Comment on “The extremal black hole bomb”

Shahar Hod
The Ruppin Academic Center, Emeq Hefer 40250, Israel
and
The Hadassah Institute, Jerusalem 91010, Israel

Oded Hod
The Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel

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Recently, we have provided an analytical treatment of the phenomena known as the 'black-hole bomb' (arXiv:0910.0734). In particular, we have determined analytically the unstable growing resonances of a massive scalar field in the rotating Kerr black hole spacetime. It was later claimed by J. G. Rosa (arXiv:0912.1780) that the analytic procedure may fail for some values of the field’s mass. This claim was based on the concern that some of the Gamma functions that are involved in the analysis may develop poles. In this comment we show, by explicit calculations, that the Gamma functions, which are used, are all well behaved near the peak of the black-hole resonances. This fact supports the validity of our previous analytical treatment. We further comment on the regime of validity of some of the approximations used by Rosa.

A bosonic field impinging on a rotating black hole can be amplified as it scatters off the hole, a phenomena known as superradiant scattering. If in addition the field has a non-zero rest mass then the mass term effectively works as a mirror, reflecting the scattered wave back towards the black hole. This physical system, known as a black-hole bomb [1, 2], the wave may bounce back and forth between the black hole and some turning point amplifying itself each time. Consequently, the massive field grows exponentially over time and is unstable.

Former analytical estimates of the instability timescale associated with the dynamics of massive scalar field in the rotating Kerr spacetime were restricted to the regimes $M \mu > 1$ [3] and $M \mu \ll 1$ [4, 5], where $M$ and $\mu$ are the masses of the black hole and the field, respectively. In these two limits the growth rate of the field (the imaginary part $\omega_f$ of the mode’s frequency) was found to be extremely weak. However, subsequent numerical investigations [3, 4] have indicated that the instability is actually greatest in the regime $M \mu = O(1)$, where the previous analytical approximations break down. Thus, a new analytical study of the instability timescale for the case $M \mu = O(1)$ is physically well motivated.

In a recent paper [7] we have studied analytically for the first time the phenomena of superradiant instability (the black-hole bomb mechanism) in the physically interesting regime $M \mu = O(1)$, the regime of greatest instability. We have shown that the resonance condition for the bound states of the field is given by

$$
\frac{1}{\Gamma(\frac{1}{2} + \beta - \kappa)} = (8i)^{2\beta} \left[ \frac{\Gamma(-2\beta)}{\Gamma(2\beta)} \right]^2 \frac{\Gamma(\frac{1}{2} + \beta - ik)}{\Gamma(\frac{1}{2} - \beta + ik)\Gamma(\frac{1}{2} - \beta - \kappa)} \left[ M r + \sqrt{\mu^2 - \omega^2(m\Omega - \omega)} \right]^{2\beta},
$$

(1)

see Ref. [7] for details. [We use here the same notations as in Ref. [7]]. Solving this resonance condition yielded an instability growth rate of $\tau^{-1} \equiv \omega_f = 1.7 \times 10^{-3} M^{-1}$ for the fastest growing mode [7]. This instability is four orders of magnitude stronger than has been previously estimated [6].

It was recently claimed [8] that the analytic treatment of Ref. [7] may fail for some values of the field’s mass. This claim was based on the concern that some of the Gamma functions that are involved in the analysis may develop poles. In order to remove this concern, we explicitly calculated all the Gamma functions which are involved in the analysis [see Eq. (1)]. We depict the results in figure [8]. It is clear that the Gamma functions are all well behaved in the vicinity of the peak black-hole resonances. Moreover, we note that at the edges of the presented mass spectrum, where some of the Gamma functions are largest, our analytical results do agree with the suggested treatment of [8], see Fig. 3a. of [8]. This clearly supports the validity of our results near the peak of the spectrum (at $M \mu \simeq 0.469$), where the Gamma functions are actually smaller (that is, further away from their poles).

It is worth noting that Strafuss and Khanna [8] have used an independent numerical technique to evaluate the instability growth rate of a massive scalar field in the spacetime of a near extremal rotating black hole. They have found an instability growth rate of $\tau^{-1} \equiv \omega_f = 2 \times 10^{-5} M^{-1}$ for the case $M \mu = 0.25$. This is two orders of magnitude stronger than the estimate of [8], a fact which indicates that the instability found in [6] is actually not the largest one possible. Since the black-hole instability is expected to be more pronounced in the regime $M \mu \simeq 0.5$ that we have analyzed in [7] (as compared to the $M \mu = 0.25$ case that was studied in [8]), one may consider the result of [8] as a lower bound on the strength of the instability for near-extremal black holes. The analytical treatment of [7] reveals...
FIG. 1: The behavior of the various Gamma functions as a function of the field’s mass $\mu$. The results are for the $l = m = 1$ mode, the mode with the greatest instability. Inset: zoom-in on the region where the largest resonance is obtained. The Gamma functions are all found to be well behaved near the spectrum’s peak.

...that this expectation is indeed correct.

Finally, we would like to take this opportunity to remark on the regime of validity of the analytical treatment used in [8]. The validity of the analytical method, first developed in [7], is restricted to the regime $\omega \simeq m\Omega \simeq \mu$ with $\Omega \simeq 1/2M$ [10]. (Fortunately, this is the regime of physical interest, where the instability is greatest.) For the most unstable mode with $m = 1$ this implies that the analytical treatment is valid in the regime $M\mu \simeq \frac{1}{2}$. Thus, using the analytical approximation down to $M\mu \simeq 0.2$ as was done in [8] is actually physically not justified.

In summary, we have shown that the Gamma functions involved in the analysis of the black-hole bomb phenomena are all well behaved at the peak resonances. This fact supports our previous findings [7] regarding the instability timescale associated with the dynamics of a massive scalar field in the Kerr spacetime. Our findings are further supported by the independent numerical analysis performed in [9].

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[10] The requirement $\omega \simeq \mu$ is needed in order to treat $a^2(\omega^2 - \mu^2)$ as a perturbation term in the equation that determines the values of the separation constants $A_{lm}$ [8]. Under this condition one can then expand the separation constants $A_{lm}$ in powers of $a^2(\mu^2 - \omega^2)$ to find $A = l(l + 1) + \sum_{k=1}^{\infty} c_k a^{2k}(\mu^2 - \omega^2)^k$. The expansion coefficients $\{c_k\}$ are given in [11]. For example, for $l = m = 1$, the case of physical interest, we find $c_1 = 1/5$, $c_2 = -4/875$, $c_3 = 8/65625$, .... Note that in [8] we had a typo in the value of $c_1$, though our calculations in [8] were performed with the correct value of that coefficient].
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