The Admissibility Theorem for the Spatial X-Ray Transform over the Two Element Field

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Abstract. We consider the Radon transform along lines in an \( n \) dimensional vector space over the two element field. It is well known that this transform is injective and highly overdetermined. We classify the minimal collections of lines for which the restricted Radon transform is also injective. This is an instance of I.M. Gelfand’s admissibility problem. The solution is in stark contrast to the more uniform cases of the affine hyperplane transform and the projective line transform, which are addressed in other papers, \[6,9\]. The presentation here is intended to be widely accessible, requiring minimum background.

1. Introduction

1.1. Dedication and two Mathematical Moments. This paper is dedicated to the memory of Leon Ehrenpreis. His colleagues were fortunate to have countless many discussions with him, after seminars (and during), in offices, hallways, and at the lounge blackboard. These served to inspire, energize and generate many new ideas. The subject of this essay may well have come up in one of these chats.

During graduate school I learned about the role of spreads in integral geometry from Ethan Bolker, via an early, handwritten version of \[2\] and when I joined Temple University the concept of linear spreads followed and came up in early conversations with Leon. He found spreads to be useful in his approach to integral geometry and he formulated a non-linear variant which he employed in framing his notion of the non-linear Radon transform. See the major work \[5\] and the review \[1\] by Carlos Berenstein.

I recall vividly a two-panel chalk board with the level sets of a homogeneous polynomial drawn on one panel, and the heat equation displayed on the other. I also recall sharing a car ride with Ethan and Leon, from San Francisco to Arcata, CA, on the way to the 1989 AMS summer conference on integral geometry and tomography, which led to \[10\]. It is safe to say that the majority of the travel time was devoted to an intensive discussion of Radon transforms (and I hope that this did not impair the safety of the ride). The beautiful California coastline was superceded by admissible line complexes.

The structure of spreads (discussed concretely below) is particularly simple in the case of the hyperplane Radon transform over finite fields, and this can
be used to solve the admissibility problem in that context. In contrast, the
structure of spreads is not as simple for transforms that integrate over planes of
larger codimension, and thus we expect the admissibility problem to have a more
complicated solution. Here we investigate the simplest higher-codimension case.

2. The Radon Transform in a finite geometry.

The theme of integral geometry, in the style introduced by P. Funk and
J. Radon and prominent in the work of Leon Ehrenpreis, involves the recovery of
functions (or data) from integrals. In applications one might imagine recovering
the density distribution of biological tissue from x-ray data. If “all” integrals
(x-ray) measurements are available then the problem is overdetermined. It is
natural to look for minimal sets of data (x-rays) with which complete recovery
is still possible (even though in applications such minimal measurements may
present stability problems). Finding and classifying such minimal families is an
instance of I.M. Gelfand’s admissibility problem [8], which initially occurred in
the context of the Plancherel formula for semi-simple Lie groups. In the con-
tinuous category, the problem depends in part on the choice of function spaces,
mapping properties of integral operators, and smoothness properties of collections
of lines. Here we focus on a finite model of integral geometry in which
analytic considerations are removed and sets of lines take center stage. In the
admissibility theory work of Gelfand and collaborators within the continuous
category ($\mathbb{R}^3$ or $\mathbb{C}^3$ and their projective and higher dimensional analogs) the
family of lines meeting a curve (the Chow variety) and the family of lines tan-
gent to a surface occur as admissible complexes [7, 12]. Here we will search
for finite analogs of these. For discussions of Radon transforms in finite geome-
tries see, e.g., [13, 15]. Recent results on admissibility in the context of finite
projective spaces may be found in [6].

Starting with the $q$-element field $\mathbb{F}_q$ one can build lines, planes, vector spaces
of dimension $n$, projective spaces, Grassmannians, and more. It is easy to
use counting measure to define the Radon transform taking functions on $\mathbb{F}_q^n$
to functions on the set of $k$-planes in $\mathbb{F}_q^n$:

$$R_k f(H) = \sum_{\{x \in H\}} f(x),$$

where $H$ is a $k$ dimensional affine plane in $\mathbb{F}_q^n$. Informally we write

$$R_k : \{\text{point functions}\} \longrightarrow \{\text{k plane functions}\}$$

It is natural to ask: is the transform $R_k$ invertible? Rather than answer the
question in this specific case, we consider a more general context, informally bor-
rowing from S.S. Chern’s (1942) formulation of integral geometry [4]. Consider
the following double fibration diagram:

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X \pi \rightarrow Z \rho \rightarrow Y
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Chern’s formulation was presented in the continuous category; he re-
sults. We think of $X$ as our space of points, $Y$ as the family of
lines or, more generally, submanifolds of $X$, and we think of $Z$ as the incidence manifold of point-line (or point-generalized line) pairs, so that the point belongs to the line:
\[
\{(x, y) | x \in y\}.
\]

The maps $\pi$ and $\rho$ are projection functions, e.g., $\pi(x, y) = x$, so that $\pi \times \rho$ is one to one. Thus thinking of $X$ as a set of points and $Y$ as a family of subsets of $X$ is manifested by $[\Pi]$:
\[
F_y = \pi \circ \rho^{-1}\{y\}.
\]

When $y$ is a line, $F_y$ is the set of points on the line. Dually, for every point $x$ we have the set of all subsets $y$ passing through $x$:
\[
G_x = \rho \circ \pi^{-1}\{x\}.
\]

With the definitions of $F_y$ and $G_x$ it is possible to relax the condition that $\pi, \rho$ be projections and consider more general diagrams, though we will not need these here. The double fibration diagram has been used extensively as a paradigm for Radon transforms and their generalizations by Gelfand and collaborators, S. Helgason, V. Guillemin & S. Sternberg, and many many others.

A double fibration diagram together with a choice of measures leads to an integral transform. In the finite category we will use counting measure and define the notion of Radon transform without making any additional choices.

Let $C(X), C(Y)$ denote ($\mathbb{R}$ or $\mathbb{C}$ valued) functions on set $X, Y$, respectively, and let $f(x), g(y)$ be functions in the appropriate spaces; then we define the Radon transform from point functions to line functions by “integrating” over points in a line and the dual Radon transform by reversing the role of points and lines:
\[
R : C(X) \longrightarrow C(Y); \quad R^t : C(Y) \longrightarrow C(X),
\]
\[
Rf(y) \equiv \sum_{\{x | x \in y\}} f(x), \quad R^t g(x) \equiv \sum_{\{y | x \in y\}} g(y).
\]

With $X, Y, Z$ and the double fibration diagram so general can anything be said about invertibility of the induced Radon transform? Surprisingly, the answer is affirmative:

**Theorem** (Bolker [2]). Assume that the double fibration diagram satisfies the following two conditions:

- $\#G_x = \alpha, \forall x \in X$ (uniform count of lines through each point)
- $\#G_{x_1} \cap G_{x_2} = \beta \forall x_1 \neq x_2$ (uniform count of lines through each pair),

for constants $\alpha, \beta$, with $0 \neq \alpha \neq \beta$. Then the Radon transform associated with the diagram is invertible, with an explicit inversion formula.

The conditions above, bundled together, are now known as the Bolker Condition, which is used extensively. The proof of the Theorem is straightforward.
Proof. We first construct a basis for $C(X)$. Let $\delta_p$ be the function on $X$ with value 1 at $p \in X$ and 0 elsewhere. Let $n$ be the cardinality of $X$. Then $\{\delta_x\}_{x \in X}$ is a basis for $C(X)$, which has dimension $n$. There is a similar basis for $C(Y)$.

The matrix of the composed transform $R' \circ R$ in this basis is:

$$
\begin{pmatrix}
\alpha & \cdots & \beta \\
\vdots & \ddots & \vdots \\
\beta & \cdots & \alpha
\end{pmatrix} = (\alpha - \beta)I + \beta1,
$$

where $I, 1$ denote the $n \times n$ identity matrix, and the $n \times n$ matrix with 1’s in every entry, resp. To invert, note that $(aI+b1) \cdot (cI+d1) = (ac)I+(ad+bc+nb)1$. □

The Bolker condition is satisfied by many geometric double fibrations, but does not hold for many others, even when the Radon transform is injective. The Radon transform on a triangle, rectangle, and pentagon can be represented by the following matrices:

![Figure 1. Some finite geometries with and without the Bolker condition, and corresponding matrices](image)

It is easy to verify the properties in the table below for the Radon transform on these geometries.

| # sides | Bolker C. Satisfied? | R injective? |
|---------|----------------------|--------------|
| 3       | Yes                  | Yes          |
| 4       | No                   | No           |
| 5       | No                   | Yes          |

The $k$-plane transform in $\mathbb{F}_q^n$ satisfies the Bolker Condition, since given points $x_1, x_2$ there is an affine map $T$ that carries $x_1$ to $x_2$ and the set of lines through $x_1$ to the set of lines through $x_2$, and given two pairs of points, there is an affine map that carries one pair to the other and the lines through one pair to the lines through the other pair. More generally, the Bolker Condition holds whenever there is a doubly transitive group action that preserves the appropriate incidence relations. When group symmetry is available it is natural consider the use of group representations. Interestingly, representation theory can be used to understand Radon transforms on the one hand, e.g., [16], and Radon transforms can be used to understand representation theory, e.g., [11, 14].
We may also inquire about a range characterization: when is a function of $k$-planes the Radon transform of a function of points? We first look at the hyperplane case, $k = n - 1$.

**Definition.** A spread of hyperplanes in $\mathbb{F}_q^n$ is a presentation of $\mathbb{F}_q^n$ as a disjoint union of hyperplanes.

**Fact.** A function $g(H)$ of hyperplanes $H$ in $\mathbb{F}_q^n$ is the Radon transform of a function of points $x \in \mathbb{F}_q^n$ only if the average of $g(H)$ over any spread is the same as the average over any other spread:

$$\sum_{\{H \in \Omega_1\}} g(H) = \sum_{\{H \in \Omega_2\}} g(H) \quad \text{(for any two spreads } \Omega_1, \Omega_2).$$

These are called the **Cavalieri conditions.** By way of illustration, In the diagram below, they state that the sum over lines with positive slope equals the sum over lines with negative slope.

![Figure 2. Two spreads leading to a Cavalieri condition](image)

**Theorem (Bolker).** The Cavalieri conditions characterize the range of the hyperplane Radon transform over a finite field.

The proof is based on a counting argument. This range condition yields an admissibility theorem.

### 3. Admissible Complexes

**Definition.** Recall that a complex of hyperplanes $\mathcal{C}$ is a collection of hyperplanes $\{H | H \in \mathcal{C}\}$ so that $|\mathcal{C}| = |\mathbb{F}_q^n| = q^n$ (there are as many hyperplanes as points). We’ll also use “complex” to denote the appropriate number of lines, curves, etc.

**Definition.** The complex $\mathcal{C}$ is said to be admissible if the Radon transform operation, restricted to planes belonging to $\mathcal{C}$ is still injective:

$$R_{\mathcal{C}} : C(\mathbb{F}_q^n) \rightarrow C(\mathcal{C}).$$

**Theorem.** (Note) A complex $\mathcal{C}$ of hyperplanes in $\mathbb{F}_q^n$ is admissible if and only if it omits precisely one plane from each spread, except for one spread, which belongs to $\mathcal{C}$ in its entirety.
To prove “if”, it suffices to show that $R_C f$ determines $R f$. A counting argument shows that every complex contains an entire spread. To evaluate $R f$ on a plane $H$ which $C$ omits, simply use the total mass of $f$ encoded in a spread that belongs to $C$ in its entirety. To prove “only if”, take two parallel hyperplanes and construct a “capacitor” charge distribution: +1 on plane, −1 on the other, and zero elsewhere. Only the two chosen planes can “see” this distribution via the Radon transform. The rest have vanishing Radon transform because of cancellation.

Thus the hyperplane case turns out to be the easy case. We now explore the next simplest: the line transform in $\mathbb{Z}_2^3$. The three dimensional vector space over $\mathbb{Z}_2$ has 8 points, 7 lines through a given point, 28 lines in all.

Figure 3. lines in $\mathbb{Z}_2^3$
Here are some ways to construct admissible complexes:

- Write $\mathbb{Z}_2^3$ as a union of two parallel planes (a spread of planes) and choose an admissible set of lines on each plane (four lines chosen in each plane).

- Choose one plane $\mathbf{p} \subset \mathbb{Z}_2^3$, choose an admissible set of (four) lines within $\mathbf{p}$, then extend four “legs” perpendicular to $\mathbf{p}$.

- Construct, if possible, admissible complexes in $\mathbb{Z}_2^3$ without using planar relatively admissible complexes.

The first two methods are illustrated below.

**Figure 4.** Some ways to construct admissible complexes.
The Radon transform for lines in $\mathbb{Z}_2^3$ can be represented by the following $28 \times 8$ matrix:

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
$$

The admissibility problems asks:

*What are the non-singular $8 \times 8$ minors of this matrix?*

We’d like an answer that is geometrically motivated. The linear algebra computer environment Octave can be used to locate all admissible complexes for this transform, as illustrated below.
The program results give:

- 3,108,105 line complexes
- 2,170,667 inadmissible complexes [later corrected to 2,170,665]
- 937,438 admissible complexes [later corrected to 937,440]

Can we describe the *moduli space* of admissible complexes? Can we enumerate them without using brute force?
There are some clear obstructions to admissibility. In particular, a complex $C$ is inadmissible if it has any of the following features.

- An omitted point.
- An isolated tree.
- An even cycle.

Clearly, a line complex that does not pass through a particular point cannot recover data at that point. Similarly, complexes with even cycles or with isolated trees are rank-deficient, as manifested by a $+1, -1$ data pattern. These contexts are illustrated below.

![Diagram](image.png)

**Figure 5.** Some inadmissible configurations

It turns out that these are the only obstructions to admissibility.

**Theorem** (Admissibility for $\mathbb{Z}_2^n$). Let $C$ be a line complex in $\mathbb{Z}_2^n$. Assume that $C$ omits no point, has no isolated trees, and does not contain an even cycle. Then $C$ is admissible.

**Proof.** Take a point $p \in \mathbb{Z}_2^n$. There’s a line $\ell \in C$ containing $P$. Expand $\ell$ to a maximal connected set of lines, $\mathcal{M}$. Then $\mathcal{M}$ cannot be a tree, so $\mathcal{M}$ contains cycles, hence odd cycles. Each odd cycle is “self inverting”. Every point in $\mathcal{M}$ is linked to an odd cycle by a contiguous path of lines, hence is solvable. \qed
4. Appendix: Counting a majority of inadmissible complexes

Here we will count two basic archetypes of inadmissible complexes, along with their intersection. This will serve to illustrate the combinatorics of the complete count.

4.1. Complexes that omit one or more points. First we enumerate complexes that are “missing points”, that is, complexes \( C \) so that there exist points \( p \in \mathbb{F}_2^3 \) so that no line \( \ell \in C \) passes through \( p \). It turns out that there are many of these. There are seven lines through \( p \), so the complexes that miss \( p \) have \( 8 \) lines chosen from the \( 28 - 7 = 21 \). Now \( \binom{21}{8} = 203,440 \). Multiplying this by the number of points, \( 8 \), and accounting for double counting (because there are complexes that omit more than one point) we obtain:

Lemma. There are \( \binom{21}{8} \times 8 = 1,627,920 \) complexes that omit points. Here, each complex is counted with multiplicity equal to the number of points in \( \mathbb{F}_2^3 \) which it misses.

4.1.1. Complexes that omit two or more points. How many complexes miss two points? There are \( 7 + 7 - 1 = 13 \) lines through one or the other or both points. So a complex that misses both points has \( 8 \) lines chosen from among \( 28 - 13 = 15 \) lines. There are \( 28 \) pairs of points, so we have double counted \( 28 \times \binom{15}{8} = 28 \times 6,435 = 180,180 \) complexes. (Note that we have double counted the double counting, because there are complexes that miss three points.)

Lemma. The number of complexes that omit a pair of points is \( 28 \times \binom{15}{8} = 28 \times 6,435 = 180,180 \). Here each complex is counted with multiplicity equal to the number of pairs of points that it misses.

4.1.2. Complexes that omit three or more points. How many lines pass through one or more of three given points? All but the \( 10 \) that form the complete graph on the remaining \( 5 \) points. Thus, to exhibit all complexes omitting three or more points, choose three points from \( 8 \) and then choose \( 8 \) lines from among \( 10 \). Thus we have:

Lemma. The number of complexes that omit precisely three points is \( \binom{10}{8} \times \binom{3}{3} = 2,520 \). There are no line complexes that miss four or more points.

Putting the above lemmas together we have

Lemma. The number of complexes that avoid one or more points is: \( 1,627,920 - 180,180 + 2,520 = 1,450,260 \). This count is without multiplicity.

4.2. Complexes with isolated lines.

4.2.1. Complexes with one or more isolated lines. Another type of non-admissible complex is one where a single line \( \ell \) is ‘isolated’, i.e., meets no other line in the complex. (This is the simplest case of an isolated tree.) How many of these are there? Well, how many lines meet \( \ell \)? \( 7 + 7 - 1 = 13 = 28 - 15 \). So the number of complexes having \( \ell \) as an isolated line is \( \binom{15}{7} = 6,435 \). Accounting for each of \( 28 \) lines, with the usual double counting reminder, we have
Lemma. There are $6,435 \times 28 = 180,180$ complexes with one or more isolated lines. Each complex is counted with multiplicity equal to the number of isolated lines it has.

4.2.2. Complexes with two or more disjoint isolated lines. If $\ell$ is a line, there are 13 lines meeting $\ell$ and 15 lines disjoint from $l$. Thus there are $(28)(15)/2 = 210$ pairs of disjoint lines. Given a complex with a pair of disjoint lines, the other 6 lines of the complex must form the complete graph on the remaining four points. Thus there are 210 complexes with precisely two disjoint isolated lines. Clearly a complex cannot have three disjoint isolated lines.

Lemma. There are $(28)(15)/2 = 210$ complexes with precisely two isolated lines, and there are no complexes with three or more isolated lines.

Lemma. There are $180,180 - 210 = 179,970$ complexes with one or more isolated lines. These complexes are counted without multiplicity.

4.3. Complexes with both omitted points and isolated lines.

4.3.1. Complexes with one or more isolated lines and one or more omitted points. There are five points disjoint from the designated omitted point and the isolated line, hence there are $\binom{5}{2} = 10$ permissible lines. We must choose 7 lines among these to form a complex, and there are $8 \times 28$ point-line pairs.

Lemma. There are no complexes with one isolated line and two omitted points.

Proof. The complement of the union of the omitted points and the isolated line has 4 points, and these form 6 lines, not enough to form a line complex. \hfill \square

Lemma. There are no complexes with two disjoint isolated lines and an omitted point.

Proof. There are five points in the union of the two lines and point, hence three points left, not enough to span a line complex. \hfill \square

Lemma. The number of complexes with one isolated line and one omitted point is $(8 \times 21)\binom{10}{7} = 20,160$. The count is multiplicity free.

Proof. There are $8 \times 21 = 168$ disjoint point-line pairs (or $28 \times 6 = 168$ disjoint line-point pairs). Given a disjoint point-line pair there are 5 remaining points and $\binom{5}{2} = 10$ lines in their complete graph. Of these we must choose 7 to obtain a line complex. Because of the preceding lemmas there are no multiplicities. Hence the claimed count is verified. \hfill \square

We have counted a majority of inadmissible complexes and illustrated the combinatorics of intersections of archetypes. If sufficient interest develops we will post a completion of this analysis.

Added in proof:
This analysis is now completed and included in a follow-up paper with Mehmet Orhon.

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REFERENCES

[1] C.A. Berenstein, review of  *The universality of the Radon transform* by Leon Ehrenpreis, Math. Reviews., MR2019604 (2007).

[2] E.D. Bolker,  *The finite Radon transform. Integral geometry*  (Brunswick, Maine, 1984), Contemp. Math., 63, Amer. Math. Soc., Providence, RI, (1987), 27-750.

[3] E.D. Bolker, E.L. Grinberg and J.P. Kung,  *Admissible complexes for the combinatorial Radon transform. A progress report.*, Integral geometry and tomography (Arcata, CA, 1989), Contemp. Math., 113, Amer. Math. Soc., Providence, RI, (1990), 17-3.

[4] S.S. Chern,  *Integral geometry in Klein Spaces*, Ann. of Math. (2) 43, (1942), 403–422.

[5] L. Ehrenpreis, Leon  *The universality of the Radon transform*, wi. appendix by P. Kuchment and E. T. Quinto, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York (2003).

[6] D.V. Feldman and E.L. Grinberg,  *Admissible Complexes for the Projective X-Ray Transform over a Finite Field*, preprint (2012)

[7] I.M. Gelfand and M.I. Graev,  *Integral Transformations Connected with Straight Line Complexes in Complex Affine Space*, Dokl. Acad. Nauk SSSR 2 (1961), 809–812.

[8] I.M. Gelfand, M.I. Graev and N. Vilenkin,  *Generalized Functions*, Vol. 5, Academic Press, New York, 1966.

[9] E.L. Grinberg,  *The Admissibility Theorem for the Hyperplane Transform over a Finite Field*, J. Comb. Theory, Series A 53, (1990), 316–320.

[10] E.L. Grinberg and E.T. Quinto, Ed.,  *Integral Geometry and Tomography* Proce. AMS-IMS-SIAM Joint Summer Research Conf.,t Humboldt State U., Arcata, CA, 1989. Contemp. Math., 113. American Mathematical Society, Providence, RI, 1990.

[11] V. Guillemin & S. Sternberg  *Geometric Asymptotics* Chp. VI and Appendix, Mathematical Surveys, No. 14. American Mathematical Society, Providence, R.I., (1977).

[12] Kirillov, A. A.  *On A Problem of I. M. Gelfand*, Dokl. Akad. Nauk SSSR 137 276–277 (Russian); translated as Soviet Math. Dokl. 2 1961 2687-269.

[13] J.P. Kung,  *The Radon Transforms of a Combinatorial Geometry I.* J. Combin. Theory Ser. A 26 (1979), no. 2, 97?-102.

[14] S. Sternberg,  *Group Theory and Physics*, Appendix C, Cambridge University Press, Cambridge, (1994).

[15] R.S. Strichartz,  *Radon Inversion–Variations on a Theme*, Amer. Math. Monthly 89 (1982), 6 , 377?384, 420?423

[16] A.V. Zelevinskii,  *Generalized Radon Transforms in Spaces of Functions on Grassmann Manifolds over a Finite field* (Russian) Uspehi Mat. Nauk 28 (1973), no. 5(173), 243-?244.

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