On Taylor coefficients of smooth functions

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Abstract We study the Borel map, which maps infinitely differentiable functions on an interval to the jets of their Taylor coefficients at a given point in the interval. Our main results include a complete description of the image of the Borel map for Beurling classes of smooth functions and a moment-type summation method which allows one to recover a function from its Taylor jet. A surprising feature of this description is an unexpected threshold at the logarithmic class. Another interesting finding is a "duality" between non-quasianalytic and quasianalytic classes, which reduces the description of the image of the Borel map for non-quasianalytic classes to the one for the corresponding quasianalytic classes, and complements classical results of Carleson and Ehrenpreis.

1 Introduction

1.1 We study the Borel map $B : C^\infty(I) \rightarrow \mathbb{C}^\mathbb{Z}_+$, which maps infinitely differentiable functions on the interval $I \subseteq \mathbb{R}$ to the jets of their Taylor coefficients at a given point $x \in I$. From here on, we assume that $I$ contains the origin, and let $x = 0$. Then $Bf = (\hat{f}(n))_{n \geq 0}$, where

$$\hat{f}(n) = \frac{f^{(n)}(0)}{n!}.$$

Given a class of smooth functions $A \subset C^\infty(I)$, the three classical problems that arise naturally, and go back to Borel and Hadamard, are as follows:

1. The uniqueness problem– When is the restriction $B|_A$ injective?
2. The punctual image problem– What is the image $BA$?
3. The summation problem– Given a sequence $a \in BA$, to recover a function $f \in A$ with $\hat{f} = a$.

These problems are usually studied for Carleman and Beurling classes of smooth functions.

Let $I \subseteq \mathbb{R}$ be an interval and let $L : [0, \infty) \rightarrow [1, \infty)$ be a non–decreasing function with $\lim_{\rho \rightarrow \infty} L(\rho) = \infty$. The Carleman class $C(L; I)$ consists of all $C^\infty(I)$–functions $f$ such that for every closed subinterval $J \subset I$, there exist constants $C, C_1 > 0$ such that

$$\max_J |f^{(n)}| \leq C \cdot (C_1 n L(n))^n, \quad n \in \mathbb{Z}_+.$$

The Beurling class $C_0(L; I)$ consists of all $C^\infty(I)$–functions $f$ such that for every closed subinterval $J \subset I$ and every $\delta > 0$, there exists a constant $C > 0$ such that

$$\max_J |f^{(n)}| \leq C \cdot (\delta n L(n))^n, \quad n \in \mathbb{Z}_+.$$

Here and elsewhere, we assume that $0^0 = 1$.

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1 Introduction

1.2 The classes $C_0(L; I)$ and $C(L; I)$ are called *quasianalytic* if the Borel map is injective in these classes. The solution of the uniqueness problem in these classes was given by Denjoy and Carleman [8]: *The classes $C(L; I)$ and $C_0(L; I)$ are quasianalytic if and only if*

$$\int^\infty \frac{du}{uL(u)} = \infty.$$  

Throughout this work we will abuse terminology and refer to the function $L$ as quasianalytic if the above integral diverges.

1.3 In 1925 [12, p.162] de la Vallée Poussin wrote: “The question of finding a criterion of consistence for the initial values of a quasi-analytic function, of a certain class, and its derivatives (i.e., a criterion for a given sequence to be the sequence of Taylor coefficients of a function in a given quasianalytic class – A.K.), is thus before us. But it seems exceedingly difficult to solve. We shall not undertake it here.” Since that the punctual image and the summation problems were studied by many authors who obtained a number of non–trivial results. Here we mention the works of Carleman [8], Bang [3], Carleson [11, 9], Ehrenpreis [15], Badalyan [2], and Écalle [14]. However, these works do not provide explicit answers to the punctual image and summation problems in what is likely the most interesting and delicate case, namely, when $L$ is slowly varying, i.e.,

$$\lim_{\rho \to \infty} \frac{L(\lambda \rho)}{L(\rho)} = 1, \ \forall \lambda > 1. \tag{1.1}$$

On the other hand, if the function $L$ grows fast, that is,

$$\liminf_{\rho \to \infty} \frac{L(2\rho)}{L(\rho)} > 1, \tag{1.2}$$

a simple description of the punctual image follows from a more general result, proven independently by Carleson [11], Ehrenpreis [15] and Mityagin [22]:

1.3.1 Put

$$\mathcal{F}_0(L) := \{(a_n)_{n \geq 0} : |a_n|^{1/n} = o(L(n)), \ n \to \infty\}. \tag{1.3}$$

Then

$$BC_0(L; I) = \mathcal{F}_0(L) \tag{1.4}$$

provided that condition (1.2) is satisfied.

Note that the inclusion

$$BC_0(L; I) \subseteq \mathcal{F}_0(L) \tag{1.5}$$

follows from the definition of the Beurling classes $C_0(L; I)$, and that a statement similar to (1.5) also holds for the Carleman classes $C(L; I)$ with $L$ satisfying condition (1.2).

1.3.2 On the other hand, for unbounded and slowly growing functions $L$ satisfying (1.1), the inclusion in (1.5) is always proper. In the quasianalytic case, this was shown by Täcklind [28] and Bang [4], while in the non-quasianalytic case, this follows from the above mentioned works of Carleson, Ehrenpreis and Mityagin.

1.4 Our goal is to give sufficiently explicit answers to the punctual image and summation problems, however, under rather restrictive smoothness assumptions imposed on the slowly growing functions $L$. Our inspiration comes from Beurling’s work [5, pp. 420–429], in which he gave a concise solution to these two problems in the logarithmic class when $L(\rho) = \log(\rho + e)$. 
1.5 Notation We shall use the symbol $C$ to denote large positive constants which may change their values at different occurrences. If a constant $C$ depends on some parameter $p$, we will write $C_p$ (again the value can change in different occurrences). Given two functions $f$ and $g$ with the same domain of definition, we write $f \lesssim g$ whenever $f(x) \leq C g(x)$ for some constant $C$. If the constant $C$ in the last inequality depends on some parameter $p$, we will write $f \lesssim_p g$ (again the value can change in different occurrences). Given two functions $f$ and $g$ with the same domain of definition, we write $f \preccurlyeq g$ whenever $f(x) \leq C g(x)$ for some constant $C$. If the constant $C$ in the last inequality depends on some parameter $p$, we will write $f \preccurlyeq_p g$. We use the notation $f \sim g$ if $f \preccurlyeq g$ and $g \preccurlyeq f$ (if $f \preccurlyeq_p g$ and $g \preccurlyeq_p f$). If for some set $\Pi$ we have $f|\Pi \preccurlyeq g|\Pi$, we will write $f \preccurlyeq g$ in $\Pi$, and we will do the same for $\sim$, $\preccurlyeq$ and $\preccurlyeq_p$.

2 Basic notions

2.1 Moment (Borel-type) summation of divergent series. Here we recall the classical moment summation method that goes back to Borel. We follow Hardy’s treatise [16]. Let $(\gamma_n)_{n \geq 0}$ be a fast growing sequence of positive numbers, $\lim_{n \to \infty} \gamma_1^n = \infty$, and suppose it is a moment sequence for a function $K$ defined on $\mathbb{R}^+$, that is

$$\int_0^\infty t^n K(t) dt = \gamma_{n+1}, \quad \forall n \in \mathbb{Z}^+.$$  \hspace{1cm} (2.1)

Let $(a_n)_{n \geq 0}$ be a sequence of complex numbers. If the series

$$A(t) = \sum_{n \geq 0} \frac{a_n t^n}{\gamma_{n+1}}$$

is convergent for $0 \leq t < t_0$, and has an analytic continuation on the whole positive ray $\mathbb{R}^+$ such that

$$\int_0^\infty A(t) K(t) dt = b,$$

then the series $\sum_{n \geq 0} a_n$ is said to be $(\gamma)$–summable to the value $b$.

2.1.1 The entire function

$$E(z) = \sum_{n \geq 0} \frac{z^n}{\gamma_{n+1}}$$

plays an important role in the theory of moment summation. Recall that a summation method is called regular if it agrees with actual limits of convergent series, and is called stable if the difference of the values it assigns to $\sum_{n \geq 0} a_n$ and $\sum_{n \geq 1} a_n$ is $a_0$. A necessary condition for regularity of the moment method is

$$\int_0^\infty E(x t) K(t) dt dy = \frac{1}{1 - x}, \quad 0 \leq x < 1,$$

while a necessary condition for its stability is

$$\int_0^\infty E(t) K(t) dt = \infty.$$

This hints at a strong connection between the growth of $E$ and the decay of $K$ on $\mathbb{R}^+$. 


2.1.2 Note that the sequence \( \gamma_n = n! \) corresponds to the classical Borel summation. In this case, \( K(t) = e^{-t} \) and \( E(z) = e^z \), that is \( E \cdot K = 1 \). The sequences \( \gamma_n = \Gamma(n\alpha) \), \( \alpha > 0 \), and \( \gamma_n = \log^n(n + e) \) correspond to the Mittag–Leffler and Beurling summations, respectively. In these cases, the asymptotics of the functions \( K \) and \( E \) are classical. In particular, there is also a very strong connection between the growth of \( E \) and the decay of \( K \) on \( \mathbb{R}_+ \) (in both cases, \( E(x)K(x) = O(\log(E(x))) \) as \( x \to \infty \), see Lindelöf [20, pp. 113–114]).

2.2 Beurling’s approach. In the preprint [9] written in 1936 and reproduced in his collected works [5, pp. 420–429], Beurling applied the idea of moment summation to the punctual image and summation problems in the logarithmic class \( C_0(L; I) \) with \( L(\rho) = \log(\rho + e) \). A somewhat similar approach to the summation problem was independently developed by Moroz [23, 24] in the context of divergent asymptotic series, which appear in mathematical physics for functions analytic in cusp domains.

2.2.1 To explain Beurling’s approach, we fix an increasing function \( L : [0, \infty) \to [1, \infty) \) with \( \lim_{\rho \to \infty} L(\rho) = \infty \) and put \( \gamma(\rho) = L(\rho)^\rho \). Given a function \( f \in C_0(L; I) \), consider the Taylor series

\[
\sum_{n \geq 0} \frac{\hat{f}(n)}{\gamma(n + 1)} z^n
\]

and note that, by the definition of the Beurling class \( C_0(L; I) \), this series has an infinite radius of convergence. Following Beurling, we call this series the singular transform of \( f \) and denote it by \( S_L f \).

2.2.2 The first observation is that \( S_L f \) depends on the sequence \( \hat{f} \) of Taylor coefficients of \( f \) at the origin, but does not depend on the values of \( f \) at other points of the interval \( I \). Therefore, studying the image of \( C_0(L; I) \) under the map \( S_L \) is equivalent to studying the punctual image of \( C_0(L; I) \).

2.2.3 We define the maps \( \hat{S}_L \) and \( \hat{R}_L \) that act on arbitrary sequences \( (a_n) \in \mathbb{C}^{\mathbb{Z}_+} \) as

\[
\left( \hat{S}_L a \right)(n) = \frac{a_n}{\gamma(n + 1)}, \quad n \in \mathbb{Z}_+,
\]

and

\[
\left( \hat{R}_L a \right)(n) = a_n\gamma(n + 1), \quad n \in \mathbb{Z}_+.
\]

Then, \( \hat{R}_L = \hat{S}_L^{-1} \) and \( \hat{S}_L \mathcal{B} = \mathcal{B} S_L \). Note that if \( a \) is an sequence in

\[
\mathcal{F}_0(L) := \left\{ (a_n)_{n \geq 0} : |a_n|^{1/n} = o(L(n)), \; n \to \infty \right\},
\]

then \( \hat{S}_L a \) are the Taylor coefficients of an arbitrary entire function. However, as Beurling observed, there are certain restrictions on the growth of entire functions in \( \hat{S}_L C_0(L; I) \) in horizontal strips. In principle, these restrictions can be used to characterize the punctual image of \( C_0(L; I) \).

2.2.4 Note that a similar idea can also be used for the Carleman classes. However, in that case, the Taylor series \([2.2]\) may have a finite radius of convergence. Thus, the description of the class \( S_L C(L; I) \) will include the fact that analytic functions from this class must have an analytic continuation to a horizontal strip. To fix ideas, we will discuss only the more transparent case of Beurling classes \( C_0(L; I) \).
2.2.5 To approach the summation problem, one needs to define the kernel $K$ that solves the moment problem

$$\int_0^{\infty} t^n K(t) dt = \gamma(n+1), \quad n \in \mathbb{Z}_+.$$ 

If the function $\gamma$ is nice (in particular, is analytic in the right half-plane), we can define $K$ as the inverse Mellin transform

$$K(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-s} \gamma(s) ds, \quad c > 0. \quad (2.3)$$

Then, one needs to check that for any entire function $F = S_L f$, $f \in C_0(L; I)$,

$$\int_0^{\infty} F(xt) K(t) dt = f(x), \quad x \in I$$

(cf. equation (2.1) in Section 2.1). Following Beurling, we call the integral

$$\int_0^{\infty} F(xt) K(t) dt$$

the regular transform of the function $F$, and denote it by $R_L F$. Note that $BR_L = \hat{R}_L B$.

2.2.6 To make this approach work, we need estimates on the asymptotic behavior of the functions $K$ and $E$. To do so, we impose certain regularity conditions on the weight function $L$ (a systematic study of the asymptotic behavior of these functions can be found in [18]).

These regularity conditions are quite technical and different in each part of this work. In order to overcome technical issues, we will first describe our results only for a particular choice of weight functions $L$ (the so-called Denjoy weights) which will satisfy all the regularity assumptions that will be imposed below. Then, before the proof of each result, we will restate it under the more general regularity assumptions.

2.2.7 Denjoy weights. A Denjoy weight is a function of the form

$$L_\alpha(s) = e^{\log^{\alpha_0}(s+1)} \prod_{k \geq 1} \log^{\alpha_k} (s + \exp_k(1)), \quad (2.4)$$

where $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots) \in \mathbb{R}_+^{\mathbb{Z}_+}$ is a multi-index with finitely many nonzero components $\alpha_k$ and with $0 \leq \alpha_0 < 1$. Here $\log_k$ is the $k$-th iterate of the logarithm function $s \mapsto \log s$, and $\exp_k$ is the $k$-th iterate of the exponential function $x \mapsto e^x$. We consider only $\alpha_0 < 1$ since, for $\alpha_0 \geq 1$, the function $L_\alpha$ is growing fast, that is, relation (1.2) holds.

3 Main results for Denjoy weights

3.1 The class $A(L; I)$ of entire functions. Following Beurling, we introduce a class of entire functions. This class is defined in terms of the function

$$E(z) = \sum_{n \geq 0} \frac{z^n}{\gamma(n+1)}, \quad \text{where} \quad \gamma(n) = L(n)^n.$$ 

The asymptotics of $E(z)$ as $z \to \infty$ will be used repeatedly in this work. Here we will only mention that if $L$ is a Denjoy weight, then the corresponding function $E$ grows very rapidly on the positive half-line ($\lim_{x \to \infty} x^{-a} \log E(x) = +\infty$ for any $a > 0$), but it is bounded on any infinite sector that does not meet the positive half-line (see Section 6 for the exact asymptotics of the function $E$).
**Theorem 1.** Suppose $L : [0, \infty) \to [1, \infty)$ be a continuous and eventually increasing function such that $\lim_{\rho \to \infty} L(\rho) = \infty$. Put $\gamma(\rho) = L(\rho)^\rho$ and $E(z) = \sum_{n \geq 0} \frac{z^n}{n^{(n+1)}}$. Let $I$ be an interval containing $0$. The class $A(L; I)$ consists of all entire functions $F$ with the property that, for every $c_+ \in I \cap (0, \infty)$, $c_- \in I \cap (-\infty, 0)$, and $Y > 0$,

$$\max_{|z| \leq Y} |F(u + iv)| \lesssim_{c_\pm, Y} E(u/c_\pm), \quad u \to \pm \infty. \quad (3.1)$$

Note that if $L(\rho) = o(L_1(\rho))$ as $\rho \to \infty$, then $A(L_1; I) \subset A(L; \mathbb{R})$ for any open interval $I$. Moreover, if $L$ is slowly growing (in particular, if $L$ is a Denjoy weight), then the RHS of (3.1) can be replaced by $\mu(u/c_\pm)$, where $\mu(r) = \max_{n \geq 0} \frac{r^n}{n^{(n+1)}}$, as well as by $\exp(L^{-1}(u/c_\pm))$, where $L^{-1}$ is the inverse function to $L$ (cf. Lemma 6.1 and its proof).

### 3.2 Extension of Beurling’s theorem

**Theorem 1.** Suppose $L$ is a Denjoy weight and $I$ is an interval containing the origin, such that $I \cap (0, \infty)$ and $I \cap (-\infty, 0)$ are open. Then, the regular transform $R_L$ maps $A(L; I)$ into $C_0(L; I)$.

**Theorem 2.** Suppose $L$ is a Denjoy weight and $I$ is an open interval containing the origin. Then, the singular transform $S_L$ maps $C_0(L; I)$ into $A(L; I) \cup A(\frac{1}{L}; \mathbb{R})$, where $\varepsilon(\rho) := \frac{\rho L(\rho)}{L(\rho)}$.

#### 3.2.1 Functions of at most logarithmic and of super-logarithmic growth

We say that a function $L$ has at most logarithmic growth if $L(\rho) = O(\log(\rho))$ as $\rho \to \infty$, and that $L$ has a super-logarithmic growth if $\log(\rho) = o(L(\rho))$ as $\rho \to \infty$. Note that any Denjoy weight must have either at most logarithmic or super-logarithmic growth (unlike more general functions $L$). Theorems 1 and 2 exhibit an essential difference between these two cases.

**Corollary 1.** Suppose that $L$ is a Denjoy weight and that $I$ is an open interval containing the origin.

1. If $L$ has at most logarithmic growth, then the singular transform $S_L$ maps $C_0(L; I)$ bijectively onto the space $A(L; I)$. Moreover, if $f \in C_0(L; I)$, then $R_L S_L f = f$.

2. If $L$ has super-logarithmic growth, then the singular transform $S_L$ maps $C_0(L; I)$ into $A(\frac{1}{L}; \mathbb{R})$.

Notice that in the case where $L$ has at most logarithmic growth we always have $L(\rho)\varepsilon(\rho) = O(1)$ as $\rho \to \infty$ and therefore, $A(L; I) \supset A(\frac{1}{L}; \mathbb{R})$. However, in the super-logarithmic case, we always have $\frac{1}{\varepsilon(\rho)} = o(L(\rho))$ as $\rho \to \infty$, and therefore $\varepsilon(\rho) A(\frac{1}{L}; \mathbb{R}) \supset A(L; I)$.

The first part of the corollary gives a full description of the punctual image and a summation method for (a divergent) Taylor series of functions in $C_0(L; I)$ when $L$ has at most logarithmic growth. In the case $L(\rho) = \log(\rho + e)$, this is the aforementioned result of Beurling. On the other hand, in the super-logarithmic case, these results do not give full answers, but still provide a non-trivial information about the punctual image and summation problem.

#### 3.2.2 Sharpness of Theorem 2

**Theorem 3.** Suppose that $L$ is a Denjoy weight with super-logarithmic growth and that $I$ is an open interval containing $0$. Then for any function $L_2$ satisfying $\frac{1}{\varepsilon(\rho)} = o(L_2(\rho))$ as $\rho \to \infty$ and any $\delta > 0$, $S_{L_2} C_0(L; (-\delta, \delta)) \not\subseteq A(L_2; \mathbb{R})$. In particular, $S_{L_2} C_0(L; (-\delta, \delta)) \not\subseteq A(L; \mathbb{R})$. 

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**Definition 1.** Let $L : [0, \infty) \to [1, \infty)$ be a continuous and eventually increasing function such that $\lim_{\rho \to \infty} L(\rho) = \infty$. Put $\gamma(\rho) = L(\rho)^\rho$ and $E(z) = \sum_{n \geq 0} \frac{z^n}{n^{(n+1)}}$. Let $I$ be an interval containing $0$. The class $A(L; I)$ consists of all entire functions $F$ with the property that, for every $c_+ \in I \cap (0, \infty)$, $c_- \in I \cap (-\infty, 0)$, and $Y > 0$,

$$\max_{|z| \leq Y} |F(u + iv)| \lesssim_{c_\pm, Y} E(u/c_\pm), \quad u \to \pm \infty. \quad (3.1)$$

Note that if $L(\rho) = o(L_1(\rho))$ as $\rho \to \infty$, then $A(L_1; I) \subset A(L; \mathbb{R})$ for any open interval $I$. Moreover, if $L$ is slowly growing (in particular, if $L$ is a Denjoy weight), then the RHS of (3.1) can be replaced by $\mu(u/c_\pm)$, where $\mu(r) = \max_{n \geq 0} \frac{r^n}{n^{(n+1)}}$, as well as by $\exp(L^{-1}(u/c_\pm))$, where $L^{-1}$ is the inverse function to $L$ (cf. Lemma 6.1 and its proof).
3.3 Duality between non-quasianalytic and quasianalytic classes. Given a non-quasianalytic function $L$, put

\[ \tilde{L}(\rho) = L(\rho) \int_{\rho}^{\infty} \frac{du}{uL(u)}, \quad \rho > 1. \]

Note that the function $\tilde{L}$ is always quasianalytic. Indeed, integration by parts yields

\[ \int_{\rho_0}^{\rho} \frac{du}{uL(u)} = \log \left( \int_{\rho_0}^{\infty} \frac{du}{uL(u)} \right) - \log \left( \int_{\rho}^{\infty} \frac{du}{uL(u)} \right) \]

and therefore

\[ \int_{\rho_0}^{\infty} \frac{du}{uL(u)} = \infty. \]

In the next table, we give some examples for non–quasianalytic functions $L$ and their duals $\tilde{L}$ (given here up to the asymptotic equivalence $\sim$).

| $I$ | $L$ | $L$ | Parameters |
|-----|-----|-----|------------|
| I   | $\log^\alpha(\rho + e)$ | $\log(\rho + e)$ | $\alpha > 1$ |
| II  | $\log(\rho + e) \log^\beta(\rho + e^e)$ | $\log(\rho + e) \log(\rho + e^e)$ | $\beta > 1$ |
| III | $\exp[\log^\alpha(\rho)]$ | $\log(1 - e^\alpha)(\rho + e)$ | $0 < \alpha < 1$ |
| IV  | $\exp[\frac{\log(\rho)}{\log(\rho + e^e)}]$ | $\log^\beta \log(\rho + e^e)$ | $\beta > 0$ |
| V   | $\rho^\alpha$ | $1$ | $\alpha > 0$ |

Our main results regarding non-quasianalytic classes are the following.

**Theorem 4.** Suppose $L$ is a non-quasianalytic Denjoy weight and $I$ is an open interval containing the origin. Then

\[ S_L C_0(L; I) = S_{\tilde{L}} C_0(\tilde{L}; \mathbb{R}). \]

There is also an analogous result for Carleman classes. We will make an exception and state it here because this is the only place in this work where the treatment of Beurling and Carleman classes requires different techniques.

We introduce the notation

\[ C(L; 0) := \bigcup_{\delta > 0} C(L; (-\delta, \delta)). \]

That is, $C(L; 0)$ consists of germs around the origin of the corresponding Carleman class.

**Theorem 5.** Suppose $L$ is a non-quasianalytic Denjoy weight and $I$ is an open interval containing the origin. Then

\[ S_L C(L; I) = S_{\tilde{L}} C(\tilde{L}; 0). \]

As we will discuss in Section 9.5.2.1, Theorems 4 and 5 are closely related to classical results by Carleson [11] and Ehrenpreis [15] and can be viewed as an improvement of their results but for a restricted class of weights.
3.3.1 Theorems [4] and [5] allow us to reduce the punctual image problem in the non-quasianalytic case to the same problem in the quasianalytic case. In particular, together with Corollary [1], we obtain the following:

**Corollary 2.** Suppose \( L \) is a Denjoy weight and \( I \) is an open interval containing the origin. If \( \log^a \rho \lesssim L(\rho) \) for some \( a > 1 \), then the singular transform \( S_L \) maps \( C_0(L; I) \) onto \( A(L; \mathbb{R}) \).

3.4 The splitting. The results presented so far describe fully the punctual image of Beurling classes, only in the cases that the Denjoy weight function \( L \) satisfies \( L \lesssim \log^a \log \rho \) for some \( a > 1 \). On the other hand, when the function \( L \) is closer to the quasianalyticity threshold, such as when \( L(\rho) = \log \rho \log^\beta \log \rho \), with \( \beta > 0 \), our results, so far, do not give a full description of the punctual image of \( C_0(L; I) \). Note that in the latter case, the inclusion \( S_L C_0(L; I) \subset A(\log; \mathbb{R}) \) of Corollary [1] part 2, is proper. So, to treat this case, new ideas are needed.

In order to overcome these difficulties, we will decompose the class \( C_0(L; I) \) into the sum of two classes \( C_0^\pm(L; I) \), in a way somewhat reminiscent to the decomposition of a Fourier series into the sum of its analytic and anti-analytic parts. Then, we will describe the image of the singular transform on each of the parts \( C_0^\pm(L; I) \). This description is valid for any Denjoy weights \( L \), both quasianalytic and non-quasianalytic, satisfying a very mild growth bound \( L(\rho) \lesssim \exp(\log^a \rho) \) with some \( a < 1/2 \) (see the discussion after Theorem [6]).

![Fig. 1: The classes \( C_0^\pm(L; I) \)](image)

**Definition 2.** Let \( I \) be an interval and \( L : [0, \infty) \to [1, \infty) \) a non-decreasing function with \( \lim_{\rho \to \infty} L(\rho) = \infty \). Then, \( C_0^+(L; I) \) is the class of all functions \( f \in C_0(L; I) \) for which there exists a domain \( \Pi = \Pi_f \subset \{ z : \text{Im}(z) > 0 \} \) with \( \mathbb{R} \cap \partial \Pi = \text{clos}(I) \) such that \( f \in C^\infty(\Pi \cup I) \cap \text{Hol}(\Pi) \). Similarly, \( C_0^-(L; I) \) is the class of all functions \( f \in C_0(L; I) \) for which there exists a domain \( \Pi = \Pi_f \subset \{ z : \text{Im}(z) < 0 \} \) with \( \mathbb{R} \cap \partial \Pi = \text{clos}(I) \) such that \( f \in C^\infty(\Pi \cup I) \cap \text{Hol}(\Pi) \).

Note that the Borel map may be injective in the classes \( C_0^+(L; I) \) even if in the original Beurling class \( C_0(L; I) \) it is not. In fact (see [19], [10] and [26]), the Borel map is injective in the classes \( C_0^\pm(L; I) \) if and only if

\[
\int_0^\infty \frac{du}{\sqrt{uL(u)}} = \infty.
\]

Also note that \( C_0^+(L; I) \cap C_0^-(L; I) = C^\omega(I) \) is the class of all real-analytic functions on \( I \), and that it is not hard to show (cf. Section 9.7.2.1 below) that

\[
C_0(L; I) = C_0^+(L; I) + C_0^-(L; I).
\]

3.4.1 Modifying the regular transform. Fix a Denjoy weight function \( L \) and put

\[
\gamma(s) = L(s)^c, \quad K(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-s} \gamma(s) ds, \quad c > 0.
\]
It follows from [18] (see also Section 6 below) that there exists $r_0 > 0$ such that $K$ is analytic in the set
\[ \Omega := \{ z : \log z = \log L(s) + \varepsilon(s), |\arg(s)| \leq \frac{\pi}{2}, |z| > r_0 \}, \quad \varepsilon(s) = \frac{sL'(s)}{L(s)} \]
and is $o(|z|^{-n})$ as $z \to \infty$ uniformly therein, for any $n > 0$. Therefore, by Cauchy’s Theorem, if $\Psi$ is any curve in the right half-plane, joining 0 and $\infty$, such that $\Psi \cap \{|z| > r_0\} \subset \Omega$, then
\[ \int_0^\infty t^n K(t) dt = \int_{\Psi} z^n K(z) dz = \gamma(n+1), \quad n \in \mathbb{Z}_+. \quad (3.2) \]

We denote by $\Psi_+$ and by $\Psi_-$ two curves joining 0 and $\infty$ in the first and fourth quadrants respectively, such that $\Psi_{\pm} \cap \{|z| > r_0\} \subset \partial \Omega$ (i.e., $\Psi_{\pm}$ coincide with the upper and lower parts of $\partial \Omega$) (see Figure 2). We modify the regular transform, putting
\[
(R_L^+ F)(t) := \int_{\Psi_+} F(tz)K(z)dz, \quad \pm t \geq 0,
\]
and
\[
(R_L^- F)(t) := \int_{\Psi_-} F(tz)K(z)dz, \quad \pm t \geq 0.
\]
whenever the integrals on the right-hand sides converge.

It follows from (3.2) that for any polynomial $P$,
\[ R_L P = R_L^+ P = R_L^- P. \]
So, we have $R_L^+ S_L = \text{Id}$ in the space of all polynomials. We will see that $R_L^\pm S_L = \text{Id}$ in $C_0^\pm (L; I)$ as well.

### 3.4.2 The spaces of entire functions $A^\pm (L; I)$.

Fix a Denjoy weight $L$. It is easy to see, that there exists $\delta > 0$ and $\rho_0 > 0$ such that for $0 < \psi < 2\delta$, there exist a unique solution to the equation
\[ \psi = \text{Im} (\log L(i\rho) + \varepsilon(i\rho)), \quad \rho > \rho_0. \]
We denote this solution by $\rho = \rho(\psi)$, and put
\[ H(\psi) = \exp \left[ \text{Re} (i\rho(\psi)\varepsilon (i\rho(\psi))) \right], \quad 0 < \psi < 2\delta. \]
By choosing the above $\delta$ sufficiently small, we can take $H$ to be positive, $C^1$, decreasing and with $H(0^+) = +\infty$. By changing the values of $H$ in the interval $(\delta, 2\delta)$, if necessary, we extend $H$ to a positive, $C^1$ and decreasing function defined in $(0, \frac{\pi}{2})$, and then we extend the domain of definition of $H$ to $(0, \pi)$, putting $H(\psi) = H(\pi - \psi)$. For example, if $L(\rho) = \log^\alpha \rho$, with $\alpha > 0$, then

$$\psi \sim \frac{\pi}{2} \cdot \frac{a}{\log \rho} \quad \text{and} \quad \log \log H(\psi) \sim \frac{\pi}{2} \cdot \frac{a}{\psi}, \quad \psi \to 0^+.$$  

**Definition 3.** Suppose that $I$ is an open interval containing the origin. The class $A^+(L; I)$ consists of all entire functions $F$ such that for any $B > 0$, $c_\pm \in I$ with $c_- < 0 < c_+$, there exists $\Delta > 0$ with

$$|F(re^{i\psi})| \lesssim_{B,c_\pm} H \left( \psi \pm \frac{2B}{r} \right) + E \left( \frac{r}{|c_\pm|} + \Delta r \sin \psi \right)$$

(3.3)

whenever $0 \leq \pm (\frac{\pi}{2} - \psi) \leq \frac{\pi}{2} + B/r$ (asymptotically this is the upper half-plane $\text{Im} \, z > -B$). We also put $A^-(L; I) = \{F : F(\bar{z}) \in A^+(L; I)\}$.

Note that in the case where $L$ has super–logarithmic growth (i.e., $\rho L'(\rho) \to \infty$ as $\rho \to \infty$), in the right-hand side of (3.3) the second term dominates above the curve $\Psi_+$, the first term dominates below the curve $\Psi_+$, while on $\Psi_+$ both terms have roughly the same growth. We also mention that in the logarithmic and sub–logarithmic cases, the first term on the RHS of (3.3) is always small compared with the second term and therefore can be discarded.

### 3.4.3 The main result of this part reads as follows.

**Theorem 6.** Suppose $L$ is a Denjoy weight with $L(\rho) \lesssim \exp(\log^a \rho)$ for some $a < 1/2$, and $I$ is an open interval containing the origin. Then, the singular transform $S_L$ maps $C_0^\infty(L; I)$ bijectively onto the space $A^\pm(L; I)$ with inverse $R_L^\pm$.

We will use the assumption $L(\rho) \lesssim \exp(\log^a \rho)$ for some $a < 1/2$ to guarantee that the estimate $\int_{x_0^+} |z^n K(z) dz| \leq C^{n+1} \gamma (n+1)$ holds for $n \in \mathbb{Z}_+$. This assumption is essential for our techniques.

The definition of the classes $A^\pm(L; I)$ can be simplified if we restrict the growth of $L$. For instance, if $L(\rho) \lesssim \log^2 \rho$, then the inequality (3.3) can be replaced by

$$|F(x + iy)| \lesssim h \left( \frac{\pi}{2} \frac{|x|}{2B + y} \right) + E \left( \frac{x}{c_\pm} + \Delta |y| \right), \quad y > -B, \, \pm x > 0,$$

where $h$ is the inverse function to $x \mapsto \frac{1}{\varepsilon(\log(x))}$. Even more explicitly, if $L(\rho) = \log^a \rho$ with $1 < a < 2$, then the inequality (3.3) can be replaced by

$$|F(x + iy)| \lesssim \exp \exp \left[ \frac{a\pi}{2} \frac{|x|}{2B + y} \right] + \exp \exp \left[ \left( \frac{x}{c_\pm} + \Delta |y| \right)^{1/a} \right], \quad y > -B, \, \pm x > 0,$$

while for $0 < a \leq 1$, inequality (3.3) can be replaced by

$$|F(x + iy)| \lesssim \exp \exp \left[ \left( \frac{x}{c_\pm} + \Delta |y| \right)^{1/a} \right], \quad y > -B, \, \pm x > 0.$$
3.4.4 Combined with the decomposition $C_0(L; I) = C_0^+(L; I) + C_0^-(L; I)$, Theorem 6 immediately yields

**Corollary 3.** Suppose that $L$ satisfies the assumptions of Theorem 6 and $I$ is an open interval containing the origin. Then

$$S_L C_0(L; I) = A^+(L; I) + A^-(L; I).$$

Moreover, if $f = f_+ + f_- \in C_0(L; I)$, with $f_\pm \in C_0^\pm(L; I)$, then

$$f = R_L^+ S_L f_+ + R_L^- S_L f_-.$$

Note that the last equality holds even if the class $C_0^0(L; I)$ is non-quasianalytic.

3.5 Structure of this work. The rest of this paper is organized as follows. In the next section we give some applications and examples of our results. In Section 5 we discuss a variety of regularity assumptions on the function $L$, which are used in the rest of this paper. Section 6 is devoted to the functions $K$ and $E$. There, we summarize the results of [18], and state additional estimates under different regularity assumptions from Section 5. In Section 7, we introduce the function $E_1$ which is the singular transform of the exponential function, and give a variety of bounds on it, to be used in the proofs of Theorems 3, 4 and 6. Section 8 is devoted to estimates of the singular transform of polynomials which are used in the proofs of Theorems 2, 4, 5 and 7. Finally in Section 9, we restate our main results in a more general form using the regularity assumptions of Section 5, and present proofs of these results.

4 Applications and examples

Here we present some applications and examples to our results.

4.1 Real-analytic functions. As a byproduct of our techniques used in the proof of Theorems 1 and 2, we also provide a description of the image of the class $C^\omega(I)$, consisting of real-analytic functions, under the singular transform.

**Definition 4.** Let $L : [0, \infty) \to [1, \infty)$ be a continuous and eventually increasing function such that $\lim_{\rho \to \infty} L(\rho) = \infty$. Put $\gamma(\rho) = L(\rho)^\rho$ and $E(z) = \sum_{n \geq 0} \frac{z^n}{\gamma(n+1)}$. Let $I$ be an interval containing 0. The class $A^\omega(L; I)$ consists of all entire functions $F$ such that, for every $c_- \in I \cap (-\infty, 0)$, $c_+ \in I \cap (0, \infty)$ there exists $\Delta = \Delta_{c_+} > 0$ so that

$$|F(u + iv)| \lesssim_{c_+, c_-} E\left(\frac{u}{c_+} + \Delta|v|\right), \quad \pm u \geq 0. \quad (4.1)$$

**Theorem 7.** Suppose that $L$ is a Denjoy weight, and that $I$ is an open interval containing 0. Then the singular transform maps the class of real-analytic functions $C^\omega(I)$ bijectively onto the set $A^\omega(L; I)$. In particular, $R_L S_L f = f$ for any $f \in C^\omega(I)$.

The proof of Theorem 7 is given in Section 9.
4.1.1 Summation in the Mittag–Leffler star. Recall that a domain \( \Omega \subseteq \mathbb{C} \) containing the origin is called star-shaped (with respect to the origin), if for any \( z \in \Omega \), \([0, z] \subset \Omega \). If \( f(z) = \sum_{n \geq 0} \hat{f}(n)z^n \) is analytic in a neighborhood of the origin, we denote by \( \Omega_f \) the largest star-shaped domain to which \( f \) can be analytically continued, and call \( \Omega_f \) the Mittag–Leffler star of \( f \). Next, we introduce a class of entire functions.

**Definition 5.** Let \( L : [0, \infty) \to [1, \infty) \) be a continuous and eventually increasing function such that \( \lim_{\rho \to \infty} L(\rho) = \infty \). Put \( \gamma(\rho) = L(\rho)^\rho \) and \( E(z) = \sum_{n \geq 0} \frac{z^n}{\gamma(n+1)} \). Let \( \Omega \subseteq \mathbb{C} \) be a star-shaped domain with respect to the origin. The class \( A^\omega(L; \Omega) \) consists of all entire functions \( F \) such that for any \( \delta > 0 \),

\[
|F(w)| \lesssim \delta \; E(H_\Omega(w) + \delta|w|),
\]

where \( H_\Omega(w) = \inf\{\lambda > 0 : w \in \lambda \Omega\} \) is the Minkowski functional of \( \Omega \).

The next result follows from Theorem \( \underline{\text{II}} \).

**Theorem 8.** Suppose \( L \) is a Denjoy weight, and \( \Omega \subseteq \mathbb{C} \) is a star–shape domain with respect to the origin. Then the singular transform maps the class \( \text{Hol}(\Omega) \) bijectively onto the class \( A^\omega(L; \Omega) \). Moreover, if \( f \in \text{Hol}(\Omega) \), then \( R_LS_f \equiv f \) in \( \Omega \).

In particular, the above theorem shows that if \( f(z) = \sum_{n \geq 0} \hat{f}(n)z^n \) has a positive radius of convergence, then \( R_LS_f(z) \) is the analytic continuation of \( \hat{f} \) to its Mittag–Leffler star \( \Omega_f \). This fact is in contrast to the classical Borel and Mittag–Leffler moment summations methods (see Section 2.1.2), which usually do not converge to \( f \) in the whole Mittag–Leffler star \( \Omega_f \). For example, the series \( \sum_{n \geq 0} z^n \) is Mittag–Leffler \( \Gamma(\alpha n + 1) \)-summable to the function \( \frac{1}{1-z} \) only in the domain

\[
\left\{z : \arg(z-1) > \frac{\pi}{2} \right\},
\]

while, if \( L \) is any function satisfying the conditions of Theorem \( \underline{\text{III}} \) then

\[
R_LS_f \left( \sum_{n \geq 0} x^n \right)(z) = \frac{1}{1-z}, \quad \text{for} \quad 0 < \arg(z-1) < 2\pi.
\]

4.2 Examples.

4.2.1 Suppose that \( L_1 \) and \( L_2 \) are two Denjoy weights satisfying \( L_1(\rho) = o(L_2(\rho)) \), as \( \rho \to \infty \). Put \( \gamma_j(\rho) = L_j(\rho)^\rho \), \( j = 1, 2 \). Then the sequence

\[
\hat{f}_0(n) = e^{i n \theta} \frac{\gamma_2(n+1)}{\gamma_1(n+1)}, \quad n \geq 0,
\]

belongs to the punctual image of the class \( C_0(L_2; \mathbb{R}) \) for \( 0 < \theta < \pi \), and to the punctual image of the class \( C_0(L_2; [0, \infty)) \) for \( \theta = \pi \).

**Proof.** Put

\[
E(z) = \sum_{n \geq 0} \frac{z^n}{\gamma_1(n+1)}, \quad E_\theta(z) = E(ze^{i \theta}).
\]

Since \( L_1 \) is a Denjoy weight, the function \( E \) is bounded on any sector that does not meet the positive ray \( \{ z : \delta < \arg(z) < 2\pi - \delta \} \) (see Theorem \( \underline{\text{III}} \) and Lemma \( \underline{\text{X}} \) in Section 6 below). Thus, for any \( 0 < \theta < \pi \) (respectively \( \theta = \pi \)) the function \( z \mapsto E_\theta(z) \) is bounded on any horizontal
For any $\delta > 0$, we can find a function $L = f(\theta) = \frac{F(n)(0)}{n!} - \gamma(n + 1)$, $n \geq 0$, not only for Denjoy weights. In fact, it is true for any weight function $L$ for instance [25]) that for any quasianalytic function $f$ cannot be extended to a function in $C_0(L; \mathbb{R})$. Let $\tilde{f} = \frac{F(n)(0)}{n!} - \gamma(n + 1)\mathbb{R}$ such that $f_\delta := R_L E_\delta \in C_0(L; \mathbb{R})$, $f_\pi := R_L E_\pi \in C_0(L; [0, \infty))$. By the definition of the regular transform, $f_\delta$ are the desired functions.

**4.2.2** Suppose that $L$ is a Denjoy weight. Then for any entire functions $F$ with a real period, the sequence

$$\hat{f}(n) = \frac{F(n)(0)}{n!} - \gamma(n + 1), \quad n \geq 0,$$

belongs to the punctual image of the class $C_0(L; \mathbb{R})$.

**Proof.** Let $F$ entire function with a real period. Clearly $F$ is bounded on any horizontal strip and therefore it belongs to $A(L; \mathbb{R})$. By Theorem [1] $f := R_L F \in C_0(L; \mathbb{R})$.

**4.3 Non-extendable Beurling classes.** Any real-analytic function $f$ in the interval $[0, 1)$ can be analytically extended to an interval $(-\delta_f, 1)$, where $\delta_f > 0$. Moreover, it is known (see for instance [25]) that for any quasianalytic function $L$, there exists $f \in C_0(L; [0, 1))$ such that $f$ cannot be extended to a function in $C_0(L_1; (-\delta, 1))$ with any $\delta > 0$ and any other quasianalytic weight function $L_1$.

Using Theorems 1 and 2, we can show that for any Denjoy weight $L$, quasianalytic or not, there exists a function $f \in C(L; [0, 1))$ such that $f$ cannot be extended to a function in $C(L; (-\delta, 1))$ with any $\delta > 0$, or, which is the same,

$$\bigcup_{\delta > 0} C_0(L; (-\delta, 1)) \not\subseteq C_0(L; [0, 1)).$$

This shows that the extendability question for Beurling or Carleman classes is not trivial also in non-quasianalytic classes.

**Proof.** Fix a Denjoy weight $L$, and put $\varepsilon(\rho) = \frac{\rho L'(\rho)}{L(\rho)}$. Let $L_1$ be another Denjoy weight satisfying $L_1(\rho) = o(L(\rho))$ and $L_1(\rho) = o\left(\frac{1}{\varepsilon(\rho)}\right)$ as $\rho \to \infty$. Put

$$E_1(z) = \sum_{n \geq 0} \frac{z^n}{L_1(n + 1)^{n+1}}, \quad E(z) = \sum_{n \geq 0} \frac{z^n}{L(n + 1)^{n+1}}, \quad \tilde{E}(z) = \sum_{n \geq 0} \varepsilon(n + 1)^{n+1}z^n.$$

For any $\delta > 0$ we then have,

$$E(x) + \tilde{E}(x) \lesssim_\delta E_1(\delta x), \quad x > 0.$$

In particular, $E_1(-z) \notin A(L; I) \cup A_{\frac{1}{2}}(\mathbb{R})$ for any open interval $I$ containing 0. On the other hand, as we have already mentioned in Section 1.2.1, it follows from Theorem [3] below that the function $E_1(-z)$ is bounded in the right half-plane. In particular, $E_1(-z) \in A(L; [0, \infty))$. Put $f = R_L (E(-z))$. By Theorem [1] $f \in C_0(L; [0, \infty))$, and by Theorem [2] $f \notin C_0(L; I)$ for any open interval containing 0.

**Remark 1.** The above assertion is true not only for Denjoy weights. In fact, it is true for any Beurling class $C_0(L; [0, 1))$, with

$$\log L(\rho) = o(\log \rho), \quad \rho \to \infty. \quad (4.2)$$

This follows from Theorems [1] and [2] of Section 9, since for any function $L$ satisfying (4.2), we can find a function $L_1$ satisfying the assumptions of Theorems [1] and [2] and such that $C_0(L; I) \subseteq C_0(L_1; I)$. The proof of this fact is the same as of the preceding assertion.
4.4 Functions of class $C_0(L; I)$ with positive or sparse Taylor series. It is well known that if the class $C_0(L; I)$ is quasianalytic and a function $f \in C_0(L; I)$ has positive (see for instance [25] or [28]) or lacunary Taylor series (see [8] and [9]), then $f$ must be analytic in some neighborhood of the origin, i.e., there exists $C > 0$ such that $|\hat{f}(n)| \lesssim C^n$. We are going to show that similar (at least in spirit) phenomena occur for arbitrary Denjoy weights.

4.4.1 Let $I = (-a, a)$ and let $L$ be a Denjoy weight. Put

$$L_*(\rho) = \frac{L(\rho)}{pL'(\rho) + 1} \quad \text{and} \quad \gamma_*(\rho) = L_*(\rho)^\rho$$

(i.e., $L_*$ is the harmonic mean of the functions $L$ and $\frac{1}{\varepsilon}$). Suppose $f \in C_0(L; I)$ satisfies one of the following two conditions:

1. $\hat{f}(n) \geq 0$, $n \in \mathbb{Z}_+$. 
2. $f \sim \sum_{n \geq 0} \hat{f}(\lambda_n) x^{\lambda_n}$, $\sum_{n \geq 0} \lambda_n^{-1} < \infty$.

Then,

$$|\hat{f}(n)| \lesssim a^{-n} \frac{\gamma(n + 1)}{\gamma_*(n + 1)^*}, \quad n \in \mathbb{Z}_+.$$ 

Proof. Fix $f \in C_0(L; I)$ and put $F = S_L f$. Since $L_*(\rho) \leq L(\rho)$, we have for any $x > 0$

$$E(x) \leq E_*(x), \quad \text{where} \quad E_*(x) := \sum_{n \geq 0} \frac{x^n}{\gamma_*(n + 1)}.$$ 

By Theorem 2 for any $\delta > 0$,

$$|F(x)| \lesssim_\delta E\left(\frac{1 + \delta}{a} x\right) \leq E_*(\frac{1 + \delta}{a} x).$$ 

Assume now that $f$ satisfies one of the conditions 1 or 2. Therefore, so does $F$, which in turns yields

$$|F(z)| \lesssim_\delta E_*(\frac{1 + \delta}{a} |z|), \quad z \in \mathbb{C}.$$ 

In the first case this follows from the fact that $|F(z)| \leq F(|z|)$ for any entire function with positive Taylor coefficients, while in the second case the estimate follows from a theorem of Anderson and Binmore [1].

Applying Theorem 8 to the Denjoy weight $L_*$, we find that

$$g := R_L, F \in \text{Hol}(\{|z| < a\}),$$

which in turn yields $|\hat{g}(n)| \lesssim a^{-n}$. The assertion follows from the relations

$$\hat{g}(n) = \gamma_*(n + 1) \hat{F}(n), \quad \hat{f}(n) = \gamma(n + 1) \hat{F}(n), \quad n \geq 0.$$ 

4.5 So far most of our results and applications may be interpreted as telling us how “small” the punctual image of $C_0(L; I)$ can be as a subset of $\mathcal{F}_0(L)$ (defined in [1.3]). The next result goes in some sense in the opposite direction.
4.5.1 Suppose that $L$ is a Denjoy weight. Then for any sequence $(a(n))_{n \geq 1} \in \mathcal{F}_0(L)$ there exist functions $f_1, f_2 \in C_0(L; \mathbb{R})$ such that

$$a(n) = \tilde{f}_1(n) + i^n \tilde{f}_2(n) \quad n \geq 0.$$  

Proof. Put $A(z) := \sum_{n \geq 0} \frac{a(n)}{7(n+1)} z^n$. By definition, this series has infinite radius of convergence, i.e., $A$ is entire. By a theorem of Ehrenpreis [15, p. 131], there exist entire functions $F_1$ and $F_2$, bounded on any horizontal strip, such that $A(z) = F_1(iz) + F_2(iz)$. In particular $F_1, F_2 \in A(L; \mathbb{R})$. Put $f_1 = R_L F_1$ and $f_2 = R_L F_2$. By Theorem 1 $f_1, f_2 \in C_0(L; \mathbb{R})$. The equality $A(z) = F_1(z) + F_2(iz)$ implies that

$$\frac{a(n)}{\gamma(n+1)} = \hat{F}_1(n) + i^n \hat{F}_2(n), \quad n \geq 0,$$

which in turn yields the assertion.

4.6 Non-quasianalytic classes with the same image under the singular transform.

Fix a non-quasianalytic Denjoy weight $L$ and denote by $\tilde{L}$ its quasianalytic dual, i.e.

$$\tilde{L}(\rho) := \int_{\rho}^{\infty} \frac{du}{u L(u)}, \quad \rho \geq 1.$$  

As we mentioned, $\tilde{L}$ is always quasianalytic. Differentiating the definition of $\tilde{L}$ yields

$$\frac{\rho \tilde{L}'(\rho) + 1}{\tilde{L}(\rho)} = \frac{\rho L'(\rho)}{L(\rho)},$$

and thus,

$$\tilde{L}(\rho) \exp \left( \int_{1}^{\rho} \frac{du}{u L(u)} \right) = CL(\rho),$$

where $C^{-1} = \int_{1}^{\infty} \frac{du}{u L(u)}$. We note that starting with an arbitrary quasianalytic function $\tilde{L}$ and defining $L$ by the above equality will always yield a non-quasianalytic $L$.

For $a > 0$, consider the function

$$L_a = \tilde{L} \left( \frac{L}{\tilde{L}} \right)^a.$$  

It follows from the above considerations that $L_a$ is again a non-quasianalytic and satisfies

$$\tilde{L}_a = a^{-1} \tilde{L}.$$  

Notice that the Beurling class $C_0(\tilde{L}; \mathbb{R})$ and the Carleman class $C(\tilde{L}; 0)$ (defined in Section 3.3) remain unchanged when we replace $\tilde{L}$ with $a^{-1} \tilde{L}$. Theorems 4 and 5 then admit.

Corollary 4. Suppose $L$ is a non–quasianalytic function that satisfies the assumptions of Theorem 4. Then for any $a > 0$,

$$S_{L_a} C_0(L_a; I) = S_L C_0(L; I), \quad S_{L_a} C(L_a; I) = S_L C(L; I).$$

For example, consider $L(\rho) = \log^2(\rho + e)$. In this case, $\tilde{L}(\rho) \asymp \log \rho$ and $L_a(\rho) \asymp \log^{1+a} \rho$. Thus

$$S_{\log^{1+a}} C_0(\log^{1+a}; I) = S_{\log^2} C_0(\log^2; I) \quad \text{for any} \quad a > 0.$$  

Note that, by Corollary 2, the classes $S_{\log^{1+a}} C_0(\log^{1+a}; I), 0 < a$, are in fact $A(\log; \mathbb{R})$. 
4.7 A Phragmén–Lindelöf type theorem. So far our applications used function theory in order to obtain results about Beurling classes. The next one goes in the opposite direction and provides a non-trivial result about the growth of entire functions in the case where $L$ is quasianalytic with super-logarithmic growth.

Corollary 5. Suppose $L$ is a function satisfying the assumptions of Theorem 6, and $I$ is an open interval containing the origin. Then, $L$ is quasianalytic if and only if

$$A^+(L; I) \cap A^-(L; I) = A^\omega(L; I).$$

For instance, consider the case, $L(\rho) = \log(\rho + e) \log \log(\rho + e^\epsilon)$, and $I = (-1, 1)$. Suppose that $F \in A^+(L; I) \cap A^-(L; I)$, i.e., for any $\delta > 0$ and $B > 0$, there exists $\Delta > 0$, such that

$$|F(x + iy)| \lesssim_{\delta,B} \exp \exp \left( \frac{\pi}{2} \frac{|x|}{|y| + B} \right) \exp \exp \left( \frac{(1 - \delta)|x| + \Delta|y|}{\log(|x| + |y|)} \right), \quad |x| + |y| \geq 2.$$

Corollary 5 implies that in fact $F \in A^\omega(L; I)$, i.e., for any $\delta > 0$ there exists $\Delta > 0$, such that

$$|F(x + iy)| \lesssim_{\delta} \exp \exp \left( \frac{(1 - \delta)|x| + \Delta|y|}{\log(|x| + |y|)} \right), \quad |x| + |y| \geq 2.$$

This conclusion is no longer true if we replace $L$ with $L(\rho) = \log(\rho + e) \log^\beta \log(\rho + e^\epsilon)$ for some $\beta > 1$ (i.e., if we replace the log term in the right-hand side of the above majorants with $\log^\beta$).

Proof of Corollary 5. The Beurling class $C_0(L; I)$ is quasianalytic if and only if $S_L : C_0(L; I) \to \text{Hol}(\mathbb{C})$ is injective. We have already mentioned that

$$C_0(L; I) = C_0^+(L; I) + C_0^-(L; I), \quad C_0^+(L; I) \cap C_0^-(L; I) = C^\omega(I).$$

By the uniqueness theorem for analytic functions, the restriction of $S_L$ to $C^\omega(I)$ is clearly injective. Since $S_L$ is also linear, we conclude that $S_L : C_0(L; I) \to \text{Hol}(\mathbb{C})$ is injective if and only if

$$S_L C_0^+(L; I) \cap S_L C_0^-(L; I) = S_L C^\omega(I).$$

By Theorems and 6 and 7, the latter holds if and only if

$$A^+(L; I) \cap A^-(L; I) = A^\omega(I).$$

5 Regularity assumptions

Here we present a list of regularity assumptions on the weight $L$ which will be used below. These assumptions are divided into two sets: the first set gathers assumptions on the behavior of $L$ for large positive numbers, while the second concerns the behavior of $L$ in the complex plane. We begin with the first set.

Let $L : [0, \infty) \to [1, \infty)$ be a $C^3$, unbounded and eventually increasing function with $L(0) = 1$. Put $\ell(t) = \log L(e^t)$ and consider the following regularity assumptions:

(R1) $\ell'(t) = o(1), \quad t \to +\infty$.

(R2) The function $\ell$ is eventually concave.

(R3) The function $\ell'$ is bounded from above and $\ell''(t) = o(\ell'(t)), \quad t \to +\infty$. 

\qed
(R4) $\ell''(t)\ell(t) = o(\ell'(t))$, $t \to +\infty$.

(R5) The function $|\ell''|$ is eventually decreasing $\ell''(t)\ell(t) = o(|\ell'(t)|)$, $t \to +\infty$.

(R6) $\ell''(t)\log \frac{1}{\ell(t)} = o(\ell'(t))$, $t \to +\infty$.

(R7) $\ell'(t)\ell(t) = o(1)$, $t \to +\infty$.

Note that assumptions (R1)–(R6) are met for any Denjoy weight $L$, while assumption (R7) is fulfilled if and only if $L(\rho) \lesssim \exp(\log^a \rho)$ for some $a < \frac{1}{2}$.

Assumptions (R1)–(R3) are standard. Assumption (R1) is equivalent to fact that the function $L$ is slowly varying (i.e., $L$ satisfies (1.1)), assumption (R2) to the fact that the function $\varepsilon(\rho) = \frac{\rho L'(\rho)}{L(\rho)}$ is eventually non-increasing, and assumption (R3) to the fact that the function $\varepsilon$ is slowly varying. On the other hand, assumptions (R4)–(R7) are less standard and are related to the notion of super-slow variation (see [7, §3.12.2]). We will use them in the context of Lemma 5.2.

We will use assumptions (R5) and (R7) in the proof of Theorem 6. These assumptions are the only ones that restrict the growth of $L$ (apart from the natural requirement that $L$ is slowly growing). Namely, if $L$ is an eventually increasing and unbounded function such that $L(\rho) \lesssim \rho^\delta$ for any $\delta > 0$, then there exists a function $L_\alpha$ satisfying assumptions (R1)–(R4) and (R6) and such that $L \lesssim L_\alpha$.

For the second set of regularity assumptions, we assume that $L$ is analytic and non-vanishing in an angle $\{s : |\arg(s)| < \alpha_0\}$ with $\frac{\pi}{2} < \alpha_0 \leq \pi$. Put $\varepsilon(s) = \frac{s L'(s)}{L(s)}$, and consider the following assumptions:

(R8) $\varepsilon(s) = (1 + o(1))\varepsilon(|s|)$ uniformly in $\{s : |\arg(s)| < \alpha_0\}$ as $s \to \infty$.

(R9) $s \varepsilon'(s) = (1 + o(1))|s|\varepsilon'(|s|)$ uniformly in $\{s : |\arg(s)| < \alpha_0\}$ as $s \to \infty$.

Note that assumptions (R8)–(R9) are met for any Denjoy weight. These assumptions can always be satisfied by regularization of the original weight function $L$ as described in [18]. For instance, if $L$ satisfies assumptions (R1) and (R3), then the function

$$L_\alpha(s) = \exp\left(s \int_0^\infty \log L(u) \frac{1}{(s+u)^2} du\right)$$

satisfies assumptions (R1), (R3) and (R8), with $L_\alpha \sim L$ (in particular the corresponding Beurling and Carleman classes coincide). Moreover, if $L$ satisfies any of the assumptions (R1)–(R7), so does $L_\alpha$.

5.1 Some lemmas about regular functions. Given a function $L : \mathbb{R}_+ \to \mathbb{R}_+$, we denote by $\Lambda_L$ the logarithm of the corresponding Ostrowski function, i.e.

$$\Lambda_L(r) := \log \sup_{x > 0} \frac{x^r}{\gamma(x)}.$$ 

Here, we summarize some properties of the functions $L$ and $\Lambda_L$ that depend on our regularity assumptions.

Lemma 5.1. Suppose that the function $L$ satisfy assumptions (R1) and (R3). Then the following hold:

1. $\Lambda_L(erL(r)) \sim r$, $r \to \infty$;
We associate with the regularity assumptions on the function $K$. This section is devoted to the asymptotics of the functions $K$.

6. The functions $K$ and $E$

This section is devoted to the asymptotics of the functions $K$ and $E$ under different types of regularity assumptions on the function $L$. Let $L$ be a function that satisfies assumptions (R3) and (R8). We associate with $L$ the functions

$$
\gamma(s) = L(s)^s, \quad E(z) = \sum_{n \geq 0} \frac{z^n}{\gamma(n+1)}, \quad K(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-s} \gamma(s) ds, \quad c > 0.
$$

Note that under assumption (R8) the asymptotic behavior in the angle $|\arg(s)| \leq \alpha_0$ is determined by the behavior on the positive ray. Namely, by Lemma 5.3 we have

$$
\log L(s) = \log L(\rho) + i\theta \varepsilon(\rho)(1 + o(1)), \quad s = \rho e^{i\theta}, \quad |\theta| \leq \alpha_0 - \delta, \quad \rho \to \infty.
$$

In particular, $K$ is well defined under these assumptions. We begin this section with a summary of the results in [18].

6.1 The saddle point equation. The asymptotics of the functions $K$ and $E$ for large $z$ are determined by the saddle–point of the function $s \mapsto \log \gamma(s) - s \log z = s \log L(s) - s \log z$, that is, by the equation

$$
\log L(s) + s \frac{L'(s)}{L(s)} = \log z.
$$

We remark that under the assumptions (R3) and (R8), the saddle–point equation can be written more explicitly (see Lemma 5.3), namely

$$
\log L(s) + s \frac{L'(s)}{L(s)} = \log L(\rho) + \varepsilon(\rho) + i(\theta + o(1)) \varepsilon(\rho), \quad s = \rho e^{i\theta}.
$$
For $0 < \alpha < \alpha_0$ and $\rho_0 > 0$, put

$$S(\alpha, \rho_0) = \{ s : |\arg(s)| < \alpha, |s| > \rho_0 \}.$$ 

Then, it is not difficult to show that under the assumptions (R3) and (R8), the LHS of the saddle-point equation (6.2) is a univalent function in $S(\alpha, \rho_0)$ (see [18, §1.3]). From here on, we assume that this is the case, and put

$$\Omega(\alpha) = \left\{ z : \log z = \log L(s) + s \frac{L'(s)}{L(s)}, s \in S(\alpha, \rho_0) \right\}.$$ 

In general, this is a domain in the Riemann surface of $\log z$, but by choosing $\rho_0$ sufficiently large, we can treat it as a subdomain of the slit plane $\mathbb{C} \setminus \mathbb{R}_-$, provided that

$$\limsup_{\rho \to \infty} \varepsilon(\rho) < \frac{\pi}{\alpha},$$

in particular, whenever $\varepsilon(\rho) = o(1)$, as $\rho \to \infty$ (which is equivalent to the fact that $L$ is slowly varying or to assumption (R1)).

In what follows, we denote by $s_z = \rho_z e^{i\theta_z}$ the unique solution of the saddle-point equation (6.2).

### 6.2 Asymptotic behavior of the functions $K$ and $E$

The next two theorems are proven in [18].

**Theorem A.** Suppose that the function $L$ satisfies assumptions (R3) and (R8). Then, for any $\delta > 0$, the function $K$ is analytic in $\Omega(\alpha_0 - \delta)$ and

$$K(z) = (1 + o(1)) \sqrt{\frac{s}{2\pi \varepsilon(s)}} \exp(-s\varepsilon(s)), \quad z \to \infty,$$

uniformly in $\Omega(\alpha_0 - \delta)$. Here $s = s_z$ and the branch of the square root is positive on the positive half-line.

**Theorem B.** Suppose that the function $L$ satisfies assumptions (R3) and (R8), and that

$$\limsup_{\rho \to \infty} \varepsilon(\rho) < 2.$$ (6.4)

Then, given a sufficiently small $\delta > 0$, we have

$$zE(z) = (1 + o(1)) \sqrt{\frac{2\pi}{\varepsilon(s)}} \exp(s\varepsilon(s)) + o(1), \quad z \to \infty,$$

uniformly in $\Omega(\pi/2 + \delta)$, and

$$zE(z) = o(1), \quad z \to \infty$$

uniformly in $\mathbb{C} \setminus \Omega(\pi/2 + \delta)$. Here, also $s = s_z$ and the branch of the square root is positive on the positive half-line.

We note that the conclusion of Theorem B is valid for the values of $z$ on the positive ray without the additional assumption (6.4). That is, if assumption (R3) holds, then

$$rE(r) = (1 + o(1)) \sqrt{\frac{2\pi}{\varepsilon(\rho)}} \exp(\rho\varepsilon(\rho)), \quad r \to \infty,$$

where $r = L(\rho)e^{\varepsilon(\rho)}$. 
6.3 Lemmata. Here we present auxiliary results regrading the asymptotics of the functions \(K\) and \(E\) needed in this work.

**Lemma 6.1.** Suppose the function \(L\) satisfies assumption (R1). Then for any \(\eta < 1\),
\[
L^{-1}(\eta r) \lesssim \eta \log E(r) \lesssim L^{-1}(r),
\]
where \(L^{-1}\) is the inverse function to \(L\) in \([C, \infty)\) for sufficiently large \(C > 0\). In particular, for any \(\delta > 0\),
\[
E^2(r) \lesssim E((1 + \delta)r).
\]

**Lemma 6.2.** Suppose the function \(L\) satisfies assumptions (R3) and (R8). Then for any \(\delta > 0\) there exists \(\delta_1 > 0\) such that
\[
E(x\delta_1)E(x(1 - \delta))|K(x)| \lesssim 1.
\]

**Lemma 6.3.** Suppose the function \(L\) satisfies assumptions (R2), (R3) and (R8). Then, there exists a \(C > 0\) such that \(\log |E(z)| = O(|z|)\), uniformly in the set
\[
\{re^{i\psi} : C\varepsilon(L^{-1}(r)) \leq |\psi| \leq \pi\}.
\]

### 6.3.1 Asymptotics of \(K\) and \(E\) in the set \(\Omega(\alpha)\).
Recall the definition of the domain \(\Omega(\alpha)\):
\[
\Omega(\alpha) = \left\{ z : \log z = \log L(s) + \frac{s'}{L(s)}, |\arg s| < \alpha, \rho > \rho_0 \right\}.
\]

Further, recall that \(\Psi_+\) and \(\Psi_-\) are two curves joining \(0\) and \(\infty\) in the first and fourth quadrants respectively, such that for sufficiently large \(r_0\), \(\Psi_\pm \cap \{|z| > r_0\} \subset \partial\Omega(\frac{\pi}{2})\) (i.e., \(\Psi_\pm\) coincide with the upper and lower parts of \(\partial\Omega(\frac{\pi}{2})\)).

**Lemma 6.4.** Suppose the function \(L\) satisfies assumptions (R1), (R2), (R3) and (R8). For any \(\delta > 0\) and \(0 < \alpha < \frac{\pi}{2}\), there exists a constant \(C > 0\), such that
\[
\int_{r_0}^{\infty} |z^n e^{-\delta_0 e^{(\rho_0)}d}|z| \leq C2^n\gamma(n + 1), \quad n \geq 0, \quad z \in \Omega(\alpha),
\]
where \(s_\pm = \rho_0 e^{\delta z}\) is related to \(z\) by the saddle–point equation.

**Lemma 6.5.** Suppose the function \(L\) satisfies assumptions (R1), (R2), (R3), (R5) and (R9). Then for any \(\delta > 0\) there exists \(\delta_1 > 0\) such that
\[
E((1 - \delta)|z|)|K(z)| \lesssim E(\delta_1|z|), \quad z \in \Psi_\pm, \quad |z| > 1.
\]

**Lemma 6.6.** Suppose the function \(L\) satisfies assumptions (R1), (R2), (R3), (R5), (R7) and (R9). Then there exists a \(C > 0\), such that
\[
\int_{\Psi_+ \cap \{|z| > 1\}} |z^n \left|E(z)\right|^{-1/|z|} d|z| \leq C^{n+1}\gamma(n + 1).
\]

Recall that \(H\) is a positive \(C^1\)-function, decreasing on \((0, \frac{\pi}{2})\), satisfying \(H(\pi - \psi) = H(\psi)\) for \(\psi \in (0, \pi)\), and defined for \(\psi \in (0, \delta)\) (with \(\delta > 0\) sufficiently small) by the equations
\[
\psi = \text{Im} \left(\log L(i\rho) + \varepsilon(i\rho)\right), \quad \rho > \rho_0,
\]
\[
H(\psi) = \text{Re} \left(i\rho \varepsilon(i\rho)\right), \quad 0 < \psi < \delta.
\]

**Lemma 6.7.** Suppose the function \(L\) satisfies assumptions (R2), (R3) and (R9). Then, there exists \(A > 0\), s.t.
\[
\log \log H \left(\psi + \frac{A}{r}\right) \leq \log \log H(\psi) - \frac{3}{r}, \quad r \geq 1, \quad 0 < \psi \leq \frac{\pi}{2}.
\]

The proofs are given in Appendix B.
7 Singular transform of the exponential, the function $E_1$

Given a function $L$, denote by $E_1$ the singular transform of the exponential function $x \mapsto \exp x$, i.e.,

$$E_1(z) = \sum_{n \geq 0} \frac{z^n}{n!\gamma(n+1)}, \quad \text{where} \quad \gamma(n) = L(n)^n.$$  

This is an entire function of zero exponential type. We will use the following estimates of the function $E_1$:

**Lemma 7.1.** Suppose that $L$ satisfies assumption (R1). Then there exists a $C > 0$ such that

$$\log |E_1(z)| \leq C\Lambda_L(|z|).$$

Here, as before, $\Lambda_L(r) = \sup_{x \geq 0} [x \log r - x \log(xL(x))]$.

**Lemma 7.2.** Suppose that $L$ satisfies assumptions (R1), (R2), (R4) and (R8). Then

$$\log |E_1(ix)| \approx \Lambda_L(x)\varepsilon(x), \quad x > 1.$$  

**Lemma 7.3.** Suppose that $L$ satisfies assumptions (R1), (R2), (R3) and (R9). For any $B > 0$, there exists $C_B > 0$, such that for any $a < c < 0 < c + b$ and any $R > \max \{R_0, |v|, \frac{n}{c_-, c_+}\}$, we have

$$\log |(S_L P)(u + iv)| \leq e^{C_B(A_L(t)+1)H} \left(\psi + \frac{B}{r}\right), \quad r > B, \quad B/r < \psi < \frac{\pi}{2}, \quad t > 1.$$  

The proofs are given in Appendix C.

8 Singular transforms of polynomials

In this section, we prove estimates on the singular transform of bounded polynomials on an interval $(a, b)$. All the estimates will follow from the next lemma.

**Lemma 8.1.** Suppose that $L$ satisfies assumptions (R2), (R3) and (R8). Let $P$ be a polynomial of degree $n$ such that $|P| \leq 1$ on $[a, b]$ ($a < 0 < b$). There exists $R_0 > 0$, such that for any $a < c_- < 0 < c_+ < b$ and any $R > \max \{R_0, |v|, \frac{n}{c_-, c_+}\}$, we have

$$\log |(S_L P)(u + iv)| \leq C_n\varepsilon \left(\frac{n}{R} \left(|u|\varepsilon (L^{-1}(R)) + |v|\right) + \log E(R) + C \log R\right).$$

In order to prove Lemma 8.1, we will need the following lemmas. The first one is a classical result of S. Bernstein. The proof of Lemma 8.3 follows by inspection of Figure 3.

**Lemma 8.2** (Bernstein). Suppose that $P$ is a polynomial of degree $n$ such that $|P| \leq 1$ on $[a, b]$. Then

$$|P(z)| < \rho^n, \quad z \in T_\rho(a, b),$$

where $T_\rho(a, b)$ is the ellipse with foci $a$ and $b$, and the sum of axes $\rho(b - a)$.

**Lemma 8.3.** Suppose that $0 < \eta < 1$ and $a < 0 < b$. For any $a < c_- < 0 < c_+ < b$, we have

$$[c_-, c_+] \times [-\eta, \eta] \subseteq T_{1+c_-, c_+}(a, b).$$

Here $T_{\cdot, \cdot}$ is the ellipse defined as in Lemma 8.2.
8 Singular transforms of polynomials

Proof of Lemma 8.1: Fix \( a < c_- < 0 < c_+ < b \) and let \([c_-, c_+] \subset (c'_-, c'_+) \subset [a, b]\). Put

\[
Q(w) := (S_LP)(w) = \sum_{k=0}^{n} \hat{P}(k) \frac{w^k}{\gamma(k+1)}.
\]

By the Cauchy integral formula,

\[
Q(w) = \frac{1}{2\pi i} \int E(s) P\left(\frac{w}{s}\right) \frac{ds}{s},
\]

where the contour of integration encloses the origin. Given a sufficiently large positive \( R \), we deform the contour to the one as in Figure 4, i.e.,

\[
\Gamma_R = \{ \Re e^{i\theta} : |\theta| \leq \theta_R \} \cup \{ R'e^{i\theta_R} : R \leq R' \leq R^2 \} \cup \{ R^2 e^{i\theta} : \theta_R \leq |\theta| \leq \pi \},
\]

where, \( \theta_R = C\varepsilon(L^{-1}(R)) \) and \( C \) is chosen as in Lemma 6.3. By Lemma 6.3,

\[
|E(s)| \lesssim |s|^a, \quad s \in \Gamma_R \setminus \{ \Re e^{i\theta} : |\theta| \leq \theta_R \}
\]

for some \( a > 0 \). Since the Taylor coefficients of \( E \) are all positive,

\[
\max_{|\theta| \leq \theta_R} |E(Re^{i\theta})| \leq E(R).
\] (8.2)

If \( w = u + iv \) is such that \( u > 0 \), \( R > \max \{ R_0, \frac{u}{c_+}, |v| \} \), where \( R_0 \) is large enough and \( s = Re^{i\theta}, |\theta| \leq \theta_R \), then

\[
\Re \frac{w}{s} = \frac{u \cos \theta + v \sin \theta}{R} < c_+ + \theta_R \leq c'_+,
\]

and

\[
\Re \frac{w}{s} > -\theta_R > -c'_-,
\]

If \( s = R^2 e^{i\theta}, \theta_R \leq |\theta| \leq \pi \), then clearly

\[
c'_- \leq \Re \frac{w}{s} \leq c'_+ \quad \text{and} \quad \left| \Im \frac{w}{s} \right| \leq \frac{u\theta_R}{R} + \frac{|v|}{R}.
\]

Since the set \([c'_-, c'_+] \times \left[ -\frac{u\theta_R + |v|}{R}, \frac{u\theta_R + |v|}{R} \right] \) is convex, we conclude that

\[
\frac{w}{s} \in \left[ c'_-, c'_+ \right] \times \left[ -\frac{u\theta_R + |v|}{R}, \frac{u\theta_R + |v|}{R} \right], \quad s \in \Gamma_R.
\] (8.3)
Therefore, by Lemma 8.3
\[ \frac{w}{s} \in T_{1+C_{c_+,c_-}(u\theta_R+|v|)/R}(a,b). \] (8.4)

Then, according to Lemma 8.2 applied to \( P(\frac{w}{s}) \), we have
\[ |Q(u)| \leq \exp \left[ C_{c_+,c_-} \frac{n}{R} (u\theta_R + |v|) \right] \int_{\Gamma_R} |E(s)| \frac{ds}{|s|}. \]

For \( R \) large enough, estimates (8.1) and (8.2) yield
\[ \int_{\Gamma_R} |E(s)| \frac{ds}{|s|} \leq 2\pi R \theta_R E(R) + CR^{2n}. \]

Therefore
\[ \log |Q(u + iv)| \leq C_{c_+,c_-} \frac{n}{R} (u\theta_R + |v|) + \log E(R) + 2(a + 1) \log R + C. \]

The last estimate finishes the proof for \( u \geq 0 \). The proof for \( u < 0 \) is similar. \qed

We finish this section with three estimates for singular transforms of polynomials, which are based on Lemma 8.1.

**Lemma 8.4.** Suppose that the function \( L \) satisfies assumptions (R1), (R2), (R4) and (R8). If \( P \) is a polynomial of degree \( n \) with \( |P| \leq 1 \) on \([a,b]\) \((a < 0 < b)\), then for any \( a < c_- < 0 < c_+ < b \), \( Y > 0 \) and \( \delta > 0 \),
\[ |(S_L P)(u + iv)| \leq e^{C_{c_+,c_-,Y,\Lambda_L}(n)} E \left( \frac{u}{c_\pm} \right) \tilde{E}(|u|), \quad 0 < |u|, |v| < Y, \]
where \( \tilde{E}(z) = \sum_{n \geq 0} \varepsilon(n+1)^{n+1} z^n. \)

**Proof.** Fix the parameters \( Y, a, b, \delta, c_- \) and \( c_+ \) and let \( \eta > 1 \) be such that \( [\eta c_-\, \eta c_+] \subseteq (a,b) \). By Lemma 8.1 applied with the parameters \( \eta c_+ \) and \( \eta c_- \) (instead of \( c_- \) and \( c_+ \)), there exists a large numerical constant \( R_0 > 0 \), such that
\[ \log |(S_L P)(u + iv)| \leq C_{c_-,c_+} \eta \frac{n}{R} (|u| \varepsilon(L^{-1}(R)) + |v|) + \log E(R) + C \log R, \]
for $R \geq \max\left\{ R_0, |v|, \frac{u}{\eta c_+}, \frac{|v|}{\eta c_+}\right\}$. Suppose that $u > 0$ and $|v| \leq Y$. First, we consider the case when $u \leq \min\left\{ \eta c_+ L(\Lambda_L(n)), \frac{2e}{\eta c_+} \right\}$. Then, we choose $R = L(\Lambda_L(n))$ and find that

$$\log |(S_L P)(u + iv)| \leq C_{c_-, c_+, \eta} \frac{n}{L(\Lambda_L(n))} (|u| \varepsilon(\Lambda_L(n)) + |v|) + 2 \log E(L(\Lambda_L(n))).$$

Since assumption (R4) holds, so does assumption (R3). Thus, by Lemma 5.1 assertion 1, we find that

$$\log |(S_L P)(u + iv)| \leq C_{Y, \varepsilon, c_+, \eta, \varepsilon} \Lambda_L(n) \left( \frac{\varepsilon(\Lambda_L(n))}{\varepsilon(n)} + 1 \right) + 2 \log E(L(\Lambda_L(n))).$$

By Lemma 5.2 assertion 1, we have $\varepsilon(n) \asymp \varepsilon(\Lambda_L(n))$ and Lemma 6.1 yields,

$$\log E(L(\Lambda_L(n))) \lesssim \Lambda_L(n).$$

Therefore, in this case

$$\log |(S_L P)(u + iv)| \lesssim_{c_-, c_+, \eta, \varepsilon} \Lambda_L(n).$$

Now, if $u \geq \min\left\{ \eta c_+ L(\Lambda_L(n)), \frac{2e}{\eta c_+} \right\}$, then we choose $R = \frac{u}{\eta c_+}$ and find that

$$\log |(S_L P)(u + iv)| \leq C_{c_-, c_+, \eta} \frac{n}{u} \left( \varepsilon(L^{-1}(\frac{u}{\eta c_+})) + |v| \right) + \log E\left( \frac{u}{\eta c_+} \right) + C_{c_-, c_+, \eta} \log u.$$

If $u > \frac{2e}{\eta c_+(n)}$, then

$$C_{c_-, c_+, \eta} \left( L^{-1} \left( \frac{u}{\eta c_+} \right) \right) n \leq C_{c_-, c_+, \eta} \left( L^{-1} \left( \frac{u}{\eta c_+} \right) \right) \left( \frac{1}{\varepsilon} \right)^{-1} \left( \frac{\delta u}{\varepsilon} \right).$$

where $\left( \frac{1}{\varepsilon} \right)^{-1}$ is the inverse function to $\rho \mapsto \frac{1}{\varepsilon}(\rho)$ (defined for $\rho > \rho_0$ large enough). Applying Lemma 6.1 to the function $\tilde{E}$ (instead of $E$), we get

$$\left( \frac{1}{\varepsilon} \right)^{-1} \left( \frac{\delta u}{2e} \right) \leq C + \log \tilde{E}(\delta u/2).$$

Therefore, in this case

$$\log |(S_L P)(u + iv)| \leq C(1 + \log u) + \log \tilde{E}(\delta u/2) + 2 \log E\left( \frac{u}{\eta c_+} \right).$$

On the other hand, if $u \leq \frac{2e}{\eta c_+(n)}$, then

$$\log |(S_L P)(u + iv)| \lesssim_{Y, \varepsilon, c_+, \eta, \varepsilon} \frac{n}{L(\Lambda_L(n))} \left( \frac{\varepsilon(\Lambda_L(n))}{\varepsilon(n)} + 1 \right) + \log E\left( \frac{u}{\eta c_+} \right) \lesssim \Lambda_L(n) + \log E\left( \frac{u}{\eta c_+} \right),$$

where we have used once again that $\varepsilon(n) \asymp \varepsilon(\Lambda_L(n))$.

We have established that

$$| (S_L P)(u + iv) | \lesssim e^{C_0 \Lambda_L(n)} E^2 \left( \frac{u}{\eta c_+} \right) \tilde{E}(\delta u/2) u^C, \quad 0 < u, |v| < Y.$$
By Lemma 6.1 applied separately to the functions $\tilde{E}$ and $E$, we find that

$$
\tilde{E}(\delta|u|/2) u^C \lesssim_{C,\delta} E(\delta|u|), \quad E^2\left(\frac{u}{\eta c_+}\right) \lesssim_{\eta} E\left(\frac{u}{c_+}\right).
$$

We conclude that

$$
| (S_L P)(u + iv)| \lesssim_{\delta,c_-,c_+} e^{\delta n} E\left(\frac{u}{c_+} + \Delta |v|\right), \quad \pm u \geq 0.
$$

This completes the proof of Lemma 8.4 for $u \geq 0$. The proof for $u \leq 0$ is similar.

**Lemma 8.5.** Suppose that the function $L$ satisfies assumptions (R1), (R2), (R4) and (R8). If $P$ is a polynomial of degree $n$ with $|P| \leq 1$ on $[a, b]$ ($a < 0 < b$), then, for any $a < c_- < 0 < c_+ < b$ and $0 < \delta < 1$, there exists $\Delta > 0$ such that

$$
| (S_L P)(u + iv)| \lesssim_{\delta,c_-,c_+} e^{\delta n} E\left(\frac{u}{c_+} + \Delta |v|\right), \quad \pm u \geq 0.
$$

**Proof.** Fix the parameters $a, b, c_-, c_+$ and $\delta$. Set $Q = S_L P$ and let $\eta > 1$ be such that $[\eta c_-, \eta c_+] \subset (a, b)$. Lemma 8.4 with parameters $\eta c_+$ and $\eta c_-$ (instead of $c_-$ and $c_+$) shows that there exists a sufficiently large numerical constant $R_0$ such that

$$
\log |Q(u + iv)| \leq C_{c_-,c_+}\eta \frac{n}{R} \left(|u| \varepsilon(L^{-1}(R)) + |v|\right) + \log E(R) + C \log R,
$$

for $R > \max \left\{ R_0, |v|, \frac{u}{\eta c_-}, \frac{u}{\eta c_+}\right\}$. Suppose that $u \geq 0$, and that $\frac{u}{\eta c_+} + |v|$ is large enough. For $\Delta > 2$ that will be chosen later, we choose $R = \frac{u}{\eta c_+} + \frac{\Delta |v|}{2}$ and find that

$$
\log |Q(u + iv)| \leq C_{c_-,c_+}\eta \frac{n}{R} \left(|u| \varepsilon\left(L^{-1}\left(\frac{u}{\eta c_+} + \frac{\Delta |v|}{2}\right)\right) + |v|\right) + \log E\left(\frac{u}{\eta c_+} + \frac{\Delta |v|}{2}\right) + C \log(u + \Delta |v|).
$$

The function $R \mapsto \varepsilon(L^{-1}(R))$ tends to zero as $R \to \infty$. Therefore, we can choose $\Delta$ so large, such that

$$
\log |Q(u + iv)| \leq \delta n + \log E\left(\frac{u}{\eta c_+} + \frac{\Delta |v|}{2}\right) + C \log(u + \Delta |v|), \quad |u + iv| > C, \quad u \geq 0,
$$

and therefore

$$
\log |Q(u + iv)| \leq \delta n + \log E\left(\frac{u}{c_+} + \Delta |v|\right), \quad |u + iv| > C, \quad u \geq 0.
$$

This finishes the proof for $u \geq 0$, the proof for $u \leq 0$ is similar.

**Lemma 8.6.** Suppose that the function $L$ satisfies assumptions (R1), (R2), (R3) and (R8). If $P$ is a polynomial of degree $n$ such that $|P| \leq 1$ on $[-1, 1]$, then there exists a constant $C > 0$ such that

$$
|S_L P(u + iv)| \leq \exp\left( C \cdot (\Lambda_{L/\varepsilon}(n + 1))\right), \quad |u| \leq 1, \quad |v| \leq \varepsilon (\Lambda_{L/\varepsilon}(n))
$$

**Proof.** Put $Q = S_L P$. Lemma 8.4 with parameters $c_\pm = \pm \frac{1}{2}$ shows that there exists a sufficiently large numerical constant $R_0$ such that

$$
\log |Q(u + iv)| \leq C \frac{n}{R} \left(|u| \varepsilon(L^{-1}(R)) + |v|\right) + \log E(R) + C \log R,
$$
for $R > \max \{ R_0, |v|, 2|u| \}$. Choosing $R = L(\Lambda_{L/\varepsilon}(n))$, and assuming that $n$ is sufficiently large, we find that
\[
\log |Q(u + iv)| \leq C \frac{n}{L(\Lambda_{L/\varepsilon}(n))} \cdot \varepsilon(\Lambda_{L/\varepsilon}(n)) + 2 \log E(L(\Lambda_{L/\varepsilon}(n))),
\]
where $|u| \leq 1$ and $|v| \leq \varepsilon(\Lambda_{L/\varepsilon}(n))$. By Lemma 6.1 assertion 1, $\frac{n}{L(\Lambda_{L/\varepsilon}(n))} \cdot \varepsilon(\Lambda_{L/\varepsilon}(n)) \leq \Lambda_{L/\varepsilon}(n)$, and by Lemma 5.1 assertion 1, $\frac{1}{\varepsilon(\Lambda_{L/\varepsilon}(n))} \leq \Lambda_{L/\varepsilon}(n)$. Therefore,
\[
\log |Q(u + iv)| \lesssim \Lambda(n), \quad |u| \leq 1 \quad |v| \leq \varepsilon(\Lambda_{L/\varepsilon}(n)),
\]
which is the desired estimate. \hfill \qed

9 Proof of theorems

Here we state and prove stronger analogues of Theorems 1–7. The theorems are stated using the regularity assumptions of Section 5, and they are somewhat stronger than the corresponding Theorems 1–7.

9.1 Theorem 1

\textbf{Theorem 1}. Suppose that the function $L$ satisfies assumptions (R3) and (R8), and that $I$ is an interval containing the origin, such that $I \cap (0, \infty)$ and $I \cap (-\infty, 0)$ are open. Then, the regular transform $R_L$ maps $A(L; I)$ into $C_0(L; I)$.

Note that Theorem 1 is valid not only for slowly growing functions $L$, but also for functions that grow faster, such as $L(\rho) = \rho^a$, for some $a > 0$ (Gevrey classes).

\textbf{Proof of Theorem 1}. Fix an interval $I$ such that $0 \in I$ and $I \cap (0, \infty)$ and $I \cap (-\infty, 0)$ are open. The proof treats the intervals $I \cap [0, \infty)$ and $I \cap [-\infty, 0)$ separately; the two cases are analogous. Thus there is no loss of generality in assuming that $I \subset [0, \infty)$. Fix $F \in A(L; I)$ and put
\[
f(x) = (R_LF)(x) = \int_0^\infty F(xt)K(t)dt
\]
(convergence will follow from the proof). We wish to show that $f \in C_0(L; I)$. Fix $0 < c_+ \in I$ and let $\eta > 1$ such that $\eta^2 c_+ \in I$. By the definition of $A(L; I),
\[
|F(u + iv)| \lesssim_{Y,c_+} E \left( \frac{u}{c_+ \eta^2} \right), \quad u > 0, \quad |v| < Y.
\]
For $u \geq 0$ and $n \geq 0$, Cauchy’s formula yields
\[
|F^{(n)}(u)| = \frac{n!}{2\pi} \left| \int_{|w-u|=\varepsilon} F(w)dw \right| \lesssim_{Y,c_+} \frac{n!}{Y^n} E \left( \frac{u + Y}{c_+ \eta^2} \right) \lesssim_{Y,c_+} \frac{n!}{Y^n} E \left( \frac{u}{c_+ \eta} \right).
\]
For $x \in [0, c_+]$, we further obtain
\[
|F^{(n)}(x)| = \int_0^\infty t^n F^{(n)}(xt)K(t)dt \lesssim_{c_+,Y} \frac{n!}{Y^n} \int_0^\infty t^n E \left( \frac{t}{\eta} \right) |K(t)|dt.
\]
By Lemma 6.2 applied with $1 - \delta = \frac{1}{\eta}$, there exists $\delta_1 := \delta_1(\eta) > 0$, such that
\[
E(\delta_1 t)E \left( \frac{t}{\eta} \right) |K(t)| \lesssim_\eta 1.
\]
Taking this into account in \([9.1]\), we obtain
\[
|f^{(n)}(x)| \lesssim_{a,+Y} \frac{n!}{Y^n} \int_0^\infty u^n \frac{dt}{E(t \delta)} = \frac{n!}{Y^n} \delta_1^{-n-1} \int_0^\infty u^n \frac{du}{E(u)} \leq \frac{n!}{Y^n} \delta_1^{-n-1} \left[ C + \int_1^\infty u^n \frac{du}{E(u)} \right].
\]
By the definition of \(E\),
\[
E(u) = \sum_{k \geq 0} \frac{u^k}{\gamma(k+1)} \geq \frac{u^{n+2}}{\gamma(n+3)}.
\]
Therefore
\[
|f^{(n)}(x)| \lesssim_{a,+Y} \frac{n!}{Y^n} \delta_1^{-n-1} \left[ C + \gamma(n+3) \right] \lesssim_{a,+Y} \frac{n!}{Y^n} \delta_1^{-n-1} \gamma(n+3).
\]
Since \(|\gamma(n+3)/\gamma(n)|^{1/n} \sim 1\) and \(Y\) can be taken arbitrarily large, the last inequality shows that \(f \in C_0(L; [0,c_+])\). Since \(c_+ \in I\) was arbitrary, we conclude that \(f \in C_0(L; I)\). This completes the proof of Theorem 1.

9.2 Theorem 2

Theorem 2'. Suppose that the function \(L\) satisfies assumptions (R1), (R2), (R4) and (R8), and that \(I\) is an open interval containing the origin. Then, the singular transform \(S_L\) maps \(C_0(L; I)\) into \(\bigcap_{\delta>0} A(\frac{L(\rho)}{\delta L(\rho)+1}; I)\)

Note that if \(L\) is a Denjoy weight, then
\[
\bigcap_{\delta>0} A(\frac{L(\rho)}{\delta L(\rho)+1}; I) = A(L; I) \cup A(\frac{1}{\varepsilon}; \mathbb{R}).
\]
So Theorem 2 follows from Theorem 2'.

9.2.1 Chebyshev polynomials expansion for functions in \(C_0(L; [-1,1])\). Denote by \(T_n(x) = \cos \left( (n \arccos(x)) \right)\) the Chebyshev polynomials. We will use the following lemma (see [21, pp.44]).

Lemma 9.1. If \(f \in C_0(L; [-1,1])\) with the Chebyshev expansion \(f = \sum_{n \geq 0} c_n T_n\). Then for any \(\delta > 0\),
\[
|c_n| \lesssim_{f,\delta} \inf_{n \leq r} \frac{n! \gamma(n+1) \delta^n}{r^n}.
\]
Moreover, if \(f \in C^\omega([-1,1])\), then there exists \(\delta = \delta_f > 0\) such that
\[
\log |c_n| \lesssim e^{-\delta n}, \quad n \geq 0
\]

The next lemma, enables us to express the majorant of the coefficients of the Chebyshev expansion in terms of the function \(\Lambda_L(r) = \sup_{x>0} [x \log r - x \log(xL(x))]\).

Lemma 9.2. Suppose that the function \(L\) satisfies assumptions (R1) and (R3). Then
\[
\Lambda_L(r) \asymp \log \left( \sup_{n \leq r} \frac{r^n}{n! \gamma(n+1) \delta^n} \right) \asymp \log \left( \sup_{n \in \mathbb{N}} \frac{r^n}{n! \gamma(n+1) \delta^n} \right).
\]
In particular, if \(f \in C_0(L; [-1,1])\) with the Chebyshev expansion \(f = \sum_{n \geq 0} c_n T_n\), then
\[
|c_n| \lesssim_{f,\delta} e^{-\delta \Lambda_L(n)}.
\]

The proof of this lemma is given in Appendix A.
9.2.2 Proof of Theorem 2

Proof. We fix an open interval $I$ and a function $f \in C_0(L; I)$. Let $[c'_-, c'_+] \subset I$, and let $a, b \in \mathbb{R}$, such that $[c'_-, c'_+] \subset (a, b) \subset [a, b] \subset I$. Denote by $\chi = \chi(a, b)$ the linear function that maps the interval $[a, b]$ onto the interval $[-1, 1]$.

The function $f \circ \chi$ belongs to $C_0([L; [-1, 1])$. By Lemma 9.2 it can be expanded into a series of Chebyshev polynomials with rapidly decaying coefficients, namely

$$f \circ \chi = \sum_{n \geq 0} c_n T_n,$$

where, for any $\delta > 0$, $|c_n| \lesssim \delta_1 e^{-\delta_1 L(n)}$.

We claim that the function $F := \sum_{n \geq 0} c_n S_n(T_n \circ \chi^{-1})$ is the singular transform of $f = \sum_{n \geq 0} c_n T_n \circ \chi^{-1}$. To show this we will use Lemma 8.4.

Note that $T_n \circ \chi^{-1}$ are polynomials of degree $n$ which are bounded by 1 on the interval $[a, b]$. Therefore, by Lemma 8.4, for any $Y > 0$ and $\delta_2 > 0$, there exists a constant $C > 0$ such that

$$|F(u + iv)| \leq E \left( \frac{u}{c_{\pm}'} \right) \tilde{E}(\delta_2 |u|) \sum_{n \geq 0} |c_n| e^{C_\delta(n)} \lesssim E \left( \frac{u}{c_{\pm}'} \right) \tilde{E}(\delta_2 |u|), \quad |v| < Y, 0 \leq u,$$

(9.2)

where we are using $c_{+}'$ when $+u \geq 0$ and $c_{-}'$ when $-u \geq 0$. The same inequality also holds for $u \leq 0$, with $c_{+}'$ replaced by $c_{-}'$. In particular, this shows that the function $F$ is analytic in any strip $|v| < Y$ (and therefore is entire). The function $S_n f$ is also entire and has the same Taylor coefficients at the origin as $F$, therefore $S_n f = F$.

Using Lemma 6.1 we find that

$$E \left( \frac{u}{c_{\pm}'} \right) \tilde{E}(\delta_2 |u|) \leq E^2 \left( \frac{u}{c_{\pm}'} \right) + \tilde{E}^2(\delta |u|) \lesssim \delta_2 \delta E \left( \frac{u}{c_{\pm}'} (1 + \delta_2) \right) + \tilde{E}(2\delta_2 |u|).$$

Since $a, b, c_{\pm}'$, $Y$, and $\delta_2$ were arbitrary, for any $\delta > 0$ and $c_\pm \in I$ (with $c_- < 0 < c_+$), we get

$$|S_n f(u + iv)| \lesssim \delta_2 \sum_{n \geq 0} \left( \frac{\delta(n + 1) L'(n + 1)}{L(n + 1)} \right)^{n+1} \left( \frac{u}{c_{\pm}'} \right)^n$$

for all $|v| < Y$ and $\pm u \geq 0$ (where we are using $c_{+}$ when $+u \geq 0$ and $c_{-}$ when $-u \geq 0$). The latter inequality yields

$$S_n f \in \bigcup_{\delta > 0} A \left( \frac{L(\rho)}{\delta \rho L'(\rho) + 1}; I \right).$$

This complete the proof of Theorem 2.

\[ \square \]

9.3 Theorem 7

Theorem 7. Suppose the function $L$ satisfy assumptions (R1), (R2), (R4) and (R8), and $I$ is an open interval containing the origin. Then the singular transform maps the class $C^\omega(I)$ of real analytic functions bijectively onto the set $A^\omega(L; I)$. In particular, $R_{L} S_{L} f = f$ for any $f \in C^\omega(I)$.

The proof uses ideas similar to the ones we used in the proofs of Theorems 1 and 2. First we show that if an entire function $F \in A^\omega(L; I)$, then its regular transform belongs to $C^\omega(I)$. Then we use Chebyshev polynomials expansion of a functions belonging to $C^\omega(I)$ together with Lemma 5 to show that $S_{L} f \in A^\omega(L; I)$, whenever $f \in C^\omega(I)$. Throughout this section, we fix a function $L$ satisfying the assumptions of Theorem 7.\[ \square \]
9.3.1 Proof that $R_L F \in C^\omega(I)$ for any entire function $F \in A^\omega(L; I)$

Proof. Fix the open interval $I$. Let $F \in A^\omega(L; I)$, that is, for every $c_- \in I \cap (-\infty, 0)$, $c_+ \in I \cap (0, \infty)$, there exists $\Delta = \Delta_{c_\pm} > 0$ such that

$$|F(u + iv)| \leq_{c_+, c_-} E \left( \frac{u}{c_\pm} + \Delta|v| \right), \quad \pm u \geq 0.$$ 

Put

$$f(x) = (R_L F)'(x) = \int_0^\infty F(xt)K(t)dt.$$ 

By Theorem 9.3, $f \in C_0(L; I)$. We will show that $f \in C^\omega(I)$ by showing that it has a positive radius of convergence at any point $x \in I$. Fix $c_-, c_+ \in I$ and $\Delta = \Delta_{c_\pm}$ as above. For any $0 < \delta < 1$, the Cauchy formula yields

$$|F^{(n)}(u)| = \frac{n!}{2\pi} \int_{|w-u| = \delta|u|+1} \frac{F(w)dw}{(w-u)^{n+1}},$$

for any $\pm u \geq 0$. For any $x \in I$,

$$|f^{(n)}(x)| = \left| \int_0^\infty t^n F^{(n)}(xt)K(t)dt \right|.$$ 

Therefore, for $x \in (0, c_+)$,

$$|f^{(n)}(x)| \leq_{c_+, c_-} n! \int_0^\infty \frac{t^n}{\delta^n(tx+1)^n} E \left( \frac{(1+\delta)|tx+1|}{c_+} + \delta\Delta tx \right) |K(t)|dt.$$ 

We choose $\delta$ so small that

$$|f^{(n)}(x)| \lesssim_{c_+, c_-} \frac{n!}{\delta^n x^n} \int_0^\infty E((1-\delta)t)|K(t)|dt. \quad (9.3)$$ 

By Lemma 6.2, there exists an $\delta_1 > 0$ such that $E((1-\delta)t)|K(t)| \lesssim \frac{1}{E(\delta t)}$. In particular, the integral in the RHS of estimate (9.3) converges. We obtain

$$|f^{(n)}(x)| \lesssim_{c_+, c_-} \frac{n!}{\delta^n x^n},$$

and hence $f \in C^\omega(0, c_+)$. To show analyticity at the point $x = 0$, we use Cauchy’s formula once again, and find that

$$|F^{(n)}(0)| = \frac{n!}{2\pi} \int_{|w|=\rho} \frac{F(w)dw}{w^{n+1}} \lesssim_{c_+, c_-} \frac{n!}{\rho^n} E \left( \frac{2\rho}{\min\{c_+, c_-, \Delta+1\}} \right), \quad \rho > 0.$$ 

Therefore, by choosing $\rho = \delta t$ with $\delta > 0$ sufficiently small, we find that

$$|f^{(n)}(0)| = \int_0^\infty t^n F^{(n)}(0)K(t)dt \lesssim_{\delta} \frac{n!}{\delta^n} \int_0^\infty E(\delta t)|K(t)|dt \lesssim_{\delta} \frac{n!}{\delta^n}.$$ 

We have shown that $f \in C^\omega(0, c_+)$. The proof that $f \in C^\omega(c_-, 0]$ is similar. Since $c_-, c_+ \in I$ were arbitrary, we conclude that $f \in C^\omega(I)$. 

\[\square\]
9.3.2 Proof that $S_L f \in A^\omega(L; I)$ for any $f \in C^\omega(I)$. Now we finish the proof of Theorem 7'.

Proof. Fix an open interval $I$ and a function $f \in C^\omega(I)$. Let $[c_-, c_+] \subset I$, $\delta > 0$ and let $a < c_- < 0 < c_+ < b$, such that $[a, b] \subset I$. Denote by $\chi = \chi(a, b)$ the linear function that maps the interval $[a, b]$ onto the interval $[-1, 1]$.

The function $f \circ \chi$ belongs to $C^\omega[-1, 1]$. By Lemma 9.1, it can be expanded into a series of Chebyshev polynomials with fast decaying coefficients, namely

$$f \circ \chi = \sum_{n \geq 0} c_n T_n, \quad |c_n| \lesssim e^{-\delta n},$$

for some $\delta > 0$.

We claim that then the function $F := \sum_{n \geq 0} c_n S_L(T_n \circ \chi^{-1})$ is the singular transform of $f = \sum_{n \geq 0} c_n T_n \circ \chi^{-1}$. To show this, we will use Lemma 8.5.

Note that $T_n \circ \chi^{-1}$ are indeed polynomials of degree $n$ which are bounded by 1 on the interval $[a, b]$. Therefore,

$$|F(u + iv)| \lesssim_{c_-, c_+} \left| \frac{u}{c_\pm} + \Delta |v| \right| \sum_{n \geq 0} |c_n| e^{\delta n/2} \lesssim_{c_+, c_-} \left| \frac{u}{c_\pm} + \Delta |v| \right|, \quad \pm u > 0.$$  

In particular, this shows that the functions $F$ entire. The function $S_L f$ is also entire and has the same Taylor coefficients at the origin as $F$, therefore $S_L f = F$. This completes the proof of Theorem 7'.

9.4 Theorem 3.

**Theorem 3**. Suppose the function $L$ satisfies assumption (R1), (R2), (R4) and that $\lim_{\rho \to +\infty} \rho L'(\rho) = +\infty$. Then for any function $L_2$ satisfying $\frac{1}{\varphi(p)} = o(L_2(\rho))$ as $\rho \to \infty$ and any $\delta > 0$, $S_L C_0(L; (-\delta, \delta)) \not\subset A(L_2; \mathbb{R})$. In particular $S_L C_0(L; (-\delta, \delta)) \not\subset A(L; \mathbb{R})$.

Here we prove Theorem 3. For a function $L$ satisfying $\lim_{\rho \to +\infty} \rho L'(\rho) = +\infty$, and another function $L_2$ with $\frac{1}{\varphi(p)} = o(L_2(\rho))$, we will construct a lacunary Fourier series

$$f(x) = \sum_{k \geq 0} e^{-\omega_{nk} \Lambda_L(nk)} e^{in_k x}, \quad n_k \in \mathbb{N},$$

where, as before, $\Lambda_L(r) := \sup_{x > 0} |x \log r - x \log(x L(x))|$. This Fourier series defines an element in the Beurling class $C_0(L; \mathbb{R})$ provided that $\omega_{nk} \to \infty$ as $k \to \infty$ (the proof is below). Then using the linearity of the singular transform, we will show that

$$F(z) = (S_L f)(z) = \sum_{k \geq 0} e^{-\omega_{nk} \Lambda_L(nk)} \left( S_L e^{in_k x} \right)(z) = \sum_{k \geq 0} e^{-\omega_{nk} \Lambda_L(nk)} E_1(in_k z),$$

where $E_1(z) := S_L(\exp)(z) = \sum_{n \geq 0} \frac{z^n}{n!(n+1)^2}$. The plan is now to choose the sequence $n_k$ so lacunary and $\omega_k$ increasing to $\infty$ so slowly, that on a special sequence of points $r_k \uparrow \infty$, we will have

$$|F(r_k)| \asymp e^{-\omega_{nk} \Lambda_L(nk)} |E_1(in_k r_k)| > e^{L_2^{-1}(r_k)}.$$
Proof. Fix a function $L$ satisfying the assumptions of Theorem 3 and another function $L_2$ with $\frac{1}{\varepsilon} = o(L_2(\rho))$ as $\rho \to \infty$. Let $L_3$ be yet another function satisfying the assumptions of Theorem 2 and such that

$$\frac{1}{\varepsilon(\rho)} = o(L_3(\rho)), \quad L_3(\rho) = o(L_2(\Lambda_L(\rho))), \quad L_3(\rho) = o(L(\rho)), \quad \rho \to \infty.$$ 

The function $L_3$ exists, since by Lemma 5.2, assertion 1, $\varepsilon(\rho) \sim \varepsilon(\Lambda_L(\rho)) = o(L_2(\Lambda_L(\rho)))$ as $\rho \to \infty$. For a sequence $\omega_n \to \infty$, and a sequence of natural numbers $n_k \uparrow \infty$ that will be chosen later, put

$$f(x) = \sum_{k \geq 0} e^{-\omega_n \Lambda_L(n_k)} e^{i n_k x}.$$ 

Clearly $f$ is a $2\pi$–periodic function in $C^\infty(\mathbb{R})$. Moreover,

$$f^{(j)}(x) = \sum_{k \geq 0} (in_k)^j e^{-\omega_n \Lambda_L(n_k)} e^{i n_k x},$$

which yields

$$\|f^{(j)}\|_\infty \leq \sum_{k \geq 0} n_k^j e^{-\omega_n \Lambda_L(n_k)}.$$ 

Since, $\omega_n \to \infty$, the definition of $\Lambda_L$ yields

$$e^{-\omega_n \Lambda(n)} \leq C_\delta \frac{\delta^{j+2} \gamma(j+2)(j+2)^{j+2}}{n^{j+2}}, \quad \delta > 0, \quad n \in \mathbb{N}.$$ 

Thus,

$$\|f^{(j)}\|_\infty \leq \delta^{j+2} \gamma(j+2)(j+2)^{j+2} \leq \delta^{j} \gamma(j) j^{j}, \quad \delta > 0,$$

which means that $f \in C_0(L; \mathbb{R})$.

Now put

$$F(z) = \sum_{k \geq 0} e^{-\omega_n \Lambda_L(n_k)} S_L(e^{i n_k z})(z) = \sum_{k \geq 0} e^{-\omega_n \Lambda_L(n_k)} E_1(in_k z).$$

By Lemmas 7.1 and 5.1, assertion 2, there exists $C > 0$ such that

$$\log |E_1(in_k z)| \leq C |z| \Lambda_L(n_k) + C, \quad k \geq 0, \quad z \in \mathbb{C}.$$ 

Since $\omega_n \to \infty$, the sum that defines $F$ converges uniformly on compact subsets of $\mathbb{C}$. Thus, $F$ is an entire function. The entire functions $F$ and $S_L f$ have the same Taylor coefficient, and therefore $F = S_L f$.

Put $r_n = L_3(n)$. By Lemma 7.2, there exists $\delta > 0$ such that for sufficiently large $n$ we have

$$\log |E_1(in_n z)| \geq 2\delta \Lambda_L(nr_n) \varepsilon(nr_n).$$

Fixing this value of $\delta > 0$, we put $\omega_n := \delta \frac{\Lambda_L(nr_n) \varepsilon(nr_n)}{\Lambda_L(n)}$. Let as verify that the sequence $\omega_n$ tends to $\infty$. Indeed, by Lemma 5.1, part 2, for sufficiently large $r$ we have,

$$\frac{\Lambda_L(nr_n) \varepsilon(nr_n)}{\Lambda_L(n)} \geq r_n \varepsilon(nr_n) \geq r_n \varepsilon(n) \to \infty, \quad n \to \infty.$$ 

We conclude that

$$-\omega_n \Lambda_L(n) + \log |E_1(in_n z)| \geq \delta \Lambda_L(nr_n) \varepsilon(nr_n).\quad (9.4)$$
where the classes $\mathcal{L}$ and $\mathcal{C}^0_0(L;\mathbb{R})$ are defined in Section 3.4.

Note that our construction of the function $f$ is analytic in the upper-half plane and therefore we actually proved that $S_L(C^+_0(L;\mathbb{R})) \varsubsetneq A(L_2;I)$, where the classes $C^+_0(L;\mathbb{R})$ are defined in Section 3.4.
9.5 Theorem 4 Throughout this section, given a non-quasianalytic eventually growing function \( L \), we put
\[
\widetilde{L}(\rho) = L(\rho) \int_{\rho}^{\infty} \frac{du}{uL(u)}, \quad \rho > 1,
\]
and
\[
\gamma(\rho) = L(\rho)^{\rho}, \quad \tilde{\gamma}(\rho) = \widetilde{L}(\rho)^{\rho}.
\]

**Theorem 4'.** Let \( L \) be a non-quasianalytic function and let \( I \) be an open interval that contains the origin. Suppose that the function \( \rho \mapsto L(\rho)/\widetilde{L}(\rho) \) satisfies assumptions (R1), (R2), (R4), (R6) and (R8), and that \( I \) is an open interval containing the origin. Then
\[
S_{\gamma}C_0(L; I) = S_{\tilde{\gamma}}C_0(\widetilde{L}; \mathbb{R}).
\]

Note that
\[
\rho \frac{d}{d\rho} \log \frac{L(\rho)}{\widetilde{L}(\rho)} = \frac{1}{L(\rho)},
\]
and so, assumption (R2) implies that \( \widetilde{L} \) is eventually increasing and part 1 of Lemma 5.2 (i.e., assumptions (R1) and (R4)) implies that \( \widetilde{L}(\rho) \sim \tilde{L}(\rho L(\rho)/\widetilde{L}(\rho)) \). In particular, under these assumptions \( \tilde{L} \) is eventually slowly growing.

In this section we will study the singular and regular transforms \( S_{L/L} \) and \( R_{L/L} \). The relevant functions associated with these transforms are
\[
E_{s}(z) = \sum_{n \geq 0} \frac{\gamma(n+1)}{\gamma(n+1)} z^n, \quad K_{s}(z) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\gamma(s)}{\gamma(s)} z^{-s} ds, \quad c > 0.
\]

9.5.1 Proof of the inclusion \( S_{L}C_0(L; I) \subseteq S_{\tilde{L}}C_0(\widetilde{L}; \mathbb{R}) \).

*Proof.* The first observation we make is that the set \( S_{L}C_0(L; I) \) does not depends on the interval \( I \). Indeed, for \( \delta > 0 \), consider the function \( \xi_{\delta} \in C_0(L; \mathbb{R}) \) which is identically 1 in the interval \((-\delta, \delta)\) and identically 0 outside the interval \((-2\delta, 2\delta)\). If \( f \in C_0(L; I) \), then \( \xi_{\delta} f \in C_0(L; I) \) and \( S_{L}(f) \equiv S_{L}(f \cdot \xi_{\delta}) \). By choosing \( \delta \) so that \([-2\delta, 2\delta] \subset I \), the function \( f \cdot \xi_{\delta} \) can be extended to an element of \( C_{0}(L; \mathbb{R}) \).

Fix \( f \in C_0(L; \mathbb{R}) \). For \( A > 0 \), we put \( f_{A}(x) = f(Ax) \). Since \( f \in C_0(L; \mathbb{R}) \), so does \( f_{A} \). We can consider the Chebyshev series expansion of the function \( f_{A} \) in the interval \([-1, 1]\),
\[
f_{A} = \sum_{k \geq 0} c_{A,k} \cdot T_{k}.
\]
By Lemma 9.1
\[
\lim_{k \to \infty} \frac{\log |c_{A,k}|}{\Lambda_{L}(k)} = -\infty,
\]
where as before
\[
\Lambda_{L}(k) = \sup_{x \geq 0} \left[ x \log k - n \log(xL(x)) \right].
\]
We put
\[
g_{A} := \sum_{k \geq 0} c_{A,k} \cdot S_{L/L}(T_{k}) = \sum_{k \geq 0} c_{A,k} \cdot Q_{k}.
\]
and notice that, formally,
\[
\frac{\widehat{f_{A}}(n)}{\gamma(n+1)} = \frac{\widehat{g_{A}}(n)}{\gamma(n+1)}, \quad n \geq 0.
\]
Next, we show that $g_A \in C(\tilde{L}; [-\frac{1}{2}, \frac{1}{2}])$. By Lemma 8.6 (applied with the functions $L/\tilde{L}$ and $1/\tilde{L}$ instead of $L$ and $\varepsilon$),

$$ |Q_k(x + iy)| \leq e^{C\lambda_L(k)}, \quad |x| \leq 1, \quad |y| \leq \frac{1}{L(\Lambda_L(k))}. $$

Therefore, the Cauchy estimates for the derivatives yield

$$ |Q_k^{(n)}(x)| \leq Cn! \cdot \tilde{L}^n (\Lambda_L(k)) \cdot e^{C\lambda_L(k)}, \quad |x| \leq \frac{1}{2}, \quad 0 \leq n \leq k. \quad (9.5) $$

By Lemma 5.1 part 4, there exists $C > 0$, such that,

$$ n \log \tilde{L}(k) - n \log \tilde{L}(n) \leq C(n + k), \quad 0 \leq n \leq k, $$

which in turn implies that

$$ n \log \tilde{L}(\Lambda_L(k)) - n \log \tilde{L}(n) \leq C(n + \Lambda_L(k)), \quad 0 \leq n \leq k. $$

Substituting this estimate into (9.5), we get

$$ |Q_k^{(n)}(x)| \leq C_{1n+1} \cdot e^{C\lambda_L(k)} \cdot n! \cdot \tilde{L}^n(n), \quad |x| \leq \frac{1}{2}, \quad 0 \leq n \leq k $$

Therefore,

$$ |g_A^{(n)}(x)| \leq C_A \cdot C_{1n} \cdot n! \cdot \tilde{\gamma}(n + 1), \quad |x| \leq \frac{1}{2}, \quad n \geq 0, $$

where the constant $C_1$ is independent of $A$, i.e., $g_A \in C(\tilde{L}; [-\frac{1}{2}, \frac{1}{2}])$.

For $x \in [-\frac{4}{2}, \frac{4}{2}]$, put $g(x) = g_A(\frac{x}{A})$. Note that $g$ belongs to the quasianalytic class $C(\tilde{L}; [-\frac{A}{2}, \frac{A}{2}])$, and

$$ \frac{\tilde{g}(n)}{\tilde{\gamma}(n + 1)} = A^{-n} \frac{\tilde{g}_A(n)}{\tilde{\gamma}(n + 1)} = A^{-n} \frac{\tilde{f}_A(n)}{\tilde{\gamma}(n + 1)} = \frac{\tilde{f}(n)}{\tilde{\gamma}(n + 1)}. $$

Since the class $C(\tilde{L}; \mathbb{R})$ is quasianalytic, we conclude that $g \in C(\tilde{L}; \mathbb{R})$.

Let $B > 0$. Taking $A > 2B$, we find that

$$ \max_{|x| \leq B} |g^{(n)}(x)| \leq C_A \cdot \left( \frac{C_1}{A} \right)^n \cdot n! \cdot \tilde{\gamma}(n + 1). $$

Since $A$ can be taken arbitrarily large, we have $g \in C_0(\tilde{L}; \mathbb{R})$, with

$$ \frac{\tilde{f}(n)}{\tilde{\gamma}(n + 1)} = \frac{\tilde{g}(n)}{\tilde{\gamma}(n + 1)}, \quad n \geq 0. $$

This finishes the proof the inclusion $S_L C_0(L; I) \subseteq S_{\tilde{L}} C_0(\tilde{L}; \mathbb{R})$.

9.5.2 The inclusion $S_L C_0(L; I) \supseteq S_{\tilde{L}} C_0(\tilde{L}; \mathbb{R})$.  

\[ \square \]
9.5.2.1 The theorem of Carleson and Ehrenpreis. Our proof relies on a theorem independently proven by Carleson [11] and Ehrenpreis [15]. Here we give the statement of this theorem and discuss its relation to our results.

Recall that for a non-decreasing function \( L : [0, \infty) \rightarrow [1, \infty) \),
\[
\Lambda_L(r) = \sup_{\rho \geq 0} (\rho \log r - \rho \log(\rho L(\rho))).
\]

Note that if the function \( \rho \mapsto \rho \log(\rho L(\rho)) \) is a convex function of \( \log \rho \), then \( L \) can be recovered from \( \Lambda_L \) by the relation
\[
\rho \log(\rho L(\rho)) = \sup_{r > 0} \rho \log r - \Lambda_L(r).
\]

**Theorem.** Suppose that \( L \) is a non-quasianalytic, eventually increasing and unbounded from above function such that \( \rho \mapsto \rho \log(\rho L(\rho)) \) is a convex function of \( \log \rho \). Let \( L_1 \) be an increasing function. Then
\[
\mathcal{F}_0(L_1) = \{ (a_n) : |a_n|^{1/n} = o(L_1(n)), \ n \to \infty \} \subseteq BC_0(L; \mathbb{R})
\]
if and only if
\[
\int_0^\infty \frac{r}{r^2 + t^2} \Lambda_L(t) dt = O(\Lambda_{L_1}(r)), \ r \to \infty.
\]

We remark that the “only if” part is due to Ehrenpreis and that Carleson proved this theorem for the Carleman classes (though his proof also works for the Beurling classes).

The assertion \([9.6]\) can be recast as
\[
\left\{ \left( \frac{a_n}{\gamma(n+1)} \right) : |a_n|^{1/n} = o(L_1(n)), \ n \to \infty \right\} \subseteq BS\overline{L}C_0(L; \mathbb{R}),
\]
which under the assumption
\[
\frac{L_1(\rho + 1)}{L_1(\rho)} < C
\]
can be also written as
\[
S_{L_1/L}Hol(\mathbb{C}) \subseteq S\overline{L}C_0(L; \mathbb{R}).
\]

The next lemma shows the connection between the theorem of Carleson and Ehrenpreis and our results.

**Lemma 9.3.** Suppose that \( L \) is non-quasianalytic and slowly growing. Then
\[
\int_0^\infty \frac{r}{r^2 + t^2} \Lambda_L(t) dt \asymp \Lambda_{L/\overline{L}}(r),
\]
where, as before, \( \overline{L}(\rho) := \int_1^\infty \frac{du}{uL(u)} \).

In particular, the above lemma and theorem of Carleson and Ehrenpreis imply that
\[
S\overline{L}Hol(\mathbb{C}) \subseteq S\overline{L}C_0(L; \mathbb{R}),
\]
and that \( \overline{L} \) in the left-hand side of \([9.7]\) cannot be replaced by any function \( L_2 \) with \( L_2(\rho) = o(\overline{L}(\rho)), \ \rho \to \infty \), while Theorem 4’ states that \( S\overline{L}C_0(L; \mathbb{R}) = S\overline{L}C_0(L; \mathbb{R}) \), but under additional regularity conditions. The proof of Lemma 9.3 is given in Appendix A.
9.5.2.2 Ehrenpreis representation. We will use the following representation of functions in the Beurling class, due to Ehrenpreis.

**Theorem.** Let $L : [0, \infty) \to (0, \infty)$ be a function such that $\lim_{\rho \to \infty} L(\rho) = \infty$ and the function $\rho \mapsto \rho \log(\rho L(\rho))$ is eventually strictly convex. If $g \in C_0(L; (R))$, then there exists a representation

$$g(t) = \int e^{iwt} \frac{d\mu(w)}{k(w)}$$

where $\mu$ is a finite complex-valued measure and $k \in C(\mathbb{C})$ is a non-negative function such that, for any $a, b > 0$,

$$\lim_{|w| \to \infty} \exp \left( a |\text{Im} w| + \Lambda_L(b|w|) \right) = \infty.$$ 

Here, as before,

$$\Lambda_L(r) = \sup_{\rho \geq 0} \left[ \rho \log r - \rho \log (\rho L(\rho)) \right].$$

From here on, we will refer to such representation of functions in $C_0(L; R)$, as the Ehrenpreis representation. Such representations are not unique (the construction of $\mu$ and $k$ uses the Hahn–Banach theorem). The proof can be found in [15, §V.6] or in [29].

It is worth mentioning that it is possible to study singular transforms of Beurling classes via the Ehrenpreis representation: observing that if $g \in C_0(L; R)$ has the representation

$$g(t) = \int e^{iwt} \frac{d\mu(w)}{k(w)},$$

then

$$(SLg)(z) = \int e^{iws} \frac{d\mu(w)}{k(w)} := \int E_1(iws) \frac{d\mu(w)}{k(s)},$$

where

$$E_1(z) = \sum_{n \geq 0} \frac{z^n}{n! \gamma(n+1)}.$$ 

For instance, in this way, one could prove Theorem 2 in the case $I = R$. The drawback of such an approach is its inability to treat intervals $I$ which are different from the whole real line. On the other hand, it has the nice feature that its easily extends to Beurling classes in several variables. We will not pursue this approach here.

9.5.2.3 The functions $K_*$ and $E_*$. We fix a non–quasianalytic $L$ satisfying the assumptions of Theorem 4. Recall the definitions of the associated functions:

$$\tilde{L}(\rho) = L(\rho) \int_\rho^\infty \frac{du}{uL(u)}, \quad \rho > 1,$$

$$\gamma(\rho) = L(\rho)^\rho, \quad \tilde{\gamma}(\rho) = \tilde{L}(\rho)^\rho$$

and

$$E_*(z) = \sum_{n \geq 0} \frac{\tilde{\gamma}(n+1)}{\gamma(n+1)} z^n, \quad K_*(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\gamma(s)}{\tilde{\gamma}(s)} z^{-s} ds, \quad c > 0.$$ 

Theorems A and B provide us with the asymptotics of $K_*$ and $E_*$. Note that

$$\rho \frac{d}{d\rho} \log \frac{L(\rho)}{L(\rho)} = \frac{1}{L(\rho)},$$
and so the corresponding saddle point equation is
\[ \log z = \log \frac{L(s)}{\tilde{L}(s)} + \frac{1}{\tilde{L}(s)}. \]
For \( z \in \Omega(\pi/2), |z| > r_0 \), we denote by \( s = s_z = \rho z e^{i\theta_z} \) the unique solution to this saddle point equation. It follows from Theorem A,
\[ \log K_*(z) \sim -\cos \theta \frac{\rho}{L(\rho)}, \quad |z| \to \infty, \quad z \in \Omega(\pi/2). \]

(9.8)

Given \( c \in [0, \frac{\pi}{2}) \), we denoted by \( \Psi_{\pm}(c) \) curves joining 0 with \( \infty \) in the 1st and 4th quadrants, respectively, and such that, for \( |z| \) sufficiently large, \( \Psi_{\pm}(c) \) coincide with the curves \( \{ z = \frac{L(s)}{L(s)} \exp \left( \frac{1}{L(s)} \right) : \arg(s) = \pm c \} \). In particular, the above asymptotic formula together with Cauchy's theorem yields
\[ \int_{\Psi_{\pm}(c)} z^n K_*(z) \, dz = \int_0^\infty r^n K_*(r) \, dr = \frac{\gamma(n+1)}{\gamma(n+1)}, \quad 0 < c < \pi/2. \]

(9.9)

9.5.2.4 Proof of the inclusion \( S_L C_0(L; I) \supseteq \tilde{S}_L C_0(\tilde{L}; \mathbb{R}) \).

Proof. As we already mentioned, since \( L \) is non-quasianalytic, the set \( S_L C_0(L; I) \) does not depend on the interval \( I \) (as long as \( I \) contains the origin). From here on, we will assume that \( I = (-1, 1) \).

Let \( g \in C_0(\tilde{L}; \mathbb{R}) \). Then, according to Ehrenpreis, there exists a representation
\[ g(t) = \int_C e^{iw t} \frac{d\mu(w)}{k(w)}, \]
where \( \mu \) is a finite measure and \( k \in C(\mathbb{C}) \) is a non-negative function such that, for every \( a, b > 0, \)
\[ \lim_{|s| \to \infty} \exp \left( a |\text{Im} \, w| + \Lambda_L(b|w|) \right) = \infty. \]

Fix a sufficiently small constant \( \delta > 0 \) that will be chosen later, set (see Figure 6)
\[ A := \{ w : |\text{Im} \, w| > \frac{2\delta}{\pi} \Lambda_L(|w|) \}, \quad B^\pm := \{ w : |\text{Im} \, w| \leq \frac{2\delta}{\pi} \Lambda_L(|w|), \pm \text{Re} \, w > 0 \} \]
and split
\[ g(t) = \int_{\mathbb{C}} e^{iwt} \frac{d\mu(w)}{k(w)} = \int_{A} + \int_{B^+} + \int_{B^-} = g_e(t) + g_+(t) + g_-(t). \]

We aim to find functions \( f_e, f_+, f_- \in C_0(L; I) \) such that

\[ \hat{f}_e(n) = \hat{g}_e(n) \frac{\gamma(n+1)}{\gamma(n+1)} \quad \text{and} \quad \hat{f}_\pm(n) = \hat{g}_\pm(n) \frac{\gamma(n+1)}{\gamma(n+1)}, \quad n \geq 0. \]

First we treat \( g_e \). By the definition of the set \( A \),

\[ g_e(t) = \int_{\mathbb{C}} e^{iwt} \frac{d\mu_e(w)}{k_e(w)}, \]

where \( \mu_e \) is a finite measure and a \( k_e \in C(\mathbb{C}) \) is a non-negative function such that

\[ \lim_{|w| \to \infty} k_e(w)e^{-a|w|} = \infty, \quad \forall a > 0. \]

Differentiation yields

\[ |g_e^{(n)}(t)| \leq C_{M,a} \int_{\mathbb{C}} |w|^n e^{-a|w|}|d\mu_e(w)|, \quad a > 0, |t| < M. \]

Therefore,

\[ |g_e^{(n)}(t)| \leq \frac{C_{M,a}n!}{a^n}, \quad a > 0, |t| < M, \]

which means that \( g_e \) is an entire function. Now, by the Carleson–Ehrenpreis Theorem (we use its corollary stated as (9.7)), there exists \( f_e \in C_0(L; \mathbb{R}) \) such that

\[ \hat{f}_e(n) = \hat{g}_e(n) \frac{\gamma(n+1)}{\gamma(n+1)}, \quad n \geq 0. \]

Now we treat the function \( g_+ \). Clearly, the function \( g_+ \) is holomorphic in the upper half-plane and smooth up to its boundary and is represented therein by the same integral. Let us estimate the derivatives of \( g_+ \) in the upper half-plane. Fix a large parameter \( b > 0 \); then for \( 0 < \psi < \pi/2 \), we have

\[ |g_+^{(n)}(re^{i\psi})| \leq \int_{B^+} |w|^n \exp \left( -\text{Im}(re^{i\psi}w) \right) \frac{|d\mu(w)|}{k(w)}. \]
By definition, \( \exp \left( 2b\Lambda_{\tilde{L}}(|w|) \right) \lesssim_b k(s) \), and for any \( w \in B_+ \), we have \( |\arg w| \leq \delta \frac{\Lambda_{\tilde{L}}(|w|)}{|w|} \). Therefore,

\[
|g_+^{(n)}(re^{i\psi})| \lesssim_b \int_{C} |w|^n \exp \left( r|w| \sin \left( \frac{\delta \Lambda_{\tilde{L}}(|w|)}{|w|} - \psi \right) - 2b\Lambda_{\tilde{L}}(|w|) \right) d|w|.
\]  

(9.10)

By Lemma 5.2 part 4 (with \( L/\tilde{L} \) and \( 1/\tilde{L} \) instead of \( L \) and \( \varepsilon \)), we have

\[
\tilde{L}(e^{\tilde{L}(\rho)}) \sim \tilde{L}(\rho), \quad \rho \to \infty.
\]  

(9.11)

Thus, by the first part of Lemma 5.1,

\[
\Lambda_{\tilde{L}}(\rho) \sim \frac{\rho}{e^{L(\rho)}}, \quad \rho \to \infty.
\]

Choosing \( \delta \) in estimate (9.10) sufficiently small, we get

\[
|g_+^{(n)}(re^{i\psi})| \lesssim_b \frac{n\tau(n+1)}{b^n} \sup_{\tau > 0} \exp \left[ \frac{r\tau}{2} \left( \frac{1}{L(\tau)} - 2\psi \right) \right].
\]

By Lemma 5.2 part 1,

\[
\tilde{L} \left( \frac{L(\rho)}{e^{L(\rho)}} \right) \sim \tilde{L}(\rho), \quad \rho \to \infty.
\]

Combining this with (9.11), we get

\[
\tilde{L}(\rho) \sim \tilde{L}(\rho L(\rho)) \sim \tilde{L}(\rho L^2(\rho)), \quad \rho \to \infty,
\]

which in turn yields

\[
|g_+^{(n)}(re^{i\psi})| \lesssim_b \frac{n\tau(n+1)}{b^n} \exp \left[ \frac{r\tau}{2\tau(\psi)} \right], \quad \tau = \sup \left\{ \tau : \tilde{L}(\tau L^2(\tau)) \leq \psi^{-1} \right\}.
\]  

(9.12)

From here on we assume that \( z = re^{i\psi} \in \Psi_{\pi/3} \) with \( r \) sufficiently large, and that \( s = \rho e^{i\pi/3} = s_z \) is related to \( s \) by the saddle point equation

\[
z = L(s) \exp \left( \frac{1}{L(s)} \right).
\]

By (6.1) (applied to the function \( L/\tilde{L} \) instead of \( L \)),

\[
\rho \asymp \frac{L(\rho)}{e^{L(\rho)}}, \quad \frac{1}{3} \frac{1}{L(\rho)} \sim \psi, \quad r \to \infty.
\]

In particular, if \( r \) is sufficiently large, then

\[
\tilde{L}(\tau_L(\rho L^2(\tau)) \leq \tilde{L}(\rho) \implies \tau_L \leq \frac{\rho}{L(\rho)}.
\]

Thus, for \( 0 \leq t \leq 1 \), the latter and inequality (9.12) yield

\[
|g_+^{(n)}(tz)| \lesssim_b \frac{n\tau(n+1)}{b^n} \exp \left[ \frac{r\rho}{L(\rho)L(\rho)} \right] \lesssim_b \frac{n\tau(n+1)}{v^n} \exp \left[ \frac{1}{4} \frac{\rho}{L(\rho)} \right].
\]
Set
\[ f_+(t) := \int_{\Psi_{\pi/3}} g_+(zt)K_+(z)dz, \quad 0 \leq t \leq 1. \]

By (9.8),
\[ \log K_+(z) \sim -\frac{1}{2} \cdot \frac{\rho}{L(\rho)}. \]

Therefore, the function \( f_+ \) is well defined and satisfies
\[ \left| f_+^{(n)}(t) \right| \leq \left| \int_{\Psi_{\pi/3}} z^n g_+^{(n)}(zt)K_+(z)dz \right| \lesssim_b \frac{n!\gamma(n+1)}{b^n} \left( 1 + \int_{0}^{\infty} r^n \exp \left[-\frac{1}{10} \cdot \frac{\rho}{L(\rho)} \right] dr \right), \]

for any \( 0 \leq t \leq 1 \). By Lemma 6.4, the integral in the right-hand side is \( \leq C 2^n \frac{\gamma(n+1)}{7(n+1)} \). Therefore,
\[ \left| f_+^{(n)}(t) \right| \lesssim_b \left( \frac{2}{b} \right)^n n!\gamma(n+1), \quad 0 \leq t \leq 1. \]

Since \( b > 0 \) can be taken arbitrarily large, we conclude \( f_+ \in C_0(L; [0, 1]) \).

For \( -1 \leq t < 0 \), we define
\[ f_+(t) := \int_{\Psi_{-\pi/3}} g_+(zt)K_+(z)dz. \]

The same reasoning as above yields, \( f_+ \in C_0(L; [-1, 0]) \). Furthermore,
\[ f_+^{(n)}(0) = g_+^{(n)}(0) \int_{\Psi_{\pi/3}} z^n K_+(z)dz \quad \text{by (9.9)} \quad g_+^{(n)}(0) \int_{0}^{\infty} z^n K_+(z)dz = g_+^{(n)}(0) \frac{\gamma(n+1)}{7(n+1)} \]

Therefore, \( f_+ \in C_0(L; [-1, 1]) \) and it is the sought for function.

For \( g_- \) we define
\[ f_-(t) := \int_{\Psi_{\pi/3}} g_+(zt)K_+(z)dz, \quad 0 \leq t \leq 1. \]

The method of estimating the derivatives of \( f_- \) is the same as the one of \( f_+ \). This finishes the proof of Theorem 5.

9.6 Carleman classes, Theorem 5

**Theorem 5’.** Let \( L \) be a non-quasianalytic function and \( I \) be an open interval that contains the origin. Suppose that the function \( \rho \mapsto L(\rho)/\tilde{L}(\rho) \) satisfies assumptions (R1), (R2), (R3) and (R8), and that \( I \) is an open interval containing the origin. Then
\[ S_L C(L; I) = S_L C(\tilde{L}; 0). \]

Throughout this section we fix a non-quasianalytic function \( L \) that satisfies the assumptions of Theorem 5.

The proof of the inclusion
\[ S_L C(L; I) \subseteq S_L C(\tilde{L}; 0) \]

follows the same lines as the analogous inclusion in the Beurling case. Therefore, we will prove only the opposite direction.
Recall that in order to prove that
\[ S_L C(L; I) \supseteq S_L C(\tilde{L}; \mathbb{R}), \]
we have used Ehrenpreis representation for functions in the Beurling class \( C_0(\tilde{L}; \mathbb{R}) \). We are not aware of an analogous representation for functions in the corresponding Carleman class. So, in order to show the inclusion
\[ S_L C(L; I) \supseteq S_L C(\tilde{L}; 0) \]
we will take a different approach, based on asymptotically holomorphic extensions of smooth functions. We begin with some preliminaries.

9.6.1 A decomposition of functions in \( C(\tilde{L}; 0) \). Denote by \( C_\omega(0) \) the sets of all functions analytic in some neighborhood of the origin.

**Lemma 9.4.** Let \( g \in C(\tilde{L}; 0) \). Then there exists functions \( g_1, g_2 \) such that
1. \( g \equiv g_1 + g_2 \) is some neighborhood of the origin;
2. \( g_1 \in C_\omega(0) \);
3. There exists \( C > 0 \) such that
\[
\sup_{x \in \mathbb{R}} |g_2^{(n)}(x)| \leq C n! \tilde{\gamma}(n+1), \quad n \geq 0.
\]

The proof is based on the theory of almost holomorphic extensions [13]. The idea to use almost holomorphic extensions was suggested by Alexander Borichev.

**Proof.** Fix \( g \in C(\tilde{L}; 0) \). Let \([-a, a] \ (a > 0) \) be an interval such that \( g \in C(\tilde{L}; [-a, a]) \). According to Dynkin [13], \( g \in C(\tilde{L}; [-a, a]) \) if and only if it can be represented as
\[
g(t) = \iint_{\mathbb{C}} G(z) \frac{dx dy}{z-t},
\]
where \( G : \mathbb{C} \to \mathbb{C} \) is a continuous and compactly supported function such that
\[
|G(z)| \leq C_1 h(C_2 \text{dist}(z, [-a, a])), \quad \text{where} \quad h(r) = \inf_{n \geq 0} \tilde{\gamma}(n+1) r^n.
\]

Let \( \delta < a/8 \) be a small parameter and \( \xi \) be a continues function such that \( \xi \equiv 0 \) for all \( z \in \{z : \text{dist}(z, [-a/4, a/4]) \leq \delta \} \) and \( \xi \equiv 1 \) for all \( z \in \{z : \text{dist}(z, [-a/4, a/4]) \geq 2\delta \} \). Finally, set
\[
g(t) = \iint_{\mathbb{C}} G(z) \frac{dx dy}{z-t} + \iint_{\mathbb{C}} G(z)(1 - \xi(z)) \frac{dx dy}{z-t} := g_1(t) + g_2(t).
\]

Clearly \( g_1 \in C_\omega([-a/4, a/4]) \), and by Dynkin’s theorem \( g_2 \in C(\tilde{L}; [-3a/4, -3a/4]) \). The function \( g_2 \) also satisfies
\[
|g_2^{(n)}(t)| \leq n! \iint |G(z)| \frac{dx dy}{|z-t|^{n+1}} \leq \frac{Cn!}{\text{dist}^n(t, [-\frac{a}{2}, \frac{a}{2}])} \leq C n! n!, \quad t \notin [-a/2, a/2], \quad n \in \mathbb{Z}_+.
\]
Since \( \tilde{L} \uparrow \infty \), the last estimate completes the proof of Lemma 9.4.
9.6.2 We will also need a statement similar to (9.7):
\[
S_L^{-} C^\omega(0) \subseteq S_L C(L; \mathbb{R}).
\] (9.13)

It follows from the theorem of Carleson and Ehrenpreis and Lemma 9.3. The proof is similar to the one given in the case of Beurling classes, so we will omit it.

Proof of the inclusion $S_L C(L; I) \supseteq S_L C(\widetilde{L}; 0)$. Fix an element in $g \in C(\widetilde{L}; 0)$, and fix functions $g_1, g_2$ as in Lemma 9.4. By (9.13), there exists a function $f_1 \in C(L; \mathbb{R})$ such that
\[
S_L^{-} g_1 = S_L f_1.
\]

Put
\[
f_2(t) = \int_0^\infty g_2(tx)K_\ast(x)dx,
\]
with the analytic function $K_\ast$ defined in the beginning of 9.5.2.3. Clearly
\[
|f_2^{(n)}(t)| = \int_0^\infty x^n|g_2^{(n)}(xt)||K_\ast(x)|dx \leq C^{n+1}n!\gamma(n + 1)\int_0^\infty x^n|K_\ast(x)|dx.
\]

Since $K_\ast$ is eventually positive, this estimate yields
\[
|f_2^{(n)}(t)| \leq C^{n+1}n!\gamma(n + 1)\frac{\gamma(n + 1)}{\gamma(n + 1)} = C^{n+1}n!\gamma(n + 1).
\]

Therefore, $f_2 \in C(L; \mathbb{R})$ and $S_L g_2 = S_L f_2$. Finally, we define $f = f_1 + f_2 \in C(L; \mathbb{R})$ and conclude that $S_L^{-} g = S_L f$. \qed

9.7 Theorem 6'.

Theorem 6'. Suppose that the function $L$ satisfies assumptions (R1), (R2), (R3), (R5), (R7) and (R9), and that $I$ is an open interval containing the origin. Then the singular transform $S_L$ maps $C_0^\pm(L; I)$ bijectively onto the space $A^\pm(L; I)$, with the inverse $R_L^\pm$.

9.7.1 The inclusion $A^\pm(L; I) \subseteq S_L C_0^\pm(L; I)$.

Proof. Fix an open interval $I$ that contains the origin, and a function $L$ that satisfies assumptions (R1), (R2), (R3), (R5), (R7) and (R9).

Let $F \in A^\pm(L; I)$ and put $f = R_L^+ F$. We want to show that $f \in C_0^\pm(L; I)$. Put $I_+ = I \cap (0, \infty)$ and $I_- = I \cap (-\infty, 0]$. The proof is broken into three parts: first we show that $f \in C_0(L; I)$, then that $f \in C_0^+(L; \text{int}(I_+))$, and finally that $f \in C_0^+(L; I)$.

Part 1. Fix $a_+ \in I_+$, and let $c_+ \in I_+$ such that $a_+ < c_+$. By the definition of the class $A^\pm(L; I)$, for any $B > 0$ there is $\Delta > 0$, such that
\[
|F(re^{i\psi})| \lesssim_B H \left( \psi + \frac{2B}{r} \right) + E \left( \frac{r}{c_+} + \Delta r \sin \psi \right)
\]
for $-\frac{B}{r} \leq \psi \leq \frac{\pi}{2} + \frac{B}{r}$. By Cauchy’s estimates, we have
\[
|F^{(n)}(z)| \lesssim \frac{n!}{B^n} \max_{|z-w|=B} |F(w)|.
\]
Since $H$ is decreasing and continuous in $(0, \frac{\pi}{2})$, we have

$$|F^{(n)}(re^{i\psi})| \lesssim_B \frac{n!}{r^n} \left[ H \left( \psi + B \frac{r}{2} \right) + E \left( \frac{r}{c_+} + \frac{B\Delta}{c_+} + \Delta r \sin \psi \right) \right]$$  \hspace{1cm} (9.14)

for $0 \leq \psi \leq \frac{\pi}{2}$. By the definition of $R_L^+$,

$$f(u) = (R_L^+ F)(u) = \int_{\Psi_+} F(uz) K(z) dz, \quad 0 \leq u < a_+.$$  

Thus,

$$|f^{(n)}(u)| \leq \int_{\Psi_+} |z^n| |F^{(n)}(uz)| |dz|, \quad 0 \leq u < a_+.$$  

By (9.14), we have

$$|f^{(n)}(u)| \lesssim_B \frac{n!}{r^n} [1 + I_1(u, n) + I_2(u, n)]$$

where

$$I_1(u, n) = \int_1^\infty r^n H \left( \psi + \frac{B}{2r \max\{u, 1\}} \right) |K(re^{i\psi})| dr, \quad z = re^{i\psi} \in \Psi_+,$$  

and

$$I_2(u, n) = \int_1^\infty r^n E \left( \frac{ru}{c_+} + \frac{B\Delta}{c_+} + \Delta ru \sin \psi \right) |K(re^{i\psi})| dr, \quad re^{i\psi} \in \Psi_+.$$  

As a result, in order to show that $f \in C_0(L; [0, a_+])$, it suffices to show that there is a $C > 0$, such that for sufficiently large $B > 0$,

$$I_1(u, n) \lesssim_B C^{n+1} \gamma(n + 1), \quad I_2(u, n) \lesssim_B C^{n+1} \gamma(n + 1), \quad 0 \leq u \leq a_+.$$  

We begin with the integral $I_1$. By Lemma 6.7 there exists $A > 0$ such that

$$H \left( \psi + \frac{A}{r} \right) \leq H(\psi) \frac{1-\frac{3}{2}}{r}.$$

By taking $B$ so large that $B > 2A \max\{1, a_+\}$, we get

$$H \left( \psi + \frac{B}{2r \max\{u, 1\}} \right) \lesssim_B H(\psi) \frac{1-\frac{3}{2}}{r} \lesssim |E(z)| \frac{1-\frac{2}{r}}{r}, \quad z = re^{i\psi} \in \Psi_+, \quad 0 \leq u \leq a_+.$$  

By Theorems A and B

$$|E(z)K(z)| \sim \frac{\rho_z}{|z|^2}, \quad z \to \infty, \quad z \in \Psi_+.$$  

In particular,

$$|E(z)K(z)| \lesssim \exp \left( |z|^{-1} \cdot \text{Re} \left( i\rho_z z(i\rho_z) \right) \right) \lesssim |E(z)| \frac{1}{|z|^2} \quad |z| > 1, \quad z \in \Psi_+.$$  

Thus,

$$I_1(u, n) \leq \int_{\Psi_+ \cap \{|z|>1\}} |z^n| |E(z)| \frac{1}{|z|^2} \, d|z|,$$

and, by Lemma 6.6

$$I_1(u, n) \lesssim_B C^{n+1} \gamma(n + 1), \quad u \in [0, a_+].$$
Now we turn to $I_2$. Since, $c_+ > a_+$, and $\psi \to 0$ as $r \to \infty$, $re^{i\psi} \in \Psi_+$, there exists a $\delta = \delta(a_+, c_+) > 0$, such that

$$I_2(u, n) \lesssim_B \int_1^\infty r^n E((1 - \delta)r) \left\| K(re^{i\psi}) \right\| dr, \quad u \in [0, a_+].$$

By Lemma 6.5, there exists $\delta_1 > 0$ such that

$$E((1 - \delta_1)|z|)K(z) \lesssim E(\delta_1|z|), \quad z \in \Psi_+, \ |z| > 1,$$

whence,

$$I_2(u, n) \lesssim_B \int_1^\infty \frac{r^n}{E(\delta_1 r)} dr.$$

By definition, $E(\delta_1 r) \geq \frac{\gamma^{(n+3)}_{(n+2)}}{\delta_1^{n+1} r^{n+2}}$. Thus,

$$I_2(u, n) \lesssim_B \delta_1^{-n} \gamma(n + 3) \leq \delta_1^{-n} \gamma(n + 1).$$

We have shown that

$$|f^{(n)}(u)| \lesssim_B \frac{n!C^n}{B^n} \gamma(n + 1), \quad 0 \leq u \leq a_+.$$

The constant $B$ can be taken arbitrarily large, so we conclude that $f \in C_0(L; [0, a_+])$. Since, $0 < a_+ \in I_+$ was arbitrary, we conclude that $f \in C_0(L; I_+)$. The proof that $f \in C_0(L; I_-)$ is similar.

**Part 2.** Now, we will show that $f \in C_0^+(L; \text{int}(I_+))$. Let $u \in \text{int}(I_+)$. We need to show that for sufficiently small $0 < u$, $f$ is analytic at $w = u + iv$.

Let $c_+ \in I_+$ be such that

$$\frac{u}{c_+} \leq 1 - 5\delta,$$

for some $\delta > 0$. By the definition of the class $A^+(L; I)$, with $B = 1$, there is a $\Delta > 0$, such that

$$|F(re^{i\psi})| \lesssim H(\psi) + E \left( \frac{r}{c_+} + \Delta r \sin \psi \right), \quad 0 < \psi \leq \frac{\pi}{2}.$$

Fix the above $\Delta$ and choose $v > 0$ so small that for $w = u + iv$, we have $\Delta \arg(w) < \delta$ and $\frac{u}{c_+} < \delta$. Denote by $D_q(w)$ the closed disk $\{w' : |w - w'| \leq q\}$, and choose $q$ so small such that that $v - q > 0$ and

$$\frac{|w'|r}{c_+} + \Delta \sin(\arg w' + \psi)r|w'| \leq (1 - \delta)r, \quad w' \in D_q(w), \ r > r_\delta, re^{i\psi} \in \Psi_+.$$

By choosing such a $q$, we have,

$$\max_{w' \in D_q(w)} |F(zw')| \lesssim_q E((1 - \delta)|z|), \quad z \in \Psi_+.$$

As a result, Lemma 6.5 yields

$$\max_{w' \in D_q(w)} \int_{\Psi_+} |F(zw')K(z)dz| < \infty.$$

Thus, the function $f = R_L^+ F$ is analytic in an upper neighborhood of $\text{int}(I_+)$, and therefore $f \in C_0^+(L; \text{int}(I_+))$, as claimed. The proof that $f \in C_0^+(L; \text{int}(I_-))$ is identical.
Part 3. So far we have shown that \( f \in C_0(L; I) \), and that \( f \in C_0^+ (L; \text{int}(I_+)) \). We need to show that \( f \in C_0^+ (L; \text{int}(I_+)) \). This will follow from Morera’s theorem, if we can show that the boundary values of \( f \) on the interval \([0, \delta]\) as an element of \( C_0^+ (L; \text{int}(I_+)) \) coincide with the boundary values of \( f \) on the same interval as an element of \( C_0^+ (L; \text{int}(I_-)) \). That is, we need to show that for \( 0 \leq v < \delta \),
\[
\int_{\Psi_+} F(ivz)K(z)dz = \int_{\Psi_-} F(ivz)K(z)dz.
\]
To do so, we fix \( a_+ \in \text{int}(I_+) \), and let \( 0 < a < \min\{a_+, -a_-\} \). Denote by \( \Omega \) the domain bounded between \( \Psi_- \) and \( \Psi_+ \) (which contains the positive ray). By the definition of the space \( A^+ (L; I) \), there exists a positive \( \delta \), such that
\[
|F(iuz)| \lesssim E(|z|/2), \quad z \in \Omega, \quad 0 \leq u < \delta.
\]
We fix this value of \( \delta \). For \( z \in \Omega \), we denote by \( z^* \) its radial projection on \( \Psi_+ \), i.e., \( |z| = |z^*|, z^* \in \Psi_+ \). By Theorem 5.3, \( |K(z)| \leq |K(z^*)|, \quad |z| > r_0, \ z \in \Omega \).
Thus, if \( \Psi \) is an arbitrary curve in \( \Omega \) joining 0 and \( \infty \), then by Lemma 6.3,
\[
\int_{\Psi} F(ivz)K(z)dz = \int_{\Psi_+} F(ivz)K(z)dz, \quad 0 \leq v < \delta.
\]
In particular, this is true with \( \Psi = \Psi_- \), which is the desired result.
We have shown that \( A^+ (L; I) \subseteq S_L C_0^+ (L; I) \), the proof of the second inclusion, \( A^- (L; I) \subseteq S_L C_0^+ (L; I) \), is the same.

**9.7.2 The inclusion \( A^\pm (L; I) \supseteq S_L C_0^+ (L; I) \)**
Here we prove the inclusion \( A^\pm (L; I) \supseteq S_L C_0^+ (L; I) \).
We begin with some preliminaries regarding the classes \( C_0^+ (L; I) \).

**9.7.2.1 A decomposition of elements in \( C_0(L; I) \).**

**Lemma 9.5.** Suppose that \( L : [0, \infty) \to [1, \infty) \) is an eventually increasing and unbounded function such that the function \( \rho \to \rho \log L(\rho) \) is eventually convex, and that \( I \) is an open interval. Then for any \( f \in C_0(L; I) \) and any closed subinterval \( J \subset I \), there exists a function \( f_\omega \in C^\omega (J) \) and \( p \in C(\mathbb{R}) \) satisfying
\[
|p(t)| \lesssim \delta^n n! \gamma(n + 1) |t|^n, \quad \delta > 0, \ t \in \mathbb{R}
\]
such that,
\[
f(x) = f_\omega(x) + \int_{\mathbb{R}} e^{ixt} p(t)dt, \quad x \in J.. \tag{9.15}
\]
For functions \( f \in C_0^+ (L; I) \), the above lemma admits.

**Corollary 6.** Suppose that \( L : [0, \infty) \to [1, \infty) \) is an eventually increasing and unbounded function such that the function \( \rho \to \rho \log L(\rho) \) is eventually convex, and that \( I \) is an open interval. Then for any \( f \in C_0^+ (L; I) \) and any closed subinterval \( J \subset I \), there exists a function \( f_\omega \in C^\omega (J) \) and \( p \in C(\mathbb{R}) \) satisfying \( 9.15 \), such that
\[
f(x) = f_\omega(x) + \int_1^\infty e^{ixt} p(t)dt, \quad x \in J.
\]
By the above, any \( f \in C_0(L; I) \) can be written as \( f = f_\omega + f_+ + f_- \), where \( f_\omega \in C^\omega(J) \) and \( f_\pm \in C^+_0(L; \mathbb{R}) \). This implies the decomposition \( C_0(L; I) = C^+_0(L; I) + C^-_0(L; I) \). The proof of the lemma, is based on the theory of almost holomorphic extensions, and resembles the proof of Lemma 9.4.

**Proof of Lemma 9.5.** Fix \( L \) and the closed interval \( J \subset I \). Let \( J' \) be a closed interval such that \( J \subset \text{int}(J') \subset J' \subset I \). According to Dynkin [13], \( f \in C_0(L; J') \) if and only if \( f \) can be represented by

\[
f(x) = \iint_C F(w) \frac{dudv}{w - x}, \quad w = u + iv,
\]

where \( F \) is a continuous and compactly supported function such that for every \( A > 0 \), there exists a \( C > 0 \) such that

\[
|F(z)| \leq C h(A \text{ dist}(z, J')) , \quad \text{where} \quad h(r) = \inf_{n\geq 0} \gamma(n) r^n.
\]

For \( \eta > 0 \), let \( \xi_\eta : C \to [0,1] \) be a continuous function such that \( \xi_\eta \equiv 0 \) on \( \{w : \text{dist}(w, J) \leq \eta\} \) and \( \xi_\eta \equiv 1 \) on \( \{w : \text{dist}(w, J) \geq 2\eta\} \). Finally, set

\[
f(t) = \iint_C F(w) \xi_\eta(w) \frac{dudv}{z - t} + \iint_C F(w)(1 - \xi_\eta(w)) \frac{dudv}{w - x} : = f_\omega(x) + f_1(x).
\]

By Dynkin’s Theorem, \( f_\omega, f_1 \in C_0(L; J') \) as well. Moreover, \( f_\omega \in \text{Hol}\{w : \text{dist}(w, J) \leq \eta\} \) and \( f_1 \in \text{Hol}\{w : \text{dist}(w, J) \geq 2\eta\} \). Choosing \( \eta \) sufficiently small, we estimate the derivatives of \( f_1 \) as follows:

\[
|f_1^{(n)}(x)| \leq n! \int |F(w)| \frac{dudv}{w - x} \leq C^{n+1} \frac{n!}{(|x| + 1)^{n+1}}, \quad n \in \mathbb{Z}_+,
\]

for any \( x \) such that \( x+1, x-1 \notin J \). It follows that, for any \( \delta > 0 \),

\[
|f_1^{(n)}(x)| \leq C_\delta \delta^n \frac{n! \gamma(n+1)}{(1 + |x|)^{n+1}}.
\]

Put

\[
p(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixt} f_1(x) dx.
\]

We will finish the proof by showing that \( p \) satisfies (9.15). Integration by parts yields

\[
p(t) = \frac{t^n}{2\pi} \int_{\mathbb{R}} e^{-ixt} f_1^{(n)}(x) dx.
\]

Therefore, for sufficiently large \( |t| \) we have

\[
|p(t)| \leq C_\delta \inf_{n>0} \delta^n \frac{n! \gamma(n+1)}{|t|^n},
\]

proving the Lemma.

**Proof of Corollary 9.6.** Fix \( f_+ \in C^+_0(L; I) \) and a closed interval \( J \subset I \). Since \( f_+ \in C_0(L; I) \), applying Lemma 9.5, we find \( f_\omega \in C^\omega(J) \) and \( p \) satisfying (9.15) such that

\[
f_+(x) = f_\omega(x) + \int_{\mathbb{R}} e^{ixt} p(t) dt, \quad x \in J.
\]
The functions \( x \mapsto \int_{-\infty}^{1} e^{ixt}p(t)dt \) and \( x \mapsto \int_{1}^{\infty} e^{ixt}p(t)dt \) are analytic in the lower and upper half-planes, respectively. Put
\[
\tilde{f}_+ := \int_{1}^{\infty} e^{ixt}p(t)dt, \quad \tilde{f}_- := \int_{-\infty}^{1} e^{ixt}p(t)dt + f_\omega(x).
\]
Then \( \tilde{f}_\pm \in C_0^+(L;I) \) and \( f_+ + 0 = \tilde{f}_+ + \tilde{f}_- \). We have found two decompositions of \( f_+ \in C_0(L;I) \) as a sum of elements in \( C_0^+(L;I) \) (one of which is the trivial one, \( \tilde{f}_+ = f_+ + 0 \)). Therefore, \( f_+ \) and \( \tilde{f}_+ \) differ by a \( C^\omega(J) \) function. This completes the proof. \( \square \)

To show that \( S_L C_0^0(L;I) \subset A^\pm(L;I) \) we will proceed by the following plan. Given \( f \in C_0^+(L;I) \) and a closed subinterval \( J \subset I \), first, we use the decomposition of Corollary 6
\[
f(x) = f_\omega(x) + \int_{1}^{\infty} e^{ixt}p(t)dt, \quad x \in J.
\]
Then by the linearity of the singular transform,
\[
S_L f = S_L f_\omega + S_L \left( \int_{1}^{\infty} e^{ixt}p(t)dt \right) = S_L f_\omega + \int_{1}^{\infty} E_1(xt)p(t)dt,
\]
where
\[
E_1(z) = S_L(\exp)(z) = \sum_{n \geq 0} \frac{z^n}{n!\gamma(n+1)}.
\]
The treatment of the summands in the RHS is different. To obtain estimates for \( S_L f_\omega \) we use Theorem 7, while upper bounds for the second summand will follow from Lemma 7.3

### 9.7.2.2 Proof of the inclusion \( A^\pm(L;I) \supseteq S_L C_0^+(L;I) \)

**Proof.** Let \( f \in C_0(L;I) \), and let \( J \) be a compact sub-interval of \( I \). By Corollary 6, there exist functions \( f_\omega \in C^\omega(J) \) and \( p \in C(\mathbb{R}) \) satisfying
\[
|p(t)| \lesssim_\delta \frac{\delta^n n!\gamma(n+1)}{|t|^n}, \quad \delta > 0, \quad t \in \mathbb{R},
\]
such that
\[
f(x) = f_\omega(x) + \int_{1}^{\infty} e^{ixt}p(t)dt, \quad x \in J.
\]
By the linearity of the singular transform,
\[
S_L f = S_L f_\omega + S_L \left( \int_{1}^{\infty} e^{ixt}p(t)dt \right) := F_\omega + F_p.
\]
We claim that
\[
F_p(z) = \int_{1}^{\infty} E_1(itz)p(t)dt, \quad \text{where} \quad E_1 = S_L(\exp).
\]
Indeed, by Lemma 9.2 for any \( \delta > 0 \),
\[
|p(t)| \lesssim_\delta e^{-\delta^{-1}\Lambda_L(t)}.
\]
Therefore, Lemma 7.1 yields that the function \( z \mapsto \int_{1}^{\infty} E_1(itz)p(t)dt \) is an entire function. This function, has the same Taylor coefficients at the origin as \( F_p \), and so these two functions coincide, as claimed.
Fix $B > 0$ and $\delta > 0$, by Lemma 7.3,
\[ |E_1(itre^{i\psi})| \leq e^{C_B(\Lambda L (t) + 1)} H \left( \psi + \frac{2B}{r} \right), \quad r > 2B, \ 2B/r < \psi \leq \frac{\pi}{2}, \ t > 1. \]
Thus,
\[ |F_p(re^{i\psi})| \lesssim H \left( \psi + \frac{2B}{r} \right), \quad 2B/r < \psi \leq \frac{\pi}{2}. \]
In order to show that $S_L f \in A^+(L; I)$, we first extend the estimate of $F_p$ to the strip $-\frac{B}{r} < \psi < \frac{2B}{r}$.

The function $F_p$ is the singular transform of a function in $C_0(\mathbb{L}; \mathbb{R})$. Thus, by Theorem 2,\[ \max_{|\psi| < \frac{3B}{r}} |F_p(re^{i\psi})| \leq \tilde{E} \left( \frac{r}{4B} \right) E (\delta r), \]
where $\tilde{E}(z) = \sum_{n \geq 0} \frac{z^n}{\sqrt{n(n+1)}}$. By Lemma 6.1 (applied both to $E$ and to $\tilde{E}$), we have,
\[ \tilde{E} \left( \frac{r}{4B} \right) E (\delta r) \lesssim \tilde{E} \left( \frac{r}{3B} \right) + E(2\delta r) \lesssim H \left( \frac{2B}{r} \right) + E(2\delta r). \]
Combining this with the previous bounds of $F_p$, we conclude that
\[ |F_p(re^{i\psi})| \lesssim H \left( \psi + \frac{2B}{r} \right) + E(2\delta r), \ -\frac{B}{r} < \psi \leq \frac{\pi}{2}, \]
Due to the symmetry, we obtain
\[ |F_p(re^{i\psi})| \lesssim H \left( \psi - \frac{2B}{r} \right) + E(2\delta r), \ \frac{\pi}{2} < \psi < \pi + \frac{B}{r}. \]

It remains to estimate the function $F_\omega$. By Theorem 7, $F_\omega \in A^\omega(L; J)$. Thus, for any $c_- < 0 < c_+ \in J$, there exists a $\Delta > 0$, such that\[ |F_\omega(re^{i\psi})| \lesssim E \left( \frac{r}{|c_+|} + \Delta r \sin \psi \right). \]
Thus, if $\delta$ is so small that $2\delta < \max\{c_+, |c_-|\}$, then,
\[ |S_L f(re^{i\psi})| \lesssim H \left( \psi \pm \frac{2B}{r} \right) + E \left( \frac{r}{|c_+|} + \Delta r \sin \psi \right) \]
whenever $0 \leq \pm (\frac{\pi}{2} - \psi) \leq \frac{\pi}{2} + \frac{B}{r}$. Since $J$ is an arbitrary compact subset of $I$, we conclude that $S_L f \in A^+(L; I)$. This finishes the proof of Theorem 6.

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Appendix A  Regular functions

A.1 Proof of Lemma 5.1 Part 1. Consider the function

$$f(x, r) = x \log(\varepsilon L(r)) - x \log(x L(x)).$$

Differentiation with respect to $x$ yields that $f_x(x, r) = 0$ if and only if

$$\varepsilon L(r) = e x L(x) \exp(\varepsilon(x)), \quad \text{where } \varepsilon(x) = \frac{x L'(x)}{L(x)}.$$  

By assumption (R1), $\varepsilon(x) = o(1)$ as $x \to \infty$. Thus

$$\sup_{x>0} f(x, r) \sim r, \quad r \to \infty.$$  

Since

$$\sup_{x>0} f(x, r) = \Lambda_L(\varepsilon L(r)),$$

we conclude part 1.

Part 2. It follows from the definition of $\Lambda_L$, that the function $r \mapsto r \Lambda'_L(r)$ is the inverse function to $\rho \mapsto \rho L(\rho) \exp(1 + \frac{\varphi L'_\rho}{L(r)})$. Denote the later function by $v(\rho)$. We therefore have,

$$\Lambda_L(r) - \Lambda_L(r_0) := \int_1^r t \Lambda'_L(t) \frac{dt}{t} = \int_{v^{-1}(r_0)}^{v^{-1}(r)} \frac{u v'(u)}{v(u)} du = \int_{v^{-1}(r_0)}^{v^{-1}(r)} 1 + \varepsilon(u) + u \varepsilon'(u) du, \quad u \to \infty.$$  

By assumption (R3), there exists $r_0 > 0$ such that $\varepsilon(u) + u \varepsilon'(u) > 0$, for $u \geq r_0$. With such choice of $r_0$, we obtain

$$\Lambda_L(r) - \Lambda_L(1) \geq \int_{v^{-1}(1)}^{v^{-1}(r)} du = r \Lambda'_L(r) - r_0 \Lambda'_L(r_0).$$

Thus, $\frac{\Lambda'_L(r)}{\Lambda_L(r)} \leq \frac{1}{r}$ for sufficiently large $r$, and hence the part 2.

Part 3. By part 1,

$$\Lambda_L(r L(r) e)) \sim r \to \infty.$$  

Since the function $L$ is slowly varying, we have

$$e(\varepsilon L)(ar) \sim a\varepsilon L(r), \quad r \to \infty$$  

for any $a > 0$. Therefore, $\Lambda_L(ar) \sim a\Lambda_L(r)$ as $r \to \infty$. This completes the proof of part 3.

Part 4. We fix a large $k > 0$, and put $g(n) = n (\log L(k) - \log L(n))$. Since, $L$ is regularly varying, we have $g(\frac{k}{2}) = O(1)$ and $g'(\frac{k}{2}) = O(1)$ as $k \to \infty$. Moreover, $g$ is eventually concave, and therefore it is bounded by its tangent line at the point $n = \frac{k}{2}$. \hfill \square

A.2 Lemma 5.2 The proof of Lemma 5.2 is based on the following lemma from [11 §3.12.2]:

Lemma A.1. Let $L_j : [0, \infty) \to [0, \infty)$, $j = 1, 2$, be two eventually non-decreasing and $C^1$ smooth functions such that the functions $\varphi_j(r) = \frac{\varphi L'_j(r)}{L_j(r)}$ are bounded in $[0, \infty)$. Then $L_1(\rho L_2(\rho)) \sim L(\rho)$, as $\rho \to \infty$, if and only if

$$\varphi_1(\rho) \log L_2(\rho) = o(1), \quad \rho \to \infty.$$
Proof of Lemma 5.2 Part 1. The assertion \( \varepsilon(\rho L(\rho)) \sim \varepsilon(\rho) \), as \( \rho \to \infty \) follows from Lemma A.1 by taking \( L_1 = 1/\varepsilon \) and \( L_2 = L \). The assertion \( \varepsilon(\Lambda_L(\rho)) \sim \varepsilon(\rho) \), as \( \rho \to \infty \) follows from the previous assertion by Lemma 5.1 part 1.

Part 2. The assertion \( L(\rho L(\rho)) \sim L(\rho) \), as \( \rho \to \infty \) follows from Lemma A.1 by taking \( L_1 = L_2 = L \). The assertion \( L(\Lambda_L(\rho)) \sim L(\rho) \), as \( \rho \to \infty \) follows from the previous assertion by Lemma 5.1, part 1.

Part 3. The assertion follows from Lemma A.1, by taking \( L_1 = L_2 = L \).

Part 4. The assertion follows from Lemma A.1, by taking \( L_1 = L_2 = 1/\varepsilon \).

A.3 Proof of Lemma 5.3

Part 1. Put \( s = \rho e^{i\theta} \).

\[
\log L(s) = \int_0^s \frac{\varepsilon(w)}{w} dw = \int_0^\rho \frac{\varepsilon(u)}{u} du + i\int_0^\theta \varepsilon(\rho e^{i\psi}) d\psi.
\]

By assumption (R8),

\[
\int_0^\theta \varepsilon(\rho e^{i\psi}) d\psi = i\theta \varepsilon(\rho)(1 + o(1)), \quad \rho \to \infty.
\]

Thus,

\[
\log L(s) = \log L(\rho) + i\theta \varepsilon(\rho)(1 + o(1)), \quad \rho \to \infty,
\]

concluding part 1.

Part 2. The proof of part 2 is the same as the proof of part 1.

A.4 Proof of Lemma 9.2

First, we show that

\[
\sup_{n \leq r} \log \frac{r^n}{n! \gamma(n+1)} \asymp \Lambda_L(r).
\]

Put

\[
f(x, r) = x \log r - x \log(xL(x)).
\]

Since \( L \) is increasing and unbounded,

\[
\sup_{n \leq r} \log \frac{r^n}{n! \gamma(n+1)} \asymp \sup_{n \in \mathbb{N}} \log \frac{r^n}{n! \gamma(n+1)}.
\]

By Stirling’s formula, the RHS is \( \asymp \sup_{n \in \mathbb{N}} f(n, r) \). Thus, it is enough to show that

\[
\sup_{n \in \mathbb{N}} f(n, r) \asymp \sup_{x > 0} f(x, r) = \Lambda_L(r).
\]

By assumption, the function \( x \mapsto x \log(xL(x)) \) is an eventually convex function of \( \log x \). Thus, for sufficiently large \( r \), the supremum in \( \sup_{x > 0} f(x, r) \) is attained in a single point, which we denote by \( x_r \). By Taylor’s theorem, there exists \( |x_r - c| \leq x_r \) such that

\[
f([x_r], r) = f(x_r, r) + \frac{f_{xx}(c, r)}{2}(x_r - [x_r]).
\]

Since, by assumption (R3),

\[
f_{xx}(c, r) = -\frac{1 + \varepsilon(x) + x\varepsilon'(x)}{x} = o(1), \quad x \to \infty,
\]
we get
\[ |f([x_r], r) - f(x_r, r)| = o(1), \quad r \to \infty. \]
Therefore,
\[ \sup_{n \leq r} \log \frac{r^n}{n! \gamma(n + 1)} \asymp \Lambda_L(r), \]
as claimed. By Lemma 5.1 part 1,
\[ \Lambda_L(\delta^{-1}r) \sim \delta^{-1} \Lambda_L(r), \quad r \to \infty. \]
Therefore, for any \( \delta \),
\[ \sup_{n \leq r} \log \frac{r^n}{n! \gamma(n + 1) \delta^n} \asymp \delta^{-1} \Lambda_L(r), \]
where the implicit constant is independent of \( \delta \). This completes the proof of Lemma 9.2.

\section*{A.5 Proof of Lemma 9.3}
Fix a a non-quasianalytic and slowly growing function \( L \). It follows from [7, §1.6] that the functions \( \tilde{L} \) and \( L/\tilde{L} \) are also slowly varying, with \( L(\rho)/\tilde{L}(\rho) \to \infty \) as \( \rho \to \infty \), and that
\[ \Lambda_L(r) = o \left( \int_0^\infty \frac{r}{t^2 + r^2} \Lambda_L(t) dt \right), \quad r \to \infty. \]
Since
\[ \Lambda_L(r) \asymp \int_0^r \frac{r}{t^2 + r^2} \Lambda_L(t) dt, \]
we have
\[ \int_0^\infty \frac{r}{t^2 + r^2} \Lambda_L(t) dt \asymp r \int_r^\infty \frac{\Lambda_L(t)}{t^2} dt. \]
Changing variables by \( t = uL(u) \) and making use of Lemma 5.1 part 1, yields
\[ \int_0^\infty \frac{r}{t^2 + r^2} \Lambda_L(t) dt \asymp r \int_{\Lambda_L(r)}^\infty \frac{du}{uL(u)} = r \frac{\tilde{L}(\Lambda_L(r))}{\tilde{L}(\Lambda_L(r))} \asymp \Lambda_L(r) \tilde{L}(\Lambda_L(r)). \]
Put \( \ell_1 = L/\tilde{L} \). It remains to show that there exists a function \( L_\ast \) such that \( L_\ast \asymp \ell_1 \) and
\[ \Lambda_{L_\ast}(r) = \Lambda_L(r) \tilde{L}(\Lambda_L(r)). \]
By Lemma 5.1 part 1, applied to the function \( L_\ast \), we have
\[ r \asymp \Lambda_L(r) \tilde{L}(\Lambda_L(r)) L_\ast(\Lambda_L(r) \tilde{L}(\Lambda_L(r))). \]
Applying the same lemma to the function \( L \) and then to the function \( \tilde{L} \) yields
\[ rL(r) \asymp r\tilde{L}(r) L_\ast(r\tilde{L}(r)) \quad \Rightarrow \quad L_\ast(r) \asymp \ell_1(\Lambda_L(r)). \]
By definition,
\[ \frac{\rho \ell_1(\rho)}{\ell_1(\rho)} \log \tilde{L}(\rho) = \frac{\log \tilde{L}(\rho)}{L(\rho)} = o(1), \quad \rho \to \infty. \]
Thus, by Lemma A.1 (applied with \( L_1 = \ell_1 \) and \( L_2 = \tilde{L} \)),
\[ \ell_1(\rho) \asymp \ell_1(\rho \tilde{L}(\rho)). \]
By Lemma 5.1 part 1, \( \Lambda_L(\rho \tilde{L}(\rho)) \asymp \rho \), and therefore
\[ \ell_1(r) \asymp \ell_1(\Lambda_L(r)) \asymp L_\ast(r). \]
This completes the proof.
Appendix B  The functions $K$ and $E$

In this section we prove the lemmas of Section 6 related to the asymptotics of the functions $K$ and $E$.

B.1 Lemma 6.1  Fix a slowly growing function $L : [0, \infty) \to [1, \infty)$ with $\lim_{\rho \to \infty} L(\rho) = \infty$\footnote{All the assertions of Lemma 6.1 are asymptotic, so it suffices to prove it for $L$ increasing and strictly positive.}

Put
\[ E(z) = \sum_{n \geq 0} \frac{z^n}{L(n+1)n+1} \]
on the positive ray. Since the function $E$ depends only on the values of $L$ on the positive integers, we can change it on non-integer values (see \cite[pp. 17–18]{27}) and assume that $L \in C^\infty[0, \infty)$ and satisfies
\[ \lim_{\rho \to \infty} \frac{\rho L'(\rho)}{L(\rho)} = \lim_{\rho \to \infty} \frac{\rho^2 L''(\rho)}{L(\rho)} = 0, \quad (B.1) \]
without changing the fact that $L$ is slowly growing. We shall assume so from here on.

Put
\[ \mu(r) = \sup_{\rho \geq 0} \frac{r^\rho}{L(\rho)^{\rho}}. \]
We claim that the function $\rho \mapsto \rho \log L(\rho)$ is an eventually convex function of $\log \rho$. Indeed, by (B.1),
\[ \lim_{\rho \to \infty} \frac{d^i}{d\rho^i} \log L(e^\rho) = 0, \quad i = 1, 2 \]
which in turns implies
\[ \frac{d^2}{d\rho^2} [e^\rho L(e^\rho)] \sim e^\rho \log L(e^\rho) > 0, \quad \rho > \rho_0, \]
as claimed.

Also note that for any fixed $r$, $\lim_{\rho \to \infty} \frac{r^\rho}{L(\rho)^{\rho}} = 0$. So, for large enough $\rho$, the supremum in the definition of $\mu$ is achieved at a single point, which we denote by $\rho_r$. A simple computation shows that $r$ and $\rho_r$ are related by
\[ r = L(\rho_r) \exp \left( \rho_r \frac{L'(\rho_r)}{L(\rho_r)} \right) \quad \text{and} \quad \log \mu_L(r) = \rho_r^2 \frac{L'(\rho_r)}{L(\rho_r)}. \quad (B.2) \]

B.1.1 Proof of the inequality $L^{-1}(\eta r) \lesssim_\eta \log E(r)$. Set $\nu(r) = \sup_{n \in \mathbb{Z}_+} \frac{r^n}{L(n+1)n+1}$. Clearly, $\nu(r) \leq E(r)$. Since $\lim_{n \to \infty} \log \frac{\log L(n+2)}{\log L(n)} = 1$, we have $\log \mu(r) \lesssim \log \nu(r)$, which in turn implies $\log \mu(r) \lesssim \log E(r)$. Now for $\eta < 1$
\[ \log \mu(\eta^{-1} r) = \sup_{\rho > 0} \left[ \rho \log \eta^{-1} r - \rho \log L(\rho) \right] \bigg|_{\rho = L^{-1}(r)} \geq L^{-1}(r) \log \eta^{-1}, \]
and therefore $L^{-1}(\eta r) \lesssim_\eta \log E(r)$.

1 All the assertions of Lemma 6.1 are asymptotic, so it suffices to prove it for $L$ increasing and strictly positive.
B.1.2 Proof of the inequality \( \log E(r) \lesssim L^{-1}(r) \). We will show that \( \log (L(r)E(L(r))) \lesssim r \).

Since \( L \) is slowly growing and positive, the function \( \rho \to L(\rho)/\rho \) is eventually decreasing. Thus, for sufficiently large \( r \),

\[
L(r)E(L(r)) = \sum_{n \geq 1} \left( \frac{L(r)}{L(n)} \right)^n \lesssim r e^r + \sum_{n > r} r^n n^n.
\]

By Stirling’s formula, \( \sum_{n > r} r^n n^n \lesssim e^{Cr} \), which establishes the lemma.

B.1.3 Proof of the inequality \( E^2(r) \lesssim_{\delta} E((1 + \delta)r) \). By the previous parts of this lemma, it is enough to show that

\[
2L^{-1}(r) \leq L^{-1}((1 + \delta)r), \quad r > r_\delta.
\]

But this follows immediately from the fact that

\[
\log \frac{L^{-1}((1 + \delta)r)}{L^{-1}(r)} = \int_r^{(1+\delta)r} \frac{1}{\varepsilon(L^{-1}(r))} \frac{du}{u} \to \infty, \quad r \to \infty.
\]

B.2 Proof of Lemma 6.2 For \( r > r_0 > 0 \), denote by \( \rho(r) \) the solution to the saddle-point equation \( r = L(\rho)e^{\varepsilon(\rho)} \). Let \( 0 < \delta < 1 \). It follows from Theorems A and B that in order to prove Lemma 6.2 it is enough to find \( 0 < \delta_1 \) such that

\[
\rho(\delta_1 r) \varepsilon (\rho(\delta_1 r)) + \rho(r(1-\delta_1)) \varepsilon (\rho(r(1-\delta_1))) - \rho(r) \varepsilon (\rho(r)) \lesssim 1, \quad r > r_0.
\]

Thus, it is sufficient to show that there exist \( \alpha > 0 \) (independent of \( \delta \)), such that

\[
\rho(r(1-\delta)) \varepsilon (\rho(r(1-\delta))) \leq (1-\delta)^\alpha \rho(r) \varepsilon (\rho(r)).
\]

Let us prove the last assertion:

\[
\log \frac{\rho(r) \varepsilon (\rho(r))}{\rho(r(1-\delta)) \varepsilon (\rho(r(1-\delta)))} = \int_{r(1-\delta)}^{r} \frac{\varepsilon (\rho(u)) + \rho(u) \varepsilon'(\rho(u))}{\rho(u) \varepsilon (\rho(u))} \rho'(u) du
\]

\[
= (1 + o(1)) \int_{r(1-\delta)}^{r} \frac{\rho'(u)}{\rho(u)} du, \quad r \to \infty,
\]

were in the last equality, we have used \( \rho |\varepsilon'(\rho)| = o(\varepsilon(\rho)) \) as \( \rho \to \infty \). Differentiating, \( \log r = \log L(\rho) + \varepsilon(\rho) \), yields

\[
\frac{\rho'(r)}{\rho(r)} = \frac{1}{r \varepsilon (\rho(r)) + \rho(r) \varepsilon'(\rho(r))} = \frac{1}{r \varepsilon (\rho(r))} (1 + o(1)), \quad r \to \infty.
\]

So,

\[
\log \frac{\rho(r) \varepsilon (\rho(r))}{\rho(r(1-\delta)) \varepsilon (\rho(r(1-\delta)))} = (1 + o(1)) \int_{r(1-\delta)}^{r} \frac{1}{\varepsilon (\rho(u))} \frac{du}{u}.
\]

Since \( \varepsilon \) is bounded from above and positive, there exists \( \alpha > 0 \), such that

\[
\log \frac{\rho(r) \varepsilon (\rho(r))}{\rho(r(1-\delta)) \varepsilon (\rho(r(1-\delta)))} > \alpha \log \frac{1}{1 - \delta}.
\]

The last inequality completes the proof.
B.2.1 Proof of Lemma 6.3. We assume that \( z = re^{i\psi} \), \( s = re^{i\theta} \), are related through the saddle-point equation
\[
\log z = \log L(s) + \varepsilon(s).
\]
Comparing the real and imaginary parts of the saddle-point equation and making use of (6.3), we find that
\[
r = L(\rho) e^{\varepsilon(\rho)} (1 + o(1)), \quad \theta \varepsilon(\rho) (1 + o(1)) = \psi, \quad r \to \infty.
\]
Let \( \delta > 0 \) be a small number. By Theorem 6.3, \( |E(z)| = O(1) \), as \( z \to \infty \), uniformly in the set \( \mathbb{C} \setminus \Omega(\frac{\pi}{2} + \delta) \). Since,
\[
\partial \Omega(\frac{\pi}{2} + \delta) := \{ z : \theta = \pm \frac{\pi}{2} + \delta, \rho > \rho_0 \},
\]
it is enough to show that
\[
\varepsilon(\rho) \lesssim \varepsilon(L^{-1}(r)), \quad r > r_0.
\]
Indeed, write,
\[
\log \frac{\varepsilon(\rho)}{\varepsilon(L^{-1}(r))} = -\int_{\rho}^{r} \frac{e^{\varepsilon(L^{-1}(\rho))} (1 + o(1))}{\varepsilon(L^{-1}(r))} \frac{\varepsilon'(u)}{\varepsilon(u)} du.
\]
By the regularity assumption (R2), the function \( \varepsilon \) is eventually non-increasing. Thus, by the regularity assumption (R3), we get
\[
-\frac{\varepsilon'(u)}{\varepsilon(u)} \lesssim \frac{1}{u}, \quad u > \rho_0.
\]
Therefore
\[
\log \frac{\varepsilon(\rho)}{\varepsilon(L^{-1}(r))} \lesssim \varepsilon(L^{-1}(\rho)) \lesssim 1,
\]
which completes the proof of Lemma 6.3. \( \square \)

B.3 An auxiliary lemma Here we give an auxiliary result that will be used in the proofs of Lemmas 6.4, 6.5, and 6.6.

Lemma B.1. Suppose that \( L_1 \) is a function that satisfies regularity assumptions (R1) and (R2), and that \( L_2 \) is a function that satisfies regularity assumptions (R1). Assume further that \( \varepsilon_1(\rho) \log L_2(\rho) = o(1) \), as \( \rho \to \infty \), where \( \varepsilon_1(\rho) = \frac{\rho L_1(\rho)}{L_1(\rho)} \). If \( \rho = \rho(r) \) satisfies
\[
r = L_1(\rho)(1 + O(\varepsilon_1(\rho))), \quad r \to \infty,
\]
then for any \( \delta > 0 \), there exists \( r_\delta > 0 \), such that
\[
L_1^{-1}((1 - \delta)r) \leq \frac{\rho}{L_2(\rho)} \leq \rho \leq L_1^{-1}((1 + \delta)r), \quad r > r_\delta.
\]
Proof. The inequalities
\[
L_1^{-1}((1 - \delta)r) \leq \rho \leq L_1^{-1}((1 + \delta)r), \quad r > r_\delta.
\]
follow immediately from
\[
L_1(L_1^{-1}((1 + \delta)r)) (1 + O(\varepsilon_1(L_1^{-1}((1 + \delta)r)))) = (1 + \delta)r + o(1), \quad r \to \infty.
\]
Since \( L_2(\rho) \to \infty \) as \( \rho \to \infty \), it suffices to show
\[
L_1^{-1}((1 - \delta)r) \leq \frac{L_1^{-1}(r)}{L_2(L_1^{-1}(r))}, \quad r > r_\delta.
\]
To show this, we begin with
\[
\frac{L_1^{-1}(r)}{L_1^{-1}((1-\delta)r)} = \exp\left(\int_{(1-\delta)r}^{r} \frac{d}{du} \left(\log L_1^{-1}(u)\right) du\right) = \exp\left(\int_{(1-\delta)r}^{r} \frac{1}{\varepsilon_1(L_1^{-1}(u))} du\right).
\]
Since the function \(\varepsilon\) is eventually decreasing,
\[
L_1^{-1}(r) \geq L_1^{-1}((1-\delta)r) \exp\left(-\frac{\log(1-\delta)}{\varepsilon_1(L_1^{-1}(r))}\right), \quad r > r_0.
\]
Thus, it is enough to show that
\[
\exp\left(-\frac{\log(1-\delta)}{\varepsilon_1(L_1^{-1}(r))}\right) > L_2 \left(L_1^{-1}(r)\right), \quad r > r_\delta.
\]
The latter follows immediately from the assumption \(\varepsilon_1(\rho) \log L_2(\rho) = o(1)\), as \(\rho \to \infty\). The proof is complete.

**B.4 Proof of Lemma 6.4.** Fix \(\alpha < \pi/2\) and \(0 < \delta < 1/3\). For \(z \in \Omega(\alpha)\) with sufficiently large \(|z|\), denote by \(s = \rho e^{i\theta}\) the unique solution to the saddle-point equation.
\[
\log z = \log L(\rho e^{i\theta}) + \varepsilon(\rho e^{i\theta}).
\]
By (6.3),
\[
\log L(\rho e^{i\theta}) + \varepsilon(\rho e^{i\theta}) = \log L(\rho) + \varepsilon(\rho) (1 + i\theta + o(1)), \quad \rho \to \infty.
\]
Thus, comparing the real parts of the saddle-point equation, we obtain
\[
|z| = L(\rho) \exp(\varepsilon(\rho)), \quad |z| \to \infty.
\]
By Lemma B.1 applied with \(L_1 = L\), \(L_2 = \frac{1}{\varepsilon}\),
\[
-\delta_1 \rho \varepsilon(\rho) \leq C - L^{-1} \left(\frac{2}{3} |z|\right)
\]
By Lemma 6.1
\[
-\rho \varepsilon(\rho) \leq C - \log E \left(\frac{3}{5} |z|\right).
\]
Thus, by the matching between the growth of \(E\) and the decay of \(K\) (i.e., Theorems A and B), we have
\[
-\delta_1 \rho \varepsilon(\rho) \leq C + \log K \left(\frac{|z|}{2}\right).
\]
Therefore,
\[
\int_{r_0}^{\infty} |z|^n e^{-\delta_1 \rho \varepsilon(\rho)} d|z| \leq C \int_0^{\infty} |z|^n K \left(\frac{|z|}{2}\right) d|z| \leq C^{2n} \frac{\gamma(n+1)}{\gamma(n+1)}.
\]
This completes the proof.
B.5 Proof of Lemma 6.6. For \( z \in \Psi_+ \) sufficiently large, let \( i\rho \) be related to \( z \) by the saddle-point equation:

\[
\log z = \log L(i\rho) + \varepsilon(i\rho).
\]

By (6.3),

\[
\log L(s) + s \frac{L'(s)}{L(s)} = \log L(\rho) + \varepsilon(\rho) + i (\theta + o(1)) \varepsilon(\rho), \quad s = \rho e^{i\theta}, \quad \rho \to \infty.
\]

Thus, comparing the real parts of the saddle point equation yields

\[
|z| = L(\rho)(1 + O(\varepsilon(\rho))), \quad |z| \to \infty. \tag{B.3}
\]

By Theorem [B] \( \log |E(z)| \sim \text{Re}(i\rho \varepsilon(i\rho)), \quad |z| \to \infty. \)

By Lemma 5.3\( \varepsilon(i\rho) = \varepsilon(\rho) + \frac{\pi}{2} \rho \varepsilon'(\rho)(1 + o(1)), \quad \rho \to \infty. \)

The function \( \varepsilon' \) is eventually negative, therefore

\[
\log |E(z)| \sim \frac{\pi}{2} \rho^2 |\varepsilon'(\rho)|, \quad z \in \Psi_+, \quad |z| \to \infty, \tag{B.4}
\]

which together and (B.3) shows that

\[
-\frac{\log |E(z)|}{z} \leq -\frac{\rho^2|\varepsilon'(\rho)|}{L(\rho)}, \quad |z| > r_0.
\]

Put \( L_2(\rho) := -\frac{L(\rho)}{\rho \varepsilon'(\rho)}. \) By assumption, \( L_2 \) is slowly growing (i.e., satisfies assumption (R1)), and \( \varepsilon(\rho) \log L_2(\rho) = o(1) \) as \( \rho \to \infty. \) Thus, Lemma B.1 yields

\[
-\frac{\rho^2|\varepsilon'(\rho)|}{L(\rho)} \leq -L^{-1}\left(\frac{3}{4}|z|\right).
\]

Finally, by Lemma 6.1 and Theorems A and B (the reasoning is the same as in the previous Lemma),

\[
-\frac{\rho^2|\varepsilon'(\rho)|}{L(\rho)} \leq -K\left(\frac{|z|}{2}\right).
\]

Hence,

\[
\int_{\Psi_+ \cap \{|z| > 1\}} |z|^n |E(z)|^{-1/|z|} d|z| \leq C + C \int_{r_0}^{\infty} K\left(\frac{|z|}{2}\right) \leq C 2^{n+2}(n + 1),
\]

which completes finishes the proof.

\[\square\]

B.6 Proof of Lemma 6.5. Let \( \delta > 0. \) By Lemma 6.1

\[
\log E((1 - \delta)|z|) \leq C + L^{-1}((1 - \delta)|z|).
\]

If \( z \in \Psi_+ \), then by theorem Theorems A and B

\[
\log |K(z)| \sim -\log |E(z)| - \frac{\pi}{2} \rho^2 |\varepsilon'(\rho)|, \quad z \in \Psi_+, \quad |z| \to \infty.
\]
where \( \rho(z) = L(|z|)(1 + O(\varepsilon(|z|))) \), as \(|z| \to \infty\). Put \( L_2(\rho) = \frac{1}{\rho \varepsilon(\rho)} \). By assumption, \( L_2 \) is slowly growing (i.e., satisfies assumption (R1)), and \( \varepsilon(\rho) \log L_2(\rho) = o(1) \) as \( \rho \to \infty \). Thus, by Lemma 3.1, \[
abla \log |K(z)| \leq C - L_1^{-1} \left( (1 - \varepsilon^2 |z|) \right).
\] Since \( L \) is slowly varying, \[
2L_1^{-1} \left( (1 - \delta) |z| \right) - L_1^{-1} \left( (1 - \frac{\varepsilon}{2}) |z| \right) \leq C
\] We obtained, \[
E^2((1 - \delta) |z|) |K(z)| \leq C, \quad z \in \Psi_+, |z| > 1,
\] and hence the lemma.

**B.7 Proof of Lemma 6.7.** Put \( q = \log \log H \). It is enough to prove the lemma for sufficiently small \( \psi > 0 \) and sufficiently large \( r \). By definition, \[
\log H(\psi) = \text{Re} \left( i\rho \varepsilon(i\rho) \right),
\] where \[
\psi = \text{Im} \left( \log L(i\rho) + \varepsilon(i\rho) \right).
\] Differentiation with respect to \( \psi \) yields \[
1 = \text{Im} \left( \frac{\varepsilon(i\rho) + \rho \varepsilon(i\rho)}{\rho} \right) \frac{d\rho}{d\psi}.
\] By Lemma 5.3, the RHS is \( \sim \frac{\pi}{2} \varepsilon'(\rho) \frac{d\rho}{d\psi} \) as \( \rho \to \infty \). The same lemma also gives \[
\text{Re} \left( i\rho \varepsilon(i\rho) \right) \sim \frac{\pi}{2} \rho^2 |\varepsilon'(\rho)|, \quad \frac{d}{d\rho} \text{Re} \left( i\rho \varepsilon(i\rho) \right) = \frac{\pi}{2} \rho |\varepsilon'(\rho)|, \quad \rho \to \infty.
\] Therefore, \[
\frac{dq}{d\psi} = \frac{dq}{d\rho} \frac{d\rho}{d\psi} = \frac{2}{\pi \rho \varepsilon'(\rho)} (1 + o(1)), \quad \rho \to \infty.
\] By assumption, the RHS tends to \( -\infty \) as \( \rho \to \infty \) and thus, as \( \psi \to 0 \). In particular for sufficiently small \( \psi \), \( \frac{dq(\psi)}{d\psi} \leq -1 \), which completes the proof.

**Appendix C The function \( E_1 \)**

**C.1 Proof of Lemma 7.1.** Put \[
\tilde{\Lambda}(r) := \log \left( \sup_{n \geq 0} \frac{r^n}{n! \gamma(n + 1)} \right).
\] By the definition of the function \( E_1 \), \[
E_1(r) = \sum_{n \geq 0} \frac{r^n}{n! \gamma(n + 1)} = \sum_{n \leq re} \frac{r^n}{n! \gamma(n + 1)} + \sum_{n > re} \frac{r^n}{n! \gamma(n + 1)}.
\] The first summand on the RHS is bounded by \( (re + 1)e^{\tilde{\Lambda}(r)} \), while the second summand is bounded by \( \sum_{n > re} \frac{r^n}{n! \gamma(n + 1)} \leq \sum_{n \geq 0} \frac{1}{\gamma(n + 1)} < \infty \). We conclude that \( \log E_1(z) \leq \tilde{\Lambda}(r) \). By Lemma 9.2, \( \tilde{\Lambda}(r) \lesssim \Lambda_L(r) \), which completes the proof.
C.2 Proof of Lemma 7.2. Let $x > 0$ be sufficiently large. We define $s = \rho e^{i\theta}$ as the solution to the saddle-point equation

$$\log ix = \frac{\Gamma'(s)}{\Gamma(s)} + \log L(s) + \varepsilon(s).$$

By (C.2),

$$x \approx \rho L(\rho), \quad \frac{\pi}{2} = \theta(1 + \varepsilon(\rho)(1 + o(1))), \quad \rho \to \infty.$$

Thus, part 1 of Lemma 5.1 yields $\rho \approx \Lambda L(x)$. Since the function $\varepsilon$ is slowly varying, we also get

$$\theta = \frac{\pi}{2} - \varepsilon(\Lambda L(x))(1 + o(1)), \quad x \to \infty.$$

By Theorem B,

$$\log E_1(ix) \sim s(1 + \varepsilon(s)), \quad x \to \infty.$$ 

Therefore,

$$\log |E_1(ix)| \sim \rho \cos \theta \approx \Lambda L(x) \varepsilon(s), \quad x \to \infty.$$

C.3 Lemma 7.3. We will use the following auxiliary result.

Lemma C.1. Suppose that $L$ satisfy assumptions (R1), (R2), (R3) and (R9). If $\delta > 0$ is sufficiently small, then there exists $R_0 > 0$, such that

$$\log |E_1(z)| \leq C + \text{Re}_+(s(1 + \varepsilon(s))), \quad |\arg z| \leq \frac{\pi}{2} + \delta, \quad |z| > R_0,$$

$$\log |E_1(z)| \leq C, \quad |\arg z| > \frac{\pi}{2} + \delta, \quad |z| > R_0,$$

where $s$ and $z$ are related by the saddle-point equation $sL(s)e^{\varepsilon(s)} = z$ and $\text{Re}_+(w) = \max\{\text{Re}(w), 0\}$.

Proof. It is easy to check that if $s \to \gamma(s)$ satisfies assumptions (R3) and (R8), then so does $s \to \Gamma(s)\gamma(s)$, where $\Gamma$ is the Euler Gamma function. Thus, Theorem B is applicable to the function

$$E_1(z) = \sum_{n \geq 0} \frac{z^n}{n!\gamma(n + 1)}.$$

The saddle-point equation for the function $E_1$ is

$$\log z = \frac{\Gamma'(s)}{\Gamma(s)} + \log L(s) + \varepsilon(s).$$

From here on we assume that $s$ is related to $z$ by the above saddle-point equation. By Stirling’s formula,

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O(|s|^{-1}),$$

uniformly in $|\arg s| < \pi - \delta$ as $s \to \infty$. Thus

$$\log z = \log s + \log L(s) + \varepsilon(s) + O(s^{-1}), \quad |s| \to \infty. \quad (C.1)$$

Combining the later with (6.3), we find that

$$\log z = \log(\rho e^{i\theta}) + \log L(\rho) + \varepsilon(\rho) + i\theta \varepsilon(\rho)(1 + o(1)) + O(\rho^{-1}), \quad s = \rho e^{i\theta}, \quad s \to \infty,$$
uniformly for $|\theta| \leq \frac{\pi}{2} + \delta$. Comparing the real and imaginary parts separately, we obtain,

$$z \sim \rho L(\rho), \quad \arg(z) = \theta(1 + \varepsilon(\rho) + o(\varepsilon(\rho))), \quad \rho \to \infty.$$  \hfill (C.2)

The function $L$ is slowly varying, therefore, $\varepsilon(\rho) \to 0$ as $\rho \to \infty$. Thus, (C.2) yields,

$$\arg(z) = \arg(\rho L(\rho)) \pm \frac{\pi}{2} + \delta.$$  

In particular, in this case, Theorem B implies that

$$|E_1(z)| \leq C \delta.$$  

On the other hand, if $|\arg(z)| \leq \frac{\pi}{2} + \delta$, then the same theorem yields

$$\log |E_1(z)| \leq C \delta + \Re_+ \left( \frac{\Gamma'(s)}{\Gamma(s)} - \frac{\log \Gamma(s)}{s} + s \varepsilon(s) \right) \leq C \delta + \Re_+ (s(1 + \varepsilon(s))).$$  

Let $s_*$ be the solution to

$$\log z = \log s_* + \log L(s_*) + \varepsilon(s_*).$$  

By (C.1), for sufficiently large $|z|$, we have

$$\Re (s(1 + \varepsilon(s))) \leq \Re (s_*(1 + \varepsilon(s_*))) + C.$$  

This completes the proof. \hfill \square

**Proof of Lemma 7.3** Fix $0 < \delta$ (small) and $\rho_0 > 0$ (big) such that the function $\rho \mapsto \Im (\log L(i\rho) + \varepsilon(i\rho))$ is decreasing and smaller then $\delta$ in the ray $[\rho_0, \infty)$. Also Fix $t \geq 1$, and $0 < \psi \leq \frac{\pi}{2}$.

If $\psi \geq \delta$, then Lemma C.1 yield

$$|E_1(itre^{i\psi})| \lesssim 1,$$

and the Lemma holds. So, we will assume from here on that $0 < \psi < \delta$.

Put

$$w(s) = (1 + \varepsilon(s))s, \quad \tilde{w}(s) = \log s + \log L(s) + \varepsilon(s).$$

Since assumptions (R3) and (R8) hold, for $r > r_0$, there exists a unique solution to the equation

$$\log t + \log r + i \left( \frac{\pi}{2} + \psi \right) = \tilde{w}(s).$$  \hfill (C.3)

We denote this solution by $s(r)$. By Lemma C.1

$$|E_1(itre^{i\psi})| \leq C + \Re_+ (w(s(r)))$$

Our goal is to show that

$$\Re (w(s(r))) \leq C + H(\psi).$$  

We begin with showing that the function $r \mapsto \Re (w(s(r)))$ has a unique maximum in the interval $[r_0, \infty)$. The functions $w$ and $\tilde{w}$ are related by

$$s\tilde{w}'(s) = w'(s).$$
By (6.3), the real and imaginary parts of (C.3) yield

$$|s(r)| \to \infty, \quad \arg(s(r))(1 + \varepsilon(|s(r)|)(1 + o(1))) = \frac{\pi}{2} + \psi$$

as $r \to \infty$. By assumption, the function $x \mapsto \varepsilon(x)$ is eventually positive and decreasing to zero as $x \to \infty$. Thus, $\text{Re} w(s(r))$ is eventually negative. Differentiation of (C.3) with respect to $r$ yields

$$s'(r)\tilde{w}'(s(r)) = \frac{1}{r}.$$ 

Thus,

$$\frac{d}{dr}w(s(r)) = w'(s(r))s'(r) = \frac{1}{r s(r)}.$$ 

In particular, $\frac{d}{dr}\text{Re}(w(s(r))) = 0$ if and only if $\text{Re}(s(r)) = 0$, or, which is the same, if and only if $s(r) = i\tau$ for some $\tau > 0$. Comparing the imaginary parts of (C.3), such a $\tau$ satisfies

$$\text{Im} \left( \log L(i\tau) + \varepsilon(i\tau) \right) = \psi.$$ 

Since, $0 < \psi < \delta$ and $r$ is sufficiently large, such a $\tau$ exists and unique. Moreover,

$$\text{Re}(w(i\tau)) = \text{Re}(i\tau \varepsilon(i\tau)) > 0.$$ 

We conclude that

$$\max_{r \geq r_0} \text{Re}(w(s(t))) = \text{Re}(i\tau \varepsilon(i\tau)) = \log H(\psi).$$ 

Given $B > 0$, we assume that $r_0$ is large enough, and that $\frac{B}{r} < \psi < \delta$. Denote by $\tau_B$ the unique solution to

$$\text{Im} \left( \log L(i\tau_B) + \varepsilon(i\tau_B) \right) = \psi + \frac{B}{r}.$$ 

By (6.3),

$$\text{Im} \left( \log (i\rho) + \varepsilon(i\rho) \right) = \frac{\pi}{2}\varepsilon(\rho)(1 + o(1)), \quad \rho \to \infty,$$

while

$$\frac{d}{d\rho} \text{Re} \left( w(\rho) \right) = \text{Re}(\varepsilon(\rho))(1 + o(1)) = o(\varepsilon(\rho)), \quad \rho \to \infty.$$ 

Thus, by mean value theorem,

$$\text{Re} \left( w(i\tau) - w(i\tau_B) \right) \leq C_B \frac{\tau}{r}.$$ 

Comparing the real parts of (C.3), and making use of (6.3), yields

$$rt = \tau L(\tau)(1 + O(1)), \quad r \to \infty.$$ 

Thus, by part 1 of Lemma 5.1

$$\frac{\tau}{r} \leq C\Lambda_L(t).$$ 

We have established

$$\max_{r \geq r_0} \text{Re}(w(s(t))) \leq C_B (\Lambda_L(u) + 1) + \log \left( \psi + \frac{B}{r} \right),$$

which completes the proof.
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