HOW HEAVY INDEPENDENT SETS HELP TO FIND ARBORESCENCES WITH MANY LEAVES IN DAGS

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Abstract. Trees with many leaves have applications on broadcasting, which is a method in networks for transferring a message to all recipients simultaneously. Internal nodes of a broadcasting tree require more expensive technology, because they have to forward the messages received. We address a problem that captures the main goal, which is to find spanning trees with few internal nodes in a given network. The Maximum Leaf Spanning Arborescence problem consists of, given a directed graph $D$, finding a spanning arborescence of $D$, if one exists, with the maximum number of leaves. This problem is known to be NP-hard in general and MaxSNP-hard on the class of rooted directed acyclic graphs. In this paper, we explore a relation between Maximum Leaf Spanning Arborescence in rooted directed acyclic graphs and maximum weight set packing. The latter problem is related to independent sets on particular classes of intersection graphs. Exploiting this relation, we derive a $7/5$-approximation for Maximum Leaf Spanning Arborescence on rooted directed acyclic graphs, improving on the previous $3/2$-approximation. The approach used might lead to improvements on the best approximation ratios for the weighted $k$-set packing problem.

1. INTRODUCTION

Broadcasting is a term used to describe the process of sending a message on a network from a root node to all other nodes of the network. A network is modeled as a directed graph in which we broadcast messages through a minimal subset of the arcs of the network. These subsets consist of what we call a spanning arborescence in the network. The internal nodes of an arborescence receive a message through an arc, and must duplicate the message and distribute its copies through the outgoing arcs of the arborescence. Thus, the internal nodes must be equipped with routers and switches, while leaves of the arborescence need only to work as message receptors. This situation motivates the search for broadcasting arborescences in networks with fewer internal nodes and more leaves [12] [15]. In what follows, we formalize this problem.

Let $D$ be a directed graph (digraph for short). A node $r$ in $D$ is a root if there is a directed path in $D$ from $r$ to every node in $D$. If $r$ is a root in $D$, then we say $D$ is $r$-rooted, or simply rooted. We say $D$ is acyclic if there is no directed cycle in $D$. For short, a directed acyclic graph is called a dag. Any rooted dag has only one root. An arborescence is an $r$-rooted dag $T$ for which there is a unique directed path from $r$ to every node in $T$. The out-degree of a node in a

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digraph is the number of arcs that start in that node, while the in-degree of a node is the number of arcs that end in that node. A node of out-degree 0 in an arborescence is called a leaf. Note that the underlying graph of an arborescence is a tree.

In the Maximum Leaf Spanning Arborescence problem, one is given a rooted digraph $D$, and the goal is to find a spanning arborescence of $D$ with the maximum number of leaves. This problem is known to be NP-hard. Indeed, its undirected version is listed as the NP-hard problem ND2 in the renowned book by Garey and Johnson [11], and one can easily reduce ND2 to Maximum Leaf Spanning Arborescence.

Maximum Leaf Spanning Arborescence was considered from the viewpoint of fixed parameter tractability [1, 4, 6, 7]. Regarding approximation algorithms, which are our interest in this paper, there exists a 92-approximation for general rooted digraphs. It came from Daligault and Thomassé’s analysis on the parameterized complexity of the problem [7]. Their work includes the proof that their algorithm has ratio 24 if the digraph has no directed cycles of length 2.

The case in which the given digraph is a rooted dag was considered in the literature [1, 10, 13, 16]. Specifically, this case was shown to be MaxSNP-hard [16] and the best known result for it is a 3/2-approximation whose analysis is tight [10]. Thus, improvements require a different algorithm. In [10], an alternative algorithm was proposed for Maximum Leaf Spanning Arborescence on rooted dags. It uses as a subroutine an approximation for maximum weight 3D-matching, whose currently best known factor is unfortunately not good enough to achieve an improvement for Maximum Leaf Spanning Arborescence on rooted dags.

The best approximations for maximum weight 3D-matchings and for maximum weight $k$-set packing in general come from the best approximations for the maximum weight independent set on 4-claw free and $(k + 1)$-claw free graphs respectively [3, 5, 14]. In this paper we explore the idea of using the latter approximations to obtain better algorithms for Maximum Leaf Spanning Arborescence on rooted dags. Concretely, we introduce a particular class of 4-claw free graphs that we call $\{2, 3\}$-intersection graphs, and we design a 7/5-approximation for the maximum weight independent set on the class of $\{2, 3\}$-intersection graphs with vertex weights defined in a particular way. The new approximation is a tuned version of Berman’s approximation for the maximum weight independent set on 4-claw free graphs [3], inspired also on ideas from the algorithm for Maximum Leaf Spanning Arborescence in [10]. From this, we derive a 7/5-approximation for Maximum Leaf Spanning Arborescence on rooted dags.

Berman’s algorithm for maximum weight independent set consists of a local improvement algorithm, as many of the algorithms for maximum (weight) independent set. Our tuned algorithm relies on restricted weights, since we are interested in a particular class of weight functions, and first modifies the criterion to decide on whether or not to apply a local improvement. Then it optimizes some particular types of improvements using a maximum matching algorithm. The way we modified the decision on when to apply an improvement step might lead to a new algorithm for maximum weight set packing, if not for general weights, maybe for particular classes of weight functions that, as the ones we used, might be of interest for other problems.
Section 2 presents the maximum weight independent set problem (wMIS) and summarizes the results known for wMIS on d-claw free graphs. In particular, we revise the algorithm of Berman known as SQUAREIMP for wMIS. Section 3 discusses the relation between Maximum Leaf Spanning Arborescence on rooted dags and wMIS on d-claw free graphs and, as an intermediate step, presents a new algorithm for Maximum Leaf Spanning Arborescence on rooted dags that uses SQUAREIMP as a subroutine. Section 4 analyzes SQUAREIMP, showing that it is a $3/2$-approximation on a subclass of weighted 4-claw free graphs that contains the weighted graphs used by the new algorithm. Section 5 concludes that the new algorithm is a $3/2$-approximation for Maximum Leaf Spanning Arborescence on rooted dags. In Section 6, we present our main result: a $7/5$-approximation for Maximum Leaf Spanning Arborescence on rooted dags that uses ingredients from the $3/2$-approximation in [10] to tune the $3/2$-approximation from Section 3. The new approximation relies on a new $7/5$-approximation on the subclass of weighted 4-claw free graphs considered in Section 4. Further directions are discussed in Section 7.

2. Independent sets in d-claw free graphs

Let $G$ be an undirected graph. A set $I$ of vertices of $G$ is independent if the vertices in $I$ are pairwise non-adjacent in $G$. A weighted graph is a pair $(G, w)$ where $G$ is an undirected graph and $w$ is a function that assigns to each vertex $v$ of $G$ a positive weight $w_v$.

The Maximum Weight Independent Set (wMIS) problem consists of, given a weighted graph $(G, w)$, finding an independent set $S$ in $G$ that maximizes $w(S)$, which is the sum of $w_v$ for all $v$ in $S$. In general, wMIS is quite hard to approximate, being Poly-APX-complete [2]. But we will consider wMIS on a well-known manageable class of graphs.

An induced subgraph $C$ of $G$ is a d-claw if it consists of an independent set $T_C$ of $d$ vertices, and a center vertex that is adjacent to all $d$ vertices in $T_C$. For convenience, a singleton is said to be a 1-claw $C$ with its unique vertex in $T_C$ and no center. Several problems were studied for the class of d-claw free graphs, which are those in which no induced subgraph is a $d$-claw. The reason is that, in many applications, d-claw free graphs arise naturally. Indeed, we will see in the next section how this class plays a role on Maximum Leaf Spanning Arborescence in rooted dags. For convenience, we will use the term claw to refer to a d-claw for an arbitrary $d$.

Berman [3] presented an algorithm for wMIS on weighted d-claw free graphs with an approximation ratio of $d/2$, enhancing on the previous work of Chandra and Halldórsson [5]. Recently Neuwohner [14] presented a variation of Berman’s algorithm, showing that its ratio is slightly less than $d/2$ on weighted d-claw free graphs. This is the currently best approximation for wMIS on weighted d-claw free graphs. Our algorithm is also based on Berman’s. In what follows, we introduce concepts that are required for the description of his algorithm and our results.

For a graph $G = (V, E)$ and a set $S \subseteq V$, we denote by $N(S)$ the neighborhood of $S$ in $G$, that is, $N(S) := \{ u \in V : vu \in E \text{ for some } v \in S \}$. We use $N(u)$ to denote $N(\{u\})$. If $A$ is an independent set and $C$ is a claw in $G$, then $(A \cup T_C) \setminus N(T_C)$ is also an independent set. So claws can be used in a greedy way to obtain a larger or heavier independent set. For a
A weighted graph \((G, w)\) and a set \(S\) of vertices of \(G\), we denote by \(w^2(S)\) the sum of the squares \(w^2_v\) for all \(v\) in \(S\). For an independent set \(A\) in \(G\), we say that a claw \(C\) improves \(w^2(A)\) if \(w^2(A) < w^2((A \cup T_C) \setminus N(T_C))\). If \(T_C = \{u\}\), we say simply that \(u\) improves \(w^2(A)\).

Algorithm \([1]\) called SQUAREIMP, was proposed by Berman for wMIS on weighted \(d\)-claw free graphs. This algorithm however might not run in polynomial time, but Berman used the strategy of Chandra and Halldórsson \([5]\) to obtain a polynomial version with a slightly increase in the approximation ratio. In particular, on weighted 4-claw free graphs, this leads to a ratio slightly more than 2.

**Algorithm 1** SQUAREIMP\((G, w)\)

**Input:** weighted graph \((G, w)\)

**Output:** an independent set in \(G\)

1. \(A \leftarrow \emptyset\)
2. \textbf{while} there is a claw \(C\) in \(G\) such that \(T_C\) improves \(w^2(A)\) \textbf{do}
   3. \(A \leftarrow (A \cup T_C) \setminus N(T_C)\)
3. \textbf{return} \(A\)

Algorithm SQUAREIMP has a better approximation ratio than its predecessor that uses improvements based on the sum of the weights \([5]\), instead of the sum of the squared weights.

We observe that, since the weights are positive and a singleton is a 1-claw by definition, the output of SQUAREIMP is always a maximal independent set. Recall also that we use claw to refer to a \(d\)-claw for an arbitrary \(d\).

3. Relation between wMIS and Maximum Leaf Spanning Arborescence

We have described in \([10]\) an algorithm for Maximum Leaf Spanning Arborescence on rooted dags called MAXLEAVES-W3DM. It uses as a black box an approximation for maximum weight 3D-matching (hence the W3DM acronym). Specifically, it builds, in one of its steps, an instance of the maximum weight 3D-matching, applies an approximation for maximum weight 3D-matching to this instance, and uses its output to extend the solution being built for Maximum Leaf Spanning Arborescence.

The best approximation for maximum weight 3D-matching comes from a reduction to wMIS on weighted 4-claw free graphs. Indeed, given an instance of maximum weight 3D-matching, which consists of a collection \(S\) of 3-sets, each with a positive weight, one can build an instance of wMIS using the intersection graph for the collection \(S\). Recall that the intersection graph of a collection of sets consists of the graph with one vertex for each set in the collection, and two vertices are adjacent if the corresponding sets intersect. For a collection \(S\) of 3-sets, the intersection graph is 4-claw free, because any set in \(S\) can intersect at most three disjoint sets. Also, in this graph, an independent set corresponds to a 3D-matching in \(S\).

The idea we will explore is to build directly an instance of wMIS and to apply an approximation for wMIS on this instance. An adapted version of a result from \([10]\) allows us to deduce that
the modified version of MAXLEAVES-W3DM that uses an approximation for wMIS instead of an approximation for maximum weight 3D-matching preserves the ratio from the approximation for wMIS, up to 4/3.

We start by translating the construction in MAXLEAVES-W3DM to wMIS instances. We will prove that, on these instances, SQUAREIMP runs in polynomial time and achieves a ratio of 3/2. Then we will deduce that the modified version of MAXLEAVES-W3DM that uses SQUAREIMP, instead of an approximation for maximum weight 3D-matching, achieves a ratio of 3/2 for MAXIMUM LEAF SPANNING ARBORESCENCE on rooted dags. This ratio matches the best known for rooted dags, given by another algorithm presented in [10].

The instances of wMIS in the translated construction use only weights 1 and 2, hence we call the resulting algorithm MAXLEAVES-12MIS. To describe the algorithm in details, we reproduce some definitions and figures from [10].

A branching in a directed graph is a collection of disjoint arborescences. Note that the underlying graph of a branching is a forest. A node that is not a leaf in a branching is called internal. For a positive integer \( t \), a \( t \)-branching is a branching for which every internal node has out-degree at least \( t \). See Figure 1.

**Figure 1.** The bold arcs show a 2-branching and a 3-branching in a same rooted dag.

Algorithm 2 describes a greedy procedure, which we call GREEDYEXPAND, that is used in our new algorithm for MAXIMUM LEAF SPANNING ARBORESCENCE. Each set \( A_v \) in Line 3 represents what we call an expansion that can be applied to the arborescence that is being built. Formally, an expansion is a set of vertices with in-degree zero in the current arborescence and that are out-neighbors of a vertex \( v \) that has out-degree zero in the current arborescence. If \( |A_v| = t \), then we call it a \( t \)-expansion. We refer to a 1-expansion as a trivial expansion.

Then algorithm MAXLEAVES-12MIS is presented in Algorithm 3 and it uses a procedure INTERSECTIONGRAPH\((V, U)\) that receives a set \( V \) and a set \( U_v \) for each \( v \) in \( V \), and returns the intersection graph for the collection \( U \) of sets. That is, the graph \( G \) whose vertex set is \( V \) and two vertices \( x \) and \( y \) are adjacent if and only if \( U_x \cap U_y \neq \emptyset \). For the analysis, we will consider \( G \) as the multigraph having \( |U_x \cap U_y| \) parallel edges between \( x \) and \( y \).

MAXLEAVES-12MIS starts by calling GREEDYEXPAND\((D, 4, F_0)\) with the empty spanning branching \( F_0 \), which outputs a maximal 4-branching \( F_1 \). Then, it considers the collection of
Algorithm 2 \textsc{GreedyExpand}(D, t, F)

\textbf{Input:} rooted dag $D$, a positive integer $t$, and a spanning $(t+1)$-branching $F$ of $D$

\textbf{Output:} a maximal spanning $t$-branching of $D$ containing $F$

1: $F' \leftarrow F$
2: \textbf{for} each $v \in V(D)$ such that $d_{F'}^+(v) = 0$ \textbf{do}
3: \hspace{1em} $A_v \leftarrow \{vu \in A(D) : d_{F'}^-(u) = 0\}$
4: \hspace{1em} \textbf{if} $|A_v| \geq t$ \textbf{then}
5: \hspace{2em} $F' \leftarrow F' + A_v$
6: \textbf{return} $F'$

Algorithm 3 \textsc{MaxLeaves-12MIS}(D)

\textbf{Input:} rooted acyclic directed graph $D$

\textbf{Output:} spanning arborescence of $D$

1: let $F_0$ be the spanning branching with no arcs
2: $F_1 \leftarrow \textsc{GreedyExpand}(D, 4, F_0)$
3: \textbf{for} each $v \in V(D)$ such that $d_{F_1}^+(v) = 0$ \textbf{do}
4: \hspace{1em} $U_v \leftarrow \{u \in V(D) : d_{F_1}^-(u) = 0$ and $vu \in A(D)\}$
5: \hspace{1em} $\text{Candidates} \leftarrow \{v \in V(D) : d_{F_1}^+(v) = 0 \text{ and } 2 \leq |U_v| \leq 3\}$
6: \textbf{for} each $v \in \text{Candidates}$ such that $|U_v| = 3$ \textbf{do}
7: \hspace{1em} \textbf{for} each $u \in U_v$ \textbf{do}
8: \hspace{2em} add to \text{Candidates} an element $v_u$
9: \hspace{2em} $U_{v_u} \leftarrow U_v \setminus \{u\}$
\hspace{1em} $\triangleright$ \text{Candidates} correspond to all non-trivial expansions that can be applied to $F_1$.
10: $G \leftarrow \textsc{IntersectionGraph}(\text{Candidates}, U)$
11: \textbf{for} each $v \in \text{Candidates}$ \textbf{do}
12: \hspace{1em} $w_v \leftarrow |U_v| - 1$
13: $I \leftarrow \textsc{SquareImp}(G, w)$
14: $F_2 \leftarrow F_1$
15: \textbf{for} each $v \in I$ \textbf{do}
16: \hspace{1em} $F_2 \leftarrow F_2 + \{vu : u \in U_v\}$
17: $T \leftarrow \textsc{GreedyExpand}(D, 1, F_2)$
18: \textbf{return} $T$

non-trivial expansions that can be applied to $F_1$. Because $F_1$ is a maximal 4-branching, this collection contains only 2-expansions and 3-expansions. Using \textsc{IntersectionGraph}, algorithm \textsc{MaxLeaves-12MIS} builds the graph $G$ whose vertex set is this collection of expansions, with two expansions adjacent if they are not compatible (that is, if both were applied to $F_1$, the result would not be a branching). It defines a weight function $w$ by assigning weight 1 to 2-expansions
and weight 2 to 3-expansions. See Figure 2. The algorithm then applies SQUAREIMP to the wMIS instance \((G, w)\) obtaining an independent set in \(G\). This independent set corresponds to a collection of compatible 2-expansions and 3-expansions that can be applied to \(F_1\), resulting in a 2-branching \(F_2\). MAXLEAVES-12MIS finishes by calling GREEDYEXPAND\((D, 1, F_2)\) to obtain a spanning arborescence in \(D\).

In our previous result for Maximum Leaf Spanning Arborescence [10], we used the collection \(S = \{U_v: v \in \text{Candidates}\}\), with the same weight function \(w\), as an instance of the maximum weight 3D-matching problem.

![Figure 2](image-url)

**Figure 2.** Illustration of a possible execution of algorithm MAXLEAVES-12MIS. (A) A 4-branching \(F_1\) in bold, obtained in Line 2, and the corresponding set \(\text{Candidates} = \{a, c, e, f, g, h, k, m, n, s, x, z\}\). (B) The corresponding intersection multigraph, obtained from the sets \(U_g = \{l, m, q\}\), \(U_gq = \{l, m\}\), \(U_{gq} = \{m, q\}\), \(U_{gmn} = \{l, q\}\), \(U_{gh} = \{l, m, n\}\), \(U_{ghn} = \{l, m\}\), \(U_{gh} = \{m, n\}\), \(U_{hmn} = \{l, n\}\), etc. (C) The 2-branching \(F_2\) in bold, obtained after Line 16 from the independent set \(I = \{a, c, e, h, k, m, n, s, x, z\}\), in bold. (D) Final arborescence \(T\) in bold, obtained from \(F_2\) in Line 17.

The intersection graph \(G\) has no 4-claws because each set \(U_v\) is either a 3-set or a 2-set. So SQUAREIMP\((G, w)\) achieves a ratio slightly greater than 2 for wMIS. However, our graph \(G\) is not only 4-claw free: it uses only weights 1 and 2, and has other particularities that we will explore in the next section.
4. Weighted \( \{2,3\} \)-intersection graphs

A pair \((V,U)\) is a hereditary \( \{2,3\} \)-collection if \( V \) is a finite set and the set \( U = \{ U_v : v \in V \} \) is a collection of 2-sets and 3-sets such that, for each \( v \) in \( V \) with \( U_v = \{a,b,c\} \), there are elements \( v_a, v_b, \) and \( v_c \) in \( V \) with \( U_{v_a} = \{b,c\}, U_{v_b} = \{a,c\}, \) and \( U_{v_c} = \{a,b\} \).

A \( \{2,3\} \)-intersection graph is the intersection multigraph associated to a hereditary \( \{2,3\} \)-collection \((V,U)\), that is, the multigraph whose vertex set is \( V \) and there are \(|U_x \cap U_y|\) parallel edges between any two vertices \( x \) and \( y \) in \( V \). A weighted \( \{2,3\} \)-intersection graph is a \( \{2,3\} \)-intersection graph whose weight for a vertex \( v \) is exactly \(|U_v| - 1\).

In a \( \{2,3\} \)-intersection graph, if \( x \) and \( y \) are neighbors and \(|U_x \cap U_y| = 1\), we say they are single neighbors. In a weighted \( \{2,3\} \)-intersection graph, every weight-2 vertex \( v \) forms a \( K_4 \) with the three weight-1 vertices \( v_a, v_b, \) and \( v_c \) where \( U_v = \{a,b,c\} \). Moreover, there are two parallel edges between \( v \) and each of \( v_a, v_b, \) and \( v_c \), and \( v_a, v_b, \) and \( v_c \) form a triangle. We use \( K^*_4 \) to refer to this \( K_4 \).

**Claim 4.1.** The weighted graph \((G,w)\) built in Lines 10-12 of Algorithm 3 is a weighted \( \{2,3\} \)-intersection graph.

**Proof.** It is enough to argue that \((\text{Candidates},U)\) is a hereditary \( \{2,3\} \)-collection. Indeed \( \text{Candidates} \) is a finite set, and every element \( v \) in \( \text{Candidates} \) is associated to a set \( U_v \) that is a 2-set or a 3-set. Moreover, for every \( v \) in \( \text{Candidates} \) such that \( U_v = \{a,b,c\} \), there are three elements \( v_a, v_b, \) and \( v_c \) in \( \text{Candidates} \) such that \( U_{v_a} = \{b,c\}, U_{v_b} = \{a,c\}, \) and \( U_{v_c} = \{a,b\} \).

Hence \((\text{Candidates},U)\) is indeed a hereditary \( \{2,3\} \)-collection, and therefore, by the definition of \( w \) in Line 12 \((G,w)\) is a weighted \( \{2,3\} \)-intersection graph. \( \square \)

There is a straightforward reduction from 3D-matching to maximum independent set in 4-claw free graphs which implies that wMIS on weighted 4-claw free graphs is NP-hard \([5,11]\). We adapt this reduction to prove the following hardness result for wMIS.

**Theorem 4.2.** wMIS is NP-hard on weighted \( \{2,3\} \)-intersection graphs.

**Proof.** We modify the reduction from 3D-matching to the maximum independent set problem so that the instance built is an instance of wMIS, specifically, is a weighted \( \{2,3\} \)-intersection graph.

An instance of 3D-matching consists of the following. Let \( X, Y, \) and \( Z \) be disjoint sets such that \(|X| = |Y| = |Z| = q\), and let \( S \) be a subset of \( X \times Y \times Z \), that is, each set in \( S \) is a triple of elements, one in \( X \), one in \( Y \), and one in \( Z \). The goal of the 3D-matching problem is to decide whether there is a subcollection of \( S \) with exactly \( q \) disjoint sets.

We build from \( S \) an enlarged collection \( S' \) that contains \( S \) and all three sets of size 2 contained in a set of \( S \). Let \( V \) be a set of size \(|S'|\) and associate to each element in \( V \) one of the sets in \( S' \). Note that the pair \((V,S')\) is a hereditary \( \{2,3\} \)-collection. Let \((G,w)\) be the weighted \( \{2,3\} \)-intersection graph associated to \((V,S')\). Recall that the weight \( w_v \) of a vertex \( v \) in \( G \) whose
associated set is $S$ in $S'$ is $|S| - 1$. We applied in Algorithm 3 a similar construction on the set Candidates.

Let us prove that there is a solution for the 3D-matching instance if and only if there is an independent set in $G$ of weight at least $2q$. If there is a solution $M \subseteq S$ for the 3D-matching problem, then let $I$ be the set of vertices of $G$ corresponding to the sets in $M$. As $M$ is a collection of disjoint sets, $I$ is an independent set in $G$. Each set in $M$ is a 3-set, therefore its corresponding vertex in $G$ has weight 2. As $|M| = q$, the weight of $I$ is $2q$.

For the other direction, let $I$ be an independent set in $G$ of weight at least $2q$. Let $q_1$ be the number of weight-1 vertices in $I$ and $q_2$ be the number of weight-2 vertices in $I$. Let us prove that $q_1 = 0$ and $q_2 = q$. This would imply that the collection of sets in $S'$ corresponding to $I$ is a subset of $S$ with exactly $q$ disjoint sets, being therefore a solution for the 3D-matching instance.

The sets corresponding to vertices in $I$ are pairwise disjoint, and their union has size $2q_1 + 3q_2$ and is contained in $X \cup Y \cup Z$, whose size is $3q$. Therefore

$$2q_1 + 3q_2 \leq 3q. \quad (1)$$

This implies that $q_2 \leq q$. Moreover, the weight of $I$ is

$$q_1 + 2q_2 \leq 3 \frac{q - q_2}{2} + 2q_2 = \frac{3}{2}q + \frac{1}{2}q_2 \leq 2q \quad (2)$$

by (1) and because $q_2 \leq q$. As the weight of $I$ is at least $2q$, the inequalities in (2) must be equalities, which means that $q_2 = q$ and $q_1 = 0$. This completes the proof. \hfill \Box

In what follows, we will present an analysis of SQUAREIMP specific for weighted \{2, 3\}-intersection graphs that shows that SQUAREIMP achieves a ratio significantly better on such graphs than on general weighted 4-claw free graphs. In Section 6, we will adapt SQUAREIMP and use ideas from [9] to obtain an even better approximation for wMIS on weighted \{2, 3\}-intersection graphs.

The following proposition states properties of weighted \{2, 3\}-intersection graphs that will be useful in our analysis.

**Proposition 4.3.** Every weighted \{2, 3\}-intersection graph $(G, w)$ has the following properties:

(i) there is no 4-claw in $G$ and every 3-claw in $G$ has a center of weight 2;
(ii) every weight-2 vertex $v$ dominates the neighborhood of $K_4^v$;
(iii) every single neighbor $u$ of a weight-2 vertex $v$ has exactly two weight-1 neighbors in $K_4^v$.

**Proof.** Let $(V, U)$ be the hereditary \{2, 3\}-collection underlying $(G, w)$. To prove (i), note first that if $C$ is a $d$-claw with center in $z$ for some $z$ in $V$, then, because $T_C$ is an independent set, each of the sets corresponding to a vertex in $T_C$ intersects $U_z$ in at least one distinct element. Since $|U_z| \leq 3$, we have that $d \leq 3$. Also, if $|U_z| = 2$, that is, $z$ has weight 1, then $d \leq 2$.

For (ii), suppose that $x$ is a neighbor of a vertex $y$ in $K_4^v$, which means $U_x \cap U_y \neq \emptyset$. Observe that $U_y \subseteq U_v$ for every $y$ in $K_4^v$. Therefore $U_x \cap U_v \supseteq U_x \cap U_y \neq \emptyset$. 


For (iii), suppose \( U_v = \{a, b, c\} \), and let \( u \) be a single neighbor of \( v \). Without loss of generality, we may assume that \( U_u \cap U_v = \{a\} \). Recall that the vertices of \( K^4_1 \) are \( v, v_a, v_b, \) and \( v_c \). Then \( u \) is a neighbor of \( v_b \) and \( v_c \), but not of \( v_a \). \( \square \)

We tailored a better and tighter analysis of SQUAREIMP for weighted \( \{2, 3\} \)-intersection graphs.

**Theorem 4.4.** SQUAREIMP is a \( \frac{3}{2} \)-approximation for \( w\text{MIS} \) on weighted \( \{2, 3\} \)-intersection graphs.

**Proof.** Let \((G, w)\) be a weighted \( \{2, 3\} \)-intersection graph. Let \( A^* \) be an independent set in \( G \) that maximizes \( w(A^*) \) and let \( A \) be the independent set produced by SQUAREIMP\((G, w)\). We shall prove that \( w(A^*) \leq \frac{3}{2} w(A) \). We will do this using the strategy of Berman \[3\]: each vertex in \( A^* \) will distribute its weight among its neighbors in \( A \) so that no vertex in \( A \) gets more than \( 3/2 \) its own weight.

For the sake of the argument in this proof, we consider \( G \) as the corresponding intersection multigraph. Vertices in \( A^* \cap A \) keep their weights. Because \( A \) is maximal, every vertex in \( A^* \setminus A \) has at least one neighbor in \( A \). Also, by Proposition 4.3(i), every vertex in \( A^* \) of weight \( z \) has at most \( z + 1 \) edges going to its neighbors in \( A \).

In Figure 3, we show how each vertex in \( A^* \setminus A \) distributes its weight to its neighbors in \( A \). We represent vertices in \( A^* \) by red squares and vertices in \( A \) by blue circles. The number on top of each vertex in \( A^* \) is its weight. The number below a vertex in \( A \) denotes its weight when that weight matters for the distribution. The number on each edge is the amount of weight distributed from the vertex in \( A^* \) to the vertex in \( A \). Recall that, in this argument, \( G \) is a multigraph, so some of the blue round vertices connected to a red square vertex might be the same, receiving some weight from the same red square vertex through two or three edges. An example of such distribution can be seen in Figure 4.

Let us argue that Configurations (c) and (e) in Figure 3 cannot happen. The red square vertex in Configuration (e) would itself improve \( w^2(A) \), so this configuration does not occur. Now, suppose, for a contradiction, that there is a weight-2 vertex \( v \in A^* \setminus A \) whose only neighbor in \( A \) is a single neighbor \( u \) of \( v \), as in Configuration (c). By Proposition 4.3(iii), vertex \( u \) must be
adjacent to exactly two weight-1 vertices in \( K^v_4 \). But then the third weight-1 vertex in \( K^v_4 \) would have no neighbor in \( A \) by Proposition 4.3(ii), and it would thus improve \( w^2(A) \), a contradiction. Hence, Configuration (c) of Figure 3 also does not occur.

Now, let us prove that no vertex in \( A \) gets more than \( 3/2 \) of its weight. First consider a weight-2 vertex \( u \) in \( A \). Such a vertex \( u \) receives weight from \( A^* \) through at most three edges, by Proposition 4.3(i). The value that \( u \) receives through each edge is in \( \{0, 1/2, 2/3, 1\} \). Thus \( u \) receives at most 3 in total. Now consider a weight-1 vertex \( u \) in \( A \). Such a vertex \( u \) receives weight from \( A^* \) through at most two edges, by Proposition 4.3(i). The value that \( u \) receives through each edge is in \( \{0, 1/2, 1\} \). The only way to receive in total more than \( 3/2 \) is by receiving 1 through the two edges. These possibilities are summarized in Figure 5 and each leads to a claw in \( G \) that improves \( w^2(A) \). Indeed, Configuration (a) in Figure 5 contains an improving 1-claw and Configuration (b) is itself an improving 2-claw. In Configuration (c), \( z \) is a single neighbor of \( v \), so there exists a weight-1 vertex \( x \) in \( K^v_4 \) (not depicted) that is not a neighbor of \( z \). Vertex \( x \) and the red square weight-1 vertex \( y \) form the set \( T_C \) of an improving 2-claw \( C \) centered at the blue round vertex \( u \). The same argument applied twice shows that there is an improving 2-claw if Configuration (d) occurs. Therefore these possibilities cannot occur and every \( u \) in \( A \) receives at most \( 3/2 \) of its weight \( w_u \).

Figure 4. Example of weight distribution for Theorem 4.4: vertex \( v \) distributes weight 1 to each \( y \) and \( t \); \( s \) distributes \( 1/2 \) to each \( t \) and \( r \); \( q \) distributes \( 2/3 \) to \( r \), \( p \), and \( o \); and \( y_u \) distributes \( 1/2 + 1/2 = 1 \) to \( y \).
We point out that this analysis is tight, since a vertex in $A$ might receive exactly $3/2$ of its weight, as shown in Figure 6 which contains no improving claw. Indeed, for these examples, SQUAREIMP might produce an independent set of weight 2 although there is one of weight 3.

Finally, let us argue that SQUAREIMP($G$, $w$) runs in polynomial time. Let $n$ be the number of vertices of $G$. Because $w^2(A)$ increases in every iteration of SQUAREIMP and all weights in $w$ are 1 or 2, SQUAREIMP($G$, $w$) does at most $4n$ iterations. In each iteration, one can test in polynomial time all $d$-claws in $G$, for $d \in \{1, 2, 3\}$, which is enough because $G$ is 4-claw free.

\section{Back to Maximum Leaf Spanning Arborescence}

In this section we go back to Maximum Leaf Spanning Arborescence, and derive a new $3/2$-approximation for rooted dags from SQUAREIMP.

Recall that in MAXLEAVES-12MIS we built the intersection graph $G$ having as vertex set the set $Candidates$, which contains vertices of out-degree 0 with two or three out-neighbors of in-degree 0. The edges were added to $G$ according to the sets $U_v$ defined for each $v \in Candidates$, while $w_v$ was set to $|U_v| - 1$. Now note that the set $S = \{U_v : v \in Candidates\}$, with the same weight function $w$, is an instance of the weighted 3D-matching problem. In the weighted 3D-matching, one wants to find a collection $S' \subseteq S$ of pairwise disjoint sets as heavy as possible. One can see that any optimal solution for the weighted 3D-matching on $(S, w)$ provides an optimal solution for the wMIS on ($G, w$) and vice-versa. In \cite{9}, we presented a theorem using the weighted 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.pdf}
\caption{Configurations that imply on an improving claw centered at $u$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.pdf}
\caption{Tight examples for the analysis of Theorem 4.4.}
\end{figure}
3D-matching nomenclature whose proof can be adapted to establish the following. We present the proof of this theorem in Appendix A (See [10, Theorem 4.3] for details.)

**Theorem 5.1.** If $A$ is an $\alpha$-approximation algorithm for the wMIS on weighted $\{2, 3\}$-intersection graphs, then algorithm MAXLEAVES-12MIS using $A$ in Line 13 instead of SQUAREIMP is a max$\{\frac{3}{4}, \alpha\}$-approximation for Maximum Leaf Spanning Arborescence on rooted directed acyclic graphs.

Applying Theorem 5.1 with SQUAREIMP as algorithm $A$, and using Theorem 4.4, we derive the following result.

**Corollary 5.2.** Algorithm MAXLEAVES-12MIS is a $3/2$-approximation for the Maximum Leaf Spanning Arborescence on rooted directed acyclic graphs.

6. **Boosting SQUAREIMP with maximum matchings**

In this section, we modify SQUAREIMP to achieve a ratio for wMIS better than $3/2$ on weighted $\{2, 3\}$-intersection graphs. The improvement depends on two ideas, each coming from one of the tight examples in Figure 6.

The example on Figure 6(a) is a claw itself, but it is not improving with the weights we assigned to the vertices of $G$. However, it ends up being a good exchange for both wMIS and Maximum Leaf Spanning Arborescence problems. Hence we decided to adapt SQUAREIMP, defining $w^2_v(A)$ to be the sum of $(w_v + 1)^2$ for every $v$ in $A$, and applying the exchange whenever $w^2_v(A)$ increases. This has the effect of doing all the previous improvements and also the one in Figure 6(a).

The example on Figure 6(b) can be generalized into a longer path on weight-1 vertices, alternating between red square and blue round vertices. So in fact it is a class of examples, and it does not contain an improving claw, but it indicates a way to increase the independent set by one: exchange the blue round vertices by the red square vertices in the path. In analogy to matching, we refer to such an exchange as an augmenting path improvement.

The idea to search for such improvements is inspired on algorithm MAXEXPAND from [9]. Let $(G, w)$ be a weighted $\{2, 3\}$-intersection graph where $G$ is the intersection graph for the pair $(V, U)$. Let $A$ be an independent set in $G$. Consider an auxiliary graph $H$ as follows. Let $X$ be the union of $U_v$ for all weight-2 vertices in $A$. We think of the elements of $X$ as forbidden. The vertex set of $H$ is the union of $U_v$ for all weight-1 vertices of $G$ such that $U_v$ has no forbidden element, that is, $U_v$ does not intersect with $X$. There is an edge between two vertices $x$ and $y$ of $H$ if there is a vertex $v$ in $G$ with $U_v = \{x, y\}$. So edges of $H$ are associated to weight-1 vertices from $G$. Let $M$ be the set of edges in $H$ corresponding to weight-1 vertices in $A$. Note that $M$ is a matching in $H$, and that $H$ and $M$ can be obtained in polynomial time from $G$ and $A$. See Figure 7 for an example.

We recall the matching nomenclature. A vertex in $H$ is $M$-covered if it is the end of an edge in $M$, and is $M$-uncovered otherwise. An $M$-alternating path in $H$ is one which alternates edges
Figure 7. The independent set \( A = \{b, d, e, g\} \) has no improving claw. Path \( \langle j, k, l, m, q, r, s, t \rangle \) is an augmenting path in \( H \). The independent set \( A' = \{a, c_n, f_p, h, e\} \), obtained from the augmenting path improvement, has an improving claw where \( T_C = \{c, f\} \).

We adapt SQUAREIMP in the following way. After a claw improvement phase is complete, we check whether there exists an augmenting path improvement. If such an improvement exists, we perform it and go back to the claw improvement phase. We repeat this until no augmenting path improvement exists. We call SQUARE+IMP the resulting algorithm, which is presented in Algorithm 4. The routine AUGMENTINGPATH\((H, M)\) returns an \( M \)-augmenting path in \( H \), if one exists, and NULL otherwise. There are polynomial-time algorithms for this in the literature [8].

The same argument that we used on SQUAREIMP\((G, w)\) assures that the SQUARE+IMP\((G, w)\) runs in polynomial time. Indeed, let \( n \) be the number of vertices of \( G \). Because \( w^2(A) \) starts from
**Algorithm 4** **SQUARE**$^+$**IMP**(G, w)

**Input:** weighted {2, 3}-intersection graph (G, w)

**Output:** an independent set in G

1: let V and U be such that $G = \text{INTERSECTIONGRAPH}(V, U)$
2: $A \leftarrow \emptyset$
3: repeat
4: while there is a claw $C$ in G such that $T_C$ improves $w^2(A)$ do
5: $A \leftarrow (A \cup T_C) \setminus N(T_C)$
6: $X \leftarrow \bigcup \{U_v : v \in A \text{ and } w_v = 2\}$
7: $V' \leftarrow \bigcup \{U_v : v \in V, w_v = 1, \text{ and } U_v \cap X = \emptyset\}$
8: $E' \leftarrow \{xy : \text{there is a vertex } v \in V \text{ such that } U_v = \{x,y\} \subseteq V'\}$
9: let $H$ be the graph $(V', E')$
10: $M \leftarrow \{xy \in E(H) : \text{there is a vertex } v \in A \text{ such that } U_v = \{x,y\}\}$
11: $P \leftarrow \text{AUGMENTINGPATH}(H, M)$
12: if $P \neq \text{Null}$ then
13: $A \leftarrow A \oplus P$
14: until $P = \text{Null}$
15: return $A$

The example in Figure 7 shows that, after an augmenting path improvement, there might be feasible claw improvements to be done.

The proof of the next result partially mimics the proof of Theorem 4.4, using a different weight distribution rule. In one of the cases, however, it bounds the average of the receiving weights, instead of the maximum receiving weight per vertex.

**Theorem 6.1.** **SQUARE**$^+$**IMP** is a $\frac{7}{5}$-approximation for wMIS on weighted {2, 3}-intersection graphs.

**Proof.** Let $(G, w)$ be a weighted {2, 3}-intersection graph. Let $A^*$ be an independent set in G that maximizes $w(A^*)$ and let $A$ be the independent set produced by **SQUARE**$^+$**IMP**(G, w). We shall prove that $w(A^*) \leq \frac{7}{5} w(A)$ by distributing differently the weight on vertices in $A^*$ among their neighbors in A so that no weight-2 vertex in $A$ gets more than 14/5 and weight-1 vertices in $A$ get, on average, no more than 7/5.

At the end of **SQUARE**$^+$**IMP**, no claw is improving for $A$. For the sake of the argument, again we consider G as the corresponding intersection multigraph. Vertices in $A^* \cap A$ keep their weights. Because $A$ is maximal, every vertex in $A^* \setminus A$ has at least one neighbor in $A$. By Proposition 4.3(i), every vertex in $A^*$ of weight $z$ has at most $z + 1$ edges going to its neighbors in $A$. 


In Figure 8, we show how each vertex in \( A^* \setminus A \) distributes its weight to its neighbors in \( A \). As in Figure 3, we represent vertices in \( A^* \) by red squares and vertices in \( A \) by blue circles. The number on top of each vertex in \( A^* \) is its weight. The number below a vertex in \( A \) denotes its weight when that weight matters for the distribution. The number next to each edge is the amount transferred from the vertex in \( A^* \) to the vertex in \( A \).

The same argument used in Theorem 4.4 shows that Configuration (e) of Figure 8 does not occur.

![Figure 8. Weight distribution for Theorem 6.1](image)

First, let us prove that no weight-2 vertex \( u \) in \( A \) gets more than \( \frac{14}{5} \). By Proposition 4.3(i), such a vertex \( u \) receives weight from \( A^* \) through at most three edges. The value that \( u \) receives through each edge is in \( \{0, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{4}{5}, 1\} \). Thus, the only way to receive in total more than \( \frac{14}{5} \) is by receiving 1 through the three edges. These possibilities are summarized in Figure 9 and, for each, there exists an improving claw, which contradicts the fact that \( A \) is the output of SQUARE+IMP.

Indeed, Configuration (a) in Figure 9 is an improving claw itself. In Configuration (b), call \( v \) the weight-2 red square vertex and \( u' \) the blue round vertex other than \( u \). Observe that \( u' \) is a single neighbor of \( v \). Because \( G \) is a weighted \( \{2, 3\} \)-intersection graph, there is a weight-1 vertex \( y \) in \( K^v_4 \) such that \( N(y) \subseteq N(v) \setminus \{u'\} \). The two weight-1 red square vertices and \( y \) form a 3-claw with \( u \) that improves \( w^2_+ (A) \). The same argument can be used to derive an improving 3-claw from Configurations (c) and (d) in Figure 9. So these possibilities cannot occur and every weight-2 vertex \( u \) in \( A \) receives at most \( 1 + 1 + \frac{4}{5} = \frac{14}{5} \).

Now we will analyze how much the weight-1 vertices in \( A \) receive from the vertices in \( A^* \). We will prove that, on average, each weight-1 vertex in \( A \) receives at most \( \frac{7}{5} \). Note that a weight-1 vertex in \( A \) receives through each edge a value in \( \{0, \frac{2}{5}, \frac{1}{2}, \frac{2}{3}, 1\} \).

Let \( Y = \bigcup \{U_v : v \in A\} \) and \( Z = \bigcup \{U_v : v \in A^*\} \). We will modify a little bit the interpretation of the weight distribution, considering that each vertex \( v \) in \( A^* \) in fact distributes its weight \( w_v \) among the elements in \( Z \cap Y \). That is, we will consider that the blue round vertices in Figure 8 represent elements in the set \( Z \cap Y \). Then, a vertex \( v \) in \( A \) receives from \( A^* \) the sum of the weights that its elements in \( U_v \cap Z \) received.
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FIGURE 9. Configurations that imply on an improving 3-claw centered in \( u \).

Note that the vertex set \( V' \) of the graph \( H \) defined in Line 10 is exactly \( Z \setminus X \). Moreover, each weight-1 vertex \( v \) in \( A \) is associated to an edge \( e_v \) in \( M \), and receives from \( A^* \) exactly the sum of the weights received by the ends of \( e_v \).

An \( A^* \)-edge is an edge of \( H \) that is contained in the set \( U_v \) for a \( v \in A^* \). Consider the spanning subgraph \( H' \) of \( H \) containing only the edges in \( M \) and the \( A^* \)-edges. See Figure 10 for an example.

FIGURE 10. Graph \( H' \) built from the intersection graph in Figure 4b. Bold blue edges are associated to vertices in \( A \). All the other are \( A^* \)-edges. Vertex \( z \) receives 1 from \( v \) in Figure 4b, while \( z' \) receives 1/2 from \( s \), as \( zz' \) is associated with vertex \( t \). Vertex \( a \) also receives 1/2 from \( s \). Each of vertices \( a' \), \( b \), and \( c \) receives 2/3 from \( q \).

Claim 6.2. There is at most one \( M \)-uncovered vertex in each component of \( H' \).

Proof. There are two types of \( A^* \)-edges in \( H' \): the isolated ones, that share no end with another \( A^* \)-edge in \( H' \), and the ones in a triangle. Indeed, as an \( A^* \)-edge \( e \) is contained in \( U_v \) for some \( v \in A^* \), either \( e = U_v \) or there is a vertex \( x \) in \( U_v \) not in \( e \). If \( x \in V' \), then the three pairs of elements in \( U_v \) correspond to \( A^* \)-edges in \( H' \), including \( e \), that form a triangle in \( H' \). If \( x \not\in V' \) or \( e = U_v \), then \( e \) is an isolated \( A^* \)-edge in \( H' \). This implies that any path in \( H' \) corresponds to an alternating path in \( H' \): if the path contains two consecutive \( A^* \)-edges, these two edges lie in a triangle in \( H' \), and the third edge in this triangle can be used to shortcut the common vertex of the consecutive \( A^* \)-edges. So, if there were two \( M \)-uncovered vertices in the same component of \( H' \), there would be an augmenting path between them, contradicting the fact that \( M \) is a maximum matching in \( H \).
\( \square \)
Now we will prove that the average weight that $A^*$ assigns to an $M$-covered vertex in $H'$ is at most $7/10$. This implies that each weight-1 vertex in $A$ receives at most $7/5$ from $A^*$. The analysis considers one connected component $H''$ of $H'$ at a time.

Every $M$-covered vertex in $H''$ that receives 1 is like the blue round vertex from Configuration 8(a), or like one of the blue round vertices in Configuration 8(f). Each of these configurations corresponds to an $M$-uncovered vertex in $H$. Thus, by Claim 6.2 in $H''$, there are at most two vertices receiving 1 from $A^*$, each one adjacent in $H''$ to the only $M$-uncovered vertex in $H''$.

If no vertex in $H''$ receives 1 from $A^*$, then every $M$-covered vertex in $H''$ would receive at most $2/3 < 7/10$ and the statement holds for the $M$-covered vertices in $H''$. The rest of the proof follows from the next two claims.

**Claim 6.3.** Let $H''$ be a connected component of $H'$ that contains an $M$-uncovered vertex $u$ incident to exactly two $A^*$-edges. In average, every $M$-covered vertex in $H''$ receives at most $7/10$.

**Proof.** Let $a$ and $b$ be the other ends of the two $A^*$-edges incident to $u$. Both $a$ and $b$ receive 1 from a vertex $v$ in $A^*$, as in Configuration 8(f). Note that $ab \not\in M$, otherwise there is a trivial augmenting claw. See Figure 11.

![Figure 11](image.png)

**Figure 11.** Here $U_v = \{a, b, u\}$. Because $u$ is $M$-uncovered, $v_u$ is the only neighbor of $v$ in $A$, thus the claw with $T_C = \{v\}$ centered at $v_u$ is improving.

Let $a'$ and $b'$ be such that $aa' \in M$ and $bb' \in M$. If each of $a'$ and $b'$ receives at most $2/5$ from $A^*$, then the claim holds. Otherwise, we may assume $a'$ receives at least $1/2$, like the blue round vertex in Configuration 8(b), or like one of the blue round vertices in Configurations 8(g), or Configuration 8(i), where its sibling of weight 2 is not in $H$. So there is at least one $A^*$-edge incident to $a'$ in $H'$.

If there are two $A^*$-edges incident to $a'$, we are in Configuration 8(g). Let $x$ be the weight-2 vertex of $G$ corresponding to the red square vertex in this configuration, and to the two $A^*$-edges incident to $a'$ in $H'$. Thus, there is an augmenting claw $C$ with $T_C = \{v, x\}$, as there are at most four weight-1 vertices in $A \cap N(T_C)$, and $9 + 9 = 18 > 16 = 4 \cdot 4$. See Figure 12a.

Otherwise, we are either in Configuration 8(b) or in Configuration 8(i). Let $x$ be the red square vertex in the corresponding configuration. If we are in Configuration 8(b), let $y = x$, that is, $y$ is the weight-1 vertex of $G$ corresponding to the only $A^*$-edge incident to $a'$ in $H'$. If we are in Configuration 8(i), let $y$ be the weight-1 vertex in $K_4$ corresponding to the only $A^*$-edge incident
The claw with $T_C = \{v, x\}$ is improving.

Figure 12. In each case, part of $H''$ is to the left while the corresponding part of $G$ is to the right. In $H''$, bold blue arcs are associated with vertices in $A$, which are bold blue in $G$. Dashed red vertices are in $A^*$.

to $a'$. Again, the claw $C$ with $T_C = \{v, y\}$ is augmenting, with at most three weight-1 vertices in $A \cap N(T_C)$, and $9 + 4 = 13 > 12 = 3 \cdot 4$. See Figure 12b.

Claim 6.4. Let $H''$ be a connected component of $H'$ that contains an $M$-uncovered vertex $u$ incident to only one $A^*$-edge $e$. In average, every $M$-covered vertex in $H''$ receives at most $7/10$.

Proof. Let $z$ be the other end of $e$. Vertex $z$ receives 1 from $A^*$, and is either like the blue round vertex in Configuration 8(a), or like one of the blue round vertices in Configuration 8(f) when its sibling has weight-2 and thus is not in $H''$. If there is a vertex $x$ in $H''$ that receives at most $2/5$ from $A^*$, then the claim holds. Indeed, every $M$-covered vertex of $H''$ other than $x$ and $z$ receives at most $2/3$ from $A^*$. Thus, because $2/3 < 7/10$ and $(1 + 2/5)/2 = 7/10$, the claim holds. So we may assume that every vertex of $H''$ receives at least $1/2$. If there is one vertex that receives $1/2$ in $H''$, then this vertex is like one of the blue round vertices in Configuration 8(b) and its sibling is also in $H''$. Therefore, there would be two vertices in $H''$ receiving $1/2$, and all the others except for $z$ receive at most $2/3$. Hence the average in $H''$ would be at most $(1 + 1/2 + 1/2)/3 = 2/3 < 7/10$. Thus we may assume that every vertex in $H''$ except for $z$ receives $2/3$.

The number of $M$-covered vertices in $H''$ is always even, and at least two because $z$ is $M$-covered. If there are at least ten $M$-covered vertices in $H''$, the claim holds because $(1 + 9 \cdot 2/3)/10 = 7/10$. So we may assume there are at most eight $M$-covered vertices in $H''$, and an odd number of them receive $2/3$.

From the configurations in Figure 8 a vertex in $A^*$ cannot send $2/3$ to only one vertex in $H''$. It either sends $2/3$ to three vertices in $H''$ (Configuration 8(g)), or to two vertices in $H''$ (Configuration 8(i)). As the number of $M$-covered vertices in $H''$ is even, but $z$ receives 1, the number
of vertices that receive $2/3$ in $H''$ is odd. Thus there must be exactly one group of three vertices $x, y, y'$ in $H''$ receiving $2/3$ from the same vertex $t$ in $A^*$ (Configuration 8(g)). If one of $xy, xy'$ or $yy' \in M$, then $t$ is an improving claw. Therefore, no two vertices in $x, y, y'$ are matched to each other in $M$.

Let $v$ be the vertex in $A^*$ corresponding to $e$. Let $z'$ be the vertex such that $zz' \in M$. Vertex $z'$ is also in $H''$. Note that $z' \notin \{x, y, y'\}$, otherwise there is an improving claw $C$ with $TC = \{v, t\}$.

Thus $H''$ must contain exactly eight $M$-covered vertices. See Figure 13 The vertices adjacent to $x, y$, and $y'$ in $M$ must be distinct and two of them, say $w$ and $w'$, receive $2/3$ from the same weight-2 vertex $s$ in $A^*$ (Configuration 8(i)). Let $s'$ be the weight-1 vertex in $K_4^s$ corresponding to the only $A^*$-edge $ww'$ in $H''$ coming from $s$. Then the claw $C$ with $TC = \{t, s'\}$ is improving (because $9 + 4 = 13 > 12 = 3 \cdot 4$).

From Claims 6.3 and 6.4 we conclude the proof of Theorem 6.1.

This analysis is tight. For the example in Figure 14 SQUARE$^+\text{IMP}$ might produce the independent set marked in bold blue, of weight 5, while the heaviest independent set is marked in dashed red, and has weight 7. Note that there is no improving claw and no augmenting path.
Applying Theorem 5.1 with SQUARE$^+$IMP as algorithm $A$, and using Theorem 6.1 we strengthen Corollary 5.2 and derive our main contribution on Maximum Leaf Spanning Arborescence:

**Theorem 6.5.** Algorithm MAXLEAVES-12MIS using SQUARE$^+$IMP instead of SQUAREIMP is a $7/5$-approximation for the Maximum Leaf Spanning Arborescence on rooted directed acyclic graphs.

7. Final remarks

One might ask whether using the sum of the weights instead of $w^{2}_{+}$ would lead to the same algorithm, that is, would induce the same improvements that SQUARE$^+$IMP does, for weighted \{2, 3\}-intersection graphs. However the claw from the example in Figure 12a would not be improving for the sum of the weights. From that example, one can construct a larger example, with a weight-1 vertex adjacent to each of $q$, $s$, and $t$, and include these in the dashed red independent set, to show that the variant using the sum of the weights does not achieve a ratio better than $7/5$.

The ideas used here to achieve a better approximation for wMIS on weighted \{2, 3\}-intersection graphs might lead to improvements for wMIS on $d$-claw free graphs, or particularly to the weighted 3D-matching problem. Also, our strategy applied to Neuwohner’s algorithm [14] might lead to a ratio better than $7/5$ for weighted \{2, 3\}-intersection graphs, which would imply an improvement for Maximum Leaf Spanning Arborescence on rooted dags.

In the Maximum Leaf Spanning Tree problem, one is given a connected undirected graph $G$ and wants to find a spanning tree of $G$ with the maximum number of leaves, where a leaf is a vertex of degree 1. The best known approximation algorithm for this problem has ratio 2 and it was proposed by Solis-Oba more than 20 years ago [17, 18]. It would be nice also if some of the ideas explored in this paper were helpful to obtain a better approximation for the maximum leaf spanning tree, or for Maximum Leaf Spanning Arborescence for general rooted digraphs. For both of these, however, the idea of using wMIS seems harder to be applied.
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Appendix A. Proof of Theorem 5.1

In a previous work, we presented a theorem for Maximum Leaf Spanning Arborescence involving approximations for the weighted 3D-matching [10, Theorem 4.3]. That theorem can be adapted to address approximations for the wMIS on weighted \{2, 3\}-intersection graphs, resulting in Theorem 5.1.

In this appendix, we present the proof for Theorem 5.1, whose proof relies on the adaptation of two lemmas from [10, Lemmas 4.1 and 4.2]. Though the statement of these three results are different from their corresponding versions in [10], their proofs depend on certain variables defined using the algorithm addressed. Once we define these variables using MaxLeaves-12MIS, the proofs are (essentially) the same. Yet, for completeness, we include them here.

We start by presenting the two adapted lemmas. For that, we establish the notation. Let us denote by \( A\)-MaxLeaves-12MIS a version of MaxLeaves-12MIS that uses an algorithm \( A \) for the wMIS on weighted \{2, 3\}-intersection graphs in Line 13 instead of SquareImp.

Let \( D \) be a rooted dag and consider a call \( A\)-MaxLeaves-12MIS\((D)\). Let \( F_1 \) and \( T \) be the branchings produced in Lines 2 and 17 respectively. Let \( F_3 \) be the state of the branching \( F_2 \) just before Line 17. In what follows, let \( F_2 \) denote the branching obtained from \( F_1 \) if Lines 15-16 were executed only for vertices \( v \) with \( w_v = 2 \). For \( i = 1, 2, 3 \), let \( k_i \) be the number of non-trivial components of \( F_i \) and \( N_i \) be the number of vertices in such components.

Lemma A.1. Let \( T \) be the arborescence produced by \( A\)-MaxLeaves-12MIS\((D)\). Then

\[
\ell(T) \geq \frac{N_1 - k_1}{12} + \frac{N_2 - k_2}{6} + \frac{N_3 - k_3}{2} + 1.
\]

Proof. Let \( n \) be the number of vertices of \( D \). Let \( T_1, \ldots, T_{k_1} \) be the non-trivial arborescences in \( F_1 \). Note that \( \ell(T_j) \geq \frac{1 + 3|V(T_j)|}{4} \) because all internal vertices of \( T_j \) have out-degree at least 4. Therefore,

\[
\ell(F_1) = n - N_1 + \sum_{j=1}^{k_1} \ell(T_j) \geq n - N_1 + \sum_{j=1}^{k_1} \frac{1 + 3|V(T_j)|}{4} = n - N_1 + \frac{3N_1}{4} + \frac{k_1}{4} = n - \frac{N_1 - k_1}{4}.
\]

The number of components in \( F_i \) is \( n - N_i + k_i \) for \( i = 1, 2, 3 \). Hence, the number of leaves lost from \( F_1 \) to \( F_2 \) is exactly

\[
\frac{(n - N_1 + k_1) - (n - N_2 + k_2)}{3} = \frac{N_2 - k_2}{3} - \frac{N_1 - k_1}{3}.
\]

Similarly, the number of leaves lost from \( F_2 \) to \( F_3 \) is exactly

\[
\frac{(n - N_2 + k_2) - (n - N_3 + k_3)}{2} = \frac{N_3 - k_3}{2} - \frac{N_2 - k_2}{2}.
\]
Also, the number of leaves lost from $F_3$ to $T$ is exactly $n - N_3 + k_3 - 1 = n - (N_3 - k_3) - 1$. Thus

$$\ell(T) \geq n - \frac{N_1 - k_1}{4} - \left( \frac{N_2 - k_2}{3} - \frac{N_1 - k_1}{3} \right) - \left( \frac{N_3 - k_3}{2} - \frac{N_2 - k_2}{2} \right) - (n - (N_3 - k_3) - 1)$$

$$= \frac{1}{12}(N_1 - k_1) + \frac{1}{6}(N_2 - k_2) + \frac{1}{2}(N_3 - k_3) + 1.$$

Now we present an upper bound on $\text{opt}(D)$ that relates to the lower bound presented in Lemma A.1.

**Lemma A.2.** If the algorithm $\mathcal{A}$ used in $\mathcal{A}$-MAXLEAVES-12MIS($D$) is an $\alpha$-approximation algorithm for the wMIS on weighted $\{2,3\}$-intersection graphs, then

$$\text{opt}(D) \leq \frac{3 - 2\alpha}{3}(N_1 - k_1) + \frac{\alpha}{6}(N_2 - k_2) + \frac{\alpha}{2}(N_3 - k_3) + 1.$$

**Proof.** Let $T^*$ be a spanning arborescence of $D$ with the maximum number of leaves. Call $R$ the set of all roots of non-trivial components of $F_1$. Call $L$ the set of leaves of $T^*$ that are isolated vertices of $F_1$. Let $Z := L \cup R \setminus \{r\}$, where $r$ is the root of $D$. The witness of a vertex $z \in Z$ is the closest proper predecessor $q(z)$ of $z$ in $T^*$ which is in a non-trivial component of $F_1$. Note that each witness is an internal vertex of $T^*$.

We will prove that the number $\psi$ of distinct witnesses is

$$\psi \geq |Z| - 2\alpha \left( \frac{N_2 - k_2}{3} - \frac{N_1 - k_1}{3} \right) - \alpha \left( \frac{N_3 - k_3}{2} - \frac{N_2 - k_2}{2} \right)$$

$$= |Z| + 2\alpha \frac{N_1 - k_1}{3} - \alpha \frac{N_2 - k_2}{6} - \alpha \frac{N_3 - k_3}{2}.$$

Because $|Z| = k_1 - 1 + |L|$ and each witness lies in a non-trivial component of $F_1$ and is internal in $T^*$, we deduce that

$$\text{opt}(D) \leq N_1 - \psi + |L|$$

$$\leq N_1 - |Z| - 2\alpha \frac{N_1 - k_1}{3} + \alpha \frac{N_2 - k_2}{6} + \alpha \frac{N_3 - k_3}{2} + |L|$$

$$= N_1 - k_1 - 2\alpha \frac{N_1 - k_1}{3} + \alpha \frac{N_2 - k_2}{6} + \alpha \frac{N_3 - k_3}{2} + 1$$

$$= \frac{3 - 2\alpha}{3}(N_1 - k_1) + \frac{\alpha}{6}(N_2 - k_2) + \frac{\alpha}{2}(N_3 - k_3) + 1.$$

It remains to prove (3).

For a witness $s$, let $Z_s := \{z \in Z : q(z) = s\}$ and let $T^*_s$ be the subarborescence of $T^*$ induced by the union of all paths in $T^*$ from $s$ to each vertex in $Z_s$. The number of such arborescences $T^*_s$ is exactly $\psi$. The only internal vertex of $T^*_s$ that is in a non-trivial component of $F_1$ is its root $s$, which is necessarily a leaf of $F_1$. So the maximum out-degree in $T^*_s$ is at most three.

Again, no $z \in Z_s$ is a predecessor in $T^*_s$ of another $z' \in Z_s$. Indeed, suppose by contradiction that $z$ is in the path from $s$ to $z'$. Then $z$ is not a leaf of $T^*$, and is in $R$, thus being in a
non-trivial component of $F_1$, which is a contradiction, because $z$, and not $s$, would be the witness for $z'$. Hence $T_s^*$ has exactly $|Z_s|$ leaves.

Let $(G, w)$ be the weighted $\{2,3\}$-intersection graph built in Lines 10-12, and let $I$ be the independent set in $G$ computed by $\mathcal{A}$ in Line 13 of $\mathcal{A}$-MAXLEAVES-12MIS($D$). For every $v$ such that $U_v = \{a,b,c\}$, the algorithm includes $v_a, v_b, v_c$ in $\textit{Candidates}$. At most one in $\{v, v_a, v_b, v_c\}$ is included in $I$. Let $B_i$ be the set of vertices $v$ in $I$ such that $|U_v| = i$, for $i = 2,3$. Note that $|B_3|$ is exactly the number of leaves lost from branching $F_1$ to $F_2$, so

$$|B_3| = \frac{N_2 - k_2}{3} - \frac{N_1 - k_1}{3}. \tag{4}$$

Also, $|B_2|$ is exactly the number of leaves lost from branching $F_2$ to $F_3$, so

$$|B_2| = \frac{N_3 - k_3}{2} - \frac{N_2 - k_2}{2}. \tag{5}$$

Finally, $|I| = |B_3| + |B_2|$ and $w(I) = 2|B_3| + |B_2|$.

Vertices with out-degree two and three in $T_s^*$ are all in the set $\textit{Candidates}$. Indeed, let $v$ be one such vertex. Either $v$ is an isolated vertex or $v$ is a leaf of a non-trivial component of $F_1$. So $d_{F_1}(v) = 0$. As the children of $v$ in $T_s^*$ have in-degree 0 in $F_1$, they are all in $U_v$. Hence $v \in \textit{Candidates}$.

For $i = 2,3$, let $C_s^i$ be the set of vertices of $\textit{Candidates}$ with out-degree $i$ in $T_s^*$, and let $C = \cup_s C_s^i$. The number of leaves in $T_s^*$ is $|Z_s| = 2|C_s^3| + |C_s^2| + 1$. The set of internal vertices of $T_s^*$ and of $T_{s'}^*$ are disjoint for distinct witnesses $s$ and $s'$. So the sets $C_s^i$ and $C_{s'}^i$ are disjoint. Note that $C$ is an independent set in $G$, thus $w(C) = 2|C_s^3| + |C_s^2| \leq w(I^*) \leq \alpha w(I)$, where $I^*$ is a maximum weight independent set in $(G, w)$. Hence

$$|Z| = \sum_s |Z_s| = \sum_s (2|C_s^3| + |C_s^2| + 1) = w(C) + \psi \leq \alpha w(I) + \psi = 2\alpha|B_3| + \alpha|B_2| + \psi.$$

Therefore,

$$\psi \geq |Z| - 2\alpha|B_3| - \alpha|B_2| = |Z| - 2\alpha \left(\frac{N_2 - k_2}{3} - \frac{N_1 - k_1}{3}\right) - \alpha \left(\frac{N_3 - k_3}{2} - \frac{N_2 - k_2}{2}\right),$$

which completes the proof of (3). \hfill \Box

Now we are ready to present the proof of Theorem 5.1.

of Theorem 5.1  First, suppose $\alpha \geq \frac{4}{3}$. In this case, $\frac{3-2\alpha}{3} \leq \frac{1}{12}$ and, by Lemmas A.1 and A.2

$$\text{opt}(D) \leq \frac{3 - 2\alpha}{3} (N_1 - k_1) + \frac{\alpha}{6} (N_2 - k_2) + \frac{\alpha}{2} (N_3 - k_3) + 1 \leq \frac{\alpha}{12} (N_1 - k_1) + \frac{\alpha}{6} (N_2 - k_2) + \frac{\alpha}{2} (N_3 - k_3) + \alpha \leq \alpha \ell(T).$$
Now, suppose $\alpha < \frac{4}{3}$, and let $\beta = \frac{4}{3} - \alpha$. By Lemma A.2,
\[
\text{opt}(D) \leq \frac{3 - 2\alpha}{3} (N_1 - k_1) + \frac{\alpha}{6} (N_2 - k_2) + \frac{\alpha}{2} (N_3 - k_3) + 1
\]
\[
\leq \left(\frac{1}{9} + \frac{2}{3} \beta\right) (N_1 - k_1) + \left(\frac{2}{9} - \frac{1}{6} \beta\right) (N_2 - k_2)
\]
\[
+ \left(\frac{2}{3} - \frac{1}{2} \beta\right) (N_3 - k_3) + 1
\]
\[
= \frac{1}{9} (N_1 - k_1) + \frac{2}{9} (N_2 - k_2) + \frac{2}{3} (N_3 - k_3) + \frac{4}{3}
\]
\[
+ \frac{2}{3} \beta (N_1 - k_1) - \frac{1}{6} \beta (N_2 - k_2) - \frac{1}{2} \beta (N_3 - k_3) - \frac{1}{3}
\]
\[
\leq \frac{4}{3} \ell(T) + \frac{2}{3} \beta \left( (N_1 - k_1) - \frac{1}{4} (N_2 - k_2) - \frac{3}{4} (N_3 - k_3) \right) - \frac{1}{3}
\]
\[
\leq \frac{4}{3} \ell(T),
\]
where (6) holds by Lemma A.1 and (7) holds because the number of components in $F_1$, $F_2$, and $F_3$ is so that $n - N_1 + k_1 \geq n - N_2 + k_2 \geq n - N_3 + k_3$, and this implies that $N_1 - k_1 \leq N_2 - k_2 \leq N_3 - k_3$, and therefore $N_1 - k_1 \leq \frac{1}{4} (N_2 - k_2) + \frac{3}{4} (N_3 - k_3)$. \qed