The general dual-polar Orlicz-Minkowski problem *

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Abstract

This paper gives a systematic study to the general dual-polar Orlicz-Minkowski problem (e.g., Problem 4.1). This problem involves the general dual volume $\tilde{V}_G(\cdot)$ recently proposed in [13, 15] in order to study the general dual Orlicz-Minkowski problem. As $\tilde{V}_G(\cdot)$ extends the volume and the $q$th dual volume, the general dual-polar Orlicz-Minkowski problem is “polar” to the recently initiated general dual Orlicz-Minkowski problem in [13, 15] and “dual” to the newly proposed polar Orlicz-Minkowski problem in [34]. The existence, continuity and uniqueness, if applicable, for the solutions to the general dual-polar Orlicz-Minkowski problem are established. Polytopal solutions and/or counterexamples to the general dual-polar Orlicz-Minkowski problem for discrete measures are also provided. Several variations of the general dual-polar Orlicz-Minkowski problem are discussed as well, in particular the one leading to the general Orlicz-Petty bodies.

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1 Introduction

Lutwak’s discovery of the $L_p$ surface area measure and the $L_p$ mixed volume [38] for $p > 1$ gave a new and thriving life to the Brunn-Minkowski theory. Among those fundamental objects related to the $L_p$ surface area measure and the $L_p$ mixed volume, the $L_p$ Minkowski problem (for $p = 1$ in [43, 44] by Minkowski and for $p \neq 1$ in [38] by Lutwak) and the $L_p$ affine surface area (for $p = 1$ in [1] by Blaschke, for $p > 1$ in [39] by Lutwak and for $p < 1$ in [50] by Schütt and Werner) arguably have the greatest influence. The former one aims to find convex bodies (i.e., convex compact sets in $\mathbb{R}^n$ with nonempty interiors) so that their $L_p$ surface area measures coincide with a pre-given nonzero finite Borel measure $\mu$ defined on the unit sphere $S^{n-1}$. The $L_p$ Minkowski problem has attracted tremendous attention in different areas, such as analysis, convex geometry, and partial differential equations (see e.g., [5, 8, 9, 22, 26, 41, 65, 67] among others). In particular, it is closely related to the far-reaching optimal mass transportation problem via the Monge-Ampère type equations. Solutions to the $L_p$ Minkowski problem have been used to develop the powerful tool of convexification for Sobolev functions and to establish the elegant $L_p$ affine Sobolev inequalities as well as the related Pólya-Szego principles, see e.g., [10, 19, 20, 40, 59, 60]. The latter one (i.e., the $L_p$ affine surface area) is more on the differential properties of convex bodies. It has many beautiful properties, including the affine invariant valuation and being 0 for polytopes (if $p > 0$); these properties make the $L_p$ affine surface area perfect geometric invariants in characterizing the affine valuations, the $L_p$ affine isoperimetric inequalities, and approximation of convex bodies by polytopes [16, 31, 32, 33, 49, 52]. The elegant integral expression for the $L_p$ affine surface area also leads to nice observations of its connection with the $f$-divergence [27, 45, 51]. It is worth to mention

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that the celebrated Blaschke-Santaló inequality was originally established as a consequence of the combination of the solutions to the $L_p$ Minkowski problem and the affine isoperimetric inequalities for the $L_p$ affine surface area (in particular, with both $p = 1$) (see e.g., [48] for details). In words, the importance of the $L_p$ Minkowski problem and the $L_p$ affine surface area cannot be over-emphasized.

The $L_p$ Minkowski problem and the $L_p$ affine surface area were apparently developed in completely different approaches, however, they were nicely connected through the $L_p$ geominimal surface area and the $L_p$ Petty bodies [39, 56, 62]. As the bridge to connect several geometries (affine, Minkowski and relative), the $L_p$ geominimal surface area is crucial in convex geometry and, in particular, share many properties similar to those for the $L_p$ affine surface area. Let $\mathcal{K}_0^n$ be the set of convex compact sets in $\mathbb{R}^n$ with the origin $o$ in their interiors. Finding the $L_p$ Petty bodies of $K \in \mathcal{K}_0^n$ for $p \in \mathbb{R} \setminus \{0, -n\}$ requires to solve the following optimization problem (with $\mu$ being the $L_p$ surface area measure of $K$):

$$\inf / \sup \left\{ \int_{S^{n-1}} h_{L^p}^p(u) \, d\mu(u) : \ L \in \mathcal{K}_0^n \text{ and } V(L) = V(B^n) \right\}, \quad (1.1)$$

where $B^n$ is the unit Euclidean ball in $\mathbb{R}^n$, $V(\cdot)$ stands for the volume, $L^o$ denotes the polar body of $L \in \mathcal{K}_0^n$, and $h_L$ is the support function of $L$ (see Section 2 for notations). As explained in [34], the $L_p$ Minkowski problem can be viewed as the “polarity” of (1.1) (in particular, for $\mu$ nice enough such as $\mu$ being even) aiming to find convex bodies (ideally in $\mathcal{K}_0^n$) to solve the optimization problem similar to (1.1), namely with $L^o$ replaced by $L$. On the other hand, the $L_p$ affine surface area of $K \in \mathcal{K}_0^n$ can be defined through a formula similar to (1.1) for $\mu$ being the $L_p$ surface area measure of $K$, but with $L \in \mathcal{K}_0^n$ and $h_{L^p}$ replaced by $L$ belong to star bodies about the origin and, respectively, $\rho_L^{-1}$ where $\rho_L$ is the radial function of $L$ (see [39, 56, 62] for more details).

The main purpose of this article is to give a systematic study to the general dual-polar Orlicz-Minkowski problem, which extends problem (1.1) in the arguably most general way: with the function $t^p$ (from the integrand of the objective functional) and $V(L)$ in problem (1.1) replaced by a (general nonhomogeneous) continuous function $\varphi : (0, \infty) \to (0, \infty)$ and, respectively, $\tilde{V}_G(L)$, the general dual volume of $L$, formulated by

$$\tilde{V}_G(L) = \int_{S^{n-1}} G(\rho_L(u), u) \, du$$

with $du$ the spherical measure of $S^{n-1}$. Namely, we pose the following problem: Under what conditions on a nonzero finite Borel measure $\mu$ defined on $S^{n-1}$, continuous functions $\varphi : (0, \infty) \to (0, \infty)$ and $G : (0, \infty) \times S^{n-1} \to (0, \infty)$ can we find a convex body $K \in \mathcal{K}_0^n$ solving the following optimization problem:

$$\inf / \sup \left\{ \int_{S^{n-1}} \varphi(h_{Q^o}(u)) \, d\mu(u) : \ Q \in \mathcal{K}_0^n \text{ and } \tilde{V}_G(Q) = \tilde{V}_G(B^n) \right\}. \quad (1.2)$$

In particular, problem (1.2) becomes problem (1.1) when $\varphi(t) = t^p$ and $G(t, u) = t^n$. Moreover, problem (1.2) also contains as a special case the recent polar Orlicz-Minkowski problem introduced in [34] by Luo, Ye and Zhu, i.e., solving the following optimization problem:

$$\inf / \sup \left\{ \int_{S^{n-1}} \varphi(h_{Q^o}(u)) \, d\mu(u) : \ Q \in \mathcal{K}_0^n \text{ and } V(Q) = V(B^n) \right\}. \quad (1.3)$$

Note that closely related to (1.3) are the Orlicz affine and geominimal surface areas, which were proposed in [57, 58, 62]. In fact, one can observe that (1.2) not only generalizes (1.3), but also is
“dual” to (1.3). This is one of our motivations to study the general dual-polar Orlicz-Minkowski problem.

Another motivation for our general dual-polar Orlicz-Minkowski problem is its relation and close connection with the recent general dual Orlicz-Minkowski problem in [13] by Gardner, Hug, Weil, Xing and Ye, and in [15] by Gardner, Hug, Xing and Ye. Indeed, the fundamental geometric invariant $V_G(\cdot)$ was mainly introduced to derive the general dual Orlicz curvature measures, the key ingredients of the general dual Orlicz-Minkowski problem. Such Minkowski type problem extends not only the $L_p$ Minkowski problem by Lutwak [38] and its Orlicz counterpart by Haberl, Lutwak, Yang and Zhang [18], but also the recently initiated dual Minkowski problem by Huang, Lutwak, Yang and Zhang [24], the $L_p$ dual Minkowski problem by Lutwak, Yang and Zhang [42], the dual Orlicz-Minkowski problem by Zhu, Xing and Ye [63], and the general dual Orlicz-Minkowski problem by Xing and Ye [55]. Here we would like to emphasize the elegance and significance of the groundbreaking work [24], where the authors, at the first time, proved the far-reaching variational formula for the $q$th dual volume (i.e., the case when $G(t, u) = t^q/n$ for $q \neq 0$) in terms of the logarithmic addition. Such variational formula can be viewed as a perfect vinculum to deeply connect the two closely related but quite different branches of convex geometry: the $L_p$ Brunn-Minkowski theory for convex bodies and its dual theory for star bodies. The variational formula has been quickly extended to other cases such as [42, 55, 63], and achieves its most generality when the $q$th volume and the logarithmic addition are replaced by the general dual volume $\tilde{V}_G(\cdot)$ and an Orlicz addition involving $\varphi$, respectively, in [13]. In many circumstance, solving the general dual Orlicz-Minkowski problem requires to find solutions to the following optimization problem:

$$\inf \left/ \sup \left\{ \int_{S^{n-1}} \varphi(hQ(u))d\mu(u) : Q \in \mathcal{K}_o^{n} \text{ and } \tilde{V}_G(Q) = \tilde{V}_G(B^n) \right\} \right..$$ (1.4)

In particular, if $G(t, u) = t^n/n$, (1.4) recovers the Orlicz-Minkowski problem [18]. In view of (1.2), one sees that (1.2) is “polar” to (1.4). It is our belief that, like the general dual Orlicz-Minkowski problem, the newly proposed general dual-polar Orlicz-Minkowski problem will constitute one of the core objectives in the rapidly developing dual Orlicz-Brunn-Minkowski theory recently started from the work [14] by Gardner, Hug, Weil and Ye, and independently the work [64] by Zhu, Zhou and Xu.

Our paper is organized as follows. Section 2 provides a brief collection of notations and well-known facts from convex geometry. In Section 3, we will introduce the homogeneous general dual volume, $\tilde{V}_G(\cdot)$, a geometric invariant sharing properties rather similar to those for the general dual volume $V_G(\cdot)$. Properties of $\tilde{V}_G(\cdot)$, such as, the homogeneity, continuity and monotonicity, are proved in Proposition 3.2. Lemma 3.3 provides reasonable conditions on $G : (0, \infty) \times S^{n-1} \to (0, \infty)$ such that, roughly speaking, if $Q_i \to Q_0$ in the Hausdorff metric with $Q_i \in \mathcal{K}_o^{n}$ for each $i \geq 1$ and $\{\tilde{V}_G(Q_i^n)\}_{i \geq 1}$ (or $\{\tilde{V}_E(Q_i^n)\}_{i \geq 1}$, respectively) as a sequence of real numbers is bounded, then $Q_0 \in \mathcal{K}_o^{n}$. This lemma is the key tool to show the existence of solutions to our general dual-polar Orlicz-Minkowski problem (i.e., (1.2)).

Section 4 dedicates to establish the continuity, uniqueness, and existence of solutions to the general dual-polar Orlicz-Minkowski problem. In particular, we first obtain the polytopal solutions to the general dual-polar Orlicz-Minkowski problem when the measure $\mu$ is discrete under certain conditions such as $\varphi$ being increasing and the infimum in (1.2) being considered; the detailed statements can be found in Theorem 4.3. In Proposition 4.4, the nonexistence of solutions to the general dual-polar Orlicz-Minkowski problem for discrete measures are proved by counterexamples if the supremum in (1.2) is considered, or if the infimum is considered with $\varphi$ being decreasing. As $\tilde{V}_G(\cdot)$ and $\tilde{V}_E(\cdot)$ are not invariant under volume-preserving linear transforms on $\mathbb{R}^n$, our calculations
in Proposition 4.4 are more delicate than those in [34] where the volume is considered. Our main results are given in Theorem 4.7 and Corollary 4.8, where the existence, uniqueness and continuity of solutions to the general dual-polar Orlicz-Minkowski problem for general nonzero finite Borel measure \( \mu \) (instead of discrete measures) are provided. Our proofs are based on the approximation of convex bodies by polytopes.

Section 5 aims to investigate several variations of the general dual-polar Orlicz-Minkowski problem, including those leading to the most general definitions extending the \( L_p \) Petty bodies (see Section 5.3). In Section 5.1, the objective functional \( \int_{S^{n-1}} \varphi(h_{Q^p}(u)) d\mu(u) \) in (1.2) will be replaced by the “Orlicz norm” \( \|h_{Q^p}\|_{\mu,\varphi} \). In this case, the continuity, uniqueness, and existence of solutions are rather similar to those in Section 4. The second variation, considered in Section 5.2, is quite different from the general dual-polar Orlicz-Minkowski problem (1.2). It replaces the general dual volume \( \tilde{V}_G(\cdot) \) by the general volume formulated as follows: for \( K \in \mathcal{K}_n \),

\[
V_G(K) = \int_{S^{n-1}} G(h_K(u), u) \, dS_K(u),
\]

where \( S_K \) denotes the surface area measure of \( K \) defined on \( S^{n-1} \). Although the geometric invariant \( V_G(\cdot) \) has most properties required to solve the related polar Orlicz-Minkowski problem, it lacks the monotonicity in terms of set inclusion, a key ingredient in the proofs of the main results in Section 4. With the help of the celebrated isoperimetric inequality, we are able to find a substitution of Lemma 3.3 for \( V_G(\cdot) \) and this will be stated in Lemma 5.9. Consequently, the existence of solutions to the related polar Orlicz-Minkowski problem is established in Theorem 5.10.

2 Preliminaries and Notations

In the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \), \( B^n \) denotes the unit Euclidean ball and \( S^{n-1} \) denotes the unit sphere. Denote by \( \{e_1, \ldots, e_n\} \) the canonical orthonormal basis of \( \mathbb{R}^n \). By \( \mathcal{K}^n \) we mean the set of all compact convex subsets of \( \mathbb{R}^n \). For each \( K \in \mathcal{K}^n \), one can define its support function \( h_K: S^{n-1} \to \mathbb{R} \) by \( h_K(u) = \max_{x \in K} \langle x, u \rangle \) for any \( u \in S^{n-1} \), where \( \langle x, y \rangle \) is the usual inner product in \( \mathbb{R}^n \). A natural metric on \( \mathcal{K}^n \) is the Hausdorff metric \( d_H \), where for \( K, L \in \mathcal{K}^n \), one has

\[
d_H(K, L) = \|h_K - h_L\|_\infty = \max_{u \in S^{n-1}} |h_K(u) - h_L(u)|.
\]

We say the sequence \( K_1, K_2, \ldots \) converges to \( K \in \mathcal{K}^n \) in the Hausdorff metric, denoted by \( K_i \to K \), if \( \lim_{i \to \infty} d_H(K_i, K) = 0 \). The Blaschke selection theorem provides a powerful machinery to solve Minkowski type problems. It asserts that if \( K_i \in \mathcal{K}^n \) and there exists a constant \( R \) such that \( K_i \subset RB^n \) for all \( i \in \mathbb{N} \), then there exist a subsequence \( \{K_{i_j}\}_{j \geq 1} \) of \( \{K_i\}_{i \geq 1} \) and \( K \in \mathcal{K}^n \) such that \( K_{i_j} \to K \) as \( j \to \infty \) in the Hausdorff metric.

Denote by \( o \) the origin of \( \mathbb{R}^n \). A convex body in \( \mathbb{R}^n \) is a compact convex subset of \( \mathbb{R}^n \) with nonempty interior. Let \( \mathcal{K}_o^n \subset \mathcal{K}^n \) denote the set of all convex bodies containing \( o \). For \( K \in \mathcal{K}_o^n \), \( h_K \) is a nonnegative function defined on \( S^{n-1} \). Besides the support function, for \( K \in \mathcal{K}_o^n \), one can also define the radial function \( \rho_K : S^{n-1} \to [0, \infty) \) by \( \rho_K(u) = \max\{\lambda \geq 0 : \lambda u \in K\} \) for \( u \in S^{n-1} \). In particular, \( \rho_K(u)u \in \partial K \), where \( \partial K \) denotes the boundary of \( K \). For convenience, in later context, we will also use \( \operatorname{int} K \) to denote the interior of \( K \). It can be easily checked that \( \rho_{sK} = s \cdot \rho_K \) and \( h_{sK} = s \cdot h_K \) for \( s > 0 \) and \( K \in \mathcal{K}_o^n \).

Associated to each \( K \in \mathcal{K}_o^n \) is the surface area measure \( S_K(\cdot) \) defined on \( S^{n-1} \) which may be formulated by \( S_K(\eta) = \mathcal{H}^{n-1}(\nu_{K}^{-1}(\eta)) \) for each Borel set \( \eta \subset S^{n-1} \) (see e.g., [48]), where \( \mathcal{H}^{n-1} \) is the \((n - 1)\) dimensional Hausdorff measure of \( \partial K \), \( \nu_K \) denotes the Gauss map of \( K \) and \( \nu_{K}^{-1} \)
denotes the reverse Gauss map of $K$. It is worthwhile to mention that for $K \in \mathcal{K}_0^n$, its volume, denoted by $V(K)$, takes the following forms:

$$V(K) = \frac{1}{n} \int_{S^{n-1}} h_K(u) dS_K(u) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^n du,$$

where $du$ denotes the spherical measure of $S^{n-1}$ (i.e., the Hausdorff measure on $S^{n-1}$).

Let $\mathcal{K}_0^n \subset \mathcal{K}_n^n$ be the set of convex bodies in $\mathbb{R}^n$ with the origin $o$ in their interiors. For each $K \in \mathcal{K}_0^n$, both $h_K$ and $\rho_K$ are strictly positive functions on $S^{n-1}$. A useful fact is that $K_i \to K$, with $K_i \in \mathcal{K}_0^n$ for all $i \in \mathbb{N}$ and $K \in \mathcal{K}_0^n$, in the Hausdorff metric is equivalent to $\rho_{K_i}$ convergent to $\rho_K$ uniformly on $S^{n-1}$. The polar body of $K \in \mathcal{K}_n^n$, denoted by $K^\circ$, may be formulated by

$$K^\circ = \{ x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for any } y \in K \}.$$ 

An easily established fact is that if $K \in \mathcal{K}_0^n$, then $K^\circ \in \mathcal{K}_0^n$ and $K = K^\circ$. Moreover, $ho_{K^\circ}(u) \cdot h_K(u) = 1$ for any $K \in \mathcal{K}_0^n$ and for any $u \in S^{n-1}$. Clearly, $S_{tK} = t^{n-1}S_K$ for any $t > 0$ and $K \in \mathcal{K}_0^n$. For more background in convex geometry, please see e.g., [13, Lemma 6.1].

Let $G : (0, \infty) \times S^{n-1} \to (0, \infty)$ be a continuous function. The general dual volume of $K \in \mathcal{K}_0^n$, denoted by $\widetilde{V}_G(K)$, was proposed in [13] as follows:

$$\widetilde{V}_G(K) = \int_{S^{n-1}} G(\rho_K(u), u) du. \quad (2.1)$$

When $G : [0, \infty) \times S^{n-1} \to [0, \infty)$, the general dual volume can be defined for $K \in \mathcal{K}_0^n$ with the formula same as (2.1). Note that the general dual volume $\widetilde{V}_G(\cdot)$ was used to derive the general dual Orlicz curvature measures and hence plays central roles in establishing the existence of solutions to the recently proposed general dual Orlicz-Minkowski problem [13, 15]. When $G(t, u) = \frac{1}{n}t^n$, one gets $\widetilde{V}_G(K) = V(K)$, and when $G(t, u) = \frac{1}{n}t^q$ for $q \neq 0, n$, $\widetilde{V}_G(K)$ becomes the $q$th dual volume $\tilde{V}_q(K)$ which plays fundamental roles in the dual Brunn-Minkowski theory [35, 36, 37] and the $L_p$ dual Minkowski problem (see e.g., [2, 4, 6, 7, 24, 25, 42, 61]). When $G(t, u) = G(t, e_1)$ for all $(t, u) \in (0, \infty) \times S^{n-1}$, $\widetilde{V}_G(K)$ becomes the dual Orlicz-quermassintegral in [63]; while if $G(t, u) = \int_0^t \phi(ru)r^{n-1} dr$ or $G(t, u) = \int_0^\infty \phi(ru)r^{n-1} dr$ for some function $\phi : \mathbb{R}^n \to (0, \infty)$, then $\widetilde{V}_G(K)$ becomes the general dual Orlicz-quermassintegral in [55]. See [13] for more special cases. It has been proved that $\widetilde{V}_G(K_i) \to \widetilde{V}_G(K)$ for $G : (0, \infty) \times S^{n-1} \to (0, \infty)$ being continuous and $K_i \to K$ with $K, K_i \in \mathcal{K}_0^n$ for all $i \in \mathbb{N}$ [13, Lemma 6.1] or $G : [0, \infty) \times S^{n-1} \to [0, \infty)$ being continuous and $K_i \to K$ with $K, K_i \in \mathcal{K}_0^n$ for all $i \in \mathbb{N}$ [15, Lemma 3.2]. It is easy to check that $\widetilde{V}_G(\cdot)$ in general is not homogeneous on $\mathcal{K}_0^n$ and/or $\mathcal{K}_n^n$. Note that the general dual volume $\widetilde{V}_G(\cdot)$ can be defined not only for convex bodies, but also for star-shaped sets, see [13] for more details.

The following property may be useful in later context. Denote by $O(n)$ the set of all orthogonal matrices on $\mathbb{R}^n$, that is, for any $T \in O(n)$, one has $TT^t = T^tT = I_n$, where $T^t$ denotes the transpose of $T$ and $I_n$ is the identity matrix on $\mathbb{R}^n$.

**Proposition 2.1.** Let $K \in \mathcal{K}_0^n$. If $G(t, u) = \phi(t)$ for all $(t, u) \in (0, \infty) \times S^{n-1}$ with $\phi : (0, \infty) \to (0, \infty)$ being a continuous function, then $\widetilde{V}_G(TK) = \widetilde{V}_G(K)$.

**Proof.** Let $G(t, u) = \phi(t)$ for all $t > 0$ and $u \in S^{n-1}$. For $K \in \mathcal{K}_0^n$ and $T \in O(n)$, then the determinant of $T$ is $\pm 1$ and

$$\widetilde{V}_G(TK) = \int_{S^{n-1}} \phi(\rho_{TK}(u)) du = \int_{S^{n-1}} \phi(\rho_K(T^t u)) du = \int_{S^{n-1}} \phi(\rho_K(u)) dv = \widetilde{V}_G(K),$$

where $du$ denotes the spherical measure of $S^{n-1}$. Note that $\phi(\cdot)$ is a continuous function on $(0, \infty)$.
if letting $T^tu = v$. This completes the proof. \hfill $\square$

In later context, we will employ Proposition 2.1 to $G(t, u) = \frac{1}{n}t^q$ for $0 \neq q \in \mathbb{R}$, which implies $\tilde{V}_q(T) = \tilde{V}_q(K)$ for all $T \in \mathcal{O}(n)$ and all $K \in \mathcal{K}^\circ$.  

The following result is an easy consequence of the weak convergence of $\mu_i \to \mu$, but plays essential roles in our later context. Its proof is simple and will be omitted.

**Lemma 2.2.** Let $\mu, \mu_i$ for each $i \in \mathbb{N}$ be nonzero finite Borel measures on $S^{n-1}$ such that $\mu_i \to \mu$ weakly. Let $f, f_i$ for each $i \in \mathbb{N}$ be continuous functions on $S^{n-1}$ such that $f_i \to f$ uniformly on $S^{n-1}$. Then,  

$$\lim_{i \to \infty} \int_{S^{n-1}} f_i d\mu_i = \int_{S^{n-1}} f d\mu.$$  

### 3 The homogeneous general dual volumes and properties

Throughout this paper, $G : (0, \infty) \times S^{n-1} \to (0, \infty)$ is always assumed to be continuous. In this section, we will define the homogeneous general dual volume and discuss related properties. For simplicity, let  

$$\mathcal{G}_I = \left\{ G : G(t, \cdot) \text{ is continuous, strictly increasing on } t, \lim_{t \to 0^+} G(t, \cdot) = 0, \lim_{t \to \infty} G(t, \cdot) = \infty \right\},$$  

$$\mathcal{G}_d = \left\{ G : G(t, \cdot) \text{ is continuous, strictly decreasing on } t, \lim_{t \to 0^+} G(t, \cdot) = \infty, \lim_{t \to \infty} G(t, \cdot) = 0 \right\}.$$  

The homogeneous general dual volume of $K \in \mathcal{K}^\circ$, denoted by $\hat{V}_G(K)$, can be formulated by  

$$\hat{V}_G(K) = \inf \left\{ \eta > 0 : \int_{S^{n-1}} G\left( \frac{\rho_K(u)}{\eta}, u \right) du \leq 1 \right\}, \quad \text{if } G \in \mathcal{G}_I, \quad (3.1)$$  

$$\hat{V}_G(K) = \inf \left\{ \eta > 0 : \int_{S^{n-1}} G\left( \frac{\rho_K(u)}{\eta}, u \right) du \geq 1 \right\}, \quad \text{if } G \in \mathcal{G}_d. \quad (3.2)$$  

The following proposition provides a more convenient formula for $\hat{V}_G(\cdot)$.

**Proposition 3.1.** Let $K \in \mathcal{K}^\circ$. For any $G \in \mathcal{G}_I \cup \mathcal{G}_d$, there exists a unique $\eta_0 > 0$ such that  

$$\int_{S^{n-1}} G\left( \frac{\rho_K(u)}{\eta_0}, u \right) du = 1. \quad (3.3)$$  

Moreover, $\eta_0 = \hat{V}_G(K)$.

**Proof.** The proof of this result is standard. For $\eta \in (0, \infty)$ and $K \in \mathcal{K}^\circ$, let $G \in \mathcal{G}_I$ and  

$$H_K(\eta) = \int_{S^{n-1}} G\left( \frac{\rho_K(u)}{\eta}, u \right) du.$$  

As $K \in \mathcal{K}^\circ$, there exist positive constants $r$ and $R$ such that $r \leq \rho_K \leq R$. Thus for any $u \in S^{n-1}$,  

$$\int_{S^{n-1}} G\left( \frac{r}{\eta}, u \right) du \leq H_K(\eta) \leq \int_{S^{n-1}} G\left( \frac{R}{\eta}, u \right) du. \quad (3.4)$$
This, together with \( G \in \mathcal{G}_I \) and Fatou's lemma, implies that
\[
\liminf_{\eta \to 0^+} H_K(\eta) \geq \liminf_{\eta \to 0^+} \int_{S^{n-1}} G \left( \frac{r}{\eta}, u \right) du \geq \int_{S^{n-1}} \liminf_{\eta \to 0^+} G \left( \frac{r}{\eta}, u \right) du = \infty.
\]
On the other hand, the dominated convergence theorem yields, by (3.4), that
\[
\lim_{\eta \to \infty} H_K(\eta) \leq \lim_{\eta \to \infty} \int_{S^{n-1}} G \left( \frac{R}{\eta}, u \right) du = \int_{S^{n-1}} \lim_{\eta \to \infty} G \left( \frac{R}{\eta}, u \right) du = 0.
\]
Thus, \( \lim_{\eta \to 0^+} H_K(\eta) = \infty \) and \( \lim_{\eta \to \infty} H_K(\eta) = 0 \). As \( G \in \mathcal{G}_I \) is continuous and strictly increasing, \( H_K(\eta) \) is clearly continuous and strictly decreasing on \( \eta \in (0, \infty) \). Hence, there exists a unique \( \eta_0 > 0 \) such that \( H_K(\eta_0) = 1 \), which proves (3.3). Clearly \( \eta_0 = \hat{V}_G(K) \) by (3.1).

The case for \( G \in \mathcal{G}_I \) follows along the similar lines as above, and its proof will be omitted. \( \square \)

Clearly, if \( G(t, u) = t^q/n \) with \( q \neq 0 \) for all \( (t, u) \in (0, \infty) \times S^{n-1} \), then
\[
\hat{V}_G(K) = \left( \frac{1}{n} \int_{S^{n-1}} \rho_K^q(u) du \right)^{1/q} = (\hat{V}_q(K))^{1/q}.
\]
Properties for \( \hat{V}_G(\cdot) \) are summarized in the following proposition.

**Proposition 3.2.** Let \( G \in \mathcal{G}_I \cup \mathcal{G}_H \). Then \( \hat{V}_G(\cdot) \) has the following properties.

i) \( \hat{V}_G(\cdot) \) is homogeneous, that is, \( \hat{V}_G(sK) = s\hat{V}_G(K) \) holds for all \( s > 0 \) and all \( K \in \mathcal{X}_n \).

ii) \( \hat{V}_G(\cdot) \) is continuous on \( \mathcal{X}_n \) in terms of the Hausdorff metric, that is, for any sequence \( \{K_i\}_{i \geq 1} \) such that \( K_i \in \mathcal{X}_n \) for all \( i \in \mathbb{N} \) and \( K_i \to K \in \mathcal{X}_n \), then \( \hat{V}_G(K_i) \to \hat{V}_G(K) \).

iii) \( \hat{V}_G(\cdot) \) is strictly increasing, that is, for any \( K, L \in \mathcal{X}_n \) such that \( K \subset L \), then \( \hat{V}_G(K) < \hat{V}_G(L) \).

**Proof.** i) The desired argument follows trivially from Proposition 3.1, and \( \rho_{sK} = s\rho_K \) for all \( s > 0 \).

ii) Let \( K_i \in \mathcal{X}_n \) for all \( i \in \mathbb{N} \) and \( K_i \to K \in \mathcal{X}_n \). Then \( \rho_{K_i} \to \rho_K \) uniformly on \( S^{n-1} \). Moreover, there exist two positive constants \( r_K < R_K \) such that \( r_K \leq \rho_K \leq R_K \) and \( r_K \leq \rho_{K_i} \leq R_K \) for all \( i \in \mathbb{N} \). For \( G \in \mathcal{G}_I \), it follows from Proposition 3.1 and (3.4) that for each \( i \in \mathbb{N} \),
\[
\int_{S^{n-1}} G \left( \frac{r_K}{\hat{V}_G(K_i)}, u \right) du \leq 1 = \int_{S^{n-1}} G \left( \frac{\rho_{K_i}(u)}{\hat{V}_G(K_i)}, u \right) du \leq \int_{S^{n-1}} G \left( \frac{R_K}{\hat{V}_G(K_i)}, u \right) du.
\]
Suppose that \( \inf_{i \in \mathbb{N}} \hat{V}_G(K_i) = 0 \), and without loss of generality, assume that \( \lim_{i \to \infty} \hat{V}_G(K_i) = 0 \). Then for any \( \varepsilon > 0 \), there exists \( i_\varepsilon \in \mathbb{N} \) such that \( \hat{V}_G(K_i) < \varepsilon \) for all \( i > i_\varepsilon \). Thus, for \( i > i_\varepsilon \),
\[
\int_{S^{n-1}} G \left( \frac{r_K}{\varepsilon}, u \right) du \leq \int_{S^{n-1}} G \left( \frac{r_K}{\hat{V}_G(K_i)}, u \right) du \leq 1.
\]
Fatou's lemma and the fact that \( \lim_{t \to \infty} G(t, \cdot) = \infty \) yield
\[
\infty = \int_{S^{n-1}} \liminf_{\varepsilon \to 0^+} G \left( \frac{r_K}{\varepsilon}, u \right) du \leq \liminf_{\varepsilon \to 0^+} \int_{S^{n-1}} G \left( \frac{r_K}{\varepsilon}, u \right) du \leq 1,
\]
a contradiction. Hence, \( A_1 = \inf_{i \in \mathbb{N}} \hat{V}_G(K_i) > 0 \). Moreover, for all \( u \in S^{n-1} \) and all \( i \in \mathbb{N} \),
\[
G \left( \frac{\rho_{K_i}(u)}{\hat{V}_G(K_i)}, u \right) \leq G \left( \frac{R_K}{A_1}, u \right).
\]
Assume that $\lim_{i \to \infty} \tilde{V}_G(K_i) > \tilde{V}_G(K)$. There exists a subsequence $\{K_{i_j}\}$ of $\{K_i\}$ such that $\lim_{j \to \infty} \tilde{V}_G(K_{i_j}) > \tilde{V}_G(K)$. Together with Proposition 3.1 and the dominated convergence theorem, one has

$$1 = \lim_{j \to \infty} \int_{S^{n-1}} G\left( \frac{\rho_{K_{i_j}}(u)}{\tilde{V}_G(K_{i_j})}, u \right) du$$

$$= \int_{S^{n-1}} \lim_{j \to \infty} G\left( \frac{\rho_{K_{i_j}}(u)}{\tilde{V}_G(K_{i_j})}, u \right) du$$

$$= \int_{S^{n-1}} G\left( \frac{\rho_K(u)}{\lim_{j \to \infty} \tilde{V}_G(K_{i_j})}, u \right) du$$

$$< \int_{S^{n-1}} G\left( \frac{\rho_K(u)}{\tilde{V}_G(K)}, u \right) du = 1.$$ 

This is a contradiction and hence $\lim_{i \to \infty} \tilde{V}_G(K_i) \leq \tilde{V}_G(K)$. Similarly, $\lim \inf_{i \to \infty} \tilde{V}_G(K_i) \geq \tilde{V}_G(K)$ also holds, which leads to $\lim_{i \to \infty} \tilde{V}_G(K_i) = \tilde{V}_G(K)$ as desired.

The case for $G \in \mathscr{B}_d$ follows along the same lines, and its proof will be omitted.

iii) Let $G \in \mathscr{B}_I$ and let $K, L \in \mathscr{M}^{n}_0$ such that $K \subseteq L$. Then, the spherical measure of the set $E = \{u \in S^{n-1} : \rho_K(u) < \rho_L(u)\}$ is positive. By Proposition 3.1, one has

$$1 = \int_{S^{n-1}} G\left( \frac{\rho_L(u)}{\tilde{V}_G(L)}, u \right) du$$

$$= \int_{S^{n-1}} G\left( \frac{\rho_K(u)}{\tilde{V}_G(K)}, u \right) du$$

$$= \int_E G\left( \frac{\rho_K(u)}{\tilde{V}_G(K)}, u \right) du + \int_{S^{n-1} \setminus E} G\left( \frac{\rho_K(u)}{\tilde{V}_G(K)}, u \right) du$$

$$< \int_E G\left( \frac{\rho_L(u)}{\tilde{V}_G(K)}, u \right) du + \int_{S^{n-1} \setminus E} G\left( \frac{\rho_L(u)}{\tilde{V}_G(K)}, u \right) du$$

$$= \int_{S^{n-1}} G\left( \frac{\rho_L(u)}{\tilde{V}_G(K)}, u \right) du.$$ 

Then $\tilde{V}_G(K) < \tilde{V}_G(L)$ follows from the fact that $G(t, \cdot)$ is strictly increasing on $t \in (0, \infty)$.

The case for $G \in \mathscr{B}_d$ follows along the same lines, and its proof will be omitted.

For $G : (0, \infty) \times S^{n-1} \to (0, \infty)$, define two families of convex bodies as follows:

$$\tilde{\mathscr{B}} = \{Q \in \mathscr{M}^{n}_0 : \tilde{V}_G(Q^\circ) = \tilde{V}_G(B^n)\};$$

$$\mathcal{B} = \{Q \in \mathscr{M}^{n}_0 : \tilde{V}_G(Q^\circ) = \tilde{V}_G(B^n)\}, \text{ if } G \in \mathscr{B}_I \cup \mathscr{B}_d.$$ 

It is obvious that both $\tilde{\mathscr{B}}$ and $\mathcal{B}$ are nonempty as they all contain the unit Euclidean ball $B^n$.

The following lemma plays essential roles in later context.

**Lemma 3.3.** Let $G : (0, \infty) \times S^{n-1} \to (0, \infty)$ be a continuous function. For $q \in \mathbb{R}$, let $G_q(t, u) = \frac{G(t, u)}{t^q}$. Suppose that there exists a constant $q \geq n - 1$ such that

$$\inf \left\{ G_q(t, u) : t \geq 1 \text{ and } u \in S^{n-1} \right\} > 0. \quad (3.5)$$
Then the following statements hold.

i) If \( \{Q_i\}_{i \geq 1} \) with \( Q_i \in \hat{B} \) for all \( i \in \mathbb{N} \) is a bounded sequence, then there exist a subsequence \( \{Q_{ij}\}_{j \geq 1} \) of \( \{Q_i\}_{i \geq 1} \) and a convex body \( Q_0 \in \hat{B} \) such that \( Q_{ij} \to Q_0 \).

ii) If in addition \( G \in \mathcal{G}_1 \), the statement in i) also holds if \( \hat{B} \) is replaced by \( \hat{B} \).

Remark. Clearly \( G(t,u) = t^q \) for some \( q \geq n - 1 \) satisfies (3.5). In particular \( G(t,u) = t^n/n \) satisfies (3.5) and hence Lemma 3.3 recovers [39, Lemma 3.2]. It can be easily checked that formula (3.5) is equivalent to: there exist constants \( c, C > 0 \), such that

\[
\inf \left\{ G_q(t,u) : \ t \geq c \text{ and } u \in S^{n-1} \right\} > C. \tag{3.6}
\]

Moreover, if \( G \in \mathcal{G}_d \), then \( G \) does not satisfy (3.5). In fact, for all \( q \geq n - 1 \) and for all \( u \in S^{n-1} \),

\[
\lim_{t \to \infty} G_q(t,u) = \lim_{t \to \infty} G(t,u) \times \lim_{t \to \infty} t^{-q} = 0.
\]

Proof. Let \( \{Q_i\}_{i \geq 1} \) be a bounded sequence with \( Q_i \in \hat{B} \) (or, respectively, \( Q_i \in \hat{B} \)) for all \( i \in \mathbb{N} \). It follows from the Blaschke selection theorem that there exist a subsequence of \( \{Q_i\}_{i \geq 1} \), say \( \{Q_{ij}\}_{j \geq 1} \), and a compact convex set \( Q_0 = \mathcal{X}^n \), such that \( Q_{ij} \to Q_0 \) in the Hausdorff metric. As \( o \in \text{int}Q_{ij} \) for all \( j \in \mathbb{N} \), one has, \( o \in Q_0 \). In order to show \( Q_0 \in \hat{B} \) (or, respectively, \( Q_0 \in \hat{B} \)), we first need to show \( o \in \text{int}Q_0 \).

i) To this end, we assume that \( o \in \partial Q_0 \) and seek for contradictions. As \( \{Q_i\}_{i \geq 1} \) is a bounded sequence, there exists a constant \( R > 0 \) such that \( Q_i \subset RB^n \) for each \( i \in \mathbb{N} \). For each \( j \in \mathbb{N} \), one can find \( u_{ij} \in S^{n-1} \) such that \( r_{ij} = h_{Q_{ij}}(u_{ij}) = \min_{u \in S^{n-1}} h_{Q_{ij}}(u) \). As \( o \in \partial Q_0 \), one sees that \( \lim_{j \to \infty} r_{ij} = 0 \). The fact that \( Q_{ij} \subset RB^n \) implies that \( \frac{1}{R} B^n \subset Q_{ij} \), and in particular, \( \rho_{Q_{ij}}(u) \geq \frac{1}{R} \) for any \( u \in S^{n-1} \).

Let the constant \( c \) in (3.6) be \( \frac{1}{R} \). For some fixed constants \( q \geq n - 1 \) and \( C > 0 \),

\[
\hat{V}_G(Q_{ij}^o) = \int_{S^{n-1}} G(\rho_{Q_{ij}}(u), u) \, du \geq C \int_{S^{n-1}} (\rho_{Q_{ij}}(u))^q \, du = Cn \hat{V}_q(Q_{ij}^o). \tag{3.7}
\]

For any \( T \in O(n) \), \( (TQ_{ij})^o = (T^{-1})^{-1}Q_{ij}^o = TQ_{ij}^o \) as \( T^{-1}T = \mathbb{I}_n \) where \( T^{-1} \) denotes the inverse map of \( T \). It follows from Proposition 2.1 that \( \hat{V}_G(Q_{ij}^o) \) is \( O(n) \)-invariant. Hence, for convenience, one can assume that \( u_{ij} = e_n \). The radial function \( \rho_{Q_{ij}}^o \) can be bounded from below by the radial function of \( C_j = \text{Cone} \left( o, \frac{1}{R}, \frac{e_n}{r_{ij}} \right) \), the cone with base \( \frac{B^{n-1}}{R} \) and the apex \( \frac{e_n}{r_{ij}} \). Note that

\[
\rho_{C_j}(u) = \begin{cases} \frac{1}{R \sin \theta + r_{ij} \cos \theta}, & \text{if } u \in S^{n-1} \text{ such that } \langle e_n, u \rangle \geq 0; \\ 0, & \text{if } u \in S^{n-1} \text{ such that } \langle e_n, u \rangle < 0, \end{cases}
\]

where \( \theta \in [0, \pi/2] \) is the angle between \( u \) and \( e_n \) (see Figure 1).

Indeed, from Figure 1, for \( u \in S^{n-1} \) such that \( \langle u, e_n \rangle \geq 0 \), one has

\[
\rho_{C_j}(u) \cdot \sin \theta = \frac{r_{ij}^{-1} - \rho_{C_j}(u) \cos \theta}{r_{ij}^{-1}} \implies \rho_{C_j}(u) = \frac{1}{R \sin \theta + r_{ij} \cos \theta}.
\]

Using the general spherical coordinate (see, e.g., [6, Page 14]) by letting

\[
u = (v \sin \theta, \cos \theta) \in S^{n-1}, \quad v \in S^{n-2} \quad \text{and} \quad \theta \in [0, \pi],
\]
we have \( du = (\sin \theta)^{n-2} d\theta \, dv \), where \( dv \) denotes the spherical measure of \( S^{n-2} \). Thus

\[
n \tilde{V}_q(C_j) = \int_{S^{n-2}} \left( \int_0^{\pi} \left( \frac{1}{R \sin \theta + r_{ij} \cos \theta} \right)^q (\sin \theta)^{n-2} \, d\theta \right) \, dv
\]

\[
= (n-1)V(B^{n-1}) \int_0^{\pi} \left( \frac{1}{R \sin \theta + r_{ij} \cos \theta} \right)^q (\sin \theta)^{n-2} \, d\theta. \quad (3.8)
\]

We will not need the precise value of \( \tilde{V}_q(C_j) \), however if \( q = n \), formula (3.8) does lead to

\[
\tilde{V}_n(C_j) = V(C_j) = \frac{V(B^{n-1})}{n R^{n-1} r_{ij}},
\]

which coincides with the calculation provided in [39, Lemma 3.2].

Together with (3.7), \( \rho_{Q_{ij}^o} \geq \rho_{C_j} \), Fatou’s lemma, and \( \lim_{j \to \infty} r_{ij} = 0 \), one has, if \( q \geq n-1 \), then \( n-2-q \leq -1 \) and

\[
\liminf_{j \to \infty} \tilde{V}_G(Q_{ij}^o) \geq \liminf_{j \to \infty} C n \tilde{V}_q(C_j)
\]

\[
= C \cdot (n-1)V(B^{n-1}) \cdot \liminf_{j \to \infty} \int_0^{\pi} \left( \frac{1}{R \sin \theta + r_{ij} \cos \theta} \right)^q (\sin \theta)^{n-2} \, d\theta
\]

\[
\geq C \cdot (n-1)V(B^{n-1}) \cdot \int_0^{\pi} \liminf_{j \to \infty} \left( \frac{1}{R \sin \theta + r_{ij} \cos \theta} \right)^q (\sin \theta)^{n-2} \, d\theta
\]

\[
= \frac{C \cdot (n-1)V(B^{n-1})}{R^q} \int_0^{\pi} (\sin \theta)^{n-2-q} \, d\theta
\]

\[
\geq \frac{C \cdot (n-1)V(B^{n-1})}{R^q} \int_0^{\pi} \frac{1}{\sin \theta} \, d\theta
\]

\[
= \frac{C \cdot (n-1)V(B^{n-1})}{R^q} \cdot \ln \tan(\theta/2) \bigg|_{\theta=\pi/2}^{\theta=\pi/2} = \infty. \quad (3.9)
\]

On the other hand, as \( Q_{ij} \in \tilde{\mathcal{B}} \) for each \( j \in \mathbb{N} \), then

\[
\tilde{V}_G(Q_{ij}^o) = \tilde{V}_G(B^n) = \int_{S^{n-1}} G(1, u) \, du < \infty.
\]
This is a contradiction and thus \( o \in \text{int} Q_0 \).

As \( Q_{ij} \in \mathcal{K}_{(o)}^n \) for each \( j \in \mathbb{N} \) and \( Q_0 \in \mathcal{K}_{(o)}^n \), \( Q_{ij} \to Q_0 \) yields \( Q_{ij}^o \to Q_0^o \). Together with the continuity of \( \tilde{V}_G(\cdot) \) (see [13, Lemma 6.1]) and the fact that \( \tilde{V}_G(Q_{ij}^o) = \tilde{V}_G(B^n) \) for each \( j \in \mathbb{N} \), one gets \( \tilde{V}_G(Q_0^o) = \lim_{j \to \infty} \tilde{V}_G(Q_{ij}^o) = \tilde{V}_G(B^n) \). This concludes that \( Q_0 \in \mathcal{B} \) as desired.

ii) Again, we assume that \( o \in \partial Q_0 \) and seek for contradictions. It follows from Proposition 3.1 that \( \tilde{V}_G(B^n) > 0 \) is a finite constant. Following notations in i), Proposition 3.1 and \( \tilde{V}_G(Q_{ij}^o) = \tilde{V}_G(B^n) \) for each \( j \in \mathbb{N} \) yield that

\[
\int_{S^{n-1}} G\left(\frac{\rho Q_{ij}^o(u)}{\tilde{V}_G(B^n)}, u\right) du = 1. \tag{3.10}
\]

As \( \frac{1}{R} B^n \subset Q_{ij}^o \) for each \( j \in \mathbb{N} \), one can take the constant \( c \) in (3.6) to be \( \frac{1}{R\tilde{V}_G(B^n)} \) and there exists a constant \( C > 0 \) such that, for all \( u \in S^{n-1} \) and some \( q \geq n - 1 \),

\[
G\left(\frac{\rho Q_{ij}^o(u)}{\tilde{V}_G(B^n)}, u\right) \geq C \cdot \left(\frac{\rho Q_{ij}^o(u)}{\tilde{V}_G(B^n)}\right)^q.
\]

Together with (3.10), one has,

\[
\int_{S^{n-1}} C \cdot \left(\frac{\rho Q_{ij}^o(u)}{\tilde{V}_G(B^n)}\right)^q du \leq 1 \implies C \cdot \int_{S^{n-1}} (\rho Q_{ij}^o(u))^q du \leq (\tilde{V}_G(B^n))^q.
\]

Similar to (3.9), one gets

\[
\infty = \liminf_{j \to \infty} C \cdot \int_{S^{n-1}} (\rho Q_{ij}^o(u))^q du \leq (\tilde{V}_G(B^n))^q,
\]

a contradiction and hence \( o \in \text{int} Q_0 \). The rest of the proof follows along the lines in i), where the continuity of \( \tilde{V}_G(\cdot) \) (see Proposition 3.2) shall be used.

\[\qed\]

4 \hspace{1em} The general dual-polar Orlicz-Minkowski problem

Motivated by the polar Orlicz-Minkowski problem proposed in [34] and by the general dual Orlicz-Minkowski problem proposed in [13, 15], we propose the following general dual-polar Orlicz-Minkowski problem:

**Problem 4.1.** Under what conditions on a nonzero finite Borel measure \( \mu \) defined on \( S^{n-1} \), continuous functions \( \varphi : (0, \infty) \to (0, \infty) \) and \( G \in \mathcal{G}_f \cup \mathcal{G}_d \) can we find a convex body \( K \in \mathcal{K}^n_{(o)} \) solving the following optimization problems:

\[
\inf \left/ \sup \left\{ \int_{S^{n-1}} \varphi(hQ(u))d\mu(u) : Q \in \mathcal{B} \right\} \right.; \tag{4.1}
\]

\[
\inf \left/ \sup \left\{ \int_{S^{n-1}} \varphi(hQ(u))d\mu(u) : Q \in \mathcal{B} \right\} \right. \tag{4.2}
\]

Although the function \( G \) in the optimization problem (4.1) can be any continuous function \( G : (0, \infty) \times S^{n-1} \to (0, \infty) \), to find its solutions, only those \( G \in \mathcal{G}_f \cup \mathcal{G}_d \) with monotonicity will be considered. One reason is that most \( G \) of interest (such as \( G(t, u) = t^q/n \) for \( 0 \neq q \in \mathbb{R} \)) are monotone. More importantly, without the monotonicity of \( G \), the set \( \mathcal{B} \) may contain only one
convex body $B^n$ (for instance, if $G(1, u) < G(t, u)$ for all $(t, u) \in (0, \infty) \times S^{n-1}$ such that $t \neq 1$). In this case, the optimization problem (4.1) becomes trivial. Note that when $G(t, u) = t^n/n$, both $\tilde{V}_G(\cdot)$ and (essentially) $\tilde{V}_G$ are volume, then Problem 4.1 becomes the polar Orlicz-Minkowski problem posed in [34].

In later context, we always assume that $\varphi : (0, \infty) \to (0, \infty)$ is a continuous function. For convenience, let $\varphi(0+) = \lim_{t \to 0^+} \varphi(t)$ and $\varphi(\infty) = \lim_{t \to \infty} \varphi(t)$ provided the above limits exist (either finite or infinite). We shall need the following classes of functions:

$$\mathcal{I} = \{ \varphi : \varphi \text{ is strictly increasing on } (0, \infty) \text{ with } \varphi(0+) = 0, \varphi(1) = 1 \text{ and } \varphi(\infty) = \infty \};$$
$$\mathcal{D} = \{ \varphi : \varphi \text{ is strictly decreasing on } (0, \infty) \text{ with } \varphi(0+) = \infty, \varphi(1) = 1 \text{ and } \varphi(\infty) = 0 \}.$$  

Note that the normalization value $\varphi(1) = 1$ is mainly for technique convenience and $\varphi(1)$ can be modified to any positive numbers.

### 4.1 The general dual-polar Orlicz-Minkowski problem for discrete measures

In this subsection, we will solve the general dual-polar Orlicz-Minkowski problem for discrete measures. Throughout this subsection, let $\mu$ be a discrete measure of the following form:

$$\mu = \sum_{i=1}^{m} \lambda_i \delta_{u_i}, \quad (4.3)$$

where $\lambda_i > 0$, $\delta_{u_i}$ denotes the Dirac measure at $u_i$, and $\{u_1, \ldots, u_m\}$ is a subset of $S^{n-1}$ which is not concentrated on any closed hemisphere (clearly $m \geq n + 1$). It has been proved in [34, Propositions 3.1 and 3.3] that the solutions to the polar Orlicz-Minkowski problem for discrete measures must be polytopes, the convex hulls of finite points in $\mathbb{R}^n$. It is well-known that all convex bodies can be approximated by polytopes, and hence to study the Minkowski type problems for discrete measures is very important and receives extensive attention, see e.g., [2, 3, 11, 15, 21, 23, 26, 29, 30, 53, 65, 66, 67].

The following lemma shows that if, when the infimum is considered, Problem 4.1 for discrete measures has solutions, then the solutions must be polytopes.

**Lemma 4.2.** Let $\varphi \in \mathcal{I}$ and $\mu$ be as in (4.3) whose support $\{u_1, \ldots, u_m\}$ is not concentrated on any closed hemisphere. Let $G \in \mathcal{G}_1$.

i) If $\tilde{M} \in \mathcal{B}$ is a solution to the optimization problem (4.1) when the infimum is considered, then $\tilde{M}$ is a polytope, and $u_1, \ldots, u_m$ are the corresponding unit normal vectors of its faces.

ii) If $\tilde{M} \in \mathcal{B}$ is a solution to the optimization problem (4.2) when the infimum is considered, then $\tilde{M}$ is a polytope, and $u_1, \ldots, u_m$ are the corresponding unit normal vectors of its faces.

**Proof.** Let $G \in \mathcal{G}_1$. For discrete measure $\mu$ and $Q \in \mathcal{K}_{(o)}^n$, one has

$$\int_{S^{n-1}} \varphi(h_Q(u))d\mu(u) = \sum_{i=1}^{m} \varphi(h_Q(u_i))\mu(\{u_i\}) = \sum_{i=1}^{m} \lambda_i \varphi(h_Q(u_i)).$$

i) Let $\tilde{M} \in \mathcal{B}$ be a solution to the optimization problem (4.1). Define the polytope $P$ as follows: $\tilde{M} \subseteq P$, $h_P(u_i) = h_{\tilde{M}}(u_i)$ for $1 \leq i \leq m$, and $u_1, \ldots, u_m$ are the corresponding unit normal vectors of the faces of $P$. As $\tilde{M} \in \mathcal{B}$, one has $\tilde{V}_G(\tilde{M}^o) = \tilde{V}_G(B^n)$ and $o \in \text{int} \tilde{M}$. Hence $P \in \mathcal{K}_{(o)}^n$. 

and \( P^o \subseteq \tilde{M}^o \). Similar to the proof of Proposition 3.2 iii), one can obtain that \( \tilde{V}_G(\cdot) \) for \( G \in \mathcal{G}_I \) is strictly increasing in terms of set inclusion. In particular, \( \tilde{V}_G(P^o) \leq \tilde{V}_G(\tilde{M}^o) = \tilde{V}_G(B^n) \). As \( \lim_{t \to \infty} G(t, \cdot) = \infty \), there exists \( t_0 \geq 1 \) such that \( \tilde{V}_G(t_0 P^o) = \tilde{V}_G(B^n) \). That is, \( P/t_0 \in \mathcal{B} \). Due to the minimality of \( \tilde{M} \) and the fact that \( \varphi \in \mathcal{I} \) is strictly increasing, one has
\[
\sum_{i=1}^{m} \lambda_i \varphi(h_P(u_i)) = \sum_{i=1}^{m} \lambda_i \varphi(h_{\tilde{M}}(u_i)) \leq \sum_{i=1}^{m} \lambda_i \varphi(h_{P/t_0}(u_i)) \leq \sum_{i=1}^{m} \lambda_i \varphi(h_P(u_i)),
\]
which yields \( t_0 = 1 \). Then, \( \tilde{V}_G(P^o) = \tilde{V}_G(B^n) = \tilde{V}_G(\tilde{M}^o) \) and hence \( P = \tilde{M} \) following from \( \tilde{M} \subseteq P \).

ii) Proposition 3.2 iii) asserts that, if \( G \in \mathcal{G}_I \), \( \tilde{V}_G(K) < \tilde{V}_G(L) \) for all \( K, L \in \mathcal{K}_n \) such that \( K \subseteq L \). The proof in this case then follows along the same lines as in i), and will be omitted. \( \square \)

The following result is for the existence of solutions to Problem 4.1 for discrete measures if the infimum is considered.

**Theorem 4.3.** Let \( \varphi \in \mathcal{I} \) and \( \mu \) be as in (4.3) whose support \( \{u_1, \ldots, u_m\} \) is not concentrated on any closed hemisphere. Let \( G \in \mathcal{G}_I \) be a continuous function such that (3.5) holds for some \( q \geq n - 1 \). Then the following statements hold.

i) There exists a polytope \( \hat{P} \in \mathcal{B} \) with \( u_1, \ldots, u_m \) being the corresponding unit normal vectors of its faces, such that
\[
\sum_{i=1}^{m} \lambda_i \varphi(h_{\hat{P}}(u_i)) = \inf \left\{ \sum_{i=1}^{m} \lambda_i \varphi(h_Q(u_i)) : Q \in \mathcal{B} \right\}. \tag{4.4}
\]

ii) There exists a polytope \( \hat{P} \in \mathcal{B} \) with \( u_1, \ldots, u_m \) being the corresponding unit normal vectors of its faces, such that
\[
\sum_{i=1}^{m} \lambda_i \varphi(h_{\hat{P}}(u_i)) = \inf \left\{ \sum_{i=1}^{m} \lambda_i \varphi(h_Q(u_i)) : Q \in \mathcal{B} \right\}.
\]

**Proof.** By Lemma 4.2, to solve (4.4), it will be enough to find a solution for the following problem:
\[
\tilde{\alpha} = \inf \left\{ \sum_{i=1}^{m} \lambda_i \varphi(z_i) : z \in \mathbb{R}_+^m \text{ such that } P(z) \in \mathcal{B} \right\}, \tag{4.5}
\]
where \( z = (z_1, \ldots, z_m) \in \mathbb{R}_+^m \) means that each \( z_i > 0 \) and
\[
P(z) = \bigcap_{i=1}^{m} \left\{ x \in \mathbb{R}^n : \langle x, u_i \rangle \leq z_i \right\} \subset \mathcal{K}_n.
\]
Clearly \( h_{P(z)}(u_i) \leq z_i \) for all \( i = 1, 2, \ldots, m \).

Let \( P_1^o = P(1, \ldots, 1) \). Then \( B^n \subseteq P_1 \) and hence \( P_1^o \subseteq B^n \). As \( G \in \mathcal{G}_I \) one has \( \tilde{V}_G(P_1^o) < \tilde{V}_G(B^n) \). The facts that \( G(t, \cdot) \) is strictly increasing on \( t \) and \( \lim_{t \to \infty} G(t, \cdot) = \infty \) imply the existence of \( t_1 > 1 \) such that \( \tilde{V}_G(t_1 P_1^o) = \tilde{V}_G(B^n) \). In other words, \( P_1/t_1 \in \mathcal{B} \) and then the infimum in (4.5) is not taken over an empty set. Moreover, due to \( \varphi \in \mathcal{I} \) (in particular, \( \varphi \) is strictly increasing and \( \varphi(1) = 1 \)) and \( 1/t_1 < 1 \), one has,
\[
\tilde{\alpha} \leq \varphi(1/t_1) \sum_{i=1}^{m} \lambda_i \leq \sum_{i=1}^{m} \lambda_i.
\]
This in turn implies that \( z \in \mathbb{R}^m \) in (4.5) can be restricted in a bounded set, for instance,
\[
z_i \leq \varphi^{-1}\left( \frac{\lambda_1 + \cdots + \lambda_m}{\min_{1 \leq i \leq m} \lambda_i} \right), \quad \text{for all } i = 1, 2, \cdots, m. \tag{4.6}
\]
Let \( z^1, \cdots, z^j \cdots \in \mathbb{R}^m \) be the limiting sequence of (4.5), that is,
\[
\alpha = \lim_{j \to \infty} \sum_{i=1}^{m} \lambda_i \varphi(z_i^j) \quad \text{and} \quad V_G(P^o(z^j)) = V_G(B^n) \quad \text{for all } j \in \mathbb{N}.
\]
Due to (4.6), without loss of generality, we can assume that \( z^j \to z^0 \) for some \( z^0 \in \mathbb{R}^m \) and hence \( P(z^j) \to P(z^0) \) in the Hausdorff metric (see e.g., [48]). Lemma 3.3 yields that \( P(z^0) \in \mathcal{B} \), i.e., \( V_G(P^o(z^0)) = V_G(B^n) \) and \( o \in \text{int} P(z^0) \). In particular, \( z_i^0 > 0 \) for all \( i = 1, 2, \cdots, m \).

On the other hand, we claim that \( h_{P(z^0)}(u_i) = z_i^0 \) for all \( i = 1, 2, \cdots, m \). To this end, assume not, then there exists \( i_0 \in \{1, 2, \cdots, m\} \) such that \( h_{P(z^0)}(u_{i_0}) < z_{i_0}^0 \). As \( \varphi \in \mathcal{I} \) is strictly increasing and \( \lambda_{i_0} > 0 \), one clearly has
\[
\alpha = \sum_{i=1}^{m} \lambda_i \varphi(z_i^0) > \sum_{i \in \{1, 2, \cdots, m\} \setminus \{i_0\}} \lambda_i \varphi(z_i^0) + \lambda_{i_0} \varphi(h_{P(z^0)}(u_{i_0})).
\]
This contradicts with the minimality of \( \alpha \).

Let \( \tilde{P} = P(z^0) \). Then \( \tilde{P} \in \mathcal{B} \) solves (4.5) and hence (4.4). This concludes the proof of i).

ii) The proof is almost identical to the one for i), and will be omitted. \( \square \)

It has been proved in [34] that the existence of solutions to Problem 4.1 for discrete measures in general is invalid when \( G(t, u) = t^n/n \), if the supremum is considered for \( \varphi \in \mathcal{I} \cup \mathcal{D} \), or the infimum is considered for \( \varphi \in \mathcal{D} \). One can also prove similar arguments for Problem 4.1 for discrete measures with more general \( G \in \mathcal{G}_I \), but more delicate calculations are required. We only state the following result as an example.

**Proposition 4.4.** Let \( \mu \) be as in (4.3) whose support \( \{u_1, \cdots, u_m\} \) is not concentrated on any closed hemisphere. Let \( G \in \mathcal{G}_I \) be such that (3.5) holds for some \( q \geq n - 1 \).

i) If \( \varphi \in \mathcal{D} \) and the first coordinates of \( u_1, u_2, \cdots, u_m \) are all nonzero, then
\[
\inf_{Q \in \mathcal{B}} \sum_{i=1}^{m} \lambda_i \varphi(h_Q(u_i)) = 0.
\]

ii) If \( \varphi \in \mathcal{I} \cup \mathcal{D} \), then
\[
\sup_{Q \in \mathcal{B}} \sum_{i=1}^{m} \lambda_i \varphi(h_Q(u_i)) = \infty.
\]

**Proof.** i) For \( 0 < \epsilon < 1 \), let \( T_\epsilon = \text{diag}(1, 1, \cdots, 1, \epsilon) \) and \( L_\epsilon = T_\epsilon B^n \). It can be checked that
\[
\rho_{L_\epsilon}(w) = \left( w_1^2 + w_2^2 + \cdots + w_{n-1}^2 + w_n^2/\epsilon^2 \right)^{-1/2}
\]
for all \( w = (w_1, \cdots, w_n) \in S^{n-1} \). Thus \( \rho_{L_\epsilon}(w) \) is increasing on \( \epsilon > 0 \) for each \( w \in S^{n-1} \) and then \( L_\epsilon \) is increasing in the sense of set inclusion on \( \epsilon > 0 \). In particular, \( L_\epsilon \subset B^n \) and \( B^n \subset L_\epsilon^0 = T_\epsilon^{-1} B^n \).

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Moreover, $L^\circ = T_{\epsilon}^{-1}B^n$ is decreasing in the sense of set inclusion on $\epsilon > 0$, and so is $\tilde{V}_G(L^\circ)$ due to $G \in \mathcal{G}$. By the homogeneity of $\tilde{V}_G(\cdot)$, one has $\tilde{V}_G(f(\epsilon)L^\circ) = \tilde{V}_G(B^n)$ if

$$f(\epsilon) = \frac{\tilde{V}_G(B^n)}{\tilde{V}_G(L^\circ)}.$$  

We now claim that $f(\epsilon) \to 0$, which is equivalent to prove $\tilde{V}_G(L^\circ) \to \infty$ as $\epsilon \to 0^+$. To this end, it is enough to prove that $\sup_{0 < \epsilon < 1} \tilde{V}_G(L^\circ) = \infty$. Assume that $\sup_{0 < \epsilon < 1} \tilde{V}_G(L^\circ) = A_0 < \infty$. By $B^n \subset L^\circ$ and (3.6) with $c = 1/A_0$, there exists a constant $C_A > 0$ such that

$$\int_{S^{n-1}} G \left( \frac{\rho L^\circ_1(u)}{A_0}, u \right) du \geq C_A \int_{S^{n-1}} \left( \frac{\rho L^\circ_2(u)}{A_0} \right)^q du \geq C_A \int_{S^{n-1}} \left( \frac{\rho C_\epsilon(u)}{A_0} \right)^q du,$$

where $C_\epsilon \subset L^\circ$ is the cone with the base $B^{n-1}$ and the apex $\epsilon^{-1}e_n$. It follows from Proposition 3.1, (3.8), (3.9) and $q \geq n - 1$ that

$$1 = \liminf_{\epsilon \to 0^+} \int_{S^{n-1}} G \left( \frac{\rho L^\circ_1(u)}{\tilde{V}_G(L^\circ)}, u \right) du \geq \liminf_{\epsilon \to 0^+} \int_{S^{n-1}} G \left( \frac{\rho L^\circ_2(u)}{A_0}, u \right) du \geq \frac{C_A}{A_0} \cdot \liminf_{\epsilon \to 0^+} \int_{S^{n-1}} (\rho C_\epsilon)^q du \geq \frac{C_A (n-1) V(B^{n-1})}{A_0} \cdot \liminf_{\epsilon \to 0^+} \int_{0}^{\frac{\pi}{2}} \left( \frac{1}{\sin \theta + \epsilon \cos \theta} \right)^q (\sin \theta)^{n-2} d\theta = \infty.$$

This is a contradiction, which yields $\sup_{0 < \epsilon < 1} \tilde{V}_G(L^\circ) = \infty$ and then $f(\epsilon) \to 0$ as $\epsilon \to 0^+$. Recall that $\tilde{V}_G(f(\epsilon)L^\circ) = \tilde{V}_G(B^n)$ and then $L^\circ / f(\epsilon) = T, B^n / f(\epsilon) \in \mathcal{H}$. It is assumed that $\alpha = \min_{1 \leq i \leq m} \{(u_i)_1\} > 0$, and hence for all $1 \leq i \leq m$ (by letting $v_2 = u_i$),

$$h_{L^\circ / f(\epsilon)}(u_i) = \max_{v_1 \in L^\circ / f(\epsilon)} \langle v_1, u_i \rangle = \max_{v_2 \in B^n} \langle T_i v_2, u_i \rangle / f(\epsilon) \geq \alpha^2 / f(\epsilon).$$

The fact that $\varphi \in \mathcal{H}$ is strictly decreasing yields

$$\inf_{Q \in \mathcal{H}} \sum_{i=1}^{m} \lambda_i \varphi(h_Q(u_i)) \leq \sum_{i=1}^{m} \varphi \left( h_{L^\circ / f(\epsilon)}(u_i) \right) \cdot \mu(\{u_i\}) \leq \varphi \left( \alpha^2 / f(\epsilon) \right) \cdot \mu(S^{n-1}) \to 0,$$

where we have used $\lim_{\epsilon \to 0^+} f(\epsilon) = 0$ and $\lim_{t \to \infty} \varphi(t) = 0$. This concludes the proof of i).

ii) Note that $\mu(\{u_i\}) > 0$. For any $0 < \epsilon < 1$, let $\tilde{L}_\epsilon = TT^tB^n$, where $T \in O(n)$ is an orthogonal matrix such that $T^t u_1 = e_1$ (indeed, this can always be done by the Gram-Schmidt process). Again $\tilde{L}_\epsilon \subset B^n$ and hence $B^n \subset \tilde{L}^\circ_\epsilon$. As in i), one can prove that

$$f(\epsilon) = \frac{\tilde{V}_G(B^n)}{\tilde{V}_G(\tilde{L}^\circ_\epsilon)} \to 0 \quad \text{as} \quad \epsilon \to 0^+.$$  

Moreover, $\tilde{V}_G(f(\epsilon)\tilde{L}^\circ_\epsilon) = \tilde{V}_G(B^n)$ and thus $\tilde{L}_\epsilon / f(\epsilon) \in \mathcal{H}$. One can check (by letting $v_2 = e_1$) that

$$h_{\tilde{L}_\epsilon / f(\epsilon)}(u_1) = f(\epsilon)^{-1} \max_{v_2 \in B^n} \langle TT^t v_2, u_1 \rangle = f(\epsilon)^{-1} \langle T^t u_1, \text{diag}(1, 1, \ldots, 1, \epsilon) \cdot e_1 \rangle = f(\epsilon)^{-1}.  $$

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Together with \( \varphi \in \mathcal{S} \) (in particular, \( \lim_{t \to \infty} \varphi(t) = \infty \)), one has

\[
\sup_{Q \in \mathcal{B}} \sum_{i=1}^{m} \lambda_i \varphi(h_Q(u_i)) \geq \sum_{i=1}^{m} \varphi\left(h_{\mathcal{L}_\epsilon/f(\epsilon)}(u_i)\right) \cdot \mu\{u_i\} \\
\geq \varphi\left(h_{\mathcal{L}_\epsilon/f(\epsilon)}(u_1)\right) \cdot \mu\{u_1\} \\
= \varphi\left(f(\epsilon)^{-1}\right) \cdot \mu\{u_1\} \to \infty,
\]
as \( \epsilon \to 0^+ \), which follows from the fact that \( \lim_{\epsilon \to 0^+} f(\epsilon) = 0 \).

When \( \varphi \in \mathcal{D} \), let \( \mathcal{L}_\epsilon = \mathcal{L}_\epsilon^0 \). Hence \( \mathcal{L}_\epsilon^0 \subset B^n \) for all \( \epsilon \in (0, 1) \). We claim that \( \hat{V}_G(\mathcal{L}_\epsilon^0) \to 0 \) as \( \epsilon \to 0^+ \). To this end, it can be checked that

\[
\rho_{\mathcal{L}_\epsilon^0}(u) = \frac{\epsilon}{\sqrt{\|(T^t u)_n\|^2 + \epsilon^2(1 - \|(T^t u)_n\|^2)}},
\]
where \( (T^t u)_n \) denotes the \( n \)-th coordinate of \( T^t u \). Clearly \( \rho_{\mathcal{L}_\epsilon^0}(u) \leq 1 \) for all \( u \in S^{n-1} \) and \( \rho_{\mathcal{L}_\epsilon^0}(u) \to 0 \) as \( \epsilon \to 0^+ \) for all \( u \in \eta \), where \( \eta = \{ u \in S^{n-1} : (T^t u)_n \neq 0 \} \). Also note that the spherical measure of \( S^{n-1} \setminus \eta \) is 0.

On the other hand, \( \mathcal{L}_\epsilon^0 \) is increasing (in the sense of set inclusion) and hence \( \hat{V}_G(\mathcal{L}_\epsilon^0) \) is strictly increasing on \( \epsilon \) due to Proposition 3.2. To show that \( \hat{V}_G(\mathcal{L}_\epsilon^0) \to 0 \) as \( \epsilon \to 0^+ \), we assume that \( \inf_{\epsilon>0} \hat{V}_G(\mathcal{L}_\epsilon^0) = \beta > 0 \) and seek for contradictions. By Proposition 3.1, one has, for all \( \epsilon \in (0, 1) \),

\[
\int_{S^{n-1}} G\left(\frac{\rho_{\mathcal{L}_\epsilon^0}(u)}{\rho_{\tilde{L}_\epsilon^0}(u)}, u\right) du \geq \int_{S^{n-1}} G\left(\frac{\rho_{\mathcal{L}_\epsilon^0}(u)}{V_G(\mathcal{L}_\epsilon^0)}, u\right) du = 1. \tag{4.7}
\]

Moreover, as \( \rho_{\mathcal{L}_\epsilon^0}(u) \leq 1 \) for all \( u \in S^{n-1} \), one has, for all \( u \in S^{n-1} \),

\[
G\left(\frac{\rho_{\mathcal{L}_\epsilon^0}(u)}{\beta}, u\right) \leq G\left(\frac{1}{\beta}, u\right).
\]

Together with (4.7) and the dominated convergence theorem, one gets that

\[
1 \leq \lim_{\epsilon \to 0^+} \int_{S^{n-1}} G\left(\frac{\rho_{\mathcal{L}_\epsilon^0}(u)}{\rho_{\tilde{L}_\epsilon^0}(u)}, u\right) du = \int_{S^{n-1}} \lim_{\epsilon \to 0^+} G\left(\frac{\rho_{\mathcal{L}_\epsilon^0}(u)}{\rho_{\tilde{L}_\epsilon^0}(u)}, u\right) du = 0.
\]

This implies \( \hat{V}_G(\mathcal{L}_\epsilon^0) \to 0 \) as \( \epsilon \to 0^+ \). Again, \( \tilde{L}_\epsilon/f(\epsilon) \in \mathcal{B} \) and \( h_{\mathcal{L}_\epsilon/f(\epsilon)}(u_1) = f(\epsilon)^{-1} \), where

\[
f(\epsilon) = \frac{\hat{V}_G(B^n)}{V_G(\mathcal{L}_\epsilon^0)} \to \infty \quad \text{as} \quad \epsilon \to 0^+.
\]

Together with \( \varphi \in \mathcal{D} \) (in particular, \( \lim_{\epsilon \to 0^+} \varphi(t) = \infty \)), one has

\[
\sup_{Q \in \mathcal{B}} \sum_{i=1}^{m} \lambda_i \varphi(h_Q(u_i)) \geq \varphi\left(h_{\mathcal{L}_\epsilon/f(\epsilon)}(u_1)\right) \cdot \mu\{u_1\} = \varphi\left(f(\epsilon)^{-1}\right) \cdot \mu\{u_1\} \to \infty,
\]
as \( \epsilon \to 0^+ \). This concludes the proof of ii). \( \square \)

It is worth to mention that the argument in Proposition 4.4 ii) for the case \( \varphi \in \mathcal{D} \) indeed works for all \( G \in \mathcal{G}_1 \) without assuming (3.5) for some \( q \geq n - 1 \). Moreover, the proof of Proposition 4.4 can be slightly modified to show similar results for the case \( \mathcal{B} \) and the details are omitted.
4.2 The general dual-polar Orlicz-Minkowski problem

In view of Proposition 4.4, in this subsection, we will provide the continuity, uniqueness, and existence of solutions to Problem 4.1 for \( \varphi \in \mathcal{I} \) and with the infimum considered. The following lemma is very useful in later context. Its proof can be found in, e.g., the proof of [34, Theorem 3.2] (slight modification is needed) and hence is omitted.

**Lemma 4.5.** Let \( \varphi \in \mathcal{I} \). Let \( \mu_i, \mu \) for \( i \in \mathbb{N} \) be nonzero finite Borel measures on \( S^{n-1} \) which are not concentrated on any closed hemisphere and \( \mu_i \to \mu \) weakly. Suppose that \( \{Q_i\}_{i \geq 1} \) is a sequence of convex bodies such that \( Q_i \in \mathcal{K}^{o}_n \) for each \( i \in \mathbb{N} \) and

\[
\sup_{i \geq 1} \left\{ \int_{S^{n-1}} \varphi(h_{Q_i}(u)) \, d\mu_i(u) \right\} < \infty.
\]

Then \( \{Q_i\}_{i \geq 1} \) is a bounded sequence in \( \mathcal{K}^{o}_n \).

The continuity of the extreme values for Problem 4.1 is given below.

**Theorem 4.6.** Let \( \mu_i, \mu \) for \( i \in \mathbb{N} \) be finite Borel measures on \( S^{n-1} \) which are not concentrated on any closed hemisphere and \( \mu_i \to \mu \) weakly. Let \( G \in \mathcal{G}_I \) be a continuous function such that (3.5) holds for some \( q \geq n-1 \) and \( \varphi \in \mathcal{I} \). The following statements hold true.

i) If for each \( i \in \mathbb{N} \), there exists \( \tilde{M}_i \in \tilde{B} \) such that

\[
\int_{S^{n-1}} \varphi(h_{\tilde{M}_i}(u)) \, d\mu_i(u) = \inf \left\{ \int_{S^{n-1}} \varphi(h_Q(u)) \, d\mu_i(u) : Q \in \tilde{B} \right\},
\]

then there exists \( \tilde{M} \in \tilde{B} \) such that

\[
\int_{S^{n-1}} \varphi(h_{\tilde{M}}(u)) \, d\mu(u) = \inf \left\{ \int_{S^{n-1}} \varphi(h_Q(u)) \, d\mu(u) : Q \in \tilde{B} \right\}.
\]

Moreover,

\[
\lim_{i \to \infty} \int_{S^{n-1}} \varphi(h_{\tilde{M}_i}(u)) \, d\mu_i(u) = \int_{S^{n-1}} \varphi(h_{\tilde{M}}(u)) \, d\mu(u).
\]

ii) If for each \( i \in \mathbb{N} \), there exists \( \hat{M}_i \in \hat{B} \) such that

\[
\int_{S^{n-1}} \varphi(h_{\hat{M}_i}(u)) \, d\mu_i(u) = \inf \left\{ \int_{S^{n-1}} \varphi(h_Q(u)) \, d\mu_i(u) : Q \in \hat{B} \right\},
\]

then there exists \( \hat{M} \in \hat{B} \) such that

\[
\int_{S^{n-1}} \varphi(h_{\hat{M}}(u)) \, d\mu(u) = \inf \left\{ \int_{S^{n-1}} \varphi(h_Q(u)) \, d\mu(u) : Q \in \hat{B} \right\}.
\]

Moreover,

\[
\lim_{i \to \infty} \int_{S^{n-1}} \varphi(h_{\hat{M}_i}(u)) \, d\mu_i(u) = \int_{S^{n-1}} \varphi(h_{\hat{M}}(u)) \, d\mu(u).
\]

**Proof.** For each \( i \in \mathbb{N} \), let

\[
\mu_i(S^{n-1}) = \int_{S^{n-1}} \, d\mu_i \quad \text{and} \quad \int_{S^{n-1}} \, d\mu = \mu(S^{n-1}).
\]
i) It can be easily checked from (4.8) and $B^n \in \widetilde{\mathcal{B}}$ that for each $i \in \mathbb{N}$,
\[
\int_{S^{n-1}} \varphi(h_{\widetilde{M}_i}(u)) d\mu_i(u) \leq \varphi(1)\mu_i(S^{n-1}).
\]
Moreover, the weak convergence of $\mu_i \to \mu$ yields $\mu_i(S^{n-1}) \to \mu(S^{n-1})$. Hence,
\[
\sup_{i \geq 1} \left\{ \int_{S^{n-1}} \varphi(h_{\widetilde{M}_i}(u)) d\mu_i(u) \right\} < \infty.
\]
By Lemma 4.5, one sees that $\{\widetilde{M}_i\}_{i \geq 1}$ is a bounded sequence in $\mathcal{K}^n_{(0)}$. As $\widetilde{M}_i \in \widetilde{\mathcal{B}}$ for each $i \in \mathbb{N}$, Lemma 3.3 implies that there exist a subsequence $\{\widetilde{M}_{i_j}\}_{j \geq 1}$ of $\{\widetilde{M}_i\}_{i \geq 1}$ and a convex body $\widetilde{M} \in \widetilde{\mathcal{B}}$ such that $\widetilde{M}_{i_j} \to \widetilde{M}$.

Now we verify that $\widetilde{M}$ satisfies the desired properties. First of all, for any given $Q \in \widetilde{\mathcal{B}}$, one has, for each $j \in \mathbb{N}$,
\[
\int_{S^{n-1}} \varphi(h_{\widetilde{M}_{i_j}}(u)) d\mu_{i_j}(u) \leq \int_{S^{n-1}} \varphi(h_Q(u)) d\mu_{i_j}(u).
\]
Together with the weak convergence of $\mu_i \to \mu$, Lemma 2.2, $\varphi \in \mathcal{S}$, and $\widetilde{M}_{i_j} \to \widetilde{M}$, one obtains that $\varphi(h_{\widetilde{M}_{i_j}}) \to \varphi(h_{\widetilde{M}})$ uniformly on $S^{n-1}$ and for each given $Q \in \widetilde{\mathcal{B}}$,
\[
\int_{S^{n-1}} \varphi(h_{\widetilde{M}}(u)) d\mu(u) = \lim_{j \to \infty} \int_{S^{n-1}} \varphi(h_{\widetilde{M}_{i_j}}(u)) d\mu_{i_j}(u)
\leq \lim_{j \to \infty} \int_{S^{n-1}} \varphi(h_Q(u)) d\mu_{i_j}(u)
= \int_{S^{n-1}} \varphi(h_Q(u)) d\mu(u).
\]
Taking the infimum over $Q \in \widetilde{\mathcal{B}}$ and together with $\widetilde{M} \in \widetilde{\mathcal{B}}$, one gets that
\[
\int_{S^{n-1}} \varphi(h_{\widetilde{M}}(u)) d\mu(u) \leq \inf_{Q \in \widetilde{\mathcal{B}}} \left\{ \int_{S^{n-1}} \varphi(h_Q(u)) d\mu(u) \right\} \leq \int_{S^{n-1}} \varphi(h_{\widetilde{M}}(u)) d\mu(u).
\]
Hence, $\widetilde{M} \in \widetilde{\mathcal{B}}$ verifies (4.9).

Now let us verify (4.10). To this end, let $\{\mu_{i_k}\}_{k \geq 1}$ be an arbitrary subsequence of $\{\mu_i\}_{i \geq 1}$. Repeating the arguments above for $\mu_{i_k}$ and $\widetilde{M}_{i_k}$ (replacing $\mu_i$ and $\widetilde{M}_i$, respectively), one gets a subsequence $\{\widetilde{M}_{i_{k_j}}\}_{j \geq 1}$ of $\{\widetilde{M}_{i_k}\}_{k \geq 1}$ such that $\widetilde{M}_{i_{k_j}} \to \widetilde{M}_0 \in \widetilde{\mathcal{B}}$ and $\widetilde{M}_0$ satisfies (4.9). Thus,
\[
\lim_{j \to \infty} \int_{S^{n-1}} \varphi(h_{\widetilde{M}_{i_{k_j}}}(u)) d\mu_{i_{k_j}}(u) = \int_{S^{n-1}} \varphi(h_{\widetilde{M}_0}(u)) d\mu(u)
= \inf \left\{ \int_{S^{n-1}} \varphi(h_Q(u)) d\mu(u) : Q \in \widetilde{\mathcal{B}} \right\}
= \int_{S^{n-1}} \varphi(h_{\widetilde{M}}(u)) d\mu(u),
\]
where the first equality follows from Lemma 2.2 and the last two equalities follow from (4.9). This concludes the proof of (4.10), i.e.,
\[
\lim_{i \to \infty} \int_{S^{n-1}} \varphi(h_{\widetilde{M}_i}(u)) d\mu_i(u) = \int_{S^{n-1}} \varphi(h_{\widetilde{M}}(u)) d\mu(u).
\]

ii) The proof of this case is almost identical to the one in i), and will be omitted. \(\square\)
The following theorem provides the existence and uniqueness of solutions to Problem 4.1 for \( \varphi \in \mathcal{F} \) and with the infimum considered.

**Theorem 4.7.** Let \( \varphi \in \mathcal{F} \) and \( \mu \) be a nonzero finite Borel measure defined on \( S^{n-1} \) which is not concentrated on any closed hemisphere. Let \( G \in \mathcal{G} \) be a continuous function such that (3.5) holds for some \( q \geq n - 1 \). Then the following statements hold.

i) There exists a convex body \( \widetilde{M} \in \mathcal{B} \) such that

\[
\int_{S^{n-1}} \varphi(h_{\widetilde{M}}(u))d\mu(u) = \inf \left\{ \int_{S^{n-1}} \varphi(h_{Q}(u))d\mu(u) : Q \in \mathcal{B} \right\}.
\]

If, in addition, both \( \varphi(t) \) and \( G(t, \cdot) \) are convex on \( t \in (0, \infty) \), then the solution is unique.

ii) There exists a convex body \( \hat{M} \in \mathcal{B} \) such that

\[
\int_{S^{n-1}} \varphi(h_{\hat{M}}(u))d\mu(u) = \inf \left\{ \int_{S^{n-1}} \varphi(h_{Q}(u))d\mu(u) : Q \in \mathcal{B} \right\}.
\]

If, in addition, both \( \varphi(t) \) and \( G(t, \cdot) \) are convex on \( t \in (0, \infty) \), then the solution is unique.

**Proof.** Let \( \mu \) be a nonzero finite Borel measure defined on \( S^{n-1} \) which is not concentrated on any closed hemisphere. Let \( \mu_i \) for all \( i \in \mathbb{N} \) be nonzero finite discrete Borel measures defined on \( S^{n-1} \), which are not concentrated on any closed hemisphere, such that, \( \mu_i \to \mu \) weakly (see e.g., [48]).

i) By Theorem 4.3, for each \( i \in \mathbb{N} \), there exists a polytope \( \hat{P}_i \in \mathcal{B} \) solving (4.11) with \( \mu_i \) replaced by \( \mu \). It follows from Theorem 4.6 that there exists a \( M \in \mathcal{B} \) such that (4.11) holds.

Now let us prove the uniqueness. Assume that \( \hat{M} \in \mathcal{B} \) and \( M_0 \in \mathcal{B} \), such that

\[
\int_{S^{n-1}} \varphi(h_{\hat{M}}(u))d\mu(u) = \int_{S^{n-1}} \varphi(h_{M_0}(u))d\mu(u) = \inf \left\{ \int_{S^{n-1}} \varphi(h_{Q}(u))d\mu(u) : Q \in \mathcal{B} \right\}.
\]

Note that both \( \hat{M} \in \mathcal{K}_{(o)}^{n} \) and \( M_0 \in \mathcal{K}_{(o)}^{n} \). Let \( K_0 = \frac{\hat{M} + M_0}{2} \in \mathcal{K}_{(o)}^{n} \). Then,

\[
h_{K_0} = \frac{h_{\hat{M}} + h_{M_0}}{2} \implies \rho_{K_0} = 2 \cdot \frac{\rho_{\hat{M}} \cdot \rho_{M_0}}{\rho_{\hat{M}} + \rho_{M_0}},
\]

following from \( h_K \cdot \rho_K = 1 \) for all \( K \in \mathcal{K}_{(o)}^{n} \). The facts that \( G(t, \cdot) \) is convex and \( G \in \mathcal{G} \) is strictly increasing, together with \( \hat{M} \in \mathcal{B} \) and \( M_0 \in \mathcal{B} \), yield that

\[
\tilde{V}_G(K_0^\circ) = \int_{S^{n-1}} G(\rho_{K_0^\circ}(u), u) du \\
\leq \int_{S^{n-1}} G \left( 2 \cdot \frac{\rho_{\hat{M}} \cdot \rho_{M_0}(u)}{\rho_{\hat{M}} + \rho_{M_0}(u)}, u \right) du \\
\leq \int_{S^{n-1}} G \left( \frac{\rho_{\hat{M}}(u) + \rho_{M_0}(u)}{2}, u \right) du \\
\leq \int_{S^{n-1}} \frac{G(\rho_{\hat{M}}(u), u) + G(\rho_{M_0}(u), u)}{2} du \\
= \frac{\tilde{V}_G(\hat{M}^\circ) + \tilde{V}_G(M_0^\circ)}{2} = \tilde{V}_G(B^n).
\]

(4.12)
Again, as $G \in \mathcal{G}_1$, one can find a constant $t_2 \geq 1$ such that $\tilde{V}_G(t_2K_0^c) = \tilde{V}_G(B^n)$ and $K_0/t_2 \in \tilde{B}$. Due to $t_2 \geq 1$ and the facts that $\varphi \in \mathcal{I}$ is convex and strictly increasing, one has

$$
\int_{S^{n-1}} \varphi(h_{K_0/t_2}(u)) \, d\mu(u) \geq \inf \left\{ \int_{S^{n-1}} \varphi(h_Q(u)) \, d\mu(u) : Q \in \tilde{B} \right\} = \frac{1}{2} \left( \int_{S^{n-1}} \varphi\left(\frac{h_{\tilde{M}}(u) + h_{\tilde{M}_0}(u)}{2}\right) \, d\mu(u) \right) \\
\geq \int_{S^{n-1}} \varphi(h_{\tilde{M}_0}(u)) \, d\mu(u) \geq \int_{S^{n-1}} \varphi(h_{K_0}(u)) \, d\mu(u).
$$

(4.13)

Hence all “$\geq$” in (4.13) become “$=$”; and this can happen if and only if $t_2 = 1$ as $\varphi$ is strictly increasing. This in turn yields that all “$\geq$” in (4.12) become “$=$” as well. In particular, as $G(t, \cdot)$ is strictly increasing, for all $u \in S^{n-1}$,

$$
2 \cdot \frac{\rho_{\tilde{M}}(u) \cdot \rho_{\tilde{M}_0}(u)}{\rho_{\tilde{M}}(u) + \rho_{\tilde{M}_0}(u)} = \frac{\rho_{\tilde{M}}(u) + \rho_{\tilde{M}_0}(u)}{2}
$$

and hence $\rho_{\tilde{M}}(u) = \rho_{\tilde{M}_0}(u)$ for all $u \in S^{n-1}$. That is, $\tilde{M} = \tilde{M}_0$ and the uniqueness follows.

ii) The proof of this case is almost identical to the one in i), and will be omitted.

The following result states that the continuity of solutions to Problem 4.1 for $\varphi \in \mathcal{I}$ and with the infimum considered.

**Corollary 4.8.** Let $\mu_i, \mu$ for $i \in \mathbb{N}$ be nonzero finite Borel measures on $S^{n-1}$ which are not concentrated on any closed hemisphere and $\mu_i \to \mu$ weakly. Let $G \in \mathcal{G}_1$ be a continuous function such that $G(t, \cdot)$ is convex on $t \in (0, \infty)$ and (3.5) holds for some $q \geq n - 1$. Let $\varphi \in \mathcal{I}$ be convex. The following statements hold true.

i) Let $\tilde{M}_i \in \tilde{B}$ for each $i \in \mathbb{N}$ and $\tilde{M} \in \tilde{B}$ be the solutions to the optimization problem (4.1) with the infimum considered for measures $\mu_i$ and $\mu$, respectively. Then $\tilde{M}_i \to \tilde{M}$ as $i \to \infty$.

ii) Let $\tilde{M}_i \in \tilde{B}$ for each $i \in \mathbb{N}$ and $\tilde{M} \in \tilde{B}$ be the solutions to the optimization problem (4.2) with the infimum considered for measures $\mu_i$ and $\mu$, respectively. Then $\tilde{M}_i \to \tilde{M}$ as $i \to \infty$.

**Proof.** i) The proof of this result follows from the combination of the proof of Theorem 4.6 and the uniqueness in Theorem 4.7. Indeed, let $\{\tilde{M}_{i_k}\}_{k \geq 1}$ be an arbitrary subsequence of $\{\tilde{M}_i\}_{i \geq 1}$. Like in the proof of Theorem 4.6, one can check that there exist a subsequence $\{\tilde{M}_{i_{kj}}\}_{j \geq 1}$ of $\{\tilde{M}_{i_k}\}_{k \geq 1}$ and a convex body $\tilde{M}_0 \in \tilde{B}$ such that $\tilde{M}_{i_{kj}} \to \tilde{M}_0$. Moreover, $\tilde{M}_0$ satisfies that

$$
\int_{S^{n-1}} \varphi(h_{\tilde{M}_0}(u)) \, d\mu(u) = \inf \left\{ \int_{S^{n-1}} \varphi(h_Q(u)) \, d\mu(u) : Q \in \tilde{B} \right\}.
$$

The uniqueness in Theorem 4.7 yields $\tilde{M}_0 = \tilde{M}$.

In other words, one shows that every subsequence $\{M_{i_k}\}_{k \geq 1}$ of $\{M_i\}_{i \geq 1}$ must have a subsequence $\tilde{M}_{i_{kj}}$ convergent to $\tilde{M}$. This concludes that $\tilde{M}_i \to \tilde{M}$.

ii) The proof of this case is almost identical to the one in i), and will be omitted.
5 Variations of the general dual-polar Orlicz-Minkowski problem

Problem 4.1 discussed in Section 4 are only typical examples of the polar Orlicz-Minkowski type problems. In this section, several variations of Problem 4.1 will be provided.

5.1 The general dual-polar Orlicz-Minkowski problem associated with the Orlicz norms

Let $\mu$ be a given nonzero finite Borel measure defined on $S^{n-1}$. For $\varphi \in \mathcal{I} \cup \mathcal{D}$ and for $Q \in \mathcal{K}^{n}_{(0)}$, the functional $\int_{S^{n-1}} \varphi(h_Q) \, d\mu$ is in general not homogeneous. However, like the definition for $\hat{V}_{G}(\cdot)$, one can define a homogeneous functional for $Q \in \mathcal{K}^{n}_{(0)}$ as follows:

$$
\|h_Q\|_{\mu,\varphi} = \inf \left\{ \lambda > 0 : \frac{1}{\mu(S^{n-1})} \int_{S^{n-1}} \varphi \left( \frac{h_Q(u)}{\lambda} \right) \, d\mu(u) \leq 1 \right\} \text{ if } \varphi \in \mathcal{I};
$$

$$
\|h_Q\|_{\mu,\varphi} = \inf \left\{ \lambda > 0 : \frac{1}{\mu(S^{n-1})} \int_{S^{n-1}} \varphi \left( \frac{h_Q(u)}{\lambda} \right) \, d\mu(u) \geq 1 \right\} \text{ if } \varphi \in \mathcal{D}.
$$

For convenience, $\|h_Q\|_{\mu,\varphi}$ is called the “Orlicz norm” of $h_Q$, although in general it may not satisfy the triangle inequality. Following the proof of Proposition 3.1, it can be checked that, for any $Q \in \mathcal{K}^{n}_{(0)}$ and $\varphi \in \mathcal{I} \cup \mathcal{D}$, $\|h_Q\|_{\mu,\varphi} > 0$ satisfies

$$
\frac{1}{\mu(S^{n-1})} \int_{S^{n-1}} \varphi \left( \frac{h_Q(u)}{\|h_Q\|_{\mu,\varphi}} \right) \, d\mu = 1. \quad (5.1)
$$

Moreover, $\|1\|_{\mu,\varphi} = 1$, $\|ch_Q\|_{\mu,\varphi} = c\|h_Q\|_{\mu,\varphi}$ for any constant $c > 0$ and for any $Q \in \mathcal{K}^{n}_{(0)}$, and $\|h_Q\|_{\mu,\varphi} \leq \|h_L\|_{\mu,\varphi}$ for $Q, L \in \mathcal{K}^{n}_{(0)}$ such that $Q \subseteq L$.

The following lemma for $\varphi \in \mathcal{I} \cup \mathcal{D}$ can be proved similar to the proof of Proposition 3.2 ii). For completeness, we provide a brief proof here. See e.g., [18, Lemma 4] and [23, Lemma 3.4 and Corollary 3.5] for similar results.

Lemma 5.1. Let $Q_i, Q \in \mathcal{K}^{n}_{(0)}$ for each $i \in \mathbb{N}$, and $\mu_i, \mu$ for each $i \in \mathbb{N}$ be nonzero finite Borel measures on $S^{n-1}$. If $Q_i \to Q$ and $\mu_i \to \mu$ weakly, then for all $\varphi \in \mathcal{I} \cup \mathcal{D}$,

$$
\lim_{i \to \infty} \|h_{Q_i}\|_{\mu_i,\varphi} = \|h_Q\|_{\mu,\varphi}.
$$

Proof. We only prove the case for $\varphi \in \mathcal{I}$ (and the case for $\varphi \in \mathcal{D}$ follows along the same lines). Let $Q_i \in \mathcal{K}^{n}_{(0)}$ for all $i \in \mathbb{N}$ and $Q_i \to Q \in \mathcal{K}^{n}_{(0)}$. Let the constants $0 < r_Q < R_Q < \infty$ be such that $r_Q \leq h_Q \leq R_Q$ and $r_Q \leq h_{Q_i} \leq R_Q$ for all $i \in \mathbb{N}$. It can be checked that

$$
r_Q \leq \inf_{i \geq 1} \|h_{Q_i}\|_{\mu_i,\varphi} \leq \sup_{i \geq 1} \|h_{Q_i}\|_{\mu_i,\varphi} \leq R_Q.
$$

Assume that $\limsup_{i \to \infty} \|h_{Q_i}\|_{\mu_i,\varphi} > \|h_Q\|_{\mu,\varphi}$. There exists a subsequence $\{Q_{i_j}\}$ of $\{Q_i\}$ such that $\lim_{j \to \infty} \|h_{Q_{i_j}}\|_{\mu_{i_j},\varphi} > \|h_Q\|_{\mu,\varphi}$. Together with (5.1), Lemma 2.2, the uniform convergence of $h_{Q_i} \to h_Q$ on $S^{n-1}$, and the weak convergence of $\mu_i \to \mu$, one has

$$
1 = \lim_{j \to \infty} \frac{1}{\mu_{i_j}(S^{n-1})} \int_{S^{n-1}} \varphi \left( \frac{h_{Q_{i_j}}(u)}{\|h_{Q_{i_j}}\|_{\mu_{i_j},\varphi}} \right) \, d\mu_{i_j}
$$

$$
= \frac{1}{\mu(S^{n-1})} \int_{S^{n-1}} \varphi \left( \frac{h_Q(u)}{\lim_{j \to \infty} \|h_{Q_{i_j}}\|_{\mu_{i_j},\varphi}} \right) \, d\mu
$$

$$
< \frac{1}{\mu(S^{n-1})} \int_{S^{n-1}} \varphi \left( \frac{h_Q(u)}{\|h_Q\|_{\mu,\varphi}} \right) \, d\mu = 1.
$$
This is a contradiction and hence \( \limsup_{i \to \infty} \| h_{Q_i} \|_{\mu_i, \varphi} \leq \| h_{Q} \|_{\mu, \varphi} \). Similarly, \( \liminf_{i \to \infty} \| h_{Q_i} \|_{\mu_i, \varphi} \geq \| h_{Q} \|_{\mu, \varphi} \) also holds, which leads to \( \lim_{i \to \infty} \| h_{Q_i} \|_{\mu_i, \varphi} = \| h_{Q} \|_{\mu, \varphi} \) as desired. \( \square \)

For the convenience of later citation, the following lemma is given, whose proof for polytopes and discrete measures has appeared in e.g., [15, 21, 23] and is similar to the proof of Lemma 4.5. A brief sketch of the proof is provided for completeness and for future reference.

**Lemma 5.2.** Let \( \varphi \in \mathcal{I} \). Let \( \mu_i, \mu \) for \( i \in \mathbb{N} \) be nonzero finite Borel measures on \( S^{n-1} \) which are not concentrated on any closed hemisphere and \( \mu_i \to \mu \) weakly. Suppose that \( \{ Q_i \}_{i \geq 1} \) is a sequence of convex bodies such that \( Q_i \in \mathcal{X}_{(o)}^n \) for each \( i \in \mathbb{N} \) and \( \sup_{i \geq 1} \| h_{Q_i} \|_{\mu_i, \varphi} < \infty \). Then \( \{ Q_i \}_{i \geq 1} \) is a bounded sequence in \( \mathcal{X}_{(o)}^n \).

**Proof.** Let \( a_+ = \max\{a,0\} \) for all \( a \in \mathbb{R} \). For each \( i \in \mathbb{N} \), let \( u_i \in S^{n-1} \) be such that \( \rho_{Q_i}(u_i) = \max_{u \in S^{n-1}} \rho_{Q_i}(u) \), and hence \( h_{Q_i}(u) \geq \rho_{Q_i}(u_i) \langle u, u_i \rangle \) for any \( u \in S^{n-1} \). Assume that \( \{ Q_i \}_{i \geq 1} \) is not bounded in \( \mathcal{X}_{(o)}^n \), i.e., \( \sup_{i \geq 1} \rho_{Q_i}(u_i) = \infty \). Without loss of generality, let \( u_i \to v \in S^{n-1} \) and \( \lim_{i \to \infty} \rho_{Q_i}(u_i) = \infty \). By formula (5.1) and \( \varphi \in \mathcal{I} \), one has for any given \( C > 0 \), there exists \( i_C \in \mathbb{N} \) such that for all \( i > i_C \),

\[
1 = \frac{1}{\mu_i(S^{n-1})} \int_{S^{n-1}} \varphi \left( \frac{h_{Q_i}(u)}{\| h_{Q_i} \|_{\mu_i, \varphi}} \right) d\mu_i(u)
\geq \frac{1}{\mu_i(S^{n-1})} \int_{S^{n-1}} \varphi \left( \frac{\rho_{Q_i}(u_i) \langle u, u_i \rangle}{\sup_{i \geq 1} \| h_{Q_i} \|_{\mu_i, \varphi}} \right) d\mu_i(u)
\geq \frac{1}{\mu(S^{n-1})} \int_{S^{n-1}} \varphi \left( \frac{C \cdot \langle u, u_i \rangle}{\sup_{i \geq 1} \| h_{Q_i} \|_{\mu_i, \varphi}} \right) d\mu(u).
\]

By Lemma 2.2, the uniform convergence of \( \langle u, u_i \rangle \to \langle u, v \rangle \) on \( S^{n-1} \) as \( u_i \to v \), the weak convergence of \( \mu_i \to \mu \), and \( \varphi \in \mathcal{I} \), one gets

\[
1 \geq \lim_{i \to \infty} \frac{1}{\mu(S^{n-1})} \int_{S^{n-1}} \varphi \left( \frac{C \cdot \langle u, u_i \rangle}{\sup_{i \geq 1} \| h_{Q_i} \|_{\mu_i, \varphi}} \right) d\mu(u)
= \frac{1}{\mu(S^{n-1})} \int_{S^{n-1}} \varphi \left( \frac{C \cdot \langle u, v \rangle}{\sup_{i \geq 1} \| h_{Q_i} \|_{\mu_i, \varphi}} \right) d\mu(u)
\geq \frac{1}{\mu(S^{n-1})} \cdot \varphi \left( \frac{C \cdot c_0}{\sup_{i \geq 1} \| h_{Q_i} \|_{\mu_i, \varphi}} \right) \cdot \int_{\{ u \in S^{n-1} : \langle u, v \rangle \geq c_0 \}} d\mu(u),
\]

where \( c_0 > 0 \) is a finite constant (which always exists due to the monotone convergence theorem and the assumption that \( \mu \) is not concentrated on any closed hemisphere) such that \( \int_{\{ u \in S^{n-1} : \langle u, v \rangle \geq c_0 \}} d\mu(u) > 0 \). Taking \( C \to \infty \), the fact that \( \lim_{t \to \infty} \varphi(t) = \infty \) then yields a contradiction as \( 1 \geq \infty \). This concludes that \( \{ Q_i \}_{i \geq 1} \) is a bounded sequence in \( \mathcal{X}_{(o)}^n \). \( \square \)

Our first variation of Problem 4.1 is the following general dual-polarr Orlicz-Minkowski problem associated with the Orlicz norms:

**Problem 5.3.** Under what conditions on a nonzero finite Borel measure \( \mu \) defined on \( S^{n-1} \), continuous functions \( \varphi : (0, \infty) \to (0, \infty) \) and \( G \in \mathcal{G}_I \cup \mathcal{G}_d \) can we find a convex body \( K \in \mathcal{X}_{(o)}^n \) solving the following optimization problems:

\[
\inf \left\{ \| h_{Q} \|_{\mu, \varphi} : Q \in \mathcal{B} \right\}; \tag{5.2}
\]

\[
\inf \left\{ \| h_{Q} \|_{\mu, \varphi} : Q \in \mathcal{B} \right\}. \tag{5.3}
\]
Due to the high similarity of properties of $\int_{S^{n-1}} \varphi(h_Q) \, d\mu$ and $\|h_Q\|_{\mu,\varphi}$, results and their proofs in Section 4 can be extended and adopted to Problem 5.3. For instance, the existence of solutions to Problem 5.3, if the infimum is considered, can be obtained.

**Theorem 5.4.** Let $\varphi \in \mathcal{I}$ and $\mu$ be a nonzero finite Borel measure defined on $S^{n-1}$ which is not concentrated on any closed hemisphere. Let $G \in \mathcal{G}$ be a continuous function such that (3.5) holds for some $q \geq n - 1$. Then the following statements hold.

i) There exists $\hat{M} \in B$ such that
\[
\|h_{\hat{M}}\|_{\mu,\varphi} = \inf \left\{ \|h_Q\|_{\mu,\varphi} : Q \in \hat{B} \right\}.
\] (5.4)
If, in addition, both $\varphi(t)$ and $G(t, \cdot)$ are convex on $t \in (0, \infty)$, then the solution is unique.

ii) There exists $\hat{M} \in \hat{B}$ such that
\[
\|h_{\hat{M}}\|_{\mu,\varphi} = \inf \left\{ \|h_Q\|_{\mu,\varphi} : Q \in \hat{B} \right\}.
\]
If, in addition, both $\varphi(t)$ and $G(t, \cdot)$ are convex on $t \in (0, \infty)$, then the solution is unique.

**Proof.** Only the brief proof for i) is provided and the proof for ii) follows along the same lines. First of all, $B^n \in \hat{B}$, and the optimization problem (5.4) is well-defined. In particular, there exists a sequence $\{Q_i\}_{i \geq 1}$ such that each $Q_i \in \hat{B}$ and
\[
\lim_{i \to \infty} \|h_{Q_i}\|_{\mu,\varphi} = \inf \left\{ \|h_Q\|_{\mu,\varphi} : Q \in \hat{B} \right\} < \infty.
\]
This further implies that $\sup_{i \geq 1} \|h_{Q_i}\|_{\mu,\varphi} < \infty$, which in turn yields the existence of a subsequence $\{Q_{i_j}\}_{j \geq 1}$ of $\{Q_i\}_{i \geq 1}$ and $\hat{M} \in \hat{B}$, such that $Q_{i_j} \to \hat{M}$, by Lemmas 3.3 and 5.2. It then follows from Lemma 5.1 that $\lim_{i \to \infty} \|h_{Q_i}\|_{\mu,\varphi} = \lim_{j \to \infty} \|h_{Q_{i_j}}\|_{\mu,\varphi} = \|h_{\hat{M}}\|_{\mu,\varphi}$. This concludes the proof, if one notices $\hat{M} \in \hat{B}$, for the existence of solutions to the optimization problem (5.4).

For the uniqueness, assume that $M \in \hat{B}$ and $M_0 \in \hat{B}$, such that
\[
\|h_{\hat{M}'}\|_{\mu,\varphi} = \|h_{\hat{M}_0}\|_{\mu,\varphi} = \inf \left\{ \|h_Q\|_{\mu,\varphi} : Q \in \hat{B} \right\}.
\] (5.5)
Note that $G(t, \cdot)$ is convex and $G \in \mathcal{G}$ is strictly increasing. Let $K_0 = \frac{\hat{M} + \hat{M}_0}{2}$. By (4.12), there is a constant $t_2 \geq 1$ such that $\hat{V}_{\hat{G}}(t_2K_0^n) = \hat{V}_{\hat{G}}(B^n)$ and hence $K_0/t_2 \in \hat{B}$. It follows from (5.1), (5.5), $t_2 \geq 1$ and $\varphi \in \mathcal{I}$ being convex and strictly increasing that
\[
\mu(S^{n-1}) = \int_{S^{n-1}} \varphi \left( \frac{h_{K_0}(u)}{\|h_{K_0}\|_{\mu,\varphi}} \right) \, d\mu
\]
\[
= \frac{1}{2} \left[ \int_{S^{n-1}} \varphi \left( \frac{h_{\hat{M}}}{\|h_{\hat{M}}\|_{\mu,\varphi}} \right) \, d\mu + \int_{S^{n-1}} \varphi \left( \frac{h_{\hat{M}_0}(u)}{\|h_{\hat{M}_0}\|_{\mu,\varphi}} \right) d\mu \right]
\]
\[
\geq \int_{S^{n-1}} \varphi \left( \frac{h_{K_0}(u)}{\|h_{K_0}\|_{\mu,\varphi}} \right) \, d\mu,
\]
and hence $\|h_{\hat{M}_0}\|_{\mu,\varphi} \geq \|h_{K_0}\|_{\mu,\varphi} \geq \|h_{K_0}/t_2\|_{\mu,\varphi} \geq \|h_{\hat{M}_0}\|_{\mu,\varphi}$. Thus, all “≥” become “=”; and this can happen if and only if $t_2 = 1$. This in turn yields that all “≥” in (4.12) become “=” as well. In particular, $\hat{M} = \hat{M}_0$ and the uniqueness follows. 

□
Our second example is the continuity for Problem 5.3 and its solutions.

**Theorem 5.5.** Let $\mu_i, \mu$ for $i \in \mathbb{N}$ be finite Borel measures on $S^{n-1}$ which are not concentrated on any closed hemisphere and $\mu_i \to \mu$ weakly. Let $G \in \mathcal{G}_I$ be a continuous function such that (3.5) holds for some $q \geq n - 1$ and $\varphi \in \mathcal{I}$. The following statements hold true.

i) Let $\hat{M}_i, \tilde{M} \in \mathcal{B}$, for all $i \in \mathbb{N}$, be solutions to the optimization problem (5.2), with the infimum considered, for measures $\mu_i$ and $\mu$, respectively. Then, $\lim_{i \to \infty} \|h_{\hat{M}_i}\|_{\mu, \varphi} = \|h_{\tilde{M}}\|_{\mu, \varphi}$.

If, in addition, both $\varphi(t)$ and $G(t, \cdot)$ are convex on $t \in (0, \infty)$, then $\hat{M}_i \to \tilde{M}$ as $i \to \infty$.

ii) Let $\hat{M}_i, \tilde{M} \in \mathcal{B}$, for all $i \in \mathbb{N}$, be solutions to the optimization problem (5.3), with the infimum considered, for measures $\mu_i$ and $\mu$, respectively. Then, $\lim_{i \to \infty} \|h_{\hat{M}_i}\|_{\mu, \varphi} = \|h_{\tilde{M}}\|_{\mu, \varphi}$.

If, in addition, both $\varphi(t)$ and $G(t, \cdot)$ are convex on $t \in (0, \infty)$, then $\hat{M}_i \to \tilde{M}$ as $i \to \infty$.

**Proof.** Only the brief proof for i) is provided and the proof for ii) follows along the same lines. It follows from $B^n \in \mathcal{B}$, (5.1), and $\varphi \in \mathcal{I}$, in particular $\varphi(1) = 1$ that

$$\sup_{i \geq 1} \|h_{\hat{M}_i}\|_{\mu, \varphi} \leq \sup_{i \geq 1} \|h_{B^n}\|_{\mu, \varphi} = 1.$$

Lemma 5.2 yields that $\{\hat{M}_i\}_{i \geq 1}$ is a bounded sequence.

Let $\{\hat{M}_{ik}\}_{k \geq 1}$ be an arbitrary subsequence of $\{\hat{M}_i\}_{i \geq 1}$. Lemma 3.3 yields the existence of a subsequence $\{\hat{M}_{ik_j}\}_{j \geq 1}$ of $\{\hat{M}_{ik}\}_{k \geq 1}$ and $\tilde{M}_0 \in \mathcal{B}$ such that $\hat{M}_{ik_j} \to \tilde{M}_0$. Together with the minimality of $\|h_{\hat{M}_{ik_j}}\|_{\mu_{ik_j}, \varphi}$, Lemma 5.1 and the weak convergence of $\mu_i \to \mu$ imply that

$$\|h_{\tilde{M}_0}\|_{\mu, \varphi} = \lim_{j \to \infty} \|h_{\hat{M}_{ik_j}}\|_{\mu_{ik_j}, \varphi} \leq \lim_{j \to \infty} \|h_{Q}\|_{\mu_{ik_j}, \varphi} = \|h_{Q}\|_{\mu, \varphi},$$

for all $Q \in \mathcal{B}$. Taking the infimum over $Q \in \mathcal{B}$ and together with $\tilde{M}_0 \in \mathcal{B}$, one gets that

$$\|h_{\tilde{M}_0}\|_{\mu, \varphi} \leq \inf_{Q \in \mathcal{B}} \|h_{Q}\|_{\mu, \varphi} = \|h_{\hat{M}}\|_{\mu, \varphi} \leq \|h_{\tilde{M}_0}\|_{\mu, \varphi}. \quad (5.6)$$

In conclusion, every subsequence $\{\hat{M}_{ik}\}_{k \geq 1}$ of $\{\hat{M}_i\}_{i \geq 1}$ has a subsequence $\{\hat{M}_{ik_j}\}_{j \geq 1}$ such that

$$\|h_{\hat{M}_i}\|_{\mu, \varphi} \leq \lim_{j \to \infty} \|h_{\hat{M}_{ik_j}}\|_{\mu_{ik_j}, \varphi},$$

which implies $\lim_{i \to \infty} \|h_{\hat{M}_i}\|_{\mu, \varphi} = \|h_{\hat{M}}\|_{\mu, \varphi}$.

Formula (5.6) asserts that $\tilde{M}_0 \in \mathcal{B}$ solves the optimization problem (5.2) with the infimum considered. If, in addition, both $\varphi(t)$ and $G(t, \cdot)$ are convex on $t \in (0, \infty)$, the uniqueness in Theorem 5.4 implies $\tilde{M}_0 = \tilde{M}$. In conclusion, every subsequence $\{\hat{M}_{ik}\}_{k \geq 1}$ of $\{\hat{M}_i\}_{i \geq 1}$ has a subsequence $\{\hat{M}_{ik_j}\}_{j \geq 1}$ such that $\hat{M}_{ik_j} \to \tilde{M}$. Hence $\hat{M}_i \to \tilde{M}$ as $i \to \infty$. \hfill \square

An argument almost identical to Lemma 4.2 shows that, if $\varphi \in \mathcal{I}$ and $G \in \mathcal{G}_I$ satisfying (3.5) for some $q \geq n - 1$, the solutions to Problem 5.3 with the infimum considered for $\mu$ being a discrete measure defined in (4.3) (whose support $\{u_1, \ldots, u_m\}$ is not concentrated on any closed hemisphere) must be polytopes with $\{u_1, \ldots, u_m\}$ being the corresponding unit normal vectors of their faces. Counterexamples in Proposition 4.4 can be used to prove that the solutions to Problem 5.3 may not exist if $\varphi \in \mathcal{I} \cup \mathcal{D}$ and the supremum is considered or if $\varphi \in \mathcal{D}$ and the infimum is considered. We leave the details for readers.
5.2 The polar Orlicz-Minkowski problem associated with the general volume

Let \( G: (0, \infty) \times S^{n-1} \to (0, \infty) \) be a continuous function. In [13], the general volume of a convex body \( K \in \mathcal{K}_n^{(0)} \), denoted by \( V_G(K) \), is proposed to be

\[
V_G(K) = \int_{S^{n-1}} G(h_K(u), u) \, dS_K(u),
\]

where \( S_K \) denotes the surface area measure of \( K \) defined on \( S^{n-1} \). Note that \( V_G(K) = V(K) \) if \( G(t, u) = t/n \) for any \((t, u) \in (0, \infty) \times S^{n-1}\).

For each \( K \in \mathcal{K}_n^{(0)} \), denote by \( S(K) \) the surface area of \( K \). A fundamental inequality for \( S(K) \) is the celebrated classical isoperimetric inequality (see e.g., [48]):

\[
S(K) \geq n[V(B^n)]^{1/n}V(K)^{n-1}. \quad (5.7)
\]

Define the homogeneous general volume of \( K \in \mathcal{K}_n^{(0)} \), denoted by \( \overline{V}_G(K) \), as follows: for \( G \in \mathcal{G}_I \cup \mathcal{G}_d \),

\[
\frac{1}{S(K)} \int_{S^{n-1}} G\left(\frac{S(K) \cdot h_K(u)}{V_G(K)}, u\right) \, dS_K(u) = 1. \quad (5.8)
\]

In particular, \( \overline{V}_G(K) = V(K) \) if \( G(t, u) = t/n \). Note that \( \overline{V}_G(K) \) has equivalent formulas similar to (3.1) and (3.2).

Problems 4.1 and 5.3 can be asked for \( V_G(\cdot) \) and \( \overline{V}_G(\cdot) \), respectively.

**Problem 5.6.** Under what conditions on a nonzero finite Borel measure \( \mu \) defined on \( S^{n-1} \), continuous functions \( \varphi: (0, \infty) \to (0, \infty) \) and \( G: (0, \infty) \times S^{n-1} \to (0, \infty) \) can we find a convex body \( K \in \mathcal{K}_n^{(0)} \) solving the following optimization problems:

\[
\inf / \sup \{ \|h_Q\|_{\mu, \varphi} : Q \in \mathcal{B} \} \quad \text{or} \quad \inf / \sup \left\{ \int_{S^{n-1}} \varphi(h_Q(u)) \, d\mu(u) : Q \in \mathcal{B} \right\};
\]

\[
\inf / \sup \{ \|h_Q\|_{\mu, \varphi} : Q \in \overline{\mathcal{B}} \} \quad \text{or} \quad \inf / \sup \left\{ \int_{S^{n-1}} \varphi(h_Q(u)) \, d\mu(u) : Q \in \overline{\mathcal{B}} \right\},
\]

where \( \mathcal{B} \) and \( \overline{\mathcal{B}} \) are given by

\[
\mathcal{B} = \{ Q \in \mathcal{K}_n^{(0)} : V_G(Q^o) = V_G(B^n) \};
\]

\[
\overline{\mathcal{B}} = \{ Q \in \mathcal{K}_n^{(0)} : \overline{V}_G(Q^o) = \overline{V}_G(B^n) \}, \quad \text{if} \quad G \in \mathcal{G}_I \cup \mathcal{G}_d.
\]

Again, when \( G = t/n \), Problem 5.6 becomes the polar Orlicz-Minkowski problem [34]. From Sections 4 and 5.1, one sees that the existence and continuity of solutions to Problems 4.1 and 5.3 are similar, and their proofs heavily depend on Lemmas 3.3, 4.5, 5.1 and 5.2. In particular, if alternative arguments of Lemma 3.3 for \( V_G(\cdot) \) and \( \overline{V}_G(\cdot) \) can be established, the desired existence and continuity of solutions, if applicable, to Problem 5.6 will follow.

Some properties for \( V_G(\cdot) \) and \( \overline{V}_G(\cdot) \) are summarized in the following two propositions.

**Proposition 5.7.** Let \( G: (0, \infty) \times S^{n-1} \to (0, \infty) \) be a continuous function. The general volume \( V_G(\cdot) \) has the following properties.

1) \( V_G(\cdot) \) is continuous on \( \mathcal{K}_n^{(0)} \) in terms of the Hausdorff metric, that is, for any sequence \( \{K_i\}_{i \geq 1} \) such that \( K_i \in \mathcal{K}_n^{(0)} \) for all \( i \in \mathbb{N} \) and \( K_i \to K \in \mathcal{K}_n^{(0)} \), then \( V_G(K_i) \to V_G(K) \).
ii) Let $K \in \mathcal{X}_n$. If $\overline{G}(t, \cdot) = t^n G(t, \cdot) \in \mathcal{G}_I$, then $V_G(tK)$ is strictly increasing on $t \in (0, \infty)$ and 
\[
\lim_{t \to 0^+} V_G(tK) = 0 \quad \text{and} \quad \lim_{t \to \infty} V_G(tK) = \infty;
\]
while if $G \in \mathcal{G}_d$, then $V_G(tK)$ is strictly decreasing on $t \in (0, \infty)$ and 
\[
\lim_{t \to 0^+} V_G(tK) = \infty \quad \text{and} \quad \lim_{t \to \infty} V_G(tK) = 0.
\]

Proof. The fact that $K_i \to K \in \mathcal{X}_n$ with $K_i \in \mathcal{X}_n$ for each $i \in \mathbb{N}$ implies that $h_{K_i} \to h_K$ uniformly on $S^{n-1}$ and $S(K_i) \to S(K)$. Moreover, there exist two positive constants $r_K < R_K$ such that 
\[
r_K \leq h_K \leq R_K \quad \text{and} \quad r_K \leq h_{K_i} \leq R_K \quad \text{for all} \quad i \in \mathbb{N}.
\]
i) As $h_{K_i} \to h_K$ uniformly on $S^{n-1}$, one has $G(h_{K_i}(u), u) \to G(h_K(u), u)$ also uniformly on $S^{n-1}$. Lemma 2.2 and the well known fact that $S_{K_i} \to S_K$ weakly yield that $V_G(K_i) \to V_G(K)$ as $i \to \infty$.

ii) Let $K \in \mathcal{X}_n$. For all $t > s > 0$ and all $u \in S^{n-1}$, if $G \in \mathcal{G}_I$ (and hence $\overline{G}(t, \cdot)$ is strictly increasing on $t > 0$), then $V_G(tK)$ is strictly increasing on $t > 0$ as follows:
\[
V_G(tK) = \int_{S^{n-1}} G(h_{tK}(u), u) \, dS_{tK}(u)
= \int_{S^{n-1}} t^{n-1} G(t \cdot h_{K}(u), u) \, dS_{K}(u)
= \int_{S^{n-1}} \overline{G}(t \cdot h_{K}(u), u) h_{1-n}^{1-n}(u) \, dS_{K}(u)
\geq \int_{S^{n-1}} \overline{G}(s \cdot h_{K}(u), u) h_{1-n}^{1-n}(u) \, dS_{K}(u) = V_G(sK).
\]
As $r_K \leq h_K(u) \leq R_K$ for all $u \in S^{n-1}$,
\[
\lim_{t \to 0^+} V_G(tK) = \lim_{t \to 0^+} \int_{S^{n-1}} \overline{G}(t \cdot h_{K}(u), u) h_{1-n}^{1-n}(u) \, dS_{K}(u)
\leq \lim_{t \to 0^+} \int_{S^{n-1}} r_K^{1-n} \overline{G}(t \cdot R_K, u) \, dS_{K}(u)
= \int_{S^{n-1}} \lim_{t \to 0^+} r_K^{1-n} \overline{G}(t \cdot R_K, u) \, dS_{K}(u) = 0,
\]
where we have used the dominated convergence theorem and the fact that $\lim_{t \to 0^+} \overline{G}(t, \cdot) = 0$. This proves that $\lim_{t \to 0^+} V_G(tK) = 0$. Similarly, $\lim_{t \to \infty} V_G(tK) = \infty$ can be proved as follows:
\[
\lim_{t \to \infty} V_G(tK) \geq \lim_{t \to \infty} \int_{S^{n-1}} \overline{G}(t \cdot r_K, u) R_K^{1-n} \, dS_K(u) \geq \int_{S^{n-1}} \lim_{t \to \infty} \overline{G}(t \cdot r_K, u) R_K^{1-n} \, dS_K(u) = \infty,
\]
where we have used Fatou’s lemma and the fact that $\lim_{t \to \infty} \overline{G}(t, \cdot) = \infty$. The desired result for the case $\overline{G} \in \mathcal{G}_d$ follows along the same lines. \hfill \Box

**Proposition 5.8.** Let $G \in \mathcal{G}_I \cup \mathcal{G}_d$. The homogeneous general volume $\nabla_G(\cdot)$ has the following properties.

i) $\nabla_G(\cdot)$ is homogeneous, that is, $\nabla_G(tK) = t^n \nabla_G(K)$ holds for all $t > 0$ and all $K \in \mathcal{X}_n$.

ii) $\nabla_G(\cdot)$ is continuous on $\mathcal{X}_n$ in terms of the Hausdorff metric, that is, for any sequence $\{K_i\}_{i \geq 1}$ such that $K_i \in \mathcal{X}_n$ for all $i \in \mathbb{N}$ and $K_i \to K \in \mathcal{X}_n$, then $\nabla_G(K_i) \to \nabla_G(K)$. 

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Proof. i) The desired argument follows trivially from (5.8), the strict monotonicity of \( G \), and the facts that \( S(tK) = t^{n-1}S(K) \) and \( h_{tK} = t \cdot h_K \) for all \( t > 0 \).

ii) Following the notations as in Proposition 5.7, we will prove the continuity for \( \nabla_G(\cdot) \) if \( G \in \mathcal{G}_I \) (and the proof for the case \( G \in \mathcal{G}_d \) is omitted). It follows from (5.8) that

\[
\int_{S^{n-1}} G \left( \frac{S(K_i) \cdot r_K}{\nabla_G(K_i)}, u \right) dS_{K_i}(u) \leq S(K_i) \leq \int_{S^{n-1}} G \left( \frac{S(K_i) \cdot R_K}{\nabla_G(K_i)}, u \right) dS_{K_i}(u).
\]

Suppose that \( \inf_{i \in \mathbb{N}} \nabla_G(K_i) = 0 \), and without loss of generality, assume that \( \lim_{i \to \infty} \nabla_G(K_i) = 0 \). Then for any \( \varepsilon > 0 \), there exists \( i_\varepsilon \in \mathbb{N} \) such that \( \nabla_G(K_i) < \varepsilon \) for all \( i > i_\varepsilon \). Hence, for \( i \geq i_\varepsilon \),

\[
\int_{S^{n-1}} G \left( \frac{S(K_i) \cdot r_K}{\varepsilon}, u \right) dS_{K_i}(u) \leq \int_{S^{n-1}} G \left( \frac{S(K_i) \cdot R_K}{\nabla_G(K_i)}, u \right) dS_{K_i}(u) \leq S(K_i).
\]

A contradiction can be obtained from Lemma 2.2, the weak convergence of \( S_{K_i} \to S_K \), the facts that \( \lim_{t \to 0^+} G(t, \cdot) = \infty \) and \( S(K_i) \to S(K) \), and Fatou’s lemma as follows:

\[
S(K) \geq \liminf_{\varepsilon \to 0^+} \left[ \lim_{i \to \infty} \int_{S^{n-1}} G \left( \frac{S(K_i) \cdot r_K}{\varepsilon}, u \right) dS_{K_i}(u) \right] \\
= \liminf_{\varepsilon \to 0^+} \int_{S^{n-1}} G \left( \frac{S(K) \cdot r_K}{\varepsilon}, u \right) dS_K(u) \\
\geq \int_{S^{n-1}} \liminf_{\varepsilon \to 0^+} G \left( \frac{S(K) \cdot r_K}{\varepsilon}, u \right) dS_K(u) = \infty.
\]

This is impossible and hence \( \inf_{i \in \mathbb{N}} \nabla_G(K_i) > 0 \). Similarly, \( \sup_{i \in \mathbb{N}} \nabla_G(K_i) < \infty \).

Now let us prove \( \lim_{i \to \infty} \nabla_G(K_i) = \nabla_G(K) \). Assume that \( \nabla_G(K) < \limsup_{i \to \infty} \nabla_G(K_i) \). There exists a subsequence \( \{K_{i_j}\} \) of \( \{K_i\} \) such that \( \nabla_G(K) < \lim_{j \to \infty} \nabla_G(K_{i_j}) \leq \sup_{i \in \mathbb{N}} \nabla_G(K_i) < \infty \).

By \( G \in \mathcal{G}_I \), (5.8), Lemma 2.2, \( S_{K_{i_j}} \to S_K \) weakly and \( h_{K_{i_j}} \to h_K > 0 \) uniformly on \( S^{n-1} \), one gets

\[
S(K) = \lim_{j \to \infty} \int_{S^{n-1}} G \left( \frac{S(K_{i_j}) \cdot h_{K_{i_j}}(u)}{\nabla_G(K_{i_j})}, u \right) dS_{K_{i_j}}(u) \\
= \int_{S^{n-1}} G \left( \frac{S(K) \cdot h_K(u)}{\lim_{j \to \infty} \nabla_G(K_{i_j})}, u \right) dS_K(u) \\
< \int_{S^{n-1}} G \left( \frac{S(K) \cdot h_K(u)}{\nabla_G(K)}, u \right) dS_K(u) = S(K).
\]

This is a contradiction and hence \( \limsup_{i \to \infty} \nabla_G(K_i) \leq \nabla_G(K) \). Similarly, \( \liminf_{i \to \infty} \nabla_G(K_i) \geq \nabla_G(K) \) and then the desired equality \( \lim_{i \to \infty} \nabla_G(K_i) = \nabla_G(K) \) holds.

The following lemma is a replacement of Lemma 3.3. Note that the monotonicity of \( V_G(\cdot) \) and \( \overline{V}_G(\cdot) \) in terms of set inclusion, in general, may be invalid. Therefore, our proof for Lemma 5.9 is quite different from the one for Lemma 3.3.

**Lemma 5.9.** Let \( G : (0, \infty) \times S^{n-1} \to (0, \infty) \) be a continuous function and \( G_q(t,u) = \frac{G(t,u)}{t_q} \) for \( q \in \mathbb{R} \).

i) Suppose that there exists a constant \( q \in (1-n,0) \), such that,

\[
\inf \left\{ G_q(t,u) : t \geq 1 \text{ and } u \in S^{n-1} \right\} > 0.
\]

(5.9)
If \(\{Q_i\}_{i\geq 1}\) with \(Q_i \in \mathcal{B}\) for all \(i \in \mathbb{N}\) is a bounded sequence, then there exist a subsequence \(\{Q_{i_j}\}_{j\geq 1}\) of \(\{Q_i\}_{i\geq 1}\) and \(Q_0 \in \mathcal{B}\) such that \(Q_{i_j} \to Q_0\).

ii) Let \(G \in \mathcal{G}_1\) satisfy (5.9) for some \(q \geq 1\). If \(\{Q_i\}_{i\geq 1}\) with \(Q_i \in \overline{\mathcal{B}}\) for all \(i \in \mathbb{N}\) is a bounded sequence, then there exist a subsequence \(\{Q_{i_j}\}_{j\geq 1}\) of \(\{Q_i\}_{i\geq 1}\) and \(Q_0 \in \overline{\mathcal{B}}\) such that \(Q_{i_j} \to Q_0\).

Proof. Let \(\{Q_i\}_{i\geq 1}\) with \(Q_i \in \mathcal{K}_n^\circ\) for each \(i \in \mathbb{N}\) be bounded. There exists a finite constant \(R > 0\) such that \(Q_i \subset RB^n\) for all \(i \in \mathbb{N}\), which in turn implies \(Q_i^0 \supset \frac{1}{R}B^n\). In particular, \(h_{Q_i^0} \geq 1/R\) for each \(i \in \mathbb{N}\) and \(S(Q_i^0) \geq R^{1-n}S(B^n)\) due to the monotonicity of surface area for convex bodies.

i) Again (5.9) is equivalent to: there exist finite constants \(c_0, C_0 > 0\) such that for \(q \in (1 - n, 0)\),

\[
\inf \left\{G_q(t, u) : t \geq c_0 \text{ and } u \in S_i^{n-1}\right\} > C_0.\tag{5.10}
\]

Let \(c_0 = 1/R\). Then \(G(t, u) \geq C_0 t^q\) for \(q \in (1 - n, 0)\) and for all \((t, u) \in [1/R, \infty) \times S_i^{n-1}\). Thus,

\[
V_G(Q_i^0) = \int_{S_i^{n-1}} G(h_{Q_i^0}(u), u) \, dS_{Q_i^0}(u)
\]

\[
\geq C_0 \cdot S(Q_i^0) \int_{S_i^{n-1}} h_{Q_i^0}^q(u) \frac{1}{S(Q_i^0)} \, dS_{Q_i^0}(u)
\]

\[
\geq C_0 \cdot S(Q_i^0) \left(\int_{S_i^{n-1}} h_{Q_i^0}(u) \frac{1}{S(Q_i^0)} \, dS_{Q_i^0}(u)\right)^q
\]

\[
= C_0 \cdot S(Q_i^0) \left(\frac{nV(Q_i^0)}{S(Q_i^0)}\right)^q
\]

\[
\geq C_0 \cdot n(V(B^n))^{\frac{1}{n}} \left(V(Q_i^0)\right)^{\frac{1}{n}} \left(V(Q_i^0)\right)^{\frac{2}{n}} \left(V(Q_i^0)\right)^{\frac{n-1+q}{n}}
\]

where we have used Jensen’s inequality and the classical isoperimetric inequality (5.7). Recall that \(V_G(Q_i^0) = V_G(B^n)\) for all \(i \in \mathbb{N}\) and \(1 - n < q < 0\), one has

\[
\sup_{i \geq 1} \left\{V(Q_i^0)\right\} \leq \left(\frac{V_G(B^n)}{C_0 \cdot n(V(B^n))^{\frac{1}{n}}}ight)^{\frac{n}{n-1+q}} < \infty.
\]

Note that \(t^n/n\) satisfies (3.5). The proof of Lemma 3.3 (in particular, (3.9)) can be used to get a subsequence \(\{Q_{i_j}\}_{j\geq 1}\) of \(\{Q_i\}_{i\geq 1}\) and \(Q_0 \in \mathcal{K}_n^\circ\) such that \(Q_{i_j} \to Q_0\) (see also [39, Lemma 3.2]). Consequently \(Q_{i_j}^0 \to Q_0^0\), and the continuity of \(V_G(\cdot)\) in Proposition 5.7 further yields that \(Q_0 \in \mathcal{B}\) following from \(Q_i \in \mathcal{B}\) for all \(i \in \mathbb{N}\).

ii) Recall that \(Q_i^0 \supset \frac{1}{R}B^n\) for each \(i \in \mathbb{N}\). As \(Q_i \in \overline{\mathcal{B}}\) for each \(i \in \mathbb{N}\), one has

\[
c_0 = R^{-n}S(B^n) \leq \frac{S(Q_i^0) \cdot h_{Q_i^0}(u)}{V_G(Q_i^0)}.
\]
It follows from (5.8), (5.10) and Jensen’s inequality for $q \geq 1$ that
\[
1 = \frac{1}{S(Q^0_i)} \int_{S^{n-1}} G\left(\frac{S(Q^0_i) \cdot h_{Q^0_i}(u)}{V_G(Q^0_i)}, u\right) dS_{Q^0_i}(u)
\geq \frac{C_0}{S(Q^0_i)} \int_{S^{n-1}} \left(\frac{S(Q^0_i) \cdot h_{Q^0_i}(u)}{V_G(B^n)}\right)^q dS_{Q^0_i}(u)
\geq C_0 \left(\int_{S^{n-1}} \frac{h_{Q^0_i}(u)}{V_G(B^n)} dS_{Q^0_i}(u)\right)^q
= C_0 \left(\frac{nV(Q^0_i)}{V_G(B^n)}\right)^q.
\]
This further implies that $V(Q^0_i) \leq n^{-1}C_0^{-1/q}V_G(B^n)$ for each $i \in \mathbb{N}$. As in i) (the last paragraph), one gets a subsequence $\{Q_{i_j}\}_{j \geq 1}$ of $\{Q_i\}_{i \geq 1}$ and $Q_0 \in \mathcal{K}^n_{(0)}$, such that, $Q^0_{i_j} \rightarrow Q^0_0$. The continuity of $V_G(\cdot)$ in Proposition 5.8 further yields that $Q_0 \in \mathcal{B}$ following from $Q_i \in \mathcal{B}$ for all $i \in \mathbb{N}$. □

**Remark.** It can be easily checked that if (5.9) holds for some $q \geq 0$, Part i) of Lemma 5.9 also holds. To this end, if (5.9) holds for $q \geq 0$, one can verify that $2q + n - 1 > 0$ and
\[
\inf \left\{ G_{\frac{1}{n-2}}(t, u) : (t, u) \in [1, \infty) \times S^{n-1}\right\} = \inf \left\{ G_q(t, u) : t \frac{2q+n-1}{2} (t, u) \in [1, \infty) \times S^{n-1}\right\}
\geq \inf \left\{ G_q(t, u) : (t, u) \in [1, \infty) \times S^{n-1}\right\} > 0.
\]
Hence, (5.9) holds for $\frac{1-n}{2} \in (1-n, 0)$ and then Part i) of Lemma 5.9 also follows. In particular, Part i) of Lemma 5.9 works for $G = t/n$ and $G = 1$ which correspond to the volume and the surface area, respectively. Similar to the remark of Lemma 3.3, if $G \in \mathcal{G}_d$, $G$ does not satisfy (5.9) for some $q \geq 1$.

The existence of solutions and the continuity of the extreme values to Problem 5.6 for $V_G$ are stated below.

**Theorem 5.10.** Let $\varphi \in \mathcal{I}$ and let $G : (0, \infty) \times S^{n-1} \rightarrow (0, \infty)$ satisfying (5.9) for some $q \in (1-n, 0)$.

i) Let $\mu$ be a nonzero finite Borel measure on $S^{n-1}$ whose support is not concentrated on any great hemisphere. Then there exist $M_1, M_2 \in \mathcal{B}$ such that
\[
\inf_{Q \in \mathcal{I}} \int_{S^{n-1}} \varphi(h_{M_1}(u)) d\mu(u) = \inf_{Q \in \mathcal{I}} \int_{S^{n-1}} \varphi(h_{Q}(u)) d\mu(u); \quad (5.11)
\|h_{M_2}\|_{\mu, \varphi} = \inf_{Q \in \mathcal{I}} \|h_{Q}\|_{\mu, \varphi}. \quad (5.12)
\]

ii) Let $\{\mu_i\}_{i=1}^\infty$ and $\mu$ be nonzero finite Borel measures on $S^{n-1}$ whose supports are not concentrated on any closed hemisphere, such that, $\mu_i \rightarrow \mu$ weakly as $i \rightarrow \infty$. Then
\[
\lim_{i \rightarrow \infty} \left( \inf_{Q \in \mathcal{I}} \int_{S^{n-1}} \varphi(h_{Q}(u)) d\mu_i(u) \right) = \inf_{Q \in \mathcal{I}} \int_{S^{n-1}} \varphi(h_{Q}(u)) d\mu(u);
\lim_{i \rightarrow \infty} \left( \inf_{Q \in \mathcal{I}} \|h_{Q}\|_{\mu_i, \varphi} \right) = \inf_{Q \in \mathcal{I}} \|h_{Q}\|_{\mu, \varphi}.
\]
**Proof.** i) Note that $B^n \in \mathcal{B}$ and hence the optimization problem in (5.11) is well defined. Let \( \{Q_i\}_{i \geq 1} \) be the limiting sequences such that $Q_i \in \mathcal{B}$ for each $i \in \mathbb{N}$ and

$$
\mu(S^{n-1}) \geq \inf_{Q \in \mathcal{B}} \int_{S^{n-1}} \varphi(h_Q(u))\,d\mu(u) = \lim_{i \to \infty} \int_{S^{n-1}} \varphi(h_{Q_i}(u))\,d\mu(u).
$$

It follows from Lemma 4.5 that $\{Q_i\}_{i \geq 1}$ is a bounded sequence in $\mathcal{K}_{(\phi)}^n$. Together with Lemma 5.9, there exist a subsequence $\{Q_{i_j}\}_{j \geq 1}$ of $\{Q_i\}_{i \geq 1}$ and $M_1 \in \mathcal{B}$ such that $Q_{i_j} \to M_1$. Lemma 2.2 and $\varphi \in \mathcal{I}$ then yield

$$
\int_{S^{n-1}} \varphi(h_{M_1}(u))\,d\mu(u) = \lim_{j \to \infty} \int_{S^{n-1}} \varphi(h_{Q_{i_j}}(u))\,d\mu(u) = \inf_{Q \in \mathcal{B}} \int_{S^{n-1}} \varphi(h_Q(u))\,d\mu(u).
$$

The existence of $M_2 \in \mathcal{B}$ that verifies (5.12) can be obtained similarly, with Lemma 4.5 and Lemma 2.2 replaced by Lemma 5.2 and Lemma 5.1, respectively, if one notices that

$$
1 \geq \inf_{Q \in \mathcal{B}} \|h_Q\|_{\mu, \varphi} = \lim_{i \to \infty} \|h_{Q_i}\|_{\mu, \varphi}.
$$

ii) First, note that from Part i), the optimization problems (5.11) and (5.12) for $\mu$ and $\mu_i$ for each $i \in \mathbb{N}$ have solutions. The rest of the proof is almost identical to those for Theorems 4.6 and 5.5, with Lemma 3.3 replaced by Lemma 5.9. \( \Box \)

Similarly, one can prove the existence of solutions and the continuity of the extreme values to Problem 5.6 for $\overline{V}_G(\cdot)$. The proof will be omitted due to the high similarity to those in e.g., Theorem 5.10.

**Theorem 5.11.** Let $\varphi \in \mathcal{I}$ and let $G \in \mathcal{G}_1$ satisfy (5.9) for some constant $q \geq 1$.

i) Let $\mu$ be a nonzero finite Borel measure on $S^{n-1}$ whose support is not concentrated on any great hemisphere. There exist $M_1, M_2 \in \mathcal{B}$ such that

$$
\int_{S^{n-1}} \varphi(h_{M_1}(u))\,d\mu(u) = \inf_{Q \in \mathcal{B}} \int_{S^{n-1}} \varphi(h_Q(u))\,d\mu(u) \text{ and } \|h_{M_1}\|_{\mu, \varphi} = \inf_{Q \in \mathcal{B}} \|h_Q\|_{\mu, \varphi}.
$$

ii) Let $\{\mu_i\}_{i=1}^\infty$ and $\mu$ be nonzero finite Borel measures on $S^{n-1}$ whose supports are not concentrated on any closed hemisphere, such that, $\mu_i \to \mu$ weakly as $i \to \infty$. Then

$$
\lim_{i \to \infty} \left( \inf_{Q \in \mathcal{B}} \int_{S^{n-1}} \varphi(h_Q(u))\,d\mu_i(u) \right) = \inf_{Q \in \mathcal{B}} \int_{S^{n-1}} \varphi(h_Q(u))\,d\mu(u);
$$

$$
\lim_{i \to \infty} \left( \inf_{Q \in \mathcal{B}} \|h_Q\|_{\mu_i, \varphi} \right) = \inf_{Q \in \mathcal{B}} \|h_Q\|_{\mu, \varphi}.
$$

### 5.3 The general Orlicz-Petty bodies

The classical geominimal surface area [46, 47] and its $L_p$ or Orlicz extensions (see e.g., [39, 56, 57, 58, 62]) are central objects in convex geometry. When studying the properties of various geominimal surface areas, the Petty body or its generalizations play fundamental roles. In short, the Orlicz-Petty bodies are the solutions to the following optimization problems [58, 62]:

$$
\inf \left\{ nV_\varphi(K, L) : L \in \mathcal{K}_{(\phi)}^n \text{ with } V(L^n) = V(B^n) \right\}; \quad (5.13)
$$

$$
\inf \left\{ \overline{V}_\varphi(K, L) : L \in \mathcal{K}_{(\phi)}^n \text{ with } V(L^n) = V(B^n) \right\}, \quad (5.14)
$$
where $\varphi \in \mathcal{I}$, and $V_\varphi(K, L)$ and $\hat{V}_\varphi(K, L)$ are the Orlicz $L_\varphi$ mixed volumes of $K, L \in \mathcal{K}_n$ defined by (see e.g., [12, 54, 62]):

$$V_\varphi(K, L) = \frac{1}{n} \int_{S^{n-1}} \varphi \left( \frac{h_L(u)}{h_K(u)} \right) h_K(u) dS_K(u) \quad \text{and} \quad \hat{V}_\varphi(K, L) = \left\| \frac{h_L}{h_K} \right\|_{S_K, \varphi}.$$  

The surface area measure $S_K$ may be replaced by other measures; for instance, Luo, Ye and Zhu in [34] obtained the $p$-capacitary Orlicz-Petty bodies where the surface area measure is replaced by the $p$-capacitary measure (see e.g., [11, 28]). As explained in [34], the polar Orlicz-Minkowski problem (i.e., Problems 4.1 and 5.3 with $G = t^n/n$) and the optimization problems (5.13) and (5.14) are quite different in their general forms; however these two problems are also very closely related. In view of their relations, we can ask the following problem aiming to find the general Orlicz-Petty bodies.

**Problem 5.12.** Let $K \in \mathcal{K}_n$ be a fixed convex body. Let $\mu_K$ be a nonzero finite Borel measure associated with $K$ defined on $S^{n-1}$, which is not concentrated on any closed hemisphere. Under what conditions on continuous functions $\varphi : (0, \infty) \to (0, \infty)$ and $G : (0, \infty) \times S^{n-1} \to (0, \infty)$ can we find a convex body $M \in \mathcal{K}_n$ solving the following optimization problems:

$$\inf \left( \sup \left\{ \left\| \frac{h_Q}{h_K} \right\|_{\mu_K, \varphi} : Q \in \mathcal{A} \right\} \right) \quad \text{or} \quad \inf \left( \sup \left\{ \int_{S^{n-1}} \varphi \left( \frac{h_Q(u)}{h_K(u)} \right) h_K(u) d\mu_K(u) : Q \in \mathcal{A} \right\} \right),$$

(5.15)

where $\mathcal{A}$ is selected from the following sets: $\widetilde{\mathcal{B}}, \hat{\mathcal{B}}, \mathcal{B}$ and $\hat{\mathcal{B}}$.

Note that the measure $\mu_K$ assumed in Problem 5.12 includes many interesting measures, such as, the surface area measure $S_K$, the $p$-capacitary measure [11, 28], the Orlicz $p$-capacitary measure [21], the $L_p$ dual curvature measures [24, 42], the general dual Orlicz curvature measures [13, 15, 55, 63], and many more.

**Definition 5.13.** Let $K \in \mathcal{K}_n$ be a fixed convex body. Let $\mu_K$ be a nonzero finite Borel measure associated with $K$ defined on $S^{n-1}$, which is not concentrated on any closed hemisphere. If $M \in \mathcal{A}$ solving the optimization problem (5.15), then $M$ is called a general Orlicz-Petty body of $K$ with respect to $\mu_K$.

Recall that if $K \in \mathcal{K}_n$, there are two constants $0 < r_K < R_K$ such that $r_K B^n \subset K \subset R_K B^n$. In view of this, the existence, continuity and uniqueness, if applicable, of the general Orlicz-Petty bodies with respect to $\mu_K$ can be obtained (almost identically) as in Sections 4, 5.1 and 5.2. Polytopal solutions and counterexamples as in Proposition 4.4, when $K$ is a polytope, can be also established accordingly, if applicable.

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