A SUBSOLUTION THEOREM FOR THE MONGE-AMPE`RE EQUATION OVER AN ALMOST HERMITIAN MANIFOLD

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Abstract Let \( \Omega \subseteq M \) be a bounded domain with a smooth boundary \( \partial \Omega \), where \((M, J, g)\) is a compact, almost Hermitian manifold. The main result of this paper is to consider the Dirichlet problem for a complex Monge-Amp`ere equation on \( \Omega \). Under the existence of a \( C^2 \)-smooth strictly \( J \)-plurisubharmonic (\( J \)-psh for short) subsolution, we can solve this Dirichlet problem. Our method is based on the properties of subsolutions which have been widely used for fully nonlinear elliptic equations over Hermitian manifolds.

Key words complex Monge-Amp`ere equation, almost Hermitian manifold, a priori estimate, subsolution, \( J \)-plurisubharmonic

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1 Introduction

Let \((M, J, g)\) be a compact almost Hermitian manifold of real dimension \(2n\), and let \(\Omega \subseteq M\) be a smooth domain with a smooth boundary \(\partial \Omega\). In what follows, we denote by \(\omega\) the K`ahler form of \(g\), i.e.,

\[
\omega(X, Y) = g(JX, Y),
\]

for all smooth vector fields \(X, Y\) on \(M\). We shall consider the subsolution theorem for the Monge-Amp`ere equation

\[
\begin{aligned}
& (\sqrt{-1} \partial \bar{\partial} u)^n = e^h \omega^n & \text{in } \Omega; \\
& u = \varphi & \text{on } \partial \Omega.
\end{aligned}
\]

(1.1)

Our main result is

**Theorem 1.1** Let \(\varphi, h \in C^\infty(\bar{\Omega})\) with \(\inf_{\Omega} h > -\infty\). Suppose that there exists a strictly \(J\)-psh subsolution \(u \in C^2(\bar{\Omega})\) for eq. (1.1), that is,

\[
\begin{aligned}
& (\sqrt{-1} \partial \bar{\partial} u)^n \geq e^h \omega^n & \text{in } \Omega; \\
& u = \varphi & \text{on } \partial \Omega.
\end{aligned}
\]

(1.2)

Then there exists a unique smooth strictly \(J\)-psh solution \(u\) for eq. (1.1).

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The study of the complex Monge-Ampère equation (1.1) (on $\mathbb{C}^n$) is closely related to certain problems in geometry and complex analysis; see, for instance, [7, 13, 16] and references therein. The equation has been studied extensively over the past several decades; see [2–5, 10, 13, 14, 16, 18, 22–24, 28, 32, 34] etc. Inspired by Guan’s work [13], it is natural to assume the existence of subsolutions in order to solve eq. (1.1).

The purpose of this paper is to study the Dirichlet problem for the complex Monge-Ampère equation on a general manifold, where the almost complex structure might not be integrable; that is, a manifold, locally, does not look like $\mathbb{C}^n$. Let us remind ourselves that when the domain $\Omega \subseteq M$ admits a strictly $J$-psh defining function, the eq. (1.1) was already solved by Plis [27]. His resolution could be understood as a generalized version of [5], but the underlying structure is only almost complex. Many interesting results were also obtained by Harvey-Lawson [17].

The Dirichlet problems regarding other related geometric PDEs also attracts the attention of many mathematicians. For instance, Wang-Zhang [35] studied the Dirichlet problem for the Hermitian-Einstein equation over an almost Hermitian manifold. In addition, the twisted quiver bundle on an almost complex manifold was researched by Zhang [36]. Very recently, Li-Zheng [25] investigated the Dirichlet problem for a class of fully nonlinear elliptic equations, and obtained the boundary second order estimates. In the aspect of real case, one can also refer [1, 20].

The structure of this paper is as follows: in Section 2 we collect some basic concepts regarding almost Hermitian manifolds. In Sections 3–5 we give the global estimates up to the second order. Once we have these estimates in hand, higher order estimates can be also obtained by the classical Evans-Krylov theory (see, for instance, [32]) and the Schauder theory. Then we can use the standard continuity method to obtain the existence; the proof of this can be found in [13], so we shall omit the standard step here. In Section 6, we obtain a strictly $J$-psh subsolution for (1.1) under the existence of a strictly $J$-psh defining function.

2 Preliminaries

Let $(M, J, g)$ be a compact manifold of real dimension $2n$ with the Riemannian metric $g$ satisfying that

$$g(Ju, Jv) = g(u, v), \ \forall u, v \in TM,$$

where $J$ is the almost complex structure. Then the complexified tangent bundle can be divided as

$$TM \otimes_{\mathbb{R}} \mathbb{C} = T_{0,1}M \oplus T_{1,0}M,$$

where $T_{0,1}M$ and $T_{1,0}M$ are the $\sqrt{-1}$ and $-\sqrt{-1}$-eigenspaces of $J$. Similarly, the induced almost complex structure $J^*$ on the cotangent bundle $T^*M$ is defined by $J^*\alpha := -\alpha \circ J$. Then we have a natural decomposition

$$T^*M \otimes_{\mathbb{R}} \mathbb{C} = T^{0,1}M \oplus T^{1,0}M.$$

For brevity, we will also denote $J^*$ by $J$, if no confusion occurs. For the decomposition of the $k$-th product of a complexified cotangent bundle,

$$A^kT^*M \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=k} A^{p,q}M.$$
Let $A^{p,q}$ be the set of smooth sections on $\Lambda^{p,q}M$ and denote that

$$A^k := \bigoplus_{p+q=k} A^{p,q}.$$  

We consider the exterior derivative $d : A^k \to A^{k+1}$ satisfying $d^2 = 0$. Let $\Pi_{p+1,q}, \Pi_{p,q+1}, \Pi_{p+2,q-1}$ and $\Pi_{p-1,q+2}$ be the projection of $A^{k+1}$ to $A^{p+1,q}$, $A^{p,q+1}$, $A^{p+2,q-1}$ and $A^{p-1,q+2}$, respectively. Thus,

$$d = \partial + \bar{\partial} + T + \bar{T},$$

where

$$\partial = \Pi_{p+1,q} \circ d, \quad \bar{\partial} = \Pi_{p,q+1} \circ d, \quad T = \Pi_{p+2,q-1} \circ d, \quad \bar{T} = \Pi_{p-1,q+2} \circ d.$$  

In particular, if $v \in C^2(M, \mathbb{R})$, then $\bar{\partial} v \in A^{0,1}$ and

$$d\bar{\partial} v = \partial \bar{\partial} v + \bar{\partial}^2 v + T\bar{\partial} v.$$  

Taking the complex conjugates and adding together,

$$\bar{T} \bar{\partial} v = -\partial^2 v, \quad \partial \bar{\partial} v = -\bar{\partial} \partial v,$$

which implies that $\sqrt{-1} \bar{\partial} \bar{\partial} v$ is a real $(1,1)$ form on $M$. Based on the notation in [26, 27], letting $e_1, \cdots, e_n$ be a local $g$-orthonormal frame of $T_{1,0}M$, we define

$$v_{ij} := e_i \bar{e}_j v - [e_i, \bar{e}_j]^{(0,1)} v.$$  

Then, in this local chart,

$$\sqrt{-1} \bar{\partial} \bar{\partial} v = \sqrt{-1} \sum_{i,j=1}^n v_{ij} \theta_i \wedge \bar{\theta}_j, \quad (2.1)$$

where $\theta_1, \cdots, \theta_n$ is a local $g$-orthonormal frame of $T^{1,0}M$ dual to $e_1, \cdots, e_n$. Thus we can rewrite the equation in (1.1) as

$$\log \det(u_{ij}) = h. \quad (2.2)$$

Let us define its linearized operator by

$$L := u^{ij} (e_i \bar{e}_j - [e_i, \bar{e}_j]^{(0,1)}),$$

where $(u^{ij}) = (u_{ij})^{-1}$ is the inverse matrix. Notice that $L$ is uniformly elliptic if $u \in C^2$ is strictly $J$-psh.

**Definition 2.1** For any $v \in C^2(M, \mathbb{R})$ and with $\Omega \subseteq M$ being an open set,

1. we say that $v$ is $J$-psh on $\Omega$ if the matrix $(u_{ij})$ is nonnegative at each point of $\Omega$;
2. we say that $v$ is strictly $J$-psh on $\Omega$ if, for each $\varphi \in C^2(\Omega)$, there exists $\varepsilon_0 > 0$ such that $u + \varepsilon \varphi$ is $J$-psh on $\Omega$ for all $0 < \varepsilon < \varepsilon_0$.

We denote the set of $J$-psh functions on $\Omega$ by $\text{PSH}(\Omega)$.

Let us recall the notion of canonical connections on almost Hermitian manifolds.

Supposing that $(M, J, g)$ is an almost Hermitian manifold, there exists a canonical connection $\nabla$ on $M$ which plays a very similar role to that of the Chern connection on the Hermitian manifold. Usually, we say that a connection on $(M, J, g)$ is an almost-Hermitian connection if $\nabla g = \nabla J = 0$. Noticing that such connection always exists [21], we have the following theorem (see [12, 33]):

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Theorem 2.2 There exists a unique almost-Hermitian connection $\nabla$ on an almost Hermitian manifold $(M, J, g)$ whose $(1, 1)$ part of the torsion vanishes.

This connection was found by Ehresmann-Libermann [9]. Sometimes it is also referred to the Chern connection, because no confusion occurs when $J$ is integrable. Under a local frame like the previous one, we have that

$$\sqrt{-1} \partial \bar{\partial} v = \sqrt{-1} \sum_{i,j=1}^{n} (\nabla_j \nabla_i v) \theta_i \wedge \bar{\theta}_j. \quad (2.3)$$

2.1 Properties of Subsolution

The following lemma is due to Guan [15], who proved it for more general fully nonlinear PDEs:

Lemma 2.3 Let $u \in C^2(\bar{\Omega})$ be a strictly $J$-psh subsolution to the eq. (1.1). There exist constants $N, \theta > 0$ such that if $\sum_{i=1}^{n} u_{i\bar{i}} \geq N$ at a point $p \in \Omega$ where $g_{ij} = \delta_{ij}$ and the matrix $\{u_{i\bar{j}}\}$ is diagonal, then

$$L(u - u) \geq \theta \left( \sum_{i=1}^{n} u_{i\bar{i}} + 1 \right) \quad \text{in } \Omega. \quad (2.4)$$

Let us remark that since $\bar{u}$ is strictly $J$-psh, there exists a uniform constant $\tau \in (0, 1)$ such that

$$\sqrt{-1} \partial \bar{\partial} \bar{u} \geq \tau \omega. \quad (2.5)$$

2.2 Maximum Principle

We have the following useful lemma.

Lemma 2.4 ([5, p. 215]) Let $\Omega \subseteq M$ be a smooth bounded domain. If $u, v \in C^2(\bar{\Omega}) \cap \text{PSH}(\Omega)$ with $u$ strictly $J$-psh and $\det(u_{i\bar{j}}) \geq \det(v_{i\bar{j}})$, then $u - v$ attains its maximum on $\partial \Omega$.

3 $C^0$ and $C^1$ Estimates

3.1 Uniform Estimate

Let $\tilde{u} \in C^2(\bar{\Omega})$ be a solution of the Dirichlet problem

$$\begin{cases}
L(u) = 0 & \text{in } \Omega; \\
u = \varphi & \text{on } \partial \Omega,
\end{cases} \quad (3.1)$$

where $\tilde{u}$ could be understood as the $L$-harmonic extension of $\varphi|_{\partial \Omega}$.

Lemma 3.1 Let $u$ (resp. $\bar{u}$) be the solution (resp. subsolution) of eq. (1.1). We have that

$$\bar{u} \leq u \leq \tilde{u}. \quad (3.2)$$

Proof On the one hand, as $\bar{u}$ is a subsolution of (1.1), the first inequality follows from Lemma 2.4. On the other hand, since $L(u) = n$, we know that $u$ is a subsolution of (3.1). By the maximum principle (for operator $L$), we also get the second inequality. \qed
3.2 Boundary Gradient Estimate

Lemma 3.2 Let $u$ (resp. $\bar{u}$) be a solution (resp. subsolution) of eq. (1.1). Then there exists a constant $C = C(\|u\|_{C^1(\Omega)}, h, \varphi)$ such that

$$\max_{\partial \Omega} |\partial u| \leq C. \tag{3.3}$$

Proof By the previous lemma, together with the fact that $u, \bar{u}$ and $h$ have the same boundary value $\varphi|_{\partial \Omega}$, we have $|\partial u| \leq \sup\{|\partial u|, |\partial \bar{u}|\}$ on $\partial \Omega$, and the lemma follows. \hfill \Box

3.3 Global Gradient Estimate

Proposition 3.3 Let $u$ (resp. $\bar{u}$) be a solution (resp. subsolution) of eq. (1.1). Then

$$\max_{\overline{\Omega}} |\partial u| \leq C \tag{3.4}$$

for some positive constant $C = C(\|u\|_{C^1(\overline{\Omega})}, \|u\|_{C^0(\overline{\Omega})}, \|u\|_{C^{0,1}(\partial \Omega)}, h)$.

Proof Let $\theta = \frac{1}{2} e^{B \eta}$ for $\eta = u - \bar{u} + \sup_{\Omega} u - \bar{u}$, where $B > 0$ is a constant to be picked up later. We will prove (3.4) by applying the maximum principle to

$$V := e^{\theta} |\partial u|^2.$$ 

Suppose that $V$ achieves its maximum at $x_0 \in \text{Int}(\Omega)$. Near $x_0$, we choose a local $g$-unitary frame $(e_1, \ldots, e_n)$ such that $g_{ij} = \delta_{ij}$. Moreover, the matrix $(u_{ij})$ is diagonal at $x_0$.

At $x_0$, it follows from the maximum principle that

$$0 \geq \frac{L(V)}{B \theta e^{\theta} |\partial u|^2} = \frac{L(e^\theta)}{B \theta e^{\theta}} + \frac{L(|\partial u|^2)}{B \theta |\partial u|^2} + 2 u \bar{u} \text{Re}(e_i(\theta) \bar{e}_i(|\partial u|^2))$$

$$= L(\eta) + B(1 + \theta) |\partial u|^2 + \frac{L(|\partial u|^2)}{B \theta |\partial u|^2} + \frac{1}{|\partial u|^2} \cdot ((*) + (**)), \tag{3.5}$$

where

$$(*) := 2 \sum_{j=1}^n e_i(\eta) \bar{e}_i(\bar{e}_j(u) \bar{e}_j(u)); \tag{3.6}$$

$$(**) := 2 \sum_{j=1}^n e_i(\eta) \bar{e}_i(\bar{e}_j(u) e_j(u)). \tag{3.7}$$

\footnote{By a straightforward calculation,}

$$L(|\partial u|^2) = u \bar{u} (e_i e_i(|\partial u|^2) - [e_i, \bar{e}_i]^{0,1}(|\partial u|^2)) := I + II + III,$$

where

$$I := u \bar{u} (e_i \bar{e}_i e_j u - [e_i, \bar{e}_i]^{0,1} e_j u); \tag{3.8}$$

$$II := u \bar{u} (e_i \bar{e}_i \bar{e} j u - [e_i, \bar{e}_i]^{0,1} \bar{e}_j u); \tag{3.9}$$

$$III := u \bar{u} (|e_i \bar{e}_i u|^2 + |e_i \bar{e}_j u|^2). \tag{3.10}$$

Differentiating (2.2) along $e_j$,

$$u \bar{u} (e_j e_i \bar{e}_i u - e_j [e_i, \bar{e}_i]^{0,1} u) = h_j,$$

\footnote{The constants $C, C'$ in the rest of the section are distinct, where $C$ is a constant depending on all the allowed data, but $C'$ further depends on a constant $B$ that we are yet to choose.}
Notice that
\[
\begin{align*}
  u^{\overline{i}}(e_i\overline{e}_j e_j u - [e_i, \overline{e}_i]^{0,1} e_j u) \\
  = u^{\overline{i}}(e_j \overline{e}_i e_j u + [e_i, \overline{e}_i] e_i u + [e_i, \overline{e}_i] e_j u - [e_i, \overline{e}_i]^{0,1} e_j u) \\
  = h_j + u^{\overline{i}} [e_i, \overline{e}_i]^{0,1} u + u^{\overline{i}}(e_i e_j u + [e_i, \overline{e}_i] e_i u - [e_i, \overline{e}_i]^{0,1} e_j u) \\
  = h_j + u^{\overline{i}} \{e_i e_j u + \overline{e}_i e_j u + [e_i, \overline{e}_i] e_i u - [e_i, \overline{e}_i]^{0,1} e_j u\}.
\end{align*}
\]

We may assume that $|\partial u| \gg 1$ (otherwise we are done), and set
\[
\mathcal{U} := \sum_{i=1}^n u^{\overline{i}}.
\]

By the Cauchy-Schwarz inequality, for each $0 < \varepsilon \leq \frac{1}{2}$,
\[
I + II \geq 2\text{Re} \left( \sum_{j=1}^n h_j u_j \right) - C|\partial u| \sum_{j=1}^n u^{\overline{i}}(e_i e_j u + |e_i\overline{e}_i|^0 u) - C|\partial u|^2 \mathcal{U}
\]
\[
\geq 2\text{Re} \left( \sum_{j=1}^n h_j u_j \right) - \frac{C}{\varepsilon} |\partial u|^2 \mathcal{U} - \varepsilon \sum_{j=1}^n u^{\overline{i}}(|e_i e_j u|^2 + |e_i\overline{e}_i|^1 u^2).
\]

It then follows from (3.5) that
\[
\frac{L(|\partial u|^2)}{B \partial |\partial u|^2} \geq \frac{-C}{B \partial |\partial u|^2} + (1 - \varepsilon) \sum_{j=1}^n u^{\overline{i}} \frac{|e_i e_j u|^2 + |e_i\overline{e}_i|^1 u^2}{B \partial |\partial u|^2} - \frac{C \mathcal{U}}{B \partial |\partial u|^2}.
\]

As $0 < \varepsilon \leq \frac{1}{2}$, $1 \leq (1 - \varepsilon)(1 + 2\varepsilon)$. Thus,
\[
(*) = 2 \sum_{j=1}^n u^{\overline{i}} \text{Re} \left( e_i(\eta)\overline{e}_j(u) \{e_j \overline{e}_i(u) - [e_j, \overline{e}_i]^{1,0}(u) - [e_j, \overline{e}_i]^{0,1}(u)\} \right)
\]
\[
= 2\text{Re} \left( \sum_{j=1}^n \eta_j u_j \right) - 2 \sum_{j=1}^n u^{\overline{i}} \text{Re} \left( e_i(\eta)\overline{e}_j(u) \{e_j \overline{e}_i(u) - [e_j, \overline{e}_i]^{1,0}(u)\} \right)
\]
\[
\geq 2\text{Re} \left( \sum_{j=1}^n \eta_j u_j \right) - \varepsilon B \partial |\partial u|^2 |\partial u|^2 - \frac{C}{B \partial |\partial u|^2} |\partial u|^2 \mathcal{U};
\]

\[
(**) \geq -\frac{(1 - \varepsilon)}{B \partial} \sum_{j=1}^n u^{\overline{i}} |e_i \overline{e}_j(u)|^2 - (1 + 2\varepsilon) B \partial |\partial u|^2 u^{\overline{i}} |\partial u|^2.
\]

It follows from (3.13) and (3.14) that
\[
\frac{1}{|\partial u|^2} \cdot ((*) + (**)) \geq \frac{2\text{Re} \left( \sum_{j=1}^n \eta_j u_j \right)}{|\partial u|^2} - (1 + 3\varepsilon) B \partial u^{\overline{i}} |\partial u|^2 - \frac{C}{B \partial |\partial u|^2} |\partial u|^2 - (1 - \varepsilon) \sum_{j=1}^n u^{\overline{i}} |e_i \overline{e}_j(u)|^2.
\]

Combining (3.5), (3.12) and (3.15) gives us
\[
0 \geq L(\eta) + B(1 - 3\varepsilon |\partial u|^2 u^{\overline{i}} |\partial u|^2 - \frac{C \mathcal{U}}{B \partial |\partial u|^2} - \frac{C}{B \partial |\partial u|^2} + \frac{2\text{Re} \left( \sum_{j=1}^n \eta_j u_j \right)}{|\partial u|^2}.
\]

Hence, if we choose $\varepsilon = \frac{1}{6\partial(x_0)} \leq \frac{1}{2}$,
\[
L(\eta) + \frac{2\text{Re} \left( \sum_{j=1}^n \eta_j u_j \right)}{|\partial u|^2} + \frac{B}{2} u^{\overline{i}} |\partial u|^2 \leq \frac{C}{B \partial |\partial u|^2} + \frac{C}{B \mathcal{U}}.
\]
Case 1 \( \sum_{i=1}^{n} u_{ii} \geq N \) for some \( N \) as in Lemma 2.3. We divide the proof into two parts.

Subcase 1(i) If \( u^{ij} \geq D \) for some \( j \), where \( D > 0 \) is a large constant to be determined shortly, then

\[
L(\eta) \geq \theta + \theta u \geq \theta + \frac{D\theta}{2} + \frac{\theta}{2} U.
\]

We may assume that \(|\partial u| \geq |\partial u|\), whence \(|\partial \eta| \leq 2|\partial u|\). Thus,

\[
2\text{Re}\left( \sum_{j=1}^{n} \eta_{j} u_{j} \right) \geq -4. \quad (3.17)
\]

Substituting this into (3.16),

\[
\theta + \frac{D\theta}{2} - 4 - \frac{\theta}{2} - \frac{C}{B} |\partial \eta| \leq \frac{C}{B|\partial u|}.
\]

We may choose \( B, D \) sufficiently large such that \( \theta \geq \frac{C}{B} \) and \( D\theta \geq 8 \), whence (3.4) follows.

Subcase 1(ii) If \( u^{ij} \leq D \) for each \( j = 1, 2, \cdots, n \), since \(|\partial u| \geq \max\{1, |\partial u|\}\),

\[
2\text{Re}\left( \sum_{j=1}^{n} \eta_{j} u_{j} \right) \geq -\frac{B}{4} u^{ii} |\eta_{i}|^2 - \frac{4}{B|\partial u|^2} \sum_{i=1}^{n} u_{ii},
\]

and it follows from (3.16) that

\[
\theta + \theta u \leq \frac{C}{B|\partial u|} + \frac{C}{B} + \frac{4}{B|\partial u|^2} \sum_{i=1}^{n} u_{ii} \quad \text{(3.18)}
\]

Notice that \( \theta \geq \frac{C}{B} \). Thus,

\[
\theta \leq \frac{C}{B|\partial u|} + \frac{4}{B|\partial u|^2} \sum_{i=1}^{n} u_{ii}.
\]

It is useful to order \( \{u_{ii}\}_{i=1}^{n} \) such that \( u_{11} \geq \cdots \geq u_{nn} \) at \( x_0 \). Thus,

\[
\eta_{11} D^{-(n-1)} \leq \prod_{i=1}^{n} u_{ii} = e^h.
\]

Then we have

\[
\sum_{i=1}^{n} u_{ii} \leq n u_{11} \leq n e^{\sup h} D^{n-1}.
\]

Substituting this into (3.18), we get that \( |\partial u| \leq C' \).

Case 2 \( \sum_{i=1}^{n} u_{ii} \leq N \), so \( u^{kk} \geq N^{-1} \) for each \( k \). We have that

\[
u^{ii} |\eta_{i}|^2 \geq N^{-1}|\partial \eta|^2. \quad (3.19)
\]

The fact that \( u \) is strictly \( J \)-psh implies that

\[
L(\eta) \geq \tau u - n. \quad (3.20)
\]

It follows from (3.16), (3.17), (3.19) and (3.20) that

\[
(\tau - \frac{C}{B}) |U| + BN^{-1}|\partial \eta|^2 \leq \frac{C}{B|\partial \eta|} + 5n.
\]

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since $|\partial u| \geq \max\{1, |\partial u|\}$. We further assume that $\tau \geq \frac{C}{B}$, so

$$BN^{-1}|\partial u|^2 \leq \frac{C}{B|\partial u|} + 5n,$$

which implies $|\partial u| \leq C'$, whence (3.4) follows.

\[\square\]

4 Interior $C^2$ Estimate

In this section we follow the arguments of [8] to estimate the largest eigenvalue $\lambda_1(\hat{\nabla}^2 u)$ of the real Hessian $\hat{\nabla}^2 u$, where $\hat{\nabla}$ is the Levi-Civita connection on $M$.

**Theorem 4.1** Let $u$ (resp. $u$) be a solution (resp. subsolution) of eq. (1.1). We have

$$\max_{\Omega} \lambda_1(\hat{\nabla}^2 u) \leq C(1 + \max_{\partial \Omega} |\nabla^2 u|), \quad (4.1)$$

where $C$ is a constant depending on $\|h\|_{C^2(\Omega)}$, $\|u\|_{C^1(\Omega)}$ and $\|u\|_{C^2(\Omega)}$.

**Proof** For brevity, we denote $\phi := u - u + \sup_{\Omega}(u - u) + 1$. Define

$$Q := \log \lambda_1(\hat{\nabla}^2 u) + \phi(|\nabla^2 u|) + e^{B\phi}$$

in $\Omega' := \{\lambda_1(\hat{\nabla}^2 u) > 0\} \subseteq \Omega$, where $B$ is a large constant to be determined later, and $\phi$ is defined by

$$\phi(s) := \frac{1}{2} \log(1 + \sup_{\Omega} |\nabla^2 u| - s).$$

Setting $K := 1 + \sup_{\Omega} |\nabla^2 u|$, we have that

$$\frac{1}{2K} \leq \phi'(|\nabla^2 u|) \leq \frac{1}{2}, \quad \phi'' = 2(\phi')^2.$$

We may assume that $\Omega'$ is a nonempty (relative) open set, otherwise we are done. As $z$ approaches $\partial \Omega \setminus \partial \Omega$, $Q \to -\infty$, if $Q$ achieves its maximum on $\partial \Omega$, then we are done, by (4.1). Thus, we may assume that $Q$ achieves its maximum in $\operatorname{Int}(\Omega')$. Near $x_0$, we choose a local $g$-unitary frame $(e_1, \ldots, e_n)$ such that, at $x_0$,

$$g_{ij} = \delta_{ij}, \quad u_{ij} = \delta_{ij} u_i \quad \text{and} \quad u_{i1} \geq u_{i2} \geq \cdots \geq u_{in}. \quad (4.2)$$

In addition, there exists a normal coordinate system $(U, \{x^\alpha\}_{i=1}^{2n})$ in a neighbourhood of $x_0$ such that

$$e_i = \frac{1}{\sqrt{2}}(\partial_{2i-1} - \sqrt{-1}\partial_{2i}) \quad \text{for} \quad i = 1, \ldots, n; \quad (4.3)$$

$$\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} = 0 \quad \text{for} \quad \alpha, \beta, \gamma = 1, \ldots, 2n, \quad (4.4)$$

where $g_{\alpha\beta} := g(\partial_{\alpha}, \partial_{\beta})$.

We define an endomorphism $\Phi = (\Phi^\gamma_\beta)$ of $TM$ by

$$\Phi^\gamma_\beta := g^{\alpha\gamma}(\hat{\nabla}^2_{\gamma\beta} u - S_{\gamma\beta}),$$

for some smooth section $S$ on $T^* M \otimes T^* M$ such that

$$\lambda_1(\Phi) \leq \lambda_1(\hat{\nabla}^2 u) \quad \text{in} \quad \Omega', \quad \square_{\text{Springer}}$$
with the equality only at $x_0$, but also $\lambda_1(\Phi) \in C^2(\Omega)$ (cf. [8, 11]). For any $\beta$, let $V_\beta$ be eigenvector of $\Phi$ with an eigenvalue $\lambda_\beta$. The proof needs the following derivatives of $\lambda_1$, which can be found in [8, 11, 29]:

**Lemma 4.2** At $x_0$, we have that

$$\frac{\partial \lambda_1}{\partial \Phi^\beta} = V_1^\alpha V_1^\beta;$$

$$\frac{\partial^2 \lambda_1}{\partial \Phi^\beta \partial \Phi^\delta} \geq \sum_{\kappa > 1} \frac{1}{\lambda_1 - \lambda_\kappa} (V_1^\alpha V_\kappa^\beta V_1^\delta + V_\kappa^\alpha V_1^\beta V_\kappa^\delta). \quad (4.5)$$

We will prove (4.1) by applying the maximum principle to the quantity

$$Q := \log \lambda_1(\Phi) + \phi(|\partial u|^2) + \phi(\omega).$$

Clearly, $Q$ attains its maximum at $x_0$. Thus, at $x_0$,

$$\frac{1}{\lambda_1} e_i(\lambda_1) = -\phi' e_i(|\partial u|^2) - Be^{B\omega} \omega_i, \quad \text{for all } 1 \leq i \leq n; \quad (4.6)$$

$$0 \geq L(Q) = \frac{L(\lambda_1)}{\lambda_1} - u^i \frac{|e_i(\lambda_1)|^2}{\lambda_1^2} + \frac{1}{\lambda_1} u^i u^{kk} |V_1(u_{ik})|^2$$

$$+ \phi' L(|\partial u|^2) + Be^{B\omega} L(\omega) + B^2 e^{B\omega} u^i |\omega_i|^2. \quad (4.7)$$

For the rest of this section we may assume that $\sum_{i=1}^n u_{ii} \geq N$ for the constant $N$ in Lemma 2.3 (otherwise we are done).

### 4.1 Lower Bound of $L(Q)$

**Proposition 4.3** For each $\varepsilon \in (0, \frac{1}{2}]$, at $x_0$, we have that

$$0 \geq L(Q)$$

$$\geq (2 - \varepsilon) \sum_{\alpha > 1} u^i \frac{|e_i(uV_\alpha V_1)|^2}{\lambda_1(1 - \lambda_\alpha)} + \frac{1}{\lambda_1} u^i u^{kk} |V_1(u_{ik})|^2$$

$$- (1 + \varepsilon) u^i \frac{|e_i(\lambda_1)|^2}{\lambda_1^2} - C \varepsilon \mathcal{U} + \phi' \varepsilon \sum_{j=1}^n u^i (|e_i e_j u|^2 + |e_i e_j|^2)$$

$$+ \phi'' u^i |e_i(|\partial u|^2)|^2 + Be^{B\omega} L(\omega) + B^2 e^{B\omega} u^i |\omega_i|^2. \quad (4.8)$$

**Proof** First, we calculate $L(\lambda_1)$. Let

$$u_{ij} = e_i e_j u - (\nabla_{e_i} e_j) u, \quad u V_\alpha V_\beta = u_{k\alpha} V_\alpha^k V_\beta^j.$$  

By Lemma 4.2 and (4.4), we can infer that

$$L(\lambda_1) = u^i \frac{\partial^2 \lambda_1}{\partial \Phi^\beta \partial \Phi^\delta} e_i(\Phi^\alpha) e_i(\Phi^\beta) + u^i \frac{\partial \lambda_1}{\partial \Phi^\beta} (e_i \bar{e}_i - [e_i, \bar{e}_i]^{0,1})(\Phi^\beta)$$

$$= u^i \frac{\partial^2 \lambda_1}{\partial \Phi^\beta \partial \Phi^\delta} e_i(u_{\alpha\delta}) e_i(\alpha_{\beta}) + u^i \frac{\partial \lambda_1}{\partial \Phi^\beta} (e_i \bar{e}_i - [e_i, \bar{e}_i]^{0,1})(\alpha_{\beta}) + u^i \frac{\partial \lambda_1}{\partial \Phi^\beta} u_{\alpha\beta} e_i(\alpha^\gamma)$$

$$\geq 2 \sum_{\alpha > 1} u^i \frac{|e_i(uV_{\alpha} V_1)|^2}{\lambda_1 - \lambda_\alpha} + u^i (e_i \bar{e}_i - [e_i, \bar{e}_i]^{0,1})(uV_1 V_1) - C \mathcal{U} \mathcal{U}. \quad (4.9)$$

Applying $V_1$ to eq. (2.2) twice,

$$u^i V_1(u_{ii}) = u^i u^{kk} |V_1(u_{ik})|^2 + V_1 V_1(h). \quad (4.10)$$
Lemma 4.4  If \( \lambda_1 \gg 1 \), then

\[
\begin{align*}
    u^{i\overline{i}}(e_i\overline{e_i} - [e_i, \overline{e_i}])^{0,1}(u V_1 V_i) & \\
    \geq u^{i\overline{i}}u^{k\overline{k}}|V_1(u_{ik})|^2 - C\lambda_1 U - 2u^{i\overline{i}}\{[V_1, \overline{e_i}]V_i e_i(u) + [V_1, e_i]V_i \overline{e_i}(u)\}. \tag{4.11}
\end{align*}
\]

Proof  By a direct calculation,

\[
\begin{align*}
    u^{i\overline{i}}(e_i\overline{e_i} - [e_i, \overline{e_i}])^{0,1}(u V_1 V_i) & \\
    = u^{i\overline{i}}e_i\overline{e_i}(V_1 V_i(u) - (\tilde{V}_1 V_i)u) - u^{i\overline{i}}[e_i, \overline{e_i}]^{0,1}(V_1 V_i(u) - (\tilde{V}_1 V_i)u) \\
    \geq u^{i\overline{i}}V_1 V_i(e_i\overline{e_i}(u) - [e_i, \overline{e_i}]^{0,1}(u)) - 2u^{i\overline{i}}\{[V_1, \overline{e_i}]V_i e_i(u) + [V_1, e_i]V_i \overline{e_i}(u)\} \\
    & \quad - u^{i\overline{i}}(\tilde{V}_1 V_i)e_i\overline{e_i}(u) + u^{i\overline{i}}(\tilde{V}_1 V_i)[e_i, \overline{e_i}]^{0,1}(u) - C\lambda_1 U \\
    \geq u^{i\overline{i}}V_1 V_i(u_{ii}) - 2u^{i\overline{i}}\{[V_1, \overline{e_i}]V_i e_i(u) + [V_1, e_i]V_i \overline{e_i}(u)\} + (\tilde{V}_1 V_i)(h) - C\lambda_1 U.
\end{align*}
\]

Then the lemma follows from (4.10) if \( \lambda_1 \gg 1 \).

It follows from (4.9) and (4.11) that

\[
L(\lambda_1) \geq 2 \sum_{\alpha > 1} \frac{u^{i\overline{i}}|e_i(u V_1 V_i)|^2}{\lambda_1 - \lambda_\alpha} + u^{i\overline{i}}u^{k\overline{k}}|V_1(u_{ik})|^2 \\
\quad - 2u^{i\overline{i}}\text{Re}([V_1, e_i]V_i \overline{e_i}(u) + [V_1, \overline{e_i}]V_i e_i(u)) - C\lambda_1 U. \tag{4.12}
\]

By (3.12), we have that

\[
L(|\partial u|^2) \geq \frac{1}{2} \sum_{j=1}^n u^{i\overline{i}}(|e_i e_j u|^2 + |e_i \overline{e_j} u|^2) - C U. \tag{4.13}
\]

Thus,

\[
L(Q) \geq 2 \sum_{\alpha > 1} \frac{u^{i\overline{i}}|e_i(u V_1 V_i)|^2}{\lambda_1(\lambda_1 - \lambda_\alpha)} + \frac{u^{i\overline{i}}u^{k\overline{k}}|V_1(u_{ik})|^2}{\lambda_1} + B^2 \frac{u^{i\overline{i}}}u |\overline{v}_1|^2 \\
\quad + B e^B \overline{u} L(\overline{z}) - 2u^{i\overline{i}}\text{Re}([V_1, e_i]V_i \overline{e_i}(u) + [V_1, \overline{e_i}]V_i e_i(u)) - C U \\
\quad - u^{i\overline{i}}|e_i(\lambda_1)|^2 \frac{\lambda_1^2}{\lambda_1^2} + \frac{\phi'}{2} \sum_{j=1}^n u^{i\overline{i}}(|e_i e_j u|^2 + |e_i \overline{e_j} u|^2) + \phi'' u^{i\overline{i}}|e_i(|\partial u|^2)|^2. \tag{4.14}
\]

Lemma 4.5  For each \( 0 < \varepsilon \leq 1/2 \), we have that

\[
2u^{i\overline{i}}\text{Re}([V_1, e_i]V_i \overline{e_i}(u) + [V_1, \overline{e_i}]V_i e_i(u)) \leq \varepsilon u^{i\overline{i}}|e_i(\lambda_1)|^2 \frac{\lambda_1^2}{\lambda_1^2} + \varepsilon \sum_{\alpha > 1} u^{i\overline{i}}|e_i(u V_1 V_i)|^2 \frac{\lambda_1(\lambda_1 - \lambda_\alpha)}{\lambda_\alpha} + \frac{C}{\varepsilon} U. \tag{4.15}
\]

Proof  Assume that

\[
[V_1, e_i] = \sum_{\beta=1}^{2n} \mu_{ij} V_\beta, \quad [V_1, \overline{e_i}] = \sum_{\beta=1}^{2n} \overline{\mu_{ij}} V_\beta,
\]

where \( \mu_{ij} \in \mathbb{C} \) are uniformly bounded constants. Then,

\[
\text{Re}([V_1, e_i]V_i \overline{e_i}(u) + [V_1, \overline{e_i}]V_i e_i(u)) \leq C \sum_{\beta=1}^{2n} |V_\beta V_i e_i(u)|. \tag{4.16}
\]
This reduces to estimate \( \frac{1}{\lambda_1} \sum_{\beta} u^{\bar{\alpha}} |V_\beta V_1 e_i(u)| \). Recalling the definition of Lie bracket \( e_i e_j - e_j e_i = [e_i, e_j] \), we have that
\[
|V_\beta V_1 e_i(u)| = |e_i V_\beta V_1 (u) + V_\beta [V_1, e_i](u) + [V_\beta, e_i]V_1(u)|
\]
\[
= |e_i(u V_\beta V_1) + e_i(\nabla V_\beta V_1)(u) + V_\beta [V_1, e_i](u) + [V_\beta, e_i]V_1(u)|
\]
\[
\leq |e_i(u V_\beta V_1)| + C \lambda_1.
\]
Therefore,
\[
\sum_{\beta=1}^{2n} u^{\bar{\alpha}} \frac{|V_\beta V_1 e_i(u)|}{\lambda_1} \leq \sum_{\beta=1}^{2n} u^{\bar{\alpha}} \frac{|e_i(u V_\beta V_1)|}{\lambda_1} + CU
\]
\[
= u^{\bar{\alpha}} \frac{|e_i(\lambda_1)|}{\lambda_1} + \sum_{\beta>1} u^{\bar{\alpha}} \frac{|e_i(u V_\beta V_1)|}{\lambda_1} + CU. \tag{4.17}
\]

For each \( \varepsilon \in (0, \frac{1}{2}] \), we deduce that
\[
u^{\bar{\alpha}} \frac{|e_i(\lambda_1)|}{\lambda_1} \leq \varepsilon u^{\bar{\alpha}} \frac{|e_i(\lambda_1)|^2}{\lambda_1} + \frac{C}{\varepsilon} U; \tag{4.18}\]
\[
\sum_{\beta>1} u^{\bar{\alpha}} \frac{|e_i(u V_\beta V_1)|}{\lambda_1} \leq \varepsilon \sum_{\beta>4} u^{\bar{\alpha}} \frac{|e_i(u V_\beta V_1)|^2}{\lambda_1(\lambda_1 - \lambda_\beta)} + \sum_{\beta>1} \frac{\lambda_1 - \lambda_\beta}{\varepsilon \lambda_1} U
\]
\[
\leq \varepsilon \sum_{\beta>1} u^{\bar{\alpha}} \frac{|e_i(u V_\beta V_1)|^2}{\lambda_1(\lambda_1 - \lambda_\beta)} + \frac{C}{\varepsilon} U, \tag{4.19}\]
where in the last inequality we have used is \( \sum_{\beta=1}^{2n} \lambda_\beta = \Delta u = \Delta^\varepsilon u + T(du) \geq -C; \) see [8]. Here \( T \) is the torsion vector field of \( (g, J) \) [31, p. 1070]. It follows from the above three inequalities that
\[
\sum_{\beta=1}^{2n} u^{\bar{\alpha}} \frac{|V_\beta V_1 e_i(u)|}{\lambda_1} \leq \varepsilon u^{\bar{\alpha}} \frac{|e_i(\lambda_1)|^2}{\lambda_1^2} + \varepsilon \sum_{\beta>1} u^{\bar{\alpha}} \frac{|e_i(u V_\beta V_1)|^2}{\lambda_1(\lambda_1 - \lambda_\beta)} + \frac{C}{\varepsilon} U.
\]
Then, by (4.16), we obtain (4.15). \( \square \)

Consequently, Proposition 4.3 follows from (4.14)–(4.15).

4.2 Proof of Theorem 4.1

We divide the proof into three cases.

Case 1 At \( x_0 \),
\[
u^{\bar{\alpha}} \leq B^3 e^{2B u} u^{11}. \tag{4.20}\]

Case 2 At \( x_0 \),
\[
\frac{\phi'}{4} \sum_{j=1}^{n} u^{\bar{\alpha}} (|e_i e_j u|^2 + |e_i e_j u|^2) > 6 \sup_\Omega (|\partial \varphi|^2) B^2 e^{2B u} U. \tag{4.21}\]
In both cases, we choose \( \varepsilon = \frac{1}{2} \). Using \( |a + b|^2 \leq 4|a|^2 + \frac{4}{3}|b|^2 \) for (4.6),
\[
-(1 + \varepsilon) u^{\bar{\alpha}} \frac{|e_i(\lambda_1)|^2}{\lambda_1^2} \geq -6 \sup_\Omega (|\partial \varphi|^2) B^2 e^{2B u} U - 2(\phi')^2 u^{\bar{\alpha}} |e_i(\partial u)|^2.
\]
\( \square \) Springer
Substituting this into (4.8),
\[ 0 \geq (2 - \varepsilon) \sum_{\alpha > 1} u^{\bar{i}} \sum_{i,j} \frac{|e_i(u \nabla_i u_i)^2|}{\lambda_1} + \frac{1}{\lambda_1} u^{\bar{i}} u^{\bar{j}} |V_i(u_{ij})|^2 \]
\[ - \left( \frac{C}{\varepsilon} + 6 \sup_{\Omega} (|\partial \varpi|^2) B^2 e^{2B \varpi} \right) U + \frac{\phi'}{2} \sum_{j=1}^{n} u^{\bar{i}} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \]
\[ + Be^{B \varpi} \varpi + B^2 e^{B \varpi} u^{\bar{i}} |\varpi|^2 - C. \]

**Proof of Case 1**  Since \( L(\varpi) \) is uniformly bounded from below, it follows from the concavity of \( L \) that
\[ 0 \geq \frac{\phi'}{2} \sum_{j=1}^{n} u^{\bar{i}} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) - C_B U. \]  (4.22)

Notice that \( \{u^{\bar{i}}\} \) are pairwisely comparable, by (4.20), so
\[ \sum_{i,j} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \leq C_B K. \]

Thus the complex covariant derivatives
\[ u_{ij} = e_i e_j u - \left( \nabla_i e_j \right) u; \quad u_{\bar{i} \bar{j}} = e_i \bar{e}_j u - \left( \nabla_i \bar{e}_j \right) u \]
satisfy
\[ \sum_{i,j} (|u_{ij}|^2 + |u_{\bar{i} \bar{j}}|^2) \leq C_B K, \]
and this proves (4.1). \( \square \)

**Proof of Case 2**  It follows from (4.8) and (4.21) that
\[ 0 \geq \frac{\phi'}{4} \sum_{j=1}^{n} u^{\bar{i}} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) - \frac{C}{\varepsilon} U + Be^{B \varpi} L(\varpi). \]  (4.23)

Using the fact that \( L(\varpi) \geq \theta (1 + U) \) (by (2.4)), we have that
\[ 0 \geq \frac{\phi'}{4} \sum_{j=1}^{n} u^{\bar{i}} (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) + \left( \frac{1}{2} \theta B e^{B \varpi} - \frac{C}{\varepsilon} \right) U + \frac{1}{2} \theta B e^{B \varpi}, \]
which yields a contradiction if we further assume that \( B \) is large enough. \( \square \)

**Case 3**  If the Cases 1 and 2 do not hold, we define
\[ I := \left\{ 1 \leq i \leq n : u^{\bar{n}}(x_0) \geq B^3 e^{2B \varpi(0)} u^{\bar{i}}(x_0) \right\}. \]
Clearly, \( 1 \in I, n \notin I \). Hence, we may let \( I = \{1, 2, \ldots, p\} \) for a certain \( p < n \).

**Lemma 4.6**  Assume that \( B \geq 6n \sup_{\Omega} (|\partial \varpi|^2) \). At \( x_0 \), we have
\[ -(1 + \varepsilon) \sum_{i \in I} u^{\bar{i}} \frac{|e_i(\lambda_1)|^2}{\lambda_1^2} \geq -U - 2(\phi') \sum_{i \in I} u^{\bar{i}} |e_i(\partial u)|. \]  (4.24)

**Proof**  It follows from (4.6) and the inequality \(|a + b|^2 \leq 4|a|^2 + \frac{4}{3}|b|^2\) that
\[ -(1 + \varepsilon) \sum_{i \in I} u^{\bar{i}} \frac{|e_i(\lambda_1)|^2}{\lambda_1^2} = \frac{3}{2} \sum_{i \in I} u^{\bar{i}} |\phi' e_i(\partial u)| + Ae^{A \varpi} \varpi_i. \]

\( ^2 \)In what follows, \( C_B \) are positive constants depending on \( B \).
where we used $B \geq 6n \sup_{\Omega}(|\partial \varphi|^2)$ in the last inequality.

Let us define a new $(1,0)$ vector field by

$$\hat{\epsilon}_1 := \frac{1}{\sqrt{2}}(V_1 - \sqrt{-1}JV_1).$$

At $x_0$, there exist $\varsigma_1, \ldots, \varsigma_n \in \mathbb{C}$ such that

$$\hat{\epsilon}_1 = \sum_{k=1}^{n} \varsigma_k \epsilon_k, \quad \sum_{k=1}^{n} |\varsigma_k|^2 = 1.$$

**Lemma 4.7** At $x_0$, $|\varsigma_k| \leq \frac{C_B}{\lambda_1}$ for all $k \notin I$.

**Proof** The proof is from [8]; we include it here for the convenience of the reader. Now we have

$$\frac{\phi^i}{4} \sum_{i \in I, j=1}^{n} u^{ij} (|e_i e_j u|^2 + |e_i e_j u|^2) \leq 6n^2 \sup_{\Omega}(|\partial \varphi|^2) B^2 e^{2 \varphi} u^\bar{n}.$$

When $u^\bar{n} \leq B^3 e^{2 \varphi} u^\bar{i}$ for each $i \notin I$, it follows that

$$\sum_{\alpha=2p+1}^{2n} \sum_{\beta=1}^{2n} |\bar{\nabla} u|^2 \leq C_B,$$

which in turn implies that $|\Phi^\alpha_\beta| \leq C_B$ for $2p + 1 \leq \alpha \leq 2n, 1 \leq \beta \leq 2n$. Since $\Phi(V_1) = \lambda_1 V_1$,

$$|V_1^\alpha| = \left| \frac{1}{\lambda_1} \Phi(V_1)^{\alpha} \right| = \frac{1}{\lambda_1} \sum_{\beta=1}^{2n} \Phi^\alpha_\beta V_1^\beta \leq \frac{C_B}{\lambda_1}.$$

This proves the lemma.

Now we estimate the first three terms in Proposition 4.3. Since $JV_1$ is $g$-unitary and $g$-orthogonal to $V_1$, there exist $\mu_1, \ldots, \mu_{2n} \in \mathbb{R}$ such that

$$JV_1 = \sum_{\alpha>1} \mu_\alpha V_\alpha, \quad \sum_{\alpha>1} \mu_\alpha^2 = 1 \text{ at } x_0.$$

**Lemma 4.8** At $x_0$, for any constant $\gamma > 0$,

$$(2 - \varepsilon) \sum_{\alpha>1} u^{ij} \left| \frac{e_i(u V_\alpha V_1)}{\lambda_1} \right|^2 + \frac{1}{\lambda_1} u^{ij} u^{kk} |V_1(u_{ik})|^2 - (1 + \varepsilon) \sum_{i \notin I} u^{ij} \left| \frac{e_i(\lambda_1)}{\lambda_i^2} \right|^2 \geq (2 - \varepsilon) \sum_{i \notin I} \sum_{\alpha>1} u^{ij} \left| \frac{e_i(u V_\alpha V_1)}{\lambda_1} \right|^2 + 2 \sum_{k \in I, i \notin I} u^{ij} u^{kk} \left| \frac{V_1(u_{ik})}{\lambda_1} \right|^2$$

$$- 3\varepsilon \sum_{i \notin I} |\frac{e_i(\lambda_1)}{\lambda_i^2}|^2 - 2(1 - \varepsilon)(1 + \gamma) \sum_{k \in I, i \notin I} u^{ij} u^{kk} \left| \frac{V_1(u_{ik})}{\lambda_i^2} \right|^2$$

$$- \frac{C_B}{\varepsilon} u^2 - (1 - \varepsilon)(1 + \frac{1}{\gamma}) \sum_{\alpha>1} \lambda_\alpha \sum_{i \notin I} \sum_{\alpha>1} u^{ij} \left| \frac{e_i(u V_\alpha V_1)}{\lambda_i^2(\lambda_1 - \lambda_\alpha)} \right|^2.$$

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Thus, we have the first term is
\[ e_i(u_{V_1}\tilde{e}_1) = e_i(V_1\tilde{e}_1u - (\nabla V_1\tilde{e}_1)u) = \tilde{e}_1e_iV_1u + O(\lambda_1) \]
where \( O(\lambda_1) \) are those terms which can be controlled by \( \lambda_1 \). The second term is
\[ e_i(u_{V_1J V_1}) = e_iV_1J V_1(u) + O(\lambda_1) = \sum_{\alpha>1} V_\alpha e_iV_1(u) + O(\lambda_1) = \sum_{\alpha>1} e_i(u_{V_\alpha V_1}) + O(\lambda_1). \]
Thus,
\[ e_i(\lambda_1) = \sqrt{2}\sum_k \lambda_k^2 V_1(u_{ik}) - \sqrt{-1}\sum_{\alpha>1} \mu_\alpha e_i(u_{V_\alpha V_1}) + O(\lambda_1). \] (4.25)

**Step 2** It follows from (4.25) and Lemma 4.7 that
\[ -(1+\varepsilon)\sum_{i\in I} u_{i}^{|e_i(\lambda_1)|} \geq - (1-\varepsilon) \sum_{i\in I} \frac{\sqrt{2}\sum_{k\in I} \lambda_k^2 V_1(u_{ik}) - \sqrt{-1}\sum_{\alpha>1} \mu_\alpha e_i(u_{V_\alpha V_1})^2}{\lambda_1^2} \]
\[ - 3\varepsilon \sum_{i\in I} u_{i}^{|e_i(\lambda_1)|} - \frac{C_B}{\varepsilon} \sum_{i\in I, k\in I} u_{i}^{|V_1(u_{ik})|} - \frac{C_B}{\varepsilon}. \] (4.26)

By the Cauchy-Schwarz inequality,
\[ \left| \sum_{\alpha>1} \mu_\alpha e_i(u_{V_\alpha V_1}) \right|^2 \leq \sum_{\alpha>1} (\lambda_1 - \lambda_\alpha \mu_\alpha^2) \sum_{\beta>1} |e_i(u_{V_\beta V_\beta})|^2; \] (4.27)
\[ \left| \sum_{k\in I} \lambda_k^2 V_1(u_{ik}) \right|^2 \leq \sum_{i\in I} \left( \sum_{k\in I} u_{ik}^2 \right) \left( \sum_{k\in I} V_1(u_{ik})^2 \right). \] (4.28)

With these, for each \( \gamma > 0 \),
\[ (1-\varepsilon) \sum_{i\in I} \frac{\sqrt{2}\sum_{k\in I} \lambda_k^2 V_1(u_{ik}) - \sqrt{-1}\sum_{\alpha>1} \mu_\alpha e_i(u_{V_\alpha V_1})^2}{\lambda_1^2} \]
\[ \leq 2(1-\varepsilon)(1+\gamma) \sum_{i\in I} u_{i}^{|e_i(\lambda_1)|} \frac{\left| \sum_{k\in I} \lambda_k^2 V_1(u_{ik}) \right|^2}{\lambda_1^2} \]
\[ + (1-\varepsilon)(1+\frac{1}{\gamma}) \sum_{i\in I} \frac{\left| \sum_{\alpha>1} \mu_\alpha e_i(u_{V_\alpha V_1}) \right|^2}{\lambda_1^2} \]
\[ \leq 2(1-\varepsilon)(1+\gamma)u_{i\bar{i}} \sum_{i\in I, k\in I} u_{i\bar{i}}^2 k^k |V_1(u_{ik})|^2 \]
\[ + (1-\varepsilon)(1+\frac{1}{\gamma})(\lambda_1 - \sum_{\alpha>1} \lambda_\alpha \mu_\alpha^2) \sum_{i\in I} \frac{\left| e_i(u_{V_\alpha V_1}) \right|^2}{\lambda_1^2(\lambda_1 - \lambda_\alpha)}. \] (4.29)
Step 3 If \( \lambda_1 \geq \frac{C_B}{\varepsilon} \) (by assumption), we know that \( u_{i1} \) is comparable to \( \lambda_1 \), whence \( \frac{C_B}{\varepsilon \lambda_1^3} \leq u_{i1} \leq u^{kk} \) for all \( k \). Thus,

\[
\bar{u}^{i} u^{kk} |v_i(u_{ik})|^2 \geq 2 \sum_{k \in I, i \notin I} \bar{u}^{i} u^{kk} |v_i(u_{ik})|^2 + \frac{C_B}{\varepsilon} \sum_{i,k \notin I} \bar{u}^{i} |v_i(u_{ik})|^2 \lambda_1^3. \tag{4.30}
\]

Then the lemma follows from (4.26), (4.29) and (4.30).

Lemma 4.9 At \( x_0 \), if \( \lambda_1 \geq \frac{C_B}{\varepsilon} \),

\[
(2 - \varepsilon) \sum_{\alpha > 1} u^{i} |e_i(u v_i v_i)|^2 \frac{\lambda_1}{\lambda_1 - \lambda_1} + \frac{1}{\lambda_1} u^{i} u^{kk} |v_i(u_{ik})|^2 - (1 + \varepsilon) \sum_{i \notin I} u^{i} |e_i(u_{i1})|^2 \lambda_1^2 \geq - 6\varepsilon B^2 e^{2B} p \sum_{i=1}^n u^{i} |v_i|^2 - 6\varepsilon (\phi')^2 \sum_{i \notin I} u^{i} |e_i(|\xi|)|^2 - \frac{C_B}{\varepsilon} U. \tag{4.31}
\]

Proof It suffices to prove that

\[
(2 - \varepsilon) \sum_{\alpha > 1} u^{i} |e_i(u v_i v_i)|^2 \frac{\lambda_1}{\lambda_1 - \lambda_1} + \frac{1}{\lambda_1} u^{i} u^{kk} |v_i(u_{ik})|^2 - (1 + \varepsilon) \sum_{i \notin I} u^{i} |e_i(u_{i1})|^2 \lambda_1^2 \geq - 3\varepsilon \sum_{i \notin I} u^{i} |e_i(u_{i1})|^2 \frac{\lambda_1^2}{\lambda_1} - \frac{C_B}{\varepsilon} U. \tag{4.32}
\]

We divide the proof into two assumptions.

Assumption 1 At \( x_0 \), we assume that

\[
\lambda_1 + \sum_{\alpha > 1} \lambda_1^2 \mu_\alpha^2 \geq 2(1 - \varepsilon) u_{i1} > 0. \tag{4.33}
\]

Proof Taking this, as well as Lemma 4.8, we get that

\[
(2 - \varepsilon) \sum_{\alpha > 1} u^{i} |e_i(u v_i v_i)|^2 \frac{\lambda_1}{\lambda_1 - \lambda_1} + \frac{1}{\lambda_1} u^{i} u^{kk} |v_i(u_{ik})|^2 - (1 + \varepsilon) \sum_{i \notin I} u^{i} |e_i(u_{i1})|^2 \lambda_1^2 \geq (2 - \varepsilon) \sum_{i \notin I} \sum_{\alpha > 1} u^{i} |e_i(u v_i v_i)|^2 \frac{\lambda_1}{\lambda_1 - \lambda_1} + \sum_{k \in I, i \notin I} \frac{2}{\lambda_1} u^{i} u^{kk} |v_i(u_{ik})|^2
\]

\[
- 3\varepsilon \sum_{i \notin I} u^{i} |e_i(u_{i1})|^2 \lambda_1^2 - (1 + \gamma)(\lambda_1 + \sum_{\alpha > 1} \lambda_1^2 \mu_\alpha^2) \sum_{k \in I, i \notin I} u^{i} u^{kk} |v_i(u_{ik})|^2
\]

\[
- \frac{C_B}{\varepsilon} U - (1 - \varepsilon)(1 + \frac{1}{\gamma})(\lambda_1 - \sum_{\alpha > 1} \lambda_1^2 \mu_\alpha^2) \sum_{i \notin I} \sum_{\alpha > 1} u^{i} |e_i(u v_i v_i)|^2 \lambda_1^2 (\lambda_1 - \lambda_1). \tag{4.34}
\]

We only choose that \( \gamma = \frac{\lambda_1 - \sum_{\alpha > 1} \lambda_1^2 \mu_\alpha^2}{\lambda_1 + \sum_{\alpha > 1} \lambda_1^2 \mu_\alpha^2} \). On the right side of (4.33), the first term cancels the last term, and the second term cancels the fourth. This proves (4.31).

Assumption 2 At \( x_0 \), we assume that

\[
\lambda_1 + \sum_{\alpha > 1} \lambda_1^2 \mu_\alpha^2 < 2(1 - \varepsilon) u_{i1}. \tag{4.35}
\]

Proof Computing at \( x_0 \), we get that

\[
u_{i1} = (\sqrt{-1} \partial u)(e_1, e_1) = \sum_{i=1}^n \{ e_i e_i(u) - [e_i, e_i]^{(0,1)}(u) \}|v_i|^2
\]
\[
\begin{align*}
&\leq \frac{1}{2} \left\{ V_1 V_1(u) + (J V_1)(J V_1)(u) + \sqrt{-1}[V_1, J V_1](u) \right\} - [\bar{e}_1, e_1]^{(0,1)}(u) + C \\
&\leq \frac{1}{2} \left\{ u_i V_1 + u_i J V_1 + (\bar{\nabla}_i V_1)(u) + (\bar{\nabla}_i J V_1)(u) + \sqrt{-1}[V_1, J V_1](u) \right\} + C \\
&\leq \frac{1}{2}(\lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2) + C.
\end{align*}
\]

It then follows from (4.34) that \( \lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 \geq -C \) and \( u_{i\bar{i}} \leq \frac{C}{\bar{\varepsilon}} \). Hence,

\[ 0 < \lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 \leq 2\lambda_1 + C \leq (2 + 2\varepsilon^2)\lambda_1, \]

provided that \( \lambda_1 \geq \frac{C}{\varepsilon} \). Choosing \( \gamma = \frac{1}{\varepsilon^2} \),

\[ (1 - \varepsilon)(1 + \frac{1}{\varepsilon^2})(\lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2) \leq 2(1 - \varepsilon)(1 + \varepsilon^2)\lambda_1 \leq (2 - \varepsilon)\lambda_1. \]

Substituting this into Lemma 4.8 yields that

\[ (2 - \varepsilon)\sum_{\alpha > 1} u_i^\alpha \bar{e}_1 V_1(u_i V_1) \geq \frac{1}{\lambda_1} u_i^\alpha \bar{e}_1 V_1(u_i V_1) - (1 + \varepsilon) \sum_{i \notin I} u_i^\alpha \bar{e}_1 V_1(u_i V_1) - (1 + \varepsilon) \sum_{i \notin I} u_i^\alpha \bar{e}_1 V_1(u_i V_1) \]

where in the last inequality we relied on the fact that \( \lambda_1 \geq \frac{C}{\bar{\varepsilon}} \). This proves (4.31), and hence the proof of the lemma is complete.

Now we complete the proof of the interior second order estimate. It follows from Lemma 4.9 and (4.8) that, at \( x_0 \),

\[ 0 \geq -6\varepsilon B^2 e^{2B\varepsilon} u_i^\alpha |\omega_i|^2 - 6\varepsilon (\phi^\prime)^2 \sum_{i \notin I} u_i^\alpha \bar{e}_i |i\partial u|^2 - \frac{C}{\varepsilon} U \]

\[ + \frac{\phi^\prime}{2} \sum_{j=1}^n u_i^\alpha (|\bar{e}_i e_j u|^2 + |e_i \bar{e}_j u|^2) + B^2 e^{2B\varepsilon} u_i^\alpha |\omega_i|^2 + Be^{B\varepsilon} L(\omega) + \phi^\prime u_i^\alpha |\bar{e}_i |i\partial u|^2| \]

(4.35)

Choosing \( \varepsilon < \frac{1}{6} \) such that \( 6\varepsilon B^2 e^{B\varepsilon}(x_0) = 1 \), and by \( \phi^\prime = 2(\phi^\prime)^2 \),

\[ 0 \geq -\frac{C}{\varepsilon} U + \frac{\phi^\prime}{2} \sum_{j=1}^n u_i^\alpha (|\bar{e}_i e_j u|^2 + |e_i \bar{e}_j u|^2) + Be^{B\varepsilon} L(\omega). \]
Thus,
\[ B\theta e^{B\varpi} + (B\theta - C)e^{B\varpi}U + \frac{\phi'}{2} \sum_{i=1}^{n} u^i (|e_i e_j u|^2 + |e_i \bar{e}_j u|^2) \leq 0. \]

We choose \( B \) sufficiently large such that \( B\theta \geq C \). This then yields a contradiction, and we have completed the proof. \( \square \)

**Remark 4.10** The interior \( C^{2,\alpha} \) estimates follow from the Evans-Krylov theorem and an extension trick introduced by Wang [34] in the study of the complex Monge-Ampère equation. Then the higher order estimates can be obtained by Schauder estimates.

## 5 Boundary \( C^2 \) Estimates

In this section we shall derive the estimate
\[ \max_{\partial \Omega} |\sqrt{-1\partial \bar{\partial} u}| \leq C \]
for a certain dependent constant \( C \).

### 5.1 Pure Tangential Estimates

Let us fix a point \( z \in \partial \Omega \), and define
\[ \rho(x) := \text{dist}_g(x, z) \quad \text{in} \ M. \]

Since \( u - \bar{u} = 0 \) on \( \partial \Omega \), we can write \( u = \bar{u} + \rho \sigma \) in a neighborhood of \( z \), where \( \sigma \) is a function defined on \( \partial \Omega \) which depends, linearly on the first order derivatives of \( u - \bar{u} \). For arbitrary vector fields \( X, Y \) which are tangential to \( \partial \Omega \),
\[ XY(u) = XY(\bar{u}) + XY(\rho) \cdot \sigma. \]

It follows from the \( C^1 \) estimate that
\[ |XY(u)(z)| \leq C. \tag{5.1} \]

Then the pure tangential estimates follow by the randomness of \( z \).

### 5.2 Mixed Direction Estimates

**Proposition 5.1** Let \( N \in T_z M \) be orthogonal to \( \partial \Omega \) such that \( N \rho = -1 \), and let \( X \) be a vector field which is tangential to \( \partial \Omega \). We have that
\[ |NX(u)(z)| \leq C, \tag{5.2} \]
where \( C \) depends on \( \| u \|_{C^1(\bar{\Omega})} \), \( h \), \( \| \bar{u} \|_{C^2} \) and other known data.

**Proof** Let \( \mathcal{O} \subseteq M \) be a local coordinate chart with \( z \in \mathcal{O} \). We may pick up real vector fields \( X_1, \cdots, X_n \) which are tangential at \( z \) to \( \partial \Omega \) such that \( X_1, JX_1, \cdots, X_n, JX_n \) is a \( g \)-orthonormal local frame near \( z \). Furthermore, we assume that \( Y_n := JX_n \) is the normal vector on \( \partial \Omega \) near \( z \).

Fixing a constant \( \delta > 0 \), we set
\[ \Omega_{\delta} := \{ x \in \Omega \mid \rho(x) \leq \delta \}. \]

Notice that \( \sqrt{-1\partial \bar{\partial} \rho^2} = \omega \) at \( z \). By continuity, we may rearrange \( \delta \ll 1 \) such that
\[ \frac{1}{2} \omega \leq \sqrt{-1\partial \bar{\partial} \rho^2} \leq 2 \omega \quad \text{in} \ \Omega_{\delta}. \]
We shall prove (5.2) by applying the maximum principle to
\[ Q_\pm = \pm X(u - \bar{u}) + \sum_{j=1}^n |X_j(u - \bar{u})|^2 + Av - B\rho^2 \]
for a negative function \( v \in C^\infty(\Omega_\delta) \), to be determined later. Let \( O' \subset O \) be a neighborhood of \( z \), and set \( S_\delta := O' \cap \Omega_\delta \).

First we choose \( B \) large enough such that \( Q_\pm \leq 0 \) on \( \partial S_\delta \). We shall prove \( Q_\pm \leq 0 \) in \( S_\delta \) for a large constant \( A \). Otherwise, suppose that \( Q_\pm \) attains its maximum at a point \( x_0 \in S_\delta \). Let \( e_1, \cdots, e_n \) with
\[ e_i := \frac{1}{\sqrt{2}}(X_i - \sqrt{-1}JX_i), \quad 1 \leq i \leq n \]
be a local \( g \)-orthonormal frame in a neighborhood of \( x_0 \) such that the matrix \((u_{ij})\) is diagonal at \( x_0 \).

The following lemma plays a significant role in our proof:

**Lemma 5.2** There exist some uniform positive constants \( t, \delta \) and \( \varepsilon \) sufficiently small, and an \( N \) sufficiently large, such that the function
\[ v := \underline{u} - u - td + Nd^2 \] satisfies \( v \leq 0 \) in \( \bar{\Omega}_\delta \) and
\[ L(v) \geq \varepsilon(1 + \mathcal{U}) \quad \text{at} \quad x_0. \]  

**Proof** As \( \underline{u} \leq u \) and \( v \leq 0 \) in \( \bar{\Omega}_\delta \), if we let \( \delta \ll t \) be small enough such that \( N\delta < t \), then by a direct calculation and the property of the mixed discriminant, at \( x_0 \),
\[ L(\underline{u} - u) \geq n\tau h^{-1}\det(I, \sqrt{-1}\partial\bar{\partial}u[n - 1]) - n = \tau\mathcal{U} - n; \]
\[ L(-td + Nd^2) = -(t - Nd)u^\delta d_{ij} + Nu^\delta d_{ij} \geq -C_1(t - Nd)\mathcal{U} + \frac{N}{2} \min_{1 \leq i \leq n} u^{\bar{i}} , \]
where we used (2.5). It follows that
\[ L(v) \geq (\tau - C_1 t)\mathcal{U} + \frac{N}{2} \min_{1 \leq i \leq n} u^{\bar{i}} - n \geq \frac{\tau'}{2} \mathcal{U} + \frac{N}{2} \min_{1 \leq i \leq n} u^{\bar{i}} - n , \] if \( t \ll 1 \). By an elementary inequality, we deduce that
\[ \frac{\tau'}{4} \mathcal{U} + \frac{N}{2} \min_{1 \leq i \leq n} u^{\bar{i}} \geq n \left( \frac{\tau'}{4} \right)^{\frac{n-1}{n}} \left( \prod_{1 \leq i \leq n} u^{\bar{i}} \right)^{\frac{1}{n}} \geq n \left( \frac{\tau'}{4} \right)^{\frac{n-1}{n}} N^{\frac{1}{n}} h^{-\frac{1}{n}} \geq C_2 \left( \frac{\tau'}{4} \right)^{\frac{n-1}{n}} N^{\frac{1}{n}} . \]
We choose \( N \) large enough such that
\[ C_2 \left( \frac{\tau'}{4} \right)^{\frac{n-1}{n}} N^{\frac{1}{n}} \geq \frac{\tau'}{4} + n. \]
Substituting this into (5.5), we get that \( L(v) \geq \frac{\tau'}{4}(1 + \mathcal{U}) \). This completes the proof. \( \square \)

Now we continue to prove Proposition 5.1. Clearly,
\[ L(\pm \underline{u} - B\rho^2) \geq -BC\mathcal{U}. \]
For each vector field \( Y \),
\[ L(Yu) = u^{\bar{j}}(e_i\bar{e}_j Y u - [e_i, \bar{e}_j]^{0,1} Y u) \]
\[ = Y(h) + u^{\bar{j}}(e_i\bar{e}_j Y u + [e_i, \bar{e}_j]^{0,1} Y u) \]
\[ - \left[ e_i, \bar{e}_j \right]^{0,1} Y u ) . \]
There exist $\alpha_{jk}, \beta_{jk} \in \mathbb{C}$ such that
\[
[e_j, Y] = \sum_{k=1}^n \alpha_{jk} e_k + \beta_{jk} X_k; \quad [\bar{e}_j, Y] = \sum_{k=1}^n \bar{\alpha}_{jk} e_k + \bar{\beta}_{jk} X_k.
\]

It follows that
\[
L(Yu) \leq Cu^\beta \left( 1 + \sum_{k=1}^n |e_i X_k u| \right),
\]
which implies that
\[
L(\pm Xu + \sum_{j=1}^n |X_j (u - w)|^2) \geq u^\beta \sum_{j=1}^n (e_i X_j (u - w))(\bar{e}_j X_j (u - w)) - Cu^\beta (1 + \sum_{j=1}^n |e_i X_j u|)
\]
\[
\geq \frac{1}{2} u^\beta \sum_{j=1}^n |e_i X_j u|^2 - Cu^\beta (1 + \sum_{j=1}^n |e_i X_j u|) \geq -CU,
\]
where in the last inequality we used the fact that $\frac{1}{2}a^2 + 2ab \geq -b^2$. It then follows from (5.4), (5.6) and (5.7) that
\[
L(Q_{\pm}(x_0) \geq A \varepsilon + (A \varepsilon - BC - C)\mathcal{U} > 0,
\]
if $A$ is large enough such that $A \varepsilon \geq (B + 1)C$, which contradicts to the fact that $Q_{\pm}$ attains its maximum at $x_0$. Consequently, $Q_{\pm} \leq 0$ in $\mathcal{S}_\delta$ and $Q_{\pm}(z) = 0$. By Hopf’s lemma, $|NXu|(z) \leq C$.

\section{5.3 Pure Normal Estimates}

\textbf{Proposition 5.3} Let $N \in T_z M$ be orthogonal to $\partial \Omega$ at $z$ such that $N\rho = -1$. We have
\[
|NN(u)|(z) \leq C,
\]
where $C$ depends on $\|u\|_{C^1(\Omega)}$, $h$, $\|\bar{u}\|_{C^2}$ and other known data.

Before proving this, let us recall some useful facts from the matrix theory. For any Hermitian matrix $A = (a_{ij})$ with eigenvalues $\lambda_i(A)$, let $\tilde{A} := (a_{\bar{m}n})$, and we denote the eigenvalues of $\tilde{A}$ by $\lambda'_\alpha(\tilde{A})$. It follows from Cauchy’s interlace inequality [19] and [6, p. 272] that when $|a_{\bar{m}n}| \to \infty$,
\[
\lambda_\alpha(A) \leq \lambda'_\alpha(\tilde{A}) \leq \lambda_{\alpha + 1}(A);
\]
\[
\lambda_\alpha(A) = \lambda'_\alpha(\tilde{A}) + O(1);
\]
\[
a_{\bar{m}n} \leq \lambda_n(A) \leq a_{\bar{m}n} \left( 1 + O\left(\frac{1}{a_{\bar{m}n}}\right)\right).
\]

\textbf{Proof} Let $U := (u_{ij})$ (resp. $\bar{U} := (\bar{u}_{ij})$) be the Hessian matrix of $u$ (resp. $\bar{u}$). We assert that there are uniform constants $c_0, R_0 > 0$ such that, for all $R \geq R_0$, $(\lambda'(\bar{U}), R) \in \Gamma_n$ and
\[
\log \det(\lambda'(\bar{U}), R) \geq h + c_0, \quad \text{on } \partial \Omega.
\]

To this end, we follow an idea of Trudinger [30] and set
\[
\tilde{m} := \liminf_{R \to \infty} \min_{\partial \Omega} \left( \log \det(\lambda'(\bar{U}), R) - h \right).
\]

---

\[\text{3In what follows, we let } \alpha, \beta = 1, 2, \cdots, n - 1; i, j = 1, 2, \cdots, n.\]
Then we are reduced to showing
\[ \hat{m} \geq c_0 > 0. \]  
(5.10)

We may assume that \( \hat{m} < \infty \), otherwise we are done. Supposing that \( \hat{m} \) is attained at a point \( x_0 \in \partial \Omega \), we pick up a local \( g \)-orthonormal frame \( (e_1, \cdots, e_n) \) as in the previous subsection such that the matrix \( (U_{\alpha \beta}(x_0)) \) is diagonal. We choose real vector fields \( X_1, \cdots, X_n \) tangential at \( x_0 \) to \( \partial \Omega \) such that \( X_1, JX_1, \cdots, X_n, JX_n \) constitute a \( g \)-orthonormal local frame near \( x_0 \), and \( Y_n := JX_n \) is the normal vector on \( \partial \Omega \) near \( x_0 \). Letting
\[ \Gamma_\infty := \{ (\lambda_1, \cdots, \lambda_{n-1}) \mid \lambda_\alpha > 0, \ 1 \leq \alpha \leq n-1 \} \]
be a positive orthant in \( \mathbb{R}^{n-1} \), we divide the proof into two cases.

**Case 1** Assume that it holds that
\[ \lim_{\lambda_n \to \infty} \sigma_n(\lambda', \lambda_n) = \infty, \quad \text{for any} \ \lambda' \in \Gamma_\infty. \]  
(5.11)
By virtue of (5.1) and (5.2), we know that
\[ \lambda'(\hat{U})(x_0) \in C, \]
where \( C \subset \Gamma_\infty \) is compact. Then there exist \( c_1, R_1 \in \mathbb{R}_{>0} \) depending on \( \lambda'(\hat{U}(x_0)) \) such that
\[ \log \det(\lambda'(\hat{U}(x_0)), R) \geq h(x_0) + c_1, \quad \text{for any} \ R \geq R_1. \]
By continuity, there exists a cone \( \hat{C} \subset \Gamma_\infty \) and a neighborhood of \( C \) such that
\[ \log \det(\lambda', R) \geq h(x_0) + \frac{c_1}{2}, \quad \text{for any} \ \lambda' \in \hat{C} \ \text{and} \ R \geq R_1. \]  
(5.12)
Now we apply (5.9) to \( U = (u_{ij}) \), and there exists a large constant \( R_2 \geq R_1 \) satisfying, if \( u_{m\hat{n}}(x_0) \geq R_2 \), then
\[ \lambda_n(U)(x_0) \geq u_{m\hat{n}}(x_0) \geq R_2 \geq R_1. \]  
(5.13)
We can shrink \( \hat{C} \) if necessary such that
\[ (\lambda_1(U)(x_0), \cdots, \lambda_{n-1}(U)(x_0)) \in \hat{C}. \]  
(5.14)
It follows from (5.12), (5.13) and (5.14) that
\[ \log \det(u_{ij})(x_0) \geq h(x_0) + \frac{c_1}{2}, \]
which yields a contradiction to (2.2). Hence (5.10) follows by letting \( c_0 := \frac{c_1}{2} \).

**Case 2** Assume that it holds that
\[ \lim_{\lambda_n \to \infty} \sigma_n(\lambda', \lambda_n) < \infty, \quad \text{for any} \ \lambda' \in \Gamma_\infty. \]  
(5.15)
We define
\[ \hat{F}(E) := \lim_{R \to \infty} \log \det(\lambda(E), R) \]
on the set of \((n-1)^2\) Hermitian matrices with \( \lambda(E) \in \Gamma_\infty \). Notice that \( \hat{F} \) is concave and finite, since the operator \( \lambda \mapsto \log \det(\lambda) \) is concave and continuous. Hence, there exists a symmetric matrix \( \hat{F}^{\alpha \beta}(\hat{U}) \) such that
\[ \hat{F}^{\alpha \beta}(\hat{U})(E_{\alpha \beta} - \hat{U}_{\alpha \beta}) \geq \hat{F}(E) - \hat{F}(\hat{U}) \]  
(5.16)
for any \((n-1)^2\) Hermitian matrix \( E \). On \( \partial \Omega \), since \( u = \underline{u} \),
\[ \hat{U}_{\alpha \beta} - \underline{U}_{\alpha \beta} = \nabla_\beta \nabla_\alpha (u - \underline{u}) = -g(Y_n, \nabla_\alpha \underline{e}_\beta)Y_n(u - \underline{u}), \]
Thus, \( \Phi(x) = [\alpha, \beta]^{0,1} \) (cf. [25]). This, together with (5.16), yield that
\[
Y_n(u - w)(x_0)\bar{F}^{\alpha\beta}(U(x_0))g(Y_n, \nabla_\alpha \bar{e}_\beta) \geq \bar{F}(\bar{U}(x_0)) - \bar{F}(\bar{U}(x_0)) = \bar{F}(\bar{U}(x_0)) - \bar{m} - h(x_0)
\[
\geq \bar{F}(\bar{U}(x_0)) - \log \det(\lambda(U))(x_0) - \bar{m}
\geq \bar{c} - \bar{m},
\]
where
\[
\bar{c} := \lim_{R \to \infty} \min_{\partial \Omega} \left[ \log \det(\lambda'((U), R) - \log \det(\lambda(U)) \right].
\]
Notice that \( 0 < \bar{c} < \infty \), since the operator \( \lambda \mapsto \log \det(\lambda) \) is strictly increasing with respect to each variable. Now we divide the proof into two cases.

**Subcase 2 (i)** Assume that at \( x_0, \)
\[
Y_n(u - w)\bar{F}^{\alpha\beta}(U)g(Y_n, \nabla_\alpha \bar{e}_\beta) \leq \frac{\bar{c}}{2}. \tag{5.17}
\]
Given this, \( \bar{m} \geq \frac{\bar{c}}{2} \), and by choosing \( c_0 = \frac{\bar{c}}{2} \), we are done.

**Subcase 2 (ii)** Assume that at \( x_0, \)
\[
Y_n(u - w)\bar{F}^{\alpha\beta}(U)g(Y_n, \nabla_\alpha \bar{e}_\beta) \geq \frac{\bar{c}}{2}. \tag{5.18}
\]
Define
\[
\eta := \bar{F}^{\alpha\beta}(U(x_0))g(Y_n, \nabla_\alpha \bar{e}_\beta) \quad \text{on } \partial \Omega.
\]
Notice that \( Y_n(u - w)(x_0) \geq 0 \), and by (5.18), is strictly positive. Thus
\[
\eta \geq \frac{\bar{c}}{2Y_n(u - w)} \geq 2\tau \bar{c} \quad \text{at } x_0
\]
for some uniform constant \( \tau > 0 \). We may assume that \( \eta \geq \tau \bar{c} \) in \( \Omega_\delta \) by shrinking \( \delta \) again if necessary.

Let us define a function in \( \Omega_\delta \) by
\[
\Phi(x) = \frac{1}{\eta(x)} \bar{F}^{\alpha\beta}(U(x_0))\left(\bar{U}_{\gamma\delta}(x) - \bar{U}_{\alpha\beta}(x)\right) - \frac{h(x) - h(x_0)}{\eta(x)} - Y_n(u - w)(x)
\]
\[
: = Q(x) - Y_n(u - w)(x).
\]
By a direct calculation,
\[
-\eta(x)Y_n(u - w)(x) = \bar{F}^{\alpha\beta}(U(x_0))\left(\bar{U}_{\alpha\beta}(x) - \bar{U}_{\alpha\beta}(x)\right).
\]
It follows from (5.16) that
\[
\eta(x)\Phi(x) = \bar{F}^{\alpha\beta}(U(x_0))\left(\bar{U}_{\alpha\beta}(x) - \bar{U}_{\alpha\beta}(x)\right) - h(x) + h(x_0)
\geq \bar{F}(\bar{U}(x)) - \bar{F}(\bar{U}(x_0)) - h(x) + h(x_0).
\]
Thus, \( \Phi(x_0) = 0 \) and \( \Phi \geq 0 \) near \( x_0 \) on \( \partial \Omega \). Define
\[
\Psi := -\sum_{j=1}^{n} |X_j(u - w)|^2 - Av + Bp^2 \quad \text{in } \Omega_\delta.
\]
One can verify that \( \Phi + \Psi \geq 0 \) on \( \partial \Omega_\delta \) and
\[
L(\Phi + \Psi) \leq 0 \quad \text{in } \Omega_\delta
\]
provided that \( A \gg B \gg 1 \). By Hopf’s lemma, we know that \( Y_n \Phi(x_0) \geq -C \), then \( Y_n Y_n u(x_0) \leq C \).

Now we are in a position where all the eigenvalues of \( U(x_0) \) are bounded, so \( \lambda(U)(x_0) \) is contained in a compact subset of \( \Gamma_n \). Since the operator \( \lambda \mapsto \log \det(\lambda) \) is strictly increasing with respect to each variable,

\[
\tilde{m} \geq m_R := \log \det(\lambda'(\tilde{U}(x_0)), R) - h(x_0) > 0
\]

when \( R \) is large enough. This proves (5.10), and the proof is complete. \( \square \)

6 Existence of Subsolutions

Suppose that \( \Omega \subseteq M \) is a smooth pseudoconvex domain, and let \( \rho \) be a strictly \( J \)-psh defining function for \( \Omega \). Then there exists a uniform positive constant \( \gamma > 0 \) such that \( \sqrt{-1} \partial \bar{\partial} \gamma \rho \geq \gamma \omega \). For each \( s > 0 \), we set

\[
u := \hat{\varphi} + s(e^{\rho} - 1),
\]

where \( \hat{\varphi} \) is an arbitrary \( J \)-psh extension of \( \varphi|_{\partial \Omega} \). Then

\[
\sqrt{-1} \partial \bar{\partial} \nu = \sqrt{-1} \partial \bar{\partial} \hat{\varphi} + s e^\rho \left( \sqrt{-1} \partial \bar{\partial} \rho + \sqrt{-1} \partial \rho \wedge \bar{\partial} \rho \right) \geq s \gamma e^\rho \omega + s e^\rho \sqrt{-1} \partial \rho \wedge \bar{\partial} \rho.
\]

Therefore,

\[
\det(u_{i\bar{j}}) \geq (s \gamma)^n e^{n \rho} \left( 1 + \frac{1}{\gamma} [\partial \rho]^2 \right).
\]

We may choose \( s \gg 1 \) such that \( \det(u_{i\bar{j}}) \geq N := \sup_{\tilde{\Omega}} h \). Notice that \( u = \varphi \) on \( \partial \Omega \), so \( u \) is a desired subsolution of eq. (1.1).

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