Topological Considerations for Tuning and Fingering Stringed Instruments

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Abstract

We present a formal language for assigning pitches to strings for fingered multi-string instruments, particularly the six-string guitar. Given the instrument’s tuning (the strings’ open pitches) and the compass of the fingers of the hand stopping the strings, the formalism yields a framework for simultaneously optimizing three things: the mapping of pitches to strings, the choice of instrument tuning, and the key of the composition. Final optimization relies on heuristics idiomatic to the tuning, the particular musical style, and the performer’s proficiency.

1. Introduction

The ‘guitar fingering problem’ is, less colloquially, the finding of an optimal assignment of pitches to strings and frets. Previous work on this problem [16, 17, 19, 20, 22, 23, 24] takes as fixed three things: the composition, the musical key, and the guitar’s tuning. From these, a fingering is found that demands little technical dexterity, or that at least obeys the mechanical limitations of the instrument.

We generalize this by fixing only the composition, and then simultaneously optimizing over three aspects: the fingering, the tuning, and the key (transposing from, say, G major to A major).

Historical motivation for exploring nonstandard tunings is found in the work of guitarists such as Muddy Waters, Chuck Berry, and Keith Richards [18], John Lennon [4], and especially Michael Hedges [5, 8]. The search for optimality is also driven by the observation that some genres such as blues or Gypsy Jazz [27] typically use only one or two fingers at a time. Strong constraints are thus critical for reconstructing fingering from audio-only recordings.

2. Definitions

A pitch is the fundamental frequency of a sound with a harmonic spectrum, such as that played by many musical instruments. Pitch is measured in log Hz. The logarithm conveniently lets us define an interval, such as an octave or semitone, as a ratio of pitches (2:1 or $\sqrt[12]{2}:1$, respectively)\(^1\)

Pitch is continuous, but as we are discussing fretted stringed instruments, we temper the octave into twelve equal semitones (equal temperament). Then we represent an interval as not a ratio but an integer counting the number of semitones between two pitches. Thus, in the familiar way, 12 represents an octave, 7 a perfect fifth, 1 a semitone. If we fix a reference pitch, then we can similarly represent any pitch as an integer, namely the integer representing the interval from the reference to the pitch in question. In the context of a particular tuning of an instrument, the reference pitch is often that of the lowest string, played unfretted (see below).

A string is defined by what it can do: sound a pitch from a contiguous range bounded at both ends by the fingerboard’s finite length. At any given moment, a string sounds either no pitch or exactly one pitch. When a string sounds its lowest pitch, we call it unfretted or open.

An instrument means a fretted multi-string musical instrument such as a guitar, arpeggione, or lute. We concentrate on the six-string guitar because it is widely played, often retuned, and used in many musical styles.

A fret constrains a string’s playable pitches to a discrete subset. We assume that adjacent frets are one semitone apart, and that all strings have the same frets (a rectangular fingerboard)\(^2\). We assume that strings are fretted by the four fingers of the left hand (not the thumb, although some advanced styles permit the thumb to fret the lowest string). One finger can fret several adjacent strings.

An instrument’s tuning is the set of pitches of its open strings. A tuning vector is that set, sorted by rising pitch. Thus, the number of strings determines the size of the tuning vector. Retuning means a change of tuning, whether or not this is to or from an instrument’s standard tuning (scordatura). We ignore unusual ‘out of order’ tunings such as Congolese mi-composé, standard tuning with the d string up an octave [21]. We also ignore tunings where several strings share the same pitch, such as variations on Lou Reed’s ‘Ostrich’ (D-D-D-D-d-d). In particular, we assume that as one moves across the fingerboard, open pitch strictly rises. Common examples for the guitar include

\(^{1}\)This is a musical interval. In context, there will be no confusion between this and ‘mathematical’ intervals like $[0, 1] \subset \mathbb{R}$.

\(^{2}\)Bending or ‘whamming’ a string to sound a pitch outside equal temperament is an issue separate to fingering, except for the detail that an open string cannot be bent.
standard tuning (E-A-d-g-b-e') and Open G (D-G-d-g-b-d'); hundreds more exist\footnote{We spell examples of tuning vectors with the older Helmholtz notation, not scientific pitch notation (E₂-A₂-D₃-G₁-B₁-E₁)\footnote{Guitarists start counting from the highest string, the one that is easiest to reach. We do the opposite to simplify our notation, and to obey the convention that a tuning’s spelling begins with the lowest string.}}. A chord is a set of pitches sounding simultaneously. (These pitches need not have begun simultaneously.) We abuse the term’s traditional meaning by letting it contain fewer than three pitches, what musicians call an interval or a unison. This is because traditional harmonic analysis concerns us less than the constraints on which strings can play which pitches.

A composition is a finite sequence of chords of finite duration. We concentrate our attention on the left hand, ignoring right-hand techniques like plucking and damping. A fingering for a composition is a mapping from its pitches to (string, fret) pairs.

A pitch value is a pitch constrained to an integer (semitone) value. A pitch class is one of the twelve traditional names C, C♯, D, ..., B. A musical key, such as E major, is a set of seven distinct pitch classes that forms a major or minor scale. One of these pitch classes is special, called the tonic. A pitch’s scale position is its distance above the next lowest tonic pitch of a musical key, measured in semitones. Thus a scale position must lie in $[0, 12)$.

3. Tuning Vectors

Consider a tuning vector $(x₁, ..., x₆)$. Because we start counting from the lowest string, the $xᵢ$ are increasing.\footnote{Values of 4 and 5 are common for guitar tunings, because of how wide the frets are spaced compared to the hand’s size.} We then define the corresponding length-5 tuning vector as:

$$(C₁, ..., C₅) = (x₂ - x₁, x₃ - x₂, ..., x₆ - x₅).$$

Because no two open strings share a pitch, the $xᵢ$ are strictly increasing: $xᵢ ≥ xᵢ₋₁$. Thus each $Cᵢ > 0$. For convenience of summation in eqs. \cite{1, 2, 4}, we also define $C₀ = 0$.

One 5-vector (of intervals) corresponds to many 6-vectors (of pitches). For example, $(5,5,5,4,5)$, abbreviated as 55545, corresponds to both E-A-d-g-b-e' and D-G-c-f-a-d'.\footnote{We abbreviate $Z/12Z$ as $Z₁₂$, since there is no confusion with the latter’s other meanings in number theory.} Converting a 5-vector to a 6-vector, by choosing a pitch for the lowest string, is what we call augmenting or evaluating a 5-vector at a pitch. For example, evaluating 75545 at D yields Drop-D tuning, D-A-d-g-b-e'.

For a given tuning, let $PVᵢ(f)$ be the pitch sounded by string $i$ when fingered at fret $f$, for $i$ from 1 to 6, for $f$ from 0 to 24 (lowest pitch highest, in both cases). As a reference for this tuning, set $PVᵢ(0) = 0$.

Since frets are spaced one semitone apart, $PVᵢ(f) = f + PVᵢ(0)$.

Combining these yields the String Changing Equation

$$PVᵢ(f) = f + \sum_{k=0}^{i-1} C_k$$

for $1 ≤ i ≤ 6$ and $0 ≤ f ≤ 24$. This equation characterizes the relationship between four concepts: pitch value $PV$, string $i$, fret $f$, and tuning vector $(C₁, ..., C₅)$. This notation derives from work by Allen, formalized by Gilles and Jennings.\footnote{To restore closure, we could have constrained the scalars to be integers. But then the scalars come from only the ring $Z$ instead of the field $R$, leaving the space as only a module over $Z$, i.e., an abelian group. This is too limited for our purposes.}

4. Tones

For notational convenience, we call $(i, f) ∈ Z²$ a guitar position, or just a position. As before, $1 ≤ i ≤ 6$ indicates string, and $0 ≤ f ≤ 24$ indicates fret. This lets us define a tone as an ordered triple: (pitch value, scale position, guitar position).

Given a particular tuning and musical key, a guitar can play some tones but not others. For example, in standard tuning in E major, playing the lowest open string yields the tone (0, 0, (1, 0)): pitch value 0, scale position 0, (string 1, fret 0). But the tone (0, 0, (2, 0)) cannot be played: string 2, fret 0 has a different pitch value and scale position than string 1, fret 0. Playing the highest open string in the context of C major yields the tone (24, 4, (6, 0)): 24 semitones higher, 4 above C, sixth string, zeroth fret.

4.1. The Vector Space of Tones

If we momentarily relax some constraints on pitch value, scale position, and the two elements of guitar position, then $Z × Z₁₂ × Z²$ describes the set of all possible tones. We still constrain scale position to lie in $[0, 12)$.

If vector addition is componentwise, and (real) scalar multiplication is multiplication on each element of the vector, then this set is not quite a vector space over the field $R$, as it lacks closure under scalar multiplication. For example, $(8, 7, 6, 5)$ is in $Z × Z₁₂ × Z²$, but not $3.14 × (8, 7, 6, 5)$. To get a vector space, we must extend the $Z$’s to $R$’s.\footnote{We abbreviate $Z/12Z$ as $Z₁₂$, since there is no confusion with the latter’s other meanings in number theory.} This demands continuous values for not only pitch, scale position, and fret number, but also for string number. (Imagine a Haken Continuum\footnote{Imagine a Haken Continuum\footnote{Programmed to change pitch along both horizontal axes.} programmed to change pitch along both horizontal axes.}

Then $Z₁₂$ becomes $R \ pmod {12}$. This is in fact a vector space over $R$. Intuitively, its vectors are directions on a horizontal axes.)

Then $Z₁₂$ becomes $R \ pmod {12}$. This is in fact a vector space over $R$. Intuitively, its vectors are directions on a
clock face and its scalars are the lengths of the clock’s hands. To prove this rigorously, we define $\mathbb{R}_{12}$ as the set of length-1 vectors whose elements lie in $[0, 12]$. For vectors $[x], [y] \in \mathbb{R}_{12}$, we define $[x] + [y] = [(x + y) - 12 \lfloor \frac{x + y}{12} \rfloor]$. For scalars $r \in \mathbb{R}$, we define multiplication as $r \cdot [x] = [rx - 12 \lfloor \frac{rx}{12} \rfloor]$. The proofs of commutativity, distributivity, etc., are then elementary.

As $\mathbb{R}$ and $\mathbb{R}^2$ are themselves vector spaces over $\mathbb{R}$ (the usual Euclidean ones), we then conclude that the set $T$ of all tones is one too, namely the direct sum $\mathbb{R} \oplus \mathbb{R}_{12} \oplus \mathbb{R}^2$.

4.2. Movement of Tones

As the set of all tones is a vector space, we can imagine the subset of \textit{playable} tones as a space within which we can move. It then becomes useful to informally discuss ‘directions of motion’ or ‘orthogonal vectors of tone movement.’ Also, instead of moving from one tone to another, we can equivalently describe one playable tone ‘changing’ to another, as a side effect of changing guitar tuning, musical key, string, or fret.

The directions then correspond to the changing of one of the three elements of a tone (pitch value, scale position, or guitar position). Insofar as these three directions have an obvious unit value, we can even call these orthogonal vectors orthonormal.

4.2.1. Intuitive Structure of Playable Tones

Given a particular guitar tuning and a particular musical key, the subset $S$ of playable tones has a certain structure relative to $T$, the vector space of all possible tones.

We can imagine $S$ as a surface or manifold within $T$. To motivate a derivation of its precise structure, consider the example of tuning in fourths, E-A-d-g-c-f’ (55555 evaluated at E), with the key of A major. To intuit the local structure of $S$, take a note somewhere in the middle of the fingerboard, such as (3, 4): d string, fourth fret, sounding $f\sharp$. Since $f\sharp$ is 9 semitones above A, the corresponding tone in $S$ is $(f\sharp, 9, (3, 4))$. To find nearby members of $S$, change this tone elementwise:

1. If we change pitch value from $f\sharp$ to g, then 9 becomes 10, and (3, 4) becomes (3, 5). More generally, as the finger moves along the string, pitch value, scale position, and fret number move along a Euclidean line.

2. If we change the scale position from 9 to 10, the same thing happens.

3. If we change fret number from 4 to 5, the same thing happens.

4. If we change string from d to g, (3, 4) to (4, 4), then $f\sharp$ becomes b and 9 becomes $9 + 5 \equiv 2 \pmod{12}$. More generally, as the finger moves transversely across the strings, pitch value and scale position (mod 12) move along a Euclidean line, while fret value remains constant.

Within this particular tuning and musical key, scale position can be derived from pitch value: a playable tone with pitch value $x$ must have scale position $(s_0 + x) \pmod{12}$, where $s_0$ is the scale position of pitch value 0, the lowest pitch of the lowest string. This removes one dimension from $S$, leaving at most three. But $S$ also needs at least three dimensions:

- From (1), pitch value is independent of string number.
- From (1), fret number is independent of string number.
- From (4), fret number is independent of pitch value.

Thus a local neighborhood (a topology) in $S$ has exactly three dimensions, or three independent vectors. We cannot call the vectors suggested by (1) through (4) orthogonal, because their dot product may be nonzero. For example, starting again from $(f\sharp, 9, (3, 4))$, moving in ‘direction’ (1) yields $(g, 10, (3, 5))$; in direction (4) yields $(b, 2, (4, 4))$. Computing the dot product,

$$((g, 10, 3, 5) - (f\sharp, 9, 3, 4)) \cdot ((b, 2, 4, 4) - (f\sharp, 9, 3, 4)) = 10 \cdot 1 - 9 \cdot 1 = 1 \cdot 1 + 1 \cdot 1 = 10 \neq 0.$$  

Since $S$ might not contain the origin, it might not be a proper vector subspace of $T$. This dimensionality argument still suggests that $S$ looks locally like a 3-dimensional affine subspace of $T$. But appearances deceive: sections 4.2.2 and 4.2.3 disprove this and find more fitting structures for $S$.

4.2.2. Playable Tones are a PL Manifold

Let $S$ be the set of playable tones in $T$, using the tuning vector 55555 from the example in section 4.2.1. Viewed discretely, each element of $S$ corresponds to one guitar position. Since a guitar position can take on 6 string values and 25 fret values, $|S| = 6 \times 25 = 150$. Viewed continuously, on the other hand, the finite fingerboard imposes bounds on $S$. Guitar position must lie in $[1, 6] \times [0, 24]$. This in turn bounds pitch values. At the bottom, position (1, 0) has pitch value 0 by definition, while at the top, eq. [1] says that position (6, 24) has pitch value $PV_6(24) = 24 + (5 + 5 + 5 + 5 + 5) = 49$.

These bounds prevent $S$ from being closed under either linear or affine combinations, so $S$ can be neither a subspace nor an affine subspace of $T$.

But when we induce a topology on $S$ from the usual topology on $\mathbb{R}^3$, the one whose basis is the set of open
spheres, then \( S \) becomes a manifold within \( T \), a manifold with corners.\(^8\) “Any closed rectangle in \( \mathbb{R}^n \) is a smooth \( n \)-manifold with corners”\(^9\). The closed rectangle here comes from the bounds themselves, \([1, 6] \times [0, 24] \times [0, 49]\). Also, since each string has comfortably more than 12 fingers, each string’s set of pitches spans more than an octave. Thus \( S \) includes all possible values of scale position.

Having jumped from linear algebra to topology, we can now dispense with the requirement of constant-interval tuning such as 44444 or 55555. When the interval between adjacent strings is nonconstant, \( S \) merely relaxes from a manifold to a piecewise linear (PL) manifold\(^{13,14,15}\).

### 4.2.3. The Set of all Playable Tones is O-Minimal on \( \mathbb{R} \)

Recall that the set of tones \( T \) is \( \mathbb{R} \oplus \mathbb{R}_{12} \oplus \mathbb{R}^2 \). Consider \( S \) as a subset of not \( T \) but rather the ‘bigger’ space \( \mathbb{R}^4 \). Relabel as \( x_1, x_2, x_3, x_4 \) the scalars that we have called pitch value, scale position, string, and fret. Then we can define \( S \) in terms of solution sets to inequalities that are in the form of polynomials in the \( x_i \). We begin with the bounds on \( S \), expressed as two inequalities each for pitch value, scale position, string, and fret:

\[
\begin{align*}
x_1 &\geq 0 \land x_1 \leq 49 \\
x_2 &\geq 0 \land x_2 < 12 \\
x_3 &\geq 1 \land x_3 \leq 6 \\
x_4 &\geq 0 \land x_4 \leq 24.
\end{align*}
\]

(Here, \( a \geq b \) abbreviates the conjunction of the equation \( a = b \) and the inequality \( a > b \).) Another Boolean expression specifies the correspondences between pitch value \( x_1 \) and (string, fret) pair \((x_3, x_4)\) found in eq. (1):

\[
\bigvee_{i=1}^{6} \left( i - 1 \leq x_3 < i \right) \land \left( x_1 - x_4 + \sum_{k=0}^{i-1} C_k = 0 \right). \quad (2)
\]

The conjunction of all these is a finite Boolean combination of sets of the forms \( \{(x_1, ..., x_4) : f(x_1, ..., x_4) > 0\} \) and \( \{(x_1, ..., x_4) : g(x_1, ..., x_4) = 0\} \), where all the \( f \)'s and \( g \)'s are polynomials (all of degree 1, incidentally). Hence, \( S \) is a semialgebraic set.

Of the many interesting corollaries that follow from this result, we note particularly that the set \( \{S\} \) of all sets of playable tones—all possible \( S \)'s, from all tunings, in all musical keys—forms an \( \alpha \)-minimal structure on \( \mathbb{R} \). This means that \( \{S\} \) is ‘nice’: it is closed under common operations such as finite unions and intersections, complements, and (via the Tarski-Seidenberg Theorem) projection to lower dimensions\(^{13,14}\).

### 4.2.4. Lines within Playable Tones

The subset of \( S \) that lies on one string has a powerful structural description, namely a line. If we call a tone’s fret position \( x \), the pitch value of the tone’s open string \( b \), and the tone’s pitch value \( y \), then the line \( y = mx + b \) relates \( x \) to \( y \) within \( S \), with the slope \( m \) being \( 1 \). Conjoining six of these equations, one per string, yields a vector equation

\[
y = x + b
\]

where \( x \) is the vector of fret positions, \( b \) is the guitar’s tuning, and \( y \) is the resulting vector of pitch values.

We call eq. (3) the pitch-position relation. Particularly when written as \( y - x - b = 0 \), it describes the shape of \( S \) for any tuning \( b \). More concretely, if we re-tune from \( b_1 \) to \( b_2 \) but wish to leave pitches unchanged, then eq. (3) shows how the fingering’s fret positions must compensate by changing from \( x_1 \) to \( x_2 \):

\[
\begin{align*}
y_1 &= x_1 + b_1 \quad \text{(old tuning)} \\
y_2 &= x_2 + b_2 \quad \text{(new tuning)} \\
y_1 &= y_2 \quad \text{(same pitches)}
\end{align*}
\]

Combining these and solving for \( x_2 \), the new fret positions are \( x_2 = x_1 + (b_1 - b_2) \).

In cryptography, such a difference of vectors, \( b_1 - b_2 \), is called a Caesar shift or Caesar cipher. Thus, we call this difference a cipher. A cipher ‘encodes’ how a change of tuning causes a change of fingering. Section 4.3.2 treats this in detail.

### 4.3. Three Directions of Tone Movement

Recall that a local neighborhood in \( S \) has three dimensions. We can describe these three dimensions functionally:

- **Transformation** means that a tone preserves its pitch value, but changes scale position.
- **Translation** means that a tone preserves its scale position, but changes pitch value.
- **Isomerization** means that a tone preserves both pitch value and scale position, but changes guitar position.

One more term will become useful: re-keying means a change of musical key (e.g., from E major to D major).

### 4.3.1. Three Corollaries

Three common applications arise when practically applying these directions:

- **Transformation usually implies retuning** (with a compensating fret change, to leave pitch unchanged). But transformation could also mean that no retuning
listening to a performance must discover the ‘hidden’ guitar

ing the fingering, The word also alludes to how someone
cipher encodes how to restore the original pitches by chang-
a key, and a fingering, if the tuning or key changes, then the
retuning, re-keying, or both. In other words, given a tuning,

Fig. 1: Iso-pitch lines for standard tuning. Fret pitch
increases from left to right; string pitch increases from
bottom (E) to top (e’). The visibly different slope of the lines
between strings 4 and 5 corresponds to the ‘4’ in standard
tuning’s 55545 tuning vector.

happened: instead, the musical key changed, landing
the same pitch on a different scale position. Transformation
could even mean both retuning and re-keying.

- Translation implies re-keying. For example, in C
major, pitch value G has scale position 7. To remain at
7, but change pitch from G to A, the musical key must
change from C major to D major. For this reason, we
sometimes call translation transposition, the everyday
term for changing the key of a musical composition.

- Isomerization demands neither retuning nor re-
keying. Because of the other two motions, transfor-
mation and translation, without loss of generality we
can constrain isomerization to preserve both guitar
tuning and musical key. Isomerization really implies
a change of string and fret, leaving everything else
constant. It is motion along iso-pitch lines, like
contour lines on a topographic map (fig. 1)9

A cipher is a vector of 6 intervals. A cipher can notate
retuning, re-keying, or both. In other words, given a tuning,
a key, and a fingering, if the tuning or key changes, then the
cipher encodes how to restore the original pitches by chang-
ing the fingering. The word also alludes to how someone
listening before the actual fingering (unravelling redundancy,
section 5.1).

As a retuning example, to notate a change from standard
to Drop-D (E-A-D-g-b-e′ to D-A-d-g-b-e′), the cipher is the
elementwise difference between these two tuning vectors
(starting arbitrarily from C=0, so E=4, A=9, etc.):

\[(4-2, 9-9, 14-14, 19-19, 23-23, 28-28) = (2, 0, 0, 0, 0, 0).\]

To leave pitches unchanged after this retuning, the cipher
indicates that fret numbers increase by 2 on the first string,
and remain unchanged on the other strings. More generally,
the number of nonzero elements in a retuning cipher indicates
how many strings were retuned. As a sign convention
we let positive elements correspond to strings retuned lower,
so elements can be directly added to fret numbers as a
compensating offset to restore pitch.

On the other hand, as a re-keying example, the cipher
\((-2, -2, -2, -2, -2, -2)\) notates a key change down two
semitones, such as from E major to D major10 Again, the
cipher’s elements work as fret offsets: to play the same
composition in the lower key, fret numbers must decrease
by 2 on all strings.

Like all vectors, ciphers add elementwise. If we
combined the previous two examples, perhaps because
the composition was in E major while Drop-D is more
playable in D major, then the resulting cipher would be

\[(0, -2, -2, -2, -2, -2).\]

Because a fret number \(f\) is bounded \((0 \leq f \leq 24)\),
adding an element of a cipher to \(f\) can push it out of range.
When this happens, we replace that tone with another that
has the same pitch value (see section 5.1). This is far more
common with negative cipher elements pushing \(f\) negative
than positive elements pushing \(f > 24\). When \(f < 0\), the
tone often moves only to the next lower string (or lower
still, until \(f\) once again reaches nonnegativity). Avoiding
excessively low strings reduces stretching and jumping
along the fingerboard (see section 5.2). When precise
voice-leading is not critical, the class of tone substitutions
can increase to include octave substitutions. Less rigorous
resolutions of out-of-range fret numbers include replacing
the tone with one whose pitch value is a different element in
the currently sounding chord (“doubling a different note”),
and even outright omission of the offending tone.

5. Constraints

When choosing a fingering, we first obey the mechanical
limits of the fingerboard and then consider more flexible
deradata. The first of these we formalize in the term
redundancy, the second in playability.

9Fig. 1 may mislead insofar as it is drawn in the plane. Its “45-degree”
appearance suggests that isomerization is a linear combination of two other
directions of motion of tones, as if \(S\) had only two dimensions, not the three
proven in section 4.2.1. But the axes of the figure are string number and
fret number, rather than directions of motion. The figure cannot represent
anything about transformation or translation, because it represents neither
tuning vectors nor musical keys. If it is to be interpreted as a picture
through which tones move, the picture is a severely restricted one.

10This same cipher could also notate a retuning up two semitones, such as
from E-A-d-g-b-e′ to F♯-B-e-a-c♯-f♯′. But in practice, a cipher with all
elements equal indicates a change of key.
These constraints apply to not just software, but also to intuitive human searching for better fingering. The ubiquity of chess-playing software that is better than almost any human player has not dulled the appeal of learning to play chess oneself. Similarly, software that finds optimal fingerings hardly makes manual search obsolete. Indeed, a guitarist might prefer an inferior fingering (that took longer to find) to one found by computer but that demanded first typing in the notes, since pleasure is to be found in the hunt itself. This preference would in fact be forced for the many competent guitarists who do not read music, or for compositions that are too improvisational to be amenable to the task of data entry.

5.1. Redundancy

Given a composition and a tuning, the most elementary choice in fingering is: which (string, fret) pair should play a given pitch value?

We formalize this choice in a pitch value’s redundancy, the number of tones that have it as a first element. Mathematically, the redundancy of a pitch value $p$ is

$$\left| \{ i \mid 1 \leq i \leq 6 \land PV_i(0) \leq p \leq PV_i(24) \} \right|$$

or, applying eq. (1),

$$\left| \left\{ i \mid 0 \leq p - \sum_{k=0}^{i-1} C_k \leq 24 \right\} \right|$$

where $p$ is relative to the tuning $\{C_k\}$, that is, pitch value zero corresponds to the lowest open string. For example, in standard tuning, e and G have a redundancy of 2, while E has a redundancy of 1. (Of course, moving a given pitch to a lower string demands a higher fret.)

If we shrink the intervals between adjacent strings of a tuning, then pitch values will increase in redundancy (fig.2). The inevitable compromise is that such a tuning has a smaller interval between its lowest and highest strings, a smaller overall pitch range. This means fewer iso-pitch lines in fig. [topographically speaking, more widely spaced contour lines, that is, less steep terrain]. The varying pitch ‘elevations’ of a composition will then increase the width of both jumps and hand-stretches.

We empirically observe that the more playable tunings are those which have redundancies slightly greater than one for a given composition’s pitch values. Redundancies of 4 or more may cost too much in terms of these jumps and stretches.

The outermost strings necessarily include some pitch values with redundancy 1. For example, the lowest string includes pitch values too low to be played on any other string. So from that redundancy-1 set of pitch values, only one at a time can sound. Chords that include multiple members of that set cannot be played. A similar constraint applies to the highest pitches on the highest string. For example, in standard tuning, both the lowest-string and highest-string redundancy-1 sets have 5 members. The 5’s come directly from the first and last digits of that tuning vector, 5545. The bottom left and top right corners of fig. [render these as groups of 5 points, where each point is a degenerately short iso-pitch line.

5.1.1. Capo

For purposes of optimization, we may ignore capo without loss of generality. A proof of this has two directions. First, say a guitarist somehow finds an acceptable tuning, key, and fingering for a composition. If that happens to use a capo, and we remove the capo, then the same fingering still works when moved down the appropriate number of frets: the musical key will have merely been transposed. Conversely, say someone wants to change musical keys to match a singer’s vocal range, without changing fingering. A retuning may naïvely tighten all strings so far that the guitar neck deforms, but a capo solves this problem more conventionally.

Fig. 3 shows how higher capos unsurprisingly reduce both redundancy and pitch range. This holds for all tunings, not just the standard tuning shown here.
5.2. Playability

In the context of performing a composition, playability is our aim of minimizing the demanded technical dexterity, the mechanical difficulty. Among other things, we wish to avoid wide stretches and large fast jumps. (This is just as true of keyboards as it is of fingerboards.)

Given a particular fingering, we assign a collection of costs to it as suggested by previous work [16] [20, p. 77] [22, p. 501]. To determine an overall cost, these costs are weighted to suit a particular guitarist’s preferences. For example, one person might dread jumps, while another jumps confidently but is limited in chord choice by injury [27] or arthritis.

5.2.1. Stretches

We prefer stretches that span fewer frets. Thus, for any chord we define its span cost. This is 0 for a barre chord (plus open strings, if applicable). It increases monotonically to 1 for a 5-fret span. For larger spans it is $+\infty$, effectively forbidding such spans. At the high end of the fingerboard, where frets are spaced more closely, this may be less important, but a virtuoso comfortable in that stratosphere needs little guidance anyways.

5.2.2. Jumps

We prefer shorter jumps. Closely related to this, we prefer keeping the hand at the low end of the fingerboard. Empirically, this lets open strings play more often; it is also more familiar to beginning players. For two consecutive chords, then, we define their jump cost as a monotonically increasing function of the jump size, that is, the number of frets moved by the index finger. (An absence of jumping yields a zero jump cost.)

5.2.3. Open Strings

We prefer open strings. Thus, for any chord we define its fret cost. This is zero if no strings are fretted, and increases monotonically to one as the number of fretted strings increases to 6. (Four fingers can fret more than four strings, as in barre chords.)

As a side effect of this preference for open strings, a constraint between tunings and musical keys arises. Given a tuning, the musical keys that are generally more playable are those whose pitch classes are found in the tuning’s open strings. This is especially so for pitch classes with harmonically important roles like the tonic, subdominant, and dominant: three or four musical keys satisfy this for most tunings.

As an elementary demonstration of this, guitar music performed in standard tuning often uses the key of E major, where four open strings play harmonically important pitches, namely E, A, b, and e'. Conversely, standard tuning almost never uses the key of E♭, where the only diatonic open strings are d and g, pitches with less important harmonic roles (leading tone and mediant).

When manually exploring fingerings and tunings, a ‘tuning-key path’ may happen: as a new tuning is tried, a new key suggests itself, which may in turn suggest another tuning, and so on.

We prefer slower jumps. For two consecutive chords, we define their speed cost as the duration expected to perform the jump divided by the duration allowed. The expected duration is predicted by Fitts’s Law [11]. Consequently, jumps towards the low end of the fingerboard, where frets are spaced more widely, are less costly. This agrees with empirical observation.

6. Notation

Once an acceptable tuning, key, and fingering is found, tablature notates this result better than a five-line staff. Tablature directly notates fingering, and is thus better for evaluating playability. Although staff notation is better suited to the tools of traditional music theory, it needs awkward diacritical marks to map pitches to particular strings and frets, and the guitar’s tuning must be extraneously indicated.

Note that without a tuning vector, tablature is as meaningless as a staff lacking a clef and key signature.

7. References

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