PARALLEL AND DUAL SURFACES OF CUSPIDAL EDGES

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ABSTRACT. We study parallel surfaces and dual surfaces of cuspidal edges. We give concrete forms of principal curvature and principal direction for cuspidal edges. Moreover, we define ridge points for cuspidal edges by using those. We clarify relations between singularities of parallel and dual surfaces and differential geometric properties of initial cuspidal edges.

1. INTRODUCTION

It is well-known that cuspidal edges and swallowtails are generic singularities of wave fronts in $\mathbb{R}^3$ (for example, see [1]). There are many studies of wave fronts from the differential geometric viewpoint ([3] [7] [8] [9] [13] [15]). In particular, various geometric invariants of cuspidal edges were studied by Martins and Saji [8]. To investigate geometric invariants of cuspidal edges, they introduced the normal form of cuspidal edges. On the other hand, parallel surfaces of a regular surface are fronts and might have singularities. Porteous, Fukui and Hasegawa studied the singularities of parallel surfaces and caustics from the viewpoint of singularity theory (cf. [3] [11] [12]) when the initial surface is regular. Porteous [11] [12] introduced the notion of ridge point for regular surfaces relative to principal curvature and principal direction. Using this notion, Fukui and Hasegawa [4] showed relations between singularities of parallel surfaces and geometric properties of initial surfaces.

In this paper, we deal with parallel surfaces when the initial surfaces have singularities. In particular, we consider parallel surfaces of cuspidal edges. Since cuspidal edges have unit normal vector fields, we can consider parallel surfaces. We show relations between singularities on parallel surfaces and geometric properties of initial cuspidal edges (Theorem 3.2). Ridge points play an important role in studying parallel surfaces of regular surfaces and also play an important role in investigating this case. Generally, mean curvature is unbounded at cuspidal edges. Thus principal curvatures might be unbounded. We give a condition for one principal curvature to be well-defined (in particular, finite) as a $C^\infty$-function at cuspidal edges (Proposition 2.2). A notion of ridge points for cuspidal edges is defined in Section 2 using principal curvature and principal direction.

In Section 3, we study parallel surfaces of a cuspidal edge from the viewpoint of differential geometry. Moreover, we study the extended distance squared functions on cuspidal edges. In the case of cuspidal edges, the extended distance squared function has $\mathcal{D}_4$ singularities or worse, unlike the case of regular surfaces. We give the conditions for distance squared functions to have $\mathcal{D}_4$ singularities (Theorem 3.3).

In Section 4, we study dual surfaces and the extended height functions. In the case of cuspidal edges, extended height functions have $\mathcal{A}_2$ singularities or worse. We define dual surfaces as a part of the discriminant set of extended height functions. We give relations between singularities of dual surfaces and geometric properties of cuspidal edges (Proposition 4.2) and show conditions for the extended height function to have $\mathcal{D}_4$ singularities. Moreover, we give relations between singularities of dual surfaces and extended height functions (Proposition 4.3).

All maps and functions considered here are of class $C^\infty$ unless otherwise stated.

2. CUSPIDAL EDGES

First we recall some properties of wave fronts and frontals. For details, see [1] [3] [7] [9] [15].

Let $f : V \to \mathbb{R}^3$ be a smooth map and $(u, v)$ be a coordinate system on $V$, where $V \subset \mathbb{R}^2$ is a domain. We call $f$ a frontal if there exists a unit vector field $\nu$ along $f$ such that $L = (f, \nu) : V \to T_f \mathbb{R}^3$ is an isotropic
map, where \( T_1 \mathbb{R}^3 \) is the unit tangent bundle of \( \mathbb{R}^3 \) equipped with the canonical contact structure, and is identified with \( \mathbb{R}^3 \times S^2 \), where \( S^2 \) is the unit sphere. If \( L \) gives an immersion, \( f \) is called a wave front or a front. The isotropity of \( L \) is equivalent to the orthogonality condition

\[ \langle df(X_p), \nu(p) \rangle, \quad (X_p \in T_p V, \ p \in V). \]

We call \( \nu \) a unit normal vector or the Gauss map of \( f \). For a frontal \( f \), the function \( \lambda : \mathbb{R} \to \mathbb{R} \) defined as \( \lambda(u,v) = \det(f_u, f_v)(u,v) \) is called the signed area density function, where \( f_u = \partial f/\partial u, f_v = \partial f/\partial v \). A point \( p \in V \) is called a singular point of \( f \) if \( f \) is not an immersion at \( p \). Let \( S(f) \) be the set of singular points of \( f \). A singular point \( p \in S(f) \) is called non-degenerate if \( d\lambda(p) \neq 0 \) holds. Let \( p \in S(f) \) be a non-degenerate singular point. Then, by the implicit function theorem, \( S(f) \) is parametrized by a regular curve \( \gamma(t) : (-\varepsilon, \varepsilon) \to V (\varepsilon > 0) \) with \( \gamma(0) = p \). We call \( \gamma \) a singular curve and the direction of \( \gamma' = d\gamma/dt \) a singular direction. Moreover, there exists a unique non-zero vector field \( \eta(t) \in T_{\gamma(t)} V \) up to non-zero functional scalar multiplications such that \( df(\eta(t)) = 0 \) on \( S(f) \). This vector field \( \eta(t) \) is called a null vector field. A non-degenerate singular point \( p \in S(f) \) is said to be of the first kind if \( \eta(0) \) is transverse to \( \gamma'(0) \). Otherwise, it is said to be of the second kind.

A cuspidal edge is a map-germ \( A \)-equivalent to \((u, v) \mapsto (u, v^2, v^3)\) at \( 0 \) and a swallowtail is a map-germ \( A \)-equivalent to \((u, v) \mapsto (u, 3v^3 + uv^2, 4v^3 + 2uv)\) at \( 0 \), where two map-germs \( f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) are \( A \)-equivalent if there exist a diffeomorphisms \( \Xi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \) on the source and \( \Xi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0) \) on the target such that \( \Xi \circ f = g \circ \Xi \). The criteria for these singularities are known.

**Theorem 2.1** ([7, Proposition 1.3]). Let \( f : V \to \mathbb{R}^3 \) be a front and \( p \in V \) a non-degenerate singular point of \( f \).

1. \( f \) at \( p \) is \( A \)-equivalent to a cuspidal edge if and only if \( \eta \lambda(p) \neq 0 \) holds.
2. \( f \) at \( p \) is \( A \)-equivalent to a swallowtail if and only if \( \eta \lambda(p) = 0 \) and \( \eta \eta \lambda(p) \neq 0 \) hold.

Let \( p = \gamma(0) \) be a non-degenerate singular point of the first kind, and set

\[ \psi_{ccr}(t) = \det(\hat{\gamma}'(t), \nu \circ \gamma(t), d\nu_{\gamma(t)}(\eta(t))) , \]

where \( \hat{\gamma} = f \circ \gamma \) and \( \hat{\gamma}' = d\hat{\gamma}/dt \). This function is originally defined in [3]. It is well-known that \( f \) at \( p \) is a front if and only if \( \psi_{ccr}(0) \neq 0 \). ([3][2])

2.1. **Normal form of cuspidal edges.** Let \( f = (f_1, f_2, f_3) : V \to \mathbb{R}^3 \) be a cuspidal edge and \( \nu = (\nu_1, \nu_2, \nu_3) \) a unit normal vector field of \( f \). Then, by using only coordinate transformations on the source and isometries on the target, we obtain the following normal form of cuspidal edges (for details, see [8]).

**Proposition 2.1** ([8, Theorem 3.1]). Let \( f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) be a map-germ and \( 0 \) a cuspidal edge. Then there exist a diffeomorphism \( \varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \) and an isometry-germ \( \Phi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0) \) satisfying that

\[ \Phi \circ f \circ \varphi(u,v) = \left( u, \frac{a_{20}}{2} u^2 + \frac{a_{30}}{6} u^3 + \frac{v^2}{2}, \frac{b_{20}}{2} u^2 + \frac{b_{30}}{6} u^3 + \frac{b_{12}}{2} u v^2 + \frac{b_{03}}{6} v^3 \right) + h(u,v), \]

where

\[ h(u,v) = (0, u^4 h_1(u), u^4 h_2(u) + u^2 v^2 h_3(u) + u v^3 h_4(u) + u^4 h_5(u,v)), \]

with \( h_i(u) \) \( (1 \leq i \leq 4) \), \( h_5(u,v) \) smooth functions.

We call this parametrization the normal form of cuspidal edges.

2.2. **Principal curvatures and principal directions.** Let \( f : V \to \mathbb{R}^3 \) be a frontal and \( p \in S(f) \) a singular point of the first kind, where \( V \subset \mathbb{R}^3 \) is a domain. A coordinate system \((U; u,v)\) centered at \( p \) is called adapted if it satisfies

1. the \( u \)-axis is the singular curve,
2. \( \eta = \partial_u \) gives a null vector field on the \( u \)-axis, and
3. there are no singular points other than the \( u \)-axis.
We fix an adapted coordinate system \((U; u, v)\) centered at \(p\). In this case, there exists a map \(\hat{\psi} : U \to \mathbb{R}^3 \setminus \{0\}\) such that \(f_u = v\hat{\psi}\). \(f_u\) and \(\hat{\psi}\) are linearly independent, and we can take a unit normal vector \(\nu\) of \(f\) as 
\(\nu = f_u \times \hat{\psi} / \|f_u \times \hat{\psi}\|\). Thus we may regard the pair \(\{f_u, \hat{\psi}, \nu\}\) as a frame of \(f\).

We now set the following functions:

\[
(2.3) \quad \hat{E} = \langle f_u, f_u \rangle, \quad \hat{F} = \langle f_u, \hat{\psi} \rangle, \quad \hat{G} = \langle \hat{\psi}, \hat{\psi} \rangle, \\
\hat{L} = -\langle f_u, \nu_u \rangle, \quad \hat{M} = -\langle \hat{\psi}, \nu_u \rangle, \quad \hat{N} = -\langle \hat{\psi}, \nu_v \rangle = -\langle \hat{\psi}, \nu \rangle.
\]

We note that \(-\langle f_u, \nu_v \rangle = v\hat{M}\) holds. If \(f\) is a normal form in \((2.2)\), we have \(\hat{E} = 1, \hat{F} = 0, \hat{G} = 1, \hat{L} = b_2, \hat{M} = b_{12}, \hat{N} = b_{03}/2\) at \(0\). In particular, \(\hat{E}\hat{G} - \hat{F}^2 \neq 0\) in a neighborhood of \(0\). We obtain the following.

**Lemma 2.1.** It holds that

\[
\nu_u = \frac{\hat{F}\hat{M} - \hat{G}\hat{L}}{\hat{E}\hat{G} - \hat{F}^2} f_u + \frac{\hat{F}\hat{L} - \hat{E}\hat{M}}{\hat{E}\hat{G} - \hat{F}^2} \hat{\psi}, \quad \nu_v = \frac{\hat{F}\hat{N} - v\hat{G}\hat{M}}{\hat{E}\hat{G} - \hat{F}^2} f_u + \frac{v\hat{F}\hat{M} - \hat{E}\hat{N}}{\hat{E}\hat{G} - \hat{F}^2} \hat{\psi}.
\]

Let \(K\) and \(H\) denote the Gaussian curvature and the mean curvature of \(f\). In this setting, we have

\[
(2.4) \quad K = \frac{\hat{L}\hat{N} - v\hat{M}^2}{v(\hat{E}\hat{G} - \hat{F}^2)}, \quad H = \frac{\hat{E}\hat{N} - 2v\hat{F}\hat{M} + v\hat{G}\hat{L}}{2v(\hat{E}\hat{G} - \hat{F}^2)}.
\]

The behavior of these functions are studied in [9, 13, 14].

We define the principal curvatures and principal directions for cuspidal edges. We recall the case of regular surfaces. Let \(g : U \to \mathbb{R}^3\) be a regular surface, where \(U \subset \mathbb{R}^2\) is a domain, and let \(E, F, G, L, M, N\) be the coefficients of the first and the second fundamental forms of \(g\). A principal curvature \(\kappa\) is a solution of

\[
(EG - F^2)\kappa^2 - (EN - 2FM + GL)\kappa + (LN - M^2) = 0.
\]

Solving this equation, we get \(2\kappa_1 = (A + B)/(EG - F^2)\) and \(2\kappa_2 = (A - B)/(EG - F^2)\), where \(A = EN - 2FM + GL\) and \(B = \sqrt{A^2 - 4(EG - F^2)(LN - M^2)}\). These functions \(\kappa_1, \kappa_2\) are smooth on \(U\). The principal directions \((\xi_1, \xi_2) \neq (0, 0)\) corresponding to \(\kappa_i (i = 1, 2)\) satisfy

\[
(2.5) \quad \left( \begin{array}{cc} L & M \\ M & N \end{array} \right) \left( \begin{array}{c} \xi_1 \\ \xi_2 \end{array} \right) = \kappa_i \left( \begin{array}{cc} E & F \\ F & G \end{array} \right) \left( \begin{array}{c} \xi_1 \\ \xi_2 \end{array} \right).
\]

We can take principal directions as \((\xi_1, \xi_2) = (N - \kappa_1 G, -M + \kappa_1 F), (\xi_2, \xi_2) = (M - \kappa_1 F, N - \kappa_1 G)\).

Next we consider the case of cuspidal edges. Let \(f : V \to \mathbb{R}^3\) be a front and \(p \in S(f)\) a singular point of the first kind. Let \((U; u, v)\) be an adapted coordinate system centered at \(p\). We define two functions

\[
(2.6) \quad \hat{k}_1 = \frac{\hat{A} + \hat{B}}{2v(\hat{E}\hat{G} - \hat{F}^2)}, \quad \hat{k}_2 = \frac{\hat{A} - \hat{B}}{2v(\hat{E}\hat{G} - \hat{F}^2)}
\]
on \(U\), where \(\hat{A} = \hat{E}\hat{N} - 2v\hat{F}\hat{M} + v\hat{G}\hat{L}, \hat{B} = \sqrt{\hat{A}^2 - 4v(\hat{E}\hat{G} - \hat{F}^2)(\hat{L}\hat{N} - v\hat{M}^2)}\). These are smooth on the regular set in \(U\). However, one of these functions might be unbounded at singular points. By \((2.4)\) and \((2.6)\), we have \(K = \hat{k}_1 \hat{k}_2, 2H = \hat{k}_1 + \hat{k}_2\). Thus we may regard \(\hat{k}_i (i = 1, 2)\) as principal curvatures of \(f\), and \(\hat{k}_i\) can be rewritten as

\[
(2.7) \quad \hat{k}_1 = \frac{2(\hat{L}\hat{N} - v\hat{M}^2)}{A - B}, \quad \hat{k}_2 = \frac{2(\hat{L}\hat{N} - v\hat{M}^2)}{A + B}.
\]

Here, \(\hat{A} \pm \hat{B} = \hat{E}(\hat{N} \pm |\hat{N}|)\) hold along the \(u\)-axis. On the other hand, the function \(\psi_{crr}\) in \((2.1)\) is

\[
\psi_{crr}(u) = \frac{||f_u(u, 0)||^2(-\langle \hat{\psi}, \nu \rangle(u, 0))}{||f_u(u, 0) \times \hat{\psi}(u, 0)||}.
\]

This implies that \(f\) is a front near \(q = (u, 0)\) if and only if \(-\langle \hat{\psi}, \nu \rangle(q) = \hat{N}(q)\) does not vanish. Thus we have the following.

**Proposition 2.2.** Under the above setting, if \(\hat{N}(q)\) is positive (resp. negative), then \(\hat{k}_2\) (resp. \(\hat{k}_1\)) in \((2.6)\) can be extended as a \(C^\infty\)-function near a singular point \(q\).
We note that the principal curvature maps for fronts are defined and discussed their behavior in [10]. For details of the principal curvature maps, see [10].

By the construction, if \( \tilde{N} \) is positive (resp. negative) along the \( u \)-axis, \( \tilde{k}_2 \) (resp. \( \tilde{k}_1 \)) can be regarded as the principal curvature of the cuspidal edge. We assume that \( \tilde{k}_2 \) is a smooth on \( U \) in the following. Let \( \hat{\nu} = (\xi, \zeta) \) be the principal direction corresponding to \( \tilde{k}_2 \). In this case, equation (2.8) is

\[
\begin{pmatrix}
\tilde{L} & \nu M \\
\nu M & v N
\end{pmatrix}
\begin{pmatrix}
\xi \\
\zeta
\end{pmatrix} = \tilde{k}_2
\begin{pmatrix}
\tilde{E} & \nu \tilde{F} \\
\nu \tilde{F} & v^2 \tilde{G}
\end{pmatrix}
\begin{pmatrix}
\xi \\
\zeta
\end{pmatrix}.
\]

We can factor out \( v \) from (2.8) and obtain

\[
\begin{pmatrix}
\tilde{L} - \tilde{k}_2 \tilde{E} & v (\tilde{M} - \tilde{k}_2 \tilde{F}) \\
v (\tilde{M} - \tilde{k}_2 \tilde{F}) & v (\tilde{N} - v \tilde{k}_2 \tilde{G})
\end{pmatrix}
\begin{pmatrix}
\xi \\
\zeta
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

Setting \( \hat{\nu} = (\xi, \zeta) = (\tilde{N} - v \tilde{k}_2 \tilde{G}, -\tilde{M} + \tilde{k}_2 \tilde{F}) \), this satisfies equation (2.9). Since \( \tilde{N} \) is a non-zero function on the \( u \)-axis, \( \hat{\nu} \) is non-zero on \( U \). This implies that \( \hat{\nu} \) can be regarded as the principal direction with respect to \( \tilde{k}_2 \).

### 2.3. Ridge Points

In this section, we introduce a notion called ridge points for cuspidal edges. First we recall the definition of ridge points and sub-parabolic points for regular surfaces (13 [11]). Let \( g : U \to \mathbb{R}^3 \) be a regular surface which has no umbilic point, and \( \kappa_i \) the principal curvatures of \( g \) and \( v_i \) the principal directions with respect to \( \kappa_i \) (\( i = 1, 2 \)). The point \( g(p) \) is called a ridge point relative to \( v_i \) if \( v_i \kappa_i(p) = 0 \), where \( v_i \kappa_i \) is the directional derivative of \( \kappa_i \) by \( v_i \). Moreover, \( g(p) \) is called a \( k \)-th order ridge point relative to \( v_i \) if \( v_i^{(m)} \kappa_i(p) \neq 0 \), where \( v_i^{(m)} \kappa_i \) is the directional derivative of \( \kappa_i \) by \( v_i \) applied \( m \) times. In addition, the point \( g(p) \) is called a sub-parabolic point relative to \( v_j \) if \( v_j \kappa_i(p) = 0 \) (\( i \neq j \)).

Let \( f : V \to \mathbb{R}^3 \) be a cuspidal edge and \( (U; u, v) \) be an adapted coordinate system centered at \( p \in S(f) \). Assume that \( \tilde{N} \) is positive on the \( u \)-axis. Then by (2.9), we can take \( \hat{\nu} = \xi \partial_u + \zeta \partial_v \) as the principal direction relative to the principal curvature \( \tilde{k}_2 \), where \( (\xi, \zeta) = (\tilde{N} - v \tilde{k}_2 \tilde{G}, -\tilde{M} + \tilde{k}_2 \tilde{F}) \). We define the ridge points for a cuspidal edge as follows:

**Definition 2.1.** Under the above setting, the point \( f(p) \) is called a ridge point for \( f \) relative to \( \hat{\nu} \) if \( \hat{\nu} \tilde{k}_2(p) = 0 \). Moreover, \( f(p) \) is called a \( k \)-th order ridge point for \( f \) relative to \( \hat{\nu} \) if \( \hat{\nu}^{(m)} \tilde{k}_2(p) = 0 \) (\( 1 \leq m \leq k \)) and \( \hat{\nu}^{(k+1)} \tilde{k}_2(p) \neq 0 \).

**Lemma 2.2.** Let \( f : U \to \mathbb{R}^3 \) be the normal form (2.22) of a cuspidal edge. Assume that \( \tilde{k} \) is the principal curvature which extends as a \( C^\infty \)-function near 0 and \( \hat{\nu} \) is the principal direction corresponding to \( \tilde{k} \). Then 0 is a first order ridge point if and only if

\[
\begin{align*}
(2.10) & \quad 4b_{12}^3 + b_{30} b_{03}^2 = 0, \\
(2.11) & \quad -2 b_{20}^4 b_{03}^3 - 3 b_{20} (4b_{12}^4 + a_{20} b_{03}^2)^2 + 24 (b_{12}^4 h_{20}(0) + 4 b_{12}^2 b_{03}^2 h_3(0) - 8 b_{12}^3 b_{03} h_4(0) + 16 b_{12} h_5(0)) \neq 0.
\end{align*}
\]

**Proof.** Without loss of generality, we assume \( \tilde{N} \) is positive near 0. Direct computations show \( (\tilde{k}_2)_{u}(0) = b_{30} - a_{20} b_{12}, \ (\tilde{k}_2)_{v}(0) = - (4b_{12}^2 + a_{20} b_{03}^2)/2b_{03}, \ \xi(0) = b_{30}/2 \) and \( \zeta(0) = -b_{12} \). Hence we get \( \hat{\nu}(\tilde{k}_2)(0) = (4b_{12}^2 + b_{30} b_{03}^2)/(2b_{03}) \), which shows (2.10). Again direct computations shows \( \xi_{u}(0) = 3 h_4(0), \zeta_{u}(0) = -b_{20} + 8 h_5(0), \xi_{v}(0) = a_{20} b_{20} - 4 h_3(0), \zeta_{v}(0) = -3 h_4(0) \). Moreover, we have

\[
\begin{align*}
\xi_{u}(0) & = -2 (a_{20} b_{20} + b_{30}^2 + a_{30} b_{12} - 12 b_{20} + 2 a_{20} h_3(0)), \\
\xi_{v}(0) & = \frac{1}{2b_{03}} \left( -a_{30} b_{03}^2 + 8 b_{12} (-4b_{03} h_3(0) + 3 b_{12} h_4(0)) + 2 a_{20} b_{03} (4 b_{20} b_{12} - 3 b_{03} h_4(0)) \right), \\
\zeta_{u}(0) & = 4 \frac{1}{b_{03}} \left( -2 b_{20} b_{12} - 6 b_{12} h_3(0) + 16 b_{12}^2 h_5(0) + b_{03}^2 (h_4(0) - 2 a_{20} h_5(0)) \right).
\end{align*}
\]

Since \( \hat{\nu}(2) \tilde{k}_2 = (\xi(\xi)_{u} + \zeta(\xi)_{v})(\tilde{k}_2)_{u} + (\xi(\zeta)_{u} + \zeta(\zeta)_{v})(\tilde{k}_2)_{v} + \xi^2(\tilde{k}_2)_{uu} + \zeta^2(\tilde{k}_2)_{vv} + 2 \xi \zeta (\tilde{k}_2)_{uv} \) and \( b_{30} = -4 b_{12}^3/b_{03}^2 \) hold, we have completed the proof. \( \square \)
3. Parallel surfaces of cuspidal edges and their singularities

In this section, we consider parallel surfaces of cuspidal edges and their singularities. First we recall the parallel surfaces of regular surfaces and their singularities. Here we treat the swallowtail singularity. In [4], how other singularities appeared in the parallel surfaces of regular surfaces was discussed.

3.1. Parallel surfaces of regular surfaces. Let \( g : U \to \mathbb{R}^3 \) be a regular surface which has no umbilic point, \( v_\nu \) the unit normal vector of \( g \), and \( \kappa_i \) \( (i = 1, 2) \) principal curvatures of \( g \). The parallel surface \( g_t \) of \( g \) is defined by \( g_t(u, v) = g(u, v) + tv_\nu(u, v) \), where \( t \in \mathbb{R} \) is a constant. We can take \( v_\nu \) as the unit normal vector along \( g_t \). Thus \( g_t \) is a front. We set \( \lambda_{g_t} = \det((g_t)_u, (g_t)_v, v_\nu) \). Using the Weingarten formula and \( \kappa_i \), we can rewrite \( \lambda_{g_t} \) as

\[
\lambda_{g_t} = (1 - t\kappa_1)(1 - t\kappa_2) \det(g_u, g_v, v_\nu).
\]

Since \( \det(g_u, g_v, v_\nu)(p) \neq 0 \) for any \( p \in U \), we have \( S(g_t) = \{ p \in U \mid t = 1/\kappa_i(p), \; i = 1, 2 \} \). The following relations are known (cf. [4, Theorem 3.4]).

**Theorem 3.1** ([4]). Let \( g : U \to \mathbb{R}^3 \) be a regular surface and \( g_t \) a parallel surface of \( g \), where \( t = 1/\kappa_i(p) \). Then \( f_{t_0} \) at \( p \) is a swallowtail if and only if \( p \) is a first order ridge with respect to \( v_i \) and not a sub-parabolic point relative to \( v_j \), where \( v_i \) is the principal direction with respect to \( \kappa_i \).

3.2. Parallel surfaces of cuspidal edges. Let \( f : V \to \mathbb{R}^3 \) be a front, \( f \) at \( p \in V \) a cuspidal edge and \( \nu \) a unit normal vector field of \( f \). Take an adapted coordinate system \((U; u, v)\) centered at \( p \). We assume that \( \kappa_2 \) is smooth on \( U \) (i.e. \( \bar{N} > 0 \) on the \( u \)-axis) and non-zero at \( p \in S(f) \). We define the parallel surface of \( f \) as \( f_t(u, v) = f(u, v) + tv(u, v) \), where \( t \in \mathbb{R} \setminus \{0\} \) is a constant. We can also take \( \nu \) as the unit normal vector field along \( f_t \). The signed area density for \( f_t \) is \( \lambda_t = \det((f_t)_u, (f_t)_v, \nu)(u, v) \). Using \( \kappa_1, \lambda_t \) can be rewritten as

\[
\lambda_t = (v - t\nu\kappa_1)(1 - t\kappa_2) \det(f_u, \psi, \nu).
\]

We note that by (2.6), \( v\kappa_1 \) is non-zero smooth function on the \( u \)-axis and \( v\kappa_1 \) is proportional to \( \bar{N} + |\bar{N}| \) along the \( u \)-axis. We set \( \lambda_t = (1 - t\kappa_2) \). Then \( S(f_t) = \{ \lambda_t = 0 \} \), and \( p \) is a singular point of \( f_t \) if and only if \( t = 1/\kappa_2(p) \). If \( f \) is a normal form (2.2) of a cuspidal edge, then \( \lambda_t = 1 - b_{20}t \). Hence we have \( t = 1/b_{20} = 1/\kappa_2(0) \). We fix \( t_0 = 1/\kappa_2(p) \). Then \( S(f_{t_0}) = \{ (u, v) \in U \mid \kappa_2(u, v) = \kappa_2(p)(= 1/t_0) \} \). Moreover, we see that \( p \in S(f_{t_0}) \) is non-degenerate if and only if \( d\lambda_{t_0}(p) = -t_0d\kappa_2(p) \neq 0 \). We consider the condition that \( f_{t_0} \) at \( p \) is a swallowtail.

To apply Theorem 2.1 (2), we need to determine the null vector field relative to \( f_{t_0} \). We set \( \eta_{t_0} = \ell_1(u, v)\partial_u + \ell_2(u, v)\partial_v \), where \( \ell_i(u, v) \) \( (i = 1, 2) \) are functions on \( U \). By Lemma 2.1

\[
df_{t_0}(\eta_{t_0}) = \left[ \ell_1 \left( 1 + t_0 \frac{\hat{F}M - \hat{G}L}{EG - F^2} \right) + \ell_2 t_0 \frac{\hat{F}N - \nu \hat{G}M}{EG - F^2} \right] f_u \\
+ \left[ \ell_1 t_0 \frac{\hat{F}L - \hat{E}M}{EG - F^2} + \ell_2 \left( v + t_0 \frac{\nu \hat{F}M - \hat{E}N}{EG - F^2} \right) \right] \psi.
\]

We set \( \ell_1 = \bar{N} - \nu \kappa_2 \hat{G}, \; \ell_2 = -\hat{M} + \kappa_2 \hat{F} \). Then \( df_{t_0}(\eta_{t_0}) = 0 \) holds for any point \((u, v) \in S(f_{t_0}) \). Thus we can take the null vector field \( \eta_{t_0} \) as \( \theta \). By Theorem 2.1 we obtain the following:

**Theorem 3.2.** Let \( f : V \to \mathbb{R}^3 \) be a smooth map and \( f \) at \( p \in V \) a cuspidal edge. Suppose that \( \kappa_2 \) can be extended as a smooth function near \( p \). Then the parallel surface \( f_{t_0} \) of \( f \), where \( t_0 = 1/\kappa_2(p) \), has a swallowtail at \( p \) if and only if \( df_{t_0}(\eta_{t_0}) \neq 0 \) and \( p \) is a first order ridge point of the initial surface \( f \).

**Proof.** Take an adapted coordinate system \((U; u, v)\) centered at \( p \in S(f) \). Then we see that \( d\lambda_{t_0} = -t_0d\kappa_2 \), and \( \eta_{t_0}\lambda_{t_0} = \psi(1 - t_0\kappa_2) = -t_0\nu\kappa_2 \). \( \eta_{t_0}\lambda_{t_0} = -t_0\nu\kappa_2 \). This completes the proof. \( \square \)

Using Lemma 2.2 and Theorem 3.2 we have the following lemma.

**Lemma 3.1.** Let \( f : V \to \mathbb{R}^3 \) be a normal form of cuspidal edge in (2.2) and \( f_{t_0} \) the parallel surface of \( f \), where \( t_0 = 1/b_{20} \). Then \( f_{t_0} \) is a swallowtail at the origin if and only if the coefficients of the normal form
satisfy
(3.4) \( b_{30} - a_{20}b_{12} \neq 0 \) or \( 4b_{12}^2 + a_{20}b_{03}^2 \neq 0 \),
(3.5) \( 4b_{12}^2 + b_{30}b_{03}^2 = 0 \),
(3.6) \(-2b_{20}b_{03}^2 - 3b_{20}(4b_{12}^2 + a_{20}b_{03}^2)^2 + 24(b_{03}b_{2}^2 + 4b_{12}^2b_{03}^2h_3(0) - 8b_{12}b_{03}b_{4}(0) + 16b_{12}h_5(0,0)) \neq 0 \).

Example 3.1. Let \( f \) be a cuspidal edge given as \( f(u,v) = (u, u^2/2 + u^3/3 + v^2/2, u^2 + v^3/3) \). The coefficients of \( f \) satisfy the conditions of Lemmas 2.2 and 3.1. The unit normal vector of \( f \) is \( \nu = (-2u + uv + u^2v, -v, 1)/\delta \), where \( \delta = \sqrt{1 + v^2 + (-2u + uv + u^2v)^2} \). Since \( b_{20} = 2 \), we take the parallel surface \( f_{t_0} \) as \( f_{t_0} = f + v/2 \).

We can see a swallowtail singularity on \( f_{t_0} \) (see Figure 1).

3.3. Extended distance squared functions on cuspidal edges. Let \( h : (R^n, 0) \rightarrow (R, 0) \) be a function-germ. A function-germ \( H : (R^n \times R^r, 0) \rightarrow (R, 0) \) is called an unfolding of \( f \) if \( H(u, 0) = h(u) \) holds. We define the discriminant set \( D_h \) of \( H \) by
\[
D_h = \{ x \in (R^r, 0) \mid H(u, x) = H_{u_1}(u, x) = \cdots = H_{u_n}(u, x) = 0 \},
\]
where \( (u, x) = (u_1, \ldots, u_n, x_1, \ldots, x_r) \in (R^n \times R^r, 0) \) and \( H_{u_i} = \partial H/\partial u_i \) (1 ≤ i ≤ n). In the case \( n = 1, r = 3, \) if \( h'(0) = h''(0) = h'''(0) = 0, h^{(4)}(0) \neq 0 \) and \( H \) is \( K \)-versal, then \( D_h \) is locally diffeomorphic to the image of swallowtail (\[2\ Section 6\]). See \[1\ Section 8\], for the \( K \)-versality. See also \[5\ [6\.

Let \( f : V \rightarrow R^3 \) be a front and \( f \) at \( p \in V \) a cuspidal edge. Take an adapted coordinate system \( (U; u, v) \) centered at \( p \). We define the following function:
\[
(3.7) \quad \Phi : (U \times R^3, (p, q)) \rightarrow R, \quad \Phi(u, v, (x, y, z)) = -\frac{1}{2} \left( ||(x, y, z) - f(u, v)||^2 - l_0^2 \right),
\]
where \( q = (x_0, y_0, z_0) = f(p) + t_0\nu(p) \in R^3 \setminus \{0\}, t_0 = 1/k_2(p) \in R \setminus \{0\} \) are constants. We call this function in (3.7) the extended distance squared function. If \( f \) is regular, it is known that \( \Phi \) is \( K \)-versal and \( D_\Phi \) is equal to the image of parallel surfaces of \( f \). Thus singularities of \( D_\Phi \) correspond to singularities of the parallel surface. Let us assume that \( f \) at \( p \) is cuspidal edge and \( (U; u, v) \) an adapted coordinate system centered at \( p \). Since \( \Phi_u = -(x, y, z) - f, \Phi_v = -v((x, y, z) - f, \psi) \), we have
\[
(3.8) \quad D_\Phi = \{ (x, y, z) \in R^3 \mid (x, y, z) = f(u, v) \pm t_0\nu(u, v) \}, \text{ for some } (u, v) \in U \cup \{ f_u \}^{-1}.
\]

Setting
\[
\phi(u, v) = \Phi(u, v, q),
\]
then \( \Phi \) is an unfolding of \( \phi \), but never a \( K \)-versal unfolding by (3.8).

Let \( p = 0 \) and \( f \) be a normal form in (2.2). Then we have
\[
\phi(u, v) = \frac{1}{6b_{20}} (b_{30}u^3 + 3b_{12}uv^2 + b_{03}v^4) + O(4),
\]
where \( O(4) = \{ h(u, v) : (R^2, 0) \rightarrow (R, 0) \mid \partial^{i+j}/\partial u^i \partial v^j h(0) = 0, \ i + j \leq 3 \} \). By a direct calculation,
\[
(3.9) \quad \Delta_\phi = (\partial^2/\partial u^2)(\phi_{uuu}^2) - 6\phi_{uuu}\phi_{uu}\phi_{evv} - 3(\phi_{uuu}^2)(\phi_{evv}^2) + 4(\phi_{uuu})^3\phi_{evv} + 4\phi_{uuu}(\phi_{evv})^3 (0)
\]
\[\]
holds. By \[13\text{ Lemma 3.1.}\], \(\varphi\) at 0 is right equivalent to \((u^3 + u^2 v)\) (resp. \((u^3 - u v^2)\)) if and only if \(\Delta_\varphi > 0\) (resp. \(\Delta_\varphi < 0\)), where function-germs \(h, k : (R^2, 0) \rightarrow (R, 0)\) are right equivalent if there exists a diffeomorphism-germ \(\theta : (R^2, 0) \rightarrow (R^2, 0)\) such that \(k = h \circ \theta\) holds. We have the following property.

**Theorem 3.3.** Under the above settings, the function \(\varphi\) has a D\(_4\) singularity at 0 if and only if the conditions that \(b_{30} \neq 0\) and 0 is not ridge hold.

The term \(b_{30}\) is called the *edge inflectional* function. See \[8\] for details. By **Theorem 3.3**, we might say that 0 satisfying the above conditions is an umbilic point of a cuspidal edge.

By \[3.7\], contact between \(f\) and the sphere whose center is \(f(p) + t_0\nu(p)\) with radius \(t_0 = 1/\kappa_2(p)\) become \(D_4\) type if the conditions in **Theorem 3.3** hold.

### 4. Dual surfaces and extended height functions

We consider the *dual surface* of a cuspidal edge and the *extended height function* on a cuspidal edge.

**4.1. Dual surfaces and height functions.** Let \(f : U \rightarrow R^3\) be a front, \(\nu\) a unit normal vector field of \(f\) and \(f\) at \(p \in U\) a cuspidal edge. Take a constant vector \(c = (c_1, c_2, c_3) \in R^3 \setminus \{0\}\) which satisfies \((\nu(p), c) \neq 0\). We set the map \(\tilde{f} : U \rightarrow R^3\) as \(\tilde{f} = f + c\), and set the function \(\rho(u, v) = (\tilde{f}(u,v), \nu(u,v))\) (\((u,v) \in U\)) We define the dual surface \(f^* : U \rightarrow R^3\) as

\[
f^*(u,v) = \rho(u,v)\nu(u,v).
\]

We define the following function:

\[
\tilde{H} : (U \times (S^2 \times R), (p, (n_0, r_0))) \rightarrow R, \quad \tilde{H}(u,v, (n,r)) = (\tilde{f}(u,v), n) - r (n_0 \in S^2, r_0 \in R).
\]

We call this function \(\tilde{H}\) the *extended height function*. It is well-known that if \(f\) is regular, then \(\tilde{H}\) is \(K\)-versal and \(D_{\tilde{H}}\) is diffeomorphic to the image of dual surfaces (cf. \[9\]). Hence singularities of \(D_{\tilde{H}}\) and the dual surface are supported. We set the function \(H(u,v) = \tilde{H}(u,v, (n_0, r_0))\). By definition, \(\tilde{H}\) is an unfolding of \(H\), but not \(K\)-versal. Since \(\tilde{H} = (f, n) - r, \quad H_u = (f_u, n)\) and \(H_v = (\psi, n)\), we see that

\[
D_{\tilde{H}} = \{(\pm \nu(u,v), \pm (\tilde{f}(u,v)\nu(u,v))) \mid (u,v) \in U\} \cup \{f_0\}.
\]

We define the map \(\Psi : S^2 \times (R \setminus \{0\}) \rightarrow R^3 \setminus \{0\}\), by \(\Psi(n, r) = r n\). Setting \(R_+ = \{r \in R \mid r > 0\}\), \(\Psi|S^2 \times R^*\) is a diffeomorphism, and we have \(\Psi(D_{\tilde{H}}) \supset f^*(U)\). Thus we can regard the image of \(f^*\) as a part of \(D_{\tilde{H}}\).

**4.2. Singularities of dual surfaces.** Let us consider singularities of dual surface. Setting a unit normal \(\nu^*\) of \(f^*\) as

\[
\nu^* = ([f^*(u)]_u \times ([f^*(v)]_v)\|([f^*(u)]_u \times ([f^*(v)]_v))\|
\]

then \(\nu^*\) is smooth on \(U\). If \(f\) is a normal form \([2.2]\), then the signed area density \(\lambda^* = \det([f^*(u)], [f^*(v)], [\nu^*])\) of \(f^*\) satisfies \(\lambda^*(0) = b_{20}a_{03}c_3|c|/2\). Thus we have the following.

**Proposition 4.1.** A point \(p \in U\) is a singular point of \(f^*\) if and only if \(\kappa_2(p) = 0\).

We assume \(b_{20}(= \kappa_2(0)) = 0\) in the following. We detect the null vector field \(\eta^*\) of \(f^*\). Let \(\eta^* = \ell_1^i \partial_u + \ell_2^i \partial_v\) be a vector field, where \(\ell_i^i\) (\(i = 1, 2\)) are functions on \(U\). By Lemma 2.1,

\[
df^*(\eta) = (\ell_1^i \rho_u + \ell_2^i \rho_v)\nu + \rho(\ell_1^i \alpha + \ell_2^i \beta) \psi.
\]

where \(\nu_u = \alpha f_u + \beta \psi, \nu_v = \alpha f_v + \beta \psi\). If \(f\) is a normal form \([2.2]\) and \(b_{20} = 0\), then rank \(df^*_\eta = 1\). Moreover,

\[
\rho_u(0) = -b_{12}c_2, \quad \rho_v(0) = -b_{03}c_2/2, \quad \alpha(0) = \alpha(0) = 0, \quad \beta(0) = -b_{12}, \quad \beta(0) = -b_{03}/2(\neq 0)
\]

holds. Setting \(\eta^* = \tilde{\beta}(u,v)\partial_u - \beta(u,v)\partial_v\), then by \([1.5]\), \(\eta^*\) is a null vector field of \(f^*\). Since, \(\eta^*\nu^* \neq 0\) holds, \(f^*\) is a front. By **Theorem 2.1** (1), we consider the conditions that make dual surface \(f^*\) a cuspidal edge. We have \(d\lambda^*(\eta^*) = \tilde{\beta}(\lambda^*)_u - \beta(\lambda^*)_v\). If \(f\) is a normal form \([2.2]\), we obtain

\[
(\lambda^*)_u(0) = \frac{1}{2}(b_{30} - a_{20}b_{12})b_{03}c_3|c|, \quad (\lambda^*)_v(0) = -\frac{1}{4}(4b_{12}^2 + a_{20}b_{03}^2)c_3|c|.
\]

Thus we have 4d\(\lambda^*_\eta(\eta^*) = -(4b_{12}^2 + b_{30}b_{03}^2)c_3|c|\). By **Lemma 2.2**, we have the following.

**Proposition 4.2.** Let \(f : U \rightarrow R^3\) be a front, \(f\) at \(p\) be a cuspidal edge and \(f^*\) a dual surface of \(f\). Then \(f^*\) has a cuspidal edge at \(p \in S(f)\) if and only if \(\kappa_2(p) = 0\) and 0 is not a ridge point.
Example 4.1. Let \( f \) be a cuspidal edge defined by \( f(u, v) = (u, 2u^3/3+v^2/2, 1+u^3/3+uv^2+v^3/6) \). The coefficients of \( f \) satisfy the conditions in Proposition 4.2. Thus \( f^\ast \) has a cuspidal edge at \( 0 \). The unit normal vector of \( f \) is \( \nu = (-u^2+4u^3+u^2v-v^2,-2u-v/2,1)/\delta \), where \( \delta = \sqrt{1+(2u+v/2)^2+(u^2-4u^3-u^2v+v^2)^2} \). The shapes of \( f \) and \( f^\ast \) are shown in Figure 2.

4.3. Singularities of extended height functions. We consider singularities of extended height function. Let \( f \) be a normal form in \([22]\). We assume \( b_{30} = \hat{\kappa}_2(0) = 0 \). Let \( n_0 = \nu(0) = (0,0,1) \) and \( r_0 = (f(0), n_0(0)) = c_3 \). Then a function \( h \) becomes \( \tilde{h}(u,v) = f_3(u,v) \), where \( f = (f_1, f_2, f_3) \). Here, \( \tilde{h} \) is

\[
\tilde{h}(u,v) = \frac{b_{30}}{6}u^3 + \frac{b_{12}}{2}uv^2 + \frac{b_{03}}{6}v^3 + u^4h_2(u) + u^2v^2h_3(u) + uv^3h_4(u) + v^4h_5(u,v).
\]

Now \( j^2\tilde{h} = 0 \) holds. By direct computation,

\[
\Delta_{\tilde{h}} = \left( \tilde{h}_{uuu}\tilde{h}_{vvv} - 6\tilde{h}_{uuu}\tilde{h}_{uuv}\tilde{h}_{vvv} - 3(\tilde{h}_{uuu})^2(\tilde{h}_{vvv})^2 + 4(\tilde{h}_{uuu})(\tilde{h}_{vvv})^3(\tilde{h}_{uuu})(\tilde{h}_{vvv})^3 \right)(0)
\]

\[
= b_{30}(b_{30}b_{03}^2 + 4b_{12}^3).
\]

By [13, Lemma 3.1], we have the following.

Lemma 4.1. Under the above conditions, the function \( \tilde{h} \) has a \( D_4 \) singularity at \( p \in S(f) \) if and only if \( \hat{\kappa}_2(p) = 0 \), the edge inflectional curvature does not vanish at \( p \) and \( p \) is not a ridge point.

By Proposition 4.2 and Lemma 4.1, we have the following.

Proposition 4.3. Let \( f : U \rightarrow \mathbb{R}^3 \) be a cuspidal edge and \( f^\ast \) the dual surface of \( f \). If the function \( \tilde{h} \) has a \( D_4 \) singularity at \( p \in S(f) \), then \( f^\ast \) has a cuspidal edge at \( p \).

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