$L^2$ estimates for the eigenfunctions corresponding to real eigenvalues of the Tricomi operator

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Abstract. For the real eigenvalues of the Tricomi operator we provide $L^2$ estimates for the corresponding eigenfunctions. In particular, provided that the elliptic boundary arc of the underlying domain $\Omega$ is the normal Tricomi curve, our result exhibits a dependence of these estimates on the length of the parabolic segment of $\Omega$.

Keywords: Spectral theory; Tricomi operator; $L^2$ eigenfunction bounds.

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1 Introduction

In this paper we deal with the problem of establishing $L^2$ estimates for the eigenfunctions corresponding to real eigenvalues of the Tricomi problem, i.e. the nontrivial solutions to

$$\begin{cases}
Tu = \lambda u & \text{in } \Omega, \\
u = 0 & \text{on } AC \cup \sigma, \\
\lambda \in \mathbb{R},
\end{cases}$$

(1.1)

where $T = -y \partial_x^2 - \partial_y^2$ is the Tricomi operator on $\mathbb{R}^2$. Here $\Omega$ is a Tricomi domain; that is a simply connected bounded region of the plane whose boundary $\partial \Omega$ consists of the elliptic arc $\sigma$ joining $A = (2x_0, 0), x_0 < 0$, to $B = (0, 0)$ in the region $y > 0$ and the two characteristics $AC$ and $BC$ for $T$ which lie in the half-plane $y \leq 0$ and meet at the point $C = (x_0, y_C)$. $y_C < 0$ (cf. Section 2 for a precise description).

Due to its physical importance, which derives from its relations with the theory of two-dimensional transonic fluid flows first observed in [9], the literature concerning the question of the unique solvability and the research of the Green’s function for the underlying Tricomi problem (1.1), with $\lambda u$ being replaced by $h \in L^2(\Omega)$, is nowadays very wide; see, for instance, the papers [1], [3]–[5], [7], [11], [15], [24] and the references therein.

On the contrary, only in quite recent times there has been a growing interest towards a development of a clear spectral theory for the Tricomi operator; an interest mainly motivated by the perspectives of making substantial progresses in the study of associated nonlinear problems, (cf. [10], [16], [17], [20] and [21]). The main results in this direction are probably those in [16] and [17] where, provided that $\Omega$ is normal in the sense that the elliptic arc $\sigma$ is perpendicular to the $x$-axis at the boundary points $A$ and $B$, it is shown that a principal eigenvalue $\lambda_0 > 0$ exists such that all the other real eigenvalues, if

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any, belong to \((\lambda_0, +\infty)\). Employing the linear solvability theory combined with nonlinear analysis tools, such a spectral information is then exploited in [17] to derive existence and uniqueness for semilinear Tricomi problems.

Differently from [17], it is our aim, here, to use the informations on the real spectrum of \(T\) to show that, if \(\sigma\) is given explicitly by the normal Tricomi curve (cf. (3.3)) and if \(u\) is an eigenfunction corresponding to \(\lambda \in [\lambda_0, +\infty)\) enjoying some further regularity on the subset \(\gamma_1 \cup \gamma_2 = BC \cup \sigma\) of \(\partial \Omega\), then the norm \(\|u\|_{L^2(\Omega)}\) is bounded by \(\lambda^{-1/2}\) times a quantity depending on \(\|u\|_{L^2(BC)}\), \(\|y^{1/2}u_x\|_{L^2(\gamma_j)}\), \(\|u_y\|_{L^2(\gamma_j)}\), \(j = 1, 2\), and \(x_0\). Our \(L^2\) eigenfunction bounds come out from an application to problem (1.1) of the Pohožaev-type identity derived in [18] for the more general semilinear problem

\[
\begin{aligned}
Tu = f(u) & \quad \text{in } \Omega, \\
u = 0 & \quad \text{on } AC \cup \sigma,
\end{aligned}
\]

where \(f \in C^0(\mathbb{R})\), and then estimating the right-hand side of such identity taking advantage from the fact that \(\sigma\) is explicitly given by the normal Tricomi curve. We stress that this choice for \(\sigma\) is motivated by two essential reasons. At first, it makes \(\Omega\) a concrete example of a domain star-shaped with respect to the vector field \(D = -3x \partial_x - 2y \partial_y\), a notion introduced in [18] only from an abstract point of view. In a certain sense, this also shows that the initial intuition of Tricomi of considering such a curve as the elliptic part of \(\partial \Omega\) (cf. [25]) was correct, even though he was unaware of the notion of \(D\)-star-shapedness. Secondly, it allows us to compute exactly the unit outer normal to \(\sigma\) entering the right-hand side of the quoted Pohožaev identity and hence to derive explicit formulae for the constants depending on \(x_0\) in our estimates. In particular, our computations exhibit the unexpected fact that the value \(x_0 = -\sqrt{3}/4\) plays the role of ‘critical” value, in the sense that our constants change according to the fact that the parabolic diameter \(2|x_0|\) of \(\Omega\) is greater or not than the value \(\sqrt{3}/2\).

It is also worth to observe that \(L^2\)-bounds for spectral projections onto eigenspaces, as those derived in [12] and [13] for the twisted Laplacian and the Hermite operator, respectively, are still lacking for the Tricomi operator. It thus seems to us that our \(L^2\) eigenfunction bounds may represent a first step in this direction.

The article is organized as follows. In Section 2 we define the weighted Sobolev spaces \(\tilde{W}^{1,2}_{AC,\mathfrak{f}}(\Omega)\) and we give an overview of the linear solvability theory for the Tricomi problem developed in [15] when \(\Omega\) is a normal Tricomi domain. This yields also to recall the before mentioned results of spectral theory for the Tricomi operator established in [10] and [17].

Section 3 is devoted to introduce the notion of \(D\)-star-shapedness, \(D = -3x \partial_x - 2y \partial_y\), and the Pohožaev identity of [18] for the semilinear problem (1.2). Moreover, recalling the basic symmetries groups that generate conservation laws for problem (1.2) we are naturally led to define the normal Tricomi curve, which constitutes the elliptic boundary arc \(\sigma\) of the domain \(\Omega\) underlying our main result.

Section 4 is the core of the paper. First, in Lemma 4.1 we show that when \(\partial \Omega = AC \cup BC \cup \sigma\), \(\sigma\) being the normal Tricomi curve, then \(\Omega\) is \(D\)-star-shaped in the sense of Section 3. We then prove the three preliminary Lemmas 4.2, 4.3 and 4.10 which supply estimates for the line integrals on the right-hand side of the Pohožaev identity. Finally, combining the quoted lemmas with the fact that when \(f(u) = \lambda u\), \(\lambda \in [\lambda_0, +\infty)\), the left-hand side of the Pohožaev identity reduces to \(4\lambda\|u\|_{L^2(\Omega)}^2\), in Theorem 4.11 we prove
our $L^2$ eigenfunctions bounds.

The paper concludes with Section 5 where we give the proofs of Lemma 4.5 and Corollaries 4.6 and 4.8 which are basic for proving Lemma 4.10. In particular, Lemma 4.5 provides the necessary estimates on the modulus of the normal vector to $\sigma$ (cf. (4.7) and (4.22)) and highlights their dependence on the length of the parabolic segment of $\Omega$. Such estimates are then used in Corollaries 4.6 and 4.8 to deduce upper and lower bounds of two functions entering the proof of Lemma 4.10. Notice that, although the functions involved in the quoted corollaries depend on one single real variable, they both elude the standard methods of calculus for finding greatest and least values, due to the computational difficulty of locating their stationary points (see Remarks 5.2 and 5.3).

2 The Tricomi problem

The Tricomi operator $T$ in two independent variable $x$ and $y$ is the second order linear partial differential operator

$$T = -y \partial_x^2 - \partial_y^2, \quad (2.1)$$

which is elliptic in the half-plane $y > 0$, parabolic along the $x$ axis, and hyperbolic in half-plane $y < 0$. A subset $\Omega \subset \mathbb{R}^2$ is said a Tricomi domain for $T$ if $\Omega$ is an open, bounded, simply connected set of $\mathbb{R}^2$ with $C^1$ piecewise boundary $\partial \Omega = AC \cup BC \cup \sigma$, where $AC$ and $BC$ are the characteristic of negative and positive slopes respectively issuing from the points $A = (2x_0, 0)$ and $B = (0, 0)$, $x_0 < 0$, and meeting at the point $C = (x_C, y_C) = (x_0, -3|x_0|/2)^{2/3}$. Due to the parabolic character of $T$ along the $x$ axis, the segment $AB = \{(x, y) \in \mathbb{R}^2 : 2x_0 < x < 0, y = 0\}$ is called the parabolic segment of $\Omega$, and its length $|AB| = 2|x_0|$ is called the parabolic diameter of $\Omega$.

For a connected subset $\Gamma$ of $\partial \Omega$ consider the following spaces of smooth real valued functions

$$C^\infty_{0, \Gamma}(\Omega; \mathbb{R}) = \{ \psi \in C^\infty(\overline{\Omega}; \mathbb{R}) : \psi \equiv 0 \text{ on } N_\epsilon \Gamma \text{ for some } \epsilon > 0 \},$$

where $N_\epsilon \Gamma$ is an $\epsilon$ neighborhood of $\Gamma$ and $C^\infty(\overline{\Omega}; \mathbb{R})$ denotes the set of all functions from $\Omega$ to $\mathbb{R}$ whose derivatives of any order are continuous in $\Omega$ and admit continuous extension up to the boundary $\partial \Omega$. To simplify notations, from now on, we shall always write $C^\infty_{0,T}(\Omega)$ in place of $C^\infty_{0,\Gamma}(\Omega; \mathbb{R})$. Then, denote by $\tilde{W}^{1,2}_1(\Omega)$ the weighted Sobolev space obtained as closure of $C^\infty_{0,T}(\Omega)$ with respect to the norm

$$\|\psi\|^{2}_{\tilde{W}^{1,2}_1(\Omega)} = \|\psi\|^{2}_{\tilde{W}^{1,2}(\Omega)} = \int_{\Omega} (|\psi|^2_x + |\psi|^2_y + \psi^2) \, dx \, dy.$$
Finally, the dual space \( \tilde{W}^{-1}_r(\Omega) \) of \( \tilde{W}^1_1(\Omega) \) is characterized as the norm closure of \( L^2(\Omega) \) with respect to the norm \( \|w\|_{\tilde{W}^{-1}_1(\Omega)} = \sup_{\|\psi\|_{\tilde{W}^1_1(\Omega)}=1} |(w, \psi)_2| \), where \( (\cdot, \cdot)_2 \) is the standard inner (real) product of \( L^2(\Omega) \). Obviously, \( \tilde{W}^1_1(\Omega) \subset L^2(\Omega) \subset \tilde{W}^{-1}_r(\Omega) \). Moreover (cf. [15, p. 538]), using the definition of the \( \tilde{W}^{-1}_r(\Omega) \)-norm it is easy to show that there exist positive constants \( c_j \), \( j = 1, 2 \), such that

\[
\|Tu\|_{\tilde{W}^{-1}_{BCU\sigma}(\Omega)} \leq c_1 \|u\|_{\tilde{W}^1_{ACU\sigma}(\Omega)}, \quad \forall u \in C^\infty_{0,ACU\sigma}(\Omega),
\]

\[
\|Tv\|_{\tilde{W}^{-1}_{ACU\sigma}(\Omega)} \leq c_2 \|v\|_{\tilde{W}^1_{BCU\sigma}(\Omega)}, \quad \forall v \in C^\infty_{0,BCU\sigma}(\Omega).
\]

The continuity estimates (2.4) and (2.5) give rise to the continuous extensions

\[
\tilde{T}_{ACU\sigma} : \tilde{W}^1_{ACU\sigma}(\Omega) \to \tilde{W}^{-1}_{BCU\sigma}(\Omega) \quad \text{and} \quad \tilde{T}_{BCU\sigma} : \tilde{W}^1_{BCU\sigma}(\Omega) \to \tilde{W}^{-1}_{ACU\sigma}(\Omega)
\]

(2.6) of the Tricomi operator \( T \) defined on the dense subspaces \( C^\infty_{0,ACU\sigma}(\Omega) \) and \( C^\infty_{0,BCU\sigma}(\Omega) \). Notice that, by denoting with \( (\tilde{T}_{ACU\sigma})^* \) and \( (\tilde{T}_{BCU\sigma})^* \) the adjoint operators of \( \tilde{T}_{ACU\sigma} \) and \( \tilde{T}_{BCU\sigma} \), respectively, from (2.6) we deduce \( (\tilde{T}_{ACU\sigma})^* = \tilde{T}_{BCU\sigma} \) and \( (\tilde{T}_{BCU\sigma})^* = \tilde{T}_{ACU\sigma} \). This implies that the problems

\[
(\text{LT}) : \begin{cases}
Tu = h & \text{in } \Omega,
\end{cases}
\]

\[
\begin{cases}
u = 0 & \text{on } \sigma, \quad \text{and} \quad (\text{LT})^* : \begin{cases}
Tv = h & \text{in } \Omega,
\end{cases}
\]

where \( h \in L^2(\Omega) \), are adjoint to each other, but they are not self-adjoint. Then, from now on, to simplify notation, we shall consider only the problem (LT). In fact, due to the adjoint character of (LT) and (LT)*, in what follows it will be suffice to replace the pair \((AC \cup \sigma, BC \cup \sigma)\) with \((BC \cup \sigma, AC \cup \sigma)\) in all the statements concerning problem (LT) for having analogous statements for problem (LT)*.

**Definition 2.1.** A function \( u \in \tilde{W}^1_{ACU\sigma}(\Omega) \) is called a generalized solution to problem (LT) if there exists a sequence \( \{u_n\}_{n \in \mathbb{N}} \subset C^\infty_{0,ACU\sigma}(\Omega) \) such that \( \|u_n - u\|_{\tilde{W}^1_{ACU\sigma}(\Omega)} \to 0 \) and \( \|Tu_n - h\|_{\tilde{W}^{-1}_{BCU\sigma}(\Omega)} \to 0 \) as \( n \to \infty \).

As shown first in [7], a necessary and sufficient condition in order to have generalized solvability of (LT) for every \( h \in L^2(\Omega) \) is to have the continuity estimates (2.4) and (2.5) as well as both the following \textit{a priori} estimates, for some positive constants \( c_j \), \( j = 3, 4 \):

\[
\|u\|_{L^2(\Omega)} \leq c_3 \|Tu\|_{\tilde{W}^{-1}_{BCU\sigma}(\Omega)}, \quad \forall u \in C^\infty_{0,ACU\sigma}(\Omega),
\]

\[
\|v\|_{L^2(\Omega)} \leq c_4 \|Tv\|_{\tilde{W}^{-1}_{ACU\sigma}(\Omega)}, \quad \forall v \in C^\infty_{0,BCU\sigma}(\Omega).
\]

Precisely, (2.8) provides the existence of a generalized solution to problem (LT) whereas (2.7) guarantees that the solution is unique. For this reason, we say that a Tricomi domain \( \Omega \) is \textit{admissible} if (2.7) and (2.8) hold. Observe also that (2.7) and (2.8) are in accordance with the result in [22] (see also [7, p. 11]) concerning the validity of a priori estimates for operators of mixed type. That is, if an inequality with a step in smoothness of two units such as \( \|\psi\|_{\tilde{W}^{-1}_{ACU\sigma}(\Omega)} \leq c_5 \|T\psi\|_{\tilde{W}^{-1}_{ACU\sigma}(\Omega)} \) would hold for every \( \psi \in C^\infty_{0,ACU\sigma}(\Omega) \), then \( T \) would be elliptic in \( \Omega \).
The class of admissible domains includes normal Tricomi domains whose elliptic boundary arc $\sigma$ is given as a graph $\{ (x, y) \in \mathbb{R}^2 : y = g(x), 2x_0 \leq x \leq 0 \}$ satisfying the following hypotheses, where $K_0$ is a positive constant:

- (g1): $g(2x_0) = g(0) = 0$ and $g(x) > 0$, $\forall x \in (2x_0, 0)$,
- (g2): $g'(2x_0) = \lim_{t \to 0^+} \frac{g(2x_0 + t)}{t} = +\infty$ and $g'(0) = \lim_{t \to 0^-} \frac{g(t)}{t} = -\infty$,
- (g3): $g \in C^2((2x_0, 0))$,
- (g4): $g''(x) \leq -K_0$, $\forall x \in (2x_0, 0)$.

We remark that condition (g2) implies that $\sigma$ is perpendicular to the $x$-axis at the boundary points $A$ and $B$. That normal Tricomi domains are admissible is a consequence of the mentioned necessary condition proved in [7] for the existence of generalized solution and of the following result (cf. [14] Theorem 2.3).

**Theorem 2.2.** Let $\Omega$ be a normal Tricomi domain. Then, for every $h \in L^2(\Omega)$, there exists a unique generalized solution $u \in \tilde{W}^{1}_{AC, \sigma}(\Omega)$ to problem (LT).

The admissibility of normal Tricomi domains allows to enlarge the class of admissible domains and lead to the following theorem (cf. [14] Theorem 2.4).

**Theorem 2.3.** Let $\Omega$ be a Tricomi domain such that: i) $\Omega$ contains a normal subdomain $\Omega_0$ having boundary $\partial\Omega_0 = AC \cup BC \cup \sigma_0$; ii) there exists an $\epsilon > 0$ such that the elliptic boundaries $\sigma$ and $\sigma_0$ of $\Omega$ and $\Omega_0$ coincide in a strip $\{ (x, y) \in \mathbb{R}^2 : 0 \leq y \leq \epsilon \}$. Then $\Omega$ is admissible.

We stress that (cf. [7] and [14]), for Tricomi domains in which $\sigma$ forms acute angles with the parabolic segment $AB$, the previous solvability theory can be developed with the pair $(\tilde{W}^1_{AC, \sigma}(\Omega), \tilde{W}^1_{BC, \sigma}(\Omega))$ being replaced by $(W^1_{AC, \sigma}(\Omega), W^1_{BC, \sigma}(\Omega))$, where $W^1_{\Gamma}(\Omega), \Gamma \in \{ AC \cup \sigma, BC \cup \sigma \}$, is defined as the closure with respect to the usual $W^{1,2}(\Omega)$-norm of the space $C^\infty_\Gamma(\Omega) = \{ \psi \in C^\infty(\Omega) : \psi \equiv 0 \text{ on } \Gamma \}$. On the contrary (cf. [15] p. 445)), when dealing with normal Tricomi domain the weight $|y|$ in the $\tilde{W}^{1,2}(\Omega)$-norm appears naturally and describes the possible lack of square integrability of the partial derivative with respect to $x$ of the solutions in a neighborhood of $A$ and $B$.

Theorem 2.2 implies the existence of a continuous right inverse $\tilde{T}^{-1}_{AC, \sigma}$ from all of $L^2(\Omega)$ onto a dense proper subspace of $\tilde{W}^1_{AC, \sigma}(\Omega)$, and such that the generalized solution is exactly $u = \tilde{T}^{-1}_{AC, \sigma}h$. Then, using Rellich’s lemma, this continuous right inverse give rise to an injective, non surjective and compact operator from $L^2(\Omega)$ to $L^2(\Omega)$ which we denote again by $\tilde{T}^{-1}_{AC, \sigma}$. It is just such a compactness of the inverse operator that permits the possibility of studying the generalized solvability of the spectral problem

$$(LTE): \begin{cases} Tu = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } AC \cup \sigma, \end{cases}$$

where $\lambda \in \mathbb{C}$. Indeed, the compactness of $\tilde{T}^{-1}_{AC, \sigma}$ combined with a maximum principle for the Tricomi problem established in [12] exploiting a slight variant of that in [12], yields the
following Theorem 2.4 which is proved in [16]. We mention that Theorem 2.4 was already announced in [10], but (cf. [15, p. 536]) that paper presented two major problems to which the proof in [16] supplies a remedy.

**Theorem 2.4.** Let \( \Omega \) be normal Tricomi domain. Then there exists an eigenvalue-eigenfunction pair \((\lambda_0, u_0)\) such that \( 0 < \lambda_0 \leq |\lambda| \) for every \( \lambda \) in the spectrum \( \sigma(\tilde{T}_{AC,\sigma}) \) of \( \tilde{T}_{AC,\sigma} \) and \( u_0 \in \tilde{W}^1_{AC,\sigma}(\Omega) \) satisfies \( u_0 \geq 0 \) almost everywhere in \( \Omega \).

Note that, since the eigenvalues of \( \tilde{T}^{-1}_{AC,\sigma} \) are the inverse of those of \( \tilde{T}_{AC,\sigma} \), the compactness of \( \tilde{T}^{-1}_{AC,\sigma} \) implies that \( \tilde{T}_{AC,\sigma} \) has a discrete spectrum composed entirely of eigenvalues of finite multiplicity with a unique accumulation point at infinity. The eigenvalue \( \lambda_0 \) of Theorem 2.4 is called a principal eigenvalue due to the positivity of the associated eigenfunction \( u_0 \) and its being of minimum modulus. However, at present, it is not known neither if the associated eigenspace is simple, nor if other eigenspaces do not contain eigenfunctions that are nonnegative almost everywhere, as it happens in the purely elliptic case. Nevertheless, what is known is that all real eigenvalue of \( \tilde{T}_{AC,\sigma} \) must be positive. This spectral information is the content of [17, Theorem 2.5(a)], and, according to Theorem 2.4, may be summarized as

\[
\sigma(\tilde{T}_{AC,\sigma}) \cap (-\infty, \lambda_0) = \emptyset. \tag{2.9}
\]

To the author’s knowledge, (2.9) is the best information on the spectrum of \( \tilde{T}_{AC,\sigma} \) compatible with the solvability theory in the space \( \tilde{W}^1_{AC,\sigma}(\Omega) \). Indeed, the results in [20] and [21], which establish that \( \sigma(\tilde{T}_{AC,\sigma}) \cap \{ \lambda \in \mathbb{C} : 2\pi/3 \leq |\arg \lambda| \leq 4\pi/3 \} = \emptyset \), require that the eigenfunctions should be at least of class \( C(\Omega) \cap C^1(\Omega) \cap C^2(\Omega_+) \cap C^2(\Omega_-) \), where \( \Omega_\pm = \{(x, y) \in \Omega : \pm y > 0 \} \). Unfortunately, the question of regularity of the eigenfunctions is still an open question, but, anyhow, one can show the existence of a continuous eigenfunction. More precisely, using the solvability result in [1, p. 64] for normal domains, in [17] it is shown the following theorem.

**Theorem 2.5.** Let \( \Omega \) be a normal Tricomi domain and let \( \lambda_0 \) be the positive eigenvalue of Theorem 2.4. Then there exists an eigenvalue-eigenfunction pair \((\tilde{\lambda}_0, \tilde{u}_0)\) such that \( \tilde{\lambda}_0 \geq \lambda_0 \) and \( \tilde{u}_0 \in \tilde{W}^1_{AC,\sigma}(\Omega) \cap C(\Omega) \) satisfies \( \tilde{u}_0 \geq 0 \) in \( \Omega \).

### 3 D-star-shaped domains and Pohožaev identity

In Section 2 we have defined the Tricomi domains so that the boundary points \( A \) and \( B \) coincide, respectively, with \((2x_0, 0)\) and \((0, 0)\), where \( x_0 < 0 \). Such a choice is made only in order to uniform our notation with that of [18], whose results we shall need later. Indeed, due to the invariance of the Tricomi operator (2.1) with respect to translations along the \( x \) axis, any other choices for \( A \) and \( B \) could be possible. To this purpose, it suffices to observe that if \( u \in C^2(\Omega) \) solves one between the problems (LT) and (LTE) in \( \Omega \), then, by setting \( x^* = x - l, \ y^* = y, \ l \in \mathbb{R} \), the function \( \tilde{u}(x^*, y^*) = u(x^* + l, y^*) \) solves the corresponding problem in the relevant translate \( \tilde{\Omega} \) of \( \Omega \).
As noticed in [19] (take there \(m = N = 1\) in the equation \(y|y|^{m-1} \sum_{i=1}^{N} u_{x_i x_i} + u_{yy} + f(u) = 0\)), translations in the \(x\) variables are the easiest symmetries that generate conservation laws associated to the semilinear problem

\[
\begin{aligned}
  &Tu = f(u) \quad \text{in } \Omega, \\
  &u = 0 \quad \text{on } AC \cup \sigma,
\end{aligned}
\]

where \(f \in C^1(\mathbb{R})\). Recall that a conservation law associated to (3.1) is a first-order equation in divergence form \(\text{div } (U) = 0\) which must be satisfied by every sufficiently regular solution of the given problem, where \(U = U(x, y, u, \nabla u, f)\) is some vector field whose dependence on \(u\) is, in general, highly nonlinear. Apart from translations, other two symmetries groups that generate conservation laws for problem (3.1) are exhibited in [19], i.e. those coming from certain anisotropic dilations and from inversion with respect to the curve

\[
9(x - x_0)^2 + 4y^3 = 9x_0^2, \quad y \geq 0.
\] (3.2)

According to [25, Chapter IV], the curve in (3.2) which joins the boundary points \(A\) and \(B\) in the elliptic region, is called the normal curve for the Tricomi operator. In particular, from (3.2) we get

\[
y = \left[9x(2x_0 - x)/4\right]^{1/3} =: g(x), \quad x \in [2x_0, 0].
\] (3.3)

Hence, a standard exercise of calculus shows that the function \(g\) in (3.3) satisfies all the conditions (\(g1\))–(\(g4\)) of Section 2 with

\[
K_0 = -g''(x_0) = \left(3x_0^4/2\right)^{-1/3}.
\] (3.4)

Since we do not need inversions in this paper, we only refer to [11] for their construction and their application to (3.1) with \(f = 0\), and to [19] for how to use inversions to derive conservation laws for (3.1) in both the cases \(f = 0\) and \(f(u) = u^\sigma\), the exponent \(\sigma = 9\) corresponding to the critical exponent obtained in [18]. Here, instead, we focus our attention to the second group of symmetries, which leads to the concept of \(D\)-star shaped domain and is strongly related to the Pohoţaev identity that we shall recall later.

Let \(\gamma > 0\) and consider the change of variable

\((x, y) \in \Omega \rightarrow (\gamma^3 x, \gamma^2 y) =: (x^*, y^*) \in \Omega^*\).

It is easy to verify that if \(u \in C^2(\Omega)\) is a solution of problem (3.1) with \(f = 0\), then, for every fixed \(\delta \geq 0\), the scaled function (cf. [19, p. 256])

\[
u_\gamma(x^*, y^*) = \gamma^{-\delta}u(\gamma^{-3}x^*, \gamma^{-2}y^*),
\] (3.5)

solved the same problem in the scaled domain \(\Omega^*\) of \(\Omega\). Thus, we have a multiplicative group \(\mathbb{R}_+\) of anisotropic dilations as a symmetry group for the linear homogeneous problem (3.1). For instance, such a dilation invariance has been applied in [3]–[5] to the research of fundamental solutions for the Tricomi operator. In the general case, the semilinear problem (3.1) does not have this symmetry group of dilations, but a straightforward
computation shows that this is true for power nonlinearities, provided \( \delta \) is opportunistically chosen in (3.3). That is, if \( f(u) = Cu^\alpha \) with \( C \in \mathbb{R} \) and \( \alpha > 1 \), then problem (3.1) has the property of dilations invariance for \( \delta = 4(\alpha - 1)^{-1} \). However, it is worth to remark that in the case \( f(u) = \lambda u \), corresponding to problem (LTE), there is no way to choose \( \delta \geq 0 \) in (3.3) such that the dilation invariance is satisfied.

The first variation of the one-parameter family of scaled functions (3.5) under the action of the one-parameter group of dilation is

\[
L u_\gamma(x^*, y^*) \big|_{\gamma = 1} = Du(x, y) - \delta u(x, y),
\]

where \( D \) is the vector field

\[
D = -3x \partial_x - 2y \partial_y. \tag{3.6}
\]

This vector field determines a flow \( \mathcal{F}_t : \mathbb{R}^2 \to \mathbb{R}^2, t \in \mathbb{R}, \) such that \( \mathcal{F}_t(\varpi, \eta) = \phi_{(\varpi, \eta)}(t) \), where, denoting by \( B^T \) the transpose of a \( p \times q \) matrix \( B \), \( \phi_{(\varpi, \eta)}^T(t) \) is the unique integral curve of the linear system

\[
\begin{align*}
V'(t) &= AV(t), \\
V(0) &= (\varpi, \eta)^T,
\end{align*}
\]

\[
V(t) = (x(t), y(t)), \quad A = \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix}; \tag{3.7}
\]

Therefore, for every \((t, \varpi, \eta) \in \mathbb{R}^3, \) we have \( \mathcal{F}_t(\varpi, \eta) = (\varpi e^{-3t}, \eta e^{-2t}) \).

**Definition 3.1.** Let \( D \) be defined by (3.6). An open set \( G \subset \mathbb{R}^2 \) is said to be \( D \)-star-shaped if for each \((\varpi, \eta) \in \overline{G} \) one has \( \mathcal{F}_t(\varpi, \eta) \subset \overline{G} \) for every \( t \in [0, +\infty] \), where \( \mathcal{F}_{+\infty}(\varpi, \eta) = \lim_{t \to +\infty} \mathcal{F}_t(\varpi, \eta) = (0, 0). \)

To make clear the importance of this definition, we recall that if \( \Omega \) is a normal Tricomi domain which is also \( D \)-star-shaped then the continuous and compact embedding \( W^{1,\infty}_{AC,\text{up}}(\Omega) \hookrightarrow L^p(\Omega) \) holds for every \( p \in [1, p^*) \), where \( p^* = 2N(N - 2)^{-1} = 10 \). Here, \( N = 5/2 \) is the so-called homogeneous dimension of \( \mathbb{R}^2 \) when equipped with a non-Euclidean metric \( d \) which is natural for the Tricomi operator as the Euclidean metric is natural for the Laplace operator (see [8], [18] and [19]).

Bounded \( D \)-star-shaped domains have \( D \)-starlike boundaries as established by the following lemma (cf. [18, Lemma 2.2]). From now on, \( \langle \cdot, \cdot \rangle \) will always denote the canonical inner product of \( \mathbb{R}^2 \).

**Lemma 3.2.** Let \( G \subset \mathbb{R}^2 \) be an open set with piecewise \( C^1 \) boundary \( \partial G \). If \( G \) is \( D \)-star-shaped, then \( \partial G \) is \( D \)-starlike in the sense that \( \langle (3x, -2y), \vec{n}(x, y) \rangle \leq 0 \) at each regular point \((x, y) \in \partial G \) where \( \vec{n}(x, y) \) is the unit outer normal to \( \partial G \) at the point \((x, y) \).

The notion of \( D \)-star-shaped domains has been used in [18] to prove the nonexistence of nontrivial regular solutions to problem (3.1) in the case \( f(u) = |u|^\alpha \) with \( \alpha > p^* - 1 \), thus showing that the homogeneous dimension of \( \mathbb{R}^2 \) is responsible for a critical-exponent phenomenon in the nonlinearity. In the quoted paper, the key tool is to combine the \( D \)-star-shapedness of \( \Omega \) with the following Pohožaev-type identity that we recall for the reader’s convenience, by referring to [18] for its proof.
Theorem 3.3. Let $\Omega$ be a Tricomi domain for $T$ and let $D$ be the vector field defined by $(\nabla \omega)$. Let $u$ be a solution of problem $(3.1)$ such that $u_y, xu_x, yu_x \in C^1(\overline{\Omega})$ and $xu \in C^2(\Omega)$. Then the following identity holds true

$$
\int_{\Omega} \left[ 10F(u) - uf(u) \right] \, dx \, dy = \int_{BC} (\omega_1 + \omega_2) \, ds + \int_{\sigma} \omega_1 \, ds. \tag{3.8}
$$

Here $F$ is a primitive of $f \in C^0(\mathbb{R})$ such that $F(0) = 0$, whereas $\omega_1$ and $\omega_2$ are defined by

$$
\omega_1 = \langle 2Du(-yu_x, -u_y) + (yu_x^2 + u_y^2)(-3x, -2y), \mathbf{n} \rangle, \tag{3.9}
$$

$$
\omega_2 = \langle -2F(u)(-3x, -2y) - u(-yu_x, -u_y), \mathbf{n} \rangle, \tag{3.10}
$$

$\mathbf{n}$ being the unit outer normal field to $\Omega$.

Remark 3.4. Observe that, according to [18, p. 420], we have formulated Theorem 3.3 in the weaker assumptions for $u$. In fact, the requirements $u_y, xu_x, yu_x \in C^1(\overline{\Omega})$ and $xu \in C^2(\Omega)$ suffice for applying the classical divergence theorem for $C^1(\overline{\Omega})$ vector fields and for exchanging the order of certain partial derivatives in the proof of $(3.8)$, and allow to weakening the original stronger condition $u \in C^2(\overline{\Omega})$.

Since the starting point for obtaining our estimates on the eigenfunctions of the Tricomi operator is the identity $(3.8)$, we conclude the section spending some words on it. In the theory of semilinear elliptic equations the first appearance of an identity between volume and surface integrals of kind $(3.8)$ goes back to [23]. There, such an identity resulted from an energy integral method consisting in multiplying the differential equation by a suitable vector field and then applying the divergence theorem. Since [23], this method for obtaining identities of type $(3.8)$ has become a standard tool in the theory of semilinear elliptic equations. On the contrary, the situation is quite different for semilinear equations of mixed elliptic-hyperbolic and degenerate types where, to our knowledge, the only remarkable results in the derivation of such identities are those in [18]. Indeed, using an argument that reproduces the original idea of [23], in [18] identities of type $(3.8)$ are derived for the semilinear problem $(3.1)$, with the Tricomi operator $T$ being replaced by the more general Gellerstedt operator $L = -y^{2k+1} \partial_x^2 - \partial_y^2$, $k \in \mathbb{N} \cup \{0\}$. In particular, the above Theorem 3.3 is obtained by taking $k = 0$ in [18, Theorem 3.1].

Usually, Pohožaev identities are applied for the proof of nonexistence results. In doing so, one has only to show that the signs of the volume and surface integrals are incompatible with the existence of nontrivial solutions. This is, for instance, the scheme followed in the quoted papers [23] and [18]. Our approach will be different. For the problem (LTE) (corresponding to $f(u) = \lambda u$ in $(3.1)$) $F(u)$ turns out to be $\lambda u^2/2$, so that the left-hand side of $(3.8)$ reduces to $4\lambda \|u\|_{L^2(\Omega)}^2$. Then, we shall get our estimates on the eigenfunctions of the Tricomi problem simply by showing that the right-hand side of $(3.8)$ is nonnegative and upper bounded by an opportune quantity.

Remark 3.5. Of course, a remark is on order about the approach summarized in the last paragraph. Indeed, the eigenfunctions of the Tricomi problem are, in general, complex valued, and we are not in position to apply Theorem 3.3 which, due to assumption $f \in C^0(R)$ and the presence of the canonical inner product of $\mathbb{R}^2$, requires a real context for its application. However, if we restrict our interest to the eigenfunctions $u$ corresponding
to real positive eigenvalues $\lambda \in [\lambda_0, +\infty)$ (cf. Theorem 2.4), then we can apply separately
our approach to their real and imaginary parts, $\Re u$ and $\Im u$. For, $T$ being a linear operator,
we have

$$Tu = \lambda u, \quad \lambda \in \mathbb{R}, \quad \iff \begin{cases} T(\Re u) = \lambda \Re u, \\ T(\Im u) = \lambda \Im u, \end{cases} \quad \lambda \in \mathbb{R}. \quad (3.11)$$

That is, $u$ is an eigenfunction corresponding to a real eigenvalue $\lambda$ if and only if its real and
imaginary parts $\Re u$ and $\Im u$ are real valued eigenfunctions corresponding to $\lambda$. Thus, once
we will have estimated $\|\Re u\|_{L^2(\Omega)}^2$ and $\|\Im u\|_{L^2(\Omega)}^2$, our estimate on the $L^2$-norm of the (pos-
sibly complex valued) eigenfunctions $u$ corresponding to positive eigenvalues will follows
from $\|u\|_{L^2(\Omega)}^2 = (u, u)_{2,\sim} = \|\Re u\|_{L^2(\Omega)}^2 + \|\Im u\|_{L^2(\Omega)}^2$, where $(v_1, v_2)_{2,\sim} = \int_{\Omega} v_1 \overline{v_2} \, dx \, dy$.

### 4 Main result

In order to perform explicit computations, from now on $\Omega$ will be a Tricomi domain
having boundary $\partial \Omega = AC \cup BC \cup \sigma$, where the characteristics $AC$ and $BC$ are as in
(2.2) and (2.3), and $\sigma$ is the normal curve (3.2). Hence, due to (3.3) and (3.4), $\Omega$ is a
normal Tricomi domain according to the definition given in Section 2.

We now parametrize the curves $AC$, $BC$ and $\sigma$ in order to give to $\partial \Omega$ the positive
orientation of leaving the interior of $\Omega$ on the left, i.e. the counterclockwise orientation. To
this purpose, denoting by $r_{\Gamma} : I \subset \mathbb{R} \to \mathbb{R}^2$, $I$ interval, the parametric curve representing
a subset $\Gamma$ of $\partial \Omega$, we have:

$$r_{AC}(-y) = (2x_0 + (2/3)(-y)^{3/2}, y), \quad y \in [y_C, 0], \quad (4.1)$$
$$r_{BC}(y) = (- (2/3)(-y)^{3/2}, y), \quad y \in [y_C, 0], \quad (4.2)$$
$$r_{\sigma}(-x) = (x, g(x)), \quad x \in [2x_0, 0], \quad (4.3)$$

where $y_C = -(3|x_0|/2)^{2/3}$ and $g$ is the function defined by (3.3). Consequently, the unit
outer normals on the characteristics and on $\sigma$ are given by

$$\bar{n}_{AC} = (1 - y)^{-1/2}(-1, -(-y)^{1/2}), \quad y \in [y_C, 0], \quad (4.4)$$
$$\bar{n}_{BC} = (1 - y)^{-1/2}(1, -(-y)^{1/2}), \quad y \in [y_C, 0], \quad (4.5)$$
$$\bar{n}_{\sigma} = \begin{cases} (1, 0), & x = 0, \\ \{[g'(x)]^2 + 1\}^{-1/2}(-g'(x), 1), & x \in (2x_0, 0), \end{cases} \quad (4.6)$$

Observe that, since from (3.3) we get $g'(x) = -(3/2)[g(x)]^{-2}(x - x_0)$ for every $x \in (2x_0, 0)$,
easy computations yield to:

$$\frac{(-g'(x), 1)}{([g'(x)]^2 + 1)^{1/2}} = \{(x - x_0)^2 + (4/9)[g(x)]^4\}^{-1/2}(x - x_0, (2/3)[g(x)]^2). \quad (4.7)$$
Then, the vector on the right-hand side of (4.7) being defined also in $x = 2x_0$ and $x = 0$ where it is equal to $(-1, 0)$ and $(1, 0)$, respectively, we can replace (4.6) with the more compact formula:

$$\tilde{n}_x = \{(x - x_0)^2 + (4/9)[g(x)]^4\}^{-1/2} (x - x_0, (2/3)[g(x)]^2), \quad x \in [2x_0, 0],$$

(4.8)

It is now an easy task to show that $\partial \Omega$ is $D$-starlike with respect to $D = -3x\partial_x - 2y\partial_y$. Indeed, from (4.1)–(4.3) and (4.7) we get:

$$\langle(-3x, -2y), \tilde{n}_{AC}\rangle = 6x_0(1 - y)^{-1/2} < 0, \quad \forall (x, y) \in AC,$$

(4.9)

$$\langle(-3x, -2y), \tilde{n}_{BC}\rangle = 0, \quad \forall (x, y) \in BC$$

(4.10)

$$\langle(-3x, -2y), \tilde{n}_x\rangle = -3xx_0\{(x - x_0)^2 + (4/9)y^4\}^{-1/2} \leq 0, \quad \forall (x, y) \in \sigma.$$  

(4.11)

There is more. That is, $\Omega$ is just $D$-star-shaped in the sense of Definition 3.1. We shall not need this fact later (all that we shall need is the already proved $D$-starlikeness of $\Omega$), but we prove it for completeness since the proof is very easy and since it gives a concrete character to the abstract notion of $D$-star-shaped domain.

**Lemma 4.1.** Let $\partial \Omega = AC \cup BC \cup \sigma$, $\sigma$ being the normal curve (3.2). Then $\Omega$ is $D$-star-shaped with respect to $D = -3x\partial_x - 2y\partial_y$.

**Proof.** As it is well-known (cf. [6, Chapter 15]), for the linear system (3.7) the origin is an improper node asymptotically stable and every orbit, except the two corresponding to the positive and negative $x$-axis, tends to the origin tangentially to the $y$-axis. Thus, it suffices to show $F_t(\tilde{x}, \tilde{y}) = (te^{-3t}, ye^{-2t}) \subset \Omega$ for every $t \in [0, +\infty]$ only for the points $(\tilde{x}, \tilde{y}) \in \partial \Omega$. Indeed, for every $(\tilde{x}, \tilde{y}) \in \Omega$ there corresponds a unique $(\tilde{x}, \tilde{y}) \in \partial \Omega$ such that $(\tilde{x}, \tilde{y}) = F_{t_0}(\tilde{x}, \tilde{y})$ for $t_0 = 3^{-1}\ln(\tilde{x}/\tilde{x}) < 0$, and, consequently, $F_t(\tilde{x}, \tilde{y}) = F_{t-t_0}(\tilde{x}, \tilde{y})$, $t \in \mathbb{R}$. Let us first assume $(\tilde{x}, \tilde{y}) \in \sigma$. To prove that $F_t(\tilde{x}, \tilde{y}) \in \Omega$, $t \in [0, +\infty]$, where $F_{+\infty}(\tilde{x}, \tilde{y}) = (0, 0) = B$, we have to show that $9(\tilde{t}e^{-3t} - x_0)^2 + 4(\tilde{y}e^{-2t})^3 \leq 9x_0^2$ for every $t \geq 0$, or, equivalently,

$$9\tilde{t}^2 - 18\tilde{t}x_0e^{3t} + 4\tilde{y}^2 \leq 0, \quad t \geq 0.$$  

(4.12)

But, if $(\tilde{x}, \tilde{y}) \in \sigma$, then $9\tilde{t}^2 + 4\tilde{y}^3 = 18\tilde{x}x_0$. Replacing this identity in (4.12) we derive $18\tilde{x}x_0(1 - e^{3t}) \leq 0$ which is true for every $t \geq 0$. Now, let us take $(\tilde{x}, \tilde{y}) \in BC$. Then $3\tilde{t}e^{-3t} + 2(-\tilde{y}e^{-2t})^{3/2} = [3\tilde{t} + 2(-\tilde{y})^{3/2}]e^{-3t} = 0$, meaning that $F_t(\tilde{x}, \tilde{y}) \in BC$ for every $t \in [0, +\infty]$. Finally, let $(\tilde{x}, \tilde{y}) \in AC$. Due to what already proved and since orbits do not intersect each other, we have that $F_t(\tilde{x}, \tilde{y})$ remains between the orbit $F_t(2x_0, 0)$ and the curve $3x + 2(-y)^{3/2} = 0$, $y \leq 0$, that is $F_t(\tilde{x}, \tilde{y}) \in \Omega_-$ for every $t \in [0, +\infty]$. This completes the proof. 

We now start to estimate the right-hand side of (3.8). For simplicity’s sake, in the sequel, for any $v : \Omega \to \mathbb{R}$, $\tilde{v}$ and $\tilde{v}$ denote its restrictions to $BC$ and $\sigma$, respectively, i.e.

$$\tilde{v} := v|_{BC}, \quad \tilde{v} := v|_{\sigma}.$$  

(4.13)

As usual, for any pair $w = (w_1, w_2)$, $|w|$ stands for its Euclidian norm $(w_1^2 + w_2^2)^{1/2}$. Then, recalling that a curve $\gamma \subset \mathbb{R}^2$ is said regular if it admits a parameterization $r_\gamma :
$I \subset \mathbb{R} \to \mathbb{R}^2$, $I$ interval, such that $r_\gamma \in C^1(I)$ and $r'_\gamma(t) \neq 0$ for every $t \in I$, we denote by $L^2(\gamma)$ the set of all real (respectively, complex) valued functions $\psi$ such that $||\psi||^2_{L^2(\gamma)} = \int_I |\psi|^2 \, ds < +\infty$ (respectively, $||\psi||^2_{L^2(\gamma)} = \int_I |\psi|^2 \, ds < +\infty$), where $ds = |r'_\gamma(t)| \, dt$, $t \in I$.

In particular, $\gamma$ is rectifiable if and only if $\psi \equiv 1 \in L^2(\gamma)$.

**Lemma 4.2.** Let $\omega_1$ be defined by formula (3.9) where $u$ is a real valued function such that $|y|^{1/2} u_x$, $u_y \in L^2(BC)$, $BC$ being defined by (2.3). Then, for every $\varepsilon > 0$, the following estimate holds:

$$0 \leq \int_{BC} \omega_1 \, ds \leq C_1(x_0, \varepsilon)||y|^{1/2} u_x||^2_{L^2(BC)} + C_2(x_0, \varepsilon)||y u_y||^2_{L^2(BC)},$$

(4.14)

where

$$C_j(x_0, \varepsilon) := \frac{6|x_0|(1 + \varepsilon^{-1})}{1 + (3|x_0|/2)^{2/3}}^{1/2}, \quad j = 1, 2.$$  

(4.15)

**Proof.** First, replacing $\tilde{n}$ with the $\tilde{n}_{BC}$ defined by (4.5) and using (4.10), formula (3.9) simplifies to give

$$\tilde{\omega}_1 = \langle 2Du(-y u_x, -u_y), \tilde{n}_{BC} \rangle$$

$$= 2(1 - y)^{-1/2}\{3xy\tilde{u}_x^2 + 2y^2 - 3x(-y)^{1/2}\tilde{u}_x\tilde{u}_y + 2(-y)^{3/2}\tilde{u}_y^2\}. $$

(4.16)

According to (4.2) we now replace $x$ with $-(2/3)(-y)^{3/2}$, where $y \in [y_C, 0]$. With a such substitution, from (4.10) we easily find

$$\tilde{\omega}_1 = 4(1 - y)^{-1/2}[(-y)^{5/2}\tilde{u}_x^2 + 2(-y)^2\tilde{u}_x\tilde{u}_y + (-y)^{3/2}\tilde{u}_y^2]$$

$$= 4(-y)^{3/2}(1 - y)^{-1/2}[(-y)^{1/2}\tilde{u}_x + \tilde{u}_y]^2 \geq 0.$$  

(4.17)

Then, using the well-know inequality $(a + b)^2 \leq (1 + \varepsilon)a^2 + (1 + \varepsilon^{-1})b^2$, $a, b \in \mathbb{R}$, $\varepsilon > 0$, and observing that the function $p(y) = (-y)^{3/2}(1 - y)^{-1/2}$ is decreasing for $y \leq 0$, from (4.17) we obtain

$$0 \leq \int_{BC} \omega_1 \, ds \leq 4 \int_{y_C}^0 (-y)^{3/2}(1 - y)^{-1/2}[(-y)^{1/2}\tilde{u}_x + \tilde{u}_y]^2 r'_BC(y) \, dy$$

$$\leq 4(-y_c)^{3/2}(1 - y_c)^{-1/2}[(1 + \varepsilon)||y|^{1/2} u_x||^2_{L^2(BC)} + (1 + \varepsilon^{-1})||y u_y||^2_{L^2(BC)}].$$

(4.18)

Replacing $y_C$ with $-(3|x_0|/2)^{2/3}$ in (4.18) the proof of (4.14) is complete. $\square$

**Lemma 4.3.** Let us replace $\omega_1$ and formula (3.9) with $\omega_2$ and formula (3.10) in the hypothesis of Lemma 4.2 and assume further that $u \in L^2(BC)$. Then, the following estimate holds:

$$\int_{BC} \omega_2 \, ds \leq C_3(x_0)||u||^2_{L^2(BC)}[(||y|^{1/2} u_x||^2_{L^2(BC)} + ||u y||^2_{L^2(BC)}],$$

(4.19)

where

$$C_3(x_0) := \frac{(3|x_0|/2)^{1/3}}{[1 + (3|x_0|/2)^{2/3}]^{1/2}}.$$  

(4.20)
Proof. As in the proof of Lemma 4.2, replacing \( \bar{n} \) with the explicit \( \bar{n}_{BC} \) given by (4.3) and using (4.10), we simplify (3.11) to
\[
\bar{\omega}_2 = \langle -\bar{u}(-y\tilde{u}_x, -\tilde{u}_y), \bar{n}_{BC} \rangle = -(-y)^{1/2}(1-y)^{-1/2}\tilde{u}[-(-y)^{1/2}\tilde{u}_x + \tilde{u}_y]. \tag{4.21}
\]
Therefore, applying Hölder inequality and observing that the function \( q(y) = -y(1-y)^{-1} \) is decreasing for \( y \leq 0 \), from (4.21) it follows
\[
\int_{BC} \omega_2 \, ds = \int_{y_C}^{0} [ -(-y)^{1/2}(1-y)^{-1/2}\tilde{u}|r'_{BC}(y)|^{1/2}] [(-y)^{1/2}\tilde{u}_x + \tilde{u}_y]|r'_{BC}(y)|^{1/2} \, dy \leq \left( \int_{y_C}^{0} (-y)(1-y)^{-1}\tilde{u}^2|r'_{BC}(y)| \, dy \right)^{1/2} \left[ \|y|^{1/2}u_x\|_{L^2(BC)} + \|u_y\|_{L^2(BC)} \right] \leq (-y_C)^{1/2}(1-y_C)^{-1/2}\|u\|_{L^2(BC)} \left[ \|y|^{1/2}u_x\|_{L^2(BC)} + \|u_y\|_{L^2(BC)} \right].
\]
This completes the proof. \( \square \)

Remark 4.4. Observe that, contrarily to Lemma 4.2 where (4.17) implies \( \int_{BC} \omega_1 \, ds \geq 0 \), in Lemma 4.3 we can not ensure \( \int_{BC} \omega_2 \, ds \geq 0 \), since (4.21) may change sign. However, as we shall see later, when \( f(u) = \lambda u \) in problem (3.1) with \( \lambda \in \sigma(\bar{T}_{ACU}) \cap [\lambda_0, +\infty) \) (cf. (2.9)), the nonnegativity of the integrals’ sum on the right-hand side of (3.8) will be a consequence of that of the left-hand side. Of course, this agrees with the obvious fact that the sum \( \int_{BC}(\omega_1 + \omega_2) \, ds + \int_{\sigma} \omega_1 \, ds \) may be nonnegative even though some of its terms are nonpositive. Notice also that we can not use the result in [18 pp. 416, 417] which establishes \( \int_{BC}(\omega_1 + \omega_2) \, ds \geq 0 \), since there it is assumed that the function \( \varphi(y) = u(r_{BC}(y)) \) belongs to \( C^2((y_C, 0)) \cap C^1([y_C, 0]) \), which is not our case.

We now turn our attention to the last term that it remains to estimate on the right-hand side of (3.4), i.e. the integral of \( \omega_1 \) along the elliptic normal arc \( \sigma \). For our purposes, we need some preliminaries results. To simplify notations, from now on we denote by \( h \) the positive continuous function
\[
h(x) = \{ (x - x_0)^2 + (4/9)\|g(x)\|^4 \}^{1/2}, \quad x \in [2x_0, 0], \tag{4.22}
\]
g being defined by (3.3). Hence, according to formula (4.8), for every \( x \in [2x_0, 0] \) we have \( \bar{n}_x = [h(x)]^{-1}(x - x_0, (2/3)\|g(x)\|^2) \). A detailed analysis of function \( h \) yields to the following Lemma 4.5 which we shall prove in Section 5 and which highlights the special role played by the value \( x_0 = -\sqrt{3}/4 \).

Lemma 4.5. Let \( h \) be the function in (4.22). i) If \( x_0 \in [-\sqrt{3}/4, 0) \), then \( h \) is a convex function such that
\[
0 < [(3/2)x_0^4]^{1/3} = h(x_0) \leq h(x) \leq h(2x_0) = h(0) = |x_0|, \quad \forall x \in [2x_0, 0]. \tag{4.23}
\]
ii) If \( x_0 < -\sqrt{3}/4 \), then
\[
0 < [x_0^2 - (3/64)]^{1/2} = h(x_+) = h(x_-) \leq h(x) \leq C_4(x_0), \quad \forall x \in [2x_0, 0]. \tag{4.24}
\]
where \( x_\pm = x_0 \pm [x_0^2 - (3/16)]^{1/2} \) and \( C_4(x_0) = \max \{ h(2x_0), h(x_0), h(0) \} \). In particular,

\[
C_4(x_0) = \begin{cases} 
  h(2x_0) = h(0) = |x_0|, & \text{if } x_0 \in (-2/3, -\sqrt{3}/4), \\
  h(2x_0) = h(x_0) = h(0) = |x_0|, & \text{if } x_0 = -2/3, \\
  h(x_0) = [(3/2)x_0^{4}]^{1/3}, & \text{if } x_0 < -2/3.
\end{cases}
\] (4.25)

Moreover, there exists a unique inflection point \( \bar{x} \in (x_0, x_+) \) such that \( h \) is convex in \([2x_0, 2x_0 - \bar{x}] \cup [\bar{x}, 0]\) and concave in \([2x_0 - \bar{x}, \bar{x}]\).

With the help of Lemma 4.5, we can now find upper and lower bounds of two functions which we shall encounter below, during the proof of Lemma 4.10. As for Lemma 4.5, the proof of the following Corollaries 4.6 and 4.8 will be furnished in Section 5.

**Corollary 4.6.** Let \( h \) be the function in (4.22), let \( x_\pm \) be the points defined in Lemma 4.5 (i) and let \( G_1(x) = [h(x)]^{-1}g_1(x) \), \( x \in [2x_0, 0] \), where \( g_1(x) = 3x(2x - 3x_0) \). i) If \( x_0 \in [-\sqrt{3}/4, 0) \), then

\[
C_5(x_0) \leq G_1(x) \leq C_6(x_0), \quad \forall x \in [2x_0, 0],
\] (4.26)

where

\[
C_5(x_0) := [h(x_0)]^{-1}g_1((3/4)x_0) = -(3/2)^{8/3}|x_0|^{2/3},
\] (4.27)

\[
C_6(x_0) := [h(3/2)x_0]^{-1}g_1(2x_0) = \frac{2^{8/3}3|x_0|}{[2^{4/3} + 9|x_0|^{2/3}]^{1/2}}.
\] (4.28)

ii) If \( x_0 < -\sqrt{3}/4 \), then

\[
C_7(x_0) \leq G_1(x) \leq C_8(x_0), \quad \forall x \in [2x_0, 0],
\] (4.29)
where
\[
C_7(x_0) := [h(x_+)]^{-1}g_1((3/4)x_0) = -\frac{(3/2)^3x_0^2}{[x_0^2 - (3/64)]^{1/2}},
\]  
(4.30)
\[
C_8(x_0) := \begin{cases}
C_6(x_0), & \text{if } x_0 \in (-1/2, -\sqrt{3}/4), \\
[h(x_-)]^{-1}g_1(2x_0) = \frac{6x_0^2}{[x_0^2 - (3/64)]^{1/2}}, & \text{if } x_0 \leq -1/2.
\end{cases}
\]  
(4.31)

**Remark 4.7.** Notice that for \( x_0 = -1/\sqrt{3} \in (-1/2, -\sqrt{3}/4) \) it holds \( x_+ = (3/4)x_0 \) and the estimate from below in (4.29) becomes \( C_7(x_0) = G_1(x_+) \leq G_1(x) \), which is sharp. On the contrary, when \( x_0 < -1/2 \) is large enough, \( x_+ \) and \( x_- \) approach 0 and 2\( x_0 \), respectively. This implies that, for \( x_0 < -1/2 \), while the lower bound \( C_7(x_0) \leq G_1(x) \) approaches \( [h(0)]^{-1}g_1((3/4)x_0) \leq G_1(x) \) and becomes less precise, the upper bound \( G_1(x) \leq C_8(x_0) \) approaches \( G_1(x) \leq G_1(2x_0) \) and becomes more accurate. Some numerical simulations made with the help of Maple 11.01 confirm this fact.

**Corollary 4.8.** Let \( h \) be the function in (4.22), let \( x_\pm \) be the points defined in Lemma 4.5, and let \( G_2(x) = [h(x)]^{-1}g_2(x) \), \( x \in [2x_0, 0] \), where \( g_2(x) = 4(2x - x_0)|g(x)|^{3/2} \). Moreover, let \( x_1 \) and \( x_2 \) be the points defined by \( x_j = [(7 + (-1)^j + 1/3)/8]x_0 \), \( j = 1, 2 \), such that \( g_2(x_1) \leq g_2(x) \leq g_2(x_2) \) for every \( x \in [2x_0, 0] \).

i) If \( x_0 \in [-\sqrt{3}/4, 0) \), then
\[
C_9(x_0) \leq G_2(x) \leq C_{10}(x_0) \quad \forall \ x \in [2x_0, 0],
\]  
(4.32)
where
\[
C_9(x_0) := [h(x_0)]^{-1}g_2(x_1) = -2^{-19/6}3^{2/3}[\sqrt{3} + 3][15 + \sqrt{33}1/2|x_0|^{2/3}],
\]  
(4.33)
\[
C_{10}(x_0) := [h(x_0/2)]^{-1}g_2(x_2) = 2^{-11/6}3[\sqrt{3} - 3][15 - \sqrt{33}]^{1/2}|x_0|^{2/3} + 9|x_0|^{2/3}]^{1/2}.
\]  
(4.34)

ii) If \( x_0 < -\sqrt{3}/4 \), then
\[
C_{11}(x_0) \leq G_2(x) \leq C_{12}(x_0), \quad \forall \ x \in [2x_0, 0],
\]  
(4.35)
where
\[
C_{11}(x_0) := [h(x_-)]^{-1}g_2(x_1) = -\frac{2^{-7/2}3[\sqrt{3} + 3][15 + \sqrt{33}]^{1/2}x_0^2}{[x_0^2 - (3/64)]^{1/2}},
\]  
(4.36)
\[
C_{12}(x_0) := \begin{cases}
C_{10}(x_0), & \text{if } x_0 \in (-1/2, -\sqrt{3}/4), \\
[h(x_+)]^{-1}g_2(x_2) = \frac{2^{-7/2}3[\sqrt{3} - 3][15 - \sqrt{33}]^{1/2}x_0^2}{[x_0^2 - (3/64)]^{1/2}}, & \text{if } x_0 \leq -1/2.
\end{cases}
\]  
(4.37)

In particular,
\[
|G_2(x)| \leq C_{13}(x_0), \quad \forall \ x \in [2x_0, 0],
\]  
(4.38)
where
\[
C_{13}(x_0) := \begin{cases}
|C_9(x_0)|, & \text{if } x_0 \in [-\sqrt{3}/4, 0), \\
|C_{11}(x_0)|, & \text{if } x_0 < -\sqrt{3}/4.
\end{cases}
\]  
(4.39)
**Remark 4.9.** Let $x_3$ and $x_4$ be the points defined by $x_j = -[(15+(-1)^j+1\sqrt{33})/32]^{1/2}$, $j = 3, 4$, such that $x_3 < x_4 < -1/2$. Then, it is worth to observe that for $x_0 = x_4$ (respectively, $x_0 = x_3$) it holds $x_+ = x_1$ (respectively, $x_- = x_2$) and the lower (respectively, upper) bound in (4.35) becomes $C_{11}(x_0) = G_2(x_-) \leq G_2(x)$ (respectively, $G_2(x) \leq G_2(x_+) = C_{12}(x_0)$), which is sharp. Unfortunately, when $x_0 < -1$ is large enough, both sides of (4.35) become less precise, as numerical simulations made with Maple 11.01 confirm. On the other side, to find the greatest and least values of $G_2$ using the standard tools of calculus is not profitable, due to the difficulties in locating its stationary points (cf. the following Remark 5.3).}

We can now proceed to estimate the line integral $\int_\sigma \omega_1 \, ds$. Due to Corollaries 4.6 and 4.8 two different estimates will be supplied, according to the fact that the parabolic diameter $|AB| = 2|x_0|$ of $\Omega$ is greater or not than the “critical” value $\sqrt{3}/2$.

**Lemma 4.10.** Let $\sigma$ be the normal elliptic arc (3.2) and let $u$ be a real valued solution of problem (3.7) which is Fréchet differentiable at each of the points of $\sigma$ and such that $|y|^{1/2}u_x, u_y \in L^2(\sigma)$. Let $\omega_1$ be defined by (3.2). Then, for every $\varepsilon > 0$, the following estimate holds:

$$0 \leq \int_\sigma \omega_1 \, ds \leq C_{14}(x_0, \varepsilon)||y|^{1/2}u_x||^2_{L^2(\sigma)} + C_{15}(x_0, \varepsilon)||u_y||^2_{L^2(\sigma)}. \quad (4.40)$$

Here

$$C_{14}(x_0, \varepsilon) := \begin{cases} C_6(x_0) + (\varepsilon/2)|C_9(x_0)|, & \text{if } x_0 \in [-\sqrt{3}/4, 0), \\ C_8(x_0) + (\varepsilon/2)|C_{11}(x_0)|, & \text{if } x_0 < -\sqrt{3}/4, \end{cases} \quad (4.41)$$

$$C_{15}(x_0, \varepsilon) := \begin{cases} -C_5(x_0) + (2\varepsilon)^{-1}|C_9(x_0)|, & \text{if } x_0 \in [-\sqrt{3}/4, 0), \\ -C_7(x_0) + (2\varepsilon)^{-1}|C_{11}(x_0)|, & \text{if } x_0 < -\sqrt{3}/4, \end{cases} \quad (4.42)$$

where the constants $C_j(x_0), j = 5, \ldots, 9, 11$, are defined by (4.27), (4.28), (4.30), (4.31), (4.33) and (4.36).

**Proof.** First, since $u_{\sigma} = 0$, the assumption that $u$ is Fréchet differentiable at each of the points of $\sigma$ implies that the directional derivative of $u$ is zero along $\sigma$. Therefore, due to the $D$-starlikeness of $\partial\Omega$ (cf. (4.9)–(4.11)), we are in position to apply the argument in [18, p. 416], to which we refer the reader for the details, to derive the lower bound $0 \leq \int_\sigma \omega_1 \, ds$. Now, recalling formulæ (3.3), (4.8), (1.11) and (1.22) and notation (4.13), from the definition (3.9) of $\omega_1$ easy computation yields:

$$\hat{\omega}_1 = [h(x)]^{-1}\left\{(6x_0\hat{u}_x + 4y\hat{u}_y)(y - x_0)y\hat{u}_x + (2/3)y^2\hat{u}_y\right\} - 3x_0(y\hat{u}_x^2 + \hat{u}_y^2)$$

$$= G_1(x)y\hat{u}_x^2 + G_2(x)y^{1/2}\hat{u}_x \hat{u}_y - G_1(x)\hat{u}_y^2, \quad (4.43)$$

where $G_1$ and $G_2$ are the functions defined in Corollaries 4.6 and 4.8. Then, using $2|a||b| \leq \varepsilon a^2 + \varepsilon^{-1}b^2, a, b \in \mathbb{R}, \varepsilon > 0$, from (4.43) we obtain:

$$\hat{\omega}_1 \leq G_1(x)y\hat{u}_x^2 + |G_2(x)||y^{1/2}\hat{u}_x||\hat{u}_y| - G_1(x)\hat{u}_y^2$$

$$\leq \{G_1(x) + (\varepsilon/2)|G_2(x)|\}y\hat{u}_x^2 + \{(2\varepsilon)^{-1}|G_2(x)| - G_1(x)\}\hat{u}_y^2. \quad (4.44)$$
It thus now suffices to apply Corollaries 4.6 and 4.8 to conclude the proof. Indeed, due to (4.26)–(4.31), (4.33), (4.36), (4.38) and (4.39), inequality (4.41) leads us to

\[ \hat{\omega}_1 \leq C_{14}(x_0, \varepsilon)y^2_u + C_{15}(x_0, \varepsilon)y^2_v, \quad \forall \varepsilon > 0, \]

where \( C_{14}(x_0, \varepsilon) \) and \( C_{15}(x_0, \varepsilon) \) are defined by (4.41) and (4.42), respectively. Hence

\[ \int_{\sigma} \omega_1 \, ds \leq C_{14}(x_0, \varepsilon)\|y|^2_{1/2} \|u_x\|^2_{L^2(\sigma)} + C_{15}(x_0, \varepsilon)\|u_y\|^2_{L^2(\sigma)}, \quad \forall \varepsilon > 0, \]

and the proof of (4.40) is complete. \( \square \)

We can now prove our main result. For brevity, in the following Theorem 4.11, the symbols \( L^2(\Omega) \), \( C^1(\Omega) \) and \( C^2(\Omega) \) are used without exception for both real and complex valued functions. Needless to say, if we have to deal with complex valued functions, then \( L^2(\Omega) \) is understood endowed with the usual complex inner product \( \langle \cdot, \cdot \rangle_{2,\sim} \) defined in Remark 3.5, whereas the spaces \( C^1(\Omega) \) and \( C^2(\Omega) \) are meant for \( C^1(\Omega; \mathbb{C}) \) and \( C^2(\Omega; \mathbb{C}) \), respectively.

**Theorem 4.11.** Let the curves \( AC \), \( BC \) and \( \sigma \) be defined by (3.5)–(3.2) and (3.4), respectively, and let \( \Omega \subset \mathbb{R}^2 \) be the normal Tricomi domain having boundary \( \partial \Omega = AC \cup BC \cup \sigma \). Let \( u = \mathfrak{R}u + i\mathfrak{I}u \), where \( \mathfrak{R}u, \mathfrak{I}u \in \hat{W}^{1,2}_{AC\cup\sigma}(\Omega) \), be a not almost everywhere vanishing solution to problem

\[ \begin{cases} T = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } AC \cup \sigma, \end{cases} \quad \lambda \in [\lambda_0, +\infty), \tag{4.45} \]

\( \lambda_0 > 0 \) being the principal eigenvalue of \( \mathfrak{F}_{AC\cup\sigma} \) defined in Theorem 2.4. Let us assume that \( u_y, xu_x, yu_x \in C^1(\Omega), xu \in C^2(\Omega), u \in L^2(BC), |y|^{1/2}u_x, u_y \in L^2(BC) \cap L^2(\sigma) \) and that \( u \) is Fréchet differentiable at each of the points of \( \sigma \). Then, for every \( \varepsilon_j > 0, j = 1, 2 \), the following estimate holds:

\[ 0 < 2\lambda^{1/2}\|u\|_{L^2(\Omega)} \leq \left\{ \begin{array}{ll} C_1(x_0, \varepsilon_1)\|y|^{1/2}u_x\|^2_{L^2(BC)} + C_2(x_0, \varepsilon_1)\|u_y\|^2_{L^2(BC)} \\ + C_3(x_0)\|\mathfrak{R}u\|_{L^2(BC)} + \|\mathfrak{I}u\|_{L^2(BC)} \| |y|^{1/2}u_x\|_{L^2(BC)} + \|u_y\|_{L^2(BC)} \]  

\[ + C_{14}(x_0, \varepsilon_2)\|y|^{1/2}u_x\|^2_{L^2(\sigma)} + C_{15}(x_0, \varepsilon_2)\|u_y\|^2_{L^2(\sigma)} \right\}^{1/2}, \tag{4.46} \]

where \( C_{j}(x_0, \varepsilon_1), j = 1, 2 \), are defined by (4.19) with \( \varepsilon = \varepsilon_1 \), \( C_3(x_0) \) is defined by (4.20), and \( C_{j}(x_0, \varepsilon_2), j = 14, 15 \), are defined by (4.41) and (4.42) with \( \varepsilon = \varepsilon_2 \).

**Proof.** First (cf. (3.11)), since \( \lambda \in [\lambda_0, +\infty) \) and \( u \) does not vanish almost everywhere, we have that the real valued functions \( v_1 = \mathfrak{R}u \) and \( v_2 = \mathfrak{I}u \) also solve (4.45), and that at least one between is not the zero element of \( L^2(\Omega) \). Moreover, our assumptions on \( u \) imply that \( (v_j)_y, x(v_j)_x, y(v_j)_x \in C^1(\Omega), xu_j \in C^2(\Omega), v_j \in L^2(BC), |y|^{1/2}(v_j)_x, (v_j)_y \in L^2(BC) \cap L^2(\sigma) \) and that \( v_j \) is Fréchet differentiable at each of the points of \( \sigma \),
Thus, setting \( f(t) = \lambda t, t \in \mathbb{R} \), we may as well suppose from the outset that \( u \in \widetilde{W}^{1}_{AC,\sigma}(\Omega), u \neq 0 \), is a real valued solution to problem (3.1) satisfying the assumptions of Theorem 3.3 and Lemmas 4.2, 4.3 and 4.10. In particular, \( u \) satisfies the identity (3.8) with \( F(t) = (\lambda/2)t^2, t \in \mathbb{R} \). Therefore, since \( 10F(u) - uf(u) = 4\lambda u^2 \), we have

\[
0 < 4\lambda \|u\|^2_{L^2(\Omega)} = \int_{\partial \Omega} (\omega_1 + \omega_2) \, ds + \int_\sigma \omega_1 \, ds,
\]

where \( \omega_1 \) and \( \omega_2 \) are defined by (3.9) and (3.10). Hence, taking \( \varepsilon = \varepsilon_1 > 0 \) in Lemma 4.2 and \( \varepsilon = \varepsilon_2 > 0 \) in Lemma 4.10 and applying estimate (4.14), (4.19) and (4.40), from (4.47) we deduce

\[
0 < 4\lambda \|u\|^2_{L^2(\Omega)} \leq C_1(x_0, \varepsilon_1)\|y\|^{1/2}u_x\|^2_{L^2(BC)} + C_2(x_0, \varepsilon_1)\|u_y\|^2_{L^2(BC)}
+ C_3(x_0)\|u\|^2_{L^2(BC)} \left[ \|y\|^{1/2}u_x\|^2_{L^2(BC)} + \|u_y\|_{L^2(BC)} \right]
+ C_{14}(x_0, \varepsilon_2)\|y\|^{1/2}u_x\|^2_{L^2(\sigma)} + C_{15}(x_0, \varepsilon_2)\|u_y\|_{L^2(\sigma)}.
\] 

This proves (4.46) in the case that \( u \) is real valued. To complete the proof in the general case it suffices to replace \( u \) in (4.48) with \( \Re u \) and \( \Im u \), respectively, and then summing up the so obtained estimate, taking into account the identities \( |y|^{1/2}(\Re u)_x = \Re(|y|^{1/2}u_x) \) and \( (\Re u)_y = \Re(u_y), \Re = \Re, \Im \), and the inequalities \( \|y|^{1/2}(\Re u)_x\|^2_{L^2(BC)} \leq \|y|^{1/2}u_x\|^2_{L^2(BC)} \) and \( \|\Re u\|_{L^2(BC)} \leq \|u\|_{L^2(BC)} \), \( \Re = \Re, \Im \).

\[\square\]

**Remark 4.12.** Notice that, if \( u \in \widetilde{W}^{1}_{AC,\sigma} \) is an eigenfunction corresponding to an eigenvalue \( \lambda \in [\lambda_0, +\infty) \) satisfying the assumption of Theorem 4.11, then (4.47) improves the inequality \( 0 \leq \int_{\partial \Omega} (\omega_1 + \omega_2) \, ds + \int_\sigma \omega_1 \, ds \) which is shown in the proof of Theorem 4.2 (take there \( k = 0 \)) under the assumption \( u \in C^2(\Omega) \).

Of course, when \( \Omega \) is as above, one can applies estimate (4.46), with the quadruplet \((\lambda, u, \Re u, \Im u)\) being replaced by \((\lambda_0, u_0, u_0, 0)\) and \((\tilde{\lambda}_0, \tilde{u}_0, \tilde{u}_0, 0)\), respectively, to the eigenfunctions \( u_0 \in \widetilde{W}^{1}_{AC,\sigma}(\Omega) \) and \( \tilde{u}_0 \in \widetilde{W}^{1}_{AC,\sigma}(\Omega) \cap C(\Omega) \) of Theorems 2.4 and 2.5 provided he can shows that they satisfy the additional regularity requirements of Theorem 4.11.

## 5 Proof of Lemma 4.5 and Corollaries 4.6 and 4.8

**Proof of Lemma 4.5.** First, from definitions (4.3) and (4.22) we immediately derive \( h(x) = h(2x_0 - x) \), so that \( h \) is an even function with respect to the line \( x = x_0 \) and it suffices to prove the lemma assuming \( x \in [x_0, 0] \). With such a convention, we change the variable from \( x \) to \( X = x - x_0 \in [0, -x_0] \) and we consider the function

\[
H(X) := h(X + x_0) = \left\{ X^2 + (4/9)[G(X)]^4 \right\}^{1/2}, \quad X \in [0, -x_0],
\]

where

\[
G(X) := g(X + x_0) = \left[ 9(x_0^2 - X^2)/4 \right]^{1/3}, \quad X \in [0, -x_0].
\]
Thus, differentiating (5.1) with respect to \(X\) and using \(G'(X) = -(3/2)[G(X)]^{-2}X\), \(X \in [0, -x_0]\), we get
\[
H'(X) = [3H(X)]^{-1}[3 - 4G(X)]X, \quad X \in [0, -x_0].
\]
(5.3)

Now, (5.2) easily yields \(3 - 4G(X) \geq 0\) for every \(X\) such that \(X^2 \geq x_0^2 - (3/16)\). Therefore, if \(x_0 \in [-\sqrt{3}/4, 0]\), from (5.3) we deduce \(H'(X) \geq 0\) for every \(X \in [0, -x_0]\). Then \(H\) is a non-decreasing function in \([0, -x_0]\) and (1.23) follows from \(h(x_0) = H(0) \leq H(X) \leq H(-x_0) = h(0)\). On the contrary, if \(x_0 < -\sqrt{3}/4\), we have \(H'(X) \leq 0\) in \([0, X_+]\) and \(H'(X) \geq 0\) in \([X_+, -x_0]\), where \(X_+ = [x_0^2 - (3/16)]^{1/2}\). Hence \(H\) is non-increasing in \([0, X_+]\) and non-decreasing in \([X_+, -x_0]\), and (1.23) follows from \(h(x_+) = H(X_+) \leq H(X) \leq \max\{H(0), H(-x_0)\} = \max\{h(x_0), h(0)\}\). By comparing the values of \(h(x_0)\) and \(h(0)\) we get (1.23), too. To complete the proof of the lemma, let us assume first that \(x_0 \in [-\sqrt{3}/4, 0]\). As we noted above, in such a case the function \(\tilde{G}(X) = 3 - 4G(X), \quad X \in [0, -x_0]\), is non-negative and, moreover, is non-decreasing due to the non-increasing character of \(G\). An easy computations taking into account formula (5.3) shows also that the function \(\tilde{H}(X) = [3H(X)]^{-1}X, \quad X \in [0, -x_0]\), is non-decreasing since
\[
\tilde{H}'(X) = 4[3H(X)]^{-3}\{[G(X)]^3 + 3X^2\}G(X) \geq 0, \quad \forall X \in [0, -x_0].
\]
Then, if we take \(0 \leq X_1 \leq X_2 < -x_0\), from \(0 \leq \tilde{G}(X_1) \leq \tilde{G}(X_2), \quad 0 \leq \tilde{H}(X_1) \leq \tilde{H}(X_2)\) and \(H'(X) = \tilde{G}(X)\tilde{H}(X)\) we derive \(0 \leq H'(X_1) \leq H'(X_2)\). Thus \(H'\) is a non-decreasing function or, equivalently, \(H\) is a convex function which completes the proof of \(i)\). Let us now assume \(x_0 < -\sqrt{3}/4\). Since in this case we have \(\tilde{G}(X) \geq 0\) if and only if \(X \in [X_+, -x_0]\), the previous argument can be used only to show that \(H\) is still a convex function in the interval \([X_+, -x_0]\). To see what happens in the interval \([0, X_+]\) we analyze the second derivative of \(H\) with respect to \(X\). From (5.3) we obtain
\[
H''(X) = [3G(X)]^{-2}[H(X)]^{-3}N(X), \quad X \in [0, X_+],
\]
(5.4)

where
\[
N(X) = 3\{6X^2 + [3 - 4G(X)][G(X)]^2\}[H(X)]^2 - \{X[3 - 4G(X)]G(X)\}^2
= 3\{6X^2 + 3[G(X)]^2 - 4[G(X)]^3\}{X^2 + (4/9)[G(X)]^4}
- 9X^2[G(X)]^2 + 24X^2[G(X)]^3 - 16X^2[G(X)]^4
= 18X^4 - 8X^2[G(X)]^4 + 12X^2[G(X)]^3 + 4[G(X)]^6 - (16/3)[G(X)]^7.
\]
(5.5)

Replacing \([G(X)]^4\) and \([G(X)]^7\) in (5.5) with \([G(X)]^3G(X)\) and \([G(X)]^6G(X)\), and using (5.2) to rewrite \([G(X)]^3 = 9(x_0^2 - X^2)/4\) and \([G(X)]^6 = [9(x_0^2 - X^2)/4]^2\), a careful computation shows that
\[
N(X) = 9\{2X^4 + (3/4)(3x_0^2 + X^2)(x_0^2 - X^2) - (3x_0^2 - X^2)(x_0^2 - X^2)G(X)\]
= (9/4)[5X^4 - 6x_0^2X^2 + 9x_0^4 - 4X^4 - 4x_0^2X^2 + 3x_0^4]G(X).
\]
(5.6)

From (5.5) it follows
\[
\begin{align*}
N(0) &= (27/4)x_0^4[3 - 4(9x_0^2/4)^{1/3}], \\
N(X_+) &= (9/4)[8x_0^4 - (15/8)x_0^2 + (9/128)],
\end{align*}
\]
so that, for \( x_0 < -\sqrt{3}/4 \), we have \( N(0) < 0 < N(X_+) \). Consequently, from (5.4) we deduce \( H''(0) < 0 < H''(X_+) \). To complete the proof of \( ii \) it then suffices to show that \( N(X) \) is an increasing function in \([0, X_+]\). Indeed, \( N(X) \) being continuous, by virtue of the Mean Value Theorem this will imply that there exists a unique \( \overline{X} \in (0, X_+) \) such that \( N(X) < 0 \) for \( X \in [0, \overline{X}) \), \( N(\overline{X}) = 0 \) and \( N(X) > 0 \) for \( X \in (\overline{X}, X_+) \). Thus, from (5.4) we shall derive \( H''(X) < 0 \) for \( X \in [0, \overline{X}) \), \( H(\overline{X}) = 0 \) and \( H''(X) > 0 \) for \( X \in (\overline{X}, X_+) \), and \( ii \) will be proved with \( \overline{x} = x_0 + \overline{X} \in (x_0, x_+) \). Now, differentiating (5.6) with respect to \( X \) and using \( G'(X) = -(3/2)X[G(X)]^{-2} \) we find

\[
N'(X) = (9/2)X[G(X)]^{-2}N_1(X), \quad X \in (0, X_+),
\]

where

\[
N_1(X) = 2(5X^2 - 3x_0^2)[G(X)]^2 - 8(X^2 - 2x_0^2)[G(X)]^3 + 3(X^4 - 4x_0^2X^2 + 3x_0^4)
= 2(5X^2 - 3x_0^2)[G(X)]^2 + 21X^4 - 66x_0^2X^2 + 45x_0^4.
\]

(5.8)

If we can show \( N_1(X) > 0 \) in \((0, X_+)\), from (5.7) we get \( N'(X) > 0 \) and \( N(X) \) is an increasing function in \([0, X_+]\), completing our proof. Therefore, to our purposes, it would be suffice that \( N_1 \) is a decreasing function and that \( N_1(X_+) > 0 \) for \( x_0 < -\sqrt{3}/4 \). To show that \( N_1 \) is decreasing we study the sign of its first derivative \( N_1'(X) \). From (5.8) it follows:

\[
N_1'(X) = 20X[G(X)]^2 + 4(5X^2 - 3x_0^2)G(X)G'(X) + 84X^3 - 132x_0^2X
= X[G(X)]^{-1}\{20[G(X)]^3 - 6(5X^2 - 3x_0^2) + (84X^2 - 132x_0^2)G(X)\}
= 3X[G(X)]^{-1}\{21x_0^2 - 25X^2 - 4(11x_0^2 - 7X^2)G(X)\}, \quad X \in (0, X_+).
\]

(5.9)

Since \( 11x_0^2 - 7X^2 > 0 \) for \( X \in (0, X_+) \subset (0, \sqrt{11/7}|x_0|) \), \( N_1'(X) \geq 0 \) is equivalent to

\[
G(X) \leq \frac{21x_0^2 - 25X^2}{4(11x_0^2 - 7X^2)} := R(X).
\]

(5.10)

Now, \( R'(X) = -64(11x_0^2 - 7X^2)^{-2}x_0^2X < 0 \) in \((0, X_+)\). Then the function \( R \) on the right-hand side of (5.10) is decreasing and satisfies \( R(X) < R(0) = 21/44 \) for every \( X \in (0, X_+) \). On the other side, the function \( G \) being decreasing, we have \( 3/4 = G(X_+) < G(X) \), so that \( R(X) < 21/44 < 3/4 < G(X) \) for every \( X \in (0, X_+) \), which is incompatible with (5.10). From (5.9) it thus follows \( N_1'(X) < 0 \) in \((0, X_+)\) and \( N_1 \) is a decreasing function. It remains only to show that \( N_1(X_+) \) is positive for \( x_0 < -\sqrt{3}/4 \). But, recalling (5.8), an easy computation leads to \( N_1(X_+) = (27/4)(x_0^2 - 3/64) \) which is positive for \( x_0 < -\sqrt{3}/8 \) and a fortiori for \( x_0 < -\sqrt{3}/4 \). This completes the proof.

**Remark 5.1.** We stress that in the previous proof the standard procedure of calculus for locating the inflection point of \( H \) is not profitable, due to the difficulty in studying the sign of \( H'' \). Indeed, from (5.1) and (5.6) we deduce that solving \( H''(X) \geq 0 \) leads us to solve \( P(X) \geq 0 \), where \( P \) is a polynomial of degree \( \deg(P) = 14 \) in the unknown \( X \in (0, -x_0) \), with variable coefficients depending on \( x_0 \).
As usual, for any function $f : I \subset \mathbb{R} \to \mathbb{R}$, $I$ interval, we denote by $f^+ = \max\{-f, 0\}$ and $f^- = \max\{f, 0\}$ its positive and negative parts, respectively, such that $f = f^+ - f^-$, $|f| = f^+ + f^-$ and $-f^- \leq f \leq f^+$.

**Proof of Corollary 4.6.** Of course, $g_1(x) = 3x(2x - 3x_0)$, $x \in [2x_0, 0]$, decreases for $x \in [2x_0, (3/4)x_0]$ and increases for $x \in [(3/4)x_0, 0]$. Moreover, it is nonnegative in $[2x_0, (3/2)x_0]$ and non positive in $[(3/2)x_0, 0]$. Then, $h$ being a positive function, we have

\[
G^+_1(x) = \begin{cases} G_1(x), & x \in [2x_0, (3/2)x_0], \\
0, & x \in [(3/2)x_0, 0], \end{cases} \\
G^-_1(x) = \begin{cases} 0, & x \in [2x_0, (3/2)x_0], \\
-G_1(x), & x \in [(3/2)x_0, 0]. \end{cases}
\]

Assume first $x_0 \in [-\sqrt{3}/4, 0)$. In this case, due to Lemma 4.5, which implies that $h$ is convex function attaining its minimum in $x = x_0$, we have that $h$ does not increase in $[2x_0, x_0]$ and hence $h((3/2)x_0) \leq h(x)$ for $x \in [2x_0, (3/2)x_0]$. Therefore

\[
G_1(x) \leq G^+_1(x) \leq [h((3/2)x_0)]^{-1}g_1(2x_0), \quad \forall x \in [2x_0, 0].
\]

Thus, using $g_1(2x_0) = 6x_0^2$ and evaluating $h((3/2)x_0) = 2^{-5/3}(2^{1/3} + 9|x_0|^{2/3})^{1/2}|x_0|$ through formula (4.22), we obtain the estimate from above in (4.26), with $C_6(x_0)$ defined by (4.28). Instead, the estimate from below, with $C_5(x_0)$ defined by (4.27), follows by combining $-(3/2)^{8/3}|x_0|^{2/3} = g_1((3/4)x_0) \leq g_1(x) \leq 0$ and $0 < [(3/2)x_0]^{1/3} = h(x_0) \leq h(x)$ for $x \in [(3/2)x_0, 0]$, which yield

\[
-(3/2)^{8/3}|x_0|^{2/3} = [h(x_0)]^{-1}g_1((3/4)x_0) \leq G^-_1(x) \leq G_1(x), \quad \forall x \in [2x_0, 0].
\]

Let us now take $x_0 < -\sqrt{3}/4$. In this case, according to Lemma 4.5, the function $h$ attains its least value $|x_0^2 - (3/64)|^{1/2}$ at both the points $x_\pm = x_0 \pm (2x_0^2 - (3/16))^{1/2}$. Also, being convex in $[2x_0, 2x_0 - x] \cup [x, 0], x \in (x_0, x_+)$, and concave in $[2x_0 - x, x]$ with a local maximum in $x = x_0$, it decreases in $[2x_0, x_-] \cup [x, 0]$. We then distinguish the two sub-cases $x_0 \in (-1/2, -\sqrt{3}/4)$ and $x_0 \leq -1/2$, corresponding to $(3/2)x_0 < x_+$ and $x_+ \leq (3/2)x_0$, respectively. Let first be $x_0 \in (-1/2, -\sqrt{3}/4)$. Since $(3/2)x_0 < x_-$ and $h$ decreases in $[2x_0, x_-]$, the same reasonings as above for the case $x_0 \in [-\sqrt{3}/4, 0)$ lead to

\[
[h(x_+)]^{-1}g_1((3/4)x_0) \leq G_1(x) \leq [h((3/2)x_0)]^{-1}g_1(2x_0), \quad \forall x \in [2x_0, 0],
\]

i.e. (4.29) with $C_7(x_0)$ and $C_8(x_0)$ being defined, respectively, by (4.30) and the first expression in (4.31). Finally, if $x_0 \leq -1/2$, since $x_- \leq (3/2)x_0$, estimate (4.29), with $C_7(x_0)$ and $C_8(x_0)$ being defined, respectively, by (4.30) and the second expression in (4.31), follows from $G^+_1(x) \leq [h(x_-)]^{-1}g_1(2x_0)$ for $x \in [2x_0, (3/2)x_0]$ and $[h(x_+)]^{-1}g_1((3/4)x_0) \leq -G^-_1(x) \leq G_1(x)$ for $x \in [(3/2)x_0, 0]$. The proof is complete.

**Remark 5.2.** Notice that, since

\[
G'_1(x) = 3[h(x)]^{-3}\{3g_1'(x)[h(x)]^2 - g_1(x)[3 - 4g(x)](x - x_0)\}, \quad (5.11)
\]

to find the greatest and least values of $G_1$ by studying its first derivative $G'_1$ it is not computationally amenable. Indeed, from (5.11) we obtain $G'_1(x) = [h(x)]^{-3}S_1(x)$, where

\[
S_1(x) = 3(2x^3 - 6x_0x^2 + 7x_0^2x - 3x_0^3) - x(4x^2 - 13x_0x + 6x_0^2)g(x). \quad (5.12)
\]
Then, the study of $G'_1$ with the consequent location of the stationary points of $G_1$ yields, more or less, to the same computational difficulties that we have highlighted in Remark 5.1 as regards to the study of $H^p$ and the location of the inflection point of $H$. In fact, from (5.12) we get that solving $G'_1(x) \geq 0$ leads us to solve $P_1(x) \geq 0$, $P_1$ being a polynomial of degree $\deg(P_1) = 11$ in the unknown $x \in (2x_0, x)$, with variable coefficients depending on $x_0$.

Proof of Corollary 4.8. Using $g'(x) = -(3/2)[g(x)]^{-2}(x - x_0)$, $x \in (2x_0, 0)$, it is easy to verify that $g_2(x) = 4(2x - x_0)[g(x)]^{3/2}$ satisfies

$$
 g'_2(x) = -9[g(x)]^{-3/2}(4x^2 - 7x_0x + x_0^2), \quad \forall x \in (2x_0, 0).
$$

Then, $g_2$ decreases for $x \in (2x_0, x_1] \cup [x_2, 0]$ and increases for $x \in [x_1, x_2]$, where the points $x_j = [(7 + (-1)^{j+1}\sqrt{33})/8]x_0$, $j = 1, 2$, satisfy $x_1 \in ((13/8)x_0, (3/2)x_0)$ and $x_2 \in (x_0/4, x_0/8)$. In addition, it is non positive in $[2x_0, x_0/2]$ and nonnegative in $[x_0/2, 0]$. We thus have

$$
 G^+_2(x) = \begin{cases} 
 G_2(x), & x \in [x_0/2, 0], \\
 0, & x \in [2x_0, x_0/2],
\end{cases}
 \quad G^-_2(x) = \begin{cases} 
 -G_2(x), & x \in [2x_0, x_0/2], \\
 0, & x \in [x_0/2, 0].
\end{cases}
$$

Let $x_0 \in [-\sqrt{3}/4, 0)$. Due to Lemma 4.3, we have $h(x_0/2) \leq h(x)$ for $x \in [x_0/2, 0]$ and $h(x_0) \leq h(x)$ for $x \in [2x_0, x_0/2]$. Therefore, from $0 \leq g_2(x) \leq g_2(x_2)$ for $x \in [x_0/2, 0]$ and $g_2(x_1) \leq g_2(x) \leq 0$ for $x \in [2x_0, x_0/2]$, we find

$$
 [h(x_0)]^{-1}g_2(x_1) \leq -G^-_2(x) \leq G_2(x) \leq G^+_2(x) \leq [h(x_0/2)]^{-1}g_2(x_2), \quad \forall x \in [2x_0, 0].
$$

Estimate (4.32), with $C_k(x_0)$, $k = 9, 10$, defined by (4.33) and (4.34), now follows by evaluating

$$
 g_2(x_1) = (-1)^{j+1}2^{-7/2}3[\sqrt{33} + (-1)^{j+1}3][15 + (-1)^{j+1}\sqrt{33}]^{1/2}x_0^2, \quad j = 1, 2,
$$

and using $h(x_0) = [(3/2)x_0^4]^{1/3}$ and $h(x_0/2) = h((3/2)x_0) = 2^{-5/3}(2^{4/3} + 9|x_0|^{2/3})^{1/2}|x_0|$. Let us now assume $x_0 < -\sqrt{3}/4$. As in Corollary 4.6 we distinguish the two sub-cases $x_0 \in (-1/2, -\sqrt{3}/4)$ and $x_0 \leq -1/2$, corresponding to $x_+ < x_0/2$ and $x_0/2 \leq x_+$, respectively. Let first $x_0 \in (-1/2, -\sqrt{3}/4)$. Since $x_+ < x_0/2$ and $h$ is increasing in $[x_+, 0]$ by virtue of Lemma 4.3 (ii), we have $h(x_0/2) \leq h(x)$ for $x \in [x_0/2, 0]$, whereas $h(x_-) = h(x_+) \leq h(x)$ for $x \in [2x_0, x_0/2]$. Then, estimate (4.35), with $C_{11}(x_0)$ and $C_{12}(x_0)$ being defined, respectively, by (4.36) and the first expression in (4.37), follows from

$$
 [h(x_-)]^{-1}g_2(x_1) = [h(x_+)]^{-1}g_2(x_1) \leq -G^-_2(x) \leq G^+_2(x) \leq [h(x_0/2)]^{-1}g_2(x_2),
$$

for $x \in [x_0/2, 0]$. Instead, if $x_0 \leq -1/2$, since $x_0/2 \leq x_+$, from $h(x_+) \leq h(x)$ for $x \in [x_0/2, 0]$ and $h(x_-) \leq h(x)$ for $x \in [2x_0, x_0/2]$ we deduce

$$
 [h(x_-)]^{-1}g_2(x_1) \leq -G^-_2(x) \leq G_2(x) \leq G^+_2(x) \leq [h(x_+)]^{-1}g_2(x_2), \quad \forall x \in [2x_0, 0],
$$

i.e. estimate (4.35) with $C_{11}(x_0)$ and $C_{12}(x_0)$ being defined, respectively, by (4.36) and the second expression in (4.37). Finally, estimate (4.38), with $C_{13}(x_0)$ defined by (4.39), follows from (4.32) and (4.35) simply by observing that $0 < g_2(x_2) < [g_2(x_1)]$ and using $h(x_3) \leq h(x_0/2)$, where $x_3 = x_0$ if $x_0 \in [-\sqrt{3}/4, 0)$ and $x_3 = x_-$ if $x_0 < -\sqrt{3}/4$. The proof is complete.
Remark 5.3. We observe that also in the case of Corollary 4.8 to find the greatest and least values of $G_2$ by locating its stationary points is not profitable. For, $G''_2(x) = -3[g(x)]^{-3/2}[h(x)]^{-3}S_2(x)$, where

\[ S_2(x) = 3(2x^4 - 8x_0x^3 + 12x_0^2x^2 - 7x_0^3x + x_0^4) - x(4x^3 - 17x_0x^2 + 17x_0^2x + 2x_0^3)g(x). \]

Then, to solve $G'_2(x) \geq 0$ one is led to solve $P_2(x) \geq 0$, where $P_2$ is a polynomial of degree \( \text{deg}(P_2) = 14 \) in the unknown $x \in (2x_0, 0)$, with variable coefficients depending on $x_0$.

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