D-branes and vacuum periodicity

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Abstract

The superstring/M-theory suggests the Born-Infeld type modification of the classical gauge field lagrangian. We discuss how this changes topological issues related to vacuum periodicity in the $SU(2)$ theory in four spacetime dimensions. A new feature, which is due to the breaking of scale invariance by the non-Abelian Born-Infeld (NBI) action, is that the potential barrier between the neighboring vacua is lowered to a finite height. At the top of the barrier one finds an infinite family of sphaleron-like solutions mediating transitions between different topological sectors. We review these solutions for two versions of the NBI action: with the ordinary and symmetrized trace. Then we show the existence of sphaleron excitations of monopoles in the NBI theory with the triplet Higgs. Soliton solutions in the constant external Kalb-Ramond field are also discussed which correspond to monopoles in the gauge theory on non-commutative space. A non-perturbative monopole solution for the non-commutative $U(1)$ theory is presented.

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1 Introduction

Recent development in the superstring theory \[1, 2\] suggests that the low-energy dynamics of a $Dp$-brane moving in a flat $D$-dimensional spacetime $z^M = z^M(x^\mu)$, $M = 0, \ldots, D - 1, \mu = 0, \ldots, p$ is governed by the Dirac-Born-Infeld (DBI) action

$$S_p = \int \left(1 - \sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})}\right) d^{p+1}x,$$  

(1)

where

$$g_{\mu\nu} = \partial_\mu z^M \partial_\nu z^N \eta_{MN},$$  

(2)

is an induced metric on the brane and $F_{\mu\nu}$ is a $U(1)$ gauge field strength. Using the gauge freedom under diffeomorphisms of the world-volume, one can choose coordinates $z^M = (x^\mu, X^i)$, where $X^i$ are transverse to the brane, and rewrite the action as

$$S_p = \int \left(1 - \sqrt{-\det(\eta_{\mu\nu} + \partial_\mu X^i \partial_\nu X^i + F_{\mu\nu})}\right) d^{p+1}x.$$  

(3)

A trivial solution to this action is $X^i = 0$, $F_{\mu\nu} = 0$, what means that the $p$-brane is flat and there is no electromagnetic field. Because of the symmetry $X^i \to -X^i$, the planar solution remains true when $F_{\mu\nu}$ does not vanish, in which case the electromagnetic field is governed by the Born-Infeld (BI) action. Moreover, as was noticed by Gibbons \[3\], the only regular static source-free solution of the BI electrodynamics which falls off at spatial infinity is a trivial one.

This is no longer true in the case of $N$ coincident $Dp$-branes whose low-energy dynamics is described by the non-Abelian generalization of the DBI action involving the $SU(N)$ Yang-Mills (YM) field. Namely, for flat $D3$-branes the regular sourceless finite energy configurations of the YM field were found to exist \[4, 5\]. The topological reason for this lies in the vacuum periodicity of the $SU(2)$ gauge field in four dimensions. Neighboring YM vacua are separated by potential barriers which in the case of the BI action are lowered down to a finite height due to the breaking of the scale invariance in the BI theory. This removes the well-known obstruction for classical glue balls \[6, 7, 8\], which can be summarized as follows. Scale invariance of the usual quadratic Yang-Mills action implies that the YM field stress–energy tensor is traceless: $T^\mu_\mu = 0 = -T_{00} + T_{ii}$, where $\mu = 0, \ldots, 3, i = 1, 2, 3$. Since the energy density is positive, $T_{00} > 0$, the sum of the principal pressures $T_{ii}$ is also everywhere positive, i.e. the Yang–Mills matter is repulsive. Consequently, mechanical equilibrium within the localized static YM field configuration is impossible \[9\]. In the spontaneously broken gauge theories scale invariance is broken by scalar fields, what opens the possibility of particle-like solutions: magnetic monopoles (in the theory with the real triplet Higgs) and sphalerons (in the theory with the complex doublet Higgs).

The role of the Higgs field in these two cases is somewhat different. For monopoles the topological significance of the Higgs field is essential: indeed, monopoles interpolate between the unbroken and broken Higgs phases. In the case of sphalerons, the Higgs field plays mostly a role of an attractive agent which is able to glue the repulsive YM matter. Historically, topological significance of the Dashen-Hasslacher-Neveu (DHN) solution in the $SU(2)$ theory with the doublet Higgs \[10\] was first explained by Manton.
as a consequence of non-triviality of the third homotopy group of the Higgs broken phase manifold $\pi_3(G/H)$. This is equivalent to existence of non-contractible loops in the space of field configurations passing through the vacuum. Then by the minimax argument one finds that a saddle point exists on the energy surface which is a proper place for the sphaleron. Later it became clear that similar solutions arise in some models without Higgs, such as Einstein-Yang-Mills [12] or Yang-Mills with dilaton [13] (for a review and further references see [14]). The main common feature of these theories is that the conformal invariance of the classical YM equations is broken, what removes the "mechanical" obstruction for existence of particle-like configurations. As far as the topological argument is concerned, it is worth noting that $H = 1$ for the DHN solution, so the same third homotopy group argument applies to the gauge group $G$ itself, that is, it works equally in the theories without Higgs.

Breaking of the scale invariance in the NBI theory also gives rise to sphaleron glueballs which mediate transitions between different topological sectors of the theory. Their mass is related to the BI field-strength parameter which for the D-branes is $2\pi\alpha'$. We will discuss here glueball solutions in two versions of the NBI theory: with the ordinary and symmetrized trace. We also show that, when the triplet Higgs field is added, the theory admits, apart from the usual magnetic monopoles, the hybrid solutions which can be interpreted as sphaleron excitations of monopoles. At the end we briefly discuss monopole solutions in gauge theories on non-commutative spaces and give an explicit solution for the $U(1)$ monopole with Higgs in the D-brane picture with the Kalb-Ramond field.

2 NBI action with ordinary and symmetrized trace

A precise definition of the NBI action was actively discussed during past few years [15, 16, 17, 18, 19, 20], for an earlier discussion see [21]. An ambiguity is encoded in specifying the trace operation over the gauge group generators. Formally a number of possibilities can be envisaged. Starting with the determinant form of the $U(1)$ Dirac-Born-Infeld action

$$S = \frac{1}{4\pi} \int \left\{ 1 - \sqrt{\det(g_{\mu\nu} + F_{\mu\nu})} \right\} d^4x,$$

one can use the usual trace, the symmetrized or antisymmetrized [15] ones, or evaluate the determinant both with respect to Lorentz and the gauge matrix indices [19]. Alternatively one can start with the 'square root' form, which is most easily derived from (4) using the identities

$$\det(g_{\mu\nu} + F_{\mu\nu}) = \det(g_{\mu\nu} - F_{\mu\nu}) = \det(g_{\mu\nu} + i\tilde{F}_{\mu\nu}) =$$

$$= \det(g_{\mu\nu} - i\tilde{F}_{\mu\nu}) = \left[ \det(g_{\mu\nu} - F_{\mu\nu}^2)(g_{\mu\nu} + \tilde{F}_{\mu\nu}^2) \right]^{1/4},$$

where $F_{\mu\nu}^2 = F_{\mu\alpha}F_{\nu}^\alpha$ (similarly for $\tilde{F}_{\mu\nu}$), and

$$F_{\mu\alpha}F^\alpha_{\nu} - \tilde{F}_{\mu\alpha}\tilde{F}^\alpha_{\nu} = \frac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta},$$

$$F_{\mu\alpha}\tilde{F}^\alpha_{\nu} = -\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}\tilde{F}^{\alpha\beta}.$$
This gives the relation

\[ \sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})} = \sqrt{-\det(g)} \sqrt{1 + \frac{1}{2} F^2 - \frac{1}{16} (F \tilde{F})^2}, \]  

(7)

with \( F^2 = F_{\mu\nu} F^{\mu\nu} \), \( F \tilde{F} = F_{\mu\nu} \tilde{F}^{\mu\nu} \).

For a non-Abelian gauge group the relations (8) are no longer valid, so there is no direct connection between the 'determinant' and the 'square root' form of the lagrangian. Therefore the latter can be chosen as an independent starting point for a non-Abelian generalization.

There is, however, a particular trace operation – symmetrized trace – under which generators commute, so both forms of the lagrangian remain equivalent. This definition is favored by the no-derivative argument, as was clarified by Tseytlin [15]. Restricting the validity of the non-Abelian effective action by the constant field approximation, one has to drop commutators of the matrix-valued \( F_{\mu\nu} \) since these can be reexpressed through the derivatives of \( F_{\mu\nu} \). This corresponds to the following definition

\[ S = \frac{1}{4\pi} \mathrm{Str} \int \left\{ 1 - \sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})} \right\} d^4x, \]  

(8)

where symmetrization applies to the field strength (not to potentials). This action reproduces an exact string theory result for non-Abelian fields up to \( \alpha'^2 \) order. Although there is no reason to believe that this will be true in higher orders in \( \alpha' \), the Str action is an interesting model providing minimal generalization of the Abelian action [14].

An explicit form of the SU(2) NBI action with the symmetrized trace for static \( SO(3) \)-symmetric magnetic type configurations was found only recently [5]. One starts with the definition

\[ L_{NBI} = \frac{\beta^2}{4\pi} \mathrm{Str} \left( 1 - \sqrt{-\det\left( g_{\mu\nu} + \frac{1}{\beta} F_{\mu\nu} \right)} \right) = \frac{k \beta^2}{4\pi} \mathrm{Str}(1 - \mathcal{R}), \]  

(9)

where

\[ \mathcal{R} = \sqrt{1 + \frac{1}{2\beta^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{16\beta^4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2}, \]  

(10)

and \( \beta \) of the dimension of length\(^{-2} \) is the BI 'critical field'. The normalization of the gauge group generators is unusual and is chosen as follows

\[ F_{\mu\nu} = F^a_{\mu\nu} t_a, \quad \mathrm{tr} t_a t_b = \delta_{ab}. \]  

(11)

The symmetrized trace of the product of \( p \) matrices is defined as

\[ \mathrm{Str}(t_{a_1} \ldots t_{a_p}) \equiv \frac{1}{p!} \mathrm{tr} \left( t_{a_1} \ldots t_{a_p} + \text{all permutations} \right), \]  

(12)

and it is understood that the general matrix function like (9) has to be series expanded. It has to be noted that under the Str operation the generators can be treated as commuting objects, and the gauge algebra should not be applied, (e.g. the square of the Pauli matrix \( \tau_x^2 \neq 1 \)) until the symmetrization in the series expansion is completed.
A general $SO(3)$ symmetric $SU(2)$ gauge field is described by the Witten’s ansatz
\[
\sqrt{2}A = a_0 t_1 \, dt + a_1 t_1 \, dr + \left\{ w_2 \, t_2 - (1 - w) \, t_3 \right\} \, d\theta + \left\{ (1 - w) \, t_2 + \tilde{w} \, t_3 \right\} \sin \theta \, d\phi, \tag{13}
\]
where the functions $a_0$, $a_1$, $w$, $\tilde{w}$ depend on $r, t$ and $\sqrt{2}$ is introduced to maintain the standard normalization. Here we use a rotating basis $t_i$, $i = 1, 2, 3$ for the $SU(2)$ generators defined as
\[
t_1 = n^a \tau^a / \sqrt{2}, \quad t_2 = \partial_\theta t_1, \quad \sin \theta t_3 = \partial_\phi t_1, \tag{14}
\]
where $n^a = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, with $\tau^a$ being the Pauli matrices. These generators obey the commutation relations $[t_i, t_j] = i \sqrt{2} \epsilon^{ijk} t_k$.

From four functions entering this ansatz one can be gauged away. In the static case we can further reduce the number of independent functions to two, while the static purely magnetic configurations are fully described by a single function $w(r)$:
\[
\sqrt{2}A_\theta = -(1 - w) t_3, \quad \sqrt{2}A_\varphi = \sin \theta (1 - w) t_2, \quad A_t = A_r = 0. \tag{15}
\]
The field strength tensor has the following non-zero components
\[
\sqrt{2}F_{\theta \varphi} = w' t_3, \quad \sqrt{2}F_{r \theta} = - \sin \theta w' t_2, \quad \sqrt{2}F_{\theta r} = \sin \theta (w^2 - 1) t_1, \tag{16}
\]
where prime denotes derivatives with respect to $r$.

For purely magnetic configurations the second term under the square root is zero, and the substitution of (16) gives
\[
\mathcal{R}^2 = 1 + \frac{(1 - w^2)^2}{\beta^2 r^4} t_1^2 + \frac{w'^2}{\beta^2 r^2} (t_2^2 + t_3^2). \tag{17}
\]
To find an explicit expression for the lagrangian one has to expand the square root in a triple series in terms of the even powers of generators $t_1, t_2, t_3$, then to calculate the symmetrized trace of the powers of generators in all orders, and finally to make a resummation of the series. The result reads
\[
L_{NBI} = \frac{\beta^2}{4\pi} \left( 1 - \frac{1 + V^2 + K^2 A}{\sqrt{1 + V^2}} \right), \tag{18}
\]
where
\[
V^2 = \frac{(1 - w^2(r))^2}{2 \beta^2 r^4}, \quad K^2 = \frac{w'^2(r)}{2 \beta^2 r^2}, \quad A = \sqrt{\frac{1 + V^2}{V^2 - K^2}} \arctanh \sqrt{\frac{V^2 - K^2}{1 + V^2}}. \tag{19}
\]
Here we assumed that $V^2 > K^2$, otherwise an arctan form is more appropriate. Note that when the difference $V^2 = K^2$ changes sign, the $k$ function $A$ remains real valued. It can be checked that when $\beta \to \infty$, the standard Yang-Mills lagrangian (restricted to monopole ansatz) is recovered. In the strong field region our expression differs essentially from the square root/ordinary trace lagrangian.

The corresponding explicit action defined in a square root form with an ordinary trace reads:
\[
L_{NBI} = \frac{\beta^2}{4\pi} \left( 1 - \sqrt{1 + V^2 + 2K^2} \right). \tag{20}
\]
3 Topological vacua and sphalerons

As is well-known, vacuum in the $SU(2)$ YM theory in the four-dimensional spacetime splits into an infinite number of disjoint classes which cannot be deformed into each other by 'small' (contractible to a point) gauge transformations. Writing the pure gauge vacuum YM potentials as $A = i UdU^{-1}$, where $U \in SU(2)$ and imposing an asymptotic condition

$$\lim_{r \to \infty} U(x^i) = 1,$$

(21)

we can interpret $U(x^i)$ as mappings $S_3 \to SU(2)$. All sets of such $U$'s falls into the sequence of homotopy classes characterized by the winding number

$$k[U] = \frac{1}{24\pi^2} \text{tr} \int_{S^3} UdU^{-1} \wedge UdU^{-1} \wedge UdU^{-1}.$$

(22)

A representative of the $k$-th class can be chosen as

$$U_k = \exp\{i\alpha(r)t_1/\sqrt{2}\}, \quad \text{where} \quad \alpha(0) = 0, \alpha(\infty) = -2\pi k.$$

(23)

The corresponding potential will be given by the Witten ansatz with $a = 0, w = \exp(i\alpha(r))$. The asymptotic condition (21) leads to the following fall-off requirements.

$$A_a = o(r^{-1}) \quad \text{for} \quad r \to \infty.$$

(24)

The representatives of different vacuum classes with different $k$ cannot be continuously deformed into each other within the class of the purely vacuum fields. But there exists an interpolating sequence of nonvacuum field configurations of finite energy (the latter can be defined on shell and then continued off-shell) satisfying the required boundary conditions (24) that connects different vacuum classes. Finite energy solutions for the actions (18) or (20) should satisfy the following boundary conditions near the origin

$$w = 1 + br^2 + O(r^4),$$

(25)

and at the infinity

$$w = \pm 1 + \frac{c}{r} + O\left(\frac{1}{r^2}\right),$$

(26)

where $b$ and $c$ are free parameters. (The value $w(\infty) = 0$ together with finiteness of the energy implies that $w \equiv 0$.) The leading terms are the same as required for the vacuum configurations. These solutions, if exists, can be shown to lie on the path in the solution space connecting two topologically distinct vacua. Consider a one-parameter sequence of field configurations (off shell generally) depending on a continuous parameter $\lambda \in [0, \pi]$

$$A[\lambda] = i \frac{1-w}{2} U_+ d U_-^{-1} + i \frac{1+w}{2} U_- d U_+^{-1},$$

(27)

where

$$U_\pm = \exp\{i\lambda(w \pm 1)t_1/\sqrt{2}\}.$$

(28)
This field vanishes for $\lambda = 0$, whereas for $\lambda = \pi$ it can be represented as

$$A[\pi] = i UdU^{-1}, \quad \text{with} \quad U = \exp\{i\pi(w - 1)t_1/\sqrt{2}\}. \quad (29)$$

In view of the above boundary condition for $w$, in the case $w(\infty) = -1$ one has the $k = 1$ vacuum. Now, the crucial thing is that for $\lambda = \pi/2$ we come back to the configuration (15). So if the solution to the classical field equations with the required asymptotics exists indeed, this can be interpreted as a manifestation of the finiteness of the potential barrier between distinct vacua.

Note that the same reasoning holds for the ordinary Yang–Mills system. But due to the scale invariance of this theory there is no function $w$ which minimizes the energy functional.

Both the analysis of the equations following from NBI lagrangians (18,20) using the methods of dynamical systems [4] and numerical experiments [5] shows that such solutions exist in both NBI models — with ordinary and symmetrized trace. They form a discrete sequence labeled by the number of nodes of the function $w(r)$, and the lower one-node solution is similar to the sphaleron of the Weinberg-Salam theory.

In the NBI theory $\beta$ is the only dimensionful parameter giving a natural scale of length, i.e. theories with different values of $\beta$ are equivalent up to rescaling. Setting $\beta = 1$ we obtain the equations of motion for the symmetrized trace NBI model

$$\frac{d}{dr} \left\{ \frac{w'}{2(V^2 - K^2)} \left( K^2 \sqrt{1 + V^2} \right) - \frac{(2V^2 - K^2)A}{\sqrt{V^2 - K^2}} \right\} = \frac{w V(K^2 A - V^2)}{(V^2 - K^2)\sqrt{1 + V^2}}. \quad (30)$$

For the ordinary trace model one has

$$\frac{d}{dr} \left\{ \frac{w'}{\sqrt{1 + V^2 + 2K^2}} \right\} = -\frac{w V}{\sqrt{1 + V^2 + 2K^2}}. \quad (31)$$

We are looking for the solutions satisfying the boundary conditions (25,26). For large $r$ both equations reduce to that of the usual YM theory, so the solutions are not much different in the far zone. Near the origin the equations are different, more careful analysis reveals that the nature of stationary points associated with the origin is different for two versions of the theory.

A trivial solution to these equations (valid for both models) is an embedded abelian monopole $w = 0$. In the BI theory it has the finite energy. From the general analysis, as discussed in [14] for the ordinary trace, one finds that $w$ can not have local minima for $0 < w < 1, w < -1$ and can not have local maxima for $-1 < w < 0, w > 1$. The same remains true for the symmetrized trace. Thus any solution which starts at the origin on the interval $-1 < w < 1$ must remain within the strip $-1 < w < 1$. Once $w$ leaves the strip, it diverges in a finite distance. Regular solutions exist for a discrete sequence of $b$ shown in the table I together with corresponding masses $M_n$ for the first six $n$ which is the number of zeroes of $w(r)$. The $n = 1$ solution is analogous to the sphaleron known in the Weinberg-Salam theory [10, 11], it is expected to have one decay mode. Higher odd-$n$ solutions may be interpreted as excited sphalerons, they are expected to have $n$ decay directions. Even-$n$ solutions are topologically trivial, they can be regarded as sphaleronic
| \( n \) | \( b_{tr} \) | \( M_{tr} \) | \( b_{str} \) | \( M_{str} \) |
|-----|--------|--------|--------|--------|
| 1   | 1.27463 \times 10^4 | 1.13559 | 1.23736 \times 10^2 | 1.20240 |
| 2   | 8.87397 \times 10^2 | 1.21424 | 5.05665 \times 10^3 | 1.234583 |
| 3   | 1.87079 \times 10^4 | 1.23281 | 1.67739 \times 10^5 | 1.235979 |
| 4   | 1.27455 \times 10^6 | 1.23572 | 7.11885 \times 10^6 | 1.236046 |
| 5   | 2.65030 \times 10^7 | 1.23603 | 4.94499 \times 10^8 | 1.2360497 |
| 6   | 1.80475 \times 10^9 | 1.23604 | 4.52769 \times 10^{10} | 1.2360497 |

Table 1: Values of \( b \) and \( M \) for first six glueball solutions in NBI models with ordinary and symmetrized traces

excitation of the vacuum. Qualitatively picture is the same as for the ordinary trace [4], but the discrete values of \( b \) are rather different.

Numerical solutions for both models are shown in the figure [4]. It is surprising that the solutions with the ordinary and the symmetrized trace are rather similar in spite of the substantial difference of the lagrangians. They have however somewhat different behavior near the origin: those with the symmetrized trace leave the vacuum value \( w = 1 \) faster and stay longer in the intermediate region where \( w(r) \) is close to zero. In this region the magnetic charge is almost unscreened, so this is the particle core. Thus for all \( n \) solutions are more compact in the ordinary trace case. For both models the parameters \( b_n \) grow infinitely with increasing node number \( n \). This means that there is no limiting solution as \( n \to \infty \) contrary to the EYM case where such solutions do exist.

### 4 Magnetic monopoles and hybrid solutions

Magnetic monopoles are associated with the deformed D3-branes with non zero transverse coordinates \( X^m \) interpreted as Higgs scalars. The deformation can be thought of as caused by an open string attached to the brane. In the BPS limit the solutions are the same as for the quadratic YM theory [17, 18] Monopoles for the ordinary trace model were constructed by Grandi, Moreno and Schaposnik [23]. For monopoles the function \( w \) monotonously varies from the value \( w = 1 \) at the origin to the asymptotic value \( w = 0 \) at infinity. Note, that assuming the asymptotic value \( w = 0 \) for pure gauge NBI theory we will get only embedded abelian solution \( w \equiv 0 \). Our aim here is to show that, in addition, there are hybrid NBI-Higgs solutions for which the function \( w(r) \) oscillates in the core region. In other words, starting from the vacuum \( w = 1 \) at the origin the function \( w(r) \) tries to follow the sphaleronic behavior, but finally turns back to the monopole regime.

Adding to the NBI action the Higgs term \( S = S_{NBI} + S_H \) where \( S_H \) is taken in the usual form

\[
S_H = \frac{1}{8\pi} \int \left( D_\mu \phi^a D^\mu \phi^a - \frac{\lambda}{2} \left( \phi^a \phi^a - v^2 \right)^2 \right),
\]  

one obtains the NBI-Higgs theory, containing, apart from \( \beta \), the second parameter \( \lambda \) (without loss of generality we put the gauge coupling constant equal to unity). For
spherically symmetric static purely magnetic configurations the YM ansatz remains the same, while for the Higgs field

\[ \phi^a = \frac{H(r)}{r} n^a. \]  

(33)

For simplicity we consider here the square root form of the NBI action (20). Performing an integration over spherical angles one obtains the energy functional (equal to minus action for static configurations)

\[ E = 4\pi \int dr \, r^2 \left\{ 2\beta^2 (\mathcal{R} - 1) + \frac{1}{2r^2} \left( (H' - \frac{H}{r})^2 + \frac{2}{r^2} H^2 w^2 \right) + V \right\}, \]

(34)

where

\[ \mathcal{R} = \sqrt{1 + \frac{1}{\beta^2 r^4} (r^2 w'^2 + \frac{1}{2} (w^2 - 1)^2)}, \quad V = \frac{\lambda}{4} \left( \frac{H^2}{r^2} - 1 \right)^2. \]

(35)

Varying this functional one finds the equations of motion

\[ r^2 w'' = w (\mathcal{R} H^2 + w^2 - 1) + r^2 \frac{\mathcal{R}'}{\mathcal{R}} w', \]

(36)

\[ r^2 H'' = 2H w^2 - \lambda H (r^2 - H^2). \]

(37)

Boundary conditions at infinity for a solution with a unit magnetic charge read

\[ \lim_{r \to \infty} w(r) = 0, \quad \lim_{r \to \infty} \frac{H(r)}{r} = 1, \]

(38)

while at the origin

\[ w(0) = 1, \quad H(0) = 0. \]

(39)

Starting with (39) one can construct the following power series solution converging in a non-zero domain around the origin:

\[ w = 1 - br^2 + \frac{\beta b^2 (22b^2 + \beta^2) + d^2 (6b^2 + \beta^2) \frac{b}{10\beta (2b^2 + \beta^2)}}{r^4} + O(r^6) \]

(40)

\[ H = d r^2 - \left( \frac{1}{10} \lambda d + \frac{2}{5} d b \right) r^4 + O(r^6), \]

(41)

where \( b \) and \( d \) are free parameters. For \( \beta \to \infty \) the theory reduces to the standard YMH-theory, admitting monopoles. In [23] it was shown that monopole solutions to the Eqs. (36, 37) continue to exist up to some limiting value \( \beta_{cr} \).

Now we have to explain why one can expect to have also the hybrid solutions. Near the origin the Higgs field is close to zero, so the influence of the term \( H^2 K \mathcal{R} \) is negligible, and the YM field behaves like in the pure NBI case. As was argued in [4], NBI theories with different \( \beta \) are equivalent up to rescaling, and so for \( \beta \) large enough the solution starts forming just near the origin. But for larger \( r \) the role of Higgs is increased, so one can expect that some solutions can be trapped to the monopole asymptotic regime. More precisely, in the region of \( r \approx 1/\sqrt{\beta} \), the function \( w(r) \) is similar to the sphaleron solution of [4]: starting with \( w = 1 \) it passes through \( w = 0 \) and then tends to the value \( w = 1. \)
After leaving this region the solution enters the region where it has properties of the NBI monopole and at $r \to \infty$ both field functions tend to their asymptotical values (38). The Higgs field $H(r)$ for these hybrid solutions behaves qualitatively in the same way as for the monopoles.

To obtain hybrid solutions numerically we introduce the logarithmic variable $t = \ln(r)$ and apply a shooting strategy to find the values of parameters $b$ and $d$ ensuring the monopole asymptotic conditions (40-41) after several oscillations of $w$. As an initial guess for $b$ one can take the (appropriately rescaled for given $\beta$) glueball values found in [4]. Another parameter $d$ turns out to be weakly sensitive on $\beta$ for $\beta$ large enough. The resulting solutions for $n = 1, 2$ and $\lambda = 1/2$ are shown on Fig. 1,2 together with the ground state monopole ($n = 0$). The masses increase with $n$ and converge rapidly to the mass of an embedded Abelian solution with frozen Higgs:

$$w(r) \equiv 0, \quad H(r) \equiv r. \quad (42)$$

Although this singular solution does not satisfy the boundary conditions (39) it has finite energy within the NBI-Higgs theory, which can be obtained by substituting the Eq. (42) into the Eq. (34):

$$E_{lim} = 2 \int \beta^2(\mathcal{R} - 1)r^2dr = \sqrt{\beta} \int \left(\sqrt{4 + \frac{2}{x^4} - 2}\right) x^2dx = 1.46738 \sqrt{\beta}. \quad (43)$$

With decreasing $\beta$, the discrete values of the parameter $b_n$ also decrease until relatively small values of $\beta$. Then, with $\beta$ further decreasing both parameters $b$ and $d$ start growing until some critical value of $\beta_{cr,n}$ is reached near which parameters $b_n$ and $d_n$ tend to infinity and monopole solutions with given number of zeroes cease to exist. The lowest of these critical values is $\beta_{cr,0} \approx 0.45$ for unexcited monopole solution. The excited solutions disappear at greater values of $\beta$. The mass of excited monopoles is well described by the formula (43), even for the lowest excited solution the difference with the exact numerical value is less then 4% for all values of $\beta$. The figure 2 shows the behavior of functions $w, H$ for some intermediate value of $\beta$. Note, that at critical $\beta$ all branches of monopole solutions (including unexcited branches) converge to the limiting Abelian solution (42) (with different rate).

The excited monopole solutions also exist in the Einstein-Yang-Mills-Higgs theory [24]. There the role of non-linear excitations is played by Bartnick-McKinnon gravitating sphalerons of EYM theory [12]. The phase diagram (regions of existence in parameter space) is somewhat different in our case, the details will be given elsewhere.

5 Non-commutative monopoles

Here we discuss one another aspect of the D-brane picture of gauge theories, which is the direct subject of the present workshop. Recently it was discovered that gauge theories on noncommutative manifolds

$$[x_\mu, x_\nu] = i\theta_{\mu\nu} \quad (44)$$
are connected with the gauge theories on D-branes with the constant background Kalb-Ramond field $B$ turned on \[ B_{\mu \nu} = -\frac{\theta_{\mu \nu}}{(2\pi\alpha')^2}. \] (45)

The relation between these two versions is non-local and is defined perturbatively through the Seiberg-Witten map \[ (26) \] (for a more recent discussion see \[ (27, 28) \]). Namely, the YM theory on a noncommutative four-dimensional space

\[
\hat{S} = \text{Tr} \int \left( \frac{1}{4g^2} \hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu} + \ldots \right) d^4x, 
\]

(46)

defined using the star-product

\[
F(x) \star G(x) = \exp \left( \frac{i\theta_{\mu\nu}}{2} \partial_{\mu} y \partial_{\nu} y \right) F(x) G(x') \big|_{x' = x}, 
\]

(47)

and the D-brane theory with $A_\mu$, $F_{\mu\nu}$ are related perturbatively via

\[
\hat{A}_\mu = A_\mu - \frac{\theta^{\alpha\beta}}{4} \{ A_\alpha, \partial_\beta A_\mu + F_{\beta\mu} \} + O(\theta^2). 
\]

(48)

The issue of magnetic monopoles in both treatments of the non-commutative YM was discussed recently in a number of papers \[ (29, 30, 31, 32, 30) \]. It was argued that BPS-saturated monopoles exist in the non-commutative case as well. Apart from the BPS bound most of the previous discussion was perturbative in terms of the non-commutativity parameter $\theta_{\mu\nu}$.

Adding the constant $B$-field spoils the spherical symmetry of monopoles and therefore their non-perturbative treatment in the D-brane picture becomes rather complicated. At best one can construct an axially symmetric model using $B_{\mu\nu}$ as a Kalb-Ramond analog of the homogeneous magnetic (electric) field. Even in this case the NBI model is still too complicated both for Tr and Str versions. Here we give a non-perturbative monopole solution in the simplest case of the $U(1)$ gauge field with Abelian Higgs. As was shown by Gibbons \[ (3) \], the system of BI $U(1)$ and Higgs fields possesses the boost symmetry (in the mixed space of coordinates and the field variables) which can be used as a solution generating technique to add a constant magnetic field to the pointlike magnetic monopole (resp. electric field to the electric Blon). Reinterpreted as the Kalb-Ramond field, this homogeneous field may be accounted for the parameter of non-commutativity.

We start with the DBI action

\[
S_{DBI} = -\int d^4x \sqrt{-\det (\eta_{\mu\nu} + \partial_{\mu} y \partial_{\nu} y + F_{\mu\nu})} 
\]

(49)

with one external coordinate $y$ (playing the role of the Higgs field) and introduce the magnetic potential $\chi$

\[
H = -\nabla \chi, 
\]

(50)

where $H$ is the magnetic field strength — canonical conjugate to the magnetic induction $B$:

\[
H = -\frac{\partial L}{\partial B}. 
\]

(51)
Performing the corresponding Legendre transformation we obtain the following Hamiltonian functional
\[
\mathcal{H} = \int d^3 x \sqrt{1 - (\nabla \chi)^2 + (\nabla y)^2 + (\nabla \chi)^2 (\nabla y)^2 - (\nabla \chi \cdot \nabla y)^2},
\] (52)
which can be interpreted as the volume of the three-dimensional hypersurface parametrized by coordinates \( x^i \) in the five-dimensional pseudoeuclidean space \( \{ x^i, y, \chi \} \) with the metric \( \text{diag}(+, +, +, +, -) \) (minus corresponds to \( \chi \)). We use the symmetries of this functional to generate first the scalar charge from the monopole charge and then to generate a constant background field which will be then interpreted as the \( B \) field. So we start with the spherically symmetric configurations. The field equations are then reduced to
\[
y'' = 2 \frac{y' (\chi'^2 - y'^2 - 1)}{r}, \quad \chi'' = 2 \frac{\chi' (\chi'^2 - y'^2 - 1)}{r},
\] (53)
where prime denotes the derivative with respect to the radial variable \( r \). It is easy to see that two potentials should be proportional. Depending on which potential dominates, one can find three different types of behaviour:

1. The spacelike vector in the \( \{ y, \chi \} \) plane. By some rotation the magnetic field can be removed. This is the catenoidal solution \([3]\). Since it does not exist for all \( r \), we will not consider it further.

2. The timelike vector in the \( \{ y, \chi \} \) plane. By a rotation it can be reduced to a \( U(1) \) monopole without excitations of the transverse degrees of freedom. The potential for this particular solution (with unit charge) is
\[
\chi_0(r) = \frac{\int r dr}{\sqrt{1 + r^4}},
\] (54)
and could be written explicitly in terms of elliptic integrals.

3. The lightlike vector \( y = \pm \chi \). This is the BPS monopole:
\[
\chi_{BPS}(r) = \pm y_{BPS}(r) = \frac{1}{r}.
\] (55)

To obtain the non-BPS monopole solution that also has a nonzero Higgs counterpart \( y(r) \) one can simply perform a boost in the \( \{ \chi, y \} \) plane:
\[
\chi(r) = \cosh \psi \chi_0(r), \quad y(r) = \sinh \psi \chi_0(r).
\] (56)
The next step is to perform a boost in the \( \{ \chi, z \} \) plane to generate the constant background magnetic field. To understand why this field may be equally interpreted as a \( B \) field one should notice that the field equations do not change if we replace \( F_{\mu \nu} \) by \( F_{\mu \nu} + B_{\mu \nu} \) with constant \( B \).

So, if we denote \( \chi = g(\rho, z) \), then after the second boost we obtain:
\[
\cosh \phi g + \sinh \phi z = \cosh \psi \chi_0 \left( \sqrt{\rho^2 + (\cosh \phi z + \sinh \phi g)^2} \right),
\] (57)
where \( \rho = \sqrt{x^2 + y^2} \) and \( \chi_0 \) is defined by the Eq.(54).

This nonlinear equation cannot be solved explicitly but it is simple to explore it numerically. The key point is to note that for a given \( g, \rho, z \), using equations (54), (57), one can find the vector \( \mathbf{F} + \mathbf{B} \) (magnetic induction plus B-field). Then the monopole field is obtained by subtracting the constant background. Note that, depending on the values of the boosts parameters \( \phi \) and \( \psi \), the solution can become double-valued. Let us consider this feature in more detail. For magnetic monopole without excitations of the transversal component the three-dimensional hypersurface \( \chi_0(x, y, z) \) is spacelike everywhere except for the origin where it touches the lightcone. When we boost in the \( \{ \chi, y \} \) directions, the surface \( \chi(x, y, z) \) acquires the timelike piece which can cause multivaluedness after boosting in the \( \{ \chi, z \} \) directions. (When treated as a hypersurface in the five-dimensional space \( \{ r, \chi, y \} \) it remains of course spacelike). This effect is interpreted from the string theory point of view as tilting the D-brane, but from the point of view of 3-dimensional field theory this multivaluedness should be interpreted as a signal that no well defined solution exists. It is worth noting that for BPS solution such multivaluedness emerges for any value of the background field.

In the figures 3, 4 the sections of level surfaces of constant \( y \) and constant \( |\mathbf{B}| \) are shown. The full solution is axially symmetric and is obtained by rotating the pictures along the symmetry axis.

## 6 Discussion

We have discussed some new issues associated with the D-brane picture of gauge theories. Apart from giving a nice geometric framework, D-branes suggest a modification of the dynamics of the YM field suggesting the Born-Infeld type lagrangian. This latter breaks the conformal invariance of the YM equations removing the obstruction for existence of classical glueballs in the SU(2) theory in four dimensions. Topological reason for existence of such glueballs lies in the vacuum periodicity which holds equally in the ordinary YM theory and in the NBI theory, with an important difference that in the latter case the potential barriers between neighboring vacua have finite heights. Classical NBI glueballs (more precisely, half of them) are sphalerons mediating the topological transitions. We have found that they exist both for the ordinary trace and the symmetrized trace versions of the NBI theory with somewhat different core structure. We have also shown that in the NBI theory with the triplet Higgs one encounters, apart from the usual magnetic monopoles, the hybrid solutions which can be regarded as sphaleronic excitations of monopoles. Finally, adding the constant Kalb-Ramond field, one is able to account for non-commutative monopoles. We presented a new nonperturbative axisymmetric solution for the U(1) non-commutative monopole with Higgs.

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Figure 1: Sphaleron glueball solutions $w_n$ for $n = 1, 2, 3$ in the symmetrized trace (red) and ordinary trace (blue) models.

Figure 2: Magnetic monopole and first two hybrid solutions in the ordinary trace model for $\beta = 30, \lambda = 1/2$. Red line — $w$, blue line — $H/r$. 
Figure 3: Non-commutative $U(1)$ monopole: constant $|\mathbf{F}|$ curves

Figure 4: Non-commutative $U(1)$ monopole: constant $y$ curves