Effective Filtering for Multiscale Stochastic Dynamical Systems Driven by Lévy Processes*

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Abstract
The work is about multiscale stochastic dynamical systems driven by Lévy processes. We prove that such a system can be approximated by a low-dimensional system on a random invariant manifold, and the original filter can be also approximated by the reduced low-dimensional filter. Finally, we investigate the reduction for $\varepsilon = 0$ and obtain that these reduced systems does not approximate these multiscale stochastic dynamical systems.

Keywords Multiscale systems driven by Lévy processes · Random slow manifolds · Dimension reduction · Efficient filtering

Mathematics Subject Classification 60H10 · 37D10 · 70K70 · 60G51

1 Introduction

Given a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ (See Subsect. 3.1 in detail). Consider the following stochastic slow-fast system on $\mathbb{R}^n \times \mathbb{R}^m$:

\[
\begin{align*}
\dot{u}^\varepsilon &= \frac{1}{\varepsilon} Au^\varepsilon + \frac{1}{\varepsilon} U(u^\varepsilon, v^\varepsilon) + \frac{\sigma_1}{\varepsilon^{1/\alpha}} \dot{L}^\alpha\pm, \\
\dot{v}^\varepsilon &= B v^\varepsilon + V(u^\varepsilon, v^\varepsilon) + \sigma_2 \dot{L}^\pm,
\end{align*}
\]

(1)

where $A$ and $B$ are $n \times n$ and $m \times m$ matrices respectively, and the interaction functions $U : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ and $V : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^m$ are Borel measurable. $L^\alpha\pm$ and $L^\pm$ are a $n$-dimensional two-sided symmetric $\alpha$-stable process and a $m$-dimensional two-sided Lévy process, respectively(c.f. Subsect. 2.3). Moreover, $\sigma_1$ and $\sigma_2$ are nonzero real noise intensities, and $\varepsilon$ is a small positive parameter representing the ratio of the two time scales. These systems like (1) are usually called multi-scale systems and have been applied to simulate many phenomena in chemistry, biology, climate, and so on [10,21,23,26].

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If Lévy processes are replaced by Brownian motions, these systems (1) have been widely observed and studied. Let us recall some references. Khasminskii and Yin [9] developed a stochastic averaging principle for systems (1). Later, Schmalfuß and Schneider [22] and Wang and Roberts [24] obtained the invariant manifold for systems (1). In the infinite dimensional framework Fu, Liu and Duan [6] and the author [17] studied the invariant manifolds of systems (1) and obtained low dimensional reduced systems. Besides, if systems (1) are driven by two symmetric \( \alpha \)-stable processes, i.e. the Lévy process \( L \) is also a symmetric \( \alpha \)-stable process, the author and two coauthors [27] showed the existence of the invariant manifold for systems (1).

The nonlinear filtering problem for systems (1) with respect to an observation process \( \{w_s^\varepsilon, 0 \leq s \leq t\} \) (See Subsect. 4.1 in details) is to evaluate the ‘filter’ \( \mathbb{E}[\Phi(u_s^\varepsilon, v_s^\varepsilon)|\mathcal{W}_s^\varepsilon]\), where \( \Phi \) is a Borel measurable function such that \( \mathbb{E}[\Phi(u_s^\varepsilon, v_s^\varepsilon)] < \infty \) for \( t \in [0, T] \), and \( \mathcal{W}_s^\varepsilon \) is the \( \sigma \)-algebra generated by \( \{w_s^\varepsilon, 0 \leq s \leq t\} \) [3,19]. And it is sometimes called data assimilation [5,10,26]. Moreover, the nonlinear filtering problems for a number of multi-scale systems have been alternatively investigated [7][10]–[13][16]–[18][26][27]. Let us mention some results. With the help of stochastic averaging, Park and his coauthors [11]–[13] studied filtering problems for two time scales systems with Brownian motions by the Zakai equations. And then Imkeller et al. [7] showed that the filtering of only the slow part converges to the homogenized filtering by double backward stochastic differential equations and asymptotic techniques when the system (1) is driven by Brownian motions. Recently, the author and two coauthors [18] reduced the system (1) with Brownian motions to a system on a random invariant manifold, and showed that the filtering of only the slow part converges to the filtering of the reduced system. Later, the author and two coauthors [27] extended the result in [18] to the case of two symmetric \( \alpha \)-stable processes.

In the paper, we consider the system (1), where \( L^\pm \) can be not only a two-sided symmetric \( \alpha \)-stable process but also a general Lévy process. First, it is proved that this system can be approximated by a low-dimensional system on a random invariant manifold. Second, we establish that the nonlinear filter of the multiscale stochastic dynamical system, rather than only the slow part, can be also approximated by that of the reduced low-dimensional system. Third, we deduce the reduced system for \( \varepsilon = 0 \) and find that the system (1) can not be approximated by the reduced system for \( \varepsilon = 0 \).

Here our motivation is three-folded. The first fold is to correct the estimate for the distance between the system (1) and the reduced system in [18, Theorem 3.2] and [27, Theorem 2]. That is, the estimate should not just depend on the initial value of the system (1) since the reduced system is on an invariant manifold (See Theorem 3.3 in detail). The second fold is to extend the result in [27, Theorem 3]. Since symmetric \( \alpha \)-stable processes are a special type of Lévy processes, the extension to general Lévy processes is necessary. Finally, we analysis the case for \( \varepsilon = 0 \). It is unfortunate to obtain that the system (1) can not be approximated by the reduced system for \( \varepsilon = 0 \).

It is worthwhile to mention our methods. First of all, note that our conditions are similar to those in [22]. There Schmalfuß and Schneider made complicated deduction in order to construct an invariant manifold and furthermore implicitly expressed the manifold. Here we construct an invariant manifold only by an integral equation. Besides, to obtain the reduced system on the invariant manifold, we only define an operator and then prove that it is contractive. Thus, a large number of computation like that in [6] is avoided.

This paper is arranged as follows. In Sect. 2, we introduce basic concepts of random dynamical systems, stationary solutions, random invariant manifolds and Lévy processes. The existence of low-dimensional systems approximating these multiscale systems is placed in Section 3. In Sect. 4, we introduce nonlinear filtering problems and prove that the nonlinear
filterings of the low-dimensional reduced systems also approximate that of the multiscale systems. Next, we analysis the case for \( \varepsilon = 0 \) in Sect. 5. In Sect. 6, we summarize all the results in the paper.

The following convention will be used throughout the paper: \( C \) with or without indices will denote different positive constants whose values may change from one place to another.

## 2 Preliminaries

In this section, we introduce basic concepts of random dynamical systems, stationary solutions, random invariant manifolds and Lévy processes.

### 2.1 Random Dynamical Systems [2]

**Definition 2.1** Let \((\Omega, \mathcal{F}, \mathbb{Q})\) be a probability space, and \((\theta_t)_{t \in \mathbb{R}}\) a family of measurable transformations from \(\Omega\) to \(\Omega\). We call \((\Omega, \mathcal{F}, \mathbb{Q}; (\theta_t)_{t \in \mathbb{R}})\) a metric dynamical system if for each \(t \in \mathbb{R}\), \(\theta_t\) preserves the probability measure \(\mathbb{Q}\), i.e.,

\[
\theta_t^* \mathbb{Q} = \mathbb{Q},
\]

and for \(s, t \in \mathbb{R}\),

\[
\theta_0 = 1_{\Omega}, \quad \theta_{t+s} = \theta_t \circ \theta_s.
\]

**Definition 2.2** Let \((U, \mathcal{U})\) be a measurable space. A mapping \(\Psi : \mathbb{R} \times \Omega \times U \mapsto U, \quad (t, \omega, x) \mapsto \Psi(t, \omega, x)\)

with the following properties is called a measurable random dynamical system (RDS in short), or a cocycle:

(i) Measurability: \(\Psi\) is \(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{U} / \mathcal{U}\)-measurable,

(ii) Càdlàg cocycle: \(\Psi(t, \omega)\) is càdlàg for \(t \in \mathbb{R}\), and further satisfies the following conditions

\[
\Psi(0, \omega) = id_U, \quad \Psi(t + s, \omega) = \Psi(t, \theta_s \omega) \circ \Psi(s, \omega),
\]

for all \(s, t \in \mathbb{R}\) and \(\omega \in \Omega\).

### 2.2 Stationary Solutions and Random Invariant Manifolds (see [2])

Let \(\Psi\) be a RDS on the normed space \((U, \|\cdot\|_U, \mathcal{U})\), where \(\|\cdot\|_U\) stands for a norm on \(U\) and \(\mathcal{U}\) is Borel \(\sigma\)-field on \(U\). We introduce stationary solutions and random invariant manifolds with respect to \(\Psi\).

A random variable \(\xi\) is called a **stationary solution** of a stochastic/random differential equation, if \(\Psi\) is defined by the solution mapping of the equation and for \(t > 0\)

\[
\Psi(t, \omega, \xi(\omega)) = \xi(\theta_t \omega), \text{ a.s.} \omega.
\]

Here, we remind that in general it is not obvious whether the solutions to stochastic differential equations define RDSs. Therefore, before mentioning stationary solutions, we need to justify that solution mappings of stochastic differential equations define RDSs.
A family of nonempty closed sets $\mathcal{M} = \{\mathcal{M}(\omega)\}_{\omega \in \Omega} \subset \mathcal{U}$ is called a random set if for every $u \in \mathcal{U}$, the mapping
$$\Omega \ni \omega \mapsto \text{dist}(u, \mathcal{M}(\omega)) := \inf_{u' \in \mathcal{M}(\omega)} ||u - u'||_{\mathcal{U}}$$
is measurable. Moreover, $\mathcal{M}$ is called a positively invariant set with respect to the random dynamical system $\Psi$ if
$$\Psi(t, \omega, \mathcal{M}(\omega)) \subseteq \mathcal{M}(\theta_t \omega), \quad t \in \mathbb{R}^+, \ \omega \in \Omega. \quad (4)$$

In the sequel, we consider a random set defined by a Lipschitz continuous graph. Concretely speaking, we define a function by
$$\Omega \times \mathbb{R}^m \ni (\omega, y) \mapsto F(\omega, y) \in \mathbb{R}^n$$such that for all $\omega \in \Omega$, $F(\omega, y)$ is globally Lipschitzian in $y$ and for any $y \in \mathbb{R}^m$, the mapping $\omega \rightarrow F(\omega, y)$ is a random vector. And set
$$\mathcal{M}(\omega) := \{(F(\omega, y), y), y \in \mathbb{R}^m\},$$and then $\mathcal{M}$ is a random set (\cite[Lemma 2.1]{22}). Moreover, the random set $\mathcal{M}(\omega)$ is called as a
*$\text{Lipschitz random invariant manifold}$* if it is (positively) invariant with respect to some random dynamical system.

### 2.3 Lévy processes (see \cite{20})

**Definition 2.3** A stochastic process $L = (L_t)_{t \geq 0}$ with $L_0 = 0$ a.s. is a $n$-dimensional Lévy process if

1. $L$ has independent increments; that is, $L_t - L_s$ is independent of $L_v - L_u$ if $(u, v) \cap (s, t) = \emptyset$;
2. $L$ has stationary increments; that is, $L_t - L_s$ has the same distribution as $L_v - L_u$ if $t - s = v - u > 0$;
3. $L_t$ is right continuous with left limit.

Its characteristic function is given by
$$\mathbb{E}\left\{\exp\{i \langle z, L_t \rangle\}\right\} = \exp\{t \varphi(z)\}, \quad z \in \mathbb{R}^n.$$The function $\varphi : \mathbb{R}^n \rightarrow \mathbb{C}$ is called as the characteristic exponent of the Lévy process $L$. By the Lévy-Khintchine formula, there exist a nonnegative-definite $n \times n$ matrix $Q, b \in \mathbb{R}^n$ and a measure $\nu$ on $\mathbb{R}^n$ satisfying
$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^n \setminus \{0\}} (|u|^2 \land 1) \nu(du) < \infty, \quad (5)$$such that
$$\varphi(z) = -\frac{1}{2} \langle z, Qz \rangle + i \langle z, b \rangle + \int_{\mathbb{R}^n \setminus \{0\}} \left(e^{i \langle z, u \rangle} - 1 - i \langle z, u \rangle 1_{|u| \leq \delta}\right) \nu(du), \quad (6)$$where $\delta$ is a positive constant. $\nu$ is called as the Lévy measure associated with $L$.

Set $\kappa_t := L_t - L_\cdot$. Then $\kappa$ defines a stationary Poisson point process with values in $\mathbb{R}^n \setminus \{0\}$ and the characteristic measure $\nu$ \cite{8}. Let $N_\kappa((0, t], du)$ be the counting measure of $\kappa_t$, i.e., for $D \in B(\mathbb{R}^n \setminus \{0\})$
$$N_\kappa((0, t], D) := \#\{0 < s \leq t : \kappa_s \in D\},$$
where \# denotes the cardinality of a set. The compensated martingale measure of \(N_κ\) is given by
\[
\tilde{N}_κ((0, t], d\xi) = N_κ((0, t], d\xi) - tν(du).
\]
The Lévy-Itô theorem states that there exist a \(n'\)-dimensional Brownian motion \(R_t, 0 ≤ n' ≤ n\) and a \(n \times n'\) matrix \(M\) such that \(L\) can be represented as
\[
L_t = bt + MR_t + \int_0^t \int_{|u|≤\delta} u \tilde{N}_κ(ds, du) + \int_0^t \int_{|u|>\delta} uN_κ(ds, du).
\]
In the sequel, we take a \(m\)-dimensional Lévy process \(L_t = MR_t + \int_0^t \int_{|u|≤\delta} u \tilde{N}_κ(ds, du), \ t ≥ 0\).
Here, for convenience of the following deduction, we require \(0 < \delta < 1\). And then set
\[
L_t^\pm := \begin{cases} 
L_t^+, & t ≥ 0, \\
L_t^{(-t)-}, & t < 0.
\end{cases}
\]
Thus, \(L_t^{\pm}\) is a two-sided \(m\)-dimensional Lévy process.

**Definition 2.4** For \(α ∈ (0, 2)\), a \(n\)-dimensional symmetric \(α\)-stable process \(L_t^α\) for \(t ≥ 0\) is a Lévy process with the characteristic exponent
\[
\varphi(u) = -C_1(n, α)|u|^α, \text{ for } u ∈ \mathbb{R}^n,
\]
where \(C_1(n, α) := π^{-\frac{1}{2}}Γ((1 + α)/2)Γ(n/2)/Γ((n + α)/2)\).
For a \(n\)-dimensional symmetric \(α\)-stable Lévy process, the diffusion matrix \(Q = 0\), the drift vector \(b = 0\), and the Lévy measure \(ν\) is given by
\[
ν(du) = \frac{C_2(n, α)}{|u|^{n+α}}du,
\]
where \(C_2(n, α) := αΓ((n + α)/2)/(2^{1-α}π^{n/2}Γ(1 - α/2))\).
In the sequel, we fix a \(n\)-dimensional symmetric \(α\)-stable process \(L_t^α\) \((1 < α < 2)\) for \(t ≥ 0\) independent of \(L_t^\pm = (L_t^\pm)_{t∈\mathbb{R}}\) and then set
\[
L_t^{α\pm} := \begin{cases} 
L_t^α, & t ≥ 0, \\
L_t^{(-t)-}, & t < 0.
\end{cases}
\]
Thus, \(L_t^{α\pm}\) is a two-sided \(n\)-dimensional symmetric \(α\)-stable process.

### 3 The Reduction System on a Random Invariant Manifold

In this section, we prove that a fast-slow system driven by Lévy processes can be approximated by a low dimensional system on a random invariant manifold.
3.1 A Metric Dynamical System

Let $D(\mathbb{R}, \mathbb{R}^n)$ be the set of all functions which are càdlàg for $t \in \mathbb{R}$, and take values in $\mathbb{R}^n$. We take the canonical sample space $\Omega^1 \triangleq D(\mathbb{R}, \mathbb{R}^n)$. It, endowed with the Skorohod metric $\rho$, can be made a complete and separable metric space. The Borel $\sigma$-algebra on the sample space $\Omega^1$ under the topology induced by $\rho$ is denoted as $\mathcal{F}^1$. Let $\theta^1_t$ be the distribution of the two-sided $n$-dimensional symmetric $\alpha$-stable Lévy process $L^\alpha = (L^\alpha_t)_{t \in \mathbb{R}}$. Set

$$\theta^1 : \mathbb{R} \times \Omega^1 \mapsto \Omega^1,$$

$$\theta^1_t \omega(\cdot) = \omega(\cdot + t) - \omega(t),$$

and then one can justify that the probability measure $Q^1$ is $\theta^1$-invariant and $\{\theta^1_t, t \in \mathbb{R}\}$ is a group. Thus, $(\Omega^1, \mathcal{F}^1, Q^1, (\theta^1_t)_{t \in \mathbb{R}})$ is a metric dynamical system.

Next let $\Omega^2 \triangleq D(\mathbb{R}, \mathbb{R}^m)$. Likewise, we define $\mathcal{F}^2$ and $\theta^2$. Again let $Q^2$ be the unique probability measure which makes the canonical process the Lévy process $L^\pm = (L^\pm_t)_{t \in \mathbb{R}}$. So, $(\Omega^2, \mathcal{F}^2, Q^2, (\theta^2_t)_{t \in \mathbb{R}})$ is another metric dynamical system.

Set $\Omega \triangleq \Omega^1 \times \Omega^2$, $\mathcal{F} \triangleq \mathcal{F}^1 \times \mathcal{F}^2$, $Q \triangleq Q^1 \times Q^2$, $\theta_t \triangleq \theta^1_t \times \theta^2_t$, and then $(\Omega, \mathcal{F}, Q, (\theta_t)_{t \in \mathbb{R}})$ is a metric dynamical system, which will be used in the following.

3.2 A Random Dynamical System

Consider the system (1), i.e.

$$\begin{cases}
\dot{u}^\varepsilon = \frac{1}{\varepsilon} Au^\varepsilon + \frac{1}{\varepsilon} U(u^\varepsilon, v^\varepsilon) + \frac{\sigma_1}{\varepsilon^{\frac{1}{\alpha}}} \dot{L}^\alpha, & u_0^\varepsilon = u_0 \in \mathbb{R}^n, \\
\dot{v}^\varepsilon = B v^\varepsilon + V(u^\varepsilon, v^\varepsilon) + \sigma_2 \dot{L}^\pm, & v_0^\varepsilon = v_0 \in \mathbb{R}^m.
\end{cases}$$

We make the following hypotheses:

**H 1** There exists a $\gamma_1 > 0$ such that for any $x \in \mathbb{R}^n$,

$$\langle Ax, x \rangle \leq -\gamma_1 |x|^2.$$

**H 2** There exist $\gamma_2, \gamma_3 \geq 0$ such that

$$\|e^{Bt}\| \leq e^{-\gamma_2 t}, \quad t \leq 0,$$

$$\|e^{Bt}\| \leq e^{-\gamma_3 t}, \quad t > 0.$$ (9) (10)

**H 3** There exists a positive constant $L$ such that for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^n \times \mathbb{R}^m$

$$|U(x_1, y_1) - U(x_2, y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|),$$

and

$$|V(x_1, y_1) - V(x_2, y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|).$$

**H 4**

$$\gamma_1 > L.$$
they are stationary solutions of two above equations. Set
\[ (a_\varepsilon) \in H_5 \]

Introduce two auxiliary systems

\[ \text{3.3 Random Invariant Manifolds} \]

Suppose that Lemma 3.1 from [17]. We start with a key lemma.

Under the assumptions \((H_1)-(H_3)\), the system (1) has a global unique solution denoted by \((u^\varepsilon(t), v^\varepsilon(t))\) [8]. Define the solution operator \(\Psi_t^\varepsilon(u_0, v_0) := (u^\varepsilon(t), v^\varepsilon(t))\) for \(t \geq 0\), and then we know that \(\Psi^\varepsilon\) is a random dynamical system.

Introduce two auxiliary systems

\[ \begin{align*}
\frac{d\zeta_t^\varepsilon}{dt} &= \frac{1}{\varepsilon} A \zeta_t^\varepsilon + \frac{\sigma_1}{\varepsilon^{1/\alpha}} dL_t^{\theta^1} + \frac{\sigma_2}{\varepsilon^{1/\alpha}} dL_t^{\theta^2}, \quad \zeta_0^\varepsilon = u \in \mathbb{R}^n, \\
\frac{d\xi_t^\varepsilon}{dt} &= B \xi_t^\varepsilon + \varepsilon \xi_t^\varepsilon + \zeta(\theta^1 \omega_1), \\
\xi_0^\varepsilon &= v \in \mathbb{R}^m.
\end{align*} \] (11)

So, by [27, Lemma 1] and [15, Example 3.7], there exist two random vectors \(\zeta^\varepsilon, \xi^\varepsilon\) such that they are stationary solutions of two above equations. Set

\[ \begin{align*}
\hat{u}^\varepsilon &= u^\varepsilon - \zeta^\varepsilon(\theta^1 \omega_1), \\
\hat{v}^\varepsilon &= v^\varepsilon - \xi(\theta^2 \omega_2),
\end{align*} \]

and then \((\hat{u}^\varepsilon, \hat{v}^\varepsilon)\) satisfy the following system

\[ \begin{align*}
\hat{u}^\varepsilon &= \frac{1}{\varepsilon} A \hat{u}^\varepsilon + \frac{1}{\varepsilon} U(\hat{u}^\varepsilon + \zeta^\varepsilon(\theta^1 \omega_1), \hat{v}^\varepsilon + \zeta(\theta^2 \omega_2)), \\
\hat{v}^\varepsilon &= B \hat{v}^\varepsilon + V(\hat{u}^\varepsilon + \zeta^\varepsilon(\theta^1 \omega_1), \hat{v}^\varepsilon + \zeta(\theta^2 \omega_2)),
\end{align*} \] (13)

Moreover, \((\hat{u}^\varepsilon_t, \hat{v}^\varepsilon_t)\) generates a random dynamical system denoted by \(\hat{\Psi}^\varepsilon\) for \(t \geq 0\).

Next, we construct a random invariant manifold with respect to \(\hat{\Psi}^\varepsilon\). The method comes from [17]. We start with a key lemma.

**Lemma 3.1** Suppose that \((H_1)-(H_5)\) are satisfied. Then for \((\bar{u}_0, \bar{v}_0) \in \mathbb{R}^n \times \mathbb{R}^m\), there exists a \(\varepsilon_0 > 0\) such that for \(0 < \varepsilon \leq \varepsilon_0\), the following integral equation has a unique solution \((\hat{u}^\varepsilon_t, \hat{v}^\varepsilon_t)\)

\[ \begin{align*}
\hat{u}^\varepsilon_t &= \left( f_t^{\infty} e^{\Delta(t-r)} \frac{1}{\varepsilon} U(\hat{u}^\varepsilon_r + \zeta^\varepsilon(\theta^1 \omega_1), \hat{v}^\varepsilon_r + \zeta(\theta^2 \omega_2)) dr \right), \\
\hat{v}^\varepsilon_t &= \left( e^{B(t-r)} \left( f_t^{\infty} e^{\Delta(t-r)} V(\hat{u}^\varepsilon_r + \zeta^\varepsilon(\theta^1 \omega_1), \hat{v}^\varepsilon_r + \zeta(\theta^2 \omega_2)) dr \right) \right),
\end{align*} \] (14)

\[ \hat{v}^\varepsilon_0 = \bar{v}_0. \]

**Proof** Firstly, we introduce two following spaces

\[ C^\varepsilon_{\tau}(\mathbb{R}^n) := \left\{ \phi \in C((-\infty, 0], \mathbb{R}^n) : \sup_{t \geq 0} e^{\frac{\tau}{\varepsilon} t} |\phi(t)| < \infty \right\}, \]

\[ C^\varepsilon_{\tau}(\mathbb{R}^m) := \left\{ \phi \in C((-\infty, 0], \mathbb{R}^m) : \sup_{t \geq 0} e^{\frac{\tau}{\varepsilon} t} |\phi(t)| < \infty \right\}, \]

\[ \hat{\Psi}^\varepsilon \]
where $\mu > 0$ is a constant and $\gamma_1 - \mu > L$. Let $C^\infty_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m) := C^\infty_{\mathcal{F}}(\mathbb{R}^n) \times C^\infty_{\mathcal{F}}(\mathbb{R}^m)$ with the norm $\|z\|_{C^\infty_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)} = \sup_{t \leq 0} e^{\frac{\mu}{\epsilon} t} |z(t)| = \sup_{t \leq 0} e^{\frac{\mu}{\epsilon} t} (|u(t)| + |v(t)|)$ for $z = (u, v) \in C^\infty_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$.

Set for $\hat{z}^\epsilon = (\hat{u}^\epsilon, \hat{v}^\epsilon) \in C^\infty_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$

$$\mathcal{I}(\hat{z}^\epsilon)(t) := \left( \begin{array}{c} \mathcal{I}_1(\hat{z}^\epsilon)(t) \\ \mathcal{I}_2(\hat{z}^\epsilon)(t) \end{array} \right) = \left( \begin{array}{c} \int_{-\infty}^t e^{\frac{\mu}{\epsilon} (t-r)} \frac{1}{\epsilon} \left( \hat{u}^\epsilon_r + \zeta^\epsilon(\theta_1^1 \omega_1), \hat{v}^\epsilon_r + \zeta^\epsilon(\theta_1^2 \omega_2) \right) dr \\ e^{B^1 t} \hat{v}_0 - \int_0^t e^{B(t-r)} V(\hat{u}^\epsilon_r + \zeta^\epsilon(\theta_1^1 \omega_1), \hat{v}^\epsilon_r + \zeta^\epsilon(\theta_1^2 \omega_2)) dr \end{array} \right),$$

and then $\mathcal{I}$ is well defined on $C^\infty_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$. In fact, by $(H_1)$ $(H_2)$ $(H_5)$ it holds that for $\hat{z}^\epsilon = (\hat{u}^\epsilon, \hat{v}^\epsilon) \in C^\infty_{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}^m)$,

$$\sup_{t \leq 0} e^{\frac{\mu}{\epsilon} t} |\mathcal{I}_1(\hat{z}^\epsilon)(t)| \leq \sup_{t \leq 0} e^{\frac{\mu}{\epsilon} t} \left| \int_{-\infty}^t e^{\frac{\mu}{\epsilon} (t-r)} \frac{1}{\epsilon} U(\hat{u}^\epsilon_r + \zeta^\epsilon(\theta_1^1 \omega_1), \hat{v}^\epsilon_r + \zeta^\epsilon(\theta_1^2 \omega_2)) dr \right| \leq \frac{M_U}{\epsilon} \sup_{t \leq 0} e^{\frac{\mu}{\epsilon} t} \int_{-\infty}^t e^{-\gamma_2 (t-r)} dr \leq \frac{M_U}{\gamma_1},$$

and

$$\sup_{t \leq 0} e^{\frac{\mu}{\epsilon} t} |\mathcal{I}_2(\hat{z}^\epsilon)(t)| \leq \sup_{t \leq 0} e^{\frac{\mu}{\epsilon} t} |e^{B^1 t} \hat{v}_0| + \sup_{t \leq 0} e^{\frac{\mu}{\epsilon} t} \left| \int_t^0 e^{B(t-r)} V(\hat{u}^\epsilon_r + \zeta^\epsilon(\theta_1^1 \omega_1), \hat{v}^\epsilon_r + \zeta^\epsilon(\theta_1^2 \omega_2)) dr \right| \leq \sup_{t \leq 0} e^{\frac{\mu}{\epsilon} t} e^{-\gamma_2 t} |\hat{v}_0| + M_V \sup_{t \leq 0} e^{\frac{\mu}{\epsilon} t} \int_t^0 e^{-\gamma_2 (t-r)} dr \leq |\hat{v}_0| + \frac{M_V}{\gamma_2}.$$
Note that $\gamma_1 - \mu > L$ and then

$$\frac{L}{\gamma_1 - \mu} < 1.$$  

Thus, there exists an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$,

$$\frac{L}{\gamma_1 - \mu} + \frac{\varepsilon L}{\mu - \varepsilon \gamma_2} < 1,$$

that is, $I$ is contractive. So, Eq. (14) has a unique solution denoted as $(\hat{u}^\varepsilon, \hat{v}^\varepsilon)$. The proof is complete.

Next, for $l \in (-\infty, 0]$, we rewrite Eq. (14) as

$$\begin{pmatrix} \hat{u}^\varepsilon_l \\ \hat{v}^\varepsilon_l \end{pmatrix} = \begin{pmatrix} e^{\frac{A}{\varepsilon}(t-l)} \hat{u}^\varepsilon_0 + \int_l^t e^{\frac{A}{\varepsilon}(t-r)} \frac{1}{\varepsilon} U(\hat{u}^\varepsilon_r + \xi^\varepsilon(\theta^1_r \omega_1), \hat{v}^\varepsilon_r + \zeta(\theta^2_r \omega_2)) dr \\ e^{B(t-l)} \hat{v}^\varepsilon_l + \int_l^t e^{B(t-r)} V(\hat{u}^\varepsilon_r + \xi^\varepsilon(\theta^1_r \omega_1), \hat{v}^\varepsilon_r + \zeta(\theta^2_r \omega_2)) dr \end{pmatrix}, \quad l \leq t \leq 0.$$  

That is, the dynamic of Eq. (14) is the same as that of the system (13). Thus, it holds that for $t \leq 0$

$$\hat{u}^\varepsilon_t(\theta_s \omega) = \hat{u}^\varepsilon_{t+s}(\omega), \quad \hat{v}^\varepsilon_t(\theta_s \omega) = \hat{v}^\varepsilon_{t+s}(\omega), \quad t + s \leq 0. \quad (16)$$

In the following, set

$$F^\varepsilon(\omega, \tilde{v}_0) = \frac{1}{\varepsilon} \int_{-\infty}^0 e^{-\frac{A}{\varepsilon}s} U(\hat{u}^\varepsilon_s + \xi^\varepsilon(\theta^1_s \omega_1), \hat{v}^\varepsilon_s + \zeta(\theta^2_s \omega_2)) ds,$$

and then we study the properties of $F^\varepsilon(\omega, \tilde{v}_0)$. First, by $(H_1)$ $(H_5)$ it holds that for $\tilde{v}_0 \in \mathbb{R}^m$

$$|F^\varepsilon(\omega, \tilde{v}_0)| \leq \frac{M_U}{\varepsilon} \int_{-\infty}^0 e^{\frac{L}{\varepsilon}s} ds = \frac{M_U}{\gamma_1}. \quad (17)$$

Second, by the proof of Lemma 3.1 one can obtain that for $0 < \varepsilon \leq \varepsilon_0$ and $\tilde{v}_0^1, \tilde{v}_0^2 \in \mathbb{R}^m$

$$|F^\varepsilon(\omega, \tilde{v}_0^1) - F^\varepsilon(\omega, \tilde{v}_0^2)| \leq \frac{L}{(\gamma_1 - \mu)} \left[1 - \frac{L}{(\gamma_1 - \mu) + \frac{\varepsilon L}{\mu - \varepsilon \gamma_2}}\right]|\tilde{v}_0^1 - \tilde{v}_0^2|$.  

Third, it follows from (16) that for $t \leq 0$,

$$F^\varepsilon(\theta_t \omega, \tilde{v}_0) = \int_{-\infty}^t e^{\frac{A}{\varepsilon}(t-r)} \frac{1}{\varepsilon} U(\hat{u}^\varepsilon_r + \xi^\varepsilon(\theta^1_r \omega_1), \hat{v}^\varepsilon_r + \zeta(\theta^2_r \omega_2)) dr = \hat{u}^\varepsilon_t. \quad (19)$$

Thus, define

$${\mathcal{M}}^\varepsilon(\omega) := \{(F^\varepsilon(\omega, y), y), y \in \mathbb{R}^m\},$$

and then by (18), it holds that $\mathcal{M}^\varepsilon(\omega)$ is a random set. And based on (19) one can justify that $\mathcal{M}^\varepsilon(\omega)$ is invariant with respect to $\hat{u}^\varepsilon$. Therefore, $\mathcal{M}^\varepsilon$ is a Lipschitz random invariant manifold with respect to $\hat{u}^\varepsilon$. Moreover, we have the following result.

**Theorem 3.2** Under the hypotheses $(H_1)$–$(H_5)$, it holds that for sufficiently small $\varepsilon$ and $(\tilde{u}_0, \tilde{v}_0) \in \mathbb{R}^n \times \mathbb{R}^m$, there exists $(\tilde{u}_0, \tilde{v}_0) \in \mathcal{M}^\varepsilon$ such that
\[|\Psi^e_t(\tilde{u}_0, \tilde{v}_0) - \Psi^e_0(\tilde{u}_0, \tilde{v}_0)| \leq \frac{e^{-\frac{L}{\gamma_1}t}}{1 - (\frac{L}{\gamma_1 - \mu} + \frac{eL}{\mu - \varepsilon \gamma_2})} \left(2(|\tilde{u}_0| + |\tilde{v}_0|) + 2\frac{M_U}{\gamma_1} + \frac{M_V}{\gamma_2}\right), \quad t \geq 0. \tag{20} \]

**Proof** To prove the theorem, we need the following spaces:
\[
C^e_{\bar{\tau}}(\mathbb{R}^n) := \left\{ \phi \in C(\mathbb{R}, \mathbb{R}^n) : \sup_{t \in \mathbb{R}} e^{\frac{\mu}{\tau}t} |\phi(t)| < \infty \right\},
\]
\[
C^e_{\bar{\tau}}(\mathbb{R}^m) := \left\{ \phi \in C(\mathbb{R}, \mathbb{R}^m) : \sup_{t \in \mathbb{R}} e^{\frac{\mu}{\tau}t} |\phi(t)| < \infty \right\},
\]
and \(C^e_{\bar{\tau}}(\mathbb{R}^n \times \mathbb{R}^m) := C^e_{\bar{\tau}}(\mathbb{R}^n) \times C^e_{\bar{\tau}}(\mathbb{R}^m)\) with the norm \(\|z\|_{C^e_{\bar{\tau}}(\mathbb{R}^n \times \mathbb{R}^m)} = \sup_{t \in \mathbb{R}} e^{\frac{\mu}{\tau}t} |z(t)| = \sup_{t \in \mathbb{R}} e^{\frac{\mu}{\tau}t} (|u(t)| + |v(t)|)\) for \(z = (u, v) \in C^e_{\bar{\tau}}(\mathbb{R}^n \times \mathbb{R}^m)\).

Next, note that by the assumption (H1), the operator \(I - |t|A\) is invertible. Thus, set
\[
\tilde{z}^e_t := \left(\tilde{u}^e_t \atop \tilde{v}^e_t\right) := \left(\begin{array}{c}
(I - |t|A)^{-1} \tilde{u}_0, \\
\tilde{u}^e(t, \omega)(\tilde{u}_0, \tilde{v}_0),
\end{array}\right), \quad t \leq 0,
\]
and
\[
\tilde{Z}_0(t) := \left(\begin{array}{c}
\tilde{z}^e_t + \mathcal{I}(\tilde{z}^e)(t), \\
-(e^{\frac{4}{\tau}t} \tilde{u}_0, e^{Bt} \tilde{v}_0) + (e^{\frac{4}{\tau}t} \mathcal{I}_1(\tilde{z}^e)(0), e^{Bt} \tilde{v}_0),
\end{array}\right), \quad t > 0,
\]
where \(\mathcal{I}\) is defined in (15). And then we consider the following integral equation
\[
\begin{pmatrix}
X^e_t \\
Y^e_t
\end{pmatrix} = \tilde{Z}_0(t) + \left(\begin{array}{c}
\int_{-\infty}^{t} e^{\frac{4}{\tau}(t-r)} \frac{1}{e} \left[U(\tilde{u}^e_r + X^e_r, \tilde{v}^e_r + Y^e_r) - U(\tilde{u}^e_r, \tilde{v}^e_r)\right] dr \\
- \int_{t}^{\infty} e^{B(t-r)} \left[V(\tilde{u}^e_r + X^e_r, \tilde{v}^e_r + Y^e_r) - V(\tilde{u}^e_r, \tilde{v}^e_r)\right] dr
\end{array}\right), \quad t \in \mathbb{R}. \tag{21}
\]
For \(Z^e := (X^e, Y^e) \in C^e_{\bar{\tau}}(\mathbb{R}^n \times \mathbb{R}^m)\), set
\[
\mathcal{J}(Z^e)(t) := \left(\begin{array}{c}
\mathcal{J}_1(Z^e)(t) \\
\mathcal{J}_2(Z^e)(t)
\end{array}\right)
\]
\[
:= \tilde{Z}_0(t) + \left(\begin{array}{c}
\int_{-\infty}^{t} e^{\frac{4}{\tau}(t-r)} \frac{1}{e} \left[U(\tilde{u}^e_r + X^e_r, \tilde{v}^e_r + Y^e_r) - U(\tilde{u}^e_r, \tilde{v}^e_r)\right] dr \\
- \int_{t}^{\infty} e^{B(t-r)} \left[V(\tilde{u}^e_r + X^e_r, \tilde{v}^e_r + Y^e_r) - V(\tilde{u}^e_r, \tilde{v}^e_r)\right] dr
\end{array}\right),
\]
and then \(\mathcal{J} : C^e_{\bar{\tau}}(\mathbb{R}^n \times \mathbb{R}^m) \mapsto C^e_{\bar{\tau}}(\mathbb{R}^n \times \mathbb{R}^m)\) is well defined. In fact, for \(Z^e = (X^e, Y^e) \in C^e_{\bar{\tau}}(\mathbb{R}^n \times \mathbb{R}^m)\), by (H1)-(H5) and (17) we compute
\[
\sup_{t \leq 0} e^{\frac{\mu}{\tau}t} |\tilde{Z}_0(t)| \leq \sup_{t \leq 0} e^{\frac{\mu}{\tau}t} |\tilde{Z}_0(t)| + \sup_{t > 0} e^{\frac{\mu}{\tau}t} |\tilde{Z}_0(t)|
\]
\[
\leq \sup_{t \leq 0} e^{\frac{\mu}{\tau}t} (|\tilde{z}^e_t| + |\mathcal{I}(\tilde{z}^e)(t)|) + \sup_{t > 0} e^{\frac{\mu}{\tau}t} e^{-\frac{\gamma_1}{\tau}t} (|\tilde{u}_0| + |\mathcal{I}_1(\tilde{z}^e)(0)|)
\]
\[
\leq \sup_{t \leq 0} e^{\frac{\mu}{\tau}t} |\tilde{z}^e_t| + \sup_{t \leq 0} e^{\frac{\mu}{\tau}t} |\mathcal{I}(\tilde{z}^e)(t)| + |\tilde{u}_0| + |\mathcal{I}_1(\tilde{z}^e)(0)|
\]
\begin{align}
&\leq |\bar{u}_0| + |\bar{v}_0| + \frac{M_U}{\gamma_1} + \frac{M_V}{\gamma_2} + |\bar{u}_0| + \frac{M_U}{\gamma_1} \\
&= 2(|\bar{u}_0| + |\bar{v}_0|) + 2\frac{M_U}{\gamma_1} + \frac{M_V}{\gamma_2}.
\end{align}

(22)

Besides, it follows from \((H_1)-(H_4)\) that

\begin{align}
&\frac{1}{\varepsilon} \sup_{t \in \mathbb{R}} e^\frac{\varepsilon}{\gamma_1} \int_{-\infty}^{t} e^{-\frac{\gamma_1}{\varepsilon}(t-r)} |U(\bar{u}_r^\varepsilon + X_r^\varepsilon, \bar{v}_r^\varepsilon + Y_r^\varepsilon) - U(\bar{u}_r^\varepsilon, \bar{v}_r^\varepsilon)| dr \\
&\leq \frac{L}{\varepsilon} \left( \sup_{t \in \mathbb{R}} e^\frac{\varepsilon}{\gamma_1} |Z_t^\varepsilon| \right) \sup_{t \in \mathbb{R}} \int_{-\infty}^{t} e^{(\frac{\varepsilon}{\gamma_1} - \frac{\mu}{\varepsilon})(t-r)} dr \\
&\leq \frac{L}{\gamma_1 - \mu} \left( \sup_{t \in \mathbb{R}} e^\frac{\varepsilon}{\gamma_1} |Z_t^\varepsilon| \right),
\end{align}

(23)

and

\begin{align}
&\sup_{t \in \mathbb{R}} e^\frac{\varepsilon}{\gamma_1} \int_{-\infty}^{t} e^{-\gamma_2(t-r)} |V(\bar{u}_r^\varepsilon + X_r^\varepsilon, \bar{v}_r^\varepsilon + Y_r^\varepsilon) - V(\bar{u}_r^\varepsilon, \bar{v}_r^\varepsilon)| dr \\
&\leq \frac{L}{\mu - \varepsilon \gamma_2} \left( \sup_{t \in \mathbb{R}} e^\frac{\varepsilon}{\gamma_1} |Z_t^\varepsilon| \right).
\end{align}

(24)

Thus, by combining (22) (23) with (24), one can obtain that

\[ \sup_{t \in \mathbb{R}} e^\frac{\varepsilon}{\gamma_1} |\mathcal{J}(Z^\varepsilon)(t)| \leq \sup_{t \in \mathbb{R}} e^\frac{\varepsilon}{\gamma_1} |\mathcal{J}_1(Z^\varepsilon)(t)| + \sup_{t \in \mathbb{R}} e^\frac{\varepsilon}{\gamma_1} |\mathcal{J}_2(Z^\varepsilon)(t)| < \infty. \]

Next, for \(Z^\varepsilon, 1, Z^\varepsilon, 2 \in C_\varepsilon(\mathbb{R}^n \times \mathbb{R}^m)\), by the similar deduction to (23) (24) it holds that

\begin{align}
&\sup_{t \in \mathbb{R}} e^\frac{\varepsilon}{\gamma_1} |\mathcal{J}_1(Z^\varepsilon, 1)(t) - \mathcal{J}_1(Z^\varepsilon, 2)(t)| \leq \frac{L}{\gamma_1 - \mu} \left( \sup_{t \in \mathbb{R}} e^\frac{\varepsilon}{\gamma_1} |Z_t^\varepsilon, 1 - Z_t^\varepsilon, 2| \right), \\
&\sup_{t \in \mathbb{R}} e^\frac{\varepsilon}{\gamma_1} |\mathcal{J}_2(Z^\varepsilon, 1)(t) - \mathcal{J}_2(Z^\varepsilon, 2)(t)| \leq \frac{\varepsilon L}{\mu - \varepsilon \gamma_2} \left( \sup_{t \in \mathbb{R}} e^\frac{\varepsilon}{\gamma_1} |Z_t^\varepsilon, 1 - Z_t^\varepsilon, 2| \right).
\end{align}

Thus, we have that

\[ \sup_{t \in \mathbb{R}} e^\frac{\varepsilon}{\gamma_1} |\mathcal{J}(Z^\varepsilon, 1)(t) - \mathcal{J}(Z^\varepsilon, 2)(t)| \leq \frac{L}{\gamma_1 - \mu} + \frac{\varepsilon L}{\mu - \varepsilon \gamma_2} \left( \sup_{t \in \mathbb{R}} e^\frac{\varepsilon}{\gamma_1} |Z_t^\varepsilon, 1 - Z_t^\varepsilon, 2| \right). \]

So, for \(0 < \varepsilon \leq \varepsilon_0\), \(\mathcal{J} : C_\varepsilon(\mathbb{R}^n \times \mathbb{R}^m) \mapsto C_\varepsilon(\mathbb{R}^n \times \mathbb{R}^m)\) is contractive. That is, Eq. (21) has a unique solution denoted as \(Z^\varepsilon = (X^\varepsilon, Y^\varepsilon)\). Moreover,

\[ \sup_{t \in \mathbb{R}} e^\frac{\varepsilon}{\gamma_1} |Z_t^\varepsilon| \leq \frac{1}{1 - \left( \frac{L}{\gamma_1 - \mu} + \frac{\varepsilon L}{\mu - \varepsilon \gamma_2} \right)} \left( 2(|\bar{u}_0| + |\bar{v}_0|) + 2\frac{M_U}{\gamma_1} + \frac{M_V}{\gamma_2} \right) \]

and then

\[ |Z_t^\varepsilon| \leq \frac{e^{-\frac{\varepsilon}{\gamma_1}}}{1 - \left( \frac{L}{\gamma_1 - \mu} + \frac{\varepsilon L}{\mu - \varepsilon \gamma_2} \right)} \left( 2(|\bar{u}_0| + |\bar{v}_0|) + 2\frac{M_U}{\gamma_1} + \frac{M_V}{\gamma_2} \right), \quad t \geq 0. \]

(25)
Set
\[ \tilde{z}^\varepsilon_t := \left( \tilde{u}^\varepsilon_t, \tilde{v}^\varepsilon_t \right) := \left( \tilde{u}^\varepsilon_t, \tilde{v}^\varepsilon_t \right) = \left( X^\varepsilon_t, Y^\varepsilon_t \right), \]
and then by simple calculation, it holds that \( \tilde{z}^\varepsilon = (\tilde{u}^\varepsilon, \tilde{v}^\varepsilon) \) solves uniquely the following equation
\[ \tilde{z}^\varepsilon = \left\{ \begin{array}{ll}
\mathcal{I}(\tilde{z}^\varepsilon)(t), & t \leq 0, \\
\tilde{\Psi}^\varepsilon(t, \omega)(\tilde{u}^\varepsilon_0, \tilde{v}^\varepsilon_0), & t > 0.
\end{array} \right. \]
Thus, by Lemma 3.1, we know that \( \tilde{z}^\varepsilon \) solves Eq. (14) for \( t \leq 0 \). In particular, \( \tilde{u}^\varepsilon_0 = F^\varepsilon(\omega, \tilde{v}^\varepsilon_0) \), which yields that \( (\tilde{u}^\varepsilon_0, \tilde{v}^\varepsilon_0) \in \mathcal{M}^\varepsilon(\omega) \). So, one can take \( \tilde{z}_0 = (\tilde{u}^\varepsilon_0, \tilde{v}^\varepsilon_0) \). Since \( \tilde{z}^\varepsilon_t = \tilde{\Psi}^\varepsilon(t, \omega)(\tilde{u}^\varepsilon_0, \tilde{v}^\varepsilon_0) \) for \( t > 0 \), \( \tilde{z}^\varepsilon_t = \tilde{\Psi}^\varepsilon(t, \omega)\tilde{z}_0 \) for \( t > 0 \). Note that \( \tilde{u}^\varepsilon_t - \tilde{u}^\varepsilon_t = X^\varepsilon_t, \tilde{v}^\varepsilon_t - \tilde{v}^\varepsilon_t = Y^\varepsilon_t \).

Thus, by (25), it holds that
\[ |\tilde{\Psi}^\varepsilon(t, \omega)\tilde{z}_0 - \tilde{\Psi}^\varepsilon(t, \omega)\tilde{z}_0| = |(\tilde{u}^\varepsilon_t, \tilde{v}^\varepsilon_t) - (\tilde{u}^\varepsilon_t, \tilde{v}^\varepsilon_t)| = |Z^\varepsilon_t| \leq \frac{e^{-\frac{\varepsilon t}{2}}}{1 - \left( \frac{L}{\gamma_1 - \mu} + \frac{\varepsilon L}{\mu - \gamma_2} \right)} \left( 2(|\tilde{u}_0| + |\tilde{v}_0|) + 2 \frac{M_U}{\gamma_1} + \frac{M_V}{\gamma_2} \right), \quad t \geq 0. \]
The proof is complete. \( \square \)

Based on the relationship between (1) and (13), it holds that the system (1) has a random invariant manifold
\[ \mathcal{M}^\varepsilon(\omega) = \{(F^\varepsilon(\omega, y) + \zeta^\varepsilon(\omega_1), y + \zeta(\omega_2), y \in \mathbb{R}^m). \}

### 3.4 A Reduction System on the Random Invariant Manifold \( \mathcal{M}^\varepsilon \)

By Theorem 3.2, we can get the following reduced system approximating the system (1).

**Theorem 3.3** Suppose that these assumptions \((H_1)-(H_5)\) hold. Then for any solution \( z^\varepsilon_t = (u^\varepsilon_t, v^\varepsilon_t) \) to the system (1) with the initial data \( z^\varepsilon_0 = (u_0, v_0) \), there exists the following reduced low dimensional system on the random invariant manifold \( \mathcal{M}^\varepsilon \)
\[ \begin{align*}
\tilde{u}^\varepsilon_t = F^\varepsilon(\theta_1 \omega, \tilde{v}^\varepsilon_t - \zeta(\theta_2^1 \omega_2)) + \zeta^\varepsilon(\theta_1^1 \omega_1), \\
d\tilde{v}^\varepsilon_t = B\tilde{v}^\varepsilon_t dt + V(\tilde{u}^\varepsilon_t, \tilde{v}^\varepsilon_t) dt + \sigma_2 dL^\varepsilon_t,
\end{align*} \quad (26)\]
such that for \( 0 < \varepsilon \leq \varepsilon_0 \), we have
\[ |z^\varepsilon(t, \omega) - \tilde{z}^\varepsilon(t, \omega)| \leq \frac{e^{-\frac{\mu t}{2}}}{1 - \left( \frac{L}{\gamma_1 - \mu} + \frac{\varepsilon L}{\mu - \gamma_2} \right)} \left( 2(|u_0 - u| + |v_0 - v|) + 2 \frac{M_U}{\gamma_1} + \frac{M_V}{\gamma_2} \right), \quad t \geq 0, \quad (27)\]
where \( \tilde{z}^\varepsilon_t = (\tilde{u}^\varepsilon_t, \tilde{v}^\varepsilon_t) \) is the solution of the low dimensional system (26) with the initial value \( \tilde{z}_0 \) and \( u, v \) are the initial values of two auxiliary systems (11) and (12), respectively.

Note that in the system (26), \( \tilde{u}^\varepsilon_t \) can be represented by \( \tilde{v}^\varepsilon_t \). Thus, the system (26) is essentially decided by \( \tilde{v}^\varepsilon_t \). That is, the dimension of the system (26) is \( m \). However, the dimension of the system (1) is \( m + n \). Therefore, the system (26) is a reduced low dimensional system.
Remark 3.4 By the estimate (27), we know that when \( \varepsilon \) is enough small or \( t \) is sufficiently large, the system (26) will approximate the system (1).

Remark 3.5 Note that the estimate (27) is different from the estimate in [18, Theorem 3.2] and (3.17) in [27]. In fact, the latter estimates omit the property of the reduced system. Besides, in the system (1), \( L^{\pm} \) can be replaced by a \( m \)-dimensional two-sided symmetric \( \alpha \)-stable process. Moreover, by the similar deduction to that in Theorem 3.3, we can also obtain the estimate (27).

4 An Approximate Filtering on the Invariant Manifold

In the section we introduce nonlinear filtering problems for the system (1) and the reduced system (26) on the random invariant manifold, and then study their relation.

4.1 Nonlinear Filtering Problems

In the subsection we introduce nonlinear filtering problems for the system (1) and the reduced system (26).

For \( T > 0 \), we take an observation system as follows

\[
\tilde{w}_t^\varepsilon = W_t + \int_0^t H(u_s^\varepsilon, v_s^\varepsilon)ds, \quad t \in [0, T]
\]

where \( W \) is a \( \ell \)-dimensional standard Brownian motion. Here \( W \) may be either independent of \( L^{\alpha \pm} \) and \( L^{\pm} \), or correlated with \( L^{\alpha \pm} \) and \( L^{\pm} \). Assume:

**H6** \( H \) is bounded and Lipschitz continuous in \((u, v)\) with the Lipschitz constant \( \|H\|_{\text{Lip}} \).

Set

\[
(\chi_t^\varepsilon)^{-1} := \exp \left\{ -\int_0^t H(u_s^\varepsilon, v_s^\varepsilon)dW_s - \frac{1}{2} \int_0^t |H(u_s^\varepsilon, v_s^\varepsilon)|^2 ds \right\},
\]

and then we know that \((\chi_t^\varepsilon)^{-1}\) is an exponential martingale under \( Q \). And define

\[
dQ^\varepsilon := (\chi_t^\varepsilon)^{-1}dQ,
\]

and it holds that \( Q^\varepsilon \) is a probability measure and \( w^\varepsilon \) is a standard Brownian motion under \( Q^\varepsilon \). Besides, set for \( \Phi \in B(\mathbb{R}^n \times \mathbb{R}^m) \) (the set of all real-valued uniformly bounded Borel-measurable functions on \( \mathbb{R}^n \times \mathbb{R}^m \))

\[
Q_t^\varepsilon(\Phi) := \mathbb{E}^\varepsilon[\Phi(u_t^\varepsilon, v_t^\varepsilon)|\mathcal{W}_t^\varepsilon],
\]

\[
\Pi_t^\varepsilon(\Phi) := \mathbb{E}[\Phi(u_t^\varepsilon, v_t^\varepsilon)|\mathcal{W}_t^\varepsilon],
\]

where \( \mathbb{E}^\varepsilon \), \( \mathbb{E} \) stand for the expectation under \( Q^\varepsilon \) and \( Q \), respectively, \( \mathcal{W}_t^\varepsilon \triangleq \sigma(w_s^\varepsilon : 0 \leq s \leq t) \cup \mathcal{N} \) and \( \mathcal{N} \) is the collection of all \( Q \)-measure zero sets. Moreover, by the Kallianpur-Striebel formula it holds that

\[
\Pi_t^\varepsilon(\Phi) = \frac{Q_t^\varepsilon(\Phi)}{Q_t^\varepsilon(1)}.
\]

Next, we study the nonlinear filtering problem for \((\tilde{u}^\varepsilon, \tilde{v}^\varepsilon)\). Set

\[
\tilde{\chi}_t^\varepsilon := \exp \left\{ \int_0^t H(\tilde{u}_s^\varepsilon, \tilde{v}_s^\varepsilon)dw_s^\varepsilon - \frac{1}{2} \int_0^t |H(\tilde{u}_s^\varepsilon, \tilde{v}_s^\varepsilon)|^2 ds \right\},
\]
and then \( \tilde{x}_t^\varepsilon \) is an exponential martingale under \( Q^\varepsilon \). Thus, we define the nonnormalized filtering for \((\tilde{u}_t^\varepsilon, \tilde{v}_t^\varepsilon)\) by

\[
\tilde{Q}_t^\varepsilon(\Phi) := E^\varepsilon[\Phi(\tilde{u}_t^\varepsilon, \tilde{v}_t^\varepsilon) \tilde{x}_t^\varepsilon | \mathcal{W}_t^\varepsilon].
\]

And set

\[
\tilde{\Pi}_t^\varepsilon(\Phi) := \frac{\tilde{Q}_t^\varepsilon(\Phi)}{\tilde{Q}_t^\varepsilon(1)},
\]

and then we will prove that \( \tilde{\Pi}_t^\varepsilon \) could be understood as the nonlinear filtering problem for \((\tilde{u}_t^\varepsilon, \tilde{v}_t^\varepsilon)\) with respect to \( \mathcal{W}_t^\varepsilon \).

### 4.2 The Relation Between \( \Pi_t^\varepsilon \) and \( \tilde{\Pi}_t^\varepsilon \)

In the subsection we will prove that \( \tilde{\Pi}_t^\varepsilon \) converges to \( \Pi_t^\varepsilon \) as \( \varepsilon \to 0 \). Here let \( C^1_b(\mathbb{R}^n \times \mathbb{R}^m) \) denote the collection of all functions which themselves and their first-order derivatives are uniformly bounded. We introduce the following norm for \( \Phi \in C^1_b(\mathbb{R}^n \times \mathbb{R}^m) \):

\[
\| \Phi \|_{C^1_b(\mathbb{R}^n \times \mathbb{R}^m)} := \max_{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m} |\Phi(x, y)| + \max_{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m} \|\nabla \Phi(x, y)\|,
\]

where \( \nabla \) stands for the gradient operator.

#### Theorem 4.1

Under \((H_1)\)–\((H_6)\), there exists a positive constant \( C \) independent of \( \varepsilon \) such that for \( 0 < \varepsilon \leq \varepsilon_0 \) and \( \Phi \in C^1_b(\mathbb{R}^n \times \mathbb{R}^m) \),

\[
E|\Pi_t^\varepsilon(\Phi) - \tilde{\Pi}_t^\varepsilon(\Phi)|^p \leq \| \Phi \|_{C^1_b(\mathbb{R}^n \times \mathbb{R}^m)}^p \left[ 1 - \left( \frac{L}{\gamma_1 - \mu} + \frac{\varepsilon L}{\mu - \gamma_2} \right) \right]^p \left( e^{-\frac{4\mu t}{p \varepsilon}} + \frac{\varepsilon}{4\mu p} \right)^{1/4}, t \in [0, T], p > 1.
\]

#### Proof

For \( \Phi \in C^1_b(\mathbb{R}^n \times \mathbb{R}^m) \), we compute that

\[
E|\Pi_t^\varepsilon(\Phi) - \tilde{\Pi}_t^\varepsilon(\Phi)|^p = E \left| \frac{\tilde{Q}_t^\varepsilon(\Phi) - \tilde{Q}_t^\varepsilon(\Phi)}{\tilde{Q}_t^\varepsilon(1)} - \frac{\Pi_t^\varepsilon(\Phi) \tilde{Q}_t^\varepsilon(1) - \tilde{Q}_t^\varepsilon(1)}{\tilde{Q}_t^\varepsilon(1)} \right|^p
\]

\[
\leq 2^{p-1} \left| \frac{\tilde{Q}_t^\varepsilon(\Phi) - \tilde{Q}_t^\varepsilon(\Phi)}{\tilde{Q}_t^\varepsilon(1)} \right|^p + 2^{p-1} \left| \Pi_t^\varepsilon(\Phi) \frac{\tilde{Q}_t^\varepsilon(1) - \tilde{Q}_t^\varepsilon(1)}{\tilde{Q}_t^\varepsilon(1)} \right|^p
\]

\[
\leq 2^{p-1} \left( E \left| Q_t^\varepsilon(\Phi) - \tilde{Q}_t^\varepsilon(\Phi) \right|^{2p} \right)^{1/2} \left( E \left| \tilde{Q}_t^\varepsilon(1) \right|^{-2p} \right)^{1/2}
\]

\[
+ 2^{p-1} \| \Phi \|_{C^1_b(\mathbb{R}^n \times \mathbb{R}^m)}^p \left( E \left| Q_t^\varepsilon(1) - \tilde{Q}_t^\varepsilon(1) \right|^{2p} \right)^{1/2} \left( E \left| \tilde{Q}_t^\varepsilon(1) \right|^{-2p} \right)^{1/2}.
\]

In the following, it is the main task to estimate \( E \left| \tilde{Q}_t^\varepsilon(\Phi) - \tilde{Q}_t^\varepsilon(\Phi) \right|^{2p} \) and \( E \left| \tilde{Q}_t^\varepsilon(1) \right|^{-2p} \).

By the similar deduction to that of [17, Lemma 5.1], it holds that

\[
E \left| \tilde{Q}_t^\varepsilon(1) \right|^{-2p} \leq \exp \left\{ (8p^2 + 2p + 1)CT/2 \right\}. \quad (30)
\]
Next, we are devoted to dealing with $E\left| Q_\varepsilon^f (\Phi) - \tilde{Q}_\varepsilon^f (\Phi) \right|^{2p}$. Following up the line in [17, Lemma 5.2], one can obtain that

$$E\left| Q_\varepsilon^f (\Phi) - \tilde{Q}_\varepsilon^f (\Phi) \right|^{2p} \leq \|\Phi_1\|_{C^b(\mathbb{R}^n \times \mathbb{R}^m)}^{2p} \left[ 1 - \left( \frac{L}{\gamma_1 - \mu} + \frac{\varepsilon L}{\mu - \gamma_2} \right) \right]^{2p} \left( e^{-\frac{4\mu t}{\varepsilon}} + \frac{\varepsilon}{4\mu p} \right)^{1/2} + \left( \frac{e^{-\frac{4\mu t}{\varepsilon}} + \frac{\varepsilon}{4\mu p}}{4\mu p} \right)^{1/2} \cdot \left( \varepsilon^{-\frac{4\mu t}{\varepsilon}} + \frac{\varepsilon}{4\mu p} \right)^{1/2}.$$

(31)

Thus, combining (29) with (30–31), we obtain (28). The proof is complete.

Remark 4.2 (28) indicates that when $\varepsilon$ goes to zero, $\tilde{\Pi}_\varepsilon$ approximates $\Pi_\varepsilon$. Therefore, $\tilde{\Pi}_\varepsilon$ could be understood as the nonlinear filtering problem for $(\tilde{u}_\varepsilon^f, \tilde{v}_\varepsilon^f)$ with respect to $\mathcal{W}_\varepsilon^f$.

Remark 4.3 If we take $\Phi(x, y) = \psi(y)$, Theorem 4.1 is Theorem 3 in [27]. Therefore, our result is more general.

5 The Reduction for $\varepsilon = 0$

In the section, we observe the system (1) with $\sigma_2 = 0$, i.e.

$$\begin{cases}
\dot{u}_\varepsilon^f = \frac{1}{\varepsilon} A u^f + \frac{1}{\varepsilon} U(u^f, v^f) + \frac{\sigma_1}{\varepsilon^{1/\alpha}} \tilde{L}^{\alpha \pm}, \\
\dot{v}_\varepsilon^f = B v^f + V(u^f, v^f).
\end{cases}$$

(32)

5.1 A Reduced System for $\varepsilon = 0$

In the subsection, we investigate the system (32) for $\varepsilon = 0$. To do this, we scale the time $t \rightarrow \varepsilon t$ and rewrite the system (32) as

$$\begin{cases}
\dot{\bar{u}}_t^f = A \bar{u}_t^f + U(\bar{u}_t^f, \bar{v}_t^f) + \sigma_1 \tilde{L}_t^{\alpha \pm}, \\
\dot{\bar{v}}_t^f = B \bar{v}_t^f + V(\bar{u}_t^f, \bar{v}_t^f),
\end{cases}$$

(33)

where $\bar{u}_t^f := u_{\varepsilon t}^f$, $\bar{v}_t^f := v_{\varepsilon t}^f$, $\tilde{L}_t^{\alpha \pm} := \frac{L_{\varepsilon t}^{\alpha \pm}}{\varepsilon^{1/\alpha}}$. By the scaling property of $\alpha$-stable processes, one can obtain that $\tilde{L}^{\alpha \pm}$ is still a two-sided $\alpha$-stable process. Introducing an auxiliary system

$$d\xi_t = A \xi_t dt + \sigma_1 d\tilde{L}_t^{\alpha \pm}, \quad \xi_0 = u \in \mathbb{R}^n,$$

by [27, Lemma 1], we know that there exists a random variable $\xi$ such that it is a stationary solution of the above equation. Set

$$\tilde{u}_t^f := \bar{u}_t^f - \xi(\theta^1, \omega_1),$$

$$\tilde{v}_t^f := \bar{v}_t^f - \xi(\theta^1, \omega_1).$$
and then \((\tilde{u}^\varepsilon, \tilde{v}^\varepsilon)\) satisfy the following system
\[
\begin{cases}
\dot{\tilde{u}}^\varepsilon = A\tilde{u}^\varepsilon + U(\tilde{u}^\varepsilon + \zeta(\theta_1^1 \omega_1), \tilde{v}^\varepsilon), & \tilde{u}_0^\varepsilon = \tilde{u}_0 \in \mathbb{R}^n, \\
\dot{\tilde{v}}^\varepsilon = \varepsilon B\tilde{v}^\varepsilon + \varepsilon V(\tilde{u}^\varepsilon + \zeta(\theta_1^1 \omega_1), \tilde{v}^\varepsilon), & \tilde{v}_0^\varepsilon = v_0 \in \mathbb{R}^m.
\end{cases}
\tag{34}
\]

Next, set
\[
\tilde{F}^\varepsilon(\omega, v_0) := \int_{-\infty}^{0} e^{-\Lambda s} U(\tilde{u}_s^\varepsilon + \zeta(\theta_1^1 \omega_1), \tilde{v}_s^\varepsilon) ds,
\]
where \(\tilde{u}_s^\varepsilon\) is the solution of an integral equation similar to Eq. (14), and then by the similar deduction to that in Subsect. 3.3, it holds that for \(0 < \varepsilon \leq \varepsilon_0\) and \(v_0^1, v_0^2 \in \mathbb{R}^m\)
\[
|\tilde{F}^\varepsilon(\omega, v_0^1) - \tilde{F}^\varepsilon(\omega, v_0^2)| \leq \frac{L}{\gamma_1 - \mu} \left[ 1 - \left( \frac{L}{\gamma_1 - \mu} + \frac{\varepsilon L}{\mu - \varepsilon_2} \right) \right] |v_0^1 - v_0^2|.
\]

Again set
\[
\tilde{\mathcal{M}}^\varepsilon(\omega) := \left\{ \left( \tilde{F}^\varepsilon(\omega, y) + \zeta(\omega_1), y \right) \in \mathbb{R}^m \right\},
\]
and then by the similar deduction to that in Theorem 3.3, we know that for any solution \(\tilde{z}^\varepsilon_t = (\tilde{u}_t^\varepsilon, \tilde{v}_t^\varepsilon)\) to the system (33) with the initial value \(\tilde{z}^\varepsilon_0 = (u_0, v_0)\), there exists the following reduced low dimensional system on the random invariant manifold \(\tilde{\mathcal{M}}^\varepsilon\)
\[
\begin{cases}
\dot{\tilde{u}}^\varepsilon_t = \tilde{F}^\varepsilon(\theta_1^1 \omega_1, \tilde{v}^\varepsilon_t) + \zeta(\theta_1^1 \omega_1), \\
\dot{\tilde{v}}^\varepsilon_t = \varepsilon B \tilde{v}^\varepsilon_t dt + \varepsilon V(\tilde{u}^\varepsilon_t, \tilde{v}^\varepsilon_t) dt,
\end{cases}
\tag{35}
\]
such that for \(0 < \varepsilon \leq \varepsilon_0\), we have
\[
|\tilde{z}^\varepsilon(t, \omega) - \tilde{z}^\varepsilon(t, \omega)| \leq \frac{e^{-\mu t}}{1 - \left( \frac{L}{\gamma_1 - \mu} + \frac{\varepsilon L}{\mu - \varepsilon_2} \right)} \left( 2|u_0 - u| + 2 \frac{M_U}{\gamma_1} + \frac{M_V}{\gamma_2} \right), \quad t \geq 0, \tag{36}
\]
where \(\tilde{z}^\varepsilon_t = (\tilde{u}_t, \tilde{v}_t)\) is the solution of the low dimensional system (35) with the initial value \(\tilde{z}^\varepsilon_0\).

As \(\varepsilon \to 0\), these systems (33) and (34) become
\[
\begin{cases}
\dot{\tilde{u}}^0 = A\tilde{u}^0 + U(\tilde{u}_0^0, \tilde{v}_0^0) + \sigma_1 \tilde{L}^{\sigma_2}, & \tilde{u}_0^0 = u_0, \\
\dot{\tilde{v}}^0 = 0, & \tilde{v}_0^0 = v_0.
\end{cases}
\tag{37}
\]
and
\[
\begin{cases}
\dot{\tilde{u}}^0 = A\tilde{u}^0 + U(\tilde{u}_0^0 + \zeta(\theta_1^1 \omega_1), \tilde{v}_0^0), & \tilde{u}_0^0 = \tilde{u}_0 \in \mathbb{R}^n, \\
\dot{\tilde{v}}^0 = 0, & \tilde{v}_0^0 = v_0 \in \mathbb{R}^m,
\end{cases}
\tag{38}
\]
respectively. Set
\[
\tilde{F}^0(\omega, v_0) := \int_{-\infty}^{0} e^{-\Lambda s} U(\tilde{u}_s^0 + \zeta(\theta_1^1 \omega_1), v_0) ds,
\]
where \(\tilde{u}_s^0\) is the solution of an integral equation similar to Eq. (14), and then by the similar deduction to that in Subsect. 3.3, we have that for \(v_0^1, v_0^2 \in \mathbb{R}^m\)
\[
|\tilde{F}^0(\omega, v_0^1) - \tilde{F}^0(\omega, v_0^2)| \leq \frac{L}{\gamma_1 - \mu - L} |v_0^1 - v_0^2|.
\tag{39}
\]
Put
\[ \mathcal{N}_0^0(\omega) := \big\{ (\tilde{F}_0^0(\omega, y) + \zeta(\omega_1), y), y \in \mathbb{R}^m \big\}, \]
and by the same deduction to that in Theorem 3.3, one can obtain that there exists the following reduced low dimensional system on the random invariant manifold \( \tilde{\mathcal{N}}_0^0 \)
\[
\begin{align*}
\tilde{u}_t^0 &= \tilde{F}_0^0(\theta_t \omega, \tilde{v}_t^0) + \zeta(\theta_t \omega_1), \\
\tilde{v}_t^0 &= 0, 
\end{align*}
\] (40)
such that
\[
|\tilde{z}^0(t, \omega) - \tilde{z}_0(t, \omega)| \leq \frac{\gamma_1 - \mu}{\gamma_1 - \mu - L} e^{-\mu t} \left( 2|u_0 - u| + 2 \frac{M_U}{\gamma_1} + \frac{M_V}{\gamma_2} \right), \quad t \geq 0, \tag{41}
\]
where \( \tilde{z}_t^0 = (\tilde{u}_t^0, \tilde{v}_t^0) \) and \( \tilde{z}_0^0 = (\tilde{u}_0^0, \tilde{v}_0^0) \) are the solutions of the systems (37) and (40) with the initial data \( \tilde{z}_0^0 = (u_0, v_0) \) and \( \tilde{z}_0^0 \), respectively.

5.2 The Relation Between the System (33) and the Reduced System (40)

In the subsection we investigate the relation between the system (33) and the system (40). Firstly, we need the following lemma.

**Lemma 5.1** Assume that \((H_1)-(H_5)\) hold. Then for \( v_0 \in \mathbb{R}^m \)
\[
|\tilde{F}_0^\varepsilon(\omega, v_0) - \tilde{F}_0^0(\omega, v_0)| \leq C \beta(\varepsilon),
\]
where \( C > 0 \) is a constant independent of \( \varepsilon \) and
\[
\beta(\varepsilon):=e^{\mu t_0}\left( \frac{1}{\gamma_1 - \varepsilon \gamma_2} e^{-\varepsilon \gamma_2 t_0} - \frac{1}{\gamma_1} \right) + \left( \frac{1}{\gamma_1 - \varepsilon \gamma_2} - \frac{1}{\gamma_1} \right), \quad t_0:=\frac{1}{\varepsilon \gamma_2} \log \left( \frac{\mu - \varepsilon \gamma_2}{\gamma_1 - \varepsilon \gamma_2} \right).\]

Since the proof of the above lemma is similar to that of [6, Theorem 5.1], we omit it.

**Theorem 5.2** Suppose that \((H_1)-(H_5)\) are satisfied. Then for \( 0 < \varepsilon \leq \varepsilon_0 \)
\[
|\tilde{z}_t^\varepsilon - \tilde{z}_t^0| \leq \frac{e^{-\mu t}}{1 - \left( \frac{L}{\gamma_1 - \mu} + \frac{\varepsilon L}{\mu - \varepsilon \gamma_2} \right)} \left( 2|u_0 - u| + 2 \frac{M_U}{\gamma_1} + \frac{M_V}{\gamma_2} \right)
+ \frac{1 - \frac{\varepsilon L}{\mu - \varepsilon \gamma_2}}{1 - \left( \frac{L}{\gamma_1 - \mu} + \frac{\varepsilon L}{\mu - \varepsilon \gamma_2} \right)} |Bv_0| + \frac{M_V}{\gamma_3} (1 - e^{-\varepsilon \gamma_3 t})
+ C \beta(\varepsilon), \quad t \geq 0, \tag{42}
\]

**Proof** By (36), it holds that
\[
|\tilde{z}_t^\varepsilon - \tilde{z}_t^0| \leq |\tilde{z}_t^\varepsilon - \tilde{z}_t^\varepsilon_0| + |\tilde{z}_t^\varepsilon - \tilde{z}_t^0|
\]
\[
\leq \frac{e^{-\mu t}}{1 - \left( \frac{L}{\gamma_1 - \mu} + \frac{\varepsilon L}{\mu - \varepsilon \gamma_2} \right)} \left( 2|u_0 - u| + 2 \frac{M_U}{\gamma_1} + \frac{M_V}{\gamma_2} \right) + |\tilde{z}_t^\varepsilon - \tilde{z}_t^0|. \tag{43}
\]
Then we estimate \( |\tilde{z}_t^\varepsilon - \tilde{z}_t^0| \). Note that \( \tilde{z}_t^\varepsilon, \tilde{z}_t^0 \) satisfy these systems (35) and (40), respectively. Thus, it follows from Lemma 5.1 that
\[ |\tilde{z}_t^\varepsilon - \tilde{z}_0^\varepsilon| = |\tilde{u}_t^\varepsilon - \tilde{u}_t^0| + |\tilde{v}_t^\varepsilon - \tilde{v}_t^0| = |\tilde{F}^\varepsilon(\theta_t\omega, \tilde{v}_t^\varepsilon) - \tilde{F}^0(\theta_t\omega, \tilde{v}_t^0)| + |\tilde{v}_t^\varepsilon - \tilde{v}_t^0| \]
\[ \leq |\tilde{F}^\varepsilon(\theta_t\omega, \tilde{v}_t^\varepsilon) - \tilde{F}^\varepsilon(\theta_t\omega, \tilde{v}_t^0)| + |\tilde{F}^\varepsilon(\theta_t\omega, \tilde{v}_t^0) - \tilde{F}^0(\theta_t\omega, \tilde{v}_t^0)| + |\tilde{v}_t^\varepsilon - \tilde{v}_t^0| \]
\[ \leq \frac{L}{(\gamma_1 - \mu)} \left[ 1 - \left( \frac{L}{\gamma_1 - \mu} + \frac{\varepsilon L}{\mu - \varepsilon \gamma_2} \right) \right] |\tilde{v}_t^\varepsilon - \tilde{v}_t^0| + C\beta(\varepsilon) + |\tilde{v}_t^\varepsilon - \tilde{v}_t^0|. \quad (44) \]

In the following, we are devoted to computing \( |\tilde{v}_t^\varepsilon - \tilde{v}_t^0| \). Based on (35), it holds that
\[ |\tilde{v}_t^\varepsilon - \tilde{v}_t^0| \leq |e^{\varepsilon Bt}v_0 - v_0| + \varepsilon \left| \int_0^t e^{\varepsilon B(t-s)} V(\tilde{u}_s^\varepsilon, \tilde{v}_s^\varepsilon) ds \right| \]
\[ \leq \varepsilon |Bv_0| \int_0^t \|e^{\varepsilon Bs}\| ds + \varepsilon M_V \int_0^t \|e^{\varepsilon B(t-s)}\| ds \]
\[ \leq \varepsilon |Bv_0| \int_0^t e^{-\varepsilon \gamma_3 s} ds + \varepsilon M_V \int_0^t e^{-\varepsilon \gamma_3(t-s)} ds \]
\[ = \frac{|Bv_0| + M_V}{\gamma_3} (1 - e^{-\varepsilon \gamma_3 t}). \quad (45) \]

Combining (43) (44) with (45), we obtain that
\[ |\tilde{z}_t^\varepsilon - \tilde{z}_t^0| \leq \frac{e^{-\mu t}}{1 - \left( \frac{L}{\gamma_1 - \mu} + \frac{e L}{\mu - e \gamma_2} \right)} \left( 2|u_0 - u| + 2 \frac{M_U}{\gamma_1} + \frac{M_V}{\gamma_2} \right) \]
\[ + \frac{1}{1 - \left( \frac{L}{\gamma_1 - \mu} + \frac{\varepsilon L}{\mu - \varepsilon \gamma_2} \right)} \frac{|Bv_0| + M_V}{\gamma_3} (1 - e^{-\varepsilon \gamma_3 t}) + C\beta(\varepsilon). \]

The proof is over. \( \square \)

**Remark 5.3** By the estimate (42), we know that when \( \varepsilon \) is smaller, \( \tilde{z}_t^\varepsilon \) is not nearer to \( \tilde{z}_t^0 \) for \( t \geq 0 \). But this does not mean that convergence of \( \tilde{z}_t^\varepsilon \) to \( \tilde{z}_t^0 \) fails, and it might just mean that the estimate was not good. Thus, we can not obtain the similar result to that in Theorem 4.1.

### 6 Conclusions

In the paper, we consider multiscale stochastic dynamical systems driven by Lévy processes. First, it is proved that these systems can approximate low-dimensional systems on random invariant manifolds. Second, we establish that nonlinear filterings of multiscale stochastic dynamical systems also approximate that of reduced low-dimensional systems. Finally, we analyze the case for \( \varepsilon = 0 \). It is unfortunate to obtain that these reduced systems does not approximate these multiscale systems.

In the future, we will investigate the possibility of doing a similar reduction and the related nonlinear filtering for the multiplicative noisy terms in the equations.

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