Quantum Affine Transformation Group and Covariant Differential Calculus

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We discuss quantum deformation of the affine transformation group and its Lie algebra in one-dimensional space. It is shown that the quantum algebra has a non-cocommutative Hopf algebra structure, simple realizations and quantum tensor operators. It is also shown that the quantum algebra does not have a universal $R$-matrix. We present a new method to construct the quantum deformation of the affine transformation group. The method is based on the quantum algebra and the adjoint representation. Furthermore, we construct a differential calculus which is covariant with respect to the action of the quantum affine transformation group.

§ 1. Introduction

Recent progress in understanding quantum integrability and non-commutative geometry has introduced the notion of quantum deformation of groups and Lie algebras. The quantum deformation of groups is characterized by the fact that the elements of its representation matrix do not mutually commute. And the quantum deformation of Lie algebras is understood that its universal enveloping algebra has the structure of non-cocommutative Hopf algebra. The former is called the quantum group and the latter the quantum algebra.

The paradigm of the quantum group is $SL_q(2)$, which is the deformation of $2 \times 2$ matrix group and the matrix elements obey certain commutation relations depending on a deformation parameter $q$. The algebra dual to $SL_q(2)$ is the quantum algebra $U_q(sl(2))$. Various aspects of $SL_q(2)$ and $U_q(sl(2))$ have been investigated by many authors. The quantum deformation of other groups and algebras of classical, exceptional and super has also been discussed. Furthermore, some generalizations to multi-parameter deformation have been attempted.

There are also some quantum deformations of the affine transformation groups, which play important roles in physics. The quantum deformation of the Euclidean group and algebra in two-dimensional space was discussed in Ref. 10). Celeghini et al. obtained the quantum Euclidean algebras and groups in two- and three-dimensional space using contraction procedures. They also succeeded to derive the universal $R$-matrices. One of the most interesting groups for physicists is the Poincaré group in four-dimensional space. Its quantum deformation was developed by Lukierski et al. in the sense of Drinfeld-Jimbo and by Ogievetsky et al. in the sense of Woronowicz. The deformation of the affine transformation group corresponding to $U(n)$ and $SO(n)$ in arbitrary dimensional space was accomplished by Schlieker et al., however their result has an additional dilatation which has no analogue in the limit of $q \to 1$. Chaichian and Demichev overcame this problem by

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considering a multiparametric quantum space.\textsuperscript{15) As an application of their procedure, they discussed the quantum Poincaré group and algebra in four-dimensional quantum Minkowski space.\textsuperscript{16) However the existence of universal $R$-matrices is still an open problem.}

However, there is no literature which discusses the quantum deformation of the affine transformation algebra in one-dimensional space. In this paper, we examine the quantum deformation of the group and the algebra of the affine transformation in one-dimensional space. The undeformed group is generated by only two generators,

\begin{equation}
\begin{align*}
\hat{p} &= -i\frac{d}{dx}, \\
\hat{d} &= \frac{i}{2} \left( x \frac{d}{dx} + \frac{d}{dx} x \right),
\end{align*}
\end{equation}

where $\hat{p}$ is the momentum operator in one-dimensional quantum mechanics and $\hat{d}$ is the dilatation operator. The one-dimensional coordinate $x$ is transformed as

\begin{equation}
\exp(-iap)x \exp(iap) = x-a,
\end{equation}

and

\begin{equation}
\exp(ilnb d)x \exp(-ilnb d) = bx.
\end{equation}

This would be a too restrictive feature to cooperate an additional structure like quantum deformation and reminds us of another simple algebra in one-dimensional space, the harmonic oscillator, which plays the important physical role as a spectrum generating algebra. The $q$-harmonic oscillator's algebra\textsuperscript{17) has not been found a 'complete' Hopf algebra structure. Also from a viewpoint of application, it is interesting to construct quantum analogues of the transformations because the undeformed transformations play important roles in various fields in physics.

We shall show that the deformation of the Lie algebra (1·1) is possible to be established as a non-cocommutative Hopf algebra. We call it the quantum affine transformation algebra (QATA). We shall also develop its representation theory and show that the QATA is not quasi-triangular. Using the QATA and the adjoint representation matrix of the affine transformation group, the quantum affine transformation group (QATG) is derived. The QATG is identical to the one-dimensional case introduced in Refs. 14) and 15), however our approach is completely different from theirs. Further we shall construct differential calculus on the quantum plane\textsuperscript{18) which is covariant under the action of the QATG.

This paper is organized as follows. In the next section, we briefly review the representation of the affine transformation group and the Lie algebra. In § 3, we discuss the QATA. Some realizations of the QATA and a quantum analogue of the adjoint representation are constructed. The quantum analogue of the tensor operator\textsuperscript{19) which carries the adjoint representation is explicitly given in terms of the generators of the QATA. It is shown that the QATA does not have the universal $R$-matrix. In § 4, using the QATA and the adjoint representation matrix of the affine transformation group, we establish the QATG. The QATG covariant differential calculus is constructed in § 5. Finally, conclusion and discussion are in § 6.
§ 2. $D=1$ affine transformation group

In this section, we give a short review of the affine transformation group in one-dimensional space. The affine transformation group is abstractly defined as the group of linear transformations without reflection on real line: $x \rightarrow bx - a$. In our particular parametrization the elements of the group are denoted by

$$U(a, b) = \exp(iap) \exp(-ilnb d),$$  \hspace{1cm} (2·1)

where the Lie algebra $A=\{p, d\}$ satisfies the commutation relation,

$$[d, p] = ip.$$  \hspace{1cm} (2·2)

The allowed region of the parameters is $-\infty < a < \infty$, $0 < b < \infty$, which requires that the group manifold is a half-plane, and so the affine transformation group is a non-Abelian and non-compact group. The group multiplication law is given by

$$U(a, b) U(a, \beta) = U(a + ab, \beta b).$$  \hspace{1cm} (2·3)

The unitary representations of the affine transformation group are found by Gel'fand and Naimark. Aslaksen and Klauder gave an alternative proof. It was shown in Refs. 20 and 21 that there exist two and only two unitary and inequivalent irreducible representations. One is the case of positive eigenvalue of $p$ and the other the case of negative. Namely, when the representation space $V$ is taken to the space of all functions $\phi(k) \in L^2(\mathbb{R})$,

$$\int_{-\infty}^{\infty} |\phi(k)|^2 dk < \infty,$$

$V$ is the direct sum of two invariant subspaces under the action of $U(a, b)$; $V = V_+ \oplus V_-$, where

$$V_+ = \{ \phi(k): \phi \in L^2(\mathbb{R}), \phi(k) = 0 \text{ for } k \leq 0 \},$$

$$V_- = \{ \phi(k): \phi \in L^2(\mathbb{R}), \phi(k) = 0 \text{ for } k \geq 0 \}. \hspace{1cm} (2·4)$$

For example, if we take the following realization of the Lie algebra $A$,

$$p = k, \hspace{1cm} d = \frac{i}{2} (k \partial_k + \partial_k k),$$  \hspace{1cm} (2·5)

then the space $V_+$ consists of only $\phi(k)$,

$$\phi(k) = Nk^{1/2} e^{-k},$$

where $N$ is the normalization factor. The action of $U(a, b)$ is given by

$$U(a, b) \phi(k) = \exp(ia k) b^{1/2} \phi(b^{1/2} k).$$

In the matrix representation of the affine transformation group, the adjoint representation is the simplest one. The adjoint representation of the algebra $A$,
\( \rho = \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \) \hspace{1cm} (2.6)

gives the adjoint representation of the affine transformation group by substitution of (2.6) into (2.1),

\[ U(a, b) = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \] \hspace{1cm} (2.7)

The adjoint representation is not a unitary one, since it is a finite dimensional representation and the affine transformation group is non-compact.

We explicitly write down the adjoint action for later convenience,

\[ \varphi(p) = U(a, b)pU^{-1}(a, b) = bp, \]
\[ \varphi(d) = U(a, b)dU^{-1}(a, b) = d + ap. \] \hspace{1cm} (2.8)

\( \varphi(p) \) and \( \varphi(d) \) also satisfy the commutation relation (2.2).

§ 3. \( D=1 \) quantum affine transformation algebra

3.1. \( QATA \) and its realization

We present a one-parameter deformation of the universal enveloping algebra of \( A \), i.e., the QATA and its realizations. First, we define QATA as the algebra which is generated by two elements \( P \) and \( D \) satisfying the commutation relation,

\[ [D, P] = i[P], \] \hspace{1cm} (3.1)

where \( [P] \equiv (q^p - q^{-p})/(q - q^{-1}) \) and \( q \) is as usual the deformation parameter. This is a non-cocommutative Hopf algebra. The Hopf algebra mappings, coproduct \( \Delta \), counit \( \epsilon \) and antipode \( S \), are given by

\[ \Delta(P) = P \otimes 1 + 1 \otimes P, \]
\[ \Delta(D) = D \otimes q^{-p} + q^p \otimes D, \]
\[ \epsilon(P) = \epsilon(D) = 0, \]
\[ S(P) = -P, \quad S(D) = -D - i(\ln q)[P], \] \hspace{1cm} (3.2)

and Eqs. (3.2) certainly satisfy the following axioms of the Hopf algebra,

\[ (id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta, \]
\[ (id \otimes \epsilon) \circ \Delta = (\epsilon \otimes id) \circ \Delta = id, \]
\[ m(id \otimes S) \circ \Delta = m(S \otimes id) \circ \Delta = 1 \epsilon, \] \hspace{1cm} (3.3)

where \( id \) denotes the identity mapping and \( m \) the product of the two terms in the tensor product; \( m(x \otimes y) = xy \). If we define the opposite coproduct \( \Delta' \) by

\[ \Delta' = \sigma \circ \Delta, \quad \sigma(x \otimes y) = y \otimes x, \] \hspace{1cm} (3.4)
\( \Delta'(P) \) and \( \Delta'(D) \) also satisfy the same commutation relation as (3·1). For the opposite coproduct, \( S(P) \) and the counit are no changed while \( S(D) \) becomes
\[
S(D) = -D + i(\ln q)[P].
\] (3·5)

Next, we show some realizations of the QATA. The generators \( P \) and \( D \) can be formally expressed in terms of the undeformed ones,
\[
P = p, \quad D = \frac{1}{2} \left( \frac{[p]}{p} [d] + d \frac{[p]}{p} \right). \tag{3·6}
\]
When the representation of \( p \) and \( d \) in the Hilbert space is considered, \( p \) and \( d \) are hermitian operators. The realization (3·6) of \( P \) and \( D \) is also chosen to be hermitian in the same representation space when \( q \) is real or \( |q|=1 \). If we require only satisfying the commutation relation (3·1), \( D \) can be simply given by
\[
D = \frac{[p]}{p} d. \tag{3·7}
\]
When the representation and the realization of \( A \) have the inverse \( p^{-1} \), or \([p]\) are proportional to \( p \), they can be transformed into those of the QATA by making use of (3·6) or (3·7).

**Examples**

1. The case of having \( p^{-1} \). The realization of Eq. (2·5) is transformed into
\[
P = k, \quad D = \frac{i}{2} \left( [k] \partial_k + \partial_k [k] \right). \tag{3·8}
\]
It is easy to verify that the commutation relation (3·1) holds.

2. On the other hand, the adjoint representation of the QATA cannot be obtained by naive use of the relation (3·6) since the adjoint representation of the undeformed generators (2·6) does not have \( p^{-1} \). However \([p]\) reduces to be proportional to \( p \), i.e.,
\[
[p] = \delta p, \quad \delta = \frac{2\ln q}{q-q^{-1}},
\]
so we get the adjoint representation of \( P \) and \( D \) with the aid of Eq. (3·6);
\[
P = \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & i\delta \end{pmatrix}. \tag{3·9}
\]
The adjoint representation of the QATA is the matrices (2·6) multiplied by \( q \)-dependent factor \( \delta \). The factor \( \delta \) becomes unity as \( q \to 1 \).

3.2. **Tensor operators**

In this subsection, we show explicit expression of the tensor operator which carries the adjoint representation of the QATA. The definition of tensor operators of a quantum algebra was given by Rittenberg and Scheunert in terms of the representation theory of the Hopf algebra. The tensor operator is generally defined through the following adjoint action of the Hopf algebra \( H \). The adjoint action of \( c \in H \) on
$i \in H$ is defined by
\[ ad(c) t = \sum_i c_i S(c_i), \] (3.10)
where the coproduct of $c$ is denoted by
\[ \Delta(c) = \sum_i c_i \otimes c_i. \]

Writing the $n \times n$ matrix representation of $c$ as $\rho_\omega(c)$, the tensor operators \{ $T_i$, $i=1, 2, \cdots, n$ \} which carry the representation $\rho(c)$ are defined by the relation,
\[ ad(c) T_i = \sum_j \rho_{ij}(c) T_j. \] (3.11)

Namely, the tensor operators \{ $T_i$ \} form a representation basis under the adjoint action.

Now in the case of the QATA, we can write down the adjoint action of \{ $P, D$ \} on an element of the QATA $T$,
\[
\begin{align*}
ad(P) T &= [P, T], \\
ad(D) T &= D T q^p - q^p T D - i (\ln q) q^p T [P].
\end{align*}
\] (3.12)

We therefore find the tensor operators which carry the adjoint representation of (3.9):
\[
\begin{align*}
T_1 &= q^{-p} D, \\
T_2 &= q^{-p} [P].
\end{align*}
\] (3.13)

It is noted that the RHS of Eq. (3.12) reduces to the commutators and Eq. (3.13) to $d$ and $\rho$ in the limit $q \rightarrow 1$.

3.3. Intertwiner and Yang-Baxter equation

As is mentioned in § 3.1, we have two coproduct; $\Delta$ and the opposite $\Delta'$. In conformity with this fact, we have two tensor product representations and are urged to question on an intertwiner between them. In this subsection, the intertwiner in the case of the adjoint representations is investigated as an example. The coproduct of two adjoint representations $\Delta$ and the opposite are $4 \times 4$ matrices and we should find a $4 \times 4$ matrix $R$ which satisfies the relations of intertwiner,
\[ R \Delta(P) R^{-1} = \Delta'(P), \quad R \Delta(D) R^{-1} = \Delta'(D). \] (3.14)

The solution of (3.14) is of the following form,
\[
R = \begin{pmatrix}
r & 0 & 0 & 0 \\
0 & r & g & 0 \\
0 & 0 & r_{33} & 0 \\
0 & r_{42} & r_{43} & r_{33}
\end{pmatrix},
\] (3.15)
where
\[
r_{33} = \frac{i - h}{i + h} r, \quad r_{43} = \frac{2h}{i + h} r,
\]
\[ r_{45} = -ihg + 2ir_{33}, \quad h = \ln q. \]  

(3·16)

The constants \( r \) and \( g \) are undetermined from Eq. (3·14). We note that one of the undetermined constants is meaningless because it is nothing but the overall multiplication factor on \( R \).

The next stage of our interests is whether the matrix \( R \) satisfies the YBE,

\[ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}, \]  

(3·17)

where \( R_{13} \), etc., mean \( R \otimes \text{id} \), etc., which act on the tensor product of three representation spaces. Substituting (3·15) into (3·17), it is not difficult to see that the YBE is satisfied if and only if \( r = 0 \). However, this case is not legitimate since the matrix \( R \) does not have the inverse.

In summary, the intertwiner for the tensor product of two adjoint representations exists, however it is incompatible with the YBE. From this result, we deduce that our QATA does not possess the universal \( R \)-matrix. It is an open question whether another kinds of deformation would have a universal \( R \)-matrix or not.

§ 4. \( D=1 \) quantum affine transformation group

In this section, we derive a quantum deformation of the affine transformation group in one-dimensional space using the quantum algebra introduced in the previous sections. Our approach exactly reproduce the one-dimensional case obtained in Refs. 14) and 15).

We have a simple matrix representation of the affine transformation group, i.e., the adjoint representation (2·7). The representation basis of the adjoint representation is the Lie algebra of the affine transformation group (see Eq. (2·1)). Hence our construction of the QATA should be based on the deformation of the Lie algebra. The deformation which is considered in the previous section is not appropriate for our purpose since the RHS of (3·1) is an infinite power series of \( P \) and so the deformed matrix becomes infinite dimensional. Our aim here is to discuss the deformation of \( 2 \times 2 \) matrix group.

To this end, we use the following new basis instead of \( P \) and \( D \),

\[ \vec{D} \vec{P} - q \vec{P} \vec{D} = i \vec{P}, \]  

(4·1)

where these new elements are transformed from those discussed in § 3 as follows:

\[ \vec{P} = f(\Lambda) P, \]

\[ \vec{D} = i q^{(A-1)/2}[A]_{1/2}. \]  

(4·2)

\( f(\Lambda) \) is an arbitrary function of \( \Lambda \) provided that \( f(\Lambda) \rightarrow 1 \) as \( q \rightarrow 1 \) and

\[ \Lambda = -i \left[ \frac{P}{[P]} \right] D, \]

\[ [A]_{1/2} = q^{A/2} - q^{-A/2} \]

\[ q^{1/2} - q^{-1/2}. \]  

(4·3)
Let us consider the situation that the new basis \( \{ \bar{D}, \bar{P} \} \) become a comodule algebra of the adjoint representation matrix, namely, under the following transformation,

\[
(\varphi(\bar{D}), \varphi(\bar{P})) = (\bar{D}, \bar{P}) \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix},
\]

(4.4)

\( \varphi(\bar{D}) \) and \( \varphi(\bar{P}) \) also satisfy the relation (4.1), where we assume that \( \bar{D} \) and \( \bar{P} \) commute with \( a \) and \( b \). This requirement deforms \( a \) and \( b \) into the non-commutative objects which satisfy the ‘quantum plane’ commutation relation,

\[
ab = qba.
\]

(4.5)

We thus obtain the QATG as the matrix,

\[
\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix},
\]

(4.6)

whose elements satisfy the relation (4.5).\(^{14,15}\) The Hopf algebra structure is given by\(^{14,15}\)

\[
\Delta \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix},
\]

(4.7)

\[
e \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

(4.8)

\[
S \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}^{-1} = b^{-1} \begin{pmatrix} b & 0 \\ -a & 1 \end{pmatrix}.
\]

(4.9)

It should be noted that \( b^{-1} \) is needed to define the antipode.

We thus complete the Hopf algebra \( \{ a, b, b^{-1} \} \) adding the following properties of \( b^{-1} \),

\[
bb^{-1} = b^{-1}b = 1, \quad ab^{-1} = a^{-1}b^{-1}a,
\]

(4.10)

\[
\Delta(b^{-1}) = b^{-1} \otimes b^{-1}, \quad e(b^{-1}) = 1, \quad S(b^{-1}) = b.
\]

(4.11)

It is emphasized that Eq. (4.6) is not a ‘subgroup’ of \( GL_q(2) \). If it were so, it should preserve commutation relation \( xy = qyx \) of the quantum plane \( (x, y) \) under the action,

\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},
\]

where \( (x, y) \) is assumed to commute with \( a \) and \( b \). It is easy to see that the commutation relation is not preserved by the action of the QATG.

It is possible to express the defining relation of the QATG in terms of the R-matrix. Considering the tensor product of (4.6) and the 2×2 unit matrix,
the commutation relation (4·5) should be expressed as
\[ RT_1 T_2 = T_2 T_1 R , \] (4·13)
where the 4×4 matrix \( R \) is the \( R \)-matrix of the QATG (we use the same notation as in § 3.3, but it will make no serious confusion). The solutions of Eq. (4·13) are given by
\[
\begin{align*}
R^{(1)} &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & q & 1-q & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, & R^{(2)} &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1-q^{-1} & q^{-1} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\end{align*}
\] (4·14)
and it is verified that both \( R^{(1)} \) and \( R^{(2)} \) satisfy the YBE,
\[ R_{12}^{(i)} R_{13}^{(j)} R_{23}^{(k)} = R_{23}^{(k)} R_{13}^{(j)} R_{12}^{(i)} , \quad i=1, 2. \] (4·15)
\( R^{(2)} \) is identical to \( D=1 \) case of the multiparametric \( R \)-matrix discussed in Ref. 15).
To clarify the relation between \( R^{(1)} \) and \( R^{(2)} \), we work with
\[
\tilde{R}^{(i)} = \sigma R^{(i)} , \quad \sigma = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\] (4·16)
and the explicit formulae are given by
\[
\begin{align*}
	ilde{R}^{(1)} &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & q & 1-q & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, & 	ilde{R}^{(2)} &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1-q^{-1} & q^{-1} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\end{align*}
\] (4·17)
From (4·17), it turns out that \( \tilde{R}^{(1)} \) is the inverse of \( \tilde{R}^{(2)} \). Both of the \( \tilde{R} \)-matrices satisfy the YBE,
\[ \tilde{R}^{(i)} \tilde{R}^{(j)} \tilde{R}^{(k)} = \tilde{R}^{(k)} \tilde{R}^{(j)} \tilde{R}^{(i)} , \quad i=1, 2. \] (4·18)
where \( \tilde{R}_{12} = \tilde{R} \otimes 1 \) and \( \tilde{R}_{23} = 1 \otimes \tilde{R} \).
For later convenience, we write down the transposed matrix of (4·6) as well,
\[
\begin{pmatrix}
1 & a \\
0 & b
\end{pmatrix},
\] (4·19)
and define
\[
T_1 = \begin{pmatrix}
1 & a \\
0 & b
\end{pmatrix} \otimes 1 , \quad T_2 = 1 \otimes \begin{pmatrix}
1 & a \\
0 & b
\end{pmatrix}.
\] (4·20)
The commutation relation (4·5) can be expressed as

$$ R' T_i T_j = T_i T_j R' \quad (4\cdot21) $$

with

$$ R'^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1-q & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, $$

$$ R'^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 1-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4\cdot22) $$

Again defining $\tilde{R}'^{(i)} = a R'^{(i)}$, it turns out that $\tilde{R}'^{(1)}$ is the inverse of $\tilde{R}'^{(2)}$ and both $\tilde{R}'^{(1)}$ and $\tilde{R}'^{(2)}$ satisfy the YBE (4·18). The relationship among $\tilde{R}'^{(i)}$ and $\tilde{R}'^{(i)}$ is summarized in Fig. 1.

In closing this section, it should be pointed out that $\varphi: F \rightarrow F \otimes \text{QATG}, F = \{D, \bar{P}\}$ is consistent with the coalgebra of (4·7) and (4·8),

$$(\text{id} \otimes D) \circ \varphi = (\varphi \otimes \text{id}) \circ \varphi,$$

$$(\text{id} \otimes \varepsilon) \circ \varphi = \text{id}. \quad (4\cdot23)$$

§ 5. QATG covariant differential calculus

In this section, we introduce a two-dimensional plane $(x^1, x^2)$ on which the QATG acts and develop a differential calculus on the plane. $GL_q(n)$ covariant differential calculus on the quantum plane was already constructed by Wess and Zumino. They considered the $n$-dimensional quantum plane and required that exterior derivative satisfy the usual properties: the nilpotency and the Leibnitz rule so that the commutation relations among coordinates, differentials and derivatives must satisfy various consistency conditions. The obtained formulae of the differential calculus are covariant with respect to the action of $GL_q(n)$.

According to Ref. 18), we wish to construct the differential calculus covariant with respect to the action of the QATG. It is natural to consider a two-dimensional quantum plane since our QATG is based on two-dimensional representation. The quantum plane which is covariant under the $D=1$ QATG reads

$$ x^1 x^2 = x^2 x^1, $$

$$ dx^1 dx^2 = -q^{-1} dx^2 dx^1, $$

$$(dx^i)^2 = 0. \quad (5\cdot1)$$
It is easy to verify that this quantum plane is comodule algebra of the QATG, namely, the action $\varphi$ of the QATG on this plane,

$$
\begin{pmatrix}
\varphi(x^1) \\
\varphi(x^2)
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
a & b
\end{pmatrix}
\begin{pmatrix}
x^1 \\
x^2
\end{pmatrix},
$$

$$
\begin{pmatrix}
\varphi(dx^1) \\
\varphi(dx^2)
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
a & b
\end{pmatrix}
\begin{pmatrix}
dx^1 \\
dx^2
\end{pmatrix},
$$

(5.2)

is consistent with the coproduct and the counit,

$$(\Delta \otimes id) \circ \varphi = (id \otimes \Delta) \circ \varphi,$$

$$(\varepsilon \otimes id) \circ \varphi = id.$$

(5.3)

Now we introduce the derivatives on our quantum plane,

$$
\partial_i = \frac{\partial}{\partial x^i}, \quad \partial_i x^j = \delta^j_i.
$$

(5.4)

Since the derivatives should be contravariant, the QATG acts on the derivatives,

$$
\begin{pmatrix}
\varphi(\partial_1) \\
\varphi(\partial_2)
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
a & b
\end{pmatrix}^T
\begin{pmatrix}
S & 0 \\
a & b
\end{pmatrix}
\begin{pmatrix}
\partial_1 \\
\partial_2
\end{pmatrix},
$$

(5.5)

where $T$ denotes the transposition. The map (5.5) preserves the relation (5.4) and satisfies the consistency conditions (5.3). The exterior derivative is then defined in the standard way,

$$
d = dx^i \partial_i,
$$

(5.6)

where the sum over $i$ is understood. It is required that $d$ satisfies the nilpotency and the Leibnitz rule,

$$
d^2 = 0,$$

$$
d(fg) = (df)g + (-1)^f f dg,$$

(5.7)

where $f$ and $g$ are $p$-form and $(-1)^f$ is $-1$ for odd (even) element. If we write (5.1) and (5.3) as

$$
x^i x^j = B^{i j} x^k x^l,
$$

$$
dx^i dx^j = -C^{i j} dx^k dx^l,
$$

(5.8)

and the commutation relations of the derivatives as operator,

$$
\partial_i \partial_j = F^{i j} \partial_k \partial_l,$$

(5.9)

the requirements of (5.7) determine the other commutation relations as follows:

$$
x^i dx^j = C^{i j} dx^k x^l,
$$

$$
\partial_i x^j = \delta^j_i + C^{i j} x^k \partial_k,
$$

$$
\partial_i dx^j = (C^{-1})^i j dx^k \partial_k,
$$

(5.10)

where the matrices $B$, $C$ and $F$ must satisfy the relations.\(^{18}\)
It is obvious that Eqs. (5.11) are satisfied by the following choice,

\[ B = F = R^{(1)}, \quad C = q^{-1} R^{(1)}, \]

where \( R^{(1)} \) is given in Eq. (4.17) and Eq. (5.8) reduces to Eq. (5.1). Of course, we have another solution,

\[ B = F = R^{(2)}, \quad C = q R^{(2)}. \]

Eq. (5.12) or (5.13) completes the differential calculus. It is not difficult to verify that all relations of the differential calculus described above are preserved under the action of the QATG, i.e., the linear transformation \( \varphi \) given in Eqs. (5.2) and (5.5).

Now we have obtained two differential calculi on the quantum plane (5.1). As an illustration, we give explicit commutation relations of the differential calculus in the case of (5.12). Denoting the coordinates by \((x, y)\) instead of \((x^1, x^2)\), the QATG covariant quantum plane is

\[ xy = yx, \]
\[ dxdy = -q^{-1} dydx, \]
\[ (dx)^2 = (dy)^2 = 0. \]

The commutation relations between the coordinates and the differentials are

\[ xdx = q^{-1} dxx, \]
\[ xdy = q^{-1} dyx, \]
\[ ydx = dxy + (q^{-1} - 1) dyx, \]
\[ ydy = q^{-1} dyy. \]

We have derivatives which satisfy the relation,

\[ \partial_x \partial_y = q^{-1} \partial_y \partial_x. \]

The commutation relations between the derivatives and the coordinates are

\[ \partial_x x = 1 + q^{-1} x \partial_x, \]
\[ \partial_x y = y \partial_x, \]
\[ \partial_y x = q^{-1} x \partial_y, \]
\[ \partial_y y = 1 + (q^{-1} - 1) x \partial_x + q^{-1} y \partial_y, \]

and those between the derivatives and the differentials are
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\[
\begin{align*}
\partial_x dx &= q dx \partial_x + (q-1) dy \partial_y, \\
\partial_x dy &= q dy \partial_x, \\
\partial_y dx &= dx \partial_y, \\
\partial_y dy &= q dy \partial_x.
\end{align*}
\]

(5.18)

Using the relation (5·17), one can calculate the derivative of the monomial,

\[
\begin{align*}
\partial_x (x^n y^m) &= \frac{1-q^{-n}}{1-q^{-1}} x^{n-1} y^m, \\
\partial_y (x^n y^m) &= q^{-m} \frac{1-q^{-m}}{1-q^{-1}} x^n y^{m-1}.
\end{align*}
\]

(5.19)

These formulae are utilized for estimating the derivative of a power series. In this way, we can calculate any higher derivatives of an arbitrary regular function of \(x\) and \(y\).

Finally, we mention another type of the quantum plane which is covariant under the action of the transposed matrix,

\[
\left( \begin{array}{c} 
\varphi(x^1) \\
\varphi(x^2) 
\end{array} \right) = \left( \begin{array}{cc}
a & 1 \\
b & 0 
\end{array} \right) \left( \begin{array}{c} 
x^1 \\
x^2 
\end{array} \right).
\]

(5.20)

The covariant quantum plane with respect to (5·20) is amount to

\[
\begin{align*}
xy &= qyx, \\
dxdy &= -dydx, \\
(dx)^2 &= (dy)^2 = 0.
\end{align*}
\]

(5.21)

Also in this case, it is possible to construct two differential calculi on this quantum plane and all their formulae are preserved by the action (5·20). The differential calculi are given by

\[
B = F = \tilde{R}^{(1)}, \quad C = q^{-1} \tilde{R}^{(1)},
\]

(5.22)

and

\[
B = F = \tilde{R}^{(2)}, \quad C = q \tilde{R}^{(2)}.
\]

(5.23)

We have, as a result, obtained four types of differential calculus. The relations among them are recognized through Fig. 1.

§ 6. Conclusions

In this paper, we discussed the quantum deformation of the affine transformation group and its Lie algebra in one-dimensional space. The Lie algebra was deformed in the sense of Drinfeld and Jimbo and it was shown that the QATA possesses the non-cocommutative Hopf algebra structure. The tensor product of two adjoint
representations of the QATA has an intertwiner, however the intertwiner was not compatible with the YBE. From this fact, it was inferred that the QATA is not quasi-triangular. The deformation, in the sense of Manin, of the affine transformation group was also accomplished by making use of the adjoint representation. Manin considered the deformation on the quantum plane whose elements commute each other in the limit $q \to 1$, while the bases of the QATG do not commute even in the limit $q \to 1$ because they form the Lie algebra. This is the main difference between Manin’s and our approaches. It was also shown that the QATG has the Hopf algebra structure.

The duality relation such as $U_q(sl(2))$ and $SL_q(2)$ does not hold in the case of the QATA and the QATG. The duality pairing of $U_q(sl(2))$ and $SL_q(2)$ are given in terms of the $R$-matrix, while on the other hand the QATA and the QATG do not have common $R$-matrices. It is an interesting problem to find a dual algebra to the QATG in arbitrary dimensional space.

In § 5, we constructed the differential calculi which are covariant with respect to the action of the QATG. The differential calculus is a QATG comodule algebra which is generated by $\{x^i, dx^i, \partial_i\}$. All the commutation relations of the differential calculus are written by using the $\bar{R}$-matrix. This would be generalized to the higher dimensional case by using the multiparametric quantum plane in Ref. 15).

The QATA and the QATG are simple in form and each algebraic structure is well established. We expect that the QATA and the QATG would be available for building blocks of other quantum algebras or groups and might play a crucial role in physics similarly in the case $q=1$.

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