Accounting for dissipation of the energy of the supple base

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Annotation. When calculating the foundations and structures on a compliant basis for the action of dynamic effects, the mathematical model of the base should reflect the real conditions for deformation of the base, due to their lamination, limited by the distribution property, weight and others. This is more, in our opinion, the axisymmetric columnar model of a compliant base is responsible, which takes into account the important properties of the ground base at a dynamic effect, as the extremity of the oscillation distribution zone and its inertia. In this case, the model allows us to determine not only the movement of the boundary plane, but also the stress-strain state of the base. With resonant phenomena, the accounting of energy dissipation during oscillations of the base is essential. The paper discusses the adaptation of the axisymmetric columnar model on a dissipative base, in which the dispersion of energy is carried out according to the complex hypothesis by E.S. Sorokin. The model in question is a cylindrical body in the form of a rod height \( H \) and radius \( R \), large enough to believe that the amplitudes of displacements on the cylindrical surface of the rod are zero, when the rod on the outer end is loading within the centrally located circular area with a radius \( P \), much smaller \( R \). The basis, obeying the three-axis linear physical law, is in the conditions of axial symmetry, therefore, only vertical \( u \) and radial \( w \) movements are observed, which are functions of variables \( x, r, t \). The solution of the dynamic Liame equations described by the deformation of the cylindrical body was carried out by the discrete method of L.V. Vinokurov, according to which the solution was found in an analytical form in the variable \( x \), and in the variable \( r \) - in the finite-difference. The use of a method of separation of variables by Fourier, a system of discrete differential equations in private derivatives was reduced to a system of differential equations in ordinary derivatives, the integration of which was carried out using Euler substitutions. Permanent integration was determined from the boundary conditions at the end of the cylindrical rod. The solutions obtained show that energy dissipation accounting causes a phase shift between the perturbing load and the movement of the base and to a significant decrease in the amplitudes of oscillations in the resonance zone.

In the calculation of foundations on an elastic basis, the reliability of the theoretical calculation results largely depends on the adopted mathematical model of the ground base, which, from our point of view, should meet two basic requirements: adequacy of its real conditions; mathematical simplicity, allowing us to use it in engineering practice.

This, in our opinion, with sufficient completeness corresponds to an axisymmetric columnar model of a compliant base [1, 2, 3], which takes into account the important properties of the ground base at a dynamic effect, as the extremity of the distribution zone of oscillations and its inertia. In this case, the model allows us to determine not only the movement of the boundary plane, but also the stress-strain state of the base.

With dynamic calculations of the foundations of industrial structures and machines with a large spectrum of frequencies, the accounting of irreversible losses of the oscillation energy at the base due to the presence of internal inelastic resistance is very significant, especially for the occurrence of resonance. Therefore, there is a generalization of an axisymmetric column model on a dissipative base.

In most cases, accounting the dissipation of energy is made according to the Foyht hypothesis of viscous friction, according to which the forces of inelastic resistance is the linear function of the deformation rate. Foyht hypothesis of viscous friction as a physical constant uses attenuation coefficient. It is convenient in mathematical terms, but contradicts experimental data. So according to
the hypothesis of viscous friction, the attenuation coefficient and the attenuation decrement are proportional to the frequency, but in reality they do not depend on the frequency of oscillations. The disadvantage of this hypothesis is also that it does not reflect the dependence of inelastic resistance in the material from the amplitude of deformation.

In our opinion, E.S. Sorokin hypothesis of the hysteresis friction more corresponds to the reality, in which the forces of inelastic resistance are proportional to the movement and as a physical constant uses the coefficient of inelastic resistance. E.S. Sorokin hypothesis provides that the forces of inelastic resistance are proportional to the elastic ones, but they shifted relative to the elastic phase at the angle $\pi/2$. According to this hypothesis, internal resistance forces are presented in a comprehensive form.

\[ S^* = (1 + i)S, \quad (*) \]

where: $S^*$ – Total (elastic + elastic) internal force;
S – elastic force;
$\gamma = \frac{\psi}{2\pi}$ – the coefficient of inelastic resistance;
$\psi$ – the absorption coefficient, which depends on the physical properties of the material;
the imaginary number $i = \sqrt{-1}$ – provides a phase shift by 90°.

The coefficient $\gamma$ is the functions of the amplitude of movements, but in most cases you can take $\gamma = \text{const}$ for a wide range of movement amplitudes.

When determining the precipitate of the base and its stress-strain state, according to the axisymmetric columnar model of the base, the zone in which the localization of movements and stresses from a uniformly distributed load occurs $q(t) = F(t)/\pi \cdot r^2$ within the centrally located platform, the radius, simulating a circular cylindrical pillar $H$ high and a finite radius $R > \tilde{r}$ in cross section (picture 1), which coincides with the radius of the attenuation of movements on the boundary plane of the base.

Connect the beginning of the cylindrical coordinate system $x$, $y$, $\theta$ with an application point of force $F(t)$, and the vertical axis $x$ will be directed according to force $F(t)$ to the lower end of the cylinder. The considered cylindrical body is in the conditions of axial symmetry, therefore only radial $w$ and vertical $u$ movements will be observed (tangential $v = 0$), which are functions of variables $x$, $r$ and $t$.

The problem of movements and stresses of the compliant layer from dynamic loads will be solving in the absence of displacements $u$ and $w$ on the outer cylindrical column surface of the radius $R$ isolated from the base. The area of the cargo platform of the $\tilde{r}$ radius is consistent with the area of sections, which is divided by the foundation base when the distribution of reactive force will possibly be represented as a step-by-step diagram.
The solution of Liame equations, which describe the deformation of the cylindrical body will carry out by L.V. Vinokurov discrete method [4]. According to the variable $x$, the solution is tried in an analytical form, and according to the variable $r$ - in the finite-difference. We approximate the volume of the cylindrical body under consideration by the prisms dividing by its longitudinal sections passing through the vertical axis $x$ and the components of the angles $\theta$, and concentric circles. The movement of the prism edges is accepted as independent indeterminates. Since the cylindrical body under consideration is in axial symmetry, then in differential equations of the discrete method, movements on vertical lines-edges (lines resulting from the intersection of a longitudinal cross section with concentric circles) located in a single diametrical plane. We divide $R$ into three equal parts (picture 2).

![Picture 2. Dividing scheme of radius R.](image)

Dynamic equilibrium of the compliant base represented by an axisymmetric columnar model, when taking into account the dissipation of energy according to the E.S. Sorokin hypothesis, is described by the following system of differential equations recorded for edges $0, j = 1, 2$

\[
(1 + i\gamma) \left[ \alpha_{\mu} \frac{\partial^2 u_0^*}{\partial \xi^2} + 2\beta_{\mu} \frac{\partial w_1^*}{\partial \xi} + 2(u_1^* - u_0^*) \right] - \frac{qb^2}{E} \rho \frac{\partial^2 u_0^*}{\partial t^2} = 0,
\]

\[
(1 + i\gamma) \left[ j^2 \frac{\partial^2 w_j^*}{\partial \xi^2} + 2\alpha_j \left[ w_{j+1}^* \left( j + \frac{1}{2} \right) + w_{j-1}^* \left( j - \frac{1}{2} \right) \right] - w_j^* \left( 2j + 1 \right) \right] +
\]

\[
+ j^2 \beta_j \frac{\partial}{\partial \xi} (u_{j+1}^* - u_{j-1}^*) \left\{ -2qj^2 \rho \frac{\partial^2 w_j^*}{\partial t^2} = 0, \right. \]

\[
(1 + i\gamma) \left[ 2\alpha_j \frac{\partial^2 u_j^*}{\partial \xi^2} + \frac{1}{2} \left( u_{j+1}^* \left( 2 + \frac{1}{j} \right) + u_{j-1}^* \left( 2 - \frac{1}{j} \right) - 4u_j^* \right) \right] +
\]

\[
\left. + 2j\beta_j \frac{\partial}{\partial \xi} \left[ \frac{1}{j^2} w_j^* + \frac{1}{2j} (w_{j+1}^* - w_{j-1}^*) \right] \right\} - \frac{2qb^2}{E} \rho \frac{\partial^2 u_j^*}{\partial t^2} = 0, \right. \]

where: $u^*$ and $w^*$ - complex movements; $\xi = \frac{x}{b}$; $q = (1 + \mu)$; $\alpha_{\mu} = \frac{1 - \mu}{1 - 2\mu}$; $\beta_{\mu} = \frac{1}{2(1 - 2\mu)}$; $\rho_{\mu} = \frac{\mu}{1 - 2\mu}$; $\mu$ - Poisson's ratio; $\rho$ - mass density of base.

Stresses at the points of edges $0$ and $j$, respectively, representing the complex harmonious stress by E.S. Sorokin in the form

\[
\sigma^* = E(1 + i\gamma)\varepsilon^*,
\]
Are defined by expressions
\[ \sigma^*_x = \frac{E(1 + iy)}{1 + \mu} \left( \alpha_{\mu} \frac{\partial u^*_0}{\partial x} + 2\rho_{\mu} \frac{w^*_1}{r_i} \right), \]
\[ \sigma^*_y = \sigma^*_z = \frac{E(1 + iy)}{1 + \mu} \left[ \alpha_{\mu} w^*_1 + \rho_{\mu} \left( \frac{1}{r_i} w^*_1 + \frac{\partial u^*_0}{\partial x} \right) \right], \]
\[ \sigma^*_x = \frac{E(1 + iy)}{2(1 + \mu)} \left[ \frac{\rho_{\mu}}{r_j} \left( 2 w^*_j + \alpha_{\mu} \rho_{\mu} \left( w^*_1 - w^*_j \right) \right) + 2 \alpha_{\mu} \frac{\partial u^*_j}{\partial x} \right], \]
\[ \sigma^*_y = \sigma^*_z = \frac{E(1 + iy)}{2(1 + \mu)} \left[ \frac{\alpha_{\mu}}{r_j} \left( 2 u^*_j + \alpha_{\mu} \rho_{\mu} \left( u^*_1 - u^*_j \right) \right) + 2 \rho_{\mu} \frac{\partial u^*_j}{\partial x} \right], \]
\[ \tau^*_x = \frac{E(1 + iy)}{2(1 + \mu)} \left[ \frac{\alpha_{\mu}}{r_j} \left( u^*_1 - u^*_j \right) + \frac{\partial w^*_j}{\partial x} \right], \]
where \( E \) – the deformation module of the base.

Accounting for the internal dispersion of energy with oscillations on the complex theory by E.S.Sorokin requires the formation of a complex excitation force \( F(t) \) for a given valid excitatory force \( F(t) \). In the case when the excitation force varies by the harmonic law \( \cos \omega t \) complex force is written as
\[ F^*(t) = F_0 \cdot e^{i\omega t}. \]  
(4)

At the same time, the imaginary part \( F(t) \) is
\[ I_m[F^*(t)] = F_0 \cdot \sin \omega t. \]  
(5)

In the future, the specified working strength is adopted for the real part of the complex force (4).

Solution of the system (1) of differential equations in private derivatives for the case of established oscillations, according to the Fourier method, we find in the form of
\[ u^*_0(\xi, t) = u^*_0(\xi) \cdot e^{i\omega t}, \]
\[ u^*_j(\xi, t) = u^*_j(\xi) \cdot e^{i\omega t}, \]
\[ w^*_j(\xi, t) = w^*_j(\xi) \cdot e^{i\omega t}, \]  
(6)

where: \( u^*_0(\xi), u^*_j(\xi), w^*_j(\xi) \) – real function variable \( \xi \).

Substitution (6) in the system (1) allows you to obtain a system of ordinary differential equations
\[ \begin{align*}
\alpha_\mu \frac{d^2 u_0}{d\xi^2} + 2\beta_\mu \frac{dw_j}{d\xi} + 2(u_i - u_0) + qk^2u_0 &= 0, \\
j^2 \frac{d^2 w_j}{d\xi^2} + 2\alpha_\mu \left[ w_{j+1} \left( j + \frac{1}{2} \right) + w_{j-1} \left( j - \frac{1}{2} \right) - w_j \left( 2j + \frac{1}{2} \right) \right] + \\
+ j^2 \beta_\mu \frac{d}{d\xi} (u_{j+1} - u_{j-1}) + 2qj^2k^2w_j &= 0, \\
2\alpha_\mu \frac{d^2 u_j}{d\xi^2} + \frac{1}{2} \left[ u_{j+1}(2 + \frac{1}{j}) + u_{j-1}(2 - \frac{1}{j}) - 4u_j \right] + \\
+ 2j\beta_\mu \frac{d}{d\xi} \left[ \frac{1}{j^2} w_j + \frac{1}{2j} (w_{j+1} - w_{j-1}) \right] + 2qk^2u_j &= 0,
\end{align*} \]

where
\[ k^2 = \frac{\rho b^2 \omega^2}{(1 + i\gamma)E}. \]

The frequency parameter \( k^2 \) is a comprehensive value.

We will integrate this system using Euler substitutions
\[ \begin{align*}
u_0(\xi) &= A_0 \cdot e^{i\xi}, \\
u_j(\xi) &= A_j \cdot e^{i\xi}, \\
w_j(\xi) &= B_j \cdot e^{i\xi}.
\end{align*} \]

After substitution (9) in (7), we obtain a system of algebraic equations, the determinant of which contains a complex frequency parameter. To solve this determinant is impossible. Therefore, from the expression (8) it is necessary to separate the real component
\[ \text{Re}(k^2) = \frac{\rho b^2 \omega^2}{(1 + \gamma^2)E}. \]

The solution of the system of homogeneous differential equations (7) is written as
\[ \begin{align*}
u_0^*(\xi, t) &= (\Sigma D^*_m \cdot e^{is\xi}) e^{iat}, \\
u_j^*(\xi, t) &= (\Sigma D^*_m \cdot a_{jm} e^{is\xi}) e^{iat}, \\
w_j^*(\xi, t) &= (\Sigma D^*_m \cdot b_{jm} e^{is\xi}) e^{iat}.
\end{align*} \]

Stress expressions (3), according to the Fourier method, is written in the form
\[ \begin{align*}
\sigma_{\xi_0}^*(\xi, t) &= (1 + i\gamma)\sigma_{\xi_0}(\xi)e^{iat}, \\
\sigma_{\xi}^*(\xi, t) &= \sigma_{\xi_0}^*(\xi, t) = (1 + i\gamma)\sigma_{\xi_0}(\xi)e^{iat}, \\
\sigma_{\gamma}^*(\xi, t) &= (1 + i\gamma)\sigma_{\gamma}(\xi)e^{iat}, \\
\sigma_{\theta}^*(\xi, t) &= (1 + i\gamma)\sigma_{\theta}(\xi)e^{iat}, \\
\tau_{\xi}^*(\xi, t) &= (1 + i\gamma)\tau_{\xi}(\xi)e^{iat}, \\
\tau_{\theta}^*(\xi, t) &= (1 + i\gamma)\tau_{\theta}(\xi)e^{iat},
\end{align*} \]
where: \( \sigma_{x0}, \sigma_{r0} \) and \( \sigma_{xj}, \sigma_{rj}, \sigma_{\theta j}, \tau_{\text{rxj}} \) are defined by expressions

\[
\sigma_{x0} = \frac{E}{1 + \mu} \left( \alpha_\mu \frac{\partial u_0}{\partial x} + 2\rho_\mu \frac{w_i}{r_i} \right),
\]

\[
\sigma_{r0} = \sigma_{\theta_0} = \frac{E}{1 + \mu} \left[ \alpha_\mu \frac{w_i}{r_i} + \rho_\mu \left( \frac{1}{r_i} w_i + \frac{\partial u_0}{\partial x} \right) \right],
\]

\[
\sigma_{xj}^* = \frac{E}{2(1 + \mu)} \left\{ \rho_\mu \left[ 2w_j + \alpha_j (w_{j+1} - w_{j-1}) \right] + 2\alpha_\mu \frac{\partial u_j}{\partial x} \right\},
\]

\[
\sigma_{rj} = \frac{E}{2(1 + \mu)} \left[ \frac{2\rho_\mu}{r_j} w_j + \frac{\alpha_\mu}{r_j} \alpha_j (w_{j+1} - w_{j-1}) + 2\rho_\mu \frac{\partial u_j}{\partial x} \right],
\]

\[
\sigma_{\theta j} = \frac{E}{2(1 + \mu)} \left[ \frac{2\alpha_\mu}{r_j} w_j + \frac{\alpha_j \rho_\mu}{r_j} (w_{j+1} - w_{j-1}) + 2\rho_\mu \frac{\partial u_j}{\partial x} \right],
\]

\[
\tau_{\text{rxj}} = \frac{E}{2(1 + \mu)} \left[ \frac{\alpha_j \rho_\mu}{2r_j} (u_{j+1} - u_{j-1}) + \frac{\partial w_j}{\partial x} \right].
\]

Integration constant in expressions (11) are determined from the boundary conditions that, after the separation of variables, are the following

for \( \xi = 0 \), and \( (1 + i\gamma) \tau_{\text{rx}} = 0 \), \( (1 + i\gamma) \sigma_{\chi} = \left\{ -\frac{F}{\pi r^2} \right\} \) at \( 0 \leq r \leq \bar{r} \),

for \( \xi = \frac{H}{b} \), \( u = 0 \), \( w = 0 \) at \( \bar{r} \leq r \leq R \).

From the boundary conditions, a system of algebraic equations is followed, which in the matrix form is the following

\[
CD^* = \overline{F},
\]

where: \( C = \{c_{ij}\} \) – matrix of coefficients with indeterminate integration constants \( D^* \);

\[
\overline{F} = \left\{ -\frac{q_b}{E(1+i\gamma)} \frac{F}{\pi r^2} , 0,0,\ldots,0 \right\} \) – column matrix of right parts.

After solving the system of equations (12), integration constants are determined by expressions

\[
D_m^* = \frac{D_m F_b}{E(1+i\gamma)\pi r^2},
\]

where \( D_m \) – numeric coefficient obtained as a result of system solution (13).

Representing an integrated expression for \( D^* \) in an exponential form and substituting it into the solution (11), we get
\[ u_0^*(\xi, t) = \left( \frac{Fb}{(1 + \gamma^2)^{1/2} E\pi r^2} \sum \overline{D_m} e^{j\omega z} \right) e^{i(\omega t + \nu)}, \]

\[ u_j^*(\xi, t) = \left( \frac{Fb}{(1 + \gamma^2)^{1/2} E\pi r^2} \sum D_m a_{jm} e^{j\omega z} \right) e^{i(\omega t + \nu)}, \] (14)

\[ w_j^*(\xi, t) = \left( \frac{Fb}{(1 + \gamma^2)^{1/2} E\pi r^2} \sum D_m b_{jm} e^{j\omega z} \right) e^{i(\omega t + \nu)} \]

where \( \nu = \arctg \beta \).

Since the force \( F \cdot \cos \alpha \) applied to the boundary plane represents the real part of the complex force (4), the solution of differential equations (1) in the real form will be introduced by the real part of the integrated solution (14).

Applying Euler identity to the latest one, we will write

\[ u_0(\xi, t) = \left( \frac{Fb}{(1 + \gamma^2)^{1/2} E\pi r^2} \sum \overline{D_m} e^{j\omega z} \right) \cos(\omega t + \nu), \]

\[ u_j(\xi, t) = \left( \frac{Fb}{(1 + \gamma^2)^{1/2} E\pi r^2} \sum D_m a_{jm} e^{j\omega z} \right) \cos(\omega t + \nu), \] (15)

\[ w_j(\xi, t) = \left( \frac{Fb}{(1 + \gamma^2)^{1/2} E\pi r^2} \sum D_m b_{jm} e^{j\omega z} \right) \cos(\omega t + \nu). \]

Accounting of the energy dissipation, as follows from expressions (15), leads to the phase shift between the perturbing load and the base movement.

For forced oscillations picture 3 shows the curves of changes in the amplitudes of vertical longitudinal movements \( u_0(0) \) of the base surface in the center of the cargo platform of radius \( \bar{r} \) depending on the change in the frequency parameter \( k^2 \), taking into account and without taking into account the energy dissipation at \( R = 6\bar{r}, \ H = 8 \) and \( \mu = 0.3 \). Change \( u_0(0) \) in the interval \( k^2 = 0.34 \div 0.39 \) is not given.
3. Changes in the amplitudes of the movements of $u_0(0)$ depending on the coefficient of inelastic resistance $\gamma$.

From picture 3 it follows that accounting the energy dissipation for values $k^2$ far from the resonance slightly affects the amplitudes of the base movements, while in the resonant zone ($k^2 = 0.3 \pm 0.4$), the accounting of energy dissipation significantly reduces the movements of the base.

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