LIMIT LINEAR SERIES MODULI STACKS IN HIGHER RANK

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Abstract. In order to prove new existence results in Brill-Noether theory for rank-2 vector bundles with fixed special determinant, we develop foundational definitions and results for limit linear series of higher-rank vector bundles. These include two entirely new constructions of “linked linear series” generalizing earlier work of the author for the classical rank-1 case, as well as a new canonical stack structure for the previously developed theory due to Eisenbud, Harris and Teixidor i Bigas. This last structure is new even in the classical rank-1 case, and yields the first proper moduli space of Eisenbud-Harris limit linear series for families of curves. We also develop results comparing these three constructions.

Contents

1. Introduction 2
Acknowledgements 4
Notational conventions 4
2. Preliminaries 5
2.1. Smoothing families 5
2.2. The almost local condition 6
2.3. Multidegrees of vector bundles 9
3. The stacks of higher-rank linked linear series 10
3.1. The underlying graphs 10
3.2. Linked linear series: type I 12
3.3. Linked linear series: type II 14
3.4. Foundational results 16
4. Stack structures on the Eisenbud-Harris-Teixidor construction 23
4.1. Statements on points 23
4.2. Stack structures 29
4.3. Comparison results 34
5. Limit series on chains of curves 37
5.1. Pairs of vanishing sequences on smooth curves 37
5.2. Chains of curves 39
6. Complementary results 43
6.1. The fixed determinant case 43
6.2. The parameter $b$ 45
6.3. Specialization 47
6.4. Stability 47
Appendix A. Prelinked Grassmannians 49
A.1. Definitions 49

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1. Introduction

Ever since the main questions of classical Brill-Noether theory were resolved in the 1980’s, the natural generalization to higher-rank vector bundles has been a subject of attention. At its most basic, this is the study of how many sections a (semistable) vector bundle of given rank and degree can have on a general curve. More precisely, one studies $g^k_{r,d,s}$, consisting of pairs $(E, V)$ of a vector bundle $E$ of rank $r$ and degree $d$, together with a $k$-dimensional space $V$ of global sections of $E$. This generalization is of fundamental interest due to the basic nature of understanding questions on vector bundles with sections, and on morphisms from curves to Grassmannians. However, despite our rather rudimentary understanding thus far, the subject has found several important applications, including work of Mukai [Muk10] classifying curves of low genus, which he in turn applied to the classification of Fano threefolds [Muk01], the work of Farkas and Popa [FP05] giving a counterexample to the slope conjecture, and most recently, work of Bhosle, Brambila-Paz, and Newstead proving the rank-1 case of a conjecture of Butler regarding stability of the kernel of the evaluation map of linear series [BBPN], which has in turn been applied by Brambila-Paz and Torres-Lopez [BPTL] to obtain new results on Chow stability of curves.

Despite extensive work and many partial results (see [GT09] for a survey), there is as yet not even a comprehensive conjecture for the case of rank-2 vector bundles. The naive generalization of the Brill-Noether theorem to higher rank fails in every possible way, with complications arising in particular from stability conditions and from the role of vector bundles with special determinant. The latter was first observed by Bertram and Feinberg [BF98] and Mukai [Muk95] for the case of bundles of rank 2 with canonical determinant, and studied more systematically in [Oss13a] and [Oss13c]. We now explain it in more detail.

The natural generalization of the classical Brill-Noether number to the higher-rank case is given by

$$
\rho := 1 + r^2(g - 1) - k(k - d + r(g - 1)).
$$

Because we will consider moduli stacks rather than coarse moduli spaces, the actual expected dimension will be $\rho - 1$ for the varying determinant case. Thus, the moduli stack $G_{r,d}^k(X)$ of $g^k_{r,d,s}$ on a general curve $X$ of genus $g$ has every component of dimension at least $\rho - 1$. In the fixed determinant case, the expected dimension remains $\rho - g$, because fixing the determinant rigidifies the moduli problem. In many cases, $G_{r,d}^k(X)$ is known to have components of dimension $\rho - 1$ (or $\rho - g$ in the fixed determinant case). However, in the case of rank 2 and fixed determinant determinant
if $h^1(L) > 0$, then the dimension of the moduli space is always at least $\rho - g + \binom{g}{2}$. Bertram, Feinberg and Mukai conjectured that the locus of $g^k_{2,d}$s with canonical determinant behaves like classical Brill-Noether loci, and the portion of their conjecture asserting non-emptiness when $\rho - g + \binom{g}{2} \geq 0$ remains open.

The primary objective of the present paper, together with [OT14b], is to develop the necessary machinery for the use degeneration techniques to prove existence results for moduli spaces of $g^k_{2,d}$s in the case of fixed special determinant. The most powerful tool for studying classical Brill-Noether theory is the Eisenbud-Harris theory of limit linear series [EH86]. In [Tei91], Teixidor i Bigas generalized this theory to the higher-rank case. The most technical part of the theory is the proof of smoothing theorems, in which one shows that the existence of families of limit linear series having the expected dimension on a given reducible curves implies the existence of linear series (or $g^k_{2,d}$s) on smooth curves. These results are proved by constructing suitable moduli spaces for families of curves, and proving dimensional lower bounds. There is no difficulty in carrying out the Eisenbud-Harris construction in the higher-rank case using the naive expected dimensions $\rho - 1$ and $\rho - g$. However, this result is useless in the case of special determinant, because the dimension will always be strictly larger than $\rho - g$ whenever $k \geq 2$. In order to obtain a usable theory in this setting, it is thus necessary to combine the dimensional lower bounds of the special determinant setting with those of the theory of limit linear series. Unfortunately, in the context of Eisenbud-Harris-Teixidor limit linear series, it is not clear how to track the symmetries which cause the higher expected dimension in the special determinant case.

In the present paper, we introduce new moduli stacks of higher-rank limit linear series (which, for the sake of avoiding confusion, we label “linked linear series”). These both generalize the construction of [Oss06a] to higher rank and to curves with more than two components. These stacks have comparison morphisms between them and to the stack of Eisenbud-Harris-Teixidor limit linear series. As in the case of [Oss06a], these morphisms are isomorphisms on certain open loci of interest, but on the boundary frequently have positive-dimensional fibers. One advantage of the linked linear series perspective is that with it, one can see the necessary symmetries in the case of special determinant. Accordingly, in [OT14b], we specialize to the setting of rank 2 and special determinant, and use these symmetries to prove the necessary dimensional lower bounds on moduli spaces to get effective smoothing theorems. Combining these with the comparison results of the present paper, we are then able to carry out limit linear series computations to prove existence of components of moduli spaces of $g^k_{2,d}$s having fixed special determinant in a large family of examples. See [OT14b] for details. Zhang [Zha14] has proved further existence results in the canonical determinant case using our smoothing theorem.

We now discuss the contents of the paper in more detail. We begin in §2 with background on “smoothing families” — the families of nodal curves which we consider for our degenerations. In §3 we introduce the central new constructions, including the definition of and construction of moduli stacks for type I and type II linked linear series. This provides two distinct generalizations of the construction of [Oss06a]. Type I linked linear series have the advantage of being more suited to universal constructions, while it is the type II linked linear series which play the crucial role in the study of loci with fixed special determinants. In §4, we provide a new perspective on Eisenbud-Harris-Teixidor limit linear series, yielding a
construction of a moduli stack which works in families. This is new even in the rank-1 case, where we obtain for the first time a proper moduli space; the previous construction for families included only refined limit linear series. In the higher-rank case, the same construction also shows that the Eisenbud-Harris-Teixidor limit linear series are locally closed in the natural ambient space, which was not clear from the original definition; see Remark 4.1.12. We then compare the linked linear series construction to the limit linear series construction, proving that they are isomorphic on certain open loci. In §5, we carry out a detailed analysis of the situation for chains of curves, focusing on the locus of “chain-adaptable” limit linear series, which occurs ubiquitously in known families of higher-rank limit linear series. This culminates in Corollary 5.2.7, which is our main comparison result and the cornerstone of [OT14b]. In §6 we briefly develop several complementary directions, including foundations for the fixed determinant case, behavior of stability conditions, and results on specialization of \( g^{k}_{r,d} \) under degeneration. In Appendix A we define and study “prelinked Grassmannians,” generalizing the linked Grassmannians of [Oss06a], and finally, in Appendix B we develop a generalization of determinantal loci to pushforwards of coherent sheaves, which is used in the new approach to constructing the moduli stack of Eisenbud-Harris-Teixidor limit linear series.

Thus, although our primary motivation is the machinery and existence results for the special determinant case addressed in [OT14b], the present paper includes a number of new ideas which should prove useful more broadly. For instance, the new description of classical Eisenbud-Harris limit linear series simplifies existing proofs, and also suggests how one might approach a theory of limit linear series for curves not of compact type.

Finally, we briefly discuss the impetus for and implications of the shift to working with stacks. The most compelling reason to work with stacks is in order to be able to work with strictly semistable vector bundles, for which coarse moduli spaces are very poorly behaved. A side benefit is that even on stable loci, arguments become cleaner, without the need to pass to etale covers in order to produce universal families of vector bundles. From the point of view of comparison results between constructions, the situation is not substantially subtler in working with stacks than it would be for schemes (or even varieties). On loci for which comparison morphisms are isomorphisms, one can typically deduce the stack isomorphism from bijectivity on points, with little additional work. Conversely, on loci for which the morphisms are not isomorphisms, the fibers are typically infinite. Dimension theory for stacks is somewhat subtler than for schemes, but this is addressed by the theory developed in [Oss13b], and consequently is handled entirely transparently in the present paper.

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Notational conventions. Because we will use graphs rather extensively for notational purposes, we state our conventions. If \( G \) is a (possibly directed) graph, we denote by \( V(G) \) and \( E(G) \) the sets of vertices and edges of \( G \), respectively. In the directed case, for \( e \in E(G) \), we denote by \( t(e) \) and \( h(e) \) the tail and head of \( e \), respectively.

If \( X \) is an \( S \)-scheme, and \( \mathcal{F} \) a sheaf on \( S \), to avoid introducing notation for structure morphisms we denote by \( \mathcal{F}|_X \) the pullback of \( \mathcal{F} \) to \( X \).
If \( X \) is a reducible curve, we will use subscripts to denote sheaves on all of \( X \) (of varying multidegrees) and spaces of global sections, and we will use superscripts to denote sheaves on individual components of \( X \) and spaces of global sections. We use script for vector bundles, and roman letter for vector spaces.

2. Preliminaries

This section is devoted to a comprehensive treatment of the families of curves we will consider, called “smoothing families.” A substantial portion of our analysis involves the introduction and development of the “almost local” condition on smoothing families, which guarantee that the dual graphs of fibers behave in a relatively simple manner. We conclude with a discussion of moduli stacks of vector bundles with prescribed multidegree.

2.1. Smoothing families. The following definition differs only slightly from that of [Oss06a]: because we work here with stacks rather than coarse moduli spaces, the hypothesis on existence of sections is unnecessary. Also, because vanishing conditions play a different role in higher rank, we omit the choice of smooth sections of the smoothing family along which to impose such conditions.

Definition 2.1.1. A morphism of schemes \( \pi : X \to B \) constitutes a smoothing family if:

(I) \( B \) is regular and connected;
(II) \( \pi \) is flat and proper;
(III) The fibers of \( \pi \) are genus-\( g \) curves of compact type;
(IV) Each connected component \( \Delta' \) of the singular locus of \( \pi \) maps isomorphically onto its scheme-theoretic image \( \Delta \) in \( B \), and furthermore \( \pi^{-1}(\Delta) \) breaks into two (not necessarily irreducible) components intersecting along \( \Delta' \);
(V) Any point in the singular locus of \( \pi \) which is smoothed in the generic fiber is regular in the total space of \( X \).

The following lemma is useful for constructing stacks of higher-rank limit linear series.

Lemma 2.1.2. Locally on the base \( B \), a smoothing family always carries a \( \pi \)-ample line bundle.

Proof. Given \( y \in B \), by Corollary 9.6.4 of [GD66], a line bundle is \( \pi \)-ample in a neighborhood if and only if it has positive degree on every component of \( X_y \). By hypothesis, \( X \) breaks into components for every singularity of \( X_y \) which is not generically smoothed, and line bundles on \( X \) are uniquely determined (up to twisting by pullbacks from \( B \)) by their restrictions to these components. In addition, the properties of a smoothing family are preserved under restriction to such a component (see Lemma 3.2 (ii) of [Oss06a]), so it is enough to treat the case that \( X \) is irreducible, or equivalently, that every singularity of \( X_y \) is smoothed generically. In this case, our hypotheses give us that \( X \) is regular. Then let \( D \) be the closure of any height-1 point of the generic fiber; we have then \( \theta(D) \) is a line bundle of strictly positive degree. Replacing \( D \) by a suitable power, we may assume the degree is at least as large as the number of components of \( X_y \). Now, for every singularity \( \Delta' \) of \( \pi \), we have by hypothesis that \( \pi^{-1}(\pi(\Delta')) \) breaks into components intersecting along \( \Delta' \); these components are each divisors on \( X \), and twisting \( \theta(D) \)
by them, we can redistribute the degree arbitrarily over the components of $X_y$, and in particular can obtain a line bundle with positive degree on each component, as desired.

We take the opportunity to state a somewhat more general version of Theorem 3.4 of [Oss06a]. The proof is the same, with appropriate considerations for working over a non-algebraically closed field as described in [Oss06b].

**Theorem 2.1.3.** Let $X_0$ be a curve of compact type over a field $k$, and $P_1, \ldots, P_n$ distinct smooth $k$-valued points. Suppose that each node of $X_0$ is a $k$-valued point, and each geometric component of $X_0$ is defined over $k$. Then $X_0$ may be placed into a smoothing family $X/B$ with sections disjoint smooth sections $P_i$, specializing to the $P_i$, where $B$ is a curve over $k$, and where the generic fiber of $X$ over $B$ is smooth.

We also have the following standard structural statement:

**Proposition 2.1.4.** If $\pi : X \to B$ is a smoothing family, and $\Delta$ is the scheme-theoretic image of a connected component of the non-smooth locus of $\pi$, then $\Delta$ is regular.

**Proof.** If $\Delta = B$, this is automatic from the hypothesis that $B$ is regular. Otherwise, let $\Delta' \subseteq X$ be the connected component of the non-smooth locus of $\pi$ with image $\Delta$: the proposition will follow from the definition of a smoothing family if we show that $\Delta'$ is regular. On the other hand, the definition states that $X$ is regular along $\Delta'$. The statement being invariant under completion, we may pass to complete local rings, in which case we have that $\pi$ looks like $\text{Spec} \, \hat{A}[u, v]/(uv - t) \to \text{Spec} \, \hat{A}$, where $\hat{A}$ is the relevant complete local ring of $B$, and $t \in \mathfrak{m}_B$ (see for instance §2.23 of [dJ96]). It is then clear that $X$ being regular along $\Delta'$ implies that $t \notin \mathfrak{m}_B^2$, and the ideal of $\Delta'$ is cut out by $(u, v)$, so the complete local ring of $\Delta'$ at the chosen point is isomorphic to $\hat{A}/(t)$, which is regular, as desired.

2.2. The almost local condition. We now devote some attention to the behavior of dual graphs in families of curves. We begin with the following:

**Lemma 2.2.1.** Suppose that $\pi : X \to B$ is a smoothing family, and $y$ specializing to $y'$ are points of $B$. Then if $\Gamma_y$ and $\Gamma_{y'}$ denote the dual graphs of the fibers $X_y$ and $X_{y'}$ respectively, there is a unique contraction map

\[ \text{cl}_{y, y'} : \Gamma_{y'} \to \Gamma_y \]

induced on vertices by associating to a component $Y'$ of $X_{y'}$ the component $Y$ of $X_y$ containing $Y'$ in its closure. The behavior of $\text{cl}_{y, y'}$ on edges is as follows: given an edge $e$ in $\Gamma_{y'}$ corresponding to a node $\Delta'$ in $X_{y'}$, if there is a node of $X_y$ specializing to $\Delta'$, then $\text{cl}_{y, y'}$ maps $e$ to the corresponding edge of $\Gamma_y$; otherwise, $e$ is contracted.

If also $y'$ specializes to some $y'' \in B$, then we have

\[ \text{cl}_{y, y''} = \text{cl}_{y, y'} \circ \text{cl}_{y', y''}. \]

**Proof.** Given $y$ specializing to $y'$ and a component $Y'$ of $X_{y'}$, we first need to see that there exists a unique component $Y$ of $X_y$ containing $Y'$ in its closure. Existence follows from flatness of $\pi$. To see uniqueness, let $Y$ and $Z$ be distinct components of $X_y$, and let $\Delta'$ be the connected component of the non-smooth locus of $\pi$ containing a node of $X_y$ separating $Y$ and $Z$. Let $\Delta$ be the image of $\Delta'$; since
y, y' ∈ Δ, we can check uniqueness of specialization after restricting to Δ. By definition of a smoothing family, we have that π⁻¹(Δ) breaks into (not necessarily irreducible) components YΔ and ZΔ with YΔ ∩ ZΔ = Δ'. Then the generic point of any component of Xy or Xy' is contained in precisely one of YΔ and ZΔ, so it follows that Y' can be in the closure of at most one of Y and Z, as desired. This gives cl_{y,y'} on the vertices of Γy' and Γy; it is then straightforward to check that we obtain a contraction map, with the claimed behavior on edges. Finally, associativity is clear from the definition.

Although the next definition is slightly complicated, the idea behind it is simple: it captures the condition that a smoothing family is combinatorially local, in the sense that there is a maximal dual graph for the fibers of the family, and the dual graph of every fiber is naturally a contraction of the maximal one.

**Definition 2.2.2.** We say a smoothing family π : X → B is **almost local** if the following condition is satisfied: if Δ1, ..., Δm are the connected components of the non-smooth locus of π, with images Δ1, ..., Δm in B, then there exists a y₀ ∈ B such that for all S, S' ⊆ {1, ..., m}, we have ∩i∈S Δi non-empty, and for any irreducible components Z of ∩i∈S Δi and Z' of ∩i∈S' Δi, every irreducible component of Z ∩ Z' contains y₀.

**Remark 2.2.3.** Observe that the almost local hypothesis is trivially satisfied in the case that π : X → B has connected non-smooth locus, as is the situation in [Oss06a]. In addition, it is always satisfied locally on B: indeed, given y ∈ B, we construct an open neighborhood on which π is almost local simply by removing any irreducible component of each ∩i∈S Δi which does not contain y, and doing likewise for intersections of pairs of components.

The following proposition says that an almost local smoothing family admits a maximal dual graph in a natural way; this is the reason for the hypothesis, as it will greatly simplify keeping track of dual graphs and multidegrees.

**Proposition 2.2.4.** Suppose that π : X → B is an almost local smoothing family. Then there exists a graph Γ, occurring as the dual graph of some fiber of π, and, for every y ∈ B, a contraction

\[ cl_y : Γ → Γ_y, \]

where Γ_y is the dual graph of the fiber X_y, satisfying the following condition: if y specializes to y', we have

\[ cl_y = cl_{y,y'} ∘ cl_{y'}. \]

Moreover, up to automorphism of Γ, we have that Γ and the contractions cl_y are unique.

Note that the uniqueness assertion in particular implies that the data of Γ and the cl_y contractions is independent of the choice of y₀ from Definition 2.2.2.

**Proof.** We first fix some notation. Let Δ1, ..., Δm be the connected components of the non-smooth locus, and Δ1, ..., Δm their images in B. For y ∈ B, set S_y ⊆ {1, ..., m} to be the subset of i such that y ∈ Δi. Observe that given y specializing to y', if S_y = S_y', the contraction cl_{y,y'} is a surjection of trees with the same number of edges, and hence is necessarily an isomorphism.
Let $y_0$ be as in Definition 2.2.2. Set $\Gamma = \Gamma_{y_0}$, with $\text{cl}_{y_0}$ being the identity. Now, for any $y \in B$, let $\tilde{y}$ be a generic point of $\cap_{e \in E} \Delta_e$ which specializes to $y$, so that $\text{cl}_{\tilde{y}}$ is an isomorphism. It follows from the definition of almost local that $\tilde{y}$ specializes to $y_0$, so we can set

$$\text{cl}_y = \text{cl}_{\tilde{y},y} \circ \text{cl}_{\tilde{y},y_0} \circ \text{cl}_{y_0}.$$  

(2.2.2)

It remains to check that given $y$ specializing to $y'$, we have (2.2.1). Let $\tilde{y}'$ be the specialization of $y'$ used to define $\text{cl}_{y'}$, and let $Z, Z'$ be the closures of $\tilde{y}$ and $\tilde{y}'$, respectively. Let $\tilde{Z}$ be a component of $Z \cap Z'$ containing $y'$, and $\tilde{y}$ its generic point. Then according to the almost local hypothesis, we have that $\tilde{y}$ specializes to $y_0$. Using associativity, we then have

$$\text{cl}_y = \text{cl}_{\tilde{y},y} \circ \text{cl}_{\tilde{y},y_0} \circ \text{cl}_{y_0}$$

as desired.

For the uniqueness assertion, observe that $\Gamma_{y_0}$ has $m$ edges, the maximal number possible among any $\Gamma_y$, so we must have $\Gamma \cong \Gamma_{y_0}$. If we fix a choice of $\text{cl}_{y_0}$, which is defined precisely up to automorphism of $\Gamma$, we then have that (2.2.1) implies that (2.2.2) must hold, so we have that $\text{cl}_y$ is uniquely determined for all $y$, as asserted.

**Corollary 2.2.5.** Let $\pi : X \to B$ be an almost local smoothing family, and $\Gamma$ and $\text{cl}_y$ for $y \in B$ as given by Proposition 2.2.4. Then there is a bijection from $E(\Gamma)$ to the connected components of the non-smooth locus of $\pi$ induced by sending an edge $e \in E(\Gamma)$ to

$$\Delta_e' := \bigcup_{y \in \text{cl}_{y_0} \text{ does not contract } e} \Delta_{\text{cl}_y(e)},$$

where $\Delta_{\text{cl}_y(e)}$ denotes the node of $X_y$ corresponding to $\text{cl}_y(e)$.

Furthermore, if $\Delta_e \subseteq B$ denotes the image of $\Delta_e'$, then given $v \in V(\Gamma)$ adjacent to $e$ there is a unique closed subset $Y_{(e,v)} \subseteq \pi^{-1}(\Delta_e)$ such that for each $y \in \Delta_e$, the fiber $(Y_{(e,v)})_y$ is equal to the union of the components of $X_y$ corresponding to the vertices of $\Gamma_y$ lying in the same connected component as $v$ in $\Gamma_y \setminus \{e\}$.

Thus, if $v, v'$ are the two vertices adjacent to an edge $e$, then $Y_{(e,v)} \cup Y_{(e,v')} = \pi^{-1}(\Delta_e)$, and $Y_{(e,v)} \cap Y_{(e,v')} = \Delta_e'$.

**Proof.** For the first assertion, we need to check that $\Delta_e'$ is in fact a connected component of the non-smooth locus of $\pi$, and that the induced map is a bijection. Let $y_0$ be as in the definition of almost local, and fix $e \in E(\Gamma)$. Let $\Delta'$ be the connected component of the non-smooth locus containing $\Delta'_{\text{cl}_{y_0}(e)}$. We want to see that $\Delta'_e = \Delta'$. Then given $y \in B$, let $\tilde{y}$ be a point generizing $y$ and $y_0$, and such that $\text{cl}_{\tilde{y},y}$ is an isomorphism. First suppose that $\text{cl}_y$ does not contract $e$. Then we
have that $\Delta_{cl_y(e)}'$ specializes to both $\Delta_{cl_y(e)}$ and to $\Delta_{cl_{\hat{y}}(e)}'$. Thus, since $\Delta'$ is a connected component of the non-smooth locus, we conclude that $\Delta_{cl_y(e)}' \subseteq \Delta'$. For the opposite containment, suppose that $y$ is in the image of $\Delta'$. Then since $\Delta'$ is a section of $\pi$ over its image, we see that $\Delta'$ has a point of $X_\hat{y}$ which specializes to $\Delta_{cl_{\hat{y}}(e)}$. It follows that $cl_{\hat{y}}$ does not contract $e$, and then neither does $cl_y$. Thus, $y$ is in the image of $\Delta', v$ and since $\Delta'$ is a section over its image and $\Delta_{cl_y(v)}' \subseteq \Delta'$, we conclude that $\Delta_{cl_y(v)}' = \Delta'$, as desired.

Next, to see that we have a bijection, we note that injectivity is trivial, since for $e \neq e'$ we have that $\Delta_{e}|_{X_{y_0}}$ and $\Delta_{e'}|_{X_{y_0}}$ are distinct points. On the other hand, the definition of almost local imposes that every connected component $\Delta'$ of the non-smooth locus of $\pi$ meets $X_{y_0}$, so meets some $\Delta_{cl_y(v)}'$. Since $\Delta_{cl_y(v)}'$ is also a connected component of the non-smooth locus, we conclude $\Delta' = \Delta_{cl_y(v)}'$ giving surjectivity.

For the assertion on $Y_{(e,v)}$, we have by hypothesis that $\pi^{-1}(\Delta_v)$ decomposes as $Y \cup Z$, with $Y \cap Z = \Delta_{e'}$; exactly one of $Y$ or $Z$ contains the component of $X_{y_0}$ corresponding to $cl_{y_0}(v)$, so if we set $Y_{(e,v)}$ equal to this subset, we need only verify that it has the desired form on every fiber. It is evident from the construction that for each $y \in \Delta_e$, we have $Y_{(e,v)}|_y$ equal to the union of components of $X_y$ corresponding to a connected component of $\Gamma_y \setminus \{e\}$, so we need only verify that the connected component in question is the one containing $cl_y(v)$. If $y$ specializes to $y_0$, this is immediate from the fact that $cl_y = cl_{y_0} \circ cl_{y_0}$, and in the general case it follows by choosing some $\hat{y}$ specializing to both $y$ and $y_0$ as above. □

2.3. Multidegrees of vector bundles. We conclude this section with some background on multidegrees of vector bundles for smoothing families, and the associated moduli stacks.

**Definition 2.3.1.** Given an almost-local smoothing family $\pi : X \to B$, let $\Gamma$ and $cl_y$ be as in Proposition 2.2.4. Given also a $B$-scheme $S$, a vector bundle $\mathcal{E}$ on $X \times_B S$ has **multidegree** $w = (i_v)_{v \in V(\Gamma)}$ if, for every $s \in S$ and every component $Y$ of the fiber $(X \times_B S)_s$, we have

$$\deg \mathcal{E}|_Y = \sum_{v \in V(\Gamma) : cl_y(v) = \imath_Y} i_v,$$

where $y$ is the image of $s$ in $B$, and $\imath_Y$ is the vertex of $\Gamma_y$ corresponding to $Y$.

We denote the groupoid of vector bundles of rank $r$ and multidegree $w$ on $X/B$ by $\mathcal{M}_{r,w}(X/B)$.

Note that our hypotheses on smoothing families imply in particular that irreducible components of fibers are geometrically irreducible, so in the above definition, the dual graphs of $(X \times_B S)_s$ and $X_y$ are canonically identified.

For lack of a suitable reference, we include the proof of the following basic fact.

**Proposition 2.3.2.** We have that $\mathcal{M}_{r,w}(X/B)$ is an open substack of the moduli stack $\mathcal{M}_{r,d}(X/B)$ of vector bundles of rank $r$ and degree $d := \sum_{v \in V(\Gamma)} i_v$. In particular, it is an Artin stack, locally of finite type over $B$.

**Proof.** Given $w = (i_v)_v$, we can associate to every pair $(e, v)$ with $e \in E(\Gamma)$ and $v \in V(\Gamma)$ adjacent to $e$ an integer $i_{(e, v)}$: let $C$ be the connected component of $\Gamma \setminus \{e\}$ containing $v$, and set $i_{(e, v)} = \sum_{v \in C} i_v$. We then observe that $\mathcal{E}$ has multidegree $w$ if and only if for every $(e, v)$ as above, and every fiber of $X \times_B S$ supported over $\Delta_e$, the bundle $\mathcal{E}$ has degree $i_{(e, v)}$ on the preimage of $Y_{(e,v)}$. But now, it is easy to
see that this is an open condition. Indeed, we have that \( Y_{e,v} \) is flat over \( \Delta_e \) by the argument for Lemma 3.2 (ii) of [Oss06a], so degree is locally constant. Thus, if the condition is satisfied on a single fiber \( X_s \) of \( X \times_B S \), it is necessarily satisfied at all points in the connected component of \( S \) except possibly for a finite union of preimages of subsets \( \Delta_e \), each of which is closed. \( \square \)

3. The stacks of higher-rank linked linear series

We now come to the foundational definitions and constructions for our new spaces of limit linear series. In order to distinguish our terminology from that of Eisenbud-Harris-Teixidor, we refer to our objects as “linked linear series.” This is consistent with [OT14a], but deviates from [Oss06a]. The idea of a linked linear series is to consider a collection of vector bundles obtained by various twists from a single vector bundle, together with spaces of global sections on each such twist satisfying a linkage condition under certain natural maps. We will introduce two variants of this idea, each generalizing the single construction of [Oss06a], but for each one the process is the same: we introduce a directed graph to parametrize the twists we wish to consider, describe twisting line bundles and associated morphisms, and then use these to define the linked linear series we wish to consider.

For the second construction, we will consider infinite graphs. This makes the definitions more transparent because we do not have to consider edges near the boundary as special cases, but it will come at the cost that in the foundational theorem, we will have to prove that the resulting construction can be described equivalently on a finite graph.

3.1. The underlying graphs. The basic situation we will consider is the following:

**Situation 3.1.1.** Let \( \pi : X \to B \) be an almost local smoothing family. Let \( \Gamma \) be the graph obtained from Proposition 2.2.4, with associated contractions \( \text{cl}_y \) for \( y \in B \). Let \( \Delta_x, \Delta_e \) and \( Y_{e,v} \) be as in Corollary 2.2.5.

Let \( r, d, k \) be positive integers, and fix also integers \( b \) and \( d_v \) for each \( v \in V(\Gamma) \), satisfying

\[
\sum_{v \in V(\Gamma)} d_v - |E(\Gamma)|rb = d.
\]

We will associate two directed graphs \( G_I \) and \( G_{II} \) to \( \Gamma \), corresponding to our two generalizations of the construction of [Oss06a]. The vertices of both graphs will correspond to choices of multidegrees, and hence lie in \( \mathbb{Z}^{V(\Gamma)} \). The choice of the \( d_v \) (which together determine \( b \)) serves two purposes: it determines a congruence class modulo \( r \) for the multidegrees in question, and it also determines in some sense where our “extremal” vertices lie. This was not necessary in the classical rank-1 case because no line bundle of negative degree has nonzero global sections. We will see in Proposition 6.2.1 below that – at least in the type II case – we can always increase the \( d_v \), and this will have the effect of imbedding the original moduli space as an open substack into the newer one.

**Definition 3.1.2.** Suppose we are in the above situation. Let \( V(G_{II}) \subseteq \mathbb{Z}^{V(\Gamma)} \) consist of vectors \((i_v)_{v \in V(\Gamma)} \) satisfying:

(I) \( \sum_{v \in V(\Gamma)} i_v = d \), and

(II) \( i_v \equiv d_v \) (mod \( r \)) for all \( v \in V(\Gamma) \),
Figure 1. An example of Definition 3.1.2, with \( b = 2 \). We have restricted \( G_{II} \) to the finite region used in the definition of \( G_{II} \) in Definition 3.4.9 below.

and let \( V(G_I) \subseteq V(G_{II}) \) be the subset on which we further have

(III) \( i_v \geq d_v - rb \) for all \( v \in V(\Gamma) \), and

(IV) \( i_v = d_v - rb \) for all but at most two \( v \in V(\Gamma) \), with the vertices on which we have strict inequality required to be adjacent in \( \Gamma \).

Let \( G_I \) be the directed graph with vertex set \( V(G_I) \), and with an edge from \( h \) to \( h' \) in \( V(G_I) \) precisely when there exist adjacent vertices \( v, v' \in V(\Gamma) \) such that \( h - h' \) is \( \pm r \) in index \( v \), \( \mp r \) in index \( v' \), and 0 elsewhere.

Let \( G_{II} \) be the directed graph with vertex set \( V(G_{II}) \), and an edge from \( h \) to \( h' \) if there is a vertex \( v \in V(\Gamma) \) of valence \( \ell \) such that \( h' - h \) is \( -\ell r \) in index \( v \), is \( r \) in index \( v' \) for each \( v' \) adjacent to \( v \), and is 0 elsewhere.

Note that although \( G_I \) is taken to be directed, for any edge from \( h \) to \( h' \) there is a corresponding edge from \( h' \) to \( h \). Thus, \( G_I \) has no more information than the underlying undirected graph, but we use the directed version for convenience of notation.

By construction, the vertices of either \( G_I \) and \( G_{II} \) naturally induce multidegrees on fibers of \( \pi \), and we will sometimes treat them as multidegrees without further comment.

For both \( G_I \) and \( G_{II} \), the edges (or more generally paths) coming out of a given vertex can be described simply in terms of \( \Gamma \), and it will be convenient to introduce notation expressing this relationship.

**Notation 3.1.3.** Given an edge \( \varepsilon \in G_I \), denote by \( v_l(\varepsilon) \) and \( e_l(\varepsilon) \) the vertex and (adjacent) edge of \( \Gamma \) such that following \( \varepsilon \) decreases the coordinate in index \( v_l(\varepsilon) \), and increases the coordinate indexed by the other vertex adjacent to \( e_l(\varepsilon) \). Given \( w \in V(G_I) \), and a sequence \( (e_1, v_1), \ldots, (e_m, v_m) \) of edges together with adjacent vertices in \( \Gamma \), denote by \( P(w, (e_1, v_1), \ldots, (e_m, v_m)) \) the path in \( G_I \) (if it exists) starting at \( w \) and consisting of edges \( \varepsilon_1, \ldots, \varepsilon_m \) such that \( v_l(\varepsilon_i) = v_i \) and \( e_l(\varepsilon_i) = e_i \) for each \( i \).

Given an edge \( \varepsilon \in G_{II} \), denote by \( v_l(\varepsilon) \) the corresponding vertex of \( \Gamma \). Given \( w \in V(G_{II}) \) and a sequence \( v_1, \ldots, v_m \) of (not necessarily distinct) vertices of \( \Gamma \), denote by \( P(w, v_1, \ldots, v_m) \) the path in \( G_{II} \) starting at \( w \) and consisting of edges \( \varepsilon_1, \ldots, \varepsilon_m \) such that \( v_l(\varepsilon_i) = v_i \) for each \( i \).
Observe that for any given collection of \( (e_1, v_1), \ldots, (e_m, v_m) \), we may not obtain a path \( P(w, (e_1, v_1), \ldots, (e_m, v_m)) \) in \( G_I \), due to our conditions on the vertices of \( G_I \). Because \( G_I \) is a tree, minimal paths between any two vertices in it are unique. On the other hand, the endpoint of \( P(w, v_1, \ldots, v_m) \) is independent of the ordering of \( v_1, \ldots, v_m \). In fact, we can be very precise about the extent to which (minimal) paths are determined by their endpoints in \( G_I \) and \( G_{II} \), as follows.

**Proposition 3.1.4.** Minimal paths are unique in \( G_I \), and \( P(w, (e_1, v_1), \ldots, (e_m, v_m)) \) is minimal if and only if no edge \( e \) of \( \Gamma \) with adjacent vertices \( v, v' \) has both \( (e, v) \) and \( (e, v') \) appearing among the \( (e_i, v_i) \).

A path \( P(w, v_1, \ldots, v_m) \) in \( G_{II} \) is minimal if and only if not every vertex of \( \Gamma \) occurs as one of the \( v_i \). In this case, the resulting path is unique up to reordering the \( v_i \).

More generally, paths \( P(w, (e_1, v_1), \ldots, (e_m, v_m)) \) and \( P(w, (e'_1, v'_1), \ldots, (e'_m, v'_m)) \) in \( G_I \) have the same endpoint if and only if the multisets \( \{ (e_1, v_1), \ldots, (e_m, v_m) \} \) and \( \{ (e'_1, v'_1), \ldots, (e'_m, v'_m) \} \) differ by unions of sets of the form \( \{ (e, v), (e, v') \} \), where \( v \) and \( v' \) are the two edges adjacent to \( e \).

Similarly, two paths \( P(w, v_1, \ldots, v_m) \) and \( P(w, v'_1, \ldots, v'_m) \) in \( G_{II} \) have the same endpoint if and only if the multisets \( \{ v_1, \ldots, v_m \} \) and \( \{ v'_1, \ldots, v'_m \} \) differ by a multiple of \( V(\Gamma) \).

The proof is straightforward, and left to the reader.

Next, observe that conditions (I) and (III) together imply the following description of the extremal vertices of \( G_I \):

**Proposition 3.1.5.** For any \( (i_v)_{v \in V(\Gamma)} \in G_I \) we have \( i_v \leq d_v \), with equality if and only if \( i_{v'} = d_{v'} - rb \) for all \( v' \neq v \) in \( V(\Gamma) \).

**Notation 3.1.6.** Given \( v \in V(\Gamma) \), denote by \( w_v \) the vertex of \( G_I \) with coordinates \( i_v = d_v \), and \( i_{v'} = d_{v'} - rb \) for all \( v' \neq v \).

Although there are no explicit extremal vertices in the definition of \( G_{II} \), the vertices \( w_v \) are in some sense extremal for \( G_{II} \) as well; see Proposition 3.4.12. We use the convention of the infinite graph \( G_{II} \) in order to keep definition as simple as possible.

**Remark 3.1.7.** Perhaps the most natural generalization of the construction of [Oss06a] is a composite of the type I and type II constructions, where the multidegrees are allowed to be arbitrary as in type II, but the edges between vertices are defined in terms of nodes of the curve as in type I. However, this approach turns out not to be well behaved, for the reason that the maps are simply too degenerate for linkage to impose enough requirements. Indeed, with this approach we see that the sort of behavior described in Example 4.3.5 below occurs even for refined \( g^0_d \)'s on curves with three components.

### 3.2. Linked linear series: type I

In the following, we will define, given a vector bundle \( \mathcal{E} \) of multidegree \( v_0 \in V(G_I) \), natural twists \( \mathcal{E}_w \) for all \( w \in V(G_I) \), along with maps (defined up to scalar, and existing locally on \( B \))

\[ f_\varepsilon : \pi_\# \mathcal{E}_w \to \pi_\# \mathcal{E}_{w'} \]

for any edge \( \varepsilon \) of \( G_I \) going from \( w \) to \( w' \). Given these, we define type-I linked linear series as follows.
Definition 3.2.1. In Situation 3.1.1, let $G_1$ be as in Definition 3.1.2, and choose a vertex $w_0 \in V(G_1)$. The moduli groupoid $G_{r,d,d}^{k,1}(X/B)$ of type-I linked linear series is the category fibered in groupoids over $B$-Sch whose objects consist of tuples $(S, \mathcal{E}, (\mathcal{Y}_w)_{w \in V(G_1)})$, where $S$ is a $B$-scheme, $\mathcal{E}$ is a vector bundle of rank $r$ and multidegree $w_0$ on $X \times_B S$, and $\mathcal{Y}_w$ is a rank-$k$ subbundle (in the sense of Definition B.2.1) of $\pi_w(\mathcal{E}_w)$, satisfying the following conditions:

(I) for every pair $(e, v)$ of an edge and adjacent vertex of $\Gamma$, and every $z \in S$ with image $y \in B$ lying in $\Delta_e$, we have

$$H^0(Y, \mathcal{E}_w|_Y(-b+1)(\Delta')) = 0,$$

where $Y$ denotes the component of the fiber $X_x$ corresponding to $\text{cl}_y(v)$, and $\Delta'$ denotes the node corresponding to $\text{cl}_y(e)$;

(II) for every edge $e$ in $G_1$, let $w$ be the tail and $w'$ the head. Then we require that

$$f_e(\mathcal{Y}_w) \subseteq \mathcal{Y}_{w'}.$$

By definition of smoothing family, the irreducible components of $X_w$ remain irreducible in $X_z$, so condition (I) makes sense. Note that, as in [Oss06a], although $f_e$ is defined only locally on $B$, the condition that $f_e(\mathcal{Y}_w) \subseteq \mathcal{Y}_{w'}$ is invariant under scalar multiplication by $\mathcal{O}_B^*$, so the resulting closed condition is defined on all of $B$. Similarly, while the definition depends \textit{a priori} on the choice of $w_0$, this is largely a matter of convenience: different choices of $w_0$ will yield equivalent groupoids.

In order to complete Definition 3.2.1, we begin by describing the relevant twisting line bundles.

Notation 3.2.2. For every pair $(e, v)$ of an edge $e \in E(\Gamma)$ and an adjacent vertex $v$, denote by $\mathcal{O}_{(e,v)}$ the line bundle on $X$ obtained as follows: write

$$X|_{\Delta_e} = Y_{(e,v)} \cup Z_{(e,v)},$$

where $Z_{(e,v)} = Y_{(e,v')}$ with $v'$ the other vertex adjacent to $e$.

Now, if $\Delta_e \neq B$, we have $Y_{(e,v)}$ a (necessarily Cartier) divisor in $X$, and we set $\mathcal{O}_{(e,v)} = \mathcal{O}_X(Y_{(e,v)})$.

On the other hand, if $\Delta_e = B$, then line bundles on $X$ are uniquely determined by their restrictions to $Y_{(e,v)}$ and $Z_{(e,v)}$, and we define $\mathcal{O}_{(e,v)}$ to be $\mathcal{O}_{Y_{(e,v)}}(-\Delta_e')$ on $Y_{(e,v)}$ and to be $\mathcal{O}_{Z_{(e,v)}}(\Delta_e')$ on $Z_{(e,v)}$.

Given $w, w' \in V(G_1)$, let $P$ be a minimal directed path in $G_1$ from $w$ to $w'$. Write $P = P(w, (e_1, v_1), \ldots, (e_m, v_m))$, and set

$$\mathcal{O}_{w,w'} = \bigotimes_{i=1}^m \mathcal{O}_{(e_i,v_i)}.$$

Finally, given $w_0 \in V(G_1)$, and $\mathcal{E}$ an $S$-valued point of $\mathcal{M}_{r,w_0}(X/B)$, for $w \in V(G_1)$ write

$$\mathcal{E}_w := \mathcal{E} \otimes \mathcal{O}_{w_0,w}|_{X \times_B S}.$$

Exactly as in [Oss06a], locally on $B$ we also have maps, unique up to non-zero scalar, defined as follows:

Notation 3.2.3. Given a pair $(e, v)$ of an edge $e \in E(\Gamma)$ and an adjacent vertex $v$, we first observe we have a canonical morphism

$$\theta_{(e,v)} : \mathcal{O}_X \rightarrow \mathcal{O}_{(e,v)}.$$
When $\Delta_e \neq B$, this is defined to be the canonical inclusion. On the other hand, when $\Delta_e = B$, this is defined to be 0 on $Y_{(e,v)}$, and the canonical inclusion on $Z_{(e,v)}$. Now, let $v'$ be the other vertex adjacent to $e$. Then (locally on $B$, in the case $\Delta_e \neq B$) we have that $\mathcal{O}_{(e,v)} \otimes \mathcal{O}_{(e,v')} \cong \mathcal{O}_X$. If we fix such an isomorphism (unique up to an element of $\mathcal{O}_B^*$), we obtain a morphism

$$\theta'_{(e,v)} : \mathcal{O}_{(e,v)} \to \mathcal{O}_{(e,v')} \to \mathcal{O}_X.$$

Finally, fix $w_0 \in V(G_1)$. For any edge $\varepsilon$ in $G_1$, let $w$ be the tail and $w'$ the head, and let $e = e_1(\varepsilon) \in E(\Gamma)$. If $\mathcal{E}$ is an $S$-valued point of $M_{r,w_0}(X/B)$, then we either have $\mathcal{E}_w = \mathcal{E}_{w'} \otimes \mathcal{O}_{(e,v)}|_{X \times_B S}$ or $\mathcal{E}_{w'} = \mathcal{E}_{w} \otimes \mathcal{O}_{(e,v)}|_{X \times_B S}$, where $v$ is a vertex adjacent to $e$. Thus, using $\theta_{(e,v)}$ or $\theta'_{(e,v)}$, and pushing forward under $\pi$, we obtain a morphism

$$f_{\varepsilon} : \pi_* \mathcal{E}_w \to \pi_* \mathcal{E}_{w'}.$$

This completes the definition of type-I linked linear series. Observe that by construction, $\theta_{(e,v)} \circ \theta'_{(e,v)}$ is equal to scalar multiplication by a scalar cutting out $\Delta_{e}$, and similarly for $f_{\varepsilon} \circ f_{\varepsilon'}$, if $\varepsilon'$ is the edge with the same adjacent vertices, but going in the opposite direction as $\varepsilon$.

### 3.3. Linked linear series: type II

We now move on to the second construction, which differs from the first construction in that it both requires more restrictive hypotheses, and also depends on additional choices. The additional hypothesis is the following.

**Situation 3.3.1.** In Situation 3.1.1, suppose further that for all $e \in E(\Gamma)$, the closed subschemes $\Delta_e \subseteq B$ agree. Denote this common closed subscheme by $\Delta$, and for every $e \in E(\Gamma)$, let $Y_e$ be the irreducible component of $\pi^{-1}(\Delta)$ corresponding to $e$.

Note that the additional condition of Situation 3.3.1 is tautologically satisfied for the case of curves with at most two components, or if $B = \text{Spec} \ F$ for some field $F$. In addition, Proposition 2.1.4 implies that if the $\Delta_e$ agree set-theoretically, they also agree scheme-theoretically, so in particular if $B$ is the spectrum of a DVR and the generic fiber is smooth, the additional condition of Situation 3.3.1 is likewise satisfied.

As in the type-I case, under the additional hypothesis of Situation 3.3.1 we will define, given a vector bundle $\mathcal{E}$ of multidegree $w_0 \in V(G_{II})$, natural twists $\mathcal{E}_w$ for all $w \in V(G_{II})$, along with maps (defined up to scalar, and existing locally on $B$)

$$f_{\varepsilon} : \pi_* \mathcal{E}_w \to \pi_* \mathcal{E}_{w'},$$

for any edge $\varepsilon$ of $G_{II}$ going from $w$ to $w'$. The additional data required to define the maps $f_{\varepsilon}$ will be a suitable collection of sections of our twisting bundles $\mathcal{O}_v$. Given the above definitions, we define type-II linked linear series as follows.

**Definition 3.3.2.** In Situation 3.3.1, let $G_{II}$ be as in Definition 3.1.2, and choose a vertex $w_0 \in V(G_{II})$. In addition, for each $v \in V(\Gamma)$, choose a morphism $\theta_v : \mathcal{O}_v \to \mathcal{O}_v$ which vanishes precisely on $Y_v$. The moduli groupoid $G_{II}^{k,d}(X/B, \theta_*)$ of **type-II linked linear series** is the category fibered in groupoids over $B$-$\text{Sch}$ whose objects consist of tuples $(S, \mathcal{E}, (\mathcal{Y}_w)_{w \in V(G_{II})})$, where $S$ is a $B$-scheme, $\mathcal{E}$ is a vector bundle of rank $r$ and multidegree $w_0$ on $X \times_B S$, and $\mathcal{Y}_w$ is a rank-$k$ subbundle (in the sense of Definition B.2.1) of $\pi_* (\mathcal{E}_w)$, satisfying the following conditions:
(I) for every pair \((e, v)\) of an edge and adjacent vertex of \(\Gamma\), and every \(z \in S\) with image \(y \in B\) lying in \(\Delta_e\), we have

\[
H^0(Y, \mathcal{O}_{w_y}|_Y(-(b+1)(\Delta'))) = 0,
\]

where \(Y\) denotes the component of the fiber \(X_z\) corresponding to \(cl_Y(v)\), and \(\Delta'\) denotes the node corresponding to \(cl_Y(e)\);

(II) for every edge \(e\) in \(G_{II}\), let \(w\) be the tail and \(w'\) the head. Then we require that

\[
f_e(\mathcal{Y}_w) \subseteq \mathcal{Y}_{w'}.
\]

In the context of this definition, the parameters \(d_e\) and \(b\) are somewhat artificial and included mainly for consistency. In particular, it is always possible to increase them and obtain open immersions on the resulting moduli spaces; see \(\S 6.2\). As in the type I case, the choice of \(w_0\) doesn’t change the resulting groupoid.

We now describe how to construct our twisting bundles and maps in order to complete Definition 3.3.2.

\textbf{Notation 3.3.3.} For every vertex \(v \in V(\Gamma)\), denote by \(\mathcal{O}_v\) the line bundle on \(X\) obtained as follows:

If \(\Delta \neq B\), we have \(Y_v\) a Cartier divisor in \(X\), and we set \(\mathcal{O}_v = \mathcal{O}_X(Y_v)\).

On the other hand, if \(\Delta = B\), let \(Z_1, \ldots, Z_m\) be the closures in \(X\) of the connected components of \(X - Y_v\), and for \(i = 1, \ldots, m\), let \(\Delta_i\) be the node \(Z_i \cap Y_v\). We then define \(\mathcal{O}_v\) to be \(\mathcal{O}_{Y_v}(-\sum_i \Delta_i)\) on \(Y_v\) and to be \(\mathcal{O}_{Z_i}(\Delta_i)\) on each \(Z_i\).

Given \(w, w' \in V(G_{II})\), let \(P = (e_1, \ldots, e_m)\) be a minimal directed path in \(G_{II}\) from \(w\) to \(w'\). Let \(v_i = v_{II}(e_i)\) for \(i = 1, \ldots, m\), and set

\[
\mathcal{O}_{w, w'} = \bigotimes_{i=1}^m \mathcal{O}_{v_i}.
\]

Finally, given \(w_0 \in V(G_{II})\), and an \(S\)-valued point \(\mathcal{E}\) of \(\mathcal{M}_{r,w_0}(X/B)\), for \(w \in V(G_{II})\) write

\[
\mathcal{E}_w := \mathcal{E} \otimes \mathcal{O}_{w_0, w}|_{X \times B S}.
\]

\textit{A priori}, the notation \(\mathcal{E}_{w, w'}\) and \(\mathcal{E}_w\) is ambiguous, since it is used in both the type-I and type-II constructions. However, we will see in Proposition 3.4.1 below that there is in fact no ambiguity.

\textbf{Notation 3.3.4.} For each \(v \in V(\Gamma)\), suppose we fix a morphism

\[
\theta_v : \mathcal{O}_X \to \mathcal{O}_v
\]

vanishing precisely on \(Y_v\).

Next, observe that \(\bigotimes_{v \in V(\Gamma)} \mathcal{O}_v \cong \mathcal{O}_X\). Fixing such an isomorphism (unique up to an element of \(\mathcal{O}_X^*\)), we obtain a induced morphism

\[
\theta'_v : \bigotimes_{v' \neq v} \mathcal{O}_{v'} \bigotimes_{v \in V(\Gamma)} \mathcal{O}_v \to \mathcal{O}_X.
\]

Finally, fix \(w_0 \in V(G_{II})\). For any edge \(e\) in \(G_{II}\), let \(w\) be the tail and \(w'\) the head, and let \(v\) be the associated edge of \(\Gamma\). If \(\mathcal{E}\) is an \(S\)-valued point of \(\mathcal{M}_{r,w_0}(X/B)\) then we either have \(\mathcal{E}_w = \mathcal{E}_w \otimes \mathcal{O}_{Y \times B S}\) or \(\mathcal{E}_w = \mathcal{E}_{w'} \otimes \bigotimes_{v' \neq v} \mathcal{O}_{Y \times B S}\). Thus, using \(\theta_v\) or \(\theta'_v\), and pushing forward under \(\pi\), we obtain a morphism

\[
f_\varepsilon : \pi_\# \mathcal{O}_w \to \pi_\# \mathcal{E}_{w'}.
\]
Note that in the case $\Delta \neq B$, we have that each $\theta_v$ and hence each $f_\varepsilon$ is unique up to scalar. However, this is not the case when $\Delta = B$, as, in the notation of Notation 3.3.3, we can scale $\theta_v$ independently on each of the $Z_i$.

### 3.4. Foundational results

The following proposition is straightforward from the definitions, and implies that our notation is consistent, and more importantly, that we have a forgetful morphism from type II linked linear series to type I linked linear series.

**Proposition 3.4.1.** In Situation 3.3.1, given $w, w' \in V(G_1)$, we have that the line bundles $\Theta_{w,w'}$ given in Notation 3.2.2 and 3.3.3 are isomorphic. Consequently, for any $w \in V(G_1)$ the vector bundles $\Theta_w$ in Notation 3.2.2 and 3.3.3 are likewise isomorphic.

In addition, if $\varepsilon \in E(G_1)$ is an edge with tail $w$ and head $w'$, for any choice of $\theta_\bullet$ as in Definition 3.3.2, and any minimal path $P = (\varepsilon_1, \ldots, \varepsilon_n)$ from $w$ to $w'$ in $G_\bullet$, we have that $f_\varepsilon$ agrees with $f_{\varepsilon_1} \circ \cdots \circ f_{\varepsilon_1}$ up to scalar multiplication by an element of $\Theta_B^*$.

Note in particular that in the last part of the proposition, each $f_\varepsilon$ depends in general on the choice of $\theta_\bullet$, but we see that the composition $f_{\varepsilon_1} \circ \cdots \circ f_{\varepsilon_1}$ does not.

We conclude:

**Corollary 3.4.2.** There is a forgetful morphism

$$\mathcal{G}^{k,\bullet}_{r,d,d_\bullet}(X/B, \theta_\bullet) \to \mathcal{G}^{k,1}_{r,d,d_\bullet}(X/B).$$

In order to state the foundational theorem, it is helpful to introduce the following open substack of $\mathcal{M}_{r,w_0}(X/B)$.

**Notation 3.4.3.** Let $X/B$ be an almost local smoothing family, and $k, r, d_\bullet$ as in Situation 3.1.1. Then $\mathcal{M}_{r,w_0,d_\bullet}(X/B)$ denotes the subgroupoid of $\mathcal{M}_{r,w_0}(X/B)$ consisting of vector bundles $\Theta$ on $X \times_B S$ such that for every pair $(e, v)$ of an edge and adjacent vertex of $\Gamma$, and every $z \in S$ with image $y \in B$ lying in $\Delta_e$, we have

$$H^0(Y, \Theta_{y,v}|- (b + 1)(\Delta_e))) = 0,$$

where $Y$ denotes the component of the fiber $X_y$ corresponding to $\text{cl}_B(v)$, and $\Delta'$ denotes the node corresponding to $\text{cl}_B(v)$.

**Lemma 3.4.4.** Let $\Theta$ on $X_y$ be a $K$-valued point of $\mathcal{M}_{r,w_0,d_\bullet}(X/B)$. Then for all $v \in V(\Gamma)$, if $Y'$ denotes the component of $X_y$ corresponding to $v$, the restriction map

$$H^0(X_{y,v}, \Theta_{w,v}) \to H^0(Y, \Theta_{w,v}|_Y)$$

is injective.

The groupoid $\mathcal{M}_{r,w_0,d_\bullet}(X/B)$ is an open substack of $\mathcal{M}_{r,w_0}(X/B)$.

**Proof.** For the first assertion, suppose we have a section in the kernel of restriction to $Y'$, which we wish to show is zero on all of $X_y$. We induct on the number of components of $X_y$, with the base case that $Y = X_y$ being trivial. Now, if $Y' \neq Y$ is another component of $X_y$, with $v'$ a corresponding vertex of $\Gamma$ being a leaf, then there is a unique node $\Delta'$ on $Y'$, and $\Theta_{w,v}'|_{Y'} = \Theta_{w,v}|- (b\Delta')$. The situation being compatible with restriction to connected subcurves, we may assume by induction that our section vanishes on all components other than $Y'$, and in particular at $\Delta'$, and thus the restriction to $Y'$ is a section of $\Theta_{w,v}'|_{Y'}|- (b + 1)(\Delta')$, which must be zero by definition of $\mathcal{M}_{r,w_0,d_\bullet}(X/B)$.
In order to prove openness, it is enough to check that the condition is constructible, and closed under generization. Constructibility is clear: considering dual graphs of fibers gives a stratification of $B$ by locally closed subschemes, and within each stratum, the condition in question fails on a closed subset, by semicontinuity of $h^0$. Note that on each stratum, each component of $X$ has irreducible fibers and is proper and flat (for flatness, see the argument for Lemma 3.2 (ii) of [Oss06a]).

In order to prove closure under generization, it is enough to consider the case that the base is the spectrum of a DVR. In this case, suppose the condition is satisfied on the special fiber $X_0$. Given $(e, v)$, let $Y_\eta$ and $\Delta'_\eta$ (respectively, $Y_0$ and $\Delta'_0$) be the relevant component and node on $X_\eta$ (respectively, on $X_0$). Let $Z$ denote the restriction to $X_0$ of the closure of $Y_\eta$ in $X$: this contains $Y_0$, but may also contain additional components of $X_0$. Since we have assumed our condition is satisfied on $X_0$, we have that $H^0(Y_0, \mathcal{E}_{w_v}|_{Y_0}(-b+1)(\Delta'_0)) = 0$, and also, by the first statement of the lemma, that the restriction map

$$H^0(Z, \mathcal{E}_{w_v}|_Z) \to H^0(Y_0, \mathcal{E}_{w_v}|_{Y_0})$$

is injective. This implies that

$$H^0(Z, \mathcal{E}_{w_v}|_Z(-b+1)(\Delta'_0)) = 0,$$

which by semicontinuity implies that

$$H^0(Y_\eta, \mathcal{E}_{w_v}|_{Y_\eta}(-b+1)(\Delta'_\eta)) = 0$$

as well, as desired. 

Thus, the natural forgetful morphism

$$\mathcal{G}_{r,d,d_\ast}^{1,k}(X/B) \to \mathcal{M}_{r,w_0}(X/B)$$

factors through $\mathcal{M}_{r,w_0,d_\ast}(X/B)$, by definition.

It will be convenient to have notation for compositions of morphisms $f_\varepsilon$, as follows.

**Notation 3.4.5.** Given a path $P = (\varepsilon_1, \ldots, \varepsilon_m)$ in $G_1$ (respectively, $G_{\ast 1}$), set

$$f_P := f_{\varepsilon_m} \circ \cdots \circ f_{\varepsilon_1}.$$

The following definition will play an important role in the type-II case.

**Definition 3.4.6.** Given a field $K$, a $K$-valued point $(\mathcal{E}', (V_w)_{w \in V(G_{\ast 1})})$ of $\mathcal{G}_{r,d,d_\ast}^{1,k}(X/B)$ is **simple** if there exist $w_1, \ldots, w_k \in V(G_{\ast 1})$ (not necessarily distinct) and $v_i \in \nu_{w_i}$ for $i = 1, \ldots, k$ such that for all $w \in V(G_{\ast 1})$, if $P_1, \ldots, P_k$ are minimal paths from $w_i$ to $w$ in $G_{\ast 1}$, the vectors $f_{P_1}(v_1), \ldots, f_{P_k}(v_k)$ form a basis of $\nu_w$.

It almost follows from Nakayama’s lemma that the simple points are an open substack of $\mathcal{G}_{r,d,d_\ast}^{1,k}(X/B, \theta_\ast)$, except that the infinite nature of $G_{\ast 1}$ requires some additional argument; nonetheless, openness follows from Proposition 3.4.12 below.

The foundational theorem on our constructions is the following:

**Theorem 3.4.7.** Let $X/B$ be an almost local smoothing family, and $k, r, d_\ast$ as in Situation 3.1.1. Then $\mathcal{G}_{r,d,d_\ast}^{1,k}(X/B)$ is an Artin stack, and the natural map

$$\mathcal{G}_{r,d,d_\ast}^{1,k}(X/B) \to \mathcal{M}_{r,w_0,d_\ast}(X/B)$$
is relatively representable by schemes which are projective, at least locally on the target. Moreover, formation of \( G_{r,d,d}^{k,\mathbb{I}}(X/B) \) is compatible with any base change \( B' \to B \) which preserves the almost local smoothing family hypotheses. In particular, if \( y \in B \) is a point with \( X_y \) smooth, then the base change to \( y \) parametrizes pairs \((\mathcal{E}, V)\) of a vector bundle \( \mathcal{E}\) of rank \( r \) and degree \( d \) on \( X_y \) together with a \( k \)-dimensional vector space \( V \subseteq H^0(X_y, \mathcal{E}) \).

Under the further hypothesis of Situation 3.3.1, all the above statements also hold for \( G_{r,d,d}^{k,\mathbb{I}}(X/B, \theta_\ast) \). Moreover, the simple locus of \( G_{r,d,d}^{k,\mathbb{I}}(X/B, \theta_\ast) \) has universal relative dimension at least \( k(d - k - r(g - 1)) \) over \( \mathcal{M}_{r,s_0,d}(X/B) \), and hence has universal relative dimension at least \( \rho - 1 \) over \( B \). In particular, if some fiber \( G_{r,d,d}^{k,\mathbb{I}}(X/y, \theta_\ast) \) has dimension exactly \( \rho - 1 \) at a simple point \( z \), then \( G_{r,d,d}^{k,\mathbb{I}}(X/B, \theta_\ast) \) is universally open over \( B \) in a neighborhood of \( z \), with pure fiber dimension \( \rho - 1 \).

In the above, the notion of having universal relative dimension at least a given number is as introduced in Definition 7.1 of [Oss13b]. While the same dimensional statements hold also for \( G_{r,d,d}^{k,\mathbb{I}}(X/B) \), they are largely vacuous in this case, since for the most part simple points occur only for type II linked linear series.

In order to prove the theorem, it will be useful to introduce additional notation.

**Notation 3.4.8.** In Situation 3.3.1, given \( w = (i_v)_v \in V(\mathcal{G}_{\mathbb{I}}) \) and a pair \((e, v)\) of an edge of \( \Gamma \) together with an adjacent vertex, let \( S \subseteq V(\Gamma) \) be the vertices not in the same connected component as \( v \) of \( \Gamma \setminus e \). Then we write

\[
\langle \rho, d_v \rangle = |S|b - \frac{1}{r} \sum_{v \in S} (d_w - i_v).
\]

Then \( t_{(e, v)}(w) \) is the number of times we twist down by \( \Delta'_w \) on \( Y_{(e, v)} \) in order to go from multidegree \( w_0 \) to multidegree \( w \).

Then we see easily from (3.1.1) that if \( v' \) is the other vertex adjacent to \( e \), we have

\[
t_{(e, v)}(w) + t_{(e, v')}(w) = b.
\]

We also see that if we start at \( w \) and go along an edge \( e \) of \( \mathcal{G}_{\mathbb{I}} \), the value of \( t_{(e, v)} \) increases by 1 whenever \( v = \nu_{\mathbb{I}}(v) \), decreases by 1 whenever \( v \) is adjacent to \( \nu_{\mathbb{I}}(v) \) in \( \Gamma \), and remains the same otherwise.

We now verify that in the definition of type II linked linear series, we could have restricted our attention to a suitable finite graph.

**Definition 3.4.9.** Consider the finite graph \( \bar{\mathcal{G}}_{\mathbb{I}} \) described as follows: the vertices of \( \bar{\mathcal{G}}_{\mathbb{I}} \) are the subset of the vertices \( w \) of \( \mathcal{G}_{\mathbb{I}} \) satisfying \( t_{(e, v)}(w) \leq b \) for all \((e, v)\), and the edges consist of all edges of \( \mathcal{G}_{\mathbb{I}} \) between vertices of \( \bar{\mathcal{G}}_{\mathbb{I}} \).

Finally, let \( \bar{\mathcal{G}}_{r,d,d}^{k,\mathbb{I}}(X/B, \theta_\ast) \) be the groupoid defined in the same manner as \( \mathcal{G}_{r,d,d}^{k,\mathbb{I}}(X/B, \theta_\ast) \), but with \( \mathcal{G}_{\mathbb{I}} \) replaced by \( \bar{\mathcal{G}}_{\mathbb{I}} \).

A priori, one might have to add extra edges to \( \bar{\mathcal{G}}_{\mathbb{I}} \) to account for paths between vertices of \( \bar{\mathcal{G}}_{\mathbb{I}} \) which are not entirely contained in \( \bar{\mathcal{G}}_{\mathbb{I}} \). However, the following proposition shows that this is not necessary.

**Proposition 3.4.10.** For any \( w, w' \in V(\bar{\mathcal{G}}_{\mathbb{I}}) \), there exists a minimal path from \( w \) to \( w' \) containing only vertices of \( \bar{\mathcal{G}}_{\mathbb{I}} \).
Proof. Let $P = (w, v_1, \ldots, v_m)$ be a minimal path from $w$ to $w'$ in $G_{II}$; it suffices to show that we can reorder the $v_i$ so that $P$ remains in $G_{II}$. In fact, it is enough to show that for some $v_j$, we have $P(w, v_j)$ in $G_{II}$, since then if $u''$ is the endpoint of $P(w, v_i)$, we have $P(u'', v_1, \ldots, v_{j-1}, v_{j+1}, v_m)$ a minimal path from $u''$ to $w'$, and we can repeat the process inductively. Now, if $P(w, v_1)$ is not in $G_{II}$, this means necessarily that $t_{(e,v)}(w) = b$ for some $e$ adjacent to $v_1$. Now, because $w' \in G_{II}$, if $v'$ is the other vertex adjacent to $e$, we see that we must have $v' = v_1$ for some $i \neq 1$.

Then again, if $P(w, v_1)$ is not in $G_{II}$, we must have some $t_{(e,v)}(w) = b$ for some $e$ adjacent to $v_1$, and since $t_{(e,v)}(w) = 0$, we have $e' \neq e$. Thus, if we continue in this way, we traverse the graph $\Gamma$ without repeating any vertices, and the process must conclude with some $v_t$ having $P(w, v_t)$ in $G_{II}$, as desired. □

The following is a trivial consequence of the first part of Lemma 3.4.4.

Corollary 3.4.11. Suppose $s'$ on $X_{y}$ is a $K$-valued point of $M_{r,w_{0},d_{*}}(X/B)$, and we have $w \in V(G_{II})$, $s \in H^{0}(X_{y}, s_{w})$, and $v \in V(\Gamma)$ such that there is a path $P_{v}$ in $G_{II}$ from $w$ to $w_{v}$ with $f_{P_{v}}(s) \neq 0$. Then $s$ does not vanish uniformly on $P_{v}$.

In particular, for any simple point of $G_{II}^{k}(X/B, \theta_{*})$, if $w_{1,\ldots,w_{k}}$ are as in Definition 3.4.6, then necessarily each $w_{i} \in V(G_{II})$.

We are now ready to prove the equivalence of the finite version of the type II construction.

Proposition 3.4.12. The canonical forgetful morphism

$$G_{II}^{k}(X/B, \theta_{*}) \to G_{II}^{k}(X/B, \theta_{*})$$

is an equivalence of groupoids.

In particular, the simple points of $G_{II}^{k}(X/B, \theta_{*})$ form an open substack.

Proof. We visibly have a forgetful morphism $G_{II}^{k}(X/B, \theta_{*}) \to G_{II}^{k}(X/B, \theta_{*})$, so in order to verify that this is an equivalence, it suffices to show that an $S$-valued point $(s', (p_{w})_{w \in V(G_{II})})$ of $G_{II}^{k}(X/B, \theta_{*})$ extends uniquely to a $S$-valued point $(s', (p_{w})_{w \in V(G_{II})})$ of $G_{II}^{k}(X/B, \theta_{*})$.

For each $w \in V(G_{II})$, choose $w_{w} \in V(G_{II})$ admitting the shortest possible path $P_{w}$ in $G_{II}$ from $w_{w}$ to $w$. We claim that $w_{w}$ is uniquely determined by this condition, and that moreover $w_{w}$ is characterized by the property that for any minimal path $P$ from $w_{w}$ to $w$, no vertex of $P$ other than $w_{w}$ lies in $G_{II}$.

For both claims, the main observation is the following: if $w \notin G_{II}$, and $w_{w}$ and $w'_{w}$ are any vertices of $G_{II}$, and $P$ and $P'$ minimal paths from $w_{w}$ and $w'_{w}$ to $w$, respectively, then we can modify $P$ and $P'$ without changing their length (or start or endpoints) so that they pass through a common intermediate vertex $w' \neq w$. Indeed, if $P = P(w_{w}, v_{1}, \ldots, v_{m})$ and $P' = P(w'_{w}, v'_{1}, \ldots, v'_{m})$, we must have some $(e, v)$ such that $t_{(e,v)}(w) > b$, and then we see that $v$ must occur both as one of the $v_i$ and one of the $v'_i$. Rearrange the $v_{i}$ and $v'_{i}$ so that $v_{m} = v'_{m} = v$ then clearly has the desired effect.

If now we suppose that $P$ and $P'$ both have minimal length among all paths from vertices of $G_{II}$ to $w$, we conclude the first claim by induction on the length of $P$: if the length is 0, there is nothing to show, while if the length is positive, the above observation allows us to consider the strictly shorter paths to $w'$, and the induction hypothesis then implies that $w_{w} = w'_{w}$. For the second claim, it is immediate that
if \( \bar{w}_w \) admits a minimal path \( P \) to \( w \) among all vertices of \( \bar{G}_1 \), then no minimal path from \( \bar{w}_w \) to \( w \) can contain a vertex of \( \bar{G}_1 \) other than \( \bar{w}_w \). Conversely, suppose that \( \bar{w}_w \) has the property that no minimal path to \( w \) contains another vertex of \( \bar{G}_1 \), let \( P \) be any minimal path to \( w \), and choose \( \bar{w}_w' \in V(\bar{G}_1) \) with \( P' \) a path from \( \bar{w}_w' \) to \( w \). We need to see that \( P' \) is at least as long as \( P \), and we argue once again by induction on the length of \( P \). If \( P \) has length zero, there is nothing to show. On the other hand, if \( P \) has positive length, then our observation implies that it and \( P' \) may be modified to pass through a common vertex \( w' \). By hypothesis, we either have \( w' = \bar{w}_w \), in which case the length of \( P \) is certainly at most the length of \( P' \), or \( w' \notin V(\bar{G}_1) \), in which case we may apply the induction hypothesis to conclude again that the length of \( P \) is at most the length of \( P' \), as desired.

Now, returning to the question of lifting points \((\bar{V}(\bar{G}_1), \bar{G}_1(\bar{G}_1))\) of \( \bar{G}_1 \) to \( \bar{V}(\bar{G}_1) \), set \( \bar{G}_w = f_{P_w}(\bar{G}_w) \) for each \( w \). We first verify that this is in fact a subbundle, and of the correct rank \( k \). Given \( w \), write \( P_w = P(\bar{w}_w, v_1, \ldots, v_m) \). We need to show that \( f_{P_w} \) is universally injective, which may be checked over each point \( y \in B \). Now, if we let \( V' \subseteq V(\Gamma) \) be the subset consisting of vertices equal to \( v_i \) for some \( i \), the kernel of \( f_{P_w} \) (over \( y \in \Delta \)) consists of sections vanishing on every \( Y' \) for \( v \notin V' \). It thus suffices to see that such a section must also vanish on every \( v \in V' \). By our second claim above, no reordering of \( P_w \) can meet \( \bar{G}_1 \) away from \( \bar{w}_w \). It follows that for any \( v \in V' \), there is some \( e \in E(\Gamma) \) adjacent to \( v \) such that \( t_{(e,v)}(\bar{w}_w) \geq b \), and because \( \bar{w}_w \in \bar{G}_1 \), we conclude that \( t_{(e,v)}(\bar{w}_w) = b \). Let \( v' \) be the other vertex adjacent to \( e \). Any section vanishing on \( Y' \) then vanishes to order at least \( b+1 \) at \( \Delta \) as a section of \( \bar{G}_w \), so by condition (I) of Definition 3.3.2, such a section vanishes also on \( v \). Thus, if \( v' \) is not in \( V' \), we conclude that any section in the kernel of \( f_{P_w} \) must vanish on every component we have traversed, and in particular on \( Y' \), giving us the desired injectivity. We thus conclude that the \( \bar{G}_w \) are subbundles of rank \( k \).

It is then clear that each \( \bar{G}_w \) is uniquely determined by the linkage conditions, and it remains to check that our construction satisfies the linkage condition. Given \( w, w' \in V(\bar{G}_1) \) connected by an edge \( e \in E(\bar{G}_1) \) from \( w \) to \( w' \), we wish to verify that \( f_e(\bar{G}_w) \subseteq \bar{G}_{w'} \). With \( \bar{w}_w \) and \( P_w \) as above, let \( P' \) be the path from \( \bar{w}_w \) to \( w' \) obtained by composing \( P_w \) with \( e \); say \( P' = P(\bar{w}_w, v_1, \ldots, v_m) \), where \( v_m = v_1(\bar{e}) \). Now, reorder the \( v_i \) so that \( P' = P'' \circ Q \), where \( Q \) ends in \( \bar{G}_1 \) and has maximum possible length (here we allow either \( P'' \) or \( Q \) to have length zero, as required). Then the second claim above implies that the starting point of \( P'' \) is equal to \( \bar{w}_{w'} \), and \( P'' = P_w \). By definition, we have \( \bar{G}_{w'} = f_{P'}(\bar{G}_w) \), and \( f_e(\bar{G}_w) = f_{P''}(f_{P''}(\bar{G}_w)) \), and the reordering of \( P' \) doesn't affect the latter because it only changes \( f_{P'} \) by an invertible scalar. It thus suffices to see that \( f_Q(\bar{G}_{w}) \subseteq \bar{G}_{w'} \), which follows from our initial linkage hypothesis.

We can now easily prove our foundational theorem on linked linear series.

**Proof of Theorem 3.4.7.** The statement being local on \( B \), we may suppose by Lemma 2.1.2 that we have a divisor \( D \) on \( X \) which is \( \pi \)-ample. We construct \( \bar{G}_w^{k\bar{G}_w}(X/B) \) (respectively, \( \bar{G}_w^{k\bar{G}_w}(X/B, \theta_*) \)) as a projective relative scheme over \( \mathcal{M}_{r,\bar{w}_w',\theta_*(X/B)} \),
yielding the first assertion of the theorem. Let $U$ be a quasicompact open substack of $\mathcal{M}_{r,d}(X/B)$, and $\mathcal{E}$ the universal bundle on $X \times B U$; then, replacing $D$ by a sufficiently high multiple, we may suppose that $R^i p_2^* \mathcal{E}_w(D) = 0$ for all $i > 0$ and all $w \in V(G_1)$. Let $\mathcal{G}$ be the prelinked Grassmannian (see Definition A.1.2) associated to $G_1$ and $p_2^* \mathcal{E}_w(D)$, with maps $p_2^* (f_\varepsilon)$ for $\varepsilon \in G_1$. Note that the condition of Definition A.1.2 is clearly satisfied in this case, due to Proposition 3.1.4.

Then, according to Corollary A.3.1, we have that $\mathcal{G}$ is relatively representable by a projective scheme over $U$, and it thus suffices to show that (the preimage of $U$ in) $\mathcal{G}_{r,d}(X/B)$ is a closed substack of $\mathcal{G}$. But, making use of Lemma B.2.3 (ii) to ensure compatibility of notions of subbundles, we see that $\mathcal{G}^k_{r,d}(X/B)$ is cut out by the closed condition that for all $w \in V(G_1)$, we have $\mathcal{V}_w$ in the kernel of the morphism

$$p_2^* \mathcal{E}_w(D) \to p_2^* (\mathcal{E}_w(D)|_D).$$

This proves the first statement of the theorem, and the same construction works also for $\mathcal{G}^k_{r,d}(X/B, \theta_\bullet)$, using $G_{II}$ in place of $G_1$ by virtue of Proposition 3.4.12.

Compatibility with base change is straightforward, although we note that the graph $\Gamma$ will typically be contracted under base change $B' \to B$. In the type-I case, arbitrary contractions are possible, depending on which of the $\Delta_\varepsilon$ miss the image of $B'$. For these $\varepsilon$, the corresponding morphisms $f_\varepsilon$ are all isomorphisms over the image of $B'$, so we obtain a natural isomorphism

$$\mathcal{G}^k_{r,d}(X/B) \times_B B' \to \mathcal{G}^k_{r,d}(X'/B'),$$

where $d' \bullet$ is induced by $d_\bullet$. Compatibility with base change is similar in the type-II case, except that $\Gamma$ is either preserved or contracted to a point, so the situation is simpler.

Finally, in the type II case, we compute the relative dimension of the simple locus. The bundles $p_2^* \mathcal{E}_w(D)$ all have rank $d + r \deg D + r(1 - g)$, so Corollary A.3.1 implies that the simple locus of $\mathcal{G}$ is smooth of relative dimension $k(d + r \deg D + r(1 - g) - k)$ over $U$, which is in turn smooth of dimension $r^2 (g - 1)$. We next claim that, at least etale locally on the base, the substack $\mathcal{G}^k_{r,d}(X/B, \theta_\bullet)$ is cut out by $kr \deg D$ equations. Etale locally, we may suppose that $D = \sum_{v \in V(\Gamma)} D_v$, where the $D_v$ are disjoint divisors supported in the smooth locus of $\pi$, and with $D_v|_{\pi_v}$ contained in the component of $X_v$ corresponding to $c\pi_v$ for every $y \in B$. We will show that $\mathcal{G}^k_{r,d}(X/B, \theta_\bullet)$ may be cut out by the conditions that for all $v \in V(\Gamma)$, we have $\mathcal{V}_w$ in the kernel of the morphism

$$p_2^* \mathcal{E}_w(D) \to p_2^* (\mathcal{E}_w(D)|_{D_v}).$$

Each $v$ will then impose $kr \deg D_v$ equations, so the claim will follow. Since the $D_v$ are disjoint, it suffices to prove that the above condition implies that for all $w \in V(G_{II})$ and all $v \in V(\Gamma)$, we have $\mathcal{V}_w$ in the kernel of the morphism

$$p_2^* \mathcal{E}_w(D) \to p_2^* (\mathcal{E}_w(D)|_{D_v}).$$

We observe that we can find a path $P = (\varepsilon_1, \ldots, \varepsilon_n)$ from $w$ to $w_v$ in $G_{II}$ such that $v_{II}(\varepsilon_i) \neq v$ for each $i$. Indeed, we can construct such a path by working inductively outwards from $v$ in $\Gamma$: for each edge $e$ adjacent to $v$, if $t_{(e,v)}(w) > 0$, and $v'$ is the other vertex adjacent to $e$, we add edges $\varepsilon_i$ with $v_{II}(\varepsilon_i) = v'$ until the new $t_{(e,v')}(w') = 0$. For each such $v'$, we then have $t_{(e,v')}(w') = b$, but if $t_{(e',v')}(w') > 0$ for some $e' \neq e$, we add edges $\varepsilon_i$ with $v_{II}(\varepsilon_i) = v''$, where $v''$ is the other vertex
adjacent to $\ell'$, and continuing in this way we will finally reach $w_v$. Thus, if we take the morphism $p_{2*}\mathcal{E}_{w}(D) \to p_{2*}\mathcal{E}_{w_v}(D)$ induced by a minimal path from $w$ to $w_v$, because $D_v$ is disjoint from $Y_v$ for any $v \neq v$ we see that the induced morphism $p_{2*}(\mathcal{E}_{w}(D)|_{D_v}) \to p_{2*}(\mathcal{E}_{w_v}(D)|_{D_v})$ is an isomorphism. Now, the composed map $\mathcal{Y}_w \to p_{2*}\mathcal{E}_{w_v}(D)|_{D_v}$ factors through $\mathcal{Y}_{w_v} \to p_{2*}(\mathcal{E}_{w_v}(D)|_{D_v})$, which vanishes by hypothesis, so we conclude that $\mathcal{Y}_w \to p_{2*}(\mathcal{E}_{w}(D)|_{D_v})$ is likewise zero, as desired. This proves the claim, and by Corollary 7.7 of [Oss13b], we conclude that the simple locus of $\mathcal{G}_{r,d,k}(X/B, \theta_\bullet)$ has universal relative dimension at least $k(d + r(1 - g) - k)$ over $\mathcal{M}_{r,w_0,d}(X/B)$, yielding the desired statement. \hfill \square

**Remark 3.4.13.** In the classical rank-1 case, it should be possible to restrict to a simpler (and smaller) bounded collection of multidegrees than the one used for $\mathcal{G}_1$. Indeed, because no line bundle of negative degree can have a global section on a smooth curve, it is more natural to restrict to $w = (w_v)_{v \in V(\Gamma)} \in V(\mathcal{G}_1)$ such that $i_v \geq 0$ for all $v$. However, in this case Proposition 3.4.10 no longer holds, so in fact the definitions and arguments involve additional complications.

We conclude with a brief study of simplicity, showing that it is enough to check the basis condition at the vertices $w_v$.

**Lemma 3.4.14.** Given a field $K$ and a $K$-valued point $(\mathcal{E}, (V_w)_{w \in V(\mathcal{G}_1)})$ of the stack $\mathcal{G}^k_{r,d,k}(X/B, \theta_\bullet)$, suppose there exists $w_1, \ldots, w_k \in V(\mathcal{G}_1)$ (not necessarily distinct) and $v_i \in V_{w_i}$ for $i = 1, \ldots, k$ such that for all $v \in V(\Gamma)$, there is a path $P_{v,v}$ such that $f_{P_{v,v}}(v_1), \ldots, f_{P_{v,v}}(v_k)$ form a basis of $V_{w_v}$. Then $(\mathcal{E}, (V_w)_{w \in V(\mathcal{G}_1)})$ is simple.

**Proof.** Let $\Delta \subseteq B$ be the image of the singular locus of $\pi$. If the point in question lies outside of $\Delta$, then all points are simple, and there is nothing to show. On the other hand, according to the hypothesis of Situation 3.3.1, all fibers of $\pi$ over $\Delta$ have dual graphs equal to $\Gamma$, so we might as well assume that $B$ is a point.

Now, we observe that if we have a path $P$ from some $w_v$ to $w_v'$ such that $f_P$ is injective (equivalently, non-zero) on $Y_v$, then $f_P(V_{w_v}) = V_{w_v'}$. Indeed, this is immediate from the fact that the restriction map to $Y_v$ is injective on $V_{w_v}$, by Lemma 3.4.4.

To prove the lemma, it suffices to prove that for any $w \in V(\mathcal{G}_1)$, if $P_1, \ldots, P_k$ are minimal paths from $w_1, \ldots, w_k$ respectively to $w$, then the $f_{P_i}(v_i)$ are linearly independent in $V_w$. Accordingly, suppose that we have $\sum_i a_i f_{P_i}(v_i) = 0$. For a given $i_0$, if we write $P_{i_0} = P(w_{i_0}, u_1, \ldots, u_m)$ with $u_i \in V(\Gamma)$, by minimality of $P_{i_0}$ and Proposition 3.1.4 there some $v \in V(\Gamma)$ not appearing among the $u_i$, and then $f_{P_{v_0}}$ is injective on $Y_v$. If $P$ is a minimal path from $w$ to $w_v$, then $f_P$ may vanish on $Y_v$, and similarly if $P$ is a minimal path from $w_v$ to $w$. However, there always exists some $w'$ such that, if $P$ and $P'$ are minimal paths from $w$ to $w'$ and from $w_v$ to $w'$ respectively, then $f_P$ and $f_{P'}$ are both nonvanishing on $Y_{w_v}$. By the above observation, we have that $f_{P'}(V_{w_v}) = V_{w_v'}$. If $P'_1, \ldots, P'_k$ are minimal paths from $w_1, \ldots, w_k$ respectively to $w_v$, then since the $f_{P'_i}(v_i)$ are assumed linearly independent, we find that the $f_{P'_{v_0}}(v_i)$ are likewise linearly independent. On the other hand, by hypothesis $\sum_i a_i f_{P_{v_0}}(v_i) = 0$. Now, for each $i$ we have that $f_{P_{v_0}}(v_i) \neq 0$. Thus, for each $i$ either $a_i = 0$ or $f_{P_{v_0}}(v_i) = 0$. Since $f_{P_{v_0}}$ was by construction injective on $Y_{w_v}$ and each $v_i$ is non-zero on every component of $X$ by Corollary 3.4.11, we have that
Remark 3.4.15. We observe that our two constructions are in fact completely equivalent in the two-component case, so that they each provide a generalization of \cite{Our06}. Indeed, we have already observed that the additional condition of Situation 3.3.1 is tautologically satisfied in the two-component case. It is clear that in this case, we have \( G_1 \) and \( G_{11} \) equal, each a chain with edges in both directions. Proposition 3.4.1 then implies that the bundles are isomorphic in both constructions, and the maps agree up to invertible scalar, so the resulting constructions agree.

This is consistent with the nature of the additional choice of \( \theta \) in the second construction, since a given \( \theta \) is determined up to scalar unless \( Y_v \) is disconnecting, which never occurs in the two-component case.

4. Stack structures on the Eisenbud-Harris-Teixidor construction

We now examine in some detail the generalization by Teixidor i Bigas of the construction of Eisenbud and Harris. We present a new point of view on this construction, which will be important for two reasons: it is an important step in our comparison theorems, and it yields a canonical stack structure on the space of Eisenbud-Harris-Teixidor limit linear series which can be placed into smoothing families. The latter is new even for the classical case of rank 1. After introducing this stack structure, we compare it to the previously constructed stacks of linked linear series.

In this section, we consider the case that \( B = \text{Spec} \, F \) is a point, and we therefore omit \( B \) from the notation. Thus, we have a single reducible curve \( X \) over \( \text{Spec} \, F \), with dual graph \( \Gamma \). In particular, the notation \( \pi_\ast \) simply corresponds to taking global sections, but we will frequently use \( \pi_\ast \), especially for constructions which generalize to families. As before, for \( v \in V(\Gamma) \), we will denote by \( Y_v \) the corresponding component of \( X \). Because much of this section is rather technical, our approach is, as much as possible, to prove general results by carrying out key calculations over fields, and then reducing the general statements to this situation.

4.1. Statements on points. We begin by recalling the definition of Eisenbud-Harris-Teixidor limit linear series, given as a subset of a certain stack which we now describe. Working on the level of points, we then give an alternate characterization of this subset which will lend itself to introducing a stack structure.

Given \( v \in V(\Gamma) \), we have the stack \( \mathcal{G}^{k}_{r,d}(Y_v) \). Given \( e \in E(\Gamma) \) adjacent to \( v \), we have also the stack \( \mathcal{M}_r(\Delta'_e) \) of rank-\( r \) vector bundles on \( \Delta'_e \), where \( \Delta'_e \) is the node corresponding to \( e \). There is a natural restriction map \( \mathcal{G}^{k}_{r,d}(Y_v) \to \mathcal{M}_r(\Delta'_e) \), but for reasons which will soon become apparent, we instead consider the map induced first by twisting the universal bundle by \( \mathcal{O}_{\pi_v, w_0} |_{Y_v} \), and then restricting to \( \Delta'_e \). Here \( w_0 \) is as in Definition 3.2.1.

Notation 4.1.1. Denote by \( \mathcal{P}^{k}_{r,d} \ast(X) \) the product of all the stacks \( \mathcal{G}^{k}_{r,d}(Y_v) \) fibered over the stacks \( \mathcal{M}_r(\Delta'_e) \) via the above maps.

Note that while \( \mathcal{M}_r(\Delta'_e) \) has a single point, that point has automorphism group \( \text{GL}_r \), so the stack is non-trivial. The purpose of fibered over it is that, due to the definition of 2-fibered products, this process precisely introduces a choice of
gluing map at each node. In fact, the moduli stack $\mathcal{M}_{r,w_0}(X)$ can be realized as the 2-fibered products of the stacks $\mathcal{M}_{r,d_r}(Y_r)$ over the stacks $\mathcal{M}_r(\Delta'_r)$ (twisting to common multidegree $w_0$ as before), so $\mathcal{P}^k_{r,d_r}(X)$ has a natural forgetful morphism to $\mathcal{M}_{r,w_0}(X)$.

We now recall the definition of Eisenbud-Harris-Teixidor limit linear series. Following Teixidor’s description as closely as possible, we give a set-theoretic description of the conditions inside $\mathcal{P}^k_{r,d_r}(X)$, but we will endow this set with a locally closed substack structure in Lemma 4.1.6 and §4.2 below. For convenience in gluing notation, we choose directions for all edges of $\Gamma$. We will also make use of the twisting line bundles we constructed in Notation 3.2.2 and 3.3.3.

**Definition 4.1.2.** Let $(\mathcal{E}^v, V^v)$ be a $K$-valued point of $\mathcal{P}^k_{r,d_r}(X)$, where $(\mathcal{E}^v, V^v)$ is the corresponding point of $\mathcal{G}^k_{r,d_r}(Y_r)$, and if $e$ is an edge from $v$ to $v'$,

$$\varphi_e : (\mathcal{E}^v \otimes (\mathcal{O}_{w_v,w_0}|_{Y_r})))|_{\Delta'_e} \xrightarrow{\sim} (\mathcal{E}^{v'} \otimes (\mathcal{O}_{w_{v'},w_0}|_{Y_{v'}}))|_{\Delta'_e}$$

is the corresponding gluing isomorphism. Then $((\mathcal{E}^v, V^v), (\varphi_e)_e)$ is an Eisenbud-Harris-Teixidor (or EHT) limit linear series if for each $e \in E(\Gamma)$ from $v$ to $v'$, we have:

(I) $H^0(Y_v, \mathcal{E}^v((b+1)\Delta'_e)) = 0$ and similarly for $v'$;

(II) if $a_1^{e,v}, \ldots, a_{k_v}^{e,v}$ and $a_1^{e,v'}, \ldots, a_{k_{v'}}^{e,v'}$ are the vanishing sequences at $\Delta'_e$ for $(\mathcal{E}^v, V^v)$ and $(\mathcal{E}^{v'}, V^{v'})$ respectively, then for every $i$ we have

$$a_i^{e,v} + a_i^{e,v'} \geq b;$$

(III) there exist bases $s_1^{e,v}, \ldots, s_{k_v}^{e,v}$ and $s_1^{e,v'}, \ldots, s_{k_{v'}}^{e,v'}$ of $V^v$ and $V^{v'}$ respectively such that $s_i^{e,v}$ has vanishing order $a_i^{e,v}$ at $\Delta'_e$ for each $i$, and similarly for $s_i^{e,v'}$, and if we have $a_i^{e,v} + a_i^{e,v'} = b$ for some $i$, then

$$\varphi_e(s_i^{e,v}) = s_i^{e,v'};$$

We say that $((\mathcal{E}^v, V^v), (\varphi_e)_e)$ is **refined** if equality always holds in (4.1.1).

**Notation 4.1.3.** We denote by $\mathcal{G}^k_{r,d_r,\text{EHT}}(X)$ the set of Eisenbud-Harris-Teixidor limit linear series, and by $\mathcal{G}^k_{r,d_r,\text{EHT,ref}}(X)$ the refined locus.

**Remark 4.1.4.** Condition (III) of Definition 4.1.2 requires some explanation. As we have already remarked, the gluings $\varphi_e$ together with the $\mathcal{E}^v$ yield a vector bundle $\mathcal{E}$ of multidegree $w_0$. As in the linked linear series constructions, our twisting bundles then yield vector bundles $\mathcal{E}_w$ of multidegree $w$ for any $w \in V(G_1)$, and this in turn induces gluing maps

$$(\mathcal{E}^v \otimes (\mathcal{O}_{w_v,w}|_{Y_r})))|_{\Delta'_e} \xrightarrow{\sim} (\mathcal{E}^{v'} \otimes (\mathcal{O}_{w_{v'},w}|_{Y_{v'}}))|_{\Delta'_e}$$

for each $w$. When $w \in V(G_1)$, we have $\mathcal{E}^v \otimes (\mathcal{O}_{w_v,w}|_{Y_r})$ and $\mathcal{E}^{v'} \otimes (\mathcal{O}_{w_{v'},w}|_{Y_{v'}})$ described explicitly in terms of twisting down by a multiple of the node $\Delta'_e$.

Now, when $a_i^{e,v} + a_i^{e,v'} = b$ we have an induced multidegree $w \in V(G_1)$, and considering $s_i^{e,v}$ as a section of $\mathcal{E}^v(-a_i^{e,v}\Delta'_e)$ and $s_i^{e,v'}$ as a section of $\mathcal{E}^{v'}(-a_i^{e,v'}\Delta'_e)$, we can apply the above gluing map to compare their values at $\Delta'_e$.

Note that this formulation of the gluing condition differs slightly from that of [Tei91], replacing a projectivized gluing condition with a gluing condition on (twists
of) the vector bundles themselves. However, our formulation is consistent with the way in which dimension counts are carried out in [Tei91] and related work of Teixidor i Bigas.

The conditions of Definition 4.1.2 are invariant under field extension, so yield a well-defined subset of $\mathcal{P}_{r,d,*}^k(X)$. However, obtaining a stack structure is subtler; see Remark 4.1.12.

We observe that the refined condition is in fact open.

**Proposition 4.1.5.** We have $\mathcal{G}_{r,d,d,*}^{k,EHT,ref}(X)$ open in $\mathcal{G}_{r,d,d,*}^{k,EHT}(X)$.

**Proof.** Given a point of $\mathcal{G}_{r,d,d,*}^{k,EHT,ref}(X)$, let $\{\alpha^{w,v}\}_{(e,v)}$ be the vanishing sequences at the nodes. Then because of condition (II) of Definition 4.1.2, we can find a refined open neighborhood of the given point in $\mathcal{G}_{r,d,d,*}^{k,EHT}(X)$ by imposing that each vanishing sequence be at most equal to the given $\{\alpha^{w,v}\}_{(e,v)}$, which is an open condition.

Given $w \in V(G_1)$ and $\theta$, as in Notation 3.3.4, if we fix a minimal path $P_e$ from $w$ to $w_v$ for each $v \in \Gamma$, then the maps $f_{P_e}$ together with restriction to $Y_v$ and modding out by $V^v$ induce a map

$$\pi_\theta \mathcal{E}_w \to \bigoplus_{v \in V(\Gamma)} (\pi_\theta \mathcal{E}^v)/V^v.$$  

(4.1.3)

The kernel of this map can be thought of as sections of $\pi_\theta \mathcal{E}_w$ gluing from sections lying in the spaces $V^v$ on each $Y_v$. If $w \in V(G_1)$, then according to Proposition 3.4.1 the maps $f_{P_e}$ and hence (4.1.3) will be independent of the choice of $\theta$.

In order to introduce our stack structures, we examine the relationship between the Eisenbud-Harris-Teixidor conditions, and the space of sections of the $V^v$ available to glue in arbitrary multidegrees. The key result is the following, which says the vanishing and gluing conditions for an EHT limit series may be reinterpreted in terms of existence, for each $w$, of $k$-dimensional subspaces of the kernel of (4.1.3).

**Lemma 4.1.6.** For a $K$-valued point $((\mathcal{E}^v, V^v)_{v \in V(\Gamma)}, (\varphi_v)_{v \in E(\Gamma)})$ of $\mathcal{P}_{r,d,*}^k(X)$ with the $\mathcal{E}^v$ satisfying (I) of Definition 4.1.2, the following are equivalent.

(a) $((\mathcal{E}^v, V^v)_{v \in V(\Gamma)}, (\varphi_v)_{v \in E(\Gamma)})$ lies in $\mathcal{G}_{r,d,d,*}^{k,EHT}(X)$.

(b) for every $w \in V(G_1)$, the map (4.1.3) has kernel of dimension at least $k$.

(c) for every $w \in V(G_1)$, the map (4.1.3) has kernel of dimension at least $k$.

Furthermore, under these equivalent conditions, the kernel of (4.1.3) always has dimension exactly $k$ if $w = w_v$ for some $v \in V(\Gamma)$, and if $((\mathcal{E}^v, V^v)_v, (\varphi_v)_v)$ is a refined EHT limit series, then (4.1.3) has kernel of dimension exactly $k$ for all $w \in V(G_1)$.

**Remark 4.1.7.** The converse of the last statement of Lemma 4.1.6 is not true even in the case of rank 1 with a two-component curve. That is, non-refined limit linear series may have kernel dimension exactly $k$ in (4.1.3) for all $w \in V(G_1)$; see Lemma 5.9 of [Oss06c].

Before proving the lemma, we introduce some notation, giving a divisor variant of Notation 3.2.2.

**Notation 4.1.8.** Given $w \in V(G_1)$ and $v \in V(\Gamma)$, let

$$D_{w,v} = \sum_{e \text{ adjacent to } v} t_{(e,v)}(w) \Delta'_e.$$
The significance of $D_{w,v}$ is that it describes the amount of vanishing at the nodes required of a section of $V^v$ in order for it to be possible that it is the restriction to $Y_v$ of a section of $\pi_v\mathcal{E}_w$. But differently, it describes how much vanishing we have to impose on sections of $V^v$ in order to be able to use them to construct sections of a space $V_w \subseteq \pi_v\mathcal{E}_w$. By definition, we see that because $w \in V(\tilde{G}_{11})$, we have $t(e,v)(w) \geq 0$ for all edges $e$ adjacent to a given $v$, so the divisor $D_{w,v}$ is effective.

**Notation 4.1.9.** Given $((\delta^v, V^v), (\varphi_e)_e) \in \mathcal{E}^{k,\text{EHT}}_{r,d,d_\ast}(X)$, and $w \in V(\tilde{G}_{11})$, for given $v \in V(\Gamma)$ denote by $\delta^v(w)$ the sheaf $\delta^v(-D_{w,v})$, and by $V^v(w)$ the subspace of $V^v$ consisting of sections lying in $\delta^v(w)$.

It will be useful to consider a component-wise variant of the map (4.1.3). Namely, using our choice of directions for each edge of $\Gamma$, given $((\delta^v, V^v)_{v \in V(\Gamma)}, (\varphi_e)_{e \in E(\Gamma)})$ as in Lemma 4.1.6, for any $w \in V(\tilde{G}_{11})$ we obtain a map

$$\bigoplus_{v \in V(\Gamma)} V^v(w) \to \bigoplus_{e \in E(\Gamma)} \left( \delta^{h(e)}(w) \right|_{\Delta_e}$$

given by the natural restriction map on each $h(e)$, and by the negative of the restriction map induced by $\varphi_e$ on each $t(e)$. We will be interested in the kernel of this map, which is independent of the choice of directions on $E(\Gamma)$.

Our first observation is that by the hypothesis on the form of $\varphi$, $\pi_v\mathcal{E}_w \to \pi_v\mathcal{E}_v$ for all $v \in V(\Gamma)$.

**Lemma 4.1.10.** The kernels of (4.1.3) and (4.1.4) are identified via the maps $\pi_v\mathcal{E}_w \to \pi_v\mathcal{E}_v$ for $v \in V(\Gamma)$.

**Proof.** $\pi_v\mathcal{E}_w$ maps isomorphically onto the kernel of

$$\bigoplus_{v \in V(\Gamma)} \pi_v(\delta^v(w)) \to \bigoplus_{e \in E(\Gamma)} \left( \delta^{h(e)}(w) \right|_{\Delta_e}$$

in general, since this is simply the restriction exact sequence. By construction, we then have that the kernel of (4.1.3) is precisely the subspace of $\pi_v\mathcal{E}_w$ which maps into the spaces $V^v(w)$ for each $v$, and is thus identified with the kernel of (4.1.4), as claimed.

We next give a description of what happens under restriction to a subcurve $X'$, when non-minimal parts of the multidegree are supported on $X'$.

**Lemma 4.1.11.** Let $X'$ be a connected subcurve of $X$, and denote by $\Gamma' \subseteq \Gamma$ the associated dual graph. Let $((\delta^v, V^v)_{v \in V(\Gamma')}, (\varphi_e)_{e \in E(\Gamma')})$ be a $K$-valued point of $\mathcal{P}^{k}r,d_\ast(X')$, fix $w \in V(\tilde{G}_{11})$ which is minimal (i.e., equal to $d_v - rb$) on all $v$ not in $\Gamma'$. Finally, let $\varphi$ be the restriction map from the kernel of (4.1.4) to the kernel of the corresponding map with $\Gamma'$ in place of $\Gamma$. Then:

(i) $\varphi$ is injective if the $\delta^v$ satisfy (I) of Definition 4.1.2;

(ii) $\varphi$ is an isomorphism if $((\delta^v, V^v)_v, (\varphi_e)_e)$ constitutes an EHT limit series.

**Proof.** The first observation is that by the hypothesis on the form of $w$, for any $v \notin \Gamma'$, then $D_{w,v} = b\Delta'$, where $\Delta'$ is the unique node of $X$ joining $Y_v$ to a component of $X$ strictly closer to $X'$. We prove both statements of the lemma inductively, noting that $\varphi$ is obtained as a composition of restriction maps which drop one component at a time. It thus suffices to consider the case that $X'$ is obtained from $X$ by dropping a single component $Y_v$, meeting $X'$ at a single node $\Delta'$. 
First, suppose condition (I) of Definition 4.1.2 is satisfied. Then the kernel of $\varphi$ consists of tuples of sections of $V^{\nu'}(w)$ for $\nu' \in \Gamma$ which glue at all nodes, and vanish uniformly away from $Y_v$. But these conditions imply that the section over $Y_v$ must vanish at $\Delta'$ as a section of $\pi_*\delta'(w) = \pi_*\delta'(w - b\Delta')$, so is a section of $\pi_*\delta'(w - (b+1)\Delta')$, and must be equal to zero by hypothesis. This proves the desired injectivity assertion.

For surjectivity, suppose also that conditions (II) and (III) of Definition 4.1.2 are satisfied, and we are given a tuple of sections of $V^{\nu'}(w)$ for $\nu' \in \Gamma'$ which glue at nodes on $X'$. Let $v'$ be the vertex adjacent to $v$ in $\Gamma$, so that $\Delta'$ is the node joining $Y_v$ to $Y_{v'}$. We wish to show that there is a section of $V^{\nu'}(w)$ which glues to the given section $s \in V^{\nu'}(w)$. If $s$ vanishes at $\Delta'$, there is nothing to show, since we can choose the zero section of $V^{\nu'}(w)$. On the other hand, if $s$ is nonvanishing at $\Delta'$, we claim that conditions (II) and (III) imply that there is a section of $V^{\nu'}(w)$ gluing to it. Let $s_1^1$ and $s_2^1$ be bases for $V^\nu$ and $V^{\nu'}$ respectively as in condition (III), and suppose $s_2^2, \ldots, s_2^2$ are the sections nonvanishing at $\Delta'$; thus, the restriction of $s$ to $\Delta'$ agrees with some linear combination of the restrictions to $\Delta'$ of the $s_2^2$ with $j \leq \ell$. Now, by condition (II) we must have that $s_{k+1-\ell}, \ldots, s_k^1$ all vanish to order at least $b$ at $\Delta'$, and by condition (I) they must vanish to order exactly $b$. Thus by condition (III), all the $s_2^2$ with $\ell \leq j$ glue to sections of $V^{\nu'}(w)$, and we conclude that $s$ likewise glues to give a section of $V^{\nu'}(w)$, as desired.

**Proof of Lemma 4.1.6.** Our first observation is that we may replace condition (b) by the same condition with $G_{11}$ in place of $G_{11}$. Indeed, if condition (b) is satisfied for all $w \in V(G_{11})$, and we have $w \in V(G_{11})$, then, following the notation of the proof of Proposition 3.4.12, let $P_w$ be a minimal length path from $G_{11}$ to $w$, with starting point $\tilde{w}_w$. Then we have that $f_{P_w}$ is universally injective, but it clearly maps the kernel of (4.1.3) in multidegree $\tilde{w}_w$ to the kernel in multidegree $w$, so we conclude that the kernel in multidegree $w$ also has dimension at least $k$, as desired.

We now prove the stated equivalence by induction on the number of components of $X$, using Lemma 4.1.10 to replace the kernel of (4.1.3) with that of (4.1.4). The base case for the induction is the case of two components. Since in this case $\Gamma$ consists of two vertices and a single edge, to simplify notation we denote the unique node by $\Delta'$, and by $d_i$, $V^i$ and $a^i_\ell$ (with $\ell = 1, \ldots, k$, and $i = 1, 2$) respectively the degrees, spaces of sections, and vanishing sequences at $\Delta'$ of the $i$th component. Now, in this case conditions (b) and (c) are trivially equivalent. We then have $w = (d_1 - rj, d_2 - r(b - j))$ for some $j$, and

$$\dim V^1(w) = \# \{ \ell : a^1_\ell \geq j \}, \quad \text{and} \quad \dim V^2(w) = \# \{ \ell : a^2_\ell \geq b - j \}. $$

Let $s$ be the rank of (4.1.4). Then the dimension of the kernel is equal to $\dim V^1(w) + \dim V^2(w) - s$. Say $a^1_1, \ldots, a^1_{\ell_1}$ are the vanishing indices equal to $j$, and $a^2_1, \ldots, a^2_{\ell_2}$ are the vanishing indices equal to $b - j$. Here, if no $a^1_\ell$ is equal to $j$, we set $\ell_2 = \ell_1 + 1$ to be maximal with $a^1_\ell \geq j$, and similarly if no $a^2_\ell$ is equal to $b - j$. Then we have $\dim V^1(w) = k + 1 - \ell_1$, and $\dim V^2(w) = k + 1 - \ell_3$, so the condition that the kernel of (4.1.4) have dimension at least $k$ is equivalent to the inequality

$$k + 2 \geq \ell_1 + \ell_3 + s. $$

(4.1.5)

Now, suppose (a) is satisfied. We first observe that condition (II) of an EHT limit series implies that $\ell_2 + \ell_4 \leq k + 1$, and $\ell_2 + \ell_3 \leq k + 1$. Indeed, if $\ell_2 > 1$, then $a^1_{\ell_2 - 1} < j$, so we have $a^2_{k+2-\ell_4} > b - j$, and hence $k + 2 - \ell_1 > \ell_4$. On the other
hand, if \( \ell_1 = 1 \), then \( \ell_4 \leq k \) immediately gives the first inequality. The second inequality follows by considering \( \ell_3 \) in place of \( \ell_1 \). To prove (4.1.5), first consider the case that \( k + 1 > \ell_2 + \ell_4 \). We necessarily have

\[
s \leq (\ell_2 + 1 - \ell_1) + (\ell_4 + 1 - \ell_3) < k + 3 - \ell_1 - \ell_3,
\]
giving the desired inequality. On the other hand, if \( k + 1 \leq \ell_2 + \ell_4 \), we obtain a range of indices \( a_{k+1-\ell_2}, \ldots, a_{\ell_4} \) complementary to \( a_{k+1-\ell_4}, \ldots, a_{\ell_2} \), and the gluing condition then gives us

\[
s \leq (\ell_2 + 1 - \ell_1) + (\ell_4 + 1 - \ell_3) - (\ell_4 + \ell_2 - k) = k + 2 - \ell_1 - \ell_3,
\]
as desired. We thus conclude that (a) implies (b).

Observe also that if (a) is satisfied and we have the refinedness condition, then following the above argument, we have have \( a_{k+1-\ell_1}^2 = b - j \), so \( k + 1 - \ell_1 \leq \ell_4 \), so we conclude that \( \ell_1 + \ell_4 = k + 1 \), and similarly \( \ell_2 + \ell_3 = k + 1 \). Thus

\[
\ell_2 + 1 - \ell_1 = \ell_4 + 1 - \ell_3 = \ell_4 + \ell_2 - k = k + 2 - \ell_1 - \ell_3,
\]
and since \( s \geq \#\{\ell : a_{\ell}^1 = j\} = \ell_2 + 1 - \ell_1 \), we conclude that (4.1.5) is satisfied with equality, and hence the kernel of (4.1.4) has dimension exactly \( k \).

Conversely, suppose that (b) is satisfied, and choose any \( j \) occurring as \( a_{\ell_1}^1 \) for some \( \ell \); setting \( \ell_1, \ldots, \ell_4 \) and \( s \) as above, by (4.1.5) we have \( s \leq k + 2 - \ell_1 - \ell_3 \). Given \( \ell \) with \( \ell_1 \leq \ell \leq \ell_2 \), we first wish to see that \( a_{\ell}^1 + a_{k+1-\ell}^2 \geq b \). Since \( s \geq \ell_2 + 1 - \ell_1 \), we obtain \( k + 1 - \ell_2 \geq \ell_3 \). We conclude that

\[
a_{\ell}^1 + a_{k+1-\ell}^2 \geq a_{k+1-\ell}^1 + a_{k+1-\ell_2}^2 \geq a_{\ell_1}^1 + a_{\ell_3}^2 = b,
\]
so condition (II) of an EHT limit series is satisfied, as desired. We next wish to verify the gluing condition. Again applying (4.1.5), the images of \( V^1(w) \) and \( V^2(w) \) at the node \( \Delta' \) must intersect in dimension at least

\[
(\ell_2 + 1 - \ell_1) + (\ell_4 + 1 - \ell_3) - (k + 2 - \ell_1 - \ell_3) = \ell_2 + \ell_4 - k.
\]

If this is non-positive, there are no indices \( \ell \) between \( \ell_1 \) and \( \ell_2 \) such that \( k + 1 - \ell \) is between \( \ell_3 \) and \( \ell_4 \), and we claim that if it is positive, it is equal to the number of such indices. Indeed, we already know that condition (II) of an EHT limit series is satisfied, so by the above observation, we have \( \ell_1 + \ell_4 \leq k + 1 \) and \( \ell_2 + \ell_3 \leq k + 1 \), and the claim follows easily. We conclude that an appropriate choice of basis elements will also satisfy the gluing condition, and hence that we have an EHT limit series.

For the induction step, assume we have at least three components. To see that (a) implies (b), the basic observation is that the restriction of an EHT limit linear series to a connected subcurve is still an EHT limit linear series. Choose a component \( Y_{v_1} \) meeting the rest of \( X \) at only a single node \( \Delta_{v_1} \). Let \( X' \) be the complement of \( Y_{v_1} \) in \( X \), and let \( Y \) be the union of \( Y_{v_1} \) and \( Y_{v_2} \), where \( v_2 \) is the other edge adjacent to \( e_1 \). Denote by \( \Gamma' \) the dual graph of \( X' \). Restricting to \( X' \) and \( Y \), we have multidegrees \( w' \) and \( w'' \) which differ from \( w \) only in index \( v_2 \): \( w' = w - w'' \) in index \( v_2 \) is \( r_{(e_2,v_2)}(w) \), while \( w'' - w \) in index \( v_2 \) is \( r_{e_1(e_2,v_2)}(w) \), where \( e' \) ranges over edges other than \( e_2 \) which are adjacent to \( v_2 \). Then, by the induction hypothesis we have that the kernels of the maps

\[
\bigoplus_{v \in V(\Gamma') \cap \Delta_{v_1}} V_v(w') \to \bigoplus_{v \in E(\Gamma') \cap \Delta_{v_1}} \left( \mathcal{O}_{h(e)}^{h(e)} \otimes (\mathcal{O}_{w,v'})_{|Y(v(e))} \right) |_{\Delta_{v_1}}
\]
and

\[
V_{v_1}(w'') \oplus V_{v_2}(w'') \to \left( \mathcal{O}_{h(e_1)}^{h(e_1)} \otimes (\mathcal{O}_{w,v}^{w,e_1})_{|Y_{v_1}} \right) |_{\Delta_{v_1}}
\]
each have dimension at least $k$. However, the kernel of (4.1.4) is the precisely the fibered product of the above two kernels over $V_{v_2}$. Since $\dim V_{v_2} = k$, we conclude that the kernel of (4.1.4) has dimension at least $k$, as desired.

It thus suffices to see that (c) implies (a). Since (a) may be checked node by node, it is enough to show that for any $v_1 \in E(\Gamma)$ from $v_1$ to $v_2$, the restriction of $(\varphi|_v, V^v)_{v_1}$ to $V_{v_1} \cup Y_{v_2}$ lies in $G^{k, \text{EHT}}_{r,d, (d,v_1,d_0)}(Y_{v_1} \cup Y_{v_2})$. But for any $w \in V(G_1)$ which is equal to $d_{v'} - rb$ for all $v' \neq v_1, v_2$, by Lemma 4.1.11 we have that restriction to $Y_{v_1} \cup Y_{v_2}$ induces an injection of the kernel of (4.1.4) into the kernel of

$$V^{v_1}(w) \oplus V^{v_2}(w) \to (\varphi|_v(w))|_{\Delta_{v_1}}$$

so by the case of two components we conclude the desired statement.

We next observe that if we have an EHT limit series, and $w = w_v$ for some $v \in V(\Gamma)$, then setting $X' = Y_v$ it follows immediately from Lemma 4.1.11 that the kernel of (4.1.3) always has dimension exactly $k$. Finally, we wish to show that if an EHT limit series is refined, then the dimension of the kernel of (4.1.3) is equal to $k$ for all $w \in V(G_1)$. We have already seen this in the case of two components. Now, suppose we have a refined EHT limit series, and $w \in V(G_1)$ which is equal to $d_{v'} - rb$ for all $v'$ except for two adjacent vertices $v, v'$. Let $X' = Y_v \cup Y_{v'}$. Then the restriction to $X'$ is itself a refined EHT limit series, so by the two-component case together with Lemma 4.1.11, we conclude that the kernel of (4.1.3) has dimension $k$, as desired.

\begin{remark}
As stated, the definition of $G^{k, \text{EHT}}_{r,d,d_0}(X)$ does not yield any obvious stack structure. Condition (I) is open, while condition (II) may be expressed as a finite union of closed conditions, but the gluing in condition (III) is subtler. A significant difficulty is that this condition cannot be applied uniformly, but only on individual strata with given vanishing sequence. However, as a consequence of Lemma 4.1.6, we will see in §4.2 below that in fact $G^{k, \text{EHT}}_{r,d,d_0}(X)$ does have a canonical stack structure.
\end{remark}

4.2. Stack structures. We now start introducing and relating various natural stack structures which arise in connection with Eisenbud-Harris-Teixidor limit linear series. First, from Lemma 4.1.6, we may use the constructions of Appendix B to define a stack structure on $G^{k, \text{EHT}}_{r,d,d_0}(X)$.

\begin{definition}
We endow $G^{k, \text{EHT}}_{r,d,d_0}(X)$ with the structure of a locally closed substack of $P^{k}_{r,d_0}(X)$ as follows: an $S$-valued point $((\varphi^v, \Upsilon^v)_{v \in V(\Gamma)}, (\varphi^e)_{e \in E(\Gamma)})$ of $P^{k}_{r,d_0}(X)$ is a point of $G^{k, \text{EHT}}_{r,d,d_0}(X)$ if it lies in the open substack satisfying (I) of Definition 4.1.2, and for each $w \in V(G_1)$, it satisfies the closed condition that the $k$th vanishing locus (Definition B.3.1) of

$$\pi_* \mathcal{E}|_w \to \bigoplus_{v \in V(\Gamma)} \pi_* \mathcal{E}^v/\Upsilon^v$$

is equal to all of $S$.

\begin{remark}
As a locally closed substack of $P^{k}_{r,d_0}(X)$, it is evident that $G^{k, \text{EHT}}_{r,d,d_0}(X)$ is an Artin stack, locally of finite type over $\text{Spec} \ F$, and the morphism

$$G^{k, \text{EHT}}_{r,d,d_0}(X) \to M_{r,w_0,d_0}(X)$$

is equal to all of $S$.
\end{remark}
Remark 4.2.3. According to Lemma 4.1.6, we could have used all $w \in V(G_{II})$ instead of $V(G_1)$ to obtain a (possibly different) stack structure on $\mathcal{G}^{k,\text{EHT}}_{r,d,d_\bullet}(X)$; we chose $V(G_1)$ largely because it seems closer in spirit to the original Eisenbud-Harris-Teixidor definition.

In order to carry out comparisons between stack structures, we now move from considering $K$-valued points to $S$-valued points. One of the pleasant aspects of EHT limit series is that, due to the component-by-component and node-by-node nature of the definition, the restriction to a subcurve of an EHT limit series is still an EHT limit series. We begin by observing that this property carries over to our stack structures.

**Proposition 4.2.4.** Let $X'$ be a connected subcurve of $X$. Then restriction to $X'$ induces a morphism

$$\mathcal{G}^{k,\text{EHT}}_{r,d,d_\bullet}(X) \to \mathcal{G}^{k,\text{EHT}}_{r,d,d_\bullet}(X'),$$

where $d_\bullet'$ denotes the restriction of $d_\bullet$ to the components lying in $X'$, and $d'$ is determined by keeping $b$ fixed.

**Proof.** Let $\Gamma'$ be the dual graph of $X'$, and let $G'_{I}$ be the associated graph as in Definition 3.1.2. Since we know the restriction morphism is defined on the set-theoretic level, the open condition (I) of Definition 4.1.2 is necessarily preserved under restriction, and it is enough to prove that given an $S$-valued point $((\varphi^v, \varphi'^v), (\varphi_e)v)$ of $\mathcal{G}^{k,\text{EHT}}_{r,d,d_\bullet}(X)$, for each $w' \in V(G'_{I})$, the $k$th vanishing locus of

$$\pi_{\bullet} \mathcal{E}_w \twoheadrightarrow \bigoplus_{v \in V(\Gamma')} \pi_{\bullet} \mathcal{E}_v / \mathcal{Y}_v$$

is equal to all of $S$. Let $w \in V(G_1)$ be the extension of $w'$ by $d_v - rb$ for all $v \notin \Gamma'$. Then by hypothesis, the $k$th vanishing locus of

$$\pi_{\bullet} \mathcal{E}_w \twoheadrightarrow \bigoplus_{v \in V(\Gamma')} \pi_{\bullet} \mathcal{E}_v / \mathcal{Y}_v$$

is equal to all of $S$. Now, we have the natural restriction morphism $\mathcal{E}_w \to \mathcal{E}_{w'}$, and the projection morphism $\bigoplus_{v \in V(\Gamma')} \mathcal{E}_v \to \bigoplus_{v \in V(\Gamma')} \mathcal{E}_v$, and these clearly satisfy (I) and (II) of Corollary B.3.5. Moreover, Lemma 4.1.11 implies that (III) of Corollary B.3.5 is likewise satisfied, so we conclude the desired statement. \hfill $\square$

Next, recall that on the refined locus, the Eisenbud-Harris-Teixidor conditions could also be used to give a stack structure. Our first comparison statement is that on the refined locus, the two stack structures agree. We begin by making precise the latter stack structure.

**Definition 4.2.5.** Let $a^\Gamma$ consist of, for each pair $(e, v)$ of an edge of $\Gamma$ together with an adjacent vertex, a sequence $a^e_{1} \leq \cdots \leq a^e_{k}$, such that for every edge $e$ with $v, v'$ the adjacent vertices, the sequences satisfy (4.1.1) with equality. Define the locally closed substack $\mathcal{G}^{k,\text{EHT}}_{r,d,d_\bullet,a^\Gamma}(X)$ of $\mathcal{P}^{k}_{r,d_\bullet}(X)$ to be the stack defined by the following conditions:

(I) the open condition (I) of Definition 4.1.2;
(II) the locally closed condition that for each adjacent pair \((e,v)\) of \(\Gamma\), the vanishing sequence on \(Y_v\) at \(\Delta_e\) is exactly equal to \(a^\Gamma_{v,v}\).

(III) the closed gluing condition that for each \(w \in G_1\) equal to \(d_{vr} - rb\) for all \(v\) other than some \(v,v'\) adjacent to an edge \(e\) of \(\Gamma\), we have that the restrictions \(\mathcal{Y}^v(w)|_{\Delta_e}\) and \(\mathcal{Y}^{v'}(w)|_{\Delta_e}\) agree under the gluing map \(\varphi_e\).

**Proposition 4.2.6.** The open substack of \(\mathcal{G}^{k,\text{EHT}}_{r,d,d_\ast}(X)\) consisting of refined \(\text{EHT}^\Gamma\) limit series is isomorphic to the (disjoint) union of the substacks \(\mathcal{G}^{k,\text{EHT}}_{r,d,d_\ast,a^\Gamma}(X)\) as \(a^\Gamma\) ranges over allowable collections of sequences.

*Proof.* Because restricting to a locus with given (refined) vanishing sequences \(a^\Gamma\) yields an open substack of \(\mathcal{G}^{k,\text{EHT}}_{r,d,d_\ast}(X)\), it is enough to fix a choice of \(a^\Gamma\) and work with the corresponding open locus.

First, suppose we are given an \(S\)-valued point \(((\mathcal{E}^v, \mathcal{Y}^v)_v, (\varphi_e)_e)\) of \(\mathcal{G}^{k,\text{EHT}}_{r,d,d_\ast,a^\Gamma}(X)\); we wish to show that it is also an \(S\)-valued point of \(\mathcal{G}^{k,\text{EHT}}_{r,d,d_\ast}(X)\), meaning that for each \(w \in V(G_1)\), it satisfies the closed condition that the \(k\)th vanishing locus of

\[
\pi_* \mathcal{E}_w \rightarrow \bigoplus_{v \in V(\Gamma)} \pi_* \mathcal{E}^v/\mathcal{Y}^v
\]

is equal to all of \(S\). According to Proposition B.3.4, it is enough to show that \(\pi_* \mathcal{E}_w\) has a rank-\(k\) subbundle contained in the kernel of the map to \(\bigoplus_{v \in V(\Gamma)} \pi_* \mathcal{E}^v/\mathcal{Y}^v\). We observe that due to the form of \(w\), we have \(D_{w,v}\) supported at a single point for every \(v \in V(\Gamma)\). Thus, the condition on vanishing sequences implies that we may view \(\mathcal{Y}^v(w)\) as a subbundle of \(\mathcal{Y}^v\) for each \(v\). Similarly, the gluing condition implies that for each \(e\) and with \(v,v'\) the adjacent vertices, \(\mathcal{Y}^v(w)\) and \(\mathcal{Y}^{v'}(w)\) have the same image (with constant rank) in \(\left(\mathcal{E}^{h(v)}(w)\right)|_{\Delta_e}\), so we conclude by inductively traversing \(\Gamma\) that the map

\[
\bigoplus_{v \in V(\Gamma)} \mathcal{Y}^v(w) \rightarrow \bigoplus_{e \in E(\Gamma)} \left(\mathcal{E}^{h(v)}(w)\right)|_{\Delta_e}
\]

has constant rank, and hence its kernel is a subbundle. We know that on points, the dimension of the kernel is equal to \(k\), so we conclude that the kernel is a subbundle of rank \(k\), which yields a rank-\(k\) subbundle of \(\pi_* \mathcal{E}_w\) contained in the kernel of the map to \(\bigoplus_{v \in V(\Gamma)} \pi_* \mathcal{E}^v/\mathcal{Y}^v\), as desired.

For the opposite containment, suppose that \(((\mathcal{E}^v, \mathcal{Y}^v)_v, (\varphi_e)_e)\) is an \(S\)-valued point of \(\mathcal{G}^{k,\text{EHT}}_{r,d,d_\ast}(X)\) lying set-theoretically in \(\mathcal{G}^{k,\text{EHT}}_{r,d,d_\ast,a^\Gamma}(X)\). As before the open condition is automatically satisfied, so we need only verify that (II) and (III) of Definition 4.1.2 are verified (stack-theoretically), for which it suffices to work with the case \(S\) local. Given \(e\), let \(v,v'\) be the adjacent vertices, and restrict to the subcurve \(Y_v \cup Y_{v'} \subseteq X\). By Proposition 4.2.4, we obtain an \(S\)-valued point of \(\mathcal{G}^{k,\text{EHT}}_{r,d',(d_v,d_{v'})}(Y_v \cup Y_{v'})\), which is to say that for any \(j\), if we set \(w = (d_v - jr, d_{v'} - (b - j)r)\), the \(k\)th vanishing locus of

\[
\pi_* \mathcal{E}_w \rightarrow \pi_* \mathcal{E}^v/\mathcal{Y}^v \bigoplus \pi_* \mathcal{E}^{v'}/\mathcal{Y}^{v'}
\]

is all of \(S\). On the other hand, since we are in the refined locus, by Lemma 4.1.6 the \((k + 1)\)st vanishing locus is empty, so by Proposition B.3.4, we conclude that the kernel \(\mathcal{K}_w\) is locally of free of rank \(k\), and commutes with base change. We
know that set-theoretically, the rank of
\[ \mathcal{V} \to (\mathcal{E})_{|\Delta'_c} \]
is constant, equal to some \( b'_w \), and similarly the rank of
\[ \mathcal{V}' \to (\mathcal{E}')_{|(b-j)\Delta'_c} \]
is constant, equal to some \( b''_w \); to prove that (II) of Definition 4.1.2 is satisfied, we wish to show that the ranks are scheme-theoretically constant, or equivalently, that the closed subschemes of \( S \) on which they are less than or equal to \( b'_w \) (respectively, \( b''_w \)) are equal to all of \( S \). We describe the argument for the first map, the second being similar. It suffices to produce a subbundle of \( \mathcal{V}' \) of rank \( k - b'_w \) contained in the kernel of the map in question. But \( \mathcal{K} \) maps into this kernel. At the closed point, we have that \( \mathcal{K} \) maps to \( \mathcal{V}' \) with rank \( k - b'_w \) so using Nakayama’s lemma we can construct a subbundle of \( \mathcal{K} \) of rank \( k - b'_w \) which yields a subbundle of \( \mathcal{V}' \) of the same rank, giving the desired rank bound and proving that (II) of Definition 4.1.2 is satisfied. Similarly, we wish to prove that the rank of
\[ \mathcal{V}'(w) \oplus \mathcal{V}'(w) \to \mathcal{E}_w|\Delta'_c \]
is at most \#\{ \ell : a_{e,v}' = j \} = \#\{ \ell : a_{e,v}' = b - j \} scheme-theoretically everywhere on \( S \). The rank of the source is \( 2k - b'_w - b''_w = k + \#\{ \ell : a_{e,v}' = j \} \), so it is enough to produce a subbundle of the kernel of rank \( k \). But again, \( \mathcal{K} \) maps to the kernel, and does so injectively at the closed point, so we obtain the desired subbundle, and conclude the asserted equality of substacks.

The stack structures we have defined thus far are in most ways fully satisfactory, but one facet is still missing: because the definition of \( \mathcal{P}^{k}_{r,d,d}(X) \) involves bundles and spaces of sections on individual components of \( X \), it does not make sense in smoothing families. We thus conclude our examination of stack structures on \( \mathcal{G}^{k}_{r,d,d}(X) \) by giving a closely related construction which is isomorphic to our original stack structure, but which works well in smoothing families.

**Notation 4.2.7.** Denote by \( \mathcal{P}^{k}_{r,d,d}(X) \) the stack over \( \mathcal{M}_{r,w_0}(X) \) parametrizing, for each \( v \in V(\Gamma) \), a choice of rank-\( k \) subbundle \( \mathcal{V}_w \) of \( \mathcal{E}_w \). Thus, the only difference between \( \mathcal{P}^{k}_{r,d,d}(X) \) and \( \mathcal{P}^{k}_{r,d,d}(X) \) is that instead of considering collections \( \mathcal{V} \) of spaces of global sections of \( \mathcal{E} = \mathcal{E}_w|_{Y_v} \) on each component \( Y_v \), we take spaces of global sections of \( \mathcal{E}_w \) on all of \( X \). Because the degrees in question are extremal, we will see that on the loci of interest to us, nothing is changed. We can then define our alternate substack structure in a manner precisely parallel to the previous one.

**Definition 4.2.8.** We define the locally closed substack \( \mathcal{G}^{k}_{r,d,d}(X) \) of \( \mathcal{P}^{k}_{r,d,d}(X) \) as follows: an \( S \)-valued point \( (\mathcal{E},(\mathcal{V}_w)_{v \in V(\Gamma)}) \) of \( \mathcal{P}^{k}_{r,d,d}(X) \) is a point of \( \mathcal{G}^{k}_{r,d,d}(X) \) if it lies in the preimage of the open substack \( \mathcal{M}_{r,w_0,d}(X) \subseteq \mathcal{M}_{r,w_0}(X) \), and for each \( w \in G_1 \), it satisfies the closed condition that the \( k \)th vanishing locus of
\[ \pi_* \mathcal{E}_w \to \bigoplus_{v \in V(\Gamma)} \pi_* \mathcal{E}_w/\mathcal{V}_w \]
is equal to all of \( S \).
Proposition 4.2.9. Restriction of the $\mathcal{Y}_w$ to $Y_v$ induces a morphism from the preimage of $\mathcal{M}_{\ast w, d_s}(X)$ in $\mathcal{P}_{r,d_s}^k(X)$ to the stack $\mathcal{P}_{r,d_s}^k(X)$. Moreover, this morphism induces an isomorphism

$$\mathcal{G}_{r,d,s}^{k,\text{EHT}}(X) \rightarrow \mathcal{G}_{r,d,s}^{k,\text{EHT}}(X).$$

Proof. All that needs to be checked to see that we obtain a morphism to $\mathcal{P}_{r,d_s}^k(X)$ is that on the open subset in question, the for each $v$ we have that $\mathcal{Y}_w$ yields a rank-$k$ subbundle of $\pi_*\mathcal{E}^v$ under composition with the restriction map $\pi_*\mathcal{E}_w \rightarrow \pi_*\mathcal{E}^v$. For this, it is enough by Lemma B.2.3 (iii) to check injectivity on points, and we obtain the desired statement from Lemma 4.1.11 with $X' = Y_v$, since the kernels of the maps in question are identified with $H^0(X, \mathcal{E}_w)$ and $H^0(Y_v, \mathcal{E}^v)$, respectively. This proves the first assertion. For convenience, we denote the induced subbundle of $\pi_*\mathcal{E}^v$ by $\mathcal{V}^v$.

Next, the image of this morphism is evidently contained in the open substack of $\mathcal{P}_{r,d_s}^k(X)$ satisfying $(I)$ of Definition 4.1.2, so in order to see we get an induced morphism

$$\mathcal{G}_{r,d,s}^{k,\text{EHT}}(X) \rightarrow \mathcal{G}_{r,d,s}^{k,\text{EHT}}(X),$$

it suffices to see that if the $k$th vanishing locus of

$$\pi_*\mathcal{E}_w \rightarrow \bigoplus_{v \in V(\Gamma)} \pi_*\mathcal{E}_w/\mathcal{V}_w$$

is equal to all of $S$, then the same is true of

$$\pi_*\mathcal{E}_w \rightarrow \bigoplus_{v \in V(\Gamma)} \pi_*\mathcal{E}^v/\mathcal{V}^v.$$ But because the second map factors through the first, this is immediate from Corollary B.3.5.

It remains to see that this induced morphism is an isomorphism, which amounts to the assertion that for a given $S$-valued collection of subbundles $\mathcal{V}^v \subseteq \pi_*\mathcal{E}^v$, there exists a unique collection of subbundles $\mathcal{Y}_w \subseteq \pi_*\mathcal{E}_w$ satisfying our vanishing condition for all $w$ and restricting to the given ones. But by definition the morphism

$$(4.2.1) \quad \pi_*\mathcal{E}_w \rightarrow \bigoplus_{v \in V(\Gamma)} \pi_*\mathcal{E}^v/\mathcal{V}^v$$

has $k$th vanishing locus equal to $S$, and by Lemma 4.1.6 its $(k+1)$st vanishing locus is empty, so Proposition B.3.4 implies that the kernel $\mathcal{K}_v$ is a rank-$k$ subbundle, which we claim is the desired $\mathcal{Y}_w$. But the restriction of $\mathcal{K}_v$ to $Y_v$ is contained in $\mathcal{V}^v$ by construction, and it follows from the injectivity of $\pi_*\mathcal{E}_w \rightarrow \pi_*\mathcal{E}^v$ on points together with Lemma B.2.3 (iii) and (iv) that in fact $\mathcal{K}_v$ restricts to $\mathcal{V}^v$. The same injectivity also implies that $\mathcal{Y}_w$ is uniquely determined by the condition that it restricts to $\mathcal{V}^v$. Finally, it follows that the morphism $\pi_*\mathcal{E}_w/\mathcal{Y}_w \rightarrow \pi_*\mathcal{E}^v/\mathcal{V}^v$ is injective, even after arbitrary base change. Thus, for any $w \in V(\Gamma)$, the last statement of Proposition B.3.2 together with the hypothesis that the $k$th vanishing locus of

$$\pi_*\mathcal{E}_w \rightarrow \bigoplus_{v \in V(\Gamma)} \pi_*\mathcal{E}^v/\mathcal{V}^v$$

is equal to all of $S$ implies that the same is true of

$$\pi_*\mathcal{E}_w \rightarrow \bigoplus_{v \in V(\Gamma)} \pi_*\mathcal{E}_w/\mathcal{Y}_w.$$
4.3. Comparison results. We now compare our linked linear series stacks to the stack of EHT limit series. We begin with the following obvious observation.

Proposition 4.3.1. There is a forgetful morphism
\[ G_{r,d,d_*}^{k,1}(X) \to P_{r,d_*}^k(X), \]
given by sending a tuple \((S, \mathcal{E}, (\mathcal{V}_w)_{w \in V(G_1)})\) to the tuple \(((\mathcal{E} \otimes \mathcal{O}_{w_0,w_v})|_{Y_v}, \mathcal{V}_{w_v}|_{Y_v})_{v \in V(\Gamma)}\), together with the natural gluing maps at the nodes induced by \(\mathcal{E}\).

The following definition will play an important role in our comparison results. The purpose of §5 below is to develop a robust criterion for when the conditions of the definition are satisfied.

Definition 4.3.2. A \(K\)-valued point \(((\mathcal{E}^v, \mathcal{V}^v)_v, (\varphi_v)_v)\) of \(G_{r,d,d_*}^{k,\text{EHT}}(X)\) is said to be constrained if there exist and \(w_1, \ldots, w_k \in V(G_{1\Gamma})\) (not necessarily distinct) and for \(i = 1, \ldots, k\) a vector \(v_i\) in the kernel of (4.1.4) (in index \(w_i\)) such that the following conditions are satisfied:

(I) for each \(w_i\), the kernel of (4.1.4) has dimension exactly \(k\);

(II) for each \(v \in V(\Gamma)\), the images of \(v_1, \ldots, v_k\) in \(V^v\) form a basis.

We then have:

Corollary 4.3.3. The constrained points form an open subset of \(G_{r,d,d_*}^{k,\text{EHT}}(X)\), and in the definition, the kernel of (4.1.4) may be replaced by the kernel of (4.1.3).

Proof. The latter statement is Lemma 4.1.10. We then conclude openness of condition (I) because the kernels have dimension at least \(k\) everywhere by Lemma 4.1.6, so condition (I) describes the complement of a closed substack by Proposition B.3.2. On the other hand, condition (II) is open by Nakayama’s lemma.

The main comparison theorem is the following:

Theorem 4.3.4. The morphism (4.3.1) factors through and surjects onto the locally closed substack \(G_{r,d,d_*}^{k,\text{EHT}}(X)\). Moreover, on the preimage of the refined locus \(G_{r,d,d_*}^{k,\text{EHT,ref}}(X)\) we have that (4.3.1) is an isomorphism.

Under the additional hypothesis of Situation 3.3.1, if we compose with the forgetful morphism of Corollary 3.4.2, the resulting morphism
\[ G_{r,d,d_*}^{k,\Gamma}(X, \theta)_* \to G_{r,d,d_*}^{k,\text{EHT}}(X), \]
is an isomorphism on the preimage of the constrained locus of \(G_{r,d,d_*}^{k,\text{EHT}}(X)\).

Proof. That (4.3.1) factors through \(G_{r,d,d_*}^{k,\text{EHT}}(X)\) follows immediately from the definitions together with Proposition B.3.4.

To check surjectivity, given a point \(((\mathcal{E}^v, \mathcal{V}^v)_v, (\varphi_v)_v)\) of \(G_{r,d,d_*}^{k,\text{EHT}}(X)\), we will construct a point of \(G_{r,d,d_*}^{k,1}(X)\) mapping to it as follows: let \(\mathcal{E}\) be the vector bundle of multidegree \(w_0\) determined by gluing (the appropriate twists of) the \(\mathcal{E}^v\) via the maps \(\varphi_v\); for \(w \in V(G_1)\), let \(\tilde{V}_w\) be the kernel of (4.1.3). Then, according to Lemma 4.1.6, we have that \(\tilde{V}_w\) has dimension at least \(k\) for all \(w\), and has dimension exactly \(k\) if \(w = w_v\) for some \(v \in V(\Gamma)\). The spaces \(\tilde{V}_w\) are visibly linked, so it is enough to show that we can produce a system of \(k\)-dimensional subspaces \(V_w \subseteq \tilde{V}_w\) which
remain linked. Because we will have \( \bar{V}_{w_v} = \bar{V}_w \) for all \( v \in V(\Gamma) \), it is enough to work with one pair \((v_1, v_2)\) of adjacent edges at a time, considering only \( w = (i_v) \) with \( i_v = d - rb \) for all \( v \neq v_1, v_2 \). We will prove the desired statement by showing that we can always reduce the dimension of some \( \bar{V}_w \) while maintaining linkage, until we reach the situation that all spaces have dimension \( k \).

Accordingly, with \((v_1, v_2)\) fixed, let \( e \in E(\Gamma) \) be the edge joining them, and take the \( w \) with \( i_{v_1} \) minimal such that \( \dim \bar{V}_w > k \). We will construct a subspace \( \bar{V}_w' \subseteq \bar{V}_w \) of dimension one less, and such that if \( w', w'' \) are the vertices adjacent to \( w \) in \( G_1 \), we still have \( \bar{V}_w' \) containing the images of \( \bar{V}_{w'} \) and \( \bar{V}_{w''} \). Iterating this process will thus yield the desired system of subspaces. There are two cases to consider. First, if \( \bar{V}_w \) contains a section \( s \) which is nonvanishing at \( \Delta'_e \), let \( \ell \subseteq \bar{E}_w|_{\Delta'_e} \) be the line generated by \( s \), and \( \bar{E}_w' \subseteq \bar{E}_w \) the kernel of the map \( \bar{E}_w \to (\bar{E}_w|_{\Delta'_e})/\ell \). Then set \( \bar{V}_w' \) to be the subspace of \( \bar{V}_w \) consisting of sections lying in \( \bar{E}_w' \). Otherwise, if \( \bar{V}_w \) consists entirely of sections vanishing at \( \Delta'_e \), suppose we have ordered \( w' \) and \( w'' \) so that the index \( i'_{v_1} \) for \( w' \) is less than \( i_{v_1} \). Let \( \bar{V}_w|_{v_1} \) (respectively, \( \bar{V}_w'|_{v_1} \)) denote the image of \( \bar{V}_w \) (respectively, \( \bar{V}_w' \)) under restriction to \( Y(e, v_i) \) for \( i = 1, 2 \).

Then, by minimality of \( i_{v_1} \), we see that we must have \( \dim \bar{V}_w|_{v_1} > \dim \bar{V}_w'|_{v_1} \), and in particular \( \bar{V}_w|_{v_1} \) strictly contains the image of \( \bar{V}_w'|_{v_1} \) under the natural inclusion; let \( V' \subseteq \bar{V}_w|_{v_1} \) be a codimension-1 subspace containing this image; then we may set \( \bar{V}_w' \) to be the preimage of \( V' \) under the restriction map. This completes the proof of surjectivity.

Next, to see that we get an isomorphism on the refined locus, suppose that \([(s^v, \mathcal{Y}^v)|_{v}, (\varphi_e)_v] \) is an \( S \)-valued point of \( G_{r,d,d}^{k,\text{EHT},\text{ref}}(X) \); we wish to show that there exists a unique \( S \)-valued point \([(s^v, \mathcal{Y}^v)_{w \in V(G_1)}], (\varphi_e)_v \) of \( G_{r,d,d}^{k,\text{I}}(X) \) mapping to it. By the last part of Lemma 4.1.6, we have that for every \( w \) in \( V(G_1) \), the kernel of (4.1.3) has dimension exactly \( k \). It follows from the definition of the stack \( G_{r,d,d}^{k,\text{EHT}}(X) \) together with Proposition B.3.4 that the kernel is a subbundle of rank \( k \), so we must have \( \mathcal{Y}_w \) equal to this kernel. This gives uniqueness, and also existence, since it is evident that the kernels of (4.1.3) as \( w \) varies must satisfy the required linkage condition.

For (4.3.2), if we have an \( S \)-valued point \([(s^v, \mathcal{Y}^v)|_{v}, (\varphi_e)_v] \) of \( G_{r,d,d}^{k,\text{EHT}}(X) \) contained in the constrained locus, we need to show that there exists a unique \( S \)-valued point \([(s^v, \mathcal{Y}^v)_{w \in V(G_1)}], (\varphi_e)_v \) of \( G_{r,d,d}^{k,\text{I}}(X, \theta_\ast) \) mapping to it. It is enough to prove this locally, so we reduce to the case that there is a fixed collection of \( v_i \) as in Definition 4.3.2 which works for every point of \( S \). For each \( i \), we first claim that the \( k \)th vanishing locus of (4.1.3) in multidegree \( w_i \) is equal to all of \( S \). The argument for this is a variant of the induction argument that (a) implies (b) in Lemma 4.1.6. Indeed, if we only have two components the statement is true tautologically, since necessarily \( w_i \in V(G_H) = V(G_1) \). If we have at least three components, choose a component \( Y_{v_1} \) meeting the rest of \( X \) at only a single node \( \Delta'_{e_1} \). Let \( X' \) be the complement of \( Y_{v_1} \) in \( X \), and let \( Y \) be the union of \( Y_{v_2} \) and \( Y_{v_1} \), where \( v_2 \) is the other edge adjacent to \( e_1 \). Denote by \( \Gamma' \) the dual graph of \( X' \). Restricting to \( X' \) and \( Y \), and letting \( w' \) and \( w_{12} \) be the multidegrees induced by \( w_i \), by the induction hypothesis we have that the \( k \)th vanishing loci of the maps

\[
\pi_w \mathcal{E}_w^v \to \bigoplus_{v \in V(\Gamma')} (\pi_w \mathcal{E}_w^v)/\mathcal{Y}^v
\]
and

\[ \pi_*E_{w|12} \rightarrow (\pi_*E^{v_1})/\mathcal{Y}^{v_1} \oplus (\pi_*E^{v_2})/\mathcal{Y}^{v_2} \]

both contain all of \( S \). However, the kernel of (4.1.3) is the precisely the fibered product of the above two kernels over \( \mathcal{Y}^{v_2} \). Since \( \mathcal{Y}^{v_2} \) has rank \( k \), we conclude from Corollary B.3.6 that the \( k \)th vanishing locus of (4.1.3) is all of \( S \), as claimed. On the other hand, the definition of constrained point implies that the \((k+1)\)st vanishing locus of (4.1.3) (still in multidegree \( w_i \)) is empty on the set of constrained points, so from Proposition B.3.4 we conclude that the kernel of (4.1.3) in multidegree \( w_i \) is a subbundle of rank \( k \). Thus if the desired lift exists, we must have \( \mathcal{Y}_{w_i} \) equal to the kernel of (4.1.3).

Next, working locally around a given point \( y \in S \) and letting \( v_i \) be as in the definition of constrained, we may lift each \( v_i \) to a \( \tilde{v}_i \in \mathcal{Y}_{w_i} \), and we claim that for any \( w \in V(G_{11}) \), locally around \( y \) we have that \( \sum f_{w,w}\tilde{v}_i \) is a rank-\( k \) subbundle of \( \pi_*E_w \). According to Lemma B.2.3 (iii), it is enough to see that the \( f_{w,w}\tilde{v}_i \) are linearly independent in \( \pi_*E_w|_y \). This follows from the same argument as Lemma 3.4.14. Thus, if we set \( \mathcal{Y}_w = \sum f_{w,w}\tilde{v}_i \), we see that we obtain the desired point of \( G_{r,d,d,x}(X,\theta_*) \). Indeed, because we are working with a fixed choice of basis vectors, the linkage condition is immediate from the fact that for two paths \( P, P' \) with the same start and endpoints, the maps \( f_P \) and \( f_{P'} \) are related by scalars. In addition, since our second description of each \( \mathcal{Y}_{w_i} \) must be contained in the first, Lemma B.2.3 (iv) implies that they agree. Finally, since any choice of \( \mathcal{Y}_w \) must contain the chosen one, again using Lemma B.2.3 (iv) we conclude the desired uniqueness of our lift.

\[ \square \]

**Example 4.3.5.** We give a simple example showing that with three or more components, the seemingly rather restrictive “constrained” hypothesis is indeed necessary. Specifically, we see that the forgetful map (4.3.2) is not an isomorphism even in the rank-1 case on the locus of refined and simple limit series.

Indeed, consider a curve with three rational components \( Y_1, Y_2, Y_3 \), with \( Y_1 \) glued to \( Y_2 \) at a node \( P_1 \), and \( Y_2 \) glued to \( Y_3 \) at a node \( P_2 \). Thus, in this case all the line bundles are uniquely determined. We consider simple linked \( g^1_1 \)'s generated by a section \( s_1 \) in multidegree \((1,0,1)\) and a section \( s_2 \) in multidegree \((0,2,0)\). Recall that in order for \( s_1 \) and \( s_2 \) to serve as generators for a simple point, neither of them can vanish uniformly on any of the \( Y_i \). The choices of \( s_1 \) may be described as follows: if \( x_1 \) denotes the section (unique up to scaling) which is nonzero on \( Y_1 \) but identically zero on \( Y_2 \) and \( Y_3 \), and \( x_2 \) denotes the section nonzero on \( Y_3 \) but identically zero on \( Y_1 \) and \( Y_2 \), then \( s_1 \) is unique up to scaling and adding multiples of \( x_1 \) and \( x_3 \). On the other hand, for a fixed choice of \( s_2 \), the image of \( s_2 \) in multidegree \((1,0,1)\) vanishes on \( Y_2 \), and is thus of the form \( c_1x_1 + c_3x_3 \) for some non-zero \( c_1, c_3 \). Thus, we see that for a fixed choice of \( s_2 \), modifying \( s_1 \) by multiples of \( x_1 \) and \( x_3 \) changes the resulting space in multidegree \((1,0,1)\), and we obtain a one-parameter family of 2-dimensional subspaces in this way. On the other hand, such modifications do not affect the subspaces in any other multidegree: the image of \( s_1 \) in multidegree \((0,2,0)\) vanishes uniformly on \( Y_1 \) and \( Y_3 \), so it is enough to check the multidegrees \((1,1,0)\) and \((0,1,1)\), and we see that modifying \( s_1 \) by multiples of \( x_1 \) and \( x_3 \) changes the image in multidegree \((1,1,0)\) by a multiple of (the image of) \( x_1 \). But the image of \( s_2 \) in multidegree \((1,1,0)\) is also a multiple of the image of \( x_1 \), so the resulting 2-dimensional subspace is unchanged. The same
holds for \((0,1,1)\), and we see that in this case the comparison morphism \((4.3.2)\) contracts curves.

5. Limit series on chains of curves

As we have discussed in Example 4.3.5, unlike the case of two components treated in [Oss06a] our comparison morphism from type II linked linear series to EHT limit linear series is not in general an isomorphism over the refined locus. Thus, for our applications it becomes very important to identify loci over which the comparison morphism is an isomorphism. It turns out that for chains of curves, there is a natural open subset, which we refer to as the “chain-adaptable” locus, over which we obtain the desired behavior. This locus contains all examples considered by Teixidor i Bigas in the existence arguments of [Tei91], [Tei05], [Tei04].

As in the previous section, here we assume throughout that \(B = \text{Spec } F\), and \(X\) is a projective curve over \(B\).

5.1. Pairs of vanishing sequences on smooth curves. We begin with some observations on smooth curves with pairs of marked points.

If \((\e,V)\) is a \(g_d^k\) on a smooth projective curve, then we denote the vanishing sequence at a point \(P\) by \(a_1^{(\e,V)}(P), \ldots, a_k^{(\e,V)}(P)\).

Definition 5.1.1. Let \(X\) be a smooth projective curve over \(\text{Spec } F\), and \((\e,V)\) a \(g_{r,d}^k\) on \(X\). Given points \(P, Q \in X(F)\), we say that a basis \(s_1, \ldots, s_k \in V\) is \((P,Q)\)-adapted if \(\text{ord}_P s_i = a_i^{(\e,V)}(P)\) and \(\text{ord}_Q s_i = a_{k+1-i}^{(\e,V)}(Q)\) for \(i = 1, \ldots, k\). We say that \((\e,V)\) is \((P,Q)\)-adaptable if there exists a \((P,Q)\)-adapted basis of \(V\).

The following lemma will be useful:

Lemma 5.1.2. Suppose \(X\) is smooth over \(\text{Spec } F\), and \((\e,V)\) is a \(g_{r,d}^k\) on \(X\).

Given \(P \in X(F)\), let \(a_1, \ldots, a_k\) be the vanishing sequence of \((\e,V)\) at \(P\), and \(s_1, \ldots, s_k \in V\) such that the order of vanishing of \(s_i\) at \(P\) is \(a_i\). Then the following are equivalent:

\(\begin{align*}
(a) & \text{ the } s_i \text{ form a basis for } V; \\
(b) & \text{ for each } a, \text{ the set of } s_i \text{ vanishing to order } a \text{ at } P \text{ is a basis for the fiber } V(-aP)/V(-(a+1)P); \\
(c) & \text{ for each } a, \text{ the set of } s_i \text{ vanishing to order } a \text{ at } P \text{ is linearly independent in the fiber } V(-aP)/V(-(a+1)P); \\
(d) & \text{ for each } a, \text{ the set of } s_i \text{ vanishing to order } a \text{ at } P \text{ spans the fiber } V(-aP)/V(-(a+1)P).
\end{align*}\)

Proof. By the definition of vanishing sequence, we have that (b), (c), and (d) are equivalent. On the other hand, given (c), and a nonzero linear combination \(s = \sum c_is_i\), if \(i_{\text{min}}\) is minimal with \(c_{i_{\text{min}}} \neq 0\), we see that \(s\) must vanish to order precisely \(a_{i_{\text{min}}}\) at \(P\), and in particular is not the zero section. Thus, (c) implies (a). Finally, suppose we have (a). We prove (d) by induction on \(a\). The statement is trivial for \(a = 0\), so suppose \(a\) is general, and we know the desired statement for all orders strictly less than \(a\). Then again using the definition of vanishing sequence, the \(s_i\) vanishing to orders \(a' < a\) form bases of \(V(-a'P)/V(-(a'+1)P)\), so no non-zero linear combination of them is in \(V(-aP)\). Since the \(s_i\) are a basis of \(V\), linear combinations of them span \(V(-aP)/V(-(a+1)P)\), and we conclude that it suffices to take linear combinations of \(s_i\) vanishing to order exactly \(a\), as desired. \(\square\)
Proposition 5.1.3. Suppose $X$ is smooth over Spec $F$, and we are given $P,Q \in X(F)$. Given also a pair $(\mathcal{E}, V)$, for any $a,b \geq 0$ we have

$$(5.1.1) \quad \dim V(-aP - bQ) \geq \# \{ i : a_{i}^{(\mathcal{E}, V)}(P) \geq a \text{ and } a_{k+1-i}^{(\mathcal{E}, V)}(Q) \geq b \}.$$ 

Moreover, $(\mathcal{E}, V)$ is $(P,Q)$-adaptable if and only if for all $a,b \geq 0$ we have that $(5.1.1)$ is satisfied with equality. In particular, the $(P,Q)$-adaptable pairs are open in the moduli stack of all pairs with given vanishing sequences at $P$ and $Q$.

Proof. We first verify that the asserted inequality holds in general. The main observation is that if the right-hand side of $(5.1.1)$ is positive, then we have

$$\{ i : a_{i}^{(\mathcal{E}, V)}(Q) < b \} \subseteq \{ i : a_{i}^{(\mathcal{E}, V)}(P) \geq a \}.$$ 

Indeed, positivity implies there exists $i_0$ such that $a_{i_0}^{(\mathcal{E}, V)}(P) \geq a$ and $a_{k+1-i_0}^{(\mathcal{E}, V)}(Q) \geq b$. Then if $a_{k+1-i}^{(\mathcal{E}, V)}(Q) < b$, we have $i > i_0$, so we get the desired statement. Now, if the right-hand side of $(5.1.1)$ is zero, there is nothing to prove. Otherwise, because $V(-aP - bQ) = V(-aP) \cap V(-bQ)$, we have

$$\dim V(-aP - bQ) \geq \dim V(-aP) + \dim V(-bQ) - k$$
$$= \# \{ i : a_{i}^{(\mathcal{E}, V)}(P) \geq a \} + \# \{ i : a_{i}^{(\mathcal{E}, V)}(Q) \geq b \} - k$$
$$= \# \{ i : a_{i}^{(\mathcal{E}, V)}(P) \geq a \} - \# \{ i : a_{i}^{(\mathcal{E}, V)}(Q) < b \}$$
$$= \# \{ i : a_{i}^{(\mathcal{E}, V)}(P) \geq a \text{ and } a_{k+1-i}^{(\mathcal{E}, V)}(Q) \geq b \},$$

where the last equality follows from our observation above.

It remains to check that equality in $(5.1.1)$ is equivalent to $(P,Q)$-adaptability. First suppose that we have a $(P,Q)$-adapted basis $s_1, \ldots, s_k$. Then by Lemma 5.1.2, there cannot be any cancellation in orders of vanishing of linear combinations of the $s_i$, so an element of $V(-aP - bQ)$ must be a linear combination of basis vectors $s_i$ with $i$ satisfying $a_{i}^{(\mathcal{E}, V)} \geq a$ and $a_{k+1-i}^{(\mathcal{E}, V)} \geq b$. We thus conclude equality in $(5.1.1)$.

Conversely, if we have equality in $(5.1.1)$, we wish to construct a $(P,Q)$-adapted basis $s_1, \ldots, s_k$. We proceed via a double downward induction on pairs $(a,b)$, with $a, b \geq 0$, and bounded above by $a_{k}^{(\mathcal{E}, V)}(P)$ and $a_{k}^{(\mathcal{E}, V)}(Q) + 1$, respectively. Given $(a,b)$, denote by $I_{(a,b)}$ the set of $i$ such that $a_{i}^{(\mathcal{E}, V)}(P) \geq a$ and $a_{k+1-i}^{(\mathcal{E}, V)}(Q) \geq b$. For each such pair $(a,b)$, and every $i$ with either $a_{i}^{(\mathcal{E}, V)}(P) > a$ or $a_{i}^{(\mathcal{E}, V)}(P) = a$ and $a_{k+1-i}^{(\mathcal{E}, V)}(Q) \geq b$, we will construct $s_i \in V$ having the orders of vanishing at $P$ and $Q$ required for a $(P,Q)$-adapted basis, and satisfying further the condition that for all $(a', b')$ with either $a' > a$ or $a' = a$ and $b' \geq b$, the set of $s_i$ with $i \in I_{(a', b')}$ is a basis of $V(-a'P - b'Q)$. If we have constructed such $s_i$ for the pair $(0,0)$, then in particular we have a $(P,Q)$-adapted basis, as desired. We start the induction with $(a,b)$ at their maximal values, so that $V(-aP - bQ) = 0$, and the desired conditions are tautologically satisfied. For each value of $a$, we decrease $b$ by 1 until we reach $b = 0$. We then decrease $a$ by 1, and reset $b$ to its maximal value. At each step, we use the $s_i$ we have constructed in the previous step, adding new ones as necessary.

In the case that $a$ remains constant and $b$ has decreased, by hypothesis we already have $s_i$ for all $i$ with either $a_{i}^{(\mathcal{E}, V)}(P) > a$ or $a_{i}^{(\mathcal{E}, V)}(P) = a$ and $a_{k+1-i}^{(\mathcal{E}, V)}(Q) > b$, and satisfying the desired basis condition for $(a', b')$ with $a' > a$ or $a' = a$ and $b' > b$. Thus, we need only extend the $s_i$ to include $i$ with $a_{i}^{(\mathcal{E}, V)}(P) = a$ and
a_k^{(\ell,V)}(Q) = b. Our claim is that the hypothesis that we have equality in (5.1.1)
implies that the number of such i is precisely equal to
\[ \dim V(-aP - bQ)/\text{span}(V(-aP - (b+1)Q), V(-(a+1)P - bQ)), \]
in which case we may choose the new s_i to induce a basis for this space. But
\[ \dim \text{span}(V(-aP - (b+1)Q), V(-(a+1)P - bQ)) = \dim V(-aP - (b+1)Q) + \dim V(-(a+1)P - bQ) - \dim V(-(a+1)P - (b+1)Q), \]
so the claim follows easily by applying equality in (5.1.1) to (a, b), (a, b+1), (a+1, b)
and (a+1, b+1). We then check the desired condition on pairs (a', b') as follows: if a' = a
and b' = b, the pairs (a, b+1) and (a+1, b) from the previously chosen s_i with i \in \hat{I}_{(a,b+1)} \cup \hat{I}_{(a+1,b)}
form a basis for \text{span}(V(-aP - (b+1)Q), V(-(a+1)P - bQ), and thus together with
the new s_i we obtain a basis for V(-aP - bQ), as desired. On the other hand, if
a' > a or a' = a and b' > b, then the new s_i are irrelevant, and we have the desired
condition from the induction hypothesis. We thus have constructed the desired s_i
in this case.

In the case that a has decreased and b is set to a_k^{(\ell,V)}(Q) + 1, there are no new
allowable values of i, so it suffices to observe that the previously chosen s_i satisfy
the desired condition on pairs (a', b') with either a' > a or a' = a and b' \geq b. But
the latter possibility is vacuous, since in this case V(-aP - bQ) = 0 and there are
no allowable indices i. On the other hand, if a' > a we are in the cases handled by
the induction hypothesis. Thus, the induction hypothesis for (a + 1, 0) implies the
desired statement for (a, a_k^{(\ell,V)}(Q) + 1), as we wished to prove. □

5.2. Chains of curves. We now return to studying reducible curves, and more
particularly, chains of smooth projective curves.

Definition 5.2.1. Let X be a curve consisting of a chain of smooth projective
curves Y_1, \ldots, Y_n over Spec F, with P_i, Q_i \in Y_i(F) for each i, and the point Q_i
on Y_i glued to P_{i+1} on Y_{i+1}. Then a refined Eisenbud-Harris-Teixidor limit series on
X is chain-adaptable if the pair induced by restriction to each Y_i (i = 2, \ldots, n-1)
is (P_i, Q_i)-adaptable.

Since imposing particular vanishing sequences at the nodes is open on the refined
locus of Eisenbud-Harris-Teixidor limit linear series, Proposition 5.1.3 implies:

Proposition 5.2.2. The locus of chain-adaptable EHT limit series is open in
\( \mathcal{G}^{k,\text{EHT}}_{r,d,d^*}(X) \).

More substantively, we have the following result.

Proposition 5.2.3. If \((\ell^i, V^i)_{i=1,\ldots,n}, (\varphi_i)_{i=1,\ldots,n-1}\) is a K-valued chain-adaptable
EHT limit series on X, then there exist bases s_i^j of V^i for each i = 1, \ldots, n,
and permutations s_i \in S_k for i = 1, \ldots, n-1, such that:

(i) for each i = 2, \ldots, n-1, the s_i^j form a \((P_i, Q_i)\)-adapted basis of V^i;
(ii) for each i = 1, \ldots, n-1, and each j = 1, \ldots, k, we have
\[ \text{ord}_{P_i} s_{\sigma_{i}(j)}^i + \text{ord}_{P_{i+1}} s_{\sigma_{i}(j)+1}^i = b, \]
and s_j^i glues to s_{\sigma_{i}(j)}^{i+1} under \varphi_i.

Proof. The argument is by induction, but requires some additional notation. For
each i, let a^i be the vanishing sequence of V^i at P_i, and let b^i be the vanishing
sequence at $Q_i$, so that $b^i_j = b - a^{i+1}_{k+1-j}$ for $i = 1, \ldots, n-1$ and $j = 1, \ldots, k$. For $i = 1, \ldots, n-1$, let $r^i_1, \ldots, r^i_\ell_i$ denote the number of repetitions in the sequence $a^{i+1}$, so that $\sum_{j=1}^{\ell_i} r^i_j = k$. It will also be convenient to write $R^i_j := \sum_{\ell=1}^{j-1} r^i_{\ell}$ (with $R^i_1 = 0$), so that the distinct values of the sequence $a^{i+1}$ are given by $a^{i+1}_{R^{i+1}_1}, \ldots, a^{i+1}_{R^{i+1}_{\ell_i}+1}$, and the distinct values of $b_i$ are given by $b^i_{k-R^{i}_1}, \ldots, b^i_{k-R^{i}_\ell_i}$. For $i = 1, \ldots, n-1$ and $j = 1, \ldots, \ell_i$, denote by $W^i_j$ the $r^i_j$-dimensional subspace of the fiber $\mathcal{E}(b - a^{i+1}_{R^{i+1}_{\ell_i}+1})|Q_i$, induced by $V^i_j$. Thus, given a $(P_i, Q_i)$-adapted basis $s^i$, we have that the restrictions of $s^i_{R^{i+1}_1}, \ldots, s^i_{R^{i+1}_{\ell_i}+r^i_j}$ to $Q_i$ form a basis of $W^i_j$.

We argue by induction on $i$, showing that for each $i$, there exist bases of $V^1, \ldots, V^i$ as in the statement, and that furthermore we have the following flexibility: for each $j = 1, \ldots, \ell_i$ there exists an ordering $\alpha_j$ of the $s^i_{R^{i+1}_1}, \ldots, s^i_{R^{i+1}_{\ell_i}+r^i_j}$ so that any basis $e^i_1, \ldots, e^i_{r^i_j}$ of $W^i_j$ obtained from the restrictions to $Q_i$ of $s^i_{R^{i+1}_1}, \ldots, s^i_{R^{i+1}_{\ell_i}+r^i_j}$ via a change of basis which is upper triangular under the ordering $\alpha_j$, without modifying the permutations $\sigma_j$ we may modify our choices of the $s^i_j$ for $i' \leq i$ so that the restriction of $s^i_{R^{i+1}_j}$ to $Q_i$ is $e^i_\ell_j(e)$ for $\ell = 1, \ldots, r^i_j$. That is, we claim that under a suitable ordering, we can realize arbitrary upper triangular change of bases to the images of the $s^i_j$ in the spaces $W^i_j$. In the case $i = 1$, there is nothing to show, as in fact the $s^1_j$ may be chosen to give arbitrary bases of the $W^1_j$, so any ordering suffices. Suppose now that the desired statement is true for $i-1$, and we wish to show it for $i$. For $j = 1, \ldots, \ell_{i-1}$, let $\alpha_j$ be the ordering on $s^i_{R^{i+1}_1}, \ldots, s^i_{R^{i+1}_{\ell_j+1}}$ supplied by induction.

In order to prove the inductive statement, we first consider all possibilities for $(P_i, Q_i)$-adapted bases of $V^i$. If $\tilde{s}^i_1, \ldots, \tilde{s}^i_k$ is one choice of $(P_i, Q_i)$-adapted basis, then in order to maintain the required vanishing orders at both $P_i$ and $Q_i$, any change of basis matrix is required to be upper and lower block triangular, with block sizes determined by $r^i_{1-1}, \ldots, r^i_{\ell_i-1}$ and $r^i_1, \ldots, r^i_\ell_i$. Considering first the fiber at $P_i$, we see that for any $W^{i-1}_j$, the allowable changes of basis for the images of $\tilde{s}^i_{R^{i-1}_j+1}, \ldots, \tilde{s}^i_{R^{i-1}_{\ell_j}+r^i_{j-1}}$ are block triangular. On the other hand, by the induction hypothesis we can also achieve upper triangular changes of basis for the images in $W^{i-1}_j$ of $\tilde{s}^{i-1}_{R^{i-1}_j+1}, \ldots, \tilde{s}^{i-1}_{R^{i-1}_{\ell_j}+r^i_{j-1}}$, so by Lemma 5.2.4 below we conclude that there exists some choices of bases $s^1, \ldots, s^i$ and permutations $\sigma_1, \ldots, \sigma_{i-1}$ as desired for the statement. It remains to prove the stronger inductive statement on the flexibility for restrictions to $Q_i$.

For each $j$, we thus describe an ordering $\alpha_j$ on $s^i_{R^{i+1}_j}, \ldots, s^i_{R^{i+1}_{\ell_i}+r^i_j}$ with the desired property. First, the ordering respects vanishing order at $P_i$, so that if $a^i_m < a^i_{m'}$ for $m, m' \in [R^i_j + 1, \ldots, R^i_j + r^i_j]$, then $\alpha_j$ places $s^i_m$ ahead of $s^i_{m'}$. On the other hand, if $a^i_m = a^i_{m'}$, then the ordering is determined by the appropriate $\alpha^i_{j'}$ and $\sigma_{i-1}$, as follows: since $a^i_m = a^i_{m'}$, we have $m, m' \in [R^{i-1}_j + 1, \ldots, R^{i-1}_{j} + r^{i-1}_{j-1}]$ for some $j'$. Then $\alpha_j$ places $s^i_m$ ahead of $s^i_{m'}$ if $s^{i-1}_{j'}$ is ahead of $s^{i-1}_{\sigma_{i-1}(m')}$ under $\alpha_{j'}$ (note that $\sigma_{i-1}$ necessarily preserves vanishing order at $Q_{i-1}$, so this makes sense). It thus remains to verify that under the ordering $\alpha_j$, we can realize any upper triangular change of basis for the images of $s^i_{R^{i+1}_j}, \ldots, s^i_{R^{i+1}_{\ell_i}+r^i_j}$ in $W^i_j$. The first observation is
that by construction, such changes of basis preserve orders of vanishing at both $P_i$ and $Q_i$, so they certainly maintain the condition of being a $(P_i, Q_i)$-adapated basis. Now, given such a change of basis, we need to verify that we can also modify the choices of $s^1, \ldots, s^{i-1}$ to maintain the gluing condition at each node. But again using the construction of the ordering, for any $j'$ such that $R_{j'}^{-1} + 1, \ldots, R_{j'}^{-1} + r_{j'}^{-1}$ overlaps with $R_{j'}^{-1} + 1, \ldots, R_{j'}^{-1} + r_{j'}^{-1}$, a change of basis of $s^{1}_{R_{1}^{-1}+1}, \ldots, s^{1}_{R_{1}^{-1}+r_{1}}$ which is upper triangular with respect to $\alpha_j$ induces in $W_{j'}^{-1}$ a change of basis which is upper triangular with respect to $\alpha'_j$ (taking into account also the reordering dictated by $\sigma_{i-1}$). Note here that although the change of basis also involves $s^m_m$ with $m > R_{j'}^{-1} + r_{j'}^{-1}$, this does not affect the image in $W_{j'}^{-1}$ because the vanishing order at $P_i$ is strictly greater. Thus, the induction hypothesis allows us modify the $s^1, \ldots, s^{i-1}$ to achieve the desired change of basis, and we conclude the statement of the proposition. \( \square \)

The following lemma is surely a standard fact from linear algebra, but for the sake of completeness we include a brief proof.

**Lemma 5.2.4.** Let $V$ be a $d$-dimensional vector space, and $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_d$ and $W_0 \subseteq W_1 \subseteq \cdots \subseteq W_d$ complete flags in $V$. Then there exists a basis $v_1, \ldots, v_d$ of $V$ such that every $U_i$ and every $W_i$ is the span of some subset of the $v_j$.

**Proof.** The proof is by induction on $d$, with the case $d = 1$ being trivial. Set $V' = U_{d-1}$. Then the chain of subspaces

$$W_0 \cap V' \subseteq W_1 \cap V' \subseteq \cdots W_d \cap V'$$

still has every quotient with dimension at most 1, and there must be precisely one index $i_0 \leq d - 1$ such that $W_{i_0} \cap V' = W_{i_0+1} \cap V'$. Further, we have $W_i \subseteq V'$ if and only if $i \leq i_0$. We thus obtain new complete flags $U'_i$ and $W'_i$ in $V'$ by setting $U'_i = U_i$ for $i = 0, \ldots, d - 1$, setting $W'_i = W_i \cap V'$ for $i = 0, \ldots, i_0$ and $W'_i = W_{i+1} \cap V'$ for $i = i_0 + 1, \ldots, d - 1$. According to the induction hypothesis, we obtain a basis $b'_1, \ldots, b'_{d-1}$ of $V'$ such that every $W'_i$ and $U'_i$ is the span of some subset of the $b'_j$. It follows that $U_i$ has the property for all $i < d$, as does $W'_i$ for $i \leq i_0$. Finally, set $b_i = b'_i$ for $i = 1, \ldots, d - 1$, and set $b_d$ to be any vector in $W_{i_0+1} \cap V'$; we see easily that the $b_i$ form a basis, and that $W_i = \text{span}(W'_i, b_d)$ for all $i > i_0$, giving us the desired property. \( \square \)

**Remark 5.2.5.** As mentioned in the proof, the conditions of the statement of Proposition 5.2.3 place strong limitations on the permutations $\sigma_i$: specifically, they imply that if $\sigma_i(j) \neq j$, we must still have $\text{ord}_{\sigma_{i+1}} s^{i+1}_{\sigma_i(j)} = \text{ord}_{\sigma_{i+1}} s^{i+1}_j = b$. Nonetheless, it is not difficult to find examples for which non-trivial permutations are necessary in order to find bases of the desired form.

In the inductive statement in the proof, it is precisely due to the necessity of these permutations that we are required to introduce the orderings with respect to which we can take upper triangular changes of basis. Thus, consideration of some such orderings is necessary. On the other hand, we are in fact being unnecessarily restrictive by looking only at upper triangular matrices. The natural level of flexibility is expressed in terms of block upper triangular matrices, or equivalently compatibility with partial flags. However, by working with the coarser statement we simplify notation considerably.

**Corollary 5.2.6.** A chain-adaptable EHT limit linear series on $X$ is constrained.
Proof. Let \(((\mathcal{E}_i^i, V_i^i)_{i=1,\ldots,n}, (\varphi_i)_{i=1,\ldots,n-1})\) be a chain-adaptable EHT limit linear series on \(X\). Then our main claim is that for any \(w \in V(G_{11})\) such that \(V_i^i(w) \neq 0\) for each \(i\), the kernel of (4.1.4) has dimension exactly \(k\).

For \(i = 1, \ldots, n\), following Notation 4.1.8, write \(D_{w,i} = a_i P_i + b_i Q_i\) for some \(a_i, b_i\); thus, we have by definition that \(b_i + a_{i+1} = b\) for \(i = 1, \ldots, n - 1\). Also, denote by \(a^{'}\) the vanishing sequence of \(V_i^i\) at \(Q_i\), so that the vanishing sequence of \(V_i^{i+1}\) at \(P_i^{i+1}\) is given by \(b - a_k^{i+1}, \ldots, b - a_1^{i+1}\). For each \(i\) from 1 to \(n - 1\), choose \(\ell_i\) minimal with \(a_\ell^{i+1} \geq b_i\) and and \(m_i\) maximal with \(a_{m_i}^{i+1} \leq b_i\). Then we have

\[
\dim V_i^i(w) = \#\{j : a_j^{i+1} \geq b_i\} = k + 1 - \ell_i, \quad \text{and}
\dim V^n(w) = \#\{j : b - a_{j-1}^{n-1} \geq a_n = b - b_{n-1}\} = m_{n-1}.
\]

By Proposition 5.1.3, for each \(i\) with \(1 < i < n\), we have

\[
\dim V_i^i(w) = \#\{j : b - a_{j-1}^{i-1} \geq a_i, a_j^{i+1} \geq b_i\} = \#\{j : a_{j-1}^{i-1} \leq b_{i-1}, a_j^{i+1} \geq b_i\} = \max\{0, m_{i-1} + 1 - \ell_i\} = m_{i-1} + 1 - \ell_i,
\]

where the last equality holds because of the hypothesis that \(V_i^i(w) \neq 0\). Adding these up, we find

\[
\sum_{i=1}^{n} \dim V_i^i(w) = k + \sum_{i=1}^{n-1} (m_i + 1 - \ell_i).
\]

At the same time, at a given node \(P_i\), we have that \(V_i^i(w)\) and \(V_i^{i+1}(w)\) each have restriction to \(P_i\) of dimension \(m_i + 1 - \ell_i\), so the rank of (4.1.4) is at least (in fact, exactly)

\[
\sum_{i=1}^{n-1} (m_i + 1 - \ell_i),
\]

and we conclude that the kernel has dimension (at most, hence equal to) \(k\), as claimed.

Now, with notation as in the statement of Proposition 5.2.3, we see that for \(i = 1, \ldots, k\), taking \(s_i^{1, \ldots, m_i, \ldots, m_i}\), \(s_i^{2, \ldots, m_i, \ldots, m_i}\), \(s_i^{3, \ldots, m_i, \ldots, m_i}\), \(\ldots\), \(s_i^{n-1, \ldots, m_i, \ldots, m_i}\), \(s_i^n, \ldots, s_i^n\), we obtain a multidegree \(w_i \in V(G_{11})\) and a vector \(v_i\) in the kernel of the corresponding map (4.1.4). Moreover, by definition for each \(j = 1, \ldots, n\) the images of the \(v_i\) in \(V_j\) are simply a permutation of \(s_j^{1, \ldots, m_i, \ldots, m_i}\), so form a basis for \(V_j\). We thus conclude that the given limit linear series is constrained.

Combining Theorem 4.3.4 with Corollary 5.2.6, we obtain the following comparison result, which may be viewed as the main result of the present paper.

**Corollary 5.2.7.** In Situation 3.3.1, the forgetful morphism

\[
G_{r, d, d^*}^{k, \text{EHT}}(X, \theta_*) \to G_{r, d, d^*}^{k, \text{EHT}}(X)
\]

is an isomorphism on the preimage of the chain-adaptable locus.

To illustrate how Corollary 5.2.7 may be applied, if we drop the hypothesis that \(B\) is a point, and use Theorem 3.4.7, we conclude the following smoothing result.
Corollary 5.2.8. In Situation 3.3.1, if $\Gamma$ is a chain, and for some fiber $X_0$ the space $\mathcal{G}^{\text{EHT}}_{r,d,*}(X_0)$ has dimension exactly $\rho - 1$ at a chain-adaptable point $z$, then $\mathcal{G}^{\text{II}}_{r,d,*}(X/B, \mathcal{O}_*)$ is universally open over $B$ at $z$, and has pure fiber dimension $\rho - 1$ in a neighborhood of $z$.

The above result is uninteresting in and of itself, since it is weaker than the corresponding result proved by Teixidor i Bigas in Theorem 2.6 of [Tei91], using an adaptation of the construction of Eisenbud and Harris. However, Corollary 5.2.7 is the key ingredient in [OT14b], in which we extend Corollary 5.2.8 to the case of special determinants. By analyzing type II linked linear series in this context, we prove new smoothing results for the modified expected dimensions arising in that case, and as a consequence obtain new results on existence of vector bundles with sections on smooth curves. This extended version of Corollary 5.2.8 should be seen as the natural culmination of the machinery developed in the present paper.

6. Complementary results

In this section, we discuss several complementary results to our main theorems, treating the fixed determinant case, as well as questions on variation of the parameter $b$, specialization in one-dimensional families, and stability.

6.1. The fixed determinant case. We now describe the case of fixed determinant. Our results hold for arbitrary determinant line bundle $\mathcal{L}$, with everything the same as in the varying determinant case except that the dimension $\rho - 1$ is replaced by $\rho - g$. The dimension lower bounds are not optimal in the case that $\mathcal{L}$ is special, but the foundational results we develop here nonetheless play a fundamental role in the generalization to special determinants presented in [OT14b].

We first fix notation for moduli spaces of vector bundles with specified determinant. It will often be convenient to specify the determinant in a given predetermined multidegree (for instance, in the case of canonical determinant), which may or may not agree with the parity possibilities for the degrees of the vector bundles under consideration, so we allow the determinant multidegree to differ from our fixed $w_0$. Also, for technical reasons, it is convenient to work with line bundles $\mathcal{L}$ not necessarily defined over the original base $B$ (see Proposition 6.2 of [OT14b]). We will want to have separate notation for twisting line bundles in the rank-1 case, which we set as follows:

Notation 6.1.1. Given $w, w' \in \mathbb{Z}^{V(\Gamma)}$ both lying in the affine hyperplane consisting of vectors summing to $d$, let $\mathcal{O}_{w,w}'$ be the line bundle defined as in Notation 3.2.2, if applied to the case $r = 1$.

Note that when $r = 1$, the hyperplane in Notation 6.1.1 is the same as $V(G_{\text{II}})$. Although Notation 3.2.2 applies a priori only to vertices in $V(G_{\text{I}})$, in fact we can between any $w$ and $w'$ by twisting with a sequence of bundles $\mathcal{O}_{e,v}$, and we express the construction this way in order to avoid imposing the additional hypothesis required for the type II construction.

Notation 6.1.2. Given a multidegree $w_0$ on $\Gamma$, a $B$-scheme $B'$, and a line bundle $\mathcal{L}$ on $X_B := X \times_B B'$ having multidegree $w$ with total degree equal to that of $w_0$, let $\mathcal{M}_{r,w_0,\mathcal{L}}(X_B'/B')$ be the moduli stack of vector bundles $\mathcal{E}$ on $X_B'$ of multidegree $w_0$, together with an isomorphism $(\det \mathcal{E}) \otimes \mathcal{O}_{w_0,w}' \cong \mathcal{L}$.
Given also $d_\bullet$, let $\mathcal{M}_{r,w_0,\mathcal{L},d_\bullet}(X_{B'}/B')$ be the open substack of $\mathcal{M}_{r,w_0,\mathcal{L}}(X_{B'}/B')$ for which the underlying vector bundles lies in $\mathcal{M}_{r,w_0,\mathcal{L}}(X/B)$.

Thus, $\mathcal{M}_{r,w_0,\mathcal{L}}(X_{B'}/B')$ is the (2-)fibered product of $\mathcal{M}_{r,w_0}(X/B)$ with $B'$ over $\mathcal{P}ic^w(X/B)$, with the first map given by taking determinant and twisting by $\psi'$, and the second map being the one induced by $\mathcal{L}$, and similarly for $\mathcal{M}_{r,w_0,\mathcal{L},d_\bullet}(X_{B'}/B')$ and $\mathcal{M}_{r,w_0,d_\bullet}(X/B)$. These stacks are algebraic, and smooth over $B'$ of relative dimension $(r^2-1)(g-1)$.

We can thus make the following definitions.

**Definition 6.1.3.** Given a line bundle $\mathcal{L}$ of degree $d$ and multidegree $w$ on $\Gamma$, as well as a $B$-scheme $B'$, we define the stack $G^{k,1}_{r,\mathcal{L},d_\bullet}(X/B)$ (respectively, $G^{k,\text{II}}_{r,\mathcal{L},d_\bullet}(X/B,\theta_\bullet)$) to be the (2-)fibered product of $G^{k,1}_{r,d_\bullet}(X/B)$ (respectively, $G^{k,\text{II}}_{r,d_\bullet}(X/B,\theta_\bullet)$) with $\mathcal{M}_{r,w_0,\mathcal{L}}(X_{B'}/B')$ over $\mathcal{M}_{r,w_0}(X/B)$.

We further define the refined and constrained loci of $G^{k,\text{II}}_{r,\mathcal{L},d_\bullet}(X/B)$ to be the preimages of the corresponding loci of $G^{k,\text{II}}_{r,d_\bullet}(X/B)$, and similarly with the simple locus of $G^{k,\text{II}}_{r,\mathcal{L},d_\bullet}(X/B,\theta_\bullet)$.

Thus, each stack is as in the varying determinant case, except that we add an isomorphism between $\mathcal{L}$ and the appropriate twist of the determinant of the underlying vector bundle. This isomorphism not only specifies the determinant, but rigidifies the groupoid, which is why the dimension in the fixed determinant case agrees with the dimension one obtains with coarse moduli spaces.

Note that each of these groupoids lives naturally over $B'$ rather than $B$, but that the family $X/B$ is used in the definition in order to define the necessary twisting line bundles, because $X_{B'}$ may not be regular.

As before, we have two main theorems on these spaces.

**Theorem 6.1.4.** Let $X/B$ be an almost local smoothing family, $k, r, d, d_\bullet$ as in Situation 3.1.1, and $\mathcal{L}$ a line bundle of degree $d$ on $X_B'$ for some $B$-scheme $B'$. Then $G^{k,1}_{r,\mathcal{L},d_\bullet}(X/B)$ is an Artin stack over $B'$, and the natural map $G^{k,1}_{r,\mathcal{L},d_\bullet}(X/B) \to \mathcal{M}_{r,w_0,\mathcal{L},d_\bullet}(X_{B'}/B')$ is relatively representable by schemes which are projective, at least locally on the target. Moreover, formation of $G^{k,1}_{r,\mathcal{L},d_\bullet}(X/B)$ is compatible with any base change $B'' \to B$ which preserves the almost local smoothing family hypotheses. In particular, if $y \in B'$ is a point with $X_y$ smooth, then the base change to $y$ parametrizes triples $(\mathcal{E}, \psi, V)$ of a vector bundle $\mathcal{E}$ of rank $r$ and degree $d$ on $X_y$ together with an isomorphism $\psi : \mathcal{E} \to \mathcal{L}|_y$ and a $k$-dimensional vector space $V \subseteq H^0(X_y, \mathcal{E})$.

Under the further hypothesis of Situation 3.1.1, all of the above statements also hold for $G^{k,\text{II}}_{r,\mathcal{L},d_\bullet}(X/B,\theta_\bullet)$. Moreover, the simple locus of $G^{k,\text{II}}_{r,\mathcal{L},d_\bullet}(X/B,\theta_\bullet)$ has universal relative dimension at least $k(d-k-(g-1))$ over $\mathcal{M}_{r,w_0,\mathcal{L},d_\bullet}(X/B)$, and therefore universal relative dimension at least $\rho - g$ over $B'$. In particular, if some fiber $G^{k,\text{II}}_{r,\mathcal{L},d_\bullet}(X_y/y,\theta_\bullet)$ has dimension exactly $\rho - g$ at a simple point $z$, then $G^{k,\text{II}}_{r,\mathcal{L},d_\bullet}(X/B,\theta_\bullet)$ is universally open at $z$, and has fibers of pure dimension $\rho - g$ in an open neighborhood of $z$.

**Remark 6.1.5.** We also see immediately that in the situation of Theorem 6.1.4, in the case that $B$ is a point, we have that $G^{k,\text{II}}_{r,\mathcal{L},d_\bullet}(X/B)$ is an Artin stack over $B'$,
and the natural map
\[ G_{r, \mathcal{L}, d_\bullet}(X/B) \to \mathcal{M}_{r, w_0, \mathcal{L}, d_\bullet}(X_{B'/B'}) \]
is relatively representable by schemes which are projective, at least locally on the target.

**Theorem 6.1.6.** Suppose that \( B \) is a point. Then the morphism (4.3.1) induces a surjective morphism
\[ (6.1.1) \quad G_{r, \mathcal{L}, d_\bullet}^{k, \text{EHT}}(X/B) \to G_{r, \mathcal{L}, d_\bullet}^{k, \text{EHT}}(X/B), \]
which is an isomorphism on the preimage of the refined locus.

In addition, under the further hypothesis of Situation 3.3.1, if we consider the composed forgetful morphism
\[ (6.1.2) \quad G_{r, \mathcal{L}, d_\bullet}^{k, \text{II}}(X, \theta_\bullet) \to G_{r, \mathcal{L}, d_\bullet}^{k, \text{EHT}}(X), \]
we find that it is an isomorphism on the preimage of the constrained locus.

Theorem 6.1.6 follows immediately from Theorem 4.3.4, since the relevant maps are simply obtained from base change of those in the latter result.

**Proof of Theorem 6.1.4.** That the groupoids are Artin stacks follows immediately from Theorem 3.4.7 and the description as fibered products. Similarly, the map
\[ G_{r, \mathcal{L}, d_\bullet}(X/B) \to \mathcal{M}_{r, w_0, \mathcal{L}, d_\bullet}(X_{B'/B'}) \]
is obtained from the map
\[ G_{r, \mathcal{L}, d_\bullet}^{k, \text{I}}(X/B) \to \mathcal{M}_{r, w_0, \mathcal{L}, d_\bullet}(X/B) \]
by base change, so its projectivity is preserved, and likewise for \( G_{r, \mathcal{L}, d_\bullet}^{k, \text{II}}(X/B, \theta_\bullet) \). Finally, since universal relative dimension is preserved under base change, the dimension statement likewise follows immediately from Theorem 3.4.7. \( \square \)

### 6.2. The parameter \( b \)
In the classical rank-1 case, Eisenbud and Harris made use of the facts that twisting acts transitively on multidegrees, and that line bundles of negative degree on a smooth curves have no nonzero global sections, in order to always work with linear series of degree \( d \) on each component. Both of these facts fail in higher rank, which is the reason for the introduction of the parameters \( b \) and \( d' \) in Teixidor’s work [Tei91]. The \( d_\bullet \) keeps track of the extremal degrees (and implicitly, the congruence class of allowable multidegrees), while \( b \) measures how much we twist to get from one extremal degree to another. Because of condition (I) in Definitions 3.2.1, 3.3.2, and 4.1.2, it is possible for \( b \) to be “too small,” but we can always increase \( b \) and \( d_\bullet \), and if we do so in certain ways, we obtain an open immersion from one stack of limit linear series to the other.

**Proposition 6.2.1.** Suppose we have \( r, d, k, b, d_\bullet \) as in Situation 3.1.1, and we are also given \( b' \) and \( d'_\bullet \) satisfying (3.1.1), and such that for each \( v \in V(\Gamma) \), we have
\[ d_v \equiv d'_v \quad (\text{mod } r) \]
and
\[ d_v \leq d'_v \leq d_v + r(b' - b). \]
Under the hypothesis of Situation 3.3.1, we have that \( G_{r, \mathcal{L}, d_\bullet}^{k, \text{II}}(X/B, \theta_\bullet) \) is an open substack of \( G_{r, \mathcal{L}, d_\bullet}^{k, \text{II}}(X/B, \theta_\bullet) \).
Similar statements appear to hold for the type I and EHT cases, but the constructions are more involved, so we do not pursue them.

Proof. The first observation is that our restrictions on \( d_{\bullet} \) and \( b' \) imply that we can go from \( d_{\bullet}, b \) to \( d'_{\bullet}, b' \) by repeatedly carrying out the following operation: for a fixed vertex \( v_0 \), increase \( d_v \) by \( r \) for all \( v \neq v_0 \), and increase \( b \) by 1. Indeed, there exists \( v_0 \) such that \( d'_{v_0} < d_{v_0} + r(b' - b) \), and for such \( v_0 \), it follows that \( d'_v > d_v \) for all \( v \neq v_0 \). If we modify \( d_{\bullet} \) and \( b \) as described above, all our restrictions are still satisfied, so we can iterate until we reach \( d'_{\bullet}, b' \). Thus, it is enough to consider the situation that, for a fixed \( v_0 \), we have

\[
d'_{v} = \begin{cases} d_v : & v = v_0 \\ d_v + r : & v \neq v_0. \end{cases}
\]

and \( b' = b + 1 \).

Now, the restriction on congruence classes means that the two versions of \( G_{11} \) arising from \( d_{\bullet} \) and \( d'_{\bullet} \) are canonically identified. For each \( v \in V(\Gamma) \), denote by \( u'_v \in V(G_{11}) \) the vertex equal to \( d'_v \) in index \( v \) and to \( d'_v - rb \) in index \( v' \) for all \( v' \neq v \). Since condition (I) of Definition 3.3.2 is an open condition, it is enough to check that the version imposed by \( d_{\bullet} \) is stronger than that imposed by \( d'_{\bullet} \). However, for each \( v \) we have that \( \mathcal{E}_{u'_v}|_Y \) is obtained from \( \mathcal{E}_{w_v}|_Y \) by twisting up by an effective divisor \( D_v \) supported at nodes, and that the degree of \( D_v \) is at most \( b' - b = 1 \). It thus follows immediately that condition (I) for \( d_{\bullet} \) is stricter than condition (I) for \( d'_{\bullet} \), as desired. □

The next proposition shows that even if we do not assume condition (I) in the definition of type II linked linear series, we can always increase \( b \) so that it will be satisfied.

Proposition 6.2.2. In the situation of Definition 3.2.1, suppose that \( S \) is a quasicompact \( B \)-scheme, and \( \mathcal{E} \) is a vector bundle of rank \( r \) and multidegree \( w_0 \) on \( X \times_B S \). Then there exist \( b', d'_{\bullet} \) as in Proposition 6.2.1 such that \( \mathcal{E} \) is in \( \mathcal{M}_{r,w_0,d'}(X/B) \).

Proof. By the quasicompactness of \( S \), there exists some \( N \) such that for every \( v \in V(\Gamma) \) and every \( y \in S \), with image \( y \in B \), there is no nonzero map from any line bundle of degree \( N \) on \( Y \) to \( \mathcal{E}_{w_v}|_Y \), where \( Y \) denotes the component of the fiber \( X_y \) corresponding to \( c_{ly}(v) \). Equivalently, if \( D \) is any divisor of degree at least \( N \) on \( Y \), then \( H^0(Y, \mathcal{E}_{w_v}|_Y(-D)) = 0 \). Note that if \( b + 1 \geq N \), condition (I) is automatically satisfied for \( \mathcal{E} \). Otherwise, if we set \( b' = b + |V(\Gamma)|(N - b - 1) - 1 \), and \( d'_{v} = d_{v} + r|E(\Gamma)|(N - b - 1) \) for all \( v \in V(\Gamma) \), then we have \( \mathcal{E}_{w_v}|_Y = \mathcal{E}_{w_v}|_Y(D) \) for a divisor \( D \) of degree \( |E(\Gamma)|(N - b - 1) \), and then \( \mathcal{E}_{w_v}|_Y(D - (b' + 1)\Delta_v) \) has no nonzero global sections, because

\[
\deg \left( (b' + 1)\Delta_v - D \right) = b + 1 + (|V(\Gamma)| - |E(\Gamma)|)(N - b - 1) = N.
\]

Remark 6.2.3. The significance of Proposition 6.2.2 is that we could have omitted \( b \) and \( d_{\bullet} \), as well as condition (I), from our definition of type II linked linear series, and obtained a single canonical moduli stack for each choice of congruence class modulo \( r \) of \( d_{\bullet} \). However, we have chosen to use the present definition in order to unify the presentation as much as possible with the type I case and the EHT case.
6.3. Specialization. Since the moduli stack of vector bundles on a reducible curve is not proper, we do not automatically obtain specialization results when we have a smoothing family with one-dimensional base. However, just as was the case with Eisenbud and Harris, and with Teixidor’s generalization thereof, we have that after base change and blowup we can always extend to a higher-rank limit linear series. In fact, our situation is better than these cases, because after base change and blow up, our extension always give a point of a relative moduli stack over the whole family, and this can be done regardless of characteristic. Previously, in the higher-rank case this had not been carried out in any characteristic, and in the rank-1 case it had only been done in characteristic 0.

Our result is the following:

Proposition 6.3.1. Suppose that $B$ is one-dimensional, let $y \in B$ be a point, set $U := B \setminus \{y\}$, and let $\pi : X \to B$ be a smoothing family such that $\pi$ is smooth on the preimage of $U$. Given $r, d, k$, suppose we have $B'$ over $B$, also one-dimensional and regular, and with a unique point $z$ lying over $y$; set $U' = B' \setminus \{z\}$. Let $\pi' : X' \to B'$ be the desingularization of $X \times_B B' \to B'$ obtained by blowing up the nodes over $z$ as necessary. Then given a pair $(E, \nu)$ on $X|_{U'}$, where $E$ is a vector bundle of degree $d$, and $\nu \subseteq \pi_*E$ is a rank-$k$ subbundle, there exist $d_*$ and $b$ as in Situation 3.1.1 such that the $U'$-valued point of $G_{r,d,k}(X'/B', \nu)_*$ determined by $(E, \nu)$ extends to a $B'$-valued point, and similarly for $G_{r,d,*}(X'/B')$ and $G_{r,d,*,\text{EHT}}(X'/B')$.

Proof. Since we have morphisms $G_{r,d,k}(X'/B', \nu)_* \to G_{r,d_*,k}(X'/B')$ and $G_{r,d_*,*}(X'/B') \to G_{r,d_*,\text{EHT}}(X'/B')$ which are isomorphisms wherever $\pi'$ is smooth, it is enough to prove the claimed statement for $G_{r,d,k}(X'/B', \nu)_*$. Let $\Gamma'$ be the graph associated to $\pi'$.

Now, since vector bundles extend over regular surfaces, there exists some $w \in \mathbb{Z}^{|\Gamma'|}$ such that $E$ extends to a vector bundle $\mathcal{E}_w$ of multidegree $w$ on $X'$. Next, by Proposition 6.2.2 we see that there exist $b$ and $d_*$ such that $\mathcal{E}_w$ satisfies condition (I) of Definition 3.3.2. Now, because $G_{r,d,k}(X'/B', \nu)_*$ is proper over $M_{r,\nu_0,\nu}(X'/B')$, we conclude that we obtain the desired $B'$-valued point of $G_{r,d_*,*}(X'/B', \nu)_*$.

6.4. Stability. We now address conditions imposed by stability of underlying vector bundles. There is substantial possibility for confusion, due in part to the fact that unlike the irreducible case, on reducible curves stability of vector bundles is not preserved by twisting by line bundles. In fact, in addition to the usual notion of stability on a reducible curve, which is most useful in specialization arguments, we will use a second notion – called $\ell$-stability – which is more powerful in the context of smoothing arguments. First, recall that on a reducible curve, stability is not in general a canonical notion, but rather depends on a polarization of the curve, as follows:

Definition 6.4.1. Let $X_0$ be a proper nodal curve with dual graph $\Gamma$; a polarization on $X_0$ consists of a weight function $\omega : V(\Gamma) \to \mathbb{Q}$ such that $\omega(v) > 0$ for all $v$, and $\sum_{v \in V(\Gamma)} \omega(v) = 1$. Given such a polarization, a vector bundle $E$ on $X_0$ is semistable, if for all nonzero subsheaves $\mathcal{F} \subseteq E$, we have

$$\frac{\sum_{v} \chi(\mathcal{F}) \omega(v) \rk \mathcal{F}|_{y_v}}{\rk \mathcal{F}} \leq \frac{\chi(E)}{\rk E}.$$  

We say $E$ is stable if we always have strict inequality above.
For an irreducible curve, this is equivalent to the usual definition in terms of degree. We see that the role of the polarization is that it determines what replaces the notion of rank for a sheaf which has different rank on different components of $X_0$.

We also have the following concept, introduced in [Oss14].

**Definition 6.4.2.** Let $X_0$ be a proper nodal curve with dual graph $\Gamma$. A vector bundle $\mathcal{E}$ on $X_0$ is $\ell$-semistable if for all nonzero subsheaves $\mathcal{F} \subseteq \mathcal{E}$ having constant rank on all component of $X_0$, we have

$$\frac{\chi(\mathcal{F})}{\text{rk} \mathcal{F}} \leq \frac{\chi(\mathcal{E})}{\text{rk} \mathcal{E}}.$$ 

We say $\mathcal{E}$ is $\ell$-stable if we always have strict inequality above.

Thus, except in the irreducible case, $\ell$-stability is visibly a weaker condition than stability. However, it is independent of polarization, and is also more robust, being stable under twists and gluing; see Propositions 1.3 and 1.6 of [Oss14]. Most importantly, it is open, and thus suffices for existence arguments using limit linear series.

We make the following definitions for (semi)stability of a limit linear series.

**Definition 6.4.3.** Let $\pi : X \to B$ be an almost-local smoothing family with associated graph $\Gamma$, and let $\mathcal{G}$ be any of $\mathcal{G}_{r,d}^k(X/B)$, $\mathcal{G}^{k,\Pi}_{r,d}(X/B, \theta_*)$, or $\mathcal{G}^{k,EHT}_{r,d}(X/B)$. Then a $K$-valued point of $\mathcal{G}$ with image $y$ in $B$ is $\ell$-semistable (respectively, $\ell$-stable) if the vector bundle $\mathcal{E}_{w_0}$ is $\ell$-semistable (respectively, $\ell$-stable). Similarly, if $\omega$ is a polarization on $\Gamma$, a $K$-valued point of $\mathcal{G}$ with image $y$ in $B$ is semistable (respectively, stable) with respect to $\omega$ if there exists some $w_1 \in V(G_{k,\Pi})$ such that the induced vector bundle $\mathcal{E}_{w_1}$ is semistable (respectively, stable) with respect to $\omega_y$, where $\omega_y$ is the polarization on the dual graph $\Gamma_y$ induced by $\omega$ and $\text{cl}_y$.

We make the same definitions for the case of fixed determinant.

Note that, unlike the case of (semi)stability with respect to a polarization, because $\ell$-(semi)stability for vector bundles is invariant under twisting, it would be equivalent to define it above in terms of existence of $w_1$ such that $\mathcal{E}_{w_1}$ is $\ell$-(semi)stable.

We first give a precise statement of the behavior of stability under smoothing.

**Proposition 6.4.4.** Let $\pi : X \to B$ be an almost-local smoothing family. Then $\ell$-stability and $\ell$-semistability are both open conditions on each of $\mathcal{G}_{r,d}^{k,1}(X/B)$, $\mathcal{G}^{k,\Pi}_{r,d}(X/B, \theta_*)$, or $\mathcal{G}^{k,EHT}_{r,d}(X/B)$, and the same holds in the case of fixed determinant.

This is an immediate consequence of Proposition 1.2 of [Oss14]. Thus, if we produce $\ell$-stable limit linear series, under smoothing we will obtain $\ell$-stable (hence stable) vector bundles on nearby smooth curves.

Next, we consider behavior under specialization.

**Proposition 6.4.5.** In the situation of Proposition 6.3.1, suppose further that $\mathcal{E}$ is semistable over the generic point of $U'$. Then given any polarization $\omega$ on the dual graph $\Gamma'$ of the special fiber of $X'$, we may choose the extension given by Proposition 6.3.1 to be semistable with respect to $\omega$. 

This is immediate from Proposition 4.1 of [Oss14].

Here, the flexibility in choice of multidegree in the definition of semistability for limit linear series weakens the conclusion of Proposition 6.4.5, in that it gives us only a single multidegree on which we have semistability. On the other hand, it is easy to see that this is the most we can hope for, since semistability on reducible curves imposes constraints on multidegrees (see for instance (**) in the proof of Proposition 2.1 of [Tei11]).

APPENDIX A. PRELINKED GRASSMANNIANS

In this appendix, we develop a partial generalization of the linked Grassmannians introduced in Appendix A of [Oss06a] for graphs consisting of chains of vertices. Linked Grassmannians act as degenerations of Grassmannians, and we expect that the full theory generalizes to graphs of the sort appearing for type I and II linked linear series, and indeed more generally. However, such a theory will be difficult, and does not yield immediate applications in the higher-rank case, so for the present we consider instead a notion of “prelinked Grassmannians,” developing easy properties which suffice for our purposes.

A.1. Definitions. We begin with the definitions underlying prelinked Grassmannians.

Situation A.1.1. Let $G$ be a finite directed graph, connected by (directed) paths, $d$ an integer, and $S$ a scheme. Suppose we are given data $\mathcal{E}_v$, consisting of vector bundles $\mathcal{E}_v$ of rank $d$ for each vertex $v \in V(G)$, and morphisms $f_e : \mathcal{E}_v \rightarrow \mathcal{E}_{v'}$ for each edge $e \in E(G)$, where $v$ and $v'$ are the tail and head of $e$, respectively. For a (directed) path $P$ in $G$, denote by $f_P$ the composition of the morphisms $f_e$ for each edge $e$ in $P$.

In the above situation, we have:

Definition A.1.2. Given an integer $r < d$, suppose further that the following condition is satisfied:

(I) For any two paths $P, P'$ in $G$ with the same head and tail, there are scalars $s, s' \in \Gamma(S, \mathcal{O}_S)$ such that

$$s \cdot f_P = s' \cdot f_{P'};$$

moreover, if $P$ (respectively, $P'$) is minimal, then $s'$ (respectively, $s$) is invertible.

Then define the **prelinked Grassmannian** $LG(r, \mathcal{E}_\bullet)$ to be the scheme representing the functor associating to an $S$-scheme $T$ the set of all collections $(\mathcal{F}_v)_{v \in V(G)}$ of rank-$r$ subbundles of the $\mathcal{E}_v$ satisfying the property that for all edges $e \in E(G)$, $f_e(\mathcal{F}_v) \subseteq \mathcal{F}_{v'}$, where $v$ and $v'$ are the tail and head of $e$ respectively.

Note that in the condition of the definition, we allow the empty path and consider $f_P = \text{id}$ in this case. Thus, the condition implies that for any loop $P$ in $G$, we have $f_P = s \cdot \text{id}$ for some scalar $s$.

The fact that the functor defining $LG(r, \mathcal{E}_\bullet)$ is representable is easy, and in fact we have the following.

Proposition A.1.3. $LG(r, \mathcal{E}_\bullet)$ is a projective scheme over $S$, and compatible with base change.
Proof. We construct $\text{LG}(r, \mathcal{E}_*)$ as a closed subscheme of the natural product of Grassmannians $G := \prod_{v \in V(G)} G(r, \mathcal{E}_v)$. For each $v$, let $\mathcal{F}_v$ denote the pullback of $\mathcal{E}_v$ to $G$, and let $\mathcal{F}'_v$ denote the pullback of the universal subbundle on $G(r, \mathcal{E}_v)$ to $G$. Then it suffices to observe that $\text{LG}(r, \mathcal{E}_*)$ is cut out as the locus on which, for every edge $e \in E(G)$, if $v$ and $v'$ are the tail and head of $e$, then the composed map of vector bundles 

$$\mathcal{F}_v \to \mathcal{F}'_v \to \mathcal{F}'_{v'}/\mathcal{F}'_{v'}$$

is zero.

That $\text{LG}(r, \mathcal{E}_*)$ is compatible with base change is evident from the definition. \qed

In this generality, we will mainly be interested in points satisfying a strong additional condition:

**Definition A.1.4.** Let $K$ be a field over $S$, and $(F_v)_v$ a $K$-valued point of a prelinked Grassmannian $\text{LG}(r, \mathcal{E}_*)$. We say that $(F_v)_v$ is **simple** if there exist $v_1, \ldots, v_r \in V(G)$ (not necessarily distinct) and $s_i \in F_{v_i}$ for $i = 1, \ldots, r$ such that for every $v \in V(G)$, there exist paths $P^{v_1}, \ldots, P^{v_r}$ with each $P^{v_i}$ going from $v_i$ to $v$, and such that $f_{P^{v_i}}(s_1), \ldots, f_{P^{v_i}}(s_r)$ form a basis for $F_v$.

Note that as a consequence of the definition of a prelinked Grassmannian, we may always take the $P^{v_i}$ in the definition of a simple point to be minimal paths from $v_i$ to $v$.

**A.2. Properties of simple points.** In [Oss06a], a notion of “exact points” of linked Grassmannians constitutes the focal point of the analysis. In the special case studied in loc. cit., exact points are the same as simple points, which substantially simplifies the situation. Exact points generalize naturally to the cases we are interested in, but are no longer the same as simple points; see Example A.2.3. We will now prove that simple points are smooth of the expected dimension, which is the relevant statement for our applications, and is much easier than analyzing exact points in general.

First, using Nakayama’s lemma we easily conclude the following:

**Proposition A.2.1.** The simple points form an open subset of $\text{LG}(r, \mathcal{E}_*)$.

More substantively, we have the following result, which can be viewed as a generalization of Lemma A.12 (ii) and Lemma A.14 of [Oss06a].

**Proposition A.2.2.** On the locus of simple points, $\text{LG}(r, \mathcal{E}_*)$ is smooth over $S$ of relative dimension $r(d - r)$.

Proof. We first prove smoothness. Let $(F_v)_v$ be a $K$-valued simple point, and fix choices of $v_i$, $s_i$ and $P^{v_i}$ as in Definition A.1.4, with each $P^{v_i}$ a minimal path. Let $A$ be a Noetherian local ring over $S$ with residue field $K$, and $A'$ a quotient ring of $A$. Suppose we are given $(F'_v)_v$ over $A'$ specializing to the given point under restriction to $K$; we wish to show that we can lift to $(F_v)_v$ over $A$. To keep notation reasonable, for any edge $e$ or path $P$ in $G$ we will still use the notation $f_e$ or $f_P$ for all base changes of the original morphisms $f_e$ and $f_P$ on $S$. For each $i$, let $\tilde{s}'_i$ be any lift of $s_i$ in $F'_v$; then by Nakayama’s lemma, we have that for each $v$, the images of $\tilde{s}'_i$ under the corresponding $f_{P^{v_i}}$, form a basis for $F'_v$. 


We claim that if we let \( \tilde{s}_i \) be any lift of \( \tilde{s}'_i \) to \( \mathcal{E}_v \) for \( i = 1, \ldots, r \), then the \( \tilde{s}_i \) uniquely determine an \( A \)-valued point of \( \text{LG}(r, \mathcal{E}) \) extending the given \( A' \)-valued one. Indeed, for each \( v \in V(G) \), we have the map
\[
A^{\otimes r} \to \mathcal{E}_v
\]
determined by \( (a_1, \ldots, a_r) \to \sum a_i f_{pv_v}(\tilde{s}_i) \). By hypothesis, this morphism has full rank at the closed point, and it follows that its image is a subbundle of \( \mathcal{E}_v \), which will be our \( \mathcal{F}_v \). It remains to check that the \( \mathcal{F}_v \) thus determined are in fact linked by \( f_e \) for each edge \( e \in E(G) \). Accordingly, let \( e \) be an edge, with tail \( v \) and head \( v' \). It suffices to observe that for each \( i \), there is some \( s \in A \) such that \( f_e(f_{pv_v}(\tilde{s}_i)) = s \cdot f_{pv'}(\tilde{s}_i) \), because we have chosen \( P_i^{\nu} \) to be a minimal path. This proves the desired lifting statement, and we conclude smoothness.

To conclude the proof of the proposition, we carry out a tangent space computation in the fiber. Since dimension isn’t affected by extension of base field, we may even assume that \( S = \text{Spec } K \) (where we are still considering the above \( K \)-valued simple point). Our claim is that the tangent space is (given the choice of \( v_i, s_i \) as above) canonically identified with \( \bigoplus_{i=1}^r E_{v_i}/P_{v_i}r \), which has dimension \( r(d-r) \), as desired. We first construct a map from \( \bigoplus_{i=1}^r E_{v_i} \) to the tangent space, as follows: setting \( A = K[x]/x^2 \), and \( A' = K \), and applying our above analysis, we see that any choices of \( \tilde{s}_i \) determine a tangent vector to \( \text{LG}(e, E) \) at the given point, and conversely, for any tangent vector \( (\mathcal{F}_v)_v \), of course there exists some choice of lifts of the \( \tilde{s}_i \) contained in the given \( \mathcal{F}_v \), and thereby inducing \( (\mathcal{F}_v)_v \). Now, the ambient spaces \( \mathcal{E}_v \) over \( A \) are by definition equal to \( E_v \oplus eE_v \), so a lift of \( s_i \) is given uniquely by an element of \( E_v \). This gives the map from \( \bigoplus_{i=1}^r E_{v_i} \) to the tangent space which we have seen to be surjective, so it remains to check that the kernel is precisely given by \( \bigoplus_{i=1}^r F_{v_i} \). It is clear that if \( s_i \) is lifted to \( s_i + tw \), with \( w \in F_{v_i} \), the resulting spaces \( \mathcal{F}_v \) are the same as if \( s_i \) were lifted to \( s_i \). Thus, the given space is contained in the kernel. Conversely, if given lifts \( s_i + tw_i \) of the \( s_i \) yield the trivial deformations \( F_v \oplus eF_v \), then in particular we see that we must have \( w_i \in F_{v_i} \) for all \( i \), so the kernel is as asserted, and we conclude the tangent space has dimension \( r(d-r) \).

**Example A.2.3.** Let \( G \) be a graph consisting of four vertices \( v_1, v_2, v_3, v_4 \) with edges \( e_{i,j} \) from \( v_i \) to \( v_j \) whenever \( i = 1 \) or \( j = 1 \). We consider vector spaces \( E_{v_i} = K^3 \) for all \( i \), and maps \( f_{e_{i,j}} \) as follows:

\[
\begin{align*}
f_{e_{1,2}} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, & f_{e_{2,1}} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} ; \\
f_{e_{1,3}} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & f_{e_{3,1}} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \\
f_{e_{1,4}} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & f_{e_{4,1}} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\end{align*}
\]

Then we set subspaces
\[
\begin{align*}
V_1 &= \langle (1, 1, 0), (1, 0, 1) \rangle , & V_2 &= \langle (1, 1, 0), (0, 0, 1) \rangle , \\
V_3 &= \langle (0, 1, 0), (1, 0, 1) \rangle , & V_4 &= \langle (1, 0, 0), (0, 1, -1) \rangle .
\end{align*}
\]
One sees that this is an exact point, but not simple. Indeed, the images of \( V_2, V_3 \) and \( V_4 \) in \( V_1 \) are three distinct lines, and this can never happen at a simple point.

Although this graph is not explicitly of the form arising from a type II linked linear series, it can easily be extended to one. Indeed, we can see the same behavior arising directly from a linked \( g^1_1 \) (of rank 1) on a curve with four rational components, having one main component glued to each of the other three. Then the image in multidegree \((1,0,0,0)\), \((0,1,0,0)\), \((0,0,1,0)\) and \((0,0,0,1)\) consists of sections vanishing at the first, second or third node respectively, so again gives three different lines in the 2-dimensional space of global sections.

### A.3. Prelinked Grassmannians over stacks.

Our results on prelinked Grassmannians can be generalized in a routine manner to stacks. In order to keep the situation more geometric, we restrict to algebraic stacks, although this is not strictly necessary. Indeed, Definition A.1.2 generalizes immediately to the case that \( S \) is replaced with a stack \( S \), yielding a groupoid \( \mathcal{L}G_{r,E_{\bullet}} \) over \( S \). Compatibility with base change in Proposition A.1.3 implies that \( \mathcal{L}G_{r,E_{\bullet}} \) is also a stack, relatively representable over \( S \) by projective schemes. The definition of simple point also generalizes to this context, and Proposition A.2.2 lets us conclude the following:

**Corollary A.3.1.** Let \( S \) be an algebraic stack, and \( \mathcal{L}G_{r,E_{\bullet}} \) a prelinked Grassmannian over \( S \). Then \( \mathcal{L}G_{r,E_{\bullet}} \) is also an algebraic stack, relatively representable by projective schemes over \( S \), and furthermore, on the locus of simple points, \( \mathcal{L}G_{r,E_{\bullet}} \) is smooth over \( S \) of relative dimension \( r(d-r) \).

### Appendix B. Generalized determinantal loci

In this appendix, we study properties of classical determinantal loci, and apply the results to generalize the definition to a relative setting. Our point of view is that frequently in applications to moduli spaces, we are interested not in the rank of a given map, but in the size of its kernel. We explore this point of view first in the context of classical determinantal ideals, and then we apply our results to develop a theory of determinal loci of pushforwards.

#### B.1. Observations on determinantal ideals.

We begin with some results in the context of classical determinantal loci.

**Definition B.1.1.** Given locally free sheaves \( \mathcal{E}, \mathcal{F} \) of finite rank on a scheme \( B \), a morphism \( f : \mathcal{E} \to \mathcal{F} \), and an integer \( k \geq 0 \), define the **kth vanishing locus** \( V_k(f) \) of \( f \) to be the closed subscheme of \( B \) cut out by the vanishing of

\[
\bigwedge^{r+1-k} f : \bigwedge^{r+1-k} \mathcal{E} \to \bigwedge^{r+1-k} \mathcal{F},
\]

where \( r \) is the rank of \( \mathcal{E} \). By convention, if \( k > r \), we set \( V_k(f) = \emptyset \).

Thus, this gives a scheme structure to the locus on which \( f \) has kernel of dimension at least \( k \). While statements in terms of such loci are often obvious on a set-theoretic level, we are concerned with canonical scheme structures, and scheme-theoretic statements require more thought. Our first observation is that this scheme structure depends only on the “universal kernel” of \( f \), in the following sense:
Lemma B.1.2. Suppose that $f : \mathcal{E} \rightarrow \mathcal{F}$ and $f' : \mathcal{E}' \rightarrow \mathcal{F}'$ are two morphisms of locally free sheaves of finite rank on a locally Noetherian scheme $B$, and suppose that for every closed subscheme $Z \subseteq B$, we have

$$\ker(f|_Z) = \ker(f'|_Z).$$

Then for each $k \geq 0$, we have $V_k(f) = V_k(f')$.

Most of the argument which follows is due to David Eisenbud.

Proof. The statement being local on $B$, we may assume $B = \text{Spec } A$ is affine, with $A$ local and $V_k(f)$ and $V_k(f')$ cut out by ideals $I$ and $I'$, respectively. Suppose that $I \neq I'$. Then we may assume without loss of generality that $I \subseteq I'$, and then replacing $A$ by $A/I'$, we have that all $(r + 1 - k) \times (r' + 1 - k)$ minors of $f'$ vanish, but not all $(r + 1 - k) \times (r + 1 - k)$ minors of $f$ vanish, where $r$ and $r'$ are the ranks of $\mathcal{E}$ and $\mathcal{E}'$, respectively. Let $g$ be one of the non-zero minors of $f$. Let $J$ be an ideal maximal among those not containing $g$; we claim that if we mod out by $J$, we obtain a Gorenstein Artin local ring. That we obtain an Artin local ring follows from the Krull Intersection Theorem. On the other hand, in the Artin case the Gorenstein property is equivalent to having simple socle (Proposition 21.5 of [Eis95]). But since the socle is by definition a vector space over the residue field, if it is not simple it contains an element linearly independent from $g$, which is not possible by the maximality of $J$.

We have thus reduced to the case $B = \text{Spec } A$, with $A$ a Gorenstein Artin local ring. We can now apply duality in this context (§21.1 of [Eis95]) to conclude that the dual sequences

$$\text{Hom}_A(\mathcal{F}(B), A) \xrightarrow{f^*} \text{Hom}_A(\mathcal{E}(B), A) \rightarrow \text{Hom}_A(\ker f(B), A) \rightarrow 0$$

and

$$\text{Hom}_A(\mathcal{F}'(B), A) \xrightarrow{(f')^*} \text{Hom}_A(\mathcal{E}'(B), A) \rightarrow \text{Hom}_A(\ker f'(B), A) \rightarrow 0$$

each give free presentations of modules which are by hypothesis isomorphic. By the theory of Fitting ideals (Corollary-Definition 20.4 of [Eis95]), we conclude that the ideals of $(r + 1 - k) \times (r + 1 - k)$ minors of $f^*$ and of $(r' + 1 - k) \times (r' + 1 - k)$ minors of $(f')^*$ are equal. But since the matrices in question are simply the transpose of those for $f$ and $f'$, this gives a contradiction.

Next, it will be helpful to know that one obtains morphisms between different vanishing loci when expected.

Proposition B.1.3. Let $f : \mathcal{E} \rightarrow \mathcal{F}$ and $f' : \mathcal{E}' \rightarrow \mathcal{F}'$ be morphisms of locally free sheaves of finite rank on $B$. Suppose we also have morphisms $g : \mathcal{E} \rightarrow \mathcal{E}'$ and $h : \mathcal{F} \rightarrow \mathcal{F}'$ such that $f' \circ g = h \circ f$, and such that at every point of $B$, after restriction to fibers we have $g$ injective on $\ker f$. Then for any $k \geq 0$, we have that $V_k(f)$ is a closed subscheme of $V_k(f')$.

Proof. The question being local, we may assume that $B = \text{Spec } A$, where $A$ is a local ring having maximal ideal $\mathfrak{m}$. Using Nakayama’s lemma, it is easy to see that we may write $\mathcal{E} \xrightarrow{f} \mathcal{F}$ as a direct sum a minimal and a trivial 2-term complex (here minimal means that the morphism is 0 modulo $\mathfrak{m}$, and trivial means that the morphism is an isomorphism). We may then replace $\mathcal{E} \xrightarrow{f} \mathcal{F}$ by the minimal subcomplex without affecting our hypotheses or the vanishing loci in question, and
in this case we see that since \( f \) is 0 modulo \( \mathfrak{m} \), then \( g \) is injective modulo \( \mathfrak{m} \), and we see we have reduced to the case that \( g \) realizes \( \mathcal{E} \) as a subbundle of \( \mathcal{E}' \).

Let \( r \) and \( r' \) denote the ranks of \( \mathcal{E} \) and \( \mathcal{E}' \), respectively. Now, if we restrict to \( V_k(f) \), we need only verify that \( V_k(f') \) is all of \( B \), or equivalently, that the morphism

\[
\bigwedge^{r'+1-k} f' : \bigwedge^{r'+1-k} \mathcal{E}' \to \bigwedge^{r'+1-k} \mathcal{F}'
\]

vanishes identically. However, one can check this easily on bases from the hypotheses that \( \bigwedge^{r+1-k} f = 0 \) and that \( \mathcal{E}' \) is a subbundle of \( \mathcal{E} \).

**Remark B.1.4.** Note the contrast between Lemma B.1.2 and Proposition B.1.3 that the latter requires a morphism of complexes, while the former does not. The morphism of complexes is very much necessary for the validity of Proposition B.1.3, as demonstrated by simple examples over a non-reduced point. By the same token, if in the statement of the proposition we assume that \( g \) induces an isomorphism of kernels at all points, it does not follow that the \( k \)th vanishing loci are the same.

The final statement of this form is a bit more complicated to state, although the proof is not difficult.

**Proposition B.1.5.** Given \( f : \mathcal{E} \to \mathcal{F} \) and \( f_i : \mathcal{E}_i \to \mathcal{F}_i \) for \( i = 1, 2 \) morphisms of locally free sheaves of finite rank on \( B \), as well as a locally free sheaf \( \mathcal{K} \) of rank \( k \) and imbeddings \( \mathcal{K} \hookrightarrow \mathcal{F}_i \) as subbundles for \( i = 1, 2 \), suppose we have a closed subscheme \( Z \) and positive integers \( m_1, m_2 \) such that:

(I) for \( i = 1, 2 \), the \( m_i \)th vanishing locus of 

\[
\mathcal{F}_i : \mathcal{E}_i \to \mathcal{F}_i/\mathcal{K}
\]

contains \( Z \);

(II) after restriction to any closed subscheme \( Z' \) of \( Z \), the kernel of \( f \) is isomorphic to the fibered product of the kernels of \( f_1' \) and \( f_2' \) over \( \mathcal{K} \).

Then

\[
V_{m_1+m_2-k}(f) \supseteq Z.
\]

**Proof.** The hypotheses and conclusion being compatible with base change, we may first restrict to \( Z \), and thus wish to show that \( V_{m_1+m_2-k}(f) = B \). Similarly, the question being local we may assume that \( B = \text{Spec } A \) is affine, so that \( \mathcal{K} \to \mathcal{F}_i \) splits, and we choose isomorphisms \( \mathcal{F}_i \cong \mathcal{F}_i/\mathcal{K} \). Making use of this isomorphism, we construct a morphism

\[
g : \mathcal{F}_1 \oplus \mathcal{F}_2 \to \mathcal{F}_1/\mathcal{K} \oplus \mathcal{F}_2/\mathcal{K} \oplus \mathcal{K}
\]

induced by \( f_1 \) and \( -f_2 \), and we observe that the kernel of \( g \) is (universally) equal to the fibered product of the kernels of \( f_1' \) and \( f_2' \) over \( \mathcal{K} \). Thus, according to Lemma B.1.2, it is enough to show that \( V_{m_1+m_2-k}(g) = B \).

But this follows immediately from the definitions: we localize further so that all modules are free, and we write \( r_1 \) and \( s_i \) for the ranks of \( \mathcal{E}_i \) and \( \mathcal{F}_i \), respectively, then the upper \( s_1 + s_2 \) rows of the matrix defining \( g \) are block-diagonal, with blocks of size \( s_1 \times r_1 \) and \( s_2 \times r_2 \). Our hypotheses imply that the \((r_i + 1 - m_i) \times (r_i + 1 - m_i)\) minors of the \( i \)th block for \( i = 1, 2 \) vanish uniformly, and the desired statement follows. □
Remark B.1.6. Suppose we have a finite complex of locally free sheaves of finite rank. Flatness implies that quasi-isomorphism is preserved under base change, so it follows from Lemma B.1.2 that the $k$th vanishing locus of the first morphism in the complex is in fact a quasi-isomorphism invariant. This is enough for most of our applications to pushforwards, and can be proved directly, but one comparison theorem will involve universally isomorphic kernels which are not naturally obtained from quasi-isomorphic complexes.

We also observe that for the quasi-isomorphism invariance, it is not enough to have a morphism of complexes inducing an isomorphism on the first two cohomology groups, or indeed on the first $n$ cohomology groups for any fixed $n$. Indeed, letting $B = \mathbb{A}^n$, consider $\mathcal{F}^\bullet$ a trivial complex, say supported on $\mathcal{F}_0$ and $\mathcal{F}_1$, and let $\mathcal{G}^\bullet$ be the Koszul complex, so that the cohomology of $\mathcal{G}^\bullet$ is nonzero only in the $n$th and final place. Then there is a morphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ inducing isomorphisms on $H^0$ through $H^{n-1}$, but the $k$th vanishing loci of $\mathcal{F}^\bullet$ for $k > 0$ are empty, while those for $\mathcal{G}^\bullet$ are not.

B.2. Subbundles of pushforwards. We now turn towards pushforwards. It turns out that it is useful to have generalizations of determinantal loci not only for maps of pushforwards, but also incorporating subbundles of pushforwards. We thus begin by recalling the following definition.

Definition B.2.1. Let $\pi : X \to B$ be a proper morphism, locally of finite presentation, and $\mathcal{E}$ a quasicoherent sheaf on $X$, locally finitely presented and flat over $B$. A subsheaf $\mathcal{V}$ is defined to be a subbundle of $\pi_\ast \mathcal{E}$ if $\mathcal{V}$ is locally free of finite rank, and for any $S \to B$, the natural map $\mathcal{V}_S \to \pi_S_\ast \mathcal{E}_S$ remains injective.

Remark B.2.2. If $B$ is locally Noetherian, our hypotheses simplify to $\pi$ being proper, and $\mathcal{E}$ being coherent and flat over $B$. However, we want to allow for $B$ not locally Noetherian, since we don’t want to restrict to the category of locally Noetherian schemes in defining our moduli stacks.

A basic fact about the situation of Definition B.2.1 is that on any affine open subset of $B$, there exists a finite complex consisting of locally free sheaves of finite rank which is a representative for $R\pi_\ast \mathcal{E}$, and remains so after arbitrary base change. In the Noetherian case, this is Theorem 6.10.5 (see also Remark 6.10.6) of [GD63], and the general case follows by Noetherian approximation and our finite presentation hypotheses, noting also that flatness implies that formation of $R\pi_\ast \mathcal{E}$ commutes with base change.

The above definition of subbundle has a number of desirable properties, which we explore in the following lemma.

Lemma B.2.3. In the situation of Definition B.2.1, we have the following statements.

1. Suppose that $\mathcal{F}^\bullet = \mathcal{F}_0 \xrightarrow{d^0} \cdots \xrightarrow{d^{n-1}} \mathcal{F}_n$ is a finite complex of locally free sheaves of finite rank on $B$ which is a representative of $R\pi_\ast \mathcal{E}$. Then subbundles of $\pi_\ast \mathcal{E}$ in our sense are the same as locally free subsheaves of $\pi_\ast \mathcal{E}$ which are subbundles of $\mathcal{F}_0$ in the usual sense.
(ii) Suppose we have $\mathcal{E}$ such that $\pi_*\mathcal{E}$ is locally free, and $R^i\pi_*\mathcal{E} = 0$ for all $i > 0$. Then our definition of subbundle of $\pi_*\mathcal{E}$ is equivalent to the usual one.

(iii) If $\mathcal{V}$ is locally free, a morphism $\mathcal{V} \to \pi_*\mathcal{E}$ yields a subbundle if and only if for all points $y \in B$, the induced map

$$\mathcal{V}|_y \to H^0(X_y, \mathcal{E}|_y)$$

is injective. Moreover, if for some $y \in B$ we have (B.2.1) injective, then $\mathcal{V} \subseteq \pi_*\mathcal{E}$ is a subbundle on an open neighborhood of $y$.

(iv) Let $\mathcal{V}_1$, $\mathcal{V}_2$ be subbundles of rank $r$ of $\pi_*\mathcal{L}$ in our sense, and suppose $\mathcal{V}_1 \subseteq \mathcal{V}_2$. Then $\mathcal{V}_1 = \mathcal{V}_2$.

Note that (iii) says that our definition of subbundle is the same as the family of $\mathfrak{g}_d^{\mathcal{V}}$’s used in §IV.3 of [ACGH85] to describe the functor represented by the classical $\mathcal{O}_d$ space for a smooth curve.

**Proof.** For (i), by hypothesis, we have an injection $\pi_*\mathcal{E} \to \mathcal{F}_0$, so the statement makes sense. Let $\mathcal{V} \subseteq \pi_*\mathcal{E}$ be a locally free subsheaf, and define $\mathcal{D}$ by the exact sequence

$$0 \to \mathcal{V} \to \mathcal{F}_0 \to \mathcal{D} \to 0$$

obtained by composing with the above injection. We then wish to show that $\mathcal{V}$ is a subbundle of $\pi_*\mathcal{E}$ in our sense if and only if $D$ is locally free. Because the $\mathcal{F}^i$ are flat over $B$, for any $S \to B$, we have that $\mathcal{F}^i|_S$ is a representative of $R\pi_*\mathcal{E}|_S$; what is relevant for our purposes is that $(\pi_*\mathcal{E})|_S \to \mathcal{F}^0|_S$ factors through the canonical morphism $(\pi_*\mathcal{E})|_S \to \pi_*\mathcal{E}|_S$, with the induced morphism $\pi_*\mathcal{E}|_S \to \mathcal{F}^0|_S$ being injective. Thus, the map $\mathcal{V}|_S \to \mathcal{F}^0|_S$ is obtained as the composition $\mathcal{V}|_S \to \pi_*\mathcal{E}|_S \to \mathcal{F}^0|_S$, and the second map is injective. We conclude that $\mathcal{V}$ is a subbundle of $\pi_*\mathcal{E}$ in our sense if and only if the induced map $\mathcal{V}|_S \to \mathcal{F}^0|_S$ is injective for all $S \to B$, which is equivalent to the condition that $\mathcal{D}$ is locally free.

For (ii), we simply observe that in this case, if we set $\mathcal{F} = \pi_*\mathcal{E}$, the hypotheses of (i) are satisfied because by [GD63, Cor. 6.9.9] (see also [GD63, 6.2.1]), the natural map $(\pi_*\mathcal{E})_S \to \pi_*\mathcal{E}|_S$ is an isomorphism for all $S \to B$.

For (iii), since restriction to $y \in B$ is a special case of base change, only the “if” direction requires argument. Suppose injectivity is satisfied for all $y \in B$, and let $S \to B$ be any morphism. Because field extensions are flat, we have that for any $s \in S$, the map $\mathcal{V}|_s \to \pi_*\mathcal{E}|_s$ is the base extension of $\mathcal{V}|_y \to \pi_*\mathcal{E}|_y$, where $y$ is the image of $s$ in $B$, and is in particular injective. By Remark B.2.2, we may choose $\mathcal{F}^i$ as in (i); then $\pi_*\mathcal{E}|_s$ is the kernel of $\mathcal{F}^0|_s \to \mathcal{F}_1|_s$, so we conclude by Nakayama’s lemma that $\mathcal{V}|_S \to \mathcal{F}^0|_S$ is injective, and hence that $\mathcal{V}|_S \to \pi_*\mathcal{E}|_S$ is likewise injective, as desired.

Similarly, if we are given only that (B.2.1) is injective for a single $y \in B$, since injectivity of $\mathcal{V}|_y \to \mathcal{F}^0|_y$ is the complement of a determinantal locus, it holds on an open neighborhood of $y$, and we conclude that $\mathcal{V}$ is a subbundle on that neighborhood.

Finally, (iv) is straightforward: let $\mathcal{D} = \mathcal{V}_2/\mathcal{V}_1$, and let $y \in B$ be any point of $B$. If we base change to $\text{Spec} \kappa(y)$, we get from the definition of subbundle that $(\mathcal{V}_1)_y \to (\mathcal{V}_2)_y$, so since both have dimension $r$, we get $\mathcal{D}_y = 0$, and by Nakayama’s lemma we conclude $\mathcal{D} = 0$ and $\mathcal{V}_1 = \mathcal{V}_2$, as asserted. □
B.3. Generalized determinantal loci. We now give the foundational definition generalizing determinantal loci to pushforwards.

**Definition B.3.1.** Let \( \pi : X \to B \) be a proper morphism, locally of finite presentation, and let \( \mathcal{E}, \mathcal{F} \) be quasicoherent sheaves on \( X \), locally of finite presentation and flat over \( B \). Suppose we are given also a morphism \( f : \mathcal{E} \to \mathcal{F} \), as well as a subbundle \( \mathcal{W} \subseteq \pi_* \mathcal{F} \). Then the \( k \)th vanishing locus of the induced map

\[
\pi_* \mathcal{E} \to \pi_* \mathcal{F} / \mathcal{W}
\]

is defined as follows: in the case that \( B \) is affine, let \( \mathcal{E}^\bullet \) and \( \mathcal{F}^\bullet \) be finite complexes of finite locally free modules on \( B \) representing \( R\pi_* \mathcal{E} \) and \( R\pi_* \mathcal{F} \) respectively, and having an induced morphism \( f^\bullet : \mathcal{E}^\bullet \to \mathcal{F}^\bullet \). Then we obtain an induced morphism

\[
\mathcal{E}^0 \to \mathcal{E}^1 \oplus \mathcal{F}^0 / \mathcal{W},
\]

and we define the desired locus to be the \( k \)th vanishing locus of this morphism. For general \( B \), we define the desired locus locally on an affine open cover.

Note that we obtain a generalization of standard determinantal loci, since we can recover the usual notion by setting \( \pi = \text{id} \) and \( \mathcal{W} = 0 \); then we have \( \mathcal{E}^\bullet = \mathcal{E} \) and \( \mathcal{F}^\bullet = \mathcal{F} \). The most basic result for our generalized version is as follows.

**Proposition B.3.2.** In the situation of Definition B.3.1, the \( k \)th vanishing locus is a well-defined closed subscheme of \( B \), stable under base change, and having as its underlying set the set of points \( y \in B \) on which the map

\[
H^0(X_y, \mathcal{E}|_y) \to H^0(X_y, \mathcal{F}|_y)/\mathcal{W}|_y
\]

has kernel of dimension at least \( k \).

Moreover, the vanishing locus is determined as a subscheme by the isomorphism classes of the kernels of

\[
\pi_*(\mathcal{E}|_Z) \to \pi_*(\mathcal{F}|_Z)/\mathcal{W}|_Z
\]

as \( Z \) varies over closed subschemes of \( B \).

**Proof.** Because the \( \mathcal{E}^\bullet \) and \( \mathcal{F}^\bullet \) are flat, they continue to compute \( R\pi_* \mathcal{E} \) and \( R\pi_* \mathcal{F} \) after arbitrary base change. Thus, after arbitrary base change \( S \to B \), the kernel of

\[
\mathcal{E}^0 \to \mathcal{E}^1 \oplus \mathcal{F}^0 / \mathcal{W}
\]

is equal to the kernel of

\[
\pi_{S*}\mathcal{E}|_S \to \pi_{S*}\mathcal{F}|_S / \mathcal{W}|_S
\]

and we see immediately that the set-theoretic support is as asserted. We further conclude from Lemma B.1.2 that our definition is independent of choices, and is determined by kernels as stated. Finally, since also formation of determinantal ideals commutes with pullback, the construction is stable under base change. \( \square \)

**Example B.3.3.** Considering the special case that \( \mathcal{F} = 0 \), we find that the \( k \)th vanishing locus is supported on the set of points of \( y \in B \) on which \( H^0(X_y, \mathcal{E}|_X) \) has dimension at least \( k \).

In particular, if \( Y \) is a variety over an algebraically closed field \( F \), and \( B \) is \( \text{Pic}(Y/F) \), and we set \( X = Y \times_F B \), and let \( \mathcal{E} \) be the universal line bundle, then we recover a new way of looking at classical Brill-Noether loci, which generalizes to higher-dimensional varieties.
We next translate some standard statements relating kernels to determinantal loci into our generalized context.

**Proposition B.3.4.** In the situation of Definition B.3.1, suppose there exists a subbundle \( \mathcal{V} \) of \( \pi_* \mathcal{E} \) of rank \( k \) which is contained in the kernel of \( \pi_* \mathcal{E} \to \pi_* \mathcal{F}/\mathcal{W} \).

Then the \( k \)th vanishing locus is equal (scheme-theoretically) to \( B \).

Conversely, if the \( k \)th vanishing locus is equal (scheme-theoretically) to \( B \), and if the \((k+1)\)st vanishing locus is empty, then the kernel \( \mathcal{K} \) of \( \pi_* \mathcal{E} \to \pi_* \mathcal{F}/\mathcal{W} \) is a rank-\( k \) subbundle of \( \pi_* \mathcal{E} \), and we have moreover that the kernel commutes with base change: that is, for all \( S \to B \), the kernel of

\[
\pi_{S*}\mathcal{E}|_S \to \pi_{S*}\mathcal{F}|_S/\mathcal{W}|_S
\]

is equal to \( \mathcal{K}|_S \).

**Proof.** Suppose that \( \mathcal{E}^0 \) has rank \( r \). For the first assertion, we have by Lemma B.2.3 that \( \mathcal{V} \) gives a subbundle of \( \mathcal{E}^0 \) in the usual sense. By definition, the vanishing locus in question is equal to the \( k \)th vanishing locus of

\[
\mathcal{E}^0 \to \mathcal{E}^1 \oplus \mathcal{F}^0/\mathcal{W},
\]

and by construction \( \mathcal{V} \) is contained in the kernel of this map. The statement thus reduces to the standard fact that if a morphism of locally free sheaves contains a subbundle of rank \( k \) in its kernel, then the \( k \)th vanishing locus is the entire scheme.

Conversely, if we have that (B.3.1) has fixed rank \( r - k \) on \( B \) (i.e., the \((r - k)\)th determinantal locus is all of \( B \), while the \((r - k + 1)\)st is empty) then the kernel is a subbundle of rank \( k \), which is universal under base change. Indeed, by Proposition 20.8 (see also comments on p. 407) of [Eis95], we have that the image is a subbundle of rank \( r - k \), and hence the kernel is a subbundle of rank \( k \), and commutes with base change. Moreover, since the kernel of (B.3.1) is equal to the kernel of \( \pi_* \mathcal{E} \to \pi_* \mathcal{F}/\mathcal{W} \) even after arbitrary base change, we obtain the desired statement.

We conclude with two corollaries translating our results on determinantal loci to the generalized context.

**Corollary B.3.5.** Suppose we are in the situation of Definition B.3.1, and we are given also \( \mathcal{E}', \mathcal{F}', \mathcal{F}' \) and \( \mathcal{W}' \) as in the same definition, as well as morphisms \( g: \mathcal{E} \to \mathcal{E}' \) and \( h: \mathcal{F} \to \mathcal{F}' \) satisfying:

(I) \( h \circ f = f' \circ g \);

(II) \( h(\mathcal{W}) \subseteq \mathcal{W}' \);

(III) at every point \( y \in B \), we have that \( g \) induces an injection

\[
\ker (\pi_* (\mathcal{E}|_y) \to \pi_* (\mathcal{F}|_y)) \to \ker (\pi_* (\mathcal{E}'|_y) \to \pi_* (\mathcal{F}'|_y)).
\]

Then for every \( k \), the \( k \)th vanishing locus of \( \pi_* \mathcal{E} \to \pi_* \mathcal{F}/\mathcal{W} \) is a closed subscheme of the \( k \)th vanishing locus of \( \pi_* \mathcal{E}' \to \pi_* \mathcal{F}'/\mathcal{W}' \).

Note that the first two conditions ensure that the kernel of \( \pi_* \mathcal{E} \to \pi_* \mathcal{F}/\mathcal{W} \) maps into the kernel of \( \pi_* \mathcal{E}' \to \pi_* \mathcal{F}'/\mathcal{W}' \), so the third is asserting simply that this map is universally injective. Under these conditions, it is clear that we have a set-theoretic inclusion of the loci in question.

**Proof.** We may obviously assume that \( B \) is affine, so that, continuing with the notation of Definition B.3.1, the vanishing loci in question are defined by determinantal loci of morphisms \( \mathcal{E}^0 \to \mathcal{E}^1 \oplus (\mathcal{F}/\mathcal{W}) \) and \( \mathcal{E}'^0 \to (\mathcal{E}')^1 \oplus ((\mathcal{F}')/\mathcal{W'}) \),
and conditions (I) and (II) imply that \( g \) and \( h \) induce a morphism from the first complex to the second. As in the proof of Proposition B.3.2, we have that formation of our complexes commutes with restriction to any point of \( B \), so (III) implies that the hypotheses of Proposition B.1.3 are satisfied, and we conclude the desired statement. □

**Corollary B.3.6.** Suppose we are in the situation of Definition B.3.1, and we are also given \( E_i, F_i, f_i \), and \( W_i \) as in the same definition for \( i = 1, 2 \). Suppose further that we have a locally free sheaf \( W' \) of rank \( k \) on \( S \), maps \( W' \to W_i \) for \( i = 1, 2 \) realizing \( W' \) as a subbundle of each, integers \( m_1, m_2 \) and a closed subscheme \( Z \) of \( B \) such that:

(I) for \( i = 1, 2 \), the \( m_i \)th vanishing locus of \( \pi_* E_i \to \pi_* F_i/W_i \) contains \( Z \);

(II) for all closed subschemes \( Z' \) of \( Z \), we have that the kernel of \( \pi_{Z'} E_i/W_i|Z' \to \pi_{Z'} F_i/W_i|Z' \) is isomorphic to the fibered product of the kernels of \( \pi_{Z'} E_i/Z\to \pi_{Z'} F_i/Z \) over \( W'|Z' \).

Then the \((m_1 + m_2 - k)\)th vanishing locus of \( \pi_* E \to \pi_* F/W \) contains \( Z \).

**Proof.** Once again, the fact that the complexes in question are compatible with base change immediately reduces the statement of the corollary to that of Proposition B.1.5. □

**Remark B.3.7.** If, in the situation of Definition B.3.1, we are given also a subbundle \( \mathcal{V} \subset \pi_* E \), one can make an analogous definition of the \( k \)th vanishing locus of the induced map

\[ \mathcal{V} \to \pi_* F/W. \]

However, in full generality we do not know that \( \pi_* E \) is a subbundle of itself, so this version is not a generalization of the stated version. We have stated the definition in the form in which we apply it.
### Index of Notation and Terminology

- \( v_1(\varepsilon) \), Notation 3.1.3
- \( e_1(\varepsilon) \), Notation 3.1.3
- \( P(w, (e_1, v_1), \ldots, (e_m, v_m)) \), Notation 3.1.3
- \( v_\Pi(\varepsilon) \), Notation 3.1.3
- \( P(w, v_1, \ldots, v_m) \), Notation 3.1.3
- \( w_v \), Notation 3.1.6
- \( t_{(\varepsilon, v)}(w) \), Notation 3.4.8
- \( G_I \), Definition 3.1.2
- \( \bar{G}_{II} \), Definition 3.4.9
- \( \mathcal{O}_{(\varepsilon, v)} \), Notation 3.2.2
- \( \mathcal{O}_v \), Notation 3.3.3
- \( \mathcal{O}_{w, w'} \), Notation 3.2.2 and 3.3.3
- \( \mathcal{O}'_{w, w'} \), Notation 6.1.1
- \( \mathcal{E}_w \), Notation 3.2.2 and 3.3.3
- \( f_x \), Notation 3.2.3 and 3.3.4
- \( f_P \), Notation 3.4.5
- \( D_{w, v} \), Notation 4.1.8
- \( \mathcal{E}^v(w) \), Notation 4.1.9
- \( V^v(w) \), Notation 4.1.9

\( (P, Q) \)-adapted, Definition 5.1.1
\( (P, Q) \)-adaptable, Definition 5.1.1

almost local, Definition 2.2.2

chain adaptable, Definition 5.2.1

constrained, Definitions 4.3.2 and 6.1.3

limit linear series, Eisenbud-Harris-Teixidor (or EHT), Definition 4.1.2

linked linear series, type-I, Definition 3.2.1

linked linear series, type-II, Definition 3.3.2

multidegree (in families), Definition 2.3.1

polarization, Definition 6.4.1

prelinked Grassmannian, Definition A.1.2

refined, Definitions 4.1.2 and 6.1.3

simple (linked linear series), Definitions 3.4.6 and 6.1.3

simple (point of linked Grassmannian), Definition A.1.4

smoothing family, Definition 2.1.1

semistable (vector bundle), Definition 6.4.1

stable (vector bundle), Definition 6.4.1

semistable (limit linear series), Definition 6.4.3

stable (limit linear series), Definition 6.4.3

\( \ell \)-semistable (vector bundle), Definition 6.4.2

\( \ell \)-stable (vector bundle), Definition 6.4.2

\( \ell \)-semistable (limit linear series), Definition 6.4.3

\( \ell \)-stable (limit linear series), Definition 6.4.3
subbundle (of a pushforward), Definition B.2.1
kth vanishing locus (of a morphism of vector bundles), Definition B.1.1
kth vanishing locus (of the pushforward of a morphism), Definition B.3.1

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