A topology for numerical analysis on manifolds

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Abstract.
Numerical approximations of differential equations are maps with domains strict subsets of the underlying space of independent variables i.e. partial maps. Such spaces admit geometric topologies defined solely in terms of the topologies of the underlying spaces. It is shown that one such, the Fell topology, is natural for the numerical analysis of differential equations on differentiable manifolds.

There is a sense in which applied mathematics is like topology, or algebra, or analysis, but there is also a sense in which applied mathematics is bad mathematics. It’s a good contribution. It serves humanity. It solves problems about waterways, sloping beaches, airplane flights, atomic bombs, and refrigerators. But just the same, much too often it is bad, ugly, badly arranged, sloppy, untrue, undigested, unorganized, and unarchitected mathematics.

Paul Halmos [10]

Their studies until that point in time would have consisted, to a large extent, of a progression of formal reasoning, the familiar sequence of axiom ⇒ theorem ⇒ proof ⇒ corollary ⇒ .... Numerical analysis does not easily fit into this straitjacket, and this goes a long way toward explaining why many students of mathematics find it so unattractive.

Arieh Iserles [12]

1 Introduction
The numerical analysis of ordinary differential equations is generally cast within the context of open subsets of \( \mathbb{R}^n \). This corresponds to the machine implementations, where almost everything is anyway a tuple of floating point numbers. However, the familiar \( \mathbb{R}^n \) structures may lead to constructs that are not deeply natural, and to unarchitected mathematics. And there is significant numerical literature specifically targeting the differentiable category — see the references within and citations to [13, 20, 21].

Consider what it means for a discrete approximation \( y_{h,k} \) to converge to a solution \( y(t) \) of an ordinary differential equation (\( k \) is an integer and the sequence \( y_{h,k} \) approximates \( y(t) \) at time \( t = kh \)). Not possible is pointwise convergence, i.e.,

\[
\lim_{h \to 0^+} y_{h}(t) = y(t), \quad t \in \text{domain}(y),
\]

where \( y_h \) is the (partial) function defined to have values \( y_{h,k} \) at \( t_{h,k} = kh \), because the \( y_h \) are not defined at every \( t \) in the continuum for which \( y(t) \) is defined. So, in general practice [1, 12, 26, 27], convergence means something more like

\[
\lim_{h \to 0^+} \max_{k=0,1,...,\lfloor t/h \rfloor} \| y_{h,k} - y(t_{h,k}) \| = 0. \tag{1}
\]

Expediently, the smooth solution has been evaluated at varying times determined by the domains of the approximations, and one of the \( \mathbb{R}^n \) norms utilized to size the differences.

In (1), an apparent dependence on some particular norm has emerged at the level of the most basic definition. The definition does not cleanly export to the coordinate-invariant context of differentiable manifolds. A somewhat more sophisticated observation is that (1) is inherently asymmetric between the limit precursors and the target, such as one has in the definition of a Cauchy sequence, which symmetrically refers to two precursors.

Had there been no distance available in the first place, then the following may have been more visible: a sequence of mesh functions \( y_n \) converges to a solution \( y \) if

\[
\lim_{n \to \infty} y_n(t_n) = y(t) \quad \text{whenever} \quad t_n \in \text{domain}(y_n) \quad \text{and} \quad t_n \to t. \tag{2}
\]

This limit-evaluation convergence (the evaluation of the limit of functions is the limit the evaluations) is metric independent. It even exports to the category of topological spaces, because it does not depend on any smooth structure. Given a uniformity, it could support a notion of a Cauchy sequence.
But there is reason to favour topologically defined convergence, as opposed to relying solely on some apriori convergence criteria. For example, it is generally sufficient in applications to use uniform convergence of derivatives on compact sets in distributional test function spaces. The natural inductive limit carrier of this convergence (see Theorem 5.9) is not a locally convex topological vector space. Of interest in the theory of distributions is the dual space, but one does not have a Hahn-Banach theorem without local convexity ([24], esp. Example 1.47). In fact, addition of test functions is not continuous unless the local convexity problem is addressed.

It is common to test a numerical method by machine verification from some fixed initial condition to some fixed final time. This notion is of course invariant. But used as a theoretical foundation of convergence, the result is not localizable, meaning that if the numerical approximations are restricted to an arbitrary open subinterval, then that restriction inherently has varying start values and initial conditions. Local theorems depending on fixed initial conditions and fixed final time do not cleanly globalize because patching inherently breaks that constraint.

## 2 The convergence criteria

Quite apart from its relation to any topology, the criteria (2) is itself somewhat dysfunctional. Consider the sequence of partial maps

\[ g_i : (-1)^{i+1}[0, 1] \to \mathbb{R}, \quad g_i(t) = (-1)^{i+1}t. \quad (3) \]

The logical predicate “\( t_i \in (-1)^{i+1}[0, 1] \) and \( t_i \to t^* \)” is empty unless \( t = 0 \), so that (2) is satisfied vacuously for all \( t \neq 0 \). Under (2), the sequence \( g_i \) converges to any function \( g \) such that \( g(0) = 0 \). An improvement may be obtained by inserting subsequences, and by restricting the domain of the limit function:

\[
(a) \text{ for all strictly increasing } i_j, \text{ if } t_j \in \text{domain}(y_{i_j}) \text{ and } t_j \to t \in \text{domain}(y) \text{ then } \lim_{j \to \infty} y_{i_j}(t_j) = y(t); \text{ and (b) for all } t \in \text{domain } y \text{ there is a sequence } t_i \in \text{domain } y_{i_k} \text{ such that } t_i \to t. \quad (4)
\]

With this, the \( g_i \) in (3) converge to \( g(x) = 0 \) with domain \( \{0\} \).

About (4) a further issue may be apparent: it allows the possibility of a subsequence \( t_j \in \text{domain}(y_{i_k}) \) such that \( t_j \) converges to \( t \notin \text{domain}(y) \) and \( y_{i_k}(t_j) \) converges. In the context of a numerical method, this would correspond to the convergence of the method where the target limit does not exist, i.e. to a false numerical prediction that a solution exists. The following modification avoids this:

\[
(a) \text{ for all strictly increasing } i_j, \text{ if } t_j \to t \text{ and } y_{i_j}(t_j) \text{ both converge then } t \in \text{domain}(y) \text{ and } \lim_{j \to \infty} y_{i_j}(t_j) = y(t); \text{ and (b) for all } t \in \text{domain } y \text{ there is a sequence } t_i \in \text{domain } y_{i_k} \text{ such that } t_i \to t. \quad (5)
\]

The criteria (5) ensures that the domain of the limit includes times for which the precursors converge. Rephrasing in terms of a topological spaces \( \mathcal{X} \) and \( \mathcal{Y} \), one arrives at

\[ f_i : A_i \subseteq \mathcal{X} \to \mathcal{Y} \text{ converges to } f : A \subseteq \mathcal{X} \to \mathcal{Y} \text{ if (a) for all strictly increasing } i_j \text{ and all } x_j \in A_i \text{ such that } x_j \text{ and } f_i(x_j) \text{ both converge, } \lim_{j \to \infty} x_j \in A \text{ and } \lim_{j \to \infty} f_i(x_j) = f(\lim_{j \to \infty} x_j); \text{ and (b) for all } x \in A \text{ there is a sequence } x_i \in A_i \text{ such that } \lim_{i \to \infty} x_i = x. \quad (6)\]

Ideally, one seeks a topology on the space of partial maps, with an explicitly known and easily visualized neighbourhood base, within which a sequence of partial maps converges if and only if (6) holds.

## 3 Geometric Hypertopologies

Spaces of partial maps are related to spaces of subsets of a set—the power set—because partial maps may be identified with their graphs. There is substantial literature about topologies of the power set of a topological space, and the related spaces of partial maps of continuous maps [2, 4, 5, 7, 9, 11, 15, 16, 17, 23, 18]. This literature may not be so well-known and accessible in the numerical analysis community; this section provides a review of the essential aspects.

Let \( \mathcal{X} \) be a topological space. A hypetopology of \( \mathcal{X} \) is a topology on a subset of \( 2^\mathcal{X} \); a hyperspace is such a topological space. The geometric hypertopologies are those defined using the only topology on \( \mathcal{X} \), as opposed to for example using a metric on \( \mathcal{X} \).

It may be useful to have in mind the following example. If by definition \( \lim_i A_i = \{\lim x_i \mid x_i \in A_i\} \) (membership in the limit is the limit of membership), then the sequence \((-1)^i[0, 1] \to \{0\}\) while the odd and even subsequences converge to \([-1, 0]\) and \([0, 1]\), respectively. Simple-minded criteria may not be so useful.
### 3.1 Kuratowski-Painlevé convergence

Although here they will be the objective because the interest is numerical analysis, sequence convergence does not generally suffice for topology. Net convergence, where the index set is generalized to a directed ordered set, does suffice. Net convergence is a elementary topic that is available in many texts e.g. [25, 28]. For convenience, some relevant aspects, particularly focused on utilizing net convergence to generate topologies, are collected in an appendix to this article.

**Definition 3.1.**

(a) \( x \in \mathcal{X} \) is a limit point of a net of subsets \( A_\lambda \subseteq \mathcal{X} \) if, for all open \( U \ni x \), \( A_\lambda \cap U \neq \emptyset \) finally i.e. for all open \( U \ni x \) there is a \( \lambda^* \) such that \( A_\lambda \cap U \neq \emptyset \) for all \( \lambda \geq \lambda^* \). The lower closed limit or Kuratowski limit inferior of a net \( A_\lambda \) is the set of its limit points, denoted \( \text{Li}_A_\lambda \).

(b) \( x \in \mathcal{X} \) is a cluster point of a net of subsets \( A_\lambda \subseteq \mathcal{X} \) if, for all open \( U \ni x \), \( A_\lambda \cap U \neq \emptyset \) cofinally i.e. for all open \( U \ni x \) and all \( \lambda^* \) there is a \( \lambda \geq \lambda^* \) such that \( A_\lambda \cap U \neq \emptyset \). The upper closed limit or Kuratowski limit superior of a net \( A_\lambda \) is the set of its cluster points, denoted \( \text{Ls}_A_\lambda \).

(c) A net of subsets \( A_\lambda \) Kuratowski-Painlevé converges to \( A \) if \( \text{Li}_A_\lambda = A \) and \( \text{Ls}_A_\lambda = A \), denoted \( A = \text{K-limit} A_\lambda \).

The notations in Definition 3.1 conform to [4]. Obviously, \( \text{Li}_A_\lambda \subseteq \text{Ls}_A_\lambda \). \( \text{Li}_A_\lambda \) and \( \text{Ls}_A_\lambda \) are both closed: If \( x \in \text{cl}(\text{Li}_A_\lambda) \) and \( U \ni x \) is open then \( U \cap \text{Li}_A_\lambda \neq \emptyset \). Choose \( y \in U \cap \text{Li}_A_\lambda \). Then \( y \in U \), so there is a \( \lambda^* \) such that \( A_\lambda \cap U \neq \emptyset \), which suffices to show that \( x \in \text{Li}_A_\lambda \). The proof for \( \text{Ls}_A_\lambda \) is similar.

Limits and cluster points are defined in terms of neighbourhoods of the underlying topology, but they have equivalent expressions in terms of net convergence.

**Proposition 1.**

If \( A_\lambda \) is a net of subsets of \( \mathcal{X} \) then

(a) \( x \in \mathcal{X} \) is a cluster point of \( A_\lambda \) if and only if there is a subnet \( A_{\lambda_*} \) and a net \( x_\mu \in A_{\lambda_*} \) such that \( x_\mu \to x \).

(b) \( x \in \mathcal{X} \) is a limit point of \( A_\lambda \) if and only if it is a cluster point of every subnet of \( A_\lambda \).

If \( \mathcal{X} \) is first countable and \( A_\lambda \) is a sequence of subsets of \( \mathcal{X} \), then

(c) \( x \) is a cluster point of \( A_i \) if and only if there is a strictly increasing \( i_j \) and \( x_j \in A_{i_j} \) such that \( x_j \to x \).

(d) \( x \) is a limit point of \( A_i \) if and only if there is a sequence \( x_i \in A_i \) such that \( x_i \to x \).

**Proof.** (a) If \( x \) is a cluster point of \( A_\lambda \) then the set pairs \( \{ (\lambda, U) \mid A_\lambda \cap U \neq \emptyset \} \) (with the ordering \( (U_1, \lambda_1) \geq (U_2, \lambda_2) \) if \( U_1 \subseteq U_2 \) and \( \lambda_1 \geq \lambda_2 \)) is directed. \( A_{\lambda_*} \) is a subnet of \( A_\lambda \) and picking \( x_\lambda \in A_{\lambda_*} \cap U \) provides a suitable net converging to \( x \). Conversely, suppose \( A_{\lambda_*} \) is a subnet of \( A_\lambda \), \( x_\mu \in A_{\lambda_*} \), and \( x_\mu \to x \), and let \( \lambda^* \in \Lambda \). Given an open \( U \ni x \) there is a \( \mu^* \) such that \( x_\mu \in U \) if \( \mu \geq \mu^* \), and there is a \( \mu^*_2 \) such that \( \lambda^* \geq \lambda^*_2 \) for \( \mu \geq \mu^*_2 \). Choose \( \mu \) such that \( \mu \geq \mu^*_1 \) and \( \mu \geq \mu^*_2 \). For that \( \mu \), \( x_\mu \in A_{\lambda^*} \cap U \), providing \( \lambda^* \geq \lambda^* \) and \( A_{\lambda^*} \cap U \neq \emptyset \).

(b) If \( x \) is a limit point of \( A_\lambda \) then it is a limit point of every subnet of \( A_\lambda \), and every limit point is a cluster point. Conversely, suppose that \( x \) is not a limit point. Then there is an open \( U \ni x \) such that for all \( \lambda^* \) there is a \( \lambda \geq \lambda^* \) such that \( A_\lambda \cap U = \emptyset \), and the set \( \{ \lambda \mid A_\lambda \cap U = \emptyset \} \) provides a subnet of \( A_\lambda \) which has no subnet that converges to \( x \).

(c) Suppose \( x \) is a cluster point of \( A_i \) and let \( U_j \) be a countable neighbourhood base at \( x \). Choose \( i_j \) strictly increasing such that \( A_{i_j} \cap U_j \neq \emptyset \) and choose \( x_{i_j} \) is that set. The converse is immediate from (1) because any subsequence of \( A_i \) is a subnet of that.

(d) Suppose \( x \) is a limit point of \( A_i \) and let \( U_j \) be a countable neighbourhood base at \( x \). For all \( j \) there is \( N_j \) such that \( i \geq N_j \) implies \( A_i \cap U_j \neq \emptyset \). Without loss of generality assume \( N_j \) is increasing. Inductively choosing a sequence \( x_i \in A_i \) such that \( x_i \in U_i \) for \( N_1 \leq i < N_2 \), \( x_i \in U_2 \) for \( N_2 \leq i < N_3 \), and so on, provides an \( x_i \in A_i \) such that \( x_i \to x \). Conversely, any such sequence obviously provides final nonempty intersection of \( A_i \) with any neighbourhood of \( x \).

□

From \( \text{Li}_A_\lambda \subseteq \text{Ls}_A_\lambda \) follows that \( A = \text{K-limit} A_i \) if and only if \( A \subseteq \text{Li}_A_\lambda \) and \( \text{Ls}_A_\lambda \subseteq A \) ([4], Lemma 5.2.4). In the case that \( \mathcal{X} \) is first countable, this obtains the convergence criteria

(a) if \( i_j \) is strictly increasing and \( x_j \in A_{i_j} \) such that \( x_j \to x \) then \( x \in A \); and (b) if \( x \in A \) then there are \( x_i \in A_i \) such that \( x_i \to x \).

Also, in that case, a sequence of subsets \( A_i \) converges if and only if every \( x_j \in A_{i_j} \) with \( x_j \to x \) admits an extension \( \tilde{x}_i \in A_i \) (i.e. \( x_j = \tilde{x}_{i_j} \)) such that \( \tilde{x}_i \to x \) (e.g. the limit of \((-1)^i[0, 1]\) does not exist): given convergence, such an \( x \) is a cluster point by Proposition 1(c), and hence is a limit point, so by Proposition 1(d) there is a \( \tilde{x}_i \in A_i \) such that \( \tilde{x}_i \to x \), and replacing \( \tilde{x}_{i_j} \) with \( x_{i_j} \) provides the required extension. Conversely, if \( x \) is a cluster point then there is
an \( x_j \in A_{ij} \) such that \( x_j \to x \), the extension provides that \( x \) is a limit point, and Kuratowski-Painlevé convergence holds. The equivalence of \( (a) \ x_j \in A_{ij} \) and \( x_j \to x \) implies \( x \in A \); and \( (b) \) there is a sequence \( x_j \in A_{ij} \) such that \( x_j \to x \). \hspace{3cm} (7)

and Kuratowski-Painlevé convergence follows because, for any strictly increasing \( i_j, x_i \in A_i \) such that \( x_i \to x \) gives by restriction \( x_{ij} \to x \).

### 3.2 The Vietoris and co-compact hypertopologies

**Definition 3.2.** Let \( \mathcal{X} \) be a topological space.

1. The lower Vietoris topology \( \{ \text{lower co-compact topology} \} \) on \( 2^\mathcal{X} \) is generated by the union of \( \{ A \subseteq \mathcal{X} \mid A \cap U \neq \emptyset \} \) over \( U \subseteq \mathcal{X} \) open \( \{ U \subseteq \mathcal{X} \text{ is co-compact} \} \).

2. The upper Vietoris topology \( \{ \text{upper co-compact topology} \} \) on \( 2^\mathcal{X} \) is generated by the union of \( \{ A \subseteq \mathcal{X} \mid A \subseteq U \} \) over \( U \subseteq \mathcal{X} \) open \( \{ U \subseteq \mathcal{X} \text{ co-compact} \} \).

There are a variety of hypertopologies and the literature is extensive, e.g., [4, 11, 16, 17, 23, 18]. [19] presents a systematization and a convenient summary table. The notation here is not quite standard: it seems that co-

compact or closed \( K \) is generally defined to mean upper co-compact and the lower co-compact topology does not appear. The appearance of compactness in these definitions does not strike as particularly compelling but it is posteriori justified by Definition 3.7.

The topologies in Definition 3.2 are often referred to as hit and miss, since they are generated by subbases which hit, or set theoretically meet, an open [co-compact] set, and miss, or are set theoretically contained in, the complement of a closed [compact] set (for the upper topologies, one can use the alternates \( \{ A \mid A \cap K = \emptyset \} \) over compact or closed \( K \), corresponding to “miss”). The Vietoris topology or exponential topology [16] is the join of the upper and lower Vietoris topologies.

If \( \mathcal{X} \) is Hausdorff then every compact set is closed, every co-compact set is open, the union of \( \{ A \subseteq \mathcal{X} \mid A \subseteq U \} \) over co-compact \( U \) is contained in the union of that over open \( U \), so the Vietoris topologies are finer than the co-compact topologies. If \( \mathcal{X} \) is a compact Hausdorff space then the co-compact subsets and the open subsets of \( \mathcal{X} \) coincide, so in that case the upper Vietoris [lower Vietoris] and the upper co-compact [lower co-compact] topologies are the same.

**Theorem 3.3.** The Vietoris \{co-compact, assuming \( \mathcal{Y} \) is closed\} topologies on \( 2^\mathcal{X} \) are natural with respect to subspaces, i.e. if \( \mathcal{Y} \subseteq \mathcal{X} \) then the topology of \( 2^\mathcal{Y} \) as a subspace of \( 2^\mathcal{X} \) is the corresponding topology on \( 2^\mathcal{Y} \).

**Proof.** The unions, over open \( U \subseteq \mathcal{X} \), of the left and right sizes of

\[
2^\mathcal{Y} \cap \{ A \subseteq \mathcal{X} \mid A \cap U \neq \emptyset \} = \{ A \subseteq \mathcal{Y} \mid A \cap (U \cap \mathcal{Y}) \neq \emptyset \}.
\]

are subbases for the subspace topology on \( 2^\mathcal{Y} \) and its lower Vietoris topology, respectively. Similarly,

\[
2^\mathcal{Y} \cap \{ A \subseteq \mathcal{X} \mid A \subseteq U \} = \{ A \subseteq \mathcal{Y} \mid A \subseteq \mathcal{Y} \cap U \}.
\]

shows the two two upper Vietoris are the same. For the upper co-compact topology, the unions of

\[
2^\mathcal{Y} \cap \{ A \subseteq \mathcal{X} \mid A \cap K \neq \emptyset \}, \quad \{ A \subseteq \mathcal{Y} \mid A \cap L \neq \emptyset \},
\]

are subbases for the subspace topology on \( 2^\mathcal{Y} \) and its lower co-compact topology, respectively. Similarly, the unions of

\[
2^\mathcal{Y} \cap \{ A \subseteq \mathcal{X} \mid A \subseteq U \} = \{ A \subseteq \mathcal{Y} \mid A \subseteq \mathcal{Y} \cap U \}.
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\]

are subbases for the subspace topology on \( 2^\mathcal{Y} \) and its lower co-compact topology, respectively. Similarly, the unions of

\[
2^\mathcal{Y} \cap \{ A \subseteq \mathcal{X} \mid A \subseteq U \} = \{ A \subseteq \mathcal{Y} \mid A \subseteq \mathcal{Y} \cap U \}.
\]
over compact $K \subseteq \mathcal{X}$ and $L \subseteq \mathcal{Y}$ are subbases for the subspace lower Vietoris topology on $2^\mathcal{Y}$, and the lower Vietoris topology on $2^\mathcal{X}$ using the subspace topology on $\mathcal{Y}$. The two collections correspond for choices of $K$ and $L$: if $\mathcal{Y}$ is closed and $K$ is compact in $\mathcal{X}$, then $L = K \cap \mathcal{Y}$ is compact in $\mathcal{Y}$, while if $L$ is compact in $\mathcal{Y}$ then it is compact in $\mathcal{X}$. Similarly, for $K \subseteq \mathcal{X}$ compact and $L \subseteq \mathcal{Y}$ compact, the two collections

$$2^\mathcal{Y} \cap \{A \subseteq \mathcal{X} \mid A \cap (\mathcal{X} \setminus K) \neq \emptyset\}, \quad \mathcal{N}_\mathcal{Y}(L) \equiv \{A \subseteq \mathcal{Y} \mid A \cap (\mathcal{Y} \setminus L) \neq \emptyset\},$$

correspond: if $\mathcal{Y}$ is closed and $K$ is compact in $\mathcal{X}$, then $K \cap \mathcal{Y}$ is compact in $\mathcal{Y}$ and $A \cap (\mathcal{X} \setminus K) = A \cap (\mathcal{Y} \setminus K) = A \cap (\mathcal{Y} \setminus (K \cap \mathcal{Y}))$, while if $L$ is compact in $\mathcal{Y}$ then $L$ is compact in $\mathcal{X}$ and $A \subseteq \mathcal{Y}$ implies $A \cap (\mathcal{X} \setminus L) = A \cap (\mathcal{Y} \setminus L)$. □

Sequences are the target here and so the countability of these hypertopologies is a focus. As it turns out, the most natural route to sequential convergence, first countability, is a deeper problem [3]. However, the (general) topologies for the underlying spaces of numerical analysis are simple: the usual assumption is at least a second countable locally compact Hausdorff space (and therefore paracompact). Second countability is stronger than first and can be relatively easily passed to hypertopologies:

**Theorem 3.4.** If $\mathcal{X}$ is a second countable [and locally compact Hausdorff] then the lower Vietoris [upper co-compact] topology is second countable.

**Proof.** Suppose $\mathcal{X}$ has a countable basis $\mathcal{B}_0$. For the lower Vietoris topology, define $\mathcal{N}^+(U) \equiv \{A \subseteq 2^\mathcal{X} \mid A \cap U \neq \emptyset\}$. It suffices to show $\mathcal{B} \equiv \{\mathcal{N}^+(U) \mid U \text{ is open}\}$ and $\mathcal{B}' \equiv \{\mathcal{N}(U) \mid U \in \mathcal{B}_0\}$ are equivalent subbases, the first being the defining subbase of the lower Vietoris and the second being countable. Indeed, $\mathcal{B}' \subseteq \mathcal{B}$, while if $\mathcal{N}(U) \in \mathcal{B}$ and $A \in \mathcal{N}(U)$ then let $x \in A$ and choose $V \in \mathcal{B}$ such that $x \in V \subseteq U$, and the result follows because $A \in \mathcal{N}(V) \subseteq \mathcal{N}(U)$.

For the upper co-compact topology, define $\mathcal{N}(K) \equiv \{A \subseteq \mathcal{X} \mid A \cap K = \emptyset\}$. Because $\mathcal{X}$ is locally compact and Hausdorff, the set $\mathcal{B}_{00}$ of relatively compact subsets of $\mathcal{B}_0$ is a countable basis of $\mathcal{X}$. It suffices to show $\mathcal{B} \equiv \{\mathcal{N}(K) \mid K \text{ is compact}\}$ and $\mathcal{B}' \equiv \{\mathcal{N}(clU) \mid U \in \mathcal{B}_{00}\}$ are equivalent subbases, the first being the defining subbase of the upper co-compact topology and the second being countable. Indeed, $\mathcal{B}' \subseteq \mathcal{B}$, while if $K$ is compact and $A \in \mathcal{N}(K)$, then, using regularity of $\mathcal{X}$ and $A$ is closed, choose finitely many $U_1, \ldots, U_n \in \mathcal{B}_{00}$ that cover $K$ with closures contained in $\mathcal{X} \setminus A$, from which $A \in \mathcal{N}(clU_1) \cap \ldots \cap \mathcal{N}(clU_n) \subseteq \mathcal{N}(K)$. □

Finding a hypertopology that fits the criteria (6) means a precise understanding of the convergence defined by those. That can be expressed in terms of Kuratowski limits:

**Lemma 3.5.** Let $\mathcal{X}$ be a topological space.

(a) If $\mathcal{X}$ is regular and $A_\lambda \to A$ (upper Vietoris) then $cl_A \supseteq Ls_A$.

(b) If $\mathcal{X}$ is locally compact and $A_\lambda \to A$ (upper co-compact) then $cl_A \supseteq Ls_A$.

(c) If $\mathcal{X}$ is compact and $A_\lambda \to A$ (upper Vietoris).

(d) If $A \supseteq Ls_A$ then $A_\lambda \to A$ (upper co-compact).

(e) $A_\lambda \to A$ (lower Vietoris) if and only if $A \subseteq Ls_A$.

(f) If $\mathcal{X}$ is Hausdorff and $A \subseteq Ls_A$ then $A_\lambda \to A$ (lower co-compact).

(g) If $\mathcal{X}$ is compact and $A_\lambda \to A$ (lower co-compact) then $A \subseteq Ls_A$.

**Proof.** (a) Suppose $A_\lambda \to A$ (upper Vietoris), $x$ is a cluster point of $A_\lambda$, $U \ni x$ is open, and $V \supseteq cl_A$. By regularity, it suffices to show $U \cap V \neq \emptyset$. Choose $\lambda^*$ such that $A_{\lambda^*} \subseteq V$ for $\lambda \geq \lambda^*$, choose $\lambda \geq \lambda^*$ such that $A_\lambda \cap U \neq \emptyset$, and note that $U \cap V \supseteq A_\lambda \cap U$.

(b) Suppose $A_\lambda \to A$ (upper co-compact). Since $\mathcal{X}$ is locally compact, if $x \notin cl_A$ then there is a compact neighbourhood $U$ of $x$ such that $A_\lambda \cap U = \emptyset$. So there is a $\lambda^*$ such that $A_{\lambda^*} \cap U = \emptyset$ for all $\lambda \geq \lambda^*$ i.e. $x$ is not a cluster point of $A_\lambda$.

(c) Suppose $A_\lambda \not\to A$ (upper Vietoris) i.e. there is an open $U$ such that $A \subseteq U$, and, for all $\lambda^*$ there is $\lambda$ such that $\lambda \geq \lambda^*$ and $A_{\lambda} \not\subseteq U$. The set $\lambda' = \{\lambda \in A \mid A_{\lambda} \not\subseteq U\}$ is directed. For each $\lambda \in \lambda'$ choose $x_\lambda \in A_{\lambda}$ such that $x_\lambda \notin U$. Since $\mathcal{X}$ is compact, a subnet $x_{\lambda'}$ converges, say to $x$; by Proposition 1(a) $x \in cl_{A_\lambda}$. But $x \notin U$ since $\mathcal{X} \setminus U$ is closed and $x_{\lambda'} \not\in U$ from which $x \notin A$ since $U \subseteq A$.

(d) Suppose $A_\lambda \not\to A$ (upper co-compact) i.e. there is a compact $K$ such that $A \cap K = \emptyset$, and, for all $\lambda^*$ there is $\lambda$ such that $\lambda \geq \lambda^*$ and $A_{\lambda} \cap K \neq \emptyset$. The set $\lambda' = \{\lambda \in A \mid A_{\lambda} \cap K \neq \emptyset\}$ is directed. For each $\lambda \in \lambda'$ choose $x_\lambda \in A_{\lambda}$ such that $x_\lambda \in K$. A subnet $x_{\lambda'}$ converges, say to $x \in K$. So $x$ is an cluster point of $A_\lambda$ and $x \notin A$, from which $A \not\supseteq Ls_A$.

(e) If $A_\lambda \to A$ (lower Vietoris) and $x \in A$ and $U \ni x$ is open then $U \cap A \neq \emptyset$ and there is a $\lambda^*$ such that $U \cap A_{\lambda} \neq \emptyset$ for all $\lambda \geq \lambda^*$, so $x$ is a limit point. The converse is similar.
By (e), \( A_\lambda \to A \) in the lower Vietoris topology. In a Hausdorff space, every co-compact set is open, so the lower Vietoris topology is finer than the co-compact topology and therefore \( A_\lambda \to A \) in the co-compact topology, while if \( \mathcal{X} \) is compact there every open set is co-compact.

**Lemma 3.6.** Let \( A_\lambda \) be a net of subsets of \( \mathcal{X} \).

(a) If \( A_{\lambda_n} \) is a subnet of \( A_\lambda \) then \( \Lambda_{A_{\lambda_n}} \supseteq \Lambda_{A_\lambda} \).

(b) If \( A = \Lambda_{A_{\lambda_n}} \) is maximal in the lower limit sets of \( A_\lambda \) then \( A_{\lambda_n} \to A \) in the upper co-compact topology.

**Proof.** (a) If \( A_{\lambda_n} \) is a subnet of \( A_\lambda \) then \( x \in \Lambda_{A_{\lambda_n}} \) and \( U \ni x \) is open then choose \( \lambda^* \) such that \( A_\lambda \cap U \neq \emptyset \) for all \( \lambda \geq \lambda^* \). Choose \( \mu^* \) so that \( A_{\lambda_n} \supseteq \lambda^* \). Then \( \mu \geq \mu^* \) implies \( \lambda_{\mu} \geq \lambda^* \) and \( A_{\lambda_{\mu}} \cap U \neq \emptyset \).

(b) Suppose \( A \) is maximal. If \( A_{\lambda_n} \not\to A \) (upper co-compact) then there is a compact set \( K \) such that \( K \cap A = \emptyset \) and such that, for all \( \mu^* \) there is a \( \mu > \mu^* \) such that \( A_{\lambda_{\mu}} \cap K \neq \emptyset \). The set \( \{ \mu \mid A_{\lambda_{\mu}} \cap K \neq \emptyset \} \) is directed; choose \( x_\mu \in A_{\lambda_{\mu}} \cap K \) for each such \( \mu \). Since \( K \) is compact, a subnet of \( x_\mu \) converges, say to \( x \in K \). The limit inferior of the corresponding subnet of \( A_{\lambda_n} \) contains \( A \) and \( x \), contradicting maximality of \( A \) since \( A \cap K = \emptyset \).

### 3.3 The Fell topology

An emphasized in the Appendix, any convergence criteria that respects subnets does generate a topology: the finest topology such that every net that satisfies the criteria converges. A convergence is called topological on the equivalence of the apriori criteria and the convergence in the generated topology. Otherwise, there may be nets that converge in the generated topology that do not satisfy such apriori criteria. Such a condition may manifest a logical inequivalence of convergence statements: convergence in the criteria implies convergence in the topology, but the converse of that involves some additional context or conditions.

This is further complicated because the generated topology is usually abstract — it is an unknown that one is seeking to identify. In that case, a putative topology may deviate in both directions of logical implication. One strategy to deal with this is to pair implications that appear to be almost equivalent and then take the union of their logical predicates, hoping that is not so restrictive as to eliminate the target application. In case of success, one arrives at a viable context in which the criteria are matched to a topology. This is not an exact science. One is sorting through logical implications that may not be the best. Nothing prevents further work from establishing a wider context that matches a given convergence criteria to a larger class of topologies.

Following these ideas, Lemma 3.5 seems to diminish the upper Vietoris topology or lower co-compact topologies, because Lemma 3.5(c) and Lemma 3.5(g) depend on compactness actual. While it may be reasonable to assume that the ambient spaces of numerical analysis are locally compact, they are not usually compact. In passing, those hypertopologies seem somewhat pathological anyway. For example, if \( \mathcal{X} = \mathbb{R}^2 \) and \( A_n = \mathbb{R} \times \{ 1/n \} \) then \( A_n \) does not converge in the upper Vietoris topology to \( \mathbb{R} \times \{ 0 \} \) because \( A_n \) is not contained in any neighbourhood of the form \( \{ (x, y) \mid -1/x < y < 1/x \} \), while \( A_n \) converges to any bounded set in the lower co-compact topology since it meets the complement of any compact set.

The condition \( A \subseteq \Lambda_{A_{\lambda_n}} \) already topological to the lower Vietoris topology, by Lemma 3.5(e). Aligning Lemma 3.5(b) and Lemma 3.5(d), the condition \( A \supseteq \Lambda_{A_{\lambda_n}} \) is topological in the context of closed sets on a locally compact Hausdorff space. By Theorem 5.8(c), the combination of \( A \subseteq \Lambda_{A_\lambda} \) and \( A \supseteq \Lambda_{A_{\lambda_n}} \), i.e. Kuratowski-Painlevé convergence, is topological to the join of the lower Vietoris and upper co-compact topologies. In the case that \( \mathcal{X} \) is also second countable, both the lower Vietoris topology and upper co-compact topologies are second countable, as is their join, and then sequences suffice, with convergence the extremely compelling (7).

**Definition 3.7.** The Fell topology is the join of the upper co-compact topology and the lower Vietoris topology. \( \text{Fell}(\mathcal{X}) \) is the set of closed subsets of \( \mathcal{X} \) with the Fell topology.

The Fell topology first occurs in [7] and [9]; see also [3]. The nomenclature is not standardized. The term Chabauty-Fell is suggested in [8], [22] calls the lower Vietoris topology the Thurston topology, and, with [6], the Fell topology is called the Chabauty topology, but the latter also use the term geometric topology. The notation \( \text{Fell}(\mathcal{X}) \) is not standard.

**Theorem 3.8.**

(a) \( \text{Fell}(\mathcal{X}) \) is compact (for any \( \mathcal{X} \)).

(b) If \( \mathcal{X} \) is locally compact then \( \text{Fell}(\mathcal{X}) \) is Hausdorff and \( A_\lambda \to A \) if and only if \( A_\lambda \to A \) (Kuratowski-Painlevé).

(c) If \( \mathcal{X} \) is locally compact, Hausdorff, and second countable then \( \text{Fell}(\mathcal{X}) \) is second countable.

**Proof.** (a) Suppose \( A_{\alpha} = \Lambda_{f_\alpha} \) is a (set theoretic) chain of lower limit sets, where \( f_\alpha : M_\alpha \to \Lambda \). The co-product \( M \equiv \bigvee M_\alpha = \{ (\alpha, \mu) \mid \mu \in M_\alpha \} \) is ordered by \( (\alpha_1, \mu_1) \geq (\alpha_2, \mu_2) \) if \( \alpha_1 = \alpha_2 \) and \( \mu_1 \geq \mu_2 \) (\( (\alpha_1, \mu_1) \) and \( (\alpha_2, \mu_2) \)}
are incomparable if $a_1 \neq a_2$). With this ordering, $(\alpha, \mu) \mapsto f_\alpha(\mu)$ is a subnet of $A_\lambda$ and $\bigcup_\alpha A_\alpha$. Thus every chain has an upper bound, Zorn’s lemma provides a maximal lower limit set, and there is a convergent subnet of $A_\lambda$ by Lemma 3.6(b).

(b) If $A_\lambda \to A$ in Fell($\mathcal{X}$) then $A_\lambda \to A$ in both the upper co-compact and the lower Vietoris topology. By Lemma 3.5, $\Lambda A_\lambda = A = \text{Ls} A_\lambda$, which establishes that limits are unique. If $\mathcal{X}$ is locally compact then by Lemma 3.5 the upper co-compact and lower Vietoris topologies separately correspond to the convergence criteria $A \supseteq \text{Ls} A_\lambda$ and $\text{Ls} A_\lambda \subseteq A$, respectively. Because of that, the join of those topologies corresponds to the logical “and” of those criteria, and that by (7) is equivalent to Kuratowski-Painlevé convergence.

(c) If $\mathcal{X}$ is second countable locally compact Hausdorff then Theorem 3.4 provides countable bases for both the lower Vietoris and upper co-compact topologies, and the union of those is a countable basis for the Fell topology. □

**Theorem 3.9.**

(a) The collection \{ $A \subseteq \mathcal{X}$ | $A \cap K \subseteq U$ and $A \cap V_i \neq \emptyset$ for all $i$ \} over all open $U, V_1, \ldots, V_n \subseteq \mathcal{X}$ and compact $K \subseteq \mathcal{X}$, is a base for Fell($\mathcal{X}$).

(b) Suppose $\mathcal{X}$ is locally compact and Hausdorff, and let $B \subseteq \mathcal{X}$ be compact subset. Then the collection \{ $A \subseteq \mathcal{X}$ | $B \subseteq A$ and $A \subseteq U$ and $A \cap V_i \neq \emptyset$ for all $i$ \} over all relatively compact open $U$, all open subsets $V_1, \ldots, V_n$ is a neighbourhood base of Fell($\mathcal{X}$) (at $B$).

**Proof.** (a) The given collection is the same as

\{ $A \subseteq \mathcal{X}$ | $A \cap K \cap L = \emptyset$ and $A \cap V_i \neq \emptyset$ for all $i$ \}

over all closed $L$, open $V_i$, and compact $K$. Since the intersection of a closed set and a compact set is compact, and since one may take $L = \mathcal{X}$, this is the same as the collection \{ $A \subseteq \mathcal{X}$ | $A \cap K = \emptyset$ and $A \cap V_i \neq \emptyset$ for all $i$ \}, which is a base for Fell($\mathcal{X}$) by definition of the join of the upper co-compact and lower Vietoris topologies.

(b) If $U$ is relatively compact then set $K = c\text{I}U$ and note that $A \cap K \subseteq U$ if and only if $A \subseteq U$, so the collection is an extraction from the base established in Theorem 3.9(a). Now if $U, V_i$ are open, $K$ is compact, and suppose a compact $B$ such that $B \cap K \subseteq U$. Then $B$ and $K \setminus U$ are disjoint compact sets, so there are disjoint open relatively compact $U' \supseteq B$ and $U'' \supseteq K \setminus U$. If $A \subseteq U''$ then $A \cap K \subseteq U' \cap K \subseteq K \cap (\mathcal{X} \setminus U'') \subseteq K \cap (\mathcal{X} \setminus (K \setminus U)) = K \cap U \subseteq U$. □

### 4 Partial maps

A **partial function** on $\mathcal{X}$ is a function with domain a (usually, but not necessarily proper) subset of $\mathcal{X}$. Since partial maps have by definition domains which are subsets, and since some topologies on partial maps are defined by replacing the partial maps with graphs, their topologies are strongly related to topologies on spaces of subsets of a topological space.

**Definition 4.1.** Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces.

(a) The topologies on the set of partial maps from $\mathcal{X}$ to $\mathcal{Y}$ with the same names as Definition 3.2 are the corresponding topologies obtained after identifying maps with their graphs.

(b) The **compact-open** topology on the set of partial maps from $\mathcal{X}$ to $\mathcal{Y}$ is the topology with subbase sets of the form $\mathcal{N}(K, U) \equiv \{f: B \to \mathcal{Y} | f(K) \subseteq U\}$, where $K \subseteq \mathcal{X}$ is compact and $U \subseteq \mathcal{Y}$ is open.

(c) The **Back topology** is the join of the compact-open topology and initial topology obtained from assignment of partial maps to domains with the lower Vietoris topology.
The first to consider spaces of partial maps was [15]. A convergence criteria occurs even at the beginning in the following form: a sequence \( f_n \) with domains \( A_n \), convergences to \( f \) with domain \( A \) if and only if \( A_n \) converges to \( A \) in the Hausdorff (\( X = \alpha \ast \varnothing \)). The Back topology is from [2]. A recent reference for topologies on the space of partial maps is [5]. The term generalized compact-open is a synonym for the Back topology, and compact-open may not be as used here but rather may be synonymous with Back topology.

If \( \mathcal{X} \) and \( \mathcal{Y} \) are second countable locally compact Hausdorff spaces then \( \mathcal{X} \times \mathcal{Y} \) is also that. Let \( \text{Fell}(\mathcal{X}, \mathcal{Y}) \) be the set of such with the subspace topology obtained by identifying partial maps with their graphs. Subspaces of second countable Hausdorff spaces are second countable Hausdorff [[28], Theorems 13.8 and 16.2 and so \( \text{Fell}(\mathcal{X}, \mathcal{Y}) \) is second countable and Hausdorff if \( \mathcal{X} \) and \( \mathcal{Y} \) are.

**Theorem 4.2.** A sequence \( (f_i : A_i \to \mathcal{Y}) \in \text{Fell}(\mathcal{X}, \mathcal{Y}) \) converges to \( (f : A \to \mathcal{Y}) \in \text{Fell}(\mathcal{X}, \mathcal{Y}) \) if and only if

(a) for all strictly increasing \( i_j \), if \( x_j \in A_{i_j} \) and both \( x_j \) and \( f_{i_j}(x_j) \) converge, then \( \lim x_j \in A \) and \( f(\lim x_j) = \lim f_{i_j}(x_j) \); and

(b) for all \( x \in A \) there is an \( x_i \in A_i \) such that \( x_i \to x \) and \( f(x_i) \) converges.

If in addition \( \mathcal{Y} \) is compact, then \( f_i \) converges if and only if

(c) for all strictly increasing \( i_j \), if \( x_j \in A_{i_j} \) and \( x_j \) converges, then \( \lim x_j \in A \) and \( f(\lim x_j) = \lim f_{i_j}(x_j) \); and

(d) for all \( x \in A \) there is an \( x_i \in A_i \) such that \( x_i \to x \).

**Proof.** Suppose \( f_i \to f \) in \( \text{Fell}(\mathcal{X}, \mathcal{Y}) \). If \( i_j \) is strictly increasing, \( x_j \in A_{i_j} \), \( x_j \to x \), and \( f_{i_j}(x_j) \to y \), then \((x_j, f_{i_j}(x_j)) \in \text{graph } f_{i_j} \) and \((x_j, f(x_j)) \to (x, y)\), so \((x, y) \in \text{graph } f \) and \( \lim f_{i_j}(x_j) = y = f(x) = f(\lim x_j) \). Also, if \( x \in A \) then \((x, f(x)) \in \text{graph } f \) and there is a \((x_i, y_i) \in \text{graph } f_i \) such that \((x_i, y_i) \to (x, y)\), implying both \( x_i \to x \) and \( f(x_i) \) converges.

Conversely, suppose (a) and (b). If \( i_j \) is strictly increasing and \((x_j, y_j) \in \text{graph } f_{i_j} \) and \((x_j, y_j) \to (x, y)\), then both \( x_j \) and \( y_j = f_{i_j}(x_j) \) converge, so \( x = \lim x_j \in A \) and \( y = \lim f_{i_j}(x_j) = f(\lim x_j) = f(x) \), i.e., \((x, y) \in \text{graph } f \). If \((x, y) \in \text{graph } f \) then \( x \in A \) and there is an \( x_i \in A_i \) such that \( x_i \to x \) and \( f(x_i) \) converges. By (a), \( y = f(x) = f(\lim x_i) = \lim f_i(x_i) \) so \((x_i, f(x_i)) \in \text{graph } f_i \) and \((x_i, f_i(x_i)) \to (x, y)\), and \( f_i \to f \) in \( \text{Fell}(\mathcal{X}, \mathcal{Y}) \) follows from (7).

Irrespective of whether \( \mathcal{Y} \) is compact or not, suppose (c) and (d). Then (a) is immediate from (c), while for (b), if \( x \in A \) then there is a \( x_i \in A_i \) such that \( x_i \to x \), and (c) implies that \( f(x_i) \) converges. Conversely, suppose (a) and (b). Then (d) is immediate, while for (c), supposing \( \mathcal{Y} \) is compact, let \( i_j \) be strictly increasing, and let \( x_j \in A_{i_j} \) be such that \( x_j \to x \). A subsequence of \( f(x_j) \) has a convergence subsequence, so by (a), \( x \in A \) and that subsequence converges to \( f(x) \). Since every subsequence of \( f(x_i) \) has a subsequence that converges to \( f(x) \), [28] (Exercise 11D) implies that \( f(x_i) \) converges, to \( f(x) \).

The graph of \( \text{tanh}(nx) \) converges to the union \( \{(x, 1) \mid x \geq 0\} \cup \{(x, -1) \mid x \leq 0\} \), which is not the graph of a function, so the set of graphs is not necessarily closed in the Fell topology in the graph space. Thus \( \text{Fell}(\mathcal{X}, \mathcal{Y}) \) is not necessarily compact even though the Fell topology on the closed subsets is compact.

For comparison, the result for the far more well-known compact-open topology follows. Note that the convergence criteria for the forward implication is essentially different from the reverse, even given local compactness.

**Proposition 2.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be topological spaces, and let \( f_x : \lambda \to \mathcal{Y} \) be a net of continuous maps.

(a) Suppose \( \mathcal{X} \) is locally compact. If \( f_x \) converges in the compact-open topology then \( x_{\mu} \to x \), such that \( x_{\mu} \in \lambda_{\mu} \) and \( x_{\mu} \in A \), implies \( f_{x_{\mu}}(x_{\mu}) \to f(x) \).

(b) Suppose \( \mathcal{X} \) is locally compact and Hausdorff. If \( x_{\mu} \in \lambda_{\mu} \) and \( x_{\mu} \to x \) implies both \( x \in A \) and \( f_{x_{\mu}}(x_{\mu}) \to f(x) \), then \( f_x \) converges to \( f \) in the compact-open topology.

**Proof.** (a) Let \( x_{\mu} \in \lambda_{\mu} \) be such that \( x_{\mu} \to x \in A \), and let \( U \ni f(x) \) be open. Choose a compact neighbourhood \( K \ni x \) such that \( f(K) \subseteq U \). Choose \( \lambda^* \) such that \( f_{\lambda}(K) \subseteq U \) for \( \lambda \geq \lambda^* \). Choose \( \mu^*_1 \) such that \( x_{\mu} \in K \) for \( \mu \geq \mu^*_1 \). Choose \( \mu^*_2 \) such that \( \mu^*_1 \geq \lambda^* \) for \( \mu \leq \mu^*_2 \). Choose \( \mu^*_2 \) such that \( \mu \geq \mu^*_2 \) implies both \( x_{\mu} \in K \) and \( f_{x_{\mu}}(K) \subseteq U \), from which \( f(x_{\mu}) \in U \). This shows \( f_{x_{\mu}}(x_{\mu}) \to f(x) \), since \( U \) was an arbitrary open set containing \( f(x) \).

(b) Assume that \( f_x \) does not converge in the compact-open topology. Assumed, then, is a compact subset \( K \) of \( \mathcal{X} \), and an open \( U \) with \( f(K) \subseteq U \), such that, for all \( \lambda^* \) there is a \( \lambda \geq \lambda^* \) such that \( f_{\lambda}(K) \subseteq U \), implying by Lemma 5.1 that \( \mathcal{X}' = \{\lambda \mid f_{\lambda}(K) \subseteq U\} \) is directed. For each \( \lambda \in \mathcal{X}' \), pick \( x_{\lambda} \in K \) such that \( f_{\lambda}(x_{\lambda}) \notin U \). There is a convergent sub-net \( x_{\mu_{x_{\lambda}}} \) of \( x_{\lambda} \), say \( x_{\mu_{x_{\lambda}}} \to x \). Then \( x \not\in A \), or \( x \in A \) and \( f_{x_{\mu_{x_{\lambda}}}}(x_{\mu_{x_{\lambda}}}) \) does not converge to \( f(x) \), where \( x_{\mu} \equiv x_{\mu_{x_{\lambda}}} \), because \( K \) is closed (since \( \mathcal{X} \) is Hausdorff) implies \( x \in K \), from which \( f(x) \in U \), whereas \( f_{x_{\mu}}(x_{\mu}) \not\in U \).
5 Conclusions

In the numerical analysis of ordinary differential equations, we deal with mesh functions of a real variable, with values in \( \mathbb{R}^n \). For a partial differential equation the domain may be such as an open subset of a manifold. Such domains are almost universally second countable locally compact Hausdorff spaces. And then, with no further assumptions, Theorem 4.2 provides that the natural invariant convergence notion (6) exactly corresponds to the well-studied Fell topology on the space of numerical approximations.

Fell-convergence is essentially equivalent to verifying machine convergence over varying start and end times and with a varying grid. The closed sets of the Fell topology are logically accessible via the convergence criteria; the open sets via the explicit and easily visualized local base provided by Theorem 3.9. The development operates at the topological level — it does not presuppose any smooth differentiable structure.

Appendix: topologies from convergence criteria

A directed set is a set \( \Lambda \) with a directed preorder: a relation \( \geq \) which satisfies: (1) reflexive: \( \lambda \geq \lambda \); (2) transitive: \( \lambda_1 \geq \lambda_2 \) and \( \lambda_2 \geq \lambda_3 \) implies \( \lambda_1 \geq \lambda_3 \); and (3) upper bounds: for all \( \lambda_1 \) and \( \lambda_2 \) there is a \( \lambda^* \) such that \( \lambda^* \geq \lambda_1 \) and \( \lambda^* \geq \lambda_2 \). A property \( P(\lambda) \) is final if there is a \( \lambda^* \in \Lambda \) such that \( P(\lambda) \) is true for all \( \lambda \geq \lambda^* \). A property \( P(\lambda) \) is cofinal if for all \( \lambda^* \in \Lambda \) there is a \( \lambda \in \Lambda \) such that \( P(\lambda) \) is true for \( \lambda \geq \lambda^* \). A set is a relation \( \Lambda \) with a set \( \lambda \in \Lambda \) where \( \lambda \) is map from a directed set \( M \) to \( \Lambda \) which satisfies (1) monotone: \( \mu_1 \geq \mu_2 \) implies \( \lambda_{\mu_1} \geq \lambda_{\mu_2} \); and (2) cofinal: for all \( \lambda \in \Lambda \) there is a \( \mu \in M \) such that \( \lambda_{\mu} \geq \lambda \). \( x \in \mathcal{X} \) is a cluster point of \( x_\lambda \) if for all neighbourhoods \( U \ni x \) there is a \( \lambda^* \) such that \( x_\lambda \in U \) for all \( \lambda > \lambda^* \), i.e., if \( x_\lambda \) is finally in any neighbourhood of \( x \). The set of cluster points of \( x_\lambda \) will be denoted by \( (x_\lambda)^* \).

The textbook [25] disposes of the monotone condition on subnets, but imposes the stronger notion of finality: for all \( \lambda^* \in \Lambda \) there is an \( \mu^* \in M \) such that \( \lambda_{\mu^*} \geq \lambda^* \) whenever \( \mu \geq \mu^* \).

Lemma 5.1. Suppose \( \Lambda \) is directed and \( \Lambda' \subseteq \Lambda \) is cofinal, i.e., for all \( \lambda \in \Lambda \) there is a \( \lambda' \in \Lambda' \) such that \( \lambda' \geq \lambda \). Then \( \Lambda' \) is directed.

Proof. If \( \lambda_1', \lambda_2' \in \Lambda' \) then choose \( \lambda_2 \in \Lambda \) such that \( \lambda_3 \geq \lambda_1' \) and \( \lambda_3 \geq \lambda_2' \). Choose \( \lambda_3' \in \Lambda' \) such that \( \lambda_3' \geq \lambda_3 \). This suffices because, by transitivity, \( \lambda_3' \geq \lambda_1' \) and \( \lambda_3' \geq \lambda_2' \).

Lemma 5.2. (Theorem 11.5 of [28]). A net \( x_\lambda \) has a cluster point \( x \) if and only if it has a subnet that converges to \( x \).

Proof. If \( x \) is a cluster point of \( x_\lambda \) then \( \{(\lambda, U)\} \) such that \( U \) is a neighbourhood of \( x \) and \( x_\lambda \in U \) is directed by \( (\lambda_1, U_1) \leq (\lambda_2, U_2) \Leftrightarrow \lambda_1 \leq \lambda_2 \) and \( U_1 \subseteq U_2 \) and \( (\lambda, U) \rightarrow \lambda \) and defines a subnet that converges to \( x \). Conversely, given a subnet \( x_{\lambda\mu} \), a neighbourhood \( U \ni x \), and a \( \lambda^* \), choose \( \mu \) such that \( \lambda_{\mu} \geq \lambda^* \) and \( x_{\lambda\mu} \in U \).

The standard result regarding convergence and topologies, which in [28] is relegated to an exercise, is Theorem 5.3.

Theorem 5.3. Net convergence on a topological space has the following properties:

(a) the constant net \( x_\lambda = x \) converges to \( x \); and
(b) if \( x_\lambda \) converges to \( x \) then every subnet of \( x_\lambda \) also converges to \( x \); and
(c) if \( x_\lambda \) converges to \( x \) and \( x_{\lambda\mu} \) converges to \( x_\lambda \) for each fixed \( \lambda \) and a lexicographic ordering on the two character words \( \lambda \mu \), then \( x_{\lambda\mu} \) has a subnet which converges to \( x \); and
(d) if every subnet of \( x_\lambda \) has a subnet that converges to \( x \), then \( x_\lambda \) converges to \( x \).

Conversely, given a convergence criteria satisfying (a)–(c), \( \{x_\lambda \mid x_\lambda \in E\} \) is a Kuratowski closure defining a topology (the convergence topology) in which a net that satisfies the convergence criteria also converges in the topology. If the convergence also satisfies (d) then every net which converges in the topology also satisfies the convergence criteria.

Proof. Net convergence on topological spaces does satisfy (a)–(d):

(a) The constant net \( x_\lambda = x \) converges because if \( U \ni x \) is open then any \( \lambda^* \) provides \( x_\lambda \in U \) for all \( \lambda \geq \lambda^* \).
(b) Suppose \( x_{\lambda\mu} \) is a subnet of \( x_\lambda \). If \( U \ni x \) is open then choose \( \lambda^* \) so that \( \lambda \geq \lambda^* \) implies \( x_\lambda \in U \). By the definition of a subnet, there is a \( \mu^* \) such that \( \mu \geq \mu^* \) implies \( \lambda_{\mu} \geq \lambda^* \), so \( \mu \geq \mu^* \) implies \( x_{\lambda\mu} \in U \).
(c) By Lemma 5.2, it suffices to show that \( x \) is a cluster point of \( x_{\lambda\mu} \). Suppose \( U \ni x \) is open, \( \lambda^* \in \Lambda \) and \( \mu^* \in M_{\lambda^*} \). Since \( x_{\lambda} \rightarrow x \), there is a \( \lambda \geq \lambda^* \) such that \( x_{\lambda} \in U \). Since \( x_{\lambda\mu} \rightarrow x_\lambda \in \mu \in M_{\lambda^*} \), and since \( x_\lambda \in U \), there is a \( \mu \in M_{\lambda^*} \) such that \( x_{\lambda\mu} \in U \) (\( \mu^* \) is irrelevant because \( \lambda > \lambda^* \) and the ordering is lexicographic).
(d) If \( x_\lambda \) does not converge to \( x \) then there is an open \( U \ni x \) such that, for all \( \lambda^* \) there is a \( \lambda \geq \lambda^* \) with \( x_\lambda \notin U \).

Then \( M = \{ \mu \in \lambda \mid x_\mu U \} \) is directed: if \( \mu_1, \mu_2 \in M \) then choose \( \lambda^* > \mu_1 \) and \( \lambda^* > \mu_2 \), and then there is a \( \mu > \lambda^* \) such that \( x_\mu U \), so \( \mu \in \lambda \) and \( \mu \geq \lambda^* \geq \mu_1 \) and \( \mu \geq \lambda^* \geq \mu_2 \). Clearly, the restriction of \( x_\lambda \) to \( M \) is a subnet which has no subnet that converges to \( x \).

Assuming (a) and (c), \( E \to cl \) \( E \) is a closure operation, and so provides a topology with closed sets exactly those \( E \) such that \( cl \ E = E \), as follows:

- \( cl \emptyset = \emptyset \); otherwise, after choosing \( x \in cl \emptyset \) there is an \( x_\lambda \to x \) with \( x_\lambda \notin \emptyset \), which is impossible.
- \( E \subseteq cl \ E \): if \( x \in E \) then any constant net is a net in \( E \) converging to \( x \), so \( x \in cl \ E \).
- \( cl cl \ E \subseteq cl \ E \): By the above, \( E \subseteq cl \ E \subseteq cl cl \ E \). On the other hand, suppose \( x_\lambda \to x \) with \( x_\lambda \in cl \ E \) and choose \( x_{\mu} \) such that \( lim_\mu x_{\lambda \mu} \lambda_\mu \in cl \ E \). Then \( x \in cl \ E \) because there is a subnet of \( x_{\lambda \mu} \) in the lexicographic ordering such that \( x_{\lambda \mu} \to x \).
- If \( x \in cl \ A \), then there is \( x_\lambda \to x \) with \( x_\lambda \in A \). Since \( x_\lambda \in A \cup B \) also, this implies \( x_\lambda \in cl(A \cup B) \). Similarly \( cl B \subseteq cl(A \cup B) \) so \( cl A \cup cl B \subseteq cl(A \cup B) \). Conversely, if \( x \in cl(A \cup B) \) then there is a net \( x_\lambda \to x \) with \( x_\lambda \in A \) or \( x_\lambda \in B \). One of \( \{ \lambda \mid x_\lambda \in A \} \) or \( \{ \lambda \mid x_\lambda \in B \} \) is directed, or else there would be \( \lambda^1 \) and \( \lambda^2 \) with \( x_{\lambda^1} \in A \), \( \lambda^2 \in A \) and \( x_{\lambda^2} \notin A \) for all \( \lambda \geq \lambda^1 \) and \( \lambda \geq \lambda^2 \), and similarly with \( A \) and \( B \) exchanged. But then choosing \( \lambda_3 \) such that all of \( \lambda_3 \geq \lambda^1 \), \( \lambda_3 \geq \lambda^2 \) implies \( x_{\lambda^3} \notin A \) and \( x_{\lambda^3} \notin B \), a contradiction. This defines a subnet either in \( A \) or \( B \) which converges to \( x \) (by (c)), so either \( x \in cl A \) or \( x \in cl B \).

Since the closed sets are more directly defined by the convergence topology than open sets, it is best to directly use the following: in a topological space, \( x_\lambda \to x \) is false if and only if there is a closed set \( K \) such that \( x \notin K \) and, for all \( \lambda^* \) there is a \( \lambda \geq \lambda^* \) such that \( x_{\lambda} \notin K \). One then shows that \( x_\lambda \to x \) (convergence topology) is false if and only if \( x_\lambda \to x \) (convergence criterion) is false. Note that, under (a)–(c), the closed sets defined by

\[ K \text{ closed iff } x_\lambda \in K \text{ and } x_\lambda \to x \text{ implies } x \in K \]

are exactly the same as the closed sets defined using the topology, by

\[ K \text{ closed iff } clK = K. \]

Assume (a)–(c), and suppose that \( x_\lambda \not\to x \) (topology), so there is a closed set \( K \) as in the statement just above. Then \( \{ \lambda \mid x_\lambda \notin K \} \) is directed and so is a subnet. If this subnet converges to \( x \) (criteria) then \( x \in K \), a contradiction. So \( x_\lambda \) cannot converge to \( x \) (criteria) because \( x_\lambda \) has a subnet which does not converge to \( x \) (criteria).

Assume (a)–(d), and suppose that \( x_\lambda \not\to x \) (criteria). Then there is a subnet \( x_{\lambda_{\nu}} \) of \( x_\lambda \), every subnet of which does not converge to \( x \) (criteria). So defining \( K = cl \{ x_{\lambda_{\nu}} \} \), it follows that \( x \notin K \). Given any \( \lambda^* \), there is a \( \mu \) such that \( \lambda_{\mu} \geq \lambda^* \), and, setting \( \lambda = \lambda_{\mu} \) this provides a \( \lambda \geq \lambda^* \) such that \( x_\lambda \in K \), from which \( x_\lambda \not\to x \) (convergence topology).

**Lemma 5.4.** Let \( C \) be a subbase for a topology \( \tau \). Then \( x_\lambda \to x \) if and only if, for all \( V \in C \), there is a \( \lambda^* \) such that \( x_\lambda \in V \) for all \( \lambda \geq \lambda^* \).

**Proof.** If \( V \in C \) then \( V \) is open so \( x_\lambda \) is finally in \( V \) by definition of net convergence. For the converse, every open \( V \) is the finite intersection of sets \( V_i \in \in C \). For each \( i \) choose \( \lambda_i^* \) such that \( x_\lambda \in V_i \) for all \( \lambda \geq \lambda_i^* \). The upper bound property of directed sets and transitivity provided an upper bound \( \lambda^* \) for all \( \lambda_i^* \). If \( \lambda \geq \lambda^* \) then \( x_\lambda \in V_i \) for all \( i \) and hence \( x_\lambda \in V \).

**Definition 5.5.** Let \( X \) be a set suppose \( f_\alpha : X \to Y_\alpha \) are maps, where \( Y_\alpha \) are topological spaces. The **weak topology defined by \( f_\alpha \)** is the topology with subbase \( \bigcup_\alpha \{ f_\alpha^{-1}(V) \mid V \subseteq Y_\alpha \text{ open} \} \).

**Proposition 3.** A net \( x_\lambda \) converges to \( x \) in the weak topology defined by \( f_\alpha : X \to Y_\alpha \) if and only if \( f_\alpha(x_\lambda) \to f(x) \) for all \( \alpha \).

**Proof.** By Lemma 5.4, \( x_\lambda \to x \) if and only if for all \( V \subseteq Y_\alpha \) there is a \( \lambda^* \) such that \( \lambda \geq \lambda^* \) implies \( x_\lambda \in f_\alpha^{-1}(V) \), and the latter is equivalent to \( f_\alpha(x_\lambda) \in V \).

There are a variety of well-known function-space topologies where a central aim seems to be the capture of some particular notion of convergence, for which the primary definition of the topology does not have an immediately obvious relationship. The topology for distributional test function spaces, the weak and Whitney fine topologies on differentiable maps between manifolds, and even the familiar compact-open topology are examples.
Theorem 5.3 suggests that, to define a topology from a convergence criteria, one should verify all of Theorem 5.3(a–d). In case of success the result is topological: not only is the topology obtained, but also a complete characterization of all convergent nets. If the aim is just to define a topology, then (b) alone suffices: define $A \subset X$ closed if $x_\lambda \in A$ and $x_\lambda \to x$ implies $x \in A$. Within this definition, it is obvious then $\emptyset$ and $X$ are closed, and that arbitrary intersections and closed sets are closed. If $A$ and $B$ are closed, $x_\lambda \in A \cup B$, and $x_\lambda \to x$, then $x_\lambda \in A$ or $x_\lambda \in B$ (depending on $\lambda$).

One of $\{ \lambda \mid x_\lambda \in A \}$ or $\{ \lambda \mid x_\lambda \in B \}$ is directed, or else there would be $\lambda_1^A$ and $\lambda_2^A$ with $x_{\lambda_1^A} \in A, \lambda_2^A \in A$ and $x_\lambda \not\in A$ for all $\lambda \geq \lambda_1^A$ and $\lambda \geq \lambda_2^A$, and analogously with $B$. But then choosing $\lambda_3$ such that all of $\lambda_3 \geq \lambda_1^A, \lambda_2^A, \lambda_3^B, \lambda_2^B$ implies $x_\lambda \not\in A$ and $x_\lambda \not\in B$, a contradiction. Thus there is a subnet either in $A$ or $B$, and that converges to $x$ by Theorem 5.3(b), so either $x \in A$ or $x \in B$ as both those are closed. If a convergence criteria $\gamma$ satisfies (b) then the topology just described will be the topology $\gamma^\top \gamma \gamma$ generated by $\gamma$, or just the convergence topology. If such a $\gamma$ also satisfies (a) then it is $T_1$, because every one point set closed.

In such a topology, there are (at least) two notions of convergence: the given one and net convergence in the topology itself. The first will be referred to as generating, while the second will be referred to as topological. If $x_\lambda \to x$ (generating) then $x_\lambda \to x$ (topological): Suppose $x_\lambda \to x$ (generating) and $x_\lambda \not\to x$ (topological). Then there is an open neighbourhood $U \ni x$ such that, for all $\lambda^*$ there is a $\lambda \geq \lambda^*$ such that $x_\lambda \not\in U$. It follows that $\Lambda = \{ \lambda \mid x_\lambda \not\in U \}$ is directed and that $x_\lambda$ with $\lambda \in \Lambda$ is a subnet of $x_\lambda$ which has no element in $U$. However by Theorem 5.3(b) this subnet converges to $x$ (generating), and $X \setminus U$ is closed, so $x \in X \setminus U$, contradicting $x \in U$.

Given a convergence criteria, the convergence topology is the finest topology such that the nets in the convergence criteria converge in the topology: Suppose $\tau$ is such a topology and $A$ is closed in $\tau$, and $x_\lambda \to x$ (generating) with $x_\lambda \in A$. Then $x_\lambda \to x$ (in $\tau$) so $x \in A$, and hence $A$ is closed in the convergence topology.

The point of all this is to facilitate the simple and transparent identification of topologies from convergence criteria: the topology $\gamma^\top \gamma \gamma$ is to be thought of as the most exact carrier of the convergence criteria $\gamma$. There would not be that much gained from that alone. Interestingly, this set-up is operationally effective, because the principle topological notions are actually captured in the usual way by the generating convergence.

**Theorem 5.6.** Let $x_\lambda \to x$ be a pre-topological convergence criteria.

1. $A \subset X$ is closed if and only if $x_\lambda \in A$ and $x_\lambda \to x$ (generating) implies $x \in A$.
2. $U \subset X$ is open if and only if, for all $x_\lambda \to x$ (generating) such that $x \in U$, there is an $\lambda^*$ such that $x_\lambda \in U$ whenever $\lambda \geq \lambda^*$.
3. $f : X \to Y$ is continuous if and only if $\lim x_\lambda = f(x)$ for all nets $x_\lambda$ such that $x_\lambda \to x$ (generating).

**Proof.** The first statement is by definition of the closed sets in the convergence topology. For the next two statements, it suffices to show the converse, i.e., it suffices to show that generating convergence suffices. For the second statement, if $x \in U$ implies that every net $x_\lambda \to x$ (generating) is eventually in $U$, then $x_\lambda \in X \setminus U$ and $x_\lambda \to x$ (generating) implies $x \in X \setminus U$ and hence $X \setminus U$ is closed, or else $x_\lambda$ is both in $X$ and $X \setminus U$ for large enough $\lambda$. For the third statement, suppose that $\lim x_\lambda = f(x)$ for all nets $x_\lambda \to x$ (generating). If $B \subset Y$ is closed and $x_\lambda \in f^{-1}(B)$ with $x_\lambda \to x$ (generating), then $f(x) = f(\lim x_\lambda) = \lim f(x_\lambda) \in B$ so $x \in f^{-1}(B)$. This shows that $f$ if continuous because $f^{-1}(B)$ is closed whenever $B$ is.

Incidentally, Theorem 5.6 explains why sequences suffice for continuity of linear maps on the test function spaces of distribution theory ([24], Theorem 6.6). The restriction to linear maps arises from the issue of local convexity referred to at the beginning: sequences do not suffice in the general for that topology.

One has to exercise care, because of the loose relationship between a convergence criteria and the topology it generates. For example, an element of $X$ may be approximable from $A \subset X$ using nets in the convergence topology, but inaccessible from $A$ via the generating convergence. It would be an easy error to assert the existence of a net $x_\lambda$ such that $x_\lambda \to x$ (generating) from the statement $x \in cl A$. In fact, there is the following: let $X = \mathbb{R}$ and use the convergence criteria $x_\lambda \to x$ if $|x - x_\lambda| \leq 1$ for all $\lambda$. If $A = [0, 1]$ then $pcl A = [-1, 2]$ while $cl A = \mathbb{R}$, where $pcl A$, or the pre-closure, denotes the limits of all convergent nets in $A$. The pre-closure is not necessarily a closed set because it only contains those limit but not necessarily limits of those limits.

If $X$ and $Y$ are topological spaces generated by convergence criteria, then the topology generated by the product criteria $(x_\lambda, y_\lambda) \to (x, y)$ if $x_\lambda \to x$ (generating) and $y_\lambda \to y$ (generating) may be strictly finer than the product topology, which is after all the coarsest topology with continuous projections to the factors. Indeed, the criteria $x_\lambda \to x$ if $|x_\lambda - x| \to 0$ through powers to $1/2$, generates a topology on $\mathbb{R}$ strictly finer than the usual. In the product topology of two copies of such, an open line segment with irrational slope does not contain any nonconstant net in the product criteria hence is closed in the topology so generated. However, open intervals are open in the factor topologies, and their product is open in the product topology. Therefore such irrational sloped open segments are not closed in the product topology because they do not contain the endpoints which are in their closure; the
product topology is strictly more coarse then the topology generated by the product criteria. This has an important operational consequence: to show that a bivariate function \( f(x, y) \) is continuous in the product topology, it is generally insufficient to show that \( f(x_\lambda, y_\lambda) \rightarrow f(x, y) \) whenever \( x_\lambda \rightarrow y \) (generating) and \( y_\lambda \rightarrow y \) (generating).

**Definition 5.7.** A convergence criteria \( \gamma \) is topological if every convergent net in \( \gamma \) satisfies \( \gamma \).

Some primitive notations are useful. Logical operations will be extended to the convergence criteria with the obvious meaning: for example, \( \gamma_1 \vee \gamma_2(x_\lambda \rightarrow x) \equiv \gamma_1(x_\lambda \rightarrow x) \vee \gamma_2(x_\lambda \rightarrow x) \), i.e., the logical “and” of \( \gamma_1 \) and \( \gamma_2 \).

**Theorem 5.8.** Suppose \( \gamma_1 \) and \( \gamma_2 \) are convergence criteria.

(a) If \( \gamma_1 \Rightarrow \gamma_2 \) then \( \gamma_1 \supseteq \gamma_2 \) (relaxed convergence criteria generate finer topologies). The convergence criteria defines as all nets converge [only the constant nets converge] generates the discrete [indiscrete] topology.

(b) \( (\gamma_1 \wedge \gamma_2) \gamma_1 \wedge \gamma_2 \), (the logical “or” of two criteria generates the intersection of the topologies generated by the criteria separately).

(c) \( (\gamma_1 \wedge \gamma_2) \gamma_1 \wedge \gamma_2 \), (the logical “and” of two criteria generates a topology finer than the join of the topologies generated by the criteria separately). If \( \gamma_1 \) and \( \gamma_2 \) are both topological then \( (\gamma_1 \wedge \gamma_2) \gamma_1 \wedge \gamma_2 \).

**Proof.** (a) Set \( \tau_i \equiv \gamma_i, i = 1, 2 \). Suppose \( E \) is closed in \( \tau_2 \). If \( \gamma_1(x_\lambda \rightarrow x) \) is true with \( x_\lambda \in E \) then \( \gamma_2(x_\lambda \rightarrow x) \) is true, from which \( x \in E \). Thus \( E \) is closed in \( \tau_1 \), so \( \tau_2 \subseteq \tau_1 \), i.e., \( \gamma_1 \supseteq \gamma_2 \). The last two statements follow from this, or directly: If every net converges then the constant net of any point in any set converges to any point not in that set, from which the only closed sets are the empty set and the whole space, and the topology generated is indiscrete. If no net converges then the condition that a set be closed is vacuously true for any set, and the topology generated is indiscrete.

(b) Suppose \( E \) is closed in \( \tau_1 \wedge \tau_2 \). Since \( \tau_1 \wedge \tau_2 \subseteq \tau_1 \wedge \tau_2 \wedge \tau_2 \), \( E \) is closed in both \( \tau_1 \) and \( \tau_2 \). So if \( \gamma_1 \wedge \gamma_2(x_\lambda \rightarrow x) \) is true then one of \( \gamma_1(x_\lambda \rightarrow x) \) or \( \gamma_2(x_\lambda \rightarrow x) \) is, and \( x \in E \) in either case. This shows \( E \) is closed in \( (\gamma_1 \wedge \gamma_2) \gamma_1 \wedge \gamma_2 \) and hence that \( \tau_1 \wedge \tau_2 \subseteq (\gamma_1 \wedge \gamma_2) \gamma_1 \wedge \gamma_2 \). Conversely, \( \gamma_1 \Rightarrow \gamma_1 \wedge \gamma_2 \) and \( \gamma_2 \Rightarrow \gamma_1 \wedge \gamma_2 \) from which \( \tau_1 \supseteq (\gamma_1 \wedge \gamma_2) \gamma_1 \wedge \gamma_2 \) and \( \tau_2 \supseteq (\gamma_1 \wedge \gamma_2) \gamma_1 \wedge \gamma_2 \).

(c) \( \gamma_1 \wedge \gamma_2 \Rightarrow \gamma_1 \) and \( \gamma_1 \wedge \gamma_2 \Rightarrow \gamma_2 \) then \( (\gamma_1 \wedge \gamma_2) \gamma_1 \wedge \gamma_2 \) contains both \( \tau_1 \) and \( \tau_2 \), so \( \tau_1 \wedge \tau_2 \subseteq (\gamma_1 \wedge \gamma_2) \gamma_1 \wedge \gamma_2 \). For the second part, let \( \hat{\gamma} \gamma_1, \hat{\gamma} \gamma_2 \) be the convergence criteria defined by the topologies \( \tau_1 \wedge \tau_2 \), \( \tau_1 \), and \( \tau_2 \), respectively. The join topology \( \tau_1 \wedge \tau_2 \) contains both \( \tau_1 \) and \( \tau_2 \), so \( \hat{\gamma} \Rightarrow \hat{\gamma}_1 \) and \( \hat{\gamma} \Rightarrow \hat{\gamma}_2 \). Since \( \hat{\gamma}_1 = \gamma_1 \) and \( \hat{\gamma}_2 = \gamma_2 \), it follows that \( \hat{\gamma} \Rightarrow \gamma_1 \wedge \gamma_2 \), so \( \gamma_1 \wedge \gamma_2 \gamma_1 \wedge \gamma_2 \gamma_1 \wedge \gamma_2 \).

With respect to (c), \( (\gamma_1 \wedge \gamma_2) \gamma_1 \wedge \gamma_2 \) is possible: consider the three point set \( \{1, 2, 3\} \) with the topology \( \tau_1 = \{\emptyset, 1, 12, 123\} \). This is one of the known 29 topologies on three point sets. Let \( \gamma_1 \) be the convergence criteria defined as every constant net converges to its value, the constant net \( 1 \) also converges to \( 2 \), and the constant net \( 2 \) also converges to \( 3 \). As is easily verified (suppress the set braces)

\[
\begin{align*}
pcl_{\tau_1}(\emptyset) &= \emptyset, & pcl_{\tau_1}(1) &= 12, & pcl_{\tau_1}(2) &= 23, & pcl_{\tau_1}(3) &= 3, \\
pcl_{\tau_1}(23) &= 23, & pcl_{\tau_1}(12) &= pcl_{\tau_1}(13) &= pcl_{\tau_1}(123) &= 123,
\end{align*}
\]

so that the closed sets are \( \emptyset, 3, 23, 123 \), and the open sets, being complements of closed sets, are \( 123, 12, 1, \emptyset \), i.e., \( \gamma_1 \) generates \( \tau_1 \). In the topology \( \tau_1 \), the only open set containing \( 3 \) is \( 123 \), so every net converges to \( 3 \), and in particular, the constant net \( 1 \) converges to \( 3 \). So \( \gamma_1 \) is not topological since that net does not converge by \( \gamma_1 \). Similarly, the convergence criteria \( \gamma_2 \) defined as every constant net converging to its value, the constant net \( 1 \) converges to \( 3 \), and the constant net \( 3 \) converges to \( 2 \) generates the topology \( \gamma_2 = 123, 13, 1, \emptyset \). The join topology \( \tau_1 \wedge \tau_2 \) is \( \tau_1 \cup \tau_2 = \emptyset, 1, 12, 13, 123 \) but in \( \gamma_1 \wedge \gamma_2 \) only the constant nets converge to themselves and \( (\gamma_1 \wedge \gamma_2) \gamma_1 \wedge \gamma_2 \) is the discrete topology.

Recall the familiar inductive limit topology on subsets: suppose \( \gamma \) is a directed set, \( X_i \subseteq X_j \) and \( X_i \) are topological spaces such that \( X_i \subseteq X_j \) with continuous inclusion whenever \( i \leq j \). The inductive limit topology is the finest topology such that every inclusion \( X_i \rightarrow X \) is continuous. A subset \( U \) is open in the inductive limit topology if and only if \( U \cap X_i \) is open for all \( i \). Consider the convergence criteria: \( x_\alpha \rightarrow x \) if there are an \( \alpha^* \) and \( i \) such that \( x_\alpha \in X_i \) whenever \( \alpha \geq \alpha^* \), and \( x_\alpha \rightarrow x \) in the topology of \( x_i \). Call this criteria eventual membership convergence. The inductive limit topology is especially convenient with respect to convergence:

**Theorem 5.9.** Eventual membership convergence generates the inductive limit topology.

**Proof.** There is a general approach for such results: to show a convergence topology is equal to another given topology, show first that the convergence criteria converges in the topology, so that Theorem 5.8(a) implies the convergence
topology is finer. In the case that the given topology is the finest satisfying some condition, then showing that the convergence topology satisfies that same condition completes the proof. So, suppose that \(x_\alpha \to x\) (eventual membership), and choose \(\alpha^*\) and \(i\) as in the definition. Then the inclusion of \(X_i \to X\) is continuous in the inductive limit topology, so \(x_i\) converges in in that. Conversely, pick any \(i\), suppose \(x_\lambda \in X_i\) and \(x_\lambda \to x\) in the topology of \(X_i\), and let \(i\) : \(X_i \to X\) be the inclusion. Then \(i(x_\lambda) = x_\lambda\) and \(x_\lambda\) satisfies the eventual membership convergence criteria, so \(i(x_\lambda)\) converges in the topology generated by that. Thus \(i\) is continuous by Theorem 5.6, but the inductive limit topology is the finest in which each such inclusion is continuous.

Suppose \(X\) is a topological space, \(\pi: X \to Y\) is onto, and define the convergence criteria \(\gamma\) by \(y_\lambda \to y\) if there is a net \(x_\lambda\) such that \(\pi(x_\lambda) = y_\lambda\), \(x_\lambda\) converges to some \(x\), and \(\pi(x) = y\). Let \(\tau\) be the quotient topology on \(Y\). Then \(\pi\) is continuous in the quotient topology on \(Y\), so \(\gamma\)-convergence implies \(\tau\)-convergence and \(\gamma \supseteq \tau\). The quotient topology \(\tau\) is the finest topology such that \(\gamma\) is continuous, and, if \(x_\lambda \to x\) in \(X\) then \(\gamma\) is true for \(\pi(x_\lambda) \to \pi(x)\) and so \(\pi\) is continuous in \(\gamma\). Hence \(\gamma\) generates the quotient topology. Let \(X = \{(x, 1/x) \mid x > 0\} \cup \{(0) \times \mathbb{R}\}\), i.e., the union of the graph of \(y = 1/x\) and the \(y\)-axis, with the subspace topology from \(\mathbb{R}^2\). Define \(Y = [0, \infty)\), with the usual topology, and \(\pi: X \to Y\) by \(\pi(x, y) = x\). Then \(\pi\) is a quotient map and \(1/n \to 0\) in \(Y\) but there is no convergent \((x_n, y_n) \in X\) and a \(y\) such that \((x_n, y_n) \to (0, y)\). Thus there are convergent nets in \(X\) that are not convergent in the \(\gamma\) topology.

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