Exact Solutions in Modified Massive Gravity and Off–Diagonal Wormhole Deformations

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Abstract
There are explored off–diagonal deformations of "prime" metrics in Einstein gravity (for instance, for wormhole configurations) into "target" exact solutions in f(R,T)–modified and massive/bi–metric gravity theories. The new classes of solutions may posses, or not, Killing symmetries and can be characterized by effective induced masses, anisotropic polarized interactions and cosmological constants. For nonholonomic deformations with (conformal) ellipsoid/ toroid and/or solitonic symmetries and, in particular, for small eccentricity rotoid configurations, we can generate wormholes like objects matching external black ellipsoid – de Sitter geometries. We conclude that there are nonholonomic transforms and/or non–trivial limits to exact solutions in general relativity when modified/ massive gravity effects are modelled by off–diagonal and/or nonholonomic parametric interactions.

Keywords: off–diagonal solutions, wormholes and solitons in modified gravity, bi–metric gravity, massive gravity, black ellipsoids and ring configurations.
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1 Introduction
The bulk of physically important exact solutions in gravity theories (for instance, defining black holes and wormholes) are described by metrics with

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two Killing symmetries, see summaries of results in monographs [1, 2]. For such solutions, there are certain "canonical" frames of reference, when the coefficients of fundamental geometric/physical objects depend generically on one or two (from maximum four, in four dimensional, 4-d, theories) spacetime coordinates. This class of metrics can be diagonalized by coordinate transforms, or contain off–diagonal terms generated by rotations. To construct generic off–diagonal solutions parameterized by metrics with six independent coefficients depending generically on three and/or, in general, on all spacetime coordinates is a very difficult technical and geometric task and the physical meaning of such generalized/modified, or Einstein, spacetimes is less clear.

In our works, see [3, 4] and references therein, we elaborated a geometric method which allows us to deform nonholonomically any "prime" diagonal metric into various classes of "target" off–diagonal solutions with one Killing and/or non-Killing symmetries. For deformations on a small parameter, the new classes of target solutions may preserve certain important physical properties of a prime metric (for instance, of a black hole/ring one, or for a wormhole) but possess also new characteristics related to anisotropic polarizations of constants, nonlinear off–diagonal interactions with new symmetries etc.

Wormhole configurations with spacetime handles (shortcuts), nontrivial topology and exotic matter [5] attract attention for theoretical probes of foundations of gravity theories and as possible objects of nature (for reviews, see [6, 7, 8] and references therein). Such solutions are determined in reverse direction when some tunneling metrics of prescribed (for instance, spherical and/or conformal) symmetry are considered and then try to find some corresponding exotic matter sources. A number of interesting and/or peculiar solutions was found when time like curves and respective causality violations are allowed, for stress–energy tensors with possible violation of the null energy conditions. The wormhole subjects where revived some times in connection to black hole solutions, coupling with gauge interactions, singularities, generalized/modified gravity theories etc.

We studied locally anisotropic wormhole and/or flux tubes in five dimensional (5-d) gravity [9, 10, 11]. Such objects can be determined by extra dimensional or warped/ trapped configurations and/or possible ellipsoidal, toroidal, bipolar, solitonic etc gravitational polarizations of vacuum and/or gravitational constants. The corresponding spacetime geometries are described by generic off–diagonal metric with coefficients depending on three or four coordinates and various types of (pseudo) Riemannian or non–Riemannian connections.

In the present paper, we address the problem of constructing deformations of prime wormhole metrics in general relativity, GR, resulting in generic off–diagonal solutions in modified gravity, MG, and theories with nonholo-
onomically induced torsion, effective masses and bi–metric and bi–connection structures. We shall work with two equivalent connections (the Levi–Civita and an auxiliary one) defined by the same metric structure and apply and extend the anholonomic frame deformation method (AFDM, see details in [3, 4], and references therein) of constructing exact solutions in gravity theories.

The idea of the AFDM is to find certain classes of nonholonomic (equivalently, anholonomic/ non-integrable) frames with conventional $2+2+...$, or $3+2+...$, splitting of dimensions on (pseudo) Riemannian spacetime when the (in general, modified) Einstein equations decouple for a correspondingly defined "auxiliary" connection. This results into systems of nonlinear partial differential equations (PDE) which can be integrated in very general forms. The corresponding solutions are with generic off–diagonal metrics and generalized connections. They may depend on all spacetime coordinates via generating and integration functions. The formalism is different from that with more "simple" diagonal ansatz when the Einstein equations are transformed into certain systems of nonlinear ordinary differential equations (ODE). For instance, for the second order ODE, we get only integration constants which are related to certain physical ones like the gravitational constant, a point particle mass and/or electric charge etc following certain asymptotic/boundary conditions.

We argue that it is possible to impose such constraints on a nonholonomic frame structure, via corresponding classes of generating/integration functions, when the "auxiliary" torsion vanishes and we can "extract" solutions for the Einstein gravity theory and various modifications. To provide a physical interpretation of certain off–diagonal exact solutions with one–Killing or non–Killing symmetries is usually a very difficult task. In general, it is not clear if any physical meaning/importance can be found for a derived new class of generalized solutions. Nevertheless, it is possible to elaborate realistic physical models with nonholonomically constrained nonlinear off–diagonal gravitational and matter field interactions if we consider deformations on a small parameter (for instance, small eccentricities for ellipsoid/rotoid configurations). This allows us to construct new classes of off–diagonal solutions determining parametric deformations of wormhole and black hole physical objects resulting into new observable physical effects and more complex spacetime configurations.

The article is organized as follows: We formulate a geometric approach to modified massive gravity theories in section 2. A proof that corresponding gravitational field equations can be decoupled and integrated in general forms with respect to certain classes of nonholonomic frames of references is provided in section 3. The method of off–diagonal deformations of wormhole – de Sitter configurations is outlined in section 4. There are considered small parametric deformations resulting in physically interesting solutions. In section 5, four classes of "locally anisotropic" deformations of original
wormhole metrics are constructed. We deduce spacetime metrics for rotoid deformations of wormholes, consider solitonic waves on such wormholes and (if possible?) black ellipsoids and explore a model with a torus ringing the throat of a wormhole. In a more general context, massive gravity and \( f \)-modifications to configurations with nonholonomically induced (by metric coefficients) torsions are considered. Section 6 is devoted to concluding remarks.

2 Nonholonomic Deformations in Modified Massive Gravity

We outline certain geometric methods on nonholonomic 2+2 spacetime splitting provided in detail in Refs. [3, 4].

2.1 Geometric preliminaries

We shall refer to gravity theories formulated on a four dimensional, 4–d, generalized pseudo–Riemannian manifold \( V \) endowed with metric structure \( g \) and a metric compatible linear connection \( \nabla \equiv \{ \Gamma^\alpha_{\beta\gamma} \} \), connections and the distortion tensor, \( Z = \{ Z^\alpha_{\beta\gamma} \} \), are completely defined by the coefficients \( g = \{ g_{\alpha\beta}(u^\gamma) \} \). To construct a natural splitting (1) following a well–defined geometric principle we can introduce a conventional horizontal (h) and vertical (v) splitting of the tangent space \( TV \), when a non–integrable (equivalently, nonholonomic, or anholonomic) distribution

\[
\mathbf{N} : TV = hV \oplus vV
\]

is determined locally via a set of coefficients \( \mathbf{N} = \{ N^\alpha_i(x, y) \} \); a 2+2 splitting can be parameterized by local coordinates \( u = (x, y) \), \( u^\mu = (x^i, y^a) \), where indices run values \( i,j,... = 1,2 \) and \( a,b,... = 3,4 \).

A h–v–splitting (2) results in a structure of \( \mathbf{N} \)-adapted local bases, \( e_\nu = (e_i, e_a) \), and cobases, \( e^\nu = (e^i, e^a) \), when

\[
e_i = \partial/\partial x^i - N_i^a(u)\partial/\partial y^a, \quad e_a = \partial_a = \partial/\partial y^a,
\]

and

\[
e^i = dx^i, \quad e^a = dy^a + N^a_i(u)dx^i.
\]

The coefficients \( \Gamma^\alpha_{\beta\gamma}, Z^\alpha_{\beta\gamma} \), and \( g_{\alpha\beta} \) are computed with respect to certain (co) frames of reference, \( e_\alpha = e^\alpha_{\nu}(u)\partial_\nu \) and \( e^\alpha = e^\alpha_{\mu}(u)du^\mu \), for \( \partial_\nu := \partial/\partial u^\nu \). The Einstein rule on summation on “up-low” cross indices will be applied if the contrary is not stated. On convenience, "primed", "underlined" etc indices will be used. The local pseudo–Euclidean signature is fixed in the form \(+ + + -\). We shall write boldface letters in order to emphasize that a nonlinear connection, \( \mathbf{N} \)-connection, structure (2) is fixed on spacetime manifold \( V \).
For such frames, there are satisfied the nonholonomy relations

$$[e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha = W^{\gamma}_{\alpha\beta} e_\gamma,$$

with nontrivial anholonomy coefficients

$$W^{\gamma}_{\alpha\beta} = \partial_a N^b \Omega^a_{\gamma b}.$$

We can distinguish the coefficients of geometric objects on $V$ with respect to $N$–adapted (co) frames (3) and (4) and call them, in brief, d–objects. For instance, a vector $Y(u) \in T V$ can be parameterized as a d–vector,

$$Y = Y_i e_i + Y_a e_a,$$

or

$$Y = (hY, vY),$$

with $hY = \{Y^i\}$ and $vY = \{Y^a\}$.

Any metric structure on $V$ can be written (up to general frame/coordinate transforms) in two equivalent forms: with respect to a dual local coordinate basis,

$$g = g^{\alpha\beta} du^\alpha \otimes du^\beta,$$

where

$$g^{\alpha\beta} = \begin{bmatrix} g_{ij} + N^a_i N^b_j g_{ab} & N^e_j g_{ae} \\ N^e_i g_{be} & g_{ab} \end{bmatrix}$$

or as a d–metric,

$$g = g_{\alpha}(u) e^\alpha \otimes e^\beta = g_i(x) dx^i \otimes dx^i + g_a(x, y) e^a \otimes e^a.$$

On a nonholonomic manifold $(V, N)$, we can consider a subclass of linear connections called distinguished connections, d–connections, $D = (hD, vD)$, preserving under parallelism the $N$–connection splitting (2). Any $D$ defines an operator of covariant derivative, $D_X Y$, for a d–vector field $Y$ in the direction of a d–vector $X$. With respect to $N$–adapted frames (3) and (4), the value $D_X Y$ can be computed as in GR but with the coefficients of the Levi–Civita connection substituted by $D = \{\Gamma^\gamma_{\alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{ba})\}$. The respective coefficients are computed for the h–v–components of $D e_\alpha e_\beta$ using $X = e_\alpha$ and $Y = e_\beta$.

A d–connection is characterized by three fundamental geometric objects: the d–torsion, $\mathcal{T}$, the nonmetricity, $\mathcal{Q}$, and the d–curvature, $\mathcal{R}$, all defined by standard formulas

$$\mathcal{T}(X, Y) := D_X Y - D_Y X - [X, Y], \mathcal{Q}(X) := D_X g,$$

$$\mathcal{R}(X, Y) := D_X D_Y X - D_Y D_X X - D_{[X,Y]}.$$

We can compute the corresponding $N$–adapted coefficients,

$$\mathcal{T} = \{T^\gamma_{\alpha\beta} = (T^i_{jk}, T^i_{ja}, T^a_{ji}, T^a_{ba}), T^a_{bc})\}, \mathcal{Q} = \{Q^\gamma_{\alpha\beta}\},$$

$$\mathcal{R} = \{R^\beta_{\gamma\alpha\beta} = (R^i_{hjk}, R^a_{hbk}, R^i_{hja}, R^i_{hja}, R^i_{hba}, R^i_{hba}, R^i_{hca})\},$$

of these geometric objects by introducing $X = e_\alpha$ and $Y = e_\beta$, and $D = \{\Gamma^\gamma_{\alpha\beta}\}$ into above formulas, see details in [3, 4].
It should be noted that the Levi–Civita connection $\nabla$ (in brief, LC) is not a d–connection because it does not preserve under general frame/coordinate transforms the N–connection splitting \footnote{it is uniquely defined by the metric structure $g$ if there are imposed two conditions: $\nabla = 0$ and $\mathcal{Q} = 0$, if $D \rightarrow \nabla$}. Nevertheless, there is a canonical d–connection $\overrightarrow{\nabla}$ also uniquely determined by any geometric data $(g, N)$ following two similar but a bit "relaxed" conditions: 1) it is metric compatible, $Dg = 0$, and 2) with zero h–torsion, $h\overrightarrow{\nabla} = \{\overrightarrow{T}_{ij}k\} = 0$, and zero v–torsion, $v\overrightarrow{\nabla} = \{\overrightarrow{T}_{a}^{b}\} = 0$. This allows us to construct a canonical distortion relation of type \footnote{The N–adapted coefficients of $\overrightarrow{D} = \{\overrightarrow{\Gamma}_{ij}^{k}, \overrightarrow{L}_{jk}^{a}, \overrightarrow{C}_{jc}^{a}, \overrightarrow{C}_{bc}^{a}\}$ and $\overrightarrow{Z}_{a}^{\cdot \cdot}$ depending only on $g_{\alpha \beta}$ and $N^{\alpha}_{\cdot \cdot}$ can be computed following formulas
\begin{align*}
\overrightarrow{L}_{jk}^{a} &= \frac{1}{2}g^{ir}(e_{g}g_{jr} + e_{j}g_{kr} - e_{r}g_{jk}) , \overrightarrow{C}_{jc}^{a} = \frac{1}{2}g^{bd}(e_{c}g_{bd} + e_{b}g_{cd} - e_{d}g_{bc}) \\
\overrightarrow{C}_{bc}^{a} &= \frac{1}{2}g^{ac}(e_{b}g_{ac} - g_{bc}e_{a}N^{c}_{\cdot \cdot} - g_{ab}e_{c}N^{d}_{\cdot \cdot}) , \overrightarrow{Z}_{a}^{\cdot \cdot}
\end{align*}.

\textup{see proofs, for instance, in [3, 4].}$} with respective splitting of N–adapted coefficients $\overrightarrow{\Gamma}_{\alpha \beta} = \Gamma_{\alpha \beta} + \overrightarrow{Z}_{\alpha \beta}$. We can work equivalently with two metric compatible connections $\overrightarrow{D}$ and $\nabla$ because both such geometric objects are completely defined by the same metric structure $g$\footnote{For the canonical d–connection, there are nontrivial d–torsions coefficients, $\overrightarrow{T}_{ij}k = \overrightarrow{L}_{jk}^{i} - \overrightarrow{L}_{kj}^{i}, \overrightarrow{T}_{ja}^{i} = \overrightarrow{C}_{ja}^{i}, \overrightarrow{T}_{ij}^{c} = \overrightarrow{L}_{aj}^{c} - e_{a}(N_{j}^{c}), \overrightarrow{T}_{bc}^{a} = \overrightarrow{C}_{bc}^{a} - \overrightarrow{C}_{cb}^{a}$.}
\footnote{This allows us to compute an "alternative" scalar curvature
\begin{align*}
\overrightarrow{R} := g^{\alpha \beta}\overrightarrow{R}_{\alpha \beta} = g^{ij}\overrightarrow{R}_{ij} + g^{ab}\overrightarrow{R}_{ab}. \tag{11}
\end{align*}

We can also introduce the Einstein d–tensor of $\overrightarrow{D}$,
\begin{align*}
\overrightarrow{E}_{\alpha \beta} := \overrightarrow{R}_{\alpha \beta} - \frac{1}{2}g_{\alpha \beta} \overrightarrow{R}. \tag{12}
\end{align*}

The values $\overrightarrow{R}, \overrightarrow{R}_{ic}$ and $\overrightarrow{R}$ for the canonical d–connection $\overrightarrow{D}$ are different from the similar ones, $\overrightarrow{R}, \overrightarrow{R}_{ic}$ and $R$, computed for the LC–connection $\nabla$.}

Nevertheless, both classes of such fundamental geometric objects are related via unique distorting relations derived from (1) for a N–connection splitting (2). To work with $\hat{D}$ is convenient for various purposes in generalized gravity theories with nontrivial torsion. The most surprising property of the Ricci d–tensor $\hat{Ric} = \{\hat{R}_{\beta\gamma}\}$ is that the corresponding modified Einstein equations of type $\hat{R}_{\beta\gamma} = \Upsilon_{\beta\gamma}$ decouple in very general forms with respect to certain classes of N–adapted frames. This property holds for generic off–diagonal ansatz of type (7) (in principle, depending on all coordinates) and for certain formally diagonalized and N–adapted sources $\Upsilon_{\beta\gamma}$. This allows us to generate various classes of exact solutions in commutative and noncommutative gravity theories with 4–d and higher dimensions spacetimes, see details and examples in Refs. [3, 4, 9, 10, 11]. Such a geometric method of constructing exact solutions in gravity is conventionally called the anholonomic frame deformation method (AFDM).

The AFDM can be used for constructing off–diagonal exact solutions in general relativity (GR) and other theories involving the LC –connection $\nabla$. In such cases, $\hat{D} = \{\hat{\Gamma}_{\alpha\beta}\}$ can be considered as an "auxiliary" connection which together with certain convenient sets of N–coefficients, $N^a_i$, are introduced with the aim to decouple certain systems of nonlinear partial differential equations (PDE) and solve them in very general forms. Such solutions are determined by corresponding classes of generating and integration functions and, in principle, on an infinite number of integration/symmetry parameters. On corresponding integral varieties of solutions, we can impose additional nonholonomic constraints when the torsion (9) vanishes and $\hat{D} \to \nabla$. Such constraints result in first order PDE equations which can be of type

$$\hat{L}^c_{a_j} = e_a(N^c_j), \hat{C}^i_{jb} = 0, \Omega^a_{ji} = 0,$$

These equations can be solved also in very general forms and allows us to extract LC–configurations. We note that if we work from the very beginning with $\nabla$, we can not decouple for general off–diagonal metrics, for instance, the Einstein equations. This is a consequence of generic nonlinearity of gravitational filed equations. The priority of $\hat{D}$ is that we can "relax" a bit the zero torsion conditions, decouple the corresponding nonlinear PDEs for certain convenient systems of reference determined by "flexible" $N^a_i$ and find general classes of solutions. At the end (after a class of generalized metrics and connections was defined), we can constrain nonholonomically/parametrically the nonlinear system and find torsionless configurations.

The main goal of this work is to show that the AFDM allows us to generate exact solutions with nonholonomic deformations of wormhole objects in modified and/or massive gravity.
2.2 Nonholonomic massive f(R,T) gravity

We study modified gravity theories derived for the action

\[ S = \frac{1}{16\pi} \int \delta u^4 \sqrt{|g_{\alpha\beta}|} [f(\hat{R}, T) - \frac{\mu_g^2}{4} U(g_{\mu\nu}, K_{\alpha\beta}) + m_L]. \]  

(14)

Such theories generalize the so–called modified f(R,T) gravity, see reviews and original results in [12, 13, 14], and the ghost–free massive gravity (by de Rham, Gabadadze and Tolley, dRGT) [15, 16, 17]. This evades from certain problems of the bi–metric theory by Hassan and Rosen, [18, 19] and connects us to various recent research in black hole physics and modern cosmology [20, 21, 22]. In this paper, we shall use the units when \( \hbar = c = 1 \) and the Planck mass \( M_{Pl} \) is defined via

\[ M_{Pl}^2 = \frac{1}{8\pi G} \]  

with 4–d Newton constant \( G \).

We write \( \delta u^4 \) instead of \( d^4 u \) because there are used N–elongated differentials (3) and consider the constant \( \mu_g \) as the mass parameter for gravity. The geometric and physical meaning of the values contained in this formula will be explained below.

There are at least three most important motivations to consider in this work such generalized models of gravity: 1) Using nonholonomic deformations described in previous section, we can transform certain classes of solutions in modified gravity into certain equivalent ones for massive gravity. 2) Via off–diagonal gravitational interactions in Einstein gravity, it is possible to mimic various classes of physical effects in modified, massive, bi–metric and bi–connection gravity. 3) The AFDM seems to be an effective geometric tool for constructing exact solutions in such "sophisticate" gravity theories.

In action (14), the Lagrangian density \( m_L \) is used for computing the stress–energy tensor of matter via variation in N–adapted form, using operators (3) and (4), on inverse metric d–tensor (8),

\[ T_{\alpha\beta} = \frac{2}{\sqrt{|g_{\mu\nu}|} m_L} \delta(\sqrt{|g_{\mu\nu}|} m_L) \delta g^{\alpha\beta}, \]

when the trace is computed \( T := g^{\alpha\beta} T_{\alpha\beta} \). The functional \( f(\hat{R}, T) \) modifies the standard Einstein–Hilbert Lagrangian (with \( R \) for the Levi–Civita connection \( \nabla \)) to that for the modified f–gravity but with dependence on \( \hat{R} \) (11) and \( T \). In a large class of generalized cosmological models, we can assume that the stress–energy tensor of the matter is given by

\[ T_{\alpha\beta} = (\rho + p) v_\alpha v_\beta - pg_{\alpha\beta}, \]  

(15)

for the approximation of perfect fluid matter with the energy density \( \rho \) and the pressure \( p \); the four–velocity \( v_\alpha \) being subjected to the conditions \( v_\alpha v^\alpha = 1 \) and \( v^\alpha \hat{D}_\beta v_\alpha = 0 \), for \( m_L = -p \) in a corresponding local N–adapted frame. For simplicity, we can parametrize

\[ f(\hat{R}, T) = \frac{1}{4} f(\hat{R}) + \frac{1}{2} f(T) \]  

(16)

and denote by \( \frac{1}{4} F(\hat{R}) := \partial \frac{1}{4} f(\hat{R})/\partial \hat{R} \) and \( 2F(T) := \partial \frac{1}{2} f(T)/\partial T \).
In addition to the usual $f$–gravity term (in particular, to the Einstein–Hilbert one) in (14), it is considered a mass term with "gravitational mass" $\mu g$ and potential

$$
U/4 = -12 + 6[\sqrt{S}] + [S] - [\sqrt{S}]^2 + \\
\alpha_3 \{ 18[\sqrt{S}] - 6[\sqrt{S}]^2 + 3[\sqrt{S}]^3 + 2[S^{3/2}] - 3[S][[\sqrt{S}] - 2] - 24 \} + \\
\alpha_4 \{ [\sqrt{S}][24 - 12[\sqrt{S}] - [\sqrt{S}]^3] - 12[\sqrt{S}][S] + 2[\sqrt{S}^2][3[S] + 2[\sqrt{S}] + 3[S][4 - [S]]] - 8[S^{3/2}][\sqrt{S} - 1] + 6[S^2] - 24 \},
$$

where the trace of a matrix $S = (S_{\mu\nu})$ is denoted by $[S] := S_{\nu\nu};$ the square root of such a matrix, $\sqrt{S} = (\sqrt{S}_{\nu\mu}),$ is understood to be a matrix for which $\sqrt{S}_\nu \sqrt{S}_\mu = S_{\nu\mu}$ and $\alpha_3$ and $\alpha_4$ are free parameters. This nonlinearly extended Fierz–Pauli type potential was shown to result in a theory of massive gravity which is free from ghost–like degrees of freedom and takes a special form of total derivative in absence of dynamics (see [16, 17] and additional arguments in [23]). The potential generating matrix $S$ is constructed in a special form to result in a d–tensor $K_\nu^\mu = \delta_\nu^\mu - \sqrt{S}_\nu \sqrt{S}_\mu$ characterizing metric fluctuations away from a fiducial (flat) 4–d spacetime. The coefficients

$$
S_\nu^\mu = g^{\alpha\nu} \eta_{\alpha\beta} e_\alpha s_\beta e_\mu s_\nu^\beta,
$$

with the Minkowski metric $\eta_{\alpha\beta} = \text{diag}(1, 1, 1, -1),$ are generated by introducing four scalar St"uckelberg fields $s_\nu,$ which is necessary for restoring the diffeomorphism invariance. Using $N$–adapted values $g^{\nu\alpha}$ and $e_\alpha$ we can always transform a tensor $S_{\mu\nu}$ into d–tensor $S_{\nu\mu}$ characterizing nonholonomically constrained fluctuations. This is possible for the values $K_\nu^\mu, S_\nu^\mu, \sqrt{S}_\nu \sqrt{S}_\mu$ etc even $s_\nu$ transforms as scalar fields under coordinate and frame transforms.

Varying the action (14) in $N$–adapted from for the coefficients of d–metric $g_{\nu\alpha}$ (3), we obtain certain effective Einstein equations, see (12), for the modified massive gravity,

$$
\hat{E}_{\alpha\beta} = \Upsilon_{\alpha\beta},
$$

with source

$$
\Upsilon_{\beta\delta} = e_f \eta G T_{\beta\delta} + e_f T_{\beta\delta} + \mu_2 K T_{\beta\delta}.
$$

The first component in such a source is determined by usual matter fields with energy momentum $T_{\beta\delta}$ tensor but with effective polarization of the gravitational constant $e_f \eta = [1 + 2F/8\pi]/1F.$ The $f$–modification of the energy–momentum tensor also results in the section term as an additional effective source

$$
e_f T_{\beta\delta} = \frac{1}{2}(1f - 1F \hat{R} + 2p^2 F + 2f)g_{\beta\delta} - (g_{\beta\delta} \hat{D}_\alpha \hat{D}^\alpha - \hat{D}_\beta \hat{D}_\delta) 1F/1F
$$

(21)
and "mass gravity" contribution (the third term) is computed as a dimensionless effective stress–energy tensor

\[ K_{\alpha\beta} := \frac{1}{4\sqrt{|g_{\mu\nu}|}} \delta(\sqrt{|g_{\mu\nu}|} \mathcal{U}) \]

\[ = -\frac{1}{12} \mathcal{U} g_{\alpha\beta}/4 - 2S_{\alpha\beta} + 2[\sqrt{S} - 3]\sqrt{S}_{\alpha\beta} + \alpha_3[3(-6 + 4[\sqrt{S}] + [\sqrt{S}]^2 - [S])\sqrt{S}_{\alpha\beta} + 6([\sqrt{S}] - 2)S_{\alpha\beta} - S_{\alpha\beta}^{3/2}] - \alpha_4[24 \left( S_{\alpha\beta}^2 - ([\sqrt{S}] - 1)S_{\alpha\beta}^{3/2} \right) + 12(2 - 2[\sqrt{S}] - [S] + [\sqrt{S}]^2)S_{\alpha\beta} + (24 - 24[\sqrt{S}] + 12[\sqrt{S}]^2 - [\sqrt{S}]^3 - 12[S] + 12[S][\sqrt{S}] - 8[S^{3/2}])\sqrt{S}_{\alpha\beta}]. \]

In "hidden" form, \( K_{\alpha\beta} \) encode a bi–metric configuration with the second (fiducial) d–metric \( f_{\mu\nu} = \eta_{\mu\nu} e_{\alpha\beta} \epsilon_{\mu\nu} s^\nu \) determined by the Stükelberg fields \( s^\nu \). The potential \( \mathcal{U} \) \( \mathcal{U} \) defines interactions between \( g_{\mu\nu} \) and \( f_{\mu\nu} \) via \( \sqrt{S}_\mu = \sqrt{g^{\nu\eta} T_{\alpha\nu}} \) and \( S^\nu := g^{\nu\mu} f_{\mu\nu} \). For simplicity, we shall study in this paper bi–metric gravity models with \( K_{\alpha\beta} = \lambda(x^k) g_{\alpha\beta} \), which can be generated by such \( s^\nu \) when \( g_{\mu\nu} = \iota^2(x^k) f_{\mu\nu} \) up to a nontrivial conformal factor \( \iota^2 \). Using \([18]\), we can compute \( S^\nu := \iota^{-2} \delta^\nu_\mu \) which allows to express the effective polarized anisotropic constant encoding the contributions of \( s^\nu \) as a functional \( \lambda(\iota^2(x^k)) \). In general, the solutions of \([19]\) depend on the type of symmetries of interactions we prescribe for \( f_{\mu\nu} \) which, in our model, are \( N \)–adapted and subjected to additional nonholonomic constraints.

The gravitational field equations \([19]\) are similar to the Einstein ones in GR but for a different metric compatible linear connection, \( \hat{\nabla} \), and with nonlinear "gravitationally polarized" coupling in effective source \( \Upsilon_{\alpha\beta} \) \([20]\). Such nonlinear systems of PDE can be integrated in general forms for any \( N \)–adapted parameterizations

\[ \Upsilon^\alpha = \text{diag} \{ \Upsilon_\alpha : \Upsilon^1_1 = \Upsilon^2_2 = \Upsilon(x^k, y^3) ; \Upsilon^3_3 = \Upsilon^4_4 = v^\alpha \Upsilon(x^k) \}, \]

in particular, if

\[ \Upsilon = v^\alpha \Upsilon = \Lambda = \text{const}, \]

for an effective cosmological constant \( \Lambda \), see details in \([3,4]\). A solution of equations \([19]\) for a source \([22]\) can be modelled effectively by certain classes of solutions generated by a \( N \)–adapted constant coefficients \([23]\) if the generating and integration functions are redefined to mimic certain classes of solutions. This is equivalent to a procedure of fixing a value for the auxiliary scalar curvature \( \tilde{R} \) \([11]\) by frame/coordinate transforms of \( N^a_i \) and related \( N \)–adapted bases which holds true not for arbitrary \( 2+2 \) splitting but for certain classes of nonholonomic frames resulting in decoupling of the generalized Einstein equations and necessary parameterizations for sources. Here we note that \( D^a_i F_{\alpha\beta} = 0 \) in \([21]\) if we prescribe a functional dependence \( \tilde{R} = \text{const} \). For rather general distributions of matter fields and effective
matter, we can prescribe such values for \( \mathbf{T}_{\beta\delta} = \hat{T}(x^k)g_{\beta\delta} \) and \( \mathbf{s}\hat{R} = \hat{\Lambda} \) in (23).

\[
\Upsilon = \Lambda = \eta G \hat{T}(x^k) + \frac{1}{2} \left[ 1 f(\hat{\Lambda}) - \hat{\Lambda} F(\hat{\Lambda}) + 2 p^2 F(\hat{T}) + \frac{2}{\pi} f(\hat{T}) + \mu^2 \lambda(x^k), \right.
\]

\[
\eta = \left[ 1 + \frac{2 F(\hat{T})}{8 \pi} \right] / F(\hat{\Lambda}).
\]

In general, any term may depend on coordinates \( x^i \) but via re-definition of generating functions they can be transformed into certain effective constants, see below the footnote 6. Prescribing values \( \hat{\Lambda}, \hat{T}, \lambda, p \) and functionals \( 1 f \) and \( 2 f \), we describe a nonholonomic matter and effective matter fields dynamics with respect to N–adapted frames.

Finally, we note that the effective source \( \Upsilon_{\beta\delta} = \Lambda \delta_{\beta\delta} \) (via nonholonomic constraints and the canonical d–connection \( \hat{\mathbf{D}} \)) encode all information on modifications of the GR theory to certain classes of \( f \)–modified and/or massive gravity theories. Imposing additional constraints when \( \hat{\mathbf{D}}_{T=0} \rightarrow \nabla \), i.e. solving the equations (13), we extract LC–configurations for above mentioned gravitational models.

3 Decoupling & Integrability of MG Field Eqs

In this section, we formulate and analyze possible conditions on the nonholonomic frame structure and matter fields and effective matter distributions when the gravitational field equations for \( f \)-modified bi–metric field equations decouple and can be integrated in very general forms. We show that such generic off–diagonal solutions depend on various classes of generating and integration functions and parameters. Such modified spacetimes describe nonholonomic deformations of a prime (fiducial and/or well defined metric in GR, for simplicity, taken in a diagonal form with two Killing symmetries) into certain "target" configurations in modified gravity theories.

There are analyzed three classes of target solutions: 1) nonvacuum off–diagonal deformations to Levi–Civita configurations with effective cosmological constants encoding contributions from massive and \( f \)-modified gravity; 2) possible generalizations to nontrivial nonholonomically induced torsion configurations; and 3) nonholonomic deformations on a small parameter.

3.1 Decoupling with respect to N–adapted frames

The local coordinates on a 4–d manifold \( V \) are parameterized in the form \( u^\alpha = (x^i, y^a) = (x^1 = r, x^2 = \theta, y^3 = \varphi, y^4 = t) \) (or, in brief, \( u = (x, y) \)), where indices run values \( i, j, ... = 1, 2 \) and \( a, b, ... = 3, 4 \) and \( t \) is a timelike coordinate. In brief, the partial derivatives \( \partial_\alpha = \partial/\partial u^\alpha \) will be labeled in the forms \( s^\bullet = \partial s/\partial x^1, s' = \partial s/\partial x^2, s^* = \partial s/\partial y^3, s^\circ = \partial s/\partial y^4 \).
We shall study nonholonomic deformations of a prime metric\(^5\)

\[
\mathbf{g} = \mathbf{g}_a(u)^{\alpha} \otimes \mathbf{e}^\beta = \mathbf{g}_i(x) dx^i \otimes dx^j + \mathbf{g}_a(x,y) \mathbf{e}^\alpha \otimes \mathbf{e}^\alpha,
\]

for

\[
\mathbf{e}^\alpha = (dx^i, \mathbf{e}^a = dy^a + \mathbf{N}^a(u) dx^i),
\]

\[
\mathbf{e}_\alpha = (\mathbf{e}_i = \partial/\partial y^a - \mathbf{N}^b(u) \partial/\partial y^b, \ e_a = \partial/\partial y^a),
\]

into a target off-diagonal one

\[
\mathbf{g} = g_a(u) \mathbf{e}^\alpha \otimes \mathbf{e}_\beta = g_i(x) dx^i \otimes dx^j + g_a(x,y) \mathbf{e}^\alpha \otimes \mathbf{e}^\alpha
\]  

(25)

where \(\mathbf{e}^\alpha\) are taken as in \((41)\). Our goal is to generate \(\mathbf{g}\) as an exact solution in a (modified) gravity theory even \(\mathbf{g}\) is not obligatory constrained to the condition to be a solution of any gravitational field equations. For certain bi-metric models, the prime metric \(\mathbf{g}\) can be considered as a fiducial one which via nonholonomic nonlinear gravitational interactions results in a solution in modified/massive gravity. In next sections, we shall take \(\mathbf{g}\) as a wormhole solution in GR and study possible off-diagonal deformations induced in generalized gravity theories. We shall study the conditions when modified gravity effects can be explained alternatively by certain effective nonlinear interactions in GR.

The nontrivial components of the Einstein equations \((19)\) with source \((22)\) parameterized with respect to \(N\)-adapted bases \((3)\) and \((4)\) for a metric ansatz \((25)\) with data \((31)\) for \(\omega = 1\) are

\[
- \hat{R}_1^1 = - \hat{R}_2^2 = \frac{1}{2g_1 g_2} [g_2^{\bullet \bullet} - \frac{g_1^{\bullet \bullet} g_2^{\bullet \bullet}}{2g_1} - \frac{(g_2^1)^2}{2g_2} + g_1^{\bullet \bullet} - \frac{(g_1^1)^2}{2g_1}] = \nu \gamma,
\]

(26)

\[
- \hat{R}_3^3 = - \hat{R}_4^4 = \frac{1}{2h_4 h_4} [h_4^{\bullet \bullet} - \frac{(h_4^1)^2}{2h_4} + \frac{h_4^1 h_4^1}{2h_4}] = \gamma,
\]

(27)

\[
\hat{R}_{3k} = \frac{w_k}{2h_4} [h_4^{\bullet \bullet} + \frac{(h_4^1)^2}{2h_4} - \frac{h_4^1 h_4^1}{4h_4} \frac{\partial h_3}{h_3} + \frac{\partial h_3}{h_3} - \frac{\partial h_3}{h_3}] = 0,
\]

(28)

\[
\hat{R}_{4k} = \frac{h_4}{2h_3} n_{k}^* + \frac{h_4}{h_3} n_{k}^* - \frac{3}{2} \frac{h_4}{h_3} n_{k}^* = 0,
\]

(29)

when the torsionless (Levi-Civita, LC) conditions \((15)\) transform into

\[
w_i^* = (\partial_i - w_i \partial_3) \ln \sqrt{h_3}, (\partial_i - w_i \partial_3) \ln \sqrt{h_4} = 0,
\]

(30)

\[
\partial_k w_i = \partial_i w_k, n_i^* = 0, \partial_i n_k = \partial_k n_i.
\]

Proofs of such formulas (but for other types of sources in GR and commutative and noncommutative Finsler like generalizations) are contained in Refs. \([3, 4]\). The above system of nonlinear PDE possesses an important decoupling property which allows us to integrate step by step such equations.

\(^5\)we consider that such a metric is with two Killing vector symmetries and that in certain systems of coordinates it can be diagonalized
3.2 Generating off–diagonal solutions

We can integrate the Einstein equations (19) for a source (22) if the N–adapted coefficients of a metric (25) are parameterized in the form

\[ g_i = e^{\psi(x^k)}, \quad g_a = \omega(x^k, y^b)h_a(x^k, y^3), \quad N^3 = w_i(x^k, y^3), \quad N^4 = n_i(x^k), \]  

(31)

considering that using frame/coordinate transforms we can satisfy the conditions \( h^*_a \neq 0, \quad \Upsilon_{2.4} \neq 0. \) In a more general context, it is possible to consider any class of metrics which via frame and coordinate transforms can be related to such an ansatz. For parameterizations (31), the system (26)–(29) transforms correspondingly into

\[ \psi^{**} + \psi'' = 2^{\Upsilon} \]  

(32)

\[ \phi^* h^*_4 = 2 h_3 h_4 \Upsilon \]  

(33)

\[ \beta w_i - \alpha_i = 0, \]  

(34)

\[ n_i^* + \gamma n_i^* = 0, \]  

(35)

\[ \partial_i \omega - (\partial_i \phi^* / \phi^*) \omega^* - n_i \omega^3 = 0, \]  

(36)

for

\[ \alpha_i = h^*_4 \partial_i \phi, \quad \beta = h^*_4 \phi^*, \quad \gamma = \left( \ln |h_4^{3/2} / |h_3| \right)^*, \]  

(37)

where

\[ \phi = \ln |h_4^* / \sqrt{|h_3 h_4|}| \]  

(38)

is considered as a generating function. The equation (36) is necessary if we introduce a nontrivial conformal (in the vertical "subspace") factor depending on all four coordinates. It will be convenient to work also with the value \( \Phi := e^\phi. \)

The above systems of nonlinear PDE can be integrated step by step in very general forms following such a procedure:

1. The (32) is just a 2–d Laplace equation which allows us to find \( \psi \) for any given source \( \Upsilon. \)

2. For \( h_a := \epsilon_a z^a_2(x^k, y^3), \) when \( \epsilon_a = \pm 1 \) depending on signature (we do not consider summation on repeating indices in this formula), the equations (33) and (38) are written correspondingly in the form

\[ \phi^* z^*_4 = \epsilon_3 z_4^2(z_3) \Upsilon \quad \text{and} \quad e^\phi z_3 = 2 \epsilon_4 z^*_4. \]  

(39)

Multiplying both equations for nonzero \( z^*_4, \phi^*, \epsilon_3 \) and introducing the result instead of the first equation, this system transforms into

\[ \Phi^* = 2 \epsilon_3 \epsilon_4 z_3 z_4 \Upsilon \quad \text{and} \quad \Phi z_3 = 2 \epsilon_4 z^*_4. \]  

(40)
Taking $z_3$ from the second equation and introducing in the first one, we obtain $[(z_4^2)]^* = \frac{\epsilon_3 \Phi^2}{4\Lambda}$. This allows us to integrate on $y^3$ and write

$$h_4 = \epsilon_4 (z_4)^2 = 0 h_4(x^k) + \frac{\epsilon_3 \epsilon_4}{4} \int dy^3 \frac{\Phi^2}{\Upsilon},$$  

(41)

for an integration function $0 h_4(x^k)$. Using the first equation in (39), we find

$$h_3 = \epsilon_3 (z_3)^2 = \frac{\Phi^*}{\Upsilon} \frac{z_4^4 z_4}{z_4^4} = \frac{1}{2\Upsilon} (\ln |\Phi|)^*(\ln |h_4|)^*.$$  

(42)

For $\Upsilon = \Lambda$, we can redefine the coordinates and $\Phi$, introduce $\epsilon_3 \epsilon_4$ in $\Lambda$ and consider solutions of type

$$h_3[\Phi] = (\Phi^*)^2/\Lambda \Phi^2 \text{ and } h_4[\Phi] = \Phi^2/4\Lambda,$$  

(43)

3. We have to solve algebraic equations for $w_i$ by introducing the coefficients $\epsilon_i$ in $\epsilon_i$ for the generating function $\phi$, or using any equivalent variables $\phi, \Phi$, and/or $\tilde{\Phi}$,

$$w_i = \partial_i \phi/\phi^* = \partial_i \Phi/\Phi^*.$$  

(44)

4. The solution of the equation (35) can be obtained by integrating two times on $y^3$,

$$n_k = 1n_k + 2n_k \int dy^3 h_3/|h_4|^3,$$  

(45)

where $1n_k(x^i), 2n_k(x^i)$ are integration functions.

5. The LC–conditions consist of a set of nonholonomic constraints which can not be solved in explicit form for arbitrary data $\Phi, \Upsilon$ and all types of integration functions $n_k$ and $2n_k$. Nevertheless, we can find explicit solutions if we consider that via frame and coordinate transforms we can chose $2n_k = 0$ and $1n_k = \partial_k n$ with a function $n = n(x^k)$. We emphasize that $(\partial_i - w_i \partial_3) \Phi = 0$ for any $\Phi(x^k, y^3)$ if $w_i$ is defined by (33). Introducing instead of $\Phi$ a new functional $H(\Phi)$, we obtain $(\partial_i - w_i \partial_3) H = \frac{\partial H}{\partial \Phi}(\partial_i - w_i \partial_3) \Phi = 0$. Using formulas (33) for functionals of type $h_4 = H(\Phi(\Phi))$, we solve always the equations $(\partial_i - w_i \partial_3) h_4 = 0$, which is equivalent to the second system of equations in (30) because $(\partial_i - w_i \partial_3) \ln |h_4| \sim (\partial_i - w_i \partial_3) h_4$. For a subclass of generating functions $\Phi = \Phi$ for which

$$(\partial_i \Phi)^* = \partial_i \Phi^*,$$  

(46)

---

*We can always re–define a generating function $\Phi(x^k, y^3) \rightarrow \tilde{\Phi}(x^k, y^3)$ and a source $\Upsilon(x^k, y^3) \rightarrow \Lambda$, reconsidering (30), in a form when $[\Phi^2]^*/4\Lambda = [\Phi^2]^*/4\Lambda$, which allows us to perform a formal integration in (31) and get $h_4 = 0 h_4(x^k) + \epsilon_3 \epsilon_4 [\Phi^2]^*/4\Lambda$. 

---
we compute for the left part of the second equation in (30), \((\partial_i - w_i\partial_3)\ln |h_{33}| = 0\). The first system of equations in (30) can be solved in explicit form if \(w_i\) are determined by formulas (44), and \(h_{3i}[\Phi, \Phi^*]\) and \(h_{4}[\Phi, \Phi^*]\) are chosen respectively for \(\Upsilon = \Lambda\). We can consider \(\Phi = \Phi(\ln |h_{33}|)\) for a functional dependence \(h_{3i}[\Phi, \Phi^*]\). This allows us to obtain the formulas \(w_i = \partial_i|\Phi^*|/|\Phi|^* = \partial_i|\ln |h_{33}||/|\ln |h_{33}||^*\). Taking derivative \(\partial_3\) on both sides of this equation, we get

\[
w_i^* = \frac{(\partial_i|\ln |h_{33}||)^*}{|\ln |h_{33}||^*} - w_i \frac{|\ln |h_{33}||^*}{|\ln |h_{33}||^*}.
\]

If the conditions (46) are satisfied, we can construct generic off–diagonal configurations with \(w_i^* = (\partial_i - w_i\partial_3) \ln |h_{33}|\) which is necessary for the zero torsion conditions. Finally, we note that the conditions \(\partial_kw_i = \partial_tw_k\) from the second line in (30) are solved for any

\[
\tilde{w}_i = \partial_i\tilde{\Phi}/\tilde{\Phi}^* = \partial_i\tilde{A}, \tag{47}
\]

with a nontrivial function \(\tilde{A}(x^k, y^3)\) depending functionally on generating function \(\Phi\).

The class of off–diagonal metrics of type (25) constructed following above steps 1-5 for \(\Upsilon = \Upsilon = \Lambda, \Phi = \Phi = \tilde{\Phi}\) and \(2n_k = 0\) in (15) are determined by quadratic elements of type

\[
ds^2 = e^{\psi(x^k)}[(dx^1)^2 + (dx^2)^2] + \frac{(\tilde{\Phi}^*)^2}{\Phi^2}dy^3 + (\partial_{\psi}\tilde{A}[\Phi])dx^1 = \frac{\tilde{w}_i^2}{4|\Lambda|}[dt + (\partial_n d_x^k)]^2. \tag{48}
\]

We can consider arbitrary generating functions but take the effective cosmological constant \(\Lambda\) for a model of \(f\)–modified massive gravity for a source (24). If \(\Upsilon = \Lambda\) (44) is for a source (22), we obtain an effective pseudo–Riemannian metric with \(N\)–adapted coefficients determined by effective sources in modified gravity. Via nonlinear off–diagonal interactions in GR, corresponding certain effective sources encoding contributions from modified gravity, we mimic both massive gravitational and/or \(f\)–functional contributions. Here we emphasize that off–diagonal configurations (of vacuum and non–vacuum type) are possible even the effective sources from modified bi–metric gravity are constrained to be zero.

For arbitrary \(\psi\) and \(\Upsilon\), and related \(\Phi\), or \(\tilde{\Phi}\), and \(\Lambda\), we can generate off–diagonal solutions of (26)–(29) with nonholonomically induced torsion,

\[
ds^2 = e^{\psi(x^k)}[(dx^1)^2 + (dx^2)^2] + (z_3)^2[dy^3 + \theta_{\Phi}/\Phi^*dx^1]^2 = (z_4)^2[dt + (\partial_n d_x^k)]^2, \tag{49}
\]

for \(\epsilon_3 = 1, \epsilon_4 = -1\), where the functions \(z_3(x^k, y^3)\) and \(z_4(x^k, y^3)\) are defined by formulas (42) and (41). In \(N\)–adapted frames, the ansatz for such solutions define a nontrivial distorting tensor as in \(\tilde{Z} = \{\tilde{Z}_{\beta\gamma}^\alpha\} \) in (1).
3.3 Formal integration via polarization functions

We can not distinguish the coefficients and multiples in a general off–diagonal solution (48) and (49) which are determined by a prime fiducial, $f$–modified and/or any diagonal exact solution in GR. Such contributions mix for general coordinate/frame transforms. Our goal is to find certain parameterizations of target metrics when the coefficients of prime metrics can be defined in explicit form together with possible "gravitational polarizations" of effective constants and nonholonomic deformations of the coefficients of metrics. For certain additional assumptions, such deformations can be parameterized on a small parameter.

3.3.1 Levi–Civita deformations in massive gravity

Metrics of type (25) can be used for constructing nonholonomic deformations $(\hat{g}, \hat{N}, \hat{\Upsilon}, \hat{\Psi}) \rightarrow (\hat{g}, \hat{N}, v^* \hat{\Upsilon}, \hat{\Psi})$, when the prime metric $\hat{g}$ may be, or not, an exact solution of the Einstein or other modified gravitational equations but the target metric $g$ positively defines a generic off–diagonal solution of field equations in a model of gravity.

We are interested in deformations of metrics $\hat{g}(x^k)$ possessing two Killing vector symmetries (in particular, such a metric may define a black hole, or wormhole solution). The $N$–adapted deformations of coefficients of metrics, frames and sources are chosen in the form

$$[g_i, h_a, w_i, n_i] \rightarrow [\tilde{g}_i = \tilde{n}_i g_i, \tilde{h}_3 = \tilde{n}_3 h_3, \tilde{h}_4 = \tilde{n}_4 h_4, \tilde{w}_i = w_i + \eta w_i, n_i = \tilde{n}_i + \eta n_i],$$

$$v^* \Upsilon = v^* \Upsilon(x^k) \Upsilon, \quad v^* \Upsilon(x^k) = \hat{\Upsilon} = \mu^2 \lambda(x^k)(h_3)^{-1}, \Phi^2 = \exp[2\varepsilon] h_3 h_4,$$

where the source $\mu^2 \lambda(x^k)$ for massive gravity is taken as in (21) and the values $\tilde{\eta}_a, \tilde{w}_i, \tilde{n}_i$ and $\varepsilon$ are functions on three coordinates $(x^k, y^3)$ and $\tilde{\eta}_i(x^k)$ depends only on $h$–coordinates. The prime data $\hat{g}_i, h_a, w_i, n_i$ (which can be determined by an exact solution in gravity theory, by any fiducial metric) are given by coefficients depending only on $(x^k)$. The value $\Upsilon$ can be defined from certain physical assumptions on matter and effective sources if $\hat{g}$ chosen as a solution of certain gravitational field equations in a theory of gravity. Conventionally, we can take $\Upsilon = 1$ if, for instance, a general pseudo–Riemannian metric $\hat{g}$ is transformed into a solution of some (generalized) field equations with source $(\Upsilon, \Psi)$.

In terms of $\eta$–functions resulting in $h_3^* \neq 0$ and $g_i = c_i e^{\psi(x^k)}$, the solutions (48) can be re–written in the form

$$ds^2 = e^{\psi(x^k)}[(dx^1)^2 + (dx^2)^2] + \frac{(\varepsilon^*)^2}{\lambda^2 h_3} dy^3 + (\partial_i \eta A) dx^i)^2 - \frac{e^{2\varepsilon}}{4\mu^2} \lambda^2 h_4 [dt + (\partial_k \eta n(x^i))dx^k]^2.$$

The gravitational polarizations $(\eta_i, \eta_a)$ and $N$–coefficients $(w_i, n_i)$ are com-
3.3.2 Induced torsion in massive gravity

The corresponding off–diagonal quadratic element is given by

\[
\eta \frac{\tilde{A}(x^k, y^\lambda)}{\lambda} \eta \psi(x^k) \psi(x^k) = \eta \tilde{A}(x^k, y^\lambda) \eta \psi(x^k) \psi(x^k),
\]

where \( \eta \tilde{A}(x^k, y^\lambda) \) is introduced via formulas and assumptions similar to (46)–(47) and \( \psi \psi + \psi'' = 2 \tilde{\Omega}(x^k) \eta \psi(\psi) \). For \( N\)-coefficients, there are used the parameterizations \( w_i = \tilde{w}_i + \eta w_i = \partial_i (\eta \tilde{A}(\psi)) \), \( n_k = \tilde{n}_k + \eta n_k = \partial_k (\eta n) \),

3.3.3 Small f–modifications and massive gravity

Additional modifications of GR are possible by \( f\)-functionals with an effective source \( \Lambda \) (24). Using two nonholonomic deformations \( (\tilde{g}, \tilde{N}, \tilde{\Omega}, \tilde{\Upsilon}) \rightarrow (g, \dot{N}, \dot{\Omega}, \dot{\Upsilon}) \rightarrow (g[\varepsilon], \dot{N}[\varepsilon], \Lambda) \), we construct off–diagonal solutions type (51) with \( g \) and \( N \) depending on a small parameter \( \varepsilon \), \( 0 < \varepsilon \ll 1 \), when the source in massive gravity \( \mu_g^2 |\lambda| \) is generalized to an effective cosmological constant \( \Lambda \) with additional contributions by matter fields and \( f\)-modifications of gravity. The corresponding \( N\)-adapted transforms are parameterized

\[
\begin{align*}
\tilde{g}_i & = (1 + \varepsilon \lambda \psi(x^k)) \tilde{g}_i, \tilde{h}_3 = (1 + \varepsilon \lambda \psi(x^k)) \tilde{h}_3, \\
\tilde{n}_i & = \tilde{n}_i + \varepsilon \tilde{n}_i,
\end{align*}
\]

with

\[
\begin{align*}
\tilde{w}_i & = \tilde{w}_i + \varepsilon \tilde{w}_i, \\
\tilde{\psi}_i & = \tilde{\psi}_i + \varepsilon \tilde{\psi}_i
\end{align*}
\]

and

\[
\begin{align*}
\mu_g^2 \lambda(x^k) & = \Lambda(1 - \varepsilon \mu \chi(x^k)), \\
\psi \psi^* & = \psi^*(1 + \varepsilon \chi(x^k, y^\lambda)),
\end{align*}
\]
where the values $\chi_i(x^k), \lambda \chi(x^k), \overline{m}_i(x^k), \omega \chi(x^k, y^3), \chi_a(x^k, y^3)$ and $\overline{w}_i(x^k, y^3)$ can be computed to define LC-configurations as solutions of the system (26)–(30).

The deformations (52) of the off–diagonal solutions (50) result in a new class of $\varepsilon$–deformed solutions if

$$\begin{align*}
\chi_3 &= \mu \chi + \omega \chi, \\
\chi_4 &= \mu \chi + \omega^{-1} \int dy^3 (\omega \chi \omega^3), \\
\overline{w}_i &= \partial_i (\omega \chi \sqrt{|\overline{h}_3 h_4|}) / \omega^\infty \sqrt{|h_3 h_4|} = \partial_i \overline{A}, \\
n_i &= \partial_i n.
\end{align*}$$

The coefficients for the $h$–metric $g_i = \exp \psi(x^i) = (1 + \varepsilon \chi_i) \tilde{\eta}_i \tilde{g}_i$ are solutions of (26) with $\tilde{\Upsilon} = \Lambda = \tilde{\Upsilon}(x^k) + \mu_2^2 \lambda$, where $\tilde{\Upsilon}(x^k)$ is determined by possible contributions of matter fields and $f$–modifications parametrized in (24).

In next sections, we shall construct such solutions in explicit form for ellipsoid, toroid and solitonic deformations. If $\varepsilon$–deformations of type (52) are considered for metrics (51), we can generate new classes of off–diagonal solutions with nonholonomically induced torsion determined both by massive and $f$–modifications of GR.

4 Off–diagonal Deformations of Wormhole Metrics

In this section, we construct and analyze two examples when a wormhole solution matching an exterior Schwarzschild – de Sitter spacetime is nonholonomically deformed into new classes of off–diagonal solutions. The target metrics are constructed for modifications of GR with effectively polarized cosmological constants and "polarization" multiples and additional terms to, respectively, diagonal and non–diagonal coefficients of metrics. The deformations resulting from massive gravity are studied for an effectively polarized cosmological constant proportional to $\mu_2^2$. Modifications determined by $f$–terms are computed for a small deformation parameter $\varepsilon$.

4.1 Prime metrics for 4–d wormholes

Let us consider a diagonal prime wormhole metric

$$\begin{align*}
\tilde{g} &= \tilde{g}_i(x^k) dx^i \otimes dx^i + \tilde{h}_a(x^k) dy^a \otimes dy^a \\
&= [1 - b(r)/r]^{-1} dr \otimes dr + r^2 (d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi) - \varepsilon^2 B(r) dt \otimes dt,
\end{align*}$$

where $B(r)$ and $b(r)$ are called respectively the red–shift and form functions, see details in [5, 6, 7, 8]. The radial coordinate has a range $r_0 \leq r < a$, where the minimum value $r_0$ is for the wormhole throat and $a$ is the distance at which the interior spacetime joins to an exterior vacuum solution ($a \rightarrow \infty$ for specific asymptotically flat wormhole geometries). Certain conditions have
to be imposed on coefficients of \( T^\mu_\nu \) and on diagonal components of the stress–energy tensor

\[
T^\mu_\nu = \text{diag}[ \tau p = \tau(r), \theta p = p(r), \varphi p = p(r), t p = \rho(r)]
\]  

in order to generate wormhole solutions of the Einstein equations in GR.

A well known class of wormhole metrics is constructed to possess conformal symmetry determined by a vector \( X = \{X^\alpha(u)\} \), when the Lie derivative \( \mathcal{L}_X g_{\mu\nu} + g_{\alpha\nu} \partial_\mu X^\alpha + g_{\alpha\mu} \partial_\nu X^\alpha = \sigma g_{\mu\nu} \), where \( \sigma = \sigma(u) \) is the conformal factor. Such solutions are parameterized by

\[
B(r) = \frac{1}{2} \ln(C^2 r^2) - \kappa \int r^{-1} (1 - b(r)/r)^{-1/2} dr, \quad b(r) = r[1 - \sigma^2(r)],
\]

\[
\tau(r) = \frac{1}{\kappa^2 r^2} (3\sigma^2 - 2\kappa\sigma - 1), \quad p(r) = \frac{1}{\kappa^2 r^2} (\sigma^2 - 2\kappa\sigma + \kappa^2 + 2\sigma\sigma^*),
\]

\[
\rho(r) = \frac{1}{\kappa^2 r^2} (1 - \sigma^2 - 2\sigma\sigma^*).
\]

The data (55) generate “diagonal” wormhole configurations determined by “exotic” matter because the null energy condition (NEC), \( T^\mu_\nu k^\mu k_\nu \geq 0 \), \((k^\nu \) is any null vector), is violated.

We shall study configurations which match the interior geometries to an exterior de Sitter one which (in general) can be also determined by an off–diagonal metric. The exotic matter and effective matter configurations are considered to be restricted to spacial distributions in the throat neighborhood which limit the dimension of locally isotropic and/or anisotropic wormhole to be not arbitrarily large.

### 4.2 Parametric deformations and exterior de Sitter space-times

The Schwarzschild – de Sitter (SdS) metric

\[
ds^2 = q^{-1}(r)(dr^2 + r^2 d\theta^2) + r^2 \sin^2 \theta \, d\varphi^2 - q(r) \, dt^2,
\]

can be re–parameterized for any \((x^1(r, \theta), x^2(r, \theta), y^3 = \varphi, y^4 = t)\) when

\[
q^{-1}(r)(dr^2 + r^2 d\theta^2) = e^{\varphi(x^h)}[(dx^1)^2 + (dx^2)^2].
\]

Such a metric defines two real static solutions of the Einstein equations with cosmological constant \( \Lambda \) if \( M < 1/3 \sqrt{|\Lambda|} \), for \( q(r) = 1 - 2M(r)/r, M(r) = M + \Lambda r^3/6 \), where \( M \) is a constant mass parameter. For diagonal configurations, we can identify \( \Lambda \) with the effective cosmological constant (24).

In this work, we study conformal, ellipsoid and/or solitonic/toroidal deformations related in certain limits to the Schwarzschild – de Sitter metric written in the form

\[
\Lambda g = d\xi \otimes d\xi + r^2(\xi) \, d\theta \otimes d\theta + r^2(\xi) \sin^2 \theta \, d\varphi \otimes d\varphi - q(\xi) \, dt \otimes dt,
\]
for local coordinates

\[ x^1 = \xi = \int dr / \sqrt{|q(r)|}, \quad x^2 = \vartheta, \quad y^3 = \varphi, \quad y^4 = t, \quad (58) \]

for a system of \( h \)-coordinates when \((r, \theta) \rightarrow (\xi, \vartheta)\) with \( \xi \) and \( \vartheta \) of length dimension. The data for this primary metric are written

\[ \tilde{g}_i = \tilde{g}_i(x^k) = e^{\tilde{\psi}(x)}(d\xi^2 + d\vartheta^2) + \left(\frac{\varpi^*}{\mu_2} \right)^2 r^2(\xi) \sin^2(\theta(\xi, \vartheta))[d\varphi + (\partial_\xi \eta_1 A) d\xi + (\partial_\vartheta \eta_2 A) d\vartheta]^2 - e^{2\varpi} \tilde{n}_i(x) = e^{2\varpi} \tilde{n}_i, \quad \tilde{n}_i = 0, \quad \tilde{n}_i = 0. \]

Let us analyze how such diagonal metrics can be off–diagonally deformed by contributions from massive and \( f \)–modified gravity:

### 4.2.1 Off–diagonal de Sitter deformations in massive gravity

Solutions resulting in the Levi–Civita configurations can be generated similarly to (50) but using data (57)

\[ ds^2 = e^{\tilde{\psi}(\xi, \vartheta)}(d\xi^2 + d\vartheta^2) + \left(\frac{\varpi^*}{\mu_2} \right)^2 r^2(\xi) \sin^2(\theta(\xi, \vartheta))[d\varphi + (\partial_\xi \eta_1 A) d\xi + (\partial_\vartheta \eta_2 A) d\vartheta]^2, \quad (59) \]

where \( e^{\tilde{\psi}(\xi, \vartheta)} = \tilde{g}_1 \tilde{g}_2 = \tilde{g}_1 \tilde{g}_2 \) are solutions of \( \tilde{\psi}^{\bullet \bullet} + \tilde{\psi} = 2 \mu_2 \lambda(\xi, \vartheta) \). The generating function \( \varpi(\xi, \vartheta, \varphi) \), effective source \( \lambda(\xi, \vartheta) \) and mass parameter \( \mu_2 \) should be fixed from physical assumptions on systems of reference, fixed prime Stükelberg fields (using algebraic conditions of type (18)) and observable effects in modern cosmology. The value \( \tilde{n}_i = \eta \tilde{n}_i(\xi, \vartheta) = \partial_\xi \eta n(\xi, \vartheta) \) is an integration function and \( \eta \tilde{A}(\xi, \vartheta, \varphi) \) is determined by \( e^{2\varpi} \) following formula (47) and

\[ \tilde{w}_i = e^{\varpi} w_i = \partial_\xi \eta \tilde{A} = \frac{\partial_\xi \eta \tilde{A}}{e^{\varpi}(\xi) \sin(\theta(\xi, \vartheta)) \sqrt{|q(\xi)|}} = \partial_\xi \eta \tilde{A}, \quad \text{for} \quad x^i = (\xi, \vartheta). \quad (60) \]

It should be noted here that the \( N \)-coefficients in (59) result in nonzero anholonomy coefficients (6) for nondolonomic relations of type (4). This proves that such solutions can not be diagonalized via frame/coordinate transforms and that, in general, they are characterized by six (from possible ten) independent coefficients of metrics. We can mimic such configurations by off–diagonal interactions in GR with corresponding effective matter source determined by terms induced by \( \mu_2 \) taken as an integration parameter. It can be related to Killing symmetries of such metrics, see details in Ref. [24].
4.2.2 Ellipsoidal $f$–modifications

Deformations (52) on a parameter $\varepsilon$, $0 \leq \varepsilon < 1$, are considered for the solutions in massive gravity (59), with

$$\chi_3 = \varpi \chi, \chi_4 = \varpi^{-1} \int d\varphi (\varpi \chi^*),$$

$$\varpi_i = \frac{\partial_i (\varpi \chi r(\xi, \vartheta) \sin \theta(\xi, \vartheta) \sqrt{|q(\xi)|})}{\varpi^* \varpi r(\xi) \sin \theta(\xi, \vartheta) \sqrt{|q(\xi)|}} = \partial_i \varpi_i = \partial_i \varpi,$$

for $x^i = (\xi, \vartheta)$ and we fix, for simplicity, $^\mu \chi = 0$ (a possible physical motivation is to consider models with constant mass gravity parameter and zero related polarization). The coefficients of the $h$–metric $g_{ij} = \exp \psi(\xi, \vartheta) = (1 + \varepsilon \chi_i) \tilde{\eta} \delta_{ij}$ are solutions of (26) with $^v \Upsilon = \Lambda = \tilde{\Upsilon}(x^k) + \mu^2 g$, where $\tilde{\Upsilon}(\xi, \vartheta)$ is determined by possible contributions of matter fields and $f$–modifications parameterized in (24). The resulting target off–diagonal quadratic element is parameterized in the form

$$d^2 s = e^{2\omega(\xi, \vartheta)} \left( d\xi^2 + d\vartheta^2 + \frac{(\varpi^*)^2}{\mu^2 \lambda(\xi, \vartheta)} [1 + \varepsilon \chi_3(\xi, \vartheta, \varphi)] r^2(\xi) \sin^2 \theta(\xi, \vartheta)(\delta \varphi)^2 \right.$$

$$- \frac{e^{2\omega}}{4 \mu^2 \lambda(\xi, \vartheta)} [1 + \varepsilon \chi_4(\xi, \vartheta, \varphi)] q(\xi)(\delta t)^2, \quad \delta \varphi = d\varphi + \tilde{w}_i(\xi, \vartheta, \varphi) d\xi^i, \quad \delta t = dt + \tilde{n}_i(\xi, \vartheta, \varphi) d\xi^i,$$

when $\tilde{w}_i$ are given by formulas (60). For such small deformations re–parameterized in $(r, \theta)$–coordinates, the coefficient

$$h_4 = - \frac{e^{2\omega}}{4 \mu^2 \lambda(\xi, \vartheta)} [1 + \varepsilon \chi_4] q \simeq - \frac{e^{2\omega}}{4 \mu^2 \lambda(\xi, \vartheta)} [1 - \frac{2M(r, \theta, \varphi)}{r}]$$

is related to small gravitational polarizations of mass coefficients,

$$M(r, \theta, \varphi) \simeq \bar{M}(r)[1 + \varepsilon (1 - \frac{r}{2M(r, \theta, \varphi)}) \chi_4(r, \theta, \varphi)].$$

We generate rotoid $f$–deformations if

$$\chi_4 = \chi_4(r, \varphi) := \frac{2\bar{M}(r)}{1 - \frac{2\bar{M}(r)}{r}} \left( 1 - \frac{2\bar{M}(r)}{r} \right)^{-1} \zeta \sin(\omega_0 \varphi + \varphi_0),$$

for some constants $\zeta, \omega_0$ and $\varphi_0$, taken as a polarization function. With respect to $N$–adapted frames, there is a smaller "ellipsoidal horizon" (when $h_4 = 0$ in (62), we get the parametric equation for an ellipse),

$$r_+ \simeq \frac{2 \bar{M}(r_+)}{1 + \varepsilon \zeta \sin(\omega_0 \varphi + \varphi_0)}.$$
where $\varepsilon$ is the eccentricity parameter. Using formulas (61) for a prescribed value $\varpi(r, \theta, \phi)$, and $\chi_3 = \frac{\varpi}{\varpi} = \frac{\partial_\varpi[\varpi_4]}{\partial_\varphi \varpi}$, we compute

$$\overline{w}_i = \frac{\partial_i \left( r(\xi) \sin \theta(\xi, \vartheta) \right) \sqrt{\varpi(\xi)} \left[ \partial_\varphi \varpi \right]}{\varpi(\xi) \sqrt{\varpi(\xi)} \left[ \partial_\varphi \varpi \right]} = \partial_i \chi_3.$$ 

The resulting solutions in massive $f$–gravity with rotoid symmetry are parameterized

$$\begin{align*}
\mathbf{ds}^2 &= e^{\tilde{\psi}(\xi, \varphi)} \left( d\xi^2 + d\vartheta^2 + \frac{(\varpi^2)^2}{\mu_g^2 \lambda^2(1 + \varepsilon \frac{\partial_\varphi \varpi}{\partial_\varphi \varpi})^2} r^2(\xi) \sin^2 \theta(\xi, \vartheta) (\delta \varphi)^2 \\
&\quad - \frac{e^{2\varpi(\xi, \varphi)}}{4\mu_g^2 \lambda^2} \left[ 1 + \varepsilon \varpi(\xi) (\delta t)^2, (64) \right], \\
\delta \varphi &= d\varphi + \left[ \partial_\xi \nabla + \varepsilon \partial_\varpi \right] dx^i, \delta t = dt + [\partial_\xi n + \varepsilon \partial_\varpi] dx^i.\end{align*}$$

Such stationary configurations are generated by nonlinear off–diagonal interactions in massive gravity with nontrivial $\mu_g^2 \lambda(\xi, \vartheta)$ terms. We note that, in general, the limit $\mu_g \rightarrow 0$ is not smooth for such classes of solutions. There are necessary additional assumptions on nonholonomic constraints resulting in diagonal metrics with two–Killing symmetries or for selecting black rotoid – de Sitter configurations. It is possible to model such solutions via locally anisotropic effective polarizations of the coefficients of metrics and physical constants (treated as integration functions and constants) in GR. For this class of solutions, the contributions related to "massive" gravity terms are very different from those generated by $f$–deformations. In the last case, there are smooth limits for $\varepsilon \rightarrow 0$, when (for instance) rotoid symmetries may transform into spherical ones.

5 Ellipsoid, Solitonic & Toroid Deformations of Wormhole Metrics

In this section, we explore rotoid deformations of wormhole configurations determined by off–diagonal effects in massive gravity and $f$–modifications. The general ansatz for such metrics is taken in the form

$$\begin{align*}
\mathbf{ds}^2 &= e^{\tilde{\psi}(\tilde{\xi}, \varphi)} \left( d\tilde{\xi}^2 + d\varphi^2 + \frac{[\partial_\varphi \varpi(\tilde{\xi}, \vartheta, \varphi)]^2}{\mu_g^2 \lambda(\tilde{\xi}, \varphi)} \left( 1 + \varepsilon \frac{\partial_\varphi \varpi(\tilde{\xi}, \vartheta, \varphi) \varpi(\tilde{\xi}, \vartheta, \varphi)}{\partial_\varphi \varpi(\tilde{\xi}, \vartheta, \varphi)} \right) r^2(\tilde{\xi}) \sin^2 \theta(\tilde{\xi}, \vartheta) (\delta \varphi)^2 \\
&\quad - \frac{e^{2\varpi(\tilde{\xi}, \varphi)}}{4\mu_g^2 \lambda(\tilde{\xi}, \varphi)} \left[ 1 + \varepsilon \varpi(\tilde{\xi}) (\delta t)^2, (65) \right], \delta \varphi = d\varphi + \left[ \partial_\tilde{\xi} \nabla + \varepsilon \partial_\varpi \right] dx^i, \delta t = dt + [\partial_\tilde{\xi} n + \varepsilon \partial_\varpi] dx^i.\end{align*}$$
where $\tilde{\xi} = \int dr/\sqrt{|1 - b(r)/r|}$ and $B(\tilde{\xi})$ are determined by the prime metric [53]. We can chose such generating and integration functions when the metrics (in corresponding limits) define exterior spacetimes [64], for coordinates [53] and $e^{2B(\tilde{\xi})} \to q(r)$, see (67).

The class of solutions (65) are for stationary configurations determined by respective general and small $\varepsilon$–parametric $\mu_\nu$– and $f$–modifications.

5.1 Ellipsoidal off–diagonal wormhole deformations

Rotoid $\varepsilon$–configurations are "extracted" from (65) if we take for the $f$–deformations

$$\chi_4 = \bar{\chi}_4(r, \varphi) := \frac{2M(r)}{r} \left(1 - \frac{2M(r)}{r}\right)^{-1} \xi \sin(\omega_0 \varphi + \varphi_0),$$

(66)

for $r$ considered as a function $r(\tilde{\xi})$. This is different from $r(\xi)$ taken in previous section but may be parameterized to have $r(\xi) \to r(\xi)$ in exterior spacetimes. Let us define

$$h_3 = \tilde{h}_3(\tilde{\xi}, \vartheta, \varphi) \left[1 + \varepsilon \chi_3(\tilde{\xi}, \vartheta, \varphi)\right] \, ^0\!T_3(\tilde{\xi}, \vartheta), \quad h_4 = \tilde{h}_4(\tilde{\xi}, \vartheta, \varphi) \left[1 + \varepsilon \chi_4(\tilde{\xi}, \vartheta, \varphi)\right] \, ^0\!T_4(\tilde{\xi}),$$

for $^0\!T_3 = r^2(\tilde{\xi}) \sin^2 \theta(\tilde{\xi}, \vartheta), \, ^0\!T_4 = q(\tilde{\xi})$ and

$$\tilde{h}_3 = \frac{[\partial_\varphi \bar{\omega}(\tilde{\xi}, \vartheta, \varphi)]^2}{\mu^2_\nu \lambda(\tilde{\xi}, \vartheta)}, \quad \tilde{h}_4 = \frac{e^{2\varepsilon(\tilde{\xi}, \vartheta, \varphi)}[\bar{\omega}(\tilde{\xi}, \vartheta, \varphi) - \varepsilon]}{4\mu^2_\nu \lambda(\tilde{\xi}, \vartheta) |q(\tilde{\xi})|} e^{2B(\tilde{\xi})},$$

(67)

where $e^{2B(\tilde{\xi})} \to q(\tilde{\xi})$ if $\tilde{\xi} \to \xi$. Introducing (66) in respective formulas (61) for any prescribed generating function $\tilde{\omega}(\tilde{\xi}, \vartheta, \varphi)$, we can compute

$$\tilde{\chi}_3 = \chi_3(\tilde{\xi}, \vartheta, \varphi) = \frac{\varepsilon}{\bar{\omega}} = \frac{\partial_{\varphi} [\bar{\omega}] / \partial_{\varphi} \bar{\omega}},$$

and

$$\bar{\omega}_i = \frac{\partial_i [r(\tilde{\xi}) \sin \theta(\tilde{\xi}, \vartheta)] \sqrt{|q(\tilde{\xi})| \partial_{\varphi} [\bar{\omega}]} = \partial_i \tilde{\omega}(\tilde{\xi}, \vartheta, \varphi),$$

for $x^i = (\tilde{\xi}, \vartheta)$. With respect to N–adapted frames, we model an ellipsoidal configuration with $r_+(\tilde{\xi} = \tilde{\xi}_+) \simeq \frac{2 \hat{\lambda}(\tilde{\xi}_+)}{1 + 2 \hat{\lambda}(\tilde{\xi}_+) + \varepsilon}$, for a corresponding value $\tilde{\xi}_+$, constants $\tilde{\xi}, \omega_0$ and $\varphi_0$ and eccentricity $\varepsilon$.

Putting together above formulas, we obtain

$$ds^2 = e^{\varepsilon(\tilde{\xi}, \vartheta)} (d\tilde{\xi}^2 + d\vartheta^2) + \frac{[\partial_{\varphi} \bar{\omega}]^2}{\mu^2_\nu \lambda} \left(1 + \frac{\partial_{\varphi} [\bar{\omega}]}{\partial_{\varphi} \bar{\omega}}\right) \, ^0\!T_3 [d\varphi + \partial_\xi (\eta \tilde{A} + \varepsilon \bar{\omega}) d\tilde{\xi} + \partial_\theta (\eta \tilde{A} + \varepsilon \bar{\omega}) d\theta]^2$$

(68)

$$- \frac{e^{2\varepsilon}}{4\mu^2_\nu \lambda} [1 + \varepsilon \bar{\omega}(\tilde{\xi}, \vartheta)] e^{2B(\tilde{\xi})} (dt + \partial_\xi (\eta n + \varepsilon \bar{\omega}) d\tilde{\xi} + \partial_\theta (\eta n + \varepsilon \bar{\omega}) d\theta]^2.$$
If the generating functions \( \tilde{\eta} \) and effective source \( \lambda \) in massive gravity are such way chosen that polarization functions (67) can be approximated \( \tilde{\eta}_a \simeq 1 \) and \( \eta_n \) are "almost constant", with respect to certain systems of radial coordinates, the metric (68) mimic small rotoid worm hole like configurations with off–diagonal terms and \( f \)–modifications of the diagonal coefficients. It is possible to chose such integration functions and constants that this class of stationary solutions define wormhole like metrics depending generically on three space coordinates with self–consistent "imbedding" in an effective massive gravity background.

For more general classes of nonholonomic deformations, we can preserve certain rotoid type symmetries but the "wormhole character" of solutions became less clear.

5.2 Solitonic waves for wormholes and black ellipsoids

Let us consider two examples of gravitational solitonic deformations in massive \( f \)–modified gravity.

5.2.1 Sine–Gordon two dimensional nonlinear waves

An interesting class of off–diagonal solutions depending on all space–time coordinates can be constructed by designing a configuration when a 1–solitonic wave preserves an ellipsoidal wormhole configuration. Such a spacetime metric can be written in the form

\[
\begin{align*}
\text{ds}^2 &= e^{\tilde{\psi}(x^i)}(d\tilde{\xi}^2 + d\vartheta^2) + \omega^2(\tilde{\xi}, t) \times \\
&\left[ \tilde{\eta}_3(1 + \varepsilon \frac{\partial\varphi}{\partial\varphi} \tilde{A}) \right] \left( \tilde{\eta}_4(\delta\varphi)^2 - \tilde{\eta}_4[1 + \varepsilon \tilde{\varphi}(\tilde{\xi}, \varphi)] \right) \left( \frac{\partial t}{\partial t} \right)^2,
\end{align*}
\]

where \( \gamma^2 = (1 - v^2)^{-1} \) and constants \( m, m_0, v \), defines a 1–soliton solution of the sine–Gordon equation \( \frac{\partial^2 \omega}{\partial t^2} - \frac{\partial^2 \omega}{\partial \tilde{\xi}^2} + \sin \omega = 0 \).

For \( \omega = 1 \), the metrics (69) are of type (68). A nontrivial value \( \omega \) depends on the time like coordinate \( t \) and has to be constrained to conditions of type (69), which can be written for \( \tilde{n}_2 = 0 \) and \( \tilde{n}_1 = \text{const} \) in the form \( \frac{\partial \omega}{\partial \xi} - \tilde{n}_n \frac{\partial \omega}{\partial t} = 0 \). A gravitational solitonic wave (70) will propagate self–consistently in a rotoid wormhole background with \( \tilde{n}_1 = v \) which solve both the sine–Gordon and constraint equations. Re–defining the system of coordinates with \( x^1 = \tilde{\xi} \) and \( x^2 = \vartheta \), we can transform any nontrivial \( \tilde{n}_1(\tilde{\xi}, \vartheta) \) into necessary \( \tilde{n}_1 = v \) and \( \tilde{n}_2 = 0 \).
5.2.2 Three dimensional solitonic waves

In general, we can construct various types of vacuum gravitational 2-d and 3-d configurations characterized by solitonic hierarchies and related bi–Hamilton structures, for instance, of Kadomtsev–Petviashvili, KdP, equations [25] with possible mixtures with solutions for 2-d and 3-d sine–Gordon equations etc, see details in Ref. [26].

Let us consider a solution of KdP equation for the $v$–conformal factor $\omega = \hat{\omega}(\bar{\xi},\varphi, t)$, when $y^4 = t$ is taken as a time like coordinate, as

$$ \pm \hat{\omega}^{**} + (\partial_t \hat{\omega} + \hat{\omega}^* + \epsilon \hat{\omega}^{***})^* = 0, $$

(71)

with dispersion $\epsilon$. In the dispersionless limit $\epsilon \to 0$, we can consider that the solutions are independent on $\varphi$ and determined by Burgers’ equation $\partial_t \hat{\omega} + \hat{\omega} \hat{\omega}^* = 0$. For 3–d solitonic configurations, the conditions (36) are written in the form

$$ e_1 \hat{\omega} = \hat{\omega}^* + w_1(\bar{\xi}, \vartheta, \varphi)\hat{\omega}^* + n_1(\bar{\xi}, \vartheta)\partial_\vartheta \hat{\omega} = 0. $$

If $\hat{\omega}' = 0$, we can fix $w_2 = 0$ and $n_2 = 0$.

Such solitonic deformations of the wormhole metrics and their massive gravity and $f$–modifications can be parameterized in the form

$$ g = e^\psi(\bar{\xi}, \vartheta) (d\bar{\xi} \otimes d\bar{\xi} + d\vartheta \otimes d\vartheta) + [\hat{\omega}(\bar{\xi}, \varphi, t)]^2 h_a(\bar{\xi}, \varphi) e^a \otimes e^a, $$

$$ e^3 = d\varphi + w_1(\bar{\xi}, \vartheta, \varphi)d\bar{\xi}, \quad e^4 = dt + n_1(\bar{\xi}, \vartheta)d\bar{\xi}. $$

This class of metrics does not have (in general) Killing symmetries but may possess symmetries determined by solitonic solutions of (71).

In a similar form, we can construct solutions for any $\hat{\omega}$ defined by any 3–d solitonic and/ or other nonlinear wave equations, or generate solitonic deformations for $\omega = \hat{\omega}(\vartheta, \varphi, t)$.

5.3 Ringed wormholes

Using the AFDM, we can generate ansatz for a rotoid wormhole plus a torus (ring) configuration,

$$ ds^2 = e^{\psi(x')} (d\bar{\xi}^2 + d\vartheta^2) + \eta_a(1 + \epsilon \frac{\partial_x F(\bar{\xi}, \varphi, \vartheta)}{\partial_x \bar{\omega}}) \delta_{a3}(\delta \varphi)^2 $$

$$ -F(\bar{\xi}, \varphi, \vartheta) \eta_a[1 + \epsilon \frac{\partial_x F(\bar{\xi}, \varphi)}{\partial_x \bar{\omega}}] \delta_{a3}(\delta t)^2 $$

$$ \delta \varphi = d\varphi + \partial_x(\eta A + \epsilon A) dx^i, \delta t = dt + \eta_n(\bar{\xi}, \vartheta) dx^i, $$

(72)

for $x^i = (\bar{\xi}, \vartheta)$ and $y^a = (\varphi, t)$, where the function $F(\bar{\xi}, \varphi, \vartheta)$ in conventional spherical coordinates can be rewritten equivalently in conventional Cartesian coordinates as $F(\bar{x}, \bar{y}, \bar{z}) = (R_0 - \sqrt{\bar{x}^2 + \bar{y}^2})^2 + \bar{z}^2 - a_0$, for $a_0 < a, R_0 < r_0$.

We get a ring around the wormhole throat. The ring configuration is defined

$^7$we can consider wormholes in the limit $\varepsilon \to 0$ and for corresponding approximations $\eta_\omega \simeq 1$ and $\eta A$ and $\eta n$ to be almost constant.

25
by the condition $F = 0$. For $F = 1$, we get a metric of type \((69)\) with $\omega = 1$.

In above formulas, $R_0$ is the distance from the center of the tube to the center of the torus/ring and $a_0$ is the radius of the tube.

If the wormhole objects exist, the variants ringed by a torus may be stable for certain nonholonomic geometry and exotic matter configurations. We omit in this work a rigorous stability analysis as well a study of issues related to cosmic censorship criteria etc.

5.4 Modified wormholes with induced torsion

The examples for wormhole nonholonomic deformations considered above are for effective LC–configurations which can be effectively modelled by non-linear off–diagonal interactions in GR. Here, we provide an example of a class of stationary off–diagonal solutions with nontrivial torsion effects resulting in addition effective rotation proportional to $\mu_g$, see a similar configuration \((51)\). The corresponding off–diagonal quadratic element is given by

$$ds^2 = e^{\tilde{\psi}(\tilde{\xi}, \theta)}(d\tilde{\xi}^2 + d\theta^2) + \frac{(\partial_\varphi \Phi)^2}{\mu_g^2 \lambda(\tilde{\xi}, \vartheta) \Phi^2} [dy^3 + \partial_\varphi \Phi dx^i]^2 + \frac{\Phi^2}{4 \mu_g^2 \lambda(\tilde{\xi}, \vartheta)} [dt + \left(1 + n_k(\tilde{\xi}, \vartheta) + 2n_k(\tilde{\xi}, \vartheta) \right) \frac{4 \mu_g (\partial_\varphi \Phi)^2}{\Phi^3} dx^k]^2, \tag{73}$$

for $x^i = (\tilde{\xi}, \theta)$ and generating function $\Phi = \exp[2\tilde{\psi}(\tilde{\xi}, \theta, \varphi)]$. The d–torsion coefficients \((9)\) for this metric are not trivial if $2n_k \neq 0$. This and other setting for more general sources $\Upsilon(\tilde{\xi}, \theta, \varphi) \tag{22}$ different classes of N–coefficients lead to different characteristic geometric and physical properties which are very different from LC–configurations.

We can parameterize \((73)\) in any form \((59), \tag{65}, \tag{68}, \tag{69}\) in order to generate off–diagonal solutions with $\mu_g$– and/or $f$–modifications possessing rotoid and/or solitonic symmetries characterized by nonholonomic torsion. If a vertical conformal factor $\omega$ similar to \((69)\) is considered, the metric and induced torsion fields might depend on all four spacetime coordinates. Toroidal configurations of type \((72)\) can be constructed if a toroidal function of type $F(x, y, \tilde{z})$ is introduced before the $v$–components of metrics in \((73)\).

6 Concluding Remarks and Discussion

Modified gravity theories with functional dependence on curvature and other traces of energy–momentum tensors for matter fields, torsion sources etc and/or with contributions by massive/bi–metric and generalized connection terms for Lagrangians belong to the most active research area oriented to solution of important problems in modern cosmology and particle physics. As we can see in recent literature, many interesting and original classical
and quantum scenarios can be elaborated for naive additions of mass terms, nontrivial geometric backgrounds and nonlinear interactions via polarized constants and quantum corrections. Such constructions are grounded on geometric models and solutions for certain (generalized) effective Einstein equations with high degrees of symmetries (for Killing vectors) and diagonalizable metrics.

In our research, we are focused on exact and approximate generic off–diagonal solutions in gravity theories with generalized symmetries and dependencies via generating and integration functions on much as possible spacetime coordinates (for instance, on three and four ones on 4–d manifolds). It is a difficult mathematical task to construct such solutions in analytic form and to provide and study certain physical important examples and interesting effects relates to outstanding issues, for instance, in cosmology and astrophysics. Nevertheless, all candidates to gravity theories are characterised by complex off–diagonal systems of nonlinear partial equations and fundamental classical and quantum properties of gravitational and matter fields interactions should be studied with regard to the found most general classes of solutions and nonlinear nonholonomically constrained configurations. Here we note that although the equations of modified gravity theories are rather involved, they became much simple in certain adapted system of reference and certain types of nonholonomic constraints on certain classes for generic off–diagonal solutions. The surprising thing is that in many cases, for well–defined geometric conditions, we can model certain classes of nonlinear solutions both in an effective Einstein like theory (with off–diagonal metrics and generalized, or the Levi–Civita, LC, connection) and in modified bi–metric/–connection gravity models, in general, with nontrivial mass terms. Hence, a generic off–diagonal solution with arbitrary generating and integration functions and constants in GR can be regarded as a possible analog of various types of similar solutions in modified gravity theories. In many cases, we can argument a quite conservative opinion: may be it is not necessary to modify the "canonical" Einstein gravity if in the framework of this theory we are able to explain many fundamental issues and observable cosmological effects via certain generalized off–diagonal solutions with generic off–diagonal interactions and nonholonomic constraints?

In order to investigate certain physical implications of off–diagonal solutions and the possibility to mimic physical effects in one theory by effective analogs of such solutions in another class of theories, a general geometric/analytic method of constructing exact solutions should be applied. For such purposes, we developed the anholonomic frame deformation method, AFDM, see [3, 4, 9, 10, 11] and references therein. Following such a geometric method, various classes of gravitational and matter field equations in modified gravity (MG) and Einstein gravity theories can be decoupled and integrated in very general forms if necessary types of adapted frame and connection structures are considered. We can impose constraints, at the
end, for extracting Levi Civita configurations. This way, a wide variety of generalized locally anisotropic wormhole and matched exterior black holes can be constructed. They can be derived for certain exotic matter and/or for off–diagonal configurations of metrics describing nonlinear gravitational and matter field interactions which may limit certain de Sitter spacetimes with effective "anisotropically" polarized cosmological and matter fields constants. The assumption on stationary properties of such locally anisotropic spacetimes is introduced from the very beginning even solitonic waves can be involved. Note that the method allows us to find a wide variety of non–stationary exact solutions.

In this paper, we focused on generic off–diagonal solutions which are constructed as nonholonomic deformations of pseudo-Riemannian metrics with two Killing vectors (in particular, they can be solutions of the Einstein equations) into certain classes of generalized exact solutions in massive gravity with possible small parametric deformations related to $f$–modified gravity. We proved that one exists a formal integration procedure via effective polarization functions which allows us to construct various classes of exact solutions depending generically on three and four coordinates on (generalized) four dimensional spacetimes. In explicit form, we constructed and studied off–diagonal deformations of wormhole solutions matching exterior (in general, nonholonomically deformed) de Sitter spacetimes with contributions by nontrivial massive gravitational terms and ellipsoidal $f$–modifications of de Sitter metrics. We also analysed solitons waves, possible "ringed wormhole" like configurations, modified wormholes "distorted" in nonholonomically induced torsion etc.

There is still much to be learned about possibilities of the AFDM and possible relations of such way constructed off–diagonal solutions with massive gravity, $f$–modified, Finsler like etc theories. Here, it should be noted that such nonholonomic structures were originally considered in Finsler like theories, fractional generalizations etc to modern cosmological scenarios [27, 28, 29, 30]. This paper and discussion provided just a glimpse to potential applications and future our work.

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