1 Introduction & Preliminaries

The Hadwiger-Finsler inequality is known in literature of mathematics as a generalization of the following

**Theorem 1.1** In any triangle $ABC$ with the side lengths $a, b, c$ and $S$ its area, the following inequality is valid

$$a^2 + b^2 + c^2 \geq 4S\sqrt{3}.$$  

This inequality is due to Weitzenbock, Math. Z, 137-146, 1919, but this has also appeared at International Mathematical Olympiad in 1961. In [7.], one can find eleven proofs. In fact, in any triangle $ABC$ the following sequence of inequalities is valid:

$$a^2 + b^2 + c^2 \geq ab + bc + ca \geq a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \geq 3\sqrt[3]{a^2b^2c^2} \geq 4S\sqrt{3}.$$

A stronger version is the one found by Finsler and Hadwiger in 1938, which states that ([2.])

**Theorem 1.2** In any triangle $ABC$ with the side lengths $a, b, c$ and $S$ its area, the following inequality is valid

$$a^2 + b^2 + c^2 \geq 4S\sqrt{3} + (a - b)^2 + (b - c)^2 + (c - a)^2.$$
In [8.] the first author of this note gave a simple proof only by using AM-GM and the following inequality due to Mitrinovic:

**Theorem 1.3** In any triangle $ABC$ with the side lengths $a, b, c$ and $s$ its semiperimeter and $R$ its circumradius, the following inequality holds

$$s \leq \frac{3\sqrt{3}}{2}R.$$ 

This inequality also appears in [3.].

A nice inequality, sharper than Mitrinovic and equivalent to the first theorem is the following:

**Theorem 1.4** In any triangle $ABC$ with sides of lengths $a, b, c$ and with inradius of $r$, circumradius of $R$ and $s$ its semiperimeter the following inequality holds

$$4R + r \geq s\sqrt{3}.$$ 

In [4.], Wu gave a nice sharpness and a generalization of the Finsler-Hadwiger inequality.

Now, we give an algebraic inequality due to I. Schur ([5.]), namely

**Theorem 1.5** For any positive real numbers $x, y, z$ and $t \in \mathbb{R}$ the following inequality holds

$$x^t(x - y)(x - z) + y^t(y - x)(y - z) + z^t(z - y)(z - x) \geq 0.$$ 

The most common case is $t = 1$, which has the following equivalent form:

$$x^3 + y^3 + z^3 + 3xyz \geq xy(x + y) + yz(y + z) + zx(z + x)$$

which is equivalent to

$$x^3 + y^3 + z^3 + 6xyz \geq (x + y + z)(xy + yz + zx).$$

Now, using the identity $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$ one can easily deduce that

$$2(xy + yz + zx) - (x^2 + y^2 + z^2) \leq \frac{9xyz}{x + y + z}. \quad (\ast)$$

Another interesting case is $t = 2$. We have
\[ x^4 + y^4 + z^4 + xyz(x + y + z) \geq xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) \]

which is equivalent to

\[ x^4 + y^4 + z^4 + 2xyz(x + y + z) \geq (x^2 + y^2 + z^2)(xy + yz + zx).(**) \]

Now, let’s rewrite theorem 1.2. as

\[ 2(ab + bc + ca) - (a^2 + b^2 + c^2) \geq 4S\sqrt{3}.(***) \]

By squaring (***) and using Heron formula we obtain

\[
4 \left( \sum_{cyc} ab \right)^2 + \left( \sum_{cyc} a^2 \right)^2 - 4 \left( \sum_{cyc} ab \right) \left( \sum_{cyc} a^2 \right) \geq 3(a + b + c) \prod(b + c - a)
\]

which is equivalent to

\[
6 \sum_{cyc} a^2b^2 + 4 \sum_{cyc} a^2bc + \sum_{cyc} a^4 - 4 \sum_{cyc} ab(a + b) \geq 3(a + b + c) \prod(b + c - a).
\]

By making some elementary calculations we get

\[
6 \sum_{cyc} a^2b^2 + 4 \sum_{cyc} a^2bc + \sum_{cyc} a^4 - 4 \sum_{cyc} ab(a + b) \geq 3(a + b + c) \left( \sum_{cyc} ab(a + b) - \sum_{cyc} a^3 - 2abc \right).
\]

We obtain the equivalent inequalities

\[
\sum_{cyc} a^4 + \sum_{cyc} a^2bc \geq \sum_{cyc} ab(a^2 + b^2)
\]

\[
a^2(a - b)(a - c) + b^2(b - a)(b - c) + c^2(c - a)(c - b) \geq 0,
\]

which is nothing else than Schur’s inequality in the particular case \( t = 2 \). In what follows we will give another form of Schur’s inequality. That is

**Theorem 1.6** For any positive reals \( m, n, p \), the following inequality holds

\[
\frac{mn}{p} + \frac{np}{m} + \frac{pm}{n} + \frac{9mnp}{mn + np + pm} \geq 2(m + n + p).
\]

**Proof.** We denote \( x = \frac{1}{m}, y = \frac{1}{n} \) and \( z = \frac{1}{p} \). We obtain the equivalent inequality

\[
\frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy} + \frac{9}{x + y + z} \geq \frac{2(xy + yz + zx)}{xyz} \Leftrightarrow 2(xy + yz + zx) - (x^2 + y^2 + z^2) \leq \frac{9xyz}{x + y + z},
\]

which is \((*)\).
2 Main results

In the previous section we stated a sequence of inequalities stronger than Weitzenbock inequality. In fact, one can prove that the following sequence of inequalities holds

\[ a^2 + b^2 + c^2 \geq ab + bc + ca \geq a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \geq 3\sqrt[3]{a^2b^2c^2} \geq 18Rr, \]

where \( R \) is the circumradius and \( r \) is the inradius of the triangle with sides of lengths \( a, b, c \). In this moment, one expects to have a stronger Finsler-Hadwiger inequality with \( 18Rr \) instead of \( 4S\sqrt{3} \). Unfortunately, the following inequality holds true

\[ a^2 + b^2 + c^2 \leq 18Rr + (a - b)^2 + (b - c)^2 + (c - a)^2, \]

because it is equivalent to

\[ 2(ab + bc + ca) - (a^2 + b^2 + c^2) \leq 18Rr = \frac{9abc}{a + b + c}, \]

which is \( (*) \) again. Now, we are ready to prove the first refinement of the Finsler-Hadwiger inequality:

**Theorem 2.1** In any triangle \( ABC \) with the side lengths \( a, b, c \) with \( S \) its area, \( R \) the circumradius and \( r \) the inradius of the triangle \( ABC \) the following inequality is valid

\[ a^2 + b^2 + c^2 \geq 2S\sqrt{3} + 2r(4R + r) + (a - b)^2 + (b - c)^2 + (c - a)^2. \]

**Proof.** We rewrite the inequality as

\[ 2(ab + bc + ca) - (a^2 + b^2 + c^2) \geq 2S\sqrt{3} + 2r(4R + r). \]

Since, \( ab + bc + ca = s^2 + r^2 + 4Rr \), it follows immediately that \( a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr) \). The inequality is equivalent to

\[ 16Rr + 4r^2 \geq 2S\sqrt{3} + 2r(4R + r). \]

We finally obtain

\[ 4R + r \geq s\sqrt{3}, \]

which is exactly theorem 1.4.

The second refinement of the Finsler-Hadwiger inequality is the following
Theorem 2.2 In any triangle ABC with the side lengths a, b, c with S its area, R the circumradius and r the inradius of the triangle ABC the following inequality is valid

\[ a^2 + b^2 + c^2 \geq 4S\sqrt{3 + \frac{4(R - 2r)}{4R + r}} + (a - b)^2 + (b - c)^2 + (c - a)^2. \]

Proof. In theorem 1.6 we put \( m = \frac{1}{2}(b+c-a), n = \frac{1}{2}(c+a-b) \) and \( p = \frac{1}{2}(a+b-c) \). We get

\[
\sum_{cyc} \frac{(b + c - a)(c + a - b)}{a + b - c} + \frac{9(b + c - a)(c + a - b)(a + b - c)}{\sum_{cyc}(b + c - a)(c + a - b)} \geq 2(a + b + c).
\]

Since \( ab + bc + ca = s^2 + r^2 + 4Rr \) (1) and \( a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr) \) (2), we deduce

\[
\sum_{cyc} (b + c - a)(c + a - b) = 4r(4R + r).
\]

On the other hand, by Heron’s formula we have \((b+c-a)(c+a-b)(a+b-c) = 8sr^2\), so our inequality is equivalent to

\[
\sum_{cyc} \frac{(b + c - a)(c + a - b)}{a + b - c} + \frac{18sr}{4R + r} \geq 4s \iff
\sum_{cyc} \frac{(s - a)(s - b)}{(s - c)} + \frac{9sr}{4R + r} \geq 2s \iff \sum_{cyc} (s - a)^2(s - b)^2 + \frac{9s^2r^3}{4R + r} \geq 2s^2r^2.
\]

Now, according to the identity

\[
\sum_{cyc} (s - a)^2(s - b)^2 = \left(\sum_{cyc}(s - a)(s - b)\right)^2 - 2s^2r^2,
\]

we have

\[
\left(\sum_{cyc}(s - a)(s - b)\right)^2 - 2s^2r^2 + \frac{9s^2r^3}{4R + r} \geq 2s^2r^2.
\]

And since

\[
\sum_{cyc} (s - a)(s - b) = r(4R + r),
\]

it follows that

\[
r^2(4R + r)^2 + \frac{9s^2r^3}{4R + r} \geq 4s^2r^2,
\]
which rewrites as
\[
\left( \frac{4R + r}{s} \right)^2 + \frac{9r}{4R + r} \geq 4.
\]
From the identities mentioned in (1) and (2) we deduce that
\[
\frac{4R + r}{s} = \frac{2(ab + bc + ca) - (a^2 + b^2 + c^2)}{4S}.
\]
The inequality rewrites as
\[
\left( \frac{2(ab + bc + ca) - (a^2 + b^2 + c^2)}{4S} \right)^2 \geq 4 - \frac{9r}{4R + r} \iff
\]
\[
\left( \frac{(a^2 + b^2 + c^2) - ((a-b)^2 + (b-c)^2 + (c-a)^2)}{4S} \right)^2 \geq 3 + \frac{4(R - 2r)}{4R + r} \iff
\]
\[
a^2 + b^2 + c^2 \geq 4S \sqrt{3 + \frac{4(R - 2r)}{4R + r}} + (a - b)^2 + (b - c)^2 + (c - a)^2.
\]

**Remark.** From Euler inequality, \( R \geq 2r \), we obtain theorem 1.2.

## 3 Applications

In this section we illustrate some basic applications of the second refinement of Finsler-Hadwiger inequality. We begin with

**Application 1.** In any triangle \( ABC \) with the sides of lengths \( a, b, c \) the following inequality holds
\[
\frac{1}{b + c - a} + \frac{1}{c + a - b} + \frac{1}{a + b - c} \geq \frac{1}{2r} \sqrt{4 - \frac{9r}{4R + r}}.
\]

**Solution.** From
\[
(b + c - a)(c + a - b)(a + b - c) = 4r(4R + r),
\]
it is quite easy to observe that
\[
\frac{1}{b + c - a} + \frac{1}{c + a - b} + \frac{1}{a + b - c} = \frac{4R + r}{2sr}.
\]
Now, applying the inequality
\[
\left( \frac{4R + r}{s} \right)^2 + \frac{9r}{4R + r} \geq 4,
\]

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we get

\[
\left( \frac{1}{b + c - a} + \frac{1}{c + a - b} + \frac{1}{a + b - c} \right)^2 = \frac{1}{4r^2} \left( \frac{4R + r}{s} \right)^2 \geq \frac{1}{4r^2} \left( 4 - \frac{9r}{4R + r} \right).
\]

The given inequality follows immediately. \(\square\)

**Application 2.** In any triangle ABC with the sides of lengths a, b, c the following inequality holds

\[
\frac{1}{a(b + c - a)} + \frac{1}{b(c + a - b)} + \frac{1}{c(a + b - c)} \geq \frac{r}{8R} \left( 5 - \frac{9r}{4R + r} \right).
\]

**Solution.** From the following identity

\[
\sum_{cyc} \frac{(s - a)(s - b)}{c} = \frac{r(s^2 + (4R + r)^2)}{4sR} = \frac{S}{4R} \left( 1 + \left( \frac{4R + r}{p} \right)^2 \right).
\]

Using the inequality

\[
\left( \frac{4R + r}{s} \right)^2 + \frac{9r}{4R + r} \geq 4,
\]

we have

\[
\sum_{cyc} \frac{(s - a)(s - b)}{c} \geq \frac{S}{4R} \left( 5 - \frac{9r}{4R + r} \right).
\]

In this moment, the problem follows easily. \(\square\)

**Application 3.** In any triangle ABC with the sides of lengths a, b, c the following inequality holds

\[
\frac{1}{(b + c - a)^2} + \frac{1}{(c + a - b)^2} + \frac{1}{(a + b - c)^2} \geq \frac{1}{r^2} \left( \frac{1}{2} - \frac{9r}{4(4R + r)} \right).
\]

**Solution.** From \((b + c - a)(c + a - b)(a + b - c) = 4r(4R + r)\), it follows that

\[
(b + c - a)^2 + (c + a - b)^2 + (a + b - c)^2 = 4(s^2 - 2r^2 - 8Rr)
\]

and

\[
(b + c - a)^2 + (c + a - b)^2 + (a + b - c)^2 = 4r^2 \left( (4R + r)^2 - 2s^2 \right).
\]

We get

\[
\frac{1}{(b + c - a)^2} + \frac{1}{(c + a - b)^2} + \frac{1}{(a + b - c)^2} = \frac{1}{4} \left( \frac{(4R + r)^2}{s^2r^2} - \frac{2}{r^2} \right).
\]
Now, applying the inequality
\[
\left( \frac{4R + r}{s} \right)^2 + \frac{9r}{4R + r} \geq 4,
\]
we have
\[
\frac{1}{(b + c - a)^2} + \frac{1}{(c + a - b)^2} + \frac{1}{(a + b - c)^2} \geq \frac{1}{4r^2} \left( 2 - \frac{9r}{4R + r} \right) = \frac{1}{r^2} \left( \frac{1}{2} - \frac{9r}{4(4R + r)} \right).
\]

\textbf{Application 4.} In any triangle $ABC$ with the sides of lengths $a, b, c$ the following inequality holds
\[
\frac{a^2}{b + c - a} + \frac{b^2}{c + a - b} + \frac{c^2}{a + b - c} \geq 3R \sqrt{4 - \frac{9r}{4R + r}}.
\]

\textit{Solution.} Without loss of generality, we assume that $a \leq b \leq c$. It follows quite easily that $a^2 \leq b^2 \leq c^2$ and $\frac{1}{b + c - a} \leq \frac{1}{c + a - b} \leq \frac{1}{a + b - c}$. Applying Chebyshev’s inequality, we have
\[
\frac{a^2}{b + c - a} + \frac{b^2}{c + a - b} + \frac{c^2}{a + b - c} \geq \left( \frac{a^2 + b^2 + c^2}{3} \right) \left( \frac{1}{b + c - a} + \frac{1}{c + a - b} + \frac{1}{a + b - c} \right).
\]
Now, the first application and the inequality $a^2 + b^2 + c^2 \geq 18Rr$ solves the problem.

\textbf{Application 5.} In any triangle $ABC$ with the sides of lengths $a, b, c$ and with the exradii $r_a, r_b, r_c$ corresponding to the triangle $ABC$, the following inequality holds
\[
\frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c} \geq 2 \sqrt{3 + \frac{4(R - 2r)}{4R + r}}.
\]

\textit{Solution.} From the well-known relations $r_a = \frac{S}{s - a}$ and the analogues, the inequality is equivalent to
\[
\frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c} = \frac{2(ab + bc + ca) - (a^2 + b^2 + c^2)}{2S} \geq 2S \sqrt{3 + \frac{4(R - 2r)}{4R + r}}.
\]
The last inequality follows from theorem 2.2 immediately.

\textbf{Application 6.} In any triangle $ABC$ with the sides of lengths $a, b, c$ and with the exradii $r_a, r_b, r_c$ corresponding to the triangle $ABC$ and with $h_a, h_b, h_c$ be the altitudes of the triangle $ABC$, the following inequality holds
\[
\frac{1}{h_ar_a} + \frac{1}{h_br_b} + \frac{1}{h_cr_c} \geq \frac{1}{S} \sqrt{3 + \frac{4(R-2r)}{4R+r}}.
\]

**Solution.** From the well-known relations in triangle \(ABC\), \(h_a = \frac{2S}{a}, r_a = \frac{S}{s-a}\) we have \(\frac{1}{h_ar_a} = \frac{a(s-a)}{2S^2}\). Doing the same thing for the analogues and adding them up we get

\[
\frac{1}{h_ar_a} + \frac{1}{h_br_b} + \frac{1}{h_cr_c} = \frac{1}{2S^2} \left( a(s-a) + b(s-b) + c(s-c) \right).
\]

On the other hand by using theorem 2.2 in the form

\[
a(s-a) + b(s-b) + c(s-c) \geq 2S \sqrt{3 + \frac{4(R-2r)}{4R+r}}.
\]

we obtain the desired inequality. 

**Application 7.** In any triangle \(ABC\) with the sides of lengths \(a, b, c\) the following inequality holds true

\[
\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} \geq \sqrt{3 + \frac{4(R-2r)}{4R+r}}.
\]

**Solution.** From the cosine law we get \(a^2 = b^2 + c^2 - 2bc \cos A\). Since \(S = \frac{1}{2}bc \sin A\) it follows that

\[
a^2 = (b-c)^2 + 4S \left( \frac{1-\cos A}{\sin A} \right).
\]

On the other hand by the trigonometric formulae \(1-\cos A = 2\sin^2 \frac{A}{2}\) and \(\sin A = 2\sin \frac{A}{2} \cos \frac{A}{2}\) we get

\[
a^2 = (b-c)^2 + 4S \tan \frac{A}{2}.
\]

Doing the same for all sides of the triangle \(ABC\) and adding up we obtain

\[
a^2 + b^2 + c^2 = (a-b)^2 + (b-c)^2 + (c-a)^2 + 4S \left( \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} \right).
\]

Now, by theorem 2.2 the inequality follows.

**Application 8.** In any triangle \(ABC\) with the sides of lengths \(a, b, c\) and with the exradii \(r_a, r_b, r_c\) corresponding to the triangle \(ABC\), the following inequality holds

\[
\frac{r_ar_a}{a} + \frac{r_br_b}{b} + \frac{r_cr_c}{c} \geq \frac{s(5R-r)}{R(4R+r)}.
\]
Solution. It is well-known that the following identity is valid in any triangle ABC

\[
\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} = \frac{(4R + r)^2 + s^2}{4Rs}.
\]

So, the inequality rewrites as

\[
\frac{(4R + r)^2}{s^2} + 1 \geq \frac{4(5R - r)}{4R + r},
\]

which is equivalent with

\[
\left(\frac{4R + r}{s}\right)^2 + \frac{9r}{4R + r} \geq 4.
\]

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