Subsequential scaling limits for Liouville graph distance

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Abstract

For $0 < \gamma < 2$ and $\delta > 0$, we consider the Liouville graph distance, which is the minimal number of Euclidean balls with Liouville quantum gravity measure at most $\delta$ whose union contains a continuous path between two endpoints. In this paper, we show that the renormalized distance is tight and thus has subsequential scaling limits at $\delta \to 0$. In particular, we show that for all $\delta > 0$ the diameter with respect to the Liouville graph distance has the same order as the typical distance between two endpoints.

1 Introduction

Let $\mathbb{R}$ be a rectangular subset of $\mathbb{R}^2$ (the Euclidean plane), let $h_{\mathbb{R}^*}$ be a Gaussian free field with Dirichlet boundary conditions on a box $\mathbb{R}^*$ centered around $\mathbb{R}$ whose sides are separated from those of $\mathbb{R}$ by twice the Euclidean diameter of $\mathbb{R}$, and $\mu_{\mathbb{R}}$ be the Liouville quantum gravity (LQG) measure on $\mathbb{R}$, at inverse temperature $\gamma \in (0, 2)$, induced by $h_{\mathbb{R}^*}$. (We precisely define these objects in Section 2; see also the surveys [41, 5, 35].) In this paper, we consider the Liouville graph distance with parameter $\delta$ on $\mathbb{R}$, which for any two points $x, y \in \mathbb{R}$ is given by $d_{\mathbb{R}, \delta}(x, y)$, the minimal number of Euclidean balls of LQG measure at most $\delta$ that it takes to cover a path between $x$ and $y$. (Again, see Section 2 for a precise definition.) Let $Q_\delta$ be the median Liouville graph distance from the left to the right side of $\mathbb{R}$. Our goal in this paper is to prove the following:

Theorem 1.1. For any sequence $\delta_n \downarrow 0$, there is a subsequence $(\delta_{n_k})$ and a limiting metric $\bar{d}$ so that

$$Q_{\delta_{n_k}}^{-1} d_{\mathbb{R}, \delta_{n_k}} \to \bar{d}$$

in distribution with respect to the uniform topology of functions $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Moreover, $\bar{d}$ is Hölder-continuous with respect to the Euclidean metric, and the Euclidean metric is Hölder-continuous with respect to $\bar{d}$.

A consequence of Theorem 1.1, but also a key step in its proof (see Proposition 6.4 below), is that for any $\delta$, the LQG diameter of a box is comparable to the left-right crossing distance:

Theorem 1.2. For any $\varepsilon > 0$, we have a $C = C(\varepsilon)$ so that, for all $\delta > 0$,

$$\mathbb{P}\left(\max_{x, y \in \mathbb{R}} d_{\mathbb{R}, \delta}(x, y) \geq C(\varepsilon)Q_\delta\right) \leq \varepsilon.$$
The normalizing constant $Q_\delta$ is poorly understood, and the results of this paper do not rely on precise estimates of its value. However, we will show the following in the course of the proof of Theorem 1.1.

**Theorem 1.3.** We have a constant $0 < C < \infty$ so that for all $\delta$ we have

$$C^{-1}\delta^{-C^{-1}} \leq Q_\delta \leq C\delta^{-C}.$$ 

### 1.1 Background and motivation

In the seminal paper [22], the Gaussian multiplicative chaos was initiated and constructed as a random measure obtained from exponentiating log-correlated Gaussian fields. The last decade has seen extensive study of Gaussian multiplicative chaos as well as Liouville quantum gravity,\(^1\) which is an important special case of Gaussian multiplicative chaos where the underlying log-correlated field is a two-dimensional Gaussian free field. See, for example, [22, 18, 34, 35, 40, 6, 36, 4].

Recently, a huge amount of effort has been devoted to the understanding for the random metric associated with LQG. Building on [32, 17], in [29, 30, 31] the authors constructed in the continuum (note that in these works there is no mention of discrete approximation of the metric) a random metric which is presumably the scaling limit of LQG distance for a specific $\gamma = \sqrt{8/3}$ and proved deep connections with the Brownian map [23, 24, 28]. In [14, 10, 20, 21, 11], various bounds on the distance as well as the geodesics were obtained; in [15, 13], some non-universality aspects (when considering underlying log-correlated fields other than GFF) for LQG distances have been demonstrated; in [27, 12] a type of equivalence between the Liouville graph distance and the heat kernel for Liouville Brownian motion was proved; and in [11] it was shown that there is a single number which determines the distance exponents for a few reasonable choices of distances associated with LQG as well as the distance exponents for random planar maps.

However, despite much dedicated effort, the two most outstanding problems related to LQG distances remain open: (1) to compute the exact distance exponents for LQG distances (note that this is now known for $\gamma = \sqrt{8/3}$ by [3, 11]); (2) to derive a scaling limit of natural discrete approximations of LQG distances.

The present paper makes some progress toward understanding the scaling limits of LQG distances.

### 1.2 Two closely related works

The present article is closely related to [9, 16], as we discuss in more detail below.

From the perspective of results, in [9] it was shown that discrete Liouville first-passage percolation (shortest path metric where the vertices are weighted by the exponential of the discrete GFF) has a subsequential scaling limit for sufficiently small $\gamma > 0$; in [16], the authors considered the case when the underlying field is a type of log-correlated Gaussian field (in the continuum) with short-range correlations (i.e., the so-called $\star$-scale invariant field) and showed that there exists a certain parameter $\gamma^* > 0$ such that the corresponding Liouville first-passage percolation has a subsequential scaling limit for all $\gamma < \min(\gamma^*, 0.4)$. The main contribution of the present article is that the result for Liouville graph distance is valid throughout the subcritical regime, i.e., $0 < \gamma < 2$. A few further remarks are in order:

- In [9, 16], the authors worked with Liouville first-passage percolation and in the present article we work with Liouville graph distance. They are both natural approximations of LQG distances and at the moment we are equally satisfied for proving results for either of these choices (or any other reasonable choices). As noted earlier, in [11], universality of the dimension exponents was proved for all reasonable choices of metric we know when we stick to GFF as the underlying field (but we

\(^1\)We note that our convention on the terminology Liouville quantum gravity follows that in [18]. This convention is a bit different from that adopted in Liouville field theory, and one shall be cautious about the underlying mathematical meaning of LQG when discussing in the context of Liouville field theory.
We now discuss similarities and differences in proof methods. In [9], we presented a framework combining multi-scale analysis and Efron–Stein-type arguments with Russo–Seymour–Welsh (RSW)-type estimates (originally introduced in [37, 39, 38]), which relate distances between boundaries of rectangles in easy and hard directions. (Here and throughout the paper, the “easy” direction across a rectangle is the crossing between the two longer sides, while the “hard” direction is the crossing between the two shorter sides; see Figure 1.1.) This is more or less the framework also used in both [16] and the present article. The key difference between [9] and [16] lies on the implementation of RSW proof: in [9] the RSW estimate was inspired by [42] and in [16] it was proved by using approximate conformal invariance. The proof in [9] draws a natural connection between random distances and percolation theory. It also presents a more widely applicable framework—based on the method of [42]—for proving RSW estimates, as it does not rely on conformal invariance (and thus for instance works in the lattice case, or potentially for more general fields). However, while the proof in [42] is simply beautiful, much of the beauty was lost when it was implemented in [9] due to the complication of considering lengths of paths rather than merely considering connectivity. In contrast, the proof method in [16], taking advantage of conformal invariance, is insightful and beautiful. While the method in [16] does suffer the drawback of relying on conformal invariance and thus may not be as widely applicable as the method in [9], this drawback is irrelevant here since at the moment we are satisfied with the case of a conformally invariant field. For the present article, while it seems likely that an RSW proof following [9] is possible (although it would require improving the analysis in [9] in a number of places), we switch to the conformal invariance-based proof as in [16] for the sake of simplicity.
In [9], since $\gamma$ is sufficiently small, in the multi-scale analysis the influence from coarse field (which one can roughly understand as circle averages with respect to macroscopic circles) is negligible, which simplifies analysis in a number of ways. (The situation is in some sense similar in [16] as there it is only shown that $\gamma^* > 0$.) Thus, the main new technical challenge in the present article is the influence from the coarse field for $0 < \gamma < 2$. (Naturally, it is most difficult when $\gamma$ is close to 2.) One manifestation of this challenge is the fact that the LQG measure only has a finite $p$th moment for $p$ approaching 1 as $\gamma$ approaching 2. While this creates challenges in the multi-scale analysis, from a conceptual point of view our proof crucially relies on the fact that the LQG measures for all Euclidean balls with radius at most $\varepsilon$ uniformly converge to 0 for $0 < \gamma < 2$ — this fact is also crucial for the diameter to have the same order as the typical distance.

1.3 Ingredients in the proof

By a chaining argument similar to the ones used in [9, 16], we can write the diameter of a box as a sum of the maximum hard crossings in dyadic subboxes at successively smaller scales. Controlling the fluctuation of box diameters uniformly in the scale is essentially enough to uniformly control the fluctuations of a Hölder norm of the metric with respect to the Euclidean metric, which yields tightness in the uniform norm. Thus, our essential goal is to bound the fluctuations of hard crossing distances, again uniformly in the scale. The form of our bound on these fluctuations will be a variance bound on the logarithm of these fluctuations, which allows us to relate different quantiles of the scale as described in Lemma A.1 below.

Our bound on the variance of the logarithm of the hard crossing distance takes place in the framework of the Efron–Stein inequality. We write the underlying Gaussian free field as a function of a space-time white noise, partition space-time into boxes, and then express the variance of the logarithm of the Liouville graph distance as the sum of the expected squared multiplicative (since we are considering the logarithm) changes in the distance when each box is resampled. The white noise decomposition is by now a widely-used way to decompose the GFF into a “fine field,” with no regularity but short-range correlations, and a “coarse field,” which is smooth but has long-range correlations. See for example [35, 40] for general descriptions of the white noise decomposition and [10, 12] for previous uses in the setting of the LQG metric.

We control the fluctuations due to the coarse field using a standard Gaussian concentration argument in Subsection 5.3.1; this is sufficient because we choose the decomposition so that the coarse field is sufficiently smooth. The more serious task is to control the fluctuations due to the fine field. Of course, the most serious effect of the fine field in a certain box is on the part of the geodesic close to the same box. Controlling the fluctuations on this part of the geodesic (done in Subsection 5.3.3) essentially has two key components:

- First, we observe that since the coarse field is smooth, it suffices to control the change in the distance for the LQG metric in boxes at a smaller scale. This fine scale is not the size of the smaller boxes, since when the coarse field is large, subboxes “look larger than they are” from the perspective of the Liouville graph distance. (This is quantified by the conformal covariance of the Liouville quantum gravity; see Proposition 2.14 and Proposition 2.39 below.) The requirement $\gamma < 2$ is exactly what is needed to ensure that, even considering the maximum effect of the coarse field, the effective size of the subboxes is strictly smaller than that the size of the overall large box.

- Second, now that we know that the subboxes have a smaller effective size, we can apply our inductive hypothesis, along with the RSW theory and a priori concentration bounds on various crossing distances, to show that the new path in the subbox (under the resampled field) is comparable in Liouville graph distance length to the original path in the subbox (under the original field). The fact that the subboxes have a smaller effective size than the larger box, along with an a priori bound on the scaling of the distance, also allows us to show that this variation is small compared to the crossing length in the overall box.
Of course, we must also control the effect of the fine field on far-away boxes, which we do by a somewhat more straightforward argument (Subsection 5.3.2).

A complication in the argument is that our multiscale analysis procedure breaks down at very small scales, which is ultimately due to the fact that, in very small boxes, a ball of LQG measure 1 may be very large compared to the size of the box. Because the choice of scales necessary to make the multiscale analysis procedure work is somewhat delicate, the failure of the multiscale analysis procedure at small scales means that we require an a priori bound on the variance, taken from [12] (see (4.7) below), to bootstrap the induction.

1.4 Organization of the paper

In Section 2 we introduce our cast of characters, collecting a number of (semi-)standard facts about Gaussian free fields, Liouville quantum gravity, and Liouville graph distance. In Section 3 we prove a conformal covariance-based RSW estimate for Liouville quantum gravity, following [16]. In Section 4 we prove concentration bounds on crossing distances using percolation arguments in a multi-scale analysis framework, and set up some quantities regarding the relationship between different quantiles of crossing distances, which will form the objects of our inductive procedure. (We defer the introduction of these quantities until that point because they depend on certain constants which are introduced in the lemmas of Section 3 and Section 4.) In Section 5, we carry out the Efron–Stein argument, and in Section 6 we show that the diameter is within a constant of the typical distance between two points, via a chaining argument, on our way to proving the tightness of the metrics which is the main ingredient in Theorems 1.1, 1.2, and 1.3. In Section 7, we prove that the the Euclidean metric is Hölder-continuous with respect to any limiting metric, completing the proof of Theorems 1.1. Finally, we have relegated several technical lemmas in analysis and probability, that do not relate particularly to the subjects of this paper, to an appendix.

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2 Preliminaries

2.1 Notation

We denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{Q}$ the set of rational numbers. If $x, y \in \mathbb{R}$, we will use the notation $x \wedge y = \min\{x, y\}, x \vee y = \max\{x, y\}, x^+ = x \vee 0, x^- = x \wedge 0$. If $x \in \mathbb{R}^2$, we will denote by $|x|$ the Euclidean norm of $x$. We denote by $B(x, r)$ the Euclidean open ball with center $x$ and radius $r$. We will often use blackboard-bold letters to refer to subsets of $\mathbb{R}^2$. If $\mathbb{X} \subset \mathbb{R}^2$, then let $\text{diam}_{E}(\mathbb{X})$ denote the Euclidean diameter of $\mathbb{X}$, and let $\overline{\mathbb{X}}$ denote the closure of $\mathbb{X}$ in the Euclidean topology.

By a box we will mean a closed rectangular subset of $\mathbb{R}^2$ whose axes are aligned with the coordinate axes. Let width$(\mathbb{B})$ and height$(\mathbb{B})$ denote the width and height of $\mathbb{B}$, respectively. We define the aspect ratio as

$$\text{AR}(\mathbb{B}) = \frac{\text{width}(\mathbb{B})}{\text{height}(\mathbb{B})}.$$ 

For a box $\mathbb{B}$ and a scalar $\lambda > 0$, we define $\lambda A$ to be the box with the same center as $A$ but $\lambda$ times the side length.

We will frequently use the notation

$$\mathbb{B}(S_1, S_2) = [-S_1/2, S_1/2] \times [-S_2/2, S_2/2]$$
Given a box \( \mathcal{B} \), let \( \text{L}_{\mathcal{B}}, \text{R}_{\mathcal{B}}, \text{T}_{\mathcal{B}}, \) and \( \text{B}_{\mathcal{B}} \) denote the left, right, top, and bottom sides of \( \mathcal{B} \), respectively. We will sometimes omit the subscript \( \mathcal{B} \) if it is clear from context. If \( \mathcal{B} \subset \mathbb{R}^2 \), we will use the notation

\[
\mathcal{B}^{(R)} = \{ x \in \mathbb{R}^2 \mid \text{dist}_E(x, \mathcal{B}) < R \} = \bigcup_{x \in \mathcal{B}} B(x, R).
\]

If \( \mathcal{B} \subset \mathbb{R}^2 \) is a box, define \( \mathcal{B}^o \) to be the smallest box containing \( \mathcal{B}^{(\text{diam}_E \mathcal{B})} \), and \( \mathcal{B}^* \) to be the smallest box containing \( \mathcal{B}^{(2 \text{diam}_E \mathcal{B})} \).

Throughout this paper, we will often work with constants, usually denoted by \( C \) or similar notation if they are to be thought of as large, and by \( c \) or similar notation if they are to be thought of as small. We will always allow such constants to change from line to line in a computation.

### 2.2 The Gaussian free field

If \( \mathbb{A} \subset \mathbb{R}^2 \), we will denote by \( h_{\mathbb{A}} \) a Gaussian free field with Dirichlet boundary conditions on \( \mathbb{A} \). (See [41, 5] for more systematic introductions to the Gaussian free field.) We recall that the Gaussian free field is a distribution-valued stochastic process with covariance function

\[
G^\mathbb{A}(x, y) = \pi \int_0^\infty p^\mathbb{A}_t(x, y) \, dt,
\]

where \( G^\mathbb{A} \) is the Green’s function for the Dirichlet problem on \( \mathbb{A} \) and \( p^\mathbb{A}_t \) is the heat kernel for a standard Brownian motion killed on \( \partial \mathbb{A} \). If \( \mathcal{B} \supset \mathbb{A} \), then we define \( h_{\mathbb{B} \setminus \mathbb{A}} \) to be the harmonic interpolation of \( h_{\mathbb{B}} \) on \( \mathbb{A} \), and simply \( h_{\mathbb{B}} \) outside of \( \mathbb{A} \). (It is not quite obvious that this harmonic interpolation makes sense, because \( h_{\mathbb{B}} \) is a distribution and not a function. See [41] or [5] for the precise construction.) We then have the following standard property. (Again see [41, 5].)

**Proposition 2.1** (Gibbs–Markov property). The field

\[
h_{\mathcal{B}} = h_{\mathbb{A}} - h_{\mathbb{A} \setminus \mathcal{B}} \tag{2.2}
\]

is a standard Gaussian free field on \( \mathcal{B} \). Moreover, the fields \( h_{\mathcal{B}} \) and \( h_{\mathbb{A} \setminus \mathcal{B}} \) are independent.

This paper will rely in crucial ways on multiscale analysis, which means that we will want to consider Gaussian free fields defined on all boxes simultaneously—indeed, in other words, we want to couple the Gaussian free fields on all boxes. We do this the same way as in [9, Section 2.2.1], which we review briefly now. The coupling is no problem if the boxes we want to consider are all subsets of a universal box \( \mathbb{A} \), because we can define \( h_{\mathcal{B}} \) simultaneously for all \( \mathcal{B} \subset \mathbb{A} \) by (2.2). But then the universal box \( \mathbb{A} \) can be taken to be arbitrarily large, and it is easy to check that the joint laws of the Gaussian free field on finitely many fixed boxes remain the same no matter how large we take \( \mathbb{A} \). Thus, by Kolmogorov’s extension theorem, we can take \( h_{\mathcal{B}} \) to be defined simultaneously on all boxes \( \mathcal{B} \subset \mathbb{R}^2 \) so that (2.2) holds for any nested pair of boxes.\(^2\)

**Proposition 2.2.** If \( \mathbb{A} \) and \( \mathbb{A}' \) are disjoint domains, then \( h_{\mathbb{A}} \) and \( h_{\mathbb{A}' \setminus \mathbb{A}} \) are independent.

A final property of the Gaussian free field, which will be crucial for our study, is that it is conformally invariant. (See [5, Theorem 1.19].)

**Proposition 2.3.** Suppose that \( F : \mathbb{V} \to \mathbb{V}' \) is a conformal map between two domains \( \mathbb{V} \) and \( \mathbb{V}' \). Then \( h_{\mathbb{V}} \) has the same law as \( h_{\mathbb{V}} \circ F^{-1} \).

\(^2\)It is not really necessary to appeal to Kolmogorov’s extension theorem here. Indeed, our work will only need to use the coupling between finitely many boxes at a time, so it would be sufficient to couple such boxes “by hand” by choosing a sufficiently large “universal box” in each construction. However, we prefer to avoid such technicalities.
2.3 Gaussian free field estimates

In this subsection we introduce certain estimates on the Gaussian free field that will be important throughout our work.

**Lemma 2.4.** There is an absolute constant $C < \infty$ so that the following holds. Let $B$ be a box such that $\text{width}(B), \text{height}(B) \in [1/3, 3]$. If $x, y \in B$, then

$$G_B^*(x, y) + \log |x - y| \leq C. \quad (2.3)$$

**Proof.** This can be computed from the explicit formula for the Green’s function of the Laplacian on the unit disk, as well as a uniform bound on the derivative on $B$ on a Riemann map from $B^*$ to the unit disk. □

**Lemma 2.5.** There is a constant $C$ so that the following holds. Let $A \subset B$ be two nested boxes such that $\text{AR}(A), \text{AR}(B) \in [1/3, 3]$, and let $x$ be the center point of $A$. Then we have

$$\text{Var}(h_B^*:A^*(x)) \leq -\log \frac{\text{diam}(A)}{\text{diam}(B)} + C. \quad (2.4)$$

**Proof.** This follows simply from writing down the integral expression for the variance of $h_B^*:A^*(x)$ and applying (2.3). □

**Lemma 2.6.** There is a constant $C < \infty$ so that if $B \subset B^* \subset A$ are boxes with aspect ratios between $1/3$ and $3$, then for all $x, y \in B^o$, we have

$$\text{Var}(h_A:B^*(x) - h_A:B^*(y)) \leq C \frac{|x - y|}{\text{diam}(B)}. \quad \text{Var}(h_A:B^*(x) - h_A:B^*(y)) \leq C \frac{|x - y|}{\text{diam}(B)}.$$  

The proof of Lemma 2.6 is given in the case when $B$ is a ball in [10, Lemma 6.1]; the proof is the same with no essential changes in our setting.

**Lemma 2.7.** There is a constant $C < \infty$ so that if $B \subset B^* \subset A$ are boxes with aspect ratios between $1/3$ and $3$, then for all $\theta > 0$ we have

$$\mathbb{P} \left[ \max_{x, y \in B^o} |h_A:B^*(x) - h_A:B^*(y)| \geq \theta \right] \leq C e^{-\theta^2/C}.$$  

**Proof.** This follows from Lemma 2.6 and [10, Lemma 3.4]. Note that the proof of [10] is routine given Dudley’s entropy bound on the expected supremum of a Gaussian process (see [1, Theorem 4.1] and [25, equation (7.4), Theorem 7.1]). □

**Corollary 2.8.** There is a constant $C < \infty$ so that the following holds. If $A, B_1, \ldots, B_J$ is a set of boxes with aspect ratios between $1/3$ and $3$, and $B_j^* \subset A$ for each $j$, then, for all $\theta > 0$, we have

$$\mathbb{P} \left( \max_{1 \leq j \leq J} \max_{x, y \in B_j^o} |h_A:B_j^*(x) - h_A:B_j^*(y)| \geq \theta \sqrt{\log(J + 1)} \right) \leq C(J + 1)^{1-\theta^2/C}.$$  

**Proof.** This comes from union-bounding the result of Lemma 2.7 over $j \in \{1, \ldots, J\}$. □
Proposition 2.9. There is a constant $C < \infty$ so that the following holds. Let $\mathbb{B}$ be a box so that $\text{AR}(\mathbb{B}) \in [1/3, 3]$. Let $\mathbb{C}_1, \ldots, \mathbb{C}_j$ be a set of subboxes of $\mathbb{B}$ so that $\text{AR}(\mathbb{C}_j) \in [1/3, 3]$ for each $1 \leq j \leq J$ and that

$$\frac{\text{diam}_E(\mathbb{C}_j)}{\text{diam}_E(\mathbb{B})} \in [K/3, 3K].$$

Then for all $\theta, \tilde{\theta} > 0$, we have

$$P \left( \max_{j=1}^{J} \max_{x \in \mathbb{C}_j} h_{\mathbb{B}^\cdot;\mathbb{C}_j}(x) \geq \theta \log K + \tilde{\theta} \sqrt{\log(J+1)} \right) \leq C(JK^{-\frac{\theta^2}{2}} + (J + 1)^{1-\tilde{\theta}^2/C}). \quad (2.5)$$

In particular, if there is a constant $Q$ so that $J \leq QK^2$, and $\iota > 0$, then we have a constant $C = C(Q, \iota) < \infty$, depending only on $Q$ and $\iota$, so that

$$P \left( \max_{j=1}^{J} \max_{x \in \mathbb{C}_j} h_{\mathbb{B}^\cdot;\mathbb{C}_j}(x) \geq (\theta + \iota) \log K \right) \leq CK^{-\frac{\theta^2}{2}}. \quad (2.6)$$

Proof. The implication from (2.5) to (2.6) is clear, so we simply must prove (2.5). This is analogous to the easy direction of the characterization of the maximum of the discrete Gaussian free field given in [8]; see also [7, Theorem 2.1] for the discrete setting. Let $x_j$ be the center of the box $\mathbb{C}_j$. We have by (2.4) that

$$\max_{j=1}^{J} \text{Var} h_{\mathbb{B}^\cdot;\mathbb{C}_j}(x) \leq \log K + C.$$

Then we see that

$$P(h_{\mathbb{B}^\cdot;\mathbb{C}_j}(x_j) > \theta \log K) \leq \exp \left( -\frac{1}{2} \frac{\theta^2(\log K)^2}{\log K + C} \right) \leq CK^{-\frac{\theta^2}{2}}$$

for some different constant $C$ as long as $K$ is sufficiently large, so a union bound yields

$$P \left( \max_{j=1}^{J} \max_{x \in \mathbb{C}_j} h_{\mathbb{B}^\cdot;\mathbb{C}_j}(x) > \theta \log K \right) \leq CJK^{-\frac{\theta^2}{2}}. \quad (2.7)$$

On the other hand, by Corollary 2.8, we also have that

$$P \left( \max_{j=1}^{J} \max_{x \in \mathbb{C}_j} \left| h_{\mathbb{B}^\cdot;\mathbb{C}_j}(x) - h_{\mathbb{B}^\cdot;\mathbb{C}_j}(x_j) \right| > \tilde{\theta} \sqrt{\log J} \right) \leq CJ\text{J}^{-\frac{\tilde{\theta}^2}{2}}. \quad (2.8)$$

Combining (2.7) and (2.8) yields (2.5). \hfill \Box

Corollary 2.10. If $\theta_0 > 2$ and $J \leq QK^2$ then there is a constant $C$ depending on $Q$ and $\theta_0$ so that the following holds. Suppose $\mathbb{B}$ is a box so that $\text{AR}(\mathbb{B}) \in [1/3, 3]$. Let $\mathbb{C}_1, \ldots, \mathbb{C}_j$ be a set of subboxes of $\mathbb{B}$ so that $\text{AR}(\mathbb{C}_j) \in [1/3, 3]$ for each $1 \leq j \leq J$ and that

$$\frac{\text{diam}_E(\mathbb{C}_j)}{\text{diam}_E(\mathbb{B})} \in [K/3, 3K].$$

Let

$$F = \max_{j=1}^{J} \max_{x \in \mathbb{C}_j} h_{\mathbb{B}^\cdot;\mathbb{C}_j}(x).$$

Then

$$P \left( e^{yF - \gamma \theta_0 \log K} \geq v \right) \leq Cv^{-2}e^{-\frac{1}{4} \log y^2 \log K}. \quad (2.9)$$

Moreover, we have, for any $B \geq 0$, a constant $C$ depending on $B$ but not on $\mathbb{B}$ or $K$ so that

$$E \left( e^{yF - \gamma \theta_0 \log K} \right)^B \leq C. \quad (2.10)$$
Proof. Let $\theta_1 = (\theta_0 + 2)/2$. By (2.6) of Proposition 2.9, we have

$$\mathbb{P}[F \geq (\theta_0 + \theta) \log K] \leq CK^{2-(\theta_1+\theta)^2/2} \leq CK^{-2\theta^2/2},$$

with the last inequality because $\theta_1 > 2$, so

$$\mathbb{P}[e^{\gamma F - \gamma \theta_1 \log K} \geq K^{\gamma \theta}] \leq CK^{-2\theta^2/2},$$

so

$$\mathbb{P}[e^{\gamma F - \gamma \theta_1 \log K} \geq \nu] \leq C\nu^{-2/\gamma}e^{-\frac{1}{2\gamma^2} \frac{(\log \nu)^2}{\log K}},$$

which is (2.9). Then we have that

$$E(e^{\gamma F - \gamma \theta_1 \log K})^B = \int_0^\infty \mathbb{P}[e^{\gamma F - \gamma \theta_1 \log K} \geq \nu^{1/B}] \, d\nu \leq C \left(1 + \int_1^\infty e^{-\frac{1}{2\gamma^2 B^2} \frac{(\log \nu)^2}{\log K}} \, d\nu\right) \leq Ce^{CB^2\sqrt{\log K}},$$

hence (2.10). \qed

2.4 Liouville quantum gravity

In this section we briefly review the properties of the Liouville quantum gravity measure. We first must define the circle average process of a Gaussian free field. If $h_A$ is a Gaussian free field on a box $A$, then, for $x \in A$ so that $\text{dist}_E(x, \partial A) > \varepsilon$, we define $h_{\varepsilon}(x)$ to be the result of integrating $h_A$ against the uniform measure on $\partial B(x, \varepsilon)$. The Liouville quantum gravity at inverse temperature $\gamma$ is then supposed to be the limit as $\varepsilon \to 0$ of the random measure

$$\mu_{h_{\varepsilon}}(dx) = e^{\frac{\gamma^2}{2} e^{\gamma h(x)}} \, dx.$$ (2.11)

Indeed, we have the following result of [18]:

**Theorem 2.11.** If $\gamma < 2$, then there is a random measure $\mu_{h_{\varepsilon}}$ such that, with probability 1, we have

$$\lim_{k \to \infty} \mu_{h_{\varepsilon}, 2^{-k}} = \mu_{h_{\varepsilon}}$$ (2.12)

weakly.

Throughout the paper, we will treat $\gamma$ as a fixed constant in $(0, 2)$. To economize on indices, we will suppress $\gamma$ in the notation for LQG and subsequently derived objects. All constants throughout the paper will implicitly depend on $\gamma$. We will also fix throughout the paper a constant $\theta_0 > 2$ so that

$$\eta := \frac{2\gamma \theta_0}{4 + \gamma^2} < 1;$$ (2.13)

$\eta$ will also be treated as a fixed constant throughout the paper. The reason for insisting that $\theta_0 > 2$ is to match with the condition in Corollary 2.10—we will treat $\theta_0 \log K$ as the cutoff below which the “coarse field” at scale $K$ must be with very high probability. The reason for insisting that $\eta < 1$ is so that, according to Proposition 2.14 below, a sub-box of a box, once the maximum coarse field is considered, will “look like” a strictly smaller box than the larger box from the perspective of the LQG measure.

We will also need the existence of positive and negative moments of the Liouville quantum gravity measure, which was proved in [22, 33]; see also [35, Theorems 2.11 and 2.12]:

\[ \]
Proposition 2.12. There is a $\nu_0 > 1$ so that if $0 \leq \nu < \nu_0$, then for all domains $A$, we have
\[ \mathbb{E} \mu_{h_A}(A)^\nu < \infty. \] (2.14)
Moreover, for any $0 \leq \nu < \infty$, we have for all domains $A$ that
\[ \mathbb{E} \mu_{h_A}(A)^\nu < \infty. \] (2.15)

Proposition 2.13. Let $A$ be a domain. It almost surely holds that for every $x \in A$ and $r > 0$ so that $B(x, r) \subset A$, we have $\mu_{h_A}(B(x, r)) > 0$.

Proof. This follows from a simple union bound and (2.15). □

The fact that the convergence in (2.12) is almost sure will be important for us, because in our multiscale analysis we will often consider the Liouville quantum gravity on different boxes, with coupling induced by the coupling we have introduced for the Gaussian free field.

If $\mu$ and $\nu$ are Radon measures on the same set $X$, we will say that $\mu \leq \nu$ if $\mu(Y) \leq \nu(Y)$ for all Borel sets $Y \subset X$. If $A \subset B$, then we have by (2.11) and the smoothness of $h_{B; A}$ that
\[ \mu_{h_B} \rvert_A = \exp \left\{ \gamma h_{B; A} \right\} \mu_{h_B} \] (2.16)
almost surely. (Here, $\mu_{h_B} \rvert_A$ denotes the measure $\mu_{h_B}$ restricted to $A$.) This implies that
\[ \mu_{h_B} \rvert_A \leq \exp \left\{ \gamma \max_{x \in A} h_{B; A}(x) \right\} \mu_{h_B}. \] (2.17)

An important property of Liouville quantum gravity is the conformal covariance of the measure; see [5, Theorem 2.8]:

Proposition 2.14. Suppose $\mathcal{Y}$ and $\mathcal{Y}'$ are domains and $F : \mathcal{Y} \to \mathcal{Y}'$ is a conformal homeomorphism. Then we have that
\[ \mu_{h_\mathcal{Y}} \circ F^{-1} = e^{(2+\gamma^2/2) \log \| F^{-1} \|} \mu_{h_\mathcal{Y}} \circ F^{-1} \mathbb{P} \text{ law } e^{(2+\gamma^2/2) \log \| F^{-1} \|} \mu_{h_\mathcal{Y}'}. \]

2.5 Metrics defined in terms of measures

In this section we describe the process we will use to construct a metric from a measure. The definitions and results in this section are purely deterministic. Of course, we plan to apply this construction to the case when the measure is a Liouville quantum gravity measure, which we will do in the next section, yielding the Liouville graph distance.

If $B$ is a box, then define the space of paths $\mathcal{P}_B$ to be the set of continuous images of $[0, 1]$ in $B$. If $x, y \in B$, then define $\mathcal{P}_B(x, y) = \{ \pi \in \mathcal{P}_B \mid x, y \in \pi \}$. Given a box $B$ and a finite measure $\mu$ on $B^{(R)}$, define
\[ \mathcal{B}(\mu, B, \delta) = \{ B(x, r) \mid r \in (0, \text{diam}(B)), x \in B, \mu(B(x, r)) < \delta \}. \] (2.18)

Here and throughout the paper, $B(x, r)$ denotes the open Euclidean ball with center $x$ and radius $r$.

Definition 2.15. Let $B$ be a domain, $\mu$ a measure on $B$, $\delta > 0$, and $R > 0$. If $\mu$ is a measure on $B^{(R)}$, then define the $\mu$-graph length of a path $\pi \in \mathcal{P}_B$ as
\[ d_{\mu, B, \delta}(\pi) = \min \left\{ n \in \mathbb{N} \mid \exists B_1, \ldots, B_n \in \mathcal{B}(\mu, B, \delta) \text{ s.t. } \pi \subset B_1 \cup \cdots \cup B_n \right\}. \]

We further define
\[ d_{\mu, B, \delta}(x, y) = \begin{cases} 0 & x = y, \\ \min_{\pi \in \mathcal{P}_B(x, y)} d_{\mu, B, \delta}(\pi) & x \neq y. \end{cases} \]
For the purposes of establishing measurability of the distances when the measure is random, we now show that the distance can be expressed as the minimum of countably many functions. This is a purely technical point.

**Lemma 2.16.** Let \( \mathbb{B} \) be a domain, \( \mu \) a measure on \( \mathbb{B} \), and \( \delta > 0 \). Define

\[
\mathcal{B}_Q(\mu, \mathbb{B}, \delta) = \{B(x, r) \mid r \in (0, \text{diam}_E(\mathbb{B})) \cap Q, x \in \mathbb{B} \cap Q^2, \mu(B(x, r)) < \delta\}.
\]

Then we have that

\[
d_{\mu, \mathbb{B}, \delta}(\pi) = \min \left\{n \in \mathbb{N} : \exists B_1, \ldots, B_n \in \mathcal{B}_Q(\mu, \mathbb{B}, \delta) \text{ s.t. } \pi \subset B_1 \cup \cdots \cup B_n \right\}.
\]

**Proof.** This follows from the fact that if \( B \) is a closed ball such that \( \mu(B) < \delta \), then there is a ball \( B' \) containing \( B \) so that \( \mu(B') < \delta \) and \( B' \) has rational center and rational radius arbitrarily close to that of \( B \). \( \square \)

It will be more convenient for us to work with a modified graph length, which has a somewhat better “continuity” property with respect to small perturbations of the field. (See equations (2.30) and (2.31) below.) We will see shortly (Proposition 2.26) that the modified graph length differs from the original one only by at most a factor of 2. Define

\[
\mathcal{B}(\mathbb{B}, R) = \{B(x, r) \mid r \in (0, R), x \in \mathbb{B}\}.
\]

**Definition 2.17.** For \( \delta > 0 \), define

\[
\kappa_\delta(t) = \max \{1, \delta^{-1}t\}. \tag{2.20}
\]

If \( \mathbb{B} \) is domain and \( \mu \) is a measure on \( \mathbb{B}^{(R)} \), then define the \( (\mu, \mathbb{B}, \delta, R) \)-graph length of a path \( \pi \in \mathcal{P}_{\mathbb{B}} \) as

\[
d_{\mu, \mathbb{B}, \delta, R}(\pi) = \inf \left\{ \sum_{k=1}^{n} \kappa_\delta(\mu(B_k)) : \exists B_1, \ldots, B_n \in \mathcal{B}(\mathbb{B}, R) \text{ s.t. } B_{\pi} \subset B_1 \cup \cdots \cup B_n \right\}.
\]

We further define

\[
d_{\mu, \mathbb{B}, \delta, R}(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \min_{\pi \in \mathcal{P}_{\mathbb{B}}(x, y)} d_{\mu, \mathbb{B}, \delta, R}(\pi) & \text{if } x \neq y. \end{cases}
\]

If \( R > \text{diam}_E(\mathbb{B}) \), then we define \( d_{\mu, \mathbb{B}, \delta, R} = d_{\mu, \mathbb{B}, \delta, \text{diam}_E(\mathbb{B})} \), since any ball of radius greater than \( \text{diam}_E(\mathbb{B}) \) can be replaced by one of radius less than \( 2 \text{diam}_E(\mathbb{B}) \) without changing the minimum.

**Remark 2.18.** The difference between the definitions of \( d \) and \( d_\delta \) is that the definition of \( d_\delta \) allows “too large” balls to be used, if one pays a “price.” The price, however, is heuristically very steep, because one has to pay for the measure of the entire large ball, rather than just the smaller region around the path that would be required if one used smaller balls. Thus we do not expect \( d \) and \( d_\delta \) to differ by very much. The reason to use \( d \) instead of \( d_\delta \) is that \( d \) changes very little under very small multiplicative perturbations of the measure, while \( d_\delta \) may change by up to a factor of 2 even if the measure is multiplied by \( 1 + \epsilon \).

**Remark 2.19.** The parameter \( R \) of \( d \) restricts the maximum Euclidean size of a ball that can be used to cover the path. If \( \mathbb{B} \) and \( R \) are of order 1, and \( \delta \ll 1 \), then we would not expect \( R \) to play any significant role, because balls with Euclidean radius of order 1 are extremely unlikely to have LQG mass \( \delta \). However, it will often simplify our analysis to work with a metric in which we know that no balls of radius greater than \( R \) are used—most importantly in order to use Proposition 2.30 below. We can then deal with the error incurred by this modification separately.

We prove a measurability lemma for \( d \), analogous to Lemma 2.16.
Lemma 2.20. Let $\mathcal{B}$ be a domain and $R > 0$. Define
\[
\mathcal{B}_Q(\mathcal{B}, R) = \{ B(x, r) \mid r \in (0, R) \cap Q, x \in \mathcal{B} \cap Q^2 \}.
\]
Then we have that
\[
d_{\mu, \mathcal{B}, \delta, R}(\pi) = \inf \left\{ \sum_{k=1}^{n} \kappa_\delta(\mu(B_k)) : \exists B_1, \ldots, B_n \in \mathcal{B}(\mathcal{B}, R) \text{ s.t. } \pi \subset B_1 \cup \cdots \cup B_n \right\}.
\]

Proof. This follows from the fact that if $B$ is a closed ball, then for any $\varepsilon > 0$ there is a ball $B'$ containing $B$ so that $\mu(B') < \mu(B) + \varepsilon$ and $B'$ has rational center and rational radius arbitrarily close to that of $B$. \hfill \Box

2.5.1 Notation for distances

We will use a flexible notation for the geometrical notions of distance that we will use. For a box $\mathcal{B}$, we recall that $L_\mathcal{B}$, $R_\mathcal{B}$, $T_\mathcal{B}$, and $B_\mathcal{B}$ denote the left, right, top, and bottom edges of $\mathcal{B}$, respectively. Given a box $\mathcal{B}$, we will call the “easy crossing” of $\mathcal{B}$ the shorter crossing between opposite sides of $\mathcal{B}$, and the “hard crossing” of $\mathcal{B}$ the longer crossing between opposite sides of $\mathcal{B}$. Formally, let
\[
(easy_\mathcal{B}, hard_\mathcal{B}) = \begin{cases} ((L_\mathcal{B}, R_\mathcal{B}), (T_\mathcal{B}, B_\mathcal{B})) & \text{width}(\mathcal{B}) < \text{height}(\mathcal{B}) \\ ((T_\mathcal{B}, B_\mathcal{B}), (L_\mathcal{B}, R_\mathcal{B})) & \text{height}(\mathcal{B}) < \text{width}(\mathcal{B}). \end{cases}
\]
In all of these notations, we will drop the subscript $\mathcal{B}$ when it is clear from context. We will often use this in notation like $d_{\mu, \mathcal{B}, \delta, R}(L, R)$, denoting the minimum $(\mu, \mathcal{B}, \delta, R)$-graph length of paths crossing from left to right in $\mathcal{B}$, or $d_{\mu, \mathcal{B}, \delta, R}(easy)$ denoting the minimum $(\mu, \mathcal{B}, \delta, \mathcal{R})$-graph length of an easy crossing of $\mathcal{B}$.

Also, define
\[
d_{\mathcal{A}, \mathcal{B}, \delta, R}(M_\mathcal{A}) = \min_{x, y \in \mathcal{A}} \{ d_{\mu, \mathcal{B}, \delta, R}(x, y) \}.
\]

Suppose that $\mathcal{A}$ is the intersection of a rectangular annulus with a rectangle $\mathcal{R}$, both of whose side lengths are greater than the diameter of the annulus, so that either $\mathcal{A} \subset \mathcal{R}$ or $\mathcal{A} \cap \partial \mathcal{R}$ has exactly two connected components. If $\mu$ is a measure on $\mathcal{A}(\mathcal{R})$, define
\[
d_{\mu, \mathcal{A}, \delta, R}(\mathcal{A}) = \min_{\pi} \{ d_{\mu, \mathcal{B}, \delta, R}(\pi) \},
\]
where $\pi$ ranges over all circuits around $\mathcal{A}$ if $\mathcal{A}$ is an annulus, or simply as
\[
d_{\mu, \mathcal{A}, \delta, R}(\mathcal{A}) = d_{\mu, \mathcal{A}, \delta, R}(X, Y),
\]
where $X$ and $Y$ are the two connected components of $\mathcal{A} \cap \partial \mathcal{R}$ if $\mathcal{A}$ is simply-connected.

2.5.2 Comparison inequalities

The following comparison inequalities are immediate consequences of Definition 2.17:

1. If $\delta \leq \delta'$, then
\[
d_{\mu, \mathcal{B}, \delta, R} \geq d_{\mu, \mathcal{B}, \delta', R}.
\]
2. If $R' \leq R$, then
\[
d_{\mu, \mathcal{B}, \delta, R} \leq d_{\mu, \mathcal{B}, \delta, R'}.
\]
3. For all $R > 0$ we have
\[ d_{\mu,\mathcal{B},\delta,R} = d_{\mu,\mathcal{B},\delta,R;\text{diam}(\mathcal{B})}. \] (2.26)

4. If $\mu|_{\overline{\mathcal{B}(r)}} \leq v|_{\overline{\mathcal{B}(r)}}$, then
\[ d_{\mu,\mathcal{B},\delta,R} \leq d_{v,\mathcal{B},\delta,R}. \] (2.27)

5. If $\mathcal{B} \subset \mathcal{B}'$, then
\[ d_{\mu,\mathcal{B}',\delta,R}(x,y) \leq d_{\mu,\mathcal{B},\delta,R}(x,y). \] (2.28)

6. For any $\alpha > 0$, we have
\[ d_{\alpha \mu,\mathcal{B},\delta,R} = d_{\mu,\mathcal{B},\alpha^{-1}\delta,R}. \] (2.29)

If $\alpha \leq 1$, then
\[ d_{\mu,\mathcal{B},\alpha\delta,R}(x,y) \leq \alpha^{-1} d_{\mu,\mathcal{B},\delta,R}(x,y). \] (2.30)

If $\alpha \geq 1$, then
\[ d_{\alpha \mu,\mathcal{B},\delta,R} \leq \alpha d_{\mu,\mathcal{B},\delta,R}(x,y). \] (2.31)

7. The triangle inequality holds: if $\pi = \pi_1 \cup \pi_2$, then
\[ d_{\mu,\mathcal{B},\delta,R}(\pi) \leq d_{\mu,\mathcal{B},\delta,R}(\pi_1) + d_{\mu,\mathcal{B},\delta,R}(\pi_2). \] (2.32)

**Remark 2.21.** All of the above properties with the exception of (2.30) and (2.31), as well as those involving $R$, also apply for $d$. Note that $d$ could have been defined with the $R$ parameter in the same way as $d$, and then (2.25) and (2.26) would hold for $d$ as well, but we will not need this in the paper.

We will need a bound in the opposite direction for (2.25), which we will prove as Lemma 2.24. This first requires the following lemma and definition.

**Lemma 2.22.** Suppose that $a_1 + \cdots + a_n \leq b$. Then we have (recalling that $\kappa_\delta$ was defined in (2.20)) that
\[ \sum_{k=1}^{n} \kappa_\delta(a_k) \leq \kappa_\delta(b) + n. \]

**Proof.** We have that $\delta^{-1} t \leq \kappa_\delta(t) \leq \delta^{-1} t + 1$ for all $t$, so
\[ \sum_{k=1}^{n} \kappa_\delta(a_k) \leq \sum_{k=1}^{n} (\delta^{-1} a_k + 1) \leq \delta^{-1} b + n \leq \kappa_\delta(b) + n. \]

**Definition 2.23.** Suppose that $\mathcal{B}$ is a domain, $\mu$ is a measure on $\mathcal{B}$, and $\delta, R > 0$. Let $x, y \in \mathcal{B}$ and suppose that $\pi$ is a path in $\mathcal{B}$ connecting $x$ and $y$. We will say that $\pi$ is a $(\mu,\mathcal{B},\delta,R)$-geodesic if $\pi$ consists of a sequence of straight line segments between successive points $x, z_1, \ldots, z_N, y \in \mathcal{B}$ and there is a sequence of radii $r_1, \ldots, r_N \in (0, R)$ so that $\sum_{j=1}^{N} \mu(B(z_j, r_j)) = d_{\mu,\mathcal{B},\delta,R}(x,y)$ and $\pi \subset \bigcup_{j=1}^{N} B(z_j, r_j)$. We will call $z_1, \ldots, z_N$ the corresponding geodesic points, $r_1, \ldots, r_N$ the corresponding geodesic radii, and $B(z_1, r_1), \ldots, B(z_N, r_N)$ the corresponding geodesic balls.

**Lemma 2.24.** There is a constant $C$ so that if $B$ is a domain, $\mu$ is a measure on $\mathcal{B}$, $\delta > 0$, and $0 < R' \leq R$, then
\[ d_{\mu,\mathcal{B},\delta,R'}(x,y) \leq d_{\mu,\mathcal{B},\delta,R}(x,y) + C \frac{R \text{Leb}(\mathcal{B}(R'))}{(R')^3}, \] (2.33)

where Leb denotes the Lebesgue measure, and also
\[ d_{\mu,\mathcal{B},\delta,R'}(x,y) \leq (1 + CR/R') d_{\mu,\mathcal{B},\delta,R}(x,y). \] (2.34)
\textbf{Proof.} Let \(x, y \in \mathbb{B}\) and let \(\pi\) be a \((\mu, \mathbb{B}, \delta, R)\)-geodesic between \(x\) and \(y\). Let \(B_1, \ldots, B_N\) be a set of geodesic balls for a geodesic between \(x\) and \(y\). We note that for any \(1 \leq i < j \leq N\), we have that \(B_i \cap B_j \cap B_k = \emptyset\): if not, then one of the balls could be eliminated to obtain a shorter distance. Therefore, if \(S = \{i \in \{1, \ldots, N\} \mid \text{diam}_E(B_i) \geq R'\}\), then we have that

\[
\frac{1}{2} |S|(R')^2 \leq \sum_{i: \text{diam}_E(B_i) \geq R'} \text{Leb}(B_i) \leq 2 \text{Leb} \left( \bigcup_{i: \text{diam}_E(B_i) \geq R'} B_i \right) \leq 2 \text{Leb}(\mathbb{B}(R)),
\]

so

\[
|S| \leq 4 \pi \frac{\text{Leb}(\mathbb{B}(R))}{(R')^2}.
\]

Now for each \(i \in S\), \(B_i\) can be replaced by a set of at most \(2\pi[R/R']\) subset balls of radius \(R'\), with centers in \(\mathbb{B}\), so that a path from \(x\) to \(y\) in \(\mathbb{B}\) is still covered, and the new balls in each older ball have total measure no larger than the measure of the original ball. This implies, by Lemma 2.22, that

\[
d_{\mu, \mathbb{B}, \delta, R'}(x, y) \leq d_{\mu, \mathbb{B}, \delta, R}(x, y) + 2\pi [R/R'] |S| \leq d_{\mu, \mathbb{B}, \delta, R}(x, y) + C \frac{R \text{Leb}(\mathbb{B}(R))}{(R')^3}
\]

for some absolute constant \(C\), which completes the proof of (2.33). On the other hand, (2.34) follows from (2.35) by the trivial bound \(|S| \leq N\). \(\Box\)

We will also want to relate our modified distances \(d\) back to our original distances of interest \(\underline{d}\). This turns out to be very simple, but we first require a definition.

**Definition 2.25.** By an \textit{admissible measure} on a set \(\mathbb{A}\) we will mean a nonatomic Radon measure \(\mu\) such that for every \(x \in \mathbb{A}\) and \(r > 0\) such that \(B(x, r) \subset \mathbb{A}\), we have

\[
\mu(B(x, r)) > 0.
\]

By Proposition 2.13, the Liouville quantum gravity measures are admissible almost surely. Note also that if \(\mu\) is an admissible measure then \(\alpha \mu\) is also an admissible measure for any \(\alpha \in (0, \infty)\). Then we can state the relationship between \(d\) and \(\underline{d}\).

**Proposition 2.26.** If \(\mu\) is an admissible measure on \(\mathbb{B}\), then

\[
d_{\mu, \mathbb{B}, \delta}(x, y) \leq d_{\mu, \mathbb{B}, \delta}(x, y) \leq 2d_{\mu, \mathbb{B}, \delta}(x, y).
\]

The first inequality is trivial by (2.20). The second follows from the following lemma.

**Lemma 2.27.** Suppose that \(\mu\) is an admissible measure on \(\mathbb{B}\), \(\alpha \in (0, 1)\), \(B\) is a closed ball so that \(\mu(B) = \delta\), and \(x, y \in \partial B\). Then there are closed balls \(B', B'' \subset B\) so that \(\mu(B_1) < \alpha \delta\), \(\mu(B_2) < (1 - \alpha) \delta\), \(B' \cup B'' \) is path-connected, and \(x, y \in B_1 \cup B_2\).

**Proof.** For \(r \in [0, 1]\), let \(B_r\) be the closed ball of radius \(r\) which is internally tangent to \(B\) at \(x\), so \(B_0 = \{x\}\) and \(B_1 = B\). Let \(\bar{B}_r\) be the closed ball which is internally tangent to \(B\) at \(y\) and externally tangent to \(B_r\). (See Figure 2.1.) Let \(f(r) = \mu(B_r)\) and let \(\overline{f}(r) = \mu(B_r)\). We note that \(f\) is increasing and \(\overline{f}\) is decreasing. Since \(\mu\) is nonatomic, we have that

\[
f(r) + \overline{f}(r) = \mu(B) = \delta.
\]

We claim that \(f\) is right-continuous. Indeed, if \(r_n \downarrow r\), then we have

\[
\lim_{n \to \infty} f(r_n) = \mu \left( \bigcap_{n=1}^{\infty} B_{r_n} \right) = \mu(B_r).
\]
Similarly, $\tilde{f}$ is left-continuous, because if $r_n \uparrow r$, then we have

$$\lim_{n \to \infty} \tilde{f}(r_n) = \mu \left( \bigcap_{n=1}^{\infty} \overline{B}_{r_n} \right) = \mu(\overline{B}_r).$$

In particular, this implies that $f$ and $\tilde{f}$ are both upper semicontinuous. Let

$$r_* = \sup \{ r \in [0,1] \mid f(r) < \alpha \delta \}. \tag{2.38}$$

Because $f$ is upper semicontinuous, we have that

$$f(r_*) \geq \alpha \delta, \tag{2.39}$$

so

$$\tilde{f}(r_*) \leq (1 - \alpha) \delta. \tag{2.40}$$

Now if $\tilde{f}(r_*) = (1 - \alpha) \delta$, then (2.39) implies that $f(r_*) = \alpha \delta$, so $\mu(B \setminus (B' \cup B'')) = 0$, contradicting the assumption that $\mu$ is admissible. Therefore, (2.40) implies that we must in fact have that $\tilde{f}(r_*) < (1 - \alpha) \delta$.

By the left-continuity of $\tilde{f}$, there is an $r' < r_*$ so that $\tilde{f}(r') < (1 - \alpha) \delta$ as well, so since $r' < r_*$, we must have that $f(r') < \alpha \delta$ by (2.38) and the fact that $f$ is increasing. Thus we can take $B' = B_{r'}$ and $B'' = \overline{B}_{r'}$ to complete the proof. \hfill \Box

**Proposition 2.28.** If $\mu$ is an admissible measure on $\mathbb{E}$ and $R > 0$, then

$$\lim_{\delta \to 0} d_{\mu, \mathbb{E}, \delta, R}(L, R) = \infty. \tag{2.41}$$

**Proof.** For any $r > 0$, there is a finite collection of balls $B_1, \ldots, B_N$ so that if $x \in \mathbb{E}$, then there is an $i \in \{1, \ldots, N\}$ so that $B_i \subset B(x, r)$. Let $\delta_0 = \min_{1 \leq i \leq N} \mu(B_i)$, which is positive by (2.36). Then if $\delta < \delta_0$, any ball of $\mu$-measure at most $\delta$ must have radius at most $r$, so

$$d_{\mu, \mathbb{E}, \delta, R}(L, R) \geq [r^{-1} \text{dist}_{E}(L_{\mathbb{E}}, R_{\mathbb{E}})],$$

which implies the desired limit. \hfill \Box

**Proposition 2.29.** Suppose that $C \subset \mathbb{R}$ are boxes and $\mu$ is a measure on $\mathbb{R}^{(R)}$. Let $x, y \in \mathbb{R}$ and let $\pi$ be a $(\mu, \mathbb{R}, \delta, R)$-geodesic between $x$ and $y$. Suppose that $w, z \in C \cap \pi$, that $w$ appears before $z$ on the path $\pi$, and that the part of $\pi$ between $w$ and $z$ is completely contained in $C$. Then

$$d_{\mu, C, \delta, R}(w, z) = d_{\mu, \mathbb{R}, \delta, R}(w, z).$$

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Figure 2.2: Construction in the proof of Proposition 2.29.

Proof. By (2.28), it is sufficient to prove that

\[ d_{\mu,C,\delta,R}(w, z) \leq d_{\mu,\mathbb{R},\delta,R}(w, z). \]

Let \( z_1, \ldots, z_d \) and \( r_1, \ldots, r_d \) be the geodesic centers and radii for \( \pi \), respectively. Let \( z_j, \ldots, z_k \) be the minimal substring of the \( z_i \)s so that the segment of \( \pi \) between \( w \) and \( z \) is contained in \( B(z_j, r_j) \cup \cdots \cup B(z_k, r_k) \). Then

\[ d_{\mu,\mathbb{R},\delta,R}(w, z) = \sum_{i=j}^{k} \mu(B(z_i, r_i)). \]

For each \( j \leq i \leq k \), if \( z_i \in C \), then

\[ B(z_i, r_i) \in \mathcal{B}(C, R). \]

We can create a closed ball \( B \) around \( u \) that is contained in \( B(z_j, r_j) \) and internally tangent to \( B(z_j, r_j) \) at the point of intersection \( \pi \cap C \cap \partial B(z_j, r_j) \). Then we have \( B \in \mathcal{B}(\mu, \mathbb{R}, \delta, R) \) so that \( \pi \cap B(z_k, r_k) \cap C \subset B' \). (See Figure 2.2.) This completes the proof, since a path from \( w \) to \( z \) is contained in \( B \cup B(z_{j+1}, r_{j+1}) \cup \cdots \cup B(z_{k-1}, r_{k-1}) \cup B' \). (If \( j = k \), then \( B' \) is not necessary and the construction still works.) \( \Box \)

Proposition 2.30. Suppose that \( C_1, \ldots, C_J \subset \mathbb{R} \) are boxes,

\[ R < \min_{1 \leq i, j \leq J} \text{dist}_E(C_i, C_j), \quad (2.42) \]

and \( \mu \) is a measure on \( \mathbb{R}(R) \). Let \( x, y \in \mathbb{R} \) and let \( \pi \) be a \( (\mu, \mathbb{R}, \delta, R) \)-geodesic between \( x \) and \( y \). Suppose that, for each \( 1 \leq i \leq j \), we have \( w_i, z_i \in C_i \cap \pi \), with \( w_i \) appearing before \( z_i \) on the path \( \pi \) and the part of \( \pi \) between \( w \) and \( z \) being completely contained in \( C_i \). Then

\[ d_{\mu,\mathbb{R},\delta,R}(x, z) \leq \sum_{i=1}^{J} d_{\mu,\mathbb{R},\delta,R}(w_i, z_i). \]

Proof: This follows from the proof of Proposition 2.29 when we note that the condition (2.42) implies that a single ball cannot intersect two distinct \( C_i \)s. \( \Box \)

2.6 Liouville graph distance

We now discuss the graph metric with respect to the Liouville quantum gravity measure: the Liouville graph distance.
**Definition 2.31.** If $A \subset \mathbb{B}$ are boxes and $\delta > 0, \gamma \in (0, 2)$, and $R \leq \text{dist}_E(\mathbb{B}, A)$, then we use the notation

$$d_{A, B, \delta, R} = d_{\mu_{A, \gamma, B, \delta, R}}.$$ 

To avoid a proliferation of subscripts, we will often abbreviate this notation for distances. If $A$ is unspecified we will mean that $A = \mathbb{B}^+$, if $R$ is unspecified then we will mean that $R = \text{diam}_E(\mathbb{B})$, and if in addition to $R$, the parameter $\delta$ is also unspecified, then we will mean that $\delta = 1$. Thus we have

$$d_A, B, \delta = d_{A, B, \delta, \text{diam}_E(\mathbb{B})},$$

$$d_{B, \delta} = d_{\mathbb{B}, B, \delta, \text{diam}_E(\mathbb{B})},$$

$$d_{A, B} = d_{A, B, 1, \text{diam}_E(\mathbb{B})},$$

$$d_B = d_{\mathbb{B}, B, 1, \text{diam}_E(\mathbb{B})}.$$

(Of course syntactically the second and third lines are ambiguous, but it will always be clear which variables represent boxes and which represent numbers, so there is no risk of confusion.)

**Definition 2.32.** We define $\Theta^{LR}_{A, B, \delta, R}(p)$, $\Theta^{\text{easy}}_{A, B, \delta, R}(p)$ and $\Theta^{\text{hard}}_{A, B, \delta, R}(p)$ to be the $p$th quantiles of the random variables $d_{A, B, \delta, R}(L, R)$, $d_{A, B, \delta, R}(\text{easy})$, $d_{A, B, \delta, R}(\text{hard})$, respectively. We will abbreviate these notations in the same way as the $d_{A, B, \delta, R}$ notations.

We now establish several properties of the Liouville graph distance.

**Proposition 2.33.** If $B \subset A$ and $\delta, R > 0$, then the processes

$$\{d_{A, B, \delta, R}(x, y)\}_{x, y \in B}$$

and

$$\{d_{A, B, \delta, R}(x, y)\}_{x, y \in B}$$

are measurable with respect to the field $h_A$.

**Proof.** This follows from Lemma 2.16 and Lemma 2.20, since both distances can be written as a minimum of countably many functions. \qed

**Proposition 2.34.** If $A_1 \cap A_2 = \emptyset$, $B_i \subset A_i$ for $i = 1, 2$, and $\delta_1, \delta_2 > 0$, then $d_{A_1, B_1, \delta_1, R}$ and $d_{A_2, B_2, \delta_2, R}$ are independent.

**Proof.** This follows from the independence of the Gaussian free field on disjoint boxes (Proposition 2.2). \qed

### 2.6.1 Covariance properties of the Liouville graph distance

The conformal covariance of the LQG measure heuristically implies conformal covariance of the Liouville graph distance. However, because the definition of Liouville graph distance specifies Euclidean balls, which are not in general taken to Euclidean balls by conformal maps, this is not exactly true. The next two propositions observe that the Liouville graph distance is covariant under similarity transformations, which do of course take Euclidean balls to Euclidean balls. The RSW result presented in Section 3 will rely on an approximate more general conformal covariance of the Liouville graph distance.

**Proposition 2.35.** If $B \subset A$, $\delta, R > 0$, and $f$ is a translation or rotation map, then

$$\{d_{A, B, \delta, R}(x, y)\}_{x, y \in B} \overset{\text{law}}{=} \{d_{f(A), f(B), \delta, R}(f(x), f(y))\}_{x, y \in B}.$$
Proposition 2.36. If $\mathbb{B} \subset \mathcal{A}$ and $\delta, R > 0$, then for any $\alpha > 0$, we have
\[
\left\{ d_{\alpha \mathbb{A}, \alpha \mathbb{B}, \alpha^{2}/2 \cdot \delta, \alpha \cdot R} (\alpha x, \alpha y) \right\}_{x, y \in \mathbb{B}} \xrightarrow{\text{law}} \left\{ d_{\mathbb{A}, \mathbb{B}, \delta, R} (x, y) \right\}_{x, y \in \mathbb{B}}.
\]

Proof. Let $F : \mathcal{A} \rightarrow \alpha \mathcal{A}$ be given by scaling by $\alpha$. Then $F$ is a conformal map, so we have that
\[
\mu_{\alpha \mathbb{A}} \xrightarrow{\text{law}} \alpha^{2 + y^{2}/2} (\mu_{\mathbb{A}} \circ F^{-1})
\]
by the conformal covariance (Proposition 2.14). Then we have, recalling (2.19), that
\[
\mathcal{B}(\alpha \mathbb{B}, \alpha R) = \{ B(x, r) \mid r \in (0, \alpha R), x \in \alpha \mathbb{B} \} = \{ \alpha B(x, \alpha^{-1} r) \mid r \in (0, \alpha R), x \in \mathbb{B} \} = \alpha \mathcal{B}(\mathbb{B}, R),
\]
and that
\[
\left\{ \kappa_{\alpha^{2 + y^{2}/2}} (\mu(B)) \right\}_{B \in \mathcal{B}(\mathbb{B}, R)} = \left\{ \kappa_{\delta} (\mu(B)) \right\}_{B \in \mathcal{B}(\mathbb{B}, R)},
\]
which means that
\[
\left\{ d_{\alpha \mathbb{A}, \alpha \mathbb{B}, \alpha^{2 + y^{2}/2} \cdot \delta, \alpha \cdot R} (\alpha x, \alpha y) \right\}_{x, y \in \mathbb{B}} \xrightarrow{\text{law}} \left\{ d_{\mathbb{A}, \mathbb{B}, \delta, R} (x, y) \right\}_{x, y \in \mathbb{B}}.
\]

We note that a simple consequence of Proposition 2.36 and (2.24) is the following monotonicity of the quantiles:

Proposition 2.37. If $\alpha \geq 1$, then for all boxes $\mathbb{B} \subset \mathcal{A}$, all $p \in (0, 1)$, and all $R > 0$ we have
\[
\Theta^{LR}_{\alpha \mathbb{A}, \alpha \mathbb{B}, 1, \alpha R} (p) = \Theta^{LR}_{\mathcal{A}, \mathcal{B}, \alpha^{2 + y^{2}/2}, R} (p) \geq \Theta^{LR}_{\mathcal{A}, \mathbb{B}, R} (p).
\]

Lemma 2.38. For any $\mathbb{B} \subset \mathcal{A}$, all $p \in (0, 1)$, and all $R > 0$, we have
\[
\lim_{\delta \to 0} \Theta^{LR}_{\mathcal{A}, \mathbb{B}, \delta, R} (p) = \infty.
\]

Proof. This follows from (2.41) and the fact that the LQG measure is admissible almost surely.

We also record a simple but useful consequence of the general facts established in Subsection 2.5.

Proposition 2.39. Suppose that $\mathbb{B} \subset \mathbb{B}^{\circ} \subset \mathcal{A} \subset \mathcal{A}'$. We have
\[
d_{\mathbb{A}, \mathbb{B}, \delta, R} \leq d_{\mathbb{A}', \mathbb{B}, \delta, R} \leq d_{\mathbb{A}, \mathbb{B}, \delta^{''}, R},
\]
where
\[
\delta' = \exp \{ -y \min_{z \in \mathbb{B}^{\circ}} h_{\mathbb{A}' : \mathbb{A}} (z) \} \delta,
\]
\[
\delta'' = \exp \{ -y \max_{z \in \mathbb{B}^{\circ}} h_{\mathbb{A}' : \mathbb{A}} (z) \} \delta.
\]
Moreover, we have
\[
d_{\mathbb{A}', \mathbb{B}, \delta, R} \leq \exp \left\{ y \left( \max_{z \in \mathbb{B}^{\circ}} h_{\mathbb{A}' : \mathbb{A}} (z) \right) \right\} d_{\mathbb{A}, \mathbb{B}, \delta, R}.
\]

Proof. This follows immediately from (2.17), (2.27), (2.29), and (2.30).

Remark 2.40. We have two ways of understanding the effect on the distance when the measure is multiplied by a constant (“coarse field”): inequalities (2.29) (generally applied through (2.44)) and (2.30) (often applied through (2.45)). As pointed out in Remark 2.18, the estimate (2.30) is poor unless $\alpha$ is very close to 1. We will thus restrict ourselves to (2.29) whenever we perform estimates that fail as $\gamma \uparrow 2$. On the other hand, the change in $\delta$ in (2.29) can cause problems if we want to compare distances for the same path with respect to two different measures—in such cases, (2.30) will play a crucial role.
3 The RSW theory

In this section we give the proof of an RSW result, relating easy crossings to hard crossings in our setting. This was previously done in the high-temperature regime by the present authors in [9], using a framework inspired by the technique used to prove an RSW result for Voronoi percolation in [42]. In the present paper we follow the much simpler RSW proof method which was developed in [16] for first-passage percolation on fields with an approximate conformal invariance property. In this section we adapt that proof to our setting. The main changes that need to be made from [16] concern the nature of the “approximate conformal invariance,” since [16] considers a random Riemannian metric while we consider a graph distance.

3.1 Approximate conformal invariance

The key ingredient of approximate conformal invariance in our setting is the following.

**Proposition 3.1.** Suppose that $\mathbb{U}$ and $\mathbb{V}$ are bounded domains (open subsets of $\mathbb{R}^2$) so that $\overline{\mathbb{U}} \subset \mathbb{V}$. Suppose further that $\mathbb{V}'$ is another domain and $F : \mathbb{V} \to \mathbb{V}'$ is a conformal homeomorphism. Finally, suppose that $R, R' > 0$. Then there is an $N = N(\mathbb{U}, \mathbb{V}, \mathbb{V}', F, R, R') \in \mathbb{N}$, such that for any closed ball $B \subset \mathbb{U}$ of radius at most $R$, any two points $y_1, y_2 \in \partial F(B)$, and any connected component $X$ of $\partial F(B) \setminus \{y_1, y_2\}$, we have that $B_1 \cup \ldots \cup B_N$ of radius at most $R'$ so that $B_1 \cup \ldots \cup B_N$ is connected, $y_1, y_2 \in B_1 \cup \ldots \cup B_N$, and $B_1 \cup \ldots \cup B_N \subset F(B)$, and

$$\sup_{x \in B_1 \cup \ldots \cup B_N} \text{dist}_E(x, X) \leq R'.$$

Moreover, we can take

$$N(\mathbb{U}, \mathbb{V}, \mathbb{V}', F, R, R') = N(\mathbb{U}, \mathbb{V}, \mathbb{V}', x \mapsto F(x/\alpha), \alpha R, R') = N(\mathbb{U}, \mathbb{V}, \mathbb{V}', x \mapsto \alpha F(x), R, \alpha R')$$

for any $\alpha > 0$.

The statement of Proposition 3.1 has an intuitive interpretation: it says that a given conformal homeomorphism, on a compact set inside its domain, cannot disturb the geometry of a ball so violently that an unbounded number of balls are required to connect two points on its boundary. This is essentially true because the derivatives of the conformal homeomorphism and its inverse must be bounded on a compact set in the interior of the domain. Quantifying this intuition, however, requires a bit of work. We first prove the following lemma.

**Lemma 3.2.** Suppose that $\xi : [-1, 1] \to \mathbb{R}^2$ is a smooth curve and $x \in \mathbb{R}^2$ is such that $x - \xi(0)$ is perpendicular to $\xi'(0)$. If

$$|x - \xi(0)| < \frac{|\xi'(0)|^2}{2\|\xi''\|_\infty},$$

then

$$|x - \xi(t)| \geq |x - \xi(0)|$$

for all

$$|t| \leq \frac{|\xi'(0)|}{2\|\xi''\|_\infty}.$$ 

**Proof.** Put $v = |\xi'(0)|$ and $d = |x - \xi(0)|$. We may change coordinates to assume that $\xi(0) = 0$, $\xi'(0) = (v, 0)$, and $x = (0, d)$. Then we have that

$$|x - \xi(t)|^2 = \xi_1(t)^2 + (d - \xi_2(t))^2 = \left(vt + \int_0^t (t-s)\xi''_1(s)\, ds\right)^2 + \left(d - \int_0^t (t-s)\xi''_2(s)\, ds\right)^2 \geq d^2 + v^2 t^2 + 2vt \int_0^t (t-s)\left[vt\xi''_1(s) - d\xi''_2(s)\right] \, ds \geq d^2 + t^2 \left(v^2 - (vt + d) \sup_{|s| \leq t} |\xi''(s)|\right).$$

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We note that
\[(vt + d) \sup_{|s| \leq t} |\xi''(t)| \leq (vt + d)\|\xi''\|_\infty < v^2,\]
where the last inequality is by (3.3) and (3.4), so we in fact have \(|x - \xi(t)|^2 > |x - \xi(0)|^2\), as claimed. \(\square\)

Now we can prove Proposition 3.1.

**Proof of Proposition 3.1.** Let \(x\) and \(r\) be the center and radius of \(B\), respectively, so \(B = B(x, r)\). Parameterize \(\partial B\) by
\[\xi(t) = x + r(\cos t, \sin t).\]
Then \(F \circ \xi\) is a parameterization of \(\partial F(B)\). We have, using the fact that \(F\) is conformal, that
\[|(F \circ \xi)'(t)| = |F'(\xi(t)) \cdot \xi'(t)| = \left[ r \|F'\|^{-1}_{L^\infty(U)} \right] F'_{L^\infty(U)} \ll 1, \] (3.5)
and
\[|(F \circ \xi)''(t)| = |F''(\xi(t)) \cdot |\xi'(t)|^2 + F'(\xi(t)) \cdot \xi''(t)| \leq r^2 \|F''\|_{L^\infty(U)} + r \|F'\|_{L^\infty(U)} \ll r \|F''\|_{L^\infty(U)} + r \|F'\|_{L^\infty(U)}. \] (3.6)

Let
\[t_0 = \max \frac{1}{r \|F''\|_{L^\infty(U)} + \|F'\|_{L^\infty(U)}}, \quad \ell_0 = \min \frac{1}{r \|F''\|_{L^\infty(U)} + \|F'\|_{L^\infty(U)}}.\]

Define
\[\omega_a(t) = F(\xi(t)) + a \frac{R(F \circ \xi)'(t)}{r \|F'\|_{L^\infty(U)}}, \] (3.7)
where \(R\) denotes counterclockwise rotation by \(\pi/2\). Let
\[b(t) = |\omega_a(t) - F(\xi(t))| = \frac{a R(F \circ \xi)'(t)}{r \|F'\|_{L^\infty(U)}} \geq \frac{|F(\xi(t))|}{r \|F'\|_{L^\infty(U)}}. \] (3.8)

Fix \(t \in [0, 2\pi]\). By (3.5), we have that \(b(t) \leq a\). Thus, by Lemma 3.2 (applied with \(x = \omega_a(t)\) and \(\xi(s) = F(\xi(s - t))\)), (3.5), and (3.6), if \(a \leq \ell_0\) and \(|t - s| \leq t_0\), then
\[|\omega_a(s) - F(\xi(t))| \geq b(t).\]
Moreover, we know that if \(x \in \partial B \setminus \xi([t - t_0, t + t_0])\), then
\[|F(x) - F(\xi(t))| \geq c \|F''\|^{-1}_{L^\infty(U)} \|F'\|^{-1}_{L^\infty(U)} t_0 \] for some absolute constant \(c\). Let \(a = ra_0\), where
\[a_0 = \min \left\{ \frac{\|F''\|^{-2}_{L^\infty(U)}}{R \|F'\|_{L^\infty(U)} + \|F'\|_{L^\infty(U)}}, \frac{1}{2} c \|F''\|^{-1}_{L^\infty(U)} t_0 R' \right\}.\]
Then we have
\[|\omega_a(s) - F(\xi(t))| \geq b(s)\]
for all \( s, t \). This implies that
\[
B(\omega_a(s), b(s)) \subseteq F(B).
\]
(3.9)

Note that \( a_0 \) depends only on \( \mathbb{U}, \mathbb{V}, F, R, R' \) and not on \( B \), and moreover is invariant under the scalings indicated in (3.2).

We can also apply the triangle inequality to (3.7) to obtain
\[
|\omega_a(s) - \omega_a(s')| \leq |F(\xi(s)) - F(\xi(s'))| + \frac{a}{r\|F'\|_{L^\infty(\mathcal{T})}} |(F \circ \xi)'(s) - (F \circ \xi)'(s')|
\leq r\|F'\|_{L^\infty(\mathcal{T})} |s - s'| + \frac{a_0}{\|F'\|_{L^\infty(\mathcal{T})}} \left( rR\|F''\|_{L^\infty(\mathcal{U})} + r\|F'\|_{L^\infty(\mathcal{U})} \right) |s - s'|
= r \left( \|F'\|_{L^\infty(\mathcal{T})} + a_0 \left( R\|F''\|_{L^\infty(\mathcal{U})}\|F'\|_{L^\infty(\mathcal{T})}^{-1} + 1 \right) \right) |s - s'|
\] 
(3.10)

by (3.5) and (3.6). On the other hand, we can compute from (3.8) and (3.5) that
\[
b(t) = a \frac{|(F \circ \xi)'(t)|}{r\|F'\|_{L^\infty(\mathcal{T})}} \geq a_0 r \frac{\|(F')^{-1}\|_{L^\infty(\mathcal{T})}}{\|F'\|_{L^\infty(\mathcal{T})}}.
\]
(3.11)

Therefore, if we define
\[
\iota = \frac{2a_0\|(F')^{-1}\|_{L^\infty(\mathcal{T})}}{\|F'\|_{L^\infty(\mathcal{T})} + a_0 \left( R\|F''\|_{L^\infty(\mathcal{U})}\|F'\|_{L^\infty(\mathcal{T})}^{-1} + 1 \right)},
\]
then whenever \( |s - s'| \leq \iota \), we have
\[
|\omega_a(s) - \omega_a(s')| \leq b(s) + b(s')
\]
by (3.10) and (3.11). This implies that
\[
B(\omega_a(s), b(s)) \cap B(\omega_a(s'), b(s')) \neq \emptyset.
\]
(3.12)

Note that, like \( a_0 \), the constant \( \iota \) depends only on \( \mathbb{U}, \mathbb{V}, F, R, R' \) and not on \( B \), and is invariant under the scalings indicated in (3.2). Now if \( s_0, s_1 \in \mathbb{R} \) with \( s_1 \in (s_0, s_0 + 2\pi) \), then we can choose a sequence of points
\[
s_0 = t_0 < t_1 < t_2 < \cdots < t_N = s_1
\]
so that \( t_i - t_{i-1} \leq \iota \) for each \( i \) and \( N \leq \frac{2\pi}{\iota} \), which is invariant under the scalings indicated in (3.2) because \( \iota \) is. Then, by (3.8), (3.9), and (3.12), we can observe that
\[
\{B(w_a(t_i), b(t_i)) \mid i \in \{0, \ldots, N\}\}
\]
is a set of balls as claimed in the lemma. (We can ensure that (3.1) is satisfied by swapping \( y_1 \) and \( y_2 \) if necessary.) \( \square \)

**Corollary 3.3.** Suppose that \( \mathbb{U} \) and \( \mathbb{V} \) are bounded open subsets of \( \mathbb{R}^2 \) so that \( \overline{\mathbb{U}} \subset \mathbb{V} \). Suppose further that \( \mathbb{V}' \) is another domain and \( F : \mathbb{V} \to \mathbb{V}' \) is a conformal homeomorphism. Let
\[
R \in (0, \frac{1}{4} \text{dist}_E(\mathbb{U}, \partial \mathbb{V}))
\]
\[
R' \in (0, \frac{1}{4} \text{dist}_E(F(\mathbb{U}), \partial \mathbb{V}')).
\]
Suppose further that \( \partial B(x, r) \cap \mathbb{U} \) is connected for every \( x \in \mathbb{R}^2, r \in (0, R) \). Then, for all \( x, y \in \mathbb{U} \), we have

\[
d_{\mu_{h_y} \circ F^{-1}, F(\mathbb{U}^{(2R')}), \delta, R'}(F(x), F(y)) \leq N d_{\mathbb{U}, \delta, R}(x, y),
\]

where

\[
N = N(\mathbb{U}, \mathbb{V}, \mathbb{V}', F, R, R')
\]
is as in the statement of Proposition 3.1.

Proof. Let \( d = d_{\mathbb{V}, \mathbb{U}}(x, y) \). There is a path \( \pi \in \mathcal{P}_{\mathbb{U}}(x, y) \) and sequence of closed balls \( B_1, \ldots, B_n \subset \mathbb{U}^{(2R)} \) so that

\[
d = \sum_{j=1}^{n} \kappa_{\delta}(\mu_{h_y}(B_j))
\]

and \( \pi \subset B_1 \cup \cdots \cup B_d \). Now by Proposition 3.1, for each \( 1 \leq j \leq d \) there is a sequence of closed balls \( B_{j,1}, \ldots, B_{j,N} \subset F(B_j) \), where \( N \) depends only on \( \mathbb{U}, \mathbb{V}, \mathbb{V}', F, R, R' \), so that if \( P_j = B_{j,1} \cup \cdots \cup B_{j,N} \) and \( P = P_1 \cup \cdots \cup P_d \), then \( P \) is connected and \( F(x), F(y) \in P \). Moreover, \( B_{j,1}, \ldots, B_{j,N} \) can be chosen to lie in \( F(\mathbb{U}^{(2R)}) \) by (3.1) and the assumption that \( \partial B_j \cap \mathbb{U} \) is connected. In particular we have that

\[
\mu_{h_y}(F^{-1}(B_{j,d})) \leq \mu_{h_y}(B_j) \leq \delta.
\]

Therefore, by Lemma 2.22, we have that

\[
d_{\mu_{h_y} \circ F^{-1}, F(\mathbb{U}^{(2R')}), \delta, R'}(F(x), F(y)) \leq N d_{\mathbb{U}, \delta, R}(x, y),
\]
as claimed. \( \square \)

3.2 The RSW result

Now we can prove our RSW result. In this section, in order to access the conformal mappings that we will need to make the argument work, it will be more convenient to work with ellipses rather than rectangles. Define \( \mathbb{E}(S_1, S_2) \) to be the filled, closed ellipse centered at 0 with horizontal axis of length \( S_1 \) and vertical axis of length \( S_2 \); that is,

\[
\mathbb{E}(S_1, S_2) = \{ (x_1, x_2) \in \mathbb{R}^2 : (x_1/S_1)^2 + (x_2/S_2)^2 \leq 1 \}.
\]

Theorem 3.4. Let \( \mathbb{R}_1 = \mathbb{E}(1, 2) \) and \( \mathbb{R}_2 = \mathbb{E}(2, 1) \). There is a constant \( c > 0 \) so that, for any \( w > 0 \), we have

\[
\mathbb{P} (d_{\mathbb{R}_2, \delta}(L, R) \leq w) \geq c \mathbb{P} (d_{\mathbb{R}_1, \delta}(L, R) \leq cw). \tag{3.13}
\]

Thus we have a constant \( C_{RSW} < \infty \) so that for any \( p \in [0, C_{RSW}] \),

\[
\Theta_{\mathbb{R}_1, \delta, R}^{\text{hard}}(p) \leq C_{RSW} \Theta_{\mathbb{R}_1, \delta, R}^{\text{easy}}(C_{RSW} p). \tag{3.14}
\]

Proof. Define

\[
p_0 = \mathbb{P} [d_{\mathbb{R}_1, \delta}(L, R) \leq w].
\]

Let \( \mathbb{E}_1 = \mathbb{E}(1, 3) \) and \( \mathbb{E}_2 = \mathbb{E}(3, 4/3) \). Let \( X \) and \( Y \) be the left and right, respectively, connected components of \( \partial \mathbb{E}_1 \cap \mathbb{R}_1 \), and let \( x_0 \) and \( x_1 \) be the bottom and top, respectively, endpoints of \( X \). By the Riemann mapping theorem and Schwartz reflection, for any \( y \in Y \) there are sets \( \mathbb{F}_1, \mathbb{F}_2(y) \) and a conformal homeomorphism \( F_y : \mathbb{F}_1 \to \mathbb{F}_2(y) \) such that the following properties hold:

1. We have \( \overline{\mathbb{E}_1} \subset \mathbb{F}_1 \subset \mathbb{E}_1^*, \mathbb{E}_{\mathbb{R}_2}^* \subset \mathbb{F}_2(y) \subset \mathbb{E}_2^*, \mathbb{R}_2^* \), and \( F_y(\mathbb{E}_1) = \mathbb{E}_2 \).
Figure 3.1: Illustration of the geometrical setup for the proof of Theorem 3.4.

2. $F_y(x_0)$ and $F_y(x_1)$ are the upper- and lower-left, respectively, points in $\partial E_2 \cap \partial \mathbb{R}_2$.

3. $F_y(y)$ is the lower-right point in $\partial E_2 \cap \partial \mathbb{R}_2$.

4. We have that

$$Q := \sup_{y \in Y} \left\{ \|F'_y\|_{L^\infty(E^1)}, \|(F'_y)^{-1}\|_{L^\infty(E^1)} \right\} < \infty. \quad (3.15)$$

(Here, $(F'_y)^{-1}(x) = (F'_y(x))^{-1}$.)

(This was essentially pointed out in [16].) We note that condition 2 implies that $F_y(X)$ is the left connected component of $E_2 \setminus \mathbb{R}_2$. See Figure 3.1 for a partial illustration of this setup.

Let $R_1 = \frac{1}{4} \text{dist}(E_1, \partial \mathbb{F}_1)$. Now, on the event $E = \left\{ \max_{z \in \mathbb{E}_1^{(R_1)}} h_{\mathbb{E}_1,\mathbb{F}_1}(z) \leq 0 \right\}$, we have that

$$d_{\mathbb{E}_1,\mathbb{E}_1,\delta,R_1}(X,Y) \leq d_{\mathbb{E}_1,\delta,R_1}(X,Y) \leq C d_{\mathbb{E}_1,\delta}(X,Y),$$

where the second inequality is by (2.34). (Here we have folded the geometrical factor in (2.34) into the constant $C$.) Now we note that the $d_{\mathbb{E}_1,\mathbb{E}_1,\delta,R_1}(X,Y)$ and $E$ are independent. Let $p_* = \mathbb{P}[E]$ (which is strictly positive because $\mathbb{E}_1^{(R_1)}$ is a positive Euclidean distance from $\partial \mathbb{F}_1$), so we have that

$$\mathbb{P}\left(d_{\mathbb{E}_1,\mathbb{E}_1,\delta,R_1}(X,Y) \leq Cw\right) \geq p_* p_0. \quad (3.16)$$

Now write $Y$ as a union of disjoint curvilinear segments $Y_1, \ldots, Y_M$ so that $\text{diam}_{\mathbb{E}}(Y_k) \leq (2Q)^{-1}$, where $Q$ is as in (3.15) and $M$ is taken to satisfy $M \leq 3Q$. By (3.16), we have that

$$p_* p_0 \leq \sum_{k=1}^{M} \mathbb{P}\left(d_{\mathbb{E}_1,\mathbb{E}_1,\delta,R_1}(X,Y_k) \leq Cw\right).$$
so there is some $1 \leq k \leq M$ so that

$$
\mathbf{P} \left( d_{\mathcal{F}_1, \mathcal{B}_1, \delta, R_1}(X, Y_k) \leq Cw \right) \geq \frac{p_*P_0}{M} \geq \frac{p_*P_0}{3Q}. \quad (3.17)
$$

Fix $y$ to be the bottom endpoint of $Y_k$. By (3.15), we observe that $F_y(Y_k)$ is contained in the right-hand connected component of $\mathcal{B}_2 \setminus \mathcal{B}_2$. Let

$$
R_2 = \min \left\{ \frac{1}{4} \text{dist}(\mathcal{B}_2, \partial \mathcal{F}_2(y)), \frac{1}{32} \right\}
$$

and define $\mathcal{F}_2 = \mathcal{B}_2^{(2R_i)}$. By Corollary 3.3, we have an $N < \infty$ so that

$$
d_{\mu_{\mathcal{F}_1} \circ F_y^{-1}, \mathcal{F}_2, \delta, R_2}(F_y(X), F_y(Y_k)) \leq Nd_{\mathcal{F}_1, \mathcal{B}_1, \delta, R_1}(X, Y_k). \quad (3.18)
$$

Note that, by Proposition 2.14,

$$
d_{\mathcal{F}_2(y), \mathcal{F}_2, \delta, R_2}(F_y(X), F_y(Y_k)) \overset{\text{law}}{=} d_{e^{(-2y^2/2) \log ||F_y^{-1}/\mu_{\mathcal{F}_1} \circ F_y^{-1}, \mathcal{F}_2, \delta, R_2}}(F_y(X), F_y(Y_k)) \quad (3.19)
$$

On the other hand, we have

$$
d_{\mathcal{F}_2(y), \mathcal{F}_2, \delta, R_2}(F_y(X), F_y(Y_k)) \geq d_{\mathcal{F}_2(y), \mathcal{F}_2, \delta, R_2}(\mathcal{B}_2 \cap L_{R_2}, \mathcal{B}_2 \cap R_{R_2}),
$$

and on the event $E' = \left\{ \max_z \in \mathcal{B}_2(y) \right\} h_{\mathcal{B}_2; \mathcal{F}_2(y)}(z) \leq 0 \right\}$ (which is independent of $d_{\mathcal{F}_2(y), \mathcal{F}_2, \delta, R_2}(F_y(X), F_y(Y_k))$) we have that

$$
d_{\mathcal{F}_2(y), \mathcal{F}_2, \delta, R_2}(\mathcal{B}_2 \cap L_{R_2}, \mathcal{B}_2 \cap R_{R_2}) \geq d_{\mathcal{F}_2(y), \mathcal{F}_2, \delta, R_2}(\mathcal{B}_2 \cap L_{R_2}, \mathcal{B}_2 \cap R_{R_2})
$$

where the second inequality is by (2.31). This implies that, if we define $p_* = \mathbf{P}[E']$ (which again is strictly positive since $\text{dist}(\mathcal{F}_2(y) \cap \mathcal{B}_2, \partial \mathcal{F}_2(y)) > 0$), then

$$
\mathbf{P} \left( d_{\mathcal{F}_2(y), \mathcal{F}_2, \delta, R_2}(\mathcal{B}_2 \cap L_{R_2}, \mathcal{B}_2 \cap R_{R_2}) \leq v \right) \geq p_* \mathbf{P} \left( d_{\mathcal{F}_2(y), \mathcal{F}_2, \delta, R_2}(F_y(X), F_y(Y_k)) \leq v \right). \quad (3.20)
$$

We can also write

$$
d_{\mathcal{F}_2(y), \mathcal{F}_2, \delta, R_2}(\mathcal{B}_2 \cap L_{R_2}, \mathcal{B}_2 \cap R_{R_2}) \geq d_{\mathcal{B}_2, \delta, R_4}(\mathcal{B}_2 \cap L_{R_2}, \mathcal{B}_2 \cap R_{R_2}) \geq d_{\mathcal{B}_2, \delta, R_4}(L, R) \geq d_{\mathcal{B}_2, \delta}(L, R). \quad (3.21)
$$

where the first inequality is by (2.28) and the third is by (2.5). Therefore, we have

$$
\mathbf{P} \left( d_{\mathcal{B}_2, \delta}(L, R) \leq Cw \right)
$$

$$
\geq \mathbf{P} \left( d_{\mathcal{F}_2(y), \mathcal{F}_2, \delta, R_2}(\mathcal{B}_2 \cap L_{R_2}, \mathcal{B}_2 \cap R_{R_2}) \leq Cw \right) \geq p_* \mathbf{P} \left( d_{\mathcal{F}_2(y), \mathcal{F}_2, \delta, R_2}(F_y(X), F_y(Y_k)) \leq Cw \right)
$$

$$
= p_* \mathbf{P} \left( d_{e^{(-2y^2/2) \log ||F_y^{-1}/\mu_{\mathcal{F}_1} \circ F_y^{-1}, \mathcal{F}_2, \delta, R_2}}(F_y(X), F_y(Y_k)) \leq Cw \right) \geq \frac{p_*P_0}{3Q}.
$$

where the first inequality is by (3.21), the second is by (3.20), the equality is by (3.19), the fourth inequality is by (3.18) and (2.31) (in light of (3.15)), and the last is by (3.17). This implies (3.13) with appropriately chosen constants. \[\square\]
4 Percolation arguments

In this section we prove concentration for crossing distances at a given scale, as well as relationships for crossing distances between different scales, using percolation arguments. Recall the definition (2.21) of $d...(M_\alpha)$.

**Proposition 4.1.** There is a $p_0 > 0$ so that for every $\theta > 2$, there is a constant $C < \infty$ so that the following holds. For any $a > 0$, any box $\mathbb{R}$ with $\text{AR}(\mathbb{R}) \in [1/3,3]$, any $R > 0$, and any $K \in [1,\infty)$, if we define

$$S = \text{diam}_{\mathbb{R}} \mathbb{R},$$

$$\omega = \frac{2\gamma\theta}{4 + \gamma^2},$$

then we have that

$$P\left( d_{\mathbb{R},\delta,R}(M_\alpha) \leq \Theta_{\mathbb{E}(K^{\gamma-1}\omega S),\delta,K^{-\omega}R}(p_0) \right) \leq C \left( K^{2-(2+\theta)^2/8} + e^{-aK/C} \right). \quad (4.2)$$

If we further assume that $R \leq K^{-1}S$, then we have that

$$P\left( d_{\mathbb{R},\delta,R}(M_\alpha) \leq \frac{aK}{C} \Theta_{\mathbb{E}(K^{\gamma-1}\omega S),\delta,K^{-\omega}R}(p_0) \right) \leq C \left( K^{2-(2+\theta)^2/8} + e^{-aK/C} \right). \quad (4.3)$$

We will prove Proposition 4.1 in Subsection 4.3. The constant $p_0$ will remain fixed throughout the remainder of the paper. Note that if $a$ is treated as a fixed constant, as it often will be in the sequel, then the second terms on the right-hand sides of (4.2) and (4.3) can be ignored.

**Proposition 4.2.** We have constants $p_1 < 1$ and $C < \infty$ so that the following holds. Let $\mathbb{R}$ be a box with $\text{AR}(\mathbb{R}) \in [1/3,3]$, let $S = \text{diam}_{\mathbb{R}} \mathbb{R}$, and let $K \in [1,\infty)$. Then we have

$$P\left( d_{\mathbb{R},\delta,\Omega}(L,R) \geq \frac{e^{K^{7/8}}}{C} \Theta_{\mathbb{E}(K^{\gamma-1}\omega S),\delta,K^{\gamma}R}(p_1) \right) \leq C e^{-K/C}. \quad (4.4)$$

Moreover, for every $p > 0$ there is a $K_0 < \infty$ so that if $K \geq K_0$, then for every $\mathbb{R}$ with $\text{AR}(\mathbb{R}) \in [1/3,3]$ and setting $S = \text{diam}_{\mathbb{R}} \mathbb{R}$, we have

$$P\left( d_{\mathbb{R},\delta,\Omega}(L,R) \geq K^C \Theta_{\mathbb{E}(K^{\gamma-1}\omega S),\delta,K^{\gamma}R}(p_1) \right) \leq p. \quad (4.5)$$

We will prove Proposition 4.2 in Subsection 4.4. Like $p_0$, the constant $p_1$ will remain fixed throughout the remainder of the paper.

Before establishing some important applications and consequences of Proposition 4.1 and Proposition 4.2, we will set up some machinery that we use to express the ratios of different quantiles to each other. In Subsection 4.2, we will use the notation to express bounds on moments of certain crossing distances, using the tools of Section 3 and Section 4. Later, in Section 5, we will show that these ratios are not too large by an inductive argument.

**Definition 4.3.** Define

$$\chi_U = \frac{\Theta_{\mathbb{E}(U)}(p_1)}{\Theta_{\mathbb{E}(U)}(p_0)}$$

and

$$\overline{\chi}_U = \sup_{V \in [0,U]} \chi_V.$$

Further define $S_{\chi} = \{U > 0 \mid \chi_U \leq \chi\}$. 

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By Theorem 3.4 and Lemma A.1, we have a constant $C < \infty$ so that

$$\chi_U \leq C \exp \left( C \sqrt{\text{Var} \left( \log d_{\Xi(U)} \right)} \right).$$

(4.6)

Thus, our inductive procedure to bound $\chi_U$ will rely on inductively bounding $\text{Var} \left( \log d_{\Xi(U)} \right)$. Also, it is proved in [12, (74)] that there is a constant $C$ so that

$$\overline{\chi}_U \leq C e^{(\log U)^{0.95}}$$

(4.7)

for all $U$. This will be important for the base case in our induction.

### 4.1 Quantile relationships

In this section, we establish consequences of Propositions 4.1 and 4.2 establishing upper and lower bounds on relationships between quantiles of crossing distances at different scales.

#### 4.1.1 Easy crossing quantile relationship

Here we establish the following proposition establishing at least power-law growth of easy-crossing quantiles.

**Proposition 4.4.** There is a $T_0 \geq 0$ and a constant $c > 0$ so that for every $S \geq T_0$ and $K \geq 1$, we have

$$\Theta_{\Xi(KS)}^{\text{easy}}(p_0) \geq c K^c \Theta_{\Xi(S)}^{\text{easy}}(p_0).$$

(4.8)

To prove Proposition 4.4, we will require the following lemma.

**Lemma 4.5.** We have constants $c > 0$ and $C < \infty$ so that for any $M \geq 1$ and any $S > 0$, we have that

$$\Theta_{\Xi(MS)}^{\text{easy}}(p_0) \geq c M^{1/2} \Theta_{\Xi(S)}^{\text{easy}}(p_0) - CM^{3/2}.$$  

(4.9)

**Proof.** Choose $\theta = \frac{4 + \gamma^2}{2 \gamma} > 2$. Note that with this choice of $\theta$ we have $\omega = 1$ in the statement of Proposition 4.1. If $K$ is large enough so that $CK^{2-(2+\theta)^2}/8 \leq p_0$, then we have that

$$\Theta_{\Xi(K^2 S),1,\sqrt{KS}}^{\text{easy}}(p_0) \geq \Theta_{\Xi(K^2 S),1,\sqrt{KS}}^{\text{easy}} \left( CK^{2-(2+\theta)^2}/8 \right) \geq \frac{K}{C} \Theta_{\Xi(S),1,\sqrt{KS}}^{\text{easy}}(p_0) = \frac{K}{C} \Theta_{\Xi(S)}^{\text{easy}}(p_0)$$

(4.10)

by (4.3) of Propositions 4.1. On the other hand, by Lemma 2.24 we have a constant $C$ so that

$$\Theta_{\Xi(K^2 S),1,\sqrt{KS}}^{\text{easy}}(p) \leq \Theta_{\Xi(K^2 S)}^{\text{easy}}(p) + CK^3.$$  

Combining this with (4.10), we have

$$\Theta_{\Xi(K^2 S)}^{\text{easy}}(p) + CK^3 \geq \frac{K}{6} \Theta_{\Xi(S)}^{\text{easy}}(p),$$

which proves (4.9) when we take $M = K^2$. □

Now we can prove Proposition 4.4 by induction.
Proof of Proposition 4.4. By Lemma 4.5, we can choose $M > 0$ so that $\Theta_{B(MS)}(p_0) \geq 2\Theta_{B(S)}(p_0) - C$ for some constant $C < \infty$ (distinct from those in (4.9)). By induction, we have

$$\Theta_{B(M^kS)}(p_0) \geq 2^k\Theta_{B(S)}(p_0) - \sum_{j=0}^{k-1} 2^j C \geq 2^k\left(\Theta_{B(S)}(p_0) - C\right). \quad (4.11)$$

Now let $k = [\log_M K]$. Then we have, by (2.43) and (4.11), that

$$\Theta_{B(KS)}(p_0) \geq \Theta_{B(M^kS)}(p_0) \geq 2^k\left(\Theta_{B(S)}(p_0) - C\right) \geq \frac{1}{2}K^{\log_M 2}\left(\Theta_{B(S)}(p_0) - C\right).$$

Now by Lemma 2.38, there is a $T_0$ so that if $S \geq T_0$, then $\Theta_{B(S)}(p_0) \geq 2C$. Therefore,

$$\Theta_{B(KS)}(p_0) \geq \frac{1}{4}K^{\log_M 2}\Theta_{B(S)}(p_0),$$

which is (4.8) with the appropriate choice of $c$. \hfill \Box

4.1.2 Hard crossing quantile relationship

Here we show that there can be at most power-law growth in the hard-crossing quantiles. We will in fact only use (4.13) of the below proposition in the sequel, but we include (4.12) as well for completeness.

Proposition 4.6. There is a $K_1 < \infty$ and a constant $C < \infty$ so that if $K \geq K_1$, then we have

$$\Theta_{B(KS)}(p_1) \leq K^C\Theta_{B(S)}(p_1) \quad (4.12)$$

and

$$\Theta_{B(KS)}(p_1) \leq K^C\Theta_{B(S)}(p_1). \quad (4.13)$$

Proof. This follows from (4.5) of Proposition 4.2, taking $K = K^{\eta-1}$. \hfill \Box

4.2 Tail and moment estimates

In this section, we establish moment estimates stemming from our percolation arguments.

4.2.1 Moment estimates for easy crossings

Proposition 4.7. For any $A, Q > 0$ there exists a constant $C = C(A, Q) < \infty$ so that if $a \in (0, 1)$, $R$ is a rectangle with $\text{AR}(R) \in [1/3, 3]$ and $S = \text{diam}(R)$, then we have

$$Ed_x(M_a)^{-A} \leq C(1 + a^{-C}\frac{A}{Q^{-1}S}\Theta_{B(S)}(p_0))^{-A}. \quad (4.14)$$

Proof. We first recall the simple formula

$$Ed_x(M_a)^{-A} = \int_0^\infty \mathbb{P}\left[d_x(M_a) \leq u^{-1/A}\right] du.$$

We have by (4.2) of Proposition 4.1 that

$$\mathbb{P}\left[d_x(M_a) \leq \Theta_{B(K^{-1-a\theta}S)}(p_0)\right] \leq C\left(K^{2-(2+\theta)^2/8} + e^{-aK/C}\right), \quad (4.15)$$
where \( \omega(\theta) = \frac{2\pi \theta}{4\pi + \gamma} \) as in (4.1). But on the other hand, we have, as long as \( K^{1+\omega(\theta)} \geq K_1 \) from Proposition 4.6,
\[
\Theta_{\mathbb{B}(K^{-1+\omega(\theta)})S_1,K^{-\omega(\theta)}}^\text{easy} (p_0) = \Theta_{\mathbb{B}(K^{-1+\omega(\theta)})S_1,K^{-\omega(\theta)}}^\text{easy} (p_0) \geq \chi_{K^{-1+\omega(\theta)}}^\text{hard} \Theta_{\mathbb{B}(K^{-1+\omega(\theta)})S_1}^\text{easy} (p_1) \\
\geq CK^{-C(1+\omega(\theta))} \chi_{K^{-1+\omega(\theta)}}^\text{hard} \Theta_{\mathbb{B}(S_1)}^\text{easy} (p_1),
\]
where the equality is by (2.25), the first inequality is by the definition of \( \chi_{K^{-1+\omega(\theta)}}^\text{hard} \), and the second inequality is by (4.13) of Proposition 4.6. Applying this to (4.15) yields
\[
P \left( \frac{d_{\mathbb{B}}(M_a)}{\Theta_{\mathbb{B}(S_1)}^\text{easy} (p_1)} \leq K^{-C(1+\omega(\theta))} \chi_{K^{-1+\omega(\theta)}}^\text{hard} \Theta_{\mathbb{B}(S_1)}^\text{easy} (p_1) \right) \leq C \left( K^{2-(2+\theta)^2/8} + e^{-aK/C} \right).
\]
This implies that
\[
P \left( \frac{d_{\mathbb{B}}(M_a)}{\Theta_{\mathbb{B}(S_1)}^\text{easy} (p_1)} \leq K^{AC(1+\omega(\theta))} \right) \leq C \left( K^{2-(2+\theta)^2/8} + e^{-aK/C} \right).
\]
Putting \( u = K^{AC(1+\omega(\theta))} \) yields, for each \( Q < \infty \), a \( u_0 \) so that if \( u \geq u_0 \) then
\[
P \left( \frac{d_{\mathbb{B}}(M_a)}{\Theta_{\mathbb{B}(S_1)}^\text{easy} (p_1)} \leq u \right) \leq C \left( u^{2-(2+\theta)^2/8} + \exp \left\{ -au \frac{A(\theta)}{CA(1+\omega(\theta))} / C \right\} \right),
\]
and thus
\[
E \left( \frac{d_{\mathbb{B}}(M_a)}{\Theta_{\mathbb{B}(S_1)}^\text{easy} (p_1)} \right)^{-A} \leq C \chi_{Q^{-1}S} \left( u_0 + \int_{u_0}^{\infty} 1 \vee u^{2-(2+\theta)^2/8} + \int_{u_0}^{\infty} \exp \left\{ -au \frac{A(\theta)}{CA(1+\omega(\theta))} / C \right\} du \right).
\]
Since the numerator of the exponent in \( u \) grows quadratically in \( \theta \) while the denominator grows only linearly, \( \theta \) can be chosen large enough (depending on \( A \)) that the first integral is finite. On the other hand, by a change of variables, the second integral is a constant times \( a^{-AC(1+\omega(\theta))} \). This implies (4.14). \( \square \)

### 4.2.2 Tail moment estimates for hard crossings

In this section we establish estimates for the tails of hard-crossing distances and for moments of crossings of annuli. We state our bound for hard-crossings in terms of a tail bound rather than in terms of moments because in the proof of Proposition 6.4 below, we will need an explicit superpolynomial concentration bound for the hard-crossing distances.

**Proposition 4.8.** There is a constant \( C \) so that, for any \( S, \theta \geq 0 \), we have
\[
P \left( d_{\mathbb{B}(S)}(\text{hard}) \geq (1 + \theta) \Theta_{\mathbb{B}(S_{\delta,R})}^\text{hard} (p_1) \right) \leq C e^{-\frac{(\log(1+\theta))^{8/7}}{C}}.
\]

**Proof.** This follows from (4.4) of Proposition 4.2, taking \( K = (\log(1+\theta))^{8/7} \), and Proposition 2.37. \( \square \)

**Corollary 4.9.** For any \( B \geq 0 \) we have a constant \( C < \infty \) so that the following holds. Suppose that \( \mathbb{B} \) is an \( S \times S \) square, \( \mathbb{R} \) is a rectangle so that either \( \mathbb{3B} \setminus \mathbb{B} \subset \mathbb{R} \) or \( (\mathbb{3B} \setminus \mathbb{B}) \cap \partial \mathbb{R} \) has exactly two connected components, and \( \mathbb{A} = (\mathbb{3B} \setminus \mathbb{B}) \cap \mathbb{R} \). Then we have
\[
E d_{\mathbb{B}(3B)}(S_{\delta,R})^B \leq C \Theta_{\mathbb{B}(S_{\delta,R})}^\text{hard} (p_1)^B.
\]
(4.16)
Proof. By the construction illustrated in Figure 4.1a, along with Proposition 4.8, (2.45) of Proposition 2.39, Corollary 2.10, and a union bound, we have

\[ P \left( d_{B_3^c, A, \delta, R}(A) \geq \theta \frac{\Theta_{B_3^c, \delta, R}(p_1)}{\Theta_{B(S), \delta, R}(p_1)} \right) \leq C e^{-\left( \log \theta \right)^{8/7}/C}. \]

Then we have

\[ E \left( \frac{d_{B_3^c, A, \delta, R}(A)}{\Theta_{B(S), \delta, R}(p_1)} \right)^B = \int_0^\infty P \left( \frac{d_{B_3^c, A, \delta, R}(A)}{\Theta_{B(S), \delta, R}(p_1)} \geq \theta^{1/B} \right) d\theta \leq C \int_0^\infty \left( e^{-\left( \log \theta \right)^{8/7}/(CB^{8/7})} \wedge 1 \right) d\theta < C, \]

where, as usual, the constant \( C \) has been allowed to change from step to step. This proves (4.16). □

### 4.3 Proof of Proposition 4.1

In this section, we prove Proposition 4.1. We do this using a percolation argument, relying on the fact that a path between two macroscopically separated points must make many easy crossings of smaller boxes. The geometry in this subsection is essentially identical to that of [9, Section 4].

Proof of Proposition 4.1. We note that there are constants \( c > 0 \) and \( C < \infty \), independent of \( a \), so that, for every \( K \in \mathbb{N} \) with \( K \geq C \), there is a collection \( C_K \) of \( K^{-1} S \times 2K^{-1} S \) and \( 2K^{-1} S \times K^{-1} S \) subboxes of \( \mathbb{R} \) and a subset \( \mathcal{C}_{a, K} \) of the power set of \( C_K \) so that the following properties hold:

1. \( |C_K| \leq CK^2 \).
2. All elements \( D \in \mathcal{C}_{a, K} \) have the same cardinality \( N \in [c K a], C K a \].
3. \( |\mathcal{C}_{a, K}| \leq C^N \).
4. For each \( C_1, C_2 \in D \in \mathcal{C}_{a, K} \), we have \( \text{dist}_E(C_1, C_2) > 0 \).
5. Whenever \( x, y \in \mathbb{R} \) and \( |x - y| \geq \alpha \text{diam}_E \mathbb{R} \), if \( \pi \) is any path in \( \mathbb{R} \) between \( x \) and \( y \) then we have a \( D \in \mathcal{C}_{a, K} \) so that \( \pi \) includes an easy crossing of \( \mathcal{C} \) for each \( C \in D \).
Now let \( x, y \in \mathbb{R} \) and let \( \pi \) be a \((\mu_{\mathbb{R}}, R, \delta, R)\)-geodesic (in the sense of Definition 2.23) between \( x \) and \( y \). Therefore, there is a \( D \in \mathcal{C}_{a,K} \) so that \( \pi \) includes an easy-crossing of every \( C \in D \). Let

\[
F = \min_{C \in \mathcal{C}_K} \min_{x \in \mathbb{C}^C} h_{\mu_{\mathbb{R}}, \mathbb{C}^C}(x).
\]

If \( R \leq K^{-1}S \), then we have

\[
d_{\mathbb{R}, \delta, R}(xy) \geq \sum_{C \in D} d_{\mathbb{R}, \delta, C, R}(\text{easy}) \geq \sum_{C \in D} d_{\mathbb{R} \cup F, \delta, R \setminus (\sqrt{S}K^{-1}S)}(\text{easy}),
\]

where the first inequality is by Proposition 2.29 and the second is by Proposition 2.30. On the other hand, we have

\[
d_{\mathbb{R}, \delta, R}(xy) \geq \max_{C \in D} d_{\mathbb{R}, \delta, C, R}(\text{easy}) \geq \max_{C \in D} d_{\mathbb{R} \cup F, \delta, R \setminus (\sqrt{S}K^{-1}S)}(\text{easy}),
\]

where the first inequality is by Proposition 2.29, the second is by (2.26), and the third is by Proposition 2.30.

Under the assumption that \( R \leq K^{-1}S \), we then have that

\[
P[\{ \exists \delta \in \mathcal{C}_{a,K} \} : d_{\mathbb{R}, \delta, R \setminus (\sqrt{S}K^{-1}S)}(\text{easy}) \leq u] \leq \frac{N}{2}
\]

(4.17) On the other hand, if we do not assume that \( R \leq K^{-1}S \), we still have

\[
P[\{ \exists \delta \in \mathcal{C}_{a,K} \} : d_{\mathbb{R}, \delta, R \setminus (\sqrt{S}K^{-1}S)}(\text{easy}) \leq u] \leq \frac{N}{2}
\]

(4.18)

Now define

\[
A = \Theta_{\mathbb{R} \cup F, \delta, R \setminus (\sqrt{S}K^{-1}S)}(\text{easy}) = \Theta_{\mathbb{R} \cup F, \delta, R \setminus (\sqrt{S}K^{-1}S)}(\omega) = \Theta_{\mathbb{R} \cup F, \delta, R \setminus (\sqrt{S}K^{-1}S)}(\text{easy}),
\]

where the second equality is by Proposition 2.34. (Recall that \( \omega \) was defined in (4.1).) Now the first terms of the right-hand sides of (4.17) and (4.18) are bounded by

\[
|\mathcal{C}_{a,K}|p^{N/2} \leq C^K(Cp)^{N/2} \leq C^N p^{N/2}
\]

(4.19) by Lemma A.2 and a union bound. On the other hand, we have

\[
P[F \leq -\theta \log K] \leq CK^{2-(\theta+2)^2/8}
\]

(4.20) by Corollary 2.10. Combining (4.17), (4.19), and (4.20), we get

\[
P[\{ \exists \delta \in \mathcal{C}_{a,K} \} : d_{\mathbb{R}, \delta, R}(M_a) \leq \frac{N}{2} \Theta_{\mathbb{R} \cup F, \delta, R \setminus (\sqrt{S}K^{-1}S)}(\text{easy})] \leq CK^2 C^N p^{N/2} + CK^{2-(\theta+2)^2/8} \leq CK^{2-(\theta+2)^2/8}
\]

as long as \( p \) is sufficiently small, where now \( C \) depends on a lower bound for the product \( aK \) but on neither \( a \) nor \( K \) individually. Similarly, combining (4.18), (4.19), and (4.20), we get

\[
P[\{ \exists \delta \in \mathcal{C}_{a,K} \} : d_{\mathbb{R}, \delta, R}(M_a) \leq \Theta_{\mathbb{R} \cup F, \delta, R \setminus (\sqrt{S}K^{-1}S)}(\omega)] \leq CK^2 C^N p^{N/2} + CK^{2-(\theta+2)^2/8} \leq CK^{2-(\theta+2)^2/8}
\]

as long as \( p \) is sufficiently small, where again \( C \) depends on a lower bound for the product \( aK \) but on neither \( a \) nor \( K \) individually.

This proves the proposition in the case where \( K \) is a sufficiently-large integer. The non-integer case follows from Proposition 2.37. □
4.4 Proof of upper bounds

In this section we prove Proposition 4.2. Although the statement of Proposition 4.2 is similar in spirit to [9, Proposition 6.1], here we need a better bound and thus use a more sophisticated percolation argument.

Proof. We assume that AR(ℝ) ∈ {1/3, 3} and that K is an integer; the general case follows from Proposition 2.37 and a simple coarse-field bound. Choose an appropriate L ∈ {K/3, 3K} and divide ℝ into a K × L grid of S × S subboxes, indexing them according to their position in the grid \( G_{K,L} = \{1, \ldots, K\} \times \{1, \ldots, L\} \) as \( (C_{k,\ell})_{(k,\ell) \in G} \), with the following layout:

\[
\begin{array}{cccccc}
C_{1,1} & \cdots & C_{K,1} \\
\vdots & \ddots & \vdots \\
C_{1,L} & \cdots & C_{K,L}
\end{array}
\]

Define \( A_{k,\ell} = (3C_{k,\ell} \setminus C_{k,\ell}) \cap \mathbb{R} \). Hence \( A_{k,\ell} \) is an intersection of a square annulus with \( \mathbb{R} \). Define \( Q \) to be the set of paths \( \omega = (\omega_1, \ldots, \omega_{J(\omega)}) \) so that \( \omega_j \in G_{K,L} \) for each \( j \), and \( \omega^{(1)}_1 = 1, \omega^{(1)}_{J(\omega)} = K \) (here the superscript \( (1) \) means to take the x-coordinate), and \( |\omega_j - \omega_{j-1}|_\infty \leq 1 \), where \( |\cdot|_\infty \) denotes the \( \ell_\infty \) norm. This is to say that the paths \( \omega \) must cross from left to right in the grid \( \{1, \ldots, K\} \times \{1, \ldots, L\} \) and must move as a chess king: one square at a time, either left, right, up, down or diagonally. Then circuits around \( A_{\omega_j}, j = 1, \ldots, J(\omega) \) can be joined together as illustrated in Figure 4.1b to form a left–right crossing of \( \mathbb{R} \), so we have

\[
d_{\mathbb{R}, \delta, R}(L, R) \leq \min_{\omega \in Q} \sum_{j=1}^{J(\omega)} d_{\mathbb{R}^*, \mathbb{B}_{\omega_j}, \delta, R}(A). \tag{4.21}
\]

(Recall the notation \( A \) defined in (2.22)–(2.23).) Define

\[
\begin{align*}
B_{k,\ell} &= \{ B \in \mathbb{B}_{k,\ell} \mid \mathbb{B} \subset \mathbb{R}\}, \\
\mathbb{B}_{k,\ell} &= \{ \sigma(C_{k+1,\ell+1} \cup C_{k,\ell+1}) \mid \sigma \in D_4 \} \cup \{ \sigma(C_{k,\ell+1} \cup C_{k,\ell+2}) \mid \sigma \in D_4 \},
\end{align*}
\]

where \( D_4 \) is the group of symmetries of the square, acting in the usual way on \( \mathbb{R}^2 \) but with the origin considered to be at the center of \( C_{k,\ell} \). We then have, by the construction illustrated in Figure 4.1a, that

\[
d_{\mathbb{R}^*, \mathbb{B}_{k,\ell}, \delta, R}(A) \leq \sum_{\mathbb{B} \in B_{k,\ell}} d_{\mathbb{R}^*, \mathbb{B}, \delta, R}(\text{hard}). \tag{4.22}
\]

Define

\[
F = \max_{1 \leq k \leq K} \max_{\mathbb{B} \in B_{k,\ell}} \max_{1 \leq \ell \leq L} h_{\mathbb{R}^*: \mathbb{B}}(x).
\]

By Corollary 2.10, we have

\[
P \left[ \omega^{F - \gamma \theta_0 \log K} \geq y \right] \leq C e^{1 \frac{-\log y^2}{\gamma \theta_0 \log K}}. \tag{4.23}
\]

(Recall that \( \theta_0 \) was fixed in Subsection 2.4.) Combining (4.21) and (4.22), we have

\[
d_{\mathbb{R}, \delta, R}(L, R) \leq \min_{\omega \in Q} \sum_{j=1}^{J(\omega)} \sum_{\mathbb{B} \in B_{k,\ell}} d_{\mathbb{R}^*, \mathbb{B}, \delta, R}(\text{hard}) \leq C e^{(F - \gamma \theta_0 \log K)^+} \min_{\omega \in Q} \sum_{j=1}^{J(\omega)} \sum_{\mathbb{B} \in B_{k,\ell}} d_{\mathbb{B}, K, \gamma \theta_0, \delta, R}(\text{hard}). \tag{4.24}
\]
Now we note that we can partition $G$ into finitely-many subsets $G_1, \ldots, G_M$, with $M$ an absolute constant not depending on $K$, so that
\[
\left\{ \sum_{B \in \mathcal{B}_{k, \ell}} d_{\mathbb{E}, K \gamma_0 \delta, R}^{(\text{hard})} : (k, \ell) \in \mathcal{G}_m \right\}
\]
is a collection of independent random variables for each $1 \leq m \leq M$. Noting that $|\mathcal{B}_{k, \ell}| \leq 12$ for all $k, \ell$, we have by a union bound
\[
P \left( \sum_{B \in \mathcal{B}_{k, \ell}} d_{\mathbb{E}, K \gamma_0 \delta, R}^{(\text{hard})} \geq w \right) \leq 12 P \left( d_{\mathbb{E}, (S)_k \gamma_0 \delta, R}^{(\text{hard})} \geq w/12 \right),
\]
where the equality is by (2.13) and Proposition 2.36. Let $R_L$ be the set of paths in $G$ which start at the top of $G$ and have length $L$, where only nearest-neighbor (horizontal and vertical) edges are allowed. We note that
\[
|R_L| \leq 4L^4 \leq 5^L. \tag{4.26}
\]
If there is no $\omega \in Q$ so that $\sum_{B \in \mathcal{B}_{k, \ell}} d_{\mathbb{E}, K \gamma_0 \delta, R}^{(\text{hard})} \leq w$ for all $(k, \ell) \in \omega$, then by planar duality there must be a $\xi \in R_L$ so that $\sum_{B \in \mathcal{B}_{k, \ell}} d_{\mathbb{E}, K \gamma_0 \delta, R}^{(\text{hard})} \geq w$. Therefore,
\[
P \left( \min_{\omega \in Q} \sum_{j=1}^{J(\omega)} \sum_{B \in \mathcal{B}_{k, \ell}} d_{\mathbb{E}, K \gamma_0 \delta, R}^{(\text{hard})} \geq KLw \right) \leq \left( \bigcup_{\xi \in R_L} \bigcap_{(k, \ell) \in \xi} \left\{ \sum_{B \in \mathcal{B}_{k, \ell}} d_{\mathbb{E}, K \gamma_0 \delta, R}^{(\text{hard})} \geq w \right\} \right) \leq |R_L| \max_{\xi \in R_L} \left( \bigcap_{(k, \ell) \in \xi} \left\{ \sum_{B \in \mathcal{B}_{k, \ell}} d_{\mathbb{E}, K \gamma_0 \delta, R}^{(\text{hard})} \geq w \right\} \right). \tag{4.27}
\]
Now for any $\xi \in R_L$, we have
\[
P \left( \bigcap_{(k, \ell) \in \xi} \left\{ \sum_{B \in \mathcal{B}_{k, \ell}} d_{\mathbb{E}, K \gamma_0 \delta, R}^{(\text{hard})} \geq w \right\} \right) \leq \max_{m=1}^{M} \left( \bigcap_{(k, \ell) \in \xi \cap G_m} \left\{ \sum_{B \in \mathcal{B}_{k, \ell}} d_{\mathbb{E}, K \gamma_0 \delta, R}^{(\text{hard})} \geq w \right\} \right) \leq \max_{m=1}^{M} \left[ 12 P \left( d_{\mathbb{E}, (K \gamma_0 S)_\delta, K \gamma_0 R}^{(\text{hard})} \geq w/12 \right) \right]^{L/M} \leq \left[ 12 P \left( d_{\mathbb{E}, (K \gamma_0 S)_\delta, K \gamma_0 R}^{(\text{hard})} \geq w/12 \right) \right]^{L/M},
\]
where in the third inequality we used (4.25). The right-hand side does not depend on $\xi$, so we have by (4.26) and (4.27) that
\[
P \left( \min_{\omega \in Q} \sum_{j=1}^{J(\omega)} \sum_{B \in \mathcal{B}_{k, \ell}} d_{\mathbb{E}, K \gamma_0 \delta, R}^{(\text{hard})} \geq 12 KLw \right) \leq \left[ 5 \left[ 12 P \left( d_{\mathbb{E}, (K \gamma_0 S)_\delta, K \gamma_0 R}^{(\text{hard})} \geq w/12 \right) \right]^{1/M} \right] L.
\]
Now we take $w = 12 \Theta^{\text{hard}}_{\mathbb{E}, (K \gamma_0 S)_\delta, K \gamma_0 R} (1 - p)$, with $p < 1/(12 \cdot 10^M)$, so
\[
P \left( \min_{\omega \in Q} \sum_{j=1}^{J(\omega)} \sum_{B \in \mathcal{B}_{k, \ell}} d_{\mathbb{E}, K \gamma_0 \delta, R}^{(\text{hard})} \geq 12 KL \Theta^{\text{hard}}_{\mathbb{E}, (K \gamma_0 S)_\delta, K \gamma_0 R} (1 - p) \right) \leq \left[ 5 \left( 12p \right)^{1/M} \right] L \leq 2^{-L}.
\]
Recalling (4.24), (4.23), and the definition of \( L \in \{ K/3, 3K \} \), we obtain

\[
P(d_{\mathbb{S}^{0}, R}(L, R) \geq 12\nu K^2 \Theta^{\text{hard}}_{\mathbb{S}^{0}, K, \nu, R}(1 - p) \leq C(2^{-K} + e^{-\frac{1}{2} \nu (\log \nu)^2 K^2}).
\]

Now taking \( \nu = e^{K^{3/4}} \) yields (4.4), while taking \( \nu = Ce^{\sqrt{\log K}} \) for an appropriate constant \( C \), and taking \( K \) sufficiently large (depending on the choice of \( p \)), yields (4.5). \( \square \)

5 Fluctuations of the crossing distance: Efron–Stein inequality and multiscale analysis

In this section we prove the following theorem, bounding the variance of the hard crossing distance at a given scale by an absolute constant.

**Theorem 5.1.** There is a constant \( C < \infty \) so that

\[
\sup_{U \in (0, \infty)} \chi_U \leq C \tag{5.1}
\]

and

\[
\text{Var}(\log d_{\mathbb{S}^{0}}(\text{hard}))) \leq C \tag{5.2}
\]

for all \( U \geq 0 \).

Before we prove Theorem 5.1, we point out some corollaries.

**Corollary 5.2.** There is an increasing function \( \Theta^{\ast}: \mathbb{R}_+ \to \mathbb{R}_+ \) so that for any \( 0 < p < 1 \) there is a constant \( C = C(p) \) so that

\[
\Theta^{\text{easy}}_{\mathbb{S}^{0}}(p), \Theta^{\text{hard}}_{\mathbb{S}^{0}}(p) \in [C^{-1}\Theta^{\ast}(U), C\Theta^{\ast}(U)].
\]

**Proof.** This follows from Theorem 5.1, Proposition 2.37, Theorem 3.4, and Lemma A.1. \( \square \)

As noted in the introduction, our proof works without precise knowledge of the growth rate of the quantiles, hence without precise knowledge of \( \Theta^{\ast}(U) \). However, we do know that the growth of \( \Theta^{\ast}(U) \) is bounded above and below by power laws.

**Corollary 5.3.** We have a constant \( C < \infty \) so that, for all \( U \geq 1 \), we have

\[
\Theta^{\ast}(U) \in [C^{-1}U^{-C^{-1}}, CU^{C}].
\]

**Proof.** This follows from Corollary 5.2 and Propositions 4.4 and 4.6. \( \square \)

We will see that Corollary 5.3 implies Theorem 1.3, as \( \Theta^{\ast}(U) \) will be the normalizing constant for the metric.

Our method of proof of Theorem 5.1 will essentially be to use the Efron–Stein inequality to inductively bound the variance. In Subsection 5.1, we express the Gaussian free field in terms of an underlying white noise, and then break this white noise into chunks, each of which we will resample independently in the Efron–Stein inequality.

Throughout the argument in this section, we will fix a scale \( S \) and let \( \mathbb{R} = \mathbb{S}(KS, LS) \), where \( K/L \in [1/3, 3] \).
5.1 Resampling the field

Let \((\mathbb{C}_k)_{k \in \mathbb{Q}}\) be a grid partition of \(\mathbb{R}^+\) into disjoint identical squares of size at most \(S\) at least \(S/100\) so that \((\mathbb{C}_i)_{i \in \mathbb{N}}\) is a partition of \(\mathbb{R}^+\) for some subset \(\mathbb{N} \subset \mathbb{Q}\). We note that there is a constant \(C\) so that \(|\mathbb{Q}| \leq CK^2\).

We note that there is a space-time white noise \(W\) on \(\mathbb{R}^+ \times (0, \infty)\) so that we can write

\[
h_{\mathbb{R}^+}(x) = \sqrt{\pi} \int_0^\infty \int_{\mathbb{R}^+} p^*_{t/2}(x, y) W(dy \, dt),
\]

where \(p^*_{t/2}\) is the transition kernel for a standard Brownian motion killed on \(\partial \mathbb{R}^+\). (This can be verified by checking that this yields the covariance function (2.1).) Note that the notation \(h_{\mathbb{R}^+}(x)\) is an abuse of notation since the Gaussian free field does not take pointwise values: \((5.3)\) should be interpreted in the sense of distributions. Now let \(\tilde{W}\) denote a two-dimensional space-time white noise on \((0, \infty) \times \mathbb{R}^+\) which is independent of \(W\). Define (again in the sense of distributions)

\[
h_{\mathbb{R}^+}^{(i)}(x) = \sqrt{\pi} \int_0^\infty \int_{C_i} p^*_{t/2}(x, y) \tilde{W}(dy \, dt) + \sqrt{\pi} \int \int_{(\mathbb{R}^+ \times (0, \infty) \setminus ((0, \mathbb{R}^+))} p^*_{t/2}(x, y) W(dy \, dt)
\]

and

\[
\tilde{h}_{\mathbb{R}^+}(x) = \sqrt{\pi} \int_{S^2} \int_{\mathbb{R}^+} p^*_{t/2}(x, y) \tilde{W}(dy \, dt) + \sqrt{\pi} \int \int_{\mathbb{R}^+} p^*_{t/2}(x, y) W(dy \, dt).
\]

Of course, \(h_{\mathbb{R}^+}^{(i)}\) for \(i \in \mathbb{Q}\) and \(\tilde{h}_{\mathbb{R}^+}\) are all Gaussian free fields, identical in law to \(h_{\mathbb{R}^+}\). As in Subsection 2.2, for all \(\mathbb{B} \subset \mathbb{R}^+\), define \(h_{\mathbb{R}^+}^{(i), \mathbb{B}}\) (resp. \(\tilde{h}_{\mathbb{R}^+}^{(i), \mathbb{B}}\)) to be the harmonic interpolation of \(h_{\mathbb{R}^+}^{(i)}\) (resp. \(\tilde{h}_{\mathbb{R}^+}\)) onto \(\mathbb{B}\), and simply \(h_{\mathbb{R}^+}^{(i)}\) (resp. \(\tilde{h}_{\mathbb{R}^+}\)) outside of \(\mathbb{B}\). Also define \(h^{(i)} = h_{\mathbb{R}^+}^{(i)} - h_{\mathbb{R}^+}^{(i), \mathbb{B}}\) and \(\tilde{h}_{\mathbb{B}} = \tilde{h}_{\mathbb{R}^+} - \tilde{h}_{\mathbb{R}^+}^{(i), \mathbb{B}}\) on \(\mathbb{B}\).

For all \(\mathbb{B} \subset \mathbb{R}^+\), let \(\mu_{\mathbb{B}}^{(i)}\) (resp. \(\tilde{\mu}_{\mathbb{B}}\)) denote the Liouville quantum gravity measure on \(\mathbb{B}\) using the GFF \(h_{\mathbb{B}}^{(i)}\) (resp. \(\tilde{h}_{\mathbb{B}}\)), and let \(d_{\mathbb{B}, \infty}^{(i)}\) (resp. \(\tilde{d}_{\mathbb{B}, \infty}\)) denote the corresponding Liouville graph distance. Thus \(d_{\mathbb{B}, \infty}^{(i)}\) and \(\tilde{d}_{\mathbb{B}, \infty}\) are identical in law to \(d_{\mathbb{B}, \infty}\)—the only difference is that they are defined using different underlying white noise fields.

We now establish a few lemmas about how distances can change when the field is resampled. Define

\[
\mathcal{N}(i) = \{j \in \mathcal{N} \mid \text{dist}_E(\mathbb{C}_i, \mathbb{C}_j) \geq 4S\}, \quad \mathcal{R}(i) = \mathcal{N} \setminus \mathcal{N}(i) = \{j \in \mathcal{N} \mid \text{dist}_E(\mathbb{C}_i, \mathbb{C}_j) < 4S\},
\]

\[
\mathbb{D}_i = \bigcup_{j \in \mathcal{R}(i)} \mathbb{C}_j, \quad \mathbb{E}_i = \mathbb{D}_i^{(2S)}, \quad A_i = (3\mathbb{E}_i \setminus \mathbb{E}_i) \cap \mathbb{R}.
\]

Further define

\[
D_{i,j} = \text{dist}_E(\mathbb{C}_i, \mathbb{C}_j^{(2S)}), \quad E_{i,j} = S^{-2}D_{i,j}^2, \quad G_{i,j} = ||h_{\mathbb{R}^+} - h_{\mathbb{R}^+}^{(i)}||_{L^\infty(\mathbb{E}_i)}, \quad G^* = \max_{i \in \mathbb{Q}} \max_{j \in \mathcal{N}(i)} G_{i,j}e^{E_{i,j}/C}
\]

where the constant \(C\) is fixed in the proof of the next lemma.

**Lemma 5.4.** For every \(\lambda > 0\), we have a constant \(C = C(\lambda)\) so that, as long as \(j \in \mathcal{N}(i)\), we have

\[
\mathbb{E}\exp\{\lambda G_{i,j}\} \leq C
\]

and

\[
\mathbb{E}\exp\{\lambda G^*\} \leq Ce^{C\sqrt{\log K}}.
\]
Proof. If \( j \in N(i) \), then \( D_{i,j} \geq S \), so we have

\[
\text{Var}\left( h^{(i)}_{R^+}(x) - h_{R^+}(x) \right) = 2\pi \int_0^S \int_{C_i} \left| p_{t/2}^R(x,y) \right|^2 \, dy \, dt \leq C \int_0^S \int_{C_i} r^{-2} \exp \left\{ \frac{|x-y|^2}{Ct} \right\} \, dy \, dt
\]

\[
\leq C \int_{C_i} \int_0^S |x-y|^2 t^{-2} e^{-1/(Ct)} \, dy \, dt \leq CS^2D_{i,j}^2 \int_0^S t^{-2} e^{-1/(Ct)} \, dt \leq Ce^{-E_{i,j}/C} \tag{5.8}
\]

and

\[
\text{Var}\left( \left( h^{(i)}_{R^+}(x) - h_{R^+}(x) \right) - \left( h^{(i)}_{R^+}(z) - h_{R^+}(z) \right) \right) = 2\pi \int_0^S \int_{C_i} \left| p_{t/2}^R(x,y) - p_{t/2}^R(z,y) \right|^2 \, dy \, dt
\]

\[
\leq C \int_{C_i} \int_0^S |x-z|^2 \left| p_{t/2}^R(x,y) \right| \left| p_{t/2}^R(z,y) \right| \, dy \, dt \leq C \int_0^S \int_{C_i} |x-z|^2 t^{-4} D_{i,j}^2 e^{-D_{i,j}^2/(Ct)} \, dy \, dt,
\]

where in the last inequality we used Lemma A.3. By Fernique’s inequality ([19]; see e.g. [1, Theorem 4.1]), this implies that

\[
\mathbb{E} \sup_{x \in C_i} \left( h^{(i)}_{R^+}(x) - h_{R^+}(x) \right) \leq CE_{i,j}e^{-E_{i,j}/C} \leq Ce^{-E_{i,j}/C}.
\]

Thus \( \{G_{i,j}e^{E_{i,j}/C} \}_{i \in Q} \) is a family of cardinality order \( K^4 \), of random variables with order-1 expectations. By the Borell–TIS inequality (see e.g. [25, Theorem 7.1], [7, Theorem 6.1], or [1, Theorem 2.1]) and (5.8), this family also has order-1 Gaussian tails, since it can be written as the maximum of many Gaussian random variables with order-1 Gaussian tails. This implies (5.6) and (5.7). \( \square \)

**Lemma 5.5.** For every \( \lambda > 0 \), we have a constant \( C \) depending on \( \lambda \) so that

\[
\mathbb{E} \exp \left\{ \lambda \sup_{x \in E_i} \left( h^{(i)}_{R^+;E_i}(x) - h_{R^+;E_i}(x) \right) \right\} \leq C. \tag{5.9}
\]

**Proof.** Inequality (5.6) of Lemma 5.4 implies that for any \( \lambda > 0 \) there is a \( C < \infty \), depending only on \( \lambda \), so that

\[
\mathbb{E} \exp \left\{ \lambda \sup_{x \in E_i \setminus E_i^c} \left( h^{(i)}_{R^+;E_i}(x) - h_{R^+;E_i}(x) \right) \right\} \leq C.
\]

By the maximum principle for harmonic functions, we have that

\[
\sup_{x \in E_i} \left| h^{(i)}_{R^+;E_i}(x) - h_{R^+;E_i}(x) \right| \leq \sup_{x \in E_i \setminus E_i^c} \left| h^{(i)}_{R^+;E_i}(x) - h_{R^+;E_i}(x) \right|,
\]

by which we obtain (5.9). \( \square \)

### 5.2 Splitting the path

When we use the Efron–Stein bound, we will resample a one part of the white noise at a time. We will use different techniques to bound the effect of this resampling on different parts of the path. In this section we describe our technique for breaking up the path into different parts which can be bounded separately.

Let \( \pi \) be a left–right \((R^+,R,1,S)\)-geodesic of \( R \) and let \( B_1, \ldots, B_N \) be the corresponding sequence of geodesic balls. Fix a parameterization of \( \pi \), which we will also call \( \pi : [0,1] \to R \). Define

\[
\mathcal{T} = \bigcup_{n=1}^{N-1} (B_n \cap B_{n+1}).
\]
Let \( \tau_0 = 0 \), and inductively define

\[
\tau_m = 1 \wedge \min\{ \tau \geq \tau_{m-1} : \tau \in \mathcal{T} \text{ and } \text{diam}_E(\pi([\tau_{m-1}, \tau_m])) \geq S \}.
\]

Let \( M \) be the first \( m \) so that \( \tau_m = 1 \). Let \( x_m = \pi(\tau_m) \) and let \( \pi_m = \pi([\tau_{m-1}, \tau_m]) \). Thus \( \pi = \bigcup_{m=1}^M \pi_m \). By the definition of \( \mathcal{T} \) and the fact that the \( B_i \)'s have radius at most \( S \), we have

\[
\text{diam}_E(\pi_m) \leq 2S. \tag{5.10}
\]

Also, by the definition of \( \mathcal{T} \) and the triangle inequality, we have

\[
\sum_{m=1}^M d_{\mathbb{R},1,S}(x_{m-1}, x_m) = d_{\mathbb{R},1,S}(L, R). \tag{5.11}
\]

For \( j \in \mathcal{N} \), define

\[
\mathcal{M}(i) = \{ 1 \leq m \leq M : \pi_m \cap D_i \neq \emptyset \}.
\]

Further define

\[
\pi_i^- = \pi \setminus \bigcup_{m \in \mathcal{M}(i)} \pi_m \tag{5.12}
\]

Note that if \( m \in \mathcal{M}(i) \), then \( \pi_m \subset \mathbb{B}_i \) by (5.10) and (5.4), so by Proposition 2.29,

\[
d_{\mathbb{R},1,S}(x_{m-1}, x_m) = d_{\mathbb{R},1,S,i}(x_{m-1}, x_m), \tag{5.13}
\]

and

\[
d_{\mathbb{R},1,S}(L, R) \leq d(i)_{\mathbb{R},1,S}(\pi_i^-) + d(i)_{\mathbb{R},1,S,A}(A) \tag{5.14}
\]

Also note that

\[
\pi_i^- \subset \bigcup_{j \in \mathcal{N}(i)} C_j \tag{5.15}
\]

by (5.12) and (5.4). We also note that we have a constant \( C \) so that for each \( m \),

\[
|\{ i \mid m \in \mathcal{M}(i) \}| \leq C, \tag{5.16}
\]

\[
|\{ j \mid \pi_m \cap C_j \}| \leq C, \tag{5.17}
\]

since each \( \pi_m \) has Euclidean diameter at most \( 2S \) by (5.10), and at most a constant number of \( C_j \)'s and \( D_i \)'s can intersect a set of diameter at most \( 2S \).

### 5.3 The Efron–Stein argument

We now describe how we apply the Efron–Stein inequality.

**Lemma 5.6.** We have

\[
\text{Var}(\log d_{\mathbb{R}}(L, R)) \leq \frac{1}{2} E \left( \frac{d_{\mathbb{R},1,S}(L, R)}{d_{\mathbb{R},1,S}(L, R)} \right)^2 + 2 \sum_{i \in Q} E \left( \frac{d(i)_{\mathbb{R},1,S}(\pi_i^-)}{d(i)_{\mathbb{R},1,S}(\pi)} - 1 \right)^2 + 2 \text{Var} \left( \frac{d_{\mathbb{R},1,S}(L, R)}{d_{\mathbb{R}}(L, R)} \right) + 2 E \max_{i \in Q} d(i)_{\mathbb{R},1,S,A}(A) E d_{\mathbb{R},1,S}(L, R)^{-3} E \left( \sum_{i \in Q} \frac{d(i)_{\mathbb{R},1,S,A}(A)}{d(i)_{\mathbb{R},1,S}(L, R)} \right)^{1/3}. \tag{5.18}
\]
Remark 5.7. Before deriving (5.18), we give a brief description of its terms. By symmetry, we only need to consider the potential increase in the distance after resampling the field. To this end, we consider a geodesic path with respect to the original field, and for each type of resampling use it to construct a slightly perturbed path with a not-too-much-larger LQG graph length with respect to the resampled field. The first term in (5.18) represents the effect of resampling the white noise at times far in the future, which corresponds to resampling a smooth “coarse field.” This has a “Lipschitz” effect on the weight of the path, so we can bound it using Gaussian concentration inequalities. The second term represents the effect of resampling the white noise at times close to 0 on the path far away from the resampled region. Here again, the resampling should be smooth, but since we are only considering the effect on part of the path, we need to use a more customized argument. The fourth term represents the effect of resampling the white noise on the path close to the resampled region. In this case, the LQG in the relevant region should change substantially, so we replace the path in this region by an optimal circuit of an annulus surrounding the region. Finally, the third term accounts for the error incurred in passing from \(d_{R,1,5}(L,R)\) to \(d_{\bar{R}}(L,R)\). This error should be small if the scale is large enough; we deal with small scales by the \textit{a priori} bound (4.7), which was proved in [12].

Proof of Lemma 5.6. By the Cauchy–Schwarz inequality, we have

\[
\text{Var}(\log d_{\bar{R}}(L,R)) \leq 2 \text{Var}(\log d_{R,1,5}(L,R)) + 2 \text{Var}\left(\frac{d_{R,1,5}(L,R)}{d_{\bar{R}}(L,R)}\right). \tag{5.19}
\]

By the Efron–Stein inequality and the decomposition of the white noise in Subsection 5.1, we have that

\[
\text{Var}(\log d_{R,1,5}(L,R)) \leq \frac{1}{2} \mathbb{E}\left(\log d_{R,1,5}(L,R) - \log d_{R,1,5}(L,R)\right)^2 + \frac{1}{2} \sum_{i \in Q} \mathbb{E}\left(\log d_{R,1,5}(L,R) - \log d_{R,1,5}(L,R)\right)^2. \tag{5.20}
\]

We note that the first term of (5.20) corresponds to the resampling of the coarse field, while the second corresponds to the resampling of the fine field. We will treat the coarse field term in Lemma 5.8 below, but further develop the fine field term now. Using the exchangeability of the resampled and unresampled random variables, and recalling the elementary inequality \((\log X - \log Y)^+ \leq (X - Y)^+/Y\) for all \(X, Y > 0\), we have

\[
\mathbb{E}\left(\log d_{R,1,5}^{(i)}(L,R) - \log d_{R,1,5}(L,R)\right)^2 \leq 2 \mathbb{E}\left(\left(\log d_{R,1,5}^{(i)}(L,R) - \log d_{R,1,5}(L,R)\right)^+\right)^2 \leq 2 \mathbb{E}\left(\frac{d_{R,1,5}^{(i)}(L,R) - d_{R,1,5}(L,R)}{d_{R,1,5}(L,R)}\right)^2. \tag{5.21}
\]

Now we have by (5.14) that

\[
\left(d_{R,1,5}^{(i)}(L,R) - d_{R,1,5}(L,R)\right)^+ \leq \left(d_{R,1,5}^{(i)}(\pi^+) + d_{R,1,5}^{(i),A_i,1,5}(A) - d_{R,1,5}(\pi)\right)^+ \leq \left(d_{R,1,5}^{(i)}(\pi^+) - d_{R,1,5}(\pi)\right)^+ + d_{R,1,5}^{(i),A_i,1,5}(A). \tag{5.22}
\]
Substituting (5.22) into (5.21), and using the fact that \( \pi \) is an \((\mathbb{R}^*, \mathbb{R}, \mathcal{S})\)-geodesic, gives us
\[
\left( \left( \log d_{\mathbb{R},1,S}^{(i)}(L,R) - \log d_{\mathbb{R},1,S}(L,R) \right)^+ \right)^2 \leq 2 \left( \left( \frac{d_{\mathbb{R},1,S}(\pi^-) - d_{\mathbb{R},1,S}(\pi)}{d_{\mathbb{R},1,S}(\pi)} \right)^+ + d_{\mathbb{R},1,S}(A) \right)^2.
\]

Now we take the sum and expectation of the second term, as in (5.20), and apply Hölder’s inequality to obtain
\[
\mathbb{E} \sum_{i \in Q} \left( d_{\mathbb{R},1,S}^{(i)}(A) \right)^2 \leq \mathbb{E} \left( \max_{i \in Q} d_{\mathbb{R},1,S}^{(i)}(A) \right) \left( \sum_{i \in Q} d_{\mathbb{R},1,S}^{(i)}(A) \right) \leq \left( \mathbb{E} \max_{i \in Q} d_{\mathbb{R},1,S}^{(i)}(A)^3 \mathbb{E} d_{\mathbb{R},1,S}(L,R)^{-3} \mathbb{E} \left( \sum_{i \in Q} d_{\mathbb{R},1,S}^{(i)}(A) \right)^3 \right)^{1/3}.
\]

Combining (5.19), (5.20), (5.23), and (5.24) yields (5.18). □

Our goal in the following subsections will be to bound the terms of (5.18).

### 5.3.1 The coarse field effect

First we address the effect of the coarse field (the first term of (5.18)) using Gaussian concentration.

**Lemma 5.8.** We have a constant \( C \) so that
\[
\mathbb{E} \left( \log \frac{d_{\mathbb{R},1,S}(L,R)}{d_{\mathbb{R},1,S}(L,R)} \right)^2 \leq C \log K.
\]

**Proof.** Let \( \mathcal{F}_{S^2} \) denote the \( \sigma \)-algebra generated by the white noise up to time \( S^2 \). We note that
\[
\frac{1}{2} \mathbb{E} \left( \log \frac{d_{\mathbb{R},1,S}(L,R)}{d_{\mathbb{R},1,S}(L,R)} \right)^2 = \mathbb{E} \text{Var} \left( \log d_{\mathbb{R},1,S}(L,R) \mid \mathcal{F}_{S^2} \right).
\]

Let
\[
h_{\text{coarse}}(x) = \sqrt{\pi} \int_{S^2} \int_{\mathbb{R}^*} \gamma \gamma(y) W(dy,dt).
\]

Now we note that \( h_{\text{coarse}} \) is a centered Gaussian process and that
\[
\sup_{x \in \mathbb{R}^*} \text{Var}(h_{\text{coarse}}(x)) \leq C \log K
\]
for some constant \( C \), according to [12, Lemma 2.7]. We claim that conditional on \( \mathcal{F}_{S^2} \), \( \log d_{\mathbb{R},1,S}(L,R) \) is a \( \gamma \)-Lipschitz function of \( h_{\text{coarse}} \). Fix the white noise up to time \( S^2 \) and write
\[
\log d_{\mathbb{R},1,S}(L,R) \mid \mathcal{F}_{S^2} =: \mathcal{D}(h_{\text{coarse}}).
\]
Thus we can compute
\[
|\mathcal{D}(h_{\text{coarse}} + \Delta) - \mathcal{D}(h_{\text{coarse}})| \leq \gamma \|\Delta\|_{L^\infty(\mathbb{R}^d)}
\]
by (2.27) and (2.31), which is the claimed Lipschitz property.

Let \( X = \{ h \mid \mathcal{D}(h) \leq X \} \). Then we have \( P(h_{\text{coarse}} \in X) \geq 1/2 \). We can apply (an infinite-dimensional version of) the Gaussian concentration inequality given in [12, Lemma 2.1], along with (5.27), to observe that
\[
P(\log d_{\mathbb{R},1,S}(L,R) \geq X + \lambda \mid \mathcal{F}_{S^2}) \leq P\left(\min_{h \in X} \|h_{\text{coarse}} - h\|_{L^\infty(\mathbb{R}^d)} \geq \gamma^{-1} \lambda \mid \mathcal{F}_{S^2}\right) \leq C e^{-\frac{(1 - C \log K)^2}{d \log K}}
\]
almost surely. A similar argument implies that
\[
P(\log d_{\mathbb{R},1,S}(L,R) \leq X - \lambda \mid \mathcal{F}_{S^2}) \leq C e^{-\frac{(1 - C \log K)^2}{d \log K}}
\]
almost surely. This implies that
\[
\text{Var}(\log d_{\mathbb{R},1,S}(L,R) \mid \mathcal{F}_{S^2}) \leq C \log K
\]
almost surely, which means that
\[
E \text{Var}(\log d_{\mathbb{R},1,S}(L,R) \mid \mathcal{F}_{S^2}) \leq C \log K,
\]
implying (5.25) by (5.26).

\[\square\]

### 5.3.2 The far fine field effect

Now we turn to the second term of (5.18). First we deal with the part of the path which is “far away” from the white noise being resampled. As mentioned above, in this case we don’t need to change the path when we resample in order to get our bound. Rather, we simply bound the increase in the weight of the path by the maximum of the change in LQG. The bound we obtain in the following lemma is in terms of the maximum annular crossing and the total crossing distance. These terms will also appear in the close fine field bound in the next section, so we wait to bound them together in Subsection 5.3.4.

**Lemma 5.9.** There is a constant \( C \) so that
\[
E \sum_{i \in Q} \left( \left( \frac{d_{\mathbb{R},1,S}(\pi_i^-)}{d_{\mathbb{R},1,S}(\pi)} - 1 \right) \right)^2 \leq C e^{C \sqrt{\log K}} \left( E \left( \max_{j \in N(i)} d_{\mathbb{R},1,S}(A) \right) \right)^3 \left( Ed_{\mathbb{R},1,S}(\pi) \right)^3.
\]

**Proof.** Note that by (5.15), (2.32), (5.5), and (2.45) of Proposition 2.39, we have
\[
d_{\mathbb{R},1,S}(\pi_i^-) \leq \sum_{j \in N(i)} d_{\mathbb{R},1,S}(\pi_i^- \cap C_j) \leq \sum_{j \in N(i)} e^{\gamma(G_i,j)^*} d_{\mathbb{R},1,S}(\pi \cap C_j).
\]
Thus we can compute
\[
\sum_{i \in Q} \left( \left( \frac{d_{\mathbb{R},1,S}(\pi_i^-)}{d_{\mathbb{R},1,S}(\pi)} - 1 \right) \right)^2 \leq d_{\mathbb{R},1,S}(\pi)^{-2} \sum_{i \in Q} \sum_{j \in N(i)} \left( e^{\gamma(G_i,j)^*} - 1 \right) d_{\mathbb{R},1,S}(\pi \cap C_j) \leq d_{\mathbb{R},1,S}(\pi)^{-2} \sum_{j_1,j_2 \in N(i)} d_{\mathbb{R},1,S}(\pi \cap C_{j_1}) d_{\mathbb{R},1,S}(\pi \cap C_{j_2}) \sum_{i \in Q} \sum_{N(i) \ni j_1,j_2} \left( e^{\gamma(G_i,j_1)^*} - 1 \right) \left( e^{\gamma(G_i,j_2)^*} - 1 \right)
\]
\[\leq d_{\mathbb{R},1,S}(\pi)^{-2} \left( \sum_{j \in N} d_{\mathbb{R},1,S}(\pi \cap C_j) \right) \left( \max_{j_1,j_2} \sum_{j \in N(i) \ni j_1,j_2} \sum_{i \in Q} \left( e^{\gamma(G_i,j_1)^*} - 1 \right) \left( e^{\gamma(G_i,j_2)^*} - 1 \right) \right).
\]
Now we have that
\[
\sum_{j \in \mathbb{N}} d_{\mathbb{R},1,S}(\pi \cap \mathcal{C}_j) \leq \sum_{j \in \mathbb{N}} \sum_{1 \leq m \leq M_{\pi \cap \mathcal{C}_j} \neq 0} d_{\mathbb{R},1,S}(\pi_m) \leq \sum_{1 \leq m \leq M} \left\{ j \in \mathbb{N} \mid \pi_m \cap \mathcal{C}_j \neq \emptyset \right\} d_{\mathbb{R},1,S}(\pi_m) \leq C d_{\mathbb{R},1,S}(\pi)
\]
by \((5.17)\) and \((5.11)\). We also have that
\[
d_{\mathbb{R},1,S}(\pi \cap \mathcal{C}_j) \leq d_{\mathbb{R},A_j,1,S}(A),
\]
because the part of \(\pi\) intersecting \(\mathcal{C}_j\) can be replaced by a circuit around the annulus \(A_j\) to produce a new crossing, so \(\pi \cap \mathcal{C}_j\) must carry less weight than the annular circuit around \(A_j\). Thus we can further simplify
\[
\sum_{i \in \mathbb{Q}} \left( \left( \frac{d_{\mathbb{R},1,S}(\pi_i^*)}{d_{\mathbb{R},1,S}(\pi)} - 1 \right) \right)^2 \leq \left( \max_{j \in \mathbb{N}} \frac{d_{\mathbb{R},A_j,1,S}(A)}{d_{\mathbb{R},1,S}(\pi)} \right) \max_{j_1 \in \mathbb{N}} \sum_{i \in \mathbb{Q}} \left( (e^{G_i} e - e_i h_i) - 1 \right) \left( (e^{G_i} e - e_i h_i) - 1 \right) \leq C \max_{j \in \mathbb{N}} \frac{d_{\mathbb{R},A_j,1,S}(A)}{d_{\mathbb{R},1,S}(\pi)} e^{G_i} e,
\]
where the third inequality is directly from the definitions \((5.5)\). The inequality \((5.28)\) then follows by taking expectations and applying Hölder’s inequality and \((5.7)\) of Lemma 5.4.

\[\square\]

### 5.3.3 The close fine field effect

Now we deal with the part of the path that is close to the part of the white noise being resampled: the last term of \((5.18)\). Here, when we resample, we have replaced the part of the path that is close to the resampled box by a crossing around an annulus. The “fine field” has been totally changed in the resampling, so there is nothing to be gained by keeping track of it. However, the “coarse field” should only be changed by a small, smooth difference (as is proved in Proposition 5.11 below), and we want to keep track of this so that the sum of all of the changes will be bounded by the total weight of the original path. Thus we define the notation (recalling that \(\theta_0\) was fixed in Subsection 2.4)
\[
F_i^\uparrow = \max_{x \in \mathbb{E}_i^{\gamma}} h^{(i)}_{\mathbb{R} \cdot \mathbb{E}_i^{\gamma}}(x), \quad F_i^\downarrow = \min_{x \in \mathbb{E}_i^{\gamma}} h^{(i)}_{\mathbb{R} \cdot \mathbb{E}_i^{\gamma}}(x), \quad F_i^2 = F_i^\downarrow \wedge (\theta_0 \log K), \quad \theta^\uparrow = \max_{i \in \mathbb{Q}_0} (F_i^\uparrow - F_i^\downarrow) \quad (5.29)
\]
and
\[
H_i = \frac{d^{(i)}_{\mathbb{R},A_i e^{-\gamma_i^1}}(A)}{d^{(i)}_{\mathbb{E}_i e^{-\gamma_i^1}}(M \mathbb{S} / \text{diam}_\mathbb{E}_i)}, \quad H = \max_{i \in \mathbb{Q}_0} H_i.
\]

Our first lemma for this part divides the sum of all of the differences divided by the total crossing length into its constituent pieces.

**Lemma 5.10.** There is a constant \(C\) so that the following holds. Then
\[
\sum_{i \in \mathbb{Q}} \frac{d^{(i)}_{\mathbb{R},A_i e^{-\gamma_i^1}}(A)}{d_{\mathbb{R},1,S}(L, R)} \leq C e^{\gamma_0} H. \quad (5.30)
\]
Proof. By (2.44) and (2.45) of Proposition 2.39 and (2.24), we can write, whenever \( m \in M(i) \),

\[
d_{R_{i}}^{(i)}(A) \leq d_{\mathbb{E}i}^{(i)}(A) \leq e^{\gamma F_{i}^{\uparrow} - f_{i}^{\downarrow}} d_{\mathbb{E}i}^{(i)}(A),
\]

\[
d_{R_{i}}^{(i)}(x_{m-1}, x_{m}) \geq d_{\mathbb{E}i}^{(i)}(x_{m-1}, x_{m}) \geq d_{\mathbb{E}i}^{(i)}(x_{m-1}, x_{m}).
\]

Thus we have

\[
d_{R_{i}}^{(i)}(A) \leq e^{\gamma (F_{i}^{\uparrow} - f_{i}^{\downarrow})} d_{\mathbb{E}i}^{(i)}(A) \leq H e^{\gamma (F_{i}^{\uparrow} - f_{i}^{\downarrow})} d_{\mathbb{E}i}^{(i)}(x_{m-1}, x_{m}) \leq H e^{\gamma (F_{i}^{\uparrow} - f_{i}^{\downarrow})} d_{R_{i},E_{i},1,S}(x_{m-1}, x_{m}).
\]

(5.31)

There is a constant \( C \) so that

\[
\sum_{i \in Q} \sum_{m \in M(i)} d_{R_{i}}^{(i)}(x_{m-1}, x_{m}) = \sum_{i \in Q} d_{R_{i}}^{(i)}(x_{m-1}, x_{m}) = \sum_{i \in Q} d_{\mathbb{E}i}^{(i)}(x_{m-1}, x_{m}) \leq C \sum_{m=1}^{M} d_{\mathbb{E}i}^{(i)}(x_{m-1}, x_{m}) = C d_{\mathbb{E}i}^{(i)}(L, R),
\]

(5.32)

where the second equality is by (5.13), the inequality is by (5.16), and the last equality is by (5.11). Therefore, we have, for each \( i \in Q \),

\[
\sum_{i \in Q} \frac{d_{\mathbb{E}i}^{(i)}(A)}{d_{\mathbb{E}i}^{(i)}(L, R)} \leq e^{\gamma \theta} H \sum_{i \in Q} \frac{d_{\mathbb{E}i}^{(i)}(x_{m(i)-1}, x_{m(i)})}{d_{\mathbb{E}i}^{(i)}(L, R)} \leq e^{\gamma \theta} H \sum_{i \in Q} \sum_{m \in M(i)} \frac{d_{\mathbb{E}i}^{(i)}(x_{m-1}, x_{m})}{d_{\mathbb{E}i}^{(i)}(L, R)} \leq C e^{\gamma \theta} H,
\]

where \( m(i) = \min M(i) \), the first inequality is by (5.31) and the third is by (5.32). This proves (5.30). \( \square \)

We now show that the change in the coarse field is small.

**Proposition 5.11.** With \( f_{i}^{\uparrow} \) and \( f_{i}^{\downarrow} \) defined as in (5.29), for any \( B > 0 \) and \( A < \infty \) we have a \( C > 0 \) so that

\[
\mathbb{E} e^{\gamma B (f_{i}^{\uparrow} - f_{i}^{\downarrow})} \leq C |Q|^{1/A}.
\]

(5.33)

Proof. First we note the inequality

\[
f_{i}^{\uparrow} - f_{i}^{\downarrow} \leq F_{i}^{\uparrow} - F_{i}^{\downarrow},
\]

(5.34)

which follows immediately from the definitions (5.29). Now we have that

\[
F_{i}^{\uparrow} - F_{i}^{\downarrow} = \max_{x \in \mathbb{E}_{i}} h_{R_{i},E_{i}}^{(i)}(x) - \min_{x \in \mathbb{E}_{i}} h_{R_{i},E_{i}}^{(i)}(x) = \max_{x, y \in \mathbb{E}_{i}} (h_{R_{i},E_{i}}^{(i)}(x) - h_{R_{i},E_{i}}^{(i)}(y)) \\
\leq \max_{x, y \in \mathbb{E}_{i}} \left| h_{R_{i},E_{i}}^{(i)}(x) - h_{R_{i},E_{i}}^{(i)}(y) \right|.
\]

(5.35)

By Lemma 2.7, we have for any \( \lambda > 0 \) that there is a \( C \) depending on \( \lambda \) so that

\[
\mathbb{E} \exp \left\{ \lambda \max_{x, y \in \mathbb{E}_{i}} \left| h_{R_{i},E_{i}}^{(i)}(x) - h_{R_{i},E_{i}}^{(i)}(y) \right| \right\} \leq C.
\]

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Therefore, by Hölder’s inequality, we have for any $A > 0$ that
\[
E \exp \left\{ \lambda \max_{x \in \mathcal{E}_i} \left| h_{R^*:\mathcal{E}_i}^{(i)}(x) - h_{R^*:\mathcal{E}_i}^{(i)}(y) \right| \right\} = \left( E \exp \left\{ \lambda \max_{x,y \in \mathcal{E}_i} \left| h_{R^*:\mathcal{E}_i}^{(i)}(x) - h_{R^*:\mathcal{E}_i}^{(i)}(y) \right| \right\} \right)^{1/A} \\
\leq \left( \sum_{i \in Q} \left( E \exp \left\{ \lambda \max_{x,y \in \mathcal{E}_i} \left| h_{R^*:\mathcal{E}_i}^{(i)}(x) - h_{R^*:\mathcal{E}_i}^{(i)}(y) \right| \right\} \right)^{1/A} \right) \leq C |Q|^{1/A} \tag{5.36}
\]
for some $C$ depending on $A$ and $\lambda$. Considering the second term of (5.35), we note that, by Lemma 5.5, we have
\[
E \exp \left\{ \lambda \max_{x \in \mathcal{E}_i} \left| h_{R^*:\mathcal{E}_i}^{(i)}(x) - h_{R^*:\mathcal{E}_i}^{(i)}(x) \right| \right\} \leq C \tag{5.37}
\]
for some $C$ depending on $\lambda$. Combining (5.34), (5.35), (5.37), and (5.36) yields (5.33). □

Finally, we use our moment bounds and our inductive quantile bound to bound the $H_i$s.

**Lemma 5.12.** For any $B < \infty$ and $A < \infty$ we have a $C < \infty$ so that the following holds. Then we have
\[
E \left( \sum_{i \in Q} \frac{d_{R^*,A_i,L,S}^{(i)}(A)}{d_{B_i,A_i,L}^{(i)}} \right)^B \leq C K^2 \frac{B}{B_i \eta S}. \tag{5.38}
\]

**Proof.** We have, by Hölder’s inequality,
\[
(E_{H_i}^B)^3 = \left( \frac{d_{R^*,A_i,L,S}^{(i)}(A)}{d_{B_i,e^{-r_i}}(M_{S/diam\mathcal{E}_i})} \right)^B \leq \left( \frac{d_{R^*,A_i,e^{-r_i}}(A)}{\Theta_{\text{hard}}(p_1)} \right)^B \left( \frac{d_{B_i,e^{-r_i}}(M_{S/diam\mathcal{E}_i})}{\Theta_{\text{easy}}(p_0)} \right)^{-3B} E \left( \frac{\Theta_{\text{hard}}}{\Theta_{\text{easy}}}(p_1) \right)^{3B}. \tag{5.39}
\]
We compute
\[
E \left( \frac{d_{R^*,A_i,e^{-r_i}}(A)}{\Theta_{\text{hard}}(p_1)} \right)^{3B} \leq E \left[ \frac{d_{R^*,A_i,e^{-r_i}}(A)}{\Theta_{\text{hard}}(p_1)} \right] \leq EC = C, \tag{5.40}
\]
by Corollary 4.9 and (2.34) of Lemma 2.24. We can also compute
\[
E \left[ \frac{d_{B_i,e^{-r_i}}(M_{S/diam\mathcal{E}_i})}{\Theta_{\text{easy}}(p_0)} \right] \leq EC \frac{B}{B_i \eta S} \leq C K^3. \tag{5.41}
\]
by Proposition 4.7 and (2.34) of Lemma 2.24.

Finally, we have that, if we put $K^* = \exp \left\{ \frac{2y_i^j}{4+y^2} \right\}$, then we have $\frac{2y_i^j}{4+y^2} \leq \eta \log K$, so $K^* \leq K^\eta$. Thus we can write, using (2.30) twice, recalling Definition 4.3, and applying (2.31),
\[
\Theta_{\text{hard}}(p_1) = \Theta_{\text{hard}}(K^* S) \geq \frac{\Theta_{\text{easy}}(p_1)}{\Theta_{\text{easy}}(p_0)} \leq \frac{\Theta_{\text{easy}}(2y_i^j e^{-r_i})}{\Theta_{\text{easy}}(2y_i^j e^{-r_i})} = C K^3. \tag{5.42}
\]
Therefore, we have

\[
\mathbb{E} \left( \frac{\Theta^{\text{hard}}_{\mathcal{S}\mathcal{K}e} \gamma_{1}^{-1}(p_1)}{\Theta^{\text{easy}}_{\mathcal{S}\mathcal{K}e} \gamma_{1}^{-1}(p_0)} \right)^{3B} \leq C \mathcal{X}^{3B}_{\mathcal{K}^{-1}} \mathcal{E} \gamma_{2}^{B} \mathcal{F} \gamma_{1}^{B} \leq C K^{2} \mathcal{X}_{\mathcal{K}^{-1}}^{2},
\]

(5.42)

where the last inequality is by Proposition 5.11 (where in the notation of that lemma we took \( A = B \)). Plugging (5.40), (5.41), and (5.42) into (5.39) yields

\[
\mathbb{E} \mathcal{H}^{B}_{i} \leq C K^{2} \mathcal{X}_{\mathcal{K}^{-1}}^{2}.
\]

(5.43)

Combining Lemma 5.10, the Cauchy–Schwarz inequality, (2.10) of Corollary 2.10, (5.43), and the fact that \(|Q| \leq C K^{2}\), we have

\[
\mathbb{E} \left( \sum_{i \in Q} \frac{d_{\mathbb{E},K}^{(i)}(A)}{d_{\mathbb{E},1}^{(i)}(L,R)} \right)^{B} \leq C \left( \mathbb{E} \gamma_{2}^{B} \mathcal{F} \gamma_{1}^{B} \right)^{1/2} \left( \mathbb{E} \mathcal{H}^{B} \right)^{1/2} \leq C \left( \sum_{i \in Q} \mathbb{E} \mathcal{H}^{B}_{i} \right)^{1/2} \leq C K^{2} \mathcal{X}_{\mathcal{K}^{-1}}^{2},
\]

which is (5.38).

\[\Box\]

5.3.4 The maximum small-box crossing weight

We now show that the maximum annular crossing is much smaller (for large \( K \)) than the crossing quantile of the large rectangle. This will be used for bounding the middle factor in (5.28) as well as the first factor in the last term of (5.18). Here, we crucially use that \( \gamma < 2 \), as this means that even after considering the coarse field, subboxes “look” smaller than the large box.

**Lemma 5.13.** Define notation as in the statement of Lemma 5.10. Suppose that \( T_{0} \leq \mathcal{K}^{-1} \), where \( T_{0} \) is as in Proposition 4.4. For any \( B \geq 0 \) there is a \( c > 0 \) so that

\[
\mathbb{E} \left( \max_{i \in Q} d_{\mathbb{E},A}^{(i)}(A) \right)^{B} \leq C K^{B} \mathcal{X}_{\mathcal{K}^{-1}}^{B} \mathbb{E} \gamma_{1}^{(i)}(p_{0})^{B},
\]

(5.44)

and

\[
\mathbb{E} \left( \max_{i \in Q} d_{\mathbb{E},A}^{(i)}(A) \right)^{B} \leq C K^{B} \mathcal{X}_{\mathcal{K}^{-1}}^{B} \mathbb{E} \gamma_{1}^{(i)}(p_{0})^{B},
\]

(5.45)

**Proof.** Let \( F_{i}^{\uparrow} = \max_{i \in Q} F_{i}^{\uparrow} \). Then we have that

\[
d_{\mathbb{E},A}^{(i)}(A) \leq d_{\mathbb{E},e^{\gamma_{2}}F_{i}^{\uparrow}}^{(i)}(A) \leq e^{(\gamma^{2}F_{i}^{\uparrow}-\theta_{0}^{2} \log K)^{B}} d_{\mathbb{E},A}^{(i),K^{-1}}(A)
\]

by (2.44). Therefore,

\[
\max_{i \in Q} d_{\mathbb{E},A}^{(i)}(A) \leq e^{(\gamma^{2}F_{i}^{\uparrow}-\theta_{0}^{2} \log K)^{B}} \max_{i \in Q} d_{\mathbb{E},A}^{(i),K^{-1}}(A).
\]

(5.46)

Taking \( B \)th moments in (5.46) and applying the Cauchy–Schwarz inequality gives us

\[
\mathbb{E} \left( \max_{i \in Q} d_{\mathbb{E},A}^{(i)}(A) \right)^{B} \leq \left( \mathbb{E} e^{2B F_{i}^{\uparrow}-\theta_{0}^{2} \log K} \right)^{1/2} \left( \sum_{i \in Q} \mathbb{E} d_{\mathbb{E},A}^{(i),K^{-1}}(A)^{2B} \right)^{1/2} \leq C |Q|^{1/2} \mathbb{E} \gamma_{2}^{B} \mathcal{F} \gamma_{1}^{B} \mathcal{K}^{-1}(p_{0})^{B},
\]

(5.47)
where the second inequality is by (2.10) of Corollary 2.10, (2.30), and Corollary 4.9, and in the last line the constant $C$ depends on $B$. (Recall that in Section 4, $p_1$ was fixed to be as in Proposition 4.2.) On the other hand, we have that

$$\Theta_{Z \in (K') S}(p_1) \leq \overline{X}_{K' S} \Theta_{Z \in (K') S}(p_0) \leq C \overline{X}_{K' S} K^{-c(1-\eta)} \Theta_{Z \in (K') S}(p_0),$$

where the first inequality is by Definition 4.3 and the second is by (4.8), where we have used the assumption that $K' S \geq T_0$. Plugging the last inequality into (5.47) yields

$$\mathbb{E} \max_{i \in Q} d^{(i)}_{Z \in (K') S}(A)^B \leq C \overline{X}_{K' S} |Q|^{1/2} K^{-c(1-\eta)} \Theta_{Z \in (K') S}(p_0)^B \leq C \overline{X}_{K' S} K^{1-c(1-\eta)} \Theta_{Z \in (K') S}(p_0)^B.$$

Choosing $B$ large enough so that

$$-c' := 1 - Bc(1-\eta) < 0,$$

this becomes

$$\mathbb{E} \max_{i \in Q} d^{(i)}_{Z \in (K') S}(A)^B \leq C \overline{X}_{K' S} K^{-c'} \Theta_{Z \in (K') S}(p_0)^B,$$

which is (5.44). The second inequality (5.45) follows in the same way, noting that (5.47) uses nothing about the correlations between different $d^{(i)}_{Z \in (K') S}(A)$s for varying $i$.\qed

### 5.3.5 The effect of requiring small balls

Our Efron–Stein argument required restricting the balls used to cover the path to be of size at most $S$. This was important for our percolation argument, because otherwise a single ball could be used to cover the path in potentially very many of the $C_i$s. However, the effect should be negligible, because at large scales, we do not expect large balls to be used: recall from (2.15) that the LQG measure has all negative moments. The next lemma quantifies this intuition.

**Lemma 5.14.** Suppose that $K S \geq T_0$. For any $Q < \infty$, there are constants $C = C(Q) < \infty$ and $c > 0$ so that

$$\mathbb{E} \left( \log \frac{d_{Z \in (K') S}(L, R)}{d_L(L, R)} \right)^2 \leq C K^c S^{-c}.$$

**Proof.** By (2.33) we have

$$\log \frac{d_{Z \in (K') S}(L, R)}{d_L(L, R)} \leq \log \left( 1 + \frac{K^3}{d_L(L, R)} \right) \leq \frac{K^3}{d_L(L, R)},$$

almost surely. Therefore, we have, as long as $K S \geq T_0$, defined as in Proposition 4.4, that

$$\mathbb{E} \left( \log \frac{d_{Z \in (K') S}(L, R)}{d_L(L, R)} \right)^2 \leq K^6 \mathbb{E} d_L(L, R)^{-2} \leq C K^6 \overline{X}^{-1} K S \Theta_{Z \in (K') S}(p_0)^{-2} \leq C K^6 \overline{X}^{-1} K S \left( \frac{K S}{T_0} \right)^{-c} \Theta_{Z \in (K') S}(p_0)^{-2} \leq C K^c S^{-c},$$

where the second inequality is by Proposition 4.7 and the third is by Proposition 4.4 and (4.7).\qed

### 5.4 The induction procedure

Finally, we have obtained all of the necessary bounds on the terms of (5.18). We combine these bounds with the objects and estimates introduced in Section 4 to provide an inductive proof of Theorem 5.1.
Proof of Theorem 5.1. We have, using Lemma 5.8, Lemma 5.9, Lemma 5.14, and to bound the first three terms of (5.18) in Lemma 5.6, that, as long as $KS \geq T_0$,

$$
\text{Var} \left( \log d_{\Xi}(L, R) \right) \leq C \log K + C e^{\sqrt{\log K}} \left( \mathbb{E} \max_{j \in \mathbb{N}} d_{\Xi, A_j, 1, S}(A)^3 \right)^{1/3} \left( \mathbb{E} d_{\Xi, 1, S}(\pi)^{-3} \right)^{1/3} + CK^c S^{-c} 
$$

$$
+ 2 \left( \mathbb{E} \max_{i \in Q} d_{\Xi, A_i, 1, S}(A)^3 \mathbb{E} d_{\Xi, 1, S}(L, R)^{-3} \mathbb{E} \left( \sum_{i \in Q} d_{\Xi, A_i, 1, S}(A)^3 \right)^{1/3} \right). 
$$

(5.48)

Now we have that, for any $B \geq 3$, there is a constant $C$ so that

$$
\left( \mathbb{E} \left( \sum_{i \in Q} d_{\Xi, A_i, 1, S}(A)^3 \right)^{1/3} \right)^{1/3} \leq \left( \mathbb{E} \left( \sum_{i \in Q} d_{\Xi, A_i, 1, S}(A)^3 \right)^{B/1} \right) \leq CK^{2/B} (K^c S)^{1/3} 
$$

(5.49)

by Jensen’s inequality and Lemma 5.12, that there is a $c > 0$ so that, as long as $K^c S \geq T_0$,

$$
\left( \mathbb{E} \max_{i \in Q} d_{\Xi, A_i, 1, S}(A)^3 \right)^{1/3} \leq C (K^c S)^{1/3} \mathbb{E}_{\Xi(KS)}(p_0) 
$$

(5.50)

by Lemma 5.13, and that

$$
\left( \mathbb{E} d_{\Xi, 1, S}(L, R)^{-3} \right)^{1/3} = \left( \mathbb{E} d_{\Xi, 1, S}(\pi)^{-3} \right)^{1/3} \leq C (K_{S/100}^c S)^{1/3} \mathbb{E}_{\Xi(KS)}(p_0)^{-1}. 
$$

(5.51)

Plugging (5.49), (5.50), and (5.50) into the last term of (5.48), and (5.50) and (5.50) into the second term, we obtain, again as long as $K^c S \geq T_0$,

$$
\text{Var} \left( \log d_{\Xi}(L, R) \right) \leq C \log K + C e^{\sqrt{\log K}} (K^c S)^{1/3} \mathbb{E}_{\Xi(KS/100)} \mathbb{E}_{\Xi(KS)} + CK^c S^{-c} + CK^{2/B} (K^c S)^{1/3} \mathbb{E}_{\Xi(KS/100)} \mathbb{E}_{\Xi(KS)} 
$$

(5.52)

with the second inequality valid as long as $B$ is chosen so large that $2/B \leq c/4$ and $K$ is large enough that $K^{c/4} \geq 1/100$ and $e^{\sqrt{\log K}} \leq K^{c/4}$.

We recall equation (4.7), which states that there is a constant $C < \infty$ so that

$$
\mathbb{X}_{S_0} \leq C e^{(\log S_0)^{0.95}}, 
$$

(5.53)

and also (4.6), which states that there is a constant $C < \infty$ so that

$$
\mathbb{X}_U \leq C \exp \left\{ C \sqrt{\text{Var} \left( \log d_{\Xi(U)}(\text{hard}) \right)} \right\}. 
$$

(5.54)

At this point we need to freeze the value of the constants $C$ and $c$ in (5.52), (5.53), and (5.54), so, in accordance with our convention that constants $C$ and $c$ can change from line to line, we will rewrite the inequalities as stating that, whenever $K \geq Z$,

$$
\text{Var} \left( \log d_{\Xi}(L, R) \right) \leq Z \left( \log K + K^{-2} \mathbb{X}_{KS/100} + K^c S^{-c} \right) 
$$

(5.55)

$$
\mathbb{X}_{S_0} \leq Z e^{(\log S_0)^{0.95}} 
$$

(5.56)

$$
\mathbb{X}_U \leq Z \exp \left\{ Z \sqrt{\text{Var} \left( \log d_{\Xi(U)}(\text{hard}) \right)} \right\}. 
$$

(5.57)
for some constants \( z > 0 \) and \( 1 \leq Z < \infty \), which are now fixed and cannot change. We want to choose \( K \) and \( S_0 \) so that

\[
K \geq Z, \tag{5.58}
\]

\[
K^{\eta-1} S_0 \geq T_0, \tag{5.59}
\]

\[
K^{-z} S_0^{-z} \leq \log K, \tag{5.60}
\]

\[
Z K^{-z} e^{3(\log S_0)^{0.95} \sqrt{3} Z \log K} \leq \log K. \tag{5.61}
\]

We can do this as follows. First, we fix

\[
S_0 = K^{Z/z+1}, \tag{5.62}
\]

which guarantees that (5.60) holds as long as \( K \) is sufficiently large. Then \( K^{\eta-1} S_0 = K^{Z/z+\eta} \), so as long as \( K \) is sufficiently large, (5.59) will hold. Finally, for (5.61) to hold, we need

\[
e^{(3(Z/z+\epsilon)(0.95)(\log K)^{0.95}) \sqrt{3} Z \log K} \leq \log K,
\]

which can be attained by taking \( K \) sufficiently large. Finally (5.58) of course holds for sufficiently large \( K \). Thus we can achieve (5.59)–(5.61) by fixing \( K \) sufficiently large and then imposing (5.62), and we assume henceforth that \( K \) and \( S_0 \) have been assigned in this way.

We note that (5.56) and (5.61) imply that if we put

\[
\chi := \frac{Z e^{Z \sqrt{3} Z \log K}}{S_0} \leq \log K,
\]

then

\[
K^{-z} \chi \leq \log K. \tag{5.63}
\]

Now note that \([0, S_0) \subset S_\chi\) by Definition 4.3. Suppose that \( S \) is such that \( K S \geq S_0 \) and \([0, KS/100] \subset S_\chi\). By (5.59), we have that

\[
K^{\eta} S = K^{\eta-1} K S \geq K^{\eta-1} S_0 \geq T_0,
\]

so by plugging (5.60) and (5.63) into (5.55), we obtain

\[
\text{Var}(\log d_{\chi}(L, R)) \leq Z \left( \log K + K^{-z} \chi^3 + K^{Z} S^{-z} \right) \leq Z \left( \log K + K^{-z} \chi^3 + K^{Z} S^{z} \right) \leq 3Z \log K.
\]

Therefore, by (5.57), we have

\[
\chi KS \leq Z e^{Z \sqrt{3} Z \log K} \leq \chi,
\]

so \( KS \in S_\chi \) as well. By induction, this implies that \( S_\chi = [0, \infty) \). From this, the first two inequalities of (5.55) imply (5.2).

\[\square\]

### 6 Subsequential scaling limits

The variance bound proved in the previous section will allow us to show that the renormalized metric is “almost” Hölder continuous. (The “almost” is because the metric takes discrete values—this point is not important.)

**Theorem 6.1.** There is a constant \( c > 0 \) and a function \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) so that

\[
\lim_{S \to \infty} f(S) = 0
\]
and the following holds. For every \( \varepsilon > 0 \), there is a constant \( C = C(\varepsilon) < \infty \) so that for every \( S \geq 0 \), with probability at least \( 1 - \varepsilon \), we have that for every \( x, y \in \mathcal{B}(S) \),

\[
\frac{d_{\mathcal{B}(S)}(x, y)}{\Theta^*(S)} \leq C(\varepsilon) \left( \frac{|x - y|}{S} \right)^{c} \vee f(S).
\]

**Corollary 6.2.** The family

\[
\left\{ \frac{d_{\mathcal{B}(1, \delta)}(\cdot, \cdot)}{\Theta^*(\delta - \frac{\gamma}{\gamma+1})} \right\}_{\delta > 0}
\]

is tight in the uniform topology on \([0, 1]^2\).

**Proof.** This follows immediately from Theorem 6.1, Proposition 2.36, and Lemma A.4 (the last being a slight generalization of the Arzelà–Ascoli theorem).

We note that Corollary 6.2 immediately implies almost all of Theorem 1.1 by Proposition 2.26 and Prokhorov’s theorem, except that the limiting objects may be pseudometrics rather than metrics. In this section we prove Theorem 6.1, and then we prove that the limiting objects are in fact metrics in the next section. We will need a preliminary lemma, which allows us to deal with the “base case” of our diameter chaining argument by showing that it is unlikely that very small boxes will have large diameter.

**Lemma 6.3.** There is a \( C < \infty \) and an \( \nu_0 > 1 \) so that if \( \nu \in (1, \nu_0) \) then there is an \( \alpha > 0 \) so that the following holds. Fix \( S, K > 0 \) and let \( \mathcal{B} = \mathcal{B}(S) \). Divide \( \mathbb{R}^d \) into \( J \)-many \( K^{-1}S \times K^{-1}S \) boxes labeled \( C_1, \ldots, C_J \), with \( J \leq CK^2 \). We have

\[
P \left( \max_{1 \leq j \leq J} \mu_{\mathcal{B}^*}(C_j) \geq u \right) \leq Cu^{-\nu}(2+\gamma^2/2)^{\nu}K^{-\alpha}.
\]

(6.1)

**Proof.** Fix \( \nu \in (1, \nu_0) \), to be specified more precisely later, where \( \nu_0 \) is as in Proposition 2.12. Define

\[
F_j = \max_{x \in C_j} h_{\mathcal{B}^*; C_j}(x).
\]

Then we note that \( F_j \) and \( \mu_{C_j}(C_j) \) are independent. Also, by (2.31), we have

\[
\mu_{\mathcal{B}^*}(C_j) \leq e^{F_j} \mu_{C_j}(C_j).
\]

(6.2)

We recall from Lemmas 2.5 and 2.7 that \( F_j \) has expectation of order 1 and tails commensurate with those of a Gaussian of variance \( \log K \). Therefore, we have

\[
E e^{F_j} \leq CK^{1+\gamma^2/2}.
\]

(6.3)

On the other hand, we have that

\[
\mu_{C_j}(C_j) \overset{\text{law}}{=} (S/K)^{2+\gamma^2/2} \mu_{\mathcal{B}^*}(\mathcal{B}),
\]

where \( \mathcal{B} = [0, 1]^2 \), so

\[
E \mu_{C_j}(C_j) = (S/K)^{2+\gamma^2/2} E \mu_{\mathcal{B}^*}(\mathcal{B})^\nu.
\]

(6.4)

Therefore, by (6.2) and the independence of the two factors on its right-hand side, along with (6.3) and (6.4), we have

\[
E \mu_{\mathcal{B}^*}(C_j)^\nu \leq CS^{2+\gamma^2/2}K^{1+\gamma^2/2+2-\gamma^2/2}E \mu_{\mathcal{B}^*}(\mathcal{B})^\nu \leq CS^{2+\gamma^2/2}K^{1+\gamma^2/2+2-\gamma^2/2}.
\]
in which we note that $E\mu_B^\gamma(\mathbb{B})^\gamma$ is a fixed absolute constant by (2.14). Then we have

$$E\max_{1 \leq j \leq J} \mu_B(\mathcal{C}_j)^\gamma \leq J\mu_B(\mathcal{C}_j)^\gamma \leq C\gamma^{2+2\gamma^2/2^\gamma+2}.$$

If we define $f(\gamma) = \frac{1}{2}\gamma^2 - (2 + \frac{1}{2}\gamma^2)\gamma + 2$, then we have that $f(1) = 0$ and $f'(1) = \frac{1}{2}\gamma^2 - 2 < 0$, since $\gamma < 2$. This implies that there is an $\nu > 1$ so that $f(\nu) < 0$, which implies that there is a $\nu_0 > 0$ so that if $\nu \in (1, \nu_0)$ then

$$E\left(\max_{1 \leq j \leq J} \mu_B(\mathcal{C}_j)^\nu\right) \leq C\gamma^{2+2\gamma^2/2^\gamma+2}.$$

Inequality (6.1) then follows from Markov’s inequality.

The following proposition is then the key to the proof of Theorem 6.1.

**Proposition 6.4.** There is a constant $c > 0$ so that for every $\varepsilon > 0$, there is a constant $C = C(\varepsilon) < \infty$ so that for every $S \geq 0$, the following holds. Let $\mathbb{R} = \mathbb{B}(S)$. For integers $t \in \mathbb{Z}_{\geq 0}$ and $0 \leq i, j \leq 2^t - 1$, define

$$\mathbb{B}_{t, i, j} = \begin{cases} (i2^{-t}S, j2^{-t+1}S) + [0, 2^{-t}S] \times [0, 2^{-t+1}S] & \text{if } t \text{ is even} \\ (i2^{-t+1}S, j2^{-t}S) + [0, 2^{-t+1}S] \times [0, 2^{-t}S] & \text{if } t \text{ is odd} \end{cases}$$

Then

$$P\left(\bigcup_{t_0 = 0}^{|\log_2 S|} \max_{t_0 \leq t \leq t - 1} \sup_{(x, y) \in \mathbb{B}_{t, i, j}} \frac{d_E(x, y)}{\Theta^*(S)} \geq C(\varepsilon)2^{-t_0}\right) < \varepsilon.$$

**Proof.** By a chaining argument illustrated in Figure 6.1, we have, for any $t, t_0 \in \mathbb{Z}_{\geq 0},$

$$\max_{i,j=0}^{2^t-1} \sup_{x,y \in \mathbb{B}_{t, i, j}} d_E(x, y) \leq 2^{t_0-1} \max_{i,j=0}^{2^t-1} \sup_{x,y \in \mathbb{B}_{t_0, i, j}} d_E(x, y) + 2 \sum_{s=t}^{t_0-1} \max_{i,j=0}^{2^s-1} d_E(x, y) \leq C \max_{i,j=0}^{2^t-1} \mu_B^*(\mathbb{B}_{t, i, j}) + 2 \sum_{s=t}^{t_0-1} \max_{i,j=0}^{2^s-1} d_E(x, y) \leq 2^{t-1} \max_{i,j=0}^{2^t-1} \mu_B^*(\mathbb{B}_{t, i, j}).$$

(See also [9, Proposition 6.7].) The second inequality in (6.5) is by the fact that all points in $\mathbb{B}_{t, i, j}$ can be connected by some fixed number of balls inside $\mathbb{B}_{t_0, i, j}$ with centers in $\mathbb{B}_{t, i, j}$.

Define

$$F_{t, i, j} = \max_{x \in \mathbb{B}_{t, i, j}} h_{\mu_B^*, \mathbb{B}_{t, i, j}}^{(t)}(x), \quad F_t = \max_{i,j=0}^{2^t-1} F_{t, i, j}.$$
Then by (2.44) of Proposition 2.39, (2.30), and Proposition 2.36, we have, defining $\mathcal{B}_s = \mathcal{B}(2^{-1-\eta})$, that
\[
d_{\mathcal{B}_s, x, j}^{\text{hard}}(\text{hard}) \leq d_{\mathcal{B}_s, x, j}^{\text{hard}}(\text{hard}) \leq e^{(\gamma F_s - \gamma \theta_0 s \log 2)^+} d_{\mathcal{B}_s, x, j}^{\text{hard}}(\text{hard}) \leq e^{(\gamma F_s - \gamma \theta_0 s \log 2)^+} d_{\mathcal{B}_s}^{\text{hard}}(\text{hard}). \tag{6.6}
\]

Let
\[
t_1 = \left[ (1 - \eta)^{-1} \log 2 \right], \tag{6.7}
\]
so $2^{-(1-\eta)t_1} S \geq 1$. By (6.5), (6.6), and a union bound, we have
\[
P \left( \bigcup_{t=0}^{t_1} \left( \max_{i,j} d_{\mathcal{B}}(x, y) \geq 2q_t + 2 \sum_{t=0}^{t_1} 2^{B_2^{1/3}} q_t \right) \right)
\leq \sum_{t=1}^{t_1} \left( 4P(d_{\mathcal{B}}(\text{hard}) \geq q_t) + P \left( \max_{i,j} \mu_{\mathcal{B}}(\mathcal{B}_{t,i,j}) \geq q_t \right) \right). \tag{6.8}
\]

for any choices of $B, q_t$. Then if we take, for $0 \leq t < t_1$,
\[
q_t = (1 + 2^{A_1(t \wedge t_1) + D}) \Theta^*(2^{-(1-\eta)t_1} S),
\]
then by Proposition 4.8 and Corollary 5.2, we have
\[
P \left( d_{\mathcal{B}}(\text{hard}) \geq q_t \right) \leq C e^{-A^{(1-\eta)/2} / 2^{2(1-\eta)/3}}. \tag{6.9}
\]

We have by Corollary 5.2, Proposition 2.37, and Proposition 4.4 (recalling the definition (6.7) of $t_1$) that there is a $c > 0$ and $C_1 < \infty$ so that, as long as $t \leq t_1$, we obtain
\[
\Theta^*(2^{-(1-\eta)t_1} S) \leq \Theta^*(2^{-(1-\eta)(t \wedge t_1)} S) \leq C_1 2^{-c(1-\eta)(t \wedge t_1)} \Theta^*(S).
\]

Therefore, we have
\[
q_t \leq C_1 2^{(A-c(1-\eta)(t \wedge t_1) + D)} \Theta^*(S).
\]

Furthermore, by (2.9) of Corollary 2.10, we have
\[
P \left( e^{\gamma F_t - \gamma \theta_0 t \log 2} \geq 2^{B_2^{1/3}} \right) \leq C e^{-B_2^{1/3} / 2^{2(1-\eta)/3}}. \tag{6.10}
\]

Substituting (6.9), (6.10), and the result of Lemma 6.3 into (6.8) yields, for any $\nu \in (1, \nu_0)$, a constant $C$ so that
\[
P \left( \bigcup_{t=0}^{t_1} \left( \max_{i,j} d_{\mathcal{B}}(x, y) \geq q_t + C_1 \Theta^*(S) 2^{D} \sum_{t=0}^{t_1} 2^{B_2^{1/3} + (A-c(1-\eta)(t \wedge t_1))} \right) \right)
\leq C \sum_{t=0}^{t_1} e^{-B_2^{1/3} / 2^{2(1-\eta)/3}} + C e^{-D^{(1-\eta)/2} / 2^{2(1-\eta)/3}} \leq C \sum_{t=0}^{t_1} e^{-B_2^{1/3} / 2^{2(1-\eta)/3}} + C q_t S^{2(1-\eta)(t \wedge t_1)} \Theta^*(S).
\]

Fix $A = c(1-\eta)/2$. We analyze the right-hand side of (6.11). To bound the third term, we take
\[
t_* = \left[ \frac{\nu}{\alpha} (2 + C^{1/2}) \log 2 \right], \tag{6.12}
\]

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so $S^{(2+γ^2/2)ν/2}2^{-t_0} ≤ 1$. We note that the first term of (6.11) goes to 0 as $B → ∞$, uniformly in all other variables (with the exception of $γ$, which we are treating as a fixed constant throughout the paper). The second term can be bounded by writing

$$
\sum_{t=0}^{t_0-1} e^t \log 4 - A_1^{8/7}(t η t_1)^{8/7} \leq C + e^{-A_1^{8/7} t_1} \sum_{t=0}^{t_0-1} 4^t
$$

$$
\leq C \left( 1 + \exp \left( -A_1^{8/7} \left( (1-η)^{-1} \log_2 S - 1 \right)^{8/7} / C + \left( \frac{\nu}{2}\left(2+\gamma^2/2\right)\log_2 S \right) \log 4 \right) \right),
$$

(6.13)

using the values (6.7) and (6.12) and $t_1$ and $t_*$. We note that (6.13) is also bounded independently of $S$. Therefore, once we have fixed the variables $A$ and $t_*$ as above, we have that

$$
\lim_{B→∞} \max_{d_{B→∞} \neq 0} \sup_{θ → ∞} \left\{ \sum_{t_0=0}^{t_0-1} \Theta'(S) \left( \frac{2^{Bt_0/3} - \frac{1}{2}e^{(1-η)(t_0+1)(t_1+1)}}{2^{Bt_0/3} - \frac{1}{2}e^{(1-η)(t_0+1)(t_1+1)}} \right) \right\} = 0,
$$

(6.14)

uniformly in $S$. By Lemma 2.38, we can fold $q_*$ into the sum, and we can also take $θ = B$ and fold that into the sum as well, so we simplify the last limit to

$$
\lim_{B→∞} \max_{d_{B→∞} \neq 0} \sup_{θ → ∞} \left\{ \sum_{t_0=0}^{t_0-1} \Theta'(S) \left( \frac{2^{Bt_0/3} - \frac{1}{2}e^{(1-η)(t_0+1)(t_1+1)}}{2^{Bt_0/3} - \frac{1}{2}e^{(1-η)(t_0+1)(t_1+1)}} \right) \right\} = 0.
$$

Now we analyze the sum in (6.14). We have that

$$
\sum_{t_0=0}^{t_0-1} 2^{Bt_0/3} - \frac{1}{2}e^{(1-η)(t_0+1)(t_1+1)} \leq C(B)e^{-c_1 t_0} + C2^{-\frac{1}{2}e^{(1-η)(t_0+1)}}
$$

(6.15)

$$
\leq C(B)e^{-c_1 t_0} + C2^{-\frac{1}{2}e^{(1-η)(t_0+1)}} \leq C(B)e^{-c_1 t_0} + C2^{-\frac{1}{2}e^{(1-η)(t_0+1)}} \leq C(B)S^{-c_2},
$$

(6.16)

and $S^{-c_2}$ as long as $t_0 ≤ \frac{\log_2 S}{c_1}$. Substituting (6.16) back into (6.15), and then (6.15) back into (6.14), proves the proposition.

Finally, we prove Theorem 6.1.

**Proof of Theorem 6.1.** By the previous proposition, for each $ε > 0$ we have a constant $C = C(ε)$ so that, with probability $1 - ε$, for each $0 ≤ t_0 ≤ c \log_2 S$,

$$
\sup_{x, y ∈ B(S)} \sup_{|x - y| ≤ 2^t_0 S} \frac{d_2(S, y)}{\Theta'(S)} \leq 2C(ε)2^{-c t_0}.
$$

Therefore, we have

$$
\sup_{x, y ∈ B(S)} \sup_{|x - y| ≤ 2^t_0 S} \frac{d_2(x, y)}{\Theta'(S)} \leq 2C(ε)
$$

with probability $1 - ε$. This proves the theorem.
7 Bi-Hölder continuity of the limiting metrics

Up to now, we have established the existence of a subsequential limiting function for our sequence of approximations of metrics. It is of course clear that any limiting function must satisfy positivity, symmetry and the triangle inequality, qualifying it for the title of “pseudo-metric.” However, it is not a priori clear that a limiting pseudometric must be a true metric—that is, that it is positive definite. That is what we will prove in this section. In fact, we will prove the stronger statement that the Euclidean metric must be Hölder continuous with respect to any limiting LGD metric. The following proposition gives a uniform upper bound on the moments of the Hölder coefficient, from which the Hölder continuity of the Euclidean metric with respect to any subsequential limit of the LQG metric follows by Fatou’s lemma.

Proposition 7.1. For every $B \in (1, \infty)$ we have an $\alpha_0 > 0$ so that for any $\alpha \in (0, \alpha_0]$, there is a constant $C < \infty$ so that for any $\delta > 0$, if $\mathbb{B} = \mathbb{B}(1)$, then

$$E \left( \max_{x, y \in \mathbb{B}} \frac{|x-y|}{d_{\mathbb{B}, \delta}(x, y)/\Theta^*(\delta^{\frac{2}{\gamma+1}})} \right)^B \leq C. \quad (7.1)$$

Proof. Fix $A \in (1, \infty)$. Note that

$$\max_{x, y \in \mathbb{B}} \frac{|x-y|}{d_{\mathbb{B}, \delta}(x, y)/\Theta^*(\delta^{\frac{2}{\gamma+1}})} = \max_{n \in \mathbb{N}} \max_{|x-y| \geq 2^{-n}} \frac{|x-y|}{d_{\mathbb{B}, \delta}(x, y)/\Theta^*(\delta^{\frac{2}{\gamma+1}})} \leq 2 \max_{n \in \mathbb{N}} 2^{-n} \left[ d_{\mathbb{B}, \delta}(M_{2^{-n}}) / \Theta^*(\delta^{\frac{2}{\gamma+1}}) \right]^{-\alpha} \quad (7.2)$$

By Proposition 4.7 and Theorem 5.1, we have for every $A < \infty$ a constant $C$ so that, for each $a \in (0, 1)$, we have

$$Ed_{\mathbb{B}}(M_a)^{-A} \leq C(1 + a^{-C}) \Theta^*(\delta^{\frac{2}{\gamma+1}})^{-A}. \quad (7.3)$$

This implies that

$$E\max_{n \in \mathbb{N}} \left[ \frac{d_{\mathbb{B}}(M_{2^{-n}})/\Theta^*(\delta^{\frac{2}{\gamma+1}})}{2^{-n/\alpha}} \right]^{-A} \leq \sum_{n=0}^{\infty} E \left( \frac{d_{\mathbb{B}}(M_{2^{-n}})/\Theta^*(\delta^{\frac{2}{\gamma+1}})}{2^{-n/\alpha}} \right)^{-A} \leq C \sum_{n=0}^{\infty} \frac{(1 + 2^n)}{2^{4n/\alpha}},$$

where the constant $C$ still depends on $A$. As long as $\alpha$ is chosen small enough that $A/\alpha \geq C$, then the last sum is finite. In particular, choose $\alpha$ so small that $A/\alpha \geq B$. This implies that

$$E \left( \max_{x, y \in \mathbb{B}} \frac{|x-y|}{d_{\mathbb{B}, \delta}(x, y)/\Theta^*(\delta^{\frac{2}{\gamma+1}})} \right)^{A/\alpha} \leq E \left( 2 \max_{n \in \mathbb{N}} 2^{-n} \left[ d_{\mathbb{B}, \delta}(M_{2^{-n}}) / \Theta^*(\delta^{\frac{2}{\gamma+1}}) \right]^{-\alpha} \right)^{A/\alpha}$$

$$= 2^{A/\alpha} E \max_{n \in \mathbb{N}} \left[ \frac{d_{\mathbb{B}, \delta}(M_{2^{-n}})/\Theta^*(\delta^{\frac{2}{\gamma+1}})}{2^{-n/\alpha}} \right]^{-A} \leq C,$$

where the first inequality is by (7.2) and the second is by (7.3). Then (7.1) follows from Jensen’s inequality.

\[ \square \]
A Technical lemmas

Lemma A.1. Let $X$ be a positive random variable and let $\Theta_X$ be its quantile function. For any $p, q \in (0, 1)$, we have

$$\Theta_X(q) \leq \exp \left\{ \sqrt{\Var(\log X)} \left( (1 - q)^{-1/2} + p^{-1/2} \right) \right\} \Theta_X(p). \tag{A.1}$$

Proof. Define $\mu = \mathbb{E} \log X$. We have, for $a \geq e^\mu$,

$$1 - F_X(a) = \Pr(X \geq a) = \Pr(\log X \geq \log a) = \Pr(\log X - \mu \geq \log a - \mu) \leq \frac{\Var(\log X)}{(\log a - \mu)^2}.$$

This means that $\Theta_X \left( 1 - \frac{\Var(\log X)}{(\log a - \mu)^2} \right) \leq a$. Now if $a = \exp \left\{ \sqrt{\frac{\Var(\log X)}{1-p}} + \mu \right\}$, then $a \geq \exp(\mathbb{E} \log X)$ and

$$1 - \frac{\Var(\log X)}{(\log a - \mu)^2} = p,$$

so

$$\Theta_X(p) \leq \exp \left\{ \sqrt{\frac{\Var(\log X)}{1-p}} + \mu \right\}.$$

Similarly, we have, if $a \leq \exp(\mathbb{E} \log X)$,

$$F_X(a) = \Pr(X \leq a) = \Pr(\log X \leq \log a) \leq \Pr(\log X - \mu \leq \log a - \mu) \leq \frac{\Var(\log X)}{(\log a - \mu)^2}.$$

Therefore, $a \leq \Theta_X \left( \frac{\Var(\log X)}{(\log a - \mu)^2} \right)$. Now if $a = \exp \left\{ -\sqrt{\frac{\Var(\log X)}{p}} + \mu \right\}$, then $a \leq \exp(\mathbb{E} \log X)$ and $\frac{\Var(\log X)}{(\log a - \mu)^2} = p$, so

$$\Theta_X(p) \geq \exp \left\{ -\sqrt{\frac{\Var(\log X)}{p}} + \mu \right\}.$$

Therefore, for any $p, q \in (0, 1)$, we have

$$\Theta_X(q) \leq \exp \left\{ \sqrt{\frac{\Var(\log X)}{1-q}} + \mu \right\} \leq \exp \left\{ \sqrt{\Var(\log X)} \left( (1 - q)^{-1/2} + p^{-1/2} \right) \right\} \exp \left\{ -\sqrt{\frac{\Var(\log X)}{p}} + \mu \right\} \leq \exp \left\{ \sqrt{\Var(\log X)} \left( (1 - q)^{-1/2} + p^{-1/2} \right) \right\} \Theta_X(p). \quad \Box$$

Lemma A.2. Suppose that $p < \frac{1}{2}$ and $X_1, \ldots, X_N$ are iid Bernoulli$(p)$ random variables. Let $S_N = \sum_{i=1}^N X_i$. Then

$$\Pr[S_N/N \geq 1/2] \leq (8p)^{N/2}.$$

Proof. We have

$$\Pr[S_N/N \geq 1/2] = \Pr[e^{iS_N} \geq e^{iN/2}] \leq \left( \frac{\mathbb{E}[e^{iX_i}]}{e^{iN/2}} \right)^N = \left( \frac{pe^i + 1 - p}{e^{i/2}} \right)^N.$$

Putting $i = \log \frac{1-p}{p}$, we obtain

$$\Pr[S_N/N \geq 1/2] \leq \left( 2(1-p) \left( \frac{1-p}{p} \right)^{-1/2} \right)^N \leq (8p)^{N/2}. \quad \Box$$
Lemma A.3. Suppose that $\mathbb{B}$ is a rectangle and $\text{dist}_E(x, \partial \mathbb{B}) \geq c \text{diam}_E(\mathbb{B})$ for some constant $c$. Then there is a constant $c$, depending on $C$, so that

$$|\nabla_y p^B_t(x, y)| \leq C \frac{|x - y|}{t^2} e^{-\frac{|x-y|^2}{2t^2}}. \quad (A.2)$$

Proof. This proof essentially appeared before in [35, Appendix 1] and was refined in [12], but as we need a somewhat different statement, we present the proof in full. We have

$$p^B_t(x, y) = \frac{e^{-\frac{|x-y|^2}{2t}}}{2\pi t} \varphi_t(x, y),$$

where

$$\varphi_t(x, y) = \mathbb{P}\left( B_s - \frac{s}{t} B_t + x + \frac{s}{t} (y - x) \in \mathbb{B} \text{ for all } s \leq t \right).$$

Therefore, we have

$$\nabla_y p^B_t(x, y) = -(x - y) e^{-\frac{|x-y|^2}{4t^2}} \varphi_t(x, y) + \frac{e^{-|y|^2/2t}}{2\pi t} \nabla_y \varphi_t(x, y). \quad (A.3)$$

Fix $y_1, y_2 \in \mathbb{R}$. Let $z^{(1)}$ be the $x$-coordinate of the right edge of $\mathbb{B}$. Let $E_1$ be the event

$$\left\{ \max_{s \leq t} (B_s - \frac{s}{t} B_t + x + \frac{s}{t} (y_1 - x))^{(1)} > z^{(1)} \right\},$$

where the superscript $(1)$ means to consider the $x$-coordinate. We note that

$$\left| (B_s - \frac{s}{t} B_t + x + \frac{s}{t} (y_1 - x))^{(1)} - (B_s - \frac{s}{t} B_t + x + \frac{s}{t} (y_2 - x))^{(1)} \right| \leq |y_1 - y_2|.$$

Therefore, if $E_1$ occurs but $E_2$ does not, then we must have that

$$\max_{s \leq t} (B_s - \frac{s}{t} B_t + x + \frac{s}{t} (y_1 - x))^{(1)} \in [z^{(1)}, z^{(1)} + |y_1 - y_2|].$$

We have that

$$\mathbb{P}\left( \max_{s \leq t} (B_s - \frac{s}{t} B_t + x + \frac{s}{t} (y_1 - x))^{(1)} \in [z^{(1)}, z^{(1)} + (y_1 - y_2)^{(1)}] \right)$$

$$= \mathbb{P}\left( \max_{s \leq t} (B_s - \frac{s}{t} B_t + x + \frac{s}{t} (y_1 - x))^{(1)} \geq z^{(1)} \right)$$

$$- \mathbb{P}\left( \max_{s \leq t} (B_s - \frac{s}{t} B_t + x + \frac{s}{t} (y_1 - x))^{(1)} \geq z^{(1)} + (y_1 - y_2)^{(1)} \right).$$

By the reflection principle, we have that

$$\mathbb{P}\left( \max_{s \leq t} (B_s - \frac{s}{t} B_t + x + \frac{s}{t} (y_1 - x))^{(1)} \geq z^{(1)} \right) = \frac{1}{2} e^{-\frac{(z^{(1)} + y_1^{(1)})^2 + (y_1^{(1)} - x)^2}{2t}}$$

$$\mathbb{P}\left( \max_{s \leq t} (B_s - \frac{s}{t} B_t + x + \frac{s}{t} (y_1 - x))^{(1)} \geq z^{(1)} + (y_1 - y_2)^{(1)} \right) = \frac{1}{2} e^{-\frac{(z^{(1)} + y_2^{(1)})^2 + (y_2^{(1)} - x)^2}{2t}},$$

so

$$\mathbb{P}(E_1 \setminus E_2) \leq \frac{1}{2} \left( e^{-\frac{(z^{(1)} + y_1^{(1)})^2 + (y_1^{(1)} - x)^2}{2t}} - e^{-\frac{(z^{(1)} + y_2^{(1)})^2 + (y_2^{(1)} - x)^2}{2t}} \right).$$

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Now we have
\[
\lim_{y_2^{(1)} \to y_1^{(1)}} \frac{1}{2(1)} \frac{1}{2} \left( e^{(\epsilon_y y - y_1^{(1)})} - e^{(\epsilon_y y - y_1^{(1)})} \right) = \frac{1}{2} \left( \epsilon(1) y_1^{(1)} - x \right) \left( \epsilon(2) y_2^{(1)} + y_1^{(1)} x \right).
\]
\[
= \frac{\left( x_1^{(1)} + y_1^{(1)} - x \right) e^{(\epsilon y e - y_1^{(1)})}}{2t}.
\]
This implies that
\[
\|\nabla \phi \|_{\Delta t} (x, y_1^{(1)}) \leq C \frac{\text{diam}_{E}(\mathbb{R})}{2t} \exp \left\{ -\frac{\text{diam}_{E}(\mathbb{R})^2}{Ct} \right\},
\]
and plugging into (A.3) we obtain
\[
\|\nabla \phi \|_{\Delta t} (x, y, \alpha) \leq \frac{|x-y|}{4\pi t^2} e^{-\frac{|x-y|^2}{4t}} + C \frac{\text{diam}_{E}(\mathbb{R})}{4\pi t^2} \exp \left\{ -\frac{\text{diam}_{E}(\mathbb{R})^2}{Ct} \right\} \leq C \frac{|x-y|}{t^2} e^{-\frac{|x-y|^2}{4t}}.
\]

**Lemma A.4.** Let $\mathcal{X}$ be a rectangular subset of $\mathbb{R}^d$ and suppose that we have a family of functions $\{f_n : \mathcal{X} \to \mathbb{R}\}$ so that, for some constant $C$, we have
\[
|f_n(x)| \leq C
\]
and
\[
|f_n(x) - f_n(y)| \leq C (|x-y|^\alpha \lor \beta_n)
\]
for some $\alpha > 0$ and some sequence $\beta_n \to 0$. Then the family $\{f_n : \mathcal{X} \to \mathbb{R}\}$ is precompact in the uniform topology.

**Proof.** Extend $f_n$ to an open subset $\mathcal{X}'$ of $\mathbb{R}^d$ containing $\mathcal{X}$ by reflecting it across the boundaries; note that this can be done in such a way that (A.5) still holds, perhaps with a larger constant $C$. Let $\rho : \mathbb{R}^d \to \mathbb{R}$ be a smooth positive function such that
\[
\text{supp } \rho \subset B(0,1)
\]
and define $\rho_\varepsilon(x) = e^{-d} \rho(e^{-1} x)$. Let $\ast$ denote convolution. We have that
\[
|\rho_\varepsilon * f_n(x) - f_n(x)| = \left| \int \rho_\varepsilon(y) f_n(x-y) \, dy - f_n(x) \right| \leq \int \rho_\varepsilon(y) |f_n(x-y) - f_n(x)| \, dy
\]
\[
\leq C \int \rho_\varepsilon(y) (|y|^\alpha \lor \beta_n) \, dy \leq C (e^\alpha \lor \beta_n).
\]
Moreover, if $|x-z|^\alpha \geq \beta_n$, then we have
\[
|\rho_\varepsilon * f_n(x) - \rho_\varepsilon * f_n(z)| = \left| \int [f_n(x-y) - f_n(z-y)] \rho_\varepsilon(y) \, dy \right| \leq C |x-z|^\alpha,
\]
while if $|x-z|^\alpha \leq \beta_n$, then we have, if $\varepsilon = \beta_n$, that
\[
|\rho_\varepsilon * f_n(x) - \rho_\varepsilon * f_n(z)| = \int f_n(y) [\rho_\varepsilon(x-y) - \rho_\varepsilon(z-y)] \, dy = \int [f_n(y) - f_n(x)] \cdot [\rho_\varepsilon(x-y) - \rho_\varepsilon(z-y)] \, dy
\]
\[
\leq C \beta_n \int |\rho_\varepsilon(x-y) - \rho_\varepsilon(z-y)| \, dy \leq C \|\rho\|_{C^1} \beta_n e^{-1} |x-z| = C |x-z|.
\]
Together, (A.7) and (A.8) imply the family \( \{ \rho_{\beta_n} * f_n \} \) is equicontinuous; since the family is evidently bounded by (A.4), the Arzelà–Ascoli theorem implies that there is a continuous function \( f \) and a subsequence \( (n_k) \) so that

\[
\lim_{k \to \infty} \rho_{\beta_n_k} * f_{n_k} = f
\]

uniformly. On the other hand, (A.6) implies that

\[
\lim_{k \to \infty} |\rho_{\beta_n} * f_n - f_n| = 0
\]

uniformly. Combining (A.9) and (A.10) implies that

\[
\lim_{k \to \infty} f_n = f.
\]

This completes the proof. \( \square \)

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