Influence of shear-flow vorticity on wave-current interaction. 
Part 1: Surface gravity waves without surface tension effect.

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Propagation of surface waves on a background shear flow with constant vorticity is studied and compared against the case when the background flow is uniform in depth. For a shear flow with the linear vertical profile, the dispersion relation of surface gravity waves is minutely analyzed for a fluid of finite depth. Under the assumption that the background flow gradually varies in the horizontal direction, the primary attention is paid to the wave blocking phenomenon; the effect of vorticity on this phenomenon is studied in details. The conditions of wave blocking are obtained and categorized for different values of the governing dimensionless parameters.

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I. INTRODUCTION

In many cases water waves propagate in a moving water which leads to the complex phenomena of wave-current interaction. Usually, currents are nonuniform both in the vertical and horizontal directions. Many examples of wave propagation on the background of spatially nonuniform shear flows may be listed in the conjunction with this phenomenon. Among them there are tidal currents, currents caused by the wind stress, oceanic currents of different scales, currents induced by long surface or internal waves, river flows, etc. It is well-known that the interaction of water waves with nonuniform currents may result in either dramatic effects such as generation of freak waves \[16, 18\] or favorable effects such as the blocking of waves by pneumatic wave-breakers \[31\].

Due to the effect of the bottom friction at the sea bed and/or of wind stress at the free surface, the currents often vary with the depth. The result of such vertical variation is the appearance of shear-flow vorticity which plays an important role in the wave variability and stability. Moreover, as has been shown in \[28\], the vorticity can even lead to the existence of new specific type of waves – the vorticity waves. Thus, the shear-flow vorticity in combination with the intensity of a current introduces an important parameter which should be taken into account in the attempts to understand the physics of the wave-current interaction. Leaving aside numerous works on the influence of shear flows on the stability of internal waves in density stratified fluid, we only refer here to some papers where the interaction of surface waves with currents has been studied (there is actually a plethora of publications on this problem; it is impossible to refer to all of them, therefore we restrict ourselves by references to the papers which are the most relevant to our work): \[2, 5, 9, 12, 13, 15, 19, 21, 27, 33, 35\]. Here, we revisit this problem following the recently developed dynamical system approach \[20, 26\]. We focus on the simplest case of a shear flow with the linear vertical profile having constant vorticity; more general case will be treated elsewhere. We show that the problem can be analyzed in terms of dimensionless parameters and introduce the ‘effective’ Froude number defined such that the flow vorticity is taken into account along with the speed of a current.

In this paper we do not consider the effect of surface tension. This issue is very interesting and important, but is much more complicated. It will be analyzed separately in the Part II, which will be published later.
II. THE BLOCKING PHENOMENON OF SURFACE GRAVITY WAVES ON
THE CURRENT SLOWLY VARYING IN HORIZONTAL DIRECTION

Let us consider one-dimensional wave propagation on a surface of moving water of a finite depth $h$. The velocity profile of the water flow $U(z)$ may be a rather complicated function of the depth, but in this paper we focus on the consideration and intercomparison of two cases, i) when the flow is uniform, $U(z) = U_0$ and ii) when it is a linear function of depth $U(z) = U_0 + \alpha z$. Here $U_0$ is the water speed at the free surface, and $\alpha$ characterizes the vorticity of the background flow. It is assumed that the axis $z$ is directed upward with zero at the unperturbed water surface. For certainty we suppose that $U_0 > 0$, i.e. we assume that the background flow is co-directed with the axis $x$. The sketch of the considering flow is shown in Fig. [1]

![Sketch of the fluid flow in the reference coordinate frame associated with the immovable bottom. Line 1 depicts the velocity with the uniform profile and line 2 – the velocity with the linear profile.](image)

FIG. 1: Sketch of the fluid flow in the reference coordinate frame associated with the immovable bottom. Line 1 depicts the velocity with the uniform profile and line 2 – the velocity with the linear profile.

In the general case, when the shear velocity is uniform in the horizontal direction, but is an arbitrary function of $z$, the dispersion relation between the frequency $\omega$ and the wave number $k$ for waves of infinitesimal amplitude can be derived in the approximate form based on the Taylor series representation (see, e.g., [15, 27, 29]). However, recently it was found [14] that in many particular cases the dispersion relation can be presented in the explicit analytical form. Two simple cases of analytical representation of the dispersion relation in the closed form are very well known (see
below), they are those which have been mentioned above and shown in Fig. [1], i.e. when the velocity profile is either depth independent or varies linearly with the depth (in the latter case the flow vorticity is constant). These two profiles were chosen for the analysis of the vorticity influence on the wave-current interaction and comparison of results obtained. Note that when the fluid velocity vanishes at the bottom (see Fig. [1]), then $\alpha = U_0/h$, but in general $\alpha$ may be considered as the independent parameter characterizing the flow vorticity.

The dispersion relation for surface gravity waves in a water with the linearly-varying current has been derived by Thompson and Biesel [2, 33]:

$$\omega^\pm(k) = U_0k - \frac{\alpha}{2} \tanh kh \pm \sqrt{\left(\frac{\alpha}{2} \tanh kh\right)^2 + gk \tanh kh},$$  \hspace{1cm} (1)

where $g$ is the acceleration due to gravity.

The dispersion relation (1) formally consists of four branches corresponding to different choices of signs of $\omega$ and $k$ and describing co- and counter-propagating waves with respect to the background flow. Clearly, there is a certain symmetry around the origin of the $(k, \omega)$ coordinate frame, so that for a given set of parameters $U_0$, $\alpha$ and $h$, we have $\omega^+(k) = -\omega^-(k)$, and the phase velocity $c \equiv \omega^+(k)/k = \omega^-(k)/(-k)$. As the physical frequency is the non-negative quantity ($\omega = 2\pi/T$, where $T$ is the wave period), therefore we will consider further only non-negative values of the dispersion relation $\omega^+(k)$ for all possible values of $k$ varying from minus to plus infinity. At such conventional agreement, $k > 0$ pertains to co-propagating surface waves with $c \geq 0$, and $k < 0$ pertains to counter-propagating surface waves with $c \leq 0$.

Note that under such conventional agreement the wave frequency in the frame moving with the certain speed $c_0 = U_0[1 - \tanh kh/(2kh)]$ is non-negative for all possible wave numbers $k$: $\omega - U_0k + \alpha \tanh kh/2 \geq 0$. Meanwhile, in the immovable coordinate system, the frequency formally may become negative for certain negative values of $k$. In such case we will consider, in accordance with the aforementioned symmetry, another branch of the dispersion relation with $\omega > 0$ and $k > 0$; this will be clarified and illustrated below in Fig. [2].

By introducing the normalized variables, $\kappa = kh$ and $\tilde{\omega} = \omega h/U_0$, Eq. (1) can be presented in the dimensionless form

$$\tilde{\omega}(\kappa) = \kappa - \frac{\Omega}{2} \tanh \kappa + \sqrt{\left(\frac{\Omega}{2} \tanh \kappa\right)^2 + \left[\left(1 - \frac{\Omega}{2}\right)^2 - \left(\frac{\Omega F}{2}\right)^2\right]} \frac{\kappa \tanh \kappa}{\frac{\Omega}{2} F^2},$$  \hspace{1cm} (2)
where $\Omega = \alpha h / U_0$ is the dimensionless parameter characterizing the vorticity of the basic flow (the actual vorticity is $-\Omega$), and $F_\Omega = (1 - \Omega/2)(\Omega^2/4 + gh/U_0^2)^{-1/2}$ is the ‘effective’ Froude number defined such that the flow vorticity is taken into account. If $\Omega = 0$, the background flow is uniform in depth with no vorticity, and the effective Froude number reduces to the conventional Froude number $F_\Omega = Fr \equiv U_0 / \sqrt{gh}$. If $\Omega = 1$ the background flow has a linear profile vanishing at the bottom as shown in Fig. 1 (the constant vorticity flow), then $F_\Omega = Fr / \sqrt{4 + Fr^2}$. The effectiveness of introduction of the new form of the Froude number is especially clear when we consider the long-wave limit of the dispersion relation (1):

$$\omega^\pm(k)_{|k \to 0} = \left(U_0 - \frac{\alpha h}{2}\right)k \pm k\sqrt{\left(\frac{\alpha h}{2}\right)^2 + gh}.$$  \hspace{1cm} (3)

In this case one can introduce the ‘effective’ uniform velocity $\bar{U} = U_0 - \alpha h/2$ and speed of long waves $\bar{c} = \sqrt{(\alpha h/2)^2 + gh}$. Then the Froude number can be defined in the traditional manner:

$$F_\Omega = \frac{\bar{U}}{\bar{c}} = \frac{U_0 - \alpha h/2}{\sqrt{(\alpha h/2)^2 + gh}} = \frac{1 - \Omega/2}{\sqrt{\Omega^2/4 + gh/U_0^2}}.$$  \hspace{1cm} (4)

Consider now the case when the current is uniform in depth ($\Omega = 0$) and present the dispersion relation (2) graphically (see Fig. 2). The co-propagating waves in accordance with our assumption that $U_0 > 0$ are those waves whose phase velocities $\tilde{c} \equiv \tilde{\omega} / \kappa$ are positive, i.e. the corresponding dispersion curves are located to the right from the vertical axis in Fig. 2. The situation is reversed for the counter-propagating waves; the corresponding dispersion curves are located to the left from the vertical axis. (Note that the group velocity, $\tilde{c}_g \equiv d\tilde{\omega}/d\kappa$, may be oppositely directed with respect to the phase velocity; we will address to this issue hereafter.)

When the Froude number goes to zero (i.e. when the current velocity vanishes), then two branches of the dispersion curve look almost symmetrical (cf. lines 1 and 1’ in Fig. 2a). Asymptotically when $\kappa \to 0$, the dispersion relation (2) reduces to $\tilde{\omega} = |\kappa| \left(1/Fr + \kappa/|\kappa|\right)$. The dispersion curves become more and more asymmetrical as the Froude number increases.

At a certain value of $\kappa = -\kappa_0$, the left branch of the dispersion curve turns to zero (see line 2’ in Fig. 2) and becomes formally negative for $\kappa < -\kappa_0$. In accordance with the aforementioned symmetry, this line reappears as the line 2'' in the right half-plane for $\kappa \geq \kappa_0$. So, the gravity water waves with sufficiently large wavelengths whose wave numbers occupy the range $-\kappa_0 < \kappa < 0$ can counter-propagate against the current, whereas shorter gravity waves cannot propagate against the
flow and they are simply pulled down by the current. The energy of such waves which are pulled down by the background current is negative in the immovable coordinate system [8, 22, 23, 30].

As shown in the Appendix, the wave energy $E$ is proportional both to the frequency $\tilde{\omega}$ (which may be formally of either sign) and the relative frequency $\tilde{\omega} - \kappa$ (which is the frequency in the frame co-moving with the fluid and is always non-negative). The waves with negative energy are potentially unstable and may grow if there is a mechanism taking away their energy (their negative energy becomes even “more negative” in this case, and the wave amplitude increases in the result).

There are several different mechanisms of energy removal from the waves leading to various types of shear flow instabilities (e.g., the dissipative instability, radiative instability, and so on), for more details see [8, 22, 23, 30].

The condition for the existence of negative-energy waves (NEWs) can be found from the dispersion relation (2) (for details see Appendix): the NEWs are waves whose frequency $\tilde{\omega}$ is formally negative, but replaced by positive value with the corresponding exchange $-\kappa \rightarrow \kappa$. In the dispersion plane of Fig. 2 the NEWs are shown by typical lines 2′′, 3′′ and 4′′. Correspondingly, waves with the positive frequency $\tilde{\omega}$ possess positive energy (positive-energy waves – PEWs). The NEWs may exist only in a moving fluid for any value of Froude number if the surface tension effect is
neglected.

The boundary between the PEWs and NEWs can be readily found in general for $\Omega \neq 0$ from the condition $\tilde{\omega} = 0$, which gives the critical value of wave number $\kappa_0$ for given parameters $F\Omega$ and $\Omega$:

$$\frac{\kappa_0}{\tanh \kappa_0} = \Omega + \frac{1}{Fr^2} \equiv \Omega \left(1 - \frac{\Omega}{4}\right) \left(1 - \frac{1}{F\Omega^2}\right) + \frac{1}{F\Omega^2}. \quad (5)$$

In the limiting cases of uniform flow ($\Omega = 0$) and a shear flow with constant vorticity like in Fig. 1 ($\Omega = 1$), Eq. (5) reduces to

$$\frac{\kappa_0}{\tanh \kappa_0} = \frac{1}{Fr^2}, \quad \Omega = 0;$$

$$\frac{\kappa_0}{\tanh \kappa_0} = \frac{1}{4} \left(3 + \frac{1}{F\Omega^2}\right), \quad \Omega = 1.$$

In Appendix we show that the same expression follows from the direct calculation of wave energy from the linearised set of hydrodynamic equations [cf. Eq. (40)].

The analysis of Eq. (5) shows that the roots of this equation may exist only when $F\Omega < 1$ (see Fig. 3). When $F\Omega = 1$, the horizontal line 2 in Fig. 3 touches line 1 at its minimum; the branch corresponding to PEWs disappears. And, at last, when $F\Omega > 1$, Eq. (5) does not have real roots, and surface waves with any wave number become NEWs including infinitely long waves with $\kappa = 0$.

When the right-hand side of Eq. (5) increases, the value of the parameter $\kappa_0$ also increases. However, when $\kappa$ becomes large enough, other factors, such as the surface tension may come into play and modify Eq. (5); we do not consider such effects in this paper leaving them for a separate publication. When the right-hand side of Eq. (5) decreases and approaches unity, the parameter $\kappa_0$ vanishes, which implies that gravity waves of any wavelengths including infinitely very long waves cannot propagate against the current any more; the current is so strong that it pulls down even very long waves.

In what follows, we will consider the propagation of quasi-sinusoidal wave train with a fixed frequency assuming that the current velocity and vorticity smoothly vary in horizontal direction so that the characteristic scale of current variation is much greater than the characteristic wavelength of the wave train. In such case, the frequency of the wave train remains constant, whereas its wave number adiabatically varies in space in accordance with the flow variation. To determine the
character of space variation of the carrier wavenumber we need to present the dispersion equation (2) in terms of $\kappa|\tilde{\omega}, F_\Omega(x)|$ assuming that the wave frequency is fixed. Unfortunately, such solution cannot be presented in the explicit analytical form in general for the arbitrary depth; however it can be obtained in the limiting cases of deep and shallow water when $|\kappa| \gg 1$ or $|\kappa| \ll 1$, correspondingly. Nevertheless, some interesting features of a wave motion can be derived even in the general case of a fluid of arbitrary constant depth.

A. Wave blocking in the general case of arbitrary water depth

If the wave train counter propagates with respect to the gradually increasing main current, the group velocity of such wave train decreases. At a certain point the wave train may stop if its group speed $\tilde{c}_g \equiv d\tilde{\omega}/d\kappa$ turns to zero (for the chosen fixed wave frequency, the dispersion relation has no more real solution for $\kappa$). In this case the “wave blocking phenomenon” occurs. There is a plethora of papers devoted to this phenomenon; it is not possible to list all of them in this paper, therefore we refer only to some of them: [2, 4, 8, 13, 15, 18, 19, 21, 24, 27, 30, 32, 33, 35].

Figure 4 illustrates schematically the blocking phenomenon with subsequent generation of re-
flected waves at the blocking point. In the geometry considered here, the incident wave denoted by 2 has positive energy and negative both group and phase velocities; it propagates to the left. All reflected waves have positive group velocities, but their phase velocities are different. For the first of them denoted by 2' the phase velocity is negative, whereas for two other reflected waves 1 and 2'' the phase velocities are positive. The energies of reflected waves are positive and negative as indicated in Fig. 4. Here the schematic picture of wave transformation is presented for pure gravity surface waves; it becomes much more complicated when the surface tension effect is taken into consideration. We plan to present our analysis with the surface tension effect in a separate publication.

![Wave transformation schematic](image)

**FIG. 4:** Schematic illustration of wave blocking phenomenon for pure gravity waves. The incident wave gives rise to three reflected waves in general. The plots are not in scale, therefore variations of amplitude and wave number within each wave train is just qualitative.

As has been mentioned above, the condition for the wave blocking phenomenon is the vanishing of the group speed of a wavetrain. Differentiating the dispersion relation (2) for counter-propagating waves over \( \kappa \), the following equation can be derived:

\[
\Omega^2 \left\{ \left( 1 - \frac{\kappa}{\tanh \kappa} \right) (1 - \Omega \text{sech}^2 \kappa) + \left[ 4 \text{sech}^2 \kappa - \left( 1 + \frac{2\kappa}{\sinh 2\kappa} \right)^2 \right] \frac{\Omega^2}{16} \right\} F_{\Omega}^4 + \\
(2 - \Omega)^2 \left\{ \frac{\kappa}{\tanh \kappa} (1 - \Omega \text{sech}^2 \kappa) - \left[ 2 \text{sech}^2 \kappa - \left( 1 + \frac{2\kappa}{\sinh 2\kappa} \right)^2 \right] \frac{\Omega^2}{8} \right\} F_{\Omega}^2 - \\
\frac{(2 - \Omega)^4}{16} \left( 1 + \frac{2\kappa}{\sinh 2\kappa} \right)^2 = 0.
\]

(6)
The critical wave number corresponding to the blocking phenomenon follows from this transcendental equation at the given values of $F_\Omega$ and $\Omega$. However, the equation is algebraic (bi-quadratic) with respect to $F_\Omega$ and $\Omega$, and by solving it one can find the Froude number $(F_\Omega)_b$ at which all wavetrains having carrier wavenumber greater than $\kappa_b$ are blocked, where $\kappa_b$ is the root of Eq. (6). Let us analyze Eq. (6) in the limiting cases of $\Omega = 0$ and $\Omega = 1$.

There is only one root of Eq. (6) when the flow is uniform, i.e. when $\Omega = 0$:

$$F_\Omega (\kappa_b, 0) = \frac{2\kappa_b \sinh 2\kappa_b}{2 \sinh 2\kappa_b} \sqrt{\frac{-\tanh \kappa_b}{\kappa_b}}. \quad (7)$$

And there are two roots for $F_\Omega^2$ when $\Omega = 1$, but only one of them has the physical meaning:

$$F_\Omega^2 (\kappa_b, 1) = \frac{4\kappa^2 - 8 \sinh^2 \kappa \left(1 + \cosh 2\kappa \sqrt{1 + 4\kappa^2}\right) + \sinh^2 2\kappa + 4\kappa \sinh 2\kappa \left(1 + 4 \sinh^2 \kappa\right)}{4\kappa^2 + \sinh^2 2\kappa - 64 \sinh^4 \kappa + 4\kappa \sinh 2\kappa + 16 \sinh^2 \kappa \left(2\kappa \sinh 2\kappa - 1\right)}. \quad (8)$$

Asymptotically at $\kappa_b \to -\infty$ these dependences reduce to

$$F_\Omega (\kappa_b, 0) \approx \frac{1}{2\sqrt{-\kappa_b}}, \quad F_\Omega (\kappa_b, 1) \approx \frac{1}{4\sqrt{-\kappa_b}}.$$

The relationships between $(F_\Omega)_b \equiv F_\Omega (\kappa_b)$ and $\kappa_b$ as per Eqs. (7) and (8) are single-valued; they are shown in Fig. 5. It follows from this figure in accordance with Fig. 3 that all counter-propagating waves including infinitely long waves are blocked if $(F_\Omega)_b \geq 1$.

Eliminating $F_\Omega$ from the dispersion relation (2), one finds the frequency of the wavetrain at the blocking point $\kappa = \kappa_b$; the dispersion curve $\tilde{\omega}(\kappa)$ has a local maximum at this point (see line $2'$ in Fig. 2). For $\Omega = 0$ and $\Omega = 1$ one can readily find

$$\tilde{\omega}_b(\kappa_b, 0) = \kappa_b - \frac{2\kappa_b \sinh 2\kappa_b}{2 \kappa_b + \sinh 2\kappa_b}; \quad (9)$$

$$\tilde{\omega}_b(\kappa_b, 1) = \kappa_b - \frac{\tanh \kappa_b}{2} \left[1 + \sqrt{1 + \frac{4\kappa_b \tanh 2\kappa_b}{\text{sech} 2\kappa_b - 2\kappa_b \tanh 2\kappa_b + \sqrt{1 + 4\kappa_b^2}}}\right]. \quad (10)$$

Combination of Eqs. (7) and (9) or Eqs. (8) and (10) gives the parametric representation of blocking wave frequency $\tilde{\omega}_b$ on the critical Froude number $(F_\Omega)_b$. The corresponding dependences are shown in Fig. 6(a) for $\Omega = 0$ and 1.

The relationship between the frequency of the blocked wave and Froude number can be also presented in terms of the dependence of critical Froude number on the normalized wave period.
FIG. 5: Dependences $F_\Omega(\kappa_b)$ as per Eqs. (7) and (8) for $\Omega = 0$ (line 1) and $\Omega = 1$ (line 2).

$\tilde{T}_b \equiv T_b \sqrt{g/h}$. It can be presented again in the parametric form, where $\kappa$ plays the role of the parameter:

$$(F_\Omega)_b = F_\Omega(\kappa_b, \Omega), \quad \tilde{T}_b = \frac{\pi}{\tilde{\omega}_b} \sqrt{\frac{2 - \Omega}{(F_\Omega)_b^2} - \Omega^2},$$

where Eqs. (7) and (9) should be used for $\Omega = 0$, and Eqs. (8) and (10) for $\Omega = 1$. These dependences are shown in Fig. 6b).

In the limiting cases of deep and shallow water the corresponding dependences can be derived in the explicit forms (see the following subsections).

**B. The deep-water case**

Within the framework of the deep-water approximation ($\kappa < 0$, but $|\kappa| \gg 1$) the dispersion relation (2) can be represented as the quadratic polynomial of $\kappa$:

$$\kappa^2 + \left[ 2\tilde{\omega} - \Omega - \frac{(2 - \Omega)^2}{4F_\Omega^2} + \frac{\Omega^2}{4} \right] |\kappa| + \tilde{\omega}(\tilde{\omega} - \Omega) = 0.$$  

Here for the characteristic spatial scale one can use the scale of variation of the background flow with depth $h$ or, if the flow is uniform in depth, the initial value of the carrier wave length $\lambda_0$.
chosen at a certain reference point.

FIG. 6: a) Dependences of dimensionless frequency $\tilde{\omega}_b$ on the critical Froude number.
b) Critical Froude number against the carrier wave period $\tilde{T}_b$.
Lines 1 pertain to $\Omega = 0$, and lines 2 – to $\Omega = 1$. Dashed lines show the corresponding dependences for the deep-water case, whereas dashed-dotted lines show the dependences for the shallow-water case (see below).
As has been aforementioned, the condition for the wave blocking phenomenon is the vanishing of the group speed of a wavetrain. This corresponds to the situation when the discriminant of Eq. (12) is zero, and two roots of this equation becomes equal [20, 26]. Figure 7 illustrates such situation for \( \Omega = 0 \) (frame a) and \( \Omega = 1 \) (frame b). Line \( g \) pertains to the greater root \( \kappa_g(\tilde{\omega}) \) of Eq. (12), and line \( s \) pertains to the smaller root \( \kappa_s(\tilde{\omega}) \) of the equation. Note that Fig. 7a) is just another representation of Fig. 2 but in the deep-water limit; lines \( g' \) and \( s' \) in Fig. 7a) correspond to line 2' in Fig. 2 whereas lines \( s \) and \( g \) correspond to lines 2 and 2'', respectively.

\[
\begin{align*}
\text{FIG. 7: Wave number versus frequency as per Eq. (12) for the particular values of } F_\Omega & \equiv Fr = 0.5 \\
\text{and } \Omega = 0 \text{ (frame a) and } 1 \text{ (frame b). Portions of curves } g \text{ and } s \text{ which correspond to } \tilde{\omega} < 0 \text{ (they are shown by dashed lines) are redrawn for } \tilde{\omega} > 0 \text{ (lines } g' \text{ and } s') \text{, but for } \kappa < 0 \text{ (for explanation see the paragraph after Eq. (1)).}
\end{align*}
\]

When the discriminant of Eq. (12) is zero (this corresponds to points \( A \) shown in Fig. 7), the dispersion relation can be presented in the factorized form:

\[
(\kappa - \kappa_b)^2 = 0, \quad \text{where} \quad \kappa_b = \tilde{\omega} - \frac{\Omega}{2} \left( \frac{2 - \Omega}{8F_\Omega^2} + \frac{\Omega^2}{8} \right).
\]

The corresponding value of the frequency when the discriminant vanishes is:

\[
\tilde{\omega}_b = \frac{1}{16(F_\Omega)_b^2} \left\{ 4 - \Omega(4 - \Omega) \left[ 1 - (F_\Omega)_b^2 \right] \right\}^2 \left[ 4 - \Omega(4 - \Omega) \left[ 1 - (F_\Omega)_b^2 \right] \right] - 4\Omega (F_\Omega)_b^2.
\]

In the particular cases, \( \Omega = 0 \) and \( \Omega = 1 \), this equation reduces to
\[
\tilde{\omega}_b|_{\Omega=0} = \frac{1}{4 \left(F_\Omega\right)_b^2}, \quad \tilde{\omega}_b|_{\Omega=1} = \frac{1}{16 \left(F_\Omega\right)_b^2} \left[ 1 + 3 \left(F_\Omega\right)_b^2 \right]^2 \left[ 1 - \left(F_\Omega\right)_b^2 \right].
\] (15)

Using this condition, one can eliminate \(\tilde{\omega}\) from the expression for \(\kappa_b\) in Eq. (13) and present it in the alternative form:

\[
\kappa_b = \frac{1}{16 \left(F_\Omega\right)_b^2} \left[ 4\Omega \left(F_\Omega\right)_b^2 \right]^2 - \frac{\left( 4(1 - \Omega) + \Omega^2 \left[ 1 - \left(F_\Omega\right)_b^2 \right] \right)^2}{4 - \Omega(4 - \Omega) \left[ 1 - \left(F_\Omega\right)_b^2 \right] - 4\Omega \left(F_\Omega\right)_b^2}. \] (16)

In two particular cases, \(\Omega = 0\) and \(\Omega = 1\), we have

\[
\kappa_b|_{\Omega=0} = -\frac{1}{4 \left(F_\Omega\right)_b^2}, \quad \kappa_b|_{\Omega=1} = -\frac{1 - 2 \left(F_\Omega\right)_b^2 - 15 \left(F_\Omega\right)_b^4}{16 \left(F_\Omega\right)_b^2 \left[ 1 - \left(F_\Omega\right)_b^2 \right]^3 / 2}. \] (17)

The normalized frequency, \(\tilde{\omega}_b\), as per Eq. (15), monotonically decreases with \(F_\Omega\) in the absence of the vorticity, i.e. when \(\Omega = 0\) [see dashed line in Fig. 6a) which asymptotically approaches solid line 1 when \((F_\Omega)_b \to 0\)]. The dependence \((F_\Omega)_b\) becomes formally non-monotonic when \(\Omega = 1\) [see another dashed line in Fig. 6b) which approaches line 2 when \((F_\Omega)_b \to 0\)]. However, the non-monotonic behavior of this function is just an artefact of the deep-water approximation \(|\kappa| \gg 1\) which is actually valid only when \((F_\Omega)_b \ll 1\). The difference between the corresponding dependences for the finite-depth water and deep-water approximation is clearly seen in Fig. 6a) when the Froude number increases.

The dependence (14) can be also presented in the form of the relationship between the critical Froude number \((F_\Omega)_b\) and normalized wave period \(\tilde{T}_b\) similar to Eq. (11); it is shown in Fig. 6a) by dashed lines for \(\Omega = 0\) and \(\Omega = 1\) when it reduces to

\[
\tilde{T}_b|_{\Omega=0} = 8\pi \left(F_\Omega\right)_b, \quad \tilde{T}_b|_{\Omega=1} = 16\pi \left(F_\Omega\right)_b \frac{\left[ 1 - \left(F_\Omega\right)_b^2 \right]^{3/2}}{\left[ 1 + 3 \left(F_\Omega\right)_b^2 \right]^2}. \] (18)

As one can see from this figure, the deep-water approximation is valid in the range of relatively small wave periods and small Froude numbers. The deviation of dashed lines from solid lines becomes noticeable when \(\tilde{T}\) increases and approaches \(3\sqrt{3}\pi/2\) for \(\Omega = 1\) or \(\tilde{T}\) becomes greater than 10 for \(\Omega = 0\). In both cases of \(\Omega = 0\) or \(\Omega = 1\) at small wave periods we have linear dependences \((F_\Omega)_b = \tilde{T}/8\pi\) when \(\Omega = 0\) and \((F_\Omega)_b = \tilde{T}/16\pi\) when \(\Omega = 1\). The linear dependence between the conventional Froude number \(Fr \equiv U_0/\sqrt{gh}\) and wave period for very short gravity waves blocked by
a uniform current with \( \Omega = 0 \) has been derived in [20, 26]. Here we have found that for the current with a constant vorticity similar linear dependence between the ‘effective’ Froude number \( F_{\Omega} \) and wave period takes place too, but with two times less gradient. However, taking into account the relationship between the conventional Froude number and ‘effective’ Froude number [see the text after Eq. (2)], we obtain that when \( Fr \to 0 \) in both case we have the same dependence \( Fr = \frac{\tilde{T}}{8\pi} \).

\[ C. \quad \text{The shallow-water case} \]

For the sake of completeness, consider also another limiting case \(|\kappa| \ll 1, \kappa < 0\), which corresponds to the shallow-water approximation. In this case the dispersion relation (2) reads:

\[ \tilde{\omega} \approx -\frac{(2 - \Omega)(1 - F_{\Omega})}{2F_{\Omega}}\kappa \left[ 1 - \frac{(2 - \Omega + \Omega F_{\Omega})^2}{6(2 - \Omega)^2(1 - F_{\Omega})^2} \right] + o(\kappa^3). \]  

(19)

From the condition of the wave blocking, \( d\tilde{\omega}/d\kappa = 0 \) we obtain:

\[ \kappa_b = -\frac{\sqrt{2}(2 - \Omega)\sqrt{1 - (F_{\Omega})_b}}{2 - \Omega + \Omega (F_{\Omega})_b}, \quad \tilde{\omega}_b = \frac{\sqrt{2}(2 - \Omega)^2 [1 - (F_{\Omega})_b]^{3/2}}{3 (F_{\Omega})_b [2 - \Omega + \Omega (F_{\Omega})_b]}. \]  

(20)

In the particular cases of uniform flow, \( \Omega = 0 \), and constant vorticity flow, \( \Omega = 1 \), these expressions reduce to

\[ \Omega = 0 : \quad \kappa_b = -\sqrt{2} \sqrt{1 - (F_{\Omega})_b}, \quad \tilde{\omega}_b = \frac{2\sqrt{2}}{3 (F_{\Omega})_b} [1 - (F_{\Omega})_b]^{3/2}; \]  

(21)

and

\[ \Omega = 1 : \quad \kappa_b = -\sqrt{2} \sqrt{1 - (F_{\Omega})_b} \frac{1}{1 + (F_{\Omega})_b}, \quad \tilde{\omega}_b = \frac{\sqrt{2}}{3 (F_{\Omega})_b} [1 + (F_{\Omega})_b]^{3/2} \frac{1}{1 - (F_{\Omega})_b}. \]  

(22)

These dependences of \( \tilde{\omega}_b \) on \( (F_{\Omega})_b \) are shown in Fig. 6b) by dotted lines. They can be also presented in the form of the relationship between the critical Froude number and the dimensionless period \( \tilde{T} \) of a carrier wave:

\[ \tilde{T}_b|_{\Omega=0} = \frac{3\pi\sqrt{2}}{2 [1 - (F_{\Omega})_b]^{3/2}}, \quad \tilde{T}_b|_{\Omega=1} = \frac{3\pi\sqrt{2}}{2} \frac{[1 + (F_{\Omega})_b]^{3/2}}{1 - (F_{\Omega})_b}. \]  

(23)

These dependences are shown in Fig. 6b) by dotted lines for large values of \( \tilde{T} \), as they are only asymptotically valid at very large wave periods (i.e. at small wave numbers \(|\kappa| \ll 1\)).

It is clearly seen from these formulae that when \( (F_{\Omega})_b \to 1 \), the wave period goes to infinity, and
when \((F_\Omega)_b\) becomes equal to one, then waves of any period will be blocked out. Note, however, when the ‘effective’ Froude number \((F_\Omega)_b \rightarrow 1\), then the conventional Froude number \((F_\Omega)_b \rightarrow \infty\) [see the text after Eq. (2)]. This means that infinitely intense surface current with linear vertical profile is required to block out the wave of infinitely long period, whereas such waves can be blocked out by a finite value of flow speed with a uniform vertical profile. Physically this is quite understandable. Indeed, as it is well known (see, e.g., [17]), the wave motion induced by a surface wave decays with the depth. The characteristic scale of decay depends on the wavelength, which, in turn, depends through the dispersion relation on the wave period. The larger the period, the larger is the characteristic vertical scale of the wave motion. When \(\Omega = 1\), the mean current velocity linearly decreases with the water depth, therefore its influence on the wave train gradually decreases and vanishes at the bottom. In the meantime, if the flow vorticity is zero, \(\Omega = 0\), then the current uniformly affects a wave train at any depth.

### III. CONCLUSION

Thus, we have demonstrated that the blocking phenomenon may occur for surface waves counter propagating with respect to gradually varying flow in horizontal direction. The flow may be uniform in depth or linearly varying. In the former case, waves of any periods can be blocked at a certain critical speed of the current, whereas in the latter case, the higher the wave period the higher current surface speed is required to block out the wave.

We focused here on study of surface gravity waves only leaving aside the effect of surface tension. With the surface tension the problem becomes more complicated as gravity-capillary waves may experience double reflection in two separate blocking points [1] [25] [34]. We plan to study this interesting phenomenon with and without main flow vorticity in our next publication.

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IV. APPENDIX. ENERGY OF SURFACE WAVES OF INFINITESIMAL AMPLITUDE IN THE UNIFORM FLOW

Following [8], consider the basic set of hydrodynamic equations in the linear approximation for surface gravity-capillary waves on the uniform flow whose velocity does not depend on depth $U(z) \equiv U_0$ (see line 1 in Fig. 1):

\[
\begin{align*}
\frac{\partial u}{\partial t} + U_0 \frac{\partial u}{\partial x} &= -\frac{1}{\rho} \frac{\partial p}{\partial x}, \\
\frac{\partial v}{\partial t} + U_0 \frac{\partial v}{\partial x} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} - g, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} &= 0,
\end{align*}
\]

where $\rho$ is the water density.

For the sake of generality we take into account the surface tension effect which is characterized by the parameter $\sigma$ (if $\sigma = 0$ we obtain pure gravity waves). In the second part of this work which will be published shortly we plan to consider the blocking phenomenon for gravity-capillary waves and pure capillary waves. Therefore, the results derived in this Appendix will be used in our next paper too.

Augment the basic set of equations by the boundary conditions:

\[
\begin{align*}
v(x,t) &= 0, \quad \text{at} \ z = -h; \\
\frac{\partial \eta}{\partial t} + U_0 \frac{\partial \eta}{\partial x} &= v, \quad \text{at} \ z = 0; \\
p &= P_a - \sigma \frac{\partial^2 \eta}{\partial x^2}, \quad \text{at} \ z = \eta,
\end{align*}
\]

where $P_a$ is the atmospheric pressure. The last boundary condition (29) should be written at the water surface $z = \eta$ and then only linear terms on wave amplitude $\eta$ should be kept in consistency with the linear approximation.
The solution of this set can be presented in the form:

\[ \eta(x,t) = \eta_0 e^{i(\omega t - kx)}, \]

\[ u(x,z,t) = \frac{\omega - U_0 k}{\sinh kh} \eta_0 \cosh k(z + h) e^{i(\omega t - kx)}, \]

\[ v(x,z,t) = i \frac{\omega - U_0 k}{\sinh kh} \eta_0 \sinh k(z + h) e^{i(\omega t - kx)}, \]

\[ p(x,z,t) = P_a - \rho g z + \rho \left( \frac{\omega - U_0 k}{k \sinh kh} \right)^2 \eta_0 \cosh k(z + h) e^{i(\omega t - kx)}. \]

One can readily proof that solution (30)–(33) automatically satisfy Eqs. (24)–(26) and boundary conditions (27)–(28), whereas substitution of Eqs. (30) and (33) into the boundary condition (29) gives the dispersion relation (1) provided that the flow is uniform in depth.

Using this solution, let us calculate the total energy of a sinusoidal wave, the sum of the potential energy \( P \) and kinetic energy \( K \): \( E = P + K \).

The potential energy does not depend on a flow; it is determined entirely by deflection of the free surface from the equilibrium position \( z = 0 \):

\[ P = \frac{\rho g}{2} \langle \eta^2 \rangle + \frac{\sigma}{2} \left( \frac{\partial \eta}{\partial x} \right)^2 \left[ \frac{\rho g}{4} \left( 1 + \frac{\sigma k^2}{\rho g} \right) \eta_0^2, \right. \]

where angular brackets stand for averaging over a wave period.

The kinetic energy of wave motion can be determined as the difference between the kinetic energy of moving fluid with a wave perturbation and without the perturbation:

\[ K = \frac{\rho}{2} \left[ \int_h^\eta \left[ (u + U_0)^2 + v^2 - U_0^2 \right] dz \right] = \frac{\rho g}{4k} \frac{\omega^2 - U_0^2 k^2}{\tanh kh} \eta_0^2. \]

Summing up \( P \) and \( K \), we obtain for the total wave energy (cf. [7])

\[ E = \frac{\rho \eta_0^2}{4k \tanh kh} \left[ \omega^2 - U_0^2 k^2 + \left( 1 + \frac{\sigma k^2}{\rho g} \right) g k \tanh kh \right] = \frac{\rho \eta_0^2}{2k \tanh kh} \omega (\omega - U_0 k). \]

Equation (36) shows that the wave energy is proportional both to the frequency \( \omega \) and relative frequency \( (\omega - U_0 k) \), which is the frequency in the frame co-moving with the fluid. Thus, as the frequency \( \omega \) is always positive in the immovable coordinate system, then we see that the energy may be negative if the relative frequency is negative; in this case we deal with the so called negative energy waves (NEWs) (see [8, 22, 23, 30]).
Substituting here the expression for the frequency

\[ \omega = U_0 k - \sqrt{(g k + \sigma k^3 / \rho) \tanh kh} \]

(37)
of gravity-capillary waves counter-current propagating \((k < 0)\) on the uniform flow \([5, 36]\), we obtain:

\[ E = \frac{\rho g}{2} \eta_0^2 \left( 1 + \frac{\sigma k^2}{\rho g} \right) \left[ 1 - \frac{U_0}{\sqrt{gh}} \sqrt{\frac{k h}{(1 + \sigma k^2 / \rho g) \tanh kh}} \right] \].

(38)

In the dimensionless variables as defined in the text, the wave energy reads:

\[ E = \frac{\rho g}{2} \eta_0^2 (1 + S \kappa^2) \left[ 1 - \text{Fr} \sqrt{\frac{\kappa}{(1 + S \kappa^2) \tanh \kappa}} \right] \],

(39)

where \( S = \sigma / (\rho gh^2) \) is the parameters which measures the strength of the capillary effect relative to the gravity effect.

As follows from Eq. (39), the energy becomes negative when

\[ \text{Fr} > \sqrt{\frac{(1 + S \kappa^2) \tanh \kappa}{\kappa}}. \]

(40)

For pure gravity waves without surface tension \((S = 0)\) this expression gives the same threshold for appearing of NEWs as Eq. (5) derived from the dispersion relation \((2)\) with \( \Omega = 0 \).

As has been explained in \([8, 30]\), the concept of negative energy waves helps to find “potentially unstable waves” which may grow if there is a mechanism taking away their energy. In the absence of such mechanism, they are neutrally stable. From the physical point of view, NEWs are waves moving slower than the fluid layer. In the “laboratory” coordinate frame where fluid flows to the right (see Fig. 1) with the velocity \( U_0 \), those waves move slower than the fluid, whose phase velocity being directed originally to the left \((k < 0, \ c_{ph} \equiv \omega / k < 0)\) are pulled down by the flow and propagate eventually to the right. As follows from the dispersion relation for the uniformly moving fluid \((37)\), this occurs when

\[ U_0 > \sqrt{\frac{g \tanh kh}{k} \left( 1 + \frac{\sigma}{\rho g k^2} \right)}, \]

(41)

where the right-hand side of this inequality represents the phase speed of gravity-capillary waves
on a calm water. This is nothing but the dimensional form of Eq. [40].

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