict the very general framework for RKHSs to concrete cases in the study of particular spectral theoretic problems; p.d. functions on groups is a case in point. Such specific issues arise in physics (e.g., [Fal74, Jor07]) where one is faced with extending p.d. functions $F$ which are only defined on a subset of a given group.

**Connections to Gaussian Processes**

By a theorem of Kolmogorov, every Hilbert space may be realized as a (Gaussian) reproducing kernel Hilbert space (RKHS), see e.g., [PS75, IM65, SNFBK10], and Theorem 1.1 below.

**Definition 1.2.** A function $c$ defined on a subset of a group $G$ is said to be **positive definite** iff

$$\sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j c(g_i^{-1}g_j) \geq 0 \quad (1.22)$$

for all $n \in \mathbb{N}$, and all $\{\lambda_i\}_{i=1}^n \subset \mathbb{C}$, $\{g_i\}_{i=1}^n \subset G$ with $g_i^{-1}g_j$ in the domain of $c$.

From (1.22), it follows that $F(g^{-1}) = F(g)$, and $|F(g)| \leq F(e)$, for all $g$ in the domain of $F$, where $e$ is the neutral element in $G$.

We recall the following theorem of Kolmogorov. One direction is easy, and the other is the deep part:

**Theorem 1.1 (Kolmogorov).** A function $c : G \rightarrow \mathbb{C}$ is positive definite if and only if there is a stationary Gaussian process $(\Omega, \mathcal{F}, \mathbb{P}, X)$ with mean zero, such that $c = c_X$, i.e., $c(g_1, g_2) = E(X_{g_1}, X_{g_2})$; see (1.19).
Proof. We refer to [PS75] for the non-trivial direction. To stress the idea, we include a proof of the easy part of the theorem: Assume \( c = c_X \). Let \( \{\lambda_i\}_{i=1}^n \subset C \) and \( \{g_i\}_{i=1}^n \subset G \), then we have

\[
\sum_i \sum_j \lambda_i \lambda_j c (g_i^{-1} g_j) = \mathbb{E}(\left| \sum \lambda_i X_{g_i} \right|^2) \geq 0,
\]
i.e., \( c \) is positive definite. \( \square \)

Example 1.2. Let \( \Omega = [0, 1] \), the closed unit interval, and let \( \mathcal{H} := \) the space of continuous functions \( \xi \) on \( \Omega \) such that \( \xi (0) = 0 \), and \( \xi' \in L^2 (0, 1) \), where \( \xi' = \frac{d}{dx} \xi \) is the weak derivative of \( \xi \), i.e., the derivative in the Schwartz-distribution sense. For \( x, y \in \Omega \), set

\[
K (x, y) = x \wedge y = \min (x, y); \quad \text{and} \quad K_x (y) = K (x, y).
\] (1.23)

Then in the sense of distribution, we have

\[
(K_x)' = \chi_{[0,x]};
\] (1.24)
i.e., the indicator function of the interval \([0,x]\), see Figure 1.3.

For \( \xi_1, \xi_2 \in \mathcal{H} \), set

\[
\langle \xi_1, \xi_2 \rangle_{\mathcal{H}} := \int_0^1 \overline{\xi_1'(x)} \xi_2'(x) dx.
\]

Since \( L^2 (0, 1) \subset L^1 (0, 1) \), and \( \xi (0) = 0 \) for \( \xi \in \mathcal{H} \), we see that

\[
\xi (x) = \int_0^x \xi' (y) dy, \quad \xi' \in L^2 (0, 1),
\] (1.25)
and \( \mathcal{H} \) consists of continuous functions on \( \Omega \).

Claim. The Hilbert space \( \mathcal{H} \) is a RKHS with \( \{K_x\}_{x \in \Omega} \) as its kernel; see (1.23).

Proof. Let \( \xi \in \mathcal{H} \), then by (1.25), we have:

\[
\xi (x) = \int_0^1 \chi_{[0,x]} (y) \xi' (y) dy = \int_0^1 K_x' (y) \xi' (y) dy = \langle K_x, \xi \rangle_{\mathcal{H}}, \quad \forall x \in \Omega.
\] \( \square \)

Remark 1.1. In the case of Example 1.2 above, the Gaussian process resulting from the p.d. kernel (1.23) is standard Brownian motion [Hid80], see Section 7.1. We shall return to the p.d. kernel (1.23) in Chapters 2, 4, and 6. It is the covariance kernel for standard Brownian motion in the interval \([0,1]\).
Subsequently, we shall be revisiting a number of specific instances of these RKHSs. Our use of them ranges from the most general case, when a continuous positive definite (p.d.) function $F$ is defined on an open subset of a locally compact group; the RKHS will be denoted $\mathcal{H}_F$. However, we stress that the associated RKHS will depend on both the function $F$, and on the subset of $G$ where $F$ is defined; hence on occasion, to be specific about the subset, we shall index the RKHS by the pair $(\Omega, F)$. If the choice of subset is implicit in the context, we shall write simply $\mathcal{H}_F$.

Depending on the context, a particular RKHS typically will have any number of concrete, hands-on realizations, allowing us thereby to remove the otherwise obtuse abstraction entailed in its initial definition.

A glance at the Table of Contents indicates a large variety of the classes of groups, and locally defined p.d. functions we consider, and subsets. In each case, both the specific continuous, locally defined p.d. function considered, and its domain are important. Each of the separate cases has definite applications. The most explicit computations work best in the case when $G = \mathbb{R}$; and we offer a number of applications in three areas: applications to stochastic processes (Sections 5.1-7.2), to harmonic analysis (Sections 10.1-10.4), and to operator/spectral theory (Section 6.1).

1.5 Earlier Papers

Below we mention some earlier papers dealing with one or the other of the two extension problems (i) or (ii) in Section 1.1. To begin with, there is a rich literature on (i), and a little on (ii), but comparatively much less is known about their interconnections.

As for positive definite (p.d.) functions, their use and applications are extensive and includes such areas as stochastic processes, see e.g., [JP13a, AJSV13, JP12, AJ12]; harmonic analysis (see [BCR84, JÓ00, JÓ98], and the references there); po-
tential theory [Fug74, KL14]; operators in Hilbert space [ADL+10, Alp92a, AD86, JN15]; and spectral theory [AH13, Nus75, Dev72, Dev59]. We stress that the literature is vast, and the above list is only a small sample.

Extensions of continuous p.d. functions defined on subsets of Lie groups $G$ was studied in [Jor91]. In our present analysis of its connections to the extension questions for associated operators in Hilbert space, we will be making use of tools from spectral theory, and from the theory of reproducing kernel Hilbert spaces, such as can be found in e.g., [Nel69, Jør81, ABDdS93, Aro50].

There is a different kind of notion of positivity involving reflections, restrictions, and extensions. It comes up in physics and in stochastic processes, and is somewhat related to our present theme. While they have several names, “reflection positivity” is a popular term. In broad terms, the issue is about realizing geometric reflections as “conjugations” in Hilbert space. When the program is successful, for a given unitary representation $U$ of a Lie group $G$, it is possible to renormalize the Hilbert space on which $U$ is acting.

Now the Bochner transform $F$ of a probability measure (e.g., the distribution of a stochastic process) which further satisfies reflection positivity, has two positivity properties: one (i) because $F$ is the transform of a positive measure, so $F$ is positive definite; and in addition the other, (ii) because of reflection symmetry (see the discussion after Corollary 2.7.) We have not followed up below with structural characterizations of this family of positive definite functions, but readers interested in the theme will find details in [JÓ00, JÓ98, Arv86, OS73], and in the references given there.

To help readers to appreciate other approaches, as well as current research, we add below citation to some recent papers covering one or the other of the many aspects of the extension problem, which is the focus of our presentation here [BT11, BD07, BT07, KL07, Sas06, Bis02].

### 1.6 Organization

We begin with a quick summary of preliminaries, making clear our setting and our choice of terminology. While we shall consider a host of variants of related extension questions for positive definite (p.d.) functions, the following theme is stressed in Chapter 2 below: Starting with a given and fixed, locally defined p.d. $F$ (say $F$ may be defined only in an open neighborhood in an ambient space), then consider first the naturally associated Hilbert space, arising from $F$ as a reproducing kernel Hilbert space (RKHS), $\mathcal{H}_F$. The question is then: When is it possible to realize an extension as a globally defined p.d. function $\tilde{F}$, i.e., $\tilde{F}$ extending $F$, in a construction which uses only spectral theory for operators in the initial RKHS, $\mathcal{H}_F$? And when will it be necessary to enlarge the initial Hilbert space, by passing to a dilation Hilbert space $\mathcal{K}$ containing an isometric copy of $\mathcal{H}_F$ itself?

The monograph is organized around the following themes, some involving dichotomies; e.g.,
(1) Abelian (sect 3.1, ch 4-6) vs non-Abelian (sect 3.2);
(2) simply connected (sect 4.5) vs connected (sect 4.4);
(3) spectral theoretic (sect 3.1, 3.2, 10.2) vs geometric (sect 4.6);
(4) extensions of p.d. functions vs extensions of systems of operators (sect 4.6); and
(5) existence (ch 2) vs computation (ch 6-7) and classification (ch 10).

Item (1) refers to the group $G$ under consideration. In order to get started, we will need $G$ to be locally compact so it comes with Haar measure, but it may be non-Abelian. It may be a Lie group, or it may be non-locally Euclidean. In the other end of this dichotomy, we look at $G = \mathbb{R}$, the real line. In all cases, in the study of the themes from (1) it is important whether the group is simply connected or not.

In order to quickly get to concrete examples, we begin the real line $G = \mathbb{R}$ (Section 2.2.1), and $G = \mathbb{R}^n$, $n > 1$ (Section 4.1); and the circle group, $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ (Section 4.2).

Of the other groups, we offer a systematic treatment of the classes when $G$ is locally compact Abelian (Section 3.1), and the case of Lie groups (Section 3.2).

We note that the subdivision into classes of groups is necessary as the theorems we prove in the case of $G = \mathbb{R}$ have a lot more specificity than their counterparts do, for the more general classes of groups. One reason for this is that our harmonic analysis relies on unitary representations, and the non-commutative theory for unitary representations is much more subtle than is the Abelian counterpart.

Taking a choice of group $G$ as our starting point, we then study continuous p.d. functions $F$ defined on certain subsets in $G$. In the case of $G = \mathbb{R}$, our choice of subset will be a finite open interval centered at $x = 0$.

Our next step is to introduce a reproducing kernel Hilbert space (RKHS) $\mathcal{H}_F$ that captures the properties of the given p.d. function $F$. The nature and the harmonic analysis of this particular RKHS are of independent interest; see Sections 1.4, 2.3, 4.1, and 6.1.

In Section 6.1, we study a certain trace class integral operator, called the Mercer operator. A Mercer operator $T_F$ is naturally associated to a given a continuous and p.d. function $F$ defined on the open interval, say $(-1, 1)$. We use $T_F$ in order to identify natural Bessel frame in the corresponding RKHS $\mathcal{H}_F$. We then introduce a notion of Shannon sampling of finite Borel measures on $\mathbb{R}$, sampling from integer points in $\mathbb{R}$. In Corollary 6.11 we then use this to give a necessary and sufficient condition for a given finite Borel measure $\mu$ to fall in the convex set $\text{Ext}(F)$: The measures in $\text{Ext}(F)$ are precisely those whose Shannon sampling recover the given p.d. function $F$ on the interval $(-1, 1)$.

In the general case for $G$, the questions we address are as follows:

(a) What (if any) are the continuous p.d. functions on $G$ which extend $F$? (Sections 2.3, 4.5, and Chapters 5, 6.) Denoting the set of these extensions $\text{Ext}(F)$, then $\text{Ext}(F)$ is a compact convex set. Our next questions are:
(b) What are the parameters for $\text{Ext}(F)$? (Section 4.5, Chapters 5 and 11.)
(c) How can we understand $\text{Ext}(F)$ from a generally non-commutative extension problem for operators in $\mathcal{H}_F$? (See especially Section 6.1.)
(d) We are further concerned with applications to scattering theory (e.g., Theorem 4.1), and to commutative and non-commutative harmonic analysis.

(e) The unbounded operators we consider are defined naturally from given p.d. functions $F$, and they have a common dense domain in the RKHS $H_F$. In studying possible selfadjoint operator extensions in $H_F$ we make use of von Neumann's theory of deficiency indices. For concrete cases, we must then find the deficiency indices; they must be equal, but whether they are $(0,0)$, $(1,1)$, $(d,d)$, $d > 1$, is of great significance to the answers to the questions from (a)-(d).

(f) Finally, what is the relevance of the solutions in (a) and (b) for the theory of operators in Hilbert space and their harmonic (and spectral) analysis? (Sections 4.1, 10.2, 10.3, and Chapter 11.)

Citations for (a)-(f) above.

(a) [Dev59, JN15, KW82, Rud63, Rud70].
(b) [JPT13, Sch38b, Pöl49, Nus75, Jor90, Jor91].
(c) [JM84, JT14a, Nel59].
(d) [LP89, JPT15, Maa10, JPT12].
(e) [JPT13, DS88, Ion01, KW82, vN32a].
(f) [Sch84, Sch85, JT14a, Nel57].
Chapter 2
Extensions of Continuous Positive Definite Functions

It is nice to know that the computer understands the problem.
But I would like to understand it too. — Eugene Wigner

We begin with a study of a family of reproducing kernel Hilbert spaces (RKHSs) arising in connection with extension problems for positive definite (p.d.) functions. While the extension problems make sense, and are interesting, in a wider generality, we restrict attention here to the case of continuous p.d. functions defined on open subsets of groups $G$. We study two questions:

(i) When is a given partially defined continuous p.d. function extendable to the whole group $G$? In other words, when does it have continuous p.d. extensions to $G$?

(ii) When continuous p.d. extensions exist, what is the structure of all continuous p.d. extensions?

Because of available tools (mainly spectral theory for linear operators in Hilbert space), we restrict here our focus as follows: In the Abelian case, to when $G$ is locally compact; and if $G$ is non-Abelian, we assume that it is a Lie group. But our most detailed results are for the two cases $G = \mathbb{R}^n$, and $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$.

Historically, the cases $n = 1$ and $n = 2$ are by far the most studied, and they are also our main focus here. Available results are then much more explicit, and the applications perhaps more far reaching. Our results and examples for the case of $n = 1$, we feel, are of independent interest; and they are motivated by such applications as harmonic analysis, sampling and interpolation theory, stochastic processes, and Lax-Phillips scattering theory. One reason for the special significance of the case $n = 1$ is its connection to the theory of unbounded Hermitian linear operators with prescribed dense domain in Hilbert space, and their extensions. Indeed, if $n = 1$, the possible continuous p.d. extensions of given partially defined p.d. function $F$ are connected with associated extensions of certain unbounded Hermitian linear operators; in fact two types of such extensions: In one case, there are selfadjoint extensions in the initial RKHS (Type I); and in another case, the selfadjoint extensions necessarily must be realized in an enlargement Hilbert space (Type II); so in a Hilbert space properly bigger than the initial RKHS associated to $F$. 
2.1 The RKHS $\mathcal{H}_F$

In our theorems and proofs, we shall make use of reproducing kernel Hilbert spaces (RKHSs), but the particular RKHSs we need here will have additional properties (as compared to a general framework); which allow us to give explicit formulas for our solutions.

Our present setting is more restrictive in two ways: (i) we study groups $G$, and translation-invariant kernels, and (ii) we further impose continuity. By “translation” we mean relative to the operation in the particular group under discussion. Our presentation below begins with the special case when $G$ is the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, or the real line $\mathbb{R}$.

For simplicity we focus on the case $G = \mathbb{R}$, indicating the changes needed for $G = \mathbb{T}$. Modifications, if any, necessitated by considering other groups $G$ will be described in the body of the book.

**Lemma 2.1.** Let $\Omega$ be an open subset of $\mathbb{R}^n$, and let $F$ be a continuous function, defined on $\Omega - \Omega$; then $F$ is positive definite (p.d.) if and only if the following holds:

$$\int_\Omega \int_\Omega \phi(x)\phi(y)F(x-y)dxdy \geq 0, \quad \forall \phi \in C^\infty_c(\Omega).$$

**Proof.** Standard. □

Consider a continuous p.d. function $F : \Omega - \Omega \to \mathbb{C}$, and set

$$F_y(x) := F(x-y), \quad \forall x, y \in \Omega.$$  (2.1)

Let $\mathcal{H}_F$ be the reproducing kernel Hilbert space (RKHS), which is the completion of

$$\left\{ \sum_{\text{finite}} c_j F_{x_j} \mid x_j \in \Omega, c_j \in \mathbb{C} \right\}$$

with respect to the inner product

$$\left\langle \sum_i c_i F_{x_i}, \sum_j d_j F_{y_j} \right\rangle_{\mathcal{H}_F} := \sum_i \sum_j c_i d_j F(x_i - y_j);$$  (2.3)

modulo the subspace of functions with zero $\| \cdot \|_{\mathcal{H}_F}$-norm.

**Remark 2.1.** Throughout, we use the convention that the inner product is conjugate linear in the first variable, and linear in the second variable. When more than one inner product is used, subscripts will make reference to the Hilbert space.

**Lemma 2.2.** The RKHS, $\mathcal{H}_F$, is the Hilbert completion of the functions

$$F_\phi(x) = \int_\Omega \phi(y)F(x-y)dy, \quad \forall \phi \in C^\infty_c(\Omega), \quad x \in \Omega$$

with respect to the inner product.
2.1 The RKHS $\mathcal{H}_F$

$$\langle F_{\varphi}, F_{\psi} \rangle_{\mathcal{H}_F} = \int_{\Omega} \int_{\Omega} \varphi(x) \psi(y) F(x-y) \, dx \, dy, \quad \forall \varphi, \psi \in C_\infty^c(\Omega).$$

(2.5)

In particular,

$$\|F_{\varphi}\|_{\mathcal{H}_F}^2 = \int_{\Omega} \int_{\Omega} \varphi(x) \psi(y) F(x-y) \, dx \, dy.$$  \hspace{1cm} (2.6)

**Proof.** Indeed, setting $\varphi_n, x(t) := n \varphi(n(t-x))$ (see Lemma 2.3), we have

$$\|F_{\varphi_n, x} - F_x\|_{\mathcal{H}_F} \to 0, \text{ as } n \to \infty.$$  \hspace{1cm} (2.7)

Hence $\{F_\varphi\}_{\varphi \in C_\infty^c(\Omega)}$ spans a dense subspace in $\mathcal{H}_F$. For more details, see [Jor86, Jor87, Jor90].

**Lemma 2.3.** For $\Omega = (\alpha, \beta)$, let $a = \beta - \alpha$. Let $\alpha < x < \beta$ and let $\varphi_{n,x}(t) = n \varphi(n(t-x))$, where $\varphi$ satisfies

1. $\text{supp}(\varphi) \subset (-a, a)$;
2. $\varphi \in C_\infty^c$, $\varphi \geq 0$;
3. $\int \varphi(t) \, dt = 1$. Note that $\lim_{n \to \infty} \varphi_{n,x} = \delta_x$, the Dirac measure at $x$.

Then

$$\lim_{n \to \infty} \|F_{\varphi_{n,x}} - F_x\|_{\mathcal{H}_F} = 0$$  \hspace{1cm} (2.8)

Hence $\{F_\varphi \mid \varphi \in C_\infty^c(\Omega)\}$ spans a dense subspace in $\mathcal{H}_F$.

Fig. 2.1: The approximate identity $\varphi_{n,x}(\cdot)$

The facts below about $\mathcal{H}_F$ follow from the general theory of RKHS [Aro50]:

- For $F$ continuous, p.d., and non zero, $F(0) > 0$, so we can always arrange $F(0) = 1$.
- $F(-x) = F(x)$, $|F(x)| \leq F(0)$
- $\mathcal{H}_F$ consists of continuous functions $\xi : \Omega \to \mathbb{C}$.
• The reproducing property holds:

$$\langle F_x, \xi \rangle_{\mathcal{H}^F} = \xi(x), \quad \forall \xi \in \mathcal{H}^F, \forall x \in \Omega.$$ 

This is a direct consequence of the definition of $\mathcal{H}^F$, see (2.3).

• If $F_{\phi_n} \to \xi$ in $\mathcal{H}^F$, then $F_{\phi_n}$ converges uniformly to $\xi$ in $\Omega$. In fact, the reproducing property yields the estimate:

$$\left| F_{\phi_n}(x) - \xi(x) \right| = \left| \langle F_x, F_{\phi_n} - \xi \rangle_{\mathcal{H}^F} \right|$$

$$\leq \| F_x \|_{\mathcal{H}^F} \| F_{\phi_n} - \xi \|_{\mathcal{H}^F}$$

$$= F(0)^{1/2} \| F_{\phi_n} - \xi \|_{\mathcal{H}^F} \xrightarrow{n \to \infty} 0.$$ 

**Theorem 2.1.** A continuous function $\xi : \Omega \to \mathbb{C}$ is in $\mathcal{H}^F$ if and only if there exists $A_0 > 0$, such that

$$\sum_i \sum_j c_i c_j \xi(x_i) \xi(x_j) \leq A_0 \sum_i \sum_j c_i c_j F(x_i - x_j) \quad (2.9)$$

for all finite system $\{c_i\} \subset \mathbb{C}$ and $\{x_i\} \subset \Omega$.

Equivalently, for all $\psi \in C^\infty_\nu(\Omega)$,

$$\left| \int_{\Omega} \psi(y) \xi(x) dy \right|^2 \leq A_0 \int_{\Omega} \int_{\Omega} \overline{\psi(x)} \psi(y) F(x - y) dxdy. \quad (2.10)$$

**Proof.** It suffices to check condition (2.10).

If (2.10) holds, then $F_\psi \mapsto \int_{\Omega} \overline{\xi(x)} \psi(x) dx$ is a bounded linear functional on the dense subspace $\{F_\psi | \psi \in C^\infty_\nu(\Omega)\}$ in $\mathcal{H}^F$ (Lemma 2.3), hence it extends to $\mathcal{H}^F$. Therefore, $\exists! l_\xi \in \mathcal{H}^F$ s.t.

$$\int_{\Omega} \overline{\xi(x)} \psi(x) dx \overset{\text{(Riesz)}}{=} \langle l_\xi, F_\psi \rangle_{\mathcal{H}^F}$$

$$\overset{\text{(Fubini)}}{=} \int_{\Omega} \psi(y) \langle l_\xi, F_y \rangle_{\mathcal{H}^F} dy$$

$$= \int_{\Omega} l_\xi(y) \psi(y) dy, \quad \forall \psi \in C^\infty_\nu(\Omega);$$

and this implies $\xi = l_\xi \in \mathcal{H}^F$.

Conversely, assume $\xi \in \mathcal{H}^F$ then $\left| \langle \xi, F_\psi \rangle_{\mathcal{H}^F} \right|^2 \leq \| \xi \|^2_{\mathcal{H}^F} \| F_\psi \|^2_{\mathcal{H}^F}$, for all $\psi \in C^\infty_\nu(\Omega)$. Thus (2.10) follows. $\square$

These two conditions (2.9)(\iff)(2.10)) are the best way to characterize elements in the Hilbert space $\mathcal{H}^F$; see also Corollary 2.1.

We will be using this when considering for example the deficiency-subspaces for skew-symmetric operators with dense domain in $\mathcal{H}^F$. 

Theorem 2.1. A continuous function $\xi : \Omega \to \mathbb{C}$ is in $\mathcal{H}^F$ if and only if there exists $A_0 > 0$, such that

$$\sum_i \sum_j c_i c_j \xi(x_i) \xi(x_j) \leq A_0 \sum_i \sum_j c_i c_j F(x_i - x_j) \quad (2.9)$$

for all finite system $\{c_i\} \subset \mathbb{C}$ and $\{x_i\} \subset \Omega$.

Equivalently, for all $\psi \in C^\infty_\nu(\Omega)$,

$$\left| \int_{\Omega} \psi(y) \xi(x) dy \right|^2 \leq A_0 \int_{\Omega} \int_{\Omega} \overline{\psi(x)} \psi(y) F(x - y) dxdy. \quad (2.10)$$

**Proof.** It suffices to check condition (2.10).

If (2.10) holds, then $F_\psi \mapsto \int_{\Omega} \overline{\xi(x)} \psi(x) dx$ is a bounded linear functional on the dense subspace $\{F_\psi | \psi \in C^\infty_\nu(\Omega)\}$ in $\mathcal{H}^F$ (Lemma 2.3), hence it extends to $\mathcal{H}^F$. Therefore, $\exists! l_\xi \in \mathcal{H}^F$ s.t.

$$\int_{\Omega} \overline{\xi(x)} \psi(x) dx \overset{\text{(Riesz)}}{=} \langle l_\xi, F_\psi \rangle_{\mathcal{H}^F}$$

$$\overset{\text{(Fubini)}}{=} \int_{\Omega} \psi(y) \langle l_\xi, F_y \rangle_{\mathcal{H}^F} dy$$

$$= \int_{\Omega} l_\xi(y) \psi(y) dy, \quad \forall \psi \in C^\infty_\nu(\Omega);$$

and this implies $\xi = l_\xi \in \mathcal{H}^F$.

Conversely, assume $\xi \in \mathcal{H}^F$ then $\left| \langle \xi, F_\psi \rangle_{\mathcal{H}^F} \right|^2 \leq \| \xi \|^2_{\mathcal{H}^F} \| F_\psi \|^2_{\mathcal{H}^F}$, for all $\psi \in C^\infty_\nu(\Omega)$. Thus (2.10) follows. $\square$

These two conditions (2.9)(\iff)(2.10)) are the best way to characterize elements in the Hilbert space $\mathcal{H}^F$; see also Corollary 2.1.

We will be using this when considering for example the deficiency-subspaces for skew-symmetric operators with dense domain in $\mathcal{H}^F$. 

Example 2.1. Let $G = T = \mathbb{R}/\mathbb{Z}$, e.g., represented as $(-\frac{1}{2}, \frac{1}{2}]$. Fix $0 < a < \frac{1}{2}$, then $\Omega - \Omega = (-a, a)$ mod $\mathbb{Z}$. So, for example, (1.1) takes the form $F : (-a, a)$ mod $\mathbb{Z} \to \mathbb{C}$. See Figure 2.2.

![Figure 2.2: Two versions of $\Omega = (0, a) \subset \mathbb{T}^1$.](image)

**An Isometry**

It is natural to extend the mapping from (2.4) to measures, i.e., extending

$$C^\infty_c(\Omega) \ni \phi \mapsto F_\phi \in \mathcal{H}_F \text{ (the RKHS)},$$

by replacing $\phi$ with a Borel measure $\mu$ on $\Omega$. Specifically, set:

$$(S\mu)(x) = F_{\mu}(x) := \int_{\Omega} F(x-y) \, d\mu(y). \quad (2.11)$$

Then the following result follows from the discussion above.

**Corollary 2.1.**

1. Let $\mu$ be a Borel measure on $\Omega$ (possibly a signed measure), then the following two conditions are equivalent:

$$\int_{\Omega} \int_{\Omega} F(x-y) \, d\mu(x) \, d\mu(y) < \infty; \quad (2.12)$$

$$\Downarrow$$

$$S\mu \in \mathcal{H}_F \text{ (see (2.11)).} \quad (2.13)$$

2. If the conditions hold, then

$$\int_{\Omega} \int_{\Omega} F(x-y) \, d\mu(x) \, d\mu(y) = \|S\mu\|^2_{\mathcal{H}_F} \quad (2.14)$$

(where the norm on the r.h.s. in (2.14) is the RKHS-norm.)
(3) And moreover, for all $\xi \in \mathcal{H}_F$, we have
\[
\langle \xi, S\mu \rangle_{\mathcal{H}_F} = \int_{\Omega} \overline{\xi}(x)d\mu(x).
\] (2.15)

(4) In particular, we note that all the Dirac measures $\mu = \delta_x$, for $x \in \Omega$, satisfy (2.12), and that
\[
S(\delta_x) = F(\cdot - x) \in \mathcal{H}_F.
\] (2.16)

(5) For every $x, y \in \Omega$, we have
\[
\langle \delta_x, \delta_y \rangle_{\mathcal{M}_2(\Omega, F)} = F(x - y),
\] (2.17)
where the r.h.s. in (2.17) refers to the $\mathcal{M}_2$-Hilbert inner product.

In other words, the signed measures $\mu$ satisfying (2.12) form a Hilbert space $\mathcal{M}_2(\Omega, F)$, and $S$ in (2.11) defines an isometry of $\mathcal{M}_2(\Omega, F)$ onto the RKHS $\mathcal{H}_F$.

Example 2.2. $\mathcal{G} = \mathbb{R}, a > 0, \Omega = (0, a)$,
\[
F(x) = e^{-|x|}, \quad |x| < a.
\] (2.18)

We now apply the isometry $S : \mathcal{M}_2(F) \xrightarrow{\mathbb{C}} \mathcal{H}_F$ (onto) to this example. The two functions
\[
\xi_+ (x) = e^{-x}, \quad \xi_- (x) = e^{x-a}, \quad x \in \Omega,
\]
will play an important role (defect-vectors) in Chapter 4 below. We have
\[
S^* \xi_+ = \delta_0 \in \mathcal{M}_2(F), \quad \text{and} \quad (2.19)
\]
\[
S^* \xi_- = \delta_a \in \mathcal{M}_2(F), \quad (2.20)
\]
where $\delta_0$ and $\delta_a$ denote the respective Dirac measures. The proof of (2.19)-(2.20) is immediate from Corollary 2.1.

Corollary 2.2. Let $F$ and $\Omega$ be as above; and assume further that $F$ is $C^\infty$, then
\[
F^{(n)}(\cdot - x) \in \mathcal{H}_F, \quad \text{for all} \ x \in \Omega.
\] (2.21)

Proof. We may establish (2.21) by induction, starting with the first derivative.

Let $x \in \Omega$; then for sufficiently small $h, |h| < \varepsilon$, we have $F(\cdot - x - h) \in \mathcal{H}_F$; and moreover,
\[
\left\| F(\cdot - x - h) - F(\cdot - x) \right\|_{\mathcal{H}_F}^2 = \frac{1}{h^2} (F'(0) + F'(0) - 2F'(-h))
\]
\[
= \frac{1}{h} \left( F'(0) - F'(-h) - \frac{F(-h) - F'(0)}{h} \right)
\]
\[
\rightarrow -F^{(2)}(0), \quad \text{as} \ h \rightarrow 0.
\] (2.22)
2.2 The Skew-Hermitian Operator $D^{(F)}$ in $H_{\bar{F}}$

Hence, the limit of the difference quotient on the r.h.s. in (2.22) exists relative to the norm in $H_{\bar{F}}$, and so the limit $-F'(\cdot-x)$ is in $H_{\bar{F}}$. □

We further get the following:

**Corollary 2.3.** If $F$ is p.d. and $C^2$ in a neighborhood of 0, then $F^{(2)}(0) \leq 0$.

**Remark 2.2.** Note that, for $\xi \in H_{\bar{F}}$, (2.13) is an integral equation, i.e.,

$$\xi(x) = \int_{\Omega} F(x-y) d\nu(y) \quad \text{for } \nu \in M_2(F).$$

Using Corollary 2.2, we note that, when p.d. $F \in C^{\infty}(\Omega-\Omega)$, then (2.23) also has distribution solutions $\nu$.

---

2.2 The Skew-Hermitian Operator $D^{(F)}$ in $H_{\bar{F}}$

In our discussion of the operator $D^{(F)}$ (Definition 2.1), we shall make use of von Neumann’s theory of symmetric (or skew-symmetric) linear operators with dense domain in a fixed Hilbert space; see e.g., [vN32a, LP85, Kre46, JLW69, dBR66, DS88].

The general setting is as follows: Let $H$ be a complex Hilbert space, and let $D \subset H$ be a dense linear subspace. A linear operator $D$, defined on $D$, is said to be skew-symmetric iff (Def)

$$\langle Df, g \rangle_H + \langle f, Dg \rangle_H = 0$$

holds for all $f, g \in D$. If we introduce the adjoint operator $D^*$, then (2.24) is equivalent to the following containment of graphs:

$$D \subseteq -D^*. \quad \text{(2.25)}$$

As in the setting of von Neumann, the domain of $D^*$, $\text{dom}(D^*)$, is as follows:

$$\text{dom}(D^*) = \left\{ g \in H \mid \exists \text{const} C = C_g < \infty \text{ s.t.} \right\}$$

$$|\langle g, Df \rangle_H| \leq C \|f\|_H, \quad \forall f \in D.$$ \quad \text{(2.26)}

And the vector $D^*g$ satisfies

$$\langle D^*g, f \rangle_H = \langle g, Df \rangle_H$$

for all $f \in D$.

An extension $A$ of $D$ is said to be skew-adjoint iff (Def)

$$A = -A^*; \quad \text{(2.28)}$$
which is “=”, not merely containment, comparing (2.25).

We are interested in skew-adjoint extensions, since the Spectral Theorem applies to them; not to the operators which are merely skew-symmetric; see [DS88]. From the Spectral Theorem, we then get solutions to our original extension problem for locally defined p.d. functions.

But, in the general Hilbert space setting, skew-adjoint extensions need not exist. However we have the following:

**Theorem 2.2 (von Neumann [DS88]).** A skew-symmetric operator \( D \) has skew-adjoint extensions \( A \), i.e.,
\[
D \subseteq A \subseteq -D^* \tag{2.29}
\]
if and only if the following two subspaces of \( \mathcal{H} \) (deficiency spaces, see Definition 2.2) have equal dimension:
\[
\operatorname{DEF}^\pm (D) = \mathcal{N} (D^* \pm I),
\]
i.e., \( \dim \operatorname{DEF}^+ (D) = \dim \operatorname{DEF}^- (D) \), where
\[
\operatorname{DEF}^\pm (D) = \{ g_\pm \in \operatorname{dom}(D^*) \mid D^* g_\pm = \mp g_\pm \}.
\]

Below, we introduce \( D^{(F)} \) systematically and illustrate how to use the corresponding skew-adjoint extensions to extend locally defined p.d. functions.

**Definition 2.1.** Set
\[
\begin{align*}
\operatorname{dom}(D^{(F)}) &= \{ F_\psi \mid \psi \in C^\infty_c (\Omega) \}, \\
D^{(F)} F_\psi &= F_\psi', \quad \forall F_\psi \in \operatorname{dom}(D^{(F)}),
\end{align*}
\]
where \( \psi' (x) = d\psi / dx \), and \( F_\psi \) is as in (2.4).

Note that the recipe for \( D^{(F)} \) yields a well-defined operator with dense domain in \( \mathcal{H}_F \). To see this, use Schwarz’ lemma to show that if \( F_\psi = 0 \) in \( \mathcal{H}_F \), then it follows that the vector \( F_\psi' \in \mathcal{H}_F \) is 0 as well. An alternative proof is given in Lemma 2.5.

**Lemma 2.4.** The operator \( D^{(F)} \) is skew-symmetric and densely defined in \( \mathcal{H}_F \).

**Proof.** By Lemma 2.2, \( \operatorname{dom}(D^{(F)}) \) is dense in \( \mathcal{H}_F \). If \( \psi \in C^\infty_c (0, a) \) and \( |t| < \operatorname{dist} (\operatorname{supp} (\psi), \text{endpoints}) \), then
\[
\| F_{\psi'(\cdot + t)} \|_{\mathcal{H}_F}^2 = \| F_\psi \|_{\mathcal{H}_F}^2 = \int_0^a \int_0^a \overline{\psi(x)} \psi(y) F(x - y) \, dx \, dy \tag{2.30}
\]
by (2.6). Thus,
\[
\frac{d}{dt} \| F_{\psi'(\cdot + t)} \|_{\mathcal{H}_F}^2 = 0
\]
which is equivalent to
2.2 The Skew-Hermitian Operator $D^{(F)}$ in $\mathcal{H}_F$

\[(D^{(F)}F\psi, F\psi)_{\mathcal{H}_F} + (F\psi, D^{(F)}F\psi)_{\mathcal{H}_F} = 0.\] (2.31)

It follows that $D^{(F)}$ is skew-symmetric.

Lemma 2.5 below shows that $D^{(F)}$ is well-defined on its dense domain in $\mathcal{H}_F$. This finishes the proof of Lemma 2.4.

Lemma 2.5. The following implication holds:

\[\psi \in C^\infty_c(\Omega), F\psi = 0 \text{ in } \mathcal{H}_F\]

\[\Downarrow\]

\[F\psi = 0 \text{ in } \mathcal{H}_F\] (2.32)

Proof. Substituting (2.32) into

\[(F\phi, F\psi')_{\mathcal{H}_F} + (F\psi', F\phi)_{\mathcal{H}_F} = 0\]

we get

\[(F\phi, F\psi')_{\mathcal{H}_F} = 0, \quad \forall \phi \in C^\infty_c(\Omega).\]

Taking $\phi = \psi'$, yields

\[(F\psi', F\psi') = \|F\psi'\|_{\mathcal{H}_F}^2 = 0\]

which is the desired conclusion (2.33). Therefore, the operator $D^{(F)}$ is well-defined. □

Definition 2.2. Let $(D^{(F)})^*$ be the adjoint of $D^{(F)}$. The deficiency spaces $DEF^\pm$ consists of $\xi^\pm \in \mathcal{H}_F$, such that $(D^{(F)})^*\xi^\pm = \pm\xi^\pm$. That is,

\[DEF^\pm = \{\xi^\pm \in \mathcal{H}_F : (D^{(F)})^*\xi^\pm = \pm\xi^\pm, \forall \psi \in C^\infty_c(\Omega)\} .\]

Elements in $DEF^\pm$ are called defect vectors. The dimensions of $DEF^\pm$, i.e., the pair of numbers

\[d^\pm = \dim DE\pm,\]

are called the deficiency indices of $D^{(F)}$. See, e.g., [vN32a, Kre46, DS88, AG93, Nel69]. Example 2.2 above is an instance of deficiency indices $(1,1)$, i.e., $d^+ = d^- = 1$.

von Neumann showed that a densely defined Hermitian operator in a Hilbert space has equal deficiency indices if it commutes with a conjugation operator [DS88]. This criterion is adapted to our setting in Lemma 2.6 and its corollary.

The Case of Conjugations

The purpose of the section below is to show that, for a large class of locally defined p.d. functions $F$, it is possible to establish existence of skew-adjoint extensions of
the corresponding operator $D(F)$ with the use of a criterion of von Neumann: It states that, if a skew-Hermitian operator anti-commutes with a conjugation (a conjugate linear period-2 operator), then it must have equal deficiency indices, and therefore have skew-adjoint extensions. In the present case, the anti-commuting properties takes the form of (2.35) below.

**Lemma 2.6.** Let $Ω = (α, β)$. Suppose $F$ is a real-valued p.d. function defined on $Ω − Ω$. The operator $J$ on $H_F$ determined by

$$JF_φ = F_φ(α + β − x), \quad φ ∈ C_c^∞(Ω)$$

is a conjugation, i.e., $J$ is conjugate-linear, $J^2$ is the identity operator, and

$$\langle JF_φ, JF_ψ \rangle_{H_F} = \langle F_ψ, F_φ \rangle_{H_F}. \quad (2.34)$$

Moreover,

$$D(F)J = −JD(F). \quad (2.35)$$

**Proof.** Let $a := α + β$ and $φ ∈ C_c^∞(Ω)$. Since $F$ is real-valued, we have

$$JF_φ(x) = \int_α^β \overline{φ(a − y)} F(x − y)dy$$

$$= \int_α^β \overline{φ(y)} F(x − y)dy$$

where $ψ(y) := \overline{φ(a − y)}$ is in $C_c^∞(Ω)$. It follows that $J$ maps the the operator domain $\text{dom}(D(F))$ onto itself. For $φ, ψ ∈ C_c^∞(Ω)$,

$$\langle JF_φ, F_ψ \rangle_{H_F} = \int_α^β F_φ(a − y)ψ(x)dx$$

$$= \int_α^β \int_α^β φ(a − y)F(x − y)ψ(x)dydx.$$  

Making the change of variables $(x, y) → (a − x, a − y)$ and interchanging the order of integration we see that

$$\langle JF_φ, F_ψ \rangle_{H_F} = \int_α^β F_φ(a − y)ψ(a − x)dydx$$

$$= \int_α^β φ(y)F_ψ(a − y)dy$$

$$= \langle JF_ψ, F_φ \rangle_{H_F},$$

establishing (2.34).

Finally, for all $φ ∈ C_c^∞(Ω)$,
Let $F$ be real-valued. Let Corollary 2.5.

Equivalently, $F$ can decide this with the use of (2.9)(2.10).

This follows from Lemma 2.6, see e.g., [AG93] or [DS88].

We proceed to characterize the deficiency spaces of $D(F)$.

**2.2 The Skew-Hermitian Operator $D(F)$ in $\mathcal{H}_F$**

$$JD^{(F)}F_0 = F_{\phi}(a-x) = -\frac{F_{\phi}}{\bar{\phi}\phi}(a-x) = -D^{(F)}JF_0,$$

hence (2.35) holds. $\square$

**Corollary 2.4.** If $F$ is real-valued, then $DEF^+$ and $DEF^-$ have the same dimension.

**Proof.** This follows from Lemma 2.6, see e.g., [AG93] or [DS88]. $\square$

If $F$ is real-valued, then $DEF^+ = DEF^- = 0$.

**Lemma 2.7.** If $\xi \in DEF^\pm$ then $\xi(y) = \text{constant} \, e^y$.

**Proof.** Specifically, $\xi \in DEF^+$ if and only if

$$\int_0^a \psi'(y) \xi(y) \, dy = \int_0^a \psi(y) \xi(y) \, dy, \quad \forall \psi \in C_0^\infty(0,a).$$

Equivalently, $y \mapsto \xi(y)$ is a weak solution to the ODE $\xi' = -\xi$, i.e., a strong solution in $C^1$. Thus, $\xi(y) = \text{constant} \, e^y$. The $DEF^-$ case is similar. $\square$

**Corollary 2.5.** Suppose $F$ is real-valued. Let $\xi_{\pm}(y) := e^y$, for $y \in \Omega$. Then $\xi_+ \in \mathcal{H}_F$ iff $\xi_- \in \mathcal{H}_F$. In the affirmative case $\|\xi_-\|_{\mathcal{H}_F} = e^{\|\xi_+\|_{\mathcal{H}_F}}$.

**Proof.** Let $J$ be the conjugation from Lemma 2.6. A short calculation:

$$\langle J\xi, F_0 \rangle_{\mathcal{H}_F} = \langle F_{\phi}(a-x), \xi \rangle_{\mathcal{H}_F} = \int \phi(a-x)\xi(x) \, dx = \int \phi(x)\xi(a-x) \, dx = \langle \xi(a-x), F_0 \rangle_{\mathcal{H}_F}$$

shows that $(J\xi)(x) = \bar{\xi(a-x)}$, for $\xi \in \mathcal{H}_F$. In particular, $J\xi_+ = e^\xi_+$. Since $\|J\xi_-\|_{\mathcal{H}_F} = \|\bar{\xi}\|_{\mathcal{H}_F}$, the proof is easily completed. $\square$

**Corollary 2.6.** The deficiency indices of $D(F)$, with its dense domain in $\mathcal{H}_F$, are $(0,0)$, $(0,1)$, $(1,0)$, or $(1,1)$.

The second case in the above corollary happens precisely when $y \mapsto e^{-y} \in \mathcal{H}_F$. We can decide this with the use of (2.9)(2.10).

In Chapter 8 we will give some a priori estimates, which enable us to strengthen Corollary 2.6. For this, see Corollary 8.7.

**Remark 2.3.** Note that deficiency indices $(1,1)$ is equivalent to

$$\sum_i \sum_j c_ic_j e^{-(x_i+x_j)} \leq A_0 \sum_i \sum_j c_ic_j F(x_i - x_j) \quad \Leftrightarrow \quad \leq A_0 \int_0^a \int_0^a \psi(y) e^{-y} \, dy dx \quad \Leftrightarrow \quad \leq A_0 \int_0^a \int_0^a \psi(x) \psi(y) F(x-y) \, dx dy$$

But it depends on $F$ (given on $(-a,a)$).
Lemma 2.8. On $\mathbb{R} \times \mathbb{R}$, define the following kernel $K_+ (x, y) = e^{-|x+y|}$, $(x, y) \in \mathbb{R} \times \mathbb{R}$; then this is a positive definite kernel on $\mathbb{R}_+ \times \mathbb{R}_+$; (see [Aro50] for details on positive definite kernels.)

Proof. Let $\{c_j\} \subset \mathbb{C}^N$ be a finite system of numbers, and let $\{x_j\} \subset \mathbb{R}^N_+$. Then

$$\sum_j \sum_k c_j c_k e^{-(x_j + x_k)} = \left| \sum_j c_j e^{-x_j} \right|^2 \geq 0.$$---

Corollary 2.7. Let $F$, $\mathcal{H}_F$, and $D(F)$ be as in Corollary 2.6; then $D(F)$ has deficiency indices $(1, 1)$ if and only if the kernel $K_+ (x, y) = e^{-|x+y|}$ is dominated by $K_F (x, y) = F (x - y)$ on $(0, a) \times (0, a)$, i.e., there is a finite positive constant $A_0$ such that

$$A_0 K_F - K_+$$

is positive definite on $(0, a) \times (0, a)$.

Proof. This is immediate from the lemma and (2.36) above. ---

In a general setting the kernels with the properties from Corollary 2.7 are called reflection positive kernels. See Example 2.3, and Section 4.5. Their structure is accounted for by Theorem 8.4. Their applications includes the study of Gaussian processes, the theory of unitary representations of Lie groups, and quantum fields. The following references give a glimpse into this area of analysis, [JÔ98, JÔ00, Kle74]. See also the papers and books cited there.

Example 2.3. The following are examples of positive definite functions on $\mathbb{R}$ which are used in a variety of applications. We discuss in Section 7.1 how they arise as extensions of locally defined p.d. functions, and some of the applications. In the list below, $F : \mathbb{R} \to \mathbb{R}$ is p.d., and $a, b > 0$ are fixed constants.

1. $F (x) = e^{-a|x|}$; Sections 4.4 (p.92), 4.5 (p.96), 7.1 (p.161).
2. $F (x) = \frac{1 - e^{-b|x|}}{b|x|}$
3. $F (x) = \frac{1}{1 + |x|}$
4. $F (x) = \frac{1}{\sqrt{1 + |x|}} e^{-\frac{|x|}{1 + |x|}}$

These are known to be generators of Gaussian reflection positive processes [JÔ98, JÔ00, Kle74], i.e., having completely monotone covariance functions. For details of these functions, see Theorem 8.4 (the Bernstein-Widder Theorem). The only one among these which is also a Markov process is the one coming from $F (x) = e^{-a|x|}, x \in \mathbb{R}$, where $a > 0$ is a parameter; it is the Ornstein-Uhlenbeck process. Note that $F (x) = e^{-|x|}, x \in \mathbb{R}$, induces the positive definite kernel in Lemma 2.8, i.e., $K_+ (x, y) = e^{-(x+y)}, (x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$.
2.2 The Skew-Hermitian Operator $D^{(F)}$ in $\mathcal{H}_F$

For these reasons Example (1) in the above list shall receive relatively more attention than the other three. Another reason for Example (1) playing a prominent role is that almost all the central questions dealing with locally defined p.d. functions are especially transparent for this example.

More generally, a given p.d. function $F$ on $\mathbb{R}$ is said to be reflection positive if the induced kernel on $\mathbb{R}^+ \times \mathbb{R}^+$, given by

$$K_+ (x, y) = F (x + y), \quad (x, y) \in \mathbb{R}^+ \times \mathbb{R}^+,$$

is positive definite.

In the case of a given p.d. function $F$ on $\mathbb{R}^n$, the reflection positivity making reference to some convex cone $C^+_n$ in $\mathbb{R}^n$, and we say that $F$ is $C^+_n$-reflection positive if

$$K_+ (x, y) := F (x + y), \quad (x, y) \in C^+_n \times C^+_n,$$

is a positive definite kernel (on $C^+_n \times C^+_n$).

**Example 2.4.** Let $\Omega = (-1, 1)$ be the interval $-1 < x < 1$. The following consideration illustrates the difference between positive definite (p.d.) kernels and positive definite functions:

1. On $\Omega \times \Omega$, set $K(x, y) := \frac{1}{1 - xy}$.
2. On $\Omega$, set $F(x) := \frac{1}{1 - x^2}$.

Then $K$ is a p.d. kernel, but $F$ is not a p.d. function.

It is well known that $K$ is a p.d. kernel. The RKHS of $K$ consists of analytic functions $\xi(x) = \sum_{n=0}^{\infty} c_n x^n$, $|x| < 1$, such that $\sum_{n=0}^{\infty} |c_n|^2 < \infty$. In fact,

$$\|\xi\|^2_{\mathcal{H}_K} = \sum_{n=0}^{\infty} |c_n|^2.$$

To see that $F$ is not a p.d. function, one checks that the $2 \times 2$ matrix

$$\begin{pmatrix} F(0) & F(x) \\ F(-x) & F(0) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{1 - x^2} \\ \frac{1}{1 - x^2} & 1 \end{pmatrix}$$

has a negative eigenvalue $\lambda(x) = -\frac{x^2}{1 - x^2}$, when $x \in \Omega \setminus \{0\}$.

The following theorem, an inversion formula (see (2.37)), shows how the full domain, $-\infty < x < \infty$, of a continuous positive definite function $F$ is needed in determining the positive Borel measure $\mu$ which yields the Bochner-inversion, i.e., $F = \hat{\mu}$.

**Theorem 2.3 (An inversion formula (see e.g., [Akh65, LP89, DM76]).** Let $F$ be a continuous positive definite function on $\mathbb{R}$ such that $F(0) = 1$. Let $\mu$ be the Borel
probability measure s.t. $F = \hat{d}\mu$ (from Bochner’s theorem), and let $(a_0, b_0)$ be a finite open interval, with the two-point set $\{a_0, b_0\}$ of endpoints. Then

$$\mu(\{(a_0, b_0)\}) + \frac{1}{2} \left( \mu(\{a_0\}) + \mu(\{b_0\}) \right) = \frac{1}{2\pi} \lim_{T \to \infty} \int_{-T}^{T} e^{-ia_0x} - e^{-ib_0x} \frac{F(x)}{ix} \, dx.$$  \hfill (2.37)

(The second term on the l.h.s. in (2.37) is $(\mu(\{a_0\}) + \mu(\{b_0\})) / 2$.)

**Proof.** The proof details are left to the reader. They are straightforward, and also contained in many textbooks on harmonic analysis. \hfill \Box

**Remark 2.4.** Suppose $F$ is continuous and positive definite, but is only known on a finite centered interval $(-a, a)$, $a > 0$. Formula (2.37) now shows how distinct positive definite extensions $\tilde{F}$ (to $\mathbb{R}$) for $F$ on $(-a, a)$ yields distinct measures $\mu_{\tilde{F}}$. However, a measure $\mu$ cannot be determined (in general) from $F$ alone, i.e., from $x$ in a finite interval $(-a, a)$.

### 2.2.1 Illustration: $G = \mathbb{R}$, correspondence between the two extension problems

Extensions of continuous p.d. functions vs extensions of operators. Illustration for the case of $(-a, a) \subset \mathbb{R}$, i.e., given $F: (-a, a) \to \mathbb{C}$, continuous and positive definite:

We illustrate how to use the correspondence

$$\text{extensions of p.d. functions} \longleftrightarrow \text{extensions of operators}$$

to get from $F(t)$ on $|t| < a$ to $\tilde{F}(t)$, $t \in \mathbb{R}$, with $\tilde{F} = \hat{d}\mu$.

Figure 2.3 illustrates the extension correspondence (p.d. function vs extension operator) in the case of Type I, but each step in the correspondence carries over to the Type II case. The main difference is that for Type II, one must pass to a dilation Hilbert space, i.e., a larger Hilbert space containing $\mathcal{H}_F$ as an isometric copy.

Notations:

- $\{U(t)\}_{t \in \mathbb{R}}$: unitary one-parameter group with generator $A^{(F)} \supset D^{(F)}$
- $P_U(\cdot)$: the corresponding projection-valued measure (PVM)
- $U(t) = \int_{\mathbb{R}} e^{it\lambda} P_U(d\lambda)$
- $d\mu(x) = \|P_U(dx)F_0\|_{\mathcal{H}_F}^2$
- An extension: $\tilde{F}(t) = \hat{d}\mu = \int_{\mathbb{R}} e^{ix} d\mu(x) = \langle F_0, U(t)F_0 \rangle_{\mathcal{H}_F}$

For the more general non-commutative correspondence (extension of p.d. vs operator extension), we refer to Section 3.2.1, the GNS-construction. Compare with Figure 3.3.
2.2 The Skew-Hermitian Operator $D^{(F)}$ in $\mathcal{H}_F$

Next, we flesh out in detail the role of $D^{(F)}$ in extending a given p.d. function $F$, defined on $\Omega - \bar{\Omega}$. This will be continued in Section 2.3 in a more general setting.

By Corollary 2.6, we conclude that there exists skew-adjoint extension $A^{(F)} \supset D^{(F)}$ in $\mathcal{H}_F$. That is, $\text{dom}(D^{(F)}) \subset \text{dom}(A^{(F)}) \subset \mathcal{H}_F$, $(A^{(F)})^* = -A^{(F)}$, and $D^{(F)} = A^{(F)} |_{\text{dom}(D^{(F)})}$.

Given a skew-adjoint extension $A^{(F)} \supset D^{(F)}$, set $U(t) = e^{tA^{(F)}} : \mathcal{H}_F \to \mathcal{H}_F$, and get the unitary one-parameter group

$$\{U(t) : t \in \mathbb{R}\}, \quad U(s + t) = U(s)U(t), \quad \forall s, t \in \mathbb{R};$$

and if

$$\xi \in \text{dom}(A^{(F)}) = \left\{ \xi \in \mathcal{H}_F \mid \text{s.t. } \lim_{t \to 0} \frac{U(t)\xi - \xi}{t} \text{ exists} \right\},$$

then

$$A^{(F)}\xi = \lim_{t \to 0} \frac{U(t)\xi - \xi}{t}. \quad \text{(2.38)}$$

Now let

$$\tilde{F}_A(t) := (F_0, U(t)F_0)_{\mathcal{H}_F}, \quad \forall t \in \mathbb{R}. \quad \text{(2.39)}$$

Using (2.8), we see that $\tilde{F}_A(t)$, defined on $\mathbb{R}$, is a continuous p.d. extension of $F$.

**Lemma 2.9.** $\tilde{F}_A(t)$ as in (2.39) is a continuous bounded p.d. function of $\mathbb{R}$, and

$$\tilde{F}_A(t) = F(t), \quad t \in (-a, a). \quad \text{(2.40)}$$

**Proof.** Since $\{U(t)\}$ is a strongly continuous unitary group acting on $\mathcal{H}_F$, we have

Fig. 2.3: Extension correspondence. From locally defined p.d $F$ to $D^{(F)}$, to a skew-adjoint extension, and the unitary one-parameter group, to spectral resolution, and finally to an associated element in $\text{Ext}(F)$.
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\[ |\tilde{F}_A(t)| = |\langle F_0, U(t)F_0 \rangle| \leq \|F_0\|_{\mathcal{H}_F} \|U(t)F_0\|_{\mathcal{H}_F} = \|F_0\|_{\mathcal{H}_F}^2 = F(0), \]

by (2.3). Recall that \( F_x = F(\cdot - x), x \in \Omega = (0, a) \). This shows that every \( F^{ext} \) is bounded and continuous.

The proof that \( \tilde{F}_A(t) \) indeed extends \( F \) to \( \mathbb{R} \) holds in a more general context, see e.g., Theorem 2.4 and [Jor89, Jor90, Jor91]. \( \square \)

Recall that \( F \) can always be normalized by \( F(0) = 1 \). Consider the spectral representation:

\[ U(t) = e^{itA^{(F)}} = \int_{-\infty}^{\infty} e^{i\lambda t} P(d\lambda) \] (2.41)

where \( P(\cdot) \) is the projection-valued measure of \( A^{(F)} \). Thus, \( P(B) : \mathcal{H}_F \to \mathcal{H}_F \) is a projection, for all Borel subsets \( B \) in \( \mathbb{R} \). Setting

\[ d\mu(\lambda) = \|P(d\lambda)F_0\|_{\mathcal{H}_F}^2 \] (2.42)

then the corresponding extension is as follows:

\[ \tilde{F}_A(t) = \int_{-\infty}^{\infty} e^{i\lambda t} d\mu(\lambda) = \tilde{d\mu}(t), \quad \forall t \in \mathbb{R}. \] (2.43)

**Conclusion.** The extension \( \tilde{F}_A(t) \) from (2.39) has nice transform properties, and via (2.43) we get

\[ \mathcal{H}_{\tilde{F}_A} \simeq L^2(\mathbb{R}, \mu) \]

with the transformation \( T_\mu \) of Theorem 2.4; where \( \mathcal{H}_{\tilde{F}_A} \) is the RKHS of the extended p.d. function \( \tilde{F}_A \). The explicit transform \( T_\mu \) realizing \( \mathcal{H}_F \to L^2(\mu) \) is given in Corollary 3.1.

**Corollary 2.8.** Let a partially defined p.d. function \( F \) be given as in Figure 2.3. Let \( D^{(F)} \) be the associated skew-Hermitian operator in \( \mathcal{H}_F \), and assume that it has a skew-adjoint extension \( A \). Let \( P_A(\cdot) \) and \( \mu_A(\cdot) \) be the measures from (2.41) and (2.42), then \( \tilde{F}_A = \tilde{d\mu}_A \) is a Type I extension, and

\[ \text{supp}(\mu_A) = i \cdot \text{spec}(A), \quad i = \sqrt{-1}. \]

**Proof.** Immediate from the proof of Lemma 2.9. \( \square \)

**Remark 2.5.** In the circle case \( G = \mathbb{T} \), the extension \( \tilde{F}_A \) in (2.39) needs not be \( \mathbb{Z} \)-periodic.

Consider \( a > 0, \Omega = (0, a) \), and a continuous p.d. function \( F \) on \( (-a, a) \). Let \( D^{(F)} \) be the corresponding skew-Hermitian operator in \( \mathcal{H}_F \). We proved that for every skew-adjoint extension \( A \supset D^{(F)} \) in \( \mathcal{H}_F \), the corresponding p.d. function

\[ F_A(t) = \langle F_0, e^{itA}F_0 \rangle_{\mathcal{H}_F} = \int_{\mathbb{R}} e^{i\lambda t} \|P_A(d\lambda)F_0\|_{\mathcal{H}_F}^2, \quad t \in \mathbb{R} \] (2.44)
is a Type I p.d. extension of $F$; see formulas (2.42)-(2.43).

**Proposition 2.1.** Let $F$ be continuous and p.d. on $(-a,a)$; and let $\tilde{F}$ be a Type I positive definite extension to $\mathbb{R}$, i.e.,

$$F(t) = \tilde{F}(t), \quad \forall t \in (-a,a).$$

Then there is a skew-adjoint extension $A$ of $D(F)$ such that $\tilde{F} = F_A$ on $\mathbb{R}$; see (2.44) above.

**Proof.** By the definition of Type I, we know that there is a strongly continuous unitary one-parameter group $\{U(t)\}_{t \in \mathbb{R}}$ acting in $H_F$ such that

$$F(t) = \langle F_0, U(t)F_0 \rangle_{H_F}, \quad \forall t, |t| < a.$$ (2.46)

By Stone’s Theorem, there is a unique skew-adjoint operator $A$ in $H_F$ such that $U(t) = e^{tA}$, $\forall t \in \mathbb{R}$; and so the r.h.s. is $F_A = \tilde{F}$; i.e., the given Type I extension $\tilde{F}$ has the form $F_A$. But differentiation, $d/dt$ at $t = 0$, in (2.46) shows that $A$ is an extension of $D(F)$ which is the desired conclusion. □

### 2.3 Enlarging the Hilbert Space

The purpose of this section is to describe the dilation-Hilbert space in detail, and to prove some lemmas which will then be used in Chapter 5. In Chapter 5, we identify extensions of the initial positive definite (p.d.) function $F$ which are associated with operator extensions in $H_F$ (Type I), and those which require an enlargement of $H_F$ (Type II).

To simplify notations, results in this section are formulated for $G = \mathbb{R}$. The modification for general Lie groups is straightforward, and are left for the reader.

Fix $a > 0$, and $\Omega = (0,a) \subset \mathbb{R}$. Let $F: \Omega - \Omega \to \mathbb{C}$ be a continuous p.d. function. Recall the corresponding reproducing kernel Hilbert space (RKHS) $H_F$ is the completion of $\text{span} \{F_x := F(\cdot - x) \mid x \in \Omega\}$ with respect to the inner product

$$\langle F_x, F_y \rangle_{H_F} := F(x) = F(x-y), \quad \forall x, y \in \Omega,$$ (2.47)

modulo elements of zero $H_F$-norm.

Following Section 2.1, let $F_\varphi = \varphi * F$ be the convolution, where

$$F_\varphi (x) = \int_{\Omega} \varphi(y) F(x-y) dy, \quad x \in \Omega, \varphi \in C_c^\infty(\Omega).$$

Setting

$$\pi(\varphi) F_0 = F_\varphi, \text{ and } \varphi^\#(x) = \overline{\varphi(-x)},$$

we may write
\[ \langle F\phi, F\psi \rangle_{\mathcal{H}} = \langle F_0, \pi(\phi^* \ast \psi) F_0 \rangle_{\mathcal{H}} = \langle \pi(\phi) F_0, \pi(\psi) F_0 \rangle_{\mathcal{H}}, \]  

(2.48)

The following theorem also holds in \( \mathbb{R}^n \) with \( n > 1 \). It is stated here for \( n = 1 \) to illustrate the “enlargement” of \( \mathcal{H}_F \) question.

**Theorem 2.4.** The following two conditions are equivalent:

1. \( F \) is extendable to a continuous p.d. function \( \tilde{F} \) defined on \( \mathbb{R} \), i.e., \( \tilde{F} \) is a continuous p.d. function defined on \( \mathbb{R} \) and \( F(x) = \tilde{F}(x) \) for all \( x \) in \( \Omega - \Omega \).

2. There is a Hilbert space \( \mathcal{K} \), an isometry \( W : \mathcal{H}_F \rightarrow \mathcal{K} \), and a strongly continuous unitary group \( U_t : \mathcal{K} \rightarrow \mathcal{K}, t \in \mathbb{R} \), such that if \( A \) is the skew-adjoint generator of \( U_t \), i.e.,

\[
\lim_{t \to 0} \frac{1}{t} (U_t k - k) = Ak, \quad \forall k \in \text{dom}(A); \tag{2.49}
\]

then \( \forall \phi \in C_0^\infty(\Omega) \), we have

\[
WF\phi \in \text{dom}(A), \quad \text{and} \quad AWF\phi = WF\phi'; \tag{2.50}
\]

The rest of this section is devoted to the proof of Theorem 2.4.

**Lemma 2.10.** Let \( W, \mathcal{H}_F, \mathcal{K} \) and \( U_t \) be as in the theorem. If \( s, t \in \Omega \), and \( \phi \in C_0^\infty(\Omega) \), then

\[
U_t W F\phi = WF\phi, \quad \text{and} \quad \langle WF\phi, U_t W F\phi \rangle_{\mathcal{H}} = \langle F_0, \pi(\phi^* \ast \phi) F_0 \rangle_{\mathcal{H}}, \tag{2.53}
\]

where \( \phi_t(\cdot) = \phi(\cdot + t) \).

**Proof.** We first establish (2.52). Consider

\[
U_{t-s} W F\phi = \begin{cases} 
U_t W F\phi & \text{at } s = 0 \\
WF\phi & \text{at } s = t
\end{cases} \tag{2.54}
\]

and

\[
\int_0^t \frac{d}{ds} (U_{t-s} W F\phi) \, ds = WF\phi - U_t W F\phi. \tag{2.55}
\]

Note that the left-side of (2.55) equals zero. Indeed, (2.49) and (2.50) imply that

\[
\frac{d}{dt} (U_{t-s} W F\phi) = -U_{t-s} AWF\phi + U_{t-s} WF\phi'. \tag{2.56}
\]

But (2.51) applied to \( \phi_t \) yields

\[
AWF\phi_t = WF\phi_t',
\]

and so (2.56) is identically zero. The desired conclusion (2.52) follows.

Moreover, we have
2.3 Enlarging the Hilbert Space

\[ l.h.s. \ (2.53) \quad \text{by} \ (2.52) \quad = \langle WF_\phi, WF_\phi \rangle_{\mathcal{H}} \]
\[ W \text{ is isom.} \quad = \langle F_\phi, F_\phi \rangle_{\mathcal{H}_F} \]
\[ \text{by} \ (2.48) \quad = \langle F_0, \pi (\varphi^\# \ast \varphi_t) F_0 \rangle_{\mathcal{H}_F} \]
\[ = r.h.s. \ (2.53). \]

□

Proof. (The proof of Theorem 2.4)

(2)⇒(1) Assume there exist \( \mathcal{H}, W, U_t \) and \( A \) as in (2). Let
\[ \tilde{F}(t) = \langle WF_0, U_t WF_0 \rangle, \quad t \in \mathbb{R}. \] (2.57)
By the Spectral Theorem, \( U_t = \int_\mathbb{R} e^{i \lambda t} P(d\lambda), \) where \( P(\cdot) \) is the corresponding projection-valued measure. Setting
\[ d\mu(\lambda) := \|P(d\lambda)WF_0\|^2_{\mathcal{H}} = \langle WF_0, P(d\lambda)WF_0 \rangle_{\mathcal{H}}, \]
then \( \tilde{F} = \hat{d}\mu, \) i.e., \( \tilde{F} \) is the Bochner transform of the Borel measure \( d\mu \) on \( \mathbb{R}. \)

Let \( \phi^{(\varepsilon)}, \varepsilon > 0, \) be an approximate identity at \( x = 0. \) (That is, \( \phi^{(\varepsilon)} \to \delta_0, \) as \( \varepsilon \to 0^+; \) see Lemma 2.3 for details.) Then
\[ \tilde{F}(t) = \langle WF_0, U_t WF_0 \rangle_{\mathcal{H}} \]
\[ = \lim_{\varepsilon \to 0^+} \langle WF_0, U_t WF_{\phi^{(\varepsilon)}} \rangle_{\mathcal{H}} \]
\[ \text{by} \ (2.52) \quad = \lim_{\varepsilon \to 0^+} \langle WF_0, WF_{\phi^{(\varepsilon)}} \rangle_{\mathcal{H}} \]
\[ = \lim_{\varepsilon \to 0^+} \langle F_0, F_{\phi^{(\varepsilon)}} \rangle_{\mathcal{H}_F} \]
\[ = \langle F_0, F_{-t} \rangle_{\mathcal{H}_F} \text{ by} \ (2.47) \quad = F(t), \quad t \in \Omega - \Omega; \] (2.58)

Therefore, \( \tilde{F} \) is a continuous p.d. extension of \( F \) to \( \mathbb{R}. \)

(1)⇒(2) Let \( \tilde{F} = \hat{d}\mu \) be a p.d. extension and Bochner transform. Define \( W : \mathcal{H}_F \to \mathcal{H}_F, \) by
\[ WF_\varphi = \tilde{F}_\varphi, \quad \varphi \in C_c^\infty(\Omega). \] (2.59)

Then \( W \) is an isometry and \( \mathcal{H}_F \preceq L^2(\mu). \) Indeed, for all \( \varphi \in C_c^\infty(\Omega), \) since \( \tilde{F} \) is an extension of \( F, \) we have
\[ \|\mathcal{F}_\varphi\|_{\mathcal{H}}^2 = \int_{\Omega} \int_{\Omega} \overline{\varphi(s)} \varphi(t) F(s-t) ds dt = \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\varphi(s)} \varphi(t) \tilde{F}(s-t) ds dt = \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(s) \varphi(t) \left( \int_{-\infty}^{\infty} e^{i\lambda(s-t)} d\mu(\lambda) \right) ds dt (\text{Fubini}) = \int_{\mathbb{R}} |\tilde{\varphi}(\lambda)|^2 d\mu(\lambda) = \|\tilde{F}_\varphi\|_{\mathcal{H}}^2. \]

Now let \( L^2(\mu) \ni f \xrightarrow{U_t} e^{i\lambda t} f(\lambda) \in L^2(\mu) \) be the unitary group acting in \( \mathcal{H} \cong L^2(\mu) \), and let \( A \) be its generator. Then,

\[ (WF_\varphi)(x) = \int e^{i\lambda x} \tilde{\varphi}(\lambda) d\mu(\lambda), \quad \forall x \in \Omega, \forall \varphi \in C_0^\infty(\Omega); \quad (2.60) \]

and it follows that

\[ (WF_\varphi')(x) = \int e^{i\lambda x} i\lambda \tilde{\varphi}(\lambda) d\mu(\lambda) = \frac{d}{dt} \bigg|_{t=0} \left( U_t WF_\varphi \right)(x) = (AWF_\varphi)(x) \]

as claimed. This proves part (2) of the theorem. \( \square \)

An early instance of dilations (i.e., enlarging the Hilbert space) is the theorem by Sz.-Nagy [RSN56, Muh74] on unitary dilations of strongly continuous semigroups.

We mention this result here to stress that p.d. functions on \( \mathbb{R} \) (and subsets of \( \mathbb{R} \)) often arise from contraction semigroups.

**Theorem 2.5 (Sz.-Nagy).** Let \( \{S_t \mid t \in \mathbb{R}_+\} \) be a strongly continuous semigroup of contractive operator in a Hilbert space \( \mathcal{H} \); then there is

1. a Hilbert space \( \mathcal{H} \),
2. an isometry \( V : \mathcal{H} \to \mathcal{H} \),
3. a strongly continuous one-parameter unitary group \( \{U(t) \mid t \in \mathbb{R}\} \) acting on \( \mathcal{H} \) such that

\[ V S_t = U(t) V, \quad \forall t \in \mathbb{R}_+ \]

(Sz.-Nagy also proved the following:

**Theorem 2.6 (Sz.-Nagy).** Let \( \{S_t, \mathcal{H}\} \) be a contraction semigroup, \( t \geq 0 \), (such that \( S_0 = I_{\mathcal{H}} \)) and let \( f_0 \in \mathcal{H} \setminus \{0\} \); then the following function \( F \) on \( \mathbb{R} \) is positive definite:

\[ F(t) = \begin{cases} (f_0, S_t f_0)_{\mathcal{H}} & \text{if } t \geq 0, \\ (f_0, S_{-t} f_0)_{\mathcal{H}} & \text{if } t < 0. \end{cases} \]

(2.62)
Corollary 2.9. Every p.d. function as in (2.62) has the form:
\[ F(t) := \langle k_0, U(t)k_0 \rangle_\mathcal{K}, \quad t \in \mathbb{R} \]  
(2.63)
where \((U(t), \mathcal{K})\) is a unitary representation of \(\mathbb{R}\).

2.4 Ext\(_1\)(F) and Ext\(_2\)(F)

Let \(G\) be a locally compact group, and \(\Omega\) an open connected subset of \(G\). Consider a continuous p.d. function \(F: \Omega^{-1} \cdot \Omega \to \mathbb{C}\).

We shall study the two sets of extensions in the title of this section.

Definition 2.3. We say that \((U, k_0) \in \text{Ext}(F)\) iff
1. \(U\) is a strongly continuous unitary representation of \(G\) in the Hilbert space \(\mathcal{K}\), containing the RKHS \(\mathcal{H}_F\); and
2. there exists \(k_0 \in \mathcal{K}\) such that
\[ F(g) = \langle k_0, U(g)k_0 \rangle_\mathcal{K}, \quad \forall g \in \Omega^{-1} \cdot \Omega. \]  
(2.64)

Definition 2.4. Let \(\text{Ext}_1(F) \subset \text{Ext}(F)\) consisting of \((U, \mathcal{K}, F_e)\) with
\[ F(g) = \langle F_e, U(g)F_e \rangle_{\mathcal{K}^e}, \quad \forall g \in \Omega^{-1} \cdot \Omega; \]  
(2.65)
where \(F_e \in \mathcal{K}_e\) satisfies \(\langle F_e, \xi \rangle_{\mathcal{K}_e} = \xi(e), \forall \xi \in \mathcal{K}_e\), and \(e\) denotes the neutral (unit) element in \(G\), i.e., \(eg = g, \forall g \in G\).

Definition 2.5. Let \(\text{Ext}_2(F) := \text{Ext}(F) \setminus \text{Ext}_1(F)\), consisting of the solutions to problem (2.64) for which \(\mathcal{K} \supset \mathcal{K}_e\), i.e., unitary representations realized in an enlargement Hilbert space.

Remark 2.6. When \(G = \mathbb{R}^n\), and \(\Omega \subset \mathbb{R}^n\) is open and connected, we consider continuous p.d. functions \(F: \Omega - \Omega \to \mathbb{C}\). In this special case, we have
\[ \text{Ext}(F) = \left\{ \mu \in \mathcal{M}_+(\mathbb{R}^n) \mid \tilde{\mu}(x) = \int_{\mathbb{R}^n} e^{i\lambda \cdot x} d\mu(\lambda) \right\} \]  
(2.66)

is a p.d. extension of \(F\).

Note that (2.66) is consistent with (2.64). In fact, if \((U, \mathcal{K}, k_0)\) is a unitary representation of \(G = \mathbb{R}^n\), such that (2.64) holds; then, by a theorem of Stone, there is a projection-valued measure (PVM) \(P_U(\cdot)\), defined on the Borel subsets of \(\mathbb{R}^n\) s.t.
\[ U(x) = \int_{\mathbb{R}^n} e^{i\lambda \cdot x} P_U(d\lambda), \quad x \in \mathbb{R}^n. \]  
(2.67)
Extensions of Continuous Positive Definite Functions

\[ d\mu (\lambda) := \| P_U (d\lambda) k_0 \|^2_{\mathcal{K}}, \quad (2.68) \]

it is then immediate that \( \mu \in \mathcal{M}_+ (\mathbb{R}^n) \), and the finite measure \( \mu \) satisfies

\[ \hat{\mu} (x) = F (x), \quad \forall x \in \Omega - \Omega. \quad (2.69) \]

The Case of \( n = 1 \)

Fix \( a > 0 \), and \( \Omega = (0, a) \subset \mathbb{R} \). Start with a local continuous p.d. function \( F : \Omega \rightarrow \mathbb{C} \), and let \( \mathcal{H}_F \) be the corresponding RKHS. Let \( \text{Ext}(F) \) be the compact convex set of probability measures on \( \mathbb{R} \) defining extensions of \( F \); see (2.66).

We see in Section 2.3 that all continuous p.d. extensions of \( F \) come from strongly continuous unitary representations. So in the case of 1D, from unitary one-parameter groups of course, say \( U(t) \). Further recall that some of the p.d. extensions of \( F \) may entail a bigger Hilbert space, say \( \mathcal{K} \). By this we mean that \( \mathcal{K} \) creates a dilation (enlargement) of \( \mathcal{H}_F \) in the sense that \( \mathcal{H}_F \) is isometrically embedded in \( \mathcal{K} \). Via the embedding we may therefore view \( \mathcal{H}_F \) as a closed subspace in \( \mathcal{K} \).

We now divide \( \text{Ext}(F) \) into two parts, say \( \text{Ext}_1(F) \) and \( \text{Ext}_2(F) \). \( \text{Ext}_1(F) \) is the subset of \( \text{Ext}(F) \) corresponding to extensions when the unitary representation \( U(t) \) acts in \( \mathcal{H}_F \) (internal extensions), and \( \text{Ext}_2(F) \) is the part of \( \text{Ext}(F) \) associated to unitary representations \( U(t) \) acting in a proper enlargement Hilbert space \( \mathcal{K} \) (if any), i.e., acting in a Hilbert space \( \mathcal{K} \) corresponding to a proper dilation. For example, the Pólya extensions in Chapter 5 account for a part of \( \text{Ext}_2(F) \).

Now consider the canonical skew-Hermitian operator \( D^{(F)} \) in the RKHS \( \mathcal{H}_F \) (Definition 2.1), i.e.,

\[ D^{(F)} (F_\varphi) = F_{\varphi'}, \quad \text{where} \]

\[ F_{\varphi'} (x) = \int_{\Omega} \varphi (y) F (x - y) dy, \quad \varphi \in C_c^\infty (\Omega). \quad (2.70, 2.71) \]

As shown in Section 2.1, \( D^{(F)} \) defines a skew-Hermitian operator with dense domain in \( \mathcal{H}_F \). Moreover, the deficiency indices for \( D^{(F)} \) can be only \((0, 0)\) or \((1, 1)\). The role of deficiency indices in the RKHS \( \mathcal{H}_F \) is as follows:

**Theorem 2.7.** The deficiency indices computed in \( \mathcal{H}_F \) are \((0, 0)\) if and only if \( \text{Ext}_1(F) \) is a singleton.

**Remark 2.7.** Even if \( \text{Ext}_1(F) \) is a singleton, we can still have non-empty \( \text{Ext}_2(F) \). In Chapter 5, we include a host of examples, including one with a Pólya extension where \( \mathcal{K} \) is infinite dimensional, while \( \mathcal{H}_F \) is 2 dimensional. (When dim \( \mathcal{H}_F = 2 \), obviously we must have deficiency indices \((0, 0)\).) In other examples we have \( \mathcal{H}_F \) infinite dimensional, non-trivial Pólya extensions, and yet deficiency indices \((0, 0)\).
Comparison of p.d. Kernels

We conclude the present section with some results on comparing positive definite kernels.

We shall return to the comparison of positive definite functions in Chapter 8. More detailed results on comparison, and in wider generality, will be included there.

**Definition 2.6.** Let \( K_i, i = 1, 2, \) be two p.d. kernels defined on some product \( S \times S \) where \( S \) is a set. We say that \( K_1 \ll K_2 \) iff there is a finite constant \( A \) such that

\[
\sum_i \sum_j c_i c_j K_1 (s_i, s_j) \leq A \sum_i \sum_j c_i c_j K_2 (s_i, s_j)
\]  

(2.72)

for all finite system \( \{c_i\} \) of complex numbers.

If \( F_i, i = 1, 2, \) are p.d. functions defined on a subset of a group, then we say that \( F_1 \ll F_2 \) iff the two kernels

\[
K_i (x, y) := K_i (x - y), i = 1, 2
\]

satisfy the condition in (2.72).

**Lemma 2.11.** Let \( \mu_i \in \mathcal{M}_+ (\mathbb{R}^n), i = 1, 2, \) i.e., two finite positive Borel measures on \( \mathbb{R}^n, \) and let \( \hat{F}_i := \hat{\mu}_i \) be the corresponding Bochner transforms. Then the following two conditions are equivalent:

(1) \( \mu_1 \ll \mu_2 \) (meaning absolutely continuous) with \( \frac{d\mu_1}{d\mu_2} \in L^1 (\mu_2) \cap L^\infty (\mu_2). \)

(2) \( F_1 \ll F_2, \) referring to the order of positive definite functions on \( \mathbb{R}^n. \)

**Proof.** (1)\( \Rightarrow \) (2) By assumption, there exists \( g \in L^2_+ (\mathbb{R}^n, \mu_2), \) the Radon-Nikodym derivative, s.t.

\[
d\mu_1 = gd\mu_2.
\]  

(2.73)

Let \( \{c_i\}_1^N \subset \mathbb{C} \) and \( \{x_i\}_1^N \subset \mathbb{R}^n, \) then

\[
\sum_i \sum_k c_i c_k F_i (x_i - x_k) = \int_{\mathbb{R}^n} \left| \sum_j c_j e^{ix_j} \right|^2 d\mu_1 (\lambda) = \int_{\mathbb{R}^n} \left| \sum_j c_j e^{ix_j} \right|^2 g (\lambda) d\mu_2 (\lambda) \quad \text{(by (2.73) & (1))}
\]

\[
\leq \|g\|_{L^\infty (\mu_2)} \int_{\mathbb{R}^n} \left| \sum_j c_j e^{ix_j} \right|^2 d\mu_2 (\lambda) \leq \|g\|_{L^\infty (\mu_2)} \sum_i \sum_k c_i c_k F_2 (x_i - x_k);
\]

which is the desired estimate in (2.72), with \( A = \|g\|_{L^\infty (\mu_2)}. \)

(2)\( \Rightarrow \) (1) Conversely, \( \exists A < \infty \) s.t. for all \( \phi \in C_c (\mathbb{R}^n), \) we have

\[
\iint \phi (x) \phi (y) F_1 (x - y) dxdy \leq A \iint \phi (x) \phi (y) F_2 (x - y) dxdy.
\]  

(2.74)
Remark 2.8. Note that (2.76) holds iff we have containment satisfies the condition in (2.76).

Lemma 2.12. Let $\mu_i \in \mathcal{M}_+ (\mathbb{R}^n)$, $i = 1, 2$, i.e., two finite positive Borel measures on $\mathbb{R}^n$, and let $F_i = d\mu_i$ be the corresponding Bochner transforms. Then the following two conditions are equivalent:

1. $\mu_1 \ll \mu_2$ (meaning absolutely continuous) with $\frac{d\mu_1}{d\mu_2} \in L^1 (\mu_2) \cap L^\infty (\mu_2)$.
2. $F_1 \ll F_2$, referring to the order of positive definite functions on $\mathbb{R}^n$.

Proof. (1) $\Rightarrow$ (2) By assumption, there exists $g \in L^2_+ (\mathbb{R}^n, \mu_2)$, the Radon-Nikodym derivative, s.t.

$$d\mu_1 = g d\mu_2.$$  \hfill (2.77)

Let $\{c_j\}_1^N \subset \mathbb{C}$ and $\{x_i\}_1^N \subset \mathbb{R}^n$, then

$$\sum_j \sum_k \overline{c_j} c_k F_1 (x_j - x_k) = \int_{\mathbb{R}^n} \left| \sum_j c_j e^{i \lambda x} \right|^2 d\mu_1 (\lambda)$$

$$= \int_{\mathbb{R}^n} \left| \sum_j c_j e^{i \lambda x} \right|^2 g (\lambda) d\mu_2 (\lambda) \quad \text{(by (2.77) & (1))}$$

$$\leq \|g\|_{L^2(\mu_2)} \int_{\mathbb{R}^n} \left| \sum_j c_j e^{i \lambda x} \right|^2 d\mu_2 (\lambda)$$

$$= \|g\|_{L^2(\mu_2)} \sum_j \sum_k \overline{c_k} c_k F_2 (x_j - x_k);$$
which is the desired estimate in (2.76), with \( A = \|g\|_{L^2(\mu_2)} \).

(2)\(\Rightarrow\)(1) Conversely, \( \exists A < \infty \) s.t. for all \( \varphi \in C_c(\mathbb{R}^n) \), we have

\[
\int \overline{\varphi(x)} \varphi(y) F_1(x - y) \, dx \, dy \leq A \int \overline{\varphi(x)} \varphi(y) F_2(x - y) \, dx \, dy.
\]  \hfill (2.78)

Using that \( F_t = \hat{d}\mu_t \), eq. (2.78) is equivalent to

\[
\int_{\mathbb{R}^n} |\hat{\varphi}(\lambda)|^2 \, d\mu_1(\lambda) \leq A \int_{\mathbb{R}^n} |\hat{\varphi}(\lambda)|^2 \, d\mu_2(\lambda),
\]  \hfill (2.79)

where \( |\hat{\varphi}(\lambda)|^2 = \varphi \ast \varphi^\dagger(\lambda), \lambda \in \mathbb{R}^n \).

Since \( \{ \varphi \mid \varphi \in C_c(\mathbb{R}^n) \} \cap L^1(\mathbb{R}^n, \mu) \) is dense in \( L^1(\mathbb{R}^n, \mu) \), for all \( \mu \in \mathcal{M}_+(\mathbb{R}^n) \),
we conclude from (2.79) that \( \mu_1 \ll \mu_2 \). Moreover, \( \frac{d\mu_1}{d\mu_2} \in L^1_+(\mathbb{R}^n, \mu_2) \cap L^{\infty}(\mathbb{R}^n, \mu_2) \)
by the argument in the first half of the proof. \( \square \)

### 2.5 Spectral Theory of \( D^{(F)} \) and its Extensions

In this section we return to \( n = 1 \), so a given continuous positive definite function \( F \), defined in an interval \((-a, a)\) where \( a > 0 \) is fixed. We shall study spectral theoretic properties of the associated skew-Hermitian operator \( D^{(F)} \) in the RKHS \( \mathcal{H}_F \) from Definition 2.1 in Section 2.2, and Theorem 2.7 above.

**Proposition 2.2.** Fix \( a > 0 \), and set \( \Omega = (0, a) \). Let \( F : \Omega - \Omega \to \mathbb{C} \) be continuous, p.d. and \( F(0) = 1 \). Let \( D^{(F)} \) be the skew-Hermitian operator, i.e., \( D^{(F)}(F\varphi) = F\varphi' \), for all \( \varphi \in C_c^\infty(0, a) \). Suppose \( D^{(F)} \) has a skew-adjoint extension \( A \supset D^{(F)} \) (in the RKHS \( \mathcal{H}_F \)), such that \( A \) has simple and purely atomic spectrum, \( \{i\lambda_n \mid \lambda_n \in \mathbb{R}\}_{n \in \mathbb{N}} \).

Then the complex exponentials

\[
e^{i\lambda_n x}, \quad x \in \Omega \tag{2.80}
\]

are orthogonal and total in \( \mathcal{H}_F \).

**Proof.** By the Spectral Theorem, and the assumption on the spectrum of \( A \), there exists an orthonormal basis (ONB) \( \{\xi_n\} \) in \( \mathcal{H}_F \), such that

\[
U_A(t) = e^{itA} = \sum_{n \in \mathbb{N}} e^{it\lambda_n} |\xi_n\rangle \langle \xi_n|, \quad t \in \mathbb{R}, \tag{2.81}
\]

where \( |\xi_n\rangle \langle \xi_n| \) is Dirac’s term for the rank-1 projection onto the \( C_0\xi_n \) in \( \mathcal{H}_F \).

Recall that \( F_0(\cdot) := F(\cdot - y), \forall x, y \in \Omega = (0, a) \). Then, we get that

\[
F_0(\cdot) = \sum_{n \in \mathbb{N}} \langle \xi_n, F_0 \rangle |\xi_n\rangle(\cdot) = \sum_{n \in \mathbb{N}} \overline{\xi_n(0)}(\cdot) \xi_n(\cdot),
\]

by the reproducing property in \( \mathcal{H}_F \); and with \( 0 < t < a \), we have:
\[ F_{-t} (\cdot) = U_A(t) F_0 (\cdot) = \sum_{n \in \mathbb{N}} e^{it \lambda_n} \xi_n (0) \xi_n (\cdot) \quad (2.82) \]

holds on \( \Omega = (0,a) \).

Now fix \( n \in \mathbb{N} \), and take the inner-product \( \langle \xi_n, \cdot \rangle_{H_F} \) on both sides in (2.82). Using again the reproducing property, we get

\[ \xi_n (t) = e^{it \lambda_n} \xi_n (0), \quad t \in \Omega; \quad (2.83) \]

which yields the desired conclusion.

Note that the functions \( \{ e^{it \lambda_n} \}_{n \in \mathbb{N}} \) in (2.80) are orthogonal, and total in \( H_F \); but they are not normalized. In fact, it follows from (2.83), that

\[ \| e^{it \lambda_n} \|_{H_F} = | \xi_n (0) |^{-1}. \]

Theorem 2.8. Let \( a > 0 \), and \( \Omega = (0,a) \). Let \( F \) be a continuous p.d. function on \( \Omega - \Omega = (-a,a) \). Let \( D = D(F) \) be the canonical skew-Hermitian operator acting in \( H_F \).

Fix \( z \in \mathbb{C} \); then the function \( \xi_z : y \mapsto e^{-zy} \), restricted to \( \Omega \), is in \( H_F \) iff \( z \) is an eigenvalue for the adjoint operator \( D^* \). In the affirmative case, the corresponding eigenspace is \( \mathbb{C} \xi_z \), in particular, the eigenspace has dimension one.

Proof. Suppose \( D^* \xi = z \xi \), \( \xi \in H_F \), then

\[ \langle DF \phi, \xi \rangle_{H_F} = \langle F \phi, z \xi \rangle_{H_F}, \quad \forall \phi \in C_\infty_c (\Omega). \]

Equivalently,

\[ \int_\Omega \phi' (y) \xi (y) dy = \int_\Omega z \phi (y) \xi (y) dy, \quad \forall \phi \in C_\infty_c (\Omega). \]

Hence, \( \xi \) is a weak solution to

\[ -\xi' (y) = z \xi (y), \quad y \in \Omega. \]

and so \( \xi (y) = \text{const} \cdot e^{-zy} \).

Conversely, suppose \( \xi_z (y) = e^{-zy} \big|_\Omega \) is in \( H_F \). It is sufficient to show \( \xi_z \in \text{dom}(D^*) \); i.e., we must show that there is a finite constant \( C \), such that

\[ \left| \langle DF \phi, \xi_z \rangle_{H_F} \right| \leq C \| F \phi \|_{H_F}, \quad \forall \phi \in C_\infty_c (\Omega). \quad (2.84) \]

But, we have

\[ \left| \langle DF \phi, \xi_z \rangle_{H_F} \right| = \left| \int_\Omega \overline{\phi' (y)} \xi_z (y) dy \right| = |z| \left| \int_\Omega \overline{\phi (y)} \xi_z (y) dy \right| = |z| \left| \langle F \phi, \xi_z \rangle_{H_F} \right| \leq |z| \| \xi_z \|_{H_F} \| F \phi \|_{H_F}; \]
which is the desired estimate in (2.84). (The final step follows from the Cauchy-Schwarz inequality.)

\[ \square \]

**Theorem 2.9.** Let \( a > 0 \), and \( \lambda_1 \in \mathbb{R} \) be given. Let \( F : (-a,a) \to \mathbb{C} \) be a fixed continuous p.d. function. Then there following two conditions (1) and (2) are equivalent:

1. \( \exists \mu_1 \in \text{Ext} (F) \) such that \( \mu_1 (\{ \lambda_1 \}) > 0 \) i.e., \( \mu_1 \) has an atom at \( \lambda_1 \); and
2. \( e_{\lambda_1} (x) := e^{i\lambda_1 x} |_{(-a,a)} \in H_F \).

**Proof.** The implication \((1) \implies (2)\) is already contained in the previous discussion.

Proof of \((2) \implies (1)\). We first consider the skew-Hermitian operator \( D(F) \) \( F \phi \) : \( H_F \to (C_c(0,a)) \), Using an idea of M. Krein \cite{Krein, KL14}, we may always find a Hilbert space \( \mathcal{F} \), an isometry \( J \) : \( H \to \mathcal{F} \), and a strongly continuous unitary one-parameter group \( U_A (t) = e^{itA}, t \in \mathbb{R} \) with \( A^* = -A \); \( U_A (t) \) acting in \( \mathcal{F} \), such that

\[ JD(F) = AJ \tag{2.85} \]

\[ \text{dom}(D(F)) = \{ F \phi \mid \phi \in C_c(0,a) \} ; \tag{2.86} \]

see also Theorem 3.6. Since \( e_1 (x) = e^{i\lambda_1 x} |_{(-a,a)} \) \( \tag{2.87} \)

is in \( H_F \), we can form the following measure \( \mu_1 \in \mathcal{M}_+ (\mathbb{R}) \), now given by

\[ d\mu_1 (\lambda) := ||P_A (d\lambda) J e_1||^2_{\mathcal{F}} , \quad \lambda \in \mathbb{R} , \tag{2.88} \]

where \( P_A (\cdot) \) is the PVM of \( U_A (t) \), i.e.,

\[ U_A (t) = \int_{\mathbb{R}} e^{it\lambda} P_A (d\lambda) , \quad t \in \mathbb{R} . \tag{2.89} \]

We claim the following two assertions:

(i) \( \mu_1 \in \text{Ext} (F) \); and

(ii) \( \lambda_1 \) is an atom in \( \mu_1 \), i.e., \( \mu_1 (\{ \lambda_1 \}) > 0 \).

This is the remaining conclusion in the theorem.

The proof of (i) is immediate from the construction above; using the intertwining isometry \( J \) from (2.85), and formulas (2.88)-(2.89).

To prove (ii), we need the following:

**Lemma 2.13.** Let \( F, \lambda_1, e_1, \mathcal{F} \), \( J \) and \( \{ U_A (t) \}_{t \in \mathbb{R}} \) be as above; then we have the identity:

\[ \langle J e_1, U_A (t) J e_1 \rangle_{\mathcal{F}} = e^{\lambda_1 t} ||e_1||^2_{\mathcal{F}} , \quad \forall t \in \mathbb{R} . \tag{2.90} \]

**Proof.** It is immediate from (2.85)-(2.88) that (2.90) holds for \( t = 0 \). To get it for all \( t \), fix \( t \), say \( t > 0 \) (the argument is the same if \( t < 0 \)); and we check that

\[ \frac{d}{ds} \left( \langle J e_1, U_A (t-s) J e_1 \rangle_{\mathcal{F}} - e^{i(t-s)\lambda_1} ||e_1||^2_{\mathcal{F}} \right) = 0 , \quad \forall s \in (0,t) . \tag{2.91} \]
But this, in turn, follow from the assertions above: First

\[ D^*_F e_1 = D^*_F J^* Je_1 = J^* A Je_1 \]

holds on account of (2.85). We get: \( e_1 \in \text{dom}(D^*_F) \), and \( D^*_F e_1 = -i\lambda_1 e_1 \).

Using this, the verification of is (2.90) now immediate. \( \square \)

As a result, we get:

\[ U_A(t) Je_1 = e^{it\lambda_1} Je_1, \quad \forall t \in \mathbb{R}, \]

and by (2.89):

\[ P_A(\{\lambda_1\}) Je_1 = Je_1 \]

where \( \{\lambda_1\} \) denotes the desired \( \lambda_1 \)-atom. Hence, by (2.88), \( \mu_1(\{\lambda_1\}) = \| Je_1 \|^2_{\mathcal{F}} = \| e_1 \|^2_{\mathcal{F}}, \) which is the desired conclusion in (2). \( \square \)
Chapter 3
The Case of More General Groups

Nowadays, group theoretical methods — especially those involving characters and representations, pervade all branches of quantum mechanics. — George Mackey.

The universe is an enormous direct product of representations of symmetry groups. — Hermann Weyl

3.1 Locally Compact Abelian Groups

We are concerned with extensions of locally defined continuous and positive definite (p.d.) functions $F$ on Lie groups, say $G$, but some results apply to locally compact groups as well. However in the case of locally compact Abelian groups, we have stronger theorems, due to the powerful Fourier analysis theory in this specific setting.

First, we fix some notations:

$G$ a given locally compact Abelian group; group operation is written additively.

$dx$ the Haar measure of $G$, unique up to a scalar multiple.

$\hat{G}$ the dual group, consisting of all continuous homomorphisms $\lambda : G \to \mathbb{T}$, s.t.

$$\lambda (x + y) = \lambda (x) \lambda (y), \quad \lambda (-x) = \overline{\lambda (x)}, \quad \forall x, y \in G.$$  

Occasionally, we shall write $\langle \lambda, x \rangle$ for $\lambda (x)$. Note that $\hat{G}$ also has its Haar measure.

The Pontryagin duality theorem below is fundamental for locally compact Abelian groups.

Theorem 3.1 (Pontryagin [Rud90]). $\hat{\hat{G}} \simeq G$, and we have the following:

$[G \text{ is compact}] \iff [\hat{G} \text{ is discrete}]$

Let $\emptyset \neq \Omega \subset G$ be an open connected subset, and let

$$F : \Omega - \Omega \to \mathbb{C} \quad (3.1)$$

be a fixed continuous p.d. function. We choose the normalization $F (0) = 1$. Set

$$F_y (x) = F (x - y), \quad \forall x, y \in \Omega.$$
The corresponding RKHS $\mathcal{H}_F$ is defined almost verbatim as in Section 2.1. Its “continuous” version is recast in Lemma 3.1 with slight modifications (see Lemma 2.2.) Functions in $\mathcal{H}_F$ are characterized in Lemma 3.2 (see Theorem 2.1.)

**Lemma 3.1.** For $\varphi \in C_c(\Omega)$, set
\[
F_\varphi(\cdot) = \int_{\Omega} \varphi(y) F(\cdot - y) \, dy,
\] (3.2)
then $\mathcal{H}_F$ is the Hilbert completion of $\{F_\varphi | \varphi \in C_c(\Omega)\}$ in the inner product:
\[
\langle F_\varphi, F_\psi \rangle_{\mathcal{H}_F} = \int_{\Omega} \int_{\Omega} \overline{\varphi(x)} \varphi(y) F(x - y) \, dx \, dy.
\] (3.3)
Here $C_c(\Omega)$ := all continuous compactly supported functions in $\Omega$.

**Lemma 3.2.** The Hilbert space $\mathcal{H}_F$ is also a Hilbert space of continuous functions on $\Omega$ as follows:
If $\xi : \Omega \to \mathbb{C}$ is a fixed continuous function, then $\xi \in \mathcal{H}_F$ if and only if $\exists K = K_\xi < \infty$ such that
\[
\left| \int_{\Omega} \xi(x) \varphi(x) \, dx \right|^2 \leq K \int_{\Omega} \int_{\Omega} \overline{\varphi(y_1)} \varphi(y_2) F(y_1 - y_2) \, dy_1 \, dy_2.
\] (3.4)
When (3.4) holds, then
\[
\langle \xi, F_\varphi \rangle_{\mathcal{H}_F} = \int_{\Omega} \xi(x) \varphi(x) \, dx, \quad \forall \varphi \in C_c(\Omega).
\]

**Proof.** We refer to the basics on the theory of RKHSs; e.g., [Aro50]. Also see Section 2.1. \qed

**Definition 3.1.** Let $\mathcal{M}(\widehat{G})$ be the set of all Borel measures on $\widehat{G}$. Given $\mu \in \mathcal{M}(\widehat{G})$, let $\widehat{\mu}$ be the Fourier transform, i.e.,
\[
\widehat{\mu}(\lambda) = \int_{\widehat{G}} \lambda(x) \, d\mu(x) = \int_{\widehat{G}} \langle \lambda, x \rangle \, d\mu(x), \quad \forall \lambda \in G.
\] (3.5)
Given $F$ as in (3.1), set
\[
\text{Ext}(F) = \left\{ \mu \in \mathcal{M}(\widehat{G}) \mid F(x) = \widehat{\mu}(x), \quad \forall x \in \Omega - \Omega \right\}.
\] (3.6)

**Theorem 3.2.** $\text{Ext}(F)$ is weak $*$-compact and convex.

**Proof.** See e.g., [Rud73]. \qed

**Remark 3.1.** We shall extend the discussion for the case of $G = \mathbb{R}^n$ to locally compact Abelian groups in general (Section 2.4, especially Remark 2.6.)
Note that $\operatorname{Ext}(F)$ may be empty. For $G = \mathbb{R}^2$, Rudin gave examples where $\operatorname{Ext}(F) = \emptyset$ [Rud70, Rud63]. See Section 3.2.2, and the example of logarithmic Riemann surface in Section 4.6.

**Question 3.1.** Suppose $\operatorname{Ext}(F) \neq \emptyset$, then what are its extreme points? Equivalently, characterize $\operatorname{ext}(\operatorname{Ext}(F))$.

The reader is referred to Section 3.2.3, especially, the discussion of $\operatorname{Ext}_1(F)$ as a set of extreme points in $\operatorname{Ext}(F)$, and the direct integral decomposition.

We now assume that $F(0) = 1$; normalization.

**Lemma 3.3.** There is a bijective correspondence between all continuous p.d. extensions $\hat{F}$ to $G$ of the given p.d. function $F$ on $\Omega \setminus \Omega$, on the one hand; and all Borel probability measures $\mu$ on $\hat{G}$, on the other, i.e., all $\mu \in \mathcal{M}(\hat{G})$ s.t.

$$F(x) = \hat{\mu}(x), \quad \forall x \in \Omega \setminus \Omega,$$

where $\hat{\mu}$ is as in (3.6).

**Proof.** This is an immediate application of Bochner’s characterization of the continuous p.d. functions on locally compact Abelian groups [BC49, BC48, Boc47, Boc46].

**Theorem 3.3.** Let $F$ and $\mathcal{H}_F$ be as above.

(1) Let $\mu \in \mathcal{M}(\hat{G})$; then there is a positive Borel function $h$ on $\hat{G}$ s.t. $h^{-1} \in L^\infty(\hat{G})$, and $h d\mu \in \operatorname{Ext}(F)$, if and only if $\exists K_\mu < \infty$ such that

$$\int_{\hat{G}} |\hat{\varphi}(\lambda)|^2 d\mu(\lambda) \leq K_\mu \int_{\Omega} \int_{\Omega} \overline{\varphi(y_1)\varphi(y_2)} F(y_1 - y_2) dy_1 dy_2,$$

for all $\varphi \in C_c(\Omega)$.

(2) Assume $\mu \in \operatorname{Ext}(F)$, then

$$\chi_{\mathcal{H}_F}(fd\mu)^\vee \in \mathcal{H}_F, \quad \forall f \in L^2(\hat{G}, \mu).$$

**Proof.** The assertion in (3.8) is immediate from Lemma 3.2. The remaining computations are left to the reader.

**Remark 3.2.** Our conventions for the two transforms used in (3.8) and (3.9) are as follows:

$$\hat{\varphi}(\lambda) = \int_{\hat{G}} \overline{\varphi(x)} dx, \quad \varphi \in C_c(\Omega); \quad \text{and}$$

$$(fd\mu)^\vee(x) = \int_{\hat{G}} \overline{\lambda, x} f(\lambda) d\mu(\lambda), \quad f \in L^2(\hat{G}, \mu).$$
Corollary 3.1.

(1) Let $F$ be as above; then $\mu \in \text{Ext}(F)$ iff the operator

$$T(F\varphi) = \hat{\varphi}, \quad \forall \varphi \in C_c(\Omega),$$

is well-defined and extends to an isometric operator $T : \mathcal{H}_F \to L^2(\widehat{G}, \mu).$

(2) If $\mu \in \text{Ext}(F)$, the adjoint operator $T^* : L^2(\widehat{G}, \mu) \to \mathcal{H}_F$ is given by

$$T^*(f) = \chi_\Omega(f d\mu)^\vee, \quad \forall f \in L^2(\widehat{G}, \mu).$$

(3.12)

Remark 3.3. Note that we have fixed $F$, and a positive measure $\mu \in \text{Ext}(F)$, both fixed at the outset. We then show that $T = T_\mu$ as defined in 1 is an isometry relative to $\mu$, i.e., mapping from $H_F$ into $L^2(\mu)$. But in the discussion below, we omit the subscript $\mu$ in order to lighten notation.

Proof. If $\mu \in \text{Ext}(F)$, then for all $\varphi \in C_c(\Omega)$, and $x \in \Omega$, we have (see (3.2))

$$F_\varphi(x) = \int_\Omega \varphi(y) F(x - y) dy = \int_\Omega \varphi(y) \widehat{\mu}(x - y) dy = \int_\Omega \varphi(y) \int_\widehat{G} \langle \lambda, x - y \rangle d\mu(\lambda) dy = \int_\widehat{G} \langle \lambda, x \rangle \hat{\varphi}(\lambda) d\mu(\lambda).$$

By Lemma 3.2, we note that $\chi_\Omega(f d\mu)^\vee \in \mathcal{H}_F$, see (3.11). Hence $\exists K < \infty$ such that the estimate (3.8) holds. To see that $T(F\varphi) = \hat{\varphi}$ is well-defined on $H_F$, we must check the implication:

$$\left( F_\varphi = 0 \text{ in } \mathcal{H}_F \right) \implies \left( \hat{\varphi} = 0 \text{ in } L^2(\widehat{G}, \mu) \right)$$

but this now follows from estimate (3.8).

Using the definition of the respective inner products in $\mathcal{H}_F$ and in $L^2(\widehat{G}, \mu)$, we check directly that, if $\varphi \in C_c(\Omega)$, and $f \in L^2(\widehat{G}, \mu)$ then we have:

$$\langle \hat{\varphi}, f \rangle_{L^2(\mu)} = \langle F_\varphi, (f d\mu)^\vee \rangle_{\mathcal{H}_F}.$$  \hspace{1cm} (3.13)

On the r.h.s. in (3.13), we note that, when $\mu \in \text{Ext}(F)$, then $\chi_\Omega(f d\mu)^\vee \in \mathcal{H}_F$. This last conclusion is a consequence of Lemma 3.2:

Indeed, since $\mu$ is finite, $L^2(\widehat{G}, \mu) \subset L^1(\widehat{G}, \mu)$, so $(f d\mu)^\vee$ in (3.11) is continuous on $G$ by Riemann-Lebesgue; and so is its restriction to $\Omega$. If $\mu$ is further assumed absolutely continuous, then $(f d\mu)^\vee \to 0$ at $\infty$. With a direct calculation, using the reproducing property in $\mathcal{H}_F$, and Fubini’s theorem, we check directly that the following estimate holds:
\[ \left| \int_{\Omega} \overline{\varphi(x)} (fd\mu)^\vee (x) \, dx \right|^2 \leq \left( \int_{\Omega} \int_{\Omega} \overline{\varphi(y_1)} \varphi(y_2) F(y_1 - y_2) \, dy_1 \, dy_2 \right) \| f \|^2_{L^2(\mu)} \]

and so Lemma 3.2 applies; we get \( \chi_{\Omega^c} (fd\mu)^\vee \in \mathcal{H}_F \).

It remains to verify the formula (3.13) for all \( \varphi \in C_c(\Omega) \) and all \( f \in L^2(\hat{G}, \mu) \); but this now follows from the reproducing property in \( \mathcal{H}_F \), and Fubini.

Once we have this, both assertions in (1) and (2) in the Corollary follow directly from the definition of the adjoint operator \( T^* \) with respect to the two Hilbert space inner products in \( \mathcal{H}_F \rightarrow L^2(\hat{G}, \mu) \). Indeed then (3.12) follows. \( \square \)

Remark 3.4. The transform \( T = T_\mu \) from Corollary 3.1 is not onto \( L^2(\mu) \) in general.

We illustrate the cases with \( G = \mathbb{R} \); and we note that then:

(i) If \( \mu \) is of compact support (in \( \mathbb{R} \)), then \( T_\mu \) maps onto \( L^2(\mu) \).

(ii) An example, where \( T_\mu \) is not onto \( L^2(\mu) \), may be obtained as follows.

Let \( F = F_2 \) in Tables 5.1-5.3, i.e.,

\[ F(x) = (1 - |x|) \left( -\frac{1}{2}, \frac{1}{2} \right) \]

(3.14)

Let

\[ d\mu(\lambda) = \frac{1}{2\pi} \left( \frac{\sin(\lambda/2)}{\lambda/2} \right)^2 \, d\lambda, \quad (\text{Table 5.2}) \]

(3.15)

where \( (d\mu)^\vee = \tilde{F} \in Ext(F) \) is the tent function on \( \mathbb{R} \) given by

\[ \tilde{F}(x) = \begin{cases} 1 - |x| & |x| \leq 1 \\ 0 & |x| > 1 \end{cases} \]

(3.16)

We say that \( \mu \in Ext(F) \). Take a function \( f \) on \( \mathbb{R} \) such that

\[ \tilde{f}(y) = \begin{cases} e^{-|y|} & |y| > 2 \\ 0 & |y| \leq 2 \end{cases} \]

(\( \tilde{f} \) denotes inverse Fourier transform.) Then (see (3.12))

\[ T_\mu^*(f) = 0 \text{ in } \mathcal{H}_F. \]

(3.18)

(Note that \( f \in L^2(\mathbb{R}, \mu) \setminus \{0\} \) on account of (3.15) and (3.17).)

Proof. (i) Fix some finite positive measure \( \mu \). Since \( Ran(T_\mu^*) = Null(T_\mu^*) \), we need only consider the homogeneous equation \( T_\mu^* f = 0 \) for \( f \in L^2(\mu) \). By (3.12), this is equivalent to

\[ \int e^{i2\pi x \lambda} f(\lambda) \, d\mu(\lambda) = 0, \quad \forall x \in \Omega, \]

(3.19)
where \( \Omega \subset \mathbb{R} \) is chosen open and bounded such that the initial p.d. function \( F \) is defined on \( \Omega - \Omega \). But, if \( \text{supp} (\mu) \) is assumed compact, then the function \( \Psi_{f,\mu} (x) \) in (3.19) has an entire analytic extension to \( \mathbb{C} \). And then (3.19) implies that \( \Psi_{f,\mu} = 0 \) in some interval; and therefore for all \( x \in \mathbb{R} \). The conclusion \( f = 0 \) in \( L^2(\mathbb{R}, \mu) \) then follows by a standard Fourier uniqueness theorem.

(ii) Let \( f, F, \Omega := (0, \frac{1}{2}) \), and \( \mu \) be as specified in (3.14)-(3.17). Now writing out \( T^\ast \mu f \), we get for all \( x \in \Omega \) (see Table 5.1):

\[
(f d\mu)^\vee (x) = (\tilde{f} * (d\mu)^\vee)(x) = \left( \tilde{f} * \tilde{F} \right)(x) = \int_{|y|<2} \tilde{f}(y) \tilde{F}(x-y) dy = 0, \quad x \in \Omega.
\] (3.20)

Note that, by (3.16), for all \( x \in \Omega \), the function \( \tilde{F}(x - \cdot) \) is supported inside \((-2, 2)\); and so the integral on the r.h.s. in (3.20) is zero. See Figure 3.1. We proved that \( \text{Null}(T^\ast \mu) \neq \{0\} \), and so \( T^\ast \mu \) is not onto.

![Fig. 3.1: The tent function. See also (3.14) and (3.16).](image)

**Remark 3.5.** In Section 5.1, we study a family of extensions in the case \( G = \mathbb{R} \), where \( F : (-a,a) \to \mathbb{R} \) is symmetric \( F(x) = F(-x), |x| < a \), and \( F\big|_{[0,a]} \) is assumed convex. By a theorem of Pólya, these functions are positive definite. The positive definite extensions \( F^{(\text{ext})} \) to \( \mathbb{R} \) of given \( F \) as above are discussed in Section 5.1. They are called spline extensions; see Figures 5.2-5.4, and Figure 5.5. They will all have \( F^{(\text{ext})} \) of compact support.

As a result, the above argument from Remark 3.4 shows (with slight modification) that the spectral transform \( T^\ast \mu = T^\ast \mu^{(\text{ext})} \) (Corollary 3.1) will not map \( \mathcal{A}_F \) onto \( L^2(\mathbb{R}, \mu^{(\text{ext})}) \). Here, with \( \mu^{(\text{ext})} \), we refer to the unique positive measure on \( \mathbb{R} \) such that \( F^{(\text{ext})} = d\mu^{(\text{ext})}, F^{(\text{ext})} \) is one of the spline extensions. For further details, see Figure 3.2 and Section 5.1.

**Example 3.1.** Let \( F(x) = e^{-|x|}, |x| < 1 \), then

\[
F^{(\text{ext})}(x) = \begin{cases} 
  e^{-|x|} & \text{if } |x| < 1, \ F \text{ itself} \\
  e^{-1}(2 - |x|) & \text{if } 1 \leq |x| < 2 \\
  0 & \text{if } |x| \geq 2
\end{cases}
\]
is an example of a positive definite Pólya spline extension; see Figure 3.2.

Theorem 3.4. Let $G$ be a locally compact Abelian group. Let $\Omega \subset G$, $\Omega \neq \emptyset$, open and connected. Let $F : \Omega \to \mathbb{C}$ be continuous, positive definite; and assume $\text{Ext}(F) \neq \emptyset$. Let $\mu \in \text{Ext}(F)$, and let $T_\mu (F_\emptyset) := \emptyset$, defined initially only for $\varphi \in C_c(\Omega)$, be the isometry $T_\mu : \mathcal{H} \to L^2(\mu) := L^2(\hat{G}, \mu)$.

Then $Q_\mu := T_\mu T_\mu^*$ is a projection in $L^2(\mu)$ onto $\text{ran}(T_\mu)$; and $\text{ran}(T_\mu)$ is the $L^2(\mu)$-closure of the span of the functions $\{e_x | x \in \Omega\}$, where $e_x(\lambda) := \langle \lambda, x \rangle$, $\forall \lambda \in \hat{G}$, and $\langle \lambda, x \rangle$ denotes the $\hat{G} \leftrightarrow G$ duality.

Proof. By Theorem 3.3, $T_\mu : \mathcal{H} \to L^2(\mu)$ is isometric, and so $Q_\mu := T_\mu T_\mu^*$ is the projection in $L^2(\mu)$ onto $\text{ran}(T_\mu)$.

We shall need the following three facts:

1. $\text{ran}(T_\mu) = (\text{null}(T_\mu^*))^\perp$, where “$\perp$” refers to the standard inner product in $L^2(\mu)$.
2. $(T_\mu f)(x) = \chi_\Omega(x)(fd\mu)^\vee(x), x \in G$.
3. Setting $e_x(\lambda) = \lambda(x) = \langle \lambda, x \rangle, \forall \lambda \in \hat{G}$, and

\[ E_\mu (\Omega) = \text{span} L^2(\mu)-\text{closure} \{e_x(\cdot) | x \in \Omega\}, \]

we note that $e_x \in \text{ran}(T_\mu)$, for all $x \in \Omega$.

Comments. (1) is general for isometries in Hilbert space. (2) is from Corollary 3.1. Finally, (3) follows from Corollary 2.1.

Now, let $f \in L^2(\mu)$; we then have the following double-implications:
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\[ f \in (\text{ran} \, (T_\mu)) \perp \]
\[ \Downarrow \text{by (1)} \]
\[ (f d\mu) \vee (x) = 0, \forall x \in \Omega \]
\[ \Downarrow \text{by (2)} \]
\[ \int_G \langle \lambda, x \rangle f (\lambda) d\mu (\lambda) = 0, \forall x \in \Omega \]
\[ \Downarrow \]
\[ (e_x, f)_{L^2(\mu)} = 0, \forall x \in \Omega. \]

Now applying “\( \perp \)” one more time, we get that

\[ \text{ran} \, (T_\mu) = \left\{ e_x \mid x \in \Omega \right\} \perp \perp = E_\mu (\Omega), \]

which is the desired conclusion. \( \square \)

3.2 Lie Groups

In this section, we consider the extension problem for continuous positive definite functions in Lie groups.

A key ingredient in our approach to extensions of locally defined positive definite functions, in the context of Lie groups, is the use of a correspondence between the following two aspects of representation theory:

(i) Unitary representations \( U \) of a particular Lie group \( G \), with \( U \) acting in some Hilbert space \( \mathcal{H} \); and

(ii) a derived representations \( dU \) of the associated Lie algebra \( \mathfrak{g} \).

This viewpoint originate with two landmark papers by Nelson [Nel59], and Nelson-Stinespring [NS59].

The following observations about (ii) will be used without further mention in our discussion below, and elsewhere: First, given \( U \), the derived representation \( dU \) of \( \mathfrak{g} \) is a representation by unbounded skew-Hermitian operators \( dU (X) \), for \( X \in \mathfrak{g} \), with common dense domain in the Hilbert space \( \mathcal{H} \).

To say that \( dU \) is a representation refers to the Lie bracket in \( \mathfrak{g} \) on one side, and the commutator bracket for operators on the other. Further, \( dU \), as a representation of \( \mathfrak{g} \), automatically induces a representation of the corresponding associative enveloping algebra to \( \mathfrak{g} \); this is the one referenced in Nelson-Stinespring [NS59]. But the most important point is that the correspondence from (i) to (ii) is automatic, while the reverse from (ii) to (i) is not. For example, the Lie algebra can only account for the subgroup of \( G \) which is the connected component of the unit element in \( G \). And, similarly, loops in \( G \) cannot be accounted for by the Lie algebra.

So, at best, a given representation \( \rho \) of the Lie algebra will give information only about possible associated unitary representations of the universal covering group to
3.2 Lie Groups

Let \( G \); the two have the same Lie algebra. But even then, there may not be a unitary representation of this group coming as an integral of a Lie algebra representation \( \rho \). This is where the Laplace operator \( \Delta \) of \( G \) comes in. Nelson’s theorem states that a given Lie algebra representation \( \rho \) is integrable (to a unitary representation) if and only if \( \rho (\Delta) \) is essentially selfadjoint.

For a systematic account of the theory of integration of representations of Lie algebras in infinite dimensional Hilbert space, and their applications to physics, see [JM84].

**Definition 3.2.** Let \( G \) be a Lie group. Assume \( \Omega \subset G \) is a non-empty, connected and open subset. A continuous \( F : \Omega^{-1}\Omega \rightarrow \mathbb{C} \) (3.21) is said to be positive definite, iff (Def) \[ \sum_i \sum_j c_i c_j F(x_i^{-1} x_j) \geq 0, \] for all finite systems \( \{c_i\} \subset \mathbb{C} \), and points \( \{x_i\} \subset \Omega \). Equivalent, \[ \int_{\Omega} \overline{\varphi(x)} \varphi(y) F(x^{-1} y) \, dx \, dy \geq 0, \] for all \( \varphi \in C_c(\Omega) \); where \( dx \) denotes a choice of left-invariant Haar measure on \( G \).

**Lemma 3.4.**

(1) Let \( F \) be as in (3.21)-(3.22). For all \( X \in \mathfrak{La}(G) = \) the Lie algebra of \( G \), set \[ (\tilde{X} \varphi)(g) = \left. \frac{d}{dt} \right|_{t=0} \varphi(\exp_G(-t X) g), \quad \varphi \in C_c^\infty(\Omega). \] (3.24)

Let \[ F_\varphi(x) = \int_{\Omega} \varphi(y) F(x^{-1} y) \, dy; \] (3.25) then \[ S_X^{(F)} : F_\varphi \mapsto F_{\tilde{X} \varphi}, \quad \varphi \in C_c^\infty(\Omega) \] (3.26) defines a representation of the Lie algebra \( \mathfrak{La}(G) \) by skew-Hermitian operators in the RKHS \( \mathcal{H}_F \), with the operator in (3.26) defined on the common dense domain \( \{ F_\varphi \mid \varphi \in C_c^\infty(\Omega) \} \subset \mathcal{H}_F \).

(2) Let \( F \) be as in part (1) above, and define \( F_\varphi \) for \( \varphi \in C_c(\Omega) \) as in (3.25), then \( F_\varphi \in C^\infty(\Omega) \), and \[ \tilde{X}(F_\varphi) = F_{\tilde{X} \varphi} \] (3.27) holds on \( \Omega \), where \( \tilde{X} \) for \( X \in \mathfrak{La}(G) \) is the vector field defined in (3.24).

**Proof.** Part (1). The arguments here follow those of the proof of Lemma 2.4 mutatis mutandis.

Part (2). Let \( F, \Omega, \varphi, \) and \( X \) be as in the statement of the theorem, and let \( g \in \Omega \) be fixed. Then the r.h.s. in (3.27) exists as a pointwise limit as follows:
Let \( t \in \mathbb{R} \setminus \{0\} \) be sufficiently small, i.e., \(|t| < a\); then
\[
\frac{1}{t} \left( F_{\varphi} \left( \exp_G (-tX) g \right) - F_{\varphi}(g) \right) \\
\text{(invariance of Haar measure)}
\]
\[
= \int_{\Omega} \frac{1}{t} \left( \varphi \left( \exp_G (-tX) y \right) - \varphi(y) \right) F \left( g^{-1}y \right) dy \\
\xrightarrow{t \to 0} \int_{\Omega} (\bar{X} \varphi)(y) F \left( g^{-1}y \right) dy = F_{\bar{X} \varphi}(g),
\]
and the desired conclusion (3.27) follows, i.e., the function \( F_{\varphi} \) is differentiable on the open set \( \Omega \), with derivative as specified w.r.t. the vector field \( \bar{X} \).

Note that \( X \in \mathcal{L}(G) \) is arbitrary. Since \( \bar{X} \varphi \in C_c^\infty(\Omega) \), the argument may be iterated; so we get \( F_{\varphi} \in C^\infty(\Omega) \) as desired. \( \square \)

**Definition 3.3.** Let \( \Omega \subset G \), non-empty, open and connected.

1. A continuous p.d. function \( F : \Omega^{-1} \Omega \to \mathbb{C} \) is said to be extendable, if there is a continuous p.d. function \( F_{\text{ex}} : G \to G \) such that
\[
F_{\text{ex}}|_{\Omega^{-1} \Omega} = F. \tag{3.28}
\]

2. Let \( U \in \text{Rep}(G, \mathcal{H}) \) be a strongly continuous unitary representation of \( G \) acting in some Hilbert space \( \mathcal{H} \). We say that \( U \in \text{Ext}(F) \) (Section 2.4), if there is an isometry \( J : \mathcal{H}_F \to \mathcal{H} \) such that the function
\[
G \ni g \mapsto \langle JF_e, U(g) JF_e \rangle_{\mathcal{H}} \tag{3.29}
\]
satisfies the condition (3.28)

**Theorem 3.5.** Every extension of some continuous p.d. function \( F \) on \( \Omega^{-1} \Omega \) as in (1) arises from a unitary representation of \( G \) as specified in (2).

**Proof.** First assume some unitary representation \( U \) of \( G \) satisfies (2), then (3.29) is an extension of \( F \). This follows from the argument in our proof of Theorem 2.4; also see Lemma 2.4 and 2.5.

For the converse; assume some continuous p.d. function \( F_{\text{ex}} \) on \( G \) satisfies (3.28).

Now apply the GNS (Theorem 3.7) to \( F_{\text{ex}} \); and, as a result, we get a cyclic representation \((U, \mathcal{H}, v_0)\) where

- \( \mathcal{H} \) is a Hilbert space;
- \( U \) is a strongly continuous unitary representation of \( G \) acting on \( \mathcal{H} \); and
- \( v_0 \in \mathcal{H} \) is a cyclic vector, \( \|v_0\| = 1 \); and

\[
F_{\text{ex}}(g) = \langle v_0, U(g) v_0 \rangle, \quad g \in G. \tag{3.30}
\]
Defining \( J : \mathcal{H} \rightarrow \mathcal{K} \) by
\[
J(F(g)) = U(g^{-1})v_0, \quad \forall g \in \Omega;
\]
and extend by limit. We check that \( J \) is isometric and satisfies the condition from (2) in Definition 3.3. We omit details as they parallel the arguments already contained in Sections 2.1-2.4.

\[ \Box \]

**Theorem 3.6.** Let \( \Omega, G, \text{La}(G), \) and \( F : \Omega^{-1} \Omega \rightarrow \mathbb{C} \) be as above. Let \( \tilde{G} \) be the simply connected universal covering group for \( G \). Then \( F \) has an extension to a p.d. continuous function on \( \tilde{G} \) iff there is a unitary representation \( U \) of \( \tilde{G} \), and an isometry \( H : \mathcal{H} \rightarrow \mathcal{K} \), such that
\[
JS_X^{(F)} = dU(X)J
\]
holds on \( \{ F\phi \mid \phi \in C^\infty_c(\Omega) \} \), for all \( X \in \text{La}(G) \); where
\[
dU(X)U(\phi)v_0 = U(\tilde{X}\phi)v_0.
\]

**Proof.** Details are contained in Sections 2.3, 4.2, and Chapter 5.

Assume \( G \) is connected. On \( C^\infty_c(\Omega) \), the Lie group \( G \) acts locally by \( \phi \mapsto \phi_g \), where \( \phi_g := \phi(g^{-1}) \) denotes translation of \( \phi \) by \( g \in G \), such that \( \phi_g \) is also supported in \( \Omega \). Note that
\[
\|F\phi\|_{\mathcal{H}} = \|F\phi_g\|_{\mathcal{H}};
\]
but only for elements \( g \in G \) in a neighborhood of \( e \in G \), and with the neighborhood depending on \( \phi \).

**Corollary 3.2.** Given
\[
F : \Omega^{-1} \Omega \rightarrow \mathbb{C}
\]
continuous and positive definite, then set
\[
L_g (F\phi) := F\phi_g, \quad \phi \in C^\infty_c(\Omega),
\]
defining a local representation of \( G \) in \( \mathcal{H} \).

**Proof.** See [Jor87, Jor86, JT14a]. Equivalently, introducing the Hilbert space \( \mathcal{M}_2(\Omega,F) \) from Corollary 2.1, i.e., all complex measures \( \nu \) on \( \Omega \) s.t.
\[
\|\nu\|^2_{\mathcal{M}_2(\Omega,F)} = \int_{\Omega} \int_{\Omega} F(x^{-1}y) \bar{\nu}(x)\nu(y) < \infty,
\]
we get the following formula for the local representation \( L \): For elements \( x, y \in \Omega \) s.t. \( xy \in \Omega \), we have
\[
L(x)\delta_y = \delta_{xy}, \quad \text{and} \quad \langle \delta_x, L(x)\delta_c \rangle_{\mathcal{M}_2(\Omega,F)} = F(x)
\]
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Corollary 3.3. Given \( F \), positive definite and continuous, as in (3.33), and let \( L \) be
the corresponding local representation of \( G \) acting on \( \mathcal{H}_F \). Then \( \text{Ext} (F) \neq 0 \) if and
only if the local representation (3.34) extends to a global unitary representation
acting in some Hilbert space \( \mathcal{K} \), containing \( \mathcal{H}_F \) isometrically.

Proof. We refer to [Jor87, Jor86] for details, as well as Chapter 6.

3.2.1 The GNS Construction

Given a group \( G \), or \( C^* \)-algebra \( \mathfrak{A} \), the Gelfand–Naimark–Segal (GNS for short)
construction establishes an explicit correspondence between representations on the
one side and positive definite functionals (states) in the case of \( C^* \)-algebras, and
positive definite functions in the case of groups. The non-trivial part is the con-
struction of the representation from the given positive definite object. In the case of
\( C^* \)-algebras the correspondence is between cyclic \( * \)-representations of \( \mathfrak{A} \) and certain
linear functionals on \( \mathfrak{A} \) (called states). The correspondence is shown by an explicit
construction of the \( * \)-representation from the state. In the case of groups \( G \), the cor-
respondence is between unitary representations of \( G \), with cyclic vectors, on the one
side, and positive definite functions \( F \) on \( G \) on the other. One further shows that the
given p.d. function \( F \) on \( G \) is continuous iff the corresponding unitary representation
is strongly continuous.

The following is the version of the GNS construction most relevant for our preset
purpose. The construction is named after Israel Gelfand, Mark Naimark, and Irving
Segal; and it is abbreviated the GNS construction [Dix96].

Theorem 3.7. Let \( G \) be a group, and let \( F \) be a positive definite (p.d.) function on
\( G \), i.e., for all \( \{x_i\}_{i=1}^n \subset G \), the \( n \times n \) matrix \( \{ F(x_i^{-1} x_j) \}_{i,j=1}^n \) is assumed positive
semidefinite. Assume \( F(e) = 1 \).

1) Then there is a Hilbert space \( \mathcal{H} \), a vector \( v_0 \in \mathcal{H} \), \( \|v_0\| = 1 \), and a unitary
representation \( U \),

\[
U : G \rightarrow (\text{unitary operators on } \mathcal{H}), \ s.t.
\]

\[
F(x) = \langle v_0, U(x)v_0 \rangle_{\mathcal{H}}, \quad \forall x \in G.
\]  

2) The triple \((\mathcal{H}, v_0, U)\) may be chosen such that \( v_0 \) is a cyclic vector for the
unitary representation in (3.35). (We shall do this.)

3) If \((\mathcal{H}_i, v_0^{(i)}, U_i), i = 1, 2, \) are two solutions to the problem in (1)-(2), i.e.,

\[
F(x) = \left\langle v_0^{(i)}, U_i(x)v_0^{(i)} \right\rangle_{\mathcal{H}_i}, \quad i = 1, 2
\]
holds for all \( x \in G \); then there is an isometric intertwining isomorphism \( W : \mathcal{H}_1 \to \mathcal{H}_2 \) (onto \( \mathcal{H}_2 \)) such that

\[
WU_1 (x) = U_2 (x) W
\]

(3.38)

holds on \( \mathcal{H}_1 \), for all \( x \in G \), the intertwining property.

**Proof.** We shall refer to [Dix96] for details. The main ideas are sketched below:

Starting with a given \( F : G \to \mathbb{C} \), p.d., as in the statement of the theorem, we build the Hilbert space (from part (1)) as follows:

Begin with the space of all finitely supported functions \( \psi : G \to \mathbb{C} \), denoted by \( \text{Fun}_{\text{fin}} (G) \). For \( \phi, \psi \in \text{Fun}_{\text{fin}} (G) \), set

\[
\langle \phi, \psi \rangle_H := \sum_x \sum_y \overline{\phi(x)} \psi(y) F(x^{-1} y).
\]

(3.39)

From the definition, it follows that

\[
\langle \psi, \psi \rangle_H \geq 0, \quad \forall \psi \in \text{Fun}_{\text{fin}} (G);
\]

and moreover, that \( N_F = \{ \psi \mid \langle \psi, \psi \rangle_H = 0 \} \) is a subspace of \( \text{Fun}_{\text{fin}} (G) \). To get \( \mathcal{H} (= \mathcal{H}_F) \) as a Hilbert space, we do the following two steps:

\[
\begin{array}{ccc}
\text{Fun}_{\text{fin}} (G) & \xrightarrow{\text{(quotient)}} & \text{Fun}_{\text{fin}} (G) / N_F \\
& \xrightarrow{\text{(Hilbert completion)}} & \mathcal{H}.
\end{array}
\]

(3.40)

For \( v_0 \in \mathcal{H} \), we choose

\[
v_0 = \text{class} (\delta_e) = (\delta_e + N_F) \in \mathcal{H}, \quad \text{see } (3.40).
\]

(3.41)

In particular, for all \( \phi, \psi \in \text{Fun}_{\text{fin}} (G) \) and \( g \in G \), we have

\[
\langle \phi (g^{-1} \cdot), \psi (g^{-1} \cdot) \rangle_H = \sum_x \sum_y \overline{\phi(x)} \psi(y) F(x^{-1} y)
\]

\[
= \langle \phi, \psi \rangle_H, \quad \text{see } (3.39).
\]

Set \( U(g) \psi = \psi (g^{-1} \cdot) \), i.e.,

\[
(U(g) \psi) (x) = \psi (g^{-1} x), \quad x \in G.
\]

(3.42)

Using (3.39)-(3.41), it is clear that \( U \) as defined in (3.42) passes to the quotient and the Hilbert completion from (3.40), and that it will be the solution to our problem.

Finally, the uniqueness part of the theorem is left for the reader. See, e.g., [Dix96].

**Corollary 3.4.** Assume that the group \( G \) is a topological group. Let \( F \) be a p.d. function specified as in Theorem 3.7 (1), and let \( (\mathcal{H}_F, v_0, U) \) be the representation
triple (cyclic, see part (2) of the theorem), solving the GNS problem. Then F is continuous if and only if the representation U is strongly continuous, i.e., if, for all \( \psi \in \mathcal{H}_F \), the function

\[
G \ni x \mapsto U(x)\psi \in \mathcal{H}_F
\]

is continuous in the norm of \( \mathcal{H}_F \); iff for all \( \varphi, \psi \in \mathcal{H}_F \), the function

\[
F_{\varphi, \psi}(x) = \langle \varphi, U(x)\psi \rangle_{\mathcal{H}_F}
\]

is continuous on G.

### 3.2.2 Local Representations

**Definition 3.4.** Let \( G \) be a Lie group and let \( \mathcal{O} \) be an open connected neighborhood of \( e \in G \). A local representation (defined on \( \mathcal{O} \)) is a function:

\[
L : \mathcal{O} \rightarrow (\text{isometric operators in some Hilbert space } \mathcal{H})
\]

such that

\[
L(e) = I_{\mathcal{H}} (= \text{the identity operator in } \mathcal{H}); \tag{3.43}
\]

and if \( x, y \in \mathcal{O} \), and \( xy \in \mathcal{O} \) as well, then

\[
L(xy) = L(x)L(y), \tag{3.44}
\]

where \( xy \) denotes the group operation from \( G \).

**Theorem 3.8 (GNS for local representations).** Let \( G \) be as above, and let \( \Omega \subset G \) be an open connected subset. Set

\[
\mathcal{O} = \Omega^{-1}\Omega = \{ x^{-1}y \mid x, y \in \Omega \}. \tag{3.45}
\]

Let \( F \) be a continuous (locally defined) p.d. function on \( \mathcal{O} = \Omega^{-1}\Omega \); and assume \( F(e) = 1 \).

1. Then there is a triple \( (\mathcal{H}, v_0, L) \), where \( \mathcal{H} \) is a Hilbert space, \( v_0 \in \mathcal{H}, \|v_0\| = 1 \), and \( L \) is an \( \mathcal{O} \)-local representation, strongly continuous, such that

\[
F(x) = \langle v_0, L(x)v_0 \rangle_{\mathcal{H}}, \quad \forall x \in \mathcal{O}. \tag{3.46}
\]

2. If \( L \) is cyclic w.r.t. \( v_0 \), then the solution \( (\mathcal{H}, v_0, L) \) is unique up to unitary equivalence.

3. Assume \( L \) is cyclic; then on the dense subspace \( \mathcal{D}_L \) of vectors spanned by

\[
L_{\varphi} v_0 = \int_{\mathcal{O}} \varphi(x)L(x)v_0 dx, \quad \varphi \in C_0^\infty(\mathcal{O}) \tag{3.47}
\]
(integration w.r.t. a choice of a left-Haar measure on $G$), we get a representation of the Lie algebra, $\mathfrak{L}(G)$ of $G$, acting as follows:

Set $\lambda := dL$, and

$$\lambda(X) L_\varphi := L_{\tilde{X} \varphi}$$

(3.48)

where $\tilde{X} = \text{the vector field (3.24)}$. Specifically, on $\mathcal{D}_L$, we have,

$$\lambda([X,Y]) = \lambda(X) \lambda(Y) - \lambda(Y) \lambda(X)$$

(3.49)

(commutator), for all $X, Y \in \mathfrak{L}(G)$, where $[X,Y]$ on the l.h.s. in (3.49) denotes the Lie bracket from $\mathfrak{L}(G)$.

**Proof.** The reader will notice that the proof of these assertions follows closely those details outlined in the proofs of the results from Sections 2.2.1, 3.2 and 3.2.1, especially Theorem 3.7 and Corollary 3.4. The reader will easily be able to fill in details; but see also [JT14a]. \[\square\]

**Corollary 3.5.** Let $G$ be a connected locally compact group, and let $\mathcal{O}$ be an open neighborhood of $e$ in $G$. Let $F$ be a continuous positive definite function, defined on $\mathcal{O}$, and let $\tilde{F}$ be a positive definite extension of $F$ to $G$; then $\tilde{F}$ is continuous on $G$.

**Proof.** Using Theorems 3.7 and 3.8, we may realize both $F$ and $\tilde{F}$ in GNS-representations, yielding a local representation $L$ for $F$, and a unitary representation $U$ for the p.d. extension $\tilde{F}$. This can be done in such a way that $U$ is an extension of $L$, i.e., extending from $\mathcal{O}$ to $G$. Since $F$ is continuous, it follows that $L$ is strongly continuous. But it is known that a unitary representation (of a connected $G$) is strongly continuous iff it is strongly continuous in a neighborhood of $e$. The result now follows from Corollary 3.4. \[\square\]

**Remark 3.6.** It is not always possible to extend continuous p.d. functions $F$ defined on open subsets $\mathcal{O}$ in Lie groups. In Section 4.6 we give examples of locally defined p.d. functions $F$ on $G = \mathbb{R}^2$, i.e., $F$ defined on some open subset $\mathcal{O} \subset \mathbb{R}^2$ for which $\text{Ext}(F) = \emptyset$. But when extension is possible, the following algorithm applies, see Figure 3.3; also compare with Figure 2.3 in the case of $G = \mathbb{R}$.

### 3.2.3 The Convex Operation in $\text{Ext}(F)$

**Definition 3.5.** Let $G$, $\Omega$, $\mathcal{O} = \Omega^{-1}\Omega$, and $F$ be as in the statement of Theorem 3.8; i.e., $F$ is a given continuous p.d. function, locally defined in $G$. Assume $F(e) = 1$.

Using Theorem 3.7, we consider “points” in $\text{Ext}(F)$ to be triples $\pi := (\mathcal{H}, U, v(0))$ specified as in (1)-2 of the theorem; i.e.,

$\mathcal{H}$: Hilbert space

$U$: strongly continuous unitary representation of $G$
Fig. 3.3: Extension correspondence. From locally defined p.d. function $F$ to local representation $L$, to representation of the Lie algebra, to a unitary representation $U$ of $G$, and finally to an element in $Ext(F)$.

$v^{(0)}$: vector in $\mathcal{H}$ s.t. $\|v^{(0)}\|_{\mathcal{H}} = 1$.

Moreover, the restriction property

$$F(x) = \langle v^{(0)}, U(x)v^{(0)} \rangle_{\mathcal{H}}, \ x \in \mathcal{O}, \quad (3.50)$$

holds. Denote the r.h.s. in (3.50), $\tilde{F}_\pi$.

Now let $\alpha \in \mathbb{R}$ satisfying $0 < \alpha < 1$, and let $\pi_i = (\mathcal{H}_i, U_i, v_i^{(0)})$, $i = 1, 2$, be two “points” in $Ext(F)$; in particular, (3.50) holds for both $\pi_1$ and $\pi_2$. The convex combination is

$$\pi = \alpha \pi_1 + (1 - \alpha) \pi_2. \quad (3.51)$$

It is realized as follows:

Set $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$, $U := U_1 \oplus U_2$, and

$$v^{(0)} = \sqrt{\alpha}v_1^{(0)} + \sqrt{1-\alpha}v_2^{(0)} \in \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2; \quad (3.52)$$

those are the standard direct sum operations in Hilbert space.

Claim. The function

$$\tilde{F}(x) = \langle v^{(0)}, U(x)v^{(0)} \rangle_{\mathcal{H}}, \ x \in G$$

satisfies (3.50), and so it is in $Ext(F)$.

Proof. Setting

$$\tilde{F}_i(x) = \langle v_i^{(0)}, U_i(x)v_i^{(0)} \rangle_{\mathcal{H}_i}, \ i = 1, 2, \text{ and } x \in G. \quad (3.53)$$
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Then, using (3.52), one easily verifies that
\[
\tilde{F}(x) = a\tilde{F}_1(x) + (1 - a)\tilde{F}_2(x)
\]
(3.54)
holds for all \(x \in G\); and the desired conclusion follows.

\[\square\]

Remark 3.7. Using Definition 3.5, one now checks that, for Examples 1-5 in Table 5.1 below, we get that
\[
\text{Ext}_1(F) \subseteq \text{ext}(\text{Ext}(F))
\]
(3.55)
where the “ext” on the r.h.s. in (3.55) denotes the set of extreme points in \(\text{Ext}(F)\), in the sense of Krein-Milman [Phe66].

In general, (3.55) is a proper containment.

Example 3.2. Below we illustrate (3.55) in Remark 3.7 with \(F = F_3\) from Table 5.1, i.e., \(F(t) = e^{-|t|}\) in \(|t| < 1\); \(G = \mathbb{G}\), and \(\Omega = (0,1)\). In this case, the skew-Hermitian operator \(D^F\) from Section 2.2 has deficiency \((1,1)\). Using the discussion above, one sees that every “point” \(\pi = (\mathcal{H}(\pi), U(\pi), v(\pi))\) in the closed convex hull, \(\text{convExt}_1(F)\), of \(\text{Ext}_1(F)\) has the following form. (See, e.g., [Phe66].)

Fix such a \(\pi\). There then is a unique probability measure \(\rho^\pi\) on \(T = \mathbb{R}/\mathbb{Z}\) such that, up to unitary equivalence, the three components of \(\pi\) are as follows:

- \(\mathcal{H}(\pi)\): measurable functions \(f\) on \(\mathbb{R} \times T\) such that
  \[
  \chi_{[0,1]}(\cdot) f(\cdot, \alpha) \in \mathcal{H}_F, \quad \alpha \in T;
  \]
  (3.56)
  \[
  (f + f')(x + n, \alpha) = \alpha^n (f - f')(x, \alpha), \quad \forall x \in \mathbb{R}, n \in \mathbb{Z}, \alpha \in T;
  \]
  (3.57)
  where \(f\) is \(C^1\) in the \(x\)-variable (in \(\mathbb{R}\)), and measurable in \(\alpha\) (in \(T\)). And
  \[
  \|f\|^2_{\mathcal{H}(\pi)} := \int_T \|\chi_{[0,1]}(\cdot) f(\cdot, \alpha)\|^2_{\mathcal{H}_F} d\alpha^{(\text{Haar})} < \infty,
  \]
  (3.58)
  where \(d\alpha\) is the Haar measure on \(T\), i.e, \(\alpha = e^{2\pi i \theta}, 0 \leq \theta \leq 1\), yields
  \[
  d\alpha^{(\text{Haar})} = d\theta.
  \]
  Set
  \[
  \mathcal{H}(\pi) = \int_T^{(1)} \mathcal{H}_F d\rho^\pi(\pi),
  \]
  (3.59)
  as a direct integral.

Remark 3.8. Eq (3.57) above is a “boundary-condition.” In detail: For fixed \(\alpha \in T\), this condition comes from the von Neumann extension corresponding to the deficiency analysis of our skew-Hermitian operator \(D^F\). More specifically, recall that, as an operator in \(\mathcal{H}_F\) (with dense domain), \(D^F\) has deficiency indices \((1,1)\) and so, by our analysis in Section 4.5, every \(\alpha \in T\) corresponds to a unique skew-adjoint extension \(A^{(\alpha)}\) of \(D^F\). The conditions in (3.57) simply allow us to realize the unitary
one-parameter group \( U(t) \) generated by \( A(\alpha) \), with \( U(t) \) realized as per the specification in (3.62). For further details on boundary conditions in reproducing kernel Hilbert spaces of type \( \mathcal{H}_F \), see [JT14b].

- \( U(\pi) \): Up to unitary equivalence, we get
  \[
  U_{t}^{(\pi)}(f(\cdot, \alpha))_{\alpha \in \mathbb{T}}(x) := (f(x + t, \alpha))_{\alpha \in \mathbb{T}}, \quad \forall t \in \mathbb{R};
  \]
  where we use the representation (3.59) above for \( \mathcal{H}(\pi) \).

- Finally, we have the following direct integral representation of the respective rank-one operators (using Dirac notation):
  \[
  \left| \left| v^{(\pi)} \right| \right| = \left( \int_{\mathbb{T}} dp^{(\pi)}(\alpha) \left| F_{0} \right\rangle \left\langle F_{0} \right| \right)
  \]
  The ket-bra notation \( |v\rangle\langle v| \) means rank-one operator.

For \( \alpha \in \mathbb{T} \) fixed, we note that
\[
\left( U_{t}^{(\alpha)} f(\cdot, \alpha) \right)(x) = f(x + t, \alpha), \quad t \in \mathbb{R},
\]
represents a unique element in \( \text{Ext}_1(F) \), and
\[
U_{t}^{(\pi)} = \int_{\mathbb{T}} U_{t}^{(\alpha)} dp^{(\pi)}(\alpha), \quad t \in \mathbb{R}
\]
is the unitary representation corresponding to \( \pi \), i.e.,
\[
\widetilde{F}^{(\pi)}(t) = \int_{\mathbb{T}} \widetilde{F}^{(\alpha)}(t) dp^{(\pi)}(\alpha), \quad t \in \mathbb{R};
\]
and \( \widetilde{F}^{(\alpha)}(\cdot) \) is the continuous p.d. function on \( \mathbb{R} \) which extends \( F \) on \((-1, 1)\), and corresponding to the “point” \( \pi \in \text{convExt}_1(F) \) [Phe66].

The respective \( \mathbb{R} \) p.d. functions in (3.64) are as follows:
\[
F^{(\pi)}(t) = \left\langle v^{(\pi)}(\cdot), U_{t}^{(\pi)} v^{(\pi)}(\cdot) \right\rangle_{\mathcal{H}(\pi)}, \quad t \in \mathbb{R};
\]
and
\[
\widetilde{F}^{(\alpha)}(t) = \left\langle F_{0}, U_{t}^{(\alpha)} F_{0} \right\rangle_{\mathcal{H}_F},
\]
defined for all \( t \in \mathbb{R} \), and \( \alpha \in \mathbb{T} \). For each \( \alpha \) fixed, we have
\[
U_{t}^{(\alpha)} = e^{A(\alpha)}, \quad t \in \mathbb{R},
\]
where \( A(\alpha) = -(A(\alpha))^* \) is a skew-adjoint extension of \( D^{(F)} \), acting in \( \mathcal{H}_F \).

Remark 3.9. In the direct-integral formulas (3.59) and (3.61), we stress that it is a direct integral of copies of the Hilbert space \( \mathcal{H}_F \); more precisely, an isomorphic
copy of $\mathcal{H}_F$ for every $\alpha \in \mathbb{T}$. For fixed $\alpha \in \mathbb{T}$, when we define our Hilbert space via eq (3.57), then we will automatically get the right boundary condition. They are built in. And on this Hilbert space, then the action of $U(t)$ will simplify; it will just be the translation formula (3.62). In fact, the construction is analogous to an induced representation [Mac88]. We must also need to justify “up to unitary equivalence.” It is on account of Mackey’s imprimitivity theorem, see [Ørs79].

Our present direct integral representation should be compared with the Zak transform from signal processing [Jan03, ZM95].
Chapter 4
Examples

It was mathematics, the non-empirical science par excellence, wherein the mind appears to play only with itself, that turned out to be the science of sciences, delivering the key to those laws of nature and the universe that are concealed by appearances.
— Hannah Arendt, The Life of the Mind (1971), p.7.

While the present chapter is made up of examples, we emphasis that the selection of examples is carefully chosen; – chosen and presented in such a way that the details involved, do in fact cover and illustrate general and important ideas, which in turn serve to bring out the structure of more general theorems (in the second half of our monograph.) Moreover, the connection from the examples to more general contexts will be mentioned inside the present chapter, on a case-by-case basis. We further emphasize that some of the examples have already been used to illustrate key ideas in Chapters 1 and 2 above. Below we flesh out the details. And our present examples will be used again in Chapters 6 through 11 below, dealing with theorems that apply to any number of a host of general settings.

4.1 The Case of $G = \mathbb{R}^n$

Of course, $G = \mathbb{R}^n$ is a special case of locally compact Abelian groups (Section 3.1), but the results available for $\mathbb{R}^n$ are more refined. We focus on this in the present section. This is also the setting of the more classical studies of the extension questions.

Let $\Omega \subset \mathbb{R}^n$ be a fixed open and connected subset; and let $F : \Omega - \Omega \to \mathbb{C}$ be a given continuous p.d. function, where

$$\Omega - \Omega := \{x - y \in \mathbb{R}^n \mid x, y \in \Omega \}. \quad (4.1)$$

Let $\mathcal{H}_F$ be the corresponding reproducing kernel Hilbert space (RKHS).

We showed (Theorem 2.4) that $\text{Ext}(F) \neq \emptyset$ if and only if there is a strongly continuous unitary representation $\{U(t)\}_{t \in \mathbb{R}^n}$ acting on $\mathcal{H}_F$ such that

$$\mathbb{R}^n \ni t \longmapsto \langle F_0, U(t)F_0 \rangle_{\mathcal{H}_F} \quad (4.2)$$

is a p.d. extension of $F$, extending from (4.1) to $\mathbb{R}^n$.

Now if $U$ is a unitary representation of $G = \mathbb{R}^n$, we denote by $P_U(\cdot)$ the associated projection valued measure (PVM) on $\mathcal{B}(\mathbb{R}^n) (= \text{the sigma–algebra of all Borel}$
subsets in $\mathbb{R}^n$). We have

$$U(t) = \int_{\mathbb{R}^n} e^{it \cdot \lambda} P_U(d\lambda), \quad \forall t \in \mathbb{R}^n; \quad (4.3)$$

where $t = (t_1, \ldots, t_n)$, $\lambda = (\lambda_1, \ldots, \lambda_n)$, and $t \cdot \lambda = \sum_{j=1}^n t_j \lambda_j$. Setting

$$d\mu(\cdot) = \|P_U(\cdot)F_0\|_{\mathcal{H}_F}^2,$$  

then the p.d. function on r.h.s. in (4.2) satisfies

$$r.h.s. \ (4.2) = \int_{\mathbb{R}^n} e^{it \cdot \lambda} d\mu(\lambda), \quad \forall t \in \mathbb{R}^n. \quad (4.5)$$

The purpose of the next theorem is to give an orthogonal splitting of the RKHS $\mathcal{H}_F$ associated to a fixed $(\Omega, F)$ when it is assumed that $\text{Ext}(F)$ is non-empty. This orthogonal splitting of $\mathcal{H}_F$ depends on a choice of $\mu \in \text{Ext}(F)$, and the splitting is into three orthogonal subspaces of $\mathcal{H}_F$, correspond a splitting of spectral types into atomic, absolutely continuous (with respect to Lebesgue measure), and singular.

**Theorem 4.1.** Let $\Omega \subset \mathbb{R}^n$ be given, $\Omega \neq \emptyset$, open and connected. Suppose $F$ is given p.d. and continuous on $\Omega - \Omega$, and assume $\text{Ext}(F) \neq \emptyset$. Let $U$ be the corresponding unitary representations of $G = \mathbb{R}^n$, and let $P_U(\cdot)$ be its associated PVM acting on $\mathcal{H}_F$.

1. Then $\mathcal{H}_F$ splits up as an orthogonal sum of three closed and $U(\cdot)$-invariant subspaces

$$\mathcal{H}_F = \mathcal{H}^{\text{atom}}_F \oplus \mathcal{H}^{\text{ac}}_F \oplus \mathcal{H}^{\text{sing}}_F,$$  

characterized by the PVM $P_U$ as follows:

(a) $P_U$ restricted to $\mathcal{H}^{\text{atom}}_F$ is purely atomic;

(b) $P_U$ restricted to $\mathcal{H}^{\text{ac}}_F$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$; and

(c) $P_U$ is continuous, purely singular, when restricted to $\mathcal{H}^{\text{sing}}_F$.

2. Case $\mathcal{H}^{\text{atom}}_F$. If $\lambda \in \mathbb{R}^n$ is an atom in $P_U$, i.e., $P_U(\{\lambda\}) \neq 0$, where $\{\lambda\}$ denotes the singleton with $\lambda$ fixed; then $P_U(\{\lambda\}) \mathcal{H}_F$ is one-dimensional. Moreover,

$$P_U(\{\lambda\}) \mathcal{H}_F = C e^{\lambda \cdot x};$$  

where $e^{\lambda \cdot x}$ is the complex exponential. In particular, $e^{\lambda \cdot x} |_{\Omega} \in \mathcal{H}_F$.

Case $\mathcal{H}^{\text{ac}}_F$. If $\xi \in \mathcal{H}^{\text{ac}}_F$, then it is represented as a continuous function on $\Omega$, and

$$\langle \xi, F \varphi \rangle_{\mathcal{H}_F} = \int_{\Omega} \overline{\xi(x)} \varphi(x) \, dx (\text{Lebesgue}), \quad \forall \varphi \in C_c(\Omega). \quad (4.8)$$
Further, there is a $f \in L^2(\mathbb{R}^n, \mu)$ ($\mu$ as in (4.4)) such that, for all $\varphi \in C_c(\Omega)$,
\[\int_{\Omega} \xi(x) \varphi(x) \, dx = \int_{\mathbb{R}^n} f(\lambda) \varphi(\lambda) \, d\mu(\lambda), \text{ and} \]
(4.9)
\[\xi = (f \, d\mu)^\vee \big|_{\Omega}. \] (4.10)
The r.h.s. of (4.10) is a $\mu$-extension of $\xi$. Consequently, every $\mu$-extension $\tilde{\xi}$ of $\xi$ is continuous on $\mathbb{R}^n$, and $\lim_{|x| \to \infty} \tilde{\xi}(x) = 0$, i.e., $\tilde{\xi} \in C_0(\mathbb{R}^n)$.

Case $\mathcal{H}_F^{(\text{sing})}$. Vectors $\xi \in \mathcal{H}_F^{(\text{sing})}$ are characterized by the following property:
The measure
\[d\mu_\xi(\cdot) := \|P_U(\cdot)\xi\|_{\mathcal{H}_F}^2\] (4.11)
is continuous and purely singular.

Proof. Most of the proof details are contained in the previous discussion. See Section 3.1, Theorem 3.3. For the second part of the theorem:

Case $\mathcal{H}_F^{(\text{atom})}$. Suppose $\lambda \in (\mathbb{R}^n)$ is an atom, and that $\xi \in \mathcal{H}_F \setminus \{0\}$ satisfies $P_U(\{\lambda\}) \xi = \xi$; then
\[U(t) \xi = e^{it \lambda} \xi, \quad \forall t \in \mathbb{R}^n. \] (4.12)
By (4.2)-(4.3), we conclude that $\xi$ is a continuous, weak solution to the elliptic system
\[\frac{\partial}{\partial x_j} \xi_j = \sqrt{-1} \lambda_j \xi \quad (\text{on } \Omega), \quad 1 \leq j \leq n. \] (4.13)
Hence $\xi = \text{const} \cdot e^{i \lambda_\Omega}$ as asserted in (4.7).

Case $\mathcal{H}_F^{(\text{ac})}$; this follows from (4.10) and the Riemann-Lebesgue theorem applied to $\mathbb{R}^n$; and the case $\mathcal{H}_F^{(\text{sing})}$ is immediate. \hfill $\square$

Example 4.1. Consider the following continuous p.d. function $F$ on $\mathbb{R}$, or on some bounded interval $(-a,a)$, $a > 0$.
\[F(x) = \frac{1}{3} \left( e^{-ix} + \prod_{n=1}^{m} \cos \left( \frac{2\pi x}{3^n} \right) + e^{ix/2} \sin \left( \frac{x}{2} \right) \right). \] (4.14)
The RKHS $\mathcal{H}_F$ has the decomposition (4.6), where all three subspaces $\mathcal{H}_F^{(\text{atom})}$, $\mathcal{H}_F^{(\text{ac})}$, and $\mathcal{H}_F^{(\text{sing})}$ are non-zero; the first one is one-dimensional, and the other two are infinite-dimensional. Moreover, the operator
\[D(F)(F_\varphi) := F_{\varphi'} \text{ on} \]
\[\text{dom}(D(F)) = \{ F_\varphi \mid \varphi \in C_c^\infty(0,a) \} \] (4.15)
is bounded, and so extends by closure to a skew-adjoint operator, satisfying
\[-(D(F))^* = D(F^*). \]
Proof. Using infinite convolutions of operators (Chapter 9), and results from [DJ12],
we conclude that $F$ defined in (4.14) is entire analytic, and $F = d\hat{\mu}$ (Bochner-
transform) where

$$d\mu(\lambda) = \frac{1}{3} (\delta_{-1} + \mu_c + \chi_{[1,2]}(\lambda) d\lambda). \quad (4.16)$$

The measures on the r.h.s. in (4.16) are as follows (Figure 4.1-4.2):

- $\delta_{-1}$ is the Dirac mass at $-1$, i.e., $\delta(\lambda + 1)$.
- $\mu_c$ is the middle-third Cantor measure; determined as the unique solution in
  $\mathcal{M}_+^{\text{prob}}(\mathbb{R})$ to

$$\int f(\lambda) d\mu_c(\lambda) = \frac{1}{2} \left( \int f\left(\frac{\lambda + 1}{3}\right) d\mu_c(\lambda) + \int f\left(\frac{\lambda - 1}{3}\right) d\mu_c(\lambda) \right), \quad \forall f \in C_c(\mathbb{R}).$$

- $\chi_{[1,2]}(\lambda) d\lambda$ is restriction to the closed interval $[1,2]$ of Lebesgue measure.

It follows from the literature (e.g. [DJ12]) that $\mu_c$ is supported in $[-\frac{1}{2}, \frac{1}{2}]$. Thus,
the three measures on the r.h.s. in (4.16) have disjoint compact supports, with the
three supports positively separately.

The conclusions asserted in Example 4.1 follow from this, in particular the prop-
erties for $D(F)$. In fact,

$$\text{spectrum}(iD(F)) \subseteq \{-1\} \cup \left[-\frac{1}{2}, \frac{1}{2}\right] \cup [1,2] \quad (i = \sqrt{-1}) \quad (4.17)$$

\[\square\]

Corollary 4.1. Let $\Omega \subset \mathbb{R}^n$ be non-empty, open and connected. Let $F: \Omega \to \mathbb{C}$
be a fixed continuous and p.d. function; and let $\mathcal{H}_F$ be the corresponding RKHS (of
functions on $\Omega$.)
4.1 The Case of $G = \mathbb{R}^n$

Fig. 4.2: Cumulative distribution $= \int_{-\infty}^{\lambda} d\mu (\lambda)$, as in Example 4.1.

(1) If there is a compactly supported measure $\mu \in \text{Ext} (F)$, then every function $\xi$ on $\Omega$, which is in $\mathcal{H}_F$, has an entire analytic extension to $\mathbb{C}^n$, i.e., extension from $\Omega \subset \mathbb{R}^n$ to $\mathbb{C}^n$.

(2) If, in addition, it is assumed that $\mu \ll d\lambda (= d\lambda_1 \cdots d\lambda_n) =$ the Lebesgue measure on $\mathbb{R}^n$, where “$\ll$” means “absolutely continuous”; then the functions in $\mathcal{H}_F$ are restrictions to $\Omega$ of $C_c (\mathbb{R}^n)$-functions.

Proof. Part (1). Let $\Omega, F, \mathcal{H}_F$ and $\mu$ be as stated. Let $\mathcal{H}_F \overset{T}{\rightarrow} L^2 (\mathbb{R}^n, \mu)$ be the isometry from Corollary 3.1, where

$$T (F \varphi) := \hat{\varphi}, \quad \varphi \in C_c (\Omega).$$

(4.18)

By (3.12), for all $f \in L^2 (\mathbb{R}^n, \mu)$, we have

$$(T^* f) (x) = \int_{\mathbb{R}^n} e^{ix \cdot \lambda} f (\lambda) d\mu (\lambda), \quad x \in \Omega;$$

(4.19)

Equivalently,

$$T^* f = \mathcal{H}_F (f d\mu)^\vee, \quad \forall f \in L^2 (\mu).$$

(4.20)

And further that

$$T^* (L^2 (\mathbb{R}^n, \mu)) = \mathcal{H}_F.$$  

(4.21)

Now, if $\mu$ is of compact support, then so is the complex measure $f d\mu$. This measure is finite since $L^2 (\mu) \subset L^1 (\mu)$. Hence the desired conclusion follows from (4.19), (4.21), and the Paley-Wiener theorem; see e.g., [Rud73].

Part (2) of the corollary follows from Theorem 4.1, case $\mathcal{H}_F^{ac}$. See also Proposition 4.2.

Details about (4.20): For all $f \in L^2 (\mu)$, and all $\varphi \in C_c (\Omega)$, we have

$$T^* (f \varphi) = \mathcal{H}_F (f d\mu)^\vee \varphi.$$
\[ \langle T(f_\phi), f \rangle_{L^2(\mu)} = \int_{\mathbb{R}} \phi(\lambda) f(\lambda) d\mu(\lambda) \]
\[ \quad = \int_{\mathbb{R}} \left( \int_{\Omega} \phi(x) e^{-i\lambda \cdot x} dx \right) f(\lambda) d\mu(\lambda) \]
\[ \quad = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \chi_{\Omega}(x) \phi(x) e^{-i\lambda \cdot x} dx \right) f(\lambda) d\mu(\lambda) \]
\[ \quad = \int_{\mathbb{R}} \chi_{\Omega}(x) \phi(x) \left( \int_{\mathbb{R}} e^{i\lambda \cdot x} f(\lambda) d\mu(\lambda) \right) dx \]
\[ \quad = \int_{\mathbb{R}} \phi(x) \chi_{\Omega}(x \cdot f d\mu) (x) dx \]
\[ \quad = \langle F_\phi, \chi_{\Omega}(x \cdot f d\mu) \rangle_{\mathcal{H}_F}, \]

which is the desired conclusion. \[ \square \]

**4.2 The Case of \( G = \mathbb{R}/\mathbb{Z} \)**

While we consider extensions of locally defined continuous and positive definite (p.d.) functions \( F \) on groups, say \( G \), the question of whether \( G \) is simply connected or not plays an essential role in the analysis, and in the possible extensions.

It turns out that the geometric issues arising for general Lie groups can be illustrated well in a simple case. To illustrate this point, we isolate below the two groups, the circle group, and its universal cover, the real line. We study extensions defined on a small arc in the circle group \( G = \mathbb{T} = \mathbb{R}/\mathbb{Z} \), versus extensions to the universal covering group \( \mathbb{R} \).

Let \( G = \mathbb{T} \) represented as \((-\frac{1}{2}, \frac{1}{2}]\). Pick \( 0 < \varepsilon < \frac{1}{2} \), and let \( \Omega = (0, \varepsilon) \), so that \( \Omega - \Omega = (-\varepsilon, \varepsilon) \mod \mathbb{Z} \).

Let \( F : \Omega - \Omega \rightarrow \mathbb{C} \) be a continuous p.d. function; \( D^{(F)} \) is the canonical skew-Hermitian operator, s.t. \( \text{dom}(D^{(F)}) = \{ F_\phi \mid \phi \in C^\infty_c(\Omega) \} \), \( D^{(F)}(F_\phi) = F_{\phi'} \), acting in the RKHS \( \mathcal{H}_F \). As shown in Section 2.1, \( D^{(F)} \) has deficiency indices \((0,0)\) or \((1,1)\).

**Lemma 4.1.** If \( D^{(F)} \) has deficiency indices \((1,1)\), there is a skew-adjoint extension of \( D^{(F)} \) acting in \( \mathcal{H}_F \), such that the corresponding p.d. extension \( \tilde{F} \) of \( F \) has period one; then \( \varepsilon \) is rational.

**Proof.** In view of Section 2.4 (also see Theorem 2.4), our assumptions imply that that the dilation Hilbert space is identical to \( \mathcal{H}_F \), thus an “internal” extension.

Since the deficiency indices of \( D^{(F)} \) are \((1,1)\), the skew-adjoint extensions of \( D^{(F)} \) are determined by boundary conditions of the form

\[ \xi(\varepsilon) = e^{i\theta} \xi(0), \quad (4.22) \]

where \( 0 \leq \theta < 1 \) is fixed.
Let $A_{\theta}$ be the extension corresponding to $\theta$, and

$$U_{A_{\theta}}(t) = e^{itA_{\theta}} = \int_{\mathbb{R}} e^{it\lambda} P_{A_{\theta}}(d\lambda), \quad t \in \mathbb{R}$$

be the corresponding unitary group, where $P_{\lambda}(\cdot)$ is the projection-valued measure of $A_{\theta}$. Set

$$F_{A_{\theta}}(t) = \langle F_0, U_{A_{\theta}}(t) F_0 \rangle_{H_{F}} = \int_{\mathbb{R}} e^{it\lambda} d\mu_{A_{\theta}}(\lambda)$$

where

$$d\mu_{A_{\theta}}(\lambda) = \| P_{A_{\theta}}(d\lambda) F_0 \|^2_{H_{F}}.$$

Repeating the calculation of the defect vectors and using that eigenfunctions of $A_{\theta}$ must satisfy the boundary condition (4.22), it follows that the spectrum of $A_{\theta}$ is given by

$$\sigma(iA_{\theta}) = \{ \theta + n\epsilon \mid n \in \mathbb{Z} \}.$$

Note that $\text{supp}(\mu_{A_{\theta}}) \subset \sigma(iA_{\theta})$, and the containment becomes equal if $F_0$ is a cyclic vector for $U_{A_{\theta}}(t)$. Since $F_{A_{\theta}}$ is assumed to have period one, $\text{supp}(\mu_{A_{\theta}})$ consists of integers. Now if $\lambda_n, \lambda_m \in \text{supp}(\mu_{A_{\theta}})$, then

$$\epsilon = \frac{n - m}{\lambda_n - \lambda_m}$$

which is rational. \qed

**Remark 4.1.** If

$$F(x) = e^{-|x|}, \quad -\pi < x \leq \pi$$

is considered a function on $\mathbb{T}$ via $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ (see Figure 4.3), then its spectral representation is

$$F(x) = \sum_{n \in \mathbb{Z}} e^{inx} \frac{1}{\pi(1 + n^2)}, \quad \forall x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}.$$ 

**Proof.** We must compute the Fourier series expansion of $F$ as given by (4.23) and the Figure 4.3 coordinate system. Using normalized Haar measure on $\mathbb{T}$, the Fourier coefficients are as follows:
Fourier \( (F,n) \) = \[ \int_{\mathbb{T}} e^{-inx} F(x) \, dx \]
\[ = \int_{\mathbb{T}} e^{-inx} \left( \int_{\mathbb{R}} e^{i\lambda x} \frac{d\lambda}{\pi(1 + \lambda^2)} \right) \, dx \]
\[ = \left( \text{by Fubini} \right) \int_{\mathbb{R}} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(\lambda - n)x} \, dx \right) \frac{d\lambda}{\pi(1 + \lambda^2)} \]
\[ = \int_{\mathbb{R}} \frac{\sin \pi(\lambda - n)}{\pi(\lambda - n)} \frac{d\lambda}{\pi(1 + \lambda^2)} \]
\[ = \frac{1}{\pi(1 + n^2)}, \quad \forall n \in \mathbb{Z}, \]

where we used Shannon’s interpolation formula in the last step, and using Figure 4.3, to establish the band-limit property required for Shannon; see also Sections 5.1, 6.1, especially Theorem 6.5.

\[ F(x) = e^{-|x|} \]

Fig. 4.3: \( F(x) = e^{-|x|} \) on \( \mathbb{R}/2\pi \mathbb{Z} \)

### 4.3 Example: \( e^{2\pi x} \)

This is a trivial example, but it is helpful to understand subtleties of extensions of positive definite functions. In a different context, one studies classes of unbounded Hermitian operators which arise in scattering theory for wave equations in physics; see e.g., [PT13, JPT12, LP89, LP85]. For a discussion, see e.g., Section 10.2.1, and Lemma 10.7.

Consider the quantum mechanical momentum operator

\[ P = -\frac{1}{2\pi i} \frac{d}{dx}, \quad \text{dom}(P) = C^\infty_c(\mathbb{T}), \]
4.3 Example: $e^{2\pi x}$

acting in $L^2(\mathbb{T})$, where $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z} \simeq (-\frac{1}{2}, \frac{1}{2}]$. $P$ is Hermitian, with a dense domain in $L^2(\mathbb{T})$. It has deficiency indices $(1,1)$, and the family of selfadjoint extensions is determined by the boundary condition (4.22).

Setting $\theta = 0$ in (4.22), the corresponding selfadjoint (s.a.) extension $\tilde{P} \supset P$ has spectrum

$$\sigma(\tilde{P}) = \{ \xi_n := e^{i2\pi n} | n \in \mathbb{Z} \};$$

so that if $U(t) = e^{it\tilde{P}}$, $t \in \mathbb{R}$, then $(U(t)f)(x) = f(x+t)$, for all $f \in L^2(\mathbb{T})$. That is, $U(t)$ is the translation group.

**Example 4.2.** Now fix $0 < \varepsilon < \frac{1}{2}$, and set

$$F(x) := e^{i2\pi [x - (\varepsilon, -\varepsilon)]}.$$  \hspace{1cm} (4.24)

**Lemma 4.2.** The RKHS $\mathcal{H}$ of $F$ in (4.24) is one-dimensional. Moreover, for all $\varphi, \psi \in C^\infty_c(0, \varepsilon)$, we have

$$\langle F\varphi, F\psi \rangle_{\mathcal{H}} = \overline{\hat{\varphi}}(1) \overline{\hat{\psi}}(1).$$  \hspace{1cm} (4.25)

In particular,

$$\|F\varphi\|_{\mathcal{H}}^2 = |\hat{\varphi}(1)|^2.$$  \hspace{1cm} (4.26)

Here, $\hat{\cdot}$ denotes Fourier transform.

**Proof.** Recall that $\mathcal{H}$ is the completion of $\text{span} \{ F\varphi : \varphi \in C^\infty_c(0, \varepsilon) \}$, where

$$F\varphi(x) = \int_0^\varepsilon \varphi(y) F(x-y) dy$$

$$= e^{i2\pi x} \int_0^\varepsilon \varphi(y) e^{-i2\pi y} dy = e^{i2\pi x} \hat{\varphi}(1);$$

it follows that $\dim \mathcal{H} = 1$.

Moreover,

$$\langle F\varphi, F\psi \rangle_{\mathcal{H}} = \int_0^\varepsilon \int_0^\varepsilon \overline{\varphi(y)} \psi(y) F(x-y) dy dx$$

$$= \overline{\hat{\varphi}}(1) \overline{\hat{\psi}}(1), \quad \forall \varphi, \psi \in C^\infty_c(0, \varepsilon);$$

which is the assertion (4.25). \hfill $\square$

Note that $F$ in (4.24) is the restriction of a p.d. function, so it has at least one extension, i.e., $e^{i2\pi x}$ on $\mathbb{T}$. In fact, this is also the unique continuous p.d. extension to $\mathbb{T}$.

**Lemma 4.3.** Let $F$ be as in (4.24). If $\tilde{F}$ is a continuous p.d. extension of $F$ on $\mathbb{T}$, then $\tilde{F}(x) = e^{i2\pi x}$, $x \in (-\frac{1}{2}, \frac{1}{2}]$. 


Proof. Assume \( \tilde{F} \) is a continuous p.d. extension of \( F \) to \( \mathbb{T} \), then by Bochner’s theorem, we have

\[
e^{2\pi x} = \sum_{n \in \mathbb{Z}} \mu_n e^{i 2\pi nx}, \quad \forall x \in (-\varepsilon, \varepsilon)
\]  

(4.27)

where \( \mu_n \geq 0 \), and \( \sum_{n \in \mathbb{Z}} \mu_n = 1 \). Now each side of the above equation extends continuously in \( x \). By uniqueness of the Fourier expansion in (4.27), we get \( \mu_1 = 1 \), and \( \mu_n = 0, \ n \in \mathbb{Z} \setminus \{1\} \). \( \square \)

Remark 4.2. In fact, Lemma 4.3 holds in a more general context. (We thank the anonymous referee who kindly suggested the following abstract argument.)

Proposition 4.1. Let \( G \) be a connected locally compact Abelian group, \( u \) a positive definite function on \( G \) and \( \chi \) a character for which \( \chi|_U = u|_U \) for some neighborhood of \( U \) of the origin \( 1 \). Then \( u = \chi \).

Proof. Indeed let \( v = \chi u \) which is \( 1 \) on \( U \). An easy argument shows that then \( v \) is also p.d. on \( G \).

We now pass to the Gelfand-Naimark-Segal (GNS) representation (see Theorem 3.7), i.e., to a Hilbert space \( \mathcal{H} \), \( \sigma \) a cyclic representation of \( G \) acting on \( \mathcal{H} \), and a unit vector \( \xi \in \mathcal{H} \) s.t.

\[
v(x) = \langle \xi, \sigma(x)\xi \rangle_{\mathcal{H}}, \quad \forall x \in G.
\]

Notice that \( \mathcal{H} = \{ x \in G : v(x) = 1 \} \) is a subgroup. Indeed, since \( v = \langle \xi, \sigma(\cdot)\xi \rangle_{\mathcal{H}} \) where \( \|\xi\| = 1 \), then \( H = \{ x \in G : \sigma(x)\xi = \xi \} \), thanks to uniform convexity. But \( H \supseteq U \) so \( H \supseteq \bigcup_{n=1}^{\infty} U^n = G \), where we used that \( G \) is assumed connected. \( \square \)

4.4 Example: \( e^{-|x|} \) in \( (-a, a) \), extensions to \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \)

Fix \( 0 < a < \frac{1}{2} \). Let \( F : \Omega - \Omega \to \mathbb{C} \) be a continuous p.d. function, where \( \Omega = (0, a) \).

For the extension problem (continuous, p.d.), we consider two cases:

\[
\Omega \subset \mathbb{R} \quad \text{vs.} \quad \Omega \subset \mathbb{T} = \mathbb{R}/\mathbb{Z}
\]

(4.28)

First, recall the Pontryagin duality (see [Rud90]) for locally compact Abelian (l.c.a) groups:

\[
\text{cont. p.d. functions } \Phi \text{ on } G \quad \xrightarrow{\text{Bochner transform}} \quad \text{finite positive Borel measures } \mu \text{ on } \hat{G}
\]

\[
\Phi(x) = \int_G \chi(x) \, d\mu(\chi), \quad \chi \in \hat{G}, \ x \in G
\]

(4.29)

Application: In our current setting, \( G = \mathbb{T} = \mathbb{R}/\mathbb{Z}, \ \hat{G} = \mathbb{Z} \), and

\[
\chi_n(x) = e^{i 2\pi nx}, \quad n \in \mathbb{Z}, \ x \in \mathbb{T}
\]

(4.30)

so that
4.4 Example: $e^{-|x|}$ in $(-a,a)$, extensions to $T = \mathbb{R}/\mathbb{Z}$

\[
\Phi(x) = \sum_{n \in \mathbb{Z}} w_n e^{2\pi i nx}, \quad w_n \geq 0, \quad \sum_{n \in \mathbb{Z}} w_n < \infty. \quad (4.31)
\]

The weights \( \{w_n\} \) in (4.31) determines a measure \( \mu_w \) on \( \mathbb{Z} \), where

\[
\mu_w(E) = \sum_{n \in E} w_n, \quad \forall E \subset \mathbb{Z}. \quad (4.32)
\]

Conclusions: Formula (4.31) is a special case of formula (4.29); and (4.29) is the general l.c.a. Pontryagin-Bochner duality. (If \( \Phi(0) = 1 \), then \( \mu_w \) is a probability measure on \( \mathbb{Z} \).)

**Remark 4.3.** Distributions as in (4.32) arise in many applications; indeed stopping times of Markov chains is a case in point [Sok13, Du12].

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, and let \(X_n : \Omega \to \mathbb{R}\) be a stochastic process indexed by \( \mathbb{Z} \). Let \( M \subset \mathbb{R} \) be a bounded Borel subset. The corresponding stopping time \( \tau_M \) is as follows:

For \( \omega \in \Omega \), set

\[
\tau_M(\omega) = \inf \{n \in \mathbb{Z} \mid X_n(\omega) \in M\}. \quad (4.33)
\]

The distribution \( \mu_{\tau_M} \) of \( \tau_M \) is an example of a measure on \( \mathbb{Z} \), i.e., an instance of the setting in (4.32).

For \( n \in \mathbb{Z} \), set

\[
w^M_n := \mathbb{P}(\{\omega \in \Omega \mid \tau_M(\omega) = n\}), \quad n \in \mathbb{Z}.
\]

Assume that the values of \( \tau_M \) are finite so \( \tau_M : \Omega \to \mathbb{Z} \). Then it follows from the definition of stopping time (4.33) that \( w^M_n \geq 0, \forall n \in \mathbb{Z} \); and \( \sum_{n \in \mathbb{Z}} w^M_n = 1 \).

Back to the extension problem (4.28). In fact, there are many more solutions to the \( \mathbb{R} \)-problem than periodic solutions on \( \mathbb{T} \). By the Pontryagin duality (see [Rud90]), any periodic solution \( \tilde{F}_{\text{per}} \) has the representation (4.31).

**Example 4.3 (The periodic extension continued).** Recall that \( e^{-|x|}, x \in \mathbb{R} \), is positive definite on \( \mathbb{R} \), where

\[
e^{-|x|} = \int_{-\infty}^{\infty} \frac{2}{1 + 4\pi^2 \lambda^2} e^{2\pi i \lambda x} d\lambda, \quad x \in \mathbb{R}. \quad (4.34)
\]

By the Poisson summation formula, we have:

\[
\tilde{F}_{\text{per}}(x) = \sum_{n \in \mathbb{Z}} e^{-|x-n|} = \sum_{n \in \mathbb{Z}} w_n e^{2\pi i nx}
\]

\[
w_n = \frac{2}{1 + 4\pi^2 n^2}
\]

\( \tilde{F}_{\text{per}}(x) \) is a continuous p.d. function on \( \mathbb{T} \); and obviously, an extension of the restriction \( \tilde{F}_{\text{per}}|_{(-a,a)} \). Note that
\[ \sum_{n \in \mathbb{Z}} w_n = \sum_{n \in \mathbb{Z}} \frac{2}{1 + 4\pi^2 n^2} = \coth(1/2) < \infty. \] \hfill (4.36)

**General Consideration**

There is a bijection between (i) functions on \( T \), and (ii) 1-periodic functions on \( \mathbb{R} \). Note that (ii) includes p.d. functions \( F \) defined only on a subset \((-a,a) \subset T\).

Given \( 0 < a < \frac{1}{2} \), if \( F \) is continuous and p.d. on \((-a,a) \subset T\), then the analysis of the corresponding RKHS \( \mathcal{H}_F \) is totally independent of considerations of periods. \( \mathcal{H}_F \) does not see global properties. To understand \( F \) on \((-a,a)\), only one period interval is needed; but when passing to \( \mathbb{R} \), things can be complicated. Hence, it is tricky to get extensions \( \tilde{F}_{\text{per}} \) to \( T \).

**Example 4.4.** Let \( F(x) = e^{-|x|} \), for all \( x \in (-a,a) \). See Figure 4.4. Then \( \tilde{F}(x) = e^{-|x|} \), \( x \in \mathbb{R} \), is a continuous p.d. extension to \( \mathbb{R} \). However, the periodic version in (4.35) is NOT a p.d. extension of \( F \). See Figure 4.5.

![Diagram](image-url)

```
Fig. 4.4: Functions on \( T = \mathbb{R}/\mathbb{Z} \) \( \leftrightsquigarrow \) (1-periodic functions on \( \mathbb{R} \)).
```

**Example 4.5.** Fix \( a \in \left(0, \frac{1}{2}\right) \). Let \( F(x) = e^{-|x|} \), \( x \in (-a,a) \). Note that

\[
\left( e^{-|\cdot|} \chi_{[-\frac{1}{2}, \frac{1}{2}]} \right)^\wedge (\lambda) = \int_{\mathbb{R}} \frac{\sin \pi (\lambda - t)}{\pi (\lambda - t)} \cdot \frac{2}{1 + 4\pi^2 t^2} dt = \frac{2(2\pi \lambda \sin(\pi \lambda) - \cos(\pi \lambda) + \sqrt{e})}{\sqrt{e(1 + 4\pi^2 \lambda^2)}}, \hfill (4.37)
\]

then, using Shannon’s interpolation formula, we get...
4.4 Example: \( e^{-|x|} \) in \((-a, a)\), extensions to \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \)

\[
\tilde{F}_{\text{per}}(x) > F(x) \quad \text{on} \quad [-\frac{1}{2}, \frac{1}{2}]
\]

\[
\tilde{F}_{\text{cir}}(x) := \sum_{n \in \mathbb{Z}} e^{-|x+n|} \chi_{[-\frac{1}{2}, \frac{1}{2}]}(x+n) = \sum_{n \in \mathbb{Z}} \hat{W}(n) e^{2\pi i n x}; \quad \text{where (4.38)}
\]

\[
\hat{W}(n) := \left( e^{-|-|} \chi_{[-\frac{1}{2}, \frac{1}{2}]} \right)^{(4.37)} (n) = \frac{2 \left( 1 - e^{-1/2} (-1)^n \right)}{1 + 4\pi^2 n^2}, \quad \forall n \in \mathbb{Z}. \quad (4.39)
\]

See Figures 4.6-4.7.

It follows from (4.34)-(4.35), that

\[
\sum_{n \in \mathbb{Z}} \hat{W}(n) = \sum_{n \in \mathbb{Z}} e^{-|n|} - e^{-1/2} \sum_{n \in \mathbb{Z}} e^{-|n-\frac{1}{2}|} = 1, \text{ and }
\]

\[
\hat{W}(n) > 0, \quad \forall n \in \mathbb{Z}.
\]

Therefore, the weights \( \{\hat{W}(n)\} \) in (4.39) determines a probability measure on \( \mathbb{Z} \) (using the normalization \( F(0) = 1 \).) By Bochner’s theorem, \( \tilde{F}_{\text{cir}} \) in (4.38) is a continuous p.d. extension of \( F \), thus a solution to the \( \mathbb{T} \)-problem.

Example 4.6. Fix \( 0 < a < \frac{1}{2} \), let \( F(x) = 1 - |x|, \ x \in (-a, a) \), a locally defined p.d. function. This is another interesting example that we will study in Chapter 5.

Using the method from Example 4.4, we have

\[
\hat{W}(\lambda) := \left( (1 - |\cdot|) \chi_{[-\frac{1}{2}, \frac{1}{2}]} \right)^{(\lambda)} (\lambda) = \frac{\pi \lambda \sin(\pi \lambda) - \cos(\pi \lambda) + 1}{2\pi^2 \lambda^2}
\]
Set $\hat{W}(\lambda) := (e^{-|\cdot|} \chi_{[-1/2,1/2]})(\lambda)$, then $(\hat{W}(n))_{n \in \mathbb{Z}}$ is a probability distribution on $\mathbb{Z}$.

Fig. 4.7: A periodic extension of $e^{-|\cdot|} \chi_{[-1/2,1/2]}$ on $\mathbb{R}$ by Shannon’s interpolation.

and so

$$\tilde{F}_{\text{cir}}(x) := \sum_{n \in \mathbb{Z}} (1 - |x + n|) \chi_{[-\frac{1}{2}, \frac{1}{2}]}(x + n) = \sum_{n \in \mathbb{Z}} \hat{W}(n) e^{2\pi i n x},$$

where

$$\hat{W}(n) = \frac{1 - (-1)^n}{2\pi^2 \lambda^2} \geq 0, \quad \forall n \in \mathbb{Z}, \quad \text{and} \quad \sum_{n \in \mathbb{Z}} \hat{W}(n) = 1.$$

Therefore, $\tilde{F}_{\text{cir}}$ is a periodic p.d. extension of $F$.

Remark 4.4. The method used in Examples 4.5-4.6 does not hold in general. For example, a periodization of the restricted Gaussian distribution, $e^{-x^2/2} \chi_{[-\frac{1}{2}, \frac{1}{2}]}$, is not positive definite on $\mathbb{R}$. The reason is that the sampling

$$\{(e^{-\cdot^2/2} \chi_{[-\frac{1}{2}, \frac{1}{2}]})^\wedge(n)\}_{n \in \mathbb{Z}}$$

does NOT give a positive measure on $\mathbb{Z}$.

4.5 Example: $e^{-|x|}$ in $(-a, a)$, extensions to $\mathbb{R}$

In this section, we consider the following partially defined p.d. function:

$$F(x) = e^{-|x|}, \quad x \in (-a, a) \quad (4.40)$$

where $0 < a < \infty$, fixed. It is the restriction of

$$e^{-|x|} = \int_{\mathbb{R}} e^{\lambda x} d\mu(\lambda), \quad x \in \mathbb{R}, \quad \text{with}$$

$$d\mu = \frac{d\lambda}{\pi (1 + \lambda^2)} (= \text{prob. measure}), \quad (4.41)$$

thus, $\mu \in \text{Ext}(F)$. (In particular, $\text{Ext}(F) \neq \emptyset$.)
Remark 4.5. This special case is enlightening in connection with deficiency indices considerations. In general, there is a host of other interesting 1D examples (Chapters 5–6), some have indices (1, 1) and others (0, 0). In case of indices (1, 1), the convex set \( \text{Ext}(F) \) is parameterized by \( T_1 \); while, in case of (0, 0), \( \text{Ext}(F) \) is a singleton. Further spectral theoretic results are given in Chapter 10, and we obtain a spectral classification of all Hermitian operators with dense domain in a separable Hilbert space, having deficiency indices (1, 1).

Now define \( D^{(F)} \) (the canonical skew-Hermitian operator), by \( D^{(F)} F_\varphi = F_{\varphi'} \), with \( \text{dom}(D^{(F)}) = \{ F_\varphi : \varphi \in C_c^\infty (0, a) \} \). We show below that \( D^{(F)} \) has deficiency indices (1, 1).

Let \( \mu \in \text{Ext}(F) \) be as in (4.41). By Corollary 3.1, there is an isometry \( \mathcal{H}_F \hookrightarrow L^2 (\mu) \), determined by

\[
\mathcal{H}_F \ni F_\varphi \mapsto \hat{\varphi} \in L^2 (\mu),
\]

where \( \mathcal{H}_F \) is the corresponding RKHS. Indeed, we have

\[
\| F_\varphi \|_{\mathcal{H}_F}^2 = \int_0^a \int_0^a \overline{\varphi(x)} \varphi(y) F(x - y) \, dx \, dy
= \int_0^a \int_0^a \overline{\varphi(x)} \varphi(y) \left( \int_{\mathbb{R}} e^{i \lambda (x-y)} \, d\mu(\lambda) \right) \, dx \, dy
= \text{Fubini}
\int_{\mathbb{R}} |\hat{\varphi}(\lambda)|^2 \, d\mu(\lambda) = \| \hat{\varphi} \|^2_{L^2(\mu)}, \quad \forall F_\varphi \in \text{dom}(D^{(F)}).
\]

Lemma 4.4. Fix \( 0 < a < \infty \), and let \( F \) be a continuous p.d. function on \((-a, a)\); assume \( F(0) = 1 \). Pick \( \mu \in \text{Ext}(F) \), i.e., a probability measure on \( \mathbb{R} \) s.t.

\[
F(x) = \tilde{d}_\mu(x), \quad \forall x \in (-a, a).
\]  
(We know that \( \text{Ext}(F) \neq \emptyset \) from Section 2.1.) Then, \( D^{(F)} \) has deficiency indices (1, 1) if and only if

\[
\left( e^{\pm i x} \chi_{(0,a)} \right)^\wedge (\cdot) \in L^2 (\mu),
\]

\[
\int_{-\infty}^\infty \frac{e^{2a} + 1 - 2 e^a \cos(\lambda a)}{1 + \lambda^2} \, d\mu(\lambda) < \infty.
\]

Proof. Recall the defect vectors of \( D^{(F)} \) are multiples of \( e^{\pm x}, x \in (0, a) \). The assertion follows from the isometry \( \mathcal{H}_F \hookrightarrow L^2 (\mu) \). \( \square \)

Corollary 4.2. Let \( F \) be as in (4.40), then the associated \( D^{(F)} \) has deficiency indices (1, 1). In particular, the defect vectors \( \xi_+ = e^{-x}, \xi_- = e^{i x} \), \( x \in (0, a) \), satisfy \( \| \xi_+ \|_{\mathcal{H}_F} = \| \xi_- \|_{\mathcal{H}_F} = 1 \).

Lemma 4.5. We have:

\[
\left( \chi_{(0,a)} (x) e^x \right)^\wedge (\lambda) = \frac{e^{2a} + 1 - 2 e^a \cos(\lambda a)}{1 + \lambda^2} \in L^2 (\mathbb{R}, d\lambda).
\]  
(4.44)
where $d\lambda$ denotes the Lebesgue measure on $\mathbb{R}$.

Proof. Direct calculation shows that
\[
(\chi_{(0,a)}(x) e^x) \wedge (\lambda) = \int_0^a e^{-i\lambda x} e^x \, dx = \frac{e^{(1-i\lambda)a} - 1}{1-i\lambda};
\]
which is the assertion in (4.44). \qed

Now return to the RKHS $\mathcal{H}_F$, defined from the function $F$ in (4.40). Recall that $\text{span} \{ F\phi \mid \phi \in C_c^\infty(0,a) \}$ is a dense subspace in $\mathcal{H}_F$ (Lemma 2.3), where
\[
F\phi(x) = \int_0^a \phi(y) e^{-|x-y|} \, dy
\]
(4.45)

Lemma 4.6. We have
\[
\|F\phi\|^2_{\mathcal{H}_F} = \int_0^a \int_0^a \phi(y) \phi(x) e^{-|x-y|} \, dx \, dy
= \int_{-\infty}^{\infty} \left| \hat{\phi}(\lambda) \right|^2 \frac{d\lambda}{\pi(1+\lambda^2)}
\]
(4.46)

Proof. Immediate. \qed

Lemma 4.7 (continuation of Example 2.2). Fix $a, 0 < a < \infty$, and let $F(\cdot) := e^{-|\cdot|} \mid_{(-a,a)}$ as in (4.40). For all $0 \leq x_0 \leq a$, let
\[
F_{x_0}(x) := F(x-x_0) \bigg|_{(0,a)} \in C(0,a).
\]
(4.47)

With
\[
\begin{align*}
\text{DEF}^+ &= \left\{ \xi : (D^F)\xi = \xi \right\} = \text{span} \left\{ \xi_+(x) := e^{-x} \bigg|_{(0,a)} \right\} \\
\text{DEF}^- &= \left\{ \xi : (D^F)^\ast \xi = -\xi \right\} = \text{span} \left\{ \xi_-(x) := e^{-a}e^x \bigg|_{(0,a)} \right\}
\end{align*}
\]
(4.48)
(4.49)
we get
\[
\|\xi_+\|^2_{\mathcal{H}_F} = \|\xi_-\|^2_{\mathcal{H}_F} = 1.
\]
(4.50)

Proof. Note that $x = 0$ and $x = a$ are the endpoints in the open interval $(0,a)$:
\[
\begin{align*}
\xi_+(x) &= e^{-x} \bigg|_{(0,a)} = F_0(x) \\
\xi_-(x) &= e^{-a}e^x \bigg|_{(0,a)} = F_a(x).
\end{align*}
\]
Let $\psi_n \in C_c^\infty(0,a)$ be an approximate identity, such that
4.5 Example: $e^{-|x|}$ in $(-a,a)$, extensions to $\mathbb{R}$

(1) $\psi_n \geq 0$, $\int \psi_n = 1$;
(2) $\psi_n \to \delta_a$, as $n \to \infty$.

Then

$$\xi_- (x) = F_a (x) = \lim_{n \to \infty} \int_0^a \psi_n (y) F_y (x) dy.$$ 

This shows that $\xi_- \in \mathcal{H}_F$. Also,

$$\|\xi_-\|_{\mathcal{H}_F}^2 = \|F_a\|_{\mathcal{H}_F}^2 = \lim_{n \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{\psi}_n (y)|^2 \hat{F} (y) dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F} (y) dy = 1.$$ 

Similarly, if instead, $\psi_n \to \delta_0$, then

$$\xi_+ (x) = F_0 (x) = \lim_{n \to \infty} \int_0^a \psi_n (y) F_y (x) dy$$

and $\|\xi_+\|_{\mathcal{H}_F}^2 = 1$.

□

**Lemma 4.8.** Let $F$ be as in (4.40). We have the following for its Fourier transform:

$$\hat{F} (y) = \frac{2 - 2e^{-a} (\cos (ay) - y \sin (ay))}{1 + y^2}.$$ 

**Proof.** Let $y \in \mathbb{R}$, then

$$\hat{F} (y) = \int_{-\infty}^{\infty} e^{iyx} e^{-|x|} dx$$

$$= \int_{-\infty}^{0} e^{iyx} dx + \int_{0}^{\infty} e^{iyx} e^{-x} dx$$

$$= \frac{1 - e^{-a(1+iy)}}{1+iy} + \frac{e^{ia(y+i)} - 1}{1-iy}$$

$$= \frac{2 - 2e^{-a} (\cos (ay) - y \sin (ay))}{1 + y^2}$$

which is the assertion. □

**Remark 4.6.** If $(F, \Omega)$ is such that $D^{(F)}$ has deficiency indices $(1,1)$, then by von Neumann’s theory [DS88], the family of selfadjoint extensions is characterized by

$$\text{dom} \left( A^{(F)}_\theta \right) = \left\{ F\psi + c \left( \xi_+ + e^{i\theta} \xi_- \right) : \psi \in C_0^\infty (0,a), c \in \mathbb{C} \right\}$$

$$A^{(F)}_\theta : F\psi + c \left( \xi_+ + e^{i\theta} \xi_- \right) \mapsto F\psi' + c i \left( \xi_+ - e^{i\theta} \xi_- \right), \text{ where } i = \sqrt{-1}.$$
Proposition 4.2. Let \( F(x) = e^{-|x|}, x \in (-a,a), \) be the locally defined p.d. function (see (4.40)), and \( \mathcal{H}_F \) be the associated RKHS. Consider \( e_\lambda(x) = \chi_{(0,a)}(x) e^{i\lambda x} \) with \( \lambda \in \mathbb{C} \) fixed. Then the function \( e_\lambda \in \mathcal{H}_F \), for all \( \lambda \in \mathbb{C} \).

Proof. For simplicity, set \( a = 1 \). Fix \( \lambda \in \mathbb{C} \), and consider two cases:

Case 1. \( \Im \{\lambda\} = 0 \), i.e., \( \lambda \in \mathbb{R} \).

Recall the isometry \( T : \mathcal{H}_F \to L^2(\mu) \), with \( d\mu \) as in (4.41); and by Corollary 4.1, the adjoint operator is given by

\[
L^2(\mu) \ni f \xrightarrow{T^*} \chi_{[0,1]}(f d\mu)^\vee \in \mathcal{H}_F.
\]

We shall show that there exists \( f \in L^2(\mu) \) s.t. \( T^*f = \chi_{[0,1]} e^{i\lambda} \).

Let

\[
\varphi(x) = (\chi_{[-1,1]} * \chi_{[0,1]})(x) = |[x-1,x]\cap [-1,1]|, \quad x \in \mathbb{R},
\]

(4.51)

where \( |\cdots| \) denotes Lebesgue measure. See Figure 4.8. Note that

\[
|\hat{\varphi}(t)|^2 = \frac{16\sin^2(t/2)\sin^2(t/2)}{t^4}, \quad t \in \mathbb{R}.
\]

(4.53)

Now, set \( e_\lambda(x) := e^{i\lambda x} \), and

\[
\psi(x) = e_\lambda(x) \varphi(x), \quad \text{and} \quad f(t) = \frac{1}{2} \hat{\psi}(t) (1+t^2), \quad t \in \mathbb{R}.
\]

(4.54)

(4.55)

Then, we have:

\[
(T^*f)(x) = (\chi_{[0,1]}(f d\mu)^\vee)(x)
\]

\[
= \chi_{[0,1]}(x) \int_{\mathbb{R}} e^{ixt} \frac{1}{2} \hat{\psi}(t) (1+t^2) \frac{dt}{\pi(1+t^2)}
\]

\[
= \chi_{[0,1]}(x) \frac{1}{2\pi} \int_{\mathbb{R}} e^{ixt} \hat{\psi}(t) dt
\]

\[
= \chi_{[0,1]}(x) \psi(x) = \chi_{[0,1]}(x) e_\lambda(x); \quad \text{and}
\]
4.5 Example: $e^{-|x|}$ in $(-a,a)$, extensions to $\mathbb{R}$

\[ \int_{\mathbb{R}} |f|^2 \, d\mu = \frac{1}{4} \int_{\mathbb{R}} |\hat{\psi}(t)|^2 \left( 1 + t^2 \right)^2 \frac{1}{\pi (1+t^2)} \, dt \]
\[ = \frac{1}{4\pi} \int_{\mathbb{R}} |\hat{\psi}(t)|^2 \left( 1 + t^2 \right) \, dt \]
\[ = \frac{4}{\pi^2} \int_{\mathbb{R}} \frac{\sin^2 \left( (t - \lambda)/2 \right) \sin^2 \left( t - \lambda \right)}{(t - \lambda)^4} \left( 1 + t^2 \right) \, dt < \infty. \]

Thus, the function $f$ in (4.55) is in $L^2(\mu)$, and satisfies $T^* f = \chi_{[0,1]} e^{i\lambda}$. This is the desired conclusion.

Case 2. $\mathbb{I} \{ \lambda \} \neq 0$.

Using general theory [DS88], it is enough to prove that $e^{i\lambda}(x)$ is in $H_F$ for $\lambda = \pm i$. So consider the following:

\[ \xi_+(x) = e^{-i} \chi_{[0,1]}(x) \]
\[ \xi_-(x) = e^{i-1} \chi_{[0,1]}(x). \] (4.56)

Let $S$ be the isometry in Corollary 2.1, then

\[ S^*(\nu * F) = \nu \in \mathcal{M}_2(F) \] (4.57)

for all signed measures $\nu \in \mathcal{M}_2(F)$. Fixing $\mu$ positive s.t. $\widehat{\mu}(x) = F(x)$ for all $x \in [-1,1]$, we get

\[ \nu * F \in H_F \iff \int_{\mathbb{R}} |\hat{\nu}(t)|^2 \, d\mu(t) < \infty, \]

and

\[ \|\nu * F\|_{H_F}^2 = \int_{\mathbb{R}} |\hat{\nu}(t)|^2 \, d\mu(t). \] (4.58)

Here, $(\nu * F)(x) = \int_{0}^{1} F(x-y) \, d\nu(y), \forall x \in [0,1]$.

In Example 2.2, we proved that

\[ (\delta_0 * F)(x) = \xi_+(x) \]
\[ (\delta_1 * F)(x) = \xi_-(x) \] (4.59)

Using (4.58), we then get

\[ \|\xi_+\|_{H_F}^2 = \|\hat{\delta}_0\|_{L^2(\mathbb{R},\mu)}^2 = 1 \]

and

\[ \|\xi_-\|_{H_F}^2 = \|\hat{\delta}_1\|_{L^2(\mathbb{R},\mu)}^2 = 1; \]

where of course $\delta_0 \equiv 1, \forall t \in \mathbb{R}$, and $\delta_1 = e^{it}$. □
Remark 4.7. In Tables 5.1-5.2, we consider six examples of locally defined continuous p.d. functions, all come from restrictions of p.d. functions on \( \mathbb{R} \). Except for the trivial case \( F_6 \), for each \( F_j, j = 1, \ldots, 5 \), we need to decide whether \( e^{i\lambda x}, \lambda \in \mathbb{C} \) are in the corresponding RKHS \( \mathcal{H}_{F_j} \). For illustration, it suffices to consider the case \( \lambda = 0 \), i.e., whether the constant functions \( \chi_{[0,a]} \) are in \( \mathcal{H}_{F_j} \). Equivalently, if there exists \( f \in L^2(\mu) \), s.t.

\[
\phi = T^* f = \chi_{[0,a]} (f d\mu)^Y, \text{ s.t.} \\
\phi \equiv 1 \text{ on } [0,a].
\]

(4.60) (4.61)

Here, \( d\mu \) denotes one of the measures introduced above. Compare with Proposition 4.2.

Note that in all cases, \( d\mu \) is absolutely continuous w.r.t. the Lebesgue measure, i.e., \( d\mu = m dx \), where \( m \) is a Radon-Nikodym derivative. Thus, a possible solution \( f \) to (4.60)-(4.61) must be given by

\[
\hat{f} = \frac{\hat{\phi}}{m}, \text{ s.t. } \int_{\mathbb{R}} \frac{|\hat{\phi}|^2(t)}{m(t)} dt < \infty.
\]

(4.62)

By the splitting, \( \phi = \phi \chi_{[0,a]} + \phi_2 \), and using (4.61), we then get

\[
\hat{\phi}(t) = \text{sinc}(t) + \hat{\phi}_2(t),
\]

(4.63)

where, in (4.63), we use that \( \phi_2 \) is supported in \( \mathbb{R} \setminus (0,a) \). In the applications mentioned above, the sinc function in (4.63) will already account for the divergence of the integral in (4.62).

Conclusion: One checks that for \( F_1, F_4 \), and \( F_5 \), the function \( \chi_{[0,a]} \) is not in the corresponding RKHS.

Example 4.7. \( F_1(x) = \frac{1}{1+x^2}, |x| < 1, d\mu_1(t) = \frac{1}{2} e^{-|t|} dt \). Then, we have

\[
\int_{\mathbb{R}} \frac{|\hat{\phi}|^2(t)}{m(t)} dt \sim \int_{\mathbb{R}} \text{sinc}^2(t) e^{|t|} dt + \text{other positive terms}
\]

which is divergent. Thus, \( \chi_{[0,1]} \notin \mathcal{H}_{F_1} \).

Remark 4.8. In the example above, we cannot have

\[
|\phi|^2(t) \sim O(e^{-(1+\epsilon)|t|}), \text{ as } |t| \to \infty;
\]

(4.64)
4.6 Example: A non-extendable p.d. function in a neighborhood of zero in $G = \mathbb{R}^2$

for some $\varepsilon > 0$. Should (4.64) hold, then By the Paley-Wiener theorem, $x \mapsto \varphi(x)$ would have an analytic continuation to a strap, $|3z| < \alpha$, $z = x + iy$, $|y| < \alpha$. But this contradicts the assumption that $\varphi(x) \equiv 1$, for all $x \in (0, 1)$; see (4.61).

4.6 Example: A non-extendable p.d. function in a neighborhood of zero in $G = \mathbb{R}^2$

Let $M$ denote the Riemann surface of the complex log $z$ function. $M$ is realized as a covering space for $\mathbb{R}^2 \setminus \{(0, 0)\}$ with an infinite number of sheets indexed by $\mathbb{Z}$, see Figure 4.9.

![Riemann Surface Diagram](image)

**Fig. 4.9:** $M$ the Riemann surface of log $z$ as an $\infty$ cover of $\mathbb{R}^2 \setminus \{(0, 0)\}$.

The 2D-Lebesgue measure lifts to a unique measure on $M$; hence $L^2(M)$. Here the two skew-symmetric operators $\frac{\partial}{\partial x_j}$, with domain $C_c^\infty(M)$, define an Abelian 2-dimensional Lie algebra of densely defined operators in the Hilbert space $L^2(M)$.

**Proposition 4.3.**

1. Each operator $\frac{\partial}{\partial x_j}$, defined on $C_c^\infty(M)$, is essentially skew-adjoint in $L^2(M)$, i.e.,

   $-\left(\frac{\partial}{\partial x_j} \big|_{C_c^\infty(M)}\right)^* = \frac{\partial}{\partial x_j} \big|_{C_c^\infty(M)}$, $j = 1, 2$; (4.65)

   where the r.h.s. in (4.65) denotes operator closure.

2. The $\left\{\frac{\partial}{\partial x_j}\right\}_{j=1,2}$ Lie algebra with domain $C_c^\infty(M) \subset L^2(M)$ does not extend to a unitary representation of $G = \mathbb{R}^2$.

3. The two skew-adjoint operators in (4.65) are NOT strongly commuting.

4. The Laplace operator
\[ L := \left( \frac{\partial}{\partial x_1} \right)^2 + \left( \frac{\partial}{\partial x_2} \right)^2, \quad \text{dom}(L) = C_c^\infty(M) \] (4.66)

has deficiency indices \((\infty, \infty)\).

**Proof.** For \( j = 1, 2 \), the operators \( \frac{\partial}{\partial x_j} \big|_{C_c^\infty(M)} \) generate unitary one-parameter groups \( U_j(t) \) on \( L^2(M) \), lifted from the coordinate translations:

\[
\begin{align*}
(x_1, x_2) & \mapsto (x_1 + s, x_2), \quad x_2 \neq 0 \quad (4.67) \\
(x_1, x_2) & \mapsto (x_1, x_2 + t), \quad x_1 \neq 0 \quad (4.68)
\end{align*}
\]

It is immediate that the respective infinitesimal generators are the closed operators \( \frac{\partial}{\partial x_j} \big|_{C_c^\infty(M)} \). Part (1) follows from this.

Part (2)-(4). Suppose \( \varphi \in C_c^\infty(M) \) is supported over some open set in \( \mathbb{R}^2 \setminus \{(0,0)\} \), for example, \((x_1 - 2)^2 + x_2^2 < 1\). If \( 1 < s, t < 2 \), then the functions

\[
U_1(s)U_2(t) \varphi \quad \text{vs.} \quad U_2(t)U_1(s) \varphi \quad (4.69)
\]

are supported on two opposite levels in the covering space \( M \); see Figure 4.10. (One is over the other; the two are on “different floors of the parking garage.”) Hence the unitary groups \( U_j(t) \) do not commute. It follows from Nelson’s theorem [Nel59] that \( L \) in (4.66) is not essentially selfadjoint, equivalently, the two skew-adjoint operators are not strongly commuting.

Since \( L \leq 0 \) (in the sense of Hermitian operators), so it has equal deficiency indices. In fact, \( L \) has indices \((\infty, \infty)\); see [JT14a, Tia11].

This completes the proof of the proposition. \( \square \)

**Fig. 4.10:** Translation of \( \varphi \) to different sheets.

**Remark 4.9.** The two unitary groups from Proposition 4.3 define a local representation of \( G = \mathbb{R}^2 \) on \( L^2(M) \), but not a global one. That is, the skew-adjoint operators \( \frac{\partial}{\partial x_j} \big|_{C_c^\infty(M)} \) in (4.65) are not strongly commuting, and so

\[
\rho : (s, t) \mapsto U_1(s)U_2(t) \quad (4.70)
\]

is not a unitary representation of \( \mathbb{R}^2 \) on \( L^2(M) \). This is different from the 1D examples.
4.6 Example: A non-extendable p.d. function in a neighborhood of zero in $G = \mathbb{R}^2$

The harmonic analysis of the Riemann surface $M$ of log $z$ is of independent interest, but it will involve von Neumann algebras and non-commutative geometry. Note that to study this, one is faced with two non-commuting unitary one-parameter groups acting on $L^2(M)$ (corresponding to the two coordinates for $M$). Of interest here is the von Neumann algebra generated by these two non-commuting unitary one-parameter groups. It is likely that this von Neumann algebra is a type III factor. There is a sequence of interesting papers by K. Schmüdgen on dealing with some of this [Sch84, SF84, Sch85, Sch86b, Sch86a].

A locally defined p.d. functions $F$ on $G = \mathbb{R}^2$ with $\text{Ext}(F) = \emptyset$.

Let $M$ be the Riemann surface of the complex log $z$; see Figure 4.9.

Denote $B_r(x) := \{y \in \mathbb{R}^2 : |x - y| < r, r > 0\}, x \in \mathbb{R}^2$, the open neighborhood of points $x \in \mathbb{R}^2$ of radius $r$. Pick $\xi \in C_\infty^c(B_{1/2}(1,1))$, such that

$$\int \xi = 1, \quad \xi \geq 0.$$  \hspace{1cm} (4.71)

Note that $\xi$ is supported on level 1 of the covering surface $M$, i.e., on the sheet $\mathbb{R}^2 \backslash \{(x, y) : x \geq 0\}$; assuming the branch cuts are along the positive $x$-axis.

Let $\pi : M \to \mathbb{R}^2 \backslash \{0\}$ be the covering mapping. Set

$$(\sigma(t)\xi)(m) := \xi(\pi^{-1}(t + \pi(m))), \quad \forall t \in B_{1/2}(0), \forall m \in M.$$ \hspace{1cm} (4.72)

Lemma 4.9. The function

$$F(t) := \langle \xi, \sigma(t)\xi \rangle_{L^2(M)}, \quad t \in B_{1/2}(0) \subset \mathbb{R}^2$$ \hspace{1cm} (4.73)

is defined on a local neighborhood of $0 \in \mathbb{R}^2$, continuous and p.d.; but cannot be extended to a continuous p.d. function $F_{\text{ext}}$ on $\mathbb{R}^2$.

Proof. If $F$ has an extension, then there exists $\rho \in \text{Rep}(\mathbb{R}^2, L^2(M))$, see (4.70), s.t. $\rho$ extends the local representation $\sigma$ in (4.72), and this implies that we get strongly commuting vector fields on $M$ generated by $\frac{\partial}{\partial x_j}|_{C_\infty^c(M)}$, $j = 1, 2$, as in (4.65). But we know that this is impossible from Proposition 4.3.

Indeed, if $U_j(t)$ are the one-parameter unitary groups from (4.69), then in our current setting, $U_2(1)U_1(1)\xi$ is supported on level 1 of the covering surface $M$, but $U_1(1)U_2(1)\xi$ is on level $-1$. \hfill $\Box$

Theorem 4.2. Let $F$ be the local p.d. function in (4.73), then $\text{Ext}(F) = \emptyset$.

Proof. The proof uses the lemma above. Assuming $F_{\text{ext}} \in \text{Ext}(F)$; we then get the following correspondences (Figure 4.11) which in turn leads to a contradiction.
The contradiction is based on the graph in Figure 4.10. We show that any $F^{ext} \in \text{Ext}(F)$ would lead to the existence of two (globally) commuting and unitary one-parameter groups of translations. Following supports, this is then shown to be inconsistent (Figure 4.9-4.10). □

\[ G = \mathbb{R}^2, \, \mathcal{O} = B_{\frac{1}{2}}(0) \subset \mathbb{R}^2 \]

\[ [F \text{ cont., p.d. in } B_{\frac{1}{2}}(0) \subset \mathbb{R}^2] \quad [F^{ext}(t) = \langle \xi, U(t) \xi \rangle_{L^2(M)}] \]

\[ \exists \text{local reprep. } \mathcal{H} = L^2(M), \, v_0 = \xi, \{\sigma(t) : t \in B_{\frac{1}{2}}(0)\} \quad \text{unitary reprep. } U \text{ of } \mathbb{R}^2 \text{ acting on } L^2(M) \]

\[ \text{reprep. of the comm. Lie algebra} \quad \text{if extendable} \quad \exists \text{two comm. unitary one-parameter groups} \]

Fig. 4.11: Extension correspondence in the log $\varepsilon$ example. From locally defined p.d. function $F$ in a neighborhood of $(0,0)$ in $G = \mathbb{R}^2$, to a representation of the 2-dimensional Abelian Lie algebra by operators acting on a dense domain in $L^2(M)$. This Lie algebra representation does not exponentiate to a unitary representation of $G = \mathbb{R}^2$. 
Chapter 5
Type I vs. Type II Extensions

“In mathematics links the abstract world of mental concepts to the real world of physical things without being located completely in either.” — Ian Stewart, Preface to second edition of What is Mathematics? by Richard Courant and Herbert Robbins

In this chapter, we identify extensions of the initially given positive definite (p.d.) functions $F$ which are associated with operator extensions in the RKHS $H_F$ itself (Type I), and those which require an enlargement of $H_F$, Type II. In the case of $G = \mathbb{R}$ (the real line) some of these continuous p.d. extensions arising from the second construction involve a spline-procedure, and a theorem of G. Pólya, which leads to p.d. extensions of $F$ that are symmetric around $x = 0$, and convex on the left and right half-lines. Further these extensions are supported in a compact interval, symmetric around $x = 0$.

A main result in this chapter (Theorem 5.4) concerns the set $\text{Ext}(F)$. Our theorem applies to any positive definite function $F$ which is defined in an interval, centered at 0, such that $F$ is also analytic in a neighborhood of 0. Let $D(F)$ be the associated skew-Hermitian operator in $H_F$. Under these assumptions, Theorem 5.4 states that the operator $D(F)$ will automatically be essentially skew-adjoint, i.e., it has indices $(0, 0)$, and, moreover we conclude that $\text{Ext}(F)$ is a singleton. In particular, under these assumptions, we get that the subset $\text{Ext}_2(F)$ of $\text{Ext}(F)$ is empty. (Note, however, that our non-trivial Pólya spline-extensions are convex on the positive and the negative half-lines. And further that, for those, the initially given p.d. function $F$ will not be analytic in a neighborhood of 0. Rather, for this class of examples, the initial p.d. function $F$ is convex on the two finite intervals, on either side of 0. In this setting, we show that there are many non-trivial Pólya spline-extensions, and they are in $\text{Ext}_2(F)$.)

5.1 Pólya Extensions

We need to recall Pólya’s theorem [Pó149] regarding positive definite (p.d.) functions. For splines and p.d. functions, we refer to [Sch83, GSS83].

Theorem 5.1 (Pólya). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function, and assume that

$$(1) \; f(0) = 1,$$

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(2) \( \lim_{t \to -\infty} f(t) = 0 \),
(3) \( f(-t) = f(t), \forall t \in \mathbb{R} \), and
(4) \( f|_{\mathbb{R}^+} \) is convex.

Then it follows that \( f \) is positive definite; and as a result that there is a probability measure \( \mu \) on \( \mathbb{R} \) such that
\[
\hat{f}(\lambda) = \int_{\mathbb{R}} e^{i\lambda t} d\mu(t), \quad \forall \lambda \in \mathbb{R}.
\]

**Proposition 5.1.** Let \( F : (-a, a) \to \mathbb{C} \) be p.d., continuous. Assume \( F \) has a Pólya extension \( F_{\text{ex}} \) supported in \([c, -c]\), with \( c > a > 0 \); then the corresponding measure \( \mu_{\text{ex}} \in \text{Ext}(F) \) has the following form:
\[
d\mu_{\text{ex}}(\lambda) = \Phi_{\text{ex}}(\lambda) d\lambda, \quad \text{where}
\]
\[
\Phi_{\text{ex}}(\lambda) = \frac{1}{2\pi} \int_{c}^{c} e^{-i\lambda y} F_{\text{ex}}(y) dy
\]
is entire analytic in \( \lambda \).

**Proof.** An application of Fourier inversion, and the Paley-Wiener theorem. \( \square \)

The construction of Pólya extensions is as follows: Starting with a convex p.d. function \( F \) on a finite interval \((-a, a)\), we create a new function \( F_{\text{ex}} \) on \( \mathbb{R} \), such that \( F_{\text{ex}}|_{\mathbb{R}^+} \) is convex, and \( F_{\text{ex}}(-x) = F_{\text{ex}}(x) \). Pólya’s theorem [Pól49] states that \( F_{\text{ex}} \) is a p.d. extension of \( F \).

As illustrated in Figure 5.1, after extending \( F \) from \((-a, a)\) by adding one or more line segments over \( \mathbb{R}^+ \), and using symmetry by \( x = 0 \), there will be a constant \( c \), with \( 0 < a < c \), such that the extension \( F_{\text{ex}} \) satisfies \( F_{\text{ex}}(x) = 0 \), for all \( |x| \geq c \).

![Fig. 5.1: An example of Pólya extension of \( F \) on \((-a, a)\). On \( \mathbb{R}^+ \), extend \( F \) by adding line segments \( L_1 \) and \( L_2 \), and then use symmetry by \( x = 0 \) to get \( F_{\text{ex}} \). By Pólya’s theorem, the extension \( F_{\text{ex}} \) is p.d. on \( \mathbb{R} \).](image)

In order to apply Pólya’s theorem, the spline extended function \( F_{\text{ex}} \) to \( \mathbb{R} \) must be convex on \( \mathbb{R}^+ \). In that case, \( F_{\text{ex}} \) will be positive definite. However, we may also start with a continuous p.d. function \( F \) on \((-a, a)\), which is concave near \( x = 0 \), and consider spline extensions that are supported in \([-c, c]\), for some \( c > a \). In Figure
5.2. F is concave in a neighborhood of 0; choose the slope of \( L_+ = F'(a) \), and the slope of \( L_- = F'(-a) = -F'(a) \), using mirror symmetry around \( x = 0 \). The extension \( F_{\text{ex}} \) does not satisfy the premise in Pólya’s theorem (not convex on \( \mathbb{R}_+ \)), and so it may not be positive definite.

Fig. 5.2: A spline extension of \( F : (-a, a) \to \mathbb{R} \), where \( F \) is concave around \( x = 0 \). The extension \( F_{\text{ex}} \) is not p.d. on \( \mathbb{R} \).

Examples 5.1-5.6 contain six cases of locally defined continuous p.d. functions \( F_i \), where \( F_i = \chi_{(-a_i, a_i)} \hat{d} \mu_i \), \( i = 1, \ldots, 6 \); i.e., each \( F_i \) is the restriction to a given finite interval \( (-a_i, a_i) \subset \mathbb{R} \) of a p.d. function \( \hat{d} \mu_i(x) \), \( x \in \mathbb{R} \). The corresponding measures \( \mu_i \) are listed in Table 5.1. In the study of the p.d. extension problem, the following hold (see Section 2.4 for the definitions):

1. Each \( F_i \) has a trivial p.d. extension to \( \mathbb{R} \), \( \hat{d} \mu_i \). In particular, \( \mu_i \in \text{Ext} \left( F_i \right) \neq \emptyset \).
2. \( F_1, F_4, F_5 \) and \( F_6 \) are concave around \( x = 0 \), so they do not yield spline extensions which are convex when restricted to \( \mathbb{R}_+ \).
3. Moreover, \( F_1, F_3, F_5 \) and \( F_6 \), are analytic in a neighborhood of 0. By Theorem 5.4, the skew-Hermitian operators \( D^{(F_i)} \) have deficiency indices \((0, 0)\), and \( \text{Ext} \left( F_i \right) = \text{Ext}_1 \left( F_i \right) = \{ \mu_i \} = \text{singleton} \). Therefore, each \( F_i \) has a unique p.d. extension to \( \mathbb{R} \), i.e., \( \hat{d} \mu_i \), which is Type I. As a result, there will be no p.d. spline extensions for these four cases. (The spline extensions \( F_{\text{ex}}^{(i)} \) illustrated below are supported in some \( [-c_i, c_i] \), but \( F_{\text{ex}}^{(i)} \) are not p.d. on \( \mathbb{R}_+ \).
4. \( F_2 \) & \( F_3 \) are convex in a neighborhood of 0, and they have p.d. spline extensions in the sense of Pólya, which are Type II. For these two cases, \( D^{(F_i)} \) has deficiency indices \((1, 1)\), and \( \text{Ext}_1 \left( F_i \right) \) contains atomic measures. As a result, we conclude that \( \mu_i \in \text{Ext}_2 \left( F_i \right) \), i.e., the trivial extensions \( \hat{d} \mu_i \) are Type II.

Example 5.1 (Cauchy distribution). \( F_1(x) = \frac{1}{1+x^2}; |x| < 1 \). Note that \( F_1 \) is concave, and analytic in a neighborhood of 0. It follows from Theorem 5.4 that the spline extension in Figure 5.3 is not positive definite.
Example 5.2. \( F_2(x) = 1 - |x|; |x| < \frac{1}{2} \). Consider the following Pólya extension

\[
F(x) = \begin{cases} 
1 - |x| & \text{if } |x| < \frac{1}{2} \\
\frac{1}{3} (2 - |x|) & \text{if } \frac{1}{2} \leq |x| < 2 \\
0 & \text{if } |x| \geq 2
\end{cases}
\]

This is a p.d. spline extension which is convex on \( \mathbb{R}_+ \). The corresponding measure \( \mu \in \text{Ext}(F) \) has the following form \( d\mu(\lambda) = \Phi(\lambda) \, d\lambda \), where \( d\lambda = \text{Lebesgue measure on } \mathbb{R} \), and

\[
\Phi(\lambda) = \begin{cases} 
\frac{3}{4\pi} & \text{if } \lambda = 0 \\
\frac{1}{3\pi\lambda^2} (3 - 2\cos(\lambda/2) - \cos(2\lambda)) & \text{if } \lambda \neq 0
\end{cases}
\]

This solution \( \mu \) is in \( \text{Ext}_2(F) \), and similarly, the measure \( \mu_2 \) from Table 5.2 is in \( \text{Ext}_2(F) \).

Example 5.3 (Ornstein-Uhlenbeck). \( F_3(x) = e^{-|x|}; |x| < 1 \). A positive definite spline extension which is convex on \( \mathbb{R}_+ \).
Remark 5.1 (Ornstein-Uhlenbeck). The p.d. function \( F(t) := e^{-|t|}, \ t \in \mathbb{R} \), is of special significance in connection with the Ornstein-Uhlenbeck process. See Figure 5.6, and Section 7.1.5.

To make the stated connection more direct, consider the standard Brownian motion \( B_t \), i.e., \( \{B_t\} \) is Gaussian with mean zero and covariance function \( \mathbb{E}(B_{t_1}B_{t_2}) = t_1 \wedge t_2 \), for all \( t_1, t_2 \geq 0 \). (See, e.g., [Hid80].) Fix \( \alpha \in \mathbb{R}_+ \), and set

\[
X_t := \frac{1}{\sqrt{\alpha}} e^{-\frac{\alpha}{2} B(e^{\alpha t})}, \quad t \in \mathbb{R}; \tag{5.2}
\]

then

\[
\mathbb{E}(X_{t_1}X_{t_2}) = \frac{1}{\alpha} e^{-\frac{\alpha}{2} |t_1 - t_2|}, \quad \forall t_1, t_2 \in \mathbb{R}. \tag{5.3}
\]

The reader will be able to verify (5.3) directly by using the covariance kernel for Brownian motion. See also Example 1.2.

\[
\begin{align*}
\text{Fig. 5.6: Simulation of the Ornstein-Uhlenbeck process. Five sample paths starting} \\
\text{at } (0, 0) \text{ with mean } \mu = 0, \text{ and standard deviation } \sigma = 0.3.
\end{align*}
\]

Example 5.4 (Shannon). \( F_4(x) = \left(\frac{\sin(x/2)}{x/2}\right)^2 \); \(|x| < \frac{1}{2} \). \( F_4 \) is concave, and analytic in a neighborhood of \( x = 0 \). By Theorem 5.4, \( \text{Ext} \ (F_4) = \text{Ext}_1 (F_4) = \text{singleton} \). Therefore, the spline extension in Figure 5.7 is not positive definite.

\[
\begin{align*}
\text{Fig. 5.7: A spline extension of } F_4(x) = \left(\frac{\sin(x/2)}{x/2}\right)^2; \ \Omega = (0, \frac{1}{2})
\end{align*}
\]
Example 5.5 (Gaussian distribution). \( F_5(x) = e^{-x^2/2}; |x| < 1 \). \( F_5 \) is concave, analytic in \(-1 < x < 1\). \( \text{Ext}(F_5) = \text{Ext}_1(F_5) \) = singleton. The spline extension in Figure 5.8 is not positive definite (see Theorem 5.4).

\[ F_5(x) \]

Fig. 5.8: A spline extension of \( F_5(x) = e^{-x^2/2}; \Omega = (0, 1) \)

Example 5.6. \( F_6(x) = \cos(x); |x| < \frac{\pi}{4} \). \( F_6 \) is concave, analytic around \( x = 0 \). By Theorem 5.4, the spline extension in Figure 5.9 is not positive definite.

\[ F_6(x) \]

Fig. 5.9: A spline extension of \( F_6(x) = \cos(x); \Omega = \left(0, \frac{\pi}{4}\right) \)

Lemma 5.1. The function \( \cos x \) is positive definite on \( \mathbb{R} \).

Proof. For all finite system of coefficients \( \{c_j\} \) in \( \mathbb{C} \), we have

\[
\sum_j \sum_k c_j c_k \cos (x_j - x_k) = \sum_j \sum_k c_j c_k (\cos x_j \cos x_k + \sin x_j \sin x_k) \\
= \left| \sum_j c_j \cos x_j \right|^2 + \left| \sum_j c_j \sin x_j \right|^2 \geq 0.
\] (5.4)

Lemma 5.2. Consider the following two functions \( F_2(x) = 1 - |x|, \) and \( F_3(x) = e^{-|x|}, \) each defined on a finite interval \((-a, a)\), possibly a different value of \( a \) from one to the next. The distributional double derivatives are as follows:

\[
F_2'' = -2 \delta_0 \\
F_3'' = F_3 - 2 \delta_0
\]

where \( \delta_0 \) is Dirac’s delta function (i.e., point mass at \( x = 0 \), \( \delta_0 = \delta(x-0). \))

Proof. The conclusion follows from a computation, making use of L. Schwartz’ theory of distributions; see [Trè06].

5.2 Main Theorems

Now, we shall establish the main result in this chapter (Theorem 5.4) concerning the set $\text{Ext}(F)$. We begin with two preliminary theorems (one is a theorem by Carleman on moments, and the other by Nelson on analytic vectors), and a lemma, which will all be used in our proof of Theorem 5.4.

**Theorem 5.2 (Carleman [Akh65]).** Let $\mu$ be a positive Borel measure on $\mathbb{R}$ such that $t^n \in L^1(\mu)$ for all $n \in \mathbb{Z}_+ \cup \{0\}$, and set $m_n := \int_{\mathbb{R}} t^n d\mu(t)$. If

$$
\sum_{k=1}^{\infty} m_{2k}^{-\frac{1}{2k}} = \infty,
$$

(5.5)

then the set of positive measures with these moments is a singleton. We say that the moment problem is determinate.

We recall E. Nelson’s theorem on analytic vectors:

**Theorem 5.3 (Nelson [Nel59]).** Let $D$ be a skew-Hermitian operator with dense domain $\text{dom}(D)$ in a Hilbert space. Suppose $\text{dom}(D)$ contains a dense set of vectors $v$ s.t. $\exists C_0, C_1 < \infty$, $\|D^n v\| \leq C_0 n! C_1^n$, $n = 0, 1, 2, \cdots$ (5.6)

where the constants $C_0, C_1$ depend on $v$; then $D$ is essentially skew-adjoint, and so it has deficiency indices $(0, 0)$. Vectors satisfying (5.6) are called analytic vectors.

**Lemma 5.3.** Let $F$ be a p.d. function defined on $(-a, a)$, $a > 0$, and assume that $F$ is analytic in a neighborhood of $x = 0$; then $D^{(F)}$ has deficiency indices $(0, 0)$, i.e., $D^{(F)}$ is essentially skew-adjoint, and so $\text{Ext}_1(F)$ is a singleton.

**Proof.** We introduce $\mathcal{H}_F$ and $D^{(F)}$ as before. Recall that $\text{dom}(D^{(F)})$ consists of

$$
F_\varphi(x) = \int_0^a \varphi(y) F(x-y) dy, \quad \varphi \in C_c^\infty(0, a)
$$

(5.7)

and

$$
D^{(F)}(F_\varphi)(x) = F_{\varphi'}(x) = \frac{d}{dx} F_\varphi(x). \text{ (integration by parts)}
$$

(5.8)

Since $F$ is locally analytic, we get

$$
(D^{(F)})^n F_\varphi = \varphi * F^{(n)}, \quad \varphi \in C_c^\infty(0, a).
$$

(5.9)

The lemma will be established by the following:

**Claim.** The vectors $\{F_\varphi : \varphi \in C_c^\infty(0, \varepsilon)\}$, for sufficiently small $\varepsilon > 0$, satisfy the condition in (5.6). Note that $\{F_\varphi\}$ is dense in $\mathcal{H}_F$ by Section 2.1, so these are analytic vectors in the sense of Nelson (see Theorem 5.3, and [Nel59]).
Proof. Since $F$ is assumed to be analytic around $x = 0$, there exists $\varepsilon > 0$ s.t.

$$F(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(y)}{n!} (x - y)^n$$  \hfill (5.10)

holds for $|x| \ll 2\varepsilon$, $|y| < \varepsilon$; and this implies that

$$|F^{(n)}(y)| \leq C_0 n! e^{-n}. \hfill (5.11)$$

We now use (5.9) to establish the estimate (5.6) for $\varepsilon$ as in (5.10)-(5.11). We pick $\varepsilon$, and consider $\varphi \in \mathcal{C}_c^\infty(0, \varepsilon)$, then

$$(D(F)^n F\varphi)(x) = \int_{0}^{\varepsilon} \varphi(y) F^{(n)}(x - y) dy$$

$$= \int_{0}^{\varepsilon} \varphi(x - y) F^{(n)}(y) dy. \hfill (5.12)$$

Using Corollary 2.2, we get

$$\left\| F^{(m)} \right\|_{H_F}^2 = -F^{(2m)}(0) \hfill (5.13)$$

and in particular $F^{(2m)}(0) \leq 0$. So,

$$\left\| (D(F)^n F\varphi) \right\|_{H_F}^2 \leq \left( \int_{0}^{\varepsilon} |\varphi| \right)^{2} \left| F^{(n)} \right|_{H_F}^2 \hfill (5.12)$$

$$\leq \left( \int_{0}^{\varepsilon} |\varphi| \right)^{2} \left(-F^{(2n)}(0) \right) \hfill (5.13)$$

$$\leq \left( \int_{0}^{\varepsilon} |\varphi| \right)^{2} C_0^2 \left( \frac{2}{\varepsilon} \right)^{2n} (n!)^2; \hfill (5.11)$$

i.e.,

$$\left\| (D(F)^n F\varphi) \right\|_{H_F} \leq \left( \int_{0}^{\varepsilon} |\varphi| \right) C_0 \left( \frac{2}{\varepsilon} \right)^{n} n!,$$

which is the Nelson estimate (5.6). This completes the proof of the claim.

Note that in Corollary 2.2, we proved (5.13) for $m = 1$, but it follows in general by induction: $m = 1$, $\|F'\|_{H_F}^2 = -F^{(2)}(0)$.

The fact that the set $\{F\varphi : \varphi \in \mathcal{C}_c^\infty(0, \varepsilon)\}$ is dense in $H_F$ follows from an argument in [Nel59, Jor87]:

Starting with $F\varphi, \varphi \in \mathcal{C}_c^\infty(0, \varepsilon)$, using a local translation, we can move $F\varphi$ towards the endpoint $x = a$. This local translation preserves the $H_F$-norm (see, e.g., the proof of Lemma 2.4), and so $F\varphi(\cdot - s)$, with sufficiently small $s$, will also satisfy the analytic estimates with the same constants $C$ as that of $F\varphi$. Thus the analytic estimates established for $F\varphi$ with $\varphi \in \mathcal{C}_c^\infty(0, \varepsilon)$ carries over to the rest of the interval.
up to the endpoint \( x = a \). It remains to note that \( \{ F_{\varphi} : \varphi \in C^\infty_c (0, a) \} \) is dense in \( H_F \); see Lemma 2.2.

Therefore, we get a dense set of analytic vectors for \( D(F) \) as claimed. By Nelson’s theorem, we conclude that \( D(F) \) is essentially skew-adjoint, so it has deficiency indices \((0, 0)\).

\[ \square \]

**Theorem 5.4.** Let \( F \) be a continuous p.d. function given in some finite interval \((-a, a)\), \( a > 0 \). Assume that \( F \) is analytic in a neighborhood of 0; then \( \text{Ext}_2(F) \) is empty.

**Proof.** Since \( F \) is assumed to be analytic in a neighborhood of 0, by Lemma 5.3, we see that \( D(F) \) has deficiency indices \((0, 0)\), i.e., it is essentially skew-adjoint; and \( \text{Ext}_1(F) = \{ \mu \} \), a singleton. Thus,

\[ F(x) = \hat{\mu}(x), \quad \forall x \in (-a, a). \quad (5.14) \]

Also, by the analytic assumption, there exists \( c > 0 \) such that

\[ F(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} x^n, \quad \forall |x| < c; \quad (5.15) \]

and using (5.14), we get that

\[ \int_{\mathbb{R}} e^{itx} d\mu(t) = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \int_{\mathbb{R}} t^n d\mu(t), \quad x \in \mathbb{R}, |x| < c. \quad (5.16) \]

Combining (5.15) and (5.16), we see that \( \exists 0 < c_1 < c \) such that the \( n^{th} \)-moment of \( \mu \), i.e.,

\[ m_n = \int_{\mathbb{R}} t^n d\mu(t) = F^{(n)}(0) (-i)^n \]

satisfies the following estimate: \( \exists c_0 < \infty \) such that

\[ \frac{|x^n|}{n! m_n} \leq c_0, \quad \forall |x| \leq c_1; \]

that is,

\[ |m_n| \leq c_0 \frac{n!}{c_1^n}, \quad \forall n = 0, 1, 2, \ldots. \quad (5.18) \]

In particular, all the moments of \( \mu \) are finite.

Now, we may apply Carleman’s condition \([Akh65]\) with the estimate in (5.18), and conclude that there is a unique measure with the moments specified by the r.h.s. of (5.17); i.e., the measure is uniquely determined by \( F \). Therefore, \( \text{Ext}(F) = \{ \mu \} \), and so \( \text{Ext}_2(F) = \text{Ext}(F) \setminus \text{Ext}_1(F) = \emptyset \), which is the desired conclusion. (To apply Carleman’s condition, we make use of Stirling’s asymptotic formula for the factorials in (5.18)).
By Carleman’s theorem (see Theorem 5.2), there is a unique measure with these moments, but by (5.15) and (5.17) these moments also determine $F$ uniquely, and vice versa. Hence the conclusion that $\text{Ext}(F)$ is a singleton. \hfill \Box

### 5.2.1 Some Applications

Below we discuss a family of examples of positive definite functions $F$, defined initially only in the interval $-1 < x < 1$, but allows analytic continuation to a complex domain,

$$\mathcal{O}^\mathbb{C} := \{z \in \mathbb{C} | \Im\{z\} > -a\},$$

see Figure 5.11. Here, $a \geq 0$ is fixed.

What is special for this family is that, for each $F$ in the family, the convex set $\text{Ext}(F)$ is a singleton. It has the following form: There is a probability measure $\mu$ on $\mathbb{R}$, such that

$$\hat{d}\mu(\lambda) = M(\lambda) d\lambda, \quad \lambda \in \mathbb{R}; \quad (5.19)$$

where $M$ is supported on $[0, \infty)$, and $\text{Ext}(F) = \{\mu\}$, the singleton.

The list of these measures $d\mu = M(\lambda) d\lambda$ includes the following distributions from statistics:

$$M(\lambda) = M_p(\lambda) = \frac{\lambda^{p-1}}{\Gamma(p)} e^{-\lambda}, \quad \lambda \geq 0; \quad (5.20)$$

where $p$ is fixed, $p > 0$, and $\Gamma(p)$ is the Gamma function. (The case when $p = \frac{1}{2}$, i.e., $p = n^2$, is called the $\chi^2$-distribution of $n$ degrees of freedom. See [AMS13].)

Another example is the log-normal distribution (see [KM04]), where

$$M(\lambda) = \frac{1}{\sqrt{2\pi}} \frac{1}{\lambda} e^{-((\log\lambda - \mu_0)^2)/2}, \quad (5.21)$$

and where $\mu_0 \in \mathbb{R}$ is fixed. The domain in $\lambda$ is $\lambda > 0$. See Figure 5.10.

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![Graphs](https://via.placeholder.com/150)

(a) The Gamma (p) distribution with parameter $p$, see (5.20).

(b) The log-normal distribution with parameter $\mu_0$, see (5.21).

Fig. 5.10: Probability densities of Gamma (p) and log-normal distributions.
Both of the families are given by parameters, but different from one to the other, see (5.20), Gamma; and 5.21, log-normal. They have in common the index \((0,0)\) conclusion. More specifically, in each of the two classes, when an initial p.d. function \(F\) is specified in a finite interval \((-a,a)\), we compute the associated skew-Hermitian operator \(D(F)\) in \(\mathcal{H}_F\), and this operator will have deficiency indices \((0,0)\).

The big difference between the two classes is that only in the first family will \(F\) be analytic in a neighborhood of \(x = 0\); not for the other. We also stress that these families are very important in applications, see e.g., [KT08, KRAA15, Lin14].

We shall consider extension theory for the p.d. functions connected with the first family of these distributions, i.e., (5.20). The conclusions we list in Example 5.7 below all follow from a direct application of Theorem 5.4.

**Example 5.7.** Fix \(p \in \mathbb{R}_+\), and let \(F = F_p\) be given by

\[
F(x) := (1 - ix)^{-p}, \quad |x| < 1;
\]  

(5.22)
i.e., \(F\) is defined in the interval \((-1,1)\). For the study of the corresponding \(\mathcal{H}_F\), we write \(\Omega = (0,1)\), so that \(\Omega - \Omega = (-1,1)\). The following properties hold:

1. \(F\) is positive definite (p.d.) in \((-1,1)\).
2. \(F\) has a unique continuous p.d. extension \(\tilde{F}\) to \(\mathbb{R}\).
3. The p.d. extension \(\tilde{F}\) in (2) has an analytic continuation, \(x \to z \in \mathbb{C}\), to the complex domain (Figure 5.11)

\[
\mathcal{O}\mathcal{C} = \{z \in \mathbb{C} | \Im\{z\} > -1\}.
\]  

(5.23)

4. \(\text{Ext}(F)\) is a singleton, and \(\text{Ext}_2(F) = \emptyset\).
5. For the skew-Hermitian operator \(D(F)\) (with dense domain in \(\mathcal{H}_F\)), we get deficiency indices \((0,0)\).
6. In case of \(p = 1\), for the corresponding p.d. function \(F\) in (5.22), we have:

\[
\Re F(x) = \frac{1}{1 + x^2}, \quad \text{and}
\]

\[
\Im F(x) = \frac{x}{1 + x^2}, \quad \text{for } |x| < 1.
\]

A systematic discussion of the real and the imaginary parts of positive definite functions, and their local properties, is included in Section 8.4.

**Remark 5.2.** While both the Gamma distribution (5.20), \(p > 0\), and the log-normal distribution (5.21) are supported on the half-line \(0 \leq \lambda < \infty\), there is an important distinction connected to moments and the Carleman condition (5.5).

The moments,

\[
m_n := \int_0^\infty \lambda^n \mu(\lambda) = \int_0^\infty \lambda^n M(\lambda) d\lambda,
\]

for the two cases are as follows:
Fig. 5.11: The complex domain $\mathcal{C} := \{z \in \mathbb{C} | \Im(z) > -a\}$, $a = 1$.

(1) Gamma ($p$):

$$m_n = (p+n-1)(p+n-2)\cdots(p+1)p, \quad n \in \mathbb{Z}_+$$

(5.24)

(2) log-normal ($\mu_0$):

$$m_n = e^{\mu_0 n + n^2/2}, \quad n \in \mathbb{Z}_+.$$ 

(5.25)

Since both are Stieltjes-problems, the Carleman condition is

$$\sum_{n=1}^{\infty} m_n^{-1/2} = \infty.$$ 

(5.26)

One checks that (5.26) is satisfied for Gamma ($p$), see (5.24); but not for log-normal ($\mu_0$) case, see (5.25). A closer inspection shows that the generating function,

$$F_{LN}(x) = \hat{d\mu_{LN}}(x), \quad x \in \mathbb{R},$$

(5.27)

for log-normal ($\mu_0$) is not analytic in a neighborhood of $x = 0$, see (5.21). In fact, we have

$$F_{LN}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left[-\frac{\lambda^2}{2} + ixe^{\lambda}\right] d\lambda,$$

(5.28)

which has a complex analytic continuation: $x \mapsto z$, $\Im(z) > 0$, the open upper half-plane (UH); but fails to be analytic on $x$-axis, the boundary of $UH$.

Note the formal power series,

$$F_{LN}(x) \sim \sum_{n=0}^{\infty} e^{\alpha^2/2} (ix)^n / n!,$$

(5.29)

is divergent. More precisely, the radius of convergence in the “power series” on the r.h.s. (5.29) is $rad = 0$. And, as a result $F_{LN}(x)$ is not analytic in a neighborhood of $x = 0$. 
5.3 The Deficiency-Indices of \( D^F \)

This section is devoted to the index problem for the skew-Hermitian operator \( D^F \) (with dense domain in the RKHS \( \mathcal{H}_F \)), but not in the general case. Rather we compute \( D^F \) for the particular choice of partially defined p.d. functions \( F \) from the list in Table 5.1. There are six of them in all, numbered \( F_1 \) through \( F_6 \). The last one \( F_6 \) is the easiest, and we begin with it. While these are only special cases (for example, they are all continuous p.d. functions defined initially only in a fixed finite interval, centered at \( x = 0 \)), a closer analysis of them throws light on much more general cases, including domains in \( \mathbb{R}^n \), and even in non-abelian groups. But even for the case of \( G = \mathbb{R} \), we stress that applications to statistics dictates a study of extension theory for much bigger families of interesting p.d. functions, defined on finite intervals; see for example Section 5.2.1.

| \( F \) : \( (-a, a) \to \mathbb{C} \) | Indices | The Operator \( D^F \) |
|--------------------------------|--------|---------------------|
| \( F_1 (x) = \frac{1}{1 + x^2}, |x| < 1 \) | (0, 0) | \( D^F \) unbounded, skew-adjoint |
| \( F_2 (x) = 1 - |x|, |x| < \frac{1}{2} \) | (1, 1) | \( D^F \) has unbounded skew-adjoint extensions |
| \( F_3 (x) = e^{-|x|}, |x| < 1 \) | (1, 1) | \( D^F \) has unbounded skew-adjoint extensions |
| \( F_4 (x) = \left( \frac{\sin(x/2)}{x/2} \right)^2, |x| < \frac{1}{2} \) | (0, 0) | \( D^F \) bounded, skew-adjoint |
| \( F_5 (x) = e^{-x^2/2}, |x| < 1 \) | (0, 0) | \( D^F \) unbounded, skew-adjoint |
| \( F_6 (x) = \cos x, |x| < \frac{\pi}{4} \) | (0, 0) | \( D^F \) is rank-one, \( \dim(\mathcal{H}_{F_6}) = 2 \) |
| \( F_7 (x) = (1 - ix)^{-\frac{3}{2}}, |x| < 1 \) | (0, 0) | \( D^F \) is unbounded, skew-adjoint, but semibounded; see Corollary (2.8) |

Table 5.1: The deficiency indices of \( D^F : F_\phi \mapsto F_\phi^*, \) with \( F = F_i, \) \( i = 1, \ldots, 7. \)

We have introduced a special class of positive definite (p.d.) extensions using a spline technique based on a theorem by Pólya [Pól49].

We then get a deficiency index-problem (see e.g., [vN32a, Kre46, DS88, AG93, Nel69]) in the RKHSs \( \mathcal{H}_{F_i}, \) \( i = 1, \ldots, 6, \) for the operator \( D^{(F_i)} F_\phi^{(i)} = F_\phi^{(i)}, \forall \phi \in C^\infty_c (0, a). \) As shown in Tables 5.1, \( D^{(F_2)} \) and \( D^{(F_3)} \) have indices \( (1, 1) \), and the other four all have indices \( (0, 0). \)

Following is an example with deficiency indices \( (0, 0) \)

**Lemma 5.4.** \( \mathcal{H}_{F_6} \) is 2-dimensional.
Proof. For all \( \varphi \in C_\infty^\infty (0, \frac{\pi}{2}) \), we have:

\[
\int_\Omega \int_\Omega \varphi(x) \varphi(y) F_0(x-y) \, dx \, dy = \left| \hat{\varphi}(c) (1) \right|^2 + \left| \hat{\varphi}(s) (1) \right|^2
\]

where \( \hat{\varphi}(c) \) is the cosine-transform, and \( \hat{\varphi}(s) \) is the sine-transform. \( \square \)

So the deficiency indices only account from some of the extension of a given positive definite function \( F \) on \( \Omega - \Omega \), the Type I extensions.

Except for \( H F_6 \) (2-dimensional), in all the other six examples, \( H F_i \) is infinite-dimensional.

In the given seven examples, we have p.d. extensions to \( \mathbb{R} \) of the following form, \( d\mu_i(\cdot) \), \( i = 1, \ldots, 7 \), where these measures are as follows:

| Measure | Description |
|---------|-------------|
| \( d\mu_1(\lambda) = \frac{1}{2} e^{-|\lambda|} d\lambda \) | Fig 10.1 (pg. 206) |
| \( d\mu_2(\lambda) = \frac{1}{\pi^2} \left( \frac{\sin(\lambda/2)}{\lambda/2} \right)^2 d\lambda \) | Fig 10.1 (pg. 206); Table 7.1 (pg. 177) |
| \( d\mu_3(\lambda) = \frac{d\lambda}{\pi (\lambda + 1)} \) | Fig 10.1 (pg. 206); Table 7.1 (pg. 177) |
| \( d\mu_4(\lambda) = \chi_{(-1,1)}(\lambda) (1 - |\lambda|) d\lambda \) | Fig 10.1 (pg. 206); Ex 8.1 (pg. 183); Ex 5.4 (pg. 111) |
| \( d\mu_5(\lambda) = \frac{1}{\sqrt{\pi}} e^{-\lambda^2/2} d\lambda \) | Fig 10.1 (pg. 206); Ex 1.1 (pg. 26) |
| \( d\mu_6(\lambda) = \frac{1}{2} (\delta_1 + \delta_{-1}) \) | Fig 10.1 (pg. 206); Lemma 5.1 (pg. 112); Thm 7.1 (pg. 165); Thm 7.2 (pg. 7.2); Rem 8.1 (pg. 180) |
| \( d\mu_7(\lambda) = \frac{\lambda_p}{\Gamma(\lambda_p)} e^{-\lambda} d\lambda, \ \lambda \geq 0 \) | Fig 10.1 (pg. 206); Fig 5.10a (pg. 116) |

Table 5.2: The canonical isometric embeddings: \( H F_i \hookrightarrow L^2(\mathbb{R}, d\mu_i) \), \( i = 1, \ldots, 7 \). In each case, we have \( \mu_i \in \text{Ext}(F_i) \), and, by Corollary 3.1 therefore, the corresponding isometric embeddings \( T_{\mu_i} \) mapping into the respective \( L^2(\mu_i) \)-Hilbert spaces.

See also Table 10.1 and Figure 10.1 below.

**Corollary 5.1.** For \( i = 1, \ldots, 6 \), we get isometries \( T^{(i)} : H F_i \rightarrow L^2(\mathbb{R}, \mu_i) \), determined by
5.3 The Deficiency-Indices of \( D^{(F)} \)

\[
T^{(i)}(F^{(i)}_\phi) = \tilde{\phi}, \quad \forall \phi \in \mathcal{C}_c^\infty(\Omega_i); \text{ where}
\]

\[
\|F^{(i)}_\phi\|^2_{\mathcal{H}_F^i} = \|\tilde{\phi}\|^2_{L^2(\mu_i)} = \int_\Omega |\tilde{\phi}|^2 \, d\mu_i.
\]

Note that, except for \( i = 6 \), \( T^{(i)} \) is only isometric into \( L^2(\mu_i) \).

The adjoint operator, \((T^{(i)})^* : L^2(\mu_i) \to \mathcal{H}_F^i\), is given by

\[
(T^{(i)})^* f = \chi_{\Omega_i}(f \, d\mu_i)^V, \quad \forall f \in L^2(\mathbb{R}, \mu_i).
\]

Proof. We refer to Corollary 3.1.

Example 5.8 (An infinite-dimensional example as a version of \( F_6 \)). Fix \( p, 0 < p < 1 \), and set

\[
F_p(x) := \prod_{n=1}^\infty \cos(2\pi px).
\]

Then \( F_p = \hat{d\mu}_p \), where \( \mu_p \) is the Bernoulli measure, and so \( F_p \) is a continuous positive definite function on \( \mathbb{R} \). Note that some of those measures \( \mu_p \) are fractal measures.

For fixed \( p \in (0, 1) \), the measure \( \mu_p \) is the law (i.e., distribution) of the following random power series

\[
X_p(\omega) := \sum_{n=1}^\infty (\pm) p^n,
\]

where \( \omega \in \prod_1^\infty \{\pm 1\} \) (= infinite Cartesian product) and where the distribution of each factor is \( \{-\frac{1}{2}, \frac{1}{2}\} \), and statically independent. For relevant references on random power series, see [Neu13, Lit99].

Pólya-extensions

The extensions we generate with the application of Pólya’s theorem are realized in a bigger Hilbert space. The deficiency indices of the skew-Hermitian operator \( D^{(F)} \) are computed w.r.t. the RKHS \( \mathcal{H}_F \), i.e., for the “small” p.d. function \( F : \Omega - \Omega \to \mathbb{C} \).

Example 5.9. \( F_2(x) = 1 - |x|, |x| < \frac{1}{2} \), has the obvious p.d. extension to \( \mathbb{R} \), i.e., \((1 - |x|) \chi_{[-1,1]}(x)\), which corresponds to the measure \( \mu_2 \) from Table 5.2. It also has other p.d. extensions, e.g., the Pólya extension in Figure 5.4. All these extensions are of Type II, realized in \( \infty \)-dimensional dilation-Hilbert spaces.

We must make a distinction between two classes of p.d. extensions of \( F : \Omega - \Omega \to \mathbb{C} \) to continuous p.d. functions on \( \mathbb{R} \).

Case 1. There exists a unitary representation \( U(t) : \mathcal{H}_F \to \mathcal{H}_F \) such that

\[
F(t) = (\xi_0, U(t) \xi_0)_{\mathcal{H}_F^i}, \quad t \in \Omega - \Omega \tag{5.31}
\]
Case 2. (e.g., Pólya extension) There exist a dilation-Hilbert space \( \mathcal{H} \), and an isometry \( J : \mathcal{H} \rightarrow \mathcal{H} \), and a unitary representation \( U(t) \) of \( \mathbb{R} \) acting in \( \mathcal{H} \), such that

\[
F(t) = \langle J\xi_0, U(t)J\xi_0 \rangle_{\mathcal{H}}, \quad t \in \Omega - \Omega
\]  

(5.32)

In both cases, \( \xi_0 = F(0 - \cdot) \in H \).

In case 1, the unitary representation is realized in \( H(F, \Omega - \Omega) \), while, in case 2, the unitary representation \( U(t) \) lives in the expanded Hilbert space \( K \).

Note that the r.h.s. in both (5.31) and (5.32) is defined for all \( t \in \mathbb{R} \).

Lemma 5.5. Let \( F_{ex} \) be one of the Pólya extensions if any. Then by the Gelfand-Naimark-Segal (GNS) construction applied to \( F_{ex} : \mathbb{R} \rightarrow \mathbb{R} \), there is a Hilbert space \( K \) and a vector \( v_0 \in K \) and a unitary representation \( \{U(t)\}_{t \in \mathbb{R}} : U(t) : \mathcal{H} \rightarrow \mathcal{H} \), such that

\[
F_{ex}(t) = \langle v_0, U(t)v_0 \rangle_{\mathcal{H}}, \quad \forall t \in \mathbb{R}.
\]  

(5.33)

Setting \( J : \mathcal{H} \rightarrow \mathcal{H} \), \( J\xi_0 = v_0 \), then \( J \) defines (by extension) an isometry such that

\[
U(t)J\xi_0 = J(\text{local translation in } \Omega)
\]  

(5.34)

holds locally (i.e., for \( t \) sufficiently close to 0.)

Moreover, the function

\[
\mathbb{R} \ni t \mapsto U(t)J\xi_0 = U(t)v_0
\]  

(5.35)

is compactly supported.

Proof. The existence of \( K, v_0 \), and \( \{U(t)\}_{t \in \mathbb{R}} \) follows from the GNS-construction.

The conclusions in (5.34) and (5.35) follow from the given data, i.e., \( F : \Omega - \Omega \rightarrow \mathbb{R} \), and the fact that \( F_{ex} \) is a spline-extension, i.e., it is of compact support; but by (5.33), this means that (5.35) is also compactly supported. \( \square \)

Example 5.2 gives a p.d. \( F \) in \( (-\frac{1}{2}, \frac{1}{2}) \) with \( D(F) \) of index \( (1, 1) \) and explicit measures in \( Ext_1(F) \) and in \( Ext_2(F) \).

We have the following:

**Deficiency** \((0, 0)\): The p.d. extension of Type I is unique; see (5.31); but there may still be p.d. extensions of Type II; see (5.32).

**Deficiency** \((1, 1)\): This is a one-parameter family of extensions of Type I; and some more p.d. extensions are Type II.

So we now divide

\[ Ext(F) = \left\{ \mu \in \text{Prob}(\mathbb{R}) \mid \hat{\mu} \text{ is an extension of } F \right\} \]

up in subsets

\[ Ext(F) = Ext_{type1}(F) \cup Ext_{type2}(F) ; \]

where \( Ext_2(F) \) corresponds to the Pólya extensions.

Return to a continuous p.d. function \( F : (-a, a) \rightarrow \mathbb{C} \), we take for the RKHS \( \mathcal{H}_F \), and the skew-Hermitian operator...
5.4 The Example 5.3, Green’s function, and an $\mathcal{H}_F$-ONB

Here, we study Example 5.3 in more depth, and we compute the spectral date of the corresponding Mercer operator. Recall that

$$F(x) := \begin{cases} 
  e^{-|x|} & |x| < 1 \\
  e^{-1} (2 - |x|) & 1 \leq |x| < 2 \\
  0 & |x| \geq 2 
\end{cases} \quad (5.36)$$

See figure 5.12 below.
Proposition 5.2. The RKHS $H_F$ for the spline extension $F$ in (5.36) has the following ONB:

$$\left\{ \frac{2}{\left(1 + \left(\frac{n\pi}{2}\right)^2\right)^{\frac{1}{2}}} \sin \left(\frac{n\pi x}{2}\right) \right\}_{n \in \mathbb{Z}^+}.$$  

Proof. The distributional derivative of $F$ satisfies

$$F'' = F - 2\delta_0 + e^{-1}(\delta_2 + \delta_2)$$

This can be verified directly, using Schwartz’ theory of distributions. See lemma 5.2 and [Trè06].

Thus, for $F_x(y) := F(x - y)$ we have the translation

$$\langle(1 - \Delta)F_x, g\rangle = \lambda g(x), 0 < x < 2.$$  

Consider the Mercer operator (see Section 6.1):

$$L^2(0, 2) \ni g \mapsto \int_0^2 F_x(y) g(y) dy = \langle F_x, g \rangle.$$  

Suppose $\lambda \in \mathbb{R}$, and

$$\langle F_x, g \rangle = \lambda g(x).$$  

Applying $(1 - \Delta)$ to both sides of (5.38), we get

$$\langle(1 - \Delta)F_x, g\rangle = \lambda \left(g(x) - g''(x)\right), 0 < x < 2.$$  

By (5.37), we also have

$$\langle(1 - \Delta)F_x, g\rangle = 2g(x), 0 < x < 2.$$  

using the fact the two Dirac masses in (5.37), i.e., $\delta_{x\pm2}$, are supported outside the open interval $(0, 2)$. 


Therefore, combining (5.39)-(5.40), we have
\[ g'' = \frac{\lambda - 2}{\lambda} g, \quad \forall x \in (0, 2). \]

By Mercer’s theorem, \(0 < \lambda < 2\). Setting
\[ k := \sqrt{\frac{2 - \lambda}{\lambda}} \quad \left( \Leftrightarrow \lambda = \frac{2}{1 + k^2} \right) \]
we have
\[ g'' = -k^2 g, \quad \forall x \in (0, 2). \]

**Boundary conditions:**
In (5.39), set \(x = 0\), and \(x = 2\), we get
\[
\begin{align*}
2g(0) - e^{-1}g(2) &= \lambda \left( g(0) - g''(0) \right) \quad (5.41) \\
2g(2) - e^{-1}g(0) &= \lambda \left( g(2) - g''(2) \right) \quad (5.42)
\end{align*}
\]
Now, assume
\[ g(x) = Ae^{ikx} + Be^{-ikx}, \]
where
\[
\begin{align*}
g(0) &= A + B \\
g(2) &= Ae^{ik2} + Be^{-ik2} \\
g''(0) &= -k^2 (A + B) \\
g''(2) &= -k^2 \left( Ae^{ik2} + Be^{-ik2} \right).
\end{align*}
\]
Therefore, for (5.41), we have
\[
2(A + B) - e^{-1} \left( Ae^{ik2} + Be^{-ik2} \right) = \lambda \left( 1 + k^2 \right) (A + B)
\]
i.e.,
\[ Ae^{ik2} + Be^{-ik2} = 0. \quad (5.43) \]
Now, from (5.42) and using (5.43), we have
\[ A = -B. \quad (5.44) \]
Combining (5.43)-(5.44), we conclude that
\[ \sin 2k = 0 \iff k = \frac{\pi n}{2}, \quad n \in \mathbb{Z}; \]
i.e.,
\[ \lambda_n := \frac{2}{1 + \left(\frac{n\pi}{2}\right)^2}, \quad n \in \mathbb{N}. \] (5.45)

The associated ONB in \( L^2(0, 2) \) is
\[ \xi_n(x) = \sin\left(\frac{n\pi x}{2}\right), \quad n \in \mathbb{N}. \] (5.46)

And the corresponding ONB in \( \mathcal{H}_F \) consists of the functions \( \left\{ \sqrt{\lambda_n} \xi_n \right\}_{n \in \mathbb{N}} \), i.e.,
\[ \left\{ \frac{\sqrt{2}}{\left(1 + \left(\frac{n\pi}{2}\right)^2\right)^{1/2}} \sin\left(\frac{n\pi x}{2}\right) \right\}_{n \in \mathbb{N}} \] (5.47)

which is the desired conclusion. \( \Box \)
Chapter 6
Spectral Theory for Mercer Operators, and Implications for $\text{Ext}(F)$

Given a continuous positive definite (p.d.) function $F$ on the open interval $(-1,1)$, we are concerned with the set $\text{Ext}(F)$ of its extensions to p.d. functions defined on all of $\mathbb{R}$, as well as a certain subset $\text{Ext}_1(F)$ of $\text{Ext}(F)$. Since every such p.d. extension of $F$ is a Bochner transform of a unique positive and finite Borel measure $\mu$ on $\mathbb{R}$, i.e., $\hat{\mu}(x), x \in \mathbb{R}$ and $\mu \in \mathcal{M}^+(\mathbb{R})$, we will speak of $\text{Ext}(F)$ as a subset of $\mathcal{M}^+(\mathbb{R})$. The purpose of this chapter is to gain insight into the nature and properties of $\text{Ext}_1(F)$.

In Section 6.1, we study the Mercer operator $T_F$ associated with $F$, which is a certain trace class integral operator. We use it (see Section 6.2) to identify a natural Bessel frame in the RKHS $\mathcal{H}_F$. We further introduce a notion of Shannon sampling of finite Borel measures on $\mathbb{R}$, then use this in Corollary 6.11 to give a necessary and sufficient condition: $\mu \in \mathcal{M}^+(\mathbb{R})$ is in $\text{Ext}(F)$ if and only if the Shannon sampling of $\mu$ recovers the p.d. function $F$ on the interval $(-1,1)$.

**Definition 6.1.** Let $F$ be a continuous positive definite (p.d.) function on a finite interval $(-a,a) \subset \mathbb{R}$. By a Mercer operator, we mean the integral operator $T_F : L^2(0,a) \to L^2(0,a)$, given by

$$
(T_F \varphi)(x) := \int_0^a \varphi(y) F(x-y) dy, \quad \varphi \in L^2(0,a), \ x \in (0,a). \quad (6.1)
$$

**Lemma 6.1.** Let $T_F$ be the Mercer operator in (6.1), and $\mathcal{H}_F$ be the RKHS of $F$.

1. Then there is a sequence $(\lambda_n)_{n \in \mathbb{N}}, \lambda_n > 0, \sum_{n \in \mathbb{N}} \lambda_n < \infty$; and a system of orthogonal functions $\{\xi_n\} \subset L^2(0,a) \cap \mathcal{H}_F$, such that

$$
F(x-y) = \sum_{n \in \mathbb{N}} \lambda_n \xi_n(x) \xi_n(y), \text{ and} \quad (6.2)
$$

$$
\int_0^a \xi_n(x) \xi_m(x) dx = \delta_{n,m}, \quad n,m \in \mathbb{N}. \quad (6.3)
$$

2. In particular, $T_F$ is trace class in $L^2(0,a)$. If $F(0) = 1$, then
Proof. This is an application of Mercer’s theorem [LP89, FR42, FM13]. Note that we must check that \( F \), initially defined on \((-a, a)\), extends uniquely by limit to a continuous p.d. function on the closed interval \([-a, a]\). This is easy to verify; also see Lemma 8.1. \(\square\)

**Corollary 6.1.** Let \( F : (-a, a) \to \mathbb{C} \) be a continuous p.d. function, assume that \( F(0) = 1 \). Let \( D^{(F)} \) be the skew-Hermitian operator in \( \mathcal{H}_F \). Let \( z \in \mathbb{C} \setminus \{0\} \), and \( \text{DEF}_F(z) \subset \mathcal{H}_F \) be the corresponding deficiency space.

Let \( \{\xi_n\}_{n \in \mathbb{N}} \subset \mathcal{H}_F \cap L^2(\Omega) \), \( \Omega := (0, a) \); and \( \{\lambda_n\}_{n \in \mathbb{N}} \) s.t. \( \lambda_n > 0 \), and \( \sum_{n=1}^{\infty} \lambda_n = a \), be one Mercer–system as in Lemma 6.1.

Then \( \text{DEF}_F(z) \neq 0 \) if and only if
\[
\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left| \int_0^a \overline{\xi_n(x)} e^{zx} \, dx \right|^2 < \infty.
\]

(6.5)

Proof. We saw in Lemma 6.1 that if \( \{\xi_n\}_{n \in \mathbb{N}}, \{\lambda_n\}_{n \in \mathbb{N}} \), is a Mercer system (i.e., the spectral data for the Mercer operator \( T_F \) in \( L^2(0, a) \)), then \( \xi_n \in \mathcal{H}_F \cap L^2(\Omega) \); and \( (\sqrt{\lambda_n} \xi_n(\cdot))_{n \in \mathbb{N}} \) is an ONB in \( \mathcal{H}_F \).

But (6.5) in the corollary is merely stating that the function \( e_z(x) := e^{zx} \) has a finite \( l^2 \)-expansion relative to this ONB. The rest is an immediate application of Parseval’s identity (w.r.t. this ONB.) \(\square\)

**Remark 6.1.** The conclusion in the corollary applies more generally: It shows that a continuous function \( f \) on \([0, a]\) is in \( \mathcal{H}_F \) iff
\[
\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left\| \int_0^a \overline{\xi_n(x)} f(x) \, dx \right\|^2 < \infty.
\]

6.1 **Groups, Boundary Representations, and Renormalization**

In this section, we study a boundary representation for the operator \( T_F \) (Corollary 6.4), and an associated renormalization for the Hilbert space \( \mathcal{H}_F \) in Corollary 6.5, in Theorems 6.1 and 6.6, with corollaries.

**Definition 6.2.** Let \( G \) be a Lie group, and let \( \Omega \) be a subset of \( G \), satisfying the following:

B1. \( \Omega \neq \emptyset \);
B2. \( \Omega \) is open and connected;
B3. the closure \( \overline{\Omega} \) is compact;
B4. the boundary of \( \overline{\Omega} \) has Haar measure zero.
Then, for any continuous positive definition function $F : \Omega^{-1} \cdot \Omega \rightarrow \mathbb{C}$, a trace class Mercer operator is defined by

$$(T_F \phi)(x) = \int_{\Omega} \phi(y) F(y^{-1}x) \, dy, \quad \phi \in L^2(\Omega)$$

(6.6)

where $dy$ is the restriction of Haar measure on $G$ to $\Omega$, or equivalently to $\Omega$.

Note that with Lemma 8.1 and assumption B3, we conclude that $H_F \subseteq C(\Omega) \subset L^2(\Omega)$.

(6.7)

It is natural to ask: “What is the orthogonal complement of $H_F$ in the larger Hilbert space $L^2(\Omega)$?” The answer is given in Corollary 6.2. We will need the following:

**Lemma 6.2.** There is a finite constant $C_1$ such that

$$\|\xi\|_{L^2(\Omega)} \leq C_1 \|\xi\|_{H_F}, \quad \forall \xi \in H_F.$$  

(6.8)

**Proof.** Since $H_F \subseteq L^2(\Omega)$ by (6.7), the inclusion mapping, $H_F \rightarrow L^2(\Omega)$, is closed and therefore bounded; by the Closed-Graph Theorem [DS88]; and so the estimate (6.8) follows. (See also Theorem 6.2 below for an explicit bound $C_1$.) □

**Corollary 6.2.** Let $T_F$ be the Mercer operator in (6.6), then

$L^2(\Omega) \ominus H_F = \ker (T_F)$;

(6.9)

and as an operator in $L^2(\Omega)$, $T_F$ takes the form:

$$T_F = \begin{pmatrix} H_F & \ker(T_F) \\ \ker(T_F) & 0 \end{pmatrix} \begin{pmatrix} H_F \\ \ker(T_F) \end{pmatrix}$$

**Proof.** Let $\xi \in H_F$, and pick $\phi_n \in C_c(\Omega)$ such that

$$\|\xi - F_{\phi_n}\|_{H_F} \rightarrow 0, \quad n \rightarrow \infty.$$  

(6.10)

Since $F_{\phi_n} = T_F \phi_n$, and so we get

$$\|\xi - T_F \phi_n\|_{L^2(\Omega)} \leq C_1 \|\xi - F_{\phi_n}\|_{H_F} \rightarrow 0, \quad n \rightarrow \infty.$$  

(6.11)

Therefore, if $f \in L^2(\Omega)$ is given, we conclude that the following properties are equivalent:
\[ \langle f, \xi \rangle_{L^2(\Omega)} = 0, \forall \xi \in \mathcal{H}_F \]
\[ \Downarrow \text{by (6.11)} \]
\[ \langle f, \varphi \rangle_{L^2(\Omega)} = 0, \forall \varphi \in C_c(\Omega) \]
\[ \Downarrow \]
\[ \langle f, T_F \varphi \rangle_{L^2(\Omega)} = 0, \forall \varphi \in C_c(\Omega) \]
\[ \Downarrow (T_F \text{ is selfadjoint, as an operator in } L^2(\Omega)). \]
\[ \langle T_F f, \varphi \rangle_{L^2(\Omega)} = 0, \forall \varphi \in C_c(\Omega) \]
\[ \Downarrow (C_c(\Omega) \text{ is dense in } L^2(\Omega)) \]
\[ T_F f = 0 \]

□

**Remark 6.2.** Note that \( \ker(T_F) \) may be infinite dimensional. This happens for example, in the cases studied in Section 4.3, and in the case of \( F_0 \) in Table 5.1. On the other hand, these examples are rather degenerate since the corresponding RKHSs \( \mathcal{H}_F \) are finite dimensional.

**Convention.** When computing \( T_F(f) \), \( f \in L^2(\Omega) \), we may use (6.7) to write \( f = f^{(F)} + f^{(K)} \), where \( f^{(F)} \in \mathcal{H}_F \), \( f^{(K)} \in \ker(T_F) \), and \( (f^{(F)}, f^{(K)})_{L^2(\Omega)} = 0 \). As a result,

\[ T_F(f) = T_F f^{(F)} + T_F f^{(K)} = T_F f^{(F)} \]

For the inverse operator, by \( T_F^{-1}(f) \) we mean \( T_F^{-1}f^{(F)} \).

**Corollary 6.3.** Let \( G, \Omega, F \) and \( \mathcal{H}_F \) be as above, assume that \( \Omega \subset G \) satisfies B1-B4. Suppose in addition that \( G = \mathbb{R}^n \), and that there exists \( \mu \in \text{Ext}(F) \) such that the support of \( \mu \), \( \text{supp}(\mu) \), contains a non-empty open subset in \( \mathbb{R}^n \); then \( \ker(T_F) = 0 \).

**Proof.** By Corollary 3.1, there is an isometry \( \mathcal{H}_F \to L^2(\mathbb{R}^n, \mu) \), by \( F_\varphi \mapsto \hat{\varphi} \), for all \( \varphi \in C_c(\Omega) \), and so \( \|F_\varphi\|_{\mathcal{H}_F} = \|\hat{\varphi}\|_{L^2(\mathbb{R}^n, \mu)} \). (Here, \( \hat{\cdot} \) denotes Fourier transform in \( \mathbb{R}^n \).)

Since \( F_\varphi = T_F(\varphi) \), for all \( \varphi \in C_c(\Omega) \), and \( C_c(\Omega) \) is dense in \( L^2(\Omega) \), we get

\[ \|T_F f\|_{\mathcal{H}_F} = \|\hat{f}\|_{L^2(\mathbb{R}^n, \mu)} \] \( (6.12) \)

holds also for a dense subspace of \( f \in L^2(\Omega) \), containing \( C_c(\Omega) \). (In fact, for all \( \xi \in \mathcal{M}_2(F, \Omega) \), we have \( \|\xi\|_{\mathcal{M}_2(F, \Omega)} = \|\hat{\xi}\|_{L^2(\mathbb{R}^n, \mu)} \); see Corollary 2.1.)

It follows that \( f \in \ker(T_F) \Rightarrow \hat{f} \equiv 0 \) on \( \text{supp}(\mu) \), by (6.12). But since \( \overline{\mathcal{D}} \) is compact, we conclude by Paley-Wiener, that \( \hat{f} \) is entire analytic. Since \( \text{supp}(\mu) \) contains a non-empty open set, we conclude that \( \hat{f} \equiv 0 \); and by (6.12), that therefore \( f = 0 \). □

**Theorem 6.1.** Let \( G, \Omega, F : \Omega^{-1} \Omega \to \mathbb{C} \), and \( \mathcal{H}_F \) be as in Definition 6.2, i.e., we assume that B1-B4 hold.
Let $T_F$ denote the corresponding Mercer operator $L^2(\Omega) \to \mathcal{H}_F$. Then $T_F^*$ is also a densely defined operator on $\mathcal{H}_F$ as follows:

\[
\begin{array}{c}
\mathcal{H}_F \\
\xrightarrow{T_F}
\end{array}
\begin{array}{c}
L^2(\Omega) \\
\xleftarrow{T_F^*}
\end{array}
\]

(6.13)

Moreover, every $\xi \in \mathcal{H}_F$ has a realization $j(\xi)$ as a bounded uniformly continuous function on $\Omega$, and

\[
T_F^* \xi = j(\xi), \quad \forall \xi \in \mathcal{H}_F.
\]

(6.14)

Finally, the operator $j$ from (1)-(2) satisfies

\[
\ker(j) = 0.
\]

(6.15)

Proof. We begin with the claims relating (6.13) and (6.14) in the theorem.

Using the reproducing property in the RKHS $\mathcal{H}_F$, we get the following two estimates, valid for all $\xi \in \mathcal{H}_F$:

\[
|\xi(x) - \xi(y)|^2 \leq 2 \|\xi\|^2_{\mathcal{H}_F} \left( F(e) - \Re \left( F(x^{-1}y) \right) \right), \quad \forall x, y \in \Omega;
\]

(6.16)

and

\[
|\xi(x)| \leq \|\xi\|_{\mathcal{H}_F} \sqrt{F(e)}, \quad \forall x \in \Omega.
\]

(6.17)

In the remaining part of the proof, we shall adopt the normalization $F(e) = 1$; so the assertion in (6.17) states that point-evaluation for $\xi \in \mathcal{H}_F$ is contractive.

Remark 6.3. In the discussion below, we give conditions which yield boundedness of the operators in (6.13). If bounded, then, by general theory, we have

\[
\|j\|_{\mathcal{H}_F \to L^2(\Omega)} = \|T_F\|_{L^2(\Omega) \to \mathcal{H}_F}
\]

(6.18)

for the respective operator-norms. While boundedness holds in “most” cases, it does not in general.

The assertion in (6.14) is a statement about the adjoint of an operator mapping between different Hilbert spaces, meaning:

\[
\int_{\Omega} j(\xi)(x) \varphi(x) \, dx = \langle \xi, T_F \varphi \rangle_{\mathcal{H}_F} = \langle \xi, F \varphi \rangle_{\mathcal{H}_F} = \langle \xi, F \varphi \rangle_{\mathcal{H}_F}
\]

(6.19)

for all $\xi \in \mathcal{H}_F$, and $\varphi \in C_c(\Omega)$. But eq. (6.19), in turn, is immediate from (6.6) and the reproducing property in $\mathcal{H}_F$.

Part (3) of the theorem follows from:
\[ \ker (j) = (\text{ran}(j^*))^\perp = (\text{ran}(T_F))^\perp = 0; \]

(by (2))

where we used that \( \text{ran}(T_F) \) is dense in \( H_F \).

\[ \square \]

Remark 6.4. The r.h.s. in (6.14) is subtle because of boundary values for the function \( \xi : \overline{\Omega} \to \mathbb{C} \) which represent some vector (also denoted \( \xi \)) in \( H_F \). We refer to Lemma 8.1 for specifics.

We showed that if \( \varphi \in C_c(\Omega) \), then (6.19) holds; but if some \( \varphi \in L^2(\Omega) \) is not in \( C_c(\Omega) \), then we pick up a boundary term when computing \( \langle \xi, T_F \varphi \rangle_{H_F} \). Specifically, one can show, with the techniques in Section 3.2, extended to Lie groups, that the following boundary representation holds:

Corollary 6.4. For a connected Lie group \( G \) and \( \Omega \subset G \) open, there is a measure \( \beta \) on the boundary \( \partial \Omega = \overline{\Omega} \setminus \Omega \) such that

\[ \langle \xi, T_F \varphi \rangle_{H_F} = \int_{\Omega} \overline{\xi(x)} \varphi(x) \, dx + \int_{\partial \Omega} \frac{\xi}{|\xi|} (\sigma) (T_F \varphi)_n (\sigma) \, d\beta(\sigma) \]  

(6.20)

for all \( \varphi \in L^2(\Omega) \).

Remark 6.5. On the r.h.s. of (6.20), we must make the following restrictions:

1. \( \overline{\Omega} \) is compact (\( \Omega \) is assumed open, connected);
2. \( \partial \Omega \) is a differentiable manifold of dimension \( \dim(G) - 1 \);
3. The function \( T_F \varphi \in H_F \subset C(\Omega) \) has a well-defined inward normal vector field; and \( (T_F \varphi)_n \) denotes the corresponding normal derivative.

It follows from Lemma 8.1 that the term \( b_\xi (\sigma) := \xi |_{\partial \Omega} (\sigma) \) on the r.h.s. of (6.20) satisfies

\[ |b_\xi (\sigma)| \leq \|\xi\|_{H_F}, \quad \forall \sigma \in \partial \Omega. \]  

(6.21)

Corollary 6.5 (Renormalization). Let \( G, \Omega, F, H_F, \) and \( T_F \) be as in the statement of Theorem 6.1. Let \( \{\xi_n\}_{n \in \mathbb{N}}, \{\lambda_n\}_{n \in \mathbb{N}} \) be the spectral data for the Mercer operator \( T_F \), i.e., \( \lambda_n > 0 \),

\[ \sum_{n=1}^\infty \lambda_n = |\Omega|, \quad \text{and} \quad \{\xi_n\}_{n \in \mathbb{N}} \subset L^2(\Omega) \cap H_F, \quad \text{satisfying} \]  

(6.22)

\[ T_F \xi_n = \lambda_n \xi_n, \int_{\Omega} \overline{\xi_n(x)} \xi_m(x) \, dx = \delta_{n,m}, \quad n, m \in \mathbb{N}, \]  

(6.23)

and

\[ F \left( x^{-1}y \right) = \sum_{n=1}^\infty \lambda_n \overline{\xi_n(x)} \xi_n(y), \quad \forall x, y \in \Omega; \]  

(6.24)

then

\[ \left\{ \sqrt{\lambda_n} \xi_n (\cdot) \right\}_{n \in \mathbb{N}} \]  

(6.25)

is an ONB in \( H_F \).
This is immediate from the theorem. To stress the idea, we include the proof that \( \| \sqrt{\lambda_n} \xi_n \|_{\mathcal{H}_F} = 1, \forall n \in \mathbb{N} \).

Clearly, \( \| \sqrt{\lambda_n} \xi_n \|_{\mathcal{H}_F}^2 = \lambda_n \langle \xi_n, \xi_n \rangle_{\mathcal{H}_F} = \langle \xi_n, T_F \xi_n \rangle_{\mathcal{H}_F} \).

But since \( \xi_n \in L^2(\Omega) \cap \mathcal{H}_F \),

\[ \langle \xi_n, T_F \xi_n \rangle_{\mathcal{H}_F} = \int_{\Omega} \overline{\xi_n(x)} \xi_n(x) \, dx = \| \xi_n \|_{L^2(\Omega)}^2 = 1 \text{ by (6.23)}, \]

and the result follows. \( \square \)

**Theorem 6.2.** Let \( G, \Omega, F, \mathcal{H}_F, \{\xi_n\}, \) and \( \{\lambda_n\} \) be as specified in Corollary 6.5.

1. Then \( \mathcal{H}_F \subseteq L^2(\Omega) \), and there is a finite constant \( C_1 \) such that

\[ \int_{\Omega} |g(x)|^2 \, dx \leq C_1 \|g\|_{\mathcal{H}_F}^2 \]  \hspace{1cm} (6.26)

holds for all \( g \in \mathcal{H}_F \). Indeed, \( C_1 = \|\lambda_1\|_\infty \) will do.

2. Let \( \{\xi_k\}, \{\lambda_k\} \) be as above, let \( N \in \mathbb{N} \), set

\[ \mathcal{H}_F(N) = \text{span} \{ \xi_k \mid k = 1, 2, 3, \ldots, N \}; \]  \hspace{1cm} (6.27)

and let \( Q_N \) be the \( \mathcal{H}_F \)-orthogonal projection onto \( \mathcal{H}_F(N) \); and let \( P_N \) be the \( L^2(\Omega) \)-orthogonal projection onto \( \text{span}_{1 \leq k \leq N} \{ \xi_k \} \); then

\[ Q_N \geq \left( \frac{1}{\lambda_1} \right) P_N \]  \hspace{1cm} (6.28)

where \( \leq \) in (6.28) is the order of Hermitian operators, and where \( \lambda_1 \) is the largest eigenvalue.

**Proof.** Part (1). Pick \( \lambda_n, \xi_n \) as in (6.23)-(6.25). We saw that \( \mathcal{H}_F \subseteq L^2(\Omega) \); recall \( \overline{\Omega} \) is compact. Hence
\[ \|g\|_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} |\langle \xi_n, g \rangle|^2 \]

\[ = \sum_{n=1}^{\infty} |\langle T_F \xi_n, g \rangle_{H_F}|^2 \]

\[ = \sum_{n=1}^{\infty} |\lambda_n \langle \xi_n, g \rangle_{H_F}|^2 \]

\[ = \sum_{n=1}^{\infty} \lambda_n \left| \left\langle \sqrt{\lambda_n} \xi_n, g \right\rangle_{H_F} \right|^2 \]

\[ \leq \left( \sup_{n\in\mathbb{N}} \{\lambda_n\} \right) \sum_{n=1}^{\infty} \left| \left\langle \sqrt{\lambda_n} \xi_n, g \right\rangle_{H_F} \right|^2 \]

\[ = \left( \sup_{n\in\mathbb{N}} \{\lambda_n\} \right) \|g\|_{H_F}^2, \text{ by (6.25) and Parseval.} \]

This proves (1) with \( C_1 = \sup_{n\in\mathbb{N}} \{\lambda_n\} \).

Part (2). Let \( f \in L^2(\Omega) \cap H_F \). Arrange the eigenvalues \( \{\lambda_n\} \) s.t.

\[ \lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots > 0. \]  

Then

\[ \langle f, Q_N f \rangle_{H_F} = \|Q_N f\|_{H_F}^2 \]

\[ = \sum_{n=1}^{N} \left| \left\langle \sqrt{\lambda_n} \xi_n, f \right\rangle_{H_F} \right|^2 \]

\[ \leq \sum_{n=1}^{N} \frac{1}{\lambda_n} \left| \lambda_n \langle \xi_n, f \rangle \right|^2 \]

\[ = \sum_{n=1}^{N} \frac{1}{\lambda_n} \left| \langle T_F \xi_n, f \rangle_{H_F} \right|^2 \]

\[ \geq \frac{1}{\lambda_1} \sum_{n=1}^{N} \left| \langle \xi_n, f \rangle_{L^2(\Omega)} \right|^2 \]

\[ \geq \frac{1}{\lambda_1} \|P_N f\|_{L^2(\Omega)}^2 = \frac{1}{\lambda_1} \langle f, P_N f \rangle_{L^2(\Omega)}; \]

and so the \textit{a priori} estimate (2) holds.

\[ \square \]

\textbf{Remark 6.6.} The estimate (6.26) in Theorem 6.2 is related to, but different from a classical Poincaré-inequality [Maz11]. The latter \textit{a priori} is as follows:

Let \( \Omega \subset \mathbb{R}^n \) satisfying in B1-B4 in Definition 6.2, and let \( |\Omega|_n \) denote the \( n \)-dimensional Lebesgue measure of \( \Omega \), i.e.,
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\[ |\Omega|_n = \int_{\mathbb{R}^n} \chi_\Omega(x) \prod_{i=1}^n dx_i; \]

let \( \nabla = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) \) be the gradient, and

\[ \|\nabla f\|_{L^2(\Omega)}^2 := \sum_{i=1}^n \int_{\Omega} \left| \frac{\partial f}{\partial x_i} \right|^2 dx. \quad (6.30) \]

Finally, let \( \lambda_1(N) = \) the finite eigenvalue for the Neumann problem on \( \Omega \) (NBP\( \Omega \)). Then,

\[ \|f - \frac{1}{|\Omega|_n} \int_{\Omega} f dx\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda_1(N)} \|\nabla f\|_{L^2(\Omega)}^2 \quad (6.31) \]

holds for all \( f \in L^2(\Omega) \) such that \( \frac{\partial f}{\partial x_i} \in L^2(\Omega) \), \( 1 \leq i \leq n \).

Since we shall not have occasion to use this more general version of the Mercer-operators we omit details below, and restrict attention to the case of finite interval in \( \mathbb{R} \).

**Remark 6.7.** Some of the conclusions in Theorem 6.1 hold even if conditions B1-B4 are relaxed. But condition B3 ensures that \( L^2(\Omega) \) has a realization as a subspace of \( \mathcal{H}_F \); see eq. (6.13). By going to unbounded sets \( \Omega \) we give up this.

Even if \( \Omega \subset G \) is unbounded, then the operator \( T_F \) in (6.6) is still well-defined; and it may be considered as a possibly unbounded linear operator as follow:

\[ L^2(\Omega) \xrightarrow{T_F} \mathcal{H}_F \quad (6.32) \]

with dense domain \( C_c(\Omega) \) in \( L^2(\Omega) \). (If \( G \) is a Lie group, we may take \( C_c^\infty(\Omega) \) as dense domain for \( T_F \) in (6.32).)

**Lemma 6.3.** Let \( (\Omega,F) \) be as above, but now assume only conditions B1 and B2 for the subset \( \Omega \subset G \).

Then the operator \( T_F \) in (6.32) is a closable operator from \( L^2(\Omega) \) into \( \mathcal{H}_F \); i.e., the closure of the graph of \( T_F \), as a subspace in \( L^2(\Omega) \times \mathcal{H}_F \), is the graph of a (closed) operator \( \overline{T_F} \) from \( L^2(\Omega) \) into \( \mathcal{H}_F \); still with dom \( \overline{T_F} \) dense in \( L^2(\Omega) \).

**Proof.** Using a standard lemma on unbounded operators, see [Rud73, ch.13], we need only show that the following implication holds:

Given \( \{f_n\} \subset C_c(\Omega) \subset L^2(\Omega) \), suppose \( \exists \xi \in \mathcal{H}_F \); and suppose the following two limits holds:

\[ \lim_{n \to \infty} \|f_n\|_{L^2(\Omega)} = 0, \quad \text{and} \]

\[ \lim_{n \to \infty} \|\xi - T_F(f_n)\|_{\mathcal{H}_F} = 0. \quad (6.34) \]

Then, it follows that \( \xi = 0 \) in \( \mathcal{H}_F \).
Now assume \( \{ f_n \} \) and \( \xi \) satisfying (6.33)-(6.34); the by the reproducing property in \( \mathcal{H}_F \), we have

\[
\langle \xi, T_F f_n \rangle_{\mathcal{H}_F} = \int_{\Omega} \overline{\xi(x)} f_n(x) \, dx
\]  

(6.35)

Using (6.34), we get

\[
\lim_{n \to \infty} l.h.s. \, (6.35) = \langle \xi, \xi \rangle_{\mathcal{H}_F} = \| \xi \|^2_{\mathcal{H}_F};
\]

and using (6.33), we get

\[
\lim_{n \to \infty} r.h.s. \, (6.35) = 0.
\]

The domination here is justified by (6.34). Indeed, if (6.34) holds, \( \exists n_0 \) s.t.

\[
\| \xi - T_F (f_n) \|_{\mathcal{H}_F} \leq 1, \, \forall n \geq n_0,
\]

and therefore,

\[
\sup_{n \in \mathbb{N}} \| T_F (f_n) \|_{\mathcal{H}_F} \leq \max_{n \leq n_0} \left( 1 + \| T_F (f_n) \|_{\mathcal{H}_F} \right) < \infty. \tag{6.36}
\]

The desired conclusion follows; we get \( \xi = 0 \) in \( \mathcal{H}_F \). \( \square \)

Remark 6.8. The conclusion in Lemma 6.3 is the assertion that the closure of the graph of \( T_F \) is again the graph of a closed operator, called the closure. Hence the importance of “closability.” Once we have existence of the closure of the operator \( T_F \), as a closed operator, we will denote this closed operator also by the same \( T_F \). This helps reduce the clutter in operator symbols to follow. From now on, \( T_F \) will be understood to be the closed operator obtained in Lemma 6.3.

Remark 6.9. If in Lemma 6.3, for \((F, \Omega)\) the set \( \Omega \) also satisfies B3-B4, then the operator \( T_F \) in (6.32) is in fact bounded; but in general it is not; see the example below with \( G = \mathbb{R} \), and \( \Omega = \mathbb{R}^+ \).

Corollary 6.6. Let \( \Omega, G \) and \( F \) be as in Lemma 6.3, i.e., with \( \Omega \) possibly unbounded, and let \( T_F \) denote the closed operator obtained from (6.32), and the conclusion in the lemma. Then we get the following two conclusions:

(1) \( T_F^* T_F \) is selfadjoint with dense domain in \( L^2(\Omega) \), and

(2) \( T_F T_F^* \) is selfadjoint with dense domain in \( \mathcal{H}_F \).

Proof. This is an application of the fundamental theorem for closed operators; see [Rud73, Theorem 13.13]. \( \square \)

Remark 6.10. The significance of the conclusions (1)-(2) in the corollary is that we may apply the Spectral Theorem to the respective selfadjoint operators in order to get that \( (T_F^* T_F)^{1/2} \) is a well-defined selfadjoint operator in \( L^2(\Omega) \); and that \( (T_F T_F^*)^{1/2} \) well-defined and selfadjoint in \( \mathcal{H}_F \).

Moreover, by the polar decomposition applied to \( T_F \) (see [Rud73, ch. 13]), we conclude that:
\[ \text{spec} \left( T_F^* T_F \right) \setminus \{0\} = \text{spec} \left( T_F T_F^* \right) \setminus \{0\}. \quad (6.37) \]

**Theorem 6.3.** Assume \( F, G, \Omega, \) and \( T_F \), are as above, where \( T_F \) denotes the closed operator \( L^2(\Omega) \overset{T}{\to} \mathcal{H}_F \). We are assuming that \( G \) is a Lie group, \( \Omega \) satisfies B1-B2. Let \( X \) be a vector in the Lie algebra of \( G \), \( X \in \mathfrak{la}(G) \), and define \( D_X^{(F)} \) as a skew-Hermitian operator in \( \mathcal{H}_F \) as follows:

\[
\text{dom} \left( D_X^{(F)} \right) = \{ T_F \phi \mid \phi \in C^\infty_c(\Omega) \}, \quad \text{and} \quad D_X^{(F)}(F\phi) = F_{\hat{X}}\phi
\]

where

\[
(\hat{X} \phi)(g) = \lim_{t \to 0} \frac{1}{t} (\phi(\exp(-tX)g) - \phi(g))
\]

for all \( \phi \in C^\infty_c(\Omega) \), and all \( g \in \Omega \).

Then \( \hat{X} \) defines a skew-Hermitian operator in \( L^2(\Omega) \) with dense domain \( C^\infty_c(\Omega) \).

(It is closable, and we shall denote its closure also by \( \hat{X} \).)

We get

\[ D_X^{(F)} T_F = T_F \hat{X} \]

on the domain of \( \hat{X} \); or equivalently

\[ D_X^{(F)} = T_F \hat{X} T_F^{-1}. \]

**Proof.** By definition, for all \( \phi \in C^\infty_c(\Omega) \), we have

\[
\left( D_X^{(F)} T_F \right) (\phi) = D_X^{(F)} F\phi = F_{\hat{X}}\phi = \left( T_F \hat{X} \right) (\phi).
\]

Since \( \{ F\phi \mid \phi \in C^\infty_c(\Omega) \} \) is a core-domain, (6.40) follows. Then the conclusions in the theorem follow from a direct application of Lemma 6.3, and Corollary 6.6; see also Remark 6.10. \( \square \)

**Corollary 6.7.** For the respective adjoint operators in (6.41), we have

\[
\left( D_X^{(F)} \right)^* = T_F^{-1} \hat{X}^* T_F^*.
\]

**Proof.** The formula (6.42) in the corollary results from applying the adjoint operation to both sides of eq (6.41), and keeping track of the domains of the respective operators in the product on the r.h.s. in eq (6.41). Only after checking domains of the three respective operators, occurring as factors in the product on the r.h.s. in eq (6.41), may we then use the algebraic rules for adjoint of a product of operators. In this instance, we conclude that adjoint of the product on the r.h.s. in eq (6.41) is the product of the adjoint of the factors, but now composed in the reverse order; so product from left to right, becomes product of the adjoints from right to left; hence the result on the r.h.s. in eq (6.42).
Now the fact that the domain issues work out follows from application of Corollary 6.6, Remark 6.10, and Theorem 6.3; see especially eqs (6.37), and (6.38). The rules for adjoint of a product of operators, where some factors are unbounded are subtle, and we refer to [Rud73, chapter 13] and [DS88, Chapter 11-12]. Care must be exercised when the unbounded operators in the product map between different Hilbert spaces. The fact that our operator \( T_F \) is closed as a linear operator from \( L^2(\Omega) \) into \( \mathcal{H}_F \) is crucial in this connection; see Lemma 6.3.

**Corollary 6.8.** Let \( G, \Omega, F, \) and \( T_F \) be as above; then the RKHS \( \mathcal{H}_F \) consists precisely of the continuous functions \( \xi \) on \( \Omega \) such that \( \xi \in \text{dom}(T_F^*T_F)^{-1/4}) \), and then

\[
\|\xi\|_{\mathcal{H}_F} = \| (T_F^*T_F)^{-1/4}\xi\|_{L^2(\Omega)}.
\]

**Proof.** An immediate application of Corollary 6.6; and the polar decomposition, applied to the closed operator \( T_F \) from Lemma 6.3.

**Example 6.1 (Application).** Let \( G = \mathbb{R}, \Omega = \mathbb{R}_+ = (0, \infty) \); so that \( \Omega - \Omega = \mathbb{R} \); let \( F(x) = e^{-|x|}, \forall x \in \mathbb{R} \), and let \( D(F) \) be the skew-Hermitian operator from Corollary 6.7. Then \( D(F) \) has deficiency indices \((1, 0)\) in \( \mathcal{H}_F \).

**Proof.** From Corollary 6.6, we conclude that \( \mathcal{H}_F \) consists of all continuous functions \( \xi \) on \( \mathbb{R}_+ (= \Omega) \) such that \( \xi \) and \( \xi' = \frac{d\xi}{dx} \) are in \( L^2(\mathbb{R}_+) \); and then

\[
\|\xi\|^2_{\mathcal{H}_F} = \frac{1}{2} \left( \int_0^\infty |\xi'(x)|^2 \, dx + \int_0^\infty |\xi''(x)|^2 \, dx \right) + \int_0^1 \xi_n \, d\beta; \tag{6.43}
\]

where \( \xi_n \) denote its inward normal derivative, and \( d\beta \) is the corresponding boundary measure. Indeed, \( d\beta = -\frac{1}{2}\delta_0 \), with \( \delta_0 := \delta(\cdot - 0) = \text{Dirac mass at } x = 0 \). See Sections 7.1.2-7.1.3 for details.

We now apply Corollary 6.7 to the operator \( D_0 = \frac{d}{dx} \) in \( L^2(\mathbb{R}_+) \) with \( \text{dom}(D_0) = C_c^\infty(\mathbb{R}_+) \). It is well known that \( D_0 \) has deficiency indices \((1, 0)\); and the space is spanned by \( \xi_+(x) := e^{-x} \in L^2(\mathbb{R}_+), \) i.e., \( x > 0 \).

Hence, using (6.42), we only need to show that \( \xi_+ \in \mathcal{H}_F \); but this is immediate from (6.43); in fact

\[
\|\xi_+\|^2_{\mathcal{H}_F} = 1.
\]

Setting \( \xi_-(x) := e^x \), the same argument shows that r.h.s. (6.43) = \infty, so the index conclusion (1, 0) follows.

We now return to the case for \( G = \mathbb{R} \), and \( F \) is fixed continuous positive definite function on some finite interval \((-a,a)\), i.e., the case where \( \Omega = (0,a) \).

**Corollary 6.9.** If \( G = \mathbb{R} \) and if \( \Omega = (0,a) \) is a bounded interval, \( a < \infty \), then the operator \( D(F) \) has equal indices for all given \( F : (-a,a) \to \mathbb{C} \) which is p.d. and continuous.
Proof. We showed in Theorem 6.1, and Corollary 6.5 that if \( \Omega = (0, a) \) is bounded, then \( T_F : L^2(0, a) \to \mathcal{H}_F \) is bounded. By Corollary 6.6, we get that \( T_F^{-1} : \mathcal{H}_F \to L^2(0, a) \) is closed. Moreover, as an operator in \( L^2(0, a) \), \( T_F \) is positive and self-adjoint.

Since

\[
D_0 = \frac{d}{dx} |_{C_c^\infty(0,a)}
\]

has indices \((1, 1)\) in \( L^2(0, a) \), it follows from (6.42) applied to (6.44) that \( D(F) \), as a skew-Hermitian operator in \( \mathcal{H}_F \), must have indices \((0, 0)\) or \((1, 1)\).

To finish the proof, use that a skew Hermitian operator with indices \((1, 0)\) must generate a semigroup of isometries; one that is non-unitary. If such an isometry semigroup were generated by the particular skew Hermitian operator \( D(F) \) then this would be inconsistent with Corollary 6.8; see especially the formula for the norm in \( \mathcal{H}_F \). □

To simplify notation, we now assume that the endpoint \( a \) in equation (6.1) is \( a = 1 \).

**Proposition 6.1.** Let \( F \) be p.d. continuos on \( I = (-1, 1) \subset \mathbb{R} \). Assume \( \mu \in \text{Ext}(F) \), and \( \mu \ll d\lambda \), i.e., \( \exists M \in L^1(\mathbb{R}) \text{ s.t.} \)

\[
d\mu(\lambda) = M(\lambda) \, d\lambda, \text{ where } d\lambda = \text{Lebesgue measure on } \mathbb{R}.
\]

Set \( \mathcal{L} = (2\pi \mathbb{Z}) \) (period lattice), and

\[
\hat{\varphi}(\xi) = \int_0^1 e^{-i\xi y} \varphi(y) \, dy, \quad \forall \varphi \in C_c(0, 1);
\]

then the Mercer operator is as follows:

\[
(T_F \varphi)(x) = \sum_{l \in \mathcal{L}} M(l) \hat{\varphi}(l) e^{ilx}.
\]

**Proof.** Let \( x \in (0, 1) \), then

\[
\text{r.h.s. (6.47)} = \sum_{l \in \mathcal{L}} M(l) \left( \int_0^1 e^{-i\xi y} \varphi(y) \, dy \right) e^{ilx}
\]

\[
= (\text{Fubini}) \int_0^1 \varphi(y) \left( \sum_{l \in \mathcal{L}} M(l) e^{i(l(x-y))} \right) dy
\]

\[
= \int_0^1 \varphi(y) F(x-y) \, dy
\]

\[
= (T_F \varphi)(x)
\]

\[
= \text{l.h.s. (6.47)},
\]
where we use that $F = \hat{\mu} \big|_{(-1,1)}$, and (6.45).

**Example 6.2.** Application to Table 5.1: $\mathcal{L} = 2\pi \mathbb{Z}$.

**Proof.** Application of Proposition 6.1. See Table 6.1 below.

| p.d. Function | $(T_F \varphi) (x)$ | $M (\lambda), \lambda \in \mathbb{R}$ |
|---------------|---------------------|-------------------------------------|
| $F_1$         | $\frac{1}{2} \sum_{l \in \mathbb{Z}} e^{-|l|} \hat{\varphi} (l) e^{ilx}$ | $\frac{1}{2} e^{-|l|}$ |
| $F_3$         | $\sum_{l \in \mathbb{Z}} \frac{1}{\pi (1 + l^2)} \hat{\varphi} (l) e^{ilx}$ | $\frac{1}{\pi (1 + l^2)}$ |
| $F_5$         | $\sum_{l \in \mathbb{Z}} \frac{1}{\sqrt{2\pi}} e^{-l^2/2} \hat{\varphi} (l) e^{ilx}$ | $\frac{1}{\sqrt{2\pi}} e^{-l^2/2}$ |

Table 6.1: Application of Proposition 6.1 to Table 5.1.

**Corollary 6.10.** Let $F : (-1,1) \to \mathbb{C}$ be a continuous positive definite function on the interval $(-1,1)$, and assume:

(i) $F (0) = 1$
(ii) $\exists \mu \in \text{Ext}_1 (F)$ s.t. $\mu \ll d\lambda$, i.e., $\exists M \in L^1 (\mathbb{R})$ s.t. $d\mu (\lambda) = M (\lambda) d\lambda$ on $\mathbb{R}$.

Now consider the Mercer operator

$$(T_F \varphi) (x) = \int_0^1 \varphi (y) F (x - y) dy, \quad \varphi \in L^2 (0,1), x \in (0,1). \quad (6.48)$$

Then the following two conditions (bd-1) and (bd-2) are equivalent, where

$$\mathcal{L} = (2\pi \mathbb{Z}) = \hat{\mathbb{T}}, \text{ and} \quad (6.49)$$

$$\begin{align*}
\text{(bd-1)} & \quad b_M := \sup_{\lambda \in [0,1]} \sum_{l \in \mathcal{L}} M (\lambda + l) < \infty, \text{ and} \\
\overset{\dagger}{\implies} & \\
\text{(bd-2)} & \quad T_F \left( L^2 (0,1) \right) \subseteq \mathcal{H}_F.
\end{align*}$$

If (bd-1) $\iff$ (bd-2) holds, then, for the corresponding operator-norm, we then have

$$\|T_F\|_{L^2(0,1) \to \mathcal{H}_F} = \sqrt{b_M} \text{ in (bd-1).} \quad (6.50)$$

**Remark 6.11.** Condition (bd-1) is automatically satisfied in all interesting cases (at least from the point of view of our present note.)
Proof. The key step in the proof of “$\iff$” was the Parseval duality,

$$\int_0^1 |f(x)|^2 \, dx = \sum_{l \in \mathcal{L}} |\hat{f}_l(l)|^2,$$

where (6.51)

$$[0, 1) \simeq \mathbb{T} = \mathbb{R}/\mathbb{Z}, \hat{\mathbb{T}} \simeq \mathcal{L}.$$

Let $F$, $T_F$, and $M$ be as in the statement of the corollary. Then for $\varphi \in C_c(0, 1)$, we compute the $H_F$-norm of

$$T_F(\varphi) = F\varphi$$

with the use of (6.48), and Proposition 6.1.

We return to

$$\hat{\varphi}_l(l) = \int_0^1 e^{-i\lambda y} \varphi(y) \, dy, \quad l \in \mathcal{L};$$

and we now compute $(T_F \varphi)_l(l), \ l \in \mathcal{L}$; starting with $T_F \varphi$ from (6.48). The result is

$$(T_F \varphi)_l(l) = M(l) \hat{\varphi}_l(l), \ \forall l \in \mathcal{L} (= 2\pi \mathbb{Z}.)$$

And further, using chapter 9, we have:

$$\left\| F\varphi \right\|_{H_F}^2 = \left\| T_F \varphi \right\|_{H_F}^2$$

$$= \int_{\mathbb{R}} |\hat{\varphi}(\lambda)|^2 M(\lambda) \, d\lambda$$

$$= \int_{0}^{1} \sum_{l \in \mathcal{L}} |\hat{\varphi}(\lambda + l)|^2 M(\lambda + l) \, d\lambda$$

$$\leq \left( \sum_{l \in \mathcal{L}} |\hat{\varphi}(l)|^2 \right) \sup_{\lambda \in [0, 1]} \sum_{l \in \mathcal{L}} M(\lambda + l)$$

$$= \left( \int_{0}^{1} |\varphi(x)|^2 \, dx \right) \cdot b_M = \left\| \varphi \right\|_{L^2(0, 1)}^2 \cdot b_M.$$

(6.53)

(6.51)

Hence, if $b_M < \infty$, (bd-2) holds, with

$$\left\| T_F \right\|_{L^2(0, 1) \to H_F} \leq \sqrt{b_M}. \quad (6.54)$$

Using standard Fourier duality, one finally sees that “$\leq$” in (6.54) is in fact “$=$”.

Remark 6.12. A necessary condition for boundedness of $T_F : L^2(0, 1) \to H_F$, is $M \in L^\infty(\mathbb{R})$ when the function $M(\cdot)$ is as specified in (ii) of the corollary.

Proof. Let $\varphi \in C_c(0, 1)$, then
\[ \left\| T_F \phi \right\|^2_{S_F} = \left\| F \phi \right\|^2_{S_F} = \int_{\mathbb{R}} |\phi(\lambda)|^2 M(\lambda) d\lambda \]
\[ \leq \|M\|_\infty \int_{\mathbb{R}} |\phi(\lambda)|^2 d\lambda \]
\[ = \|M\|_\infty \int_{\mathbb{R}} |\phi(x)|^2 d\lambda \quad \text{(Parseval)} \]
\[ = \|M\|_\infty \|\phi\|^2_{L^2(0,1)}. \]

□

**Theorem 6.4.** Let \( F \) be as in Proposition 6.1, and \( \mathcal{H}_F \) the corresponding RKHS. Define the Hermitian operator \( D^F(F\phi) \) on \( \text{dom}(D^F(F\phi)) = \{ F\phi \mid \phi \in C^\infty_c(0,1) \} \subset \mathcal{H}_F \) as before. Let \( A \supset D^F(F\phi) \) be a selfadjoint extension of \( D^F(F\phi) \), i.e.,
\[ D^F(F\phi) \subset A \subset (D^F(F\phi))^* , \quad A = A^*. \]

Let \( P_A \) be the projection valued measure (PVM) of \( A \), and
\[ U^A_t = e^{tA} = \int_{\mathbb{R}} e^{it\lambda} P_A(d\lambda) , \quad t \in \mathbb{R} \quad (6.55) \]
be the one-parameter unitary group; and for all \( f \) measurable on \( \mathbb{R} \), set (the Spectral Theorem applied to \( A \))
\[ f(A) = \int_{\mathbb{R}} f(\lambda) P_A(d\lambda) ; \quad (6.56) \]
them we get the following
\[ (T_F \phi)(x) = (M\tilde{\phi})(A)U^A_x = U^A_x(M\tilde{\phi})(A) \quad (6.57) \]
for the Mercer operator \( (T_F \phi)(x) = \int_0^1 F(x-y) \phi(y), \, \phi \in L^2(0,1). \)

**Proof.** Using (6.56), we get
\[ \tilde{\phi}_t(A) U^A_x = \int_{\mathbb{R}} \int_0^1 \phi(y) e^{-i\lambda y} P_A(d\lambda) U^A(x) \]
\[ = (\text{Fubini}) \int_0^1 \phi(y) \left( \int_{\mathbb{R}} e^{i\lambda(x-y)} P_A(d\lambda) \right) dy \]
\[ = (6.55) \int_0^1 U^A(x-y) \phi(y) dy \]
\[ = U^A_x U^A(\phi) , \quad \text{where} \quad (6.58) \]
6.2 Shannon Sampling, and Bessel Frames

The notion “sampling” refers to the study of certain function spaces, consisting of functions that allow effective “reconstruction” from sampled values, usually from discrete subsets; and where a reconstruction formula is available. The best known result of this nature was discovered by Shannon (see e.g., [DM76, DM70]), and Shannon’s variant refers to spaces of band-limited functions. “Band” in turn refers to a bounded interval in the Fourier-dual domain. For time functions, the Fourier dual is a frequency-variable. Shannon showed that band-limited functions allow perfect reconstruction from samples in an arithmetic progression of time-points. After rescaling, the sample points may be chosen to be the integers \( \mathbb{Z} \); referring to the case when the time-function is defined on \( \mathbb{R} \). Eq (6.69) below illustrates this point. Our purpose here is to discuss sampling as a part of the central theme of our positive definite (p.d.) extension analysis.

However, there is a rich literature dealing with sampling in a host of other areas of both pure and applied mathematics.

**Theorem 6.5.** Let \( F : (-1, 1) \to \mathbb{C} \) be a continuous p.d. function, and

\[
T_F : L^2(0, 1) \to \mathcal{H}_F \subset L^2(0, 1) \tag{6.62}
\]

be the corresponding Mercer operator. Let \( \mu \in \text{Ext}(F) \); then the range of \( T_F \), \( \text{ran}(T_F) \), as a subspace of \( \mathcal{H}_F \), admits the following representation:

\[
\text{ran}(T_F) = \left\{ \sum_{n \in \mathbb{Z}} c_n f_n \mid (c_n) \in l^2(\mathbb{Z}) \text{, } f_n \in \mathcal{H}_F \right\}, \tag{6.63}
\]

where \( f_n \) in the r.h.s. of (6.63) is as follows:
\[ f_n(x) = \int_{\mathbb{R}} e^{i2\pi \lambda x} \frac{\sin \pi (\lambda - n)}{\pi (\lambda - n)} e^{-i\pi (\lambda - n)} d\mu(\lambda), \quad x \in [0, 1]. \] (6.64)

Setting
\[ \text{Sha}(\xi) = e^{i\xi \sin \xi}, \] (6.65)
then
\[ f_n(x) = \int_{\mathbb{R}} \text{Sha} (\pi (\lambda - n)) e^{i2\pi \lambda x} d\mu(x). \] (6.66)

**Proof.** We first show \( \subseteq \) in (6.63). Suppose \( \varphi \in L^2(0, 1) \), then it extends to a 1-periodic function on \( \mathbb{R} \), so that
\[ \varphi(x) = \sum_{n \in \mathbb{Z}} \hat{\varphi}(n) e^{i2\pi nx}, \quad \forall x \in (0, 1); \] where
\[ \hat{\varphi}(\lambda) = \int_0^1 e^{-i2\pi \lambda x} \varphi(x) dx, \quad \lambda \in \mathbb{R}. \] (6.67)

Applying Fourier transform to (6.67) yields
\[ \hat{\varphi}(\lambda) = \sum_{n \in \mathbb{Z}} \hat{\varphi}(n) \frac{\sin \pi (\lambda - n)}{\pi (\lambda - n)} e^{-i\pi (\lambda - n)}, \quad \lambda \in \mathbb{R}, \] (6.69)
and by Parseval’s identity, we have
\[ \int_0^1 |\varphi(x)|^2 dx = \sum_{n \in \mathbb{Z}} |\hat{\varphi}(n)|^2. \]

Note the r.h.s. in (6.69) uses the Shannon integral kernel (see, e.g., [KT09].)
But, for \( \mu \in \text{Ext}(F) \), we also have
\[ (T_F \varphi)(x) = \int_{\mathbb{R}} e^{i2\pi \lambda x} \hat{\varphi}(\lambda) d\mu(\lambda), \] (6.70)
which holds for all \( \varphi \in L^2(0, 1), x \in [0, 1] \). Now substituting (6.69) into the r.h.s. of (6.70), we get
\[ (T_F \varphi)(x) = \sum_{n \in \mathbb{Z}} \hat{\varphi}(n) \int_{\mathbb{R}} e^{i2\pi \lambda x} \frac{\sin \pi (\lambda - n)}{\pi (\lambda - n)} e^{-i\pi (\lambda - n)} d\mu(\lambda) \]
\[ = \sum_{n \in \mathbb{Z}} \hat{\varphi}(n) \int_{\mathbb{R}} e^{i2\pi \lambda x} \text{Sha} (\pi (\lambda - n)) d\mu(\lambda), \] (6.71)
where “Sha” in (6.71) refers to the Shannon kernel from (6.65). Thus, the desired conclusion (6.63) is established. (The interchange of integration and summation is justified by (6.70) and Fubini.)
It remains to prove \( \supseteq \) in (6.63). Let \((c_n) \in l^2(\mathbb{Z})\) be given. By Parseval's identity, it follows that
\[
\varphi(x) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x} \in L^2(0, 1),
\]
and that (6.70) holds. Now the same argument (as above) with Fubini shows that
\[
(T_F \varphi)(x) = \sum_{n \in \mathbb{Z}} c_n f_n(x), \quad x \in [0, 1];
\]
with \(f_n\) given by (6.64).

\[\Box\]

Remark 6.13. While the system \(\{f_n\}_{n \in \mathbb{Z}}\) in (6.63) is explicit, it has a drawback compared to the eigenfunctions for the Mercer operator, in that such a system is not orthogonal.

Example 6.3. Let \(F(x) = e^{-|x|}, |x| < 1, i.e., F = F_3\) in Table 5.1. Then the generating function system \(\{f_n\}_{n \in \mathbb{Z}}\) in \(H_F\) (from Shannon sampling) is as follows:
\[
\Re \{f_n\}(x) = \frac{e^{-1} + e^{-3} - 2 \cos(2\pi nx)}{1 + (2\pi n)^2}, \quad \text{and}
\]
\[
\Im \{f_n\}(x) = \frac{(e^{-1} - e^{-3}) 2\pi n - 2 \sin(2\pi nx)}{1 + (2\pi n)^2}, \quad \forall n \in \mathbb{Z}, x \in [0, 1].
\]
The boundary values of \(f_n\) are as in Table 6.2.

| \(x\) | \(\Re \{f_n\}(x)\) | \(\Im \{f_n\}(x)\) |
|------|----------------|------------------|
| 0    | \(\frac{e^{-1} + 1 - 2}{1 + (2\pi n)^2}\) | \(\frac{-2\pi n (1 - e^{-1})}{1 + (2\pi n)^2}\) |
| 1    | \(\frac{e^{-1} + 1 - 2}{1 + (2\pi n)^2}\) | \(\frac{2\pi n (1 - e^{-1})}{1 + (2\pi n)^2}\) |

Table 6.2: Boundary values of the Shannon functions, s.t. \(\Re \{f_n\}(1) = \Re \{f_n\}(0)\), and \(\Im \{f_n\}(1) = -\Im \{f_n\}(0)\).

Corollary 6.11. Let \(F : (-1, 1) \to \mathbb{C}\) be continuous and positive definite, and assume \(F(0) = 1\). Let \(\mu \in M_+(\mathbb{R})\). Then the following two conditions are equivalent:

(1) \(\mu \in \text{Ext}(F)\)
(2) For all \( x \in (-1, 1) \),

\[
\sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} e^{i2\pi \lambda x} \text{Sha}(\pi (\lambda - n)) \, d\mu(\lambda) = F(x). \tag{6.72}
\]

**Proof.** The implication (2) \( \Rightarrow \) (1) follows since the l.h.s. of (6.72) is a continuous p.d. function defined on all of \( \mathbb{R} \), and so it is an extension as required. Indeed, this is (6.71), with \( \hat{\varphi}(n) = 1 \), for all \( n \in \mathbb{Z} \).

Conversely, assume (1) holds. Let \( \varphi_{\varepsilon} \xrightarrow{\varepsilon \to 0^-} \delta_0 \) (Dirac mass at 0) in (6.71), and then (2) follows. \( \square \)

**Definition 6.3.** A system of functions \( \{f_n\}_{n \in \mathbb{Z}} \subset \mathcal{F} \) is said to be a *Bessel frame* [CM13] if there is a finite constant \( A \) such that

\[
\sum_{n \in \mathbb{Z}} \left| \langle f_n, \xi \rangle_{\mathcal{F}} \right|^2 \leq A \left\| \xi \right\|^2_{\mathcal{F}}, \quad \forall \xi \in \mathcal{F}. \tag{6.73}
\]

**Theorem 6.6.** Let \( F : (-1, 1) \to \mathbb{C} \) be continuous and p.d., and let \( \{f_n\}_{n \in \mathbb{Z}} \subset \mathcal{F} \) be the system (6.64) obtained by Shannon sampling in Fourier-domain; then \( \{f_n\} \) is a Bessel frame, where we may take \( A = \lambda_1 \) as frame bound in (6.73). (\( \lambda_1 \) = the largest eigenvalue of the Mercer operator \( T_F \).)

**Proof.** Let \( \xi \in \mathcal{F} \), then

\[
\sum_{n \in \mathbb{Z}} \left| \langle f_n, \xi \rangle_{\mathcal{F}} \right|^2 = \sum_{n \in \mathbb{Z}} \left| \langle T e_n, \xi \rangle_{\mathcal{F}} \right|^2 = \sum_{n \in \mathbb{Z}} \left| \langle e_n, \xi \rangle_{L^2(0,1)} \right|^2 = \lambda_1 \left\| \xi \right\|^2_{\mathcal{F}}
\]

which is the desired conclusion.

At the start of the estimate above we used the Fourier basis \( e_n(x) = e^{i2\pi nx}, n \in \mathbb{Z} \), an ONB in \( L^2(0,1) \); and the fact that \( f_n = T_F(e_n), n \in \mathbb{Z} \); see the details in the proof of Theorem 6.5. \( \square \)

**Corollary 6.12.** For every \( f \in L^2(\Omega) \cap \mathcal{F} \), and every \( x \in \overline{\Omega} \) (including boundary points), we have the following estimate:

\[
|f(x)| \leq \|f\|_{\mathcal{F}} \tag{6.74}
\]

(Note that this is independent of \( x \) and of \( f \).)

**Proof.** The estimate in (6.74) follows from Lemma 8.1; and (6.28) in part (2) of Theorem 6.2. \( \square \)
Corollary 6.13. Let \( \{ \xi_n \}, \{ \lambda_n \}, T_F, \mathcal{H}_F \) and \( L^2(\Omega) \) be as above; and assume \( \lambda_1 \geq \lambda_2 \geq \cdots \); then we have the following details for operator norms

\[
\| T_F^{-1} \|_{\text{span}\{\xi_k: k=1, \ldots, N\} \to \mathcal{H}_F} = \| T_F^{-1} \|_{L^2(\Omega) \to L^2(\Omega)} = \frac{1}{\lambda_N} \to \infty, \text{ as } N \to \infty.
\]

6.3 Application: The Case of \( F_2 \) and Rank-1 Perturbations

In Theorem 6.7 below, we revisit the example \( F_2 \) (in \( |x| < \frac{1}{2} \)) from Table 5.1. This example has a number of intriguing properties that allow us to compute the eigenvalues and the eigenvectors for the Mercer operator from Lemma 6.1.

Theorem 6.7. Set \( E(x, y) = x \wedge y = \min(x, y) \), \( x, y \in (0, \frac{1}{2}) \), and

\[
(T_E \varphi)(x) = \int_0^x \varphi(y) x \wedge y dy
\]

then the spectral resolution of \( T_E \) in \( L^2(0, \frac{1}{2}) \) is as follows:

\[
E(x, y) = \sum_{n=1}^{\infty} \frac{4}{(\pi (2n - 1))^2} \sin((2n - 1) \pi x) \sin((2n - 1) \pi y)
\]

for all \( \forall (x, y) \in (0, \frac{1}{2}) \times (0, \frac{1}{2}) \).

Setting

\[
u(x) = (T_E \varphi)(x), \text{ for } \varphi \in C_c^\infty(0, \frac{1}{2}) \text{, we get}
\]

\[
u'(x) = \int_x^1 \varphi(y) dy, \text{ and}
\]

\[
u''(x) = -\varphi(x).
\]

moreover, \( u \) satisfies the boundary condition

\[
\begin{cases}
u(0) = 0 \\
u'\left(\frac{1}{2}\right) = 0
\end{cases}
\]

Note that \( E(x, y) \) is a p.d. kernel, but not a positive definite function in the sense of Definition 1.1.

In particular, \( E \) is a function of two variables, as opposed to one. The purpose of Theorem 6.8, and Lemmas 6.4, and 6.5 is to show that the Mercer operator \( T_F \) defined from \( F = F_2 \) (see Table 5.1) is a rank-1 perturbation of the related operator
$T_E$ defined from $E(x,y)$. The latter is of significance in at least two ways: $T_E$ has an explicit spectral representation, and $E(x,y)$ is the covariance kernel for Brownian motion; see also Figure 7.2. By contrast, we show in Lemma 7.2 that the Mercer operator $T_F$ is associated with pinned Brownian motion.

The connection between the two operators $T_F$ and $T_E$ is reflecting a more general feature of boundary conditions for domains $\Omega$ in Lie groups; – a topic we consider in Chapters 5, and 10.

Rank-one perturbations play a role in spectral theory in different problems; see e.g., [Yos12, DJ10, Ion01, DRSS94, TW86].

Proof. (Theorem 6.7) We verify directly that (6.79)-(6.80) hold.

Consider the Hilbert space $L^2(0,1)$. The operator $\triangle_E := T_E^{-1}$ is a selfadjoint extension of $-(\frac{d}{dx})^2|_{C^\infty_c(0,1)}$ in $L^2(0,1)$; and under the ONB

$$f_n(x) = 2\sin((2n-1)\pi x), \ n \in \mathbb{N},$$

and the boundary condition (6.81), $\triangle_E$ is diagonalized as

$$\triangle_E f_n = ((2n-1)\pi)^2 f_n, \ n \in \mathbb{N}. \tag{6.82}$$

We conclude that

$$\triangle_E = \sum_{n=1}^{\infty} ((2n-1)\pi)^2 |f_n\rangle \langle f_n| \tag{6.83}$$

$$T_E = \triangle_E^{-1} = \sum_{n=1}^{\infty} \frac{1}{((2n-1)\pi)^2} |f_n\rangle \langle f_n| \tag{6.84}$$

where $P_n := |f_n\rangle \langle f_n| = \text{Dirac’s rank-1 projection in } L^2(0,1)$, and

$$(P_n \varphi)(x) = \langle f_n, \varphi \rangle f_n(x)$$

$$= \left( \int_0^1 f_n(y) \varphi(y) \, dy \right) f_n(x)$$

$$= 4 \sin((2n-1)\pi x) \int_0^1 \sin((2n-1)\pi y) \varphi(y) \, dy. \tag{6.85}$$

Combing (6.84) and (6.85), we get

$$(T_E \varphi)(x) = \sum_{n=1}^{\infty} \frac{4 \sin((2n-1)\pi x)}{((2n-1)\pi)^2} \int_0^1 \sin((2n-1)\pi y) \varphi(y) \, dy. \tag{6.86}$$

Note the normalization considered in (6.77) and (6.86) is consistent with the condition:

$$\sum_{n=1}^{\infty} \lambda_n = \text{trace } (T_E)$$
in Lemma 6.1 for the Mercer eigenvalues \( (\lambda_n)_{n\in\mathbb{N}} \). Indeed,

\[
\text{trace } (T_E) = \int_0^{\frac{1}{2}} x \wedge x \, dx = \sum_{n=1}^{\infty} \frac{1}{((2n-1)\pi)^2} = \frac{1}{8}.
\]

\[\square\]

**Theorem 6.8.** Set \( F(x-y) = 1 - |x-y|, x, y \in (0, \frac{1}{2}) \); \( K^{(E)}(x,y) = x \wedge y = \min(x,y) \), and let

\[
(T_E \varphi)(x) = \int_0^{\frac{1}{2}} \varphi(y) K^{(E)}(x,y) \, dy;
\]

then

\[
K^{(E)}(x,y) = \sum_{n=1}^{\infty} \frac{4 \sin((2n-1)\pi x) \sin((2n-1)\pi y)}{(\pi(2n-1))^2} \quad (6.88)
\]

and

\[
F(x-y) = 1 - x - y + 2 \sum_{n=1}^{\infty} \frac{4 \sin((2n-1)\pi x) \sin((2n-1)\pi y)}{(\pi(2n-1))^2}. \quad (6.89)
\]

That is,

\[
F(x-y) = 1 - x - y + 2K^{(E)}(x,y). \quad (6.90)
\]

**Remark 6.14.** Note the trace normalization \( \text{trace } (T_E) = \frac{1}{2} \) holds. Indeed, from (6.89), we get

\[
\text{trace } (T_E) = \int_0^{\frac{1}{2}} \left( 1 - 2x + 2 \sum_{n=1}^{\infty} \frac{4 \sin^2((2n-1)\pi x)}{(\pi(2n-1))^2} \right) \, dx
\]

\[
= \frac{1}{2} - \frac{1}{4} + 2 \sum_{n=1}^{\infty} \frac{1}{(\pi(2n-1))^2} = \frac{1}{2} - \frac{1}{4} + 2 \cdot \frac{1}{8} = \frac{1}{2}.
\]

where \( \frac{1}{2} \) on the r.h.s. is the right endpoint of the interval \( [0, \frac{1}{2}] \).

**Proof.** The theorem follows from lemma 6.4 and lemma 6.5. \(\square\)

**Lemma 6.4.** Consider the two integral kernels:

\[
F(x-y) = 1 - |x-y|, x, y \in \Omega; \quad (6.91)
\]

\[
K^{(E)}(x,y) = x \wedge y = \min(x,y), x, y \in \Omega. \quad (6.92)
\]

We take \( \Omega = (0, \frac{1}{2}) \). Then

\[
F(x-y) = 2K^{(E)}(x,y) + 1 - x - y; \quad \text{and} \quad (6.93)
\]
\[
\left( F_x - 2K^{(E)}_x \right)''(y) = -2\delta (0 - y). \tag{6.94}
\]

**Proof.** A calculation yields:
\[x \wedge y = \frac{x + y - 1 + F(x - y)}{2},\]
and therefore, solving for \(F(x - y)\), we get (6.93).

To prove (6.94), we calculate the respective Schwartz derivatives (in the sense of distributions). Let \(H_x\) = the Heaviside function at \(x\), with \(x\) fixed. Then
\[
\left( K^{(E)}_x \right)' = H_0 - H_x, \quad \text{and} \quad (6.95)
\]
\[
\left( K^{(E)}_x \right)'' = \delta_0 - \delta_x \tag{6.96}
\]
combining (6.96) with \((F_x)'' = -2\delta_x\), we get
\[
\left( F_x - 2K^{(E)}_x \right)'' = -2\delta_x - 2(\delta_0 - \delta_x) = -2\delta_0
\]
which is the desired conclusion (6.94). \(\square\)

**Remark 6.15.** From (6.93), we get the following formula for three integral operators
\[
T_F = 2T_E + L, \quad \text{where} \tag{6.98}
\]
\[
(L \phi)(x) = \int_0^1 \phi(y)(1 - x - y) \, dy. \tag{6.99}
\]
Now in Lemma 6.4, we diagonalize \(T_E\), but the two Hermitian operators on the r.h.s. in (6.98), do not commute. But the perturbation \(L\) in (6.98) is still relatively harmless; it is a rank-1 operator with just one eigenfunction: \(\phi(x) = a + bx\), where \(a\) and \(b\) are determined from \((L \phi)(x) = \lambda \phi(x)\); and
\[
(L \phi)(x) = (1 - x) \frac{a}{2} - \frac{1}{8} \left( a + \frac{b}{3} \right)
\]
\[
= \left( \frac{3}{8}a - \frac{b}{24} \right) - \left( \frac{a}{2} \right)x = \lambda (a + bx)
\]
thus the system of equations
\[
\begin{cases}
\left( \frac{3}{8} - \lambda \right)a - \frac{1}{24}b = 0 \\
-\frac{1}{2}a - \lambda b = 0.
\end{cases}
\]
It follows that
6.3 Application: The Case of $F_2$ and Rank-1 Perturbations

\[ \lambda = \frac{1}{48} \left( 9 + \sqrt{129} \right) \]
\[ b = -\frac{1}{2} \left( \sqrt{129} - 9 \right) a. \]

**Remark 6.16 (A dichotomy for integral kernel operators).** Note the following dichotomy for the two integral kernel-operators:

(5) one with the kernel $L(x, y) = 1 - x - y$, a rank-one operator; and the other $T_F$ corresponding to $F = F_2$, i.e., with kernel $F(x-y) = 1 - |x-y|$.

And by contrast:

(6) $T_F$ is an infinite dimensional integral-kernel operator. Denoting both the kernel $L$, and the rank-one operator, by the same symbol, we then establish the following link between the two integral operators: The two operators $T_F$ and $L$ satisfy the identity $T_F = L + 2T_E$, where $T_E$ is the integral kernel-operator defined from the covariance function of Brownian motion. For more applications of rank-one perturbations, see e.g., [Yos12, DJ10, Ion01, DRSS94, TW86].

**Lemma 6.5.** Set

\[ (T_E \varphi)(x) = \int_0^{\frac{1}{2}} K^{(E)}(x, y) \varphi(y) \, dy, \quad \varphi \in L^2 \left( 0, \frac{1}{2} \right), x \in (0, \frac{1}{2}) \] (6.100)

Then $s_n(x) := \sin \left( (2n-1) \pi x \right)$ satisfies

\[ T_E s_n = \frac{1}{(2n-1) \pi^2} s_n; \] (6.101)

and we have

\[ K^{(E)}(x, y) = \sum_{n \in \mathbb{N}} \frac{4}{(2n-1) \pi^2} \sin \left( (2n-1) \pi x \right) \sin \left( (2n-1) \pi y \right) \] (6.102)

**Proof.** Let $\Omega = \left( 0, \frac{1}{2} \right)$. Setting

\[ s_n(x) := \sin \left( (2n-1) \pi x \right); \quad x \in \Omega, n \in \mathbb{N}. \] (6.103)

Using (6.96), we get

\[ (T_E s_n)(x) = \frac{1}{(2n-1) \pi^2} s_n(x), \quad x \in \Omega, n \in \mathbb{N}, \]

where $T_E$ is the integral operator with kernel $K^{(E)}$, and $s_n$ is as in (6.103). Since

\[ \int_0^{\frac{1}{2}} \sin^2 \left( (2n-1) \pi x \right) \, dx = \frac{1}{4} \]
the desired formula (6.102) holds. □

**Corollary 6.14.** Let $T_F$ and $T_E$ be the integral operators in $L^2(0, \frac{1}{2})$ defined in the lemmas; i.e., $T_F$ with kernel $F(x - y)$; and $T_E$ with kernel $x \wedge y$. Then the selfadjoint operator $(T_F - 2T_E)^{-1}$ is well-defined, and it is the Friedrichs extension of

$$- \frac{1}{2} \left( \frac{d}{dx} \right)^{-1} \bigg|_{C^\infty_c(0, \frac{1}{2})}$$

as a Hermitian and semibounded operator in $L^2(0, \frac{1}{2})$.

**Proof.** Formula (6.94) in the lemma translates into

$$(T_F \varphi - 2T_E \varphi)'' = -2\varphi \quad (6.104)$$

for all $\varphi \in C^\infty_c(0, \frac{1}{2})$. Hence $(T_F - 2T_E)^{-1}$ is well-defined as an unbounded selfadjoint operator in $L^2\left(0, \frac{1}{2}\right)$; and

$$(T_F - 2T_E)^{-1} \varphi = -\frac{1}{2} \varphi'', \quad \forall \varphi \in C^\infty_c(0, \frac{1}{2}).$$

Since the Friedrichs extension is given by Dirichlet boundary condition in $L^2(0, \frac{1}{2})$, the result follows. □

### 6.4 Positive Definite Functions, Green’s Functions, and Boundary

In this section, we consider a correspondence and interplay between a class of boundary value problems on the one hand, and spectral theoretic properties of extension operators on the other.

Fix a bounded domain $\Omega \subset \mathbb{R}^n$, open and connected. Let $F: \Omega \rightarrow \mathbb{C}$ be a continuous positive definite (p.d.) function. We consider a special case when $F$ occurs as the Green’s function of certain linear operator.

**Lemma 6.6.** Let $\mathcal{D}$ be a Hilbert space, a Fréchet space or an LF-space (see [Tré06]), such that $\mathcal{D} \hookrightarrow L^2(\Omega)$; and such that the inclusion mapping $j$ is continuous relative to the respective topologies on $\mathcal{D}$, and on $L^2(\Omega)$. Let $\mathcal{D}^* :=$ the dual of $\mathcal{D}$ when $\mathcal{D}$ is given its Fréchet (LF, or Hilbert) topology; then there is a natural “inclusion” mapping $j^*$ from $L^2(\Omega)$ to $\mathcal{D}^*$, i.e., we get

$$\mathcal{D} \hookrightarrow L^2(\Omega) \hookrightarrow \mathcal{D}^*. \quad (6.105)$$

**Proof.** It is immediate from the assumptions, and the fact that $L^2(\Omega)$ is its own dual. See also [Tré06]. □
Remark 6.17. In the following we shall use Lemma 6.6 in two cases:

1. Let $A$ be a selfadjoint operator (unbounded in the non-trivial cases) acting in $L^2(\Omega)$; and with dense domain. For $D = D_A$, we may choose the domain of $A$ with its graph topology.

2. Let $D$ be a space of Schwartz test functions, e.g., $C^\infty_c(\Omega)$, given its natural LF-topology, see [Tré06]; then the inclusion $C^\infty_c(\Omega) \hookrightarrow L^2(\Omega)$ satisfies the condition in Lemma 6.6.

Corollary 6.15. Let $D \subset L^2(\Omega)$ be a subspace satisfying the conditions in Lemma 6.6; and consider the triple of spaces (6.105); then the inner product in $L^2(\Omega)$, here denoted $\langle \cdot, \cdot \rangle_2$, extends by closure to a sesquilinear function $\langle \cdot, \cdot \rangle : L^2(\Omega) \times D^* \to \mathbb{C}$.

Proof. This is a standard argument based on dual topologies; see [Tré06]. □

Example 6.4 (Application). If $D = C^\infty_c(\Omega)$ in (6.105), then $D^* = \text{the space of all Schwartz-distributions on } \Omega$, including the Dirac masses. Referring to (6.107), we shall write $\langle \delta_x, f \rangle_2$ to mean $f(x)$, when $f \in C(\overline{\Omega}) \cap L^2(\Omega)$.

Adopting the constructions from Lemma 6.6 and Corollary 6.15, we now turn to calculus of positive definite functions:

Definition 6.4. If $F : \Omega \to \mathbb{C}$ is a function, or a distribution, then we say that $F$ is positive definite iff

$$\langle F, \overline{\varphi} \otimes \varphi \rangle \geq 0$$

for all $\varphi \in C^\infty_c(\Omega)$. The meaning of (6.108) is the distribution $K_F := F(x - y)$ acting on $(\overline{\varphi} \otimes \varphi)(x,y) := \overline{\varphi(x)} \varphi(y)$, $x,y \in \Omega$.

Let

$$\Delta := \sum_{j=1}^n \left( \frac{\partial}{\partial x_j} \right)^2$$

and consider an open domain $\Omega \subset \mathbb{R}^n$.

In $\mathcal{H}_F$, set

$$D^{(F)}_j (F\varphi) := F_{\partial_j \varphi} , \varphi \in C^\infty_c(\Omega) , \ j = 1, \ldots, n.$$
where $\triangle F$ on the r.h.s. in (6.111) is in the sense of distributions. Then $M$ is also positive definite and

$$\langle M \varphi, M \psi \rangle_{\mathcal{H}^M} = \sum_{j=1}^{n} \langle D_j^{(F)} F \varphi, D_j^{(F)} F \psi \rangle_{\mathcal{H}^F}$$

(6.112)

for all $\varphi, \psi \in C_c^\infty(\Omega)$. In particular, setting $\varphi = \psi$ in (6.112), we have

$$\|M \varphi\|^2_{\mathcal{H}^M} = \sum_{j=1}^{k} \|D_j^{(F)} F \varphi\|^2_{\mathcal{H}^F}.$$ 

(6.113)

**Proof.** We must show that $M$ satisfies (6.108), i.e., that

$$\langle M, \varphi \otimes \varphi \rangle \geq 0;$$

(6.114)

and moreover that (6.112), or equivalently (6.111), holds.

For l.h.s of (6.114), we have

$$\langle M, \varphi \otimes \varphi \rangle = \langle (-\triangle F, \varphi \otimes \varphi \rangle = -\sum_{j=1}^{n} \langle \left( \frac{\partial}{\partial x_j} \right)^2 F, \varphi \otimes \varphi \rangle$$

$$= \sum_{j=1}^{n} \langle F, \frac{\partial \varphi}{\partial x_j} \otimes \frac{\partial \varphi}{\partial x_j} \rangle = \sum_{j=1}^{n} \|D_j^{(F)} F \varphi\|^2_{\mathcal{H}^F} \geq 0,$$

using the action of $\frac{\partial}{\partial x_j}$ in the sense of distributions. This is the desired conclusion. □

**Example 6.5.** For $n = 1$, consider the functions $F_2$ and $F_3$ from Table 5.1.

1. Let $F = F_2$, $\Omega = (-\frac{1}{2}, \frac{1}{2})$, then

$$M = -F'' = 2\delta$$

(6.115)

where $\delta$ is the Dirac mass at $x = 0$, i.e., $\delta = \delta(x-0)$.

2. Let $F = F_3$, $\Omega = (-1, 1)$, then

$$M = -F'' = 2\delta - F$$

(6.116)

**Proof.** The proof of the assertions in the two examples follows directly from Sections 7.1.2 and 7.1.3. □

Now we return to the p.d. function $F : \Omega \to \mathbb{C}$. Suppose $A : L^2(\Omega) \to L^2(\Omega)$ is an unbounded positive linear operator, i.e., $A \geq c > 0$, for some constant $c$. Further assume that $A^{-1}$ has the integral kernel (Green’s function) $F$, i.e.,

$$A^{-1} f \big( x \big) = \int_{\Omega} F(x - y) f(y) \, dy, \; \forall f \in L^2(\Omega).$$

(6.117)
For all \( x \in \Omega \), define
\[
F_x (\cdot) := F(x - \cdot). \tag{6.118}
\]

Here \( F_x \) is the fundamental solution to the following equation
\[
Au = f
\]
where \( u \in \text{dom}(A) \), and \( f \in L^2(\Omega) \). Hence, in the sense of distribution, we have
\[
AF_x (\cdot) = \delta_x \upharpoonright \mathcal{A} \left( \int_{\Omega} F(x,y) f(y) \, dy \right) = \int (AF_x(y)) f(y) \, dy = \int \delta_x(y) f(y) \, dy = f(x).
\]

Note that \( A^{-1} \geq 0 \) iff \( F \) is a p.d. kernel.

Let \( \mathcal{H}_A = \) the completion of \( C_c^\infty (\Omega) \) in the bilinear form
\[
\langle f, g \rangle_A := \langle Af, g \rangle_{L^2}; \tag{6.119}
\]
where the r.h.s. extends the inner product in \( L^2(\Omega) \) as in \( (6.107) \).

**Lemma 6.8.** \( \mathcal{H}_A \) is a RKHS and the reproducing kernel is \( F_x \).

**Proof.** Since \( A \geq c > 0 \), in the usual ordering of Hermitian operator, \( (6.119) \) is a well-defined inner product, so \( \mathcal{H}_A \) is a Hilbert space. For the reproducing property, we check that
\[
\langle F_x, g \rangle_A = \langle AF_x, g \rangle_{L^2} = \langle \delta_x, g \rangle_{L^2} = g(x).
\]

**Lemma 6.9.** Let \( \mathcal{H}_F \) be the RKHS corresponding to \( F \), i.e., the completion of \( \text{span} \{ F_x : x \in \Omega \} \) in the inner product
\[
\langle F_x, F_y \rangle_F := F_x(y) = F(x - y) \tag{6.120}
\]
extending linearly. Then we have the isometric embedding \( \mathcal{H}_F \hookrightarrow \mathcal{H}_A \), via the map,
\[
F_x \mapsto F_x. \tag{6.121}
\]

**Proof.** We check directly that
\[
\| F_x \|_F^2 = \langle F_x, F_x \rangle_F = F_x(x) = F(0)
\]
\[
\| F_x \|_A^2 = \langle F_x, F_x \rangle_A = \langle AF_x, F_x \rangle_{L^2} = \langle \delta_x, F_x \rangle_{L^2} = F_x(x) = F(0).
\]

\( \square \)
Remark 6.18. Now consider $\mathbb{R}$, and let $\Omega = (0,a)$. Recall the Mercer operator

$$
T_F : L^2(\Omega) \rightarrow L^2(\Omega),
$$

by

$$
(T_F g)(x) := \int_0^a F(x)(y)g(y)
dy (6.122)
= \langle F_x, g \rangle_2, \ \forall g \in L^2(0,a).
$$

By Lemma 6.1, $T_F$ can be diagonalized in $L^2(0,a)$ by

$$
T_F \xi_n = \lambda_n \xi_n, \ \lambda_n > 0
$$

where $\{\xi_n\}_{n \in \mathbb{N}}$ is an ONB in $L^2(0,a)$; further $\xi_n \subset \mathcal{H}_{F}$, for all $n \in \mathbb{N}$.

From (6.122), we then have

$$
\langle F_x, \xi_n \rangle_2 = \lambda_n \xi_n(x). (6.123)
$$

Applying $A$ on both sides of (6.123) yields

$$
\text{l.h.s. } (6.123) = \langle AF_x, \xi_n \rangle_2 = \langle \delta_x, \xi_n \rangle_2 = \xi_n(x)
$$

$$
\text{r.h.s. } (6.123) = \lambda_n \langle A \xi_n \rangle_2(x)
$$

therefore, $A \xi_n = \frac{1}{\lambda_n} \xi_n$, i.e.,

$$
A = T_F^{-1}. (6.124)
$$

Consequently,

$$
\langle \xi_n, \xi_m \rangle_A = \langle A \xi_n, \xi_m \rangle_2 = \frac{1}{\lambda_n} \langle \xi_n, \xi_m \rangle_2 = \frac{1}{\lambda_n} \delta_{n,m}.
$$

And we conclude that $\{\sqrt{\lambda_n} \xi_n\}_{n \in \mathbb{N}}$ is an ONB in $\mathcal{H}_A = \mathcal{H}_{T_F^{-1}}$.

See Section 5.4, where $F = \text{Pólya extension of } F_3$, and a specific construction of $\mathcal{H}_{T_F^{-1}}$.

6.4.1 Connection to the Energy Form Hilbert Spaces

Now consider $A = 1 - \triangle$ defined on $C^\infty_c(\Omega)$. There is a connection between the RKHS $\mathcal{H}_A$ and the energy space as follows:

For $f, g \in \mathcal{H}_A$, we have (restricting to real-valued functions),
\[ \langle f, g \rangle_A = \langle (1 - \Delta) f, g \rangle_{L^2} \]
\[ = \int_{\Omega} fg - \int_{\Omega} (\Delta f) g \]
\[ = \int_{\Omega} fg + \int_{\Omega} Df \cdot Dg + \text{boundary corrections; energy inner product} \]

So we define
\[ \langle f, g \rangle_{\text{Energy}} := \int_{\Omega} fg + \int_{\Omega} Df \cdot Dg; \quad (6.125) \]
and then
\[ \langle f, g \rangle_A = \langle f, g \rangle_{\text{Energy}} + \text{boundary corrections}. \quad (6.126) \]

**Remark 6.19.** The A-inner product on the r.h.s. of (6.126) incorporates the boundary information.

**Example 6.6.** Consider \( L^2(0, 1), F(x) = e^{-|x|} |_{[-1,1]}, \) and \( A = \frac{1}{2} (1 - (\frac{d}{dx})^2) \). We have
\[ \langle f, g \rangle_A = \frac{1}{2} \langle f - f'', g \rangle_{L^2} \]
\[ = \frac{1}{2} \int_0^1 fg - \frac{1}{2} \int_0^1 f'' g \]
\[ = \frac{1}{2} \left( \int_0^1 fg + \int_0^1 f'' g \right) + \frac{(f' g)(0) - (f' g)(1)}{2} \]
\[ = \langle f, g \rangle_{\text{Energy}} + \frac{(f' g)(0) - (f' g)(1)}{2}. \]

Here, the boundary term
\[ \frac{(f' g)(0) - (f' g)(1)}{2} \quad (6.127) \]
contains the inward normal derivative of \( f' \) at \( x = 0 \) and \( x = 1. \)

(1) We proceed to check the reproducing property w.r.t. the A-inner product:
\[ 2 \left\langle e^{-|x-y|}, g \right\rangle_{\text{Energy}} = \int_0^1 e^{-|x-y|} g(y) dy + \int_0^1 \left( \frac{d}{dy} e^{-|x-y|} \right) g'(y) dy \]
where
\[
\int_{0}^{1} \left( \frac{d}{dy} e^{-|x-y|} \right) g'(y) dy \\
= \int_{0}^{x} e^{-(x-y)} g'(y) dy - \int_{x}^{1} e^{-(y-x)} g'(y) dy \\
= 2g(x) - g(0) e^{-x} - g(1) e^{-(1-x)} - \int_{0}^{1} e^{-|x-y|} g(y) dy;
\]

it follows that
\[
\langle e^{-|x-\cdot|}, g \rangle_{\text{Energy}} = g(x) + \frac{g(0) e^{-x} + g(1) e^{-(1-x)}}{2} 
\]

(2) It remains to check the boundary term in (6.128) comes from the inward normal derivative of \(e^{-|x-\cdot|}\). Indeed, set \(f(\cdot) = e^{-|x-\cdot|}\) in (6.127), then
\[
f'(0) = e^{-x}, \quad f'(1) = -e^{-(1-x)}
\]

therefore,
\[
\frac{(f'g)(0) - (f'g)(1)}{2} = \frac{e^{-x} g(0) + e^{-(1-x)} g(1)}{2}.
\]

**Example 6.7.** Consider \(L^2(0, \frac{1}{2})\), \(F(x) = 1 - |x|\) with \(|x| < \frac{1}{2}\), and let \(A = -\frac{1}{2} \left( \frac{d}{dx} \right)^2\). Then the \(A\)-inner product yields
\[
\langle f, g \rangle_A = \frac{1}{2} \langle f'', g \rangle_{L^2} \\
= \frac{1}{2} \int_{0}^{1} \frac{1}{2} f''g' - \frac{(f'g) \left( \frac{1}{2} \right) - (f'g)(0)}{2} \\
= \langle f, g \rangle_{\text{Energy}} + \frac{(f'g)(0) - (f'g) \left( \frac{1}{2} \right)}{2}
\]

where we set
\[
\langle f, g \rangle_{\text{Energy}} := \frac{1}{2} \int_{0}^{1} f''g';
\]

and the corresponding boundary term is
\[
\frac{(f'g)(0) - (f'g) \left( \frac{1}{2} \right)}{2} (6.129)
\]

(1) To check the reproducing property w.r.t. the \(A\)-inner product: Set \(F_{x}(y) := 1 - |x-y|, x, y \in \left( 0, \frac{1}{2} \right)\); then
6.4 Positive Definite Functions, Green’s Functions, and Boundary

\[ \langle F_x, g \rangle_{\text{Energy}} = \frac{1}{2} \int_0^1 F_x(y) \, g'(y) \, dy \]
\[ = \frac{1}{2} \left( \int_0^x g'(y) \, dy - \int_x^1 g'(y) \, dy \right) \]
\[ = g(x) - \frac{g(0) + g\left(\frac{1}{2}\right)}{2}. \quad (6.130) \]

(2) Now we check the second term on the r.h.s. of (6.130) contains the inward normal derivative of \( F_x \). Note that

\[ F'_x(0) = \left. \frac{d}{dy} \right|_{y=0} (1 - |x - y|) = 1 \]
\[ F'_x\left(\frac{1}{2}\right) = \left. \frac{d}{dy} \right|_{y=\frac{1}{2}} (1 - |x - y|) = -1 \]

Therefore,

\[ \frac{(f'g)(0) - (f'g)\left(\frac{1}{2}\right)}{2} = \frac{g(0) + g\left(\frac{1}{2}\right)}{2}, \]

which verifies the boundary term in (6.129).
Chapter 7
Green’s Functions

The focus of this chapter is a detailed analysis of two specific positive definite functions, each one defined in a fixed finite interval, centered at \( x = 0 \). Rationale: The examples serve to make explicit some of the many connections between our general theme (locally defined p.d. functions and their extensions), on the one hand; and probability theory and stochastic processes on the other.

7.1 The RKHSs for the Two Examples \( F_2 \) and \( F_3 \) in Table 5.1

In this section, we revisit cases \( F_2 \), and \( F_3 \) (from Table 5.1) and their associated RKHSs. The two examples are

\[
F_2(x) = 1 - |x|, \text{ in } |x| < \frac{1}{2}; \text{ and }
\]

\[
F_3(x) = e^{-|x|}, \text{ in } |x| < 1.
\]

We show that they are (up to isomorphism) also the Hilbert spaces used in stochastic integration for Brownian motion, and for the Ornstein-Uhlenbeck process (see e.g., [Hid80]), respectively. As reproducing kernel Hilbert spaces, they have an equivalent and more geometric form, of use in for example analysis of Gaussian processes. Analogous results for the respective RKHSs also hold for other positive definite function systems \((F, \Omega)\), but for the present two examples \( F_2 \), and \( F_3 \), the correspondences involved are explicit. As a bonus, we get an easy and transparent proof that the deficiency-indices for the respective operators \( D(F) \) are \((1, 1)\) in both these examples.

The purpose of the details below are two-fold. First we show that the respective RKHSs corresponding to \( F_2 \) and \( F_3 \) in Table 5.1 are naturally isomorphic to more familiar RKHSs which are used in the study of Gaussian processes, see e.g., [AJL11, AL10, AJ12]; and secondly, to give an easy (and intuitive) proof that the deficiency indices in these two cases are \((1, 1)\). Recall for each p.d. function \( F \) in an interval \((-a, a)\), we study
\[
\mathcal{D}(F) \left( F\varphi \right) := F\varphi ', \varphi \in C^\infty_c (0,a) \tag{7.1}
\]
as a skew-Hermitian operator in \( \mathcal{H}_F \); see Lemma 3.1.

### 7.1.1 Green’s Functions

The term “Green’s function” (also called “fundamental solution,” see e.g., [Trè06]) is used generally in connection with inverses of elliptic operators; but, more importantly, for the solution of elliptic boundary value problems. We show that for the particular case of one dimension, for the two elliptic operators (7.2) and (7.4), and for the corresponding boundary value problems for finite intervals, the two above mentioned p.d. functions play the role of Green’s functions.

**Lemma 7.1.**

1. For \( F_2(x) = 1 - |x|, |x| < \frac{1}{2} \), let \( \varphi \in C^\infty_c \left( 0, \frac{1}{2} \right) \), then \( u(x) := (T_{F_2} \varphi)(x) \) satisfies

\[
\varphi = -\frac{1}{2} \left( \frac{d}{dx} \right)^2 u. \tag{7.2}
\]

Hence,

\[
T_{F_2}^{-1} \supset -\frac{1}{2} \left( \frac{d}{dx} \right)^2 \bigg|_{C^\infty_c \left( 0, \frac{1}{2} \right)}. \tag{7.3}
\]

2. For \( F_3(x) = e^{-|x|}, |x| < 1 \), let \( \varphi \in C^\infty_c \left( 0, 1 \right) \), then

\[
\varphi = \frac{1}{2} \left( I - \left( \frac{d}{dx} \right)^2 \right) u. \tag{7.4}
\]

Hence,

\[
T_{F_3}^{-1} \supset \frac{1}{2} \left( I - \left( \frac{d}{dx} \right)^2 \right) \bigg|_{C^\infty_c \left( 0, 1 \right)}. \tag{7.5}
\]

**Proof.** The computation for \( F = F_2 \) is as follows: Let \( \varphi \in C^\infty_c \left( 0, \frac{1}{2} \right) \), then

\[
u(x) = (T_{F_2} \varphi)(x) = \int_0^x \varphi(y) (1 - |x-y|) dy
\]

\[
= \int_0^x \varphi(y) (1 - (x-y)) dy + \int_x^1 \varphi(y) (1 - (y-x)) dy;
\]

and

\[
\int_0^x \varphi(y) (1 - (x-y)) dy = \int_0^x \varphi(y) dy - x \int_0^x \varphi(y) dy + x \varphi(y) dy
\]
7.1 The RKHSs for the Two Examples \( F_2 \) and \( F_3 \) in Table 5.1

\[
u'(x) = -\int_0^x \varphi(y) + \varphi(x) + \int_x^1 \varphi(y) dy - \varphi(x)
\]
\[
u''(x) = -2\varphi(x).
\]

Thus, \( \varphi = -\frac{1}{2}u'' \), and the desired result follows.

For \( F_3 \), let \( \varphi \in C^\infty_c(0, 1) \), then

\[
u(x) = (T_{F_3} \varphi)(x) = \int_0^1 e^{-|x-y|} \varphi(y) dy
\]
\[
= \int_0^x e^{-(x-y)} \varphi(y) dy + \int_x^1 e^{-(y-x)} \varphi(y) dy.
\]

Now,

\[
u'(x) = (T_{F_3} \varphi)'(x) = -e^{-x} \int_0^1 e^{-y} \varphi(y) dy + \varphi(x)
\]
\[
+ e^x \int_x^1 e^{-y} \varphi(y) dy - \varphi(x)
\]
\[
= -e^{-x} \int_0^x e^y \varphi(y) dy + e^x \int_x^1 e^{-y} \varphi(y) dy
\]
\[
u'' = e^{-x} \int_0^x e^y \varphi(y) dy - \varphi(x)
\]
\[
+ e^x \int_x^1 e^{-y} \varphi(y) dy - \varphi(x)
\]
\[
= -2\varphi + \int_0^1 e^{-|x-y|} \varphi(y) dy
\]
\[
= -2\varphi + T_{F_3}(\varphi);
\]

and then

\[
u''(x) = e^{-x} \int_0^x e^y \varphi(y) dy - \varphi(x)
\]
\[
+ e^x \int_x^1 e^{-y} \varphi(y) dy - \varphi(x)
\]
\[
= -2\varphi + \int_0^1 e^{-|x-y|} \varphi(y) dy
\]
\[
= -2\varphi + u(x).
\]

Thus, \( \varphi = \frac{1}{2}(u - u'') = \frac{1}{2} \left( I - \frac{1}{2} \left( \frac{d}{dx} \right)^2 \right) u \). This proves (7.4).

Summary: Conclusions for the two examples.

The computation for \( F = F_2 \) is as follows: If \( \varphi \in L^2 \left( 0, \frac{1}{2} \right) \), then \( u(x) := (T_F \varphi)(x) \) satisfies
\( (F_2) \quad \varphi = \frac{1}{2} \left( -\left( \frac{d}{dx} \right)^2 \right) u; \)

while, for \( F = F_3 \), the corresponding computation is as follows: If \( \varphi \in L^2(0,1) \), then \( u(x) = (T_F \varphi)(x) \) satisfies

\( (F_3) \quad \varphi = \frac{1}{2} \left( I - \left( \frac{d}{dx} \right)^2 \right) u; \)

For the operator \( D(F) \), in the case of \( F = F_2 \), it follows that the Mercer operator \( T_F \) plays the following role: \( T_F^{-1} \) is a selfadjoint extension of \( -\frac{1}{2} \left( D(F) \right)^2 \). In the case of \( F = F_3 \) the corresponding operator \( T_F^{-1} \) (in the RKHS \( \mathcal{H}_F \)) is a selfadjoint extension of \( \frac{1}{2} \left( I - \left( D(F) \right)^2 \right) \); in both cases, they are the Friedrichs extensions.

**Remark 7.1.** When solving boundary values for elliptic operators in a bounded domain, say \( \Omega \subset \mathbb{R}^n \), one often ends up with Green’s functions (= integral kernels) which are positive definite kernels, so \( K(x,y) \), defined on \( \Omega \times \Omega \), not necessarily of the form \( K(x,y) = F(x-y) \).

But many of the questions we ask in the special case of p.d. functions, so when the kernel is \( K(x,y) = F(x-y) \) will also make sense for p.d. kernels.

### 7.1.2 The Case of \( F_2(x) = 1 - |x|, x \in (-\frac{1}{2}, \frac{1}{2}) \)

Let \( F = F_2 \). Fix \( x \in (0, \frac{1}{2}) \), and set

\[ F_x(y) = F(x-y), \text{ for } x,y \in (0,\frac{1}{2}); \quad (7.6) \]

where \( F_x(\cdot) \) and its derivative (in the sense of distributions) are as in Figure 7.1 (sketched for two values of \( x \)).

Consider the Hilbert space

\[ \mathcal{H}_F := \left\{ h; \text{ continuous on } (0,\frac{1}{2}), \text{ and } h' = \frac{dh}{dx} \in L^2(0,\frac{1}{2}) \right\} \quad \text{where the derivative is in the weak sense} \quad (7.7) \]

modulo constants; and let the norm, and inner-product, in \( \mathcal{H}_F \) be given by

\[ ||h||_{\mathcal{H}_F}^2 = \frac{1}{2} \int_0^{\frac{1}{2}} |h'(x)|^2 \, dx + \int_{\partial \Omega} h_n h \, d\beta. \quad (7.8) \]

On the r.h.s. of (7.8), \( d\beta \) denotes the corresponding boundary measure, and \( h_n \) is the inward normal derivative of \( h \). See Theorem 7.1 below.

Then the reproducing kernel property is as follows:

\[ (F_x,h)_{\mathcal{H}_F} = h(x), \forall h \in \mathcal{H}_F, \forall x \in (0,\frac{1}{2}) ; \quad (7.9) \]

and it follows that \( \mathcal{H}_F \) is naturally isomorphic to the RKHS for \( F_2 \) from Section 3.1.
Theorem 7.1. The boundary measure for $F = F_2$ (see (7.8)) is

$$\beta = \frac{1}{2} (\delta_0 + \delta_{1/2}).$$

Proof. Set

$$\delta' (\xi) := \frac{1}{2} \int_0^\frac{1}{2} |\xi' (x)|^2 dx, \forall \xi \in \mathcal{H}_F. \tag{7.10}$$

And let $F_x (\cdot) := 1 - |x - \cdot|$ defined on $[0, \frac{1}{2}]$, for all $x \in (0, \frac{1}{2})$. Then

$$E (F_x, \xi) = \frac{1}{2} \int_0^{\frac{1}{2}} F_x' (y) \xi' (y) dy$$

$$= \frac{1}{2} \left( \int_0^x \xi' (y) dy - \int_0^{\frac{1}{2}} \xi' (y) dy \right)$$

$$= \xi (x) - \frac{\xi (0) + \xi (\frac{1}{2})}{2}. \text{ (see Fig. 7.1)}$$

Since
\[ \| \xi \|_{\mathcal{H}_F}^2 = E(\xi) + \int |\xi|^2 \, d\beta \]
we get
\[ \langle F, \xi \rangle_{\mathcal{H}_F} = \xi(x), \forall \xi \in \mathcal{H}_F. \]
We conclude that
\[ \langle \xi, \eta \rangle_{\mathcal{H}_F} = E(\xi, \eta) + \int_{\partial\Omega} \xi \eta \, d\beta; \tag{7.11} \]
note the boundary in this case consists two points, \( x = 0 \) and \( x = \frac{1}{2} \).

Remark 7.2. The energy form in (7.10) also defines a RKHS (Wiener’s energy form for Brownian motion [Hid80], see Figure 1.2) as follows:

On the space of all continuous functions, \( \mathcal{C}([0, \frac{1}{2}]) \), set
\[ \mathcal{H}_E := \{ f \in \mathcal{C}([0, \frac{1}{2}]) \mid \mathcal{E}(f) < \infty \} \tag{7.12} \]
modulo constants, where
\[ \mathcal{E}(f) = \int_0^{\frac{1}{2}} |f'(x)|^2 \, dx. \tag{7.13} \]

For \( x \in [0, \frac{1}{2}] \), set
\[ E_x(y) = x \wedge y = \min(x, y), \quad y \in (0, \frac{1}{2}); \]
see figure 7.2; then \( E_x \in \mathcal{H}_E \), and
\[ \langle E_x, f \rangle_{\mathcal{H}_E} = f(x), \forall f \in \mathcal{H}_E, \forall x \in [0, \frac{1}{2}]. \tag{7.14} \]

Fig. 7.2: The covariance function \( E_x(\cdot) = \min(x, \cdot) \) of Brownian motion.

Proof. For the reproducing property (7.14): Let \( f \) and \( x \) be as stated; then
7.1 The RKHSs for the Two Examples $F_2$ and $F_3$ in Table 5.1

\[
\langle E_x, f \rangle_{\mathcal{H}_E} = \int_0^1 E'_x(y) f'(y) \, dy
\]

Fig. (7.2) $\int_0^1 \chi_{[0,1]}(y) f'(y) \, dy$

\[
= \int_0^1 f'(y) \, dy = f(x) - f(0).
\]

Note in (7.12) we define $\mathcal{H}_E$ modulo constants; alternatively, we may stipulate $f(0) = 0$. □

Note that the Brownian motion RKHS is not defined by a p.d. function, but rather by a p.d. kernel. Nonetheless the remark explains its connection to our present RKHS $\mathcal{H}_F$ which is defined by the p.d. function, namely the p.d. function $F_2$.

**Pinned Brownian Motion.**

We illustrate the boundary term in Theorem 7.1, eq. (7.11) with pinned Brownian motion (also called “Brownian bridge.”) In order to simplify constructions, we pin the Brownian motion at the following two points in $(t,x)$ space, $(t,x) = (0,0)$, and $(t,x) = (1,1)$; see figure 7.3. To simplify computations further, we restrict attention to real-valued functions only.

**Remark 7.3.** The literature on Gaussian processes, Ito-calculus, and Brownian motion is vast, but for our purposes, the reference [Hid80] will do. For background on stochastic processes, see especially [IW89, p.243-244].

For the pinning down the process $X_t$ at the two points $(0,0)$ and $(1,1)$ as in figure 7.3, we have

**Proposition 7.1.** The Brownian bridge $X_t$, connecting (or pinning down) the two points $(0,0)$ and $(1,1)$, satisfies $X_0 = 0, X_1 = 1$, and it has the following covariance function $c$,

\[
c(s,t) = s \wedge t - st.  \tag{7.15}
\]

Moreover,

\[
X_t = t + (1 - t) \int_0^t \frac{dB_s}{1 - s}, 0 < t < 1,  \tag{7.16}
\]

where $dB_s$ on the r.h.s. of (7.16) refers to the standard Brownian motion $dB_s$, and the second term in (7.16) is the corresponding Ito-integral.

**Proof.** We have

\[
E \left( \int_0^t \frac{dB_s}{1 - s} \right) = 0, \quad \text{and}  \tag{7.17}
\]

\[
E \left( \left. \int_0^t \frac{dB_s}{1 - s} \right|^2 \right) = \int_0^t \frac{ds}{(1 - s)^2} = \frac{t}{1 - t}.  \tag{7.18}
\]

where $E(\cdots) = \int_{\Omega} \cdots \, d\mathbb{P} =$ expectation with respect to the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Hence, for the mean and covariance function of the process $X_t$ in
Green’s Functions

(7.16), we get

\[ \mathbb{E}(X_t) = t, \quad \text{and} \]

\[ \text{Cov}(X_t, X_s) = \mathbb{E}((X_t - t)(X_s - s)) = s \wedge t - st; \]

where \( s \wedge t = \min(s, t) \), and \( s, t \in (0, 1) \).

And it follows in particular that the function on the r.h.s. in eq (7.20) is a positive definite kernel. Its connection to \( F_2 \) is given in the next lemma.

Now return to the p.d. function \( F = F_2 \); i.e., \( F(t) = 1 - |t| \), and therefore, \( F(s-t) = 1 - |s-t|, s, t \in (0, 1) \), and let \( \mathcal{H}_F \) be the corresponding RKHS.

Let \( \{F_t\}_{t \in (0,1)} \) denote the kernels in \( \mathcal{H}_F \), i.e., \( \langle F_t, g \rangle_{\mathcal{H}_F} = g(t), t \in (0,1), g \in \mathcal{H}_F \). We then have the following:

**Lemma 7.2.** Let \( (X_t)_{t \in (0,1)} \) denote the pinned Brownian motion (7.16); then

\[ \langle F_s, F_t \rangle_{\mathcal{H}_F} = \text{Cov}(X_s, X_t); \]  

see (7.20); and

\[ \langle F_s, F_t \rangle_{\text{energy}} = s \wedge t; \]

while the boundary term

\[ bdr(s,t) = -st. \]  

**Proof.** With our normalization from Figure 7.3, we must take the energy form as follows:

\[ \langle f, g \rangle_{\text{energy}} = \int_0^1 f'(x)g'(x) \, dx. \]

Set \( F_s(y) = s \wedge y \), see Figure 7.2. For the distributional derivative we have

\[ F_s'(y) = \chi_{[0,s]}(y) = \begin{cases} 1 & \text{if } 0 \leq y \leq s, \\ 0 & \text{else,} \end{cases} \]
7.1 The RKHSs for the Two Examples $F_2$ and $F_3$ in Table 5.1

then

$$\langle F_s, F_t \rangle_{\text{energy}} = \int_0^1 F'_s(y) F'_t(y) \, dy$$

$$= \int_0^1 \chi_{[0,s]}(y) \chi_{[0,t]}(y) \, dy$$

$$= |[0,s] \cap [0,t]| \text{Lebesgue measure}$$

$$= s \wedge t.$$ 

The desired conclusions (7.22)-(7.23) in the lemma now follow. See Remark 7.2.

The verification of (7.21) uses Ito-calculus \[Hid80\] as follows: Note that (7.16) for $X_t$ is the solution to the following Ito-equation:

$$dX_t = \left( X_t - 1 \right) \frac{1}{t-1} \, dt + dB_t; \quad (7.25)$$

and by Ito's lemma \[Hid80\], therefore,

$$(dX_t)^2 = (dB_t)^2. \quad (7.26)$$

As a result, if $f : \mathbb{R} \to \mathbb{R}$ is a function in the energy-Hilbert space defined from (7.24), then for $0 < T < 1$, we have,

$$\mathbb{E}\left( |f(X_T)|^2 \right) = \mathbb{E}\left( |f(B_T)|^2 \right) = \int_0^T |f'(t)|^2 \, dt = \mathcal{E}_2(f),$$

the energy form of the RKHS (7.24).

\[ \square \]

Remark 7.4. In (7.25)-(7.26), we use standard conventions for Brownian motions $B_t$ (see, e.g., \[Hid80\]): Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a choice of probability space for \{\{B_t\}_{t \in \mathbb{R}}, \text{or} t \in [0,1]\}.) With $E(\cdots) = \int_{\Omega} \cdots \, d\mathbb{P}$, we have

$$E(B_s B_t) = s \wedge t = \min(s,t), \quad t \in [0,1]. \quad (7.27)$$

If $f : \mathbb{R} \to \mathbb{R}$ is a $C^1$-function, we write $f(B_t)$ for $f \circ B_t$; and $df(B_t)$ refers to Ito-calculus.

7.1.3 The Case of $F_3(x) = e^{-|x|}, x \in (-1, 1)$

Recall, if $\{B(x)\}_{x \in \mathbb{R}}$ denotes the standard Brownian motion (see, e.g., Sections 1.3, 5.1, and 6.1); then if we set

$$X(x) := e^{-x}B(e^{2x}), \quad x \in \mathbb{R}; \quad (7.28)$$

then $X(x)$ is the Ornstein-Uhlenbeck process with
\[ E(X(x)X(y)) = e^{-|x-y|}, \forall x, y \in \mathbb{R}. \]  
\( (7.29) \)

Let \( F_i(y) = F(x-y) \), for all \( x, y \in (0, 1) \); and consider the Hilbert space

\[ \mathcal{H}_F := \left\{ h; \text{continuous on } (0,1), h \in L^2(0,1), \text{ and the weak derivative } h' \in L^2(0,1) \right\}; \]  
\( (7.30) \)

and let the \( \mathcal{H}_F \)-norm, and inner product, be given by

\[ \| h \|_{\mathcal{H}_F}^2 = \frac{1}{2} \left( \int_0^1 |h(x)|^2 \, dx + \int_0^1 |h'(x)|^2 \, dx \right) + \int_{\partial \Omega} h_n h \, d\beta. \]  
\( (7.31) \)

Here, \( d\beta \) on the r.h.s. of (7.31) denotes the corresponding boundary measure, and \( h_n \) is the inward normal derivative of \( h \). See Theorem 7.2 below.

Then a direct verification yields:

\[ \langle F_i, h \rangle_{\mathcal{H}_F} = h(x), \forall h \in \mathcal{H}_F, \forall x \in (0,1); \]  
\( (7.32) \)

and it follows that \( \mathcal{H}_F \) is naturally isomorphic to RKHS for \( F_i \) (see Section 3.1).

For details of (7.32), see also [Jør81].

**Corollary 7.1.** In both \( \mathcal{H}_F, i = 2,3 \), the deficiency indices are \((1,1)\).

**Proof.** In both cases, we are referring to the skew-Hermitian operator \( D^{(F_i)} \) in \( \mathcal{H}_{F_i}, i = 2,3 \); see (7.1) above. But it follows from (7.8) and (7.31) for the respective inner products, that the functions \( e^{\pm x} \) have finite positive norms in the respective RKHSs. \( \square \)

**Theorem 7.2.** Let \( F = F_3 \) as before. Consider the energy-Hilbert space

\[ \mathcal{H}_E := \left\{ f \in C[0,1] \mid f, f' \in L^2(0,1) \text{ where } f' \text{ is the weak derivative of } f \right\}; \]  
\( (7.33) \)

with

\[ \langle f, g \rangle_E = \frac{1}{2} \left( \int_0^1 f'(x)g'(x) \, dx + \int_0^1 f'(x)g(x) \, dx \right); \text{ and so} \]  
\( (7.34) \)

\[ \| f \|_E^2 = \frac{1}{2} \left( \int_0^1 |f'(x)|^2 \, dx + \int_0^1 |f(x)|^2 \, dx \right), \forall f, g \in \mathcal{H}_E. \]  
\( (7.35) \)

Set

\[ P(f,g) = \int_0^1 f_n(x)g(x) \, d\beta(x), \text{ where} \]  
\( (7.36) \)

\[ \beta := \frac{\delta_0 + \delta_1}{2}, \text{ i.e., Dirac masses at endpoints} \]  
\( (7.37) \)

where \( g_n \) denotes the inward normal derivative at endpoints.
7.1 The RKHSs for the Two Examples \( F_2 \) and \( F_3 \) in Table 5.1

Let

\[
\mathcal{H}_F := \left\{ f \in C[0,1] \left| \| f \|_2^2 + P_2(f) < \infty \text{ where} \right. \right\}
\]

(7.38)

Then we have following:

(1) As a Green-Gauss-Stoke principle, we have

\[
\| f \|_F^2 = \| f \|_E^2 + P_2(f), \text{ i.e.,}
\]

(7.39)

(7.40)

(2) Moreover,

\[
\langle F_x, g \rangle_E = g(x) - \frac{e^{-x}g(0) + e^{-(1-x)}g(1)}{2}.
\]

(7.41)

(3) As a result of (7.41), eq. (7.39)-(7.40) is the reproducing property in \( \mathcal{H}_F \). Specifically, we have

\[
\langle F_x, g \rangle_{\mathcal{H}_F} = g(x), \forall g \in \mathcal{H}_F. \text{ (see (7.38))}
\]

(7.42)

Proof. Given \( g \in \mathcal{H}_F \), one checks directly that

\[
P(F_x, g) = \int_0^1 F_x(y) g(y) d\beta(y)
\]

\[
= \frac{F_x(0)g(0) + F_x(1)g(1)}{2}
\]

\[
= \frac{e^{-x}g(0) + e^{-(1-x)}g(1)}{2}.
\]

and, using integration by parts, we have

\[
\langle F_x, g \rangle_E = g(x) - \frac{e^{-x}g(0) + e^{-(1-x)}g(1)}{2}
\]

\[
= \langle F_x, g \rangle_F - P(F_x, g).
\]

Now, using

\[
\int_{\partial \Omega} (F_x)_n g d\beta := \frac{e^{-x}g(0) + e^{-(1-x)}g(1)}{2},
\]

(7.43)

and \( \langle F_x, g \rangle_{\mathcal{H}_F} = g(x), \forall x \in (0,1) \), and using (7.41), the desired conclusion

\[
\langle F_x, g \rangle_{\mathcal{H}_F} = \langle F_x, g \rangle_E + \int_{\partial \Omega} (F_x)_n g d\beta
\]

(7.44)

follows. Since the \( \mathcal{H}_F \)-norm closure of the span of \( \{ F_x \mid x \in (0,1) \} \) is all of \( \mathcal{H}_F \), from (7.44), we further conclude that
\begin{align}
\langle f, g \rangle_{\mathcal{H}_F} = \langle f, g \rangle_E + \int_{\partial \Omega} T_n g d\beta 
\end{align}  \tag{7.45}

holds for all \( f, g \in \mathcal{H}_F \).

In (7.44) and (7.45), we used \( f_n \) to denote the inward normal derivative, i.e.,
\( f_n(0) = f'(0) \), and \( f_n(1) = -f'(1), \forall f \in \mathcal{H}_E \). □

**Example 7.1.** Fix \( p, 0 < p \leq 1 \), and set
\begin{align}
F_p(x) := 1 - |x|^p, \quad x \in \left( -\frac{1}{2}, \frac{1}{2} \right) ;
\end{align}  \tag{7.46}

see figure 7.4. Then \( F_p \) is positive definite and continuous; and so \( -F_p''(x - y) \) is a
p.d. kernel, so \( -F_p'' \) is a positive definite distribution on \( \left( -\frac{1}{2}, \frac{1}{2} \right) \).

We saw that if \( p = 1 \), then
\begin{align}
-F_1'' = 2\delta
\end{align}  \tag{7.47}

where \( \delta \) is the Dirac mass at \( x = 0 \). But for \( 0 < p < 1 \), \( -F_p'' \) does not have the form
(7.47). We illustrate this if \( p = \frac{1}{2} \). Then
\begin{align}
-F_{\frac{1}{2}}'' = \chi_{\{x \neq 0\}} \left( \frac{1}{4} |x|^{-\frac{3}{2}} + \frac{1}{4} \delta'' \right),
\end{align}  \tag{7.48}

where \( \delta'' \) is the double derivative of \( \delta \) in the sense of distributions.

![Fig. 7.4: \( F_p(x) = 1 - |x|^p, x \in \left( -\frac{1}{2}, \frac{1}{2} \right) \) and \( 0 < p \leq 1 \).](image)

**Remark 7.5.** There is a notion of boundary measure in potential theory. Boundary
measures exist for any potential theory. In our case it works even more generally,
whenever p.d. \( F_p \) for bounded domains \( \Omega \subset \mathbb{R}^n \), and even Lie groups. But in the
example of \( F_3 \), the boundary is two points.

In all cases, we get \( \mathcal{H}_F \) as a perturbation of the energy form:
\begin{align}
\| \cdot \|^2_{\mathcal{H}_F} = \text{energy form} + \text{perturbation} \tag{7.49}
\end{align}

It is a Green-Gauss-Stoke principle. There is still a boundary measure for \( \text{ALL} \)
bounded domains \( \Omega \subset \mathbb{R}^n \), and even Lie groups.
7.1 The RKHSs for the Two Examples $F_2$ and $F_3$ in Table 5.1

And RKHS form

$$\|\cdot\|^2_{\mathcal{H}_{F}} = \text{energy form} + \int_{\partial\Omega} f_n f \, d\mu_{bd \text{ meas.}}$$

(7.50)

For the general theory of boundary measures and their connection to the Green-Gauss-Stoke principle, we refer readers to [JP13a, JP13b, Mov12, Bat90, Tel83, ACF09].

The approach via $\|\cdot\|^2_{\mathcal{H}_{F}} = \text{"energy term + boundary term"}$, does not fail to give a RKHS, but we must replace "energy term" with an abstract Dirichlet form; see Refs [HS12, Tre88].

### 7.1.4 Integral Kernels and Positive Definite Functions

Let $0 < H < 1$ be given, and set

$$K_H(x, y) = \frac{1}{2} \left( |x|^{2H} + |y|^{2H} - |x - y|^{2H} \right), \quad \forall x, y \in \mathbb{R}. \quad (7.51)$$

It is known that $K_H(\cdot, \cdot)$ is the covariance kernel for fractional Brownian motion [AJ12, AJL11, AL08, Aur11]. The special case $H = \frac{1}{2}$ is Brownian motion; and if $H = \frac{1}{2}$, then

$$K_{\frac{1}{2}}(x, y) = |x| \wedge |y| = \min(|x|, |y|)$$

if $x$ and $y$ have the same signs. Otherwise the covariance vanishes. See Figure 7.5.

Set

$$\tilde{F}_H(x, y) := 1 - |x - y|^{2H}$$
$$F_H(x) := 1 - |x|^{2H}$$

and we recover $F(x) = 1 - |x| = F_2$ as a special case of $H = \frac{1}{2}$.

![Integral kernel](image_url)  

**Fig. 7.5:** The integral kernel $K_{\frac{1}{2}}(x, y) = |x| \wedge |y|$.  

Solving for $F_H$ in (7.51) ($0 < H < 1$ fixed), we then get
\[ F_H (x - y) = \frac{1 - |x|^{2H} - |y|^{2H}}{L_H} + 2K_H (x, y); \]  
(7.52)

and specializing to \( x, y \in [0, 1] \); we then get

\[ F (x - y) = 1 - x - y + 2 (x \wedge y); \]

or written differently:

\[ K_F (x, y) = L (x, y) + 2E (x, y) \]  
(7.53)

where \( E (x, y) = x \wedge y \) is the familiar covariance kernel for Brownian motion.

Introducing the Mercer integral kernels corresponding to (7.52), we therefore get:

\[ (T_F \varphi) (x) = \int_0^1 \varphi (y) F_H (x - y) \, dy \]  
(7.54)

\[ (T_L \varphi) (x) = \int_0^1 \varphi (y) L_H (x, y) \, dy \]  
(7.55)

\[ (T_K \varphi) (x) = \int_0^1 \varphi (y) K_H (x, y) \, dy \]  
(7.56)

for all \( \varphi \in L^2 (0, 1) \), and all \( x \in [0, 1] \). Note the special case of (7.56) for \( H = \frac{1}{2} \) is

\[ (T_E \varphi) (x) = \int_0^1 \varphi (y) (x \wedge y) \, dy, \varphi \in L^2 (0, 1), x \in [0, 1]. \]

We have the following lemma for these Mercer operators:

**Lemma 7.3.** Let \( 0 < H < 1 \), and let \( F_H, L_H \) and \( K_H \) be as in (7.52), then the corresponding Mercer operators satisfy:

\[ T_F = T_L + 2T_K. \]  
(7.57)

**Proof.** This is an easy computation, using (7.52), and (7.54)-(7.56). \( \square \)

### 7.1.5 The Ornstein-Uhlenbeck Process Revisited

The reproducing kernel Hilbert space \( \mathcal{H}_F \) in (7.7) is used in computations of Itôintegrals of Brownian motion; while the corresponding RKHS \( \mathcal{H}_F \) from (7.30)-(7.31) is used in calculations of stochastic integration with the Ornstein-Uhlenbeck process.

Motivated by Newton’s second law of motion, the Ornstein-Uhlenbeck velocity process is proposed to model a random external driving force. In 1D, the process is
the solution to the following stochastic differential equation

\[ dv_t = -\gamma v_t dt + \beta dB_t, \quad \gamma, \beta > 0. \quad (7.58) \]

Here, \(-\gamma v_t\) is the dissipation, \(\beta dB_t\) denotes a random fluctuation, and \(B_t\) is the standard Brownian motion.

Assuming the particle starts at \(t = 0\). The solution to \((7.58)\) is a Gaussian stochastic process such that

\[
\mathbb{E}[v_t] = v_0 e^{-\gamma t}
\]

\[
\text{var}[v_t] = \frac{\beta^2}{2\gamma} (1 - e^{-2\gamma t})
\]

with \(v_0\) being the initial velocity. See Figure 7.6. Moreover, the process has the following covariance function

\[
c(s, t) = \frac{\beta^2}{2\gamma} (e^{-\gamma|t-s|} - e^{-\gamma|s+t|}).
\]

If we wait long enough, it turns to a stationary process such that

\[
c(s, t) \sim \frac{\beta^2}{2\gamma} e^{-\gamma|s-t|}.
\]

This corresponds to the function \(F_3\). See also Remark 5.1.
7.1.6 An Overview of the Two Cases: \( F_2 \) and \( F_3 \).

In Table 7.1 below, we give a list of properties for the particular two cases \( F_2 \) and \( F_3 \) from table 5.1. Special attention to these examples is merited; first they share a number of properties with much wider families of locally defined positive definite functions; and these properties are more transparent in their simplest form. Secondly, there are important differences between cases \( F_2 \) and \( F_3 \), and the table serves to highlight both differences and similarities. A particular feature that is common for the two is that, when the Mercer operator \( T_F \) is introduced, then its inverse \( T_F^{-1} \) exists as an unbounded positive and selfadjoint operator in \( \mathcal{H}_F \). Moreover, in each case, this operator coincides with the Friedrichs extension of a certain second order Hermitian semibounded operator (calculated from \( D^{(F)} \)), with dense domain in \( \mathcal{H}_F \).

7.2 Higher Dimensions

Our function above for \( F_3 \) (Sections 7.1.1 and 7.1.3) admits a natural extension to \( \mathbb{R}^n, n > 1 \), as follows.

Let \( \Omega \subset \mathbb{R}^n \) be a subset satisfying \( \Omega \neq \emptyset \), \( \Omega \) open and connected; and assume \( \bar{\Omega} \) is compact. Let \( \triangle = \sum_{i=1}^{n} (\partial / \partial x_i)^2 \) be the usual Laplacian in \( n \)-variables.

**Lemma 7.4.** Let \( F: \Omega \rightarrow \mathbb{C} \) be continuous and positive definite; and let \( \mathcal{H}_F \) be the corresponding RKHS. In \( \mathcal{H}_F \), we set

\[
L^{(F)}(F\varphi) := F\varphi, \quad \forall \varphi \in C^\infty_c(\Omega), \quad \text{with}
\]

\[
\text{dom}(L^{(F)}) = \{F\varphi \mid \varphi \in C^\infty_c(\Omega)\};
\]

then \( L^{(F)} \) is semibounded in \( \mathcal{H}_F \), \( L^{(F)} \leq 0 \) in the order of Hermitian operators.

Let \( T_F \) be the Mercer operator; then there is a positive constant \( c = c_{(F,\Omega)} > 0 \) such that \( T_F^{-1} \) is a selfadjoint extension of the densely defined operator \( c \left( I - L^{(F)} \right) \)

in \( \mathcal{H}_F \).

**Proof.** The ideas for the proof are contained in Sections 7.1.1 and 7.1.3 above. To get the general conclusions, we may combine these considerations with the general theory of Green’s functions for elliptic linear PDEs (partial differential equations); see also [Nel57, JLW69]. \( \square \)
7.2 Higher Dimensions

1. \( F_2(x) = 1 - |x|, |x| < \frac{1}{2} \)

2. Mercer operator

\[ T_{F_2} : L^2 \left( 0, \frac{1}{2} \right) \rightarrow L^2 \left( 0, \frac{1}{2} \right) \]

3. \( T_{F_2}^{-1} \) is unbounded, selfadjoint

**Proof.** Since \( T_{F_2} \) is positive, bounded, and trace class, it follows that \( T_{F_2}^{-1} \) is positive, unbounded, and selfadjoint. \( \Box \)

4. \( T_{F_2}^{-1} \) Friedrichs extension of \( -\frac{1}{2} \left( \frac{d}{dx} \right)^2 \bigg|_{C^\infty_0(0, \frac{1}{2})} \)

as a selfadjoint operator on \( L^2 \left( 0, \frac{1}{2} \right) \).

**Sketch of proof.**

Setting

\[ u(x) = \int_0^{\frac{x}{2}} \varphi(y) (1 - |x - y|) dy \]

then

\[ u'' = -2\varphi \]

\[ \Downarrow \]

\[ u = \left( -\frac{1}{2} \left( \frac{d}{dx} \right)^2 \right)^{-1} \varphi \]

and so

\[ T_{F_2} = \left( -\frac{1}{2} \left( \frac{d}{dx} \right)^2 \right)^{-1}. \]

And we get a selfadjoint extension

\[ T_{F_2}^{-1} \supset -\frac{1}{2} \left( \frac{d}{dx} \right)^2 \]

in \( L^2 \left( 0, \frac{1}{2} \right) \), where the containment refers to operator graphs.

---

1. \( F_3(x) = e^{-|x|}, |x| < 1 \)

2. Mercer operator

\[ T_{F_3} : L^2 (0, 1) \rightarrow L^2 (0, 1) \]

3. \( T_{F_3}^{-1} \) is unbounded, selfadjoint

**Proof.** Same argument as in the proof for \( T_{F_2}^{-1} \); also follows from Mercer’s theorem. \( \Box \)

4. \( T_{F_3}^{-1} \) Friedrichs extension of \( \frac{1}{2} \left( I - \left( \frac{d}{dx} \right)^2 \right) \bigg|_{C^\infty_0(0, 1)} \)

as a selfadjoint operator on \( L^2 (0, 1) \).

**Sketch of proof.**

Setting

\[ u(x) = \int_0^1 \varphi(y) e^{-|x-y|} dy \]

then

\[ u'' = u - 2\varphi \]

\[ \Downarrow \]

\[ u = \left( \frac{1}{2} \left( 1 - \left( \frac{d}{dx} \right)^2 \right) \right)^{-1} \varphi \]

and so

\[ T_{F_3} = \left( \frac{1}{2} \left( I - \left( \frac{d}{dx} \right)^2 \right) \right)^{-1}. \]

Now, a selfadjoint extension

\[ T_{F_3}^{-1} \supset \frac{1}{2} \left( I - \left( \frac{d}{dx} \right)^2 \right) \]

in \( L^2 (0, 1) \).

---

Table 7.1: An overview of two cases: \( F_2 \) v.s. \( F_3 \).
Chapter 8
Comparing the Different RKHSs $\mathcal{H}_F$ and $\mathcal{H}_K$

In the earlier chapters we have studied extension problems for particular locally defined positive definite functions $F$. In each case, the p.d. function $F$ was fixed. Many classes of p.d. functions were studied but not compared. Indeed, the cases we studied have varied between different levels of generality, and varied between cases separated by different technical assumptions. All our results, in turn, have been motivated by explicit applications.

In the present chapter, we turn to a comparison between the results we obtain for pairs of locally defined positive definite functions. We answer such questions as this: Given an open domain $\Omega$ (in an ambient group $G$), and two continuous positive functions $F$ and $K$, both defined on $\Omega^{-1}\Omega$, how do the two associated RKHSs compare? (Theorem 8.1.) How to compare spectral theoretic information for the two? How does the operator theory compare, for the associated pair of skew-Hermitian operators $D(F)$ and $D(K)$? (Theorem 8.2.) What if $F$ is fixed, and $K$ is taken to be the complex conjugate of $F$? Theorem 8.5.) What are the properties of the imaginary part of $F$? (Section 8.4.)

Before we turn to comparison of pairs of RKHSs, we will prove a lemma which accounts for two uniformity principles for positive definite continuous functions defined on open subsets in locally compact groups:

**Lemma 8.1.** Let $G$ be a locally compact group, and let $\Omega \subset G$ be an open and connected subset, $\Omega \neq \emptyset$. Let $F : \Omega^{-1}\Omega \to \mathbb{C}$ be a continuous positive definite function satisfying $F(e) = 1$, where $e \in G$ is the unit for the group operation.

1. Then $F$ extends by limit to a continuous p.d. function $F^{(ex)} : \Omega^{-1}\Omega \to \mathbb{C}$. (8.1)

2. Moreover, the two p.d. functions $F$ and $F^{(ex)}$ have the same RKHS consisting of continuous functions $\xi$ on $\Omega$ such that, $\exists 0 < A < \infty$, $A = A_\xi$, s.t.

$$\left| \int_{\Omega^{-1}\Omega} \xi(x) \varphi(x) \, dx \right|^2 \leq A \| F \varphi \|^2_{H_F}, \forall \varphi \in C_c(\Omega) \quad (8.2)$$
where \( dx = \text{Haar measure on } G \),
\[
F_\phi (\cdot) = \int_\Omega \phi (y) F (y^{-1} \cdot) \, dy, \quad \phi \in C_c (\Omega);
\]
and \( \|F_\phi\|_{\mathcal{H}_\phi} \) denotes the \( \mathcal{H}_\phi \)-norm of \( F_\phi \).

(3) Every \( \xi \in \mathcal{H}_\phi \) satisfies the following a priori estimate:
\[
|\xi (x) - \xi (y)|^2 \leq 2 \|\xi\|_{\mathcal{H}_\phi}^2 (1 - \Re \{ F (x^{-1} y) \})
\]
for all \( \xi \in \mathcal{H}_\phi \), and all \( x, y \in \overline{\Omega} \).

**Proof.** The arguments in the proof only use standard tools from the theory of reproducing kernel Hilbert spaces. We covered a special case in Section 6.1, and so we omit the details here. \( \square \)

Now add the further assumption on the subset \( \Omega \subset G \) from the lemma: Assume in addition that \( \Omega \) has compact closure, so \( \overline{\Omega} \) is compact. Let \( \partial \Omega = \overline{\Omega} \setminus \Omega \) be the boundary. With this assumption, we get automatically the inclusion
\[
C (\overline{\Omega}) \subset L^2 (\overline{\Omega})
\]
since continuous functions on \( \overline{\Omega} \) are automatically uniformly bounded, and Haar measure on \( G \) has the property that \( |\overline{\Omega}| < \infty \).

**Definition 8.1.** Let \( (\Omega, F) \) be as above, i.e.,
\[
F : \overline{\Omega}^{-1} \Omega \rightarrow \mathbb{C}
\]
is continuous and positive definite. Assume \( G \) is a Lie group. Recall, extension by limit to \( \overline{\Omega} \) is automatic by the limit. Let \( \beta \in \mathcal{M} (\partial \Omega) \) be a positive finite measure on the boundary \( \partial \Omega \). We say that \( \beta \) is a boundary measure iff
\[
\langle T_F f, \xi \rangle_{\mathcal{H}_\phi} - \int_\Omega f (x) \xi (x) \, dx = \int_{\partial \Omega} \overline{(T_F f)_n (\sigma)} \xi (\sigma) \, d\beta (\sigma)
\]
holds for all \( f \in C (\overline{\Omega}) \), \( \forall \xi \in \mathcal{H}_\phi \), where \( (\cdot)_n = \text{normal derivative computed on the boundary } \partial \Omega \) of \( \Omega \).

**Remark 8.1.** For the example \( G = \mathbb{R}, \overline{\Omega} = [0, 1], \partial \Omega = \{0, 1\} \), and \( F = F_3 \) on \([0, 1]\), where
\[
F_3 (x) = e^{-|x|}, \quad \forall x \in [-1, 1],
\]
the boundary measure is
\[
\beta = \frac{1}{2} (\delta_0 + \delta_1).
\]
Let \( G \) be a locally compact group with left-Haar measure, and let \( \Omega \subset G \) be a non-empty subset satisfying: \( \Omega \) is open and connected; and is of finite Haar measure;
write \(|\Omega| < \infty\). The Hilbert space \(L^2(\Omega) = L^2(\Omega, dx)\) is the usual \(L^2\)-space of measurable functions \(f\) on \(\Omega\) such that
\[
\|f\|^2_{L^2(\Omega)} := \int_{\Omega} |f(x)|^2 \, dx < \infty.
\] (8.6)

**Definition 8.2.** Let \(F\) and \(K\) be two continuous and positive definite functions defined on
\[
\Omega^{-1} \cdot \Omega := \left\{ x^{-1}y \mid x, y \in \Omega \right\}.
\] (8.7)

We say that \(K \ll F\) if and only if there is a finite constant \(A\) such that for all finite subsets \(\{x_i\}_{i=1}^N \subset \Omega\), and all systems \(\{c_i\}_{i=1}^N \subset \mathbb{C}\), we have:
\[
\sum_i \sum_j c_i c_j K(x_i^{-1} x_j) \leq A \sum_i \sum_j c_i c_j F(x_i^{-1} x_j).
\] (8.8)

**Lemma 8.2.** Let \(F\) and \(K\) be as above; then \(K \ll F\) if and only if there is a finite constant \(A \in \mathbb{R}_+\) such that
\[
\int_{\Omega} \int_{\Omega} \overline{\varphi(x)} \varphi(y) K(x^{-1}y) \, dx \, dy \leq A \int_{\Omega} \int_{\Omega} \overline{\varphi(x)} \varphi(y) F(x^{-1}y) \, dx \, dy
\] (8.9)

holds for all \(\varphi \in C_c(\Omega)\). The constant \(A\) in (8.8) and (8.9) will be the same.

**Proof.** Easy; use an approximate identity in \(G\), see e.g., [Rud73, Rud90].

Setting
\[
F_{\varphi}(x) := \int_{\Omega} \varphi(y) F(y^{-1} x) \, dy,
\] (8.10)
and similarly for \(K_{\varphi} = \int_{\Omega} \varphi(y) K(y^{-1} \cdot) \, dy\), we note that \(K \ll F\) if and only if:
\[
\|K_{\varphi}\|_{\mathcal{H}_K} \leq \sqrt{A} \|F_{\varphi}\|_{\mathcal{H}_F}, \forall \varphi \in C_c(\Omega).
\] (8.11)

Further, note that, if \(G\) is also a Lie group, then (8.11) follows from checking it only for all \(\varphi \in C_c^\infty(G)\). See Lemma 2.12.

**Theorem 8.1.** Let \(\Omega, F\) and \(K\) be as in Definition 8.2, i.e., both continuous and p.d. on the set \(\Omega^{-1} \cdot \Omega\) in (8.7); then the following two conditions are equivalent:

(i) \(K \ll F\), and
(ii) \(\mathcal{H}_K\) is a closed subspace of \(\mathcal{H}_F\).

**Proof.** \(\Downarrow\) Assume \(K \ll F\), we then define a linear operator \(l : \mathcal{H}_F \to \mathcal{H}_K\), setting
\[
l(F_{\varphi}) := K_{\varphi}, \forall \varphi \in C_c(\Omega).
\] (8.12)

We now use (8.11), plus the fact that \(\mathcal{H}_F\) is the \(\|\cdot\|_{\mathcal{H}_F}\)-completion of
\[
\left\{ F_{\varphi} \mid \varphi \in C_c(\Omega) \right\}.
\]
Comparing the Different RKHSs $\mathcal{H}_F$ and $\mathcal{H}_K$

As a result of (8.12) and (8.11), we get a canonical extension of $l$ to a bounded linear operator, also denoted $l : \mathcal{H}_F \to \mathcal{H}_K$, and

$$\|l(\xi)\|_{\mathcal{H}_K} \leq \sqrt{A} \|\xi\|_{\mathcal{H}_F}, \text{ for all } \xi \in \mathcal{H}_F. \tag{8.13}$$

We interrupt the proof to give a lemma:

**Lemma 8.3.** Let $F, K, \Omega$ be as above. Assume $K \ll F$, and let $l : \mathcal{H}_F \to \mathcal{H}_K$ be the bounded operator introduced in (8.12) and (8.13). Then the adjoint operator $l^* : \mathcal{H}_K \to \mathcal{H}_F$ satisfies:

$$(l^*(\xi))(x) = \xi(x), \text{ for all } \xi \in \mathcal{H}_K, x \in \Omega. \tag{8.14}$$

And of course, $l^*(\xi) \in \mathcal{H}_F$.

**Proof.** By the definition of $l^*$, as the adjoint of a bounded linear operator between Hilbert spaces, we get

$$\|l^*\|_{\mathcal{H}_K \to \mathcal{H}_F} = \|l\|_{\mathcal{H}_F \to \mathcal{H}_K} \leq \sqrt{A} \tag{8.15}$$

for the respective operator norms; and

$$\langle l^*(\xi), F \varphi \rangle_{\mathcal{H}_F} = \langle \xi, K \varphi \rangle_{\mathcal{H}_K}, \forall \varphi \in C_c(\Omega). \tag{8.16}$$

Using now the reproducing property in the two RKHSs, we get:

$\text{l.h.s.}\ (8.16) = \int_{\Omega} l^*(\xi)(x) \varphi(x) \, dx, \text{ and}$

$\text{r.h.s.}\ (8.16) = \int_{\Omega} \xi(x) \varphi(x) \, dx, \text{ for all } \varphi \in C_c(\Omega).$

Taking now approximations in $C_c(\Omega)$ to the Dirac masses $\{\delta_x \mid x \in \Omega\}$, the desired conclusion (8.14) follows. \hfill \Box

**Proof.** (of Theorem 8.1 continued.) Assume $K \ll F$, the lemma proves that $\mathcal{H}_K$ identifies with a linear space of continuous functions $\xi$ on $\Omega$, and if $\xi \in \mathcal{H}_K$, then it is also in $\mathcal{H}_F$.

We claim that this is a closed subspace in $\mathcal{H}_F$ relative to the $\mathcal{H}_F$-norm.

**Step 1.** Let $\{\xi_n\} \subset \mathcal{H}_K$ satisfying

$$\lim_{n,m \to \infty} \|\xi_n - \xi_m\|_{\mathcal{H}_F} = 0.$$

By (8.13) and (8.14), the lemma; we get

$$\lim_{n,m \to \infty} \|\xi_n - \xi_m\|_{\mathcal{H}_K} = 0.$$

**Step 2.** Since $\mathcal{H}_K$ is complete, we get $\chi \in \mathcal{H}_K$ such that
Step 3. We claim that this $\mathcal{H}_K$-limit $\chi$ also defines a unique element in $\mathcal{H}_F$, and it is therefore the $\mathcal{H}_F$-limit.

We have for all $\phi \in C_c(\Omega)$:

$$
\left| \int_{\Omega} \overline{\chi(x)} \phi(x) \, dx \right| \leq \|\chi\|_{\mathcal{H}_K} \|K\phi\|_{\mathcal{H}_K} \\
\leq \|\chi\|_{\mathcal{H}_K} \sqrt{A} \|F\phi\|_{\mathcal{H}_F};
$$

and so $\chi \in \mathcal{H}_F$.

We now turn to the converse implication of Theorem 8.1:

Assume $F$ and $K$ are as in the statement of the theorem; and that $\mathcal{H}_K$ is a close subspace in $\mathcal{H}_F$ via identification of the respective continuous functions on $\Omega$. We then prove that $K \ll F$.

Now let $P_K$ denote the orthogonal projection of $\mathcal{H}_F$ onto the closed subspace $\mathcal{H}_K$. We claim that

$$
P_K (F\phi) = K\phi, \ \forall \phi \in C_c(\Omega). \tag{8.18}
$$

Using the uniqueness of the projection $P_K$, we need to verify that $F\phi - K\phi \in \mathcal{H}_F \ominus \mathcal{H}_K$; i.e., that

$$
\langle F\phi - K\phi, \xi_K \rangle_{\mathcal{H}_F} = 0, \text{ for all } \xi_K \in \mathcal{H}_K. \tag{8.19}
$$

But since $\mathcal{H}_K \subset \mathcal{H}_F$, we have

$$
l.h.s. \ (8.19) = \int_{\Omega} \overline{\phi(x)} \xi_K(x) \, dx - \int_{\Omega} \overline{\phi(x)} \xi_K(x) \, dx = 0,
$$

for all $\phi \in C_c(\Omega)$. This proves (8.18).

To verify $K \ll F$, we use the criterion (8.11) from Lemma 8.2. Indeed, consider $K\phi \in \mathcal{H}_K$. Since $\mathcal{H}_K \subset \mathcal{H}_F$, we get

$$
l(F\phi) = P_K (F\phi) = K\phi, \text{ and}

\|K\phi\|_{\mathcal{H}_K} = \|l(F\phi)\|_{\mathcal{H}_F} \leq \sqrt{A} \|F\phi\|_{\mathcal{H}_F},
$$

which is the desired estimate (8.11). \hfill \Box

Example 8.1. To illustrate the conclusion in Theorem 8.1, take

$$
K(x) = \left( \frac{\sin x} {x} \right)^2, \text{ and } F(x) = \frac{1}{1 + x^2};\tag{8.20}
$$

both defined on the fixed interval $-\frac{1}{2} < x < \frac{1}{2}$; and so the interval for $\Omega$ in both cases can be take to be $\Omega = (0, \frac{1}{2})$; compare with $F_i$ and $F_i$ from Table 5.1.

Claim. $K \ll F$, and $\mathcal{H}_K \subseteq \mathcal{H}_F$.  

Proof. Using Table 5.2, we can identify measures $\mu_K \in \text{Ext}(K)$, and $\mu_F \in \text{Ext}(F)$.

With the use of Corollary 2.1, and Corollary 3.1, it is now easy to prove that $K \ll F$; i.e., that condition (i) in Theorem 8.1 is satisfied. Also, recall from Lemma 2.2 that the RKHSs are generated by the respective kernels, i.e., the functions $K$ and $F$:

For $x_0 \in \left[0, \frac{1}{4}\right] = \Omega$, set

$$K_{x_0}(y) = \left(\frac{\sin(y - x_0)}{y - x_0}\right)^2, \quad y \in \Omega.$$  \hspace{1cm} (8.21)

To show directly that $\mathcal{H}_K$ is a subspace of $\mathcal{H}_F$, we use Corollary 2.1: We find a signed measure $v_{x_0} \in \mathcal{M}_2(F, \Omega)$ such that

$$K_{x_0}(\cdot) = (v_{x_0} * F)(\cdot) \text{ on } \Omega \text{ holds.}$$  \hspace{1cm} (8.22)

The r.h.s. in (8.22) is short hand for

$$K_{x_0}(x) = \int_0^\frac{1}{2} \left(\frac{1}{1 + (x-y)^2} dv_{x_0}(y)\right),$$  \hspace{1cm} (8.23)

using Fourier transform, eq (8.23) is easily solved by the signed measure $v_{x_0}$ s.t.

$$\hat{v}_{x_0}(\lambda) = \chi_{[-\frac{1}{2}, \frac{1}{2}]}(\lambda) (1 - 2|\lambda|) e^{i\xi_0 \lambda} d\lambda.$$  \hspace{1cm} (8.24)

Note that the r.h.s. in (8.24) is of compact support; and so an easy Fourier inversion then yields the signed measure $v_{x_0} \in \mathcal{M}_2(F, \Omega)$ which solves eq (8.22). Hence $K_{x_0}(\cdot) \in \mathcal{H}_F$; and so $\mathcal{H}_K$ is a subspace of $\mathcal{H}_F$. □

8.1 Applications

Below we give an application of Theorem 8.1 to the deficiency-index problem, and to the computation of the deficiency spaces; see also Lemma 4.7, and Lemma 10.5.

As above, we will consider two given continuous p.d. functions $F$ and $K$, but the group now is $G = \mathbb{R}$: We pick $a, b \in \mathbb{R}_+, 0 < a < b$, such that $F$ is defined on $(-b, b)$, and $K$ on $(-a, a)$. The corresponding two RKHSs will be denoted $\mathcal{H}_F$ and $\mathcal{H}_K$. We say that $K \ll F$ iff there is a finite constant $A$ such that

$$\|K\varphi\|^2_{\mathcal{H}_K} \leq A \|F\varphi\|^2_{\mathcal{H}_F}$$  \hspace{1cm} (8.25)

for all $\varphi \in C_c(0,a)$. Now this is a slight adaptation of our Definition 8.2 above, but this modification will be needed; for example in computing the indices of two p.d. functions $F_2$ and $F_3$ from Table 5.1; see also Section 7.1 below. In fact, a simple direct checking shows that

$$F_2 \ll F_3 \quad (\text{see Table 5.1}),$$  \hspace{1cm} (8.26)
and we now take \( a = \frac{1}{2}, b = 1 \).

Here, \( F_2(x) = 1 - |x| \) in \(|x| < \frac{1}{2}\); and \( F_3(x) = e^{-|x|} \) in \(|x| < 1\); see Figure 8.1.

![Fig. 8.1: The examples of \( F_2 \) and \( F_3 \).](image)

We wish to compare the respective skew-Hermitian operators, \( D^{(F)} \) in \( \mathcal{H}_F \); and \( D^{(K)} \) in \( \mathcal{H}_K \) (Section 4.1), i.e.,

\[
D^{(F)}(F\phi) = F\phi', \quad \forall \phi \in C_c^{\infty}(0,b); \quad \text{and} \quad (8.27)
\]

\[
D^{(K)}(K\phi) = K\phi', \quad \forall \phi \in C_c^{\infty}(0,a). \quad (8.28)
\]

Let \( z \in \mathbb{C} \); and we set

\[
EF_F(z) = \{ \xi \in \text{dom}(D^{(F)}^*) \mid (D^{(F)}^*)^\ast \xi = z\xi \}, \quad \text{and} \quad (8.29)
\]

\[
EF_K(z) = \{ \xi \in \text{dom}(D^{(K)}^*) \mid (D^{(K)}^*)^\ast \xi = z\xi \}. \quad (8.30)
\]

**Theorem 8.2.** Let two continuous p.d. functions \( F \) and \( K \) be specified as above, and suppose \( K \ll F; \) (8.31) then

\[
EF_K(z) = EF_F(z) \big|_{(0,a)} \quad (8.32)
\]

i.e., restriction to the smaller interval.

**Proof.** Since (8.31) is assumed, it follows from Theorem 8.1, that \( \mathcal{H}_K \) is a subspace of \( \mathcal{H}_F \).

If \( \phi \in C_c^{\infty}(0,b), \) and \( \xi \in \text{dom}(D^{(F)}^*) \), then

\[
\langle (D^{(F)}^*)^\ast \xi, F\phi \rangle_{\mathcal{H}_F} = \langle \xi, F\phi' \rangle_{\mathcal{H}_F} = \int_0^b \overline{\xi(x)} \phi'(x) \, dx;
\]

and it follows that functions \( \xi \) in \( EF_F(z) \) must be multiples of
Comparing the Different RKHSs $\mathcal{H}_F$ and $\mathcal{H}_K$

$$(0,h) \ni x \mapsto e_z(x) = e^{-zx}. \quad (8.33)$$

Hence, by Theorem 8.1, we get

$$\text{DEF}_K(z) \subseteq \text{DEF}_F(z),$$

and by $(8.33)$, we see that $(8.32)$ must hold.

Conversely, if $\text{DEF}_F(z) \neq 0$, then $I(\text{DEF}_F(z)) \neq 0$, and its restriction to $(0,a)$ is contained in $\text{DEF}_K(z)$. The conclusion in the theorem follows. \(\square\)

**Remark 8.2.** The spaces $\text{DEF}_F(z), z \in \mathbb{C}$, are also discussed in Theorem 2.8.

**Example 8.2 (Application).** Consider the two functions $F_2$ and $F_3$ in Table 5.1. Both of the operators $D^{(F_i)}, i = 2, 3,$ have deficiency indices $(1, 1)$.

**Proof.** One easily checks that $F_2 \ll F_3$. And it is also easy to check directly that $D^{(F_2)}$ has indices $(1, 1)$. Hence, by $(8.32)$ in the theorem, it follows that $D^{(F_3)}$ also must have indices $(1, 1)$. (The latter conclusion is not as easy to verify by direct means!) \(\square\)

### 8.2 Radially Symmetric Positive Definite Functions

Among other subclasses of positive definite functions we have radially symmetric p.d. functions. If a given p.d. function happens to be radially symmetric, then there are a number of simplifications available, and the analysis in higher dimension often simplifies. This is to a large extend due to theorems of I. J. Schöenberg and D. V. Widder. Below we sketch two highpoints, but we omit details and application to interpolation and to geometry. These themes are in the literature, see e.g. [Sch38c, SW53, Sch64, Wid41, WW75].

**Remark 8.3.** In some cases, the analysis in one dimension yields insight into the possibilities in $\mathbb{R}^n, n > 1$. This leads for example for functions $F$ on $\mathbb{R}^n$ which are radial, i.e., of the form $F(x) = \Phi(\|x\|^2)$, where $\|x\|^2 = \sum_{i=1}^{n} x_i^2$.

A function $q$ on $\mathbb{R}_+, q : \mathbb{R}_+ \to \mathbb{R}$, is said to be completely monotone iff $q \in \mathcal{C}([0, \infty)) \cap \mathcal{C}^\infty((0, \infty))$ and

$$(-1)^n q^{(n)}(r) \geq 0, \quad r \in \mathbb{R}_+, n \in \mathbb{N}_0. \quad (8.34)$$

**Example 8.3.**

$$q(r) = e^{-ar}, \quad \alpha \geq 0;$$

$$q(r) = \frac{\alpha}{r^{1-a}}, \quad \alpha \leq 1;$$

$$q(r) = \frac{1}{(r + \alpha^2)^\beta}, \quad \alpha > 0, \beta \geq 0.$$
8.2 Radially Symmetric Positive Definite Functions

Theorem 8.3 (Schöenberg (1938)). A function \( q : \mathbb{R}_+ \to \mathbb{R} \) is completely monotone iff the corresponding function \( F_q(x) = q(\|x\|^2) \) is positive definite and radial on \( \mathbb{R}^n \) for all \( k \in \mathbb{N} \).

Proof. See, e.g., [BCR84, Sch38a, Sch38b]. We omit details, but the proof uses.

Theorem 8.4 (Bernstein-Widder). A function \( q : \mathbb{R}_+ \to \mathbb{R} \) is completely monotone iff there is a finite positive Borel measure on \( \mathbb{R}_+ \) s.t. \( q(r) = \hat{\mu}(r) \) for \( r \in \mathbb{R}_+ \), i.e., \( q \) is the Laplace transform of a finite positive measure \( \mu \) on \( \mathbb{R}_+ \).

Proof. See, e.g., [BCR84, Wid41, Ber46, Ber29, Wid64, BW40]. □

Remark 8.4. The condition that the function \( q \) in (8.34) be in \( C^\infty(\mathbb{R}_+) \) may be relaxed; and then (8.34) takes the following alternative form:

\[
\sum_{k=1}^{n} (-1)^k \binom{n}{k} q(r + k\delta) \geq 0
\]

for all \( n \in \mathbb{N} \), all \( \delta > 0 \), and \( x \in [0, \infty) \); i.e.,

\[
q(r) - q(r + \delta) \geq 0
\]

\[
q(r) - 2q(r + \delta) + q(r + 2\delta) \geq 0 \text{ e.t.c.}
\]

It is immediate that every completely monotone function \( q \) on \([0, \infty)\) is convex.

Remark 8.5. A related class of functions \( q : \mathbb{R}_+ \to \mathbb{R} \) is called \( \Phi_n \); the functions \( q \) such that \( \mathbb{R}^n \ni x \mapsto q(\|x\|) \) is positive definite on \( \mathbb{R}^n \). Schönberg [Sch38a] showed that \( q \in \Phi_n \) if and only if there is a finite positive Borel measure \( \mu \) on \( \mathbb{R}_+ \) such that

\[
\Omega_n(s) = \Gamma \left( \frac{n}{2} \right) \left( \frac{2}{\lambda} \right)^{\frac{n-1}{2}} J_{\frac{n-1}{2}}(s) = \sum_{j=0}^{\infty} \left( -\frac{s^2}{4} \right)^j \frac{\Gamma \left( \frac{n}{2} \right)}{j! \Gamma \left( \frac{n}{2} + j \right)}
\]

Indeed, if \( \nu_n \) denotes the normalized uniform measure on \( S := \{ x \in \mathbb{R}^n \mid \|x\| = 1 \} \), then

\[
\Omega_n(\|x\|) = \int_{S} e^{ix \cdot y} d\nu_n(y), \quad \forall x \in \mathbb{R}^n.
\]
8.3 Connecting $F$ and $\overline{F}$ When $F$ is a Positive Definite Function

Let $F : (-1, 1) \to \mathbb{C}$ be continuous and positive definite, and let $\overline{F}$ be the complex conjugate, i.e., $\overline{F}(x) = F(-x)$, $\forall x \in (-1, 1)$. Below, we construct a contractive-linking operator $\mathcal{H}_F \to \mathcal{H}_{\overline{F}}$ between the two RKHSs.

**Lemma 8.4.** Let $\mu$ and $\mu^{(s)}$ be as before, $\mu^{(s)} = \mu \circ s$, $s(x) = -x$; and set

$$ g = \sqrt{\frac{d\mu^{(s)}}{d\mu}}; \quad (8.37) $$

(the square root of the Radon-Nikodym derivative) then the $g$-multiplication operator is isometric between the respective Hilbert spaces: $L^2\left(\mu^{(s)}\right)$ and $L^2(\mu)$ as follows:

$$ L^2\left(\mathbb{R}, \mu^{(s)}\right) \xrightarrow{M_g \circ \overline{\lambda} = \lambda g} L^2(\mathbb{R}, \mu) . \quad (8.38) $$

**Proof.** Let $h \in L^2\left(\mu^{(s)}\right)$, then

$$ \int_{\mathbb{R}} |gh|^2 \, d\mu = \int_{\mathbb{R}} |h|^2 \frac{d\mu^{(s)}}{d\mu} \, d\mu = \int_{\mathbb{R}} |h|^2 \, d\mu^{(s)} = \|h\|^2_{L^2(\mu^{(s)})}. $$

□

**Lemma 8.5.** If $F : (-1, 1) \to \mathbb{C}$ is a given continuous p.d. function, and if $\mu \in \text{Ext}(F)$, then

$$ \mathcal{H}_F \ni F\phi \xrightarrow{V(F)} \widehat{\phi} \in L^2(\mathbb{R}, \mu) \quad (8.39) $$

extends by closure to an isometry.

**Proof.** For $\phi \in C_c((0,1)$, we have:

$$ \|F\phi\|_{\mathcal{H}_F}^2 = \int_0^1 \int_0^1 \overline{\phi(x)} \phi(y) F(x-y) \, dx \, dy $$

$$ = \int_{\text{Ext}(F)} \int_0^1 \int_0^1 \overline{\phi(x)} \phi(y) \left( \int_{\mathbb{R}} e^{i(x-y)\lambda} \, d\mu(\lambda) \right) \, dx \, dy $$

$$ = \left(\text{Fubini}\right) \int_{\mathbb{R}} |\phi(\lambda)|^2 \, d\mu(\lambda) $$

$$ = (8.39) \left\| V(F) (F\phi) \right\|_{L^2(\mathbb{R}, \mu)}^2. $$

□
8.3 Connecting $F$ and $\mathcal{F}$ When $F$ is a Positive Definite Function

**Definition 8.3.** Set

$$ (\varphi \ast g^\vee)(x) := \int_0^1 \varphi(x) g^\vee(x-y) \, dy \quad (8.40) $$

$$ = (g^\vee \ast \varphi)(x), \; x \in (0,1), \varphi \in C_c(0,1). $$

**Theorem 8.5.** We have

$$ \left( (V^{(F)})^* M_g V^{(F)} \right)(F \varphi) = T_F \left( g^\vee \ast \varphi \right), \; \forall \varphi \in C_c(0,1). \quad (8.41) $$

**Proof.** Let $\varphi \in C_c(0,1)$, we will then compute the two sides in (8.41), where $g^\vee :=$ inverse Fourier transform:

$$ l.h.s. \quad (8.41) = V^{(F)^*} (g\hat{\varphi}); $$

$(\hat{\varphi} \in L^2(\mu^{(s)}))$, and using that $g\hat{\varphi} \in L^2(\mu)$ by lemma 8.4 we get:

$$ l.h.s. \quad (8.41) = \left( (V^{(F)})^* \left( g^\vee \ast \varphi \right) \right)_{L^2(\mu)} $$

$$ = T_F \left( g^\vee \ast \varphi \right) $$

where $T_F$ is the Mercer operator $T_F : L^2(\Omega) \rightarrow \mathcal{H}_F$ defined using

$$ T_F (\varphi)(x) = \int_0^1 \varphi(x) F(x-y) \, dy $$

$$ = \chi_{[0,1]}(x) (\hat{g} \, d\mu)^\vee(x). $$

$\square$

**Corollary 8.1.** Let $\mu$ and $\mu^{(s)}$ be as above, with $\mu \in Ext(F)$, and $\mu^{(s)} \ll \mu$. Setting $g = \sqrt{\frac{d\mu^{(s)}}{d\mu}}$, we get

$$ \| (V^{(F)^*} M_g V^{(F)}) \|_{\mathcal{H}_F \rightarrow \mathcal{H}_F} \leq 1. \quad (8.42) $$

**Proof.** For the three factors in the composite operator $\left( (V^{(F)})^* M_g V^{(F)} \right)$ in (8.42), we have two isometries as follows:

$$ V^{(F)} : \mathcal{H}_F \rightarrow L^2(\mu^{(s)}), \text{ and} $$

$$ M_g : L^2(\mu^{(s)}) \rightarrow L^2(\mu), $$

and both isometries; while

$$ (V^{(F)^*} : L^2(\mu) \rightarrow \mathcal{H}_F $$

is co-isometric, and therefore contractive, i.e.,
\[ \|(V(F))^*\|_{L^2(\mu)\to \mathcal{H}_F} \leq 1. \quad (8.43) \]

But then:
\[ \left\| (V(F))^* M \right\|_{\mathcal{H}_F \to \mathcal{H}_F} \leq \left\| (V(F))^* \right\| \left\| M \right\|_{\mathcal{H}_F} \]
\[ = \left\| (V(F))^* \right\| \leq 1, \text{ by (8.43)}. \]

\[ \Box \]

8.4 The Imaginary Part of a Positive Definite Function

**Lemma 8.6.** Let \( F : (-1, 1) \to \mathbb{C} \) be a continuous p.d. function. For \( \phi \in C_c^\infty(0,1) \) let
\[ (t(\phi))(x) = \phi(1-x), \quad \text{for all } x \in (0,1). \]

The operator \( F_\phi \to F(t(\phi)) \) is bounded in \( \mathcal{H}_F \) iff
\[ F \ll F \]
where \( F \) is the complex conjugate of \( F \), and \( \ll \) is the order on p.d. functions, i.e., there is an \( A < \infty \) such that
\[ \sum \sum c_j c_k F(x_j - x_k) \leq A \sum \sum c_j c_k F(x_j - x_k), \quad (8.45) \]
for all finite systems \( \{ c_j \} \) and \( \{ x_j \} \), where \( c_j \in \mathbb{C}, x_j \in (0,1) \).

**Proof.** It follows from (8.45) that \( F \ll F \) iff there is an \( A < \infty \) such that
\[ \|F_\phi\|_{\mathcal{H}_F} \leq \sqrt{A} \|F_\phi\|_{\mathcal{H}_F}, \quad (8.46) \]
for all \( \phi \) in \( C_c^\infty(0,1) \). Since
\[ \|F(t(\phi))\|^2_{\mathcal{H}_F} = \int_0^1 \int_0^1 \overline{\phi(1-x)} \phi(1-y) F(x-y) dx dy \]
\[ = \int_0^1 \int_0^1 \overline{\phi(x)} \phi(y) F(y-x) dx dy \]
\[ = \|F_\phi\|^2_{\mathcal{H}_F} \quad (8.47) \]
we have established the claim. \( \Box \)

Let \( M = (M_{jk}) \) be an \( N \times N \) matrix over \( \mathbb{C} \). Set
\[ \mathcal{R} \{ M \} = (\mathcal{R} \{ M_{jk} \}), \quad \mathcal{I} \{ M \} = (\mathcal{I} \{ M_{jk} \}). \]
Assume $M^* = M$, where $M^*$ is the conjugate transpose of $M$, and $M \geq 0$. Recall, $M \geq 0$ iff all eigenvalues of $M$ are $\geq 0$ iff all sub-determinants $\det M_n \geq 0$, $n = 1, \ldots, N$, where $M_n = (M_{jk})_{j,k\leq n}$.

**Definition 8.4.** Let $s(x) = -x$. For a measure $\mu$ on $\mathbb{R}$, let $\mu^s = \mu \circ s$.

**Lemma 8.7.** If $F = \hat{\mu}$ then $F^* = \hat{\mu^s}$.

**Proof.** Suppose $F = \hat{\mu}$, then the calculation

$$
F(x) = F(-x) = \int_{\mathbb{R}} e^\lambda (-x) d\mu(\lambda) \\
= \int_{\mathbb{R}} e^{-\lambda}(x) d\mu(\lambda) \\
= \int_{\mathbb{R}} e^\lambda (x) d\mu^s(\lambda) = \hat{\mu^s}(x)
$$

establishes the claim. □

**Corollary 8.2.** If $F = \hat{\mu}$, then (8.46) takes the form

$$
\left| \hat{\phi}(\lambda) \right|^2 d\mu^s(\lambda) \leq A \int_{\mathbb{R}} \left| \hat{\phi}(\lambda) \right|^2 d\mu(\lambda),
$$

for all $\phi$ in $C^\infty_c(0,1)$.

**Proof.** A calculation show that

$$
\|F_\phi\|_{\mathcal{H}_F}^2 = \int_{\mathbb{R}} \left| \hat{\phi}(\lambda) \right|^2 d\mu(\lambda)
$$

and similarly $\|F^*_\phi\|_{\mathcal{H}_F}^2 = \int_{\mathbb{R}} \left| \hat{\phi}(\lambda) \right|^2 d\mu^s(\lambda)$, where we used Lemma 8.7. □

**Example 8.4.** If $\mu = \frac{1}{2} (\delta_{-1} + \delta_2)$, then $\mu^s = \frac{1}{2} (\delta_{-1} + \delta_2)$. Set

$$
F(x) = \hat{\mu}(x) = \frac{1}{2} \left( e^{-ix} + e^{ix^2} \right),
$$

then

$$
F(x) = \hat{\mu^s}(x) = \frac{1}{2} \left( e^{ix} + e^{-ix^2} \right).
$$

It follows from Corollary 8.2 and Lemma 8.6 that $\mathcal{F} \nless F$ and $F \nless \mathcal{F}$. In fact,

$$
\|F_\phi\|_{\mathcal{H}_F}^2 = \frac{1}{2} \left( \left| \hat{\phi}(-1) \right|^2 + \left| \hat{\phi}(2) \right|^2 \right) \\
\|F^*_\phi\|_{\mathcal{H}_F}^2 = \frac{1}{2} \left( \left| \hat{\phi}(1) \right|^2 + \left| \hat{\phi}(-2) \right|^2 \right).
$$

Fix $f \in C^\infty_c$, such that $f(0) = 1, f \geq 0$, and $\int f = 1$. Considering $\phi_n(x) = \frac{1}{2} \left( e^{-ix} + e^{ix^2} \right) f \left( \frac{x}{n} \right)$, and $\psi_n(x) = \frac{1}{2} \left( e^{ix} + e^{-ix^2} \right) f \left( \frac{x}{n} \right)$, completes the verification, since $\phi_n \to \mu$ and $\psi_n \to \mu^s$. 

And similarly, $F \not\ll F$ and $F \not\ll \mathcal{F}$, where $F$ is as in Example 8.5.

**Remark 8.6.** In fact, $F \ll F \iff \mu^s \ll \mu$ with Radon-Nikodym derivative $\frac{d\mu^s}{d\mu} \in L^\infty(\mu)$. See, Section 8.3.

**Corollary 8.3.** If $F = \hat{d}\mu$, then

$$
\Re \{F\} = \frac{1}{2}(\mu + \mu^s), \text{ and }
\Im \{F\} = \frac{1}{2i}(\mu - \mu^s).
$$

We can rewrite the corollary in the form: If $F = \hat{d}\mu$, then

$$
\Re \{F\}(x) = \int_{\mathbb{R}} \cos(\lambda x) \, d\mu(\lambda), \text{ and } (8.48)
$$

$$
\Im \{F\}(x) = \int_{\mathbb{R}} \sin(\lambda x) \, d\mu(\lambda). \quad (8.49)
$$

**Remark 8.7.** $(8.48)$ simply states that if $F$ is positive definite, so is its real part $\Re \{F\}$. But $(8.49)$ is deeper: If the function $\lambda$ is in $L^1(\mu)$, then

$$
\frac{d}{dx} \Im \{F\}(x) = \int_{\mathbb{R}} \cos(\lambda x)\lambda \, d\mu(\lambda)
$$

is the cosine transform of $\lambda \, d\mu(\lambda)$.

Suppose $F$ is p.d. on $(-a,a)$ and $\mu \in \text{Ext}(F)$, i.e., $\mu$ is a finite positive measure satisfying

$$
F(x) = \int_{\mathbb{R}} e^\lambda(x) \, d\mu(x).
$$

For a finite set $\{x_j\} \in (-a,a)$ let

$$
M := (F(x_j-x_k)).
$$

For $e_j$ in $\mathbb{C}$ consider

$$
\overline{e^T}Mc = \sum_j \sum_k \overline{e_j}c_kM_{jk}.
$$

The for $\Re \{F\}$, we have

$$
\sum_j \sum_k \overline{e_j}c_k \Re \{F\}(x_j-x_k)
= \sum_j \sum_k \overline{e_j}c_k \int_{\mathbb{R}} (\cos(\lambda x_j) \cos(\lambda x_k) + \sin(\lambda x_j) \sin(\lambda x_k)) \, d\mu(\lambda)
= \int_{\mathbb{R}} \left( |C(\lambda)|^2 + |S(\lambda)|^2 \right) \, d\mu(\lambda) \geq 0,
$$

where
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\[ C(\lambda) = C(\lambda, (x_j)) = \sum_j c_j \cos (\lambda x_j) \]
\[ S(\lambda) = S(\lambda, (x_j)) = \sum_j c_j \sin (\lambda x_j) \]
for all \( \lambda \in \mathbb{R} \). Similarly, for \( \Im \{ F \} \), we have

\[ \sum_j \sum_k c_j c_k \Im \{ F \}(x_j - x_k) \]
\[ = \sum_j \sum_k c_j c_k \int_{\mathbb{R}} (\sin (\lambda x_j) \cos (\lambda x_k) - \cos (\lambda x_j) \sin (\lambda x_k)) d\mu(\lambda) \]
\[ = \int_{\mathbb{R}} \left( \overline{S(\lambda)C(\lambda)} - \overline{C(\lambda)S(\lambda)} \right) d\mu(\lambda) \]
\[ = 2i \int_{\mathbb{R}} \mathcal{A} \left( \overline{S(\lambda)C(\lambda)} \right) d\mu(\lambda). \]

If \( \{ c_j \} \subset \mathbb{R} \), then \( S(\lambda), C(\lambda) \) are real-valued and

\[ \sum_j \sum_k c_j c_k \Im \{ F \}(x_j - x_k) = 0. \]

8.4.1 Connections to, and applications of, Bochner’s Theorem

In this section, we study complex valued positive definite functions \( F \), locally defined, so on a fixed finite interval \((-a,a)\). The purpose of the present section is to show how the real and the imaginary parts of \( F \) are related, when studied as individual functions on \((-a,a)\).

**Lemma 8.8.** Let \( F \) be a continuous positive definite function on some open interval \((-a,a)\). Let \( K \) be the real part \( \Re \{ F \} \) of \( F \) and let \( L \) be the imaginary part \( \Im \{ F \} \) of \( F \), hence \( K \) and \( L \) are real-valued, \( K \) is a continuous positive definite real-valued function, in particular \( K \) is an even function, and \( L \) is an odd function.

**Proof.** The even/odd claims follow from \( F(-x) = \overline{F(x)} \) for \( x \in (-a,a) \). For a finite set of points \( \{ x_j \}_{j=1}^N \) in \((-a,a)\) form the matrices

\[ M_F = (F(x_j - x_k))_{j,k=1}^N, M_K = (K(x_j - x_k))_{j,k=1}^N M_L = (L(x_j - x_k))_{j,k=1}^N. \]

Let \( c = (c_j) \) be a vector in \( \mathbb{R}^N \). Since \( L \) is an odd function it follows that \( c^T M_L c = 0 \), consequently,

\[ c^T M_K c = c^T M_F c \geq 0. \quad (8.50) \]

It follows that \( K \) is positive definite over the real numbers and therefore also over the complex numbers \([\text{Aro50}]) \).
Definition 8.5. We say a signed measure \( \mu \) is even, if \( \mu(B) = \mu(-B) \) for all \( \mu \)-measurable sets \( B \), where \( -B = \{ -x : x \in B \} \). Similarly, we say \( \mu \) is odd, if \( \mu(B) = -\mu(-B) \) for all \( \mu \)-measurable sets \( B \).

Remark 8.8. Let

\[
\mu_K(B) := \frac{\mu(B) + \mu(-B)}{2} \quad \text{and} \quad \mu_L(B) := \frac{\mu(B) - \mu(-B)}{2}
\]

for all \( \mu \)-measurable sets \( B \). Then \( \mu_K \) is an even probability measure and \( \mu_L \) is an odd real-valued measure. If \( F = \hat{d} \mu \), \( K = \hat{d} \mu_K \), and \( iL = \hat{d} \mu_L \), then \( K \) and \( L \) are real-valued continuous functions, \( F \) and \( K \) are continuous positive definite functions, \( L \) is a real-valued continuous odd function and \( F = K + iL \).

Lemma 8.9. Suppose \( K \) as the Fourier transform of some even probability measure \( \mu_K \) and \( iL \) as the Fourier transform of some odd measure \( \mu_L \), then \( F := K + iL \) is positive definite iff \( \mu := \mu_K + \mu_L \) is a probability measure, i.e., iff \( \mu(B) \geq 0 \) for all Borel set \( B \).

Proof. This is a direct consequence of Bochner’s theorem. See e.g., [BC49, BC48, Boc47, Boc46].

Corollary 8.4. If \( F \) is positive definite, and \( \Im \{ F \} \neq 0 \), then

(i) \( F_m := \Re \{ F \} + \Im \{ F \} \), is positive definite for all \( -1 \leq m \leq 1 \) and

(ii) \( F_m \) is not positive definite for sufficiently large \( m \).

Proof. (i) We will use the notation from Remark 8.8. If \( 0 < m \) and \( \mu_L(B) < 0 \), then

\[
\mu_m(B) := \mu_K(B) + m \mu_L(B) \geq \mu(B) \geq 0.
\]

The cases where \( m < 0 \) are handled by using that \( F \) is positive definite.

(ii) Is established using a similar argument.

Corollary 8.5. Let \( K \) and \( L \) be real-valued continuous functions on \( \mathbb{R} \). Suppose \( K \) positive definite and \( L \) odd, and let \( \mu_K \) and \( \mu_L \) be the correspond even and odd measures. If \( K + iL \) is positive definite for some real \( m \neq 0 \), then the support of \( \mu_L \) is a subset of the support of \( \mu_K \).

Proof. Fix \( m \neq 0 \). If the support of \( \mu_L \) is not contained in the support of \( \mu_K \), then \( \mu_K(B) + m \mu_L(B) < 0 \) for some \( B \).

Remark 8.9. The converse fails, support containment does not imply \( \mu_K + m \mu_L \) is positive for some \( m > 0 \) since \( \mu_K \) can “decrease” much faster than \( \mu_L \).

Example 8.5. Let \( d \mu(\lambda) := \delta_{-1} + \chi_{\mathbb{R}^+}(\lambda) e^{-\lambda} d\lambda \) and set \( F := \hat{d} \mu \bigg|_{(-1,1)} \). Then
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$$\Re \{F\}(x) = \cos(x) + \frac{1}{1+x^2}$$

$$\Im \{F\}(x) = -\sin(x) + \frac{x}{1+x^2};$$

and $D^F$ in $\mathcal{H}_F$ has deficiency indices $(0,0)$.

**Proof.** By construction:

$$F(x) = \int e^{i\lambda x} \, d\mu(\lambda) = e^{-ix} + \int_0^\infty e^{\lambda x} - \lambda \, d\lambda$$

establishing the first claim.

Consider $u = T_F \phi$, for some $\phi \in C_c^\infty(0,1)$. By (8.51)

$$u(x) = \widehat{\phi}(-1)e^{-ix} + \int_0^1 \phi(y) \frac{1}{1-i(x-y)} \, dy.$$ 

Taking two derivatives we get

$$u''(x) = -\widehat{\phi}(-1)e^{-ix} + \int_0^1 \phi(y) \frac{-2}{(1-i(x-y))^3} \, dy.$$ 

It follows that $u'' + u \to 0$ as $x \to \pm\infty$, i.e.,

$$\lim_{|x| \to \infty} |u''(x) + u(x)| = 0. \quad (8.52)$$

A standard approximation argument shows that (8.52) holds for all $u \in \mathcal{H}_F$.

Equation (8.52) rules out that either of $e^{\pm x}$ is on $\mathcal{H}_F$, hence the deficiency indices are $(0,0)$ as claimed. \qed

If $\mu_K$ is an even probability measure and $f(x)$ is an odd function, s.t. $-1 \leq f(x) \leq 1$, then $d\mu(x) := (1 + f(x)) \, d\mu_K(x)$ is a probability measure. Conversely, we have

**Lemma 8.10.** Let $\mu$ be a probability measure on the Borel sets. There is an even probability measure $\mu_K$ and an odd real-valued $\mu-$measurable function $f$ with $|f| \leq 1$, such that $d\mu(\lambda) = (1 + f(\lambda)) \, d\mu_K(\lambda)$.

**Proof.** Let $\mu_K := \frac{1}{2}(\mu + \mu^*)$ and $\mu_L := \frac{1}{2}(\mu - \mu^*)$. Clearly, $\mu_K$ is an even probability measure and $\mu_L$ is an odd real-valued measure. Since $\mu_K(B) + \mu_L(B) = \mu(B) \geq 0$, it follows that $\mu_K(B) \geq \mu_L(B)$ for all Borel sets $B$.

Applying the Hahn decomposition theorem to $\mu_L$ we get sets $P$ and $N$ such that $P \cap N = \emptyset$, $\mu_L(B \cap P) \geq 0$ and $\mu_L(B \cap N) \leq 0$ for all $B$. Let
Comparing the Different RKHSs \( \mathcal{H}_F \) and \( \mathcal{H}_K \)

\[
P' := \{ x \in P : -x \in N \}
\]
\[
N' := \{ x \in N : -x \in P \}
\]
\[
O' := (P \setminus P') \cup (N \setminus N'),
\]
then \( N' = -P' \) and \( \mu_L(B \cap O') = 0 \) for all \( B \). Write

\[
\mu_L(B) = \mu_L(B \cap P') + \mu_L(B \cap N').
\]

Then \( \mu_K(B) \geq \mu_K(B \cap P') \geq \mu_L(B \cap P') \) and

\[
0 \leq -\mu_L(B \cap N') = \mu_L(- (B \cap N'))
= \mu_L(-B \cap P')
\leq \mu_K(-B \cap P') \leq \mu_K(B).
\]

Hence, \( \mu_L \) is absolutely continuous with respect to \( \mu_K \). Setting \( f := \frac{d\mu_L}{d\mu_K} \), the Radon-Nikodym derivative of \( \mu_L \) with respect to \( \mu_K \), completes the proof. \( \square \)

**Corollary 8.6.** Let \( F = \hat{\mu} \) be a positive definite function with \( F(0) = 1 \). Let \( \mu_K := \frac{1}{2}(\mu + \mu^t) \) then \( \Re\{F\}(x) = \hat{\mu}_K(x) \) and there is an odd function

\[-1 \leq f(\lambda) \leq 1,\]

such that \( \Im\{F\}(x) = \hat{f}\mu_K(x) \).

**Corollary 8.7.** Let \( F \) be a continuous p.d. function on \((-a, a)\). Let \( \Re\{F\} \) be the real part of \( F \). Then \( \mathcal{H}_F \) is a subset of \( \mathcal{H}_{\Re\{F\}} \). In particular, if \( D(\Re\{F\}) \) has deficiency indices \((1, 1)\) so does \( D(F) \).

**Proof.** Recall, a continuous function \( \xi \) is in \( \mathcal{H}_F \) iff

\[
\left| \int_0^1 \psi(y)\xi(y)dy \right|^2 \leq A \int_0^1 \int_0^1 \phi(x)\phi(y)F(x-y)dxdy.
\]

Since,

\[
\int_0^1 \int_0^1 \phi(x)\phi(y)F(x-y)dxdy = \int_0^1 \int_0^1 \phi(x)\phi(y)e^{-i\lambda(x-y)}d\mu(\lambda)dxdy
= \int_\mathbb{R} |\phi(\lambda)|^2 d\mu(\lambda)
\leq 2 \int_\mathbb{R} |\phi(\lambda)|^2 d\mu_K(\lambda)
= 2 \int_0^1 \int_0^1 \phi(x)\phi(y)K(x-y)dxdy
\]

it follows that \( \mathcal{H}_F \) is contained in \( \mathcal{H}_{\Re\{F\}} \). \( \square \)
A source of interesting measures in probability are constructed as product measures or convolutions; and this includes infinite operations; see for example [IM65, Jor07, KS02, Par09].

Below we study these operations in the contest of our positive definite functions, defined on subsets of groups. For example, most realizations of fractal measures arise as infinite convolutions, see e.g., [DJ10, JP12, JKS12, DJ12, JKS11, JKS08]. Motivated by these applications, we show below that, given a system of continuous positive definite functions $F_1, F_2, \ldots$, defined on an open subset of a group, we can form well defined products, including infinite products, which are again continuous positive definite. We further show that if finite positive measures $\mu_i, i = 1, 2, \ldots$, are given, $\mu_i \in \Ext(F_i)$ then the convolution of the measures $\mu_i$ is in $\Ext(F)$ where $F$ is the product of the p.d. functions $F_i$. This will be applied later in the note.

**Definition 9.1.** Let $F$ be a continuous positive definite function defined on a subset in $G$ (a locally compact Abelian group). Set

$$\Ext(F) = \left\{ \mu \in \mathcal{M}(\hat{G}) \mid \hat{d}\mu \text{ is an extension of } F \right\}.$$  \hfill (9.1)

In order to study the set $\Ext(F)$ from above, it helps to develop tools. One such tool is convolution, which we outline below. It is also helpful in connection with the study of symmetric spaces, such as the case $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ (the circle group), versus extensions to the group $\mathbb{R}$.

Let $G$ be a locally compact group, and let $\Omega$ be a non-empty, connected and open subset in $G$. Now consider systems of p.d. and continuous functions on the set $\Omega^{-1}\Omega$. Specifically, let $F_i$ be two or more p.d. continuous functions on $\Omega^{-1}\Omega$; possibly an infinite family, so $F_1, F_2, \ldots$, all defined on $\Omega^{-1}\Omega$. As usual, we normalize our p.d. functions $F_i(e) = 1$, where $e$ is the unit element in $G$.

**Lemma 9.1.** Form the point-wise product $F$ of any system of p.d. functions $F_i$ on $\Omega^{-1}\Omega$; then $F$ is again p.d. and continuous on the set $\Omega^{-1}\Omega$. 

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Proof. This is an application of a standard lemma about p.d. kernels, see e.g., [BCR84]. From this, we conclude that $F$ is again a continuous and positive definite function on $\Omega^{-1}\Omega$. \hfill \Box

If we further assume that $G$ is also Abelian, and so $G$ is locally compact Abelian, then the spectral theory takes a more explicit form.

**Lemma 9.2.** Assume $\text{Ext}(F_i)$ for $i = 1, 2, \ldots$ are non-empty. For any system of individual measures $\mu_i \in \text{Ext}(F_i)$ we get that the resulting convolution-product measure $\mu$ formed from the factors $\mu_i$ by the convolution in $G$, is in $\text{Ext}(F)$.

**Proof.** This is an application of our results in Sections 3.1-4.1. \hfill \Box

**Remark 9.1.** In some applications the convolution $\mu_1 * \mu_2$ makes sense even if only one of the measures is finite.

**Application.** The case $G = \mathbb{R}$. Let $\mu_1$ be the Dirac-comb ([Ric03, Cór89])

$$d\mu_1 := \sum_{n \in \mathbb{Z}} \delta(\lambda - n), \ \lambda \in \mathbb{R};$$

let $\Phi \geq 0, \Phi \in L^1(\mathbb{R})$, and assume $\int_\mathbb{R} \Phi(\lambda) \, d\lambda = 1$. Set $d\mu_2 = \Phi(\lambda) \, d\lambda$, where $d\lambda$ = Lebesgue measure on $\mathbb{R}$; then $\mu_1 * \mu_2$ yields the following probability measure on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$: Set

$$\Phi_{\text{per}}(\lambda) = \sum_{n \in \mathbb{Z}} \Phi(\lambda - n);$$

then $\Phi_{\text{per}}(\lambda) \in L^1(\mathbb{T}, dt)$, where $dt = \text{Lebesgue measure on} \ \mathbb{T}$, i.e., if $f(\lambda + n) = f(n), \forall n \in \mathbb{Z}, \forall \lambda \in \mathbb{R}$, then $f$ defines a function on $\mathbb{T}$, and $\int_{\mathbb{T}} f \, dt = \int_0^1 f(t) \, dt$. We get

$$d(\mu_1 * \mu_2) = \Phi_{\text{per}}(\cdot) \, dt \text{ on } \mathbb{T}.$$

**Proof.** We have

$$1 = \int_{-\infty}^{\infty} \Phi(\lambda) \, d\lambda = \sum_{n \in \mathbb{Z}} \int_{n}^{n+1} \Phi(\lambda) \, d\lambda$$

$$= \int_{0}^{1} \sum_{n \in \mathbb{Z}} \Phi(\lambda - n) \, d\lambda$$

$$= \int_{\mathbb{T}} \Phi_{\text{per}}(t) \, dt.$$ \hfill \Box

We now proceed to study the relations between the other items in our analysis, the RKHSs $\mathcal{H}_i$, for $i = 1, 2, \ldots$, and computing $\mathcal{H}_f$ from the RKHSs $\mathcal{H}_i$. 
We further study the associated unitary representations of $G$ when $\text{Ext}(F_i), i = 1, 2, \ldots$ are non-empty?

As an application, we get infinite convolutions, and they are fascinating; include many fractal measures of course.

In the case of $G = \mathbb{R}$ we will study the connection between deficiency index values in $\mathcal{H}_F$ as compared to those of the factor RKHSs $F_i$. 
Chapter 10
Models for, and Spectral Representations of, Operator Extensions

A special case of our extension question for continuous positive definite (p.d.) functions on a fixed finite interval \( |x| < a \) in \( \mathbb{R} \) is the following: It offers a spectral model representation for ALL Hermitian operators with dense domain in Hilbert space and with deficiency indices \((1, 1)\). (See e.g., [vN32a, Kre46, DS88, AG93, Nel69].)

Specifically, on \( \mathbb{R} \), all the partially defined continuous p.d. functions extend, and we can make a translation of our p.d. problem into the problem of finding all \((1, 1)\) restrictions selfadjoint operators.

By the Spectral theorem, every selfadjoint operator with simple spectrum has a representation as a multiplication operator \( M_\lambda \) in some \( L^2(\mathbb{R}, \mu) \) for some probability measure \( \mu \) on \( \mathbb{R} \). So this accounts for all Hermitian restrictions operators with deficiency indices \((1, 1)\).

So the problem we have been studying for just the case of \( G = \mathbb{R} \) is the case of finding spectral representations for ALL Hermitian operators with dense domain in Hilbert space having deficiency indices \((1, 1)\).

10.1 Model for Restrictions of Continuous p.d. Functions on \( \mathbb{R} \)

Let \( \mathcal{H} \) be a Hilbert space, \( A \) a skew-adjoint operator, \( A^* = -A \), which is unbounded; let \( v_0 \in \mathcal{H} \) satisfying \( \|v_0\|_{\mathcal{H}} = 1 \). Then we get an associated p.d. continuous function \( F_A \) defined on \( \mathbb{R} \) as follows:

\[
F_A(t) := \langle v_0, e^{tA}v_0 \rangle = \langle v_0, U_A(t)v_0 \rangle, \ t \in \mathbb{R},
\]

where \( U_A(t) = e^{tA} \) is a unitary representation of \( \mathbb{R} \). Note that we define \( U(t) = U_A(t) = e^{tA} \) by the Spectral Theorem. Note (10.1) holds for all \( t \in \mathbb{R} \).

Let \( P_U(\cdot) \) be the projection-valued measure (PVM) of \( A \), then

\[
U(t) = \int_{-\infty}^{\infty} e^{i\lambda t} P_U(d\lambda), \ \forall t \in \mathbb{R}.
\]

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Lemma 10.1. Setting

\[ d\mu = \| P_{U}(d\lambda) v_0 \|^2 \]  

we then get

\[ F_A(t) = \hat{d\mu}(t), \forall t \in \mathbb{R} \]  

Moreover, every probability measure \( \mu \) on \( \mathbb{R} \) arises this way.

Proof. By (10.1),

\[
F_A(t) = \int e^{it\lambda} \langle v_0, P_{U}(d\lambda) v_0 \rangle \\
= \int e^{it\lambda} \| P_{U}(d\lambda) v_0 \|^2 \\
= \int e^{it\lambda} d\mu(\lambda)
\]

\[ \square \]

Lemma 10.2. For Borel functions \( f \) on \( \mathbb{R} \), let

\[ f(A) = \int_{\mathbb{R}} f(\lambda) P_{U}(d\lambda) \]  

be given by functional calculus. We note that

\[ v_0 \in \text{dom}(f(A)) \iff f \in L^2(\mu) \]  

where \( \mu \) is the measure in (10.3). Then

\[ \| f(A) v_0 \|^2 = \int_{\mathbb{R}} |f|^2 d\mu. \]  

Proof. The lemma follows from

\[
\| f(A) v_0 \|^2 = \left\| \int_{\mathbb{R}} f(\lambda) P_{U}(d\lambda) v_0 \right\|^2 \quad \text{(by (10.5))} \\
= \int |f(\lambda)|^2 \| P_{U}(d\lambda) v_0 \|^2 \\
= \int |f(\lambda)|^2 d\mu(\lambda). \quad \text{(by (10.3))}
\]

\[ \square \]

Now we consider restriction of \( F_A \) to, say \((-1, 1)\), i.e.,

\[ F(\cdot) = F_A \big|_{(-1,1)} (\cdot) \]  

(10.8)
Lemma 10.3. Let \( \mathcal{H}_F \) be the RKHS computed for \( F \) in (10.4); and for \( \varphi \in C_c (0, 1) \), set \( F_\varphi \) the generating vectors in \( \mathcal{H}_F \), as usual. Set

\[
U (\varphi) := \int_0^1 \varphi (y) U (-y) dy
\]

(10.9)

where \( dy \) = Lebesgue measure on \( (0, 1) \); then

\[
F_\varphi (x) = \langle v_0, U (x) U (\varphi) v_0 \rangle, \forall x \in (0, 1).
\]

(10.10)

Proof. We have

\[
F_\varphi (x) = \int_0^1 \varphi (y) F (x - y) dy
\]

\[
= \int_0^1 \varphi (y) \langle v_0, U_A (x - y) v_0 \rangle dy \quad \text{(by (10.1))}
\]

\[
= \langle v_0, U_A (x) \int_0^1 \varphi (y) U_A (-y) v_0 dy \rangle
\]

\[
= \langle v_0, U_A (x) U (\varphi) v_0 \rangle \quad \text{(by (10.9))}
\]

\[
= \langle v_0, U (\varphi) U_A (x) v_0 \rangle
\]

for all \( x \in (0, 1) \), and all \( \varphi \in C_c (0, 1) \).

\( \square \)

Corollary 10.1. Let \( A, U (t) = e^{itA}, v_0 \in \mathcal{H}, \varphi \in C_c (0, 1) \), and \( F \) p.d. on \( (0, 1) \) be as above; let \( \mathcal{H}_F \) be the RKHS of \( F \); then, for the inner product in \( \mathcal{H}_F \), we have

\[
\langle F_\varphi, F_\psi \rangle_{\mathcal{H}_F} = \langle U (\varphi) v_0, U (\psi) v_0 \rangle_{\mathcal{H}_F}, \forall \varphi, \psi \in C_c (0, 1).
\]

(10.11)

Proof. Note that

\[
\langle F_\varphi, F_\psi \rangle_{\mathcal{H}_F} = \int_0^1 \int_0^1 \overline{\varphi (x)} \psi(y) F (x - y) dxdy
\]

\[
= \int_0^1 \int_0^1 \overline{\varphi (x)} \psi(y) \langle v_0, U_A (x - y) v_0 \rangle_{\mathcal{H}_F} dxdy \quad \text{(by (10.8))}
\]

\[
= \int_0^1 \int_0^1 \langle \varphi (x) U_A (-y) v_0, \psi(y) U_A (-y) v_0 \rangle_{\mathcal{H}_F} dxdy
\]

\[
= \langle U (\varphi) v_0, U (\psi) v_0 \rangle_{\mathcal{H}_F} \quad \text{(by (10.9))}
\]

\( \square \)

Corollary 10.2. Set \( \varphi^\# (x) = \overline{\varphi (-x)}, x \in \mathbb{R} \), \( \varphi \in C_c (\mathbb{R}) \), or in this case, \( \varphi \in C_c (0, 1) \); then we have:

\[
\langle F_\varphi, F_\psi \rangle_{\mathcal{H}_F} = \langle v_0, U (\varphi^\# * \psi) v_0 \rangle_{\mathcal{H}_F}, \forall \varphi, \psi \in C_c (0, 1).
\]

(10.12)

Proof. Immediate from (10.11) and Fubini.

\( \square \)
Corollary 10.3. Let $F$ and $\varphi \in C_c(0,1)$ be as above; then in the RKHS $\mathcal{H}_F$ we have:

$$\left\| F \varphi \right\|_{\mathcal{H}_F}^2 = \left\| U(\varphi)v_0 \right\|_{\mathcal{H}^F}^2 = \int |\hat{\varphi}(\lambda)|^2 d\mu$$

(10.13)

where $\mu$ is the measure in (10.3). $\hat{\varphi} =$ Fourier transform: $\hat{\varphi}(\lambda) = \int_0^1 e^{-i\lambda x} \varphi(x) dx$, $\lambda \in \mathbb{R}$.

Proof. Immediate from (10.12); indeed:

$$\left\| F \varphi \right\|_{\mathcal{H}_F}^2 = \int_0^1 \int_0^1 \overline{\varphi(x)} \varphi(y) \int_{\mathbb{R}} e_{\lambda}(x-y) d\mu(\lambda)$$

$$= \int_{\mathbb{R}} |\hat{\varphi}(\lambda)|^2 d\mu(\lambda), \forall \varphi \in C_c(0,1).$$

□

Corollary 10.4. Every Borel probability measure $\mu$ on $\mathbb{R}$ arises this way.

Proof. We shall need to following:

Lemma 10.4. Let $A$, $\mathcal{H}$, $\{U_A(t)\}_{t \in \mathbb{R}}$, $v_0 \in \mathcal{H}$ be as above; and set

$$d\mu = d\mu_A(\cdot) = \|P_{U_A(\cdot)}v_0\|^2$$

(10.14)

as in (10.3). Assume $v_0$ is cyclic; then $W_\mu f(A)v_0 = f$ defines a unitary isomorphism $W_\mu : \mathcal{H} \rightarrow L^2(\mu)$; and

$$W_\mu U_A(t) = e^{it}W_\mu$$

(10.15)

where $e^{it}$ is seen as a multiplication operator in $L^2(\mu)$. More precisely:

$$(W_\mu U(t)\xi)(\lambda) = e^{it\lambda}(W_\mu \xi)(\lambda), \forall t, \lambda \in \mathbb{R}, \forall \xi \in \mathcal{H}.$$ (10.16)

(We say that the isometry $W_\mu$ intertwines the two unitary one-parameter groups.)

Proof. Since $v_0$ is cyclic, it is enough to consider $\xi \in \mathcal{H}$ of the following form: $\xi = f(A)v_0$, with $f \in L^2(\mu)$, see (10.6) in Lemma 10.2. Then

$$\|\xi\|_{\mathcal{H}^F} = \int_{\mathbb{R}} |f(\lambda)|^2 d\mu(\lambda), \text{so}$$

(10.17)

$$\|W_\mu \xi\|_{L^2(\mu)} = \|\xi\|_{\mathcal{H}^F} \iff (10.17))$$

For the adjoint operator $W_\mu^* : L^2(\mathbb{R},\mu) \rightarrow \mathcal{H}$, we have

$$W_\mu^* f = f(A)v_0,$$

see (10.5)-(10.7). Note that $f(A)v_0 \in \mathcal{H}$ is well-defined for all $f \in L^2(\mu)$. Also $W_\mu^* W_\mu = I_{\mathcal{H}}$, $W_\mu W_\mu^* = I_{L^2(\mu)}$. 

Proof of (10.16). Take $\xi = f(A) v_0$, $f \in L^2(\mu)$, and apply the previous lemma, we have

$$W_\mu U(t) \xi = W_\mu U(t) f(A) v_0 = W_\mu \left(e^{it} f(\cdot)\right)(A) v_0 = e^{it} f(\cdot) = e^{it} W_\mu \xi;$$

or written differently:

$$W_\mu U(t) = M e^{it} W_\mu, \forall t \in \mathbb{R}$$

where $M e^{it}$ is the multiplication operator by $e^{it}$. □

Remark 10.1. Deficiency indices $(1,1)$ occur for probability measures $\mu$ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} |\lambda|^2 d\mu(\lambda) = \infty. \quad (10.18)$$

See examples below.

| p.d. function $F$ | measure | condition (10.18) | deficiency indices |
|-------------------|---------|--------------------|--------------------|
| $F_1(x) = \frac{1}{1+x^2}$, $|x| < 1$ | $\mu_1$ | $\int_{\mathbb{R}} |\lambda|^2 \frac{1}{2} e^{-|\lambda|} d\lambda < \infty$ | $(0,0)$ |
| $F_2(x) = 1 - |x|$, $|x| < \frac{1}{2}$ | $\mu_2$ | $\int_{\mathbb{R}} |\lambda|^2 \left(\frac{\sin(|\lambda|/2)}{|\lambda|/2}\right)^2 d\lambda = \infty$ | $(1,1)$ |
| $F_3(x) = e^{-|x|}$, $|x| < 1$ | $\mu_3$ | $\int_{\mathbb{R}} |\lambda|^2 \frac{d\lambda}{\pi(1+\lambda^2)} = \infty$ | $(1,1)$ |
| $F_4(x) = \left(\frac{\sin(|x|/2)}{|x|/2}\right)^2$, $|x| < \frac{1}{2}$ | $\mu_4$ | $\int_{\mathbb{R}} |\lambda|^2 \chi_{(-1,1)}(\lambda) (1 - |\lambda|) d\lambda < \infty$ | $(0,0)$ |
| $F_5(x) = e^{-x^2/2}$, $|x| < 1$ | $\mu_5$ | $\int_{\mathbb{R}} |\lambda|^2 \frac{1}{\sqrt{2\pi}} e^{-\lambda^2/2} d\lambda = 1 < \infty$ | $(0,0)$ |
| $F_6(x) = \cos x$, $|x| < \frac{\pi}{4}$ | $\mu_6$ | $\int_{\mathbb{R}} |\lambda|^2 \frac{1}{2} (\delta_1 + \delta_{-1}) d\lambda = 1 < \infty$ | $(0,0)$ |
| $F_7(x) = (1 - ix)^{-p}$, $|x| < 1$ | $\mu_7$ | $\int_{0}^{\infty} \lambda^2 \frac{\lambda^p}{\Gamma(p)} e^{-\lambda} d\lambda < \infty$ | $(0,0)$ |

Table 10.1: Application of Theorem 10.1 to the functions from Tables 5.1 and 5.3. From the results in Chapter 5, we conclude that $\text{Ext}_1(F) = \{\text{a singleton}\}$ in the cases $i = 1,4,5,6,7$. 

Remark 10.2. For the two p.d. functions $F_2$ and $F_3$ in Table 10.1; i.e., the cases of deficiency indices $(1, 1)$, one shows that, in each case, $Ext_1(F)$ is a one-parameter family $\{F(\theta)\}$ indexed by $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $F(\theta) = d\mu(\theta)$, where the associated probability measures $\{\mu(\theta)\}$ constitute a system of mutually singular measures, i.e., for $\theta_1 \neq \theta_2$, the two measures are mutually singular.

The proof of these conclusions is based on the von Neumann formula for the elements in $Ext_1(F)$, together with our results for the index $(1, 1)$ operator $D^{(F)}$; see Chapters 2-5.

Summary: restrictions with deficiency indices $(1, 1)$.

Theorem 10.1. If $\mu$ is a fixed probability measure on $\mathbb{R}$, then the following two conditions are equivalent:

1. $\int_{\mathbb{R}} \lambda^2 d\mu(\lambda) = \infty$;
2. The set
   \[
   dom(S) = \left\{ f \in L^2(\mu) \mid \lambda f \in L^2(\mu) \text{ and } \int_{\mathbb{R}} (\lambda + i) f(\lambda) d\mu(\lambda) = 0 \right\}
   \]
   is the dense domain of a restriction operator $S \subset M_\lambda$ with deficiency indices $(1, 1)$, and the deficiency space $DEF_+ = \mathbb{C}1$, ($1$ = the constant function $1$ in $L^2(\mu)$).

Remark 10.3. By Table 10.1, the theorem applies to $\mu_2$ and $\mu_3$. 

Fig. 10.1: The measures $\mu_i \in Ext(F_i)$ extending p.d. functions $F_i$ in Table 5.1, $i = 1, 2, \ldots 7$. 

- $\mu_1$, $\mu_5$
- $\mu_2$, $\mu_6$
- $\mu_3$, $\mu_7$
- $\mu_4$

![Graphs of measures](image)
10.2 A Model of ALL Deficiency Index-(1, 1) Operators

Lemma 10.5. Let $\mu$ be a Borel probability measure on $\mathbb{R}$, and denote $L^2(\mathbb{R}, d\mu)$ by $L^2(\mu)$. then we have TFAE:

(1) \[ \int_{\mathbb{R}} |\lambda|^2 d\mu(\lambda) = \infty \] (10.19)

(2) the following two subspaces in $L^2(\mu)$ are dense (in the $L^2(\mu)$-norm):
\[
\left\{ f \in L^2(\mu) \left| \left( (\lambda \pm i) f(\lambda) \right) \in L^2(\mu) \text{ and } \int (\lambda \pm i) f(\lambda) d\mu(\lambda) = 0 \right. \right\}
\] (10.20)

where $i = \sqrt{-1}$.

Proof. See [Jør81]. \qed

Remark 10.4. If (10.19) holds, then the two dense subspaces $\mathcal{D}_\pm \subset L^2(\mu)$ in (10.20) form the dense domain of a restriction $S$ of $M_\lambda$ in $L^2(\mu)$; and this restriction has deficiency indices $(1, 1)$. Moreover, all Hermitian operators having deficiency indices $(1, 1)$ arise this way.

Assume (10.19) holds; then the subspace
\[
\mathcal{D} = \left\{ f \in L^2(\mu) \left| (\lambda + i) f(\lambda) \in L^2(\mu) \text{ and } \int (\lambda + i) f(\lambda) d\mu(\lambda) = 0 \right. \right\}
\]
is a dense domain of a restricted operator of $M_\lambda$, so $S \subset M_\lambda$, and $S$ is Hermitian.

Lemma 10.6. With $i = \sqrt{-1}$, set
\[
\text{dom}(S) = \left\{ f \in L^2(\mu) \left| (\lambda + i) f(\lambda) \in L^2(\mu) \text{ and } \int (\lambda + i) f(\lambda) d\mu(\lambda) = 0 \right. \right\}
\] (10.21)

then $S \subset M_\lambda \subset S^*$; and the deficiency subspaces $\text{DEF}_\pm$ are as follow:
\[
\text{DEF}_+ = \text{the constant function in } L^2(\mu) = C^1
\] (10.22)
\[
\text{DEF}_- = \text{span} \left\{ \frac{\lambda - i}{\lambda + i} \right\}_{\lambda \in \mathbb{R}} \subseteq L^2(\mu)
\] (10.23)

where $\text{DEF}_-$ is also a 1-dimensional subspace in $L^2(\mu)$.

Proof. Let $f \in \text{dom}(S)$, then, by definition,
\[
\int_{\mathbb{R}} (\lambda + i) f(\lambda) d\mu(\lambda) = 0 \text{ and so}
\]
\[
\langle 1, (S + iI) f \rangle_{L^2(\mu)} = \int_{\mathbb{R}} (\lambda + i) f(\lambda) d\mu(\lambda) = 0
\] (10.24)
hence \((10.22)\) follows.

Note we have formula \((10.21)\) for \(\text{dom}(S)\). Moreover \(\text{dom}(S)\) is dense in \(L^2(\mu)\) because of \((10.20)\) in Lemma 10.5.

Now to \((10.23)\): Let \(f \in \text{dom}(S)\); then
\[
\langle \lambda - i, (S - iI) f \rangle_{L^2(\mu)} = \int_{\mathbb{R}} \langle \lambda + i, (\lambda - i) f(\lambda) \rangle d\mu(\lambda) = \int_{\mathbb{R}} (\lambda + i) f(\lambda) d\mu(\lambda) = 0
\]
again using the definition of \(\text{dom}(S)\) in \((10.21)\).

We have established a representation for all Hermitian operators with dense domain in a Hilbert space, and having deficiency indices \((1,1)\). In particular, we have justified the answers in Table 5.1 for \(F_i, i = 1, \ldots, 5\).

To further emphasize the result we need about deficiency indices \((1,1)\), we have the following:

**Theorem 10.2.** Let \(\mathcal{H}\) be a separable Hilbert space, and let \(S\) be a Hermitian operator with dense domain in \(\mathcal{H}\). Suppose the deficiency indices of \(S\) are \((d,d)\); and suppose one of the selfadjoint extensions of \(S\) has simple spectrum. Then the following two conditions are equivalent:

(1) \(d = 1\);
(2) for each of the selfadjoint extensions \(T\) of \(S\), we have a unitary equivalence between \((S, \mathcal{H})\) on the one hand, and a system \((S_\mu, L^2(\mathbb{R}, \mu))\) on the other, where \(\mu\) is a Borel probability measure on \(\mathbb{R}\). Moreover,

\[
\text{dom}(S_\mu) = \left\{ f \in L^2(\mu) \mid \lambda f(\cdot) \in L^2(\mu), \text{ and } \int_{\mathbb{R}} (\lambda + i) f(\lambda) d\mu(\lambda) = 0 \right\},
\]

\(\text{dom}(S_\mu)\) (10.25)

and

\[
(S_\mu f)(\lambda) = \lambda f(\lambda), \forall f \in \text{dom}(S_\mu), \forall \lambda \in \mathbb{R}.
\]

(10.26)

In case \(\mu\) satisfies condition (10.26), then the constant function \(1\) (in \(L^2(\mathbb{R}, \mu)\)) is in the domain of \(S_\mu^*\), and

\[
S_\mu^* 1 = i 1
\]

(10.27)

i.e., \((S_\mu^* 1)(\lambda) = i, \, \text{a.a.} \, \lambda \, \text{w.r.t.} \, d\mu\).

**Proof.** For the implication (2)⇒(1), see Lemma 10.6.

(1)⇒(2). Assume that the operator \(S\), acting in \(\mathcal{H}\), is Hermitian with deficiency indices \((1,1)\). This means that each of the two subspaces \(\text{DEF}_\pm \subset \mathcal{H}\) is one-dimensional, where

\[
\text{DEF}_\pm = \left\{ h_\pm \in \text{dom}(S^*) \mid S^* h_\pm = \pm ih_\pm \right\}.
\]

(10.28)

Now pick a selfadjoint extension, say \(T\), extending \(S\). We have
where “⊆” in (10.29) means “containment of the respective graphs.”

Now set \( U(t) = e^{itT}, t \in \mathbb{R} \), and let \( P_U(\cdot) \) be the corresponding projection-valued measure, i.e., we have:

\[
U(t) = \int_{\mathbb{R}} e^{it \lambda} P_U(d \lambda), \quad \forall t \in \mathbb{R}. \tag{10.30}
\]

Using the assumption (1), and (10.28), it follows that there is a vector \( h_+ \in \mathcal{H} \) such that \( \|h_+\|_{\mathcal{H}} = 1 \), \( h_+ \in \text{dom}(S^*) \), and \( S^* h_+ = ih_+ \). Now set

\[
d\mu(\lambda) := \|P_U(d \lambda) h_+\|^2_{\mathcal{H}}. \tag{10.31}
\]

Using (10.30), we then verify that there is a unitary (and isometric) isomorphism of \( L^2(\mu) \) onto \( \mathcal{H} \) given by

\[
Wf = f(T)h_+, \quad \forall f \in L^2(\mu); \quad (10.32)
\]

where \( f(T) = \int_{\mathbb{R}} f(T) P_U (d \lambda) \) is the functional calculus applied to the selfadjoint operator \( T \). Hence

\[
\|Wf\|^2_{\mathcal{H}} = \|f(T)h_+\|^2_{\mathcal{H}} = \int_{\mathbb{R}} |f(\lambda)|^2 \|P_U(d \lambda) h_+\|^2 \]
\[
= \int_{\mathbb{R}} |f(\lambda)|^2 d\mu(\lambda) \quad \text{(by 10.31)}
\]
\[
= \|f\|^2_{L^2(\mu)}. \]

To see that \( W \) in (10.32) is an isometric isomorphism of \( L^2(\mu) \) onto \( \mathcal{H} \), we use the assumption that \( T \) has simple spectrum.

Now set

\[
S_\mu := W^* SW \tag{10.33}
\]
\[
T_\mu := W^* TW. \tag{10.34}
\]

We note that \( T_\mu \) is then the multiplication operator \( M \) in \( L^2(\mathbb{R}, \mu) \), given by

\[
(Mf)(\lambda) = \lambda f(\lambda), \quad \forall f \in L^2(\mu) \tag{10.35}
\]

such that \( \lambda f \in L^2(\mu) \). This assertion is immediate from (10.32) and (10.31).

To finish the proof, we compute the integral in (10.25) in the theorem, and we use the intertwining properties of the isomorphism \( W \) from (10.32). Indeed, we have
\[
\int_{\mathbb{R}} (\lambda + i) f(\lambda) \, d\mu(\lambda) = \langle 1, (M + iI) f \rangle_{L^2(\mu)}
\]

\[
= \langle W1, W (M + iI) f \rangle_{\mathcal{H}}
\]

\[
(10.31) = \langle h_+, (T + iI) W f \rangle_{\mathcal{H}}. \tag{10.36}
\]

Hence \( W f \in \text{dom}(S) \iff f \in \text{dom}(S_\mu) \), by (10.33); and, so for \( W f \in \text{dom}(S) \), the r.h.s. in (10.36) yields \( \langle (S^* - iI) h_+, W f \rangle_{\mathcal{H}} = 0 \); and the assertion (2) in the theorem follows. \( \Box \)

### 10.2.1 Momentum Operators in \( L^2(0, 1) \)

What about our work on momentum operators in \( L^2(0, 1) \)? They have deficiency indices \((1, 1)\); and there is the family of measures \( \mu \) on \( \mathbb{R} \):

Fix \( 0 \leq \theta < 1 \), fix \( w_n > 0 \), \( \sum_{n \in \mathbb{Z}} w_n = 1 \); \( \tag{10.37} \)

and let \( \mu = \mu(\theta,w) \) as follows:

\[
d\mu = \sum_{n \in \mathbb{Z}} w_n \delta_{\theta+n}
\]

\( \tag{10.38} \)

such that

\[
\sum_{n \in \mathbb{Z}} n^2 w_n = \infty. \tag{10.39}
\]

In this case, there is the bijective \( L^2(\mathbb{R}, d\mu) \leftrightarrow \{ \xi_n \}_{n \in \mathbb{Z}}, \xi \in \mathbb{C} \) s.t.

\[
\sum_{n \in \mathbb{Z}} |\xi_n|^2 w_n < \infty.
\]

We further assume that

\[
\sum_{n \in \mathbb{Z}} n^2 w_n = \infty.
\]

The trick is to pick a representation in which \( v_0 \) in Lemma 10.2, i.e., (10.5)-(10.7):

\( \mathcal{H}, \) cyclic vector \( v_0 \in \mathcal{H}, \|v_0\|_{\mathcal{H}} = 1, \{ U_A(t) \}_{t \in \mathbb{R}}. \)

**Lemma 10.7.** The \((1,1)\) index condition is the case of the momentum operator in \( L^2(0, 1) \). To see this we set \( v_0 = e^t \), or \( v_0 = e^{2\pi t}. \)

**Proof.** Defect vector in \( \mathcal{H} = L^2(0, 1) \), say \( e^{2\pi t}: \)

\[
v_0(x) = \sum_{n \in \mathbb{Z}} c_n e^{\theta n} (x)
\]

with

\[
w_n \sim \frac{(\cos(2\pi(\theta+n)) - e^{-2\pi})^2 + \sin^2(2\pi(\theta+n))}{1 + (\theta+n)^2}
\]
and so the \((1,1)\) condition holds, i.e.,

\[
\sum_{n \in \mathbb{Z}} n^2 w_n = \infty.
\]

\[\square\]

We specialize to \(\mathcal{H} = L^2 (0,1)\), and the usual momentum operator in \((0,1)\) with the selfadjoint extensions. Fix \(\theta, 0 \leq \theta < 1\), we have an ONB in \(\mathcal{H} = L^2 (0,1)\),

\[
e^\theta_k (x) := e^{i 2 \pi (\theta + k) x},
\]

and

\[
v_0 (x) = \sum_{k \in \mathbb{Z}} c_k e^\theta_k (x), \sum_{k \in \mathbb{Z}} |c_k|^2 = 1. \tag{10.40}
\]

**Fact.** \(v_0\) is cyclic for \(\{U_{\theta_0} (t) \}_t \epsilon \mathbb{R} \iff c_k \neq 0, \forall k \in \mathbb{Z}\); then set \(w_k = |c_k|^2\), and the conditions \((10.37)-(10.39)\) holds. Reason: the measure \(\mu = \mu_{v_0, \theta}\) depends on the choice of \(v_0\). The non-trivial examples here \(v_0 \iff \{c_k\}_{k \in \mathbb{Z}}, c_k \neq 0, \forall k\), is cyclic, i.e.,

\[
c l \ span \ \{U_{\theta_0} (t) v_0 | t \in \mathbb{R}\} \ (= \mathcal{H} = L^2 (0,1))
\]

**Lemma 10.8.** The measure \(\mu\) in \((10.38)\) is determined as:

\[
F_{\theta} (t) = \left< v_0, U_{\theta_0} (t) v_0 \right> = \sum_{k \in \mathbb{Z}} e^{it 2 \pi (\theta + k)} w_k = \int_{\mathbb{R}} e^{it \lambda} d \mu (\lambda)
\]

where \(\mu\) is as in \((10.38)\); so purely atomic.

Note that our boundary conditions for the selfadjoint extensions of the minimal momentum operator are implied by \((10.40)\), i.e.,

\[
f_0 (x + 1) = e^{i 2 \pi \theta} f_0 (x), \ \forall x \in \mathbb{R}. \tag{10.41}
\]

It is implied by choices of \(\theta\) s.t.

\[
f_0 = \sum_{n \in \mathbb{Z}} B_n e^n_\theta (x), \ \sum_{n \in \mathbb{Z}} |B_n|^2 < \infty.
\]

### 10.2.2 Restriction Operators

In this representation, the restriction operators are represented by sequence \(\{f_k\}_{k \in \mathbb{Z}}\) s.t. \(\sum |f_k|^2 w_k < \infty\), so use restriction the selfadjoint operator corresponds to the \(\theta\)-boundary conditions \((10.41)\), \(dom (S)\) where \(S\) is the Hermitian restriction operator, i.e., \(S \subset s.a. \subset S'\). It has its dense domain \(dom (S)\) as follows: \((f_k) \in dom (S) \iff (k f_k) \in l^2 (\mathbb{Z}, w)\), and
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\[ \sum_{k \in \mathbb{Z}}(\theta + k + i)f_k w_k = 0; \quad i = \sqrt{-1}. \]

Comparison with (10.21) in the general case. What is special about this case \( \mathcal{H} = L^2(0, 1) \) and the usual boundary conditions at the endpoints is that the family of measures \( \mu \) on \( \mathbb{R} \) are purely atomic; see (10.38)

\[ d\mu = \sum_{k \in \mathbb{Z}}w_k \delta_{k+\theta}. \]

10.3 The Case of Indices \((d, d)\) where \(d > 1\)

Let \( \mu \) be a Borel probability measure on \( \mathbb{R} \), and let

\[ L^2(\mu) := L^2(\mathbb{R}, \mathcal{B}, \mu). \] (10.42)

The notation \( \text{Prob}(\mathbb{R}) \) will be used for these measures.

We saw that the restriction/extension problem for continuous positive definite (p.d.) functions \( F \) on \( \mathbb{R} \) may be translated into a spectral theoretic model in some \( L^2(\mu) \) for suitable \( \mu \in \text{Prob}(\mathbb{R}) \). We saw that extension from a finite open \((\neq 0)\) interval leads to spectral representation in \( L^2(\mu) \), and restrictions of

\[ (M_\mu f)(\lambda) = \lambda f(\lambda), \quad f \in L^2(\mu) \] (10.43)

having deficiency-indices \((1, 1)\); hence the case \( d = 1 \).

**Theorem 10.3.** Fix \( \mu \in \text{Prob}(\mathbb{R}) \). There is a 1-1 bijective correspondence between the following:

1. certain closed subspaces \( \mathcal{L} \subset L^2(\mu) \)
2. Hermitian restrictions \( S_\mathcal{L} \) of \( M_\mu \) (see (10.43)) such that

\[ \text{DEF}_+(S_\mathcal{L}) = \mathcal{L}. \] (10.44)

The closed subspaces in (1) are specified as follows:

(i) \( \dim(\mathcal{L}) = d < \infty \)

(ii) the following implication holds:

\[ g \neq 0, \text{ and } g \in \mathcal{L} \implies [\lambda \mapsto \lambda g(\lambda)] \notin L^2(\mu) \] (10.45)

Then set

\[ \text{dom}(S_\mathcal{L}) := \left\{ f \in \text{dom}(M_\mu) \left| \int g(\lambda)(\lambda + i)f(\lambda)d\mu(\lambda), \forall g \in \mathcal{L} \right. \right\} \] (10.46)

and set
10.4 Spectral Representation of Index (1, 1) Hermitian Operators

\[ S_{\mathcal{L}} := M_\mu \bigg|_{\text{dom}(S_{\mathcal{L}})} \]  

(10.47)

where \( \text{dom}(S_{\mathcal{L}}) \) is specified as in (10.46).

**Proof.** Note that the case \( d = 1 \) is contained in the previous theorem.

Proof of (1) \( \Rightarrow \) (2). We will be using an idea from [Jør81]. With assumptions (i)-(ii), in particular (10.45), one checks that \( \text{dom}(S_{\mathcal{L}}) \) as specified in (10.46) is dense in \( L^2(\mu) \). In fact, the converse implication is also true.

Now setting \( S_{\mathcal{L}} \) to be the restriction in (10.47), we conclude that

\[ S_{\mathcal{L}} \subseteq M_\mu \subseteq S_{\mathcal{L}}^* \]  

(10.48)

where

\[ \text{dom}(S_{\mathcal{L}}^*) = \left\{ h \in L^2(\mu) \mid \text{s.t. } \exists C < \infty \text{ and } \int_{\mathbb{R}} \left| \frac{h(\lambda)}{\lambda} f(\lambda) \right|^2 d\mu(\lambda) \leq C \int_{\mathbb{R}} |f(\lambda)|^2 d\mu(\lambda) \right\} \]

holds \( \forall f \in \text{dom}(S_{\mathcal{L}}) \) for every densely defined restriction \( S \) of \( M_\mu \).

The assertions in (2) now follow from this.

Proof of (2) \( \Rightarrow \) (1). Assume that \( S \) is a densely defined restriction of \( M_\mu \), and let \( \text{DEF}_+(S) = \) the (+) deficiency space, i.e.,

\[ \text{DEF}_+(S) = \left\{ g \in \text{dom}(S^*) \mid S^* g = ig \right\} \]  

(10.49)

Assume \( \dim(\text{DEF}_+(S)) = d \), and \( 1 \leq d < \infty \). Then set \( \mathcal{L} := \text{DEF}_+(S) \). Using [Jør81], one checks that (1) then holds for this closed subspace in \( L^2(\mu) \).

The fact that (10.45) holds for this subspace \( \mathcal{L} \) follows from the observation:

\[ \text{DEF}_+(S) \cap \text{dom}(M_\mu) = \{0\} \]

for every densely defined restriction \( S \) of \( M_\mu \). \( \square \)

10.4 Spectral Representation of Index (1, 1) Hermitian Operators

In this section we give an explicit answer to the following question: How to go from any index (1, 1) Hermitian operator to a \( (\mathcal{H}_f, D(f)) \) model; i.e., from a given index (1, 1) Hermitian operator with dense domain in a separable Hilbert space \( \mathcal{H} \), we
build a p.d. continuous function $F$ on $\Omega - \Omega$, where $\Omega$ is a finite interval $(0,a)$, $a > 0$.

So far, we have been concentrating on building transforms going in the other direction. But recall that, for a given continuous p.d. function $F$ on $\Omega - \Omega$, it is often difficult to answer the question of whether the corresponding operator $D^{(F)}$ in the RKHS $\mathscr{H}_F$ has deficiency indices $(1,1)$ or $(0,0)$.

Now this question answers itself once we have an explicit transform going in the opposite direction. Specifically, given any index $(1,1)$ Hermitian operator $S$ in a separable Hilbert space $\mathscr{H}$, we then to find a pair $(F,\Omega)$, p.d. function and interval, with the desired properties. There are two steps:

1. Step 1, writing down explicitly, a p.d. continuous function $F$ on $\Omega - \Omega$, and the associated RKHS $\mathscr{H}_F$ with operator $D^{(F)}$.

2. Step 2, constructing an intertwining isomorphism $W : \mathscr{H} \to \mathscr{H}_F$, having the following properties. $W$ will be an isometric isomorphism, intertwining the pair $(\mathscr{H},S)$ with $(\mathscr{H}_F,D^{(F)})$, i.e., satisfying $WS = D^{(F)}W$; and also intertwining the respective domains and deficiency spaces, in $\mathscr{H}$ and $\mathscr{H}_F$.

Moreover, starting with any $(1,1)$ Hermitian operator, we can even arrange a normalization for the p.d. function $F$ such that $\Omega = (0,1)$ will do the job.

We now turn to the details:

We will have three pairs $(\mathscr{H},S)$, $(L^2(\mathbb{R},\mu)$, restriction of $M_\mu)$, and $(\mathscr{H}_F,D^{(F)})$, where:

(i) $S$ is a fixed Hermitian operator with dense domain $\text{dom}(S)$ in a separable Hilbert space $\mathscr{H}$, and with deficiency indices $(1,1)$.

(ii) From the given information in (i), we will construct a finite Borel measure $\mu$ on $\mathbb{R}$ such that an index-$(1,1)$ restriction of $M_\mu : f \mapsto \lambda f(\lambda)$ in $L^2(\mathbb{R},\mu)$, is equivalent to $(\mathscr{H},S)$.

(iii) Here $F : (-1,1) \to \mathbb{C}$ will be a p.d. continuous function, $\mathscr{H}_F$ the corresponding RKHS; and $D^{(F)}$ the usual operator with dense domain

$$\left\{ F_\varphi ~|~ \varphi \in C^\infty_c(0,1) \right\},$$

and

$$D^{(F)}(F_\varphi) = \frac{1}{i}F_{\varphi'}, \quad \varphi' = \frac{d\varphi}{dx}.$$  \hspace{1cm} (10.50)

We will accomplish the stated goal with the following system of intertwining operators: See Figure 10.2.

But we stress that, at the outset, only (i) is given; the rest $(\mu, F$ and $\mathscr{H}_F)$ will be constructed. Further, the solutions $(\mu,F)$ in Figure 10.2 are not unique; rather they depend on choice of selfadjoint extension in (i): Different selfadjoint extensions of $S$ in (i) yield different solutions $(\mu,F)$. But the selfadjoint extensions of $S$ in $\mathscr{H}$ are parameterized by von Neumann’s theory; see e.g., [Rud73, DS88].
Remark 10.5. In our analysis of (i)-(iii), we may without loss of generality assume that the following normalizations hold:

1. \( \mu(\mathbb{R}) = 1 \), so \( \mu \) is a probability measure;
2. \( F(0) = 1 \), and the p.d. continuous solution
3. \( F: (-1,1) \to \mathbb{C} \) is defined on \((-1,1)\); so \( \Omega := (0,1) \).

Further, we may assume that the operator \( S \) in \( \mathcal{H} \) from (i) has simple spectrum.

Theorem 10.4. Starting with \((\mathcal{H},S)\) as in (i), there are solutions \((\mu,F)\) to (ii)-(iii), and intertwining operators \( W_\mu, T_\mu \) as in Figure 10.2, such that

\[
W := T_\mu^* W_\mu \quad \text{(10.51)}
\]

satisfies the intertwining properties for \((\mathcal{H},S)\) and \((\mathcal{H}_F, D^{(F)})\).

Proof. Since \( S \) has indices \((1,1)\), \( \dim \text{DEF}_\pm(S) = 1 \), and \( S \) has selfadjoint extensions indexed by partial isometries \( \text{DEF}_+ \xrightarrow{\nu} \text{DEF}_- \); see [Rud73, DS88]. We now pick \( g_+ \in \text{DEF}_+, \|g_+\| = 1 \), and partial isometry \( \nu \) with selfadjoint extension \( S_\nu \), i.e.,

\[
S \subset S_\nu \subset S_\nu^* \subset S^*. \quad \text{(10.52)}
\]

Hence \( \{ U_\nu(t) \mid t \in \mathbb{R} \} \) is a strongly continuous unitary representation of \( \mathbb{R} \), acting in \( \mathcal{H} \), \( U_\nu(t) := e^{itS_\nu}, t \in \mathbb{R} \). Let \( P_{S_\nu}(\cdot) \) be the corresponding projection valued measure (PVM) on \( \mathcal{B}(\mathbb{R}) \), i.e., we have

\[
U_\nu(t) = \int_{\mathbb{R}} e^{it\lambda} P_{S_\nu}(d\lambda); \quad \text{(10.53)}
\]

and set

\[
d\mu(\lambda) := d\mu_\nu(\lambda) = \|P_{S_\nu}(d\lambda) g_+\|^2_{\mathcal{H}_F}. \quad \text{(10.54)}
\]

For \( f \in L^2(\mathbb{R}, \mu_\nu) \), set
W_{\mu} (f (S_v) g_+) = f;
\quad (10.55)
then \( W_{\mu} : \mathcal{H} \to L^2 (\mathbb{R}, \mu_v) \) is isometric onto; and
\[
W_{\mu^*} (f) = f (S_v) g_+,
\quad (10.56)
\]
where
\[
f (S_v) g_+ = \int_{\mathbb{R}} f (\lambda) P_{S_v} (d\lambda) g_+.
\quad (10.57)
\]
For justification of these assertions, see e.g., [Nel69]. Moreover, \( W_{\mu} \) has the intertwining properties sketched in Figure 10.2.

Returning to (10.53) and (iii) in the theorem, we now set \( F = \) the restriction to \((-1, 1)\) of
\[
F_{\mu} (t) := \langle g_+, U_v (t) g_+ \rangle
\quad (10.58)
\]
\[
= \langle g_+, \int_{\mathbb{R}} e^{it\lambda} P_{S_v} (d\lambda) g_+ \rangle
\quad (by \ (10.53))
\]
\[
= \int_{\mathbb{R}} e^{it\lambda} \| P_{S_v} (d\lambda) g_+ \|^2
\]
\[
= \int_{\mathbb{R}} e^{it\lambda} d\mu_v (\lambda)
\quad (by \ (10.54))
\]
\[
= \hat{d\mu_v} (t), \ \forall t \in \mathbb{R}.
\]

We now show that
\[
F := F_{\mu} \big|_{(-1,1)}
\quad (10.59)
\]
has the desired properties.

From Corollary 3.1, we have the isometry \( T_{\mu} (F_\phi) = \hat{\phi}, \phi \in C_c (0, 1) \), with adjoint
\[
T_{\mu}^* (f) = \chi_{\mathcal{F}} (f d\mu)^\vee, \quad (10.60)
\]
see also Figure 10.2.

The following properties are easily checked:
\[
W_{\mu} (g_+) = 1 \in L^2 (\mathbb{R}, \mu), \quad (10.61)
\]
\[
T_{\mu}^* (1) = F_0 = F (\cdot - 0) \in \mathcal{H}, \quad (10.62)
\]
as well as the intertwining properties stated in the theorem; see Figure. 10.2 for a summary.
10.4 Spectral Representation of Index (1, 1) Hermitian Operators

Proof of (10.61) We will show instead that \( W^*_\mu (1) = g_+ \). From (10.57) we note that if \( f \in L^2 (\mathbb{R}, \mu) \) satisfies \( f = 1 \), then \( f(S_v) = I = \text{the identity operator in } \mathcal{H} \). Hence
\[
W^*_\mu (1) = 1 (S_v) g_+ = g_+ ,
\]
which is (10.61).

Proof of (10.62) For \( \varphi \in C_c (0, 1) \) we have \( \hat{\varphi} \in L^2 (\mathbb{R}, \mu) \), and
\[
T^*_\mu T_\mu (F \varphi) = T^*_\mu (\hat{\varphi}) = (\text{by } (10.60)) \chi_{\Omega} (\hat{\varphi} d\mu)^\vee = F \varphi .
\]
Taking now an approximation \( (\varphi_n) \subset C_c (0, 1) \) to the Dirac unit mass at \( x = 0 \), we get (10.62).

□

Corollary 10.5. The deficiency indices of \( D(F) \) in \( \mathcal{H}_F \) for \( F(x) = e^{-|x|} \), \( |x| < 1 \), are \((1, 1)\).

Proof. Take \( \mathcal{H} = L^2 (\mathbb{R}) = \{ f \text{ measurable on } \mathbb{R} \mid \int_{\mathbb{R}} |f(x)|^2 \, dx < \infty \} \), where \( dx = \text{Lebesgue measure} \).

Take \( g_+ := \left( \frac{1}{\lambda + i} \right)^\vee (x), x \in \mathbb{R} \); then \( g_+ \in L^2 (\mathbb{R}) =: \mathcal{H} \) since
\[
\int_{\mathbb{R}} |g_+(x)|^2 \, dx \quad \text{Parseval} \quad = \int_{\mathbb{R}} \left| \frac{1}{\lambda + i} \right|^2 d\lambda
\]
\[
= \int_{\mathbb{R}} \frac{1}{1 + \lambda^2} d\lambda = \pi .
\]
Now for \( S \) and \( S_v \) in Theorem 10.4, we take
\[
S_v h = \frac{1}{i} \frac{d}{dx} h \text{ on } \left\{ h \in L^2 (\mathbb{R}) \mid h' \in L^2 (\mathbb{R}) \right\} \quad \text{and} \quad (10.63)
\]
\[
S = S_v \left\{ h \mid h, h' \in L^2 (\mathbb{R}), h(0) = 0 \right\} ,
\]
then by [Jør81], we know that \( S \) is an index \((1, 1)\) operator, and that \( g_+ \in DEF_+ (S) \).

The corresponding p.d. continuous function \( F \) is the restriction to \( |t| < 1 \) of the p.d. function:
\[
\langle g_+, U_v (t) g_+ \rangle_{\mathcal{H}} = \int_{\mathbb{R}} \frac{1}{\lambda - i} e^{\mu \lambda} d\lambda
\]
\[
= \left( \frac{1}{1 + \lambda^2} \right)^\vee (t) = \pi e^{-|t|} .
\]

□
Example 10.1 (Lévy-measures). Let $0 < \alpha \leq 2, -1 < \beta < 1, \nu > 0$; then the Lévy-measures $\mu$ on $\mathbb{R}$ are indexed by $(\alpha, \beta, \nu)$, so $\mu = \mu(\alpha, \beta, \nu)$. They are absolutely continuous with respect to Lebesgue measure $d\lambda$ on $\mathbb{R}$; and for $\alpha = 1$,

$$F_{(\alpha, \beta, \nu)}(x) = \hat{\mu}(\alpha, \beta, \nu)(x), \ x \in \mathbb{R}, \quad (10.64)$$

satisfies

$$F_{(\alpha, \beta, \nu)}(x) = \exp \left( -\nu |x| \cdot \left( 1 + \frac{2i\beta}{\pi} - \text{sgn}(x) \ln |x| \right) \right). \quad (10.65)$$

The case $\alpha = 2, \beta = 0$, reduces to the Gaussian distribution.

The measures $\mu(1, \beta, \nu)$ have infinite variance, i.e.,

$$\int_{\mathbb{R}} \lambda^2 d\mu(1, \beta, \nu) = \infty.$$

As a Corollary of Theorem 10.4, we therefore conclude that, for the restrictions,

$$F_{(1, \beta, \nu)}^{(\text{res})}(x) = F_{(1, \beta, \nu)}(x), \text{ in } x \in (-1, 1), \text{ (see (10.64) – (10.65))}$$

the associated Hermitian operator $D^{F(\nu)}$ all have deficiency indices $(1, 1)$.

In connection Lévy measures, see e.g., [ST94].
Chapter 11
Overview and Open Questions

11.1 From Restriction Operator to Restriction of p.d. Function

The main difference between our measures on \( \mathbb{R} \), and the measures used in fractional Brownian motion and related processes is that our measures are finite on \( \mathbb{R} \), but the others aren’t; instead they are what is called tempered (see [AL08]). If \( \mu \) is a tempered positive measure, then the function \( F = \tilde{d}\mu \) is still positive definite, but it is not continuous, unless \( \mu(\mathbb{R}) < \infty \).

This means that they are unbounded but only with a polynomial degree. For example, Lebesgue measure \( dx \) on \( \mathbb{R} \) is tempered. Suppose \( 1/2 < g < 1 \), then the positive measure \( d\mu(x) = |x|^g \, dx \) is tempered, and this measure is what is used in accounting for the p.d. functions discussed in the papers on Fractional Brownian motion.

But we could ask the same kind of questions for tempered measures as we do for the finite positive measures.

Following this analogy, one can say that the paper [Jør81] (by one of the co-authors) was about models of index-\((1,1)\) operators, admitting realizations in \( L^2(\mathbb{R}) \), i.e., \( L^2 \) of Lebesgue measure \( dx \) on \( \mathbb{R} \). This work was followed up subsequently also for the index-\((m,m)\) case, \( m > 1 \).

11.2 The Splitting \( \mathcal{H}_F = \mathcal{H}_F^{(\text{atom})} \oplus \mathcal{H}_F^{(\text{ac})} \oplus \mathcal{H}_F^{(\text{sing})} \)

Let \( \Omega \) be as usual, connected and open in \( \mathbb{R}^n \); and let \( F \) be a p.d. continuous function on \( \Omega - \Omega \). Suppose \( \text{Ext} (F) \) is non-empty. We then get a unitary representation \( U \) of \( G = \mathbb{R}^n \), with associated projection valued measure (PVM) \( P_U \), acting on the RKHS \( \mathcal{H}_F \). This gives rise to an orthogonal splitting of \( \mathcal{H}_F \) into three parts, atomic, absolutely continuous, and continuous singular, defined from \( P_U \).

**Question 11.1.** What are some interesting examples illustrating the triple splitting of \( \mathcal{H}_F \), as in Theorem 4.1, eq (4.6)?

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In Section 4.1, we constructed an interesting example for the splitting of the RKHS $\mathcal{H}_F$: All three subspaces $\mathcal{H}_F^{(\text{atom})}$, $\mathcal{H}_F^{(\text{ac})}$, and $\mathcal{H}_F^{(\text{sing})}$ are non-zero; the first one is one-dimensional, and the other two are infinite-dimensional. See Example 4.1 for details.

For some recent developments, see, also, [JPT15, JPT14, PT13].

11.3 The Case of $G = \mathbb{R}^1$

We can move to the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, or stay with $\mathbb{R}$, or go to $\mathbb{R}^n$; or go to Lie groups. For Lie groups, we must also look at the universal simply connected covering groups. All very interesting questions.

A FEW POINTS, about a single interval and asking for p.d. extensions to $\mathbb{R}$:

1. The case of $\mathbb{R}$, we are asking to extend a (small) p.d. continuous function $F$ from an interval to all of $\mathbb{R}$, so that the extension to $\mathbb{R}$ is also p.d. continuous; – in this case we always have existence. This initial interval may even be “very small.” If we have several intervals, existence may fail, but the problem is interesting. We have calculated a few examples, but nothing close to a classification!

2. Computable ONBs in $\mathcal{H}_F$ would be very interesting. We can use what we know about selfadjoint extensions of operators with deficiency indices $(1, 1)$ to classify all the p.d. extension to $\mathbb{R}$. While both are “extension” problems, operator extensions and p.d. extensions is subtle, and non-intuitive. For this problem, there are two sources of earlier papers; one is M. Krein [Kre46], and the other L. deBranges. Both are covered beautifully in a book by Dym and McKean [DM76].

11.4 The Extreme Points of $\text{Ext} \{ F \}$ and $\Im \{ F \}$

Given a locally defined p.d. function $F$, i.e., a p.d. function $F$ defined on a proper subset in a group, then the real part $\Re \{ F \}$ is also positive definite. Can anything be said about the imaginary part, $\Im \{ F \}$? See Section 8.4.

Assuming that $F$ is also continuous, then what are the extreme points in the compact convex set $\text{Ext} \{ F \}$, i.e., what is the subset $\text{ext} \{ \text{Ext} \{ F \} \}$? How do properties of $\text{Ext} \{ F \}$ depend on the imaginary part, i.e., on the function $\Im \{ F \}$? How do properties of the skew-Hermitian operator $D(F)$ (in the RKHS $\mathcal{H}_F$) depend on the imaginary part, i.e., on the function $\Im \{ F \}$?
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