DENSE STABLE RANK AND RUNGE TYPE APPROXIMATION THEOREMS FOR $H^\infty$ MAPS

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Abstract. Let $H^\infty(D \times \mathbb{N})$ be the Banach algebra of bounded holomorphic functions defined on the disjoint union of countably many copies of the open unit disk $D \subset \mathbb{C}$. We show that the dense stable rank of $H^\infty(D \times \mathbb{N})$ is one and using this fact prove some nonlinear Runge-type approximation theorems for $H^\infty(D \times \mathbb{N})$ maps. Then we apply these results to obtain a priori uniform estimates of norms of approximating maps in similar approximation problems for algebra $H^\infty(D)$.

1. Formulation of Main Results

1.1. Let $H^\infty(U)$ denote the Banach algebra of bounded holomorphic functions on a complex manifold $U$ equipped with pointwise multiplication and supremum norm. In this paper we continue to study properties of the algebra $H^\infty(D \times \mathbb{N})$, where $D \subset \mathbb{C}$ is the open unit disk, started in [Br2]. In particular, we prove that the dense stable rank of $H^\infty(D \times \mathbb{N})$ is one and using this fact prove some nonlinear Runge-type approximation theorems for this algebra. As a consequence, we obtain a priori uniform estimates of norms of approximating maps in similar approximation problems for $H^\infty$ maps defined on subsets of $D$.

To formulate our results we recall some definitions and notations.

Let $A$ be an associative ring with identity 1. An element $a = (a_1, \ldots, a_n) \in A^n$ is unimodular if there exists $b = (b_1, \ldots, b_n) \in A^n$ such that $\sum_{i=1}^n b_i a_i = 1$. By $U_n(A) \subset A^n$ we denote the set of unimodular elements of $A^n$. An element $a = (a_1, \ldots, a_n) \in U_n(A)$ is said to be reducible if there exist $c_1, \ldots, c_{n-1} \in A$ such that

$$(a_1 + c_1 a_n, \ldots, a_{n-1} + c_{n-1} a_n) \in U_{n-1}(A).$$

Ring $A$ is said to have a stable rank at most $n - 1$ if every $a \in U_n(A)$ is reducible. The stable rank of $A$, denoted by $sr(A)$, is the least $n - 1$ with this property. We write $sr(A) = \infty$ if such $n$ does not exist.

The concept of the stable rank introduced by Bass [B] plays an important role in some stabilization problems of algebraic $K$-theory analogous to that of dimension in topology. Despite a simple definition, $sr(A)$ is often quite difficult to calculate even for relatively uncomplicated rings $A$ (cf. [V]). An example of such calculation is the classical Treil...
Runge type approximation theorems for $H^\infty$ maps

Theorem (T, Theorem 1). Let $f, g \in H^\infty$, $\|f\|_{H^\infty} \leq 1$, $\|g\|_{H^\infty} \leq 1$ and

$$\inf_{z \in \mathbb{D}} (|f(z)| + |g(z)|) =: \delta > 0.$$

Then there exists a function $G \in H^\infty$ such that the function $\Phi = f + gG$ is invertible in $H^\infty$, and moreover $\|G\|_{H^\infty} \leq C$, $\|\Phi^{-1}\|_{H^\infty} \leq C$, where the constant $C$ depends only on $\delta$.

It was observed in [To, Thm. 2] that uniform estimates of norms of $G$ and $\Phi$ of the theorem imply that $sr(H^\infty(\mathbb{D} \times \mathbb{N})) = 1$ and de facto the latter statement is equivalent to the Treil theorem. In the present paper, we show that some other results for $H^\infty(\mathbb{D} \times \mathbb{N})$ lead to analogs of the above theorem with estimates of norms of the corresponding ‘solutions’ depending only on numerical data. Such estimates are of importance in many nonlinear problems for algebra $H^\infty$, see, e.g., [Ga] for the references.

Using some arguments of the proof of [T, Thm. 1], Suárez [S1, Cor. 4.6] proved that $\text{dsr}(H^\infty(\mathbb{D} \times \mathbb{N})) = 1$. Based on this result we prove:

Theorem 1.1. $\text{dsr}(H^\infty(\mathbb{D} \times \mathbb{N})) = 1.$

Notation. In what follows, $\mathfrak{M}(A)$ stands for the maximal ideal space of a commutative complex unital Banach algebra $A$ (i.e., the set of nonzero homomorphisms $A \to \mathbb{C}$ equipped with the Gelfand topology). Also, if $X$ and $Y$ are topological spaces, then $[X]$ denotes the set of connectivity components of $X$ and $[X, Y]$ denotes the set of homotopy classes of continuous maps from $X$ to $Y$. Theorem 1.1 asserts that each dense image morphism of complex unital Banach algebras $\varphi : H^\infty(\mathbb{D} \times \mathbb{N}) \to A$ induces a dense image map $U_n(H^\infty(\mathbb{D} \times \mathbb{N})) \to U_n(A)$, $n \in \mathbb{N}$, see [CS2, Prop. 3.2]. Hence, it induces a surjective map $[U_n(H^\infty(\mathbb{D} \times \mathbb{N}))] \to [U_n(A)]$. For $n = 1$ due to the Arens-Royden theorem the latter implies that $\varphi$ induces an epimorphism of Čech cohomology groups $H^1(\mathfrak{M}(H^\infty(\mathbb{D} \times \mathbb{N})), \mathbb{Z}) \to H^1(\mathfrak{M}(A), \mathbb{Z})$. In fact, this statement is equivalent to Theorem 1.1 as well as to the next ‘quantitative’ Runge-type approximation theorem similar to [T, Thm. 1]. In the formulation of the theorem we use the following notation.
In the sequel, a set of the form

\[ \Pi_{c_0}^n := \left\{ z \in \mathbb{D} : \max_{1 \leq i \leq n} |f_i(z)| < \nu, f_i \in H^\infty, \|f_i\|_{H^\infty} \leq c, 1 \leq i \leq n \right\} \]

is called an open n-polyhedron. We set \( \Pi_c^n := \Pi_{c_1}^n \).

The group of invertible elements of a unital algebra \( A \) is denoted by \( A^{-1} \).

We write \( C = C(\alpha_1, \alpha_2, \ldots) \) if the constant \( C \) depends only on \( \alpha_1, \alpha_2, \ldots \).

**Theorem 1.2.** Suppose \( g \in (H^\infty(\Pi_c^n))^{-1} \) satisfies

\[ \|g_{\pm 1}\|_{H^\infty(\Pi_c^n)} \leq M. \]

Given \( 0 < \delta < 1 \) and \( \varepsilon > 0 \), there exist a constant \( C = C(M, c, n, \delta, \varepsilon) \) and a function \( h \in (H^\infty)^{-1} \) such that

\[ \|h_{\pm 1}\|_{H^\infty} \leq C \quad \text{and} \quad \|(h - g)\|_{H^\infty(\Pi_c^n)} \leq \varepsilon. \]

**1.2.** In this part, we present some extensions of Theorems 1.1 and 1.2. To this end, we recall some concepts of the Novodvorski–Taylor theory \( [Ta] \).

Let \( A \) be a commutative complex unital Banach algebra. By definition, \( \mathfrak{M}(A) \) is a weak* compact subset of the dual space \( A^* \). If \( U \subseteq A^* \) is a weak* open neighbourhood of \( \mathfrak{M}(A) \), we denote by \( \mathcal{O}(U, \mathbb{C}^n) \) the vector space of weak* continuous holomorphic maps of \( U \) in \( \mathbb{C}^n \). (Recall that a continuous in the norm topology on \( A^* \) map \( U \to \mathbb{C}^n \) is holomorphic if all its complex directional derivatives exist.) We equip \( \mathcal{O}(U, \mathbb{C}^n) \) with the topology of uniform convergence on weak* compact subsets of \( U \). Then we define \( \mathcal{O}(\mathfrak{M}(A), \mathbb{C}^n) \) to be the inductive limit of spaces \( \mathcal{O}(U, \mathbb{C}^n) \) as \( U \) ranges over all weak* open neighbourhoods of \( \mathfrak{M}(A) \) equipped with the inductive limit topology. We regard \( \mathcal{O}(\mathfrak{M}(A), \mathbb{C}^n) \) as a topological algebra with the product induced by coordinatewise multiplication of maps in \( \mathcal{O}(U, \mathbb{C}^n) \). Similarly, we consider \( A^n \) as the Banach algebra equipped with coordinatewise multiplication. Then there exists a continuous algebra epimorphism, the **holomorphic functional calculus**, \( T_A^n : \mathcal{O}(\mathfrak{M}(A), \mathbb{C}^n) \to A^n \) such that

(i) The composition \( (^n) \circ T_A^n : \mathcal{O}(\mathfrak{M}(A), \mathbb{C}^n) \to C(\mathfrak{M}(A), \mathbb{C}^n) \), where \( ^n : A^n \to C(\mathfrak{M}(A), \mathbb{C}^n) \),

\[ (^n)(a_1, \ldots, a_n)(\xi) := (\xi(a_1), \ldots, \xi(a_n)), \quad \xi \in \mathfrak{M}(A), \]

is the \( n \)-fold product of the Gelfand transform, is the restriction map \( f \mapsto f|_{\mathfrak{M}(A)} \);

(ii) The composition of \( T_A^n \) and the \( n \)-fold product of the injective map \( A \to \mathcal{O}(\mathfrak{M}(A), \mathbb{C}) \) defined by the natural embedding \( A \hookrightarrow A^{**} \) is \( \text{id}_{A^n} \), the identity map of \( A^n \).

Next, for a complex submanifold \( \mathcal{M} \subset \mathbb{C}^n \) we define \( \mathcal{O}(\mathcal{U}, \mathcal{M}) \subset \mathcal{O}(U, \mathbb{C}^n) \) as the subset of maps with images in \( \mathcal{M} \). Applying to the family of spaces \( \mathcal{O}(\mathcal{U}, \mathcal{M}) \) the inductive limit construction we get a subspace \( \mathcal{O}(\mathfrak{M}(\mathcal{A}), \mathcal{M}) \subset \mathcal{O}(\mathfrak{M}(A), \mathbb{C}^n) \). Then we set

\[ A_{\mathcal{M}} := T_A^n(\mathcal{O}(\mathfrak{M}(\mathcal{A}), \mathcal{M})) (\subset A^n). \]

It is known that if \( \mathcal{M} \subset \mathbb{C}^n \) is a complex submanifold and a homogeneous manifold (i.e., it is equipped with a holomorphic and transitive action of a complex Lie group, see, e.g.,
For the references, then $A_M$ is locally path-connected and the map $[A_M] \to [\mathfrak{M}(A), \mathcal{M}]$ induced by $^\sim^n$ is a bijection, see [13, Sec. 2.7]. In fact, the same is true for the class of manifolds $\mathcal{M}$ subject to the following definition.

A complex manifold $\mathcal{M}$ is said to be Oka if every holomorphic map $f : K \to \mathcal{M}$ from a neighbourhood of a compact convex set $K \subset \mathbb{C}^k$, $k \in \mathbb{N}$, can be approximated uniformly on $K$ by entire maps $\mathbb{C}^k \to \mathcal{M}$.

The class of Oka manifolds includes, in particular, complex homogeneous manifolds, complements in $\mathbb{C}^k$, $k > 1$, of complex algebraic subvarieties of codimension $\geq 2$ and of compact polynomially convex sets, Hopf manifolds (i.e., nonramified holomorphic quotients of $\mathbb{C}^n \setminus \{0\}$). Also, holomorphic fibre bundles whose bases and fibres are Oka manifolds are Oka manifolds as well. (We refer to the book [F2] and the paper [K] for other examples and basic results of the theory of Oka manifolds.)

**Proposition 1.3.** Let $\mathcal{M} \subset \mathbb{C}^n$ be a complex submanifold and an Oka manifold. Then $A_M$ is locally path-connected and the map $[A_M] \to [\mathfrak{M}(A), \mathcal{M}]$ induced by $^\sim^n$ is a bijection.

Now we move on to the formulation of the nonlinear Runge-type approximation theorem for commutative complex unital Banach algebras.

Let $r : A_1 \to A_2$ be a dense image morphism of commutative complex unital Banach algebras. Then the transpose map $r^*$ embeds $\mathfrak{M}(A_2)$ into $\mathfrak{M}(A_1)$. In turn, the pullback with respect to $r^*|_{\mathfrak{M}(A_2)}$ denoted by $r_c$ maps $C(\mathfrak{M}(A_1))$ surjectively onto $C(\mathfrak{M}(A_2))$.

Let $\mathcal{M} \subset \mathbb{C}^n$ be a complex submanifold. Then the $n$-fold product of $r$ (denoted by $r^n$) maps $(A_1)_M$ in $(A_2)_M$ (see Proposition 1.1 below). Let $r^n((A_1)_M)$ be the closure of $r^n((A_1)_M)$ in $(A_2)_M$.

**Theorem 1.4.** Assume that $\mathcal{M}$ is an Oka manifold. The following is true:

1. $r^n((A_1)_M) = \{a \in (A_2)_M : r^n(a) \in \text{range } (r_c)^n\}$.
2. If $r$ is an epimorphism, then $r^n((A_1)_M)$ is a closed subset of $(A_2)_M$.

Thus, identifying $\mathfrak{M}(A_2)$ with its image under $r^*$ in $\mathfrak{M}(A_1)$ we obtain that an element $a \in (A_2)_M$ is approximated by images under $r^n$ of elements from $(A_1)_M$ if and only if its Gelfand transform $^\sim^n(a) \in C(\mathfrak{M}(A_2), \mathcal{M})$ is extended to a map from $C(\mathfrak{M}(A_1), \mathcal{M})$.

**Remark 1.5.** Let $\mathcal{M}$ be a commutative complex unital Banach algebra. We set for a complex submanifold $\mathcal{M} \subset \mathbb{C}^n$,

$$A^\mathcal{M} := \{(a_1, \ldots, a_n) \in A^n : (\xi(a_1), \ldots, \xi(a_n)) \in \mathcal{M} \ \forall \xi \in \mathfrak{M}(A)\}.$$ 

Clearly, $A_M \subseteq A^\mathcal{M}$ and these sets coincide if either $\mathcal{M}$ is an open subset of $\mathbb{C}^n$ or $A$ is semisimple (i.e., the Gelfand transform $^\sim : A \to C(\mathfrak{M}(A))$ is injective), see [13, Sec. 2.8].

Let $R_A \subset A$ be the Jacobson radical, i.e., the kernel of the Gelfand transform of $A$. Then $j_A : A \to A/R_A := A_s$ is an epimorphism of complex unital Banach algebras and $A_s$ is semisimple. Moreover, the transpose map $(j_A)^*$ maps $\mathfrak{M}(A_s)$ homeomorphically onto $\mathfrak{M}(A)$. Applying Theorem 1.4 to $A_1 := A$, $A_2 := A_s$ we obtain:
Corollary 1.7. Following result.

(A) Suppose $\mathcal{M} \subset \mathbb{C}^n$ is a complex submanifold and an Oka manifold. Then $(j_A)^n$ maps $A_\mathcal{M}$ surjectively onto $(A_1)^{\mathcal{M}}$. Moreover, the map $[A_\mathcal{M}] \rightarrow [(A_1)^{\mathcal{M}}]$ of sets of connectivity components induced by $(j_A)^n$ is a bijection.

Let $\rho : A_1 \rightarrow A_2$ be a dense image morphism of commutative complex unital Banach algebras. Clearly, $\rho(R_{A_1}) \subset R_{A_2}$. Hence, there is a dense image morphism $r_s : (A_1)_s \rightarrow (A_2)_s$ such that $j_{A_2} \circ r = j_{A_1} \circ r_s$. Now Theorem 1.4 leads to the following statement.

(B) Suppose that $\mathcal{M} \subset \mathbb{C}^n$ is a complex submanifold and an Oka manifold. An element $a \in (A_2)_\mathcal{M}$ is approximated by images under $r^n$ of elements from $(A_1)_\mathcal{M}$ if and only if $j_{A_2}(a) \in ((A_2)_s)^{\mathcal{M}}$ is approximated by images under $(r_s)^n$ of elements from $((A_1)_s)^{\mathcal{M}}$.

This reduces the general Runge-type theorem for commutative complex unital Banach algebras to the case of semisimple algebras.

As a corollary of Theorem 1.4 we obtain an extension of Theorem 1.1. To formulate the result, we recall the following definition.

(V) A path-connected topological space $X$ is $i$-simple if for each $x \in X$ the fundamental group $\pi_i(X, x_0)$ acts trivially on the $i$-homotopy group $\pi_i(X, x)$ (see, e.g., [Hu1, Ch. IV.16] for the corresponding definitions and results).

For instance, $X$ is $i$-simple if group $\pi_i(X) = 0$ and 1-simple if and only if group $\pi_i(X)$ is abelian. Also, every path-connected topological group is $i$-simple for all $i$. The same is true for a complex manifold biholomorphic to the quotient of a connected complex Lie group by a connected closed Lie subgroup, see, e.g., [Hu1, (3.2)].

Let $\varphi : H^\infty(\mathbb{D} \times \mathbb{N}) \rightarrow A$ be a dense image morphism of complex unital Banach algebras and let $\mathcal{M} \subset \mathbb{C}^n$ be a connected submanifold and an Oka manifold.

Theorem 1.6. Assume that a finite unbranched covering of $\mathcal{M}$ is $i$-simple for $i = 1, 2$. Then the image of $(H^\infty(\mathbb{D} \times \mathbb{N}))^{\mathcal{M}}$ under $\varphi^n$ is a dense subset of $A_\mathcal{M}$.

If, in addition, $\varphi$ is an epimorphism, then this image coincides with $A_\mathcal{M}$.

Since each connected component of a topological space is open, Theorem 1.6 implies the following result.

Corollary 1.7. Under hypotheses of Theorem 1.6, the map $[(H^\infty(\mathbb{D} \times \mathbb{N}))^{\mathcal{M}}] \rightarrow [A_\mathcal{M}]$ induced by $\varphi^n$ is a surjection.

Remark 1.8. (1) As was mentioned in Section 1.1, Theorem 1.4 follows from Corollary 1.7 with $n = 1$ and $\mathcal{M} = \mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

(2) Let $\mathcal{O}$ be the class of connected Oka manifolds $\mathcal{M}$ embeddable as complex submanifolds into complex Euclidean spaces and having $i$-simple for $i = 1, 2$ finite unbranched coverings. This class contains e.g., complements in $\mathbb{C}^k$, $k > 1$, of complex algebraic subvarieties of codimension $\geq 2$ and of compact polynomially convex sets (these manifolds are simply connected, see [F1]), connected Stein Lie groups, quotients of connected reductive complex Lie groups by Zariski closed subgroups (these manifolds are quasi-affine algebraic, see, e.g., [A, Thm. 5.6] for the references; they have $i$-simple finite unbranched coverings because Zariski closed subgroups have finitely many connected components and quotients.
of connected complex Lie groups by connected closed Lie subgroups are \( i \)-simple for all \( i \), see, e.g., [Hu1, (3.2)]. Also, direct products of manifolds from class \( \mathcal{O} \) belongs to \( \mathcal{O} \), etc.

In the next two results, \( \mathcal{M} \subset \mathbb{C}^n \) is a connected complex submanifold from class \( \mathcal{O} \), \( K \subset \mathcal{M} \) is a compact subset, and \( \Pi^k \subset \mathbb{D} \) is an open \( k \)-polyhedron, see (1.2).

As a corollary of Theorem 1.9 we obtain the following extension of Theorem 1.2.

**Theorem 1.9.** Suppose \( g \in \mathcal{O}(\Pi^k_\mathcal{C}, \mathcal{M}) \) is such that \( g(\Pi^k_\mathcal{C}) \subset K \). Given \( 0 < \delta < 1 \) and \( \varepsilon > 0 \) there exist a constant \( C = C(\mathcal{M}, K, n, c, k, \delta, \varepsilon) > 0 \) and a map \( h \in (H^\infty)^\mathcal{M} \) such that

\[
\|h\|_{(H^\infty)^n} \leq C \quad \text{and} \quad \|(h - g)\|_{\Pi^k_\mathcal{C}, \delta} \leq \varepsilon. 
\]

Further, if \( \Pi^k_\mathcal{C} \) is determined by functions \( f_1, \ldots, f_k \in H^\infty(\mathbb{D}) \), then we define an open set \( \hat{\Pi}^k_\mathcal{C} \subset \mathcal{M}(H^\infty) \) by the formula

\[
\hat{\Pi}^k_\mathcal{C} := \left\{ \xi \in \mathcal{M}(H^\infty) : \max_{1 \leq i \leq k} |\hat{f}_i(\xi)| < \nu \right\}, \quad \hat{\Pi}^k_\mathcal{C} := \hat{\Pi}^k_\mathcal{C,1}.
\]

Clearly, \( \Pi^k_\mathcal{C} = \hat{\Pi}^k_\mathcal{C} \cap \mathbb{D} \) and by the corona theorem \( \Pi^k_\mathcal{C} \) is an open dense subset of \( \hat{\Pi}^k_\mathcal{C} \). Moreover, due to [S1, Thm. 3.2] each \( f \in H^\infty(\Pi^k_\mathcal{C}) \) admits an extension \( \hat{f} \in C(\Pi^k_\mathcal{C}) \).

Let \( J \subset H^\infty(\mathbb{D}) \) be a closed ideal and let

\[
hull J := \{ \xi \in \mathcal{M}(H^\infty) : \hat{f}(\xi) = 0 \quad \forall f \in J \}.
\]

Let \( \Pi^k_\mathcal{C} \) be a polyhedron such that

\[
hull J \subset \hat{\Pi}^k_\mathcal{C,\delta} \quad \text{for some} \quad 0 < \delta < 1.
\]

(Such polyhedra always exist, see Remark 1.11 below.)

**Theorem 1.10.** Let \( g \in (H^\infty)^n \) be such that

\[
\|g\|_{(H^\infty)^n} \leq b \quad \text{and} \quad \hat{\sigma}^n(g)(\xi) \in K \quad \forall \xi \in \hull J.
\]

Suppose that there is \( f \in \mathcal{O}(\Pi^k_\mathcal{C}, \mathcal{M}) \), \( f(\Pi^k_\mathcal{C}) \subset K \), such that

\[
\hat{f}|_{\hull J} = \hat{g}|_{\hull J}.
\]

Then there exist a constant \( C = C(\mathcal{M}, K, n, b, c, k, \delta) > 0 \) and a map \( h \in (H^\infty)^\mathcal{M} \) such that

\[
\|h\|_{(H^\infty)^n} \leq C \quad \text{and} \quad \hat{\sigma}^n(h)|_{\hull J} = \hat{\sigma}^n(g)|_{\hull J}.
\]

A particular case of Theorem 1.10 for \( \mathcal{M} = \mathbb{C}^* \) follows from Treil’s theorem [T] via its cohomological interpretation due to Suárez [S1, Thm. 1.3] and the Arens-Royden theorem.

**Remark 1.11.** A compact subset \( S \subset \mathcal{M}(H^\infty) \) is called \textit{holomorphically convex} if for each \( \xi \not\in S \) there is \( f \in H^\infty \) such that

\[
\max_{S} |\hat{f}| < |\hat{f}(\xi)|.
\]
If $U$ is an open neighbourhood of such $S$, then there exists an open polyhedron $\Pi C \subset \mathbb{D}$ such that $S \subset \Pi C \subset U$, see, e.g., [Br3] Lm. 5.1. Hence, Theorems [1.9] and [1.10] imply the following result.

Let $S \subset \mathfrak{M}(H^\infty)$ be holomorphically convex, $U \subset \mathfrak{M}(H^\infty)$ be an open neighbourhood of $S$ and $K \subset \mathcal{M}$ be as in the above theorems.

**Corollary 1.12.** (1) Suppose $g \in C(U, K)$ is such that $g|_{U \cap \mathbb{D}}$ is holomorphic. Given $0 < \delta < 1$ and $\varepsilon > 0$ there exist a constant $c = c(\mathcal{M}, K, n, S, U, \delta, \varepsilon) > 0$ and a map $h \in (H^\infty)^{\mathcal{M}}$ such that

$$
\|h\|_{(H^\infty)^n} \leq c \quad \text{and} \quad (\hat{\mathcal{O}}(h) - g)|_S \leq \varepsilon.
$$

(2) Let $S = \text{hull } J$ for some closed ideal $J \subset H^\infty$ and let $g \in (H^\infty)^n$ satisfy

$$
\|g\|_{(H^\infty)^n} \leq b \quad \text{and} \quad (\hat{\mathcal{O}}(g))(\xi) \in K \quad \forall \xi \in S.
$$

Suppose that there is $f \in C(U, K)$ such that $f|_{U \cap \mathbb{D}}$ is holomorphic and $f|_S = \hat{g}|_S$.

Then there exist a constant $c = c(\mathcal{M}, K, n, b, S, U) > 0$ and a map $h \in (H^\infty)^{\mathcal{M}}$ such that

$$
\|h\|_{(H^\infty)^n} \leq c \quad \text{and} \quad \hat{\mathcal{O}}(h)|_{\text{hull } J} = \hat{\mathcal{O}}(g)|_{\text{hull } J}.
$$

The previous result partially extends Theorems 3.7 and 3.8 of [Br3] and give a priori uniform estimates of norms of approximating and interpolating maps there.

### 2. Auxiliary Results

**2.1.** Due to the Carleson corona theorem $\mathbb{D}$ can be regarded as a dense open subset of the maximal ideal space $\mathfrak{M}(H^\infty)$ equipped with the Gelfand topology. Then the Gelfand transform extends each $f \in H^\infty$ to a unique continuous function $\hat{f}$ on $\mathfrak{M}(H^\infty)$. For an open subset $U \subset \mathfrak{M}(H^\infty)$ the corona theorem implies that $U \cap \mathbb{D}$ is an open dense subset of $U$. A function in $C(U)$ is called holomorphic if its restriction to $U \cap \mathbb{D}$ is holomorphic in the standard sense. The set of such functions is denoted by $\mathcal{O}(U)$. Each $f \in H^\infty(U \cap \mathbb{D})$ extends (uniquely) to a bounded function $f \in \mathcal{O}(U)$, see [S1] Thm. 3.2. A compact subset $K \subset U$ is called $\mathcal{O}(U)$-convex if for each $x \notin K$ there is $f \in \mathcal{O}(U)$ such that

$$
\max_K |f| < |f(x)|.
$$

**2.2.** In what follows, $\mathbb{D}_r(c) := \{z \in \mathbb{C} : |z - c| < r\}$, $r > 0$, $c \in \mathbb{C}$, i.e., $\mathbb{D} := \mathbb{D}_1(0)$. For a subset $S$ of a topological space we denote by $S$ its closure.

Let $\Omega := \left(\mathbb{D} \cup \mathbb{D}_1(\frac{1}{2})\right) \setminus \left\{\frac{3}{2}\right\}$.

The fundamental group $\pi_1(\Omega)$ of $\Omega$ is isomorphic to $\mathbb{Z}$, i.e., $\pi_1(\Omega) = \{a^n\}_{n \in \mathbb{Z}}$ for some $a \in \pi_1(\Omega)$. Let $r : \mathbb{D} \to \Omega$ be the universal covering of $\Omega$. The deck transformation group $\pi_1(\Omega)$ acts discretely on $\mathbb{D}$ by Möbius transformations. Since $r \in H^\infty$, it extends to a
function \( \hat{r} \in \mathcal{O}(\mathbb{M}(H^\infty)) \) such that \( \hat{r}(\mathbb{M}(H^\infty)) = \overline{\Omega} \). Let \( U := r^{-1}(\mathbb{D}) \subset \mathbb{D} \). Since each loop in \( \mathbb{D} \) is contractible in \( \Omega \),

\[
U = \bigsqcup_{g \in \pi_1(\Omega)} g(U')
\]

for some \( U' \subset \mathbb{D} \) biholomorphic to \( \mathbb{D} \) via \( r \). In particular, the map \( s : U \to \mathbb{D} \times \mathbb{Z} \),

\[
s(z) := (r(z), n), \quad z \in a^n(U'), \quad n \in \mathbb{Z},
\]

is biholomorphic.

By the corona theorem \( U \) is dense in

\[
\tilde{U} := \{ x \in \mathbb{M}(H^\infty) : |\hat{r}(x)| < 1 \}.
\]

and due to [S1, Thm. 3.2] each \( f \in H^\infty(U) \) extends to a (unique) \( \hat{f} \in \mathcal{O}(\tilde{U}) \).

Let us consider \( h := \text{Re} \ r \) and its extension \( \hat{h} := \text{Re} \hat{r} \in C(\mathbb{M}(H^\infty)) \). Clearly, \( N := \{ x \in \mathbb{M}(H^\infty) : \hat{h}(x) \leq 0 \} \) is a \( \mathcal{O}(\mathbb{M}(H^\infty)) \)-convex compact subset of \( \tilde{U} \).

**Lemma 2.1.** Every \( \mathcal{O}(\tilde{U}) \)-convex subset \( K \subset N \) is \( \mathcal{O}(\mathbb{M}(H^\infty)) \)-convex.

**Proof.** Let \( x \not\in K \) and \( f \in \mathcal{O}(\tilde{U}) \) satisfy (2.1). Clearly, we can assume that \( x \in N \). We set \( W := \{ x \in \mathbb{M}(H^\infty) : \hat{h}(x) < \frac{1}{2} \} \).

Then \( W \Subset \tilde{U} \) is an open neighbourhood of \( N \). Since \( N \) is \( \mathcal{O}(\mathbb{M}(H^\infty)) \)-convex, \( f|_W \) can be uniformly approximated on \( N \) by functions in \( \mathcal{O}(\mathbb{M}(H^\infty)) \), see [S2, Cor. 2.6] or [Br1, Thm. 1.7]. Hence, there is \( f' \in \mathcal{O}(\mathbb{M}(H^\infty)) \) sufficiently close to \( f|_N \) such that (2.1) holds for \( f \) replaced by \( f' \). This proves the lemma. \( \square \)

2.3. Let \( A(\mathbb{D}) \) be the disk algebra of holomorphic functions in \( \mathbb{D} \) continuous on \( \overline{\mathbb{D}} \). A compact set \( K \subset \mathbb{C} \) is called polynomially convex if for each \( z \not\in K \) there is a holomorphic polynomial \( p \) on \( \mathbb{C} \) such that

\[
\max_K |p| < |p(z)|.
\]

**Lemma 2.2.** Let \( K \subset \overline{\mathbb{D}} \) be a proper polynomially convex set. Then there is a Möbius transformation \( g : \mathbb{D} \to \mathbb{D} \) such that

\[
g(K) \subset D_- := \{ z \in \overline{\mathbb{D}} : \text{Re} \ z \leq 0 \}.
\]

**Proof.** Since \( K \) is proper and polynomially convex, there exist a point \( c \in S := \overline{\mathbb{D}} \setminus \mathbb{D} \) and an open disk \( \mathbb{D}_r(c) \) such that \( \mathbb{D}_r(c) \cap K = \emptyset \). We choose some \( t \in (1-r, 1) \) such that the disk \( \mathbb{D}_{r'}(c') \) of radius \( r' := \frac{1}{2} (\frac{1}{\sqrt{t}} - tc) \) centered at \( c' := \frac{1}{2} (\frac{1}{\sqrt{t}} + tc) \) satisfies

\[
\mathbb{D}_{r'}(c') \cap \overline{\mathbb{D}} \subset \mathbb{D}_r(c).
\]
By the definition, the part of the boundary of $D_{\nu}(c')$ in $D$ is a geodesic in the Poincaré metric on $D$ passing through $tc$. Then the Möbius transformation

$$g(z) := \frac{z - tc}{1 - tcz}, \quad z \in \overline{D},$$

maps this geodesic to the interval $\{\text{Re } z = 0\} \cap \overline{D}$ and $D \setminus D_{\nu}(c')$ to $D_-$. Hence, $g(K) \subset D_-$ as required.

3. Proofs of Theorems 1.1 and 1.2

3.1. Proof of Theorem 1.2

Recall that

$$(3.1) \quad \Pi_{c,\delta}^n := \left\{ z \in D : \max_{1 \leq i \leq n} |f_i(z)| < \delta, f_i \in H^\infty, \|f_i\|_{H^\infty} \leq c, 1 \leq i \leq n \right\}, \quad \Pi_c^n := \Pi_{c,1}^n.$$

We also use notation $\Pi_{c,\delta}^n[F]$ and $\Pi_c^n[F]$, where $F = \{f_i\}_{1 \leq i \leq n}$, to emphasize dependence of the polyhedron on the family of functions determining it.

We have to prove that given $g \in (H^\infty(\Pi_c^n))^{-1}$ satisfying

$$\|g^\pm 1\|_{H^\infty(\overline{D})} \leq M$$

and every $\varepsilon > 0$ there exist a constant $C = C(c, M, n, \delta, \varepsilon)$ and a function $h \in (H^\infty)^{-1}$ such that

$$\|h^\pm 1\|_{H^\infty} \leq C \quad \text{and} \quad \|(h - g)|_{\Pi_{c,\delta}^n}\|_{H^\infty(\Pi_{c,\delta}^n)} \leq \varepsilon.$$

It suffices to prove the lemma for $n$-polyhedra $\Pi_c^n$ determined by functions from the disk algebra $A(D)$. Then a standard normal family argument produces the required result in the general case. Clearly, we may assume that closures of such polyhedra are proper subsets of $D$ (for otherwise, the statement is trivial). In turn, applying Lemma 2.2 we may assume that $\Pi_c^n \subset D_-$. If $\Pi_c^n$ is determined by $F = \{f_i\}_{1 \leq i \leq n} \subset H^\infty$, then the set $\mathcal{D} = \{F, c, M, n, \delta, \varepsilon\}$ is called the data.

According to [S2, Cor. 2.6] each function in $H^\infty(\Pi_c^n)$ can be uniformly approximated on subsets $\Pi_{c,\nu}^n, 0 < \nu < 1$, by functions from $H^\infty$. Then according to [S1, Cor. 4.6] applied to the dense morphism $H^\infty \to A, f \mapsto f|_{\Pi_{c,\delta}^n}$, where $A$ is the uniform closure of $H^\infty|_{\Pi_{c,\delta}^n}$, for each $g$ satisfying (3.2) there exists a function $h \in (H^\infty)^{-1}$ (depending on $g$ and the data) such that

$$\|(h - g)|_{\Pi_{c,\delta}^n}\|_{H^\infty(\Pi_{c,\delta}^n)} \leq \varepsilon.$$

By $\mathcal{H}_{g, \mathcal{D}}$ we denote the class of such functions $h$.

We have to prove that

$$C = C(c, M, n, \delta, \varepsilon) := \sup_{F, g} \inf_{h \in \mathcal{H}_{g, \mathcal{D}}} \max\{\|h\|_{H^\infty}, \|h^{-1}\|_{H^\infty}\}$$

is finite.
To this end, let \( \{ \Pi^n[F_i] \}_{i \in \mathbb{N}} \subset D_-, \) \( F_i = \{ f_{ki} \}_{k=1}^n \subset A(\mathbb{D}) \), and \( \{ g_i \in (\mathcal{H}^\infty(\Pi^n[F_i]))^{-1} \}_{i \in \mathbb{N}} \) be sequences of polyhedra and functions satisfying assumptions of the theorem such that
\[
(3.6) \quad C = \lim_{i \to \infty} \inf_{h \in \mathcal{H}_{g_i, D_i}} \max \{ \| h \|_{\mathcal{H}^\infty}, \| h^{-1} \|_{\mathcal{H}^\infty} \};
\]
here \( D_i := \{ F_i, c, M, n, \delta, \varepsilon \} \).

We define functions \( f_k \in H^\infty(\mathbb{D} \times \mathbb{N}) \), \( 1 \leq k \leq n \), by the formulas
\[
(3.7) \quad f_k(z, i) := f_{ki}(z), \quad (z, i) \in \mathbb{D} \times \mathbb{N}.
\]

Next, we set \( F = \{ f_k \}_{k=1}^n \) and define polyhedra \( \Pi_\nu[F] \subset \mathbb{D} \times \mathbb{N} \),
\[
(3.8) \quad \Pi_\nu[F] := \{ x \in \mathbb{D} \times \mathbb{N} : \max_{1 \leq k \leq n} |f_k(x)| < \nu \} \quad (= \{ (z, i) \in \mathbb{D} \times \mathbb{N} : z \in \Pi^n_{c, \nu[F_i]} \}).
\]

Finally, we define \( g \in H^\infty(\Pi_1[F]) \),
\[
(3.9) \quad g(z, i) := g_i(z), \quad (z, i) \in \Pi_1[F].
\]

Let \( \tau : \mathbb{Z} \to \mathbb{N} \) be a bijection. Consider the biholomorphic map
\[
(3.10) \quad b := (id_\mathbb{D} \times \tau) \circ s : U \to \mathbb{D} \times \mathbb{N},
\]
see (2.3), (2.4) in Section 2.2. We pull back functions \( f_k \) and \( g \) to \( U \) by \( b \). Hence, \( b^* f_k := f_k \circ b \in H^\infty(U) \) and \( b^* g \in H^\infty(b^{-1}(\Pi_1[F])) \). By definition, \( b^{-1}(\Pi_\nu[F]) \) are polyhedra in \( U \) determined by functions \( b^* f_k \), \( 1 \leq k \leq n \), cf. (3.8). Let \( \hat{b}^* f_k \in \mathcal{O}(\hat{U}) \) be the extension of \( b^* f_k \) to \( \hat{U} = \hat{\tau}^{-1}(\mathbb{D}) \subset \mathcal{M}(H^\infty) \), see (2.5). We define open polyhedra \( \hat{\Pi}_\nu \subset \hat{U} \) by the formulas
\[
(3.11) \quad \hat{\Pi}_\nu := \{ x \in \hat{U} : \max_{1 \leq k \leq n} |\hat{b}^* f_k(x)| < \nu \}.
\]

Then the open polyhedron \( b^{-1}(\Pi_\nu[F]) \) is dense in \( \hat{\Pi}_\nu \). In particular, by [S1] Thm. 3.2, \( b^* g \) is extended to a function \( \hat{b}^* g \in \mathcal{O}(\hat{\Pi}_1) \). By assumptions of the theorem \( (\hat{b}^* g)^{-1} \in \mathcal{O}(\hat{\Pi}_1) \) as well.

Further, since each \( \Pi^n_{c}[F_i] \subset D_- \), the open polyhedron \( \hat{\Pi}_1 \) lies in the \( \mathcal{O}(\mathcal{M}(H^\infty)) \)-convex compact set \( N \), see (2.6). Moreover, every open polyhedron \( \hat{\Pi}_\nu \), \( 0 < \nu < 1 \), is a relatively compact subset of \( \hat{\Pi}_1 \). Hence, due to Lemma 2.1 the \( \mathcal{O}(\hat{U}) \)-convex set \( K_\delta := \bigcap_{\nu > \delta} \hat{\Pi}_\nu \) (the \( \mathcal{O}(\hat{U}) \)-convex hull of \( \hat{K}_\delta \)) is \( \mathcal{O}(\mathcal{M}(H^\infty)) \)-convex. Since function \( \hat{b}^* g \) is defined on an open neighbourhood of \( K_\delta \), [S1] Cor. 2.6 implies that \( \hat{b}^* g \) can be uniformly approximated on \( K_\delta \) by invertible functions from \( \mathcal{O}(\mathcal{M}(H^\infty)) \). Thus there is an invertible function \( H \in \mathcal{O}(\mathcal{M}(H^\infty)) \) such that
\[
(3.12) \quad \sup_{x \in \hat{K}_\delta} |H(x) - \hat{b}^* g(x)| \leq \varepsilon.
\]

We set
\[
(3.13) \quad h_i(z) := (H|_U \circ b^{-1})(z, i), \quad (z, i) \in \mathbb{D} \times \mathbb{N}.
\]

Then \( h_i \in (\mathcal{H}^\infty)^{-1} \) and
\[
\|(h_i - g_i)|_{\Pi^n_{c, \cdot}[F_i]}\|_{\mathcal{H}^\infty} \leq \varepsilon \quad \text{for all} \quad i \in \mathbb{N}.
\]
Moreover,
\[
\sup_{i \in \mathbb{N}} \max \{ \| h_i \|_{H^\infty}, \| h_i^{-1} \|_{H^\infty} \} \leq \max \{ \| H \|_{H^\infty}, \| H^{-1} \|_{D^\infty} \}.
\]

Hence,
\[
C \leq \max \{ \| H \|_{H^\infty}, \| H^{-1} \|_{D^\infty} \}
\]
as well, see (3.6).

This completes the proof of the theorem. \(\square\)

3.2. Proof of Theorem 1.1. The corona theorem for \(H^\infty(\mathbb{D} \times \mathbb{N})\) follows from Carleson estimates for solutions of the corona problem for \(H^\infty\), see, e.g., [Be, Lm. 1]. Hence, \(\mathbb{D} \times \mathbb{N}\) is embedded into \(\mathcal{M}(H^\infty(\mathbb{D} \times \mathbb{N}))\) as an open dense subset in the Gelfand topology.

A compact subset \(K \subset \mathcal{M}(H^\infty(\mathbb{D} \times \mathbb{N}))\) is called holomorphically convex if for each \(x \notin K\) there is \(f \in H^\infty(\mathbb{D} \times \mathbb{N})\) such that
\[
(3.14) \quad \max_{K} |\hat{f}| < |\hat{f}(x)|;
\]

here \(^* : H^\infty(\mathbb{D} \times \mathbb{N}) \to C(\mathcal{M}(H^\infty(\mathbb{D} \times \mathbb{N}))\) is the Gelfand transform.

Let \(\varphi : H^\infty(\mathbb{D} \times \mathbb{N}) \to A\) be a dense morphism of complex unital Banach algebras. Then the transpose map \(\varphi^*\) establishes a homeomorphism from \(\mathcal{M}(A)\) onto a holomorphically convex subset of \(\mathcal{M}(H^\infty(\mathbb{D} \times \mathbb{N}))\). Without loss of generality we identify \(\mathcal{M}(A)\) with its image under \(\varphi^*\). In order to prove the theorem we have to show that for every \(g \in H^\infty(\mathbb{D} \times \mathbb{N})\) such that \(\hat{g}\) is zero free on \(\mathcal{M}(A)\) there is a sequence \(\{h_i\}_{i \in \mathbb{N}} \subset (H^\infty(\mathbb{D} \times \mathbb{N}))^{-1}\) such that \(\{\hat{h}_i\}_{\mathcal{M}(A)}\) converges uniformly to \(\hat{g}_{\mathcal{M}(A)}\), see, e.g., [SI page 262] and references therein.

To this end, let \(O \subset \mathcal{M}(H^\infty(\mathbb{D} \times \mathbb{N}))\) be an open neighbourhood of \(\mathcal{M}(A)\) such that \(\hat{g}\) is yet zero free on its closure \(\bar{O}\). Since \(\mathcal{M}(A)\) is holomorphically convex, there exist functions \(f_k \in H^\infty(\mathbb{D} \times \mathbb{N})\), \(1 \leq k \leq n\), \(n \in \mathbb{N}\), and \(\delta \in (0, 1)\) such that \(\mathcal{M}(A) \subset \Pi_{\delta} \subset \Pi_1 \subset O\), where
\[
(3.15) \quad \Pi_{\nu} := \{ x \in \mathcal{M}(H^\infty(\mathbb{D} \times \mathbb{N})) : \max_{1 \leq k \leq n} |\hat{f}_k(x)| < \nu \},
\]

see [Br1, Lm. 5.1] for a similar argument.

We set
\[
g_i(z) := g(i, z), \quad f_{ki}(z) := f_k(i, z), \quad (i, z) \in \mathbb{D} \times \mathbb{N}.
\]

If \(F_i = \{f_{ki}\}_{1 \leq k \leq n} \subset H^\infty\), then
\[
\Pi_{\nu} \cap (\mathbb{D} \times \{i\}) = \Pi^o_{\nu \cdot p}[F_i],
\]

where \(c := \max_{1 \leq k \leq n} \|f_k\|_{H^\infty(\mathbb{D} \times \mathbb{N})}\), see definition (3.1).

Moreover,
\[
\max \left\{ \|g_i\|_{H^\infty(\Pi^o_{\nu \cdot p}[F_i])}, \|g_i^{-1}\|_{H^\infty(\Pi^o_{\nu \cdot \delta}[F_i])} \right\} \leq M := \max_{\bar{O}} \{|\hat{g}|, |\hat{g}|^{-1}\}.
\]

Thus we can apply Theorem 1.2 to functions \(g_i\). According to this theorem, there exist sequences \(\{h_{ij}\}_{j \in \mathbb{N}} \in (H^\infty)^{-1}, \ i \in \mathbb{N}\), such that \(\{h_{ij}\}_{\Pi^o_{\nu \cdot \delta}[F_i]}\) converges uniformly to
by restriction map \( \hat{\cdot} \), see, e.g., \([Ta]\).

Each functional \( \hat{\cdot} \) of the set of variables of all \( p \) \( \alpha \) \( a \in A \), determines an element of \( \mathcal{O}(\mathfrak{M}(A)) \), see Section 1.2. We denote by \( P(A^*) \) the algebra (under pointwise operations) generated by the collection of linear functions \( \hat{\cdot} \), \( a \in A \). Then the image of the natural monomorphism \( P(A^*) \to \mathcal{O}(\mathfrak{M}(A)) \) is a dense subset of \( \mathcal{O}(\mathfrak{M}(A)) \) (see, e.g., \([Ta]\) Sect. 2.3).

Each \( p \in P(A^*) \) can be presented in the form \( p = \sum_{i,\alpha} c_{\alpha_i} \hat{\alpha}_i \), where all \( c_{\alpha_i} \in \mathbb{C} \) and the family \( \{ \hat{\alpha}_i \} \subset A^* \) is linearly independent. Elements of this family are referred to as variables of \( p \) so we write \( p = p(\hat{\alpha}_1, \hat{\alpha}_2, \ldots) \). If \( \{p_j\} \subset P(A^*) \) is a finite family of polynomials, then its set of variables is defined as the maximal linear independent subset of the set of variables of all \( p_j \).

Let \( K \subset A^* \) be a weak* compact subset. The polynomially convex hull of \( K \) is the set

\[
\hat{K} := \{ y \in A^* : |p(y)| \leq \max_K |p| \quad \forall p \in P(A^*) \}.
\]

It is a weak* compact subset of \( A^* \) as well. Such set \( K \) is said to be polynomially convex if \( \hat{K} = K \). For instance, \( \mathfrak{M}(A) \subset A^* \) is a polynomially convex set.

Further, a polynomial polyhedron is a set of the form

\[
\Pi = \{ y \in A^* : \max_{1 \leq i \leq n} |p_i(y)| < 1 \} \quad \text{for} \quad p_1, \ldots, p_n \in P(A^*).
\]

For each weak* open neighbourhood \( U \) of a polynomially convex set \( K \) there is an open polynomial polyhedron \( \Pi \) such that \( K \subset \Pi \subset U \) (see, e.g., \([Ta]\) Sec. 2.2).

Recall that \( T^\alpha_A : \mathcal{O}(\mathfrak{M}(A), \mathbb{C}^n) \to A^n \) denotes the holomorphic functional calculus and \( A_{\mathcal{M}} := T^\alpha_A(\mathcal{O}(\mathfrak{M}(A), \mathcal{M})) \) for a complex submanifold \( \mathcal{M} \subset \mathbb{C}^n \), see \([La]\).

Let \( r : A_1 \to A_2 \) be a dense image morphism of commutative complex unital Banach algebras. Then the transpose map \( r^* : A_2^* \to A_1^* \) is a bounded linear injection. Without loss of generality we identify \( A_2^* \) with its image under \( r^* \) so that \( r \) is identified with the restriction map \( \hat{\alpha} \mapsto \hat{\alpha}|_{A_2^*} \), \( a \in A_1 \). The density of the image of \( r \) implies that \( \mathfrak{M}(A_2) \) is a
polynomially convex subset of $\mathcal{M}(A_1)$. Also, the restriction map to $A_2^{*}$ maps $O(\mathcal{M}(A_1), \mathbb{C}^n)$ in $O(\mathcal{M}(A_2), \mathbb{C}^n)$.

**Proposition 4.1.** We have for all $f \in O(\mathcal{M}(A_1), \mathbb{C}^n)$,

$$(r^n \circ T^n_{A_1^*})(f) = T^n_{A_2^*}(f|_{A_2^*}).$$

*Proof.* It suffices to prove the result for $n = 1$ and $f \in P(A_1^*)$ (because $P(A_1^*)$ forms a dense subset of $O(\mathcal{M}(A_1))$). So assume that $f$ is a polynomial in variables $v_1, \ldots, v_k$ for some $v_1, \ldots, v_k \in A_1$. Then

$$(r \circ T_{A_1^*}^1)(f(v_1, \ldots, v_k)) = f(r(v_1), \ldots, r(v_k)) = T_{A_2^*}^1(v_1|_{A_2^*}, \ldots, v_k|_{A_2^*}),$$

as required. \hfill $\square$

**4.3.** Let $\mathcal{M} \subset \mathbb{C}^n$ be a complex submanifold and an Oka manifold. Proposition 4.1 implies that $r^n$ maps $(A_1)_\mathcal{M}$ in $(A_2)_\mathcal{M}$. Let $r^n((A_1)_\mathcal{M})$ be the closure of $r^n((A_1)_\mathcal{M})$ in $(A_2)_\mathcal{M}$. Theorem 1.4 (1) asserts that

$$r^n((A_1)_\mathcal{M}) = \{T^n_{A_2^*}(f), f \in O(\mathcal{M}(A_2), \mathcal{M}) : \exists \hat{f} \in C(\mathcal{M}(A_1), \mathcal{M}), \hat{f}|_{\mathcal{M}(A_2)} = f|_{\mathcal{M}(A_2)}\}.$$

The most difficult part of the proof is to establish that

(e) If $f \in O(\mathcal{M}(A_2), \mathcal{M})$ is such that $f|_{\mathcal{M}(A_2)}$ is extended to a map $\hat{f} \in C(\mathcal{M}(A_1), \mathcal{M})$, then $T^n_{A_2^*}(f) \in r^n((A_1)_\mathcal{M})$.

In this part, we show that it suffices to prove this statement for some special $f$.

Without loss of generality we may assume that $f \in O(\Pi, \mathcal{M})$ for an open polynomial polyhedron $\Pi \subset \mathcal{M}(A_2)$ determined by polynomials from $P(A_2^*)$ in variables $\hat{a}_1, \ldots, \hat{a}_m$. Then $\Pi = \pi_m^{-1}(V)$, where $\pi_m = (\hat{a}_1, \ldots, \hat{a}_m) : A_2^* \to \mathbb{C}^m$ is surjective and $V$ is an ordinary open polynomial polyhedron in $\mathbb{C}^m$ containing $\pi_m(\mathcal{M}(A_2))$. Shrinking $\Pi$, if necessary, we may assume that $f$ is a bounded function and there is a bounded function $g \in O(V, \mathcal{M})$ such that $\pi_m^*(g) = f$, see, e.g., Proposition in [13] page 159.

Further, by the density of the image of $r$, there is a sequence $\{(b_{ij}, \ldots, b_{mj})\}_{j \in \mathbb{N}} \subset A_1^m$ such that the sequence $\{(r(b_{ij}), \ldots, r(b_{mj}))\}_{j \in \mathbb{N}} \subset A_2^m$ converges to $(a_1, \ldots, a_m)$. Then the sequence of maps $\{\pi_{mj}|_{A_2^*}\}_{j \in \mathbb{N}}$, where $\pi_{mj} := (b_{ij}, \ldots, b_{mj}) : A_2^* \to \mathbb{C}^m$, converges to $\pi_m$ in the norm topology of the space of bounded linear maps $A_2^* \to \mathbb{C}^n$. Hence, there is $j_0 \in \mathbb{N}$ such that $\pi_{mj}(\mathcal{M}(A_2)) \subset V$ and maps $\pi_{mj}|_{A_2^*}$ are surjective for all $j \geq j_0$. In particular, $g(\pi_{mj}(\mathcal{M}(A_2))) \subset \mathcal{M}$ for such $j$. Let us consider functions $\pi_m^*(g) \in O(\Pi_j, \mathcal{M}), j \geq j_0$; here $\Pi_j := \pi_{mj}^{-1}(V) \subset A_1^*$ are open polynomial polyhedra containing $\mathcal{M}(A_2)$. Since the map $r : A_2^* \to A_1^*$ is weak* continuous, $\Pi_j \cap A_2^*$ are weak* open neighbourhoods of $\mathcal{M}(A_2)$ in $A_2^*$. Hence, $\pi_m^*(g)|_{A_2^*} \in O(\mathcal{M}(A_2), \mathcal{M})$ and $T^n_{A_2^*}(\pi_m^*(g)|_{A_2^*}) \in (A_2)_\mathcal{M}$.

**Lemma 4.2.** The sequence $\{T^n_{A_2^*}(\pi_m^*(g)|_{A_2^*})\}_{j \geq j_0}$ converges to $T^n_{A_2^*}(f)$. 


Thus given \( \varepsilon > 0 \) there is \( n_0 \in \mathbb{N} \) such that for each \( n \geq n_0 \),

\[
\| T^n_{A_2} (\pi^*_m (g_i)) |_{A_2} \| \leq \frac{\varepsilon}{3}.
\]

Next, since \( \lim_{j \to \infty} r(b_{kj}) = a_k, 1 \leq k \leq m \), and \( g_i \) is a polynomial, \( \lim_{j \to \infty} g_i(r(b_{kj}), \ldots, r(b_{mj})) = g_i(a_1, \ldots, a_m) \). Hence, there is \( j_1 \in \mathbb{N}, j_1 \geq j_0 \), such that for each \( j \geq j_1 \),

\[
\| g_i(r(b_{kj}), \ldots, r(b_{mj})) - g_i(a_1, \ldots, a_m) \|_{A_2} \leq \frac{\varepsilon}{3}.
\]

Finally,

\[
\lim_{i \to \infty} g_i(a_1, \ldots, a_m) = \lim_{i \to \infty} T^n_{A_2} (\pi^*_m (g_i)) = T^n_{A_2} \left( \pi^*_m \left( \lim_{i \to \infty} g_i \right) \right) = T^n_{A_2} (f).
\]

Hence, there is \( i_1 \geq i_0 \) such that for each \( i \geq i_1 \)

\[
\| g_i(a_1, \ldots, a_m) - T^n_{A_2} (f) \|_{A_2} \leq \frac{\varepsilon}{3}.
\]

Adding inequalities \((4.3)\)-\((4.5)\) we obtain for each \( j \geq j_1 \)

\[
\| T^n_{A_2} (\pi^*_m (g)) |_{A_2} \| - T^n_{A_2} (f) \|_{A_2} \leq \varepsilon.
\]

This proves the required statement. \( \square \)

Next, we prove the following result.

**Lemma 4.3.** If \( f\rceil_{\partial \mathcal{M}(A_2)} \) is extended to a map \( \tilde{f} \in C(\partial \mathcal{M}(A_1), \mathcal{M}) \), then for all sufficiently large \( j \) each map \( \pi^*_m (g)\rceil_{\partial \mathcal{M}(A_2)} \) is extended to a map from \( C(\partial \mathcal{M}(A_1), \mathcal{M}) \).

**Proof.** By the definition of a complex submanifold of \( \mathbb{C}^n \), for each \( z \in \mathcal{M} \) there is an open complex Euclidean ball \( B_z \subset \mathbb{C}^n \) centered at \( z \) such that \( B_z \cap \mathcal{M} \) is a closed complex submanifold of \( B_z \). This implies that \( \mathcal{M} \) is a closed complex submanifold of the open set \( \bigcup_{z \in \mathcal{M}} B_z \subset \mathbb{C}^n \). Indeed, we have

\[
\left( \bigcup_{z \in \mathcal{M}} B_z \right) \setminus \mathcal{M} = \left( \bigcup_{z \in \mathcal{M}} B_z \setminus (B_z \cap \mathcal{M}) \right).
\]

Then since each \( B_z \setminus (B_z \cap \mathcal{M}) \) is an open subset of \( B_z \), the complement of \( \mathcal{M} \) in \( \bigcup_{z \in \mathcal{M}} B_z \) is open as claimed.
We obtain from here that since $\mathcal{M}$ is an absolute neighbourhood retract, there is an open neighbourhood $U \subset \bigcup_{z \in \mathcal{M}} B_z$ of $\mathcal{M}$ and a continuous retraction $R : U \to \mathcal{M}$.

Further, by our definition each $\pi_{m_j}(g)$, $j \geq j_0$, maps the weak* open neighbourhood $\Pi_j \subset A_1^*$ of $\mathcal{M}(A_2)$ into $\mathcal{M}$. Also, Lemma 4.2 implies that the sequence $\{\pi_{m_j}(g)|_{A_2}\}_{j \geq j_0}$ converges to $f$ uniformly on $\mathcal{M}(A_2)$. Hence, given $\varepsilon > 0$ there is $j_1 \in \mathbb{N}$, $j_1 \geq j_0$, and for each $j \geq j_1$ there is an open in the Gelfand topology of $\mathcal{M}(A_1)$ neighbourhood $O_{j,\varepsilon} \subset \mathcal{M}(A_1)$ of $\mathcal{M}(A_2)$ contained in $\Pi_j$ such that

\begin{equation}
\|\pi_{m_j}(g)(y) - f(y)\|_{C^\alpha} \leq \varepsilon \quad \text{for all} \quad y \in O_{j,\varepsilon}.
\end{equation}

Let $\rho_1 \in C(O_{j,\varepsilon})$, $\rho_2 \in C(\mathcal{M}(A_1) \setminus \mathcal{M}(A_2))$ be a continuous partition of unity subordinate to the open cover $\{O_{j,\varepsilon}, \mathcal{M}(A_1) \setminus \mathcal{M}(A_2)\}$ of the compact space $\mathcal{M}(A_1)$. Then (4.6) implies

\begin{equation}
\sup_{y \in \mathcal{M}(A_1)} \|\rho_1(y)\pi_{m_j}(g)(y) + \rho_2(y)f(y) - f(y)\|_{C^\alpha} \leq \varepsilon.
\end{equation}

Since $f(\mathcal{M}(A_1)) \subset \mathcal{M}$ is a compact subset, (4.7) implies that for a sufficiently small $\varepsilon$, $(\rho_1\pi_{m_j}(g) + \rho_2f)(y) \in U$ for all $y \in \mathcal{M}(A_1)$. We define

$$h_j := R \circ (\rho_1\pi_{m_j}(g) + \rho_2f).$$

Then $h_j \in C(\mathcal{M}(A_1), \mathcal{M})$ and since $\rho_2 = 0$ on $\mathcal{M}(A_2)$,

$$h_j|_{\mathcal{M}(A_2)} = R \circ (\pi_{m_j}(g)|_{\mathcal{M}(A_2)}) = \pi_{m_j}(g)|_{\mathcal{M}(A_2)}.$$

This completes the proof of the lemma.

The above arguments show that it suffices to prove statement (c) for elements $\pi_{m_j}(g)|_{A_2^*}$ for all sufficiently large $j$ in place of $f$.

4.4. Proof of Theorem 1.4(1). Let $f \in \mathcal{O}(\mathcal{M}(A_2), \mathcal{M})$ be such that $\hat{f}|_{\mathcal{M}(A_2)} = f|_{\mathcal{M}(A_2)}$ for some $\hat{f} \in C(\mathcal{M}(A_1), \mathcal{M})$. We have to prove that $T_{A^*}(f) \in \mathcal{M}(A_1).$

Due to the reduction presented in the previous section, it suffices to prove this result for a bounded function $f \in \mathcal{O}(\Pi, \mathcal{M})$, where $\Pi$ is an open polynomial polyhedron $A_1^*$ containing $\mathcal{M}(A_2)$. Shrinking $\Pi$, if necessary, we may assume that $f(\Pi)$ is a relatively compact subset of $\mathcal{M}$. By the Stone-Weierstrass theorem, the uniform algebra generated by restrictions of $P(A_1^*)$ and its complex conjugate $\bar{P}(A_1^*)$ to $\mathcal{M}(A_1)$ coincides with $C(\mathcal{M}(A_1))$. Hence, given $\varepsilon > 0$ there is a (nonholomorphic) polynomial map $p_\varepsilon : A_1^* \to \mathbb{C}^n$ such that

\begin{equation}
\sup_{x \in \mathcal{M}(A_1)} \|p_\varepsilon(x) - \hat{f}(x)\|_{C^n} < \varepsilon.
\end{equation}

Since by our hypothesis the closure $\overline{f(\Pi)}$ of $f(\Pi)$ is a compact subset of $\mathcal{M}$, we can choose $\varepsilon > 0$ so small that the open $\varepsilon$-neighbourhood of $\overline{f(\Pi)}$,

$$\overline{f(\Pi)}_\varepsilon := \{z \in \mathbb{C}^n : \exists x \in \Pi, \|f(x) - z\|_{C^n} < \varepsilon\},$$

is contained in the open neighbourhood $U$ of Lemma 4.3. Since the map $p_\varepsilon|_{\Pi} - f \in C(\Pi, \mathbb{C}^n)$ is weak* continuous and $\Pi$ is a weak* open subset of $A_1^*$, the set $(p_\varepsilon|_{\Pi} - f)^{-1}(B_\varepsilon)$ is a weak* open subset of $A_1^*$ containing $\mathcal{M}(A_2)$; here $B_\varepsilon := \{z \in \mathbb{C}^n : \|z\|_{C^n} < \varepsilon\}$. In particular,
\( \Pi \cap ((p_\varepsilon \Pi - f)^{-1}(B_\varepsilon)) \) is a weak* open neighbourhood of \( \mathfrak{M}(A_2) \) and so it contains an open polynomial polyhedron \( \Pi_\varepsilon \) with the same property. Then we have
\[
\lambda p_\varepsilon(x) + (1 - \lambda)f(x) \in U \quad \text{for all} \quad x \in \Pi_\varepsilon, \ \lambda \in [0, 1].
\]

Further, due to (4.8), \( p_\varepsilon^{-1}(U) \) is a weak* open neighbourhood of \( \mathfrak{M}(A_1) \). Then there is an open polynomial polyhedron \( \Pi_\varepsilon \subseteq p_\varepsilon^{-1}(U) \) containing \( \mathfrak{M}(A_1) \) such that \( p_\varepsilon \in C(\Pi_\varepsilon, \mathbb{C}^n) \) is a bounded map. In turn, \( \Pi_\varepsilon' := \Pi_\varepsilon \cap \Pi_\varepsilon \) is an open polynomial polyhedron containing \( \mathfrak{M}(A_2) \). Suppose that \( \Pi_\varepsilon' \) is determined by polynomials in variables \( \hat{v}_1, \ldots, \hat{v}_l \) for some \( v_1, \ldots, v_l \in A_1 \) (so that the variables of polynomials defining \( \Pi_\varepsilon \) are linear combinations of these variables). Consider the bounded linear surjective map \( \pi_l = (\hat{v}_1, \ldots, \hat{v}_l) : A_1^l \to \mathbb{C}^l \). Then there are ordinary open polynomial polyhedra \( Q'_\varepsilon \subset Q_\varepsilon \subset \mathbb{C}^l \) such that \( \pi_l^{-1}(Q'_\varepsilon) = \Pi_\varepsilon' \) and \( \pi_l^{-1}(Q_\varepsilon) = \Pi_\varepsilon \). Also, there are bounded maps \( g_\varepsilon \in C(Q'_\varepsilon, \mathbb{C}^n) \) and \( g_\varepsilon \in O(Q'_\varepsilon, \mathbb{C}^n) \) such that
\[
\pi_l^*(q_\varepsilon) = p_\varepsilon \quad \text{and} \quad \pi_l^*(g_\varepsilon) = f|_{\Pi_\varepsilon'}.
\]

Note that \( K := \pi_l(\mathfrak{M}(A_2)) \) is a compact subset of \( Q'_\varepsilon \). Hence the polynomially convex hull \( \tilde{K} \) of \( K \) is a compact subset of \( Q'_\varepsilon \) as well. We choose a compact polynomially convex subset \( S \subseteq Q'_\varepsilon \) whose interior contains \( \tilde{K} \). Let \( \rho_1 \in C(Q'_\varepsilon) \) and \( \rho_2 \in C(Q_\varepsilon \setminus S) \) be a continuous partition of unity subordinate to the open cover \( \{Q'_\varepsilon, Q_\varepsilon \setminus S\} \) of \( Q_\varepsilon \) such that \( \rho_1 = 1 \) on an open neighbourhood of \( S \). Then by our choice of \( \varepsilon \), see (4.3), the map
\[
h_\varepsilon := R \circ (\rho_1 g_\varepsilon + \rho_2 g_\varepsilon) \in C(Q_\varepsilon, \mathcal{M})
\]
and coincides with \( g_\varepsilon \) on an open neighbourhood of \( S \); in particular, \( h_\varepsilon \) is holomorphic there.

Recall that each Oka manifold \( Y \) satisfies the basic Oka property with approximation. This means that every continuous map \( f_0 : X \to Y \) from a Stein space \( X \) that is holomorphic on \( (a \text{ neighbourhood of}) \) a compact \( \mathcal{O}(X)\)-convex subset \( C \subset X \) can be deformed to a holomorphic map \( f_1 : X \to Y \) by a homotopy of maps that are holomorphic near \( C \) and arbitrary close to \( f_0 \) on \( C \), [P2, Sect. 5.15].

We apply this property to our case with \( X := Q_\varepsilon, Y := \mathcal{M}, C := S \) and \( f_0 := h_\varepsilon \). Then we obtain that there is a sequence of holomorphic maps \( \{F_i\}_{i \in \mathbb{N}} \subset \mathcal{O}(Q_\varepsilon, \mathcal{M}) \) such that
\[
\lim_{i \to \infty} \max_{z \in S} \|F_i(z) - g_\varepsilon(z)\|_{\mathbb{C}^n} = 0.
\]

Hence, the sequence \( \{\pi_l^*(F_i)\}_{i \in \mathbb{N}} \subset \mathcal{O}(\Pi_\varepsilon, \mathcal{M}) \) converges uniformly to \( f \) on the open neighbourhood \( \pi_l^{-1}(S) \) of \( \mathfrak{M}(A_2) \). (Here \( S \) is the interior of \( S \).) We set
\[
c_i := T_{A_2}^n(F_i), \quad i \in \mathbb{N}.
\]

Then each \( c_i \in (A_1)_\mathcal{M} \) and
\[
\lim_{i \to \infty} r^n(c_i) = \lim_{i \to \infty} T_{A_2}^n(F_i|_{A_2^c}) = T_{A_2}^n\left(\lim_{i \to \infty} F_i|_{A_2^c}\right) = T_{A_2}^n(f).
\]

This shows that \( T_{A_2}^n(f) \in r^n((A_1)_\mathcal{M}) \) which completes the proof of the first part of Theorem L4(1).
Conversely, suppose that \( T^n_{A_2}(f), f \in \mathcal{O}(\mathcal{M}(A_2), \mathbb{C}^n) \), belongs to \( \overline{r^n((A_1),\mathcal{M})} \). We have to show that there is \( \tilde{f} \in C(\mathcal{M}(A_1), \mathbb{C}^n) \) such that \( \tilde{f}|_{\mathcal{M}(A_2)} = f \).

Indeed, since \( T^n_{A_2}(f) \in \overline{r^n((A_1),\mathcal{M})} \), there is a sequence \( \{f_i\}_{i \in \mathbb{N}} \subset \mathcal{O}(\mathcal{M}(A_1), \mathbb{C}^n) \) such that the sequence \( \{r^n(T^n_{A_1}(f_i))\}_{i \in \mathbb{N}} \) converges to \( T^n_{A_2}(f) \). This implies that \( \{f_i\}_{i \in \mathbb{N}} \) converges to \( f \) uniformly on \( \mathcal{M}(A_2) \). Since \( \mathcal{M} \) is an absolute neighbourhood retract, \( f \) can be extended to a map \( f' \in C(\mathcal{O}, \mathcal{M}) \), where \( \mathcal{O} \subset \mathcal{M}(A_1) \) is an open (in the Gelfand topology of \( \mathcal{M}(A_1) \)) neighbourhood of \( \mathcal{M}(A_2) \). Hence, given \( \varepsilon > 0 \) there is \( i_0 \in \mathbb{N} \) and an open neighbourhood \( O_{i_0,\varepsilon} \subset \mathcal{M}(A_1) \) of \( \mathcal{M}(A_2) \) contained in \( \mathcal{O} \) such that

\[
\sup_{y \in O_{i_0,\varepsilon}} \|f_{i_0}(y) - f'(y)\|_{\mathbb{C}^n} \leq \varepsilon.
\]

Let \( \rho_1 \in C(O_{i_0,\varepsilon}), \rho_2 \in C(\mathcal{M}(A_1) \setminus \mathcal{M}(A_2)) \) be a continuous partition of unity subordinate to the open cover \( \{O_{i_0,\varepsilon}, \mathcal{M}(A_1) \setminus \mathcal{M}(A_2)\} \) of the compact space \( \mathcal{M}(A_1) \). Then due to (4.10),

\[
\sup_{y \in \mathcal{M}(A_1)} \|\rho_1(y)f'(y) + \rho_2(y)f_{i_0}(y) - f_{i_0}(y)\|_{\mathbb{C}^n} \leq \varepsilon.
\]

Since \( f_{i_0}(\mathcal{M}(A_1)) \subset \mathcal{M} \) is a compact set, (4.11) implies that for a sufficiently small \( \varepsilon \),

\[
\rho_1(y)f'(y) + \rho_2(y)f_{i_0}(y) \in U \quad \text{for all } y \in \mathcal{M}(A_1).
\]

We define

\[
\tilde{f} := R \circ \rho_1 f' + \rho_2 f_{i_0} \in C(\mathcal{M}(A_1), \mathcal{M}).
\]

Since \( \rho_2 = 0 \) on \( \mathcal{M}(A_2) \),

\[
\tilde{f}|_{\mathcal{M}(A_2)} = R \circ (f'|_{\mathcal{M}(A_2)}) = f.
\]

This completes the proof of part (1) of Theorem 1.4.

4.5. Proof of Theorem 1.4(2). We retain notation of the previous sections. Our goal is to prove that under hypotheses of the theorem

\[
\overline{r^n((A_1),\mathcal{M})} = \overline{r^n((A_1),\mathcal{M})}.
\]

To this end, let \( T^n_{A_2}(f), f \in \mathcal{O}(\mathcal{M}(A_2), \mathcal{M}) \), belong to \( \overline{r^n((A_1),\mathcal{M})} \). We have to prove that there exists \( \tilde{f} \in \mathcal{O}(\mathcal{M}(A_1), \mathcal{M}) \) such that

\[
\overline{r^n(T^n_{A_1}(\tilde{f}))} = T^n_{A_2}(f).
\]

As before, we assume that \( A_2^* \) is a vector subspace of \( A_1^* \). Since \( r \) is a surjective map, the Banach open mapping theorem implies that \( A_2^* \) is a weak* closed subspace of \( A_1^* \).

Without loss of generality we may assume that \( f \in \mathcal{O}(\Pi, \mathcal{M}) \) for an open polynomial polyhedron \( \Pi \subset \mathcal{M}(A_2) \) determined by polynomials from \( P(A_2^*) \) in variables \( \hat{a}_1, \ldots, \hat{a}_m \). Then \( \Pi = \pi^{-1}_m(V) \), where \( \pi_m = (\hat{a}_1, \ldots, \hat{a}_m) : A_2^* \rightarrow \mathbb{C}^m \) is surjective and \( V \) is an ordinary open polynomial polyhedron in \( \mathbb{C}^m \) containing \( \pi_m(\mathcal{M}(A_2)) \). Shrinking \( \Pi \), if necessary, we may assume that \( f \) is a bounded function and there is a bounded function \( g \in \mathcal{O}(V, \mathcal{M}) \) such that \( \pi_m^*(g) = f \).

Further, by surjectivity of \( r \), there is an \( m \)-tuple \( (b_1, \ldots, b_m) \subset A_1^m \) such that

\[
(r(b_1), \ldots, r(b_m)) = (a_1, \ldots, a_m).
\]
We set $\tilde{\pi}_m := (\hat{b}_1, \ldots, \hat{b}_m) : A^*_1 \to \mathbb{C}^m$ and consider the function $\tilde{\pi}_m^*(g) \in \mathcal{O}(\tilde{\Pi}, \mathcal{M})$; here $\tilde{\Pi} := \tilde{\pi}_m^{-1}(V) \subset A^*_1$ is an open polynomial polyhedron containing $\mathcal{M}(A_2)$. Since, $\hat{b}_1|_{A_2^*} = \hat{a}_1, 1 \leq i \leq m$,

\begin{equation}
\tilde{\pi}_m^*(g)|_{A_2^*} = f.
\end{equation}

Next, since $T^n_{A_2}(f) \in \tilde{r}^n((A_1), \mathcal{M})$, according to Theorem 1.4 (1), see Section 4.4, for a sufficiently small $\epsilon > 0$ we can find a (nonholomorphic) polynomial map $p_\epsilon : A^*_1 \to \mathbb{C}^n$ and open polynomial polyhedra $\tilde{\Pi}_\epsilon \subset \tilde{\Pi}$ containing $\mathcal{M}(A_2)$ and $\tilde{\Pi}_\epsilon$ containing $\mathcal{M}(A_1)$ such that $p_\epsilon \in C(\tilde{\Pi}_\epsilon, \mathbb{C}^n)$ is a bounded map and

\begin{equation}
\sup_{x \in \mathcal{M}(A_1)} \|p_\epsilon(x) - f'(x)\|_{\mathbb{C}^n} < \epsilon, \quad \text{and}
\end{equation}

\begin{equation}
\lambda p_\epsilon(x) + (1 - \lambda)\tilde{\pi}_m^*(g)(x) \in U \quad \forall x \in \tilde{\Pi}_\epsilon, \lambda \in [0, 1].
\end{equation}

Without loss of generality we assume that

\begin{equation}
K := \mathcal{M}(A_1) \setminus \tilde{\Pi}_\epsilon \neq \emptyset.
\end{equation}

(For otherwise, $\tilde{f} := \tilde{\pi}_m^*(g) \in \mathcal{O}(\mathcal{M}(A_1), \mathcal{M})$ is the required function satisfying (4.12), see Proposition 4.1.)

By definition, $K$ is a compact subset of $\mathcal{M}(A_1) \setminus \mathcal{M}(A_2)$; hence, $K \cap A^*_1 = \emptyset$. Then since $A^*_2$ is a weak* closed subset of $A^*_1$, there is a weak* open neighbourhood $\tilde{O} \subset A^*_1$ of $K$ such that its weak* closure $\tilde{O}$ does not interest $A^*_1$ as well. In particular, $\tilde{\Pi}_\epsilon \cap (O \cup \tilde{\Pi}_\epsilon) \subset A^*_1$ is a weak* open neighbourhood of $\mathcal{M}(A_1)$. Hence, there is an open polynomial polyhedron $\Pi_\epsilon$ which is contained in this neighbourhood and contains $\mathcal{M}(A_1)$.

Continuing as in the proof of Theorem 1.4 (1), we define an open polynomial polyhedron $\Pi'_\epsilon := \tilde{\Pi}_\epsilon \cap \Pi_\epsilon$ in $A^*_1$ containing $\mathcal{M}(A_2)$. Then there are a bounded weak* continuous linear surjective map $\pi_l : A^*_1 \to \mathbb{C}^l$ and ordinary open polynomial polyhedra $Q'_\epsilon \subset Q_\epsilon \subset \mathbb{C}^l$ such that $\pi_l^{-1}(Q'_\epsilon) = \Pi'_\epsilon$ and $\pi_l^{-1}(Q_\epsilon) = \Pi_\epsilon$. Also, there are bounded maps $q_\epsilon \in C(Q'_\epsilon, \mathbb{C}^n)$ and $g_\epsilon \in \mathcal{O}(Q'_\epsilon, \mathbb{C}^n)$ such that

\begin{equation}
\pi_l^*(q_\epsilon) = p_\epsilon \quad \text{and} \quad \pi_l^*(g_\epsilon) = \tilde{\pi}_m^*(g)|_{\Pi'_\epsilon}.
\end{equation}

Let

\begin{equation}
L := \pi_l(A^*_2) \subset \mathbb{C}^l.
\end{equation}

**Lemma 4.4.** $L$ is a proper linear subspace of $\mathbb{C}^l$. Moreover,

\begin{equation}
L \cap Q'_\epsilon = L \cap Q_\epsilon.
\end{equation}

**Proof.** By our assumption, $\Pi_\epsilon \setminus \Pi'_\epsilon \neq \emptyset$, see (4.15), and

\begin{equation}
\pi_l^{-1}(Q_\epsilon \setminus Q'_\epsilon) = \Pi_\epsilon \setminus \Pi'_\epsilon.
\end{equation}

Hence, $Q_\epsilon \setminus Q'_\epsilon \neq \emptyset$ as well and so it suffices to prove only (4.17).
Suppose, on the contrary, that \((L \cap Q_\varepsilon) \setminus (L \cap Q'_\varepsilon) \neq \emptyset\). Since 
\[(L \cap Q_\varepsilon) \setminus (L \cap Q'_\varepsilon) = L \cap (Q_\varepsilon \setminus Q'_\varepsilon),\]
there is a point \(v \in A_2^*\) such that \(\pi_l(v) \in Q_\varepsilon \setminus Q'_\varepsilon\). Then 
\[v \in \pi_l^{-1}(Q_\varepsilon \setminus Q'_\varepsilon) = \Pi_\varepsilon \cap \Pi_\varepsilon' = \Pi_\varepsilon \cap \bar{\Pi}_\varepsilon \subset (O \cup \bar{\Pi}_\varepsilon) \setminus \bar{\Pi}_\varepsilon = O \setminus \bar{\Pi}_\varepsilon \subset O.\]
However, \(O \cap A_2^* = \emptyset\) by the choice of \(O\). This contradiction proves the lemma. 

The lemma shows that \(Z := L \cap Q'_\varepsilon\) is a closed complex submanifold of \(Q_\varepsilon\). 

Further, let \(\rho_1 \in C(Q'_\varepsilon)\) and \(\rho_2 \in C(Q_\varepsilon \setminus Z)\) be a continuous partition of unity subordinate to the open cover \(\{Q'_\varepsilon, Q_\varepsilon \setminus Z\}\) of \(Q_\varepsilon\) (in particular, \(\rho_1 = 1\) on an open neighbourhood of \(Z\)). Then by our choice of \(\varepsilon\), see \([4,14]\), the map 
\[h_\varepsilon := R \circ (\rho_1 g_\varepsilon + \rho_2 q_\varepsilon) \in C(Q_\varepsilon, \mathcal{M})\]
and coincides with \(g_\varepsilon\) on an open neighbourhood of \(Z\); in particular, \(h_\varepsilon\) is holomorphic there. 

Recall that each Oka manifold \(Y\) satisfies the basic Oka property with interpolation. That is, every continuous map \(f_0 : X \to Y\) from a Stein space \(X\) holomorphic on an open neighbourhood of a closed complex subvariety \(X' \subset X\) can be deformed to a holomorphic map \(f_1 : X \to Y\) by a homotopy of maps that is fixed on \(X'\), \([P2\, \text{Sect. 5.15}]\). 

We apply this property to our case with \(X := Q_\varepsilon\), \(X' = Z\), \(Y := \mathcal{M}\) and \(f_0 := h_\varepsilon\). Then we obtain a map \(F \in \mathcal{O}(Q_\varepsilon, \mathcal{M})\) such that 
\[F|_Z = g_\varepsilon|_Z.\]
Hence \(\pi_l^*(F) \in \mathcal{O}(\Pi_\varepsilon, \mathcal{M})\) satisfies 
\[\pi_l^*(F)|_{A_2^*} = \pi_l^*(g_\varepsilon)|_{A_2^*} = \pi_l^*(g)|_{\Pi_\varepsilon \cap A_2^*} = f|_{\Pi_\varepsilon \cap A_2^*}.\]
Since \(\Pi_\varepsilon \cap A_2^* \subset A_2^*\) is an open polyhedron containing \(\mathfrak{M}(A_2)\), the latter and Proposition \([4,1]\) imply 
\[(r^n \circ T_{A_2}^n)(\pi_l^*(F)) = T_{A_2}^n(\pi_l^*(F)|_{A_2^*}) = T_{A_2}^n(f).\]
This gives \([4,12]\) with \(\bar{f} := \pi_l^*(F)\) and completes the proof of Theorem \([1,4](2).\)

5. Proof of Theorem \([1,6]\)

5.1. First, we prove the following result.

Let \(K \subset \mathfrak{M}(H^\infty(D \times \mathbb{N}))\) be a holomorphically convex set, see \([3,14]\), and let \(f \in C(K, \mathcal{M})\) be a continuous map into a manifold \(\mathcal{M}\). Let \(r : \mathcal{M}' \to \mathcal{M}\) be a finite unbranched covering of \(\mathcal{M}\).

**Proposition 5.1.** There exists a continuous map \(f' \in C(K, \mathcal{M}')\) such that \(f = r \circ f'\).

**Proof.** Since \(\mathcal{M}\) is an absolute neighbourhood retract, the map \(f : K \to \mathcal{M}\) admits a continuous extension to a map of an open neighbourhood \(O \subset \mathfrak{M}(H^\infty(D \times \mathbb{N}))\) of \(K\) into \(\mathcal{M}\). We retain symbol \(f\) for the extended map.

Next, we regard \(r : \mathcal{M}' \to \mathcal{M}\) as a locally trivial fibre bundle on \(\mathcal{M}\) with a finite fibre \(F\). We denote by \(\rho : S_d \to \text{Aut}(F)\) the natural action of the permutation group \(S_d\).
Let $(U_i)_{i \in I}$ be a cover of $\mathcal{M}$ by simply connected open sets. Then each $r^{-1}(U_i)$ is homeomorphic to $U_i \times F$. Let $s_i : U_i \to \mathcal{M}'$ be a continuous map (section) such that $r \circ s_i = \text{id}_{U_i}, i \in I$. Then there exist locally constant continuous maps $g_{ij} : U_i \cap U_j \to S_d, i, j \in I$, such that
\begin{equation}
(5.1) \quad s_i(x) = \rho(g_{ij}(x))(s_j(x)) \quad \text{for all} \quad x \in U_i \cap U_j (\neq \emptyset).
\end{equation}
We have by the definition,
\begin{equation}
g_{ij} \cdot g_{jk} \cdot g_{ki} = 1 \quad \text{if} \quad U_i \cap U_j \cap U_k \neq \emptyset \quad \text{and} \quad g_{ii} = 1, \quad g_{ij} = g_{ij}^{-1};
\end{equation}
here $1$ is the identity of $S_d$.

Thus $\{g_{ij}\}_{i,j \in I}$ is a locally constant cocycle on the cover $(U_i)_{i \in I}$ with values in $S_d$. In particular, it determines a principal bundle $E \to \mathcal{M}$ on $\mathcal{M}$ with fibre $S_d$ (see, e.g., [Hus] for basic definitions and results of fibre bundles theory). The pullback of $E$ by $f$ is a principal bundle $f^*E \to O$ on $O$ with fibre $S_d$ defined on the open cover $(f^{-1}(U_i))_{i \in I}$ by the locally constant cocycle $\{f^*g_{ij}\}_{i,j \in I}$.

Let $\Pi \subset O$ be an open polyhedron containing $K$ (existing because $K$ is holomorphically convex), see (3.15). We prove the following result.

**Lemma 5.2.** The bundle $f^*E$ admits a continuous section over $\Pi$.

**Proof.** Suppose that
\begin{equation}
\Pi = \left\{ x \in \mathfrak{M}(H^\infty(\mathbb{D} \times \mathbb{N})) : \max_{1 \leq k \leq n} |\hat{f}_k(x)| < 1 \right\}
\end{equation}
for some $f_k \in H^\infty(\mathbb{D} \times \mathbb{N}), 1 \leq k \leq n$. Then $\Pi \cap (\mathbb{D} \times \{i\})$ is an open polyhedron determined by functions $f_{ki} := f(\cdot, i) \in H^\infty, 1 \leq k \leq n, i \in \mathbb{N}$. By the maximum modulus principle for the subharmonic function $\max_{1 \leq k \leq n} |f_{ki}|$, each connected component of $\Pi \cap (\mathbb{D} \times \{i\})$ is an open simply connected set (hence, it is contractible). In particular, the bundle $f^*E$ is trivial over each such component and so it is trivial over $\Pi \cap (\mathbb{D} \times \mathbb{N})$ which is the disjoint union of these components. Therefore there is a continuous section $g : \Pi \cap (\mathbb{D} \times \mathbb{N})$ of $f^*E$. In local coordinates of $f^*E$ on $(f^{-1}(U_i))_{i \in I}$ this section is given by continuous maps $g_i : f^{-1}(U_i) \cap (\mathbb{D} \times \mathbb{N}) \to S_d, i \in I$, such that
\begin{equation}
(5.2) \quad g_j(x) = g_i(x) \cdot (f^*g_{ij})(x) \quad \text{for all} \quad x \in (f^{-1}(U_i) \cap (\mathbb{D} \times \mathbb{N})) \cap (f^{-1}(U_j) \cap (\mathbb{D} \times \mathbb{N})), \quad i, j \in I.
\end{equation}

Let $\tau : S_d \subset \mathbb{N}$ be an embedding. Then each $\tau \circ g_i : f^{-1}(U_i) \cap (\mathbb{D} \times \mathbb{N}) \to \mathbb{N}$ is a locally constant continuous function with range in $\tau(S_d)$ and, in particular, it is a bounded holomorphic function. Since each $f^{-1}(U_i)$ is an open subset of $\mathfrak{M}(H^\infty(\mathbb{D} \times \mathbb{N}))$ and $f^{-1}(U_i) \cap (\mathbb{D} \times \mathbb{N})$ is dense in $f^{-1}(U_i)$ by the corona theorem, each $\tau \circ g_i$ admits a continuous extension to a function on $f^{-1}(U_i)$ with range in $\tau(S_d)$, see [Br2] Lm. 8.1. Applying to this extension the inverse map $\tau^{-1} : \tau(S_d) \to S_d$ we obtain that each $g_i$ admits a continuous extension $\hat{g}_i : f^{-1}(U_i) \to S_d$. Since the set $(f^{-1}(U_i) \cap (\mathbb{D} \times \mathbb{N})) \cap (f^{-1}(U_j) \cap (\mathbb{D} \times \mathbb{N}))$ is dense in the open set $f^{-1}(U_i) \cap f^{-1}(U_j)$ by the corona theorem, equation (5.2) and continuity of the extensions imply that
\begin{equation}
(5.3) \quad \hat{g}_j(x) = \hat{g}_i(x) \cdot (f^*g_{ij})(x) \quad \text{for all} \quad x \in f^{-1}(U_i) \cap f^{-1}(U_j), \quad i, j \in I.
\end{equation}
The latter shows that the family \( \{ \hat{g}_i \}_{i \in I} \) determines a continuous section \( \hat{c} \) of \( f^*E \) over \( \Pi \), as required.

Using the lemma let us complete the proof of the proposition.

We define a map \( f' : \Pi \to \mathcal{M}' \) by the formula
\[
(5.4) \quad f'(x) = \rho(\hat{g}_i(x))(s_i(f(x))), \quad x \in f^{-1}(U_i), \; i \in I.
\]
Since by \((5.1), (5.3)\)
\[
\rho(\hat{g}_i(x))(s_i(f(x))) = \rho(\hat{g}_i(x))(\rho(g_{ij}(f(x))(s_j(f(x)))) = \rho(\hat{g}_i(x)(f^*g_{ij})(x))(s_j(f(x))
\]
\[
\quad = \rho(\hat{g}_j(x))(s_j(f(x))) \quad \text{for all} \quad x \in f^{-1}(U_i) \cap f^{-1}(U_j) \neq \emptyset,
\]
the map \( f' \) is well defined. Also, by the definition \( r \circ f' = f \).

The proof of the proposition is complete. \( \Box \)

\section{5.2.}

Let \( K \subset \mathcal{M}(H^\infty(\mathbb{D} \times \mathbb{N})) \) be a holomorphically convex set and let \( G \) be an abelian group. In this part we prove the following result.

\textbf{Lemma 5.3.} The homomorphism
\[
(5.5) \quad H^1(\mathcal{M}(H^\infty(\mathbb{D} \times \mathbb{N})), G) \to H^1(\mathcal{M}(A), G)
\]
induced by the restriction map to \( K \) is surjective.

\textbf{Proof.} As was mentioned in the Introduction, Theorem 1.3 is equivalent to the fact that the above homomorphism is surjective for \( G = \mathbb{Z} \). Hence, it is surjective for \( G = \mathbb{Z}^k \), \( k \in \mathbb{N} \), as well (because \( H^1(X, \mathbb{Z}^k) = (H^1(X, \mathbb{Z}))^k \) for every topological space \( X \)).

In general, let \( g = \{ g_{ij} \in C(U_i \cap U_j, G) \}_{i,j \in I} \) be a 1-cocycle defined on an open finite cover \( U = \{ U_i \}_{i \in I} \) of \( K \). Passing to a refinement of \( U \), if necessary, we may assume that each \( g_{ij} \) is defined on the closure of \( U_i \cap U_j \). Since the latter is a compact subset of \( K \) and \( G \) is considered with the discrete topology, the image of each \( g_{ij} \) is finite. Then the subgroup of \( G \) generated by elements of images of all \( g_{ij} \) is a finitely generated abelian group \( G' \). Hence, \( g \) is a cocycle with values in \( G' \). Next, group \( G' \) being finitely generated is isomorphic to \( G'_f \oplus G'_t \), where \( G'_f \cong \mathbb{Z}^k \) for some \( k \in \mathbb{Z}_+ \) and \( G'_t \) is the finite torsion subgroup of \( G' \). Hence, we can write \( g = g_1 + g_2 \), where \( g_1 \) and \( g_2 \) are cocycles on \( U \) with values in \( G'_f \) and \( G'_t \), respectively. Cocycle \( g_1 \) determines an element \( \{ g_1 \} \in H^1(K, G'_f) \cong H^1(K, \mathbb{Z}^k) \).

Thus as explained above, \( \{ g_1 \} \) lies in the image of homomorphism \((5.5)\) for \( G = G'_f \). In turn, cocycle \( g_2 \) determines an element \( \{ g_2 \} \in H^1(K, G'_t) \). In particular, \( \{ g_2 \} \) determines an isomorphism class of principal bundles on \( K \) with finite fibre \( G'_t \). According to Lemma 5.2 every such bundle is trivial. This implies that \( \{ g_2 \} = 0 \); hence, we obtain that the cohomology class \( \{ g \} \) of the cocycle \( g \) coincides with \( \{ g_1 \} \). In particular, it lies in the image of homomorphism \((5.5)\), as required. \( \Box \)

\section{5.3.}

\textbf{Proof of Theorem 1.6.} Let \( \varphi : H^\infty(\mathbb{D} \times \mathbb{N}) \to A \) be a dense image morphism of complex unital Banach algebras and let \( \mathcal{M} \subset \mathbb{C}^n \) be a connected complex submanifold and an Oka manifold. Assuming that a finite unbranched covering \( \mathcal{M}' \) of \( \mathcal{M} \) is \( i \)-simple for \( i = 1, 2 \), we have to prove that the image of \((H^\infty(\mathbb{D} \times \mathbb{N}))^{\mathcal{M}'} \) under \( \varphi^n \) is a dense subset...
of \(A_M\). Retaining notation of Theorem \([1.4]\) we assume that \(A^* \subset (H^\infty(\mathbb{D} \times N))^*\) so that \(\mathfrak{M}(A)\) is a compact holomorphically convex subset of \(\mathfrak{M}(H^\infty(\mathbb{D} \times N))\). Then since

\[
(H^\infty(\mathbb{D} \times N))^M = (H^\infty(\mathbb{D} \times N))_M
\]

because the algebra \(H^\infty(\mathbb{D} \times N)\) is semisimple, due to this theorem, it suffices to prove that each \(f \in C(\mathfrak{M}(A), M)\) can be extended to a map \(\tilde{f} \in C(\mathfrak{M}(H^\infty(\mathbb{D} \times N)), M)\).

Let \(r : M' \to M\) be the covering map. According to Proposition \([5.1]\) there is a map \(f' \in C(\mathfrak{M}(A), M')\) such that \(f = r \circ f'\). Next, we use that \(\dim(\mathfrak{M}(H^\infty(\mathbb{D} \times N))) = 2\), see \([Br2\ Thm.2.6]\). (Recall that for a normal space \(X\), \(\dim X \leq n\) if every open cover of \(X\) can be refined by an open cover whose order \(\leq n + 1\). If \(\dim X \leq n\) and the statement \(\dim X \leq n - 1\) is false, then \(\dim X = n\).) Since \(M'\) is 1-simple, the fundamental group \(\pi_1(M')\) is abelian. Then Due to Lemma \([5.3]\) the homomorphism

\[
H^1(\mathfrak{M}(H^\infty(\mathbb{D} \times N)), \pi_1(M')) \to H^1(\mathfrak{M}(A), \pi_1(M'))
\]

is surjective. This and \([Hu2\ (10.4)]\) with \(n = 1\) and \(m = 2\) imply that \(f'\) can be extended to a map \(\tilde{f}' \in C(\mathfrak{M}(H^\infty(\mathbb{D} \times N)), M')\). We set \(\tilde{f} := r \circ \tilde{f}' \in C(\mathfrak{M}(H^\infty(\mathbb{D} \times N)), M)\). Then \(\tilde{f}\) is the required extension of \(f\).

The proof of the theorem is complete. \(\square\)

6. PROOFS OF THEOREMS \([1.9]\) AND \([1.10]\)

**Proof of Theorem \([1.9]\)** Suppose \(\Pi^{k}_c = \Pi^{k}_c[F]\), that is, it is determined by a family \(F = \{f_i\}_{1 \leq i \leq r} \subset H^\infty\), see \([1.2]\). The set \(\mathcal{D} = \{F, M, K, n, c, k, \delta, \varepsilon\}\) is called the data. Since \(\Pi^{k}_c\) is the disjoint union of open sets biholomorphic to \(\mathbb{D}\), the corona theorem is valid for \(H^\infty(\Pi^{k}_c)\). Let \(A \subset H^\infty(\Pi^{k}_c)\) be the uniform closure of the restriction of \(H^\infty\) to \(\Pi^{k}_c\). According to \([S2\ Cor.2.6]\) the restriction of \(H^\infty(\Pi^{k}_c)\) to \(\Pi^{k}_c\) lies in \(A\). Hence, the transpose of the restriction embeds \(M(A)\) into \(\mathfrak{M}(H^\infty(\Pi^{k}_c))\). Since by the hypotheses of the theorem \(g \in (H^\infty(\Pi^{k}_c))^M\), the latter implies that \(g|_{\Pi^{k}_c} \in A^M (= A_M)\). Then according to Theorem \([1.6]\) given \(\varepsilon > 0\) there exists \(h \in (H^\infty)^M\) (depending on \(g\) and the data) such that

\[
(6.1) \quad \|(h - g)|_{\Pi^{k}_c}\|_{(H^\infty)^{\Pi^{k}_c}} \leq \varepsilon.
\]

By \(\mathcal{H}_{g, D}\) we denote the class of such maps \(h\). We have to prove that

\[
(6.2) \quad C = C(M, K, n, c, k, \delta, \varepsilon) := \sup_{F, g} \inf_{h \in \mathcal{H}_{g, D}} \|h\|_{(H^\infty)^n}
\]

is finite.

To this end, let \(\{\Pi^{k}_c[F_i]\}_{i \in \mathbb{N}} \subset \mathcal{D}\), \(F_i = \{f_{ij}\}_{j=1}^{k_i} \subset H^\infty\) and \(\{g_i\}_{i \in \mathbb{N}} \subset H^\infty(\Pi^{k}_c, M)\), \(g_i(\Pi^{k}_c) \subset K\), be sequences satisfying assumptions of the theorem such that

\[
(6.3) \quad C = \lim_{i \to \infty} \inf_{h \in \mathcal{H}_{g_i, D_i}} \|h\|_{(H^\infty)^n};
\]

here \(D_i := \{F_i, M, K, n, c, k, \delta, \varepsilon\}\).
As in the proof of Theorem 1.2 we define \( F = \{ f_j \}_{1 \leq j \leq k} \subset H^\infty(\mathbb{D} \times \mathbb{N}), \Pi_\nu[F] \subset \mathbb{D} \times \mathbb{N} \) and \( g \in \mathcal{O}(\Pi_1[F], \mathcal{M}), g(\Pi_1[F]) \subset K, \) by the formulas
\[
(6.4) \quad f_j|_{\mathbb{D} \times \{i\}} := f_{ji}, \quad 1 \leq j \leq k,
\]
\( \Pi_\nu[F] \cap (\mathbb{D} \times \{i\}) := \Pi_{\nu_i}[F], \quad g|_{\Pi_\nu[F]} := g_i, \quad i \in \mathbb{N}. \)
Since \( \Pi_1[F] \) is biholomorphic to \( \mathbb{D} \times \mathbb{N}, \) the corona theorem is valid for \( H^\infty(\Pi_1[F]) \). This implies that \( g \in (H^\infty(\Pi_1[F]), \mathcal{M}). \)

By \( A_\delta[F] \) we denote the uniform closure of the restriction of \( H^\infty(\mathbb{D} \times \mathbb{N}) \) to \( \Pi_\delta[F]. \)

**Lemma 6.1.** The restriction of \( H^\infty(\Pi_1[F]) \) to \( \Pi_\delta[F] \) forms a dense subalgebra of \( A_\delta[F] \).

**Proof.** Let \( f \in H^\infty(\Pi_1[F]). \) Then \( f-c \in (H^\infty(\Pi_1[F]))^{-1}, \) where \( c := \|f\|_{H^\infty(\Pi_1[F])} + 1, \) and
\[
\|(f-c)^\pm\|_{H^\infty(\Pi_\delta[F])} \leq 2c - 1.
\]
Using this estimate and applying Theorem 1.2 to each function \( (f-c)|_{\Pi_\delta[F]} \), we obtain that for every \( \epsilon > 0 \) there exist a constant \( C \) and a function \( f_\epsilon \in H^\infty(\mathbb{D} \times \mathbb{N}) \) such that
\[
\|f_\epsilon\|_{H^\infty(\mathbb{D} \times \mathbb{N})} \leq C \quad \text{and} \quad \|(f_\epsilon - (f-c))|_{\Pi_\delta[F]}\|_{H^\infty(\Pi_\delta[F])} \leq \epsilon.
\]
This shows that \( f \) can be approximated uniformly on \( \Pi_\delta[F] \) by restrictions of functions from \( H^\infty(\mathbb{D} \times \mathbb{N}) \), as required.

From the lemma we obtain that the transpose of the restriction map \( \Pi_1[F] \to \Pi_\delta[F] \) embeds \( \mathfrak{M}(A_\delta[F]) \) into \( \mathfrak{M}(H^\infty(\Pi_1[F])). \) Then since \( g \in (H^\infty(\Pi_1[F]), \mathcal{M}), \) the restriction \( g|_{\Pi_\delta[F]} \in (A_\delta[F], \mathcal{M}). \) In particular, due to Theorem 1.6 given \( \epsilon > 0 \) there exists \( f \in (H^\infty(\mathbb{D} \times \mathbb{N}))^\mathcal{M} \) such that
\[
\|(f-g)|_{\Pi_\delta[F]}\|_{(H^\infty(\Pi_\delta[F]))^\mathcal{M}} \leq \epsilon.
\]
This implies that \( C \) can be estimated from above by \( \|f\|_{(H^\infty(\mathbb{D} \times \mathbb{N}))^\mathcal{M}} \) which completes the proof of the theorem.

**Proof of Theorem 1.10.** We retain notation of the proof of the previous theorem.

Let \( \Pi_\epsilon[F] = \Pi_{\epsilon_i}[F] \subset \mathbb{D}. \) Our data in this case is the set \( \mathcal{D} = \{ F, \mathcal{M}, K, n, c, k, J, b, \delta \}. \) Let \( r : H^\infty \to C(\text{hull} J), \) \( r(f) := \tilde{f}|_{\text{hull} J}, \) be the restriction homomorphism and let \( I = \ker r. \)

Then \( I \subset H^\infty \) is a proper closed ideal containing \( J. \) We naturally identify the complex unital Banach algebra \( A_I := H^\infty/I \) with the algebra \( r(H^\infty) \) equipped with the quotient norm. Since \( \text{hull} I = \text{hull} J, \) the transpose of \( r \) maps the maximal ideal space \( \mathfrak{M}(A_I) \) homeomorphically onto \( \text{hull} J. \) We identify these two spaces so that each homomorphism from \( \mathfrak{M}(A_I) \) is the evaluation homomorphism at a point of \( \text{hull} J. \) By the hypothesis of the theorem \( r^\delta(g) \in (A_I, \mathcal{M}), \) see (1.11); hence, Theorem 1.6 implies that there exists \( h \in (H^\infty)^\mathcal{M} \) (depending on \( g \) and the data) such that
\[
(6.6) \quad r^\delta(h) = r^\delta(g).
\]

By \( \mathcal{H}_{g, \mathcal{D}} \) we denote the class of such maps \( h. \) We have to prove that
\[
(6.7) \quad C = C(\mathcal{M}, K, n, b, c, k, \delta) := \sup_{F, \mathcal{M}, I, h \in \mathcal{H}_{g, \mathcal{D}}} \inf_{F, I, h} \|h\|_{(H^\infty)^\mathcal{M}}
\]
is finite.
Let \( \{ \Pi_i^k[F_i] \}_{i \in \mathbb{N}} \subset \mathbb{D} \), \( J_i \subset H^\infty \), \( \{ g_i \}_{i \in \mathbb{N}} \subset (H^\infty)^n \), be sequences satisfying assumptions of the theorem such that

\[
(6.8) \quad C = \lim_{i \to \infty} \inf_{h \in H_{\alpha_i}, \nu_i} \| h \|_{(H^\infty)^n};
\]

where \( \mathcal{D}_i := \{ F_i, \mathcal{M}, K, n, c, k, J_i, b, \delta \} \).

By \( r_i : H^\infty \to C(\hull J_i) \) we denote the corresponding restriction homomorphisms, by \( I_i \) their kernels, and by \( A_{I_i} \) the corresponding quotient algebras. The hypotheses of the theorem provide also some maps \( f_i \in \mathcal{O}(\Pi_i^k[F_i], \mathcal{M}) \) with images in \( K \) such that

\[
(6.9) \quad \hat{f}_i|_{\hull I_i} = \hat{g}_i|_{\hull I_i}.
\]

As in the proof of the previous theorem, see \([6.3]\), we define \( F \subset H^\infty(\mathbb{D} \times \mathbb{N}), \Pi_\nu[F] \subset \mathbb{D} \times \mathbb{N}, \)
\( g \in H^\infty(\mathbb{D} \times N), \) and \( f \in \mathcal{O}(\Pi_1[F], \mathcal{M}), f(\Pi_1[F]) \subset K \), by the formulas

\[
(6.10) \quad F|_{\mathbb{D} \times \{ i \}} := F_i, \quad \Pi_\nu[F] \cap (\mathbb{D} \times \{ i \}) := \Pi_{\text{cir}}^k[F_i], \quad g|_{\mathbb{D} \times \{ i \}} := g_i, \quad f|_{\Pi_\nu^k[F_i]} := f_i, \quad i \in \mathbb{N}.
\]

Since \( \Pi_1[F] \) is biholomorphic to \( \mathbb{D} \times \mathbb{N} \), the corona theorem is valid for \( H^\infty(\Pi_1[F]) \). This implies that \( f \in (H^\infty(\Pi_1[F]), \mathcal{M}) \). Moreover, according to \([6.2] \text{ Lm. 8.1}\), \( f \) admits a continuous extension \( \hat{f} \) to an open polyhedron \( \hat{\Pi}_1[F] \subset \mathfrak{M}(H^\infty(\mathbb{D} \times \mathbb{N})) \) determined by extension (denoted by \( \hat{F} \)) of the family \( F \) to \( \mathfrak{M}(H^\infty(\mathbb{D} \times \mathbb{N})) \) by the Gelfand transform (cf. \([6.11]\)). Also, \( \Pi_1(F) \) is an open dense subset of \( \hat{\Pi}_1[F] \) by the corona theorem for \( H^\infty(\mathbb{D} \times \mathbb{N}) \). This implies that \( \hat{f} \) maps \( \hat{\Pi}_1[F] \) into \( K \).

Next, by the definition,

\[
\mathbb{D} \times \mathbb{N} \subset \mathfrak{M}(H^\infty) \times \mathbb{N} \subset \mathfrak{M}(H^\infty(\mathbb{D} \times \mathbb{N})).
\]

Let

\[
(6.11) \quad r : H^\infty(\mathbb{D} \times \mathbb{N}) \to C_b \left( \bigcup_{i \in \mathbb{N}} \hull J_i \right), \quad r(f)|_{\hull J_i} := r_i(f|_{\mathbb{D} \times \{ i \}}), \quad i \in \mathbb{N}.
\]

Clearly, \( r \) is a well-defined morphism of complex Banach algebras of norm \( \leq 1 \) (Here \( C_b(X) \subset C(X) \) is the Banach algebra of bounded continuous functions on \( X \) equipped with supremum norm.) We set \( I := \ker r \). Then

\[
(6.12) \quad u \in I \iff u|_{\mathbb{D} \times \{ i \}} \in I_i \quad \forall i \in \mathbb{N}.
\]

We define \( A_I := H^\infty(\mathbb{D} \times \mathbb{N})/I \) and naturally identify this algebra with \( r(H^\infty(\mathbb{D} \times \mathbb{N})) \) equipped with the quotient norm. Now the transpose of \( r \) maps the maximal ideal space \( \mathfrak{M}(A_I) \) homeomorphically onto the set

\[
\hull I := \{ \xi \in \mathfrak{M}(H^\infty(\mathbb{D} \times \mathbb{N})) : \hat{u}(\xi) = 0 \quad \forall u \in I \}.
\]

**Lemma 6.2.** It is true that

\[
\hull I \subset \hat{\Pi}_1[F].
\]
Proof. Let $\tilde{A}_I$ be the uniform closure of $A_I$ in $C_b(X)$, $X := \bigsqcup_{i \in \mathbb{N}} \text{hull}_j J_i$. Then the transpose of the embedding $A_I \hookrightarrow \tilde{A}_I$ maps $\mathcal{M}(\tilde{A}_I)$ homeomorphically onto $\mathcal{M}(A_I) = \text{hull} I$.

Let $\hat{F} = \{f_j\}_{1 \leq j \leq k}, f_j \in H^\infty(D \times \mathbb{N})$. Suppose on the contrary that there is $\xi \in \text{hull} I \setminus \Pi_1[\hat{F}]$. Then since each $\xi \in \mathcal{M}(\tilde{A}_I)$ has norm $\leq 1$, by assumption (1.10) of the theorem we obtain

$$1 \leq \max_{1 \leq j \leq k} |\hat{f}_j(\xi)| \leq \max_{1 \leq j \leq k} \sup_X |\hat{f}_j| \leq \delta < 1.$$ 

This contradiction proves the required implication. \hfill \Box

Lemma 6.2 and the fact that $\hat{f}$ maps $\hat{\Pi}_1[\hat{F}]$ into $K$ and $\hat{f}|_X = r^n(g)$, see (6.9), show that $r^n(g) \in (A_I)_M (= (A_I)^M)$.

Finally, applying Theorem 1.6 to the epimorphism $r : H^\infty(D \times \mathbb{N}) \to A_I$ and the element $r^n(g)$ we find an element $h \in H^\infty(D \times \mathbb{N})^M$ such that $r^n(h) = r^n(g)$.

Then by our definition, see (6.8), $h|_{D \times \{i\}} \in \mathcal{H}_{g_i, \mathcal{P}}$, and

$$C \leq \sup_{i \in \mathbb{N}} \|h|_{D \times \{i\}}\|_{(H^\infty)^n} =: \|h\|_{(H^\infty(D \times \mathbb{N}))^n} < \infty.$$ 

This completes the proof of the theorem. \hfill \Box

References

[A] D. N. Akhiezer, Homogeneous complex manifolds, in Enc. Math. Sci., vol. 10, Several Complex Variables IV, S. G. Gindikin and G. M. Khenkin (Eds.), Springer-Verlag, New York, 1990.

[B] H. Bass, K-theory and stable algebra, Publ. Mat. I. H. E. S. 22 (1964), 5–60.

[Be] M. Behrens, The corona conjecture for a class of infinitely connected domains, Bull. Amer. Math. Soc. 76 (1970), no. 2, 387–391.

[Br1] A. Brudnyi, Stein-like theory for Banach-valued holomorphic functions on the maximal ideal space of $H^\infty$, Invent. math. 193 (2013), 187–227.

[Br2] A. Brudnyi, Structure of the maximal ideal space of $H^\infty$ on the countable disjoint union of open disks, Preprint 2019, 12 pp.

[Br3] A. Brudnyi, Oka principle on the maximal ideal space of $H^\infty$, St. Petersbourg J. Math., 31 (5) (2019), 24–89.

[CL] G. Corach and A. R. Larotonda, Stable range in Banach algebras, J. Pure Appl. Algebra 32 (1984), 289–300.

[CS1] G. Corach and F. D. Suárez, Extension problems and stable rank in commutative Banach algebras, Topology Appl. 21 (1985), 1–18.

[CS2] G. Corach and F. D. Suárez, Dense morphisms in commutative Banach algebras, Trans. Amer. Math. Soc. 304 (1987), 537–547.

[F1] F. Forstnerič, Complements of Runge domains and holomorphic hulls, Michigan Math. J. 41 (1993), 297–308.

[F2] F. Forstnerič, Stein manifolds and holomorphic maps, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 56, Springer, Heidelberg, 2011.

[Ga] J. B. Garnett, Bounded analytic functions, Academic Press, New York, 1981.

[GR] R. C. Gunning and H. Rossi, Analytic functions of several complex variables, Prentice Hall, Englewood Cliffs, N.J., 1965.
[Hu1] S.-T. Hu, Some homotopy properties of topological groups and homogeneous spaces, Ann. of Math. 49 (1948), 67–74.

[Hu2] S.-T. Hu, Mappings of a normal space into an absolute neighborhood retract, Trans. Amer. Math. Soc. 64 (1948), 336–358.

[Hu3] S.-T. Hu, Homotopy theory, Academic Press, New York, 1959.

[Hus] D. Husemoller, Fibre bundles, Springer-Verlag, New York, 1994.

[K] Y. Kusakabe, Oka properties of complements of holomorphically convex sets, arXiv:2005.08247, 2020.

[S1] D. Suárez, Čech cohomology and covering dimension for the $H^\infty$ maximal ideal space, J. Funct. Anal. 123 (1994) 233–263.

[S2] D. Suárez, Approximation by ratios of bounded analytic functions, J. Funct. Anal. 160 (1998), 254–269.

[Ta] J. L. Taylor, Topological invariants of the maximal ideal space of a Banach algebra, Adv. Math. 19 (1976), 149–206.

[To] V. Tolokomnikov, Stable rank of $H^\infty$ in multiply connected domains, Proc. Amer. Math. Soc. 123 (10) (1995), 3151–3156.

[T] S. Treil, Stable rank of $H^\infty$ is equal to one, J. Funct. Anal. 109 (1992), 130–154.

[V] L. N. Vaserstein, The stable range for rings and the dimension of topological spaces, Functional Anal. Appl., vol. 5 (1971), 102–110.

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