On intriguing sets of the Penttila-Williford association scheme

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Abstract. We investigate intriguing sets of an association scheme introduced by Penttila and Williford (2011) that was the basis for their construction of primitive cometric association schemes that are not $P$-polynomial nor the dual of a $P$-polynomial scheme. In particular, we give examples and characterisation results for the four types of intriguing sets that arise in this scheme.

1. Introduction

A celebrated result of Beniamino Segre [13, n. 91] is that if there exists a nonempty proper subset $S$ of the point set $P$ of the elliptic quadric $Q^-(5, q)$, $q$ odd, such that every generator meets $S$ in a constant number $m$ of points, then $m = (q + 1)/2$. Such a set $S$ is called a hemisystem, and by the work of Cossidente and Penttila [7], these configurations exist for every odd prime power $q$. Hemisystems garnered interest from the algebraic combinatorics community, as they give rise to cometric $Q$-antipodal association schemes [14]. Segre’s result was extended to $q$ even by Bruen and Hirschfeld [3], who showed that no such subsets $S$ can exist. Thus for $q$ even, there is an absence of such interesting configurations. However, Penttila and Williford [12] introduced the notion of a relative hemisystem of $Q^-(5, q)$, $q > 2$ even, which give rise to atypical and rare association schemes; primitive cometric association schemes that do not arise from distance regular graphs. Their idea was to consider a non-tangent hyperplane $H$ for $Q^-(5, q)$ and to only regard the points $X$ that lie outside of $H$. Now, a nonempty proper subset $S$ of $X$ such that every generator meets $S$ in a constant number $m$ of points must have $m = q/2$, according to [1, Theorem 1]. In particular, $q$ is even. Such a configuration is a relative hemisystem.

In the background, there is a 4-class cometric association scheme, which we call the Penttila-Williford scheme. It arises from taking the natural relations that are invariant under the stabiliser of $H$ in the full similarity group of $Q^-(5, q)$, and it exists for all prime powers $q$, odd or even (but greater than 2). Table 1 summarises the relations of this scheme, and full details will be given in Section 3.

| Relations | Description |
|-----------|-------------|
| $R_0$     | equality    |
| $R_1$     | noncollinear but collinear to the conjugate of the other |
| $R_2$     | noncollinear and noncollinear to the conjugate of the other |
| $R_3$     | collinear and not conjugate |
| $R_4$     | conjugate   |

Table 1. The Penttila-Williford Scheme

A relative hemisystem, if it exists, provides an example of an intriguing set for this association scheme, in the language of De Bruyn and Suzuki [9]. Little is known about other intriguing sets for this association scheme, apart from the devices used in Melissa Lee’s MPhil thesis [10] and in [1]. In this paper, we begin an investigation into the various intriguing sets for this scheme and we derive characterisation results.

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Since there are four non-principal eigenspaces, there are four ‘types’ \( i \in \{1, 2, 3, 4\} \) of intriguing sets depending on the index \( i \) for which its characteristic vector belongs to the sum \( V_0 \oplus V_1 \) of eigenspaces for the Penttila-Williford scheme. There is a particular involution \( \sigma \) that acts fixed-point-freely on \( X \), and we say that two points \( x, y \in X \) are conjugate if \( x = y^\sigma \). A subset \( S \) of \( X \) is \( \sigma \)-invariant, if \( x^\sigma \in X \) for all \( x \in X \). If an intriguing set \( S \) is not \( \sigma \)-invariant, then the image \( S^\sigma \) of \( S \) under \( \sigma \) is disjoint \( S \), and we shall see that \( |S| = \frac{1}{2}|X| \).

**Theorem 1.1** (Paraphrase of Theorem 3.3). Suppose that \( Y \) is an intriguing set of type \( i \), where \( i \in \{1, 2, 3, 4\} \).

(a) If \( i = 1 \) or \( i = 3 \), then \( |Y| = \frac{1}{2}|X| \) and \( |\{p, p^\sigma\} \cap Y| = 1 \) for all \( p \in X \).
(b) If \( i = 2 \) or \( i = 4 \), then \( Y^\sigma = Y \).

We show in Section 4 that an intriguing set of type 2 has size at least \( 4(q + 1) \) (see Lemma 4.9), and this bound is sharp. In Theorem 4.12, we characterise the smallest examples for when \( q \geq 59 \). We also show in Section 5 that an intriguing set of type 4 has size at least \( q^2(q - 1) \), and this bound is sharp. The following result characterises the smallest examples.

**Theorem 1.2** (Paraphrase of Theorem 5.5). Suppose \( Y \) is an intriguing set of type 4 of the Penttila-Williford scheme. Then \( |Y| \geq q^2(q - 1) \), and in the case of equality, there exists a point \( p \in H \cap Q^{-}(5, q) \) such that \( Y \) consists of the \( q^2(q - 1) \) points of \( Q^{-}(5, q) \setminus H \) that are collinear to \( p \).

**2. Preliminaries**

Let \( \Gamma = (X, R) \) be a finite connected regular graph, and we will also assume throughout that \( \Gamma \) is nontrivial; neither empty nor complete. A subset \( Y \) of \( X \) is called an intriguing set if and only if there are integers \( h_1, h_2 \geq 0 \) such that every vertex of \( Y \) is adjacent to exactly \( h_1 \) vertices of \( Y \) and every vertex of \( X \setminus Y \) is adjacent to exactly \( h_2 \) vertices of \( Y \). We will use the symbol \( j \) to denote the ‘all-ones’ row vector. We always assume a given numbering \( X = \{x_1, \ldots, x_n\} \) of the vertices of \( X \) where \( n = |X| \). The adjacency matrix of \( \Gamma \) is the real \((n \times n)\)-matrix \( A \) with \( A_{ij} = 1 \) when \( x_i \) and \( x_j \) are adjacent and \( A_{ij} = 0 \) otherwise. Notice that \( A \) is a real symmetric matrix, so the row space \( \mathbb{R}^n \) decomposes into the orthogonal sum of the eigenspaces of \( A \), which we also call the eigenspaces of \( \Gamma \). For every subset \( Y \) of \( X \) we denote by \( \chi_Y \) its characteristic vector, that is the real row vector of length \( n \) whose \( i \)-th entry is 1, if \( x_i \in Y \), and 0 otherwise. The following theorem is from [9].

**Result 2.1** ([9]). Let \( \Gamma = (X, R) \) be a finite connected nontrivial regular graph with eigenspaces \( V_0, \ldots, V_s \) where \( V_0 = \langle j \rangle \) and adjacency matrix \( A \). Let \( Y \) be a non-empty proper subset of \( X \).

(i) \( Y \) is an intriguing set of \( \Gamma \) if and only if \( \chi_Y \in V_0 \oplus V_i \) for some integer \( 1 \leq i \leq s \).

(ii) If \( Y \) is an intriguing set, \( i \geq 1 \) is the integer with \( \chi_Y \in V_0 \oplus V_i \), and if \( \theta \) is the eigenvalue of \( A \) on \( V_i \), then every vertex of \( X \setminus Y \) is adjacent to exactly

\[
\frac{k - \theta}{|X|} \cdot |Y|
\]

elements of \( Y \) and every vertex of \( Y \) is adjacent to exactly

\[
\theta + \frac{k - \theta}{|X|} \cdot |Y|
\]

elements of \( Y \).

If \( Y \) is an intriguing set with \( Y \neq \emptyset \), and \( i \) is the unique index such that \( \chi_Y \in V_0 \oplus V_i \), then we say that \( Y \) is an intriguing set of type \( i \), or for the eigenspace \( V_i \). Notice that \( \chi_Y = c_j + v \) with \( c := |Y|/|X| \) and \( v \in V_i \). Since eigenvectors for distinct eigenvalues have inner product zero, the following well-known result follows easily.

**Result 2.2.** Let \( \Gamma = (X, R) \) be a finite connected regular graph. Let \( Y_1 \) and \( Y_2 \) be intriguing sets of \( \Gamma \) for different eigenspaces. Then \( |Y_1 \cap Y_2| \cdot |X| = |Y_1| \cdot |Y_2| \).

We give an example that will be used later.
Example 2.3. The collinearity graph of $Q^-(5, q)$ is a strongly regular graph with eigenvalues $(q^2 + 1)q$, $q - 1$ and $-(q^2 + 1)$ where $(q^2 + 1)q$ has eigenspace $V_0 := \langle j \rangle$. Intriguing sets for the eigenvalue $-(q^2 + 1)$ are hemisystems (by Segre’s theorem), that is, sets of points of size $(q^3 + 1)(q+1)/2$ such that every generator meets in $(q+1)/2$ elements. The intriguing sets for the eigenvalue $q - 1$ are usually called ‘tight sets’ in the literature. It follows from the above results that for any tight set $Y$ we have $|Y| = c(q + 1)$, where $c$ is the number of points in $Y$ collinear to a given point not in $Y$. There exist many different examples of tight sets of $Q^-(5, q)$. The $q + 1$ points of a line form a tight set. The union of two perpendicular conics is a tight set. Moreover, any union of disjoint tight sets is a tight set, so a large number of examples can be obtained by taking a disjoint union of lines and pairs of perpendicular conics. If $c$ is sufficiently small, it was proven in [11, Theorem 2.15] that this construction describes all tight sets of size $c(q + 1)$.

3. The Penttila–Williford scheme

We consider a finite 5-dimensional projective space $P = PG(5, q)$, $q > 2$, and in there, an elliptic quadric $Q = Q^-(5, q)$. We let $\perp$ be the associated polarity of $P$ whose absolute points are the points of $Q$. Let $H$ be a hyperplane of $P$ meeting $Q$ in a parabolic quadric $Q(4, q)$.

We consider the point set $X := Q \setminus (Q \cap H)$ obtained from $Q$ by removing the points of $H \cap Q$. Note that $|X| = q^2(q^2 - 1)$. We define $\sigma: X \to X$ as follows. Let $p$ be a point of $Q$ not in $H$. Then the line joining $H^\perp$ and $p$ is a hyperbolic line and so contains a unique second point $p'$ of $Q$. Let $\sigma$ be the central collineation of $PG(5, q)$ having axis $H$ and centre $H^\perp$, mapping $p$ to $p'$. Then $\sigma$ commutes with the polarity $\perp$ and so stabilises $Q$. Moreover, since $pp'$ has only two points of $Q$ on it, we have $\sigma(p') = p$ and hence $\sigma^2$ is the identity. Via the Klein correspondence, we can map the points of $X$ to lines of the Hermitian surface $H(3, q^2)$ that lie outside of a symplectic subgeometry $W(3, q)$. This subgeometry can alternatively be defined as the fixed subspaces of a Baer involution, and this involution corresponds to our map $\sigma$.

Two points $p, p'$ of $Q$ are said to be collinear when $p \neq p'$ and $p' \in q^\perp$, that is, $p$ and $p'$ are distinct points that span a line of $PG(5, q)$ contained in $Q$. In this case we write $p \sim p'$. On $X$ the following symmetric relations $R_i$, $0 \leq i \leq 4$, define a cometric association scheme on $X$ (see [12, Theorem 1] and [10, Section 5.4]).

$$R_0 = \{(u, v) \in X \times X \mid u = v\},$$
$$R_1 = \{(u, v) \in X \times X \mid v \neq u \sim v^\sigma\},$$
$$R_2 = \{(u, v) \in X \times X \mid v \neq u \not\sim v^\sigma\},$$
$$R_3 = \{(u, v) \in X \times X \mid v \sim u \not\sim v^\sigma\},$$
$$R_4 = \{(u, v) \in X \times X \mid u \sim v^\sigma\}.$$

We will frequently refer to the ‘association scheme $X$’, which will mean the association scheme on $X$ given by the above relations. The matrix of eigenvalues $P$ and matrix of dual eigenvalues $Q$ of this association scheme are as follows:

$$P = \begin{pmatrix}
1 & (q - 1) & (q^2 + 1) & (q - 2) & q & (q^2 + 1) & (q - 1) & (q^2 + 1) & 1 \\
1 & q^2 + 1 & 0 & - (q^2 + 1) & -1 \\
1 & q - 1 & -2q & q - 1 & 1 \\
1 & -(q - 1) & 0 & q - 1 & -1 \\
1 & -(q - 1)^2 & 2 & (q - 2) & q & -(q - 1)^2 & 1
\end{pmatrix},$$

$$Q = \begin{pmatrix}
1 & \frac{(q - 1)^2 q}{2} & \frac{(q - 2)(q + 1)(q^2 + 1)}{2} & \frac{(q - 1)q(q^2 + 1)}{2} & \frac{q(q^2 + 1)}{2} \\
1 & \frac{(q - 1) q}{2} & \frac{(q - 2)(q + 1)}{2} & \frac{(q - 1)q}{2} & \frac{q}{2} \\
1 & 0 & -q - 1 & 0 & q \\
1 & \frac{(q - 1)q}{2} & \frac{(q - 2)(q + 1)}{2} & \frac{(q - 1)q}{2} & \frac{q(q^2 + 1)}{2} \\
1 & \frac{(q - 1)^2 q}{2} & \frac{(q - 2)(q + 1)(q^2 + 1)}{2} & \frac{(q - 1)q}{2} & \frac{q(q^2 + 1)}{2}
\end{pmatrix}.$$
If $A_i$ denotes the adjacency matrix of the graph $(X, R_i)$, this means that we can number the eigenspaces $V_0, \ldots, V_4$ of the association scheme in such a way that $A_i v = P_{ij} v$ for $v \in V_i$, where the entries of $P$ are numbered $P_{ij}$ with $i, j = 0, \ldots, 4$. Notice that this implies that $V_0 = \langle \chi_X \rangle$.

The reason to exclude the case when $q = 2$ is that the relation $R_2$ is empty when $q = 2$ as can be seen from the middle entry in the first line of $P$.

Let $Y$ be a non-empty proper subset of $X$. We call $Y$ an intriguing set of this association scheme, if $\chi_Y \in V_0 \oplus V_i$ for some $i \in \{1, 2, 3, 4\}$. Result 2.1 shows that an intriguing set $Y$ of the association scheme is an intriguing set of each graph $(V, R_i)$.

**Example 3.1 (Intriguing sets of type 1).** It was shown in [1] that an intriguing set of type 1, if it exists, is a relative hemisystem and $q$ is even. Moreover, Penttila and Williford [12] provided an infinite family of examples that exist for each even prime power $q > 2$. Further examples were constructed in [2, 4, 5, 6].

**Example 3.2 (Some intriguing sets of type 4).** Let $w$ be a point of $H \cap Q$ and let $Y$ be the set consisting of all points of $X$ that are in $Q$ collinear to $w$. Then $|Y| = (q^2 - q)q$, every point of $Y$ is collinear to $q - 1$ points of $Y$ and every point of $X \setminus Y$ is collinear to $q^2 - q$ points of $Y$. Hence $Y$ is an intriguing set of $(X, R_3)$ corresponding to the eigenvalue $q - 1 - (q^2 - q) = -(q - 1)^2$. Since $V_4$ is the eigenspace of this graph for the eigenvalue $-(q - 1)^2$, it follows that $\chi_Y \in V_0 \oplus V_4$, so $Y$ is also an intriguing set of the association scheme $X$.

Examples of intriguing sets of types 2 and 3 will appear in Section 4. In the following result, we show that an intriguing set is either $\sigma$-invariant or has its image under $\sigma$ disjoint from it.

**Theorem 3.3.** Suppose that $Y$ is a subset of $X$ such that $\chi_Y \in V_0 \oplus V_i$ for some $i \in \{1, 2, 3, 4\}$.

(a) If $i = 1$ or $i = 3$, then $|Y| = \frac{1}{2} |X|$ and $|\{p, p^\sigma\} \cap Y| = 1$ for all $p \in X$.

(b) If $i = 2$ or $i = 4$, then $Y^\sigma = Y$.

**Proof.** Let $\theta$ be the eigenvalue of the relation $R_3$ on $V_i$. Result 2.1 shows that every point of $X \setminus Y$ is in relation $R_3$ to $a_3 := \frac{|Y|}{|X|} (k - \theta)$ points of $Y$ (where $k$ is the valency of $R_3$), and every point of $Y$ is in relation $R_3$ to $b_3 := a_3 + \theta$ points of $Y$. Inspection of the first matrix of eigenvalues shows that $(-1)^i \theta$ is the eigenvalue of $R_1$ on $V_i$. Result 2.1 thus shows that every point of $X \setminus Y$ is in relation $R_1$ to $a_1 := \frac{|Y|}{|X|} (k - (-1)^i \theta)$ points of $Y$, and every point of $Y$ is in relation $R_1$ to $b_1 := a_1 + (-1)^i \theta$ points of $Y$. Hence, if $p \in X \setminus Y$, then $p^\sigma$ is collinear to $a_1$ points of $Y$, and for every point $p \in Y$, the point $p^\sigma$ is collinear to exactly $b_1$ points of $Y$. Since $\theta \neq 0$, we have $a_1 \neq b_1$ and $a_3 \neq b_3$.

If $i$ is even, that is $i = 2$ or $i = 4$, then $a_1 = a_3$ and $b_1 = b_3$, and it follows immediately that for every point $p$, both points $p$ and $p^\sigma$ are in relation $R_3$ to the same number of points of $Y$, hence both points lie in $Y$ or both points do not. Hence $Y = Y^\sigma$ in this case.

If $i$ is odd, then $\theta = -(q^2 + 1)$ or $\theta = q - 1$ and an easy calculation gives $a_1 = b_3$ and $a_3 = b_1$. This time it follows for every point $p \in X$ that $p$ and $p^\sigma$ are in relation $R_3$ to a different number of points of $Y$ and hence, exactly one of these two points lies in $Y$.

As a corollary, we obtain the following result of [1].

**Theorem 3.4.** If $Y$ is a non-trivial $m$-cover of $H(3, q^2) \setminus W(3, q)$, then $q$ is even, $m = \frac{q}{2}$, and from any pair of conjugate lines exactly one lies in $Y$.

**Proof.** We do this in the dual setting, that is we consider a set $Y$ of points of $Q^-(5, q) \setminus Q(4, q)$ such that every line of $Q^-(5, q)$ that is not contained in $Q(4, q)$ has exactly $m$ points in $Y$. Hence, in the graph $(X, R_3)$, every point of $Y$ is collinear to exactly $(q^2 + 1)(m - 1)$ points of $Y$, and every point of $X \setminus Y$ is collinear to exactly $(q^2 + 1)m$ points of $Y$. Therefore Result 2.1 implies that $Y$ is an intriguing set for the eigenvalue $-(q^2 + 1)$ of $(X, R_3)$. Since the eigenspace of the graph $(X, R_3)$ for this eigenvalue is $V_1$, which is an eigenspace of the association scheme, it follows that $Y$ is an intriguing set of the scheme. Theorem 3.3 shows that $|Y| = |X|/2$ and that $Y$ contains exactly one point of every pair of conjugate points. Since each point of $X \setminus Y$ is collinear to $m(q^2 + 1)$ points of $Y$, part (ii) of Result 2.1 gives $m = q/2$.  □
4. A connection to tight sets of $Q^-(5, q)$

In this section, we consider intriguing sets of the graph $(X, R_3)$ corresponding to the eigenvalue $q - 1$. Notice that the eigenspace for this eigenvalue of $(X, R_3)$ is $V_2 \oplus V_3$. Hence a subset $Y$ of $X$ is an intriguing set of $(X, R_3)$ if and only if $\chi_Y \in V_0 \oplus V_2 \oplus V_3$. We will investigate when $Y$ is also an intriguing set of the association scheme $X$, that is when $\chi_Y \in V_0 \oplus V_2$ or $\chi_Y \in V_0 \oplus V_3$.

By Result 2.1, every point $p \in X \setminus Y$ is collinear to exactly $\alpha := |Y|/(q + 1)$ points of $X$ and every point of $Y$ is collinear to exactly $q - 1 + \alpha$ points of $Y$. Recall that $X$ is the set of points of the elliptic quadric $Q$ that lie outside the non-tangent hyperplane $H$ of the ambient projective space.

**Lemma 4.1.** If $Y$ is an intriguing set of the graph $(X, R_3)$ corresponding to the eigenvalue $q - 1$, then $Y$ is a tight set of $Q^-(5, q)$.

**Proof.** Let $p$ be a point of $H \cap Q$. We have seen in Example 3.2 that the set $\mathcal{F}$ consisting of the $q^2(q - 1)$ points of $X$ that are in $Q$ collinear to $p$ is an intriguing set of the association scheme with $\chi_{\mathcal{F}} \in V_0 \oplus V_4$. Applying Result 2.2 to the intriguing sets $Y$ and $\mathcal{F}$ of $(X, R_3)$, it follows that

$$|Y \cap \mathcal{F}| = \frac{|Y| \cdot |\mathcal{F}|}{|X|} = \frac{|Y|}{q + 1}.$$  

Hence $p$ is collinear to exactly $|Y|/(q + 1)$ points of $Y$. Since this is true for all points of $H \cap Q$ and since the points of $X \setminus Y$ are collinear to the same number $|Y|/(q + 1)$ points of $Y$, it follows that $Y$ is an intriguing set of the collinearity graph of $Q^-(5, q)$ for the eigenvalue $q - 1$. Hence $Y$ is a tight set of $Q^-(5, q)$; see Example 2.3.

**Remark 4.2.** The algebraic reason behind this phenomenon is the following. If $\mathcal{P}$ is the set of points of $Q^-(5, q)$, then we have the following decomposition into eigenspaces of the collinearity relation:

$$\mathbb{C}\mathcal{P} = \langle \chi_{\mathcal{P}} \rangle \oplus V^+ \oplus V^-$$

| Eigenspace | $\langle \chi_{\mathcal{P}} \rangle$ | $V^+$ | $V^-$ |
|------------|-----------------|-------|-------|
| Dimension  | 1               | $q^2(q^2 + 1)$ | $q(q^2 - q + 1)$ |

Whereas for the association scheme on $X$, we have

$$\mathbb{C}X = \langle \chi_X \rangle \oplus V_1 \oplus V_2 \oplus V_3 \oplus V_4$$

| Eigenspace | $\langle \chi_X \rangle$ | $V_1$ | $V_2$ | $V_3$ | $V_4$ |
|------------|-----------------|-------|-------|-------|-------|
| Dimension  | 1               | $q(q-1)^2$ | $(q-2)(q+1)(q^2+1)$ | $(q-1)q(q^2+1)$ | $q(q^2+1)$ |

We have two linear maps: (1) the natural inclusion map $\iota: \mathbb{C}X \rightarrow \mathbb{C}\mathcal{P}$; (2) the projection map $\rho: \mathbb{C}\mathcal{P} \rightarrow \mathbb{C}X$ which maps $\sum_{x \in \mathcal{P}} \alpha_x \chi_x$ to $\sum_{x \in X} \alpha_x \chi_x$. Note that $\iota$ is injective, $\rho$ is surjective, and $\rho \circ \iota$ is the identity on $\mathbb{C}X$. The kernel of $\rho$ is clearly $K := \langle \chi_h : h \in H \cap \mathcal{P} \rangle$. Let $S$ be the $|X| \times |\mathcal{P}|$ inclusion matrix for $X$ in $\mathcal{P}$. Let $(A_0, A_1, A_2, A_3, A_4)$ be the adjacency matrices for the association scheme on $X$, and let $(B_0, B_1, B_2)$ be the adjacency matrices for the association scheme on $\mathcal{P}$, where $B_1$ represents the collinearity relation. So $SB_0S^T = A_0$, $SB_1S^T = A_3$, and $SB_2S^T = A_1 + A_2 + A_4$.

From the matrix of dual eigenvalues\(^1\) for the association scheme on $\mathbb{C}\mathcal{P}$, we have

$$(q + 1)^2E^- = qB_0 - B_1 + \frac{1}{q}B_2$$

and hence

$$SE^-S^T = \frac{1}{(q + 1)^2} \left( qA_0 - A_3 + \frac{1}{q}(A_1 + A_2 + A_4) \right)$$

$$= \frac{1}{q^2(q^2 - 1)} \left( \frac{q(q-1)^2}{2}A_0 + \frac{q(q-1)}{2}A_1 - \frac{q(q-1)}{2}A_1 + \frac{q(q-1)}{2}A_3 - \frac{q(q-1)^2}{2}A_4 \right)$$

$$+ \frac{1}{q^2(q^2 + 1)^2} \left( \frac{q(q^2+1)}{2}A_0 - \frac{q(q-1)}{2}A_1 + qA_2 - \frac{q(q-1)}{2}A_3 + \frac{q(q^2+1)}{2}A_4 \right)$$

$$= E_1 + \frac{1}{q^2}E_4.$$  

\(^1\)which is

$$\begin{bmatrix}
1 & q(q^2 + 1) & q^2q^2 - q + 1 \\
1 & q(q-1) & -(q^2 - q + 1) \\
1 & -\frac{1}{q}(q^2 + 1) & \frac{1}{q}(q^2 - q + 1)
\end{bmatrix}.$$
Therefore, $\rho(V^-)$ contains the rowspace of $E_1 + \frac{q-4}{q+1}E_4$. Since $E_1 E_4 = 0$, the rowspace of $E_1 + \frac{q-4}{q+1}E_4$ is the sum of the rowspaces of $E_1$ and $E_4$; that is, $V_1 \perp V_4$. Hence $\rho(V^-)$ contains $V_1 \perp V_4$. As $\dim(V^-) = \dim(V_1) + \dim(V_4)$, it follows that $\rho$ restricted to $V^-$ is a bijective map from $V^-$ onto $V_1 \perp V_4$.

For each $v \in \langle \chi_X \rangle \perp V_2 \perp V_3$, we have

$$\langle v E^\top \rangle (v E^\top) = \langle v E^\top \rangle (v E^\top) \top = v (E^\top (v E^\top)) \top = v (E^\top E (v E^\top)) \top = v (E^\top E) v \top = 0.$$

Therefore, $\rho(v) = 0$ and consequently $\rho(v) \in \langle \chi_P \rangle \perp V^+$. Hence $\rho$ maps $\langle \chi_X \rangle \perp V_2 \perp V_3$ into $\langle \chi_P \rangle \perp V^+$, which affirms Lemma 4.1. So we have the following diagram:

Now we investigate when $Y$ is an intriguing set of the association scheme $X$, that is when $\chi_Y \in V_0 \perp V_2$ or $\chi_Y \in V_0 \perp V_3$. We already know that $\chi_Y \in V_0 \oplus V_2$ implies $Y = Y^\sigma$ and that $\chi_Y \in V_0 \oplus V_3$ implies $Y \cap Y^\sigma = \emptyset$. The following observation is crucial in proving that the converse is also true.

**Lemma 4.3.** We have $\langle \chi_p - \chi_p^\sigma \mid p \in X \rangle = V_1 \oplus V_3$ and $\langle \chi_p + \chi_p^\sigma \mid p \in X \rangle = V_0 \oplus V_2 \oplus V_4$.

**Proof.** We number the rows and columns of $Q$ from 0 to 4. Inspection of $Q$ shows that $Q_{ij} = (-1)^{i+j}Q_{i-j}$ for all $i, j = 0, \ldots, 4$. The definition of the relations $R_i$ shows that $A_i \chi_p^\sigma = A_{i+1} \chi_p$. It follows for each $j \in \{0, 1, 2, 3, 4\}$ that

$$E_j \chi_p^\sigma = \frac{1}{|X|} \sum_i Q_{ij} A_i \chi_p^\sigma = \frac{1}{|X|} \sum_i (-1)^{i} Q_{ij} A_{i-1} \chi_p^\sigma = (-1)^{i} E_j \chi_p.$$

Hence, for each $p \in X$ we have $E_j (\chi_p - \chi_p^\sigma) = 0$ for $j = 0, 2, 4$ and $E_j (\chi_p + \chi_p^\sigma) = 0$ for $j = 1, 3$. Hence $\langle \chi_p - \chi_p^\sigma \mid p \in X \rangle \subseteq V_1 \oplus V_3$ and $\langle \chi_p + \chi_p^\sigma \mid p \in X \rangle \subseteq V_0 \oplus V_2 \oplus V_4$. Since $\{\chi_p \mid p \in X\}$ forms a basis of $V_0 \oplus V_1 \oplus V_2 \oplus V_3 \oplus V_4$, we have equality.

**Lemma 4.4.** If $S \subseteq X$, then $S = S^\sigma$ if and only if $\chi_S \in V_0 \oplus V_2 \oplus V_4$.

**Proof.** We have $S = S^\sigma$ if and only if $$(\chi_p - \chi_p^\sigma) \cdot \chi_S = 0$$
for all $p \in \mathcal{P}$. Therefore the statement follows from Lemma 4.3.

**Theorem 4.5.** Let $Y$ be a proper nonempty subset of $X$ with $\chi_Y \in V_0 \oplus V_2 \oplus V_3$, that is, $Y$ is an intriguing set of $(X, R_3)$. Then $\chi_Y \in V_0 \oplus V_2$ if and only if $Y^\sigma = Y$, and $\chi_Y \in V_0 \oplus V_3$ if and only if $Y$ contains exactly one of every two conjugate points of $X$.

**Proof.** If we assume that $\chi_Y \in V_0 \oplus V_2$ or $\chi_Y \in V_0 \oplus V_3$, then Theorem 3.3 proves the statement. If we assume that $Y^\sigma = Y$, then $\chi_Y \in V_0 \oplus V_2 \oplus V_1$ by Lemma 4.4 and hence $\chi_Y \in V_0 \oplus V_2$. Finally assume that $Y$ contains one point of any pair of conjugate points. Then $(\chi_p + \chi_p^\sigma) \cdot (2 \chi_Y - j) = 0$ for every point $p \in X$. Therefore Lemma 4.3 shows that $2 \chi_Y - j \in V_1 \oplus V_3$. Hence $2 \chi_Y - j \in V_3$, which shows that $\chi_Y \in V_0 \oplus V_3$. 

\[ \square \]
Intriguing sets of type 3. We construct an intriguing set \( Y \) of the association scheme \( X \) with \( \chi_Y \in \mathbb{V}_0 + \mathbb{V}_3 \). Recall that \( X = \mathcal{Q} \setminus (H \cap \mathcal{Q}) \) where \( \mathcal{Q} \) is an elliptic quadric \( \mathcal{Q} \) in \( \text{PG}(5,q) \) and \( H \) is a non-tangent hyperplane of \( \text{PG}(5,q) \), that is a hyperplane that meets \( \mathcal{Q} \) in a parabolic quadric \( \mathcal{Q}(4,q) \).

Example 4.6 (Intriguing sets of type 3, \( q \) even).
Assume \( q \) is even and \( q > 2 \). Consider a solid \( S \) of \( H \) such that \( S \cap \mathcal{Q} \) is a hyperbolic quadric. Then \( S^\perp \) is an elliptic line of \( \text{PG}(5,q) \) and \( H^\perp \) is a point on this line. Since \( q \) is even, \( H^\perp \in H \) and \( \sigma \) has \( q/2 \) orbits of length two in its action on the remaining \( q \) points of \( S^\perp \).

Consider \( p \in S^\perp \) with \( p \neq H^\perp \). Then \( p^\perp \) is a non-tangent hyperplane to \( \mathcal{Q} \) and hence \( p^\perp \cap \mathcal{Q} \) is a parabolic quadric \( \mathcal{Q}(4,q) \) of \( \mathcal{Q} \). This implies that \( p^\perp \cap \mathcal{Q} \) is a tight set of \( \mathcal{Q} \). As \( p \in S^\perp \) we have \( p^\perp \cap H = S \). As \( S \cap \mathcal{Q} \) is a hyperbolic quadric \( \mathcal{Q}^\perp(3,q) \), which is the union of its lines of each of its reguli, the set \( S \cap \mathcal{Q} \) is also a tight set of \( \mathcal{Q} \) and it is contained in \( p^\perp \cap \mathcal{Q} \). Hence \( T_p := (p^\perp \setminus S) \cap \mathcal{Q} \) is a tight set of \( \mathcal{Q} \). For different points \( p \) the different hyperplanes \( p^\perp \) meet within \( S \) and hence the corresponding tight sets \( T_p \) are disjoint. As \( \sigma \) has order two and \( H^\perp \) is the only point of \( S^\perp \) fixed by \( \sigma \), we can partition the set of \( q \) points of \( S^\perp \) into two sets \( M \) and \( M' \) of cardinality \( q/2 \) such that \( M^\sigma = M' \). Then the union \( T_M \) of the tight sets \( T_p \) with \( p \in M \) is a tight set of \( \mathcal{Q} \) and the union \( T_{M'} \) of the tight sets \( T_p \) with \( p \in M' \) is a tight set of \( \mathcal{Q} \). Moreover every point of \( X \) is contained in exactly one of the two sets \( T_M \) and \( T_{M'} \). Since \( M^\sigma = M' \), we have \( (T_M)^\sigma = T_{M'} \). Theorem 4.5 now shows that \( T_M \) is an intriguing set of \( X \) with \( \chi_{T_M} \in \mathbb{V}_0 + \mathbb{V}_3 \).

Example 4.7 (Intriguing sets of type 3, \( q \) odd). Here we assume that \( q \) is odd, and hence \( H^\perp \notin H \).
Let \( m \) be a line of \( H \cap \mathcal{Q} \). Then \( H^\perp \) and \( m \) generate a plane \( \pi \) that meets the quadric \( \mathcal{Q} := \mathcal{Q} \) only in the line \( m \). Set \( z = H^\perp \) and let \( u \) be a point of \( m \). Then \( (zu)^\perp \) is a 3-space that is contained in \( H \) and meets \( \mathcal{Q} \) in a cone \( C \) with vertex \( u \) over a conic. In particular \( |C| = q^2 + q + 1 \).

Let \( S_1 \) be any set consisting of \( \frac{q}{2}(q - 1) \) points of the line \( uz \) such that \( u, z \notin S_1 \), let \( S_2 \) be any set consisting of \( \frac{q}{2}(q - 1) \) lines of \( \pi \) on \( u \) such that \( uz, m \notin S_2 \), and let \( T \) be the set consisting of all points of \( \mathcal{Q} \setminus H \) that are perpendicular to a point in \( S_1 \) or to a line in \( S_2 \). For \( y \in S_1 \), the hyperplane \( y^\perp \) meets \( \mathcal{Q} \) in a quadric \( \mathcal{Q}(4,q) \), and \( y^\perp \cap H \) meets \( \mathcal{Q} \) in the cone \( C \). For \( \ell \in S_2 \), the solid \( \ell^\perp \) meets \( \mathcal{Q} \) in a cone with vertex \( u \) over a conic, and \( \ell^\perp \cap H \) meets \( \mathcal{Q} \) in the line \( m \). It follows that

\[
|T| = \frac{q - 1}{2}(q^2 + q + 1 - |C|) + \frac{q - 1}{2}(q^2 + q + 1 - |m|) = \frac{1}{2}q^2(q^2 - 1).
\]

Let \( p \) be a point of \( X \). We want to determine \( |p^\perp \cap T| \). First notice that \( p^\perp \) meets the line \( uz \) in exactly one point and this point is different from \( z \). Also, if \( p^\perp \cap uz \) is the point \( u \), then \( p^\perp \cap \pi \) is one of the lines of \( \pi \) on \( u \) that is different from \( m \) and \( uz \). For \( y \in S_1 \) we have \( y^\perp \cap H \cap \mathcal{Q} = C \) and thus

\[
|p^\perp \cap y^\perp \cap T| = |p^\perp \cap y^\perp \cap \mathcal{Q}| - |p^\perp \cap C|.
\]

For \( \ell \in S_2 \) we have \( \ell^\perp \cap H \cap \mathcal{Q} = m \) and thus

\[
|p^\perp \cap \ell^\perp \cap T| = |p^\perp \cap \ell^\perp \cap \mathcal{Q}| - |p^\perp \cap m|.
\]

As \( |S_1| = |S_2| = \frac{1}{2}(q - 1) \), it follows that

\[
|p^\perp \cap T| = \sum_{y \in S_1} |p^\perp \cap y^\perp \cap T| + \sum_{\ell \in S_1} |p^\perp \cap \ell^\perp \cap T| = \sum_{y \in S_1} |p^\perp \cap y^\perp \cap \mathcal{Q}| + \sum_{\ell \in S_1} |p^\perp \cap \ell^\perp \cap \mathcal{Q}| - \frac{1}{2}(q - 1)(|p^\perp \cap C| + |p^\perp \cap m|).
\]

Case 1. \( p^\perp \cap uz \notin S_1 \cup \{u\} \):

Then \( p^\perp \cap C \) is a conic and \( p^\perp \cap m \) is a single point. If \( y \in S_1 \), then \( y^\perp \cap \mathcal{Q} \) is a \( \mathcal{Q}(4,q) \) not containing \( p \), so \( p^\perp \) meets this \( \mathcal{Q}(4,q) \) in a \( \mathcal{Q}^\perp(3,q) \). If \( \ell \in S_2 \), then \( \ell^\perp \cap \mathcal{Q} \) is a cone with vertex \( u \) over a conic and since \( p \notin \ell^\perp \), the hyperplane \( p^\perp \) meets this cone in a conic. Hence

\[
|p^\perp \cap T| = |S_1|(q^2 + 1) + |S_2|(q + 1) - \frac{1}{2}(q - 1)((q + 1) + 1) = \frac{1}{2}(q - 1)q^2.
\]
Case 2. $p^\perp \cap uz$ is a point $y_0$ of $S_1$:
The only difference to Case 1 occurs for the point $y_0 \in S_1$, namely $p^\perp \cap y_0^\perp$ is a solid that meets $Q$ in a cone with vertex $p$ over a conic, so $|p^\perp \cap y_0^\perp \cap Q| = q^2 + q + 1$. It follows that $|p^\perp \cap T| = \frac{1}{2}(q - 1)q^2 + q$.

Case 3. $p^\perp \cap uz = \{u\}$ and the line $p^\perp \cap \pi$ is not in $S_2$:
Then $p^\perp \cap C = p^\perp \cap m = \{u\}$. If $y \in S_1$, then $p^\perp \cap y^\perp \cap Q$ is a $Q^-(3, q)$ as before. As $pu$ is a line of $Q$ we have $(pu)^\perp \cap Q = pu$. If $\ell \in S_2$, then $\ell$ and $p$ are not perpendicular and therefore $\ell^\perp \cap p^\perp \cap Q = \ell^\perp \cap (pu)^\perp \cap Q = \ell^\perp \cap pu = \{u\}$. Hence $|p^\perp \cap T| = \frac{1}{2}(q - 1)q^2$.

Case 4. $p^\perp \cap uz = \{u\}$ and the line $\ell_0 := p^\perp \cap \pi$ is a line of $S_2$:
The only difference to Case 3 is that now $p^\perp \cap \ell_0^\perp$ meets $Q$ in the line $pu$, so that $|p^\perp \cap \ell_0^\perp \cap Q| = q + 1$. Hence $|p^\perp \cap T| = \frac{1}{2}(q - 1)q^2 + q$.

Notice that Case 2 or 4 occurs if and only if $p \in T$. Hence

$$|p^\perp \cap T| = \begin{cases} \frac{1}{2}(q - 1)q^2 & \text{if } p \notin T, \\ \frac{1}{2}(q - 1)q^2 + q & \text{if } p \in T. \end{cases}$$

This shows that $T$ is a tight set of the graph $(X, R_3)$. Since $\sigma$ has orbits of length two on the points not equal to $u, z$ of $uz$, and orbits of length two on the lines not equal to $m, uz$ of $\pi$, we can choose $S_1$ and $S_2$ in such a way that $S_1^\perp \cap S_1 = \emptyset$ and $S_2^\perp \cap S_2 = \emptyset$. As the polarity of $Q$ commutes with $\sigma$, this implies that $T^\sigma \cap T = \emptyset$. Since $|T| = \frac{1}{2}|X|$, it follows that $|T \cap \{p, p^\sigma\}| = 1$ for every point $p \in X$.

Therefore Theorem 4.5 shows that $\chi_T \in V_0 \oplus V_3$.

**Intriguing sets of type 2**. Now we focus on the intriguing sets $Y$ with $\chi_Y \in V_0 \oplus V_2$. We will prove a lower bound on the size of $Y$ and provide an example that shows that the bound is sharp.

**Lemma 4.8.** If $S$ is the union of two perpendicular conics of $Q^-(5, q)$ with $S \cap H = \emptyset$, then $S^\sigma \neq S$.

**Proof.** Assume $S$ is the union of conics $C_1 = \pi_1 \cap Q^-(5, q)$ and $C_2 = \pi_2 \cap Q^-(5, q)$ where $\pi_1$ and $\pi_2$ are perpendicular planes. Then $\pi_1 \cap \pi_2 = \emptyset$. Also $\pi_1 \cap H$ and $\pi_2 \cap H$ are elliptic lines. Suppose that $S^\sigma = S$. As $\pi_1^\sigma$ is a plane on $\pi_1 \cap H$, then either $\pi_1^\sigma = \pi_1$ or $\pi_1^\sigma \cap S \subseteq \pi_1^\sigma \cap \pi_2$. As $\pi_1^\sigma \cap S = \pi_1^\sigma \cap S^\sigma = (\pi_1 \cap S)^\sigma$ is a conic, the second case is impossible. Hence $\pi_1^\sigma = \pi_1$. As $\sigma$ is a non-trivial central collineation with centre $H^\perp$, it follows that $H^\perp \in \pi_1$. The same argument shows $H^\perp \in \pi_2$. But $\pi_1 \cap \pi_2 = \emptyset$, a contradiction. □

**Lemma 4.9.** If $Y$ is a non-empty subset of $X$ with $\chi_Y \in V_0 \oplus V_2$, then $|Y| = \alpha(q + 1)$ where $\alpha$ is an even integer such that $\alpha \geq 4$.

**Proof.** The set $Y$ is a tight set of $Q^-(5, q)$ (by Lemma 4.1), and so $|Y| = \alpha(q + 1)$ for some positive integer $\alpha$. Let $\mathcal{F}$ be an intriguing set of type 4 of size $q^2(q - 1)$ as in Example 3.2. Then by Result 2.1,

$$|Y \cap \mathcal{F}| = \frac{\alpha(q + 1) \cdot q^2(q - 1)}{q^2(q^2 - 1)} = \alpha.$$

Now $Y$ and $\mathcal{F}$ are both $\sigma$-invariant (by Theorem 3.3) and so $Y \cap \mathcal{F}$ is $\sigma$-invariant. So $Y \cap \mathcal{F}$ is a union of $\sigma$-orbits, each of size 2. Therefore, $|Y \cap \mathcal{F}| = \alpha$ is even.

A 2-tight set is either a union of two lines or two perpendicular conics (c.f., [11, Theorem 1.1]). As $Y$ is disjoint from the hyperplane $H$, we have that $Y$ contains no line. As $Y^\sigma = Y$, Lemma 4.8 shows that $Y$ is not a union of two perpendicular conics. Hence $\alpha \neq 2$ and thus $\alpha \geq 4$. □

**Example 4.10** (Intriguing sets of type 2). Recall that $X = Q \setminus H$. Consider a solid $S$ of $H$ such that $S \cap Q$ is a hyperbolic quadric $Q^+(3, q)$. Then $S^\perp$ is an elliptic line of $PG(5, q)$, that is a line of $PG(5, q)$ with no point in $Q$, and $H^\perp$ is a point on this line. Consider an elliptic line $\ell_1$ of $S$. Then $\ell_2 := \ell_1^\perp \cap S$ is also an elliptic line. Let $p_1$ and $p_2$ be two perpendicular points of the line $S^\perp$ that are different from $H^\perp$ (and hence of $S^\perp \cap H$). Then $(p_1, \ell_1)$ and $(p_2, \ell_2)$ are perpendicular conic planes that have all their conic points outside $H$. Let $Y$ be the set consisting of the 2(q + 1) conic points in these two planes. As the conics are perpendicular, $Y$ is a 2-tight set of $Q$ (cf. Example 2.3). The points $p_1^\sigma$ and $p_2^\sigma$ are perpendicular points of $S^\perp$ and as before, $(p_1^\sigma, \ell_1)$ and $(p_2^\sigma, \ell_2)$ are conic planes and the 2(q + 1) points of their conics form a 2-tight set $Y'$ of $Q$. We have $Y' = Y^\sigma$ and $Y = (Y')^\sigma$, so $Y \cup Y'$ is $\sigma$-invariant. Notice that either
that shows that 3.2 4.11 8 4.5 that lines span a solid Y since C and Q are both contained in \( q \), we have seen in Example 35 [11, Theorem 2.15] that \( Z \) is the union of four conics \( C_1, C_2, C_3 \) and \( C_4 \) where \( C_1 \) and \( C_2 \), as well as \( C_3 \) and \( C_4 \), are perpendicular. Let \( \pi_1 \) be the plane spanned by \( C_1 \). Since \( Z \subseteq X \) and hence \( Z \cap H = \emptyset \), the lines \( \ell_x := \pi_1 \cap H, i = 1, 2, 3, 4 \), are elliptic lines. Since \( \pi_1 \) and \( \pi_2 \) are perpendicular and hence skew, the lines \( \ell_1 \) and \( \ell_2 \) are skew and perpendicular. Hence these two lines span a solid \( S \) that meets \( Q \) in a hyperbolic quadric. Hence \( \pi_1 \) and \( \pi_2 \) are contained in \( S \) and \( S^\perp \). Now \( \ell_1 \) and \( \ell_2 \) are perpendicular and hence skew, the lines \( \ell_1 \) and \( \ell_2 \) are skew and perpendicular. Hence these two lines span a solid \( S \) that meets \( Q \) in a hyperbolic quadric \( Q^+ \) (by Lemma 4.11). We have \( S \subseteq H \) and \( S^\perp \) is an elliptic line. Since \( S^\perp \) and \( \pi_1 \) are contained in \( \ell_2 \), it follows that \( \pi_1 \) and \( S^\perp \) meet in a point \( p_1 \), and similarly \( \pi_2 \) and \( S^\perp \) meet in a point \( p_2 \). Lemma 4.8 shows that \( C_1 \cup C_2 \) is not invariant under \( \sigma \), and Theorem 4.5 shows that \( Z^\sigma = Z \). Since \( C_1 \cup C_2 \subseteq Z \), it follows that \( \pi_1^\sigma \neq \pi_1 \) or \( \pi_2^\sigma \neq \pi_2 \). We may assume without loss of generality that \( \pi_1^\sigma \neq \pi_1 \). Then \( \pi_1^\sigma \) is a conic plane on \( \ell_1 \). Since the conic \( \sigma \pi_1^\sigma \cap Q \) is contained in \( Z^\sigma = Z \) and hence in the union of the conics \( C_2, C_3 \) and \( C_4 \), it follows from \( q \geq 59 \) that \( \pi_1^\sigma \) must be one of the planes \( \pi_2, \pi_3 \) or \( \pi_4 \). Now \( \ell_1 \subseteq \pi_1^\sigma \), and so \( \pi_1^\sigma \neq \pi_2 \) and hence we may assume that \( \pi_1^\sigma = \pi_3 \). Therefore, \( \pi_2^\sigma = (\pi_1^\sigma)^\sigma = (\pi_1^\sigma)^\perp = \pi_3 = \pi_4 \). Thus with \( Y := C_1 \cup C_2 \) and \( Y' = C_3 \cup C_4 \), we have \( Y^\sigma = Y' \). So, \( Z = Y \cup Y' \) has the structure described in Example 4.10.

5. Intriguing sets that are not tight; type 4

Let \( Y \) be an intriguing set of the association scheme \( X \) corresponding to \( V_4 \). Since the relation \( R_3 \) has the eigenvalue \( \theta := -(q-1)^2 \) on \( V_4 \), Result 2.1 shows that every point of \( X \setminus Y \) is collinear to

\[
\alpha := \frac{k-\theta}{|L|} \cdot \frac{|Y|}{q} \leq 1
\]

points of \( Y \), and every point of \( Y \) is collinear to exactly \( \theta + \alpha \) points of \( Y \). Since \( \alpha + \theta \geq 0 \), we have \( \alpha \geq -\theta = (q-1)^2 \) and hence \( |Y| \geq q(q-1)^2 \).

Lemma 5.1. Let \( Y \) be an intriguing set of type 4. Then \( |Y| \geq q^2(q-1) \).

Proof. We have seen in Example 4.10 that there exists an intriguing set \( S \) of size \( 4(q+1) \) with \( \chi_S \in V_0 \oplus V_2 \). Hence

\[
|S \cap Y| = \frac{|S|}{|X|} \cdot \frac{|Y|}{q} = \frac{4(q+1)|Y|}{q^2(q^2-1)} = \frac{4|Y|}{q^2(q-1)}.
\]

that is \( q^2(q-1) \) divides \( 4|Y| \). We also have \( |Y| \geq q(q-1)^2 \). As \( q \geq 3 \), it follows that either \( |Y| \geq q^2(q-1) \) or otherwise that \( q = 4 \) and \( |Y| = q(q-1)^2 = 36 \). However, \( |Y| = q^2(q-1) \) and \( q = 4 \) implies that \( \alpha + \theta = 0 \), that is \( Y \) is a partial ovoid. It is known that \( Q^- \) (5, 4) has no partial ovoids of size larger than 35 [8, Theorem 4.2].

We have seen in Example 3.2 that this bound is tight. In the rest of this section we show that every intriguing set \( Y \) of type 4 with \( |Y| = q^2(q-1) \) has the form described in Example 3.2. Recall that \( H \) is the hyperplane of \( PG(5, q) \) such that \( X = Q \setminus H \), and that \( H \cap Q \) is a parabolic quadric \( Q(4, q) \).
Lemma 5.2. Suppose that $Y$ is an intriguing set of $X$ with $q^2(q - 1)$ points such that every point in $Y$ is collinear to $q - 1$ points of $Y$, and such that every point of $X \setminus Y$ is collinear to $q^2 - q$ points of $Y$. Then we have.

(a) For each $p \in H \cap Q$ there exists an integer $s_p$ such that $|\ell \cap Y| = s_p$ for every line of $Q$ that contains $p$ and is not contained in $H$. For every subset $M$ of $H \cap Q$ we set $s_M := \sum_{p \in M} s_p$.

(b) $s_{H \cap Q} = q^3 + q$ and $\sum_{p \in H \cap Q} s_p(s_p - 1) = q(q - 1)$.

(c) If $\ell$ is a line of $H \cap Q$, then $s_\ell = q$.

(d) Let $U$ be a solid of $H$ such that $\mathcal{E} := U \cap Q$ is an elliptic quadric $Q^{-}(3,q)$ and suppose that $s_p = 0$ for at least one point $p \in \mathcal{E}$. Then $s_\mathcal{E} = q^2 - q$.

(e) Let $U$ be a solid of $H$ such that $\mathcal{H} := U \cap Q$ is a hyperbolic quadric $Q^{+}(3,q)$. Then $s_\mathcal{H} = q^2 + q$.

(f) Suppose that $q$ is odd. Let $\pi$ be a plane of $H$ such that $C := \pi \cap Q$ is a conic and suppose that $s_x = 0$ for at least one point $x$ of $C$. If the line $\ell := \pi^\perp \cap H$ is external to $Q$, then $s_C = q - 1$, and if $\ell$ meets $Q$ in two points $p$ and $r$, then $s_C = q + 1 - s_p - s_r$.

(g) Suppose that $q$ is even. Let $\pi$ be a plane of $H$ such that $C := \pi \cap Q$ is a conic.

If $H^\perp \notin \pi$ and $s_p = 0$ for at least one point $p$ of $C$, then $\pi^\perp \cap H$ is a line on $H^\perp$ meeting $Q$ in exactly one point $u$ and we have $s_C + s_u = q$.

If $H^\perp \in \pi$, then $\pi^\perp \cap H$ is a plane that meets $Q$ in a conic $C'$ and we have $s_C + s_{C'} = 2q$.

Proof.

(a) Let $\ell$ be a line of $Q$ that contains $p$ and is not contained in $H$. Let $s$ be the number of points of $Y$ on $\ell$. Let $r$ be a point of $\ell$ with $r \neq p$. If $r \in Y$, then $r$ is collinear to $q - 1$ points of $Y$, of which $s - 1$ are on $\ell$, and if $r \notin Y$, then $r$ is collinear to $q^2 - q$ points of $Y$, of which $s$ are on $\ell$. Hence there exist exactly $s(q - s) + (q - s)(q^2 - q - s) = (q - s)(q^2 - q)$ points in $Y$ that are collinear to exactly one of the points $\notin p$ of $\ell$. The remaining points of $X$ must therefore be collinear to $p$. This shows that $s$ depends only on $p$ but not on the choice of $\ell$.

(b) Each point $p \in H \cap Q$ is collinear to $(q^2 - q)s_p$ points of $Y$. On the other hand, every point of $Y$ is collinear to $q^2 + 1$ points of $H \cap Q$. A double counting argument thus gives $\sum_{p \in H \cap Q} s_p(q^2 - q) = |Y|(q^2 + 1)$. Since $|Y| = q^2(q - 1)$, this gives $\sum_{p \in H \cap Q} s_p = q^3 + q$.

For the second equality in (b) we count triples $(p, r_1, r_2) \in (H \cap Q) \times Y \times Y$ of collinear points with $r_1 \neq r_2$. Each point $r_1 \in Y$ is collinear to $q - 1$ points of $Y$ and thus occurs in $q - 1$ such triples in the middle position. Each point $p \in H \cap Q$ occurs in $(q^2 - q)s_p(s_p - 1)$ such triples. It follows that

$$\sum_{p \in H \cap Q} s_p(s_p - 1)(q^2 - q) = |Y|(q - 1).$$

Since $|Y| = q^2(q - 1)$, this implies the second equation of statement (b).

(c) A point $p$ of $\ell$ is collinear to $s_p(q^2 - q)$ points of $Y$. Since every point of $Y$ is collinear to exactly one point of $\ell$, it follows that $s_\ell(q^2 - q) = |Y|$, hence $s_\ell = q$.

(d) The line $U^\perp$ meets $Q$ in two points $r$ and $r'$ and these points do not lie in $H$. Since $s_p = 0$ for some point of $\mathcal{E}$, we see that $r, r' \notin Y$. Thus $r$ is collinear to exactly $q^2 - q$ points of $Y$ and these are exactly the points of $Y$ on the lines $rz$ with $z \in \mathcal{E}$. Since $r \notin Y$, it follows that $s_\mathcal{E} = q^2 - q$.

(e) Because a hyperbolic quadric is the union of the $q + 1$ lines of any of its two reguli, this follows from part (c).

(f) As $q$ is odd, $\pi^\perp \cap H$ is in fact a line and moreover this line is skew to $\pi$. First consider the case that $\ell$ has no point on the quadric. Then each solid $T$ of $H$ on $\pi$ meets $Q$ in an elliptic quadric $Q^{-}(3,q)$ or a hyperbolic quadric $Q^{+}(3,q)$, and there are $\frac{1}{2}(q + 1)$ solids of each kind. Counting the sum $\sum_{p \in H \cap Q} s_p$
using the solids on \( \pi \), it follows from (b), (d) and (e) that
\[
q^3 + q + qs_C = \frac{1}{2}(q + 1)(q^2 - q) + \frac{1}{2}(q + 1)(q^2 + q)
\]
This implies that \( s_C = q - 1 \).

Now consider the case that the \( \pi^\perp \cap H \) is a secent line to the quadric, and let \( p \) and \( r \) be the two points of \( Q \) on this line. The solids \( \langle \pi, p \rangle \) and \( \langle \pi, r \rangle \) meet \( Q \) in a cone \( C_p \) resp. \( C_r \) with vertex \( p \) resp. \( r \) over the conic \( C \). From the remaining \( q - 1 \) solids of \( H \) through \( \pi \) one half meets \( Q \) in an elliptic quadric \( Q^{-}(3, q) \) and one half meets \( Q \) in a hyperbolic quadric \( Q^{+}(3, q) \). It follows from (c) that \( s_{C_p} = q(1 + s_p) \) and \( s_{C_r} = q(q + 1 - s_r) \). Then a similar counting as in the first case gives
\[
q^3 + q + qs_C = q(q + 1) + q(q + 1 - s_r) + \frac{1}{2}(q - 1)(q^2 - q) + \frac{1}{2}(q - 1)(q^2 + q).
\]
This implies that \( s_C = q + 1 - s_p - s_r \).

(g) First consider the case when \( \pi \) does not contain \( H^\perp \). Then \( \langle \pi, H^\perp \rangle \) meets \( Q \) in a cone \( U \) with vertex \( u \) over the conic \( C \). The vertex \( u \) is the unique point on the line joining \( H^\perp \) to the nucleus of the conic in \( \pi \). Part (c) shows that \( s_U = q(1 + s_u) \). Every other solid on \( \pi \) meets \( Q \) in a \( Q^{-}(3, q) \) or \( Q^{+}(3, q) \) and there are \( q/2 \) solids of each type. If \( C \) has a point \( p \) with \( s_p = 0 \), then the previous parts imply similarly as in the proof of (f) that
\[
q^3 + q + qs_C = q(q + 1 - s_u) + \frac{1}{2}(q^2 - q) + \frac{1}{2}(q^2 + q)
\]
and this gives \( s_C + s_u = q \).

Now consider the case when \( \pi \) contains \( H^\perp \). Then \( \tau := \pi^\perp \cap H \) is a conic plane with \( \pi \cap \tau = H^\perp \).
Let \( C' \) be the conic \( \tau \cap Q \). The solids on \( \tau \) are spanned by \( \pi \) and a point of \( C' \). Hence all these solids meet \( Q \) in cones with vertex a point of \( C' \). If \( u \) is a point of \( C' \), then the solid \( \tau := \langle \pi, u \rangle \) meets \( Q \) in the union of \( q + 1 \) lines on \( u \), so part (c) shows that \( s_{\tau} = q(1 + s_u) \). It follows that
\[
q^3 + q + qs_C = \sum_{u \in C'} q(q + 1 - s_u) = q(q + 1)^2 - qs_{C'},
\]
which gives \( s_C + s_{C'} = 2q \).

**Lemma 5.3.** Suppose that \( Y \) is an intriguing set of \( X \) with \( q^2(q - 1) \) points such that every point in \( Y \) is collinear to \( q - 1 \) points of \( Y \), and every point of \( X \setminus Y \) is collinear to \( q^2 - q \) points of \( Y \). Let \( \ell \) be a line of \( H \) that meets \( Q \) in exactly two points \( p \) and \( r \), and let \( C \) be the conic \( \ell^\perp \cap H \cap Q \). Suppose that \( s_p = 0 \). Then \( s_C = q(2 - s_r) \).

**Proof.** Put \( C = \{z_0, \ldots, z_q\} \) and \( \pi := \ell^\perp \cap H \). Then \( C = \pi \cap Q \).

Case 1. We assume that \( q \) is odd.

The planes of \( H \) on \( \ell \) are the planes \( \tau \) spanned by \( \ell \) and a point \( z \) of \( \pi \). For \( z = z_i \in C \), the plane \( \tau := \langle \ell, z_i \rangle \) meets \( Q \) in the union of the lines \( z_ip \) and \( z_ir \), so we have \( s_\tau = 2q - s_z \). Now consider one of the \( q^2 \) points \( z \) of \( \pi \cap C \) and the plane \( \tau_z := \langle \ell, z \rangle \). Then \( \tau_z \) meets \( Q \) in a conic. The line \( \tau_z^\perp \cap H = z^\perp \cap \pi \) is either an exterior line or a secant line of the conic \( \pi \cap Q \), depending on whether or not \( z \) is an interior or exterior point of the conic \( C \). Thus, exactly \( \binom{q}{2} \) planes \( \tau_z \) have the property that \( \tau_z^\perp \cap H \) is an exterior line of the quadric and these planes satisfy \( s_{\tau_z} = q - 1 \) by Lemma 5.2 (f). The remaining \( \binom{q+1}{2} \) planes \( \tau_z \) of \( H \) on \( \ell \) can be indexed \( \tau_{ij} \) with \( i < j \) where \( z_i \) and \( z_j \) are the two points of \( C \) that lie on the line \( \langle z, \ell \rangle^\perp \cap \pi \). Then \( s_{\tau_{ij}} = q + 1 - s_z - s_{z_i} \), again by Lemma 5.2 (f). Counting the sum \( \sum_{p \in H \cap Q} s_p \) by considering the planes of \( H \) on \( \ell \) we thus find
\[
q^3 + q + (q^2 + q)s_\ell = \sum_{i=0}^{q} (2q - s_{z_i}) + \binom{q}{2}(q - 1) + \sum_{i,j} (q + 1 - s_{z_i} - s_{z_j}).
\]
For each $i$ the term $s_{z_i}$ occurs $q + 1$ times in this equation. It follows thus that
\[
q^3 + q + (q^2 + q)s_\ell = (q + 1) \cdot 2q + \left(\frac{q}{2}\right)(q - 1) + \left(\frac{q + 1}{2}\right)(q + 1) - (q + 1) \sum_{i=0}^{q} s_{z_i}.
\]
As $s_C = \sum_i s_{z_i}$ this gives $qs_\ell = 2q - s_C$. By Lemma 5.2(f), we have
\[
q s_C = q(q + 1 - s_\ell) = q^2 + q - (2q - s_C) = q^2 - q + s_C
\]
and hence $s_C = q$ and $s_\ell = 1$. Therefore, $s_r = s_p + s_r = s_\ell = 1$ and $s_C = q(2 - s_r)$.

Case 2. We assume that $q$ is even.

As before the planes of $H$ on $\ell$ are the planes spanned by $\ell$ and a point of $\tau = (\ell, z)$. Let $z$ be a point of $\tau$ and $\tau = (\ell, z)$. If $z = z_i \in C$, then $s_r = 2q - s_{z_i}$ as in Case 1. If $z = H^\bot$, then $\tau^\bot \cap H = \pi$, so $s_r = 2q - s_{z_i}$ by Lemma 5.2(g). If $z$ is one of the $q^2 - 1$ points $z$ with $z \neq H^\bot$ and $z \notin C$, then the line on $H^\bot$ and $z$ meets $C$ in a unique point $z_i$ and Lemma 5.2(g) shows $s_\tau = q - s_{z_i}$. Notice that every point $z_i$ of $C$ occurs for exactly $q - 1$ choices for $z$ of $\pi$ with $z \neq H^\bot$ and $z \notin C$. Counting the sum $\sum_{p \in H \cap Q} s_p$ by considering the planes of $H$ on $\ell$ we thus find
\[
q^3 + q + (q^2 + q)s_\ell = \sum_i (2q - s_{z_i}) + (2q - s_C) + (q - 1) \sum_i (q - s_{z_i}).
\]
Using $s_C = \sum_i s_{z_i}$, this simplifies to $(q^2 + q)s_\ell = 2q(q + 1) - (q + 1)s_C$. Since $s_\ell = s_p + s_r = s_r$, this proves the statement. □

**Lemma 5.4.** Suppose that $q$ is odd and that $Y$ is an intriguing set with $q^2(q - 1)$ points such that every point in $Y$ is collinear to $q - 1$ points of $Y$, and every point of $X \setminus Y$ is collinear to $q^2 - q$ points of $Y$. Then $s_r \in \{0, 1\}$ for every point $r \in H \cap Q$.

**Proof.** Since $\sum_{w \in H \cap Q} s_w (s_w - 1) = q(q - 1)$, we have $s_r \leq q$. We may assume that $0 < s_r < q$. Let $\ell$ be a line of $H \cap Q$ on $r$. It follows from Lemma 5.2(c) that $\ell$ contains a point $z$ with $z \neq r$ and $s_z > 0$. Let $h$ be a line of $H \cap Q$ on $z$ with $h \neq \ell$. Lemma 5.2(c) shows that $h$ contains a point $p$ with $s_p = 0$. Then $pr$ is a secant line and $z$ is a point of the conic $C := (pr)^\bot \cap H \cap Q$. Hence $s_C \geq s_z > 0$. Lemma 5.3 shows that $s_C = q(2 - s_r)$. Hence $s_r < 2$, that is $s_r = 1$. □

**Theorem 5.5.** Suppose $Y$ is an intriguing set of the Pentilla-Williford scheme $(X, \{R_0, \ldots, R_4\})$ with $|Y| = q^2(q - 1)$ such that every point in $Y$ is collinear to $q - 1$ points of $Y$, and every point of $X \setminus Y$ is collinear to $q^2 - q$ points of $Y$. Then there exists a point $p \in H \cap Q$ such that $Y$ consists of the $(q^2 - q)$ points of $X = Q \setminus H$ that are collinear to $p$.

**Proof.** Since $s_p \in \{0, 1, q\}$ for all points $p \in H \cap Q$, Lemma 5.2(b) implies that there exists a unique point $p$ with $s_p = q$. Then each of the $q^2 - q$ lines of $Q$ on $p$ that is not contained in $H$ meets $Y$ in $q$ points. Hence, $Y$ consists of the points of $X$ that are collinear to $p$. □

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