A GROSS-KOHNEN-ZAGIER FORMULA FOR HEEGNER-DRINFELD CYCLES

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ABSTRACT. Let $F$ be the field of rational functions on a smooth projective curve over a finite field, and let $\pi$ be an unramified cuspidal automorphic representation for $\text{PGL}_2$ over $F$. We prove a variant of the formula of Yun and Zhang relating derivatives of the $L$-function of $\pi$ to the self-intersections of Heegner-Drinfeld cycles on moduli spaces of shtukas.

In our variant, instead of a self-intersection, we compute the intersection pairing of Heegner-Drinfeld cycles coming from two different quadratic extensions of $F$, and relate the intersection to the $r$-th derivative of a product of two toric period integrals.

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1. INTRODUCTION

Let $X$ be a smooth, projective, geometrically connected curve over a finite field $k$, and let $F = k(X)$ be the field of rational functions on $X$. Any finite étale cover $f : X' \to X$ of degree 2, with $X'$ geometrically connected, determines a quadratic extension $F' = k(X')$ of $F$, and hence determines a nonsplit torus $T = (\text{Res}_{F'/F}\mathbb{G}_m)/\mathbb{G}_m$ of rank one.

Let $r \geq 0$ be an even integer, and suppose that $\pi$ is an unramified cuspidal automorphic representation of the adelic points of $G = \text{PGL}_{2/F}$. Yun and Zhang have recently proved a remarkable formula for the $r$-th central derivative of the normalized base-change $L$-function $L(\pi_{F'}, s)$ in terms of the self-intersection of a Heegner-Drinfeld cycle $[\text{Sht}^r_G]_\pi$. The latter is an algebraic cycle of dimension $r$ on the $2r$-dimensional $k$-stack

$$\text{Sht}^r_G = \text{Sht}^r_G \times_{X'} X'^r,$$

B.H. was supported in part by NSF grant DMS-0901753.
where \( \text{Sht}_G^r \to X^r \) is the moduli stack of \( G \)-shtukas with \( r \) modifications. In particular, they prove that

\[
L^{(\pi)}(1/2) = \kappa \cdot \langle [\text{Sht}_G^r], [\text{Sht}_T^r] \rangle,
\]

for a certain non-zero real number \( \kappa \). The right hand side is the intersection pairing on \( \text{Ch}_{c,r}(\text{Sht}_G^r)_{\mathbb{R}} \), the middle dimensional Chow group with proper support and \( \mathbb{R} \)-coefficients.

Our goal is to prove a variant of this result, where the self-intersection of a Heegner-Drinfeld cycle is replaced by the intersection pairing of two different Heegner-Drinfeld cycles, in the spirit of the Gross-Kohnen-Zagier theorem [GKZ87].

1.1. Statement of the results. Suppose we are instead given two nonisomorphic finite étale double covers

\[
f_1 : Y_1 \to X, \quad f_2 : Y_2 \to X,
\]

where \( Y_1 \) and \( Y_2 \) are projective and geometrically connected. It is natural to wonder how the corresponding Heegner-Drinfeld cycles are related, and whether their intersection pairing is also related to \( L \)-values. We answer these questions by giving a formula (Theorem A) for the intersection pairing of two different Heegner-Drinfeld cycles on \( \text{Sht}_G^r \) in terms of certain toric period integrals.

To state this result, we must introduce some notation. Denote by \( \sigma_i \in \text{Aut}(Y_i/X) \) the nontrivial automorphism. The fiber product

\[
Y = Y_1 \times_X Y_2
\]

is again a smooth geometrically connected curve, and \( X \) is its quotient by the action of the Klein four group \( \{\text{id}, \tau_1, \tau_2, \tau_3\} = \text{Aut}(Y/X) \) with nontrivial elements

\[
(y_1, y_2)^{\tau_1} = (y_1, y_2^{\sigma_2}), \quad (y_1, y_2)^{\tau_2} = (y_1^{\sigma_1}, y_2), \quad (y_1, y_2)^{\tau_3} = (y_1^{\sigma_1}, y_2^{\sigma_2}).
\]

The quotient of \( Y \) by the action of \( \tau_3 \) is a geometrically connected étale double cover of \( X \), which we denote by \( Y_3 \). The picture is

\[
\begin{tikzcd}
Y \\
Y_1 \\
Y_2 \\
Y_3 \\
X \\
Y \arrow[Rightarrow]{r} & Y_1 \arrow[Rightarrow]{r} & Y_2 \arrow[Rightarrow]{r} & Y_3 \arrow[Rightarrow]{r} & X.
\end{tikzcd}
\]
Taking fields of rational functions, the corresponding picture is

\[ K \xrightarrow{\tau_1} K_1 \quad K \xrightarrow{\tau_2} K_2 \quad K \xrightarrow{\tau_3} K_3 \]

\[ K_1 \xrightarrow{\sigma_1} F, \quad K_2 \xrightarrow{\sigma_2} K, \quad K_3 \xrightarrow{\sigma_3} K \]

where the labels indicate the nontrivial automorphisms.

There is a fourth quadratic algebra lurking in the background, corresponding to the trivial double cover \( Y_0 = X \sqcup X \). Let \( f_0 : Y_0 \rightarrow X \) be the natural étale double cover, and let \( \sigma_0 \in \text{Aut}(Y_0/X) \) be the nontrivial automorphism. The algebra of rational functions on \( Y_0 \) is \( K_0 = F \oplus F \).

For \( i \in \{0, 1, 2, 3\} \) there is a natural closed immersion

\[ \tilde{T}_i = \text{Res}_{Y_i/X} \mathbb{G}_m \]

\[ \tilde{G}_i = \text{Aut}_{\mathcal{O}_X}(f_0_* \mathcal{O}_{Y_i}) \]

of group schemes over \( X \). The group scheme \( \tilde{G}_i \) is Zariski locally isomorphic to \( \text{GL}_2 \), and \( \tilde{T}_i \subset \tilde{G}_i \) is a maximal torus. Let \( T_i \subset G_i \) be obtained from this pair by quotienting out by the central \( \mathbb{G}_m \). It is Zariski locally isomorphic to a maximal torus in \( \text{PGL}_2 \) (over \( X \)). On \( F \)-points the picture is

\[ \tilde{T}_i(F) = K_i^\times \quad T_i(F) = K_i^\times / F^\times \]

\[ \tilde{G}_i(F) = \text{Aut}_F(K_i) \quad G_i(F) = \text{Aut}_F(K_i) / F^\times. \]

Note that the canonical isomorphism \( f_0_* \mathcal{O}_{Y_0} \cong \mathcal{O}_X \oplus \mathcal{O}_X \) identifies

\[ \tilde{G}_0 \cong \text{GL}_2, \quad G_0 \cong \text{PGL}_2, \]

and identifies \( \tilde{T}_0 \) and \( T_0 \) with their diagonal tori.

Let \( \mathbb{O} \subset \mathbb{A} \) be the subring of integral elements in the adele ring of \( F \). Similarly, for \( i \in \{0, 1, 2, 3\} \), denote by \( \mathbb{A}_i \) the adele ring of \( K_i \), and by \( \mathbb{O}_i \subset \mathbb{A}_i \) its subring of integral elements. Define \( U_i = G_i(\mathbb{O}) \). The pair \( U_i \subset G_i(\mathbb{A}) \) is isomorphic to \( \text{PGL}_2(\mathbb{O}) \subset \text{PGL}_2(\mathbb{A}) \), but there is no canonical such isomorphism except when \( i = 0 \).

There is, however, a canonical isomorphism of spaces of unramified cuspidal automorphic forms

\[ \mathcal{A}_{\text{cusp}}(G_i)^{U_i} \cong \mathcal{A}_{\text{cusp}}(G_0)^{U_0} \]
by Lemma 3.3. These are finite dimensional \( \mathbb{C} \)-vector spaces, and the space on the right carries a natural action of the Hecke algebra \( \mathcal{H} \) of \( \mathbb{Q} \)-valued compactly supported \( U_0 \)-bi-invariant functions on \( G_0(\mathbb{A}) \).

**Remark 1.1.** Note that our automorphic forms are complex valued, as in [BJ79, §5], as opposed to the \( \mathbb{Q} \)-valued automorphic forms of [YZ17, §1.2].

To set up the analytic side of our formula, for any \( \phi \in \mathcal{A}_{\text{cusp}}(G_0)^{U_0} \) we define the \( T_0 \)-period integral
\[
\mathcal{P}_0(\phi, s) = \int_{T_0(F) \setminus T_0(\mathbb{A})} \phi(t_0) |t_0|^{2s} dt_0,
\]
which is absolutely convergent for all \( s \in \mathbb{C} \). Viewing \( \phi \) in \( \mathcal{A}_{\text{cusp}}(G_3)^{U_3} \), we also define a \( T_3 \)-period integral
\[
\mathcal{P}_3(\phi, \eta) = \int_{T_3(F) \setminus T_3(\mathbb{A})} \phi(t_3) \eta(t_3) dt_3.
\]

Here \( \eta : \mathbb{A}_3^\times \to \{\pm 1\} \) is the quadratic character determined by the extension \( K/K_3 \). It descends to the quotient \( T_3(\mathbb{A}) = \mathbb{A}_3^\times / A^\times \) by Lemma 3.6.

Now suppose that \( \pi \) is an unramified cuspidal automorphic representation of \( G_0(\mathbb{A}) \), and define
\[
C(\pi, s) = \frac{\mathcal{P}_0(\phi, s) \mathcal{P}_3(\phi, \eta)}{\langle \phi, \phi \rangle_{\text{Pet}}}
\]
for any nonzero \( \phi \in \pi^{U_0} \). If \( \lambda_{\pi} : \mathcal{H} \to \mathbb{C} \) denotes the character giving the action of \( \mathcal{H} \) on the line \( \pi^{U_0} \), we prove in Proposition 5.6 that
\[
C_r(\pi) \overset{\text{def}}{=} (\log q)^{-r} \cdot \frac{d^r}{ds^r} C(\pi, s) \big|_{s=0},
\]
lies in totally real number field \( E_{\pi} = \lambda_{\pi}(\mathcal{H}) \). Here \( q \) is the cardinality of \( k \).

To define the geometric side of our formula, recall from [YZ17] the stacks
\[
\text{Sht}^r_{T_1} \to Y_1^r, \quad \text{Sht}^r_{T_2} \to Y_2^r
\]
of \( T_i \)-shtukas with \( r \) modifications, and the stack
\[
\text{Sht}^r_{G_0} \to X^r
\]
of \( \text{PGL}_2 \)-Shtukas with \( r \) modifications. For \( i \in \{1, 2\} \) the stacks \( \text{Sht}^r_{T_i} \) are proper over \( k \), and admit finite morphisms \( \text{Sht}^r_{T_i} \to \text{Sht}^r_{G_0} \). Denote by
\[
[Sht^r_{T_i}] \in \text{Ch}_{c,r}(\text{Sht}^r_{G_0})
\]
the pushforward of the fundamental class under the induced map on Chow groups with \( \mathbb{Q} \)-coefficients.

**Remark 1.2.** The definitions of these stacks require the choice of a tuple \( \mu = (\mu_i) \in \{\pm 1\}^r \) satisfying the parity condition \( \sum_{i=1}^r \mu_i = 0 \), and in particular we must assume that \( r \) is even. We suppress the choice of \( \mu \) from the notation.
Denote by $\tilde{W}_i \subset \text{Ch}_{c,r}(\text{Sht}_{G_0}^r)$ the $H$-submodule generated by the class $[\text{Sht}_{G_0}^r]$. Restricting the intersection pairing on the Chow group defines a pairing $\langle \cdot, \cdot \rangle : \tilde{W}_1 \times \tilde{W}_2 \to \mathbb{Q}$. If we define

$$W_1 = \tilde{W}_1 / \{ c \in \tilde{W}_1 : \langle c, \tilde{W}_2 \rangle = 0 \}$$

$$W_2 = \tilde{W}_2 / \{ c \in \tilde{W}_2 : \langle c, \tilde{W}_1 \rangle = 0 \},$$

this pairing obviously descends to $\langle \cdot, \cdot \rangle : W_1 \times W_2 \to \mathbb{Q}$, and we extend it to an $\mathbb{R}$-bilinear pairing

$$W_1(\mathbb{R}) \times W_2(\mathbb{R}) \to \mathbb{R}.$$ 

We prove in §5.4 that there is a decomposition into isotypic components

$$W_i(\mathbb{R}) = W_{i,\text{Eis}} \oplus \left( \bigoplus_{\pi} W_{i,\pi} \right),$$

where the sum is over all unramified cuspidal $\pi$, and $\mathcal{H}$ acts on the summand $W_{i,\pi}$ via $\lambda_{\pi} : \mathcal{H} \to \mathbb{R}$. Denote by $[\text{Sht}_{G_0}^r]_{\pi} \in W_{i,\pi}$, the projection of $[\text{Sht}_{G_0}^r] \in W_i(\mathbb{R})$ to this summand. Our main result is the following intersection formula.

**Theorem A.** For any unramified cuspidal automorphic representation $\pi$, and for any even $r \geq 0$,

$$\langle [\text{Sht}_{G_0}^r]_{\pi}, [\text{Sht}_{G_0}^r]_{\pi} \rangle = C_r(\pi).$$

Theorem A is the function field analogue of the Gross-Kohnen-Zagier formula [GKZ87, Theorem B], but for higher order derivatives. The original Gross-Kohnen-Zagier formula corresponds to the case $r = 1$ over $\mathbb{Q}$. Their result allows for ramified $\pi$, and we expect our formula can be extended to the mildly ramified case as well.

The proof of Theorem A is very different from that of [GKZ87], and is much closer to the relative trace formula arguments of [Jac86, YZ17].

As an application of Theorem A, we obtain a criterion for the non-vanishing of certain special values of $L$-functions and their derivatives.

**Theorem B.** Let $L(\pi, s)$ be the standard $L$-function attached to $\pi$, and let $\chi_1$ and $\chi_2$ be the quadratic characters of $\mathbb{A}^\times$ corresponding to the extensions $K_1/F$ and $K_2/F$. The Heegner-Drinfeld cycles satisfy

$$\langle [\text{Sht}_{G_0}^r]_{\pi}, [\text{Sht}_{G_0}^r]_{\pi} \rangle = 0$$

if and only if

$$L^{(r)}(\pi, 1/2) \cdot L(\pi \otimes \chi_1, 1/2) \cdot L(\pi \otimes \chi_2, 1/2) = 0,$$

where $L^{(r)}(\pi, s)$ is the $r$th derivative of $L(\pi, s)$.

There are three main difference between our results and the main result of [YZ17]. First, of course, we are computing the intersection of two distinct cycles, as opposed to a self-intersection. Second, our intersection pairing is on the $X^r$-stack $\text{Sht}_{G_0}^r$, not its base-change to the $r$-fold product of a double
cover as in [YZ17]. More precisely, our Heegner-Drinfeld cycle $[\text{Sht}^r_{T_i}]$ is the pushforward via the $2^r$-fold cover

$$\text{Sht}^r_{G_0} \times_{X^r} Y^r_i \to \text{Sht}^r_{G_0}$$

of the Heegner-Drinfeld cycle of [YZ17]. Finally, our formula relates the intersection pairing to the standard $L$-function $L(\pi, s)$, not the base-change to a quadratic extension.

Finally, we also prove the following result on the intersection of the “bare” Heegner-Drinfeld cycles without any projection.

**Theorem C.** Let $r \geq 0$ be an even integer, and let

$$\langle \cdot, \cdot \rangle : \text{Ch}_{c,r}(\text{Sht}^r_{G_0}) \times \text{Ch}_{c,r}(\text{Sht}^r_{G_0}) \to \mathbb{Q}$$

be the intersection pairing.

(a) If $\text{char}(k) = 2$, then $\langle [\text{Sht}^r_{T_1}], [\text{Sht}^r_{T_2}] \rangle = 0$.
(b) If $\text{char}(k) \neq 2$, then

$$\langle [\text{Sht}^r_{T_1}], [\text{Sht}^r_{T_2}] \rangle = \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{if } r > 0. \end{cases}$$

1.2. **The case** $r = 0$. Assume that $r = 0$, and omit it from the notation.

On $k_{\text{alg}}$-points, the morphisms $\text{Sht}_{T_i} \to \text{Sht}_{G_0}$ become

$\begin{array}{c}
\text{Sht}_{T_1}(k_{\text{alg}}) \\
\downarrow \\
\text{T}_1(F)/\text{T}_1(\mathbb{A})/\text{T}_1(\mathbb{Q}) \\
\downarrow \\
\text{G}_1(F)/\text{G}_1(\mathbb{A})/\text{U}_1
\end{array}
\quad
\begin{array}{c}
\downarrow \\
\text{Sht}_{G_0}(k_{\text{alg}}) \\
\downarrow \\
\text{G}_0(F)/\text{G}_0(\mathbb{A})/\text{U}_0
\end{array}
\quad
\begin{array}{c}
\text{Sht}_{T_2}(k_{\text{alg}}) \\
\downarrow \\
\text{T}_2(F)/\text{T}_2(\mathbb{A})/\text{T}_2(\mathbb{Q}) \\
\downarrow \\
\text{G}_2(F)/\text{G}_2(\mathbb{A})/\text{U}_2
\end{array}$

where each “$=$” is a canonical isomorphism (see Lemma 3.3 for example).

Suppose that $\pi$ is an unramified cuspidal automorphic representation of $G_0(\mathbb{A}_f)$, and that $\phi \in \pi^{U_0}$ is a spherical vector. As noted earlier, Lemma 3.3 provides canonical isomorphisms

$$\mathcal{A}_{\text{cusp}}(G_0)^{U_0} \cong \mathcal{A}_{\text{cusp}}(G_i)^{U_i}$$

for $i \in \{1, 2, 3\}$, and we view $\phi$ as an automorphic form on any of the four groups $G_i$. The following result, relating the periods of this form along the four tori $T_i \subset G_i$, is equivalent to the $r = 0$ case of Theorem A.
Theorem D. For any $\phi \in \pi_{U_0}$ as above, we have

$$\left( \int_{T_1(F) \backslash T_1(\mathbb{A})} \phi(t_1) dt_1 \right) \left( \int_{T_2(F) \backslash T_2(\mathbb{A})} \overline{\phi}(t_2) dt_2 \right) = \left( \int_{T_0(F) \backslash T_0(\mathbb{A})} \phi(t_0) dt_0 \right) \left( \int_{T_3(F) \backslash T_3(\mathbb{A})} \overline{\phi}(t_3) \eta(t_3) dt_3 \right).$$

Remark 1.3. If one takes the absolute value of both sides of the equality, then Waldspurger’s formula [Wal85, CW16] relates all four toric periods to $L$-values, and the equality of absolute values follows from a formal manipulation of Euler products. Thus, as both sides of the stated equality are easily seen to be real numbers (the $\mathbb{C}$-line $\pi_{U_0}$ is generated by some $\mathbb{R}$-valued $\phi$), the only new information is that the signs agree.

Just as Waldspurger’s period formula generalizes to higher rank unitary and orthogonal groups, as in the conjectures of Gan-Gross-Prasad and the work of Zhang [Zha14], one could hope that there are analogues of Theorem D also on higher rank groups. The precise form that such analogues might take is not at all clear to the authors.

1.3. The case of self-intersection. Our methods should also be applicable to the case where $Y_1 \cong Y_2$, with relatively minor adjustments, and we expect that Theorems A and B hold verbatim (but not Theorem C).

If we denote by $X'$ the curve $Y_1 \cong Y_2$, and by $T$ the torus $T_1 \cong T_2$, then degenerates to

and our Heegner-Drinfeld cycle $[\text{Sht}_T^r] \in \text{Ch}_{c,r}(\text{Sht}_{G_0})$ is, as was noted before, the pushforward via the $2^r$-fold cover

$$\text{Sht}_{G_0}^r \times_{X'} X'' \to \text{Sht}_{G_0}^r$$

of the Heegner-Drinfeld cycle of $[YZ17]$.

The analogue of Theorem B in this setting says that

$$\langle [\text{Sht}_T^r]_\pi, [\text{Sht}_T^r]_\pi \rangle = 0 \iff L^{(r)}(\pi, 1/2) \cdot L(\pi \otimes \chi, 1/2) = 0,$$

where $\chi$ is the quadratic character of $\mathbb{A}^X$ determined by $X' \to X$. We have learned that the case $Y_1 \cong Y_2$ will be treated in detail in forthcoming work of Yun and Zhang (private communication). As omitting this case leads to some simplification of the arguments, we always assume that $Y_1 \not\cong Y_2$. 
1.4. **Notation.** After [2] which contains results about quaternion algebras over arbitrary fields, the curves and functions fields of (1.1) and (1.2) will remain fixed, as will the group schemes \( T_i \rightarrow G_i \) over \( X \). The Haar measures on \( T_i(A) \) and \( G_i(A) \) will always be chosen so that \( T_i(O) \) and \( G_i(O) \) have volume 1.

Let \( |X| \) be the set of closed points of \( X \). We normalize the absolute value

\[ |\cdot| = \prod_{x \in |X|} |\cdot|_x : A^X \rightarrow \mathbb{Z} \]

so that a uniformizer \( \pi_x \in F_x \) with residue field \( k_x \) satisfies \( |\pi_x|_x = q^{-[k_x : k]} \).

Unless otherwise specified, all Chow groups have \( \mathbb{Q} \)-coefficients.

2. Biquadratic algebras and quaternion algebras

In this section alone we allow \( F \) to be any field whatsoever, and let \( K_1 \) and \( K_2 \) be quadratic étale \( F \)-algebras. In other words, each \( K_i \) is either a degree two Galois extension, or \( K_i \cong F \oplus F \). In particular, the results below apply both to the global fields of (1.2) and to their completions.

2.1. Invariants of quaternion embeddings. As usual, a quaternion algebra over \( F \) is a central simple \( F \)-algebra of dimension four.

**Definition 2.1.** A quaternion embedding of \((K_1, K_2)\) is a triple \((B, \alpha_1, \alpha_2)\) consisting of a quaternion algebra \( B \) over \( F \), and \( F \)-algebra embeddings \( \alpha_1 : K_1 \rightarrow B \) and \( \alpha_2 : K_2 \rightarrow B \). Such a triple is regular if \( \alpha_1(K_1) \cup \alpha_2(K_2) \) generates \( B \) as an \( F \)-algebra.

Our goal is to parametrize the set

\[ Q(K_1, K_2) = \{ \text{isomorphism classes of quaternion embeddings } (B, \alpha_1, \alpha_2) \} \]

along with its subset

\[ Q_{reg}(K_1, K_2) \subset Q(K_1, K_2) \quad (2.1) \]

of regular quaternion embeddings.

For \( i \in \{1, 2\} \), denote by \( \sigma_i \in \text{Aut}(K_i/F) \) the nontrivial automorphism. The automorphism group of the quartic \( F \)-algebra \( K = K_1 \otimes_F K_2 \) contains the Klein four subgroup

\[ \{ \text{id}, \tau_1, \tau_2, \tau_3 \} \subset \text{Aut}(K/F) \]

with nontrivial elements

\[ (x \otimes y)^{\tau_1} = x \otimes y^{\sigma_2}, \quad (x \otimes y)^{\tau_2} = x^{\sigma_1} \otimes y, \quad (x \otimes y)^{\tau_3} = x^{\sigma_1} \otimes y^{\sigma_2}. \]

Denote by \( K_3 \subset K \) the subalgebra of elements fixed by \( \tau_3 \). Elementary algebra shows that \( K_1 \cong K_2 \) if and only if \( K_3 \cong F \oplus F \). The picture, along with generators of the automorphism groups, is (1.2).

Fix a triple \((B, \alpha_1, \alpha_2) \in Q(K_1, K_2)\), and define an \( F \)-linear map \( \alpha : K \rightarrow B \) by

\[ \alpha(x_1 \otimes x_2) = \alpha_1(x_1)\alpha_2(x_2). \]
Clearly \( \alpha|_{K_1} = \alpha_1 \) and \( \alpha|_{K_2} = \alpha_2 \) are \( F \)-algebra homomorphisms, but \( \alpha \) is an \( F \)-algebra homomorphism if and only if \( \alpha_1(K_1) = \alpha_2(K_2) \).

**Lemma 2.2.** The image of \( \alpha \) is the smallest \( F \)-subalgebra of \( B \) containing both \( \alpha_1(K_1) \) and \( \alpha_2(K_2) \). In particular, \((B, \alpha_1, \alpha_2)\) is regular if and only if \( \alpha(K) = B \).

**Proof.** It is clear from the definition that

\[
\alpha(K) = \text{Span}_F \{ \alpha_1(x)\alpha_2(y) : x \in K_1, y \in K_2 \},
\]

and so it suffices to prove that \( \alpha(K) \) is a subalgebra of \( B \).

If \( \alpha(K) \) has dimension 2, then \( \alpha_1(K_1) = \alpha(K) = \alpha_2(K_2) \), and we are done. If \( \alpha(K) \) has dimension 4, then \( \alpha(K) = B \), and again we are done. Now suppose that \( \alpha(K) \) has dimension 3. As \( \alpha(K) \) is stable under left multiplication by \( \alpha_1(K_1) \) and right multiplication by \( \alpha_2(K_2) \), these quadratic algebras cannot be fields. Thus each is isomorphic to \( F \times F \), and also \( B \cong M_2(F) \). This latter isomorphism may be chosen so that \( \alpha_1(K_1) \subset M_2(F) \) is the algebra of diagonal matrices. This implies that \( \alpha(K) \subset M_2(F) \) is either the space of upper triangular matrices or the space of lower triangular matrices, as these are the only three dimensional subspaces of \( M_2(F) \) stable under left multiplication by the diagonal subalgebra. In either case \( \alpha(K) \) is a subalgebra of \( B \).

Denote by \( b \mapsto b^\dagger \) the main involution on \( B \), and by \( \text{Nrd}_B(b) = bb^\dagger \) and \( \text{Trd}_B(b) = b + b^\dagger \) the reduced norm and reduced trace.

**Proposition 2.3.** There is a unique \( \xi \in K_3 \) satisfying

\[
\text{Nrd}_B(\alpha(x)) = \text{Tr}_{K_3/F}(\xi x x^\dagger) \tag{2.2}
\]

for all \( x \in K \). It further satisfies \( \text{Tr}_{K_3/F}(\xi) = 1 \) and

\[
\text{Trd}_B(\alpha(x)\alpha(y)^\dagger) = \text{Tr}_{K/F}(\xi x^\dagger y) \tag{2.3}
\]

for all \( x, y \in K \). Moreover, \((B, \alpha_1, \alpha_2)\) is regular if and only if \( \xi \in K_3^\times \).

**Proof.** Endow \( B \) with the structure of a \( K \)-module via the action

\[
(x_1 \otimes x_2) \bullet b = \alpha_1(x_1) \cdot b \cdot \alpha_2(x_2),
\]

so that the \( F \)-linear map \( \alpha : K \to B \) defined above is \( \alpha(x) = x \bullet 1 \).

The action of \( K \) on \( B \) induces an action of \( K = K \otimes_F K_3 \) on \( B = B \otimes_F K_3 \), and the orthogonal idempotents in \( K = K \times K \) determines a splitting \( B = B_+ \oplus B_- \). More precisely,

\[
B_+ = \{ b \in B : \forall x \in K_3, x \bullet b = b \cdot x \}
\]

\[
B_- = \{ b \in B : \forall x \in K_3, x \bullet b = b \cdot x^\dagger \}.
\]

The projection map \( B \to B_\pm \) is denoted \( b \mapsto b_\pm \). Each summand is \( K \)-stable, the projections are \( K \)-linear, and the decomposition is orthogonal with respect to the reduced norm \( \text{Nrd}_B : B \to K_3 \).
By restricting the action of $K$, we view $B$ as a free $K_3$-module of rank 2, and define a quadratic form $\nr_B^\dagger : B \to K_3$ by $\nr_B^\dagger(b) = \nr_B(b_+)$. We leave it as an exercise to the reader to verify the relations

$$\nr_B^\dagger(b \cdot b) = xx^\tau_3 \nr_B^\dagger(b),$$

$$\nr_B(b) = \tr_{K_3/F}(\nr_B^\dagger(b)).$$

for all $x \in K$ and $b \in B$.

Setting $\xi = \nr_B^\dagger(1)$, the first of these relations implies

$$\xi xx^\tau_3 = xx^\tau_3 \nr_B^\dagger(1) = \nr_B^\dagger(x \cdot 1) = \nr_B^\dagger(\alpha(x)),$$

and hence the second implies

$$\nr_B(\alpha(x)) = \tr_{K_3/F}(\xi xx^\tau_3).$$

This proves the equality of quadratic forms (2.2), which then implies the equality (2.3) of corresponding bilinear forms. Taking $x = 1$ in (2.2) shows that $\tr_{K_3/F}(\xi) = 1$.

If $\xi' \in K_3$ also satisfies (2.2), then it also satisfies (2.3), and hence $\zeta = \xi - \xi'$ satisfies $\tr_{K/F}(\zeta xy) = 0$ for all $x, y \in K$. Thus $\zeta = 0$ and $\xi = \xi'$.

Finally, it is easy to see from Lemma 2.2 that each of the following statements is equivalent to the next.

- The quaternion embedding $(B, \alpha_1, \alpha_2)$ is regular.
- The $F$-linear map $\alpha : K \to B$ is an isomorphism.
- The radical of the bilinear form (2.3) is trivial.
- The element $\xi \in K_3$ is a unit.

This completes the proof of Proposition 2.3. \hfill \Box

Proposition 2.3 attaches to each $(B, \alpha_1, \alpha_2) \in \mathcal{Q}(K_1, K_2)$ an element $\xi \in K_3$. This defines the invariant

$$\text{inv} : \mathcal{Q}(K_1, K_2) \to \{ \xi \in K_3 : \tr_{K_3/F}(\xi) = 1 \},$$

which satisfies

$$\mathcal{Q}_{\text{reg}}(K_1, K_2) = \{(B, \alpha_1, \alpha_2) \in \mathcal{Q}(K_1, K_2) : \text{inv}(B, \alpha_1, \alpha_2) \in K_3^\times \}.$$ 

**Theorem 2.4.** The invariant (2.4) restricts to a bijection

$$\text{inv} : \mathcal{Q}_{\text{reg}}(K_1, K_2) \cong \{ \xi \in K_3^\times : \tr_{K_3/F}(\xi) = 1 \}.$$ 

**Proof.** The strategy is to reduce to the case where $K_1$ and $K_2$ are split.

Abbreviate (2.1) to $\mathcal{Q}_{\text{reg}} \subset \mathcal{Q}$. Let $F/F$ be a Galois extension large enough that both $F$-algebras $K_1 = K_1 \otimes_F F$ and $K_2 = K_2 \otimes_F F$ are isomorphic to $F \times F$. Applying $\otimes_F F$ to the diagram (1.2) gives a new diagram of
and we consider the set $Q$ of isomorphism classes of quaternion embeddings of $(K_1, K_2)$. That is to say, isomorphism classes of triples $(B, \alpha_1, \alpha_2)$ where $B$ is a quaternion algebra over $F$, and $\alpha_1 : K_1 \to B$ and $\alpha_2 : K_2 \to B$ are $F$-algebra embeddings. Of course such embeddings can only exist if $B \cong M_2(F)$. Again let $Q_{\text{reg}} \subset Q$ denote the subset of regular triples.

The Galois group $\text{Gal}(F/F)$ acts on the set $Q_{\text{reg}}$ and on the rings (2.6) in a natural way, and the construction of (2.5) defines a $\text{Gal}(F/F)$-invariant function

$$\text{inv} : Q_{\text{reg}} \to \{\xi \in K_3^\times : \text{Tr}_{K_3/F}(\xi) = 1\}. \quad (2.7)$$

Extension of scalars from $F$ to $F$ defines the vertical arrows in the commutative diagram

$$Q_{\text{reg}} \xrightarrow{\text{inv}} \{\xi \in K_3^\times : \text{Tr}_{K_3/F}(\xi) = 1\} \quad (2.8)$$

$$Q_{\text{reg}} \xrightarrow{\text{inv}} \{\xi \in K_3^\times : \text{Tr}_{K_3/F}(\xi) = 1\} \text{Gal}(F/F).$$

**Lemma 2.5.** The function $Q_{\text{reg}} \to Q_{\text{reg}}^{\text{Gal}(F/F)}$ is a bijection.

**Proof.** This is immediate from the theory of Galois descent. The only thing to check is that a regular quaternion embedding $(B, \alpha_1, \alpha_2) \in Q_{\text{reg}}$ has no nontrivial automorphisms. Indeed, any $F$-algebra automorphism of $B$ is given by conjugation by some $b \in B^\times$. If $b$ defines an automorphism of the triple $(B, \alpha_1, \alpha_2)$, then $b$ centralizes both $\alpha_1(K_1)$ and $\alpha_2(K_2)$. These subalgebras are equal to their own centralizers, and so $b$ lies in the intersection $\alpha_1(K_1)^\times \cap \alpha_2(K_2)^\times$. The regularity of $(B, \alpha_1, \alpha_2)$ implies that this intersection is equal to $F^\times$, and so conjugation by $b$ is trivial. □

**Lemma 2.6.** The function (2.7) is a bijection.

**Proof.** Fix isomorphisms $K_i \cong F \times F$ for $i \in \{1, 2, 3\}$, and an isomorphism $K \cong F \times F \times F \times F$, in such a way that

$$\begin{align*}
(x_1, x_2, x_3, x_4)^{\tau_1} &= (x_3, x_4, x_1, x_2) \\
(x_1, x_2, x_3, x_4)^{\tau_2} &= (x_4, x_3, x_2, x_1) \\
(x_1, x_2, x_3, x_4)^{\tau_3} &= (x_2, x_1, x_4, x_3),
\end{align*}$$
and the inclusions of $K_1 \cong K_2 \cong K_3$ into $K$ are identified (respectively) with

$$(u, v) \mapsto (u, v, u, v), \quad (u, v) \mapsto (u, v, u, v), \quad (u, v) \mapsto (u, u, v, v).$$

Define embeddings $\alpha_1 : K_1 \to M_2(F)$ and $\alpha_2 : K_2 \to M_2(F)$ by

$$(u, v) \mapsto \begin{pmatrix} u & v \\ v & u \end{pmatrix}, \quad (u, v) \mapsto \begin{pmatrix} u & v \\ v & u \end{pmatrix}.$$

If we let $A \subset M_2(F)$ denote the subalgebra of diagonal matrices, there is an induced bijection $A \times \mathbb{GL}_2(F)/A \cong \mathbb{Q}$. In particular, $\gamma$ determines an $F$-linear map $F \times F \times F \times F = K \to M_2(F)$ and an element $\text{inv}(\gamma) \in K_3$ of trace 1. Direct calculation shows that

$$\alpha(x_1, x_2, x_3, x_4) = \frac{1}{ad - bc} \begin{pmatrix} ax_1 - bcx_3 & ab(x_3 - x_1) \\ cd(x_4 - x_2) & adx_2 - bcx_4 \end{pmatrix}$$

and

$$\text{inv}(\gamma) = \left( \frac{ad}{ad - bc}, \frac{-bc}{ad - bc} \right) \in F \times F = K_3.$$

In [Jac86, §1.3] one can find a complete set of representatives for the double coset space (2.9). There are six non-regular elements, represented by

$$(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}), \quad (\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}), \quad (\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}), \quad (\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}), \quad (\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}).$$

(Compare with the proof of Lemma 2.2. For the first two double cosets the corresponding map $\alpha : K \to M_2(F)$ has image of dimension two. For the remaining four double cosets the image has dimension three.) A complete set of representatives for the regular elements is given by

$$\left\{ \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} : x \in F \setminus \{0, 1\} \right\} \cong \mathbb{Q}_{\text{reg}}.$$

For this set of coset representatives, the function $\text{inv} : \mathbb{Q}_{\text{reg}} \to \{(u, v) \in F^\times \times F^\times : u + v = 1\}$ of (2.7) takes the explicit form

$$\begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -x \\ 1 - x & 1 - x \end{pmatrix},$$

and the reader will have no difficulty in checking that this is a bijection. □

The bijectivity of (2.5) is clear from the two lemmas above. Indeed, the left vertical arrow in (2.8) is bijective by Lemma 2.5, the right vertical arrow is obviously bijective, and the bottom horizontal arrow is bijective by Lemma 2.6. Therefore the top horizontal arrow is a bijection. This completes the proof of Theorem 2.4. □
Remark 2.7. Although we will not explicitly need to do so, one can understand Theorem 2.4 as a parametrization of certain double coset spaces, in the spirit of [Jac86]. Call a quaternion algebra $B$ relevant if it admits embeddings of both $K_1$ and $K_2$. For each relevant $B$, fix such embeddings once and for all. These choices determine bijections

$$B^\times/K_i^\times \cong \{ \text{embeddings } K_i \to B \}$$

and

$$\bigsqcup_{\text{relevant } B} K_1^\times \backslash B^\times/K_2^\times \cong \bigsqcup_{\text{relevant } B} B^\times \backslash (B^\times/K_1^\times \times B^\times/K_2^\times) \cong Q(K_1, K_2).$$

With these identifications, the invariant (2.4) determines a function

$$\bigsqcup_{\text{relevant } B} K_1^\times \backslash B^\times/K_2^\times \to \{ \xi \in K_3 : \Tr_{K_3/F}(\xi) = 1 \}.$$  

The content of Theorem 2.4 is that this is very nearly a bijection. In fact, if $K_1 \not\cong K_2$ then $K_3$ is a field, $Q(K_1, K_2) = Q_{reg}(K_1, K_2)$, and Theorem 2.4 asserts the bijectivity of this function.

2.2. Invariants of double cosets. Let $\text{Iso}(K_2, K_1)$ be the set of isomorphisms of $F$-vector spaces $K_2 \to K_1$. The group $K_1^\times$ acts on the left by postcomposition, and the group $K_2^\times$ acts on the right by precomposition. We use similar notation if $K_1$ and $K_2$ are replaced by other $F$-algebras. Our goal is to define a function

$$K_1^\times \backslash \text{Iso}(K_2, K_1)/K_2^\times \overset{\text{inv}}{\longrightarrow} \{ \xi \in K_3 : \Tr_{K_3/F}(\xi) = 1 \},$$

intimately related to the invariant (2.4).

There are canonical $F$-algebra isomorphisms

$$K_1 \otimes_F K \cong K \oplus K, \quad K_2 \otimes_F K \cong K \oplus K,$$

the first defined by $x \otimes y \mapsto (xy, x^{\sigma_1}y)$, and the second defined similarly. Any $\phi \in \text{Hom}(K_2, K_1)$ therefore induces a $K$-linear map

$$K \oplus K \cong K_2 \otimes_F K \xrightarrow{\phi \otimes \text{id}} K_1 \otimes_F K \cong K \oplus K,$$

represented by a matrix $(a \ b \ c \ d) \in M_2(K)$ whose entries are related by

$$b = a^{\tau_1}, \quad c = a^{\tau_2}, \quad d = a^{\tau_3}.$$

Define a one-dimensional $F$-vector space

$$\Delta = \text{Hom}_F(\det_F(K_2), \det_F(K_1)),$$

and define $\det^\sharp(\phi) \in \Delta \otimes_F K$ by

$$\det_F(K_2) \otimes_F K \xrightarrow{\det_F(K_2) \otimes_F K} \det_K(K_2 \otimes_F K) \xrightarrow{\det_K(K_2 \otimes_F K)} \det_K(K \oplus K) \xrightarrow{ad} \det_F(K_1) \otimes_F K \xrightarrow{\det_F(K_1) \otimes_F K} \det_K(K_1 \otimes_F K) \xrightarrow{\det_K(K_1 \otimes_F K)} \det_K(K \oplus K)$$

where $ad$ is the map $e_1 \wedge e_2 \mapsto (ae_1) \wedge (de_2)$.  

Using the relations (2.11), one can check that \( \det^\sharp(\phi) \) actually lies in \( \Delta \otimes_F K_3 \), and its trace to \( \Delta \) is the usual determinant \( \det(\phi) \in \Delta \). In other words, we have a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_F(K_2, K_1) & \xrightarrow{\det^\sharp} & \Delta \otimes_F K_3 \\
\downarrow \text{det} & & \downarrow \text{Tr}_{K_3/F} \\
\Delta & & \Delta,
\end{array}
\]

and the desired function (2.10) is defined by

\[
\text{inv}(\phi) = \frac{\det^\sharp(\phi)}{\det(\phi)} \in \{ \xi \in K_3 : \text{Tr}_{K_3/F}(\xi) = 1 \}.
\]

**Proposition 2.8.** If we define

\[
\text{Iso}_{\text{reg}}(K_2, K_1) = \{ \phi \in \text{Iso}(K_2, K_1) : \text{inv}(\phi) \in K_3^\times \},
\]

the function (2.10) restricts to an injection

\[
K_1^\times \backslash \text{Iso}_{\text{reg}}(K_2, K_1)/K_2^\times \xrightarrow{\text{inv}} \{ \xi \in K_3^\times : \text{Tr}_{K_3/F}(\xi) = 1 \}.
\]

**Proof.** Any pair

\[
(a_1, a_2) \in \text{Iso}(K_1, F \oplus F) \times \text{Iso}(K_2, F \oplus F)
\]

determines an \( F \)-linear isomorphism \( a_1^{-1} a_2 : K_2 \to K_1 \), as well as \( F \)-algebra embeddings \( \alpha_1 : K_1 \to M_2(F) \) and \( \alpha_2 : K_2 \to M_2(F) \). These constructions induce canonical identifications

\[
K_1^\times \backslash \text{Iso}_{\text{reg}}(K_2, K_1)/K_2^\times \overset{(2.12)}{=} \text{GL}_2(F) \backslash \left( \text{Iso}(K_1, F \oplus F)/K_1^\times \times \text{Iso}(K_2, F \oplus F)/K_2^\times \right)
\]

\[
\cong \text{GL}_2(F) \backslash \left\{ \begin{array}{c}
\alpha_1 : K_1 \to M_2(F) \\
\alpha_2 : K_2 \to M_2(F)
\end{array} \right\},
\]

which realize

\[
K_1^\times \backslash \text{Iso}(K_2, K_1)/K_2^\times \subset \mathcal{Q}(K_1, K_2)
\]
as the set of all quaternion embeddings whose underlying quaternion algebra is \( M_2(F) \). A tedious but elementary calculation shows that the invariant (2.10) is simply the restriction of (2.4) to this subset, and so the claim follows from the bijectivity of (2.5). \( \square \)

### 2.3. The dual picture

In this subsection and the next, we let

\[
K_0 = F \oplus F.
\]

The initial input to the constructions of §2.1 and §2.2 was a pair of quadratic algebras \( K_1 \) and \( K_2 \), from which we produced a diagram (1.2).
We may repeat these constructions, but now take the initial input to be $K_0$ and $K_3$. In this case $K_0 \otimes_F K_3 \cong K_3 \times K_3$, and the diagram (1.2) is replaced by

$$
\begin{array}{ccc}
K_3 \times K_3 \\
K_0 \downarrow & & \downarrow K_3 \\
K_3 & & K_3 \\
F. & & \\
\end{array}
$$

Here the upper left inclusion is just $K_0 = F \times F \subset K_3 \times K_3$, the middle inclusion $K_3 \to K_3 \times K_3$ is the diagonal embedding, and the inclusion $K_3 \to K_3 \times K_3$ on the right is the twisted diagonal $y_3 \mapsto (y_3, y_3^{F})$.

Repeating the construction of (2.10) in this new setting yields a function

$$
K_0 \setminus \text{Iso}(K_3, K_0)/K_3^{\times} \xrightarrow{\text{inv}} \{ \xi \in K_3 : \text{Tr}_{K_3/F}(\xi) = 1 \}.
$$

In fact, the construction of this function simplifies slightly because $K_0$ is split. For comparison with later constructions (see especially the proof of Lemma 3.18) we now make this completely explicit.

There are canonical isomorphisms of $F$-algebras

$$
K_0 \otimes_F K_3 \cong K_3 \oplus K_3, \quad K_3 \otimes_F K_3 \cong K_3 \oplus K_3.
$$

The first is the $K_3$-linear extension of $K_0 = F \oplus F \subset K_3 \oplus K_3$, and the second is $x \otimes y \mapsto (xy, x^{F}y)$. Any $\phi \in \text{Hom}_F(K_3, K_0)$ induces a $K_3$-linear map

$$
K_3 \oplus K_3 \cong K_3 \otimes_F K_3 \xrightarrow{\phi \otimes \text{id}} K_0 \otimes_F K_3 \cong K_3 \oplus K_3
$$

represented by a matrix $(a \; b) \in M_2(K_3)$ satisfying $b = a^{F}$ and $c = d^{F}$.

Define a one-dimensional $F$-vector space

$$
\Delta = \text{Hom}_F(\det_F(K_3), \det_F(K_0)),
$$

and define $\det^2(\phi) \in \Delta \otimes_F K_3$ by

$$
\begin{array}{ccc}
\det_F(K_3) \otimes_F K_3 & \xrightarrow{\det^2(\phi)} & \det_K(K_3 \otimes_F K_3) \\
\downarrow & & \downarrow \text{ad} \\
\det_F(K_0) \otimes_F K_3 & \xrightarrow{\det^2(\phi)} & \det_K(K_0 \otimes_F K_3) \\
\end{array}
$$

Here $\text{ad}$ denotes the map $e_1 \wedge e_2 \mapsto (ae_1) \wedge (de_2)$. Exactly as before, we have a commutative diagram

$$
\begin{array}{ccc}
\text{Hom}_F(K_3, K_0) & \xrightarrow{\det^2} & \Delta \otimes_F K_3 \\
\downarrow \text{det} & & \downarrow \text{Tr}_{K_3/F} \\
\Delta & & \Delta
\end{array}
$$

and (2.13) is given by $\text{inv}(\phi) = \det^2(\phi)/\det(\phi)$. 
Proposition 2.9. If we define

$$\text{Iso}_{\text{reg}}(K_3, K_0) = \{ \phi \in \text{Iso}(K_3, K_0) : \text{inv}(\phi) \in K_3^\times \},$$

the function (2.13) restricts to a bijection

$$K_0^\times \backslash \text{Iso}(K_3, K_0) / K_3^\times \overset{\text{inv}}{\rightarrow} \{ \xi \in K_3^\times : \text{Tr}_{K_3/F}(\xi) = 1 \}.$$ 

Proof. Exactly as in §2.1, denote by $Q(K_0, K_3)$ the set of isomorphism classes of quaternion embeddings $(B, \alpha_0, \alpha_3)$. The subset of regular pairs is again denoted by $Q_{\text{reg}}(K_0, K_3) \subset Q(K_0, K_3)$.

Exactly as in (2.12), there is a canonical bijection

$$K_0^\times \backslash \text{Iso}(K_3, K_0) / K_3^\times \cong \text{GL}_2(F) \setminus \left\{ \begin{array}{l}
\text{pairs of embeddings} \\
\alpha_0 : K_0 \rightarrow M_2(F) \\
\alpha_3 : K_3 \rightarrow M_2(F)
\end{array} \right\}.$$ 

As $K_0$ is split, any quaternion embedding $(B, \alpha_0, \alpha_3)$ must have $B \cong M_2(F)$. Thus we obtain a canonical bijection

$$K_0^\times \backslash \text{Iso}(K_3, K_0) / K_3^\times \cong Q(K_0, K_3).$$

Applying the construction of (2.4) with the triple of algebras $(K_1, K_2, K_3)$ replaced by $(K_0, K_3, K_3)$ yields a function

$$Q(K_0, K_3) \overset{\text{inv}}{\rightarrow} \{ \xi \in K_3^\times : \text{Tr}_{K_3/F}(\xi) = 1 \},$$

which, as a tedious but elementary calculation shows, agrees with (2.13) under the above identification. Thus the claim follows from the bijectivity of (2.5). □

3. The analytic calculation

For the remainder of the paper we return to the situation of the introduction, so that $F = k(X)$ and all $F$-algebras appearing in (1.2) are Galois field extensions of $F$.

3.1. Automorphic forms. Denote by $A(G_0)$ the space of automorphic forms [BJ79 §5] on $G_0(\mathbb{A})$, and by $A_{\text{cusp}}(G_0) \subset A(G_0)$ the subspace of cuspidal automorphic forms. The subspace of unramified (that is, $U_0$-invariant) cuspforms is finite-dimensional, and admits a decomposition

$$A_{\text{cusp}}(G_0)_{U_0} = \bigoplus_{\text{unr. cusp. } \pi} \pi_{U_0}$$

as a direct sum of lines, where the sum is over the unramified cuspidal automorphic representations $\pi \subset A_{\text{cusp}}(G_0)$.

Denote by $\mathcal{H}$ the (commutative) Hecke algebra of all compactly supported functions $f : U_0 \backslash G_0(\mathbb{A}) / U_0 \rightarrow \mathbb{Q}$. The $\mathcal{H}$-module of compactly supported unramified $\mathbb{Q}$-valued automorphic forms is denoted

$$\mathcal{A} = C^\infty_c(G_0(F) \backslash G_0(\mathbb{A}) / U_0, \mathbb{Q}).$$
We let $\mathcal{A}_C = \mathcal{A} \otimes \mathbb{C}$ denote the corresponding complex space, so that
\[ \mathcal{A}_{\text{cusp}}(G_0)^{U_0} \subset \mathcal{A}_C \subset \mathcal{A}(G_0)^{U_0}. \] (3.1)

Note that the first inclusion follows from Harder’s theorem [BJ79, Proposition 5.2] that every cuspidal automorphic form on $G_0(\mathbb{A})$ is compactly supported.

According to [YZ17, §4.1], the Satake transform induces a canonical $\mathbb{Q}$-algebra surjection $a_{\text{Eis}} : \mathcal{H} \to \mathbb{Q}[\text{Pic}_X(k)]^{\text{Pic}}$, for a particular involution $\iota_{\text{Pic}}$ of $\mathbb{Q}[\text{Pic}_X(k)]$. The Eisenstein ideal is defined by
\[ I_{\text{Eis}} = \ker (a_{\text{Eis}} : \mathcal{H} \to \mathbb{Q}[\text{Pic}_X(k)]^{\text{Pic}}). \] (3.2)

As in [YZ17, §7.3], define $\mathbb{Q}$-algebras
\[ \mathcal{H}_{\text{aut}} = \text{Image} (\mathcal{H} \to \text{End}_\mathbb{Q}(\mathcal{A}) \times \mathbb{Q}[\text{Pic}_X(k)]^{\text{Pic}}) \]
\[ \mathcal{H}_{\text{cusp}} = \text{Image} (\mathcal{H} \to \text{End}_\mathbb{C}(\mathcal{A}_{\text{cusp}}(G_0)^{U_0})). \]

It follows from (3.1) that the quotient map $\mathcal{H} \to \mathcal{H}_{\text{cusp}}$ factors through $\mathcal{H}_{\text{aut}}$, and the resulting map
\[ \mathcal{H}_{\text{aut}} \to \mathcal{H}_{\text{cusp}} \times \mathbb{Q}[\text{Pic}_X(k)]^{\text{Pic}} \] (3.3)
is an isomorphism by [YZ17, Lemma 7.16].

For each unramified cuspidal automorphic representation $\pi \subset \mathcal{A}_{\text{cusp}}(G_0)$, denote by
\[ \lambda_\pi : \mathcal{H} \to \mathbb{C} \]
the character through which the Hecke algebra acts on the line $\pi^{U_0}$. As in [YZ17, §7.5.1], the $\mathbb{Q}$-algebra $\mathcal{H}_{\text{cusp}}$ is isomorphic to a finite product of number fields, and the product of all characters $\lambda_\pi$ induces an isomorphism
\[ \mathcal{H}_{\text{cusp}} \otimes \mathbb{C} \cong \bigoplus_{\text{unr. cusp. } \pi} \mathbb{C}. \]

Remark 3.1. There is an action of $\text{Aut}(\mathbb{C}/\mathbb{Q})$ on the finite set of unramified cuspidal automorphic representations $\pi \subset \mathcal{A}_{\text{cusp}}(G_0)$, characterized by the relation $\lambda_\pi^\sigma = \sigma \circ \lambda_\pi$.

Remark 3.2. The Petersson inner product identifies the contragredient of $\pi \subset \mathcal{A}_{\text{cusp}}(G_0)$ with the space $\overline{\pi}$ of complex conjugate functions. As $\pi$ has trivial central character, multiplicity one implies that $\pi = \pi^\vee = \overline{\pi}$. From this it is easy to see first that each character $\lambda_\pi : \pi \to \mathbb{C}$ is real-valued, and then that $\mathcal{H}_{\text{cusp}}$ is isomorphic to a product of totally real number fields.

All of the above was for the group scheme $G_0 \cong \text{PGL}_2$ over $X$, but the same discussion holds word-for-word if $G_0$ is replaced by $G_1$, $G_2$, or $G_3$.

Lemma 3.3. For any $i \in \{1, 2, 3\}$ there is a canonical bijection
\[ G_0(F) \backslash G_0(\mathbb{A})/U_0 \to G_i(F) \backslash G_i(\mathbb{A})/U_i. \]
It induces an isomorphism of $\mathbb{C}$-vector spaces
\[ \mathcal{A}(G_0)^{U_0} \cong \mathcal{A}(G_i)^{U_i} \]
respecting the subspaces of cusp forms.

**Proof.** Fix an isomorphism \( \rho : K_0 \to K_i \) of \( F \)-vector spaces. The induced isomorphism \( \rho : \mathbb{A}_0 \to \mathbb{A}_i \) satisfies \( \rho(O_0) = hO_3 \) for some \( h \in \tilde{G}_i(A) \). The choice of \( \rho \) also determines an isomorphism \( \rho : G_{0F} \to G_{iF} \) by \( \rho(g) = \rho \circ g \circ \rho^{-1} \), and the desired bijection is

\[
G_{0F}/G_0(A)/U_0 \xrightarrow{g \mapsto \rho(g)h} G_{iF}/G_i(A)/U_i.
\]

This is easily seen to be independent of the choices of \( \rho \) and \( h \).

**3.2. The analytic distribution.** The \( X \)-scheme

\[
\tilde{J} = \text{Iso}_{\mathcal{X}}(f_3, O_{Y_3}, f_0, O_{Y_0})
\]

is both a left \( \tilde{G}_0 \)-torsor and a right \( \tilde{G}_3 \)-torsor, and similarly \( J = \tilde{J}/\mathbb{G}_m \) is both a left \( G_0 \)-torsor and a right \( G_3 \)-torsor. There are canonical identifications

\[
T_0(F)/J(F)/T_3(F) = \tilde{T}_0(F)/\tilde{J}(F)/\tilde{T}_3(F) = K_0^\times/\text{Iso}(K_3, K_0)/K_3^\times,
\]

and so (2.13) defines a bijective function

\[
T_0(F)/J(F)/T_3(F) \xrightarrow{\text{inv}} \{ \xi \in K_3 : \text{Tr}_{K_3/F}(\xi) = 1 \}.
\]

The bijectivity follows from Proposition 2.9, as \( K_3 \) is now a field.

**Lemma 3.4.** There is a canonical homeomorphism

\[
U_0 \backslash J(A)/U_3 \cong U_0 \backslash G_0(A)/U_0,
\]

**Proof.** As \( O_0 \) and \( O_3 \) are both free \( \mathcal{O} \)-modules of rank two, we may choose an isomorphism of \( \mathbb{A} \)-modules \( a : A_0 \to A_3 \) in such a way that \( a(O_0) = O_3 \). This determines a homeomorphism

\[
\tilde{J}(A) = \text{Iso}(A_3, A_0) \xrightarrow{\phi_3 \circ \phi_0} \text{Iso}(A_0, A_0) = \tilde{G}_0(A).
\]

It is easy to check that this descends to a homeomorphism (3.5), which is independent of the choice of \( a \).

Now fix \( f \in \mathcal{H} \). Slightly switching the point of view, we use the bijection of Lemma 3.4 to instead view \( f : U_0 \backslash J(A)/U_3 \to \mathbb{Q} \), and define a function on \( G_0(A) \times G_3(A) \) by

\[
\mathbb{K}_f(g_0, g_3) = \sum_{\gamma \in J(F)} f(g_0^{-1}\gamma g_3).
\]

**Remark 3.5.** Using Lemma 3.3, we may instead view \( \mathbb{K}_f \) as a function on \( G_0(A) \times G_0(A) \). This agrees with the kernel function defined in [YZ17, (2.3)].

We adopt the usual notation

\[
[T_i] = T_i(F)/T_i(A), \quad [G_i] = G_i(F)/G_i(A),
\]
and define a distribution on $\mathcal{H}$ by
\[
\mathcal{J}(f, s) = \int_{[T_3] \times [T_3]}^{\text{reg}} \mathbb{K}_f(t_0, t_3)|t_0|^{2s} \eta(t_3) \, dt_0 \, dt_3.
\]
(3.7)

We will explain what this regularized integral means momentarily. For the other notation, recalling that $T_0 \subset G_0 = \text{PGL}_2$ is the diagonal torus, let
\[
|\cdot| : T_0(\mathbb{A}) \to \mathbb{R}^\times
\]
be the homomorphism $|a/d| = |a/d|$. The character $\eta$ is defined by the following lemma.

**Lemma 3.6.** The quadratic character $\eta : \mathbb{A}_3^\times \to \{\pm 1\}$ determined by $Y/Y_3$ factors through the quotient $T_3(\mathbb{A}) = \mathbb{A}_3^\times / \mathbb{A}_3^\times$, and hence defines a character $\eta : [T_3] \to \{\pm 1\}$.

**Proof.** Denote by $\chi_i : \mathbb{A}^\times \to \{\pm 1\}$ the quadratic character determined by $Y_i/X$. An exercise in class field theory shows that $\chi_1(\text{Nm}(x)) = \eta(x) = \chi_2(\text{Nm}(x))$ for all $x \in \mathbb{A}_3^\times$, where $\text{Nm} : \mathbb{A}_3^\times \to \mathbb{A}^\times$ is the norm. The claim is immediate from this, and the fact that $\text{Nm}(t) = t^2$ for all $t \in \mathbb{A}^\times$. \qed

The integral in (3.7) need not converge absolutely, and we now explain how it is regularized. First define
\[
T_0(\mathbb{A})_n = \{t_0 \in T_0(\mathbb{A}) : |t_0| = q^{-n}\}
\]
and $[T_0]_n = T_0(F) \backslash T_0(\mathbb{A})_n$, and set
\[
\mathcal{J}_n(f, s) = \int_{[T_0]_n \times [T_3]} \mathbb{K}_f(t_0, t_3)|t_0|^{2s} \eta(t_3) \, dt_0 \, dt_3
\]
(3.8)
\[
= q^{-2ns} \int_{[T_0]_n \times [T_3]} \mathbb{K}_f(t_0, t_3) \eta(t_3) \, dt_0 \, dt_3
\]
This integral is absolutely convergent. Indeed, the set $[T_0]_n$ is compact, and the finiteness of
\[
T_3(F) \backslash T_3(\mathbb{A}) / U_3 \cong K_3^\times \backslash \mathbb{A}_3^\times / \mathbb{A}_3^\times \cong \text{Pic}(Y_3) / f_3^* \text{Pic}(X)
\]
implies that $[T_3]$ is also compact.

**Proposition 3.7.** The integral $\mathcal{J}_n(f, s)$ vanishes for $|n|$ sufficiently large.

**Proof.** For any $\gamma \in J(F)$, and for any $g_0 \in [G_0]$ and $g_3 \in [G_3]$, we set
\[
\mathbb{K}_{f, \gamma}(g_0, g_3) = \sum_{\delta \in T_0(F) \gamma T_3(F)} f(g_0^{-1} \delta g_3)
\]
(3.9)
so that
\[
\mathbb{K}_f(g_0, g_3) = \sum_{\gamma \in T_0(F) \backslash J(F) / T_3(F)} \mathbb{K}_{f, \gamma}(g_0, g_3).
\]
We also set
\[
\mathbb{J}_n(\gamma, f, s) = \int_{[T_0]_n \times [T_3]} \mathbb{K}_{f, \gamma}(t_0, t_3) |t_0|^2 \eta(t_3) \ dt_0 \ dt_3, \quad (3.10)
\]
so that
\[
\mathbb{J}_n(f, s) = \sum_{\gamma \in T_0(F) \setminus J(F)/T_3(F)} \mathbb{J}_n(\gamma, f, s).
\]

**Lemma 3.8.** For all but finitely many \( \gamma \in T_0(F) \setminus J(F)/T_3(F) \), the function \((3.9)\) vanishes identically on \([T_0] \times [T_3]\).

**Proof.** The function \((3.4)\) extends in a natural way to a continuous function on adelic points
\[
\text{inv} : T_0(\mathbb{A}) \setminus J(\mathbb{A})/T_3(\mathbb{A}) \to \{ \xi \in \mathbb{A}_3 : \text{Tr}_{K_3/F}(\xi) = 1 \}.
\]
Indeed, the function \((3.4)\) was defined using the constructions of \((2.3)\) which can be applied locally at every place of \(F\). (Of course, locally \(K_3\) need not be a field, and so this function need not be a bijection).

Let \(C \subset \mathbb{A}_3\) be the image under \(\text{inv} : J(\mathbb{A}) \to \mathbb{A}_3\) of the support of \(f\). This is a compact set, and so has finite intersection with the ( discrete) image of the injection
\[
T_0(F) \setminus J(F)/T_3(F) \xrightarrow{\text{inv}} K_3 \subset \mathbb{A}_3.
\]
Thus there are only finitely many \(\gamma \in T_0(F) \setminus J(F)/T_3(F)\) with \(\text{inv}(\gamma) \in C\).

It is clear from the definitions that if \(\text{inv}(\gamma) \not\in C\) then \((3.9)\) vanishes identically on \([T_0] \times [T_3]\), proving the claim. \(\square\)

To unfold the integral \((3.10)\), we need to understand the stabilizer
\[
S_\gamma = \{(t_0, t_3) : t_0^{-1} \gamma t_3 = \gamma\} \subset T_{0F} \times T_{3F}, \quad (3.11)
\]
which is an algebraic group over \(F\).

**Lemma 3.9.** The algebraic group \(S_\gamma\) is trivial.

**Proof.** Fix a lift \(\gamma \in \tilde{J}(F) = \text{Iso}(K_3, K_0)\) and use this to identify \(K_0 \cong F \oplus F \cong K_3\) as \(F\)-vector spaces. This identifies \(T_{0F}\) and \(T_{3F}\) as tori inside of a common algebraic group \(G_{0F} \cong \text{PGL}_2 \cong G_{3F}\) over \(F\), and further identifies
\[
S_\gamma \cong \{(t_0, t_3) : t_0^{-1} t_3 = \text{id}_{\text{PGL}_2}\} \subset T_{0F} \times T_{3F}.
\]
In other words, \(S_\gamma \cong T_{0F} \cap T_{3F} \subset \text{PGL}_2\). As \(T_{0F}\) is split but \(T_{3F}\) is not, this intersection is trivial. \(\square\)

**Lemma 3.10.** For fixed \(\gamma\) and \(f\) as above, the integral \((3.10)\) vanishes for all but finitely many \(n\).

**Proof.** Since \(S_\gamma\) is trivial, we may unfold the integral \((3.10)\) to obtain
\[
\mathbb{J}_n(\gamma, f, s) = \int_{T_0(\mathbb{A})_n \times T_3(\mathbb{A})} f(t_0^{-1} \gamma t_3) \ |t_0|^2 \eta(t_3) \ dt_0 \ dt_3.
\]
The map $i: T_0(\mathbb{A}) \times T_3(\mathbb{A}) \to J(\mathbb{A})$ given by $(t_0, t_3) \mapsto t_0^{-1} \gamma t_3$ is a closed embedding, so $f \circ i$ has compact support. It follows that $\mathbb{J}_n(\gamma, f, s)$ vanishes for $|n|$ large enough.

We now complete the proof of Proposition 3.7. By Lemma 3.8,

$$\mathbb{J}_n(f, s) = \sum_{\gamma \in \Gamma} \mathbb{J}_n(\gamma, f, s)$$

for some finite subset $\Gamma_f \subset T_0(F) \backslash J(F) / T_3(F)$ independent of $n$. Lemma 3.10 now implies that $\mathbb{J}_n(f, s)$ vanishes for $|n| \gg 0$.

Using Proposition 3.7, the regularized integral (3.7) is defined as

$$\mathbb{J}(f, s) = \sum_{n \in \mathbb{Z}} \mathbb{J}_n(f, s).$$

This is a Laurent polynomial in $q^s$. Recalling (3.10), we also define

$$\mathbb{J}(\gamma, f, s) = \sum_{n \in \mathbb{Z}} \mathbb{J}_n(\gamma, f, s),$$

so that there are decompositions

$$\mathbb{J}(f, s) = \sum_{\gamma \in T_0(F) \backslash J(F) / T_3(F)} \mathbb{J}(\gamma, f, s) = \sum_{\xi \in K_3 \backslash T_3(F) / (\xi = 1)} \mathbb{J}(\xi, f, s). \quad (3.12)$$

Here, in the final expression, we use (3.4) to define

$$\mathbb{J}(\xi, f, s) = \mathbb{J}(\gamma, f, s)$$

for the unique double coset $\gamma \in T_0(F) \backslash J(F) / T_3(F)$ satisfying $\text{inv}(\gamma) = \xi$.

### 3.3. Spectral decomposition.

For any $\phi \in \mathcal{A}_{\text{cusp}}(G_0)^{U_0}$ we define a period integral

$$\mathcal{P}_0(\phi, s) = \int_{[T_0]} \phi(t_0) |t_0|^{2s} dt_0.$$  

This integral is absolutely convergent for all $s \in \mathbb{C}$. Using the isomorphism of Lemma 3.4 to view $\phi \in \mathcal{A}_{\text{cusp}}(G_3)^{U_3}$, we define another period integral

$$\mathcal{P}_3(\phi, \eta) = \int_{[T_3]} \phi(t_3) \eta(t_3) dt_3.$$  

As $[T_3]$ is compact, this integral is also absolutely convergent.

Recall the Eisenstein ideal $\mathcal{I}^\text{Eis} \subset \mathcal{H}$ of (3.2).

**Proposition 3.11.** Every $f \in \mathcal{I}^\text{Eis}$ satisfies

$$\mathbb{J}(f, s) = \sum_{\text{unr. cusp. } \pi} \lambda_\pi(f) \frac{\mathcal{P}_0(\phi, s) \mathcal{P}_3(\phi, \eta)}{\langle \phi, \phi \rangle}, \quad (3.13)$$

where the sum is over all unramified cuspidal automorphic representations $\pi \subset \mathcal{A}_{\text{cusp}}(G_0)$, and $\phi \in \pi^{U_0}$ is any nonzero vector. Moreover, $\mathbb{J}(f, s)$ only depends on the image of $f$ under the quotient map $\mathcal{H} \to \mathcal{H}_{\text{aut}}$. 
Proof. If we use Remark 3.5 to view (3.6) as a function on $G_0(\mathbb{A}) \times G_0(\mathbb{A})$, then invoke the decomposition

$$K_f(x, y) = K_{f, \text{cusp}}(x, y) + K_{f, \text{sp}}(x, y)$$

of [YZ17, Theorem 4.3], and then convert all three terms back into functions on $G_0(\mathbb{A}) \times G_0(\mathbb{A})$, the result is a decomposition

$$K_f(g_0, g_3) = \sum_{\text{unr. cusp. } \pi} \lambda_\pi(f) \cdot \frac{\phi(g_0)\phi(g_3)}{\langle \phi, \phi \rangle} + \sum_{\text{unr. quad. } \chi} \lambda_\chi(f) \cdot \chi(\det(g_0)) \cdot \chi(\det(g_3)).$$

The first sum is over all unramified cuspidal representations $\pi$, and $\phi \in \pi^{T_0}$ is any nonzero vector. The second sum is over all unramified quadratic characters

$$\text{Pic}(X) \cong F^\times \backslash \mathbb{A}^\times / \mathbb{Q}^\times \xrightarrow{\lambda} \{\pm 1\},$$

and

$$\lambda_\chi(f) = \int_{G_0(\mathbb{A})} f(g) \chi(\det(g)) \, dg.$$

The distribution (3.8) now decomposes as

$$J_{n}(f, s) = \sum_{\text{unr. cusp. } \pi} J_{n, \pi}(f, s) + \sum_{\text{unr. quad. } \chi} J_{n, \chi}(f, s),$$

where we have set

$$J_{n, \pi}(f, s) = \frac{\lambda_\pi(f)}{\langle \phi, \phi \rangle} \left( \int_{[T_0]_n} \phi(t_0) |t_0|^{2s} \, dt_0 \right) \left( \int_{[T_3]} \phi(t_3) \eta(t_3) \, dt_3 \right)$$

and

$$J_{n, \chi}(f, s) = \lambda_\chi(f) \left( \int_{[T_0]_n} \chi(\det(t_0)) |t_0|^{2s} \, dt_0 \right) \left( \int_{[T_3]} \chi(\det(t_3)) \eta(t_3) \, dt_3 \right).$$

In fact, $J_{n, \chi}(f, s) = 0$ for all $n$ and all $\chi$. The proof is similar to that of [YZ17, Lemma 4.4]. Briefly, if the restriction of $\chi$ to

$$\text{Pic}^0(X) = F^\times \backslash \mathbb{A}^\times / \mathbb{Q}^\times$$

is nontrivial then the integral over $[T_0]_n$ vanishes. If the restriction of $\chi$ to $\text{Pic}^0(X)$ is trivial then the integral over $[T_3]$ vanishes. This leaves us with

$$J_{n}(f, s) = \sum_{\text{unr. cusp. } \pi} J_{n, \pi}(f, s),$$

and (3.13) follows by summing both sides over $n$.

For the final claim, suppose $f$ has trivial image under $\mathcal{H} \to \mathcal{H}_{\text{aut}}$. This implies that $f$ annihilates $\mathcal{L}_{\mathcal{C}}$, and lies in $T^{\text{Eis}}$. The first inclusion in (3.1) implies that $\lambda_\pi(f) = 0$ for all $\pi$, and so $J(f, s) = 0$ by (3.13). $\square$
The Hecke algebra $\mathcal{H}$ has a $\mathbb{Q}$-basis $\{f_D\}$ indexed by the effective divisors $D \in \text{Div}(X)$, and defined as follows (see also [YZ17, §3.1]). Let $S_D$ be the image of the set 

$$\{X \in M_2(\mathbb{O}) : \text{div}(\det X) = D\}$$

in $\text{PGL}_2(\mathbb{A}) = G_0(\mathbb{A})$. Then $f_D : U_0 \setminus G_0(\mathbb{A}) / U_0 \to \mathbb{Q}$ is the characteristic function of $S_D$.

The remainder of §3 is devoted to interpreting the $r$-th derivative

$$\frac{d^r}{ds^r} \mathbb{J}(f_D, s)|_{s=0} = \sum_{\xi \in K_3} \text{Tr}_{K_3/F}(\xi) = 1 \frac{d^r}{ds^r} \mathbb{J}(\xi, f_D, s)|_{s=0},$$

in terms of algebraic geometry, for each such $D$. This interpretation can be found in §3.6 below (see especially Proposition 3.17), after the definitions and preliminary results of §3.4 and §3.5.

### 3.4. Some moduli spaces

Fix an integer $d$. Our goal is to describe a commutative diagram of $k$-schemes

$$\begin{array}{ccc}
N_{(d_1, d_2)} & \longrightarrow & \Sigma_{d_1}(Y_3) \times_k \Sigma_{d_2}(Y_3) \\
\downarrow & & \downarrow \otimes \\
A_{d} & \longrightarrow & \Sigma_{2d}(Y_3) \\
\downarrow & \downarrow \text{Tr} \\
\Sigma_{d}(X) & \longrightarrow &
\end{array} \tag{3.14}$$

for any pair of non-negative integers with $d_1 + d_2 = 2d$, in such a way that the square is cartesian. We will define these schemes by specifying their functors of points. Let $S$ be a $k$-scheme.

Denote by $\Sigma_d(X)(S)$ the set of isomorphism classes of pairs $(\Delta, \zeta)$ consisting of

- a line bundle $\Delta$ on $X_S = X \times_k S$ of degree $d$,
- a nonzero section $\zeta \in H^0(X_S, \Delta)$.

Here (and hereafter) these conditions should be interpreted fiber-by-fiber: for every $s \in S$ we assume $\deg(\Delta_s) = d$ and $\zeta_s \neq 0$. With this convention, it follows from [Mil80, Proposition I.2.5] that $\text{div}(\zeta)$ is flat over $S$. Moreover, this divisor determines the pair $(\Delta, \zeta)$ up to isomorphism, and so $\Sigma_d(X)$ is identified with the Hilbert scheme parametrizing effective relative Cartier divisors on $X$ of degree $d$. It follows from the discussion of [BLR90, §9.3] that there is a canonical isomorphism

$$\Sigma_d(X) \cong \text{Sym}^d(X) \cong S_d \setminus X^d \tag{3.15}$$

(the rightmost scheme is the GIT quotient), and that $\Sigma_d(X)$ is a smooth projective $k$-scheme. The schemes $\Sigma_{2d}(Y_3)$ and $\Sigma_{d_1}(Y_3)$ are defined similarly, and the vertical arrow in (3.14) labeled $\otimes$ has the obvious meaning.

Let $A_d(S)$ be set of isomorphism classes of pairs $(\Delta, \zeta)$ consisting of
• a line bundle $\Delta$ on $X_S$ of degree $d$,
• a section $\xi \in H^0(Y_{3S}, f_3^*\Delta)$ with nonzero trace
  \[ \text{Tr}_{Y_{3S}/X}(\xi) = \xi + \xi^{\sigma_3} \in H^0(X_S, \Delta) . \]

The arrows in (3.14) emanating from $A_d$ are
  \[ \text{Tr}(\Delta, \xi) = (\Delta, \text{Tr}_{Y_{3S}/X}(\xi)) \quad \text{and} \quad f_3^*(\Delta, \xi) = (f_3^*\Delta, \xi). \]

It is easy to check that $\text{Tr}$ is a quasi-projective morphism, and hence $A_d$ is a quasi-projective $k$-scheme.

Now define $\tilde{N}_{d_1,d_2}(S)$ to be the groupoid of quadruples $(L_1, L_2, L_3, \phi)$ consisting of

• line bundles $L_1, L_2 \in \text{Pic}(X_S)$ and $L_3 \in \text{Pic}(Y_{3S})$ satisfying
  \[ 2\deg(L_1) - d_1 = \deg(L_3) = 2\deg(L_2) - d_2, \]
• a morphism $\phi : f_3^*L_3 \to L_1 \oplus L_2$ of rank 2 vector bundles on $X_S$ with nonzero determinant.

The functor $\tilde{N}_{d_1,d_2}$ from $k$-schemes to groupoids is represented by an Artin stack. The Picard group $\text{Pic}(X_S)$ acts on $\tilde{N}_{d_1,d_2}(S)$ by twisting
  \[ (L_1, L_2, L_3, \phi) \otimes L = (L_1 \otimes L, L_2 \otimes L, L_3 \otimes f_3^*L, \phi \otimes \text{id}), \]

defining an action of the Picard stack $\text{Pic}_X$ on $\tilde{N}_{d_1,d_2}$. The representability of the quotient stack
  \[ N_{(d_1,d_2)} = \tilde{N}_{d_1,d_2}/\text{Pic}_X \]
by a scheme is part of the following result, which also defines the arrows in (3.14) emanating from $N_{(d_1,d_2)}$. (See [Ngo06, §4] for a discussion of the meaning of such a quotient.)

**Proposition 3.12.** There is a canonical isomorphism
  \[ N_{(d_1,d_2)} \cong A_d \times_{\Sigma_{2d}(Y_3)} \left( \Sigma_{d_1}(Y_3) \times_{k} \Sigma_{d_2}(Y_3) \right) . \]

**Proof.** Suppose $(L_1, L_2, L_3, \phi) \in \tilde{N}_{(d_1,d_2)}(S)$. The pullback of $\phi$ via $f_3 : Y_{3S} \to X_S$ is a morphism
  \[ L_3 \oplus L_3^{\sigma_3} \to f_3^*L_1 \oplus f_3^*L_2, \]
encoded by four maps
  \[ \begin{align*}
  L_3 & \xrightarrow{a} f_3^*L_1 \\
  L_3^{\sigma_3} & \xrightarrow{b=a^{\sigma_3}} f_3^*L_1 \\
  L_3 & \xrightarrow{c} f_3^*L_2 \\
  L_3^{\sigma_3} & \xrightarrow{d=c^{\sigma_3}} f_3^*L_2.
  \end{align*} \]

By viewing $a$ and $d$ as global sections of the line bundles
  \[ K_1 = \text{Hom}(L_3, f_3^*L_1), \quad K_2 = \text{Hom}(L_3^{\sigma_3}, f_3^*L_2) \]
(3.17)
of degree $d_1$ and $d_2$, respectively, we obtain $S$-points of $\Sigma_{d_1}(Y_3)$ and $\Sigma_{d_2}(Y_3)$. This defines a morphism

$$\tilde{N}_{(d_1,d_2)} \to \Sigma_{d_1}(Y_3) \times_k \Sigma_{d_2}(Y_3),$$

which is easily seen to descend to the quotient $N_{(d_1,d_2)}$.

Define a degree $d$ line bundle

$$\Delta = \text{Hom}(\det(f_3, \mathcal{L}_3), \det(\mathcal{L}_1 \oplus \mathcal{L}_2))$$

on $X_S$. Its pullback to $Y_{3S}$ is

$$f_3^* \Delta \cong \text{Hom}(\det(\mathcal{L}_3 \oplus \mathcal{L}_3^{a_3}), \det(f_3^* \mathcal{L}_1 \oplus f_3^* \mathcal{L}_2))$$

which has a global section $a \cdot d$ defined by $e_1 \wedge e_2 \mapsto a(e_1) \wedge d(e_2)$ for local sections $e_1$ and $e_2$ of $\mathcal{L}_3$ and $\mathcal{L}_3^{a_3}$, respectively. The equality of global sections

$$\det(\phi) = \text{Tr}_{Y_3/X}(a \cdot d) \in H^0(X_S, \Delta),$$

shows that $\text{Tr}_{Y_3/X}(a \cdot d)$ is nonzero. Thus the pair $(\Delta, a \cdot d)$ defines an $S$-point of $A_d$, and we have constructed a morphism

$$\tilde{N}_{(d_1,d_2)} \to A_d.$$

Again, this is easily seen to descend to the quotient $N_{(d_1,d_2)}$.

Combining the above constructions defines a morphism

$$N_{(d_1,d_2)} \to A_d \times_{\Sigma_{2d}(Y_3)} (\Sigma_{d_1}(Y_3) \times_k \Sigma_{d_2}(Y_3)).$$

(3.19)

To show it is an isomorphism we construct the inverse. An $S$-point of the fiber product consists of a line bundle $\Delta$ on $X_S$ of a degree $d$, line bundles $\mathcal{K}_1$ and $\mathcal{K}_2$ on $Y_{3S}$ of degrees $d_1$ and $d_2$, global sections

$$\xi \in H^0(Y_{3S}, f_3^* \Delta), \quad a \in H^0(Y_{3S}, \mathcal{K}_1), \quad d \in H^0(Y_{3S}, \mathcal{K}_2),$$

such that $\text{Tr}_{Y_3/X}(\xi) \in H^0(X_S, \Delta)$ is nonzero, and an isomorphism $f_3^* \Delta \cong \mathcal{K}_1 \otimes \mathcal{K}_2$ identifying $\xi = a \otimes d$.

Starting from this data we define $(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \phi)$ as follows. Let $\mathcal{L}_1$ be any line bundle on $X_S$. If we define $\mathcal{L}_3 = \mathcal{K}_1^{-1} \otimes f_3^* \mathcal{L}_1$, there are canonical isomorphisms

$$(\mathcal{L}_3^{a_3} \otimes \mathcal{K}_2)^{a_3} \cong \mathcal{L}_3 \otimes (\mathcal{K}_1^{-1} \otimes f_3^* \Delta)^{a_3}$$

$$\cong \mathcal{L}_3 \otimes (\mathcal{L}_3 \otimes f_3^* \mathcal{L}_1^{-1} \otimes f_3^* \Delta)^{a_3}$$

$$\cong \mathcal{L}_3^{a_3} \otimes (\mathcal{L}_3 \otimes f_3^* \mathcal{L}_1^{-1} \otimes f_3^* \Delta)$$

$$\cong \mathcal{L}_3^{a_3} \otimes (\mathcal{K}_1^{-1} \otimes f_3^* \Delta)$$

$$\cong \mathcal{L}_3^{a_3} \otimes \mathcal{K}_2.$$

Viewing this as descent data relative to $Y_3/X$, we obtain a line bundle $\mathcal{L}_2$ on $X_S$ endowed with an isomorphism $f_3^* \mathcal{L}_2 \cong \mathcal{K}_2 \otimes \mathcal{L}_3^{a_3}$. We now view $a$ and $d$ as global sections of the line bundles

$$\text{Hom}(\mathcal{L}_3, f_3^* \mathcal{L}_1) \cong \mathcal{K}_1, \quad \text{Hom}(\mathcal{L}_3^{a_3}, f_3^* \mathcal{L}_2) \cong \mathcal{K}_2,$$
and use (3.16) to define two more global sections $b$ and $c$. These four sections define a morphism
\[ L_3 \oplus L_3^\sigma \to f_3^* L_1 \oplus f_3^* L_2 \]
of line bundles on $Y_3$, which descends to a morphism $\phi : f_3^* L_3 \to L_1 \oplus L_2$ of vector bundles on $X$. The composition
\[ f_3^* \Delta \cong K_1 \otimes K_2 \]
is compatible with the descent data on both sides, and descends to an isomorphism
\[ \Delta \cong \text{Hom}(\det(f_3^* L_3), \det(L_1 \oplus L_2)) \]
sending $\text{Tr}_{Y_3/X}(\xi) \mapsto \det(\phi)$. Thus $\det(\phi)$ is nonzero.

This construction defines the desired inverse to (3.19). □

**Proposition 3.13.** Let $g$ and $g_3$ be the genera of $X$ and $Y_3$, respectively, so that $g_3 = 2g - 1$.

1. The vertical morphisms in (3.14) labeled $\beta$ and $\otimes$ are finite.
2. If $d \geq 2g_3 - 1$ then $N_{(d_1,d_2)}$ is smooth over $k$ of dimension $2d - g + 1$.

**Proof.** For the first claim, the morphism in (3.14) labeled $\otimes$ can be identified with the canonical morphism
\[ \text{Sym}^d Y_3 \times \text{Sym}^d Y_3 \to \text{Sym}^d Y_3, \]
using the obvious analogues of (3.15). This is obviously finite. Proposition 3.12 which asserts that the square in (3.14) is cartesian, therefore implies the finiteness of $\beta$.

The proof of the second claim is similar to [YZ17, Proposition 3.1(2)]. Examining the proof of Proposition 3.12 yields a cartesian diagram
\[ \begin{array}{ccc}
N_{(d_1,d_2)} & \rightarrow & \Sigma_{d_1}(Y_3) \\
\downarrow & & \downarrow \\
\text{Pic}^d_X \times_k \Sigma_{d_2}(Y_3) & \rightarrow & \text{Pic}^d_{Y_3}.
\end{array} \]

Here $\text{Pic}^d_X$ is the Picard stack of line bundles of degree $d$. It is a smooth Artin stack over $k$ of dimension $g - 1$. The stack $\text{Pic}^d_{Y_3}$ is defined similarly. To define the various morphisms, start with a quadruple $(L_1, L_2, L_3, \phi) \in N_{(d_1,d_2)}(S)$. Recall the line bundles
\[ \Delta \in \text{Pic}^d(X), \quad \kappa_1 \in \text{Pic}^{d_1}(Y_3), \quad \kappa_2 \in \text{Pic}^{d_2}(Y_3) \]
of (3.18) and (3.17), which are related by
\[ f_3^* \Delta \cong \text{Hom}(L_3 \otimes L_3^\sigma, f_3^* L_1 \otimes f_3^* L_2) \cong \kappa_1 \otimes \kappa_2, \]
and the global sections $a \in H^0(Y_3, \kappa_1)$ and $b \in H^0(Y_3, \kappa_2)$. The top horizontal arrow sends $(L_1, L_2, L_3, \phi) \mapsto (\kappa_1, a)$. The vertical arrow on
the left sends \((L_1, L_2, L_3, \phi) \mapsto (\Delta, K_2, d)\), and the bottom horizontal arrow sends \((\Delta, K_2, d) \mapsto f_3^* \Delta \otimes K_2^{-1}\). The right vertical arrow sends \((L_1, a) \mapsto L_1\).

Now assume that \(d \geq 2g_3 - 1\). This implies that either \(d_1 \geq 2g_3 - 1\) or \(d_2 \geq 2g_3 - 1\), and without loss of generality we assume the former inequality. This implies that the vertical arrow on the right in (3.20) is smooth of relative dimension \(d_1 - g_3 + 1\), and hence the same is true of the vertical arrow on the left. As the target of the left arrow is smooth over \(k\) of dimension \(g - 1 + d_2\), we deduce that \(N(d_1, d_2)\) is smooth of dimension \(2d - g + 1\).

3.5. A local system. Let \(d\) be any non-negative integer. By identifying \(\{\pm 1\}^d \cong \text{Aut}(Y/Y_3)^d\), the group \(\Gamma_d = \{\pm 1\}^d \rtimes S_d\) acts on \(Y_d\). As in (3.15), there are canonical isomorphisms

\[
\Sigma_d(Y_3) \cong \text{Sym}^d(Y_3) \cong \Gamma_d\backslash Y^d.
\]

Denote by \(\eta_d : \{\pm 1\}^d \to \{\pm 1\}\) the product character, extend it to a character \(\eta_d : \Gamma_d \to \{\pm 1\}\) trivial on \(S_d\), and form the GIT quotient

\[
\Sigma_d(Y/Y_3) = \text{Ker}(\eta_d)\backslash Y^d.
\]

**Proposition 3.14.** The canonical morphism

\[
\Sigma_d(Y/Y_3) \to \Sigma_d(Y_3)
\]

is a finite étale double cover.

**Proof.** The finiteness claim is clear. Étaleness can be verified on the level of completed étale local rings. Accordingly, let \(A\) be the completed étale local ring at a point of \(\Sigma_d(Y_3)(\bar{k})\), and let \(B\) be the product of the completed étale local rings at all points of \(\Sigma_d(Y/Y_3)(\bar{k})\) above it. We now have a cartesian diagram

\[
\begin{array}{c}
\text{Spec}(B) \\
\downarrow \\
\text{Spec}(A)
\end{array} \quad \begin{array}{c}
\rightarrow \\
\downarrow \\
\rightarrow
\end{array} \quad \begin{array}{c}
\Sigma_d(Y/Y_3) \\
\downarrow \\
\Sigma_d(Y_3)
\end{array}
\]

and it suffices to prove that \(B \cong A \oplus A\) as \(A\)-algebras.

Set \(D = \bar{k}[[x]]\), and consider the \(D\)-algebra \(E = D \oplus D\). Let \(D_d\) and \(E_d\) be the \(d\)-fold completed tensor products (over \(\bar{k}\)) of \(D\) and \(E\) with themselves. These are complete local \(\bar{k}\)-algebras carrying actions of \(S_d\) and \(\Gamma_d\), respectively, and \(D_d^S = E_d^\Gamma\). As \(Y \to Y_3\) is an étale double cover of smooth curves, we may identify \(A \to B\) with

\[
D_d^S \to E_d^\text{Ker}(\eta_d).
\]

Label the orthogonal idempotents in \(E\) as \(e_+, e_- \in E\). Each tuple of signs \(x = (x_1, \ldots, x_d) \in \{\pm 1\}^d\) determines an idempotent

\[
e_x = e_{x_1} \otimes \cdots \otimes e_{x_d} \in E_d.
\]
Each of the orthogonal idempotents
\[ f_+ = \sum_{x \in \{1\}^d, \eta_d(x) = 1} e_x, \quad f_- = \sum_{x \in \{1\}^d, \eta_d(x) = -1} e_x \]
in \( E_d \) is fixed by the action of the subgroup \( \text{Ker}(\eta_d) \subset \Gamma_d \), and the composition \( D_d \to E_d \to f_\pm E_d \) restricts to an isomorphism
\[
D_d^{S_d} \cong (f_\pm E_d)^{\text{Ker}(\eta_d)}.
\]
Thus we obtain the desired decomposition
\[
E_d^{\text{Ker}(\eta_d)} = (f_+ E_d)^{\text{Ker}(\eta_d)} \oplus (f_- E_d)^{\text{Ker}(\eta_d)} \cong D_d^{S_d} \oplus D_d^{S_d}.
\]
\[\square\]

Choose any prime \( \ell \) different from the characteristic of \( k \). The \( \acute{e}tale \) double cover \((3.21)\) determines a quadratic character of the \( \acute{e}tale \) fundamental group of \( \Sigma_d(Y_3) \), which then determines a rank one \( \acute{e}tale \) local system \( L_d \) of \( \mathbb{Q}_\ell \)-vector spaces on \( \Sigma_d(Y_3) \).

Now fix \( d_1, d_2 \geq 0 \), and let
\[
j : N_{(d_1,d_2)} \to \Sigma_{d_1}(Y_3) \times_k \Sigma_{d_2}(Y_3)
\]
be as in \((3.14)\). Define a rank one local system
\[
L_{(d_1,d_2)} = j^*(L_{d_1} \boxtimes \mathbb{Q}_\ell)
\]
of \( \mathbb{Q}_\ell \)-vector spaces on \( N_{(d_1,d_2)} \).

**Proposition 3.15.** Suppose \( z = (L_1, L_2, L_3, \phi) \in N_{(d_1,d_2)}(k) \) is a \( k \)-point, and \( \bar{z} \to z \) is a geometric point above it. Recalling the character
\[
\text{Pic}(Y_3) \cong K_3^X \setminus A_3^X / \mathbb{Q}_3^X \to \{ \pm 1 \}
\]
of Lemma 7.6, the Frobenius \( \text{Frob}_z \in \text{Aut}(\bar{z} / z) \) acts on \( L_{(d_1,d_2),\bar{z}} \) via the scalar \( \eta(L_3) \).

**Proof.** If \( y = (\mathcal{K}, a) \in \Sigma_{\eta}(Y_3)(k) \) is a \( k \)-point, and \( \bar{y} \to y \) is a geometric point above it, it is easy to see from the definitions that \( \text{Frob}_y \) acts on the fiber \( L_{n,\bar{y}} \) as \( \eta(\mathcal{K}) \). Recalling the construction of the map \( j \) from the proof of Proposition 3.12, especially \((3.17)\), it follows that \( \text{Frob}_z \) acts on \( L_{(d_1,d_2),\bar{z}} \) by the scalar \( \eta(K_1) = \eta(L_3) \eta(f_3^* L_1) \). The proof of Lemma 3.6 shows that \( \eta(f_3^* L_1) = \chi_1(\mathcal{L}_1^\otimes 2) = 1 \), completing the proof. \(\square\)

### 3.6. Geometric interpretation of orbital integrals.

Let \( D \) be an effective divisor on \( X \) of degree \( d \). The constant function \( 1 \) defines a global section of \( \mathcal{O}_X(D) \), defining a point \((\mathcal{O}_X(D), 1) \in \Sigma_d(X)(k) \). We define \( A_D \) as the fiber product

\[
\begin{array}{ccc}
A_D & \xrightarrow{\text{triv}} & A_d \\
\downarrow & & \downarrow \\
\text{Spec}(k) & \xrightarrow{\mathcal{O}_X(D), 1} & \Sigma_d(X).
\end{array}
\]
Proposition 3.16. There is a canonical bijection

\[ A_D(k) \cong \left\{ \xi \in K_3 : \text{Tr}_{K_3/F}(\xi) = 1, \text{div}(\xi) + f_3^*D \geq 0 \right\}. \]

Proof. A k-point of \( A_D \) consists of a line bundle \( \Delta \in \text{Pic}^d(X) \) and a global section \( \xi \in H^0(Y_3, f_3^*\Delta) \), together with an isomorphism \( \Delta \cong \mathcal{O}_X(D) \) identifying \( \text{Tr}_{Y_3/X}(\xi) = 1 \). In other words, a trace 1 element of

\[ H^0(Y_3, f_3^*\mathcal{O}_X(D)) = H^0(Y_3, \mathcal{O}_{Y_3}(f_3^*D)) = \{ \xi \in K_3 : \text{div}(\xi) + f_3^*D \geq 0 \}. \]

\[ \square \]

Recall the canonical Q-basis \( \{ f_D \} \subset \mathcal{H} \) indexed by effective divisors \( D \in \text{Div}(X) \). We are now ready to give a geometric interpretation of the orbital integral \( I(\xi, f_D, s) \) appearing in (3.12).

Using the homeomorphism of Lemma 3.4, we change the point of view and regard \( f_D \) as a compactly supported function

\[ f_D : U_0 \setminus J(\mathbb{A})/U_3 \rightarrow \mathbb{Q}. \] (3.23)

To make this more explicit, define a free rank one \( \mathcal{O} \)-module

\[ \Delta = \text{Hom}(\text{det}(\mathcal{O}_3), \text{det}(\mathcal{O}_0)), \]

and define \( \hat{\Omega}_D \) as the set of all \( \phi \in \hat{J}(\mathbb{A}) = \text{Iso}(\mathbb{A}_3, \mathbb{A}_0) \) such that

- \( \phi(\mathcal{O}_3) \subset \mathcal{O}_0, \)
- \( \text{det}(\phi) \in \Delta \) generates \( a_D \Delta \) as an \( \mathcal{O} \)-module.

Here \( a_D \in \mathbb{A}^\times/\mathcal{O}^\times \cong \text{Dix}(X) \) represents the divisor \( D \). Now let \( \Omega_D \) be the image of \( \hat{\Omega}_D \) under the quotient map \( \hat{J}(\mathbb{A}) \rightarrow J(\mathbb{A}) \). The function (3.23) is the characteristic function of \( \Omega_D \).

Recall that local system \( L_{(d_1, d_2)} \) of (3.22). Its pushforward via the finite morphism \( \beta \) of (3.11) is a constructible sheaf on \( A_d \). The following result relates this sheaf to the distribution (3.12).

Proposition 3.17. Fix \( \xi \in K_3 \) with \( \text{Tr}_{K_3/F}(\xi) = 1 \), and view \( A_D(k) \subset K_3 \) using Proposition 3.16.

(1) If \( \xi \not\in A_D(k) \) then \( I(\xi, f_D, s) = 0 \).

(2) If \( \xi \in A_D(k) \) then

\[ I(\xi, f_D, s) = \sum_{d_1, d_2 \geq 0 \atop d_1 + d_2 = 2d} q^{(d_1 - d_2)s} \cdot \text{Trace}((\beta_\xi L_{(d_1, d_2)})^s), \]

where \( \xi \) is a geometric point above \( \xi : \text{Spec}(k) \rightarrow A_D \).

Proof. Recalling (3.4), let \( \gamma \in T_0(F) \setminus J(F)/T_3(F) \) be the unique element with \( \xi = \text{inv}(\gamma) \), and fix a lift

\[ \gamma \in \hat{J}(F) = \text{Iso}(K_3, K_0). \]

We view \( \gamma \) as a rational map

\[ f_{3*}\mathcal{O}_{Y_3} \xrightarrow{\gamma} f_{0*}\mathcal{O}_{Y_0} = \mathcal{O}_X \oplus \mathcal{O}_X. \]
of rank two vector bundles. Each triple
\[ (E_1, E_2, E_3) \in \text{Div}(X) \backslash (\text{Div}(X) \times \text{Div}(X) \times \text{Div}(Y_3)) \] (3.24)
determines a rational map \( \phi_\gamma \) by the commutativity of
\[
\begin{array}{c}
\begin{array}{c}
\longrightarrow \\
\phi_\gamma \\
\longrightarrow \\
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
O_Y(-E_3) \\
O_X(-E_1) \oplus O_X(-E_2)
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
O_X \oplus O_X
\end{array}
\end{array}
\]
Here the vertical arrows are the canonical rational maps.

Denote by \( \mathfrak{N}_{D,\gamma} \) the set of all triples (3.24) such that \( \phi_\gamma \) is a morphism of vector bundles (as opposed to just a rational map) with \( \deg(\text{det}(\phi_\gamma)) = D \).

If the canonical bijection
\[
\mathbb{A}^\times \backslash (\hat{T}_0 \times \hat{T}_3)(\mathbb{A})/(\hat{T}_0 \times \hat{T}_3)(\mathbb{O})
\cong \text{Div}(X) \backslash (\text{Div}(Y_0) \times \text{Div}(Y_3))
\cong \text{Div}(X) \backslash (\text{Div}(X) \times \text{Div}(X) \times \text{Div}(Y_3))
\]
sends \( (t_0, t_3) \mapsto (E_1, E_2, E_3) \), then
\[
\hat{f}_D(t_0^{-1} \gamma t_3) = \begin{cases} 
1 & \text{if } (E_1, E_2, E_3) \in \mathfrak{N}_{D,\gamma} \\
0 & \text{otherwise.}
\end{cases} \] (3.25)
Here, as in (3.23), we define \( \hat{f}_D \) as the characteristic function of the subset
\[
\hat{\Omega}_D = \prod_{x \in |X|} \hat{\Omega}_{D,x} \subset \hat{J}(\mathbb{A}).
\]

Any triple \((E_1, E_2, E_3) \in \mathfrak{N}_{D,\gamma}\) defines a quadruple
\[ (O_X(-E_1), O_X(-E_2), O_{Y_3}(-E_3), \phi_\gamma) \in N_{(d_1, d_2)}(k) \] (3.26)
where \( d_i = \deg(E_i) - 2 \deg(E_i) \). The image of this quadruple under
\[
N_{(d_1, d_2)}(k) \rightarrow A_d(k) \xrightarrow{\text{Tr}} \Sigma_d(X)(k)
\]
is the pair \((\Delta, \zeta)\) defined by the line bundle
\[
\Delta = \text{Hom}(\text{det}(f_3, O_{Y_3}(-E_3)), \text{det}(O_X(-E_1) \oplus O_X(-E_2)))
\]
on \( X \) and its global section \( \zeta = \text{det}(\phi_\gamma) \) with divisor \( D \). In other words, the image of (3.26) under \( N_{(d_1, d_2)}(k) \rightarrow A_d(k) \) lies in the fiber over
\[
(\Delta, \zeta) \equiv (O_X(D), 1) \in \Sigma_d(X)(k),
\]
and so defines an element of \( A_D(k) \subset K_3 \). Tracing through the definitions, this element is precisely \( \xi \), and hence \( \xi \in A_D(k) \).

In particular, if \( \xi \notin A_D(k) \) then \( \mathfrak{N}_{D,\gamma} = \emptyset \). It then follows from (3.25) that
\[
\hat{f}_D(t_0^{-1} \gamma t_3) = 0 \text{ for all } (t_0, t_3) \in \hat{T}_0(\mathbb{A}) \times \hat{T}_3(\mathbb{A}),
\]
which in turn implies
\[ f_D(t_0^{-1} \gamma t_3) = 0 \] for all \((t_0, t_3) \in T_0(A) \times T_3(A)\).

The relation \( \mathbb{J}(\gamma, f_D, s) = \mathbb{J}(\xi, f_D, s) = 0 \) now follows directly from the definitions of \( \mathbb{J} \), giving the first claim.

From now on we assume that \( \xi \in A_D(k) \), and form the fiber product
\[ N_{(d_1, d_2)}(\xi) \rightarrow N_{(d_1, d_2)} \rightarrow \mathcal{O}_X(−E_3), \phi \gamma) \]
\[ \text{Spec}(k) \rightarrow A_d. \]

\[ (E_1, E_2, E_3) \rightarrow (\mathcal{O}_X(−E_1), \mathcal{O}_X(−E_2), \mathcal{O}_Y(−E_3), \phi \gamma) \]
defines a function
\[ \mathbb{N}_{D, \gamma} \rightarrow \bigsqcup_{d_1, d_2 \geq 0, d_1 + d_2 = 2d} N_{(d_1, d_2), \xi}(k). \]

We interrupt the proof of Proposition 3.17 for a lemma.

**Lemma 3.18.** The function \( (3.28) \) is a bijection.

**Proof.** To construct the inverse, start with a quadruple
\[ (L_1, L_2, L_3, \phi) \in \bigsqcup_{d_1, d_2 \geq 0, d_1 + d_2 = 2d} N_{(d_1, d_2), \xi}(k). \]

For \( i \in \{1, 2, 3\} \), let \( V_i \) be the space of rational sections of \( L_i \), and set \( V_0 = V_1 \oplus V_2 \). We view \( V_0 \) and \( V_3 \) as rank one modules over \( K_0 \) and \( K_3 \), respectively. The morphism \( \phi : f_3L_3 \rightarrow L_1 \oplus L_2 \) of vector bundles induces an \( F \)-linear isomorphism \( \phi : V_3 \rightarrow V_0 \).

Start by choosing any \( K_0 \)-linear isomorphism \( g_0 : V_0 \cong K_0 \), and any \( K_3 \)-linear isomorphism \( g_3 : V_3 \cong K_0 \). These choices determine a bijection
\[ \text{Iso}(V_3, V_0) \cong \text{Iso}(K_3, K_0), \]
and it is easy to see that the induced bijection
\[ K_0^\times \backslash \text{Iso}(V_3, V_0)/K_3^\times \cong K_0^\times \backslash \text{Iso}(K_3, K_0)/K_3^\times \]
does not depend on the initial choices of \( g_0 \) and \( g_3 \). Let \( \gamma' \) denote the image of \( \phi \) under this bijection.

We claim that \( \gamma' = \gamma \). This follows by unwinding the construction of the morphism \( N_{(d_1, d_2)} \rightarrow A_d \), and comparing with the constructions of \( \mathbb{J} \). As our initial triple \( (L_1, L_2, L_3, \phi) \) lies in the fiber \((3.27)\), the invariant
\[ K_0^\times \backslash \text{Iso}(K_3, K_0)/K_3^\times \xrightarrow{\text{inv}} K_3 \]
of \((2.13)\) satisfies \( \xi = \text{inv}(\gamma') \). On the other hand, we defined \( \xi = \text{inv}(\gamma) \).

As we are assuming that \( K_3 \) is a field, Proposition 2.9 implies that \( \text{inv} \) is a bijection, and so \( \gamma' = \gamma \).
The fact that (3.29) identifies $\phi$ with $\gamma$ implies that we may choose the isomorphisms $g_0$ and $g_3$ so that the diagram

$$
\begin{array}{ccc}
V_3 & \xrightarrow{\phi} & V_0 \\
\downarrow g_3 & & \downarrow g_0 \\
K_3 & \xrightarrow{\gamma} & K_0
\end{array}
$$

commutes. Moreover, this commutativity determines $(g_0, g_3)$ uniquely up to simultaneous rescaling by $F^\times$. Indeed, any other pair making the diagram commute would have the form $(t_0 g_0, t_3 g_3)$ for some $(t_0, t_3) \in K_0^\times \times K_3^\times$ satisfying $t_0^{-1} \gamma t_3 = \gamma$. The image of this pair under $K_0 \times K_3 = \tilde{T}_0(F) \times \tilde{T}_3(F) \to T_0(F) \times T_3(F)$ lies in the subgroup $S_\gamma$ of (3.11), which is trivial by Lemma 3.9. Thus $t_0, t_3 \in F^\times$, and the condition $t_0^{-1} \gamma t_3 = \gamma$ implies that $t_0 = t_3$.

The $K_0$-linearity of $g_0$ allows us to rewrite the above diagram as

$$
\begin{array}{ccc}
V_3 & \xrightarrow{\phi} & V_1 \oplus V_2 \\
\downarrow g_3 & & \downarrow g_1 \oplus g_2 \\
K_3 & \xrightarrow{\gamma} & F \oplus F
\end{array}
$$

for isomorphisms $g_1$ and $g_2$. Let $(s_1, s_2, s_3) \in V_1 \oplus V_2 \oplus V_3$ be defined by $g_i(s_i) = 1$. These three vectors are (tautologically) rational sections of $L_1$, $L_2$, and $L_3$, respectively, and we at last define

$$(E_1, E_2, E_3) = (-\text{div}(s_1), -\text{div}(s_2), -\text{div}(s_3)).$$

We leave it as an exercise to the reader to verify that this construction is inverse to (3.28), completing the proof of the lemma. $\square$

Now we complete the proof of Proposition 3.17. First use Lemma 3.9 to rewrite

$$J(\xi, f_D, s) = \int_{(T_0 \times T_3)(\mathbb{A})} f_D(t_0^{-1} \gamma t_3) |t_0|^{2s} \eta(t_3) \, dt_0 \, dt_3$$

$$= \int_{\mathbb{A}^\times \setminus (\tilde{T}_0 \times \tilde{T}_3)(\mathbb{A})} \tilde{f}_D(t_0^{-1} \gamma t_3) |t_0|^{2s} \eta(t_3) \, dt_0 \, dt_3.$$ 

Using (3.25) this may be rewritten as

$$J(\xi, f_D, s) = \sum_{(E_1, E_2, E_3) \in N_D, \gamma} q^{-\deg(E_1 - E_2)^2} \eta(E_3),$$
and Lemma 3.18 allows us to rewrite this as
\[ J(\xi, f_D, s) = \sum_{d_1, d_2 \geq 0, d_1 + d_2 = 2d} q^{(2 \deg(L_1) - 2 \deg(L_2))s} \eta(L_3) \]
\[ = \sum_{d_1, d_2 \geq 0, d_1 + d_2 = 2d} q^{(d_1-d_2)s} \eta(L_3) \]

Combining the Grothendieck-Lefschetz trace formula with Proposition 3.15 shows that
\[ \sum_{(L_1, L_2, L_3, \phi) \in N(d_1, d_2), \xi(k)} \eta(L_3) = \text{Trace}(\text{Frob}_\xi; (R\beta_* L_{(d_1, d_2)})_\xi). \]

As the morphism \( \beta \) is finite, the complex \( R\beta_* L_{(d_1, d_2)} \) is supported in degree 0, completing the proof. \( \square \)

4. Intersection theory and moduli spaces of shtukas

Fix an integer \( r \geq 0 \), and an \( r \)-tuple \( \mu = (\mu_1, \ldots, \mu_r) \in \{\pm 1\}^r \) satisfying the parity condition \( \sum_{i=1}^r \mu_i = 0 \). In particular, \( r \) is even.

4.1. Shtukas and Heegner-Drinfeld cycles. We rapidly recall some notation from [YZ17]. Recall that \( G_0 \cong \text{PGL}_2/X \).

Let \( \text{Bun}_{G_0} \) be the Artin stack parametrizing \( G_0 \)-torsors on \( X \), and let \( \text{Hk}_{G_0}^\mu \) be the Hecke stack parameterizing \( G_0 \)-torsors on \( X \) with \( r \) modifications of type \( \mu \). It comes equipped with morphisms
\[ p_0, \ldots, p_r : \text{Hk}_{G_0}^\mu \to \text{Bun}_{G_0} \]
and \( p_X : \text{Hk}_{G_0}^\mu \to X^r \). For the definitions, see [YZ17, §5.2].

Define the moduli stack of \( G_0 \)-Shtukas of type \( \mu \) by the cartesian diagram
\[ \begin{array}{ccc}
\text{Sht}_{G_0}^\mu & \to & \text{Hk}_{G_0}^\mu \\
\downarrow & & \downarrow (p_0, p_r) \\
\text{Bun}_{G_0} & \to & \text{Bun}_{G_0} \times \text{Bun}_{G_0}.
\end{array} \]

It is a Deligne-Mumford stack, locally of finite type over \( k \), and the morphism
\[ \pi_{G_0} : \text{Sht}_{G_0}^\mu \to X^r \]
induced by \( p_X \) is separated and smooth of relative dimension \( r \).

Recall that our two étale double covers \( f_1 : Y_1 \to X \) and \( f_2 : Y_2 \to X \) determine rank one tori \( T_1 \) and \( T_2 \) over \( X \). Fix \( i \in \{1, 2\} \) and let \( \text{Bun}_{T_i} \) be the moduli stack of \( T_i \)-torsors on \( X \). Denote by \( \text{Hk}_{T_i}^\mu \) the Hecke stack parameterizing \( T_i \)-torsors with \( r \) modifications of type \( \mu \). It comes with morphisms
\[ p_1, \ldots, p_r : \text{Hk}_{T_i}^\mu \to \text{Bun}_{T_i}, \]
and \( p_Y : H^\mu_{T_i} \to Y_i^r \). See [YZ17, §5.4] for the definitions.

The stack of \( T_i \)-Shtukas of type \( \mu \) is defined by the cartesian diagram

\[
\begin{array}{ccc}
\text{Sht}^\mu_{T_i} & \to & H^\mu_{T_i} \\
\downarrow & & \downarrow (p_0,p_r) \\
\text{Bun}_{T_i} & \times & \text{Bun}_{T_i}.
\end{array}
\]

(4.1)

It is a smooth and proper Deligne-Mumford stack over \( k \), and the morphism \( \pi_{T_i} : \text{Sht}^\mu_{T_i} \to Y_i^r \) induced by \( p_Y \) is finite étale. In particular, \( \text{Sht}^\mu_{T_i} \) is smooth and proper over \( k \) of dimension \( r \).

The closed immersions of \( T_1 \) and \( T_2 \) into \( G_0 \) induce finite morphisms

\[
\begin{array}{ccc}
\text{Sht}^\mu_{T_1} & \to & \text{Sht}^\mu_{G_0} \\
\downarrow \theta^\mu_1 & & \downarrow \theta^{\mu*}_2 \\
\text{Sht}^\mu_{G_0} & \to & \text{Ch}_{c,r}(\text{Sht}^\mu_{G_0})
\end{array}
\]

which then induce push-forwards

\[
\begin{array}{ccc}
\text{Ch}_r(\text{Sht}^\mu_{T_1}) & \to & \text{Ch}_r(\text{Sht}^\mu_{T_2}) \\
\downarrow \theta^{\mu*}_1 & & \downarrow \theta^{\mu*}_2 \\
\text{Ch}_{c,r}(\text{Sht}^\mu_{G_0}) & \to & \text{Ch}_{c,r}(\text{Sht}^\mu_{G_0})
\end{array}
\]

on Chow groups with \( \mathbb{Q} \)-coefficients [Kre99, Theorem 2.1.12]. We obtain cycle classes

\[
[Sht^\mu_{T_1}], [Sht^\mu_{T_2}] \in \text{Ch}_{c,r}(\text{Sht}^\mu_{G_0})
\]

by pushing forward the fundamental classes.

As \( \text{Sht}^\mu_{G_0} \) has dimension \( 2r \), there is an intersection pairing

\[
\langle \cdot, \cdot \rangle : \text{Ch}_{c,r}(\text{Sht}^\mu_{G_0}) \times \text{Ch}_{c,r}(\text{Sht}^\mu_{G_0}) \to \mathbb{Q}
\]

as in [YZ17, §A.1]. Recall the Hecke algebra \( \mathcal{H} \) of §3.4. For any \( f \in \mathcal{H} \) define

\[
\mathbb{I}_r(f) = \langle [\text{Sht}^\mu_{T_1}], f * [\text{Sht}^\mu_{T_2}] \rangle \in \mathbb{Q},
\]

(4.2)

where * is the action of \( \mathcal{H} \) on \( \text{Ch}_{c,r}(\text{Sht}^r_{G_0}) \) defined in [YZ17, §5.3].

**Remark 4.1.** The isomorphism class of \( H^\mu_{G_0} \) is independent of \( \mu \), and so we sometimes call this stack \( H^\mu_{G_0} \). Similarly, we sometimes write \( \text{Sht}^r_{G_0} \) instead of \( \text{Sht}^\mu_{G_0} \).
4.2. Some moduli spaces. Fix an integer \(d\). The purpose of this subsection is to construct a commutative diagram of \(k\)-schemes

\[
\begin{array}{ccc}
M_d & \longrightarrow & \Sigma_{2d}(Y) \\
\alpha & & \downarrow \text{Nm} \\
A_d & \longrightarrow & \Sigma_{2d}(Y_3) \\
\downarrow \text{Tr} & & \downarrow \\
\Sigma_d(X) & &
\end{array}
\] (4.3)

in such a way that the square is cartesian. Let \(S\) be any \(k\)-scheme.

Recall from (3.14) that \(\Sigma_d(X)(S)\) is the set of pairs \((\Delta, \zeta)\) consisting of

- a degree \(d\) line bundle \(\Delta\) on \(X_S\),
- a nonzero section \(\zeta \in H^0(X_S, \Delta)\).

The schemes \(\Sigma_{2d}(Y_3)\) and \(\Sigma_{2d}(Y)\) are defined similarly. The morphism labeled \(\text{Nm}\) is \((\Delta, \zeta) \mapsto (\text{Nm}_{Y/Y_3}(\Delta), \text{Nm}_{Y/Y_3}(\zeta))\).

Recall also that \(A_d(S)\) is the set of all pairs \((\Delta, \xi)\) consisting of

- a degree \(d\) line bundle \(\Delta\) on \(X_S\),
- a \(\xi \in H^0(Y_3 S, f_2^* \Delta)\) with nonzero trace \(\text{Tr}_{Y_3/X}(\xi) \in H^0(X_S, \Delta)\).

Denote by \(\tilde{M}_d(S)\) the groupoid of triples \((\mathcal{L}_1, \mathcal{L}_2, \phi)\) consisting of

- a line bundle \(\mathcal{L}_1 \in \text{Pic}(Y_1 S)\),
- a line bundle \(\mathcal{L}_2 \in \text{Pic}(Y_2 S)\),
- a morphism \(\phi : f_2_2^* \mathcal{L}_2 \to f_1_1^* \mathcal{L}_1\) of rank two vector bundles on \(X_S\).

We require further that the line bundle

\[
\Delta = \text{Hom}(\det(f_2_2^* \mathcal{L}_2), \det(f_1_1^* \mathcal{L}_1))
\]
on \(X_S\) has degree \(d\), and that \(\det(\phi) \in H^0(X_S, \Delta)\) is nonzero. The functor \(\tilde{M}_d\) from \(k\)-schemes to groupoids is represented by an Artin stack over \(k\).

The Picard group \(\text{Pic}(X_S)\) acts on \(\tilde{M}_d(S)\) by twisting

\[
(\mathcal{L}_1, \mathcal{L}_2, \phi) \otimes \mathcal{L} = (\mathcal{L}_1 \otimes f_2_2^* \mathcal{L}, \mathcal{L}_2 \otimes f_2_2^* \mathcal{L}, \phi \otimes \text{id}),
\]

inducing an action of the Picard stack \(\text{Pic}_X\) on \(\tilde{M}_d\). The representability of the quotient stack

\[
M_d = \tilde{M}_d/\text{Pic}_X
\]

by a scheme is part of the following proposition, which also defines the two arrows in (4.3) emanating from \(M_d\).

**Proposition 4.2.** There is a canonical isomorphism

\[
M_d \cong A_d \times_{\Sigma_{2d}(Y_3)} \Sigma_{2d}(Y).
\]

**Proof.** Start with a \(k\)-scheme \(S\) and a triple \((\mathcal{L}_1, \mathcal{L}_2, \phi) \in \tilde{M}_d(S)\). Define line bundles

\[
\tilde{\mathcal{L}}_1 = \mathcal{L}_1|_{Y_S}, \quad \tilde{\mathcal{L}}_2 = \mathcal{L}_2|_{Y_S}
\] (4.4)
on $Y_S$. The pullback of $\phi$ via $Y_S \to X_S$ is a morphism

$$\widetilde{\mathcal{L}}_2 \oplus \widetilde{\mathcal{R}}_2^\tau_3 \cong (f_2, \mathcal{L}_2)|_{Y_S} \xrightarrow{\phi|_{Y_S}} (f_1, \mathcal{L}_1)|_{Y_S} \cong \widetilde{\mathcal{L}}_1 \oplus \widetilde{\mathcal{R}}_1^\tau_3$$

of rank two vector bundles on $Y_S$, encoded by four morphisms

\[
\begin{align*}
\widetilde{\mathcal{L}}_2 & \xrightarrow{a} \widetilde{\mathcal{L}}_1 \\
\widetilde{\mathcal{L}}_2^\tau_3 & \cong \widetilde{\mathcal{L}}_2^\tau_3 \\
\widetilde{\mathcal{L}}_2 & \cong \widetilde{\mathcal{L}}_1^\tau_3 \\
\widetilde{\mathcal{L}}_2^\tau_3 & \cong \widetilde{\mathcal{L}}_1^\tau_3.
\end{align*}
\]

(4.5)

The assumption $\det(\phi) \neq 0$ implies that $a \neq 0$, and setting $\mathcal{K} = \text{Hom}(\widetilde{\mathcal{L}}_2, \widetilde{\mathcal{L}}_1)$ defines a point $(\mathcal{K}, a) \in \Sigma_{2d}(Y)(S)$. We have now constructed a morphism

$$\bar{M}_d \to \Sigma_{2d}(Y),$$

which is easily seen to descend to the quotient $M_d$.

Consider the map

$$\det(f_2, \mathcal{L}_2)|_{Y_S} = \det(\widetilde{\mathcal{L}}_2 \oplus \widetilde{\mathcal{R}}_2^\tau_3) \xrightarrow{\det^\tau(\phi)} \det(\widetilde{\mathcal{L}}_1 \oplus \widetilde{\mathcal{R}}_1^\tau_3) = \det(f_1, \mathcal{L}_1)|_{Y_S},$$

where the arrow labeled $\det^\tau(\phi)$ sends $s \wedge t \mapsto a(s) \wedge d(t)$ for local sections $s$ and $t$ of $\widetilde{\mathcal{L}}_2$ and $\widetilde{\mathcal{L}}_2^\tau_3$, respectively. When viewed as a section of $\Delta|_{Y_S}$, this map is $\tau_3$-equivariant. Hence it admits a canonical descent to

$$\det^\tau(\phi) \in H^0(Y_{3\sigma}, f_3^* \Delta)$$

whose trace $\text{Tr}_{Y_{3\sigma}/X}(\det^\tau(\phi)) = \det(\phi)$ is nonzero. Thus $(\Delta, \det^\tau(\phi)) \in A_d(S)$, and we have have constructed a morphism

$$\bar{M}_d \to A_d.$$

After this is easily seen to descend to the quotient $M_d$.

The canonical isomorphism

$$\mathcal{K} \otimes \mathcal{K}^\tau_3 \cong \text{Hom}(\widetilde{\mathcal{L}}_2 \otimes \widetilde{\mathcal{R}}_2^\tau_3, \widetilde{\mathcal{L}}_1 \otimes \widetilde{\mathcal{R}}_1^\tau_3) \cong \Delta|_{Y_S}$$

on $Y_S$ descends to an isomorphism $\text{Nm}_{Y/Y_3}(\mathcal{K}) \cong f_3^* \Delta$, and this isomorphism sends $\text{Nm}_{Y/Y_3}(a) \mapsto \det^\tau(\phi)$. In other words,

$$(f_3^* \Delta, \det^\tau(\phi)) \cong \text{Nm}(\mathcal{K}, a)$$

define the same element of $\Sigma_{2d}(Y_3)(S)$, and so the two morphisms constructed above define a map

$$M_d \to A_d \times \Sigma_{2d}(Y_3) \Sigma_{2d}(Y).$$

(4.6)

We will show that (4.6) is an isomorphism by constructing the inverse. An $S$-point of $A_d \times \Sigma_{2d}(Y_3) \Sigma_{2d}(Y)$ consists of two pairs

$$(\Delta, \zeta) \in A_d(S), \quad (\mathcal{K}, a) \in \Sigma_{2d}(Y)(S)$$

along with an isomorphism $\text{Nm}_{Y/Y_3}(\mathcal{K}) \cong f_3^* \Delta$ satisfying $\text{Nm}_{Y/Y_3}(a) \mapsto \zeta$. 
The isomorphism $\text{Nm}_{Y/X}(K) \cong f_3^* \Delta$ endows the line bundle $K \otimes K^{\tau_3}$ with descent data relative to $Y_S/X_S$. In other words, we are given isomorphisms between this line bundle and all of its Aut($Y/X$)-conjugates, and hence an isomorphism $K \otimes K^{\tau_3} \cong K^{\tau_1} \otimes K^{\tau_2}$. This induces the first isomorphism in

$$\text{Hom}(K, K^{\tau_1}) \cong \text{Hom}(K^{\tau_2}, K^{\tau_3}) \cong \text{Hom}(K, K^{\tau_1})^{\tau_2},$$

and, by viewing the composition as descent data relative to $Y_S/Y_2S$, we obtain a degree 0 line bundle $M$ on $Y_2S$ endowed with an isomorphism $M|_{Y_S} \cong \text{Hom}(K, K^{\tau_1})^{\tau_2}$.

The canonical trivialization $(M \otimes M^{\sigma_2})|_{Y_S} \cong \text{Hom}(K, K^{\tau_1}) \otimes \text{Hom}(K, K^{\tau_1})^{\tau_2}
\cong \text{Hom}(K \otimes K^{\tau_3}, K^{\tau_1} \otimes K^{\tau_2})
\cong \mathcal{O}_Y$

is compatible with the natural descent data relative to $Y_S/X_S$ on the source and target, and so the line bundle $\text{Nm}_{Y_2/X}(\mathcal{M})$ is trivial. As in the proof of [YZ17, Proposition 6.1(1)], this implies the existence of an $S$-point $L_2$ of the quotient Pic$_Y^{\tau_2}$/Pic$_X$ satisfying

$L_2 \cong M \otimes L^{\sigma_2}.$

Viewing the isomorphism

$$(K \otimes L_2|_{Y_S})^{\tau_1} \cong K^{\tau_1} \otimes L^{\sigma_2}_2|_{Y_S}
\cong K^{\tau_1} \otimes (\mathcal{M}^{-1} \otimes L_2)|_{Y_S}
\cong K^{\tau_1} \otimes \text{Hom}(K^{\tau_3}, K) \otimes L_2|_{Y_S}
\cong K \otimes L_2|_{Y_S}$$

as descent data relative to $Y_S/Y_1S$, we obtain a line bundle $L_1$ on $Y_1S$ endowed with an isomorphism $L_1|_{Y_S} \cong K \otimes L_2|_{Y_S}$. If we set $\tilde{L}_1 = L_1|_{Y_S}$ and $\tilde{L}_2 = L_2|_{Y_S}$ as in (4.4), this isomorphism can be rewritten as

$K \cong \text{Hom}(\tilde{L}_2, \tilde{L}_1).$ (4.7)

Now view $a$ as a global section of (4.7), and define global sections $b$, $c$, and $d$ using (4.5). These four global sections define a global section $\phi$ of

$\text{Hom}(\tilde{L}_2 \oplus \tilde{L}_2^{\tau_1}, \tilde{L}_1 \oplus \tilde{L}_1^{\tau_1}) \cong \text{Hom}(f_2^* L_2, f_1^* L_1)|_{Y_S},$

which, by construction, is invariant under Aut($Y/X$). Thus $\phi$ descends to

$\phi \in \text{Hom}(f_2^* L_2, f_1^* L_1).$

The triple $(L_1, L_2, \phi)$ defines an object of $M_\delta(S)$, completing the construction of the inverse of (4.6). \hfill $\square$

**Proposition 4.3.** Let $g$ and $g_3$ be the genera of $X$ and $Y_3$.

1. The morphisms $\alpha$ and $\text{Nm}$ in (4.3) are finite.
2. If $d \geq 2g_3 - 1$ then $M_\delta$ is smooth over $k$ of dimension $2d - g + 1$. 


Proof. The map $Nm$ in (4.3) is clearly finite, and therefore $\alpha$ is as well. This proves the first claim. For the proof of the second claim, recall the cartesian diagram

$$
\begin{array}{ccc}
M_d & \longrightarrow & \Sigma_{2d}(Y) \\
\rho \downarrow & & \downarrow Nm \\
\text{Pic}_X^d & \longrightarrow & \text{Pic}_{Y_3}^d
\end{array}
$$

of Proposition 4.2. In the notation used there, the map $\pi$ sends $(L_1, L_2, \phi)$ to $(K, a)$, and $\rho$ sends the same data to $\Delta$.

The vertical arrow on the right factors as

$$
\Sigma_{2d}(Y) \xrightarrow{AJ} \text{Pic}_Y^d \xrightarrow{Nm} \text{Pic}_{Y_3}^d.
$$

Letting $g_Y$ denote the genus of $Y$, the Abel-Jacobi morphism $AJ$ is smooth by our hypothesis

$$
2d \geq 2g_Y - 1 = 2(2g_3 - 1) - 1.
$$

The norm map $\text{Pic}_Y^d \to \text{Pic}_{Y_3}^d$ is smooth by the following lemma.

**Lemma 4.4.** The norm $Nm: \text{Pic}_Y \to \text{Pic}_{Y_3}$ is a smooth morphism.

**Proof.** We use the infinitesimal lifting criterion for smoothness. Suppose $A \to B$ is a surjection of local Artinian $k$-algebras with kernel $I$ satisfying $I^2 = 0$. Suppose also that we have morphisms $\alpha$ and $\beta$ making the square

$$
\begin{array}{ccc}
\text{Spec}(B) & \xrightarrow{\alpha} & \text{Pic}_Y \\
\downarrow & & \downarrow Nm \\
\text{Spec}(A) & \xrightarrow{\beta} & \text{Pic}_{Y_3}
\end{array}
$$

commute. We must prove the existence of a morphism $\gamma$ making the two triangles commute.

Let $g: Y \to Y_3$ be the étale double cover of (1.1). The trace morphism $\text{Tr}: g_*\mathcal{O}_Y \to \mathcal{O}_{Y_3}$ induces a short exact sequence

$$
0 \longrightarrow \ker(\text{id} \otimes \text{Tr}) \longrightarrow I \otimes_B g_*\mathcal{O}_Y \xrightarrow{\text{id} \otimes \text{Tr}} I \otimes_B \mathcal{O}_{Y_3} \longrightarrow 0.
$$

of coherent sheaves on $Y_{3B}$. As $Y_{3B}$ is a curve, the induced map

$$
H^1(Y_B, I \otimes_B \mathcal{O}_{Y_B}) \cong H^1(Y_{3B}, I \otimes_B g_*\mathcal{O}_{Y_B}) \xrightarrow{\text{id} \otimes \text{Tr}} H^1(Y_{3B}, I \otimes_B \mathcal{O}_{Y_{3B}})
$$

on cohomology is surjective.

As the closed immersion $Y_B \hookrightarrow Y_A$ is an isomorphism on the underlying topological spaces, the category of sheaves on these two spaces are canonically identified. Thus we have an exact sequence of sheaves

$$
0 \longrightarrow I \otimes_B \mathcal{O}_{Y_B} \xrightarrow{j} \mathcal{O}_X^\wedge \longrightarrow \mathcal{O}_Y^\wedge \longrightarrow 1
$$
on $Y_B$, where $j(r \otimes f) = 1 + rf$, and a similar exact sequence on $Y_{3A}$. Taking cohomology yields a commutative diagram

$$
\begin{array}{ccc}
H^1(Y_B, I \otimes_B \mathcal{O}_{Y_B}) & \longrightarrow & \text{Pic}(Y_A) \\
\downarrow \text{id} \otimes \text{Tr} & & \downarrow \text{Nm} \\
H^1(Y_{3B}, I \otimes_{3B} \mathcal{O}_{Y_{3B}}) & \longrightarrow & \text{Pic}(Y_{3A})
\end{array}
$$

with exact rows and columns. Note that the surjectivity of $\text{Pic}(Y_A) \to \text{Pic}(Y_B)$ follows from the smoothness of the Picard stack $\text{Pic}_Y$ over $k$, and similarly with $Y$ replaced by $Y_3$.

The maps $\alpha$ and $\beta$ determine elements of $\text{Pic}(Y_B)$ and $\text{Pic}(Y_{3A})$, having the same image in $\text{Pic}(Y_{3B})$. A diagram chase shows that they come from a common element of $\text{Pic}(Y_A)$, which is the desired $\gamma$.

We have now shown that $\text{Nm} : \Sigma_{2d}(Y) \to \text{Pic}^d_{Y_3}$ is a smooth morphism between stacks of dimension $2d$ and $g_3 - 1$, and hence $\rho : M_d \to \text{Pic}^d_X$ is smooth of relative dimension

$$2d - g_3 + 1 = 2d - 2g + 2.$$

As $\text{Pic}^d_X$ is smooth over $k$ of dimension $g - 1$, $M_d$ is smooth and of dimension $2d - g + 1$ over $k$.

4.3. **Interpretation of the intersection number.** We define a correspondence

as follows. We first define a stack $\tilde{\text{Hk}}_{M_d}$ whose $S$-points classify

- two $S$-points $(\mathcal{L}_1^{(0)}, \mathcal{L}_2^{(0)}, \phi^{(0)})$ and $(\mathcal{L}_1^{(1)}, \mathcal{L}_2^{(1)}, \phi^{(1)})$ of $\tilde{M}_{dS}$,
- one $S$-point $y = (y_1, y_2)$ of $Y_S = Y^1_S \times X_S Y^2_S$,
- injective morphisms $s_1 : \mathcal{L}_1^{(0)} \to \mathcal{L}_1^{(1)}$ and $s_2 : \mathcal{L}_2^{(0)} \to \mathcal{L}_2^{(1)}$. 

\[\text{Hk}_{M_d}\]
We require that the cokernels of $s_1$ and $s_2$ are invertible sheaves on the graph of $y_1$ and $y_2$, respectively, and that the diagram

$$
\begin{array}{c}
\phi^{(0)} \\
\phi^{(1)}
\end{array}
\begin{array}{c}
\downarrow f_2^*L_2^{(0)} \\
\downarrow f_2^*L_2^{(1)}
\end{array}
\begin{array}{c}
\downarrow f_1^*L_1^{(0)} \\
\downarrow f_1^*L_1^{(1)}
\end{array}
$$

of $\mathcal{O}_{X_S}$-modules commutes. Then $\text{Pic}_X$ acts on $\tilde{\text{H}}_k^{M_d}$ by simultaneously twisting the $\mathcal{L}_i^{(j)}$, and we define $\text{Hk}_{M_d} = \tilde{\text{H}}_k^{M_d}/\text{Pic}_X$.

Using the top horizontal arrow of (4.3), we may realize the above correspondence on $M_d$ as the pullback of a correspondence on $\Sigma_{2d}(Y)$. More precisely, there is a correspondence

$$
\begin{array}{c}
H_{2d}(Y) \\
\Sigma_{2d}(Y) \\
\Sigma_{2d}(Y)
\end{array}
\begin{array}{c}
\leftarrow \\
\cong \\
\to
\end{array}
\begin{array}{c}
\Sigma_{2d}(Y) \\
\Sigma_{2d}(Y)
\end{array}
\begin{array}{c}
\leftarrow \\
\cong \\
\to
\end{array}
\begin{array}{c}
\Sigma_{2d}(Y_3),
\end{array}

(4.9)

where, for any $k$-scheme $S$, the set $H_{2d}(Y)(S)$ classifies

- a pair of $S$-points $(\mathcal{K}^{(0)}, a^{(0)})$ and $(\mathcal{K}^{(1)}, a^{(1)})$ of $\Sigma_{2d}(Y)$,
- an $S$-point $y \in Y(S)$,
- an isomorphism

$$
s : \mathcal{K}^{(0)}(y^{\tau_1} - y^{\tau_2}) \cong \mathcal{K}^{(1)}
$$

of line bundles on $Y_S$ such that $s(a^{(0)}) = a^{(1)}$, where we view the global section $a^{(0)}$ of $\mathcal{K}^{(0)}$ as a rational section of $\mathcal{K}^{(0)}(y^{\tau_1} - y^{\tau_2})$.

Note that all of the data is determined by the pair $(\mathcal{K}^{(0)}, a^{(0)})$ and the point $y \in Y(S)$, for from this we may recover the line bundle $\mathcal{K}^{(1)} = \mathcal{K}^{(0)}(y^{\tau_1} - y^{\tau_2})$ and its rational section $a^{(1)} = a^{(0)}$. The condition that $a^{(1)}$ be a section of $\mathcal{K}^{(1)}$, as opposed to merely a rational section, is equivalent to

$$
\text{div}(a^{(0)}) + y^{\tau_1} - y^{\tau_2} \geq 0.
$$

This is in turn equivalent to the condition that the effective Cartier divisor $y^{\tau_2}$ appears in the support of $\text{div}(a^{(0)})$. In other words, we may realize

$$
H_{2d}(Y) \hookrightarrow \Sigma_{2d}(Y) \times_k Y
$$

as the closed subscheme of triples $(\mathcal{K}^{(0)}, a^{(0)}, y)$ for which $y^{\tau_2}$ appears in the support of $\text{div}(a^{(0)})$. 

Proposition 4.5. The diagram (4.8) is canonically identified with the diagram

\[
\begin{array}{ccc}
A_d \times \Sigma_{2d(Y_3)} & \overset{H_2d(Y)}{\longrightarrow} & A_d \\
\downarrow & & \downarrow \\
A_d \times \Sigma_{2d(Y_3)} & \overset{\Sigma_{2d(Y)}}{\longrightarrow} & A_d \times \Sigma_{2d(Y_3)}
\end{array}
\]

obtained from (4.9) by base change along the arrow \(A_d \to \Sigma_{2d(Y_3)}\) in (4.3).

Proof. The proof amounts to carefully tracing through the constructions in the proof of Proposition 4.2.

It is enough to show that \(H_{k_Md} \cong A_d \times \Sigma_{2d(Y_3)} H_{2d(Y)}\). We define a map

\[\widetilde{H_{k_Md}} \to A_d \times \Sigma_{2d(Y_3)} H_{2d(Y)}\]

as follows. Given a quintuple

\[(L^{(0)}_1, L^{(0)}_2, \phi^{(0)}), (L^{(1)}_1, L^{(1)}_2, \phi^{(1)}), (y_1, y_2), (s_1, s_2) \in \widetilde{H_{k_Md}}(S),\]

we obtain from the first two pieces of data, points \((\mathcal{K}^{(0)}, a^{(0)})\) and \((\mathcal{K}^{(1)}, a^{(1)})\) of \(\Sigma_{2d(Y)}(S)\). For \(i = 1, 2\), we have \(\mathcal{K}^{(i)} \cong \text{Hom}(L^{(i)}_2|_{Y, S}, L^{(i)}_1|_{Y, S})\), and so the isomorphisms \(\mathcal{L}^{(0)}_1(y^{(1)}) \cong \mathcal{L}^{(1)}_1\) and \(\mathcal{L}^{(0)}_2(y^{(2)}) \cong \mathcal{L}^{(1)}_2\) induce an isomorphism \(\mathcal{K}^{(0)}(y^{(1)} - y^{(2)}) \cong \mathcal{K}^{(1)}\) sending \(a^{(0)}\) to \(a^{(1)}\). This gives a map

\[\widetilde{H_{k_Md}} \to A_d \times \Sigma_{2d(Y_3)} H_{2d(Y)},\]

which factors through \(H_{k_Md}\).

To construct a map in the other direction, suppose given an \(S\)-point

\[(\Delta, \xi, \mathcal{K}^{(0)}, a^{(0)}, \mathcal{K}^{(1)}, a^{(1)}, y, s) \in A_d \times \Sigma_{2d(Y_3)} H_{2d(Y)}\].

The proof of Proposition 4.2 constructs a point \((L^{(i)}_1, L^{(i)}_2, \phi^{(i)}) \in M_d(S)\) corresponding to \((\Delta, \xi, \mathcal{K}^{(i)}, a^{(i)})\), and we must show that that there are isomorphisms \(L^{(1)}_1 \cong L^{(0)}_1(y^{(1)})\) and \(L^{(2)}_1 \cong L^{(0)}_2(y^{(2)})\) inducing the given isomorphism

\[s : \mathcal{K}^{(0)}(y^{(1)} - y^{(2)}) \cong \mathcal{K}^{(1)}\].

The line bundle \(\mathcal{M}^{(i)} \in \text{Pic}(Y_{2S})\), used to construct \(L^{(i)}_2\), is a descent of \(\widetilde{\text{Hom}}(\mathcal{K}^{(i)}, \mathcal{K}^{(i)\tau})\) via the isomorphism \(\text{Nm}_{Y/\Delta Y}(\mathcal{K}^{(i)}) \cong f^{\ast}_3 \Delta\). The isomorphism \(s\) therefore induces a canonical isomorphism

\[\mathcal{M}^{(1)} \cong \mathcal{M}^{(0)}(y^{(2)} - y^{(2)}).\]  \hspace{1cm} (4.10)

By construction, \(L^{(i)}_2\) is the unique \(S\)-point of \(\text{Pic}_{Y_{2S}}/\text{Pic}_X\) such that

\[L^{(i)}_2 \cong \mathcal{M}^{(i)} \otimes L^{(i)\sigma_2}.\]
It then follows from (4.10) that $L_2(0)(y_2) \cong L_2(1)(y_1)$. One constructs the isomorphism $L_1(1) \cong L_0(0)(y_1)$ in a similar fashion. □

**Corollary 4.6.** The maps $\gamma_0, \gamma_1 : H^k_{M_d} \to M_d$ are finite and surjective. In particular, by Proposition 4.3, if $d \geq 2g_3 - 1$ then $\dim H^k_{M_d} = 2g - d + 1$.

The correspondence $H^k_{M_d}$ induces an endomorphism

$[H^k_{M_d}] : \alpha_* \mathbb{Q}_\ell \to \alpha_* \mathbb{Q}_\ell$

of sheaves on $A_d$, given by the composition

$\alpha_* \mathbb{Q}_\ell \to \alpha_* \gamma_0^* \gamma_0^* \mathbb{Q}_\ell \cong \alpha_* \gamma_1^* \mathbb{Q}_\ell \to \alpha_* \mathbb{Q}_\ell$.

The first and last maps are induced by adjunction, using that $\gamma_0$ and $\gamma_1$ are finite. Denote by $[H^k_{M_d}]^r$ the $r$-fold composition of this endomorphism with itself. The remainder of §4 is devoted to the proof of the following proposition.

**Proposition 4.7.** Fix an effective divisor $D \in \text{Div}(X)$ of degree $d \geq 2g_3 - 1$, and recall the closed subscheme $A_D \subset A_d$ and the inclusion $A_D(k) \subset K_3$ of Proposition 4.16. The intersection multiplicity (4.2) satisfies

$I_r(f_D) = \sum_{\xi \in K_3} I_r(\xi, f_D),$

where

$I_r(\xi, f_D) = \begin{cases} \text{Trace}([H^k_{M_d}]^r_\xi \circ \text{Frob}_\xi; (\alpha_* \mathbb{Q}_\ell)_\xi) & \text{if } \xi \in A_D(k) \\ 0 & \text{otherwise.} \end{cases}$

Here $\xi$ is any geometric point above $\xi : \text{Spec}(k) \to A_D$.

4.4. **Correspondences with multiple paws.** As a first step toward proving Proposition 4.7, we want to interpret the $r$-fold iterated endomorphism $[H^k_{M_d}]^r$ as the endomorphism associated with a single correspondence.

To this end, we define a stack $H^\mu_{M_d}$, sitting in a commutative diagram

\[
\begin{array}{ccc}
\text{H}^\mu_{M_d} & \xrightarrow{\gamma_0} & \text{M}_d \\
\downarrow \gamma_r & & \downarrow \alpha \\
\text{M}_d & \xrightarrow{\alpha} & \text{A}_d \\
\end{array}
\]

First define a stack $\tilde{H}^\mu_{M_d}$ whose $S$-points are given by:

- For each $0 \leq i \leq r$, points $(L_1^{(i)}, L_2^{(i)}, \phi_i) \in \tilde{M}_d(S)$.
- For each $1 \leq i \leq r$, points $y^{(i)} = (y_1^{(i)}, y_2^{(i)}) \in Y(S) = (Y_1 \times X Y_2)(S)$. 

\[
(4.11)
\]
• For each \(1 \leq i \leq r\), rational maps
\[
s_1^{(i)} : L_1^{(i-1)} \rightarrow L_1^{(i)} \quad \text{and} \quad s_2^{(i)} : L_2^{(i-1)} \rightarrow L_2^{(i)},
\]
such that \((L_1^{(i)}, s_1^{(i)}, y_1^{(i)})\) and \((L_2^{(i)}, s_2^{(i)}, y_2^{(i)})\) give points of \(H^\mu_{T_1}(S)\) and \(H^\mu_{T_2}(S)\), respectively, and such that the following diagram commutes
\[
\begin{array}{ccc}
& f_1^* L_1^{(0)} & \\
\phi_0 & \downarrow & \phi_1 \\
& f_1^* L_1^{(1)} & \\
\end{array} \quad \cdots \quad \begin{array}{ccc}
& f_1^* L_1^{(r)} & \\
\phi_r & \downarrow & \phi_r \\
& f_1^* L_1^{(0)} & \\
\end{array}
\]
(4.12)

We then set
\[
H^\mu_{M_d} = \tilde{H}^\mu_{M_d}/\text{Pic}_X.
\]

For \(0 \leq i \leq r\), we have morphisms \(\gamma_i : H^\mu_{M_d} \rightarrow M_d\) which remember the \(i\)th column in (4.12), and this gives the diagram (4.11).

Exactly as in [YZ17, Lemma 6.2], there is an isomorphism of \((M_d \times A_d M_d)\)-schemes
\[
H^\mu_{M_d} \cong H^\mu_{M_d} \times_{\gamma_0, \gamma_0} H^\mu_{M_d} \times_{\gamma_1, \gamma_0} \cdots \times_{\gamma_1, \gamma_0} H^\mu_{M_d}, \quad (4.13)
\]
where the fiber products are with respect to the morphisms of (4.8).

**Corollary 4.8.** If \(d \geq 2g_3 - 1\), then \(\dim H^\mu_{M_d} = \dim H^\mu_{M_d} = 2g - d + 1\).

Define a \(k\)-scheme \(\text{Sht}^\mu_{M_d}\) as the fiber product
\[
\begin{array}{ccc}
\text{Sht}^\mu_{M_d} & \rightarrow & H^\mu_{M_d} \\
\downarrow & \downarrow & \downarrow \\
M_d & \rightarrow & M_d \times M_d. \\
\end{array}
\]
(4.14)

**Proposition 4.9.** The scheme \(\text{Sht}^\mu_{M_d}\) has dimension 0, and the image of the composition
\[
\text{Sht}^\mu_{M_d}(\bar{k}) \rightarrow M_d(\bar{k}) \xrightarrow{\alpha} A_d(\bar{k})
\]
is a finite subset of \(A_d(\bar{k})\).

**Proof.** By Proposition 4.6 and (4.13), the map \((\gamma_0, \gamma_r)\) is finite. On the other hand, from the cartesian diagram we see that any \(\bar{k}\)-point of \(\text{Sht}^\mu_{M_d}\) lies over a \(k\)-point of \(M_d\). Thus, the map \(\text{Sht}^\mu_{M_d} \rightarrow M_d\) is finite and factors through a 0-dimensional subscheme of \(M_d\). It follows that \(\text{Sht}^\mu_{M_d}\) is 0-dimensional. \(\square\)
For any $\xi \in A_d(k)$ we form the fiber product

\[
\begin{array}{ccc}
\text{Sh}_M^\mu(\xi) & \longrightarrow & \text{Spec}(k) \\
\downarrow & & \downarrow \\
\text{Sh}_M^\mu & \longrightarrow & M_d \\
\alpha & \longrightarrow & A_d,
\end{array}
\]

and so obtain a decomposition

\[
\text{Sh}_M^\mu = \bigsqcup_{\xi \in A_d(k)} \text{Sh}_M^\mu(\xi)
\]

into finitely many open and closed 0-dimensional subschemes. On the level of point sets there is a decomposition $A_d(k) = \bigsqcup D A_D(k)$, where the disjoint union runs over all effective degree $d$ divisors

\[
D \in \text{Sym}^d(X) \cong \Sigma_d(X)(k),
\]

and $A_D$ is as in §3.6 Setting

\[
\text{Sh}_M^\mu = \bigsqcup_{\xi \in A_d(k)} \text{Sh}_M^\mu(\xi),
\] (4.15)

we obtain a decomposition of the Chow group

\[
\text{Ch}_0(\text{Sh}_M^\mu) = \bigoplus_{D \in \Sigma_d(X)(k)} \text{Ch}_0(\text{Sh}_M^\mu).
\] (4.16)

4.5. The refined Gysin map. As $M_d$ is smooth (by Proposition 4.3), the morphism $(\text{id}, \text{Fr}_{M_d})$ of (4.14) is a regular local immersion. We therefore have a refined Gysin map

\[
(\text{id}, \text{Fr}_{M_d})^! : \text{Ch}_{2d-g+1}(\text{Hk}_M^\mu) \to \text{Ch}_0(\text{Sh}_M^\mu)
\] (4.17)

defined as in [Kre99 §3.1].

Proposition 4.10. Suppose $D$ is an effective divisor on $X$ of degree $d \geq 2g_3 - 1$. The composition

\[
\begin{array}{ccc}
\text{Ch}_{2d-g+1}(\text{Hk}_M^\mu) & \xrightarrow{(4.17)} & \text{Ch}_0(\text{Sh}_M^\mu) \\
\xrightarrow{(4.16)} & & \xrightarrow{\text{deg}} \mathbb{Q}
\end{array}
\]

sends the fundamental class $[\text{Hk}_M^\mu] \in \text{Ch}_{2d-g+1}(\text{Hk}_M^\mu)$ to the intersection multiplicity $\mathbb{L}_f(f_D)$ defined by (4.2).
Proof. As in [YZ17, 6.3] we consider an octahedral diagram:

\[
\begin{array}{cccccccc}
\text{Hk}_T^\mu \times \text{Hk}_T^\mu & \xrightarrow{\Pi^\mu \times \Pi^\mu} & \text{Hk}_G^r \times \text{Hk}_G^r & \xrightarrow{(\vec{s}, \vec{t})} & \text{Hk}_{G,d}^r \\
(\gamma_0, \gamma_r) \downarrow & & (\gamma_0, \gamma_r) \downarrow & & (\gamma_0, \gamma_r) \downarrow \\
(\text{Bun}_T)^2 \times (\text{Bun}_T)^2 & \xrightarrow{\Pi \times \Pi \times \Pi} & (\text{Bun}_G)^2 \times (\text{Bun}_G)^2 & \xrightarrow{s^2 \times t^2} & H_d \times H_d \\
(id, Fr) \downarrow & & (id, Fr) \downarrow & & (id, Fr) \downarrow \\
\text{Bun}_T \times \text{Bun}_T & \xrightarrow{\Pi_1 \times \Pi_2} & \text{Bun}_G \times \text{Bun}_G & \xrightarrow{(s, t)} & H_d \\
\end{array}
\]

(4.18)

The stack $H_d$ is defined exactly as in [YZ17]. In particular, $H_d = \overline{H}_d/\text{Pic}_X$, where $\overline{H}_d$ parameterizes colength $d$ injections $\phi: \mathcal{E} \to \mathcal{E}'$ of rank two vector bundles on $X$. The map $(s, t): H_d \to \text{Bun}_G^2$ appearing in the bottom row of (4.18) takes the map $\phi$ to $(\mathcal{E}, \mathcal{E}')$.

The stack $\text{Hk}_{G,d}^r$ is defined, as in [YZ17, 6.3.3], to be $\overline{\text{Hk}}_{G,d}^r/\text{Pic}_X$. Here, $\overline{\text{Hk}}_{G,d}^r$ parameterizes colength $d$ injections $\phi: \mathcal{E} \to \mathcal{E}'$ of rank two vector bundles on $X$, together with $r$ modifications $f_i: \mathcal{E}_i \to \mathcal{E}_{i+1}$ and $f'_{i}: \mathcal{E}'_i \to \mathcal{E}'_{i+1}$ of $\mathcal{E}$ and $\mathcal{E}'$, compatible with $\phi$. The modifications $f_i$ and $f'_i$ are required to be of type $\mu$ and above the same points $(x_1, \cdots, x_r) \in X^r$. The isomorphism class of $\text{Hk}_{G,d}^r$ is independent of $\mu$. The map

\[ (\vec{s}, \vec{t}): \text{Hk}_{G,d}^r \to \text{Hk}_{G,d}^r \times \text{Hk}_{G,d}^r \]

in (4.18) is $\{\phi, f_i, f'_i\} \mapsto (\{\mathcal{E}, f_i\}, \{\mathcal{E}', f'_i\})$.

The following two lemmas follow immediately from the definitions.

Lemma 4.11. The fiber product of the bottom row in (4.18) is $M_d$.

Lemma 4.12. The fiber product of the top row in (4.18) is $\text{Hk}_{M,d}^\mu$, i.e.

\[
\begin{array}{cccc}
\text{Hk}_{M,d}^\mu & \xrightarrow{\text{Hk}_{G,d}^r} & \text{Hk}_{G,d}^r \\
\downarrow & & \downarrow \\
\text{Hk}_{T_1}^\mu \times \text{Hk}_{T_2}^\mu & \xrightarrow{\Pi^\mu \times \Pi^\mu} & \text{Hk}_{G_0}^r \times \text{Hk}_{G_0}^r, \\
\end{array}
\]

(4.19)

is cartesian.

We will have to work around the singularities of $\text{Hk}_{G,d}^r$ in order to compute various intersection pairings. Let $\text{Hk}_{G,d}^{r,0} \subset \text{Hk}_{G,d}^r$ be the open substack consisting of those $\{\phi, f_i, f'_i, x_i\}$ such that the support of the divisor of $\text{det}(\phi)$ is disjoint from the $x_i$. Then $\text{Hk}_{G,d}^{r,0}$ is smooth of dimension $2d + 2r + 3g - 3$ [YZ17 6.10(1)]. Let $\text{Hk}_{M,d}^{\mu,0}$ be the preimage of $\text{Hk}_{G,d}^{r,0}$ in $\text{Hk}_{M,d}^\mu$.

Lemma 4.13. If $d \geq 2g_3 - 1$, then

\[ \dim(\text{Hk}_{M,d}^\mu - \text{Hk}_{M,d}^{\mu,0}) < 2d - g + 1 = \dim \text{Hk}_{M,d}^\mu. \]
Proof. From Proposition [4.5] and [4.13] we have

$$Hk^\mu_{M_d} \cong A_d \times_{\Sigma_{2d}(Y_3)} H^r_{2d}(Y),$$

where $H^r_{2d}(Y)$ parameterizes tuples $(E_0, E_1, \cdots, E_r)$ of effective divisors on $Y$ of degree $2d$, such that for each $1 \leq i \leq r$, $E_i$ is obtained from $E_{i-1}$ by changing a point $y_i \in E_{i-1}$ to $\tau_3(y_i)$. If we write $g_3 : Y \to Y_3$ for the double cover, then the divisor $E = g_3^*(E_i) \in \text{Div}(Y_3)$ is independent of $i$.

The locus $Hk^\mu_{M_d} - Hk^\mu_{M_d}$ consists of those triples

$$(\Delta, \xi, (E_i)) \in A_d \times_{\Sigma_{2d}(Y_3)} H^r_{2d}(Y)$$

such that the divisors $\text{div}(\text{Tr}(\xi)) = \text{div}(\xi + \xi^{\sigma_3})$ and $\text{div}(\xi^{\sigma_3}) = f_3^*(E)$ have a point in common. For such triples, there exists $x \in |Y_3|$ such that $x$ and $x^{\sigma_3}$ are in $E$. Thus, the image of $(\Delta, \xi, (E_i))$ in $A_d$ lies in the subscheme $C_d \subset A_d$ consisting of pairs $(\Delta, \xi)$ such that $\text{div}(\xi)$ and $\text{div}(\xi^{\sigma_3})$ have a point in common. Since there is a surjective map $X \times A_{d-1} \to C_d$, we have

$$\dim C_d \leq \dim(X \times A_{d-1}) = 2d - g.$$

As the composition $Hk^\mu_{M_d} \to M_d \to A_d$ is a finite morphism, we deduce the desired inequality: $\dim(Hk^\mu_{M_d} - Hk^\mu_{M_d}) \leq 2d - g$. □

**Lemma 4.14.** The refined Gysin map

$$(\Pi_1^{\mu} \times \Pi_2^{\mu})^! : \text{Ch}_{2d+2r+3g-3}(Hk_{r,G_0,d}) \to \text{Ch}_{2d-g+1}(Hk^\mu_{M_d})$$

associated to the diagram $$(4.19)$$ is defined. Moreover,

$$(\Pi_1^{\mu} \times \Pi_2^{\mu})^! [Hk_{r,G_0,d}] = [Hk^\mu_{M_d}], \quad (4.20)$$

**Proof.** For the first statement, it is enough to verify the two conditions in [YZ17, A.2.8]. The first condition is satisfied since $Hk^\mu_{M_d}$ is a scheme (Corollary 4.6). For the second condition, it is enough to show that for $i = 1, 2$, the map $\Pi_i^{\mu} : Hk^r_{T_i} \to Hk^r_{G_0}$ can be factored as a regular local immersion followed by a smooth relative Deligne-Mumford type morphism. Since $X^r \to X^r$ is étale, it is enough to prove this for the base change $Hk^r_{T_i} \to Hk^r_{G_0} \times_{X^r} X^r$, and this is proved in [YZ17, Lem. 6.11(1)].

For the second statement, note that

$$\text{Ch}_{2d+2r+3g-3}(Hk^r_{G_0,d}) \cong \text{Ch}_{2d+2r+3g-3}(Hk^r_{G_0,d}).$$

By Lemma 4.13 we also have

$$\text{Ch}_{2d-g+1}(Hk^\mu_{M_d}) \cong \text{Ch}_{2d-g+1}(Hk^\mu_{M_d}).$$

Both of these isomorphisms preserve fundamental classes. On the other hand, $(\Pi_1^{\mu} \times \Pi_2^{\mu})^! [Hk^r_{G_0,d}] = [Hk^\mu_{M_d}]$, since $Hk^r_{G_0,d}$ is smooth. The equality $(4.20)$ follows. □
The stack $\text{Sht}_{G_0,d}^r$ is defined to be the fiber product of the third column in (4.18):

$$\begin{array}{ccc}
\text{Sht}_{G_0,d}^r & \longrightarrow & \text{HK}_{G_0,d}^r \\
\downarrow & & \downarrow \text{(id,Fr)} \\
H_d & \longrightarrow & H_d \times H_d.
\end{array}$$

By [YZ17, Lem. 6.12], there is a canonical isomorphism

$$\text{Sht}_{G_0,d}^r \cong \bigsqcup_{D \in \Sigma_d(X)\langle k \rangle} \text{Sht}_{G_0}^r(f_D),$$

where $\text{Sht}_{G_0}^r(f_D)$ is the stack of Hecke correspondences between Shtukas with $r$ paws, defined in [YZ17, 5.3.1]. Recall that it is these Hecke correspondences which give the action of $\mathcal{H}$ on $\text{Ch}_{c,r}(\text{Sht}_{G_0}^r)$, and which allowed us to define $\mathbb{I}_r(f_D)$.

**Lemma 4.15.** Let $D$ be an effective divisor of degree $d$, and recall the definition of $\text{Sht}^\mu_{MD}$ in (4.17). Then the following diagram is cartesian

$$\begin{array}{ccc}
\text{Sht}^\mu_{MD} & \longrightarrow & \text{Sht}_{G_0}^r(f_D) \\
\downarrow & & \downarrow \\
\text{Sht}^\mu_{T_1} \times \text{Sht}^\mu_{T_2} & \longrightarrow & \text{Sht}_{G_0}^r \times \text{Sht}_{G_0}^r.
\end{array}$$

**Proof.** The fiber products of the three rows in the octahedron are

$$\text{HK}_{MD}^\mu \xrightarrow{\text{(id,Fr)}} M_d \times M_d \xleftarrow{\text{(id,Fr)}} M_d,$$

and the fiber product of this resulting diagram is by definition

$$\text{Sht}^\mu_{MD} \cong \bigsqcup_{D \in \Sigma_d(X)\langle k \rangle} \text{Sht}^\mu_{MD}.$$

By (4.1), the fiber products of the three columns in the octahedron are

$$\text{Sht}^\mu_{T_1} \times \text{Sht}^\mu_{T_2} \xrightarrow{\theta_1^\mu \times \theta_2^\mu} \text{Sht}_{G_0}^r \times \text{Sht}_{G_0}^r \xleftarrow{(s,t)} \text{Sht}_{G_0,d}^r.$$

By [YZ17] Lem. A.9], the fiber product of (4.24) is canonically isomorphic, as an $A_d$-stack, to the fiber product of (4.26). We therefore have the cartesian square

$$\begin{array}{ccc}
\text{Sht}^\mu_{MD} & \longrightarrow & \text{Sht}_{G_0,d}^r \\
\downarrow & & \downarrow \\
\text{Sht}^\mu_{T_1} \times \text{Sht}^\mu_{T_2} & \longrightarrow & \text{Sht}_{G_0}^r \times \text{Sht}_{G_0}^r.
\end{array}$$

Taking the fiber over $D \in \Sigma_d(X)\langle k \rangle$ in (4.27) and using the decompositions (4.22) and (4.25), we obtain the desired cartesian square (4.28).  \qed
Lemma 4.15 allows us to define two different maps

\[ \text{Ch}_{2d+2r+3g-3}(\text{Hk}_{G_0,d}) \to \text{Ch}_0(\text{Sh}_M), \]

each obtained by composing two refined Gysin morphisms. Specifically, the

cartesian squares (4.21) and (4.27) induce the composition

\[
\begin{align*}
\text{Ch}_{2d+2r+3g-3}(\text{Hk}_{G_0,d}) & \xrightarrow{(\text{id, Fr}_{Hd})^i} \text{Ch}_{2r}(\text{Sh}_G) \\
& \xrightarrow{(\theta_1^\mu \times \theta_2^\mu)^i} \text{Ch}_0(\text{Sh}_M),
\end{align*}
\]

whereas the cartesian squares (4.19) and (4.14) give

\[
\begin{align*}
\text{Ch}_{2d+2r+3g-3}(\text{Hk}_{G_0,d}) & \xrightarrow{(\Pi_1^\mu \times \Pi_2^\mu)^i} \text{Ch}_{2d-g+1}(\text{Hk}_M) \\
& \xrightarrow{(\text{id, Fr}_{Md})^i} \text{Ch}_0(\text{Sh}_M).
\end{align*}
\]

This is assuming that \((\theta_1^\mu \times \theta_2^\mu)^i\) and \((\text{id, Fr}_{Hd})^i\) are well-defined, which we justify next. The other technical result that we need is that these two compositions agree on the fundamental cycle:

**Lemma 4.16.** The refined Gysin maps \((\theta_1^\mu \times \theta_2^\mu)^i\) and \((\text{id, Fr}_{Hd})^i\) are well-defined. Moreover, we have

\[
(\theta_1^\mu \times \theta_2^\mu)^i(\text{id, Fr}_{Hd})^i[\text{Hk}_{G_0,d}] = (\text{id, Fr}_{Md})^i(\Pi_1^\mu \times \Pi_2^\mu)^i[\text{Hk}_{G_0,d}].
\]

**Proof.** This is the octahedron lemma [YZ17, Thm. A.10] applied to the
diagram (4.18). We must show that hypotheses (1)-(4) of that lemma are satisfied. The smoothness of all stacks in (4.18) aside from \(\text{Hk}_{G_0,d}\) was proven in [YZ17], so (1) is verified. That the middle and left rows and middle and bottom columns have expected fiber product dimensions is an easy computation using Proposition 4.3 and that \(\text{dim } H_d = 2d + 3g - 3\), \(\dim \text{Bun}_{G_0} = 3g - 3\), and \(\dim \text{Bun}_{T_i} = g - 1\). This proves (2). Hypothesis (3) was verified in the proof of Lemma 4.14 and [YZ17, Lem. 6.14].

To verify (4), we must check that the two conditions in [YZ17, A.2.8] are satisfied for the cartesian diagrams (4.14) and (4.27). First note that the fiber product in both of these diagrams is \(\text{Sh}_M\), which is a scheme by Proposition 4.13 and (4.14). Thus, it remains to check that the bottom row in each of the cartesian squares (4.14) and (4.27) can be factored as regular local immersion followed by a smooth relative Deligne-Mumford type

We finally put everything together to prove Proposition 4.10. Recalling that \([\text{Sh}_{T_i}^\mu]\) is the pushforward of the fundamental class along the map

\[ \theta_i^\mu : \text{Sh}_{T_i} \to \text{Sh}_{G_0}, \]

we have

\[
\mathbb{I}_r(f_D) = \left( [\text{Sh}_{T_1}^\mu] * [\text{Sh}_{T_2}^\mu] \right)_{\text{Sh}_{G_0}} = \text{deg} \left( (\theta_1^\mu \times \theta_2^\mu)^i [\text{Sh}_{G_0}(f_D)] \right).
\]
By Lemma 4.15, the class \((\theta_1^\mu \times \theta_2^\mu)^!\text{[Sht}^r_{G_0}(f_D)]\) is the \(D\)th component of 
\[
(\theta_1^\mu \times \theta_2^\mu)^!\text{[Sht}^r_{G_0,d}]
\] 
So to prove Proposition 4.10 for all \(D\) of degree \(d\), it is enough to show that 
\[
(\theta_1^\mu \times \theta_2^\mu)^!\text{[Sht}^r_{G_0,d}] = (\text{id}, \text{Fr}_{M_d})^!\text{[Hk}^\mu_{M_d}].
\]
For this we compute:
\[
(\theta_1^\mu \times \theta_2^\mu)^!\text{[Sht}^r_{G_0,d}] = (\text{id}, \text{Fr}_{M_d})^!\text{[Hk}^\mu_{M_d}] = (\text{id}, \text{Fr}_{M_d})^!\text{[Hk}^\mu_{M_d}].
\]
This completes the proof of Proposition 4.10. □

4.6. Completion of the proof of Proposition 4.7. The proof of Proposition 4.7 is obtained by combining Proposition 4.10 with the Lefschetz-Verdier trace formula, as we now explain.

Proof of Proposition 4.7. Consider the composition
\[
\text{Ch}_{2d-g+1}(\text{Hk}^\mu_{M_d}) \xrightarrow{(4.17)} \text{Ch}_0(\text{Sht}^\mu_{M_d}) \xrightarrow{(4.16)} \text{Ch}_0(\text{Sht}^\mu_{M_d}) = \bigoplus_{\xi \in AD(k)} \text{Ch}_0(\text{Sht}^\mu_{M_d}(\xi)),
\]
where the final decomposition is induced by (4.15). If the image of the fundamental class \([\text{Hk}^\mu_{M_d}] \in \text{Ch}_{2d-g+1}(\text{Hk}^\mu_{M_d})\) in the component indexed by \(\xi\) is denoted \(C_\xi\), the trace formula of [YZ17, A.12] implies
\[
\deg(C_\xi) = \text{Trace}([\text{Hk}^\mu_{M_d}]_\xi \circ \text{Frob}_\xi; (\alpha_*\mathbb{Q}_\ell)_\xi),
\]
The isomorphism (4.13) implies the equality \([\text{Hk}^\mu_{M_d}] = [\text{Hk}^\mu_{M_d}]^r\) of endomorphisms of \(\alpha_*\mathbb{Q}_\ell\), and so
\[
\deg(C_\xi) = \text{Trace}([\text{Hk}^\mu_{M_d}]^r_\xi \circ \text{Frob}_\xi; (\alpha_*\mathbb{Q}_\ell)_\xi).
\]
On the other hand, Proposition 4.10 gives the first equality in
\[
\mathbb{I}_r(f_D) = \sum_{\xi \in AD(k)} \deg(C_\xi) = \sum_{\xi \in AD(k)} \text{Trace}([\text{Hk}^\mu_{M_d}]^r_\xi \circ \text{Frob}_\xi; (\alpha_*\mathbb{Q}_\ell)_\xi),
\]
completing the proof. □

5. Completion of the proofs
Fix an integer \(r \geq 0\), and an \(r\)-tuple \(\mu = (\mu_1, \ldots, \mu_r) \in \{\pm 1\}^r\) satisfying the parity condition \(\sum_{i=1}^r \mu_i = 0\). In particular, \(r\) is even.
5.1. **Representations of symmetric groups.** Fix a positive integer $d$. We will prove some elementary facts about the representation theory of the finite group

$$\Gamma_{2d} = \{\pm 1\}^{2d} \rtimes S_{2d}.$$ 

These facts will be used in the proof of Proposition 5.3 below.

Denote by $1$ the trivial representation of $S_{2d}$ on $Q_\ell$, so that

$$\text{Ind}_{S_{2d}}^{\Gamma_{2d}} 1 = \{ \Phi : S_{2d} \to \text{Hom}(\Gamma_{2d}, Q_\ell) \}. \quad (5.1)$$

For each $x \in \{\pm 1\}^{2d}$ denote by $\Phi_x$ the characteristic function of the coset $S_{2d} \cdot x \subset \Gamma_{2d}$. As $x$ varies these form a basis for (5.1). Setting $e_i = (1, \ldots, 1, -1, \ldots, 1) \in \{\pm 1\}^{2d}$,

define a $\Gamma_{2d}$-linear endomorphism of (5.1) by

$$H \cdot \Phi_x = \sum_{i=1}^{2d} \Phi_{e_i x}.$$ 

**Proposition 5.1.** There is a unique $\Gamma_{2d}$-stable decomposition

$$\text{Ind}_{S_{2d}}^{\Gamma_{2d}} 1 = \bigoplus_{d_1, d_2 \geq 0} V_{(d_1, d_2)} \quad (5.2)$$

such that $H$ acts on $V_{(d_1, d_2)}$ as the scalar $d_1 - d_2$. Moreover, each $V_{(d_1, d_2)}$ is an irreducible representation of $\Gamma_{2d}$, and $V_{(d_1, d_2)} \cong V_{(d_2, d_1)}$.

**Proof.** For each character $\chi : \{\pm 1\}^{2d} \to \{\pm 1\}$ define

$$\Psi_\chi = \sum_{x \in \{\pm 1\}^{2d}} \chi(x) \Phi_x.$$ 

As $\chi$ varies these form a basis for (5.1), and an easy calculation shows that

$$H \cdot \Psi_\chi = \left( \text{pos}(\chi) - \text{neg}(\chi) \right) \cdot \Psi_\chi,$$

where $\text{pos}(\chi) = \# \{ e_i : \chi(e_i) = 1 \}$ and $\text{neg}(\chi) = \# \{ e_i : \chi(e_i) = -1 \}$. It is now easy to see that the subspaces

$$V_{(d_1, d_2)} = \text{Span} \left\{ \Psi_\chi : \begin{array}{l} \text{pos}(\chi) = d_1 \\ \text{neg}(\chi) = d_2 \end{array} \right\}$$

are irreducible subrepresentations satisfying (5.2), on which $H$ acts by $d_1 - d_2$. The uniqueness of the decomposition follows from the characterization of $V_{(d_1, d_2)}$ as the $d_1 - d_2$ eigenspace of $H$.

It only remains to show that $V_{(d_1, d_2)} \cong V_{(d_2, d_1)}$. Let $\eta : \{\pm 1\}^{2d} \to \{\pm 1\}$ be the character

$$\eta(x_1, \ldots, x_{2d}) = x_1 \cdots x_{2d},$$

and define an order two automorphism of (5.1) by $\Psi_\chi \mapsto \Psi_{\eta \chi}$. Direct calculation shows that this automorphism commutes with the action of $\Gamma_{2d}$, and interchanges the subspaces $V_{(d_1, d_2)}$ and $V_{(d_2, d_1)}$. \qed
Now fix a pair \((d_1, d_2)\) of non-negative integers such that \(d_1 + d_2 = 2d\). For \(i \in \{1, 2\}\), set \(\Gamma_{d_i} = \{\pm 1\}^{d_i} \times S_{d_i}\), and define a character

\[\eta_{d_i} : \{\pm 1\}^{d_i} \to \{\pm 1\}\]

by \((x_1, \ldots, x_{d_i}) \mapsto x_1 \cdots x_{d_i}\). It extends uniquely to a character of \(\Gamma_{d_i}\), trivial on the subgroup \(S_{d_i}\).

**Proposition 5.2.** There are isomorphisms of \(\Gamma_{2d}\)-representations

\[
\text{Ind}_{\Gamma_{d_1} \times \Gamma_{d_2}}^{\Gamma_{2d}}(\eta_{d_1} \boxtimes 1) \cong V_{(d_1,d_2)} \cong \text{Ind}_{\Gamma_{d_1} \times \Gamma_{d_2}}^{\Gamma_{2d}}(1 \boxtimes \eta_{d_2}).
\]

**Proof.** Define a character \(\chi : \{\pm 1\}^{2d} \to \{\pm 1\}\) by

\[
\chi(x_1, \ldots, x_{2d}) = \eta_{d_1}(x_1, \ldots, x_{d_1}).
\]

Using the notation of the proof of Proposition 5.1 it is easy to see that the subgroup \(\Gamma_{d_1} \times \Gamma_{d_2} \subset \Gamma_{2d}\) acts on the vector \(\Psi_\chi\) via the character \(\eta_{d_1} \boxtimes 1\), and that \(\Psi_\chi\) generates \(V_{(d_1,d_2)}\) as a \(\Gamma_{2d}\)-representation. Thus Frobenius reciprocity provides a surjection

\[
\text{Ind}_{\Gamma_{d_1} \times \Gamma_{d_2}}^{\Gamma_{2d}}(\eta_{d_1} \boxtimes 1) \to V_{(d_1,d_2)},
\]

which is an isomorphism by dimension counting. The construction of the other isomorphism is entirely similar. \(\square\)

### 5.2. Comparison of étale sheaves.

The constructions of \([4.3]\) provide us with a constructible \(\ell\)-adic sheaf \(\alpha_\star \mathbb{Q}_\ell\) on \(A_d\), endowed with an endomorphism \([\text{Hk}_{M_d}]\). On the other hand, for every pair of non-negative integers \((d_1, d_2)\) with \(d_1 + d_2 = 2d\) the constructions of \([5.3]\) provide us with a constructible \(\ell\)-adic sheaf \(\beta_\star L_{(d_1,d_2)}\) on \(A_d\). The goal of this section is to prove the following result, relating these étale sheaves.

**Proposition 5.3.** Assume that \(d \geq 2g_3 - 1\). There is an isomorphism

\[
\alpha_\star \mathbb{Q}_\ell \cong \bigoplus_{d_1, d_2 \geq 0 \atop d_1 + d_2 = 2d} \beta_\star L_{(d_1,d_2)} \tag{5.3}
\]

of \(\ell\)-adic sheaves on \(A_d\). Each summand is stable under the Hecke correspondence \([\text{Hk}_{M_d}]\), which acts on \(\beta_\star L_{(d_1,d_2)}\) via the scalar \(d_1 - d_2\).

**Proof.** Let \(U_{2d}(Y_3) \subset Y_3^{2d}\) be the open subscheme parametrizing 2\(d\)-tuples of distinct points on \(Y_3\), and let \(U_{2d}^\prime(Y) \subset Y^{2d}\) be its preimage under the morphism \(Y^{2d} \to Y_3^{2d}\). Thus we have a cartesian diagram

\[
\begin{array}{ccc}
U_{2d}^\prime(Y) & \longrightarrow & Y^{2d} \\
\downarrow & & \downarrow \\
U_{2d}(Y_3) & \longrightarrow & Y_3^{2d},
\end{array}
\]

in which the horizontal arrows are open immersions with dense image, and the vertical arrows are finite étale. Taking the GIT quotients by the action of
Throughout, and using the isomorphisms of (3.15), we obtain a cartesian diagram

\[
\begin{array}{ccc}
S_{2d} \setminus U_{2d}(Y) & \longrightarrow & \Sigma_{2d}(Y) \\
\downarrow a & \downarrow \Nm \\
S_{2d} \setminus U_{2d}(Y_3) & \longrightarrow & \Sigma_{2d}(Y_3),
\end{array}
\]

(5.4)
in which the horizontal arrows are open immersions, the vertical arrows are finite, and \(a\) is étale.

By identifying \(\{\pm 1\}^{2d} \cong \text{Aut}(Y/Y_3)^{2d}\), the semi-direct product \(\Gamma_{2d}\) of §5.1 acts on \(U_{2d}(Y)\). In fact \(\Gamma_{2d}\) is the automorphism group of the Galois cover

\[
U_{2d}(Y) \to S_{2d} \setminus U_{2d}(Y_3),
\]

and the local system \(a_*\mathbb{Q}_{\ell}\) on \(S_{2d} \setminus U_{2d}(Y_3)\) corresponds to the induced representation (5.1). Each summand on the right hand side of (5.2) also determines a local system, denoted the same way, and so we have a decomposition

\[
a_*\mathbb{Q}_{\ell} = \bigoplus_{d_1, d_2 \geq 0, \ d_1 + d_2 = 2d} V(d_1, d_2)
\]

(5.5)
of local systems on \(S_{2d} \setminus U_{2d}(Y_3)\).

We interrupt the proof of Proposition 5.3 for a lemma.

**Lemma 5.4.** If we define \(A^o_d\) as the cartesian product

\[
\begin{array}{ccc}
A^o_d & \longrightarrow & A_d \\
\downarrow \pi & & \downarrow f_d^2 \\
S_{2d} \setminus U_{2d}(Y_3) & \longrightarrow & \Sigma_{2d}(Y_3),
\end{array}
\]

there are isomorphisms local systems

\[
v^*\alpha_*\mathbb{Q}_\ell \cong \pi^* a_*\mathbb{Q}_\ell \quad \quad \quad (5.6)
v^*\beta_* L(d_1, d_2) \cong \pi^* V(d_1, d_2).
\]

(5.7)

**Proof.** Applying the proper base change theorem to (4.3) and (5.4) provides the first and third isomorphisms, respectively, in

\[
v^*\alpha_*\mathbb{Q}_\ell \cong (f_d^2 \circ v)^* \Nm_*\mathbb{Q}_\ell \cong (u \circ \pi)^* \Nm_*\mathbb{Q}_\ell \cong \pi^* a_*\mathbb{Q}_\ell.
\]

This proves (5.6).

The proof of (5.7) is very similar. Consider the canonical morphism

\[
\Sigma_{d_1}(Y_3) \times_k \Sigma_{d_2}(Y_3) \xrightarrow{g} \Sigma_{2d}(Y_3).
\]

(This is the morphism labeled \(\otimes\) in (3.14), but we temporarily change the name to avoid the awkward notation \(\otimes\) for the pushforward.) By the first isomorphism of Proposition 5.2 the local system \(L_{d_1} \boxtimes \mathbb{Q}_\ell\) on \(\Sigma_{d_1}(Y_3) \times_k \Sigma_{d_2}(Y_3)\) defined in (3.5) satisfies \(u^* g_* (L_{d_1} \boxtimes \mathbb{Q}_\ell) \cong V(d_1, d_2)\). On the other hand, applying proper base change to the diagram (3.14) and recalling (3.22), there
is an canonical isomorphism \( \beta_* L_{(d_1,d_2)} \cong (f_3^*)^* g_*(L_{d_1} \boxtimes \mathbb{Q}_\ell) \). The desired isomorphism (5.7) is the composition

\[
v^* \beta_* L_{(d_1,d_2)} \cong (f_3^* \circ v)^* g_*(L_{d_1} \boxtimes \mathbb{Q}_\ell) \cong (u \circ \pi)^* g_*(L_{d_1} \boxtimes \mathbb{Q}_\ell) \cong \pi^* V_{(d_1,d_2)}. \]

\[ \square \]

Combining (5.5) with the isomorphisms (5.6) and (5.7) yields an isomorphism of local systems

\[
v^* \alpha_* \mathbb{Q}_\ell \cong \bigoplus_{d_1,d_2 \geq 0 \atop d_1 + d_2 = 2d} v^* \beta_* L_{(d_1,d_2)} \tag{5.8} \]

over the open subscheme \( A^o_d \subset A_d \). We claim that this isomorphism extends uniquely to an isomorphism of constructible sheaves (5.3). In fact, this is a formal consequence of the theory of intermediate extensions of perverse sheaves [KW01, Chapter 3].

To make this a bit more explicit, recall from Proposition 4.3 (and our assumption \( d \geq 2g_3 - 1 \)) that the \( k \)-scheme \( M_d \) is smooth of dimension \( n = 2d - g + 1 \), and that the morphism \( \alpha : M_d \to A_d \) is finite. According to [KW01, Lemma 7.5] the push-forward of the shifted constant sheaf \( \mathbb{Q}_\ell[n] \) on \( M_d \) is a perverse sheaf \( R\alpha_* \mathbb{Q}_\ell[n] \) on \( A_d \), and is equal to the intermediate extension of its restriction to the dense open subscheme \( A^o_d \subset A_d \). In other words

\[
R\alpha_* \mathbb{Q}_\ell[n] \cong v_* (v^* R\alpha_* \mathbb{Q}_\ell[n]).
\]

By Proposition 3.13 the same argument applies to \( \beta : N_{(d_1,d_2)} \to A_d \), and so

\[
R\beta_* L_{(d_1,d_2)}[n] \cong v_* (v^* R\beta_* L_{(d_1,d_2)}[n]).
\]

Using the finiteness of \( \alpha \) and \( \beta \), we may view (5.8) as an isomorphism

\[
v^* R\alpha_* \mathbb{Q}_\ell[n] \cong \bigoplus_{d_1,d_2 \geq 0 \atop d_1 + d_2 = 2d} v^* R\beta_* L_{(d_1,d_2)}[n], \tag{5.9}
\]

and apply the functor \( v_* \) to both sides to obtain (5.3).

We now turn to the endomorphism \( [H_{kM_d}] : \alpha_* \mathbb{Q}_\ell \to \alpha_* \mathbb{Q}_\ell \). The endomorphism \( H \) of (5.1) induces an endomorphism \( H : a_* \mathbb{Q}_\ell \to a_* \mathbb{Q}_\ell \) of the local system (5.5). After restricting to the open subscheme \( A^o_d \subset A_d \) there is a commutative diagram

\[
v^* \alpha_* \mathbb{Q}_\ell \xrightarrow{[H_{kM_d}]} v^* \alpha_* \mathbb{Q}_\ell \xrightarrow{\pi^* a_* \mathbb{Q}_\ell} \pi^* a_* \mathbb{Q}_\ell.
\]

Commutativity of the diagram follows by direct comparison of the definition of \( H \) with Proposition 4.5 which characterizes \( H_{kM_d} \) in terms of the correspondence (4.9).
By Proposition 5.1, the endomorphism $H$ acts via the scalar $d_1 - d_2$ on the summand $V(d_1, d_2)$ in (5.5). This implies that the endomorphism $[Hk_M]$ in the top row of the above diagram acts via the scalar $d_1 - d_2$ on the summand $v^* R_{\beta} L(d_1, d_2)$ in (5.8), or, equivalently, on the summand $v^* R_{\beta} L(d_1, d_2)[n]$ in (5.9). It now follows from [KW01, Corollary 5.11] that $[Hk_M]$ acts via the scalar $d_1 - d_2$ on the summand $v^* \beta L(d_1, d_2)$ in (5.3).

5.3. The intersection pairing in cohomology. Fix an auxiliary prime $\ell \neq \text{char}(k)$. Before stating and proving our main results, we must summarize some results from [YZ17] on various quotients of the $\ell$-adic analogue $H_{\ell} = H \otimes \mathbb{Q}_\ell$ of the $\mathbb{Q}$-algebra $H$ of §3.1.

The Hecke algebra $H_{\ell}$ acts on the $\ell$-adic cohomology group

$$V = H_{\ell}^{2r}(\text{Sh}_{G_0}(k \bar{k}), \mathbb{Q}_\ell)(r),$$

as in [YZ17, §7.1]. The cycle class map $\text{cl}: \text{Ch}_{c,r}(\text{Sh}_{G_0}) \to V$ is $H$-equivariant, and the cup product $\langle \cdot, \cdot \rangle: V \times V \to \mathbb{Q}_\ell$ (5.10) pulls back to the intersection pairing on the Chow group.

Recalling the map $H \to \mathbb{Q}[\text{Pic}_X(k)]^\text{pic}$ appearing in (3.2), define

$$\widetilde{H}_{\ell} = \text{Image}(H_{\ell} \to \text{End}_{\mathbb{Q}_\ell}(V) \times \text{End}_{\mathbb{Q}_\ell}(A_{\ell}) \times \mathbb{Q}_\ell[\text{Pic}_X(k)]^\text{pic}),$$

$$\overline{H}_{\ell} = \text{Image}(H_{\ell} \to \text{End}_{\mathbb{Q}_\ell}(V) \times \mathbb{Q}_\ell[\text{Pic}_X(k)]^\text{pic}),$$

$$\mathcal{H}_{\text{aut},\ell} = \text{Image}(H_{\ell} \to \text{End}_{\mathbb{Q}_\ell}(A_{\ell}) \times \mathbb{Q}_\ell[\text{Pic}_X(k)]^\text{pic}).$$

These are finite type $\mathbb{Q}_\ell$-algebras, related by surjections

$$\frac{\mathbb{Q}_{\ell}[\text{Pic}_X(k)]^\text{pic}}{\mathbb{Q}_{\ell}[\text{Pic}_X(k)]^\text{pic}},$$

Recalling the $\mathbb{Q}$-algebra $\mathcal{H}_{\text{aut}}$ of §3.1, there is a canonical isomorphism

$$\mathcal{H}_{\text{aut}} \otimes \mathbb{Q}_\ell \cong \mathcal{H}_{\text{aut},\ell}.$$

For any $f \in \mathcal{H}$ the function $\mathcal{J}(f, s)$ of (5.7) is a Laurent polynomial in $q^s$ with rational coefficients. Setting

$$\mathcal{J}_r(f) = (\log q)^{-r} \left. \frac{d^r}{ds^r} \mathcal{J}(f, s) \right|_{s=0},$$

we obtain a linear functional $\mathbb{J}_r: \mathcal{H} \to \mathbb{Q}$. The following result shows that this agrees with the linear functional $\mathbb{I}_r: \mathcal{H} \to \mathbb{Q}$ defined by (4.2).
Proposition 5.5. The equality
\[ \mathbb{I}_r(f) = \mathbb{J}_r(f) \]
holds for every \( f \in \mathcal{H} \). Moreover, the \( \mathbb{Q}_\ell \)-linear extensions of \( \mathbb{I}_r \) and \( \mathbb{J}_r \) to \( \mathcal{H}_\ell \to \mathbb{Q}_\ell \) factor through \( \tilde{\mathcal{H}}_\ell \).

Proof. The compatibility of the cup product pairing (5.10) with the intersection pairing on the Chow group implies that the \( \mathbb{Q}_\ell \)-linear extension \( \mathbb{I}_r: \mathcal{H}_\ell \to \mathbb{Q}_\ell \) factors through \( \mathcal{H}_\ell \). The final claim of Proposition 3.11 implies that the \( \mathbb{Q}_\ell \)-linear extension \( \mathbb{J}_r: \mathcal{H}_\ell \to \mathbb{Q}_\ell \) factors through \( \mathcal{H}_{\text{aut,}\ell} \).

It follows that both \( \mathbb{I}_r \) and \( \mathbb{J}_r \) factor through the quotient \( \tilde{\mathcal{H}}_\ell \).

It remains to prove that \( \mathbb{I}_r(f) = \mathbb{J}_r(f) \) for all \( f \in \mathcal{H} \). Assume first that \( f = f_D \) for some effective divisor \( D \in \text{Div}(X) \) of degree \( d \geq 2g_3 - 1 \).

Combining the decomposition (3.12) with Proposition 3.17, we find
\[ \mathbb{J}_r(f_D) = \sum_{\xi \in A_D(k)} \sum_{d_1, d_2 \geq 0, d_1 + d_2 = 2d} (d_1 - d_2)^r \cdot \text{Trace}(\text{Frob}_\xi; (\beta_* L_{d_1, d_2})_{\xi}). \]

On the other hand, Proposition 4.7 tells us that
\[ \mathbb{I}_r(f_D) = \sum_{\xi \in A_D(k)} \text{Trace}([H_{kM_d}]^r_{\xi} \circ \text{Frob}_\xi; (\alpha_* \mathbb{Q}_\ell)_{\xi}). \]

These two expressions are equal, by Proposition 5.3.

The proof of [YZ17, Theorem 9.2] shows that the image of \( \mathcal{H}_\ell \to \tilde{\mathcal{H}}_\ell \) is generated as \( \mathbb{Q}_\ell \)-vector space by the images of \( f_D \in \mathcal{H} \) as \( D \) ranges over all effective divisors on \( X \) of degree \( d \geq 2g_3 - 1 \). Therefore \( \mathbb{I}_r = \mathbb{J}_r \). \( \square \)

According to [YZ17, (9.5)], there is a canonical \( \mathbb{Q}_\ell \)-algebra decomposition
\[ \mathcal{H}_\ell = \mathcal{H}_{\text{Eis}} \oplus \bigoplus_m \mathcal{H}_{\ell,m}, \tag{5.11} \]
where \( m \) runs over the finitely many maximal ideals \( m \subset \mathcal{H}_\ell \) that do not contain the kernel of the projection
\[ \mathcal{H}_\ell \to \mathbb{Q}_\ell[\text{Pic}_X(k)]^\text{Pic}. \tag{5.12} \]
For each such \( m \) the localization \( \mathcal{H}_{\ell,m} \) is a finite (hence Artinian) \( \mathbb{Q}_\ell \)-algebra. If we denote by \( E_m \) its residue field, then Hensel’s lemma implies that the quotient map \( \mathcal{H}_{\ell,m} \to E_m \) admits a unique section, which makes \( \mathcal{H}_{\ell,m} \) into an Artinian local \( E_m \)-algebra.

The decomposition (5.11) induces a decomposition of \( \tilde{\mathcal{H}}_\ell \)-modules
\[ V = V_{\text{Eis}} \oplus \bigoplus_m V_m, \tag{5.13} \]
in which each localization \( V_m \) is a finite-dimensional \( E_m \)-vector space. It follows from [YZ17, Corollary 7.15] that this decomposition is orthogonal with respect to the cup product pairing. Moreover, the self adjointness of
the action of $\mathcal{H}_\ell$ with respect to the cup product pairing (5.10) implies that there is a unique symmetric $E_m$-bilinear pairing

$$\langle \cdot, \cdot \rangle_{E_m} : V_m \times V_m \to E_m$$

such that $\text{Trace}_{E_m/Q_\ell} \langle \cdot, \cdot \rangle_{E_m} = \langle \cdot, \cdot \rangle$.

For $i \in \{1, 2\}$, we define $[\text{Sht}_{T_i}^]\in V_m$ to be the projection of the cycle class $\text{cl}([\text{Sht}_{T_i}^]) \in V$, and form the intersection pairing

$$\langle [\text{Sht}_{T_1}^], [\text{Sht}_{T_2}^] \rangle_{E_m} \in E_m.$$

Some of the maximal ideals $m \subset \widetilde{\mathcal{H}}_\ell$ appearing in (5.11) are attached to cuspidal automorphic forms, as we now explain. Fix an unramified cuspidal automorphic representation $\pi \subset A_{\text{cusp}}(G_0)$. As in §3.1, such a representation determines a homomorphism

$$\mathcal{H}_{\text{aut}} \to \mathcal{H}_{\text{cusp}} \xrightarrow{\lambda_\pi} \mathbb{C}$$

whose image is a number field $E_\pi$. The induced map

$$\widetilde{\mathcal{H}}_\ell \to \mathcal{H}_{\text{aut}, \ell} \xrightarrow{\lambda_\pi} E_\pi \otimes \mathbb{Q}_\ell \cong \prod_{l|\ell} E_{\pi,l},$$

determines, for every prime $l | \ell$ of $E_\pi$, a surjection $\lambda_{\pi,l} : \mathcal{H}_\ell \to E_{\pi,l}$ whose kernel is one of those maximal ideals

$$m = \ker(\lambda_{\pi,l})$$

appearing in the decomposition (5.14). This is a consequence of the isomorphism (3.3).

Recalling the period integrals $\mathcal{P}_0$ and $\mathcal{P}_3$ of §3.3, for every cuspidal automorphic representation $\pi \subset A_{\text{cusp}}(G_0)$ define

$$C(\pi, s) = \frac{\mathcal{P}_0(\phi, s) \mathcal{P}_3(\phi, \eta)}{\langle \phi, \phi \rangle_{\text{Pet}}}$$

Here $\phi \in \pi^{U_0}$ is any nonzero vector. Recall from Remark 3.1 that $\text{Aut}(\mathbb{C}/\mathbb{Q})$ acts on the set of all unramified cuspidal automorphic representations, in such a way that stabilizer of $\pi$ is the subgroup $\text{Aut}(\mathbb{C}/E_\pi)$.

**Proposition 5.6.** The complex number

$$C_r(\pi) = (\log q)^{-r} \cdot \frac{d^r}{ds^r} C(\pi, s) \big|_{s=0}$$

satisfies $C_r(\pi^\sigma) = C_r(\pi)$ for all $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$. In particular, it lies in $E_\pi$.

**Proof.** Proposition 3.11 implies that

$$\mathbb{J}_r(f) = \sum_{\text{unr. cusp. } \pi} C_r(\pi) \cdot \lambda_\pi(f)$$

for all $f \in I_{\text{Eis}}$, and both sides factor through the quotient

$$\mathcal{H} \to \mathcal{H}_{\text{aut}} \cong \mathcal{H}_{\text{cusp}} \times \mathbb{Q}[\text{Pic}_X(k)]^{\text{Pic}}.$$
appearing in (3.3). In other words, we may view (5.15) as an equality of linear functionals on the cuspidal subalgebra $\mathcal{H}_{\text{cusp}} \subset \mathcal{H}_{\text{aut}}$.

It follows from what was said in §3.1 that $\mathcal{H}_{\text{cusp}}$ is a finite product of number fields, where the factors are indexed by the $\text{Aut}(\mathbb{C}/\mathbb{Q})$-orbits of unramified cuspidal automorphic representations. Restricting (5.15) to the factor $\mathcal{H}_{\text{cusp}}(\pi) \cong E_\pi$ indexed by the Galois orbit of $\pi$ yields the equality

$$J_r(f) = \sum_{\sigma : E_\pi \to \mathbb{C}} C_r(\pi^\sigma) \cdot \sigma(f)$$

for all $f \in E_\pi$. The sum is over all $\mathbb{Q}$-algebra embeddings $\sigma : E_\pi \to \mathbb{C}$, and for each such embedding we fix an extension to $\text{Aut}(\mathbb{C}/\mathbb{Q})$. On the other hand, we know that $J_r(f)$ is $\mathbb{Q}$-valued, so the right hand side must be fixed by the action of $\text{Aut}(\mathbb{C}/\mathbb{Q})$. The claim follows easily from this and the linear independence of $\{\sigma : E_\pi \to \mathbb{C}\}$. □

**Theorem 5.7.** Let $m \subset \widehat{\mathcal{H}}_\ell$ be a maximal ideal that does not contain the kernel of (5.12).

1. If $m$ is of the form (5.14) for an unramified cuspidal automorphic representation $\pi$ and a place $l | \ell$ of $E_\pi$, the equality

$$\langle [\text{Sh}_{T_1}]_m, [\text{Sh}_{T_2}]_m \rangle_{E_m} = C_r(\pi)$$

holds in $E_m = E_\pi, l$.

2. If $m$ is not of the form (5.14) then

$$\langle [\text{Sh}_{T_1}]_m, [\text{Sh}_{T_2}]_m \rangle_{E_m} = 0.$$

**Proof.** Given Proposition 5.5, the proof is essentially the same as that of [YZ17, Theorem 1.6]. Briefly, restrict both

$$\mathbb{I}_r, \mathbb{J}_r : \widehat{\mathcal{H}}_\ell \to \mathbb{Q}_\ell$$

to the $E_m$-algebra $\widehat{\mathcal{H}}_{\ell, m}$ in (5.11), and then further restrict to $E_m$ itself. Directly from its definition (4.2), the resulting $\mathbb{I}_r : E_m \to \mathbb{Q}_\ell$ satisfies

$$\mathbb{I}_r(f) = \langle [\text{Sh}_{T_1}]_m, f * [\text{Sh}_{T_2}]_m \rangle_{E_m} = \text{Trace}_{E_m/\mathbb{Q}_\ell}(f \cdot \langle [\text{Sh}_{T_1}]_m, [\text{Sh}_{T_2}]_m \rangle_{E_m}),$$

where the first pairing is (5.10). As for $\mathbb{J}_r : E_m \to \mathbb{Q}_\ell$, an argument similar to that used in Proposition 5.6 shows that

$$\mathbb{J}_r(f) = \text{Trace}_{E_m/\mathbb{Q}_\ell}(f \cdot C_r(\pi))$$

if $m$ has the form (5.14), and otherwise $\mathbb{J}_r(f) = 0$. The claim now follows from $\mathbb{I}_r = \mathbb{J}_r$ and the nondegeneracy of the trace pairing. □
5.4. The proofs of Theorems A and B. As in the introduction, for \( i \in \{1, 2\} \) we let \([\text{Sht}^r_T]_i\) be the pushforward of the fundamental class under \( \theta^i : \text{Sht}^r_T \to \text{Sht}_G \), and let \( \tilde{W}_i \subset \text{Ch}_{c,r}(\text{Sht}^r_G) \) be the \( \mathcal{H} \)-submodule generated by it. Define quotients

\[
W_1 = \tilde{W}_1 / \{ c \in \tilde{W}_1 : \langle c, \tilde{W}_2 \rangle = 0 \}
\]

\[
W_2 = \tilde{W}_2 / \{ c \in \tilde{W}_2 : \langle c, \tilde{W}_1 \rangle = 0 \},
\]

so that the intersection pairing descends to \( \langle \cdot, \cdot \rangle : W_1 \times W_2 \to \mathbb{Q} \).

**Proposition 5.8.** The actions of \( \mathcal{H} \) on \( W_1 \) and \( W_2 \) factor through the quotient

\[
\mathcal{H} \to \mathcal{H}_{\text{aut}} \cong \mathcal{H}_{\text{cusp}} \times \mathbb{Q}[\text{Pic}_X(k)]^{[\text{Pic}_X]}
\]

defined in §3.1.

**Proof.** By Proposition 3.11 the distribution \( J_r(f) \) only depends on the image of \( f \) under \( \mathcal{H} \to \mathcal{H}_{\text{aut}} \). By Proposition 5.5 the same is true of the distribution \( I_r(f) \) defined by (4.2), and the claim follows exactly as in [YZ17, Corollary 9.4]. \( \square \)

It follows from the discussion of §3.1 that \( \mathcal{H}_{\text{cusp,}\mathbb{R}} = \mathcal{H}_{\text{cusp}} \otimes_{\mathbb{Q}} \mathbb{R} \) is isomorphic to a product of copies of \( \mathbb{R} \), indexed by the unramified cuspidal automorphic representations \( \pi \). For each such \( \pi \), let \( e_{\pi} \in \mathcal{H}_{\text{cusp,}\mathbb{R}} \) be the corresponding idempotent. Using Proposition 5.8 these idempotents induce a decomposition

\[
W_i(\mathbb{R}) = W_{i,\text{cusp}} \oplus W_{i,\text{Eis}} = \bigoplus_{\pi} W_{i,\pi} \oplus W_{i,\text{Eis}}
\]

where the sum is over all unramified cuspidal \( \pi \), and \( W_{i,\pi} \subset W_i(\mathbb{R}) \) is the \( \lambda_{\pi} \)-eigenspace of \( \mathcal{H} \).

The following is Theorem A of the introduction.

**Theorem 5.9.** If \([\text{Sht}^r_T]_\pi\) denotes the projection of the image of \([\text{Sht}^r_T]_1\) to the summand \( W_{i,\pi} \), then

\[
\langle [\text{Sht}^r_T]_1, [\text{Sht}^r_T]_\pi \rangle = C_r(\pi).
\]

**Proof.** It follows from the discussion of §3.1 that \( \mathcal{H}_{\text{cusp}} \) decomposes as a product of totally real fields, indexed by the \( \text{Aut}(\mathbb{C}/\mathbb{Q}) \)-orbits of unramified cuspidal automorphic representations. Let \( \Pi \) denote the \( \text{Aut}(\mathbb{C}/\mathbb{Q}) \)-orbit of \( \pi \), and let \( \mathcal{H}_\Pi \subset \mathcal{H}_{\text{cusp}} \) be the corresponding summand.

For each \( \pi' \in \Pi \), the corresponding \( \lambda_{\pi\pi'} : \mathcal{H}_{\text{cusp}} \to \mathbb{C} \) restricts to an isomorphism \( \lambda_{\pi\pi'} : \mathcal{H}_\Pi \to \mathbb{C}_{\pi'} \). As \( \sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}) \) varies, it follows from Proposition 5.6 that we may collect together the constants \( C_r(\pi') \) into a single

\[
C_r(\Pi) \in \mathcal{H}_\Pi
\]
such that $\lambda_\pi(C_r(\Pi)) = C_r(\pi^\sigma)$.

The idempotent $e_\Pi \in \mathcal{H}_\Pi \subset \mathcal{H}_{\text{aut}}$ cuts out an $\mathcal{H}_\Pi$-vector space

$$e_\Pi W_1 \subset W_i.$$  

The action of $\mathcal{H}$ on the Chow group is self-adjoint relative to the $\mathbb{Q}$-bilinear intersection pairing, and it follows that there is a unique $\mathcal{H}_\Pi$-bilinear pairing

$$\langle \cdot, \cdot \rangle_{\Pi}: e_\Pi W_1 \times e_\Pi W_2 \rightarrow \mathcal{H}_\Pi$$

whose trace is the $\mathbb{Q}$-valued intersection form.

Using Theorem 5.7 and the compatibility of the cycle class map with intersection pairings, we obtain the equality

$$\langle e_\Pi[Sht_{T_1}], e_\Pi[Sht_{T_2}] \rangle_{\Pi} = C_r(\Pi)$$

in $\mathcal{H}_\Pi$, and applying the isomorphism $\lambda_\pi: \mathcal{H}_\Pi \rightarrow E_\pi$ to both sides proves the claim. □

Let $L(\pi, s)$ be the standard $L$-function attached to $\pi$ and recall that $\chi_1$ and $\chi_2$ are quadratic characters corresponding to $K_1$ and $K_2$, respectively. The following is Theorem B of the introduction.

**Theorem 5.10.** In the notation of Theorem 5.9,

$$\langle [Sht_{T_1}], [Sht_{T_2}] \rangle_\pi = 0$$

if and only if

$$L^{(r)}(\pi, 1/2) L(\pi \otimes \chi_1, 1/2) L(\pi \otimes \chi_2, 1/2) = 0.$$  

**Proof.** By Theorem 5.9, it suffices to show that

$$\mathcal{P}_0^{(r)}(\phi, 0) = 0 \quad \text{if and only if} \quad L^{(r)}(\pi, 1/2) = 0 \quad (5.16)$$

and

$$\mathcal{P}_3(\bar{\phi}, \eta) = 0 \quad \text{if and only if} \quad L(\pi \otimes \chi_1, 1/2)L(\pi \otimes \chi_2, 1/2) = 0. \quad (5.17)$$

The equivalence (5.16) is clear since $\mathcal{P}_0(\phi, s) = c(s)L(\pi, s + 1/2)$ for some function $c(s)$ not vanishing at $s = 0$ [YZ17, pg. 27].

To prove (5.17), we use the following general result of Waldspurger. Let $P_3^\pi: \pi \otimes \pi \rightarrow \mathbb{C}$ be the linear functional

$$\phi_1 \otimes \phi_2 \mapsto P_3^\pi(\phi_1 \otimes \phi_2) \overset{\text{def}}{=} \mathcal{P}_3(\phi_1, \eta)\mathcal{P}_3(\phi_2, \eta).$$

For each place $x \in |X|$, there is a $(T_3(F_x) \times T_3(F_x))$-invariant linear functional $P_3^\pi_x: \pi_x \otimes \pi_x \rightarrow \mathbb{C}$, which can be written down explicitly using a toric period integral. Waldspurger [Wal85, CW16] proves the equality of linear functionals

$$P_3^\pi = L(\pi_{K_3} \otimes \eta, 1/2) \prod_{x \in |X|} P_3^\pi_x. \quad (5.18)$$

Since $\pi$ is everywhere unramified and $\phi \in \pi^{U_0}$ is spherical, it follows from [GP91, Prop. 2.3] that $P_3^\pi_x(\phi \otimes \bar{\phi}) \neq 0$ for all $x \in |X|$. We conclude from
that the period integral $\mathcal{P}_3(\phi, \eta)$ vanishes if and only if $L(\pi_K \otimes \eta, 1/2)$ vanishes. The equivalence (5.17) then follows from the factorization

$$L(\pi_K \otimes \eta, s) = L(\pi \otimes \chi_1, s)L(\pi \otimes \chi_2, s).$$

5.5. The proof of Theorem C

The goal of this section is to prove that

$$\langle [\text{Sht}^\mu_{T_1}], [\text{Sht}^\mu_{T_2}] \rangle = 0$$

when $r > 0$. This will be deduced from the following result.

**Proposition 5.11.** Let $f \in \mathscr{H}$ be the characteristic function of $U_0 \subset G_0(\mathbb{A})$, and recall the function $\mathcal{I}(f, s)$ of (3.7).

(1) If $\text{char}(k) = 2$ then $\mathcal{I}(f, s) = 0$ for all $s \in \mathbb{C}$.

(2) If $\text{char}(k) \neq 2$ then $\mathcal{I}(f, s) = 1$ for all $s \in \mathbb{C}$.

**Proof.** Using Lemma 3.3, we view $f$ as a compactly supported function

$$f : U_0 \setminus J(\mathbb{A})/U_3 \to \mathbb{Q}.$$ 

In other words, $f$ is the characteristic function of the image of

$$\text{Iso}(\mathbb{O}_3, \mathbb{O}_0) \subset \text{Iso}(\mathbb{A}_3, \mathbb{A}_0) = \tilde{J}(\mathbb{A})$$

under $\tilde{J}(\mathbb{A}) \to J(\mathbb{A})$.

**Lemma 5.12.** Fix $\gamma \in J(F)$, and let $\xi \in K_3$ be its image under (3.4). If there exist $t_0 \in T_0(\mathbb{A})$ and $t_3 \in T_3(\mathbb{A})$ such that $f(t_0^{-1}\gamma t_3) \neq 0$, then $\xi \in k$ and $2\xi = 1$.

**Proof.** By hypothesis there is some $\phi \in \tilde{J}(\mathbb{A})$ lifting $t_0^{-1}\gamma t_3 \in J(\mathbb{A})$, and satisfying the integrality condition $\phi(\mathbb{O}_3) = \mathbb{O}_0$. As in the proof of Proposition 2.9 there is a canonical bijection

$$\mathbb{A}_0^\times \setminus \text{Iso}(\mathbb{A}_3, \mathbb{A}_0)/\mathbb{A}_3^\times \cong \text{GL}_2(\mathbb{A}) \setminus \left\{ \begin{array}{l}
\text{pairs of embeddings} \\
\alpha_0 : \mathbb{A}_0 \to M_2(\mathbb{A}) \\
\alpha_3 : \mathbb{A}_3 \to M_2(\mathbb{A})
\end{array} \right\},$$

and the image of $\phi$ under this bijection is represented by a pair of $\mathbb{O}$-algebra embeddings $\alpha_0 : \mathbb{O}_0 \to M_2(\mathbb{O})$ and $\alpha_3 : \mathbb{O}_3 \to M_2(\mathbb{O})$.

As both $K_0$ and $K_3$ are unramified over $F$, the quartic $\mathbb{O}$-algebra

$$R = \mathbb{O}_0 \otimes_{\mathbb{O}} \mathbb{O}_3$$

is self-dual with respect to the bilinear form $(x, y) \mapsto \text{Tr}_{R/\mathbb{O}}(xy)$. If we define an $\mathbb{O}$-linear map $\alpha : R \to M_2(\mathbb{O})$ by $\alpha(x_0 \otimes x_3) = \alpha_0(x_0)\alpha_3(x_3)$, then tracing the construction of the invariant (3.4) all the way back to Proposition 2.9 shows that $\xi \in K_3$ satisfies

$$\text{Trd}_{M_2(\mathbb{O})}(\alpha(x)\alpha(y)^t) = \text{Tr}_{R/\mathbb{O}}(\xi x \overline{y})$$

for all $x, y \in R$. Here $i$ is the main involution on the quaternion order $M_2(\mathbb{O})$, and $y \mapsto \overline{y}$ is the involution on $R$ defined by $x_0 \times x_3 \mapsto x_0^{\sigma_0} \otimes x_3^{\sigma_3}$. 

The left hand side clearly lies in $\mathcal{O}$ for all choices of $x$ and $y$, and hence $\xi \in R$, by the self-duality of $R$ noted above.

Recalling that $K_3 = k(Y_3)$ is the field of rational functions on a projective and geometrically connected curve,

$$\xi \in K_3 \cap R = K_3 \cap \mathcal{O}_3 = k,$$

and the condition $2\xi = 1$ then follows from $\text{Tr}_{K_3/F}(\xi) = 1$. \hfill $\square$

Returning to the main proof, fix a $\gamma \in J(F)$ and recall from \S 3.2 the notation

$$\mathbb{J}(\gamma, f, s) = \int_{T_0(k) \times T_3(k)} f(t_0^{-1} \gamma t_3) |t_0|^{2s} \eta(t_3) \, dt_0 \, dt_3. \quad (5.19)$$

If $(5.19)$ nonzero, the lemma implies that the invariant $\xi = \text{inv}(\gamma) \in K_3$ lies in the field of constants $k$ and satisfies $2\xi = 1$. If $\text{char}(k) = 2$ there is no such $\xi$, and so $(5.19)$ vanishes for all $\gamma \in J(F)$. The first claim of Theorem 5.11 follows from this and the decomposition (3.12).

From now on we assume that $\text{char}(k) > 2$, and let

$$\gamma \in T_0(F) \backslash J(F) / T_3(F)$$

be the unique element with $\text{inv}(\gamma) = 1/2$. Thus, by the discussion above,

$$\mathbb{J}(f, s) = \int_{T_0(k) \times T_3(k)} f(t_0^{-1} \gamma t_3) |t_0|^{2s} \eta(t_3) \, dt_0 \, dt_3. \quad (5.20)$$

Fix an $\epsilon \in K_3^\times$ satisfying $\text{Tr}_{K_3/F}(\epsilon) = 0$, and define an $F$-linear isomorphism $\phi : K_3 \to K_0$ by $\phi(x + ye) = (x, y)$. By carefully unwinding the definition of the invariant (3.4), one can see that $\phi \mapsto \gamma$ under the canonical bijection

$$K_0^\times \backslash \text{Iso}(K_3, K_0) / K_3^\times = T_0(F) \backslash J(F) / T_3(F).$$

**Lemma 5.13.** If we factor $f = \prod_{x \in X} f_x$ and $\eta = \prod_{x \in X} \eta_x$, then

$$\int_{T_0(F_x) \times T_3(F_x)} f_x(t_0^{-1} \gamma t_3) |t_0|^{2s} \eta_x(t_3) \, dt_0 \, dt_3 = |\epsilon|_x^{-2s} \quad (5.21)$$

for every place $x$ of $F$.

**Proof.** As $K_3/F$ is unramified, we may choose $c \in F_x^\times$ so that $ce \in \mathcal{O}_x^\times$. For any such choice we have $\mathcal{O}_{K_3,x} = \mathcal{O}_{F,x} \oplus c \mathcal{O}_{F,x}$, and hence

$$\phi(\mathcal{O}_{K_3,x}) = (1, c) \cdot \mathcal{O}_{K_0,x}.$$

Suppose first that $x$ is inert in $K_3$. The integral over $T_3(F_x)$ can be replaced by a sum over the singleton set $F_x^\times \backslash K_3^\times F_x / \mathcal{O}_{K_3,x}^\times = \{1\}$, while the integral over $T_0(F_x)$ can be replaced by a sum over

$$F_x^\times \backslash (F_x^\times \times F_x^\times) / (\mathcal{O}_{F_x}^\times \times \mathcal{O}_{F_x}^\times) = \{(1, \varpi_k) : k \in \mathbb{Z}\}.$$
for any uniformizer $\varpi \in F_x$. Moreover, $f_x((1, \varpi^{-k}) \cdot \gamma)$ is equal to 1 if the $O_{F_x}$-lattices $\phi(O_{K_{3,x}})$ and $(1, \varpi^k) \cdot O_{K_{0,x}}$ agree up to scaling by $F_x^\times$, and is 0 otherwise. In other words
\[
f_x((1, \varpi^{-k}) \cdot \gamma) = \begin{cases} 1 & \text{if } |\varpi|^k = |c| \\
0 & \text{otherwise}, \end{cases}
\]
and the integral $\text{(5.21)}$ reduces to
\[
\sum_{k \in \mathbb{Z}} |\varpi|^{2ks} f_x((1, \varpi^{-k}) \cdot \gamma) = |c|^{2s} = |\epsilon|^{-2s}.
\]

Now suppose that $x$ is split in $K_3$. In this case we can choose $c$ in such a way that $(e + \varpi^f) \cdot O_{K_{3,x}} = \{e + \varpi^f : \ell \in \mathbb{Z}\}$. Moreover, $f_x((1, \varpi^{-k}) \cdot \gamma \cdot (e + \varpi^f))$ is equal to 1 if the $O_{F_x}$-lattices
\[
\phi((e + \varpi^f) \cdot O_{K_{3,x}}) = \{(x + \varpi^f y, cx - \varpi^f cy) \in K_{0,x} : x, y \in O_{F_x}\}
\]
and $(1, \varpi^k) \cdot O_{K_{0,x}}$ agree up to scaling by $F_x^\times$, and is 0 otherwise. After some elementary linear algebra, this simplifies to
\[
f_x((1, \varpi^{-k}) \cdot \gamma \cdot (e + \varpi^f)) = \begin{cases} 1 & \text{if } |\varpi|^k = |c| \text{ and } \ell = 0 \\
0 & \text{otherwise}, \end{cases}
\]
and the integral $\text{(5.21)}$ again reduces to
\[
\sum_{k, \ell \in \mathbb{Z}} |\varpi|^{2ks} f_x((1, \varpi^{-k}) \cdot \gamma \cdot (e + \varpi^f)) \eta_x(e + \varpi^f) = |c|^{2s} = |\epsilon|^{-2s}.
\]
This proves the lemma.

Combining $\text{(5.20)}$ with the preceding lemma shows that $\mathcal{J}(f, s) = |\epsilon|^{-2s}$, completing the proof of the second claim of Theorem $5.11$. \hfill $\square$

The following is Theorem $\text{C}$ of the introduction.

**Theorem 5.14.** Let $\langle \cdot, \cdot \rangle : \text{Ch}_{c,r}(\text{Sht}_{G_0}) \times \text{Ch}_{c,r}(\text{Sht}_{G_0}) \to \mathbb{Q}$ be the intersection pairing.

(a) If $\text{char}(k) = 2$, then $\langle [\text{Sht}^r_{T_1}], [\text{Sht}^r_{T_2}] \rangle = 0$.
(b) If $\text{char}(k) \neq 2$, then
\[
\langle [\text{Sht}^r_{T_1}], [\text{Sht}^r_{T_2}] \rangle = \begin{cases} 1 & \text{if } r = 0 \\
0 & \text{if } r > 0. \end{cases}
\]
Proof. Let $f \in \mathcal{H}$ be the characteristic function of $U_0$. That is, $f = 1 \in \mathcal{H}$. Proposition 5.11 implies that $\mathcal{J}(f, s) = 1$, and so the $r$th derivative of $\mathcal{J}(f, s)$ vanishes identically for any $r > 0$. Combining this with Proposition 5.5 shows that for $r = 0$, we have $\langle [Sht_{T_1}], [Sht_{T_2}] \rangle = 1$ whereas for $r > 0$, we have

$$\langle [Sht_{T_1}^\mu], [Sht_{T_2}^\mu] \rangle = \mathcal{I}_r(f) = (\log q)^{-r} \frac{d^r}{ds^r} \mathcal{J}(f, s) \bigg|_{s=0} = 0.$$ 

\[ \Box \]

5.6. The proof of Theorem D. Let $\pi$ be an unramified cuspidal automorphic representation of $G_0(\mathbb{A})$, as usual.

The following is Theorem D of the introduction.

**Theorem 5.15.** For any $\phi \in \pi^{U_0}$ as above, we have

$$\left( \int_{T_1(F)\backslash T_1(\mathbb{A})} \phi(t_1) dt_1 \right) \left( \int_{T_2(F)\backslash T_2(\mathbb{A})} \overline{\phi}(t_2) dt_2 \right) = \left( \int_{T_0(F)\backslash T_0(\mathbb{A})} \phi(t_0) dt_0 \right) \left( \int_{T_3(F)\backslash T_3(\mathbb{A})} \overline{\phi}(t_3) \eta(t_3) dt_3 \right).$$

**Proof.** We consider the $r = 0$ case of Theorem A and drop the $r$ from the superscripts, as in the introduction. The stacks $Sht_{G_0}$, $Sht_{T_1}$, and $Sht_{T_2}$ are then disjoint unions of stack quotients of $\text{Spec}(k)$, and the latter two have only finitely many connected components. On $k^\text{alg}$-points, the maps $\theta_i : Sht_{T_i} \to Sht_{G_0}$ fit into the diagram (1.3).

The Chow group of 0-cycles with proper support $\text{Ch}_{c,0}(Sht_{G_0})$ is identified with the space

$$\mathcal{A}_R = \mathcal{C}_c^\infty(G_0(F)\backslash G_0(\mathbb{A})/U_0, \mathbb{R})$$

of compactly supported, $\mathbb{R}$-valued, unramified automorphic forms. Under this identification, the intersection pairing becomes

$$\langle \phi_1, \phi_2 \rangle = \int_{G_0(F)\backslash G_0(\mathbb{A})} \phi_1(g) \phi_2(g) \, dg,$$

where the Haar measure on $G_0(\mathbb{A})$ is normalized so that $\text{vol}(U_0) = 1$.

The class $[Sht_{T_i}]$ is then the push-forward $\theta_{i*}1$ of the constant function 1 on $T_i(F)\backslash T_i(\mathbb{A})/T_i(\mathcal{O})$. It suffices to prove the theorem for any $\phi \in \pi^{U_0}$ which is $\mathbb{R}$-valued. Then the projection of $[Sht_{T_i}]$ onto the $\pi$-isotypic component is

$$[Sht_{T_i}]_\pi = \frac{\langle \theta_{i*}1, \phi \rangle}{\langle \phi, \phi \rangle_\text{Pet}} \phi,$$

and so we compute

$$\langle [Sht_{T_1}]_\pi, [Sht_{T_2}]_\pi \rangle = \frac{1}{\langle \phi, \phi \rangle_\text{Pet}} \left( \int_{[T_1]} \phi(t_1) dt_1 \right) \left( \int_{[T_2]} \overline{\phi}(t_2) dt_2 \right).$$
On the other hand, by Theorem A we have

$$\langle [\text{Sht}_{T_1}], [\text{Sht}_{T_2}] \rangle = \frac{1}{\langle \phi, \phi \rangle_{\text{Pet}}} \left( \int_{[T_0]} \phi(t_0) dt_0 \right) \left( \int_{[T_3]} \overline{\phi(t_3)} \eta(t_3) dt_3 \right).$$

The theorem follows from these two equalities. \hfill \Box

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