MULTIVARIATE DETERMINATENESS

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To Ciprian Foias, on the occasion of his seventy-fifth birthday

Abstract. The uniqueness question of the multivariate moment problem is studied by different methods: Hilbert space operators, complex function theory, polynomial approximation, disintegration, integral geometry. Most of the known results in the multi-dimensional case are reviewed and reproved, and a number of new determinacy criteria are developed.

1. Introduction

A moment problem represents a quintessential inverse problem, continuously reborn and rediscovered on the basis of novel, surprising applied necessities. In the restricted sense we adopt in this article, a moment problem revolves around reconstructing, approximating or only estimating a positive Borel measure on a closed subset of the Euclidean space from its power moments, or a finite selection of them. Among all mathematical aspects of moment problems, the uniqueness, or, following circulating terminologies, the determinateness or determinacy question (that is, under which conditions a sequence of moments corresponds to a single measure) has attracted analysts for more than a century. With all efforts, this very topic remains largely unfinished in the case of multivariate moment problems. Our paper aims at offering a complete picture of the present status of the determinateness question for multivariate moment problems.

Let $\mu$ be a positive measure on the real line, having all power moments finite. The asymptotic expansion in wedges $0 < \delta < \arg z < \pi - \delta$ of the associated Cauchy transform

$$
\int_{\mathbb{R}} \frac{d\mu(x)}{x - z} \sim -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots, \quad \Im(z) > 0,
$$


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and the well defined continued fraction
\[-\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots = - \frac{s_0}{\beta_0^2} \cdot \frac{1}{z - \alpha_0 - \frac{1}{\beta_0^2} \cdot \frac{1}{z - \alpha_1 - \frac{1}{\beta_1^2} \cdot \frac{1}{z - \alpha_2 - \frac{1}{\beta_2^2} \cdots}}, \quad \alpha_k, \beta_k \in \mathbb{R},\]
depend solely on the sequence of moments
\[s_k = \int_{\mathbb{R}} x^k \, d\mu(x), \quad k \geq 0.\]
The moment sequence \(\{s_k; k \geq 0\}\) is called determinate if there exists a unique representing positive measure.

Determinateness criteria for the above (so called Hamburger) moment problem are classical. One can start with the observation that two measures \(\mu\) and \(\nu\) with the same moments coincide if and only if their Cauchy transforms are equal on a set of uniqueness for analytic functions in the upper half-plane. For instance the above moment problem is determinate precisely when the associated continued fraction converges at non-real points \(z\). Originally, all determinateness criteria were obtained via elaborated computations involving continued fractions; for an excellent presentation of these aspects see the old monograph by Perron [28], Hamburger’s work [17, 18] or even the original memoir by Stieltjes [40] which defeats time by its universality and freshness.

The orthogonal polynomials of the first and second kind \(P_k(z)\), respectively \(Q_k(z)\), \(k \geq 0\), again depending only on the moment sequence \(\{s_k\}\), introduce into the picture the Jacobi matrix \(J\), a three diagonal infinite, formally symmetric matrix representing the multiplication by \(z\) in the basis \(\{P_k(z)\}\):
\[zP_k(z) = b_{k-1}P_{k-1}(z) + c_kP_k(z) + b_kP_{k+1}(z), \quad k \in \mathbb{N}.\]
The (unbounded) self-adjoint extensions \(A\) of \(J\) on a possibly larger Hilbert space than \(l^2(\mathbb{N})\) give a parametrization of all solutions to the moment problem, via the resolvent identity
\[\int_{\mathbb{R}} \frac{d\nu(x)}{x - z} = \langle (A - z)^{-1} 1, 1 \rangle, \quad \Im z > 0.\]
From here one derives M. Riesz’ criterion [33]: the moment problem associated to the measure \(\mu\) is determinate if and only if the constant function 1 is in the closure of the polynomial ideal \((x + z)\) in \(L^2(\mu)\), for a single non-real complex number \(z\), and hence for all \(z\). For these aspects of the theory of moments the reader can consult the monographs [1, 37, 41] and the recent survey [38].

The link between moment problems and complex analytic function theory is not less exciting and rewarding. It was Carleman [9] who approached
Stieltjes moment problem (on the semi-axis) with Laplace transform methods. Specifically, if $\mu$ is a positive measure on $\mathbb{R}_+ = [0, \infty)$, then the analytic continuation of the Laplace transform
\[ \tilde{\mu}(z) = \int_0^\infty e^{-xz}d\mu(x), \quad \Re z > 0, \]
provides another effective uniqueness criterion: the quasi-analyticity of $\tilde{\mu}(x) - \tilde{\nu}(x), x \geq 0$, where $\nu$ is another measure possessing the same moments, yields the determinacy of the original moment problem.

In another direction, we owe to Nevanlinna a thorough analysis of indeterminate moment problems on the line, via a canonical representation of the Cauchy transforms of all solutions $\nu$:
\[ \int_\mathbb{R} \frac{d\nu(x)}{x-z} = -\frac{C(z)\Phi(z) + A(z)}{D(z)\Phi(z) + B(z)}, \quad \Im z > 0. \]
The entire functions $A, B, C, D$ depend only on the orthogonal polynomials, and hence only on the sequence of moments. The parameter $\Phi$ runs over all analytic functions in the upper half-plane with values in the closure of it in the Riemann sphere. A canonical parametrization of all solutions $\nu$ of the moment problem is thus derived, see for details [1, 38].

In sharp contrast to the one-dimensional case the uniqueness question for multivariate moment problems is much less understood. Riesz' density criterion has several counterparts (see e.g. [16]); however, all of them provide only sufficient uniqueness conditions. Hilbert space methods lead to rather strong determinateness results (cf. [12, 26]), but again they remain far from being also necessary. Integral transforms give uniqueness results via quasi-analyticity and the geometry of the support of the measure can also play a decisive role. Disintegration methods and integral geometry techniques naturally come into discussion and allow possible dimension reductions. Integral geometry leads even to a parametrization of all solutions of the moment problem supported by a convex wedge.

The aim of this paper is to develop in a self-contained text the above ideas. We expose (almost) all known facts and prove a number of new results related to the multi-dimensional determinacy problem. While we will always take the one-dimensional case for granted, we shall give complete proofs of all main theorems referring to the multi-dimensional case. These proofs are inserted into a new, more natural, context; many of them depart from the original ones and in this way we have corrected some existent unnoticed errors in the literature.

The contents of this paper is the following. Section 2 deals with three classical integrals: the Laplace, Fantappiè and respectively Cauchy transforms of a positive measure in $\mathbb{R}^d$. Their asymptotic expansions depend on the moments of the original measure, and a discussion of the correspondence between moment sequences (including the uniqueness question) and the function spaces where their transforms belong is included.
Section 3 is devoted to the Hilbert space approach to moment problems. We begin with the Gelfand-Naimark-Segal construction and the spectral theorem for strongly commuting self-adjoint extensions of the multiplication operators and treat various operator-theoretic determinacy notions in this context. We propose a new concept, called strict determinacy, by requiring determinacy together with the density of polynomials in $L^2(\mu)$.

In Section 4 we derive a series of determinacy criteria based on polynomial or rational approximations. They include generalizations and applications of Petersen’s theorem \[29\] which are useful for verifying determinacy in many situations, but also an approximation result obtained in \[31\].

Section 5 is centered around partial determinateness. If determinacy of ”sufficiently many” 1-subsequences of a positive semi-definite multi-sequence is assumed, then the multi-sequence is even a moment sequence and determinate. Pioneering results belong to A. Devinatz \[12\], G.I. Eskin \[14\], J. Friedrich \[15\] and others. In this section we develop some general results of this kind and offer new proofs, again relying essentially on self-adjointness theory.

In Section 6 the partial determinacy results of Section 5 are used to derive another classical result due to A.E. Nussbaum \[26\] on quasi-analyticity: a positive semi-definite multi-sequence is a determinate moment sequence if all marginal sequences satisfy Carleman’s quasi-analyticity condition.

Section 7 is an essay on how a parametrization of all solutions to the moment problem on an octant $\mathbb{R}^d_+$ in $\mathbb{R}^d$ should look like. To this aim we combine a natural geometric push forward on a pencil of lines with a characterization of the Fantappiè transforms of positive measures on $\mathbb{R}^d_+$ due to Henkin and Shananin \[20\].

In Section 8 we apply disintegration techniques to the determinacy problem. Roughly speaking, we study how determinateness assumptions on the factors appearing in the disintegration formula

$$\int_X f(x) \, d\nu(x) = \int_T d\mu(t) \int_X f(x) \, d\lambda_t(x)$$

imply the determinateness of the measure $\nu$. Our main result is a new reduction theorem which shifts the determinacy question to lower dimensions. A number of corollaries and applications show the usefulness of this result.

Finally, in Section 9 we reproduce the main result from \[31\] which gives geometric criteria of determinateness. That is, quite unexpectedly, there are closed and unbounded real algebraic subsets of $\mathbb{R}^d$ which support only determined moment problems. Again, a variety of geometric examples of low degree/low dimension illustrate this phenomenon.

As mentioned above, we have tried to present the status of the art of, and to develop new results referring to the multivariate determinacy problem. We hope that this article will stir interest and stimulate further research towards a deeper understanding of the multi-dimensional moment problem.
Or equivalently when a given linear functional \( L \) (that is, \( L \in M_\mu \)) and \( s \in M_\sigma \) are such that \( L \{ s \} = \sigma \), between multi-sequences and the moment functional \( K \) we mean that \( \{ s \} \) and \( K \) have the same moments or equivalently if \( \alpha \in \mathbb{N}_0^d \) and \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), we use the multi-index notation \( x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d} \), where \( x_0^0 := 1 \). Let \( \mathbb{R}_+ := [0, \infty) \).

Let \( \mathcal{M}(\mathbb{R}^d) \) be the set of \( \alpha \)-th moment of \( \mu \) is the number \( s_\alpha = \int x^\alpha d\mu(x) \) and the moment functional \( L_\mu \) is the linear functional on \( C[\mathbb{R}] \) defined by \( L_\mu(p) = \int p(x)d\mu(x) \) for \( p \in C[\mathbb{R}] \). Throughout we freely use the one-to-one correspondence between multi-sequences \( \{ s_\alpha; \alpha \in \mathbb{N}_0^d \} \) and linear functionals \( L \) on \( C[\mathbb{R}] \) given by \( L(x^\alpha) = s_\alpha, \alpha \in \mathbb{N}_0 \). The moment problem is the question when a given multi-sequence \( s = \{ s_\alpha; \alpha \in \mathbb{N}_0^d \} \) is the moment sequence of some measure \( \mu \in \mathcal{M}(\mathbb{R}^d) \) or equivalently when a given linear functional \( L \) is the moment functional of some \( \mu \in \mathcal{M}(\mathbb{R}^d) \). A necessary condition for this is that the sequence is positive semi-definite (that is, \( \sum s_\alpha + \beta \xi_\alpha \xi_\beta \geq 0 \) for any finite complex sequence \( \{ \xi_\alpha; \alpha \in \mathbb{N}_0^d \} \)) or equivalently the corresponding functional \( L \) is positive (that is, \( L(p\overline{p}) \geq 0 \) for all \( p \in C[\mathbb{R}] \), but this condition is not sufficient when \( d \geq 2 \).

We say that two measures \( \mu, \nu \in \mathcal{M}(\mathbb{R}^d) \) are equivalent and write \( \mu \equiv \nu \) if they have the same moments or equivalently if \( L_\mu = L_\nu \). Let \( V_\mu \) denote the set of all measures \( \nu \in \mathcal{M}(\mathbb{R}^d) \) such that \( \mu \equiv \nu \). The set \( V_\mu \) is convex and compact in the weak-* topology.

A measure \( \mu \in \mathcal{M}(\mathbb{R}) \) (and likewise its moment sequence \( s \) and its moment functional \( L_\mu \)) is called determinate on \( K \) if any other measure \( \nu \in \mathcal{M}(\mathbb{R}) \) such that \( \mu \equiv \nu \) is equal to \( \mu \). Let \( V_\mu(K) \) be the set of all positive measures belonging to \( V_\mu \) and having support contained in \( K \).
We say that $\mu$ and $L_\mu$ are \textit{strongly determinate} if the associated multiplication operators $X_1, \ldots, X_d$ by the coordinates are essentially self-adjoint on the Hilbert space completion of polynomials. We call $\mu$ and $L_\mu$ \textit{strictly determinate} if $\mu$ and $L_\mu$ is determinate and the polynomials $\mathbb{C}[x]$ are dense in $L^2(\mu)$. Finally we say that $\mu$ and $L_\mu$ are called \textit{ultradeterminate} if the polynomials $\mathcal{C}_\mu[x]$ are dense in $L^2((1 + ||x||^2)\mu)$.

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2. \textbf{Review of integral transforms}

We focus on the retrieval of a positive measure $\mu$ from its moments $\{s_\alpha; \alpha \in \mathbb{N}_0^d\}$. It is well known that this is an ill posed problem, and that only under additional assumptions the representing measure $\mu$ associated to a prescribed moment sequence is unique.

Traditionally, the retrieval of $\mu$ is achieved by the exact or approximative inversion of certain integral transforms of $\mu$. These transforms contain the moment data in a closed form or an asymptotic expansion form. We discuss below three such classical examples.

2.1. \textbf{Laplace transform.} Suppose that $\mu \in \mathcal{M}(\mathbb{R}_+^d)$. Then the function

$$\tilde{\mu}(z) = \int_{\mathbb{R}_+^d} e^{-t \cdot z} d\mu(t),$$

is analytic in the tube domain $\Re z \in \text{int} \mathbb{R}_+^d$. Remark that

$$\left. \frac{\partial^\alpha}{\partial z^\alpha} \tilde{\mu}(z) \right|_{z=0} = (-1)^{|\alpha|} \int_{\mathbb{R}_+^d} t^\alpha e^{-t \cdot z} d\mu(t),$$

whence

$$\sup_{\Re z \in \text{int} \mathbb{R}_+^d} \left| \left. \frac{\partial^\alpha}{\partial z^\alpha} \tilde{\mu}(z) \right| \right| \leq s_\alpha, \quad \alpha \in \mathbb{N}_0^d.$$  

It is well known that the Laplace transform is injective, that is $\tilde{\mu} = 0$ implies $\mu = 0$, see for instance [13].

Regarded as a function of real variables, $\tilde{\mu} \in C^\infty(\mathbb{R}_+^d)$ and we have

$$(-1)^{|\alpha|} \left. \frac{\partial^\alpha}{\partial x^\alpha} \tilde{\mu}(x) \right|_{x=0} = s_\alpha, \quad \alpha \in \mathbb{N}_0^d.$$  

The function $\tilde{\mu}$ is \textit{completely monotonic}, in the sense that

$$(-1)^{|\alpha|} \left. \frac{\partial^\alpha}{\partial x^\alpha} \tilde{\mu}(x) \right| \geq 0, \quad x \in \mathbb{R}_+^d, \quad \alpha \in \mathbb{N}_0^d.$$  

The following remarkable characterization of Laplace transforms goes back to the works of S. Bernstein, V. Gilbert and S. Bochner. For the proof we refer to [20].
**Theorem 1.** A function $F \in C^\infty (\mathbb{R}_d^+)$ is the Laplace transform of a measure from $\mathcal{M}(\mathbb{R}_d^+)$, if and only if $F$ is completely monotonic.

Variants of this result for exponentially decaying measures, or simply for all positive measures on $\mathbb{R}_d^+$ are also well known, see [20].

Thus, for the uniqueness problem, we are naturally led to consider two representing measures $\mu_1, \mu_2$ and take

$$f(x) = \tilde{\mu}_1(x) - \tilde{\mu}_2(x), \quad x \in \mathbb{R}_d^+.$$ 

Then

1. $f \in C^\infty (\mathbb{R}_d^+),$
2. $\sup_{x \in \mathbb{R}_d^+} |f^{(\alpha)}(x)| \leq 2s_\alpha, \quad \alpha \in \mathbb{N}_0^d,$
3. $f^{(\alpha)}(0) = 0, \quad \alpha \in \mathbb{N}_0^d.$

Uniqueness holds if $f$ is identically zero, that is this function belongs to a quasi-analytic class. This was the original path followed by Carleman in his application of quasi-analyticity to the study of the uniqueness in the one variable moment problem [9]. In the same spirit, we will discuss at the end of this section how quasi-analyticity conditions can be applied to find uniqueness criteria in the multi-variate moment problem on the positive octant.

**2.2. Fantappiè transform.** Closer to Stieljes original study [40] of the moment problem on the semi-axis is the Fantappiè transform:

$$\mathcal{F}(\mu)(p_0, p) = \int_{\mathbb{R}_d^+} \frac{d\mu(x)}{p_0 + p \cdot x}, \quad p_0 > 0, \quad p \in \mathbb{R}_d^+.$$

A similar result to the characterization of Laplace transforms holds in this case.

**Theorem 2.** A function $F \in C^\infty (\text{int} \mathbb{R} \times \mathbb{R}_d^+)$ represents the Fantappiè transform of a measure from $\mathcal{M}(\mathbb{R}_d^+)$ if and only if $F$ is completely monotone and homogeneous of degree $-1$.

For a proof and variants, see again [20]. Note that the moments $s_\alpha$ appear in the asymptotic expansion

$$\mathcal{F}(\mu)(p_0, p) \sim \sum_{\alpha \in \mathbb{N}^d} \frac{(-1)^{|\alpha|}}{|\alpha| \cdot p_0^{|\alpha|}} \left( \frac{|\alpha|}{\alpha} \right) p^\alpha s_\alpha.$$

Again, it is well known that $\mathcal{F}(\mu) = 0$ implies $\mu = 0$.

**2.3. Cauchy transform.** Among the various choices of Cauchy’s integral transforms we prefer the following one. The Cauchy transform of a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ is

$$\mathcal{C}_\mu(z) = \int_{\mathbb{R}^d} \frac{d\mu(x)}{(x_1 - z_1)\ldots(x_d - z_n)}, \quad z \in \mathbb{C}^n \setminus \mathbb{R}^d.$$
This is an analytic function in the variable $z$, which determines the measure $\mu$.

**Proposition 3.** If $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $C\mu = 0$, then $\mu = 0$.

**Proof.** Let $\hat{\mu}(u) = \int_{\mathbb{R}^d} e^{-iu \cdot x} d\mu(x)$ be the Fourier transform of $\mu$. Due to the decay assumptions, $\hat{\mu} \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $\hat{\mu}$ is uniformly bounded on $\mathbb{R}^d$.

Let $y \in \mathbb{R}^d$ be a vector with positive entries. Then Laplace’s transform is well defined, and Fubini’s Theorem yields:

\[
\int_{\mathbb{R}^d} e^{-uy} \hat{\mu}(u) du = \int_{\mathbb{R}^d_+} e^{-uy} \int_{\mathbb{R}^d} e^{-iu \cdot x} d\mu(x) \, du = \int_{\mathbb{R}^d_+} \frac{d\mu(x)}{(y_1 + ix_1) \cdots (y_d + ix_d)} = (-i)^n C\mu(-iy) = 0.
\]

Therefore $\hat{\mu}(u) = 0$ for all $u \in \mathbb{R}^d_+$. By repeating the same computations to $u$ in a different octant, with the proper choice of the signs of $y_k$, we find $\hat{\mu} = 0$, whence $\mu = 0$.

\[\square\]

The characterization of all Cauchy transforms is available via standard techniques of harmonic analysis in a product of half planes, cf. [22, 39]. We briefly indicate below the details.

**Theorem 4.** An analytic function $F(z_1, \ldots, z_d)$, $z_i \notin \mathbb{R}$ for all $i = 1, \ldots, n$, satisfies $F = C\mu$ for a positive, finite measure $\mu$ on $\mathbb{R}^d$ if and only if the functions $f_k(z)$ defined inductively as

\[
f_1(z_1, \ldots, z_d) = (F(z_1, \ldots, z_{d-1}, 0) - F(z_1, \ldots, z_{d-1}, x_d))/(2i),
\]

\[
f_{k+1}(z) = (f_k(z_1, \ldots, z_{k-1}, z_k, z_{k+1}, \ldots, z_d) - f_k(z_1, \ldots, z_{k-1}, x_k, z_{k+1}, \ldots, z_d))/(2i),
\]

for all $1 \leq k \leq d$, fulfill the conditions:

1. $f_d(z) \geq 0$, $\Im z_i > 0$, $1 \leq i \leq d$,
2. $\sup_y \int_{\mathbb{R}^d} |f_d(x_1 + iy_1, \ldots, x_d + iy_d)| dx_1 \cdots dx_d < \infty$,
3. $\lim_{y_k \to \infty} F(z_1, \ldots, z_{k-1}, x_k + iy_k, z_{k+1}, \ldots, z_d) = 0$, $1 \leq k \leq d$.

In this case $w^n - \lim_{y \to 0} f_n(x+iy)/(x+iy) \to d\mu$.

The case $n = 1$ corresponds to the well studied class of Cauchy transforms of positive measures on the line, see for instance [1].

**Proof.** We outline the main steps of the proof in the case of two variables $(z, w)$. The passage to higher dimensions does not require additional arguments.

Suppose that $F = C\mu$. The recursively generated function is in this situation an iterated Poisson transform:

\[
f_2(x + iy, u + iv) = \int_{\mathbb{R}^2} \frac{y}{(x - s)^2 + y^2} \frac{v}{(u - t)^2 + v^2} d\mu(s, t).
\]
Thus \( f_2 \) is non-negative for \( y, v > 0 \) and
\[
\int f_2(x + iy, u + iv)dxdu = \pi^2 \int_{\mathbb{R}^2} d\mu(s, t) < \infty.
\]
Moreover, \( w^* - \lim_{v \downarrow 0} \frac{vdu}{w^2 + v^2} = \pi \delta(u) \), whence the stated \( w^* \)-convergence of \( f_2 \) holds true.

Conversely, assume that the analytic function \( F(z, w) \) satisfies the conditions in the statement. The function \( f_2 \) is then harmonic in every variable and by a repeated application of Theorem II.2.5 of \[39\] there exists a finite, positive measure \( \mu \) on \( \mathbb{R}^2 \) such that \( f_2 \) is its iterated Poisson transform. Thus, for all \( z, w \not\in \mathbb{R} \) we have:
\[
F(z, w) = \int_{\mathbb{R}^2} \frac{1}{(t - z)(s - w)} d\mu(s, t).
\]
By polarization, we obtain the identity
\[
F(z, w) - F(z, w') - F(z', w) + F(z', w') = \int_{\mathbb{R}^2} \frac{1}{(t - z)(s - w)} d\mu(s, t).
\]
By letting \( z' \to \infty \) and then \( w' \to \infty \) we obtain
\[
F(z, w) = \int_{\mathbb{R}^2} \frac{d\mu(s, t)}{(t - z)(s - w)}.
\]

2.4. **Determinateness and quasi-analyticity.** Without aiming at obtaining optimal results (they will be improved later via Hilbert space methods) we exemplify below the link between quasi-analyticity of a Fourier-Laplace transform of a measure supported by the positive octant in the Euclidean space and its determinateness. In this direction we go back to some old theorems due to Bochner-Taylor and Bochner \[4, 5\].

We state a result of \[4\] for the convenience of the reader. This is only a first generic result. Quite a few ramifications of it are presented in the same article as well as in \[5, 13, 21\].

**Theorem 5.** Let \( f \) be a \( C^{\infty} \) function defined on a domain \( \Omega \) of \( \mathbb{R}^d \). Let \( x_0 \in \Omega \) and let \( m_n \) be a sequence of positive numbers. Assume that
\begin{enumerate}
  \item \( \sum_{|\alpha|=n} |\frac{\partial^\alpha}{\partial x^\alpha} f(x)|^2 \leq m_n^2 \) for all \( n \geq 0 \) and \( x \in \Omega \);
  \item \( \frac{\partial^\alpha}{\partial x^\alpha} f(x_0) = 0 \) for all \( \alpha \in \mathbb{N}^d \);
  \item \( \sum_n m_n^{-1/n} = \infty \).
\end{enumerate}
Then \( f \) is identically zero on \( \Omega \).

The application to moment problems is now transparent:
Theorem 6. Let \( \{s_\alpha\} \) be the moment sequence of a measure \( \mu \in \mathcal{M}(\mathbb{R}^d_+) \). If
\[
\sum_{k=0}^{\infty} \left( \sum_{|\alpha|=k} |s_\alpha|^2 \right)^{-\frac{1}{k}} = \infty,
\]
then the moment problem is determinate on \( \mathbb{R}^d_+ \).

Proof. Denote by \( \mu_1, \mu_2 \) two measures having the prescribed moment sequence \( \{s_\alpha; \alpha \in \mathbb{N}^d_0\} \). Let \( \mu = \mu_1 - \mu_2 \) and take the Laplace transform \( \tilde{\mu} \). This is a \( C^\infty \) function on the closed octant \( \mathbb{R}^d_+ \), and by assumption \( \frac{\partial^{\alpha}}{\partial x^{\alpha}} \tilde{\mu}(0) = 0 \) for all \( \alpha \in \mathbb{N}^d_0 \). Note that
\[
|\frac{\partial^{\alpha}}{\partial x^{\alpha}} \tilde{\mu}(x)| \leq 2s_\alpha, \quad \alpha \in \mathbb{N}^d_0.
\]

If \( n = 1 \), then \( \tilde{\mu} \) extends smoothly to the negative semi-axis and the statement follows from a standard quasi-analyticity criterion \([5]\). We prove the general statement by induction on \( d \).

Due to the induction hypothesis, Laplace transform \( \tilde{\mu} \) vanishes with all partial derivatives along the boundary of \( \mathbb{R}^n_+ \). Whence we can extend \( C^\infty \) it by zero to the whole \( \mathbb{R}^n \). Then one applies directly Theorem 1 of \([4]\). \( \square \)

The article \([4]\) and its companion \([5]\) contain similar criteria of quasi-analyticity, in terms of the growth of \( L(x, \partial)^k \tilde{\mu} \), where \( L(x, \partial) \) is a linear partial differential operator. The multivariate quasi-analyticity theme is also amply discussed in \([13]\). A link between the rate of rational approximation and quasi-analyticity is developed in \([30]\). It is worth mentioning that all these authors obtain their quasi-analyticity conditions via Bernstein’s theorem on the real axis, by restriction to lines or one-dimensional curves.

Next we give an example of a variation of the above result, derived from Theorem 10 of \([4]\) with the same proof as above.

Corollary 7. If \( \{s_\alpha\} \) is the moment sequence of a measure \( \mu \in \mathcal{M}(\mathbb{R}^d_+) \) such that
\[
\sum_{k=0}^{\infty} \left( \sum_{|\alpha|=k} \binom{k}{\alpha} s_{2\alpha} \right)^{-\frac{1}{k}} = \infty,
\]
then \( \mu \) is determinate on \( \mathbb{R}^d_+ \).

3. Hilbert space methods

The spectral theory of self-adjoint operators in Hilbert space is well suited and provides powerful techniques for the study of the the moment problem and in particular of the determinacy question. The present section is devoted to this approach. A thorough treatment of the one dimensional moment problem in terms of the self-adjoint extension theory of symmetric operators can be found in the recent survey by B. Simon \([38]\).
Let $L$ be a linear functional on the $*$-algebra $\mathbb{C}[x] \equiv \mathbb{C}[x_1, \ldots, x_d]$. Throughout this section we assume that $L$ is positive functional, that is, $L(p\overline{p}) \geq 0$ for all $p \in \mathbb{C}[x]$ or equivalently $L(q^2) \geq 0$ for all $q \in \mathbb{R}[x]$.

The technical tool that relates the moment problem to operator theory in Hilbert space is the GNS-construction. We briefly develop this construction. (Note that it works for positive linear functionals on arbitrary unital $*$-algebras.) Since $L$ is positive, the Cauchy-Schwarz inequality
\[ |L(p_1\overline{p_2})|^2 \leq L(p_1\overline{p_1})L(p_2\overline{p_2}), \quad p_1, p_2 \in \mathbb{C}[x], \]
holds and implies that $\mathcal{N}_L := \{ p \in \mathbb{C}[x] : L(p\overline{p}) = 0 \}$ is an ideal of the algebra $\mathbb{C}[x]$. Hence there exist a scalar product $\langle \cdot, \cdot \rangle_L$ on the quotient space $\mathcal{D}_L = \mathbb{C}[x]/\mathcal{N}_L$ and an algebra homomorphism $\pi_L$ of $\mathbb{C}[x]$ into the linear operators on $\mathcal{D}_L$ defined by
\[ \langle p + \mathcal{N}_L, q + \mathcal{N}_L \rangle_L = L(p\overline{q}) \quad \text{and} \quad \pi_L(p)(q + \mathcal{N}_L) = pq + \mathcal{N}_L, \quad p, q \in \mathbb{C}[x]. \]

Let $\mathcal{H}_L$ denote the Hilbert space completion of the pre-Hilbert space $\mathcal{D}_L$. If no confusion can arise we omit all subscripts $L$ and write $q$ for $q + \mathcal{N}_L$ and $X_k$ for $\pi(x_k), \ k = 1, \ldots, d$. Then we have $\pi(p)q = pq$ and
\[ \langle \pi(p)p_1, p_2 \rangle = \langle p_1, \pi(p) p_2 \rangle = L(pp_1\overline{p_2}), \quad p, p_1, p_2 \in \mathbb{C}[x]. \]

In particular, $L(p) = \langle \pi(p)1, 1 \rangle$ for $p \in \mathbb{C}[x]$ and $X_1, \ldots, X_d$ are pairwise commuting symmetric operators on the dense invariant domain $\mathcal{D} = \pi(\mathbb{C}[x])1$ of the Hilbert space $\mathcal{H}$.

The next proposition relates the moment problem to spectral measures of strongly commuting $d$-tuples of self-adjoint operators extending the $d$-tuple $(X_1, \ldots, X_d)$. This connection goes back to the early days of functional analysis.

**Proposition 8.** A positive linear functional $L$ of $\mathbb{C}[x_1, \ldots, x_d]$ is a moment functional if and only if there exists a $d$-tuple $(A_1, \ldots, A_d)$ of strongly commuting self-adjoint operators $A_1, \ldots, A_d$ acting on a Hilbert space $\mathcal{K}$ such that $\mathcal{H}$ is a subspace of $\mathcal{K}$ and $X_1 \subseteq A_1, \ldots, X_d \subseteq A_d$.

If $L$ is a moment functional, then all solutions $\mu$ of the moment problem for $L$ are of the form $\mu(\cdot) = \langle E_{(A_1, \ldots, A_d)}(\cdot)1, 1 \rangle$, where $(A_1, \ldots, A_d)$ is such a $d$-tuple and $E_{(A_1, \ldots, A_d)}$ denotes its spectral measure.

Let us first recall the notions occurring in this proposition. A $d$-tuple $(A_1, \ldots, A_d)$ of self-adjoint operators on a Hilbert space $\mathcal{K}$ is called strongly commuting if the resolvents $(A_k - iI)^{-1}$ and $(A_l - iI)^{-1}$ commute or equivalently if the corresponding spectral measures $E_{A_k}$ and $E_{A_l}$ commute (that is, $E_{A_k}(M)E_{A_l}(N) = E_{A_l}(N)E_{A_k}(M)$) for all Borel subsets $M, N$ of $\mathbb{R}^d$ for all $k, l = 1, \ldots, d, k \neq l$. Such a $d$-tuple has a unique spectral measure $E_{(A_1, \ldots, A_d)}$ on $\mathbb{R}^d$ determined by
\[ E_{(A_1, \ldots, A_d)}(M_1 \times \cdots \times M_d) = E_{A_1}(M_1) \cdots E_{A_d}(M_d) \]
for arbitrary Borel subsets $M_1, \ldots, M_d$ of $\mathbb{R}$. 


Proof of Proposition 8.
Let $L$ be a moment functional for a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$. Then it is easily checked that the multiplication operators $A_k$, $k = 1, \ldots, d$, by the coordinate functions $x_k$ form a $d$-tuple of strongly commuting self-adjoint operators on the Hilbert space $\mathcal{K} = L^2(\mu)$ such that $\mathcal{H} \subseteq \mathcal{K}$ and $X_k \subseteq A_k$ for $k = 1, \ldots, d$. The spectral measure $E \equiv E_{(A_1, \ldots, A_d)}$ of this $d$-tuple is given by $E(M) = \chi_M \cdot f$, $f \in L^2(\mu)$, where $\chi_M$ is the characteristic function of the Borel set $M \subseteq \mathbb{R}^d$. Hence we have $\langle E(M)1, 1 \rangle = \mu(M)$.

Conversely, let $(A_1, \ldots, A_d)$ be such a $d$-tuple and let $E_{(A_1, \ldots, A_d)}$ be its spectral measure. Put $\mu(\cdot) = \langle E(\cdot)1, 1 \rangle$. Let $p \in \mathbb{C}[x]$. Since $X_k \subseteq A_k$, we have $p(X_1, \ldots, X_d) \subseteq p(A_1, \ldots, A_d)$. The polynomial $1$ belongs to the domain of $p(X_1, \ldots, X_d)$ and hence of $p(A_1, \ldots, A_d)$. From the spectral calculus we obtain
\[
\int p(\lambda) \, d\mu(\lambda) = \int p(\lambda) \, d\langle E_{(A_1, \ldots, A_d)}(\lambda)1, 1 \rangle = \langle p(A_1, \ldots, A_d)1, 1 \rangle = \langle p(X_1, \ldots, X_d)1, 1 \rangle = L(p(x_1, \ldots, x_d)),
\]
where the last equality follows from (3.1). This shows that $\mu$ is a solution of the moment problem for $L$. \hfill $\square$

For the determinacy problem we have the following important sufficient criterion which was first noticed by A. Devinatz [12], see also [16].

**Proposition 9.** Suppose $L$ is a moment functional and $\mu$ is a representing measure for $L$. If each symmetric operator $X_k$, $k = 1, \ldots, d$, is essentially self-adjoint (that is, $X_k = X_k^*$) or equivalently if $\mathbb{C}[x_1, \ldots, x_d]$ is dense in $L^2((1 + x_k^2)\mu)$ for each $k = 1, \ldots, d$, then the moment problem for $L$ is determinate and the polynomials $\mathbb{C}[x_1, \ldots, x_d]$ are dense in $L^2(\mu)$.

Our proof of Proposition 9 given below is different from the ones in [12] and [16]. It is essentially based on the following lemma.

**Lemma 10.** Let $A$ be a closed symmetric operator on a Hilbert space $\mathcal{K}$ and let $P$ denote the orthogonal projection of $\mathcal{K}$ onto its closed subspace $\mathcal{H}$. Suppose that $\mathcal{D}$ is a dense linear subspace of $\mathcal{H}$ such that $\mathcal{D} \subseteq \mathcal{D}(A)$, $A\mathcal{D} \subseteq \mathcal{H}$ and $X := A|\mathcal{D}$ is an essentially self-adjoint operator on $\mathcal{H}$. Then we have $PA \subseteq AP$. Moreover, if $Y$ denotes the restriction of $A$ to $(I - P)\mathcal{D}(A)$, then $A = X \oplus Y$ on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$.

Proof. Suppose that $\varphi \in \mathcal{D}(A)$. Let $\psi \in \mathcal{D}$. Using the facts that $\psi$ and $X\psi$ are in $\mathcal{H}$ and the relation $X \subseteq A$, it follows that
\[
\langle X\psi, P\varphi \rangle = \langle PX\psi, \varphi \rangle = \langle X\psi, \varphi \rangle = \langle A\psi, \varphi \rangle = \langle \psi, A\varphi \rangle = \langle P\psi, A\varphi \rangle = \langle \psi, PA\varphi \rangle.
\]
Since $\psi \in \mathcal{D}$ was arbitrary, $P\varphi \in \mathcal{D}(X^*)$ and $X^*P\varphi = AP\varphi$. Since $X$ is essentially self-adjoint, $X^* = X$ and so $AP\varphi = XP\varphi = PA\varphi$. That is, $PA \subseteq AP$. 


If \( \varphi \in \mathcal{D}(A) \), we have \( P \varphi \in \mathcal{D}(X) \) and \( (I - P) \varphi \in \mathcal{D}(A) \) as just shown. Hence \( \varphi = P \varphi + (I - P) \varphi \in \mathcal{D}(X) \oplus \mathcal{D}(Y) \) and \( A \varphi = X \varphi + Y (I - P) \varphi \), so \( A \subseteq X \oplus Y \). By definition the converse inclusion is clear. \( \square \)

Proof of Proposition \([9]\).

Suppose that the operators \( X_1, \ldots, X_d \) are essentially self-adjoint on \( \mathfrak{H} \).

Let \( (A_1, \ldots, A_d) \) be a \( d \)-tuple of strongly commuting self-adjoint operators on a larger Hilbert space \( K \) such that \( X_j \subseteq A_j \), \( j = 1, \ldots, d \). By Lemma \([10]\) there is a decomposition \( A_j = X_j \oplus Y_j \) on \( K = \mathfrak{H} \oplus \mathfrak{H}^\perp \). Obviously, \( Y_j \) is also self-adjoint and \( E_{A_j} = E_{X_j} \oplus E_{Y_j} \) for the corresponding spectral measures.

Hence the spectral measures of \( X_k \) and \( X_l \) commute and formula \([3.2]\) yields \( E_{(X_1, \ldots, X_d)} \subseteq E_{(A_1, \ldots, A_d)} \). Therefore, \( \mu(\cdot) := \langle E_{(X_1, \ldots, X_d)}(\cdot) 1, 1 \rangle \) is equal to \( \langle E_{(A_1, \ldots, A_d)}(\cdot) 1, 1 \rangle \) by Proposition \([8]\) and so coincides with \( \mu \) by the preceding, the moment problem is determinate.

Let \( M \) be a Borel set of \( \mathbb{R}^d \). By definition the polynomials are dense in \( \mathfrak{H}_L \), so there is a sequence \( \{p_n\} \) of polynomials \( p_n \in \mathbb{C}[x] \) such that \( p_n(x) \to E_{(X_1, \ldots, X_d)}(M)1 \) in \( \mathfrak{H}_L \) and or equivalently

\[
|| (p_n(x) - E_{(X_1, \ldots, X_d)}(M)) 1 \|^2 = \int_{\mathbb{R}^d} |p_n(\lambda) - \chi_M(\lambda)|^2 d(E_{(X_1, \ldots, X_d)}(\lambda)(1, 1)) \to 0
\]

as \( n \to \infty \). Hence the closure of polynomials \( \mathbb{C}[x] \) in \( L^2(\mu) \) contains all characteristic functions \( \chi_M \), so it is equal to \( \mathfrak{H}_L = L^2(\mu) \).

Since \( X_k \) is essentially self-adjoint, \( (X_k \pm i) \mathbb{C}[x] \) is dense in \( \mathfrak{H}_L = L^2(\mu) \) which obviously implies the density of \( \mathbb{C}[x] \) in \( L^2((1 + x_k^2)\mu) \).

Conversely, if \( \mathbb{C}[x] \) is dense in \( L^2((1 + x_k^2)\mu) \), it follows that \( (x_k \pm i) \mathbb{C}[x] = (X_k \pm i) \mathbb{C}[x] \) is dense in \( L^2(\mu) \) and so in its subspace \( \mathfrak{H}_L \). Hence \( X_k \) is essentially self-adjoint on the Hilbert space \( \mathfrak{H}_L \).

The preceding Proposition \([9]\) suggests the following definitions.

**Definition 1.** Let \( \mu \in \mathcal{M}(\mathbb{R}^d) \) and let \( L \) be its moment functional, that is, \( L(p) = \int p \, d\mu \) for \( p \in \mathbb{C}[x] \). We say that \( \mu \), respectively \( L_\mu \), is strongly determinate if the symmetric operators \( X_1, \ldots, X_d \) are essentially self-adjoint on the Hilbert space \( \mathfrak{H}_L \). We call \( \mu \) resp. \( L_\mu \) strictly determinate if \( \mu \) resp. \( L_\mu \) is determinate and the polynomials \( \mathbb{C}[x_1, \ldots, x_d] \) are dense in \( L^2(\mu) \).

Using these notions we can restate Proposition \([9]\) as follows: The measure \( \mu \) is strongly determinate if and only if \( \mathbb{C}[x_1, \ldots, x_d] \) is dense in \( L^2((1 + x_k^2)\mu) \) for \( k = 1, \ldots, d \). Strong determinacy always implies strict determinacy and so determinacy.

Strong determinacy and another concept called ultradeterminacy have been introduced by B. Fuglede, see [10], p. 57. A measure \( \mu \in \mathcal{M}(\mathbb{R}^d) \) resp. its moment functional \( L_\mu \) is called ultradeterminate if the polynomials \( \mathbb{C}[x_1, \ldots, x_d] \) are dense in \( L^2((1 + ||x||^2)\mu) \). Since the norm of \( L^2((1 + ||x||^2)\mu) \) is obviously stronger than the norm of \( L^2((1 + x_k^2)\mu) \) for \( k = 1, \ldots, d \), each ultradeterminate measure is strongly determinate.
From the Weierstrass approximation theorem it follows that each measure with compact support is ultradeterminate. Further, if \( \mu \in \mathcal{M}(\mathbb{R}^d) \) and \( \mathbb{C}[x_1, \ldots, x_d] \) is dense in \( L^p(\mu) \) for some \( p > 2 \), then \( \mu \) is ultradeterminate, see [15], p.61.

The equivalence (i)\( \Leftrightarrow \) (iii) of Proposition 11 below shows that in the one-dimensional case all four concepts ultradeterminacy, strong determinacy, strict determinacy, and determinacy are the same. However, in dimension \( d \geq 2 \), they are all different. Examples of a strongly determinate measure that is not ultradeterminate and of a strictly determinate measure that is not strongly determinate have been given in [35]. By Theorem 5.4 in [3] there exist (rotation invariant) determinate measures on \( \mathbb{R}^d, d \geq 2 \), which are not strictly determinate.

Examples show that ultradeterminacy and of strong determinacy are rather strong concepts. In many situations they are even too strong. It seems to us that strict determinacy is a more important and fundamental concept. One reason is that strict determinacy is crucial for the reduction theorem proved below. Another reason stems from the theory of orthogonal polynomials, because for a strict determinate measure \( \mu \) any sequence of orthonormal polynomials is an orthonormal basis of \( L^2(\mu) \).

Let us briefly discuss the preceding Propositions 8 and 9 in the one-dimensional case \( d = 1 \). From Proposition 8 it follows at once that a positive functional on \( \mathbb{C}[x_1] \) is always a moment functional, because each symmetric operator has a self-adjoint extension in a larger Hilbert space (see e.g. [2], Nr. 111). A more careful analysis (see [1] or [38]) shows that the operator \( X_1 \) is either essentially self-adjoint or has deficiency indices \((1,1)\). For the determinacy in dimension one we have the following fundamental result.

**Proposition 11.** For a measure \( \mu \in \mathcal{M}(\mathbb{R}) \) and its moment functional \( L_\mu \) the following statements are equivalent:

(i) \( \mu \) is determinate.

(ii) The symmetric operator \( X_1 \) is essentially self-adjoint on \( \mathcal{H}_{L_\mu} \).

(iii) \( \mathbb{C}[x_1] \) is dense in \( L^2((1 + x_1^2)\mu) \).

(iv) 1 is in the closure of \((x_1 + z)\mathbb{C}[x_1] \) in \( \mathcal{H}_{L_\mu} \) for some (resp. all) \( z \in \mathbb{C} \setminus \mathbb{R} \).

(v) 1 is in the closure of \((x_1 + z)\mathbb{C}[x_1] \) in \( L^2(\mu) \) for some (resp. all) \( z \in \mathbb{C} \setminus \mathbb{R} \).

**Proof.** See [1] Chapter 3, or [38].

The striking difference between the one-dimensional moment problem and the multi-dimensional case \( d \geq 2 \) appears also in the Hilbert space approach. First, there exist positive linear functionals on \( \mathbb{C}[x_1, \ldots, x_d], d \geq 2 \), which are not moment functionals. Probably the simplest example of this kind is Example 6 in [36]. To explain the difference to the one-dimensional case, let \( L \) be a positive linear functional on \( \mathbb{C}[x_1, \ldots, x_d], d \geq 2 \). Since the complex conjugation on \( \mathbb{C}[x_1, \ldots, x_d] \) commutes with the operator \( X_k \), it is
a bijective map of ker $(X_k^* - \bar{z} I)$ on ker $(X_k^* - \bar{z} I)$. Therefore, each symmetric operator $X_k$ has equal deficiency indices and so a self-adjoint extension on the Hilbert space $\mathcal{H}_L$. However, in order to apply Proposition 8, one needs strongly commuting self-adjoint extensions of $X_1, \ldots, X_d$ in a possible larger Hilbert space. To find reasonable results for the existence of such extensions is a very difficult task. This makes the Hilbert space approach much more complicated and so less powerful than in the one-dimensional case. Secondly, as noted above there are various different determinacy concepts in case $d \geq 2$. In particular, we don’t know of any useful general necessary operator-theoretic criterion for determinacy in the multi-dimensional case.

4. POLYNOMIAL AND RATIONAL APPROXIMATION

In this section we develop a number of criteria for determinacy which are useful by their flexibility in applications. All of them are based on density conditions of polynomials.

Let us begin with two simple results concerning the following general question: Given $\mu \in \mathcal{M}(\mathbb{R}^d)$, what can be said about the set $V_\mu$ of equivalent measures ?

The next proposition is a classical result of M.A. Naimark [21] for the one-dimensional case. The proof given in [1], p. 47, carries over verbatim to the multi-dimensional case.

Proposition 12. Suppose that $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\nu \in V_\mu$. Then $\mathbb{C}[x_1, \ldots, x_d]$ is dense in $L^1(\nu)$ if and only if $\nu$ is an extreme point of the convex set $V_\mu$. In particular, $\mathbb{C}[x_1, \ldots, x_d]$ is dense in $L^1(\mu)$ if $\mu$ is determinate.

Proposition 13. Suppose that $\nu, \mu \in \mathcal{M}(\mathbb{R}^d)$ and $\nu \asymp \mu$.

(i) If $C \subseteq \mathbb{R}^d$ is a real algebraic variety and $\text{supp } \mu \subseteq C$, then $\text{supp } \nu \subseteq C$.

(ii) If $p \in \mathbb{C}[x_1, \ldots, x_d]$ is bounded on $\text{supp } \mu$, then $p$ is also bounded on $\text{supp } \nu$ and we have $\sup \{|p(x)|; x \in \text{supp } \mu\} = \sup \{|p(x)|; x \in \text{supp } \nu\}$.

Proof. (i): By definition there are polynomials $p_1, \ldots, p_r \in \mathbb{C}[x]$ such that $C = \{x \in \mathbb{R}^d : p_1(x) = \cdots p_r(x) = 0\}$. Since $\nu \asymp \mu$ and $C \subseteq \text{supp } \mu$, we have

$$-\int p_j(x)^2 d\nu(x) = -\int p_j^2(x) \ d\mu(x) = -\int_C p_j(x)^2 \ d\mu(x) = 0$$

which implies that $p_j(x) = 0$ on $\text{supp } \nu$, $j = 1, \ldots, r$. Therefore, $\text{supp } \nu \subseteq C$.

(ii): In this proof we repeat an argument which has been used in [36], p. 228. Let $K_\nu$ and $K_\mu$ denote the supremum of $|p(x)|$ over the sets $\text{supp } \nu$ and $\text{supp } \mu$, respectively. For $\lambda > K_\mu$, put $M_\lambda := \{x \in \mathbb{R}^d : |p(x)| \geq \lambda\}$. Then we obtain

$$\lambda^2 \nu(M_\lambda) \leq \int_{M_\lambda} (pp^*)^n \ d\nu \leq \int_{\mathbb{R}^d} (pp^*)^n \ d\nu = \int_{\mathbb{R}^d} (pp^*)^n \ d\mu \leq K_\mu^2 \mu(\mathbb{R}^d)$$
for \( n \in \mathbb{N} \). Since \( \lambda > K_\mu \), the preceding inequality can be only valid for arbitrary \( n \) when \( \nu(M_\lambda) = 0 \). Therefore, \( \text{supp } \nu \subseteq \{ x\in \mathbb{R}^d : |p(x)| \leq K_\mu \} \) which in turn implies that \( K_\nu \leq K_\mu \). Interchanging the role of \( \nu \) and \( \mu \), we get \( K_\nu = K_\mu \).

Now we return to the determinacy problem.

For a Borel mapping \( \varphi : \mathbb{R}^d \to \mathbb{R}^m \) and a Borel measure \( \mu \) on \( \mathbb{R}^d \), we denote by \( \varphi(\mu) \) the image of \( \mu \) under the mapping \( \varphi \), that is, \( \varphi(\mu)(M) := \mu(\varphi^{-1}(M)) \) for any Borel set \( M \) of \( \mathbb{R}^m \). Then the transformation formula

\[
(4.1) \quad \int_{\mathbb{R}^m} f(y) \, d\varphi(\mu)(y) = \int_{\mathbb{R}^d} f(\varphi(x)) \, d\mu(x)
\]

holds for any function \( f \in L^1(\varphi(\mu)) \).

Let \( \pi_j(x_1, \ldots, x_d) = x_j \) be the \( j \)-th coordinate mapping of \( \mathbb{R}^d \) into \( \mathbb{R} \). Then \( \pi_j(\mu) \) is the \( j \)-th marginal measure of \( \mu \). The following basic result was obtained by L. C. Petersen [29].

**Theorem 14.** If all marginal measures \( \pi_1(\mu), \ldots, \pi_d(\mu) \) are determinate, then \( \mu \) is determinate.

In order to make our exposition complete in the multidimensional case as promised above, we repeat the proof of Theorem 14 from [29].

Proof. Suppose that \( \nu \in V_\mu \). Let \( \chi_1, \ldots, \chi_d \) be characteristic functions of Borel subsets of \( \mathbb{R} \). If \( p_1, \ldots, p_d \) are polynomials in one variable, we estimate

\[
||\chi_1(x_1) \ldots \chi_d(x_d) - p_1(x_1) \ldots p_d(x_d)||_{L^1(\nu)} \leq ||(\chi_1(x_1) - p_1(x_1))\chi_2(x_2) \ldots \chi_d(x_d)||_{L^1(\nu)} + ||p_1(x_1)(\chi_2(x_2) - p_2(x_2))\chi_3(x_3) \ldots \chi_d(x_d)||_{L^1(\nu)} + \cdots + ||p_1(x_1) \ldots p_d(x_d)||_{L^1(\nu)}
\]

\[
\leq ||(\chi_2(x_2) - p_2(x_2))\chi_3(x_3) \ldots \chi_d(x_d)||_{L^2(\nu)} + ||\chi_2(x_2)(\chi_3(x_3) - p_3(x_3)) \ldots \chi_d(x_d)||_{L^2(\nu)} + \cdots + ||p_1(x_1) \ldots p_{d-1}(x_{d-1})||_{L^2(\nu)} ||\chi_d(x_d) - p_d(x_d)||_{L^2(\nu)}.
\]

Let \( j \in \{1, \ldots, d\} \). Clearly, \( \nu \cong \mu \) implies that \( \pi_j(\nu) \cong \pi_j(\mu) \), so that \( \pi_j(\nu) = \pi_j(\mu) \), because \( \pi_j(\mu) \) is determinate by assumption. Hence we have

\[
(4.2) \quad ||\chi_j(x_j) - p_j(x_j)||_{L^2(\nu)} = ||\chi_j(x_j) - p_j(x_j)||_{L^2(\pi_j(\mu))} = ||\chi_j(x_j) - p_j(x_j)||_{L^2(\pi_j(\mu))}.
\]

Since \( \pi_j(\mu) \) is determinate, the polynomials \( \mathbb{C}[x_j] \) are dense in \( L^2(\pi_j(\mu)) \). Therefore, by (4.2), we can first choose \( p_1 \) such that \( ||\chi_1(x_1) - p_1(x_1)||_{L^2(\nu)} \) becomes arbitrarily small, then \( p_2 \) such that \( ||\chi_2(x_2) - p_2(x_2)||_{L^2(\nu)} \) becomes small and finally \( p_d \) such that \( ||\chi_d(x_d) - p_d(x_d)||_{L^2(\nu)} \) becomes small. By the above inequality, \( ||\chi_1(x_1) \ldots \chi_d(x_d) - p_1(x_1) \ldots p_d(x_d)||_{L^1(\nu)} \) becomes as small as we want. Since the span of functions \( \chi_1(x_1) \ldots \chi_d(x_d) \) is dense in
$L^1(\nu)$, this shows that the polynomials are dense in $L^1(\nu)$, so $\nu$ is an extreme point of $V_\mu$ by Proposition [12]. Because $\nu \in V_\mu$ was arbitrary, $\nu = \mu$. \hfill \Box

As shown by a simple example in [29], there exist determinate measures for which not all marginal measures are determinate.

Now we derive two theorems which are corollaries of Proposition [14]. They provide some general determinacy criteria in terms of polynomial approximations and contain, of course, Theorem [14] as special cases.

Let $\varphi = (\varphi_1, \ldots, \varphi_m) : \mathbb{R}^d \to \mathbb{R}^m$ be a polynomial mapping, where $\varphi_1, \ldots, \varphi_m \in \mathbb{R}[x_1, \ldots, x_d]$. Since $\mu \in \mathcal{M}(\mathbb{R}^d)$, we have $\varphi(\mu) \in \mathcal{M}(\mathbb{R}^m)$ and $\varphi_k(\mu) \in \mathcal{M}(\mathbb{R})$ by (1.1). Combining Proposition [11] and formula (4.1) it follows that the following three conditions are equivalent:

(a) $\varphi_k(\mu)$ is determinate.
(b) There exist a sequence $\{q_{kn}; n \in \mathbb{N}\}$ of polynomials $q_{kn} \in \mathbb{C}[y_1]$ such that $(\varphi_k(x) + z)q_{kn}(\varphi_k(x)) \to 1$ as $n \to \infty$ in $\mathcal{H}_{L_\mu}$ for some $z \in \mathbb{C} \setminus \mathbb{R}$.
(c) $(\varphi_k(x) + z)\mathbb{C}[\varphi_k(x)]$ is dense in $L^2(\varphi_k(\mu))$ for some $z \in \mathbb{C} \setminus \mathbb{R}$.

Conditions (b) and (c) can be rephrased by saying that 1 resp. elements of $L^2(\varphi_k(\mu))$ can be approximated by (certain) polynomials. It should be emphasized that condition (b) is "intrinsic" to the moment problem, that is, it depends only on the moment functional $L_\mu$, but not on the particular representing measure $\mu$.

**Theorem 15.** Let $K$ be a closed subset of $\mathbb{R}^d$ and $\mu \in \mathcal{M}(K)$. Suppose that the map $\varphi : K \to \mathbb{R}^m$ is injective and that for each $k = 1, \ldots, m$ one of the equivalent conditions (a)–(c) is satisfied. Then $\mu$ is determinate on $K$.

Proof. Let $\nu$ be another measure such that $\nu \cong \mu$ and $\text{supp} \nu \subseteq K$. Since $\varphi$ is a polynomial mapping and $\nu \cong \mu$, it follows from (1.1) that $\varphi(\nu) \cong \varphi(\mu)$. By assumption, all marginal measures $\pi_k(\varphi(\mu)) = \varphi_k(\mu)$ are determinate. Therefore, $\varphi(\mu)$ is determinate by Proposition [14] so that $\varphi(\nu) = \varphi(\mu)$. Since $\text{supp} \ \varphi(\mu) \subseteq \varphi(K)$ and $\varphi : K \to \mathbb{R}^m$ is injective by assumption, we conclude that $\nu = \mu$. \hfill \Box

For the second result we consider $r$ polynomial mappings $\varphi^j : \mathbb{R}^d \to \mathbb{R}^{m_j}$, $j = 1, \ldots, r$, and define a polynomial mapping $\varphi = (\varphi^1, \ldots, \varphi^r) : \mathbb{R}^d \to \mathbb{R}^m$, where $m = m_1 + \cdots + m_r$.

**Theorem 16.** Let $\mu$ and $K$ be as in Theorem [12]. If $\varphi : K \to \mathbb{R}^m$ is injective and all measures $\varphi^1(\mu), \ldots, \varphi^r(\mu)$ are strictly determinate, then $\mu$ is determinate on $K$.

Proof. The proof is based on a generalization of Proposition [14] which is obtained by replacing the map $\pi_j$ onto a single coordinate by a map $\pi^j$ onto a finite set of coordinates. To be more precise, we write $y \in \mathbb{R}^m$ as $y = (y_1, \ldots, y_{m_1}, y_{m_1+1}, \ldots, y_{m_r})$ and define mappings $\pi^j : \mathbb{R}^d \to \mathbb{R}^{m_j}$ by $\pi^j(y) = (y_{mj_1}, \ldots, y_{mj_1})$. Then we have the following result: If $\nu \in \mathcal{M}(\mathbb{R}^m)$ and all measures $\pi^1(\nu), \ldots, \pi^r(\nu)$ are strictly determinate, then $\nu$ is determinate.
A proof of this statement can be given by repeating the arguments of the above proof of Proposition 14 and using that the polynomials $C \mathcal{L}$ are dense in $L^2(\pi^j(\nu))$ for the strictly determinate measure $\pi^j(\nu)$. We do not carry out the details.

Using this generalization instead of Proposition 14 we can argue as in the proof of the preceding theorem. □

The presence of "sufficiently many" bounded polynomials on the set supp $\mu$ allows us to prove stronger results that the plain determinacy. As a sample we consider subsets of cylinders with compact base sets.

**Proposition 17.** Let $K$ be a closed subset of $\mathbb{R}^d$, $d \geq 2$, such that $K$ is a subset of $K_0 \times \mathbb{R}$, where $K_0$ is a compact set of $\mathbb{R}^{d-1}$. Suppose that $\mu \in \mathcal{M}(K)$. If the marginal measure $\pi_d(\mu)$ is determinate, then $\mu$ is ultradeterminate and hence strongly determinate.

Proof. We shall write $x \in \mathbb{R}^d$ as $x = (y, x_d)$ with $y \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$. Suppose that $f(y)$ and $g(x_d)$ are given continuous functions with compact support. Let us abbreviate $M_1 = \sup \{||g(x_d)||; x_d \in \mathbb{R}\}$, $M_2 = \sup \{|p(y)|; y \in K_0\}$ and $M_3 = \sup \{|y|; y \in K\}$. Moreover we denote by $|| \cdot ||_1$ the norm of $L^2((1 + ||x||^2)\mu)$, by $|| \cdot ||_2$ the norm of $L^2(\mu)$ and by $|| \cdot ||_3$ the norm of $L^2((1 + x_d^2)\pi_d(\mu))$. Let $p \in \mathbb{C}[y]$ and $q \in \mathbb{C}[x_d]$. Using the assumption supp $\mu \subseteq K \subseteq K_0 \times \mathbb{R}$ and formula (4.1), we estimate

$$||f(y)g(x_d) - p(y)q(x_d)||_1 \leq ||(f(y) - p(y))g(x_d)||_1 + ||p(y)(g(x_d) - q(x_d))||_1$$

$$\leq ||f(y) - p(y)|| M_1 + M_2 M_3 ||(g(x_d) - q(x_d))(1 + x_d^2)||_2$$

$$\leq \sup |f(y) - p(y)|; y \in K_0 \} M_1 \mu(\mathbb{R}^d) + M_2 M_3 ||g(x_d) - q(x_d)||_3.$$

By the Weierstrass approximation theorem, $p \in \mathbb{C}[y]$ can be chosen such that the supremum of $|f(y) - p(y)|$ over the compact set $K_0$ is arbitrary small. Since the marginal measure $\pi_d(\mu)$ is determinate, $\mathbb{C}[x_d]$ is dense in $L^2((1 + x_d^2)\pi_d(\mu))$ by Proposition 14. Hence we can choose $q \in \mathbb{C}[x_d]$ such that $||g - q||_3$ is sufficiently small. Therefore, we have shown that the function $f(y)g(x_d)$ is in the closure of $\mathbb{C}[x]$ in $L^2((1 + ||x||^2)\mu)$. Since the span of such functions is obviously dense, it follows that the polynomials $\mathbb{C}[x]$ are dense in $L^2((1 + ||x||^2)\mu)$, so $\mu$ is ultradeterminate. □

We reproduce below from [31] another general determinateness criterion related to polynomial approximation.

**Proposition 18.** Let $K$ be a closed subset of $\mathbb{R}^d$, and let $\mu, \nu \in \mathcal{M}(K)$. Let $f \in \mathcal{C}(K, \mathbb{R})$ be a function satisfying $f \geq 1$, a.e. on $K$ (with respect to both $\mu$ and $\nu$).

Assume that there exists a sequence of polynomials $p_n$ in $\mathbb{R}[x]$ such that $p_n \to \frac{1}{f}$ under the norm $|| \cdot || = || \cdot ||_{L^2(\mu)} + || \cdot ||_{L^2(\nu)}$, and let

$$\mathcal{A}_0 := \mathcal{A}_0(K, f) := \left\{ \frac{p}{f} : p \in \mathbb{R}[x], k \geq 0, \frac{p(x)}{f(x)x^k} \to 0 \text{ for } |x| \to \infty, x \in K \right\}.$$
If $A_0$ separates the points of $K$, then $\mu = \nu$.

**Proof.** All fractions $\frac{p}{f^k}$ (with $p \in \mathbb{R}[x]$ and $k \geq 0$) are integrable with respect to both $\mu$ and $\nu$. We prove by induction on $k$ that

$$\int \frac{p}{f^k} d\mu = \int \frac{p}{f^k} d\nu.$$

For $k = 0$ this holds, since $L_\mu = L_\nu$ by assumption. For the induction step $k \to k + 1$ note that

$$\frac{p}{f^{k+1}} = \lim_{n \to \infty} \left( p_n \cdot \frac{p}{f^k} \right),$$

both in $L^1(\mu)$ and $L^1(\nu)$ (since $\|fg\|_2^2 = \langle |f|, |g|^2 \rangle \leq \|f\|_2^2 \cdot \|g\|_2^2$ by the Cauchy-Schwarz inequality), and hence

$$\int \frac{p}{f^{k+1}} d\mu = \int \frac{p}{f^{k+1}} d\nu.$$

Let $B$ denote the subalgebra of $\mathcal{C}(K, \mathbb{R})$ consisting of the functions $\phi$ for which the limit of $\phi(x)$, for $x \in K$ and $|x| \to \infty$, exists in $\mathbb{R}$. So $B = \mathcal{C}(K^+, \mathbb{R})$ where $K^+ = K \cup \{\infty\}$ is the one-point compactification of $K$. (If $K$ is already compact, put $K^+ = K$.) By assumption, the subalgebra $A := \mathbb{R}1 \oplus A_0$ of $B$ separates the points of $K^+$. Therefore, by the Stone-Weierstraß theorem, $A$ is dense in $B$ under uniform convergence. Therefore the measures $\mu$ and $\nu$ coincide as linear functionals on $B$, because if $b \in B$ and $a_n \in A$ with $a_n \to b$ under $\| \cdot \|_\infty$, then $\int_K a_n \to \int_K b$ for both $\mu$ and $\nu$ since $\mu(K) = \nu(K) < \infty$. It follows that $\mu = \nu$. □

The proposition applies to a variety of situations. For instance:

1. Assume we have $f \in \mathbb{R}[x]$ with $f \geq 1$ on $K$ for which there is a sequence $\{p_n\}$ in $\mathbb{R}[x]$ with $\|1 - fp_n\|_{L,2} \to 0$. If $A_0(K, f)$ separates the points of $K$, then the moment problem on $K$ given by $L$ is determinate. (Indeed, $\|\frac{1}{f} - p_n\|_2 \leq \|1 - fp_n\|_2$ since $f \geq 1$.) Note that, contrary to the preceding proposition, the condition $\|1 - fp_n\|_{L,2} \to 0$ is intrinsic in $L$ and its values on polynomials.

2. Assume that an entire function is given in form of its Taylor series

$$\frac{1}{f} = \sum_{\alpha} c_\alpha x^\alpha.$$

Assume that the algebra $A_0(K, f)$ fulfills the separation condition in the statement of the Proposition, and let $\mu$ be a positive measure on $K$ with moments $\{s_\alpha\}$. The normal convergence condition

$$\limsup_{\alpha} \left( |c_\alpha| \cdot s_{2\alpha} \right)^{1/|\alpha|} < 1$$
will assure that the partial sums converge to $1/f$ in $L^2(\mu)$, and hence the above result is applicable.

3. There are natural choices of continuous functions $f \geq 1$ for which $A_0(K, f)$ separates the points of $K$, like $f = 1 + \sum_i x_i^2$ or $f = e^{x^2}$.

4. In the one-dimensional case we know from Proposition 11 that the converse of the assertion of Proposition 18 holds as well: The moment functional $L$ is determinate if and only if there exists a sequence of polynomials $\{p_n\}$ for which $||1 - (1 + x^2)p_n||_{\delta_L} \to 0$.

5. Partial determinacy, moment functionals and determinacy

In this section we show how partial determinacy can be used to conclude that a positive linear functional is even a moment functional and to ensure determinacy. The pioneering result on partial determinacy is due A. Devinatz [12]. A slightly stronger theorem was obtained later by G.I. Eskin [14], see e.g. [26] and the references in [14].

For technical reasons we first develop all results in the case $d = 2$. The case $d \geq 3$ is more complicated and will be treated afterwards. The following theorem is closely related to Theorem 22 in [15].

**Theorem 19.** Let $Q \subseteq \mathbb{C}[x_1, x_2]$ be a set of polynomials such that $\mathbb{C}[x_1, x_2]$ is the linear span of polynomials $p(x_1)q(x_1, x_2)$, where $p \in \mathbb{C}[x_1]$ and $q \in Q$. Suppose that $L$ is a positive linear functional on $\mathbb{C}[x_1, x_2]$ such that for each $q \in Q$ the (positive) linear functional $L_q(p) := L(p(x_1)(x_2^2 + 1)q)$ on $\mathbb{C}[x_1]$ is determinate. Then $L$ is a moment functional.

The special case $Q = \{x_2^k; k \in \mathbb{N}_0\}$ of Theorem 19 is due to Eskin [14]. It is the first assertion of the following corollary.

**Corollary 20.** Suppose $s = \{s_{n,k}; n \in \mathbb{N}_0\}$ is a positive semidefinite 2-sequence such that for each $k \in \mathbb{N}_0$ the 1-sequence

\begin{equation}
\{s_{(n,2(k+1))} + s_{(n,2k)}; n \in \mathbb{N}_0\}
\end{equation}

is determinate. Then $s$ is a moment sequence. If in addition the sequence $\{s_{0,n}; n \in \mathbb{N}_0\}$ is determinate, then $s$ is determinate.

Proof. Put $Q = \{x_2^k; k \in \mathbb{N}_0\}$. Since $s_{(n,m)} = L(x_1^n x_2^m)$, $L_{x_2^k}$ has the moment sequence (5.1), so $L$ is a moment functional by Theorem 19.

The determinacy of the sequence $\{s_{(n,2)} + s_{(n,0)}\}$ clearly implies that of the sequence $\{s_{(n,0)}\}$. Hence both marginal sequences of $s$ are determinate, so $s$ is determinate by Petersen’s Theorem 14. □

The main technical parts of the proof of Theorem 19 are contained in the following two lemmas. If $A$ is a self-adjoint operator, the unitary operator $U_A := (A - iI)(A + iI)^{-1}$ is called the Cayley transform of $A$. 
Lemma 21. Let $A_j, j \in J$, be a family of pairwise strongly commuting self-adjoint operators $A_j$ and let $B$ be a densely defined symmetric operator acting on a Hilbert space $\mathcal{H}$. Suppose that $U_{A_j}B\tilde{U}_{A_j}^* = B$ for all $j \in J$. Then there exists a family $\tilde{A}_j, j \in J$, of strongly commuting self-adjoint operators $\tilde{A}_j$ and a self-adjoint operator $\tilde{B}$ on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that $\tilde{B} \supseteq B$, $A_j$ and $B$ strongly commute and $\tilde{A}_j \supseteq A_j$ for all $j \in J$.

Proof. The proof is based on the so-called "doubling trick" which is often used in operator theory. Define $\tilde{A}_j = A_j \oplus A_j$ for $j \in J$ and $B_0 = B \oplus (-B)$ on the Hilbert space $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$. Obviously, the operators $\tilde{A}_j, j \in J$, strongly commute as the operators $A_j, j \in J$, do by assumption. In the proof we freely use the self-adjoint extension theory of symmetric operators (see [2] or [32]). Without loss of generality we can assume that the operator $B$ is closed. Let $\mathcal{N}_{\pm}(T) = \ker(T^* \pm iI)$ denote the deficiency spaces of a symmetric operator $T$. From the definition of $B_0$ we obtain $\mathcal{N}_{\pm}(B_0) = \mathcal{N}_{\pm}(B) \oplus \mathcal{N}_{\pm}(B)$. Hence $V(\varphi, \psi) := (\psi, \varphi)$, where $\varphi \in \mathcal{N}_+(B)$, $\psi \in \mathcal{N}_-(B)$, defines an isometric linear map $V$ of $\mathcal{N}_+(B_0)$ onto $\mathcal{N}_-(B_0)$. The restriction $\tilde{B}$ of $(B_0)^*$ to the domain $\mathcal{D}(\tilde{B}) := \mathcal{D}(B_0) + (I-V)\mathcal{N}_+(B_0)$ is a self-adjoint extension of $B_0$.

From the assumption $U_{A_j}B\tilde{U}_{A_j}^* = B$ we derive $U_{A_j}B^*\tilde{U}_{A_j}^* = B^*$ which in turn yields $U_{A_j}\mathcal{N}_{\pm}(B) = \mathcal{N}_{\pm}(B)$. From $\mathcal{N}_{\pm}(B_0) = \mathcal{N}_{\pm}(B) \oplus \mathcal{N}_{\pm}(B)$ and the definition of $V$ it follows that the Cayley transform $U_{\tilde{A}_j} = U_{A_j} \oplus U_{A_j}$ of $\tilde{A}_j$ maps $(I-V)\mathcal{N}_+(B_0)$ onto itself. Since $U_{\tilde{A}_j}^*\mathcal{D}(B) = \mathcal{D}(B)$ by assumption and hence $U_{A_j}^*\mathcal{D}(B) = \mathcal{D}(B)$, we get $U_{\tilde{A}_j}^*\mathcal{D}(B) = \mathcal{D}(\tilde{B})$. Combined with the relation

$$U_{\tilde{A}_j}(B_0)^*U_{\tilde{A}_j}^* = U_{A_j}B^*\tilde{U}_{A_j}^* \oplus (-U_{A_j}B^*\tilde{U}_{A_j}^*) = B^* \oplus (-B^*) = (B_0)^*$$

the latter implies that $U_{\tilde{A}_j}^*\tilde{B}U_{A_j}^* = \tilde{B}$ and so $U_{\tilde{A}_j}^*\tilde{B}U_{A_j}^* = U_{\tilde{B}}$. Hence the resolvents of $\tilde{A}_j$ and $\tilde{B}$ commute. Therefore, $\tilde{A}_j$ and $\tilde{B}$ strongly commute. □

Lemma 22. Let $A$ and $B$ be closed symmetric linear operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{D} \subseteq \mathcal{D}(A) \cap \mathcal{D}(B)$ be a linear subspace such that $A\mathcal{D} \subseteq \mathcal{D}$, $B\mathcal{D} \subseteq \mathcal{D}$, and $AB\psi = BA\psi$ for $\psi \in \mathcal{D}$. Let $\mathcal{Q}$ be a subset of $\mathcal{D}$ such that the linear span of vectors $p(A)\varphi$, where $p \in \mathbb{C}[x]$ and $\varphi \in \mathcal{Q}$, is dense in $\mathcal{H}$ and a core for $B$. For $\varphi \in \mathcal{Q}$ we denote by $A_\varphi$ the restriction of $A$ to the invariant linear subspace $\mathcal{D}_\varphi := \mathbb{C}[A](B + i)\varphi$. Suppose that for each $\varphi \in \mathcal{Q}$ the symmetric operator $A_\varphi$ is essentially self-adjoint in the Hilbert space obtained as the closure of $\mathcal{D}_\varphi$ in $\mathcal{H}$. Then the operator $A$ on $\mathcal{H}$ is self-adjoint and we have $U_{A}B\tilde{U}_{A}^* = B$.

Proof. Let $\varphi \in \mathcal{Q}$, $z \in \mathbb{C}\setminus \mathbb{R}$ and $p \in \mathbb{C}[x]$. Since $A_\varphi$ is essentially self-adjoint, there exists a sequence $\{p_n\}$ of polynomials $p_n \in \mathbb{C}[x]$ such that $(A + z)p_n(A)(B + i)\varphi \to p(A)(B + i)\varphi$ in $\mathcal{H}_\varphi$. Using that $B$ is a symmetric
operator commuting with $A$ on $\mathcal{D}$ we compute
\[
|| (A + z)p_n(A)(B + i)\varphi - p(A)(B + i)\varphi ||^2 =
\]
\[
|| (B + i)((A + z)p_n(A)\varphi - p(A)\varphi) ||^2 =
\]
\[
||B((A + z)p_n(A)\varphi - p(A)\varphi)||^2 + ||(A + z)p_n(A)\varphi - p(A)\varphi||^2 =
\]
\[
||(A + z)Bp_n(A)\varphi - Bp(A)\varphi||^2 + ||(A + z)p_n(A)\varphi - p(A)\varphi||^2
\]
From these equations we conclude that $(A + z)Bp_n(A)\varphi \rightarrow Bp(A)\varphi$ and
$(A + z)p_n(A)\varphi \rightarrow p(A)\varphi$ as $n \rightarrow \infty$. Since the span of vectors $p(A)\varphi$ is
dense in $\mathcal{H}$, the preceding shows that the range of $(A + z)$ is dense in $\mathcal{H}$, so
the operator $A$ on $\mathcal{H}$ is self-adjoint. Further, because $(A + z)^{-1}$ is bounded, it
follows that $p_n(A)\varphi \rightarrow (A + z)^{-1}p(A)\varphi$ and $Bp_n(A)\varphi \rightarrow (A + z)^{-1}Bp(A)\varphi$
as $n \rightarrow \infty$. Since the operator $B$ is closed, the latter yields $B(A + z)^{-1}p(A)\varphi =
(A + z)^{-1}Bp(A)\varphi$. By assumption the span of vectors $p(A)\varphi$ is a core for
$B$. Therefore, we obtain that $B(A + z)^{-1}\psi = (A + z)^{-1}B\psi$ for all vectors
$\psi \in \mathcal{D}(B)$. Setting $z = i$ and $z = -i$, it follows that $U_A = I - 2i(A + i)^{-1}$
and $U_A^{-1} = I + 2i(A - i)^{-1}$ map the domain $\mathcal{D}(B)$ into itself. Therefore, we
conclude that $U_A\mathcal{D}(B) = \mathcal{D}(B)$. Hence the relation
\[
U_A B\psi = (I - 2i(A + i)^{-1})B\psi = B(I - 2i(A - i)^{-1})\psi = BU_A\psi
\]
for $\psi \in \mathcal{D}(B)$ implies that $U_A BU_A^* = B$. \hfill \Box

Proof of Theorem 19. We shall apply Lemma 22. Let $A$ and $B$ be the closures of the
multiplication operators by $x_1$ and $x_2$, respectively, on the Hilbert space $(\mathcal{H}_L, \langle \cdot, \cdot \rangle)$. For $q \in \mathcal{Q}$, we denote by $(\mathcal{H}_q, \langle \cdot, \cdot \rangle_q)$ the Hilbert
space of the moment functional $L_q$ on $\mathbb{C}[x_1]$. From the definitions of the
functional $L_q$ and of the scalar products (see formula (3.1)) we obtain
\[
\langle p_1(x_1), p_2(x_1) \rangle_q = L(p_1(x_1)p_2(x_1)(x_2^2 + 1)q) = \langle p_1(A)(B+q), p_2(A)(B+q) \rangle
\]
for $p_1, p_2 \in \mathbb{C}[x_1]$. Since $L_q$ is determinate by assumption, the multiplication
operator $X_1$ is essentially self-adjoint on $\mathcal{H}_q$ by Proposition 11(i) $\rightarrow$ (ii).
Therefore, by the preceding equality the operator $A_q$ is essentially self-adjoint on $\mathcal{D}_q$
for each $q \in \mathcal{Q}$. (Here we retain the notation of Lemma 22) Hence the assumptions of Lemma 22 are fulfulled, so we have $U_A BU_A^* = B$. Therefore, combining the conclusion of Lemma 21 applied to the single
operator $A$, with Proposition 8 we infer that $L$ is a moment functional. \hfill \Box

Now we turn to the case when the dimension $d$ is larger than 2. The following
theorem is the counter-part of Corollary 20.

**Theorem 23.** Let $s = \{s_n; n \in \mathbb{N}_0\}$ be a positive semidefinite $d$-sequence,
where $d \geq 3$. Suppose that for arbitrary numbers $j = 1, \ldots, d-1$ and $k_1, \ldots,$
$k_{j-1}, k_{j+1}, \ldots, k_d \in 2\mathbb{N}_0$ the 1-sequence
\[
(5.2) \quad \{s_{(k_1, \ldots, k_{j-1} - n, k_{j+1}, \ldots, k_{d-1} - 1, k_d + 2)} + s_{(k_1, \ldots, k_{j-1} - n, k_{j+1}, \ldots, k_{d-1} - 1, k_d)}; n \in \mathbb{N}_0\},
\]
is determinate. Further, suppose that for all numbers $j, l \in \{1, \ldots, d-1\}$,
j $< l$, all sequences of one of the following two sets of 1-sequences indexed
over \( n \in \mathbb{N}_0 \)

\[
(5.3) \quad \{s(k_1,\ldots,k_{j-1},n,k_{j+1},\ldots,k_{l-1},k_l+1,\ldots,k_d) + s(k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_{l-1},k_l,k_{l+1},\ldots,k_d)\},
\]

\[
(5.4) \quad \{s(k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_{l-1},n,k_{l+1},\ldots,k_d) + s(k_1,\ldots,k_{j-1},k_j,k_{j+1},\ldots,k_{l-1},n,k_{l+1},\ldots,k_d)\},
\]

where \( k_1,\ldots,k_d \in 2\mathbb{N}_0 \), are determinate. Then the \( d \)-sequence \( s \) is a moment sequence. If in addition the \( 1 \)-sequence \( \{s(0,\ldots,0,n); n \in \mathbb{N}_0\} \) is determinate, then \( s \) is also determinate.

Proof. The proof is given by some modifications of the above proof of Theorem 19. Let \( A_j, j = 1,\ldots,d \), denote the closure of the multiplication operator by \( x_j \) in the Hilbert space \( \mathcal{H}_{L_2} \). First we fix \( j \in \{1,\ldots,d-1\} \) and apply Lemma 22 in the case where \( A = A_j \), \( B = A_d \), and \( Q \) is the set of all monomials \( x_1^{k_1}\cdots x_{j-1}^{k_{j-1}}x_j^{k_j+1}\cdots x_d^{k_d} \). Since all sequences (5.2) are determinate, the assumptions of Lemma 22 are fulfilled. Therefore, by Lemma 22 the operator \( A_j \) is self-adjoint and we have \( U_{A_j}A_dU_{A_j}^* = A_d \). In order to show that the self-adjoint operators \( A_1,\ldots,A_{d-1} \) strongly commute we apply Lemma 22 once more. Let \( j, l \in \{1,\ldots,d-1\} \), \( j < l \). Assume that all sequences (5.3) are determinate. Then we set \( A = A_j \), \( B = A_l \), and let \( Q \) be the set of monomials \( x_1^{k_1}\cdots x_{j-1}^{k_{j-1}}x_j^{k_j+1}\cdots x_d^{k_d} \). From Lemma 22 we obtain that \( U_{A_j}A_lU_{A_j}^* = A_l \). Since \( A_j \) and \( A_l \) are both self-adjoint, the latter yields \( U_{A_j}U_{A_j}A_lU_{A_j}^* = U_{A_j} \). Hence the resolvents of \( A_j \) and \( A_l \) strongly commute. If the sequences (5.3) of the second set are determinate, we interchange the role of \( j \) and \( l \) and proceed in a similar manner. Thus, \( A_1,\ldots,A_{d-1} \) is a family of strongly commuting self-adjoint operators such that \( U_{A_j}A_dU_{A_j}^* = A_d \) for \( j = 1,\ldots,d-1 \). That is, the assumptions of Lemma 21 are satisfied with \( B := A_d \). From Lemma 21 and Proposition 8 we conclude that \( s \) is a moment sequence. As in proof of Corollary 5.1 the determinacy assertion follows from Petersen’s result (Theorem 14).

\[ \square \]

6. Quasi-analyticity

Growth conditions on the moments are another kind of assumptions that yield determinacy criteria for the moment problem. However, it is well known, even in the one variable case, that growth conditions cannot characterize determinateness. In this short section we will elaborate this method in the case of quasi-analyticity.

If \( T \) is a symmetric operator on a Hilbert space, a vector \( \varphi \in \cap_{n=1}^{\infty} D(T^n) \) is called a quasi-analytic vector for \( T \) if

\[
(6.1) \quad \sum_{n=1}^{\infty} ||T^n\varphi||^{-1/n} = +\infty.
\]
In terms of the moments \( s_n := \langle T^n \varphi, \varphi \rangle \), (6.1) is obviously equivalent to the so-called Carleman condition
\[
\sum_{n=1}^{\infty} s_n^{-1/2n} = +\infty. \tag{6.2}
\]

The importance of this condition for the moment problem stems from the following classical result due to T. Carleman [9] in dimension one. See also [21].

**Theorem 24.** If \( \{s_n; n \in \mathbb{N}_0\} \) is a moment sequence satisfying Carleman’s condition (6.2), then the corresponding moment problem is determinate.

Carleman’s result is the main technical ingredient of the following theorem.

**Theorem 25.** Suppose that \( s = \{s_n; n \in \mathbb{N}_0\} \) is a positive semi-definite \( d \)-sequence such that the first \( d-1 \) marginal 1-sequences
\[
\{s_{(n,0,...,0)}; n \in \mathbb{N}_0\}, \{s_{(0,n,...,0)}; n \in \mathbb{N}_0\}, \ldots, \{s_{(0,...,0,n,0)}; n \in \mathbb{N}_0\} \tag{6.3}
\]
satisfies the Carleman condition (6.2). Then \( s \) is a moment sequence. If in addition the last marginal 1-sequence \( \{s_{(0,...,0,n)}; n \in \mathbb{N}_0\} \) satisfies (6.2), then the moment sequence \( s \) is also determinate.

**Proof.** That the sequences in (6.3) satisfy condition (6.2) means that the vector 1 of the corresponding Hilbert space \( \mathcal{H} \) is quasi-analytic for the symmetric operators \( X_1, \ldots, X_{d-1} \). Assume first that \( d = 2 \). Since the operator \((X_2 + i)X_2^k\) commutes with \( X_1 \), the vector \((X_2 + i)X_2^k 1\) is also quasi-analytic for \( X_1 \) according to Theorem 4 in [26]. Using (3.1) we see that the moment 1-sequence for \( X_1 \) corresponding to this vector is just the sequence in (5.1), so it fulfills (6.2) as well and is determinate by Carleman’s Theorem 24. Therefore, \( s \) is a moment sequence by Corollary 20.

In the case \( d \geq 3 \) the proof is similar. Using again the fact that commuting operators preserve quasi-analytic vectors it follows that all sequences in equations (5.2), (5.3) and (5.4) satisfy Carleman’s condition, so they are determinate by Theorem 24. Hence \( s \) is a moment sequence by Theorem 23.

If in addition (6.2) holds for the sequence \( \{s_{(0,...,0,n)}\} \), then \( s \) is determinate by Petersen’s Theorem [14].

The preceding theorem contains the following fundamental result due to A.E. Nussbaum [26]: A positive semi-definite \( d \)-sequence is a determinate moment sequence if all \( d \) marginal 1-sequences satisfy Carleman’s condition.

Other classical papers on quasi-analytic or analytic vectors are [10], [11], [23], [25] and [27].

7. Towards a Parametrization of Solutions

The parametrization of all solutions of an indeterminate one variable moment problem on the line or on the semi-axis is well understood from at least two perspectives: via a canonical linear-fractionar representation of
the Cauchy transforms of the respective measures, or via an analysis of the
generalized self-adjoint extensions of a symmetric operators. See for details
\[1, 8, 38, 37\].

The aim of the present section is to parametrize all solutions to an inde-
terminate moment problem in the positive octant of \(\mathbb{R}^d\), by ”integrating”
a family of Stieltjes 1D moment problems with respect to a pencil of lines
spanning the octant. Although rather theoretical, at this point, our ap-
proach combines for the first time Nevanlinna’s parametrization with the
characterization of Fantappié transforms of positive measures.

We recall first some known facts about Nevalinna parametrization of so-
lutions to the Hamburger and Stieltjes moment problems, following the con-
ventions in [38]. Let \(\mu\) be a rapidly decaying at infinity positive measure on
the real line. For every \(z \notin \mathbb{R}\) and every equivalent measure \(\nu \in V_\mu\), that is
in our notation \(\nu \equiv \mu\), the values of the Cauchy transform

\[
\int_{\mathbb{R}} \frac{d\nu(x)}{x - z}
\]

fill a disk \(D(\mu, z)\), known as the Weyl disk. Denoting by \(P_k(x), k \geq 0\), the
sequence of orthogonal polynomials associated to the measure \(\mu\) (or any
equivalent measure, as these polynomials depend solely on the sequence of
moments) and by \(Q_k\) the associated orthogonal polynomials of the second
kind:

\[
Q_k(x) = \int_{\mathbb{R}} \frac{P_k(x) - P_k(u)}{x - u} d\mu(u), \quad k \geq 0,
\]

the equation of the boundary of \(D(\mu, z)\), in the case \(\Im z > 0\) is

\[
D(\mu, z) = \{w \in \mathbb{C}; \quad \frac{w - \overline{w}}{z - \overline{z}} = \sum_{n=0}^{\infty} |wP_k(z) + Q_k(z)|^2\}\.
\]

The radius of this circle is

\[
\rho(z) = \frac{1}{|z - \overline{z}|} \frac{1}{|\sum_{n=0}^{\infty} |P_n(z)|^2|}
\]

and \(\rho(z_0) > 0\) for an arbitrary non-real \(z_0\), if and only if the moment problem
associated to \(\mu\) is indeterminate. If this is the case, then

\[
A(z) = z \sum_{k=0}^{\infty} Q_k(0)Q_k(z),
\]

\[
B(z) = -1 + z \sum_{k=0}^{\infty} Q_k(0)P_k(z),
\]

\[
C(z) = 1 + z \sum_{k=0}^{\infty} P_k(0)Q_k(z),
\]

\[
D(z) = z \sum_{k=0}^{\infty} P_k(0)P_k(z),
\]
are entire functions fully constructible from the moments of $\mu$.

The convex set $V_\mu$ is then parametrized by all analytic maps $\Phi$ from the upper half-plane to the closure of the upper half-plane in the Riemann sphere, as follows:

$$\int_{\mathbb{R}} \frac{d\nu(x)}{x-z} = -\frac{C(z)\Phi(z) + A(z)}{D(z)\Phi(z) + B(z)}.$$ 

The functions $\Phi$, known as Nevanlinna, or Herglotz functions, provide a free parameter into the bijective description of all $\nu \in V_\mu$.

The case of an indeterminate Stieltjes problem requires additional care. Specifically, assume that $\mu$ is a positive measure on $\mathbb{R}_+$, having all moments finite. The orthogonal polynomials $P_k, Q_k, k \geq 0$, as well as the entire functions $A, B, C, D$ are the same as above. It is only the parametrization of all measures $\nu \in V_\mu(\mathbb{R}_+)$ which is more restrictive, in the following precise sense. There exists a bijective correspondence between the set $\nu \in V_\mu(\mathbb{R}_+)$ and all Nevanlinna functions of the form

$$\Psi(z) = d_0 + \int_0^\infty \frac{d\sigma(t)}{t-z}, \quad \Im z > 0,$$

where

$$d_0 = -\lim_{k \to \infty} \frac{Q_k(0)}{P_k(0)},$$

and $\sigma$ is an arbitrary positive measure, rapidly decaying at infinity. The formula relating the measure $\nu$ with the function $\Psi$ is the same as in the case of Hamburger problem:

$$\int_{\mathbb{R}_+} \frac{d\nu(x)}{x-z} = -\frac{C(z)\Psi(z) + A(z)}{D(z)\Psi(z) + B(z)}.$$ 

Equivalently, the functions $\Psi$ entering into the above representation can be characterized by the fact that either they are constant, or they are analytic in $\mathbb{C} \setminus \mathbb{R}_+$, and in any case

$$\Psi(t) \geq d_0, \quad \text{for all } t < 0.$$ 

For details see [38].

All these known facts being recalled, we return now to the multivariate moment problem in the positive octant of $\mathbb{R}^d$, where we will combine the above structure and the theory of the Fantappiè transform. To this aim, start with a measure $\mu \in \mathcal{M}(\mathbb{R}^d_+)$. For every vector $a \in \text{int}\mathbb{R}^d_+$ we consider the push-forward measure $(a \cdot)\mu = (a \cdot)_*\mu$ defined by the identity

$$\int_{\mathbb{R}_+^d} \phi(t)d[(a \cdot)_*\mu](t) = \int_{\mathbb{R}_+^d} \phi(a \cdot x)d\mu(x).$$ 

For an equivalent measure $\nu \in V_\mu(\mathbb{R}^d_+)$ the moments of these push-forward measures agree, that is $(a \cdot)\nu \in V_\mu(\mathbb{R}_+)$. Therefore, for a fixed $z \notin \mathbb{R}_+$ we
have
\[ \int_{\mathbb{R}_+^d} \frac{d\mu(x)}{a \cdot x - z} = \int_{\mathbb{R}_+} \frac{d[(a \cdot \nu)](t)}{t - z} \in \Delta((a \cdot \nu, z) \in \Delta((a \cdot \nu, z)) \}
and these Cauchy transforms depend on a "free" parameter \( \Psi \) as explained above.

Next we reverse this process, and analyze the characteristic properties of a family of positive measures \( \sigma_a \in \mathcal{M}(\mathbb{R}_+^d) \) indexed over \( a \in \text{int} \mathbb{R}_+^d \) which satisfy \( \sigma_a \in \mathcal{V}_a(\mathbb{R}_+^d) \) and "integrate" to a measure \( \nu \in \mathcal{V}_\mu(\mathbb{R}_+^d) \). First of all these measures should satisfy a smoothness dependence on the parameter \( a \). Second, they should fulfill a compatibility condition with respect to the homothety \( a \mapsto \lambda a \) where \( \lambda > 0 \):
\[
(7.1) \quad \sigma_{\lambda a} = \lambda \sigma_a, \quad a \in \text{int} \mathbb{R}_+^d, \lambda > 0.
\]
In other terms, for every bounded continuous functions \( \phi \) we have
\[
\int_{\mathbb{R}_+} \phi(t) d\sigma_{\lambda a}(t) = \int_{\mathbb{R}_+} \phi(\lambda t) d\sigma_a(t).
\]
To such a family of measures \( \sigma_a \) we associate the function
\[
F(a, z) = \int_{\mathbb{R}_+} \frac{d\sigma_a(t)}{t - z}, \quad a \in \text{int} \mathbb{R}_+^d, \quad z \notin \mathbb{R}_+.
\]
Notice that for every \( \lambda > 0 \)
\[
F(\lambda a, \lambda z) = \int_{\mathbb{R}_+} \frac{d\sigma_{\lambda a}(t)}{t - \lambda z} = \int_{\mathbb{R}_+} \frac{d\sigma_a(t)}{\lambda t - \lambda z} = \lambda^{-1} F(a, z).
\]
Hence, in view of the characterization of Fantapp\'e transforms of positive measures (see Theorem 2) we are led to the following rather abstract observation.

**Theorem 26.** Let \( \mu \in \mathcal{M}(\mathbb{R}_+^d) \). For every \( a \in \text{int} \mathbb{R}_+^d \) let \( P_k(a, z), Q_k(a, z) \) be the orthogonal polynomials of the first and second kind associated to the push forward measure \( (a \cdot \mu) \) on \( \mathbb{R}_+^d \). Assume that all these measures \( (a \cdot \mu) \), \( a \in \text{int} \mathbb{R}_+^d \) are indeterminate, so that the associated entire functions \( A(a, z), B(a, z), C(a, z), D(a, z) \) and limit \( d_0(a) \) exist.

Then there exists a bijective correspondence between all measures \( \nu \in \mathcal{V}_\mu(\mathbb{R}_+^d) \) and the \((-1)\text{-homogeneous,} \quad \mathcal{C}^\infty \text{cross sections in the Weyl disks bundle of the form}
\[
F(a, z) \in \Delta((a \cdot \mu), z),
\]
\[
F(a, z) = \frac{C(a, z)\Psi_a(z) + A(a, z)}{D(a, z)\Psi_a(z) + B(a, z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}_+,
\]
where \( \Psi_a \) are Nevanlinna functions analytic in \( \mathbb{C} \setminus \mathbb{R}_+ \) satisfying \( \Psi_a(t) \geq d_0(a), \quad t < 0, \) and such that the function \( F(a, -t), \quad a \in \text{int} \mathbb{R}_+^d, t > 0 \) is completely monotonic.
Proof. One direction of the correspondence was established before. In the other direction, the assumptions in the statement imply, in virtue of Nevanlinna parametrization theorem, the existence of a positive measure $\sigma_a$ on $\mathbb{R}_+$ with the property

$$F(a, z) = \int_{\mathbb{R}_+} \frac{d\sigma_a(t)}{t - z}.$$ 

Moreover, the dependence of $\sigma_a$ on the parameter $a$ is smooth, as stated. In virtue of Theorem 2 there exists a positive measure $\nu \in \mathcal{M}(\mathbb{R}^d_+)$ satisfying

$$F(a, -t) = \int_{\mathbb{R}^d_+} \frac{d\nu(x)}{a \cdot x + t}.$$ 

On the other hand, for all $a$ and $t$,

$$F(a, -t) = \int_{\mathbb{R}_+} \frac{d\sigma_a(u)}{u + t}.$$ 

Consequently the measures $(a \cdot \nu)$ and $\sigma_a$ have the same one dimensional moments, that is $(a \cdot \nu)$ and $(a \cdot \mu)$ have the same moments. In particular, for every fixed vector $a$ and positive integer $k$ we obtain

$$\int (a \cdot x)^k d\mu = \int (a \cdot x)^k d\nu.$$ 

A Vandermonde determinant argument implies then

$$\int x^\alpha d\mu = \int x^\alpha d\nu, \quad |\alpha| = k.$$ 

In other terms $\nu \in V_\mu(\mathbb{R}^d_+)$. □

A similar analysis can be carried out for an indeterminate moment problem in $\mathbb{R}^d$ provided that a characterization of Fantappiè transforms of positive measures $\mu$

$$F(a, z) = \int_{\mathbb{R}^d} \frac{d\mu(x)}{a \cdot x - z}, \quad a \in \mathbb{R}^n, \ z \in \mathbb{C} \setminus \mathbb{R},$$

is developed. So far, we are not aware of the existence of such a result.

As a direct application of the above discussion we can state the following uniqueness criterion.

**Corollary 27.** Let $\mu$ be a positive measure supported by the positive octant $\mathbb{R}^d_+$. If all push-forward measures $\pi_\ell(\mu)$ are determinate, where $\ell$ is an arbitrary semi-axis contained in $\mathbb{R}^d_+$ passing through the origin and $\pi_\ell$ denotes the orthogonal projection onto $\ell$, then $\mu$ is determinate.

The following section is devoted to an elaboration of the latter observation.
8. Disintegration techniques

The present section focuses on disintegration of measures techniques as another efficient tool for studying the determinacy problem.

The starting point is the following disintegration theorem for measures. Recall that \( p(\nu) = p_* (\nu) \) denotes the push-forward of a measure \( \nu \) by a map \( p \).

**Proposition 28.** Suppose that \( X \) and \( T \) are a closed subsets of Euclidean spaces and \( \nu \) is a finite positive Borel measure on \( X \). Let \( p : X \to T \) be a \( \nu \)-measurable mapping and let \( \mu : = p(\nu) \). Then there exist a mapping \( t \mapsto \lambda_t \) of \( T \) into the set of positive Borel measures on \( X \) satisfying the following conditions:

(i) \( \text{supp} \lambda_t \subseteq p^{-1}(t) \),

(ii) \( \lambda_t(p^{-1}(t)) = 1 \mu \text{-a.e.} \),

(iii) \( \int_X f(x) \, d\nu(x) = \int_T d\mu(t) \int_X f(x) \, d\lambda_t(x) \).

This is a special case of Proposition 2.7.13 from [7], Chapter IX. Note that the measure \( \nu \) is moderate, since \( X \) and \( T \) are closed subsets of Euclidean spaces and hence locally compact, and the map \( p \) is \( \nu \)-proper, since \( \nu \) is finite. Let us retain the assumptions and notations of Proposition 28.

Let \( A \) be a countably generated unital \( * \)-algebra of \( \nu \)-integrable functions on \( X \). Let \( f \in A \) and \( t \in T \). We define linear functionals \( L_t \) and \( L \) on \( A \) by

\[
L_t(f) := \int_X f(x) \, d\lambda_t(x), \quad L(f) \equiv L_\nu(f) := \int_X f(x) \, d\nu(x).
\]

**Lemma 29.** For \( f \in A \), the function \( t \mapsto L_t(f) \) is in \( L^2(T, \mu) \).

Proof. We use freely the properties (i)–(iii) from the preceding proposition. In particular, (iii) implies that \( L_t(1) = 1 \). In view of the above definitions and the Cauchy-Schwarz inequality we obtain

\[
\int_T d\mu(t) |L_t(f)|^2 = \int_T d\mu(t) \left( \int_X |f| \, d\lambda_t(x) \right)^2 \leq \\
\int_T d\mu(t) \left( \int_X |f|^2 \, d\lambda_t(x) \right) \left( \int_X 1 \, d\lambda_t(x) \right) = \\
\int_T d\mu(t) L_t(f \bar{f}) L_t(1) = \int_T d\mu(t) L_t(f \bar{f}) = L(f \bar{f}). \quad \square
\]

Suppose now that \( B \) is a countably generated unital \( * \)-algebra of functions on \( T \) such that \( p^* B \) is a \( * \)-subalgebra of \( A \). That is all functions \( (p^* f)(x) := f(p(x)) \), where \( f \in B \), belong to \( A \) and the involution commutes with the pull-back operation. Here and in what follows we put \( t = p(x) \) and write \( \tilde{f}(t) = f(p(x)) \). (Note that \( p^* B \) might be equal to \( A \)). We assume that the following three conditions are satisfied:
(1) The measure $\mu$ is $\mathcal{B}$-determinate (that is, if $\mu'$ is another measure on $T$ such that $\int d\mu(t)f(t) = \int f d\mu'(t)$ for all $f \in \mathcal{B}$, then $\mu = \mu'$).

(2) $\mathcal{B}$ is dense in $L^2(T, \mu)$.

(3) For $\mu$-almost all $t \in T$, the measure $\lambda_t$ is $\mathcal{A}$-determinate on $p^{-1}(t)$ (that is, if $\lambda_t'$ is another measure on the fibre $p^{-1}(t)$ such that $\int f d\lambda_t = \int f d\lambda_t'$ for all $f \in \mathcal{A}$, then $\lambda_t = \lambda_t'$.)

Our main result in this section is the following theorem.

**Theorem 30.** Under the assumptions (1)–(3) the measure $\nu$ is $\mathcal{A}$-determinate on $X$.

Proof: Suppose that $\nu'$ is another measure on $X$ such that $\int f d\nu = \int f d\nu'$ for all $f \in \mathcal{A}$.

Let $\mu = p(\nu)$, $\lambda_t$ and $\mu' = p(\nu')$, $\lambda_t'$, respectively, be the corresponding measures from the disintegration theorem and let $L_t$ and $L_t'$, respectively, be the corresponding linear functionals for $\nu$ and $\nu'$, respectively. For $f \in \mathcal{B}$ we compute

$$
\int d\mu'(t) \hat{f}(t) = \int d\mu(t) \hat{f}(t) \int d\chi_t'(x) = \int d\mu'(t) f(p(x)) d\chi_t'(x) = \int f(p(x)) d\nu'(x) = L_{\nu'}(f(p(x))) = L_{\nu}(f(p(x))) = \int d\mu(t) \hat{f}(t).
$$

Since $\mu$ is $\mathcal{B}$-determinate by assumption (1), it follows that $\mu = \mu'$.

Since $\mathcal{B}$ is countably generated and dense in $L^2(T, \mu)$ by assumption (2), there are functions $\varphi_n = p^*(\varphi_n) \in \mathcal{A}$, $n \in \mathbb{N}$, such that $\{\varphi_n; n \in \mathbb{N}\}$ is an orthonormal basis of $L^2(T, \mu)$. Fix $f \in \mathcal{A}$. We compute the Fourier coefficients of the function $L_t(f)$ of $L^2(T, \mu)$ with respect to this orthonormal basis by

$$
\int_T d\mu(t) \hat{\varphi}_n(t) L_t(f) = \int_T d\mu(t) \varphi_n(t) \int f(x) d\lambda_t(x) = \int d\mu(t) \int \varphi(p(x)) f(x) d\lambda_t(x) = L(\varphi_n(p(x)) f(x)).
$$

Since $\mu = \mu'$, the same reasoning with $\lambda_t$ replaced by $\lambda_t'$ shows that

$$
\int_T d\mu(t) \hat{\varphi}_n(t) L'_t(f) = L(\varphi_n(p(x)) f(x)).
$$

Therefore, both functions $L_t(f)$ and $L'_t(f)$ from $L^2(T, \mu)$ (by Lemma [29]) have the same developments

$$
L_t(f) = \sum_n L(\varphi_n(p(x)) \varphi_n(t), \quad L'_t(f) = \sum_n L(\varphi(p(x)) \varphi_n(t)
$$

in $L^2(T, \mu)$. Consequently, we have $L_t(f) = L'_t(f)$ $\mu$-a.e. on $T$. That is, there is a $\mu$-null subset $M_f$ of $T$ such that $\int f(x) d\lambda_t(x) = \int f(x) d\lambda_t'(x)$ for $t \in T \setminus M_f$. Since $\mathcal{A}$ is countably generated, there is a $\mu$-null subset $M$ of $T$ such that the latter holds for all $f \in \mathcal{A}$ and for $t \in T \setminus M$. From assumption (3) it follows therefore that $\lambda_t = \lambda_t'$ $\mu$-a.e. on $T$. From the disintegration
We now specialize the preceding general theorem to the moment problem and derive a "reduction procedure" for proving determinacy.

Let $X$ be a closed subset of $\mathbb{R}^d$, $A := \mathbb{C}[x_1, \ldots, x_d]$ and $B := \mathbb{C}[t_1, \ldots, t_m]$. Let $p_1, \ldots, p_m \in \mathbb{R}[x_1, \ldots, x_d]$ be polynomials and define a mapping $p : X \to T$ by $p(x) = (p_1(x), \ldots, p_m(x))$, where $T$ is a closed subset of $\mathbb{R}^m$ such that $p(X) \subseteq T$. Note that in the case $X = \mathbb{R}^m$ the fibres are just the real algebraic varieties $$\tilde{p}^{-1}(t) = \{ x \in \mathbb{R}^n : p_1(x) = t_1, \ldots, p_k(x) = t_k \}, \quad t \in T.$$ Suppose that $\nu \in \mathcal{M}(X)$. Let $\mu$ and $\lambda_t$ denote the corresponding measures from Proposition 28. Since $L_\nu(\tilde{f}(p(x))) = \int \tilde{f}(t) \tilde{f}(t) < \infty$ for all $\tilde{f} \in \mathbb{R}[t_1, \ldots, t_m]$, we also have that $\mu \in \mathcal{M}(X)$. From Lemma 29 it follows that $\lambda_t \in \mathcal{M}(X) \mu$-a.e. on $T$. Note that each measure $\lambda_t$ is supported on the fibre $\tilde{p}^{-1}(t)$. Recall that a measure $\mu \in \mathcal{M}(T)$ is strictly determinate if it is determinate on $T$ and if the polynomials $C[t_1, \ldots, t_m]$ are dense in $L^2(\mu)$.

We now specialize the preceding general theorem to the moment problem and derive a "reduction procedure" for proving determinacy. Theorem 31 can be restated as follows:

**Theorem 31.** If $\mu$ is strictly determinate on $T$ and $\lambda_t$ is determinate on the fibre $\tilde{p}^{-1}(t)$ for $\mu$-almost all $t \in T$, then $\nu$ is determinate on $X$.

Theorem 31 combined with the fact that measures with compact support are always (strictly) determinate gives the following two corollaries.

**Corollary 32.** If $\mu$ is strictly determinate on $T$ and the fibre $\tilde{p}^{-1}(t)$ is bounded for $\mu$-almost all $t \in T$, then $\nu$ is determinate on $X$.

**Corollary 33.** If $T$ is compact and the measure $\lambda_t$ is determinate on the fibre $\tilde{p}^{-1}(t)$ for $\mu$-almost all $t \in T$, then $\nu$ is determinate on $X$.

There is a large number of applications of Theorem 31 and Corollaries 32 and 33 by specifying the set $X$ and the polynomials $p_1, \ldots, p_m$ occurring therein. We mention three such results and retain the notations and the setup introduced before Theorem 31.

1. Let $p_1(x) = x_1, \ldots, p_m(x) = x_m, m < d$, and let $X$ and $T$ be closed subsets of $\mathbb{R}^d$ and $\mathbb{R}^m$, respectively, such that $p(X) \subseteq T$. Then Theorem 31 yields:

   The measure $\nu$ is determinate on $X$ if $\mu = p(\nu)$ is strictly determinate on $T$ for $\mathbb{C}[t_1, \ldots, t_k]$ and the fibre measure $\lambda_{(t_1, \ldots, t_m)}$ is determinate on $\tilde{p}^{-1}(t)$ for $\mathbb{C}[x_1, \ldots, x_d]$ and $\mu$-almost all $t \in T$.

2. Set $p(x) = x_1^2 + \cdots + x_m^2$. Let $X = \mathbb{R}^n$ and $T := p(X) = [0, \infty)$. Since strict determinacy is the same as determinacy in dimension one, Corollary 32 gives:
The measure $\nu$ is determinate if $\mu = p(\nu)$ is determinate on $[0, \infty)$.

For rotation invariant measures on $\mathbb{R}^d$ this assertion has been obtained in [3].

3. Suppose that $p_1, \ldots, p_m \in \mathbb{R}[x_1, \ldots, x_d]$ are polynomials which are bounded on the closed subset $X$ of $\mathbb{R}^d$. Put

$$\alpha_j = \inf \{ p_j(x); x \in X \}, \beta_j = \sup \{ p_j(x); x \in X \}, T = [\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m].$$

An immediate consequence of Corollary 33 is the following assertion:

The measure $\nu$ is determinate on $X$ if the fibre measures $\lambda_t$ are determinate on $p^{-1}(t)$ for $\mu$-almost all $t \in T$.

If the closed set $X$ has non-constant bounded polynomials, it is often possible to reduce question about the moment problem on $X$ to their counterparts for the moment problem on their (in general lower dimensional) fibre sets. An existence theorem of this kind was proved in [36]. The preceding assertion is a similar result for the determinacy question.

9. Geometric determinateness

We reproduce below in a condensed and almost identical form a result from [31] which illustrates the thesis that the geometry of the support of a measure implies, under specific conditions, its determinateness. The notation and terminology, standard in real algebraic geometry, are explained in the same article [31] or in [34].

The simplest example is of course a compact support, in which case Stone-Weierstrass theorem leads immediately to the uniqueness conclusion. The next proposition generalizes this observation.

Proposition 34. Let $X$ be an affine $\mathbb{R}$-variety, and let $K$ be a closed subset of $X(\mathbb{R})$. If the algebra $\mathcal{H}(K) = \{ p \in \mathbb{R}[X]: p \text{ is bounded on } K \}$ separates the points of $K$, then every $K$-moment problem is determinate.

Proof. First assume that $H = \mathcal{H}(K)$ is generated by finitely many elements $h_1, \ldots, h_m$ as an $\mathbb{R}$-algebra. The map $h := (h_1, \ldots, h_m): K \to \mathbb{R}^m$ is injective, and the subset $h(K)$ of $\mathbb{R}^m$ is compact. From Theorem 15 we infer that every $K$-moment problem is determinate. The situation when $\mathcal{H}(K)$ is not finitely generated can be reduced to the previous case via a finite chain condition and an inductive limit argument, see for details [31].

We are now discussing cases to which Proposition 34 applies by first considering one dimensional varieties.

Let $X$ be an affine curve over $\mathbb{R}$, and let $\overline{X}$ be its (good) completion. That is, the unique (up to unique isomorphism) projective curve which contains $X$ as a Zariski dense open subset and whose points in the complement of $X$ are nonsingular. Let $S = \overline{X} - X$ (a finite set), and let $K$ be a closed
subset of $X(\mathbb{R})$. For simplicity, assume that $X$ is irreducible. Following [34] we’ll say that $K$ is virtually compact if $S$ contains at least one point which is either non-real or does not lie in the closure $\bar{K}$, the closure being taken in $\mathbb{R}(\mathbb{R})$.

Let $\mathcal{H} = \mathcal{H}(K)$ be the subring of $\mathbb{R}[X]$ consisting of all regular functions which are bounded on $K$. Regarding elements $p \in \mathbb{R}[X]$ as rational functions on $X$, $p$ is bounded on $K$ if and only if none of the points of $\bar{K} \cap S$ is a pole of $p$. So $\mathcal{H} = \mathcal{O}(\bar{X} - T)$, where $T$ is the set of points in $S$ which do not lie in $\bar{K}$. Therefore we see (cf. [34], Lemma 5.3) that $K$ is virtually compact if and only if $\mathcal{H} \neq \mathbb{R}$, and that in this case $\mathcal{H}$ separates the points of $X(\mathbb{R})$.

Hence:

**Theorem 35.** Let $X$ be an irreducible affine curve over $\mathbb{R}$, and let $K$ be a closed subset of $X(\mathbb{R})$. If $K$ is virtually compact then every moment problem on $K$ is determinate. □

The condition that $X$ is irreducible can be removed (see [34], Definition 5.1 and Lemma 5.3).

For the case of one-dimensional sets $K$, and for the determinateness question, this leaves us with the case where $K$ is not virtually compact. In other words, the case where every polynomial which is bounded on $K$ is constant on $K$. Going back to Proposition 18, the following observation is in order.

**Lemma 36.** Assume that the affine curve $X$ is irreducible, and let $K \subset X(\mathbb{R})$ be a closed subset. Then $\mathcal{A}_0(K,f)$ separates the points of $X(\mathbb{R})$ for every non-constant $f$ in $\mathbb{R}[X]$ with $f \geq 1$ on $X(\mathbb{R})$.

(Instead of $f \geq 1$ it is only needed here that $f$ vanishes nowhere on $X(\mathbb{R})$.)

**Proof.** Let $Y \subset \bar{X}$ be the open set where $f$ is regular. So $Y$ is affine, contains $X$, and the points in $\bar{X} - Y$ are nonsingular on $\bar{X}$. Since the rational function $\frac{1}{f}$ vanishes in the points of $\bar{X} - Y$, there exists for every $q \in \mathbb{R}[Y]$ an integer $k \geq 1$ such that $\frac{q}{f^k}$ vanishes at all points of $\bar{X} - Y$. Let

$$I = \{q \in \mathbb{R}[Y] : \forall y \in Y - X, \ q(y) = 0\},$$

an ideal of $\mathbb{R}[Y]$. If $q \in I$, and if $k \geq 1$ is chosen for $q$ as before, the rational function $\frac{q}{f^k}$ lies in $\mathcal{A}_0(K,f)$. Since the elements of $I$ separate the points of $X(\mathbb{R})$, the lemma follows. □

Consequently the following application is obtained.

**Corollary 37.** Let $X$ be an irreducible affine curve and $K \subset X(\mathbb{R})$ a closed set. If there are a non-constant $f \in \mathbb{R}[X]$ with $f \geq 1$ on $K$ and a sequence $p_n$ in $\mathbb{R}[X]$ with $fp_n \to 1$ under $|| \cdot ||_{L^2}$, the $K$-moment problem $L$ is determinate. □

There are higher-dimensional cases as well which are non-compact and to which Proposition [34] applies. Here is a class of examples.
Example 1. Let $K_1$ be a compact subset of $\mathbb{R}^n$, let $f: K_1 \to \mathbb{R}$ be a continuous function, and let $K = \{(x,t) \in K_1 \times \mathbb{R} : tf(x) = 1\}$, a closed subset of $\mathbb{R}^{n+1}$. Then $\mathcal{H}(K)$ separates the points of $K$. By Proposition [34], therefore, any $K$-moment problem is determinate.

To illustrate Theorem [35] we give a couple of examples of one-dimensional sets which are virtually compact but not compact. For simplicity we stick to subsets of the plane. Let $p \in \mathbb{R}[x,y]$ be an irreducible polynomial, let $X$ denote the plane affine curve $p = 0$.

*If the leading form (i.e., highest degree form) of $p(x,y)$ is not a product of linear real factors, then every closed subset $K$ of $X(\mathbb{R})$ is virtually compact. Thus, every $K$-moment problem is determinate.*

But also if the leading form of $p$ is a product of real linear forms, $X(\mathbb{R})$ may be virtually compact (let alone closed subsets $K$). The reason is that, although the Zariski closure of the curve $X$ in the projective plane contains only real points at infinity, some of them may be singular and may blow up to one or more non-real points. An example is given by the curve $p = x + xy^2 + y^4 = 0$.

Finally, even if the entire curve $X(\mathbb{R})$ itself fails to be virtually compact, suitable non-compact closed subsets $K$ may still be. For example, this is so for the hyperelliptic curves $y^2 = q(x)$, where $q$ is monic of even degree, not a square. For example, one easily checks that if $\deg(q)$ is divisible by 4, then $K$ is virtually compact if (and only if) the coordinate function $y$ is bounded on $K$ from above or from below.

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