Abstract. For a ring $A$, the following conditions are equivalent.
1) $A$ is a right strongly semiprime ring.
2) Every right $A$-module which is injective with respect to some essential right ideal of the ring $A$, is an injective module;
3) Every quasi-injective right $A$-module which is injective with respect to some essential right ideal of the ring $A$ is an injective module.

The study is supported by Russian Scientific Foundation (project 16-11-10013).

Key words: injective module; strongly semiprime ring; quasi-injective module

1. Introduction and preliminaries

All rings are assumed to be associative and with zero identity element; all modules are unitary and, unless otherwise specified, all modules are right modules. This paper is a continuation of [7].

For a module $Y$, a module $X$ is said to be injective with respect to $Y$ or $Y$-injective if for each submodule $Y_1$ of $Y$, every homomorphism $Y_1 \to X$ can be extended to a homomorphism $Y \to X$. A module is said to be injective if it is injective with respect to each module. A module $X$ is said to be quasi-injective if $X$ is injective with respect to $X$. Every finite cyclic group is a quasi-injective non-injective module over the ring of integers $\mathbb{Z}$.

Remark 1.1. The following Baer’s criterion is well-known: if $A$ is a ring and $X$ is a right $A$-module, then $X$ is injective if and only if $X$ is injective with respect to some non-zero right ideal of the ring $A$.

A ring $A$ is said to be right strongly semiprime [2] if each ideal of $A$ which is an essential right ideal contains a finite subset with zero right annihilator. A ring $A$ is said to be right strongly prime [3] if every non-zero ideal of $A$ contains a finite subset with zero right annihilator. It is clear that every right strongly prime ring is right strongly semiprime. The direct product of two finite fields is a finite commutative strongly semiprime ring which is not strongly prime.

Remark 1.2. Let $A$ be a right strongly prime ring and let $X$ be a right $A$-module. In [7], it is proved that $X$ is injective if and only if $X$ is injective with respect to some non-zero right ideal of the ring $A$.

For a module $X$, a submodule $Y$ of $X$ is said to be essential in $X$ if $Y \cap Z \neq 0$ for each non-zero submodule $Z$ of $X$. A right module $X$ over the ring $A$ is said to be non-singular if the right annihilator $r(x)$ of any non-zero element $x \in X$ is not essential right ideal of the ring $A$. For a module $X$, we denote by $G(X)$ or $\text{Sing}_2 X$ the intersection of all submodules $Y$ of the module $X$ such that the factor module $X/Y$ is non-singular. The submodule $G(X)$ is a a fully

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\footnote{For example, see [6, Proposition 10.3].}
invariant submodule of $X$ and is called the Goldie radical or the second singular submodule of the module $X$.

In connection to Remark 1.1 and Remark 1.2, we prove Theorem 1.3 and Theorem 1.4 which are the main results of this paper.

**Theorem 1.3.** For a given ring $A$ with right Goldie radical $G(A_A)$, the following conditions are equivalent.

1) Every non-singular right $A$-module $X$ which is injective with respect to some essential right ideal of the ring $A$ is an injective module.

2) $A/G(A_A)$ is a right strongly semiprime ring.

**Theorem 1.4.** For a given ring $A$, the following conditions are equivalent.

1) $A$ is a right strongly semiprime ring;

2) Every right $A$-module which is injective with respect to some essential right ideal of the ring $A$, is an injective module and $A$ is right non-singular.

**Remark 1.5.** In connection to Theorem 1.3 and Theorem 1.4, we note that there exist a finite commutative ring $A$, an essential ideal $B$ of the ring $A$, and a non-injective $B$-injective $A$-module $X$. We denote by $A$, $B$ and $X$ the finite commutative ring $\mathbb{Z}/4\mathbb{Z}$, the ideal $2\mathbb{Z}/4\mathbb{Z}$ and the module $B_A$, respectively. Then $B$ is an essential ideal and the module $X$ is injective with respect to $B_A$. Since $X$ is not a direct summand of $A_A$, the module $X$ is not injective.

**Remark 1.6.** A ring without non-zero nilpotent ideals is called a semiprime ring. Every right strongly semiprime ring is a right non-singular semiprime ring [2]. The direct product of infinitely many fields is an example of a commutative semiprime non-singular ring which is not strongly semiprime. All finite direct products of rings without zero-divisors and all finite direct products of simple rings are right and left strongly semiprime rings.

**Remark 1.7.** A ring $A$ is called a right Goldie ring if $A$ is a ring with the maximum condition on right annihilators which does not contain the direct sum of infinitely many non-zero right ideals. If $A$ is a semiprime right Goldie ring, then it is well known [3] that every essential right ideal of the ring $A$ contains a non-zero-divisor. Therefore, all semiprime right Goldie rings are right strongly semiprime. In particular, all right Noetherian semiprime rings are right strongly semiprime.

We denote by $\text{Sing} X$ the singular submodule of the right $A$-module $X$, that is $\text{Sing} X$ is the fully invariant submodule of $X$ consisting of all elements $x \in X$ such that $r(x)$ is an essential right ideal of the ring $A$. A module $X$ is said to be singular if $X = \text{Sing} X$. A module $X$ is called a Goldie-radical module if $X = G(X)$. The relation $G(X) = 0$ is equivalent to the property that the module $M$ is non-singular. We use well known properties of $\text{Sing} X$ and $G(X)$; e.g., see [1]. A submodule $Y$ of the module $X$ is said to be closed in $X$ if $Y = Y'$ for each submodule $Y'$ of $X$ which is an essential extension of the module $Y$.

[3] For example, see [6, Theorem 3.2.14].
2. The proof of Theorem 1.3 and Theorem 1.4

The proof of Theorem 1.3 and Theorem 1.4 is decomposed into a series of assertions, some of which are of independent interest.

Lemma 2.1. Let $A$ be a ring and let $X$ be a right $A$-module.

1. If $B$ is a right ideal of the ring $A$ and the module $X$ is injective with respect to the module $B_A$, then $X$ is injective with respect to the module $(AB)_A$, where $AB$ is the ideal generated by the right ideal $B$. In addition, if the ideal $AB$ contains a finite subset $C$ with $r(C) = 0$, then the module $X$ is injective.

2. If $X \neq G(X)$, then there exists non-zero right ideal $B$ of the ring $A$ such that the module $B_A$ is isomorphic to a submodule of the module $X$.

3. Let $Y$ be a right $A$-module, $\{Y_i\}_{i \in I}$ be some set of right modules such that the module $X$ is injective with respect to $Y_i$ for each $i \in I$ and let $\{f_i \in \text{Hom}(Y_i, Y)\}_{i \in I}$ be some set of homomorphisms. Then the module $X$ is injective with respect to the submodule $\sum_{i \in I} f_i(Y_i)$ of the module $Y$. In addition, if there exists a monomorphism $A_A \rightarrow \sum_{i \in I} f_i(Y_i)$, then the module $X$ is injective.

4. Let $Y$ be a right $A$-module and let $\{Y_i\}_{i \in I}$ be some set of submodules of the module $Y$ such that the module $X$ is injective with respect to $Y_i$ for each $i \in I$. Then the module $X$ is injective with respect to the submodule $\sum_{i \in I} Y_i$ of the module $Y$. In addition, if there exists a monomorphism $A_A \rightarrow \sum_{i \in I} Y_i$, then the module $X$ is injective.

Proof. 1, 2, 3. The assertions are proved in Lemma 3, Lemma 4 and Lemma 2.2 from [7], respectively.

4. The assertion follows from 3 if we denote by $f_i$ the natural embeddings $Y_i \rightarrow Y$. □

Lemma 2.2. Let $A$ be a ring, $X$ be a non-singular non-zero right $A$-module, $\{C_i \mid i \in I\}$ be the set of all non-zero right ideals of the ring $A$ such that every non-zero submodule of the $A$-module $C_i$ is not isomorphic to a submodule of the module $X$, and let $\{D_j \mid j \in J\}$ be the set of all non-zero right ideals $D_j$ of the ring $A$ such that $D_j$ is isomorphic to a submodule of the module $X$. We set $C = \sum_{i \in I} C_i$, $D = \sum_{j \in J} D_j$, and $B = C + D$.

1. For any submodule $C'$ of the module $C_A$, every homomorphism $f : C_A \rightarrow X$ is the zero homomorphism.

2. The module $X$ is injective with respect to the module $C_A$.

3. $B$ is an essential right ideal of the ring $A$.

4. If the module $X$ is quasi-injective, then $X$ is injective with respect to the essential right ideal $B$.

Proof. 1. Let us assume that $f \neq 0$. Since $X$ is a non-singular module and $C'/\text{Ker} f \cong f(C') \subseteq X$, we have that $\text{Ker} f$ is not an essential submodule of
There exists a non-zero element $c \in C'$ with $cA \cap \text{Ker} \ f = 0$. Then the non-zero submodule $cA$ of the module $C'$ is isomorphic to the non-zero submodule $f(cA)$ of the module $X$. Therefore, $f(c) \neq 0$. There exists a finite subset $K$ in $I$ such that $c = \sum_{k \in K} c_k$ and $c_k \in C_k$ for all $k \in K$. Since $f(c) \neq 0$, we have that $f(c_k) \neq 0$ for some $k \in K \subseteq I$. Therefore, $c_k A$ is a non-zero submodule of the $A$-module $C_k$ which is isomorphic to a non-zero submodule of the module $X$. This contradicts to the property that $C_k \in \{C_i \mid i \in I\}$.

2. The assertion follows from 1.

3. Let us assume that $B$ is not an essential right ideal. Then $B \cap E = 0$ for some non-zero right ideal $E$. Then $C \cap E = 0$ and $D \cap E = 0$. Since $C \cap E = 0$, we have that $E \notin \{C_i \mid i \in I\}$. Therefore, there exists a non-zero submodule $E_1$ of the module $E$ which is isomorphic to a submodule of the module $X$. Then $E_1 \in \{D_j \mid j \in J\}$. Therefore, $E_1 \subseteq D \cap E = 0$. This is a contradiction.

4. Since $X$ is a quasi-injective module, $X$ is injective with respect to any module which is isomorphic to a submodule of the module $X$. Therefore, $X$ is injective with respect to each of the $A$-module $D_i$. By Lemma 2.1(4), the module $X$ is injective with respect to the module $D_A$. In addition, $X$ is injective with respect to the module $C_A$ by 2. By Lemma 2.1(4), the module $X$ is injective with respect to the module $C + D = B$. □

**Proposition 2.3.** Let $A$ be a right strongly semiprime ring and $X$ be a right $A$-module. If there exists an essential right ideal $B$ of the ring $A$ such that $X$ is injective with respect to the module $B_A$, then $X$ is an injective module.

**Proof.** By Lemma 2.1(1), $X$ is injective with respect to the module $(AB)_A$, where $AB$ is the ideal generated by the right ideal $B$. Since $B$ is an essential right ideal and $B \subseteq AB$, the ideal $AB$ is an essential right ideal. Since $A$ is a right strongly semiprime ring, the ideal $AB$ contains a finite subset $K = \{k_1, \ldots, k_n\}$ with zero right annihilator $r(K)$. Since $r(K) = r(k_1) \cap \ldots \cap r(k_n) = 0$, the module $A_k$ is isomorphic to a submodule of the direct sum of $n$ copies of the module $(AB)_A$. In addition, module $X$ is injective with respect to the module $(AB)_A$. By Lemma 2.1(3), the module $X$ is injective. □

For completeness, we briefly prove the following familiar lemma.

1. If $B$ is an essential right ideal of the ring $A$, then $h(B)$ is an essential right ideal of the ring $h(A)$.

2. If $B$ is a right ideal of the ring $A$ such that $G \subseteq B$ and $h(B)$ is an essential right ideal of the ring $h(A)$, then $B$ is an essential right ideal of the ring $A$.

3. $MG \subseteq G(M)$ for each right $A$-module $M$.

4. $XG = 0$ and a natural $h(A)$-module $X$ is non-singular. In addition, if $Y$ be an arbitrary non-singular right $A$-module, then $YG = 0$ and the $h(A)$-module homomorphisms $Y \to X$ coincide with the $A$-module homomorphisms $Y \to X$. Therefore, $X$ is an $Y$-injective $A$-module if and only if $X$ is an $Y$-injective $h(A)$-module.

5. $X$ is an injective $h(A)$-module if and only if $X$ is an injective $A$-module.
6. $X_{h(A)}$ is an essential extension of a direct sum of uniform modules if and only if $X_A$ is an essential extension of a direct sum of uniform modules.

**Proof.** 1. Let us assume that $h(B)$ is not an essential right ideal of the ring $h(A)$. Then there exists a right ideal $C$ of the ring $A$ such that $C$ properly contains $G$ and $h(B) \cap h(C) = h(0)$. Since $h(B) \cap h(C) = h(0)$, we have $B \cap C \subseteq G$. Since $C$ properly contains the closed right ideal $G$, then $C_A$ contains a non-zero submodule $D$ with $D \cap G = 0$. Since $B$ is an essential right ideal, $B \cap D \neq 0$; in addition, $(B \cap D) \cap G = 0$. Thus $h(0) \neq h(B \cap D) \subseteq h(B) \cap h(C) = h(0)$. This is a contradiction.

2. Let us assume that $B$ is not an essential right ideal of the ring $A$. Then $B \cap C = 0$ for some non-zero right ideal $C$ of the ring $A$ and $G \cap C \subseteq B \cap C = 0$. Therefore, $h(C) \neq h(0)$. Since $h(B)$ is an essential right ideal of the ring $h(A)$, we have $h(B) \cap h(C) \neq h(0)$. Let $h(0) \neq h(b) = h(c) \in h(B) \cap h(C)$, where $b \in B$ and $c \in C$. Then $c - b \in G \subseteq B$. Therefore, $c \in B \cap C = 0$, whence we have $h(c) = h(0)$. This is a contradiction.

3. For any element $m \in M$, the module $mG_A$ is a Goldie-radical module, since $mG_A$ is a homomorphic image of the Goldie radical module $G$. Therefore, $mG \subseteq G(M)$ and $MG \subseteq G(M)$.

4. By 3, $XG = 0$. Let us assume that $x \in X$ and $xb(B) = 0$ for some essential right ideal $h(B)$, where $B = h^{-1}(h(B))$ is the complete pre-image of $h(B)$ in the ring $A$. By 2, $B$ is an essential right ideal of the ring $A$. Then $xB = 0$ and $x \in \text{Sing } X = 0$. Therefore, $X$ is a non-singular $h(A)$-module. The remaining part of 4 is directly verified.

5. Let $R$ be one of the rings $A$, $h(A)$ and let $M$ be a right $R$-module. By Lemma 2.1(4), the module $M$ is injective if and only if $M$ is injective with respect to the module $R_R$. Now the assertion follows from 4.

6. The assertion follows from 4.

**Proposition 2.5.** Let $A$ be a ring and let $G = A/G(A_A)$. The following conditions are equivalent.

1) Every non-singular right $A$-module $X$ which is is injective with respect to some essential right ideal of the ring $A$ is an injective module.

2) Every quasi-injective non-singular right $A$-module $X$ which is is injective with respect to some essential right ideal of the ring $A$ is an injective module.

3) Every quasi-injective non-singular right $A$-module is an injective module.

4) $A/G(A_A)$ is a right strongly semiprime ring.

**Proof.** The implication 1) $\Rightarrow$ 2) is obvious.

The implication 2) $\Rightarrow$ 3) follows from Lemma 2.2(4).

The equivalence of 3) and 4) is proved in [4].

4) $\Rightarrow$ 1). Let $R$ be one of the rings $A$, $A/G(A_A)$ and let $M$ be a right $R$-module. By Lemma 2.1(4), the module $M$ is injective if and only if $M$ is injective with respect to the module $R_R$. 

Let \( h: A \to A/G \) be a natural ring epimorphism and let \( X \) be a non-singular right \( A \)-module which is injective with respect to some essential right ideal \( B \) of the ring \( A \). By Lemma 2.4(4), \( XG = 0 \) and \( X \) is a natural non-singular \( h(A) \)-module. By Lemma 2.4(1), \( h(B) \) is an essential ideal of the ring \( h(A) \). By Lemma 2.4(4), the module \( X \) is injective with respect to \( h(B) \). By Proposition 2.3, \( X \) is an injective \( h(A) \)-module. By Lemma 2.4(5), \( X \) is an injective \( A \)-module. \( \square \)

**Remark 2.6.** The completion of the proof of Theorem 1.3 and Theorem 1.4. Theorem 1.3 follows from Proposition 2.5. Theorem 1.4 follows from Proposition 2.3 and the property that every right strongly semiprime ring is right non-singular [2].

**References**

[1] Goodearl K.R. Ring Theory. Non-singular Rings and Modules. – Marcel Dekker, New York, 1976.

[2] Handelman D. Strongly semiprime rings // Pacific J. Math. – 1975. – Vol. 211. – P. 209-223.

[3] Handelman D., Lawrence J. Strongly prime rings // Trans. Amer. Math. Soc. – 1975. – Vol. 211. – P. 209-223.

[4] Kutami M., Oshiro K. Strongly semiprime rings and non-singular quasi-injective modules // Osaka J. Math. – 1980. – Vol. 17. – P. 41-50.

[5] Lawrence J. A singular primitive ring // Trans. Amer. Math. Soc. – 1974. – Vol. 45, no. 1. – P. 59-62.

[6] Rowen L.H. Ring Theory. Volume I. – Academic Press, Boston, 1988.

[7] Tuganbaev A.A. Characteristic submodules of injective modules over strongly prime rings // Discrete Math. Appl. – 2014. – Vol. 24, no. 4. – P. 253–256.

[8] Wisbauer R. Foundations of Module and Ring Theory. – Gordon and Breach, Philadelphia. – 1991.