AN ERROR ESTIMATE FOR COUNTING $S_3$-SEXTIC NUMBER FIELDS

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Abstract. In this note, we prove a power-saving remainder term for the function counting $S_3$-sextic number fields. We also give a prediction on the second main term.

In addition, we present numerical data on counting functions for $S_3$-sextic number fields. The data indicates that our prediction is likely to be correct, and it also suggests the existence of additional lower order terms which we have not yet been able to explain.

1. Statement

We call a sextic number field $\tilde{K}$ $S_3$-sextic if $\tilde{K}$ is Galois over $\mathbb{Q}$ with $\text{Gal}(\tilde{K}/\mathbb{Q})$ isomorphic to the symmetric group $S_3$. Let $N^\pm_6(X; S_3)$ be the number of $S_3$-sextic fields $\tilde{K}$ with $0 < \pm \text{Disc}(\tilde{K}) < X$. The primary term of $N^\pm_6(X; S_3)$ was obtained in independent works of Belabas-Fouvry [2] and Bhargava-Wood [3], and in this article we prove the following power-saving remainder term.

Theorem 1.1. We have

\begin{equation}
N^\pm_6(X; S_3) = \frac{C^\pm}{12} \prod_p c_p \cdot X^{1/3} + O(X^{1/3 - 5/447 + \epsilon}),
\end{equation}

where $C^+ = 1, C^- = 3$, the product is over all primes, and

\[ c_p = \begin{cases} (1 - p^{-1})(1 + p^{-1} + p^{-4/3}) & p \neq 3, \\ (1 - \frac{1}{3})(1 + \frac{1}{3} + \frac{2}{3}) & p = 3. \end{cases} \]

Moreover, under a natural but rather strong assumption of uniformity estimates for certain counting functions of cubic fields, we obtain

\begin{equation}
N^\pm_6(X; S_3) = \frac{C^\pm}{12} \prod_p c_p \cdot X^{1/3} + \frac{4K^\pm \zeta(1/3)}{5\Gamma(2/3)^3} \prod_p k_p \cdot X^{5/18} + o(X^{5/18}),
\end{equation}

where $K^+ = 1, K^- = \sqrt{3}$ and

\[ k_p = \begin{cases} 1 + \frac{p^{1/3} - (1 + p^{-1})}{p^{1/3}} \left( \frac{1 - 1}{p^{1/3}} - \frac{1}{p^{3/9}} - \frac{1}{p^{2/3}} \right) & p \neq 3, \\ \frac{1}{4} \left( \frac{11}{3} - \frac{1}{3^{2/3}} + \frac{1}{3^{5/9}} + \frac{2}{3^{13/9}} - \frac{1}{3^{19/9}} - \frac{2}{3} \right) & p = 3. \end{cases} \]

As in [2] and [3], we relate counting $S_3$-sextic fields to counting non-cyclic cubic fields with certain local completions. These cubic fields may then be counted using our previous work [11]. We may obtain a power saving error term simply by quoting our previous results, but we improve on this by applying the methods used in the proofs in [11]. This amounts to computing the Fourier transform of a function related to these local completions, and this computation was essentially carried out in [10].

We also present numerical data for $N^\pm_6(X; S_3)$ for $X \leq 10^{23}$, computed by Cohen and the second author in [5], and verified independently by the present authors for $X \leq 5 \cdot 10^{18}$ by a second method. Interestingly, our computations suggest that (1.2) is likely to be correct, but with additional lower order terms which we were not able to explain. Our data also suggests the existence of surprising biases in arithmetic progressions, for example modulo 5, which cannot be explained by any heuristic of which we are aware.

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2. Proof

For a non-cyclic cubic field $K$, let $\bar{K}$ denote its Galois closure. Then the map $K \mapsto \bar{K}$ gives a canonical bijection between the set of isomorphism classes of non-cyclic cubic fields and the set of isomorphism classes of $S_3$-sextic fields. Let us compare $\text{Disc}(K)$ and $\text{Disc}(\bar{K})$. They have the same sign, and if we write

$$\text{Disc}(K) = \pm \prod p^{e_p(K)}, \quad \text{Disc}(\bar{K}) = \pm \prod p^{e_p(\bar{K})},$$

we have the following.

**Lemma 2.1.** (1) If $K$ is not totally ramified at $p$, then $e_p(\bar{K}) = 3e_p(K)$.
(2) If $K$ is totally ramified at $p$ and $p \neq 3$, then $e_p(\bar{K}) = 2e_p(K) = 4$.
(3) If $K$ is totally ramified at $p = 3$, then $e_p(\bar{K}) = 7, 8$ or $11$ according as $e_p(K) = 3, 4$ or $5$.

*Proof.* Equivalent statements appear in [2] and [3], and we give a proof for the convenience of the reader. Let $F = \mathbb{Q}(\sqrt{\text{Disc}(K)})$ be the quadratic resolvent field of $K$ (equivalently, the unique quadratic subfield of $\bar{K}$). We have the classical formula (see, e.g. Theorem 2.5.1 and Lemma 10.1.27 of [4])

$$\text{Disc}(\bar{K}) = \text{Disc}(K)\text{Disc}(F).$$

Therefore, for $p > 2$, $e_p(\bar{K})$ is equal to $2e_p(K) + a$, where $a = 0$ or $1$ depending on whether $e_p(K)$ is even or odd.

For $p = 2$, observe that $2$ can ramify in $F$ only if it ramifies in $K$. If $(2) = p^2q$ in $K$, then in $\bar{K}$, $(2)$ must split into three ideals with ramification index $2$. Therefore $2$ must ramify in $F$ with $e_2(F) = e_2(K)$ so that $e_2(\bar{K}) = 3e_2(K)$. If $(2) = p^3$ in $K$, then $2$ is tamely ramified in $K$, and therefore $\bar{K}$, so that $(2)$ splits into two ideals of ramification index $3$ in $\bar{K}$. This implies that $(2)$ is unramified in $F$, so that $e_2(\bar{K}) = 2e_2(K)$.

In particular, $e_p(K)$ determines $e_p(\bar{K})$ uniquely except for the case $p = 2$ and $e_2(K) = 2$, while in this case $e_2(\bar{K})$ is either $6$ or $4$ according as $K$ is partially or totally ramified at $2$.

Let us briefly explain our approach. If we ignore the ramification over the prime $3$, then Lemma 2.1 implies

$$\text{Disc}(\bar{K}) = * r^{-2}\text{Disc}(K)^3,$$

where $r$ is the product of all primes where $K$ is totally ramified.\footnote{We have starred two equalities which are not actually true as stated. We correct them in (2.4) and (2.6) respectively.}

Denoting by $N_3^\pm(X; r)$ the number of non-cyclic cubic fields $K$ with $r$ as above and $0 < \pm\text{Disc}(K) < X$, then

$$N_6^\pm(X; S_3) = * \sum_r N_3^\pm(r^{2/3}X^{1/3}; r).$$

Here the sum is over all square-free integers $r$. However, (2.3) may not be true because of the ramification at $3$, so we specify the completion $A$ of $K$ at $3$ and count for each $A$.

Let $A$ denote an étale cubic algebra over $\mathbb{Q}_3$ (i.e., a direct product of field extensions of $\mathbb{Q}_3$ whose degrees add to $3$) and $r$ a square-free integer coprime to $3$. Let $K_3(A, r)$ be the set of non-cyclic cubic fields $K$ satisfying (i) $K \otimes \mathbb{Q}_3 \cong A$, (ii) $K$ is totally ramified at all prime divisors of $r$, and (iii) $K$ is not totally ramified at any prime $p \nmid 3r$. Let $\bar{A}$ be the sextic algebra over $\mathbb{Q}_3$ isomorphic to $\bar{K} \otimes \mathbb{Q}_3$.
for $K \in \mathcal{K}_3(A, r)$, which does not depend on $K$. Let $\text{Disc}_3(A)$ and $\text{Disc}_3(\overline{A})$ be the 3-parts of their discriminants; e.g., write $\text{Disc}(A) = u\text{Disc}_3(A)$, where $u$ is a 3-adic unit\footnote{Observe that $\text{Disc}(A)$ and $u$ are only defined up to squares of 3-adic units, but $\text{Disc}_3(A)$ is well defined.}, and similarly for $\text{Disc}(\overline{A})$. Then for $K \in \mathcal{K}_3(A, r)$, instead of (2.2) we have

\begin{equation}
(2.4) \quad \text{Disc}(K) = r^{-2}m_A^{-1}\text{Disc}_3(A)^2\text{Disc}(K)^3,
\end{equation}

with $m_A := \text{Disc}_3(A)^3/\text{Disc}_3(\overline{A})$. Let $N_3^\pm(X; A, r)$ denote the number of $K \in \mathcal{K}_3(A, r)$ with $0 < \pm \text{Disc}(K) < X$. We will use a formula of the form

\begin{equation}
N_3^\pm(X; A, r) = \eta_3(A)\theta_3(A)\prod_p\left(1 - p^{-2}\right)^{C^\pm_2}X
\end{equation}

\begin{equation}
+ \theta_3(A)\theta(r)\prod_{p\neq 3}\left(1 - p^{1/3} + 1\right)\frac{4K_3^\pm(1/3)}{5\Gamma(2/3)^3}X^{5/6} + O(r^\alpha X^\beta).
\end{equation}

Here $\eta_3(A)$ and $\theta_3(A)$ are “local densities” of $A$ computed in $[1]$, $\eta$ and $\theta$ are multiplicative functions satisfying

$$
\eta(p) = \frac{1}{p^2(1 + p^{-1})}, \quad \theta(p) = \frac{1}{p^2(1 + p^{-2/3} + p^{-1} + p^{-4/3})}
$$

for any prime $p$, and $\alpha, \beta$ are certain real constants. By Theorem 1.2 in $[1]$, (2.5) is true with $\alpha = 40/23, \beta = 18/23 + \epsilon$ and this suffices to obtain (1.1) with a larger error term of $O(X^{1/3 - 5/744 + \epsilon})$. In this paper, we improve the estimate as follows.

**Theorem 2.2.** The formula (2.5) is true for $\alpha = 7/23 + \epsilon, \beta = 18/23 + \epsilon$.

We postpone its proof to the next section, and continue the proof of (1.1) and (1.2). Let $N_6^\pm(X; A)$ be the number of $S_3$-sextic fields $K$ such that $K \otimes \mathbb{Q}_3 \cong A$. Then by (2.4),

\begin{equation}
(2.6) \quad N_6^\pm(X; A) = \sum_{3|r}N_3^\pm(r^{2/3}m_A^{1/3}X^{1/3}; A, r)
\end{equation}

where the sum is over all square-free integers coprime to 3. Therefore, our results follow from (2.6), (2.4), and a computation, the details of which follow.

We choose $Q$ and split this sum into $r < Q$ and $r \geq Q$. By $[1]$ Lemma 3.4 we have the estimate $N_3^\pm(X; A, r) = O(r^{-2+\epsilon}X)$. Hence the latter sum is bounded by $O(Q^{-1/3+\epsilon}X^{1/3})$. On the other hand it is easy to see that

$$
\sum_{3|r, r < Q} \eta(r)r^{2/3} = \prod_{p\neq 3}(1 + \eta(p)p^{2/3}) + O(Q^{-1/3+\epsilon}),
$$

$$
\sum_{3|r, r < Q} \theta(r)r^{5/9} = \prod_{p\neq 3}(1 + \theta(p)p^{5/9}) + O(Q^{-4/9+\epsilon}).
$$

We define

$$
c_p := (1 + \eta(p)p^{2/3})(1 - p^{-2}), \quad k_p := (1 + \theta(p)p^{5/9})\left(1 - \frac{p^{1/3} + 1}{p(p + 1)}\right) \quad (p \neq 3),
$$

which coincide with the constants given in Section $[4]$. We also put

$$
\eta_3'(A) := (1 - 3^{2/3})\eta_3(A)m_A^{1/3}, \quad \theta_3'(A) := \left(1 - \frac{3^{1/3} + 1}{3(3 + 1)}\right)\theta_3(A)m_A^{5/18}.
$$
Then by (2.6) and (2.5), and ignoring a negligible $O(Q^{-4/9+\varepsilon}X^{5/18})$ term, we have

$$N^\pm_6(X; A) = \eta_3'(A) \prod_{p \neq 3} C_p \pm \theta_3(A) \prod_{p \neq 3} k_p \cdot \frac{4K^\pm \zeta(1/3)}{5\Gamma(2/3)^3}X^{5/18} + O(X^{\beta/3} \sum_{r<Q} r^{\alpha+2\beta/3}) + O(Q^{-1/3+\varepsilon}X^{1/3}).$$

(2.7)

The first $O$-term is $O(Q^{\alpha+2\beta/3+1}X^{\beta/3})$, and we choose $Q = X^{\frac{1}{3\alpha+2\beta+\varepsilon}}$ to obtain an error of $O(X^{\frac{1}{3\alpha}+(1-\frac{1}{3\alpha+2\beta+\varepsilon})+\varepsilon})$ in (2.7). With our constants $\alpha = 7/23 + \varepsilon$ and $\beta = 18/23 + \varepsilon$, this is $O(X^{\frac{1}{3}\alpha+(1-\frac{1}{3\alpha})+\varepsilon})$. If (2.5) is true for e.g., $\alpha = -1, \beta = 1/2$, this is $O(X^{1/4+\varepsilon})$ and we would obtain the second main term. Such an estimate might be true, but it seems difficult to prove; moreover, our numerical data (see Section 5) suggests that perhaps such a strong estimate is not true.

Recall that

$$N^\pm_6(X; S_3) = \sum_A N^\pm_6(X; A)$$

where $A$ in the right hand side runs through all the étale cubic algebras over $\mathbb{Q}_3$. (There are finitely many, as there are finitely many field extensions of $\mathbb{Q}_3$ of degree $\leq 3$.) Hence the contribution to the main term of $N^+_6(X; S_3)$ from the prime 3 is given by

$$c_3 := \sum_A \eta_3'(A) = (1 - 3^{-2}) \sum_A \eta_3(A)m_A^{1/3}.$$  

The local density $\eta_3(A)$ is given in the tables in Section 6.2 of [11]. Also, $m_A$ is equal to 1, 9, or 81 depending on whether the 3-adic valuation of $\text{Disc}(A)$ is less than 3, equal to 3, or greater than 3. We therefore compute that

$$c_3 = (1 - 3^{-2}) \cdot \frac{1 + \frac{1}{3} + \frac{2}{27} \cdot 3^{2/3} + \frac{1}{27} \cdot 3^{4/3}}{1 + \frac{4}{3}},$$

which is equal to the quantity given in Section 1. Similarly, the contribution to the secondary term is given by

$$\sum_A \theta_3'(A) = \left(1 - \frac{3^{1/3} + 1}{3(3 + 1)}\right) \sum_A \theta_3(A)m_A^{5/18},$$

and a similar calculation yields the value of $k_3$ given in Section 1.

3. Proof of Theorem 2.2

In this section, we prove Theorem 2.2 by following the arguments of [10] and [11].

A brief sketch of our proof is as follows. In [11], we counted cubic fields in terms of contour integrals of certain zeta functions introduced by Shintani [9], associated to the space of binary cubic forms. Our method is naturally compatible with “local specifications” such as those appearing in (2.5), and the error terms of (2.5) depend on the “shape” of these local specifications – in particular, on the Fourier transforms of their indicator functions. We establish fairly sharp bounds for these Fourier transforms on average, which lead to reasonably good bounds on the error terms in (2.5) (in $\alpha$-aspect) and therefore in Theorem 1.1.

We follow the notations of [10] and [11], but recall the most basic ones. Let

$$V(\mathbb{Z}) := \{x = (x_1,x_2,x_3,x_4) = x_1u^2 + x_2u^2v + x_3uv^2 + x_4v^3 \mid x_1,x_2,x_3,x_4 \in \mathbb{Z}\}$$

be the lattice of integral binary cubic forms, with its usual action of $\text{GL}_2(\mathbb{Z})$. Then there is a discriminant preserving bijection between $\text{GL}_2(\mathbb{Z}) \backslash V(\mathbb{Z})$ and the set of isomorphism classes of cubic rings.$^3$

$^3$A cubic ring is a commutative ring which is free of rank 3 as a $\mathbb{Z}$-module.
We now define these indicator functions. Let $p \neq 3$ be a prime. We define $\Phi_p : V(\mathbb{Z}) \to \mathbb{C}$ to be the characteristic function of those $x \in V_3$ whose corresponding cubic ring is either nonmaximal at $p$, or maximal and totally ramified at $p$. We similarly define $\Psi_p$ by requiring the cubic ring to be both maximal and totally ramified at $p$. These two functions factor through the reduction map $V(\mathbb{Z}) \to V(\mathbb{Z}/p\mathbb{Z})$, and we also write $\Phi_p$, $\Psi_p$ for these functions on $V(\mathbb{Z}/p\mathbb{Z})$.

The prime 3 demands a special treatment. We fix an étale cubic algebra $A$ over $\mathbb{Q}_3$ throughout this section; note that since there are only finitely many $A$, uniformity in our error terms with respect to $A$ is automatic. Let $\Phi_A$ be the characteristic function on $V(\mathbb{Z})$ or $V(\mathbb{Z}/27\mathbb{Z})$ corresponding to cubic rings $R$ such that $R \otimes \mathbb{Z}_3 \cong O_A$, where $O_A$ is the integral closure of $\mathbb{Z}_3$ in $A$. This $\Phi_A$ factors through $V(\mathbb{Z}) \to V(\mathbb{Z}/27\mathbb{Z})$.

Let $r$ and $q$ be squarefree integers satisfying $(q,r) = (qr,3) = 1$. We put $\Phi_q = \prod_{p|q} \Phi_p$ and $\Psi_r = \prod_{p|r} \Psi_p$, and define the zeta functions

\[
\xi_{r,q}^\pm(s) := \sum_{x \in \text{GL}_2(\mathbb{Z}) \setminus V(\mathbb{Z})} \Phi_A(x) \Psi_r(x) \Phi_q(x) \frac{|\text{Stab}(x)|^{-1}}{|\text{Disc}(x)|^s}.
\]

As in [11], Theorem 2.2 follows from uniform estimates for the zeta functions $\hat{\xi}_{r,q}^\pm(s)$ which are dual to $\xi_{r,q}^\pm(s)$.

Let $V^* = \text{Hom}(V(\mathbb{Z}), \mathbb{Z})$ be the dual space of $V$. The (finite) Fourier transform of $\Psi_r$, a function of $y \in V^*(\mathbb{Z}/r^2\mathbb{Z})$, is defined by

\[
\hat{\Psi}_r(y) := \frac{1}{r^8} \sum_{x \in V^*(\mathbb{Z}/r^2\mathbb{Z})} \Psi_r(x) \exp \left(2\pi \sqrt{-1} \cdot \frac{[x,y]}{r^2}\right), \quad y \in V^*(\mathbb{Z}/r^2\mathbb{Z}),
\]

where $[x,y] = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$ (using the coordinate system on $V^*(\mathbb{Z})$ induced by the canonical pairing), and we define $\hat{\Phi}_A$ and $\hat{\Phi}_q$ similarly. Then the dual zeta function is defined by

\[
\hat{\xi}_{r,q}^\pm(s) := \sum_{y \in \text{GL}_2(\mathbb{Z}) \setminus V^*(\mathbb{Z})} \hat{\Phi}_A(y) \hat{\Psi}_r(y) \hat{\Phi}_q(y) |\text{Stab}(y)|^{-1} \frac{|\text{Disc}(y)|^{s}}{|P^*(y)/3^2r^8q^8|^s}.
\]

The ‘dual discriminant’ $P^*(y)$ is the same as the ‘ordinary’ discriminant $P(x)$ or $\text{Disc}(x)$ on $V(\mathbb{Z})$ apart from some 3-adic factors; we refer to Section 2 of [10] for details.

Because of the functional equation

\[
\left(\frac{\xi_{r,q}^+(1-s)}{\xi_{r,q}^-(1-s)}\right) = \frac{3^{6s-2}}{2\pi^{1s}} \Gamma(s) \Gamma(s - \frac{1}{6}) \Gamma(s + \frac{1}{6}) \left(\sin 2\pi s \sin \pi s \right) \left(\xi_{r,q}^+(s) \xi_{r,q}^-(s)\right),
\]

the estimate of the $O$-term in (2.5) is reduced to estimates for these dual zeta functions $\hat{\xi}_{r,q}^\pm(s)$ which are uniform with respect to $r$ and $q$. We write

\[
\hat{\xi}_{r,q}^\pm(s) := \sum_{\mu_n} \frac{c_{\xi_{r,q}^\pm}(\mu_n)}{\mu_n^s},
\]

the sum being over $\mu_n \in \mathbb{Z}/3^{2r^2q^8}\mathbb{Z}$. We fix a choice of sign and drop $\pm$ from our notation. The following bound essentially follows from Theorem 4.1 in [11].

\[\text{If } A \text{ is of the form } A = \prod A_i \text{ where } A_i/\mathbb{Q}_3 \text{ are field extensions, then } O_A = \prod O_{A_i} \text{ where each } O_{A_i} \text{ is the integer ring of } A_i.\]
Proposition 3.1. For any fixed \( \epsilon > 0 \), we have the bounds
\[
\sum_{\mu_n < X} |c_{r,q}(\mu_n)| \ll (rq)^{2+\epsilon}X,
\]
\[
\sum_{\mu_n < X} |c_{r,q}(\mu_n)| \ll (rq)^{1+\epsilon}X + (rq)^{-1+\epsilon},
\]
uniformly for all \( r, q \) and \( X \).

Proof. In [10], we gave explicit formulas for the Fourier transforms \( (\Phi_p - \Psi_p)^\wedge = \hat{\Phi}_p - \hat{\Psi}_p \) and \( \hat{\Phi}_p \) in Theorems 6.3 and 6.4, respectively, so a formula for \( \hat{\Psi}_p \) follows by linearity. We introduce a function \( \Phi_p^* \) on \( V^*(\mathbb{Z}/p^2\mathbb{Z}) \) by
\[
\Phi_p^*(b) = \begin{cases} 
1 & \text{if } b \text{ is of type } (1^{3\max}), \\
|\Phi_p(b)| & \text{otherwise}.
\end{cases}
\]
Then we have \( |\hat{\Phi}_p| \leq \Phi_p^* \) and \( |\hat{\Psi}_p| \leq (1 + p^{-2})\Phi_p^* \). Let \( c = \prod_p (1 + p^{-2}) \). Note the trivial bound \( |\hat{\Phi}_A| \leq 1 \). Therefore \( \xi_{r,q}^\pm(s) \) is bounded coefficientwise by
\[
\sum_{y \in \text{GL}_2(\mathbb{Z})/V^*(\mathbb{Z})} c \Phi_p^*(y) \frac{\text{Stab}(y)^{-1}}{|P^*(y)/3!2^r8^q|^s}.
\]
Here \( \Phi_p^* = \prod_{p|rq} \Phi_p^* \). If \( \Phi_p^*(y) \) in above were replaced with \( |\hat{\Phi}_r(y)| \), then (3.11) is, in the notation of Section 4 in [11], given by
\[
c \sum_{\mu_n} \frac{b_{r,q}(\mu_n)}{(\mu_n/3!2^r)^s}.
\]
So the bounds of this proposition follow from Theorem 4.1 in [11]. Our actual (3.11) is slightly different from (3.11) because of (3.9), but we can nevertheless easily modify the proof of Theorem 4.1 in [11] for our case and obtain the same estimate. We omit the detail.

Similarly to Proposition 4.2 in [11], we have the following corollary.

Proposition 3.2. Let \( z \geq r^{-2}q^{-2} \). For a fixed \( 0 < \delta < 1 \) (and \( \epsilon > 0 \)), we have the bounds
\[
\sum_{\mu_n < z} |c_{r,q}(\mu_n)|/\mu_n^\delta \ll (rq)^{3\delta-1+\epsilon} + (rq)^{1+\epsilon}z^{1-\delta}.
\]
We also have, for any fixed \( \delta > 1 \),
\[
\sum_{\mu_n > z} |c_{r,q}(\mu_n)|/\mu_n^\delta \ll (rq)^{1+\epsilon}z^{1-\delta}.
\]

We are ready to prove Theorem 2.2.

Proof of Theorem 2.2. From exactly the same argument as of Section 5.3 in [11], the difference of the counting functions and the corresponding two main terms in (2.5) are, for any parameter \( Q \leq X \) and \( y \geq X^{3/5} \), bounded by
\[
\ll \sum_{q < Q} E_q(r, y, X) + y^{1+\epsilon} + X/Q^{1-\epsilon},
\]
where
\[
E_q(r, y, X) = X^{3/8} \sum_{\mu_n < z_q} \frac{|c_{r,q}(\mu_n)|}{\mu_n^{\delta/8}} + X^{3/8} \sum_{\mu_n \geq z_q} \frac{|c_{r,q}(\mu_n)|}{\mu_n^{\delta/8+p/4}}
\]
for our case and obtain the same estimate. We omit the detail.

\[
\sum_{\mu_n < X} |c_{r,q}(\mu_n)| \ll (rq)^{2+\epsilon}X,
\]
\[
\sum_{\mu_n < X} |c_{r,q}(\mu_n)| \ll (rq)^{1+\epsilon}X + (rq)^{-1+\epsilon},
\]
Here $\rho \geq 3$ is a positive integer and $z_q$ is another parameter which we can choose freely for each $q$. By Proposition 3.2 for $z_q \geq r^{-2}q^{-2}$,

$$E_q(r, y, X) \ll X^{3/8}r^{7/8+\epsilon}q^{7/8+\epsilon} + X^{3/8}r^{1+\epsilon}q^{1+\epsilon}z_q^{3/8} + X^{3/8}r^{1+\epsilon}q^{1+\epsilon}z_q^{3/8} \left( \frac{X^3}{y^4z_q} \right)^{\rho/4}.$$  

For $q$ satisfying $X^3/y^4 \geq r^{-2}q^{-2}$, we choose $z_q = X^3/y^4$ and get the bound

$$E_q(r, y, X) \ll X^{3/8}r^{7/8+\epsilon}q^{7/8+\epsilon} + X^{3/2}r^{1+\epsilon}q^{1+\epsilon}y^{-3/2}.$$

If $X^3/y^4 \leq r^{-2}q^{-2}$, we choose $z_q = r^{-2}q^{-2}$. Then $\frac{X^3}{y^4z_q} \leq 1$, and so the latter two terms in the right hand side of (3.16) are bounded by the first, so that (3.17) holds for such $q$ as well. Hence (3.14) is

$$\ll X^{3/8}r^{7/8+\epsilon}Q^{15/8+\epsilon} + X^{3/2}r^{1+\epsilon}Q^{2+\epsilon}y^{-3/2} + y^{1+\epsilon} + X/Q^{1-\epsilon}.$$

Our theorem follows by choosing $y = X/Q$ and $Q = X^{5/23}r^{-7/23}$. \hfill \Box

4. Remarks

We give some remarks. First, we counted $S_3$-sextic fields $\tilde{K}$ with specifying the 3-adic completion $A$ of $K$, and by the same method we may specify any finite number of local completions of $K$. In particular for a fixed prime $p \neq 3$, the ratio of the contributions of $S_3$-sextic fields whose splitting type of $p$ is (111111), (222), (33), (121212) and (1313) for the first and second main terms of (1.2) are respectively given by

$$\frac{1}{6} : \frac{1}{2} : \frac{1}{3} : \frac{1}{p^{4/3}} \text{ and } \frac{1 - \frac{1}{p^{r/3}}}{6} : \frac{1 + \frac{1}{p^{r/3}}}{2} : \frac{1 - \frac{1}{p^{r/3}}}{3} : \frac{1 + \frac{1}{p^{r/3}}}{p} : \frac{1 + \frac{1}{p^{13/9}}}{p^{13/9}}.$$

For $p = 3$ the last term should be replaced by $3^{-5/3} + 2 \cdot 3^{-7/3}$ and $3^{-17/9} + 2 \cdot 3^{-22/9}$ respectively. For the splitting types (121212) and (1313) there are often multiple possibilities for $K \otimes \mathbb{Q}_p$, depending on $p$, and the terms above can be further subdivided following the tables in Section 6.2 of [11]. Note that for any $p$ the sum of the first three entries (corresponding to fields unramified at $p$) is 1 and $1 + p^{-2/3}$ respectively. All of this also follows from the methods of [2] or [3]. In our case the error term remains the same, except that now it also depends (polynomially) on the prime(s) $p$.

In addition, by the same method, we can prove the analogue of the power-saving remainder term (1.1) for relative $S_3$-extensions over an arbitrary base number field $F$. This would use the generalization of (2.5) over $F$, whose proof will appear elsewhere. The exponent of $X$ in the $O$-term depends (only) on the degree $[F : \mathbb{Q}]$.

5. Numerical experiments

Finally, we compared our result (1.2) for $N_6^+(X; S_3)$ to numerical data. Our data weakly confirms (1.2), but it suggests the presence of one or more additional secondary terms. Indeed, our data will demonstrate several curious phenomena for which we don’t have a satisfactory explanation.

We computed tables of $N_6^+(X; S_3)$ using two distinct methods:

- We began with a direct approach, which allowed us to tabulate $N_6^+(X; S_3)$ for $X \leq 5 \cdot 10^{18}$. We used Belabas’s cubic program [1] to generate a list of all cubic fields $K$ with $|\text{Disc}(K)| < (5/3)^{1/2}10^9$, including generating polynomials. We have $\text{Disc}(\tilde{K}) = \text{Disc}(K)^2|\text{Disc}(F)|$, where $F$ is the quadratic resolvent of $K$, and as $|\text{Disc}(F)| \geq 3$ we were able to tabulate $S_3$-fields with discriminant bounded by $5 \cdot 10^{18}$.

  We used Lemma 2.1 to compute $\text{Disc}(\tilde{K})$ in terms of $\text{Disc}(K)$. In particular, $\text{Disc}(\tilde{K})$ is determined by $\text{Disc}(K)$ apart from the power of 2, which depends on whether or not $K$ is totally ramified at 2. For the power of 2, Belabas’s program outputs a binary cubic form.
\( f = av^3 + bv^2v + cuv^2 + dv^3 \) which corresponds to the maximal order \( \mathcal{O}_K \), and 2 is totally ramified in \( K \) if and only if \( f \) has a triple root \( \pmod 2 \), i.e., if

\[
(a, b, c, d) \pmod 2 \in \{(1, 1, 1, 1), (1, 0, 0, 0), (0, 0, 0, 1)\}.
\]

We used this condition to check the ramification at 2 and therefore to compile our list of \( S_3 \)-sextic extensions.

This approach is somewhat inefficient: we also obtained many fields \( \tilde{K} \) with larger discriminant, which we had no choice but to discard.

• Recently, Cohen and the second author [5] proved an explicit formula enumerating cubic fields by their quadratic resolvents. As the referee (of the present paper) suggested, the approach of [5] is ideal for counting \( N^\pm_6(X; S_3) \), and so we computed \( N^\pm_6(X; S_3) \) for \( X \leq 10^{23} \); we refer to [5] for details.

Code implementing each of these algorithms, in Java and in PARI/GP [8] respectively, is available from the second author’s website; to reproduce our results using either program, the reader must also download and run cubic. (Belabas informs us that this functionality may be incorporated into a future version of PARI/GP; this has the potential to make computations beyond \( 10^{18} \) practical.)

As we will see, our data is a rather odd match to our theoretical investigations, and the reader might be forgiven for speculating that our data is in error. To that end we note that implementing redundant algorithms for \( X \leq 5 \cdot 10^{18} \) allowed us to double check our results.

This brings us to the actual data, which we quote from [5]. The tables below list \( N^\pm_6(X; S_3) \) for various \( X \) between \( 10^{12} \) and \( 3 \cdot 10^{23} \). The columns labeled (1.2) give the values predicted by (1.2), which are consistently too high. (The bare main terms of Theorem 1.1 are still higher.)

| \( X \) | \( N^+_6(X; S_3) \) | (1.2) | (5.4) | Error |
|-------|-------------------|-------|-------|-------|
| \( 10^{12} \) | 690 | 756 | 709 | .031 |
| \( 10^{13} \) | 1650 | 1762 | 1682 | .027 |
| \( 10^{14} \) | 3848 | 4045 | 3910 | .025 |
| \( 10^{15} \) | 8867 | 9181 | 8955 | .021 |
| \( 10^{16} \) | 20062 | 20658 | 20276 | .021 |
| \( 10^{17} \) | 45054 | 46159 | 45513 | .021 |
| \( 10^{18} \) | 100335 | 102555 | 101460 | .022 |
| \( 10^{19} \) | 222939 | 226801 | 224943 | .020 |
| \( 10^{20} \) | 492335 | 499647 | 496490 | .020 |
| \( 10^{21} \) | 1083761 | 1097214 | 1091842 | .020 |
| \( 10^{22} \) | 2378358 | 2402995 | 2393842 | .019 |
| \( 10^{23} \) | 5207310 | 525840 | 5235221 | .018 |

To explain the apparent discrepancy between the data and (1.2), we tried an amended heuristic. If \( |\text{Disc}(\tilde{K})| < X \), then \( \tilde{K} \) cannot be totally ramified at any prime \( > X^{1/4} \). This suggests multiplying the main term by a factor

\[
\prod_{p > X^{1/4}} \frac{1 + p^{-1}}{1 + p^{-1} + p^{-4/3}} \sim 1 - \sum_{p > X^{1/4}} p^{-4/3} \sim 1 - \int_{X^{1/4}}^\infty \frac{t^{-4/3}}{\log t} \, dt \sim 1 - \frac{12X^{-1/12}}{\log X}.
\]

(The approximations above are rather simple, so we verified numerically that improving any of them leads only to minor differences.) Similarly, for the secondary term we incorporate the correction term

\[
\prod_{p > X^{1/4}} \frac{1 + p^{-2/3} + p^{-1} + p^{-4/3}}{1 + p^{-2/3} + p^{-1} + p^{-4/3} + p^{-13/9}} \sim 1 - \sum_{p > X^{1/4}} p^{-13/9} + O(p^{-19/9}) \sim 1 - 9\frac{X^{-1/9}}{\log X}.
\]
This suggests the asymptotic formula

\[
N_6^\pm(X;S_3) \sim \frac{C^\pm}{12} \prod_p c_p \cdot X^{1/3} \left(1 - \frac{12X^{-1/12}}{\log X}\right) + \frac{4K^\pm \zeta(1/3)}{5\Gamma(2/3)^3} \prod_p k_p \cdot X^{5/18} \left(1 - 9X^{-1/9}/\log X\right).
\]

With these corrections, we obtained the values listed under (5.4) in our tables. These values are more accurate, but still do not seem to closely match the data.

The final column labeled ‘Error’ gives the relative error estimate \[\frac{12 - N_6^\pm(X;S_3)}{X^{5/18}}.\] This column suggests that the secondary term in (1.2) is likely to be relevant, but the evidence is not overwhelming. Our heuristics also do not explain why the relative error is larger for negative discriminants, but (apparently) converges faster.

We tried other variations of our heuristics as well. As described earlier, we experimented with improving the estimates in (5.2) and (5.3) (e.g. evaluating the integrals in (5.2) and (5.3) numerically instead of using the approximation \(\log t \sim \log X\) and evaluating them). This made only a very minor difference, and it adjusted our counts upward rather than downward. Also, we observed that in fact no prime larger than \(X^{4/3}\) can totally ramify (as an \(S_3\)-sextic field has a nontrivial quadratic resolvent), and we tried an accordingly modified version of (5.2) and (5.3). These modified heuristics still produced data which were too high.

**Arithmetic progressions.** Our work in [11] found and explained interesting discrepancies in the distribution of cubic field discriminants in arithmetic progressions. For example, the following table lists the number of cubic fields \(K\) with \(0 < \text{Disc}(K) < 2 \cdot 10^6\) and \(\text{Disc}(K) \equiv a \pmod{m}\) for \(m = 5\) and 7. The “predicted” row is the sum of the \(X\) and \(X^{5/6}\) terms of the asymptotic formula proved in [11].

| Discriminant modulo 5 | 0  | 1  | 2  | 3  | 4  |
|-----------------------|----|----|----|----|----|
| Actual count          | 21277 | 22887 | 22751 | 22748 | 22781 |
| Predicted             | 21307 | 22757 | 22757 | 22757 | 22757 |

| Discriminant modulo 7 | 0  | 1  | 2  | 3  | 4  | 5  | 6  |
|-----------------------|----|----|----|----|----|----|----|
| Actual count          | 15330 | 17229 | 14327 | 15323 | 17027 | 18058 | 15150 |
| Predicted             | 15316 | 17209 | 14277 | 15316 | 17024 | 18063 | 15131 |

The results (mod 5) could have been predicted by Davenport and Heilbronn [7]. In contrast, the \(X^{5/6}\) term of the asymptotic is different for every residue class \(a\) (mod 7). We proved this in [11]; these results are explained by the existence of nontrivial sextic characters (mod 7), a phenomenon that could have been predicted earlier by Datskovsky and Wright [6].

We briefly investigated analogous questions for \(S_3\)-sextic field discriminants, and we quickly found interesting behavior which our methods could not explain.

For example, \(S_3\)-sextic field discriminants seem to not be equidistributed modulo 5! Using the algorithm of [5], we computed the following data for \(S_3\)-sextic fields \(\tilde{K}\) of negative discriminant (where there are no cyclic cubic fields), unramified at 2 and 3 (to eliminate wild ramification), and with \(0 < -\text{Disc}(\tilde{K}) < X\):
Each entry counts the number of $\overline{K}$ with $\text{Disc}(\overline{K}) \pmod{5}$; note that the discriminants are negative. The last two rows are predictions from (1.2), modified as described in Section 4 for the primes 2, 3, and 5. For $p = 5$ the 0 column is the contribution from fields ramified at 5; the remainder is divided into four equal parts, as predicted by our methods above and in [11].

The surplus of discriminants divisible by 5 is predicted by Lemma 2.1: for any cubic field $K$ totally ramified at 5, we know that $\text{Disc}(\overline{K}) \leq \frac{1}{25} \text{Disc}(K)^3$, and so many such fields have small discriminant. However, we were surprised to observe a surplus of field discriminants $\equiv 2, 4 \pmod{5}$. Certainly this is not predicted by any analysis involving the Shintani zeta function. We looked for other heuristic explanations, for example using the fact that

$$
\text{Disc}(\overline{K}) \equiv \pm (p_1 p_2 \cdots p_m)^{-1} \pmod{5},
$$

where $p_1, p_2, \cdots, p_m > 5$ are the primes ramified but not totally ramified in $K$, but we did not find any convincing explanation.

In conclusion, (1.2) and probably also its generalization to arithmetic progressions, appear to be correct – but our experiments have uncovered additional phenomena which call for explanation. Naturally we hope to see further work on this topic in the future!

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$^{5}$In particular, the secondary terms of counting functions for cubic field discriminants, twisted by nontrivial Dirichlet characters (mod 5), vanish; see Section 6.4 of [11].
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