CALLS, ZONOIDS, PEACOCKS AND LOG-CONCAVITY

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Abstract. The main results are two characterisations of log-concave densities in terms of the collection of lift zonoids corresponding to a peacock. These notions are recalled and connected to arbitrage-free asset pricing in financial mathematics.

1. Statement of results

The main mathematical contributions of this short note are the following two observations. Let $f$ be a positive and differentiable probability density function, let $F$ be the corresponding cumulative distribution function and let $F^{-1}$ its quantile function.

**Theorem 1.1.** There exists a martingale $(S_t)_{t \geq 0}$ such that
\[
\inf_{K \in \mathbb{R}} \{ E[(S_t - K)^+] + pK \} = \sqrt{tf}(F^{-1}(p)) \text{ for all } t \geq 0, \ 0 < p < 1,
\]
if and only if $f$ is log-concave.

**Theorem 1.2.** There exists a positive martingale $(S_t)_{t \geq 0}$ such that
\[
\inf_{K \in \mathbb{R}} \{ E[(S_t - K)^+] + pK \} = F(F^{-1}(p) + \sqrt{t}) \text{ for all } t \geq 0, \ 0 < p < 1,
\]
if and only if $f$ is log-concave.

Note that for any increasing bijection $Y$ on the interval $[0, \infty)$, the process $(S_t)_{t \geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ if and only if $(S_{Y(t)})_{t \geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_{Y(t)})_{t \geq 0}$. In particular, the $\sqrt{t}$ appearing in both theorems could be replaced with any increasing bijection on $[0, \infty)$.

The motivation for the choice of the $\sqrt{t}$ time-parametrisation can be found in the following statements. We will use the notation
\[
\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}
\]
to denote the standard normal density function, $\Phi(z) = \int_{-\infty}^{z} \varphi(u)du$ the standard normal cumulative distribution function and $\Phi^{-1}$ its quantile function. We will let $(W_t)_{t \geq 0}$ be a standard Brownian motion.

**Proposition 1.3.**
\[
\min_{K \in \mathbb{R}} \{ E[(W_t - K)^+] + pK \} = \sqrt{t} \varphi(\Phi^{-1}(p)) \text{ for all } t \geq 0, \ 0 < p < 1.
\]
Proposition 1.4.

\[ \min_{K \in \mathbb{R}} \left\{ \mathbb{E}[e^{W_{-t/2}} - K] + pK \right\} = \Phi(\Phi^{-1}(p) + \sqrt{t}) \text{ for all } t \geq 0, \ 0 < p < 1. \]

The proof of Propositions 1.3 and 1.4 are straightforward calculations. Note that the linear and geometric Brownian motions are martingales in their common filtration. It was an attempt to generalise these propositions which lead the author to discover the above (seemingly novel) characterisations of log-concavity in one dimension found in Theorems 1.1 and 1.2.

To motivate interest in expressions of the form

\[ C(t, K) = \mathbb{E}[(S_t - K)^+], \]

where \( S \) is a martingale, we appeal to the arbitrage theory of asset pricing. Recall that in a typical financial market model with zero risk-free interest and dividend rates, there is no arbitrage if the prices of all traded assets are martingales. A converse statement, that the absence of arbitrage implies the existence of an equivalent measure under which the asset prices are martingales, is true in discrete time by the theorem of Dalang, Morton & Willinger [2]; however, it is worth noting that formulations of a ‘correct’ converse in continuous time are considerably more involved. See, for instance, the book of Delbaen & Schachermayer [3] for precise details.

Now consider a market with a stock with time \( u \) price \( S_u \) and a call option written on the stock with maturity date \( t \) and strike price \( K \). In the classical arbitrage theory recalled above, it is natural to assume that \( S \) is a martingale and that the initial price of the call is \( C(t, K) \).

Recall that in the Bachelier model, the stock price is given by

\[ S_t = S_0 + \sigma W_t \]

for constants \( S_0 \in \mathbb{R} \) and \( \sigma > 0 \) from which the call price is computed as

\[ \mathbb{E}[(S_t - K)^+] = \sigma \sqrt{t} \phi \left( \frac{S_0 - K}{\sigma \sqrt{t}} \right) + (S_0 - K) \Phi \left( \frac{S_0 - K}{\sigma \sqrt{t}} \right). \]

Note that the linear Brownian motion in Proposition 1.3 corresponds to the Bachelier model for the stock price, with zero initial price and unit linear volatility.

Along similar lines, recall also that in the Black–Scholes model, the the stock price is given by

\[ S_t = S_0 e^{\sigma W_t - \sigma^2 t/2} \]

for constants \( S_0 > 0 \) and \( \sigma > 0 \) from which the call price is computed as

\[ \mathbb{E}[(S_t - K)^+] = \Phi \left( \frac{\log(S_0/K) + \sigma \sqrt{T}}{\sigma \sqrt{T}/2} \right) - K \Phi \left( \frac{\log(S_0/K) - \sigma \sqrt{T}}{\sigma \sqrt{T}/2} \right) \]

for \( K > 0 \). The geometric Brownian motion in Proposition 1.3 corresponds to the Black–Scholes model (with unit initial price and unit geometric volatility). We note here that Proposition 1.4 appears in [13] and is employed to derive upper bounds on Black–Scholes implied volatility. See section 2.6 for some brief details.

Note that, in Theorems 1.1 and 1.2 only the marginal laws of the random variables \( S_t \) appear explicitly, but not the joint law of the process \((S_t)_{t \geq 0}\). Indeed, the filtration \((\mathcal{F}_t)_{t \geq 0}\) for which the martingale property is defined is only implicit. Therefore, we find it useful
to recall the definition of a term popularised by Hirsh, Profeta, Roynette & Yor [4]: a peacock is a collection of random variables \((S_t)_{t \geq 0}\) with the property that there exists a filtered probability space on which a martingale \((\tilde{S}_t)_{t \geq 0}\) is defined such that \(S_t \sim \tilde{S}_t\) for all \(t \geq 0\). The term peacock is derived from the French acronym PCOC, Processus Croissant pour l’Ordre Convexe. Peacocks have a useful characterisation in terms of the prices of call options thanks to a theorem of Kellerer [6].

**Theorem 1.5.** The family \((S_t)_{t \geq 0}\) of integrable random variables is a peacock if and only if the following holds: the map \(t \mapsto \mathbb{E}(S_t)\) is constant and the map \(t \mapsto \mathbb{E}[(S_t - K)^+]\) is increasing for all \(K \in \mathbb{R}\).

See the paper [5] of Hirsh & Roynette for a proof.

Finally, to see why one might want to compute what the Legendre transform of a call price \(C(t, K)\) with respect to the strike parameter \(K\), we recall that the zonoid of an integrable random \(d\)-vector \(X\) is the set

\[
Z_X = \{ \mathbb{E}[Xg(X)] \text{ measurable } g : \mathbb{R}^d \to [0, 1] \} \subseteq \mathbb{R}^d,
\]

and that the lift zonoid of \(X\) is the zonoid of the \((1 + d)\)-vector \((1, X)\) given by

\[
\hat{Z}_X = \{ (\mathbb{E}[g(X)], \mathbb{E}[Xg(X)]) \text{ measurable } g : \mathbb{R}^d \to [0, 1] \} \subseteq \mathbb{R}^{1+d}.
\]

The notion of lift zonoid was introduced in the paper of Koshevoy & Mosler [8]. We note that the calculation of the lift zonoid of a Gaussian measure can be found in another paper of Koshevoy & Mosler [7, Example 6.3]. We will see that this calculation is essentially our Proposition 1.3.

In the case \(d = 1\), the lift zonoid \(\hat{Z}_X\) is a convex set contained in the rectangle

\[
[0, 1] \times [-m_-, m_+].
\]

where \(m_\pm = \mathbb{E}(X^\pm)\). We can define the upper boundary of the lift zonoid by the function \(\hat{C}_X : [0, 1] \to \mathbb{R}\) given by

\[
\hat{C}_X(p) = \sup \{ q : (p, q) \in \hat{Z}_X \} = \sup \{ \mathbb{E}[Xg(X)] \text{ measurable } g : \mathbb{R} \to [0, 1] \text{ with } \mathbb{E}[g(X)] = p \}.
\]

Note that by replacing \(g\) with \(1 - g\) in the definition, we see that \(\hat{Z}_X\) is symmetric about the point \((\frac{1}{2}, \frac{1}{2}m)\) where \(m = m_+ - m_- = \mathbb{E}(X)\). Hence, we can recover \(\hat{Z}_X\) from its upper boundary from the formula

\[
\hat{Z}_X = \left\{ (p, q) : 0 \leq p \leq 1, m - \hat{C}_X(1 - p) \leq q \leq \hat{C}_X(p) \right\}.
\]

Our interest in the notion of lift zonoid is explained by the following result:

**Proposition 1.6.** We have \(\hat{C}_X(0) = 0\), \(\hat{C}_X(1) = \mathbb{E}(X)\) and

\[
\hat{C}_X(p) = \min_{K \in \mathbb{R}} [C_X(K) + pK] \text{ for all } 0 < p < 1
\]

where \(C_X(K) = \mathbb{E}[(X - K)^+]\). Furthermore, we have

\[
C_X(K) = \max_{0 \leq p \leq 1} [\hat{C}_X(p) - pK] \text{ for all } K \in \mathbb{R}.
\]
Note that if we let
\[ \Theta(K) = \mathbb{P}(X \geq K) \]
then we have
\[ C_X(K) = \int_K^\infty \Theta(\kappa)d\kappa \]
by Fubini’s theorem. Also if we define the inverse function \( \Theta^{-1} \) for \( 0 < p < 1 \) by
\[ \Theta^{-1}(p) = \inf\{K : \Theta(K) \geq p\} \]
then by a result of Koshevoy & Mosler [9, Lemma 3.1] we have
\[ \hat{C}_X(p) = \int_0^p \Theta^{-1}(\phi)d\phi. \]
These representations could be used to prove Proposition 1.6. However, since the result can be viewed as an application of the Neyman–Pearson lemma, we include a short proof for completeness.

**Proof.** For any measurable function \( g \) valued in \([0, 1]\) we have
\[ Xg(X) \leq (X - K)^+ + Kg(X) \]
with equality when \( g \) is of the form
\[ g = \lambda \mathbf{1}_{(K,\infty)} + (1 - \lambda) \mathbf{1}_{[K,\infty)} \]
where \( \lambda \in [0, 1] \). Computing expectations and optimising over \( g \) yields
\[ \hat{C}_X(p) \leq C_X(K) + Kp \]
with equality if
\[ \mathbb{P}(X > K) \leq p \leq \mathbb{P}(X \geq K). \]

\[ \square \]

We remark that the explicit connection between lift zonoids and the price of call options has been noted before, for instance in the paper of Mochanov & Schmutz [11].

It is interesting to observe that a consequence of Proposition 1.6 is that for two integrable random variables \( X \) and \( Y \) with the same mean \( \mathbb{E}(X) = \mathbb{E}(Y) \), that the following are equivalent, as noted by Koshevoy & Mosler [9, Theorem 5.2],
- \( X \) is dominated by \( Y \) with respect to the lift zonoid order, in the sense that \( \hat{Z}_X \subseteq \hat{Z}_Y \),
- \( \hat{C}_X(p) \leq \hat{C}_Y(p) \) for all \( 0 \leq p \leq 1 \),
- \( C_X(K) \leq C_Y(K) \) for all real \( K \),
- \( X \) is dominated by \( Y \) with respect to the convex order, in the sense that \( \mathbb{E}[\psi(X)] \leq \mathbb{E}[\psi(Y)] \) for all convex \( \psi \) for which the expectations are defined.

When \( d > 1 \), things are slightly more subtle. In particular, see the paper of Koshevoy [8] for an example of random vectors \( X \) and \( Y \) such that \( X \) is dominated by \( Y \) with respect to the lift zonoid order, and yet \( X \) is not dominated by \( Y \) with respect to the convex order.

We now briefly look at the case where the random \( X \) is strictly positive:
Proposition 1.7. Suppose $\mathbb{P}(X > 0) = 1$ and that $\mathbb{E}(X) = m$. Then the upper boundary of its lift zonoid is a strictly increasing continuous function $\hat{C}_X : [0, 1] \rightarrow [0, m]$. Its inverse is given by $\hat{C}_X^{-1}(0) = 0$ and $\hat{C}_X^{-1}(m) = 1$, and

$$\hat{C}_X^{-1}(q) = \max_{K > 0} \frac{q - C_X(K)}{K} \text{ for all } 0 < q < m.$$  

Since Proposition 1.7 appears to be new, or at least its statement does not seem to be easy to find in the literature, we now offer a proof.

Proof. Since $\mathbb{P}(X > 0) = 1$ almost surely we can conclude

$$\Theta^{-1}(p) > 0$$

for all $0 < p < 1$. This shows that $\hat{C}_X$ is strictly increasing.

Now, let $Y$ be a positive random variable such that

$$\mathbb{P}(Y \leq K) = \mathbb{E}\left[\frac{X}{m} 1\{Y \geq 1/K\}\right] \text{ for all } K > 0.$$  

That is to say, the distribution of $Y$ is given by the distribution of $1/X$ under the equivalent measure with density $X/m$. Note that

$$\hat{C}_X^{-1}(q) = \inf\{\mathbb{E}[g(X)], \text{ measurable } g : \mathbb{R} \rightarrow [0, 1] \text{ with } \mathbb{E}[Xg(X)] = q\}$$

$$= 1 - \sup\{\mathbb{E}[g(X)], \text{ measurable } g : \mathbb{R} \rightarrow [0, 1] \text{ with } \mathbb{E}[Xg(X)] = m - q\}$$

$$= 1 - \sup\{m\mathbb{E}[Yg(Y)], \text{ measurable } g : \mathbb{R} \rightarrow [0, 1] \text{ with } \mathbb{E}[g(Y)] = 1 - q/m\}$$

$$= 1 - m\hat{C}_Y(1 - q/m)$$

$$= 1 - \min_K [mC_Y(K) + (m - q)K],$$

where the minimisation can be restricted to positive $K$. The proof is concluded by noting that

$$mC_Y(K) = \mathbb{E}[(1 - XK)^+]$$

$$= \mathbb{E}[1 - XK + (XK - 1)^+]$$

$$= 1 - Km + KC_X(1/K)$$

for $K > 0$.  \[\square\]

To prove Theorems 1.1 and 1.2, we now need to know how to characterise the call price function $C_X(\cdot)$ of an integrable random variable $X$, as well as the upper boundary $\hat{C}_X(\cdot)$ of its lift zonoid. The following fact is well known. The proof is well-known, and can be found in the paper of Hirsh & Roynette [5, Proposition 2.1], for instance. In the financial context, the link from the call price function $C_X$ to and the random variable $X$ is sometimes called the Breeden–Litzenberger formula.

Proposition 1.8. Suppose that the function $C : \mathbb{R} \rightarrow \mathbb{R}_+$ is decreasing, convex and satisfies

$$C(K) \rightarrow 0 \text{ as } K \rightarrow \infty$$

and

$$C(K) + K \rightarrow m \text{ as } K \rightarrow -\infty$$
for some finite constant \( m \). There exists a (unique in law) integrable random variable \( X \) such that
\[
m = \mathbb{E}(X)
\]
and
\[
C(K) = C_X(K) \text{ for all } K \in \mathbb{R}.
\]

The next result is the lift zonoid version of Proposition 1.8. Its proof can be found in the paper of Koshevoy & Mosler [9, Theorem 3.5].

**Proposition 1.9.** Suppose that \( \hat{C} : [0, 1] \to \mathbb{R} \) is concave function and such that \( \hat{C}(0) = 0 \) and \( \hat{C}(1) = m \). Then there exists a (unique in law) integrable random variable \( X \) such that
\[
m = \mathbb{E}(X)
\]
and
\[
\hat{C}(p) = \hat{C}_X(p) \text{ for all } 0 \leq p \leq 1.
\]

In light of the proceeding discussion, we now see that Theorem 1.1 is equivalent to

**Proposition 1.10.** The map
\[
p \mapsto f(F^{-1}(p))
\]
is concave if and only if \( f \) is log-concave.

while Theorem 1.2 is equivalent to

**Proposition 1.11.** The map
\[
p \mapsto F(F^{-1}(p) + y)
\]
is concave for all \( y \geq 0 \) if and only if \( f \) is log-concave.

**Proof of Propositions 1.10 and 1.11.** First, let
\[
G(p) = f(F^{-1}(p)).
\]
Note that the derivative is given by the formula
\[
G'(p) = \frac{f'(a)}{f(a)}
\]
where \( a = F^{-1}(p) \). Therefore the function \( G' \) is decreasing if and only log \( f \) is concave.

Now fix \( y \geq 0 \) and let
\[
H_y(p) = F(F^{-1}(p) + y).
\]
Note that
\[
H'_y(p) = \frac{f(F^{-1}(p) + y)}{f(F^{-1}(p))}.
\]
Therefore, the function \( H'_y \) is decreasing if and only if
\[
\log f(b + y) - \log f(b) \leq \log f(a + y) - \log f(a)
\]
for all \( b \geq a \). This last condition holds for all \( y \geq 0 \) if and only if log \( f \) is concave. \[\square\]
2. Various remarks

We conclude this note with various remarks expanding on the main results. As before, let \( f \) be a strictly positive probability density, and \( F \) its cumulative distribution function and \( F^{-1} \) its quantile function. In this section, we further assume that \( f \) is log-concave. We will use the notation

\[
G = f \circ F^{-1}
\]

and

\[
H_y = F(F^{-1}(\cdot) + y) .
\]

2.1. Group property. To better understand the connection between the functions \( G \) and \( H_y \) introduced in the proofs of Propositions 1.10 and 1.11 note that the family of functions \( (H_y)_{y \in \mathbb{R}} \) on \([0, 1]\) form a group with respect to composition

\[
H_{y_1 + y_2} = H_{y_1} \circ H_{y_2} .
\]

Note that \( H_0 = \text{Id} \), and that the generator of this group is given by \( G \) in the sense that

\[
\frac{H_y - \text{Id}}{y} \to G
\]
as \( y \to 0 \). The above observations appear in the paper of Kulik & Tymoshkevych [10] in the case where \( f = \varphi \) is the standard normal density. In this case, the function \( G = \varphi \circ \Phi^{-1} \) is called the Gaussian isoperimetric function.

2.2. Recovering \( F \). Given the function \( G \) we can solve for the distribution function \( F \) and hence the density \( f \). Moreover, the solution is unique up to a free location parameter. Indeed, fix \( p_0 \) and declare \( F(a) = p_0 \). Then

\[
\int_{p_0}^{p} \frac{dq}{G(q)} = \int_{p_0}^{p} \frac{dq}{f(F^{-1}(q))} = F^{-1}(p) - a
\]

from which \( F \) can be recovered.

Furthermore, given the family of functions \( (H_y)_{y \geq 0} \) we can recover \( F \) in two different ways, where again we fix \( p_0 \) and set \( F(a) = p_0 \). Firstly, note that

\[
\partial_y H_y(p)|_{y=0} = G(p)
\]
to recover \( F \) as described above. Secondly, we can simply observe that

\[
F(x) = H_{x-a}(p_0) \text{ for all } x \in \mathbb{R} .
\]

2.3. Symmetries. If the integrable random variable \( X \) has arithmetic symmetry, in the sense that \(-X\) has the same law as \( X \), then its call function satisfies

\[
C_X(K) = \mathbb{E}[(X - K)^+]
\]

\[
= \mathbb{E}[X - K + (-X + K)^+]
\]

\[
= -K + C_X(-K) .
\]
The upper boundary of its lift zonoid satisfies
\[
\hat{C}_X(p) = \min_K [C_X(K) + pK] \\
= \min_K [C_X(-K) - (1-p)K] \\
= \hat{C}_X(1-p).
\]

Note that if \( f \) is an even function and \( \hat{C}_X(p) = f(F^{-1}(p)) \), then \( X \) has arithmetic symmetry since in this case \( F^{-1}(1-p) = -F^{-1}(p) \).

A strictly positive random variable \( X \) has geometric symmetry if
\[
\mathbb{E}[\psi(X)] = \mathbb{E}\left[X\psi\left(\frac{1}{X}\right)\right]
\]
for all non-negative \( \psi \). In particular, geometric symmetry implies \( \mathbb{E}(X) = 1 \) and that the call function satisfies the put-call symmetry formula
\[
C_X(K) = \mathbb{E}[(X - K)^+] \\
= \mathbb{E}[X - K + (K - X)^+] \\
= 1 - K + K\mathbb{E}\left[X\left(\frac{1}{X} - \frac{1}{K}\right)^+\right] \\
= 1 - K + KC_X(1/K)
\]
for \( K > 0 \). In this case, the upper boundary of its lift zonoid satisfies
\[
\hat{C}_X(p) = \min_K [C_X(K) + pK] \\
= 1 - \max_{K>0} K(1-p - C_X(1/K)) \\
= 1 - \hat{C}_X^{-1}(1-p)
\]
by Proposition 1.7. Since the lift zonoid of \( X \) is given by
\[
\hat{Z}_X = \{(p, q) : 0 \leq p \leq 1, 1 - \hat{C}_X(1-p) \leq q \leq \hat{C}_X(p)\}
\]
we see that another way to characterise geometric symmetry of \( X \) is that the lift zonoid is symmetric about the line \( p = q \).

Note that if \( f \) is even and \( \hat{C}_X(p) = F(F^{-1}(p) + y) \), then \( X \) has geometric symmetry thanks to the calculation
\[
\hat{C}_X^{-1}(q) = F(F^{-1}(q) - y) = 1 - \hat{C}_X(1-q).
\]

Applications of arithmetic and geometric symmetries to construct semi-static hedging strategies for certain barrier options is explored in the paper of Carr & Lee [1].

2.4. The initial stock price. Note that \( G(1) = 0 \). Hence, if the martingale \( S \) is such that
\[
\min_K \{\mathbb{E}[(S_t - K)^+ + pK]\} = Y(t)G(p)
\]
for some increasing function \( Y \), then \( E(S_t) = S_0 = 0 \). To consider models with non-zero initial prices, let \( S_t = s + S_t \) for some constant \( s \). Then

\[
\min_K \{E(\tilde{S}_t - K) + pK\} = \min_K \{E((S_t - (K - s)) + p(K - s) + ps\}
= ps + Y(t)G(p).
\]

Similarly, note that \( H_y(1) = 1 \) for all \( y \geq 0 \). Hence if

\[
\min_K \{E((\tilde{S}_t - K) + pK\} = H_Y(t)(p)
\]

then \( S_0 = 1 \). To consider more general initial prices, let \( \tilde{S}_t = sS_t \). Then

\[
\min_K \{E((\tilde{S}_t - K) + pK\} = s \min_K \{E((S_t - K/s) + pK/s\}
= sH_Y(t)(p).
\]

2.5. **The call function.** Given the upper boundary of the lift zonoïde \( \hat{C}_X \), we can compute the corresponding call function \( C_X \) by Proposition [1.6]. We now explore these representations when \( \hat{C}_X \) has the specific forms appearing in Theorems [1.1] and [1.2].

First suppose

\[
\hat{C}_X(p) = sp + G(p)
= sp + f(F^{-1}(p))
\]

for some constant \( s \). Our first calculation is then

\[
C_X(K) = \max_{0 \leq p \leq 1} [\hat{C}_X(p) - pK]
= f(U(K - s)) - F(U(K - s))(K - s),
\]

where \( U \) is inverse of the decreasing function \( f'/f = (\log f)' \). Furthermore, we have the calculation

\[
P(X \geq K) = -C'_X(K)
= F(U(K - s)).
\]

In the special case when \( f = \varphi \) is the standard normal density, we have \( U(x) = -x \) and we see that \( X \) has the standard normal distribution in agreement with the Bachelier model and Proposition [1.3].

Similarly, if

\[
\hat{C}_X(p) = sH_y(p)
= sF(F^{-1}(p) + y)
\]

for some \( y > 0 \) and \( s > 0 \), then we have

\[
C_X(K) = F(V_y(K/s) + y) - F(V_y(K/s))K,
\]

where \( V_y \) is the inverse of the decreasing function \( f(\cdot + y)/f \). Furthermore, we have

\[
P(X \geq K) = F(V_y(K/s)).
\]

In the special case when \( f = \varphi \) and \( V_y(x) = -\log x/y - y/2 \), we see that \( \log X \) has the normal distribution with mean \(-y^2/2\) and variance \( y^2 \), in agreement with the Black–Scholes model and Proposition [1.4].
2.6. **Implied volatility.** Let

\[ C_F(y, K) = \max_{0 \leq p \leq 1} [H_y(p) - pK]. \]

This corresponds to a call price on a stock with initial price \( S_0 = 1 \) and strike \( K \), or equivalently the call price normalised by the initial stock price and \( K \) is the strike price normalised by the initial stock price.

We now show for fixed \( K \) that \( C_F(\cdot, K) \) takes values in the interval \( [(1 - K)^+ , 1) \). Note that \( H_0(p) = p \) and so \( C_F(0,K) = \max_{0 \leq p \leq 1} (1 - K)^+ \).

Also, note for \( y > 0 \) that we have

\[ C_F(y, K) = F(V_y(K) + y) - F(V_y(K))K. \]

where \( V_y \) is the inverse of the decreasing function \( f(\cdot + y)/f \). It is clear from the formula that \( C_F(y, K) < 1 \).

Now, one can verify by differentiation that

\[ C_F(y, K) = (1 - K)^+ + \int_0^y f(V_u(K) + u)du. \]

Since the quantity \( y \) corresponds to \( \sigma \sqrt{t} \) in the Black–Scholes model, the above formula can be seen as a generalisation of the formula for the vega, the sensitivity of the call price with respect to the Black–Scholes volatility. In particular, we see that \( y \mapsto C_F(y, K) \) is continuous and strictly increasing. We have for every \( 0 < p < 1 \) the inequality

\[ C_F(y, K) \geq F(F^{-1}(p) + y) - pK. \]

By taking \( y \uparrow \infty \) and then \( p \downarrow 0 \) we see that \( C_F(y, K) \to 1 \). In particular, for every \( c \in [(1 - K)^+, 1) \) there is a unique \( y \) such that

\[ c = C_F(y^*, K). \]

This \( y^* \) generalises the notion of Black–Scholes implied volatility.

We now show that \( y^* \) can be recovered from \( c \) by the formula

\[ y^* = \min_{0 \leq p \leq 1} [F^{-1}(c + pK) - F^{-1}(p)]. \]

Indeed, we can rearrange the inequality

\[ c \leq F(F^{-1}(p) + y^*) - pK, \]

which holds for all \( 0 \leq p \leq 1 \), to yield the bound

\[ y^* \leq F^{-1}(c + pK) - F^{-1}(p). \]

Since there is equality above when \( p = F(V_y(K)) \), the claim is proven.

The above representation of implied volatility as the value of a minimisation problem was exploited in \([13]\) to obtain upper bounds on Black–Scholes implied volatility.
2.7. **Local volatility.** Suppose the martingale $S$ evolves according to the stochastic differential equation

$$dS_t = \sigma(t, S_t) dW_t$$

where $W$ is a Brownian motion, and the local volatility function $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is continuous. It is well known that the linear volatility $\sigma$ can be recovered from the call prices $C(t, K) = \mathbb{E}[(S_t - K)^+]$ by Dupire's formula

$$\sigma(t, K)^2 = \frac{2}{\partial_t C} \bigg|_{(t, K)} \frac{\partial K}{\partial C} \bigg|_{(t, K)}.$$

See, for instance, the book of Musiela & Rutkowski [12, Proposition 7.3.1] for a precise statement and proof.

Hence by Proposition 1.6, the function $\sigma$ can be recovered from the upper boundary of the lift-zonoid

$$\hat{C}(t, p) = \sup \{ \mathbb{E}[S_t g(S_t)], \text{ measurable } g : \mathbb{R} \to [0, 1] \text{ with } \mathbb{E}[g(S_t)] = p \}$$

via

$$\sigma(t, \partial_p \hat{C}|_{(t, p)})^2 = -2 \partial_t \hat{C} \partial_{pp} \hat{C}|_{(t, p)}.$$

In the case where $f$ is log-concave and $\hat{C}(t, p) = S_0 p + Y(t) f(F^{-1}(p))$ for an increasing function $Y$, we get

$$\sigma(t, K)^2 = -2Y(t) \dot{Y}(t) \left( (\log f)'(Y(t)) U \left( \frac{K - S_0}{\dot{Y}(t)} \right) \right)$$

where $U$ is the inverse of the decreasing function $(\log f)' f = (\log f)'$. Note that when we specialise to $Y(t) = \sigma \sqrt{t}$ and $f = \varphi$ the standard normal density, the right side is equal to the constant $\sigma^2$, in agreement with the Bachelier model and Proposition 1.3.

Finally, in the case where $f$ is log-concave and

$$\hat{C}(t, p) = S_0 F(F^{-1}(p) + Y(t))$$

we get

$$\bar{\sigma}(t, K)^2 = 2\dot{Y}(t) \left( (\log f)'(V_Y(Y(t) (K/S_0)) - (\log f)'(V_Y(Y(t) (K/S_0) + Y(t))) \right)$$

where $\bar{\sigma}(t, K) = \sigma(t, K)/K$ for $K > 0$ is the geometric volatility, and $V_y$ is the inverse of the decreasing function $(\cdot + y)/f$ for $y > 0$. Again, when we specialise to the case $Y(t) = \sigma \sqrt{t}$ and $f = \varphi$ we see that the right side is the constant $\sigma^2$, in line with the Black–Scholes model and Proposition 1.4.

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