ON SOME QUESTIONS OF EYMARD AND BEKKA CONCERNING AMENABILITY OF HOMOGENEOUS SPACES AND INDUCED REPRESENTATIONS

VLADIMIR PESTOV

Abstract. Let $F \subseteq H \subseteq G$ be closed subgroups of a locally compact group. In response to a 1972 question by Eymard, we construct an example where the homogeneous factor space $G/F$ is amenable in the sense of Eymard–Greenleaf, while $H/F$ is not. (In our example, $G$ is discrete.) As a corollary which answers a 1990 question by Bekka, the induced representation $\text{ind}^G_H(\rho)$ can be amenable in the sense of Bekka even if $\rho$ is not amenable. The second example, answering another question by Bekka, shows that $\text{ind}^G_H(\rho)$ need not be amenable even if both the representation $\rho$ and the coset space $G/H$ are amenable.

Résumé. Soient $F \subseteq H \subseteq G$ deux sous-groupes fermés d’un groupe localement compact $G$. En réponse à une question posée en 1972 par Eymard, nous construisons un groupe discret $G$ tel que l’espace homogène $G/F$ est moyennable au sens d’Eymard et Greenleaf, bien que $H/F$ n’est pas moyennable. On obtienne un corollaire qui répond à un problème posé par Bekka en 1990: une représentation induite $\text{ind}^G_H(\rho)$ peut être moyennable au sens de Bekka même si $\rho$ n’est pas moyennable. Le deuxième exemple, qui répond à une autre question de Bekka, montre que $\text{ind}^G_H(\rho)$ n’est pas nécessairement moyennable même si la représentation $\rho$ et l’espace homogène $G/H$ sont l’un et l’autre moyennables.

1. Introduction

Let $H$ be a closed subgroup of a locally compact group $G$. The homogeneous factor space $G/H$ is amenable in the sense of Eymard and Greenleaf [5, 6], if $L^\infty(G/H)$ supports a $G$-invariant mean. If $H = \{e\}$, one obtains the classical concept of an amenable locally compact group.

A unitary representation $\rho$ of a group $G$ in a Hilbert space $\mathcal{H}$ is amenable in the sense of Bekka [4] if there exists a state, $\phi$, on the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators that is $\text{Ad} G$-invariant: $\phi(\pi(g)T\pi(g)^{-1}) = \phi(T)$ for every $T \in \mathcal{B}(\mathcal{H})$ and every $g \in G$. For instance, the homogeneous space $G/H$ is Eymard–Greenleaf amenable if and only if the quasi-regular representation $\lambda_{G/H}$ of $G$ in $L^2(G/H)$ is amenable.

Let $F$ and $H$ be closed subgroups of a locally compact group $G$, such that $F \subseteq H \subseteq G$. In 1972 Eymard had asked ([5], p. 55):

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Q 1. Suppose the space $G/F$ is amenable. Is then $H/F$ amenable?

This is of course a classical result in the case $F = \{e\}$.

Let $\pi$ be a strongly continuous unitary representation of $H$. In 1990 Bekka has shown [1] that, if the unitarily induced representation $\text{ind}_H^G(\pi)$ is amenable, then $G/H$ is amenable. He asked the following:

Q 2. Does amenability of $\text{ind}_H^G(\pi)$ imply that of $\pi$?

(This is a more general version of Eymard’s question Q 1.)

Q 3. Suppose both $G/H$ and $\pi$ are amenable. Is then $\text{ind}_H^G(\pi)$ amenable?

Question 3 was partially answered in the positive by Bekka himself [1] under extra assumptions on $H$. Further discussion can be found in [2].

We show that in general the answer to all three questions above is negative.

Remark. In a recent independent work Nicolas Monod and Sorin Popa have also solved Eymard’s problem (and thus Bekka’s Q 2 as well). (See their note [3] in the same issue.)

2. Reminders and simple facts

2.1. A unitary representation $\pi$ of a locally compact group $G$ in a Hilbert space $\mathcal{H}$ is said to almost have invariant vectors if for every compact $K \subseteq G$ and every $\varepsilon > 0$ there is a $\xi \in \mathcal{H}$ satisfying $\|\xi\| = 1$ and $\|\pi_g(\xi) - \xi\| < \varepsilon$ for all $g \in K$.

2.2. If a unitary representation $\pi$ of a group $G$ (viewed as discrete) almost has invariant vectors, then $\pi$ is amenable.

This follows from Corollary 5.3 of [1], because $\pi$ weakly contains the trivial one-dimensional representation, $1_G < \pi$. Alternatively, choose for every finite $K \subseteq G$ and every $\varepsilon > 0$ an almost invariant vector $\xi_{K,\varepsilon}$ as above, and set $\phi_{K,\varepsilon}(T) := \langle T\xi_{K,\varepsilon}, \xi_{K,\varepsilon} \rangle$. Every weak* cluster point, $\phi$, of the net of states $(\phi_{K,\varepsilon})$ is a $G$-invariant state.

Note that the converse of 2.2 is not true, as for example every unitary representation of an amenable group is amenable by Theorem 2.2 of [1].

2.3. Let $H$ be a closed subgroup of a locally compact group $G$. The following statements are equivalent.

(i) The homogeneous space $G/H$ is amenable.

(ii) The left quasi-regular representation $\lambda_{G/H}$ of $G$ in $L^2(G/H)$, formed with respect to any (equivalently: every) quasi-invariant measure $\nu$ on $G/H$, almost has invariant vectors.

(iii) The left quasi-regular representation $\lambda_{G/H}$ is amenable.

(iv) If $G$ acts continuously by affine transformations on a convex compact set $C$ in such a way that $C$ contains an $H$-fixed point, then $C$ contains a $G$-fixed point.
The equivalences between (i), (ii), and (iv) are due to Eymard [3], while (iii) was added by Bekka (Th. 2.3.(i) of [1]). The condition (ii) is an analogue of Reiter’s condition $(P_2)$ for homogeneous spaces. (Interestingly, a natural analogue of Følner’s condition for homogeneous spaces fails [3].)

2.4. Let the homogeneous space $G/H$ be amenable, and let $\pi$ be a strongly continuous unitary representation of $G$. If the restriction of $\pi$ to $H$ is an amenable representation, then $\pi$ is amenable as well.

To prove this statement, denote, following Bekka (Section 3 of [1]), by $X(H)$ the $C^*$-subalgebra of $B(\mathcal{H})$ formed by all operators $T$ with the property that the orbit map of the adjoint action,

$$G \ni g \mapsto \pi(g)T\pi(g)^{-1} \in B(\mathcal{H}),$$

is norm-continuous. Then $G$ acts upon $X(\mathcal{H})$ in a strongly continuous way by isometries. It follows that the dual adjoint action of $G$ on the state space of $X(\mathcal{H})$, equipped with the weak$^*$ topology, is continuous. Because of the assumed amenability of $\pi|_H$, there is an $H$-invariant state, $\psi$, on $B(\mathcal{H})$ and consequently on $X(\mathcal{H})$. Eymard’s conditional fixed point property 2.3.(iv) allows us to conclude that there is a $G$-invariant state, $\phi$, on $X(\mathcal{H})$. This in turn implies amenability of $\pi$ by Theorem 3.5 of [1].

2.5. Let $H$ be a closed subgroup of a locally compact group $G$, and let $\pi$ be a strongly continuous unitary representation of $H$ in a Hilbert space. By $\text{ind}^G_H(\pi)$ we will denote, as usual, the unitarily induced representation of $G$.

2.6. The following is Mackey’s generalization of the Frobenius Reciprocity Theorem (Theorem 5.3.3.5 of [3]).

Let $H$ be a closed subgroup of a locally compact group $G$, and let $\pi$ and $\rho$ be finite-dimensional irreducible unitary representations of $H$ and $G$, respectively. If $G/H$ carries a finite invariant measure, then $\text{ind}^G_H(\pi)$ contains $\rho$ as a discrete direct summand exactly as many times as $\rho|_H$ contains $\pi$ as a discrete direct summand.

2.7. Let $G$ be a locally compact group, and let $H, F$ be closed subgroups of $G$ such that $F \subseteq H$. Set $\pi = \text{ind}^H_F(1_F)$, where $1_F$ stands for the trivial one-dimensional representation of $F$. Then $\pi = \lambda_{H/F}$ is unitarily equivalent to the quasi-regular representation of $H$ in $L^2(H/F)$ ([3], Corollary 5.1.3.6). By the theorem on induction in stages (ibid., Proposition 5.1.3.5), the induced representation $\text{ind}^G_H(\lambda_{H/F})$ is unitarily equivalent to $\text{ind}^G_F(1_F)$, which is just the representation $\lambda_{G/F}$, the quasi-regular representation of $G$ in $L^2(G/F)$.

Now assume that the homogeneous factor space $G/F$ is amenable. By Bekka’s result mentioned above (2.3.(iii)), this amounts to the amenability of $\lambda_{G/F}$.

This argument shows that Eymard’s Q 1 is a particular case of Bekka’s Q 2.
3. First example

3.1. Let $H = F_{\infty}$, where $F_{\infty}$ denotes the free non-abelian group on infinitely many free generators $x_i$, $i \in \mathbb{Z}$. For every $n \in \mathbb{Z}$, let $\Gamma_n$ denote the normal subgroup of $F_{\infty}$ generated by $x_i$, $i \leq n$. We will set $F = \Gamma_0$.

As the factor space $H/F = F_{\infty}/\Gamma_0$ is a free group, it is not amenable.

3.2. Let the group $\mathbb{Z}$ act on $F_{\infty}$ by group automorphisms via shifting the generators:

$$\tau_n(x_m) := x_{m+n},$$

where $m, n \in \mathbb{Z}$ and $\tau$ denotes the action of $\mathbb{Z}$ on $F_{\infty}$.

Denote by $G = \mathbb{Z} \ltimes_{\tau} F_{\infty}$ the semi-direct product formed with respect to the action $\tau$. That is, $G$ is the Cartesian product $\mathbb{Z} \times F_{\infty}$ equipped with the group operation $(m,x)(n,y) = (m+n, x\tau_m y)$, the neutral element $(0,e)$ and the inverse $(n,x)^{-1} = (-n, \tau_{-n} x^{-1})$.

Let us show that the homogeneous space $G/F$ is Eymard–Greenleaf amenable.

3.3. For every $n \in \mathbb{Z}$,

$$(n,e)F = \{(n,e)(0,y) : y \in F\} = \{(n,\tau_n y) : y \in \Gamma_0\} = \{n\} \times \Gamma_n.$$

Let $S \subset F_{\infty}$ be an arbitrary finite subset. For $n \in \mathbb{Z}$ sufficiently large, $S \subset \Gamma_n$.

Now for each $(0,s) \in S$,

$$(0,s)(n,e)F = \{(0,s)(n,z) : z \in \Gamma_n\} = \{n\} \times (s\Gamma_n) = \{n\} \times \Gamma_n = (n,e)F;$$

that is, the left $F$-coset $(n,e)F \in G/F$ is $S$-invariant.

Consequently, the unit vector $\delta_{(n,e)F} \in \ell^2(G/F)$ is $S$-invariant. We have proved that the $H$-module $\ell^2(G/F)$ almost has invariant vectors, and therefore the restriction to $H$ of the left regular representation $\lambda_{G/F}$ is amenable. Since $G/H$ is an amenable homogeneous space ($H$ is normal in $G$ and the factor-group $G/H$ is isomorphic to $\mathbb{Z}$), we conclude by 2.4 that the left regular representation of $G$ in $\ell^2(G/F)$ is amenable, that is, that $G/F$ is an amenable homogeneous space.

3.4. The constructed triple of groups $F \subset H \subset G$ provides a negative answer to the question of Eymard (Q 1).

In view of the remarks in 2.7, it provides a negative answer to Bekka’s question (Q 2) as well. Take as $\pi = \text{ind}_{H/F}^H(1_F) = \lambda_{H/F}$ the left quasi-regular representation of $H$ in $\ell^2(H/F)$. While $\pi$ is not amenable, the induced representation $\text{ind}_{H/F}^G(\pi) = \lambda_{G/F}$ is amenable.
3.5. The following way of viewing our example may be instructive.

If considered as a unitary $F_\infty$-module, $\ell^2(G/F)$ decomposes, up to unitary equivalence, into the orthogonal sum

$$\ell^2(G/F) \cong \bigoplus_{n \in \mathbb{Z}} \ell^2(F_\infty/\Gamma_n).$$

None of the unitary $F_\infty$-modules $\ell^2(F_\infty/\Gamma_n)$ is amenable, yet their orthogonal sum is amenable, because the family $(\ell^2(F_\infty/\Gamma_n))_{n \in \mathbb{Z}}$ “asymptotically has invariant vectors”: every finite $S \subset F_\infty$ acts trivially on the Hilbert space $\ell^2(F_\infty/\Gamma_n)$, provided $n$ is large enough so that $S \subset \Gamma_n$. (Indeed, $\Gamma_n$ are normal in $F_\infty$.)

4. Second example

4.1. Let $G = \text{SL}(n, \mathbb{R})$ and $H = \text{SL}(n, \mathbb{Z})$, $n \geq 3$. Since $H$ is a (non-uniform) lattice in $G$ (cf. Exercise 7, §2, Chapitre VII of [3]), the homogeneous space $G/H$ is amenable.

4.2. The group $\text{SL}(n, \mathbb{Z})$ is maximally almost periodic (for instance, homomorphisms to the groups $\text{SL}(n, \mathbb{Z}_p)$, where $p$ is a prime number, separate points in $\text{SL}(n, \mathbb{Z})$). Let $\pi$ be a non-trivial (of dimension $> 1$) irreducible unitary finite-dimensional representation of $H$. Being finite-dimensional, $\pi$ is an amenable representation; cf. [1], Theorem 1.3.(i).

4.3. Applying Mackey’s Reciprocity Theorem (quoted above, [2,4]) with $\rho$ equal to $1_G$, a trivial one-dimensional representation of $G$, we conclude that the unitarily induced representation $\text{ind}_H^G(\pi)$ has no non-zero invariant vectors.

At the same time, the Lie group $G = \text{SL}(n, \mathbb{R})$, $n \geq 3$, has property (T) (see e.g. [4], Chapitre 2.a, Théorème 4), and the only amenable representations of non-compact simple Lie groups with Kazhdan’s property (T), such as $G$, are those having a non-zero invariant vector. ([1], Remark 5.10; cf. also the principal result of [2].)

4.4. We conclude: $\text{ind}_H^G(\pi)$ is non-amenable, even if the unitary representation $\pi$ and the homogeneous space $G/H$ are both amenable.

This answers in the negative another question posed by Bekka (Q 3).

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF OTTAWA, 585 KING EDWARD AVE., OTTAWA, ONTARIO, CANADA K1N 6N5.

E-mail address: vpest283@science.uottawa.ca