Stock Price Prediction using Fractional Gradient-Based Long Short Term Memory

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Abstract. Deep Learning is considered one of the most effective strategies used by hedge funds to maximize profits. But Deep Neural Networks (DNN) lack theoretical analysis of memory exploitation. Some traditional time series methods such as Auto-Regressive Integrated Moving Average (ARIMA) and Generalized Auto-Regressive Conditional Heteroskedasticity (GARCH) work only when the entire series is pre-processed or when the whole data is available. Thus, it fails in a live trading system. So, there is a great need to develop techniques that give more accurate stock/index predictions. This study has exploited fractional-order derivatives' memory property in the backpropagation of LSTM for stock predictions. As the history of previous stock prices plays a significant role in deciding the future price, fractional-order derivatives carry the past information along with itself. So, the use of Fractional-order derivatives with neural networks for this time series prediction is meaningful and helpful.

1. Introduction

Stock equity plays a significant role in a nation's economy. The stock market prediction focuses on the extraction of future movement of the stock price. The accurate forecast of share price movement results in great profit avenues. Thus, this makes the investors eager about the direction of market prices. The stock market is highly dynamic, and it is a challenging task to predict the movement, as it depends on numerous factors such as politics, economic growth, etc. There are two conventional ways of predicting the market the one is technical, and the other is fundamental. The technical analysis investigates price change's direction to predict the future stock values, while fundamental analysis depends on analysing the unstructured textual information.

These traditional approaches are now considered inferior due to increased computational power, where larger data sets can be analysed more accurately in a short duration. With these approaches, new artificial intelligence approaches such as machine learning, computational intelligence, deep learning, and others originate. Linear regression, moving average, support vector machine, k-nearest neighbour algorithm [1], ARIMA [2], and LSTM [3] are machine learning algorithms widely used for stock price prediction. Most of these methods are not able to reflect the significant changes in the stock price. Linear models like ARIMA work only for a particular time-series data. It does not work for a different set of data. LSTM works fine for this problem, but the results can still be further improved. Predicting a stock price is a challenging task on account of its history dependency and high volatility.
This study predicts the stock price movement in the future using the integrated approach involving neural network variant LSTM and fractional derivatives. The customer will have a certain number of years for which the money is invested. To predict the portfolio's value, we have to expect each stock's price given the number of years (with/without dividends). Our task is Sequence Learning (Time Series Analysis). From a sequence of Historical Prices, we will train our model to predict the future price.

2. Literature Review
In the recent decade, enormous deep learning-based techniques have been introduced for stock prediction [4]. Before that, Artificial Neural Network (ANN) was used extensively for stock market prediction [5]. Deep learning works significantly better than simple ANN. Several other techniques are carried out to analyse the market, some of which are statistical methods, fractional calculus-based [6,7,8], and multi-criteria decision-making techniques [9,10]. Pritam et al. [9] constructed a portfolio considering the customer's risk preferences using the fuzzy technique for order of preference by similarity to ideal solution. The fuzzy analytic hierarchy process has been used in [10] to select Bombay Stock Exchange Sensitive Index (BSE SENSEX) sectors’ dominance for optimal equity. In [6], the stock market has been analysed using fractional differential equations. Fractional Calculus has also been applied to ANNs due to the long-term memory or non-locality of fractional derivatives [11-14]. Wang et al. [11] used the fractional steepest descent algorithm for training three-layered neural networks and proved its monotonicity and convergence. Stochastic gradient descent algorithm has been generalized to Fractional-order Gradient Descent Algorithm (FGD), the fractional-order derivative replaces the integer-order derivative, and the update rule becomes

\[ \Delta w = -\eta \frac{\partial^{\nu} E}{\partial w^{\nu}} \]

where \( 0 < \eta < 1 \) is the learning rate, and \( \nu > 0 \) is the fractional order of differentiation. The fractional derivative was used for the backpropagation algorithm for Feed Forward Neural Networks (FNNs) by Chen et al. [14] in 2013. The simulation results demonstrated that the convergence speed based on fractional-order FNNs was much faster than integer-order FNNs. In 2015, Pu [15] paid attention to the fractional-order gradient method. It was observed that this method might not lead to the actual extreme point. This defect was rectified by Chen [16,17] by using the truncation and short memory principle in the fractional-order gradient method. Khan et al. [13,18] proposed fractional-order backpropagation through a time algorithm for Recurrent Neural Networks (RNNs) and Radial Basis Function neural networks. In [18], FGD is the amalgamation of the conventional and the modified Riemann–Liouville derivative-based fractional gradient descent method. In 2018, Bao et al. [19] proposed a deep fractional-order BP neural network with \( L_2 \) regularisation term and the order \( \alpha \) can be any positive real number. Caputo's derivative-based fractional gradient method for backpropagation of Convolutional Neural Networks has been introduced, which successfully converges to real extreme point [20].

RNNs are capable of processing sequential data. However, it is challenging to train long term dependencies in these kinds of structures. LSTM is a special kind of networks capable of learning long term dependencies using dedicated circuits. Memory or history plays a significant role in the training of RNNs/LSTMs. Hence, the application of fractional derivatives that inherently incorporate history in computation seems very meaningful and beneficial for training these types of Neural Networks. Due to fractional derivatives' non-locality property, fractional-order LSTM/RNN networks are expected to learn long-term dependencies more easily than the integer-order gradient-based LSTM/RNN. For dealing with the complexity of stock market data, we are using fractional-order LSTM for its rigorous analysis. In the following sections, the concept of LSTM and the basics of fractional derivatives are briefly described.

3. Preliminaries
3.1. Basics of Fractional Calculus

Fractional calculus was originated around three centuries ago. The inventor of calculus, Leibniz, was asked by L’Hospital about derivatives' behaviour when order is non-integer. At that moment, Leibniz replied that "It will lead to some useful consequences in the future." This response caught the attention of other great mathematicians, Lacroix, Riemann, Liouville, Abel, and many others. These mathematicians started working in fractional derivatives and gave many definitions for fractional integrals and derivatives. Abel was the first mathematician to incorporate the use of fractional calculus in solving the tautochrone problem. But still, fractional calculus was considered a field of abstract research involving complicated mathematical calculations. Due to the advent of computational support, fractional calculus started emerging and applied to several areas of Science, Engineering, and Economics [21-31]. Various other definitions of fractional derivatives were introduced, which can be used for solving real-life problems [33]. The most used definitions are as follows:

3.1.1. Riemann-Liouville Fractional Integral Operator: Let $\alpha > 0$ and $g$ be the function which is piecewise continuous on $G = (0, \infty)$ and integrable on any finite subinterval of $G = [0, \infty)$. Then for $t > a$ and $t, a \in \mathbb{R}$ the following equation

$$a D_t^{-\alpha}g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\xi)^{\alpha-1}g(\xi)d\xi$$

represents the Riemann-Liouville fractional integral of function $g$ of order $\alpha$. This is the most generally used variant of fractional integration. Gamma function is not defined at zero and negative integers. Thus, this definition cannot be generalized for non-integer differentiation. So separate definitions were introduced for fractional differentiation. The three most frequently used definitions of the fractional derivative are:

3.1.2. Riemann-Liouville Fractional Derivative. Suppose that $\alpha > 0$, $t > a$ and $m \in \mathbb{N}$ such that $m - 1 \leq \alpha < m$, then to get $\alpha^{th}$-derivative of $g(t)$ we first integrate $g(t)$ fractionally up to $m- \alpha$, then differentiate $m$ times, i.e.

$$a D_t^{\alpha}g(t) = \begin{cases} 
\frac{d^m}{dt^m} \left[ \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\xi)^{m-\alpha-1}g(\xi)d\xi \right] & \text{When } m - 1 \leq \alpha < m. \\
\frac{d^m}{dt^m}g(t) & \text{When } \alpha = m.
\end{cases}$$

where $\Gamma$ is the Gamma function, is called the Riemann-Liouville fractional derivative of function $g$ of order $\alpha$. This definition requires only a continuous function for its application, but this definition has some limitations. Firstly, it does not give the differentiation of a constant function to be zero. Secondly, while solving fractional differential equations involving this derivative, Laplace transform is used, and Laplace transform of this derivative involves initial conditions at fractional orders that are not having any physical interpretations. Caputo introduced another definition of derivative for dealing with such problems, as explained below [34]. But it requires the function to be differentiable.

3.1.3. Caputo Fractional Derivative. Suppose that $\alpha > 0$, $t > a$ and $m \in \mathbb{N}$ such that $m - 1 \leq \alpha < m$, then to get $\alpha^{th}$-derivative of $g(t)$ we first integrate $g(t)$ fractionally up to $m- \alpha$, then differentiate $m$ times, i.e.
From the above equation, we can also say that is summarised as:

\[ \frac{d^m g(t)}{dt^m}, \quad \alpha = m. \]  

(4)

is called the Caputo fractional derivative of function \( g \) of order \( \alpha \).

### 3.1.4. Grünwald-Letnikov (G-L) Fractional Derivative

The \( \alpha > 0 \) order derivative of a function \( g \) is expressed as the limit of a sum by

\[
D^\alpha g(t) = \lim_{h \to 0} \sum_{r=0}^{\frac{n}{h}} (-1)^r \left( \frac{\alpha}{r} \right) g(t - rh)
\]

(5)

where \( \left( \frac{\alpha}{r} \right) = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - r + 1) \Gamma(r + 1)} \). This discrete version introduced by Grünwald and Letnikov [35,36] does not require the function to be differentiable or continuous.

It is visible in the definition that evaluation of fractional derivative at \( t \) considers the value of the function at all past values \( t - nh \). But while evaluating integer-order derivative at \( t \) considers the value of the function at \( t \) and \( t - h \). The main advantage of using fractional derivatives is their memory property. These can consider history dependency and non-local distribution of the system. Another advantage of fractional derivatives is that the order of differentiation acts as an additional degree of freedom. On these grounds, fractional calculus has been exploited in anomalous diffusion [37], viscoelasticity [38], signal and image processing [39], and biology [40].

### 3.1.5. Fractional Chain Rule

Due to computational complexity, an approximated chain rule has been used, which is given in [41], obtained by using the fractional Taylor’s series on a differentiable function and that is summarised as:

\[
g(x + h) = g(x) + \frac{h^\alpha}{\Gamma(1 + \alpha)} g^{(\alpha)}(x) + \frac{h^{2\alpha}}{\Gamma(2 + \alpha)} g^{(2\alpha)}(x) + \frac{h^{3\alpha}}{\Gamma(3 + \alpha)} g^{(3\alpha)}(x) + \cdots
\]

(6)

\[
\Rightarrow \frac{\Delta g(x)}{h^\alpha} = \frac{1}{\Gamma(1 + \alpha)} g^{(\alpha)}(x) + \frac{h^\alpha}{\Gamma(2 + \alpha)} g^{(2\alpha)}(x) + \frac{h^{2\alpha}}{\Gamma(3 + \alpha)} g^{(3\alpha)}(x) + \cdots
\]

(7)

Taking limit \( h \to 0 \), we get

\[
\lim_{h \to 0} \frac{\Delta g(x)}{h^\alpha} = \frac{1}{\Gamma(1 + \alpha)} g^{(\alpha)}(x)
\]

(8)

Also, \( \lim_{h \to 0} \frac{\Delta^\alpha g(x)}{h^\alpha} = g^{(\alpha)}(x) \Rightarrow \Delta^\alpha g(x) \approx \Gamma(1 + \alpha) \Delta g(x) \quad 0 < \alpha < 1. \)

From the above equation, we can also say \( d^\alpha g(x) \approx \Gamma(1 + \alpha) d g(x), \quad 0 < \alpha < 1. \) Using this result,

\[
\frac{d^\alpha g(u(x))}{dx^\alpha} = \frac{d^\alpha g(u(x))}{du^\alpha} \frac{du}{dx} = \frac{\Gamma(1 + \alpha) d g(u(x))}{\Gamma(1 + \alpha) du} u_x^\alpha(x) = g_u^\alpha(u) u_x^\alpha(x)
\]

(9)
Hence,

\[ D_x^\alpha f(u(x)) = D_1^\alpha f(u) D_x^\alpha u(x). \] (10)

3.2. Long Short Term Memory (LSTM)

After reviewing the state-of-the-art literature, we noticed the various kinds of neural networks that could be used for this prediction. As for this prediction, we are using time series data. For such data, either Deep Neural Networks (DNN) or Recurrent Neural Networks (RNN) are used. But DNN struggles when there are abrupt changes in the data. Also, DNN does not have any memory. RNNs cover the entire sequence starting from the beginning and then help predict the next term. These networks have loops in them, allowing the information to persist. But as the gap increases, Vanilla RNNs fail to connect the information (resulting in less accuracy in predictions). Also, vanilla RNN sometimes suffers from the problem of vanishing gradient. Hence, we have chosen the LSTM variant for price prediction, which was introduced for solving this problem of the gradient.

4. Proposed Approach

The algorithm employed for predicting the stock price involves the following steps: 1.) Data Exploration 2.) Data Pre-processing 3.) Forward Propagation 4.) Fractional Gradient Descent Algorithm for training. The approach involves coding the LSTM from scratch.

4.1. Data Exploration and Processing

In this study, we have used the data of Google.com for a decade, January 2007 - June 2017. These data points are indexed in time order. The goal of the study is to predict the closing price for any given date after training. The data was obtained from the Google finance application programming interface, which is shown in Table 1.

| Date       | Open     | High      | Low       | Close     | Adjusted close | value    |
|------------|----------|-----------|-----------|-----------|----------------|---------|
| 2015-11-25 | 107.510002 | 107.660004 | 107.250000 | 107.470001 | 101.497200 | 1820300 |
| 2015-11-27 | 107.589996 | 107.760002 | 107.320001 | 107.629997 | 101.648300 | 552400  |
| 2015-11-30 | 107.779999 | 107.849998 | 107.110001 | 107.169998 | 101.213867 | 3618100 |
| 2015-12-01 | 107.589996 | 108.209999 | 107.370003 | 108.180000 | 102.167740 | 2443600 |
| 2015-12-02 | 107.099998 | 108.269997 | 106.879997 | 107.050003 | 101.100533 | 2937200 |
| 2015-12-03 | 107.290001 | 107.480003 | 105.059998 | 105.449997 | 99.589470  | 3345600 |
| 2015-12-04 | 107.099998 | 107.540001 | 105.620003 | 107.389999 | 101.421646 | 4520000 |
| 2015-12-07 | 107.230003 | 107.269997 | 106.059998 | 106.550003 | 100.628342 | 3000500 |
| 2015-12-08 | 107.940002 | 106.400002 | 105.269997 | 105.910004 | 100.023895 | 3149600 |

4.2. Neural Network Architecture and Training

LSTM is used for obtaining better predictions as these are capable of learning long-term dependencies. We have optimized the model by tuning the following parameters. The model consists of three layers of nodes, and each layer contains 64 nodes. The training and testing data used are 80% and 20%, respectively. The validation sets are kept 0.5% of the training data. Random values initialize the network parameters, and the batch size is 512. We train 20 epochs for our fractional-order LSTM.

4.3. Backward Propagation

To learn the optimal parameters, we have used fractional derivatives to differentiate loss function concerning each of the parameters, and the update rule becomes
\[ \Delta w = -\eta \frac{\partial^\alpha E}{\partial w^\alpha} \]  

where \( \alpha \) is the fractional order of differentiation and \( \eta \) is the learning rate. The error function is a composite function. Thus, the above-mentioned fractional chain rule is employed. Hence

\[ D^\alpha_w E(j(w)) = D^\frac{1}{\alpha} E(j) D^\alpha_w j(w) \]  

The backpropagation code for evaluation of G-L derivative of order 'alpha' is as follows, where we have run the process for \( \alpha = 0.9 \).

```python
GL_previous = f_values[1]
for index in range(2, num_points):
    GL_current = GL_previous*(num_points-alpha-index-1)/(num_points-index) + f_values[index]
    GL_previous=GL_current
return GL_current*(num_points/(domain_end-domain_start))**alpha
```

4.4. Results and Analysis

This section presents the prediction results of the proposed algorithm. The results have been obtained from the test data of Google.com for the past decade. The model also had a validation split, while training phase, to avoid overfitting. Firstly, we have analysed the different order of differentiation \( \alpha \) as shown in the graph in Figure 3. We have observed the predicted price for \( 0 < \alpha < 1 \). In Figure 1, the green, blue, orange, and green plots depict the predicted prices at order 1.2, 0.9, 0.7, and 0.5, respectively. The error is obtained least at \( \alpha = 0.9 \). For estimating the error, we have used the root mean squared error.

![Figure 1: Comparison of prediction for the different order of differentiation](image)

Then we have analysed the results of LSTM obtained at order 0.9 and the adjusted close value of the stock. The graph in Figure 2 depicts that the fractional-order LSTM has successfully predicted the stock's future price trend, i.e., upward, or downward. But the proposed model fails to predict the price with the required accuracy. In Figure 2, the blue and green plots depict the predicted and adjusted close prices, respectively. The proposed LSTM (with fractional derivative) has an error of 0.030% in
the training dataset and 0.107% for the test dataset at order 0.9. It is evident from the graph as well that the direction of price movements is well predicted.

![Graph showing predicted vs actual stock prices](image)

**Figure 2:** Comparison of predicted result at order 0.9 with the adjusted close stock value

5. Conclusion
In this work, a novel fractional backpropagation learning algorithm for the LSTMs is proposed and is applied in portfolio management. The proposed method deployed the fractional gradient descent algorithm involving Caputo's derivative in the learning of LSTM. It is observed that fractional derivatives incorporated in the neural network help predict stock price direction. The method is not precise in indicating the exact Stock Price. Still, it successfully suggests whether the investor should remain invested in the stock for the long-term depending upon stock price movements' direction. The accuracy of fractional-order backpropagation can be studied upon in the future.

Acknowledgment
Sugandha is grateful to University Grants Commission for the financial assistance with UGC Ref No 1034/ (CSIR-UGC NET JUNE 2017).

Conflicts of Interest
The authors declare that they have no conflicts of interest to report regarding the present study.

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