Zeckendorf representation of multiplicative inverses modulo a Fibonacci number

Gessica Alecci\(^1\) · Nadir Murru\(^2\) · Carlo Sanna\(^1\)

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Abstract
Prempreesuk, Noppakaew, and Pongsriiam determined the Zeckendorf representation of the multiplicative inverse of 2 modulo \(F_n\), for every positive integer \(n\) not divisible by 3, where \(F_n\) denotes the \(n\)th Fibonacci number. We determine the Zeckendorf representation of the multiplicative inverse of \(a\) modulo \(F_n\), for every fixed integer \(a \geq 3\) and for all positive integers \(n\) with \(\gcd(a, F_n) = 1\). Our proof makes use of the so-called base-\(\phi\) expansion of real numbers.

Keywords
Base-\(\phi\) expansion · Fibonacci number · Multiplicative inverse · Zeckendorf representation

Mathematics Subject Classification
Primary 11B39 · Secondary 11A67, 11A99

1 Introduction
Let \((F_n)_{n \geq 1}\) be the sequence of Fibonacci numbers, which is defined by the initial conditions \(F_1 = F_2 = 1\) and by the linear recurrence \(F_n = F_{n-1} + F_{n-2}\) for \(n \geq 3\). It is well known [22] that every positive integer \(n\) can be written as a sum of distinct non-

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\(\text{Nadir Murru}\)
nadir.murru@unitn.it

Gessica Alecci
gessica.alecci@polito.it

Carlo Sanna
carlo.sanna@polito.it

\(^1\) Department of Mathematical Sciences, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

\(^2\) Department of Mathematics, Università degli Studi di Trento, Via Sommarive 14, I-38123 Povo (Trento), Italy
consecutive Fibonacci numbers, that is, \( n = \sum_{i=1}^{m} d_i F_i \), where \( m \in \mathbb{N} \), \( d_i \in \{0, 1\} \), and \( d_i d_{i+1} = 0 \) for all \( i \in \{1, \ldots, m-1\} \). This is called the Zeckendorf representation of \( n \) and, apart from the equivalent use of \( F_1 \) instead of \( F_2 \) or vice versa, is unique.

The Zeckendorf representation of integer sequences has been studied in several works. For instance, Filipponi and Freitag [6, 7] studied the Zeckendorf representation of numbers of the form \( F_{kn}/F_n \), \( F_2 n/d \) and \( L_2 n/d \), where \( L_n \) are the Lucas numbers and \( d \) is a Lucas or Fibonacci number. Filipponi, Hart, and Sanchis [8, 13, 14] analyzed the Zeckendorf representation of numbers of the form \( mF_n \). Filipponi [8] determined the Zeckendorf representation of \( mF_n F_{n+k} \) and \( mL_n L_{n+k} \) for \( m \in \{1, 2, 3, 4\} \). Bugeaud [3] studied the Zeckendorf representation of smooth numbers. The study of Zeckendorf representations has been also approached from a combinatorial point of view [1, 9, 12, 21]. Moreover, generalizations of the Zeckendorf representation to linear recurrences other than the sequence of Fibonacci numbers have been considered [4, 5, 10, 11, 16].

For all integers \( a \) and \( m \geq 1 \) with \( \gcd(a, m) = 1 \), let \( (a^{-1} \mod m) \) denote the least positive multiplicative inverse of \( a \) modulo \( m \), that is, the unique \( b \in \{1, \ldots, m\} \) such that \( ab \equiv 1 \pmod{m} \). Prempreesuk, Noppakaew, and Pongsriiam [17] determined the Zeckendorf representation of \( (2^{-1} \mod F_n) \), for every positive integer \( n \) that is not divisible by 3. (The condition \( 3 \nmid n \) is necessary and sufficient to have \( \gcd(2, F_n) = 1 \).) In particular, they showed [17, Theorem 3.2] that

\[
(2^{-1} \mod F_n) = \begin{cases} 
\sum_{k=0}^{(n-7)/2} F_{n-3k-2} + F_3 & \text{if } n \equiv 1 \mod 3; \\
\sum_{k=0}^{(n-8)/2} F_{n-3k-2} + F_4 & \text{if } n \equiv 2 \mod 3;
\end{cases}
\]

for every integer \( n \geq 8 \). We extend their result by determining the Zeckendorf representation of the multiplicative inverse of \( a \) modulo \( F_n \), for every fixed integer \( a \geq 3 \) and every positive integer \( n \) with \( \gcd(a, F_n) = 1 \). Precisely, we prove the following result.

**Theorem 1.1** Let \( a \geq 3 \) be an integer. Then there exist integers \( M, n_0, i_0 \geq 1 \) and periodic sequences \( z^{(0)}, \ldots, z^{(M-1)} \) and \( w^{(1)}, \ldots, w^{(i_0)} \) with values in \( \{0, 1\} \) such that, for all integers \( n \geq n_0 \) with \( \gcd(a, F_n) = 1 \), the Zeckendorf representation of \( (a^{-1} \mod F_n) \) is given by

\[
(a^{-1} \mod F_n) = \sum_{i=i_0}^{n-1} z^{(n \mod M)}_i F_i + \sum_{i=1}^{i_0-1} w^{(i)} F_i.
\]

From the proof of Theorem 1.1 it follows that \( M, n_0, i_0, z^{(0)}, \ldots, z^{(M-1)} \), and \( w^{(1)}, \ldots, w^{(i_0)} \) can be computed from \( a \) (see also Remark 4.1 at the end of the paper).
2 Preliminaries on Fibonacci numbers

Let us recall that for every integer \( n \geq 1 \) it holds the Binet formula

\[
F_n = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}},
\]

where \( \varphi := (1 + \sqrt{5})/2 \) is the Golden ratio and \( \overline{\varphi} := (1 - \sqrt{5})/2 \) is its algebraic conjugate. Furthermore, it is well known that for every integer \( m \geq 1 \) the Fibonacci sequence \( (F_n)_{n \geq 1} \) is (purely) periodic modulo \( m \). Let \( \pi(m) \) denote its period length, or the so-called Pisano period.

The next lemma gives a formula for the inverse of \( a \) modulo \( F_n \).

**Lemma 2.1** For all integers \( a \geq 1 \) and \( n \geq 3 \) with \( \gcd(a, F_n) = 1 \), we have that

\[
(a^{-1} \mod F_n) = \frac{bF_n + 1}{a},
\]

where \( b := (-F_r^{-1} \mod a) \) and \( r := (n \mod \pi(a)) \).

**Proof** Since \( r \equiv n \pmod{\pi(a)} \), we have that \( F_r \equiv F_n \pmod{a} \). In particular, it follows that \( \gcd(a, F_r) = \gcd(a, F_n) = 1 \). Hence, \( F_r \) is invertible modulo \( a \), and consequently \( b \) is well defined. Moreover, we have that

\[
bF_n + 1 \equiv -F_r^{-1} F_r + 1 \equiv 0 \pmod{a},
\]

and thus \( c := (bF_n + 1)/a \) is an integer. On the one hand, we have that

\[
ac \equiv bF_n + 1 \equiv 1 \pmod{F_n}.
\]

On the other hand, since \( b \leq a - 1 \) and \( n \geq 3 \), we have that

\[
0 \leq c \leq \frac{(a - 1)F_n + 1}{a} = F_n - \frac{F_n - 1}{a} < F_n.
\]

Therefore, we get that \( c = (a^{-1} \mod F_n) \), as desired.

3 Preliminaries on base-\( \varphi \) expansion

We need some basic results regarding the so-called base-\( \varphi \) expansion of real numbers, which was introduced by Bergman [2] in 1957 (see also [19]), and which is a particular case of non-integer base expansion (see, e.g., [15, 18]). Let \( \mathcal{O} \) be the set of sequences in \( \{0, 1\} \) that have no two consecutive terms equal to 1, and that are not ultimately equal to the periodic sequence 0, 1, 0, 1, \ldots. Then for every \( x \in [0, 1) \) there exists a unique sequence \( \delta(x) = (\delta_i(x))_{i \in \mathbb{N}} \) in \( \mathcal{O} \) such that \( x = \sum_{i=1}^{\infty} \delta_i(x) \varphi^{-i} \). Precisely, \( \delta_i(x) = \lfloor T^{(i)}(x) \rfloor \) for every \( i \in \mathbb{N} \), where \( T^{(i)} \) denotes the \( i \)th iterate of the map
The function $T : [0, 1) \to [0, 1)$ defined by $T(\hat{x}) := (\varphi \hat{x} \mod 1)$ for every $\hat{x} \in [0, 1)$. Furthermore, letting $\mathcal{F} := \mathbb{Q}(\varphi) \cap [0, 1)$, if $x \in \mathcal{F}$ then $\delta(x)$ is ultimately periodic. In particular, if $x \in \mathcal{F}$ is given as $x = x_1 + x_2\varphi$, where $x_1, x_2 \in \mathbb{Q}$, then the preperiod and the period of $\delta(x)$ can be effectively computed by finding the smallest $i \in \mathbb{N}$ such that $T^{(i)}(x) = T^{(j)}(x)$ for some $j \in \mathbb{N}$ with $j < i$. Conversely, for every ultimately periodic sequence $d = (d_i)_{i \in \mathbb{N}}$ in $\mathcal{D}$ we have that the number $x = \sum_{i=1}^{\infty} d_i \varphi^{-i}$ belongs to $\mathcal{F}$, and $x_1, x_2 \in \mathbb{Q}$ such that $x = x_1 + x_2\varphi$ can be effectively computed in terms of the preperiod and period of $d$ by using the formula for the sum of the geometric series. Moreover, in the case that $x$ is a rational number in $[0, 1)$ then $\delta(x)$ is purely periodic [20].

The next lemma collects two easy inequalities for sums involving sequences in $\mathcal{D}$.

**Lemma 3.1** For every sequence $(d_i)_{i \in \mathbb{N}}$ in $\mathcal{D}$ and for every $m \in \mathbb{N} \cup \{\infty\}$, we have:

1. $\sum_{i=1}^{m} d_i \varphi^{-i} \in [0, 1)$ and
2. $\sum_{i=1}^{\infty} d_i (-\varphi)^{-i} \in (-1, \varphi^{-1})$.

**Proof** Since $(d_i)_{i \in \mathbb{N}}$ belongs to $\mathcal{D}$, there exists $k \in \mathbb{N}$ such that $d_k = d_{k+1} = 0$. Let $k$ be the minimum integer with such property. Then

$$
\sum_{i=1}^{\infty} d_i \varphi^{-i} = \sum_{i=1}^{k-1} d_i \varphi^{-i} + \sum_{i=k+2}^{\infty} d_i \varphi^{-i} < \sum_{j=1}^{[k/2]} \varphi^{-(2j-1)} + \sum_{i=k+2}^{\infty} \varphi^{-i} = \left(1 - \varphi^{-2\lfloor k/2 \rfloor}\right) + \varphi^{-k} \leq 1,
$$

and (1) is proved. Let us prove (2). On the one hand, we have

$$
\sum_{i=1}^{m} d_i (-\varphi)^{-i} \leq \sum_{j=1}^{m} d_{2j} \varphi^{-2j} < \sum_{j=1}^{\infty} \varphi^{-2j} = \varphi^{-1},
$$

where the second inequality is strict because $\mathcal{D}$ does not contain sequences that are ultimately equal to $(0, 1, 0, 1, \ldots)$. On the other hand, similarly, we have

$$
\sum_{i=1}^{m} d_i (-\varphi)^{-i} \geq -\sum_{j=1}^{m} d_{2j-1} \varphi^{-(2j-1)} > -\sum_{j=1}^{\infty} \varphi^{-(2j-1)} = -1.
$$

Thus (2) is proved.

The following lemma relates base-$\varphi$ expansion and Zeckendorf representation.

**Lemma 3.2** Let $N$ be a positive integer and write $N = x\varphi^m / \sqrt{5}$ for some $x \in \mathcal{F}$ and some integer $m \geq 2$. Then the Zeckendorf representation of $N$ is given by

$$
N = \sum_{i=1}^{m-1} \delta_{m-i}(x) F_i.
$$

Moreover, we have $\delta_m(x) = 0$. 
Proof Let $R := N - \sum_{i=1}^{m-1} \delta_{m-i}(x) F_i$. We have to prove that $R = 0$. Since $R$ is an integer, it suffices to show that $|R| < 1$. We have

$$\sqrt{5}N = x \varphi^m = \sum_{i=1}^{\infty} \delta_i(x) \varphi^{m-i} = \sum_{i=1}^{m} \delta_i(x) \varphi^{m-i} + \sum_{i=m+1}^{\infty} \delta_i(x) \varphi^{m-i}$$

$$= \sum_{i=0}^{m-1} \delta_{m-i}(x) \varphi^{i} + \sum_{i=1}^{\infty} \delta_{i+m}(x) \varphi^{-i}$$

$$= \sum_{i=0}^{m-1} \delta_{m-i}(x) (\varphi^i - \varphi^i) + \sum_{i=0}^{m-1} \delta_{m-i}(x) \varphi^{i} + \sum_{i=1}^{\infty} \delta_{i+m}(x) \varphi^{-i}$$

$$= \sqrt{5} \sum_{i=1}^{m-1} \delta_{m-i}(x) F_i + \sum_{i=0}^{m-1} \delta_{m-i}(x) (\varphi^{-i}) + \sum_{i=1}^{\infty} \delta_{i+m}(x) \varphi^{-i}.$$

Hence, we get that

$$\sqrt{5}R = \sum_{i=0}^{m-1} \delta_{m-i}(x) (-\varphi)^{-i} + \sum_{i=1}^{\infty} \delta_{i+m}(x) \varphi^{-i}.$$

For the sake of contradiction, suppose that $\delta_m(x) = 1$. Then $\delta_{m+1}(x) = 0$ and, by Lemma 3.1, it follows that

$$\sqrt{5}R = 1 + \sum_{i=1}^{m-1} \delta_{m-i}(x) (-\varphi)^{-i} + \sum_{i=2}^{\infty} \delta_{i+m}(x) \varphi^{-i} \in (1 - 1 + 0, 1 + \varphi^{-1} + \varphi^{-1}) = (0, \sqrt{5}),$$

which is a contradiction, since $R$ is an integer.

Therefore, $\delta_m(x) = 0$ and, again by Lemma 3.1, we have

$$\sqrt{5}R = \sum_{i=1}^{m-1} \delta_{m-i}(x) (-\varphi)^{-i} + \sum_{i=1}^{\infty} \delta_{i+m}(x) \varphi^{-i} \in (-1 + 0, \varphi^{-1} + 1) \subseteq (-\sqrt{5}, \sqrt{5}),$$

so that $|R| < 1$, as desired.

The next lemma regards the base-$\varphi$ expansions of the sum of two numbers.

Lemma 3.3 Let $x, y \in [0, 1), m \in \mathbb{N}$, and put $v := x + y \varphi^{-m}$. Suppose that there exists $\lambda \in \mathbb{N}$ such that $\lambda + 2 \leq m$ and $\delta_\lambda(x) = \delta_{\lambda+1}(x) = 0$. Then, putting

$$w := \sum_{i=\lambda+2}^{\infty} \delta_i(x) \varphi^{-i} + \sum_{i=m+1}^{\infty} \delta_{i-m}(y) \varphi^{-i},$$
we have that $v, w \in [0, 1)$ and

$$
\delta_i(v) = \begin{cases} 
\delta_i(x) & \text{if } i \leq \lambda, \\
\delta_i(w) & \text{if } i > \lambda,
\end{cases}
$$

for every $i \in \mathbb{N}$.

**Proof** From Lemma 3.1(1), we have that

$$
0 \leq w < \varphi^{-(\lambda+1)} + \varphi^{-m} < \varphi^{-(\lambda+1)} + \varphi^{-(\lambda+2)} = \varphi^{-\lambda}.
$$

Hence, $w \in [0, \varphi^{-\lambda}) \subseteq [0, 1)$ and so $w = \sum_{i=\lambda+1}^{\infty} \delta_i(w)\varphi^{-i}$. Therefore, recalling that $\delta_{\lambda+1}(x) = 0$, we get that

$$
v = x + y\varphi^{-m} = \sum_{i=1}^{\infty} \delta_i(x)\varphi^{-i} + \sum_{i=1}^{\infty} \delta_i(y)\varphi^{-i-m} = \sum_{i=1}^{\infty} \delta_i(x)\varphi^{-i} + \sum_{i=m+1}^{\infty} \delta_i(y)\varphi^{-i}
$$

$$
= \sum_{i=1}^{\lambda} \delta_i(x)\varphi^{-i} + w = \sum_{i=1}^{\lambda} \delta_i(x)\varphi^{-i} + \sum_{i=\lambda+1}^{\infty} \delta_i(w)\varphi^{-i},
$$

which is the base-$\varphi$ expansion of $v$. (Note that $\delta_{\lambda}(x) = 0$.) In particular, by Lemma 3.1(1), we have that $v \in [0, 1)$. Thus (1) follows.

### 4 Proof of Theorem 1.1

Fix an integer $a \geq 3$. Let us begin by defining $M, n_0, i_0, \text{ and } z^{(0)}, \ldots, z^{(M-1)}$. Put $M := \pi(a)$. For each $r \in \{0, \ldots, M-1\}$ with $\gcd(a, F_r) = 1$, let $b_r := (-F_r^{-1} \mod a), x_r := b_r/a, \text{ and } z^{(r)} := \delta(x_r)$. Note that $x_r \in (0, 1)$. Since $x_r$ is a positive rational number, we have that $z^{(r)}$ is a (purely) periodic sequence belonging to $\mathcal{D}$. Let $\ell$ be the least common multiple of the period lengths of $z^{(0)}, \ldots, z^{(M-1)}$, and put $i_0 := \ell + 3$.

Finally, let $n_0 := \max\{i_0 + 1, \lceil \log(2a)/\log \varphi \rceil \}$.

Pick an integer $n \geq n_0$ with $\gcd(a, F_n) = 1$ and, for the sake of brevity, put $r := (n \mod M)$. From Lemma 2.1 and Binet’s formula (2), we get that

$$
(a^{-1} \mod F_n) = \frac{b_r F_n + 1}{a} = \frac{b_r (\varphi^n - \bar{\varphi}^n)}{\sqrt{5}a} + \frac{1}{a} = (x_r + y_n \varphi^{-n}) \frac{\varphi^n}{\sqrt{5}},
$$

where

$$
y_n := \frac{\sqrt{5}}{a} - x_r (-\varphi)^{-n}.
$$
Since \( n \geq n_0 \), it follows that \( y_n \in (0, 1) \) and \( x_r + y_n \varphi^{-n} \in (0, 1) \). Therefore, from (2) and Lemma 3.2, we get that

\[
(a^{-1} \mod F_n) = \sum_{i=1}^{n-1} \delta_{n-i}(x_r + y_n \varphi^{-n}) F_i.
\]

Since \( \delta(x_r) \) is (purely) periodic and belongs to \( \mathcal{D} \), we have that \( \delta(x_r) \) contains infinitely many pairs of consecutive zeros. Furthermore, since the period length of \( \delta(x_r) \) is at most \( \ell \), we have that among every \( \ell + 1 \) consecutive terms of \( \delta(x_r) \) there are two consecutive zero. In particular, there exists \( \lambda = \lambda(r) \) such that \( n - \ell - 3 \leq \lambda \leq n - 2 \) and \( \delta_{\lambda}(x_r) = \delta_{\lambda+1}(x_r) = 0 \). Consequently, by Lemma 3.3, we get that \( \delta_{i}(x_r + y_n \varphi^{-n}) = \delta_{i}(x_r) \) for each positive integer \( i \leq \lambda \) and, a fortiori, for each positive integer \( i \leq n - i_0 \). Therefore, we have that

\[
(a^{-1} \mod F_n) = \sum_{i=i_0}^{n-1} \delta_{n-i}(x_r + y_n \varphi^{-n}) F_i + \sum_{i=1}^{i_0-1} \delta_{n-i}(x_r + y_n \varphi^{-n}) F_i
\]

where \( w^{(1)}, \ldots, w^{(i_0)} \) are the sequences defined by \( w^{(i)} := \delta_{n-i}(x_r + y_n \varphi^{-n}) \). Note that, by construction,

\[
z^{(r)}_{i_0}, z^{(r)}_{i_0-1}, \ldots, z^{(r)}_{1}, y^{(i_0-1)}, y^{(i_0-2)}, \ldots, y^{(1)}
\]

is a string in \( \{0, 1\} \) with no consecutive zeros. Hence, (3) is the Zeckendorf representation of \( (a^{-1} \mod F_n) \).

It remains only to prove that \( w^{(1)}, \ldots, w^{(i_0)} \) are periodic. By (3) and the uniqueness of the Zeckendorf representation, it suffices to prove that

\[
R(n) := (a^{-1} \mod F_n) - \sum_{i=i_0}^{n-1} z^{(r)}_{n-i} F_i = \sum_{i=1}^{i_0-1} w^{(i)}_{n} F_i
\]

is a periodic function of \( n \). From the last equality in (4), we have that \( 0 \leq R(n) < \sum_{i=1}^{i_0-1} F_i \). (Actually, one can prove that \( 0 \leq R(n) < F_{i_0} \), but this is not necessary for our proof.) Fix a prime number \( p > \max\{a, \sum_{i=1}^{i_0-1} F_i\} \). It suffices to prove that \( R(n) \) is periodic modulo \( p \). Recalling that \( (a^{-1} \mod F_n) = (b_r F_n + 1)/a \) and that the sequence of Fibonacci numbers is periodic modulo \( p \), it follows that \( (a^{-1} \mod F_n) \) is periodic modulo \( p \). Hence, it suffices to prove that \( R'(n) := \sum_{i=i_0}^{n-1} z^{(r)}_{n-i} F_i \) is periodic modulo \( p \). Using that \( z^{(r)} \) has period length dividing \( \ell \), we get that
\[ R'(n + \ell M) - R'(n) = \sum_{i = i_0}^{n+\ell M-1} z_{n+i M-1}^{(n+\ell M \mod M)} F_i - \sum_{i = i_0}^{n-1} z_{n-i}^{(r)} F_i \]
\[ = \sum_{i = i_0}^{n+\ell M-1} z_{n+i M-1}^{(r)} F_i - \sum_{i = i_0}^{n-1} z_{n-i}^{(r)} F_i \]
\[ = \sum_{i = n}^{n+\ell M-1} z_{n+i M-1}^{(r)} F_i + \sum_{i = i_0}^{n-1} (z_{n+i M-1}^{(r)} - z_{n-i}^{(r)}) F_i \]
\[ = \sum_{j = 1}^{\ell M} z_j^{(r)} F_{n+\ell M-j}, \]

which is a linear combination of sequences that are periodic modulo \( p \). Hence \( R'(n) \) is periodic modulo \( p \). The proof is complete.

**Remark 4.1** The proof of Theorem 1.1 provides a way to compute the positive integers \( M, i_0, n_0 \) and the periods of the periodic sequences \( z^{(0)}, \ldots, z^{(M-1)} \) and \( w^{(1)}, \ldots, w^{(i_0)} \). Indeed, going through the proof, we have that: \( M = \pi(a) \) is the Pisano period of \( a \), which can be computed in an obvious way; \( z^{(r)} = \delta((-F_r^{-1} \mod a)/a) \) and so the period of \( z^{(r)} \) can be computed as explained at the beginning of Section 3; \( i_0 \) and \( n_0 \) have simple formulas in terms of \( \ell \), which is the least common multiple of the period lengths of \( z^{(0)}, \ldots, z^{(M-1)} \). Finally, the periods of \( w^{(1)}, \ldots, w^{(i_0)} \) can be computed from (4) and the fact that \( R(n) \) is periodic with period length at most \( \pi(p)^2 \ell M \), which follows from the arguments after (4). However, note that proceeding in this way might be impractical, since \( \ell \) might be exponential in \( M \), and thus \( p \) might be double exponential in \( M \); making the search for the periods of \( w^{(1)}, \ldots, w^{(i_0)} \) extremely long.

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