Asymptotic uniform boundedness of energy solutions to the Penrose-Fife model

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Abstract. We study a Penrose-Fife phase transition model coupled with homogeneous Neumann boundary conditions. Improving previous results, we show that the initial value problem for this model admits a unique solution under weak conditions on the initial data. Moreover, we prove asymptotic regularization properties of weak solutions.

1. Introduction

The Penrose-Fife system, proposed by Penrose and Fife in [25,26], represents a thermodynamically consistent model for the description of the kinetics of phase transition and phase separation processes in binary materials. It couples a singular heat equation (cf. (2.3) below) for the absolute temperature $\theta$ with a nonlinear relation describing the evolution of the phase variable $\chi$, which represents the local proportion of one of the two components. This can be of the fourth order in space (cf. (2.6)–(2.7) below), in case the physical process preserves the total mass of $\chi$ (conserved Penrose-Fife model, describing phase separation) or of the second order in space (cf. (2.5) below), in case the total mass of $\chi$ is admitted to vary (non-conserved Penrose-Fife model, describing phase transition). In the conserved case, the equation for $\chi$ is usually written as a system by introducing an auxiliary variable $w$ called chemical potential. We refer to the next section for a detailed presentation of the equation and to the papers [10,11,32] for further mathematical background.

Due to physical considerations, it is generally accepted to consider no-flux boundary conditions for $\chi$ and, in the conserved case, also for $w$. On the other hand, various types of boundary conditions (for instance, no-flux, non-homogeneous Dirichlet, or Robin conditions) make sense for the heat equation (2.3), which give rise to different mathematical scenarios. We refer the reader to [11–13,21,22,32] for the case of Robin (or “third type”) conditions, to [14,15] for the Dirichlet case, and to [10,17,34] for the homogeneous Neumann case, which is probably the most difficult one due to lower coercivity properties of the elliptic operator in (2.3).

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Since also the present paper is devoted to the Neumann case, it is worth describing here in some detail the related results proved in [10,17,34]. In the pioneering paper [34], the non-conserved model is studied in one space dimension, and well-posedness is proved for smooth initial data (in particular, the author investigates the properties of “classical” solutions). The one-dimensional restriction has been removed in [17], where well-posedness is proved, both in the non-conserved and in the conserved case, for initial data satisfying the equivalent of assumptions (2.11) and (2.18) below, plus the additional summability condition $\vartheta_0 \in L^2(\Omega)$. The authors of [17] can also consider the case when a heat source with zero spatial mean is present on the right hand side of (2.3). This restriction is overcome in [10], where a general heat source is considered, paying, however, the price of having a less general class of admissible functions $f$ in (2.5) (excluding in particular the “singular” nonlinearities corresponding to the case of a bounded domain $I$ in (2.9) below). It is also worth mentioning that the heat flux law considered in [10] is slightly different from ours, in the sense that our (2.4) is replaced by a relation like $u \sim \vartheta - \vartheta^{-1}$ (or a generalization of this).

Our aim in this paper is that of extending the above-described results and proving further properties of solutions to the homogeneous Neumann problem for the Penrose-Fife model, both in the non-conserved and in the conserved case. Actually, the two situations can be treated together, with some variants in the proofs (which happen to be slightly more involved for the conserved model). For simplicity, we limit our analysis to the homogeneous problem, but our results can be easily adapted to the case of a (suitably regular) heat source with zero spatial mean (on the other hand, it does not seem possible to remove the zero-mean condition in our setting). As a first property, we will show that the problem admits a unique solution under weak and physically natural assumptions on the initial data. In particular, we can remove the requirement $\vartheta_0 \in L^2(\Omega)$ taken in [10,17]. Actually, noting that the system admits a Liapounov functional representing the total energy, we will prove that a (unique) weak solution exists for any initial datum $(\vartheta_0, \chi_0)$ having finite energy (this corresponds exactly to hypotheses (2.11) and (2.18) below). As a side effect, we pay the price that, for these solutions (called $V'$-solutions in the sequel; indeed, we will note $V := H^1(\Omega)$), the heat equation has to be interpreted in the generalized $(H^1)'$-framework developed by Damlamian and Kenmochi in [13]. Namely, the (thermal part of the) energy has to be intended as a (relaxed) functional operating on the negative order Sobolev space $H^1(\Omega)'$ (cf. (2.20) below), and also, the relation linking $\vartheta$ to its inverse has to be stated properly. If, in addition, we assume $\vartheta_0 \in L^1(\Omega)$, then we obtain a class of more regular solutions (called $V' \cap L^1$-solutions) and we can prove that, in this setting, $\vartheta(t) \in L^1(\Omega)$ for all $t \geq 0$; moreover, both the energy functional and the relation between $\vartheta$ and its inverse can be written in the usual (pointwise) sense (cf. (2.19) below). The proof of this property, especially in the conserved case, requires a careful combination of Hilbert and $L^1$-techniques.

After analyzing well-posedness in the weak setting, we pass to what we can consider the main results of the paper, regarding uniform time-regularization properties of weak solutions.