Abstract

In this paper, we study the Hölder-type interpolation inequality and observability inequality from measurable sets in time for parabolic equations either with $L^p$ unbounded potentials or with electric potentials. The parabolic equations under consideration evolve in bounded $C^{1,1}$ domains of $\mathbb{R}^N (N \geq 3)$ with homogeneous Neumann boundary conditions. The approach for the interpolation inequality is based on a modified reduction method and some stability estimates for the corresponding elliptic operator.

Keywords. Interpolation inequality, observability inequality, controllability, quantitative unique continuation, stability estimate

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1 Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be a bounded domain with a $C^{1,1}$ boundary $\partial \Omega$ and such that $0 \in \Omega$. For any $T > 0$, consider the following parabolic equation with time-independent coefficients and homogeneous conormal Neumann boundary condition

$$
\begin{cases}
  u_t - \text{div}(A(x)\nabla u) + b(x)u = 0 \quad \text{in } \Omega \times (0,T), \\
  A\nabla u \cdot \nu = 0 \quad \text{on } \partial \Omega \times (0,T), \\
  u(\cdot,0) = u_0 \in L^2(\Omega),
\end{cases}
$$

where $\nu$ is the exterior unit normal vector on $\partial \Omega$, the symmetric matrix-valued function $A: \overline{\Omega} \to \mathbb{R}^{N \times N}$ is Lipschitz continuous and satisfies the uniform ellipticity condition, i.e., there is a constant $\Lambda_1 > 1$ such that

$$
\begin{cases}
  |a_{ij}(x) - a_{ij}(y)| \leq \Lambda_1|x - y| \quad \text{for all } x, y \in \Omega \text{ and each } i,j = 1, \ldots, N, \\
  \Lambda_1^{-1}||\xi||^2 \leq A(x)\xi \cdot \xi \leq \Lambda_1||\xi||^2 \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^N,
\end{cases}
$$

the unbounded potential $b(\cdot)$ verifies one of the following two assumptions:

$$
\begin{cases}
  (i) \quad ||b(\cdot)||_{L^{N+\delta}(\Omega)} \leq \Lambda_2 \text{ for some } \delta > 0; \\
  (ii) \quad |b(x)| \leq \frac{\Lambda_2}{|x|} \quad \text{for a.e. } x \in \Omega
\end{cases}
$$

with $\Lambda_2 > 0$.

The first goal of the present paper is to establish a H"older-type interpolation inequality at one time point for all solutions $u$ to (1.1). Roughly speaking, for any $t > 0$, there exist constants $C > 0$ and $\theta \in (0,1)$ such that

$$
||u(\cdot, t)||_{L^\theta(\Omega)} \leq C||u(\cdot, t)||_{L^2(B_{\nu(t\theta)}(\Omega))}^\theta||u_0||_{L^2(\Omega)}^{1-\theta} \quad \text{for all } u_0 \in L^2(\Omega).
$$

Such a kind of interpolation inequality have been established for solutions of parabolic equations either in convex bounded domains or in bounded $C^2$-smooth domains but with homogeneous Dirichlet boundary conditions; See for instance [5, 23, 24, 25, 26, 31]. In these papers, the approach for the desired interpolation inequality is mainly based on the parabolic-type Almgren frequency function method, which is essentially adapted from [12, 27].

The second goal of this paper is to deduce an observability inequality from measurable sets in time. This can be immediately obtained from the above-mentioned interpolation inequality combined with the telescoping series method developed in [4, 25].

More precisely, the main results of this paper can be stated as follows.

**Theorem 1.1.** Let $T > 0$ and $\omega \subset \Omega$ be a non-empty open subset. Then there are constants $C = C(\Lambda_1, \Lambda_2, N, \delta, \Omega, \omega) > 0$ and $\sigma = \sigma(\Lambda_1, \Lambda_2, N, \delta, \Omega, \omega) \in (0,1)$ such that for any solution $u$ of (1.1) with the initial value $u_0 \in L^2(\Omega)$, \begin{equation}
||u(\cdot, t_0)||_{L^2(\Omega)} \leq C e^{C(t^{\frac{2+1}{\sigma}})} ||u(\cdot, t_0)||_{L^2(\omega)}^\sigma ||u_0||_{L^2(\Omega)}^{1-\sigma} \quad \text{for all } t_0 \in (0,T).
\end{equation}
Remark 1.1. In [25], the authors have obtained the global interpolation inequality (1.4) for the heat equation with zero Dirichlet boundary condition and $L^\infty(0,T;L^p(\Omega))$ potential under the assumption $p > N$. This is coincident with the assumption (i) in (1.3). However, in view of the electric potential $O(|x|^{-1})$, one could see that the assumption $p > N$ is not optimal (see also [31]).

Theorem 1.2. Assume $\omega \subset \Omega$ is a non-empty open subset. Let $T > 0$ and $E \subset [0,T]$ be a subset of positive measure. Then there is a constant $C = C(\Lambda_1, \Lambda_2, N, \delta, \Omega, \omega, T, E) > 0$ such that for any solution $u$ of (1.1) with the initial value $u_0 \in L^2(\Omega)$,

$$
\|u(\cdot, T)\|_{L^2(\Omega)} \leq C \int_E \|u(\cdot, t)\|_{L^2(\omega)} dt.
$$

In particular, when $E = [0,T]$, the constant $C$ in the above inequality can be taken the form

$$
C(\Lambda_1, \Lambda_2, N, \delta, \Omega, \omega) e^{C(\Lambda_1, \Lambda_2, N, \delta, \Omega, \omega)(T^2+1)}.
$$

It follows from the classical Hilbert uniqueness method (HUM) that (see, e.g., [4])

Corollary 1.1. Let $T > 0$. Assume $\omega \subset \Omega$ is a nonempty open subset and $E \subset [0,T]$ is a subset of positive measure. Then, for any $u_0 \in L^2(\Omega)$, there is a control $f \in L^\infty(0,T;L^2(\Omega))$, with

$$
\|f\|_{L^\infty(0,T;L^2(\Omega))} \leq C\|u_0\|_{L^2(\Omega)}
$$

for the same constant $C$ appeared in (1.5), such that the solution of

$$
\begin{cases}
\begin{aligned}
u t - \text{div}(A(x)\nabla u) + b(x)u &= \chi_E \times \omega f &\text{in } \Omega \times (0,T), \\
A\nabla u \cdot \nu &= 0 &\text{on } \partial\Omega \times (0,T), \\
u(\cdot, 0) &= u_0 \in L^2(\Omega)
\end{aligned}
\end{cases}
$$

satisfies $u(x,T) = 0$ for a.e. $x \in \Omega$.

The interpolation inequality (1.4) at one time point in Theorem 1.1 is a quantitative form of strong unique continuation for the equation (1.1). The study of unique continuation property for parabolic equation has a long history. For the works in this topic, one can see [2, 3, 7, 9, 11, 12, 13, 14, 15, 19, 20, 27, 28, 30] and references therein. Among these papers, it is worth mentioning particularly [19] and [7]. In the paper [19], F. H. Lin showed the strong unique continuation property for the equation (1.1) when the potential $b(\cdot) \in L^{(N+1)/2}(\Omega)$. Although it is a qualitative form of unique continuation, F. H. Lin constructed an important and smart strategy that deduces a strong unique continuation of parabolic equations with time-independent coefficients to the elliptic counterparts. Later, by following and quantifying this strategy, B. Canuto, E. Rosset and S. Vessella proved in [7] the local quantitative unique continuation for time-independent parabolic equations but without potentials (i.e., $b(\cdot) = 0$). It seems to us that the results in [7] are not enough to derive the interpolation inequality in Theorem 1.1; See more discussions in Remark 2.1 below. Further, the presence of potential term will lead to some difficulties if one follows the same argument used in [7]. These difficulties force us to slightly improve the strategy used by B. Canuto, E. Rosset and S. Vessella (see Section 3 below).

When the boundary condition in (1.1) is homogeneous Dirichlet-type, through using the frequency function method, the global interpolation inequality in Theorem 1.1 has been studied in [5, 23, 24, 25, 26, 31]. However, to the best of our knowledge, this approach seems to be not applicable for the case of homogeneous Neumann boundary condition (at least we do not know). This forces us to find a new method to obtain the corresponding interpolation inequality.
In order to overcome these difficulties mentioned above, in this paper we shall adopt and slightly modify the reduction method, as well as Carleman estimates of elliptic operators. Roughly speaking, the reduction method [19] is to reduce a parabolic equation into an elliptic equation by using the Fourier transformation and adding one more spatial variable. However, because of the appearance of potential term, we shall adopt a sinh-type weighted Fourier transformation, which is slightly different to the strategy used in [19, 7]. Moreover, for the proof of stability estimate (see Lemma 3.3 below), the authors of [7] reduced the elliptic equation to a hyperbolic equation and used harmonic measure. This strategy, in our opinion, cannot be applied when the potential is nonzero. Instead, in this paper we shall use suitable Carleman estimates to deduce the corresponding stability estimate. Note that the reduction method is based on a representation formula for solutions of parabolic equations in terms of eigenfunctions of the corresponding elliptic operators, and therefore cannot be applied to general parabolic equations with time-dependent coefficients.

We emphasis that in the case of heat equation with homogeneous Dirichlet boundary conditions, the authors in [4] first observed that the observability estimate at one time point is in fact equivalent to a type of spectral inequality in [17] (see also [23]). This type of spectral inequality, roughly speaking, is an observability inequality from a partial region on the finite sum of eigenfunctions of the principal elliptic operator. For related works, we refer the reader to [8, 16, 18, 21, 22] and references therein. Therefore, if one could establish a type of spectral inequality as in [17] (see also [23]), the global interpolation inequality can also be deduced by the technique utilized in [4]. We refer [21] for the spectral inequality of elliptic equation with Neumann boundary condition and without any potential term.

Meanwhile, we also refer [12, 28] for quantitative estimates of unique continuation of parabolic equations with time dependent coefficients, in which some parabolic-type Carleman estimates were established. We believe that the Carleman method developed in [28] (or [12]) may provide a possible approach for proving the corresponding interpolation inequality. However, this issue escapes the study of the present paper and is deserved to be investigated in the continued work.

Last but not least, we would like to stress that the observability estimate from measurable sets in the time variable established in Theorem 1.2 has several applications in control theory. In particular, it implies bang-bang properties of minimal norm and minimal time optimal control problems (see for instance [25, 29]).

The structure of this paper is organized as follows. In Section 2, we first present two quantitative estimates of unique continuation needed for proof of the main results, and then we prove Theorems 1.1 and 1.2, respectively. In Section 3, we are devoted to the proofs of the above-mentioned two quantitative estimates of unique continuation. In Appendix, the proofs of some results used in Section 3 are given.

**Notation.** Throughout the paper, $\triangle_R(x_0)$ stands for a ball in $\mathbb{R}^N$ with the center $x_0$ and of radius $R > 0$, $B_R(x_0, 0)$ stands for a ball in $\mathbb{R}^{N+1}$ with the center $(x_0, 0)$ and of radius $R > 0$. Denote by $\partial \triangle_R(x_0)$ the boundary of $\triangle_R(x_0)$, by $\rho_0 = \sup\{|x-y| : x, y \in \Omega\}$ and $\Omega_\rho = \{x \in \Omega : d(x, \partial \Omega) \geq \rho\}$ with $\rho \in (0, \min\{1, \rho_0\})$. Write $\bar{z}$ for the complex conjugate of a complex number $z \in \mathbb{C}$. The letter $C$ denotes a generic positive constant that depends on the a priori data but not on the solution and may vary from line to line. Moreover, we shall denote by $C(\cdot)$ a positive constant if we need to emphasize the dependence on some parameters in the brackets.
2 Proofs of main results

2.1 Unique continuation estimates

In order to present the proof of Theorem 1.1, we first state two results concerning quantitative estimates of unique continuation: The first one is local, and the second one is global. Their proofs are postponed to give in Section 3.

Proposition 2.1. Let \( T > 0 \). Suppose \( \rho \in (0, \min\{1, \rho_0\}) \) such that \( \Omega_\rho \neq \emptyset \). Then there exist \( R \in (0, \rho) \) and \( \kappa \in (0, 1/2) \) such that for any \( r \in (0, \kappa R) \), any \( t_0 \in (0, T/2) \) and any \( x_0 \in \Omega_\rho \), we have

\[
\|u(\cdot, t_0)\|_{L^2(\Delta_{2r}(x_0))} \leq Ce^{C\frac{r^2}{t_0}} \|u(\cdot, t_0)\|_{L^2(\Delta_{r}(x_0))} \left( \sup_{s \in [0, T]} \|u(\cdot, s)\|_{H^1(\Delta_R(x_0))} \right)^{1-\sigma}
\]

with some constants \( C = C(\Lambda_1, \Lambda_2, N, \delta, r, R) > 0 \) and \( \sigma = \sigma(\Lambda_1, \Lambda_2, N, \delta, r) \in (0, 1) \), where \( u \in C([0, 2T]; H^1_{loc}(\Omega)) \) satisfies

\[
l(x)\partial_t u - \text{div}(A(x)\nabla u) + b(x)u = 0 \quad \text{in} \quad \Omega \times (0, 2T).
\]

(2.1)

Here \( A \) and \( b \) are the same as in (1.1), and \( l : \Omega \to \mathbb{R}^+ \) verifies

\[
\Lambda_3^{-1} \leq l(x) \leq \Lambda_3, \quad |l(x) - l(y)| \leq \Lambda_3|x - y| \quad \text{for a.e. } x, y \in \Omega
\]

(2.2)

with a constant \( \Lambda_3 > 1 \).

Proposition 2.2. Let \( T > 0 \) and \( \omega \subset \Omega \) be a non-empty open subset. Then there are constants \( C = C(\Lambda_1, \Lambda_2, N, \delta, \Omega, \omega) > 0 \) and \( \sigma = \sigma(\Lambda_1, \Lambda_2, N, \delta, \Omega, \omega) \in (0, 1) \) such that for any solution \( u \in C([0, T]; H^1(\Omega)) \) of (1.1), we have

\[
\|u(\cdot, t_0)\|_{L^2(\Omega)} \leq Ce^{C\frac{1}{t_0}} \|u(\cdot, t_0)\|_{L^2(\omega)} \left( \sup_{s \in [0, T]} \|u(\cdot, s)\|_{H^1(\Omega)} \right)^{1-\sigma}
\]

for all \( t_0 \in (0, T/2) \).

Remark 2.1. The local interpolation inequality established in Proposition 2.1 is slightly different from the two spheres and one cylinder inequality established in [7, Theorem 3.1.1']. Actually, in [7], the bound for the parameter \( r \) depends on the instant \( t_0 \). This, however, will lead some difficulties when one applies it to prove the global interpolation and observability inequalities.

Remark 2.2. Equations of type (2.1) appear when one transforms the parabolic operator via a linear mapping from \( \mathbb{R}^N \) into \( \mathbb{R}^N \). It is also worth mentioning that parabolic equations of form (2.1) with positive coefficients in front of the time derivative are much more nature from the physical point of view. They model the heat diffusion of the temperature in a non-isotropic and non-homogeneous material. In fact, there are two relevant physical quantities in heat diffusion processes: the conductivity coefficients and the specific heat capacity. The latter appears in the equation in front of the time derivative.

2.2 Proof of Theorem 1.1

We first recall the following well-known Hardy inequality (see e.g. [10]) and Sobolev interpolation theorem (see e.g. [1, Theorem 5.8]), which will be used frequently in our argument below.
Lemma 2.1. Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$ $(N \geq 3)$. Then, it holds that

(i) (Hardy’s inequality)
\[ \int_{\Omega} |x|^2 |f|^2 \, dx \leq \frac{4}{(N-2)^2} \int_{\Omega} |\nabla f|^2 \, dx \quad \text{for any } f \in H^1_0(\Omega). \]

(ii) (Sobolev’s interpolation theorem) For each $p \in \left[ 2, \frac{2N}{N-2} \right]$, there is a constant $\Gamma_1(\Omega, N, p) > 0$ such that
\[ \|f\|_{L^p(\Omega)} \leq \Gamma_1(\Omega, N, p) \|f\|_{H^1(\Omega)}^\theta \|f\|_{L^2(\Omega)}^{1-\theta} \quad \text{for any } f \in H^1(\Omega). \]

where $\theta = N\left(\frac{1}{2} - \frac{1}{p}\right)$.

As a simple consequence of the above Sobolev interpolation theorem, we have

Corollary 2.1. Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$ $(N \geq 3)$. For each case of (1.3) and for every $\epsilon \in (0, \frac{1}{2}]$, it holds that
\[ b(\cdot) \in L^{\frac{N}{2} + \epsilon}(\Omega). \]

Further, for each $\eta > 0$ there is a constant $\Gamma_2(\Omega, N, \eta) > 0$ such that, for any $h(\cdot) \in L^{\frac{N}{2} + \eta}(\Omega)$ and $f(\cdot) \in H^1(\Omega)$,
\[ \int_{\Omega} |h| |f|^2 \, dx \leq \Gamma_2(\Omega, N, \eta) \|h\|_{L^{\frac{N}{2} + \eta}(\Omega)} \|f\|_{L^2(\Omega)}^\eta \|f\|_{H^1(\Omega)}^{1-\eta}. \]  

Proof of Theorem 1.1. The proof is divided into two steps.

Step 1. Energy estimates. In this step, we shall prove the following two claims:

- If $u(\cdot, 0) \in L^2(\Omega)$, then for each $t \in [0, 6T]$ we have
\[ \|u(\cdot, t)\|_{L^2(\Omega)} \leq e^{Ct} \|u(\cdot, 0)\|_{L^2(\Omega)} \]  

and
\[ \|u(\cdot, t)\|_{H^1(\Omega)} \leq \frac{Ce^{Ct}}{\sqrt{t}} \|u(\cdot, 0)\|_{L^2(\Omega)}. \]

- If $u(\cdot, 0) \in H^1(\Omega)$, then we have
\[ \|u(\cdot, t)\|_{H^1(\Omega)} \leq e^{Ct} \|u(\cdot, 0)\|_{H^1(\Omega)} \]  

for each $t \in [0, 6T]$.

Indeed, multiplying the first equation in (1.1) by $u$ and then integrating by parts over $\Omega$, we get
\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 \, dx + \int_{\Omega} \nabla u \cdot (A\nabla u) \, dx = - \int_{\Omega} b|u|^2 \, dx. \]

Note that, from (2.3) (by letting $\eta = \frac{1}{2}$ there) and (1.2), we have
\[- \int_{\Omega} b|u|^2 \, dx \leq C\|b\|_{L^{\frac{N}{2} + 1}(\Omega)} \|u\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)} \leq CA_2 \|u\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)} \]
\[ \leq C e^{-N} A_2^{N+1} \|u\|_{L^2(\Omega)}^2 + \epsilon \|u\|_{H^1(\Omega)}^2 = (C e^{-N} A_2^{N+1} + \epsilon) \|u\|_{L^2(\Omega)}^2 + \epsilon \|\nabla u\|_{L^2(\Omega)}^2 \]
\[ \leq (C e^{-N} A_2^{N+1} + \epsilon) \|u\|_{L^2(\Omega)}^2 + \epsilon A_1 \int_{\Omega} \nabla u \cdot (A\nabla u) \, dx. \]
Taking $\epsilon = \frac{1}{2\Lambda_1}$ in the above inequality, we obtain that

$$- \int_{\Omega} b|u|^2 dx \leq \left[ C\Lambda_1 N^2 + \frac{1}{2}\Lambda_1 \right] \int_{\Omega} |u|^2 dx + \frac{1}{2} \int_{\Omega} \nabla u \cdot (A\nabla u) dx.$$  

This, along with (2.7), yields

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 dx + \frac{1}{2} \int_{\Omega} \nabla u \cdot (A\nabla u) dx \leq \left[ C\Lambda_1 N^2 + \frac{1}{2}\Lambda_1 \right] \int_{\Omega} |u|^2 dx.$$  

(2.8)

Then

$$\frac{d}{dt} \left( e^{-[C\Lambda_1 N^2 + \Lambda_1^{t-1}]} t \int_{\Omega} |u|^2 dx \right) \leq 0.$$  

(2.9)

This gives (2.4). Moreover, by (2.8) and (2.9), we obtain

$$\int_{0}^{t} \int_{\Omega} \nabla u \cdot (A\nabla u) dxds \leq \left[ t \left( C\Lambda_1 N^2 + \Lambda_1^{-1} \right) e^{(C\Lambda_1 N^2 + \Lambda_1^{t-1})t} + 1 \right] \|u(\cdot,0)\|_{L^2(\Omega)}^2.$$  

(2.10)

Next, we show (2.6). Here we divide our proof into two cases based on the assumptions in (1.3).

Case I. $|b(x)| \leq \frac{\Lambda_0}{|\Omega|}$ a.e. $x \in \Omega$. In this case, we take $r_0 \in (0, d(0, \partial\Omega))$ and $\eta \in C^\infty(\mathbb{R}^N; [0, 1])$ such that

$$\begin{align*}
&\{ \frac{\Delta_x (0)}{r_0} < \Omega, \\
&\eta = 1 \quad \text{in} \quad \Delta_x (0), \\
&\eta = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Delta_x (0) \\
&|\nabla \eta| \leq \frac{C}{r_0} \quad \text{in} \quad \mathbb{R}^N.
\end{align*}$$

Multiplying the first equation of (1.1) by $-\text{div}(A\nabla u)\eta^2$ and integrating by parts over $\Omega$, by Lemma 2.1 we have

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \nabla u \cdot (A\nabla u)\eta^2 dx + \int_{\Omega} |\text{div}(A\nabla u)|^2\eta^2 dx \leq 2 \int_{\Omega} |\text{div}(A\nabla u)||\nabla u| \cdot \eta dx + 2 \int_{\Omega} |b||u||\nabla u| \cdot \eta dx + \int_{\Omega} |b||u|\eta^2|\text{div}(A\nabla u)| dx$$

$$\leq \frac{1}{2} \int_{\Omega} |\text{div}(A\nabla u)|^2\eta^2 dx + 5 \int_{\Omega} |\nabla u| \cdot \eta dx^2 + 2 \int_{\Omega} |b|^2|u|^2\eta^2 dx$$

$$\leq \frac{1}{2} \int_{\Omega} |\text{div}(A\nabla u)|^2\eta^2 dx + \frac{5C\Lambda_1}{r_0} \int_{\Omega} \nabla u \cdot (A\nabla u) dx + \frac{16C^2\Lambda_2^2}{(N-2)^2} \int_{\Omega} \nabla u \cdot (A\nabla u)\eta^2 dx$$

$$+ \frac{16C^2\Lambda_2^2}{(N-2)^2} \int_{\Omega} |u|^2 dx.$$  

(2.11)

Further, multiplying the first equation of (1.1) by $-\text{div}(A\nabla u)(1 - \eta^2)$ and integrating by parts over $\Omega$, we get

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \nabla u \cdot (A\nabla u)(1 - \eta^2) dx + \int_{\Omega} |\text{div}(A\nabla u)|^2(1 - \eta^2) dx \leq 2 \int_{\Omega} |\text{div}(A\nabla u)||\nabla u| \cdot \eta dx + 2 \int_{\Omega} |b||u||(A\nabla u) \cdot \nabla \eta| dx + \int_{\Omega} |b||u|(1 - \eta^2)|\text{div}(A\nabla u)| dx$$

$$\leq \frac{1}{4} \int_{\Omega} |\text{div}(A\nabla u)|^2\eta^2 dx + 5 \int_{\Omega} |\nabla \eta \cdot (A\nabla u)|^2 dx + \int_{\Omega} |b|^2|u|^2\eta^2 dx.$$
\[
\begin{align*}
&+ \frac{\Lambda_2}{r_0} \int_\Omega |u(1 - \eta^2)| |\text{div}(A \nabla u)| dx \\
\leq & \quad \frac{1}{4} \int_\Omega |\text{div}(A \nabla u)|^2 \eta^2 dx + \frac{5CA_1}{r_0^2} \int_\Omega \nabla u \cdot (A \nabla u) dx + \frac{8A_1 \Lambda_2^2}{(N - 2)^2} \int_\Omega |\nabla u \cdot (A \nabla u)\eta^2 dx \\
&+ \frac{8CA_2^2}{(N - 2)^2 r_0^2} \int_\Omega |u|^2 dx + \frac{\Lambda_2^2}{3r_0^2} \int_\Omega |u|^2 (1 - \eta^2) dx + \frac{3}{4} \int_\Omega |\text{div}(A \nabla u)|^2 (1 - \eta^2) dx.
\end{align*}
\]

This, together with (2.11), gives that
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_\Omega |u|^2 dx + \int_\Omega \nabla u \cdot (A \nabla u) dx & \leq C(r_0) \left( \int_\Omega |u|^2 dx + \int_\Omega \nabla u \cdot (A \nabla u) dx \right),
\end{align*}
\]

where
\[
C(r_0) := CA_1 \Lambda_2^{N+1} + \frac{1}{2A_1} + \left( \frac{24C}{(N - 2)^2} + \frac{1}{3} \right) \Lambda_2^2 \frac{10CA_1}{r_0^2} + \frac{24A_2^2}{(N - 2)^2}.
\]

Therefore, we have
\[
\begin{align*}
\frac{d}{dt} & \left[ e^{-2C(r_0)t} \int_\Omega (|u|^2 + \nabla u \cdot (A \nabla u)) dx \right] \leq 0. \quad (2.13)
\end{align*}
\]

This gives
\[
\|u(\cdot, t)\|_{H^1(\Omega)}^2 \leq \Lambda_2^2 e^{2C(r_0)t} \|u(\cdot, 0)\|_{H^1(\Omega)}^2. \quad (2.14)
\]

Hence (2.6) holds in this case.

**Case II.** \(b(\cdot) \in L^{N+8}(\Omega)\) and \(\|b(\cdot)\|_{L^{N+8}(\Omega)} \leq \Lambda_2\). Multiplying the first equation of (1.1) by \(-\text{div}(A \nabla u)\) and integrating by parts over \(\Omega\), by Lemma 2.1, we get
\[
\begin{align*}
&\frac{1}{2} \frac{d}{dt} \int_\Omega \nabla u \cdot (A \nabla u) dx + \frac{1}{2} \int_\Omega |\text{div}(A \nabla u)|^2 dx \\
\leq & \quad 2 \|b\|^2_{L^N(\Omega)} \|u\|^2_{L^{N+8}(\Omega)} \leq 2\Gamma_1(\Omega, N, 2) \|b\|^2_{L^N(\Omega)} \|u\|_{H^1(\Omega)}^2 \\
\leq & \quad CA_1 \|b\|^2_{L^N(\Omega)} \int_\Omega (|u|^2 + \nabla u \cdot (A \nabla u)) dx.
\end{align*}
\]

This, along with (2.8), gives that
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_\Omega (|u|^2 + \nabla u \cdot (A \nabla u)) dx & \leq \left( CA_1 \Lambda_1 + CA_1 \Lambda_2^{N+1} + \frac{1}{2A_1} \right) \int_\Omega (|u|^2 + \nabla u \cdot (A \nabla u)) dx.
\end{align*}
\]

This implies that
\[
\begin{align*}
\frac{d}{dt} & \left[ e^{-\left( CA_1 \Lambda_1 + CA_1 \Lambda_2^{N+1} + \frac{1}{2A_1} \right)t} \int_\Omega (|u|^2 + \nabla u \cdot (A \nabla u)) dx \right] \leq 0. \quad (2.15)
\end{align*}
\]

Similar to the proof of (2.14), we obtain (2.6) in this case.
Moreover, using (2.13) and (2.15) respectively in each case analyzed above, we obtain that there exists \( C > 0 \) such that
\[
\int_0^t \int_\Omega (|u|^2 + \nabla u \cdot (A \nabla u)) \, dx \, ds \geq \int_0^t e^{-C(t-s)} \|u(\cdot, t)\|_{H^1(\Omega)}^2 \geq \Lambda_1^{-1} t e^{-Ct} \|u(\cdot, t)\|_{H^1(\Omega)}^2.
\]
This, together with (2.10) and (2.4), yields (2.5).

**Step 2. Completing the proof.** We arbitrarily fixed \( t_0 \in (0, T) \) and consider the following equation
\[
\begin{cases}
\nu \cdot (A(x) \nabla v) + bv &= 0 \quad \text{in } \Omega \times (0, 4T), \\
A \nabla v \cdot \nu &= 0 \quad \text{on } \partial \Omega \times (0, 4T), \\
v(\cdot, 0) &= u(\cdot, \frac{t_0}{2}) \quad \text{in } \Omega.
\end{cases}
\]
It is obvious that \( v(\cdot, t) = u(\cdot, t + \frac{t_0}{2}) \) when \( t \in [0, 4T] \). Moreover, by (2.5) we have \( u(\cdot, \frac{t_0}{2}) \in H^1(\Omega) \), which means that \( v \in C([0, 4T]; H^1(\Omega)) \). From Proposition 2.2, it follows that there are \( C > 0 \) and \( \sigma \in (0,1) \) such that
\[
\|v(\cdot, \frac{t_0}{2})\|_{L^2(\Omega)} \leq Ce^{\frac{C(T^2+1)}{t_0}}\|v(\cdot, \frac{t_0}{2})\|_{L^2(\omega)}^\sigma \left( \sup_{s \in [0,T]} \|v(\cdot, s)\|_{H^1(\Omega)} \right)^{1-\sigma}.
\]
This, along with (2.6), gives that
\[
\|v(\cdot, \frac{t_0}{2})\|_{L^2(\Omega)} \leq Ce^{\frac{C(T^2+1)}{t_0}}\|v(\cdot, \frac{t_0}{2})\|_{L^2(\omega)}^\sigma \|v(\cdot, 0)\|_{H^1(\Omega)}^{1-\sigma}.
\]
Which is
\[
\|u(\cdot, t_0)\|_{L^2(\Omega)} \leq Ce^{\frac{C(T^2+1)}{t_0}}\|u(\cdot, t_0)\|_{L^2(\omega)}^\sigma \|u(\cdot, \frac{t_0}{2})\|_{H^1(\Omega)}^{1-\sigma}.
\]
This, together with (2.5), implies (1.4) and completes the proof. \( \square \)

### 2.3 Proof of Theorem 1.2

To make the paper self-contained, we here provide the proof of Theorem 1.2 in detail, although it is almost the same as the proof of [25, Theorem 1.1] or [4, Theorem 1].

**Lemma 2.2.** ([25, Proposition 2.1]) Let \( E \subset (0, T) \) be a measurable set of positive measure, \( \ell \) be a density point of \( E \). Then for each \( z > 1 \), there exists \( \ell_1 \in (\ell, T) \) such that \( \{\ell_m\}_{m \in \mathbb{N}^+} \) given by
\[
\ell_{m+1} = \ell + \frac{1}{2^m}(\ell_1 - \ell)
\]
verifies
\[
\ell_m - \ell_{m+1} \leq 3|E \cap (\ell_{m+1}, \ell_m)|.
\]

**Proof of Theorem 1.2.** By (1.4), one can show that, for arbitrary fixed \( \epsilon > 0 \) and any \( t_0 \in (0, T) \),
\[
\|u(\cdot, t_0)\|_{L^2(\Omega)} \leq Ce^{\frac{C(T^2+1)}{t_0}}\|u(\cdot, t_0)\|_{L^2(\omega)} + \epsilon\|u(\cdot, 0)\|_{L^2(\Omega)},
\]
where $\gamma > 0$ is a constant. By a translation in time, one has for each $0 \leq t_1 < t_2 < T$,
\[ \| u(\cdot, t_2) \|_{L^2(\Omega)} \leq C e^{\frac{C(t^2_2 - t^2_1)}{t^2_1}} \| u(\cdot, t_2) \|_{L^2(\Omega)} + \epsilon \| u(\cdot, t_1) \|_{L^2(\Omega)} \] for all $\epsilon > 0$.

Let $0 < \ell_{m+2} < \ell_{m+1} \leq t < \ell_m < T$, by (2.3), we get
\[ \| u(\cdot, t) \|_{L^2(\Omega)} \leq C e^{\frac{C(t^2_2 + 3\ell_{m+2})}{t}} \| u(\cdot, t) \|_{L^2(\Omega)} + \epsilon \| u(\cdot, \ell_{m+2}) \|_{L^2(\Omega)} \] for all $\epsilon > 0$. (2.18)

Noting that, by (2.4),
\[ e^{-C\ell} \| u(\cdot, \ell_m) \|_{L^2(\Omega)} \leq \| u(\cdot, t) \|_{L^2(\Omega)}. \]

This, along with (2.18), yields that for any $\epsilon > 0$,
\[ \| u(\cdot, \ell_m) \|_{L^2(\Omega)} \leq C e^{\frac{C(t^2_2 + 3\ell_{m+2})}{t}} \| u(\cdot, t) \|_{L^2(\Omega)} + \epsilon \| u(\cdot, \ell_{m+2}) \|_{L^2(\Omega)}. \]

Integrating over $E \cap (\ell_{m+1}, \ell_m)$, we get
\[ \| u(\cdot, \ell_m) \|_{L^2(\Omega)} \leq \frac{C e^{\frac{C(t^2_2 + 3\ell_{m+2})}{t}}}{|E \cap (\ell_{m+1}, \ell_m)| e^t} \int_{\ell_{m+1}}^{\ell_m} \chi_E \| u(\cdot, t) \|_{L^2(\Omega)} dt + \epsilon \| u(\cdot, \ell_{m+2}) \|_{L^2(\Omega)}. \]

This, together with (2.16) and (2.17), gives
\[ e^\gamma e^{-\eta z^{m+2}} \| u(\cdot, \ell_m) \|_{L^2(\Omega)} - e^{1+\gamma} e^{-\eta z^{m+2}} \| u(\cdot, \ell_{m+2}) \|_{L^2(\Omega)} \leq C \int_{\ell_{m+1}}^{\ell_m} \chi_E \| u(\cdot, t) \|_{L^2(\Omega)} dt, \]
where $\eta := \frac{C(t^2 + 1)}{z(2+1)\ell(t_1 - t)}$. Letting $\epsilon := e^{-\eta z^{m+2}}$ and $z := \sqrt{\frac{2+3}{1+\gamma}}$, we have
\[ e^{-\eta(2+\gamma)z^{m+2}} \| u(\cdot, \ell_{m+2}) \|_{L^2(\Omega)} \leq C \int_{\ell_{m+1}}^{\ell_m} \chi_E \| u(\cdot, t) \|_{L^2(\Omega)} dt. \] (2.19)

By taking $m = 2m'$ and then summing the estimate (2.19) from $m' = 1$ to infinity, we obtain
\[ \sum_{m' = 1}^{\infty} \left[ e^{-\eta(2+\gamma)z^{2m'}} \| u(\cdot, \ell_{2m'}) \|_{L^2(\Omega)} - e^{-\eta(2+\gamma)z^{2m'+2}} \| u(\cdot, \ell_{2m'+2}) \|_{L^2(\Omega)} \right] \leq C \int_{E \cap (\ell_2, \ell_2)} \| u(\cdot, t) \|_{L^2(\Omega)} dt \leq C \int_{E \cap (\ell_2, \ell_2)} \| u(\cdot, t) \|_{L^2(\Omega)} dt. \]

Note that $e^{-\eta(2+\gamma)z^{2m'}} \to 0$ as $m' \to \infty$. Therefore,
\[ \| u(\cdot, \ell_2) \|_{L^2(\Omega)} \leq C e^{\eta(2+\gamma)z^2} \int_E \| u(\cdot, t) \|_{L^2(\Omega)} dt. \]

This, along with (2.4), leads to the desired observability inequality.

Finally, when $E = [0, T]$, we can take $\ell = 0$ and $\ell_1 = T$ in the above argument to conclude the desired result. \(\square\)
3 Proofs of quantitative estimates of unique continuation

3.1 Preliminary lemmas

3.1.1 Local energy estimates and exponential decay

Suppose \( \rho \in (0, \min\{1, \rho_0\}) \) such that \( \Omega_\rho \neq \emptyset \), \( T > 0 \), \( t_0 \in (0, T) \) and \( x_0 \in \Omega_\rho \). Let \( u \in C([0, 2T]; H^1(\Delta_\rho(x_0))) \) be a solution of

\[
\begin{cases}
  l(x)u_t - \text{div}(A(x)\nabla u) + b(x)u = 0 & \text{in } \Delta_\rho(x_0) \times (0, 2T), \\
  u(\cdot, 0) = 0 & \text{in } \Delta_\rho(x_0).
\end{cases}
\]  

(3.1)

Assume \( \eta \in C^{\infty}(\mathbb{R}^+; [0, 1]) \) is a cutoff function satisfying

\[
\begin{cases}
  \eta \equiv 1 & \text{in } (0, t_0), \\
  \eta \equiv 0 & \text{in } [T, +\infty), \\
  |\eta_t| \leq \frac{C}{t-t_0} & \text{in } (t_0, T)
\end{cases}
\]

with a generic positive constant \( C \) independent of \( t_0 \) and \( T \). Set

\[
R_0 = \left\{ \begin{array}{ll}
\Theta_N^N \left( 8\sqrt{2}A_1A_2(\Delta_1(0), N, \frac{N}{2}) \right)^{-\frac{N}{N-2}} & \text{if (i) in (1.3) holds,} \\
\frac{N(N-2)}{2\sqrt{2}A_1A_2} & \text{if (ii) in (1.3) holds,}
\end{array} \right.
\]  

(3.3)

and take \( R \in (0, \min\{R_0, \rho\}) \). Here \( \Theta_N = |\Delta_1(0)| \). Let \( v \) be the solution of

\[
\begin{cases}
  l(x)v_t - \text{div}(A(x)\nabla v) + b(x)v = 0 & \text{in } \Delta_R(x_0) \times \mathbb{R}^+, \\
  v = \eta u & \text{on } \partial \Delta_R(x_0) \times \mathbb{R}^+, \\
  v(\cdot, 0) = 0 & \text{in } \Delta_R(x_0),
\end{cases}
\]  

(3.4)

where \( u \) satisfies (3.1) and \( \eta \) verifies (3.2). Then, we have the following exponential decay estimate of \( H^1 \)-energy for (3.4).

**Lemma 3.1.** There exists a generic constant \( C > 0 \) such that

\[
\|v(\cdot, t)\|_{L^1(\Delta_R(x_0))} \leq C T^{-1} e^{\frac{C R^{-N} T (1 + t^{-1})}{r_0}} - \frac{C R^{-2} (T - t)^+}{R^2} \|F(R)\| \text{ for all } t \in \mathbb{R}^+,
\]

where \( (T - t)^+ = \max\{0, t - T\} \) and \( F(R) = \sup_{s \in [0, T]} \|u(\cdot, s)\|_{H^1(\Delta_R(x_0))} \).

**Proof.** We proceed the proof into two steps as follows.

**Step 1. To prove (3.1) when \( t \in [0, T] \).** Setting \( w = v - \eta u \) in \( \Delta_R(x_0) \times \mathbb{R}^+ \), we find that \( w \) verifies that

\[
\begin{cases}
l(x)w_t - \text{div}(A(x)\nabla w) + b(x)w = -l(x)\eta_t u & \text{in } \Delta_R(x_0) \times \mathbb{R}^+, \\
w = 0 & \text{on } \partial \Delta_R(x_0) \times \mathbb{R}^+, \\
w(\cdot, 0) = 0 & \text{in } \Delta_R(x_0).
\end{cases}
\]  

(3.5)

We now prove that for each \( t \in [0, T] \),

\[
\|w(\cdot, t)\|_{L^2(\Delta_R(x_0))} \leq \frac{C T}{T - t_0} e^{\frac{C (r_1 - N + \frac{N}{2})}{t_0}} \sup_{s \in [0, T]} \|u(\cdot, s)\|^2_{L^2(\Delta_R(x_0))}
\]  

(3.6)
with a generic constant $C > 0$. We divide our proof into two cases.

**Case I.** $|b(x)| \leq \frac{\Delta}{|x|}$ a.e. $x \in \Omega$. Indeed, multiplying first (3.5) by $w$ and integrating by parts over $\Delta_R(x_0) \times (0, t)$, along with the Hardy inequality in Lemma 2.1, we have

$$
\frac{1}{2} \int_{\Delta_R(x_0)} l|w(\cdot, t)|^2 dx + \int_0^t \int_{\Delta_R(x_0)} \nabla w \cdot (A \nabla w) dx ds
\leq \int_0^t \int_{\Delta_R(x_0)} |b||w|^2 dx ds + \int_0^t \int_{\Delta_R(x_0)} l_{\eta} w dx ds
\leq \Lambda_2 \int_0^t \int_{\Delta_R(x_0)} \Delta |w|^2 dx ds + \frac{1}{2} \int_0^t \int_{\Delta_R(x_0)} l|\eta||w|^2 dx ds + \frac{1}{2} \int_0^t \int_{\Delta_R(x_0)} l|\eta||w|^2 dx ds
\leq \frac{1}{2} \epsilon \int_0^t \int_{\Delta_R(x_0)} \Delta |w|^2 dx ds + \frac{CT}{T - t_0} \sup_{s \in [0, T]} \|u(\cdot, s)\|_{L^2(\Delta_R(x_0))}^2 + \frac{1}{2} \left( -\Lambda_2^2 + \frac{C}{T - t_0} \right) \int_0^t \int_{\Delta_R(x_0)} |w|^2 dx ds
\leq 2 \Lambda_1 \int_0^t \int_{\Delta_R(x_0)} \Delta w \cdot (A \nabla w) dx ds + \frac{CT}{T - t_0} \sup_{s \in [0, T]} \|u(\cdot, s)\|_{L^2(\Delta_R(x_0))}^2 + \frac{1}{2} \left( -\Lambda_2^2 + \frac{C}{T - t_0} \right) \int_0^t \int_{\Delta_R(x_0)} |w|^2 dx ds.
$$

Taking $\epsilon = \frac{(N-2)^2}{2\Lambda_1}$ in the above inequality, we obtain

$$
\|w(\cdot, t)\|_{L^2(\Delta_R(x_0))}^2 \leq C \left( 1 + \frac{1}{T - t_0} \right) \int_0^t \|u(\cdot, s)\|_{L^2(\Delta_R(x_0))}^2 ds + \frac{CT}{T - t_0} \sup_{s \in [0, T]} \|u(\cdot, s)\|_{L^2(\Delta_R(x_0))}^2
$$

with a generic constant $C > 0$. By the Gronwall inequality, we get (3.6) immediately.

**Case II.** $b(\cdot) \in L^{N+\delta}(\Omega)$ and $\|b(\cdot)\|_{L^{N+\delta}(\Omega)} \leq \Lambda_2$. We first note that, by using a standard scaling technique to (2.3), without lose of generality, one has

$$
\Gamma_2(\Delta R(x_0), N, \eta) = \Gamma_2(\Delta_1(0), N, \eta) r^{-\frac{\Delta}{2\varepsilon + \eta}} \text{ for each } r \in (0, 1).
$$

(3.7)

Multiplying first (3.5) by $w$ and then integrating by parts over $\Delta_R(x_0) \times (0, t)$, along with (2.3) (by letting $\epsilon = \frac{1}{2}$ there) and (3.7), we have

$$
\frac{1}{2} \int_{\Delta_R(x_0)} l(x)|w(x, t)|^2 dx + \int_0^t \int_{\Delta_R(x_0)} \nabla w \cdot (A \nabla w) dx ds
\leq \int_0^t \int_{\Delta_R(x_0)} \eta \frac{\Delta}{|x|} w \eta dx ds + \int_0^t \int_{\Delta_R(x_0)} \eta w dx ds
\leq \int_0^t \int_{\Delta_R(x_0)} \eta \frac{\Delta}{|x|} w \eta dx ds + \int_0^t \int_{\Delta_R(x_0)} \eta w dx ds
\leq \left( C^{-N} R^{N+1} \Lambda_2^{N+1} + \frac{C}{2(T - t_0)} + \epsilon R^{-2} \right) \int_0^t \|w\|_{L^2(\Delta_R(x_0))}^2 ds
$$

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+\epsilon R^{-2} A_1 \int_0^t \int_{\triangle_R(x_0)} \nabla w \cdot (A \nabla w) \, dx \, ds + \frac{CT}{T - t_0} \sup_{s \in [0, T]} \|u(\cdot, s)\|_{L^2(\triangle_R(x_0))}^2 \quad \text{for any } \epsilon > 0.

Taking $\epsilon = \frac{R^2}{\Lambda_1}$ in the above inequality, we obtain

$$
\|w(\cdot, t)\|_{L^2(\triangle_R(x_0))}^2 \leq C \left( R^{1-N} + \frac{1}{T - t_0} \right) \int_0^t \|w(\cdot, s)\|_{L^2(\triangle_R(x_0))}^2 \, ds + \frac{CT}{T - t_0} \sup_{s \in [0, T]} \|u(\cdot, s)\|_{L^2(\triangle_R(x_0))}^2
$$

with a generic constant $C > 0$. By the Gronwall inequality, we get (3.6) immediately. Hence, from (3.6) and the definition of $w$, we know that for any $t \in [0, T]$,

$$
\|v(\cdot, t)\|_{L^2(\triangle_R(x_0))}^2 \leq C \left( \frac{CT}{T - t_0} \right) e^{C \left( R^{1-N} + \frac{1}{T - t_0} \right) T} \sup_{s \in [0, T]} \|u(\cdot, s)\|_{L^2(\triangle_R(x_0))}^2
$$

with a generic constant $C > 0$.

Next, we show that

$$
\|\nabla w(\cdot, t)\|_{L^2(\triangle_R(x_0))}^2 \leq \left( 1 + \frac{1}{T - t_0} \right) \frac{CT}{T - t_0} e^{C \left( R^{1-N} + \frac{1}{T - t_0} \right) T} \sup_{s \in [0, T]} \|u(\cdot, s)\|_{H^1(\triangle_R(x_0))}^2
$$

(3.9)

Which, along with the definition of $w$, gives that for each $t \in [0, T]$,

$$
\|\nabla v(\cdot, t)\|_{L^2(\triangle_R(x_0))}^2 \leq 2\|\nabla w(\cdot, t)\|_{L^2(\triangle_R(x_0))}^2 + 2\|\nabla u(\cdot, t)\|_{L^2(\triangle_R(x_0))}^2
\leq \left( 1 + \frac{1}{T - t_0} \right) \frac{CT}{T - t_0} e^{C \left( R^{1-N} + \frac{1}{T - t_0} \right) T} \sup_{s \in [0, T]} \|u(\cdot, s)\|_{H^1(\triangle_R(x_0))}^2
$$

(3.10)

with a generic constant $C > 0$. Hence, the desired estimate (3.1) follows from (3.8) and (3.10) when $t \in [0, T]$. We also divide the proof of (3.9) into two cases under the assumptions in (1.3).

Case I. $|b(x)| \leq \frac{\Lambda_3}{|x|} \text{ a.e. } x \in \Omega$. Multiplying first (3.5) by $w_t$ and then integrating by parts over $\Omega \times (0, T)$, we find

$$
\int_0^t \int_{\triangle_R(x_0)} |w_t|^2 \, dx \, ds + \frac{1}{2} \int_0^t \int_{\triangle_R(x_0)} |\nabla w \cdot (A \nabla w)| \, dx \, ds
\leq \frac{1}{2} \int_0^t \int_{\triangle_R(x_0)} |x|^{-2} |w|^2 \, dx \, ds + \frac{\Lambda_3^2 + \frac{C}{2|\Lambda_3|}}{2} \int_0^t \int_{\triangle_R(x_0)} l(x) |w_t|^2 \, dx \, ds + \frac{C}{2(T - t_0)} \int_0^t \int_{\triangle_R(x_0)} |u|^2 \, dx \, ds
$$

for any $\epsilon > 0$

with a generic constant $C > 0$. Letting $\epsilon = \frac{\Lambda_3^2 + \frac{C}{2|\Lambda_3|}}{2}$ in the inequality above, combined with the Hardy inequality in Lemma 2.1, leads to

$$
\int_{\triangle_R(x_0)} \nabla w(x, t) \cdot (A(x) \nabla w(x, t)) \, dx
\leq C \left( 1 + \frac{1}{T - t_0} \right) \int_0^t \int_{\triangle_R(x_0)} |\nabla w|^2 \, dx \, ds + \left( 1 + \frac{1}{T - t_0} \right) \frac{CT}{T - t_0} \sup_{s \in [0, T]} \|u(\cdot, s)\|_{L^2(\triangle_R(x_0))}^2,
$$
for a generic constant $C > 0$. This, together with the uniform ellipticity condition (1.2), means that
\[
\|\nabla w(\cdot, t)\|_{L^2(\Delta_R(x_0))}^2 \leq C \left( 1 + \frac{1}{T - t_0} \right) \int_0^t \|\nabla w(\cdot, s)\|_{L^2(\Delta_R(x_0))}^2 ds + \frac{C T}{T - t_0} \sup_{s \in [0, T]} \|w(\cdot, s)\|_{L^2(\Delta_R(x_0))}^2.
\]
By the Gronwall inequality, we get (3.9).

Case II. $b(\cdot) \in L^{N+\delta}(\Omega)$ and $||b(\cdot)||_{L^{N+\delta}(\Omega)} \leq \Lambda_2$. Multiplying (3.5) by $w_t$ and integrating by parts over $\Delta_R(x_0) \times (0, t)$, we have
\[
\int_0^t \int_{\Delta_R(x_0)} l|w_t|^2 dxds + \frac{1}{2} \int_0^t \int_{\Delta_R(x_0)} |\nabla w \cdot (A \nabla w)| dxds \\
\leq \frac{\epsilon}{2} \int_0^t \int_{\Delta_R(x_0)} |b|^2 |w|^2 dxds + \frac{\Lambda_3 + \frac{C}{2\epsilon}}{2} \int_0^t \int_{\Delta_R(x_0)} l|w_t|^2 dxds \\
+ \frac{C \epsilon}{2(T - t_0)} \int_0^t \int_{\Delta_R(x_0)} |u|^2 dxds + \frac{C \epsilon R^{-2} \Lambda_2^2}{2} \int_0^t \int_{\Delta_R(x_0)} \nabla w \cdot (A \nabla w) dxdt \\
+ \frac{\Lambda_3 + \frac{C}{2\epsilon}}{2} \int_0^t \int_{\Delta_R(x_0)} l|w_t|^2 dxds + \frac{C \epsilon}{2(T - t_0)} \int_0^t \int_{\Delta_R(x_0)} |w|^2 dxds.
\]
Here, we used (2.3) and (3.7). Taking $\epsilon = \frac{\Lambda_3 + \frac{C}{2\epsilon}}{2}$ in the above inequality, by (3.6) we get
\[
\int_{\Delta_R(x_0)} \nabla w(x, t) \cdot (A \nabla w(x, t)) dx \\
\leq CR^{-2} \left( 1 + \frac{1}{T - t_0} \right) \int_0^t \int_{\Delta_R(x_0)} (|w|^2 + \nabla w \cdot (A \nabla w)) dxds \\
+ \frac{C T}{T - t_0} \left( 1 + \frac{1}{T - t_0} \right) \sup_{s \in [0, T]} \|u(\cdot, s)\|_{L^2(\Delta_R(x_0))}^2 \\
\leq CR^{-N} \left( 1 + \frac{1}{T - t_0} \right) \int_0^t \int_{\Delta_R(x_0)} \nabla w \cdot (A \nabla w) dxds \\
+ C \frac{T}{T - t_0} \left( 1 + \frac{1}{T - t_0} \right) e^{C\left(R^{-N} + \frac{1}{T - t_0}\right)T} \sup_{s \in [0, T]} \|u(\cdot, s)\|_{L^2(\Omega)}^2.
\]
By the Gronwall inequality, we get (3.9).

Step 2. To prove (3.1) when $t \geq T$.

Define for any $f, g \in C_0^\infty(\Delta_R(x_0))$, $(f, g)_{L^2(\Delta_R(x_0))} := \int_{\Delta_R(x_0)} l(x)f(x)g(x) dx$ and $\|f\|_{L^2(\Delta_R(x_0))} := (f, f)_{L^2(\Delta_R(x_0))}^{1/2}$.

Set $L^2(\Delta_R(x_0)) = C_0^\infty(\Delta_R(x_0))^{\|\cdot\|_{L^2(\Delta_R(x_0))}}$. Since $l$ is positive, it is clear that $L^2(\Delta_R(x_0)) = L^2(\Delta_R(x_0))$ with an equivalent norm. Denoting $A = -l^{-1}[\text{div}(A \nabla) - b]$, we claim that there is a
generic constant \( C > 0 \) \( \) (independent of \( R \)) such that
\[
\langle Af, f \rangle_{L^2(\Delta_R(x_0))} \geq CR^{-2}\|f\|_{L^2(\Delta_R(x_0))}^2 \quad \text{for each} \quad f \in H^1_0(\Delta_R(x_0)) \cap H^2(\Delta_R(x_0)). \tag{3.11}
\]
We also divide its proof into two cases.

**Case I.** \( |b(x)| \leq \frac{\Lambda_1}{\epsilon} \) a.e. \( x \in \Omega \). In this case, by Lemma 2.1, we find that for each \( \epsilon > 0 \),
\[
\langle Af, f \rangle_{L^2(\Delta_R(x_0))} \geq \Lambda_1^{-1} \int_{\Delta_R(x_0)} \|\nabla f\|^2 \, dx - \frac{2\epsilon}{(N-2)^2} \int_{\Delta_R(x_0)} \|\nabla f\|^2 \, dx - \frac{\Lambda_3^2}{2\epsilon} \int_{\Delta_R(x_0)} |f|^2 \, dx,
\]
for any \( f \in H^1_0(\Delta_R(x_0)) \cap H^2(\Delta_R(x_0)) \). Letting \( \epsilon = \frac{(N-2)^2}{4\Lambda_1} \) in the above inequality, by the Poincaré inequality
\[
\int_{\Delta_R(x_0)} |f(x)|^2 \, dx \leq \left(\frac{2R}{N}\right)^2 \int_{\Delta_R(x_0)} |\nabla f(x)|^2 \, dx \quad \text{for each} \quad f \in H^1_0(\Delta_R(x_0)), \tag{3.12}
\]
we derive
\[
\langle Af, f \rangle_{L^2(\Delta_R(x_0))} \geq \Lambda_1^{-1} \left[ \frac{1}{2} - \frac{8\Lambda_1\Lambda_3^2 R^2}{(N-2)^2 N^2} \right] \int_{\Delta_R(x_0)} |\nabla f|^2 \, dx.
\]
From the definition of \( R_0 \) given in (3.3) and (3.12), we can conclude the claim (3.11).

**Case II.** \( b(\cdot) \in L^{N+\delta}(\Omega) \) and \( \|b(\cdot)\|_{L^{N+\delta}(\Omega)} \leq \Lambda_2 \). By using (2.3), (3.7) and (3.12), we have
\[
\int_{\Delta_R(x_0)} |f|^2 \, dx \leq \Gamma_2 \left( \Delta_R(x_0), N, N \right) \|b\|_{L^N(\Delta_R(x_0))} \|f\|_{L^2(\Delta_R(x_0))} \|f\|_{H^1_0(\Delta_R(x_0))}
\]
\[
\leq \sqrt{2\Gamma_2} \left( \Delta_1(0), N, N \right) \|b\|_{L^N(\Delta_R(x_0))} \|\nabla f\|_{L^2(\Delta_R(x_0))}^2
\]
\[
\leq \sqrt{2\Gamma_2} \frac{\Lambda_2}{N} \Gamma_2 \left( \Delta_1(0), N, N \right) R^{-\frac{N+\delta}{2}} \|f\|_{L^{N+\delta}(\Omega)} \|\nabla f\|_{L^2(\Delta_R(x_0))}.
\]
From the definition of \( R_0 \), we have
\[
\int_{\Delta_R(x_0)} |f|^2 \, dx \leq \frac{\Lambda_1^{-1}}{8} \|\nabla f\|_{L^2(\Delta_R(x_0))}^2.
\]
This implies
\[
\langle Af, f \rangle_{L^2(\Delta_R(x_0))} \geq \frac{7\Lambda_1^{-1}}{8} \int_{\Delta_R(x_0)} |\nabla f|^2 \, dx.
\]
and then (3.11) holds.

As a consequence of (3.11), we see that the inverse of \( A \) is positive, self-adjoint and compact in \( L^2(\Delta_R(x_0)) \). By the spectral theorem for compact self-adjoint operators, there are eigenvalues \( \{\mu_i\}_{i \in \mathbb{N}^+} \subset \mathbb{R}^+ \) and eigenfunctions \( \{f_i\}_{i \in \mathbb{N}^+} \subset H^1_0(\Delta_R(x_0)) \), which make up an orthogonal basis of \( L^2(\Delta_R(x_0)) \), such that
\[
\begin{align*}
-\Lambda_1 f_i = \mu_i f_i \quad \text{and} \quad \|f_i\|_{L^2(\Delta_R(x_0))} = 1 \quad \text{for each} \quad i \in \mathbb{N}^+ \quad \text{and} \quad CR^{-2} < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_i \to +\infty \quad \text{as} \quad i \to +\infty. 
\end{align*} \tag{3.13}
\]
Then, by the formula of Fourier decomposition, the solution \( w \) of (3.5) in \([T, +\infty)\) is given by
\[
w(\cdot, t) = \sum_{i=1}^{\infty} \langle w(\cdot, T), f_i \rangle_{L^2(\Delta_R(x_0))} e^{-\mu_i(t-T)} f_i \quad \text{in} \quad \Delta_R(x_0) \quad \text{for each} \quad t \in [T, +\infty).
Hence, we deduce that for each \( t \in [T, +\infty) \),
\[
\|w(\cdot, t)\|_{L^2(\triangle_R(x_0))}^2 \leq e^{-C_R^{-2}(t-T)}\|w(\cdot, T)\|_{L^2(\triangle_R(x_0))}^2
\]  
(3.14)
and
\[
w_t(\cdot, t) = -\sum_{i=1}^{\infty} \mu_i \langle w(\cdot, T), f_i \rangle_{L^2(\triangle_R(x_0))} e^{-\mu_i(t-T)} f_i.
\]

It follows that
\[
-\langle w(\cdot, t), w_t(\cdot, t) \rangle_{L^2(\triangle_R(x_0))} = \sum_{i \in \mathbb{N}^+} \mu_i \langle w(\cdot, T), f_i \rangle_{L^2(\triangle_R(x_0))}^2 e^{-2\mu_i(t-T)},
\]
(3.15)
for each \( t \in [T, +\infty) \). In particular, taking \( t = T \) in the above identity leads to
\[
-\langle w(\cdot, T), w_t(\cdot, T) \rangle_{L^2(\triangle_R(x_0))} = \sum_{i \in \mathbb{N}^+} \mu_i \langle w(\cdot, T), f_i \rangle_{L^2(\triangle_R(x_0))}^2.
\]
(3.16)

Meanwhile, it follows from (3.5) and Lemma 2.1 that
\[
-\langle w(\cdot, T), w_t(\cdot, T) \rangle_{L^2(\triangle_R(x_0))} \leq C\|w(\cdot, T)\|_{H^1_0(\triangle_R(x_0))}^2
\]
(3.17)
with a generic constant \( C > 0 \). From (3.16) and (3.17), we have
\[
\sum_{i=1}^{\infty} \mu_i \langle w(\cdot, T), f_i \rangle_{L^2(\triangle_R(x_0))}^2 \leq C\|w(\cdot, T)\|_{H^1_0(\triangle_R(x_0))}^2.
\]
This, together with (3.15), gives
\[
-\langle w(\cdot, t), w_t(\cdot, t) \rangle_{L^2(\triangle_R(x_0))} \leq C e^{-C_R^{-2}(t-T)}\|w(\cdot, T)\|_{H^1_0(\triangle_R(x_0))}^2,
\]
(3.18)
for each \( t \in [T, +\infty) \). On the other hand, by (3.5) and (3.11), we see that for each \( t \in [T, +\infty) \),
\[
-\langle w(\cdot, t), w_t(\cdot, t) \rangle_{L^2(\triangle_R(x_0))} = \langle w(\cdot, t), -Aw(\cdot, t) \rangle_{L^2(\triangle_R(x_0))} \geq C\|\nabla w(\cdot, t)\|_{L^2(\triangle_R(x_0))}^2.
\]
(3.19)

By (3.18) and (3.19), we find that for each \( t \in [T, +\infty) \),
\[
\|\nabla w(\cdot, t)\|_{L^2(\triangle_R(x_0))} \leq C e^{-C_R^{-2}(t-T)}\|w(\cdot, T)\|_{H^1_0(\triangle_R(x_0))}^2.
\]
This, together with (3.14), means that
\[
\|w(\cdot, t)\|_{H^1_0(\triangle_R(x_0))} \leq C e^{-C_R^{-2}(t-T)}\|w(\cdot, T)\|_{H^1_0(\triangle_R(x_0))}^2.
\]
By the fact that \( w(\cdot, t) = v(\cdot, t) \) for each \( t \geq T \), we conclude the desired result.

We next define
\[
\tilde{v}(\cdot, t) = \begin{cases} 
v(\cdot, t) & \text{if } t \geq 0, \\
0 & \text{if } t < 0,
\end{cases}
\]
where \( v \) is the solution of (3.4). By Lemma 3.1, we can take the Fourier transform of \( \tilde{v} \) with respect to the time variable \( t \in \mathbb{R} \)
\[
\hat{v}(x, \mu) = \int_\mathbb{R} e^{-i\mu t}\tilde{v}(x, t)dt \quad \text{for } (x, \mu) \in \triangle_R(x_0) \times \mathbb{R}.
\]
Then, we have
Lemma 3.2. There exists a generic constant $C > 0$ such that, for each $\mu \in \mathbb{R}$, the following two estimates hold:

$$\|\nabla \hat{v}(\cdot, \mu)\|_{L^2(\Delta_r(x_0))} \leq \frac{C(1 + \sqrt{|\mu|})}{R - 2r} \|\hat{v}(\cdot, \mu)\|_{L^2(\Delta_{\frac{r}{2}}(x_0))} \quad \text{for all } 0 < r < R/2,$$

(3.20)

and

$$\|\hat{v}(\cdot, \mu)\|_{L^2(\Delta_{\frac{r}{2}}(x_0))} \leq C T^{-\frac{1}{4}} e^{C R^{1-N} (1 + \frac{1}{\sqrt{|\mu|}}) T - \frac{N}{4}} F(R)$$

(3.21)

with a positive constant $\Pi$.

Proof. By (3.4), we have that for each $\mu \in \mathbb{R}$,

$$i \mu(x)\hat{v}(x, \mu) - \text{div}(A(x)\nabla \hat{v}(x, \mu)) + b(x)\hat{v}(x, \mu) = 0 \quad \text{in } \Delta_R(x_0).$$

(3.22)

Take arbitrarily $r \in (0, \frac{R}{2})$ and define a cutoff function $\psi \in C^\infty(\mathbb{R}^N; [0, 1])$ verifying

$$\begin{cases}
\psi = 1 & \text{in } \Delta_r(x_0), \\
\psi = 0 & \text{in } \mathbb{R}^N \setminus \Delta_{\frac{r}{2}}(x_0), \\
|\nabla \psi| \leq \frac{C}{R - 2r} & \text{in } \mathbb{R}^N.
\end{cases}$$

(3.23)

Multiplying first (3.22) by $\bar{v}\psi^2$ and then integrating by parts over $\Delta_{\frac{r}{2}}(x_0)$, we have

$$\int_{\Delta_{\frac{r}{2}}(x_0)} \nabla \hat{v} \cdot (A \nabla \hat{v}) \psi^2 dx + 2 \int_{\Delta_{\frac{r}{2}}(x_0)} \nabla \psi \cdot (A \nabla \hat{v}) \hat{v} \psi dx = -i \int_{\Delta_{\frac{r}{2}}(x_0)} \mu|\hat{v}|^2 \psi^2 dx - \int_{\Delta_{\frac{r}{2}}(x_0)} b|\hat{v}|^2 \psi^2 dx.$$

We divide the proof of (3.20) into two cases.

Case I. $|b(x)| \leq \frac{\Lambda_3}{|x|}$ a.e. $x \in \Omega$. By (1.2), (2.2) and the Hardy inequality in Lemma 2.1, we derive that for each $\epsilon_1 > 0$ and $\epsilon_2 > 0$,

$$\begin{align*}
\Lambda_1^{-1} & \int_{\Delta_{\frac{r}{2}}(x_0)} |\nabla \hat{v}|^2 \psi^2 dx \\
& \leq 2 \Lambda_1 \int_{\Delta_{\frac{r}{2}}(x_0)} |\nabla \hat{v}| |\hat{v}| |\nabla \psi| |\psi| dx + \Lambda_3 |\mu| \int_{\Delta_{\frac{r}{2}}(x_0)} |\hat{v}|^2 \psi^2 dx + \int_{\Delta_{\frac{r}{2}}(x_0)} |b| |\hat{v}|^2 \psi^2 dx \\
& \leq \epsilon_1 \int_{\Delta_{\frac{r}{2}}(x_0)} |\nabla \hat{v}|^2 \psi^2 dx + \frac{\Lambda^2_2}{\epsilon_1} \int_{\Delta_{\frac{r}{2}}(x_0)} |\hat{v}|^2 |\nabla \psi|^2 dx + \epsilon_2 \int_{\Delta_{\frac{r}{2}}(x_0)} |x|^{-2} |\hat{v}| \psi^2 dx \\
& \quad + \left( \frac{\Lambda^2_3}{4 \epsilon_2} + \Lambda_3 |\mu| \right) \int_{\Delta_{\frac{r}{2}}(x_0)} |\hat{v}|^2 \psi^2 dx \\
& \leq \left( \epsilon_1 + \frac{8 \epsilon_2}{(N - 2)^2} \right) \int_{\Delta_{\frac{r}{2}}(x_0)} |\nabla \hat{v}|^2 \psi^2 dx + \left( \frac{8 \epsilon_2}{(N - 2)^2} + \frac{\Lambda^2_2}{\epsilon_1} \right) \int_{\Delta_{\frac{r}{2}}(x_0)} |\hat{v}|^2 |\nabla \psi|^2 dx \\
& \quad + \left( \frac{\Lambda^2_3}{4 \epsilon_2} + \Lambda_3 |\mu| \right) \int_{\Delta_{\frac{r}{2}}(x_0)} |\hat{v}|^2 \psi^2 dx.
\end{align*}$$

Taking $\epsilon_1 = \frac{1}{4 \Lambda_1}$ and $\epsilon_2 = \frac{(N - 2)^2}{16 \Lambda_1}$ in the above inequality, we derive (3.20).
Case II. $b(\cdot) \in L^{N+5}(\Omega)$ and $\|b(\cdot)\|_{L^{N+3}(\Omega)} \leq \Lambda_2$. By (1.2), (2.2) and (2.3) with $\eta = \frac{N}{2}$, we get

$$
\Lambda_1^{-1} \int_{\Delta_{\mathcal{A}}(x_0)} |\nabla \hat{v}|^2 \psi^2 dx
\leq 2 \Lambda_1 \int_{\Delta_{\mathcal{A}}(x_0)} |\nabla \hat{v}| \psi |\nabla \psi| |\psi| dx + \Lambda_3 |\mu| \int_{\Delta_{\mathcal{A}}(x_0)} |\hat{v}|^2 \psi^2 dx + \int_{\Delta_{\mathcal{A}}(x_0)} |b| |\hat{v}|^2 \psi^2 dx
\leq 2 \Lambda_1 \int_{\Delta_{\mathcal{A}}(x_0)} |\nabla \hat{v}| \psi |\nabla \psi| |\psi| dx + \Lambda_3 |\mu| \int_{\Delta_{\mathcal{A}}(x_0)} |\hat{v}|^2 \psi^2 dx
+ \Gamma_2 \left( \triangle_1(0), N, \frac{N}{2} \right) \|b\|_{L^5(\Delta_{\mathcal{A}}(x_0))} \left( \frac{R}{2} \right)^{-1} \|\hat{\psi}\|_{L^2(\Delta_{\mathcal{A}}(x_0))} \|\hat{\psi}\|_{H^1(\Delta_{\mathcal{A}}(x_0))}.
$$

Then for any $\epsilon > 0$, we have

$$
\Lambda_1^{-1} \int_{\Delta_{\mathcal{A}}(x_0)} |\nabla \hat{v}|^2 \psi^2 dx
\leq \epsilon \int_{\Delta_{\mathcal{A}}(x_0)} |\nabla \hat{v}|^2 \psi^2 dx + \Lambda_1^2 \frac{\Lambda_3^2}{\epsilon} \int_{\Delta_{\mathcal{A}}(x_0)} |\hat{v}|^2 |\nabla \psi|^2 dx + \Lambda_3 |\mu| \int_{\Delta_{\mathcal{A}}(x_0)} |\hat{v}|^2 \psi^2 dx
+ 2\sqrt{2} \Gamma_2 \left( \triangle_1(0), N, \frac{N}{2} \right) \|b\|_{L^5(\Delta_{\mathcal{A}}(x_0))} \left( \frac{R}{2} \right)^{-1} \|\hat{\psi}\|_{L^2(\Delta_{\mathcal{A}}(x_0))} \|\hat{\psi}\|_{H^1(\Delta_{\mathcal{A}}(x_0))}.
$$

Here, we used (3.7) and the definition of $R_0$. Taking $\epsilon = \frac{\Lambda_3^2}{4}$ in the above inequality and using (3.23) lead to (3.20).

Note that, when $\mu = 0$, by Lemma 3.1 we have

$$
\|\hat{v}(\cdot, 0)\|_{L^2(\Omega)} \leq C T^{-\frac{\lambda}{4}} e^{C R^{1-N} \left( 1 + \frac{1}{1-m} \right) T} F(R).
$$

Thus it suffices to prove (3.21) in the case that $\mu \neq 0$. To this end, define for each $\mu \in \mathbb{R} \setminus \{0\}$,

$$
p(x, \xi, \mu) = e^{i \sqrt{\|\mu\|} |\xi|} \hat{\tilde{v}}(x, \mu) \quad \text{for a.e.} \quad (x, \xi) \in \Delta_R(x_0) \times \mathbb{R}.
$$

Then, $p(\cdot, \cdot, \mu)$ verifies

$$
\text{div}(A \nabla p(\cdot, \cdot, \mu)) + i\text{sign}(\mu) \partial_\xi p(\cdot, \cdot, \mu) - b p(\cdot, \cdot, \mu) = 0 \quad \text{in} \quad \Delta_R(x_0) \times \mathbb{R}.
$$
Here

\[ \text{sign}(\mu) := \begin{cases} 
1 & \text{if } \mu > 0, \\
-1 & \text{if } \mu < 0.
\end{cases} \]

Let \( m \in \mathbb{N}^+ \) and \( a_j = 1 - \frac{1}{2m} \) for \( j = 0, 1, \ldots, m + 1 \). For each \( j \in \{0, 1, \ldots, m\} \), we define a cutoff function

\[ h_j(s) := \begin{cases} 
0 & \text{if } |s| > a_j, \\
\frac{1}{2} \left[ 1 + \cos \left( \frac{\pi(a_{j+1}-s)}{a_{j+1}-a_j} \right) \right] & \text{if } a_{j+1} \leq |s| \leq a_j, \\
1 & \text{if } |s| < a_{j+1}.
\end{cases} \]

Clearly,

\[ |h'_j(s)| \leq m\pi \text{ for any } s \in \mathbb{R}. \]

Denote \( p_j = \frac{\partial q_j}{\partial q_j}, j = 0, 1, \ldots, m \). Then \( p_j \) verifies

\[ \text{div}(A \nabla p_j(\cdot, \cdot, \mu)) + \text{sign}(\mu)|p_{j+2}(\cdot, \cdot, \mu) - b p_j(\cdot, \cdot, \mu)| = 0 \text{ in } \Delta(R(x_0) \times \mathbb{R}). \quad (3.25) \]

Let

\[ \eta_j(x, \xi) = h_j \left( \frac{|x - x_0|}{R} \right) h_j \left( \frac{\xi}{R} \right) \text{ for } (x, \xi) \in \Delta(R(x_0) \times \mathbb{R}). \]

Multiplying first (3.25) by \( \bar{p}_j \eta_j^2 \) and then integrating by parts over \( D_j = \Delta(a_j R(x_0) \times (-a_j R, a_j R)) \), we obtain

\[
- \int_{D_j} \nabla \bar{p}_j \cdot (A \nabla p_j) \eta_j^2 \, dxd\xi - \text{sign}(\mu) \int_{D_j} l|p_{j+1}|^2 \eta_j^2 \, dxd\xi = \int_{D_j} b|p_j|^2 \eta_j^2 \, dxd\xi + \int_{D_j} (A \nabla p_j) \bar{p}_j \eta_j^2 \, dxd\xi + \text{sign}(\mu) \int_{D_j} l p_{i+1} \partial \xi \eta_j^2 \bar{p}_j \eta_j^2 \, dxd\xi. \quad (3.26)
\]

Since \( \nabla \bar{p}_j \cdot (A \nabla p_j) \) and \( |p_{j+1}|^2 \) are real-valued, by (3.26) we get

\[
\left( \int_{D_j} \nabla \bar{p}_j \cdot (A \nabla p_j) \eta_j^2 \, dxd\xi \right)^2 + |\text{sign}(\mu)|^2 \left( \int_{D_j} l|p_{j+1}|^2 \eta_j^2 \, dxd\xi \right)^2 \leq 3 \left( \int_{D_j} b|p_j|^2 \eta_j^2 \, dxd\xi \right)^2 + 3 \left( \int_{D_j} |\nabla \eta_j^2 \cdot (A \nabla p_j)||p_j| \, dxd\xi \right)^2 + 3 \left( \int_{D_j} l|p_{j+1}||p_j| \partial \xi \eta_j^2 \, dxd\xi \right)^2 := 3 \sum_{i=1}^3 I_i. \quad (3.27)
\]

Next, we will estimate \( I_i \) (\( i = 1, 2, 3 \)) one by one. For the term \( I_1 \), we shall prove that

\[ I_1 \leq \frac{A_1^2}{16} \left( \int_{D_j} |\nabla p_j|^2 \eta_j^2 \, dxd\xi \right)^2 + \frac{C(1 + m^4)}{R^4} \int_{D_j} |p_j|^2 \, dxd\xi. \quad (3.28) \]

We divide its proof into two cases.

**Case 1.** \(|b(x)| \leq \frac{A_2}{|x|} \text{ a.e. } x \in \Omega \). By the Hardy inequality, we derive that

\[
\int_{D_j} |b||p_j|^2 \eta_j^2 \, dxd\xi \leq \frac{A_4}{(N-2)^2} \int_{D_j} (|\nabla p_j|^2 \eta_j^2 + |p_j|^2|\nabla \eta_j|^2) \, dxd\xi + \frac{A_2^2}{2\epsilon_1} \int_{D_j} |p_j|^2 \eta_j^2 \, dxd\xi
\]

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Further, we have
\[ I_1 \leq \frac{2^5 \epsilon_1^2}{(N-2)^2} \left( \int_{D_j} |\nabla p_j|^2 \eta_j^2 \, dx \, d\xi \right)^2 + \frac{8\pi^2 m^2 \epsilon_1^4 + \Lambda_1^2 R^2 (N-2)^2}{2\epsilon_1 R^2 (N-2)^2} \left( \int_{D_j} |p_j|^2 \, dx \, d\xi \right)^2, \quad \forall \epsilon_1 > 0. \]

Therefore,
\[ I_1 \leq \frac{2^5 \epsilon_1^2}{(N-2)^2} \left( \int_{D_j} |\nabla p_j|^2 \eta_j^2 \, dx \, d\xi \right)^2 + \frac{8\pi^2 m^2 \epsilon_1^4 + \Lambda_1^2 R^2 (N-2)^2}{2\epsilon_1 R^2 (N-2)^4} \left( \int_{D_j} |p_j|^2 \, dx \, d\xi \right)^2. \]

Let \( \epsilon_1 = \frac{(N-2)^2}{2\pi \Lambda_1} \), we derive (3.28).

Case II. \( b(\cdot) \in L^{N+\delta}(\Omega) \) and \( \|b(\cdot)\|_{L^{N+\delta}(\Omega)} \leq \Lambda_2 \). By (2.3), (3.7) and the definition of \( R_0 \), we have
\[
\int_{D_j} |b| |p_j|^2 \eta_j^2 \, dx \leq \Gamma_2 \left( \Delta_{a_j R}(x_0), N, \frac{N}{2} \right) \int_{-a_j R}^{a_j R} \int_{-a_j R}^{a_j R} \left| \nabla (p_j \eta_j) \right| \left| \eta_j \right| \, dx \, dy \]
\[
\leq \sqrt{2} \Theta_N^{\frac{N}{N+\delta}} \Gamma_2 \left( \Delta_1(0), N, \frac{N}{2} \right) R \frac{\pi^\delta}{\delta^{\frac{4}{N}}} \int_{-a_j R}^{a_j R} \left| \nabla (p_j \eta_j) \right| \left| \eta_j \right| \, dx \, dy \]
\[
\leq 2\sqrt{2} \Theta_N^{\frac{N}{N+\delta}} \Gamma_2 \left( \Delta_1(0), N, \frac{N}{2} \right) \Lambda_2 R_0 \frac{\pi^\delta}{\delta^{\frac{4}{N}}} \int_{D_j} \left( |\nabla p_j|^2 \eta_j^2 + \frac{m^2 \pi^2}{R^2} |p_j|^2 \right) \, dx \, d\xi \]
\[
\leq \frac{\Lambda_1^{-1}}{4} \int_{D_j} |\nabla p_j|^2 \eta_j^2 \, dx + \frac{\Lambda_1^{-1} m^2 \pi^2}{4R^2} \int_{D_j} |p_j|^2 \, dx, \]

which gives (3.28).

Moreover,
\[
\int_{D_j} |\nabla \eta_j^2 \cdot (A \nabla p_j)| |p_j| \, dx \, d\xi \leq 2 \Lambda_1 \int_{D_j} |\nabla \eta_j| |\eta_j| |\nabla p_j| |p_j| \, dx \, d\xi
\leq \epsilon_2 \Lambda_1 \int_{D_j} |\nabla p_j|^2 \eta_j^2 \, dx \, d\xi + \frac{\Lambda_1}{\epsilon_2} \int_{D_j} |p_j|^2 |\nabla \eta_j|^2 \, dx \, d\xi
\leq \epsilon_2 \Lambda_1 \int_{D_j} |\nabla p_j|^2 \eta_j^2 \, dx \, d\xi + \frac{\Lambda_1 \pi^2 m^2}{R^2 \epsilon_2} \int_{D_j} |p_j|^2 \, dx \, d\xi
\]

and
\[ I_2 \leq 2\Lambda_1^2 \epsilon_2^2 \left( \int_{D_j} |\nabla p_j|^2 \eta_j^2 \, dx \, d\xi \right)^2 + \frac{2\Lambda_1^4 \pi^4 m^4}{R^4 \epsilon_2^4} \left( \int_{D_j} |p_j|^2 \, dx \, d\xi \right)^2, \quad \forall \epsilon_2 > 0. \]

Further,
\[
\int_{D_j} |p_{j+1}| |p_j| |\partial_z \eta_j| \, dx \, d\xi \leq \epsilon_3 \int_{D_j} |p_{j+1}|^2 \eta_j^2 \, dx \, d\xi + \frac{1}{\epsilon_3} \int_{D_j} |p_j|^2 |\partial_z \eta_j|^2 \, dx \, d\xi
\leq \epsilon_3 \int_{D_j} |p_{j+1}|^2 \eta_j^2 \, dx \, d\xi + \frac{m^2 \pi^2 \Lambda_3}{R^2 \epsilon_3} \int_{D_j} |p_j|^2 \, dx \, d\xi
\]
and
\[ I_3 \leq 2\epsilon_3^2 \left( \int_{D_j} |p_{j+1}|^2 \eta_j^2 \, dx \, d\xi \right)^2 + \frac{2\Lambda_3 \pi^4 m^4}{R^4 \epsilon_3^4} \left( \int_{D_j} |p_j|^2 \, dx \, d\xi \right)^2, \quad \forall \epsilon_3 > 0. \]
Taking $\epsilon_2 = \frac{\sqrt{2}}{4m_1}, \epsilon_3 = \frac{1}{4}$ in (3.29) and (3.30), respectively, by (3.28), we derive that
\[
\sum_{i=1}^{3} I_i \leq \frac{\Lambda_1^{-2}}{8} \left( \int_{D_j} |\nabla \phi_j|^2 \eta_j^2 dx \right)^2 + \frac{1}{8} \left( \int_{D_j} |p_{j+1}|^2 \eta_j^2 dx \right)^2 + \frac{M_1 + M_2 m^4}{R^4} \left( \int_{D_j} |p_j|^2 \eta_j^2 dx \right)^2
\]
with two positive constants $M_1$ and $M_2$. On the other hand, by the uniform ellipticity condition (1.2), we find that
\[
\left( \int_{D_j} \nabla \phi_j \cdot (A \nabla \phi_j) \eta_j^2 dx \right)^2 \geq \Lambda_1^{-2} \left( \int_{D_j} |\nabla \phi_j|^2 \eta_j^2 dx \right)^2.
\]
This, together with (2.2), (3.27) and (3.31), gives that for each $j \in \{0, 1, \cdots, m-1\}$,
\[
\int_{D_{j+1}} |p_{j+1}|^2 \eta_j^2 dx \leq \frac{2\Lambda_1 \sqrt{2(M_1 + M_2 m^4)}}{R^2} \int_{D_j} |p_j|^2 \eta_j^2 dx \leq \frac{\Pi(1 + m^2)}{R^2} \int_{D_j} |p_j|^2 \eta_j^2 dx,
\]
where $\Pi = 2\Lambda_1 \sqrt{2(M_1 + M_2)}$. Here, we used the definition of $D_j$. Iterating (3.32) for each $j \in \{0, 1, \cdots, m-1\}$, by the fact that $p_0 = p = \hat{v}$ we obtain
\[
\int_{\Delta_F(x_0) \times (-\frac{R}{2}, \frac{R}{2})} |p_m|^2 dx \leq 2R \left[ \frac{\Pi(1 + m^2)}{R^2} \right]^m \int_{\Delta_N(x_0)} |\hat{v}(x, \mu)|^2 dx.
\]
By Lemma 3.1, we get that for each $\mu \in \mathbb{R}$,
\[
||\hat{v}(\cdot, \mu)||_{L^2(\Delta_N(x_0))} \leq CT^{-\frac{\beta}{2}}e^{CR^{1-N} \left( 1 + \frac{R}{4m} \right) \frac{\nu}{2} R} F(R).
\]
Therefore, by (3.32) and (3.33), we get that for each $m \in \mathbb{N}^+$,
\[
\int_{\Delta_F(x_0) \times (-\frac{R}{2}, \frac{R}{2})} |p_m|^2 dx \leq CT^{-1} \left[ \frac{\Pi(1 + m^2)}{R^2} \right]^m \frac{R}{2} e^{CR^{1-N} \left( 1 + \frac{R}{4m} \right) \frac{\nu}{2} R^2} F(R).
\]
For any $\varphi \in L^2(\Delta_F(x_0); \mathbb{C})$, we define
\[
P_{\mu}(\xi) := \int_{\Delta_F(x_0)} p(x, \xi, \mu) \varphi(x) dx, \quad \xi \in \left( -\frac{R}{2}, \frac{R}{2} \right).
\]
It is well known that the following interpolation inequality holds (See a proof in Appendix)
\[
\|f\|_{L^\infty(I)} \leq C \left( |I| \|f'\|_{L^2(I)}^2 + \frac{1}{|I|} \|f\|_{L^2(I)}^2 \right)^{\frac{1}{2}} \quad \text{for each } f \in H^1(I),
\]
where $I$ is an bounded nonempty interval of $\mathbb{R}$ and $|I|$ is the length. Therefore, by (3.34) we have that for any $\xi \in (-\frac{R}{2}, \frac{R}{2})$ and $m \in \mathbb{N}^+$,
\[
|P_{\mu}(\xi)| \leq C \left( R \int_{-\frac{R}{2}}^{\frac{R}{2}} |P_{\mu}(\xi)|^2 d\xi + \frac{1}{R} \int_{-\frac{R}{2}}^{\frac{R}{2}} |P_{\mu}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.
\]
\[
\leq C \left( R \int_{\Delta R(x_0) \times (-\frac{R}{4}, \frac{R}{4})} |p_{m+1}|^2 \, dx \, d\xi + \frac{1}{R} \int_{\Delta R(x_0) \times (-\frac{R}{4}, \frac{R}{4})} |p_m|^2 \, dx \, d\xi \right)^{\frac{1}{2}} 
\leq CT^{-\frac{3}{4}} e^{CR^{1-N}(1 + \frac{1}{m})} F(R) \|\varphi\|_{L^2(\Delta R(x_0))}.
\] (3.36)

This implies that \( P_{\mu}(\cdot) \) can be analytically extended to the complex plane (still denoted by the same notation)

\[
E_0 := \left\{ \xi \in \mathbb{C} : \text{Re} \xi \in \left( -\frac{R}{2}, \frac{R}{2} \right) \text{ and } \text{Im} \xi \in (-L_0, L_0) \right\},
\]

where \( L_0 := \frac{R}{2\pi}. \) Then,

\[
|P_{\mu}(\xi)| \leq \sum_{m=0}^{\infty} \frac{|P^{(m)}(0)|}{m!} |\xi|^m,
\]

when \( \xi \in i\mathbb{R} \cap E_0. \) Taking \( \xi_0 = -\frac{iR}{2\pi} \), by (3.36), we get that

\[
|P_{\mu}(\xi_0)| \leq C \sum_{m=0}^{\infty} \frac{(m + 1)^{m+1}}{m!(2e)^m} T^{-\frac{3}{4}} e^{CR^{1-N}(1 + \frac{1}{m})} F(R) \|\varphi\|_{L^2(\Delta R(x_0))}. \] (3.37)

While, by the definition,

\[
P_{\mu}(\xi_0) = e \frac{\xi_0}{m!} \int_{\Delta R(x_0)} \hat{v}(x, \mu) \hat{\varphi}(x) \, dx.
\]

This, together with (3.37), means that,

\[
\|\hat{v}(\cdot, \mu)\|_{L^2(\Delta R(x_0))} \leq CT^{-\frac{3}{4}} e^{CR^{1-N}(1 + \frac{1}{m})} T^{-\frac{3}{4}} F(R).
\] (3.38)

By (3.38) and (3.24), we derive (3.21) and complete the proof. \(\square\)

### 3.1.2 Stability estimate and three-ball inequality for elliptic equations

Suppose \( T > 0, L > 0 \) and \( \Delta R(x_0) \subset \Omega \) with \( x_0 \in \Omega. \) Let \( g \in H^1(\Delta R(x_0) \times (-L, L)) \) be a solution of the following elliptic equation

\[
\begin{aligned}
\text{div}(A(x) \nabla g) + l(x)g_{x,N+1}x_{N+1} - b(x)g &= 0 \quad \text{in } \Delta R(x_0) \times (-L, L), \\
g(x,0) &= f_1(x) \quad \text{in } \Delta R(x_0), \\
g_{x,N+1}(x,0) &= f_2(x) \quad \text{in } \Delta R(x_0),
\end{aligned}
\] (3.39)

where \( f_1 \in H^1(\Delta R(x_0)), f_2 \in L^2(\Delta R(x_0)), A, b \) and \( l \) satisfy the same assumptions as before.

**Lemma 3.3** (Stability estimate). There is \( \gamma \in (0,1) \) such that for any \( r \in (0, \min(\{R, L\})/3)), \)

\[
\|g\|_{H^1(B_r(x_0,0))} \leq C r^{-4} \|g\|_{H^1(B_{2r}(x_0,0))} \left( \|f_1\|_{L^2(\Delta R(x_0))} + \|f_2\|_{L^2(\Delta R(x_0))} \right)^{1-\gamma}.
\] (3.40)

The proof of Lemma 3.3 is based on a point-wise estimate (see Lemma 3.4 below). Here and in the sequel, for simplicity we denote

\[
\tilde{A}(x, x_{N+1}) = \left[ \tilde{a}^{ij}(x, x_{N+1}) \right]_{(N+1) \times (N+1)} := \text{diag}(A(x), l(x)),
\]

\[
\nabla = (\nabla_x, \partial_{x,N+1}), \quad \text{div} = \text{div}_x + \partial_{x,N+1}
\]

when they do not arise any confusion in the context.
Lemma 3.4. Let $s > 0$, $\lambda > 0$, $\varphi \in C^2(B_R(x_0, 0))$ and set $\alpha = e^{\lambda \varphi}$, $\theta = e^{s\alpha}$. If $V \in C^2(\triangle_R(x_0) \times (-L, L))$ and $W = \theta V$, then the following inequality holds:

$$
\theta^2 |\text{div}(\bar{A} \nabla V)|^2 + \mathcal{D} \\
\geq B_1 |W|^2 + B_2 \nabla W \cdot (\bar{A} \nabla W) + 2s\lambda^2 \omega^2 \nabla [\omega \nabla \varphi \cdot (\bar{A} \nabla \varphi)] \cdot (\bar{A} \nabla W) + 2s\lambda^2 \alpha |\nabla W \cdot (\bar{A} \nabla \varphi)|^2 \\
+ 2s\lambda \omega (\bar{A} \nabla W) \cdot [D^2 \varphi(\bar{A} \nabla W)] + 2s\lambda \omega \left( \sum_{i,j=1}^{N+1} \partial_i W \nabla \bar{a}^j \partial_j \varphi \right) \cdot (\bar{A} \nabla W) \\
- s\lambda \omega \left( \sum_{i,j=1}^{N+1} \partial_i W \nabla \bar{a}^j \partial_j W \right) \cdot (\bar{A} \nabla \varphi),
$$

where

$$
\begin{align*}
B_1 &= s^3 \lambda^4 \alpha^3 |\nabla \varphi \cdot (\bar{A} \nabla \varphi)|^2 + s^3 \lambda^3 \alpha^2 \text{div}(\bar{A} \nabla \varphi) \cdot (\bar{A} \nabla \varphi)]^2 - 2s^2 \lambda^2 \alpha^2 |\nabla \varphi \cdot (\bar{A} \nabla \varphi)|^2 \\
B_2 &= s^2 \lambda^2 \alpha \nabla \varphi \cdot (\bar{A} \nabla \varphi) - s\lambda \omega \text{div}(\bar{A} \nabla \varphi) \\
\mathcal{D} &= 2s^2 \lambda \text{div}[\omega W \bar{A} \nabla W \nabla \varphi \cdot (\bar{A} \nabla \varphi)] + 2s\lambda \text{div}[\omega \bar{A} \nabla W \nabla \varphi \cdot (\bar{A} \nabla \varphi)]
\end{align*}
$$

Proof of Lemma 3.3. With the same notation as above, (3.39) can be rewritten as

$$\text{div}(\bar{A} \nabla g) - bg = 0 \text{ in } \triangle_R(x_0) \times (-L, L),$$

where $\bar{A}$ satisfies

$$
\Lambda_1^{-1} |\xi|^2 \leq \bar{A}(x,x_{N+1}) \xi \cdot \xi \leq \Lambda_4 |\xi|^2 \text{ for each } (x, \xi) \in (\triangle_R(x_0) \times (-L, L)) \times \mathbb{R}^{N+1},
$$

with $\Lambda_4 = \max \{\Lambda_1, \Lambda_3\}$.

We next divide the proof into two steps as follows.

**Step 1.** For each $r < \min \{R, L\}$, let us set

$$
r_1 = r, \quad r_2 = \frac{3r}{2}, \quad r_3 = 2r, \quad r_4 = 3r
$$

and

$$
\omega_1 = B_{r_1}(x_0, 0), \quad \omega_2 = B_{r_2}(x_0, 0), \quad \omega_3 = B_{r_3}(x_0, 0), \quad \omega_4 = \triangle_{r_4}(x_0) \times (0, 3r).
$$

Let $\varphi \in C^2(\mathbb{R}; [0, 4])$ be such that

$$
\begin{cases}
3 < \varphi < 4 & \text{in } \omega_1, \\
0 < \varphi < 1 & \text{in } \omega_4 \backslash \omega_2, \\
|\nabla \varphi| > 0 & \text{in } \omega_3.
\end{cases}
$$

Take a cutoff function $\eta \in C^\infty(\mathbb{R}^{N+1}; [0, 1])$ to be such that

$$
\begin{align*}
\eta &= 1 \text{ in } \omega_2, \\
\eta &= 0 \text{ in } \omega_1 \backslash \omega_3, \\
|\text{div}(\bar{A} \nabla \eta)| + |\nabla \eta|^2 \leq \frac{C}{r} \text{ in } \mathbb{R}^{N+1},
\end{align*}
$$

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where $C$ is a generic constant independent of $r$. Setting $V = \eta g$, we have
\[
\begin{cases}
\text{div} (A \nabla V) - bV = \text{div} (A \nabla \eta) g + 2 \nabla \eta \cdot (A \nabla g) & \text{in } \omega_4, \\
|\nabla V| = V = 0 & \text{on } \partial \omega_4 \setminus (\triangle r_\epsilon(x_0) \times \{0\}).
\end{cases}
\]
(3.42)

It follows from Lemma 3.4 that
\[
\begin{align*}
& \int_{\omega_4} \theta^2 |\text{div}(A \nabla V)|^2 dxdx_{N+1} + \int_{\omega_4} D dxdx_{N+1} \\
& \quad \geq \int_{\omega_4} B_1 |W|^2 dxdx_{N+1} + \int_{\omega_3} B_2 \nabla W \cdot (A \nabla W) dxdx_{N+1} \\
& \quad \quad + 2s\lambda^2 \int_{\omega_4} W \nabla [\alpha \nabla \varphi] \cdot (A \nabla \varphi) dxdx_{N+1} \\
& \quad \quad + 2s\lambda \int_{\omega_4} \alpha (A \nabla W) \cdot [D^2 \varphi (A \nabla W)] dxdx_{N+1} + 2s\lambda^2 \int_{\omega_4} \alpha |\nabla W \cdot (A \nabla \varphi)|^2 dxdx_{N+1} \\
& \quad \quad + 2s \lambda \int_{\omega_4} \alpha \left( \sum_{i,j=1}^{N+1} \partial_i W \nabla \bar{a}^j \partial_j \varphi \right) \cdot (A \nabla \varphi) dxdx_{N+1} \\
& \quad \quad - s \lambda \int_{\omega_4} \alpha \left( \sum_{i,j=1}^{N+1} \partial_i W \nabla \bar{a}^j \partial_j W \right) \cdot (A \nabla \varphi) dxdx_{N+1}.
\end{align*}
\]
(3.43)

By the Cauchy-Schwarz inequality, we find
\[
2s\lambda^2 |W \nabla [\alpha \nabla \varphi] \cdot (A \nabla \varphi)| \leq C \lambda^2 (s^2 \lambda^2 \alpha |W|^2 + \alpha |\nabla W|^2),
\]
(3.44)
\[
2s \lambda \alpha |(A \nabla W) \cdot [D^2 \varphi (A \nabla W)]| \leq C s \lambda \alpha |\nabla W|^2,
\]
(3.45)
\[
2s \lambda \alpha \left( \sum_{i,j=1}^{N+1} \partial_i W \nabla \bar{a}^j \partial_j \varphi \right) \cdot (A \nabla \varphi) \leq C s \lambda \alpha |\nabla W|^2
\]
(3.46)
and
\[
s \lambda \alpha \left( \sum_{i,j=1}^{N+1} \partial_i W \nabla \bar{a}^j \partial_j W \right) \cdot (A \nabla \varphi) \leq C s \lambda \alpha |\nabla W|^2.
\]
(3.47)

By definitions of $B_1$ and $B_2$, we get
\[
B_1 |W|^2 \geq (s^3 \lambda^4 \alpha^3 \Lambda^{-1} |\nabla \varphi|^2 + s^3 \alpha^3 O(\lambda^3) + s^2 \alpha^2 O(\lambda^4)) |W|^2
\]
(3.48)
and
\[
B_2 |\nabla W|^2 \geq (s \lambda^2 \alpha \Lambda^{-1} |\nabla \varphi|^2 + s \alpha O(\lambda)) |\nabla W|^2.
\]
(3.49)

From (3.43)–(3.49) and the positivity of $|\nabla \varphi|$, we have
\[
\begin{align*}
& \int_{\omega_4} \theta^2 |\text{div}(A \nabla V)|^2 dxdx_{N+1} + \int_{\omega_4} D dxdx_{N+1} \\
& \quad \geq C \int_{\omega_4} (s^3 \lambda^4 \alpha^3 + s^3 \alpha^3 O(\lambda^3) + s^2 \alpha^2 O(\lambda^4) - C s^2 \lambda^4 \alpha) |W|^2 dxdx_{N+1} \\
& \quad + C \int_{\omega_4} [s \lambda^2 \alpha + s \alpha O(\alpha) - C (\lambda^2 + s \lambda) \alpha] |\nabla W|^2 dxdx_{N+1}.
\end{align*}
\]
Based on the case of the potential $b$, by Lemma 2.1, (2.3) with $\epsilon = 0$, the first equation in (3.42) and the fact that $\omega_4$ is a rectangle domain, we have

$$
\int_{\omega_4} \theta^2 |\text{div}(\tilde{A}\nabla V)|^2 dx_{N+1} = \int_{\omega_4} \theta^2 |\text{div}(\tilde{A}\nabla \eta) + 2\nabla \eta \cdot (\tilde{A}\nabla g) + bV|^2 dx_{N+1} \\
\leq C \int_{\omega_4} \theta^2 |\text{div}(\tilde{A}\nabla \eta)|^2 dx_{N+1} + C \int_{\omega_4} \theta^2 |bV|^2 dx_{N+1} + C \int_{\Delta_{\omega_4}(x_0) \times \{0\}} |bV|^2 dx_N \\
\leq C \int_{\omega_4} \theta^2 |\text{div}(\tilde{A}\nabla \eta) + 2\nabla \eta \cdot (\tilde{A}\nabla g)|^2 dx_{N+1} + C \int_{\Delta_{\omega_4}(x_0) \times \{0\}} |bV|^2 dx_N + C \int_{\omega_4} (|\nabla W|^2 + |W|^2) dx_{N+1}.
$$

(3.54)

By the definition of $\mathcal{D}$, we obtain

$$
\int_{\omega_4} \mathcal{D} dx_{N+1} = 2s\lambda^2 \int_{\Delta_{\omega_4}(x_0) \times \{0\}} \alpha W(\tilde{A}\nabla W) \cdot \tilde{n} \nabla \varphi \cdot (\tilde{A}\nabla \varphi) d\Gamma \\
+ 2s\lambda \int_{\Delta_{\omega_4}(x_0) \times \{0\}} \alpha (\tilde{A}\nabla W) \cdot \tilde{n} \nabla W \cdot (\tilde{A}\nabla \varphi) d\Gamma \\
- s\lambda \int_{\Delta_{\omega_4}(x_0) \times \{0\}} \alpha \nabla W \cdot (\tilde{A}\nabla W)(\tilde{A}\nabla \varphi) \cdot \tilde{n} d\Gamma \\
+ s\lambda^3 \int_{\Delta_{\omega_4}(x_0) \times \{0\}} \alpha^3 |W|^2 (\tilde{A}\nabla \varphi) \cdot \tilde{n} \nabla \varphi \cdot (\tilde{A}\nabla \varphi) d\Gamma \\
\leq C\alpha \int_{\Delta_{\omega_4}(x_0) \times \{0\}} \alpha |\nabla W|^2 d\Gamma + C\alpha^3 \lambda^3 \int_{\Delta_{\omega_4}(x_0) \times \{0\}} \alpha^3 |W|^2 d\Gamma.
$$

(3.51)

From (3.50) and (3.51), we have

$$
C\alpha \int_{\omega_4} \alpha^3 |W|^2 dx_{N+1} + C\alpha^3 \lambda^3 \int_{\omega_4} \alpha^3 |W|^2 dx_{N+1} \\
\leq \int_{\omega_4} \theta^2 |\text{div}(\tilde{A}\nabla V)|^2 dx_{N+1} + C\alpha |W|^2 d\Gamma \\
+ C\alpha^3 \lambda^3 \int_{\Delta_{\omega_4}(x_0) \times \{0\}} \alpha^3 |W|^2 d\Gamma.
$$

(3.52)

**Step 2.** Now, we return $W$ in (3.52) to $V$. Note that

$$
\frac{1}{C} \theta^2 (|\nabla V|^2 + s^2 \lambda^2 \alpha^2 |V|^2) \leq |\nabla W|^2 + s^2 \lambda^2 \alpha^2 |W|^2 \leq C\theta^2 (|\nabla V|^2 + s^2 \lambda^2 \alpha^2 |V|^2).
$$

(3.53)

Based on the case of the potential $b$, by Lemma 2.1, (2.3) with $\epsilon = 0$, the first equation in (3.42) and the fact that $\omega_4$ is a rectangle domain, we have
Therefore, by (3.52)-(3.54) and taking $\lambda_0 > 1$ large enough, we get
\[
C s^3 \lambda^4 \int_{\omega_1} \alpha^3 \theta^2 |V|^2 dx d\omega_{N+1} + C s \lambda^2 \int_{\omega_1} \alpha \theta^2 |\nabla V|^2 dx d\omega_{N+1}
\leq C \int_{\omega_1} \theta^2 \text{div}(\bar{A} \nabla \eta) g + 2 \nabla \eta \cdot (\bar{A} \nabla g)^2 dx d\omega_{N+1}
+ C s \lambda \int_{\Delta \omega_4(x_0) \times \{0\}} \alpha \theta^2 |\nabla V|^2 d\Gamma + C s^3 \lambda^3 \int_{\Delta \omega_4(x_0) \times \{0\}} \alpha^3 \theta^2 |V|^2 d\Gamma. \tag{3.55}
\]

By the definition of $\varphi$ (see (3.41)), we know that
\[
\begin{cases}
\alpha \geq e^{3 \lambda} \text{ and } \theta \geq e^{3 e^{3 \lambda}} \text{ in } \omega_1, \\
\alpha \leq e^\lambda \text{ and } \theta \leq e^{3 e^\lambda} \text{ in } \overline{\omega_2 \setminus \omega_3}.
\end{cases}
\]

Moreover, by the definition of $\eta$, we have
\[
\nabla \eta = 0 \text{ in } \omega_2 \cup (\overline{\omega_1 \setminus \omega_3}).
\]

By the fact that $V = \eta g$, one can get
\[
C s^3 \lambda^4 \int_{\omega_1} \alpha^3 \theta^2 |V|^2 dx d\omega_{N+1} + C s \lambda^2 \int_{\omega_1} \alpha \theta^2 |\nabla V|^2 dx d\omega_{N+1}
\geq C s^3 \lambda^4 \int_{\omega_1} \alpha^3 \theta^2 |g|^2 dx d\omega_{N+1} + C s \lambda^2 \int_{\omega_1} \alpha \theta^2 |\nabla g|^2 dx d\omega_{N+1}
\geq C s^3 \lambda^4 e^{3 \lambda} e^{2 e^{3 \lambda}} \int_{\omega_1} |g|^2 dx d\omega_{N+1} + C s \lambda^2 e^{3 \lambda} e^{2 e^{3 \lambda}} \int_{\omega_1} |\nabla g|^2 dx d\omega_{N+1}. \tag{3.56}
\]

Moreover,
\[
\int_{\omega_1} \theta^2 |\text{div}(\bar{A} \nabla \eta) g + 2 \nabla \eta \cdot (\bar{A} \nabla g)\eta| dx d\omega_{N+1}
\leq \frac{C}{r^4} \int_{\omega_1 \setminus \omega_2} \theta^2 (|g|^2 + |\nabla g|^2) dx d\omega_{N+1} \leq \frac{C}{r^4} e^{2 e^{3 \lambda}} \int_{\omega_1 \setminus \omega_2} (|g|^2 + |\nabla g|^2) dx d\omega_{N+1}. \tag{3.57}
\]

Further,
\[
C s \lambda \int_{\Delta \omega_4(x_0) \times \{0\}} \alpha \theta^2 |\nabla V|^2 d\Gamma + C s^3 \lambda^3 \int_{\Delta \omega_4(x_0) \times \{0\}} \alpha^3 \theta^2 |V|^2 d\Gamma
\leq \frac{C s^3 \lambda^3}{r^2} e^{3 \lambda} e^{2 e^{3 \lambda}} \int_{\Delta \omega_4(x_0) \times \{0\}} |g|^2 d\Gamma + C s \lambda e^{3 \lambda} e^{2 e^{3 \lambda}} \int_{\Delta \omega_4(x_0) \times \{0\}} |\nabla g|^2 d\Gamma. \tag{3.58}
\]

Combining (3.55)-(3.58), we have
\[
C e^{3 \lambda} e^{2 e^{3 \lambda}} \int_{\omega_1} (|g|^2 + |\nabla g|^2) dx d\omega_{N+1} \leq \frac{1}{r^4} e^{2 e^\lambda} \int_{\omega_3} (|g|^2 + |\nabla g|^2) dx d\omega_{N+1}
+ \frac{s^3 \lambda^3}{r^2} e^{3 \lambda} e^{2 e^{3 \lambda}} \int_{\Delta \omega_4(x_0) \times \{0\}} (|g|^2 + |\nabla g|^2) d\Gamma.
\]

Hence,
\[
C \int_{\omega_1} (|g|^2 + |\nabla g|^2) dx d\omega_{N+1} \leq \frac{1}{r^4} e^{-3 \lambda} e^{2 (e^\lambda - e^{3 \lambda})} \int_{\omega_3} (|g|^2 + |\nabla g|^2) dx d\omega_{N+1}
\]

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and the proof is completed.

Combining (3.59) and (3.60), we get that

\[
\|g\|_{H^2(B^+_{r_0}(x_0,0))} \leq C r^{-4} \frac{\mu}{\lambda_0^{1-n}} \left( \|g\|_{H^2(B^+_{r_0}(x_0,0))}^2 + \|\partial_{N+1} g(\cdot,0)\|_{L^2(\Delta_{r_0}(x_0))}^2 \right)^{\frac{1}{2}}. 
\]

Note that \( g \) is an even function with respect to the variable \( x_{N+1} \). So, by (3.63), we have (3.40) and the proof is completed. 

Fix \( \lambda := \lambda_0 > 1 \) and define

\[
e := e^{-3\lambda_0 r_0 e^{3\lambda_0}}. \quad \mu := \frac{2s(e^{4\lambda_0} - e^{3\lambda_0}) + 3(\ln s + \ln \lambda_0)}{2s(e^{3\lambda_0} - e^{\lambda_0}) + 3\lambda_0},
\]

\[
\epsilon_0 := e^{-3\lambda_0 r_0 e^{3\lambda_0}}.
\]

So, (3.59) can be rewritten by

\[
C \int_{\omega_1} (|g|^2 + |\nabla g|^2) dx_N + \frac{\epsilon}{r^4} \int_{\omega_3} (|g|^2 + |\nabla g|^2) dx_N + \frac{\epsilon}{r^2} \int_{\Delta_{r_0}(x_0) \times [0]} (|g|^2 + |\nabla g|^2) d\Gamma. 
\]

We treat two cases separately.

- If

\[
\left( \frac{\int_{\Delta_{r_0}(x_0) \times [0]} (|g|^2 + |\nabla g|^2) d\Gamma}{\int_{\omega_3} (|g|^2 + |\nabla g|^2) dx_N + \frac{\epsilon}{r^4} \int_{\omega_3} (|g|^2 + |\nabla g|^2) dx_N + \frac{\epsilon}{r^2} \int_{\Delta_{r_0}(x_0) \times [0]} (|g|^2 + |\nabla g|^2) d\Gamma} \right)^{\frac{1}{1+p}} > \epsilon_0.
\]

Then

\[
C \int_{\omega_1} (|g|^2 + |\nabla g|^2) dx_N + \frac{\epsilon}{r^4} \int_{\omega_3} (|g|^2 + |\nabla g|^2) dx_N + \frac{\epsilon}{r^2} \int_{\Delta_{r_0}(x_0) \times [0]} (|g|^2 + |\nabla g|^2) d\Gamma.
\]

- If

\[
\left( \frac{\int_{\Delta_{r_0}(x_0) \times [0]} (|g|^2 + |\nabla g|^2) d\Gamma}{\int_{\omega_3} (|g|^2 + |\nabla g|^2) dx_N + \frac{\epsilon}{r^4} \int_{\omega_3} (|g|^2 + |\nabla g|^2) dx_N + \frac{\epsilon}{r^2} \int_{\Delta_{r_0}(x_0) \times [0]} (|g|^2 + |\nabla g|^2) d\Gamma} \right)^{\frac{1}{1+p}} \leq \epsilon_0.
\]

In this case, we choose a \( s > s_0 \) such that

\[
e = \frac{\int_{\Delta_{r_0}(x_0) \times [0]} (|g|^2 + |\nabla g|^2) d\Gamma}{\int_{\omega_3} (|g|^2 + |\nabla g|^2) dx_N + \frac{\epsilon}{r^4} \int_{\omega_3} (|g|^2 + |\nabla g|^2) dx_N + \frac{\epsilon}{r^2} \int_{\Delta_{r_0}(x_0) \times [0]} (|g|^2 + |\nabla g|^2) d\Gamma} \left( \|g\|_{H^2(B^+_{r_0}(x_0,0))}^2 + \|\partial_{N+1} g(\cdot,0)\|_{L^2(\Delta_{r_0}(x_0))}^2 \right)^{\frac{1}{2}}.
\]

Combining (3.61) and (3.62), we get that

\[
\|g\|_{H^2(B^+_{r_0}(x_0,0))} \leq C r^{-4} \|g\|_{H^2(B^+_{r_0}(x_0,0))} \left( \|g\|_{H^2(B^+_{r_0}(x_0,0))}^2 + \|\partial_{N+1} g(\cdot,0)\|_{L^2(\Delta_{r_0}(x_0))}^2 \right)^{\frac{1}{2}}.
\]

Note that \( g \) is an even function with respect to the variable \( x_{N+1} \). So, by (3.63), we have (3.40) and the proof is completed. 

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Lemma 3.5 (Three-ball inequality). There is $\beta \in (0, 1)$ such that for any $r \in (0, \frac{1}{12} \min \{R, L\})$, the inequality
\[
\|g\|_{L^\beta(B_{r^2}(x_0, 0))} \leq C(r) \|g\|_\infty^{1-\beta} \|g\|_{L^\beta(B_{r^2}(x_0, 0))}^{\beta}
\]  
(3.64)
holds for all solutions of
\[
\text{div}(A(x)\nabla g) + l(x)g_{x_{N+1}x_{N+1}} - b(x)g = 0 \quad \text{in} \quad \triangle R(x_0) \times (-L, L).
\]

Proof. We divide the proof into the following two steps.

Step 1. For any $r < \min \{R, L\}$, let us set
\[
r_1 = r, \quad r_2 = 6r, \quad r_3 = 8r, \quad r_4 = 12r.
\]
Take
\[
\varphi(x, x_{N+1}) = r_1^2 - |x - x_0|^2 - x_{N+1}^2, \quad (x, x_{N+1}) \in B_{r_4}(x_0, 0),
\]
and set a cutoff function $\eta \in C^\infty(\mathbb{R}^{N+1}; [0, 1])$ to be such that
\[
\begin{cases}
\eta = 0 & \text{in } B_{2\frac{r_4}{3}}(x_0, 0), \\
\eta = 1 & \text{in } B_{2\frac{r_4}{3}}(x_0, 0) \setminus B_{2\frac{r_4}{3}}(x_0, 0), \\
\eta = 0 & \text{in } B_{4}(x_0, 0) \setminus B_{2\frac{r_4}{3}}(x_0, 0)
\end{cases}
\]
\[
|\text{div} \tilde{A} \nabla \eta| + |\nabla \eta|^2 \leq \frac{C}{r_4}
\]
in $\mathbb{R}^{N+1}$, where $C > 0$ is a positive constant independent of $r$. Let $V = \eta g$. Then,
\[
\begin{cases}
\text{div} (\tilde{A} \nabla V) - b \nabla V = \text{div} (\tilde{A} \nabla \eta)g + 2 \nabla \eta \cdot (\tilde{A} \nabla g) & \text{in } B_{r_4}(x_0, 0), \\
|\nabla V| = V = 0 & \text{on } \partial B_{r_4}(x_0, 0).
\end{cases}
\]
Taking $W := \theta V$ and repeating the proof of Step 1 in Lemma 3.3, one can claim that there is $\lambda_0(r) > 0$ such that for any $\lambda \geq \lambda_0(r)$, one can find $s_0(r) > 1$ such that $s \geq s_0$,
\[
C(r)s^3 \lambda^4 \int_{B_{r_4}(x_0, 0)} \alpha^3 |W|^2 dx dx_{N+1} + C(r)s \lambda^2 \int_{B_{r_4}(x_0, 0)} \alpha |\nabla W|^2 dx dx_{N+1} \\
\leq \int_{B_{r_4}(x_0, 0)} \theta^2 |\text{div}(\tilde{A} \nabla V)|^2 dx dx_{N+1}.
\]
Similar to the proof of (3.55), we can get
\[
C(r)s^3 \lambda^4 \int_{B_{r_4}(x_0, 0)} \alpha^3 \theta^2 |W|^2 dx dx_{N+1} + C(r)s \lambda^2 \int_{B_{r_4}(x_0, 0)} \alpha \theta^2 |\nabla V|^2 dx dx_{N+1} \\
\leq \int_{B_{r_4}(x_0, 0)} \theta^2 |\text{div} (\tilde{A} \nabla \eta)g + 2 \nabla \eta \cdot (\tilde{A} \nabla g)|^2 dx dx_{N+1}.
\]

Step 2. By the definition of $\varphi$ (see (3.65)), we have
\[
\begin{cases}
\alpha \geq e^{108\lambda r^2} & \geq 1, \quad \theta \geq e^{6e^{108\lambda r^2}} & \text{in } B_{r_2}(x_0, 0) \setminus B_{r_1}(x_0, 0), \\
\theta \leq e^{6e^{144\lambda r^2}} & \text{in } B_{2r_3}(x_0, 0), \\
\theta \leq e^{6e^{93\lambda r^2}} & \text{in } B_{4r_3}(x_0, 0) \setminus B_{2r_3}(x_0, 0).
\end{cases}
\]
Further,  
\(|\text{div}(A \nabla \eta)| = |\nabla \eta| = 0 \text{ in } B_{\frac{1}{2}}(x_0, 0) \bigcup \left( B_{\frac{1}{4}+\epsilon}(x_0, 0) \right) \bigcup \left( B_{\frac{1}{4}}(x_0, 0) \right) \bigcup \left( B_{\frac{1}{4}+\epsilon}(x_0, 0) \right).

Hence, from the fact \( V = \eta g \), we have

\[
C(r) s^3 \lambda \alpha^3 \gamma^2 |V|^2 dx_{N+1} \geq C(r) s^3 \lambda^4 e^{2s e^{105\lambda r^2}} \int_{B_{2r}(x_0, 0) \setminus B_{r_1}(x_0, 0)} |g|^2 dx_{N+1},
\]

and

\[
\begin{align*}
\int_{B_{r_1}(x_0, 0)} \theta^2 |\text{div}(\tilde{A} \nabla \eta)| g + 2V \eta \cdot (\tilde{A} \nabla g)|^2 dx_{N+1} &
\leq e^{2s e^{144\lambda r^2}} \int_{B_{r_1}(x_0, 0) \setminus B_{r_2}(x_0)} \left( \frac{1}{r^2} |g|^2 + \frac{1}{r^2} |\nabla g|^2 \right) dx_{N+1} \\
&+ e^{2s e^{95\lambda r^2}} \int_{B_{r_2}(x_0, 0) \setminus B_{r_2}(x_0)} \left( \frac{1}{r^2} |g|^2 + \frac{1}{r^2} |\nabla g|^2 \right) dx_{N+1}. \tag{3.68}
\end{align*}
\]

By the interior estimate of elliptic equations

\[
\begin{align*}
\int_{B_{r_1}(x_0, 0) \setminus B_{r_2}(x_0, 0)} |\nabla g|^2 dx_{N+1} &\leq \frac{C}{r^2} \int_{B_{r_1}(x_0, 0)} |g|^2 dx_{N+1} \\
&= \frac{C}{r^2} \int_{B_{r_1}(x_0, 0) \setminus B_{r_2}(x_0, 0)} |g|^2 dx_{N+1}. 
\end{align*}
\]

These, along with (3.68), yield that

\[
\begin{align*}
\int_{B_{r_1}(x_0, 0)} \theta^2 |\text{div}(\tilde{A} \nabla \eta)| g + 2V \eta \cdot (\tilde{A} \nabla g)|^2 dx_{N+1} &
\leq C \frac{1}{r^4} e^{2s e^{144\lambda r^2}} \int_{B_{r_1}(x_0, 0)} |g|^2 dx_{N+1} + C \frac{1}{r^4} e^{2s e^{95\lambda r^2}} \int_{B_{r_2}(x_0, 0) \setminus B_{r_2}(x_0, 0)} |g|^2 dx_{N+1}. \tag{3.69}
\end{align*}
\]

From (3.66), (3.67) and (3.69), we get

\[
C(r) \int_{B_{r_2}(x_0, 0) \setminus B_{r_1}(x_0, 0)} |g|^2 dx_{N+1} \leq e^{2s e^{144\lambda r^2} - 105\lambda r^2} \int_{B_{r_1}(x_0, 0)} |g|^2 dx_{N+1} \\
+ e^{2s e^{95\lambda r^2} - 105\lambda r^2} \int_{B_{r_2}(x_0, 0)} |g|^2 dx_{N+1}. \tag{3.70}
\]

Fix \( \lambda := \lambda_0 > 0 \) and denote

\[
\epsilon := e^{2s e^{95\lambda_0 r^2} - 105\lambda_0 r^2}, \quad \epsilon_0 := e^{2s e^{105\lambda_0 r^2} - 105\lambda_0 r^2}
\]

and

\[
\mu := \min_{r > 0} \frac{e^{144\lambda_0 r^2} - e^{105\lambda_0 r^2}}{e^{105\lambda_0 r^2} - e^{95\lambda_0 r^2}} > 0.
\]

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Note that, this minimum can be taken by the fact
\[
\lim_{r \to 0} \frac{e^{144 \lambda_0 r^2} - e^{108 \lambda_0 r^2}}{e^{95 \lambda_0 r^2} - e^{95 \lambda_0 r^2}} = \frac{36}{13}.
\]

Then, it follows from (3.70) that
\[
C(r) \int_{B_r(x_0) \setminus B_{\rho r}(x_0)} |g|^2 \, dx + \epsilon \int_{B_r(x_0)} |g|^2 \, dx 
\leq \epsilon^{-\mu} \int_{B_r(x_0)} |g|^2 \, dx + \epsilon \int_{B_r(x_0)} |g|^2 \, dx.
\]

We treat in two cases separately.

- If
  \[
  \left( \frac{\int_{B_r(x_0)} |g|^2 \, dx}{\int_{B_{\rho r}(x_0)} |g|^2 \, dx} \right)^{\frac{1}{1+\mu}} > \epsilon_0,
  \]
then
  \[
  C(r) \int_{B_r(x_0) \setminus B_{\rho r}(x_0)} |g|^2 \, dx 
  \leq C \left( \frac{\int_{B_{\rho r}(x_0)} |g|^2 \, dx}{\int_{B_{\rho r}(x_0)} |g|^2 \, dx} \right)^{1 - \frac{1}{1+\mu}} \left( \frac{\int_{B_r(x_0)} |g|^2 \, dx}{\int_{B_{\rho r}(x_0)} |g|^2 \, dx} \right)^{\frac{1}{1+\mu}}.
  \]

- If
  \[
  \left( \frac{\int_{B_r(x_0)} |g|^2 \, dx}{\int_{B_{\rho r}(x_0)} |g|^2 \, dx} \right)^{\frac{1}{1+\mu}} \leq \epsilon_0,
  \]
we choose \( s \geq s_0 \) such that \( \epsilon = \left( \frac{\int_{B_{\rho r}(x_0)} |g|^2 \, dx}{\int_{B_{\rho r}(x_0)} |g|^2 \, dx} \right)^{\frac{1}{1+\mu}} \). Then, by (3.71), we have
\[
C(r) \int_{B_r(x_0) \setminus B_{\rho r}(x_0)} |g|^2 \, dx \leq 2 \left( \frac{\int_{B_r(x_0)} |g|^2 \, dx}{\int_{B_{\rho r}(x_0)} |g|^2 \, dx} \right)^{\frac{1}{1+\mu}} \left( \frac{\int_{B_r(x_0)} |g|^2 \, dx}{\int_{B_{\rho r}(x_0)} |g|^2 \, dx} \right)^{\frac{1}{1+\mu}}.
\]

So, by (3.72) and (3.73), we get (3.64) with \( \beta = \frac{1}{1+\mu} \). The proof is completed.

### 3.2 Proof of Proposition 2.1

**Proof of Proposition 2.1.** Arbitrarily take \( R \in (0, \min\{R_0, \rho\}) \). Let \( u_1 \) and \( u_2 \) be accordingly the solution to
\[
\begin{cases}
  l(x)\partial_t u_1 - \text{div}(A(x)\nabla u_1) + b(x)u_1 &= 0 &\text{in } \triangle_R(x_0) \times (0, 2T), \\
  u_1 &= u &\text{on } \partial\triangle_R(x_0) \times (0, 2T), \\
  u_1(\cdot, 0) &= 0 &\text{in } \triangle_R(x_0).
\end{cases}
\]
and
\[
\begin{aligned}
&l(x)\partial_t u_2 - \text{div}(A(x) \nabla u_2) + b(x)u_2 = 0 \quad \text{in } \triangle_R(x_0) \times (0, 2T), \\
u_2 = 0 \quad &\text{on } \partial \triangle_R(x_0) \times (0, 2T), \\
u_2(\cdot, 0) = u(\cdot, 0) \quad &\text{in } \triangle_R(x_0).
\end{aligned}
\]

It is clear that \( u = u_1 + u_2 \) in \( \triangle_R(x_0) \times [0, 2T] \). By a standard energy estimate for parabolic equations, we have
\[
\sup_{t \in [0, T]} \|u_2(\cdot, t)\|_{H^1(\triangle_R(x_0))} \leq C e^{CT} \|u(\cdot, 0)\|_{H^1(\triangle_R(x_0))}. \tag{3.74}
\]

Hence
\[
\sup_{t \in [0, T]} \|u_1(\cdot, t)\|_{H^1(\triangle_R(x_0))} \leq C(1 + e^{CT}) \sup_{t \in [0, T]} \|u(\cdot, t)\|_{H^1(\triangle_R(x_0))}. \tag{3.75}
\]

Fix arbitrarily \( t_0 \in (0, \frac{T}{2}) \) and let \( v_1 \) be the solution of
\[
\begin{aligned}
l(x)\partial_t v_1 - \text{div}(A(x) \nabla v_1) + b(x)v_1 &= 0 \quad \text{in } \triangle_R(x_0) \times \mathbb{R}^+, \\
v_1 = \eta u_1 &\quad \text{on } \partial \triangle_R(x_0) \times \mathbb{R}^+, \\
v_1(\cdot, 0) = 0 \quad &\text{in } \triangle_R(x_0),
\end{aligned}
\]

where \( \eta \) is given by (3.2). Clearly, \( u = v_1 + u_2 \) in \( \triangle_R(x_0) \times [0, t_0] \). In particular,
\[
u(\cdot, t_0) = v_1(\cdot, t_0) + u_2(\cdot, t_0) \quad \text{in } \triangle_R(x_0).
\]

Define
\[
\tilde{v}_1(\cdot, t) := \begin{cases} v_1(\cdot, t) & \text{if } t \geq 0, \\
0 & \text{if } t < 0,
\end{cases}
\]

and
\[
\tilde{v}_1(x, \mu) = \int_\mathbb{R} e^{-i\mu t} \tilde{v}_1(x, t) dt \quad \text{for } (x, \mu) \in \triangle_R(x_0) \times \mathbb{R}.
\]

Note from Lemma 3.1 that \( \tilde{v}_1 \) is well defined.

Let
\[
\kappa := \min \left\{ \frac{1}{2}, \sqrt{\frac{2}{4\pi}} \right\} \quad \text{with } \Pi \text{ given in Lemma 3.2}.
\]

We define
\[
V = V_1 + V_2 \quad \text{in } \triangle_R(x_0) \times (-\kappa R, \kappa R),
\]

where
\[
V_1(x, y) = \frac{1}{2\pi} \int_\mathbb{R} e^{i\mu y} \tilde{v}_1(x, \mu) \frac{\sinh(\sqrt{-\mu} y)}{\sqrt{-\mu}} d\mu \quad \text{in } \triangle_R(x_0) \times (-\kappa R, \kappa R) \tag{3.76}
\]

and
\[
V_2(x, y) = \sum_{i=1}^\infty \alpha_i e^{-\mu_i t_0} f_i(x) \frac{\sinh(\sqrt{\mu_i} y)}{\sqrt{\mu_i}} \quad \text{in } \triangle_R(x_0) \times (-\kappa R, \kappa R) \tag{3.77}
\]

with
\[
\alpha_i = \int_{\triangle_R(x_0)} l(x) u_2(x, 0) f_i(x) dx \quad \text{for each } i \in \mathbb{N}^+,
\]

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Again, by the interior estimate, there is a constant for any \( Q \). As a simple corollary of (3.81), Lemma 9.9, page 315], we have the following trace theorem

\[
\int_{B_{r/2}(x_0,0)} |V|^2 dx \geq C r^2 \int_{B_{11r/2}(x_0,0)} \left( |\nabla V|^2 + |V_y|^2 \right) dx dy
\]

By Lemma 3.5, we have for any \( r \in (0, \frac{1}{10} \kappa R) \),

\[
\|V\|_{L^2(B_{16r}(x_0,0))} \leq C r \|V\|_{L^2(B_{r}(x_0,0))} \|V\|_{L^2(B_{5r}(x_0,0))}^{1-\beta}. \tag{3.79}
\]

Since \( V_y \) also satisfies the first equation of (3.78), by the interior estimate of elliptic equations we find

\[
\int_{B_{r/2}(x_0,0)} |Q(x,0)|^2 dx \leq C \left( \frac{1}{r} \int_{B_{r}(x_0,0)} |Q|^2 dx + r \int_{B_{5r}(x_0,0)} |Q|^2 dx \right)
\]

As a simple corollary of [6, Lemma 9.9, page 315], we have the following trace theorem

\[
\int_{\Delta_r(x_0)} |Q(x,0)|^2 dx \leq C \left( \frac{1}{r} \int_{B_{r}(x_0,0)} |Q|^2 dx + r \int_{B_{5r}(x_0,0)} |Q|^2 dx \right)
\]

for any \( Q \in H^1(B_{5r}(x_0,0)) \). Hence, by (3.78) and (3.80) we have

\[
Cr^3 \int_{\Delta_r(x_0)} |u(x,0)|^2 dx \leq \int_{B_{r}(x_0,0)} |V|^2 dx dy. \tag{3.81}
\]

By Lemma 3.3, we obtain that there is \( \gamma \in (0,1) \) such that for any \( r \in (0, \frac{1}{10} \kappa R) \),

\[
\|V\|_{L^2(B_{r}(x_0,0))} \leq Cr^{-2} \|V\|_{L^2(B_{2r}(x_0,0))} \|u(\cdot,0)\|_{L^2(\Delta_{2r}(x_0))}^{1-\gamma}. \tag{3.82}
\]

Again, by the interior estimate, there is a constant \( C > 0 \) such that

\[
\|V\|_{H^1(B_{2r}(x_0,0))} \leq Cr^{-1} \|V\|_{L^2(B_{3r}(x_0,0))}. \tag{3.83}
\]

Hence, it follows from (3.82) and (3.83) that

\[
\|V\|_{L^2(B_{r}(x_0,0))} \leq Cr^{-3} \|V\|_{L^2(B_{3r}(x_0,0))} \|u(\cdot,0)\|_{L^2(\Delta_{2r}(x_0))}^{1-\gamma}. \tag{3.84}
\]

It follows from (3.81), (3.79) and (3.84) that

\[
\|u(\cdot,0)\|_{L^2(\Delta_{4r}(x_0))} \leq C(r) r^{-3(\frac{1}{3} + \beta)} \|u(\cdot,0)\|_{L^2(\Delta_{2r}(x_0))}^{(1-\gamma)\beta} \|V\|_{L^2(B_{5r}(x_0,0))}^{1-(1-\gamma)\beta}. \tag{3.85}
\]
To finish the proof, it suffices to bound the term $\|V\|_{L^2(B_{r}(x_0,0))}$. Recall that $V = V_1 + V_2$, we will treat $V_1$ and $V_2$ separately.

In fact, we derive from (3.76) that for each $x \in \Delta_{\delta r}(x_0) \subset \Delta_{\frac{3}{2}r}(x_0)$ and $|y| < \frac{\kappa R}{8}$,

\[
|V_1(x,y)| = \left| \frac{1}{2\pi} \int_{\mathbb{R}} e^{it_0 \mu} \hat{\delta}_1(x,\mu) \int_{-y}^{y} e^{x^2 \mu^2} dsd\mu \right| \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{\delta}_1(x,\mu)| \int_{-y}^{y} |e^{x^2 \mu^2}| dsd\mu \\
\leq \frac{\kappa R}{8\pi} \int_{\mathbb{R}} |\hat{\delta}_1(x,\mu)| e^{\frac{x^2 \kappa^2}{R^2} \mu R} dsd\mu \\
\leq \frac{\kappa R}{8\pi} \left( \int_{\mathbb{R}} |\hat{\delta}_1(x,\mu)|^2 e^{\frac{x^2 \kappa^2}{R^2} \mu R} dsd\mu \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} e^{\frac{x^2 \kappa^2}{R^2} \mu R} dsd\mu \right)^{\frac{1}{2}} \\
= \frac{\sqrt{2}}{4\pi} \left( \int_{\mathbb{R}} |\hat{\delta}_1(x,\mu)|^2 e^{\frac{x^2 \kappa^2}{R^2} \mu R} dsd\mu \right)^{\frac{1}{2}}.
\]

Hence, by Lemma 3.2 and (3.75), we have for each $r < \frac{R}{32}$,

\[
\int_{\Delta_{\kappa r}(x_0)} |V_1(x,y)|^2 dx \leq \frac{C T^{-1} e^{CR^{1-N}(1+\frac{1}{1-\sigma})T} G^2(R)}{R^2} \int_{\mathbb{R}} e^{\frac{x^2 \kappa^2}{R^2} \mu R} dsd\mu
\]

While, by (3.77) and (3.74) we obtain

\[
\int_{B_{r}(x_0,0)} |V_2|^2 dx dy \leq \Lambda_3 \int_{-8r}^{8r} \int_{\Delta_{\kappa r}(x_0,0)} l(x)|V_2|^2 dx dy \leq \Lambda_3 \int_{-8r}^{8r} \sum_{i=1}^{\infty} \alpha_i^2 e^{-2\mu_i t_0} \left| \frac{\sinh(\sqrt{\mu_i} y)}{\sqrt{\mu_i}} \right|^2 dy \\
\leq 2^8 r^2 \Lambda_3 \left( 1 + e^{\frac{8r^2}{10}} \right) \sum_{i=1}^{\infty} \alpha_i^2 \leq C r^2 e^{\frac{C(1+T\tau)}{6t_0}} \int_{\Delta_{\kappa r}(x_0)} |u(x,0)|^2 dx \\
\leq C e^{\frac{C(1+T\tau)}{6t_0}} G^2(R).
\]

Therefore, by (3.86) and (3.87) we conclude that

\[
\|V\|_{L^2(B_{r}(x_0,0))} \leq C R^{-2} e^{\frac{C R^{1-N}(1+\gamma)\beta}{t_0}} G(R).
\]

This, together with (3.85), means that

\[
\|u(\cdot,t_0)\|_{L^2(\Delta_{\delta r})} \leq C(r) R^{-2[1-(1-\gamma)\beta]} e^{\frac{C R^{1-N}(1+\gamma)\beta}{t_0} (1-(1-\gamma)\beta)} \|u(\cdot,t_0)\|_{L^2(\Delta_{\delta r})} G^{1-(1-\gamma)\beta}(R).
\]

Taking $\sigma = (1 - \gamma)\beta$ and using a scaling technique, the proof is immediately achieved. \qed

### 3.3 Proof of Proposition 2.2

**Proof of Proposition 2.2.** We proceed the proof with three steps as follows.

**Step 1. In the interior.** Let $K_1$ and $K_2$ be two compact subsets with non-empty interior of $\Omega$. Denoting $G_\Omega = \sup_{t \in [0,T]} \|u(\cdot,t)\|_{H^1(\Omega)}$, we shall show that

\[
\|u(\cdot,t_0)\|_{L^2(K_1)} \leq e^{\frac{C(1+\gamma)\beta}{t_0}} \|u(\cdot,t_0)\|_{L^2(K_2)}^{\sigma} G_\Omega^{1-\sigma}.
\]

(3.88)
In fact, there exists a sequence of balls \( \{\triangle_r(x_i)\}^p_{i=0} \) such that

\[
K_1 \subset \bigcup_{i=1}^p \triangle_r(x_i) \subset \Omega, \quad \triangle_r(x_0) \subset K_2,
\]

and for each \( 1 \leq i \leq p \), there exists a chain of balls \( \triangle_r(x^i_1) \), \( 1 \leq j \leq n_i \), such that

\[
\triangle_r(x^i_1) = \triangle_r(x_i), \quad \triangle_r(x^i_{n_i}) = \triangle_r(x_0),
\]

\[
\triangle_r(x^i_j) \subset \triangle_{2r}(x^{i+1}_j) \subset \Omega, \quad 1 \leq j \leq n_i - 1.
\]

By Proposition 2.1, we obtain that there are constants \( N^i_j = N^i_j(r, p) \geq 1 \) and \( \theta_i^j = \theta_i^j(r, p) \in (0, 1) \) such that

\[
\|u(\cdot, t_0)\|_{L^2(\triangle_r(x^i_1))} \leq \|u(\cdot, t_0)\|_{L^2(\triangle_{2r}(x^{i+1}))} \leq C \|u(\cdot, t_0)\|_{L^2(\triangle_r(x_j))}^{\theta_i^j} G_\Omega^{1-\theta_i^j}.
\]

Iterating the above procedure, we derive that there are constants \( N_i = N_i(K_1, K_2, p) \geq 1 \) and \( \theta_i = \theta_i(K_1, K_2, p) \in (0, 1) \) such that

\[
\|u(\cdot, t_0)\|_{L^2(\triangle_r(x^i_1))} \leq e \frac{N_i(\gamma^2+1)}{6} \|u(\cdot, t_0)\|_{L^2(\triangle_r(x_j))}^{\theta_i} G_\Omega^{1-\theta_i}.
\]

Hence, (3.88) follows.

**Step 2. Flattening the boundary and taking the even reflection.** Arbitrarily fix \( x_0 \in \partial \Omega \). Without loss of generality, we may assume that \( \hat{A}(x_0) = I \). Following the arguments to flatten locally the boundary as in [2] (see also [7]), we have that there exists a \( C^1 \)-diffeomorphism \( \Phi \) from \( \triangle_{r_2}(0) \) to \( \triangle_{r_1}(x_0) \) such that

\[
\Phi(y^0, 0) \in \partial \Omega \cap \triangle_{r_1}(x_0) \text{ for each } y^0 \in \triangle^+_{r_2}(0),
\]

\[
\Phi(\triangle^+_{r_2}(0)) \subset \triangle_{r_1}(x_0) \cap \Omega,
\]

\[
C^{-1} \leq \det J \Phi(y) \leq C \quad \text{for each } y \in \triangle_{r_2}(0),
\]

\[
|\det J \Phi(y) - \det J \Phi(\tilde{y})| \leq C |y - \tilde{y}| \quad \text{for each } y, \tilde{y} \in \triangle_{r_2}(0),
\]

\[
C^{-1} |y - \tilde{y}| \leq |\Phi(y) - \Phi(\tilde{y})| \leq C |y - \tilde{y}| \quad \text{for each } y, \tilde{y} \in \triangle_{r_2}(0),
\]

\[
\tilde{a}_{ij}(y^0, 0) = \tilde{a}_{ij}(y^0, 0) = 0 \quad \text{for each } y^0 \in \triangle^+_{r_2}(0), \quad j = 1, \ldots, N - 1,
\]

where

\[
\hat{A}(y) = [\tilde{a}_{ij}]_{N \times N} = \det J \Phi(y)(J \Phi^{-1})(\Phi(y))^{tr} A(\Phi(y))(J \Phi^{-1})(\Phi(y)), \quad y \in \triangle^+_{r_2}(0).
\]

By (3.89) and (3.90), one can check that \( \hat{A}(\cdot) \) satisfies the uniform ellipticity condition and the Lipschitz condition in \( \triangle^+_{r_2}(0) \). Denoting

\[
z(y, t) = u(\Phi(y), t), \quad \tilde{b}(y) = \det J \Phi(y)b(\Phi(y)) \quad \text{for each } y \in \triangle^+_{r_2}(0), \quad t \in (0, 2T),
\]

by (3.89) we have

\[
\tilde{b}(\cdot) \text{ satisfies (1.3) in } \triangle^+_{r_2}(0),
\]

\[
\begin{aligned}
det J \Phi(y)z(y, t) - \text{div}(\hat{A}(y)\nabla z(y, t)) + \tilde{b}(y) = 0 \quad & \text{in } \triangle^+_{r_2}(0) \times (0, 2T), \\
\frac{\partial z}{\partial y_N} = 0 \quad & \text{on } (\triangle^+_{r_2}(0) \times \{0\}) \times (0, 2T).
\end{aligned}
\]
For any \( y = (y', y_N) \in \Delta_{r_2}(0) \), using the even reflection and denoting \( \hat{A}(y) = [\hat{a}^{ij}(y)]_{N \times N} \) by
\[
\begin{cases}
\hat{a}_{ij}(y', y_N) = \hat{a}_{ij}(y', |y_N|), & \text{if } 1 \leq i, j \leq N - 1, \text{ or } i = j = N, \\
\hat{a}_{Nj}(y', y_N) = \text{sign}(y_N)\hat{a}_{Nj}(y', |y_N|), & \text{if } 1 \leq j \leq N - 1,
\end{cases}
\]
and
\[
\hat{b}(y', y_N) = \hat{b}(y', |y_N|), \quad \hat{I}(y', y_N) = \text{det}\hat{J}\Phi(y', |y_N|),
\]
and
\[
Z(y, t) = z(y', |y_N|, t) \text{ for each } (y, t) \in \Delta_{r_2}(0) \times (0, 2T).
\]
By (3.91), we see that \( \hat{A} \) verifies the uniform ellipticity condition and the Lipschitz condition in \( \Delta_{r_2}(0) \), \( \hat{b}(\cdot) \) verifies (1.3) in \( \Delta_{r_2}(0) \),
\[
C^{-1} \leq \hat{l}(y) \leq C, \quad |\hat{l}(y) - \hat{l}(\tilde{y})| \leq C|y - \tilde{y}| \quad \text{for each } y, \tilde{y} \in \Delta_{r_2}(0),
\]
and that
\[
\hat{l}(y)Z_t(y, t) - \text{div}(\hat{A}(y)\nabla Z(y, t)) + \hat{b}(y) = 0 \text{ in } \Delta_{r_2}(0) \times (0, 2T). \tag{3.92}
\]
Let \( \tilde{y} = (y', r_2/2) \). For each \( 0 < r \leq r_2/8 \), by applying Proposition 2.1 to the solution \( Z \) of (3.92), similar to the proof of Step 1, we obtain
\[
\|Z(\cdot, t_0)\|_{L^2(\triangle_{r_2}(0))} \leq C(r)e^{-\int_{t_0}^t \frac{C(r)^2}{8}\|Z(\cdot, s)\|_{H^1(\triangle_{r_2}(0))}^2 G_1 G_2^{-\sigma_1}(\triangle_{r_2}(0))},
\]
where \( G_1(\triangle_{r_2}(0)) = \sup_{s \in [0, T]} \|Z(\cdot, s)\|_{H^1(\triangle_{r_2}(0))} \). Hence,
\[
\|Z(\cdot, t_0)\|_{L^2(\partial \triangle_{r_2}(0))} \leq C(r)e^{-\int_{t_0}^t \frac{C(r)^2}{8}\|Z(\cdot, s)\|_{L^2(\partial \triangle_{r_2}(0))}^2 G_2^{-\sigma_1}(\triangle_{r_2}(0))},
\]
where \( G_2(\triangle_{r_2}(0)) = \sup_{s \in [0, T]} \|Z(\cdot, s)\|_{H^1(\partial \triangle_{r_2}(0))} \). Since the map \( \Phi \) is \( C^1 \)-diffeomorphism, we obtain that there exist \( r_3 > 0 \) and \( \rho > 0 \) such that
\[
\|u(\cdot, t_0)\|_{L^2(\Omega \cap \triangle_{r_3}(x_0))} \leq C(r)e^{-\int_{t_0}^t \frac{C(r)^2}{8}\|u(\cdot, s)\|_{L^2(\Omega \cap \triangle_{r_3}(x_0))}^2 G_2^{-\sigma_1}(\triangle_{r_2}(0))},
\]
where \( \rho > 0 \) is selected, we have
\[
\|u(\cdot, t_0)\|_{L^2(\triangle_{r_2}(0))} \leq C(r)e^{-\int_{t_0}^t \frac{C(r)^2}{8}\|u(\cdot, s)\|_{L^2(\Omega \cap \triangle_{r_3}(x_0))}^2 G_2^{-\sigma_1}(\triangle_{r_2}(0))}.
\]

**Step 3. Completing the proof.** When \( \Gamma \) is a neighborhood of \( \partial \Omega \) in \( \Omega \), there are a sequence \( \{x_j\}_{j=1}^p \subset \partial \Omega \) and a sequence \( \{\triangle_{r_j}(x_j)\}_{j=1}^p \) such that
\[
\Gamma \subset \bigcup_{j=1}^p (\Omega \cap \triangle_{r_j}(x_j)).
\]
By the result in Step 2 and a finite covering argument, we first have
\[
\|u(\cdot, t_0)\|_{L^2(\Gamma)} \leq C e^{-\int_{t_0}^t \frac{C(r)^2}{8}\|u(\cdot, s)\|_{L^2(\Omega \cap \triangle_{r_3}(x_0))}^2 G_2^{-\sigma_1}(\triangle_{r_2}(0))_\Omega}, \tag{3.93}
\]
with some \( \rho > 0 \). By the result in Step 1, we then have
\[
\|u(\cdot, t_0)\|_{L^2(\Omega \cap \triangle_{r_3}(x_0))} \leq C e^{-\int_{t_0}^t \frac{C(r)^2}{8}\|u(\cdot, s)\|_{L^2(\Omega \cap \triangle_{r_3}(x_0))}^2 G_2^{-\sigma_1}(\triangle_{r_2}(0))_\Omega},
\]
This, together with (3.93), indicates that
\[
\|u(\cdot, t_0)\|_{L^2(\Gamma)} \leq C e^{-\int_{t_0}^t \frac{C(r)^2}{8}\|u(\cdot, s)\|_{L^2(\Omega \cap \triangle_{r_3}(x_0))}^2 G_2^{-\sigma_1}(\triangle_{r_2}(0))_\Omega}.
\]
Which, combined with the result in Step 1 again, implies the desired estimate and completes the proof. \( \square \)
4 Appendix

4.1 Proof of Lemma 3.4

By the definition of $W$, we have

$$\nabla V = \nabla (\theta^{-1} W) = W \nabla \theta^{-1} + \theta^{-1} \nabla W$$

Therefore,

$$- \theta \text{div}(\bar{A} \nabla V) = s \lambda^2 \alpha W \nabla \varphi \cdot (\bar{A} \nabla \varphi) - s^2 \lambda^2 \alpha^2 W \nabla \varphi \cdot (\bar{A} \nabla \varphi)$$

$$+ 2s \lambda \alpha W \cdot (\bar{A} \nabla \varphi) + s \lambda \alpha W \text{div}(\bar{A} \nabla \varphi) - \text{div}(\bar{A} \nabla W).$$  \hspace{2cm} (4.1)

Let

$$I_1 := - \text{div}(\bar{A} \nabla W) - s^2 \lambda^2 \alpha^2 W \nabla \varphi \cdot (\bar{A} \nabla \varphi),$$

$$I_2 := 2s \lambda \alpha \nabla W \cdot (\bar{A} \nabla \varphi) + s^2 \lambda \alpha W \nabla \varphi \cdot (\bar{A} \nabla \varphi),$$

$$I_3 := - \theta \text{div}(\bar{A} \nabla V) - s \lambda \alpha W \text{div}(\bar{A} \nabla \varphi) + s \lambda^2 \alpha W \nabla \varphi \cdot (\bar{A} \nabla \varphi).$$

By (4.1), it is clear that $I_1 + I_2 = I_3$. Then

$$I_1 I_2 \leq \frac{1}{2} |I_3|^2.$$  \hspace{2cm} (4.2)

For the term $|I_3|^2$, we have

$$\frac{1}{2} |I_3|^2 \leq \theta^2 |\text{div}(\bar{A} \nabla V)|^2 + 2s^2 \lambda^2 \alpha^2 |W|^2 |\text{div}(\bar{A} \nabla \varphi)|^2$$

$$+ 2s^2 \lambda^4 \alpha^2 |\nabla \varphi \cdot (\bar{A} \nabla \varphi)|^2 |W|^2.$$  \hspace{2cm} (4.3)

By (4.1), we have

$$I_1 I_2 = - 2s \lambda \alpha [\text{div}(\bar{A} \nabla W) + s^2 \lambda \alpha^2 W \nabla \varphi \cdot (\bar{A} \nabla \varphi)][\nabla W \cdot (\bar{A} \nabla \varphi) + \lambda W \nabla \varphi \cdot (\bar{A} \nabla \varphi)]$$

$$= -2s \lambda \alpha W [\text{div}(\bar{A} \nabla W)] + s^2 \lambda \alpha^2 W \nabla \varphi \cdot (\bar{A} \nabla \varphi) [\nabla \varphi \cdot (\bar{A} \nabla \varphi)] - 2s \lambda \alpha \text{div}(\bar{A} \nabla W) \nabla W \cdot (\bar{A} \nabla \varphi) - 2s \lambda^3 \alpha^3 W \nabla W \cdot (\bar{A} \nabla \varphi) \nabla \varphi \cdot (\bar{A} \nabla \varphi)$$

$$:= \sum_{i=1}^{3} J_i.$$  \hspace{2cm} (4.4)

Next, we compute the terms $J_i$ one by one. For the term $J_1$, we have

$$J_1 = - 2s^2 \lambda^2 \alpha^3 |W|^2 |\nabla \varphi \cdot (\bar{A} \nabla \varphi)|^2 + 2s \lambda^2 \alpha \nabla W \cdot (\bar{A} \nabla W) \nabla \varphi \cdot (\bar{A} \nabla \varphi)$$

$$+ 2s \lambda^2 W \nabla [\alpha \nabla \varphi \cdot (\bar{A} \nabla \varphi)] \cdot (\bar{A} \nabla W) - 2s \lambda^2 \text{div}[\alpha W \bar{A} \nabla W \nabla \varphi \cdot (\bar{A} \nabla \varphi)].$$  \hspace{2cm} (4.5)

Moreover,

$$J_2 = 2s \lambda \alpha [\nabla W \cdot (\bar{A} \nabla \varphi)] \cdot (\bar{A} \nabla W) + 2s \lambda^2 \alpha |\nabla W \cdot (\bar{A} \nabla \varphi)|^2$$

$$- 2s \lambda \text{div}[\alpha \bar{A} \nabla W \nabla W \cdot (\bar{A} \nabla \varphi)].$$

Note that

$$\nabla [\nabla W \cdot (\bar{A} \nabla \varphi)] \cdot (\bar{A} \nabla W)$$

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Further, \( J_2 \) is given by

\[
J_2 = 2s\lambda^2|\nabla W \cdot (\bar{A} \nabla \varphi)|^2 - s\lambda^2 \alpha \nabla \varphi \cdot (\bar{A} \nabla \varphi) \nabla W \cdot (\bar{A} \nabla \varphi) - 2s\lambda \alpha \nabla (\bar{A} \nabla \varphi) \nabla W \cdot (\bar{A} \nabla \varphi) + 2s\lambda \alpha \nabla (\bar{A} \nabla \varphi) \nabla W \cdot (\bar{A} \nabla \varphi) - s\lambda \alpha \left( \sum_{i,j=1}^{N+1} \partial_i W \nabla a_{ij} \partial_j \varphi \right) \cdot (\bar{A} \nabla \varphi).
\]

Hence

\[
J_2 = 2s\lambda^2|\nabla W \cdot (\bar{A} \nabla \varphi)|^2 - s\lambda^2 \alpha \nabla \varphi \cdot (\bar{A} \nabla \varphi) \nabla W \cdot (\bar{A} \nabla \varphi) - s\lambda \alpha \nabla (\bar{A} \nabla \varphi) \nabla W \cdot (\bar{A} \nabla \varphi) + s\lambda \alpha \nabla (\bar{A} \nabla \varphi) \nabla W \cdot (\bar{A} \nabla \varphi).
\]

Further,

\[
J_3 = 3s^3 \lambda^4 |W|^2 \nabla \varphi \cdot (\bar{A} \nabla \varphi)|^2 + s^3 \lambda^3 \alpha |W|^2 \nabla \varphi \cdot (\bar{A} \nabla \varphi) - s^3 \lambda^3 \nabla \varphi \cdot (\bar{A} \nabla \varphi) |W|^2 (\bar{A} \nabla \varphi).
\]

Finally, by (4.2)–(4.7) we obtain the desired identity and complete the proof.

### 4.2 Proofs of some useful inequalities

#### 4.2.1 Proof of (3.7)

For each \( h \in L^{p} \Delta_{r}(\Delta_{r}(x_0)) \) and \( f \in H^{1}(\Delta_{r}(x_0)) \), we let \( \theta = \frac{1}{r}(x - x_0) \in \Delta_{1}(0) \), \( \tilde{h}(\theta) = h(r\theta + x_0) = h(x) \) and \( \tilde{f}(\theta) = f(x) \) similarly. One can check that

\[
\int_{\Delta_{1}(0)} |\tilde{h}| |\tilde{f}|^2 d\theta = r^{-N} \int_{\Delta_{r}(x_0)} |h||f|^2 dx,
\]

\[
\|\tilde{h}\|_{L^{p} \Delta_{1}(0)} = r^{-\frac{2N}{p-2}} \|h\|_{L^{p} \Delta_{r}(x_0)}.
\]

Moreover, when \( r \in (0, 1) \),

\[
|\tilde{f}|_{H^{1}(\Delta_{1}(0))}^2 = r^N \int_{\Delta_{1}(0)} \left( \frac{1}{r^2} |\nabla_{\theta} \tilde{f}|^2 + |\tilde{f}|^2 \right) d\theta
\]

\[
\geq r^N \int_{\Delta_{1}(0)} (|\nabla_{\theta} \tilde{f}|^2 + |\tilde{f}|^2) d\theta = r^N \|\tilde{f}\|_{H^{1}(\Delta_{1}(0))}^2,
\]

i.e.,

\[
\|\tilde{f}\|_{H^{1}(\Delta_{1}(0))} \leq r^{-\frac{4}{N}} \|f\|_{H^{1}(\Delta_{r}(x_0))}.
\]
Therefore, by (2.3), we have
\[ r^{-N} \int_{\Delta_r(x_0)} |h||f|^2 dx = \int_{\Delta_1(0)} |\tilde{h}|\tilde{f}^2 d\theta \]
\[ \leq \Gamma_2(\Delta_1(0), N, \eta) \|h\|_{L^{\frac{4N}{N-\eta}}(\Delta_1(0))} \|f\|_{L^2(\Delta_1(0))} \|\tilde{f}\|_{H^1(\Delta_1(0))} \]
\[ \leq \Gamma_2(\Delta_1(0), N, \eta) r^{-\frac{2N}{N-\eta}} \|h\|_{L^{\frac{4N}{N-\eta}}(\Delta_r(x_0))} \|f\|_{L^2(\Delta_r(x_0))} \|\tilde{f}\|_{H^1(\Delta_r(x_0))}. \]
This implies that
\[ \int_{\Delta_r(x_0)} |h||f|^2 dx \leq \Gamma_2(\Delta_1(0), N, \eta) r^{-\frac{2N}{N-\eta}} \|h\|_{L^{\frac{4N}{N-\eta}}(\Delta_r(x_0))} \|f\|_{L^2(\Delta_r(x_0))} \|\tilde{f}\|_{H^1(\Delta_r(x_0))}. \]

Then (3.7) holds.

4.2.2 Proof of (3.35)
Indeed, for any \( f \in H^1(I) \), taking any \( x \in I \), we have
\[ |f(x)|^2 \leq 2 \left| \int_y^x f'(s)ds \right|^2 + 2|f(y)|^2 \leq 2|I| \int_I |f'(s)|^2 ds + 2|f(y)|^2 \]
for each \( y \in I \). Integrating it with respect to \( y \) over \( I \), we get
\[ |f(x)|^2 \leq 2|I| \int_I |f'(s)|^2 ds + \frac{2}{|I|} \int_I |f(s)|^2 ds. \]
This implies (3.35).

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