On Realisations of $W$ Algebras

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ABSTRACT

It has been known for some time that $W$ algebras can be realised in terms of an energy-momentum tensor together with additional free scalar fields. Some recent results have shown that more general realisations are also possible. In this paper, we consider a wide class of realisations that may be obtained from the Miura transformation, related to the existence of canonical subalgebras of the Lie algebras on which the $W$ algebras are based. We give explicit formulae for all realisations of this kind, and discuss their applications in $W$-string theory.

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1) Introduction

Two-dimensional conformal field theories with non-linear local symmetry algebras, namely \( W \) algebras, have applications in extensions of string theories [1-6]. The physical interpretation of the corresponding string theory depends upon the specific realisation of the \( W \) symmetry. Since the \( W \) algebras, which are higher-spin extensions of the Virasoro algebra, are generically non-linear, realisations are not easy to come by. Unlike the Virasoro algebra, one cannot obtain new realisations simply by tensoring together old ones. In fact, it has recently been found that owing to the non-linearity of \( W \) algebras, there can exist intrinsically different types of realisations of the same \( W \) algebra [6,7]. They will lead to different string theories. Thus it is interesting to explore all the possible realisations for the \( W \) algebras. One way to obtain realisations is by the free-field construction via the Miura transformation [8]. The Miura transformations constructed hitherto have been restricted to the classical Lie algebras \( A_n, D_n \) and \( B_n \), which give rise to realisations for \( WA_n, WD_n \) and \( WB_n \) algebras respectively [9,10,11]. The \( WA_n \) algebra is also known as \( W^{n+1} \).

First we consider \( WA_n \) and \( WD_n \), based on the classical simply-laced Lie algebras \( A_n \) and \( D_n \). The corresponding Miura transformations give rise to realisations in terms of \( n \) free scalars \( \vec{\varphi}^{(n)} \equiv (\varphi_1, \ldots, \varphi_n) \). It was noticed in [12,1,2,4] that one special free scalar, say \( \varphi_1 \), appears in the currents of the corresponding \( W \) algebra only via its energy-momentum tensor. The contribution from \( \varphi_1 \) can then be replaced by an arbitrary energy-momentum tensor with the same central charge. This leads to realisations of \( WA_n \) and \( WD_n \) in terms of an arbitrary energy-momentum tensor, together with \( (n-1) \) free scalars. In [6] new realisations were obtained from the Miura transformation for \( n \geq 3 \) in terms of two arbitrary commuting energy-momentum tensors with the same central charge, together with \( (n-2) \) free scalars.

In this paper we shall review how these realisations can be obtained from Miura transformations by using certain specific reduction procedures, i.e. by expressing the currents of \( WA_n \) or \( WD_n \) in terms of those of \( WA_{n-1} \) or \( WD_{n-1} \) respectively, together with an extra free scalar field. We shall then show that these reductions are special cases of more general reductions, which lead to realisations in terms of multiple numbers of arbitrary commuting energy-momentum tensors with the same central charge, together with a certain number of free scalars. It was observed in [7] by using the free-field construction and screening charges that such general reductions are possible in principle. An example of a realisation of \( WA_4 \) in terms of \( WA_2 \) and \( WA_1 \) together with an extra free scalar was given in [7]. In this paper we shall straightforwardly derive the general reductions directly from Miura transformations and give explicit formulae for all cases.

Secondly, we generalise the results to \( WB_n \) algebras, obtaining realisations in terms of a super energy-momentum tensor and bosonic energy momentum tensors, together with some necessary free scalars. In all the cases \( WA_n, WD_n \) and \( WB_n \), the essential characteristic of the reductions obtained in this paper is that one can remove any desired vertex in the
Dynkin diagram for $A_n$, $D_n$ or $B_n$, and realise the associated $W$ algebra in terms of the two commuting $W$ algebras corresponding to the two factors in the product subalgebra. At the end of the paper, we comment on the applications of these realisations to $W$ strings.

2) Review of Previous Results

$WA_n$ algebras are generated by primary currents of spins $3, 4, \ldots, (n+1)$, together with an energy-momentum tensor. A realisation of the $WA_n$ algebra in terms of $n$ free scalars $\vec{\varphi}^{(n)} ≡ (\varphi_1, \ldots, \varphi_n)$ is given by the Miura transformation for $A_n ≡ su(n+1)$ [9]

$$\prod_{k=1}^{n+1} (\alpha_0 \partial + \vec{h}^{(n)}_k \cdot (\partial \vec{\varphi}^{(n)})) = (\alpha_0 \partial)^{n+1} + \sum_{\ell=2}^{n+1} W^{(n)}_\ell (\alpha_0 \partial)^{n+1-\ell},$$

(1)

where the $\vec{h}^{(n)}_k$ are $n$-component vectors satisfying

$$\vec{h}^{(n)}_i \cdot \vec{h}^{(n)}_j = \delta_{ij} - \frac{1}{n+1}.$$  

(2)

The quantities $W^{(n)}_\ell$ in (1) with $2 \leq \ell \leq (n+1)$ are spin-$\ell$ currents that generate the $WA_n$ algebra. These higher-spin currents are not yet primary with respect to the energy-momentum tensor $T ≡ W^{(n)}_2$, but they can be made so by adding composites and derivatives of the lower-spin currents. The closure of the algebra only requires that the vectors $\vec{h}^{(n)}_i$ satisfy the conditions (2), for which many solutions exist. If one just considers the $n$-scalar realisation, all these solutions are equivalent; an orthonormal transformation of the free scalars $\vec{\varphi}^{(n)}$ will map one solution to another. However, we shall see later that certain choices of $\vec{h}^{(n)}_i$ will make it possible to express the currents of the $WA_n$ algebra in terms of the currents of arbitrary $WA_k$ and $WA_{n-k-1}$ algebras with $0 \leq k \leq (n-1)$, together with one extra scalar field.

In [2,4] a specific choice for the vectors $\vec{h}^{(n)}_i$ is proposed for which they have the nice property that one can express $\vec{h}^{(n)}_i$ in terms of $\vec{h}^{(n-1)}_i$, viz.

$$\vec{h}^{(n)}_i = \left(\frac{1}{\sqrt{n(n+1)}}, 0, \ldots, 0\right), \quad 1 \leq i \leq n,$$

$$\vec{h}^{(n)}_{n+1} = \left(0, \ldots, 0, -\frac{n}{\sqrt{n(n+1)}}\right),$$

(3)

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where the \((n - 1)\)-component vectors \(\vec{h}_i^{(n-1)}\), also satisfying (2) with \(n \to (n - 1)\), are those for \(WA_{n-1}\). Substituting (3) into (1), the left-hand side of equation (1) can be rewritten as

\[
\left(\alpha_0 \partial - n(\partial \phi_n)\right) \prod_{\ell=1}^{n} \left(\alpha_0 \partial + \vec{h}_\ell^{(n-1)} \cdot (\partial \phi^{(n-1)}) + (\partial \phi_n)\right)
\]

\[= \left(\alpha_0 \partial - n(\partial \phi_n)\right) e^{-\phi_n/\alpha_0} \prod_{\ell=1}^{n} \left(\alpha_0 \partial + \vec{h}_\ell^{(n-1)} \cdot (\partial \phi^{(n-1)})\right) e^{\phi_n/\alpha_0}, \tag{4}\]

where we have defined \(\phi_n \equiv \frac{1}{\sqrt{n(n+1)}} \phi_n\). Using the Miura transformation (1) for \(WA_{n-1}\), we can write the \(\prod_{\ell=1}^{n}\) factors in the right-hand side of the equation (4) in terms of the currents of the \(WA_{n-1}\) algebra. Since the currents of \(WA_{n-1}\), differential polynomials of free scalars \(\vec{\varphi}^{(n-1)}\), commute with \(\varphi_n\), we can realise the currents of \(WA_n\) in terms of those of an arbitrary \(WA_{n-1}\) and an extra scalar \(\varphi_n\). Applying this procedure recursively leads to a realisation of the \(WA_n\) algebra in terms of an arbitrary energy-momentum tensor and \((n - 1)\) additional scalar fields \((\varphi_2, \ldots, \varphi_n)\).

In [5] a similar reduction procedure is proved for the \(WD_n\) algebras. One can realise the currents of the \(WD_n\) algebra in terms of those of \(WD_{n-1}\) and an extra free scalar. Since \(D_2 \cong A_1 \times A_1\), the algebra of \(WD_2\) is isomorphic to the direct product of two independent Virasoro algebras but with the same central charge. Exploiting the fact that \(A_3 \cong D_3\) and hence \(WA_3 \cong WD_3\), one can therefore realise \(WA_n\) and \(WD_n\) for \(n \geq 3\) in terms of either one arbitrary energy-momentum tensor together with \((n - 1)\) additional free scalars [5] or two arbitrary energy-momentum tensors together with \((n - 2)\) additional free scalars [6].

3) General Reduction for \(WA_n\)

The reduction \(WA_n \to WA_{n-1}\) described above is dictated by the specific choice of the vectors \(\vec{h}_i^{(n)}\) given in (3). This choice can be easily generalised so that the vectors \(\vec{h}_i^{(n)}\) can be expressed in terms of \(\vec{h}_i^{(n-k-1)}\) and \(\vec{h}_i^{(k)}\), with \(0 \leq k \leq (n - 1)\), viz.

\[
\vec{h}_i^{(n)} = \left(h_i^{(n-k-1)}, 0, \ldots, 0, \sqrt{\frac{k+1}{(n-k)(n+1)}} \right), \quad 1 \leq i \leq (n - k),
\]

\[
\vec{h}_j^{(n)} = \left(0, \ldots, 0, \sqrt{\frac{n-k}{(k+1)(n+1)}} \right), \quad (n - k + 1) \leq j \leq (n + 1), \tag{5}\]

where \(\vec{h}_i^{(n-k-1)}\) and \(\vec{h}_i^{(k)}\) are the corresponding vectors for \(WA_{n-k-1}\) and \(WA_k\), satisfying (2) with \(n \to (n - k - 1)\) and \(n \to k\) respectively. It is easy to check that the vectors \(\vec{h}_i^{(n)}\) given in (5) satisfy the condition (2). In particular, when \(k = 0\), equation (5) will reduce to
the special case given in equation (3). Substituting (5) into the Miura transformation (1), the left-hand side of the equation reads

\[
\prod_{\ell = n-k+1}^{n+1} \left( \alpha_0 \partial + \bar{h}_{\ell+k-n}^{(k)} \cdot (\partial \bar{\varphi}^{(k)}) - (n-k)(\partial \phi_n) \right)
\]

\[
\times \prod_{\ell = 1}^{n-k} \left( \alpha_0 \partial + \bar{h}_{\ell}^{(n-k-1)} \cdot (\partial \bar{\varphi}^{(n-k-1)}) + (k+1)(\partial \phi_n) \right)
\]

\[
= e^{(n-k)\phi_n/\alpha_0} \prod_{\ell = n-k+1}^{n+1} \left( \alpha_0 \partial + \bar{h}_{\ell+k-n}^{(k)} \cdot (\partial \bar{\varphi}^{(k)}) \right) e^{-(n-k)\phi_n/\alpha_0}
\]

\[
\times e^{-(k+1)\phi_n/\alpha_0} \prod_{\ell = 1}^{n-k} \left( \alpha_0 \partial + \bar{h}_{\ell}^{(n-k-1)} \cdot (\partial \bar{\varphi}^{(n-k-1)}) \right) e^{(k+1)\phi_n/\alpha_0},
\]

where we have defined \( \phi_n \equiv \frac{1}{\sqrt{(k+1)(n-k)(n+1)}} \varphi_n \) and we also have split \( \bar{\varphi}^{(n)} \) into

\[
\bar{\varphi}^{(n)} = (\bar{\varphi}^{(n-k-1)}, \bar{\varphi}^{(k)}, \varphi_n)
\]

with \( \bar{\varphi}^{(k)} \equiv (\varphi_{n-k},...,\varphi_{n-1}) \). Using the Miura transformation (1), we can write the \( \prod_{\ell = 1}^{k} \) factors in the right-hand side of equation (6) in terms the currents of \( WA_k \) and write the \( \prod_{\ell = n-k+1}^{n+1} \) factors in terms of the currents of \( WA_{n-k-1} \). We therefore obtain a realisation for the \( WA_n \) algebra in terms of commuting \( WA_k \) and \( WA_{n-k-1} \) realisations with an additional scalar \( \varphi_n \). The integer \( k \) takes values \( 0,...,(n-1) \). When \( k = 0 \) or \( k = n-1 \), we will recover the result given in (4). This reduction can be understood schematically from the Dynkin diagram

\[
A_k \rightarrow WA_k \quad \partial \varphi \quad A_{n-k-1} \rightarrow WA_{n-k-1}
\]

One can obtain a subalgebra by removing any vertex in the Dynkin diagram for \( A_n \); consequently, one can realise the \( WA_n \) in terms of the corresponding lower \( W \) algebras based on the direct-product subalgebra with an extra scalar corresponding to the vertex removed. Applying this reduction procedure recursively, one can realise \( WA_n \) algebra in terms of \( k \) energy-momentum tensors with \( 0 \leq k \leq \left[ \frac{n+1}{2} \right] \), together with \( (n-k) \) additional scalars. Since the energy-momentum tensors commute with each other and with the additional scalars, they can be arbitrary and independent of one another. However the central charges for all these energy-momentum tensors must be same. This follows from the fact that the vectors \( \bar{h}_i^{(1)} \) for \( WA_1 \equiv \text{Virasoro} \) are unique. Consequently, the background charges for these energy-momentum tensors, originally derived from the Miura transformation, all take the same value \( \alpha_0/\sqrt{2} \), which leads to the same central charge \( c_{\text{eff}} \) given by

\[
c_{\text{eff}} = 1 + 6\alpha_0^2.
\]
The total central charge of the $WA_n$ algebra is independent of which of the above realisations is used. It is given by
\[
c(WA_n) = n + 12\alpha_0^2(\vec{p})^2
\equiv n + 12\alpha_0^2\left(\sum_{j=2}^{n+1}(1-j)\vec{h}_j^{(n)}\right)^2
= n\left(1 + (n+1)(n+2)\alpha_0^2\right).
\] (10)

4) General Reduction for $WD_n$

Now we turn our attention to the $WD_n$ algebras, which are generated by currents $W^{(n)}(z)$ of spin $s = 2j$, with $j = 1, \ldots, (n-1)$, and a current $U^{(n)}(z)$ of spin $n$. A similar reduction also works in this case. One can realise the $WD_n$ in terms of $WA_k$ and $WD_{n-k-1}$ and an extra scalar, with $0 \leq k \leq (n-1)$. To see this, we write down the $n$-scalar realisation from the Miura transformation for the $WD_n$ algebra. The spin-$n$ current is given by the Miura-type transformation \[10,11\]
\[
U^{(n)}(z) = \prod_{\ell=1}^{n} \left(\alpha_0 \partial - \vec{\sigma}^{(n)}_{\ell} \cdot (\partial \varphi^{(n)})\right) \cdot \mathbb{1},
\] (11)
where the $\vec{\sigma}^{(n)}_{\ell}$ are $n$-component vectors satisfying
\[
\vec{\sigma}^{(n)}_{i} \cdot \vec{\sigma}^{(n)}_{j} = \delta_{ij}.
\] (12)
The $W^{(n)}_{2j}$ currents can then be read off from the operator-product expansion $U^{(n)}(z)U^{(n)}(w)$:
\[
U^{(n)}(z)U^{(n)}(w) \sim \frac{a_n}{(z-w)^{2n}} + \sum_{j=1}^{n-1} \frac{a_{n-j}}{(z-w)^{2n-2j}} \left(W^{(n)}_{2j}(z) + W^{(n)}_{2j}(w)\right),
\] (13)
where $a_j$ are normalisation constants given by $a_j = (-1)^{j+1} \prod_{\ell=1}^{j-1} (1 + 2\ell(2\ell + 1)\alpha_0^2)$.

The closure of the $WD_n$ algebra requires that the vectors $\vec{\sigma}^{(n)}_{i}$ satisfy the condition (12). In \[10,11\] the simplest solution to (12) was used, namely $\vec{\sigma}^{(n)}_{i} = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 occurs in the $i$-th entry. This solution enables us to express the currents of the $WD_n$ algebra in terms of those of $WD_{n-1}$ and an extra scalar \[5\]. In general, one can achieve condition (12) by expressing the $\vec{\sigma}^{(n)}_{i}$ in terms of $\vec{\sigma}^{(n-k-1)}_{i}$ and $\vec{h}^{(k)}_{i}$, viz.
\[
\vec{\sigma}^{(n)}_{i} = \left(\vec{\sigma}^{(n-k-1)}_{i}, 0, \ldots, 0\right), \quad 1 \leq i \leq (n-k-1),
\]
\[
\vec{\sigma}^{(n)}_{j} = \left(0, \ldots, 0, \vec{h}^{(k)}_{j-n+k+1}, \frac{1}{\sqrt{k+1}}\right), \quad (n-k) \leq j \leq n,
\] (14)
where \( \hat{\sigma}_i^{(n-k-1)} \) and \( \hat{h}_j^{(k)} \) are the vectors for \( WD_{n-k-1} \) and \( WA_k \), satisfying (2) and (12) respectively. Substituting (14) into (11), we can write the spin-\( n \) current \( U^{(n)} \) in terms of the spin-\( (n-k-1) \) current \( U^{(n-k-1)} \) of \( WD_{n-k-1} \) and the currents of \( WA_k \), together with an extra scalar. Consequently, it follows from equation (14) that we can express all the currents of \( WD_n \) in terms of those of \( WD_{n-k-1} \) and \( WA_k \) together with the extra field. This reduction procedure can be summarised by the Dynkin diagram

\[
\begin{array}{c}
A_k \rightarrow WA_k \\
\cdots \quad \cdots \\
D_{n-k-1} \rightarrow WD_{n-k-1}
\end{array}
\]  

Applying this reduction, and the reduction for \( WA_n \), recursively gives rise to realisations of the \( WD_n \) algebra in terms of \( k \) energy-momentum tensors together with \( (n-k) \) free scalars, with \( 0 \leq k \leq \left\lfloor \frac{n+1}{2} \right\rfloor \). Like the case of \( WA_n \), these energy-momentum tensors are arbitrary and independent, but with the same central charge given in (9). The total central charge is given by

\[
c(WD_n) = n(1 + (2n-1)(2n-2)\alpha_0^2).
\]  

Note that taking \( k = n-1 \) in the above reduction defined by (14) corresponds to reducing \( WD_n \) to \( WA_{n-1} \), represented by the deletion of one of the “ears” in the Dynkin diagram (15). When \( k = n-2 \), the corresponding subalgebra of \( D_n \) is \( A_{n-2} \times D_1 \). Although \( D_1 \) is isomorphic to \( A_1 \), the \( WD_1 \) algebra, as defined by the Miura transformation (11), is not isomorphic to \( WA_1 \cong Virasoro \); instead it is an algebra with just a spin-1 current \( \partial \phi_1 \). Thus when \( k = n-2 \) we get \( WA_n \) realised in terms of \( WA_{n-2} \) and 2 additional free scalars.

5) General Reduction for \( WB_n \)

Having obtained new realisations for bosonic \( WA_n \) and \( WD_n \) based on the classical simply-laced Lie algebras \( A_n \) and \( D_n \), we should like to study the case of \( WB_n \), which is based on the classical non-simply-laced Lie algebra \( B_n \). The \( WB_n \) algebra has bosonic currents of spins \( s = 2, 4, \ldots, 2n \), and an additional fermionic current with spin \( (n + \frac{1}{2}) \). The special case of \( WB_1 \) is linear, and in fact is simply the \( N = 1 \) super-Virasoro algebra. The higher-\( n \) \( WB_n \) algebras can thus be thought of as higher-spin extensions of the \( N = 1 \) super-Virasoro algebra; as such, they may turn out to be the most suitable candidates for generating higher-spin extensions of superstring theory. The first non-trivial (\( i.e. \) non-linear) example, \( WB_2 \), was constructed explicitly in [13]. The fermionic spin-\( (n + \frac{1}{2}) \) current \( Q^{(n)}(z) \) of \( WB_n \) plays an analogous rôle to the bosonic spin-\( n \) current \( U^{(n)}(z) \) in the \( WD_n \) algebra. It is given by the Miura-type transformation [10]

\[
Q^{(n)}(z) = \left( \prod_{\ell=1}^{n} (\alpha_0 \partial - \hat{\sigma}_\ell^{(n)} \cdot (\partial \phi^{(n)})) \right) \psi ,
\]  

7
where the $\tilde{\sigma}_i^{(n)}$ are $n$-component vectors satisfying condition (12) and $\psi$ is a real free fermion field. The bosonic currents $W_{2\ell}^{(n)}$ can be read off from the operator-product expansion $Q^{(n)}(z)Q^{(n)}(w)$:

$$Q^{(n)}(z)Q^{(n)}(w) \sim \frac{b_n}{(z-w)^{2n+1}} + \sum_{j=1}^{n} \frac{b_{n-j}}{(z-w)^{2n+1-2j}} (W_{2j}^{(n)}(z) + W_{2j}^{(n)}(w))$$

with $b_j = \prod_{\ell=1}^{j} (1 + 2\ell(2\ell - 1)\alpha_0^2)$. Like the case of $WD_n$, we can express the vectors $\tilde{\sigma}_i^{(n)}$ in terms of $\tilde{\sigma}_i^{(n-k-1)}$ and $\tilde{h}_j^{(k)}$ given in (14). Thus we can realise the $WB_n$ algebra in terms of $WB_{n-k-1}$ and $WA_k$ together with an extra scalar, with $0 \leq k \leq (n-1)$. The result for the case $k = 0$ was first obtained in [5]. Schematically, the general case is summarised by the Dynkin diagram

$$A_k \rightarrow WA_k \quad \cdots \quad \partial \phi \quad B_{n-k-1} \rightarrow WB_{n-k-1}$$

(19)

Applying this reduction procedure and the reduction for $WA_n$ recursively, one can realise the $WB_n$ algebra in terms of $WB_1 \cong$ super-Virasoro, $k$ arbitrary energy-momentum tensors, and $(n-k-1)$ additional scalars, with $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$. The total central charge for $WB_n$ is given by

$$c(WB_n) = (n + \frac{1}{2})(1 + 2n(2n - 1)\alpha_0^2) .$$

(20)

The central charges for each arbitrary energy-momentum tensor are again the same and given by (9). The central charge $c_{\text{sup}}^{\text{eff}}$ for the super energy-momentum tensor is

$$c_{\text{sup}}^{\text{eff}} = \frac{3}{2} + 3\alpha_0^2 .$$

(21)

Taking $k = n - 1$ in (14) corresponds here to the reduction of $WB_n$ to $WA_{n-1}$, represented by deleting the black vertex in the Dynkin diagram (19). In this case there is no super energy-momentum tensor, and $WB_n$ is realised in terms of the currents of $WA_{n-1}$ together with a free scalar and a free fermion.

6) Application in String Theories

String theory is two-dimensional gravity coupled to a critical matter system that includes free scalar fields which are interpreted as coordinates on the target spacetime. One may construct generalisations of two-dimensional gravity, by gauging matter systems with $W$ symmetries. If the matter systems are critical, and include free scalars, then one obtains $W$-string theories. The realisations for $W$ algebras obtained above all have two types of elements: some arbitrary energy-momentum tensors (or in the case of $WB_n$, energy-momentum tensors and a super energy-momentum tensor) and some necessary additional
free scalars. Calculations for specific examples [1-6] show that the physical-state conditions for higher-spin currents just serve the purpose of “freezing” these additional coordinates, i.e. the momenta conjugate to these coordinates are frozen to certain specific values by the physical state conditions. If any of these (super) energy-momentum tensors comprises \( D \) free scalars, these scalars form the \( D \)-dimensional physically-observable coordinates on the target spacetime. Thus a \( W \)-string theory is effectively described by a set of independent ordinary (super) Virasoro-like strings with a non-standard effective central charge and a set of non-standard spin-2 intercept values. We shall solve these central charges and exhibit a numerological connection with (super) Virasoro minimal models, for the \( W_{A_n}, W_{D_n} \) and \( W_{B_n} \) strings.

Anomaly freedom of a \( W \) string requires that the total central charge take a certain specific value in order to cancel the anomalous contributions from the ghosts. The critical central charges for \( W \) strings are given by [5]

\[
\begin{align*}
c^* &= 2n(2n^2 + 6n + 5), \quad \text{for } W_{A_n}, \\
c^* &= 2n(8n^2 - 12n + 5), \quad \text{for } W_{D_n}, \\
c^* &= (2n + 1)(8n^2 - 4n + 1), \quad \text{for } W_{B_n}.
\end{align*}
\]

(22)

Substituting these values into (10), (16) and (20), one can solve for the critical background-charge parameter \( \alpha^*_0 \) for \( W_{A_n}, W_{D_n} \) and \( W_{B_n} \) respectively. One can then, from (9) and (21), obtain the effective central charges for the corresponding \( W \) strings:

\[
\begin{align*}
c^{\text{eff}} &= 26 - \left( 1 - \frac{6}{(n + 1)(n + 2)} \right), \quad \text{for } W_{A_n}, \\
c^{\text{eff}} &= 26 - \left( 1 - \frac{6}{(2n - 2)(2n - 1)} \right), \quad \text{for } W_{D_n}, \\
c^{\text{eff}} &= 26 - \left( 1 - \frac{6}{2n(2n - 1)} \right), \quad \text{for } W_{B_n}, \\
c^{\sup} &= 15 - \left( 3/2 - \frac{12}{4n(4n - 2)} \right),
\end{align*}
\]

(23)

These effective central charges are all equal to the critical central charge of the (super) Virasoro string minus the central charges of certain unitary (super) Virasoro minimal models. Such connections between \( W \) strings and corresponding unitary minimal models are strengthened by the fact that if one writes the effective spin-2 intercepts \( L^{\text{eff}}_0 = 1 - L^{\min}_0 \) for \( W_{A_n} \) and \( W_{D_n} \), then the values of \( L^{\min}_0 \) are precisely the dimensions of primary fields of the corresponding minimal models. In the case of \( W_{B_n} \), one similarly writes and \( L^{\text{eff}}_0 = 1 - L^{\min}_0 \) for the energy-momentum tensors, and \( (L^{\text{eff}}_0) = 1/2 - L^{\min}_0 \) for the super energy-momentum tensor. It has been shown in [4,5,6] that connections with minimal models occur in the \( W \) strings with realisation in terms of one (or two) arbitrary energy-momentum tensors for
WA_n and WD_n, and one super energy-momentum tensor for WB_n. We may expect that this connection exists also in the more general realisations obtained here.

7) Summary

In this paper, we have studied general classes of realisations for W algebras. The known realisations all originate from Miura transformations. The Miura transformations for the WA_n, WD_n and WB_n algebras all share the common feature that they can be “factorised” as a product of two Miura transformations for two smaller commuting W algebras. Specifically, the possible factorisations are dictated precisely by the canonical subalgebras of the underlying A_n, D_n or B_n Lie algebras obtained by removing any vertex from the corresponding Dynkin diagram. Thus for example the currents of WA_n can be realised in terms of those for WA_p and WA_q algebras such that p + q = n − 1, together with an additional free scalar field (corresponding to the deleted vertex). Similarly, WD_n may be realised in terms of WD_p and WA_q, and WB_n may be realised in terms of WB_p and WA_q, with p + q = n − 1 in each case. Again, there is one additional free scalar in each realisation.

A straightforward extension of the above results is to apply the procedure recursively, so that one is effectively deleting more non-adjacent vertices in the original Dynkin diagram. The maximal case is when one deletes all alternate vertices. One can therefore obtain realisations of WA_n or WD_n in terms of ℓ mutually-commuting energy-momentum tensors and (n − ℓ) additional free scalars, for any ℓ in the range 0 ≤ ℓ ≤ ⌊n+1/2⌋. In the non-simply-laced case of WB_n, one can realise it in terms of an N = 1 super energy-momentum tensor, ℓ bosonic energy-momentum tensors, and (n − ℓ − 1) additional scalars, for any ℓ in the range 0 ≤ ℓ ≤ ⌊n−1/2⌋. Alternatively, in view of the remark at the end of section 5, it can be realised in terms of ℓ energy-momentum tensors, (n − ℓ − 1) free scalars and a free fermion, for ℓ in the range 0 ≤ ℓ ≤ ⌊n/2⌋.

An application of these realisations is to the corresponding W-string theories. Results for the previously-known realisations of WA_n or WD_n with ℓ = 1 and ℓ = 2, and WB_n with ℓ = 0, have revealed connections with certain unitary minimal models. It appears, by looking at the central charges for the effective energy-momentum tensors appearing in the more general realisations considered here, that these connections should persist for all the new realisations. There is an accumulation of evidence that all W-string theories are related to Virasoro-type strings and minimal models. The underlying significance of this relation remains to be understood.
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