COHOMOLOGY AND ANDRÉ MOTIVES OF HYPERKÄHLER ORBIFOLDS

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Abstract. One of the main tools for the study of compact hyperkähler manifolds is the natural action of the Looijenga–Lunts–Verbitsky Lie algebra on the cohomology of such manifolds. This also applies to the mildly singular holomorphic symplectic varieties—hyperkähler orbifolds, allowing us to generalize the results of [18] to this setting.

1. Introduction

It has recently been established that the theory of irreducible holomorphic symplectic manifolds may be generalized in a reasonable way to the situation when the symplectic varieties are singular. The basics of the theory have been developed by Beauville, Kaledin, Namikawa and others (see e.g. [5], [12], [16] and references therein), and more recently a version of the global Torelli theorem for singular symplectic varieties has been proven by Bakker and Lehn [4]. More can be said when the the symplectic varieties have only quotient singularities. In this case the varieties possess metric properties similar to the smooth case, namely they admit hyperkähler metrics, see [6]. This fact leads to a number of consequences that have not been fully covered in the literature yet. The purpose of this paper is to develop some of these aspects of the theory of hyperkähler orbifolds.

Existence of hyperkähler metrics on symplectic orbifolds implies that their singular cohomology with rational coefficients admits a natural Lie algebra action, similar to the smooth case. Moreover, purity of the Hodge structures allows us to define André motives of hyperkähler orbifolds and to generalize the results of [18] to the orbifold setting. The main result is Theorem 4.2 which allows us to prove that many of the known examples of projective irreducible hyperkähler orbifolds with $b_2 \geq 4$ have abelian André motives. The importance of this statement stems from its relation to the Hodge and Mumford–Tate conjectures, see [18], [8], [9], [10] for a detailed discussion of these relations in the case of smooth hyperkähler varieties.

2. Hyperkähler orbifolds

2.1. Complex orbifolds. For a complex analytic variety $X$ its singular locus will be denoted by $X^{\text{sg}}$ and its smooth locus by $X^{\text{sm}}$. For the purposes of this paper the term orbifold will be synonymous with variety that has only quotient singularities. More precisely, we will use the following definition.
Definition 2.1. Let $X$ be a normal, connected, complex analytic variety of dimension $n$. We will call $X$ an orbifold if any point of $X$ has an open neighbourhood isomorphic to $U/G$, where $G \subset \text{GL}_n(\mathbb{C})$ is a finite group and $U \subset \mathbb{C}^n$ is an open $G$-invariant neighbourhood of zero.

Remark 2.2. If $G_0 \subset G$ is the subgroup generated by all pseudoreflections contained in $G$, then $G_0$ is normal in $G$ and $U/G_0$ is smooth. Therefore we may assume in the above definition that no $g \in G, g \neq 1$ fixes a hyperplane. Under this assumption for any point $x \in X$ the group $G$ is uniquely recovered as the local fundamental group of $X^{\text{sm}}$ around $x$.

For a local chart $U/G \to X$ of an orbifold $X$ denote by $U^o$ the preimage of $X^{\text{sm}}$ in $U$. A differential form on $X$ is a differential form on $X^{\text{sm}}$ whose pull-back to any local chart $U^o$ extends to a differential form on $U$. A two-form is called Kähler if this extension is a Kähler form on $U$ for any chart. It has been shown by Satake [17] that a number of results analogous to the smooth case hold for orbifolds. In particular, de Rham theorem, Poincaré duality with rational coefficients, Hodge and Lefschetz decompositions of cohomology are among such results.

2.2. Beauville-Bogomolov decomposition and hyperkähler orbifolds. By the fundamental group of an orbifold $X$ we will mean the group $\pi_1(X^{\text{sm}})$, and we will call $X$ simply connected if this group is trivial. A morphism of complex analytic spaces $f: Y \to X$, where $X$, $Y$ are orbifolds, is called an orbifold covering if $f$ is a topological covering over $X^{\text{sm}}$.

We recall the orbifold version of the Beauville–Bogomolov decomposition theorem. A simply connected $n$-dimensional orbifold $X$ will be called Calabi–Yau if it admits a Kähler metric with holonomy equal to $SU(n)$ and hyperkähler if it admits a Kähler metric with holonomy equal to $Sp(n/2)$.

Theorem 2.3 ([6, Theorem 6.4]). Let $X$ be a compact Kähler orbifold with $c_1(X) = 0$. Then there exists a finite orbifold covering $\pi: \tilde{X} \to X$ with

$$\tilde{X} \simeq T \times \prod_i Y_i \times \prod_j Z_j,$$

where $T$ is a compact complex torus, $Y_i$ are Calabi–Yau orbifolds and $Z_j$ are hyperkähler orbifolds.

Assume that $X$ is a hyperkähler orbifold with hyperkähler metric $g$. Let $I$ denote its complex structure. Since the standard representation of $Sp(n/2)$ is quaternion-ionic, the holonomy principle implies that $X$ admits two more complex structures $J$ and $K$ with $IJ = -JI = K$. Integrability of these structures follows from the fact that they are preserved by the Levi–Civita connection which is torsion-free. We denote the corresponding Kähler forms by $\omega_I, \omega_J$ and $\omega_K$. The form $\sigma_I = \omega_J + i\omega_K$ is holomorphic symplectic with respect to $I$.

A typical example of a hyperkähler orbifold is a singular K3 surface obtained by contracting $(-2)$-curves on a smooth K3 surface. Such orbifold admits a small deformation that is again smooth. Hence it is more natural to think of such an example as of a degeneration of a smooth hyperkähler manifold, and we would like to focus our attention on the case when the singularities can not be removed by deformation. Therefore we introduce the following more restrictive notion.
Definition 2.4. A simply connected hyperkähler orbifold $X$ satisfying the condition $\text{codim}_X X^{\text{sm}} \geq 4$ will be called irreducible.

The singularities of any compact complex orbifold are $\mathbb{Q}$-factorial (see [4, Proposition 2.15] for the discussion of $\mathbb{Q}$-factoriality in the complex analytic setting). The singularities of any hyperkähler orbifold are rational [5]. For an irreducible hyperkähler orbifold the singularities are moreover terminal [4, Theorem 3.4].

2.3. Deformations and moduli of irreducible hyperkähler orbifolds. Assume that $X$ is an irreducible hyperkähler orbifold. A deformation of $X$ is a proper flat morphism of complex analytic spaces $\pi: \mathcal{X} \to B$ with connected base $B$ and such that some fibre of $\pi$ is isomorphic to $X$. We will denote the fibre of $\pi$ over a point $t \in B$ by $X_t$. The condition on the codimension of $X^{\text{sm}}$ implies (see [11, Lemma 3.3]) that any deformation of $X$ is locally trivial in the following sense.

Definition 2.5. A morphism $\pi: \mathcal{X} \to B$ of complex analytic spaces is called locally trivial if for any point $x \in \mathcal{X}$ some neighbourhood of $x$ in $\mathcal{X}$ is biholomorphic to $V_1 \times V_2$ for some open neighbourhood $V_1$ of $x$ in the fibre $\pi(x)$ and some open neighbourhood $V_2$ of $\pi(x)$ in $B$, and $\pi|_{V_1 \times V_2}$ is the projection to the second factor.

Fix a basepoint $t_0$ such that $X_{t_0} \simeq X$. It is known that locally trivial deformations of complex spaces are trivial in the real analytic category, see [2, Proposition 5.1]. This means, in particular, that all fibres $X_t$ are isomorphic as real analytic varieties and hence we have the monodromy action of $\pi_1(B, t_0)$ on $H^*(X, \mathbb{Q})$.

The moduli theory of hyperkähler manifolds has been generalized to the orbifold setting in [15]. As a consequence, it has been shown that $H^2(X, \mathbb{Q})$ carries a canonical symmetric bilinear form $q$ called the BBF form, [15, section 3.4]. This form has the following property: there exists a constant $C_X \in \mathbb{Q}$ such that for any $a \in H^2(X, \mathbb{Q})$ we have

$$q(a)^n = C_X \int_X a^{2n},$$

where we use the intersection product on the right hand side. The form $q$ is normalized so that it is integral and primitive on the image of $H^2(X, \mathbb{Z})$ in $H^2(X, \mathbb{Q})$. The signature of $q$ is $(3, b_2(X) - 3)$.

We fix the lattice $\Lambda = H^2(X, \mathbb{Z})/(\text{torsion})$ with the BBF form and define the moduli space $\mathcal{M}$ of marked hyperkähler orbifolds $(Y, \phi)$ deformation equivalent to $X$, where $\phi: H^2(Y, \mathbb{Z})/(\text{torsion}) \to \Lambda$ is a lattice isomorphism called marking, see [15, section 3.5]. For a symplectic form $\sigma \in H^0(Y, \Omega^2_Y)$ we have $q(\sigma) = 0$ and $q(\sigma, \bar{\sigma}) > 0$. Let $\mathcal{D} \subset \mathbb{P}(\Lambda \otimes \mathbb{C})$ be the period domain:

$$\mathcal{D} = \{ x \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid q(x) = 0, q(x, \bar{x}) > 0 \}.$$ 

Then the period of $(Y, \phi)$ is the point $\rho(Y, \phi) = [\phi(\sigma)] \in \mathcal{D}$.

The moduli space $\mathcal{M}$ is the set of isomorphism classes of marked hyperkähler orbifolds deformation equivalent to $X$ and $\rho$ defines the period map $\mathcal{M} \to \mathcal{D}$. The construction and the properties of $\mathcal{M}$ in the orbifold setting are completely analogous to the smooth case. In particular, $\mathcal{M}$ is a non-Hausdorff complex manifold of dimension $h^{1,1}(X)$ and $\rho$ is a local isomorphism.

Recall the definition of the Hausdorff reduction of $\mathcal{M}$. We call two points $x, y \in \mathcal{M}$ inseparable if for any open neighbourhoods $U_x$ of $x$ and $U_y$ of $y$ we have $U_x \cap U_y \neq \emptyset$. This turns out to be an equivalence relation, and we define $\overline{\mathcal{M}}$ to be the set of
equivalence classes. Then \( \overline{\mathcal{M}} \) is a Hausdorff complex manifold, and clearly \( \rho \) factors through \( \overline{\mathcal{M}} \). For a fixed connected component \( \mathcal{M} \) of \( \mathcal{M} \) denote by \( \overline{\mathcal{M}} \) its Hausdorff reduction.

**Theorem 2.6** ([15, Theorem 5.9]). The period map \( \rho \) defines an isomorphism between \( \overline{\mathcal{M}} \) and \( \mathcal{D} \).

Below we will need to consider special subvarieties in the period domain of the following form. Fix an element \( h \in \Lambda \) such that \( q(h) > 0 \). Let

\[
\mathcal{D}_h = \{ x \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid q(x) = q(x, h) = 0, q(x, \bar{x}) > 0 \} \subset \mathcal{D}.
\]

The points of \( \mathcal{D}_h \) are periods of orbifolds that admit a line bundle with first Chern class \( h \). All these orbifolds are projective by [15, Theorem 1.2]. Denote by \( \mathcal{M}_h \) and \( \mathcal{M}_h \) the preimages of \( \mathcal{D}_h \) in \( \mathcal{M} \) and \( \mathcal{M} \) respectively.

2.4. The Looijenga–Lunts–Verbitsky Lie algebra action on cohomology. If \( X \) is an irreducible hyperkähler orbifold with Kähler forms \( \omega_I \), \( \omega_J \) and \( \omega_K \), let us denote the corresponding Lefschetz operators by \( L_I \), \( L_J \) and \( L_K \). Let \( \Lambda_I \), \( \Lambda_J \) and \( \Lambda_K \) be the dual Lefschetz operators. A pointwise computation identical to the case when \( X \) is smooth (see e.g. [19]) shows that the dual Lefschetz operators commute with each other and we have \( [\Lambda_I, L_K] = W_I \) (and similarly for the cyclic permutations of \( I, J, K \)), where \( W_I \) is the operator that acts on the differential forms of \( I \)-type \((p, q)\) as multiplication by \( i(p - q) \). Verbitsky [21] deduces from this that the Lie subalgebra of \( \text{End}(\Lambda^* T^* X) \) generated by the above operators is isomorphic to \( \mathfrak{so}(4,1) \).

The Lefschetz operators and their duals act on the cohomology of \( X \), so we have an embedding of Lie algebras \( \mathfrak{so}(4,1) \hookrightarrow \text{End}(H^*(X, \mathbb{C})) \). Let \( \mathfrak{g}_{\text{tot}}(X) \) be the subalgebra of \( \text{End}(H^*(X, \mathbb{C})) \) generated by all such embeddings for all hyperkähler metrics on \( X \). This is the “total Lie algebra” of Looijenga–Lunts [14] and its structure was determined in [22].

We recall the description of \( \mathfrak{g}_{\text{tot}}(X) \). Let \( V = H^2(X, \mathbb{Q}) \) and let \( \tilde{V} = \langle e_0 \rangle \oplus V \oplus \langle e_4 \rangle \) be graded with \( e_k \) of degree \( k \) and \( V \) in degree 2. Define \( \tilde{q} \in S^2 \tilde{V}^* \) such that \( \tilde{q}|_V = q \), \( \langle e_0, e_4 \rangle \) orthogonal to \( V \) with \( q(e_0) = q(e_4) = 0 \) and \( q(e_0, e_4) = 1 \).

**Proposition 2.7** ([22], see also [19, Proposition 2.9]). The graded Lie algebra \( \mathfrak{g}_{\text{tot}}(X) \) is isomorphic to \( \mathfrak{so}(\tilde{V}, \tilde{q}) \). The subalgebra \( \mathfrak{so}(V, q) \subset \mathfrak{g}_{\text{tot}}(X) \) acts on \( H^*(X, \mathbb{Q}) \) by derivations.

2.5. Monodromy action on cohomology. For an irreducible hyperkähler orbifold \( X \) denote by \( \text{Aut}^+(H^*(X, \mathbb{Q})) \) the group of graded algebra automorphisms that act trivially in the top degree \( H^{4n}(X, \mathbb{Q}) \). It follows from Proposition 2.7 that there exists a representation

\[
\lambda: \text{Spin}(V, q) \to \text{Aut}^+(H^*(X, \mathbb{Q})).
\]

Let \( \pi: X \to B \) be a deformation of \( X \) with \( X_{t_0} \simeq X \) for a basepoint \( t_0 \in B \). For a covering \( B' \to B \) we denote by \( t'_0 \in B' \) some preimage of \( t_0 \). The action of \( \pi_1(B, t_0) \) on the cohomology of \( X \) clearly preserves the top degree cohomology classes.

**Proposition 2.8.** There exists a finite covering \( B' \to B \) such that the monodromy representation \( \pi_1(B', t'_0) \to \text{Aut}^+(H^*(X, \mathbb{Q})) \) factors through \( \lambda \).
Proof. Let $\text{Aut}^\circ(\mathcal{H}^\bullet(X,\mathbb{Q}))$ be the connected component of the identity in the group $\text{Aut}^+(\mathcal{H}^\bullet(X,\mathbb{Q}))$. After passing to a finite covering of $B$ we may assume that the monodromy representation factors through $\text{Aut}^\circ(\mathcal{H}^\bullet(X,\mathbb{Q}))$. Consider the action of $\text{Aut}^\circ(\mathcal{H}^\bullet(X,\mathbb{Q}))$ on $H^2_i(X,\mathbb{Q})$. It is clear from (1) that this action preserves the BBF form $q$, hence a homomorphism $\alpha: \text{Aut}^\circ(\mathcal{H}^\bullet(X,\mathbb{Q})) \rightarrow SO(V,q)$.

Let $\Gamma \subset \text{Aut}^\circ(\mathcal{H}^\bullet(X,\mathbb{Q}))$ be a torsion-free arithmetic subgroup, where the integral structure is determined by the integral cohomology of $X$. The action of $\ker(\alpha)$ on $H^\bullet(X,\mathbb{Q})$ preserves the Hodge structure and the Hodge–Riemann bilinear forms. Therefore it preserves a positive definite scalar product on $H^\bullet(X,\mathbb{Q})$, and the real Lie group $\ker(\alpha)(\mathbb{R})$ is compact. Since $\Gamma$ is torsion-free, it is mapped injectively into $SO(V,q)$ by $\alpha$. Passing to a finite index subgroup of $\Gamma$ we may assume that $\alpha(\Gamma)$ is contained in the image of the morphism $\text{Spin}(V,q)(\mathbb{Q}) \rightarrow SO(V,q)(\mathbb{Q})$. As a conclusion, we may assume that $\Gamma$ is contained in the image of $\lambda$.

Since the monodromy action preserves the integral cohomology of $X$, the image of $\rho: \pi_1(B,t_0) \rightarrow \text{Aut}^\circ(\mathcal{H}^\bullet(X,\mathbb{Q}))$ is contained in an arithmetic subgroup. Then $\rho^{-1}(\Gamma)$ is a finite index subgroup of $\pi_1(B,t_0)$, and we define $B'$ to be the corresponding finite covering of $B$. \qed

3. André motives of varieties with quotient singularities

3.1. Motivated cycles. Let $X$ be a non-singular complex projective variety, $n = \dim_{\mathbb{C}}(X)$. If $\mathcal{L}$ is an ample line bundle on $X$ with the first Chern class $h \in H^2(X,\mathbb{Z})$, the Lefschetz operator $L_h \in \text{End}(H^\bullet(X,\mathbb{C}))$ induces isomorphisms

$$L_h^k: H^{n-k}(X,\mathbb{C}) \rightarrow H^{n+k}(X,\mathbb{C})$$

for every $k = 0, \ldots, n$. Denote by $H^k_{\text{pr}}(X,\mathbb{C}) \subset H^k(X,\mathbb{C})$ the kernel of $L_h^k$ for $k = 0, \ldots, n$. Denote by $*_h \in \text{End}(H^\bullet(X,\mathbb{C}))$ the Lefschetz involution: for $x \in H^k_{\text{pr}}(X,\mathbb{C})$ and $i = 0, \ldots, n-k$ we have $*_h(L_h^i x) = L_h^{n-k-i} x$.

Consider two non-singular complex projective varieties $X$, $Y$ with two ample line bundles $\mathcal{L}_1 \in \text{Pic}(X)$, $\mathcal{L}_2 \in \text{Pic}(Y)$. Let $h = c_1(p^*_X \mathcal{L}_1 \otimes p^*_Y \mathcal{L}_2) \in H^2(X \times Y,\mathbb{Z})$. For arbitrary classes of algebraic cycles $\alpha, \beta \in H^\bullet_{\text{alg}}(X \times Y,\mathbb{Q})$, consider the class

$$p_{X*}(\alpha \cup *_h \beta),$$

where $p_X: X \times Y \rightarrow X$ is the projection. Let $H^\bullet_{\text{alg}}(X,\mathbb{Q})$ be the subspace of $H^\bullet_{\text{alg}}(X,\mathbb{Q})$ spanned by the classes (3) for all $Y$, $\mathcal{L}_1$, $\mathcal{L}_2$, $\alpha$, $\beta$ as above. Elements of $H^\bullet_{\text{alg}}(X,\mathbb{Q})$ will be called motivated cycles.

3.2. André motives. We recall the definition of André motives from [3]. For two non-singular connected projective varieties $X$, $Y$ let $\text{Cor}^k_M(X,Y) = H^{k+\dim(X)}_M(X \times Y,\mathbb{Q})$ be the space of motivated correspondences. Define the $\mathbb{Q}$-linear category $(\text{Mot}_\mathbb{A})$ whose objects are triples $(X,p,n)$, where $X$ is a variety as above, $p \in \text{Cor}^n_M(X,X)$, $\circ p = p$, and $n \in \mathbb{Z}$. Define the morphisms from $(X,p,n)$ to $(Y,q,m)$ to be the subspace

$q \circ \text{Cor}^{m-n}(X,Y) \circ p \subset \text{Cor}^m_M(X,Y).$

Cartesian product of varieties provides a tensor product in $(\text{Mot}_\mathbb{A})$, and it is known [3] that $(\text{Mot}_\mathbb{A})$ is a semi-simple graded neutral Tannakian category. We will denote the André motive of a variety $X$ as follows: $M(X)(n) = (X,[\Delta X],n) \in (\text{Mot}_\mathbb{A})$, where $\Delta X$ is the diagonal in $X \times X$. The Künneth components $\delta_k$ of the diagonal $[\Delta X]$ are motivated cycles and we have the decomposition $M(X)(n) =$
where $H^k(X)(n) = (X, \delta_k, n) \in (\text{Mot}_A)$. The motives $M(A)(n)$ for all abelian varieties $A$ generate the full Tannakian subcategory of $(\text{Mot}_A)$ that we denote $(\text{Mot}_A^A)$ and call the category of abelian motives.

Assume now that $X$ is a projective orbifold. Let $r : Y \to X$ be a resolution of singularities. Since Poincaré duality with rational coefficients holds for $X$, the Hodge structure on the cohomology $H^k(X, \mathbb{Q})$ is pure and the pull-back morphism $r^* : H^k(X, \mathbb{Q}) \to H^k(Y, \mathbb{Q})$ is injective, see [7, Theorem 8.2.4(iv) and Proposition 8.2.5]. Therefore $H^k(X, \mathbb{Q})$ is a sub Hodge structure in the cohomology of a smooth projective variety. We define the André motive of $X$ as a submotive of $M(Y)$ in a similar way.

Namely, we consider a simplicial resolution $\mathcal{Y} \to X$ with $\mathcal{Y}_i$ smooth projective varieties and $\mathcal{Y}_0 = Y$, see [7, 6.2.5]. We have the exact sequence [7, Proposition 8.2.5]

$$0 \to H^k(X, \mathbb{C}) \xrightarrow{r^*} H^k(Y, \mathbb{C}) \xrightarrow{\delta_0 - \delta_1} H^k(\mathcal{Y}_1, \mathbb{C}),$$

where $\delta_0, \delta_1 : \mathcal{Y}_1 \to \mathcal{Y}_0 = Y$ are the face maps. Note that $\delta_i^*$ define morphisms of motives $H^k(Y) \to H^k(\mathcal{Y}_1)$.

**Definition 3.1.** In the above setting we define

$$H^k(X) = \ker \left( H^k(Y) \xrightarrow{\delta_0 - \delta_1} H^k(\mathcal{Y}_1) \right).$$

Strictly speaking, the motive $H^k(X)$ in the above definition depends on the choice of the resolution $Y$. But if $r' : Y' \to X$ is another resolution, we can find a third resolution $Y''$ that dominates both $Y$ and $Y'$. Using this, we see that the submotives of $H^k(Y)$ and $H^k(Y')$ in the above definition are canonically isomorphic. Therefore $H^k(X)$ is well defined up to a canonical isomorphism and we may ignore dependence on the resolution.

## 4. Deformation Principle for Hyperkähler Orbifolds

### 4.1. Constructing families of hyperkähler orbifolds with a given central fibre.

We start this section from the key technical result of the paper. Assume that we have a projective family $\pi : \mathcal{X} \to B$ of hyperkähler varieties over a quasi-projective base $B$. A fibre $X_0$ of this family represents a point $p \in \mathfrak{M}$ in the moduli space. Given any other point $p' \in \mathfrak{M}$ that is non-separated from $p$, we may replace the fibre $X_0$ by the corresponding variety $X'_0$. This gives a new family $\mathcal{X}'$, possibly non-projective. This operation involves gluing two families over a punctured disc, and after such gluing $\mathcal{X}'$ may a priori not even be Moishezon (a typical situation when this happens is the Shafarevich–Tate twist, see [1] and [20]). However, in the case of families of hyperkähler varieties this does not happen and the total space $\mathcal{X}'$ of the new family is Moishezon, i.e. an algebraic space in the sense of Artin.

**Proposition 4.1.** Let $X$ be an irreducible hyperkähler orbifold and $h \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ a cohomology class with $q(h) > 0$. Then there exists a flat locally trivial family $\pi : \mathcal{X} \to C$ over a connected quasi-projective curve $C$ and a line bundle $\mathcal{L} \in \text{Pic}(\mathcal{X})$ such that $\mathcal{L}|_{\mathcal{X}_t}$ is ample for a general $t \in C$ and for some $t_0 \in C$ we have $\mathcal{X}_{t_0} \simeq X$ and $c_1(\mathcal{L}|_{\mathcal{X}_{t_0}}) = h$.

**Proof.** We pick a marking $\varphi : H^2(X, \mathbb{Z})/(\text{torsion}) \xrightarrow{\sim} \Lambda$ and identify $h$ with its image in $\Lambda$. Denote by $p \in \mathcal{M}_h$ the point corresponding to $(X, \varphi)$, where $\mathcal{M}_h$ is the
and obtain

we construct a family of Kuga–Satake abelian varieties

allows us to construct enough families of


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has Picard rank one, generated by an ample line bundle with first Chern class

isomorphic to

is general, so that for a general \( t \in C \) the fibre \( X_t \) has Picard rank one, generated by an ample line bundle with first Chern class \( h \).

The fibre \( X_{t_0} \) may be not isomorphic to \( X \). Let \( \Delta \subset C \) be a small disc around \( t_0 \) and \( \Delta^* = \Delta \setminus \{ t_0 \} \). We use [18, Lemma 6.2] to replace \( X_{\Delta} \) with another family, isomorphic to \( X_{\Delta} \) over \( \Delta^* \) and with central fibre isomorphic to \( X \). This completes the proof.

4.2. The main result. Proposition 4.1 allows us to construct enough families of irreducible hyperkähler orbifolds to connect any two points in the moduli space. Then we can apply deformation principle for motivated cycles to these families. Using the Kuga–Satake construction we obtain the following result.

Theorem 4.2. Assume that \( X_1 \) and \( X_2 \) are deformation equivalent projective irreducible hyperkähler orbifolds with \( b_2 \geq 4 \). If \( \mathsf{M}(X_1) \in (\mathsf{Mot}^{ab}_h) \) then \( \mathsf{M}(X_2) \in (\mathsf{Mot}^{ab}_h) \).

Proof. We choose two markings \( \varphi_1: H^2(X_1, \mathbb{Z})/(\text{torsion}) \sim \Lambda \) such that \((X_1, \varphi_1)\) and \((X_2, \varphi_2)\) are in the same connected component \( \mathcal{M} \) of the moduli space. Using the construction [18, section 6.2], we reduce to the case when \((X_1, \varphi_1)\) and \((X_2, \varphi_2)\) lie in \( \mathcal{M}_h \) for some \( h \in \Lambda \) with \( q(h) > 0 \). Then we apply Proposition 4.1 and obtain a locally trivial family \( \pi: \mathcal{X} \to C \) such that \( X_1 \) and \( X_2 \) are two fibres, and there exists a line bundle \( L \in \text{Pic}(\mathcal{X}) \) that is relatively ample over a general point of \( C \).

Using Proposition 2.8 we construct a family of Kuga–Satake abelian varieties \( \alpha: \mathcal{A} \to C \) attached to \( \pi: \mathcal{X} \to C \), analogously to [18, Lemma 6.3]. Let \( \rho: \tilde{X} \to X \) be a simultaneous resolution of singularities (see [4, Lemma 4.9]), where \( \tilde{\pi}: \tilde{X} \to C \) is a smooth family. Since the fibres of \( \pi \) have quotient singularities, we have an embedding of variations of Hodge structures:

\[
R^\pi_! \mathcal{Q} \hookrightarrow R^\rho_\tilde{\pi} \mathcal{Q}.
\]

We also have the Kuga–Satake embedding for the family \( \mathcal{X} \), see [13] and [18, Corollary 3.3]. This embedding is given by a section of the local system \( R^\rho_! \psi_* \mathcal{Q} \), where \( \psi: \tilde{X} \times_C \mathcal{A} \to C \). Since the André motive of \( X_1 \) is abelian, the value of that section at one point is a motivated cycle. By a version of the deformation principle proven in [18, Proposition 5.1], the value of the section at any point of \( C \) is a motivated
cycle, hence the motive of $X_2$ is embedded into the motive of an abelian variety. This concludes the proof. □

4.3. Examples of irreducible hyperkähler orbifolds. We apply Theorem 4.2 to two families of irreducible hyperkähler orbifolds.

For the first example consider a $K3$ surface $S$ with a symplectic involution $f$. Then $f$ has 8 fixed points. Consider the induced involution $f^{[2]}$ of the Hilbert square $S^{[2]}$. The involution $f^{[2]}$ has 28 isolated fixed points and a fixed surface $S' \subset S^{[2]}$ that is isomorphic to the resolution of $S/f$.

Let $Y$ be the blow-up of $S^{[2]}$ in $S'$ and $X$ be the quotient of $Y$ by the action of $f^{[2]}$. Note that by [3] the motives of $S^{[2]}$, $S'$ and $Y$ are abelian, hence $M(X) \in (\text{Mot}_A^\text{ab})$. It is clear that $X$ is a symplectic orbifold with only isolated singularities. Since the involution $f^{[2]}$ fixes the exceptional divisor $E \subset Y$ of the blow-up $Y \to S^{[2]}$, the orbifold fundamental group of $X$ is trivial, hence $X$ is irreducible. Note also that $b_2(X) = b_2(S^{[2]}/f^{[2]}) + 1$, so a general deformation of $X$ is not obtained from a deformation of $S^{[2]}$.

A similar construction applies to the generalized Kummer variety $K^2A$ if we start from an abelian surface $A$ with an involution $f$. We obtain an irreducible orbifold $X'$ as a partial resolution of the quotient of $K^2A$ by an involution induced by $f$.

Corollary 4.3. Any projective hyperkähler orbifold deformation equivalent either to $X$ or to $X'$ described above has an abelian André motive.

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