Closing in on Time and Space Optimal Construction of Compressed Indexes

Dominik Kempa

Department of Computer Science,
University of Helsinki, Finland
dkempa@cs.helsinki.fi

Abstract. The invention of the FM-index [FOCS 2000] and the compressed suffix array [STOC 2000] have revolutionized the field of string algorithms for nearly two decades. These indexes are now mature and widespread, both in theory where they provide an off-the-shelf small space indexing structures, and in practice (particularly bioinformatics) where they achieved massive success. Their time-and-space optimal construction, however, still remains an open problem. The currently fastest algorithms, due to Belazzougui [STOC 2014] and Munro et al. [SODA 2017], operate in \(O(n)\) time and optimal space of \(O(n \log \sigma)\) bits, where \(n\) is the text length and \(\sigma\) is the alphabet size. Thus, they are optimal only for large alphabets \(\sigma = \Omega(n^\epsilon)\). In this paper we propose algorithms constructing the Burrows-Wheeler transform (BWT), the permuted longest-common-prefix (PLCP) array, and the LZ77 parsing of the input text in \(O(n/\log \sigma n + r \text{polylog } n)\) time and space, where \(r\) is the number of runs in the Burrows-Wheeler of the input text. These are the essential components of nearly every index developed in the last two decades (in particular, the CSA and the FM-index, but also grammar and LZ77-based indexes). Thus, we obtain a time-and-space optimal construction algorithms for inputs satisfying a weak assumption \(n/r \in \Omega(\text{polylog } n)\) on the repetitiveness. Our result has a particularly important implications for bioinformatics, where most of the data is highly-repetitive and over small (DNA) alphabet. Furthermore, our techniques imply a time-and-space optimal solutions on highly repetitive data for a range of fundamental problems arising in sequence analysis and data compression such as: Lyndon factorization, construction of run-length compressed suffix arrays, and a number of simple “textbook” problems such as computing the longest substring occurring at least some fixed number of times.

1 Introduction

The invention of FM-index [18,17] and the compressed suffix array [26,27] at the turn of the millennium have revolutionized the field of string algorithms for nearly two decades. For the first time the algorithms could use the full potential of powerful indexing structures in space asymptotically equal to that needed by the text, \(O(n \log \sigma)\) bits or \(O(n/\log \sigma n)\) words, where \(n\) is the text length and \(\sigma\) is its alphabet size. Dozens of papers followed the two seminal papers, proposing various improvements, generalizations, and practical implementations (see [51,52,8] for excellent surveys). These indexes are now widespread, both in theory where they provide an “off-the-shelf” small space indexing structures, and in practice (particularly bioinformatics) where they achieved massive success: to date, the FM-index-based read aligner “Bowtie” [46] has been cited over ten thousand times.

The other approach to indexing, recently gaining popularity due to quick increase in the amount of highly repetitive data such as software repositories and genomic databases (see [24] for the recent survey), is designing indexes specialized for repetitive strings. The first such index [39] was based on the Lempel-Ziv (LZ77) parsing [59], one of the most popular compression algorithms (used, e.g., in gzip or 7-zip) also with numerous applications outside of data compression such as detecting regularities [15,28,44], approximating the smallest grammar [55,11], or the smallest string attractor [41]. Many improvements to the basic scheme were proposed since then [22,9,6,21,3,2,1], and now the performance of LZ-based indexes is often on par with the FM-index or CSA [16]. In recent years, other compression paradigms have crossed over into the field of indexing. Most notably, Mäkinen et al. [47] have shown that applying run-length compression to FM-index and CSA yields excellent practical performance (see also [58]). The most recent addition to this family
is the version of run-length compressed CSA (RLCSA) of Gagie et al. [24] who showed how to effectively store the samples necessary to report occurrences during pattern matching queries, and allowed them to achieve optimal pattern matching and substring extraction query times within $O(r \log \log n)$ or even $O(r)$ space, where $r$ is the number of runs in the Burrows-Wheeler transform (BWT) [10] of the text. The value of $r$ is (next to $z$, the size of the LZ77 parsing of the text) a common measure of the repetitiveness of the text. In theory it allows achieving polynomial or even exponential compression ratio, and in practice it compresses similarly to LZ77 [24]. Furthermore, Mäkinen [47] proved that in the model where several identical copies of a given text are subject to point mutations (such as indels), the value of $r$ is provably small under probabilistic assumptions matching the biological processes in the genome.

Although the literature on indexes is very rich, comparatively few results exists for efficient index construction. A gradual improvement [45,29,30] in the construction of compressed suffix arrays/trees culminated with the recent work of Belazzougui [5] who described the (randomized) linear-time construction with optimal working space of $O(n/\log \sigma n)$. Shortly after, an alternative (and deterministic) construction was proposed by Munro at al. [50]. This solves the problem for large alphabets (i.e., when $\sigma = \Omega(n')$) but leaves open the existence of the algorithm running in $O(n/\log \sigma n)$ time, which can be up to $\Theta(\log n)$ smaller for small alphabets.

Our contribution. In this paper we propose the first algorithms constructing the Burrows-Wheeler transform (BWT), the permuted longest-common-prefix (PLCP) array, and the LZ77 parsing, in optimal time and space for highly repetitive strings. More precisely, all our construction algorithms achieve runtime and construction space of $O(n/\log \sigma n + r \log \log n)$, where $r$ is the number of runs in the BWT of the input text.

Virtually every index developed during the last 20 years is based on one or more of these components: all variants of (run-length compressed) FM-index rely on BWT, (run-length)compressed suffix arrays/trees rely on $\Psi$ (which is easily derived from BWT). These indexes also often include the PLCP array [57]. On the other hand, LZ77-based and grammar-based indexes rely on the computation of LZ77 parsing.

Our results have particularly important implications for bioinformatics, where most of the data is highly-repetitive and over small (DNA) alphabet. Furthermore, our results and techniques imply a time- and space-optimal solution for a range of fundamental problems arising in sequence analysis and data compression:

- We show how to compute the Lyndon factorization [12] (also known as Standard factorization), a fundamental tool in string combinatorics [4]. More precisely, we for this problem we obtain an algorithm running in $O(n/\log \sigma n + r \log^9 n + z \log^9 n)$ time and $O(n/\log \sigma n + r \log^8 n + z \log^2 n)$ space (this includes the output factorization). Since $z = O(r \log n)$ [20], the above is indeed of the form $O(n/\log \sigma n + r \log \log n)$.

- We then show how to construct the RLCSA of Gagie et al. [24]. More precisely we demonstrate how to construct a data structure that grants a $O(\ell + \log \tau(n/r))$-time random-access to (any a segment of $\ell$ consecutive values of) suffix array in $O(n/\log \sigma n + r(\tau \log^6 n + \log^7 n))$ time and $O(n/\log \sigma n + r(\tau \log \log n/r + \log^5 \tau))$ working space. We also show that our construction algorithm can be generalized to other arrays (e.g., ISA, LCP, PLCP, T).

- Lastly, we show a time and space-optimal solutions to a number of “textbook” problems. As two typical problems we consider computing the longest substring occurring at least $k$ times, and the problem of computing the number of distinct substrings of the input text. For example, for the latter we obtain an algorithm running in $O(n/\log \sigma n + r \log^{11} n)$ time and $O(n/\log \sigma n + r \log^{10} n)$ working space.

Note that all the algorithms are time- and space-optimal (i.e., they achieve $O(n/\log \sigma n)$ running time and space usage) under the assumption $n/r \in \Omega(\log \log n)$ on the input text. Since $r$ can be polynomially or even exponentially smaller than $n$, our assumption on the required (polylogarithmic) compression ratio of
the input text is very weak. Furthermore, we remark that for the sake of presentation, we have not optimized the exponents in the log-terms.

2 Preliminaries

We assume a word-RAM model with a word of \( w = \Theta(\log n) \) bits and with all usual arithmetic and logic operations taking constant time. Unless explicitly specified otherwise, all space complexities are in given in words. All our algorithms are deterministic.

Throughout we consider a string \( T[1..n] \) of \( |T| = n \) symbols drawn from an integer alphabet \([1..\sigma]\) of size \( \sigma = O(2^w) \). For \( i \in [1..n] \), we write \( T[i..n] \) to denote the suffix of \( T \), that is \( T[i..n] = T[i]T[i + 1] \ldots T[n] \). We will often refer to suffix \( T[i..n] \) simply as “suffix \( i \)”. We define the rotation of \( T \) as a string \( T[i..n]T[1..i - 1] \), for some \( i \in [1..n] \).

The suffix array \([48,25]\) \( SA \) of \( T \) is an array \( SA[1..n] \) which contains a permutation of the integers \([1..n]\) such that \( T[SA[1..n]] \prec T[SA[2..n]] \prec \cdots \prec T[SA[n]..n] \), where \( \prec \) denotes the lexicographical order. In other words, \( SA[j] = i \) if \( T[i..n] \) is the \( j \)th suffix of \( T \) in ascending lexicographical order. The inverse suffix array \( ISA \) is the inverse permutation of \( SA \), that is \( ISA[i] = j \) if \( SA[j] = i \). Conceptually, \( ISA[i] \) tells us the position of suffix \( i \) in \( SA \). The array \( \Phi[1..n] \) (see \([34]\)) is defined by \( \Phi[i] = SA[ISA[i] - 1] \), that is, the suffix \( \Phi[i] \) is the immediate lexicographical predecessor of the suffix \( i \).

Let \( lcp(i, j) \) denote the length of the longest-common-prefix (LCP) of suffix \( i \) and suffix \( j \). The longest-common-prefix array \([48,40]\) \( \text{LCP}[1..n] \), is defined such that \( \text{LCP}[i] = \text{lcp}(SA[i], SA[i - 1]) \) for \( i \in [2..n] \) and \( \text{LCP}[1] = 0 \). The permuted LCP array \([34]\) \( \text{PLCP}[1..n] \) is the LCP array permuted from the lexicographical order into the text order, i.e., \( \text{PLCP}[SA[j]] = \text{LCP}[j] \) for \( j \in [1..n] \). Then \( \text{PLCP}[i] = \text{lcp}(i, \Phi[i]) \) for all \( i \in [1..n] \).

The succinct PLCP array \([56,34]\) \( \text{PLCP}_{\text{suc}}[1..2n] \) represents the PLCP array using \( 2n \) bits. Specifically, \( \text{PLCP}_{\text{suc}}[j] = 1 \) if \( j = 2i + \text{PLCP}[i] \) for some \( i \in [1..n] \), and \( \text{PLCP}_{\text{suc}}[j] = 0 \) otherwise. Notice that the value \( 2i + \text{PLCP}[i] \) is unique for each \( i \). Any lcp value can be recovered by the equation \( \text{PLCP}[i] = \text{select} (\text{PLCP}_{\text{suc}}, i) - 2i \), where \( \text{select}(\text{PLCP}_{\text{suc}}, i) \) returns the location of the \( i \)th 1-bit in \( \text{PLCP}_{\text{suc}} \). The select query can be answered in \( O(1) \) time given a precomputed data structure of \( o(n) \) bits \([13,49]\).

The Burrows–Wheeler transform \([10]\) \( \text{BWT}[1..n] \) of \( T \) is defined by \( \text{BWT}[i] = T[SA[i] - 1] \) if \( SA[i] > 1 \) and otherwise \( \text{BWT}[i] = $ \), where \$ is a special symbol that does not appear elsewhere in the text, smaller than all other symbols in the text. Let \( \mathcal{M} \) denote the \( n \times n \) matrix, whose rows are all the rotations of \( T \) in lexicographical order. We denote the rows by \( \mathcal{M}[i], i \in [1..n] \). The alternative and equivalent definition of \( \text{BWT} \) is to define \( \text{BWT} \) as the last column of \( \mathcal{M} \).

We define \( \text{LF}[i] = j \) iff \( SA[j] = SA[i] - 1 \), except when \( SA[i] = 1 \), in which case \( \text{LF}[i] = ISA[n] \). By \( \Psi \) we denote the inverse of \( \text{LF} \). Let \( C[c] \), for symbol \( c \), be the number of symbols in \( T \) lexicographically smaller than \( c \). The function rank\((T, c, i)\), for string \( T \), symbol \( c \), and integer \( i \), returns the number of occurrences of \( c \) in \( T[1..i - 1] \). Whenever clear from the context we will omit the string from the list of arguments to the rank function. It is well known that \( \text{LF}[i] = C[\text{BWT}[i]] + \text{rank}(\text{BWT}, \text{BWT}[i], i) \). From the formula for \( \text{LF} \) we obtain the following fact.

Lemma 1. Assume that \( \text{BWT}[b..e] \) is a run of the same symbol and let \( i, j \in [b, e] \). Then, \( \text{LF}[j] = \text{LF}[i] + (j - i) \).

If \( k \) is rank (i.e., the number of smaller suffixes) of string \( P \) among all suffixes of \( T \), then \( C[c] + \text{rank}(\text{BWT}, c, k) \) is the rank of string \( cP \) among the suffixes of \( T \). This is called backward search \([17]\).

We say that an lcp value \( \text{LCP}[i] = \text{PLCP}[SA[i]] \) is reducible if \( \text{BWT}[i] = \text{BWT}[i - 1] \) and irreducible otherwise. The significance of reducibility is summarized in the following two lemmas.
Lemma 2 ([34]). If PLCP[i] is reducible, then PLCP[i] = PLCP[i] - 1 and Φ[i] = Φ[i] - 1 + 1.

Lemma 3 ([34][33]). The sum of all irreducible lcp values is ≤ n log n.

By r we denote the number of equal-letter runs in BWT. By the above, r is also the number of irreducible lcp values. It is well known that for highly repetitive string, the value of r is notably smaller than n.

3 Augmenting RLBWT

In this section we present a number of extensions of run-length compressed BWT that we will use later. Each extension expands the functionality of RLBWT while maintaining small space usage and, crucially, admits a low construction time and working space.

3.1 Rank and select support

One of the basic operations we will need during construction are rank and select queries on BWT. We now show that a run-length compressed BWT (potentially over large alphabet) can be quickly augmented with a data structure capable of answering these queries in BWT-runs space.

Theorem 1. Let T[1..n] be a string over alphabet of size σ = O(2^w) such that BWT of T contains r runs. Given RLBWT of T we can add O(r) space so that, given any i ∈ [1..n] and any c ∈ [1..σ], values rank(c, i) and select(c, i) on BWT can each be computed in O(log r) time. The data structure can be constructed in O(r log r) time using O(r) working space.

Proof. Assume that the input RLBWT contains the starting position and the symbol associated with each run. We start by augmenting each run with its length. We then sort all BWT-runs using the head-symbol as the primary key, and the start of the run in the BWT as the secondary key. We then augment each run [b..e] of symbol c with the value rank(c, b), i.e., the total length of all runs of c preceding [b..e] in the sorted order. Using the list, both queries can be easily answered in O(log r) time using binary search. The list and all the auxiliary information can be computed in O(r log r) time and O(r) extra space. □

3.2 LF/Ψ and backward search support

We now show that with the help of the above rank/select data structures we can support more complicated navigational queries, namely, given any i ∈ [1..n] such that SA[i] = j we can compute ISA[j - 1] (i.e., LF[i]) and ISA[j + 1] (i.e., Ψ[i]). Note that none of the queries will require the knowledge of j. As a consequence of the previous section we also obtain an efficient support for backwards-search on RLBWT.

Theorem 2. Let T[1..n] be a string over alphabet of size σ = O(2^w) such that BWT of T contains r runs. Given RLBWT of T we can add O(r) space so that, given any i ∈ [1..n], values LF[i] and Ψ[i] can each be computed in O(log r) time. The data structure can be constructed in O(r log r) time using O(r) working space.

Proof. To support LF-queries it suffices to sample the LF-value for the first position of each run. Then to compute LF[i] we first binary-search in O(log r) time the list of BWT-runs for the run containing position i and then return the answer using Lemma[1]. To compute the samples we first create a sorted list of characters occurring in T by sorting the head-symbols of all runs and removing duplicates. We then perform two scans of the runs, first to compute the frequency of all symbols (on which we then perform the prefix sum), and
second in which we compute LF-values for run-heads. The total time spent is $\mathcal{O}(r \log r)$ and the working space does not exceed $\mathcal{O}(r)$ words.

To support $\Psi$-queries we first need a list containing, for each symbol $c$ occurring in $T$, the total frequency of all symbols smaller than $c$. This list can be computed as above in $\mathcal{O}(r \log r)$ time. At query time it allows determining in $\mathcal{O}(\log r)$ time the symbol $c$ following BWT$[i]$ in text and the number $k$ such that this $c$ is the $k$-th occurrence of $c$ in the first column of BWT-matrix. To compute $\Psi[i]$ it then remains to find $k$-th occurrence of $c$ in BWT. Thus, applying Theorem 1 finalizes the construction. \hfill $\square$

**Theorem 3.** Let $T[1..n]$ be a string over alphabet of size $\sigma = \mathcal{O}(2^w)$ such that BWT of $T$ contains $r$ runs. Given RLBWT of $T$ we can add $\mathcal{O}(r)$ space so that, given a rank $i \in [1..n]$ (i.e., a number of smaller suffixes) of some string $P$ among the suffixes of $T$, for any $c \in [0..\sigma]$ we can compute in $\mathcal{O}(\log r)$ time the rank of string $cP$. The data structure can be constructed in $\mathcal{O}(r \log r)$ time using $\mathcal{O}(r)$ working space.

**Proof.** First we augment RLBWT with rank-support using Theorem 1. It then remains to build a data structure that allows to determine, for any $c \in [1..\sigma]$ the total number of occurrences of symbols $c' < c$ in $T$. To achieve this we construct a sorted list (similar as in Theorem 2) of symbols occurring in $T$, together with their counts, and perform a prefix-sum on the counts. The construction altogether takes $\mathcal{O}(r \log r)$ time and $\mathcal{O}(r)$ extra space, and enables performing a step of the backward search in $\mathcal{O}(\log r)$ time. \hfill $\square$

### 3.3 Suffix-rank support

In this subsection we describe a non-trivial extension of RLBWT that will allow us to efficiently merge two RLBWTs during the BWT construction algorithm. We start by defining a generalization of BWT-runs and stating their basic properties.

Let $\text{lcs}(x, y)$ denote the length of the longest common suffix of strings $x$ and $y$. We define the LCS$[1..n]$ array \footnote{LCS$_{\sigma} = \mathcal{LCS}(\mathcal{M}[i], \mathcal{M}[i - 1])$ for $i \in [2..n]$ and LCS$[1] = 0$, where $\mathcal{M}$ is the BWT-matrix (i.e., a matrix containing sorted rotations of string $T$).} as LCS$[i] = \text{lcs}(\mathcal{M}[i], \mathcal{M}[i - 1])$ for $i \in [2..n]$ and LCS$[1] = 0$, where $\mathcal{M}$ is the BWT-matrix (i.e., a matrix containing sorted rotations of string $T$). Let $\tau \geq 1$ be an integer. We say that a range $[b..e]$ of BWT is a $\tau$-run if LCS$[b] < \tau$, LCS$[e + 1] < \tau$, and for any $j \in [b + 1..e]$, LCS$[j] \geq \tau$. By this definition, a BWT run is a 1-run. Let $Q_\tau = \{i \in [1..n] \ | \ \text{LCS}[i] = \tau\}$ and $R_\tau = \bigcup_{j=0}^{\tau-1} Q_j$. Then, $R_\tau$ is exactly the set of starting positions of $\tau$-runs.

**Lemma 4 (\cite{37}).** For any $i \in [1..n]$, 

$$\text{LCS}[i] = \begin{cases} 0 & \text{if } i = 1 \text{ or } \text{BWT}[i] \neq \text{BWT}[i - 1] \\ \text{LCS}[\text{LF}[i]] + 1 & \text{otherwise} \end{cases}$$

Since $\Psi$ is the inverse of LF we obtain that for any $\tau \geq 1$, $Q_\tau = \{\Psi[i] \ | \ i \in Q_{\tau-1} \text{ and } \Psi[i] \notin Q_0\}$. Thus, the set $R_\tau$ can be efficiently computed by iterating each of the starting positions of BWT-runs $\tau - 1$ times using $\Psi$ and taking a union of all visited positions. From the above we see that $|Q_{i+1}| \leq |Q_i|$, which implies that the number of $\tau$-runs satisfies $|R_\tau| \leq |Q_0|\tau = r\tau$.

**Theorem 4.** Let $S[1..m]$, $S'[1..m']$ be two strings over alphabet $[1..\sigma]$ for $\sigma = \mathcal{O}(2^w)$ such that BWT of $S$ (resp. $S'$) contains $r$ (resp. $r'$) runs. Given RLBWTs or $S$ and $S'$ it is possible, for any parameter $\tau \geq 1$, to build a data structure of size $\mathcal{O}(\log r + r + r')$ that can, given a rank (i.e., the number of smaller suffixes) $i \in [1..m]$ of some suffix $S[j..m]$ among suffixes of $S$, compute the rank of $S[j..m]$ among suffixes of $S'$ in $\mathcal{O}\left(\tau(\log \frac{m}{r} + \log r + \log r')\right)$ time. The data structure can be constructed in $\mathcal{O}\left(\tau^2(r + r')\log(r\tau + r'\tau) + \frac{m}{r}(\log(r\tau) + \log(r'\tau) + \log \frac{m}{r})\right)$ time and $\mathcal{O}\left(\tau^2(r + r') + \frac{m}{r}\right)$ working space.
Proof. We start by augmenting both RLBWTs with \( \Psi \) and LF support (Theorem 2). We furthermore augment RLBWT of \( S' \) with the backward-search support (Theorem 3). This requires \( \mathcal{O}(r \log r + r' \log r') \) time and \( \mathcal{O}(r) \) space.

We then compute a (sorted) set of starting positions of \( \tau \)-runs for both RLBWTs. For \( S \) this requires answering \( \tau T \Psi \)-queries which takes \( \mathcal{O}(\tau r \log r) \) time in total, and then sorting the resulting set of positions in \( \mathcal{O}(\tau r \log(\tau r)) \) time. Analogous processing for \( S' \) takes \( \mathcal{O}(\tau (r' \tau) \log(\tau r')) \) time. The starting positions of all \( \tau \)-runs require \( \mathcal{O}((r + r') \tau) \) space in total.

Next, for any \( \tau \)-run \([b..e]\) we compute and store the associated \( \tau \) symbols. Furthermore we also store the value \( \text{LF}^T[b] \), but only for \( \tau \)-runs of \( S \). Due to simple generalization of Lemma 1 this will allow us to compute the value \( \text{LF}^T[i] \) for any \( i \). In total this requires answering \( (r + r') \tau \) LF-queries and hence takes \( \mathcal{O}(\tau (r + r') \log r) \) time. The space needed to store all symbols is \( \mathcal{O}(\tau^2 (r + r')) \).

We then lexicographically sort all length-\( \tau \) strings associated with \( \tau \)-runs (henceforth called \( \tau \)-substrings) and assign to each run the rank of the associated substring in the sorted order. Importantly, \( \tau \)-substrings of \( S \) and \( S' \) are sorted together. These ranks will be used as order-preserving names for \( \tau \)-substrings. We use an LSD-radix sort with a stable comparison-based sort for each digit hence the sorting takes \( \mathcal{O}\left(\tau^2 (r + r') \log(\tau + r' \tau)\right) \) time. The working space does not exceed \( \mathcal{O}(\tau (r + r')) \). After the names are computed, we discard the substrings.

We now observe that order-preserving names for \( \tau \)-substrings allow us to perform backward-search \( \tau \) symbols at a time. We build a rank-support data structure analogous to the one from Theorem 1 for names of \( \tau \)-substrings of \( S' \). We also add support for computing the total number of occurrences of names smaller than a given name. This takes \( \mathcal{O}(r' \tau \log(r' \tau)) \) time and \( \mathcal{O}(\tau r') \) space. Then, given a rank \( i \) of suffix \( S[j..m] \) among suffixes of \( S' \), we can compute the rank of suffix \( S[j - \tau..m] \) among suffixes of \( S' \) in \( \mathcal{O}(\log(r' \tau)) \) time by backwards-search on \( S' \) using \( i \) as a position, and the name of \( \tau \)-substring preceding \( S[j..m] \) as a symbol.

We now use the above multi-symbol backward search to compute the rank of every suffix of the form \( S[m - k \tau..m] \) among suffixes of \( S' \). We start from the shortest suffix and increase the length by \( \tau \) in every step. During the computation we also maintain the rank of the current suffix of \( S \) among suffixes of \( S \). This allows us to efficiently compute the name of the preceding \( \tau \)-substring. The rank can be updated using values \( \text{LF}^T \) stored with each \( \tau \)-run of \( S \). Thus, for each of the \( m/\tau \) suffixes of \( S \) we obtain a pair of integers \((i_S, i_{S'})\), denoting its rank among the suffixes of \( S \) and \( S' \). We store these pairs as a list sorted by \( i_S \). Computing the list takes \( \mathcal{O}\left( \frac{m}{\tau} (\log(r \tau) + \log(r' \tau)) + \frac{m}{\tau} \log \frac{m}{\tau} \right) \) time. After the list is computed we discard all data structures associated with \( \tau \)-runs.

Using the above list of ranks we can answer the query from the claim as follows. Starting with \( i \), we compute a sequence of \( \tau \) positions in the BWT of \( S \) by iterating \( \Psi \) on \( i \). For each position we can check in \( \mathcal{O}(\log \frac{m}{\tau}) \) time whether that position is in the list of ranks. Because we evenly sampled text positions, one of these positions has to correspond to the suffix of \( S \) for which we computed the rank in the previous step. Suppose we found such position after \( \Delta \leq \tau \) steps, i.e., we now have a pair \((i_S, i_{S'})\) such that \( i_{S'} \) is the rank of \( S[j + \Delta..m] \) among suffixes of \( S' \). Then we perform \( \Delta \) steps of the standard backward search starting from rank \( i_{S'} \) in the BWT of \( S' \) using symbols \( S[j+\Delta-1], \ldots, S[j] \). This takes \( \mathcal{O}\left( \Delta (\log r + \log r') \right) = \mathcal{O}\left( \tau (\log r + \log r') \right) \) time.

4 Constructing BWT

In this section we show that given the non-wasteful encoding of text \( T[1..n] \) over alphabet \( \Sigma = [1..\sigma] \) of size \( \sigma \leq n \) using \( \mathcal{O}(n / \log_\sigma n) \) words of space, we can compute the non-wasteful encoding of BWT of \( T \) in
\( O(n / \log \sigma n + r \log^7 n) \) time and \( O(n / \log \sigma n + r \log^5 n) \) space, where \( r \) is the number of equal-letter runs in the BWT of \( T \).

### 4.1 Algorithm overview

The basic scheme of our algorithm follows the BWT construction algorithm of Hon et al. \([30]\) but we perform the main steps differently. Assume for simplicity that \( w / \log \sigma = 2^k \) for some integer \( k \). The algorithm works in \( k + 1 \) rounds, where \( k = \log \log \sigma n \). In the \( i \)-th round, \( i \in [0..k] \), we interpret the string \( T \) as a string over supersymbol \( \Sigma_i = [1..\sigma_i] \) of size \( \sigma_i = \sigma^{2^i} \), i.e., we group symbols of the original string \( T \) into supersymbols consisting of \( 2^i \) original symbols. We denote this string as \( T_i \). Our working encoding of BWT will be run-length encoding. The input to the \( i \)-th round, \( i \in [0..k-1] \), is the run-length compressed BWT of \( T_{i+1} \), and the output is the run-length compressed BWT of \( T_i \). The rounds are executed in decreasing order of \( i \). The final output is the run-length compressed BWT of \( T_0 = T \), which we then convert into non-wasteful encoding taking \( O(n / \log \sigma n) \) words of space.

For the \( k \)-th round, we observe that \( |\Sigma_k| = \Theta(n) \) and \( |T_k| = \Theta(n / \log \sigma n) \) hence to compute BWT of \( T_k \) it suffices to first run any of the linear-time algorithms for constructing the suffix array \([35, 44, 43, 42]\) for \( T_k \) and then naively compute the RLBWT from the suffix array. This takes \( O(n / \log \sigma n) \) time and space.

Let \( S = T_i \) for some \( i \in [0..k-1] \) and suppose we are given the RLBWT of \( T_{i+1} \). Let \( S_e \) be the string of length \(|S|/2\) created by grouping together symbols \( S[2i-1..S[2i] \) for all \( i \), and let \( S_o \) be the analogously constructed string for pairs \( S[2i], S[2i+1] \). Clearly we have \( S_e = T_{i+1} \). Furthermore, it is easy to see that the BWT of \( S \) can be obtained by merging BWTs of \( S_e \) and \( S_o \), and discarding (more significant) half of the bits in the encoding of each symbol.

The construction of RLBWT for \( S \) consists of two steps: (1) first we compute the RLBWT of \( S_o \) from RLBWT of \( S_e \), and then (2) merge RLBWT of \( S_e \) and \( S_o \) into RLBWT of \( S \).

### 4.2 Computing BWT of \( S_o \)

In this section we assume that \( S = T_i \) for some \( i \in [0..k] \) and that we are given the RLBWT of \( S_e = T_{i+1} \) of size \( r_e = r_{i+1} \). Denote the size of RLBWT of \( S_o \) by \( r_o \). We will show that RLBWT of \( S_o \) can be computed in \( O(r_o + r_e \log r_e) \) time using \( O(r_o + r_e) \) working space.

Recall that both \( S_e \) and \( S_o \) are over alphabet \( \Sigma_{i+1} \). Each of the symbols in that alphabet can be interpreted as a concatenation of two symbols in the alphabet \( \Sigma_i \). Let \( c \) be the symbol of either \( S_o \) or \( S_e \) and assume that \( c = S[j]S[j+1] \) for some \( j \in [1..|S|-1] \). By major subsymbol of \( c \) we denote a symbol (equal to \( S[j] \)) from \( \Sigma_i \) encoded by the more significant half of bits encoding \( c \), and by minor subsymbol we denote symbol encoded by remaining bits (equal to \( S[j+1] \)).

We first observe that by enumerating all runs of the RLBWT of \( S_e \) in increasing order of their minor subsymbols (and in case of ties, in the increasing order of run beginnings), we obtain (on the remaining bits) the minor subsymbols of the BWT of \( S_o \) in correct order. Such enumeration could easily be done in \( O(r_e \log r_e) \) time and \( O(r_e) \) working space. To obtain the missing (major) part of the encoding of symbols in the BWT of \( S_o \), it suffices to perform the LF-step for each of the runs in the BWT of \( S_e \) in the sorted order above (i.e., by minor subsymbol), and look up the minor subsymbols in the resulting range of BWT of \( S_e \).

The problem with the above approach is the running time. While it indeed produces correct RLBWT of \( S_o \), having to scan all runs in the range of BWT of \( S_e \) obtained by performing the LF-step on each of the runs of \( S_e \) could be prohibitively high. To address this we first construct a run-length compressed sequence of minor subsymbols extracted from BWT of \( S_e \) and use it to extract minor subsymbols of BWT of \( S_e \) in total time proportional to the number of runs in the BWT of \( S_o \).
Theorem 5. Given RLBWT of $S_e = T_{i+1}$ of size $r_e$ we can compute RLBWT of $S_o$ in $O(r_o + r_e \log r_e)$ time and $O(r_o + r_e)$ working space, where $r_o$ is the size of RLBWT of $S_o$.

Proof. The whole process requires scanning the BWT of $S_e$ to create a run-length compressed encoding of minor subsymbols, adding the LF support to (the original) RLBWT of $S_e$, sorting the runs in RLBWT of $S_e$ by the minor subsymbol, and executing $r_e$ LF-queries on the BWT of $S_e$, which altogether takes $O(r_e \log r_e)$. All other operations take time proportional to $O(r_e + r_o)$. The space never exceeds $O(r_e + r_o)$. □

4.3 Merging BWTs of $S_o$ and $S_e$

As in the previous section, we assume that $S = T_i$ for some $i \in [0..k]$ and that we are given the RLBWT of $S_e = T_{i+1}$ of size $r_e = r_{i+1}$ and RLBWT of $S_o$ of size $r_o$. We will show how to use these to efficiently compute the RLBWT of $S$ in $O(|S|/\log |S| + (r_o + r_e) \log_2 |S|)$ time and space.

We start by observing that to obtain BWT of $S$ it suffices to merge the BWT of $S_e$ and BWT of $S_o$ and discard all major subsymbols in the resulting sequence. The original algorithm of Hon et al. \cite{Hon2008} achieves this by performing backward search. This requires $\Omega(|S|)$ time and hence is too expensive in our case.

Instead we employ the following observation. Suppose we have already computed the first $t$ runs of the BWT of $S$ and let the next unmerged character in the BWT of $S_o$ be a part of a run of symbol $c_o$. Let $c_e$ be the analogous symbol from the BWT of $S_e$. Further, let $c'_e$ (resp. $c'_o$) be the minor subsymbol of $c_e$ (resp. $c_o$).

If $c'_e = c'_o$ then either all symbols in the current run in the BWT of $S_o$ (restricted to minor subsymbols) or all symbols in the current run in the (also restricted) BWT of $S_e$ will belong to the next run in the BWT of $S$. Assuming we can determine the order between any two arbitrary suffixes of $S_o$ and $S_e$ given their ranks in the respective BWTs, we could consider both cases and in each perform a binary search to find the exact length of $(t+1)$-th run in the BWT of $S$. We first locate the end of the run of $c'_e$ (resp. $c'_o$) in the BWT of $S_o$ (resp. $S_e$) restricted to minor subsymbols; this can be done after preprocessing input BWTs without increasing the time/space of the merging. We then find the largest suffix of $S_e$ (resp. $S_o$) not greater than the suffix at the end of the run in the BWT of $S_o$. Importantly, the time to compute the next run in the BWT of $S$ does not depend on the number of times the suffixes in that run alternate between $S_o$ and $S_e$. The case $c'_e \neq c'_o$ is handled similarly, except we do not need to locate the end of each run. The key property of this algorithm is that the number of patterns searches is $O(r_i)$.

Thus, the merging problem can be reduced to the problem of efficient comparison of suffixes of $S_e$ and $S_o$. To achieve that we augment both RLBWTs of $S_e$ and $S_o$ with the suffix-rank support data structure from Section 3.3. This will allow us to determine, given a rank of any suffix of $S_o$, the number of smaller suffixes of $S_e$ and vice-versa, thus eliminating even the need for binary search. Our aim is to achieve $O(|S|/\log |S|)$ space and construction time, thus we apply Theorem 4 with $\tau = \log^2 |S|$.

Theorem 6. Given the RLBWT of $S_e = T_{i+1}$ of size $r_e$ and the RLBWT of $S_o$ of size $r_o$ we can compute the RLBWT of $S = T_i$ in $O((r_o + r_e) \log^6 |S| + |S|/\log |S| + r_i \log^3 |S|)$ time and using $O(|S|/\log^2 |S| + (r_e + r_o) \log^4 |S| + r_i)$ working space, where $r_i$ is the size of the output RLBWT of $S$.

Proof. Constructing the suffix-rank support for $S_o$ and $S_e$ with $\tau = \log^2 |S|$ takes $O((r_o + r_e) \log^5 |S| + |S|/\log |S|)$ time and $O((r_o + r_e) \log^4 |S| + |S|/\log^2 |S|)$ working space. The resulting data structures occupy $O(|S|/\log^2 |S| + r_e + r_o)$ space and answer suffix-rank queries in $O(\log^3 |S|)$ time. To compute the RLBWT of $S$ we perform $2r_i$ suffix-rank queries for a total of $O(r_i \log^3 |S|)$ time. □
4.4 Putting things together

To bound the size of RLBWTs in intermediate rounds, consider the $i$-th round where for $d = 2^i$ we group each $d$ symbols of $T$ to obtain the string $S = T_i$ of length $|T|/d$ and let $r_i$ be the number of runs in the BWT of $S$. Recall now the construction of generalized BWT-runs from Section 3.3 and observe that the symbols of $T$ comprising each supersymbol $S[j]$ are the same as the substring corresponding to $d$-run containing suffix $T[jd..n]$ in the BWT of $T$. It is easy to see that the corresponding suffixes of $T$ are in the same lexicographic order as the suffixes of $S$. Thus, $r_i$ is bounded by the number of $d$-runs in the BWT of $T$, which by Section 3.3 is bounded by $rd$. Hence, the size of the output RLBWT of the $i$-th round does not exceed $r2^i = \mathcal{O}(r \log n)$. The analogous analysis shows that the size of RLBWT of $S_0$ has the same upper bound $r2^{i+1}$ as $S_e = T_{i+1}$.

**Theorem 7.** Given a non-wasteful encoding of string $T[1..n]$ over alphabet $[0..\sigma]$ of size $\sigma \leq n$ using $\mathcal{O}(n/\log \sigma n)$ words, the non-wasteful encoding of BWT of $T$ can be computed in $\mathcal{O}(n/\log \sigma n + r \log^7 n)$ time and $\mathcal{O}(n/\log \sigma n + r \log^5 n)$ working space, where $r$ is the number of runs in the BWT of $T$.

**Proof.** The $k$-th round of the algorithm takes $\mathcal{O}(n/\log \sigma n)$ time working space and produces a BWT of taking $\mathcal{O}(n/\log \sigma n)$ words of space. Consider the $i$-th round of the algorithm for $i < k$ and let $S = T_i$, and $r_s$ and $r_o$ denote the sizes of RLBWT of $S_e$ and $S_o$ respectively. By the above discussion we have $r_o, r_e = \mathcal{O}(r \log n)$. Thus, by Theorem 5 and Theorem 6 the $i$-th round takes $\mathcal{O}(n_i/\log n_i + r \log^6 n_i) = \mathcal{O}(n/2^i \log n + r \log^6 n)$ time and the working space does not exceed $\mathcal{O}(n_i/\log^2 n + r \log^5 n)$ words, where $n_i = |T_i| = n/2^i$, and we used the fact that for $i < k$, $\log n_i = \Theta(\log n)$. Hence over all rounds we spend $\mathcal{O}(n/\log \sigma n + r \log^7 n)$ time and never use more than $\mathcal{O}(n/\log \sigma n + r \log^5 n)$ space. Finally, it is easy to convert RLBWT into the non-wasteful encoding in $\mathcal{O}(n/\log \sigma n + r \log^5 n)$ time. \)

Thus, we obtained a time- and space-optimal construction algorithm for BWT under a weak assumption on repetitiveness of the input: $n/r = \Omega(\text{polylog } n)$.

5 Constructing PLCP

In this section we show that given the run-length compressed representation of BWT of $T$ it is possible to compute the PLCP succ bitvectors in $\mathcal{O}(n/\log n + r \log^{11} n)$ time and $\mathcal{O}(n/\log n + r \log^{10} n)$ working space. The resulting bitvector takes $2n$ bits, or $\mathcal{O}(n/\log n)$ words of space. This result complements the result from the previous section. After we construct RLBWT, before converting to non-wasteful encoding, we can use it to construct the PLCP bitvector — another component of CSA and CST.

The key observation used to construct the PLCP values is that it suffices to only compute the irreducible LCP values. Then, by Lemma 2 all other values can be quickly deduced. This significantly simplifies the problem because it is known (Lemma 3) that the sum of irreducible LCP values is bounded by $\mathcal{O}(n \log n)$.

The main idea of the construction is to compute (as in Theorem 4) names of $\tau$-runs for $\tau = \log^5 n$. This will allow us to compare $\tau$ symbols at a time and thus quickly computing a lower bound for large irreducible LCP values. Before we can use this, we need to augment the BWT with the support for SA/ISA queries. Note that in the next section we show how to efficiently build (in similar time and space) a support for the same queries but only taking up space $\mathcal{O}(r \text{ polylog } n)$ space.

5.1 Computing SA/ISA support

Suppose that we are given a run-length compressed BWT of $T[1..n]$ taking $\mathcal{O}(r)$ space. Let $\tau \geq 1$ be an integer. Assume for simplicity that $n$ is a multiple of $\tau$. We start by computing the sorted list of starting
We start by computing the lower-bound for $LCP[i]$. This requires augmenting the RLBWT with the LF/$\psi$ support first and in total takes $O((\tau r \log(\tau r))$ time and $O(\tau r)$ working space. We then compute and store, for the first position of each $\tau$-run $[b..e]$, the value of $LF^T[b]$. This will allow us to efficiently compute $LF^T[i]$ for any $i \in [1..n]$.

We then locate the occurrence $i_0$ of the symbol $\$ in BWT and perform $n/\tau$ iterations of $LF^T$ on $i_0$. By definition of LF, the position $i$ visited after $j$ iterations of $LF^T$ is equal to ISA$[n - j\tau]$, i.e., ISA$[i] = n - j\tau$. For any such $i$ we save the pair $(i, n - j\tau)$ into a list. When we finish the traversal we sort the list by the first component (assume this list is called $L_{SA}$). We then create the copy of the list (call it $L_{ISA}$) and sort it by the second component. Creating the lists takes $O((n/\tau)(\log r + \log(n/\tau)))$ time and they occupy $O(n/\tau)$ space. After the lists are computed we discard $LF^T$ samples associated with all runs. Having these lists allows us to efficiently compute SA$[i]$ or ISA$[i]$ for any $i \in [1..n]$ as follows.

To compute ISA$[i]$ we find (in constant time, as we in fact can store $L_{ISA}$ in an array) the pair $(p, j)$ in $L_{ISA}$ such that $j = \lceil i/\tau \rceil \tau$. We then perform $j - i < \tau$ steps of LF on position $p$. The total query time is thus $O(\tau \log r)$.

To compute SA$[i]$ we perform $\tau$ steps of LF (each taking $O(\log r)$ time) on position $i$. Due to the way we sampled SA/ISA values, one of the visited positions has to be the first component in the $L_{SA}$ list. For each position we can check this in $O(\log(n/\tau))$ time. Suppose we found a pair after $\Delta < \tau$ steps, i.e., a pair $(LF^A[i], p)$ is in $L_{SA}$. This implies $SA[LF^A[i]] = p$, i.e., $SA[i] = p + \Delta$. The query time is $O(\tau(\log r + \log(n/\tau)))$.

\textbf{Theorem 8.} Given RLBWT of text $T[1..n]$ of size $O(r)$ we can, for any $\tau \geq 1$, build a data structure taking $O(r + n/\tau)$ space that, for any $i \in [1..n]$, can answer SA$[i]$ query in $O(\tau \log r)$ time and ISA$[i]$ query in $O(\tau(\log r + \log(n/\tau)))$ time. The construction takes $O((n/\tau)(\log r + \log(n/\tau)) + (r\tau) \log(r\tau))$ time and $O(n/\tau + r\tau)$ working space.

### 5.2 Computing irreducible LCP values

We start by augmenting the RLBWT with the SA/ISA support as explained in the previous section using $\tau_1 = \log^2 n$. The resulting data structure answers SA/ISA queries in $O(\log^3 n)$ time. We then compute $\tau_2$-runs and their names using the technique introduced in Theorem 4 for $\tau_2 = \log^5 n$.

Given any $j_1, j_2 \in [1..n]$ we can check if $T[j_1..j_1 + \tau_2 - 1] = T[j_2..j_2 + \tau_2 - 1]$ using the above names as follows. First compute $i_1 = ISA[j_1 + \tau_2]$ and $i_2 = ISA[j_2 + \tau_2]$ using the ISA support. Then compare the names of $\tau_2$-substrings preceding these two suffixes. Thus, comparing two arbitrary substrings of $T$ of length $\tau_2$, given their text positions, takes $O(\log^3 n)$ time.

The above toolbox allows computing all irreducible LCP values as follows. For any $i \in [1..n]$ such that LCP$[i]$ is irreducible (such $i$ can be recognized by checking if BWT$[i - 1]$ belongs to a BWT-run different than BWT$[i]$) we compute $j_1 = SA[i - 1]$ and $j_2 = SA[i]$. We then have LCP$[i] = lcp(T[j_1..n], T[j_2..n])$. We start by computing the lower-bound for LCP$[i]$ using the names of $\tau_2$-substrings. Since the sum of irreducible LCP values is bounded by $O(n \log n)$, over all irreducible LCP values this will take $O(r \log^3 n + \log^3 n \cdot (n \log n)/\tau_2) = O(r \log^3 n + n/\log n)$ time. Finishing the computation of each LCP value requires at most $\tau_2$ symbol comparisons. This can be done by following $\psi$ for both pointers as long as the preceding symbols (found in the BWT) are equal. Over all irreducible LCP values, finishing the computation takes $O(r\tau_2 \log n) = O(r \log^6 n)$ time.

\textbf{Theorem 9.} Given the RLBWT of $T[1..n]$ of size $r$, the PLCP$_{abcuv}$ bi-vector (or the list of pairs storing all irreducible LCP values in text-order) can be computed in $O(n / \log n + r \log^{11} n)$ time and $O(n/\log n + r \log^{10} n)$ working space.
Proof. Adding the SA/ISA support using $\tau_1 = \log^2 n$ takes $O((n/\log n + r \log^3 n)$ time and $O(n/\log^2 n + r \log^2 n)$ working space (Theorem 8). The resulting structure needs $O(r + n/\log^2 n)$ space and answers SA/ISA queries in $O(\log^3 n)$ time.

Computing the names takes $O(\tau_2 r \log(\tau_2 r)) = O(r \log^{11} n)$ time and $O(\tau_2^2 r) = O(r \log^{10} n)$ working space (see the proof of Theorem 4). The names need $O(\tau_2 r) = O(r \log^5 n)$ space. Then, by the above discussion, computing all irreducible LCP values takes $O(n/\log n + r \log^6 n)$ time.

Converting the list of irreducible LCP values (in text order) into $PLCP_{suc}$ requires $r$ SA-queries and hence can be done in $O(n/\log n + r \log^3 n)$ time.

By combining with Theorem 7 we obtain the following result.

Theorem 10. Given a non-wasteful encoding of string $T[1..n]$ over alphabet $[0..\sigma]$ of size $\sigma \leq n$ using $O(n/\log \sigma n)$ words, the $PLCP_{suc}$ bitvector (or the list of pairs storing all irreducible LCP values in text-order) can be computed in $O(n/\log \sigma n + r \log^{11} n)$ time and $O(n/\log \sigma n + r \log^{10} n)$ working space, where $r$ is the number of runs in the BWT of $T$.

6 Optimal construction of run-length compressed CSA

In this section we show how our techniques can be used to build, in optimal time and space, the run-length compressed suffix array (RLCSA) recently proposed by Gagie et al. [24]. They observed that both the text and BWT have a bidirectional parse of size $O(r)$, where $r$ is the number of runs on the BWT, and that this property transfers to other arrays (such as SA/ISA or LCP/PLCP) except they first need to be differentially encoded. They use a locally-consistent parsing to grammar-compress these arrays and describe the necessary augmentations to achieve fast decoding of the original values. They also describe a data structure called a block tree (see also [23] for an early, unidirectional, version) that allows random-access to any length-$\ell$ substring of text in $O(\log(n/r) + \ell \log(\sigma)/w)$ time. The block tree takes $O(r \log(n/r))$ space.

The structures described below are slightly different than the original index proposed by Gagie et al. [24]. Rather than compressing the differentially-encoded arrays, they directly exploit the structure of bidirectional-parse present in all the arrays. They can be thought of as a multi-ary block trees modified to work with arrays indexed in “lex-order” instead of the original “text-order”. Our main contribution, however, is the time- and space-optimal construction of the above data structures for highly repetitive inputs.

6.1 Small-space SA support

The data structure. The data structure is parametrized by an integer parameter $\tau > 1$. Assume we are given RLBWT of size $r$ for text $T[1..n]$. The data structure is organized into $\log_r(n/r)$ levels. The main idea is, at any given level, to store $2\tau$ pointers for each BWT-run that will reduce the SA-access query for positions nearby that BWT-run boundary into the SA-access query at some different position but closer (by at least a factor of $\tau$) to some run boundary. Level describes the allowed proximity of the query, so a pointer from level $i$ will always lead to level $i + 1$. Upon reaching the last level, the SA value at each run boundary is stored explicitly.

More precisely, for level $1 \leq k \leq \lceil \log_r(n/r) \rceil$, let $b_k = n/(r\tau^k)$ and let $BWT[b..e]$ be one of the runs in the BWT (and $BWT[b'..e']$ be the preceding run, $e' + 1 = b$). Consider $2\tau$ non-overlapping blocks of size $b_k$, evenly spread around position $b$, i.e., $BWT[b + ib_k..b + (i + 1)b_k - 1]$, $i = -\tau, \ldots, \tau - 1$, except we omit blocks not contained in $BWT[b'..e]$. For each block $BWT[s..t]$ we store the smallest $d$ (called $LF$-distance) such that there exists are least one $i \in [s..t]$ such that $LF[i]$ is the beginning of the run in the BWT of $T$. 


We also store the value $\text{LF}^{d}[s]$ (called LF-shortcut), both as an absolute value in $[1..n]$ and as a pointer to the BWT-run that contains it. Due to simple generalization of Lemma 1, this allows us to compute $\text{LF}^{d}[i]$ for any $i \in [s..t]$. 

To access $\text{SA}[i]$ we proceed as follows. Assume first that $i$ is not more than $n/r$ positions from the closest run boundary. We first find the run that contains $i$. We then follow the LF-shortcuts starting at level 1 down to the last level. After every step the distance to the closest run boundary is reduced by a factor $\tau$. Thus, after $\log\tau(n/r)$ steps the current position is equal to boundary $b$ of some run BWT$[b..e]$. Let $d_{\text{sum}}$ denote the total lengths of LF-distances of the used shortcuts. Since $\text{SA}[b]$ is stored we can now answer the query as $\text{SA}[i] = \text{SA}[b] + d_{\text{sum}}$. To handle positions $i$ further than $n/r$ from the nearest run boundary, we add a lookup table $LT[1..r]$ such that $LT[i]$ stores the LF-shortcut for block BWT$[(i-1)(n/r)+1..(i+1)(n/r)]$.

The above data structure can be generalized to extract segments of $\text{SA}[p..p+\ell-1]$, for any $p$ and $\ell$, faster than $\ell$ single SA-accesses, that would cost $O(\ell \log\tau(n/r))$. The main modification is that at level $k$ we instead consider $4\tau-1$ blocks of size $b_k$, evenly spread around position $b$, each overlapping the next by exactly $b_k/2$ symbols, i.e., BWT$[b + ib_k/2..b + (i+2)b_k/2 - 1]$, $i = -2\tau, ..., 2(\tau - 1)$. This guarantees that any segment-access to SA of length at most $b_k/2$ at level $i$ can be transformed into the segment-access at level $i+1$. We also truncate the data structure at level $k$ where $k$ is the smallest integer with $b_k < \log\tau(n/r)$. At that level we store a segment of $2 \log\tau(n/r)$ SA values around each BWT run. These values take $O(r \log\tau(n/r))$ space, and hence these two modifications do not increase the space needed by the data structure. This way we can extract $\text{SA}[p..p+\alpha-1]$, where $\alpha = \log\tau(n/r)$ in $O(\alpha)$ time, and consequently a segment $\text{SA}[p..p+\ell-1]$ in $O((\ell/\alpha + 1)\alpha) = O(\ell + \log\tau(n/r))$ time.

**Construction algorithm.** Assume that we have the run-length compressed BWT of $T[1..n]$ of size $r$. Consider any block BWT$[s..t]$. Let $d$ be the corresponding LF-distance and let $\text{LF}^{d}[i] = b$ for some $i \in [s..t]$ be the beginning of a BWT-run $[b..e]$. We observe that this implies LCP$[b]$ is irreducible and LCP$[b] \geq d$.

We start by augmenting the RLBWT with the SA/ISA support from Section 5.1 using $\tau_1 = \log^2 n$. This, by Theorem 8, takes $O(n/\log n + r \log^3 n)$ time and $O(n/\log^2 n + r \log^2 n)$ working space. The resulting structure needs $O(r + n/\log^2 n)$ space and answers SA/ISA queries in $O(\log^3 n)$ time.

Consider now the sequence $Q$ containing every position $i$ in $T$ such that PLCP$[i]$ is irreducible. Such list can be obtained by computing value $\text{SA}[b]$ for every BWT run $[b..e]$ and sorting the resulting values. Computing the list $Q$ takes $O(r \log^3 n)$ time and $O(r)$ working space. The list itself is stored in plain form using $O(r)$ space. Next, for any irreducible value LCP$[i] = \ell$ where $\text{SA}[i] = j$, we compute, for any $t = 1, ..., [\ell'/\tau_2]$ a pair containing ISA$[j + t\tau_2]$ (as key) and $t\tau_2$ (as value), where $\tau_2 = \log^4 n$, and $\ell' \leq \ell$ is such that the distance to the successor of $\text{SA}[i]$ in $Q$ is $\ell'$ (the value $\ell'$ is found by binary-searching the list $Q$ in $O(\log r)$ time). Since the sum of all considered $\ell'$ values is $O(n)$, computing all pairs takes $O(\log^3 n \cdot (r + n/\tau_2)) = O(n/\log n + r \log^3 n)$ time and $O(n/\tau_2) = O(n/\log^4 n)$ working space. The resulting pairs need $O(n/\log^4 n)$ space.

We then sort all the computed pairs by the keys and build a static RMQ data structure (e.g., the static balanced BST) over the associated values. This can be done in $O((n/\tau_2) \log(n/\tau_2)) = O(n/\log^3 n)$ time so that an RMQ query takes $O(\log n)$ time.

Having the above samples augmented with the RMQ allows us to compute LF-shortcuts as follows. Let BWT$[s..t]$ be one of the blocks. We perform $\tau_2$ LF-steps on the position $s$. Then, one of two things must occur: (1) either for some $\Delta < \tau_2$ we found that the interval $[\text{LF}^{\Delta}[s]..\text{LF}^{\Delta}[s] + (t - s)]$ contains a boundary of a BWT-run (this is easy to check in $O(\log r)$ time), or (2) for some $\Delta < \tau_2$ the interval $[\text{LF}^{\Delta}[s]..\text{LF}^{\Delta}[s] + (t - s)]$ contains the key of at least one pair computed above (this can be checked in $O(\log n)$ time). This holds due to the way we computed the pairs.
The construction takes \(O(n \log n + \tau \log \log n)\) time and \(O(n/\log \log n + r(\log \log n + \log^2 n))\) working space.

For \(\tau = 2\) the above data structure matches the space and query time of [24]. For \(\tau = \log^\epsilon n\), where \(\epsilon > 0\) is an arbitrary constant it achieves \(O(r \log^2 n \log(n/r))\) space and \(O(\log n / \log \log n)\) query time. We conjecture this query time to be optimal within \(O(r \text{polylog } n)\) space. Finally, for \(\tau = (n/r)^\epsilon\) it achieves \(O(r^{1-\epsilon} n^\epsilon)\) space and constant-time query. In particular, if \(r = o(n)\) the data structure takes \(o(n)\) space and is able to access (any segment of) SA in optimal time.

By combining with Theorem 7 we obtain the following theorem.

**Theorem 12.** Given a non-wasteful encoding of string \(T[1..n]\) over alphabet \([0..\sigma]\) of size \(\sigma \leq n\) using \(O(n/\log \sigma n)\) words, we can build, for any \(\tau > 1\), a data structure of size \(O(\tau \log \log n)\) that, for any \(p \in [1..n]\) and \(\ell \geq 1\), can compute \(\text{SA}[p..p + \ell - 1]\) in \(O(\ell + \log \log n)\) time, in \(O(n/\log \sigma n + r(\log \log n + \log^2 n))\) time and \(O(n/\log \sigma n + r(\log \log n + \log^2 n))\) working space, where \(r\) is the number of runs in the BWT of \(T\).

### 6.2 Small-space ISA support

We now describe a data structure orthogonal to the SA support data structure from section 6.1. The SA support used a natural partitioning of BWT (i.e., text in lex-order) into \(r\) blocks. Here we utilize a naturally induced (albeit somewhat less straightforward) \(r\)-size text-order partitioning of \(T\).

We define a position \(i \in [1..n]\) to be the beginning of a block if PLCP\([i]\) is irreducible. The key property of such partitioning is that if \(T[b..e]\) is a block, then for any \(i \in [b+1..e]\) it holds \(\Phi[i] = \Phi[e] - (e - i)\), i.e., it suffices to know \(\Phi[e]\) to compute any other \(\Phi\) value inside the block. This is because by the definition of our partitioning, PLCP\([b+1..e]\) are all reducible and hence by Lemma 2, \(\Phi[i] = \Phi[i+1] - 1 = \ldots = \Phi[e] + (e - i)\). Gagie et al. [24] used this natural text partition to obtain a data structure supporting efficient queries to access ISA, but they again first differentially encode ISA and then apply grammar compression.

We instead, analogously to our data structure supporting SA queries, consider, at each level \(k \geq 1\), \(2\tau\) non-overlapping bt-blocks (which stands for “block tree”-blocks, to distinguish from the elements of the text partition defined above), of size \(b_k = n/((\tau + k)\tau)\), evenly spread around position \(b\), i.e., \(T[b + ib_k..b + (i + 1)b_k - 1]\), \(i = -\tau, \ldots, \tau - 1\), where \(T[b..e]\) is one the text blocks introduced above. As before we omit bt-blocks not contained inside \(T[b'..e]\) where \(T[b'..e]\) precedes \(T[b..e]\). For each block \(T[s..t]\) we store the smallest \(d\) (called \(\Phi\)-distance) such that there exists \(i \in [s..t]\) such that \(\Phi[i]\) is the beginning of the block in our partition of \(T\), i.e., such that PLCP\(\Phi[i]\) is irreducible. We also store the value \(\Phi[i]\) (called \(\Phi\)-shortcut), both as an absolute value in \([1..n]\) and as a pointer (and offset) to the block in text-partition that contains it.

All the remaining details of the data structure as well as construction are translated, via “orthogonal” symmetry of the two size-\(r\) partitionings of BWT and \(T\), to the data structure supporting access to SA. This
includes the generalization of Theorem 13 to support $\Phi$ queries in $O(\log r)$ time and $O(r)$ space. Thus, we obtain:

**Theorem 13.** Given the size-$r$ RLBWT of text $T[1..n]$, we can build, for any $\tau > 1$, a data structure of size $O(r\tau \log_r (n/r))$ that, for any $p \in [1..n]$ and $\ell \geq 1$, can compute $\text{ISA}[p..p+\ell-1]$ in $O(\ell + \log_r (n/r))$ time. The construction takes $O(n/\log n + r\tau \log_6 n)$ time and $O(n/\log^2 n + r(\tau \log_r (n/r) + \log^2 n))$ working space.

**Theorem 14.** Given a non-wasteful encoding of string $T[1..n]$ over alphabet $[0..\sigma]$ of size $\sigma \leq n$ using $O(n/\log_\sigma n)$ words, we can build, for any $\tau > 1$, a data structure of size $O(r\tau \log_r(n/r))$ that, for any $p \in [1..n]$ and $\ell \geq 1$, can compute $\text{ISA}[p..p+\ell-1]$ in $O(\ell + \log_r (n/r))$ time, in $O(n/\log_\sigma n + r(\tau \log_6 n + \log^7 n))$ time and $O(n/\log_\sigma n + r(\tau \log_r (n/r) + \log^5 n))$ working space, where $r$ is the number of runs in the BWT of $T$.

### 6.3 Other arrays

We remark that the above construction algorithm can be easily adapted for other arrays. More precisely, for any “text-order” array (such as PLCP, $T$, or $\Psi$) we can generalize Theorem 13 and for any “lex-order” array (such as BWT, LCP, LF, or $\Phi$) we can generalize Theorem 11. Typically, however, only few of the arrays need to be encoded, as other values can be obtained using known relations. For example, if we only have support for random-access on SA, ISA, PLCP and $T$ we can obtain all other values using relations:

\[
\begin{align*}
\text{BWT}[i] &= T[\text{SA}[i] - 1] \\
\Phi[i] &= \text{ISA}[\text{SA}[i] - 1] \\
\text{LCP}[i] &= \text{PLCP}[\text{SA}[i]] \\
\Psi[i] &= \text{ISA}[\text{SA}[i] + 1]
\end{align*}
\]

### 7 Optimal construction of LZ77 parsing

In this section we show how to use some of the tools introduced in this paper to obtain a time and space optimal LZ77 factorization algorithm for highly repetitive strings.

#### 7.1 Definitions

The LZ77 factorization uses the notion of a longest previous factor (LPF). The LPF at position $i$ (denoted $\text{LPF}[i]$) in $T$ is a pair $(p_i, \ell_i)$ such that $p_i < i$, $T[p_i..p_i+\ell_i-1] = T[i..i+\ell_i - 1]$ and $\ell_i > 0$ is maximized. In other words, $T[i..i+\ell_i - 1]$ is the longest prefix of $T[i..n]$ which also occurs at some position $p_i < i$ in $T$. If $T[i]$ is the leftmost occurrence of a symbol in $T$ then such a pair does not exist. In this case we define $p_i = T[i]$ and $\ell_i = 0$. Note that there may be more than one potential $p_i$, and we do not care which one is used.

The LZ77 factorization (or LZ77 parsing) of a string $T$ is then just a greedy, left-to-right parsing of $T$ into longest previous factors. More precisely, if the $j$th LZ factor (or phrase) in the parsing is to start at position $i$, then we output $(p_i, \ell_i)$ (to represent the $j$th phrase), and then the $(j+1)$th phrase starts at position $i + \ell_i$, unless $\ell_i = 0$, in which case the next phrase starts at position $i + 1$. For the example string $T = zzzzzzipzip$, the LZ77 factorization produces:

$$(z, 0), (1, 4), (i, 0), (p, 0), (5, 3).$$
We denote the number of phrases in the LZ77 parsing of \( T \) by \( z \). The following theorem shows that LZ77 parsing can be encoded in \( \mathcal{O}(n \log \sigma) \) bits, thus the lower bound for its construction time is \( \mathcal{O}(n/\log \sigma n) \).

**Theorem 15** (e.g. [36]). The number of phrases \( z \) in the LZ77 parsing of a text of \( n \) symbols over an alphabet of size \( \sigma \) is \( \mathcal{O}(n/\log \sigma n) \).

The LPF pairs can be computed using next and previous smaller values (NSV/PSV) defined as

\[
\text{NSV}_{\text{lex}}[i] = \min\{j \in [i + 1..n] \mid \text{SA}[j] < \text{SA}[i]\}
\]
\[
\text{PSV}_{\text{lex}}[i] = \max\{j \in [1..i - 1] \mid \text{SA}[j] < \text{SA}[i]\}.
\]

If the set on the right hand side is empty, we set the value to 0. We further define

\[
\text{NSV}_{\text{lex}}[i] = \text{SA}[\text{NSV}_{\text{lex}}[\text{ISA}[i]]] \quad (1)
\]
\[
\text{PSV}_{\text{lex}}[i] = \text{SA}[\text{PSV}_{\text{lex}}[\text{ISA}[i]]]. \quad (2)
\]

If \( \text{NSV}_{\text{lex}}[\text{ISA}[i]] = 0 \) (\( \text{PSV}_{\text{lex}}[\text{ISA}[i]] = 0 \)) we set \( \text{NSV}_{\text{lex}}[i] = 0 \) (\( \text{PSV}_{\text{lex}}[i] = 0 \)).

The usefulness of the NSV/PSV values is summarized by the following easily proved lemma due to [14].

**Lemma 5** ([14]). For \( i \in [1..n] \), let \( i_{\text{nsv}} = \text{NSV}_{\text{lex}}[i] \), \( i_{\text{psv}} = \text{PSV}_{\text{lex}}[i] \), \( \ell_{\text{nsv}} = \text{lcp}(i, i_{\text{nsv}}) \) and \( \ell_{\text{psv}} = \text{lcp}(i, i_{\text{psv}}) \). Then

\[
\text{LPF}[i] = \begin{cases} 
(i_{\text{nsv}}, \ell_{\text{nsv}}) & \text{if } \ell_{\text{nsv}} > \ell_{\text{psv}} \\
(i_{\text{psv}}, \ell_{\text{psv}}) & \text{if } \ell_{\text{psv}} = \max(\ell_{\text{nsv}}, \ell_{\text{psv}}) > 0 \\
(T[i], 0) & \text{if } \ell_{\text{nsv}} = \ell_{\text{psv}} = 0.
\end{cases}
\]

### 7.2 Algorithm overview

The general approach of our algorithm follows the lazy LZ77 factorization algorithms of [38]. Namely, we opt out of computing all LPF values which takes \( \Omega(n) \) time, in favor of computing the \( \text{LPF}[j] \) only whenever there is LZ factor starting at position \( j \). This will allow us to achieve the parsing time and space of \( \mathcal{O}(n/\log \sigma n + r \, \text{polylog } n) \).

Suppose we have already computed the parsing of \( T[1..j - 1] \). To compute the factor starting at position \( j \) we first compute \( i = \text{ISA}[j] \). We will then compute (using a small-space data structure) values \( i_{\text{nsv}} = \text{NSV}_{\text{lex}}[i] \) and \( i_{\text{psv}} = \text{PSV}_{\text{lex}}[i] \). By Lemma 5 it then suffices to compute the longest common prefix of \( T[j..n] \) and each of the two suffixes starting at positions \( \text{SA}[i_{\text{psv}}] \) and \( \text{SA}[i_{\text{nsv}}] \).

It is easy to see that the total length of computed lcps will be \( \mathcal{O}(n) \), since after each step we increase \( j \) by the longest of the two lcps. To perform the lcp computation efficiently we will employ the technique from Section 5 which will allow us to compare multiple symbols at a time. This will allow us to spend \( \mathcal{O}(z + n/\log n) = \mathcal{O}(n/\log \sigma n) \) time in the lcp computation. The problem is thus reduced to being able to quickly answer \( \text{PSV}_{\text{lex}}/\text{NSV}_{\text{lex}} \) queries.

### 7.3 Computing NSV/PSV support for SA

Assume that we are given RLBWT of size \( \mathcal{O}(r) \) for text \( T[1..n] \). We will show how to quickly build a small-space data structure that, given any \( i \in [1..n] \) can compute \( \text{PSV}_{\text{lex}} \) or \( \text{NSV}_{\text{lex}} \) in \( \mathcal{O}(\text{polylog } n) \) time.

The basic idea is to split \( \text{BWT}[1..n] \) into blocks of size \( \tau = \mathcal{O}(\text{polylog } n) \) and for each \( i \in [1..n/\tau] \) compute the minimal value in \( \text{SA}[i \tau..(i + 1) \tau - 1] \) as well as its position. We then build a balanced binary
search tree over the array of minimas and augment each internal node with the minimal value in its subtree. This allows, for any \( i \in [1..n] \), to find the maximal (resp. minimal) \( j < i \) (resp. \( j > i \)) such that block \( SA[j..(j+1)\tau - 1] \) contains the value smaller than \( SA[i] \). To finish the computation of PSV\(_{\text{lex}}\) (resp. NSV\(_{\text{lex}}\)) it then suffices to scan all SA values inside the found block. Assuming it takes \( O(\log^3 n) \) time to compute SA value (Theorem 8), answering a single NSV\(_{\text{lex}}\)/PSV\(_{\text{lex}}\) query will take \( O(\tau \log^3 n) \).

To compute the minimum for each of the \( n/\tau \) blocks of SA we observe that, up to a shift by a constant, there is only \( \tau \tau \) different blocks. More specifically, consider a block \( SA[i\tau..(i+1)\tau - 1] \). Let \( k \) be the smallest integer such that for some \( t \in [i\tau..(i+1)\tau - 1] \), \( LF^k[t] \) is the beginning of a run in BWT. It is easy to see that, due to Lemma 14, \( SA[i\tau..(i+1)\tau - 1] = k + SA[LF^k[i\tau]..LF^k[i\tau + \tau - 1]] \), in particular the equality holds for the minimal element. Thus, it suffices to precompute the minimal value and its position for each of the \( \tau \tau \) minimal elements in each of the \( \tau \tau \) different blocks. More specifically, consider a block \( SA[i..n] \). To achieve the \( O(\tau \log^3 n) \) time and \( O(\tau \tau \log^3 n) \) working space. The resulting values need \( O(r\tau \tau) \) space.

It thus remains to compute the “LF-distance” for each of the \( n/\tau \) blocks of SA, i.e., the smallest \( k \) such that for at least one position \( t \) inside the block, \( LF^k[t] \) is the beginning of a BWT-run. To achieve this we utilize the technique used in Section 6.1. There we presented a data structure of size \( O(n/\log^2 n + n/\tau_1) \) that can be built in \( O(n/\log n + r \log^3 n + (n/\log^3 n)/\tau_1) \) time and \( O(n/\log^2 n + r \log^2 n + n/\tau_1) \) working space, and is able to compute the LF-shortcut for any interval \( [s..t] \) in SA in \( O(\tau_1 \log n) \) time.

**Theorem 16.** Given RLBWT of size \( r \) for text \( T[1..n] \), we can build a data structure of size \( O(r + n/\log^2 n) \) that can answer PSV\(_{\text{lex}}\)/NSV\(_{\text{lex}}\) queries in \( O(\log^3 n) \) time. The data structure can be built in \( O(n/\log n + r \log^3 n) \) time and \( O(n/\log^2 n + r \log^6 n) \) working space.

**Proof.** We first by augmenting the RLBWT with SA/ISA support. Thus takes (Theorem 8) \( O(n/\log n + r \log^3 n) \) time and \( O(n/\log^2 n + r \log^2 n) \) working space. The resulting data structure takes \( O(r + n/\log^2 n) \) space and answers SA/ISA queries in \( O(\log^3 n) \) time.

To achieve the \( O(n/\log n) \) construction time for the data structure from Section 6.1 we set \( \tau_1 = \log^4 n \). Then, computing the LF-shortcut for any interval in SA takes \( O(\log^5 n) \) time. Since we have \( n/\tau \) intervals to query, we set \( \tau = \log^6 n \) to achieve a total construction time of \( O(n/\log n) \). Answering a single NSV\(_{\text{lex}}\)/PSV\(_{\text{lex}}\) query then takes \( O(\tau \log^3 n) = O(\log^9 n) \).

The RMQ data structure built on top of the minimas of the intervals of SA takes \( O(n/\tau) = O(n/\log^7 n) \) space, hence the space of the final data structure is dominated by SA/ISA support taking \( O(r + n/\log^2 n) \) words.

The construction time is split between precomputation of the minimas in each of the \( n \tau \) intervals crossing boundaries of BWT-runs in \( O(\tau \log^3 n) = O(r \log^9 n) \) time, and other steps introducing term \( O(n/\log n) \).

The working space is maximized when building the SA/ISA support and during the precomputation of minimas in each of the \( n \tau \) intervals, for a total of \( O(n/\log^2 n + r \log^6 n) \). \( \square \)

### 7.4 Algorithm summary

**Theorem 17.** Given RLBWT of size \( r \) of \( T[1..n] \), the LZ77 factorization of \( T \) can be computed in \( O(n/\log n + r \log^9 n + z \log^9 n) \) time and \( O(n/\log^2 n + z + r \log^8 n) = O(n/\log^9 n + r \log^8 n) \) working space, where \( z \) is the size of the LZ77 parsing of \( T \).

**Proof.** We start by augmenting the RLBWT with the SA/ISA support from Section 5.1 using \( \tau_1 = \log^2 n \). This, by Theorem 8 takes \( O(n/\log n + r \log^3 n) \) time and \( O(n/\log^2 n + r \log^2 n) \) working space. The resulting structure needs \( O(r + n/\log^2 n) \) space and answers SA/ISA queries in \( O(\log^3 n) \) time.
The data structure can be built in \( O(n \log n + r \log^9 n) \) time and \( O(n/\log^2 n + r \log^6 n) \) working space. Over the course of the whole algorithm we ask \( O(z \log^9 n) \) time.

Lastly, we compute \( \tau_3 \)-runs and their names using the technique introduced in Section 5.2 for \( \tau_3 = \log^4 n \). This takes \( O(\tau_3^2 r \log(\tau_3^2 r)) = O(r \log^9 n) \) time and \( O(\tau_3^2 r) = O(r \log^8 n) \) working space (see the proof of Theorem 9). The names need \( O(\tau_3 r) = O(r \log^4 n) \) space. The names allow, given any \( j_1, j_2 \in [1..n] \), to compute \( \ell = \text{lcp}(j_1, j_2) \) in \( O(\log^2 n(1 + \ell/\tau_3) + \tau_3 \log n) = O(\log^5 n + \ell/\log n) \) time. Thus, over the course of the whole algorithm we will spend \( O(z \log^5 n + n/\log n) \) time computing lcp values.

By combining with Theorem 9 we obtain the following result.

**Theorem 18.** Given string \( T[1..n] \) over alphabet \([0..\sigma]\) of size \( \sigma \leq n \) stored in \( O(n/\log_\sigma n) \) words of space, we can compute the LZ77 factorization of \( T \) in \( O(n/\log_\sigma n + r \log^9 n + z \log^9 n) \) time and \( O(n/\log_\sigma n + r \log^8 n) \) working space, where \( r \) is the number of runs in the BWT of \( T \) and \( z \) is the size of the LZ77 parsing of \( T \).

Since \( z = O(r \log n) \) [20], the above algorithm is time and space optimal under a weak assumption \( n/r \in \Omega(\text{polylog } n) \).

## 8 Optimal construction of Lyndon factorization

In this section we show another application of our techniques. Namely, we show that our algorithms imply a time- and space-optimal construction of Lyndon factorization for highly repetitive strings.

### 8.1 Definitions

A string \( S \) is called a Lyndon word if \( S \) is lexicographically smaller than all its non-empty proper suffixes. The Lyndon factorization (also called Standard factorization) of a string \( T \) is its unique (see [12]) factorization \( T = f_1^{e_1} \cdots f_m^{e_m} \) such that each \( f_i \) is a Lyndon word, \( e_i \geq 1 \), and \( f_i > f_{i+1} \) for all \( 1 \leq i < m \). We call each \( f_i \) a Lyndon factor of \( T \), and each \( F_i = f_i^{e_i} \) a Lyndon run of \( T \). The size of the Lyndon factorization is \( m \), the number of distinct Lyndon factors, or equivalently, the number of Lyndon runs.

Each Lyndon run can be encoded as a triple of integers storing the boundaries of some occurrence of \( f_i \) in \( T \) and the exponent \( e_i \). Since, for any string, it holds \( m < 2z \) [32] and \( z = O(n/\log_\sigma n) \) [36], where \( z \) is the number of phrases in the LZ77 parsing, it follows that Lyndon factorization can be stored in \( O(n \log \sigma) \) bits, and thus the lower bound for its construction time is \( O(n/\log_\sigma n) \).

### 8.2 Algorithm overview

Our algorithm utilizes many of the algorithms from the long line of research on algorithms operating on compressed representations such as grammars or LZ77 parsing:

- Furuya et al. [19] have shown that given an SLP (i.e., a grammar in Chomsky normal form generating a single string) of size \( g \) generating string \( T \) of length \( n \), the Lyndon factorization of \( T \) can be computed in \( O(P(g, n) + Q(g, n) \log \log n) \) time and \( O(g \log n + S(g, n)) \) space, where \( P(g, n) \), \( S(g, n) \), \( Q(g, n) \) are respectively the pre-processing time, space, and query time of a data structure for longest common extensions (LCE) queries on SLPs. The LCE query, given two positions \( i \) and \( j \) in the string \( T \), returns \( \text{lcp}(i, j) \), i.e., the length of the longest common prefix of suffixes \( T[i..n] \) and \( T[j..n] \).
– On the other hand, Nishimoto et al. [55, Thm 3] have shown how, given an SLP of size \( g \) generating string \( T \) of length \( n \), to convert the LZ77 parsing of \( T \) of size \( z \) to an SLP of size \( g \) in \( O(\log g \log \log n) \) time and \( O(n\log g + r\log^9 n + z\log^9 n) \) working space, where \( r \) is the number of distinct substrings of \( T \). The resulting SLP has size \( n/g \) and \( \text{LCE}(S) = \text{LCE}(T) \), and \( \text{LCP}(S) = \text{LCP}(T) \).

9.1 Number of distinct substrings

It is a folklore knowledge that the number \( d \) of distinct substrings of a given string \( T \) of length \( n \) is given by the formula:

\[
d = \frac{n(n + 1)}{2} - \sum_{i=1}^{n} \text{LCP}[i].
\]
Suppose we are given a (sorted) list \((i_1, \ell_1), \ldots, (i_r, \ell_r)\) of irreducible lcp values (i.e., PLCP\([i_k] = \ell_k\)) of string \(T\). Since all other lcp values can be derived from this list using Lemma\(^2\) we can rewrite the above formula (letting \(i_{r+1} = n + 1\)) as:

\[
d = \frac{n(n+1)}{2} - \sum_{i=1}^{r} f(\ell_i, i_{i+1} - i_i),
\]

where

\[
f(v, d) = \begin{cases} \frac{v(v+1)}{2} & \text{if } v < d \\ d(v - d) + \frac{d(d+1)}{2} & \text{otherwise} \end{cases}
\]

Thus, by Theorem\(^{10}\) we immediately obtain the following result.

**Theorem 20.** Given string \(T[1..n]\) over alphabet \([1..\sigma]\) of size \(\sigma \leq n\) stored in \(O(n/\log_\sigma n)\) space, we can compute the number \(d\) of distinct substrings of \(T\) in \(O(n/\log_\sigma n + r \log^{11}n)\) time and \(O(n/\log_\sigma n + r \log^{10}n)\) working space, where \(r\) is the number of runs in the BWT of \(T\).

### 9.2 Longest substring occurring \(k\) times

We now show how to utilize the techniques from this paper to solve time- and space-optimally a slightly less trivial problem. Suppose that we want to find the length \(\ell\) of the longest substring of \(T\) that occurs in \(T\) at least \(2 \leq k = \Omega(1)\) times. This amounts to simply computing

\[
\ell = \max_{i=1}^{n-k+2} \min_{j=0}^{k-2} \text{LCP}[i + j].
\]

For \(k = 2\) the above formula can be evaluated by only looking at irreducible lcp values, i.e., using the definition from previous section, \(\ell = \max_{i=1}^{\tau-1} \ell_i\). For \(k > 2\), this does not work, since we have to inspect blocks of LCP values of size \(k-1\) in “lex-order”. Even if we build the small-space LCP-access structure from Section\(^{5}\) this still requires \(\Omega(n)\) queries. We can, however, utilize observations from previous sections.

More precisely, recall from Section\(^{7}\) that for any \(\tau\), up to a shift by a constant, there is only \(r \tau\) different blocks of size \(\tau\) in SA, i.e., for any block block \(SA[i..i + \tau - 1]\) there exists \(k\) such that \(SA[i..i + \tau - 1] = k + SA[j..j + \tau - 1]\) and BWT\([j..j + \tau - 1]\) contains a BWT-run boundary. We now observe that analogous property holds also for the LCP array: for any block LCP\([i..i + \tau - 1]\) there is \(k\) (the same as above) such that LCP\([i..i + \tau - 1] = \text{LCP}[j..j + \tau - 1] - k\) and BWT\([j..j + \tau - 1]\) contains boundary of some BWT-run. This implies that we only need to precompute and store the minimal value inside blocks of LCP of length \(k-1\) that are not further than \(\tau\) positions from the closest BWT-run boundary. All other blocks of LCP can be handled using the above observation and the structure for computing the LF-shortcut for any block of BWT from Section\(^{7}\) after a suitable overlap (by at least \(k\)) of blocks of size \(\tau = \Omega(\text{polylog } n)\), we can get the answer for all remaining blocks in \(O(n/\text{polylog } n + r \text{ polylog } n)\) time. We omit the details and state the result as follows:

**Theorem 21.** Given text \(T[1..n]\) over alphabet \([1..\sigma]\) of size \(\sigma \leq n\) stored in \(O(n/\log_\sigma n)\) words of space, we can find the length \(\ell\) of the longest substring occurring at least \(k = \Omega(1)\) times in \(T\) in optimal time and space under a weak assumption \(n/r = \Omega(\text{polylog } n)\) on the repetitiveness of the input.
10 Concluding remarks

In this paper we proposed the first algorithms constructing all essential components of nearly every index developed in the last two decades: the Burrows-Wheeler transform (BWT), the permuted longest-common-prefix (PLCP) array, and the LZ77 parsing, in optimal time and space for highly repetitive strings, i.e., $O(n/\log_\sigma n + r \text{polylog } n)$. Our results have particularly important implications for bioinformatics, where most of the data is highly-repetitive and over small (DNA) alphabet. Furthermore, our techniques imply a time- and space-optimal solutions for a range of fundamental problems arising in sequence analysis and data compression such as: Lyndon factorization, construction of run-length compressed suffix arrays, and a number of simple “textbook” problems such as computing the longest substring occurring at least $k$ times.

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