Enhancement of network synchronizability via two oscillatory system

Harpertap Singh
Division of Strategic Research and Development, Graduate School of Science and Engineering, Saitama University, Shimo-okubo 255, Sakura-ku, Saitama 338-8570, Japan
E-mail: harpartap@mail.saitama-u.ac.jp

Abstract
Network’s synchronization destabilizes at large coupling strength when its nodes exchange information by scalar coupling (few coordinates) instead of using vector coupling (all coordinates). This issue is commonly tackled by modifying the network topology like a ring network to Small World or Scale Free networks which increases the coupling cost and decreases the robustness. In the present work, we show that without any structural alteration even a large ring network in the nearest neighborhood configuration using only a single coordinate in the coupling, could be made synchronizable. The primary condition is that the node dynamics should be given by a pair of oscillators (say, two oscillatory system TOS) rather than by a conventional way of single oscillator (say, single oscillatory system). It has been found that TOS not only stabilizes the chaotic synchronization but also the hyperchaotic synchronization manifold (a major challenge in the field of secure communication wherein multi parameter BK method is needed). The frameworks of drive-response system and master stability function have been used to study the TOS effect by varying TOS parameters with and without feedback (feedback means quorum sensing conditions). The TOS effect has been found numerically both in the chaotic (Rössler, Chua and Lorenz) and hyperchaotic (electrical circuit) systems. However, since threshold also increases as a side effect of TOS, the extent of β enhancement depends on the choice of oscillator model like larger for Rössler, intermediate for Chua and smaller for Lorenz.

1. Introduction
Synchronization is not always an obvious emergent behavior of the interacting dynamical systems even in the case when they are identical limit cycle oscillators. For an example, N identical Rössler oscillators (periodic/chaotic) coupled via a single coordinate (scalar coupling) on a ring network in the nearest neighborhood configuration exhibit desynchronization at large coupling strength [1, 2]. This desynchronization behavior (known as short wavelength bifurcation [1]) arises because the minimum coupling strength (threshold) requires for the synchronization increases with the increase in number of oscillators whereas the synchronization manifold in case of scalar coupling mode, loses its stability (riddle basin behavior [3, 4]) as the coupling strength (γ) augments towards a critical value (overload-tolerance). Thus, the small values of overload-tolerance (say γdyn) limit the size of a synchronizable network, e.g. the synchronized state of a ring network having x1-coupled chaotic Rössler becomes unstable with the increase in N from 18 to 19 at γdyn = 1.5. To tackle this issue of limited values of γdyn, the network modification methods are generally used such as (1) adding the additional edges between the nodes deterministically (Pristine World) or and stochastically (Small World) [5], (2) modifying the ring network topology to a synchronizable topology like a unidirectional tree network or a star network (a hub of Scale Free network) [6–8]. Practically these modifications could be considered as the distribution of overload among the nodes and theoretically these alterations imply the minimization of eigen-ratio (R) of the Laplacian/coupling matrix (largest eigenvalue to the smallest nonzero eigenvalue). The concept of R minimization comes from the theory of master stability function (MSF) [9] which says that a complex network of size N is
synchronizable if $R < \beta$ where $\beta = \gamma_{\text{dyn}} / \gamma_{\text{syn}}$ ($\gamma_{\text{syn}}$ means threshold) [5]. Therefore, to study the complete synchronization behavior in the complex systems, finding the different pathways that could reduce $R$ remain a primary goal for the researchers [5–22]. However, modification (1) increases the coupling cost of synchronization whereas the modification (2) decreases the robustness by increasing the centralization in a network. Hence, finding the non-modification methods to enhance the network synchronizability, have recently gained the attention of the researchers. This can be seen in the recent works [23, 24] wherein the concept of switch coupling based on the phase space is introduced. These works show that if one can locate the synchronization supportive region of the phase space then the chaotic networks can be made synchronizable even if the original topology is nonsynchronizable, i.e. independent from the network topology. But since tracing the desired small region of phase space in the presence of noise is a difficult as well as time consuming task, the practical implementation of this switch coupling method is quite challenging. Moreover, it should be noted that the switch coupling (on-off mechanism) does not increase $\gamma_{\text{dyn}}$ (overload-tolerance) because it stabilizes synchronization manifold just by cutting the overload rather than absorbing the overload. This means that only stabilization via continuous coupling can be considered as the scenario of increment of $\gamma_{\text{dyn}}$ which has not been developed so far (to the best of my knowledge).

In summary, the primary question which has not been answered yet is 'how to achieve the finitely large overload-tolerance in case of scalar coupling as similar to the vector coupling scenario (wherein all coordinates are used), i.e., maximization of $\gamma_{\text{dyn}}$ via scalar coupling?' The importance of this question lies on the fact that getting stability by using all the coordinates is neither useful (as in case of secure communication) nor realistic (as in case of complex system). Moreover, this theoretical problem also appears in many different forms such as: is it possible to stabilize the chaotic/hyperchaotic synchronization manifold? or is it possible to make a large non-centralized chaotic/hyperchaotic synchronizable network? or is it possible to surely stabilize the chaotic/hyperchaotic sub system under local/global parameter fluctuations?, etc. The answer to this question could be considered as a possible solution to the real problems such as overload failure in the real networks like Internet system and power grid system [25, 26], the stability issue of chaotic/hyperchaotic transmitter-receiver system in the field of secure communication [4, 27, 28], etc. It should be noted that since large coupling strength means infinite coupling ($\gamma = \infty$) in drive-response system [4, 27, 28], MSF complements the drive-response formulation by explicitly incorporating $\gamma$ dependence which results into an elegant relation between the node property ($\beta$) of a just two node system (coupled bidirectionally) with the structural property ($R$) of an arbitrary network having $N$ nodes (provided coupling matrix has zero row sum). In other words, if a system is stable in the drive-response framework then at the large coupling strength MSF shows negative values and vice versa (discussed later).

In the present work, an attempt has been made to answer the primary question of maximization of $\gamma_{\text{dyn}}$ via scalar coupling in case of linear interactions under small perturbations and global parameter fluctuations (each node experiences same fluctuations, i.e., identical node scenario). We argue that $\gamma_{\text{syn}}$ could be maximized, if the node dynamics are given by a pair of oscillators (say, two oscillatory system TOS) rather than by a conventional way of single oscillator (say, single oscillatory system SOS) as schematic shown in figure 1 on a ring network (i.e., a basic non-centralized network). In addition, since the stabilization of synchronization manifold happens only due to the emergence of dissipative factors via TOS (explained later by drive-response framework), to maximize $\gamma_{\text{dyn}}$ the following two conditions should be met: 1. TOS should be in mTOS configuration as dTOS configuration behaves same as SOS (discussed later), 2. only that coordinate can be employed whose self dissipation could stabilize the unstable fixed point by adding its linear dissipative term in the autonomous

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**Figure 1.** Schematics of a ring network (nearest neighborhood configuration bidirectionally coupled) for two scenarios: each node behaves as (a) conventional single oscillatory system (SOS), (b) proposed two oscillatory system TOS. The subplots (b1) and (b2) depict the implementation of TOS, i.e., by using both the oscillators (dTOS) and one of the oscillator (mTOS), respectively, where ‘d’ means ‘di’ and ‘m’ means ‘mono’. Oscillators $x$ and $y$ are coupled via all the coordinates (vector coupling) whereas oscillators $x$ and $y$ are coupled via a single coordinate (scalar coupling).
system (discussed later). According to condition 2, all the three coordinates of Chua [29] and Lorenz [30] oscillators fulfill this criteria whereas only two out of the three coordinates of Rössler [31] (i.e., $x_1$ and $x_2$) and also only two out of the four coordinates of piecewise electronic hyperchaotic system [32] (i.e., $x_1$ and $x_2$), satisfy this condition. However, there also exist a possibility when none of the coordinates of an oscillator model like hyperchaotic Rössler [33], could meet the condition and hence can not be employed as the oscillatory dynamics for TOS. Furthermore, since threshold increases with the increase in number of oscillators, $\gamma_{\text{syn}}$ also increases as the side effect of mTOS along with the incrementation of $\gamma_{\text{syn}}$. This makes $\beta$ enhancement dependent on the choice of oscillator model like larger for Rössler, intermediate for Chua and smaller for Lorenz. Furthermore, it is worth noticing that since TOS alters only the node dynamics not the network structure, TOS could be applied to any arbitrary network topology.

The motivation behind TOS (pair of oscillators) comes from the phenomenon of superconductivity wherein the coherent state (same energy level) is the consequence of electrons pairing, i.e., Cooper pairs [34]. Despite of the fact that Cooper pair is a quantum element and TOS is purely a classical formulation, they could be realized on some common platforms such as both execute far from equilibrium (Cooper pair is a low temperature behavior and TOS is an oscillatory unit) and leading to the stabilization (condensation in case of Cooper pair and synchronization in case of TOS) via the emergence of negative potential due to pairing (in case of Cooper pair the oppositely moving electrons overcome the positive Coulomb potential whereas mTOS configuration induces negative MSF, a potential energy function). This analogy would be more clear if one would see the electron–phonon interactions (the basis of Cooper pairs) wherein a pair of electrons interact via the lattice ions (say environment), as the case of indirect coupling like quorum sensing mechanism [35, 36] (oscillators interact via medium) exhibited by TOS due to its parameters (discussed later).

The paper is organized as follows. In section 2, the stability issue of scalar coupling and the theory of TOS are given in terms of MSF and drive–response frameworks along with the equations of the employed oscillator models. The stabilization via TOS for the scenarios of chaotic (Rössler, Chua and Lorenz) as well as hyperchaotic (electrical circuit) oscillators are presented in section 3. Finally, the paper is concluded in section 4.

2. Theory

In this paper, the enhancement of network synchronizability via TOS has been studied using MSF and drive–response frameworks. Before explaining these employed frameworks in the context of TOS, MSF and drive–response have been discussed briefly in case of conventional node dynamics, i.e. SOS, along with the scalar coupling issues of complex systems and secure communication. In addition, we have also discussed the equivalence between MSF and drive–response system at large coupling strength, i.e., if a chaotic/hyperchaotic system is stable in the drive–response framework then at the large coupling strength MSF shows negative values and vice versa (shown in section 3). In fact, MSF under small perturbations (linear analysis) serves as the necessary and sufficient condition to judge the basin stability at large coupling strength. This is analogous to the drive–response system wherein the conditional Lyapunov exponents surely determine the stability of a subsystem (response) by running the system from the different initial conditions. Therefore, the reason behind the emergence of TOS effect has been given in terms of the sub-Jacobian (Jacobian of response) method.

2.1. MSF

To understand the formulism of MSF [9], consider the equations of motion for a complex network having $N$ identical nodes for the conventional SOS:

$$\dot{x}^{i} = F(x^{i}) + \gamma \sum_{j=1}^{N} G_{ij} H(x^{j}), \quad i = 1, \ldots, N$$

where $x^{i}$ is an $m$-dimensional vector of $i$’th node whose autonomous behavior is described by $F(x^{i})$ ($R^{m} \rightarrow R^{m}$). Here, $H(x)$ ($R^{m} \rightarrow R$) represents $m \times m$ matrix (linear coupling function) which gives the information about the coordinates of $x$ involved in the coupling, i.e., its all other elements are zero except a diagonal element corresponding to the employed coordinate (for more details see [9]). In equation (1), $G$ is a $N \times N$ coupling matrix (captures network’s architect) which could be symmetric or not but it must be real with zero row sum [8] so that synchronous state ($x^{i} = s, i = 1, \ldots, N, s = F(s)$) could become a solution of equation (1). For a symmetric case (Laplacian), $G_{ii} = -g_{i}$ ($g_{i}$ is the degree/connections of $i$’th node) and $G_{ij} = 1$ if $i$’th node is connected to $j$’th node otherwise $G_{ij} = 0$. The parameter $\gamma(>0)$ represents the coupling strength.

The $N$ block diagonalized linear variational equations for equation (1), that depict the stability of the synchronous state, i.e., evolution of perturbation ($\eta$), are [9]:
\[ \dot{\eta}^k = (DF(s) - \gamma \delta_l DH(s))\eta^k. \]  

(2)

Here \( DF(s) \) and \( DH(s) \) are the Jacobian matrices evaluated at \( s = F(s) \). In equation (2), \( \delta_l (k = 1, 2, ..., N) \) show \( N \) non-negative real eigenvalues of \( G \) (symmetric), such that \( 0 = \delta_1 \leq \delta_2 \leq \ldots \leq \delta_N \). So, the structural property becomes \( R = \delta_N / \delta_2 \), as \( \delta_2 \) is minimum non-zero eigenvalue \( (\delta_{\text{min}}) \) and \( \delta_N \) is maximum eigenvalue \( (\delta_{\text{max}}) \). Now to check the network’s synchronizability, we need to evaluate node property \( \beta \) from equation (2). For this, we proceed as follows.

Corresponding to the \( N \) eigenvalues, there are \( N \) maximum Lyapunov exponents, \( \lambda(\gamma_{\delta_k}) \), wherein \( \lambda(\gamma_{\delta_l}) \) \((\lambda(0) \text{ or } \lambda_0)\) describes the state of the synchronized regime as \( \lambda_0 = 0 \) implies periodic and \( \lambda_0 > 0 \) means chaotic. The linear stability of this synchronized state is decided by the remaining \( N - 1 \) maximum Lyapunov exponents (transverse), i.e., the given synchronous state is stable if \( \lambda(\gamma_{\delta_k}) < 0 \) \((k > 1)\), and these \( N - 1 \) exponents can be found from a single variational equation by using scaling relation \( (\gamma_{\delta_l}) \) [2]. This means that at fixed \( \gamma \), the variational equation in the presence of \( \delta_{\text{max}} \) may experience stronger coupling and hence its stabilization destabilization may occur earlier than the other modes. In other words, \( \lambda(\gamma_{\delta_{\text{max}}}) \) plays a vital role in the stability condition wherein all the modes should be simultaneously stabilized or one could say that \( \lambda(\gamma_{\delta_{\text{max}}}) \) can solely provide the stability of a network which is the case and hence it is called MSF for given \( DF(s) \) and \( DH(s) \). This is because, as long as \( G \) is diagonalizable its eigenvalues corresponding to different network topologies as well as sizes can be found from any topology using scaling relation (for more details see [4, 9]). Therefore, MSF of two bidirectionally coupled nodes, i.e., \( \lambda(2\gamma) \), yields same \( \beta(= \gamma_{\text{dyn}} / \gamma_{\text{syn}}) \) as an arbitrary network having \( N \) nodes does, i.e., \( \beta(= \gamma_{\text{syn}} / \gamma_{\text{syn}}) \) where \( \gamma = \gamma_{\delta_{\text{max}}} \). Now we can define \( \gamma_{\text{syn}} \) and \( \gamma_{\text{dyn}} \) (similarly \( \sigma_{\text{syn}}, \sigma_{\text{syn}} \)), i.e., \( 2\gamma_{\text{syn}} \) is the minimum coupling strength at which \( \lambda(2\gamma) = 0 \) or \( \lambda(2\gamma) \rightarrow 0^+ \) (emergence of synchrony) and \( 2\gamma_{\text{dyn}} \) is the maximum coupling strength at which \( \lambda(2\gamma) = 0 \) or \( \lambda(2\gamma) \rightarrow 0^- \) (emergence of desynchrony).

2.1.1. Issue of scalar coupling in complex systems
In contrast to the vector coupling scenario \((\lambda(\sigma \rightarrow \infty) < 0)\), in case of scalar coupling wherein \( \lambda(\sigma) > 0 \) as \( \sigma > \sigma_{\text{dyn}} (\sigma_{\text{syn}} \text{ small}) \), it becomes a challenge to synchronize a ring network topology (figure 1(a)) because its \( R \) (i.e., \( 1 / \sin^2(\pi / N) \)) grows faster with the increase in \( N \) than any other topology, e.g. for a star network \( R = N \). Therefore, the previous works on the enhancement of network’s synchronizability using network modification methods could be considered as the work on the minimization of \( R = 1 / \sin^2(\pi / N) \).

2.2. Drive-response
To understand the framework of drive-response [27, 28], consider the equations of motion for two unidirectionally coupled identical nodes for the conventional SOS (figure 2(a)):

\[
\begin{align*}
\dot{x}^3 &= F(x^3), \\
\dot{x}^4 &= F(x^4) + \gamma \Gamma(x^4 - x^3),
\end{align*}
\]

(3)

where \( \Gamma \) depicts same information as \( H(x) \) does in equation (1), e.g. for \( m = 3 \), i.e., \( x = (x_1, x_2, x_3) \); \( \Gamma = \text{diag}(1,0,0)_{3 \times 3} \) or \( \text{diag}(0,1,0)_{3 \times 3} \) or \( \text{diag}(0,0,1)_{3 \times 3} \) (scalar coupling). Now we choose \( x_1, x_2 \) coordinate for the interactions and assume that the coupling term vanishes, i.e., \( x_2^j = x_1^j \), as \( \gamma \rightarrow \infty \) which is only possible if \( x_2^j = x_1^j \). This means that \( x^2 \) node instead of generating its own \( x^2 \) signal must use \( x^1 \) signal from \( x^1 \) node in order to ensure \( x_2^j = x_1^j \) \((\gamma = \infty)\) for all the time, i.e., even when \( x_2^j \neq x_1^j \) \((j = 2, ..., m)\). Hence, equation (3), becomes \( \dot{x}^2 = F(x^2, x_1^1) \) and \( \dot{x}^4 = (\dot{x}^2, ..., \dot{x}^4) \). In this scenario, equation (3), depicts \( x_1 \)-driving system wherein \( x^4 \) is drive and \( x^4 \) is a response sub system. The maximum conditional Lyapunov exponent \( \lambda(X) \) of sub Jacobian (sub Jacobian) decides the stability of synchronization manifold, i.e., \( X' < 0 \) implies stabilization (for more details see [4]). Thus, this becomes analogous to the MSF behavior of two bidirectionally \( x_1 \)-coupled nodes wherein the condition of stabilized synchronization manifold is \( \lambda(2\gamma) < 0 \). Therefore, by using scaling relation one could also relate with the \( N \) nodes scenario, i.e., \( \lambda(\gamma_{\delta_{\text{max}}}) < 0 \) (as discussed above).
2.2.1. Issue of scalar coupling in secure communication

Since the practical version of drive-response system is transmitter-receiver system in the field of secure communication, the information transfer via a single coordinate remains a primary concern. But in case of SOS, only the specific coordinates of $x$ can be used as a drive (depending upon the choice of oscillator) which sometimes can not provide the stabilization against the parameter fluctuation as the scenario of $x_2$-driving chaotic Rössler (shown in [28]). Secondly, to stabilize a hyperchaotic drive-response usually a scalar signal is generated by using BK method [37] which requires many parameters (2m). Hence, this necessitates some simpler mechanism which could work for both chaotic as well as hyperchaotic scenarios.

2.3. TOS

In contrast to SOS ($\dot{x} = F(x)$), the proposed TOS is made up of two bidirectionally coupled oscillators (identical/nonidentical) by employing their all coordinates (vector coupling) so that the stable synchronization behavior (complete/generalized) could be ensured. Here, one should not confuse the used terminology of two oscillators with two nodes, i.e., in TOS scenario, the dynamics of a single node are generated by a pair of oscillators as:

\[
\dot{x} = F(x, \mu) + \gamma_1 \Gamma_i(y - x),
\]
\[
\dot{y} = F(y, \mu') + \gamma_2 \Gamma_i(x - y).
\]

Similar to $x, y$ is also a $m$-dimensional vector whose autonomous behavior is described by same function $F(y)$ ($R^m \rightarrow R^m$) where $\mu$ and $\mu'$ are the intrinsic parameters of the models. The parameters $\gamma_1$ and $\gamma_2$ represent the intra-coupling constants of TOS and $\Gamma_i$ shows the vector coupling scenario between $x$ and $y$, i.e., $\Gamma_i = \text{diag}(1, \ldots, 1)_{m \times m}$. For the scenario of $\gamma_2 = \theta \gamma_1$ (feedback), TOS could be considered as a two oscillatory representation of the quorum sensing network of $x$ oscillators interacting with each other via a $y$ oscillator (medium) [38]. The parameter $\theta$ represents the population density of $x$ at each node of a network whereas $y$ may have periodic/chaotic dynamics, i.e., other than the conventional steady state behavior used in quorum sensing [35, 36, 38].

Further, the nodes having TOS dynamics may interact with each other in two ways, i.e., (i) each node uses either its $x$ or $y$ (mTOS, also quorum sensing type) and (ii) each node uses both $x$ and $y$ (dTOS), as shown in figures 1(b) and 2(b). It has been found that only former way (mTOS) could lead to TOS effect since the latter way (dTOS) behaves same as SOS (found analytically as well as numerically). Moreover, since threshold increases with the increase in number of oscillators, $\gamma_{\text{syn}}$ also increases as the side effect of mTOS along with the maximization of $\gamma_{\text{dyn}}$. This situation is more clear in terms of a two node network (figure 2) wherein SOS is the case of two oscillators (figure 2(a)) whereas mTOS depicts the scenario of four oscillators (figure 2(b2)).

To demonstrate the TOS effect in terms of MSF ($\lambda(\sigma \rightarrow \infty) < 0$) and drive-response system ($X < 0$) for scalar coupling, we use a ring network of size $N$ (Figure 1) and a two node network (figure 2), respectively. Furthermore, it should be noted that TOS could be applied to any arbitrary network topology since TOS alters only the node dynamics not the network structure, e.g. a ring network of TOS has same R as for SOS, i.e.,

\[
R = 1/\sin^2(\pi/N)\] and $\delta_{\text{min}}^{\text{ring}} = 4\sin^2(\pi/N)$ and $\delta_{\text{max}}^{\text{ring}} = 4$ ($N$ is even [2]).

2.3.1. MSF in case of TOS

The equations of motion for the nodes of a ring network having TOS dynamics (figure 1(b)), are:

\[
\begin{align*}
\dot{x}^i &= F(x^i, \mu) + \gamma_1 \Gamma_i(y^i - x^i) + \gamma \Gamma (x^{i-1} - 2x^i + x^{i+1}), \\
\dot{y}^i &= F(y^i, \mu') + \gamma_2 \Gamma_i(x^i - y^i) + \gamma \Gamma (y^{i-1} - 2y^i + y^{i+1}),
\end{align*}
\]

\[(4)\]

\[
\begin{align*}
\dot{x}^i &= F(x^i, \mu) + \gamma_1 \Gamma_i(y^i - x^i), \\
\dot{y}^i &= F(y^i, \mu') + \gamma_2 \Gamma_i(x^i - y^i) + \gamma \Gamma (y^{i-1} - 2y^i + y^{i+1}).
\end{align*}
\]

\[(5)\]

Corresponding to figures 1(b1) and (b2), equations (4), and (5), respectively, represent the scenarios of $N$ identical diffusively coupled dTOS and mTOS. The bold terms of equations (4) and (5), show the nearest neighborhood interactions form of $\gamma \sum_{j=1}^{N} G_{ij} H(x^i)$ (equation (1)) wherein $\Gamma$ represents scalar coupling scenario (same as equation (3)).

To understand the different behavior of dTOS and mTOS, consider the block diagonalized linear variational equations of equations (4) and (5), by using technique of spatial Fourier modes (given in [39]):

\[
\begin{align*}
\eta_x^k &= (DF(x, \mu) - \gamma_1 \Gamma_i - 4\gamma \sin^2(\pi k/N) \Gamma) \eta_x^k + \gamma_1 \Gamma_i \eta_x^k, \\
\eta_y^k &= (DF(y, \mu') - \gamma_2 \Gamma_i - 4\gamma \sin^2(\pi k/N) \Gamma) \eta_y^k + \gamma_2 \Gamma_i \eta_y^k.
\end{align*}
\]

\[(6)\]
\[ \eta^k = (DF(s, \mu) - \gamma_1 \Gamma_s) \eta^k_x + \gamma_1 \Gamma_s \eta^k_y + \gamma_2 \Gamma_s \eta^k_w, \]
\[ \eta^k = (DF(y_s, \mu') - \gamma_1 \Gamma_s - 4\gamma \sin^2(\pi k/N) \Gamma) \eta^k_x + \gamma_2 \Gamma_s \eta^k_w, \]

where \( \xi^k = (1/N) \sum_{i=1}^{N-1} \xi_{i,j} e^{2\pi ij/2/N}, \eta^k = (1/N) \sum_{i=1}^{N-1} \eta_{i,j} e^{2\pi ij/2/N} \) and \( \xi^k = x^i - s, \eta^k = y^i - y^j. \) Moreover, \( DF(s, \mu) \) and \( DF(y_s, \mu') \) are the Jacobian matrices evaluated at the synchronization manifold: \( x = s, y = y^j; \)
\( \hat{s} = F(s, \mu) + \gamma_1 \Gamma_s (y^j - s), \hat{y} = F(y_s, \mu') + \gamma_2 \Gamma_s (s - y^j). \) The bold terms of equations (6) and (7), depict the form of \( \gamma \delta_1 \) in (equation 2) in case of ring network topology. Equations (6) and (7), are corresponding to equations (4) (dTOS) and (5) (mTOS), respectively. By using \( \eta^k = \gamma_2 \eta^k_x + \gamma_1 \eta^k_y, \) equations (6) and (7), become:

\[ \dot{\eta}^k = (DF(s, \mu) - \gamma_1 \Gamma_s + \gamma_2 \Gamma_s) \eta^k_x + \gamma_2 \Gamma_s \eta^k_w - 4\gamma \sin^2(\pi k/N) \Gamma \eta^k_y, \]
\[ \dot{\eta}^k = (DF(s, \mu) - \gamma_1 \Gamma_s - \gamma_2 \Gamma_s) \eta^k_x + \gamma_2 \Gamma_s \eta^k_w - 4\gamma \sin^2(\pi k/N) \Gamma \eta^k_y, \]

To simplify, we assume \( DF(s, \mu) \approx DF(y_s, \mu') = D \) since TOS due to vector coupling (\( \Gamma_s \)) and strong values of intra-coupling constants (\( \gamma_1, \gamma_2 \)), could behave as a system whose divergence rates from the synchronization manifold (complete/generalized) for both the oscillators could be equal even when \( \delta \mu \neq 0 (\delta \mu = \mu' - \mu). \) This assumption is valid for every oscillator model for a given range of \( \delta \mu \) (similar type of assumption had also been used previously [40]). Therefore using this assumption, equation (8) (dTOS), becomes:

\[ \dot{\eta}^k = (D - 4\gamma \sin^2(\pi k/N) \Gamma) \eta^k_y, \]

and equation (9) (mTOS), becomes:

\[ \dot{\eta}^k = (D - 4\gamma \sin^2(\pi k/N) \Gamma) \eta^k_y + \gamma_2 4\gamma \sin^2(\pi k/N) \Gamma \eta^k_x, \]
\[ \dot{\eta}^k = (D - (\gamma_1 + \gamma_2) \Gamma_s) \eta^k_x + \Gamma_s \eta^k_y. \]

The form of equation (10), evidently shows that dTOS behaves same as SOs (equation 2)). On the contrary, equation (11), depicts that in case of mTOS, the perturbation \( \eta \) also depends upon its x-component (\( \eta^k_x \)) in addition to coupling strength (\( \gamma \)) and eigenmodes (\( k \)). Moreover, since the evolution of \( \eta^k_x \) does not depend on \( \gamma \) and \( k \) (equation (11b)), it could act as a stability factor. However, this analysis (without numerics) does not provide any hint that \( \gamma \)-independent equation (11b), could lead to the maximization of \( \gamma_{\text{dyn}} \). Thus, we need to further investigate by using drive-response framework.

Furthermore, it should be noted that since each \( k \) is twice degenerate in case of ring network, MSF \( \lambda(\gamma_{\text{max}}) \) is obtained by solving the perturbation equations (equations (10) and (11)) for \( k = N/2 \) (maximum eigenvalue), i.e., \( \lambda(4\gamma) = \lambda(\gamma_{\text{max}}). \)

### 2.3.2. Drive-response case in TOS

The equations of motion for two unidirectionally coupled identical nodes having TOS dynamics are:

\[ \dot{x}^1 = F(x^1, \mu) + \gamma_1 \Gamma_s (y^2 - x^1), \dot{y}^1 = F(y^1, \mu') + \gamma_2 \Gamma_s (x^1 - y^1) \]
\[ \dot{x}^2 = F(x^2, \mu) + \gamma_1 \Gamma_s (y^2 - x^2) + \gamma_2 \Gamma_s (y^1 - y^2), \]
\[ \dot{y}^2 = F(y^2, \mu') + \gamma_2 \Gamma_s (x^2 - y^2) + \gamma_1 \Gamma_s (y^1 - y^2), \]
\[ \dot{x}^3 = F(x^3, \mu) + \gamma_1 \Gamma_s (y^1 - x^1), \dot{y}^3 = F(y^3, \mu') + \gamma_2 \Gamma_s (x^1 - y^1), \]

Similar to equation (3), the node \( x^1 - y^1 \) is drive whereas the node \( x^2 - y^2 \) is response. Equations (12) and (13), depict the scenarios of dTOS and mTOS, respectively. Now we choose \( x_1, y_1 \) coordinates for dTOS whereas \( x_1 \) coordinate in case of mTOS for the interactions and assume that the coupling term vanishes at \( \gamma \to \infty \). Hence, equations (12) and (13), become equations (14) and (15), and equations (16) and (17), respectively (same as discussed for equation (3));

\[ \dot{x}^2 = F(x^2, x_1, \mu) + \gamma_1 \Gamma_s (y^2 - x^2), \]
\[ \dot{x}^3 = F(x^3, y_1, \mu') + \gamma_2 \Gamma_s (x^1 - y^2), \]
\[ \dot{x}^3 = F(x^3, x_1, \mu) + \gamma_2 \Gamma_s (y^1 - x^1), \]
\[ \dot{x}^2 = F(y^2, y_1, \mu') + \gamma_2 \Gamma_s (x^2 - y^2). \]

Equations (14) and (15), depict that for dTOS, \( x^2 \) and \( y^3 \) depend only on their respective missing components as \( x_1 \) only drives \( x^2 \) and \( y_1 \) only drives \( y^2 \) which is similar to equation (3). On the other hand, equation (16), shows that in case of mTOS both \( x^2 \) and \( y^2 \) are driven by same \( y_1 \). Now we need to analyze \( y_1 \) driving effect on \( x^3 \) (which is missing in dTOS). For this we decompose only equation (16) (because equation (17), is same as equation (15)): 
The bifurcation parameter and \( F \) becomes the source of TOS effect, i.e., \( g \) depends on the choice of oscillator model. In the present work, the lower bound of the diagonal elements gets balanced by the positive terms \( (\gamma_1/\gamma_2) \) of non-diagonal elements. Therefore, we can say that equation (18) demonstrates the role of \( \gamma \)-independent equation (11b), towards the maximization of \( \gamma_{\text{det}} \). In addition, equation (18), also explains the need of condition 2 (section 1), i.e., \( x_1(y_1) \) should stabilize the unstable fixed point of \( y \) since the TOS effect emerges only due to the additional linear dissipative term \((\gamma_1, x_1)\). Moreover, this also shows the relevance of \( \gamma_1 \), i.e., to induce the TOS effect \( \gamma_{\text{det}} \leq \gamma_1 \leq \gamma_{\text{bd}} \), where \( \gamma_{\text{det}} \) and \( \gamma_{\text{bd}} \) depend on the choice of oscillator model. In the present work, the lower bound \( \gamma_{\text{det}} \) has been found by exploiting the quorum sensing conditions (the upper bound \( \gamma_{\text{bd}} \) has been located just by scanning the intra-coupling parameter space).

Furthermore, it should also be noted that in case of scalar coupling (like \( \Gamma = \text{diag}(1, 0, 0, 0, 0)_{m \times m} \)), the sub-Jacobian matrices sizes for equation (3) (SOS), equation (12) (dTOS) and equation (13) (mTOS) are: \((m - 1) \times (m - 1), (2m - 2) \times (2m - 2)\) and \((2m - 1) \times (2m - 1)\), respectively.

### 2.4. Oscillator models

#### 2.4.1. Chaotic

**Rössler oscillator** [31]: \( F(x, \mu) = [-x_2 - x_3; x_1 + 0.15x_2; 0.15 + x_3(x_1 - d)]' \), \( F(y, \mu') = [-y_2 - y_3; y_1 + \mu_2y_2; \mu_3 + y_3(y_1 - d)]' \) where \( d \) and \( \mu_2 \) are the bifurcation parameters.

**Chua oscillator** [29]: \( F(x, \mu) = [18.5(x_2 - x_3 - g(x_1)); x_1 - x_2 + x_3; -14.97x_2]' \) (chaotic behavior), \( F(y, \mu') = [\mu_2y_2 - y_1 - g(y_1)]; y_1 - y_3 + y_2; -14.97y_3]' \) where \( \mu_2 \) is the bifurcation parameter and \( g(x_1/y_1) = mx_1/y_1 + 0.5(m_0 - m)(abs(x_1/y_1) + 1) - abs(x_1/y_1 - 1) \) (\( m = -0.68; m_0 = -1.31 \)).

**Lorenz oscillator** [30]: \( F(x, \mu) = [10(x_2 - x_3); -x_1x_2 + 28x_3 - x_3; x_1x_2 - 2x_3]' \) (chaotic behavior), \( F(y, \mu') = [10(y_2 - y_1); -y_1y_2 + 28y_3 - y_2; y_1y_2 - \mu_2y_2]' \) where \( \mu_2 \) is the bifurcation parameter.

#### 2.4.2. Hyperchaotic

The employed four-dimensional electronic system [32] (piecewise linear form of hyperchaotic Rössler [33]): \( F(x, \mu) = [-0.05x_1 - 0.5x_2 - 0.62x_3; x_1 + 0.15x_2 + 0.4x_3; -2x_3 + f(x_1); -1.5x_3 + 0.18x_4 + \mu(x_4)]' \), \( F(y, \mu') = [-0.05y_1 - 0.5y_2 - 0.62y_3; y_1 + 0.15y_2 + 0.4y_3; -2y_3 + f(y_1); -1.5y_3 + 0.18y_4 + \mu(x_4)]' \)

1 If \( y \) exhibits steady behavior then for \( \gamma_1 = \gamma_{\text{bd}} \) and \( \gamma_2 = 0, \lambda_0 < 0 \), i.e., the given network should also exhibit steady state behavior [35, 36] where the magnitude of \( \gamma_{\text{det}} \) depends on the chosen model as \( \gamma_{\text{det}}^\text{Lorenz} > \gamma_{\text{det}}^\text{Chua} > \gamma_{\text{det}}^\text{Rössler} \).
where
\[
f(x_1/y_1) = 10(x_1/y_1 - 0.68)\Theta(x_1/y_1 - 0.68) \quad \text{and} \quad h(x_4/y_4) = -0.41(x_4/y_4 - 3.8)\Theta(x_4/y_4 - 3.8).
\]
Here, \(\Theta(\cdot)\) is the Heaviside step function.

### 3. Results and discussions

In the present work, we employ 5 different scenarios: \(x_1\) and \(x_2\)-coupled/driving chaotic Rössler, \(x_3\)-coupled/driving chaotic Chua, \(x_3\)-coupled/driving chaotic Lorenz and \(x_3\)-coupled/driving hyperchaotic oscillator, to generalize the TOS effect, i.e., the stabilization of chaotic/hyperchaotic synchronization manifold \((\lambda < 0)\) or the maximization of \(\gamma_{\text{dyn}}(\lambda(4\gamma \rightarrow \infty) < 0)\) in case of scalar coupling. To reiterate, since these different scenarios implies different DF and/or DH (equation (2)), each scenario has its own MSF. In addition, ‘coupled’ term depicts explicit \(\gamma\) dependence, i.e., the stability is shown by MSF using \(\lambda(4\gamma)\) (ring network) whereas ‘driving’ term shows the drive–response framework wherein the stabilization is given by maximum conditional Lyapunov exponent \(\lambda\) of sub-Jacobian. Furthermore, the robustness of TOS effect has also been shown by varying the TOS parameters \((\gamma_1, \gamma_2)\) and the model parameters of Rössler \((\mu_R)\), Chua \((\mu_C)\) and Lorenz \((\mu_L)\). It has been found that the intra-coupling feedback situation of the TOS parameters (discussed in section 2.3) provides more robustness but lesser \(\beta\) enhancement, in comparison to no feedback situation.

It should be noted that the Lyapunov exponents are calculated by using Wolf’s algorithm [41] and the model equations are integrated by using fourth order Runge–Kutta algorithm (integration step-size = 0.01).

#### 3.1. Rössler

The emergent effect of TOS for Rössler dynamics is given in figures 3–7 wherein figure 3 depicts the response of \(x_2\)-driving Rössler using \(X\) against the variation of its bifurcation parameter \(d\) for both SOS and TOS.
The Lyapunov exponent parallel to synchronization manifold, $\lambda_0$, is same for both dTOS and mTOS since for $k = 0$ mode, dTOS and mTOS follow same equation, i.e, $\dot{\eta}^m = D\eta^m$. 

\(2\) The Lyapunov exponent parallel to synchronization manifold, $\lambda_0$, is same for both dTOS and mTOS since for $k = 0$ mode, dTOS and mTOS follow same equation, i.e, $\dot{\eta}^m = D\eta^m$. 

configurations. Figure 3(a) shows the scenario (which had also been previously reported in figure 10 of [28]) wherein the subsystem ($\lambda_1$-driven Rössler) loses its stability close to the bifurcation points especially at the periodic windows in case of SOS. In contrast to figures 3(a), figure 3(b) evidently demonstrates that the mTOS configuration stabilizes the subsystem at all the domains whereas the dTOS configuration exhibits qualitatively similar behavior as of SOS (consistent with the theory given in section 2.3). Moreover, figure 4 shows the MSF

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**Figure 6.** Plot depicts the robustness and the side effect of mTOS in case of figure 5 at fixed $\gamma_2 = 5$ (no-feedback), i.e., the alterations of $\gamma_{\text{syn}}$ (a), $\beta$ (b), $\gamma_{\text{syn}}$ (c), $\gamma_{\text{syn}}$ (d) and $\lambda_0$ (see footnote 2) (e) against the variation of TOS parameter $\gamma_1$ and the model parameter $\mu_k$. In subplot (a), the yellow region implies mTOS effect ($\lambda(4\gamma \rightarrow \infty) < 0$), blue region shows weak-mTOS effect ($\lambda_{\text{syn}} > 1.5$, $\lambda(4\gamma \rightarrow \infty) > 0$) and no-mesh region represents no-mTOS effect ($\lambda_{\text{syn}} \leq 1.5$, i.e., at $\gamma_1 < \gamma_{\text{syn}}$). In subplot (b), yellow region means strong-enhancement ($\beta > 37$) and blue region represents $\beta$ enhancement ($\beta > 37$) where no-mesh region depicts no-enhancement ($\beta \leq 37$). Subplot (c) depicts the incrementation of $\gamma_{\text{syn}}$ via mTOS (no-mesh for large $\gamma_{\text{syn}}$ at $\gamma_1 \rightarrow 0$, $\gamma_{\text{syn}} \rightarrow \infty$). Since $\beta = \gamma_{\text{syn}}^{-1}$, subplot (d) demonstrates that the enhancement as well as no-enhancement regions of subplot (b) depend on the variation of $\gamma_{\text{syn}}$. Subplots (e) and (a) show that the state of synchronization manifold $\lambda_0$ does not effect $\gamma_{\text{syn}}$.

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**Figure 7.** Plot depicts the robustness and the side effect of mTOS in case of figure 5 at varying $\gamma_2 = 5$ $\gamma_1$ (feedback), i.e., the alterations of $\gamma_{\text{syn}}$ (a), $\beta$ (b), $\gamma_{\text{syn}}$ (c), $\gamma_{\text{syn}}$ (d) and $\lambda_0$ (see footnote 2) (e) against the variation of TOS parameter $\gamma_1$ and the model parameter $\mu_k$. In subplot (a), the yellow region implies mTOS effect ($\lambda(4\gamma \rightarrow \infty) < 0$), blue region shows weak-mTOS effect ($\lambda_{\text{syn}} > 1.5$, $\lambda(4\gamma \rightarrow \infty) > 0$) and no-mesh region represents no-mTOS effect ($\lambda_{\text{syn}} \leq 1.5$, i.e., at $\gamma_1 < \gamma_{\text{syn}}$). In subplot (b), yellow region means strong-enhancement ($\beta > 37$) and blue region represents $\beta$ enhancement ($\beta > 37$) where no-mesh region depicts no-enhancement ($\beta \leq 37$). Subplot (c) depicts the incrementation of $\gamma_{\text{syn}}$ via mTOS (no-mesh for large $\gamma_{\text{syn}}$ at $\gamma_1 \rightarrow 0$, $\gamma_{\text{syn}} \rightarrow \infty$). Since $\beta = \gamma_{\text{syn}}^{-1}$, subplot (d) demonstrates that the enhancement as well as no-enhancement regions of subplot (b) depend on the variation of $\gamma_{\text{syn}}$. Subplots (e) and (a) show that the state of synchronization manifold $\lambda_0$ does not effect $\gamma_{\text{syn}}$. 

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behavior for $x_2$-coupled chaotic Rössler (for SOS and TOS) corresponding to $d = 6.8$ (chaos) of figure 3 at which synchronization manifold is stable for mTOS ($\lambda' < 0$) and unstable for SOS ($\lambda' > 0$). Now it is evident from figure 4 that $\lambda(4\gamma \rightarrow \infty) < 0$ only happens for mTOS configuration (where ‘$\infty$’ means finitely large value).

This also proves that at the large coupling strength MSF exhibits same information as the corresponding drive-response system does, i.e., $\lambda(4\gamma \rightarrow \infty) \approx X$. Analogous to figure 4, MSF behavior shown in figure 5 for $x_1$-coupled chaotic Rössler also illustrates that only mTOS stabilizes the chaotic synchronization manifold or maximizes $\gamma_{\textrm{syn}}$.

Moreover, the insets of figures 4 and 5 show the incrementation of $\gamma_{\textrm{syn}}$ as the side effect of mTOS configuration.

Furthermore, the mTOS effect shown in figure 5 is studied under parameter fluctuations, i.e., by varying TOS’s critical parameter $\gamma_1$ (discussed in section 2.3.2) and the model parameter $\mu_R$ (nonidentical TOS), for no-feedback (figure 6) and feedback (figure 7) scenarios. In contrast to figure 6(a) wherein weak-mTOS effect observed at both small and large values of $\gamma_1$, figure 7(a) evidently demonstrates the robustness of mTOS effect $\lambda(4\gamma \rightarrow \infty) < 0$ over the broader range of parameter space in case of feedback scenario. Moreover, it also suggests that for the induction of mTOS effect at large $\gamma_1$, $\gamma_1$ should be less than $\gamma_2$ (one can say that the upper bound of $\gamma_1$ $\gamma_2$ also depends on $\gamma_2$ along with the model parameters). However, the behavior of incrementation of $\gamma_{\textrm{syn}}$ increases with the increase in $\gamma_2$ which reduces $\beta$ enhancement as depicted by figures 7(b)–(d) in comparison to figures 6(b)–(d)
is

does not effect
g
\(\gamma\) (feedback), i.e., the alterations of \(\gamma_{\text{syn}}\), \(\beta\), \(\gamma_{\text{syn}}\), and \(\lambda_0\) (see footnote (2)) against the variation of TOS parameter \(\gamma\) and the model parameter \(\mu_C\). In subplot (a), the yellow region implies mTOS effect (\(\lambda(4\gamma \to \infty) < 0\)), blue region shows weak-mTOS effect (\(\gamma_{\text{syn}} > 1.3\), \(\lambda(4\gamma \to \infty) > 0\)) and no-mesh region represent no-mTOS effect (\(\gamma_{\text{syn}} \leq 1.3\), i.e., at \(\gamma_1 < \gamma_{\text{crit}}\)). In subplot (b), yellow and blue regions represent \(\beta\) enhancement (\(\beta > 15\)) where no-mesh region depicts no-enhancement (\(\beta \leq 15\)). Subplot (c) depicts the incrementation of \(\gamma_{\text{syn}}\) via mTOS (no-mesh for large \(\gamma_{\text{syn}}\) as \(\gamma_1 \to 0\), \(\gamma_{\text{syn}} \to \infty\)). Since \(\beta = \gamma_{\text{crit}}^{-1}\), subplot (d) demonstrates that the enhancement as well as no-enhancement regions of subplot (b) depend on the variation of \(\gamma_{\text{syn}}\). Subplots (e) and (a) show that the state of synchronization manifold \(\lambda_0\) does not effect \(\gamma_{\text{syn}}\).

Figure 11. For \(x_1\)-coupled chaotic Lorenz, plot depicts the stability of a ring network of \(N\) identical nodes (figure 1) using \(\lambda(4\gamma)\) (MSF) with the increase in coupling strength \(\gamma\) for SOS (dotted black), dTOS (solid black) and mTOS (solid red) scenarios. Figure shows the stabilization via mTOS, i.e., the two values of \(\gamma_{\text{syn}}\) (0.35 and 12.81) in the cases of SOS and dTOS, become a single value (\(= 2.63\)) and first \(\gamma_{\text{syn}}(2.4)\) is maximized. Here, the used TOS parameters are: \(\gamma_1 = 4\), \(\gamma_2 = 20\) and the model parameter of \(\gamma_{\text{crit}} = 2\) (i.e., \(\gamma = x\), identical TOS scenario). Solid blue line depicts zero base line, i.e., at which \(\lambda(4\gamma) = 0\).

wherein \(\gamma_2\) is fixed. Moreover, it should be noted that the extra strong-enhancement observed only in no-feedback scenario (figure 6(b)) is the consequence of decrement of \(\gamma_{\text{syn}}\) due to the stable periodic synchronization manifold, \(\lambda_0 = 0\) (figure 6(e)), which emerge at \(\gamma_1 \to \gamma_2\) for \(\mu_R < 0\) (Rössler’s steady state domain). But this behavior is missing in case of feedback scenario (figure 7(b)) because \(\gamma_1\) increases with the increase in \(\gamma_1\) which results into chaotic dynamics, \(\lambda_0 > 0\) (figure 7(e)), even at \(\mu_R < 0\).

Furthermore, it is worth noticing that analogous to figure 3(b), the independence of \(\gamma_{\text{syn}}\) maximization (figures 6(a) and 7(a)) from the state of synchronization manifold (figures 6(e) and 7(e)) clearly depict the potency of mTOS configuration.

3.2. Chua

Analogous to figure 5, figure 8 demonstrates the mTOS effect for \(x_1\)-coupled chaotic Chua and the robustness of this effect is given in figures 9 and 10 (similar to figures 6 and 7). Here also, the feedback scenario (Figure 10) provides more robustness but lesser \(\beta\) enhancement than the case of no-feedback (figure 9). Moreover, by comparing figure 9(a) with figure 6(a), one could realize the dependence of \(\gamma_{\text{crit}}\) on the oscillator model whereas figure 10(a) depicts that this dependence gets reduced in the presence of feedback (similar to figure 7(a)). But the major difference arises due to change in oscillator model from Rössler to Chua, is the extent of \(\beta\) enhancement as shown in figures 9(b) and 10(b) which is lesser in contrast to figures 6(b) and 7(b). This is because of more augmentation of \(\gamma_{\text{syn}}\) in case of
最大化从同步映射的条件，类似于Rössler的情况。

\( g \) 由于 \( g \) 取定了同步映射 \( g \). 在子图 2.2 中，类似于 Rössler 和 Chua，情况 \( g \) 给出了同步映射的鲁棒性，但存在一定的分歧。

图 12. 图描绘了 mTOS 在图 11 的情况下的鲁棒性和副作用。具体的，当 \( l \) 为定值时，\( g_{\text{syn}} \) 和 \( g_{\text{syn}} \) 与 \( g \) (见脚注 2) (c) 由 \( g \) 变化而 \( g \) 的变化。

图 13. 图描绘了 mTOS 在图 11 的情况下的鲁棒性和副作用。具体的，当 \( l \) 为定值时，\( g_{\text{syn}} \) 和 \( g_{\text{syn}} \) 与 \( g \) (见脚注 2) (c) 由 \( g \) 变化而 \( g \) 的变化。

Chua (figures 9(c), (d) and 10(c), (d)) 与 Rössler (figures 6(c), (d) and 7(c), (d)) 的情况不同。这种差异，figures 9(a) 和 10(a) 与 figures 9(e) 和 10(e) 显示了 \( g_{\text{syn}} \) 的最大化从同步映射的鲁棒性，这与 Rössler 的情况相似。

3.3. Lorenz

Lorenz 与 Rössler 和 Chua 相似，对于 3-coupled chaotic Lorenz 给出的图 11–13 也表现出更多的鲁棒性但更小的 \( \beta \) 增强在反馈（图 13）情况下，而非无反馈（图 12）。
addition, consistent with Rössler and Chua, figures 12(a) and 13(a) in conjunction with figures 12(e) and 13(e), evidently demonstrate the independence of $\gamma_{\text{syn}}$ maximization from the state of synchronization manifold. Again the difference emerge in the extent of $\beta$ enhancement (figures 12(b) and 13(b)) which is much smaller in contrast to Rössler because $\gamma_{\text{syn}}$ augments much more in case of Lorenz.

3.4. Hyperchaotic

Figure 14 demonstrates the mTOS effect, $\lambda(4\gamma \to \infty) < 0$, in case of $x_1$-coupled hyperchaotic electronic system. Moreover, it is worth noticing that since $\lambda(4\gamma \to \infty) < 0$ implies $X < 0$ (as shown for Rössler), we can say that a hyperchaotic drive-response system could also be stabilized without employing BK method [37]. Furthermore, similar to the above chaotic oscillators, in case of hyperchaotic oscillator feedback scenario yields more robustness but lesser $\beta$ enhancement than the scenario of no-feedback and the state of synchronization manifold does not effect the maximization of $\gamma_{\text{syn}}$ (results not shown).

4. Conclusions

In the present work, without altering the network topology and without employing multi parameter BK method (previous approaches), an effort has been made to stabilize the chaotic as well as hyperchaotic synchronization manifold at large coupling strength in case of scalar coupling, just by modifying the node dynamics from a single oscillator (SOS) to a pair of oscillators (TOS). The impact of modified node configuration for both chaotic and hyperchaotic dynamics, has been studied using MSF (for a fundamental non-centralized ring network) along with the drive-response framework (for a two node network). The presented results evidently demonstrate the potency of TOS in mTOS configuration, i.e., $\lambda(4\gamma \to \infty)/X < 0$ or maximization of $\gamma_{\text{syn}}$ under the broad range of parametric fluctuations. Thus, since TOS can be formed just by conventional diffusive coupling, mTOS could be considered as a simple possible solution to the real problems such as overload failure in the real networks like Internet system and power grid system [25, 26], the stability issue of chaotic/hyperchaotic transmitter-receiver system in the field of secure communication [4, 27, 28], etc. Moreover, since TOS alters only the node dynamics not the network structure, TOS could be applied to any arbitrary network topology. It should also be noted that though TOS increases the dimensionality of a node by twice as compared to SOS, the TOS effect could be simply explained by the emergence of unbalanced linear dissipative term in case of mTOS. Furthermore, results also show the enhancement of $\beta$ even when the threshold ($\gamma_{\text{syn}}$) augments to large values. Therefore, we believe that the proposed TOS would play a vital role both in the fields of complex system as well as secure communication wherein the network synchronizability at large coupling strength in case of scalar coupling, remains a primary concern especially for the hyperchaotic systems.

Furthermore, it is also worth to notice that the concept of enhancement of synchronizability or robustness against the perturbations via a node of two functionally identical coupled elements (like identical/, nonidentical Rössler) could be considered as a validation of the theory of biological robustness wherein the functional redundancy plays a vital role as suggested by Kitano in his papers [42].

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ORCID iDs

Harpartap Singh 🏰 https://orcid.org/0000-0003-1970-0800

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