Puzzles on the Duality between
Heterotic and Type IIA Strings

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ABSTRACT

We discuss the possibility of the extension of the duality between the webs of heterotic string and the type IIA string to Calabi-Yau three-folds with another K3 fiber by comparing the dual polyhedron of Calabi-Yau three-folds given by Candelas, Perevalov and Rajesh.

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1 Introduction

Some Calabi-Yau three-folds (=CY 3-folds) with base, \( \mathbb{CP}^1(1, s) \) and fiber, \( K3 = \mathbb{CP}^3(u_1, u_2, u_3, u_4)[d] \) are represented in hypersurface in weighted projective 4-space, \( \mathbb{CP}^4 \).

\[
\text{CY3-fold} = \mathbb{CP}^4(u_1, su_1, (s + 1)u_2, (s + 1)u_3, (s + 1)u_4)[(s + 1)d],
\]

with \( d = \sum_{i=1}^{4} u_i \).

The type IIA string on CY3-folds which have K3 fiber and \( T^2 \) fiber with at least one section is dual to the heterotic string on \( K3 \times T^2 \) as pointed out in \([1, 2]\). Much research has been explored on CY3-folds with \( s = 1 \) and \( u_1 = 1 \) cases and \( F_0 \) based case \([1, 2, 4, 3, 5, 6, 7]\). \( s = 1 \) and \( u_1 = 1 \) cases are given by the extremal transitions or by the conifold transitions from \( F_0 \) based CY3-folds \([5, 6, 7]\).

Thus far, CY3-folds which have been studied are constructed by using dual polyhedra. Some of them may be related to the hypersurface representations of eq. (1) with \( s \geq 2 \). However, identification between them has some ambiguities and is complicated \([8, 10, 11, 12]\). Therefore, the relation between these results and the duality between the type IIA string on CY3-folds with \( s \geq 2 \) to heterotic string on \( K3 \times T^2 \) is not clear yet. There have been three types of web sequences of heterotic - type IIA string duality from the terminal CY3-fold in A series \([4, 3, 8]\).

(iii) in \([8]\) may be the subset of (i)\(^{†}\) in \([6]\), though the properties of dual polyhedron are slightly different \([8, 10, 11, 12]\). (i)\(^{†}\) means the modified (i) in \([6]\) with extra tensor multiplets.

\(^{†}\) The difference between dual polyhedron of (i)\(^{†}\) and (iii) is as follows \([4, 6, 7]\). The dual polyhedra of case (i)\(^{†}\) have the modified dual polyhedra of K3 fiber. For the case (iii) in A series, the dual polyhedron of K3 fiber is not modified. The highest point in the additional points is represented by the weights of the K3 fiber of the terminal A series in this base. One point such as \((0, *, *, *)\) is also represented by the part of the weight of the terminal K3 fiber. The following element in SL(4, \( \mathbb{Z} \)) can transform these polyhedra into the dual polyhedron given by the ref. of \([6]\).

\[
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & 2 & 3 \\
0 & 0 & 1 & 2 \\
0 & 0 & -1 & -1
\end{pmatrix}.
\]

In the base of \([8]\), the additional points make a line with \( x_4 = -1 \).
(i) \( G_2^0 = I \big|_{G_2^0,n_T^0} \) with \( n^0 = \{0, 1, 2\} \) \( \rightarrow G_2^0 = \hat{G}_2 \big|_{G_2^0,n_T^0} \) with \( n^0 \geq 3 \) \( \rightarrow \cdots \),

\[
\{(i)^+ \} \quad (G_2^0, n_T^0) \big|_{G_2^0,n_T^0} \rightarrow \cdots.
\]

(ii) \( G_1^0 = I \big|_{G_1^0,n_T^0} \) with \( n^0 = j \) \( \rightarrow G_1^0 = \hat{G}_1 \big|_{G_1^0,n_T^0} \) with \( n^0 = j \) \( \rightarrow \cdots \),

\[
\{(ii)^+ \} \quad (G_1^0, n_T^0) \big|_{G_1^0,n_T^0} \rightarrow \cdots.
\]

(iii) \( (G_2^0 = I, n_T^0) \big|_{G_1^0=I} \) with \( n^0 = 0 \) \( \rightarrow (G_2, n_T) \big|_{G_1=I} \rightarrow \cdots \),

\[
(2)
\]

where \( \hat{G}_1 \) and \( \hat{G}_2 \) are non-Abelian gauge symmetries and \( 12 \geq j \geq 0 \), see Fig. 1 and Fig. 2. In this paper, we investigate the possibility of following sequence where we picked up CY3-folds and Hodge numbers from the list by Hosono et al. \[8\].

(iv) \( (G_2^0 = I, s = 1, n_T^0) \big|_{G_1^0=I} \) with \( n^0 = 2 \) \( \rightarrow (G_2, s = 2, n_T) \big|_{G_1=I} \rightarrow \cdots \) \[3\].

\[
(3)
\]

\( G_1^0 \) (or \( G_1 \)) : the gauge symmetry with charged matter, whose information may come from the bottom of dual polyhedron, \( \nabla \)
of appropriate K3 fiber for the series (ii) (or (ii\(^+\))) \[5\],

\( G_2^0 \) (or \( G_2 \)) : the terminal gauge symmetry with no charged matter, whose information may come from the top of dual polyhedron,

\( \nabla \) of appropriate K3 fiber for the series (i) (or (i\(^+\))) \[3\], \[4\].

For A series, the non-Abelian instanton numbers of the vector bundles on

K3 are denoted as \( (k_1^0, k_2^0) \); \( k_1^0 + k_2^0 = 24 \) \[3\]. The integer of \( n^0 \) is defined by \( k_1^0 = 12 + n^0 \) and \( k_2^0 = 12 - n^0 \) \[3\]. The suffix 0 denotes terminal case with \( n_T = 1 \) in each series, where “terminal case” means \( G_1^0 = I \) or \( G_1 = 1 \).

\[2\] Let \( m_i^0 = m_i, m_{iA}^0 = m_{iA} \) and \( m_{iB}^0 = m_{iB} \) \( (i = 1, 2) \) be Abelian instanton numbers.

For B series, non-Abelian instanton numbers and Abelian instanton numbers are denoted as \( (k_1^0, m_{1A}^0 : k_2^0, m_{2B}^0); k_1^0 + k_2^0 = 18, m_1^0 = m_2^0 = 3 \). For C series, they are denoted as \( (k_1^0, m_{1A}^0, m_{2A}^0 : k_2^0, m_{1B}^0, m_{2B}^0); k_1^0 + k_2^0 = 14, m_{1A}^0 = m_{1B}^0 = 3, m_{2A}^0 = m_{2B}^0 = 2 \).

\[3\] The number of non-Abelian instantons should be greater than three except zero.

The terminal group should change to avoid this situation, which causes a correction or a modification \[3\], \[4\]. For C series, the non-Abelian and Abelian instanton numbers are modified as follows \[3\]. In \( n^0 = 5 \) case, \( (k_1^0, m_{1A}^0, m_{1B}^0 : k_2^0, m_{2A}^0, m_{2B}^0) = (12, 3, 2; 0, 3, 3) \).

In \( n^0 = 6 \) case, \( (k_1^0, m_{1A}^0, m_{1B}^0 : k_2^0, m_{2A}^0, m_{2B}^0) = (13, 3, 2; 0, 3, 3) \).
I. When $\Delta n_T$ small instantons shrink, in the terminal gauge symmetry side, the non-Abelian instanton numbers should be modified as follows \[4, 7\]:

\[(k_1, k_2), k_1 + k_2 + \Delta n_T = 24, \quad k_1 = k_1^0 = 12, \quad k_2 = k_2^0 - \Delta n_T = 12 - \Delta n_T\]

The additional $\Delta n_T$ numbers of tensor multiplets are created and 29 $\Delta n_T$ appears in the anomaly cancellation condition \[5\]. $n_T = n_T^0 + \Delta n_T$ with $n_T^0 = 1$ is the number of the tensor multiplets in D=6 N=1 compactification \[5\].

The puzzles which we would like to consider are as follows:

I: Why (iii) and (iv) have the same Hodge numbers for A series? Do (iv) satisfy the duality? Though K3 fibers of CY3-folds in (iv) are different from those in (iii), is there any gauge enhancement for (iv)?

II: Is it possible to obtain B or C versions of (iii) which satisfy the duality by means of conifold transitions? Already, the method to construct dual polyhedron was given by \[6, 7\] about this question. They pointed out that their Hodge numbers are obtained by shrinking of tensor multiplets \[6\].

III: Is it possible to obtain B or C versions of (iv) which satisfy the duality by means of conifold transitions?

IV: Do the Hodge numbers in B or C series of (iii) coincide with those of (iv)?

V: Are there any hypersurface representing CY3-folds with $s \geq 2$ which satisfy the duality from $n^0 \geq 4$?

In section 2, we investigate prob. I. We also discuss prob. III by using the results of \[4\] and \[7\] in section 3.

\[4\] Similarly to A, for B and C, the modifications may follow
B-chain : $k_1 + m_1 = k_1^0 + m_1^0 = 12, \quad k_2 + m_2 = 12 - \Delta n_T$,
C-chain : $k_1 + m_{1A} + m_{1B} = k_1^0 + m_{1A}^0 + m_{1B}^0 = 12, \quad k_2 + m_{2A} + m_{2B} = 12 - \Delta n_T$,
which we will discuss in this article.

\[5\] 29 is the dual coxetor number of $E_8$–one \[4, 5\]. The subtraction of one means the freedom of fixing the place where a tensor multiplet is created. The dual coxetor number of $E_7$–one = 17 works for B-chain case when a tensor multiplet is created and that of $E_6$–one = 11 for the C-chain case \[4, 5\].

\[6\] In D=4 and N=2 case, these tensor multiplets become vector multiplets.
2 The duality in K3 fibered Calabi-Yau 3-fold

We treat the extension from A-chain $n^0 = 2$ terminal case. $\text{CY}_3 = \mathbb{CP}^4(1, s, (s + 1)(1, 4, 6))|12(s + 1)$ with $\text{K3} = \mathbb{CP}^3(1, 1, 4, 6)[12]$ fiber have the same Hodge numbers as those examples given by [6, 8]. To see why they coincide, we compare dual polyhedron of CY 3-folds. They are not $\text{SL}(4, \mathbb{Z})$ equivalent. We represent them in the base where one of vertices of dual polyhedron in the bottom denotes the part of its weight for the hypersurface representation. It is $(-s, -s - 1, -4(s + 1), -6(s + 1))$ in this case. The base manifold under the elliptic fibration is $F_2$ for $s = 1$ case. $F_i$ denotes Hirzebruch surface. The dual polyhedron of $F_i$ has three vertices : $\{\vec{v}_1, \vec{v}_2, -\vec{v}_1 - i\vec{v}_2\}$. An example of integral points in the dual polyhedron of $F_i$ is given by $\{\vec{v}_1 = (1, 0), (0, -1), \vec{v}_2 = (0, 1), (0, 0), (-1, -i)\}$.

The Hodge numbers and the dualities in case (iii) are derived by investigating the extremal transition of the dual polyhedron of $F_0$ based CY3-fold in [6].

By using dual polyhedra, we can find a fibrations and base manifolds in some cases [4, 7, 11, 12]. The upper suffix in $\nabla$ denotes the dimension of the lattice of a polyhedron. $\{(x_1, x_2, x_3, x_4)\} \subset 4\nabla$ forms the dual polyhedron of CY3-fold. $\{(x_1, x_2)\} \subset 2\nabla \subset 4\nabla$ represents the dual polyhedron of the base under the elliptic fibration of CY3-folds. In this paper, they are the blown up Hirzebruch surfaces. $\{(x_2, x_3, x_4)\} |_{x_1=0} \subset 3\nabla \subset 4\nabla$ represents the dual polyhedron of K3 fiber of CY3-fold. $\{(x_3, x_4)\} |_{x_1=x_2=0} \subset 2\nabla \subset 4\nabla$ are the dual polyhedron of common elliptic fiber of CY3-folds and K3fiber. However, this is not a sufficient condition of having elliptic fibration. The result of fibrations in CY3-folds is given in table 1.

In $s \geq 2$ case, the additional integral points to $F_2$ are $\{(-i, -i - 1)\}$ with $2 \leq i \leq s$, $(-1, -1)$ within the base. Thus, for $s \geq 2$ case, the base manifolds remain blown up $F_2$ since $\frac{s+1}{s} \leq 2$. We can see the Hodge number of the base under the elliptic fibration and $n_T$ from dual polyhedron, $n_T + 1 = h^{1, 1}(F_2 \text{ blown up}) = d_1 - 2d_0 = s + 2$ ($s \geq 2$), where $d_i$ is the number of i-dimensional cones [13]. $n_T$ coincides with case (iii) [6]. On the other side, K3 fibrations of case (iii) change so that the base manifolds under elliptic fibrations also alter to $F_i$ blown up ($i \geq 2$) with including $F_0$ and $F_2$. In any
Table 1: The kinds of fibrations of CY3-folds

| $\Delta n_T$ | the base under $T^2$ | K3 fiber | $s$ | the base under $T^2$ | K3 fiber |
|--------------|----------------------|----------|-----|----------------------|----------|
| 0            | $F_0$                | $\mathbb{C}P^3(1,1,4,6)[12]$ | 1   | $F_2$                | $\mathbb{C}P^3(1,1,4,6)[12]$ |
| 2            | $F_2$ blown up       | $\mathbb{C}P^3(1,1,4,6)[12]$ | 2   | $F_2$ blown up       | $\mathbb{C}P^3(1,1,4,6)[12]$ |
| 3            | $F_3$ blown up       | $\mathbb{C}P^3(1,2,6,9)[18]$ | 3   | $F_2$ blown up       | $\mathbb{C}P^3(1,1,4,6)[12]$ |
| 4            | $F_4$ blown up       | $\mathbb{C}P^3(1,2,6,9)[18]$ | 4   | $F_2$ blown up       | $\mathbb{C}P^3(1,1,4,6)[12]$ |
| 6            | $F_6$ blown up       | $\mathbb{C}P^3(1,3,8,12)[24]$ | 6   | $F_2$ blown up       | $\mathbb{C}P^3(1,1,4,6)[12]$ |
| 8            | $F_8$ blown up       | $\mathbb{C}P^3(1,4,10,15)[30]$ | 8   | $F_2$ blown up       | $\mathbb{C}P^3(1,1,4,6)[12]$ |
| 12           | $F_{12}$ blown up    | $\mathbb{C}P^3(1,5,12,18)[36]$ | 12  | $F_2$ blown up       | $\mathbb{C}P^3(1,1,4,6)[12]$ |

N.B. ; The dual polyhedron of $\mathbb{C}P^3(1,5,12,18)[36]$ coincides with that of $\mathbb{C}P^3(1,6,14,21)[42]$.

case, the bases under elliptic fibrations in case (iv) are birationally equivalent to those in case (iii), which leads to the following identification. (iii) and (iv) are connected by the extremal transitions within each sequences. $\Delta n_T = 0$ cases are deformed to each other by the change of base manifolds, i.e., $F_0 \leftrightarrow F_2$ by using a non-polynomial freedom of deformation \[6, 14\]. Both CY 3-folds with $\Delta n_T = 0$ are the same manifold with double K3 fibrations \[14\].

For $\Delta n_T > 0$, blown up $F_2$ based CY3-folds in case (iv) also can be deformed to those of case (iii) in \[8\] by using a non-polynomial freedom of deformation. $\hat{\nabla}_{s=i, \Delta n_T=i} \leftrightarrow \nabla_{\Delta n_T=i}$, where $\hat{\nabla}$ denotes dual polyhedron of case (iv) and $\nabla$ denotes those of case (iii) \[8\]. The CY 3-folds constructed by \[8\] and those with $s$ are the same manifolds with double K3 fibrations \[8\]. Each CY 3-fold with $\Delta n_T = i$ can be represented as a K3 fibration in two inequivalent ways as table 1.

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7They are represented by the same dual polyhedra in \[8\].
The CY3-fold with $s = 1$ is the terminal case of A-chains with $n^0 = 2$ in the duality web [4]. If duality exists, then their Hodge numbers must satisfy the following conditions which comes from D=6 and D=4 theories as the anomaly cancellation [1, 5, 6].

$$h^0_{2,1} |_{G_2^0 = I} \text{ in (i)} = - (a - b \Delta n_T) + \dim G_2 - 29 \Delta n_T, \quad (6)$$

$$h^0_{1,1} |_{G_2^0 = I} \text{ in (i)} = + \text{rank} G_2 + \Delta n_T,$$

$$h^0_{2,1} |_{G_2^0 = I} \text{ in (i)} = 243, \quad h^0_{1,1} |_{G_2^0 = I} \text{ in (i)} = 3.$$

where $G_1^0 = G_1 = I$ up to U(1) in these cases.

This duality between heterotic string and Type IIA string is summarized in the table 4, which coincides with case (iii) [6].

For CY3-folds side, $G_1$ and $G_2$ symmetries are due to the quotient singularities of K3 fiber in the first column of table 1 rather than the quotient singularities of CY 3-folds with $s$ in table 2.

| $s$ | $s = 1$ | $s = 2$ | $s = 3$ | $s = 4$ | $s = 6$ | $s = 8$ | $s = 12$ |
|-----|---------|---------|---------|---------|---------|---------|---------|
| quot.sing. | $A_1$ | $A_1A_2$ | $A_1^2A_3$ | $A_1^3A_4$ | $A_1^3A_6$ | $A_1^4A_8$ | $A_1^6A_{12}$ |

Table 2: The quotient singularities in K3=$\mathbb{CP}^3(1,1,4,6)$[12] fibered CY3-fold

For K3× T$^2$ side, they are the subgroups of the $E_8 \times E_8$ group for case (iii) [9]. $a + bn$ is calculated by the index theorem and denotes the number of $G_1$ (or $G_2$) charged hyper multiplet fields, which is summarized in table 3 [3, 4, 5, 7]. $n = n^0$ for $G_1^0 = \hat{G}_1$ case. The number of the charged hyper multiples of $G_2$ vanishes. We can see this by substituting $n = -n^0$ for $G_2^0 = \hat{G}_2$ case and $n = -\Delta n_T$ for $G_2 = \hat{G}_2$ for A series [1, 7].

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8 This may imply heterotic - heterotic string duality.

9 We compared some superpotentials of type IIB side in case (iii) and those in case (iv) and examined the possibility to derive the terminal gauge symmetry by the method of [13]. We will give dual polyhedra of case (iv) in the next paper to report about superpotentials more precisely.

10 They are perturbative gauge symmetries for case (iii) and may be non-perturbative ones for case (iv) as $\Delta n_T = 0$ case [4].
| \(G_i\) | A series | B series | C series |
|---|---|---|---|
| \(A_1\) | \(12n + 32\) | \(6k_1 - 39\) | \(4k_1 - 17\) |
| \(A_2\) | \(18n + 54\) | \(10k_1 - 64\) | \(6k_1 - 26\) |
| \(A_3\) | \(22n + 76\) | \(12k_1 - 75\) | \(8k_1 - 33\) |
| \(A_4\) | \(25n + 100\) | \(14k_1 - 84\) | \(9k_1 - 35\) |
| \(D_4\) | \(24n + 96\) | | |
| \(D_5\) | \(26n + 124\) | \(15k_1 - 90\) | \(10k_1 - 39\) |
| \(E_6\) | \(27n + 162\) | \(16k_1 - 94\) | | |
| \(E_7\) | \(28n + 224\) | | |

Table 3: The review of the number of the charged hyper multiplets.

| \(s\) | \(U(1)^4 \times G_2\) | \(h^{1,1}\) | \(h^{1,2}\) | \(k_1\) | \(k_2\) | \(n_T^0\) | \(\Delta n_T\) | \(n_T\) | \(n^0\) |
|---|---|---|---|---|---|---|---|---|---|
| \(s = 1\) | \(U(1)^4 \times (G_2^0 = I)\) | 3 | 243 | 12 | 12 | 1 | 0 | 1 | 2 |
| \(s = 2\) | \(U(1)^4 \times I\) | 5 | 185 | 12 | 12 - 2 | 1 | 2 | 3 |
| \(s = 3\) | \(U(1)^4 \times A_2\) | 8 | 164 | 12 | 12 - 3 | 1 | 3 | 4 |
| \(s = 4\) | \(U(1)^4 \times D_4\) | 11 | 155 | 12 | 12 - 4 | 1 | 4 | 5 |
| \(s = 6\) | \(U(1)^4 \times E_6\) | 15 | 147 | 12 | 12 - 6 | 1 | 6 | 7 |
| \(s = 8\) | \(U(1)^4 \times E_7\) | 18 | 144 | 12 | 12 - 8 | 1 | 8 | 9 |
| \(s = 12\) | \(U(1)^4 \times E_8\) | 23 | 143 | 12 | 12 - 12 | 1 | 12 | 13 |

Table 4: The duality of (iv) about \(K3=\mathbb{CP}^3(1,1,4,6)[12\) fibered CY3-fold

### 3 Discussion and Conclusion

In this article, we have studied the property of CY3-folds with \(s \geq 2\) case from A series. They are the same CY3 folds as those constructed in [1], which have double K3 fibrations.
For the puzzle III, the conclusion is that the hypersurface representations with $s \geq 2$ cases from B series and C series in $n^0 = 2$ satisfy the duality of terminal B and C series with extra tensor multiplets. For example, B and C series in $s=2$, their Hodge numbers can be interpreted as those with shrinking of two instantons from $n^0 = 2$ case as the ref. of [7]. $s \geq 3$ cases are also explained by shrinking $\Delta n_T$ instantons from the terminal case with the same $\hat{G}_2$ symmetry.

$$
\Delta h_{2,1} = -h_{2,1} \mid _{G_2^2=\hat{G}_2 \text{ terminal in (i)}} + h_{2,1} \mid _{G_2=\hat{G}_2 \text{ terminal in (iv)}},
$$

$$
\Delta h_{1,1} = -h_{1,1} \mid _{G_2^2=\hat{G}_2 \text{ terminal in (i)}} + h_{1,1} \mid _{G_2=\hat{G}_2 \text{ terminal in (iv)}},
$$

B series : $-\Delta h_{2,1} = 17\Delta h_{1,1}$, C series : $-\Delta h_{2,1} = 11\Delta h_{1,1}.$

(7)

This situation is quite similar to that of A series. They are interpreted as the dualities obtained by unhiggsing of $U(1)$ and $U(1)^2$ from (iv) in A series.

For puzzle IV, Candelas, Perevalov and Rajesh seemed to construct B and C versions of (iii) and derived the Hodge numbers already, though they do not write them explicitly. According to the description, Hodge numbers of them are the same as table 5 and table 7.

For the puzzle V, we examine K3 fiberd CY3-folds whose $s=1$ case is the terminal case in the A-chains with $n^0 = 4$. Their K3 fiber is

| $s$ | $U(1)^4 \times U(1) \times G_2$ | $h^{1,1}$ | $h^{1,2}$ | $k_1$ | $k_2$ | $n_T^0$ | $\Delta n_T$ | $n_T$ | $n^0$ |
|-----|--------------------------------|----------|----------|-------|-------|---------|-----------|-------|-------|
| 1   | $U(1)^4 \times U(1) \times (G_2^0 = I)$ | 4        | 148      | 9     | 9     | 1       | 0         | 1     | 2     |
| 2   | $U(1)^4 \times U(1) \times I$       | 6        | 114      | 9     | 9 - 2 | 1       | 2         | 3     |       |
| 3   | $U(1)^4 \times U(1) \times A_2$     | 6        | 101      | 9     | 9 - 3 | 1       | 3         | 4     |       |
| 4   | $U(1)^4 \times U(1) \times D_4$     | 9        | 96       | 9     | 9 - 4 | 1       | 4         | 5     |       |
| 6   | $U(1)^4 \times U(1) \times E_6$     | 16       | 92       | 9     | 9 - 6 | 1       | 6         | 7     |       |

Table 5: The duality of (iv) about K3=$\mathbb{CP}^3(1,1,2,4)[10]$ fiberd CY3-fold

For B and C sequences, the modification of eq. (6) is necessary. The change in the Hodge numbers, $h_{2,1}$ according to the change of the terminal group is not only the difference of dim. $G_2^0$ (or dim. $G_2$) [4, 5]. Thus we compare the Hodge numbers of CY3-folds which have the same terminal gauge symmetry.

\footnote{For B and C sequences, the modification of eq. (6) is necessary. The change in the Hodge numbers, $h_{2,1}$ according to the change of the terminal group is not only the difference of dim. $G_2^0$ (or dim. $G_2$) [4, 5]. Thus we compare the Hodge numbers of CY3-folds which have the same terminal gauge symmetry.}
Table 6: The duality of (i) about terminal CY3-folds in B series

| n^0 | U(1)^4 × U(1) × G_2^0 | h_{1,1} | h_{1,2} | k_1 | k_2 | n_T | ∆n_T | n_T |
|-----|------------------------|---------|---------|-----|-----|------|--------|------|
| n^0 = 0 | U(1)^4 × U(1) × I | 4 | 148 | 9 | 1 | 0 | 1 |
| n^0 = 3 | U(1)^4 × U(1) × A_2 | 6 | 152 | 9 + 3 | 9 - 3 | 1 | 0 | 1 |
| n^0 = 4 | U(1)^4 × U(1) × D_4 | 8 | 164 | 9 + 4 | 9 - 4 | 1 | 0 | 1 |
| n^0 = 6 | U(1)^4 × U(1) × E_6 | 10 | 194 | 9 + 6 | 9 - 6 | 1 | 0 | 1 |

Table 7: The duality of (iv) about K3=CP^3(1,1,2,2)[6] fiberd CY3-fold

| s | U(1) × U(1)^2 × G_2 | h_{1,1} | h_{1,2} | k_1 | k_2 | n_T | ∆n_T | n_T | n^0 |
|---|-------------------|---------|---------|-----|-----|------|--------|------|-----|
| s = 1 | U(1)^4 × U(1)^2 × (G_2^0 = I) | 5 | 101 | 7 | 1 | 0 | 1 | 1 | 2 |
| s = 2 | U(1)^4 × U(1)^2 × I | 7 | 79 | 7 | 7 - 2 | 1 | 2 | 3 |
| s = 3 | U(1)^4 × U(1)^2 × A_2 | 10 | 70 | 7 | 7 - 3 | 1 | 3 | 4 |
| s = 4 | U(1)^4 × U(1)^2 × D_5 | 13 | 67 | 7 | 7 - 4 | 1 | 4 | 5 |
| s = 5 | U(1)^4 × U(1)^2 × E_6 | 17 | 65 | 7 | 0 | 1 | 5 | 6 |
| s = 6 | U(1)^4 × U(1)^2 × E_6 | 17 | 65 | 7 | 0 | 1 | 6 | 7 |

K3=CP^3(1,2,6,9)[18]. The dual polyhedron of s = 2 case has following additional points (−2, −6, −18, −27), (−1, −3, −10, −15) and (−1, −3, −9, −14), which imply ∆n_T=1. h_{1,1} |_{s=2} − h_{1,1}^0 |_{s=1} = 3. The terminal group should change from D_4 to G_2 with rank 6. We can not find G_2 which satisfies the extension of eq. (6). It seems that it needs another idea to see the duality in this case.

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Table 8: The duality of (i) about terminal CY3-folds in C series

| $n^0$ | $U(1) \times U(1)^2 \times G_2^0$ | $h^{1,1}$ | $h^{1,2}$ | $k_1$ | $k_2$ | $n_T^{\text{II}}$ | $\Delta n_T$ | $n_T$ |
|-------|----------------------------------|----------|----------|-------|-------|----------------|-------------|-------|
| $n^0 = 0$ | $U(1)^4 \times U(1)^2 \times I$ | 5        | 101      | 7     | 7     | 1              | 0           | 1     |
| $n^0 = 3$ | $U(1)^4 \times U(1)^2 \times A_2$ | 7        | 103      | 7 + 3 | 7 - 3 | 1              | 0           | 1     |
| $n^0 = 4$ | $U(1)^4 \times U(1)^2 \times D_5$ | 10       | 110      | 7 + 4 | 7 - 4 | 1              | 0           | 1     |
| $n^0 = 5$ | $U(1)^4 \times U(1)^2 \times E_6$ | 12       | 120      | 7 + 5 |    0  | 1              | 0           | 1     |
| $n^0 = 6$ | $U(1)^4 \times U(1)^2 \times E_6$ | 11       | 131      | 7 + 6 |    0  | 1              | 0           | 1     |

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| $G_2^0 \setminus G_1^0$ | $G_1^0$ | (ii) | $G_1^0 = \hat{G}_1$ |
|------------------------|---------|------|------------------|
| $G_2^0 = I$            | $I \times I$ | $\cdots$ | $G_1 \times I$ |
| $(n^0 = 0, 1, 2)$      | $\cdots$ | $\cdots$ | $\cdots$ |
| ↓ (i)                  | $\cdots$ | $\cdots$ | $\cdots$ |
| $G_2^0 = \hat{G}_2$    | $I \times \hat{G}_2$ | $\cdots$ | $\hat{G}_1 \times \hat{G}_2$ |
| $(12 \geq n^0 \geq 3)$| $\cdots$ | $\cdots$ | $\cdots$ |

Fig. 1 The gauge symmetries from (i) and (ii)

| $G_2^0$ or $G_2^d \setminus G_1^0$ or $G_1^d$ | $G_1^0 = G_1 = I$ | |
|-----------------------------------------------|-----------------------------------------------|---|
| $G_2^0 = I$                                  | $G_1^0 \times G_2^0 = I \times I$           | . |
| $(n^0 = 0, \Delta n_T = 0)$                  |                                              | . |
| ↓ (iii)                                      |                                              | . |
| $G_2 = \hat{G}_2$                            |                                              | . |
| $(\Delta n_T \geq 3)$                       | $G_1 \times G_2 = I \times \hat{G}_2$       | . |

Fig. 2 The gauge symmetries from (iii)