Global Well-Posedness of Free Interface Problems for the Incompressible Inviscid Resistive MHD

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Abstract: We consider the plasma-vacuum interface problem in a horizontally periodic slab impressed by a uniform non-horizontal magnetic field. The lower plasma region is governed by the incompressible inviscid and resistive MHD, the upper vacuum region is governed by the pre-Maxwell equations, and the effect of surface tension is taken into account on the free interface. The global well-posedness of the problem, supplemented with physical boundary conditions, around the equilibrium is established, and the solution is shown to decay to the equilibrium almost exponentially. Our results reveal the strong stabilizing effect of the magnetic field as the global well-posedness of the free-boundary incompressible Euler equations, without the irrotational assumption, around the equilibrium is unknown. One of the key observations here is an induced damping structure for the fluid vorticity due to the resistivity and transversal magnetic field. A similar global well-posedness for the plasma-plasma interface problem is obtained, where the vacuum is replaced by another plasma.

1. Introduction

1.1. Formulation in Eulerian coordinates. We consider the plasma-vacuum interface problem in the slab \(\Omega = \mathbb{T}^2 \times (-1, 1)\) impressed by a uniform transversal magnetic field \(\vec{B}, i.e., B_3 \neq 0\), where the slab is assumed to be horizontally periodic for \(\mathbb{T} = \mathbb{R}/\mathbb{Z}\). Let \(\Sigma_{\pm} = \mathbb{T}^2 \times \{\pm 1\}\) be the upper and lower fixed boundaries, respectively. The plasma moves in the lower domain

\[
\Omega_-(t) = \left\{ y = (y_h, y_3) = (y_1, y_2, y_3) \in \mathbb{T}^2 \times \mathbb{R} \mid -1 < y_3 < \eta(t, y_h) \right\}, \quad (1.1)
\]

the vacuum occupies the upper domain

\[
\Omega_+(t) = \left\{ y \in \mathbb{T}^2 \times \mathbb{R} \mid \eta(t, y_h) < y_3 < 1 \right\}, \quad (1.2)
\]
and the interface $\Sigma(t):=\{y \in \mathbb{T}^2 \times \mathbb{R} \mid y_3 = \eta(t, y_h)\}$ is free to move, where the graph function $\eta: \mathbb{R}_+ \times \mathbb{T}^2 \rightarrow \mathbb{R}$ is unknown. We assume that the velocity $u$, the pressure $p$ and the magnetic field $B$ of the plasma satisfy the incompressible inviscid and resistive magnetohydrodynamic equations (MHD):

$$
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p &= \text{curl } B \times B \quad \text{in } \Omega_-(t) \\
\text{div } u &= 0 \quad \text{in } \Omega_-(t) \\
\partial_t B &= \text{curl } E, \quad E = u \times B - \kappa \text{curl } B \quad \text{in } \Omega_-(t) \\
\text{div } B &= 0 \quad \text{in } \Omega_-(t),
\end{align*}
$$

(1.3)

where $E$ is the electric field of the plasma and $\kappa > 0$ is the magnetic diffusion coefficient, the inverse of the electric conductivity. The magnetic field $\hat{B}$ and the electric field $\hat{E}$ in vacuum are assumed to satisfy the pre-Maxwell equations:

$$
\begin{align*}
\text{curl } \hat{B} &= 0, \quad \text{div } \hat{B} = 0 \quad \text{in } \Omega_+ \\
\partial_t \hat{B} &= \text{curl } \hat{E}, \quad \text{div } \hat{E} = 0 \quad \text{in } \Omega_+ 
\end{align*}
$$

(1.4)

The interface $\Sigma(t)$ is advected with the plasma through the kinematic boundary condition:

$$
\partial_t \eta = u \cdot \mathcal{N} \quad \text{on } \Sigma(t),
$$

(1.5)

where $\mathcal{N} = (-\nabla_h \eta, 1)$ is the upward non-unit normal vector to $\Sigma(t)$ with $\nabla_h = (\partial_1, \partial_2)$ the horizontal gradient. Note that the equations (1.3) and (1.4) are derived by neglecting the displacement current in the full-Maxwell equations, and one may refer to the books [7,14,19,22,39] for the physical backgrounds and applications.

To solve (1.3) and (1.4), one needs to impose certain physical boundary conditions. First, due to $\text{curl } B \times B = -\text{div}(\frac{1}{2} |B|^2 I - B \otimes B)$, the dynamic boundary condition of the balance of the normal stresses on the free interface reads as

$$
\left(pI + \frac{1}{2} |B|^2 I - B \otimes B\right)\mathcal{N} = \left(\frac{1}{2} |\hat{B}|^2 I - \hat{B} \otimes \hat{B}\right)\mathcal{N} - \sigma H \mathcal{N} \quad \text{on } \Sigma(t),
$$

(1.6)

where $I$ is the $3 \times 3$ identity matrix, $\sigma > 0$ is the surface tension coefficient and $H$ is the mean curvature of $\Sigma(t)$ given by

$$
H = \text{div}_h \left(\frac{\nabla_h \eta}{\sqrt{1 + |\nabla_h \eta|^2}}\right).
$$

(1.7)

Here $\text{div}_h$ is the horizontal divergence, and also the notation $\Delta_h = \text{div}_h \nabla_h$ will be used later. Next, the classical jump conditions for the magnetic and electric fields, which follow from the Maxwell equations (see [19,22]), are

$$
B \cdot \mathcal{N} = \hat{B} \cdot \mathcal{N}, \quad (E - \hat{E}) \times \mathcal{N} = u \cdot \mathcal{N}(B - \hat{B}) \quad \text{on } \Sigma(t).
$$

(1.8)

Due to the consideration in this paper that the problem is around the uniform traversal magnetic field $\bar{B}$, $B \cdot \mathcal{N} = \hat{B} \cdot \mathcal{N} \neq 0$ on $\Sigma(t)$, and hence the tangential components
of the jump condition (1.6) imply in particular that \( B \times \mathcal{N} = \mathbf{\hat{B}} \times \mathcal{N} \) on \( \Sigma(t) \), and one then finds that (1.6) and (1.8) are equivalent to the following boundary conditions:

\[
p = -\sigma H, \quad B = \mathbf{\hat{B}}, \quad E \times \mathcal{N} = \mathbf{\hat{E}} \times \mathcal{N} \quad \text{on} \quad \Sigma(t).
\]

(1.9)

Finally, we impose the impermeable condition on the lower fixed boundary:

\[
u \cdot e_3 = 0 \quad \text{on} \quad \Sigma_-, \quad (1.10)
\]

with \( e_3 = (0, 0, 1) \). It should be emphasized that the boundary conditions of the magnetic and electric fields on a fixed boundary depend on the nature of the boundary, see [7, 14, 19, 22, 39]; we assume that the lower fixed boundary is a perfect conducting wall, so

\[
B \cdot e_3 = \bar{\mathbf{B}} \cdot e_3, \quad E \times e_3 = 0 \quad \text{on} \quad \Sigma_-, \quad (1.11)
\]

while the upper fixed boundary is a perfect insulating wall, then

\[
\mathbf{\hat{B}} \times e_3 = \bar{\mathbf{B}} \times e_3, \quad \mathbf{\hat{E}} \cdot e_3 = 0 \quad \text{on} \quad \Sigma_+. \quad (1.12)
\]

One may refer to [48] for more physical and mathematical discussions on the boundary conditions of the magnetic and electric fields and related literature.

Mathematically, the electric field in vacuum \( \mathbf{\hat{E}} \) could be regarded as a second variable, see Ladyzhenskaya and Solonnikov [32, 33]. Indeed, one can eliminate \( \mathbf{\hat{E}} \) from the problem under consideration, i.e., (1.3)–(1.5) and (1.9)–(1.12), and arrive at the following system for \( (u, p, \eta, b, \mathbf{\hat{b}}) \), with \( b = B - \bar{B} \) and \( \mathbf{\hat{b}} = \mathbf{\hat{B}} - \bar{B} \),

\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p &= \text{curl} \ b \times (\bar{\mathbf{B}} + b) \quad \text{in} \quad \Omega_-(t) \\
\text{div} \ u &= 0 \quad \text{in} \quad \Omega_-(t) \\
\partial_t b &= \text{curl} \ E, \quad E \times e_3 = 0 \quad \text{in} \quad \Omega_-(t) \\
\text{div} \ b &= 0 \quad \text{in} \quad \Omega_-(t) \\
\text{curl} \ \mathbf{\hat{b}} &= 0, \quad \text{div} \ \mathbf{\hat{b}} = 0 \quad \text{in} \quad \Omega_+(t) \quad (1.13) \\
\partial_t \eta &= u \cdot \mathcal{N} \quad \text{on} \quad \Sigma(t) \\
p &= -\sigma H, \quad b = \mathbf{\hat{b}} \quad \text{on} \quad \Sigma(t) \\
u_3 = 0, \quad b_3 = 0, \quad E \times e_3 = 0 \quad \text{on} \quad \Sigma_- \\
\mathbf{\hat{b}} \times e_3 &= 0 \quad \text{on} \quad \Sigma_+.
\end{align*}
\]

Once (1.13) is solved, then \( \mathbf{\hat{E}} \) can be recovered from the following problem:

\[
\begin{align*}
\text{curl} \ \mathbf{\hat{E}} &= \partial_t \mathbf{\hat{b}}, \quad \text{div} \ \mathbf{\hat{E}} = 0 \quad \text{in} \quad \Omega_+(t) \\
\mathbf{\hat{E}} \times \mathcal{N} &= E \times \mathcal{N} \quad \text{on} \quad \Sigma(t) \\
\mathbf{\hat{E}}_3 &= 0 \quad \text{on} \quad \Sigma_+. \quad (1.14)
\end{align*}
\]

Note also that the magnetic field in vacuum \( \mathbf{\hat{b}} \) is completely determined by \( b \cdot \mathcal{N} \) on \( \Sigma(t) \) via the following problem:

\[
\begin{align*}
\text{curl} \ \mathbf{\hat{b}} &= 0, \quad \text{div} \ \mathbf{\hat{b}} = 0 \quad \text{in} \quad \Omega_+(t) \\
\mathbf{\hat{b}} \cdot \mathcal{N} &= b \cdot \mathcal{N} \quad \text{on} \quad \Sigma(t) \\
\mathbf{\hat{b}} \times e_3 &= 0 \quad \text{on} \quad \Sigma_+. \quad (1.15)
\end{align*}
\]
Then the jump condition \( b = \hat{b} \) on \( \Sigma(t) \) could be regarded as a nonlocal boundary condition for \( b \) (see [32, 33]):

\[
b \times \mathcal{N} = B^t (b \cdot \mathcal{N}) \times \mathcal{N} \quad \text{on} \quad \Sigma(t),
\]

where \( B^t (b \cdot \mathcal{N}) \) is the solution to (1.20). Thus one could further formally suppress \( \hat{b} \) in (1.13).

To complete the statement of the problem (1.13), one must specify the initial conditions. Suppose that the initial interface \( \Sigma_0(0) \) is given by the graph of the function \( \eta(0) = \eta_0 : \mathbb{T}^2 \to \mathbb{R} \), which yields the initial lower domain \( \Omega_-(0) \) on which the initial velocity \( u(0) = u_0 : \Omega_-(0) \to \mathbb{R}^3 \) and the initial magnetic field \( b(0) = b_0 : \Omega_-(0) \to \mathbb{R}^3 \) are specified.

1.2. Physical energy-dissipation law. The problem (1.13) possesses a natural physical energy-dissipation law. First, as for the free-surface incompressible Euler equations, one has

\[
\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega_-(t)} |u|^2 \, dy + \int_{\mathbb{T}^2} 2\sigma \left( \sqrt{1 + |\nabla_h \eta|^2} - 1 \right) \, dy_h \right) = \int_{\Omega_-(t)} \nabla \times (\widehat{\mathbb{B}} + b) \cdot u \, dy = - \int_{\Omega_-(t)} u \times (\widehat{\mathbb{B}} + b) \cdot \nabla \times b \, dy.
\]

Next, to handle the magnetic system in (1.13), making use of the electric field in vacuum \( \hat{E} \) satisfying (1.14), one obtains

\[
\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega_-(t)} |b|^2 \, dy + \int_{\Omega_+(t)} |\hat{b}|^2 \, dy \right) \quad \text{in} \quad \Omega_-(t), \quad \text{div} \, b = 0 \quad \text{in} \quad \Omega_-(t)\]

\[
= \int_{\Omega_-(t)} \nabla \times b \cdot \hat{E} \, dy + \int_{\Omega_+(t)} \nabla \times \hat{b} \cdot \hat{b} \, dy = \int_{\Omega_+(t)} \nabla \times \hat{b} \cdot \hat{b} \, dy = \int_{\Omega_-(t)} \nabla \times (\widehat{\mathbb{B}} + b) \cdot \nabla \times b \, dy.
\]

It then follows from (1.17) and (1.18) that

\[
\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega_-(t)} (|u|^2 + |b|^2) \, dy + \int_{\Omega_+(t)} |\hat{b}|^2 \, dy + \int_{\mathbb{T}^2} 2\sigma \left( \sqrt{1 + |\nabla_h \eta|^2} - 1 \right) \, dy_h \right)
\]

\[
+ \int_{\Omega_-(t)} (\nabla \times b)^2 \, dy = 0.
\]

This structure of the energy evolution equation is the basis of the energy method we will use to analyze the problem (1.13).

Note that (1.19) can be derived in an alternative way, motivated by Ladyzhenskaya and Solonnikov [32, 33], that does not involve the electric field in vacuum \( \hat{E} \). The idea is to introduce instead a virtual magnetic field \( b \) in \( \Omega_-(t) \) as the solution to

\[
\begin{cases}
\nabla \times b = 0, & \text{in} \quad \Omega_-(t) \\
\text{div} \, b = 0 & \text{in} \quad \Omega_-(t) \\
b \times \mathcal{N} = \hat{b} \times \mathcal{N} & \text{on} \quad \Sigma(t) \\
b_3 = 0 & \text{on} \quad \Sigma_-. 
\end{cases}
\]
Then one may write \( \hat{b} = \nabla \hat{\phi} \) and \( b = \nabla \phi \) with \( \hat{\phi} \) solving
\[
\Delta \hat{\phi} = 0 \text{ in } \Omega_+(t), \quad \nabla \hat{\phi} \cdot \mathcal{N} = b \cdot \mathcal{N} \text{ on } \Sigma(t), \quad \hat{\phi} = 0 \text{ on } \Sigma_+ \tag{1.21}
\]
and \( \phi \) satisfying
\[
\Delta \phi = 0 \text{ in } \Omega_-(t), \quad \phi = \hat{\phi} \text{ on } \Sigma(t), \quad \partial_3 \phi = 0 \text{ on } \Sigma_- \tag{1.22}
\]
Note that \( b \times \mathcal{N} = \mathcal{N} \times b \) on \( \Sigma_+(t) \). Then one has
\[
\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega_-(t)} |b|^2 \, dy + \int_{\Omega_+(t)} |\hat{b}|^2 \, dy \right) = \int_{\Omega_-(t)} \partial_t b \cdot (b - \hat{b}) \, dy + \int_{\Omega_-(t)} \partial_t \mathcal{N}_b \cdot \partial_3 \hat{b} \, dy
\]
\[
= \int_{\Omega_-(t)} \text{curl } E \cdot (b - \hat{b}) \, dV + \int_{\Omega_+(t)} \partial_3 \hat{b} \cdot \nabla \phi \, dy + \int_{\Omega_+(t)} \partial_3 \hat{b} \cdot \nabla \phi \, dy
\]
\[
= \int_{\Omega_-(t)} E \cdot \text{curl } b \, dy. \tag{1.23}
\]
This yields again (1.18) and hence (1.19).

1.3. Related works. Free boundary problems in fluid mechanics have attracted huge attention in the mathematical community. Unlike the Euler equations (see Nalimov [37], Wu [55, 56]) and the Navier–Stokes equations (see Solonnikov [43], Beale [2]), the free boundary problems for MHD have been studied only more recently. The free boundary problems for MHD arise typically when a plasma is surrounded by the vacuum and when two plasmas are separated by a free interface, which are known as the plasma-vacuum interface problem and the plasma-plasma interface problem, respectively. In this paper we focus on the incompressible MHD.

For the ideal (inviscid and non-resistive) MHD, the magnetic field is required to be tangential on the free interface, which is transformed to be the constraint on the initial magnetic field, and in this case the dynamic boundary condition on the free interface is then reduced to the balance of the total pressure of the hydrodynamic part and the magnetic part. For the incompressible plasma-vacuum interface problem, under the non-colinearity condition of the magnetic fields on the free interface, which yields a regularizing effect for the free interface, Morando et al. [36] showed the well-posedness of the linearized problem and Sun et al. [50] proved the local well-posedness of the nonlinear problem; under the Taylor condition of the total pressure on the free interface, when the magnetic field in the vacuum is trivial Hao and Luo [27] established an a priori estimate and Gu and the first author [23] proved the local well-posedness, while when the magnetic field in the vacuum is nontrivial the local well-posedness is still unknown. For the incompressible plasma-plasma interface problem, Sun et al. [49] proved the local well-posedness under the Syrovatskij stability condition, and previously, Coulombel et al. [11] showed an a priori estimate under a stronger condition. For the incompressible Euler equations, it is known that either the Taylor condition of the pressure or the effect of surface tension on the free surface is required for the local well-posedness of the one-phase problem (see Christodoulou and Lindblad [10], Lindblad [34], Coutand and Shkoller [13], Shatah and Zeng [40] and Zhang and Zhang [59]), and the effect of surface
tension is necessary for the local well-posedness of the two-phase problem (see Cheng et
al. [8] and Shatah and Zeng [41,42]); otherwise, one has the ill-posedness of the problem
(see Ebin [16,17] and Caflisch and Orellana [4]). Thus the works [11,36,49,50] show
the stabilizing effect of the magnetic field on the local well-posedness for inviscid fluids.

It is natural to consider the question whether there is a global well-posedness for free
boundary problems or not. The recent works, Castro et al. [5,6], Fefferman et al. [18]
and Coutand [12], imply the development of singularities in finite time of free boundary
problems for some large initial data. For the irrotational incompressible Euler equations
in the horizontally nonperiodic setting, certain dispersive effects can be used to establish
the global well-posedness for the small initial data; we refer to Wu [57,58], Germain et
al. [20,21], Ionescu and Pusateri [29,30], Alazard and Delort [1] and Deng et al. [15].
We refer to Beale [3], Solonnikov [44], Hataya [28], Guo and Tice [24–26], Wang [51]
and Wang et al. [53] for the global well-posedness of the incompressible Navier–Stokes
equations. Despite these, it is still not clear whether the free-boundary incompressible
Euler equations for the general small initial data admits a global unique solution or not.
It is then interesting and important to study the effect of the magnetic field on the global
well-posedness for inviscid fluids.

Note that the global well-posedness of free boundary problems for the ideal MHD
is unknown, and it is reasonable to expect the global well-posedness of the viscous and
resistive MHD. We may refer to Padula and Solonnikov [38] and Solonnikov [45,46] for
the local well-posedness of the incompressible plasma-vacuum interface problem and
Solonnikov and Frolova [47] and Solonnikov [46] for the global well-posedness around
the zero magnetic field. In [52], the first author proved the global well-posedness of
the incompressible viscous and non-resistive plasma-plasma interface problem around a
traversal uniform magnetic field. These results [46,47,52] of the global well-posedness
rely heavily on the dissipation and regularizing effects of the viscosity for the velocity
field. In this paper, we will prove the global well-posedness of free interface problems
for the incompressible inviscid and resistive MHD around a traversal uniform magnetic
field. It seems more subtle and difficult to prove the global well-posedness for the inviscid
and resistive MHD since the flow is transported by the velocity. Indeed, even the local
well-posedness theory is much involved and technically difficult (see Sect. 8) and the
global existence of classical solutions to the Cauchy problem in 2D is unknown. Our
analysis here depends on the finite depth of the fluid in our setting, which allows the use
of the Poincaré-type inequality.

There are a huge amount of mathematical works for free boundary problems in fluid
mechanics, and it is impossible to provide a thorough survey of the literature here. We
may refer to the references cited in these works above for more proper survey of the
literature.

2. Main Results

2.1. Reformulation in flattening coordinates. As usual for free boundary problems in
fluid mechanics, we use a coordinate transformation in which the interface stays fixed
in time. Set
\[ \Omega_-:=\mathbb{T}^2 \times (-1,0) \] and 
\[ \Omega_+:=\mathbb{T}^2 \times (0,1), \] (2.1)
and denote by \( \Sigma:=\mathbb{T}^2 \times \{0\} \) for the interface. The domains can be flattened by the
mapping
\[ \Omega_\pm \ni x \mapsto (x_h, \varphi(t,x)) := x_3 + \tilde{\eta}(t,x))=:\Phi(t,x) = y \in \Omega_\pm(t), \] (2.2)
where $\tilde{\eta} = \chi P\eta$ for $\chi = \chi(x_3)$ a smooth function in $\mathbb{R}$ that satisfies $\chi(0) = 1$ and $\chi(\pm 1) = 0$ and $P\eta$ the specialized harmonic extension of $\eta$ onto $\mathbb{R}^3$ with $P$ defined by (A.1).

If $\eta$ is sufficiently small and regular, then the mapping $\Phi$ is a diffeomorphism. This allows one to transform the problem in $\Omega_\pm(t)$ to one in $\Omega_\pm$ for each $t \geq 0$. Set
\[
\bar{\partial}_i^\varphi = \partial_i - \partial_3 \eta \bar{\partial}_3^\varphi, \quad \bar{\partial}_3^\varphi = \frac{1}{\partial_3 \varphi} \partial_3.
\]
(2.3)

For the jump conditions on $\Sigma$, define the interfacial jump as
\[
[b] := \tilde{b} |_{\Sigma} - b |_{\Sigma}.
\]
(2.4)

etc. Then the problem (1.13) is equivalent to the following problem in new coordinates:
\[
\begin{align*}
\bar{\partial}_i^\varphi u + u \cdot \nabla^\varphi u + \nabla^\varphi p &= \text{curl}^\varphi b \times (\tilde{B} + b) & \text{in } \Omega_- \\
\text{div}^\varphi u &= 0 & \text{in } \Omega_- \\
\bar{\partial}_i^\varphi b &= \text{curl}^\varphi E, \quad E = u \times (\tilde{B} + b) - \kappa \text{curl}^\varphi b & \text{in } \Omega_- \\
\text{div}^\varphi b &= 0 & \text{in } \Omega_- \\
\text{curl}^\varphi \tilde{b} &= 0, \quad \text{curl}^\varphi \tilde{b} = 0 & \text{in } \Omega_+ \\
\partial_3 \eta &= u \cdot \mathcal{N} & \text{on } \Sigma \\
p &= -\sigma H, \quad [b] = 0 & \text{on } \Sigma \\
u_3 = 0, \quad b_3 = 0, \quad E \times e_3 = 0 & \text{on } \Sigma_- \\
\tilde{b} \times e_3 = 0 & \text{on } \Sigma_+ \\
(u, b, \eta) |_{t=0} &= (u_0, b_0, \eta_0).
\end{align*}
\]
(2.5)

Here $(\nabla^\varphi)_i = \bar{\partial}_i^\varphi$, $i = 1, 2, 3$, $\text{div}^\varphi = \nabla^\varphi \cdot$ and $\text{curl}^\varphi = \nabla^\varphi \times$. Also the notation $\Delta^\varphi = \text{div}^\varphi \nabla^\varphi$ will be used later.

The energy-dissipation law (1.19) in the new coordinates reads as
\[
\frac{1}{2} \frac{d \left( \int_{\Omega_-} \left( |u|^2 + |b|^2 \right) d\mathcal{V}_t + \int_{\Omega_+} |\tilde{b}|^2 d\mathcal{V}_t + \int_{\mathbb{T}^2} 2\sigma \left( \sqrt{1 + |\nabla_h \eta|^2} - 1 \right) \right) }{dt} + \kappa \int_{\Omega_-} |\text{curl}^\varphi b|^2 d\mathcal{V}_t = 0.
\]
(2.6)

Here $d\mathcal{V}_t := \partial_3 \varphi \ dx$ is the volume element induced by the change of variables (2.2).

2.2. Statement of the results. One of the aims of this paper is to show the global well-posedness of the problem (2.5) around the trivial equilibrium state when $\tilde{B}_3 \neq 0$.

Before stating the main results, we first mention the issue of compatibility conditions for the initial data $(u_0, b_0, \eta_0)$ since the problem (2.5) is considered in a domain with boundary. We will work in a high-regularity context, essentially with regularity up to $2N$ temporal derivatives. This requires one to use $(u_0, b_0, \eta_0)$ to construct the initial data $\partial_j^i \eta(0)$ for $j = 1, \ldots, 2N + 1$, $\partial_j^i u(0)$ and $\partial_j^i b(0)$ for $j = 1, \ldots, 2N$, $\partial_j^i p(0)$ for $j = 0, \ldots, 2N - 1$ and $\partial_j^i \tilde{b}(0)$ for $j = 0, \ldots, 2N$. These data need to satisfy various conditions, which in turn require $(u_0, b_0, \eta_0)$ to satisfy the necessary compatibility conditions that are natural for the local well-posedness of (2.5) in the functional framework.
below. The construction of these data is technically quite involved and will be given in
details in Sect. 8.1, and these compatibility conditions will be described explicitly as the
2N-th order compatibility conditions (8.6). We will also show in Sect. 8.1 that the set
of the initial data \((u_0, b_0, \eta_0)\) satisfying the compatibility conditions (8.6) is not empty.
For the global well-posedness of (2.5), it is assumed further that

\[
\int_{\mathbb{T}^2} \eta_0 = 0. \tag{2.7}
\]

For sufficiently regular solutions, the condition (2.7) persists in time, \textit{i.e.},

\[
\int_{\mathbb{T}^2} \eta = 0. \tag{2.8}
\]

Indeed, one has

\[
\frac{d}{dt} \int_{\mathbb{T}^2} \eta = \int_{\mathbb{T}^2} \partial_t \eta = \int_{\mathbb{T}^2} u \cdot \mathcal{N} = \int_{\Omega} \text{div} u \, u \, \text{d} \Omega = 0. \tag{2.9}
\]

Let \(H^k(\Omega_\pm), k \geq 0\) and \(H^s(\mathbb{T}^2), s \in \mathbb{R}\) be the usual Sobolev spaces with norms
denoted by \(\|\cdot\|_m\) and \(|\cdot|_s\), respectively. For an integer \(N \geq 4\), we define the high-order
energy as

\[
\mathcal{E}_{2N} := \sum_{j=0}^{2N} \left\| \partial_t^j u \right\|^2_{2N-j} + \sum_{j=0}^{2N-1} \left\| \partial_t^j b \right\|^2_{2N-j+1} + \left\| \partial_t^{2N} p \right\|^2_0 + \sum_{j=0}^{2N-1} \left\| \partial_t^{2N+1} \eta \right\|^2_{j+1/2}.
\]

\[
\mathcal{D}_n := \sum_{j=0}^{n-1} \left\| \partial_t^j u \right\|^2_{n-j-1} + \sum_{j=0}^{n-2} \left\| \partial_t^j b \right\|^2_{n-j} + \sum_{j=0}^{n} \left\| \partial_t^j b \right\|^2_{1,n-j} + \sum_{j=0}^{n} \left\| \partial_t^j \hat{b} \right\|^2_{n-j+1} + \sum_{j=0}^{n-2} \left\| \partial_t^j p \right\|^2_{n-j-1} + \sum_{j=0}^{n-2} \left\| \partial_t^j \eta \right\|^2_{n-j+1/2} + \left\| \partial_t^{n-1} \eta \right\|^2_1 + \left\| \partial_t^n \eta \right\|^2_0. \tag{2.11}
\]
Here the anisotropic Sobolev norm $\| \cdot \|_{m, \ell}$ is defined as

$$
\| f \|_{m, \ell} := \sum_{\alpha \in \mathbb{N}^2, |\alpha| \leq \ell} \| \partial^\alpha f \|_m ,
$$

(2.12)

where $\partial^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2}$ for $\alpha \in \mathbb{N}^2$; we refer to Sect. 2.4 for more conventions for notations. Note that the dissipation $\mathcal{D}_N$ can not control the energy $\mathcal{E}_N$. Furthermore, it is the following energy which is involved in the derivation of the dissipation estimates of $\mathcal{D}_n$:

$$
\mathcal{E}_n := \| u \|_{n-1}^2 + \| u \|_{0, n}^2 + \frac{\pi}{\nu} \left( \sum_{j=1}^{n-1} \| \partial_j u \|_{n-j}^2 + \| b \|_n^2 + \sum_{j=1}^{n-1} \| \partial_j b \|_{n-j+1}^2 + \| \partial_1 b \|_0^2 \right)
$$

(2.13)

$$
+ \| \hat{b} \|_n^2 + \sum_{j=1}^{n-1} \| \partial_j \hat{b} \|_{n-j+1}^2 + \| \partial_1 \hat{b} \|_0^2 + \sum_{j=0}^{n-1} \| \partial_j \eta \|_{n-j+3/2}^2 + \| \partial_1 \eta \|_{1}^2 + \| \partial_{n+1} \eta \|_{-1/2}^2 .
$$

Now the main results of this paper are stated as follows.

**Theorem 2.2.** Assume that $\kappa > 0$, $\vec{B}_3 \neq 0$ and $\sigma > 0$ and let $N \geq 8$ be an integer. Assume that $u_0 \in H^{2N}(\Omega_-)$, $b_0 \in H^{2N+1}(\Omega_-)$ and $\eta_0 \in H^{2N+3/2}(\Sigma)$ are given such that $\mathcal{E}(0) < +\infty$ and that the $2N$-th order compatibility conditions (8.6) as well as the zero average condition (2.7) are satisfied. There exists a universal constant $\varepsilon_0 > 0$ such that if $\mathcal{E}_N(0) \leq \varepsilon_0$, then there exists a global unique solution $(u, p, \eta, b, \hat{b})$ to (2.5). Moreover, for all $t \geq 0$,

$$
\mathcal{E}_N(t) + \int_0^t \mathcal{D}_N(s) \, ds \lesssim \mathcal{E}_N(0) \quad (2.14)
$$

and

$$
\sum_{j=0}^{N-6} (1 + t)^{N-5-j} \mathcal{E}_{N+4+j}(t) + \sum_{j=0}^{N-6} \int_0^t (1 + s)^{N-5-j} \mathcal{D}_{N+4+j}(s) \, ds \lesssim \mathcal{E}_N(0) .
$$

(2.15)

**Remark 2.3.** Theorem 2.2 implies in particular that $\sqrt{\mathcal{E}_{N+4}(t)} \lesssim (1 + t)^{(N-5)/2}$, which is integrable in time for $N \geq 8$. Since $N$ may be taken to be arbitrarily large, this decay result can be regarded as an “almost exponential” decay rate. Since $\eta$ is such that the mapping $\Phi(t, \cdot)$, defined in (2.2), is a diffeomorphism for each $t \geq 0$, one may change coordinates to $y \in \Omega_\pm(t)$ to produce a global-in-time, decaying solution to (1.13).

**Remark 2.4.** In contrast with the works [11,36,49,50] which show the stabilizing effect of the tangential magnetic field on the local well-posedness of the ideal MHD, the global well-posedness in Theorem 2.2 relies crucially on that the magnetic field is traversal (see also [52]). Indeed, the analysis here cannot be applied to the case when $\vec{B}$ is horizontal; for example, if $\vec{B} = e_1 := (1, 0, 0)$, $B = \vec{B}$, $\vec{B} = \vec{B}$ and $u_1 = 0$, then the problem under consideration reduces to the free-boundary incompressible Euler equations in 2D for which the global well-posedness is unknown. To avoid the technical difficulties
and to present the key ideas, here we show only the strong stabilizing effect of the 
traversal magnetic field for strip domains. Yet we believe that the ideas developed here 
will shed light on the physically important case of more general domains, such as a 
toroidal geometry of a tokamak, which is left for future study.

Remark 2.5. It turns out that to solve (2.5) with the desired regularity of $b$ (and $\hat{b}$) in 
(2.10), even local in time, one needs $\eta \in H^{2N+1/2}$ due to the magnetic diffusion term 
$\nabla b \cdot \nabla b$. For the case without surface tension, i.e., $\sigma = 0$, it seems that only $H^{2N}$ 
regularity for $\eta$ is available. Hence $\sigma > 0$ is necessary here even for the local well-
posedness. This is different from the viscous and resistive problem [46], where the 
viscosity has a regularizing effect of $1/2$ order for $\eta$ and so $\sigma > 0$ is not necessary. 
Also, it can be checked readily from the proof of Theorem 2.2 that our result holds also 
when the gravitational force $-ge_3$ with gravity constant $g > 0$ is included, which leads 
to an additional positive energy $\frac{1}{2}g |\eta|^2_0$ in the basic energy-dissipation law (2.6); indeed, 
the result holds even for $g < 0$ provided that $\sigma$ is sufficiently large, which follows easily 
by the Poincaré inequality (see [53]).

Remark 2.6. The local well-posedness of (2.5), which is of independent interest, will 
be proved in Sect. 8. It should be noted that the ideas in the works [23,36,50] for the 
local well-posedness of the ideal MHD do not work in our case. Indeed, even though the 
magnetic diffusion here has a regularized effect for the magnetic field, one of the main 
difficulties in constructing solutions to (2.5) lies in solving the magnetic system due to 
the nonlocal boundary condition for the magnetic field. For the viscous and resistive 
MHD, Padula and Solonnikov [38] solved the magnetic system in the framework of 
full parabolic regularity theory, which unfortunately can not be applied to the inviscid 
problem here due to the less regularity of the velocity. Our way is to solve the magnetic 
system in the framework of energy method, which is naturally consistent with the Euler 
equations, and the solution is constructed as the limit of approximate solutions to an 
appropriate regularization as described in the next subsection.

Remark 2.7. The main ideas and strategies for the the plasma-vacuum interface problem 
can be modified to study the plasma-plasma interface problem to obtain its global well-
posedness. This will be given in Sect. 10. To our best knowledge, the results in this 
paper are the first ones on the global well-posedness of free boundary problems for the 
incompressible inviscid rotational fluids around the equilibrium. This is due to the strong 
coupling between the fluid and the diffusive magnetic field.

2.3. Strategy of the proof. Theorem 2.2 will be proved in Sect. 9 by combining the 
local well-posedness of (2.5), Theorem 8.8, and the global-in-time a priori estimates, 
Theorem 7.3, with a standard continuity argument.

Note that for free boundary problems in fluid mechanics, even with the necessary 
a priori estimates of the solutions ready, it is often still highly nontrivial to construct 
such solutions, especially for inviscid fluids. So we consider first the construction of 
local solutions to (2.5). As the Lorentz force is of lower order regularity compared to 
the magnetic diffusion term, one may decompose (2.5) into the hydrodynamic part in 
$\Omega_-$ and the magnetic part in $\Omega$ and then construct the solutions by an iteration. For the 
force $F = \nabla b \times (\bar{B} + b)$ with $\bar{\eta}$ and $b$ given, the hydrodynamic part (cf. (8.7)) is the 
free-surface incompressible Euler equations with surface tension, which can be solved 
in spirit of Coutand and Shkoller [13]. It then remains to handle the magnetic part for 
$G = u \times (\bar{B} + \bar{b})$ with $u$, $\bar{b}$ and $\eta$ given (cf. (8.22)). The magnetic part (8.22) was solved
in Padula and Solonnikov [38] in a different setting by treating (8.22) with \( \eta \) small as a perturbation of the “flat interface” problem, the one obtained by setting \( \eta = 0 \) in (8.22). The flat interface problem can be solved by employing the Galerkin method, see Ladyzhenskaya and Solonnikov [32,33]. The solution to (8.22) is then produced from solutions to the flat interface problem and an iteration argument in [38] by employing the full parabolic regularity, which works in the anisotropic space-time Sobolev spaces (cf. (8.39)). Such spaces were used extensively in studying the nonhomogeneous boundary value problem for parabolic systems, see Lions and Magenes [35]. A subtle point of using such spaces in [38] is that it allows for the control of the resulting forcing terms when one adjusts the inhomogeneous terms, e.g. \( \text{curl } \hat{b} \neq 0 \) in \( \Omega_1 \) here, see also [2,44] for the study of the free-surface incompressible Navier–Stokes equations. However, such full parabolic regularity of solving (8.22) is not consistent in the iteration scheme of constructing solutions to (2.5) as the hyperbolic Euler equations could not provide such higher regularity of \( u \) and \( \eta \). To get around this, we will solve (8.22) in the functional framework of using the energy structure of the problem (cf. (2.6)). The main strategy is to first construct approximate solutions to an appropriately regularized problem by following the arguments of [38] and then derive the uniform estimates independent of the smoothing parameter of the solutions in the framework of energy method to pass to the limit to produce a solution to (8.22). More precisely, we will consider first the following regularized problem:

\[
\begin{align*}
\partial_t \phi^\varepsilon b^\varepsilon + \kappa \text{curl}^\varepsilon \text{curl}^\varepsilon b^\varepsilon &= \text{curl}^\varepsilon (G^\varepsilon - \Psi^\varepsilon) \quad &\text{in } \Omega_- \\
\text{div}^\varepsilon b^\varepsilon &= 0 \quad &\text{in } \Omega_- \\
\text{curl}^\varepsilon \hat{b}^\varepsilon &= 0, \quad \text{div}^\varepsilon \hat{b}^\varepsilon = 0 \quad &\text{in } \Omega_+ \\
\{ b^\varepsilon \} &= 0 \quad &\text{on } \Sigma \\
b_3^\varepsilon &= 0, \quad \kappa \text{curl}^\varepsilon b^\varepsilon \times e_3 = G^\varepsilon \times e_3 \quad &\text{on } \Sigma_- \\
\hat{b}^\varepsilon \times e_3 &= 0 \quad &\text{on } \Sigma_+ \\
b^\varepsilon \bigr|_{t=0} &= b_0^\varepsilon.
\end{align*}
\]

(2.16)

Here \( \phi^\varepsilon = \phi(\eta^\varepsilon) \) as in (2.2), \( \eta^\varepsilon \) and \( G^\varepsilon \) are the smooth approximations of \( \eta \) and \( G \), respectively, where \( \varepsilon > 0 \) is the smoothing parameter. It should be pointed out that the introduction of both the so-called corrector \( \Psi^\varepsilon \) in (2.16) and a sequence of correctors \( \phi^\varepsilon_j \) in (8.28) is crucial here, which allows one to construct the smooth approximation \( b_0^\varepsilon \) of \( b_0 \) satisfying the corresponding compatibility conditions for (2.16). We then follow the arguments of [38] to solve (2.16), in a higher order regularity context. To derive the uniform estimates independent of \( \varepsilon > 0 \) of the solutions to (2.16), with the desired regularity in our functional framework, we will make an important use of the corresponding regularized electric field in vacuum, \( \hat{E}^\varepsilon \), which solves

\[
\begin{align*}
\text{curl}^\varepsilon \hat{E}^\varepsilon &= \partial_t^\varepsilon \hat{b}^\varepsilon, \quad \text{div}^\varepsilon \hat{E}^\varepsilon = 0 \quad &\text{in } \Omega_+ \\
\hat{E}^\varepsilon \times N^\varepsilon &= (-\kappa \text{curl}^\varepsilon b^\varepsilon + G^\varepsilon - \Psi^\varepsilon) \times N^\varepsilon \quad &\text{on } \Sigma \\
\hat{E}_3^\varepsilon &= 0 \quad &\text{on } \Sigma_+.
\end{align*}
\]

(2.17)

The solvability of (2.17) is classical, see Cheng and Shkoller [9] and the references therein for instance. Indeed, the first, second, fourth and fifth equations in (2.16) provide the necessary conditions for solving (2.17). The solution to the original problem (8.22) is then obtained as the limit of the sequence of solutions to (2.16) as \( \varepsilon \to 0 \) after deriving the uniform estimates for the approximate solutions on a time interval independent of \( \varepsilon \),
by a slight variant of the derivation of the estimates of (2.5) as outlined below. Note that in our way of solving the magnetic system (8.22), the electric field in vacuum $\hat{E}$ could be viewed as an auxiliary variable, rather than the secondary variable as in [38,46,49,50]. We remark that one could also use the virtual magnetic field $b$ as an auxiliary variable instead of $\hat{E}$; yet we choose to work with $\hat{E}$ here as it is a physical variable. Finally, one can then construct solutions to (2.5) by the method of successive approximations, based on the solvability of the problems (8.7) and (8.22).

Now we turn to the derivation of the a priori estimates for the solutions to (2.5). Our derivation involves the electric field in vacuum $\hat{E}$ which solves (4.3), and the estimates of $\hat{E}$ are provided in Sect. 4.1. The basic ingredient in our analysis is to use the energy-dissipation structure (2.6). Since the higher order energy functionals are needed to control the nonlinear terms, one applies the temporal and horizontal spatial derivatives $\partial^\alpha$ of $\hat{E}$ instead of $\hat{b}$.

The derivation involves the electric field in vacuum $\hat{E}$ which solves (4.3), and the estimates of $\hat{E}$ are provided in Sect. 4.1. The basic ingredient in our analysis is to use the energy-dissipation structure (2.6). Since the higher order energy functionals are needed to control the nonlinear terms, one applies the temporal and horizontal spatial derivatives $\partial^\alpha$ for $\alpha \in \mathbb{N}^{1+2}$ with $|\alpha| \leq 2N$ to (2.5) and (4.3) to derive the tangential energy evolution

$$
\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega_-} (|\partial^\alpha u|^2 + |\partial^\alpha b|^2) \, d\mathcal{V}_t + \int_{\Omega_+} |\partial^\alpha \hat{b}|^2 \, d\mathcal{V}_t + \int_\Sigma \sigma |\nabla \partial^\alpha \eta|^2 \right) \\
+ \kappa \int_{\Omega_-} |\text{curl}^e \partial^\alpha b|^2 \, d\mathcal{V}_t \\
= \int_{\Omega_-} \partial^\alpha p F^{2,\alpha} \, d\mathcal{V}_t - \int_\Sigma \sigma \partial^\alpha HF^{5,\alpha} + \int_{\Omega_+} \partial^\alpha \hat{E} \cdot \hat{F}^{4,\alpha} \, d\mathcal{V}_t + \sum \mathcal{R}.
$$

(2.18)

Here the nonlinear terms $F^{2,\alpha}$, $F^{5,\alpha}$ and $\hat{F}^{4,\alpha}$ are defined by (4.21), (4.26) and (4.25), respectively, and $\sum \mathcal{R}$ denotes terms involving other nonlinearities, which, after some delicate arguments, can be controlled well in the sense that a term in $\sum \mathcal{R}$ is either bounded by $\sqrt{\mathcal{E}_{N+4}} (\mathcal{E}_{2N} + \mathcal{D}_{2N})$, or its time integration is bounded by $(\mathcal{E}_{2N})^{3/2}$, as $\mathcal{E}_{2N}$ is small. When $\alpha_0 \leq 2N - 1$, the first three terms in the right hand side of (2.18) can be shown to be also of $\sum \mathcal{R}$. When $\alpha_0 = 2N$, the difficulty is that $\partial^{2N} p$, $\partial^{2N} H$ and $\partial^{2N} \hat{E}$ seem to be out of control. Integrating by parts in time shows that the third term is of $\sum \mathcal{R}$, while integrating by parts in both time and space in an appropriate order and then employing a crucial cancelation between $\partial^{2N} p$ and $\sigma \partial^{2N} H$ on $\Sigma$ by using the dynamic boundary condition as what we have done in [54], one can show that the first two terms are of $\sum \mathcal{R}$. These lead to the following tangential energy evolution estimates:

$$
\hat{\mathcal{E}}_{2N}(t) + \int_0^t \hat{\mathcal{D}}_{2N} \lesssim \mathcal{E}_{2N}(0) + (\mathcal{E}_{2N}(t))^{3/2} + \int_0^t \sqrt{\mathcal{E}_{N+4}} (\mathcal{E}_{2N} + \mathcal{D}_{2N}) ,
$$

(2.19)

where $\hat{\mathcal{E}}_n$ and $\hat{\mathcal{D}}_n$ represent the tangential energy and dissipation functionals to be defined by (5.1) and (5.2) respectively.

Note that, as already seen from the estimate (2.19), to close the estimates, since the energy can not be dominated by the dissipation, one needs to show that $\sqrt{\mathcal{E}_{N+4}}(t)$ is integrable in time. The key here is to show that $\mathcal{E}_{N+4}(t)$ decays sufficiently fast in time.

To this end, employing an elaborate argument, we are able to derive a related set of tangential energy evolution estimates different from (2.19):

$$
\frac{d}{dt} (\hat{\mathcal{E}}_n + B_n) + \hat{\mathcal{D}}_n \lesssim \sqrt{\mathcal{E}_{2N} \mathcal{D}_n} , \quad n = N + 4, \ldots, 2N - 2,
$$

(2.20)

with $B_n$ satisfying $|B_n| \lesssim \sqrt{\mathcal{E}_{2N}} \mathcal{E}_n$. 

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With the control of the tangential energy estimates, one proceeds to derive the full energy estimates by exploiting further the structures of (2.5). First, one may improve the curl-estimates of $b$ in the tangential dissipation $\mathcal{D}_n$ to be the $H^1$-estimates of $b$ and derive the desired dissipation estimates of $\hat{b}$ by applying the Hodge-type estimates. Note that such dissipation estimates control only $b$ and $\hat{b}$. To get the tangential dissipation estimates for $u$, it is crucial for us to derive the dissipation estimates of the following $\hat{B} \cdot \nabla-$terms from the control of $\mathcal{D}_n$ for $\kappa > 0$,

$$
\sum_{j=0}^{n-1} \left\| \hat{B} \cdot \nabla \phi_j u_3 \right\|_{0,n-j-1}^2 + \sum_{j=0}^{n-1} \left\| \hat{B} \cdot \nabla \phi_j (\kappa \partial_3 b_h + \hat{B}_3 u_h) \right\|_{0,n-j-1}^2. \tag{2.21}
$$

Here we have used the notation that $v_h = (v_1, v_2)$ for any vector $v$. This follows by projecting the magnetic equations onto the vertical and horizontal components, respectively. Then one uses the Poincaré-type inequalities related to $\hat{B} \cdot \nabla$ for $\hat{B}_3 \neq 0$ together with the boundary conditions on $\Sigma_\pm$ to derive the tangential dissipation estimates of $u$.

Now the heart of the analysis is to derive the estimates involving the normal derivatives of $\phi$ means plus some nonlinear terms. One of the key observations here is the treatment of the linear term in the right hand side of (2.22): by using the third and second equations in (2.5), one finds

$$
\hat{B} \cdot \nabla \phi (\text{curl} \phi b)_1 \equiv \hat{B}_h \cdot \nabla \phi (\text{curl} \phi b)_1 + \hat{B}_3 \partial_1 \phi (\text{curl} \phi b)_3 + \hat{B}_3 (\text{curl} \phi \text{curl} \phi b)_2
$$

$$
= \hat{B}_h \cdot \nabla \phi (\text{curl} \phi b)_1 + \hat{B}_3 \partial_1 \phi (\text{curl} \phi b)_3 + \frac{\hat{B}_3}{\kappa} (-\partial_1 \phi b_2 + \hat{B} \cdot \nabla \phi u_2 + \cdots). \tag{2.23}
$$

On the other hand, one has

$$
\hat{B} \cdot \nabla \phi u_2 \equiv \hat{B}_h \cdot \nabla \phi u_2 - \hat{B}_3 (\text{curl} \phi u)_1 + \hat{B}_3 \partial_2 \phi u_3. \tag{2.24}
$$

Carrying out the similar computations for $\hat{B} \cdot \nabla \phi (\text{curl} \phi b)_2$, one then deduces from (2.22) the following equation of $(\text{curl} \phi u)_h$: for $i = 1, 2$,

$$
\partial_i \phi (\text{curl} \phi u)_i + u \cdot \nabla \phi (\text{curl} \phi u)_i + \frac{\hat{B}_3^2}{\kappa} (\text{curl} \phi u)_i
$$

$$
= \hat{B}_h \cdot \nabla \phi (\text{curl} \phi b)_1 + \hat{B}_3 \partial_i (\text{curl} \phi b)_3
$$

$$
+ (-1)^{i+1} \frac{\hat{B}_3}{\kappa} (-\partial_i b_{3-i} + \hat{B}_h \cdot \nabla h u_{3-i} + \hat{B}_3 \partial_3 u_{3} + \cdots). \tag{2.25}
$$

One thus sees again the key roles of the positivity of the magnetic diffusion coefficient $\kappa > 0$ and the non-vanishing of $\hat{B}_3 \neq 0$; they induce the damping term in (2.25), which provides the mechanism for the global-in-time estimates of $(\text{curl} \phi u)_h$. Note that for the estimates of $u$ in the energy $\mathcal{E}_{2N}$, one can estimate the linear $\nabla h \mu$ terms in the right hand of (2.25) by the control of $\mathcal{E}_{2N}$; for the estimates of $u$ in the dissipation $\mathcal{D}_n$, one has to estimate these terms by using instead the tangential dissipation estimates of $u$ derived from the $\hat{B} \cdot \nabla$-estimate (2.21). Making use of these estimates, the transport-damping structure of $(\text{curl} \phi u)_h$ in (2.25) and the Hodge-type estimates, one can derive the desired
estimates of $u$ in a recursive way in terms of the number of normal derivatives of $u$; the
desired estimates of $b$ and $\hat{b}$ can be derived along by employing the elliptic estimates.

With these estimates above, one may then derive the desired estimates for $p$ and $\eta$
by using directly the equations (2.5). The conclusion is that one can thus improve (2.19)
and (2.20) to be, respectively, since $E_{2N}$ is small,

$$E_{2N}(t) + \int_0^t D_{2N} \lesssim E_{2N}(0) + \int_0^t \sqrt{E_{N+4}} E_{2N}$$

(2.26)

and

$$\frac{d}{dt} E_n + N \leq 0, \ n = N + 4, \ldots, 2N - 2.$$  (2.27)

Note that if $E_{N+4}(t)$ decays at a sufficiently fast rate, then the estimate (2.26) can lead
to (2.14). This will be achieved by using (2.27). One does not have that $E_n \lesssim D_n$, which
rules out the exponential decay; also, $D_n$ can not control $E_n$ with respect to not only
the spatial regularity but also the temporal regularity, which prevents one from using the
spatial regularity Sobolev interpolation argument to bound $E_n \lesssim E_{2N}^{-\theta} D_n^\theta$, $0 < \theta < 1$
so as to derive the algebraic decay. Our key ingredient to get around this here is to
observe that $E \lesssim D_{\ell+1}$ and employ a time weighted inductive argument to (2.27). The conclusion is

$$\sum_{j=0}^{N-6} (1 + s)^{-5-j} E_{N+4+j}(t) + \sum_{j=0}^{N-6} \int_0^t (1 + s)^{-5-j} D_{N+4+j}(s) ds$$

$$\lesssim E_{2N}(0) + \int_0^t D_{2N-1}(s) ds.$$  (2.28)

This together with (2.14) yields (2.15) and hence a decay of $E_{N+4}$ with the rate $(1 + t)^{-N+5}$. Consequently, this scheme of the a priori estimates can be closed by requiring
$N \geq 8$.

2.4. Notation. We now list the conventions for notations. $C > 0$ denotes a generic
constant independent of the data and time, but may depend on the parameters of the
problem, $\kappa$, $\sigma$, $\tilde{B}$ and $N$, which is referred to as “universal”. Such constants are allowed
to change from line to line. To indicate some constants in some places so that they can
be referred to later, they will be denoted in particular by $C_1$, $C_2$, etc. $A_1 \lesssim A_2$ means
that $A_1 \leq C A_2$, and $A_1 \lesssim A_2 + A_3$ means that $A_1 \leq A_2 + C A_3$, for a universal
constant $C > 0$. To avoid the constants in various time differential inequalities, we use the
following two conventions:

$$\partial_t A_1 + A_2 \lesssim A_3 \text{ means } \partial_t \tilde{A}_1 + A_2 \lesssim A_3 \text{ for } A_1 \lesssim \tilde{A}_1 \lesssim A_1,$$  (2.29)

$$\partial_t (A_1 + A_2) + A_3 \lesssim A_4 \text{ means } \partial_t (A_1 + A_2) + A_3 \lesssim A_4.$$  (2.30)

Also, $N = \{0, 1, 2, \ldots\}$ denotes for the collection of non-negative integers. When
using differential multi-indices, we write $\mathbb{N}^3 = \{\alpha = (\alpha_1, \alpha_2, \alpha_3)\}$ for spatial derivatives
and $\mathbb{N}^2 = \{\alpha = (\alpha_1, \alpha_2)\}$ for horizontal spatial derivatives, while we write $\mathbb{N}^{1+d} = \{\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d)\}$, $d = 2, 3$, to emphasize that the $0-$index term is related to temporal
derivatives; that is, for $\alpha \in \mathbb{N}^d$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$, and for $\alpha \in \mathbb{N}^{1+d}$, $\partial^\alpha = \partial^{\alpha_0} \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$.

We define the standard commutator
\[
[\partial^\alpha, f, g] = \partial^\alpha (fg) - f \partial^\alpha g
\]
and the symmetric commutator
\[
[\partial^\alpha, f, g] = \partial^\alpha (fg) - f \partial^\alpha g - \partial^\alpha fg.
\]

We omit the differential elements $dx$ and $dx_h$ of the integrals over $\Omega_\pm$ and $\Sigma$ and also sometimes the differential elements $ds$ of the time integrals.

2.5. Organization of the paper. The rest of the paper is organized as follows. Section 3 concerns several Hodge-type elliptic problems to be used often later. Section 4 contains some preliminary results for the a priori estimates. We derive first the tangential energy evolution estimates in Sect. 5, then the full energy and dissipation estimates in Sect. 6, and finally the global a priori estimates in Sect. 7. Section 8 contains the proof of the local well-posedness. The global well-posedness is proved in Sect. 9. Section 10 considers the plasma-plasma interface problem. Some analytic tools are collected in “Appendix A”.

3. Hodge-Type Elliptic Systems

In this section we will consider the solvability and regularity of several Hodge-type elliptic problems to be used later.

First, we consider the following one-phase Hodge-type elliptic problem:
\[
\begin{align*}
\operatorname{curl}^\varphi v &= f^1, & \operatorname{div}^\varphi v &= f^2 \quad &\text{in } \tilde{\Omega} \\
v \times \mathcal{N} &= f^3 \quad &\text{on } \tilde{\Sigma}_1 \\
v \cdot \mathcal{N} &= f^4 \quad &\text{on } \tilde{\Sigma}_2.
\end{align*}
\]

Here $\tilde{\Omega}$ is either $\Omega_-$ or $\Omega_+$ or $\Omega$, $\tilde{\Sigma}_1$, $\tilde{\Sigma}_2$ are the two boundaries of $\tilde{\Omega}$ and we have extended $\mathcal{N}$ to be $(-\nabla_h \tilde{\eta}, 1)$, which reads as $(-\nabla_h \eta, 1)$ on $\Sigma$ and $e_3$ on $\Sigma_\pm$, respectively.

**Proposition 3.1.** Assume $\eta \in H^{k+1/2}$ for an integer $k > 3/2$ with $\partial_3 \varphi \geq \theta > 0$. Let $r = 1, \ldots, k$ and $f^1 \in H^{r-1}(\tilde{\Omega})$, $f^2 \in H^{r-1}(\tilde{\Omega})$, $f^3 \in H^{r-1/2}(\tilde{\Sigma}_1)$ and $f^4 \in H^{r-1/2}(\tilde{\Sigma}_2)$ be given such that
\[
\operatorname{div}^\varphi f^1 = 0 \quad \text{in } \tilde{\Omega}, \quad f^3 \cdot \mathcal{N} = 0 \quad \text{and} \quad f^1 \cdot \mathcal{N} = \operatorname{div}_h f^3_h \quad \text{on } \tilde{\Sigma}_1.
\]

There exists a unique solution $v$ to (3.1) satisfying
\[
\|v\|_{r, \tilde{\Omega}} \lesssim_\eta \|f^1\|_{r-1, \tilde{\Omega}} + \|f^2\|_{r-1, \tilde{\Omega}} + |f^3|_{r-1/2, \tilde{\Sigma}_1} + |f^4|_{r-1/2, \tilde{\Sigma}_2}.
\]

Hereafter $\lesssim_\eta$ stands for $\leq C_\eta$ for a constant $C_\eta$ depending on $|\eta|_{k+1/2}$ and $\theta$. 

Proof. This can be proved similarly as for Theorem 1.1 in Cheng and Shkoller [9], which established the solvability and regularity for a Hodge-type elliptic system in a Sobolev-class bounded domain, where the curl and divergence inside the domain and either the normal component or tangential components on the boundary of a vector field are prescribed. In order to understand the compatibility conditions in (3.2) and for reader’s convenience, we still sketch the construction of the solution to (3.1), for the case that $\tilde{\Omega} = \Omega_+$, $\Sigma_1 = \Sigma$ and $\Sigma_2 = \Sigma_+$ for instance.

First, by the first condition in (3.2), according to the assertion (1) of Theorem 1.1 in [9], one can define $\tilde{v}$ as the solution to

$$
\begin{aligned}
&\text{curl}\phi\tilde{v} = f^1, \quad \text{div}\phi\tilde{v} = f^2 \quad \text{in} \Omega_+ \\
&\tilde{v} \cdot N = \int_{\Sigma_+} f_4 - \int_{\Omega_+} f^2 \partial_3\phi \quad \text{on} \Sigma \\
&\tilde{v}_3 = f^4 \quad \text{on} \Sigma_+.
\end{aligned}
$$

(3.4)

Then the solution to (3.1) can be constructed as $v = \tilde{v} + \bar{v}$ with $\tilde{v}$ the solution to

$$
\begin{aligned}
&\text{curl}\phi\bar{v} = 0, \quad \text{div}\phi\bar{v} = 0 \quad \text{in} \Omega_+ \\
&\bar{v} \times N = f^3 := f^3 - \tilde{v} \times N \quad \text{on} \Sigma \\
&\bar{v}_3 = 0 \quad \text{on} \Sigma_+.
\end{aligned}
$$

(3.5)

Next, note that the last two conditions in (3.2) and the first equation in (3.4) yield

$$
\begin{aligned}
\bar{f}^3 \cdot N &= f^3 \cdot N = 0 \quad \text{and} \\
\text{div}_h f_3 &:= \text{div}_h f_3^3 - \text{curl}^h \tilde{v} \cdot N = \text{div}_h f_3^3 - f^1 \cdot N = 0 \quad \text{on} \Sigma.
\end{aligned}
$$

(3.6)

This implies that there exists a $\psi = \psi(x_1, x_2)$ such that

$$
\bar{f}^3 = (\partial_2\psi, -\partial_1\psi, \partial_1\eta\partial_2\psi - \partial_2\eta\partial_1\psi).
$$

(3.7)

Then the solution to (3.5) can be constructed as $\bar{v} = \nabla^\phi\phi$ with $\phi$ the solution to

$$
\begin{aligned}
\Delta^\phi \phi &= 0 \quad \text{in} \Omega_+ \\
\phi &= \psi \quad \text{on} \Sigma \\
\partial_3\phi &= 0 \quad \text{on} \Sigma_+.
\end{aligned}
$$

(3.8)

The construction of the solution to (3.1) is thus completed. □

Next, we consider the following two-phase Hodge-type elliptic problem:

$$
\begin{aligned}
&\text{curl}^\phi v = f^1, \quad \text{div}^\phi v = f^2 \quad \text{in} \Omega_- \\
&\text{curl}^\phi \hat{v} = f^1, \quad \text{div}^\phi \hat{v} = f^2 \quad \text{in} \Omega_+ \\
&[v] = 0 \quad \text{on} \Sigma \\
v_3 = 0 \quad \text{on} \Sigma_- \\
\hat{v} \times e_3 = 0 \quad \text{on} \Sigma_+.
\end{aligned}
$$

(3.9)
Proposition 3.2. Assume \( \eta \in H^{k+1/2} \) for an integer \( k \geq 3/2 \) with \( \partial_3 \varphi \geq \theta > 0 \). Let \( r = 1, \ldots, k + 1 \) and \( f^1, f^2 \in H^{r-1}(\Omega_-), \hat{f}^1, \hat{f}^2 \in H^{r-1}(\Omega_+) \) be given such that

\[
\begin{align*}
\text{div}^\varphi f^1 &= 0 \text{ in } \Omega_-, \quad \text{div}^\varphi \hat{f}^1 = 0 \text{ in } \Omega_+, \\
\left\| f^1 \right\|_{r-1} &\leq C \left\| f^1 \right\|_{r-1}, \quad \left\| \hat{f}^1 \right\|_{r-1} + \left\| \hat{f}^2 \right\|_{r-1} \geq C \eta (\left\| v \right\|_\ell + \left\| \hat{v} \right\|_\ell).
\end{align*}
\]

(3.10)

There exists a unique solution \( (v, \hat{v}) \) to (3.9) satisfying

\[
\left\| v \right\|_r + \left\| \hat{v} \right\|_r \lesssim \eta \left\| f^1 \right\|_{r-1} + \left\| f^2 \right\|_{r-1} + \left\| \hat{f}^1 \right\|_{r-1} + \left\| \hat{f}^2 \right\|_{r-1}.
\]

(3.11)

Proof. This can be proved similarly as for Theorem 2 in Padula and Solonnikov [38], which concerns the bounded domain case when \( \Omega_- \) is surrounded by \( \Omega_+ \) and the normal component of \( \hat{v} \) was prescribed on \( \partial \Omega \). Here we sketch an alternative proof based on Proposition 3.1.

To this end, one may define \( F^1 \) in \( \Omega \) with \( F^1 = f^1 \) in \( \Omega_- \) and \( F^1 = \hat{f}^1 \) in \( \Omega_+ \) and \( F^2 \) in \( \Omega \) with \( F^2 = f^2 \) in \( \Omega_- \) and \( F^2 = \hat{f}^2 \) in \( \Omega_+ \). By the first three conditions in (3.10), one has that \( \text{div}^\varphi F^1 \) exists and vanishes in \( \Omega \), and the last condition implies \( F^1 \cdot e_3 = 0 \) on \( \Sigma_+ \). Hence, by Proposition 3.1, there exists a unique solution \( V \in H^1(\Omega) \) to

\[
\begin{align*}
\text{curl}^\varphi V &= F^1, \quad \text{div}^\varphi V = F^2 \quad \text{in } \Omega, \\
V_3 &= 0 \quad \text{on } \Sigma_- \\
\hat{V} \times e_3 &= 0 \quad \text{on } \Sigma_+.
\end{align*}
\]

(3.12)

Taking \( v = V \) in \( \Omega_- \) and \( \hat{v} = V \) in \( \Omega_+ \) yields the conclusion for \( r = 1 \).

Now we derive the higher regularity estimates (3.11) for \( 2 \leq r \leq k+1 \) by an induction argument. Suppose that \( \ell \in [1, r-1] \) and (3.11) holds for \( \ell \). One then applies \( \partial^\alpha \) for \( \alpha \in \mathbb{N}^2 \) with \( |\alpha| \leq \ell \) to (3.9) to find that

\[
\begin{align*}
\text{curl}^\varphi \partial^\alpha v &= f^{1,\alpha}, \quad \text{div}^\varphi \partial^\alpha v = f^{2,\alpha} \quad \text{in } \Omega_- \\
\text{curl}^\varphi \partial^\alpha \hat{v} &= \hat{f}^{1,\alpha}, \quad \text{div}^\varphi \partial^\alpha \hat{v} = \hat{f}^{2,\alpha} \quad \text{in } \Omega_+ \\
\partial^\alpha v_3 &= 0 \quad \text{on } \Sigma_- \\
\partial^\alpha \hat{v} \times e_3 &= 0 \quad \text{on } \Sigma_+,
\end{align*}
\]

(3.13)

where

\[
\begin{align*}
f^{1,\alpha} := \partial^\alpha f^1 - \left[ \partial^\alpha, \text{curl}^\varphi \right] v, \quad f^{2,\alpha} := \partial^\alpha f^2 - \left[ \partial^\alpha, \text{div}^\varphi \right] v, \\
f^{1,\alpha} := \partial^\alpha \hat{f}^1 - \left[ \partial^\alpha, \text{curl}^\varphi \right] \hat{v}, \quad f^{2,\alpha} := \partial^\alpha \hat{f}^2 - \left[ \partial^\alpha, \text{div}^\varphi \right] \hat{v}.
\end{align*}
\]

(3.14)

It is routine to check that

\[
\begin{align*}
\text{div}^\varphi f^{1,\alpha} &= 0 \text{ in } \Omega_- , \quad \text{div}^\varphi \hat{f}^{1,\alpha} = 0 \text{ in } \Omega_+, \\
\left[ f^{1,\alpha} \right] \cdot \mathbf{N} &= 0 \text{ on } \Sigma \text{ and } \hat{f}^{1,\alpha} \cdot e_3 = 0 \text{ on } \Sigma_+.
\end{align*}
\]

(3.15)

The conclusion for \( r = 1 \) then yields that

\[
\left\| v \right\|_{1,\ell} + \left\| \hat{v} \right\|_{1,\ell} \lesssim \eta \left\| f^1 \right\|_\ell + \left\| f^2 \right\|_\ell + \left\| \hat{f}^1 \right\|_\ell + \left\| \hat{f}^2 \right\|_\ell + C_\eta \left( \left\| v \right\|_\ell + \left\| \hat{v} \right\|_\ell \right).
\]

(3.16)
This together with the trace theory and the induction assumption imply
\[
|v|_{\ell+1/2} + |\hat{v}|_{\ell+1/2} \lesssim \|v\|_{1,\ell} + \|\hat{v}\|_{1,\ell} \lesssim \eta \|f^1\|_\ell + \|f^2\|_\ell + \|\hat{f}^1\|_\ell + \|\hat{f}^2\|_\ell.
\] (3.17)

Hence, by Proposition 3.1, one has
\[
\|v\|_{\ell+1} + \|\hat{v}\|_{\ell+1} \lesssim \eta \|f^1\|_\ell + \|f^2\|_\ell + |v|_{\ell+1/2} + |\hat{v}|_{\ell+1/2} \\
\lesssim \eta \|f^1\|_\ell + \|f^2\|_\ell + \|\hat{f}^1\|_\ell + \|\hat{f}^2\|_\ell.
\] (3.18)

This implies that (3.11) holds for \(\ell + 1\). (3.11) is thus proved. \(\square\)

Finally, we consider the following mixed-phase Hodge-type elliptic problem:

\[
\begin{aligned}
curl^\theta \curl^\theta v &= f^1 \quad \text{in } \Omega_-

div^\varphi v &= f^2 \quad \text{in } \Omega_-
\curl^\theta \hat{v} &= \hat{f}^1, \quad \text{div}^\varphi \hat{v} = \hat{f}^2 \quad \text{in } \Omega_+
[\! [v] \! ] &= 0 \quad \text{on } \Sigma
v_3 &= 0, \quad \curl^\theta v \times e_3 = f^3 \quad \text{on } \Sigma_-
\hat{v} \times e_3 &= 0 \quad \text{on } \Sigma_+.
\end{aligned}
\] (3.19)

**Proposition 3.3.** Assume \(\eta \in H^{k+1/2}\) for an integer \(k > 3/2\) with \(\partial_3 \varphi \geq \theta > 0\). Let \(r = 2, \ldots, k+1\) and \(f^1 \in H^{r-2}(\Omega_-), f^2 \in H^{r-1}(\Omega_-), \hat{f}^1, \hat{f}^2 \in H^{-1}(\Omega_+)\) and \(f^3 \in H^{-3/2}(\Sigma_-)\) be given such that

\[
div^\varphi f^1 = 0 \text{ in } \Omega_-, \quad f^3 \cdot e_3 = 0 \text{ on } \Sigma_- \text{ and } f^1 \cdot e_3 = \text{div}_h \hat{f}^3 \text{ on } \Sigma_-
\] (3.20)

and

\[
div^\varphi \hat{f}^1 = 0 \text{ in } \Omega_+ \text{ and } \hat{f}^1 \cdot e_3 = 0 \text{ on } \Sigma_+.
\] (3.21)

There exists a unique solution \((v, \hat{v})\) to (3.19) satisfying

\[
\|v\|_r + \|\hat{v}\|_r \lesssim \eta \|f^1\|_{r-2} + \|f^2\|_{r-1} + \|\hat{f}^1\|_{r-1} + \|\hat{f}^2\|_{r-1} + \|f^3\|_{r-3/2}.
\] (3.22)

**Proof.** Motivated by the works of Ladyzhenskaya and Solonnikov [32,33], the solution to (3.19) can be constructed as follows. First, it follows from (3.20) and Proposition 3.1 that

\[
\begin{aligned}
curl^\theta w &= f^1, \quad \text{div}^\varphi \ w = 0 \quad \text{in } \Omega_+
\ \ \ \ w \cdot \mathcal{N} &= \hat{f}^1 \cdot \mathcal{N} \quad \text{on } \Sigma
w \times e_3 &= f^3 \quad \text{on } \Sigma_+
\end{aligned}
\] (3.23)

has a unique solution \(w\) which satisfies, by the trace theory,

\[
\|w\|_{r-1} \lesssim \eta \|f^1\|_{r-2} + \|\hat{f}^1 \cdot \mathcal{N}\|_{r-3/2} + \|f^3\|_{r-3/2} \\
\lesssim \eta \|f^1\|_{r-2} + \|\hat{f}^1\|_{r-1} + \|f^3\|_{r-3/2}.
\] (3.24)
Then by the second and third equations in (3.23), (3.21) and Proposition 3.2, one can define \((v, \hat{v})\) as the solution to

\[
\begin{align*}
\text{curl} \phi v &= w, \quad \text{div} \phi v = f^2 \quad \text{in } \Omega_-
\text{curl} \phi \hat{v} &= \hat{f}^1, \quad \text{div} \phi \hat{v} = f^2 \quad \text{in } \Omega_+
[v] &= 0 \quad \text{on } \Sigma
v_3 &= 0 \quad \text{on } \Sigma_-
\hat{v} \times e_3 &= 0 \quad \text{on } \Sigma_+
\end{align*}
\]

which satisfies

\[
\|v\|_r + \|\hat{v}\|_r \leq \eta \|w\|_{r-1} + \|f^2\|_{r-1} + \|\hat{f}^1\|_{r-1} + \|\hat{f}^2\|_{r-1}. \tag{3.26}
\]

It is easy to check that \((v, \hat{v})\) solves (3.19), and (3.22) follows from (3.24) and (3.26). \(\square\)

4. Preliminaries for the A Priori Estimates

In this section we give some preliminary results to be used in the derivation of the a priori estimates for solutions to (2.5). It will be assumed throughout Sects. 4–6 that the solution is given on the interval 

\[
[0, T],
\]

and obeys the a priori assumption

\[
\mathcal{E}_{2N}(t) \leq \delta, \quad \forall t \in [0, T] \tag{4.1}
\]

for \(N \geq 4\) and a sufficiently small constant \(\delta > 0\). This implies in particular that

\[
\frac{1}{2} \leq \partial_3 \varphi(t, x) \leq \frac{3}{2}, \quad \forall (t, x) \in [0, T] \times \bar{\Omega}. \tag{4.2}
\]

We remark that (4.1) and (4.2) are always used; in particular, the smallness (4.1) is used in many nonlinear estimates so that the various polynomials of \(\mathcal{E}_{2N}\) are bounded by \(C\mathcal{E}_{2N}\).

4.1. Estimates of \(\hat{E}\). Our derivation of the estimates of the solutions to (2.5) will involve the electric field in vacuum \(\hat{E}\), which solves

\[
\begin{align*}
\text{curl} \phi \hat{E} &= \partial_t^\phi \hat{b}, \quad \text{div} \phi \hat{E} = 0 \quad \text{in } \Omega_+
\hat{E} \times \mathcal{N} &= E \times \mathcal{N} \quad \text{on } \Sigma
\hat{E}_3 &= 0 \quad \text{on } \Sigma_+
\end{align*}
\]

Remark 4.1. Note that by the seventh, fifth, tenth and third equations in (2.5), one has

\[
\text{div} \phi \partial_t^\phi \hat{b} = 0 \quad \text{in } \Omega_+ \quad \text{and} \quad \partial_t^\phi \hat{b} \cdot \mathcal{N} = \partial_t^\phi b \cdot \mathcal{N} = \text{curl} \phi E \cdot \mathcal{N} \equiv \text{div}_h(E \times \mathcal{N})_h \quad \text{on } \Sigma. \tag{4.4}
\]

Thus Proposition 3.1 guarantees the existence of a unique solution \(\hat{E}\) to (4.3).
Now we estimate $\hat{E}$. For $N \geq 4$, define

$$\mathcal{E}_{2N}(\hat{E}):= \sum_{j=0}^{2N-1} \| \partial_t^j \hat{E} \|_{2N-j}^2$$

(4.5)

and that for $n = N + 4, \ldots, 2N$,

$$\mathcal{E}_n(\hat{E}):= \| \hat{E} \|_{n-1}^2 + \sum_{j=1}^{n-1} \| \partial_t^j \hat{E} \|_{n-j}^2$$

(4.6)

and

$$\mathcal{D}_n(\hat{E}):= \sum_{j=0}^{n-2} \| \partial_t^j \hat{E} \|_{n-j-1}^2.$$  

(4.7)

The equations (4.3) can be rewritten as

$$\begin{cases}
\text{curl } \hat{E} = P^1, & \text{div } \hat{E} = P^2 \quad \text{in } \Omega_+ \\
\hat{E} \times e_3 = P^3 & \quad \text{on } \Sigma \\
\hat{E}_3 = 0 & \quad \text{on } \Sigma_+,
\end{cases}$$

(4.8)

where

$$P^1 = \partial_t^\varphi \hat{b} + \nabla \bar{\eta} \times \partial_3^\varphi \hat{E},$$

(4.9)

$$P^2 = \nabla \bar{\eta} \cdot \partial_3^\varphi \hat{E},$$

(4.10)

$$P^3 = E \times N + \hat{E} \times (e_3 - N).$$

(4.11)

Here one has used the fact that $\partial_t^\varphi = \partial_t - \bar{\eta} \partial_t^\varphi$ for $i = t, 1, 2, 3$ due to (2.3).

The terms $P_i$ are estimated as follows.

**Lemma 4.2.** It holds that

$$\sum_{j=0}^{2N-1} \| \partial_t^j P^1 \|_{2N-j-1}^2 + \sum_{j=1}^{2N-1} \| \partial_t^j P^2 \|_{2N-j-1}^2 + \sum_{j=0}^{2N-1} \| \partial_t^j P^3 \|_{2N-j-1/2}^2$$

$$\lesssim \mathcal{E}_{2N} + \mathcal{E}_{2N} \mathcal{E}_{2N}(\hat{E})$$

(4.12)

and that for $n = N + 4, \ldots, 2N$,

$$\| P^1 \|_{n-2}^2 + \sum_{j=1}^{n-1} \| \partial_t^j P^1 \|_{n-j-1}^2 + \| P^2 \|_{n-2}^2 + \sum_{j=1}^{n-1} \| \partial_t^j P^2 \|_{n-j-1}^2$$

$$+ \| P^3 \|_{n-3/2}^2 + \sum_{j=1}^{n-1} \| \partial_t^j P^3 \|_{n-j-1/2}^2 \lesssim \mathcal{E}_n + \mathcal{E}_{2N} \mathcal{E}_n(\hat{E})$$

(4.13)

and

$$\sum_{j=0}^{n-2} \| \partial_t^j P^1 \|_{n-j-2}^2 + \sum_{j=0}^{n-2} \| \partial_t^j P^2 \|_{n-j-2}^2 + \sum_{j=0}^{n-2} \| \partial_t^j P^3 \|_{n-j-3/2}^2$$

$$\lesssim \mathcal{D}_n + \mathcal{E}_{2N} \mathcal{D}_n(\hat{E}).$$

(4.14)
Proof. These can be checked similarly as for Lemmas 4.4 and 4.5 below. □

The estimates of \( \hat{E} \) are given as follows.

**Proposition 4.3.** It holds that

\[
\mathcal{E}_{2N}(\hat{E}) \lesssim \mathcal{E}_{2N} \tag{4.15}
\]

and that for \( n = N + 4, \ldots, 2N, \)

\[
\mathcal{E}_n(\hat{E}) \lesssim \mathcal{E}_n \tag{4.16}
\]

and

\[
\mathcal{D}_n(\hat{E}) \lesssim \mathcal{D}_n. \tag{4.17}
\]

**Proof.** For \( j = 0, \ldots, 2N - 1, \) it follows from the Hodge-type estimates (3.3) of Proposition 3.1 (setting \( \eta = 0 \)) with \( r = 2N - j \geq 1 \) that, by (4.8) and (4.12),

\[
\left\lVert \hat{a}_i^j \hat{E} \right\rVert_{2N-j}^2 \lesssim \left\lVert \hat{a}_i^j \text{curl } \hat{E} \right\rVert_{2N-j-1}^2 + \left\lVert \hat{a}_i^j \text{div } \hat{E} \right\rVert_{2N-j-1}^2 + \left\lVert \hat{a}_i^j \times e_3 \right\rVert_{H^{2N-j-1/2}(\Sigma)}^2
\]

\[
\lesssim \left\lVert \hat{a}_i^j P \right\rVert_{2N-j}^2 + \left\lVert \hat{a}_i^j P^2 \right\rVert_{2N-j-1}^2 + \left\lvert \hat{a}_i^j P^3 \right\rvert_{2N-j-1/2}^2
\]

\[
\lesssim \mathcal{E}_{2N} + \mathcal{E}_{2N} \mathcal{E}_{2N}(\hat{E}). \tag{4.18}
\]

Summing (4.18) over \( j = 0, \ldots, 2N - 1 \) yields the estimate (4.15) since \( \mathcal{E}_{2N} \) is small. The estimates (4.16) and (4.17) follow similarly by using instead (4.13) and (4.14), respectively. □

In light of Proposition 4.3, in the nonlinear estimates to be carried out in Sects. 5–6, we can simply use \( \mathcal{E}_{2N} \) to bound terms which are controlled by \( \mathcal{E}_{2N}(\hat{E}) \), etc.

### 4.2. Geometric perturbed formulation

In order to use the energy-dissipation structure (2.6) to derive the tangential energy evolution estimates for solutions to (2.5), as usual for free boundary problems in fluid mechanics, it is natural to utilize the geometric structure given in (2.5). For this, one applies the temporal and horizontal spatial derivatives \( \partial^\alpha \) for \( \alpha \in \mathbb{N}^{1+2} \) with \( |\alpha| \geq 1 \) to (2.5) and (4.3) to find that

\[
\begin{aligned}
\partial^\alpha u + u \cdot \nabla^\alpha u + \nabla^\alpha p &= \text{curl}^\alpha \partial^\alpha b \times (\hat{B} + b) + F^{1,\alpha} \quad \text{in } \Omega_- \\
\text{div}^\alpha \partial^\alpha u &= F^{2,\alpha} \quad \text{in } \Omega_- \\
\partial^\alpha \partial^\alpha b &= \text{curl}^\alpha \partial^\alpha E + F^{3,\alpha}, \quad \partial^\alpha E = \partial^\alpha u \times (\hat{B} + b) - \kappa \text{curl}^\alpha \partial^\alpha b + F^{4,\alpha} \quad \text{in } \Omega_- \\
\partial^\alpha \partial^\alpha \hat{b} &= \text{curl}^\alpha \partial^\alpha \hat{E} + \hat{F}^{3,\alpha}, \quad \text{curl}^\alpha \partial^\alpha \hat{b} = \hat{F}^{4,\alpha} \quad \text{in } \Omega_+ \\
\partial^\alpha \eta &= \partial^\alpha u \cdot \mathcal{N} + F^{5,\alpha} \quad \text{on } \Sigma \\
\partial^\alpha p &= -\sigma \partial^\alpha H, \quad \partial^\alpha H = \text{div}_h \left( \frac{\nabla_h \partial^\alpha \eta}{\sqrt{1 + |\nabla_h \eta|^2}} - \frac{\nabla_h \eta \cdot \nabla_h \partial^\alpha \eta}{\sqrt{1 + |\nabla_h \eta|^2}^3} \nabla_h \eta \right) + F^{6,\alpha} \quad \text{on } \Sigma \\
\left[ \partial^\alpha b \right] &= 0, \quad \left[ \partial^\alpha E \right] \times \mathcal{N} = F^{7,\alpha} \quad \text{on } \Sigma \\
\partial^\alpha u_3 &= 0, \quad \partial^\alpha E \times e_3 = 0 \quad \text{on } \Sigma_- \\
\partial^\alpha \hat{b} \times e_3 &= 0 \quad \text{on } \Sigma_+.
\end{aligned}
\tag{4.19}
\]
where, recalling the commutator notations (2.31) and (2.32),
\[
F^{1,\alpha} = - \left[ \partial^\alpha, (\vec{B} + b) \times \text{curl}^{\psi} \right] b - \left[ \partial^\alpha, \partial_i^{\psi} + u \cdot \nabla^{\psi} \right] u - \left[ \partial^\alpha, \nabla^{\psi} \right] p, \tag{4.20}
\]
\[
F^{2,\alpha} = - \left[ \partial^\alpha, \text{div}^{\psi} \right] u, \tag{4.21}
\]
\[
F^{3,\alpha} = \left[ \partial^\alpha, \text{curl}^{\psi} \right] E - \left[ \partial^\alpha, \partial_i^{\psi} \right] b, \tag{4.22}
\]
\[
F^{4,\alpha} = - \left[ \partial^\alpha, b \right] \times u - \kappa \left[ \partial^\alpha, \text{curl}^{\psi} \right] b, \tag{4.23}
\]
\[
F^{5,\alpha} = \hat{F}^3,\alpha = \left[ \partial^\alpha, \text{curl}^{\psi} \right] \hat{E} - \left[ \partial^\alpha, \partial_i^{\psi} \right] \hat{b}, \tag{4.24}
\]
\[
\hat{F}^{4,\alpha} = - \left[ \partial^\alpha, \text{curl}^{\psi} \right] \hat{b}, \tag{4.25}
\]
\[
F^{6,\alpha} = \text{div}_h \left( - \left[ \partial^{\alpha-\alpha'}, \frac{\nabla_h \eta}{\sqrt{1 + |\nabla_h \eta|^2}} \right] \cdot \nabla_h \partial^{\alpha'} \eta \nabla_h \eta + \left[ \partial^{\alpha}, \frac{1}{\sqrt{1 + |\nabla_h \eta|^2}}, \nabla_h \eta \right] \right), \tag{4.27}
\]
\[
F^{7,\alpha} = \left[ \partial^\alpha, \mathcal{N} \right] \times E. \tag{4.28}
\]

Here \(\alpha'\) in (4.27) is any \(\alpha' < \alpha\) with \(|\alpha'| = 1\). Note that the rest of equations in (2.5) and (4.3) that are not considered in (4.19) will be not needed in the tangential energy evolution estimates.

**Lemma 4.4.** It holds that for \(|\alpha| \leq 2N\) are estimated as follows.

\[
\left\| F^{1,\alpha} \right\|_0^2 + \left\| F^{2,\alpha} \right\|_0^2 + \left\| F^{3,\alpha} \right\|_0^2 + \left\| \hat{F}^{3,\alpha} \right\|_0^2 + \left\| F^{4,\alpha} \right\|_0^2 + \left\| \hat{F}^{4,\alpha} \right\|_0^2
+ \left\| F^{5,\alpha} \right\|_0^2 + \left\| F^{6,\alpha} \right\|^{1/2}_0 + \left\| F^{7,\alpha} \right\|^{1/2}_0 \lesssim \mathcal{E}_{N+4} \mathcal{E}_{2N}, \tag{4.29}
\]

and for \(|\alpha| \leq 2N, \alpha_0 \leq 2N - 1,\)

\[
\left\| F^{5,\alpha} \right\|^{1/2}_0 \lesssim \mathcal{E}_{N+4} \mathcal{E}_{2N} \tag{4.30}
\]

and

\[
\left\| \partial_t \hat{F}^{4,(2N,0)} \right\|_{-1}^2 \lesssim \mathcal{E}_{N+4} \left( \mathcal{E}_{2N} + \mathcal{D}_{2N} \right). \tag{4.31}
\]

where \(\|\cdot\|_{-1}\) denotes the norm of \((H^1(\Omega))^*\).

**Proof.** To estimate \(F^{1,\alpha}\), it follows from the Leibniz rule and the Sobolev embeddings that, by the definition (2.3) of \(\partial_i^{\psi}\) and Lemma A.5,

\[
\left\| \left[ \partial^{\alpha}, \partial_i^{\psi} \right] u \right\|_0^2 \leq \sum_{\alpha' \leq \alpha} \left\| \partial^{\alpha'} \left( \frac{\partial_i \eta}{\partial_3 \psi} \right) \partial_3 u \right\|_0^2 \lesssim \sum_{\alpha' \leq \alpha} \left\| \partial^{\alpha'} \left( \frac{\partial_i \eta}{\partial_3 \psi} \right) \partial_3 u \right\|_0^2.
\]

\[
\lesssim \sum_{\alpha' \leq \alpha} \left\| \partial^{\alpha'} \left( \frac{\partial_i \eta}{\partial_3 \psi} \right) \right\|_2^2 \left\| \partial^{\alpha'-\alpha'} \partial_3 u \right\|_2^2 + \sum_{\alpha' \leq \alpha} \left\| \frac{\partial^{\alpha'-\alpha'} \partial_3 u}{\partial_3 \psi} \right\|_2^2 \left\| \partial^{\alpha'} \left( \frac{\partial_i \eta}{\partial_3 \psi} \right) \right\|_0^2.
\]

\[
(4.32)
\]
To bound the $H^0$ norms in the right hand side of (4.32), one may check directly that the terms of the highest order derivatives involved are $\partial^{\alpha-\alpha'} \partial_3 u$ for $\alpha' \in \mathbb{N}^{1+2}$ with $|\alpha'| = 1$ and $\partial^{\alpha+\beta'} \bar{\eta}$ for $\beta' \in \mathbb{N}^{1+3}$ with $|\beta'| = 1$. Noting that the term $|\partial_t^{2N+1} \eta|_{-1/2}^2$ is included in $\mathcal{E}_{2N}$ so that when $\partial^{\alpha+\beta'} = \partial_t^{2N+1}$, one gets by Lemma A.1 that

$$\left\| \partial_t^{2N+1} \eta \right\|_0^2 \lesssim \left\| \partial_t^{2N+1} \bar{\phi} \right\|_0^2 \lesssim \left\| \partial_t^{2N+1} \eta \right\|_{-1/2}^2 \leq \mathcal{E}_{2N}. \quad (4.33)$$

Then the $H^0$ norms in the right hand side of (4.32) are bounded by $\mathcal{E}_{2N}$, due to Lemmas A.1 and A.5. On the other hand, by Lemmas A.1 and A.5 again along with the definition (2.13) of $H$, one notes that the extra 4 derivatives in $\mathcal{E}_{N+4}$ have been chosen so that those $H^2$ norms in the right hand side of (4.32) can be bounded by $\mathcal{E}_{N+4}$. Hence \[ \left\| \left[ \partial^\alpha, \partial^\beta, \partial^\gamma \right] u \right\|_0^2 \lesssim \mathcal{E}_{N+4}\mathcal{E}_{2N}. \] Estimating the other terms in $F^{1,\alpha}$ in the same way, one may conclude that

$$\left\| F^{1,\alpha} \right\|_0^2 \lesssim \mathcal{E}_{N+4}\mathcal{E}_{2N}. \quad (4.34)$$

Similarly, one has that

$$\left\| F^{2,\alpha} \right\|_0^2 + \left\| \hat{F}^{3,\alpha} \right\|_0^2 + \left\| F^{4,\alpha} \right\|_0^2 + \left\| \hat{F}^{4,\alpha} \right\|_0^2 \lesssim \mathcal{E}_{N+4}\mathcal{E}_{2N}. \quad (4.35)$$

To estimate $F^{3,\alpha}$, one may argue again as $F^{1,\alpha}$ to derive the desired estimates except the term $\kappa [\partial^\alpha, \text{curl}^\phi] \text{curl}^\phi b$, which involves new types of terms of the highest order derivatives: $\partial^\alpha \partial_3 \nabla \bar{\eta}$ and $\partial^\alpha \partial_3 \nabla b$ for $\alpha' \in \mathbb{N}^{1+2}$ with $|\alpha'| = 1$. By Lemma A.1, one has

$$\left\| \partial^{\alpha-\alpha'} \partial_3 \nabla \bar{\eta} \right\|_0^2 \lesssim \left\| \partial^{\alpha-\alpha'} \bar{\phi} \right\|_2^2 \lesssim \left\| \partial^{\alpha-\alpha'} \eta \right\|_{3/2}^2 \leq \mathcal{E}_{2N} \quad (4.36)$$

and since $\alpha_0 - \alpha'_0 \leq 2N - 1$,

$$\left\| \partial^{\alpha-\alpha'} \partial_3 \nabla b \right\|_0^2 \leq \left\| \partial^{\alpha-\alpha'} b \right\|_2^2 \leq \mathcal{E}_{2N}. \quad (4.37)$$

Hence, one can get

$$\left\| F^{3,\alpha} \right\|_0^2 \lesssim \mathcal{E}_{N+4}\mathcal{E}_{2N}. \quad (4.38)$$

To estimate $F^{5,\alpha}$, since the terms of the highest order derivatives are $\partial^{\alpha-\alpha'} u_h$ for $\alpha' \in \mathbb{N}^{1+2}$ with $|\alpha'| = 1$ and $\partial^{\alpha} \nabla_h \eta$, one then separates the cases $\alpha_0 = 2N$ and $\alpha_0 \leq 2N - 1$. It follows from Lemma A.5 and the trace theory that

$$\left\| F^{5,\alpha} \right\|_0^2 \lesssim \mathcal{E}_{N+4}\mathcal{E}_{2N} \text{ for } \alpha_0 = 2N \text{ and } \left\| F^{5,\alpha} \right\|_{1/2}^2 \lesssim \mathcal{E}_{N+4}\mathcal{E}_{2N} \text{ for } \alpha_0 \leq 2N - 1. \quad (4.39)$$

To estimate $F^{6,\alpha}$, since the terms of the highest order derivatives are $\partial^{\alpha-\alpha'} \nabla_h^2 \eta$ for $\alpha' \in \mathbb{N}^{1+2}$ with $|\alpha'| = 1$, similarly, one obtains

$$\left\| F^{6,\alpha} \right\|_{1/2}^2 \lesssim \mathcal{E}_{N+4}\mathcal{E}_{2N}. \quad (4.40)$$
Next, for $F^{7, \alpha}$, the new terms of the highest order derivatives here are $\partial^{\alpha - \alpha'} \nabla b$ for $\alpha' \in \mathbb{N}^{1+2}$ with $|\alpha'| = 1$. It follows from the trace theory that

$$\left| \partial^{\alpha - \alpha'} \nabla b \right|^2_0 \leq \left| \partial^{\alpha - \alpha'} b \right|^2_2 \leq E_{2N} \text{ and } \left\| \partial^{\alpha - \alpha'} \hat{F} \right\|^2_{L^2(\Sigma)} \leq \left\| \partial^{\alpha - \alpha'} \hat{E} \right\|^2_1 \leq E_{2N}. \tag{4.41}$$

Hence, one can get

$$\left| F^{7, \alpha} \right|^2_0 \lesssim \mathcal{E}_{N+4} \mathcal{E}_{2N}. \tag{4.42}$$

Finally, in $\partial_t \hat{F}^{4,(2N,0)}$, the new terms of the highest order derivatives are $\partial_t^{2N+1} \nabla h \eta$ and $\partial_t^{2N} \partial_3 \hat{b}$. Note that

$$\left\| \partial_t^{2N+1} \nabla \eta \right\|^2_{-1} \lesssim \left\| \partial_t^{2N+1} \eta \right\|^2_{-1/2} \leq E_{2N} \text{ and } \left\| \partial_t^{2N} \partial_3 \hat{b} \right\|^2_0 \leq \mathcal{D}_{2N}. \tag{4.43}$$

Hence, by Lemma A.6 one can get

$$\left\| \partial_t \hat{F}^{4,(2N,0)} \right\|^2_{-1} \lesssim \mathcal{E}_{N+4} \left( E_{2N} + \mathcal{D}_{2N} \right) \tag{4.44}$$

Consequently, the estimates (4.29)–(4.31) follow. \qed

We now present some specialized estimates of $F^{i, \alpha}$ when $|\alpha| \leq 2N - 2$.

**Lemma 4.5.** For $|\alpha| \leq 2N - 2$, it holds that

$$\left\| F^{1, \alpha} \right\|^2_1 + \left\| \partial_t F^{1, \alpha} \right\|^2_0 + \left\| F^{2, \alpha} \right\|^2_1 + \left\| \partial_t F^{2, \alpha} \right\|^2_0 + \left\| F^{3, \alpha} \right\|^2_0 + \left\| F^{4, \alpha} \right\|^2_1 + \left\| \partial_t F^{4, \alpha} \right\|^2_0 + \left\| F^{5, \alpha} \right\|^2_{3/2} + \left\| \partial_t F^{5, \alpha} \right\|^2_{1/2}$$

$$+ \left\| F^{6, \alpha} \right\|^2_1 + \left\| \partial_t F^{6, \alpha} \right\|^2_0 + \left\| F^{7, \alpha} \right\|^2_0 \lesssim \mathcal{D}_{N+4} E_{2N} \tag{4.45}$$

and

$$\left\| F^{1, \alpha} \right\|^2_0 + \left\| F^{2, \alpha} \right\|^2_0 + \left\| \partial_t F^{2, \alpha} \right\|^2_0 + \left\| F^{4, \alpha} \right\|^2_0 + \left\| \partial_t F^{4, \alpha} \right\|^2_0$$

$$+ \left\| F^{5, \alpha} \right\|^2_0 + \left\| \partial_t F^{5, \alpha} \right\|^2_0 + \left\| F^{6, \alpha} \right\|^2_0 \lesssim \mathcal{E}_{N+4} \mathcal{E}_{2N}. \tag{4.46}$$

**Proof.** This follows similarly as for Lemma 4.4, by checking the terms of the highest order derivatives involved. \qed
4.3. **Linear perturbed formulation.** In order to use the linear structure of (2.5), it is more convenient to write it as a perturbation of the linearized equations:

\[
\begin{align*}
\partial_t u + \nabla p &= \text{curl } b \times \tilde{B} + G^1 \\
\text{div } u &= G^2 \\
\partial_t b + \kappa \text{ curl } b &= \text{curl}(u \times \tilde{B}) + G^3 \\
\text{div } b &= G^4 \\
\text{curl } \hat{b} &= \hat{G}^3, \quad \text{div } \hat{b} = \hat{G}^4
\end{align*}
\]

in \( \Omega_- \)

\[
\begin{align*}
\partial_t \eta &= u_3 + G^5 \\
p &= -\sigma \Delta_h \eta + G^6, \quad \llbracket b \rrbracket = 0
\end{align*}
\]

in \( \Omega_+ \)

(4.47)

where

\[
G^1 = \partial_t \tilde{\eta} \partial_2^\theta u - u \cdot \nabla^\varphi u + \nabla \tilde{\eta} \partial_3^\theta p - (\nabla \tilde{\eta} \times \partial_3^\theta b) \times \tilde{B} + \text{curl}^\varphi b \times b,
\]

(4.48)

\[
G^2 = \nabla \tilde{\eta} \cdot \partial_2^\theta u,
\]

(4.49)

\[
G^3 = \partial_t \tilde{\eta} \partial_3^\theta b - \kappa (\text{curl}^\varphi \text{curl}^\varphi b - \text{curl } b) + (\text{curl}^\varphi - \text{curl})(u \times \tilde{B}) + \text{curl}^\varphi (u \times b),
\]

(4.50)

\[
\begin{align*}
G^4 &= \nabla \tilde{\eta} \cdot \partial_3^\theta b, \\
\hat{G}^3 &= \nabla \tilde{\eta} \times \partial_3^\theta \hat{b}, \\
\hat{G}^4 &= \nabla \tilde{\eta} \cdot \partial_3^\theta \hat{b}, \\
G^5 &= -u_h \cdot \nabla_h \eta,
\end{align*}
\]

(4.51 - 4.53)

\[
G^6 = -\sigma \text{ div}_h \left((1 + |\nabla_h \eta|^2)^{-1/2} - 1\right) \nabla_h \eta,
\]

(4.54)

\[
G^7 = \kappa (\text{curl } b - \text{curl}^\varphi b) \times e_3.
\]

(4.55)

The nonlinear terms \( G^i \) are estimated as follows.

**Lemma 4.6.** It holds that

\[
\begin{align*}
\sum_{j=0}^{2N-1} \left\| \partial^j \tilde{G}^1 \right\|_{2N-j-1}^2 + \sum_{j=0}^{2N-1} \left\| \partial^j \tilde{G}^2 \right\|_{2N-j-1}^2 + \sum_{j=0}^{2N-1} \left\| \partial^j \tilde{G}^3 \right\|_{2N-j-1}^2 \\
+ \sum_{j=0}^{2N-1} \left\| \partial^j \tilde{G}^4 \right\|_{2N-j}^2 + \sum_{j=0}^{2N-1} \left\| \partial^j \tilde{G}^3 \right\|_{2N-j}^2 + \sum_{j=0}^{2N-1} \left\| \partial^j \tilde{G}^4 \right\|_{2N-j}^2 \\
+ \sum_{j=0}^{2N} \left\| \partial^j \tilde{G}^5 \right\|_{2N-j-1/2}^2 + \sum_{j=0}^{2N-1} \left\| \partial^j \tilde{G}^6 \right\|_{2N-j-1/2}^2 + \sum_{j=0}^{2N-1} \left\| \partial^j \tilde{G}^7 \right\|_{2N-j-1/2}^2
\end{align*}
\]

\[
\lesssim \min\{C_{N+4}, D_{N+4}\} E_{2N}
\]

(4.57)

and

\[
\sum_{j=0}^{2N} \left\| \partial^j \tilde{G}^4 \right\|_{2N-j}^2 + \sum_{j=0}^{2N} \left\| \partial^j \tilde{G}^3 \right\|_{2N-j}^2 + \sum_{j=0}^{2N} \left\| \partial^j \tilde{G}^4 \right\|_{2N-j}^2 \lesssim E_{N+4} (E_{2N} + D_{2N}).
\]

(4.58)
Proof. (4.57) can be proved similarly as Lemmas 4.4 and 4.5, and (4.58) follows similarly by noting that the new term of the highest order derivatives is $\partial_t^{2N} \partial_3 b$ and estimating $\| \partial_t^{2N} \partial_3 b \|_0^2 \leq \mathcal{O}_{2N}$. □

The following estimates on the difference between $\partial_i^\alpha$ and $\partial_t^\alpha$ will be used later.

Lemma 4.7. It holds that for $|\alpha| \leq 2N - 1$,

$$\| \partial_i^\alpha (\partial_t^\alpha u - \partial_t u) \|_0^2 \lesssim \min \{ \mathcal{E}_{N+4}, \mathcal{D}_{N+4} \} \mathcal{E}_{2N} \tag{4.59}$$

and that for $|\alpha| \leq n$ with $n = N + 4, \ldots, 2N$,

$$\| \partial_i^\alpha \partial^\alpha b - \partial_t \partial^\alpha b \|_0^2 \lesssim \mathcal{E}_{N+4} \mathcal{D}_n. \tag{4.60}$$

Proof. The proof follows in the same way as for Lemmas 4.4 and 4.5. □

4.4. Vorticity equations. For the estimates of the normal derivatives of $u$, as for the incompressible Euler equations, a natural way is to estimate first the vorticity curl $\phi$ (to get rid of the pressure term $\nabla^\phi p$ and avoid the loss of derivatives) and then to use the Hodge-type estimates. Applying curl $\phi$ to the first equation in (2.5) yields that

$$\partial_t^\alpha \text{curl}^\phi u + u \cdot \nabla^\phi \text{curl}^\phi u = \bar{B} \cdot \nabla^\phi \text{curl}^\phi b + \text{curl}^\phi u \cdot \nabla^\phi u + \text{curl}^\phi (\text{curl}^\phi b \times b). \tag{4.61}$$

The difficulty is that there is a linear forcing term $\bar{B} \cdot \nabla^\phi \text{curl}^\phi b$ on the right hand side of (4.61), and one cannot use the equations of curl $\phi b$ to balance this term as for the tangential energy evolution estimates, due to the usual difficulties caused by the diffusion term $\kappa \text{curl}^\phi \text{curl}^\phi b$. This is harmful for the global-in-time uniform estimate of curl $\phi u$. But, on the other hand, if there were without this term, then it would be difficult to derive the global-in-time uniform estimates of the higher order derivatives for curl $\phi u$ just as for the incompressible Euler equations. The crucial observation here is that there is a new damping structure for the vorticity curl $\phi u$. Indeed, it follows from the second component of the third equation and the second equation in (2.5) that

$$\bar{B} \cdot \nabla^\phi (\text{curl}^\phi b)_1 \equiv \bar{B}_h \cdot \nabla^\phi_h (\text{curl}^\phi b)_1 + \bar{B}_3 (\text{curl}^\phi \text{curl}^\phi b)_2 + \bar{B}_3 \partial_1^\phi (\text{curl}^\phi b)_3$$

$$= \bar{B}_h \cdot \nabla^\phi_h (\text{curl}^\phi b)_1 + \bar{B}_3 \partial_1^\phi (\text{curl}^\phi b)_3 + \frac{\bar{B}_3}{\kappa} (-\partial_t^\phi b_2 + \bar{B} \cdot \nabla^\phi u_2 + (\text{curl}^\phi (u \times b))_2). \tag{4.62}$$

On the other hand, one can write

$$\bar{B} \cdot \nabla^\phi u_2 \equiv \bar{B}_h \cdot \nabla^\phi_h u_2 - \bar{B}_3 (\text{curl}^\phi u)_1 + \bar{B}_3 \partial_2^\phi u_3. \tag{4.63}$$

Hence, as a consequence of (4.62) and (4.63), the first component of (4.61) can be rewritten as

$$\partial_t^\phi (\text{curl}^\phi u)_1 + u \cdot \nabla^\phi (\text{curl}^\phi u)_1 + \frac{\bar{B}_3^2}{\kappa} (\text{curl}^\phi u)_1$$

$$= \bar{B}_h \cdot \nabla^\phi_h (\text{curl}^\phi b)_1 + \bar{B}_3 \partial_1^\phi (\text{curl}^\phi b)_3 + \frac{\bar{B}_3}{\kappa} (-\partial_t^\phi b_2 + \bar{B}_h \cdot \nabla^\phi u_2 + \bar{B}_3 \partial_2^\phi u_3)$$

$$+ \frac{\bar{B}_3}{\kappa} (\text{curl}^\phi (u \times b))_2 + \text{curl}^\phi u \cdot \nabla^\phi u_1 + (\text{curl}^\phi (\text{curl}^\phi b \times b))_1. \tag{4.64}$$
Similarly, one has

\[
\partial_t \phi^\nu (\text{curl}^\nu u) + u \cdot \nabla (\text{curl}^\nu u) + \frac{\tilde{B}_3^2}{\kappa} (\text{curl}^\nu u)
\]

\[
= \tilde{B}_h \cdot \nabla \phi^\nu (\text{curl}^\nu b) + \tilde{B}_3 \partial_t \phi^\nu (\text{curl}^\nu b) - \frac{\tilde{B}_3}{\kappa} (-\partial_t b_1 + \tilde{B}_h \cdot \nabla \phi^\nu u_1 + \tilde{B}_3 \partial_1^\nu u_3)
\]

\[
- \frac{\tilde{B}_3}{\kappa} (\text{curl}^\nu (u \times b))_1 + \text{curl}^\nu u \cdot \nabla \phi^\nu u_2 + (\text{curl}^\nu (\text{curl}^\nu b \times b))_2.
\]

(4.65)

The equations (4.64) and (4.65) yield a transport-damping evolution structure for \((\text{curl}^\nu u)_h\), and one then sees the key roles of the positivity of the magnetic diffusion coefficient \(\kappa > 0\) and the non-vanishing of \(\tilde{B}_3 \neq 0\).

Applying \(\partial^\alpha\) for \(\alpha \in \mathbb{N}^{1+3}\) with \(|\alpha| \geq 1\) to (4.64) and (4.65) gives that

\[
\partial_t^\alpha (\text{curl}^\nu u)_h + u \cdot \nabla^\alpha (\text{curl}^\nu u)_h + \frac{\tilde{B}_3^2}{\kappa} \partial^\alpha (\text{curl}^\nu u)_h = \partial^\alpha L_h + \Phi_h^\alpha,
\]

(4.66)

where for \(i = 1, 2,\)

\[
L_i = \tilde{B}_h \cdot \nabla (\text{curl} b)_i + \tilde{B}_3 \partial_i (\text{curl} b)_3 + (-1)^i+1 \frac{\tilde{B}_3}{\kappa} (-\partial_i b_{3-i})
\]

\[
+ \tilde{B}_h \cdot \nabla u_{3-i} + \tilde{B}_3 \partial_{3-i} u_3
\]

(4.67)

and

\[
\Phi_h^\alpha = \partial^\alpha, \partial_1^\alpha + u \cdot \nabla^\alpha (\text{curl}^\nu u)_h + \partial^\alpha \Phi_h
\]

(4.68)

with that for \(i = 1, 2,\)

\[
\Phi_i = - \tilde{B}_h \cdot \nabla \tilde{\eta} \partial_1^\nu (\text{curl} b)_i - \tilde{B}_3 \partial_i \tilde{\eta} \partial_3^\nu (\text{curl} b)_3 - \tilde{B}_h \cdot \nabla \phi^\nu (\nabla \tilde{\eta} \times \partial_3^\nu b)_i
\]

\[
- \tilde{B}_3 \partial_i (\nabla \tilde{\eta} \times \partial_3^\nu b)_3 + (-1)^{i+1} \frac{\tilde{B}_3}{\kappa} (\partial_i \tilde{\eta} \partial_3^\nu b_{3-i} - \tilde{B}_h \cdot \nabla \tilde{\eta} \partial_3^\nu u_{3-i} - \tilde{B}_3 \partial_{3-i} \tilde{\eta} \partial_3^\nu u_3)
\]

\[
+ (-1)^{i+1} \frac{\tilde{B}_3}{\kappa} (\text{curl}^\nu (u \times b))_{3-i} + \text{curl}^\nu u \cdot \nabla \phi^\nu u_i + (\text{curl}^\nu (\text{curl}^\nu b \times b))_i.
\]

(4.69)

The nonlinear term \(\Phi_h^\alpha\) can be estimated as follows.

**Lemma 4.8.** It holds that for \(|\alpha| \leq 2N - 1,\)

\[
\|\Phi_h^\alpha\|_0^2 \lesssim \min(\mathcal{E}_{N+4}, \mathcal{D}_{N+4}) \mathcal{E}_{2N}.
\]

(4.70)

**Proof.** The proof follows in the same way as for Lemmas 4.4 and 4.5. □
5. Tangential Energy Evolution

In this section we will derive the tangential energy evolution estimates for solutions to (2.5). For a generic integer \( n \geq 3 \), we define the tangential energy that involves the temporal and horizontal spatial derivatives by, employing the anisotropic Sobolev norm (2.12),

\[
\bar{E}_n := \sum_{j=0}^{n} \| \partial_j u \|_{0,n-j}^2 + \sum_{j=0}^{n} \| \partial_j b \|_{0,n-j}^2 + \sum_{j=0}^{n} \| \partial_j \tilde{b} \|_{0,n-j}^2 + \sum_{j=0}^{n} \| \partial_j \eta \|_{n-j+1}^2
\]  
(5.1)

and the corresponding dissipation by

\[
\bar{D}_n := \sum_{j=0}^{n} \| \text{curl} \partial_j b \|_{0,n-j}^2.
\]  
(5.2)

5.1. Energy evolution at the \( 2N \) level. We start with the following time-integrated tangential energy evolution estimate at the \( 2N \) level.

Proposition 5.1. It holds that

\[
\bar{E}_{2N}(t) + \int_0^t \bar{D}_{2N} \lesssim \bar{E}_{2N}(0) + (\bar{E}_{2N}(t))^{3/2} + \int_0^t \sqrt{\bar{E}_{N+4}} (\bar{E}_{2N} + \bar{D}_{2N}).
\]  
(5.3)

Proof. Let \( \alpha \in \mathbb{N}^{1+2} \) such that \( 1 \leq |\alpha| \leq 2N \). Taking the inner product of the first equation in (4.19) with \( \partial^\alpha u \) and the third equation with \( \partial^\alpha b \), respectively, integrating by parts over \( \Omega_- \) by using the second, eighth and eleventh equations in (2.5), and then adding the resulting equations together, one has

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_-} \left( |\partial^\alpha u|^2 + |\partial^\alpha b|^2 \right) d\mathcal{V}_i - \frac{1}{2} \int_{\Sigma} \partial_t \eta \ |\partial^\alpha b|^2 + \int_{\Omega_-} \nabla \phi \partial^\alpha p \cdot \partial^\alpha u d\mathcal{V}_i
\]
\[
= \int_{\Omega_-} \left( \text{curl} \phi \partial^\alpha b \times (\bar{B} + b) \cdot \partial^\alpha u + \text{curl} \phi \partial^\alpha E \cdot \partial^\alpha b \right) d\mathcal{V}_i
\]
\[
+ \int_{\Omega_-} \left( F_{1,\alpha} \cdot \partial^\alpha u + F_{3,\alpha} \cdot \partial^\alpha b \right) d\mathcal{V}_i.
\]  
(5.4)

The integration by parts over \( \Omega_- \) shows that, by using the thirteenth, fourth, eleventh and tenth equations in (4.19),

\[
\int_{\Omega_-} \left( \text{curl} \phi \partial^\alpha b \times (\bar{B} + b) \cdot \partial^\alpha u + \text{curl} \phi \partial^\alpha E \cdot \partial^\alpha b \right) d\mathcal{V}_i
\]
\[
= \int_{\Omega_-} \left( (\bar{B} + b) \times \partial^\alpha u + \partial^\alpha E \right) \cdot \text{curl} \phi \partial^\alpha b d\mathcal{V}_i + \int_{\Sigma} \mathcal{N} \times \partial^\alpha E \cdot \partial^\alpha b
\]
\[
= \int_{\Omega_-} \left( -\kappa \text{curl} \phi \partial^\alpha b + F_{4,\alpha} \right) \cdot \text{curl} \phi \partial^\alpha b d\mathcal{V}_i + \int_{\Sigma} \mathcal{N} \times \partial^\alpha \tilde{E} \cdot \partial^\alpha b + \int_{\Sigma} F_{7,\alpha} \cdot \partial^\alpha b.
\]  
(5.5)
The integration by parts over $\Omega_+$ yields, by using the fifth, sixth, and fourteenth equations in (4.19),

$$
\int_{\Sigma} \mathcal{N} \times \partial^\alpha \mathbf{\hat{E}} \cdot \partial^\alpha \mathbf{\hat{b}}
= - \int_{\Omega_+} \text{curl} \, \partial^\alpha \mathbf{\hat{E}} \cdot \partial^\alpha \mathbf{\hat{b}} \, dv_t + \int_{\Omega_+} \partial^\alpha \mathbf{\hat{E}} \cdot \text{curl} \, \partial^\alpha \mathbf{\hat{b}} \, dv_t + \int_{\Sigma} \mathbf{e}_3 \times \partial^\alpha \mathbf{\hat{E}} \cdot \partial^\alpha \mathbf{\hat{b}}
= - \int_{\Omega_+} (\partial^t \partial^\alpha \mathbf{\hat{b}} - \mathbf{\hat{F}}^{3,\alpha}) \cdot \partial^\alpha \mathbf{\hat{b}} \, dv_t + \int_{\Omega_+} \partial^\alpha \mathbf{\hat{E}} \cdot \mathbf{\hat{F}}^{4,\alpha} \, dv_t + \int_{\Sigma} \partial^\alpha \mathbf{\hat{b}} \times \mathbf{e}_3 \cdot \partial^\alpha \mathbf{\hat{E}}
= - \frac{1}{2} \frac{d}{dt} \int_{\Omega_+} \left| \partial^\alpha \mathbf{\hat{b}} \right|^2 \, dv_t - \int_{\Omega_+} \frac{1}{2} \sum \partial_t |\partial^\alpha \mathbf{\hat{b}}|^2 + \int_{\Omega_+} \left( \mathbf{\hat{F}}^{3,\alpha} \cdot \partial^\alpha \mathbf{\hat{b}} + \partial^\alpha \mathbf{\hat{E}} \cdot \mathbf{\hat{F}}^{4,\alpha} \right) \, dv_t.
$$

(5.6)

By the twelfth, eighth, seventh and second equations in (4.19), one integrates by parts over $\Omega_-$ to obtain

$$
\int_{\Omega_-} \nabla^\psi \partial^\alpha p \cdot \partial^\alpha u \, dv_t = \int_{\Omega_-} \partial^\alpha p \partial^\alpha u \cdot \mathcal{N} - \int_{\Omega_-} \partial^\alpha p \partial^\alpha p \, dv_t
= \int_{\Omega_-} (\partial^\alpha \partial^\alpha \eta - F^{5,\alpha}) - \int_{\Omega_-} \partial^\alpha \partial^\alpha \eta F^{2,\alpha} \, dv_t.
$$

(5.7)

By the ninth equation in (4.19), one may write

$$
- \int_{\Sigma} \sigma \partial^\alpha H \partial_t \partial^\alpha \eta = - \int_{\Sigma} \sigma \left( \text{div}_h \left( \frac{\nabla_h \partial^\alpha \eta}{\sqrt{1 + |\nabla_h \eta|^2}} \right) - \frac{\nabla_h \eta \cdot \nabla_h \partial^\alpha \eta}{\sqrt{1 + |\nabla_h \eta|^2}} \nabla_h \eta \right) + F^{6,\alpha} \partial_t \partial^\alpha \eta.
$$

(5.8)

Integrating by parts in both $x_h$ and $t$ yields

$$
- \int_{\Sigma} \sigma \text{div}_h \left( \frac{\nabla_h \partial^\alpha \eta}{\sqrt{1 + |\nabla_h \eta|^2}} \right) - \frac{\nabla_h \eta \cdot \nabla_h \partial^\alpha \eta}{\sqrt{1 + |\nabla_h \eta|^2}} \nabla_h \eta \partial_t \partial^\alpha \eta
= \int_{\Sigma} \sigma \left( \frac{\nabla_h \partial^\alpha \eta}{\sqrt{1 + |\nabla_h \eta|^2}} - \frac{\nabla_h \eta \cdot \nabla_h \partial^\alpha \eta}{\sqrt{1 + |\nabla_h \eta|^2}} \nabla_h \eta \right) \partial_t \nabla_h \partial^\alpha \eta
= \frac{1}{2} \frac{d}{dt} \int_{\Sigma} \sigma \left( \frac{|\nabla_h \partial^\alpha \eta|^2}{\sqrt{1 + |\nabla_h \eta|^2}} - \frac{|\nabla_h \eta \cdot \nabla_h \partial^\alpha \eta|^2}{\sqrt{1 + |\nabla_h \eta|^2}} \right) - T^\alpha,
$$

(5.9)

where

$$
T^\alpha = \frac{1}{2} \int_{\Sigma} \sigma \partial_t \left( \frac{1}{\sqrt{1 + |\nabla_h \eta|^2}} \right) \left| \nabla_h \partial^\alpha \eta \right|^2 - \frac{1}{2} \int_{\Sigma} \sigma \partial_t \left( \frac{1}{\sqrt{1 + |\nabla_h \eta|^2}} \right) \left| \nabla_h \eta \cdot \nabla_h \partial^\alpha \eta \right|^2
- \int_{\Sigma} \frac{\nabla_h \eta \cdot \nabla_h \partial^\alpha \eta}{\sqrt{1 + |\nabla_h \eta|^2}} \partial_t \nabla_h \eta \cdot \nabla_h \partial^\alpha \eta.
$$

(5.10)
Consequently, in light of (5.5)–(5.9), (5.4) yields the following energy identity:

\[
\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega_-} \left( |\partial^\alpha u|^2 + |\partial^\alpha b|^2 \right) dV_t + \int_{\Omega_+} |\partial^\alpha b|^2 dV_t + \int \sigma \left( \frac{|\nabla_h \partial^\alpha \eta|^2}{\sqrt{1 + |\nabla_h \eta|^2}} - \frac{|\nabla_h \eta \cdot \nabla_h \partial^\alpha \eta|^2}{\sqrt{1 + |\nabla_h \eta|^2}} \right) \right) \\
+ \kappa \int_{\Omega_-} |\text{curl} \ \partial^\alpha b|^2 dV_t \\
= \mathcal{I}^\alpha + \int_{\Omega_+} \left( F^{1,\alpha} \cdot \partial^\alpha u + F^{3,\alpha} \cdot \partial^\alpha b + F^{4,\alpha} \cdot \text{curl} \ \partial^\alpha b \right) dV_t \\
+ \int \left( \hat{F}^{3,\alpha} \cdot \partial^\alpha \hat{b} + \partial^\alpha \hat{H} \cdot \hat{F}^{4,\alpha} \right) dV_t + \int \left( -F^{5,\alpha} \sigma \partial^\alpha H + \sigma F^{6,\alpha} \partial^\alpha \eta + F^{7,\alpha} \cdot \partial^\alpha b \right) .
\]

(5.11)

We now estimate the most delicate three remaining terms. As explained in Sect. 2, one needs to consider the cases \( \alpha_0 \leq 2N - 1 \) and \( \alpha_0 = 2N \) separately. For the case \( \alpha_0 \leq 2N - 1 \), one has that by (4.29) and (4.30),

\[
\int_{\Omega_-} \partial^\alpha p F^{2,\alpha} dV_t + \int_{\Omega_+} \partial^\alpha \hat{E} \cdot \hat{F}^{4,\alpha} dV_t - \int_{\Omega} \sigma \partial^\alpha H F^{5,\alpha} \\
\lesssim \|\partial^\alpha p\|_0 \|F^{2,\alpha}\|_0 + \|\partial^\alpha \hat{E}\|_0 \|\hat{F}^{4,\alpha}\|_0 + \|\partial^\alpha H\|_{-1/2} \|F^{5,\alpha}\|_{1/2} \\
\lesssim \sqrt{\mathcal{E}_N} \sqrt{\mathcal{E}_N} \sqrt{\mathcal{D}_N}.
\]

(5.15)

For the case \( \alpha_0 = 2N \), one integrates by parts in \( t \) to have, by (4.29) and (4.31),

\[
\int_{\Omega_+} \partial_t^{2N} \hat{E} \cdot \hat{F}^{4,(2N,0)} dV_t \\
= \frac{d}{dt} \int_{\Omega_+} \partial_t^{2N-1} \hat{E} \cdot \hat{F}^{4,(2N,0)} dV_t - \int_{\Omega_+} \partial_t^{2N-1} \hat{E} \cdot \partial_t \left( \hat{F}^{4,(2N,0)} \partial_3 \phi \right) \\
\lesssim \frac{d}{dt} \int_{\Omega_+} \partial_t^{2N-1} \hat{E} \cdot \hat{F}^{4,(2N,0)} dV_t + \|\partial_t^{2N-1} \hat{E}\|_1 \left( \|\partial_t \hat{F}^{4,(2N,0)}\|_{-1} + \|\hat{F}^{4,(2N,0)}\|_0 \right) \\
\lesssim \frac{d}{dt} \int_{\Omega_+} \partial_t^{2N-1} \hat{E} \cdot \hat{F}^{4,(2N,0)} dV_t + \sqrt{\mathcal{E}_N} \sqrt{\mathcal{E}_N + \mathcal{D}_N}.
\]

(5.17)
The treatment of the remaining two terms is more involved. The difficulty is that there is no any estimate of $\partial_t^2 p$ and so one needs to integrate by parts in $t$ for the pressure term, and also there is a $1/2$ regularity loss of $\partial_t^2 H$ so that it is insufficient to control the surface tension term. The crucial observation here is that these two terms will enjoy some cancellation by performing some careful computations. We start with the integration by parts in $t$ for the pressure term, and make use of a variant of the expression of $F^2, (2N, 0)$ defined by (4.21). Indeed, $\text{div}^\psi u = 0$ yields

$$ \text{div}^\psi u \partial_3 \phi = N \cdot \partial_3 u + \partial_3 \phi \text{div}_h u_h = 0. $$

(5.18)

Applying $\partial_t^2$ to (5.18) and using the second equation in (4.19), one gets that

$$ -\partial_3 \phi F^2, (2N, 0) = \left[ \partial_t^{2N}, -\nabla_h \vec{y} \right] \cdot \partial_3 u_h + \left[ \partial_t^{2N}, \partial_3 \phi \right] \text{div}_h u_h. $$

(5.19)

Moreover, one needs to single out the highest $2N - 1$ order time derivative terms of $u_h$ and the highest $2N$ order time derivative terms of $\eta$ as

$$ -\partial_3 \phi F^2, (2N, 0) = \sum_{i=1}^{5} F_i^{2, (2N, 0)}, $$

(5.20)

where

$$ F_1^{2, (2N, 0)} = -2N \partial_t \nabla_h \vec{y} \cdot \partial_t^{2N-1} \partial_3 u_h, $$

(5.21)

$$ F_2^{2, (2N, 0)} = 2N \partial_t \partial_3 \vec{y} \partial_t^{2N-1} \text{div}_h u_h, $$

(5.22)

$$ F_3^{2, (2N, 0)} = -\partial_t^{2N} \nabla_h \vec{y} \cdot \partial_3 u_h, $$

(5.23)

$$ F_4^{2, (2N, 0)} = \partial_t^{2N} \partial_3 \vec{y} \text{div}_h u_h, $$

(5.24)

$$ F_5^{2, (2N, 0)} = \sum_{\ell=2}^{2N-1} C_{2N}^\ell \left( -\partial_t^\ell \nabla_h \vec{y} \cdot \partial_t^{2N-\ell} \partial_3 u_h + \partial_t^\ell \partial_3 \vec{y} \partial_t^{2N-\ell} \text{div}_h u_h \right). $$

(5.25)

Accordingly,

$$ \int_{\Omega^-} \partial_t^{2N} p F^2, (2N, 0) \, d\mathcal{V} = -\sum_{i=1}^{5} \int_{\Omega^-} \partial_t^{2N} p F_i^{2, (2N, 0)}. $$

(5.26)

Integrating by parts in $t$ for the last four terms yields

$$ -\sum_{i=2}^{5} \int_{\Omega^-} \partial_t^{2N} p F_i^{2, (2N, 0)} $$

$$ = -\sum_{i=2}^{5} \frac{d}{dt} \int_{\Omega^-} \partial_t^{2N-1} p F_i^{2, (2N, 0)} + \sum_{i=2}^{5} \int_{\Omega^-} \partial_t^{2N-1} p \partial_t F_i^{2, (2N, 0)}. $$

(5.27)

One can check easily that $\left\| \partial_t F_5^{2, (2N, 0)} \right\|_0 \lesssim \sqrt{E_{N+4}E_{2N}}$ as in Lemma 4.4. Thus

$$ \int_{\Omega^-} \partial_t^{2N-1} p \partial_t F_5^{2, (2N, 0)} \leq \left\| \partial_t^{2N-1} p \right\|_0 \left\| \partial_t F_5^{2, (2N, 0)} \right\|_0 \lesssim \sqrt{E_{2N}} \sqrt{E_{N+4}E_{2N}}. $$

(5.28)
Upon an integration by parts in $x_3$ and estimating as in Lemma 4.4, one has
\[
\int_{\Omega^-} \partial_t^{2N-1} p \partial_t F_4^{2,(2N,0)}
= \int_{\Omega^-} \partial_t^{2N-1} p \left( \partial_t^{2N+1} \partial_3 \tilde{\eta} \operatorname{div}_h u_h + \partial_t^{2N} \partial_3 \tilde{\eta} \partial_t \operatorname{div}_h u_h \right)
= \int_{\Omega} \partial_t^{2N-1} p \partial_t^{2N+1} \eta \operatorname{div}_h u_h
- \int_{\Omega} \left( \partial_t^{2N+1} \eta \partial_3 \left( \partial_t^{2N-1} p \operatorname{div}_h u_h \right) + \partial_t^{2N-1} p \partial_t^{2N} \partial_3 \tilde{\eta} \partial_t \operatorname{div}_h u_h \right)
\preceq \left| \partial_t^{2N+1} \eta \right|_{1/2} \left\| \partial_t^{2N-1} p \operatorname{div}_h u_h \right\|_{H^{1/2}(\Sigma)} + \left\| \partial_t^{2N+1} \tilde{\eta} \right\|_0 \left\| \partial_3 \left( \partial_t^{2N-1} p \operatorname{div}_h u_h \right) \right\|_0
+ \left\| \partial_t^{2N} \partial_3 \tilde{\eta} \right\|_0 \left\| \partial_t^{2N-1} p \partial_t \operatorname{div}_h u_h \right\|_0
\preceq \sqrt{\mathcal{E}_{2N}} \sqrt{\mathcal{E}_{N+4}} \sqrt{\mathcal{E}_{2N}},
\tag{5.29}
\]
while by integrating by parts in $x_h$, one deduces
\[
\int_{\Omega^-} \partial_t^{2N-1} p \partial_t F_2^{2,(2N,0)} = \int_{\Omega^-} \partial_t^{2N-1} p \left( 2N \partial_t^2 \partial_3 \tilde{\eta} \partial_t^{2N-1} \operatorname{div}_h u_h + 2N \partial_t \partial_3 \tilde{\eta} \partial_t^{2N} \operatorname{div}_h u_h \right)
= \int_{\Omega^-} \left( \partial_t^{2N-1} p 2N \partial_t^2 \partial_3 \tilde{\eta} \partial_t^{2N-1} \operatorname{div}_h u_h - \nabla_h (\partial_t^{2N-1} p 2N \partial_t \partial_3 \tilde{\eta}) \cdot \partial_t^{2N} u_h \right)
\preceq \left\| \partial_t^{2N-1} p \partial_t^2 \partial_3 \tilde{\eta} \right\|_0 \left\| \partial_t^{2N-1} \operatorname{div}_h u_h \right\|_0 + \left\| \nabla_h (\partial_t^{2N-1} p \partial_t \partial_3 \tilde{\eta}) \right\|_0 \left\| \partial_t^{2N} u_h \right\|_0
\preceq \sqrt{\mathcal{E}_{2N}} \sqrt{\mathcal{E}_{N+4}} \sqrt{\mathcal{E}_{2N}}.
\tag{5.30}
\]
Similarly, one has
\[
\int_{\Omega^-} \partial_t^{2N-1} p \partial_t F_2^{3,(2N,0)} \preceq \sqrt{\mathcal{E}_{2N}} \sqrt{\mathcal{E}_{N+4}} \sqrt{\mathcal{E}_{2N}}.
\tag{5.31}
\]
It remains to deal with the most difficult term, the first term involving $F_1^{2,(2N,0)}$ in (5.26). One integrates by parts in $x_3$ first to get
\[
- \int_{\Omega^-} \partial_t^{2N} \partial_1^{2,(2N,0)}
= \int_{\Omega^-} \partial_t^{2N} p 2N \partial_t \nabla_h \eta \cdot \partial_t^{2N-1} u_h - \int_{\Omega^-} \partial_3 \left( \partial_t^{2N} p 2N \partial_t \nabla_h \eta \right) \cdot \partial_t^{2N-1} u_h.
\tag{5.32}
\]
Then integrating by parts in $t$ for the second term in the right hand side of (5.32) yields
\[
- \int_{\Omega^-} \partial_3 \left( \partial_t^{2N} p 2N \partial_t \nabla_h \eta \right) \cdot \partial_t^{2N-1} u_h
= - \frac{d}{dt} \int_{\Omega^-} \partial_3 \left( \partial_t^{2N-1} p 2N \partial_t \nabla_h \eta \right) \cdot \partial_t^{2N-1} u_h
+ \int_{\Omega^-} \left( \partial_3 \left( \partial_t^{2N-1} p 2N \partial_t \nabla_h \eta \right) \cdot \partial_t^{2N} u_h + \partial_3 \left( \partial_t^{2N-1} p 2N \partial_t^2 \nabla_h \eta \right) \cdot \partial_t^{2N-1} u_h \right)
\leq - \frac{d}{dt} \int_{\Omega^-} \partial_3 \left( \partial_t^{2N-1} p 2N \partial_t \nabla_h \eta \right) \cdot \partial_t^{2N-1} u_h + C \sqrt{\mathcal{E}_{N+4}} \sqrt{\mathcal{E}_{2N}} \sqrt{\mathcal{E}_{2N}}.
\tag{5.33}
\]
Note carefully that we integrate by parts in \( x_3 \) here first rather than in \( t \) since there are no estimates for \( \partial_t^{2N} u_h \) on the boundary. This also indicates the difficulty in controlling the first term in the right hand side of (5.32) since one can no longer integrate by parts in \( t \). Recall here that there is also another term out of control, which is the surface tension term in the right hand side of (5.11) when \( \alpha = (2N, 0) \). Our crucial observation is that there is a cancellation between them since \( \partial_t^{2N} p = -\sigma \partial_t^{2N} \eta \) on \( \Sigma \). Indeed, one has

\[
- \int_\Sigma \sigma \partial_t^{2N} H F^{5,(2N,0)} + \int_\Sigma \partial_t^{2N} p 2N \partial_t \nabla \eta \cdot \partial_t^{2N-1} u_h \\
= - \int_\Sigma \sigma \partial_t^{2N} H \left( F^{5,(2N,0)} + 2N \partial_t \nabla \eta \cdot \partial_t^{2N-1} u_h \right) \\
+ \int_\Sigma (\sigma \partial_t^{2N} H + \partial_t^{2N} p) 2N \partial_t \nabla \eta \cdot \partial_t^{2N-1} u_h \\
= \int_\Sigma \sigma \partial_t^{2N} H \left( u_h \cdot \nabla \eta \partial_t^{2N} \eta + \tilde{F}^{5,(2N,0)} \right) \quad (5.34)
\]

where

\[
\tilde{F}^{5,(2N,0)} = \sum_{\ell=2}^{2N-1} C_{2N}^\ell \partial_t^\ell \nabla \eta \cdot \partial_t^{2N-\ell} u_h. \quad (5.35)
\]

Note that \( \left| \tilde{F}^{5,(2N,0)} \right|_1 \lesssim \sqrt{\mathcal{E}_{N+4} \mathcal{E}_{2N}} \) as in Lemma 4.4. So integrating by parts in \( x_h \) yields

\[
\int_\Sigma \sigma \partial_t^{2N} H \tilde{F}^{5,(2N,0)} = - \int_\Sigma \sigma \partial_t^{2N} \left( \frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right) \cdot \nabla \tilde{F}^{5,(2N,0)} \\
\lesssim \left| \partial_t^{2N} \left( \frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right) \right|_0 \left| \tilde{F}^{5,(2N,0)} \right|_1 \lesssim \sqrt{\mathcal{E}_{2N} \sqrt{\mathcal{E}_{N+4} \mathcal{E}_{2N}}}. \quad (5.36)
\]

It follows from the ninth equation in (4.19) that

\[
\int_\Sigma \sigma \partial_t^{2N} H u_h \cdot \nabla \eta \partial_t^{2N} \eta \\
= \int_\Sigma \sigma \left( \text{div}_h \left( \frac{\nabla \eta \partial_t^{2N} \eta}{\sqrt{1 + |\nabla \eta|^2}} \right) - \frac{\nabla \eta \cdot \nabla \eta \partial_t^{2N} \eta}{\sqrt{1 + |\nabla \eta|^2}^3} \nabla \eta \right) + F^{6,(2N,0)} \right) u_h \cdot \nabla \eta \partial_t^{2N} \eta. \quad (5.37)
\]

Then (4.29) implies

\[
\int_\Sigma \sigma F^{6,(2N,0)} u_h \cdot \nabla \eta \partial_t^{2N} \eta \lesssim \left| F^{6,(2N,0)} \right|_0 \left| \partial_t^{2N} \eta \right|_1 \lesssim \sqrt{\mathcal{E}_{N+4} \mathcal{E}_{2N} \sqrt{\mathcal{E}_{2N}}}. \quad (5.38)
\]
Integrating by parts in $x_h$, one can deduce that
\[
\int_{\Sigma} \sigma \text{div}_h \left( \frac{\nabla_h \partial_t^2 \eta}{\sqrt{1 + |\nabla h|^2}} \right) u_h \cdot \nabla_h \partial_t^2 \eta = - \int_{\Sigma} \sigma \left( \frac{\nabla_h \partial_t^2 \eta}{\sqrt{1 + |\nabla h|^2}} \cdot \nabla_h u_h \cdot \nabla_h \partial_t^2 \eta - \frac{1}{2} \text{div}_h \left( \frac{u_h}{\sqrt{1 + |\nabla h|^2}} \right) |\nabla_h \partial_t^2 \eta|^2 \right)
\leq \sqrt{E_{N+4}} \left| \partial_t^2 \eta \right| L \lesssim \sqrt{E_{N+4} E_{2N}}. \tag{5.39}
\]

Similarly, one has
\[
- \int_{\Sigma} \sigma \left( \frac{\nabla h \cdot \nabla h \partial_t^2 \eta}{\sqrt{1 + |\nabla h|^2}} \nabla h \right) u_h \cdot \nabla h \partial_t^2 \eta \lesssim \sqrt{E_{N+4} E_{2N}}. \tag{5.40}
\]

Hence, it follows from (5.36)–(5.40) and (5.34) that
\[
- \int_{\Sigma} \sigma \partial_t^2 \eta F_{5,2N,0} + \int_{\Omega} \partial_t^2 \eta \text{div} (-2 \partial_t \nabla_h \partial_t^2 \eta + \partial_t \nabla_h \bar{\eta}) \partial_t^2 \eta \lesssim \sqrt{E_{N+4} E_{2N}}. \tag{5.41}
\]

This together with (5.26)–(5.33) implies that
\[
- \int_{\Sigma} \sigma \partial_t^2 \eta F_{5,2N,0} + \int_{\Omega} \partial_t^2 \eta \text{div} (-2 \partial_t \nabla_h \partial_t^2 \eta + \partial_t \nabla_h \bar{\eta}) \partial_t^2 \eta \leq - \frac{d}{dt} B_{2N} + C \sqrt{E_{N+4} E_{2N}}. \tag{5.42}
\]

where
\[
B_{2N} := \sum_{i=2}^5 \int_{\Omega} \partial_t^2 \eta \text{div} (-2 \partial_t \nabla_h \partial_t^2 \eta + \partial_t \nabla_h \bar{\eta}) \partial_t^2 \eta. \tag{5.43}
\]

As a consequence of the estimates (5.12)–(5.17) and (5.42), one deduces from (5.11) with summing over such $\alpha$ and (2.6) that, by (4.60) with $n = 2N$ and Cauchy’s inequality and then integrating in time from 0 to $t$,
\[
\hat{E}_{2N}(t) + \int_0^t \hat{D}_{2N} \lesssim \hat{E}_{2N}(0) + B_{2N}(0) - B_{2N}(t) + \int_0^t \sqrt{E_{N+4} (E_{2N} + \mathcal{D}_{2N})}. \tag{5.44}
\]

Note that $\left\| F_{i,2N,0} \right\|_0 \lesssim \sqrt{E_{N+4} E_{2N}}, \ i = 2, \ldots, 5$, as Lemma 4.4. Thus
\[
\sum_{i=2}^5 \left\| \partial_t^2 \eta \text{div} (-2 \partial_t \nabla_h \partial_t^2 \eta + \partial_t \nabla_h \bar{\eta}) \partial_t^2 \eta \right\|_0 \left\| F_{i,2N,0} \right\|_0 + \sqrt{E_{N+4} E_{2N}} \sqrt{E_{2N}} \lesssim (\mathcal{E}_{2N})^{3/2}. \tag{5.45}
\]

Then the estimate (5.3) follows. \qed
5.2. Energy evolution at N + 4, . . . , 2N − 2 levels. Now we present the following time-differential tangential energy evolution estimate, at N + 4, . . . , 2N − 2 levels.

**Proposition 5.2.** For n = N + 4, . . . , 2N − 2, it holds that

\[
\frac{d}{dt} (\mathcal{E}_n + B_n) + \mathcal{D}_n \lesssim \sqrt{E_{2N}} \mathcal{D}_n,
\]

where \( B_n \) is defined by (5.61) below and satisfies the estimate

\[
|B_n| \lesssim \sqrt{E_{2N} E_n}.
\]

**Proof.** Let \( n \) denote \( N + 4, . . . , 2N - 2 \) throughout the proof and \( \alpha \in \mathbb{N}^{1+2} \) such that \( 1 \leq |\alpha| \leq n \). The equality (5.11) in the proof of Proposition 5.1 holds also here, and we will estimate the right hand side in a quite different way from the arguments that lead to the estimates (5.12)–(5.17) and (5.42).

First, one has

\[
\mathcal{I}_\alpha \lesssim \sqrt{\mathcal{D}_{N+4}} \left| \nabla_h \partial^\alpha \eta \right|_1 \left| \nabla_h \partial^\alpha \eta \right|_{-1} \lesssim \sqrt{\mathcal{D}_{N+4}} \sqrt{E_{2N}} \sqrt{\mathcal{D}_n}.
\]

It follows from (4.45) and (4.60) that

\[
\int_{\Omega_-} \left( F^{3,\alpha} \cdot \partial^\alpha b + F^{4,\alpha} \cdot \text{curl} \partial^\alpha b \right) d\mathcal{V}_t + \int_{\Omega_+} \hat{F}^{3,\alpha} \cdot \partial^\alpha \hat{b} d\mathcal{V}_t \lesssim \left\| F^{3,\alpha} \right\|_0 \left\| \partial^\alpha b \right\|_0 + \left\| F^{4,\alpha} \right\|_0 \left\| \text{curl} \partial^\alpha b \right\|_0 + \left\| \hat{F}^{3,\alpha} \right\|_0 \left\| \partial^\alpha \hat{b} \right\|_0 \lesssim \sqrt{\mathcal{D}_{N+4}} \sqrt{E_{2N}} \sqrt{\mathcal{D}_n}
\]

and by trace theory,

\[
\int_{\Sigma} F^{7,\alpha} \cdot \partial^\alpha b \lesssim \left\| F^{7,\alpha} \right\|_0 \left\| \partial^\alpha b \right\|_{L^2(\Sigma)} \lesssim \sqrt{\mathcal{D}_{N+4}} \sqrt{E_{2N}} \sqrt{\mathcal{D}_n}.
\]

Next, we consider the terms involving \( F^{1,\alpha} \) and \( F^{6,\alpha} \). If \( |\alpha| \leq n - 1 \), then by (4.45), one has

\[
\int_{\Omega_-} F^{1,\alpha} \cdot \partial^\alpha u d\mathcal{V}_t + \int_{\Sigma} \sigma F^{6,\alpha} \partial_t \partial^\alpha \eta \lesssim \left\| F^{1,\alpha} \right\|_0 \left\| \partial^\alpha u \right\|_0 + \left\| F^{6,\alpha} \right\|_0 \left\| \partial_t \partial^\alpha \eta \right\|_0 \lesssim \sqrt{\mathcal{D}_{N+4}} \sqrt{E_{2N}} \sqrt{\mathcal{D}_n}.
\]

If \( |\alpha| = n, \alpha_1 + \alpha_2 \geq 1 \), then by (4.45), one obtains

\[
\int_{\Omega_-} F^{1,\alpha} \cdot \partial^\alpha u d\mathcal{V}_t + \int_{\Sigma} \sigma F^{6,\alpha} \partial_t \partial^\alpha \eta \lesssim \left\| F^{1,\alpha} \right\|_1 \left\| \partial^\alpha u \right\|_{-1} + \left\| F^{6,\alpha} \right\|_1 \left\| \partial_t \partial^\alpha \eta \right\|_{-1} \lesssim \sqrt{\mathcal{D}_{N+4}} \sqrt{E_{2N}} \sqrt{\mathcal{D}_n}.
\]

The remaining case is that when \( \alpha_0 = n \), and integrating by parts in \( t \) and using (4.45) show that

\[
\int_{\Omega_-} F^{1,(n,0)} \cdot \partial_t^n u d\mathcal{V}_t = \frac{d}{dt} \int_{\Omega_-} F^{1,(n,0)} \cdot \partial_t^{n-1} u d\mathcal{V}_t - \int_{\Omega_-} \partial_t \left( F^{1,(n,0)} \partial_3 \psi \right) \cdot \partial_t^{n-1} u \lesssim \frac{d}{dt} \int_{\Omega_-} F^{1,(n,0)} \cdot \partial_t^{n-1} u d\mathcal{V}_t + \left\| F^{1,(n,0)} \right\|_0 + \left\| \partial_t F^{1,(n,0)} \right\|_0 \left\| \partial_t^{n-1} u \right\|_0 \lesssim \frac{d}{dt} \int_{\Omega_-} F^{1,(n,0)} \cdot \partial_t^{n-1} u d\mathcal{V}_t + \sqrt{\mathcal{D}_{N+4}} \sqrt{E_{2N}} \sqrt{\mathcal{D}_n}
\]

(5.53)
and
\[
\int \sigma F^6,(n,0) \partial_t \partial^n \eta = \frac{d}{dt} \int \sigma F^6,(n,0) \partial^n \eta - \int \sigma \partial_t F^6,(n,0) \partial^n \eta \\
\lesssim \frac{d}{dt} \int \sigma F^6,(n,0) \partial^n \eta + \left| \partial_t F^6,(n,0) \right|_0 \left| \partial^n \eta \right|_0 \\
\lesssim \frac{d}{dt} \int \sigma F^6,(n,0) \partial^n \eta + \sqrt{\mathcal{D}_n \mathcal{E}_N^2} \sqrt{\mathcal{D}_n} 
\] (5.54)

Next, we treat the terms involving \( F^{2,0} \) and \( \hat{F}^{4,0} \). If \( |\alpha| \leq n-1, \alpha_0 \leq n-2 \), then (4.45) implies
\[
\int_{\Omega_-} \partial^\alpha p F^{2,\alpha} d\mathcal{V}_t + \int_{\Omega_+} \partial^\alpha \hat{E} \cdot \hat{F}^{4,\alpha} d\mathcal{V}_t \lesssim \left\| \partial^\alpha p \right\|_0 \left\| F^{2,\alpha} \right\|_0 + \left\| \partial^\alpha \hat{E} \right\|_0 \left\| \hat{F}^{4,\alpha} \right\|_0 \\
\lesssim \sqrt{\mathcal{D}_n \mathcal{D}_N \mathcal{E}_2 \mathcal{E}_N}. 
\] (5.55)

If \( \alpha = (n-1, 0) \), then one integrates by parts in \( t \) to get
\[
\int_{\Omega_-} \partial^{n-1}_{t} p F^{2,(n-1,0)} d\mathcal{V}_t + \int_{\Omega_+} \partial^{n-1}_{t} \hat{E} \cdot \hat{F}^{4,(n-1,0)} d\mathcal{V}_t \\
= \frac{d}{dt} \left( \int_{\Omega_-} \partial^{n-2}_{t} p F^{2,(n-1,0)} d\mathcal{V}_t + \int_{\Omega_+} \partial^{n-2}_{t} \hat{E} \cdot \hat{F}^{4,(n-1,0)} d\mathcal{V}_t \right) \\
- \int_{\Omega_-} \partial^{n-2}_{t} p \partial_t \left( F^{2,(n-1,0)} \partial_3 \varphi \right) - \int_{\Omega_+} \partial^{n-2}_{t} \hat{E} \cdot \partial_t \left( \hat{F}^{4,(n-1,0)} \partial_3 \varphi \right) \\
\lesssim \frac{d}{dt} \left( \int_{\Omega_-} \partial^{n-2}_{t} p F^{2,(n-1,0)} d\mathcal{V}_t + \int_{\Omega_+} \partial^{n-2}_{t} \hat{E} \cdot \hat{F}^{4,(n-1,0)} d\mathcal{V}_t \right) \\
+ \left\| \partial^{n-2}_{t} p \right\|_0 \left( \left\| F^{2,(n-1,0)} \right\|_0 + \left\| \partial_t F^{2,(n-1,0)} \right\|_0 \right) \\
+ \left\| \partial^{n-2}_{t} \hat{E} \right\|_0 \left( \left\| \hat{F}^{4,(n-1,0)} \right\|_0 + \left\| \partial_t \hat{F}^{4,(n-1,0)} \right\|_0 \right) \\
\lesssim \frac{d}{dt} \left( \int_{\Omega_-} \partial^{n-2}_{t} p F^{2,(n-1,0)} d\mathcal{V}_t + \int_{\Omega_+} \partial^{n-2}_{t} \hat{E} \cdot \hat{F}^{4,(n-1,0)} d\mathcal{V}_t \right) \\
+ \sqrt{\mathcal{D}_n \mathcal{D}_N \mathcal{E}_2 \mathcal{E}_N}. 
\] (5.56)

If \( |\alpha| = n \) and \( \alpha_1 + \alpha_2 \geq 2 \), then
\[
\int_{\Omega_-} \partial^\alpha p F^{2,\alpha} d\mathcal{V}_t + \int_{\Omega_+} \partial^\alpha \hat{E} \cdot \hat{F}^{4,\alpha} d\mathcal{V}_t \lesssim \left\| \partial^\alpha p \right\|_{-1} \left\| F^{2,\alpha} \right\|_1 + \left\| \partial^\alpha \hat{E} \right\|_{-1} \left\| \hat{F}^{4,\alpha} \right\|_1 \\
\lesssim \sqrt{\mathcal{D}_n \mathcal{D}_N \mathcal{E}_2 \mathcal{E}_N}. 
\] (5.57)

If \( |\alpha| = n \) and \( \alpha_1 + \alpha_2 = 1 \), then one writes \( \alpha = (n-1, 0) + \alpha' \) for \( \alpha' \in \mathbb{N}^2 \) with \( \alpha' \leq \alpha \) and \( |\alpha'| = 1 \) and then integrates by parts in \( t \) to have, by (4.45),
\[\int_{\Omega_{-}} \partial_{t}^{n-1} \partial^a p F^{2,\alpha} \, dV_l + \int_{\Omega_{+}} \partial_{t}^{n-1} \partial^a \hat{E} \cdot \hat{F}^{4,\alpha} \, dV_l \]
\[= \frac{d}{dt} \left( \int_{\Omega_{-}} \partial_{t}^{n-2} \partial^a p F^{2,\alpha} \, dV_l + \int_{\Omega_{+}} \partial_{t}^{n-2} \partial^a \hat{E} \cdot \hat{F}^{4,\alpha} \, dV_l \right) \]
\[- \int_{\Omega_{-}} \partial_{t}^{n-2} \partial^a p \partial_{t} \left( F^{2,\alpha} \partial_{3}\phi \right) - \int_{\Omega_{+}} \partial_{t}^{n-2} \partial^a \hat{E} \cdot \partial_{t} \left( \hat{F}^{4,\alpha} \partial_{3}\phi \right) \]
\[\lesssim \frac{d}{dt} \left( \int_{\Omega_{-}} \partial_{t}^{n-2} \partial^a p F^{2,\alpha} \, dV_l + \int_{\Omega_{+}} \partial_{t}^{n-2} \partial^a \hat{E} \cdot \hat{F}^{4,\alpha} \, dV_l \right) \]
\[+ \left\| \partial_{t}^{n-2} p \right\|_1 \left( \left\| F^{2,\alpha} \right\|_0 + \left\| \partial_{t} F^{2,\alpha} \right\|_0 \right) + \left\| \partial_{t}^{n-2} \hat{E} \right\|_1 \left( \left\| \hat{F}^{4,\alpha} \right\|_0 + \left\| \partial_{t} \hat{F}^{4,\alpha} \right\|_0 \right) \]
\[\lesssim \frac{d}{dt} \left( \int_{\Omega_{-}} \partial_{t}^{n-2} \partial^a p F^{2,\alpha} \, dV_l + \int_{\Omega_{+}} \partial_{t}^{n-2} \partial^a \hat{E} \cdot \hat{F}^{4,\alpha} \, dV_l \right) + \sqrt{D_n \sqrt{D_{N+4} E_{2N}}} \cdot \]

(5.58)

The remaining case, \( \alpha_0 = n \), can be handled by the integration by parts in \( t \) twice and using (4.45) as

\[\int_{\Omega_{-}} \partial_{t}^{n} p F^{2,(n,0)} \, dV_l + \int_{\Omega_{+}} \partial_{t}^{n} \hat{E} \cdot \hat{F}^{4,(n,0)} \, dV_l \]
\[= \frac{d}{dt} \left( \int_{\Omega_{-}} \partial_{t}^{n-1} p F^{2,(n,0)} \, dV_l + \int_{\Omega_{+}} \partial_{t}^{n-1} \hat{E} \cdot \hat{F}^{4,(n,0)} \, dV_l \right) \]
\[- \int_{\Omega_{-}} \partial_{t}^{n-1} p \partial_{t} \left( F^{2,(n,0)} \partial_{3}\phi \right) - \int_{\Omega_{+}} \partial_{t}^{n-1} \hat{E} \cdot \partial_{t} \left( \hat{F}^{4,(n,0)} \partial_{3}\phi \right) \]
\[= \frac{d}{dt} \left( \int_{\Omega_{-}} \partial_{t}^{n-1} p F^{2,(n,0)} \, dV_l - \int_{\Omega_{-}} \partial_{t}^{n-2} p \partial_{t} \left( F^{2,(n,0)} \partial_{3}\phi \right) \right) \]
\[+ \int_{\Omega_{+}} \partial_{t}^{n-2} \hat{E} \cdot \hat{F}^{4,(n,0)} \, dV_l - \int_{\Omega_{+}} \partial_{t}^{n-1} \hat{E} \cdot \partial_{t} \left( \hat{F}^{4,(n,0)} \partial_{3}\phi \right) \]
\[\lesssim \frac{d}{dt} \left( \int_{\Omega_{-}} \partial_{t}^{n-1} p F^{2,(n,0)} \, dV_l - \int_{\Omega_{-}} \partial_{t}^{n-2} p \partial_{t} \left( F^{2,(n,0)} \partial_{3}\phi \right) \right) \]
\[+ \left\| \partial_{t}^{n-2} p \right\|_0 \left( \left\| F^{2,(n,0)} \right\|_0 + \left\| \partial_{t} F^{2,(n,0)} \right\|_0 + \left\| \partial_{t}^2 F^{2,(n,0)} \right\|_0 \right) \]
\[+ \left\| \partial_{t}^{n-2} \hat{E} \right\|_0 \left( \left\| \hat{F}^{4,(n,0)} \right\|_0 + \left\| \partial_{t} \hat{F}^{4,(n,0)} \right\|_0 + \left\| \partial_{t}^2 \hat{F}^{4,(n,0)} \right\|_0 \right) \]
\[\lesssim \frac{d}{dt} \left( \int_{\Omega_{-}} \partial_{t}^{n-1} p F^{2,(n,0)} \, dV_l - \int_{\Omega_{-}} \partial_{t}^{n-2} p \partial_{t} \left( F^{2,(n,0)} \partial_{3}\phi \right) \right) \]
\[+ \int_{\Omega_{+}} \partial_{t}^{n-1} \hat{E} \cdot \hat{F}^{4,(n,0)} \, dV_l - \int_{\Omega_{+}} \partial_{t}^{n-2} \hat{E} \cdot \partial_{t} \left( \hat{F}^{4,(n,0)} \partial_{3}\phi \right) \]
\[+ \sqrt{D_n \sqrt{D_{N+4} E_{2N}}} \cdot \]

(5.59)
As a consequence of the estimates (5.48)–(5.59), one deduces from (5.11) with sum-
ing over $1 \leq |\alpha| \leq n$ and (2.6) that, by (4.60) and since $n \geq N + 4$,

$$
\frac{d}{dt} (\tilde{E}_n + B_n) + \tilde{D}_n \lesssim \sqrt{\mathcal{D}_{N+4} \mathcal{E}_{2N}^2 - \mathcal{D}_n + \mathcal{E}_{2N} \mathcal{D}_n} \lesssim \sqrt{\mathcal{E}_{2N} \mathcal{D}_n},
$$

(5.60)

where

$$
B_n := - \int_{\Omega_-} F^{1,(n,0)} \cdot \partial_{n-2} \partial_\tau \eta \, dV_t - \int_{\Sigma} \sigma F^{6,(n,0)} \partial_\tau \eta - \int_{\Omega_-} \partial_{n-2} \partial_\tau \sigma \partial F^{2,(n-1,0)} \, dV_t
- \int_{\Omega_-} \partial_{n-2} \partial_\tau \partial F^{2,(n,0)} \, dV_t + \int_{\Omega_-} \partial_{n-2} \partial_\tau \partial F^{2,(n,0)} \, dV_t
- \int_{\Omega_+} \partial_{n-2} \partial_\tau \partial F^{2,(n,0)} \, dV_t - \int_{\Omega_+} \partial_{n-2} \partial_\tau \partial F^{2,(n,0)} \, dV_t
+ \int_{\Omega_+} \partial_{n-2} \partial_\tau \partial F^{2,(n,0)} \, dV_t
$$

(5.61)

By (4.46), one has

$$
|B_n| \lesssim \sqrt{\mathcal{E}_{N+4} \mathcal{E}_{2N}^2 - \mathcal{E}_n} \lesssim \sqrt{\mathcal{E}_{2N} \mathcal{E}_n}.
$$

(5.62)

Then the estimates (5.46) and (5.47) follow. □

6. Improved Estimates

In this section, making use of the tangential energy evolution estimates derived in Sect. 5, we established the full energy and dissipation estimates by exploiting the important damping structure of (2.5) and some elaborate elliptic analysis.

6.1. Primary improvement of the dissipation estimates. In this subsection we first give certain improvements of the tangential dissipation $\tilde{D}_n$, defined by (5.2).

6.1.1. $H^1$-dissipation estimates of $b$ and full dissipation estimates of $\hat{b}$ One may first apply the Hodge-type estimates to improve the dissipation estimates of the magnetic fields $b$ and $\hat{b}$ from the assumed control of $\tilde{D}_n$. Define

$$
\tilde{\mathcal{D}}_n := \sum_{j=0}^n \| \partial^j b \|_{1,n-j}^2 + \sum_{j=0}^n \| \partial^j \hat{b} \|_{n-j+1}^2.
$$

(6.1)

**Proposition 6.1.** It holds that

$$
\tilde{\mathcal{D}}_{2N} \lesssim \tilde{\mathcal{D}}_{2N} + \mathcal{E}_{N+4} (\mathcal{E}_{2N} + \mathcal{D}_{2N})
$$

(6.2)

and for $n = N + 4, \ldots, 2N - 1$,

$$
\tilde{\mathcal{D}}_n \lesssim \tilde{\mathcal{D}}_n + \mathcal{D}_{N+4} \mathcal{E}_{2N}.
$$

(6.3)
Proof. Assume that \( n = N + 4, \ldots, 2N \). First, the magnetic part of (4.47) yields that

\[
\begin{aligned}
\text{curl } b &= \text{curl } b, \quad \text{div } b = G^4 \quad \text{in } \Omega_-
\text{curl } \hat{b} &= \hat{G}^3, \quad \text{div } \hat{b} = \hat{G}^4 \quad \text{in } \Omega_+
[b] &= 0 \quad \text{on } \Sigma
b_3 &= 0 \quad \text{on } \Sigma_-
\hat{b} \times e_3 &= 0 \quad \text{on } \Sigma_+.
\end{aligned}
\] (6.4)

It then follows from the Hodge-type estimates (3.11) of Proposition 3.2 (setting \( \eta = 0 \)) with \( r = 1 \) that for \( j = 0, \ldots, n \),

\[
\begin{aligned}
\| \partial_j b \|_{1,n-j}^2 + \| \partial_j \hat{b} \|_{1,n-j}^2
\lesssim
\| \partial_j \text{curl } b \|_{0,n-j}^2 + \| \partial_j G^4 \|_{0,n-j}^2 + \| \partial_j \hat{G}^3 \|_{0,n-j}^2 + \| \partial_j \hat{G}^4 \|_{0,n-j}^2.
\end{aligned}
\] (6.5)

On the other hand, employing the Hodge-type estimates (A.14) of Lemma A.9 with \( r = n - j + 1 \geq 1 \) in \( \Omega_+ \), one deduces that, by the third and fourth equations in (6.4),

\[
\begin{aligned}
\| \partial_j b \|_{n-j+1}^2
\lesssim
\| \partial_j \text{curl } b \|_{0,n-j}^2 + \| \partial_j G^4 \|_{0,n-j}^2 + \| \partial_j \hat{G}^3 \|_{n-j}^2 + \| \partial_j \hat{G}^4 \|_{n-j}^2.
\end{aligned}
\] (6.6)

It follows from (6.5) and (6.6) that

\[
\bar{D}_n \lesssim \bar{D}_n + \sum_{j=0}^n \| \partial_j G^4 \|_{n-j}^2 + \sum_{j=0}^n \| \partial_j \hat{G}^3 \|_{n-j}^2 + \sum_{j=0}^n \| \partial_j \hat{G}^4 \|_{n-j}^2.
\] (6.7)

By using (4.58) when \( n = 2N \) and (4.57) when \( n = N + 4, \ldots, 2N - 1 \), one obtains (6.2) and (6.3), respectively, from (6.7). \( \square \)

Remark 6.2. Note that one can derive the desired boundary regularity of \( b \) in the dissipation estimates. Indeed, it follows from the trace theory that

\[
\sum_{j=0}^n \| \partial_j b \|_{H^{n-j+1/2}(\Sigma \cup \Sigma_-)}^2 \lesssim \sum_{j=0}^n \| \partial_j b \|_{1,n-j}^2 \lesssim \bar{D}_n.
\] (6.8)

6.1.2. \( \hat{B} \cdot \nabla \)-dissipation estimates of \( u \) Note that by now the dissipation estimates only control the magnetic fields \( b \) and \( \hat{b} \). The dissipation estimates for the velocity \( u \) rely on the coupling between the fluid and the magnetic field and \( \hat{B}_3 \neq 0 \), and one first has the following.

Proposition 6.3. For \( n = N + 4, \ldots, 2N \), it holds that

\[
\sum_{j=0}^{n-1} \left( \| \partial_j u \|_{0,n-j-1}^2 + \| \partial_j u_3 \|_{H^{n-j-1}(\Sigma)}^2 \right) \lesssim \bar{D}_n + \bar{D}_{N+4} \mathcal{E}_{2N}.
\] (6.9)
Lemma A.7 and the tenth equation in (4.47), that

\[ \partial_t b - \kappa \Delta b = \vec{B} \cdot \nabla u + G^3 + \kappa \nabla G^4 \text{ in } \Omega_- \]  

(6.10)

By the vertical component of (6.10) and the fourth equations in (4.47), one has

\[ \vec{B} \cdot \nabla u_3 = \partial_t b_3 - \kappa \Delta b_3 - G_3^3 - \kappa \partial_3 G^4 \]
\[ = \partial_t b_3 - \kappa \Delta_h b_3 + \kappa \partial_3 \text{div}_h b_h - G_3^3 - 2\kappa \partial_3 G^4. \]  

(6.11)

It then follows from (6.11) and (4.57) that for \( j = 0, \ldots, n-1, \)

\[ \left\| \vec{B} \cdot \nabla \partial_j^i u_3 \right\|_{0,n-j-1}^2 \lesssim \left\| \partial_t^{j+1} b_3 \right\|_{0,n-j-1}^2 + \left\| \partial_t^j b \right\|_{1,n-j}^2 + \left\| \partial_t^j G_3 \right\|_{n-j-1}^2 + \left\| \partial_t^j \nabla G^4 \right\|_{n-j}^2 \]
\[ \lesssim \tilde{D}_n + {\mathcal{O}}_{N+4}{\mathcal{E}}_2N. \]  

(6.12)

This implies, since \( \vec{B}_3 \neq 0, \) by the Poincare-type inequalities (A.11) and (A.12) of Lemma A.7 and the tenth equation in (4.47), that

\[ \left\| \partial_t^j u_3 \right\|_{0,n-j-1}^2 + \left\| \partial_j^i u_3 \right\|_{H^{n-j-1}(\Sigma)}^2 \lesssim \left\| \vec{B} \cdot \nabla \partial_j^i u_3 \right\|_{0,n-j-1}^2 \lesssim \tilde{D}_n + {\mathcal{O}}_{N+4}{\mathcal{E}}_2N. \]  

(6.13)

Next, the tenth to twelfth equations in (4.47) imply

\[ \kappa \partial_3 b_h + \vec{B}_3 u_h = G_h^7 \text{ on } \Sigma_. \]  

(6.14)

This motivates one to consider the quantity \( \kappa \partial_3 b_h + \vec{B}_3 u_h. \) It then follows from the horizontal components of the third equation in (4.47) that

\[ \vec{B} \cdot \nabla (\kappa \partial_3 b_h + \vec{B}_3 u_h) = \vec{B}_h \cdot \nabla_h (\kappa \partial_3 b_h) + \vec{B}_3 (\kappa \partial_3^2 b_h + \vec{B} \cdot \nabla u_h) \]
\[ = \vec{B}_h \cdot \nabla_h (\kappa \partial_3 b_h) + \vec{B}_3 (\kappa \Delta_h b_h - \partial_t b_h + G_h^3 + \kappa \nabla_h G^4). \]  

(6.15)

(6.15) and (4.57) imply that for \( j = 0, \ldots, n-1, \)

\[ \left\| \vec{B} \cdot \nabla \partial_j^i (\kappa \partial_3 b_h + \vec{B}_3 u_h) \right\|_{0,n-j-1}^2 \lesssim \left\| \partial_j^i b_h \right\|_{1,n-j}^2 + \left\| \partial_j^{i+1} b_h \right\|_{0,n-j-1}^2 + \left\| \partial_j^i G_h^3 \right\|_{0,n-j-1}^2 + \left\| \partial_j^i \nabla G^4 \right\|_{0,n-j}^2 \]
\[ \lesssim \tilde{D}_n + {\mathcal{O}}_{N+4}{\mathcal{E}}_2N. \]  

(6.16)

By (A.11) and (A.12) again, it follows from (6.16), (6.14) and (4.57) that

\[ \left\| \partial_j^i (\kappa \partial_3 b_h + \vec{B}_3 u_h) \right\|_{0,n-j-1}^2 + \left\| \partial_j^i (\kappa \partial_3 b_h + \vec{B}_3 u_h) \right\|_{H^{n-j-1}(\Sigma_\kappa)}^2 \lesssim \left\| \vec{B} \cdot \nabla \partial_j^i (\kappa \partial_3 b_h + \vec{B}_3 u_h) \right\|_{0,n-j-1}^2 + \left\| \partial_j^i G_h^7 \right\|_{n-j-1}^2 \]
\[ \lesssim \tilde{D}_n + {\mathcal{O}}_{N+4}{\mathcal{E}}_2N. \]  

(6.17)
Hence, by (6.17) and since $\tilde{B}_3 \neq 0$ again, one has
\[
\left\| \partial_t^j u_h \right\|_{0,n-j-1}^2 \lesssim \left\| \partial_t^j \partial_3 b_h \right\|_{0,n-j-1}^2 + \left\| \partial_t^j (\kappa \partial_3 b_h + \tilde{B}_3 u_h) \right\|_{0,n-j-1}^2 \lesssim \tilde{D}_n + \mathcal{D}_{N+4} \mathcal{E}_{2N}.
\]
(6.18)

Consequently, collecting the estimates (6.12), (6.13) and (6.18) yields (6.9).

\section*{6.2. Estimates of $u$, $b$ and $\hat{b}$.}

In this subsection we will complete the estimates of the velocity $u$ and the magnetic field $b$.

\subsection*{6.2.1. Estimates of $u$, $b$ and $\hat{b}$ at the $2N$ level}

We first derive the normal estimates of $u$, $b$ and $\hat{b}$ at the $2N$ level.

\begin{proposition}
It holds that
\[
\frac{d}{dt} \left\| (\text{curl}^\nu u)_h \right\|_{2N-1}^2 + \left\| (\text{curl}^\nu u)_h \right\|_{2N-1}^2 + \sum_{j=0}^{2N} \left\| \partial_t^j u \right\|_{2N-j}^2 \\
+ \sum_{j=0}^{2N-1} \left\| \partial_t^j b \right\|_{2N-j+1}^2 + \sum_{j=0}^{2N-1} \left\| \partial_t^j \hat{b} \right\|_{2N-j+1}^2 \lesssim \tilde{E}_{2N} + \tilde{D}_{2N} + \mathcal{E}_{N+4} \mathcal{E}_{2N}
\]
and that
\[
\sum_{j=0}^{2N} \left\| \partial_t^j u \right\|_{2N-j}^2 + \sum_{j=0}^{2N-1} \left\| \partial_t^j b \right\|_{2N-j+1}^2 + \left\| \partial_t^{2N} b \right\|_0^2 + \sum_{j=0}^{2N-1} \left\| \partial_t^j \hat{b} \right\|_{2N-j+1}^2 + \left\| \partial_t^{2N} \hat{b} \right\|_0^2 \lesssim \tilde{E}_{2N} + \left\| (\text{curl}^\nu u)_h \right\|_{2N-1}^2 + \mathcal{E}_{N+4} \mathcal{E}_{2N}.
\]
\end{proposition}

\begin{proof}
Fix $\ell = 0, \ldots, 2N-1$. Let $\alpha \in \mathbb{N}^3$ with $|\alpha| \leq 2N-1$ such that $\alpha_3 \leq 2N-1-\ell$. Taking the inner product of the equations (4.66) with $\partial^\alpha (\text{curl}^\nu u)_h$ and integrating by parts over $\Omega_-$, by using the second, eighth and eleventh equations in (2.5), one obtains
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_-} |\partial^\alpha (\text{curl}^\nu u)_h|^2 dV_t + \frac{\tilde{B}_3^2}{\kappa} \int_{\Omega_-} |\partial^\alpha (\text{curl}^\nu u)_h|^2 dV_t \lesssim \left( \left\| \partial^\alpha L_h \right\|_0^2 + \left\| \Phi^\alpha_h \right\|_0 \right) \left\| \partial^\alpha (\text{curl}^\nu u)_h \right\|_0 \,
\]
(6.21)

It then follows from (6.21), Cauchy’s inequality, (4.67), (4.70) and (4.59) that
\[
\frac{d}{dt} \left\| \partial^\alpha (\text{curl}^\nu u)_h \right\|_0^2 + \left\| \partial^\alpha (\text{curl}^\nu u)_h \right\|_0^2 + \left\| \partial^\alpha (\text{curl}^\nu u)_h \right\|_0^2 \\
\lesssim \left\| \partial^\alpha L_h \right\|_0^2 + \left\| \Phi^\alpha_h \right\|_0^2 + \left\| \partial^\alpha (\text{curl}^\nu u - \text{curl} u)_h \right\|_0^2 \\
\lesssim \left\| \partial^\alpha u \right\|_{0,1}^2 + \left\| \partial^\alpha b \right\|_{1,1}^2 + \left\| \partial^\alpha \partial_\ell b \right\|_0^2 + \mathcal{E}_{N+4} \mathcal{E}_{2N}.
\]
(6.22)
Summing (6.22) over such \( \alpha \) yields
\[
\frac{d}{dt} \| (\text{curl}^{\alpha} u)_{h} \|_{2N-1-\ell,\ell}^{2} + \| (\text{curl}^{\alpha} u)_{h} \|_{2N-1-\ell,\ell}^{2} + \| (\text{curl} u)_{h} \|_{2N-1-\ell,\ell}^{2} \leq \| u \|_{2N-1-\ell+1,\ell}^{2} + \| b \|_{2N-\ell,\ell+1}^{2} + \| \partial_{t} b \|_{2N-1-\ell,\ell}^{2} + \mathcal{E}_{N+\mathcal{E}_{2N}}. \tag{6.23}
\]

On the other hand, employing the Hodge-type estimates (A.14) of Lemma A.9 with \( r = 2N - \ell \geq 1 \) and using the second equation in (4.47) and (4.57), one obtains
\[
\| u \|_{2N-\ell,\ell}^{2} \lesssim \| u \|_{0,2N-\ell+\ell}^{2} + \| (\text{curl} u)_{h} \|_{2N-1-\ell,\ell}^{2} + \| \text{div} u \|_{2N-1-\ell,\ell}^{2} \leq \| u \|_{2N}^{2} + \| (\text{curl} u)_{h} \|_{2N-1-\ell,\ell}^{2} + \| G^{2} \|_{2N-1}^{2} \lesssim \| u \|_{2N}^{2} + \| (\text{curl} u)_{h} \|_{2N-1-\ell,\ell}^{2} + \mathcal{E}_{N+\mathcal{E}_{2N}}. \tag{6.24}
\]

Then one deduces from (6.23) and (6.24) that, recalling the notation (2.12),
\[
\frac{d}{dt} \| (\text{curl}^{\alpha} u)_{h} \|_{2N-1-\ell,\ell}^{2} + \| (\text{curl}^{\alpha} u)_{h} \|_{2N-1-\ell,\ell}^{2} + \| u \|_{2N-\ell,\ell}^{2} \leq \| u \|_{2N-1-\ell+1,\ell}^{2} + \| b \|_{2N-\ell,\ell+1}^{2} + \| \partial_{t} b \|_{2N-1-\ell,\ell}^{2} + \mathcal{E}_{N+\mathcal{E}_{2N}}. \tag{6.25}
\]

Noting (6.8) in Remark 6.2, we consider the following elliptic problem ((6.10)):
\[
\begin{aligned}
-\kappa \Delta b &= \tilde{B} \cdot \nabla u - \partial_{t} b + G^{3} + \kappa \nabla G^{4} \quad \text{in } \Omega_{-} \\
b &= b \quad \text{on } \Sigma \cup \Sigma_{-}
\end{aligned}
\tag{6.26}
\]

It then follows from the standard \( H^{r} \) elliptic estimates with \( r = 2N - \ell + 1 \geq 2 \), (4.57) and (6.8) with \( n = 2N \) that
\[
\| b \|_{2N-\ell+1,\ell}^{2} \lesssim \| u \|_{2N-\ell,\ell}^{2} + \| \partial_{t} b \|_{2N-1-\ell,\ell}^{2} + \| G^{2} \|_{2N-1-\ell,\ell}^{2} + \| b \|_{H^{2N+1/2}(\Sigma \cup \Sigma_{-})}^{2} + \mathcal{E}_{2N} + \mathcal{E}_{N+\mathcal{E}_{2N}}. \tag{6.27}
\]

Then it follows from (6.25) and (6.27) that
\[
\frac{d}{dt} \| (\text{curl}^{\alpha} u)_{h} \|_{2N-1-\ell,\ell}^{2} + \| (\text{curl}^{\alpha} u)_{h} \|_{2N-1-\ell,\ell}^{2} + \| u \|_{2N-\ell,\ell}^{2} + \| b \|_{2N-\ell+1,\ell}^{2} \leq \| u \|_{2N-1-\ell+1,\ell}^{2} + \| b \|_{2N-\ell,\ell+1}^{2} + \mathcal{E}_{2N} + \mathcal{E}_{N+\mathcal{E}_{2N}}. \tag{6.28}
\]

A suitable linear combination of (6.28) for \( \ell = 0, \ldots, 2N - 1 \) yields that, recalling the conventional notation (2.29) and the definition of \( \mathcal{D}_{2N}, \)
\[
\frac{d}{dt} \| (\text{curl}^{\alpha} u)_{h} \|_{2N-1}^{2} + \| (\text{curl}^{\alpha} u)_{h} \|_{2N-1}^{2} + \| u \|_{2N}^{2} + \| b \|_{2N+1}^{2} \lesssim \| u \|_{2N}^{2} + \| b \|_{2N}^{2} + \| \partial_{t} b \|_{2N-1}^{2} + \mathcal{D}_{2N} + \mathcal{E}_{N+\mathcal{E}_{2N}} \lesssim \| u \|_{2N}^{2} + \| \partial_{t} b \|_{2N-1}^{2} + \mathcal{D}_{2N} + \mathcal{E}_{N+\mathcal{E}_{2N}}. \tag{6.29}
\]

Next, applying curl to the first equation in (4.47) yields
\[
\partial_{t} \text{curl } u = \tilde{B} \cdot \nabla \text{curl } b + \text{curl } G^{1}. \tag{6.30}
\]
For \( j = 1, \ldots, 2N - 1 \), employing the Hodge-type estimates (A.14) of Lemma A.9 with \( r = 2N - j \geq 1 \), by (6.30), the second equation in (4.47) and (4.57), one obtains

\[
\left\| \partial_j^r u \right\|_{2N-j}^2 \lesssim \left\| \partial_j^r u \right\|_{0,2N-j}^2 + \left\| \partial_j^{r+1} b \right\|_{2N-j}^2 + \sum_{j=N}^{2N-1} \left\| \partial_j^r G^j \right\|_{2N-j}^2 + \left\| \partial_j^r G^2 \right\|_{2N-j}^2 + \mathcal{E}_{N+4} \mathcal{E}_{2N}. \tag{6.31}
\]

On the other hand, applying \( \partial_j^r \), \( j = 1, \ldots, 2N - 1 \), to the problem (6.26) and the standard \( H^r \) elliptic estimates with \( r = 2N - j + 1 \geq 2 \), (4.57) and (6.8) with \( n = 2N \) show that

\[
\left\| \partial_j^r b \right\|_{2N-j+1}^2 \lesssim \left\| \partial_j^r u \right\|_{2N-j}^2 + \sum_{j=N}^{2N-1} \left\| \partial_j^{r+1} b \right\|_{2N-j}^2 + \sum_{j=0}^{N} \left\| \partial_j^r G^j \right\|_{2N-j}^2 + \sum_{j=0}^{N} \left\| \partial_j^r G^2 \right\|_{2N-j}^2 + \tilde{D}_{2N} + \mathcal{E}_{N+4} \mathcal{E}_{2N}. \tag{6.32}
\]

Combining (6.31) and (6.32) and then summing over \( j = 1, \ldots, 2N - 1 \) yield that

\[
\sum_{j=1}^{2N-1} \left\| \partial_j^r u \right\|_{2N-j}^2 + \sum_{j=1}^{2N-1} \left\| \partial_j^r b \right\|_{2N-j+1}^2 \lesssim \sum_{j=1}^{2N-1} \left\| \partial_j^r u \right\|_{0,2N-j}^2 + \sum_{j=0}^{N} \left\| \partial_j^r b \right\|_{2N-j}^2 + \tilde{D}_{2N} + \mathcal{E}_{N+4} \mathcal{E}_{2N}. \tag{6.33}
\]

Now combining (6.29) and (6.33) leads to

\[
\frac{d}{dt} \left\| \text{curl} u \right\|_{2N-1}^2 + \left\| \text{curl} u \right\|_{2N-1}^2 \lesssim \sum_{j=0}^{2N-1} \left\| \partial_j^r u \right\|_{0,2N-j}^2 + \sum_{j=0}^{N} \left\| \partial_j^r b \right\|_{2N-j}^2 + \tilde{D}_{2N} + \mathcal{E}_{N+4} \mathcal{E}_{2N}. \tag{6.34}
\]

This together with the Sobolev interpolation implies that

\[
\frac{d}{dt} \left\| \text{curl} u \right\|_{2N-1}^2 + \left\| \text{curl} u \right\|_{2N-1}^2 \lesssim \sum_{j=0}^{2N-1} \left\| \partial_j^r u \right\|_{0,2N-j}^2 + \sum_{j=0}^{N} \left\| \partial_j^r b \right\|_{2N-j}^2 + \tilde{D}_{2N} + \mathcal{E}_{N+4} \mathcal{E}_{2N}
\]

\[
\lesssim \sum_{j=0}^{2N-1} \left\| \partial_j^r u \right\|_{2N-j}^2 + \tilde{D}_{2N} + \mathcal{E}_{N+4} \mathcal{E}_{2N}. \tag{6.35}
\]
Finally, since $\sum_{j=0}^{2N-1} \| \partial_j^j u \|_{0,2N-j}^2 \lesssim \tilde{E}_{2N}$, (6.35) yields the estimate (6.19) as $\| \partial_{2N}^N u \|_0^2 \leq \tilde{E}_{2N}$ and $\| \partial_{2N}^N b \|_0^2 + \sum_{j=0}^{2N} \| \partial_j^j b \|_{2N-j+1}^2 \leq \tilde{D}_{2N}$.

We now prove (6.20). First, one recalls from (6.24) with $\ell = 0$ that

$$\| u \|_{2N}^2 \lesssim \| u \|_{0,2N}^2 + \| (\text{curl} u)_h \|_{2N-1}^2 + \mathcal{E}_{N+4} \mathcal{E}_{2N}$$

and from (6.31) that for $j = 1, \ldots, 2N-1$,

$$\| \partial_j^j u \|_{2N-j}^2 \lesssim \| \partial_j^j u \|_{0,2N-j}^2 + \| \partial_j^{j-1} b \|_{2N-(j-1)}^2 + \mathcal{E}_{N+4} \mathcal{E}_{2N}.$$  

Now consider the following two-phase elliptic problem, which follows from (4.47),

$$\begin{cases}
\kappa \text{curl curl } b = \hat{B} \cdot \nabla u - \partial_t b + G^3 & \text{in } \Omega_-
\
\text{div } b = G^4 & \text{in } \Omega_-
\
\text{curl } \hat{b} = \hat{G}^3, \text{ div } \hat{b} = \hat{G}^4 & \text{in } \Omega_+
\
[b] = 0 & \text{on } \Sigma
\
b_3 = 0, \quad \kappa \text{curl } b \times e_3 = (u \times \hat{B}) \times e_3 + G^7 & \text{on } \Sigma_-
\
b \times e_3 = 0 & \text{on } \Sigma_+.
\end{cases}$$

Applying $\partial_j^j$, $j = 0, \ldots, 2N-1$ to (6.38) and using the Hodge-type estimates (3.22) of Proposition 3.3 (with $\eta = 0$) with $r = 2N - j + 1 \geq 2$, (4.57) and the trace theory, one can get

$$\begin{align*}
\| \partial_j^j b \|_{2N-j+1}^2 + \| \partial_j^j \hat{b} \|_{2N-j+1}^2 \\
\lesssim \| \partial_j^j u \|_{2N-j}^2 + \| \partial_j^{j+1} b \|_{2N-j-1}^2 + \| \partial_j^j G^3 \|_{2N-j-1}^2 + \| \partial_j^j G^4 \|_{2N-j}^2 \\
+ \| \partial_j^j \hat{G}^3 \|_{2N-j}^2 + \| \partial_j^j \hat{G}^4 \|_{2N-j}^2 + \| \partial_j^j u \|_{H^{2N-j-1/2}(\Sigma_-)}^2 + \| \partial_j^j G^7 \|_{2N-j-1/2}^2 \\
\lesssim \| \partial_j^j u \|_{2N-j}^2 + \| \partial_j^{j+1} b \|_{2N-(j+1)}^2 + \mathcal{E}_{N+4} \mathcal{E}_{2N}.
\end{align*}$$

Hence, collecting (6.36), (6.37) for $j = 1, \ldots, 2N-1$ and (6.39) for $j = 0, \ldots, 2N-1$ leads to

$$\begin{align*}
\sum_{j=0}^{2N-1} \| \partial_j^j u \|_{2N-j}^2 + \sum_{j=0}^{2N-1} \| \partial_j^j b \|_{2N-j+1}^2 + \sum_{j=0}^{2N-1} \| \partial_j^j \hat{b} \|_{2N-j+1}^2 \\
\lesssim \sum_{j=0}^{2N-1} \| \partial_j^j u \|_{0,2N-j}^2 + \sum_{j=0}^{2N} \| \partial_j^j b \|_{2N-j}^2 + \| (\text{curl} u)_h \|_{2N-1}^2 + \mathcal{E}_{N+4} \mathcal{E}_{2N} \\
\lesssim \tilde{E}_{2N} + \sum_{j=0}^{2N} \| \partial_j^j b \|_{2N-j}^2 + \| (\text{curl} u)_h \|_{2N-1}^2 + \mathcal{E}_{N+4} \mathcal{E}_{2N}.
\end{align*}$$
This together with the Sobolev interpolation implies that

\[
\sum_{j=0}^{2N-1} \left\| \partial_j^t u \right\|_{2N-j}^2 + \sum_{j=0}^{2N-1} \left\| \partial_j^t \hat{b} \right\|_{2N-j+1}^2 + \sum_{j=0}^{2N-1} \left\| \partial_j^t b \right\|_{2N-j+1}^2 
\]

\[
\lesssim \tilde{E}_{2N} + \sum_{j=0}^{2N} \left\| \partial_j^t \hat{b} \right\|_0^2 + \| (\text{curl}^\varphi u)_h \|_{2N-1}^2 + \mathcal{E}_{N+4} \mathcal{E}_{2N} 
\]

\[
\lesssim \tilde{E}_{2N} + \| (\text{curl}^\varphi u)_h \|_{2N-1}^2 + \mathcal{E}_{N+4} \mathcal{E}_{2N}. 
\] (6.41)

This yields the estimate (6.20) as \( \left\| \partial_j^{2N} u \right\|_0^2 + \left\| \partial_j^{2N} b \right\|_0^2 + \left\| \partial_j^{2N} \hat{b} \right\|_0^2 \leq \tilde{E}_{2N}. \)

6.2.2. Estimates at \( N+4, \cdots, 2N \) levels We now derive the following energy-dissipation estimates of \( u, b \) and \( \hat{b} \) at \( N+4, \cdots, 2N \) levels.

**Proposition 6.5.** For \( n = N+4, \ldots, 2N \), it holds that

\[
\frac{d}{dt} \left( \| (\text{curl}^\varphi u)_h \|_{n-2}^2 + \sum_{j=0}^{n-1} \left\| \partial_j^t u \right\|_{n-j-1}^2 + \sum_{j=0}^{n-2} \left\| \partial_j^t b \right\|_{n-j}^2 + \sum_{j=0}^{n} \left\| \partial_j^t \hat{b} \right\|_{n-j+1}^2 \right) 
\]

\[
\lesssim \tilde{\mathcal{D}}_n + \mathcal{D}_{N+4} \mathcal{E}_{2N} 
\] (6.42)

and that

\[
\| u \|_{n-1}^2 + \left\| u \right\|_{0,n}^2 + \sum_{j=1}^{n-1} \left\| \partial_j^t u \right\|_{n-j}^2 + \| b \|_{n}^2 + \sum_{j=1}^{n-1} \left\| \partial_j^t b \right\|_{n-j+1}^2 + \left\| \partial_j^t \hat{b} \right\|_{n-j+1}^2 
\]

\[
+ \left\| \partial_j^t \hat{b} \right\|_{n-j+1}^2 + \left\| \partial_j^t \hat{b} \right\|_{n-j+1}^2 \lesssim \tilde{E}_n + \| (\text{curl}^\varphi u)_h \|_{n-2}^2 + \mathcal{E}_{N+4} \mathcal{E}_{2N}. 
\] (6.43)

**Proof.** Assume that \( n = N+4, \ldots, 2N \). It follows similarly as the derivation of (6.35) in the proof of Proposition 6.4, with \( 2N \) replaced by \( n - 1 \) in (6.29), (6.31) and (6.32), that

\[
\frac{d}{dt} \left( \| (\text{curl}^\varphi u)_h \|_{n-2}^2 + \sum_{j=0}^{n-2} \left\| \partial_j^t u \right\|_{n-j-1}^2 + \sum_{j=0}^{n-2} \left\| \partial_j^t b \right\|_{n-j}^2 \right) 
\]

\[
\lesssim \sum_{j=0}^{n-2} \left\| \partial_j^t u \right\|_{0,n-j-1}^2 + \tilde{\mathcal{D}}_{n-1} + \mathcal{D}_{N+4} \mathcal{E}_{2N}. 
\] (6.44)

Different from the derivation of (6.19), here one can estimate the term \( \sum_{j=0}^{n-2} \| \partial_j^t u \|_{0,n-j-1}^2 \) in the right hand side of (6.44) by the dissipation rather than the energy. One thus appeals to (6.9) to obtain the estimate (6.42) from (6.44) by the definition of \( \tilde{\mathcal{D}}_n \).
We now prove (6.43). It follows similarly as the derivation of (6.41), with $2N$ replaced by $n - 1$ in (6.36) and (6.39) with $j = 0$, and $2N$ replaced by $n$ in (6.37) and (6.39) with $j = 1, \ldots, n - 1$, that
\[
\left\| u \right\|_{n-1}^2 + \sum_{j=1}^{n-1} \left\| \partial^j_t u \right\|_{n-j}^2 + \left\| b \right\|_n^2 + \sum_{j=1}^{n-1} \left\| \partial^j_t b \right\|_{n-j+1}^2 + \left\| \tilde{b} \right\|_n^2 + \sum_{j=1}^{n-1} \left\| \partial^j_t \tilde{b} \right\|_{n-j+1}^2
\leq \tilde{E}_n + \left\| (\text{curl}^p u)_h \right\|_{n-2}^2 + \mathcal{E}_{N+4}\mathcal{E}_{2N}.
\] (6.45)
This yields the estimate (6.43) by noting that $\left\| u \right\|_{0,n}^2 + \left\| \partial^m_t u \right\|_0^2 + \left\| \partial^m_t b \right\|_0^2 + \left\| \partial^m_t \tilde{b} \right\|_0^2 \leq \tilde{E}_n$. \hfill \square

6.3. Estimates of $p$ and $\eta$. In this subsection we shall complete the estimates on the pressure $p$ and the free interface function $\eta$.

6.3.1. Energy. We begin with the estimates in the energy.

**Proposition 6.6.** It holds that for $n = N + 4, \ldots, 2N$,
\[
\sum_{j=0}^{n-1} \left\| \partial^j_t p \right\|_{n-j}^2 + \sum_{j=0}^{n-1} \left| \partial^j_t \eta \right|_{n-j+3/2}^2 + \left| \partial^m_t \eta \right|_{1}^2 + \left| \partial^{m+1}_t \eta \right|_{-1/2}^2
\leq \tilde{E}_n + \sum_{j=1}^{n} \left\| \partial^j_t u \right\|_{n-j}^2 + \sum_{j=0}^{n-1} \left\| \partial^j_t b \right\|_{n-j}^2 + \mathcal{E}_{N+4}\mathcal{E}_{2N}.
\] (6.46)

**Proof.** Assume that $n = N + 4, \ldots, 2N$. Let $\mathcal{X}_n$ denote the two sums in the right hand side of (6.46). It follows from the first equation in (4.47) and (4.57) that for $j = 0, \ldots, n - 1$,\[
\nabla \partial^j_t p \right\|_{n-j-1}^2 \leq \left\| \partial^{j+1}_t u \right\|_{n-j-1}^2 + \left\| \partial^j_t b \right\|_{n-j}^2 + \left\| \partial^j_t G^1 \right\|_{n-j-1}^2 \leq \mathcal{X}_n + \mathcal{E}_{N+4}\mathcal{E}_{2N}.
\] (6.47)
By the eighth equation in (4.47) and (4.57), one obtains that for $j = 0, \ldots, n - 1$,
\[
\left\| \partial^j_t p \right\|_{L^2(\Sigma)}^2 \leq \left\| \partial^j_t \eta \right\|_{2}^2 + \left\| \partial^j_t G^6 \right\|_{0}^2 \leq \tilde{E}_n + \mathcal{E}_{N+4}\mathcal{E}_{2N}.
\] (6.48)
It then follows from (6.47) and (6.48) that for $j = 0, \ldots, n - 1$, by Poincaré’s inequality,
\[
\left\| \partial^j_t p \right\|_{n-j}^2 \leq \nabla \partial^j_t p \right\|_{n-j-1}^2 + \left\| \partial^j_t p \right\|_{L^2(\Sigma)}^2 \leq \tilde{E}_n + \mathcal{X}_n + \mathcal{E}_{N+4}\mathcal{E}_{2N}.
\] (6.49)
Now, we improve the estimates of $\eta$ by using the estimates of $p$ derived in (6.49). The standard elliptic theory on the eighth equation in (4.47) yields that for $j = 0, \ldots, n - 1$, by the trace theory, (4.57) and (6.49),
\[
\left| \partial^j_t \eta \right|_{n-j+3/2}^2 \leq \left| \partial^j_t \eta \right|_{0}^2 + \left\| \partial^j_t p \right\|_{H^{n-j-1/2}(\Sigma)}^2 + \left| \partial^j_t G^6 \right|_{n-j-1/2}^2
\leq \tilde{E}_n + \left\| \partial^j_t p \right\|_{n-j}^2 + \mathcal{E}_{N+4}\mathcal{E}_{2N} \leq \tilde{E}_n + \mathcal{X}_n + \mathcal{E}_{N+4}\mathcal{E}_{2N}.
\] (6.50)
Finally, using the normal trace estimate (A.13) of Lemma A.8, by the second equation in (4.19) and (4.29), one obtains
\[ \| \partial_t^n u \cdot \mathcal{N} \|^2_{H^{-1/2}(\Sigma)} \lesssim \| \partial_t^n u \|^2_0 + \| \text{div}^\theta \partial_t^n u \|^2_0 \lesssim \tilde{\mathcal{E}}_n + \| F^{2,(n,0)} \|^2_0 \lesssim \mathcal{E}_{N+4}\mathcal{E}_{2N}. \] (6.51)

It then follows from the seventh equation in (4.19), (6.51) and (4.29) that
\[ \left| \partial_t^{n+1} \eta \right|^2_{-1/2} \lesssim \| \partial_t^n u \cdot \mathcal{N} \|^2_{H^{-1/2}(\Sigma)} + \left| F^{5,(n,0)} \right|^2_{-1/2} \lesssim \tilde{\mathcal{E}}_n + \mathcal{E}_{N+4}\mathcal{E}_{2N}. \] (6.52)

Hence, summing (6.49), (6.50) over \( j = 0, \ldots, n - 1 \) and (6.52) yields the estimate (6.46) due to that \( \left| \partial_t^n \eta \right|^2_1 \leq \tilde{\mathcal{E}}_n. \) \( \square \)

6.3.2. Dissipation
Now we consider the estimates in the dissipation.

**Proposition 6.7.** For \( n = N + 4, \ldots, 2N, \) it holds that
\[ \sum_{j=0}^{n-2} \left\| \partial_t^j p \right\|^2_{n-j-1} + \sum_{j=0}^{n-2} \left| \partial_t^j \eta \right|^2_{n-j+1/2} + \left| \partial_t^n \eta \right|^2_0 \lesssim \mathcal{D}_n + \sum_{j=1}^{n-1} \left\| \partial_t^j u \right\|^2_{n-j-1} + \sum_{j=0}^{n-2} \left\| \partial_t^j b \right\|^2_{n-j-1} + \mathcal{D}_{N+4}\mathcal{E}_{2N}. \] (6.53)

**Proof.** Assume that \( n = N+4, \ldots, 2N. \) Let \( \mathcal{Y}_n \) denote the two sums in the right hand side of (6.53). It follows from the first equation in (4.47) and (4.57) that for \( j = 0, \ldots, n - 2, \)
\[ \left\| \nabla \partial_t^j p \right\|^2_{n-j-2} \lesssim \left| \partial_t^{j+1} u \right|^2_{n-j-2} + \left| \partial_t^j b \right|^2_{n-j-1} + \left| \partial_t^j G^1 \right|^2_{n-j-2} \lesssim \mathcal{Y}_n + \mathcal{D}_{N+4}\mathcal{E}_{2N}. \] (6.54)

To estimate \( \partial_t^j p \) on the interface \( \Sigma, \) one may estimate \( \eta \) first. Indeed, by the seventh equation in (4.47), (6.9) and (4.57), one obtains that for \( j = 1, \ldots, n, \)
\[ \left| \partial_t^j \eta \right|^2_{n-j} \lesssim \left| \partial_t^{j-1} u_3 \right|^2_{H^{n-j}(\Sigma)} + \left| \partial_t^{j-1} G^5 \right|^2_{n-j} = \left| \partial_t^{j-1} u_3 \right|^2_{H^{n-(j-1)-1}(\Sigma)} + \left| \partial_t^{j-1} G^5 \right|^2_{n-(j-1)-1} \lesssim \mathcal{D}_n + \mathcal{D}_{N+4}\mathcal{E}_{2N}. \] (6.55)

Hence by the eighth equation in (4.47), (6.55) and (4.57), one has that for \( j = 1, \ldots, n - 2, \)
\[ \left\| \partial_t^j p \right\|^2_{L^2(\Sigma)} \lesssim \left| \partial_t^{j} \eta \right|^2_2 + \left| \partial_t^j G^6 \right|^2_0 \lesssim \mathcal{D}_n + \mathcal{D}_{N+4}\mathcal{E}_{2N}. \] (6.56)

For \( \| p \|^2_{L^2(\Sigma)} \), a different argument is needed since one has not controlled \( \| \eta \|^2_0 \) yet. Note that by the trace theory, the estimate (6.54) with \( j = 0 \) implies in particular that
\[ \left\| \nabla_h p \right\|^2_{H^{n-5/2}(\Sigma)} \lesssim \left\| \nabla_h p \right\|^2_{n-2} \lesssim \left\| \partial_t u \right\|^2_{n-2} + \left\| b \right\|^2_{n-1} + \mathcal{D}_{N+4}\mathcal{E}_{2N}. \] (6.57)
Then it follows from the eighth equation in (4.47), (6.57) and (4.57) that
\[ |\nabla h \eta|_{n-1/2}^2 \lesssim \|\nabla h p\|_{H^{n-5/2}(\Sigma)}^2 + \left| \nabla h G^6 \right|_{n-5/2}^2 \lesssim \tilde{\mathcal{D}}_n + \mathcal{Y}_n + \mathcal{D}_{N+4}\mathcal{E}_{2N}. \] (6.58)
Since \( \int_{\Sigma} \eta = 0 \), so Poincaré’s inequality and (6.58) yield
\[ |\eta|_{n+1/2}^2 \lesssim |\nabla h \eta|_{n-1/2}^2 \lesssim \tilde{\mathcal{D}}_n + \mathcal{Y}_n + \mathcal{D}_{N+4}\mathcal{E}_{2N}, \] (6.59)
which in turn implies that, by the eighth equation in (4.47) and (4.57) again,
\[ \|p\|_{L^2(\Sigma)}^2 \lesssim |\eta|_2^2 + \left| G^6 \right|_0^2 \lesssim \tilde{\mathcal{D}}_n + \mathcal{Y}_n + \mathcal{D}_{N+4}\mathcal{E}_{2N}. \] (6.60)
Now by Poincaré’s inequality (A.11), one deduces from (6.54), (6.56) and (6.60) that
\[ \left\| \partial_j t p \right\|_{2}^2 \lesssim \left\| \nabla \partial_j t p \right\|_{2}^2 + \left\| \partial_j t p \right\|_{L^2(\Sigma)}^2 \lesssim \tilde{\mathcal{D}}_n + \mathcal{Y}_n + \mathcal{D}_{N+4}\mathcal{E}_{2N}. \] (6.61)
This in turn, by the trace theory, the eighth equation in (4.47) and (4.57), improves the estimates of \( \partial_j t \eta \) so that for \( j = 1, \ldots, n-2 \), it holds that
\[ \left| \partial_j t \eta \right|_{n-j+1/2}^2 \lesssim \left\| \partial_j t p \right\|_{H^{n-j-3/2}(\Sigma)}^2 + \left| \partial_j t G^6 \right|_{n-j-3/2}^2 \lesssim \left\| \partial_j t p \right\|_{n-j-1}^2 + \mathcal{D}_{N+4}\mathcal{E}_{2N} \lesssim \tilde{\mathcal{D}}_n + \mathcal{Y}_n + \mathcal{D}_{N+4}\mathcal{E}_{2N}. \] (6.62)
Consequently, collecting (6.61) with \( j = 0, \ldots, n-2 \), (6.59), (6.62) with \( j = 1, \ldots, n-2 \) and (6.55) with \( j = n-1 \) and \( n \) yields the estimate (6.53). \( \square \)

7. Global Energy Estimates

In this section we will derive the global-in-time full energy estimates by making use of the estimates derived in Sects. 5 and 6. Set
\[ \mathcal{E}^w_{N+4}(t) := \sup_{0 \leq s \leq t} (1 + s)^{N-5} \mathcal{E}_{N+4}(s). \] (7.1)

7.1. Boundedness estimates of \( \mathcal{E}_{2N} \) and \( \mathcal{D}_{2N} \). We first show the the boundedness of \( \mathcal{E}_{2N} \) and \( \mathcal{D}_{2N} \).

**Theorem 7.1.** Let \( N \geq 8 \). There exists a universal constant \( \delta > 0 \) such that if
\[ \mathcal{E}_{2N}(t) \leq \delta, \quad \forall t \in [0, T], \] (7.2)
then
\[ \mathcal{E}_{2N}(t) + \int_0^t \mathcal{D}_{2N}(s) \, ds \lesssim \mathcal{E}_{2N}(0) + \sqrt{\sup_{0 \leq s \leq t} \mathcal{E}_{2N}(s) \mathcal{E}^w_{N+4}(t)}, \quad \forall t \in [0, T]. \] (7.3)
Proof. First, the estimate (5.3) and Cauchy’s inequality imply that for $N \geq 8$,
\[
\tilde{E}_{2N}(t) + \int_{0}^{t} \tilde{D}_{2N}(s) \, ds \\
\leq E_{2N}(0) + (E_{2N}(t))^{3/2} + \sup_{0 \leq s \leq t} E_{2N}(s) \sqrt{e_{N+4}^{w}(t)} \int_{0}^{t} (1 + s)^{-(N-5)/2} \, ds \\
+ \sqrt{\sup_{0 \leq s \leq t} E_{2N}(s)} \int_{0}^{t} D_{2N}(s) \, ds \\
\leq E_{2N}(0) + \max_{0 \leq s \leq t} E_{2N}(s) + \int_{0}^{t} D_{2N}(s) \, ds.
\]
(7.4)

Next, it follows from the estimates (6.19) and (6.2) that
\[
\frac{d}{dt} \| (\text{curl} u)_{h} \|_{2}^{2} + \| (\text{curl} u)_{h} \|_{2}^{2} \leq \tilde{E}_{2N} + \tilde{D}_{2N} + \mathcal{E}_{N+4}(E_{2N} + D_{2N}).
\]
(7.5)

A Gronwall type argument on (7.5) yields
\[
\| (\text{curl} u)_{h}(t) \|_{2}^{2} \leq E_{2N}(0) + \max_{0 \leq s \leq t} E_{2N}(s) + \int_{0}^{t} \tilde{D}_{2N}(s) \, ds \\
+ \max_{0 \leq s \leq t} E_{2N}(s) + \int_{0}^{t} D_{2N}(s) \, ds.
\]
(7.6)

Now combining the estimates (6.20) and (6.46) with $n = 2N$ yields
\[
E_{2N} \leq \tilde{E}_{2N} + \| (\text{curl} u)_{h} \|_{2}^{2} + \mathcal{E}_{N+4} E_{2N}.
\]
(7.7)

Hence, it follows from (7.4), (7.6) and (7.7) that for $E_{2N} \leq \delta$ small,
\[
\sup_{0 \leq s \leq t} E_{2N}(s) + \int_{0}^{t} \tilde{D}_{2N}(s) \, ds \leq E_{2N}(0) + \sqrt{\sup_{0 \leq s \leq t} E_{2N}(s)} \left( \int_{0}^{t} D_{2N}(s) \, ds + \mathcal{E}_{N+4}^{w}(t) \right).
\]
(7.8)

On the other hand, taking $n = 2N$ in the estimates (6.42) and (6.53) and using the estimate (6.2), one deduces that for $E_{2N}$ small,
\[
\frac{d}{dt} \| (\text{curl} u)_{h} \|_{2}^{2} + \mathcal{D}_{2N} \leq \tilde{D}_{2N} + \mathcal{E}_{N+4} E_{2N} + \mathcal{D}_{N+4} E_{2N}.
\]
(7.9)

which implies
\[
\frac{d}{dt} \| (\text{curl} u)_{h} \|_{2}^{2} + \mathcal{D}_{2N} \leq \tilde{D}_{2N} + \mathcal{E}_{N+4} E_{2N}.
\]
(7.10)

Integrating (7.10) in time gives in particular that
\[
\int_{0}^{t} \mathcal{D}_{2N}(s) \, ds \leq E_{2N}(0) + \int_{0}^{t} \tilde{D}_{2N}(s) \, ds + \sup_{0 \leq s \leq t} E_{2N}(s) \mathcal{E}_{N+4}(t).
\]
(7.11)
Hence, one may improve (7.8) to be, since $\mathcal{E}_{2N} \leq \delta$ is small,

$$\sup_{0 \leq s \leq t} \mathcal{E}_{2N}(s) + \int_0^t \mathcal{D}_{2N}(s) \, ds \lesssim \mathcal{E}_{2N}(0) + \sqrt{\sup_{0 \leq s \leq t} \mathcal{E}_{2N}(s)} \mathcal{E}_{N+4}(t).$$

(7.12)

This implies (7.3). □

7.2. Decay estimates of $\mathcal{E}_n$ and $\mathcal{D}_n$. Next we derive the energy-dissipation estimates with respect to $\mathcal{E}_n$ and $\mathcal{D}_n$ and show the decay estimates for $n = N + 4, \ldots, 2N - 2$.

**Theorem 7.2.** Let $N \geq 8$. There exists a universal constant $\delta > 0$ such that if

$$\mathcal{E}_{2N}(t) \leq \delta, \quad \forall t \in [0, T],$$

(7.13)

then

$$\sum_{j=0}^{N-6} (1 + t)^{N-5-j} \mathcal{E}_{N+4+j}(t) + \int_0^t (1 + s)^{N-5-j} \mathcal{D}_{N+4+j}(s) \, ds \lesssim \mathcal{E}_{2N}(0) + \int_0^t \mathcal{D}_{2N}(s) \, ds, \quad \forall t \in [0, T].$$

(7.14)

**Proof.** First, it follows from the estimates (6.42), (6.53) and (6.3) that for $n = N + 4, \ldots, 2N - 2$,

$$\frac{d}{dt} \| (\text{curl } u)_h \|^2_{n-2} + \mathcal{D}_n \lesssim \tilde{\mathcal{D}}_n + \mathcal{D}_{N+4}\mathcal{E}_{2N}.$$  

(7.15)

This together with the estimate (5.46) implies that, since $n \geq N + 4$ and $\mathcal{E}_{2N} \leq \delta$ is small,

$$\frac{d}{dt} \left( \tilde{\mathcal{E}}_n + \mathcal{B}_n + \| (\text{curl } u)_h \|^2_{n-2} \right) + \mathcal{D}_n \leq 0.$$  

(7.16)

On the other hand, it follows from the estimates (6.43) and (6.46) that for $n = N + 4, \ldots, 2N - 2$,

$$\mathcal{E}_n \lesssim \tilde{\mathcal{E}}_n + \| (\text{curl } u)_h \|^2_{n-2} + \mathcal{E}_{N+4}\mathcal{E}_{2N}.$$  

(7.17)

This together with (5.47) implies that

$$\mathcal{E}_n \lesssim \tilde{\mathcal{E}}_n + \mathcal{B}_n + \| (\text{curl } u)_h \|^2_{n-2} \lesssim \mathcal{E}_n.$$  

(7.18)

Hence, combining (7.16) and (7.18) yields that for $n = N + 4, \ldots, 2N - 2$,

$$\frac{d}{dt} \mathcal{E}_n + \mathcal{D}_n \leq 0.$$  

(7.19)

Note that $\mathcal{D}_n$ can not control $\mathcal{E}_n$, which can be seen by checking both the spatial and the temporal regularities in their definitions. This rules out not only the exponential decay of $\mathcal{E}_n$ but also prevents one from using the spatial Sobolev interpolation as [25,26] to bound $\mathcal{E}_n \lesssim \mathcal{E}_{2N}^{1-\theta} \mathcal{D}_n^\theta, 0 < \theta < 1$ so as to derive the algebraic decay. Observe that
\[ E_{\ell} \leq D_{\ell+1}. \] Then we will employ a time weighted inductive argument here. To begin with, we may rewrite (7.19) as that for \( j = 0, \ldots, N - 6, \)

\[
\frac{d}{dt} E_{N+4+j} + D_{N+4+j} \leq 0.
\]  

(7.20)

Multiplying (7.20) by \((1 + t)^{N-5-j}, \) one has that, by using \( E_{N+4+j} \leq D_{N+5+j}, \)

\[
\frac{d}{dt} \left( (1 + t)^{N-5-j} E_{N+4+j} \right) + (1 + t)^{N-5-j} D_{N+4+j} \\
\leq (N - 5 - j)(1 + t)^{N-6-j} E_{N+4+j} \lesssim (1 + t)^{N-5-(j+1)} D_{N+4+(j+1)}. 
\]  

(7.21)

Integrating (7.21) in time directly, by a suitable linear combination of the resulting inequalities, one obtains

\[
\sum_{j=0}^{N-6} (1 + t)^{N-5-j} E_{N+4+j}(t) + \sum_{j=0}^{N-6} \int_0^t (1 + s)^{N-5-j} D_{N+4+j}(s) \, ds \\
\lesssim E_{2N}(0) + \int_0^t D_{2N-1}(s) \, ds.
\]  

(7.22)

This implies (7.14). \( \square \)

### 7.3. The a priori estimates

Now we can arrive at the ultimate energy estimates.

**Theorem 7.3.** Let \( N \geq 8. \) There exists a universal constant \( \tilde{\delta} > 0 \) such that if

\[
E_{2N}(t) \leq \tilde{\delta}, \quad \forall t \in [0, T],
\]  

(7.23)

then

\[
E_{2N}(t) + \int_0^t D_{2N}(s) \, ds \leq \tilde{C}_1 E_{2N}(0), \quad \forall t \in [0, T]
\]  

(7.24)

and

\[
\sum_{j=0}^{N-6} (1 + t)^{N-5-j} E_{N+4+j}(t) + \sum_{j=0}^{N-6} \int_0^t (1 + s)^{N-5-j} D_{N+4+j}(s) \, ds \\
\lesssim E_{2N}(0), \quad \forall t \in [0, T].
\]  

(7.25)

**Proof.** Let \( \tilde{\delta} \) be smaller than those \( \delta \) in Theorems 7.1 and 7.2. The estimate (7.14) of Theorem 7.2 implies in particular that

\[
E_{N+4}(t) \lesssim E_{2N}(0) + \int_0^t D_{2N}(s) \, ds.
\]  

(7.26)

Then combining the estimates (7.3) and (7.26) yields that

\[
E_{2N}(t) + \int_0^t D_{2N}(s) \, ds \leq C_1 E_{2N}(0) + C_1 \tilde{\delta}^{1/2} \int_0^t D_{2N}(s) \, ds,
\]  

(7.27)

which implies (7.24) if \( C_1 \tilde{\delta}^{1/2} \leq 1/2. \) Finally, (7.25) follows from (7.14) and (7.24). \( \square \)
8. Local Well-Posedness

In this section we will prove the local well-posedness of (2.5), as stated in Theorem 8.8. As mentioned already in Sect. 2, despite the a priori energy estimates established in Sect. 7, the local well-posedness of (2.5) is still highly nontrivial due to the structure of our energy functionals, even the set of initial data with the high order compatibility conditions has to be examined. Based on the a priori energy estimates in Sect. 7, we will use an iteration scheme to construct solutions to (2.5) by solving a free-surface Euler equations with surface tension (the hydrodynamic part) and a two-phase magnetic system in moving domains (the magnetic part). Note that the hydrodynamic part can be handled, yet great cares are needed to construct solutions to the magnetic part due to the nonlocal boundary condition and the less regularity of the velocity. We will construct elaborate approximate solutions for the magnetic part by modifying the analysis of Padula and Solonnikov [38] and obtain the solution as the limit of these approximate solutions. To this end, our a priori energy estimates independent of the regularization parameter are crucial.

8.1. The initial data and the compatibility conditions. Since our energy functional framework requires high order regularity, we have to specify explicitly the high order compatibility conditions of the initial data. Thus, for given initial data, \((u_0, b_0, \eta_0)\), one needs to construct the data \(\partial_j^{\ell} \eta(0)\) for \(j = 1, \ldots, 2N + 1\), \(\partial_j^{\ell} u(0)\) and \(\partial_j^{\ell} b(0)\) for \(j = 1, \ldots, 2N\), \(\partial_j^{\ell} p(0)\) for \(j = 0, \ldots, 2N - 1\), and \(\partial_j^{\ell} b(0)\) for \(j = 0, \ldots, 2N\), and state the 2N-th order compatibility conditions. The construction of the data can be given as follows.

First, one writes \((\partial_j^{0} u(0), \partial_j^{0} b(0), \partial_j^{0} \eta(0)) = (u_0, b_0, \eta_0)\) and constructs \(\partial \eta(0) = u_0 \cdot N_0\), hereafter \(N_0 = N(0)\), etc. Now, suppose that \(j \in [0, 2N - 1]\) and that \(\partial_j^{\ell} u(0), \partial_j^{\ell} b(0)\) are known for \(\ell = 0, \ldots, j\), \(\partial_j^{\ell} p(0)\) are known for \(\ell = 0, \ldots, j - 1\) (with the understanding that nothing is known of \(p(0)\) when \(j = 0\) and \(\partial_j^{\ell} \eta(0)\) are known for \(\ell = 0, \ldots, j + 1\), then \(\partial_j^{j+1} u(0), \partial_j^{j+1} b(0), \partial_j^{j} p(0)\) and \(\partial_j^{j+2} \eta(0)\) can be obtained as follows. First, let \(\partial_j^{j} p(0)\) be the solution to

\[
\begin{cases}
\Delta^{\varphi_0} \partial_j^j p(0) = -\Delta^{\varphi_0} \left((\partial_j^{\varphi})^j - \partial_j^j\right) p(0) - (\partial_j^{\varphi})^j (\nabla \varphi u : \nabla \varphi u^j)(0) \\
\quad + \text{div}^{\varphi_0}(\partial_j^{\varphi} (\text{curl} \varphi b \times (\bar{B} + b)))(0) \\
\partial_j^j p(0) = -\sigma \partial_j^j H(0) \\
\partial_3^{\varphi_0} \partial_j^j p(0) = \partial_3^{\varphi_0} \left((\partial_j^{\varphi})^j - \partial_j^j\right) p(0) + \partial_j^j (\text{curl} \varphi b \times (\bar{B} + b))(0) \cdot e_3 \quad \text{on } \Sigma_-
\end{cases}
\tag{8.1}
\]

Next, define \(\partial^{j+1} u(0)\) as

\[
\begin{align*}
\partial^{j+1} u(0) &= - \left((\partial_j^{\varphi})^{j+1} - \partial_j^{j+1}\right) u(0) \\
&\quad + (\partial_j^{\varphi})^j \left(-u \cdot \nabla \varphi u - \nabla \varphi p + \text{curl} \varphi b \times (\bar{B} + b)\right)(0)
\tag{8.2}
\end{align*}
\]

and \(\partial^{j+1} b(0)\) as

\[
\begin{align*}
\partial^{j+1} b(0) &= - \left((\partial_j^{\varphi})^{j+1} - \partial_j^{j+1}\right) b(0) + \text{curl}^{\varphi_0} (\partial_j^{\varphi})^j E(0).
\tag{8.3}
\end{align*}
\]

Finally, set \(\partial^{j+2} \eta(0)\) to be

\[
\partial^{j+2} \eta(0) = \partial^{j+1} (u \cdot N) (0).
\tag{8.4}
\]
Note that now one has the data \( \partial^j \eta(0) \) for \( j = 0, \ldots, 2N + 1, \partial^j u(0) \) and \( \partial^j b(0) \) for \( j = 0, \ldots, 2N \) and \( \partial^j p(0) \) for \( j = 0, \ldots, 2N - 1 \), one then can obtain \( \partial^j \tilde{b}(0) \) for \( j = 0, \ldots, 2N \) as the solution to, iteratively,

\[
\begin{align*}
\text{curl}^e \partial^j \tilde{b}(0) &= -\text{curl}^e \left( (\partial^j \eta - \partial^j \eta) \right) \tilde{b}(0) & \text{in } \Omega_+ \\
\text{div}^e \partial^j \tilde{b}(0) &= -\text{div}^e \left( (\partial^j \eta - \partial^j \eta) \right) \tilde{b}(0) & \text{in } \Omega_+ \\
\partial^j \tilde{b}(0) \cdot \mathcal{N}_0 &= -\left[ \partial^j \mathcal{N}, \mathcal{N}_0 \right] \cdot \tilde{b}(0) + \partial^j (b \cdot \mathcal{N}) (0) & \text{on } \Sigma \\
\partial^j \tilde{b}(0) \times e_3 &= 0 & \text{on } \Sigma_+.
\end{align*}
\]  

(8.5)

Note that \( \text{curl}^e ((\partial^j \eta - \partial^j \eta) \tilde{b}(0) \cdot e_3 = 0 \) on \( \Sigma_+ \), which follows by the fact that \( e_3 \cdot \text{curl}^e = e_3 \cdot \text{curl} \) and \( (\partial^j \eta - \partial^j \eta) \tilde{b}(0) \) on \( \Sigma_+ \), and thus, by Proposition 3.1, guarantees the solvability of (8.5). The construction of the data is thus completed. In order for \( (u_0, b_0, \eta_0) \) to be taken as the initial data for the local well-posedness of (2.5) in our energy functional framework, these data constructed above need to satisfy the following \( 2N \)-th order compatibility conditions:

\[
\begin{align*}
\text{div}^e u_0 &= \text{div}^e b_0 = 0 \text{ in } \Omega_-, \quad u_{0,3} = b_{0,3} = 0 \text{ on } \Sigma_- \,, \\
\left[ \partial^j b(0) \right] \times \mathcal{N}_0 &= 0 \text{ on } \Sigma \text{ and } \partial^j E(0) \times e_3 = 0 \text{ on } \Sigma_-, \quad j = 0, \ldots, 2N - 1. \quad (8.6)
\end{align*}
\]

We shall now show that the set of the initial data satisfying the compatibility conditions (8.6) is not empty. In principle, this is highly technical since the problem (2.5) is nonlinear and nonlocal. Our key observation here is that since the nonlinear problem is a small perturbation of the linearized one, so their compatibility conditions for the initial data should be close to each other. Then our idea is to first construct the initial data for the linearized problem that satisfies the corresponding linear compatibility conditions, and then we obtain a family of initial data satisfying the compatibility conditions for the nonlinear problem, which are close to the initial data of the linearized problem, by a perturbation argument. For the linearized problem of (2.5) (i.e., (4.47) with \( G^i \) and \( \hat{G}_i \) being zero), given the initial data \( (u_0, b_0, \eta_0) \), the construction of the data \( \partial^j \eta(0) \) for \( j = 1, \ldots, 2N + 1, \partial^j u(0) \) and \( \partial^j b(0) \) for \( j = 1, \ldots, 2N \), \( \partial^j p(0) \) for \( j = 0, \ldots, 2N - 1 \) and \( \partial^j \tilde{b}(0) \) for \( j = 0, \ldots, 2N \) is similar to that of the nonlinear problem (2.5), and the linear compatibility conditions is the one obtained by setting \( \eta = 0 \) (and \( E = u \times \bar{B} - \kappa \text{curl} b \)) in (8.6). Note that the set of the initial data satisfying the linear compatibility conditions is not empty; indeed, any triple of \( u_0 \in C^\infty(\Omega_-) \) with \( \text{div} u_0 = 0, b_0 \in C^\infty(\Omega_-) \) with \( \text{div} b_0 = 0 \) and \( \eta_0 = 0 \) satisfies the linear compatibility conditions, which follows from the fact that, from the construction, \( \partial^j \eta(0) = 0 \) for \( j = 1, \ldots, 2N + 1, \partial^j u(0) = -\bar{B} \cdot \partial^j b(0) \) for \( j = 0, \ldots, 2N - 1, \partial^j p(0) = -\nabla \partial^j - 1 p(0) + \text{curl} \partial^j - 1 b(0) \times \bar{B} \) for \( j = 1, \ldots, 2N \) and \( \partial^j b(0) = -\kappa \text{curl} \partial^j - 1 b(0) + \bar{B} \cdot \nabla \partial^j - 1 u(0) \) for \( j = 1, \ldots, 2N \) all belong to \( C^\infty(\Omega_-) \) and \( \partial^j \tilde{b}(0) = 0 \) for \( j = 0, \ldots, 2N \). Now given any smooth initial data satisfying the linear compatibility conditions, denoted by \( (u^L_0, b^L_0, \eta^L_0) \), one may then employ the abstract argument before Lemma 5.3 of [31] of using the implicit function theorem to show that there exist a constant \( \iota_0 > 0 \) and a family of smooth initial data of the form \( (u^\iota_0, b^\iota_0, \eta^\iota_0) = \iota (u^L_0, b^L_0, \eta^L_0) + \iota^2 (\tilde{u}(\iota), \tilde{b}(\iota), \tilde{\eta}(\iota)) \) for \( \iota \in [0, \iota_0) \) and some smooth \( (\tilde{u}(\iota), \tilde{b}(\iota), \tilde{\eta}(\iota)) \) so that \( (u^\iota_0, b^\iota_0, \eta^\iota_0) \) satisfies the nonlinear compatibility conditions (8.6).
8.2. The free-surface Euler equations with surface tension. In this subsection we consider the following free-surface Euler equations that for given $F$,

$$
\begin{aligned}
\frac{\partial}{\partial t} u + u \cdot \nabla u + \nabla p &= F \quad \text{in } \Omega_-
\\
\text{div}^\theta u &= 0 \quad \text{in } \Omega_-
\\
\partial_t \eta &= u \cdot \mathcal{N} \quad \text{on } \Sigma
\\
p &= -\sigma H \quad \text{on } \Sigma
\\
u_3 &= 0 \quad \text{on } \Sigma_-
\\
(u, \eta)|_{t=0} &= (u_0, \eta_0).
\end{aligned}
$$

(8.7)

Given the initial data $(u_0, \eta_0)$, let the data $\partial_t^j \eta(0)$ for $j = 1, \ldots, 2N + 1$, $\partial_t^j u(0)$ for $j = 1, \ldots, 2N$ and $\partial_t^j p(0)$ for $j = 0, \ldots, 2N - 1$ be constructed similarly as in Sect. 8.1. The initial data are required to satisfy the following compatibility conditions

$$\text{div}^\theta u_0 = 0 \quad \text{in } \Omega_- \quad \text{and } u_{0,3} = 0 \quad \text{on } \Sigma_-.$$  

(8.8)

Recall the definition (2.10) of $\mathcal{E}_{2N}$, and denote the $u$-parts of $\mathcal{E}_{2N}$ by $\mathcal{E}_{2N}(u)$, etc. Set

$$
\tilde{\mathcal{Z}}_{2N}^2(F) := \int_0^T \sum_{j=0}^{2N} \left\| \partial_t^j F \right\|_{2N-j}^2,
$$

and

$$
\mathcal{Z}_{0}^{2N}(F) := \sum_{j=0}^{2N-1} \left\| \partial_t^j (F(0)) \right\|_{2N-j-1}^2.
$$

(8.9)

(8.10)

Note that

$$\mathcal{Z}_{\infty}^{2N}(F) \lesssim \mathcal{Z}_{0}^{2N}(F) + T \mathcal{Z}_{2}^{2N}(F).$$

(8.11)

Now the local well-posedness of (8.7) can be stated as follows.

**Proposition 8.1.** Let $N \geq 4$ be an integer. Assume that $\mathcal{Z}_{0}^{2N}(F) + \mathcal{Z}_{2}^{2N}(F) < \infty$ for any $0 < T \leq 1$, $u_0 \in H^{2N}(\Omega_-)$ and $\eta_0 \in H^{2N+3/2}(\Sigma)$ are given such that $\mathcal{E}_{2N}(u, p, \eta)(0) < \infty$ and the compatibility conditions (8.8) are satisfied. There exist a universal constant $\delta_1 > 0$ such that if $\mathcal{E}_{2N}(u, p, \eta)(0) + \mathcal{Z}_{0}^{2N}(F) + \mathcal{Z}_{2}^{2N}(F) \leq \delta_1$, then there exists a unique solution $(u, p, \eta)$ to (8.7) on $[0, T]$ satisfying

$$
\sup_{[0,T]} \mathcal{E}_{2N}(u, p, \eta) \lesssim \mathcal{E}_{2N}(u, p, \eta)(0) + \mathcal{Z}_{0}^{2N}(F) + T \mathcal{Z}_{2}^{2N}(F).
$$

(8.12)

**Proof.** The problem (8.7) could be solved similarly as in Coutand and Shkoller [13]. In fact, the arguments here would be reasonably easier since the geometry of (8.7) is simpler than those of [13] which treats more general domains in Lagrangian coordinates. So we may omit the details, and focus only on the derivation of the a priori estimate (8.12). The proof follows similarly as that of the a priori estimates for (2.5), not involving the magnetic part, and thus we provide only the necessary modifications.

Assume that the solution $(u, p, \eta)$ to (8.7) is given on the interval $[0, T]$ and satisfies

$$\mathcal{E}_{2N}(t) \leq \delta, \quad \forall t \in [0, T]$$

(8.13)
for sufficiently small $\delta > 0$. First, one may modify the proof of Proposition 5.1 easily to deduce

$$
\tilde{E}_{2N}(u, \eta)(t) \lesssim E_{2N}(u, p, \eta)(0) + (E_{2N}(u, p, \eta)(t))^{3/2} + \int_0^t \left( E_{2N}(u, p, \eta) \right)^{3/2} + \sum_{j=0}^{2N} \int_0^t \sqrt{\tilde{E}_{2N}(u)} \left\| \partial_j F \right\|_{2N-j}.
$$

(8.14)

By Cauchy’s inequality and the Cauchy-Schwarz inequality, one deduces that for $T \leq 1$,

$$
\sup_{[0,T]} \tilde{E}_{2N}(u, \eta) \lesssim E_{2N}(u, p, \eta)(0) + \sup_{[0,T]} \left( E_{2N}(u, p, \eta) \right)^{3/2} + T \tilde{\delta}_{2}^{2N}(F).
$$

(8.15)

Next, following a variant of the proof of Proposition 6.4 by using the vorticity equations

$$
\partial_t \psi \text{curl} \psi u + u \cdot \nabla \psi \text{curl} \psi u = \text{curl} \psi u \cdot \nabla \psi u + \text{curl} \psi F,
$$

(8.16)

one obtains

$$
\sup_{[0,T]} E_{2N}(u) \lesssim E_{2N}(u, \eta)(0) + \sup_{[0,T]} \tilde{E}_{2N}(u)
+ \sup_{[0,T]} \left( E_{2N}(u, p, \eta) \right)^{3/2} + T \tilde{\delta}_{2}^{2N}(F) + \tilde{\delta}_{\infty}^{2N}(F).
$$

(8.17)

Following the proof of Proposition 6.6 leads to

$$
E_{2N}(p, \eta) \lesssim \tilde{E}_{2N}(\eta) + E_{2N}(u) + \left( E_{2N}(u, p, \eta) \right)^2 + \tilde{\delta}_{\infty}^{2N}(F).
$$

(8.18)

Now, collecting the estimates (8.15), (8.17) and (8.18) yields that, by (8.11),

$$
\sup_{[0,T]} E_{2N}(u, p, \eta) \lesssim E_{2N}(u, p, \eta)(0) + \sup_{[0,T]} \left( E_{2N}(u, p, \eta) \right)^{3/2} + \tilde{\delta}_{0}^{2N}(F) + T \tilde{\delta}_{2}^{2N}(F).
$$

(8.19)

This implies in particular that, for $\delta > 0$ small,

$$
\sup_{[0,T]} E_{2N}(u, p, \eta) \leq C_1 \left( E_{2N}(u, p, \eta)(0) + \tilde{\delta}_{0}^{2N}(F) + T \tilde{\delta}_{2}^{2N}(F) \right),
$$

(8.20)

which closes the a priori assumption (8.13) if one has assumed that

$$
E_{2N}(u, p, \eta)(0) + \tilde{\delta}_{0}^{2N}(F) + \tilde{\delta}_{2}^{2N}(F) \leq \delta_1 := \delta/C_1.
$$

(8.21)

This in turn implies that the solution $(u, p, \eta)$ exists on $[0, T]$ for any $0 < T \leq 1$ and the estimate (8.12) holds, and the proposition is thus proved. □
8.3. The two-phase magnetic system in moving domains. In this subsection we consider the following magnetic system that for given \( \eta \) and \( G \),

\[
\begin{align*}
\partial_t^\psi b + \kappa \text{curl}^\psi \text{curl}^\psi b &= \text{curl}^\psi G & \text{in } \Omega_- \\
\text{div}^\psi b &= 0 & \text{in } \Omega_- \\
\text{curl}^\psi \hat{b} &= 0, \quad \text{div}^\psi \hat{b} &= 0 & \text{in } \Omega_+ \\
[b] &= 0 & \text{on } \Sigma \\
b_3 &= 0, \quad \kappa \text{curl}^\psi b \times e_3 &= G \times e_3 & \text{on } \Sigma_- \\
\hat{b} \times e_3 &= 0 & \text{on } \Sigma_+ \\
b |_{t=0} &= b_0.
\end{align*}
\]

(8.22)

Given the initial data \( b_0 \), let the data \( \partial_t^j b(0) \) for \( j = 1, \ldots, 2N \) and \( \partial_t^j \hat{b}(0) \) for \( j = 0, \ldots, 2N \) be constructed similarly as in Sect. 8.1. For the later use, we may reformulate the construction of these data and the corresponding \( 2N \)-th order compatibility conditions as follows. For \( j = 0, \ldots, 2N \), denote \( P_j(f_0, \ldots, f_{j-1}) \) (depending on \( \varphi \)) to be the corresponding expression \( (\partial_t^\psi)^j - \partial_t^j f(0) \) with \( \partial_t^j f(0) \) replaced by \( f_\ell \) for \( \ell = 0, \ldots, j - 1 \) (with the understanding that \( P_0 = 0 \) when \( j = 0 \)). For the given initial data \( b(0) = b_0 \), one can define iteratively that for \( j = 0, \ldots, 2N - 1 \),

\[
\partial_t^{j+1} b(0) = -P_{j+1}(b(0), \ldots, \partial_t^j b(0)) - \kappa \text{curl}^\psi \text{curl}^\psi \left( \partial_t^j b(0) + P_j(b(0), \ldots, \partial_t^{j-1} b(0)) \right)
\]

\[
+ \text{curl}^\psi (\partial_t^\psi)^j G(0)
\]

and for \( j = 0, \ldots, 2N \), \( \partial_t^j \hat{b}(0) \) is the solution to

\[
\begin{align*}
\text{curl}^\psi \partial_t^j \hat{b}(0) &= -\text{curl}^\psi P_j(\hat{b}(0), \ldots, \partial_t^{j-1} \hat{b}(0)) & \text{in } \Omega_+ \\
\text{div}^\psi \partial_t^j \hat{b}(0) &= -\text{div}^\psi P_j(\hat{b}(0), \ldots, \partial_t^{j-1} \hat{b}(0)) & \text{in } \Omega_+ \\
\partial_t^j b(0) \cdot N_0 &= -\left[ \partial_t^j, N \right] \cdot \hat{b}(0) + \partial_t^j (b \cdot N)(0) & \text{on } \Sigma \\
\partial_t^j \hat{b}(0) \times e_3 &= 0 & \text{on } \Sigma_+.
\end{align*}
\]

(8.24)

The \( 2N \)-th order compatibility conditions for (8.22) are the following:

\[
\left\{ \begin{array}{l}
\text{div}^\psi b_0 = 0 \text{ in } \Omega_- , \quad b_{0,3} = 0 \text{ on } \Sigma_- , \quad \left[ \partial_t^j b(0) \right] \times N_0 = 0 \text{ on } \Sigma , \quad j = 0, \ldots, 2N - 1, \\
\kappa \text{curl}^\psi \partial_t^j b(0) \times e_3 = \left( -\kappa \text{curl}^\psi P_j(b(0), \ldots, \partial_t^{j-1} b(0)) + \partial_t^j G(0) \right) \times e_3 \text{ on } \Sigma_- , \quad j = 0, \ldots, 2N - 1.
\end{array} \right.
\]

(8.25)

Note that it follows from (8.23), the last two lines in (8.25) and the third equation in (8.24) that

\[
\left\{ \begin{array}{l}
\text{div}^\psi \partial_t^j b(0) = -\text{div}^\psi P_j(b(0), \ldots, \partial_t^{j-1} b(0)) \text{ in } \Omega_+ , \quad j = 1, \ldots, 2N, \\
\partial_t^j b_3(0) = 0 \text{ on } \Sigma_- , \quad j = 1, \ldots, 2N \quad \text{and } \left[ \partial_t^j b(0) \right] \cdot N_0 = 0 \text{ on } \Sigma , \quad j = 0, \ldots, 2N.
\end{array} \right.
\]

(8.26)

The problem (8.22) was solved in Padula and Solonnikov [38] in a slightly different setting by using the full parabolic regularity of the problem. However, it should be noted that one key subtle point in [38] is that the iteration scheme of constructing the solutions to the viscoresistive plasma-vacuum interface problem therein requires high order regularities of \( u \) and \( \eta \) guaranteed by the viscosity, which unfortunately is not the case here for solving the inviscid and resistive plasma-vacuum interface problem (2.5). Our way to get around this difficulty is to regularize (8.22), and to solve the regularized
problem by modifying the arguments of [38]. Then we derive the uniform estimates of the approximate solutions independent of the smoothing parameter, which enable us to take the limit to solve the original problem (8.22). To this end, we will make an important use of the corresponding regularized electric field in vacuum. As our energy functional is different from the parabolic one of [38] and is of high order, we need to solve the regularized problem in a higher regularity counterpart of that of [38], which requires us also to smooth out the initial data \( b_0 \). Such a smoothing procedure is highly technical as it needs to guarantee the high order compatibility conditions for the regularized problem. It seems extremely difficult for us to apply directly the usual standard regularization needs to guarantee the high order compatibility conditions for the regularized problem. More precisely, we will regularize (8.22) as follows:

\[
\begin{align*}
\partial_t \psi^e \cdot b^e + \kappa \text{curl} \psi^e \cdot \text{curl} \psi^e b^e &= \text{curl} \psi^e \cdot (G^e - \Psi^e) & \text{in } \Omega_- \\
\text{div} \psi^e b^e &= 0 & \text{in } \Omega_- \\
\text{curl} \psi^e \cdot b^e &= 0, \quad \text{div} \psi^e \cdot \hat{b}^e = 0 & \text{in } \Omega_+ \\
\hat{b}^e &= 0 & \text{on } \Sigma \\
b^e &= 0, \quad \kappa \text{curl} \psi^e \cdot e_3 = G^e \cdot e_3 & \text{on } \Sigma_- \\
b^e \cdot e_3 &= 0 & \text{on } \Sigma_+ \\
b^e \big|_{t=0} &= b_0^e.
\end{align*}
\]

Here \( \psi^e = \psi(\eta^e) \) with \( \eta^e = (\eta)_{t,x,h} \) and \( G^e = (G)_{t,x} \), where \( (\cdot)^e \) is the usual smooth approximation in time through a mollifier, etc., and the corrector \( \Psi^e \) and the smooth data \( b_0^e \) are constructed simultaneously as follows. For the given \( \partial_t^j b(0) = b_0 \), the initial date of the original problem (8.22), let \( \partial_t^j b(0) \) for \( j = 1, \ldots, 2N \) and \( \partial_t^j \hat{b}(0) \) for \( j = 0, \ldots, 2N \) be constructed by (8.23) and (8.24), respectively. For \( j = 0, \ldots, 2N \), let \( P_j^e \) be \( P_j \) with \( \varphi \) replaced by \( \varphi^e \). Note that \( P_j^e \) is the solution to (8.28) and (8.29), and that \( \hat{w}_j^e \) and \( \hat{w}_j^e \) are known for \( \ell = 0, \ldots, j - 1 \) (with the understanding that nothing is known for \( w_0^e \) and \( \hat{w}_0^e \) when \( j = 0 \), we define \( w_j^e \) and \( \hat{w}_j^e \) as the solution to

\[
\begin{align*}
\kappa \text{curl} \psi^e \cdot \text{curl} \psi^e w_j^e &= -\kappa \text{curl} \psi^e \cdot \text{curl} \psi^e P_j^e (w_0^e, \ldots, w_{j-1}^e) \\
&+ \kappa \text{curl} \psi^e \cdot \text{curl} \psi^e ((\partial_t^j)^\epsilon b(0))^\epsilon - \kappa \text{curl} \psi^e \phi_j^e & \text{in } \Omega_- \\
\text{div} \psi^e w_j^e &= -\text{div} \psi^e P_j^e (w_0^e, \ldots, w_{j-1}^e) & \text{in } \Omega_- \\
\text{curl} \psi^e \hat{w}_j^e &= -\text{curl} \psi^e P_j^e (\hat{w}_0^e, \ldots, \hat{w}_{j-1}^e) & \text{in } \Omega_+ \\
\text{div} \psi^e \hat{w}_j^e &= -\text{div} \psi^e P_j^e (\hat{w}_0^e, \ldots, \hat{w}_{j-1}^e) & \text{in } \Omega_+ \\
\left[ w_j^e \right] &= 0 & \text{on } \Sigma \\
w_j^e \cdot e_3 &= 0, \quad \kappa \text{curl} \psi^e \cdot e_3 = (\kappa \text{curl} \psi^e P_j^e (w_0^e, \ldots, w_{j-1}^e) + \partial_t^j G^e (0)) \cdot e_3 & \text{on } \Sigma_- \\
\hat{w}_j^e \cdot e_3 &= 0 & \text{on } \Sigma_+ \\
\phi_j^e \text{ is a sequence of correctors satisfying} & \phi_j^e, \hat{w}_j^e = \left( \kappa \text{curl} \psi^e \left( (\partial_t^j)^\epsilon b(0) \right)^\epsilon - \partial_t^j G^e (0) \right) & \text{on } \Sigma_-, \quad j = 0, \ldots, 2N - 1.
\end{align*}
\]

This can be constructed by the harmonic extension, similarly as Lemma A.1. It should be noted here that the introduction of the correctors \( \phi_j^e \) is crucial in order to guarantee
the solvability of (8.28) according to Proposition 3.3. Indeed, without $\phi^\epsilon_j$, the solvability of (8.28) would require $\kappa \text{curl}^\epsilon \left( (\partial_i^\epsilon)^j b(0) \right)^\epsilon \times e_3 = \partial_i^j G^\epsilon(0) \times e_3$ on $\Sigma_-$, which is not valid in general even that the last line in the $2N$-th order compatibility conditions (8.25) hold. Next, by the second, sixth and seventh equations in (8.28), according to Proposition 3.1, for $j = 0, \ldots, 2N - 1$ one can define $\psi^\epsilon_j$ as the solution to

$$
\begin{align*}
\text{curl}^\epsilon \psi^\epsilon_j = -\kappa \text{curl}^\epsilon \text{curl}^\epsilon \left( w^\epsilon_j + P^\epsilon_j (w^\epsilon_0, \ldots, w^\epsilon_{j-1}) \right) + \text{curl}^\epsilon \left( \partial_i^\epsilon \right)^j G^\epsilon(0) - w^\epsilon_{j+1} - P^\epsilon_{j+1} (w^\epsilon_0, \ldots, w^\epsilon_j) & \quad \text{in } \Omega_- \\
\text{div}^\epsilon \psi^\epsilon_j = 0 & \quad \text{in } \Omega_- \quad (8.30) \\
\psi^\epsilon_j \cdot \mathcal{N}^\epsilon_0 = 0 & \quad \text{on } \Sigma \\
\psi^\epsilon_j \times e_3 = 0 & \quad \text{on } \Sigma_-,
\end{align*}
$$

where $\mathcal{N}^\epsilon = (-\nabla b^\epsilon, 1)$. Now we can set $b_0^\epsilon = w_0^\epsilon$ and $\Psi^\epsilon(t)$, by the time extension similarly as Lemmas A.2–A.4, such that

$$
(\partial_t^\epsilon)^j \Psi^\epsilon(0) = \psi^\epsilon_j, \quad j = 0, \ldots, 2N - 2 \quad \text{and } (\partial_t^\epsilon)^{2N-1} \Psi^\epsilon(0) = 0. \quad (8.31)
$$

It follows from the fact that $(\partial_t^\epsilon)^j = \partial_t^j$ on $\Sigma_-$ and the last equation in (8.30) that $\partial_t^j \Psi^\epsilon(0) \times e_3 = 0$ on $\Sigma_-, \quad j = 0, \ldots, 2N - 1$, hence one can further choose to have that (see Chapter 4 in Lions and Magenes [35])

$$
\Psi^\epsilon \times e_3 = 0 \quad \text{on } \Sigma_-, \quad (8.32)
$$

Note that (8.32) is required for the solvability of (8.27).

Now having constructed smooth $b^\epsilon(0) = b_0^\epsilon$ and $\Psi^\epsilon(t)$, we can construct the data $\partial_t^j b^\epsilon(0)$ for $j = 1, \ldots, 2N$ and $\partial_t^j \hat{b}^\epsilon(0)$ for $j = 0, \ldots, 2N$ inductively that for $j = 0, \ldots, 2N - 1$,

$$
\partial_t^{j+1} b^\epsilon(0) = -P^\epsilon_{j+1} (b^\epsilon(0), \ldots, \partial_t^j b^\epsilon(0)) \\
- \kappa \text{curl}^\epsilon \text{curl}^\epsilon \left( \partial_t^j b^\epsilon(0) + P^\epsilon_j (b^\epsilon(0), \ldots, \partial_t^{j-1} b^\epsilon(0)) \right) \\
+ \text{curl}^\epsilon \left( (\partial_t^\epsilon)^j G^\epsilon(0) - (\partial_t^\epsilon)^j \Psi^\epsilon(0) \right)
$$

and that for $j = 0, \ldots, 2N$,

$$
\begin{align*}
\text{curl}^\epsilon \partial_t^j \hat{b}^\epsilon(0) &= -\text{curl}^\epsilon P^\epsilon_j (\hat{b}^\epsilon(0), \ldots, \partial_t^{j-1} \hat{b}^\epsilon(0)) \quad \text{in } \Omega_+ \\
\text{div}^\epsilon \partial_t^j \hat{b}^\epsilon(0) &= -\text{div}^\epsilon P^\epsilon_j (\hat{b}^\epsilon(0), \ldots, \partial_t^{j-1} \hat{b}^\epsilon(0)) \quad \text{in } \Omega_+ \\
\partial_t^j \hat{b}^\epsilon(0) \cdot \mathcal{N}^\epsilon_0 &= - \left[ \partial_t^j \hat{b}^\epsilon(0), \mathcal{N}^\epsilon_0 \right] \cdot \hat{b}^\epsilon(0) + \partial_t^j (b^\epsilon \cdot \mathcal{N}^\epsilon_0)(0) \quad \text{on } \Sigma \\
\partial_t^j \hat{b}^\epsilon(0) \times e_3 &= 0 \quad \text{on } \Sigma_+.
\end{align*}
$$

where $\mathcal{N}^\epsilon = (-\nabla b^\epsilon, 1)$.

We now claim that $\partial_t^j b^\epsilon(0) = w_0^\epsilon, \quad j = 1, \ldots, 2N - 1$ and $\partial_t^j \hat{b}^\epsilon(0) = \hat{w}_0^\epsilon, \quad j = 0, \ldots, 2N - 1$. First, since $\hat{b}^\epsilon(0) = b_0^\epsilon = w_0^\epsilon$, it follows from (8.28) with $j = 0$ that $\hat{w}_0^\epsilon$ solves (8.34) with $j = 0$, and hence by the uniqueness one has $\hat{b}^\epsilon(0) = \hat{w}_0^\epsilon$. Now, suppose that $j \in [0, 2N - 2]$ and that $\partial_t^j b^\epsilon(0) = w_\epsilon^j$ and $\partial_t^j \hat{b}^\epsilon(0) = \hat{w}_\epsilon^j$ have been
verified for \( \ell = 0, \ldots, j \), one finds that, by (8.33), (8.30) and the first equation in (8.30) and (8.31),
\[
\partial_t^{j+1} b^\epsilon (0) = -P^\epsilon_{j+1} (w_0^\epsilon, \ldots, w_j^\epsilon) - \kappa \text{curl}^\epsilon_0 \text{curl}^\epsilon_0 \left( w_j^\epsilon + P^\epsilon_j (w_0^\epsilon, \ldots, w_{j-1}^\epsilon) \right) \\
+ \text{curl}^\epsilon_0 (\partial_t^{j+1} \psi^\epsilon (0)) = w_{j+1}^\epsilon.
\tag{8.35}
\]

It then follows from (8.28) with \( j \) replaced by \( j + 1 \) and the induction assumption that \( \hat{w}_{j+1}^\epsilon \) solves (8.34) with \( j \) replaced by \( j + 1 \), and hence by the uniqueness one has \( \partial_t^{j+1} b^\epsilon (0) = \hat{w}_{j+1}^\epsilon \). The claim is thus proved. Note then that by (8.28), one finds that the corresponding 2\( N \)-th order compatibility conditions for (8.27) are satisfied, i.e.,
\[
\begin{align*}
&\text{div}^\epsilon b_0^\epsilon = 0 \text{ in } \Sigma_-, \quad b_0^\epsilon \in \Sigma_-
\text{and}
&\left\{ \begin{array}{ll}
&\text{for } j = 0, \ldots, 2N - 1, \quad \left[ \partial_t^j b^\epsilon (0) \right] \times N_0^\epsilon = 0 \text{ on } \Sigma_\text{ and}
&\kappa \text{curl}^\epsilon_0 \partial_t^j b^\epsilon (0) \times e_3 \left( -\kappa \text{curl}^\epsilon_0 P^\epsilon_j (b^\epsilon (0), \ldots, \partial_t^{j-1} b^\epsilon (0)) + \partial_t^j G^\epsilon (0) \right) \times e_3 \text{ on } \Sigma_-
\end{array} \right.
\tag{8.36}
\end{align*}
\]

In general, \( b_0^\epsilon \) constructed above does not converge to \( b_0 \) and \( \Psi^\epsilon \) does not vanish as \( \epsilon \to 0 \). To ensure such convergence, additional conditions are required as shown in the following lemma, where
\[
\tilde{\mathfrak{S}}_2^{2N} (G) := \int_0^T \sum_{j=0}^{2N} \left\| \partial_t^j G \right\|_{2N-j}^2, \quad \tilde{\mathfrak{S}}_\infty^{2N} (G) := \sup_{[0,T]} \sum_{j=0}^{2N-1} \left\| \partial_t^j G \right\|_{2N-j}^2 \tag{8.37}
\]
and
\[
\tilde{\mathfrak{S}}_2^{2N} (\Psi) := \int_0^T \sum_{j=0}^{2N} \left\| \partial_t^j \Psi \right\|_{2N-j+1/2}^2, \quad \tilde{\mathfrak{S}}_\infty^{2N} (\Psi) := \sup_{[0,T]} \sum_{j=0}^{2N-1} \left\| \partial_t^j \Psi \right\|_{2N-j}^2 \tag{8.38}
\]

\begin{lemma}
Suppose that \( \sup_{[0,T]} \mathcal{E}_2^N (\eta) < \infty, \tilde{\mathfrak{S}}_\infty^{2N} (G) < \infty, \mathcal{E}_2^N (b, \hat{b}) (0) < \infty \) and the 2\( N \)-th order compatibility conditions (8.25) are satisfied. Then as \( \epsilon \to 0 \), \( (b_\epsilon^0, \hat{b}_\epsilon^0) \to (b_0, \hat{b}_0) \) in the norms of \( \mathcal{E}_2^N (b, \hat{b}) (0) \) and \( \Psi^\epsilon \to 0 \) in the norms of \( \tilde{\mathfrak{S}}_2^{2N} (\Psi) + \tilde{\mathfrak{S}}_\infty^{2N} (\Psi) \).
\end{lemma}

\begin{proof}
First, it follows from the usual properties of mollifiers that if \( \sup_{[0,T]} \mathcal{E}_2^N (\eta) < \infty \), then \( \sup_{[0,T]} \mathcal{E}_2^N (\eta^\epsilon) \lesssim \sup_{[0,T]} \mathcal{E}_2^N (\eta) \lesssim \mathcal{E}_2^N (\eta) \to 0 \) in the norms of \( \mathcal{F}(j) \mathcal{E}_2^N (\eta) \) and \( \eta^\epsilon \to 0 \) in the norms of \( \mathcal{E}_2^N (\eta) \) as \( \epsilon \to 0 \). Similarly, if \( \tilde{\mathfrak{S}}_\infty^{2N} (G) < \infty \), then \( \tilde{\mathfrak{S}}_\infty^{2N} (G^\epsilon) \lesssim \tilde{\mathfrak{S}}_\infty^{2N} (G) + \tilde{\mathfrak{S}}_2^{2N} (G^\epsilon) \lesssim \tilde{\mathfrak{S}}_\infty^{2N} (G) < \infty \) as \( \epsilon \to 0 \).

Now suppose that \( \mathcal{E}_2^N (b, \hat{b}) (0) < \infty \) and (8.25) holds and recall (8.26). Then by (8.29) and the last line in (8.25), according to the trace theory and the estimates of the harmonic extension similarly as Lemma A.1, one has that \( \phi_j^\epsilon \to 0 \) in \( H^{2N-j} (\Omega_-) \), \( j = 0, \ldots, 2N - 1 \), as \( \epsilon \to 0 \). Hence, by Propositions 3.3 and 3.1, it is then routine to check from (8.28) for \( j = 0, \ldots, 2N - 1 \), (8.33) with \( j = 2N - 1 \), (8.34) with \( j = 2N \) and (8.30) for \( j = 0, \ldots, 2N - 2 \) that as \( \epsilon \to 0 \), \( (b_\epsilon^0, \hat{b}_\epsilon^0) \to (b_0, \hat{b}_0) \) in the norms of \( \mathcal{E}_2^N (b, \hat{b}) (0) \) and \( \psi_j^\epsilon \to 0 \) in \( H^{2N-j} (\Omega_-) \), \( j = 0, \ldots, 2N - 2 \). Finally, according to (8.31) and the estimates of the time extension similarly as Lemmas A.2–A.4, one has that \( \Psi^\epsilon \to 0 \) in the norm of \( \tilde{\mathfrak{S}}_\infty^{2N} (\Psi) + \tilde{\mathfrak{S}}_2^{2N} (\Psi) \) as \( \epsilon \to 0 \).
\end{proof}
We now establish the well-posedness of the regularized problem (8.27). Recall the $L^2$ anisotropic space-time Sobolev spaces

$$
H^{r/2}((0, T) \times \Omega) = L^2(0, T; H^r(\Omega)) \cap H^{r/2}(0, T; L^2(\Omega)), \ r \geq 0,
$$

(8.39)
eq

et., see Lions and Magenes [35]. Define

$$
\mathcal{R}^n(b^\epsilon, \hat{b}^\epsilon) := \|b^\epsilon\|_{H^{n+\ell, n+\ell/2}((0,T)\times\Omega_-)} + \|\hat{b}^\epsilon\|_{H^{n+\ell, n+\ell/2}((0,T)\times\Omega_+)}. \tag{8.40}
$$

**Lemma 8.3.** Let $N \geq 4$ be an integer and $1/2 < \ell < 1$ or $1 < \ell < 3/2$. Assume that the smooth initial data $b^0_\epsilon$ satisfies the $2N$-th order compatibility conditions (8.36). There exists a universal constant $\delta_2 > 0$ such that if $\sup_{[0,T]} \mathcal{E}_2 N(\eta) \leq \delta_2$ for any $0 < T \leq 1$, then for any $\epsilon > 0$ there exists a unique strong solution $(b^\epsilon, \hat{b}^\epsilon)$ to (8.27) on $[0, T]$ satisfying

$$
\mathcal{R}^{2N}(b^\epsilon, \hat{b}^\epsilon) < \infty. \tag{8.41}
$$

**Proof.** Note that a similar problem as (8.27) was solved in Padula and Solonnikov [38], that is, the problem when an isolated plasma surrounded by a vacuum which are bounded from the outside by a perfectly conducting wall. But the main scheme in [38] can be modified slightly to the case here. We utilize the results of [38] and repeat some main steps, but refer to [38] for the full details.

As in Beale [3] and Padula and Solonnikov [38], one may introduce the following change of unknowns:

$$
b^\epsilon = J^\epsilon(J^\epsilon)^{-1} b^\epsilon \text{ in } \Omega_- \quad \text{and} \quad \hat{b}^\epsilon = J^\epsilon(J^\epsilon)^{-1} \hat{b}^\epsilon \text{ in } \Omega_+, \tag{8.42}
$$

where $J^\epsilon = \nabla \Phi^\epsilon$ with $\Phi^\epsilon = \Phi(\eta^\epsilon)$ in (2.2) and $J^\epsilon$ is its determinant. The advantages of introducing $b^\epsilon$ and $\hat{b}^\epsilon$ are that $\text{div} \ b^\epsilon = 0$ in $\Omega_-$, $\text{div} \ \hat{b}^\epsilon = 0$ in $\Omega_+$ and that it also keeps the boundary conditions of $b^\epsilon$ and $\hat{b}^\epsilon$ same as those of $b$ and $\hat{b}$. By (8.42), one may rewrite (8.27) with $\eta^\epsilon$ small in terms of $(b^\epsilon, \hat{b}^\epsilon)$ in the following perturbed form:

$$
\begin{aligned}
\partial_t b^\epsilon + \kappa \text{curl} \text{curl} b^\epsilon &= Q^{1, \epsilon} \quad \text{in } \Omega_-
\text{div} b^\epsilon &= 0 \quad \text{in } \Omega_-
\text{curl} \hat{b}^\epsilon &= \text{curl} Q^{2, \epsilon}, \quad \text{div} \hat{b}^\epsilon = 0 \quad \text{in } \Omega_+
\|b^\epsilon\| &= 0 \quad \text{on } \Sigma
\int l=0 b^\epsilon = 0, \quad \kappa \text{curl} b^\epsilon \times e_3 = Q^{3, \epsilon} \times e_3 \quad \text{on } \Sigma_-
\hat{b}^\epsilon \times e_3 &= 0 \quad \text{on } \Sigma_+
\int b^\epsilon \big|_{l=0} = b^\epsilon_0 = J_0^\epsilon(J_0^\epsilon)^{-1} b^\epsilon_0,
\end{aligned} \tag{8.43}
$$

where

$$
Q^{1, \epsilon} = \text{curl} \left( (J^\epsilon)^T (G^\epsilon - \Psi^\epsilon) + \kappa (\text{curl} b^\epsilon - (J^\epsilon)^T (J^\epsilon)^{-1} J^\epsilon \text{curl} (J^\epsilon)^T (J^\epsilon)^{-1} J^\epsilon b^\epsilon) \right)
- J^\epsilon(J^\epsilon)^{-1} \partial_l ((J^\epsilon)^{-1} J^\epsilon b^\epsilon) + (J^\epsilon)^{-1} \partial_l \eta^\epsilon \partial_3 ((J^\epsilon)^{-1} J^\epsilon b^\epsilon), \tag{8.44}
$$

$$
Q^{2, \epsilon} = \left( I - (J^\epsilon)^T (J^\epsilon)^{-1} J^\epsilon \right) \hat{b}^\epsilon, \tag{8.45}
$$

$$
Q^{3, \epsilon} = (J^\epsilon)^T G^\epsilon + \kappa (\text{curl} b^\epsilon - (J^\epsilon)^T (J^\epsilon)^{-1} J^\epsilon \text{curl} (J^\epsilon)^T (J^\epsilon)^{-1} J^\epsilon b^\epsilon). \tag{8.46}
$$
It is straightforward to check that
\[
\text{div } Q^{1,\varepsilon} = 0 \text{ in } \Omega_-, \quad Q^{1,\varepsilon} \cdot e_3 = \text{div}_h (Q^{3,\varepsilon} \times e_3)_h \text{ on } \Sigma_- \text{ and } Q^{2,\varepsilon} \times e_3 = 0 \text{ on } \Sigma_+,
\]
where one has used (8.32). The \(n\)-th order compatibility conditions for (8.43) read as
\[
\begin{cases}
\text{div } \mathbf{b}_0^\varepsilon = 0 \text{ in } \Omega_-, \quad \mathbf{b}_{0,3}^\varepsilon = 0 \text{ on } \Sigma_-, \quad \left[ \partial_i^j \mathbf{b}^\varepsilon (0) \right] \times e_3 = 0 \text{ on } \Sigma, \quad j = 0, \ldots, n - 1, \\
\kappa \text{ curl } \partial_i^j \mathbf{b}^\varepsilon (0) \times e_3 = \partial_i^j Q^{3,\varepsilon} \times e_3 \text{ on } \Sigma_-, \quad j = 0, \ldots, n - 1.
\end{cases}
\]

The problem (8.43) with \(Q^{2,\varepsilon} = 0, \ Q^{3,\varepsilon} = 0\) and \(Q^{1,\varepsilon}\) given and satisfying (8.47) could be solved by employing the Galerkin method as in Ladyzhenskaya and Solonnikov [32,33]. For the general \(Q^{2,\varepsilon} \neq 0\) or \(Q^{3,\varepsilon} \neq 0\), (8.43) could be solved as in Padula and Solonnikov [38] by making use of the full parabolic regularity of the problem, which works in the anisotropic space-time Sobolev spaces; such spaces allow for the control of the resulting forcing terms when one adjusts \(Q^{2,\varepsilon}\) and \(Q^{3,\varepsilon}\) to be zero. Indeed, by (8.47), according to Proposition 3.3 (setting \(\eta = 0\)), one can define \((\mathbf{b}^\varepsilon, \tilde{\mathbf{b}}^\varepsilon)\) as the solution to

\[
\begin{cases}
\kappa \text{ curl curl } \mathbf{b}^\varepsilon = Q^{1,\varepsilon} \quad &\text{in } \Omega_- \\
\text{div } \mathbf{b}^\varepsilon = 0 \quad &\text{in } \Omega_- \\
\text{curl } \tilde{\mathbf{b}}^\varepsilon = \text{curl } Q^{2,\varepsilon} \quad &\text{in } \Omega_+ \\
\left[ \mathbf{b}^\varepsilon \right] = 0 \quad &\text{on } \Sigma \\
\mathbf{b}_{3}^\varepsilon = 0, \quad \kappa \text{ curl } \mathbf{b}^\varepsilon \times e_3 = Q^{3,\varepsilon} \quad &\text{on } \Sigma_-
\end{cases}
\]

Similarly as for the Corollary on page 151 of [38], in a higher regularity, one can show that
\[
\mathcal{R}^n (\mathbf{b}^\varepsilon, \tilde{\mathbf{b}}^\varepsilon) \lesssim \mathcal{R}^n (Q^{1,\varepsilon}, Q^{2,\varepsilon}, Q^{3,\varepsilon}),
\]
where
\[
\mathcal{R}^n (Q^{1,\varepsilon}, Q^{2,\varepsilon}, Q^{3,\varepsilon}) := \left\| Q^{1,\varepsilon} \right\|_{H^{2n-2+\ell,n-1+\ell/2}(0,T) \times \Omega_-} + \left\| Q^{2,\varepsilon} \right\|_{H^{2n+\ell,n+\ell/2}(0,T) \times \Omega_+}
\]
\[
\quad + \left\| Q^{3,\varepsilon} \right\|_{H^{2n-3/2+\ell,n-3/4+\ell/2}(0,T) \times \Sigma_-}. \tag{8.51}
\]

Then one finds that \((\mathbf{b}^\varepsilon, \tilde{\mathbf{b}}^\varepsilon) := (\mathbf{b}^\varepsilon - \mathbf{b}, \hat{\mathbf{b}}^\varepsilon - \tilde{\mathbf{b}})\) solves
\[
\begin{cases}
\partial_i^j \mathbf{b}^\varepsilon + \kappa \text{ curl curl } \mathbf{b}^\varepsilon = Q^{1,\varepsilon} := - \partial_i \mathbf{b}^\varepsilon \quad &\text{in } \Omega_- \\
\text{div } \mathbf{b}^\varepsilon = 0 \quad &\text{in } \Omega_- \\
\text{curl } \tilde{\mathbf{b}}^\varepsilon = 0, \quad \text{div } \mathbf{b}^\varepsilon = 0 \quad &\text{in } \Omega_+ \\
\left[ \mathbf{b}^\varepsilon \right] = 0 \quad &\text{on } \Sigma \\
\mathbf{b}_{3}^\varepsilon = 0, \quad \kappa \text{ curl } \mathbf{b}^\varepsilon \times e_3 = Q^{3,\varepsilon} \quad &\text{on } \Sigma_- \\
\mathbf{b} \times e_3 = 0 \quad &\text{on } \Sigma_+
\end{cases}
\]
\[
\mathbf{b}^\varepsilon \big|_{t=0} = \mathbf{b}^\varepsilon \big|_{t=0} = \mathbf{b}_0^\varepsilon - \tilde{\mathbf{b}}^\varepsilon (0). \tag{8.52}
\]
Note that by (8.50), one has
\[ R^n(\tilde{Q}^{1,\epsilon}) := \|\tilde{Q}^{1,\epsilon}\|_{H^{2n-2+\ell,n-1+\ell/2}(0,T) \times \Omega_\omega} \lesssim \|\tilde{b}^\epsilon\|_{H^{2n+\ell,n+\ell/2}(0,T) \times \Omega_\omega} \]
\[ \lesssim R^n(Q^{1,\epsilon}, Q^{2,\epsilon}, Q^{3,\epsilon}) \]
and by the trace theory (see Chapter 4 in Lions and Magenes [35]),
\[ R_0^n(b^\epsilon) := \|\tilde{b}^\epsilon\|_{H^{2n-1+\ell}(\Omega_\omega)} \lesssim R^n(b^\epsilon) + \|\tilde{b}^\epsilon\|_{H^{2n+\ell,n+\ell/2}(0,T) \times \Omega_\omega} \]
\[ \lesssim R^n(b^\epsilon) + R^n(Q^{1,\epsilon}, Q^{2,\epsilon}, Q^{3,\epsilon}). \]

We may now apply the results of Theorem 4 in [38], with a slight modification, in a higher regularity context as follows. Assume that \( b_0^\epsilon \) and \( Q^{1,\epsilon}, Q^{2,\epsilon}, Q^{3,\epsilon} \) are given such that \( R^n(b^\epsilon) < \infty, R^n(Q^{1,\epsilon}, Q^{2,\epsilon}, Q^{3,\epsilon}) < \infty \) and the \( n \)-th compatibility conditions (8.48) are satisfied. Then there exists a unique strong solution \( (b^\epsilon, \tilde{b}^\epsilon) \) to (8.52), and hence \( (\tilde{b}^\epsilon, \tilde{b}^\epsilon) \) solves (8.43) on \([0,T]\) satisfying, by (8.50), (8.53) and (8.54),
\[ R^n(b^\epsilon, \tilde{b}^\epsilon) \lesssim R^n(b^\epsilon, \tilde{b}^\epsilon) + R^n(\tilde{b}^\epsilon, \tilde{b}^\epsilon) \lesssim R^n(\tilde{b}^\epsilon) + R^n(Q^{1,\epsilon}, Q^{2,\epsilon}, Q^{3,\epsilon}) \]
\[ \lesssim R^n(b^\epsilon) + R^n(Q^{1,\epsilon}, Q^{2,\epsilon}, Q^{3,\epsilon}). \]

Indeed, the case that when \( n = 1, 1/2 < \ell < 1 \) and the first order compatibility conditions (i.e., (8.48) with \( n = 1 \)) are satisfied was proved in [38]; the restriction \( \ell < 1 \) can be relaxed to include the case of \( 1 < \ell < 3/2 \), see the last paragraph on page 578 of Solonnikov [45]. The restriction of \( \ell, 1/2 < \ell < 3/2 \) with \( \ell \neq 1 \), is required so that the trace operator of \( H^{2+\ell,1+\ell/2}(0,T) \times \Omega_\omega \) onto the set of the initial data in (8.52) satisfying the first order compatibility conditions has a bounded right inverse, see Lemma 2.1 in Beale [2] or Chapter 4 in Lions and Magenes [35]. This then allows one to adjust the initial data to be zero, see Theorem 4 in [38]. The general cases for \( n \geq 1 \) follow by an induction argument under the assumption (8.48).

We now construct solutions to (8.43) with \( Q^{1,\epsilon} = Q^{1,\epsilon}(b^\epsilon, \eta^\epsilon, G^\epsilon, \Psi^\epsilon) \), \( Q^{2,\epsilon} = Q^{2,\epsilon}(\tilde{b}^\epsilon, \eta^\epsilon) \) and \( Q^{3,\epsilon} = Q^{3,\epsilon}(b^\epsilon, \eta^\epsilon, G^\epsilon) \) defined by (8.44)–(8.46), respectively. We will use an iteration argument by making use of the smallness of \( \mathcal{E}_{2N}(\eta^\epsilon) \); it is crucial for our later use to not assume the higher order norm of \( \eta^\epsilon, \|\eta^\epsilon\|_{H^{2N-1/2+\ell,2N-1/4+\ell/2}(0,T) \times \Sigma} \), to be small. Our key point here is to apply (8.55) in two levels of regularity, i.e., \( n = 3 \) and \( 2N \), respectively. One may use the following well-known fact (see Lions and Magenes [35]) that for \( l > (d+2)/2 \) with \( d \) the spatial dimension,
\[ \|fg\|_{H^{l,r/2}} \lesssim \|f\|_{H^{l,r/2}} \|g\|_{H^{l,r/2}} + \|g\|_{H^{l+1/2}} \|f\|_{H^{l,r/2}}. \]

Recall that \( \eta^\epsilon, G^\epsilon, \Psi^\epsilon \) and \( b_0^\epsilon \) are smooth. By (8.56), it is direct to check that
\[ R^3(Q^{1,\epsilon}, Q^{2,\epsilon}, Q^{3,\epsilon}) \lesssim C_\epsilon + \sup_{[0,T]} \mathcal{E}_{2N}(\eta) R^3(b^\epsilon, \tilde{b}^\epsilon) \]
\[ \text{(8.57)} \]
and
\[ R^{2N}(Q^{1,\epsilon}, Q^{2,\epsilon}, Q^{3,\epsilon}) \lesssim C_\epsilon + \sup_{[0,T]} \mathcal{E}_{2N}(\eta) R^{2N}(b^\epsilon, \tilde{b}^\epsilon) + C_\epsilon R^3(b^\epsilon, \tilde{b}^\epsilon), \]
\[ \text{(8.58)} \]
where \( C_\epsilon \) is a positive constant depending on \( \eta^\epsilon, G^\epsilon, \Psi^\epsilon \) and \( b_0^\epsilon \). The solution to (8.43) is obtained as the limit of a sequence of approximate solutions to be constructed below. We
first extend the initial data \((\partial_t^j b^\epsilon(0), \partial_t^j \hat{b}^\epsilon(0))\) to a time-dependent function \((b^{\epsilon,0}, \hat{b}^{\epsilon,0})\) such that \((\partial_t^j b^{\epsilon,0}(0), \partial_t^j \hat{b}^{\epsilon,0}(0)) = (\partial_t^j b^\epsilon(0), \partial_t^j \hat{b}^\epsilon(0)), \ j = 0, \ldots, 2N - 1\), see for instance Lemmas A.2–A.4. Next, we claim that there exist two constants \(M_2 > M_1 > 0\), independent of \(m\), such that for \(m \geq 0\), if \((b^{\epsilon,m}, \hat{b}^{\epsilon,m})\) satisfies
\[
(\partial_t^j b^{\epsilon,m}(0), \partial_t^j \hat{b}^{\epsilon,m}(0)) = (\partial_t^j b^\epsilon(0), \partial_t^j \hat{b}^\epsilon(0)), \ j = 0, \ldots, 2N - 1
\]
and
\[
\mathcal{R}^3(b^{\epsilon,m}, \hat{b}^{\epsilon,m}) \leq M_1 \quad \text{and} \quad \mathcal{R}^{2N}(b^{\epsilon,m}, \hat{b}^{\epsilon,m}) \leq M_2,
\]
then there exists a unique solution \((b^{\epsilon,m+1}, \hat{b}^{\epsilon,m+1})\) to (8.43) with \(Q^{1,\epsilon} = Q^{1,\epsilon}(b^{\epsilon,m}, \eta^{\epsilon}, G^{\epsilon}, \Psi^{\epsilon}), Q^{2,\epsilon} = Q^{2,\epsilon}(b^{\epsilon,m}, \eta^{\epsilon})\) and \(Q^{3,\epsilon} = Q^{3,\epsilon}(b^{\epsilon,m}, \eta^{\epsilon}, G^{\epsilon})\) satisfying
\[
\mathcal{R}^3(b^{\epsilon,m+1}, \hat{b}^{\epsilon,m+1}) \leq M_1 \quad \text{and} \quad \mathcal{R}^{2N}(b^{\epsilon,m+1}, \hat{b}^{\epsilon,m+1}) \leq M_2.
\]
To prove the claim, note first that (8.57)–(8.60) imply \(\mathcal{R}^{2N}(Q^{1,\epsilon}, Q^{2,\epsilon}, Q^{3,\epsilon}) < \infty\) and that the corresponding \(2N\)-th order compatibility conditions are satisfied. Hence, one has the existence of \((b^{\epsilon,m+1}, \hat{b}^{\epsilon,m+1})\). Moreover, by (8.55) with \(n = 3\), (8.57) and the first assumption in (8.60), one obtains
\[
\mathcal{R}^3(b^{\epsilon,m+1}, \hat{b}^{\epsilon,m+1}) \leq C_2 (C_\epsilon + \delta_2 M_1).
\]
So if \(\delta_2 \leq 1/(2C_2)\) and taking \(M_1 = 2C_2 C_\epsilon\), then one has
\[
\mathcal{R}^3(b^{\epsilon,m+1}, \hat{b}^{\epsilon,m+1}) \leq M_1/2 + M_1/2 = M_1.
\]
On the other hand, by (8.55) with \(n = 2N\), (8.58) and the second assumption in (8.60), one has
\[
\mathcal{R}^{2N}(b^{\epsilon,m+1}, \hat{b}^{\epsilon,m+1}) \leq C_2 (C_\epsilon + \delta_2 M_2 + C_\epsilon M_1).
\]
Hence, taking \(M_2 = 2C_2 C_\epsilon (1 + M_1)\), one then gets
\[
\mathcal{R}^{2N}(b^{\epsilon,m+1}, \hat{b}^{\epsilon,m+1}) \leq M_2/2 + M_2/2 = M_2.
\]
Thus the claim is proved. Note that if one has taken \(M_1 \geq \mathcal{R}^{2N}(b^{\epsilon,0}, \hat{b}^{\epsilon,0})\), then (8.60) holds for \(m = 0\). Consequently, one can then iterate from \(m = 0\) to construct the sequence of approximate solutions \(\{(b^{\epsilon,m}, \hat{b}^{\epsilon,m})\}_{m=1}^\infty\).

The uniform estimates in (8.60) imply that as \(m \to \infty\), up to a subsequence, the sequence \((b^{\epsilon,m}, \hat{b}^{\epsilon,m})\) converges to a limit \((b^\epsilon, \hat{b}^\epsilon)\) in the weak or weak-* sense of the norms \(\mathcal{R}^{2N}\). Moreover, according to the weak lower semicontinuity, one has
\[
\mathcal{R}^{2N}(b^\epsilon, \hat{b}^\epsilon) \leq M_2.
\]
Now we prove the contraction of the approximate sequence \(\{(b^{\epsilon,m}, \hat{b}^{\epsilon,m})\}_{m=0}^\infty\). For \(m \geq 1\), set \(B^{\epsilon,m} = b^{\epsilon,m} - b^{\epsilon,m-1}\) and \(\hat{B}^{\epsilon,m} = \hat{b}^{\epsilon,m} - \hat{b}^{\epsilon,m-1}\). Then \((B^{\epsilon,m+1}, \hat{B}^{\epsilon,m+1})\) solves (8.43) with \(Q^{1,\epsilon} = Q^{1,\epsilon}(B^{\epsilon,m}, \eta^{\epsilon}, 0, 0), Q^{2,\epsilon} = Q^{2,\epsilon}(\hat{B}^{\epsilon,m}, \eta^{\epsilon}), Q^{3,\epsilon} = Q^{3,\epsilon}(B^{\epsilon,m}, \eta^{\epsilon}, 0)\) and \(B^{\epsilon,m+1}(0) = 0\). Hence, in the same way as for (8.62), one has
\[
\mathcal{R}^3(B^{\epsilon,m+1}, \hat{B}^{\epsilon,m+1}) \leq C_2 \delta_2 \mathcal{R}^3(B^\epsilon, \hat{B}^\epsilon) \leq \frac{1}{2} \mathcal{R}^3(b^{\epsilon,m}, \hat{b}^{\epsilon,m}).
\]
This implies that the sequence \( \{(b^\epsilon_m, \hat{b}^\epsilon_m)\}_{m=0}^\infty \) is contractive in the norm \( \mathcal{F}^3 \) and then converges to the limit \( (b^\epsilon, \hat{b}^\epsilon) \), strongly in the norm \( \mathcal{F}^3 \), which is a strong solution to the original problem (8.43) on \([0, T]\) satisfying (8.66). The uniqueness of solutions to (8.43) satisfying (8.66) can be obtained by a similar argument as for the contraction.

Note that with the \((b^\epsilon, \hat{b}^\epsilon)\) in hand, \((b^\epsilon, \hat{b}^\epsilon) = (J^\epsilon)^{-1} J^\epsilon (b^\epsilon, \hat{b}^\epsilon)\) is then the unique solution to (8.27) on \([0, T]\) satisfying (8.41), which follows from (8.66).

**Remark 8.4.** Since \( \eta^\epsilon \) and \( G^\epsilon \) are smooth, one may employ a standard parabolic regularization argument (see for instance [3]) to show that the solution \((b^\epsilon, \hat{b}^\epsilon)\) to (8.27) constructed in Lemma 8.3 is indeed smooth for any positive time \( t > 0 \).

Now we shall derive the uniform estimates of the approximate solutions, independent of the smoothing parameter \( \epsilon > 0 \), to take the limit as \( \epsilon \to 0 \). Recall \( \mathcal{E}_{2N}(b, \hat{b}) \), and define

\[
\mathbb{D}_{2N}(b, \hat{b}) := \sum_{j=0}^{2N} \left\| \partial_j^2 b \right\|_{2N-j+1}^2 + \sum_{j=0}^{2N} \left\| \partial_j^2 \hat{b} \right\|_{2N-j+1}^2. \tag{8.68}
\]

**Proposition 8.5.** Let \( N \geq 4 \) be an integer. Assume that for \( 0 < T \leq 1 \), \( \sup_{[0, T]} \mathcal{E}_{2N}(\eta) < \infty \), \( \mathfrak{M}_{2N}(G) + \mathfrak{M}_{2N}(G) < \infty \) and \( b_0 \in H^{2N+1}(\Omega_-) \) are given such that \( \mathcal{E}_{2N}(b, \hat{b})(0) < \infty \) and the \( 2N\)-th order compatibility conditions (8.25) are satisfied. There exists a universal constant \( \delta_3 > 0 \) such that if \( \sup_{[0, T]} \mathcal{E}_{2N}(\eta) \leq \delta_3 \), then there exists a unique solution \((b, \hat{b})\) to (8.22) on \([0, T]\) satisfying

\[
\sup_{[0, T]} \mathcal{E}_{2N}(b, \hat{b}) + \int_0^T \mathbb{D}_{2N}(b, \hat{b}) \lesssim \mathcal{E}_{2N}(b, \hat{b})(0) + \mathfrak{M}_{2N}(G) + \mathfrak{M}_{2N}(G). \tag{8.69}
\]

**Proof.** For each \( \epsilon > 0 \), let \((b^\epsilon, \hat{b}^\epsilon)\) be the solution to the regularized problem (8.27) on \([0, T]\) constructed in Lemma 8.3. As (8.27) is solved on \([0, T]\), similarly as Remark 4.1, it follows that there exists solutions \( \hat{E}^\epsilon \) on \([0, T]\) to the corresponding regularized electric system in vacuum:

\[
\begin{align*}
\text{curl}^\epsilon \hat{E}^\epsilon &= \hat{\nabla}^\epsilon \hat{b}^\epsilon, & \text{div}^\epsilon \hat{E}^\epsilon &= 0 \quad \text{in } \Omega_+ \\
\hat{E}^\epsilon \times N^\epsilon &= (-\kappa \text{curl}^\epsilon b^\epsilon + G^\epsilon - \Psi^\epsilon) \times N^\epsilon \quad \text{on } \Sigma \\
\hat{E}_3^\epsilon &= 0 \quad \text{on } \Sigma_+.
\end{align*} \tag{8.70}
\]

Since \((b^\epsilon, \hat{b}^\epsilon)\) is smooth for \( t > 0 \) (see Remark 8.4), \( \hat{E}^\epsilon \) is indeed smooth for \( t > 0 \).

We now derive the \( \epsilon \)-independent estimates of \((b^\epsilon, \hat{b}^\epsilon)\) (and \( \hat{E}^\epsilon \)). The proof follows similarly as that of the a priori estimates of (2.5), not involving to the hydrodynamic part, and we provide only the necessary modifications.

First, one may follow the proof of Proposition 4.3 to deduce

\[
\mathcal{E}_{2N}(\hat{E}^\epsilon) \lesssim \mathcal{E}_{2N}(b^\epsilon, \hat{b}^\epsilon) + \mathfrak{M}_{2N}(G^\epsilon) + \mathfrak{M}_{2N}(\Psi^\epsilon). \tag{8.71}
\]

Next, following the proof of Proposition 5.1 yields

\[
\begin{align*}
\sup_{[0, T]} \mathcal{E}_{2N}(b^\epsilon, \hat{b}^\epsilon)(t) + \int_0^T \mathcal{D}_{2N}(b^\epsilon) \\
\lesssim \mathcal{E}_{2N}(b^\epsilon, \hat{b}^\epsilon)(0) + \int_0^T \sqrt{\mathcal{E}_{2N}(\eta)} \left( \mathcal{D}_{2N}(b^\epsilon, \hat{b}^\epsilon) + \mathcal{E}_{2N}(\hat{E}^\epsilon) \right) + \mathfrak{M}_{2N}(G^\epsilon) + \mathfrak{M}_{2N}(\Psi^\epsilon). \tag{8.72}
\end{align*}
\]
Following a variant in the proof of Proposition 6.4 of applying the elliptic theory yields

\[
\mathcal{E}_{2N}(b^e, \hat{b}^e) \lesssim \mathcal{E}_{2N}(b^e, \hat{b}^e) + \mathcal{E}_{2N}(\eta)\mathcal{E}_{2N}(b^e, \hat{b}^e) + \delta_{2N}^{\mathcal{E}}(G^e) + \delta_{\infty}^{2N}(\Psi^e)
\]  
(8.73)

and

\[
\int_0^T \mathbb{D}_{2N}(b^e, \hat{b}^e) \lesssim \int_0^T \mathbb{D}_{2N}(b^e, \hat{b}^e) + \int_0^T \mathcal{E}_{2N}(\eta)\mathbb{D}_{2N}(b^e, \hat{b}^e) + \delta_{2N}^{\mathcal{E}}(G^e) + \delta_{\infty}^{2N}(\Psi^e)
\].
(8.74)

Since \(\delta_3\) is small, combining (8.71)–(8.74) yields

\[
\sup_{[0,T]} \mathcal{E}_{2N}(b^e, \hat{b}^e) + \int_0^T \mathbb{D}_{2N}(b^e, \hat{b}^e) \lesssim \mathcal{E}_{2N}(b^e, \hat{b}^e)(0) + \delta_{\infty}^{2N}(G) + \delta_{\infty}^{2N}(G) + \delta_{\infty}^{2N}(\Psi^e). 
\]  
(8.75)

It follows from Lemma 8.2 and (8.75) that

\[
\sup_{[0,T]} \mathcal{E}_{2N}(b^e, \hat{b}^e) + \int_0^T \mathbb{D}_{2N}(b^e, \hat{b}^e) \lesssim \mathcal{E}_{2N}(b, \hat{b})(0) + \delta_{\infty}^{2N}(G) + \delta_{\infty}^{2N}(G) + \epsilon. 
\]  
(8.76)

The estimate (8.76) allows one to conclude that as \(\epsilon \to 0\), up to extraction of a subsequence, the sequence \((b^e, \hat{b}^e)\) converges to a limit \((b, \hat{b})\) in the weak or weak-* sense of the norms in the left hand side of (8.76), which makes it possible to take the limit in (8.27) to find that \((b, \hat{b})\) solves (8.22). The estimate (8.69) follows from the the weak lower semicontinuity of the left hand side of (8.76) and passing to the limit in the right hand side. \(\square\)

8.4. Sequence of approximate solutions. The solution to the problem (2.5) will be obtained by the method of successive approximations. The sequence of approximate solutions, \(\{(u^m, p^m, \eta^m, b^m, \hat{b}^m)\}_{m=0}^\infty\), is constructed as follows. First, one constructs \((u^0, b^0, \eta^0)\) achieving the initial data. Second, assuming that \((u^m, b^m, \eta^m)\) for \(m \geq 0\) achieves the initial data and satisfies suitable estimates to be specified later, we define \((u^{m+1}, p^{m+1}, \eta^{m+1}, b^{m+1}, \hat{b}^{m+1})\) as the solution to

\[
\begin{align*}
\partial_t^m u^{m+1} + u^{m+1} \cdot \nabla u^{m+1} + \nabla \psi^{m+1} p^{m+1} &= \text{curl} \psi \ b^{m} \times (\hat{B} + b^{m}) & & \text{in } \Omega_- \\
\text{div} \psi^{m+1} u^{m+1} &= 0 & & \text{in } \Omega_- \\
\partial_t^m b^{m+1} + \kappa \text{curl} \psi^{m} \text{curl} \psi^{m} b^{m+1} &= \text{curl} \psi^{m} (u^{m+1} \times (\hat{B} + b^{m})) & & \text{in } \Omega_- \\
\text{div} \psi^{m} b^{m+1} &= 0 & & \text{in } \Omega_- \\
\text{curl} \psi^{m} \hat{b}^{m+1} &= 0, \text{ div} \psi^{m} \hat{b}^{m+1} &= 0 & & \text{in } \Omega_+ \\
\partial_t \eta^{m+1} &= u^{m+1} \cdot \nabla \eta^{m+1} & & \text{on } \Sigma_+ \\
p^{m+1} &= -\sigma H^{m+1}, \text{ } [b^{m+1}] = 0 & & \text{on } \Sigma \\
u_3^{m+1} &= 0, b_3^{m+1} = 0, \text{ } \kappa \text{curl} \psi^{m} b^{m+1} \times e_3 = (u^{m+1} \times (\hat{B} + b^{m})) \times e_3 & & \text{on } \Sigma_- \\
b^{m+1} \times e_3 &= 0 & & \text{on } \Sigma_+ \\
(u^{m+1}, b^{m+1}, \eta^{m+1}) |_{t=0} &= (u_0, b_0, \eta_0). 
\end{align*}
\]  
(8.77)
Here \( \varphi^m = \varphi(\eta^m), N^m = (-\nabla_h \eta^m, 1) \) and \( H^m = H(\eta^m) \) as in (1.7).

This construction and the corresponding estimates are recorded in the following proposition.

**Proposition 8.6.** There exist universal positive constants \( \tilde{\delta}_1 \) and \( T_1 \) such that if \( \mathcal{E}_{2N}(0) \leq \tilde{\delta}_1 \) and \( 0 < T \leq T_1 \), then there exists a sequence \( \{ (u^m, p^m, \eta^m, b^m, \hat{b}^m) \}_{m=0}^\infty \) that solves (8.77) on \([0, T]\) and satisfies the following estimates

\[
\sup_{[0, T]} \mathcal{E}_{2N}(u^m, p^m, \eta^m, b^m, \hat{b}^m) + \int_0^T \mathbb{D}_{2N}(b^m, \hat{b}^m) \lesssim \mathcal{E}_{2N}(0). \tag{8.78}
\]

**Proof.** First, extend the initial data \((\partial^j u(0), \partial^j b(0), \partial^j \eta(0))\), \( j = 0, \ldots, 2N - 1 \), to the time-dependent functions \((u^0, b^0, \eta^0)\) so that \((\partial^j u^0(0), \partial^j b^0(0), \partial^j \eta^0(0)) = (\partial^j u(0), \partial^j b(0), \partial^j \eta(0))\). This can be done by applying Lemmas A.2–A.4, and one has in particular that

\[
\sup_{[0, \infty]} \mathcal{E}_{2N}(u^0, b^0, \eta^0) + \int_0^\infty \mathbb{D}_{2N}(b^0) \leq C_1 \mathcal{E}_{2N}(0). \tag{8.79}
\]

Next, we claim that there exist \( \gamma_1, \gamma_2 > 0 \) and \( T > 0 \) such that if \((u^m, b^m, \eta^m)\) achieves the initial data and satisfies

\[
\sup_{[0, T]} \mathcal{E}_{2N}(u^m, \eta^m) \leq \gamma_1 \mathcal{E}_{2N}(0) \tag{8.80}
\]

and

\[
\sup_{[0, T]} \mathcal{E}_{2N}(b^m) + \int_0^T \mathbb{D}_{2N}(b^m) \leq \gamma_2 \mathcal{E}_{2N}(0), \tag{8.81}
\]

then there exists a unique solution \((u^{m+1}, p^{m+1}, \eta^{m+1}, b^{m+1}, \hat{b}^{m+1})\) to (8.77) on \([0, T]\) satisfying

\[
\sup_{[0, T]} \mathcal{E}_{2N}(u^{m+1}, p^{m+1}, \eta^{m+1}) \leq \gamma_1 \mathcal{E}_{2N}(0) \tag{8.82}
\]

and

\[
\sup_{[0, T]} \mathcal{E}_{2N}(b^{m+1}, \hat{b}^{m+1}) + \int_0^T \mathbb{D}_{2N}(b^{m+1}, \hat{b}^{m+1}) \leq \gamma_2 \mathcal{E}_{2N}(0). \tag{8.83}
\]

To prove the claim, one may first use \((b^m, \eta^m)\) to construct \((u^{m+1}, p^{m+1}, \eta^{m+1})\) as the solution to (8.7) with \( F = \text{curl}^m b^m \times (\hat{B} + b^m) \). Recall the notations (8.9) and (8.10). Note that

\[
\tilde{\mathcal{E}}^{2N}_0(F) \lesssim \{1 + \mathcal{E}_{2N}(b^m, \eta^m)(0)\} \mathcal{E}_{2N}(b)(0) \lesssim \mathcal{E}_{2N}(0), \tag{8.84}
\]

and by (8.80) and (8.81),

\[
\tilde{\mathcal{E}}^{2N}_2(F) \lesssim \int_0^T \{1 + \mathcal{E}_{2N}(\eta^m, b^m)\} \mathbb{D}_{2N}(b^m) \lesssim (1 + (\gamma_1 + \gamma_2) \mathcal{E}_{2N}(0)) \gamma_2 \mathcal{E}_{2N}(0). \tag{8.85}
\]
Hence,

\[
\delta_0^2 N(F) + T \delta_2^2 N(F) \leq C_2 (1 + T \gamma_2 (1 + (\gamma_1 + \gamma_2) E_{2N}(0))) E_{2N}(0). \tag{8.86}
\]

If \( \delta_1 \leq \min\{1/(\gamma_1 + \gamma_2), \delta_1/(1 + C_2(1 + 2\gamma_2))\} \) with \( \delta_1 \) given in Proposition 8.1, then

\[
E_{2N}(0) + \delta_0^2 N(F) + T \delta_2^2 N(F) \leq (1 + C_2 (1 + T \gamma_2 (1 + (\gamma_1 + \gamma_2) E_{2N}(0))) E_{2N}(0)
\leq (1 + C_2 (1 + 2T \gamma_2)) E_{2N}(0) \leq \delta_1, \quad \forall T \leq 1. \tag{8.87}
\]

Hence, Proposition 8.1 guarantees the existence of a unique \((u^{m+1}, p^{m+1}, \eta^{m+1})\) solving (8.7) on \([0, T]\) for any \(T \leq 1\). Moreover, (8.12) and (8.87) imply

\[
\sup_{[0,T]} E_{2N}(u^{m+1}, p^{m+1}, \eta^{m+1}) \leq C_3 C_2 (1 + 2T \gamma_2) E_{2N}(0). \tag{8.88}
\]

If \( \gamma_1 \geq 2C_3C_2 \) and \( T \leq 1/(2\gamma_2) \), then (8.88) yields (8.82).

Now one uses \((u^{m+1}, b^m, \eta^m)\) to construct \((b^{m+1}, \hat{b}^{m+1})\) as the solution to (8.22) with \( \varphi = \varphi^m \) and \( G = u^{m+1} \times (\hat{B} + b^m) \). It follows from (8.82) and (8.81) that for \( T > 0 \) in the above,

\[
\delta_\infty^2 N(G) + \delta_2^2 N(G) \lesssim (1 + T) \sup_{[0,T]} E_{2N}(u^{m+1}) \left(1 + E_{2N}(b^{m+1})\right) \lesssim \gamma_1 E_{2N}(0) (1 + \gamma_2 E_{2N}(0)). \tag{8.89}
\]

If \( \tilde{\delta}_1 \leq \delta_3/\gamma_1 \) with \( \delta_3 \) given in Proposition 8.5 and hence by (8.80), \( E_{2N}(\eta^m) \leq \delta_3 \), then Proposition 8.5 guarantees the existence of a unique \((b^{m+1}, \hat{b}^{m+1})\) solving (8.22) on \([0, T]\). In addition, (8.69) and (8.89) imply

\[
\sup_{[0,T]} E_{2N}(b^{m+1}, \hat{b}^{m+1}) + \int_0^T D_{2N}(b^{m+1}, \hat{b}^{m+1}) \leq C_4 (E_{2N}(0) + \gamma_1 E_{2N}(0) (1 + \gamma_2 E_{2N}(0))). \tag{8.90}
\]

Hence if \( \tilde{\delta}_1 \leq 1/\gamma_2 \) and \( \gamma_2 \geq C_4(1 + 2\gamma_1) \), then (8.90) yields (8.83).

Consequently, the claim is proved. Note that if \( \gamma_1, \gamma_2 \geq C_1 \), then (8.79) implies that (8.80) and (8.81) hold for \( m = 0 \). Hence, one can then iterate from \( m = 0 \) to construct the sequence \((u^m, p^m, \eta^m, b^m, \hat{b}^m)_{m=0}^\infty \) satisfying the conclusions. \( \square \)

Now we prove the contraction of the sequence \((u^m, p^m, \eta^m, b^m, \hat{b}^m)_{m=0}^\infty \). For \( m \geq 1 \), define

\[
U^m = u^m - u^{m-1}, \quad Q^m = p^m - p^{m-1}, \quad \xi^m = \eta^m - \eta^{m-1},
\]

\[
B^m = b^m - b^{m-1}, \quad \hat{B}^m = \hat{b}^m - \hat{b}^{m-1}. \tag{8.91}
\]
Then it follows from (8.77) that

\[
\begin{align*}
\begin{cases}
\partial_t \psi^m U^{m+1} + u^m \cdot \nabla \psi^m U^{m+1} + \nabla \psi^m Q^{m+1} = F^{1,m} & \text{in } \Omega_-
\div \psi^m U^{m+1} = F^{2,m} & \text{in } \Omega_-
\partial_t \psi^m B^{m+1} + \kappa \curl \psi^m \curl \psi^m B^{m+1} = F^{3,m} & \text{in } \Omega_-
\div \psi^m \hat{B}^{m+1} = F^{4,m} & \text{in } \Omega_-
\curl \psi^m \hat{B}^{m+1} = F^{5,m} & \text{in } \Omega_-
\partial_t \zeta^{m+1} = U^{m+1} \cdot N^m + F^{6,m} & \text{on } \Sigma
Q^{m+1} = -\sigma \Delta_h \zeta^{m+1} + F^{6,m}, \quad \left[ B^{m+1} \right] = 0 & \text{on } \Sigma
U_3^{m+1} = 0, \quad B_3^{m+1} = 0, \quad \curl \psi^m B^{m+1} \times e_3 = F^{7,m} & \text{on } \Sigma_-
\hat{B}^{m+1} \times e_3 = 0 & \text{on } \Sigma_+
\left( U^{m+1}, B^{m+1}, \zeta^{m+1} \right) |_{t=0} = (0, 0, 0),
\end{cases}
\end{align*}
\]

where

\[
\begin{align*}
F^{1,m} &= - (\partial_t \psi^m - \partial_t \hat{\psi}^m) u^{m+1} - \left( u^{m+1} \cdot \nabla \psi^m - u^m \cdot \nabla \psi^m \right) u^{m+1} - \left( \nabla \psi^m - \nabla \hat{\psi}^m \right) p^{m+1} + \curl \psi^m b^m \times (\tilde{B} + b^m) - \curl \psi^{m-1} b^{m-1} \times (\tilde{B} + b^{m-1}), \\
F^{2,m} &= - \left( \div \psi^m - \div \hat{\psi}^m \right) u^{m+1}, \\
F^{3,m} &= - (\partial_t \psi^m - \partial_t \hat{\psi}^m) b^{m+1} - \kappa \left( \curl \psi^m \curl \psi^m - \curl \psi^{m-1} \curl \psi^{m-1} \right) b^{m+1} + \curl \psi^m u^{m+1} \times (\tilde{B} + b^m) - \curl \psi^{m-1} u^m \times (\tilde{B} + b^{m-1}), \\
F^{4,m} &= - \left( \div \psi^m - \div \hat{\psi}^m \right) b^{m+1}, \\
\hat{F}^{3,m} &= - \left( \curl \psi^m - \curl \psi^{m-1} \right) \hat{b}^{m+1}, \\
\hat{F}^{4,m} &= - \left( \div \psi^m - \div \hat{\psi}^m \right) \hat{b}^{m+1}, \\
F^{5,m} &= \left( u^{m+1} \cdot N^m + N^m \right), \\
F^{6,m} &= - \sigma \div_h \left( \frac{\nabla_h \eta^{m+1}}{\sqrt{1 + |\nabla_h \eta^{m+1}|^2}} - \nabla_h \eta^{m+1} \right) + \sigma \div_h \left( \frac{\nabla_h \eta^m}{\sqrt{1 + |\nabla_h \eta^m|^2}} - \nabla_h \eta^m \right), \\
F^{7,m} &= \left( \kappa (-\curl \psi^m + \curl \psi^{m-1}) b^m + u^{m+1} \times (\tilde{B} + b^m) - u^m \times (\tilde{B} + b^{m-1}) \right) \times e_3.
\end{align*}
\]

The contraction of the sequence in the lower-order energy, say, $\mathcal{E}_N$, is given as follows.

**Proposition 8.7.** There exist universal positive constants $\tilde{\delta}_2 \leq \tilde{\delta}_1$ and $T_2 \leq T_1$ such that if $\mathcal{E}_N(0) \leq \tilde{\delta}_2$ and $T \in (0, T_2]$, then it holds that

\[
\begin{align*}
\sup_{[0,T]} \mathcal{E}_N(U^{m+1}, Q^{m+1}, \zeta^{m+1}, B^{m+1}, \hat{B}^{m+1}) + \int_0^T \mathbb{D}_2 N(B^{m+1}, \hat{B}^{m+1}) &= \left( \sup_{[0,T]} \mathcal{E}_N(U^m, \zeta^m, B^m) + \int_0^T \mathbb{D}_2 N(B^m) \right).
\end{align*}
\]
Proof. By (8.78), one has the uniform smallness that
\[ \mathcal{E}_{2N}(u^m, p^m, \eta^m, b^m, \hat{b}^m) \lesssim \mathcal{E}_{2N}(0) \leq \tilde{\delta}_2. \] (8.103)

Then following the proof of Proposition 8.1 with slight modifications, one may conclude that
\[ \sup_{[0,T]} \mathcal{E}_N(U^{m+1}, Q^{m+1}, \zeta^{m+1}) \lesssim \tilde{\delta}_2 \sup_{[0,T]} \mathcal{E}_N(U^{m+1}, Q^{m+1}, \zeta^{m+1}, \zeta^m, B^m) \]
\[ + \sqrt{\sup_{[0,T]} \mathcal{E}_{2N}(U^{m+1}, \zeta^{m+1}) T} \int_0^T \mathbb{D}_N(B^m) + T \int_0^T \mathbb{D}_N(B^m). \] (8.104)

On the other hand, modifying the proof of Proposition 8.5 slightly yields
\[ \sup_{[0,T]} \mathcal{E}_N(B^{m+1}, \hat{B}^{m+1}) + \int_0^T \mathbb{D}_N(B^{m+1}, \hat{B}^{m+1}) \]
\[ \lesssim \sup_{[0,T]} \mathcal{E}_N(U^{m+1}) + \tilde{\delta}_2 \sup_{[0,T]} \mathcal{E}_N(U^{m+1}, B^{m+1}, \hat{B}^{m+1}, \zeta^m, B^m) \]
\[ + \tilde{\delta}_2 \int_0^T \mathbb{D}_N(B^{m+1}, \hat{B}^{m+1}, B^m). \] (8.105)

Hence, combining (8.104) and (8.105) shows, by the smallness of \( \tilde{\delta}_2 \) and Cauchy’s inequality,
\[ \sup_{[0,T]} \mathcal{E}_N(U^{m+1}, Q^{m+1}, \zeta^{m+1}, B^{m+1}, \hat{B}^{m+1}) + \int_0^T \mathbb{D}_N(B^{m+1}, \hat{B}^{m+1}) \]
\[ \leq C_1 \left( \tilde{\delta}_2 \sup_{[0,T]} \left( \mathcal{E}_N(U^m, \zeta^m, B^m) \right) + (\tilde{\delta}_2 + T) \int_0^T \mathbb{D}_N(B^m) \right). \] (8.106)

Consequently, if \( \tilde{\delta}_2 \leq 1/(4C_1) \) and \( T \leq 1/(4C_1) \), then (8.106) implies in particular (8.102). \( \Box \)

8.5. Local well-posedness of (2.5). We are now ready to state the local well-posedness result for (2.5).

Theorem 8.8. Assume that \( \kappa > 0 \) and \( \sigma > 0 \) and let \( N \geq 4 \) be an integer. Assume that \( u_0 \in H^{2N}(\Omega) \), \( b_0 \in H^{2N+1}(\Omega) \) and \( \eta_0 \in H^{2N+3/2}(\Sigma) \) are given such that \( \mathcal{E}_{2N}(0) < \infty \) and that the compatibility conditions (8.6) are satisfied. There exist universal positive constants \( \delta_0 \) and \( T_0 \) such that if \( \mathcal{E}_{2N}(0) \leq \delta_0 \) and \( 0 < T \leq T_0 \), then there exists a unique solution \((u, p, \eta, b, \hat{b})\) to (2.5) on \([0, T]\) satisfying
\[ \sup_{[0,T]} \mathcal{E}_{2N} + \int_0^T \mathbb{D}_{2N}(b, \hat{b}) \leq \tilde{C}_2 \mathcal{E}_{2N}(0). \] (8.107)
Proof. Take $\delta_0 = \tilde{\delta}$ and $T_0 = T_2$ as given in Proposition 8.7. Let $\{(u^m, p^m, \eta^m, b^m, \hat{b}^m)\}_{m=1}^{\infty}$ on $[0, T]$ with $0 < T \leq T_0$ be the sequence constructed in Proposition 8.6 that solves (8.77). The uniform estimate (8.78) implies that as $m \to \infty$, up to extraction of a subsequence, the sequence $(u^m, p^m, \eta^m, b^m, \hat{b}^m)$ converges to a limit $(u, p, \eta, b, \hat{b})$ in the weak or weak-$*$ sense of the norms in (8.78). Then the weak lower semicontinuity shows that $(u, p, \eta, b, \hat{b})$ satisfies the estimate (8.107). On the other hand, the contractive estimate (8.102) shows that the whole sequence $(u^m, p^m, \eta^m, b^m, \hat{b}^m)$ converges strongly to the limit $(u, p, \eta, b, \hat{b})$ in the norms of $E_N$, which is sufficient for passing to the limit in (8.77). Then one finds that the limit $(u, p, \eta, b, \hat{b})$ is a strong solution to (2.5) on $[0, T]$. The uniqueness of solutions to (2.5) satisfying (8.107) follows similarly as in the proof of the contraction. □

Remark 8.9. It is possible to remove the smallness assumption of $u_0$ and $b_0$ in Theorem 8.8 by restricting the local existence time of the solution to be smaller, depending on the initial data, see Padula and Solonnikov [38]. However, much more work is required if one would like to relax the smallness of $\eta_0$, owing to the way of solving the magnetic part; yet the local well-posedness recorded in Theorem 8.8 is sufficient to be adapted in the proof of our main theorem of global well-posedness.

9. Global Well-Posedness

In this section we prove the global well-posedness of (2.5) as follows.

Proof of Theorem 2.2. Recall the constants $\tilde{\delta}$ and $\tilde{C}_1$ in Theorem 7.3 and $\delta_0$, $T_0$ and $\tilde{C}_2$ in Theorem 8.8, and, without loss of generality, assume $\tilde{C}_1, \tilde{C}_2 \geq 1$. Assume that $E_{2N}(0) \leq \epsilon_0$ for

$$\epsilon_0 = \min \left\{ \frac{\delta_0}{\tilde{C}_1}, \frac{\tilde{\delta}}{2\tilde{C}_1\tilde{C}_2} \right\}. \quad (9.1)$$

Set

$$T^* := \sup_T \left\{ T > 0 \mid \text{There exists a unique solution to (2.5) on } [0, T] \text{ satisfying } \sup_{[0,T]} E_{2N} \leq 2\tilde{C}_1\tilde{C}_2\epsilon_0 \right\}. \quad (9.2)$$

It follows from Theorem 8.8 that $T^* \geq T_0 > 0$. We will show that $T^* = \infty$ by contradiction. Assume that $T^* < \infty$. Then for any $0 < T < T^*$, it follows from (9.2) and (9.1) that

$$\sup_{[0,T]} E_{2N} \leq 2\tilde{C}_1\tilde{C}_2\epsilon_0 \leq \tilde{\delta}. \quad (9.3)$$

Then Theorem 7.3 shows

$$\sup_{[0,T]} E_{2N} \leq \tilde{C}_1E_{2N}(0) \leq \tilde{C}_1\epsilon_0 \leq \delta_0. \quad (9.4)$$

Now taking the $T$ above as the initial time, one can apply Theorem 8.8 again to find that there is a unique solution to (2.5) on $[T, T + T_0]$ satisfying

$$\sup_{[T,T+T_0]} E_{2N} \leq \tilde{C}_2E_{2N}(T) \leq \tilde{C}_1\tilde{C}_2\epsilon_0. \quad (9.5)$$

This contradicts the definition of $T^*$, and so $T^* = \infty$. Hence, the solution is global and the estimates (2.14) and (2.15) follow from Theorem 7.3. □
10. The Plasma-Plasma Interface Problem

In this section we will present a similar global well-posedness for the plasma-plasma interface problem for the incompressible inviscid and resistive MHD, where the two immiscible plasmas occupy the two regions $\Omega_{\pm}(t)$ respectively. Assume that the velocities $u_{\pm}$, the pressures $p_{\pm}$ and the magnetic fields $B_{\pm}$ satisfy the following problem:

\[
\begin{align*}
\partial_t u_\pm + u_\pm \cdot \nabla u_\pm + \nabla p_\pm &= \text{curl} B_\pm \times B_\pm & \text{in} \Omega_{\pm}(t) \\
\text{div} u_\pm &= 0 & \text{in} \Omega_{\pm}(t) \\
\partial_t B_\pm &= \text{curl} E_\pm, \quad E_\pm = u_\pm \times B_\pm - \kappa \text{curl} B_\pm & \text{in} \Omega_{\pm}(t) \\
\text{div} B_\pm &= 0 & \text{in} \Omega_{\pm}(t) \\
\partial_\eta \eta &= u_\pm \cdot \mathcal{N} & \text{on} \Sigma(t) \\
p_+ &= p_- + \sigma H, \quad B_+ = B_-, \quad E_+ \times \mathcal{N} = E_- \times \mathcal{N} & \text{on} \Sigma(t) \\
\begin{cases}
u_+ \cdot e_3 = 0, & \quad B_+ \times e_3 = \tilde{B} \times e_3 \\
u_- \cdot e_3 = 0, & \quad B_- \cdot e_3 = \tilde{B} \cdot e_3, \quad E_- \times e_3 = 0
\end{cases} & \text{on} \Sigma.
\end{align*}
\]

(10.1)

Here $E_{\pm}$ are the electric fields of the plasmas and $\kappa_{\pm} > 0$ are the magnetic diffusion coefficients. Note that we have set the problem to be around the traversal magnetic field $\tilde{B}$ with $\tilde{B}_3 \neq 0$.

Use again the mapping $\Phi$ (cf. (2.2)) to transform the problem (10.1) to the one in $\Omega_{\pm}$. Since the domains $\Omega_{\pm}$ are fixed, one may simplify notation by writing $f$ to refer to $f_{\pm}$. In the following, an equation for $f$ on $\Omega$ means that the equation holds for $f_{\pm}$ on $\Omega_{\pm}$, and an equation involving $f$ on $\Sigma$ means that the equation holds for both $f = f_+$ and $f = f_-$. For a quantity $f = f_{\pm}$, the interfacial jump on $\Sigma$ is defined as

\[
[f] := f_+|_{\Sigma} - f_-|_{\Sigma}.
\]

(10.2)

Then the problem (10.1) is equivalent to the following problem for $(u, p, \eta, b, \tilde{b})$, with $b = B - \tilde{B}$,

\[
\begin{align*}
\partial_t^{\nu} u + u \cdot \nabla^{\nu} u + \nabla^{\nu} p &= \text{curl}^{\nu} b \times (\tilde{B} + b) & \text{in} \Omega \\
\text{div}^{\nu} u &= 0 & \text{in} \Omega \\
\partial_t^{\nu} b &= \text{curl}^{\nu} E, \quad E = u \times (\tilde{B} + b) - \kappa \text{curl}^{\nu} b & \text{in} \Omega \\
\text{div}^{\nu} b &= 0 & \text{in} \Omega \\
\partial_\eta \eta &= u \cdot \mathcal{N} & \text{on} \Sigma \\
[p] &= \sigma H, \quad [b] = 0, \quad [E] \times \mathcal{N} = 0 & \text{on} \Sigma \\
u_3 = 0, & \quad b \times e_3 = 0 & \text{on} \Sigma_+ \\
u_3 = 0, & \quad b_3 = 0, \quad E \times e_3 = 0 & \text{on} \Sigma_- \\
(u, b, \eta) \mid_{t=0} = (u_0, b_0, \eta_0).
\end{align*}
\]

(10.3)

As the pressure $p$ is uniquely determined up to constants (constant in space), to guarantee the uniqueness of $p$ and without loss of generality, one may require $\int_{\Sigma} p = 0$.

Given the initial data $(u_0, b_0, \eta_0)$, one needs to use the equations (10.3) to construct the data $\partial_t^{j} u(0)$ and $\partial_t^{j} b(0)$ for $j = 1, \ldots, 2N$, $\partial_t^{j} p(0)$ for $j = 0, \ldots, 2N - 1$ and $\partial_t^{j} \eta(0)$ for $j = 1, \ldots, 2N + 1$. These data need to satisfy the $2N$-th order compatibility conditions:
\[ \begin{align*}
\text{div}^{\sigma_0} u_0 &= 0 \text{ in } \Omega, \quad [u_0] \cdot N_0 = 0 \text{ on } \Sigma, \quad u_{0,3} = 0 \text{ on } \Sigma_\pm, \\
\text{div}^{\sigma_0} b_0 &= 0 \text{ in } \Omega, \quad [b_0] = 0 \text{ on } \Sigma, \quad b_0 \times e_3 = 0 \text{ on } \Sigma_+, \quad b_{0,3} = 0 \text{ on } \Sigma_-, \\
\left[ \partial_t^j b(0) \right] \times N_0 &= 0 \text{ on } \Sigma \text{ and } \partial_t^j b(0) \times e_3 = 0 \text{ on } \Sigma_+, \quad j = 1, \ldots, 2N - 1, \\
\partial_t^j (\{E\} \times N) (0) &= 0 \text{ on } \Sigma \text{ and } \partial_t^j E(0) \times e_3 = 0 \text{ on } \Sigma_-, \quad j = 0, \ldots, 2N - 1.
\end{align*} \]

(10.4)

For \( f = f_\pm \), denote \( \| f \|_2^2 = \| f_+ \|_{H^k(\Omega_+)}^2 + \| f_- \|_{H^k(\Omega_-)}^2 \) and \( |f|_2^2 = \| f_+ \|_{H^k(\Sigma)}^2 + \| f_- \|_{H^k(\Sigma)}^2 \).

For an integer \( N \geq 4 \), we define the high-order energy as
\[
\mathcal{E}_{2N} := \sum_{j=0}^{2N} \left| \partial_t^j u \right|_{2N-j}^2 + \sum_{j=0}^{2N-1} \left| \partial_t^j b \right|_{2N-j+1}^2 + \left| \partial_t^j f \right|_0^2
\]

(10.5)

For \( n = N + 1, \ldots, 2N \), we define a set of energies as
\[
\mathcal{E}_n := \| u \|_{n-1}^2 + \| u \|_{0,n}^2 + \sum_{j=0}^{n-1} \left| \partial_t^j u \right|_{n-j}^2 + \sum_{j=0}^{n-1} \left| \partial_t^j b \right|_{n-j+1}^2 + \left| \partial_t^n b \right|_0^2
\]

(10.6)

and the corresponding dissipations as
\[
\mathcal{D}_n := \sum_{j=0}^{n-2} \left| \partial_t^j u \right|_{n-j-1}^2 + \sum_{j=0}^{n-2} \left| \partial_t^j b \right|_{n-j}^2 + \left| \partial_t^n b \right|_{1,n-j}^2
\]

(10.7)

Now the main results of this section are stated as follows.

**Theorem 10.1.** Assume that \( \kappa > 0 \), \( \bar{B}_3 \neq 0 \) and \( \sigma > 0 \) and let \( N \geq 8 \) be an integer. Let \( u_0 \in H^{2N}(\Omega) \), \( b_0 \in H^{2N+1}(\Omega) \) and \( \eta_0 \in H^{2N+3/2}(\Sigma) \) be given such that \( \mathcal{E}_{2N}(0) < \infty \) and the compatibility conditions (10.4) as well as the zero average condition (2.7) are satisfied. There exists a universal constant \( \varepsilon_0 > 0 \) such that if \( \mathcal{E}_{2N}(0) \leq \varepsilon_0 \), then there exists a global unique solution \( (u, p, \eta, b) \) to (10.3). Moreover, for all \( t \geq 0 \),
\[
\mathcal{E}_{2N}(t) + \int_0^t \mathcal{D}_{2N}(s) \, ds \lesssim \mathcal{E}_{2N}(0)
\]

(10.8)

and
\[
\sum_{j=0}^{N-6} (1 + t)^{N-5-j} \mathcal{E}_{N+4+j}(t) + \sum_{j=0}^{N-6} \int_0^t (1 + s)^{N-5-j} \mathcal{D}_{N+4+j}(s) \, ds \lesssim \mathcal{E}_{2N}(0).
\]

(10.9)
Proof. The main approach is similar to that of Theorem 2.2. We will not repeat all the details of the proof but sketch only the main differences.

For the a priori estimates, the main difference here lies in deriving the energy evolution estimates of the highest order time derivative of the solution to (10.3). Indeed, compared to the plasma-vacuum interface problem, the new and most technical difficulty is the control of the following nonlinear terms for the hydrodynamic part:

\[- \int_{\Sigma} \partial_t^{2N-1} p_- \partial_t^{2N+1} [u] \cdot \mathcal{N} \]

\[= \int_{\Sigma} \partial_t^{2N-1} p_- \left( (2N+1) \partial_t^{2N} [u] \cdot \partial_t \mathcal{N} + [u] \cdot \partial_t^{2N+1} \mathcal{N} \right) + \sum R, \quad (10.10)\]

where one used the fact that \([u] \cdot \mathcal{N} = 0\) on \(\Sigma\) by the fifth equation in (10.3). The first term in the right hand side of (10.10) is controlled by using \(\left| \partial_t^{2N} u \right|^{2/2} \lesssim \mathcal{E}_{2N}\) (cf. (10.5)), while the treatment of the second term is much more involved: one needs to introduce the “material” derivative \(D^- = \partial^\theta + U_- \cdot \nabla^\theta\), where \(U_-\) is an extension of \(u_-\) onto \(\Omega\), and the key point lies in that when considering \(D^- \partial_t^{2N-1} u\) instead of \(\partial_t^{2N}\), certain troublesome nonlinear terms will be canceled. One may refer to Cheng et al. [8] for the details. With this, the rest of the estimates can be derived more or less similarly as those for the plasma-vacuum interface problem (2.5).

For the construction of local solutions, we also decompose (10.3) into the hydrodynamic part and the magnetic part. The hydrodynamic part is the two-phase incompressible Euler equations with surface tension, which can be solved in spirit of Cheng et al. [8]. For the other part of the two-phase magnetic system in moving domains, we can employ an argument different from the one for (2.5). Indeed, by using again the change of unknown:

\[b = J \mathcal{J}^{-1} b, \quad (10.11)\]

for \(\eta\) small we can solve the transformed system of \(b\) in our energy functional framework by using directly the Galerkin method (recall the reason that we resorted to a smoothing argument for solving (2.5) is merely due to that \(\text{curl } b \neq 0\) in \(\Omega_+\); the fact that \(\text{curl } b \times e_3 \neq 0\) on \(\Sigma_-\) is indeed not an obstacle, see Guo and Tice [24] for the study of the incompressible Navier–Stokes equations). However, the validity of (10.11) requires that \(\eta\) is of half regularity index higher than \(b\), which is not the case for the pair of \(\partial_t^{2N} \eta\) and \(\partial_t^{2N} b\) in our energy functional setting. Our way to overcome this difficulty is to regularize the hydrodynamic part by considering the following approximate problem

\[
\begin{align*}
\partial_t^\theta u + u \cdot \nabla^\theta u - (\epsilon \Lambda \tilde{\eta} - \epsilon \Lambda \Psi) \partial_t^\theta u + \nabla^\theta p &= \text{curl}^\theta b \times (\bar{B} + b) \quad \text{in } \Omega \\
\text{div}^\theta u &= 0 \\
\partial_t^\theta b &= \text{curl}^\theta E, \quad E = u \times (\bar{B} + b) - \kappa \text{curl}^\theta b \quad \text{in } \Omega \\
\text{div}^\theta b &= 0 \\
\partial_t \eta + \epsilon \Lambda \eta - \epsilon \Lambda \Psi &= u \cdot \mathcal{N} \quad \text{on } \Sigma \quad (10.12) \\
[p] &= \sigma H, \quad [b] = 0, \quad [E] \times \mathcal{N} = 0 \quad \text{on } \Sigma \\
u_3 &= 0, \quad b \times e_3 = 0 \quad \text{on } \Sigma_+ \\
u_3 &= 0, \quad b_3 = 0, \quad E \times e_3 = 0 \quad \text{on } \Sigma_-.
\end{align*}
\]

Here \(\epsilon > 0\) is the artificial viscosity coefficient, \(\Lambda := (-\Delta_h)^{1/2}\) and \(\Psi\) is the so-called compensator satisfying \(\partial_t^j \Psi(0) = \mathcal{P}(\partial_t^j \tilde{\eta}(0)), \ j = 0, \ldots, 2N - 2\), where the initial
data $\partial^j_t \eta(0)$ are those from the original problem (10.3). By the introduction of such $\Psi$, at time $t = 0$, one essentially adds nothing on the equations and its time derivatives up to $2N - 2$ order. This allows one to take the initial data for the problem (10.3) exactly as the one for the regularized problem (10.12) (the compatibility conditions to (10.12) are same as (10.3)). It is crucial that the regularized problem (10.12) is asymptotically consistent with the a priori estimates of (10.3). Indeed, compared to (10.3), the artificial viscosity term leads to the gain of regularities for $\eta$ through the following (cf. the first term in the right hand side of (5.7)): for $\alpha \in \mathbb{N}^{1+2}$ with $|\alpha| \leq 2N$,

$$ - \int_{\Sigma} \left[ \partial^\alpha p \right] \epsilon \partial^\alpha \Lambda \eta = - \int_{\Sigma} \sigma \partial^\alpha H \epsilon \partial^\alpha \Lambda \eta \leq - \sigma \epsilon \int_{\Sigma} \left| \nabla_h \Lambda^{1/2} \partial^\alpha \eta \right|^2 + C \sqrt{E_{2N} \epsilon} \mathbb{D}_{2N}(\eta), $$

(10.13)

where

$$ \mathbb{D}_{2N}(\eta) := \sum_{j=0}^{2N} \left| \partial^j \eta \right|_{2N-j+3/2}^2. $$

(10.14)

On the other hand, all the new nonlinear terms resulting from the artificial viscosity term can be controlled by $\sum_{R}$ with an exception (cf. the second term in the right hand side of (10.10)), which is estimated as follows: by the trace theory,

$$ - \int_{\Sigma} \partial_{t}^{2N-1} p_{-} \left[ u_{h} \right] \cdot \nabla_h \epsilon \partial_{t}^{2N-1} \Lambda \eta \lesssim \left\| \partial_{t}^{2N-1} p_{-} \right\|_{H^{1/2}(\Sigma)} \sqrt{E_{2N}} \left\| \nabla_h \epsilon \partial_{t}^{2N} \Lambda \eta \right\|_{-1/2} \lesssim E_{2N} \epsilon \sqrt{\mathbb{D}_{2N}(\eta)}. $$

(10.15)

The uniform in $\epsilon$ a priori estimates of (10.12) can be thus closed for $E_{2N}$ small. It is also then routine to check that the regularized hydrodynamic part of (10.12) can be solved as the original one of (10.3). Note that for each fixed $\epsilon > 0$, the gained regularity of $\eta$ (cf. (10.14)) in particular validates the change of unknown in (10.11). Hence, one can first construct the solutions to the nonlinear $\epsilon$-approximate problem (10.12) by the method of successive approximations basing on the solvability of the hydrodynamic and magnetic parts, whose limit as $\epsilon \to 0$ yields the desired solution to (10.3). □

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Appendix A. Analytic tools

In this appendix we will collect the analytic tools which are used throughout the paper.
A.1. Harmonic extension. Define the specialized Poisson sum in $\mathbb{T}^2 \times \mathbb{R}$ by (see \cite{53})

$$\mathcal{P} f(x) := \begin{cases} 
\sum_{\xi \in \mathbb{Z}^2} e^{2\pi i \xi \cdot x_h} \sum_{j=0}^{m} \alpha_j e^{-|\xi| \lambda_j x_3} \hat{f}(\xi), & x_3 > 0 \\
\sum_{\xi \in \mathbb{Z}^2} e^{2\pi i \xi \cdot x_h} e^{2\pi |\xi| x_3} \hat{f}(\xi), & x_3 \leq 0,
\end{cases} \tag{A.1}$$

where

$$\hat{f}(\xi) = \int_{\mathbb{T}^2} f(x_h) e^{-2\pi i \xi \cdot x_h}, \quad \xi \in \mathbb{Z}^2. \tag{A.2}$$

Here $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_m < \infty$ for $m \in \mathbb{N}$, and $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m)^T$ is the solution to

$$V(\lambda_0, \lambda_1, \ldots, \lambda_m) \alpha = (1, 1, \ldots, 1)^T, \tag{A.3}$$

where $V$ is the $(m+1) \times (m+1)$ Vandermonde matrix. The Poisson sum (A.1) is specialized in that $\mathcal{P} f$ is differentially continuous across $\Sigma$ up to any order as needed provided that $m$ is sufficiently large. Moreover, the following estimate holds.

**Lemma A.1.** It holds that for all $s \in \mathbb{R}$,

$$\|\mathcal{P} f\|_s \lesssim |f|_{s-1/2}. \tag{A.4}$$

**Proof.** One may refer to Lemma A.9 of \cite{24} for instance. \hfill $\square$

A.2. Time extension. The following lemmas allow one to extend the initial data to be time-dependent functions, which are “hyperbolic” versions of the “parabolic” ones in \cite{24} with some minor modifications.

**Lemma A.2.** Suppose that $\partial_t^j \eta(0) \in H^{2N-j+3/2}(\Sigma)$ for $j = 0, \ldots, 2N - 1$. There exists an extension $\eta^0$ defined on $[0, \infty)$, achieving the initial data, so that

$$\sum_{j=0}^{2N+1} \sup_{[0, \infty]} \left| \partial_t^j \eta^0 \right|_{2N-j+3/2}^2 + \sum_{j=0}^{2N+2} \int_0^\infty \left| \partial_t^j \eta^0 \right|_{2N-j+2}^2 \lesssim \sum_{j=0}^{2N-1} \left| \partial_t^j \eta(0) \right|_{2N-j+3/2}^2. \tag{A.5}$$

**Proof.** For each $j = 0, \ldots, 2N - 1$, we denote $f_j = \partial_t^j \eta(0)$ and let $\varphi_j \in C^\infty_0(\mathbb{R})$ be such that $\varphi_j^{(k)}(0) = \delta_{j,k}$ for $k = 0, \ldots, 2N - 1$. Then $\eta^0$ is constructed as a sum $\eta^0 = \sum_{j=0}^{2N-1} F_j$, where $F_j$ is defined via its Fourier coefficients:

$$\hat{F}_j(\xi, t) = \varphi_j(t \langle \xi \rangle) \hat{f}_j(\xi) \langle \xi \rangle^{-j}, \quad j = 0, \ldots, 2N - 1,$$

where $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$. It follows by modifying the proof of Lemma A.5 of \cite{24} suitably that $\eta^0$ satisfies the conclusion. \hfill $\square$
Lemma A.3. Suppose that $\partial^j_t u(0) \in H^{2N-j}(\Omega_-)$ for $j = 0, \ldots, 2N - 1$. There exists an extension $u^0$ defined on $[0, \infty)$, achieving the initial data, so that
\[
\sum_{j=0}^{2N} \left( \sup_{[0,\infty]} \left\| \partial^j_t u^0 \right\|_{H^{2N-j}}^2 \right) + \sum_{j=0}^{2N} \int_0^\infty \left( \left\| \partial^j_t u^0 \right\|_{H^{2N-j+1/2}} \right) \lesssim \sum_{j=0}^{2N-1} \left\| \partial^j_t u(0) \right\|_{H^{2N-j}}^2. \tag{A.6}
\]

Proof. It follows similarly as Lemma A.2, by using additionally the usual theory of extensions and restrictions in Sobolev spaces between $H^k(\Omega_-)$ and $H^k(\mathbb{R}^3)$ for $k \geq 0$. \hfill $\square$

Lemma A.4. Suppose that $\partial^j_t b(0) \in H^{2N-j+1}(\Omega_-)$ for $j = 0, \ldots, 2N - 1$. There exists an extension $b^0$ defined on $[0, \infty)$, achieving the initial data, so that
\[
\sum_{j=0}^{2N+1} \left( \sup_{[0,\infty]} \left\| \partial^j_t b^0 \right\|_{H^{2N-j+1}}^2 \right) + \sum_{j=0}^{2N+1} \int_0^\infty \left( \left\| \partial^j_t b^0 \right\|_{H^{2N-j+3/2}} \right) \lesssim \sum_{j=0}^{2N-1} \left\| \partial^j_t b(0) \right\|_{H^{2N-j+1}}^2. \tag{A.7}
\]

Proof. It follows in the same way as Lemma A.3. \hfill $\square$

A.3. Product estimates. The following standard estimates in Sobolev spaces are needed.

Lemma A.5. Let the domains below be either $\Omega_{\pm}$, $\Sigma_{\pm}$ or $\Sigma_{\pm}$, and $d$ be the dimension.

1. Let $0 \leq r \leq s_1 \leq s_2$ be such that $s_1 > d/2$. Then
\[
\left\| fg \right\|_{H^r} \lesssim \left\| f \right\|_{H^{s_1}} \left\| g \right\|_{H^{s_2}}. \tag{A.8}
\]

2. Let $0 \leq r \leq s_1 \leq s_2$ be such that $s_2 > r + d/2$. Then
\[
\left\| fg \right\|_{H^r} \lesssim \left\| f \right\|_{H^{s_1}} \left\| g \right\|_{H^{s_2}}. \tag{A.9}
\]

Lemma A.6. It holds that for $s > 5/2$,
\[
\left\| fg \right\|_{H^{s-1}} \lesssim \left\| f \right\|_{H^{s-1}} \left\| g \right\|_{s}. \tag{A.10}
\]

A.4. Poincaré-type inequality. The following Poincaré-type inequality related to $\tilde{B} \cdot \nabla$ holds.

Lemma A.7. For any constant vector $\tilde{B} \in \mathbb{R}^3$ with $\tilde{B}_3 \neq 0$, it holds that
\[
\left\| f \right\|_{0}^2 \leq \frac{1}{\tilde{B}_3^2} \left\| (\tilde{B} \cdot \nabla) f \right\|_{0}^2 + \left| f \right|_{0}^2 \tag{A.11}
\]
and
\[
\left| f \right|_{0}^2 \leq \frac{1}{\tilde{B}_3^2} \left\| (\tilde{B} \cdot \nabla) f \right\|_{0}^2 + \left\| f \right\|_{0}^2. \tag{A.12}
\]

Proof. It follows by the fundamental theorem of calculus, see Lemma A.4 in [52]. \hfill $\square$
A.5. Normal trace estimates. The following $H^{-1/2}$ boundary estimate holds for functions satisfying $v \in L^2$ and $\text{div}^\varphi v \in L^2$.

**Lemma A.8.** Assume that $\|\nabla \varphi\|_{L^\infty} \leq C$, then

$$|v \cdot \mathcal{N}|_{-1/2} \lesssim \|v\|_0 + \|\text{div}^\varphi v\|_0.$$  \hspace{1cm} (A.13)

**Proof.** One may refer to Lemma 3.3 in [24]. ⊓⊔

A.6. Hodge-type estimates. The following Hodge-type estimate holds, when the boundary conditions are not specified. Let the domain be either $\Omega_-$ or $\Omega_+$ or $\Omega$.

**Lemma A.9.** Let $r \geq 1$ be an integer. Then it holds that

$$\|v\|_r \lesssim \|v\|_{0,r} + \|(\text{curl} v)\|_{r-1} + \|\text{div} v\|_{r-1}.$$  \hspace{1cm} (A.14)

**Proof.** Notice that (A.14) follows easily for $r = 1$. Now for $r \geq 2$, applying the previous estimate of $r = 1$ gives that for $\ell = 1, \ldots, r$,

$$\|v\|_{\ell,r-\ell} \lesssim \|v\|_{\ell-1,r-\ell+1} + \|(\text{curl} v)\|_{\ell-1,r-\ell} + \|\text{div} v\|_{\ell-1,r-\ell} \lesssim \|v\|_{\ell-1,r-\ell+1} + \|(\text{curl} v)\|_{r-1} + \|\text{div} v\|_{r-1}.$$  \hspace{1cm} (A.15)

By an induction argument on $\ell = 1, \ldots, r$, one gets (A.14). ⊓⊔

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