From PDE Systems and Metrics to Generalized Field Theories

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Abstract

Let \((T, h)\) and \((M, \varphi)\) be two Riemannian manifolds and
\[
(J^1(T, M), S = h + \varphi + h^{-1} \ast \varphi)
\]
the first-order jet fibre bundle, endowed with Sasakian-like metric \(S\), associated to these manifolds. Developing our ideas from [7], [20], we show that a given first-order PDEs system on \(J^1(T, M)\), and the Riemannian metric \(S\), determine an ElectroDynamics Metrical Multi-Time Lagrange Space
\[
PDEsEDML^p_n = (J^1(T, M), L),
\]
where \(L : J^1(T, M) \rightarrow \mathbb{R}\) is a quadratic Lagrangian function of electrodynamics type [4]. In this new geometrical structure, all \(C^2\) solutions of the starting PDEs system become harmonic maps, being extremals of a least squares problem of variational calculus. Our ideas are structured in the following way:

1) we find a suitable geometrical structure on \(J^1(T, M)\) that convert the solutions of a given PDEs system into harmonic maps (Section 1);
2) we build a natural geometry induced by a such PDEs system (Section 2);
3) we construct a field theory, in a general setting, naturally attached to this PDEs system (Section 3).

Consequently, we give a complete answer to our generalization [6], [14] of a problem rised by Poincaré, studied by others, but unfinalized for a long time because of the absence of a suitable geometrical structure. The Poincaré problem was solved recently by the first author [18], using the Lagrangian Geometry [4].

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1 Generalized Lorentz-Udriște world-force law and a generalized approach of inverse problem

Let \(T\) and \(M\) be two smooth, real, connected manifolds of dimensions \(p\) and \(n\), with coordinates \((t^\alpha)_{\alpha=1}^p\) and \((x^i)_{i=1}^n\). Greek, respectively Latin, letters will be used for indexing the components of geometrical objects attached to the manifold \(T\), respectively \(M\).

From physical point of view, \(T\) is regarded like a "multi-temporal" manifold while the manifold \(M\) like a "spatial" one. Let us consider the jet fibre bundle of order one
$J^1(T, M)$, which is well known as a basic object in the study of classical and quantum field theories. The local coordinates on $J^1(T, M)$ are $(t^n, x^i, x^i_\alpha)$, where $x^i_\alpha$ have the physical meaning of "partial directions" or "partial derivatives".

Let us start with a given d-tensor $X_{(\alpha)}^{(i)}(t^n, x^k)$ on $J^1(T, M)$, which defines the PDEs system of order one

$$\tag{1.1} x^i_\alpha = X_{(\alpha)}^{(i)}(t^n, x^k(t^n)),$$

where $x^i_\alpha = \frac{\partial x^i}{\partial t^\alpha}$. Obviously, the complete integrability conditions,

$$\frac{\partial X_{(\alpha)}^{(i)}}{\partial t^\beta} + \frac{\partial X_{(\alpha)}^{(i)}}{\partial x^m} X_{(\beta)}^{(m)} = \frac{\partial X_{(\beta)}^{(i)}}{\partial t^\alpha} + \frac{\partial X_{(\beta)}^{(i)}}{\partial x^m} X_{(\alpha)}^{(m)},$$

are required by the existence of solutions.

The Lorentz world-force law, initially stated for particles in non-quantum relativity [14], was generalized by Udriște, using the notion of potential map [19], [20]. In this direction, starting with $h = (h_\alpha(t^n))$ and $\varphi = (\varphi_{ij}(x^k))$ as semi-Riemannian metrics on $T$, respectively $M$, the first author proved the following

**Theorem 1.1 (Lorentz-Udriște World-Force Law)**

Every $C^2$ solution of the PDEs system (1.1) is a horizontal potential map of the semi-Riemannian-Lagrange manifold

$$(T \times M, h + \varphi, M_{(\alpha)\beta}^{(i)} = -H_\alpha^\mu x^\mu_\beta, N_{(\alpha)j}^{(i)} = \gamma_{jm} x^m_\alpha - F_{(\alpha)j}^{(i)}),$$

where $H_\alpha^\mu$ (resp. $\gamma_{jm}^{(i)}$) are the Christoffel symbols of $h_\alpha$ (resp. $\varphi_{ij}$), and

$$F_{(\alpha)j}^{(i)} = \nabla^i_j X_{(\alpha)}^{(i)} - \varphi^{ir}_{(\alpha)} \varphi_{js} \nabla^s_r X_{(\alpha)}^{(i)}$$

is the external distinguished tensor field which characterizes the helicity of the distinguished tensor field $X_{(i)}^{(i)}$.

A new version of the Lorentz-Udriște World-Force Law was formulated by the second author. Its advantage is the possibility of constructing a generalized field theory, naturally attached to the PDEs system (1.1).

In order to give this version, we suppose that the metrics $h$ and $\varphi$ are just Riemannian metrics and we endow the jet fibre bundle $J^1(T, M)$ with the canonical nonlinear connection $\Gamma_0 = (M_{(\alpha)\beta}^{(i)}, N_{(\alpha)j}^{(i)})$ associated to the Riemannian metrics pair $(h, \varphi)$, where

$$M_{(\alpha)\beta}^{(i)} = -H_\alpha^\mu x^\mu_\beta,$$

$$N_{(\alpha)j}^{(i)} = \gamma_{jm} x^m_\alpha.$$

At the same time, let us consider $B\Gamma_0 = (H_\alpha^\gamma, 0, \gamma_{jk}^i, 0)$ the Berwald $\Gamma_0$-linear connection on $J^1(T, M)$ and $\circ \mu_{\beta}^\alpha, \partial_{\beta}^\mu, \partial_{\beta_{\alpha}^{(i)}}$ its local covariant derivatives [10].

Taking into account the $x^m_\mu$-independence of the d-tensor field $X_{(\alpha)}^{(i)}$, by a simple calculation, we obtain
Theorem 1.3. Theorem 1.3

Theorem 1.3 (Generalized Lorentz-Udrište World-Force Law)
The $C^2$ solutions of the PDEs system (1.1) are harmonic maps of the multi-time dependent spray $(H, G)$ on $J^1(T, M)$, defined by the temporal components

$$(1.2) \quad H^{(i)}_{(\alpha)\beta} = -\frac{1}{2} H^{\gamma}_{\alpha\beta} x^{(i)}_{\gamma},$$

and the local spatial components

$$G^{(i)}_{(\alpha)\beta} = \frac{1}{2} \gamma^{ij} x^j + h_{\alpha\beta} F^i,$$

where,

$$F^i = \frac{h^{\mu\nu}}{2p} \left\{ \varphi^{i(\nu)} x^{(r)}_{(\nu)} \left[ x^{(r)}_{(\alpha)} - x^{(i)}_{(\alpha)} \right] + x^{(i)}_{(\alpha)} x^m_{\mu} + x^{(i)}_{(\alpha)\mu} \right\}, \quad p = \dim T.$$

In other words, the $C^2$ solutions of (1.1) verify the harmonic map equations $[9], [11], h^{\alpha\beta} \left\{ x_{\alpha\beta} + 2 H^{(i)}_{(\alpha)\beta} + 2 G^{(i)}_{(\alpha)\beta} \right\} = 0.$

Proof. Let us consider the multi-time least squares Lagrangian

$$\mathcal{L} = \| C - X \|^2 \sqrt{T} = \left\{ h^{\alpha\beta} (t^\gamma) \varphi_{ij} (x^k) \left[ x^i_{\alpha} - X^{(i)}_{(\alpha)} \right] \left[ x^j_{\beta} - X^{(j)}_{(\beta)} \right] \right\} \sqrt{T} = \left\{ \gamma^{ij} (t^\gamma) \varphi_{ij} (x^k) x^i_{\alpha} x^j_{\beta} + U^{(i)} (t^\gamma, x^k) x^i_{\alpha} + \Phi (t^\gamma, x^k) \right\} \sqrt{T},$$

where

$$C = x^i_{\alpha} \frac{\partial}{\partial x^i_{\alpha}}, \quad U^{(i)} = -2 h^{\alpha\mu} \varphi_{im} X^{(m)}_{(\mu)},$$

$$X = X^{(i)}_{(\alpha)} \frac{\partial}{\partial x^i_{\alpha}} \quad \Phi = h^{\mu\nu} \varphi_{rs} X^{(r)}_{(\mu)} X^{(s)}_{(\nu)},$$

and let $\mathcal{E}_\mathcal{L} : C^2(T, M) \to R_+$ (note that the Riemannian metrics $h_{\alpha\beta}$ and $\varphi_{ij}$ are positive definite) be the least squares energy of $\mathcal{L}$, i.e.,

$$\mathcal{E}_\mathcal{L} = \int_T \mathcal{L} dt^1 \wedge dt^2 \wedge \ldots \wedge dt^p = \int_T \| C - X \|^2 \sqrt{T} dt^1 \wedge dt^2 \wedge \ldots \wedge dt^p \geq 0.$$

It is obvious now that $f \in C^2(T, M)$, locally expressed by $(t^\gamma) \to (x^i(t^\gamma))$, is a solution of the PDEs system (1.1) iff the map $f$ is a global minimum point for the
least squares energy functional $E_L$. Therefore, a $C^2$ solution $f$ of (1.1) verifies the Euler-Lagrange equations,

$$\frac{\partial L}{\partial x^i} - \frac{\partial}{\partial t^\alpha} \left( \frac{\partial L}{\partial x^i_\alpha} \right) = 0, \quad \forall \ i = 1, n.$$  

(1.4)

But, we proved in the paper [9] that the Euler-Lagrange equations (1.4) can be rearranged in the general Poisson form,

$$h^{\alpha\beta} \left\{ x^i_{\alpha\beta} + 2H^{(i)}(\alpha)\beta + 2G^{(i)}(\alpha)\beta \right\} = 0,$$

where the temporal spray components $H^{(i)}(\alpha)$ are given by (1.2), while the spatial spray components have the expressions

$$G^{(i)}(\alpha)\beta = \frac{1}{2} \epsilon^{jk} x^j_{\alpha\beta} x^k_{\alpha\beta} + \frac{h_{\alpha\mu} g^{il}}{4p} \left[ U^{(\mu)}_{m\nu} x^m_{\mu\nu} + \frac{\partial U^{(\mu)}_{m\nu}}{\partial t^\mu} + U^{(\mu)}_{m\nu} H^\gamma_{\mu\nu} - \frac{\partial \Phi}{\partial x^l} \right],$$

where $p = \text{dim} T$ and

$$U^{(\alpha)}_{m\nu} = \frac{\partial U^{(\alpha)}_{m\nu}}{\partial x^j} - \frac{\partial U^{(\alpha)}_{m\nu}}{\partial x^i}.$$ 

Now, using the relations (1.3) and direct computations, it follows

$$U^{(\alpha)}_{m\nu} = -2h^\alpha\mu \left[ \varphi_{\nu m} X^{(m)}_{(\mu)\nu)} - \varphi_{\mu m} X^{(m)}_{(\nu)i} \right],$$

$$\frac{\partial U^{(\alpha)}_{m\nu}}{\partial t^\alpha} + U^{(\nu)}_{(i)} H^\gamma_{\mu\nu} = -2h^\mu\nu \varphi_{\nu m} X^{(m)}_{(\mu)\nu)},$$

$$\frac{\partial \Phi}{\partial x^l} = 2h^\mu\nu \varphi_{m\nu} X^{(m)}_{(\mu)\nu)} X^{(r)}_{(\nu)ll}. \quad \blacksquare$$

**Remark 1.1** The method used in the proof of the theorem 1.3 is a general one and may be called the least squares variational calculus method. For example, let us consider $X$ a d-tensor field on the jet fibre bundle of $r$-order $J^r(T, M)$, supposed as being endowed with an "a priori" Riemannian metric $<,>$, and let $X(t, x^{(r)}) = 0$ be the associated $r$-order PDEs system. Automatically, we can attach a least squares problem of variational calculus, using the Lagrangian function $L = \|X\|^2$, where the norm $\|\|$ is builded by the metric $<,>$. This remark works on any DEs or PDEs system. As example, the Yang-Mills functional

$$\mathcal{YM}(\nabla) = \frac{1}{2} \int_M \|R^\nabla\|^2 dV_g,$$

on the space of connections $\nabla$, is the least squares functional attached to the PDEs system $R^\nabla = 0$. In this sense, $\mathcal{YM}(\nabla)$ measures the deviation from flatness.

The theorem 1.3 shows that the $C^2$ solutions of the PDEs system (1.1) can be viewed like some harmonic maps on $J^1(T, M)$, via two "a priori" fixed Riemannian metrics $h$ on $T$, respectively $\varphi$ on $M$. 
We recall the well known fact that paracompactness of a real manifold implies the existence of a Riemannian metric. So, we can restrict the imposed condition concerning the "a priori" existence of the Riemannian metrics \( h \) and \( \varphi \), supposing the paracompactness of the manifolds \( T \) and \( M \).

In conclusion, we can formulate the following

**Corollary 1.4** If \( T \) and \( M \) are paracompact manifolds, then the \( C^2 \) solutions of the PDEs system of order one (1.1) can be regarded like harmonic maps on the jet fibre bundle of order one \( J^1(T, M) \), in the sense of theorem 1.3.

In the sequel, let us denote \( J^s(T, M) \) the total space of the jet fibre bundle of order \( s \), whose local coordinates are

\[
(t, x^{(s)}) = (t^i, x^k, x^k_{\gamma_1}, x^k_{\gamma_1\gamma_2}, \ldots, x^k_{\gamma_1\gamma_2\ldots\gamma_s}),
\]

where the coordinates \( x^k_{\gamma_1\gamma_2\ldots\gamma_l}, l \in \{1, 2, \ldots, s\}, k \in \{1, 2, \ldots, n\} \), have the meaning of "partial derivatives of order \( l \) of the spatial variables \( x^k \), with respect to the temporal variables \( t^{\gamma_1}, t^{\gamma_2}, \ldots, t^{\gamma_l} \)."

**Lemma 1.5** The dimension of the total space of the jet fibre bundle \( J^s(T, M) \) is

\[
p + n \left[ C^0_p + C^1_p + \ldots + C^{s-1}_p \right],
\]

where \( \dim T = p \) and \( \dim M = n \).

Let us consider the PDEs system of order \( r \geq 1 \) on \( J^r(T, M) \), locally expressed by

\[
(1.5) \quad x_{\alpha_1\alpha_2\ldots\alpha_{r-1}\alpha_r}^{(i)}(1) = X^{(i)}_{\alpha_1\alpha_2\ldots\alpha_{r-1}(\alpha_r)}(t, x^{(r-1)}),
\]

where \( X^{(i)}_{\alpha_1\alpha_2\ldots\alpha_{r-1}(\alpha_r)} \) is a \( d \)-tensor field on \( J^r(T, M) \), with respect to the indices \( i \) and \( \alpha_r \).

Let \( \hat{M} \) be the submanifold of \( J^r(T, M) \), whose coordinates are only \( x^{(r-1)} = (x^k, x^k_{\gamma_1}, x^k_{\gamma_1\gamma_2}, \ldots, x^k_{\gamma_1\gamma_2\ldots\gamma_{r-1}}) \). In these conditions, we deduce the following important result.

**Theorem 1.6** (Geometrical Solution of Generalized Inverse Problem)

If the manifolds \( T \) and \( M \) are paracompact, then the \( C^{r+1} \) solutions of the PDEs system (1.5) can be viewed like harmonic maps on the jet fibre bundle of order one \( J^1(T, \hat{M}) \), whose dimension is

\[
p + (p + 1) \left[ C^0_p + C^1_p + \ldots + C^{r-1}_p \right] n,
\]

where \( \dim T = p \) and \( \dim \hat{M} = n \).

**Proof.** Obviously, the paracompactness of the manifolds \( T \) and \( M \) implies the paracompactness of the jet fibre bundle \( J^r(T, M) \) and, implicitly the paracompactness of \( \hat{M} \). Moreover, if \( x^{(r-1)} = (x^k, x^k_{\gamma_1}, x^k_{\gamma_1\gamma_2}, \ldots, x^k_{\gamma_1\gamma_2\ldots\gamma_{r-1}}) \) are the coordinates of \( \hat{M} \) and \( x^{(r-1)} = (x^k_{\alpha_1\alpha_2\ldots\alpha_{r-1}}) \), then the PDEs system (1.5) writes in the form,

\[
x^{(i(r-1))}_{\alpha_r} = X^{(i)}_{(\alpha_r)}(t, x^{(r-1)}).
\]

Now, the theorem 1.3 implies the required result. ■
Remark 1.2 In a general approach of the well known inverse problem, we consider that the previous theorem gives a final geometrical answer for this open problem. From our point of view, the old classical open problem was not solved because of two reasons:

i) Firstly, it was incorrectly formulated;

ii) Secondly, almost all attempts of solving this problem used an unsuitable Lagrangian function space of possible solutions.

In our opinion, the inverse problem must be formulated in the following way:

Generalized Inverse Problem: Find a space of Lagrangian functions such that the $C^2$ solutions of a given PDEs system to become solutions of the Euler-Lagrange equations of a Lagrangian on this space.

2 From PDEs system of order one and metrics to generalized Maxwell and Einstein equations

Let us return to the PDEs system of order one, given by (1.1). The theorem 1.3 ensures us that a $C^2$ map $f$ is a solution of PDEs system (1.1) iff $f$ is a global minimum point of the energy functional produced by the multi-time Lagrangian $L = L \sqrt{h}$, whose least squares Lagrangian function $L$ is expressed by

\begin{equation}
L = \|C - X\|^2 = h^\alpha\beta(t)\varphi_{ij}(x^k)x^i_\alpha x^j_\beta + U^{(\alpha)}(t, x^k)x^i_\alpha + \Phi(t^\gamma, x^k),
\end{equation}

where $U^{(\alpha)}(t) = -2h^{\alpha\mu}\varphi_im_{(i)}^{(m)}$ and $\Phi = h^{\mu\nu}\varphi_{rs}X^{(r)}_{(\mu)}X^{(s)}_{(\nu)}$.

But, the Lagrangian function $L$ is a natural generalization of the classical Lagrangian of electrodynamics [6], which governs the movement law of a particle placed concomitantly into a gravitational field and an electromagnetic one. For that reason, in our generalized context, the Riemannian metric $h_{\alpha\beta}$ (resp. $\varphi_{ij}$) will have the temporal (resp. spatial) gravitational potential physical meaning, the components $X^{(i)}_{(j)}$ that of electromagnetic potentials on $J^1(T, M)$, and $F$ that of the potential function.

In this geometrical-physical context, following the terminology used in [9], we can introduce

Definition 2.1 The pair $(J^1(T, M), L)$, which consists of jet fibre bundle of order one and a Lagrangian function of the form (2.1), is called the canonical autonomous metrical multi-time Lagrange space of electrodynamics, attached to the PDEs system (1.1) and to the Riemannian metrics $h$ and $\varphi$, and is denoted $PDEsEDML^n_p$.

On the jet fibre bundle of order one $J^1(T, M)$, a natural generalized theory of physical fields, attached to a given multi-time Lagrangian function of electrodynamics type was recently developed by the second author [9]. Consequently, via the canonical autonomous metrical multi-time Lagrange space of electrodynamics $PDEsEDML^n_p$, we can construct a natural physical field theory, in a general setting, arising only from the PDEs system (1.1) and two "a priori" given Riemannian metrics $h_{\alpha\beta}$ and $\varphi_{ij}$.

Definition 2.2 The multi-time dependent spray $(H, G)$ used in the theorem 1.3 is called the canonical multi-time dependent spray attached to $PDEsEDML^n_p$.
Using the canonical multi-time dependent spray \((H,G)\), one naturally induces a nonlinear connection \(\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})\) on \(J^1(T,M)\), which is also called the canonical nonlinear connection of the autonomous metrical multi-time Lagrange space of electrodynamics \(\text{PDEsEDML}_n\). In this direction, denoting \(G^i = h^{\alpha\beta}G^{(i)}_{(\alpha)\beta}\), we establish the following

**Theorem 2.1** The canonical nonlinear connection of the autonomous metrical multi-time Lagrange space of electrodynamics \(\text{PDEsEDML}_n\) is determined by the temporal components

\[
M^{(i)}_{(\alpha)\beta} = 2H^{(i)}_{(\alpha)\beta} = M^{0}_{(\alpha)\beta}
\]

and the local spatial components

\[
N^{(i)}_{(\alpha)j} = \frac{\partial G^i}{\partial x^j} h_{\alpha\gamma} = \delta_{\alpha j}^i - F^i_{\alpha j},
\]

where \(M^{0}_{(\alpha)\beta} = -H^{\gamma}_{\alpha\beta} x^i_{\gamma}\), \(N^{0}_{(\alpha)j} = \gamma^i_{jk} x^k_{\alpha}\), and the \(d\)-tensor field

\[
F^i_{\alpha j} = \frac{1}{2} [X^{(i)}_{(\alpha)j} - \phi^{ir} X^{(s)}_{(\alpha)jr} \phi_{sj}]
\]

characterizes the helicity of the \(d\)-tensor field \(X^{(i)}_{(\alpha)}\).

To determine the generalized Cartan canonical connection \(\text{CT}\) of the autonomous metrical multi-time Lagrange space of electrodynamics \(\text{PDEsEDML}_n\), together with its torsion and curvature local \(d\)-tensors, we use the adapted dual bases \(\{\delta \delta t^\alpha, \delta \delta x^i, \delta \partial x^i\} \subset \mathcal{X}(E)\) and \(\{dt^\alpha, dx^i, \delta x^i\} \subset \mathcal{X}^*(E)\) of the nonlinear connection \(\Gamma\), via the formulas

\[
\delta \delta t^\alpha = \frac{\partial}{\partial t^\alpha} - M^{(i)}_{(\beta)\alpha} \frac{\partial}{\partial x^j},
\]

\[
\delta \delta x^i = \frac{\partial}{\partial x^i} - N^{(i)}_{(\beta)j} \frac{\partial}{\partial x^j},
\]

\[
\delta x^i_{\alpha} = dx^i_{\alpha} + M^{(i)}_{(\alpha)\beta} dt^\beta + N^{(i)}_{(\alpha)j} dx^j.
\]

In this context, by simple computations, we find

**Theorem 2.2** i) The generalized Cartan canonical connection \(\text{CT}\) of the metrical multi-time Lagrange space \(\text{PDEsEDML}_n\) has the adapted components

\[
H^{\gamma}_{\alpha\beta} = H^{\gamma}_{\alpha\beta}, \quad G^k_{\gamma} = 0, \quad I^{i}_{jk} = \gamma^i_{jk}, \quad C^{(i)}_{j(k)} = 0.
\]

ii) The torsion \(\text{T}\) of the generalized Cartan canonical connection of the autonomous metrical multi-time Lagrange space of electrodynamics \(\text{PDEsEDML}_n\) is
determined by three adapted local $d$-tensors, namely,

\[ R^{(i)}_{(\alpha)\beta\gamma} = -H_{\alpha\beta\gamma}^\mu x^i_{\mu}, \]

\[ R^{(i)}_{(\alpha)\beta j} = \frac{1}{2} \left[ X^{(s)}_{(\alpha)\|j/\beta} \varphi^{ir} X^{(s)}_{(\alpha)\|r/\beta}\varphi_{sj}\right], \]

\[ R^{(i)}_{(\alpha)jk} = r^{i}_{\ jkm} x^m_{\alpha} - \frac{1}{2} \left[ X^{(i)}_{(\alpha)\|j/k} \varphi_{\|k}\varphi_{sj}\right], \]

where $H_{\alpha\beta\gamma}^\mu$ (resp. $r^{i}_{\ jmk}$) are the local curvature tensors of the Riemannian metric $h_{\alpha\beta}$ (resp. $\varphi_{ij}$), and

\[ \nabla^i = \left( \frac{\partial X^{(i)}_{(\alpha)\|j}}{\partial t^j} - X^{(i)}_{(\alpha)\|j} H_{\alpha\beta}^\mu, \right. \]

\[ \left. \frac{\partial X^{(i)}_{(\alpha)\|j}}{\partial x^k} + X^{(m)}_{(\alpha)\|j} \gamma^i_{mk} - X^{(i)}_{(\alpha)\|m}\gamma^m_{jk}. \right] \]

iii) The curvature $\mathbf{R}$ of the generalized Cartan canonical connection of the autonomous metrical multi-time Lagrange space of electrodynamics $PDEsEDML^n_p$ is determined by two adapted local $d$-tensors, namely, $H_{\alpha\beta\gamma}^\mu$ and $R^{(i)}_{\|jk} = r^{i}_{\ jmk}$, that is, exactly the curvature tensors of the Riemannian metrics $h_{\alpha\beta}$ and $\varphi_{ij}$.

**Remarks 2.1**

i) Both Cartan and Berwald connections on $J^1(T, M)$ have the same adapted components. However, they are two distinct linear connections. This is because the Cartan connection is a $\Gamma$-linear connection while the Berwald connection is a $\Gamma$-linear one.

ii) Let $''/\beta''$, $''\|j''$ and $''(\beta)''$ be the local covariant derivatives of the Cartan connection $CT$, and let us consider a distinguished tensor field $D = (D_{\gamma k(\beta)(l)}^{ai(j)}...)$, whose components do not depend on the partial directions $x^m_j$. In these conditions, we obtain without difficulties that

\[ D_{\gamma k(\beta)(l)}^{ai(j)}...A = D_{\gamma k(\beta)(l)}^{ai(j)}...A, \]

where $''!A''$, respectively $''!\pi A''$, is one of the local operators $''/\beta''$, $''\|j''$ or $''(\beta)''$, respectively $''/\beta''$, $''\|j''$ or $''(\beta)''$.

iii) The only geometrical objects of $PDEsEDML^n_p$, effectively dependent of the PDEs system (1.1), are the spatial nonlinear connection components $N^{(i)}_{(\alpha)j}$ and the adapted local torsion $d$-tensors $R^{(i)}_{(\alpha)\beta j}$, $R^{(i)}_{(\alpha)jk}$.

In the sequel, following the paper [9], we shall write the generalized Maxwell and Einstein equations produced by $PDEsEDML^n_p$.

**Theorem 2.3**

i) The electromagnetic distinguished 2-form of $PDEsEDML^n_p$ is

\[ F = F^{(\alpha)}_{(i)j} \delta x^i_{\alpha} \wedge dx^j, \]

where

\[ F^{(\alpha)}_{(i)j} = \frac{h^{i\mu}}{2} \left[ \varphi_{im} X^{(m)}_{(\mu)\|j} - \varphi_{jm} X^{(m)}_{(\mu)\|i} \right]. \]
ii) The electromagnetic local components $F_{(i)j//\beta}^{(\alpha)}$ of the autonomous metrical multi-time Lagrange space of electrodynamics $\text{PDEsEDML}_p^n$ are governed by the following generalized Maxwell equations,

$$
\begin{align*}
&\left\{ F_{(i)j//\beta}^{(\alpha)} = \frac{1}{4} A_{(i,j)} \left\{ h^{\alpha\mu} \varphi_{im} \left[ X_{(m)j//\beta}^{(s)} - \varphi_{mr} X_{(r)j//\beta}^{(s)} \right] \right\}
\right. \\
&\sum_{(i,j,k)} F_{(i)j//\beta}^{(\alpha)} = 0 \\
&\sum_{(i,j,k)} F_{(i)j//\beta}^{(\alpha)} = 0,
\end{align*}
$$

where $A_{(i,j)}$ represents an alternate sum and $\sum_{(i,j,k)}$ means a cyclic sum.

**Theorem 2.4**

i) The gravitational $h$-potential on $J^1(T, M)$, induced by the metrical multi-time Lagrange space $\text{PDEsEDML}_p^n$, is given by the Sasakian-like metric

$$
G = h_{\alpha\beta} dt^\alpha \otimes dt^\beta + \varphi_{ij} dx^i \otimes dx^j + h_{\alpha\beta} \varphi_{ij} \delta x^\alpha \otimes \delta x^\beta.
$$

ii) The generalized Einstein equations which govern the gravitational $h$-potential $G$ of $\text{PDEsEDML}_p^n$ are locally expressed by

$$
\begin{align*}
(H_{\alpha\beta} - \frac{H + r}{2} h_{\alpha\beta} &= \mathcal{K} T_{\alpha\beta} \\
(r_{ij} - \frac{H + r}{2} \varphi_{ij} &= \mathcal{K} T_{ij} \\
-H + r h_{\alpha\beta} \varphi_{ij} &= \mathcal{K} T_{(i)j}^{(\alpha)(\beta)},
\end{align*}
$$

where $T_{AB}, A, B \in \{\alpha, i, (\alpha) \}$ represent the components of an intrinsic stress-energy d-tensor $T$ of matter on $J^1(T, M)$, $H_{\alpha\beta}$ (resp. $r_{ij}$) are the components of Ricci tensor of $h_{\alpha\beta}$ (resp. $\varphi_{ij}$), and $H$ (resp. $r$) is the scalar curvature of the Riemannian metric $h_{\alpha\beta}$ (resp. $\varphi_{ij}$).

iii) The components $T_{\mu\beta}$ and $T_{m\beta}$ of the stress-energy d-tensor $T$ of the metrical multi-time Lagrange space $\text{PDEsEDML}_p^n$ satisfy the classical conservation laws,

$$
T_{\beta//\mu} = 0, \quad T_{\beta//\mu} = 0.
$$

**Remark 2.2** The theorems 2.3 and 2.4 emphasize that only the electromagnetic field $F$ of the $\text{PDEsEDML}_p^n$ is effectively dependent by the PDEs system (1.1), via the electromagnetic potentials $X_{(i)}^{(\alpha)}$. Consequently, in order to obtain some information upon the PDEs system (1.1), via the canonical attached generalized field theory, it is interesting to study only the electromagnetism generated by this system.
3 Generalized electromagnetic theory induced by remarkable PDE systems

In this section we will consider some important particular PDE systems of order one and we apply them the previous general geometrical and physical results. Taking into account the remarks 2.1 and 2.2, we will study only the geometrical and physical objects which are effectively determined by these systems (i.e., the nonlinear connection, the torsion d-tensors and the electromagnetic field). From this point of view, we will ignore their attached generalized gravitational theory, because it is dependent only of the Riemannian metrics used in the construction of the metrical multi-time Lagrange space $PDEsEDML^n$.

3.1 Orbits

Let $T = [a, b] \subset R$, $h_{11}(t) = 1$ and $X^{(i)}(t, x^k) = \xi^i(x^k)$, where $\xi^i(x^k)$ is a d-vector field on $J^1([a, b], M)$. The PDEs system (1.1) becomes

$$\frac{dx^i}{dt} = \xi^i(x^k(t)),$$

that is, the DEs system which gives the orbits of the spatial vector field $\xi$.

In this context, the $C^2$ orbits of $\xi$ are the global minimum points of the least squares energy functional

$$E_\xi = \int_a^b \left[ \phi_{ij}(x^k)y^i y^j - 2\xi_j(x^k)y^j + \|\xi\|^2(x^k) \right] dt,$$

where $y^i = \frac{dx^i}{dt}$, $\xi_j(x^k) = \varphi_m(x^k)\xi^m(x^k)$ and $\|\xi\|^2(x^k) = \xi^m(x^k)\xi_m(x^k)$.

Theorem 3.1 i) The canonical nonlinear connection, induced by the system $(O)$ on $J^1([a, b], M) \equiv TM$, is given by the components

$$M^{(i)}_{(1)1} = 0, \quad N^{(i)}_{(1)j} = \frac{1}{2} \left[ \xi_{ij} - \varphi^m \xi_m \varphi_{sj} \right].$$

ii) The torsion $T$ of the generalized Cartan canonical connection, induced by the differential system $(O)$, is determined only by the local components

$$R^{(i)}_{(1)jk} = r^i_{jkm}y^m - \frac{1}{2} \left[ \xi_{ij}\delta^k - \varphi^m \xi_m \varphi_{sj}\delta^k \right].$$

iii) The electromagnetic components of the metrical time-dependent Lagrange space of electrodynamics, induced by the orbits system $(O)$, have the expressions

$$F^{(1)}_{(i)kj} = \frac{1}{2} \left[ \xi_j\delta^i - \xi_i\delta^j \right].$$

Moreover, they are governed by the following generalized Maxwell equations:

$$\sum_{\{i,j,k\}} F^{(1)}_{(i)kj} = 0, \quad \sum_{\{i,j,k\}} F^{(1)}_{(i)j}|^i(k) = 0.$$
3.2 Pfaffian systems

Let $M = R$, $\varphi_{11}(x) = 1$ and $X_{\alpha}^{(1)}(t^\gamma, x) = A_\alpha(t^\gamma)$, where $A_\alpha(t^\gamma)$ is a distinguished 1-form on $J^1(T, R)$. The $C^2$ solutions of the Pfaffian system

\((P)\)

\[ x_\alpha = A_\alpha(t^\gamma), \]

are the global minimum points of the least squares energy functional

\[ E_A = \int_T \left[ h^{\alpha\beta}(t^\gamma)x_\alpha x_\beta - 2A^\alpha(t^\gamma)x_\alpha + \|A\|^2(t^\gamma) \right] \sqrt{h} dt, \]

where $x_\alpha = x^k_\alpha = \partial x^k / \partial t^\alpha$, $A^\alpha(t^\gamma) = h^{\alpha\mu}(t^\gamma)A_\mu(t^\gamma)$ and $\|A\|^2(t^\gamma) = A_\alpha(t^\gamma)A^\alpha(t^\gamma)$.

**Theorem 3.2** i) The nonlinear connection on $J^1(T, R)$, induced by the Pfaffian system $(P)$, has the components

\[ M_{(\alpha)\beta}^{(i)} = -h_{\alpha\beta}^\mu x_\mu, \quad N_{(\alpha)1}^{(1)} = 0. \]

ii) The only local torsion d-tensor, induced by $(P)$, which does not vanish is

\[ R_{(\alpha)\beta\gamma}^{(1)} = -h_{\alpha\beta\gamma}^\mu x_\mu. \]

iii) The electromagnetic components $F_{(\alpha)1}^{(a)}$, induced by the Pfaffian system $(P)$, vanish.

3.3 Continuous groups of transformations

The fundamental PDEs system of a transformations group having the infinitesimal generators \(\{\xi_a(x^k)\}_{a=1}^c\), as d-vector fields on $J^1(T, M)$, is

\((G)\)

\[ x^i_\alpha = \sum_{a=1}^c \xi^i_a(x^k(t^\gamma))A^a_\alpha(t^\gamma), \]

where \(\{A^a(t^\gamma)\}_{a=1}^c\) are d-forms on $J^1(T, M)$.

**Theorem 3.3** The $C^2$ solutions of the PDEs system $(G)$ are the global minimum points of the least squares energy functional

\[ E_{\xi A} = \int_T \left[ h^{\alpha\beta}\varphi_{ij}x^i_\alpha x^j_\beta - 2A^\alpha x_\alpha + <A^\alpha, A^b><\xi_a, \xi_b> \right] \sqrt{h} dt, \]

where

\[ A^\alpha(t^\gamma) = h^{\alpha\mu}(t^\gamma)A_\mu(t^\gamma), \]

\[ \xi_a(x^k) = \varphi_{jm}(x^k)\xi^m_a(x^k), \]

\[ <A^\alpha, A^b>(t^\gamma) = h^{\alpha\beta}(t^\gamma)A_\alpha^a(t^\gamma)A_\beta^b(t^\gamma), \]

\[ <\xi_a, \xi_b>(x^k) = \varphi_{ij}(x^k)\xi^i_a(x^k)\xi^j_b(x^k). \]
Theorem 3.4  
\( i) \) The canonical nonlinear connection on \( J^1(T, M) \), induced by the PDEs system \( (G) \), is determined by the components

\[
M^{(i)}_{\alpha\beta} = -H^\mu_{\alpha\beta} x^i_\mu, \quad N^{(i)}_{(\alpha)j} = \gamma^i_{j\mu} x^m_\alpha - \frac{A^a}{2} \left[ \xi^i_{a||j} - \varphi^{ir} \xi^s_{a||r} \xi^j_{s||} \right].
\]

\( ii) \) The local d-torsions which characterize the geometry of \( J^1(T, M) \), attached to the PDEs system \( (G) \), have the expressions

\[
R^{(i)}_{(\alpha)\beta\gamma} = -H^\mu_{\alpha\beta\gamma} x^i_\mu, \\
R^{(i)}_{(\alpha)\beta j} = \frac{A^a_{\alpha/\beta}}{2} \left[ \xi^i_{a||j} - \varphi^{ir} \xi^s_{a||r} \xi^j_{s||} \right], \\
R^{(i)}_{(\alpha)jk} = \frac{A^a_{\alpha/1}}{2} \left[ \xi^i_{a||j} - \varphi^{ir} \xi^s_{a||r} \xi^j_{s||} \right].
\]

Theorem 3.5  
\( i) \) The components of the electromagnetic field \( F \) of the metrical multi-time Lagrange space attached to the PDEs system \( (G) \), are

\[
F^{(\alpha)}_{(i)j} = \frac{A^a_{\alpha/\alpha}}{2} \left[ \xi^i_{a||j} - \xi^i_{a||j} \right].
\]

\( ii) \) The following generalized Maxwell equations, derived from the PDEs system \( (G) \), hold good:

\[
\sum_{\{i, j, k\}} F^{(\alpha)}_{(i)j||k} = 0, \quad \sum_{\{i, j, k\}} F^{(\alpha)}_{(i)j||k} = 0.
\]

3.4 Constrained potentials of physical fields

Let \( (E, \pi, T) \) be a vector bundle of rank \( q \) over the Riemannian temporal manifold \( (T, h) \), whose structural group is the compact Lie orthogonal subgroup \( G \subset O_q(R) \). Let \( \{\sigma_a\}_{a=1}^q \) be local basis of cross-sections on \( E \), and let us consider \( g = (g_{ab})_{a,b=1}^q \in G \), the coordinate transformations of the bundle \( E \).

Let \( F \in \Gamma(A^2T^*T \otimes \text{End}(E)) \) be a fixed \( \text{End}(E) \)-valued 2-form on \( T \), locally expressed by

\[
F = \sum_{1 \leq \alpha < \beta \leq p} F_{\alpha\beta} dt^\alpha \wedge dt^\beta,
\]

where the components \( F_{\alpha\beta} = (F_{\alpha\beta})_{a,b=1}^q \) are skew-symmetric matrices (i. e. , \( F_{\alpha\beta} \in L(G) \subset L(O_q(R)) = o(q), \forall \alpha, \beta \in \{1, \ldots, p\} \)). Physically, the cross-section \( F \) is a geometrical model for a given physical field, in Yang-Mills sense. For example, \( F \) can be one of the gravitational, electromagnetic, strong or weak nuclear fields.

Let \( \nabla \in \Gamma(A^1T^*T \otimes \text{End}(E)) \) be a \( G \)-covariant derivative on \( E \), whose locally described by

\[
\nabla = \nabla dt^\alpha,
\]

where \( \nabla_a = (\nabla_{a\alpha})_{a=1}^q \in L(G) \subset o(q), \forall \alpha \in \{1, \ldots, p\} \). From a physical point of view, the \( G \)-connection \( \nabla \) can be viewed like a possible potential of the physical field.
$F$. We recall that a potential of the physical field $F$, in the sense of Yang-Mills, is a $G$-connection $\nabla$ on $E$, which verifies the local identity

$$F_{\alpha\beta} = \frac{\partial \nabla_\beta}{\partial t^\alpha} - \frac{\partial \nabla_\alpha}{\partial t^\beta} + \nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha, \quad \forall \alpha < \beta \in \{1, \ldots, p\}. \tag{3.1}$$

In the sequel, we will show that the $G$-potentials $\nabla$ of the given physical field $F$ are harmonic maps on the jet space $J^1(T, M)$, where $M = \Lambda^1 T^* T \otimes \text{End}(E)$.

To reach this aim, let us suppose that the set of partial derivatives $\frac{\partial \nabla_\alpha}{\partial t^\beta}, \forall \alpha \leq \beta$ verifies the constraints on $J^1(T, M)$,

$$(C) \quad \frac{\partial \nabla_\alpha}{\partial t^\beta} = f_{\alpha\beta}(t^\gamma, \nabla_\mu), \quad \forall \alpha \leq \beta,$$

where $f_{\alpha\beta}(t^\gamma, \nabla_\mu), \forall \alpha, \beta \in \{1, 2, \ldots, p\}$, are "a priori" given local components of a geometrical object on $J^1(T, M)$, which, under a multi-temporal reparametrization $(t^\nu) \leftrightarrow (t^\mu)$, transform like the local components $\frac{\partial \nabla_\alpha}{\partial t^\beta}, \forall \alpha, \beta = 1, p$. In this context, from PDE equations (3.1) we obtain

$$\frac{\partial \nabla_\alpha}{\partial t^\beta} = \nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha + f_{\beta\alpha}(t^\gamma, \nabla_\mu) - F_{\alpha\beta}(t^\gamma), \quad \forall \alpha > \beta.$$

Consequently, the PDE equations (3.1), constrained by (C), can be rewritten in the equivalent PDEs system form on $J^1(T, M)$,

$$(CPS) \quad \frac{\partial \nabla_\alpha}{\partial t^\beta} = F_{(\alpha)(\beta)}(t^\gamma, \nabla_\mu), \quad \forall \alpha, \beta = 1, p,$$

where

$$F_{(\alpha)(\beta)}(t^\gamma, \nabla_\mu) = \begin{cases} f_{\alpha\beta}(t^\gamma, \nabla_\mu), & \forall \alpha \leq \beta \\ \nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha + f_{\beta\alpha}(t^\gamma, \nabla_\mu) - F_{\alpha\beta}(t^\gamma), & \forall \alpha > \beta. \end{cases}$$

Obviously, the PDEs system (CPS) is equivalent to

$$(CPS') \quad \frac{\partial \nabla_{\alpha\beta}}{\partial t^\alpha} = F_{b(\alpha)(\beta)}(t^\gamma, \nabla_\mu), \quad \forall \alpha, \beta = 1, p, \forall a, b = 1, q.$$}

Taking into account the Riemannian structure induced from $T$ on $\Lambda^1 T^* T$,

$$<dt^\alpha \wedge dt^\beta, dt^\mu \wedge dt^\nu> = h^{\alpha\mu} h^{\beta\nu} - h^{\beta\mu} h^{\alpha\nu},$$

and the well known Riemannian metric on the Lie algebra $o(q)$,

$$<A, B> = -Tr(AB^t), \quad \forall A, B \in o(q),$$

we can endow the vector bundle $M = \Lambda^1 T^* T \otimes \text{End}(E) \to T$ with a natural Riemannian structure, on each fibre.

Now, applying the ideas of this paper to the PDEs system (CPS'), we obtain the following important result:
Theorem 3.6 Every $G$-potential $\nabla$ of the physical field $F$, which verifies the constraints $(C)$, is a harmonic map on $J^1(T,M)$, being global minimum point of the least squares Lagrangian $\mathcal{L} = L\sqrt{h}$, where the Lagrangian function $L$ is defined by

$$L = h^{\alpha\beta} h^{\mu\nu} \left\{ \frac{\partial \xi^b_{a\alpha}}{\partial t^\mu} - F^b_{a(\alpha)(\mu)} \right\} \left\{ \frac{\partial \xi^a_{b\beta}}{\partial t^\nu} - F^a_{b(\beta)(\nu)} \right\}.$$ 

Remarks 3.1 i) The least squares Lagrangian function $L$ from the previous theorem is more general than the Yang-Mills Lagrangian function, whose expression is

$$YM = h^{\alpha\beta} h^{\mu\nu} F^n_{a\alpha\beta} F^n_{b\mu\nu}.$$ 

Consequently, the Euler-Lagrange equations of the least squares Lagrangian $\mathcal{L}$ may be called the jet-extension of the Yang-Mills equations. Moreover, the $G$-connections, that verify the PDEs system $(CP S')$, may be called jet-harmonic Yang-Mills connections.

ii) The use of the least squares Lagrangian in the study of PDEs systems, allows a natural approach of the multi-time geometric dynamics, in terms of computer simulation [13], [21]. Via the multi-time geometric dynamics and its computer simulation induced by a given PDEs system of order one, it is possible to obtain an original geometrical classification of these systems. This study of classification is in our research group attention.

Open problem. What is the real physical meaning of our geometrical theory?

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References

[1] R. G. Beil, *Comparison of unified theories*, Tensor N. S. , Vol 56 (2), 175-183, (1995).

[2] J. P. Bourguignon, H. B. Lawson Jr., *Yang-Mills Theory: Its Physical Origins and Differential Geometric Aspects*, Sem. on Diff. Geom., Princeton University Press, New Jersey, 395-421, (1982).

[3] F. M. Crampin, *A Linear Connection Associated with Any Second Order Differential Equation Field*, New Developments in Geometry, L. Tamassy, J. Szenthe, Eds., Kluwer Academic Publishers, 77-86, (1996).

[4] J. Eells, L. Lemaire, *A Report on Harmonic Maps*, Bull. London Math. Soc. 10, 1-68, (1978).

[5] T. Kawaguchi, R. Miron, *A Lagrangian Model for Gravitation and Electromagnetism*, Tensor N. S. Vol 48, 153-166, (1989).

[6] R. Miron, M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Academic Publishers, (1994).
[7] M. Neagu, *Harmonic Maps between Generalized Lagrange Spaces*, Southeast Asian Bulletin of Mathematics, Springer-Verlag, (2001), in press; [http://xxx.lanl.gov/math.DG/0009138](http://xxx.lanl.gov/math.DG/0009138) (2000).

[8] M. Neagu, *Solutions of Inverse Problems for Variational Calculus*, Workshop on Diff. Geom., Global Analysis, Lie Algebras, Aristotle University of Thessaloniki, Greece, June 24-28, (1998); BSG Proceedings 4, 180-187, Editor: Prof. Dr. Grigoris Tsagas, Geometry Balkan Press, Bucharest, (2000); [http://xxx.lanl.gov/math.DG/0009139](http://xxx.lanl.gov/math.DG/0009139) (2000).

[9] M. Neagu, *The Geometry of Autonomous Metrical Multi-Time Lagrange Space of Electrodynamics*, [http://xxx.lanl.gov/math.DG/0010091](http://xxx.lanl.gov/math.DG/0010091) (2000).

[10] M. Neagu, *Upon h-Normal Γ-Linear Connection on J¹(T, M)*, (2000), [http://xxx.lanl.gov/math.DG/0009070](http://xxx.lanl.gov/math.DG/0009070).

[11] M. Neagu, C. Udrişte, *Multi-Time Dependent Sprays and Harmonic Maps on J¹(T, M)*, Third Conference of Balkan Society of Geometers, Politehnica University of Bucharest, Romania, July 31-August 3, (2000); [http://xxx.lanl.gov/math.DG/0009049](http://xxx.lanl.gov/math.DG/0009049) (2000).

[12] V. Obâdeanu, C. Vernic, *Systèmes Dynamiques sur des Éspaces de Riemann*, Balkan Journal of Geometry and Its Applications 2 (1), 73-82, (1997).

[13] D. Opriş, C. Giulvezan, *Geometric Integrators and Applications*, Ed. Mirton, Timişoara, (1999), in Romanian.

[14] R. K. Sachs, H. Wu, *General Relativity for Mathematicians*, Springer-Verlag, New-York, (1977).

[15] S. Sasaki, *Almost Contact Manifolds, I, II, III*, Mathematical Institute of Tohoku University, (1965), (1967), (1968).

[16] D. Saunders, *The Geometry of Jet Bundle*, Cambridge University Press, New York, London, (1989).

[17] Z. Shen, *Geometric Methods for Second Order Ordinary Differential Equations*, preprint, (2000).

[18] C. Udrişte, *Geometric Dynamics*, Southeast Asian Bulletin of Mathematics, Springer-Verlag, 24 (2000), 313-322; Kluwer Academic Publishers, (2000).

[19] C. Udrişte, *Solutions of DEs and PDEs as potential maps using first order Lagrangians*, Centennial Vâlceanu, Romanian Academy, University of Bucharest, June 30-July 4, 2000; [http://xxx.lanl.gov/math.DS/0007067](http://xxx.lanl.gov/math.DS/0007067) (2000).

[20] C. Udrişte, M. Neagu, *Geometrical Interpretation of Solutions of Certain PDEs*, Balkan Journal of Geometry and Its Applications, 4,1, 145-152, (2000).

[21] C. Udrişte, M. Postolache, *Least Squares Problem and Geometric Dynamics*, preprint, (2000).
[22] C. Udrişte, A. Udrişte, *From flows and metrics to dynamics*, Proc. of International Symposium on Mathematics and Mathematical Sciences, Calcutta Mathematical Society, India, Jan. 22-24, 2000; [http://xxx.lanl.gov/math.DS/0007062](http://xxx.lanl.gov/math.DS/0007062).

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