Intersections of $\psi$ classes on Hassett Spaces for genus 0 with all weights $\frac{1}{2}$

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Abstract
Hassett spaces are moduli spaces of weighted stable pointed curves. In this work, we consider such spaces of curves of genus 0 with weights all $\frac{1}{2}$. These spaces are interesting as they are isomorphic to $\overline{M}_{0,n}$ but have different universal families and different intersection theory. We develop a closed formula for intersections of $\psi$-classes on such spaces. In our main result, we encode the formula for top intersections in a generating function obtained by applying a differential operator to the Witten-potential.

1 Introduction
The moduli space of algebraic curves of genus 0 with $n$ marked points, $\overline{M}_{0,n}$ (with the Deligne-Mumford compactification [5]) has been an important topic of research in algebraic geometry. These spaces provide an algebro-geometric tool to study how pointed rational curves vary in families, and are of fundamental importance in areas like Gromov-Witten theory and topological quantum field theories [9].

In [7], Hassett constructed a new class of modular compactifications $\overline{M}_{0,A}$ of the moduli space $M_{0,n}$ of smooth curves with $n$ marked points parameterized by an input datum $A$, consisting of a collection $A = (a_1, \ldots, a_n)$ of weights $a_i \in \mathbb{Q} \cap (0, 1]$ such that $a_1 + \ldots + a_n > 2$. We call these spaces $\overline{M}_{0,A}$ the Hassett spaces of rational curves.

A lot of work is being done on Hassett spaces including developing its tautological intersection theory and weighted Gromov-Witten theory, e.g. in [1], [3] and [10]. In this work, we contribute to the tautological intersection theory of a special case of such spaces- Hassett spaces of rational curves with weights all $\frac{1}{2}$, denoted $\overline{M}_{0,\frac{1}{2}}$. These spaces provide for interesting spaces for combinatorial results in its intersection theory because

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of the symmetry of weights and its connections with intersection theory for $\overline{M}_{0,n}$. These spaces are interesting also because they are fine moduli spaces, are isomorphic to $\overline{M}_{0,n}$, but have different universal families and different intersection theory. Exploring these differences and developing some results in its tautological intersection theory is the contribution of this work.

For this work, the following notations are used: a $\psi$ class on $\overline{M}_{0,n}$ is denoted as $\psi_i$; a $\psi$ class on $\overline{M}_{0,(\frac{1}{2})^n}$ is denoted as $\bar{\psi}_i$, and the pullback of a $\psi$ class under the reduction morphism from $\overline{M}_{0,n}$ to $\overline{M}_{0,(\frac{1}{2})^n}$ is denoted $\hat{\psi}_i$.

In our first result 3.1, we develop a closed formula (13) for the monomials in $\hat{\psi}$ classes in terms of cycles on $\overline{M}_{0,n}$. This closed formula is derived using the relation (9) between the $\hat{\psi}$ classes and $\psi$ classes on $\overline{M}_{0,n}$, in which $\psi_i$ is corrected by all boundary divisors where the $i$-th mark is on a twig that gets contracted when pushed forward to $\overline{M}_{0,(\frac{1}{2})^n}$. The proof uses this relation to obtain the $\hat{\psi}$ monomials as monomials in $\psi$ classes and boundary divisors on $\overline{M}_{0,n}$. So, the summands in the resulting expansion correspond to modified $\psi$ monomials on certain boundary strata on $\overline{M}_{0,n}$ that are the intersections of these boundary divisors. The dual graphs of these strata are all ‘forked’ graphs- graphs with a ‘central’ node and some ‘forks’, e.g. figure (3.5). We then establish a bijection between summands in the expansion corresponding to these graphs that we call ‘$P$’-graphs and the unordered partitions of $[n]$, such that cardinality of each subset in the partition is either 2 or 1. The resulting formula (13) has the pullback of monomials in $\bar{\psi}$ classes on $\overline{M}_{0,(\frac{1}{2})^n}$ as a sum of the intersections of monomials in $\psi$ classes and boundary strata corresponding to $P$-graph on $\overline{M}_{0,n}$. Then we derive two corollaries (4.1 and 4.2) of this result to calculate the top intersections. These give the top intersections of $\hat{\psi}$ classes as a sum of top intersections of $\psi$ classes on $\overline{M}_{0,n}$, $\overline{M}_{0,n-1}$, $\overline{M}_{0,n-2}$, ... with some multiplicities. We point out here that our corollaries (4.1 and 4.2) can also be deduced from theorem 7.9 in [1]. For our work, we develop specific and explicit closed formulas for our special case of all weights $\frac{1}{2}$ and base our combinatorial analysis closely on the structure of dual graphs.

The main theorem 4.1 of this work encodes the closed formula (4.1) for top intersections in a generating function $G(t)$ obtained by applying a differential operator to the Witten-potential $F(t)$ [8]. This operator $\hat{\mathcal{L}}$ takes the form of an exponential partial differential operator and provides a very nice compact way to describe these top intersections. The proof of this formula also is based on a bijection between the ‘forked’ graphs and the summands in the coefficient of the appropriate term in $\hat{\mathcal{L}}(F(t))$. But in this, the bijection is not with graphs corresponding to the partitions of $[n]$, but with ‘$P_k$-graphs’ that are defined by replacing the $i$-th mark with $k_i$ on a $P$-graph. Here $k_i$ is the exponent of $\hat{\psi}_i$ in the $\hat{\psi}$ monomial (4.2). As expected, there is a surjection between $P$-graphs and $P_k$-graphs. For the proof, we
write a new version of our closed formula in terms of these \( P_k \)-graphs \( \{24\} \).
Then we show a bijection between the summands in this formula and the summands in the coefficient of the appropriate term in \( \hat{L}(F(t)) \). And the resulting coefficient, as a sum of all these summands, corresponds to the top intersections of \( \psi \) classes.

The paper is organized as follows. In section \( \{2\} \) we give the background required for this work which consists of a brief introduction to \( \overline{M}_{0,n} \), \( \psi \) classes and Hassett Spaces, with some relevant facts and lemmas on these topics. In section \( \{3\} \) we prove our first result which gives the closed formula for the intersections of \( \psi \) classes. In section \( \{4\} \) we give the results for top intersections, and encode the formula for top intersections in the generating function that we obtain by applying a partial differential operator to the Witten-potential.

## 2 Background

For background on for this work, the author has mainly used \( \{9\} \), \( \{4\} \), \( \{8\} \), \( \{6\} \) and introductory sections of \( \{2\} \). Here we will recall some selected facts we explicitly use in this work.

\( \overline{M}_{0,n} \) denotes the moduli space of stable, \( n \) pointed rational curves, with at worst nodal singularities. The boundary of \( \overline{M}_{0,n} \) is defined to be the complement of \( M_{0,n} \) in \( \overline{M}_{0,n} \). It consists of all points parameterizing nodal stable curves.

Given a rational, stable \( n \)-pointed curve \((C, p_1, \ldots, p_n)\), its dual graph is defined to have:

- a vertex for each irreducible component of \( C \);
- an edge for each node of \( C \), joining the appropriate vertices;
- a labeled half edge for each mark, emanating from the appropriate vertex.

Figure below gives an example of the dual graphs of some strata in \( \overline{M}_{0,5} \).

The closures of the codimension 1 boundary strata of \( \overline{M}_{0,n} \) are called the irreducible boundary divisors; they are in one-to-one correspondence with all ways of partitioning \([n] = A \cup A^c \) with the cardinality of both \( A \) and \( A^c \) strictly greater than 1. We denote \( D(A) = D(A^c) \) the divisor corresponding to the partition \( A, A^c \).

For \( i = 1, \ldots, n \), we define the class \( \psi_i \in A^1(\overline{M}_{0,n}) \). Let \( L_i \to \overline{M}_{0,n} \) be a line bundle whose fiber over each point \((C, p_1, \ldots, p_n) \) is canonically identified with \( T_{p_i}^* (C) \). The line bundle \( L_i \) is called the \( i \)-th cotangent (or tautological) line bundle. Then

\[
\psi_i := c_1(L_i)
\]
where $c_1$ is the first Chern class of the line bundle $L_i$.

Some properties of $\psi$ classes on $M_{0,n}$ that we use are the following lemmas. Interested reader can find their proofs in [8].

**Lemma 2.1.** Consider the gluing morphism $gl : M_{0,\{i\}} \times M_{0,\{i\}} \to M_{0,n}$. Assume that $i \in I$ and denote by $\pi_1 : M_{0,\{i\}} \times M_{0,\{i\}} \to M_{0,\{i\}}$ the first projection. Then:

$$gl^*(\psi_i) = \pi_1^*(\psi_i).$$

(2)

**Lemma 2.2.** Consider the forgetful morphism $\pi_{n+1} : M_{0,n+1} \to M_{0,n}$. Then, for every $i = 1, \ldots, n$,

$$\psi_i = \pi_{n+1}^*(\psi_i) + D(\{i, n+1\}).$$

(3)

**Lemma 2.3.** For any choice of $i, j, k$ distinct, we have the following equation in $A^1(M_{0,n})$:

$$\psi_i = \sum_{i \in I, j, k \notin I} D(I).$$

(4)

**Lemma 2.4** (String Equation). Consider the forgetful morphism $\pi_{n+1} : M_{0,n+1} \to M_{0,n}$. Then

$$\pi_{n+1*} \left( \prod_{i=1}^{n} \psi_i^{k_i} \right) = \sum_{j|k_j \neq 0} \psi_j^{k_j-1} \prod_{i \neq j} \psi_i^{k_i}.$$  

(5)

Let $\sum k_i = n - 3$. Then

$$\int_{\overline{M}_{0,n}} \prod_{i=1}^{n+1} \psi_i^{k_i} = \binom{n-3}{k_1, \ldots, k_{n+1}},$$

(6)

where the integral sign denotes push-forward to the class of a point.
2.1 Hassett spaces

In [7], Hassett constructed a new class of modular compactifications $\overline{M}_{g,A}$ of the moduli space $M_{g,n}$ of smooth curves with $n$ marked points parameterized by an input datum $(g, A)$. Here $g$ is the genus of the curves and $A = (a_1, \ldots, a_n)$ is the weight data of weights $a_i \in \mathbb{Q} \cap (0, 1]$ satisfying the inequality $2g - 2 + a_1 + \ldots + a_n > 0$.

$\overline{M}_{g,A}$ that we call Hassett space parameterizes curves $(C, p_1, \ldots, p_n)$ with $n$ marked non-singular points on $C$ that are $A$-stable if the following two conditions are fulfilled.

1. The twisted canonical divisor $K_C + a_1p_1 + \ldots + a_np_n$ is ample.
2. A subset $p_{i_1}, \ldots, p_{i_k}$ of the marked points is allowed to coincide only if the inequality $a_{i_1} + \ldots + a_{i_k} \leq 1$ holds.

For $g = 0$, the stability condition means that a rational $n$-pointed curve $(C, p_1, \ldots, p_n)$ is $A$-stable if on every irreducible component of $C$ the number of nodes plus the sum of the weights of the marks lying on the component is strictly greater than 2, with $a_1 + \ldots + a_n > 2$.

In the case $(a_1, \ldots, a_n) = (1, \ldots, 1)$, this condition is nothing but the traditional notion of an $n$-marked stable curve, and so the compactification $\overline{M}_{g,A}$ is exactly the well-known Deligne-Mumford compactification $\overline{M}_{g,n}$ of $M_{g,n}$.

**Definition 2.1.** Given two weight data $A, B$, we say that $B \leq A$ if for every $i$, $b_i \leq a_i$. Then there exists a regular reduction morphism:

$$c_{B,A} : \overline{M}_{0,A} \to \overline{M}_{0,B}$$

s.t. $c_{B,A}(C, p_1, \ldots, p_n)$ is obtained by contracting twigs that become unstable when the weights of the points are “lowered” from $a_i$ to $b_i$.

Moduli spaces of weighted stable rational curves also have psi classes, which are defined in the same way as in $\overline{M}_{0,n}$. A $\psi$ class on $\overline{M}_{0,A}$ will be denoted as $\overline{\psi}_i$.

**Lemma 2.5.** Consider the reduction morphism $c : \overline{M}_{0,n} \to \overline{M}_{0,A}$. For $i = 1, \ldots, n$, we have:

$$\psi_i = c^*(\overline{\psi}_i) + \sum_{j \neq i, \sum_j a_j \leq 1} D(I).$$

**Proof.** Consider the following commutative diagram:

$$\begin{array}{ccc}
\overline{M}_{0,n} & \xrightarrow{c} & \overline{M}_{0,A} \\
\downarrow{\sigma} & & \downarrow{\pi} \\
\overline{M}_{0,n} & \xrightarrow{c} & \overline{M}_{0,A}
\end{array}$$
Here $\overline{U}_{0,n} \text{ and } \overline{U}_{0,A} \text{ are the universal families over } \overline{M}_{0,n} \text{ and } \overline{M}_{0,A} \text{ respectively; } \pi \text{ and } \overline{\pi} \text{ are the forgetful morphisms, and } \sigma_i \text{ and } \overline{\sigma}_i \text{ are the } i \text{-th tautological sections of the corresponding universal families; } c \text{ and } C \text{ are the reduction morphisms. Let } S_i = Im(\sigma_i) \text{ and } \overline{S}_i = Im(\overline{\sigma}_i). \text{ Then,}

\psi_i = \pi_*(-S_i^2)
\overline{\psi}_i = \overline{\pi}_*(-\overline{S}_i^2)

Now, $C^*(\overline{S}_i) = S_i + \sum_I E_I$, such that $i \in I, j \in I$ if $\sum a_j \leq 1$, and $E_I$ is the exceptional divisor in $\text{Blow}_{\gamma_i \in \sigma_i}$ such that $\pi_*(E_I) = D(I)$. Then,

\[ c^*\overline{\psi}_i = c^*\pi_*(-\overline{S}_i^2) = \pi_*C^*(-\overline{S}_i^2) = \pi_*(-(S_i + \sum_I E_I)^2) = \pi_*(-S_i^2 - \sum_I S_i E_I + \sum_I E_I^2)) = \psi_i - 2\sum I D(I) + \sum I D(I) = \psi_i - \sum I D(I) = \psi_i - \sum_{I \ni i, \sum_{j \in I} a_j \leq 1} D(I) \]

Informally, for the pullback of a $\overline{\psi}_i$, a $\psi_i$ is corrected by all boundary divisors where the $i$-th mark is on a twig that gets contracted by $c$. In particular we will use the above special case of reduction morphism for this work.

**Definition 2.2.** We define the $\hat{\psi}_i$ class as the pullback of a $\overline{\psi}$ class under the reduction morphism $c : \overline{M}_{0,n} \rightarrow \overline{M}_{0,A}$.

\[ \hat{\psi}_i := c^*\overline{\psi}_i \]

**Corollary 2.1.** For the reduction morphism $c : \overline{M}_{0,n} \rightarrow \overline{M}_{0,A}$, where $A = \{ \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} \}$. For $i = 1, \ldots, n$, we have:

\[ \hat{\psi}_i = \psi_i - \sum_{j,j \neq i} D(\{i, j\}) \]

(9)

**Proof.** This follows from [8] by observing that $a_i + a_j = 1$ for all $i, j$ when $a_i = \frac{1}{2} \forall \ i$. 

\[ \square \]
3 Closed Formula for intersections of $\psi$-classes

For the work that follows, $A = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ for $\overline{M}_{0,A}$, and we denote it by $\overline{M}_{0,\frac{1}{2}}$. In this, we develop the closed formula for integrals of $\hat{\psi}$ monomials corresponding to the $\bar{\psi}$ monomials on $\overline{M}_{0,\frac{1}{2}}$.

We denote by $P = \{P_1, P_2, \ldots, S_1, S_2, \ldots, S_t\}$ an unordered partition of $[n]$, such that cardinality of each subset in the partition is either 2 or 1. Further, if the cardinality of such subsets is 2, we denote them with $P_j$ and with $S_j$ if the cardinality is 1. Denote by $F$ the set of all $P_j$’s, and by $S$ the set of all $S_j$’s. Denote by $\mathcal{P}$ the set of all such partitions $P$.

**Definition 3.1.** Given a $P \in \mathcal{P}$, we define the graph $\Gamma_P$ as follows:

1. The $\Gamma_P$ has a ‘central’ node, with $|F|$ number of edges with nodes on ends
2. Attach to each non-‘central’ node two half-edges forming a ‘fork’; to the ‘central’ node, attach $|S|$ number of half-edges
3. Label a half-edges on forks with $P_j$ and $S_j$’s, and half-edges on central node with $S_j$’s.

So, each $P_j$ corresponds to a fork and $S_j$’s to half-edges on the central node. We call this a $P$-graph. $|F|$ gives the number of forks on the graph. Each such graph is a dual graph of a stratum in $\overline{M}_{0,n}$.

Clearly, the set of all $P$-graphs as defined above are in bijection with the set of all partitions $P \in \mathcal{P}$.

**Definition 3.2.** Given a $\hat{\psi}$-monomial $m = \hat{\psi}_1^{k_1} \hat{\psi}_2^{k_2} \hat{\psi}_3^{k_3} \ldots \hat{\psi}_r^{k_r}$, a decorated $P$-graph $\Gamma_P^d$ is obtained by coloring a half-edge corresponding to point $t \in P_j$ or $t \in S_j$ if $k_t \neq 0$.

**Example 3.1.** Given $m = \hat{\psi}_1^2 \hat{\psi}_2$ on $\overline{M}_{0,6}$, for $P = \{1, 2\}, \{3\}, \{4\}, \{5\}, \{6\}$, we get the decorated graph $\Gamma_P^d$ as in figure 3.1.

for $P = \{1, 3\}, \{2, 4\}, \{5\}, \{6\}$, we get the decorated graph $\Gamma_P^d$ as in figure 3.2.

and for $P = \{1, 3\}, \{5, 6\}, \{2\}, \{4\}$, we get the decorated graph $\Gamma_P^d$ as in figure 3.3.

**Definition 3.3.** Consider the decorated $P$-graph in figure 3.4. In $\overline{M}_{0,n}$, this represents the dual graph of a boundary stratum that is the image of the following gluing morphism:

$$gl : \overline{M}_{0,P_1 \cup \cdots \cup P_s} \times \ldots \overline{M}_{0,S_1 \cup \cdots \cup S_t} \rightarrow \overline{M}_{0,n}.$$
where for each $P_j$, the half-edge $\bullet P_j$ corresponds to node pulled back from the factor corresponding to the central node, and $\ast P_j$ corresponds to node pulled back from the factor corresponding to the respective fork. For this decorated $P$-graph, define the following $\psi$-function:

$$\phi_p(\psi) = \psi_{P_1}^{k_1+1} \ast \psi_{P_2}^{k_2} \ast \cdots \ast \psi_{P_t}^{k_t}$$

where $\psi_{P_j}$ is the $\psi$-class at the $\bullet P_j$ node on the factor corresponding to the central node.

**Lemma 3.1.** Let $D_i$ be a divisor $D(\{i, j\})$ where $j \in [n] \setminus \{i\}$; then on $\overline{M}_{0,n}$, $s \prod_{i=1}^{s} D_i$, $s \leq n - 3$, is supported on a $P$-graph $\Gamma_p$. And the number of forks on $\Gamma_p$ can vary from $\lfloor \frac{n}{2} \rfloor$ to $\min(\lfloor \frac{n}{2} \rfloor, s)$.

**Proof.** We prove by induction. $D_1 = \left( \begin{array}{c} i_1 \\ 1 \end{array} \right)$ is clearly a $P$-graph. Now,

$$D_1 D_2 = \left( \begin{array}{c} i_1 \\ 1 \end{array} \right) \left( \begin{array}{c} i_2 \\ 2 \end{array} \right)$$
\[
\begin{cases}
0, & \text{if } |\{1, i_1\} \cap \{2, i_2\}| = 1 \\
\left(\begin{array}{c}
0 \\
1 \\
2
\end{array}\right), & \text{if } |\{1, i_1\} \cap \{2, i_2\}| = 0 \\
-\left(\begin{array}{c}
2 \\
1 \\
0
\end{array}\right), & \text{if } \{1, i_1\} = \{2, i_2\}
\end{cases}
\]

So, all the graphs that support \( D_1D_2 \) are \( \mathcal{P} \)-graphs. Now suppose \( \prod_{i=1}^{s} D_i \) is supported on a \( \mathcal{P} \)-graph, and suppose that \( \mathcal{P} \)-graph has \( j \) number of forks \( P_i \)’s, and denote by \( S \) the set of all half-edges not on forks. Then,

\[
(\prod_{i=1}^{s} D_i)D_{s+1} = \left(\begin{array}{c}
P_0 \\
P_1 \\
P_2
\end{array}\right) \left(\begin{array}{c}
0 \\
\text{if}\{s+1,k\} \not\subseteq P_i, S \forall i \\
\text{if}\{s+1,k\} \subseteq S \\
\text{if}\{s+1,k\} \subseteq P_i \text{ for some } i
\end{array}\right)
\]

So, all graphs that we get for non-zero intersections are in fact \( \mathcal{P} \)-graphs.

For the number of forks on the graphs, there are 2 cases to consider: 1) \( s > \left\lceil \frac{n}{2} \right\rceil \), and 2) \( s \leq \left\lceil \frac{n}{2} \right\rceil \). In the first case, there are at least \( s - \left\lfloor \frac{n}{2} \right\rfloor \) forks with both half-edges colored, and maximum number of forks is \( \left\lceil \frac{n}{2} \right\rceil \); as \( s < n \), minimum number of forks is \( \left\lceil \frac{s}{2} \right\rceil \). In the second case, the minimum number of forks is \( \left\lceil \frac{s}{2} \right\rceil \) and the maximum \( s \). So, the number of forks on \( \Gamma_{\mathcal{P}} \) can vary from \( \left\lceil \frac{s}{2} \right\rceil \) to \( \min(\left\lceil \frac{n}{2} \right\rceil, s) \).

\[\Box\]

**Lemma 3.2.** With \( D_i = \sum_j D(\{i,j\}) \) on \( \overline{M}_{0,n} \), and \( \bullet \mathcal{P}_j \) as in Definition 3.3, then for \( \overline{M}_{0,\left\lfloor \frac{n}{2} \right\rfloor} \)

\[
\hat{\psi}_i^{k_i} = \psi_i^{k_i} - \sum_j \psi_i^{k_i-1} D(\{i,j\})
\]
Proof. Using Corollary 2.1

\[ \hat{\psi}_i^{k_i} = (\psi_i - \sum_j D(\{i, j\}))^{k_i} \]

\[ = \psi_i^{k_i} + \ldots + (-1)^{k_i} \sum_j D(\{i, j\})^{k_i} \]

\[ = \psi_i^{k_i} + \ldots + (-1)^{k_i}(1-1)^{k_i} \sum_j D(\{i, j\}) \]

\[ = \psi_i^{k_i} - \sum_j \psi_i^{k_i} \sum_j D(\{i, j\}) \]

The second equality happens as all the terms in the expansion except the first and the last vanish due to dimension reasons. And it’s only the self intersections that are non-zero in the last term in the expansion of \((\psi_i - \sum_j D(\{i, j\}))^{k_i}\). The third and fourth equalities follow from the \(k_i\) self intersections of \(\sum_j D(\{i, j\})\).

For the result in the above lemma, we will use the following notation for brevity.

\[ \hat{\psi}_i^{k_i} = \psi_i^{k_i} - \psi_i^{k_i-1} D_i \]

where

\[ D_i := \sum_j D(\{i, j\}) \]

and

\[ \psi_i^{k_i-1} D_i := \sum_j \psi_i^{k_i-1} D(\{i, j\}) \]

Also, again for brevity, if the \(P\)-graph in figure [3.4] corresponds to partition \(P = \{P_1, P_2, ..., P_s, S_1, S_2, ..., S_q\}\), then we will denote \(\psi_{*P_j}\) also as \(\psi_{P_j}\) in what follows.

**Theorem 3.1.** With \(P, F, S, P_j, P_{j1}, P_{j2}, S_i\) as defined above, for \(n \geq 5\) we have for \(M_{0,\langle \frac{1}{2}, \ldots, \frac{1}{2} \rangle}\):

\[ \hat{\psi}_1^{k_1} \hat{\psi}_2^{k_2} \ldots \hat{\psi}_n^{k_n} = \sum_{\mathcal{P} \in \Psi} (-1)^{|F|} [\Gamma \mathcal{P}] \prod_{S_i \in S} \psi_i^{k_{S_i}} \prod_{P_j \in F} \psi_{P_j}^{k_{P_{j1}}+k_{P_{j2}}+1} \]

where \([\Gamma \mathcal{P}]\) is the class of boundary stratum in \(\overline{M}_{0,n}\) with dual graph \(\Gamma \mathcal{P}\).

**Proof.** Let \(D_i = \sum_j D(\{i, j\})\). Omitting \(\hat{\psi}\)'s with 0-exponents, and assuming VLOG that the first \(r\) \(\hat{\psi}\)'s remain with non-zero exponents,

\[ \hat{\psi}_1^{k_1} \hat{\psi}_2^{k_2} \ldots \hat{\psi}_n^{k_n} = \hat{\psi}_1^{k_1} \hat{\psi}_2^{k_2} \hat{\psi}_3^{k_3} \ldots \hat{\psi}_r^{k_r} \]

Then,

\[ \hat{\psi}_1^{k_1} \hat{\psi}_2^{k_2} \hat{\psi}_3^{k_3} \ldots \hat{\psi}_r^{k_r} = (\psi_1 - D_1)^{k_1} (\psi_2 - D_2)^{k_2} \ldots (\psi_r - D_r)^{k_r} \]
using relation (9)

\[ = (\psi_1^{k_1} - \psi_1^{k_1-1} D_1)(\psi_2^{k_2} - \psi_2^{k_2-1} D_2) \ldots (\psi_r^{k_r} - \psi_r^{k_r-1} D_r) \]

using relation (12)

\[ = (-1)^s \sum_{s=0}^{r} \psi_1^{k_1-1} \psi_2^{k_2-1} \ldots \psi_s^{k_s-1} D_1 D_2 \ldots D_s \psi_{s+1}^{k_{s+1}-1} \ldots \psi_r^{k_r-1} \] (14)

with \(1 \leq i_j \leq r\) and \(i_1 < i_2 < \ldots < i_s\). Now, each term in the expansion of the expression above is supported on a \(P\)-graph from lemma 3.1.

Pick a \(P\)-graph with \(s\) number of colored half-edges on forks and create the corresponding \(\Gamma_{dP}\). There are two possibilities: 1) at least one fork has both half-edges uncolored, or 2) all forks of \(\Gamma_{dP}\) have at least one colored half-edge. Let the second type of graph (figure 3.5) have \(t\) number of forks with both half-edges colored as shown below.

![Figure 3.5: No uncolored fork](image)

Intersect both types of \(P\)-graph with \(\phi_P(\psi)\) as defined in (10):

\[ \phi_P(\psi) = \psi_1^{k_1} \psi_2^{k_2-1} \ldots \psi_s^{k_s-1} \psi_{s+1}^{k_{s+1}-1} \ldots \psi_r^{k_r-1} \] (15)

Claim: The intersection of \(\phi_P(\psi)\) with the first type of graphs gives 0. Proof: WLOG, suppose the first type of \(P\)-graph have \(i_1\) and \(i_2\) on a fork with \(k_{i_1} = k_{i_2} = 0\). Then \(k_{i_1} + k_{i_2} = -1\) and

\[ \phi_P(\psi) = \psi_1^{k_1} \psi_2^{k_2-1} \ldots \psi_{s+1}^{k_{s+1}-1} \psi_{s+2}^{k_{s+2}-1} \ldots \psi_r^{k_r-1} \]

is 0 as negative power of a \(\psi\) class by standard convention is 0.

Now, consider the second type of \(P\)-graph (figure 3.5). Claim: \(\phi_P(\psi)\) \(\Gamma_{dP}\), where \(\Gamma_{dP}\) is the second type of graph, uniquely determines a term in the expansion (14) above.

Proof: Define a map from the set of \(P\)-graphs to the terms in the expansion (14) as follows. For each \(i_l\) in \(\Gamma_{dP}\) where \(i_l\) is a colored half-edge on a fork,
assign a \( D_i \), to form the product \( \prod_{i=1}^s D_i \). And for each fork on the graph with half-edges \( i_l \) and \( i_q \), assign \( \psi_{i_l}^{k_{i_l} + k_{i_q} - 1} \) and form their product; the result is \( \phi_P(\psi) \). Then \( \phi(\psi) \cdot \prod_{i=1}^s D_i \) is precisely the term in the expansion (14) that \( \Gamma_d^d \) maps to. Further, reversing the process, we get the preimage of a term in expansion (14) the unique \( P \)-graph (figure 3.5). So, the map is in fact a bijection. And,

\[
\phi_P(\psi) \cdot \Gamma_d^d =
\]

\[
(-1)^{s+\frac{t}{2}} (\Gamma_P) \psi_{i_1}^{k_{i_1} + k_{i_2} - 1} \psi_{i_{t-1}}^{k_{i_{t-1}} + k_{i_t} - 1} \psi_{i_{t+1}}^{k_{i_{t+1}} - 1} \psi_{i_s}^{k_{i_s} - 1} \psi_{i_{s+1}}^{k_{i_{s+1}} - 1} \cdots \psi_{i_r}^{k_{i_r} - 1}
\]

(16)

\[
= (-1)^{|F|} [\Gamma_P] \prod_{S_i \in S} \psi_{S_i}^{k_{S_i}} \prod_{P_j \in F} \psi_{P_j}^{k_{P_{j_1} + P_{j_2} - 1}}
\]

\[
\square
\]

4 Numerical intersections

In this section, we develop two corollaries of theorem \((3.1)\) to develop two versions of a closed formula for top intersections of \( \hat{\psi} \)-classes on \( \overline{M}_{0,n} \). Then we encode this formula in a generating function obtained by applying a differential operator to the Witten-potential. As pointed earlier, these corollaries \((4.1 \text{ and } 4.2)\) can also be deduced from theorem 7.9 in [1]. For our work, we develop specific and explicit closed formulas here and base our combinatorial analysis closely on the structure of dual graphs.

**Corollary 4.1.** With \( P, F, P_j \), \( P_{j_1}, P_{j_2}, S_i \) as defined above, for \( n \geq 5 \) we have:

\[
\int_{\overline{M}_{0,n}} \hat{\psi}_1^{k_1} \hat{\psi}_2^{k_2} \cdots \hat{\psi}_n^{k_n} = \sum_{P \in \Psi} (-1)^{|F|} \left( \binom{n-3-|F|}{k_{P_{j_1}} + k_{P_{j_2}} - 1}, (k_S) \right)
\]

(17)

where \( \sum k_i = n - 3 \), and for a \( P = P_1, P_2, \ldots, P_s, S_1, \ldots, S_q \),

\( k_{P_{j_1}} + k_{P_{j_2}} - 1) = k_{P_{j_1}} + k_{P_{j_2}} - 1, \ldots, k_{P_{s_1}} + k_{P_{s_2}} - 1, \text{ and } (k_S) = k_{S_1}, \ldots, k_{S_q} \).

**Proof.** The proof of this is same as for theorem \((3.1)\) except in the last part of evaluation of \( \phi_P(\psi) \cdot \Gamma_d^d \). Here when \( \sum k_i = n - 3 \) this evaluation gives

\[
\phi_P(\psi) \cdot \Gamma_d^d =
\]

\[
= (-1)^{s+\frac{t}{2}} \left( \binom{n-3-(s-\frac{t}{2})}{k_{i_1} + k_{i_2} - 1, \ldots, k_{i_{s-1}} - 1, k_{i_s} - 1, k_{i_{s+1}} - 1, \ldots, k_{i_r}} \right)
\]

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Proof.

Corresponding to a partition graph for \( P \) getting \( \{ \phi \} \) as described above. Also we can collect together terms corresponding to permutations of the set \( \{ j_1, j_2, ..., j_{s-t} \} \) as all terms evaluate to the same value as \( k_{j_i} = 0 \) for all these \( j_i \).

Form a new set \( \mathcal{P}' \) in the following way: Make the powerset \( \mathcal{R} \) of \([r]\), where \( r \) denotes the number of \( \psi \)'s with non-zero exponent in the \( \psi \)-monomial. For each set \( \mathcal{R} \in \mathcal{R} \), create all subsets \( \mathcal{P}' \) of \( \mathcal{R} \) whose elements are subsets of \( \mathcal{R} \) of cardinality 2 or 1 with upper bound of number of subsets of cardinality 2 fixed at \( \lfloor \frac{r}{2} \rfloor \). Call \( \mathcal{P}' \) the set of all \( \mathcal{P}' \). This set \( \mathcal{P}' \) can also be obtained from \( \mathcal{P} \) via the following map: Given a partition \( \mathcal{P} \), project to a \( \mathcal{P}' \) by forgetting all points \( S_i \in \mathcal{P} \) and in a \( P_j \in \mathcal{P} \) forget a point \( P_{ji} \) if \( k_{P_{ji}} = 0 \). More formally, \( \mathcal{P}' = \{ P'_1, P'_2, ..., P'_i, S'_1, S'_2, ..., S' \} \) where \( P'_i = P_i \) if \( k_{P_{i1}} > 0 \) and \( k_{P_{i2}} > 0 \), \( S'_i = P_i \setminus \{ P_{ji} \} \) if \( k_{P_{ji}} = 0 \). This is an onto map. Each \( P'_j \) has cardinality 2, and each \( S'_i \) has cardinality 1. Denote by \( \mathcal{P}' \) the set \([r]\) \( \setminus P'_1 \cup ... \cup P'_i \cup S'_1 \cup S'_2 \cup ... \cup S' \).

Corollary 4.2. With \( \mathcal{P}' \) as defined above, for \( n \geq 5 \) we have:

\[
\int_{M_0,n} \hat{\psi}_{k_1} \hat{\psi}_{k_2} \hat{\psi}_{k_3} ... \hat{\psi}_{k_t} = \sum_{\mathcal{P}' \subseteq \mathcal{P}} (-1)^{s+t} \frac{(n-r)!}{(n-r-s+t)!} \left( k_{i_1} + k_{i_2} - 1, ..., k_{i_{s-t}} - 1, ..., k_{i_j} - 1, k_{i_{s-t}}, ..., k_{i_t} \right) \\
\text{where } t = 2|\{ P'_i \}|, \quad s = |\{ S'_i \}| + t \quad \text{and} \quad \{ i_1, ..., i_t \} = \mathcal{P}' \cap \mathcal{P}
\]

Proof. Corresponding to a partition \( \mathcal{P} \), form a corresponding decorated graph for \( \mathcal{P}' \) by uncoloring any half-edges on the central node, and forgetting the \( j_i \)'s on uncolored half-edges on the forks as discussed above. This corresponds to \( \mathcal{P}' = \{ P'_1, P'_2, ..., P'_{s-2}, S'_1, ..., S'_{s-t} \} \), where \( P'_i \) correspond to nodes with both half-edges colored, and \( S'_i = i_{s-t} \) in the decorated graph of \( \mathcal{P} \) in figure (3.5). Now consider intersection of this graph with \( \hat{\phi}(\psi) \) as defined earlier in (10) :

\[
= (-1)^{|\mathcal{F}|} \left( k_{i_1} + k_{i_2} - 1, ..., k_{i_{s-t}} - 1, k_{i_{s-t}+1}, ..., k_{i_t} \right) \\
as k_{i_{s-t}+1} = ... = k_{i_t} = 0 \\
= (-1)^{|\mathcal{F}|} \left( n - 3 - |\mathcal{F}| \right)
\]
\[ \phi p(\psi) = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ j_1 & j_2 & j_3 & j_4 \end{pmatrix} \]

\[ \cong \overline{M}_{0,3} \times \overline{M}_{0,3}, \ldots, \overline{M}_{0,3} \times \phi(\psi). \overline{M}_{0,n-3-(s-\frac{t}{2})} \]

\[ = (-1)^{s+\frac{t}{2}} \left( n - 3 - (s - \frac{t}{2}) \right) \]

As there are \( (n-r)! \) ways of choosing the uncolored half-edges on the forks, corresponding to \( j_i \)'s, the term evaluates to

\[ = (-1)^{s+\frac{t}{2}} \frac{(n-r)!}{(n-r-s+t)!} \left( n - 3 - (s - \frac{t}{2}) \right) \]

\[ \left( k_{i_1} + k_{i_2} - 1, \ldots, k_{i_{s-1}} + k_{i_s} - 1, k_{i_{s+1}} - 1, \ldots, k_{i_t}, \ldots, k_{i_t}, \ldots, k_{i_{s-1}}, k_{i_0} \right) \]

\[ \square \]

### 4.1 Generating Function for the top intersections

We start with the generating function- Witten-potential (8). The correlation functions are defined as intersection numbers on the moduli space of stable n-pointed curves (here for genus 0) as

\[ \langle \tau_{k_1} \cdots \tau_{k_n} \rangle := \int_{\overline{M}_{0,n}} \psi_{k_1}^{i_1} \psi_{k_2}^{i_2} \cdots \psi_{k_n}^{i_n} \]

Collecting all tau’s with equal exponent, we can write \( \langle \tau_{k_1} \cdots \tau_{k_n} \rangle = \langle \tau_0^{s_0} \tau_1^{s_1} \tau_2^{s_2} \cdots \tau_m^{s_m} \rangle \). Now, define \( s = (s_0, s_1, \ldots) \), and \( \langle \tau^s \rangle := \langle \tau_0^{s_0} \tau_1^{s_1} \tau_2^{s_2} \cdots \tau_m^{s_m} \rangle \). So, for each sequence \( s \), there is a correlation function \( \langle \tau^s \rangle \); and \( |s| := \sum s_i \) is the number of marks \( n \). For the generating function, all these correlation functions are collected and used as coefficients in a formal power series.

Using notation \( t^s = \prod_{i=0}^{\infty} t_i^{s_i} \), and \( s! = \prod_{i=0}^{\infty} s_i! \), the generating function is

\[ F(t) := \sum_{s} \frac{t^s}{s!} \langle \tau^s \rangle \]

\[ (1) \frac{t_0^3}{3!} + (1) \frac{t_0^2 t_1}{3!} + (1) \frac{t_0^4 t_2}{4!} + (2) \frac{t_0^3 t_1^2}{3! 2!} + (1) \frac{t_0^5 t_3}{5!} + (3) \frac{t_0^4 t_1 t_2}{4!} + (6) \frac{t_0^3 t_1^3}{3! 3!} + \ldots \]

where the coefficients of appropriate terms give the intersection numbers
\[ \int_{\hat{M}_{0,n}} \hat{\psi}_{k_1} \hat{\psi}_{k_2} \ldots \hat{\psi}_{k_n}. \] Observe that the total codimension of the integrand in \( \langle \tau^s \rangle \) is \( \sum i s_i \), so \( |s| - 3 = \sum i s_i \). With this generating function, the String equation for \( \hat{M}_{0,n} \) is the differential equation

\[ \frac{\partial}{\partial t} F = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial}{\partial t_i} F. \]

**Definition 4.1.** Define a new generating function

\[ G(t) := \sum_s t^s \langle \hat{\tau}^s \rangle \]

where \( \langle \hat{\tau}^s \rangle = \int_{\hat{M}_{0,(k_1^{i_1})^{k_1} \ldots k_n^{i_n}}} \hat{\psi}_{k_1} \hat{\psi}_{k_2} \ldots \hat{\psi}_{k_n} \) as defined earlier, \( \sum k_i = n - 3 \).

So, \( G(t) \) has as coefficients of the monomials \( t^s/s! \) the intersection numbers \( \int_{\hat{M}_{0,n}} \hat{\psi}_{k_1} \hat{\psi}_{k_2} \ldots \hat{\psi}_{k_n} \) for any value of \( n \) and any values of \( k_i \)'s, with \( \sum k_i = n - 3 \).

**Theorem 4.1.** With \( G(t) \) as defined above,

\[ G(t) = \hat{\mathcal{L}}(F(t)) - \frac{t_0^3}{3!} \]

where \( \hat{\mathcal{L}} = : e^{-\mathcal{L}} : \), and \( : e^{-\mathcal{L}} : \) denotes the operator with normal ordering\(^1\).

and

\[ \mathcal{L} = \frac{1}{2} \sum_{i,j=0}^{\infty} t_i t_j \frac{\partial}{\partial t_{i+j-1}} \]

Before we prove the above theorem, consider the \( \hat{\psi} \)- monomial \( \int_{\hat{M}_{0,n}} \hat{\psi}_{k_1}^{k_1} \hat{\psi}_{k_2}^{k_2} \ldots \hat{\psi}_{k_n}^{k_n} \). It is the coefficient of \( t \)-monomial \( \frac{t_{0}^{(n-1)} t_{n-3}}{(n-1)!} \) in \( G(t) \). When we apply the operator \( \hat{\mathcal{L}} = 1 - \mathcal{L} + : \frac{L^2}{2!} : \ldots \) to \( F(t) \), only the terms with the following \( t \)-monomials in \( F(t) \) contribute to the term with \( t \)-monomial \( \frac{t_{0}^{(n-1)} t_{n-3}}{(n-1)!} \) in \( \hat{\mathcal{L}}(F(t)) \): \( \frac{t_{0}^{(n-1)} t_{n-3}}{(n-1)!} \) and \( \frac{t_{0}^{(n-2)} t_{n-4}}{(n-2)!} \).

\(^1\)by normal ordering of the operator, we mean that we treat the \( t_i \)'s and \( \frac{\partial}{\partial t_i} \)'s as commuting variables, and bring all \( t_i \)'s to the left of \( \frac{\partial}{\partial t_i} \)'s. E.g., if \( \mathcal{J} = t_i t_j \frac{\partial}{\partial t_{i+j-1}} \),

\[ \mathcal{J}^2 = t_i t_j \frac{\partial}{\partial t_{i+j-1}} t_i t_j \frac{\partial}{\partial t_{i+j-1}}, \]

but

\[ : \mathcal{J}^2 : = t_i^2 t_j^2 \frac{\partial}{\partial t_{i+j-1}} \frac{\partial}{\partial t_{i+j-1}}. \]
The first term of \( \tilde{L} \) is 1, which when acts on \( F(t) \) produces the \( t \)-monomial \( \frac{t_0^{(n-1)}t_{n-3}}{(n-1)!} \) as it is; the second operates as follows:

\[
-t_0t_{n-3}\frac{\partial}{\partial t_{n-4}} \left( \frac{t_0^{(n-2)}t_{n-4}}{(n-2)!} \right) = -(n-1)\frac{t_0^{(n-1)}t_{n-3}}{(n-1)!}.
\]

No other term in \( F(t) \) contributes to the coefficient of the \( t \)-monomial \( \frac{t_0^{(n-1)}t_{n-3}}{(n-1)!} \) in \( \tilde{L}(F(t)) \).

As the monomial \( \frac{t_0^{(n-1)}t_{n-3}}{(n-1)!} \) has as coefficient \( \int_{M_{0,n}} \hat{\psi}_1^{n-3} \), and the monomial \( \frac{t_0^{(n-2)}t_{n-4}}{(n-2)!} \) has as coefficient \( \int_{M_{0,n-1}} \hat{\psi}_1^{n-4} \) in \( F(t) \), the coefficient of \( \frac{t_0^{(n-1)}t_{n-3}}{(n-1)!} \) in \( \tilde{L}(F(t)) \) is \( \int_{M_{0,n}} \psi_1^{n-3} - (n-1) \int_{M_{0,n-1}} \psi_1^{n-4} \) which equals \( \int_{M_{0,(k_1)n}} \hat{\psi}_1^{n-3} \) from corollary (4.1).

Observe that both the contributions correspond to the two types of \( \mathcal{P} \)-graphs that make non-zero contributions to \( \int_{M_{0,n}} \hat{\psi}_1^{n-3} \) in corollary (4.1). The first are of type with no forks; the second of type with one fork.

**Definition 4.2.** For each \( \mathcal{P} \)-graph, the corresponding \( \mathcal{P}_k \)-graph is defined by replacing each \( i \in [n] \) on the \( \mathcal{P} \)-graph by \( k_i \).

Clearly the map \( \{ \mathcal{P} \text{-graphs} \} \rightarrow \{ \mathcal{P}_k \text{-graphs} \} \) is a surjection.

**Lemma 4.1.** In genus 0, for a given \( \langle \tau_{k_1}, \ldots, \tau_{k_n} \rangle = \int_{M_{0,n}} \psi_1^{k_1} \psi_2^{k_2} \ldots \psi_n^{k_n} \) as defined above, let \( s = (s_0, s_1, \ldots, s_n) \), and \( \langle \tau_{k_1}, \ldots, \tau_{k_n} \rangle = \langle \tau_s^{0}, \tau_s^{1}, \tau_s^{2}, \ldots, \tau_s^{n} \rangle \). Consider a \( \mathcal{P}_k \)-graph with \( m \) forks with \( q \) distinct \( k_i \)'s appearing on the forks; let such \( k_i \)'s be \( \{k_1, k_2, \ldots, k_q\} \). Let \( l_i \) be the number of times a given \( k_i \) appears on any fork on the \( \mathcal{P}_k \)-graph. Then the number of \( \mathcal{P} \)-graphs that map to this \( \mathcal{P}_k \)-graph is given by:

\[
\frac{1}{|Aut(\mathcal{P}_k)|} \frac{s_{k_1}!}{(s_{k_1} - l_1)!} \frac{s_{k_2}!}{(s_{k_2} - l_2)!} \cdots \frac{s_{k_q}!}{(s_{k_q} - l_q)!} =: C_{\mathcal{P}_k}
\]

where \( |Aut(\mathcal{P}_k)| \) is the number of automorphisms of the subgraph of \( \mathcal{P}_k \)-graph obtained from removing half-edges on the central node.

**Proof.** Consider a \( \mathcal{P}_k \)-graph with \( m \) number of forks s.t \( q \) number of \( k_i \)'s appear on the forks as defined above. Then if all \( n \) half-edges are given ordering, the number of \( \mathcal{P} \)-graphs where half-edges are ordered would be

\[
s_1!s_2! \ldots s_n!.
\]

Now, we divide by the permutations of half-edges on the central node to get

\[
\frac{s_1!}{(s_1 - l_1)!} \frac{s_2!}{(s_2 - l_2)!} \cdots \frac{s_n!}{(s_n - l_n)!}.
\]
As only $q$ number of $k_i$'s appear on the forks, $\ell_i = 0$ for $j > q$, so

$$\frac{s_1!}{(s_1 - \ell_1)!} \frac{s_2!}{(s_2 - \ell_2)!} \cdots \frac{s_n!}{(s_n - \ell_n)!} = \frac{s_{k_1}!}{(s_{k_1} - \ell_1)!} \frac{s_{k_2}!}{(s_{k_2} - \ell_2)!} \cdots \frac{s_{k_q}!}{(s_{k_q} - \ell_q)!}$$

Further we need to divide by permutations of half-edges on the forks. Let $j_1, j_2, \ldots, j_f$ be the number of forks with the same set of $k_i$'s on them; and let $d$ be the number of forks with both $k_i$'s same on that fork. Then, we divide by $2^d (j_1! j_2! \cdots j_f!)$ to get

$$\left( \frac{1}{2^d (j_1! j_2! \cdots j_f!)} \right) \frac{s_{k_1}!}{(s_{k_1} - \ell_1)!} \frac{s_{k_2}!}{(s_{k_2} - \ell_2)!} \cdots \frac{s_{k_q}!}{(s_{k_q} - \ell_q)!}$$

Observe that the number $2^d (j_1! j_2! \cdots j_f!)$ is the number of automorphisms of the subgraph of $P_k$-graph consisting of only the forks; denote this sub-graph as $P_k$. Then the number of $P$-graphs that map to this $P_k$-graph can be rewritten as :

$$\frac{1}{|\text{Aut}(P_k)|} \frac{s_{k_1}!}{(s_{k_1} - \ell_1)!} \frac{s_{k_2}!}{(s_{k_2} - \ell_2)!} \cdots \frac{s_{k_q}!}{(s_{k_q} - \ell_q)!}$$

☐

The reason for organizing $C_{P_k}$ as in (23) will become clear in the proof of theorem 4.1.

Lemma 4.2. With the definitions and notations above, corollary 4.1 can be rewritten as :

$$\int_{M_0, (\frac{1}{n})^n} \hat{\psi}_1^{k_1} \hat{\psi}_2^{k_2} \cdots \hat{\psi}_n^{k_n} = \sum_{P_k \in \Omega} (-1)^m C_{P_k} \int_{M_0, (n-m)} \psi_{i_1}^{k_{i_1}+\cdots+k_{i_{k_1}}} \cdots \psi_{i_{m}+\cdots+k_{i_{k_2}}}^{k_{i_{m}}+\cdots+k_{i_{k_2}}} \cdots \psi_{i_{m+l-1}}^{k_{i_{m+l-1}}} \cdots \psi_{i_{m+l}}^{k_{i_{m+l}}} \cdots \psi_{i_{r}}^{k_{i_{r}}}$$

(24)

where $m$ is number of forks on the $P_k$-graph and $C_{P_k}$ is the number of $P$-graphs that map to this $P_k$-graph, and $\Omega$ is the set of all $P_k$-graphs.

Proof. This version of closed formula for $\int_{M_0, (\frac{1}{n})^n} \hat{\psi}_1^{k_1} \hat{\psi}_2^{k_2} \cdots \hat{\psi}_n^{k_n}$ is just a reorganization of (17) using $P_k$-graphs instead of $P$-graphs. As the map $\{P$-graphs$\} \to \{P_k$-graphs$\}$ is a surjection, we get all the terms in (17). ☐

Now, for a general $P_k$-graph with $m$ number of forks shown below, define the following operator (which appears in $\mathcal{L}$):

$$D_{P_k} := t_{k_1} t_{k_2} \cdots t_{k_{i_2m-1}} t_{k_{i_2m}} \frac{\partial}{\partial t_{k_{i_2m-1}+k_{i_2m-1}}} \cdots \frac{\partial}{\partial t_{k_{i_2m-1}+k_{i_2m-1}}}$$

(25)
and the following term in (24):

$$(-1)^m C_{\mathcal{P}_k} \int_{\mathcal{M}_{0,(n-m)}} \hat{\psi}_{k_1}^{k_1} \hat{\psi}_{k_2}^{k_2+1} \ldots \hat{\psi}_{m}^{k_{i_1}+k_{i_2}+1} \ldots \hat{\psi}_{ir}^{k_{ir}+1} \hat{\psi}_{ir+1}^{k_{ir+1}} \ldots \hat{\psi}_{ir+m-1}^{k_{ir+m-1}+1} \hat{\psi}_{ir+m}^{k_{ir+m}+1} \ldots \hat{\psi}_{ir+m+k_{ir+m+1}}^{k_{ir+m+k_{ir+m+1}}+1} \ldots \hat{\psi}_{ir+m+k_{ir+m+k_{ir+m+1}}}^{k_{ir+m+k_{ir+m+k_{ir+m+1}}}}$$

(26)

By construction, the terms (25) are in bijection with the $\mathcal{P}_k$-graphs. Furthermore, the term (25) arises in $\hat{\mathcal{L}}$ as a summand in $(-1)^m \frac{1}{|\text{Aut}(\mathcal{P}_k)|}$ with some multiplicity. As part of the proof of theorem, we will see that this multiplicity is $(-1)^m \frac{1}{|\text{Aut}(\mathcal{P}_k)|}$ as defined in Lemma (4.1). And the term (26) is a summand in (24) corresponding to this $\mathcal{P}_k$-graph.

Strategy of proof of theorem (4.1): we will show that for a general t-monomial $t_0^{s_0} t_1^{s_1} \ldots t_l^{s_l}$, its coefficients in $G(t)$ and $\hat{\mathcal{L}}(F(t)) - \frac{t_0^3}{3!}$ are equal. And that both equal $\int_{\mathcal{M}_{0,(n)}} \hat{\psi}_1 \hat{\psi}_2 \ldots \hat{\psi}_n$. To show that, we show a bijection between $\mathcal{P}_k$ graphs and the terms (26) which are the summands in the formula (24). To show this bijection, we pick a $\mathcal{P}_k$ graph, and find the term (25) in $\hat{\mathcal{L}}$ (with some multiplicity); then we find the term $\mathcal{T}_{\mathcal{P}_k}$ in $F(t)$, such that the term (25) when applied to $\mathcal{T}_{\mathcal{P}_k}$ gives the t-monomial $t_0^{s_0} t_1^{s_1} \ldots t_l^{s_l}$ with coefficient the intersection number defined in (26). As that exactly is the summand in (24) corresponding to the chosen $\mathcal{P}_k$-graph, this proves the theorem in one direction. In the other direction, we pick a term (26) which is a summand in the coefficient of t-monomial $t_0^{s_0} t_1^{s_1} \ldots t_l^{s_l}$, and show that this maps to the same $\mathcal{P}_k$-graph.

Before the proof, here is an example that illustrates the idea.

**Example 4.1.** Consider

$$\int_{\mathcal{M}_{0,(\frac{1}{2})}} \hat{\psi}_1 \ldots \hat{\psi}_8 \hat{\psi}_{9} \hat{\psi}_{10} \hat{\psi}_{11} \hat{\psi}_{12} \hat{\psi}_{13} \hat{\psi}_{14} \hat{\psi}_{15} \hat{\psi}_{16} \hat{\psi}_{17} \hat{\psi}_{18} \hat{\psi}_{19} \hat{\psi}_{20} \hat{\psi}_{21} \hat{\psi}_{22}$$

The corresponding t-monomial in $G(t)$ is

$$\frac{t_0^{39} t_1^8 t_2^5 t_3^4 t_4^3 t_5^2 t_6^2}{s_0 s_1! \ldots s_l!} = \frac{39895!}{32!2!} =: \mathcal{T}_{\mathcal{P}_k}$$
Now consider the following $P_k$-graph with $m = 7$ forks as in the following figure.

![Figure 4.2: $P_k$-graph with $m = 7$ forks](image)

The corresponding operator (25) is

$$D_{P_k} = t_1^7 t_2 t_3 t_4 \frac{\partial^3}{\partial t_1^3} \frac{\partial^2}{\partial t_2^2} \frac{\partial}{\partial t_3} \frac{\partial}{\partial t_4}$$

The coefficient of this term in $(-1)^7 \frac{\mathcal{L}_T}{7!}$ is given by

$$(-1)^7 \left( \frac{1}{2^7 7!} \right) \left( 2^4 \left( \frac{7}{3, 2, 1, 1} \right) \right) = (-1)^7 \left( \frac{1}{2^3 3! 2!} \right)$$

The corresponding unique term in $F(t)$ is

$$\frac{t_0^3 t_1^3 t_2^3 t_3^2 t_4^1 t_5^2 t_6^3}{39! 3! 2! 1! 2! 3!} =: \tilde{T}_{P_k}$$

In $\hat{L}(F(t))$, the corresponding term is

$$(-1)^7 \left( \frac{1}{2^3 3! 2!} \right) \left( \langle \tau^s \rangle \right) D_{P_k}(\tilde{T}_{P_k})$$

where

$$\langle \tau^s \rangle = \int_{\mathcal{M}} \psi_1 \psi_2 \psi_3 \psi_4 \psi_5 \psi_6 \psi_7 \psi_8 \psi_9 \psi_{10} \psi_{11} \psi_{12} \psi_{13} \psi_{14} \psi_{15}$$

Observe that the coefficient $(-1)^7 \left( \frac{1}{2^3 3! 2!} \right)$ of $D_{P_k}$ in $(-1)^7 \frac{\mathcal{L}_T}{7!}$ is exactly $(-1)^m \frac{1}{|\text{Aut}(P_k)|}$ as claimed earlier. Now,

$$(-1)^7 \left( \frac{1}{2^3 3! 2!} \right) \left( \langle \tau^s \rangle \right) D_{P_k}(\tilde{T}_{P_k})$$

$$= (-1)^7 \left( \frac{1}{2^3 3! 2!} \right) \left( \frac{8! 5! 2! 2! 3!}{(8 - 7)! (5 - 5)! (2 - 1)! (2 - 1)! (3 - 1)!} \right) \left( \langle \tau^s \rangle \right) T_{P_k}$$

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\[ = (-1)^7 \left( \frac{1}{2^33!^2} \right) \left( \frac{8!5!2!2!3!}{2!} \right) ((\tau^s)) T_{P_k} \]
\[ = (-1)^7 \left( \frac{1}{\text{Aut}(P_k)} \right) \left( \frac{8!5!2!2!3!}{2!} \right) ((\tau^s)) T_{P_k} \]
\[ = (-1)^7 C_{P_k} ((\tau^s)) T_{P_k} \]

which term in \( \hat{\mathcal{L}}(F(t)) \) has as coefficient of \( T_{P_k} \) exactly the term \( [26] \) which is the summand in \( [24] \) corresponding to the chosen \( P_k \)-graph.

Proof. (of theorem (4.1))

Consider a general term in \( G(t) \) with the corresponding \( t \)-monomial \( \frac{t_0^{s_0}t_1^{s_1}...t_k^{s_k}}{s_0!s_1!...s_k!} \), with coefficient \( \int_{\mathbb{M}_0(n-1)} \psi_1^{k_1}\psi_2^{k_2}...\psi_n^{k_n} \). Now, we will show that we get all the terms of formula (24) in \( \hat{\mathcal{L}}(F(t)) = (1 - : L : + : \frac{\ell_2^2}{2} : - ... : (-1)^m \frac{m!}{m!} : + ...)(F(t)) \) and that each term corresponds to a \( P_k \)-graph.

1. Pick a \( P_k \)-graph with no fork. Associated operator \( [25] \) in \( \hat{\mathcal{L}} \) is 1, the first term in \( \hat{\mathcal{L}} \), which when applied to \( F(t) \) results in coefficient 1 for \( \frac{t_0^{s_0}t_1^{s_1}...t_k^{s_k}}{s_0!s_1!...s_k!} \) in \( \hat{\mathcal{L}}(F(t)) \).

2. Pick a \( P_k \)-graph with one fork. WLOG, assume a \( P_k \)-graph with \( k_1, k_2 \) on a single fork, with \( k_1, k_2 \) not simultaneously 0.

Case 1 : \( k_1 \neq k_2 \). Then, the corresponding operator \( [25] \) as \( t_{k_1}t_{k_2}\frac{\partial}{\partial t_{k_1+k_2-1}} \). In \( \hat{\mathcal{L}} \), this term has coefficient \(-1\). The term in \( F(t) \) that it operates on to produce \( \frac{t_0^{s_0}...t_{k_1}^{s_{k_1}}...t_{k_2}^{s_{k_2}}...t_{k_1+k_2-1}^{s_{k_1+k_2-1}}}{s_0!...s_{k_1-1}!...s_{k_2-1}!...s_1!} \) has \( t \)-monomial

\[ \frac{t_0^{s_0}...t_{k_1}^{s_{k_1}}...t_{k_2}^{s_{k_2}}...t_{k_1+k_2-1}^{s_{k_1+k_2-1}}}{s_0!...s_{k_1-1}!...s_{k_2-1}!...s_1!} \]

The result of applying in \( -t_{k_1}t_{k_2}\frac{\partial}{\partial t_{k_1+k_2-1}} \) in \( \hat{\mathcal{L}} \) to \( F(t) \) is the following :

\[ -t_{k_1}t_{k_2}\frac{\partial}{\partial t_{k_1+k_2-1}} \left( \frac{t_0^{s_0}...t_{k_1}^{s_{k_1}}...t_{k_2}^{s_{k_2}}...t_{k_1+k_2-1}^{s_{k_1+k_2-1}}}{s_0!...s_{k_1-1}!...s_{k_2-1}!...s_1!} \right) \]
\[ = -s_{k_1}s_{k_2} \frac{t_0^{s_0}...t_{k_1}^{s_{k_1}}...t_{k_2}^{s_{k_2}}}{s_0!s_1!...s_1!} \]

So, the coefficient contributed by this operator to \( \frac{t_0^{s_0}t_1^{s_1}...t_k^{s_k}}{s_0!s_1!...s_k!} \) in \( \hat{\mathcal{L}}(F(t)) \) is

\[ -s_{k_1}s_{k_2} \int_{\mathbb{M}_0(n-1)} \psi_1^{k_1}\psi_2^{k_2-1}\psi_3^{k_3}...\psi_r^{k_r} \]

and \( C_{P_k} = s_{k_1}s_{k_2} \) is the number of \( \mathcal{P} \)-graphs that map to this kind of \( \mathcal{P}_k \)-graph.
Case 2 : \( k_1 = k_2 \). In this case we get the corresponding term (25) as 
\[ t^2 k_1 \frac{\partial}{\partial t_{2k_1-1}} \] 
The coefficient of this term in \( \hat{L} \) is \( -\frac{1}{2} \). When \( -\frac{1}{2} t^2 k_1 \frac{\partial}{\partial t_{2k_1-1}} \) is applied to \( F(t) \), the only terms that produce \( \frac{t_0^{s_0} t_1^{s_1} \ldots t_k^{s_k}}{s_0! \ldots s_{k-1}! \ldots s_l!} \) is 
\[ \frac{t_0^{s_0} t_1^{s_1} \ldots t_{k_1}^{s_{k_1-2}} \ldots t_{k_1}^{s_{k_1}}}{s_0! \ldots s_{k_1-2}! \ldots s_l!} t_{2k_1-1} \]
And
\[ -\frac{1}{2} t^2 k_1 \frac{\partial}{\partial t_{2k_1-1}} \left( \frac{t_0^{s_0} t_1^{s_1} \ldots t_{k_1}^{s_{k_1}}}{s_0! \ldots s_{k_1-2}! \ldots s_l!} t_{2k_1-1} \right) \]
\[ = -\frac{1}{2} s_{k_1} (s_{k_1} - 1) \frac{t_0^{s_0} t_1^{s_1} \ldots t_{k_1}^{s_{k_1}}}{s_0! s_1! \ldots s_l!} \]
So, the coefficient contributed by this operator to \( \frac{t_0^{s_0} t_1^{s_1} \ldots t_{k_1}^{s_{k_1}}}{s_0! s_1! \ldots s_l!} \) in \( \hat{L}(F(t)) \) is 
\[ -\frac{1}{2} s_{k_1} (s_{k_1} - 1) \int_{\mathcal{P}_0(n-1)} \psi_1^{2k_1-1} \psi_2^{k_2} \ldots \psi_r^{k_r} \]
and \( C_{P_k} = \frac{1}{2} s_{k_1} (s_{k_1} - 1) \) is the number of \( P \)-graphs that that map to to this kind of \( P_k \)-graph.

In both cases, the terms contribute \( -C_{P_k} \int_{\mathcal{P}_0(n-1)} \psi_1^{k_1+j_2-1} \psi_2^{k_3} \ldots \psi_r^{k_r} \) to the coefficient of t-term \( \frac{t_0^{s_0} t_1^{s_1} \ldots t_{k_1}^{s_{k_1}}}{s_0! s_1! \ldots s_l!} \) in \( \hat{L}(F(t)) \). So, we get both terms in (23) corresponding to two \( P_k \)-graphs with one fork. Also, observe that if \( k_1 = k_2 = 0 \), the term \( -t^2 k_1 \frac{\partial}{\partial t_{2k_1-1}} \) contributes nothing.

Now consider a \( P_k \)-graph with \( m \) number of forks. WLOG, let the \( k_i \)'s on the forks be \( \{k_1, k_2, \ldots, k_{2m}\} \) as shown in figure below. Let \( l_i \) be the number of times a given \( k_i \) appears on any fork on the \( P_k \)-graph, and let \( j_1, j_2, \ldots, j_f \) be the number of forks with the same set of \( k_i \)'s on them; and let \( d \) be the number of forks with both \( k_i \)'s same on that fork.

Then the corresponding operator (25) is 
\[ t_{k_1} t_{k_2} \cdots t_{k_{2m-1}} t_{k_{2m}} \frac{\partial}{\partial t_{k_1 + k_2 - 1}} \ldots \frac{\partial}{\partial t_{k_{2m-1} + t_{2m} - 1}} := D_{P_k} \]
The coefficient of this term in \( \hat{L} \) as a summand in \(( -1)^m \frac{L^m}{m!} \) is given by

\[
( -1)^m \left( \frac{1}{2^m m!} \right) \left( \begin{array}{l} 2^m-d \\ t_1, t_2, \ldots, t_n \end{array} \right) = ( -1)^m \left( \frac{1}{2^d t_1! t_2! \ldots t_n!} \right)
\]

The corresponding term in \( F(t) \) is

\[
t_0 s_0 \ldots t_k \left( \frac{1}{s_1 \ldots s_{k_1} \ldots s_{k_2m-l_{k_2m}}} \right) \frac{1}{s_{k_1} - l_{k_1}! \ldots s_{k_2m} - l_{k_2m}!} t_{k_1+k_2-1} \ldots t_{k_2m-1} = : \hat{T}_{p_k}
\]

In \( \hat{L}(F(t)) \), the corresponding term is

\[
( -1)^m \left( \frac{1}{2^d t_1! t_2! \ldots t_n!} \right) \langle \tau^s \rangle D_{p_k}(\hat{T}_{p_k})
\]

where \( \langle \tau^s \rangle \) is the appropriate \( \psi \)-monomial on \( M_{0,\{1\}}(n-m) \) that appears as coefficient of \( \hat{T}_{p_k} \) in \( F(t) \). Observe that the coefficient \( ( -1)^m \left( \frac{1}{2^d t_1! t_2! \ldots t_n!} \right) \) of \( D_{p_k} \) in \(( -1)^m \frac{L^m}{m!} \) is exactly \(( -1)^m \frac{1}{|Aut(\hat{T}_{p_k})|} \) as claimed earlier. Now,

\[
( -1)^m \left( \frac{1}{2^d t_1! t_2! \ldots t_n!} \right) \langle \tau^s \rangle D_{p_k}(\hat{T}_{p_k})
\]

\[
= ( -1)^m \left( \frac{1}{|Aut(\hat{T}_{p_k})|} \right) \left( \frac{s_1!}{(s_1 - l_1)! (s_2 - l_2)!} \ldots \frac{s_n!}{(s_n - l_n)!} \right) \langle \tau^s \rangle \n = ( -1)^m \left( \frac{1}{|Aut(\hat{T}_{p_k})|} \right) \left( \frac{s_1!}{(s_1 - l_1)! (s_2 - l_2)!} \ldots \frac{s_n!}{(s_n - l_n)!} \right) \langle \tau^s \rangle \n = ( -1)^m C_{p_k} \langle \tau^s \rangle \hat{T}_{p_k}
\]

which term in \( \hat{L}(F(t)) \) has as coefficient of \( T_{p_k} \) exactly the term \([26] \) which is the summand in \([24] \) corresponding to the chosen \( p_k \)-graph.

So, one direction is proved. To show bijection in the other direction, we pick a summand in the coefficient of \( T_{\frac{t_0}{n^m t_1 \ldots t_n}} \) = \( T_{p_k} \) in \( \hat{L}(F(t)) \) that comes from term \( \frac{(( -1)^m L^m)}{m!} \). Let this summand come from the following summand in \( \frac{(( -1)^m L^m)}{m!} \):

\[
( -1)^m \left( \frac{1}{2^d t_1! t_2! \ldots t_n!} \right) \langle \tau^s \rangle D_{p_k}(\hat{T}_{p_k})
\]

where

\[
D_{p_k} = t_{k_1} t_{k_2} \ldots t_{k_{2m-1}} t_{k_{2m}} \frac{\partial}{\partial t_{k_1+k_2-1}} \ldots \frac{\partial}{\partial t_{k_{2m-1}+t_{2m}-1}}
\]
\( \langle \tau^s \rangle \) is uniquely determined by \( \hat{T}_{P_k} \), and \( \hat{T}_{P_k} \) is uniquely determined by

\[
D_{P_k}(\hat{T}_{P_k}) = T_{P_k}
\]

As this \( D_{P_k} \) is in bijection with the \( P_k \)-graph as in the figure by construction, we get the term in \( (24) \) corresponding to this \( P_k \)-graph as the chosen summand in \( \hat{L}(F(t)) \). So the coefficient of \( \frac{t_0^{s_0}t_1^{s_1}\cdots t_l^{s_l}}{s_0!s_1!\cdots s_l!} \) in \( \hat{L}(F(t)) \) equals the coefficient of \( \frac{t_0^{s_0}t_1^{s_1}\cdots t_l^{s_l}}{s_0!s_1!\cdots s_l!} \) in \( G(t) \).

\[
\square
\]

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