Intrinsic limitations on the size of quantum database

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It is found that Grover’s quantum search algorithm is not robust against phase inversion and Hadmard transformation inaccuracies. Imperfect phase inversions and Hadmard-Walsh transformations in Grover’s quantum search algorithm lead to reductions in the maximum probability of the marked state and affect the efficiency of the algorithm, even in the absence of decoherence. Given the degrees of inaccuracies, we find that to guarantee half rate of success, the size of the database should be in the order of $O(\frac{1}{\delta^2})$, where $\delta$ is the uncertainty.

I. INTRODUCTION

Grover’s quantum search algorithm is a remarkable achievement in quantum computing [1]. There have been intensive interests in Grover’s quantum search algorithm recently [2 –18]. It uses only two simple gate operations, the controlled phase rotations and Hadamard transformations. It has been successfully demonstrated in solution NMR bulk quantum computers with a few qubits [18]. However, the inevitable quantum state decoherence and gate inaccuracies can introduce errors [18,19], which accumulate through the computation and make long computation unreliable. While, in order to find out the marked state with high probability, it still requires exponential number of iterations. Then, the error probability of the complete algorithm may be as exponentially large as the error probability of each iteration. In other words, even with small imperfection per step, large scale quantum search may be difficult.

Fortunately, recent study in quantum error correction shows that in principle, whenever the noise rates are below a constant threshold, an arbitrary long quantum operations can be performed reliably through fault-tolerant quantum computation [20]. Experimentally, different types of faults can occur with different rates and will affect the efficiencies of the algorithm differently. For example, the effect of quantum state decoherence and operational errors on the efficiency of quantum algorithms have been studies in [21] with ion trap quantum computers. A good understanding of the effect from different noises on the algorithms can help us look for specific potential physical realizations of quantum computers.

In this paper, we address the problem of influences of imperfect gate operations in the quantum search algorithm, in the absence of decoherence and error corrections. We will show that systematic phase mismatching and random errors in the Walsh-Hadamard transformation lead to exponential reduction in the maximum success probability when $n$ is linearly increased. Therefore, if we can not avoid them completely, to ensure a large success rate in a quantum searching machine, the size of the database should be limited. This limitation is due to the intrinsic vulnerability of the algorithm to imperfect gate operations. In designing a quantum searching machine, this limitation should be taken into account.

The paper is organized as follows: Section 2 is devoted to the description of different error models in phase mismatching and the corresponding simulation results. In section 3, we present the consequence of imperfect Hadamard transformation. Section 4 gives a short summary.

II. EFFECTS OF IMPERFECT PHASE INVERSIONS

Grover’s algorithm consists of essentially four steps in an iteration [5]: (1) a Walsh-Hadamard transformation $U = W$; (2) a phase inversion of the prepared state $|\gamma\rangle$, usually $|\gamma\rangle = |0\rangle$, $I_\gamma = I - 2|\gamma\rangle\langle\gamma|$; (3) a phase inversion of the marked state $|\tau\rangle$, $I_\tau = I - 2|\tau\rangle\langle\tau|$; and (4) an inverse of the Walsh-Hadamard transformation $U^{-1} = W$ (W is self-inverse.). The operator for one Grover iteration is $Q = -I_\gamma U^{-1}I_\tau U$.

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In this section, we focus on the imperfection in phase inversions and therefore choose $U$ to be the ideal Hadamard transformation. We consider the imperfections in the phase inversion to be systematic, such that

$$
I_x = I - (1 - e^{i\theta})|\gamma\rangle\langle\gamma|, \\
I_x = I - (1 - e^{i\varphi})|\tau\rangle\langle\tau|,
$$

(1)

where $\theta = \pi + \theta_0$, $\varphi = \pi + \varphi_0$ with $\theta_0$ and $\varphi_0$ constant and small. When $\theta_0 = \varphi_0 = 0$, we recover the original Grover’s algorithm. The generalized quantum search algorithm is a rotation in a 2-dimensional space spanned by $|\gamma\rangle$ and $|\tau\rangle$. In the following two orthonormal basis

$$
|1\rangle = \frac{(|\gamma\rangle - U_{\gamma\gamma}U^{-1}|\tau\rangle)}{\sqrt{1 - |U_{\gamma\gamma}|^2}}, \\
|2\rangle = U^{-1}|\tau\rangle.
$$

(2)

with $U_{\gamma\gamma} = \langle\tau|U|\gamma\rangle = 1/\sqrt{N}$, the operator $Q$ is represented by

$$
Q \approx \cos \delta I + i \sin \delta \sigma_z + \beta' \sigma_y + o(\beta'),
$$

(4)

where $\sigma_x, \sigma_y$ and $\sigma_z$ are Pauli operators and $I$ is the identity operator in dimension 2. $\beta' = 2\beta + O(\theta_0\beta) = 2\beta + o(\beta)$ with $\beta = \frac{\sqrt{N-1}}{N}$. For small $\delta$, we can further simplify operator $Q$ as

$$
Q \approx I + i G \approx e^{iG},
$$

with $G = \sin \delta \sigma_z + \beta' \sigma_y$. Using $G^2 = (\delta^2 + \beta'^2)I$, we obtain

$$
Q^j = \begin{bmatrix}
\cos j\lambda + \frac{i\delta \sin j\lambda}{\lambda} & \frac{\beta' \sin j\lambda}{\lambda} \\
-\frac{\beta' \sin j\lambda}{\lambda} & \cos j\lambda - \frac{i\delta \sin j\lambda}{\lambda}
\end{bmatrix},
$$

(5)

with $\lambda = \sqrt{\delta^2 + \beta'^2}$. Then, starting from the prepared state $|\gamma\rangle = \sqrt{1 - |U_{\gamma\gamma}|^2}|1\rangle + U_{\gamma\gamma}|2\rangle = \cos \beta|1\rangle + \sin \beta|2\rangle \approx |1\rangle$, after $j$ number of iterations, the norm of the amplitude of the marked state in the quantum computer is

$$
|B_j| \approx \frac{\beta'}{\lambda} \sin(j\lambda).
$$

(6)

and the maximum probability of the marked state in the algorithm is

$$
P_{max} \approx \frac{\beta'^2}{\beta'^2 + \delta^2} \leq 1.
$$

(7)

Therefore, for large $N$, Grover’s algorithm is efficient only when $\delta = 0$. When $\delta \neq 0$, we find

$$
P_{max} \approx \frac{\beta'^2}{\delta^2} \sim \frac{4}{N\delta^2}.
$$

(8)

Thus, $P_{max}$ decreases linearly with $N$ or exponentially with $n = \log_2 N$. This concludes our proof that systematic phase mismatching results in exponential reduction in the success probability and consequently gives an upper bound on the size of the database. If half rate of success is required, that is $P_{max} \geq 1/2$, $N$ cannot exceed $8/\delta^2$.

So far, we have assumed that the errors in the phase inversions are systematic such that $\delta$ is constant. We now extend this simple model (EM1) to another two error models. The second error model (EM2) assumes $\delta$ in each
step is a Gaussian random variable with mean $\delta_0 = 0$ and standard deviation $s$. Such an error is conventionally defined as random error. Finally, we let $\delta$ be a Gaussian random variable with mean $\delta_0 \neq 0$ and standard deviation $s$ (EM3). The exact effect of EM2 and EM3 are difficult to compute analytically due to their randomness. Hence, we only present the simulation results. We vary $n = \log_2 N$ and run the algorithm with sufficient number of iterations so that a maximum probability is found. Since $\delta$ in EM2 and EM3 are random variables, we adopt the random sampling techniques in the simulation. The relationships between the maximum success probabilities and the size of the database are shown in Fig. 2 and Fig. 3 for EM2 and EM3 respectively. For comparisons, we also provide the simulation result from EM1 in Fig.1.

Our simulation results are consistent with mathematical predictions. First, both systematic and random errors cause reduction in the maximum probability. Second, the success probability drops quickly after a transition point, which is determined by the error parameter $\delta_0$ and $s$. When $n$ is large, the probabilities decreases exponentially. Third, the different effects of systematic errors and random errors also meet our expectations. Mathematically, systematic errors cause the error amplitudes to grow exponentially with the number of gates applied; while the random errors cause the error probabilities to grow linearly. This difference is clearly demonstrated in our simulation results. Fig. 2 shows that random errors give a much larger transition point than systematic errors. Fig. 3 shows that the average success probability from EM3 is nearly identical to EM1 except some small fluctuations.

It is shown in this section that systematic errors in the phase inversions lead to reduction in the maximum probability of finding the marked state. Random errors also affect this success rate, but in a lesser degree. In practice, we should make $\delta_0$ as small as possible. However, due to imperfection, nonzero $\delta_0$ occur inevitably. Due to imperfection, nonzero $\delta$ occur inevitably. For instance, systematic errors arises from imperfect calibration and inhomogeneity in the radio frequency pulses in NMR realization. Random errors are always present in a realistic environment. These errors will reduce the maximum probability of the algorithm. To make an estimate on the combined effect of systematic and random errors[EM3], we assume that random errors affect the algorithm just like the systematic errors. Then we can treat $\Delta = 2\delta$ as the uncertainty due to both systematic errors and random errors and use this to derive an upper bound for the size of a quantum database: any phase inversion operation is imperfect, there is an uncertainty, and this uncertainty sets an upper bound on the size of the database $N$. For half rate success, the dimension of the database should be less than $\frac{1}{\Delta^2}$.

### III. IMPERFECT HADAMARD TRANSFORMATION

Hadamard-Walsh transformations are also subject to errors. To study the effect of the imperfect Hadamard-Walsh transformation, let’s take $\delta = 0$ in eqn. (3). Then the maximum probability for finding the marked state is approximately $\sin^2(2\beta)$ for perfect unitary transformation. For perfect Hadamard-Walsh transformation, $\beta = \arcsin(|U_{r\gamma}|)$, and $|U_{r\gamma}| = \sqrt{1/N}$. For systematic errors in the Hadamard-Walsh transformation, the matrix elements of $U$ is no longer equal to $\sqrt{1/N}$. If $|U_{r\gamma}|$ is larger than $\sqrt{1/N}$, then the algorithm will require less steps in reaching the desired state compared with the standard Grover’s algorithm. If it is smaller than $\sqrt{1/N}$, the algorithm will require more steps of iteration. In this case, the searching algorithm still can give a probability quite close to unity. But if one makes a measurement at the normal optimal number of iteration, one will get a reduction in the successful rate. This difficulty can be overcome by using the algorithm several times with measurements made around the optimal iteration which is similar to the one used in Ref. [12].

Here, we can give a simple interpretation why Grover’s algorithm is optimal. The rigorous proof has been given in Refs. [12]. Grover’s algorithm can be seen as a rotation of the state vector in a 2-dimensional space span by $|\tau\rangle$ and $|\gamma\rangle$. Each iteration rotates an angle $\lambda = \beta = 2\sin(\theta/2)\beta$. $\theta = \phi = \pi$ gives the largest angle $2\beta = 2\arcsin(|U_{r\gamma}|)$. So one has to choose phase inversions. As for the unitary transformation $U$, at first glance one maybe attempted to think that a larger $|U_{r\gamma}|$ will constitute a faster search algorithm. However, since $U$ is unitary, its matrix elements satisfy the normalization relation $\sum_{\tau} |U_{r\gamma}|^2 = 1$, where $\tau$ runs through all the $N$ basis states. The mean value of the matrix element is $\sqrt{\frac{1}{N}}$. If some of the matrix elements are larger than the average, some other matrix elements will be less than this average. In other words, while making the search for some marked states in less steps, the modified algorithm has to search the rest of the basis states in more steps. In contrast, the original Grover algorithm searches all possible marked state with the same optimal number of iterations. Together with its simplicity and easy implementation, the Walsh-Hadamard transformation lend itself the best choice.

We discuss the effects of random errors in the Walsh-Hadamard transformation in a simple model. In this case, the algorithm is no longer a simple rotation in 2 dimensions. Though in each iteration, the operator can be approximately written as
\[
Q = \begin{pmatrix}
  \cos \beta & \sin \beta \\
  -\sin \beta & \cos \beta 
\end{pmatrix},
\]

the basis states in each iteration has been changed, that is, the 2 dimensional space in each iteration is no longer the same. This is apparent by inspecting the expressions in eqn. (3). Suppose in the first iteration, the unitary transformation is \( U \) and in the following iteration, the operator becomes \( V \). Then after the first iteration, the state vector of the quantum computer is

\[
|\psi_1\rangle = \cos \beta |1\rangle - \sin \beta |2\rangle \approx \cos \beta |1'\rangle - \sin \beta U^{-1} V |2'\rangle,
\]

where \( |2'\rangle = V^{-1} |\tau\rangle \). Because \( U \neq V, U^{-1} V \) is no longer the identity operator. Expanding \( U^{-1} V |2'\rangle = (U^{-1} V)_{22} |2'\rangle + \ldots \), we see that the Grover search operator acts only on the subspace span by \( |1'\rangle \) and \( |2'\rangle \), and the other terms are leaked out the 2 dimensional space. To make an estimate, let’s assume that in each iteration, \((U^{-1} V)_{22} |2'\rangle \approx (1 - \delta_1) |2\rangle + \) higher order terms. Then in this model, the matrix for a Grover search operator becomes

\[
Q = \begin{pmatrix}
  \cos \beta & \sin \beta (1 - \delta_1) \\
  -\sin \beta & \cos \beta (1 - \delta_1)
\end{pmatrix}.
\]

Starting from initial state \( |\gamma\rangle \approx |1\rangle \), after \( j \) iterations, the amplitude of the state \( |2\rangle \) becomes

\[
| (1 - j - \frac{1}{2} \delta_1) \sin(j\beta) |,
\]

where only first order in \( \delta_1 \) is retained. With optimal number of iterations, \( j \approx \frac{\pi \sqrt{N}}{2} \), \( \sin(j\beta) \approx 1 \), the successful rate is

\[
P \approx \left( 1 - \frac{\pi \sqrt{N} \delta_1}{8} \right)^2 \approx 1 - \frac{\pi \sqrt{N} \delta_1}{4}.
\]

For half success rate, one must have \( N \leq \frac{4}{\pi^2 \delta_1} \), which is similar to the limitation on the size of the database in the phase inversion inaccuracies. However, the mechanism is different. Here the random errors play a more important role than the systematic errors, whereas in the phase inversion case, it is just the opposite.

### IV. SUMMARY

In summary, we find that the dominating gate imperfection in Grover’s algorithm is the systematic phase mismatching and the random errors in the Walsh-Hadamard transformation. Using the results obtained in this work, it is easy to understand the simulating results of Ref. [21]. In Fig.1a of [21] is the results with only random errors (both in phase inversions and Hadmard transformation), we see that the peak in the probability curve drops down as random errors grow. But the position of the peak is relatively fixed. Random errors in the phase inversions does not affect the algorithm very seriously. Random errors in the Hadmard transformation reduce the maximum probability. The optimal iteration number remain more or less the same. When there is only systematic errors as shown in Fig.2b of Ref. [21], we see a drop in the maximum probability and also a shifting of the peak position to the left, as is shown in Fig.2b of Ref. [21]. The drop in maximum probability is caused by phase mismatching. The shift of the peak position is due to the systematic errors in the Walsh-Hadamard transformation.

These gate inaccuracies set an upper bound on the size of the database. We estimate that the upper bound is inversely proportional to the quadrature of the uncertainty in the phase mismatching or in the Walsh-Hadamard transformation. In real quantum computation, imperfect gate operations exist all the time at constant rate while decoherence increases rapidly with computing time. At the early stage of a quantum computation, gate imperfection is dominant in affecting a quantum algorithm. As the computation continues, decoherence increases and then dominates. Suitable quantum correction codes and in particular fault-tolerant quantum computation can reduce the decoherence, and ease the stringent requirement on gate accuracies. The limitations on the quantum datasize can then be greatly relieved.

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Figure captions

Fig.1. EM1 with $\delta_0 = 10^{-2}, 10^{-3}, 10^{-4}$.
Fig.2. EM2 with $\delta_0 = 0, s = 10^{-2}$ and $\lambda = \beta'$
Fig.3. EM3 with $\delta_0 = 10^{-2}, 10^{-3}, 10^{-4}, s = 10^{-3}$
\[ n = \log_2 N, \quad \beta = \frac{1}{\sqrt{N}} \]
$n = \log_2 N$, $\beta = 1/\sqrt{N}$, $s = 0.01$
\[\delta_0 = 10^{-2}, 10^{-3}, 10^{-4}, \quad s = 10^{-2}\]