Floquet Topological States in Shaking Optical Lattices

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In this letter we propose realistic schemes to realize topologically nontrivial Floquet states by shaking optical lattices, using both one-dimension lattice and two-dimensional honeycomb lattice as examples. The topological phase in the two-dimensional model exhibits quantum anomalous Hall effect. The transition between topological trivial and nontrivial states can be easily controlled by shaking frequency and amplitude. Our schemes have two major advantages. First, both the static Hamiltonian and the shaking scheme are sufficiently simple to implement. Secondly, it requires relatively small shaking amplitude and therefore heating can be minimized. These two advantages make our scheme much more practical.

Topological state of matters such as quantum Hall effect and topological insulator have been extensively studied in equilibrium systems. Recently, topological classification of quantum states in a periodically driven nonequilibrium system has been proposed [11,12], in which the topologically nontrivial states are named as “Floquet topological insulator” [11]. Floquet topological band has been first realized in photonic crystal and the edge state of light has been observed [3]. While so far it has not been realized in any solid-state or cold-atom system.

Realizing and studying topological state of matter is also one of the major trends for cold atom physics nowadays, for which Raman laser coupling [17] and shaking optical lattice [8,10] have been developed as two major schemes. In several recent experiments, it has been demonstrated that fast shaking optical lattices can generate synthetic abelian gauge field and magnetic flux [8,9]. In this letter we propose that shaking optical lattice is also a powerful tool to realize Floquet topological state in cold atom systems.

We first demonstrate that in one-dimension lattice it realizes a system equivalent to Su-Schrieffer-Heeger model [11] with nonzero Zak phase; Then, we show that in two-dimension honeycomb lattice [12] it realizes a system equivalent to the Haldane model which exhibits quantum anomalous Hall effect [13]. So far, quantum anomalous Hall effect has only been found in chromium-doped (Bi,Sb)$_3$Te$_3$, and growing this material is extremely challenging [14]. It is therefore highly desirable that one can quantum simulate this effect with cold atom system. However, despite of several proposals [15] this effect has not yet been successfully simulated in cold atom setup. Our scheme has two major advantages for which it becomes much practical.

The first is its simplicity. To realize a topological state in a static system, it usually requires particular form of hopping term. For instance, in order to realize the Haldane model [15], one needs to generate a special next-nearest range hopping term, which usually requires engineering laser-assisted tunneling in cold atom system [4–7]. In contrast, in our scheme, the static Hamiltonian is quite simple (it only contains normal nearest neighboring hopping without extra phase factor) and has been realized in different laboratories already. The beauty of this scheme is that such a simple static Hamiltonian can result in a topological nontrivial state when a proper shaking term is turn on, and such a shaking can also be very easily implemented by time-periodically modulating the relative phase of two counter-propagating laser beams.

The second is minimizing heating. In our proposals, it only demands quite small shaking amplitude in order to reach topological phase, and consequently heating is minimized. In fact, our proposals are inspired by recent experiment in Chicago group, in which s-band and p-band of a one-dimensional optical lattice is resonantly coupled by shaking. It is shown that the band dispersion can be qualitatively changed even with small shaking amplitude and the accompanying heating is insignificant [10].

**General Method.** Our theoretical treatment of shaking optical lattices is based on the Floquet theory. For a periodical driven Hamiltonian $\hat{H}(t)$ with period $T$, its Floquet operator can be defined as

$$\hat{F} = \hat{U}(T_0 + T, T_0) = \hat{T} \exp \left\{ -i \int_{T_0}^{T_0 + T} dt \hat{H}(t) \right\}, \tag{1}$$

where $\hat{T}$ denotes time-order, and $T_0$ is the initial time. The eigenvalue and eigenstates of $\hat{F}$ is given by

$$\hat{F} |\varphi_n\rangle = e^{-i\epsilon_n T} |\varphi_n\rangle, \tag{2}$$

where $\epsilon_n = \epsilon_n(T) = \pi/T < \epsilon_n < \pi/T$ is the quasi-energy. In this work we shall use two different methods listed below to show how simple shaking schemes can result in nontrivial topology in optical lattice systems, because each method has its own advantage.

**Method I.** We can numerically evaluate Floquet operator $\hat{F}$ according to Eq. 1 and determine its eigenvalues and eigen-wave-functions from Eq. 2. If a periodically driven system exhibits nontrivial topological, there must be in-gap quasi-energies $\epsilon$ and their corresponding wave functions $\varphi$ spatially well localized at the edge of the system $2$. The advantage of this method is that once $\hat{H}(t)$ is given, it is free from any further approximations.
FIG. 1: (a) The typical energy structure under consideration. (b) The laser setup of the one-dimensional shaking optical lattice. Solid and dashed line represent lattice potential at two different time. (c) The laser setup of the two-dimensional honeycomb optical lattice. The dashed circle with arrow indicates how lattice potential rotates in time.

Method II. We can introduce a time-independent effective Hamiltonian \( \hat{H}_{\text{eff}} \) via \( \hat{F} = e^{-i\hat{H}_{\text{eff}}T} \). Expanding \( \hat{F}(t) \) as \( \hat{F}(t) = \sum_{n=-\infty}^{\infty} \hat{H}_n(t) e^{i\omega nt} \) with \( \omega = 2\pi/T \), we consider a situation as shown in Fig.1(a), that is, the static component \( \hat{H}_0 \) contains \( m \)-bands within an energy range of \( \Delta \) and \( \omega \gg \Delta \). The two concrete examples discussed later either belong to this situation or can be transferred into this situation by a rotating wave transformation, with \( m = 2 \). Under this condition, it is straightforward to show that to the leading order of \( \Delta/\omega \), \( \hat{H}_{\text{eff}} \) can be deduced as

\[
\hat{H}_{\text{eff}} = \hat{H}_0 + \sum_{n=1}^{\infty} \left\{ \frac{[\hat{H}_n, \hat{H}_{-n}]}{n\omega} - \frac{[\hat{H}_n, \hat{H}_0]}{e^{-2\pi i n\omega} n\omega} + \frac{[\hat{H}_{-n}, \hat{H}_0]}{e^{2\pi i n\omega} n\omega} \right\}
\]

(3)

where \( T_i \) is taken as \( \alpha T \) with \( 0 \leq \alpha < 1 \). Since \( \hat{F} \) with different choices of initial time \( T_i \) relate to each other by a unitary transformation, quasi-energy is independent of the choice of \( T_i \). Practically, we can choose an optimal \( \alpha \) that simplifies \( \hat{H}_{\text{eff}} \). Then we can apply schemes developed for a time-independent Hamiltonian to classify topology of \( \hat{H}_{\text{eff}} \). Although this method involves further approximations, it has the advantage that it is physically more transparent and can bring out the connection to topological phenomena in equilibrium systems.

One-Dimensional Case. A one-dimensional lattice is formed by two counter-propagating lasers. As one time-periodically modulates the relative phase \( \theta \) between two lasers, it will result in a time-dependent lattice potential, as shown in Fig.1(b),

\[
H(t) = \frac{k_x^2}{2m} + V \cos^2[k_x x + \theta(t)]
\]

(4)

where \( \theta(t) = k_x b \cos(\omega t) \), and \( b \) is the maximum lattice displacement. By transferring to the moving frame, \( x \rightarrow x + b \cos(\omega t) \), the Hamiltonian acquires a time-dependent vector potential term as

\[
H(t) = \frac{k_x^2}{2m} + V \cos^2[k_x x - b\omega \sin(\omega t) \cdot \hat{k}_x].
\]

(5)

The first two static terms give a static band structure with Bloch wave function \( \phi_\lambda(k_x) \). In this basis, by only keeping \( s- \) and \( p- \) band (for the reason which will be clear later), we can write down a tight-binding Hamiltonian as

\[
\hat{H}(t) = \sum_i \hat{\Psi}_i^\dagger K(t) \hat{\Psi}_i + \sum_i \hat{\Psi}_i^\dagger J(t) \hat{\Psi}_i+1 + h.c.
\]

(6)

where \( \hat{\Psi}_i = (\hat{a}_{i,s}^\dagger, \hat{a}_{i,s-1}^\dagger) \) are creation operators for \( s- \) and \( p- \) orbitals. And

\[
K(t) = \begin{pmatrix}
\epsilon_p & i h_{1p}^s \sin(\omega t) \\
-i h_{1p}^s \sin(\omega t) & \epsilon_s \\
\end{pmatrix},
\]

(7)

\[
J(t) = \begin{pmatrix}
t_p - i h_{2p}^{sp} \sin(\omega t) & i h_{1p}^{sp} \sin(\omega t) \\
-i h_{1p}^{sp} \sin(\omega t) & t_s - i h_{2p}^{sp} \sin(\omega t) \\
\end{pmatrix},
\]

(8)

where \( \epsilon_s \) and \( \epsilon_p \) is the onsite energy, \( t_s \) and \( t_p \) are the hopping amplitude from the static part. \( h_{0p}^s \) denotes shaking-induced on-site coupling between the \( s- \) and \( p- \) band, and \( h_{1p}^{sp}, h_{1s}^s, h_{1p}^{sp} \) denote shaking-induced nearest neighboring hopping within \( s- \) and \( p- \) band, between \( s- \) and \( p- \) band, respectively \( \text{[17]} \). For a given lattice depth \( V \), \( \epsilon_s, \epsilon_p, t_s, t_p \) are fixed, and \( h_{0p}^s, h_{1s}^s, h_{1p}^{sp} \) scale linearly with \( k_x b \).

With the Hamiltonian Eq. (6) and Method I, we find phase transitions between topological trivial and nontrivial phase, by changing frequency via \( \Delta_0 = (\epsilon_p - \epsilon_s - 2\omega)/2 \) and shaking amplitude \( k_x b \). A phase diagram is shown in Fig.2(d). The topological nontrivial state possess a pair of in-gap states in the quasi-energy spectrum of a finite size lattice, as shown in Fig.2(b), whose corresponding wave functions are well localized in the edges, see Fig.2(c). In contrast, in the topological trivial regime, there is no in-gap state in the quasi-energy spectrum, see Fig.2(a). As one can see clearly in Fig.2(d), even for relatively small shaking amplitude \( k_x b \approx 0.1 \), there is a quite large regime for topological nontrivial phase in the phase diagram.

To understand the emergence of topological nontrivial phase, we write the Hamiltonian into momentum space as \( H = \sum_{k_x} \hat{\Psi}_k^\dagger \hat{H}_{k_x} \hat{\Psi}_{k_x} \) (\( k_x \) in unit of \( 1/a \), \( a \) is lattice spacing) and \( \hat{H}_{k_x} \) is given by

\[
\hat{H}_{k_x} = \begin{pmatrix}
\epsilon_p + 2t_p \cos k_x & 0 & 0 \\
0 & \epsilon_s + 2t_s \cos k_x & \sin(\omega t) \\
2h_{1p}^{sp} \sin k_x & i(h_{1p}^{sp} + 2h_{1s}^{sp} \cos k_x) & 2h_{1s}^{sp} \sin k_x \\
-i(h_{1p}^{sp} + 2h_{1s}^{sp} \cos k_x) & 0 & 2h_{1s}^{sp} \sin k_x \\
\end{pmatrix}
\]

(9)
With two-phonon resonance condition $2\omega \approx \epsilon_p - \epsilon_s$, $p$-band with dispersion $\epsilon_p + 2t_p \cos(k_x)$ and two-phonon dressed $s$-band with dispersion $\epsilon_s + 2\omega + 2t_s \cos(k_x)$ form two close bands as schematized in Fig. 1(a). Therefore, we shall first apply a unitary rotation $O(t) = \exp(i\omega t\sigma_z)$ that leads to

$$\hat{H}_{\text{rot}}(t) = \hat{H}_0 + \sum_{n=\pm1,\pm3} \hat{H}_n e^{i\omega nt}.$$  \hspace{1cm} (10)

Here $\hat{H}_0 = (\Delta_0 + 2t \cos k_x\sigma_z)$, $\hat{H}_1 = -i\hbar_1 \sin k_x \sigma_z - (h_0^{\text{pp}} + 2h_1^{\text{sp}} \cos k_x) \sigma_z/2$, $\hat{H}_2 = (h_0^{\text{pp}} + 2h_1^{\text{sp}} \cos k_x) \sigma_z/2$, $\hat{H}_{-n} = \hat{H}_n^\dagger$, where $2t = t_p - t_s$, $2\hbar_1 = h_1^{\text{pp}} - h_1^{\text{sp}}$. In the tight-binding regime, we have $\omega \gg \Delta_0, t, h_0^{\text{pp}}, h_1^{\text{sp}}$. Thus, it fulfills the condition to apply Method II. By choosing $T_i = T/4$ and following the formula Eq. 3 the effective Hamiltonian can be deduced as $H_{\text{eff}} = B(k_x) \cdot \sigma$ where $B_x = 0$ and

$$B_y = \frac{2}{\omega} \left( h_0^{\text{pp}} + 2h_1^{\text{sp}} \cos k_x \right) \left[ \hbar_1 \sin k_x - \frac{2}{3} \left( \Delta_0 + 2t \cos k_x \right) \right]$$

$$B_z = \Delta_0 + 2t \cos k_x + \frac{2}{3\omega} \left( h_0^{\text{pp}} + 2h_1^{\text{sp}} \cos k_x \right)^2.$$  \hspace{1cm} (11)

This describes a momentum-dependent magnetic field in the $yz$ plane of the Bloch sphere, which is analogous to momentum space representation of Su-Schrieffer-Heeger model \[11\]. Su-Schrieffer-Heeger model exhibits topological nontrivial phase characterized by non-zero Zak phase \[13\], which has been realized and measured recently in double-well optical lattice \[19\]. Whether the system is topologically trivial or not depends on whether $B(k_x)$ has a nonzero winding number in the $yz$ plane as $k_x$ changes from $-\pi$ to $\pi$. As shown in the inset of Fig. 2(a) and (b), for topological trivial case of Fig. 2(a), $B(k_x)$ has a winding number zero; while for topological nontrivial case of Fig. 2(b), the winding number of $B(k_x)$ equals to one. From Eq. 17 it is easy to see that when $|\Delta_0|$ is large enough, $B_z$ is dominated by the constant term and therefore $B(k_x)$ has no winding. That explains why the topological nontrivial phase occurs around $\Delta_0 \approx 0$.

Before ending this part, it is worth to note that the two-phonon resonance condition plays a crucial role here. In contrast, if we consider one-phonon resonance condition $\omega \sim \epsilon_p - \epsilon_s$, it is straightforward to show by similar analysis that there will be no topological nontrivial phase \[17\].

**Two-Dimensional Case.** We employ the laser setup for a two-dimensional honeycomb lattice used by ETH group \[12\], as shown in Fig. 1(a). The interference of $X$ and $Y$ beams gives a chequerboard of spacing $\lambda/\sqrt{2}$. $X$ gives an additional standing wave with spacing $\lambda/2$. When $V_X \gg V_Y \gg V_X$, it leads to a honeycomb lattice as shown in Fig. 1(c). Using the same method as in one-dimensional case, optical lattice can be shaken in both $x$ and $y$ directions with a phase difference $\pi/2$. This gives rise to a time-dependent potential are:

$$V(x, y, t) = -V_X \cos^2[k_r(x + b \cos \omega t) + \theta/2] - V_Y \cos^2[k_r(x + b \cos \omega t) - \theta/2] - V_Z \cos^2[k_r(y + b \sin \omega t)] - 2\alpha \sqrt{V_X V_Y} \cos[k_r(x + b \cos \omega t)] \cos[k_r(y + b \sin \omega t)].$$

Here $\theta$ controls the energy offset $M$ between AB sublattice. Similar as in one-dimensional case, transferring into the comoving frame $x \rightarrow x + \lambda \cos(\omega t)$, and $y \rightarrow y + b \sin(\omega t)$, one obtains a Hamiltonian with time-dependent vector potential term

$$H(t) = \frac{1}{2m} [k - A(t)]^2 + V(x, y),$$  \hspace{1cm} (12)

where $A(x, t) = m\omega b \sin(\omega t)$ and $A_y(t) = -m\omega b \cos(\omega t)$. It is equivalent to an ac electrical field in the two-dimensional plane $E(t) = m\omega b \cos(\omega t, \sin(\omega t))$. With the tight-binding approximation and Peierls substitution, the Hamiltonian is given by

$$\hat{H}(t) = \sum_{\langle i j \rangle} \left( \hat{a}_{A,j}^\dagger, \hat{a}_{B,j}^\dagger \right) \left( \begin{array}{c} M\delta_{j,i} \cr h.c. \end{array} \right) \left( \begin{array}{c} a_{A,j} \cr a_{B,j} \end{array} \right) \left( \begin{array}{c} a_{A,i} \cr a_{B,i} \end{array} \right)$$  \hspace{1cm} (13)

where $d_{ji}$ is the vector from site $i$ pointing to site $j$, and $t_j$ is the hopping amplitude.

Applying Method I to this model, we find a similar phase diagram that contains topological trivial and nontrivial phases, as shown in Fig. 3(d). In this case, the phase diagram is controlled by parameter $M/E_r$ and
shaking amplitude \( k_r b \). Similarly, the topological trivial phase has no in-gap states in quasi-energy spectrum (Fig. 3(a)), and topological nontrivial phase has a pair in-gap states (Fig. 3(b)), whose corresponding wave function (Fig. 3(c)) is localized at the edge of the two-dimensional sample. Same as one-dimensional case, even for small shaking amplitude of \( k_r b \approx 0.1 \), the topological nontrivial regime occupies a large parameter space.

To illustrate the relation of this topological nontrivial phase with the Haldane model and the quantum anomalous Hall effect, we first expand Hamiltonian Eq. 13 as \( H(k, t) = \sum_{n=-\infty}^{\infty} \hat{H}_n(k) e^{intt} \). \( \hat{H}_0 \) gives rise to a static honeycomb lattice structure, which contains two bands with band-width \( \sim 2t_f \) and band-gap \( \sim 2M \). When \( \omega \gg 2M, 2t_f \), the condition for applying Method II is satisfied, and it yields an effective Hamiltonian \( H_{\text{eff}}(k) = B(k) \cdot \sigma \). The explicit form of \( B(k) \) is given in supplementary material 17. This effective Hamiltonian can be compared with Haldane model. If \( B(k) \) fully covers the Bloch sphere as \( k \) goes over the Brillouin zone, this phase is topologically nontrivial and exhibits quantum anomalous Hall effect 18.

For small shaking amplitude, at the leading order of \( k_r b, B_x(k) \) and \( B_y(k) \) are given by the static part of the honeycomb lattice Hamiltonian. Due to the Dirac point structure, \( \{ B_x, B_y \} \) has desired winding structure in the \( xy \) plane. \( B_z(k) \) can be written as \( M + D(k) \), and for small shaking amplitude, \( D(k) \) scales linearly with \( k_r b \). If \( |M| > D(k) \) for all \( k \), either due to small \( k_r b \) or large \( |M| \), \( B_z \) always has the same sign as \( M \) and therefore spin can only point to half of the Bloch sphere, the resulting state will still be topological trivial, as shown in Fig. 3(d).

As \( k_r b \) increases, \( D(k) \) will become larger than \( M \) in certain regime of \( k \) space. In particular, for our model, similar as the case of Haldane mode, \( D(k) \) takes opposite sign between two Dirac points (where both \( B_x \) and \( B_y \) vanish), and its absolute value is larger than \( |M| \). Thus, \( B_z \) takes opposite values between two Dirac points and the spin vector points to north and south poles, respectively, at two Dirac points. This feature, together with nontrivial winding of \( \{ B_x, B_y \} \) in the \( xy \) plane, gives rise to a topologically nontrivial coverage of spin vector in the Bloch sphere. Consequently, it enters topological nontrivial phase, with a non-zero Chern number and chiral edge state, as shown in Fig. 3. With noninteracting fermions in this setup, it will exhibit quantum anomalous Hall effect with quantized Hall conductance, which can be measured by various methods 15 20.

Final Remark: We believe the schemes and examples presented in this work open a new route toward topological states in cold atom systems. It will be more interesting to generalize the current work to three-dimension and the case with interactions.

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See supplementary materials for (i) explicit definition of parameters in the one-dimensional model; (ii) one-phonon transition regime of one-dimensional optical lattice case and (iii) the explicit definition for \( B(k) \) for two-dimensional honeycomb model.

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**SUPPLEMENTARY MATERIALS**

**Definition of Parameters**

The shaking induced hopping parameters in Eq. (6) of main text are defined as

\[
\begin{align*}
  h_{0}^{sp} &= b\omega \int dx \phi_{s}(x) \frac{\partial}{\partial x} \phi_{p}(x), \\
  h_{1}^{sp} &= b\omega \int dx \phi_{s}(x-a) \frac{\partial}{\partial x} \phi_{p}(x), \\
  h_{1}^{ss} &= b\omega \int dx \phi_{s}(x-a) \frac{\partial}{\partial x} \phi_{s}(x), \\
  h_{1}^{pp} &= b\omega \int dx \phi_{p}(x-a) \frac{\partial}{\partial x} \phi_{p}(x).
\end{align*}
\]

Here \( \phi_{s}(x) \) and \( \phi_{p}(x) \) are the wave functions of s-orbit and p-orbit. \( a = \pi/k \) is the lattice constant in one dimensional optical lattice.

**One-Phonon Transition**

Similarly, we shall first apply a unitary rotation \( O(t) = \exp(i\omega t\sigma_{z}/2) \) to make p-band and one-phonon dressed s-band nearly degenerate. Then, following formula we reach an effective Hamiltonian as:

\[
\begin{align*}
  H_{\text{eff}} &= \left[ \Delta_{0} + \frac{\omega}{2} + 2t \cos k_{x} - \frac{(h_{0}^{sp} + 2h_{1}^{sp} \cos k_{x})^{2}}{8\omega} \right] \sigma_{z} \\
  &\quad + \frac{h_{0}^{pp} + 2h_{1}^{sp} \cos k_{x}}{\omega} \left( \Delta_{0} + \frac{3\omega}{2} + 2t \cos k_{x} \right) \sigma_{y}
\end{align*}
\]  

(14)

It is obviously that the constant term in \( B_{y} \) is about \( \Delta_{0} + 3\omega/2 \sim \omega \), which dominates the \( k_{x} \)-dependent terms, \( \omega \gg t \). In this situation, \( B_{y} \) cannot change the sign as \( k_{x} \) changes from \( -\pi \) to \( \pi \), and gives zero winding. So it is in the topological trivial phase. As mentioned previously, this is confirmed by the Method I, where no edge state is found in the one-phonon resonance regime.

The effective Hamiltonian of shaked honeycomb lattice

Transforming the Hamiltonian (12) in the main text into the momentum space, one obtains:

\[
\hat{H}(t) = \sum_{k} \left( a_{A}^{\dagger}(k), a_{B}^{\dagger}(k) \right) H(k, t) \left( a_{A}(k), a_{B}(k) \right),
\]  

(15)
Employing the Bessel functions, one can expanded this Hamiltonian as $H(k, t) = \left( \begin{array}{c} M t_1 e^{-i[k_x - A_x(t)]a_1} + t_2 e^{i[k_x - A_x(t)]a_2 + i[k_y - A_y(t)]a_3} + t_2 e^{i[k_x - A_x(t)]a_2 - i[k_x - A_x(t)]a_1} \\ -M \end{array} \right)$.

Employing the Bessel functions, one can expanded this Hamiltonian as $H(k, t) = \sum_{n=-\infty}^{\infty} H_n(k) e^{i n \omega t}$. For simplicity, we only keep to $n = 0, \pm 1$ terms. Such an approximation is valid for the small shaking amplitude. Then one obtains

$$H_0(k) = M \sigma_z + B_x(k) \sigma_x + B_y(k) \sigma_y$$

$$H_1(k) = -[u_x(k) + iv_x(k)] \sigma_x + [u_y(k) + iv_y(k)] \sigma_y$$

and $H_{-1}(k) = H_1^*(k)$, where

$$B_x(k) = t_1^0 \sin(kx a_1) - t_2^0 \sin(k_y a + k_x a_2) + t_2^2 \sin(k_y a - k_x a_2),$$

$$B_y(k) = t_1^0 \cos(kx a_1) + t_2^0 \cos(k_y a + k_x a_2) + t_2^2 \cos(k_y a - k_x a_2),$$

$$u_x(k) = 2t_1^0 \sin(k_y a) \cos(k_x a_2),$$

$$u_y(k) = 2t_2^0 \sin(k_y a) \sin(k_x a_2),$$

$$v_x(k) = t_1^0 \sin(kx a_1) + 2t_2^0 \cos(k_y a) \sin(k_x a_2),$$

$$v_y(k) = t_1^0 \cos(kx a_1) - 2t_2^0 \cos(k_y a) \cos(k_x a_2).$$

and

$$t_1^0 = t_1 J_0(a_1 \omega b),$$

$$t_2^0 = t_2 J_0 \left( \sqrt{a^2 + a_2^2} \omega b \right),$$

$$t_1'' = t_1 J_1(a_1 \omega b/2),$$

$$t_2'' = \frac{a}{\sqrt{a^2 + a_2^2}} t_2 J_1 \left( \sqrt{a^2 + a_2^2} \omega b \right),$$

$$t_2'' = \frac{a_2}{\sqrt{a^2 + a_2^2}} t_2 J_1 \left( \sqrt{a^2 + a_2^2} \omega b \right).$$

Using Eq. (3) one obtained the effective Hamiltonian (13)

$$H_{\text{eff}}(k) = H_0 + \frac{[H_1, H_{-1}]}{\omega} - \frac{[H_1, H_0]}{\omega} + \frac{[H_{-1}, H_0]}{\omega}$$

$$= [M + D(k)] \sigma_z + B_x(k) \sigma_x + B_y(k) \sigma_y,$$

where

$$D(k) = -\frac{4}{\omega} [u_x(k) v_y(k) - u_y(k) v_x(k) + v_x(k) B_y(k) + v_y(k) B_x(k)].$$

Here for convenience we choose $\alpha = 0$. One notes that $D(k)$ breaks the time reversal symmetry, and $D^*(-k) = -D(k)$. 

