The Jacobi last multiplier for linear partial difference equations

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Abstract

We present a discretization of the Jacobi last multiplier, with some applications to the computation of solutions of linear partial difference equations.

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1. Introduction

The Jacobi last multiplier (JLM) \([1, 7]\) plays, in first-order linear partial differential equations, a role similar to the integrating factor in first-order ordinary differential equations. If one can guess a JLM, it is possible to find the general solution of the equation or for equations with more than two variables to reduce the number of variables. However, in the case of quasilinear first-order partial differential equations, we can always integrate them going over to the characteristics. So the role of JLM is certainly not crucial as an integrating tool of PDEs, but it has an important role in many other applications. For example, the JLM has recently received a great deal of attention in the theory of \(\lambda\)-symmetries of differential equations \([9, 15]\).

In this work, we present a first approach to an equivalent concept for difference equations. In the case of partial linear difference equations no integration technique equivalent to the use of the characteristics exists \([3, 4]\) so the use of the JLM can be very proficuous.

Section 2 is devoted to a short review of JLM in first-order partial differential equations, in particular the method to obtain the equation satisfied by a JLM. In section 3, the case of a difference equation in a two-dimensional lattice is fully developed together with some examples. The conclusions presented in section 4 are devoted to a summary of the results obtained, showing the difficulties that appear when extending the method to a higher number of variables, and some future perspectives.
2. The continuous JLM: a review

Let us consider a first-order linear partial differential equation:

\[ Xu = 0, \quad X = \sum_{i=1}^{N} f^{(i)} \partial_{x^{i}}, \]  

(1)

where \( f^{(i)} \) are some smooth functions on the variables \( x^{(i)} \). If \( N - 1 \) functionally independent solutions of this equation are known, i.e. \( u^{(1)}, \ldots, u^{(N-1)} \), we can write the following determinant:

\[
\begin{vmatrix}
\frac{\partial u}{\partial x^{1}} & \cdots & \frac{\partial u}{\partial x^{n}} \\
\cdots & \cdots & \cdots \\
\frac{\partial u^{(N-1)}}{\partial x^{1}} & \cdots & \frac{\partial u^{(N-1)}}{\partial x^{n}}
\end{vmatrix}
= \det
\begin{pmatrix}
\frac{\partial u}{\partial x^{1}} & \cdots & \frac{\partial u}{\partial x^{n}} \\
\cdots & \cdots & \cdots \\
\frac{\partial u^{(N-1)}}{\partial x^{1}} & \cdots & \frac{\partial u^{(N-1)}}{\partial x^{n}}
\end{pmatrix}
\]  

(2)

for any function \( u(x^{(1)}, \ldots, x^{(n)}) \). Determinant (2) is zero only if \( u \) is a solution of equation (1), as can be easily shown, since the elements of the matrix are the coefficients of the forms \( du^{(i)} \) and the functions \( u, u^{(1)}, \ldots, u^{(N-1)} \) are functionally dependent if \( u \) is a solution of the differential equation. Then, the equations

\[ Xu = 0, \quad \frac{\partial (u, u^{(1)}, \ldots, u^{(N-1)})}{\partial (x^{(1)}, \ldots, x^{(n)})} = 0 \]  

(3)

have the same set of solutions and must be proportional as \( N - 1 \) functionally independent solutions fix the coefficients of the linear first-order PDE up to a global factor:

\[
\frac{\partial (u, u^{(1)}, \ldots, u^{(N-1)})}{\partial (x^{(1)}, \ldots, x^{(n)})} = M X u.
\]  

(4)

The multiplicative factor \( M \) is called the JLM.

If we expand determinant (2) using the first row, we find that equation (4) can be written as

\[
A^{(1)} \frac{\partial u}{\partial x^{(1)}} + \cdots + A^{(N)} \frac{\partial u}{\partial x^{(n)}} = M \sum_{i=1}^{N} f^{(i)} \frac{\partial u}{\partial x^{(i)}},
\]  

(5)

where \( A^{(k)} \) are the corresponding cofactors of the first row of the matrix in (2). Comparing the coefficients of the derivatives of \( u \) in (5), we obtain

\[
A^{(k)} = M f^{(k)}, \quad k = 1, \ldots, N.
\]  

(6)

We can now write a differential equation satisfied by the functions \( A^{(k)} \)

\[
(-1)^{k-1} \det \begin{pmatrix}
\frac{\partial u^{(1)}}{\partial x^{(1)}} & \cdots & \frac{\partial u^{(1)}}{\partial x^{(n)}} \\
\cdots & \cdots & \cdots \\
\frac{\partial u^{(N-1)}}{\partial x^{(1)}} & \cdots & \frac{\partial u^{(N-1)}}{\partial x^{(n)}}
\end{pmatrix} = A^{(k)}, \quad k = 1, \ldots, N.
\]  

(7)

This is a general relation arising from the particular form of the functions \( A^{(k)} \), written as cofactors of matrix (2), and does not depend on the fact that the functions \( u^{(k)} \) are the solutions of the linear homogeneous first-order PDE (1) with coefficients \( f^{(k)} \). In fact, the equation for \( A^{(k)} \) can be considered as a consistency condition for the system of equations in \( u^{(j)} \) with \( k = 1, \ldots, N, \ j = 1, \ldots, N - 1 \).

Let us write (7) as

\[
A^{(k)} = (-1)^{k-1} \det(B_{1}, \ldots, \hat{B}_{k}, \ldots, B_{N}),
\]  

(8)
where by \( B_i \) we denote the \( i \)th column of matrix (2) and the symbol \( \hat{B}_k \) means that the corresponding \( k \)th column of matrix (2) is removed. Deriving (8) with respect to \( x^{(k)} \) and summing over \( k \), we obtain
\[
\sum_{k=1}^{N} \frac{\partial A^{(k)}}{\partial x^{(k)}} = \sum_{i,k=1,i\neq k}^{N} (-1)^{k-1} \det(B_1, \ldots, \frac{\partial B_i}{\partial x^{(k)}}, \ldots, \hat{B}_k, \ldots, B_N).
\] (9)

Since
\[
\frac{\partial B_i}{\partial x^{(k)}} = \left( \frac{\partial^2 u^{(1)}}{\partial x^{(i)} \partial x^{(k)}}, \ldots, \frac{\partial^2 u^{(N-1)}}{\partial x^{(i)} \partial x^{(k)}} \right) = \frac{\partial B_k}{\partial x^{(i)}},
\] (10)
we have
\[
\det(B_1, \ldots, \frac{\partial B_i}{\partial x^{(k)}}, \ldots, \hat{B}_k, \ldots, B_N) = (-1)^{k-1} \det(B_1, \ldots, \frac{\partial B_k}{\partial x^{(i)}}, \ldots, \hat{B}_k, \ldots, B_N).
\] (11)

Finally, since \((-1)^{k}(-1)^{k-1} = -(-1)^{i}\) the sum of all terms in (10) is zero, and consequently, we obtain the equation
\[
\sum_{k=1}^{N} \frac{\partial A^{(1)}}{\partial x^{(1)}} + \cdots + \frac{\partial A^{(N)}}{\partial x^{(N)}} = 0.
\] (12)

In three dimensions, this expression is the classical formula of vector calculus stating that the divergence of the cross product of two gradient vectors is zero. In an arbitrary dimension, it can be written using exterior products and differential forms (see for instance [6]).

We can derive an equation for \( M \), differentiating (6) with respect to \( x^{(k)} \) and summing over all \( k \) between 1 and \( N \). We obtain
\[
\sum_{k=1}^{N} \frac{\partial A^{(k)}}{\partial x^{(k)}} = M \sum_{k=1}^{N} \frac{\partial f^{(k)}}{\partial x^{(k)}} + \sum_{k=1}^{N} f^{(k)} \frac{\partial M}{\partial x^{(k)}},
\] (13)
and consequently, taking into account (12), we obtain
\[
\sum_{k=1}^{N} f^{(k)} \frac{\partial \log M}{\partial x^{(k)}} + \sum_{k=1}^{N} f^{(k)} \frac{\partial M}{\partial x^{(k)}} = 0.
\] (14)

This equation depends only on the differential equation (1), and thus, \( M \) does not depend on any particular solution. From a practical point of view, equation (14) is an inhomogeneous version of the original equation (1). However, as in the method of integrating factors, if we know a particular solution of (14), we could use it to compute a solution of (1). This is exploited in the examples presented in the next subsection where we consider a few examples and we compare the results obtained by the use of JLM with those obtained through other methods of integration of the linear first-order partial differential equations.

Although we have followed in this paper the original algebraic approach of Jacobi, a geometrical point of view can also be used (see for instance [6]) and give an alternative and interesting insight into the theory.

### 2.1. Examples

As a simple illustration of the method, let us consider the following examples of partial differential equations in two and three independent variables.
2.1.1. Two independent variables

\[ yu_x + xu_y = 0. \]  
(15)

From (14), the equation for \( M \) is

\[ y\partial_x \log M + x\partial_y \log M = 0. \]  
(16)

An obvious solution of this equation is \( M = 1 \). Then,

\[ A^{(1)} = Mf_1 = y, \quad A^{(2)} = Mf_2 = x \]  
(17)

and (6) reduces to the compatible overdetermined system of equations

\[ u_x = -x, \quad u_y = y. \]  
(18)

Solving this system we find a non-trivial solution of the original partial differential equation

\[ u(x, y) = \frac{1}{2}(y^2 - x^2). \]  
(19)

The general solution can be obtained by computing the characteristic variable \( \xi = y^2 - x^2 \) and is given by

\[ u(x, y) = F(\xi), \]  
(20)

where \( F \) is an arbitrary function of its argument defined by the initial conditions or from the boundary values.

2.1.2. Three independent variables

\[ x(x + y)u_x - y(x + y)u_y + z(x - y)u_z = 0. \]  
(21)

The equation satisfied by a JLM \( M \) is

\[ x(x + y)\frac{\partial}{\partial x} \log M - y(x + y)\frac{\partial}{\partial y} \log M + z(x - y)\frac{\partial}{\partial z} \log M + 2(x - y) = 0. \]  
(22)

Looking for a particular solution of this equation, e.g., \( M = M(z) \), we find \( M = \frac{1}{z} \), and the system of equations we have to solve is (with \( u^{(1)} \equiv u, u^{(2)} \equiv v \)):

\[ u_yv_z - u_zv_y = \frac{x(x + y)}{z^2}, \quad u_xv_z - u_zv_x = -\frac{y(x + y)}{z^2}, \quad u_xv_y - u_yv_x = \frac{x - y}{z}. \]  
(23)

Given a solution \( u \) of (21), (23) is an overdetermined system for \( v \). For instance, we can take \( u(x, y, z) = xy \) and the new solution \( v \) is obtained from the following overdetermined system of equations:

\[ v_x = \frac{x + y}{z^2}, \quad yv_y - xv_x = \frac{x - y}{z}. \]  
(24)

A solution of (24) is

\[ v = -\frac{x + y}{z}. \]  
(25)

From the method of characteristics, we find that any solution of (21) is a function of the two particular solutions that we have found:

\[ u(x, y, z) = F\left(xy, \frac{x + y}{z}\right). \]  
(26)
3. Difference equations

A difference equation for one dependent variable $u$ is a relation between the function in various points of a lattice. If the lattice is $r$-dimensional, it can be put in correspondence with the points of an $r$-dimensional space.

An ordinary difference equation (ODE) is a difference equation on a one-dimensional lattice. In this case, the lattice is given by an ordered sequence of points on a line characterized by their relative distance (see figure 1). If $x_i$ and $x_{i+1}$ are two subsequent points, their distance $|x_{i+1} - x_i|$ will be $h_i$. We can then introduce an $x$-shift operator $T_x$ such that $T_x x_i = x_{i+1}$, and in terms of this, we can construct delta operators which in the continuous limit, when $h_i \to 0$, go over to the derivative. An example of such delta operator is given by the right-shifted discrete derivative

$$\Delta_1^x u(x_i) = u(x_{i+1}) - u(x_i).$$

(27)

In some instances, it may be more convenient to introduce symmetric delta operators [11], e.g., such as

$$\Delta^s_1 u(x_i) = u(x_{i+1}) - u(x_{i-1}).$$

(28)

An ODE of order $n$, i.e. involving $n + 1$ points of the lattice, can thus be written as

$$E(x_i, u_i, \Delta_1^x u_i, \Delta^2_1 x u_i, \ldots, \Delta^n_1 x u_i) = 0,$$

(29)

or, equivalently, in terms of the operator delta as

$$F(x_i, u_i, \Delta_1 x u_i, \Delta^2_1 x u_i, \ldots, \Delta^n_1 x u_i) = 0.$$  

(30)

However, equation (29) (or (30)) is not completely defined unless we specify the values of the distance between the various lattice points $h_j, j = i, \ldots, i + n$, involved in the equation. This implies that an ODE will be defined only if we attach to it a second equation that defines the lattice. The set of these two equations is called a difference scheme.

In many instances, when the equation comes from some physical problem, the lattice is a priori given, such as, for example, when all the points are equidistant so that $h_i = h$. In this case, the lattice equation is trivial $x_{i+1} - x_i = h$. However, there may be situations, as, for example, discretizing a continuous differential equation to solve it on the computer, when we want to take advantage of the freedom of the lattice to preserve in the discretization other properties of the continuous system like its symmetries [14]. In such a situation, the lattice may be defined by a non-trivial equation maybe also depending on the dependent variable so as to have a denser grid when the solution varies rapidly.

In the case of ODEs, there is at least one natural parametrization of the differences, which in the continuous limit goes to the corresponding derivatives and simplifies the discretization procedure [12]. Such parametrization gives a one-to-one transformation between the lattice differences, discrete approximations of the derivatives and the lattice points.

A similar situation exists also in the case of partial difference equations (PDEs); however, in this case, the definition of the lattice must be given by compatible equations as the
independent variables depend on several indices (see figure 2 for the two-dimensional case, where \( x_{n,m} \) and \( y_{n,m} \) depend on two indices). In general, the difference scheme will be given, apart from the P\( \Delta \)E, by a set of equations, which define the lattice and depend on the number of independent variables and on the problem that we are solving [13]. However, in this case, up to now, no natural parametrization exists that in the continuous limit goes to the corresponding derivatives and simplifies the discretization procedure. Work on this is in progress [10, 16].

The solution of linear O\( \Delta \)Es follows the standard technique of solving ordinary differential equations. The solution is given by a linear combination with arbitrary coefficients of powers of the independent variable and the exponents are defined by a characteristic polynomial. In the case of linear P\( \Delta \)Es, the situation is more complicated (as is also the case for partial differential equations). As one can read in [3]: the method of trial and error is still one of the basic methods for obtaining explicit solutions. If the P\( \Delta \)E has constant coefficients, then a few techniques can be found in [8], such as the Laplace method of generating functions, or the methods of Fourier, Lagrange and Ellis [5]. If the P\( \Delta \)E does not depend explicitly on one of the two independent variables, then Boole’s symbolic method can be applied [2].

Consequently, it seems particularly important to extend the last Jacobi multiplier technique to the case of linear P\( \Delta \)Es as this will provide solutions also in the case of non-constant coefficient P\( \Delta \)Es. In the following, for the sake of simplicity, we will consider the case of an orthogonal lattice, where the independent variables \( x_{n,m} \) and \( y_{n,m} \) can be written in terms of just one index, i.e. \( x_n \) and \( y_m \), and we will just concentrate on the solution of the difference equation.

### 3.1. JLM on a two-dimensional lattice

Let us write an equivalent discrete expression defined on four lattice points of the linear first-order partial differential equation (1). We could write this expression in terms of the shift operators but we will use the difference operators (27) to follow closely the continuous case and to limit the number of points involved. Let us consider a two-dimensional orthogonal
We consider now a few examples of the calculus of the solution of linear difference equations using the JLM. 

### 3.2. Examples

#### 3.2.1. First example. We consider the equation

$$f_{n,m}(x_n, x_{n+1}, y_n, y_{m+1}) \Delta_x u_{n,m} + f_{n,m}^{(2)}(x_n, x_{n+1}, y_n, y_{m+1}) \Delta_y u_{n,m} = 0.$$  \hspace{1cm} (31)

As in the continuous case, if $u_{n,m}^{(1)}$ is a solution of (31), the $2 \times 2$ matrix

$$\frac{\Delta (u_{n,m}, u_{n,m}^{(1)})}{\Delta (x_n, y_m)} = \begin{pmatrix} \Delta_x u_{n,m} & \Delta_y u_{n,m} \\ \Delta_x u_{n,m}^{(1)} & \Delta_y u_{n,m}^{(1)} \end{pmatrix}$$  \hspace{1cm} (32)

has a determinant equal to zero if and only if the function $u_{n,m}$ is also a solution of the difference equation (31). The cofactors of the first row of (32) are

$$A_{n,m}^{(1)} = \Delta_x u_{n,m}^{(1)} \quad A_{n,m}^{(2)} = -\Delta_x u_{n,m}^{(1)}$$  \hspace{1cm} (33)

and it is trivial to check by direct computation that $A_{n,m}^{(1)}$ and $A_{n,m}^{(2)}$ satisfy the equation

$$\Delta_x A_{n,m}^{(1)} + \Delta_y A_{n,m}^{(2)} = 0.$$  \hspace{1cm} (34)

Then, as in the continuous case, there exists a function $M_{n,m}$, the JLM, such that

$$A_{n,m}^{(1)} = M_{n,m} f_{n,m}^{(1)} \quad A_{n,m}^{(2)} = M_{n,m} f_{n,m}^{(2)}.$$  \hspace{1cm} (35)

Consequently, the JLM $M_{n,m}$ satisfies a difference equation, which is the discrete analog of the differential one (14)

$$\frac{1}{M_{n,m}} \left[ (\Delta_x M_{n,m}) f_{n+1,m}^{(1)} + (\Delta_y M_{n,m}) f_{n,m+1}^{(2)} \right] + \Delta_x f_{n,m}^{(1)} + \Delta_y f_{n,m}^{(2)} = 0.$$  \hspace{1cm} (36)

Given any particular, even trivial, solution $M_{n,m}$ of the PDE (36), the solution of the overdetermined system (35) provides a solution of (31). As in the case of PDEs, the solution of an overdetermined compatible system is always simpler than that of the original equation.

We consider now a few examples of the calculus of the solution of linear difference equations using the JLM.

#### 3.2.2. Uniform lattice

In such a case, (1) reads

$$f_{n,m}(x_n, x_{n+1}, y_n, y_{m+1}) \Delta_x u_{n,m} + f_{n,m}^{(2)}(x_n, x_{n+1}, y_n, y_{m+1}) \Delta_y u_{n,m} = 0.$$  \hspace{1cm} (37)

The equation for the JLM $M_{n,m}$ is the same as that for $u_{n,m}$:

$$y_n \Delta_x M_{n,m} + x_n \Delta_y M_{n,m} = 0,$$  \hspace{1cm} (38)

and a particular solution is obviously $M_{n,m} = 1$ that gives, taking into account (35), the following system of equations for $u_{n,m}$:

$$\Delta_x u_{n,m} = -x_n, \quad \Delta_y u_{n,m} = y_m.$$  \hspace{1cm} (39)

To solve this system of difference equations, we need to specify the lattice. If we consider a uniform lattice in both variables:

$$x_{n+1} - x_n = \delta_1, \quad y_{m+1} - y_m = \delta_2,$$  \hspace{1cm} (40)

so that $x_n = \delta_1 n + x_0$ and $y_m = \delta_2 m + y_0$, where $x_0$ and $y_0$ are arbitrary initial points, the system (39) reduces to a system of OΔEs, one for each direction:

$$u_{n+1,m} = u_{n,m} - n \delta_1^2 - \delta_1 x_0, \quad u_{n,m+1} = u_{n,m} + m \delta_2^2 + \delta_2 y_0.$$  \hspace{1cm} (41)

Using the well-known procedures for solving OΔEs [8], we obtain a particular solution of (37), depending on three arbitrary constants:

$$u_{n,m} = u_{0,0} - \frac{1}{2} (x_n + x_0) (x_n - x_0 - \delta_1) + \frac{1}{2} (y_m + y_0) (y_m - y_0 - \delta_2).$$  \hspace{1cm} (42)
3.2.2. Second example. We choose the equation
\[ y_m x_{n+1} \Delta_x u_{n,m} + x_n y_{m+1} \Delta_y u_{n,m} = 0, \] (43)
which corresponds to (31) with
\[ f_{n,m}^{(1)} = y_n x_{n+1}, \quad f_{n,m}^{(2)} = x_n y_{m+1}. \] (44)

The equation for the JLM can be written as
\[ y_n x_{n+1} \left\{ M_{n,m+1} x_{n+2} - x_{n+1} \right\} + x_n y_{m+1} \left\{ M_{n,m} y_{m+2} - y_{m+1} \right\} = 0. \] (45)

A particular solution of equation (45) can be obtained by requiring that both its curly brackets should identically be zero. In such a case, we obtain a particular solution
\[ M_{n,m} = \frac{\alpha}{y_m + x_{n+1}}. \] (46)

where \( \alpha \) is an arbitrary constant. If we introduce this result in (35) with \( A_{n,m}^{(1)} \) and \( A_{n,m}^{(2)} \) given by (33), we obtain the following system of compatible equations:
\[ u_{n+1,m} - u_{n,m} = -\alpha x_n \left( 1 - \frac{x_n}{x_{n+1}} \right), \]
\[ u_{n,m+1} - u_{n,m} = \alpha y_m \left( 1 - \frac{y_m}{y_{m+1}} \right). \] (47)

As in the previous example, we need the lattice equations to solve equations (47). Using again a uniform lattice in each variable, we obtain
\[ x_n = x_0 + h_x n, \quad y_m = y_0 + h_y m, \] (48)
\[ u_{n+1,m} - u_{n,m} = -\alpha x_n \left( 1 - \frac{h_x}{x_0 + (n+1)h_x} \right), \]
\[ u_{n,m+1} - u_{n,m} = \alpha y_m \left( 1 - \frac{h_y}{y_0 + (m+1)h_y} \right). \] (49)

To solve the first equation, we define
\[ v_{n,m} = \frac{1}{\alpha h_x} u_{n,m} + n \] (50)
and the equation satisfied by \( v_{n,m} \) is
\[ v_{n+1,m} = v_{n,m} + \frac{1}{1 + \frac{h_x}{n} + n}, \] (51)
which is the recursion equation for the Euler digamma function \( \psi \). Then,
\[ v_{n,m} = a_m + \psi \left( n + \frac{x_0}{h_x} + 1 \right). \] (52)

Substituting the second equation into (49), we easily obtain an equation for \( a_m \):
\[ a_{m+1} = a_m - \frac{h_x}{y_0 + (m+1)h_y} \left( 1 - \frac{h_y}{y_0 + (m+1)h_y} \right). \] (53)

Solving this equation as in (49),
\[ a_m = \frac{h_x}{h_y} \left( c + \psi \left( m + \frac{y_0}{h_y} + 1 \right) - m \right), \] (54)
where \( c \) is a constant. Then, the complete solution is
\[ u_{n,m} = u_{0,0} + \alpha \left[ x_0 - y_0 - h_x \psi \left( 1 + \frac{x_0}{h_x} \right) + h_y \psi \left( 1 + \frac{y_0}{h_y} \right) \right. \]
\[ - x_n + y_m + h_x \psi \left( 1 + \frac{x_n}{h_x} \right) - h_y \psi \left( 1 + \frac{y_m}{h_y} \right) \right]. \] (55)
4. Conclusions

In this paper, we have extended the results presented by Jacobi in 1844 to obtain solutions of linear partial differential equations to the case of partial difference equations in two independent variables.

If we consider an \( N \)-dimensional lattice of independent coordinates, i.e. the lattice coordinates \( x^{(i)} \) depend only on one index \( n_i, i = 1, \ldots, N \), then it is easy to see that we get into trouble as the cofactors are nonlinear functions and the difference operator, in contrast with the differential one, does not satisfy the Leibniz rule. In fact denoting by \( u_{n_1, \ldots, n_N} \) the value of \( u \) at the point \( (x_n^{(1)}, \ldots, x_n^{(N)}) \) and using the following notation:

\[
\mathbf{n} = (n_1, \ldots, n_N), \quad \mathbf{x}_n = (x_n^{(1)}, \ldots, x_n^{(N)}), \quad \mathbf{e}_i = (0, \ldots, 1(i), \ldots, 0),
\]

the discrete derivatives reads

\[
\Delta_i u_n = \frac{u_{n+\epsilon_i} - u_n}{x_{n_i+1} - x_{n_i}}
\]

The difference equation is

\[
\sum_{i=1}^{N} f_n^{(i)} \Delta_i u_n = 0,
\]

where \( f_n^{(i)} \) are some functions depending on a finite number of points in the lattice. If we know \( N - 1 \) particular solutions of the equation, i.e. \( u_n^{(1)}, \ldots, u_n^{(N-1)} \), we can construct the matrix

\[
\Delta(u_n, u_n^{(1)}, \ldots, u_n^{(N-1)}) = \begin{pmatrix}
\Delta_1 u_n & \cdots & \Delta_N u_n \\
\Delta_1 u_n^{(1)} & \cdots & \Delta_N u_n^{(1)} \\
\vdots & \ddots & \vdots \\
\Delta_1 u_n^{(N-1)} & \cdots & \Delta_N u_n^{(N-1)}
\end{pmatrix}
\]

and define the cofactors \( A_n^{(k)} \), \( k = 1, \ldots, N \), corresponding to the first line of the matrix above (the column \( k \) is removed):

\[
A_n^{(k)} = (-1)^{k-1} \det \begin{pmatrix}
\Delta_1 u_n^{(1)} & \cdots & \Delta_N u_n^{(1)} \\
\vdots & \ddots & \vdots \\
\Delta_1 u_n^{(N-1)} & \cdots & \Delta_N u_n^{(N-1)}
\end{pmatrix}.
\]

As in the continuous case (see section 2), the \( N - 1 \) solutions fix the coefficients of the difference equation (58) up to a factor. Then, using Cramer’s rule, we obtain

\[
A_n^{(k)} = M_n f_n^{(k)}, \quad k = 1, \ldots, N.
\]

To find the compatibility relation we will closely follow the argument we used in the continuous case. Then, writing

\[
A_n^{(k)} = (-1)^{k-1} \det(B_1, \ldots, \hat{B}_k, \ldots, B_N),
\]

where \( B_i \) is the \( i \)th column of (60) and \( \hat{B}_k \) means that the \( k \)th column is absent, we obtain that the sum

\[
\sum_{i,k=1,i\neq k}^{N} (-1)^{k-1} \det(B_1, \ldots, \Delta_i B_i, \ldots, \hat{B}_k, \ldots, B_N)
\]

is equal to zero. In fact

\[
\Delta_k B_i = (\Delta_k \Delta_i u_n^{(1)}, \ldots, \Delta_k \Delta_i u_n^{(N-1)})^T = \Delta_i B_k,
\]
since $\Delta_k \Delta_i = \Delta_i \Delta_k$ as the independent variables commute. Then,

$$\det(B_1, \ldots, \Delta_k B_i, \ldots, \hat{B}_i, \ldots, B_N) = \det(B_1, \ldots, \Delta_i B_k, \ldots, \hat{B}_k, \ldots, B_N)$$

$$= (-1)^{k-i-1} \det(B_1, \ldots, \hat{B}_i, \ldots, \Delta_i B_k, \ldots, B_N)$$ \hspace{1cm} (65)

and the sum (63) is zero. This is exactly the same equation we found in the continuous case. However, since the Leibniz rule does not apply in the case of difference operators, expression (63) is not equal to $\sum_{k=1}^{N} \Delta_k A_n^{(k)}$, as the difference operator does not follow the same rules as the differential operator when it is applied to a determinant. There are some additional terms, as $\Delta(fg) = f \Delta g + g \Delta f + h(\Delta f)(\Delta g)$, whose consequences have to be analyzed and possibly overcome. At the moment, we do not know how to overcome these difficulties, which we mentioned above for completeness. This will be the content of a future work.

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