IIB matrix model: Extracting the spacetime metric

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Abstract

The large-$N$ master field of the Lorentzian IIB matrix model is of course not known, but we can assume that we already have it and investigate how the emerging spacetime metric could be extracted. We show that, in principle, it is possible to obtain both the Minkowski metric and the spatially flat Robertson–Walker metric.

PACS numbers: 98.80.Bp, 11.25.-w, 11.25.Yb
Keywords: origin and formation of the Universe, strings and branes, M theory

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I. INTRODUCTION

The ten-dimensional IIB matrix model [1, 2] in the Lorentzian version has provided some hints that a classical spacetime may emerge with three “large” spatial dimensions and six “small” spatial dimensions [3, 4]. Still, the conceptual origin of such a classical spacetime has not been addressed satisfactorily.

Recently, it has been suggested that the large-$N$ master field [5] may play a crucial role for the emergence of a classical spacetime manifold from the IIB matrix model (see App. B in the preprint version [6] of Ref. [7]). This suggestion has now been clarified in a triple of follow-up papers.

The first paper [8] gives a general discussion of how the classical spacetime points can be extracted from the bosonic master field of the IIB matrix model and how the corresponding classical spacetime metric is obtained.

The second paper [9] presents an explicit calculation of how the spacetime points appear in the bosonic master field, under the assumption that master field has been calculated exactly or that a reliable approximation of the master field has been found.

The current paper, the third in the series, aims to present a direct calculation of the spacetime metric, under the assumption that the master field (or an approximation of it) is at hand. In fact, we will try to see what would be required from the master field, in order to obtain the Robertson–Walker metric.

All calculations of this paper are analytic and MATHEMATICA 5.0 [10] is used.

II. EMERGENT SPACETIME METRIC

Adapting Eq. (4.16) of Ref. [2] to our master-field approach, we obtained in Ref. [8] the following expression for the emergent inverse metric:

$$g^{\mu\nu}(x) \sim \int d^D y \langle\langle \rho(y) \rangle\rangle (y - x)^\mu (y - x)^\nu f(y - x) r(x, y),$$

(2.1)

with continuous spacetime coordinates $x^\mu$ having the dimension of length and spacetime dimension $D = 10$ for the original model. The meaning of the average $\langle\langle \ldots \rangle\rangle$ will be discussed later. We refer to Refs. [8, 9] for the details of how the discrete spacetime points $\hat{x}_k^\mu$, with index $k \in \{1, \ldots, K\}$, are extracted from the bosonic master field $\hat{A}^\mu$, which corresponds to ten $N \times N$ traceless Hermitian matrices for $N = Kn$, with positive integers
$K$ and $n$. The limit $K \to \infty$ entails the limit $N \to \infty$, as long as $n$ stays constant or increases.

The quantities entering the above integral are, first, the density function

$$\rho(y) \equiv \sum_{k=1}^{\mathcal{K}} \delta^{(10)}(y - \hat{x}_k)$$

(2.2)

for the emergent spacetime points $\hat{x}_k$ as obtained in Refs. [8, 9] and, second, the dimensionless density correlation function $r(x, y)$ which is defined by

$$\langle\langle \rho(x) \rho(y) \rangle\rangle \equiv \langle\langle \rho(x) \rangle\rangle \langle\langle \rho(y) \rangle\rangle r(x, y).$$

(2.3)

In addition, there is a sufficiently localized symmetric function $f(y - x)$, which appears in the effective action of a low-energy scalar degree of freedom $\phi$ “propagating” over the discrete spacetime points $\hat{x}_k$; see Refs. [2, 8] for further details. As this function $f(x) = f(x^0, x^1, \ldots, x^9)$ has the dimension of $1/(\text{length})^2$, the inverse metric $g^{\mu\nu}(x)$ from (2.1) is seen to be dimensionless. The metric $g^{\mu\nu}$ is obtained as the matrix inverse of $g_{\mu\nu}$.

The average $\langle\langle \ldots \rangle\rangle$ entering (2.1) and (2.3) corresponds, for the extraction procedure of the discrete spacetime points $\hat{x}_k$ from Refs. [8, 9], to averaging over different block sizes $n$ and block positions along the diagonal in the master field. But it is not really necessary to do this additional averaging in the integrand of (2.1), as that is already taken care of by the limit $N \to \infty$, with appropriate block dimension $n \gtrsim \Delta N$ for width $\Delta N$ of the band-diagonal master-field matrices. In the following, we will just use $\rho(y)$ in the integrand of (2.1), so that we have, for the emergent inverse metric,

$$g^{\mu\nu}(x) \sim \int d^D y \rho(y) (y - x)^\mu (y - x)^\nu f(y - x) r(x, y).$$

(2.4)

The goal of the current paper is to investigate the integral (2.4) and to determine what would be required of the unknown functions $\rho$, $f$, and $r$ [all three tracing back to the IIB-matrix-model master field], so that the integral gives, in particular, the Robertson–Walker inverse metric.

III. QUESTION AND SETUP

In this article, we address the following concrete question: is it at all possible to get the Minkowski inverse metric and the spatially flat Robertson–Walker inverse metric from the expression (2.4)?
We restrict ourselves to four “large” spacetime dimensions, setting

\[ D = 4 \tag{3.1} \]

in (2.4), and also use length units that normalize the IIB-matrix-model length scale \( \ell \) as introduced in Ref. [8],

\[ \ell = 1. \tag{3.2} \]

Next, define

\[ r(x, y) \equiv \tilde{r}(y - x) \tau(x, y), \tag{3.3a} \]

\[ h(y - x) \equiv f(y - x) \tilde{r}(y - x), \tag{3.3b} \]

where \( \tau(x, y) \) has a more complicated (but still symmetric) dependence on \( x \) and \( y \) than just the combination \( x - y \) [a trivial example would be \( r(x, y) = (x^0)^2 + (y^0)^2 \)].

Now, change the integration variables \( y^\mu \) in the integral (2.4) to \( z^\mu \equiv y^\mu - x^\mu \) and get

\[ g^{\mu\nu}(x) \sim \int_{-\infty}^{\infty} dz^0 \int_{-\infty}^{\infty} dz^1 \int_{-\infty}^{\infty} dz^2 \int_{-\infty}^{\infty} dz^3 \rho(z + x) z^\mu z^\nu h(z) \tau(x, z + x). \tag{3.4} \]

Observe that \( x^\mu \) enters the right-hand side only via the density function \( \rho \) and the correlation function \( \tau \). This observation plays a crucial role for obtaining the constant Minkowski inverse metric, as will become clear in Sec. [IV].

Finally, recall that the ten-dimensional Lorentzian IIB matrix model has coupling constants \( \tilde{\eta}_{KL} \) (with indices \( K, L \in \{0, 1, \ldots, 9\} \)), which give the usual components \( \tilde{\eta}_{\mu\nu} \) for the four-dimensional case:

\[ \tilde{\eta}_{\mu\nu} = \begin{cases} -1, & \text{for } \mu = \nu = 0, \\ +1, & \text{for } \mu = \nu = m \in \{1, 2, 3\}, \\ 0, & \text{otherwise}. \end{cases} \tag{3.5} \]

**IV. MINKOWSKI METRIC**

There are several ways to obtain the Minkowski inverse metric from (3.4). One way has been mentioned in the toy-model calculation of Sec. V in Ref. [8]. But perhaps the simplest
recipe is to take the following Ansatz:

\[ \rho(z + x) = 1, \quad (4.1a) \]
\[ \varphi(x, z + x) = 1, \quad (4.1b) \]
\[ h(z) = \xi \exp \left[ - (z^0)^2 - (z^1)^2 - (z^2)^2 - (z^3)^2 \right] \left( \eta_{00} [\zeta(z^0)^2 - 1] + \eta_{11} [\zeta(z^1)^2 - 1] + \eta_{22} [\zeta(z^2)^2 - 1] + \eta_{33} [\zeta(z^3)^2 - 1] \right), \quad (4.1c) \]

where the exponential function in (4.1c) provides a symmetric cutoff on the integrals of (3.4) and the IIB-matrix-model coupling constants \( \eta_{\mu\nu} \) are given by (3.5). The expression (4.1c) could be simplified somewhat, but it is more instructive to keep the various contributions \([\zeta(z^\mu)^2 - 1]\) separate. In fact, this \( \zeta \) factor in square brackets gives, for the special value \( \zeta = 2 \), the following definite integrals:

\[ I_n \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dz \ z^n \left[ 2 z^2 - 1 \right] e^{-z^2} = \begin{cases} 0, & \text{for } n = 0, 1, \\ 1, & \text{for } n = 2, \end{cases} \quad (4.2) \]

which are an essential ingredient of our Ansatz.

With the resulting expression from (3.4),

\[ g^{\mu\nu}(x) \sim \int_{-\infty}^{\infty} dz^0 \int_{-\infty}^{\infty} dz^1 \int_{-\infty}^{\infty} dz^2 \int_{-\infty}^{\infty} dz^3 \ z^\mu z^\nu h(z), \quad (4.3) \]

in terms of the \( h \) function (4.1c), and the particular numerical values

\[ \zeta = 2, \quad (4.4a) \]
\[ \xi = 1/\pi^2, \quad (4.4b) \]

we get

\[ g_{\text{Mink}}^{\mu\nu}(x) \sim \begin{cases} -1, & \text{for } \mu = \nu = 0, \\ +1, & \text{for } \mu = \nu = m \in \{1, 2, 3\}, \\ 0, & \text{otherwise}. \end{cases} \quad (4.5) \]

which is the standard inverse metric of Minkowski spacetime with Cartesian coordinates. Observe that this inverse metric is independent of \( x^\mu \), because \( x^\mu \) has disappeared from the integral on the right-hand side of (4.3). Essentially the same observation has been made in the sentence below Eq. (4.17) of Ref. 2.
Note, finally, that we get the Minkowski metric by taking the matrix inverse of (4.5). The resulting covariant tensor \( g_{\mu\nu}^{(\text{Mink})}(x) \) has the same components as the contravariant tensor (4.5).

V. ROBERTSON–WALKER METRIC

Consider the spacetime points with coordinates

\[
x^\mu = (x^0, x^1, x^2, x^3),
\]

\[
x^0 = \tilde{c}t = t,
\]

where \( t \) is interpreted as the cosmic-time coordinate and \( \tilde{c} \) is set to unity by an appropriate choice of the time unit. We will try to get the spatially flat Robertson–Walker (RW) inverse metric by choosing appropriate Ansatz parameters \( r_n \) (see below for their definition), so that

\[
g^{00}_{\text{RW}}(x) \sim -1,
\]

\[
g^{mm}_{\text{RW}}(x) \sim +1 + c_2 t^2 + c_4 t^4 + \ldots,
\]

for \( m \in \{1, 2, 3\} \) and constants \( c_2 \) and \( c_4 \), and with all off-diagonal components vanishing.

We can obtain this result by making one change in the previous calculation of Sec. IV, namely, by letting the Ansatz density function \( \rho(y) \) be a nontrivial even function of the time-component of the coordinate \( y^\mu = x^\mu + z^\mu \):

\[
\rho(y) = \rho(y^0) \neq 1.
\]

Specifically, we take an even polynomial (distinguished by an overbar) of order \( 2K_0 \),

\[
\overline{\rho}(y^0) = \sum_{k=0}^{K_0} r_{2k} (y^0)^{2k},
\]

with arbitrary constants \( r_{2k} \). The resulting expression from (3.4) reads

\[
g^{\mu\nu}(x) \sim \int_{-\infty}^\infty dz^0 \int_{-\infty}^\infty dz^1 \int_{-\infty}^\infty dz^2 \int_{-\infty}^\infty dz^3 \overline{\rho}(z + x) \ z^\mu z^\nu h(z) \ T(x, z + x),
\]

with the \( h \) function (1.1c) for numerical values (4.4) and, for the moment, \( T(x, z + x) = 1 \). Observe that, with \( r_0 = 1 \) and \( r_{2k} = 0 \) for \( k \geq 1 \), we recover the integral (4.3), from which the Minkowski inverse metric (4.5) was derived.
Fixing, for definiteness, the \( \mathcal{P} \) polynomial (5.4) to be tenth order, we obtain from (5.5) for \( \mathcal{T} = 1 \), with an appropriate choice of Ansatz parameters,

\[
\begin{align*}
    r_0 &= 1, \\
    r_2 &= \frac{7371}{64} r_{10}, \\
    r_4 &= -\frac{4599}{16} r_{10}, \\
    r_6 &= \frac{11151}{80} r_{10}, \\
    r_8 &= -\frac{108}{5} r_{10},
\end{align*}
\]

the following inverse metric:

\[
g^{\mu\nu}(x) \sim \begin{cases} 
    -1 + O(t^6), & \text{for } \mu = \nu = 0, \\
    +1 + c_2 t^2 + O(t^4), & \text{for } \mu = \nu = m \in \{1, 2, 3\}, \\
    0, & \text{otherwise}.
\end{cases}
\]

\[
c_2 = -\frac{567}{8} r_{10}.
\]

In this way, we can get any Taylor coefficient \( C_2 \) (denoted by an upper-case symbol) for the space-space components of the RW inverse metric (5.7) by choosing an appropriate input value for \( r_{10} \). Three technical remarks are in order:

1. The off-diagonal components of (5.7) vanish because of an extra spatial factor \( z^m \) in the integrand of (5.5) for \( \mathcal{T} = 1 \), where the rest of the integral is symmetric in \( z^m \).

2. We can focus on the \( 2 \times 2 \) block for \( \mu, \nu \in \{0, 1\} \), as the components \( g^{22} \) and \( g^{33} \) both equal \( g^{11} \) and all off-diagonal components vanish, \( g^{\mu\nu} \sim 0 \) for \( \mu \neq \nu \).

3. The inverse metric (5.7) is \( x^m \)-independent because \( \mathcal{P}(z + x) \) in the integrand of (5.5), for \( \mathcal{T} = 1 \), only depends on \( x^0 \), according to the Ansatz (5.4).

We can obtain the Taylor coefficients beyond \( c_2 \) by going to higher orders in the \( \mathcal{P} \) polynomial. For example, by going to twelfth order, we get, with appropriate values of the Ansatz parameters \( \{r_0, r_2, r_4, r_6, r_8\} \), the Taylor coefficients \( c_2 = c_2(r_{10}, r_{12}) \) and \( c_4 = c_4(r_{10}, r_{12}) \). However, the mapping \( (r_{10}, r_{12}) \rightarrow (c_2, c_4) \) is noninvertible and it is not possible to get arbitrary values of \( (c_2, c_4) \) from appropriate values of \( (r_{10}, r_{12}) \). In order to be able to obtain arbitrary values of \( (c_2, c_4) \) from appropriate parameters, our Ansatz needs to be extended.
In fact, return to the tenth-order polynomial \( \overline{\rho} \), but now take a nontrivial Ansatz for \( \overline{r} \):

\[
\overline{r}(x, z + x) = 1 + s_4 (x^0)^4 (z^0 + x^0)^4,
\] (5.8)

with an arbitrary constant \( s_4 \). Inserting the Ansatz (5.8) in the integrand of (5.5), we obtain, with \( \overline{\rho} \) parameters

\[
\begin{align*}
r_0 &= 1, \\
r_2 &= \frac{45 (152 s_4 + 63 r_{10} [416 + 563 s_4])}{16 (640 + 777 s_4)}, \\
r_4 &= -\frac{45 (44 s_4 + 7 r_{10} [1168 + 1569 s_4])}{2 (640 + 777 s_4)}, \\
r_6 &= \frac{9 (76 s_4 + 21 r_{10} [944 + 1233 s_4])}{2 (640 + 777 s_4)}, \\
r_8 &= -\frac{6 (32 s_4 + 63 r_{10} [256 + 323 s_4])}{7 (640 + 777 s_4)},
\end{align*}
\] (5.9a)(5.9b)(5.9c)(5.9d)(5.9e)

the following inverse metric:

\[
g^{\mu\nu}(x) \sim \begin{cases} 
-1 + O(t^6), & \text{for } \mu = \nu = 0, \\
+1 + c_2 t^2 + c_4 t^4 + O(t^6), & \text{for } \mu = \nu = m \in \{1, 2, 3\}, \\
0, & \text{otherwise},
\end{cases}
\] (5.10a)

\[
c_2 = -\frac{540 (42 r_{10} [2 + 3 s_4] + s_4)}{640 + 777 s_4},
\] (5.10b)

\[
c_4 = \frac{(945 r_{10} [80 + 78 s_4 - 63 s_4^2] - 6 s_4 [10 + 273 s_4])}{2 (640 + 777 s_4)}. 
\] (5.10c)

Now, we can get arbitrary Taylor coefficients \( C_2 \) and \( C_4 \) (denoted by upper-case symbols) for the space-space components of the RW inverse metric (5.10) by choosing appropriate input values for \( r_{10} \) and \( s_4 \). Specifically, we obtain by inverting (5.10b) and (5.10c):

\[
r_{10}^{(\text{input})} = \frac{1855 C_2^2 - C_2 (60 - 1554 C_4) + 1080 C_4}{17010 (4 - 9 C_2 - 8 C_4)}, \\
s_4^{(\text{input})} = -\frac{8 (5 C_2 + 6 C_4)}{36 - 21 C_2}.
\] (5.11a)(5.11b)
Note, finally, that we obtain the spatially flat Robertson–Walker metric by taking the matrix inverse of \((5.2)\). In this way, we get the following covariant tensor:

\[
g^{(RW)}_{\mu\nu}(x) \sim \begin{cases} 
-1, & \text{for } \mu = \nu = 0, \\
1/(1 + c_2 t^2 + c_4 t^4 + \ldots), & \text{for } \mu = \nu = m \in \{1, 2, 3\}, \\
0, & \text{otherwise}, 
\end{cases} 
\]

with, for example, explicit coefficients \(c_2\) and \(c_4\) from \((5.10b)\) and \((5.10c)\).

VI. DISCUSSION

In the present article, we have considered in some detail how a Robertson–Walker spacetime metric could arise from the IIB-matrix-model master field, assuming that the master field (or a reliable approximation of it) is known. The relevant expression \([2, 8]\) for the emergent spacetime metric is given by a ten-dimensional integral \((2.4)\), with functions \(\rho\), \(f\), and \(r\) that follow from the emerged discrete spacetime points \(\hat{x}_k^\mu\).

The particular construction we have performed starts from the Minkowski metric obtained in Sec. IV where we now understand how a constant \((x^\mu\)-independent) inverse metric may come about. By a deformation, we have then found, in Sec. VI the spatially flat \((k = 0)\) Robertson–Walker metric. This immediately raises two questions.

First, is it possible to obtain, in the same way, a Robertson–Walker metric with positive \((k = +1)\) or negative \((k = -1)\) spatial curvature? *A priori*, we would expect that this is impossible. Yet, recall that, for example, the \(k = 1\) Robertson–Walker metric may not really have an underlying \(\mathbb{R} \times S^3\) topology but can be described by strong gravitational fields over Minkowski spacetime with \(\mathbb{R}^4\) topology (see Ref. [11] and references therein).

Second, is it possible to modify the construction of Sec. VI in order to obtain the regularized-big-bang metric \([12]\)? This question is not quite trivial, as the regularized-big-bang inverse-metric component \(g^{00}\) diverges at cosmic-time coordinate \(t = 0\), where a spacetime defect has replaced the big bang singularity. Perhaps it is possible to modify the functions entering the integrand of \((5.5)\) in such a way that the effective cutoff on the integrals for \(\mu = \nu = 0\) disappears at \(x^0 = t = 0\), making the inverse-metric component \(g^{00}\) diverge at that time slice. We hope to address this issue in a future publication.
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