A regularity criterion for the Navier Stokes equation involving only the middle eigenvalue of the strain tensor

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Abstract

This manuscript derives an evolution equation for the symmetric part of the gradient (the strain tensor) in the incompressible Navier Stokes equation on $\mathbb{R}^3$, and proves the existence of $L^2$ mild solutions to this equation. We use this equation to obtain a simplified identity for the growth of enstrophy for mild solutions that depends only on the strain tensor, not on the nonlocal interaction of the strain tensor with the vorticity. The resulting identity allows us to prove a new family of scale-critical necessary and sufficient conditions for blow-up of the solution at time $T_{\text{max}} < +\infty$, which depend only on the history of the positive part of the second eigenvalue of the strain matrix. Since this matrix is trace-free, this severely restricts the geometry of any finite-time blow-up. This regularity criterion provides analytical evidence of the numerically observed tendency of the vorticity to align with the eigenvector corresponding to the middle eigenvalue of the strain matrix. This regularity criterion allows us to prove as a corollary a new scale critical one component type regularity criterion for a range of exponents for which there were previously no known critical one component type regularity criteria. Our analysis also permits us to extend the known time of existence of smooth solutions with initial enstrophy $E_0 = \frac{1}{2}\|\nabla \otimes v^0\|_{L^2}^2$ fixed by a factor of 4,920.75—which the previous constant in the literature was not expected to be close to optimal, so this improvement is less drastic than it sounds, especially compared with numerical results. Finally, we will prove some properties about blowup for a toy model ODE of the strain tensor evolution equation.

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1 Introduction

The Navier Stokes equations are the fundamental equations of fluid dynamics. The incompressible Navier Stokes equations are given by

\[ u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \]
\[ \nabla \cdot u = 0. \]  

(1)

where \( u \in \mathbb{R}^3 \) denotes the velocity and \( p \) the pressure. For incompressible flow with no external force, the pressure is defined in terms of \( u \), by taking the divergence of both sides of the equation, which yields

\[- \Delta p = \sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}. \]  

(2)

Much about their solutions is unknown; as the only estimates presently available are supercritical in three spatial dimensions with respect to the scaling \( u^\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t) \) that preserves the solution set of (1).

To begin with, we will define a few matrices. The gradient tensor will be given by \( (\nabla \otimes u)_{ij} = \frac{\partial u_i}{\partial x_j} \). The symmetric part \( S \), which we will refer to as the strain tensor, will be given by \( S_{ij} = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \), and the anti-symmetric part \( A \), given by \( A_{ij} = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \right) \). It is immediately clear that \( S \) is symmetric, \( A \) is anti-symmetric, and that \( \nabla \otimes u = S + A \). We’ll also define the transformation \( \nabla_{sym} \) to map a vector to the symmetric part of it’s gradient tensor: \( \nabla_{sym}(v)_{ij} = \frac{1}{2} \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) \). We will also note that in three spatial dimensions the anti-symmetric matrix \( A \) can be represented as a vector, which we will call the vorticity. The vorticity, given by the curl of the flow \( \omega = \nabla \times u \), plays a very important role in fluid mechanics. It is related to \( A \) by

\[ A = \frac{1}{2} \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}. \]  

(3)

The equation for the evolution of vorticity \( \omega \) is written as follows:

\[ \omega_t - \Delta \omega + (u \cdot \nabla)\omega - S \omega = 0. \]  

(4)

The vortex stretching term \( S \omega \) is often written \( (\omega \cdot \nabla)u \), but it is clear from (3) that \( A \omega = 0 \), therefore,

\[ (\omega \cdot \nabla)u = (S + A) \omega = S \omega. \]  

(5)

The vorticity equation (4) is also a nonlocal equation because \( u = \nabla \times (-\Delta)^{-1} \omega \). Here vorticity is a vector representation of the antisymmetric part of \( \nabla \otimes u \), which we can represent as a vector only in the case of \( \mathbb{R}^3 \). The vortex stretching term \( S \omega \) is the quadratic nonlinearity, but does not have a very natural structure relative to the space of divergence free vector fields. For this reason, we will consider the evolution equation for the strain tensor. In three spatial dimensions the evolution equation for the strain is given by

\[ S_t + (u \cdot \nabla)S - \Delta S + S^2 + \frac{1}{4} \omega \otimes \omega - \frac{1}{4} |\omega|^2 I_3 + \text{Hess}(p) = 0, \]  

(6)

as we will show in the next section.
The strain tensor has already been studied \cite{12,15,33} in terms of its importance for enstrophy production in the vorticity equation (4). The evolution equation for the strain tensor, while it requires additional terms, has a quadratic nonlinearity whose structure is far better from an algebraic point of view. This is because a vector cannot be squared, and the square of an anti-symmetric matrix (the other representation of vorticity) is a symmetric matrix, while the square of a symmetric matrix is again a symmetric matrix.

It is well known that
\[
\partial_t \frac{1}{2} ||\omega(\cdot, t)||_{L^2}^2 = -||\omega||_{H^1}^2 + \langle S, \omega \otimes \omega \rangle.
\]
Using the Sobolev embedding of $\dot{H}^1 (\mathbb{R}^3)$ into $L^6 (\mathbb{R}^3)$ it follows from (7) that
\[
\partial_t ||\omega(\cdot, t)||_{L^2}^2 \leq C||\omega(\cdot, t)||_{L^2}^6,
\]
which is sufficient to guarantee regularity at least locally in time, but cannot prevent blowup because it is a cubic differential inequality. Enstrophy can be defined equivalently as
\[
E(t) = \frac{1}{2} ||\nabla \otimes u(\cdot, t)||_{L^2}^2 = \frac{1}{2} ||\omega(\cdot, t)||_{L^2}^2 = ||S||_{L^2}^2.
\]
We will prove the equivalence of these definitions in section 3. In this paper, we will prove a new identity for enstrophy growth:
\[
\partial_t ||S(\cdot, t)||_{L^2}^2 = -2||S||_{H^1}^2 - \frac{4}{3} \int \text{tr}(S^3).
\]
The nonlinearity in (10) is still of the same degree as in (7). Both nonlinearities are of degree 3, and so cannot be controlled by the dissipation in either case, however the identity (10) does have several advantages. First, unlike (7), this identity is entirely local. The identity (7) is nonlocal with a singular integral kernel, because $S$ can be determined in terms of $\omega$ with a zeroth order pseudo-differential operator, $S = \nabla_{\text{sym}} (\Delta)^{-1} \nabla \times \omega$. The identity (10) also reveals very significant information about the eigenvalues of the strain tensor $S$ and their relation to blowup. Note that throughout this paper $\dot{H}^s$ will refer to the homogeneous Sobolev space with the norm
\[
||f||_{\dot{H}^s}^2 = \langle f, (-\Delta)^s f \rangle,
\]
and we will take the magnitude of a matrix, $M \in \mathbb{R}^{3 \times 3}$, to be the Euclidean norm
\[
|M|^2 = \sum_{i,j=1}^{3} M_{ij}^2.
\]
Finally, this will lead to a new regularity criterion that encodes information about the geometric structure of potential blow-up solutions. The main result of this paper is:

**Theorem** (Middle eigenvalue of strain characterizes blowup time). Let $u \in C \left([0, T]; \dot{H}^1 (\mathbb{R}^3)\right)$ for all $T < T_{\text{max}}$ be a mild solution to the Navier Stokes equation, and let $\lambda_1(x) \leq \lambda_2(x) \leq \lambda_3(x)$ be the eigenvalues of the strain tensor $S(x) = \nabla_{\text{sym}} u(x)$. Let $\lambda^+_2(x) = \max \{\lambda_2(x), 0\}$. If $\frac{3}{p} + \frac{2}{q} = 2$, with $\frac{3}{2} < q \leq +\infty$, then
\[
||u(\cdot, T)||_{\dot{H}^1}^2 \leq ||u^0||_{\dot{H}^1}^2 \exp \left(C_0 \int_0^T ||\lambda^+_2(\cdot, t)||_{L^q(\mathbb{R}^3)}^p dt\right),
\]
with the constant $C_0$ depending only on $p$ and $q$. In particular if $T_{\text{max}} < +\infty$, where $T_{\text{max}}$ is the maximal existence time for a smooth solution, then

$$\int_0^{T_{\text{max}}} \|\lambda_2^+(\cdot, t)\|_{L^p(\mathbb{R}^3)}^p dt = +\infty.$$  \hspace{1cm} (14)

It goes back to the classic work of Kato and Fujita [11] that smooth solutions must exist locally in time for any initial data $u^0 \in L^q(\mathbb{R}^3)$, $\nabla \cdot u^0 = 0$, when $q > 3$. In particular, this implies that a smooth solution of the Navier Stokes equations developing singularities in finite time requires that the $L^q$ norm of $u$ must blow up for all $q > 3$. This was extended to the case $q = 3$ by Escauriaza, Seregin, and Šverák [9]. The regularity criteria implied by the local existence of smooth solutions for initial data in $L^q(\mathbb{R}^3)$ when $q > 3$ are all subcritical with respect to the scaling that preserves the solution set of the Navier Stokes equations:

$$u^\lambda(x, t) = u(\lambda x, \lambda^2 t).$$  \hspace{1cm} (15)

If $u$ is a solution to the Navier Stokes equations on $\mathbb{R}^3$, then so is $u^\lambda$ for all $\lambda > 0$, although the time interval may have to be adjusted, depending on what notion of a solution (Leray-Hopf [23], mild, strong [11]) we are using. $L^3(\mathbb{R}^3)$ is the scale critical Lebesgue space for the Navier Stokes equations, so the Escauriaza-Seregin-Šverák condition is scale critical.

Critical Regularity criteria for solutions to the Navier Stokes equations go back to the work of Prodi, Serrin, and Ladyzhenskaya [21, 25, 29], who proved that if a smooth solution blows up in finite time, then

$$\int_0^{T_{\text{max}}} \|u\|_{L^p}^p dt = +\infty,$$  \hspace{1cm} (16)

where $\frac{2}{p} + \frac{3}{q} = 1$, and $3 < q \leq +\infty$. This result was then extended in the aforementioned Escauriaza-Seregin-Šverák paper [9] to the endpoint case $p = +\infty, q = 3$. They proved that if a smooth solution $u$ of the Navier Stokes equation blows up in finite time, then

$$\limsup_{t \to T_{\text{max}}} \|u(\cdot, t)\|_{L^3(\mathbb{R}^3)} = +\infty.$$  \hspace{1cm} (17)

Gallagher, Koch, and Planchon [13] also proved the above statement using a different approach based on profile decomposition. The other endpoint case of this family of criteria is the Beale-Kato-Majda criterion [2], which holds for Euler as well as for Navier Stokes, and states that if a smooth solution to either the Euler or Navier Stokes equations develops singularities in finite time, then

$$\int_0^{T_{\text{max}}} \|\omega(\cdot, t)\|_{L^\infty} dt = +\infty.$$  \hspace{1cm} (18)

This result was also extended to the strain tensor [16].

The family of regularity criteria in (16) has since been generalized to the critical Besov spaces $\dot{B}^{\frac{2}{p}}_{pq}(\mathbb{R}^3)$ [6, 14, 18, 19]. These criteria have also been generalized to criteria controlling the pressure $\dot{B}_{pq}(\mathbb{R}^3)$ [27, 30, 32]. In addition to strengthening regularity criteria to larger spaces, there have also been results not involving all the components of $u$, for instance regularity criteria on the gradient of one component $\nabla u_j$ [34], involving only the derivative in one direction, $\partial_{x_i} u$ [20], involving only one component $u_j$ [4, 5] and involving only one component of the gradient tensor $\partial u_j / \partial x_i$ [3]. For a comprehensive overview of the literature on regularity criteria for solutions to the Navier Stokes Equations see Chapter 11 in [22]. We will discuss the relationship between these results and our main theorem.
It remains to compute the Ladyzhenskaya regularity criterion, the regularity criteria we prove on \( \lambda_2 \) are critical with respect to scaling. The reason we require that \( \frac{2}{p} + \frac{3}{q} = 2 \), with \( \frac{3}{2} < q \leq +\infty \), then

\[
\|u(\cdot, T)\|_{\dot{H}^1}^2 \leq \left( \|u^0\|_{\dot{H}^1}^2 \right) \exp \left( C_0 \int_0^T \|\partial_3 u(\cdot, t) + \nabla u_3(\cdot, t)\|_{L^p(S)}^p \, dt \right),
\]

with the constant \( C_0 \) depending only on \( p \) and \( q \). In particular if the maximal existence time for a smooth solution \( T_{\text{max}} < +\infty \), then

\[
\int_0^{T_{\text{max}}} \|\partial_3 u(\cdot, t) + \nabla u_3(\cdot, t)\|_{L^p(S)}^p \, dt = +\infty.
\]

In fact, we will also prove a stronger version of this theorem that allows for us to consider different directions in different regions in \( \mathbb{R}^3 \). Finally, we will note that like the Prodi-Serrin-Ladyzhenskaya regularity criterion, the regularity criteria we prove on \( \lambda_2 \) and \( \partial_3 u + \nabla u_3 \) are critical with respect to scaling. The reason we require that \( \frac{2}{p} + \frac{3}{q} = 2 \), not \( \frac{2}{p} + \frac{3}{q} = 1 \) is because \( \lambda_2 \) is an eigenvalue of \( S \), and therefore scales like \( \nabla \otimes u \), not like \( u \). In addition, both regularity criteria can be generalized to the Navier Stokes equation with an external force \( f \in L^2_t L^2_x \), which will be discussed in section 5, but is left out of the introduction for the sake of brevity.

## 2 Evolution equation for the Strain Tensor

We will begin this section by deriving the Navier Stokes strain equation in three spatial dimensions.

**Proposition 2.1** (Strain reformulation of the dynamics). Suppose \( u \) is a classical solution to the Navier Stokes equation. Then \( S = \nabla_{\text{sym}}(u) \) is a classical solution to the Navier Stokes strain equation

\[
S_t + (u \cdot \nabla)S - \Delta S + S^2 + \frac{1}{4} \omega \otimes \omega - \frac{1}{4} \omega^2 I_3 + \text{Hess}(p) = 0.
\]

**Proof.** We begin by applying the operator \( \nabla_{\text{sym}} \) to the Navier Stokes Equation (1); we find immediately that

\[
S_t - \Delta S + \text{Hess}(p) + \nabla_{\text{sym}} ((u \cdot \nabla)u) = 0.
\]

It remains to compute \( \nabla_{\text{sym}} ((u \cdot \nabla)u) \).

\[
\nabla_{\text{sym}} ((u \cdot \nabla)u)_{ij} = \frac{1}{2} \partial_{x_i} \sum_{k=1}^3 u_k \frac{\partial u_j}{\partial x_k} + \frac{1}{2} \partial_{x_j} \sum_{k=1}^3 u_k \frac{\partial u_i}{\partial x_k}
\]

\[
\nabla_{\text{sym}} ((u \cdot \nabla)u)_{ij} = \sum_{k=1}^3 u_k \partial_{x_k} \left( \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \right) + \frac{1}{2} \sum_{k=1}^3 \partial_{x_i} u_k \frac{\partial u_j}{\partial x_k} + \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_j}.
\]

We can see from our definitions of \( S \) and \( A \) that

\[
S^2_{ij} = \frac{1}{4} \sum_{k=1}^3 \left( \frac{\partial u_k}{\partial x_i} + \frac{\partial u_k}{\partial x_j} \right) \left( \frac{\partial u_j}{\partial x_k} + \frac{\partial u_i}{\partial x_k} \right) = \frac{1}{4} \sum_{k=1}^3 \frac{\partial u_k}{\partial x_i} \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \frac{\partial u_i}{\partial x_k} + \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}.
\]
and
\[
A^2_{ij} = \frac{1}{4} \sum_{k=1}^{3} \left( \frac{\partial u_k}{\partial x_i} - \frac{\partial u_i}{\partial x_k} \right) \left( \frac{\partial u_j}{\partial x_k} - \frac{\partial u_k}{\partial x_j} \right) = \frac{1}{4} \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_i} \frac{\partial u_j}{\partial x_k} + \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_j} - \frac{\partial u_k}{\partial x_i} \frac{\partial u_i}{\partial x_j}.
\]  
(26)

Taking the sum of these two equations, we find that
\[
(S^2 + A^2)_{ij} = \frac{1}{2} \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_i} \frac{\partial u_j}{\partial x_k} + \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_j} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_i}{\partial x_j}.
\]  
(27)

From this we can conclude that
\[
\nabla_{sym} ((u \cdot \nabla) u) = (u \cdot \nabla) S + S^2 + A^2.
\]  
(28)

Recall that
\[
A = \frac{1}{2} \begin{pmatrix}
0 & \omega_3 & -\omega_2 \\
-\omega_3 & 0 & \omega_1 \\
\omega_2 & -\omega_1 & 0
\end{pmatrix},
\]
(29)
so we can express \(A^2\) as
\[
A^2 = \frac{1}{4} \omega \otimes \omega - \frac{1}{4} |\omega|^2 I_3.
\]  
(30)
This concludes the proof.

We also can see that \(\text{tr}(S) = \nabla \cdot u = 0\), so in order to maintain the divergence free structure of the flow, we require that the strain tensor be trace free. For the vorticity the only consistency condition is that the vorticity be divergence free. Any divergence free vorticity can be inverted back to a unique velocity field, assuming suitable decay at infinity, with \(u = \nabla \times (-\Delta)^{-1}w\). This is not true of the strain tensor, for which an additional consistency condition is required.

If we know the strain tensor \(S\), this is enough for us to reconstruct the flow. We take
\[
-2 \text{div}(S) = -\Delta u - \nabla (\nabla \cdot u) = -\Delta u
\]  
(31)
Therefore we find that
\[
u = -2 \text{div}(-\Delta)^{-1} S.
\]  
(32)
This allows us to reconstruct the flow \(u\) from the strain tensor \(S\), but it doesn’t guarantee that if we start with a general trace free symmetric matrix, the \(u\) we reconstruct will actually have this symmetric matrix as its strain tensor. We will need to define a consistency condition guaranteeing that the strain tensor is actually the symmetric part of the gradient of some divergence free vector field. This condition for the strain equation will play the same role that the divergence free condition plays in the vorticity equation. We will now define the subspace of strain matrices \(L^2_{st} \subset L^2(\mathbb{R}^3; S^3 \times S^3)\) as follows:

**Definition 2.2** (Strain subspace).
\[
L^2_{st} = \left\{ \frac{1}{2} \nabla \otimes u + \frac{1}{2} (\nabla \otimes u)^* : u \in \dot{H}^1 (\mathbb{R}^3; \mathbb{R}^3), \nabla \cdot u = 0 \right\}
\]  
(33)
This subspace of \(L^2(\mathbb{R}^3; S^3 \times S^3)\) can in fact be characterized by a partial differential equation, although in this case, it is significantly more complicated than the equation \(\nabla \cdot u = 0\), that characterizes the space of divergence free vector fields.
Proposition 2.3 (Characterization of the strain subspace). Suppose $S \in L^2(\mathbb{R}^3; S^{3\times3})$. Then $S \in L^{2}_{st}$ if and only if
\begin{equation}
\text{tr}(S) = 0
\end{equation}
and
\begin{equation}
-\Delta S + 2\nabla_{sym} \left( \text{div}(S) \right) = -\Delta S + (\nabla \otimes \nabla)S + ((\nabla \otimes \nabla)S)^* = 0.
\end{equation}
Note that because by hypothesis we only have $S \in L^2$, we will consider $S$ to be a solution to (35) if the condition is satisfied pointwise almost everywhere in Fourier space, that is if
\begin{equation}
|\xi|^2 \hat{S}(\xi) - (\xi \otimes \xi) \hat{S}(\xi) - \hat{S}(\xi)(\xi \otimes \xi) = 0,
\end{equation}
almost everywhere $\xi \in \mathbb{R}^3$. The partial differential equation (35) can be written out in components as
\begin{equation}
-\Delta S_{ij} + 3 \sum_{k=1}^{3} \partial_{x_i} \partial_{x_k} S_{kj} + \partial_{x_j} \partial_{x_k} S_{ki} = 0
\end{equation}
Proof. First suppose $S \in L^2_{st}$, so there exists a $u \in \dot{H}^1$, $\nabla \cdot u = 0$, such that
\begin{equation}
S = \nabla_{sym} u.
\end{equation}
As we have already shown, $\text{tr}(S) = \nabla \cdot u = 0$. Next we will take the divergence of (38), and find that,
\begin{equation}
-2 \text{div}(S) = -2 \text{div}(\nabla_{sym} u) = -\Delta u - \nabla(\nabla \cdot u) = -\Delta u
\end{equation}
Applying $\nabla_{sym}$ to (39) we find that
\begin{equation}
-2 \nabla_{sym}(\text{div}(S)) = \nabla_{sym}(-\Delta u) = -\Delta S,
\end{equation}
so the condition (35) is also satisfied.
Now suppose $\text{tr}(S) = 0$ and $-\Delta S + 2\nabla_{sym}(\text{div}(S)) = 0$. Define $u$ by
\begin{equation}
u = (-\Delta)^{-1}(-2 \text{div}(S))
\end{equation}
Applying $\nabla_{sym}$ to this definition we find that
\begin{equation}
\nabla_{sym} u = (-\Delta)^{-1}(-2 \nabla_{sym}(\text{div}(S))) = (-\Delta)^{-1}(-\Delta S) = S.
\end{equation}
Clearly $u \in \dot{H}^1$ because $S \in L^2$ and $(-\Delta)^{-1}(-2 \text{div})$ is a pseudo-differential operator with order $-1$. It only remains to show that $\nabla \cdot u = 0$. Next we will take the trace of (37) and find that
\begin{equation}
(\text{div})^2(S) = 3 \sum_{i,j=1}^{3} \partial_{x_i} \partial_{x_j} S_{ij} = 0.
\end{equation}
Using this we compute that
\begin{equation}
\nabla \cdot u = (-\Delta)^{-1}(-2(\text{div})^2(S)) = 0.
\end{equation}
This completes the proof. \qed
Note that the the consistency condition (35) is linear, so the set of matrices satisfying it form a subspace of $L^2$. The Navier Stokes equation (11) and the vorticity equation (10) can best be viewed not as systems of equations, but as evolution equations on the space of divergence free vector fields. Similarly, we can view the Navier Stokes strain equation (6) as an evolution equation on $L^2_{st}$.

The Navier Stokes strain equation has already been examined in [8,12], however the consistency condition (35) does not play a role in this analysis. In [12], the authors focus on the relationship between vorticity and the strain tensor in enstrophy production, as the strain tensor and vorticity are related by a linear zero order pseudo-differential operator, $S = \nabla_{sym}(\Delta)^{-1} \nabla \times \omega$. However, the consistency condition is actually very useful in dealing with the evolution of the strain tensor, because a number of the terms in the evolution equation (6) are actually in the orthogonal complement of $L^2_{st}$ with respect to the $L^2$ inner product. This will allows us to prove an identity for enstrophy growth involving only the strain, where previous identities involved the interaction of the strain and the vorticity. We will now make an observation about what matrices in $L^2(R^3; S^{3\times 3})$ are in the orthogonal complement of $L^2_{st}$ with respect to the $L^2$ inner product.

**Proposition 2.4** (Orthogonal subspaces). For all $f \in H^2(R^3)$, for all $g \in L^2(R^3)$, and for all $S \in L^2_{st}$

$$\langle S, gI_3 \rangle = 0 \quad \text{(45)}$$

and

$$\langle S, \text{Hess}(f) \rangle = 0 \quad \text{(46)}$$

**Proof.** First we’ll consider the case of $gI_3$. Fix $S \in L^2_{st}$ and we’ll take the inner product

$$\langle gI_3, S \rangle = \int_{R^3} 3 \sum_{i,j=1}^3 g_{ij} S_{ij} = \int_{R^3} tr(S)g = 0 \quad \text{(47)}$$

In order to show that $\text{Hess}(f) \in (L^2_{st})^\perp$, we will use the property that for $S \in L^2_{st}$

$$\text{tr}((\nabla \otimes \nabla)S) = \sum_{i,j=1}^3 \partial x_i \partial x_j S_{ij} = 0 \quad \text{(48)}$$

Because $S \in L^2$ and therefore $\hat{S} \in L^2$, the above condition can be expressed as

$$\sum_{i,j=1}^3 \xi_i \xi_j \hat{S}_{ij}(\xi) = 0 \quad \text{(49)}$$

almost everywhere $\xi \in R^3$. Using the fact that the Fourier transform is an isometry on $L^2$, and $\text{Hess}(f), S \in L^2$ we compute that

$$\langle \text{Hess}(f), S \rangle = \langle \text{Hess}(f), \hat{S} \rangle = -4\pi^2 \int_{R^3} \check{f}(\xi) \sum_{i,j=1}^3 \xi_i \xi_j \hat{S}_{ij}(\xi) d\xi = 0 \quad \text{(50)}$$

This means that as long as $u$ is sufficiently regular, $\text{Hess}(p)$ and $-\frac{1}{4}\vert\omega\vert^2 I_3$ are in the orthogonal compliment of $L^2_{st}$. This fact will play a key role in the new identity for enstrophy growth that we will prove in section 4.
3 The Relationship between Strain and Vorticity

We have already established in \[432\] that \(u = -2 \text{div}(-\Delta)^{-1} S\). The antisymmetric part of the gradient tensor then, can be reconstructed applying a zeroth order pseudo-differential operator to \(S\). We find that
\[
A = (\nabla \otimes \nabla)(-\Delta)^{-1} S - ((\nabla \otimes \nabla)(-\Delta)^{-1} S)^*.
\]
Because this is a zeroth order operator related to the Riesz transform, it is bounded from \(L^p\) to \(L^p\) for \(1 < p < +\infty\), but we will only have Calderon-Zygmund type estimates, so our control will be very bad. We can say something much stronger in the case of \(L^2\), and in fact for every Hilbert space \(H^\alpha\), \(-\frac{3}{2} < \alpha < \frac{3}{2}\).

**Proposition 3.1** (Hilbert space isometries). For all \(-\frac{3}{2} < \alpha < \frac{3}{2}\),
\[
||S||_{H^\alpha}^2 = ||A||_{H^\alpha}^2 = \frac{1}{2}||\omega||_{H^\alpha}^2 = \frac{1}{2}||u||_{H^{\alpha+1}}^2
\]
**Proof.** First fix \(s, -\frac{3}{2} < s < \frac{3}{2}\). We will begin relating the \(H^s\) norms of the anti-symmetric part and the vorticity. Recall that
\[
A = \frac{1}{2} \begin{pmatrix}
0 & \omega_3 & -\omega_2 \\
-\omega_3 & 0 & \omega_1 \\
\omega_2 & -\omega_1 & 0
\end{pmatrix}
\]
Therefore, for all \(x \in \mathbb{R}^3\),
\[
|(-\Delta)^{\frac{s}{2}} A(x)|^2 = \frac{1}{2} |(-\Delta)^{\frac{s}{2}} \omega(x)|^2.
\]
Because in general we have that \(||f||_{H^s} = ||(-\Delta)^{\frac{s}{2}} f||_{L^2}||\), it immediately follows that
\[
||A||_{H^s}^2 = \frac{1}{2}||\omega||_{H^s}^2.
\]
Because the vorticity is divergence free, in Fourier space
\[
|\hat{\omega}(\xi)| = |2\pi i \xi \times \hat{u}(\xi)| = 2\pi |\xi||\hat{u}(\xi)| = |\hat{\nabla \otimes u}(\xi)|.
\]
From this we can conclude that
\[
||\omega||_{H^s}^2 = ||\nabla \otimes u||_{H^s}^2 = ||u||_{H^{s+1}}^2.
\]
Finally we will compute
\[
\left|(-\Delta)^{\frac{s}{2}} (\nabla \otimes u)^2\right| = \text{tr} \left((-\Delta)^{\frac{s}{2}} S + (-\Delta)^{\frac{s}{2}} A \right) \left((-\Delta)^{\frac{s}{2}} S^* + (-\Delta)^{\frac{s}{2}} A^*\right)
\]
\[
\left|(-\Delta)^{\frac{s}{2}} (\nabla \otimes u)^2\right| = \left|(-\Delta)^{\frac{s}{2}} S\right|^2 + \left|(-\Delta)^{\frac{s}{2}} A\right|^2 + \text{tr} \left((-\Delta)^{\frac{s}{2}} A(-\Delta)^{\frac{s}{2}} S\right) - \text{tr} \left((-\Delta)^{\frac{s}{2}} S(-\Delta)^{\frac{s}{2}} A\right)
\]
But we know that the trace of the product of a symmetric matrix and an antisymmetric matrix is always zero, so we can immediately see that
\[
\left|(-\Delta)^{\frac{s}{2}} (\nabla \otimes u)^2\right| = \left|(-\Delta)^{\frac{s}{2}} S\right|^2 + \left|(-\Delta)^{\frac{s}{2}} A\right|^2.
\]
From this it follows that
\[
||\nabla \otimes u||_{H^s}^2 = ||S||_{H^s}^2 + ||A||_{H^s}^2,
\]
but we have already established that

\[ ||A||_{H^s}^2 = \frac{1}{2}||\nabla \otimes u||_{H^s}^2, \]  

so we can conclude that

\[ ||A||_{H^s}^2 = ||S||_{H^s}^2 = \frac{1}{2}||\nabla \otimes u||_{H^s}^2. \]  

This concludes the proof. \(\square\)

We have now established all the necessary basics and will proceed to considering enstrophy growth in terms of the strain and vorticity equations.

4 Enstrophy and the \(L^2\) Growth of the Strain Tensor

Before proceeding, further, however, we need to show the existence of solutions in a suitable space. Leray solutions, first developed in the classic paper \[23\] are not the most well adapted to the Navier Stokes strain equation, so we will work with mild solutions instead \[11\]. We will begin by defining mild solutions in \(\hat{H}^1\) to the Navier Stokes equations, and then adapt this definition for mild solutions in \(L^2\) to the Navier Stokes strain equation and the vorticity equation.

**Definition 4.1** (Mild velocity solutions). Suppose \(u \in C([0,T];\hat{H}^1(\mathbb{R}^3))\) and \(\nabla \cdot u = 0\) in \(L^2\).

Then \(u\) is a mild solution to the Navier Stokes equation if

\[ u(x,t) = \int_{\mathbb{R}^3} K(x-y,t)u^0(y)dy - \int_0^t \int_{\mathbb{R}^3} K(x-y,t-\tau)((u \cdot \nabla)u + \nabla p)(y,\tau)dyd\tau, \]  

where \(K\) is the heat kernel,

\[ K(x,t) = \frac{1}{(4\pi t)^{\frac{3}{2}}} \exp \left( -\frac{|x|^2}{4t} \right), \]  

and \(p\) is defined in terms of \(u\) by convolution with the Poisson kernel

\[ p = (-\Delta)^{-1} \sum_{i,j=1}^3 \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j}. \]  

This simply says that \(u\) solves the heat equation with a source, where the source is taken to be \(-u \cdot \nabla)u - \nabla p\).

Note that the equality of the function and the convolution should be understood as equality in \(\hat{H}^1\), not pointwise. It is written in terms of \(u(x,t)\) only for the sake of clarity. Now we can define a mild solution to the Navier Stokes strain equation accordingly.

**Definition 4.2** (Mild strain solutions). Suppose \(S \in C([0,T]; L^2_{\text{st}})\). Then we will call \(S\) a mild solution to the Navier Stokes Strain equation \([6]\) if and only if for all \(0 < t \leq T\),

\[ S(x,t) = \int_{\mathbb{R}^3} K(x-y,t)S^0(y)dy - \int_0^t \int_{\mathbb{R}^3} K(x-y,t-\tau)
\left( (u \cdot \nabla)S + S^2 + \frac{1}{4} \omega \otimes \omega - \frac{1}{4} |\omega|^2 I_3 + \text{Hess}(p) \right)(y,\tau)dyd\tau \]  

where here equality is in \(L^2\).
It is a classical result that $\dot{H}^1$ mild solutions to the Navier Stokes equation exist locally in time. We will state this result precisely and then use the result to establish local in time existence of $L^2$ mild solutions to the Navier Stokes Strain equation.

**Theorem 4.3** (Mild velocity solutions exist for short times). There exists a constant $C > 0$, such that for all $u^0 \in \dot{H}^1(\mathbb{R}^3), \nabla \cdot u^0 = 0$, for all $0 < T < \frac{C}{\|u^0\|^2_{H^1}}$, there exists a mild solution to the Navier Stokes equation $u \in C\left([0,T];\dot{H}^1(\mathbb{R}^3)\right)$. Furthermore for all $0 < \epsilon < T$, $u \in C\left([\epsilon,T];\dot{H}^{\alpha}(\mathbb{R}^3)\right)$ for all $\alpha > 1$, and therefore $u \in C^\infty\left((0,T] \times \mathbb{R}^3\right)$.

**Proof.** This is the classical result [11] of Fujita and Kato. It has since been generalized to weaker spaces [4, 17]. The result is proven using an iteration scheme. First define

$$v^0(x,t) = \int_{\mathbb{R}^3} K(x-y,t)u^0(y)dy. \quad (68)$$

Then define a sequence $\{v^n\}_{n \in \mathbb{N}} \subset C\left([0,T];\dot{H}^1(\mathbb{R}^3)\right)$ iteratively by

$$v^{n+1}(x,t) = \int_{\mathbb{R}^3} K(x-y,t)u^0(y)dy - \int_0^t \int_{\mathbb{R}^3} K(x-y,t-\tau)((v^n \cdot \nabla)v^n + \nabla p^n)(y,\tau)dyd\tau, \quad (69)$$

where $p^n$ is the pressure associated to the divergence free vector field $v^n$. It can be shown that $v^n$ converges to a mild solution $u$ strongly in $C\left([0,T];\dot{H}^1(\mathbb{R}^3)\right)$, but we will not go through the technical details here. For a detailed exposition and an excellent overview of more recent developments in mild solutions see Chapter 7 in [22]. This result does not hold only for initial data in $\dot{H}^1$, but also for initial data in $\dot{H}^s$ for all $s > \frac{3}{2}$, for all the Hilbert spaces which are subcritical with respect to scaling.

To prove the smoothness of mild solutions, we make use of a bootstrapping argument. The smoothing of the heat kernel means that a mild solution in $C\left((0,T);L^q\right)$ with $q > 3$ must also be in $L^{q+\epsilon}$ for all $0 < t \leq T$. Iterating such estimates is enough to show that any mild solution in $C\left((0,T);L^3\right)$ with $q > 3$ must also be in $L^\infty$ for all $0 < t \leq T$, and similar arguments allow us to show that the mild solution must be in arbitrarily higher order Sobolev spaces, and therefore must be smooth. \[\square\]

**Theorem 4.4** (Mild strain solutions exist for short times). There exists a constant $C > 0$ such that for all $S^0 \in L^2_{\mbox{str}},$ if $T < \frac{C}{4\|S^0\|^2_{L^2}}$, then there exists a mild solution to the Navier Stokes Strain equation $S \in C\left([0,T];L^2_{\mbox{str}}\right).$ Furthermore for all $0 < \epsilon < T$, $S \in C\left([\epsilon,T];\dot{H}^{\alpha}\right)$ for all $\alpha > 0$, and therefore $S \in C^\infty\left((0,T] \times \mathbb{R}^3\right)$.

**Proof.** We begin by inverting the strain tensor $S^0$ to recover the initial velocity:

$$u^0 = -2 \mbox{div}(-\Delta)^{-1}S^0. \quad (70)$$

We can see from the pseudo-differential operator used to obtain $u^0$, that $S^0 \in L^2$ implies $u^0 \in \dot{H}^1$. This means that we can apply the theorem above to show that there exists $T > 0$, such that there is a $\dot{H}^1$ mild solution $u$ on $[0,T]$ with $u(\cdot,0) = u^0$, that is, for all $0 < t \leq T$,

$$u(x,t) = \int_{\mathbb{R}^3} K(x-y,t)u^0(y)dy - \int_0^t \int_{\mathbb{R}^3} K(x-y,t-\tau)((u \cdot \nabla)u + \nabla p)(y,\tau)dyd\tau. \quad (71)$$
Next we will compute $S = \nabla_{sym} u$. When taking the derivative of a convolution, the derivative can be applied to either of the functions being convolved; in this case, we will apply the differential operator $\nabla_{sym}$ to $u^0$ and $(u \cdot \nabla)u + \nabla p$, and find that

$$S(x, t) = \int_{\mathbb{R}^3} K(x - y, t)S^0(y)dy - \int_0^t \int_{\mathbb{R}^3} K(x - y, t - \tau) \left( (u \cdot \nabla)S + S^2 + \frac{1}{4} \omega \otimes \omega - \frac{1}{4} |\omega|^2 I_3 + \text{Hess}(p) \right) (y, \tau)dyd\tau. \quad (72)$$

Finally the higher order regularity of a mild solution $u$ proved by Kato and Fujita in [11] immediately implies higher order regularity for $S = \nabla_{sym} u$. This completes the proof. □

Now that existence of mild solutions in a suitable space is ensured, we can use the Navier Stokes Strain equation to simplify our estimates for enstrophy growth.

**Theorem 4.5** (Enstrophy growth identity). Suppose $S \in C \left([0, T]; L^2_{st}\right)$ is a mild solution to the Navier Stokes strain equation. Then for all $0 < t \leq T$

$$\partial_t ||S(\cdot, t)||^2_{L^2} = -2||S||^2_{H^1} - \frac{4}{3} \int_{\mathbb{R}^3} \text{tr}(S^3). \quad (73)$$

**Proof.** Using (74), we can compute the rate of change of enstrophy

$$\partial_t \frac{1}{2} ||\omega(\cdot, t)||^2_{L^2} = -\langle -\Delta \omega, \omega \rangle - \langle (u \cdot \nabla)\omega, \omega \rangle + \langle S \omega, \omega \rangle = -||\omega||^2_{H^1} + \langle S; \omega \otimes \omega \rangle \quad (74)$$

This is the standard identity for enstrophy growth, based on the interaction of the Strain matrix and the vorticity, with the $\langle \omega, (u \cdot \nabla)\omega \rangle = 0$ from integration by parts. See chapter 7 in [22] for more details. We can use the isometry in Proposition 2.4 to restate (74) in terms of strain:

$$\partial_t ||S(\cdot, t)||^2_{L^2} = -2||S||^2_{H^1} + \langle S; \omega \otimes \omega \rangle. \quad (75)$$

However we can also calculate the $L^2$ growth of the strain tensor directly from our evolution equation for the strain tensor (6),

$$\partial_t ||S(\cdot, t)||^2_{L^2} = -2 \langle -\Delta S, S \rangle - 2 \langle (u \cdot \nabla)S, S \rangle - 2 \langle S^2, S \rangle - \frac{1}{2} \langle \omega \otimes \omega; S \rangle - 2 \langle \text{Hess}(p), S \rangle + \frac{1}{2} \langle |\omega|^2 I_3, S \rangle \quad (76)$$

Integrating by parts we know that $\langle (u \cdot \nabla)S, S \rangle = 0$. Note that $S \in C \left([0, T], L^2 \right) \cap C \left((0, T], H^1 \right)$. In particular this implies that $S(\cdot, t), \omega(\cdot, t) \in L^2 \cap L^6$ for all $0 < t \leq T$. This means that $S(\cdot, t), \omega(\cdot, t) \in L^2$, so $\langle S; \omega \otimes \omega \rangle$ and $\int \text{tr}(S^3)$ are both well defined. This also means that $|\omega(\cdot, t)|^2, \text{Hess}(p)(\cdot, t) \in L^2$ for all $0 < t \leq T$. Therefore we can apply Proposition 2.4 and find that $|\omega|^2 I_3, \text{Hess}(p) \in L^2_{st}$, so

$$\langle S, \frac{1}{2} |\omega|^2 I_3 \rangle = 0 \quad (77)$$

and

$$\langle \text{Hess}(p), S \rangle = 0. \quad (78)$$
Now we can use the fact that \( S \) is symmetric to compute that \( \langle S^2, S \rangle = \int_{\mathbb{R}^3} \text{tr}(S^3) \), so
\[
\partial_t ||S(\cdot, t)||_{L^2}^2 = -2 ||S||_{H^1}^2 - \frac{1}{2} \langle S; \omega \otimes \omega \rangle - 2 \int_{\mathbb{R}^3} \text{tr}(S^3). \tag{79}
\]
We now will add \( \frac{1}{3} \) \( (75) \) to \( \frac{2}{3} \) \( (79) \) to cancel the term \( \langle S, \omega \otimes \omega \rangle \), and we find
\[
\partial_t ||S(\cdot, t)||_{L^2}^2 = -2 ||S||_{H^1}^2 - \frac{4}{3} \int_{\mathbb{R}^3} \text{tr}(S^3). \tag{80}
\]
Finally we will note that because the subcritical quantity \( ||S(\cdot, t)||_{L^2} \) is controlled uniformly on \([0, T]\), the smoothing due to the heat kernel guarantees that \( S \) is smooth, so the identity \( (73) \) can be understood as a derivative of a smooth quantity in the classical sense.

Now we have improved the estimate for enstrophy growth from one that involved the interaction of the vorticity and the strain tensor to an estimate that only involves the strain tensor. We can still extract more geometric information about the flow, however. The identity for enstrophy growth in Theorem \( \text{[4.5]} \) can also be expressed in terms of \( \text{det}(S) \).

**Corollary 4.6** (Alternative enstrophy growth identity). Suppose \( S \in C([0, T]; L^2_{\text{str}}) \) is a mild solution to the Navier Stokes strain equation. Then for all \( 0 < t \leq T \)
\[
\partial_t ||S(\cdot, t)||_{L^2}^2 = -2 ||S||_{H^1}^2 - 4 \int_{\mathbb{R}^3} \text{det}(S). \tag{81}
\]

**Proof.** Because \( S \) is symmetric it will be diagonalizable with three real eigenvalues, and because \( S \) is trace free, we have \( \text{tr}(S) = \lambda_1 + \lambda_2 + \lambda_3 = 0 \). This allows us to relate \( \text{tr}(S^3) \) to \( \text{det}(S) \) by
\[
\text{tr}(S^3) = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = \lambda_1^3 + \lambda_2^3 + (-\lambda_1 - \lambda_2)^3 = -3\lambda_1^2\lambda_2 - 3\lambda_1\lambda_2^2 \tag{82}
\]
\[
\text{tr}(S^3) = -3(\lambda_1 + \lambda_2)\lambda_1\lambda_2 = 3\lambda_1\lambda_2\lambda_3 = 3 \text{det}(S).
\]
So we can write our growth estimate as:
\[
\partial_t ||S(\cdot, t)||_{L^2}^2 = -2 ||S||_{H^1}^2 - 4 \int_{\mathbb{R}^3} \text{det}(S) \tag{83}
\]
This completes the proof.

The identity \( (73) \), gives us a significantly better understanding of enstrophy production than the classical enstrophy growth identity \( (74) \), because we now have the growth controlled solely in terms of the strain tensor, rather than both the strain tensor and the vorticity. This estimate also provides analytical confirmation of the well known result that the vorticity tends to align with the eigenvector corresponding to the intermediate eigenvalue of the strain matrix \([12, 33]\). Comparing the identities in \( (73), (74), \) and \( (81) \) we see that
\[
\langle S, \omega \otimes \omega \rangle = -4 \int_{\mathbb{R}^3} \text{det}(S) = -\frac{4}{3} \int \text{tr}(S^3). \tag{84}
\]
When \( \text{det}(S) \) tends to be positive, it means there are two negative eigenvalues and one positive eigenvalue, so \( \langle S, \omega \otimes \omega \rangle \) being negative means the vorticity tends to align, on average when integrating over the whole space, with the negative eigenspaces. Likewise, when \( \text{det}(S) \) tends to be negative, it means there are two positive eigenvalues and one negative eigenvalue, so \( \langle S, \omega \otimes \omega \rangle \)
being positive means the vorticity tends to align, on average when integrating over the whole space, with the positive eigenspaces. When $\det(S)$ tends to be zero when integrated over the whole space, the vorticity tends clearly to be aligned with the intermediate eigenvalue, as well. Growth in all cases geometrically corresponds to the strain matrix $S$ stretching in two directions, while strongly contracting in the third direction. If this is true globally, then the vorticity will tend to align with the stretching eigenvectors, the eigenvectors associated to the two positive eigenvalues, in order to satisfy (84).

Finally we will bound the growth rate of enstrophy (73) in terms of the size of the strain matrix, and see what this matrix looks like in the sharp case of this bound.

**Proposition 4.7 (Determinant bound).** Let $M$ be a symmetric, trace free matrix, then

$$-4 \det(M) \leq \frac{2}{9} \sqrt{6} |M|^3,$$  

(85)

with equality if and only if $-\frac{1}{2} \lambda_1 = \lambda_2 = \lambda_3$, where $\lambda_1 \leq \lambda_2 \leq \lambda_3$ are the eigenvalues of $M$.

**Proof.** In the case where $M = 0$, it holds trivially. In the case where $M \neq 0$, then we have $\lambda_1 < 0, \lambda_3 > 0$. This allows us to define a parameter $r = -\frac{\lambda_1}{\lambda_3}$. The two parameters $\lambda_3$ and $r$ completely define the system because $\lambda_1 = -r \lambda_3$ and $\lambda_2 = -\lambda_1 - \lambda_3 = (r - 1) \lambda_3$. We must now say something about the range of values the parameter $r$ can take on. $\lambda_1 \leq \lambda_2 \leq \lambda_3$ implies that $-r \leq r - 1 \leq 1$, so therefore $\frac{1}{2} \leq r \leq 2$. Now we can observe that

$$-4 \det(M) = -4 \lambda_1 \lambda_2 \lambda_3 = 4r(r - 1)\lambda_3^3,$$  

(86)

and that

$$|M|^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = (r^2 + (r - 1)^2 + 1)\lambda_3^2 = (2r^2 - 2r + 2)\lambda_3^2.$$  

(87)

We can combine the two equations above to find that

$$-4 \det(M) = \sqrt{2} \frac{r^2 - r}{(r^2 - r + 1)^2} |M|^3$$  

(88)

Next we will observe that

$$\sqrt{2} \frac{r^2 - r}{(r^2 - r + 1)^2} |_{r=2} = \sqrt{2} \frac{2}{3\sqrt{3}} = \frac{2}{9} \sqrt{6}.$$  

(89)

This is exactly as we want, as $r = 2$ is the case that we want to correspond to equality. Finally we observe that for all $\frac{1}{2} \leq r < 2$, we have that

$$\sqrt{2} \frac{r^2 - r}{(r^2 - r + 1)^2} < \frac{2}{9} \sqrt{6}.$$  

(90)

This completes the proof. $\square$

The structure of the quadratic term in relation to $r = -\frac{\lambda_1}{\lambda_3} = 2$, the extremal case, will be investigated further in section 6 when we consider blow up for a toy model ODE for the Navier Stokes strain equation. It is an interesting open question whether or not there is a strain matrix which saturates this inequality globally in space. More precisely, is there a nonzero $S \in L^2_{st}$ such that $-\frac{4}{3} tr(S^3) = \frac{2}{3} \sqrt{6} |S|^3$ almost everywhere in $\mathbb{R}^3$?
Corollary 4.8. Suppose $S \in C \left((0, T); L^2_{ad}\right)$ mild solution to the Navier Stokes strain equation. Then for all $0 < t \leq T$,
\[
\partial_t ||S(\cdot, t)||_{L^2}^2 \leq -2 ||S||_{H^1}^2 + \frac{2}{9} \sqrt{6} \int_{\mathbb{R}^3} |S|^3.
\] (91)

Proof. This corollary follows immediately from Proposition [4.7] and Corollary [4.6] □

This corollary allows us to derive an a priori estimate on the growth of enstrophy, which will then give us a minimum blow up time $T$.

Theorem 4.9 (Enstrophy Growth Differential Inequality). Suppose $S \in C \left((0, T); L^2_{ad}\right)$ is a mild solution to the Navier Stokes strain equation. Then for all $0 < t \leq T$,
\[
\partial_t ||S(\cdot, t)||_{L^2}^2 \leq \frac{1}{1458 \pi} ||S(\cdot, t)||_6^6
\] (92)

and

\[
||S(\cdot, t)||_{L^2}^2 \leq \left( \frac{1}{||S||_{L^2}^2 - \frac{1}{729 \pi t}} \right)^{\frac{2}{3}}
\] (93)

Proof. For the first portion (92) we begin by applying the interpolation inequality for $L^p$ to bound the $L^3$ norm by the $L^2$ and $L^6$ norms:
\[
\int_{\mathbb{R}^3} |S|^3 \leq ||S||_{L^2}^2 ||S||_{L^6}^3
\] (94)

The sharp Sobolev inequality [7, 31] states that:
\[
||S||_{L^6} \leq \left( \frac{1}{3 \pi} \right)^{\frac{2}{3}} ||S||_{H^1}
\] (95)

Observing that $(\frac{2}{3})^3 = \frac{2}{\sqrt{6}}$, we can combine the above equations to find that:
\[
\partial_t ||S(\cdot, t)||_{L^2}^2 \leq -2 ||S||_{H^1}^2 + \left( \frac{2}{3} \right)^{\frac{2}{3}} \frac{1}{3 \pi} ||S||_{H^1}^2 ||S||_{L^2}^3.
\] (96)

We will regroup terms and write this as
\[
\left( \frac{2}{3} \right)^{\frac{2}{3}} \frac{1}{3 \pi} ||S||_{H^1}^2 ||S||_{L^2}^3 = \left( \frac{8}{3} \right)^{\frac{2}{3}} ||S||_{H^1}^2 \left( \frac{2^\frac{2}{3}}{2^\frac{2}{3} + 3^2 \pi} ||S||_{L^2}^3 \right)
\] (97)

We now apply Young’s inequality using the conjugate exponents $\frac{4}{3}$ and 4 to show that
\[
\left( \frac{8}{3} \right)^{\frac{2}{3}} ||S||_{H^1}^2 \left( \frac{2^\frac{2}{3}}{2^\frac{2}{3} + 3^2 \pi} ||S||_{L^2}^3 \right) \leq \frac{3}{4} \left( \frac{8}{3} \right)^{\frac{2}{3}} ||S||_{H^1}^2 + \frac{1}{4} \left( \frac{2^\frac{2}{3}}{2^\frac{2}{3} + 3^2 \pi} ||S||_{L^2}^3 \right)^4
\] (98)

Therefore we can conclude that
\[
\partial_t ||S(\cdot, t)||_{L^2}^2 \leq \frac{1}{2 \pi^3} ||S(\cdot, t)||_{L^6}^6 = \frac{1}{1458 \pi^4} ||S(\cdot, t)||_{L^2}^6
\] (99)

This completes the proof of the first part of the theorem (92); the second portion of the theorem follows immediately from integrating this differential inequality. □
This is a significant improvement on the best known estimates for enstrophy growth. If we take the enstrophy to be the square of the $L^2$ norm of the vorticity $E(t) = \frac{1}{2}||w(\cdot, t)||_{L^2}^2 = ||S(\cdot, t)||_{L^2}^2$, then this means
\[ \partial_t E(t) \leq \frac{1}{1458\pi^4} E^3 \]

The previous best known estimate for enstrophy growth \cite{124,26} was
\[ \partial_t E(t) \leq \frac{27}{8\pi^4} E(t)^3. \]

This work has recently been extended in \cite{1}, which shows numerically that solutions which are sharp for this inequality locally in time, actually tend to decay fairly quickly, and so are not good candidates for blowup. This work also suggested numerically that the constant $\frac{27}{8\pi^4}$ is non-optimal, but we will note that this work was on the torus, which may result in a different constant than the whole space as, for example, the sharp Sobolev constant may not be the same. Nonetheless, the fairly drastic improvement in the constant is not too surprising, because $\frac{27}{8\pi^4}$ was not expected to be sharp.

Previous papers investigating the growth of enstrophy, such as in \cite{24} simply apply the sharp Gadliargo-Nirenberg inequality to bound $\int_{\mathbb{R}^3} -\Delta u \cdot ((u \cdot \nabla) u)$. A finer analysis of the role of the strain matrix, using both the vorticity equation and the strain equation, allows us to drastically improve this estimate by a factor of 4,920.75. This is a huge quantitative improvement on the estimate for enstrophy growth, although the estimate is of course still cubic, so an improvement in the constant only increases the minimum time until a solution might blow up, it cannot rule out blowup. The advantage of the estimate \cite{73} is that the sign of $\det(S)$ is a much more straightforward geometric question than the alignment of the vorticity $\omega$ with the eigenvalues of $S$, which is complicated by the nonlocal dependence of $S$ on $\omega$.

Finally, this provides almost immediately as a corollary an estimate for the quickest possible blowup time in terms of the initial enstrophy. First, we must define $T_{\text{max}}$, the maximal time of existence for a mild solution corresponding to some initial data $S^0 \in L^2_{\text{st}}$.

**Definition 4.10 (Maximal time of existence).** Suppose $S^0 \in L^2_{\text{st}}$. Then the maximal time of existence for a smooth solution corresponding to $S^0$ is
\[ T_{\text{max}} = \sup \{ T > 0 : S \in C \left( [0, T], L^2_{\text{st}} \right), S(\cdot, 0) = S^0 \}, \]

where $S \in C \left( [0, T], L^2_{\text{st}} \right)$ is a mild solution to the Navier Stokes strain equation.

We will note here that $T_{\text{max}}$ for to mild solutions to the Navier Stokes strain equation with initial data $S^0 \in L^2_{\text{st}}$ is equivalent to $T_{\text{max}}$ for mild solutions to the Navier Stokes equations corresponding to initial data $u^0 = -2 \text{div}(-\Delta)^{-1} S^0 \in \dot{H}^1$. We will note that $T_{\text{max}} = +\infty$ corresponds to a global smooth solution, whereas $T_{\text{max}} < +\infty$ corresponds to solutions that develop singularities in finite time. Whether or not there smooth solutions that develop singularities in finite time is one of the biggest open problems in partial differential equations, and is one of the Millennium Problems put forward by the Clay Mathematics Institute \cite{10}. Definition 4.10 is directly related to the Navier Stokes regularity problem as put forward by the Clay Mathematics Institute \cite{10}. The Millennium Problem could be stated as: show $T_{\text{max}} = +\infty$ for all $S^0 \in L^2_{\text{st}}$ or provide a counterexample.

We will now prove a lower bound on $T_{\text{max}}$ based on the growth estimate in Theorem 4.9.
Corollary 4.11 (Lower bound on time to blowup). Suppose $S^0 \in L^2_{st}$. Then

$$T_{\text{max}} \geq \frac{729\pi^4}{||S^0||^4_{L^2}}.$$  

(103)

In other words, for all $T < \frac{729\pi^4}{||S^0||^4_{L^2}}$ there exists a mild solution $S \in C ([0,T];L^2_{st})$ to the Navier Stokes strain equation, and this solution is smooth.

Proof. Fix $T < \frac{729\pi^4}{||S^0||^4_{L^2}}$. In Theorem 4.4 we showed that mild solutions must exist locally in time for any initial data $S^0 \in L^2_{st}$. From this it follows that unless $||S(\cdot,t)||_{L^2}$ becomes unbounded as $t \to \tau$, then a mild solution $S \in C ((0,\tau);L^2_{st})$ can be extended beyond $\tau$ to some $\tau' > \tau$. It is clear from Theorem 4.9 that $||S(\cdot,t)||_{L^2}$ remains bounded for all $t \leq T < \frac{729\pi^4}{||S^0||^4_{L^2}}$, so this establishes the existence of a mild solution $S \in C ([0,T];L^2_{st})$. In particular, this implies that $||S(\cdot,t)||_{L^2}$ must become unbounded as $t \to T_{\text{max}}$ if $T_{\text{max}} < +\infty$. We already showed in Theorem 4.4 that mild solutions $S \in C ([0,T];L^2_{st})$ are smooth, so this completes the proof.

Note that this is similar to the estimate in the initial theorem establishing the local in time existence of mild solutions; the only difference is that we have improved the constant. The estimate on local existence in [17] does not actually state the value of the constant $C > 0$. It is shown in [1] that

$$T_{\text{max}} \geq \frac{4\pi^4}{27||S^0||^4_{L^2}},$$

(104)

although their statement is in terms of $E_0 = \frac{1}{2}||\omega^0||^2_{L^2}$, as their analysis does not focus on the strain. The detailed analysis of the evolution of the minimal blowup time by a factor of 4,920.75 from previous estimates, which were derived in using standard harmonic analysis methods, and did not take full advantage of more of the structures that incompressible flow imposes on the Navier Stokes problem, for instance that the strain matrix must be trace free.

Now that we have finished outlining the main estimates that can be derived from the Navier Stokes strain equation, we will go on to use these estimates to prove a new regularity criterion in terms of the middle eigenvalue of the strain matrix. We will then consider a toy model ODE that captures some of the features of the quadratic term and tells us a little bit about the local structure of blow up solutions, in particular what the distribution of eigenvalues will tend towards. First, however, we will outline the equivalent results for mild solutions to the Navier Stokes equations in the presence of an external force.

Thus far, we have only considered the Navier Stokes equation with no external force; therefore is is necessary to state some definitions involving the external force $f$. For the incompressible Navier Stokes equations with a force we still require that $\nabla \cdot u = 0$, but now the evolution equation is given by

$$u_t + (u \cdot \nabla)u - \Delta u + \nabla p = f.$$  

(105)

The evolution equation for the vorticity is now given by

$$\omega_t + (u \cdot \nabla)\omega - \Delta \omega - S\omega = \nabla \times f,$$  

(106)

and the evolution equation for the strain is given by

$$S_t + (u \cdot \nabla)S - \Delta S + S^2 + \frac{1}{4}\omega \otimes \omega + \frac{1}{4}||\omega||^2 I_3 + Hess(p) = \nabla_{\text{sym}} f.$$  

(107)
We will define a mild solution to the Navier Stokes equation with an external force as follows.

**Definition 4.12 (Mild solution with a force).** Suppose \( u \in C \left( [0, T]; \dot{H}^1 (\mathbb{R}^3) \right) \cap L^2 \left( [0, T]; \dot{H}^2 (\mathbb{R}^3) \right) \), with \( \nabla \cdot u = 0 \) in \( L^2 \), and suppose \( f \in L^2 \left( (0, T); L^2 (\mathbb{R}^3) \right) \). Then \( u \) is a mild solution to the Navier Stokes equation with external force \( f \), if

\[
  u(x, t) = \int_{\mathbb{R}^3} K(x - y, t) u^0(y) dy - \int_0^t \int_{\mathbb{R}^3} K(x - y, t - \tau) (\langle u \cdot \nabla \rangle u + \nabla p + f)(y, \tau) dy d\tau,
\]

where \( p \) is defined in terms of \( u \) and \( f \) by convolution with the Poisson kernel

\[
  p = (-\Delta)^{-1} \left( \sum_{i,j=1}^3 \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j} - \nabla \cdot f \right).
\]

**Theorem 4.13 (Mild solutions with a force exist for short times).** Suppose \( f \in L^2_{\text{loc}} \left( (0, T^*); L^2 (\mathbb{R}^3) \right) \) with \( T^* \leq +\infty \) and \( u^0 \in \dot{H}^1 (\mathbb{R}^3) \), \( \nabla \cdot u = 0 \). Then there exists \( 0 < T_{\text{max}} \leq T^* \) and a mild solution \( u \in C \left( [0, T]\dot{H}^1 (\mathbb{R}^3) \right) \cap L^2 \left( [0, T]; \dot{H}^2 (\mathbb{R}^3) \right) \), for all \( T < T_{\text{max}} \) with \( u(\cdot, 0) = u^0 \).

**Proof.** Like Theorem 4.3, this is proven in Kato and Fujita’s class work [11]. The proof relies on precisely the same iteration scheme that we outlined in the description of the proof of Theorem 4.3, but with \( f \) now included as in Definition 4.12. Note here that the mild solution is not necessarily smooth. In order to get higher regularity, we would need stronger smoothness assumptions on the force than \( f \in L^2 \left( (0, T^*); L^2_{\text{loc}} (\mathbb{R}^3) \right) \).

We can also prove an identity for the growth of enstrophy analogous to the Theorem 4.5, but in this case involving the external force \( f \).

**Theorem 4.14 (Enstrophy growth identity with a force).** Suppose \( u \in C \left( [0, T]; \dot{H}^1 (\mathbb{R}^3) \right) \cap L^2 \left( \dot{H}^2 (\mathbb{R}^3) \right) \), with \( \nabla \cdot u = 0 \) in \( L^2 \), is a mild solution to the Navier Stokes strain equation with force \( f \in L^2 \left( (0, T), L^2 (\mathbb{R}^3) \right) \) and \( S = \nabla_{\text{sym}} u \). Then almost everywhere \( t \in (0, T) \),

\[
  \partial_t ||S(\cdot, t)||^2_{L^2} = -2||S||^2_{\dot{H}^1} - 4 \int \det(S) + \langle -\Delta u, f \rangle.
\]

**Proof.** First we observe that because \( u \) is a mild solution to the Navier Stokes equation, we have that

\[
  \omega_t + (u \cdot \nabla) \omega - \Delta \omega - S\omega = \nabla \times f,
\]

where \( \omega = \nabla \times u \), and

\[
  S_t + (u \cdot \nabla) S - \Delta S + S^2 + \frac{1}{4} \omega \otimes \omega + \frac{1}{4} ||\omega||^2 I_3 + Hess(p) = \nabla_{\text{sym}} f.
\]

As in the case where there was no external force, we can see that \( \langle (u \cdot \nabla) \omega, \omega \rangle = 0 \), by integrating by parts, because \( \nabla \cdot u = 0 \) and we have sufficient regularity. Therefore,

\[
  \partial_t \frac{1}{2} ||\omega(\cdot, t)||^2_{L^2} = -||\omega||^2_{\dot{H}^1} + \langle S; \omega \otimes \omega \rangle + \langle \omega, \nabla \times f \rangle.
\]

Note that this holds almost everywhere \( t \in (0, T] \) because \( \omega \in L^2 \left( [0, T]; \dot{H}^1 (\mathbb{R}^3) \right) \) and because \( \omega(\cdot, t), S(\cdot, t) \in L^2 \cap L^6 \) almost everywhere \( t \in (0, T] \), so \( \langle S, \omega \otimes \omega \rangle \) is well defined at almost every
time. Finally \( \omega(\cdot, t) \in \dot{H}^1 \) and \( \nabla \times f \in \dot{H}^{-1} \) for almost every time, so \( \langle \omega, \nabla \times f \rangle \) is also well defined for almost every \( t \in (0, T] \). Integrating by parts using Stokes Theorem we can see that

\[
\langle \omega, \nabla \times f \rangle = \langle \nabla \times \omega, f \rangle = \langle -\Delta u, f \rangle .
\] (114)

Applying the isometry form Proposition 3.1, we find that

\[
\partial_t \|S(\cdot, t)\|^2_{L^2} = -2\|S\|^2_{H^1} + \langle S; \omega \otimes \omega \rangle + \langle -\Delta u, f \rangle
\] (115)

Observe that \( S(\cdot, t), \omega(\cdot, t) \in L^2 \cap L^6 \) almost everywhere \( t \in (0, T] \) implies that \( \text{Hess}(p), \frac{1}{2} |\omega|^2 I_3 \in L^2 \), so we can apply Proposition 2.4 and find that

\[
\langle S, \text{Hess}(p) \rangle = 0
\] (116)

and

\[
\left( S, \frac{1}{4} |\omega|^2 I_3 \right) = 0,
\] (117)

almost everywhere \( t \in (0, T] \). We can also see that

\[
2 \langle S, \nabla_{sym} f \rangle = \langle -2 \text{ div } S, f \rangle = \langle -\Delta u, f \rangle .
\] (118)

Therefore we find that

\[
\partial_t \|S(\cdot, t)\|^2_{L^2} = -2\|S\|^2_{H^1} - 2 \int \text{ tr}(S^3) - \frac{1}{2} \langle S; \omega \otimes \omega \rangle + \langle -\Delta u, f \rangle ,
\] (119)

almost everywhere in times, noting that \( \int \text{ tr}(S^3) \) is also well defined because almost everywhere \( t \in (0, T] \) because \( S(\cdot, t) \in L^3 \subset L^2 \cap L^6 \). Adding \( \frac{2}{3} \) (123) to \( \frac{1}{2} \) (115) we find

\[
\partial_t \|S(\cdot, t)\|^2_{L^2} = -2\|S\|^2_{H^1} - \frac{4}{3} \int \text{ tr}(S^3) + \langle -\Delta u, f \rangle ,
\] (120)

almost everywhere \( t \in (0, T] \). Observing that \( \int \text{ tr}(S^3) = 3 \det(S) \), as we have shown in Corollary 4.6, this completes the proof. \( \square \)

5 Strain Eigenvalue Regularity Criteria

Before we can state the regularity criteria, we will need to prove a lemma bounding the derivative of enstrophy in terms of \( \lambda_2^+ \).

Lemma 5.1 (Middle eigenvalue determinant bound). Suppose \( S \in C \left( (0, T); L^2_{sa} \right) \) is a mild solution to the Navier Stokes strain equation and \( S(x) \) has eigenvalues \( \lambda_1(x) \leq \lambda_2(x) \leq \lambda_3(x) \). Define

\[
\lambda_2^+ (x) = \max \{ \lambda_2(x), 0 \} .
\] (121)

Then

\[
- \det(S) \leq \frac{1}{2} |S|^2 \lambda_2^+ .
\] (122)

and for all \( 0 < t \leq T \)

\[
\partial_t \|S(\cdot, t)\|^2_{L^2} \leq -2\|S\|^2_{H^1} + 2 \int_{\mathbb{R}^3} \lambda_2^+ |S|^2 .
\] (123)
Next we can apply Young’s Inequality to show that
\[- \lambda_1 \lambda_3 \leq \frac{1}{2}(\lambda_1^2 + \lambda_3^2) \leq \frac{1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) = \frac{1}{2}|S|^2. \tag{125}\]

Finally, we can combine these inequalities and conclude that
\[- \det(S) \leq \frac{1}{2}|S|^2 \lambda_2^+. \tag{126}\]

Applying Corollary 4.6, this completes the proof. □

With this bound, we are now ready to prove the main result of the paper.

**Theorem 5.2** (Middle eigenvalue of strain characterizes the blow-up time). Let \( u \in C \left( [0, T]; \dot{H}^1(\mathbb{R}^3) \right) \) for all \( T < T_{\max} \) be a mild solution to the Navier Stokes equation. If \( \frac{2}{p} + \frac{3}{q} = 2 \), with \( \frac{3}{2} < q \leq +\infty \), then
\[ ||u(\cdot, T)||^2_{\dot{H}^1} \leq ||u^0||^2_{\dot{H}^1} \exp \left( C_0 \int_0^T ||\lambda^+_{\frac{2}{3}}(\cdot, t)||^p_{L^p(\mathbb{R}^3)} dt \right), \tag{127}\]
with the constant \( C_0 \) depending only on \( p \) and \( q \). In particular if the maximal existence time for a smooth solution \( T_{\max} < +\infty \), then
\[ \int_0^{T_{\max}} ||\lambda^+_{\frac{2}{3}}(\cdot, t)||^p_{L^p(\mathbb{R}^3)} dt = +\infty. \tag{128}\]

**Proof.** First we will note that \( ||u(\cdot, t)||_{\dot{H}^1} \) must blow up if our solution is to develop singularities in finite time, so it suffices to prove the bound (127). Applying the isometry (3.1) it suffices to show that
\[ ||S(\cdot, T)||^2_{L^2} \leq ||S^0||^2_{L^2} \exp \left( C_0 \int_0^T ||\lambda^+_{\frac{2}{3}}(\cdot, t)||^p_{L^p(\mathbb{R}^3)} dt \right). \tag{129}\]

To begin we recall the conclusion in Lemma 5.1 (123)
\[ \partial_t ||S(\cdot, t)||^2_{L^2} \leq -2||S||^2_{\dot{H}^1} + 2 \int_{\mathbb{R}^3} \lambda^+_{\frac{2}{3}} ||S||^2. \tag{130}\]

First we will consider the case \( q = +\infty \). Applying Holder’s inequality with exponents 1 and \( +\infty \) we see that,
\[ \partial_t ||S(\cdot, t)||^2_{L^2} \leq 2||\lambda^+_{\frac{2}{3}}||_{L^\infty} ||S||^2_{L^2}. \tag{131}\]

Now we can apply Gronwall’s inequality and find that
\[ ||S(\cdot, T)||^2_{L^2} \leq ||S^0||^2_{L^2} \exp \left( 2 \int_0^T ||\lambda^+_{\frac{2}{3}}||_{L^\infty} dt \right). \tag{132}\]

Now we will consider the case \( \frac{3}{2} < q < +\infty \). We will begin by applying Holder’s inequality to (128), so take \( \frac{1}{q} + \frac{1}{a} = 1 \), and so
\[ \partial_t ||S(\cdot, t)||^2_{L^2} \leq -2||S||^2_{\dot{H}^1} + 2||\lambda^+_{\frac{2}{3}}||_{L^a} ||S||^2_{L^{2a}}. \tag{133}\]
Applying the Sobolev inequality we find
\[
\partial_t \|S(\cdot,t)\|_{L^2}^2 \leq -C\|S\|^2_{L^6} + 2\|\lambda^+_{L^2}\|_{L^6}\|S\|_{L^2}^2. \tag{134}
\]
Noting that \( q > \frac{3}{2}, \) it follows that \( a < 3, \) so \( 2a < 6. \) Take \( \sigma \in (0,1), \) such that \( \frac{1}{2a} = \sigma \frac{1}{2} + (1-\sigma)\frac{1}{6}. \) Then interpolating between \( L^2 \) and \( L^6 \) we find that
\[
\partial_t \|S(\cdot,t)\|_{L^2}^2 \leq -C\|S\|^2_{L^6} + 2\|\lambda^+_{L^2}\|_{L^6}\|S\|^{2(1-\sigma)}_{L^2} \tag{135}
\]
We know that \( \frac{\sigma}{3} + \frac{1}{6} = \frac{1}{2a}, \) so \( \sigma = \frac{3}{2a} - \frac{1}{2} \cdot \frac{1}{a} = 1 - \frac{1}{q}, \) so \( \sigma = 1 - \frac{3}{2q}. \) Therefore we conclude that
\[
\partial_t \|S(\cdot,t)\|_{L^2}^2 \leq -C\|S\|^2_{L^6} + 2\|\lambda^+_{L^2}\|_{L^6}\|S\|^2_{L^2} \tag{136}
\]
Now take \( b = \frac{2q}{3}. \) That means \( 1 < b < +\infty. \) Define \( p \) by \( \frac{1}{p} + \frac{1}{b} = 1, \) and apply Young’s inequality with exponents \( p \) and \( b, \) and we find that
\[
\partial_t \|S(\cdot,t)\|_{L^2}^2 \leq -C\|S\|^2_{L^6} + C_0 \left( \|\lambda^+_{L^2}\|_{L^6}\|S\|^2_{L^2} \right)^p + C\|S\|^{\frac{b^2}{p}}_{L^6}. \tag{137}
\]
Note that \( \frac{1}{p} = 1 - \frac{1}{b} = 1 - \frac{3}{2q}. \) This means that \( p(2 - \frac{3}{q}) = 2 \) and that \( \frac{2}{p} + \frac{3}{q} = 2, \) and we know by definition that \( b^2 = 2, \) so
\[
\partial_t \|S(\cdot,t)\|_{L^2}^2 \leq C_0\|\lambda^+_{L^2}\|_{L^6}^p\|S\|^2_{L^2}. \tag{138}
\]
Applying Gronwall’s inequality we find that
\[
\|S(\cdot,T)\|_{L^2}^2 \leq \|S^0\|_{L^2}^2 \exp \left( C_0 \int_0^T \|\lambda^+_{L^2}\|_{L^6}^p dt \right). \tag{139}
\]
This completes the proof. \( \square \)

We will note here that the case \( p = 1, q = +\infty \) corresponds to the Beale-Kato-Majda criterion, so it may be possible to show that in this case the regularity criterion holds for the Euler equations as well as the Navier Stokes equations. Note in particular that we did not use the dissipation to control the enstrophy, so there is a natural path to extend the result to solutions of Euler. There is more work to do however, as bounded enstrophy is not sufficient to guarantee regularity for solutions to the Euler equations.

There is also an open question at the other boundary case, \( p = +\infty, q = +\infty. \) This would likely be quite difficult as the methods used in [9][11] to extend the Prodi-Serrin-Ladyzhenskaya regularity criterion to the boundary case \( p = +\infty, q = 3 \) were much more technical than the methods in [21][25][29]. In particular, when \( p = +\infty \) it is no longer adequate to rely on the relevant Sobolev embeddings, because we cannot apply Gronwall’s inequality. Nonetheless, it is natural to suspect based on Theorem 5.2 that if \( u \) is a smooth solution to the Navier Stokes equation with a maximal time of existence, \( T_{\text{max}} < +\infty, \) then
\[
\limsup_{t \to T_{\text{max}}} \|\lambda^+_{L^2}(\cdot,t)\|_{L^2}^2 = +\infty. \tag{140}
\]
Note that the boundary case in our paper is \( q = \frac{3}{2}, \) not \( q = 3. \) This is because the regularity criterion in [9][11] is on \( u, \) whereas our regularity criterion is on an eigenvalue of the strain matrix, which scales like \( \nabla \otimes u. \) This is directly related to the Sobolev embedding \( W^{1,\frac{3}{2}}(\mathbb{R}^3) \subset \mathcal{L}^3(\mathbb{R}^3). \)
Theorem 5.2 is one of few regularity criteria for the Navier Stokes equations involving a signed quantity, which is not too surprising, given that the Navier Stokes equation is a vector valued equation. Even the scalar regularity criteria based on only one component of \( u \) do not involve signed quantities [4]. The only other regularity criterion for the Navier Stokes equation involving a signed quantity—at least to the knowledge of the author—is the regularity criterion proved by Seregin and Šverák [27] that for a smooth solution to the Navier Stokes equation to blowup in finite time, \( p \) must become unbounded below and \( p + \frac{1}{2} |u|^2 \) must become unbounded above.

We will also make a remark about the relationship between this result and the regularity criterion on one component of the gradient tensor \( \frac{\partial u_i}{\partial x^j} \) in [3]. A natural question to ask in light of this regularity criterion is whether it is possible to prove a regularity criterion on just one entry of the strain tensor \( S_{ij} \). This paper does not answer this question, however we do prove a regularity criterion on just one diagonal entry of the diagonalization of the strain tensor.

**Corollary 5.3** (Any eigenvalue of strain characterizes the blow-up time). Let \( u \in C \left( [0, T]; H^1(\mathbb{R}^3) \right) \) for all \( T < T_{\text{max}} \) be a mild solution to the Navier Stokes equation, and let \( \lambda_i(x) \) be an eigenvalue of \( S(x) \). If \( \frac{2}{p} + \frac{3}{q} = 2 \), with \( \frac{2}{3} < q \leq +\infty \), then

\[
||u(\cdot, T)||_{H^1}^2 \leq ||u^0||_{H^1}^2 \exp \left( C_0 \int_0^T ||\lambda_i(\cdot, t)||_{L^p(\mathbb{R}^3)}^p dt \right),
\]

with the constant \( C_0 \) depending only on \( p \) and \( q \). In particular if the maximal existence time for a smooth solution \( T_{\text{max}} < +\infty \), then

\[
\int_0^{T_{\text{max}}} ||\lambda_i(\cdot, t)||_{L^p(\mathbb{R}^3)}^p dt = +\infty.
\]

**Proof.** \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \) and \( \lambda_1 + \lambda_2 + \lambda_3 = 3 \) implies that \( |\lambda_1|, |\lambda_3| \geq |\lambda_2| \geq |\lambda_2^+| \). Therefore

\[
\int_0^T ||\lambda_2^+(\cdot, t)||_{L^p}^p dt \leq \int_0^T ||\lambda_i(\cdot, t)||_{L^p}^p dt.
\]

Applying this inequality to both conclusions in Theorem 5.2, this completes the proof. \( \square \)

We will also note that there is a gap to be closed in the regularity criterion on \( \frac{\partial u_i}{\partial x^j} \), because it is not the optimal result with respect to scaling and requires subcritical control on \( \frac{\partial u_i}{\partial x^j} \). That is, the result only holds for \( \frac{2}{p} + \frac{3}{q} = \frac{n+3}{2q} < 2 \), for \( i \neq j \) and \( \frac{2}{p} + \frac{3}{q} = \frac{3q+6}{4q} < 2 \), for \( i = j \), whereas the regularity criterion on one of the eigenvalues in Corollary 5.3 is critical with respect to the scaling. It is natural, however, to ask whether the main theorem in this paper can be extended to the critical Besov space, so in that sense the result may be pushed further.

Corollary 5.3 is not really new, however, for \( \lambda_1 \) or \( \lambda_3 \). This is because \( |\lambda_1| \) and \( |\lambda_3| \) both control \( |S| \). As we will see from the following proposition, it is possible to prove regularity criteria in terms of \( \lambda_1 \) or \( \lambda_3 \) without any reference to the strain equation at all.

**Proposition 5.4** (Lower bounds on the magnitude of the external eigenvalues). Suppose \( M \in S^{3\times 3} \) is a symmetric trace free matrix with eigenvalues \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \). Then

\[
\lambda_3 \geq \frac{1}{\sqrt{6}} |S|
\]

(144)
with equality if and only if \(-\frac{1}{2}\lambda_1 = \lambda_2 = \lambda_3\), and
\[
\lambda_1 \leq -\frac{1}{\sqrt{6}}|S|,
\]  
(145)
with equality if and only if \(\lambda_1 = \lambda_2 = -\frac{1}{2}\lambda_3\).

Furthermore, for all \(S \in L^2_{st}\) and for all \(1 \leq q \leq +\infty\)
\[
||S||_{L^q} \leq \sqrt{6}||\lambda_1||_{L^q}
\]  
(146)
and
\[
||S||_{L^q} \leq \sqrt{6}||\lambda_3||_{L^q}.
\]  
(147)

Proof. We will prove the statement for \(\lambda_3\). The proof of the statement for \(\lambda_1\) is entirely analogous and is left to the reader. First observe that if \(-\frac{1}{2}\lambda_1 = \lambda_2 = \lambda_3\), then
\[
|S|^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 6\lambda_3^2,
\]  
(148)
So we have proven that if \(\lambda_2 = \lambda_3\), then \(\lambda_3 = \frac{1}{\sqrt{6}}|S|\). Now suppose \(\lambda_2 < \lambda_3\). Recall that
\[
\text{tr}(M) = \lambda_1 + \lambda_2 + \lambda_3 = 0,
\]  
(149)
so
\[
\lambda_1 = -\lambda_2 - \lambda_3.
\]  
(150)
Therefore we find that
\[
|S|^2 = (-\lambda_2 - \lambda_3)^2 + \lambda_2^2 + \lambda_3^2 = 2\lambda_2^2 + 2\lambda_3^2 + 3\lambda_2\lambda_3.
\]  
(151)
Applying Young’s Inequality we can bound
\[
2\lambda_2\lambda_3 \leq \lambda_2^2 + \lambda_3^2,
\]  
(152)
so
\[
|S|^2 \leq 3\lambda_2^2 + 3\lambda_3^2 < 6\lambda_3^2.
\]  
(153)
\(\lambda_3 \geq 0\), so this completes the proof. We leave the analogous proof for \(\lambda_1\) to the reader. The \(L^q\) bounds follow immediately from integrating these bounds pointwise when one recalls that \(\text{tr}(S) = 0\). We will note here that the \(L^q\) norms may be infinite, as by hypothesis we only have \(S \in L^2\). □

In particular this implies that regularity criteria involving \(\lambda_1\) or \(\lambda_3\) follow immediately from regularity criteria involving \(S\), so while the regularity criteria on \(\lambda_1\) and \(\lambda_3\) in Corollary 5.3 do not appear in the literature to my knowledge, these criteria do not offer a real advance over the Prodi-Serrin-Ladyzhenskaya criterion [21][25][29], as the critical norm on \(u\) can be controlled by the critical norm on \(S\) using Sobolev embedding, which can in turn be bounded by the critical norm on \(\lambda_1\) or \(\lambda_3\) using Proposition 5. That is
\[
||u||_{L^{q^*}} \leq C||S||_{L^q} \leq \sqrt{6}C||\lambda_3||_{L^q}.
\]  
(154)
It is the regularity criterion in terms of \(\lambda_3^+\) that is really significant, because it encodes geometric information about the strain beyond just its size.

We will also note that none of the regularity criteria involving \(\nabla u_j\) [31], \(\partial_x u\) [20], or \(\partial_x u_j\) [3], have been proven for the Navier Stokes equation with an external force. However, the regularity criterion in Theorem 5.2 can be generalized for the Navier Stokes equation with an external force.
**Theorem 5.5** (Middle eigenvalue of strain characterizes the blow-up time with force). Let \( u \in C \left( [0, T]; \dot{H}^1 \left( \mathbb{R}^3 \right) \right) \cap L^2 \left( [0, T]; \dot{H}^2 \left( \mathbb{R}^3 \right) \right) \), for all \( T < T_{\text{max}} \) be a mild solution to the Navier Stokes equation with force \( f \in L^2_{\text{loc}} \left( (0, T^*); L^2 \left( \mathbb{R}^3 \right) \right) \). If \( \frac{2}{p} + \frac{3}{q} = 2 \), with \( \frac{3}{2} < q \leq +\infty \), then

\[ ||u(\cdot, T)||_{\dot{H}^1}^2 \leq \left( ||u^0||_{\dot{H}^1}^2 + 2 \int_0^T ||f(\cdot, t)||_{L^2}^2 \right) \exp \left( C_0 \int_0^T ||\lambda_2^+(\cdot, t)||_{L^q(\mathbb{R}^3)}^p dt \right), \]  

with the constant \( C_0 \) depending only on \( p \) and \( q \). In particular if the maximal existence time for a mild solution \( T_{\text{max}} < T^* \), then

\[ \int_0^{T_{\text{max}}} ||\lambda_2^+(\cdot, t)||_{L^q(\mathbb{R}^3)}^p dt = +\infty. \]

**Proof.** Because it was shown in [11] that mild solutions to the Navier Stokes strain equation must exist locally in time as long as \( f \in L^2_{\text{loc}} \) and \( u^0 \in \dot{H}^1 \) it is sufficient to prove the bound (155).

Recall that for a mild solution to the Navier Stokes strain equation with external force \( f \in L^2_{\text{loc}} \) we showed in Theorem [11] that

\[ \partial_t ||S(\cdot, t)||_{L^2}^2 = -2||S||_{\dot{H}^1}^2 - 4 \int \det(S) + \langle -\Delta u, f \rangle, \]  

almost everywhere \( t \in (0, T_{\text{max}}) \). Observe that

\[ \langle -\Delta u, f \rangle \leq || -\Delta u||_{L^2} ||f||_{L^2}. \]  

(158)

Recall that \( || -\Delta u||_{L^2} = 2||S||_{\dot{H}^1}^2 \), so

\[ \langle -\Delta u, f \rangle \leq 2||S||_{\dot{H}^1} ||f||_{L^2}. \]  

(159)

Next we see that

\[ -\det(S) \leq \frac{1}{2} ||S||_{\dot{H}^1}^2. \]  

(160)

as in Lemma 5.1. Therefore, almost everywhere \( t \in (0, T_{\text{max}}) \),

\[ \partial_t ||S(\cdot, t)||_{L^2}^2 \leq -2||S||_{\dot{H}^1}^2 + 2 \int ||S||_{\dot{H}^1}^2 + 2||S||_{\dot{H}^1} ||f||_{L^2}. \]  

(161)

Applying Hölder’s inequality and Sobolev embedding, and interpolating between \( L^2 \) and \( L^{2\alpha} \) as in Theorem 5.2 we find that

\[ \partial_t ||S(\cdot, t)||_{L^2}^2 \leq -2||S||_{\dot{H}^1}^2 + C||\lambda_2^+||_{L^q} ||S||_{L^2}^{2-\frac{2}{q}} ||S||_{\dot{H}^1}^\frac{2}{q} + 2||S||_{\dot{H}^1} ||f||_{L^2}. \]  

(162)

Applying Young’s Inequality to the second term as in Theorem 5.2 and to the third term with exponents \( \frac{1}{2} \) we find that

\[ \partial_t ||S(\cdot, t)||_{L^2}^2 \leq C_0||\lambda_2^+||_{L^q} ||S||_{L^2}^2 + ||f||_{L^2}^2. \]  

(163)

Applying Gronwall’s inequality we find that

\[ ||S(\cdot, T)||_{L^2}^2 \leq \left( ||S^0||_{L^2}^2 + \int_0^T ||f(\cdot, t)||_{L^2}^2 \right) \exp \left( C_0 \int_0^T ||\lambda_2^+(\cdot, t)||_{L^q}^p dt \right). \]  

(164)

This completes the proof.
Lemma 5.6 (The middle eigenvector is minimal). Suppose $S \in L^2_{\text{ad}}$ and $v \in L^\infty(\mathbb{R}^3; \mathbb{R}^3)$ with $|v(x)| = 1$ almost everywhere $x \in \mathbb{R}^3$. Then

$$|\lambda_2(x)| \leq |S(x)v(x)|$$

almost everywhere $x \in \mathbb{R}^3$.

Proof. By the spectral theorem, we know that there is an orthonormal eigenbasis for $\mathbb{R}^n$. In particular, take $v_1(x), v_2(x), v_3(x)$ to be eigenvectors of $S(x)$ corresponding to eigenvalues $\lambda_1(x), \lambda_2(x), \lambda_3(x)$ such that $|v_1(x)|, |v_2(x)|, |v_3(x)| = 1$ almost everywhere $x \in \mathbb{R}^3$. Then from the spectral theorem we know that $\{v_1(x), v_2(x), v_3(x)\}$ is an orthonormal basis for $\mathbb{R}^3$ almost everywhere $x \in \mathbb{R}^3$. Therefore

$$Sv = \lambda_1(v \cdot v_1) v_1 + \lambda_2(v \cdot v_2) v_2 + \lambda_3(v \cdot v_3) v_3.$$ (166)

Therefore we can see that

$$|Sv|^2 = \lambda_1^2(v \cdot v_1)^2 + \lambda_2^2(v \cdot v_2)^2 + \lambda_3^2(v \cdot v_3)^2$$ (167)

We know that $|\lambda_2| \leq |\lambda_1|, |\lambda_3|$, so

$$|Sv|^2 \geq \lambda_2^2 ((v \cdot v_1)^2 + (v \cdot v_2)^2 + (v \cdot v_3)^2)$$ (168)

Because $\{v_1(x), v_2(x), v_3(x)\}$ is an orthonormal basis for $\mathbb{R}^3$ almost everywhere $x \in \mathbb{R}^3$, we conclude that

$$(v \cdot v_1)^2 + (v \cdot v_2)^2 + (v \cdot v_3)^2 = |v|^2 = 1.$$ (169)

Therefore

$$|Sv|^2 \geq \lambda_2^2.$$ (170)

This concludes the proof. 

Theorem 5.7 (Blowup requires the strain to blow up in every direction). Let $u \in C \left([0, T]; \dot{H}^1(\mathbb{R}^3)\right) \cap L^2 \left([0, T]; \dot{H}^2(\mathbb{R}^3)\right)$, for all $T < T_{\text{max}}$ be a mild solution to the Navier Stokes equation with force $f \in L^2_{\text{loc}}((0, T^*); L^2(\mathbb{R}^3))$, and let $v \in L^\infty(\mathbb{R}^3 \times [0, T_{\text{max}}]; \mathbb{R}^3)$, with $|v(x, t)| = 1$ almost everywhere. If $\frac{2}{p} + \frac{2}{q} = 2$, with $\frac{3}{2} < q \leq +\infty$, then

$$\|u(\cdot, T)\|^2_{\dot{H}^1} \leq \left(\|u(\cdot, T)\|^2_{\dot{H}^1} + 2 \int_0^T \|f(\cdot, t)\|^2_{L^2} \exp \left(C_0 \int_0^T \|S(\cdot, t)v(\cdot, t)\|_{L^q(\mathbb{R}^3)}^p \right) dt \right),$$ (171)

with the constant $C_0$ depending only on $p$ and $q$. In particular if the maximal existence time for a mild solution $T_{\text{max}} < T^*$, then

$$\int_0^{T_{\text{max}}} \|S(\cdot, t)v(\cdot, t)\|_{L^q(\mathbb{R}^3)}^p dt = +\infty.$$ (172)

Proof. This follows immediately from Lemma 5.6 and Theorem 5.5

We can use Theorem 5.7 to prove a new one component regularity criterion involving the sum of the derivative of the whole velocity in one direction, and the gradient of the component in the same direction.
Corollary 5.8 (Global one direction regularity criterion). Let $u \in C \left( [0, T]; \dot{H}^1(\mathbb{R}^3) \right) \cap L^2 \left( [0, T]; \dot{H}^2(\mathbb{R}^3) \right)$, for all $T < T_{\text{max}}$ be a mild solution to the Navier Stokes equation with force $f \in L^2_{\text{loc}} \left( (0, T^*); L^2(\mathbb{R}^3) \right)$. If $\frac{2}{p} + \frac{3}{q} = 2$, with $\frac{3}{2} < q \leq +\infty$, then

$$
||u(\cdot, T)||_{\dot{H}^1}^2 \leq \left( ||u^0||_{\dot{H}^1}^2 + 2 \int_0^T ||f(\cdot, t)||_{L^2}^2 \right) \exp \left( C_0 \int_0^T ||\partial_3 u(\cdot, t) + \nabla u_3(\cdot, t)||_{L^q(\mathbb{R}^3)}^p \, dt \right), \quad (173)
$$

with the constant $C_0$ depending only on $p$ and $q$. In particular if the maximal existence time for a mild solution $T_{\text{max}} < T^*$, then

$$
\int_0^{T_{\text{max}}} ||\partial_3 u(\cdot, t) + \nabla u_3(\cdot, t)||_{L^q(\mathbb{R}^3)}^p \, dt = +\infty. \quad (174)
$$

Proof. Take $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Then $Sv = \frac{1}{T} \partial_3 u + \frac{1}{T} \nabla u_3$. This completes the proof.

Corollary 5.8 significantly extends the range of exponents for which we have a critical regularity criterion involving one component. For instance, Kukavica and Zaine [20] that if $T_{\text{max}} < +\infty$, and if $\frac{2}{p} + \frac{3}{q}$, with $\frac{3}{4} \leq q \leq 3$, then

$$
\int_0^{T_{\text{max}}} ||\partial_3 u(\cdot, t)||_{L^q(\mathbb{R}^3)}^p \, dt = +\infty. \quad (175)
$$

Corollary 5.8 is not immediately implied by this result even in the range $\frac{3}{4} \leq q \leq 3$, because the control in Corollary 5.8 is in terms of the norm of the sum, not the sum of the norms. That is to say, it is not obvious in general that

$$
||\partial_3 u||_{L^q} \leq C ||\partial_3 u + \nabla u_3||_{L^q}. \quad (176)
$$

Nonetheless, in the range $\frac{3}{4} \leq q \leq 3$ the the result in [20] is stronger in the sense that it involves only control of $\partial_3 u$, not $\partial_3 u + \nabla u_3$. It was recently proven by Chemin, Zhang, and Zhang [4,5] that if $T_{\text{max}} < +\infty$ then

$$
\int_0^{T_{\text{max}}} ||u_3||_{H^{\frac{1}{2} + \frac{3}{p}}}^p \, dt = +\infty, \quad (177)
$$

for $4 < p < \infty$. Note this criterion is also critical—and stronger than Corollary 5.8 in the sense that

$$
||u_3||_{H^{\frac{1}{2} + \frac{3}{p}}} \leq C ||\nabla u_3||_{L^q}, \quad (178)
$$

by the Sobolev embedding $\dot{H}^{\frac{1}{2} + \frac{3}{p}}(\mathbb{R}^3) \subset W^{1,q}(\mathbb{R}^3)$ when $\frac{2}{p} + \frac{3}{q} = 2$. However, Corollary 5.8 is not implied by the results in [4,5] even in the range $4 < p < +\infty$ because it is not true in general that

$$
||u_3||_{H^{\frac{1}{2} + \frac{3}{p}}} \leq C ||\nabla u_3 + \partial_3 u||_{L^q}. \quad (179)
$$

Note that $4 < p < +\infty$ corresponds to $\frac{3}{2} < q < 6$. This means that there are other critical one component regularity criteria than Corollary 5.8 for $\frac{3}{2} < q < 6$, but for $6 \leq q \leq +\infty$ Corollary 5.8 is the only critical one component regularity criterion available.
Finally, while the regularity criterion in \textbf{[4.5]} involve a fixed direction, which can be taken to be $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ without loss of generality because of the rotational invariance of the Navier Stokes equation, Theorem \textbf{5.7} allows us to prove a one component blow up criterion that involves different directions in different regions of $\mathbb{R}^3$. First off, for any unit vector $v \in \mathbb{R}^3$, $|v| = 1$ we define $\partial_v = v \cdot \nabla$ and $u_v = u \cdot v$.

\textbf{Corollary 5.9} (Local one direction regularity criterion). Let $\{v_n(t)\}_{n \in \mathbb{N}} \subset \mathbb{R}^3$ with $|v_n(t)| = 1$. Let $\{\Omega_n(t)\}_{n \in \mathbb{N}} \subset \mathbb{R}^3$ be Lesbesgue measurable sets such that for all $m \neq n$, $\Omega_m(t) \cap \Omega_n(t) = \emptyset$, and $\mathbb{R}^3 = \bigcup_{n \in \mathbb{N}} \Omega_n(t)$. Let $u \in C \left([0, T]; \dot{H}^1(\mathbb{R}^3)\right) \cap L^2([0, T]; \dot{H}^2(\mathbb{R}^3))$, for all $T < T_{\text{max}}$ be a mild solution to the Navier Stokes equation with force $f \in L^2_{\text{loc}}((0, T^*); L^2(\mathbb{R}^3))$. If $\frac{2}{p} + \frac{3}{q} = 2$, with $\frac{3}{2} < q \leq +\infty$, then

$$||u(\cdot, T)||^2_{H^1} \leq \left(||u^0||^2_{H^1} + 2 \int_0^T ||f(\cdot, t)||^2_{L^2} \right) \exp \left(C_0 \int_0^{T_{\max}} \left(\sum_{n=1}^\infty \left\| \frac{1}{2} \partial_{v_n} u(\cdot, t) + \frac{1}{2} \nabla u_{v_n}(\cdot, t) \right\|_{L^q(\Omega_n(t))}^q \right)^\frac{p}{q} dt \right),$$

with the constant $C_0$ depending only on $p$ and $q$. In particular if the maximal existence time for a mild solution $T_{\text{max}} < T^*$, then

$$\int_0^{T_{\max}} \left(\sum_{n=1}^\infty \left\| \frac{1}{2} \partial_{v_n} u(\cdot, t) + \frac{1}{2} \nabla u_{v_n}(\cdot, t) \right\|_{L^q(\Omega_n(t))}^q \right)^\frac{p}{q} dt = +\infty. \quad (181)$$

\textbf{Proof.} Let $v(x, t) = \sum_{n=1}^\infty v_n(t) I_{\Omega_n(t)}(x)$, where $I_\Omega$ is the indicator function $I_\Omega(x) = 1$ for all $x \in \Omega$ and $I_\Omega(x) = 0$ otherwise. Note that in this case we clearly have

$$S(x, t)v(x, t) = \sum_{n=1}^\infty I_{\Omega_n(t)}(x)S(x, t)v_n(t)$$

(182)

Because $\{\Omega_n\}_{n \in \mathbb{N}}$ are disjoint, we have

$$||S(\cdot, t)v(\cdot, t)||^p_{L^q(\mathbb{R}^3)} = \sum_{n=1}^\infty ||S(\cdot, t)v_n(t)||^p_{L^q(\Omega_n(t))}. \quad (183)$$

Therefore we find that

$$||S(\cdot, t)v(\cdot, t)||^p_{L^q(\mathbb{R}^3)} = \left(\sum_{n=1}^\infty ||S(\cdot, t)v_n(t)||^q_{L^q(\Omega_n(t))}\right)^\frac{p}{q}. \quad (184)$$

Finally observe that

$$S(x, t)v_n(t) = \frac{1}{2} \partial_{v_n} u(x, t) + \frac{1}{2} \nabla u_{v_n}(x, t), \quad (185)$$

so

$$||S(\cdot, t)v(\cdot, t)||^p_{L^q(\mathbb{R}^3)} = \left(\sum_{n=1}^\infty \left\| \frac{1}{2} \partial_{v_n} u(\cdot, t) + \frac{1}{2} \nabla u_{v_n}(\cdot, t) \right\|_{L^q(\Omega_n(t))}^q \right)^\frac{p}{q}. \quad (186)$$

Applying Theorem \textbf{5.7} this completes the proof. \qed

27
Corollary 5.8 essentially says that for a solution to blowup in finite time, this blowup must be essentially three dimensional. For a two dimensional flow

\[ u(x) = \begin{pmatrix} u_1(x_1, x_2) \\ u_2(x_1, x_2) \\ 0 \end{pmatrix}, \quad (187) \]

it is clear that both \( u_3 = 0 \) and \( \partial_3 u = 0 \). Note, that we don’t require \( u \in L^2(\mathbb{R}^3) \) in this case, as this would imply \( u = 0 \). Therefore \( \partial_3 u + \nabla u_3 \) remaining bounded in \( L^p_t L^q_x \) can be seen as a statement that the flow is approximately two dimensional, in the sense that it is not blowing up in all directions. Corollary 5.8 is a statement that a solution \( u \) must be smooth as long as it is approximately two dimensional globally. Corollary 5.9 strengthens this to a statement that a solution \( u \) must be smooth as long as it is approximately two dimensional locally. Note that every smooth solution is “approximately two dimensional” in the sense we described, as fully three dimensional in this sense corresponds to fully three dimensional blowup. Corollary 5.8 says blowup must be three dimensional globally, whereas Corollary 5.9 says blowup must be three dimensional locally.

This is directly tied to the regularity criterion on \( \lambda_2 \) in Theorem 5.2, because if the flow is two dimensional as in \( (187) \), then \( \lambda_2 = 0 \), so the magnitude of \( \lambda_2 \) can be seen as characterizing how fully three dimensional the flow is. The fact that the regularity criterion in Theorem 5.2 is on the positive part of the of the intermediate eigenvalue also has geometric significance for the flow. This regularity criterion shows that \( \lambda_2 \) must be large and positive in order for blowup to occur. Enstrophy growth requires two positive eigenvalues—ideally equal—and one very negative eigenvalue. This corresponds physically to the strain tensor stretching in two directions, while contracting even more strongly in a third.

The regularity criterion in Theorem 5.2 also offers analytical evidence of the numerically observed tendency [12] of the vorticity to align with the eigenvector corresponding to the intermediate eigenvalue \( \lambda_2 \). If it is true that the vorticity tends to align with the intermediate eigenvalue we would heuristically expect that

\[ tr(S(x)\omega(x) \otimes \omega(x)) \sim \lambda_2(x)|\omega(x)|^2. \quad (188) \]

We would then heuristically expect that

\[ \langle S; \omega \otimes \omega \rangle \sim \int_{\mathbb{R}^3} \lambda_2(x)|\omega(x)|^2 dx, \quad (189) \]

and so we would expect that there would be some inequality of the form

\[ \langle S; \omega \otimes \omega \rangle \leq C \int_{\mathbb{R}^3} \lambda_2^+(x)|\omega(x)|^2 dx \quad (190) \]

This is all, of course, entirely heuristic, but it is interesting that the regularity criterion we have proven is precisely of the form that would be predicted by the observed tendency of the vorticity to align with the eigenvector associated with the intermediate eigenvalue. This suggests that significant information about the geometric structure of incompressible flow is encoded in the identity for enstrophy growth in Theorem 4.5 and the regularity criterion in Theorem 5.2.

6 Blowup for a Toy Model ODE

The main advantage of the strain equation formulation of the Navier Stokes equation compared with the vorticity formulation is that the quadratic term \( S^2 + \frac{1}{4} w \otimes w \) has a much nicer structure
then the quadratic term $S \omega$ in the vorticity formulation. The price we pay for this is that there are additional terms, particularly Hess$(p)$ which are not present in the vorticity formulation. There is also the related difficulty that the consistency condition in the strain formulation is significantly more complicated than in the vorticity formulation. We will now examine a toy model ODE, to prove a few results about blow up, and the stability and asymptotic behavior of blowup. The simplest toy model equation would be to keep only the local part of the quadratic term (vorticity depends non-locally on $S$), and to study the ODE $M_t + M^2 = 0$. As long as the initial condition $M(0)$ is an invertible matrix, this has the solution $(M(t))^{-1} = (M(0))^{-1} + tI_3$. This equation will blow up in finite time assuming that $M(0)$ has at least one negative eigenvalue. Blowup is unstable in general, because any small perturbation into the complex plane will mean there will not be blowup. However, if we restrict to symmetric matrices, then blowup is stable, because then the eigenvalues must be real valued, so a small perturbation will remain on the negative real axis. The negative Real axis is an open set of $R$, but not of $C$, so blowup is stable only when we are restricted to matrices with real eigenvalues, which is the case we are concerned with as the strain tensor is symmetric. This equation does not preserve the family of trace free matrices however, because $tr(M^2) = |M|^2 \neq 0$, and therefore doesn’t really capture any of the features of the Strain equation (6). We will instead take our toy model ODE on the space of symmetric, trace free matrices to be

$$M_t + M^2 - \frac{1}{3}|M|^2 I_3 = 0. \quad (191)$$

Because every symmetric matrix is diagonalizable over the Reals, and every diagonalizable matrix is mutually diagonalizable with the identity matrix, this equation can be treated as a system of ODEs for the evolution of the eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3$ with for every $1 \leq i \leq 3$,

$$\partial_t \lambda_i = -\lambda_i^2 + \frac{1}{3}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2). \quad (192)$$

This equation has two families of solutions with a type of scaling invariance. Let $S(0) = Cdiag(-2, 1, 1)$, with $C > 0$ then $S(t) = f(t)diag(-2, 1, 1)$, where $f_t = f^2, f(0) = C$. Therefore we have blowup in finite time, with $S(t) = \frac{1}{t^2 + t} diag(-2, 1, 1)$. The reverse case, one positive eigenvalue and two equal negative eigenvalues, also preserves scaling, but decays to zero as $t \to \infty$. Let $S(0) = Cdiag(-1, -1, 2)$, with $C > 0$. Then $S(t) = \frac{1}{t^2 + t} diag(-1, -1, 2)$.

We will show that the blow up solution is stable, while the decay solution is unstable. Furthermore the blow up solution is asymptotically a global attractor except for the unstable family of solutions that decay to zero (i.e two equal negative eigenvalues and the zero solution). To prove this we will begin by rewriting our system. First of all, we will assume without loss of generality, that $S \neq 0$, because clearly if $S(0) = 0$, then $S(t) = 0$, is the solution. If $S \neq 0$, then clearly $\lambda_1 < 0$ and $\lambda_3 > 0$. Our system of equations really only has two degrees of freedom, because of the condition $tr(S) = \lambda_1 + \lambda_2 + \lambda_3 = 0$, but because we are interested in the ratios of the eigenvalues asymptotically, we will reduce the system to the two parameters $\lambda_3$ and $r = -\frac{\lambda_1}{\lambda_3}$. These two parameters completely determine our system because $\lambda_1 = -r \lambda_3$ and $\lambda_2 = -\lambda_1 - \lambda_3 = (r - 1)\lambda_3$. We now will rewrite our system of ODEs:

$$\partial_t \lambda_3 = \frac{1}{3}(\lambda_1^2 + \lambda_2^2 - 2\lambda_3^2) = \frac{1}{3}\lambda_3^2 (r^2 + (r - 1)^2 + 2) = \frac{1}{3}\lambda_3^2 (2r^2 - 2r - 1)$$

$$\partial_r = \frac{\lambda_1 \partial_t \lambda_3 - \lambda_3 \partial_t \lambda_1}{\lambda_3^2} = \lambda_3 \left(-r \left(-\frac{1}{3} - \frac{2}{3}r + \frac{2}{3}r^2\right) + \frac{2}{3}r + \frac{1}{3}r^2\right) = \frac{1}{3}\lambda_3(-2r^3 + 3r^2 + 3r - 2) \quad (193)$$
At this point it will be useful to remark on the range of values our two variables can take. Clearly the largest eigenvalue \( \lambda_3 \geq 0 \), and \( \lambda_3 = 0 \) if and only if \( \lambda_1, \lambda_2, \lambda_3 = 0 \). Now we turn to the range of values for \( r \). Recall that \( \lambda_2 = (r - 1)\lambda_3 \), and that \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \). Therefore \(-r \leq r - 1 \leq 1\), so \( \frac{1}{2} \leq r \leq 2 \). If we take \( f(r) = -2r^3 + 3r^2 + 3r - 2 \), we find that \( f(r) \) is positive for \( \frac{1}{2} < r < 2 \) with \( f(\frac{1}{2}), f(2) = 0 \). This is the basis for the blowup solution being the asymptotic attractor. We are now ready to state our theorem on the existence and algebraic structure of finite time blow up solutions.

**Theorem 6.1** (Toy model dynamics). Suppose \( \lambda_3(0) > 0 \) and \( r(0) > \frac{1}{2} \), then there exists \( T > 0 \) such that \( \lim_{t \to T} \lambda_3(t) = +\infty \), and furthermore \( \lim_{t \to T} r(t) = 2 \).

**Proof.** We’ll start by showing that finite time blow up exists, and then we will show that \( r \) goes to 2 as we approach the blow up time. First we observe that \( g(r) = 2r^2 - 2r - 1 \), has a zero at \( \frac{1+\sqrt{3}}{2} \). \( g(r) < 0 \), for \( \frac{1}{2} \leq r < \frac{1+\sqrt{3}}{2} \), and \( g \) is both positive and increasing on \( \frac{1+\sqrt{3}}{2} < r \leq 2 \). We will begin with the case where \( r(0) = r_0 > \frac{1+\sqrt{3}}{2} \). Clearly \( \partial_t r \geq 0 \), so \( r(t) > r_0 \), and \( g(r(t)) > g(r_0) \). Let \( C = \frac{1}{3}g(r_0) \), then we find that:

\[
\partial_t \lambda_3 = \frac{1}{3} g(r(t)) \lambda_3^2 \geq C \lambda_3^2.
\]

From this differential inequality, we find that

\[
\lambda_3(t) \geq \frac{1}{\lambda_3(0) - Ct},
\]

so clearly there exists a time \( T \leq \frac{1}{C\lambda_3(0)} \), such that \( \lim_{t \to T} \lambda_3(t) = +\infty \).

Now we consider the case where \( \frac{1}{2} < r_0 \leq \frac{1+\sqrt{3}}{2} \). It suffices to show that there exists a \( T_a > 0 \) such that \( r(T_a) > \frac{1+\sqrt{3}}{2} \), then the proof above applies. Note that \( g \) is increasing on the interval \([-\frac{1}{2}, 2]\), so \( g(r(t)) > g(r_0) \). Let \( B = -\frac{1}{3}g(r_0) > 0 \), and let \( C = \frac{1}{\min(f(r_0), f(\frac{1+\sqrt{3}}{2}))} \). Suppose towards contradiction that for all \( t > 0 \), \( r(t) \leq \frac{1+\sqrt{3}}{2} \). Then we will have the differential inequalities,

\[
\partial_t r \geq C \lambda_3, \tag{196}
\]

\[
\partial \lambda_3 \geq -B \lambda_3^2. \tag{197}
\]

From (197) it follows that

\[
\lambda_3(t) \geq \frac{1}{\lambda_3(0) + Bt}. \tag{198}
\]

Plugging (198) into (196), we find that

\[
r(t) \geq r_0 + C \int_0^t \frac{1}{\lambda_3(0) + B\tau} d\tau = r_0 + \frac{C}{B} \log \left(1 + B\lambda_3(0)t\right). \tag{199}
\]

However, this estimate (199) clearly contradicts our hypothesis that \( r(t) \leq \frac{1+\sqrt{3}}{2} \) for all \( t > 0 \). Therefore, we can conclude that there exists \( T_a > 0 \), such that \( r(T_a) > \frac{1+\sqrt{3}}{2} \), and then we have reduced the problem to the case that we have already proven.
Now we will show that \( \lim_{t \to T} r(t) = 2 \). Suppose toward contradiction that \( \lim_{t \to T} r(t) = r_1 < 2 \).

First take \( a(t) = \frac{1}{3} f(r(t)) \). Observe that \( a(t) > 0 \) for \( 0 \leq t \leq T \). Our differential equation is now given by \( \partial_t \lambda_3 = a(t) \lambda_3^2 \), which must satisfy

\[
\frac{1}{\lambda_3(t_1)} - \frac{1}{\lambda_3(t_2)} = \int_{t_1}^{t_2} a(\tau) d\tau. \tag{200}
\]

If we take \( t_2 = T \), the blow up time, then (200) reduces to

\[
\frac{1}{\lambda_3(t)} = \int_t^T a(\tau) d\tau. \tag{201}
\]

Let \( A(t) = \int_t^T a(\tau) d\tau \). Clearly \( A(T) = 0, A'(T) = -a(T) < 0 \). By the fundamental theorem of calculus, for all \( m > a(T) \), there exists \( \delta > 0 \), such that for all \( t, T - \delta < t < T \),

\[
A(t) \leq -m(t - T) = m(T - t). \tag{202}
\]

Using the definition of \( A \) and plugging in to (201) we find that for all \( T = \delta < t < T \),

\[
\lambda_3(t) \geq \frac{1}{m(T - t)}. \tag{203}
\]

Let \( B = \frac{1}{3} \min(f(r_0), f(r_1)) \). It then follows from our hypothesis that

\[
\partial_t r \geq B\lambda_3. \tag{204}
\]

Therefore we can apply the estimate (203) to the differential inequality (204) to find that for all \( T - \delta < t < T \),

\[
r(t) \geq r(T - \delta) + B \int_{T-\delta}^t \frac{1}{m(T - \tau)} d\tau = r(T - \delta) + \frac{B}{M} \log \left( \frac{\delta}{T - t} \right). \tag{205}
\]

However, it is clear from (205) that \( \lim_{t \to T} r(t) = +\infty \), contradicting our hypothesis that \( \lim_{t \to T} r(t) < 2 \), so we can conclude that \( \lim_{t \to T} r(t) = 2 \). \( \square \)

This toy model ODE shows that the local part of the quadratic nonlinearity tends to drive the intermediate eigenvalue \( \lambda_2 \) upward to \( \lambda_3 \), unless \( \lambda_1 = \lambda_2 \). Given the nature of the regularity criterion on \( \lambda_2^+ \), the dynamics of the eigenvalues of the strain matrix are extremely important. The fact that the toy model ODE blows up from all initial conditions where \( \lambda_1 < \lambda_2 \), and that \( \lambda_2 = \lambda_3 \) is a global attractor on all initial conditions where \( \lambda_1 < \lambda_2 \), provides a mechanism for blowup, but of course the very complicated nonlocal effects make it impossible to say anything definitive about blowup for the full Navier Stokes strain equation without a much more detailed analysis.
References

[1] Diego Ayala and Bartosz Protas. Extreme vortex states and the growth of enstrophy in three-dimensional incompressible flows. *J. Fluid Mech.*, 818:772–806, 2017.

[2] J. T. Beale, T. Kato, and A. Majda. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.*, 94(1):61–66, 1984.

[3] Chongsheng Cao and Edriss S. Titi. Global regularity criterion for the 3D Navier-Stokes equations involving one entry of the velocity gradient tensor. *Arch. Ration. Mech. Anal.*, 202(3):919–932, 2011.

[4] J.-Y. Chemin. Remarques sur l’existence globale pour le système de Navier-Stokes incompressible. *SIAM J. Math. Anal.*, 23(1):20–28, 1992.

[5] Jean-Yves Chemin and Ping Zhang. On the critical one component regularity for 3-D Navier-Stokes systems. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(1):131–167, 2016.

[6] Qionglei Chen and Zhifei Zhang. Space-time estimates in the Besov spaces and the Navier-Stokes equations. *Methods Appl. Anal.*, 13(1):107–122, 2006.

[7] A. Cianchi, N. Fusco, F. Maggi, and A. Pratelli. The sharp Sobolev inequality in quantitative form. *J. Eur. Math. Soc. (JEMS)*, 11(5):1105–1139, 2009.

[8] Peter Constantin. Note on loss of regularity for solutions of the 3-D incompressible Euler and related equations. *Comm. Math. Phys.*, 104(2):311–326, 1986.

[9] L. Escauriaza, G. A. Seregin, and V. Šverák. $L_{3,\infty}$-solutions of Navier-Stokes equations and backward uniqueness. *Uspekhi Mat. Nauk*, 58(2(350)):3–44, 2003.

[10] Charles L. Fefferman. Existence and smoothness of the Navier-Stokes equation. In *The millennium prize problems*, pages 57–67. Clay Math. Inst., Cambridge, MA, 2006.

[11] Hiroshi Fujita and Tosio Kato. On the Navier-Stokes initial value problem. I. *Arch. Rational Mech. Anal.*, 16:269–315, 1964.

[12] B. Galanti, J. D. Gibbon, and M. Heritage. Vorticity alignment results for the three-dimensional Euler and Navier-Stokes equations. *Nonlinearity*, 10(6):1675–1694, 1997.

[13] Isabelle Gallagher, Gabriel S. Koch, and Fabrice Planchon. A profile decomposition approach to the $L_t^\infty(L_x^3)$ Navier-Stokes regularity criterion. *Math. Ann.*, 355(4):1527–1559, 2013.

[14] Isabelle Gallagher, Gabriel S. Koch, and Fabrice Planchon. Blow-up of critical Besov norms at a potential Navier-Stokes singularity. *Comm. Math. Phys.*, 343(1):39–82, 2016.

[15] Peter E. Hamlington, Jörg Schumacher, and Werner J. A. Dahm. Local and nonlocal strain rate fields and vorticity alignment in turbulent flows. *Phys. Rev. E (3)*, 77(2):026303, 8, 2008.

[16] Tosio Kato and Gustavo Ponce. Commutator estimates and the Euler and Navier-Stokes equations. *Comm. Pure Appl. Math.*, 41(7):891–907, 1988.
D. Q. Khai and N. M. Tri. On the initial value problem for the Navier-Stokes equations with the initial datum in critical Sobolev and Besov spaces. *J. Math. Sci. Univ. Tokyo*, 23(2):499–528, 2016.

Hideo Kozono, Takayoshi Ogawa, and Yasushi Taniuchi. The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations. *Math. Z.*, 242(2):251–278, 2002.

Hideo Kozono and Yasushi Taniuchi. Bilinear estimates and critical Sobolev inequality in BMO, with applications to the Navier-Stokes and the Euler equations. *Surikaisekikenkyusho Kōkyūroku*, (1146):39–52, 2000. Mathematical analysis of liquids and gases (Japanese) (Kyoto, 1999).

Igor Kukavica and Mohammed Ziane. Navier-Stokes equations with regularity in one direction. *J. Math. Phys.*, 48(6):065203, 10, 2007.

O. A. Ladyzhenskaya. Uniqueness and smoothness of generalized solutions of Navier-Stokes equations. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 5:169–185, 1967.

Pierre Gilles Lemarié-Rieusset. *The Navier-Stokes problem in the 21st century*. CRC Press, Boca Raton, FL, 2016.

Jean Leray. Sur le mouvement d’un liquide visqueux emplissant l’espace. *Acta Math.*, 63(1):193–248, 1934.

Lu Lu and Charles R. Doering. Limits on enstrophy growth for solutions of the three-dimensional Navier-Stokes equations. *Indiana Univ. Math. J.*, 57(6):2693–2727, 2008.

Giovanni Prodi. Un teorema di unicità per le equazioni di Navier-Stokes. *Ann. Mat. Pura Appl. (4)*, 48:173–182, 1959.

Jörg Schumacher, Bruno Eckhardt, and Charles R. Doering. Extreme vorticity growth in Navier-Stokes turbulence. *Phys. Let. A*, 374(6):861–865, 2010.

G. Seregin and V. Šverák. Navier-Stokes equations with lower bounds on the pressure. *Arch. Ration. Mech. Anal.*, 163(1):65–86, 2002.

Gregory Seregin. A note on necessary conditions for blow-up of energy solutions to the Navier-Stokes equations. In *Parabolic problems*, volume 80 of *Progr. Nonlinear Differential Equations Appl.*, pages 631–645. Birkhäuser/Springer Basel AG, Basel, 2011.

James Serrin. On the interior regularity of weak solutions of the Navier-Stokes equations. *Arch. Rational Mech. Anal.*, 9:187–195, 1962.

Michael Struwe. On a Serrin-type regularity criterion for the Navier-Stokes equations in terms of the pressure. *J. Math. Fluid Mech.*, 9(2):235–242, 2007.

Giorgio Talenti. Best constant in Sobolev inequality. *Ann. Mat. Pura Appl. (4)*, 110:353–372, 1976.

Chuong V. Tran and Xinwei Yu. Pressure moderation and effective pressure in Navier-Stokes flows. *Nonlinearity*, 29(10):2990–3005, 2016.
[33] A. Tsinober. Is concentrated vorticity that important. *Eur. J. Mech. B/Fluids*, 17(4):421–449, 1998.

[34] Yong Zhou and Milan Pokorný. On a regularity criterion for the Navier-Stokes equations involving gradient of one velocity component. *J. Math. Phys.*, 50(12):123514, 11, 2009.