Few Cycle Optical Pulse Propagation: a detailed calculation

Paul Kinsler
Department of Physics*, Imperial College, Prince Consort Road, London SW7 2BW, United Kingdom.
(Dated: September 5, 2018)

This document contains my detailed calculation of the Generalised Few-cycle Envelope Approximation (GFEA) propagation equation reported and used in Phys. Rev. A67, 023813 (2003) [1] and its associated longer version at arXiv.org [2]. This GFEA propagation equation is intended to be applicable to optical pulses only a few cycles long, a regime where the standard Slowly Varying Envelope Approximation (SVEA) fails.

The calculation is intended to be as complete as possible, but is still a “work in progress”, and so may, despite my best efforts, contain occasional mistakes. It is an edited version of a longer document from which on-going work has been excised. Please contact me if you have any comments, corrections or queries.

[*] I worked at this institution while doing the bulk of this calculation. My main project was with Prof. G.H.C. New on few-cycle optical pulses, and I was funded with money from the EPSRC.

IMPORTANT NOTE: this calculation is of historical interest only: the current state of the art is summarized in arXiv:0707.0982. Please note that the approach taken here is now entirely redundant.

WWW: QOLS Group http://www.qols.ph.ic.ac.uk/
WWW: Physics Dept. http://www.ph.ic.ac.uk/
WWW: Imperial College http://www.ic.ac.uk/
Email: Paul Kinsler Dr.Paul.Kinsler@physics.org
Email: G.H.C. New g.new@ic.ac.uk

Contents

I. Introduction 3

References 3

II. Envelopes and Carriers 4

A. A Phase Function 5
B. A Single Envelope Represents a Set of Pulses 5
C. The so-called “carrier phase”, i.e. the pulse phase 6

III. Extending Brabec and Krausz: the post-transform envelope 6

A. The linear electric susceptibility 7
B. The envelope and carrier 8
C. Scaled co-moving variables 9
D. Aside: Brabec and Krausz approximation criteria 11
E. The Generalised Few-Cycle Envelope equation 11
F. The nonlinear “few cycle” term 13

IV. Approximations: SEWA, SEEA, and GFEA 14

1. Note on the use of moduli 14
A. Evolution: \( \partial^2_t \) approximation 15
B. Dispersion: \( \partial_{\omega} \) Approximation 16
C. Diffraction: \( \nabla^2_\perp \) approximations 18
D. Nonlinearity: $B(\vec{r}_\perp, \xi, \tau; A)$ approximations
I. INTRODUCTION

This report is a calculation intended to generate an evolution equation for an envelope approximation description of pulse propagation in the few-cycle regime. I try to fully investigate each step and approximation, include all the algebra, and discuss any subtleties that may arise.

The calculation parallels and extends that in T. Brabec, F. Krausz, “Nonlinear optical pulse propagation in the single-cycle regime” [3]. I also (amongst other things) make some comments about their slowly-evolving-wave-approximation (SEWA).

The calculation includes all the steps taken from the Brabec and Krausz[3] starting point of (their eqn.(1))

\[
(\partial^2_t + \nabla^2_{\perp}) E(\vec{r}, t) - \frac{1}{c^2} \partial^2_t \int_{-\infty}^{t} dt' \epsilon(t - t') E(\vec{r}, t') = \frac{4\pi}{c^2} \partial^2_t P(\vec{r}, t),
\]

through to their basic nonlinear envelope equation (NEE) (their eqn.(6))

\[
\partial_\tau A = -\frac{\alpha_0}{2} A + i \hat{D} A + \frac{i}{2\beta_0} \left(1 + \frac{i\beta_1}{\beta_0} \partial_\tau\right)^{-1} \nabla^2_{\perp} A + \frac{2\pi\beta_0}{n_0} \left(1 + \frac{i\omega_0}{\omega_0} \partial_\tau\right) B.
\]

This report also derives the NEE equation found by M.A. Porras, “Propagation of single-cycle pulsed light beams in dispersive media” [4] which extends the Brabec and Krausz[3] theory so as to treat diffraction and self-focussing better, but which neglects the nonlinearity, giving their SEEA equation:

\[
\partial_\tau A = -\frac{\alpha_0}{2} A + i \hat{D} A + \frac{i}{2\beta_0} \left(1 + \frac{i\beta_1}{\beta_0} \partial_\tau\right)^{-1} \nabla^2_{\perp} A.
\]

There seem to be hints of the Brabec and Krausz[3] result in the early paper by J.A. Fleck, “Ultra-short pulse generation by Q-sqitched lasers” [5]. Fleck[5] has an envelope eqn.(2.3a,b) of (not using eqn.(2.2))

\[
\eta c \partial_\tau E^+ + \partial_\tau E^+ = -(1 + i\delta) P' + c.c.,
\]

\[
\eta c \partial_\tau E^- - \partial_\tau E^- + 2i k = -(1 + i\delta) P' + c.c.,
\]

where the 1 + i\delta results from a 1 + \partial_t/\omega term acting on the polarization, which has a linearly decaying memory with characteristic time T_2 (and \delta = (\omega T_2)^{-1}); dispersion and diffraction are not included. It is seems plausible that we can get from eqn (1.2) to (1.5) – we scale the variables (the \partial_\tau will disappear with a co-moving frame), neglect diffusion, neglect dispersion, and introduce the same polarization model. Etc. This will be clearer as we follow the full calculation in the next sections.

An early review of attempts to describe ultra-short pulse propagation was given by J. G.L. Lamb [6].

[1] P. Kinsler and G. H. C. New, Phys. Rev. A 67, 023813 (2003), URL http://link.aps.org/abstract/PRA/v67/e023813
[2] P. Kinsler and G. H. C. New, arXiv.org physics, 0212016 (2002), URL http://arXiv.org/physics/0212016
[3] T. Brabec and F. Krausz, Phys. Rev. Lett. 78, 3282 (1997), URL http://link.aps.org/abstract/PRL/v78/p3282
[4] M. A. Porras, Phys. Rev. A 60, 5069 (1999), URL http://link.aps.org/abstract/PRA/v60/p5069
[5] J. A. Fleck, Phys. Rev. B 1, 84 (1970), URL http://link.aps.org/abstract/PRB/v1/p84
[6] J. G.L. Lamb, Rev. Mod. Phys. 43 (1971), URL http://link.aps.org/abstract/RMP/v43/p99
[7] E. G. Kanetsyan, IQEC Techn. Digest p. 465 (2002), abstract number QTHL20.
[8] J. Xiao, Z. Wang, and Z. Xu, Phys. Rev. A 65, 031402(R) (2002), URL http://link.aps.org/abstract/PRA/v65/e031402
[9] P. Kinsler, Two level atoms and the few-cycle regime (Personal Report, 2002), URL file:twolevelatom.dvi
[10] S. Chelkowski and A. D. Bandrauk, Phys. Rev. A 65, 061802(R) (2002), URL http://link.aps.org/abstract/PR/v65/e061802
The substitution used in the “envelope approximation” is just a splitting of a general waveform (of e.g. the electric field amplitude) into two parts, an “envelope”, and a “carrier”: where the carrier part is intended to carry almost all of the oscillatory part of the waveform. It is usual to describe a plane-polarized wave, with the B field perpendicular to, in phase with, and proportional to (in amplitude) the E field. It is also possible to do such a calculation with circularly polarized field variables \( \mathbf{r} \) (using \( E_x = E_x + i E_y \) rather than just \( E_z \)).

For a forwardly propagating carrier \( \exp[i(\beta_0 z - \omega_0 t + \psi_0)] \) and an envelope \( A(\mathbf{r}_\perp, z, t) \), the substitution is

\[
E(\mathbf{r}, t) = A(\mathbf{r}_\perp, z, t)e^{i(\beta_0 z - \omega_0 t + \psi_0)} + A^*(\mathbf{r}_\perp, z, t)e^{-i(\beta_0 z - \omega_0 t + \psi_0)}
\]

\[= A(\mathbf{r}_\perp, z, t)e^{i\Xi} + A^*(\mathbf{r}_\perp, z, t)e^{-i\Xi}, \quad \Xi = (\beta_0 z + \omega_0 t + \psi_0). \tag{2.1} \]

Note that the carrier is forward propagating because of the chosen signs on the wavevector \( (\beta_0) \) and frequency \( (\omega_0) \) parts; both terms in eqn. (2.1) are forward propagating (c.f. a wave described by e.g. \( f(x - vt) \)). Also note that the Poynting vector for the carrier field also has a direction, given by \( E \times B \) – and here \( E \) is determined by \( E \) (c.f. Fleck’s approach). Using a forward carrier means that any backward propagating components that happen to be in the \( E \) waveform need to be contained in the envelope – unless extra backward carrier terms are added to the substitution above. However, since in many useful situations an initially forwardly propagating wave does not develop a significant backward propagating component, we can use approximations to work in a regime where backward contributions are negligible, rather than complicate our representation of the \( E \) waveform: a forward+backward carrier equation might look like this:

\[
E(\mathbf{r}, t) = A(\mathbf{r}_\perp, z, t)e^{i(\beta_0 z - \omega_0 t + \psi_0)} + A^*(\mathbf{r}_\perp, z, t)e^{-i(\beta_0 z - \omega_0 t + \psi_0)}
\]

\[+ B(\mathbf{r}_\perp, z, t)e^{i(\beta_0 z + \omega_0 t + \psi_-)} + B^*(\mathbf{r}_\perp, z, t)e^{-i(\beta_0 z + \omega_0 t + \psi_-)}. \tag{2.4} \]

Typically we then try to factor the carrier part out of our equations of motion for the waveform, and simplify the envelope equation of motion by making approximations based on assumptions of (e.g.) smoothness of the envelope, the small contributions from backward propagating terms, and so on. This then leaves us with a (hopefully) manageable approach. Using a forward carrier means that any backward propagating components that happen to be in the \( E \) waveform need to be contained in the envelope – unless extra backward carrier terms are added to the substitution above. However, since in many useful situations an initially forwardly propagating wave does not develop a significant backward propagating component, we can use approximations to work in a regime where backward contributions are negligible, rather than complicate our representation of the \( E \) waveform: a forward+backward carrier equation might look like this:

\[
E(\mathbf{r}, t) = A(\mathbf{r}_\perp, z, t)e^{i(\beta_0 z - \omega_0 t + \psi_0)} + A^*(\mathbf{r}_\perp, z, t)e^{-i(\beta_0 z - \omega_0 t + \psi_0)}
\]

\[+ B(\mathbf{r}_\perp, z, t)e^{i(\beta_0 z + \omega_0 t + \psi_-)} + B^*(\mathbf{r}_\perp, z, t)e^{-i(\beta_0 z + \omega_0 t + \psi_-)}. \tag{2.4} \]

Typically we then try to factor the carrier part out of our equations of motion for the waveform, and simplify the envelope equation of motion by making approximations based on assumptions of (e.g.) smoothness of the envelope, the small contributions from backward propagating terms, and so on. This then leaves us with a (hopefully) manageable equation for just the envelope function. Note that our choice of carrier frequency is only constrained by the need to keep the approximations manageable.

One point with the use of the envelope function substitution is that any phase-like properties of the waveform become obscured. This is because they can be contained in either (or both) the carrier and the envelope. An alternative choice in the phase of the carrier (e.g. replacing \( \psi_0 \) with some \( \psi_0' \neq \psi_0 \) will in fact mean that the envelope function is different. Plotted on a graph, this can seem to have a large effect, particularly for a few-cycle pulse, where the peak of the waveform moves about noticeably if the phase of the carrier is changed in an envelope-carrier pair – see fig. 1 and comments in Brabec and Krausz for some analysis.

Note however that the envelope (and carrier) are complex – and a phase shift of \( \exp[i(\psi_0' - \psi_0)] \) in the carrier can be exactly matched by a fixed shift of \( \exp[-i(\psi_0' - \psi_0)] \) in the envelope. This means there is no calculation or simulation problem associated with shifting the carrier phase, as long as the envelope is also adjusted. What matters is that the real \( E \) field waveform resulting from a envelope-carrier pair is correct according to the boundary/initial conditions. Confusion can only occur from the point of view of an envelope-only picture of the pulse shape.

In the usual case of a pulse that is long compared to its natural carrier frequency, the envelope should be smooth and so the relationship between the underlying carrier phase and the electric field \( E() \) is often ignored, or, if not ignored, then can be regarded as specified by the many oscillations in \( E \) at the carrier frequency. This tend to lead one towards an “envelope only” view of the pulse, which can later cause confusion for few-cycle pulses where a different choice of carrier phase leads to a different looking envelope on a \( E \) vs \( z \) (or \( t \)) graph, where only the real part is plotted. For some kind of complex plot, we would clearly see that these different looking envelopes are complex rotations of each other, and their “differences” are merely an artifact (but see the following subsection).
FIG. 1: \textbf{\textit{F-envcar}}: Diagram showing how a field \( E_i \) is decomposed into an envelope \( A_i \) and a carrier \( \Xi \), the envelope \( A \) is propagated to its final state \( A_f \), then the final field \( E_f \) is reconstructed. It also shows that an alternate field \( E'_i \) can have the same initial envelope when a different carrier \( \Xi' \) is used, and that the same propagation can be used to extract \( E'_i \)’s final state \( E'_f \).

\begin{itemize}
  \item[A. A Phase Function]

Some authors (e.g. \cite{6,8}) use a separate phase function \( \psi(\vec{r}, t) \) in order to ensure their envelope function remains real. In this case the definition in eqn.(2.3) looks like

\begin{equation}
E(\vec{r}, t) = A_R(\vec{r}_\perp, z, t)e^{i(\beta_0 z - \omega_0 t + \psi(\vec{r}, t))} + A_R(\vec{r}_\perp, z, t)e^{-i(\beta_0 z - \omega_0 t + \psi(\vec{r}, t))}.
\end{equation}

Whilst this might look like a good idea, it complicates any kind of propagation equation for the pulse envelope that we might derive: it would now contain additional derivative terms (of the phase function) and also we would need a propagation equation for the phase function itself. Further, this phase function is ambiguous (recall \( \psi + \theta \) is the same angle as \( \theta \)), would be undefined (or any value) when \( A_R \) is zero, and numerically difficult to handle when \( A_R \) is small.

I do not use this sort of phase function in this document.

\item[B. A Single Envelope Represents a Set of Pulses]

For a given set of initial conditions (usually just the electric field profile of the input pulse(s), we might pick any value of carrier phase we liked. Each different value of carrier phase would result in a different pulse envelope – but each of the resulting combinations of carrier phase and envelope will specify the same initial conditions.

So, starting with a fixed field, you can use a variety of carrier choices, and end up with a variety of envelopes. After solving for the propagation of the chosen “initial state” envelope, we get a “final state” envelope. From this we can reconstruct a unique final state \( E \) field. This parallels the left hand side of Fig. 1.

But note since a given envelope may be turned back into an electric field by applying any values for carrier phase, this final state envelope can be used to generate a range of final state electric fields. Each of these fields corresponds to the initial condition specified by the initial state envelope and that same choice of carrier phase.

So one envelope simulation provides a range of \( E_i \rightarrow E_f \) solutions, as indicated diagrammatically on fig. 1.

Note that for polarization terms with their own dynamics (e.g. a two level atom, see my report \textit{Two level atoms and the few cycle regime} \cite{9}), the choice of carrier phase alters not only the pulse envelopes but also the representation of the initial polarization state.
\end{itemize}
C. The so-called “carrier phase”, i.e. the pulse phase

Many authors publishing on short pulses refer to “the carrier phase”. By this they seem to mean something derived by comparing an inferred field envelope to the peaks and troughs of the actual (oscillating) electric field. For example, they might take the distance between the peak of the envelope as their zero (i.e. the point of reference), and the peak of the nearest electric field oscillation as giving the field phase (see fig 2). A good example of this is the paper by Chelkowski and Bandrauk [10]. To work, this “maximum amplitude” procedure assumes a number of things:

(a1) The envelope has a single peak: Of course this is not always true. Further, even if it does happen in some particular case to be true, that peak is not always well localised – such as in flat-topped pulses.

(a2) The pulse does not have a complex phase structure: Of course it is straightforward to generate real valued envelopes from an oscillating electric field profile, as the electric field is real valued. However, if (e.g.) the pulse is chirped, the electric field oscillations will no longer have a fixed frequency, so it is not possible to use and envelope-peak to field-peak distance to guess a phase without additional assumptions.

I believe it is unhelpful to talk of “carrier phase” in the contexts the term is usually applied. From a mathematical point of view, there are two phases: the carrier phase, and the envelope phase; further, the envelope phase may well have a complicated structure that obscures or overrides the role of the carrier phase. The carrier phase should be fixed according to some spatio-temporal reference point, then further discussions along the lines of “the phase of the pulse” should refer to the envelope phase, and so the phase structure of the pulse will not so easily be ignored or sidestepped. However, in the event that a single phase parameter is appropriate or desirable, a clearly defined method should be used to extract it, avoiding unreliable and subjective arguments about which value of the phase profile of the envelope is “the phase”.

III. EXTENDING BRABEC AND KRAUSZ: THE POST-TRANSFORM ENVELOPE

Brabec and Krausz[3] consider the case of small transverse inhomogeneities of the polarization, and so start with the three dimensional wave equation

$$(\partial_z^2 + \nabla^2_{\perp}) E(\vec{r}, t) - \frac{1}{c^2} \partial_t^2 \int_{-\infty}^{t} dt' \epsilon(t - t') E(\vec{r}, t') = \frac{4\pi}{c^2} \left( \partial_t \otimes \frac{qc}{n} \partial_z \right) \partial_t P_{nl}(\vec{r}, t). \quad (3.1)$$

Here (as in Brabec and Krausz[3]) $\nabla^2_{\perp}$ is the transverse Laplace operator, $\partial_\alpha$ is used as a shorthand notation for $\partial/\partial \alpha$, $\epsilon(t) = (2\pi)^{-1} \int_{-\infty}^{\infty} d\omega \tilde{\epsilon}(\omega)e^{i\omega t}$, $\tilde{\epsilon}(\omega) = 1 + 4\pi \chi(\omega)$, and $\chi(\omega)$ is the linear electric susceptibility. The electric field $E$ propagates along the $z$ direction. Both $E$ and the nonlinear polarization $P_{nl}$ are polarized parallel to the $x$ axis.
The $\varphi \frac{\partial}{\partial z}$ is a new term include to allow the calculation to apply to the $G^{\pm}$ Fleck\cite{Fleck} variables. For normal cases set $g = 0$ and forget it. Here I use an alternative “±” sign: $\varphi$ – this is to distinguish this variable sign from a later (and independent) variable sign ± cause by the choice of carrier direction.

A. The linear electric susceptibility

Now we need to treat the effects of the linear electric susceptibility. We start by fourier transforming the equation using $\exp (-\omega t)$; but neglecting to keep track of the normalisation, since this will take care of itself when we transform back. Using the correspondence $\partial_t \leftrightarrow -\omega$, the first LH term is simple, and transforms to:

$$\left(\partial_z^2 + \nabla_z^2\right) \tilde{E}(\vec{r}_\perp, z, \omega). \quad (3.2)$$

The RH term is also simple, and transforms to:

$$\frac{4\pi}{c^2} \left( -\i \omega \sum \frac{qc}{n} \frac{\partial}{\partial z} \right) \left( -\i \omega \right) \hat{P}_{nl}(\vec{r}_\perp, z, \omega). \quad (3.3)$$

The second LH term (with the $dt'$ integral) is more complicated:

$$- \int_{-\infty}^{+\infty} dt \ e^{-\i \omega t} \frac{1}{c^2} \partial_t^2 \int_{-\infty}^{+\infty} dt' \ \epsilon(t-t') E(\vec{r}_\perp, z, t') \quad (3.4)$$

$$= - \frac{(-\i \omega)^2}{c^2} \int_{-\infty}^{+\infty} dt \ e^{-\i \omega t} \int_{-\infty}^{+\infty} dt' \ \epsilon(t-t') E(\vec{r}_\perp, z, t'). \quad (3.5)$$

If the upper limit of the $dt'$ integral was $t$ and not $\infty$, the $dt'$ integral part would be the normal convolution integral; hence we could convert it into the product of the Fourier transforms of its constituents. This could be justified by saying $\epsilon$ must be causal, and so is $0$ for any $t' > t$, hence the limits of the integral can be extended to $+\infty$.

Extending the upper limit of the $dt'$ integral to $\infty$ gives

$$+ \frac{\omega^2}{c^2} \int_{-\infty}^{+\infty} dt \ e^{-\i \omega t} \int_{-\infty}^{+\infty} dt' \ \epsilon(t-t') E(\vec{r}_\perp, z, t') \quad (3.6)$$

$$= + \frac{\omega^2}{c^2} \left\{ \int_{-\infty}^{+\infty} dt \ e^{-\i \omega t} \epsilon(t) \right\} \left\{ \int_{-\infty}^{+\infty} dt \ e^{-\i \omega t} E(\vec{r}_\perp, z, t) \right\} \quad (3.7)$$

$$= + \frac{\omega^2}{c^2} \tilde{\epsilon}(\omega) \tilde{E}(\vec{r}_\perp, z, \omega). \quad (3.8)$$

The resulting transformed version of eqn. \ref{eq:3.1} is

$$\left(\partial_z^2 + \nabla_z^2\right) \tilde{E}(\vec{r}_\perp, z, \omega) + \frac{\omega^2}{c^2} \tilde{\epsilon}(\omega) \tilde{E}(\vec{r}_\perp, z, \omega) = \frac{4\pi}{c^2} \left( -\i \omega \sum \frac{qc}{n} \frac{\partial}{\partial z} \right) \left( -\i \omega \right) \hat{P}_{nl}(\vec{r}_\perp, z, \omega). \quad (3.9)$$

Now I might want to expand $\tilde{\epsilon}(\omega)$ in powers of $\omega$, but to make things easier I'll replace it with the $k^2(\omega)$ and expand $k$ about $\omega_c$ instead. Using $\tilde{\epsilon}(\omega) = c^2 k(\omega)^2 / \omega^2$ (as do Brabec and Krausz\cite{Brabec}) and then

$$k(\omega) = \sum_{n=0}^{\infty} \frac{\gamma_n (\omega - \omega_c)^n}{n!}; \quad \gamma_n = \partial^\alpha_n k(\omega)|_{\omega_c} = \beta_n + \i \alpha_n \quad (3.10)$$

Note that Brabec and Krausz\cite{Brabec} have $\alpha_n/2$ in their equations where my definitions will give $\alpha_n$. This is because the definition Brabec and Krausz\cite{Brabec} give for $\alpha_n$ (below their eqn. (BK3)) is not the one they actually use. The one they use is consistent with $\alpha$ corresponding to the decay in the intensity, not the field; and in fact Porras\cite{Porras} alters
his definition of \( \alpha_n \) from that stated by Brabec and Krausz\(^{[3]} \) (and me) in order to have terms like \( \alpha_n/2 \) appear. Porras\(^{[3]} \), despite his different definition, is consistent with my calculations. The \( \alpha_n \) that Brabec and Krausz\(^{[3]} \) use is the same as that defined by Porras\(^{[4]} \); and both Brabec and Krausz\(^{[3]} \) and Porras\(^{[4]} \) use \( \omega_c = \omega_0 \).

Using this expansion, the equation becomes

\[
(\partial_z^2 + \nabla_\perp^2) \tilde{E}(\vec{r}_\perp, z, \omega) + \frac{\omega^2}{c^2} \epsilon^2 k(\omega)^2 \tilde{E}(\vec{r}_\perp, z, \omega) = \frac{4\pi}{c^2} \left( -i\omega \mp \frac{g_c}{n} \partial_z \right) (-\omega) \tilde{P}(\vec{r}_\perp, z, \omega) \quad (3.11)
\]

\[
(\partial_z^2 + \nabla_\perp^2) \tilde{E}(\vec{r}_\perp, z, \omega) + \left[ \sum_{n=0}^{\infty} \frac{\gamma_n (\omega - \omega_n)^n}{n!} \right] \tilde{E}(\vec{r}_\perp, z, \omega) = \frac{4\pi}{c^2} \left( -i\omega \mp \frac{g_c}{n} \partial_z \right) (-\omega) \tilde{P}_n(\vec{r}_\perp, z, \omega). \quad (3.12)
\]

This can then be transformed back into the time domain (NB: \( \partial_z \leftrightarrow -i\omega \), \( i\partial_t \leftrightarrow \omega \); \( \omega - \omega_c \) \( \rightarrow (i\partial_t + i^2 \omega_c) = i(\partial_t + i\omega_c) \))

\[
(\partial_z^2 + \nabla_\perp^2) E(\vec{r}_\perp, z, t) + \left[ \sum_{n=0}^{\infty} \frac{i^n \gamma_n (\partial_t + i\omega_c)^n}{n!} \right] E(\vec{r}_\perp, z, t) = \frac{4\pi}{c^2} \left( \partial_t \mp \frac{g_c}{n} \partial_z \right) \partial_t P_n(\vec{r}_\perp, z, t). \quad (3.13)
\]

**B. The envelope and carrier**

Now I split the field up into an envelope part and a forwardly propagating carrier-wave part using the substitution

\[
E(\vec{r}, t) = A(\vec{r}_\perp, z, t) e^{i(\beta_0 z - \omega_0 t + \psi_0)} + A^*(\vec{r}_\perp, z, t) e^{-i(\beta_0 z - \omega_0 t + \psi_0)} \quad (3.14)
\]

\[
\Rightarrow e^{i\Xi^\mp} \left[ [i\beta_0 + \partial_z]^2 + \nabla_\perp^2 + \epsilon n \right] A(\vec{r}_\perp, z, t) + e^{-i\Xi^\mp} \left[ [i\beta_0 + \partial_z]^2 + \nabla_\perp^2 + \epsilon n \right] A^*(\vec{r}_\perp, z, t)
\]

and similarly for \( P_n(\vec{r}, t) = B(\vec{r}_\perp, z, t; A) e^{i\Xi^\mp} + B^*(\vec{r}_\perp, z, t; A) e^{-i\Xi^\mp} \). The symbol \( \Xi^\mp \) is introduced purely as a convenient shorthand notation for the terms in the exponential; and the minus sign (i.e. \( \Xi^- \)) refers to a forwardly propagating carrier, and plus sign (i.e. \( \Xi^+ \)) a backwardly propagating carrier. With these envelope-carrier substitutions, the equation of motion becomes

\[
e^{i\Xi^\mp} \left[ [i\beta_0 \mp \partial_z]^2 + \nabla_\perp^2 + \epsilon n \right] A(\vec{r}_\perp, z, t) + e^{-i\Xi^\mp} \left[ [i\beta_0 \pm \partial_z]^2 + \nabla_\perp^2 + \epsilon n \right] A^*(\vec{r}_\perp, z, t)
\]

\[
e^{i\Xi^\mp} \left[ [i\beta_0 \pm \partial_z]^2 + \nabla_\perp^2 + \epsilon n \right] A(\vec{r}_\perp, z, t) + e^{-i\Xi^\mp} \left[ [i\beta_0 \pm \partial_z]^2 + \nabla_\perp^2 + \epsilon n \right] A^*(\vec{r}_\perp, z, t)
\]

\[
e^{i\Xi^\mp} \left[ [i\beta_0 - \partial_z]^2 + \nabla_\perp^2 + \epsilon n \right] \nabla \left( \frac{\nabla^2}{\nabla^2} \right) A(\vec{r}_\perp, z, t) + e^{-i\Xi^\mp} \left[ [i\beta_0 + \partial_z]^2 + \nabla_\perp^2 \right] A^*(\vec{r}_\perp, z, t)
\]

Here \( \Xi^\mp = (\omega_c \pm \omega_0) / \omega_0 \), which is a quantity which usually would be set to zero – but retaining it allows me to expand the dispersion around a frequency other than \( \omega_0 \). Note the usage of \( \Xi^\mp \) and \( \omega_0 \) is clumsy, because we need to alter its sign under complex conjugation; carrier direction reversal is taken care of with the \( \pm \) notation.
I now split eqn (3.17) into two separate equations, the first “A equation” containing the terms like $e^{i\Xi^+}$, and the second “$A^*$ equation” containing the terms like $e^{-i\Xi^+}$. These two equations are simply the complex conjugates of one another, and so writing down only the first one is sufficient:

$$
e^{i\Xi^+} \left( [i\beta_0 + \partial_z]^2 + \nabla_\perp^2 \right) A(\vec{r}_\perp, z, t) + e^{i\Xi^+} \left[ \sum_{n=0}^\infty \frac{\gamma_n (\omega_0)^n}{n!} \left( \frac{\Xi^+}{\omega_0} - i \frac{\partial_z}{\omega_0} \right)^n \right]^2 A(\vec{r}_\perp, z, t) = 0 \quad (3.18)$$

$$= e^{i\Xi^+} \frac{4\pi}{c^2} \left[ (\mp i\omega_0 + \partial_t) \mp gc \left( i\beta_0 + \partial_z \right) \right] (\mp i\omega_0 + \partial_t) B(\vec{r}_\perp, z, t; A). \quad (3.19)$$

This is simplified with number of minor steps: dividing by the $e^{i\Xi^+}$ factors (which are conveniently never zero), extracting factors of $\mp i\omega_0$ from the RHS ($\Rightarrow (\mp i\omega_0 + \partial_t) = (\mp i\omega_0 \mp i\omega_0 \partial_t/(\mp i\omega_0)) = \mp i\omega_0 (1 + \partial_t/(\mp i\omega_0)) = \mp i\omega_0 (1 \mp \partial_t/(\mp i\omega_0))$, then preparing to use $1 = c\beta_0/n\omega_0$, leaves

$$\left( [i\beta_0 + \partial_z]^2 + \nabla_\perp^2 \right) A(\vec{r}_\perp, z, t) + \left[ \sum_{n=0}^\infty \frac{\gamma_n (\omega_0)^n}{n!} \left( \frac{\Xi^+}{\omega_0} - i \frac{\partial_z}{\omega_0} \right)^n \right]^2 A(\vec{r}_\perp, z, t) = 0 \quad (3.20)$$

$$= - \frac{4\pi\omega_0^2}{c^2} \left[ 1 \mp \frac{i}{\omega_0} \partial_t \right] \mp \frac{gc\beta_0}{n\omega_0} \left( 1 \mp \frac{i}{\beta_0} \partial_z \right) \left( 1 \mp \frac{i}{\omega_0} \partial_t \right) B(\vec{r}_\perp, z, t; A). \quad (3.21)$$

This appears to differ slightly from Brabec-Krausz eqn.(2) in that it has the opposite sign on the RHS – however, agreement is recovered later in eqns (3.15).

If $A_0$ is a solution of the $A$ equation, then its conjugate $A^*_0$ is a solution of the conjugate $A^*$ equation. This means that solving one solves the other, and a total waveform can then be easily reconstructed using eqn (3.15). There are no approximations made in doing this, but it may be that there are complicated (or subtle) cases where solutions of the full equation are not expressible in terms of solutions of the two separate ones.

Note that for the forward propagating carrier ($\Xi^+$, the upper sign choice in $\Xi^\pm$), there are no explicitly backward propagating terms, despite the fact that we have not excluded them in any way – this is because they do not arise spontaneously, but need to be created. Such an effect could occur in the case of multi-field systems or exotic polarization behaviour, where there may well be spatially oscillating terms from the nonlinear polarization term $B$ (e.g. exp$(\pm ik_B z)$) which could force a (possibly backward propagating) oscillation onto $A$. This would very likely violate some approximation we will want to make later, e.g. a “smooth” or slowly varying envelope function $A$. We can only neglect backward propagating components if (a) there were none to start with, and (b) by verifying (or assuming) that the nonlinear polarization has convenient properties – although we could extend eqn (3.15) to include backward carrier terms as already discussed (see section (11)).

At the equivalent point to eqns (4.21) in the Brabec-Krausz paper, they already claim to have neglected backward propagating waves (after their eqn(1)): “the neglect of backward propagating waves is consistent with the approximations that will be made in the following derivation of the envelope equation and will be commented on later” – their comment being that “excessive” change in the envelope can lead to backwardly propagating components to the envelope (see e.g. Shen (11)). I, however, leave any approximations relating to the neglect of backward terms to later on – it is still the case that (in principle) that the envelope function might contain backwardly propagating components.

This differs from Fleck (8) in that his $E^\pm$ and $E^\mp$ ($G^+$ and $G^-$ in my notation) are constructed as explicitly forward and backward propagating, and hence should not really be compared directly to my $E$ or $A$. My $E$ is in fact Fleck’s $E^+ + E^-$ ($G^+ + G^-$), and somewhere in the approximations used to get to my starting point eqn (3.11) the magnetic field parts (retained by Fleck(8)) have been assumed to be solely dependent on (derivable from) the electric field. See my derivation “A second-order wave equation using Fleck field variables” (12, 13) for more information.

C. Scaled co-moving variables

I now change into a scaled co-moving reference frame, but one slightly different to that of Brabec-Krausz – the difference being that I scale space by $\beta_0$ and the time by $\omega_0$ as well as shifting the origin. Brabec-Krausz and Porros use $\xi = z$ and $\tau = t \mp \beta_1 z$, but I instead put
\[ \xi = \beta'_0 z, \]  
\[ \tau = \omega_0 (t \mp \beta'_1 z). \]  
(3.22)  
(3.23)

Here I use \( \beta'_0 \) and \( \beta'_1 \) rather than just \( \beta_0 \) and \( \beta_1 \) because it may not always be convenient to use the natural scaled co-moving reference frame; which might well be the case for solving multi-mode problems. For an \( A^* \) equation, the signs of both \( z \) and \( t \) should be reversed so that \( \xi = -\beta'_0 z \) and \( \tau = -\omega_0 (t \mp \beta'_1 z) \). Do not think that this \( A^* \) frame (for \( \Sigma^* \), upper sign) is backwardly propagating, since although \( \xi \sim -z \), time has also reversed \( \tau \sim -t \). The derivatives for the \( A \) equation transform like

\[
\begin{align*}
\partial_t &\equiv \frac{d}{dt} = \frac{d\tau}{dt} \frac{d}{d\tau} + \frac{d\xi}{dz} \frac{d}{d\xi} = \omega_0 \frac{d}{d\tau} \equiv \omega_0 \partial_{\tau} \\
\partial_z &\equiv \frac{d}{dz} = \frac{d\xi}{dz} \frac{d}{d\xi} + \frac{d\tau}{d\xi} \frac{d}{d\tau} = \beta_0 \frac{d}{d\xi} + \omega_0 \beta'_1 \frac{d}{d\tau} \equiv \beta'_0 \partial_{\xi} + \omega_0 \beta'_1 \partial_{\tau},
\end{align*}
\]
(3.24)  
(3.25)

(for \( A^* \), \( \partial_t = -\omega_0 \partial_{\tau} \) and \( \partial_z = -\beta'_0 \partial_{\xi} \pm \omega_0 \beta'_1 \partial_{\tau} \)). The scaled co-moving \( A \) equation is then (with \( q = \beta'/\beta_0 \)). Note that now a condition that \( \xi \gg 1 \) refers to lengths \( \gg 1/\beta_0 \) (i.e. “long”), and \( \tau \gg 1 \) refers to times \( \gg 1/\omega_0 \) (i.e. “slow”). Similarly, \( \xi \ll 1 \) refers to lengths \( \ll 1/\beta_0 \) (i.e. “short”), and \( \tau \ll 1 \) refers to times \( \ll 1/\omega_0 \) (i.e. “fast”).

\[
0 = \left\{ (i\beta_0 + \beta'_0 \partial_{\xi} \mp \omega_0 \beta'_1 \partial_{\tau})^2 + \nabla_\perp^2 + \left[ \sum_{n=0}^\infty \frac{\gamma_n (-\omega_0)^n}{n!} (\nabla_+ t - i\partial_{\tau})^n \right]^2 \right\} A(\bar{r}_\perp, \xi, \tau) 
+ \frac{4\pi\omega_0^2}{c^2} \left( 1 + i\partial_{\tau} \right) g \frac{q\beta_0}{\omega_0} (1 - i\eta \partial_{\xi} - i\sigma \partial_{\tau}) \left( 1 + i\eta \partial_{\xi} \right) B(\bar{r}_\perp, \xi, \tau; A) \quad \text{... now divide through by } \beta_0^2 \quad \text{(3.26)}
\]

\[
= \left\{ \left( 1 + i\eta \partial_{\xi} \right) \left( 1 + i\eta \partial_{\xi} \right) \left( 1 + i\eta \partial_{\xi} \right) B(\bar{r}_\perp, \xi, \tau; A) \quad \text{... now expand, prepare for } D' \quad \text{(3.27)}
\right.
\]

\[
= \left\{ -1 + 2i \left( \frac{q\partial_{\xi} + \omega_0 \beta'_1 \partial_{\tau}}{\beta_0} \right) + \left( \frac{q\partial_{\xi} + \omega_0 \beta'_1 \partial_{\tau}}{\beta_0} \right) \frac{1}{\beta_0^2} \nabla_\perp^2 + \left[ \frac{\beta_0}{\beta_0} \frac{\omega_0 \beta'_1}{\beta_0} \left( \nabla_+ t - i\partial_{\tau} \right) \frac{\omega_0 \beta'_1}{\beta_0} \right]^2 \right\} A(\bar{r}_\perp, \xi, \tau) 
+ \frac{4\pi\omega_0^2}{c^2} \left[ 1 + i\eta \partial_{\xi} \right] \left[ 1 + i\eta \partial_{\xi} \right] \left( 1 + i\eta \partial_{\xi} \right) B(\bar{r}_\perp, \xi, \tau; A) \quad \text{(3.28)}
\]

\[
= \left\{ -1 + 2i \left( \frac{q\partial_{\xi} + \omega_0 \beta'_1 \partial_{\tau}}{\beta_0} \right) + \left( \frac{q\partial_{\xi} + \omega_0 \beta'_1 \partial_{\tau}}{\beta_0} \right) \frac{1}{\beta_0^2} \nabla_\perp^2 + \left[ 1 + i \left( \frac{\omega_0 \beta'_1}{\beta_0} \left( \nabla_+ t - i\partial_{\tau} \right) + \frac{\alpha_0}{\beta_0} - iD' \right) \right]^2 \right\} A(\bar{r}_\perp, \xi, \tau) 
+ \frac{4\pi\omega_0^2}{c^2} \left[ 1 + i\eta \partial_{\xi} \right] \left[ 1 + i\eta \partial_{\xi} \right] \left( 1 + i\eta \partial_{\xi} \right) B(\bar{r}_\perp, \xi, \tau; A) \quad \text{(3.29)}
\]

\[
= \left\{ \left( q\partial_{\xi} + \omega_0 \beta'_1 \partial_{\tau} \right) + \frac{1}{2i} \left( q\partial_{\xi} + \omega_0 \beta'_1 \partial_{\tau} \right) \frac{1}{2i\beta_0^2} \nabla_\perp^2 + \left[ \frac{\omega_0 \beta'_1}{\beta_0} \left( \nabla_+ t - i\partial_{\tau} \right) + \frac{\alpha_0}{\beta_0} - iD' \right]^2 \right\} A(\bar{r}_\perp, \xi, \tau) 
+ \frac{2\pi\omega_0^2}{c^2} \left[ 1 + i\eta \partial_{\xi} \right] \left[ 1 + i\eta \partial_{\xi} \right] \left( 1 + i\eta \partial_{\xi} \right) B(\bar{r}_\perp, \xi, \tau; A) \quad \text{(3.30)}
\]

\[
= \left\{ \left( q\partial_{\xi} + \omega_0 \beta'_1 \partial_{\tau} \right) + \frac{1}{2i} \left( q\partial_{\xi} + \omega_0 \beta'_1 \partial_{\tau} \right) \frac{1}{2i\beta_0^2} \nabla_\perp^2 + \left[ \frac{\omega_0 \beta'_1}{\beta_0} \left( \nabla_+ t - i\partial_{\tau} \right) + \frac{\alpha_0}{\beta_0} - iD' \right]^2 \right\} A(\bar{r}_\perp, \xi, \tau) 
+ \frac{2\pi\omega_0^2}{c^2} \left[ 1 + i\eta \partial_{\xi} \right] \left[ 1 + i\eta \partial_{\xi} \right] \left( 1 + i\eta \partial_{\xi} \right) B(\bar{r}_\perp, \xi, \tau; A) \quad \text{(3.31)}
\]
\[ + \frac{2\pi}{m_0} [1 \pm i\partial_r \oplus i g \sigma' \partial_r \oplus g (1 - i g \partial_k)] (1 \pm i \partial_r) B(\vec{r}_\perp, \xi, \tau; A). \] (3.32)

Here I have introduced the dimensionless \( \sigma = \omega_0 \beta_1 / \beta_0 = (\omega_0 / \beta_0)/(1 / \beta_1) = v_f / v_g \), \( \sigma' = \omega_0 \beta_1 / \beta_0 \), and used the fact that the refractive index at \( \omega_0 \) is \( n_0 = c \beta_0 / \omega_0 \). I also define a dispersion term \( \hat{D} \) in a similar way to Brabec and Krausz [3], but instead use a scaled (dimensionless) version \( \hat{D}' \) in following equations:

\[ \hat{D}' = \frac{n_0}{\beta_0} \hat{D} = - \frac{n_0}{\beta_0} \left[ i \omega_1 (\hat{\omega}^+ - i \partial_r) + \sum_{n=2}^\infty \gamma_n' \left( -i \omega_0 \right)^{n-1} \left( \hat{\omega}^+ - i \partial_r \right)^n \right], \] (3.33)

with \( \gamma_n' = \gamma_n \) for \( n \geq 2 \); otherwise \( \gamma_0' = 0; \gamma_1' = i \omega_1 \): the parameters \( \alpha_0, \beta_0, \beta_1 \) are handled separately from \( \hat{D} \) because of their important role.

D. Aside: Brabec and Krausz approximation criteria

Brabec and Krausz [3] introduce some criteria designed to motivate approximations to their equations (BK5a,b,c). Since I use the same variable names, but differently scaled, I here write the Brabec and Krausz criteria in their form on the LHS, and indicate with an arrow my form on the RHS.

\[
\begin{align*}
(5a) & \quad |\partial_\xi A| \ll \beta_0 |A| \quad \implies \quad |\partial_\xi A| \ll A, \\
(5b) & \quad |\partial_r A| \ll \omega_0 |A| \quad \implies \quad |\partial_r A| \ll A, \\
(5c) & \quad \left| \frac{\beta_0 - \omega_0 \beta_1}{\beta_0} \right| \ll 1. 
\end{align*}
\]

Note that \( g = 0 \) here. The motivation for the first two (5a,b) are obvious from eqn. (3.31), in my scaled co-moving frame – they allow me to say certain quantities are small, and hence I could choose to neglect them. In contrast, it is not clear how the third condition will make my equations simpler beyond removing a single prefactor (since \( |\omega_0 \beta_1 / \beta_0| \approx 1 \)); but \( |\omega_0 \beta_1 / \beta_0| \ll 1 \) might seem better still, if perhaps not as physically relevant. However, Brabec and Krausz collect their terms together differently, and indeed the situation becomes clearer after I rearrange the equations.

E. The Generalised Few-Cycle Envelope equation

I keep \( (\hat{\omega}^+ - i \partial_r) \) terms intact because for the usual case where \( \omega_r = \omega_0 \) is chosen, they will simplify to \(-i \partial_r\). Note also I have divided the equation through by \( \beta_0^2 \) rather than the single \( \beta_0 \) of Brabec and Krausz [3]. Still retaining all terms in the equation, I have

\[
0 = \left\{ (q \partial_k + \sigma' \partial_r) + \frac{1}{2t} (q \partial_k + \sigma' \partial_r)^2 + \frac{1}{2i \beta_0^2} \nabla_\perp^2 \right\} A(\vec{r}_\perp, \xi, \tau)
+ \frac{i}{2} \left[ \sigma (\hat{\omega}^+ - i \partial_r) + \left( \frac{\alpha_0}{\beta_0} - i \hat{D}' \right) \right] A(\vec{r}_\perp, \xi, \tau)
+ \frac{2\pi}{m_0} [1 \pm i\partial_r \oplus i g \sigma' \partial_r \oplus g (1 - i g \partial_k)] (1 \pm i \partial_r) B(\vec{r}_\perp, \xi, \tau; A)
\]

(a) \[
0 = \left\{ (q \partial_k + \sigma' \partial_r) + \frac{1}{2t} (q \partial_k + \sigma' \partial_r)^2 + \sigma (\hat{\omega}^+ - i \partial_r) + \left( \frac{\alpha_0}{\beta_0} - i \hat{D}' \right) \right\}
+ \frac{i}{2} \left[ -\sigma^2 (\hat{\omega}^+ - i \partial_r)^2 + 2i \sigma (\hat{\omega}^+ - i \partial_r) \left( \frac{\alpha_0}{\beta_0} - i \hat{D}' \right) \right]
+ \frac{2\pi}{m_0} [1 \pm i\partial_r \oplus i g \sigma' \partial_r \oplus g (1 - i g \partial_k)] (1 \pm i \partial_r) B(\vec{r}_\perp, \xi, \tau; A)
\]

\]

(3.37)
\[ (a) = \left\{ (q \partial q + \sigma^2 \partial_r) + \frac{i}{2} \left( q \partial q - \sigma^2 \partial_r \right)^2 + \sigma \left( \nabla^2 q - i \partial_r \right) - \frac{i}{2} \sigma^2 \left( \nabla^2 q - i \partial_r \right)^2 \right\} \]

\[-\sigma \left( \nabla^2 q - i \partial_r \right) + \left( \frac{\alpha}{\beta_0} - i \dot{D} \right)^2 + \frac{i}{2} \left( \frac{\alpha}{\beta_0} - i \dot{D} \right)^2 + \frac{1}{2 \beta_0^2} \nabla^2 \right\} A(r, \xi, \tau) + \frac{2\pi}{\mu_0} \left[ \frac{1 + i \partial_r \odot v \sigma^2 \partial_r \odot g (1 - u q \partial_q \partial_q) g (1 - u q \partial_q \partial_q) \right] B(r, \xi, \tau; A) \tag{3.39} \]

\[ (b) = \left\{ (q \partial q + \sigma^2 \partial_r) + \frac{1}{2} \left( q^2 \partial^2 q + \sigma^2 \partial^2 r \right) + \frac{1}{2} \sigma^2 \partial^2 r + \sigma \nabla^2 q + \sigma \partial_r - \frac{\sigma^2}{2} \left( \nabla^2 q - i \partial_r \right)^2 - \sigma \left( \nabla^2 q - i \partial_r \right) \left( \frac{\alpha}{\beta_0} - i \dot{D} \right) \right\} \]

\[ + \left( \frac{\alpha}{\beta_0} - i \dot{D} \right)^2 + \frac{i}{2} \left( \frac{\alpha}{\beta_0} - i \dot{D} \right)^2 + \frac{1}{2 \beta_0^2} \nabla^2 \right\} A(r, \xi, \tau) + \frac{2\pi}{\mu_0} \left[ \frac{1 + i \partial_r \odot v \sigma^2 \partial_r \odot g (1 - u q \partial_q \partial_q) g (1 - u q \partial_q \partial_q) \right] B(r, \xi, \tau; A) \tag{3.40} \]

\[ (c) = \left\{ (1 + \sigma^2 \partial_r) q \partial q + \sigma \nabla^2 q + \frac{1}{2} \left( q^2 \partial^2 q + \sigma^2 \partial^2 r \right) + (\sigma \neq \sigma^+ \partial_r) - \frac{\sigma^2}{2} \left( \nabla^2 q - i \partial_r \right)^2 - \sigma \left( \nabla^2 q - i \partial_r \right) \left( \frac{\alpha}{\beta_0} - i \dot{D} \right) \right\} \]

\[ + \left( \frac{\alpha}{\beta_0} - i \dot{D} \right)^2 + \frac{i}{2} \left( \frac{\alpha}{\beta_0} - i \dot{D} \right)^2 + \frac{1}{2 \beta_0^2} \nabla^2 \right\} A(r, \xi, \tau) + \frac{2\pi}{\mu_0} \left[ \frac{1 + i \partial_r \odot v \sigma^2 \partial_r \odot g (1 - u q \partial_q \partial_q) g (1 - u q \partial_q \partial_q) \right] B(r, \xi, \tau; A) \tag{3.41} \]

\[ (d) = \left\{ (1 + \sigma^2 \partial_r) q \partial q + \sigma \nabla^2 q + \frac{1}{2} \left( q^2 \partial^2 q + \sigma^2 \partial^2 r \right) + (\sigma \neq \sigma^+ \partial_r) - \frac{\sigma^2}{2} \left( \nabla^2 q - i \partial_r \right)^2 - \sigma \left( \nabla^2 q - i \partial_r \right) \left( \frac{\alpha}{\beta_0} - i \dot{D} \right) \right\} \]

\[ + \left( \frac{\alpha}{\beta_0} - i \dot{D} \right)^2 + \frac{i}{2} \left( \frac{\alpha}{\beta_0} - i \dot{D} \right)^2 + \frac{1}{2 \beta_0^2} \nabla^2 \right\} A(r, \xi, \tau) + \frac{2\pi}{\mu_0} \left[ \frac{1 + i \partial_r \odot v \sigma^2 \partial_r \odot g (1 - u q \partial_q \partial_q) g (1 - u q \partial_q \partial_q) \right] B(r, \xi, \tau; A). \tag{3.42} \]

This equation \(3.32\) has a nice \(1 + i (\alpha_0 / \beta_0 - i \dot{D})\) term which may be useful in a generic “small dispersion” case.

However, we want to reach the more typical Brabec and Krausz\(3\) (or Porras\(4\)) form; and so instead continue with the algebra from eqn.\(3.41\). Like Porras\(4\), I retain the natural choice of \(1 + \sigma \partial_r\) multiplier for the \(\partial_q\) term. This differs from the \(1 + i \partial_r\) used by Brabec and Krausz\(3\), and is why Porras\(4\) claims that the Brabec and Krausz\(3\) SEWA equation has the space-time focussing “slightly falsified”, mentioning Rothenburg\(14\) — it is because Brabec and Krausz\(3\) had already introduced a \(\sigma = 1\) approximation, and as such the “missing” \(\sigma\) is an approximation, not anything “slightly falsified”.

Now ruthlessly shift all the \(\nabla^2 q, \partial^2 q, \nabla^2 r, \partial^2 r\) terms to the RHS...

\[ \left\{ (1 + i \sigma^2 \partial_r) q \partial q + (\sigma \neq \sigma^+ \partial_r) + (1 + i \sigma \partial_r) \left( \frac{\alpha}{\beta_0} - i \dot{D} \right) \right\} A(r, \xi, \tau) + \frac{1}{2 \beta_0^2} \nabla^2 \right\} A(r, \xi, \tau) + \frac{2\pi}{\mu_0} \left[ \frac{1 + i \partial_r \odot v \sigma^2 \partial_r \odot g (1 - u q \partial_q \partial_q) g (1 - u q \partial_q \partial_q) \right] B(r, \xi, \tau; A) \]

\[ = \left[ \sigma \nabla^2 q - \sigma \left( \frac{\alpha}{\beta_0} - i \dot{D} \right)^2 - \frac{\sigma^2}{2} \left( \nabla^2 q - i \partial_r \right)^2 + \left( \frac{1}{2 \beta_0^2} \nabla^2 \right)^2 \right] A(r, \xi, \tau). \tag{3.43} \]

Presumably Brabec and Krausz\(3\) used their eqn.\( (BK4)\) to motivate their \(BK5c)\) \(\sigma \simeq 1\) to get a \((1 + i \partial_r)\) term — because they wanted to cancel the same factor from the nonlinear term multiplying \(B(r, \xi, \tau; A)\). This can be contrasted with Porras\(4\), who was not interested in the nonlinear case but wanted instead to cancel terms in the “\(\nabla^2 \)” diffraction (or self-focusing) term. In fact using Brabec and Krausz\(3\) eqn.\( (BK4)\) as a midpoint on the way to Brabec and Krausz\(3\) eqn.\( (BK6)\) is not the best path, even for Brabec and Krausz\(3\) — their early selection of a \((1 + i \partial_r)\) prefactor for \(\partial_q\) is unnecessary. I show this here, but first hide all the \(r\) dependent terms in \(T_r\) (as they are usually zero anyway), and the rest of the RHS terms in \(T_r\) because they remain unchanged.

After some sign changes caused by moving factors of \(i\) from the denominator to the numerator, and by moving one inside a set of \(()\) brackets, we get...
\[ 0 = (1 \pm ıσ' \partial_r) q \partial_ξ A(\vec{r}_⊥, ξ, τ) + (σ \mp σ') \partial_r A(\vec{r}_⊥, ξ, τ) - (1 + ıσ \partial_r) \left( -\frac{α_0}{β_0} + ı\hat{D}' \right) A(\vec{r}_⊥, ξ, τ) \]
\[ - \frac{i}{2β_0^2} \nabla^2_⊥ A(\vec{r}_⊥, ξ, τ) - \frac{2πı}{n_0^2} \left[ 1 \pm ıσ' \partial_r \mp ıσ \partial_r \right] g(1 - ıσ \partial_r) B(\vec{r}_⊥, ξ, τ; A) - T_τ - T_{RHS}, \quad (3.44) \]

\[ T_τ = -σ \nabla^2_⊥ \left[ 1 - \left( \frac{α_0}{β_0} - ı\hat{D}' \right) - \frac{ıσ}{2} \nabla^2_⊥ + ıσ \partial_r \right] A(\vec{r}_⊥, ξ, τ) \quad (3.45) \]
\[ T_{RHS} = \frac{i}{2} \left[ -q^2 \partial^2_ξ + (σ^2 - σ'^2) \partial^2_r + \left( \frac{α_0}{β_0} - ı\hat{D}' \right)^2 \right] A(\vec{r}_⊥, ξ, τ). \quad (3.46) \]

Hence
\[ q \partial_ξ A(\vec{r}_⊥, ξ, τ) = -\frac{σ \mp σ'}{1 \pm ıσ' \partial_r} \partial_r A(\vec{r}_⊥, ξ, τ) + \frac{1 + ıσ \partial_r}{1 \pm ıσ' \partial_r} \left( -\frac{α_0}{β_0} + ı\hat{D}' \right) A(\vec{r}_⊥, ξ, τ) + \frac{i}{2β_0^2 (1 \pm ıσ' \partial_r)} \nabla^2_⊥ A(\vec{r}_⊥, ξ, τ) \]
\[ + \frac{2πı (1 + ıσ' \partial_r)^2}{n_0^2 (1 \pm ıσ' \partial_r)} B(\vec{r}_⊥, ξ, τ; A) + \frac{T_τ + T_{RHS}}{1 \pm ıσ' \partial_r}. \quad (3.47) \]

Now to avoid overly complex equations I set \( \sigma' = ±σ \) (i.e. \( β'_1 = β_1 \)), and a carrier appropriate to the group velocity by choosing the upper sign), forcing the \( τ \) scaling for the field to match the material rather than (e.g.) another field. Note that in the case of multiple field components with different group velocities, it may be necessary to have \( β'_1 \neq β_1 \) in order to keep all of the co-moving frames aligned.

One final step now gives us a Generalised Few-Cycle Envelope equation for a propagating pulse in a nonlinear medium. It is in the style of Brabec and Krausz\[3\], but unlike that of Brabec and Krausz\[3\] (and of Porras\[4\]), it has no approximations beyond that from the starting point, the permittivity convolution, and the separation of \( \hat{A} \) and \( A^* \) propagation equations. Remembering to expand the \( T_{RHS} \) and \( T_τ \) terms as necessary, with \( g = 0 \), we have the Generalised Few-Cycle Envelope Approximation (GFEA) equation:
\[ q \partial_ξ A(\vec{r}_⊥, ξ, τ) = \left( -\frac{α_0}{β_0} + ı\hat{D}' \right) A(\vec{r}_⊥, ξ, τ) + \frac{i}{2β_0^2 (1 \pm ıσ' \partial_r)} \nabla^2_⊥ A(\vec{r}_⊥, ξ, τ) \]
\[ + \frac{2πı (1 + ıσ' \partial_r)^2}{n_0^2 (1 \pm ıσ' \partial_r)} B(\vec{r}_⊥, ξ, τ; A) + \frac{T_τ + T_{RHS}}{1 \pm ıσ' \partial_r}. \quad (3.48) \]

NOTE: If \( g \neq 0 \), replace one of the \( (1 + ıσ' \partial_r) \) numerator terms multiplying \( B \) in eqn. (3.48) with \([1 \pm ıσ' \partial_r \mp ıσ \partial_r \pm ıσ \partial_r] g(1 - ıσ \partial_r)\).

### F. The nonlinear “few cycle” term

We might prefer to handle the nonlinear term by basing it on \( (1 + ıσ \partial_r) \), for example if we intended to neglect the diffraction term entirely (e.g. as in Brabec and Krausz\[3\]). By starting at eqn (3.48) we see the possibility of a nice expansion in \( δ = 1 - σ \) by simply replacing the entire \( B \) prefactor term in eqn (3.48) with:
\[ \frac{(1 + ıσ' \partial_r)^2}{(1 + ıσ \partial_r)} B(\vec{r}_⊥, ξ, τ; A) = \frac{(1 + ıσ \partial_r)[(1 + ıσ \partial_r) + ı(1 - σ) \partial_r]}{(1 + ıσ \partial_r)} B(\vec{r}_⊥, ξ, τ; A) \]
\[ = (1 + ıσ \partial_r) \left[ 1 + ı(1 - σ) \partial_r \frac{(1 + ıσ \partial_r)^2}{(1 + ıσ \partial_r)^2} \right] B(\vec{r}_⊥, ξ, τ; A). \quad (3.49) \]

We could instead rearrange the \( B \) term by aiming at the \( 1 + ıσ \partial_r \) form of Porras\[4\]:
We need to consider these terms in detail, and understand in what limits these terms might be simplified or neglected. Criteria that need to hold for their SEWA to be valid, all of which are that various quantities must be slowly varying.

We see how my generalised equation (3.48) reduces to Brabec and Krausz’s

\[ T = \text{the “extra” terms.} \]

It would be better, of course, to expand the few cycle term to a fixed order explicitly – there will be many expansions like the above, that when truncated to first order, are correct to within terms of second order; but which differ from each other by amounts that are also of second order. So:

\[
\frac{(1 + i\sigma\dot{r})^2}{(1 + i\sigma\dot{r})} = \frac{(1 + 2i\sigma\dot{r} - \sigma^2)}{(1 + i\sigma\dot{r})^{-1}}
\]

It is interesting that this systematic expansion gives yet another first order correction to the nonlinear polarization – but of course it only differs from Brabec and Krausz’s and Porras’s by terms like \( (1 - \sigma) \dot{r} \), which are second order corrections. Note again that Brabec and Krausz work in the case \( \sigma = 1 \), whereas Porras allows \( \sigma \neq 1 \), but does not consider nonlinear processes.

### IV. Approximations: SEWA, SEEA, and GFEA

The full generalised few-cycle equation (3.48) has a rather complicated prefactor for the polarization term, and also the “extra” \( T_{RHS} \) term. If we want to make approximations that reduce it to Brabec and Krausz’s SEWA (slowly evolving wave approximation), Porras’s SEEA (slowly evolving envelope approximation), or some other form then we need to consider these terms in detail, and understand in what limits these terms might be simplified or neglected. Note that setting \( T_Y = 0 \) is a matter of chosen convention, and is not an approximation of any kind.

For example, Brabec and Krausz have a discussion in their PRL on p3284, after their eqn.(8) about the various criteria that need to hold for their SEWA to be valid, all of which are that various quantities must be slowly varying as the pulse propagates along \( \xi \). It is also instructive to see what parameter values or what terms are neglected to see how my generalised equation (3.48) reduces to Brabec and Krausz’s SEWA (slowly evolving wave approximation), Porras’s SEEA (slowly evolving envelope approximation), or to my GFEA (generalised few-cycle envelope approximation), which is in some sense equivalent to a “best of” combination of the two others.

1. **SEWA:** Brabec and Krausz’s eqn.(6). In eqn. (3.48), set \( \sigma = 1 \), use \( T_Y = 0 \) (since \( \omega_i = \omega_0 \)) and ignore \( T_{RHS} \):

\[
g\dot{\xi}A(\vec{r}_\perp, \xi, \tau) = \left( -\frac{\alpha_0}{\beta_0} + i\dot{D} \right) A(\vec{r}_\perp, \xi, \tau) + \frac{i}{2\beta_0^2 (1 + i\sigma_\tau)} \nabla_\perp^2 A(\vec{r}_\perp, \xi, \tau) + \frac{2\pi}{n_0} (1 + i\sigma_\tau) B(\vec{r}_\perp, \xi, \tau; A) (4.1)
\]

2. **SEEA:** Porras’s eqn.(2). In eqn. (3.48), set \( B(\vec{r}_\perp, \xi, \tau; A) = 0 \), use \( T_Y = 0 \) and ignore \( T_{RHS} \):
over some finite timescale relevant to the dynamics, without having to deal directly with instantaneous derivatives. This could then give us constraints valid for small $\delta$ if the following condition holds:

$$\left| (1 + \sigma \partial_\tau)^{-1} \left[ \partial_\xi^2 - \left( \frac{\alpha_0}{\beta_0} - i \hat{D} \right)^2 \right] A(\bar{r}_\perp, \xi, \tau) \right| \ll |\partial_\xi A(\bar{r}_\perp, \xi, \tau)|. \tag{4.5}$$

A nice way of dealing with the presence of the $\partial_\tau$ terms is to Fourier transform into the frequency domain, where

$$i \partial_\tau \rightarrow \Omega. \tag{4.6}$$

This enables us to avoid speculation about the possible time derivatives of $A$, and instead constrain its frequency components. However, this assumes knowlege of the all-time behaviour of the terms under consideration, so when being careful it might be better to use a time-windowed transform or similar. This could then give us constraints valid over some finite timescale relevant to the dynamics, without having to deal directly with instantaneous derivatives.

The condition (4.5) can be broken into two parts, which are

$$|\partial_\xi^2 A(\bar{r}_\perp, \xi, \tau)| \ll |(1 + \sigma \partial_\tau) \partial_\xi A(\bar{r}_\perp, \xi, \tau)|, \tag{4.7}$$

$$\left| \left( \frac{\alpha_0}{\beta_0} - i \hat{D} \right)^2 A(\bar{r}_\perp, \xi, \tau) \right| \ll |\partial_\xi A(\bar{r}_\perp, \xi, \tau)|. \tag{4.8}$$

The second of these I assume holds as a further consequence of the “first order” dispersion condition [4.14] [SeeRef:d0] below.

1. Note on the use of moduli

The constraints we are attempting to apply are that the RHS term(s) has negligible effect on the propagation compared to that of the LHS term(s); the specific mathematical expression of this comparison is up to us. The

$$q \partial_\xi A(\bar{r}_\perp, \xi, \tau) = \left( -\frac{\alpha_0}{\beta_0} + i \hat{D}' \right) A(\bar{r}_\perp, \xi, \tau) + \frac{1}{2 \beta_0^2 (1 + \sigma \partial_\tau)} \nabla_\perp^2 A(\bar{r}_\perp, \xi, \tau). \tag{4.2}$$

3. GFEA: Keeps the accuracies of both Brabec and Krausz[3] and Porras[4], whilst avoiding the more complicated parts of the full generalised eqn. (3.47). In eqn. (3.48), set $1 - \sigma \ll 1$ and use $T' = 0$ and ignore $T_{RHS}$:

$$q \partial_\xi A(\bar{r}_\perp, \xi, \tau) = \left( -\frac{\alpha_0}{\beta_0} + i \hat{D}' \right) A(\bar{r}_\perp, \xi, \tau) + \frac{2 \pi}{n_0^2} \frac{(1 + \sigma \partial_\tau)^2}{(1 + \sigma \partial_\tau)} B(\bar{r}_\perp, \xi, \tau; A) + \frac{1/ \beta_0^2}{2 (1 + \sigma \partial_\tau)} \nabla_\perp^2 A(\bar{r}_\perp, \xi, \tau). \tag{4.3}$$

The advantage of the generalised eqn. (3.48) over these is that $I_2$ could easily put $\delta$ and go beyond both Brabec and Krausz[3] and Porras[4]. Since the equation would be considerably easier to solve if $T_{RHS}$ were negligible, I will examine it in carefully in order to see what justification or constraints are required to do so. The term is:

$$\frac{T_{RHS}}{1 + \sigma \partial_\tau} = -\frac{i}{2} \left( 1 + \sigma \partial_\tau \right)^{-1} \left[ -\partial_\xi^2 + \left( \frac{\alpha_0}{\beta_0} - i \hat{D}' \right)^2 \right] A(\bar{r}_\perp, \xi, \tau). \tag{4.4}$$

Clearly this $T_{RHS}$ needs to be small compared to the other terms in eqn. (3.48) if it is to be neglected. If it happens that it is small, then the other, non-negligible, terms sum to close to $\partial_\xi \partial_\xi A(\bar{r}_\perp, \xi, \tau)$. So we can self-consistently ignore $T_{RHS}$ if the following condition holds:

$$\left| (1 + \sigma \partial_\tau)^{-1} \left[ \partial_\xi^2 - \left( \frac{\alpha_0}{\beta_0} - i \hat{D}' \right)^2 \right] A(\bar{r}_\perp, \xi, \tau) \right| \ll |\partial_\xi A(\bar{r}_\perp, \xi, \tau)|. \tag{4.5}$$

This enables us to avoid speculation about the possible time derivatives of $A$, and instead constrain its frequency components. However, this assumes knowlege of the all-time behaviour of the terms under consideration, so when being careful it might be better to use a time-windowed transform or similar. This could then give us constraints valid over some finite timescale relevant to the dynamics, without having to deal directly with instantaneous derivatives.

The condition (4.5) can be broken into two parts, which are

$$|\partial_\xi^2 A(\bar{r}_\perp, \xi, \tau)| \ll |(1 + \sigma \partial_\tau) \partial_\xi A(\bar{r}_\perp, \xi, \tau)|, \tag{4.7}$$

$$\left| \left( \frac{\alpha_0}{\beta_0} - i \hat{D}' \right)^2 A(\bar{r}_\perp, \xi, \tau) \right| \ll |\partial_\xi A(\bar{r}_\perp, \xi, \tau)|. \tag{4.8}$$

The second of these I assume holds as a further consequence of the “first order” dispersion condition [4.14] [SeeRef:d0] below.

1. Note on the use of moduli

The constraints we are attempting to apply are that the RHS term(s) has negligible effect on the propagation compared to that of the LHS term(s); the specific mathematical expression of this comparison is up to us. The
situation is complicated by the fact that either side can be complex, and will likely have a different complex argument (i.e. phase). Clearly the \textit{largest} number we can make with the (hopefully small) LHS is given by the modulus, so that will give us an appropriate value for the LHS. What to do with the RHS is less obvious, because we would instead (to be cautious) want to pick a smallest reasonable value; but (e.g.) picking the minimum value of either the real or imaginary part would miss the point: indeed if the term were real, the smallest imaginary part would be zero; thus leading the condition to always fail. This leaves us with little choice (as far as I can see) but to use the modulus again; which in any case this would be the typical physicist’s approach.

Perhaps the best justification for applying the modulus to both sides in the comparison follows from the fact that a rotation in the complex plane applied equally to both terms should not affect the outcome. Hence we can rotate both terms so that the (hopefully large) RHS term becomes real-valued; now, since only the LHS might be complex, taking its modulus gives a useful upper bound on the significance of its contribution to the dynamics. This is equivalent to just taking the moduli of both terms.

\subsection{A. Evolution: $\partial_\xi^2$ approximation}

I can now constrain the evolution of the pulse in $\xi$ by evaluating how to ensure that the $\partial_\xi^2$ terms is negligible. Starting with eqn. (4.7),

\begin{align}
|\partial_\xi^2 A(\vec{r}_\perp,\xi,\tau)| & \ll |(1 + i\sigma \partial_\tau) \partial_\xi A(\vec{r}_\perp,\xi,\tau)| \quad (4.9) \\
\Rightarrow |\partial_\xi^2 A(\vec{r}_\perp,\xi,\tau)| & \ll |(1 + i\sigma \partial_\tau) \tilde{A}(\vec{r}_\perp,\xi,\Omega)| \quad (4.10) \\
\Rightarrow |\partial_\xi \tilde{A}(\vec{r}_\perp,\xi,\Omega)| & \ll |(1 + i\sigma \partial_\tau) \tilde{A}(\vec{r}_\perp,\xi,\Omega)| \quad (4.11) \\
\Rightarrow |\partial_\xi \tilde{A}(\vec{r}_\perp,\xi,\Omega)| & \ll |\tilde{A}(\vec{r}_\perp,\xi,\Omega)| \quad (4.12)
\end{align}

because (i) I assume I can cancel (hopefully with no side effects) a $\partial_\xi$ derivative term from either side [SeeRef:d0]; and (ii) I tighten the constraint somewhat by relying on $\sigma \sim 1$ \textit{not} $\sigma \simeq 1$) and that only considering positive frequencies means that $\Omega > 0$. Eqn (4.7) is therefore revealed as a condition that the envelope function only changes slightly when propagated over distances $\xi \sim 1$ (i.e. $z \sim 1/\beta_0$), and as such will depend on other terms in the evolution equation.

More carefully, if we assume we know the $\xi$ behaviour of the envelope $A$, can use it to see how these constraints might hold in spatial-frequency ($\kappa$) space (since $\xi \leftrightarrow -i\kappa$),

\begin{align}
|\partial_\xi^2 A(\vec{r}_\perp,\xi,\tau)| & \ll |(1 + i\sigma \partial_\tau) \partial_\xi A(\vec{r}_\perp,\kappa,\tau)| \quad (4.13) \\
\Rightarrow |i^2\kappa^2 \tilde{A}(\vec{r}_\perp,\kappa,\tau)| & \ll | -i\kappa (1 + i\sigma \partial_\tau) \tilde{A}(\vec{r}_\perp,\kappa,\tau)| \quad (4.14) \\
\text{cancelling }i\text{ and }\kappa \text{ gives} & \Rightarrow |i\kappa \tilde{A}(\vec{r}_\perp,\kappa,\tau)| \ll | - (1 + i\sigma \partial_\tau) \tilde{A}(\vec{r}_\perp,\kappa,\tau)| \quad (4.15) \\
\text{using moduli} & \Rightarrow |\kappa \tilde{A}(\vec{r}_\perp,\kappa,\tau)| \ll | (1 + i\sigma \partial_\tau) \tilde{A}(\vec{r}_\perp,\kappa,\tau)| \quad (4.16)
\end{align}

This gives us an expression rather like that as if the propagation gave us different wavelength (wavevector) to that specified by the carrier exponential. We want the envelope to propagate with its spatially ($\xi$) behaviour to be peaked around small values of $\kappa$, so that the bulk of the spatial variation is included in the carrier exponential.”

Note that the constraint will \textit{always} be violated somewhere if the spatial bandwidth of the propagation is too large (even if it has an e.g. exponential fall off). The approximation therefore amounts to ignoring such violations of the constraint in the (spatial-frequency) wings of the propagation, on the basis they are “negligible” (which indeed seems perfectly reasonable). In a simulation, the LHS and RHS can be calculated as it progresses, and the frequency counted, and possible significance assessed (I have done this in OPA simulations).

\textbf{SeeNote:d0:} There may be complications I do not see, because these terms ($\tilde{D}'$, etc) do contain derivative terms. However, since SVEA treatments ignore such issues, I do also.

The spatial evolution sub-condition (4.12), the dispersion sub-condition (4.17) (c.f. condition (4.8)); and hence the \textit{total} condition (4.13) will only hold if the other non-negligible terms in eqn. (5.48) are similarly small compared to $|A(\vec{r}_\perp,\xi,\tau)|$, viz.
\[
\left| \frac{\alpha_0}{\beta_0} - i D' \right| A(\vec{r}_\perp, \xi, \tau) \ll |A(\vec{r}_\perp, \xi, \tau)| \quad (4.17)
\]
\[
\frac{1}{2 \beta_0^2 (1 + i \sigma \partial_\tau)} \nabla^2_\perp A(\vec{r}_\perp, \xi, \tau) \ll |A(\vec{r}_\perp, \xi, \tau)| \quad (4.18)
\]
\[
\frac{2 \pi (1 + i \sigma \partial_\tau)^2}{\mu_0^2 (1 + i \sigma \partial_\tau)} B(\vec{r}_\perp, \xi, \tau; A) \ll |A(\vec{r}_\perp, \xi, \tau)| \quad (4.19)
\]

Note that we could use the alternative eqn \((3.50)\) in condition \((4.19)\).

I will assume condition \((4.17)\) implies that \((4.8)\) also holds \([\text{See Refd}\ 0]\), leaving us with four conditions in total \((4.12, 4.17, 4.18, 4.19)\). Note that Brabec and Krausz\[^{3}\] split \((4.17)\) into multiple pieces, which will be discussed in the following subsection.

**B. Dispersion: \(\partial_\tau\) Approximation**

I will now treat the dispersion condition \((4.17)\) in the above approximations and associated conditions. Note that Brabec and Krausz\[^{3}\] claim that their SEWA “does not explicitly impose a limitation on the pulse width”; however this is rather misleading as shortly afterward they introduce a pulse duration \(\tau_p\) which is used in the inequalities constraining the material parameters, which then give the region of validity of the SEWA. Even the weakest statement we might make about \(\tau_p\) needs to state that it does constrain the SEWA, because it (further) constrains the material parameters! For a given set of material parameters, there will be some pulse width limitation, although it might well be the few-cycles we hope to describe.

I now break up eqn. \((4.17)\) into parts containing single factors of \(\gamma_m\). Note that I need to exclude any terms including \(\beta_0\) and \(\beta_1\) in \(\gamma_m\) as they were treated separately in the analysis. I thus write \(\gamma'_m = \gamma_m\) for \(m \geq 2\), and \(\gamma'_0 = \alpha_0\), \(\gamma'_1 = \alpha_1\). The inequality is then

\[
\left| \left( \frac{\omega_0^{m-1} \gamma_m}{\beta_0 m!} \right) \partial_\tau^m A(\vec{r}_\perp, \xi, \tau) \right| \ll |A(\vec{r}_\perp, \xi, \tau)| \quad (4.20)
\]
\[
\left| \left( \frac{\omega_0^{m-1} \gamma_0}{\beta_0 m!} \right) \partial_\tau^m \tilde{A}(\vec{r}_\perp, \xi, \Omega) \right| \ll |\tilde{A}(\vec{r}_\perp, \xi, \Omega)| \quad (4.21)
\]

where the second line has been fourier transformed in time. Note that while both \(\omega_0\) and \(\gamma'\) have units, \(\Omega\) does not, as it is the counterpart of the dimensionless (scaled) \(\tau\) (see \((4.6)\)).

**To Do:** Now get from \(\Omega\) to a \(\tau_5\) or Brabec and Krausz\[^{3}\] \(\tau_p\), by some physical motivation justifying \(\tau_5 = 2\pi/\Omega\).

I could treat the time-domain condition \((4.20)\) qualitatively as follows: introduce \(\tau_5 \simeq |A|/|\partial_\tau A|\), the time for which a rate of change of \(\partial_\tau A\) would accumulate (in absolute value) to \(A\); and is thus some measure of how much time it takes the envelope to change significantly, and is thus hopefully something we can relate to the pulse width (or at least the width of one “bump” on the pulse envelope). Also, we assume that the higher derivatives of \(A\) are related, with \(|\partial_\tau^m A/m!|/A \approx \tau_5^{-m}\) \([\text{See Refx1}\] This is essentially the same parameter as Brabec and Krausz\[^{3}\]'s \(\tau_p\), but scaled into my dimensionless picture: \(\tau_p \equiv \tau_5/\omega_0\). Using this \(\tau_5\) we can rewrite eqn. \((4.20)\) as

\[
\left| \frac{\omega_0^m \gamma_m}{\beta_0 m!} \tau_5^{-m} \right| \ll 1 \quad (4.22)
\]
\[
\frac{\tau_5^m}{\omega_0^m \gamma_m} \equiv \frac{\tau_5}{\gamma_m} \gg \beta_0^{-1} \sim L_{\gamma,m} \quad (4.23)
\]

This final condition is the same as those for \(\alpha_m\) and \(\beta_m\) in Brabec and Krausz\[^{3}\], since \((\beta_0\) and wavelength comment), and \((L_{\gamma,m}\) justification of BK).

**SeeNote1:** The factoring of the \(m!\) into \(\partial_\tau^m A\) seem the most mathematically most sensible thing to do by analogy to the terms in Taylor expansions, and expansions of exponential functions (etc).

**Note:** Is it possible to derive a true time-domain treatment of the constraints by replacing the \(\gamma_m\) by \(\partial_\tau k(\omega)\), and relate it back to the time domain \(\epsilon(t - t_0)\) (or moments thereof).

**Note:** The eqn. \((4.8)\) \(D'^2\) terms.
C. Diffraction: \( \nabla^2_\perp \) approximations

Treating the \( \partial_\tau \) in condition (4.18) by Fourier transform, as above, we can use the fact that for Gaussian beams with a beam waist \( w_0 \), we have \( \nabla^2_\perp A(\vec{r}_\perp, \xi, \tau) \sim w_0^{-2} \); similar statements could be made for other typical beam profiles. This leads to the diffraction constraint on the SEWA becoming

\[
(1 + \sigma \Omega) \beta_0^2 w_0^2 \gg 1. \tag{4.24}
\]

Comparing this to the comparable condition in Brabec and Krausz\(^8\) after their eqn (8) (i.e. in my units \( \beta_0^2 w_0^2 \gg 1 \)), we see that they are the same except for the \( \Omega \) term, so that my condition is in fact somewhat less restrictive than theirs. This is because considering positive frequencies only means that \( \Omega > 0 \); and \( \Omega \sim 1 \) for \( \partial_\tau \) modulations much less than the variation of the carrier frequency.

D. Nonlinearity: \( B(\vec{r}_\perp, \xi, \tau; A) \) approximations

The nonlinearity constraint (4.19) is very complicated, so I ignore the term in square brackets \([...]\). Then, treating the \( \partial_\tau \) in condition (4.19) by Fourier transform, as above, and using \( \sigma \simeq 1 \), the nonlinearity constraint on the SEWA becomes

\[
\frac{2\pi}{n_0^2} \frac{(1 + \sigma \Omega)}{(1 + \Omega)^2} \gg \left| \frac{\tilde{B}(\vec{r}_\perp, \xi, \Omega; A)}{\tilde{A}(\vec{r}_\perp, \xi, \Omega)} \right|. \tag{4.25}
\]

Comparing this to the comparable condition in Brabec and Krausz\(^8\) after their eqn (BK8), we see that the same comments as for the diffraction hold – my condition is somewhat less restrictive than theirs. Of course it may be convenient to simplify the LHS of eqn. (4.25) with various small \( \Omega \) expansions.

In both cases I could instead include the \( \partial_\tau \) corrections qualitatively by replacing \( \partial_\tau \to \tau_\delta^{-1} \) before proceeding from (4.18) and (4.19); but even with \( \tau_\delta \simeq 1 \) this will not alter the constraints greatly.