STATISTICALLY CONVEX-COCOMPACT ACTIONS OF GROUPS WITH CONTRACTING ELEMENTS

WEN-YUAN YANG

ABSTRACT. This paper presents a study of the asymptotic geometry of groups with contracting elements, with emphasis on a subclass of statistically convex-cocompact (SCC) actions. The class of SCC actions includes relatively hyperbolic groups, CAT(0) groups with rank-1 elements and mapping class groups, among others. We exploit an extension lemma to prove that a group with SCC actions contains large free sub-semigroups, has purely exponential growth and contains a class of barrier-free sets with a growth-tight property. Our study produces new results and recovers existing ones for many interesting groups through a unified and elementary approach.

CONTENTS

1. Introduction 1
2. An extension lemma 11
3. Large free sub-semigroups 22
4. Growth-tightness theorem 24
5. Purely exponential growth 30
6. Constructing SCC actions 32
References 35

1. INTRODUCTION

1.1. Background. Suppose that a group $G$ admits a proper and isometric action on a proper geodesic metric space $(Y, d)$. The group $G$ is assumed to be non-elementary: there is no finite-index subgroup isomorphic to the integers $\mathbb{Z}$ or to the trivial group. The goal of this paper is to study the asymptotic geometry of the group action in the presence of a contracting element. Through a unified approach, we shall present a series of applications to the following classes of groups with contracting elements:

- $\text{Hyp} = \{ \text{a group } G \text{ acts properly and cocompactly on a } \delta \text{-hyperbolic space } (Y, d) \}$. See [10], [22] for general references.
- $\text{RelHyp} = \{ \text{a relatively hyperbolic group } G \text{ acts on a Cayley graph } (Y, d) \text{ with respect to a generating set } S \}$. See [10], [61], and [31].
- $\text{Floyd} = \{ \text{a group } G \text{ with non-trivial Floyd boundary acts on a Cayley graph } (Y, d) \text{ with respect to a generating set } S \}$. See [37], [50], [39], [40].
- $\text{CAT}(0)^1 = \{ \text{a group } G \text{ acts properly and cocompactly on a CAT(0) space } (Y, d) \text{ with rank-1 elements} \}$. See [12], [6], [19].

Date: July 2, 2017.

2000 Mathematics Subject Classification. Primary 20F65, 20F67.

Key words and phrases. Contracting elements, critical exponent, convex-compactness, purely exponential growth, growth tightness.
$\text{GSC} = \{ \text{a Gr}'(1/6)$-labeled graphical small cancellation group $G$ with finite components labeled by a finite set $S$ acts on the Cayley graph $(Y, d)$ with respect to the generating set $S \}$. See [3], [17].

$\text{Mod} = \{ \text{the mapping class group $G$ of a closed orientable surface with genus greater than two acts on Teichmüller space $(Y, d)$ equipped with the Teichmüller metric} \}$. See [36], [34].

Let us give a definition of a contracting element (cf. Definition 2.1). First, a subset $X$ is called contracting if any metric ball disjoint with $X$ has a uniformly bounded projection to $X$ (cf. [57], [12]). An element $g \in G$ of infinite order is contracting if for some basepoint $o \in Y$, an orbit $\{ g^n \cdot o : n \in \mathbb{Z} \}$ is contracting, and the map $n \in \mathbb{Z} \rightarrow h^n o \in Y$ is a quasi-isometric embedding.

The prototype of a contracting element is a hyperbolic isometry on hyperbolic spaces, but more interesting examples are furnished by the following:

- hyperbolic elements in $\text{RelHyp}$ and $\text{Floyd}$, cf. [41], [40];
- rank-1 elements in $\text{CAT}^2_0$, cf. [6], [12];
- certain infinite order elements in $\text{GSC}$, cf. [4];
- pseudo-Anosov elements in $\text{Mod}$, cf. [57].

We shall demonstrate in §1.2 that a proper action with a contracting element already admits interesting consequences. With appropriate hypothesis, more information can be obtained on a subclass of statistically convex-cocompact actions (SCC) to be discussed in §1.3. This is a central concept of the paper since it allows to generalize dynamical aspects of the theory of convex-cocompact Kleinian groups to the above list ($\text{Hyp} - \text{Mod}$) of groups.

Following the groundbreaking work of Masur and Minsky [53], [54], the study of mapping class groups from the point of view of geometric group theory has received much attention. Indeed, one of the motivations behind the present study is the application of the approach presented here to $\text{Mod}$. Most mapping class groups are not relatively hyperbolic, cf. [9]. Nevertheless, their action on Teichmüller spaces exhibits very interesting negative behavior, thanks to the following two facts from Teichmüller geometry:

1. Minsky’s result that a pseudo-Anosov element is contracting [57];
2. the fact that the group action of mapping class groups on Teichmüller space is SCC, which follows from a deep theorem of Eskin, Mirzakhani and Rafi [33, Theorem 1.7], as observed in [2, Section 10].

This study therefore considers some applications in $\text{Mod}$. We emphasize that most of arguments, once SCC actions are provided, are completely general without appeal to specific theory of groups ($\text{Hyp} - \text{Mod}$) under consideration.

We are now formulating the setup of main questions to be addressed in next subsections. Fixing a basepoint $o \in Y$, one expects to read off information from the growth of orbits in the ball of radius $n$:

$$N(o, n) := \{ g \in G : d(o, go) \leq n \}.$$  

For instance, a celebrated theorem of Gromov in [45] says that the class of virtually nilpotent groups is characterized by the polynomial growth of $N(o, n)$ in Cayley graphs. Most groups that one encounters in fact admit exponential growth: for instance, thanks to the existence of non-abelian free subgroups. This is indeed the case in our study, where a well-known ping-pong game played by contracting elements gives rise to many free subgroups. In this regard, an asymptotic quantity called the critical exponent (also called the growth rate) is associated with the growth function. In practice, it is useful to consider the critical exponent $\omega(\Gamma)$ for a subset $\Gamma \subset G$:

$$\omega(\Gamma) = \limsup_{n \rightarrow \infty} \frac{\log \| N(o, n) \cap \Gamma \|}{n},$$ (1)
which is independent of the choice of \( o \in Y \). The following alternative definition of \( \omega(\Gamma) \) is very useful in counting problems:

\[
\limsup_{n \to \infty} \frac{\log \#(A(o, n, \Delta) \cap \Gamma)}{n} = \omega(\Gamma),
\]

where we consider the annulus

\[
A(o, n, \Delta) = \{ g \in G : |d(o, go) - n| \leq \Delta \}
\]

for \( \Delta > 0 \).

The remainder of this introductory section is based on consideration of the following questions:

**Question.**

1. When is the critical exponent \( \omega(\Gamma) \) \[^1\] a true limit? Can the value \( \omega(\Gamma) \) be realized by some geometric subgroup?

2. Under what conditions does a group action have so-called purely exponential growth, i.e., there exists a constant \( \Delta > 0 \) such that

\[
\| A(o, n, \Delta) \| \geq \exp(\omega(G)n)
\]

3. Which subsets \( X \) in \( G \) are growth-tight: \( \omega(X) < \omega(G) \)? This shall admit several applications to genericity problems studied in a subsequent paper \[^76\].

1.2. Large free semigroups. At first glance, it seems somewhat of a leap of faith to anticipate that a general and non-trivial result can be obtained for a proper group action with a contracting element. Therefore, our first objective is to convince the reader that it is indeed fruitful to work with this aim in mind. We are going to introduce two general and interesting results that recover many existing results in a unified manner. The first group of results concerns the existence of large free semi-subgroups in various interesting classes of groups.

To be concrete, let us motivate our discussion by considering the class of Schottky groups among discrete groups in \( \text{Isom}(\mathbb{H}^n) \), called Kleinian groups in the literature. By definition, a Schottky group is a free, convex-cocompact, Kleinian group in \( \text{Isom}(\mathbb{H}^n) \).

A seminal work of Phillips and Sarnak \[^67\] showed that the critical exponent of any classical Schottky group in \( \text{Isom}(\mathbb{H}^n) \) is uniformly bounded away from \( n - 1 \) for \( n > 3 \), with the case \( n = 3 \) being established later by Doyle \[^30\]. In \( n \)-dimensional quaternionic hyperbolic spaces, a well-known result of Corlette \[^23\] implies that the critical exponent of any subgroup in lattices is at most \( 4n \), uniformly different from the value \( 4n + 2 \) for lattices. Recently, Bowen \[^18\] proved an extension of Phillips and Sarnak’s work in higher even dimensions, showing that the critical exponent of any free discrete group in \( \text{Isom}(\mathbb{H}^n) \) for \( n \geq 2 \) is uniformly bounded away from \( 2n - 1 \). However, he also proved in \[^17\] that the \( \pi_1 \) of a closed hyperbolic 3-manifold contains a free subgroup with critical exponent arbitrarily close to 2. Let us close this discussion by mentioning a very recent result of Dahmani, Futer and Wise \[^20\] that for free groups \( \mathbb{F}_n \) (\( n \geq 2 \)) there exists a sequence of finitely generated subgroups with critical exponents tending to \( \log(2n - 1) \). Hence, an intriguing question arises concerning the conditions under which there exists a gap of critical exponents for free subgroups of ambient groups. Although this question remains unanswered for free groups, if we consider the class of free semigroups, then we are indeed able to obtain a general result.

**Theorem A** (large free semigroups). Let \( G \) admit a proper action on a geodesic space \( (Y,d) \) with a contracting element. Fix a basepoint \( o \in Y \). Then there exists a sequence of free semigroups \( \Gamma_n \subset G \) such that

1. \( \omega(\Gamma_n) < \omega(G) \) but \( \omega(\Gamma_n) \to \omega(G) \) as \( n \to \infty \).

2. The standard Cayley graph of \( \Gamma_n \) is sent by a natural orbital map to a quasi-geodesic tree rooted at \( o \) such that each branch is a contracting quasi-geodesic.
The property in Assertion (1) will be referred to as the large free semigroup property.

The notion of quasi-convexity have been studied in the literature, some of which we shall consider in the sequel. Let’s first introduce the strong one. A subset $X$ is called $\sigma$-quasi-convex for a function $\sigma: \mathbb{R}_+ \to \mathbb{R}_+$ if, given $c \geq 1$, any $c$-quasi-geodesic with endpoints in $X$ lies in the neighborhood $N_{\sigma(c)}(X)$. It is clear that this notion of quasi-convexity is preserved up to a finite Hausdorff distance. A subset $\Gamma$ of $G$ is called $\sigma$-quasi-convex if the set $\Gamma \cdot o$ is $\sigma$-quasi-convex for some (or any) point $o \in Y$.

We usually speak of a purely contracting sub-semigroup $\Gamma$ if every non-trivial element is contracting. Thus, the theorem can be rephrased as follows:

**Corollary 1.1.** Any proper action on a geodesic space with contracting element has purely contracting, quasi-convex, large free semigroups.

Construction of large quasi-geodesic trees can be traced back at least to the work of Bishop and Jones [14] on the Hausdorff dimension of limit sets of Kleinian groups. Generalizing earlier results of Patterson [63] and Sullivan [75], they constructed such trees to give a lower bound on the Hausdorff dimension in a very general setting. Later, their construction was implemented for discrete groups acting on $\delta$-hyperbolic spaces by Paulin [64].

Except the growth-tightness property, Mercat [56] has proved the other properties of $\Gamma_n$ in Theorem A for (semi)groups acting properly on a hyperbolic space. It should be noted here that all of these constructions make essential use of Gromov’s hyperbolicity and its consequences, whence a direct generalization à la Bishop and Jones fails in non-hyperbolic spaces, such as Cayley graphs of a relatively hyperbolic group. In this relative case, the present author [77] has introduced the notion of a “partial” cone, which allows one to construct the desired trees by iterating partial cones. This approach has been applied to large quotients and Hausdorff dimensions in [77] and [68]. Nevertheless, the current construction given in §3 follows more closely the construction à la Bishop and Jones.

Our method here relies only on the existence of a contracting element, which is usually thought of as a hyperbolic direction, a very partial negative curvature in a metric space. This sole assumption allows our result to be applied to many more interesting class of groups: see the list at the beginning of this paper. In the interests of clarity and brevity, we mention below some corollaries that we believe to be of particular interest.

**Theorem 1.2** ($\text{RelHyp}$). A relatively hyperbolic group $G \in \text{RelHyp}$ has quasi-convex, large free sub-semigroups.

This result does not follow from Mercat’s theorem, since the Cayley graphs of a relatively hyperbolic group are not $\delta$-hyperbolic. Further applications of this corollary to Hausdorff dimension will be given elsewhere; cf. [68].

**The class of irreducible subgroups.** Given a proper action of $G$ on a geodesic metric space $(Y, d)$, it is natural to look at an irreducible subgroup $\Gamma$ which, by definition, contains at least two independent contracting elements (cf. §2.4). Equivalently, it is the same as a non-elementary subgroup with at least one contracting element.

In the context of mapping class groups, a sufficiently large subgroup was studied first by McCarthy and Papadopoulos [55], as an analog of non-elementary subgroups of Kleinian groups. By definition, a sufficiently large subgroup is one with at least two independent pseudo-Anosov elements, so it coincides with the notion of an irreducible subgroup by Lemma 2.13. We refer the reader to [55] for a detailed discussion. The interesting examples include convex-cocompact subgroups [35], the handlebody group, among many others. Hence, we deduce the following from Theorem A.
Theorem 1.3 (Mod). Consider a sufficiently large subgroup \( \Gamma \) of \( G \) \( \in \) Mod. Then there exists a sequence of purely pseudo-Anosov, free semi-subgroups \( \Gamma_n \subset \Gamma \) such that
\[
\omega(\Gamma_n) < \omega(\Gamma) \text{ but } \omega(\Gamma_n) \to \omega(\Gamma)
\]
as \( n \to \infty \).

1.3. Statistically convex-cocompact actions. In this subsection, we focus on a subclass of proper actions akin to a cocompact action in a statistical sense. By abuse of language, a geodesic between two sets \( A \) and \( B \) is a geodesic \([a, b]\) between \( a \in A \) and \( b \in B \).

A significant part of this paper is concerned with studying a statistical version of convex-cocompact actions. Intuitively, the reason that an action fails to be convex-cocompact is the existence of a concave region formulated as follows. Given constants \( 0 \leq M_1 \leq M_2 \), let \( O_{M_1, M_2} \) be the set of elements \( g \in G \) such that there exists some geodesic \( \gamma \) between \( B(o, M_2) \) and \( B(go, M_2) \) with the property that the interior of \( \gamma \) lies outside \( N_{M_1}(Go) \); see Figure 1.

![Figure 1. Definition of \( O_{M_1, M_2} \)](image)

Definition 1.4 (statistically convex-cocompact action). If there exist two positive constants \( M_1, M_2 > 0 \) such that \( \omega(O_{M_1, M_2}) < \omega(G) \), then the action of \( G \) on \( Y \) is called statistically convex-cocompact (SCC).

Remark. (1) This underlying concept was introduced by Arzhantseva et al. in [2] as a generalization of a parabolic gap condition due to Dal’bo et al. [28]. We propose the particular terminology used here since we shall prove (in a forthcoming work) that a SCC action with contracting elements is statistically hyperbolic, a notion introduced earlier by Duchin et al. [32].

(2) (about parameters) We have chosen two parameters \( M_1, M_2 \) (differing from [2]) so that the definition is flexiable and easy to verify. In practice, however, it is enough to take \( M_1 = M_2 \) without losing anything, since \( O_{M_2, M_2} \subset O_{M_1, M_2} \). Henceforth, we set \( O_M := O_{M, M} \) for ease of notation.

(3) Moreover, when the SCC action contains a contracting element, the definition will be proven independent of the basepoint in Lemma 6.2.

The list of groups in \( (\mathcal{Hyp} - \mathcal{Mod}) \) all admit SCC actions on the interesting spaces. Here we emphasize some prototype examples motivating the notion of SCC actions.

Examples. (1) Any proper and cocompact action on a geodesic metric space. In this case, \( O_M \) is empty.

(2) The class of relatively hyperbolic groups with parabolic gap property (cf. [28]); in this case, the set \( O_M \) is the union of finitely many parabolic groups, up to a finite Hausdorff distance.

(3) The action of mapping class groups on Teichmüller spaces is SCC (cf. [2]).

Analogous to irreducible subgroups in a proper action, it appears to be interesting to study the class of statistically convex-cocompact subgroups (SCC subgroups) in a given SCC action.
This should be regarded as a generalization of convex-cocompact subgroups studied in some groups.

The notion of convex-cocompact subgroups in Mod was introduced by Farb and Mosher \[35\]. This is conceived as an analog of the well-studied class of convex-cocompact Kleinian groups, with applications to surface group extensions. We believe that SCC subgroups are a useful generalization of their notion from a dynamical point of view. Moreover, the notion of SCC subgroups is strictly bigger than that of convex-cocompact subgroups in Mod:

**Proposition 1.5 (cf. 6.6).** In mapping class groups, there exist, free and non-free, non-convex-cocompact subgroups which admit SCC actions on Teichmüller spaces.

It is an interesting question to determine to which extent a SCC subgroup generalizes the notion of a geometrically finite subgroup in the following classes of groups:

1. In hyperbolic groups, does there exist a SCC subgroup which is not quasiconvex?
2. In relatively hyperbolic groups, does there exist a SCC subgroup which is not relative quasiconvex? The point is that whether the concave region \(O_M\) is always coming from the union of finitely many parabolic subgroups.

The second main result of this paper concerns the quantitative behavior of the orbital growth function. A group action has so-called purely exponential growth if there exists \(\Delta > 0\) such that

\[ \# A(o, n, \Delta) \asymp \exp(\omega(G)n) \]

for any \(n \geq 1\). This property admits several interesting applications, for instance, to statistical hyperbolicity \[32\] \[60\], to counting conjugacy classes and automatic structures \[1\], and to the finiteness of Bowen–Margulis–Sullivan measure \[70\] \[79\].

**Theorem B (exponential growth).** Let \(G\) admit a proper action on a geodesic space \((Y, d)\) with a contracting element. Then the following holds:

1. The critical exponent is a true limit:

\[ \omega(\Gamma) = \lim_{n \to \infty} \frac{\log \#(N(o, n) \cap \Gamma)}{n} \]

2. For some \(\Delta > 0\), we have

\[ \# A(o, n, \Delta) < \exp(\omega(G)n) \]

for any \(n \geq 1\).

3. If the action is SCC, then \(G\) has purely exponential growth.

**Remark.** Assertion (1) was established by Roblin \[69\] for CAT(\(-1\)) spaces, whose method used conformal density in a crucial way. For Kleinian groups, the proof of Assertion (2) via the use of Patterson–Sullivan measures is well known, with the most general form being due to Coornaert \[22\] for a discrete group acting on \(\delta\)-hyperbolic spaces.

In \[70\], Roblin showed the equivalence of purely exponential growth and finiteness of Bowen–Margulis–Sullivan measure for discrete groups on CAT(\(-1\)) spaces. In a coarse setting, we gave in \[79\] a characterization of purely exponential growth in the setting of cusp-uniform actions on \(\delta\)-hyperbolic spaces.

We emphasize that our elementary proof does not use the machinery of Patterson–Sullivan theory and, more importantly, it is valid in a very general setting.

First of all, we give some corollaries to the class of an irreducible subgroup of the first two Assertions in the theorem.

The following application appears to be new even in \(\text{Hyp}\), which was proved recently for free groups by Olshanskii \[59\]. Indeed, an infinite subgroup of a hyperbolic group always contains a hyperbolic element that is contracting.
Corollary 1.6. Assume that $G$ acts properly on a geodesic space $(Y, d)$. Then the limit

$$\lim_{n \to \infty} \frac{\log \#(N(o, n) \cap \Gamma)}{n}$$

exists for any irreducible subgroup $\Gamma$.

Moreover, any irreducible subgroup $\Gamma$ has an upper exponential growth function as well:

$$\#(A(o, n, \Delta) \cap \Gamma) < \exp(\omega(\Gamma)n)$$

for some $\Delta > 0$. Specializing to the class of mapping class groups, the notion of an irreducible subgroup coincides with a sufficiently large subgroup. Therefore, we obtain the following result which appears to be new:

Corollary 1.7 ($\Mod$). For any sufficiently large subgroup $\Gamma$ of $G \in \Mod$, we have

$$\#(A(o, n, \Delta) \cap \Gamma) < \exp(\omega(\Gamma)n)$$

for some $\Delta > 0$.

As a matter of fact, Theorem B gives an elementary and unified proof of the following class of groups with purely exponential growth, which were established by different methods:

1. hyperbolic groups (Coornaert [22]);
2. groups acting on CAT(-1) space, with finite Bowen–Margulis–Sullivan measure (Roblin [70]);
3. fundamental groups of compact rank-1 manifolds (Kneiper [51]);
4. relatively hyperbolic groups, with the word metric [77] and the hyperbolic metric [79];
5. mapping class groups, with the Teichmüller metric (Athreya et al. [5]).

Let us comment on the last of these. Our general methods work for mapping class groups as well as further other applications, with the only assumption that the action of mapping class groups on Teichmüller spaces (with Teichmüller metric) is SCC with contracting elements. On the other hand, we need point out that computation of the precise value of the critical exponent, the entropy of Teichmüller geodesic flows which is $6g - 6$, is out of scope of this approach.

In addition to recovering the above well-known results, the following new classes of groups with purely exponential growth have been established as direct consequences of Theorem B:

Theorem 1.8. The following class of groups have purely exponential growth:

1. The action on their Cayley graphs of $Gr'(1/6)$-labeled graphical small cancellation groups with finite components labeled by a finite set $S$;
2. CAT(0) groups with rank-1 elements;
3. the action on the Salvetti complex of right-angled Artin groups that are not direct products;
4. the action on the Davis complex of a right-angled Coxeter group that is not virtually a direct product of non-trivial groups.

The items (3) and (4) are two important classes of CAT(0) groups with rank-1 elements. A detailed proof is given in the subSection 5.2.

1.4. Tools: Growth-tightness and Extension Lemma. As the title indicates, we are now going to describe two basic tools throughout this study. The first is a growth-tightness theorem addressing the question that which subsets are growth-tight.

Let us first give some historical background on the notion of growth-tightness. It was introduced by Grigorchuk and de la Harpe in [44] for a group: roughly speaking, a group is called growth-tight if the growth rate strictly decreases under taking quotients. Their main motivation is perhaps that if, for some generating set, the growth rates of a growth-tight group achieve an infimum, called the entropy (the entropy realization problem), then the group is Hopfian.
In practice, it appears to be quite difficult to solve the entropy realization problem, whereas the Hopfian nature of a group is relatively easy to establish (for instance as a consequence of residual finiteness). So conversely, Sambusetti [71] constructed the first examples of groups with unrealized entropy: indeed, he showed the growth-tightness of a free product of any two non-Hopfian groups.

Since their introduction, the property of growth-tightness was then established by Arzhantseva and Lysenok [3] for hyperbolic groups, by Sambusetti [71, 72] for free products and co-compact Kleinian groups, and by Dal’bo, Peigné, Picaud and Sambusetti [29] for geometrically finite Kleinian groups with parabolic gap property. The present author [78] realized their most compact Kleinian groups, and by Dal’bo, Peigné, Picaud and Sambusetti [29] for geometrically

The next main theorem of this study provides a class of growth-tight sets called barrier-free elements. With a basepoint \( o \) fixed, an element \( h \in G \) is called \( (\epsilon, M, g) \)-barrier-free if there exists an \( (\epsilon, g) \)-barrier-free geodesic \( \gamma \) with \( \gamma_\epsilon \in B(o, M) \) and \( \gamma_o \in B(ho, M) \): there exists no \( t \in G \) such that \( d(t \cdot o, \gamma), d(t \cdot go, \gamma) \leq \epsilon \). We refer to Definition 4.1 for more details.

**Theorem C (Growth-tightness).** Suppose that \( G \) has a SCC action on a geodesic space \((Y, d)\) with a contracting element. Then there exist constants \( \epsilon, M > 0 \) such that for any given \( g \in G \), we have

\[
\varpi(\mathcal{V}_{\epsilon, M, g}) < \varpi(G)
\]

where \( \mathcal{V}_{\epsilon, M, g} \) denotes the set of \((\epsilon, M, g)\)-barrier-free elements.

**Remark.** (about the constants \( \epsilon, M \)) Any constant \( M \) satisfying Definition 1.4 works here. By Lemma 6.1, the constant \( M \) can be chosen as large as possible in applications. The constant \( \epsilon = \epsilon(M, \mathcal{F}) \) depends on the choice of a contracting system \( \mathcal{F} \) (Convention 2.5) by Lemma 4.4. The bigger the constant \( \epsilon > 0 \), the smaller the set \( \mathcal{V}_{\epsilon, M, g} \) is.

It is easy to see that this result generalizes all the above-mentioned results for growth-tightness of groups:

**Corollary 1.9 (4.6).** Under the same assumption as in **Theorem C**, we have

\[
\varpi(\bar{G}) < \varpi(G),
\]

for any quotient \( \bar{G} \) of \( G \) by an infinite normal subgroup \( N \), where \( \varpi(\bar{G}) \) is computed with respect to the proper action of \( \bar{G} \) on \( Y/N \) endowed with the quotient metric defined by \( \bar{d}(Nx, Ny) := d(Nx, Ny) \).

We shall consider two applications to the growth-tightness of a weakly quasi-convex subgroup. A subset \( X \) is called weakly \( M \)-quasi-convex for a constant \( M > 0 \) if for any two points \( x, y \) in \( X \), there exists a geodesic \( \gamma \) between \( x \) and \( y \) such that \( \gamma \subset NM(X) \). A subgroup \( \Gamma \) of \( G \) is called weakly quasi-convex if the set \( \Gamma \cdot o \) is weakly quasi-convex for some point \( o \in Y \). A sample application of **Theorem C** establishes the following.

**Theorem 1.10 (4.8).** Let \( G \) admit a SCC action on a geodesic space \((Y, d)\) with a contracting element. Then any weakly quasiconvex subgroup \( \Gamma \) of infinite index is growth-tight.

**Remark (on the proof).** Roughly speaking, the strategy when utilizing **Theorem C** is to find interesting sets with certain negative or non-negative curvatures. These sets will be “barrier-free”, thanks to the exclusivity of negative curvature and non-negative curvature. We refer the reader to Theorem 4.8 and Lemma 6.2. This idea turns out to be very fruitful, which we shall pursue in a subsequent paper [76].
The first corollary considers the class of convex-cocompact subgroups in $\mathcal{M}_\mathcal{M}$. Free convex-cocompact subgroups exist in abundance ([35, Theorem 1.4]), but one-ended ones are unknown at present. Although we cannot determine whether a given mapping class group has any large free convex-cocompact subgroups (cf. Theorem A), the growth-tightness part is indeed true:

**Corollary 1.11 (4.9).** Any convex-cocompact subgroup $\Gamma$ in $G \in \mathcal{M}_\mathcal{M}$ is growth-tight: $\omega(\Gamma) < \omega(G)$.

The next corollary applies to the class of *cubulated* groups which acts properly on a non-positively curved cubical complex. Recall that a subgroup is *cubically convex* if it acts cocompactly on a convex subcomplex. The following therefore answers positively [26, Problem 9.7].

**Corollary 1.12 (4.10).** Suppose that a group $G$ acts properly and cocompactly on a CAT(0) cube complex $Y$ such that $Y$ does not decompose as a product. Then any weakly quasi-convex subgroup of infinite index in $G$ is growth-tight. In particular, any cubically convex subgroup is growth-tight if it is of infinite index.

The second tool is a so-called *extension lemma*, which is a simple but quite useful consequence of the existence of a contracting element. To illustrate this, we emphasize that proofs of Theorem A, Theorem B, and Theorem C are constructed via repeated applications of extension lemmas. For convenience, we state here a simplified version and refer the reader to §2.5 for other versions.

**Lemma 1.13 (Extension Lemma).** There exist $\epsilon_0 > 0$ and a set $F$ of three elements in $G$ with the following property. For any two $g, h \in G$, there exists $f \in F$ such that $g \cdot f \cdot h$ is almost a geodesic:

$$|d(o, gfh \cdot o) - d(o, go) - d(o, ho)| \leq \epsilon_0.$$ 

Remark. This result is best illustrated for free groups with standard generating sets. In this case, we choose $F = \{a, b, a^{-1}\}$ and $\epsilon_0 = 1$. To the best of our knowledge, this result was first proved by Arzhantseva and Lysenok [3, Lemma 3] for hyperbolic groups, and reproved later in [38, Lemma 4.4] and [43, Lemma 2.4]. In joint work with Potyagailo [68], we have proved a version for relatively hyperbolic groups. The proof generalizes to the current setting with more advanced versions.

To finish this introductory section, we compare with the study of acylindrically hyperbolic groups formulated in [62]. By a result of Sisto [74] (see also [78, Appendix]), the existence of a contracting element produces a hyperbolically embedded subgroup in the sense of [27]. Thus, they all belong to the category of acylindrically hyperbolic groups, which are studied previously in various guises in [12, 11, 48], and [27] and in a continually growing body of literature. The emphasis of our study is, however, on understanding the asymptotic geometry of these groups, relying on their concrete actions rather than on the actions as tools.

Furthermore, the reader should distinguish our definition of a contracting element from others in the literature (e.g., [7, 74]): the projection in definition is meant to be a nearest point projection, whereas some authors take a more flexible one. As a consequence, a contracting element in their sense is a quasi-isometric invariant, whereas this is not the case for our definition, as shown by a recent example in [4]. However, it is this definition which brings the extension lemma into play.

**The structure of this paper.** As a prerequisite, §2 introduces the extension lemma and gives two immediate applications to the positive density of contracting elements (cf. Proposition 2.21), the finite depth of dead-ends (cf. Proposition 2.22). The next Sections 3, 4, and 5, which are mutually independent, contain respectively the proofs of Theorem A, Theorem C, and Theorem B. In the final §6 we give a way to construct non-convex-compact SCC actions.
This, the first of a series of papers, lays the foundation for the further study of groups with contracting elements. In a subsequent paper [76], we make use of the results established here to investigate the genericity of contracting elements in groups (Hyperbolic). Acknowledgment. The author is grateful to Jason Behrstock for pointing out Theorem 5.7. Laura Ciobanu, Thomas Koberda, Leonid Potyagailo, Mahan Mj and Dani Wise for helpful conversations. Thanks also go to Lewis Bowen for pointing out an error in the argument in proving (stronger) Theorem A. Ilya Gekhtman gave him valuable feedback and shared with him lots of knowledge in dynamics of Teichmüller geometry.

2. An extension lemma

This section introduces the basic tool: an extension lemma, in which a notion of a contracting element plays an important role. We first fix some notations and conventions.

2.1. Notations and conventions. Let $(Y,d)$ be a proper geodesic metric space. Given a point $y \in Y$ and a subset $X \subset Y$, let $\pi_X(y)$ be the set of points $x$ in $X$ such that $d(y,x) = d(y,X)$. The projection of a subset $A \subset Y$ to $X$ is then $\pi_X(A) := \cup_{a \in A} \pi_X(a)$.

Denote $d_{X}^n(Z_1,Z_2) := \text{diam}(\pi_X(Z_1 \cup Z_2))$, which is the diameter of the projection of the union $Z_1 \cup Z_2$ to $X$. So $d_{X}^n(\cdot,\cdot)$ satisfies the triangle inequality

$$d_X^n(A,C) \leq d_X^n(A,B) + d_X^n(B,C).$$

We always consider a rectifiable path $\alpha$ in $Y$ with arc-length parameterization. Denote by $\ell(\alpha)$ the length of $\alpha$, and by $\alpha_-, \alpha_+$ the initial and terminal points of $\alpha$ respectively. Let $x,y \in \alpha$ be two points which are given by parameterization. Then $[x,y]_\alpha$ denotes the parameterized subpath of $\alpha$ going from $x$ to $y$. We also denote by $[x,y]$ a choice of a geodesic in $Y$ between $x,y \in Y$.

**Entry and exit points.** Given a property (P), a point $z$ on $\alpha$ is called the entry point satisfying (P) if $\ell([\alpha_-,z]_\alpha)$ is minimal among the points $z$ on $\alpha$ with the property (P). The exit point satisfying (P) is defined similarly so that $\ell([w,\alpha_+]_\alpha)$ is minimal.

A path $\alpha$ is called a $c$-quasi-geodesic for $c \geq 1$ if the following holds

$$\ell(\beta) \leq c \cdot d(\beta_-,\beta_+) + c$$

for any rectifiable subpath $\beta$ of $\alpha$.

Let $\alpha, \beta$ be two paths in $Y$. Denote by $\alpha \cdot \beta$ (or simply $\alpha\beta$) the concatenated path provided that $\alpha_+ = \beta_-$. Let $f,g$ be real-valued functions with domain understood in the context. Then $f \prec_{c_i} g$ means that there is a constant $C > 0$ depending on parameters $c_i$ such that $f \prec Cg$. The symbols $\succ_{c_i}$ and $\sim_{c_i}$ are defined analogously. For simplicity, we shall omit $c_i$ if they are universal constants.

2.2. Contracting property.

**Definition 2.1** (Contracting subset). Let $QG$ denote a preferred collection of quasi-geodesics in $Y$. For given $C \geq 1$, a subset $X \subset Y$ is called $C$-contracting with respect to $QG$ if for any quasi-geodesic $\gamma \in QG$ with $d(\gamma,X) \geq C$, we have

$$d_{X}^n(\gamma) \leq C.$$ 

A collection of $C$-contracting subsets is referred to as a $C$-contracting system (with respect to $QG$).

**Example 2.2.** We note the following examples in various contexts.

(1) Quasi-geodesics and quasi-convex subsets are contracting with respect to the set of all quasi-geodesics in hyperbolic spaces.
(2) Fully quasi-convex subgroups (and in particular, maximal parabolic subgroups) are contracting with respect to the set of all quasi-geodesics in relatively hyperbolic groups (see Proposition 8.2.4 in [41]).

(3) The subgroup generated by a hyperbolic element is contracting with respect to the set of all quasi-geodesics in groups with non-trivial Floyd boundary. This is described in [78, Section 7].

(4) Contracting segments in CAT(0)-spaces in the sense of in Bestvina and Fujiwara are contracting here with respect to the set of geodesics (see Corollary 3.4 in [12]).

(5) The axis of any pseudo-Anosov element is contracting relative to geodesics by Minsky [57].

(6) Any finite neighborhood of a contracting subset is still contracting with respect to the same $QG$.

Convention. In view of Examples [2.2], the preferred collection $QG$ in the sequel will always be the set of all geodesics in $Y$.

In fact, the contracting notion is equivalent to the following one considered by Minsky [57]. A proof given in [12, Corollary 3.4] for CAT(0) spaces is valid in the general case. Despite this equivalence, we always work with the above definition of the contracting property.

Lemma 2.3. A subset $X$ is contracting in $Y$ if and only if any open ball $B$ missing $X$ has a uniformly bounded projection to $X$.

We collect a few properties that will be used often later on. The proof is straightforward applications of contracting property, and is left to the interested reader.

Proposition 2.4. Let $X$ be a contracting set.

(1) (Quasi-convexity) $X$ is $\sigma$-quasi-convex for a function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$: given $c \geq 1$, any $c$-quasi-geodesic with endpoints in $X$ lies in the neighborhood $N_{\sigma(c)}(X)$.

(2) (Finite neighborhood) Let $Z$ be a set with finite Hausdorff distance to $X$. Then $Z$ is contracting.

There exists $C > 0$ such that the following holds:

(3) For any geodesic segment $\gamma$, the following holds

$$\left| d_X^\gamma(\{\gamma_-, \gamma_+\}) - d_X^\gamma(\gamma) \right| \leq C.$$

(4) (1-Lipschitz projection) $d_X^\gamma(\{y, z\}) \leq d(y, z) + C$.

(5) (Projection point) Let $\gamma$ be a geodesic segment such that $\gamma_- \in X$, and $x \in X$ be a projection point of $\gamma_+$ to $X$. Then $d(x, \gamma) \leq C$.

(6) (Coarse projections) For any two points $x \in X, y \notin X$, we have

$$\left| d(x, \pi_X(y)) - (d(x, y) - d(y, X)) \right| \leq C.$$

In most cases, we are interested in a contracting system with a $R$-bounded intersection property for a function $R : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ if the following holds

$$\forall X \neq X' \in \mathcal{X} : \text{diam}(N_r(X) \cap N_r(X')) \leq R(r)$$

for any $r \geq 0$. This property is, in fact, equivalent to a bounded intersection property of $\mathcal{X}$: there exists a constant $B > 0$ such that the following holds

$$d_{X'}^\tau(X) \leq B$$

for $X \neq X' \in \mathcal{X}$. See [78] for further discussions.

Remark. Typical examples include sufficiently separated quasi-convex subsets in hyperbolic spaces, and parabolic cosets in relatively hyperbolic groups (see [31]).
2.3. Admissible paths. The notion of an admissible path is defined relative to a contracting system $X$ in $Y$. Roughly speaking, an admissible path can be thought of as a concatenation of quasi-geodesics which travels alternatively near contracting subsets and leave them in an orthogonal way.

**Definition 2.5** (Admissible Path). Given $D, \tau \geq 0$ and a function $R : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, a path $\gamma$ is called $(D, \tau)$-admissible in $Y$, if the path $\gamma$ contains a sequence of disjoint geodesic subpaths $p_i$ ($0 \leq i \leq n$) in this order, each associated to a contracting subset $X_i \in X$, with the following called Long Local and Bounded Projection properties:

1. Each $p_i$ has length bigger than $D$, except that $(p_i)_- = \gamma_-$ or $(p_i)_+ = \gamma_+$.
2. For each $X_i$, we have
   \[ d_{X_i}^X((p_i)_+, (p_{i+1})_-) \leq \tau \]
   and
   \[ d_{X_i}^X((p_{i-1})_+, (p_i)_-) \leq \tau \]
   when $(p_{i-1})_+ \neq \gamma_-$ and $(p_{n+1})_- \neq \gamma_+$ by convention.

**Saturation.** The collection of $X_i \in X$ indexed as above, denoted by $X(\gamma)$, will be referred to as contracting subsets for $\gamma$. The union of all $X_i \in X(\gamma)$ is called the saturation of $\gamma$.

The set of endpoints of $p_i$ shall be referred to as the vertex set of $\gamma$. We call $(p_i)_-$ and $(p_i)_+$ the corresponding entry vertex and exit vertex of $\gamma$ in $X_i$, (compare with entry and exit points in subsection 2.1).

**Remark.** In [78], an admissible path is defined so that the subpath between $p_i$ and $p_{i+1}$ is a geodesic and $d_{X_i}^X([((p_{i-1})_+, (p_i)_-) \leq \tau$. By Proposition 2.4.8, this condition is equivalent to (BP). Up to replace them by corresponding geodesics, we obtain a notion of admissible path originally defined in [78]. We allow a non-geodesic path so it is easier to verify (BP).

**Remark** (Bounded intersection). In most applications, the contracting system relative to which we consider admissible paths has bounded intersection. Hence, it suffices to show that $X_i$ and $X_{i+1}$ are distinct in the verification of Condition (LL2).

By definition, a sequence of points $x_i$ in a path $\alpha$ is called linearly ordered if $x_{i+1} \in [x_i, \alpha_+]$ for each $i$.

**Definition 2.6** (Fellow travel). Assume that $\gamma = p_0q_1p_1q_2p_2\ldots q_np_n$ is a $(D, \tau)$-admissible path, where each $p_i$ has two endpoints in $X_i \in X$. The paths $p_0, p_n$ could be trivial.

Let $\alpha$ be a path such that $\alpha_- = \gamma_-, \alpha_+ = \gamma_+$. Given $\epsilon > 0$, the path $\alpha \epsilon$-fellow travels $\gamma$ if there exists a sequence of linearly ordered points $z_i, w_i$ ($0 \leq i \leq n$) on $\alpha$ such that $d((z_i, (p_i)_-)) \leq \epsilon, d((w_i, (p_i)_+)) \leq \epsilon$.

The basic fact is that a “long” admissible path is a quasi-geodesic.

**Proposition 2.7.** Let $C$ be the contraction constant of $X$. For any $\tau > 0$, there are constants $B = B(\tau), D = D(\tau), \epsilon = \epsilon(\tau), \epsilon = \epsilon(\tau) > 0$ such that the following holds.

Let $\gamma$ be a $(D, \tau)$-admissible path and $\alpha$ a geodesic between $\gamma_-$ and $\gamma_+$. Then

1. For a contracting subset $X_i \in X(\gamma)$ with $0 \leq i \leq n$,
   \[ d_{X_i}^X(\beta_1) \leq B, d_{X_i}^X(\beta_2) \leq B \]
   where $\beta_1 = [\gamma_-, (p_i)_-], \beta_2 = [(p_i)_+, \gamma_+]$.
2. $\alpha \cap N_C(X) \neq \emptyset$ for every $X_i \in X(\gamma)$.
3. $\alpha \epsilon$-fellow travels $\gamma$. In particular, $\gamma$ is a $c$-quasi-geodesic.
Sketch of the proof. The content of this proposition was proved in [78 Proposition 3.3]. The constant $D > 0$ is taken to be sufficiently large but independent of $n$, and the first statement was proved by induction on $n$ as [78 Corollary 3.7]. Assuming Assertion (1), the second and third statements follow as consequences. For instance, if $D > 2B + C$, then we must have $\alpha \cap N_C(X) \neq \emptyset$ by the contracting property. Moreover, we can set $\epsilon := 2C + B$. We refer the interested reader to [78 Section 3] for more details.

The next result generalizes [78 Lemma 4.4] by a similar proof. The main use of this lemma (the second statement) is to construct the following type of paths in verifying that an element is contracting.

Definition 2.8. Let $L, \Delta > 0$. With notations in definition of a $(D, \tau)$-admissible path $\gamma$, if the following holds

$$|d((p_{i+1})_{\gamma}, (p_i)) - L| \leq \Delta$$

for each $i$, we say that $\gamma$ is a $(D, \tau, L, \Delta)$-admissible path.

Proposition 2.9. Assume that $X$ has bounded intersection in admissible paths considered in the following statements. For any $\tau > 0$ there exists $D = D(\tau) > 0$ with the following properties.

1. For any $L, \Delta > 0$, there exists $C = C(L, \Delta) > 0$ such that the saturation of a $(D, \tau, L, \Delta)$-admissible path is $C$-contracting.

2. For any $L, \Delta, K > 0$, there exists $C = C(L, \Delta, K) > 0$ such that if the entry and exit vertices of a $(D, \tau, L, \Delta)$-admissible path $\gamma$ in each $X \in \mathcal{X}(\gamma)$ has distance bounded above by $K$, then $\gamma$ is $C$-contracting.

Proof. Consider a $(D, \tau, L, \Delta)$-admissible path $\gamma = p_0q_1p_1\cdots p_{n-1}q_n$, where $p_i$ are geodesics with two endpoints in $X_i \in \mathcal{X}(\gamma)$ and $q_i$ are geodesics. Denote by $C_0$ the contraction constant for all $X_i$. For $\tau > 0$, let $D = D(\tau), B = B(\tau)$ provided by Proposition 2.7.

(1) Denote the saturation $A := \bigcup_{X \in \mathcal{X}(\gamma)} X_i$. Consider a geodesic $\alpha$ such that

$$\alpha \cap N_C(A) = \emptyset$$

where the constant $C > C_0$ (given below in (3)) depend $L, \Delta$ and $C_0$. Let $z \in Z, w \in W$ be projection points of $\alpha, \alpha$, respectively to $A$, where $Z, W \in \mathcal{X}(\gamma)$ appear on $\gamma$ in this order. Without loss of generality, assume that $d(z, w) = d_A^\tau(\alpha)$. The purpose of the proof is to prove $d(z, w) \leq C$.

If $Z = W$, the conclusion then follows from the contracting property of $Z$. So, assume that $Z \neq W$ below. Since $X$ has bounded projection, there exists a constant, for simplicity, the same $B > 0$ from Proposition 2.7 such that

$$d_A^\tau(X') \leq B$$

for $X \neq X' \in \mathcal{X}$.

We first observe that

$$\max\{\text{diam}([z, \alpha] \cap N_{C_0}(X)), \text{diam}([w, \alpha] \cap N_{C_0}(X))\} \leq C_0$$

for any $X \in \mathcal{X}(\gamma)$. Indeed, consider the entry point of $[z, \alpha]$ in $N_{C_0}(X)$ which is $C_0$-close to $X \subset A$, so the exit point must be within a $C_0$-distance to the entry point, since $z$ is a shortest point in $A$ to $\alpha$. The (3) thus follows.

Let $z' \in Z$ and $w' \in W$ be the corresponding exit and entry vertices (cf. Definition 2.5) of $\gamma$ in $Z$ and $W$.

Applying (3) to $X = W$, we see $d_W^\tau([w, \alpha]) \leq 5C_0$ by the contracting property. By assumption, $\alpha \cap N_{C_0}(Z) = \emptyset$ and then $d_Z^\tau(\alpha) \leq C_0$. Using (2), we obtain:

$$d(z, z') \leq d_Z^\tau(\alpha) + d_Z^\tau([w, \alpha]) + d_Z^\tau(W) + d_Z^\tau([z', w'], \gamma) \leq 6C_0 + 2B,$
where \( d^\pi_z([z', w'], \gamma) \leq B \) follows by Proposition 2.7. Proceeding similarly, we obtain that
\[
d(w, w') \leq 6C_0 + 2B.
\]

In order to bound the distance \( d(z, w) \), it remains to prove that the subpath \([z', w'], \gamma\) between 
\( z', w' \in \gamma \) contains no other vertices than \( z', w' \). Indeed, by definition of \( \gamma \) being a \((D, \tau, L, \Delta)\)-admissible path, we have \( |d(z', w') - L| \leq \Delta \). Consequently, we obtain
\[
d(z, w) \leq d(z', w') + d(z, z') + d(w, w') \leq C := 4(3C_0 + B) + L + \Delta,
\]
concluding the proof of the assertion (1).

Assume, by way of contradiction, that \([z', w'], \gamma\) contains a geodesic segment \( p \) associated to a contracting set \( X \in \mathcal{X}(\gamma) \). By definition, \( \ell(p) > D \). It is obvious that \( Z \neq X \neq W \).

On the other hand, noting that \( \alpha \) lies outside \( N_r(A) \) and \( X \subset A \), we have \( \alpha \cap N_{C_0}(X) = \emptyset \) for \( \epsilon > C_0 \) so \( d^\pi_X(\alpha) \leq C_0 \). By 3, we see \( d^\pi_X([z, \alpha]), d^\pi_X([w, \alpha]) \leq 5C_0 \). Thus the length of \( p \) gets bounded as follows:
\[
d(p_-, p_+) \leq d^\pi_X([p_-, z'], \gamma) + d^\pi_X(Z) + d^\pi_X([z, \alpha]) + d^\pi_X([\alpha, w]) + d^\pi_X(W) + d^\pi_X([w', z], \gamma) \\
\leq 4B + 11C_0,
\]
where \( d^\pi_X([p_-, z'], \gamma), d^\pi_X([w', z], \gamma) \leq B \) by Proposition 2.7(1), and \( d^\pi_X(Z), d^\pi_X(W) \leq B \) by 2. Consequently, this gives a contradiction, by further setting
\[
D > 4B + 11C_0,
\]
whence the proof of (1) is complete.

(2). We follow the same line as (1): the notation \( A \) now denotes the set of vertices of \( \gamma \). Note also that \( z \) and \( w \) are vertices of \( \gamma \). The main difference comes from treating the case that \( Z = W \): the contracting property now follows by the assumption that \( d(z, w) \) is uniformly bounded by \( K \). So the contraction constant \( C \) depends on \( K \) as well. We leave the details to the interested reader. \( \square \)

Before passing to further discussions, let us introduce a few ways to manipulate existing admissible paths to produce new ones.

**Subpath**: let \( \gamma \) be a \((D, \tau)\)-admissible path. An *admissible subpath* \( \beta \) is a subpath between two vertices in \( \gamma \). It is clear that, an admissible subpath is \((D, \tau)\)-admissible.

**Concatenation**: let \( \alpha, \beta \) be two \((D, \tau)\)-admissible paths. Suppose the last contracting subset associated to the geodesic segment \( p \) of \( \beta \) is the same as the first contracting subset to the geodesic segment \( q \) of \( \gamma \). A new path \( \gamma \) can be thus formed by concatenating paths
\[
\gamma := [\alpha \ldots p_-, q_+]q_+ \beta_+ \beta.
\]
A useful observation is that if \( d(p_-, q_+) > D \), then \( \gamma \) is \((D, \tau)\)-admissible.

**Path label convention**. We conclude this subsection with the following terminology, which is designed to be consistent with the common one – paths labeling by words in Cayley graphs.

Let \( \{g_i \in G : 1 \leq i \leq n; \ n \leq \infty\} \) be a (possibly infinite) sequence of elements. Fixing a basepoint \( o \in X \), we plot a sequence of points \( x_i = h_i \cdot o \) \((i = 0, \ldots, n)\) where \( h_i = g_0 \cdots g_{i-1} \) with \( g_0 = 1 \), and then connect \( x_i, x_{i+1} \) by a geodesic to define a piecewise geodesic path \( \gamma \). We call such a path \( \gamma \) *labeled* by the sequence \( \{g_i\} \), and \( x_i \) the *vertices* of \( \gamma \). If \( n \) is finite, the path \( \gamma \) is also said to be *labeled* by the (product form of) element \( g_1g_2\cdots g_n \), it represents the element \( g_1g_2\cdots g_n \) in \( G \).

Very often, we need to write the path \( \gamma \) explicitly as follows
\[
\gamma := g_1g_2\cdots g_{n-1}g_n \ldots
\]
where \( g_i \) denotes a choice of a geodesic between \( x_i, x_{i+1} \).
By abuse of language, we say that any translate of a path $\gamma$ by an element in $G$ is also labeled by $\{g_i\}$.

By definition, a labeled path by a bi-infinite sequence of elements is defined as the union of two labeled paths by one-sided infinite sequence of elements.

2.4. Contracting subgroups. We first setup a few definitions. An infinite subgroup $H$ in $G$ is called contracting if for some (hence any by Proposition 2.4.2) $o \in Y$, the subset $Ho$ is contracting in $Y$. In fact, we usually deal with a contracting subgroup $H$ with bounded intersection: the collection of subsets

$$\{gH \cdot o : g \in G\}$$

is a contracting system with bounded intersection in $Y$. (In [78], a contracting subgroup $H$ with bounded intersection was called strongly contracting.)

**Lemma 2.10.** Assume that $G$ acts properly on $(Y, d)$. If $H$ is a contracting subgroup, then $[N(H) : H] < \infty$, where $N(H)$ is the normalizer of $H$ in $G$. In particular, $N(H)$ is contracting.

**Proof.** Let $g \in N(H)$ so $gH = Hg$. It follows that $gH \cdot o \subset N_D(H \cdot o)$ for $D := d(o, go)$. Let $C$ be the contraction constant of $Ho$, and also satisfy Proposition 2.4.1 such that any geodesic with two endpoints in $gHo$ lies in $N_C(gHo)$. Since $gHo$ is unbounded, choose a geodesic $\gamma$ of length $\geq 2D + C$ with two endpoints in $gHo \subset N_D(H \cdot o)$. By contracting property, $\gamma$ has to intersect $N_C(Ho)$: if not, we have $d_{H,o}(\gamma) \leq C$ and thus $\ell(\gamma) \leq d_{H,o}(\gamma) + d(\gamma, Ho) + d(\gamma, Ho) \leq C + 2D$, a contradiction. Together with $\gamma \subset N_C(gHo)$, we obtain $N_C(gHo) \cap N_C(Ho) \neq \emptyset$: there exists $h \in H$ such that $hg \in N(o, 2C)$. The finiteness of the set $N(o, 2C)$ shows that $[N(H) : H] < \infty$. The finite neighborhood of a contracting set is contracting by Proposition 2.4.2 so $N(H)$ is a contracting subgroup as well.

An element $h \in G$ is called contracting if the subgroup $\langle h \rangle$ is contracting, and the orbital map

$$(6) \quad n \in \mathbb{Z} \to h^n o \in Y$$

is a quasi-isometric embedding. The set of contracting elements is preserved under conjugacy.

Given a contracting subgroup $H$, define a group $E(H)$ as follows:

$$E(H) := \{g \in G : \exists r > 0, gHo \subset N_r(Ho)\}.$$ (7)

For a contracting element $h$, the structure of $E(h) := E(\langle h \rangle)$ could be made precise as follows.

**Lemma 2.11.** Assume that $G$ acts properly on $(Y, d)$. For a contracting element $h$, the following statements hold:

1. $[E(h) : \langle h \rangle] < \infty$, and $E(h)$ is a contracting subgroup with bounded intersection.
2. $E(h) = \{g \in G : \exists n > 0, (gh^n g^{-1} = h^n) \lor (gh^n g^{-1} = h^{-n})\}$.

**Proof.** (1). Since $n \in \mathbb{Z} \to h^n o \in Y$ is a quasi-isometric embedding, the path $\gamma$ obtained by connecting consecutive dots is a quasi-geodesic which is contracting. Hence, for any $r \gg 0$, the following unbounded intersection

$$\text{diam}(N_r(\langle h \rangle o) \cap N_r(g(h) o)) = \infty$$

implies that there exists a uniform constant $C$ such that

$$\langle h \rangle o \subset N_C(g(h) o) \quad \text{and} \quad g(h) o \subset N_C(\langle h \rangle o)$$

yielding $g \in E(h)$. As a consequence, the constant $r$ can be made uniform in definition of $E(h)$. So, the assertion $[E(h) : \langle h \rangle] < \infty$ follows by a similar argument as in Lemma 2.10.

Furthermore, if $g \notin E(h)$, then $gE(h) o$ and $E(h) o$ have bounded intersection. The proper action also implies the uniformity of bounded intersection for all $g \notin E(h)$. Thus, $E(h)$ is contracting with bounded intersection.
(2). The right-hand set is contained in $E(h)$ as a subgroup. For given $g \in E(h)$, there exists some $r > 0$ such that
\[ g(h)o \subset N_r((h)o). \]
Since $(h)o \subset N_D((h)go)$ for $D := d(o, go)$, we have $g(h)o \subset N_{r+D}((h)go)$. There exists a sequence of distinct pairs of integers $(n_i, m_i)$ such that $d(g^{n_i}o, h^{m_i}go) \leq r + D$. It follows that $h^{-n_i}g^{-1}h^{m_i}g \in N(o, r + D)$, which is a finite set by the proper action. So there exist two distinct pairs $(n_i, m_i)$ and $(n_j, m_j)$ such that $h^{-n_i}g^{-1}h^{m_i}g = h^{-n_j}g^{-1}h^{m_j}g$ and so $gh^{n_i-n_j}g^{-1} = h^{m_i-m_j}$. An induction argument then proves
\[ g^l h^{(n_i-n_j)l} g^{-l} = h^{(m_i-m_j)l} \]
for any $l \in \mathbb{N}$. So if $|n_i - n_j| \neq |m_i - m_j|$, a straightforward calculation gives a contradiction since the map $n \mapsto h^n o$ is a quasi-isometric embedding. Letting $n = n_i - n_j$, we obtain $gh^n g^{-1} = h^{\pm n}$.

The description of $E(h)$ thus follows.

\[ \square \]

In what follows, the contracting subset
\[ Ax(h) = \{ f \cdot o : f \in E(h) \} \]
shall be called the axis of $h$. Two contracting elements $h_1, h_2 \in G$ are called independent if the collection $\{ g \cdot Ax(h_i) : g \in G ; i = 1, 2 \}$ is a contracting system with bounded intersection.

The following result could be thought of as an analog of the well-known fact in $\text{Isom}(\mathbb{H}^2)$ that two hyperbolic isometries in a discrete group have either disjoint fixed points or the same fixed points.

**Lemma 2.12.** Assume that a group $G$ acts properly on $(Y, d)$. For two contracting elements $h_1, h_2 \in G$ are called independent if the collection $\{ g \cdot Ax(h_i) : g \in G ; i = 1, 2 \}$ is a contracting system with bounded intersection, or they have finite Hausdorff distance and $h_1 \in E(h_2)$.

In particular, if $G$ is non-elementary, then there are infinitely many pairwise independent contracting elements.

**Proof.** Assume that $(h_1)o$ and $(h_2)o$ have unbounded intersection: there exists a constant $r > 0$ such that
\[ \text{diam}(N_r((h_1)o) \cap N_r((h_2)o)) = \infty. \]
There exists a sequence of distinct pairs of integers $(n_i, m_i)$ such that $d(h_1^{n_i}o, h_2^{m_i}o) \leq 2r$, so $h_1^{n_i} h_2^{m_i} \in N(o, 2r)$. The set $N(o, 2r)$ is thus finite by properness of the action. Hence, there exist two distinct pairs $(n_i, m_i)$ and $(n_j, m_j)$ such that $h_1^{n_i-n_j} = h_2^{m_i-m_j}$. This means that $E(h_1)$ and $E(h_2)$ are commensurable, i.e. they have finite index subgroups which are isomorphic. So $Ax(h_1)$ and $Ax(h_2)$ have finite Hausdorff distance. The lemma is proved.

\[ \square \]

Finally, we record the following elementary well-known fact: a proof is given for completeness. We refer the reader to [34] for the classification of periodic, reducible, and pseudo-Anosov elements.

**Lemma 2.13.** In mapping class groups, a contracting element coincides with a pseudo-Anosov element with respect to the Teichmüller metric.

**Proof.** A pseudo-Anosov element $g$ is contracting by Minsky’s theorem [37] that the axis of $g$ is contracting. A contracting element is of infinite order by definition. So suppose that $g$ is reducible. By definition, some power $g^n$ of a reducible element fixes a finite set of simple closed curves and thus commutes with Dehn twists around these curves. Hence, $g^n$ is of infinite index in its centralizer. On the other hand, a contracting element has to be of finite index in its centralizer by Lemma 2.10. So we got a contradiction, completing the proof of the lemma.

\[ \square \]
2.5. **An extension lemma.** In this subsection, we do not demand that the action of $G$ on $Y$ is proper. The only requirement is the existence of three pairwise independent contracting elements.

**Lemma 2.14 (Extension Lemma).** Suppose that a group $G$ acts on a geodesic metric space $(Y, d)$ with contracting elements. Consider a collection of subsets

$$ F = \{ g \cdot \text{Ax}(h_i) : g \in G \}, $$

where $h_i$ $(1 \leq i \leq 3)$ are three pairwise independent contracting elements. Then there exist $\epsilon_0, \tau, D > 0$ depending only on $F$ with the following property.

Let $F$ be a set consisting of a choice of an element $f_i$ from each $E(h_i)$ such that $d(o, f_i o) > D$ $(0 \leq i \leq 3)$.

1. For any two elements $g, h \in G$, there exist an element $f \in F$ such that $g f h \neq 1$ and the path labeled by $g f h$ is a $(D, \tau)$-admissible path.
2. As a consequence, the following holds

$$(9) \quad \max\{ d(go, \gamma), d(gfo, \gamma) \} \leq \epsilon_0,$$

for any geodesic $\gamma := [o, gfo]$. 

**Remark.**

1. We emphasize an arbitrary choice $F$ of three elements would satisfy the lemma.
2. We do allow the possibility of $g$ or $h$ to be the identity.
3. The group action is not assumed to be a proper action. Another important source of examples comes from the acylindrical action in [27, 62].

The following simplified version of the extension lemma explains its name.

**Corollary 2.15 (Extension Lemma: simplified version).** Under the same assumption as Lemma 2.14. Then for any two $g, h \in G$, there exist $f \in F$ such that the above (9) holds.

We are going to prove a more general version which could deal with any number of elements.

**Lemma 2.16 (Extension Lemma: infinite version).** Under the same assumption as Lemma 2.14, there exist $\epsilon_0, \tau, D, c > 0$ with the following property.

1. For any $g_i, \epsilon_i, f_i, f_i+1, g_i+2, \ldots$ is a $(D, \tau)$-admissible path, and is a $c$-quasi-geodesic.
2. The product of any finite sub-sequence of consecutive elements in

$$ (...) g_1 f_1 g_1+1 f_1+1 g_1+2 \ldots $$

is non-trivial in $G$.
3. Let $\alpha$ be a geodesic between two vertices in $\gamma$. Then each vertex in $[\alpha_-, \alpha_+]$, has a distance at most $\epsilon_0$ to $\alpha$.
4. Consider two finite sequences $\{ g_i : 1 \leq i \leq m \}, \{ g'_j : 1 \leq j \leq n \}$ with $f_i, f'_i \in F$ provided by the statement (1) satisfying that

$$ g_1 f_1 g_2 \ldots f_{m-1} g_m o = g'_1 f'_1 g'_2 \ldots f'_{n-1} g'_n o. $$

If $g_1 = g'_1$, then $f_1 = f'_1$. 

Assume that $F$ is a $C$-contracting system with $R$-bounded intersection for some $C > 0$ and $R : \mathbb{F}_{20} \rightarrow \mathbb{F}_{20}$. For simplicity we denote $A_k = \text{Ax}(h_k)$ below. The proof replies on the following observation.

**Lemma 2.17.** There exists a constant $\tau$ such that for any element $g \in G$, the following holds

$$(10) \quad \min\{ d_{\lambda_1}(o, go), d_{\lambda_2}(o, go) \} \leq \tau.$$
Proof. By the contracting property, any geodesic segment with two endpoints within \( C \)-distance from \( A_i \) lies in the \( 3C \)-neighborhood of \( A_i \) for \( i = 1, 2 \). We set
\[
\tau = \mathcal{R}(3C) + 2C + 1.
\]
(11)
Suppose, by contradiction, that there exist some \( g \in G \) such that
\[
\max\{d_{A_i}^\tau(\alpha), d_{A_2}^\tau(\alpha)\} > \tau
\]
where \( \alpha := [o, go] \). Let \( z_1 \in A_1 \) and \( z_2 \in A_2 \) be the corresponding projection points of \( g \) such that \( \min\{d(o, z_1), d(o, z_2)\} > \tau \). Consider the exit points \( w_1, w_2 \) of \( \alpha \) in \( N_C(A_1) \) and \( N_C(A_2) \) respectively. By the contracting property, we obtain that \( \max\{d(z_1, w_1), d(z_2, w_2)\} \leq 2C \). By the choice of \( \tau \) \((11)\), we have
\[
\min\{d(o, w_1), d(o, w_2)\} \geq \mathcal{R}(3C) + 1.
\]
On the other hand, assume that \( d(o, w_1) \leq d(o, w_1) \) for concreteness. Since \( o, w_2 \in N_C(A_2) \), by the quasi-convexity of \( A_2 \) stated above, we see that \([o, w_2]\) lies in the \( 3C \)-neighborhood of \( A_2 \). This implies
\[
o, w_1 \in N_{3C}(A_1) \cap N_{3C}(A_2).
\]
Since \( d(o, w_1) > \mathcal{R}(3C) \), we get a contradiction since the pair \( A_1, A_2 \) has \( \mathcal{R} \)-bounded intersection. Thus, \( (10) \) is proved. \( \square \)

We are in a position to give a proof of Lemma 2.16

Proof of Lemma 2.16. The proof consists in proving the statement (1), from which the statements (2) and (3) follow by Proposition 2.7 in a straightforward way.

Let \( \tau \) given by Lemma 2.17 First of all, for any \( g, h \in G \), there exists at least one \( A \in \{A_k : 1 \leq k \leq 3\} \) such that
\[
\max\{d_{A}^\tau([o, go]), d_{A}^\tau([o, ho])\} \leq \tau.
\]
Indeed, by Lemma 2.17 , each \( g \) and \( h \) has a bounded projection by \( \tau \) to at least two sets from \( \{A_k : 1 \leq k \leq 3\} \). Thus, there is at least one \( A \) in common such that \((12)\) holds.

Now we may choose the collection of the constants. For a constant \( D = D(\tau) \) given by Proposition 2.7 choose one element \( f_k \) from each \( E(h_k) \) such that \( d(o, f_k o) > D \). Denote \( F = \{f_1, f_2, f_3\} \).

Construction of admissible paths. Let \( \{g_i \in G\} \) be a (finite, or infinite, or bi-infinite) sequence of elements. The goal is to choose \( A_i \in \{A_k : 1 \leq k \leq 3\} \) for each pair \((g_i, g_{i+1})\) such that
\[
\max\{d_{A_i}^\tau([o, g_i^{-1}o]), d_{A_i}^\tau([o, g_{i+1}o])\} \leq \tau.
\]
and
\[
A_i \neq g_{i+1}A_{i+1}
\]
for verifying the condition \((\text{LL}2)\).

We start with a fixed pair \((g_1, g_{i+1})\), for which \( A_i \) is chosen as above satisfying \((12)\) so \((13)\) is immediate. For subsequent pairs, we need to be careful when applying \((12)\) to obtain \( A_i \neq g_{i+1}A_{i+1} \) for the following reason. It is possible that \( g_{i+1} \) belong to \( E(h_i) \). If it happens, the choice of \( A_{i+1} \) by application of \((12)\) may not satisfy \( A_i \neq g_{i+1}A_{i+1} \). A simple example to bear in mind is that \( G \) splits a direct product of two groups one of which is finite. The next paragraph thus treats this possibility.

If \( g_{i+1} \) does not belong to \( E(h_i) \), then by \((12)\) there exists \( A_{i+1} \in \{A_k : 1 \leq k \leq 3\} \) such that \((13)\) holds for the index \( "i + 1" \) and consequently, \( A_i \neq g_{i+1}A_{i+1} \). Otherwise, assume that \( g_{i+1} \) lies in \( E(h_i) \), then by \((13)\), we have \( d(o, g_{i+1}o) \leq \tau \) so \( d_{A_i}^\tau([o, g_{i+1}^{-1}o]) \leq \tau \). We choose \( A_{i+1} \in \{A_k : 1 \leq k \leq 3\} \setminus A_i \) which exists by the Claim above with the following property
\[
d_{A_{i+1}}^\tau([o, g_{i+2}o]) \leq \tau
\]
which shows (13) for the case “i + 1” and $A_i \neq g_{i+1}A_{i+1}$. Hence, the property (14) is fullfilled.

In this manner, we construct an admissible path $\gamma$ labeled by $(\ldots, g_i, f_i, g_{i+1}, f_{i+1}, \ldots)$, where $f_i \in F$ is the chosen element as above from $E(h_k)$. Since $A_i \neq g_{i+1}A_{i+1}$, the condition (LL2) follows from bounded intersection of $F$. The condition (BP) follows from (13), and (LL1) by the choice of $f_i$ with $d(o, f_i o) > D$. Hence, the path $\gamma$ is $(D, \tau)$-admissible with respect to the contracting system $F$.

Let $\epsilon_0 = \epsilon(\tau), c = c(\tau) > 0$ be given by Proposition 2.7 from which all assertions of this lemma follows as a consequence.

Now, it remains to prove the statement (4). We are looking at their associated admissible paths $\gamma$ and $\gamma'$ with the same endpoints by the hypothesis. If $f_i \neq f'_i \in F$, then by definition of $F$, they belong to different subgroups in $\{E(h_k) : 1 \leq k \leq 3\}$. Denote by $X_1, X'_1$ their axis sets so $f_1 o \in X_1, f'_1 o \in X'_1$. Connect the two endpoints of $\gamma$ (or $\gamma'$) by a geodesic $\sigma$. By the statement (3), we have $d(g o, \sigma), d(g f_1 o, \sigma), d(g f'_1 o, \sigma) \leq \epsilon_0$ for $g := g_1 = g'_1$ by assumption. We thus obtain a constant $\sigma$ depending on $\epsilon_0$ from the quasi-convexity of $X_1$ and $X'_1$ that a subsegment of $\sigma$ of length at least $\min\{d(o, f_1 o), d(o, f'_1 o)\}$ is contained in a $\sigma$-neighborhood of both $gX_1$ and $gX'_1$. Hence, it suffices to make $D > R(\sigma)$ larger so the bounded intersection of $X_1, X'_1$ would imply a contradiction. The constant $D$ is still uniform, and so the choice of $F$ can be made for the statement (4) as well. Hence all statements are proved.

Convention. Choose three pairwise independent contracting elements $f_i$ $(1 \leq i \leq 3)$ in $G$ so that $F = \{gAx(f_i) : g \in G\}$ is a contracting system with bounded intersection. Let $F$ be a finite set and $\epsilon_0, \tau, D > 0$ constants supplied by lemma 2.7.

The extension map. In order to facilitate the use of extension lemmas, it is useful to construct a kind of maps as described as follows.

Given an alphabet set $A$, denote by $\mathcal{W}(A)$ the set of all (finite) words over $A$. Consider an evaluation map $\iota : A \rightarrow G$. We are going to define an extension map $\Phi : \mathcal{W}(A) \rightarrow G$ as follows: given a word $W = a_1 a_2 \cdots a_n \in \mathcal{W}(A)$, set

$$\Phi(W) = \iota(a_1) \cdot f_1 \cdot \iota(a_2) \cdot f_2 \cdots \iota(a_{n-1}) \cdot f_{n-1} \cdot \iota(a_n) \in G,$$

where $f_i \in F$ is supplied by the extension lemma 2.14 for each pair $(a_i, a_{i+1})$. The product form as above of $\Phi(W)$ labels a $(D, \tau)$-admissible path as follows

$$\gamma = \iota(a_1^1) \cdot f_1^1 \cdot \iota(a_1^2) \cdot f_1^2 \cdots \iota(a_{n-1}^1) \cdot f_{n-1}^1 \cdot \iota(a_n^1).$$

Lemma 2.18. For any $\Delta > 0$, there exists $R > 0$ with the following property.

Let $W = a_1 a_2 \cdots a_n$ and $W' = a'_1 a'_2 \cdots a'_n$ be two words in $\mathcal{W}(A)$ such that $\Phi(W) = \Phi(W')$. If $|d(o, \iota(a_1)) - d(o, \iota(a'_1))| \leq \Delta$, then $d(\iota(a_1) o, \iota(a'_1) o) \leq R$.

In particular, if $\iota : A \rightarrow G$ is injective such that $\iota(A) o$ is $R$-separated in $A(o, L, \Delta)$ for some $L > 0$, then the extension map $\Phi$ defined as above is injective with the image consisting entirely of contracting elements.

Proof. Consider the $(D, \tau)$-admissible path labeled by $\Phi(W')$:

$$\gamma' = \iota(a'_1) \cdot f'_1 \cdot \iota(a'_2) \cdot f'_2 \cdots \iota(a_{n-1}^1) \cdot f_{n-1}^1 \cdot \iota(a_n^1).$$

By Proposition 2.7 a common geodesic $\alpha = [o, \iota(W) o]$ $\epsilon_0$-fellow travels both $\gamma$ and $\gamma'$ so we have

$$d(\iota(a_1) o, o, \iota(a'_1) o, o) \leq \epsilon_0.$$

By the hypothesis, it follows that $|d(o, \iota(a_1)) - d(o, \iota(a'_1))| \leq \Delta$. A standard argument shows that $d(g o, g' o) \leq 2(2\epsilon_0 + \Delta)$. Setting $R := 2(2\epsilon_0 + \Delta)$ thus completes the proof of the first assertion.

By (4) of Lemma 2.16 the injectivity part of the last statement follows as a consequence of the first one. Observe that the set of powers $\{\Phi(W)^n : n \in \mathbb{Z}\}$ labels a $(D, \tau, L, \Delta)$-admissible path and thus is contracting by Proposition 2.9. By definition, the element $\Phi(W)$ is contracting, thereby concluding the proof of the lemma.
The construction of an extension map is by no means unique. Here is another way to construct it. This will be a key ingredient to prove the existence of large free sub-semigroups in §

Lemma 2.19. There exist a finite set $F$ in $G$ and for any $\Delta > 0$, there exists $R = R(\Delta) > 0$ with the following property.

Let $Z$ be an $R$-separated subset in $A(o, L, \Delta)$ for any given $L > 0$. Then there exist an element $f \in F$ and a subset $A \subset Z$ of cardinality greater than $2^{-4} \|Z\|$ such that the extension map $\Phi: \hex(A) \rightarrow G$ given by

$$a_1a_2\cdots a_n \rightarrow a_1f^{\epsilon_1}a_2f^{\epsilon_2}\cdots a_nf^{\epsilon_n}$$

for any $\epsilon_i \in \{1, 2\}$ is injective with the image consisting entirely of contracting elements.

Proof. Let $\tau > 0$ given by Lemma 2.17 and $D = D(\tau)$ given by Proposition 2.7. The idea of the proof is to find an appropriate element $f$ such that for each $\epsilon_i \in \{1, 2\}$, the element $a_1f^{\epsilon_1}a_2f^{\epsilon_2}\cdots a_nf^{\epsilon_n}$ labels a $(D, \tau)$-admissible path. The injectivity statement then follows the same line as in Lemma 2.13.

The first observation is follows: there exist a pair $(A_i, A_j)$ and a subset $Z' \subset Z$ such that $4\|Z' \geq \|Z$, and for each $g \in Z'$ we have

$$\max\{d_{A_i}^\pi([o, go]), d_{A_j}^\pi([o, go])\} \leq \tau.$$ 

Indeed, this is achieved by applying Lemma 2.17 twice: for a fixed pair, say $(A_1, A_2)$, we first apply it for every $g \in Z$, then there exists half of elements $Z'$ in $Z$ and one, say $A_i$, of $(A_1, A_2)$ such that $d_{A_i}^\pi([o, go]) \leq \tau$ for all $g \in Z'$. On the second time, applying Lemma 2.17 to $\{A_1, A_2, A_3\} \setminus A_i$, we reduce $Z'$ in half and find another, say $A_j \neq A_i$ such that $d_{A_j}^\pi([o, go]) \leq \tau$ for all $g \in Z'$. This thus completes the proof of the observation.

Repeating the above argument to the pair $(A_i, A_j)$ and all elements $g \in Z'\!\!\!'$. We eventually find $A \subset Z'$ and $A_k \in \{A_i, A_j\}$ such that $4\|A \geq \|Z'$ and

(15) $$\forall g \in A, \max\{d_{A_k}^\pi([o, go]), d_{A_k}^\pi([o, g^{-1}o])\} \leq \tau.$$ 

The Condition (BP) is verified by (15). It suffices to show that $gA_k \neq A_k$ for any $g \in A$. Indeed, if $gA_k = A_k$, then (15) implies $d([o, go]) \leq \tau$. Since $A$ is $R$-separated, we choose $R > \tau$ so that $gA_k \neq A_k$ for any $g \in A$. Choose an element $f_k$ from the corresponding subgroup of $A_k$ such that $d([o, f]) > D$ and $d([o, f^2]) > D$. Consequently, we have shown that $a_1f^{\epsilon_1}a_2f^{\epsilon_2}\cdots a_nf^{\epsilon_n}$ labels a $(D, \tau)$-admissible path. By Proposition 2.7, it is thus a quasi-geodesic. Moreover, noting that $a_i \in A(o, L, \Delta)$, we see that it is $(D, \tau, L, \Delta)$-admissible path by Definition 2.8. Hence, the path is contracting by Proposition 2.9. This shows that $a_1f^{\epsilon_1}a_2f^{\epsilon_2}\cdots a_nf^{\epsilon_n}$ is a contracting element, thereby completing the proof of the lemma.

Positive density of contracting elements. By the above proof, we can prove the following lemma, which owns its existence to a recent result of M. Cumplido and B. Wiest in [25, Theorem 2]. Their result was proved for mapping class groups, but our result works for any sufficiently large subgroup as well as any other proper action with a contracting element.

Lemma 2.20. Let $G$ be a group acting properly on a geodesic metric space $(Y, d)$ with a contracting element. Then there exists a finite set of elements $F$ with the following property. For any $g \in G$ there exists $f \in F$ such that $gf$ is a contracting element.

Proof. Following the same line as in Lemma 2.19, we prove that for any $g \in G$, there exists $A_k \in \{A_1, A_2, A_3\}$ such that (15) holds. If $g \notin E(h_k)$, then the condition (BP) is satisfied; $gA_k \neq A_k$. Thus, $af$ is a contracting element for some $f \in E(h_k)$. Now we consider the case $g \in E(h_k)$. By Lemma 2.11, the contracting subgroup $\{h_k\}$ is of finite index in $E(h_k)$, so the conclusion is already satisfied for $E(h_k)$. Therefore, the proof is complete.\[\square\]
With Lemma 2.20, the following result follows by an argument as in [23, Corollary]. Simultaneously, this is also obtained independently by M. Cumplido [24] for Artin-Tits groups on certain hyperbolic spaces.

**Proposition 2.21** (Positive density of contracting elements). Under the hypothesis as Lemma 2.20. If $G$ is generated by a finite set, then for $R > 0$,

\[
\frac{\# \{ g \in N(1, R) : g \text{ is contracting} \}}{\# N(1, R)} > 0,
\]

where the ball $N(1, R)$ is defined using the corresponding word metric.

**Remark.** This strengthens a similar result in a previous version of this paper, stating that the growth rate of contracting element equals to $\omega(G)$ computed using word metric.

We remark that a similar statement holds for loxodromic elements in a group $G$ acting on a hyperbolic space with WPD loxodromic elements. Indeed, a loxodromic WPD element in a (not necessarily proper) group action gives rise to a contracting subgroup with bounded intersection (see Theorem 6.8 in [27]). This is the only ingredient of the above two lemmas so their proofs show the positive density of loxodromic elements for any word metric.

**Finiteness of dead-end depth.** To conclude this subsection, we mention a straightforward application of the extension lemma to dead-ends. In Cayley graphs, the dead-depth of a vertex roughly describes the “detour” cost to get around a dead-end (i.e. an endpoint of a maximal unextendable geodesic). In this context, the finiteness of dead-end depth was known in hyperbolic groups [15], and in relative case [78].

Let $(Y, d)$ be a proper geodesic metric space with a basepoint $o$. The dead-end depth of a point $x$ in $Y$ is the infimum of a non-negative real number $r \geq 0$ such that there exists a geodesic ray $\gamma$ satisfying $\gamma_\infty = o$ and $\gamma \cap B(x, r) \neq \emptyset$. If the dead-end depth is non-zero, then $g$ is called a dead end (i.e. the geodesic $[o, x]$ could not be further extended). Such elements exist, for instance, in the Cayley graph of $G \times \mathbb{Z}_2$ with respect to a particular generating set. See [14] for a brief discussion and references therein.

The first, straightforward consequence of the extension lemma is the following.

**Proposition 2.22.** Let $o$ be a fixed basepoint. If $G$ acts properly on $Y$ with a contracting element, then the dead-end depth of all points in $G o$ is uniformly finite.

As a corollary, the dead-end depth of an element in $\text{Gr}'(1/6)$-labeled graphical small cancellation group as described in [5,3] is uniformly finite. This result appears to be not recorded in the literature.

It is worth to point out that the conclusion of the extension lemma holds, as long as there exist at least two independent contracting elements. In particular, the lemma applies to the action of $\text{Mod}$ on a curve graph ([53,54]) associated to orientable surfaces with negative Euler number. Consequently, the dead-depth of vertices in curve graph is uniformly finite. This result was proved by Schleimer [73], and by Birman and Menasco [13] with a more precise description. Here our arguments are completely general, without appeal to the construction of curve graphs.

2.6. **A critical gap criterion.** A proper action of $G$ on a metric space $(Y, d)$ naturally induces a proper left-invariant pseudo-metric $d_G$ on $G$ as follows:

\[
\forall g, h \in G : d_G(g, h) = d(go, ho)
\]

for a basepoint $o \in Y$. Using the pseudo-metric $d_G$, we can define the ball set $N(o, n)$, the critical exponent $\omega(\Gamma)$ of a subset $\Gamma$ equivalently. To emphasize the metric, we use the notation $\omega_{d_G}(\Gamma)$ only in this subsection. In other places, the metric should be clear in context.

To present the criterion, one needs to introduce the Poincaré series

\[
\mathcal{P}_{\Gamma, d_G}(s) = \sum_{g \in \Gamma} \exp(-s \cdot d_G(1, g)), \ s \geq 0
\]
which clearly diverges for \( s < \omega_{d_G}(\Gamma) \) and converges for \( s > \omega_{d_G}(\Gamma) \).

Dalbo et al. [29] presented a very useful criterion to differentiate the critical exponents of two Poincaré series, which is the key tool to establish growth-tightness. We formulate it with purpose to exploit the extension map.

Let \( A, B \) be two sets in \( G \). Denote by \( \mathcal{W}(A, B) \) the set of all words over the alphabet set \( A \cup B \) with letters alternating in \( A \) and \( B \). We consider a left-invariant pseudo-metric \( d_G \) on \( G \) (which might not be coming from the pullback via a proper action).

**Lemma 2.23.** Assume that there exists an injective map \( \iota : \mathcal{W}(A) \to \mathcal{W}(A, B) \) such that the evaluation map \( \Phi : \mathcal{W}(A, B) \to G \) is injective on the subset \( \iota(\mathcal{W}(A)) \) as well. Denote \( X := \Phi(\iota(\mathcal{W}(A))) \subset G \). If \( B \) is finite, then \( \mathcal{P}_{A,d_G}(s) \) converges at \( s = \omega_{d_G}(X) \). In particular, we have the critical gap:

\[
\omega_{d_G}(X) > \omega_{d_G}(A)
\]

provided that \( \mathcal{P}_{A,d_G}(s) \) diverges at \( s = \omega_{d_G}(A) \).

**Proof.** Since \( \Phi : \mathcal{W}(A, B) \to G \) is injective, each element in the image \( X \) has a unique alternating product form over \( A \cup B \). For a word \( W = a_1b_1a_2\cdots a_nb_n \in \mathcal{W}(A, B) \), we have

\[
d_G(1,a_1b_1\cdots a_nb_n) \leq \sum_{1 \leq i \leq n} (d_G(1,a_i) + D),
\]

where \( D = \max\{d_G(1,b) : b \in B\} < \infty \) since \( B \) is a finite set. As a consequence, we estimate the Poincaré series \( \mathcal{P}_{X,d_G}(s) \) of \( X \) as follows:

\[
\sum_{g \in X} \exp(-s \cdot d_G(1,g))
\]

\[
\geq \sum_{n=1}^{\infty} \left( \sum_{a_1,\ldots,a_n \in A} \exp(-s \cdot d_G(1,a_1b_1\cdots a_nb_n)) \right)
\]

\[
\geq \sum_{n=1}^{\infty} \left( \sum_{a \in A} \exp(-s \cdot d_G(1,a)) \right)^n \cdot \exp(-sD)^n.
\]

By way of contradiction, assume that \( \mathcal{P}_{A,d_G}(s) \) diverges at \( s = \omega_{d_G}(X) \) so there exists some \( s_0 > \omega_{d_G}(X) \) such that \( \sum_{a \in A} \exp(-s_0 \cdot d_G(1,a)) > 1/D \). Consequently, the above series \( \mathcal{P}_{X,d_G}(s) \) diverges at \( s = s_0 \), so implies that \( \omega_{d_G}(X) > s_0 \). This contradiction concludes the proof of lemma. \( \square \)

**Remark.** From now on, we shall always consider the metric \( d_G \) as the pullback of the metric \( d \) on \( Y \) via the proper action. Hence, the subindex \( d_G \) is omitted for simplicity.

Finally, the following lemma will be frequently used, cf. [68 Lemma 3.6]. Recall that a metric space \( (X,d) \) is \( C \)-separated if \( d(x,x') > C \) for any two \( x \neq x' \in X \).

**Lemma 2.24.** Let \( (Y,d) \) be a proper metric space on which a group \( G \subset \text{Isom}(Y) \) acts properly. For any orbit \( Go \ (o \in Y) \) and \( R > 0 \) there exists a constant \( \theta = \theta(Go,R) > 0 \) with the following property.

For any finite set \( X \) in \( Go \), there exists an \( R \)-separated subset \( Z \subset X \) such that \( \|Z\| > \theta \cdot \|X\| \).

3. Large free sub-semigroups

This section is devoted to the proof of \[ \text{Theorem A} \]

Recall

\[
\limsup_{n \to \infty} \frac{\log \| A(o,n,\Delta) \|}{n} = \omega(G).
\]

We first prove the existence of large free semigroups.

**Lemma 3.1.** For any \( 0 < \omega < \omega(G) \), there exists a free sub-semigroup \( \Gamma \) such that \( \omega(\Gamma) > \omega \).
Proof. Fix $\Delta > 0$. Let $R = R(\Delta) > 0$ given by Lemma 2.19 and $\theta = \theta(R)$ obtained by Lemma 2.24. For given $\omega < \omega(G)$, choose a large number $k_\omega > 4$ and then a constant $\omega \leq \delta < 2\omega(G)$ such that

$$\delta \geq \frac{(k_\omega + \Delta)\omega - 2^{-k_\omega}\theta}{k_\omega} \geq \omega,$$

and at the same time, the following is true:

$$\|A(o, k_\omega, \Delta)\| > \exp(k_\omega \delta).$$

Indeed, when $k_\omega$ is sufficiently large, the fraction in (16) lies between $\omega$ and $\omega(G)$.

By Lemma 2.24 there exists $Z \in A(o, k_\omega, \Delta)$ such that $Zo$ is $R$-separated set and (17)

$$\|Z\| \geq \theta \cdot \exp(k_\omega \delta).$$

Furthermore, the lemma 2.19 gives a subset $A$ of $Z$ with $2^4\|A\| \geq \theta\|Z\|$ and an element $f \in G$ such that

$$\Phi : \mathcal{W}(A) \rightarrow G$$

defined by

$$a_1a_2\cdots a_n \rightarrow a_1fa_2f\cdots a_nf$$

is injective. Setting $S = A \cdot f$, the injectivity of $\Phi$ is amount to saying that the semigroup $\Gamma$ generated by $S$ is free with base $S$.

To complete the proof, it suffices to estimate the critical exponent. Note that the ball $N(o, n \cdot (k_\omega + \Delta))$ contains at least a set of elements of $\Phi(\mathcal{W}_n)$ where $\mathcal{W}_n$ is the set of words of length $n$ in $\mathcal{W}$. By the injectivity of $\Phi$, we have $\|\Phi(\mathcal{W}_n)\| \geq (\|A\|)^n$. So by (17) we obtain:

$$\|N(o, n \cdot (k_\omega + \Delta)) \cap \Gamma\| \geq \|\mathcal{W}_n\| \geq (\|A\|)^n \geq (2^{-4}\theta n) \exp(n \cdot k_\omega \delta).$$

By definition of $\delta$, (16), the critical exponent can be estimated below:

$$\omega(T) \geq \limsup_{n \rightarrow \infty} \frac{\log \|N(o, n \cdot (k_\omega + \Delta)) \cap \Gamma\|}{n \cdot (k_\omega + \Delta)} \geq \frac{2^{-4}\theta + k_\omega \delta}{k_\omega} \geq \omega.$$

The proof is complete.

\[\square\]

Quasi-geodesic contracting tree. The Cayley graph of a semigroup $G$ with respect to a generating set $S$ can be defined in the same way as the case of groups. The vertex set is $G$, and two vertices $g_1, g_2 \in G$ are connected by an oriented edge labeled by $s \in S$ if $g_2 = g_1s$. Consider the orbital map

$$\Psi : \mathcal{G}(G, S) \rightarrow Y$$

which sends vertices $g$ to $go$ and the edges $[1, s]$ to a geodesic $[o, so]$ and other edges by translations. If the semigroup $G$ is freely generated by $S$, then the Cayley graph is a tree. Moreover, the image $\Psi(\mathcal{G}(G, S))$ is a quasi-geodesic rooted tree so that each branch is contracting. This is just a re-interpretation of Lemma 2.19. Indeed, each branch was proved to be a $(D, \tau, L, \Delta)$-admissible path, so it is contracting by Proposition 2.9. And the quasi-geodesic statement follows from Proposition 2.7 for a $(D, \tau, L, \Delta)$-admissible path.

To complete the proof of Theorem A, it suffices to verify the following.

Lemma 3.2. The free semigroup $A := \Gamma_n$ is growth-tight.

Proof. To prove growth-tightness of $A$, we shall apply the criterion 2.23 to $A$ and $B := \{f\}$. Consider the set $\mathcal{W}(A, B)$ of words with letters alternating in $A$ and $B$. Hence, each word in $\mathcal{W}(A, B)$ has the form $a_1f^{e_1}a_2f^{e_2}\cdots a_nf^{e_n}$ for $e_i \in \{1, 2\}$. As in the proof of Lemma 3.1 we obtain by Lemma 2.19 that $\mathcal{W}(A, B)$ is sent into $G$ injectively.

Hence, it suffices to verify that the Poincaré series $\mathcal{P}_A(s)$ of $A$ diverges at $s = \omega(A)$. In turn, we shall prove that $\|S_{n, \Delta} \geq \exp(n \cdot \omega(A))$, where $S_{n, \Delta} := A \cap A \cap (o, n, \Delta)$. Apparently, this implies the divergence of $\mathcal{P}_A(s)$ at $s = \omega(A)$. 


Since each branch of the tree associated to \( \Gamma_n = A \) is contracting with a uniform contraction constant, it is thus a quasi-geodesic path. A similar argument as in the Case 1 of proof of Lemma \[12\] shows that \( \| S_{n+m, \Delta} \| \leq \| S_{n, \Delta} \| \cdot \| S_{m, \Delta} \| \) for some \( \Delta > 0 \). By Fekete’s Lemma, we have \( \| S_{n, \Delta} \| \geq \exp(n \cdot \omega(A)) \), thereby completing the proof of the result. \( \square \)

So all the statements in Theorem A are proved.

4. Growth-tightness theorem

Recall that a subset \( X \) in \( G \) is growth-tight if \( \omega(X) < \omega(G) \). The union of two growth-tight sets is growth-tight. The main result, Theorem C, of this section shall provide a class of growth-tight sets.

**Definition 4.1**. Fix constants \( \epsilon, M > 0 \) and a set \( P \) in \( G \).

1. (Barrier/Barrier-free geodesic) Given \( \epsilon > 0 \) and \( f \in P \), we say that a geodesic \( \gamma \) contains an \((\epsilon, f)\)-barrier if there exists an element \( h \in G \) so that

\[
\max\{d(h \cdot o, \gamma), d(h \cdot f o, \gamma)\} \leq \epsilon.
\]

If no such \( h \in G \) exists so that \((19)\) holds, then \( \gamma \) is called \((\epsilon, f)\)-barrier-free.

2. (Barrier-free element) An element \( g \in G \) is \((\epsilon, P)\)-barrier-free if it is \((\epsilon, f)\)-barrier-free for some \( f \in P \). An obvious fact is that any subsegment of \( \gamma \) is also \((\epsilon, P)\)-barrier-free.

The definition \[13\] of statistically convex-cocompact actions in Introduction replies on the formulation of a concave region. Let us repeat it here for convenience. For constants \( M_1, M_2 \geq 0 \), let \( \mathcal{O}_{M_1, M_2} \) be the set of elements \( g \in G \) such that there exists some geodesic \( \gamma \) between \( B(o, M_1) \) and \( B(go, M_2) \) with the property that the interior of \( \gamma \) lies outside \( N_{M_1}(Go) \).

In applications, since \( \mathcal{O}_{M_1, M_2} \subset \mathcal{O}_{M_1, M_2} \), we can assume that \( M_1 = M_2 \) and henceforth, denote \( \mathcal{O}_M := \mathcal{O}_{M_1, M_2} \) for easy notations.

4.1. Sub-multiplicative inequality. In this subsection, we establish a variant of sub-multiplicative inequality for SCC actions. This idea was first introduced by F. Dal’bo et al. \[29\]. We here adapt their argument in our context.

For a constant \( \Delta > 0 \) and a subset \( P \subset G \), consider the annulus sets as follows

\[
\mathcal{V}_{\epsilon, M, P}(n, \Delta) := \mathcal{V}_{\epsilon, M, P} \cap A(o, n, \Delta),
\]

and

\[
\mathcal{O}_M(n, \Delta) := \mathcal{O}_M \cap A(o, n, \Delta) \cup \{1\}
\]

where \( \mathcal{O}_M(n, \Delta) \) is used in the proof of the following lemma.

**Lemma 4.2.** There exists \( \Delta > 0 \) such that the following holds

\[
\| \mathcal{V}_{\epsilon, M, P}(n + m, \Delta) \| \leq \sum_{1 \leq k \leq n, 1 \leq j \leq m} \| \mathcal{V}_{\epsilon, M, P}(k, \Delta) \| \cdot \mathcal{O}_M(n + m - k - j, 2\Delta) \cdot \| \mathcal{V}_{\epsilon, M, P}(j, \Delta) \|
\]

for any \( n, m \geq 0 \).

**Proof.** By definition, an element \( g \) belongs to \( \mathcal{V}_{\epsilon, M, P} \), if and only if, there exists a geodesic \( \gamma = [x, y] \) for some \( x \in B(o, M) \) and \( y \in B(go, M) \) such that \( \gamma \) is \((\epsilon, P)\)-barrier-free. Note that any subpath of \( \gamma \) is \((\epsilon, P)\)-barrier-free as well.

Set \( \Delta = 4M \). Let \( g \in \mathcal{V}_{\epsilon, M, P}(n + m, \Delta) \), so there exists a geodesic \( \gamma = [x, y] \) with properties stated as above. Since \( |d(o, go) - n - m| \leq \Delta \) and thus \( |d(x, go) - n - m| \leq \Delta + 2M = 6M \), we can write

\[
d(x, go) = m + n + 2\Delta_1,
\]

for some \( |\Delta_1| \leq 3M \). Consider a point \( z \) in \([x, y]\) such that \( d(o, z) = n + \Delta_1 \).
Case 1. Assume that $z \in N_M(Go)$ so there exists $h \in G$ such that $d(z, ho) \leq M$. Thus,

$$\max\{d(o, ho) - n, |d(ho, go) - m|\} \leq 4M \leq \Delta.$$ 

Note that $[x, z]$, and $[z, w]$, as subsegments of $\gamma$ are $(\epsilon, P)$-barrier-free, so we obtain that $h \in V_{\epsilon, M, P(n, \Delta)}$ and $h^{-1}g \in V_{\epsilon, M, P}(m, \Delta)$. Since we included $1 \in \mathcal{O}_M(0, 2\Delta)$ by definition, the element $g$ belongs to the union set in the right-hand of (20).

Case 2. Otherwise, consider the maximal open segment $(z_1, z_2)$ of $[x, y]$ which contains $z$ but lies outside $N_M(Go)$. Hence, there exist $g_1, g_2 \in G$ such that $d(z_i, g_i o) \leq M$ for $i = 1, 2$. By definition, we have $t := g_1^{-1}g_2 \in \mathcal{O}_M$. Similarly as above, we see that

$$g_1 \in V_{\epsilon, P}(k, \Delta), \ g_2^{-1}g \in V_{\epsilon, M, P}(l, \Delta)$$

where $k := d(o, z_1)$ and $l := d(z_2, go)$. On the other hand,

$$d(z_1, z_2) = d(x, y) - d(x, z_1) - d(z_2, y)$$

$$= z_{2M} d(x, y) - d(o, z_1) - d(z_2, go)$$

$$= z_{2M + \Delta} n + m - k - l.$$ 

It follows from $|d(g_1 o, g_2 o) - d(z_1, z_2)| \leq 2M$ that

$$|d(o, to) - n - m + k + l| \leq 2\Delta.$$ 

That is, $t \in \mathcal{O}_M(n + m + k - j, 2\Delta)$. Writing $g = g_1 \cdot t \cdot (g_2^{-1}g)$ completes the proof of (20) in the Case (2). The lemma is proved.

For $\omega > 0$, we define

$$a^{\omega}(n, \Delta) = \exp(-\omega n) \cdot \|V_{\epsilon, M, P}(n, \Delta).$$

Lemma 4.3. Assume that $\omega(V_{\epsilon, M, P}) > \omega(\mathcal{O}_M).$ Then given any $\omega(V_{\epsilon, M, P}) > \omega > \omega(\mathcal{O}_M)$, there exist $\Delta, c_0 > 0$ such that the following holds

$$(21) \quad a^{\omega}(n + m, \Delta) \leq c_0 \left( \sum_{1 \leq k \leq n} a^{\omega}(k, \Delta) \right) \cdot \left( \sum_{1 \leq j \leq m} a^{\omega}(j, \Delta) \right),$$

for any $n, m \geq 0$. Moreover, the Poincaré series $P_{V_{\epsilon, P}}(s)$ diverges at $s = \omega(V_{\epsilon, P})$.

Proof. Note that there exists $c_0 > 0$ such that $\|\mathcal{O}_M(i, 2\Delta) \leq c_0 \exp(i \cdot \omega)$ for any $i \geq 1$. Consequently, a re-arrangement of (20) gives rise to the form of (21) (see [28, Proposition 4.1] for instance). The “moreover” statement follows by [28, Lemma 4.3].

Remark. The above proof of Lemma 4.2 still works for the set $V_{\epsilon, P}(n, \Delta)$ replaced by $A(o, n, \Delta)$: in fact, the proof gets greatly simplified. So the inequality (21) holds also for $a^{\omega}(n, \Delta) := \exp(-\omega n) \cdot \|A(o, n, \Delta)$. See the proof of Theorem 5.3 below.

4.2. The main construction. For given $g \in G$ and $\epsilon > 0$, denote by $V_{\epsilon, M, g}$ the set of $(\epsilon, M, g)$-barrier-free elements in $Go$ from $o$. Denote by $\mathcal{V}(A)$ the free monoid over the alphabet set $A := V_{\epsilon, M, g}$.

To be clear, we fix some constants at the beginning (the reader is encourage to read the proof first and return here until the constant appears).

Setup. Let $\mathcal{F}$ be a contracting system satisfying Convention 2.5

1. We denote by $C > 0$ the contraction constant for the contracting system $\mathcal{F}$. For easy notations, we assume that $C$ satisfies Proposition 2.4 as well.
2. Constants $\tau, D > 0$ in admissible paths are given by Lemma 2.14
3. Let $B > 0$ be the constant by Proposition 2.7 for $(D, \tau)$-admissible paths relative to the contracting system $\mathcal{F}$. 

Choose by Lemma 2.14 a finite set \( F \) such that
\[
d(o, ho) > 2B + C + M
\]
for each \( h \in F \).

**The extension map** \( \Phi : \mathcal{W}(A) \to G \). To each word \( W = w_1w_2\cdots w_n \in \mathcal{W}(A) \), we associate an element \( \Phi(W) \in G \) defined as follows:
\[
\Phi(w_1w_2\cdots w_n) = w_1 \cdot (f_1gh_1) \cdot w_2 \cdot (f_2gh_2) \cdots (f_ngh_n) \cdot w_n
\]
where \( f_i, h_i \in F \) are chosen by the extension lemma 2.14.

As before, consider the path labeled by \( \Phi(W) \) which is a \((D, \tau)\)-admissible path:
\[
\gamma = w_1 \cdot (f_1 \cdot g_1 \cdot h_1) \cdot w_2 \cdot (f_2 \cdot g_2 \cdot h_2) \cdots (f_n \cdot g_n \cdot h_n) \cdot w_n.
\]
Note that the geodesics \( \gamma \) are all labeled by the same element \( g \).

**Lemma 4.4.** There exist constants \( \epsilon = \epsilon(\mathbb{F}, M), R = R(\mathbb{F}, M) > 0 \) with the following property.

Let \( W = w_1w_2\cdots w_n, W' = w'_1w'_2\cdots w'_m \in \mathcal{W}(A) \) such that \( \Phi(W) = \Phi(W') \). Then \( d(w_1o, w'_1o) \leq R \).

**Proof.** Let us look at their labeled paths:
\[
\gamma = w_1 \cdot (f_1 \cdot g_1 \cdot h_1) \cdot w_2 \cdot (f_2 \cdot g_2 \cdot h_2) \cdots (f_n \cdot g_n \cdot h_n) \cdot w_n,
\]
and
\[
\gamma' = w'_1 \cdot (f'_1 \cdot g'_1 \cdot h'_1) \cdot w'_2 \cdot (f'_2 \cdot g'_2 \cdot h'_2) \cdots (f'_m \cdot g'_m \cdot h'_m) \cdot w'_n.
\]
Fix a geodesic \( \alpha \) between \( o \) and \( \Phi(W)o \), where \( \Phi(W) = \Phi(W') \). It is possible that \( m \neq n \).

First, set
\[
R_0 := 2\max\{d(o, f o) : f \in F\} + d(o, go)
\]
which implies
\[
d(o, w_1fgh_1o) \leq d(o, w_1o) + R_0.
\]
Without loss of generality, assume that \( d(o, w'_1o) \geq d(o, w_1o) \).

Set \( R := R_0 + 3(2C + B) + M \). Our goal is to prove that
\[
d(o, w'_1o) \leq d(o, w_1o) + R,
\]
which clearly concludes the proof of the lemma.

By way of contradiction, let us assume that \( [24] \) fails, and go to prove that \( w'_1 \) contains an \((\epsilon, g)\)-barrier. To this end, consider a geodesic \( \beta = [x, y] \) where \( x \in B(o, M) \) and \( y \in B(w'_1o, M) \).

Since \( \gamma \) is \((D, \tau)\)-admissible, by Proposition 2.7 there exists \( B > 0 \) such that
\[
\max\{d_X(\beta_1), d_X(\beta_2)\} \leq B
\]
where \( \beta_1 := [o, (h_1)_-] \), and \( \beta_2 := [(h_1)_+, \gamma_+] \). See Figure 2.

![Figure 2. Proof of Lemma 4.4](image-url)
Denote by \( X \) the contracting set associated to \( h_1 \). Let \( z \) be the corresponding exit point of \( \alpha \) in \( N_C(X) \), which exists by Proposition 2.7. By contracting property and (25) we obtain
\[
d(z, (h_1)_{+}) \leq d^X_\alpha(\beta_2) + d^X_N([\alpha_+, z]_\alpha) + d(z, X) \leq 2C + B.
\]
Note that \((h_1)_{+} = w_1 f_1 g h_1 o\). So by (23) and (26), we have
\[
d(o, z) \leq d(o, w_1 o) + R_0 + 2C + B.
\]

We claim that
\[\beta \cap N_C(X) \neq \emptyset.\]

**Proof of the Claim.** We consider the contracting set \( Y \) associated to \( h_1' \) and the entry point \( y' \) of \( \alpha \) in \( N_C(Y) \). A similar estimate as (26) shows
\[
d((w_1')_{+}, y') \leq 2C + B.
\]
Noticing that \((w_1')_{+} = w'_1 o\) and [24] was assumed to be false, we get the following from (27) and (24):
\[
d(o, y') - d(o, z) \geq d(o, w'_1 o) - d(o, w_1 o) - R_0 - 2(C + B) > 2C + B + M > 0,
\]
implying \( y' \in [z, \alpha_+]_\alpha \). Since \( d(y', y) \leq d(y, (w_1')_{+}) + d((w_1')_{+}, y) \leq 2C + B + M \), we have \( d(y', z) > d(y', y) \). This shows that the geodesic \([y, y']\) is disjoint with \( N_C(X) \); indeed, since \( z \) is the exit point of \( \alpha \) in \( N_C(X) \), we would obtain \( d(y', z) \leq d(y', y) \), a contradiction. The contracting property thus implies
\[
d^X_\alpha([y, y']) \leq C.
\]
We are now ready to prove the claim: \( \beta \cap N_C(X) \neq \emptyset \). Indeed, if it is false, then it follows \( d^X_\beta(\beta) \leq C \) by contracting property. Moreover, since \( d(o, x) \leq M \), we have
\[
d^X_\beta([o, x]) \leq M + C,
\]
by Proposition 24. We now estimate by projection:
\[
\ell(h_1) \leq d^X_N(\beta_1) + d^X_\alpha([o, x]) + d^X_\alpha(\beta) + d^X_\alpha([y, y']) + d^X_N([\alpha_+, \alpha_+]_\alpha) + d^X_N(\beta_2)
\]
\[
\leq 2B + 3C + M,
\]
where (25) and (28) are used. This inequality contradicts to the choice of \( h_1 \in F \) satisfying (22). So the claim is proved. \( \square \)

Let us return to the proof of the lemma. Consider the entry point \( w \) of \( \beta \) in \( N_C(X) \), which exists by the above claim. So
\[
d((g_1)_{+}, w) \leq d^X_\alpha(\beta_1) + d^X_\alpha([o, x]) + d^X_\alpha([x, w]_\beta) + d(w, X)
\]
\[
\leq B + M + 3C.
\]
Similarly, we proceed the above analysis for the contracting set associated to \( f_1 \) and we can prove that \( d((g_1)_{+}, \beta) \leq B + M + 3C \).

By setting
\[\epsilon = B + M + 3C,\]
we have proved that the two endpoints of \( g_1 \) lie within at most an \( \epsilon \)-distance to \( \beta \). By definition of barriers, we have that \( \beta \) contains an \((\epsilon, g)\)-barrier. This contradicts to the choice of \( w'_1 \in A \), where \( A = V_{\epsilon, M, g} \).

In conclusion, we have showed that (24) is true: \( d(w_1 o, w'_1 o) \leq R \), completing the proof of lemma. \( \square \)
4.3. Proof of Theorem C. By definition of a SCC action, there exist a constant \( M > 0 \) such that \( \omega(O_M) < \omega(G) \).

For the set \( A := V_{r,M,g} \), there exist \( \epsilon = \epsilon(\mathbb{F}, M) \) and \( R = R(\mathbb{F}, M) \) satisfying the conclusion of Lemma 4.3.

Without loss of generality, assume that \( \omega(A) > \omega(O_M) \) and we shall prove that \( \omega(A) < \omega(G) \).

Let \( B \) be a maximal \( R \)-separated subset in \( A \) so that

- for any distinct \( a, a' \in B \), \( d(aa,a'a') > R \), and
- for any \( x \in V_{r,M,g} \), there exists \( a \in B \) such that \( d(xa,ao) < R \).

Taking Lemma 2.10(4) into account, the map \( \Phi : 0(B) \rightarrow G \) defined before Lemma 4.4 is injective.

On the other hand, an elementary argument shows that \( P_B(s) \simeq P_A(s) \), whenever they are finite, and so

\[
\omega(B) = \omega(A).
\]

By Lemma 4.3 \( P_A(s) \) and thus \( P_B(s) \) are divergent at \( s = \omega(A) \).

Consider, \( X = \Phi(0(B)) \), the image of \( 0(B) \) under the map \( \Phi \) in \( G \). The criterion 2.23 implies \( \omega(X) > \omega(B) \) and so \( \omega(G) \geq \omega(X) > \omega(A) \). Thus, the barrier set \( V_{r,M,g} \) is growth-tight, thereby concluding the proof of theorem.

For a proper action, the above proof actually shows the following general fact. We consider a weaker notion of growth-negligible subsets \( X \) in \( G \) are proven to be useful in a further study:

\[
\frac{\#(N(o,n) \cap X)}{\exp(\omega(G)n)} \rightarrow 0
\]
as \( n \rightarrow \infty \).

Corollary 4.5. Suppose that a group \( G \) acts properly on a geodesic space \((Y,d)\). Then, under the same quantifiers as Theorem C, \( V_{r,M,g} \) is a growth-negligible set.

Proof. Indeed, the assumption of SCC actions is used by Lemma 4.3 to guarantee the divergence of \( P_A(s) \). Except this place, the proper action suffices to prove Lemma 4.4. So the criterion 2.23 shows that \( V_{r,M,g} \) is a growth-negligible set. \( \square \)

4.4. Some applications. In this subsection, we collect some sample applications of Theorem C by demonstrating some ways to embed interesting subsets into a barrier-free set. More applications shall be presented in the paper [76].

Firstly, Theorem C generalizes the growth-tightness for groups introduced by Grigorchuk and de la Harpe in [44].

Corollary 4.6. Under the same assumption as Theorem C, we have

\[
\omega(\tilde{G}) < \omega(G),
\]

for any quotient \( \tilde{G} \) of \( G \) by an infinite normal subgroup \( N \). Here \( \omega(\tilde{G}) \) is computed with respect to the proper action of \( G \) on \( Y/N \).

Proof. Indeed, choose a shortest representative \( h \) in \( G \) for each element \( \tilde{h} = Nh \) in a quotient group: \( d(o,ho) = d(o,Nh \cdot o) \). It is clear that the set \( \Gamma \) of these representatives has growth rate (computed with metric \( d \)) equal to \( \omega(\tilde{G}) \). It suffices to see that \( \Gamma \) is contained in a set of \((\epsilon,M,g)\)-barrier-free elements for a fixed “long” element \( g \) in the kernel \( N \). We can first find an element \( \bar{g} \) in \( N \) such that \( d(o,go) > 4\epsilon + 1 \), since \( N \) is infinite.

We claim that \( \Gamma \subset V_{r,M,g} \). If not, then the geodesic \( \gamma = [o,ho] \) contains an \((\epsilon,g)\)-barrier \( t \in G: d(t \cdot o,\gamma), d(t \cdot ho,\gamma) \leq \epsilon \). Since \( N \) is normal, we have \( t\bar{g} = \bar{g}t \) for some \( \bar{g} \in N \). So

\[
d(o, \bar{g}^{-1}h \cdot o) \leq d(o, to) + d(to, \bar{g}^{-1}ho) \\
\leq d(o, to) + d(tg \cdot o, ho) \\
\text{(triangle inequality)} \leq d(o, ho) - d(o, go) + 2d(t \cdot o, \gamma) + 2d(t \cdot ho, \gamma) \\
\leq d(o, ho) - d(o, go) + 4\epsilon < d(o, ho),
\]

where the last inequality follows from the growth rate of \( \omega(\tilde{G}) \). This contradicts the assumption that \( \Gamma \) is an \((\epsilon,M,g)\)-barrier-free set.
this contradicts to the minimal choice of $h$. So the claim is proved and the corollary follows from Theorem C.

Notice that the set $O_M$ is barrier-free. Informally, Theorem C can be interpreted as follows: $O_M$ is growth-tight if and only if any barrier-free set is growth-tight. On the other hand, we could produce a barrier-free set without growth-tightness in a geometrically finite group, when its parabolic gap property fails. In this sense, Theorem C is best possible.

Proposition 4.7. There exists a divergent group action of $G$ on a geodesic space $(Y,d)$ with a contracting element such that for any $\epsilon,M>0$ we have

$$\omega(V_{\epsilon,M,g}) = \omega(G)$$

for some $g \in G$.

Remark. These groups were constructed by Peigné in [60]: an exotic Schottky group $G$ acts on Cartan–Hadamard manifolds such that $G$ is of divergent type and has no parabolic gap property. A convergent-type case without the parabolic gap property was constructed earlier by Dal’bo et al. [28].

Sketch of the proof. Let $G$ be an exotic Schottky group constructed by Peigné in [60]. It admits a geometrically finite action of divergent type on a Cartan–Hadamard manifold and $G$ has no parabolic gap property: there exists a maximal parabolic subgroup $P$ such that $\omega(P) = \omega(G)$.

We draw on a result from [77, Proposition 1.5]. For $M,\epsilon>0$ fixed, we can invoke the Dehn filling in [27] to kill a “long” hyperbolic element so that $P$ is “almostly” preserved in the quotient $G$: it projects to be a maximal parabolic subgroup $\tilde{P}$ with $\omega(P) = \omega(\tilde{P})$. Hence, $\omega(G) = \omega(G)$.

Lift all the elements from $\tilde{G}$ to $G$ to their shortest representatives in $G$. By the same argument as in Corollary 4.6 they are contained in $V_{\epsilon,M,g}$ so that $\omega(V_{\epsilon,M,g}) = \omega(G)$. This concludes the proof.

Recall that a subset $X$ in $Y$ is called weakly $M$-quasi-convex for a constant $M>0$ if for any two points $x,y$ in $X$ there exists a geodesic $\gamma$ between $x$ and $y$ such that $\gamma \subset N_M(X)$.

Theorem 4.8. Suppose $G$ admits a SCC action on $(Y,d)$ with a contracting element. Then every infinite index weakly quasi-convex subgroup $\Gamma$ of $G$ has the property $\omega(\Gamma) < \omega(G)$.

Proof. Let $\epsilon,M$ be the constants in Theorem C and $M$ is also the quasi-convexity constant of $H$. The idea of proof is to find an element $g \in G$ such that every element $h \in H$ is $(\epsilon,M,g)$-barrier-free. The existence of such $g$ is guaranteed by the following claim.

We claim that for any finite set $F$, the set $G \setminus F \cdot H \cdot F$ is infinite. Suppose to the contrary that $G \setminus F \cdot H \cdot F$ is finite for some finite $F$. By enlarging $F$ by a finite set, we can assume that $G = F \cdot H \cdot F$ so $G \subset (\cup_{f \in F} fHf^{-1})F$. This contradicts to a result of Neumann [55] states that a group $G$ cannot be a finite union of right cosets of infinite index subgroups. Our claim thus follows.

Let $F$ be the set of elements $f \in G$ such that $d(fo,o) \leq M + \epsilon$, where $M$ is the quasi-convexity constant of $H$. Since $G \setminus F \cdot H \cdot F$ is infinite, let us choose one element $g \notin F \cdot H \cdot F$. In the remainder, we prove $H \subset V_{\epsilon,M,g}$.

Indeed, suppose to contrary that there exist some $h$ in $H$ which is not $(\epsilon,M,g)$-barrier-free. By weak quasi-convexity, there exists a geodesic $\gamma = [o,ho]$ such that $\gamma \subset N_M(Ho)$. By definition of barriers, any geodesic between $B_M(o)$ and $B_M(ho)$ contains an $(\epsilon,g)$-barrier, so for the geodesic $\gamma \subset N_M(Ho)$, there exist $b \in G$ and $h_1,h_2 \in H$ such that $d(b,o,h_1o),d(b,o,h_2o) \leq \epsilon + M$. By definition of $F$, we have $b^{-1}h_1b^{-1}h_2 \in F$. Note that $g \in b^{-1}h_2F = b^{-1}h_1h_1^{-1}h_2F$ and so $g \in F \cdot H \cdot F$ gives a contradiction. The subgroup $H$ is therefore contained in a growth-tight $V_{\epsilon,M,g}$, which allows Theorem C to conclude the proof.

Remark. In the proof, we can choose $g$ to be a contracting element by Lemma 2.21.
We first give a corollary in mapping class groups.

**Proposition 4.9.** Any convex-cocompact subgroup $\Gamma$ in $G \in \text{Mod}$ is growth-tight: $\omega(\Gamma) < \omega(G)$.

**Proof.** By [35, Proposition 3.1], a convex-cocompact subgroup $\Gamma$ in $G \in \text{Mod}$ is purely pseudo-Anosov and thus is of infinite index: otherwise any element in $G$ would have some power being pseudo-Anosov. Any orbit of $\Gamma$ on Teichmüller space is weakly quasi-convex by [35, Theorem 1.1]. Since the action of $G$ on the Teichmüller space is statistically convex-cocompact [2], the result thus follows from Theorem 4.8. 

Here is another corollary for cubulated groups.

**Proposition 4.10.** Suppose that a group $G$ acts properly and cocompactly on a CAT(0) cube complex $Y$ so that $Y$ does not decompose as a product. Then any weakly quasi-convex subgroup of infinite index in $G$ is growth-tight. In particular, any cubically convex subgroup is growth-tight, if it is of infinite index.

**Proof.** In [20, Theorem A], Caprace and Sageev showed that if $Y$ does not decompose as a product, then $G$ contains a rank-1 element which is contracting in our sense. The conclusion therefore follows from Theorem 4.8.

## 5. Purely exponential growth

In this section, we first give a proof of Theorem B and then furnish more details on purely exponential growth of the class of groups listed in Theorem 1.8.

### 5.1. Proof of Theorem B

We remark that the elementary approach presented here is greatly inspired by the notes of Peigné [65]. We first recall an elementary lemma.

**Lemma 5.1.** [65, Fait 1.0.4] Given $k > 1$, let $a_n$ be a sequence of positive real numbers such that $a_n a_m \leq \sum_{|j| \leq k} a_{n+m-j}$. Then the following limit

$$\omega := \lim_{n \to \infty} \frac{\log(a_1 + a_2 + \cdots + a_n)}{n}$$

exists and $a_n < \exp(n \cdot \omega)$ for $n \geq 1$.

We now prove the upper bound for any proper action.

**Proposition 5.2 (Upper bound).** Assume that $G$ acts properly on a geodesic metric space $(Y, d)$ with a contracting element. Then the following hold for given $\Delta > 0$,

1. $\omega(G) = \lim_{n \to \infty} \frac{\log \| N(o,n,\Delta) \|}{n}$
2. $\| A(o,n,\Delta) \| < \Delta \exp(\omega(G)n)$

for any $n \geq 1$.

**Proof.** Let $R = R(\epsilon, \Delta)$ be a constant given by Lemma 2.18. By Lemma 2.24, there exist a constant $\theta = \theta(R) > 0$ and two subsets $B_1 \subseteq A(n, \Delta)$ and $B_2 \subseteq A(m, \Delta)$ such that $B_1, o$ are both $R$-separated and

$$\theta \cdot \| B_1 \| \geq \| A(o,n,\Delta) \|, \quad \theta \cdot \| B_2 \| \geq \| A(o,m,\Delta) \|.$$

The proof proceeds by establishing the following variant of a super-multiplicative inequality: there exists an integer $k > 0$ such that

$$\theta \cdot \| A(o,n,\Delta) \| \cdot \| A(o,m,\Delta) \| \leq \theta \sum_{|j| \leq k} \| A(o,n+m+j,\Delta) \|.$$
For this purpose, we now define a map $\Phi : B_1 \times B_2 \to G$. Given $W = (b_1, b_2)$, define $\Phi(W) = b_1 f b_2$ for some $f \in F$ provided by the extension lemma 2.14. Thus by Lemma 2.18 the map $\Phi$ is injective. Moreover, any geodesic $[o, b_1 f b_2 o]$ $\epsilon_0$ -fellow travels the path labeled by $b_1 f b_2$, so 

$$d(o, b_1 o) + d(o, b_2 o) + L \geq d(o, b_1 f b_2 o) \geq d(o, b_1 o) + d(o, b_2 o) - 2\epsilon_0$$

where $L = \max\{d(o, f o) : f \in F\} < \infty$. Noting that $B_1 \subset A(n, \Delta)$ and $B_2 \subset A(m, \Delta)$, we have $|d(o, b_1 o) - n| \leq \Delta$ and $|d(o, b_2 o) - m| \leq \Delta$. Setting $k := \Delta + L + 2\epsilon_0$, it then follows 

$$b_1 f b_2 \in A(o, n + m, \Delta + k).$$

Since $\Phi$ is injective, we obtain 

$$\|B_1 \cdot B_2 \| \leq \sum_{|j| \leq k} \|A(o, n + m, \Delta + j)\|,$$

with [29], which establishes the inequality (30). Denoting $a_n = \theta \|A(o, n, \Delta)\|$, the proposition then follows from Lemma 5.1.

We now prove the last claim of Theorem B.

**Theorem 5.3 (Purely exponential growth).** Assume that $G$ admits a SCC action on $Y$ with a contracting element. Then there exists $\Delta > 0$ such that 

$$\|A(o, n, \Delta)\| \asymp \exp(\omega(G)n)$$

for $n \geq 1$.

**Proof.** By Proposition 5.2, it suffices to prove the lower bound. For $\omega > 0$, we define: 

$$b^{\omega}(n, \Delta) = \exp(-\omega n) \cdot \|A(o, n, \Delta)\|.$$ 

The following claim follows by a simpler argument as in the proof of Lemma 4.2.

**Claim.** Given $\omega(G) > \omega > \omega(O_R)$, there exist $\Delta, c_0 > 0$ such that the following holds

$$b^{\omega}(n + m, \Delta) \leq c_0 \left( \sum_{1 \leq k \leq n} b^{\omega}(k, \Delta) \right) \left( \sum_{1 \leq j \leq m} b^{\omega}(j, \Delta) \right),$$

for any $n, m \geq 0$.

Denote $V_k := \sum_{1 \leq i \leq k} b^{\omega}(i, \Delta)$ and $\tilde{V}_n := c_0 \cdot \sum_{1 \leq i \leq n} V_k$. By [28, Lemma 4.3], it is proved that 

$$\tilde{V}_{n+m} \leq \tilde{V}_n \tilde{V}_m$$

for $n, m \geq 1$. By Feketa’s Lemma, it follows that

$$\limsup_{n \to \infty} \frac{\log \tilde{V}_n}{n} = \inf_{n \geq 1} \left\{ \frac{\log \tilde{V}_n}{n} \right\} = L$$

for some $L \in \mathbb{R} \cup \{-\infty\}$. Take into account the elementary fact

$$\limsup_{n \to \infty} \frac{\log \tilde{V}_n}{n} = \limsup_{n \to \infty} \frac{\log V_n}{n} = \limsup_{n \to \infty} \frac{\log b^{\omega}(n, \Delta)}{n},$$

which implies that $L = \omega(G) - \omega > 0$. Hence, $\tilde{V}_n \geq \exp(Ln)$. On the other hand, by Proposition 5.2, it follows that $b^{\omega}(n, \Delta) < \exp(Ln)$. From definition of $V_n$ and $\tilde{V}_n$, it implies that $V_n < \exp(Ln)$ and then $\tilde{V}_n < \exp(Ln)$ for $n \geq 1$. Since $\tilde{V}_n \geq \exp(Ln)$, an elementary argument produces a constant $K > 0$ such that

$$\sum_{0 \leq k < K} V_{n+k} = \tilde{V}_{n+K} - \tilde{V}_n > \exp(Ln),$$

which yields $V_n \geq \exp(Ln)$, due to $V_n \leq V_{n+1}$. We repeat the argument for $V_{n-K}$ by making use of $V_n < \exp(Ln)$, and show that $b^{\omega}(n, \Delta) > \exp(Ln)$. By the fact $L = \omega(G) - \omega$, we have

$$\|A(o, n, \Delta)\| \asymp \exp(n \cdot \omega(G)),$$

completing the proof. \qed
Let us record the following useful corollary.

**Corollary 5.4.** Any SCC action with a contracting element is of divergent type: $P_G(s)$ diverges at $s = \omega(G)$. 

5.2. **Proof of Theorem 1.8.** We explain in details the proof of each assertion in Theorem 1.8.

The class of graphical small cancellation groups was introduced by Gromov [47] for building exotic groups (the “Gromov monster”). In [4], Arzhantseva et al. have made a careful study of contracting phenomena in a $Gr'(1/6)$-labeled graphical small cancellation group $G$. Such a group $G$ is given by a presentation obtained from a labeled graph $\mathcal{G}$ such that under a certain small cancellation hypothesis, the graph $\mathcal{G}$ embeds into the Cayley graph of $G$. It is proved that if $\mathcal{G}$ has only finitely many components labeled by a finite set $S$, then the action of $G$ on Cayley graphs contains a contracting element. Therefore, the following holds by Theorem B:

**Theorem 5.5.** A $Gr'(1/6)$-labeled graphical small cancellation group $G$ with finite components labeled by a finite set $S$ has purely exponential growth for the corresponding action.

The class of CAT(0) groups with rank-1 elements admits a geometric (and thus SCC) action with a contracting element. In particular, consider the class of right-angled Artin group (RAAG) whose presentation is obtained from a finite simplicial graph $\Gamma$ as follows:

\[(31) \quad G = \langle V(\Gamma) \mid v_1 v_2 = v_2 v_1 \iff (v_1, v_2) \in E(\Gamma) \rangle\]

See [52] for a reference on RAAGs. It is known that an RAAG acts properly and cocompactly on a non-positively curved cube complex called the Salvetti complex.

It is known that the defining graph is a join if and only if the RAAG is a direct product of non-trivial groups. In [8, Theorem 5.2], Behrstock and Charney proved that any subgroup of an RAAG $G$ that is not conjugated into a join subgroup (i.e., obtained from a join subgraph) contains a contracting element. Therefore, we obtain the following:

**Theorem 5.6 (RAAG).** Right-angled Artin groups that are not direct products have purely exponential growth for the action on their Salvetti complex.

The class of right-angled Coxeter groups (RACGs) can be defined as in (31) with additional relations $v^2 = 1$ for each $v \in V(\Gamma)$. An RACG also acts properly and cocompactly on a CAT(0) cube complex called the Davis complex (which is equal to the Salvetti complex of the corresponding RAAG). In [10, Proposition 2.11], Behrstock et al. characterized an RACG $G$ of linear divergence as virtually a direct product of groups. By [21, Theorem 2.14], Charney and Sultan proved that the existence of rank-1 elements is equivalent to a superlinear divergence. Hence, we have the following.

**Theorem 5.7 (RACG).** If a right-angled Coxeter group is not virtually a direct product of non-trivial groups, then it has purely exponential growth for the action on the Davis complex.

6. **Constructing SCC actions**

In this section, we shall present a simple method to produce a statistically convex-cocompact action. The main result is constructing examples in $\mathbb{Mod}$ of irreducible subgroups with a SCC action on Teichmüller space but which are not convex-cocompact.

6.1. **Independence of basepoints for SCC actions.** In this subsection we show that SCC actions with a contracting element do not depend on the choice of basepoints. The ingredient of the proof is the growth-tightness [Theorem C]. It is not clear whether the assumption of the existence of a contracting element is removable.

**Lemma 6.1.** If $\omega(O_M) < \omega(G)$ for some $M > 0$, then $\omega(O_{M_1, M_2}) < \omega(G)$ for any $M_2 \geq M_1 \gg M$. 

Proof. Fix a contracting element \( f \in G \). By Theorem C, the set \( \mathcal{V}_{e,M,f} \) is growth-tight for any \( m > 0 \). The proof consists in verifying \( \mathcal{O}_{M_1,M_2} \subset \mathcal{V}_{e,M,f} \) for appropriate constants \( M_1, M_2 \).

Let \( g \in \mathcal{O}_{M_1,M_2} \) so there exists a geodesic \( \gamma \) between \( B(o,M_2) \) and \( B(go,M_2) \) such that the interior of \( \gamma \) lies outside \( N_{M_1}(Go) \). Assume, to the contrary, that \( g \notin \mathcal{V}_{e,M,f} \), then any geodesic \( \beta \) between \( B(o,M) \) and \( B(go,M) \) has an \((\epsilon,f^m)\)-barrier so there exists \( b \in G \) such that

\[
\text{diam}(bAx(f) \cap \beta) \geq d(o,f^m o) - 2\epsilon
\]

where the right side tends to \( \infty \) as \( m \to \infty \). Denote by \( C \) the contraction constant of \( Ax(f) \). Noticing that \( d(\gamma-,\beta_+), d(\gamma_+\beta_-) \leq M + M_2 \), a contracting argument implies that if \( d(o,f^m o) \) is sufficiently large comparable with \( M + M_2 \), then \( \gamma \cap N_C(bAx(f)) \neq \emptyset \). By setting \( M_1 > C \), we obtain \( \gamma \cap N_{M_1}(Go) \neq \emptyset \), a contradiction with the assumption of \( \hat{\gamma} \) outside \( N_{M_1}(Go) \). Hence, it is proved that \( \mathcal{O}_{M_1,M_2} \subset \mathcal{V}_{e,M,f} \) for large \( m \); the proof is done.

Lemma 6.2. The definition of a SCC action with a contracting element is independent of the choice of basepoints.

Proof. Consider different basepoints \( o, o' \in Y \). We choose two constants \( M_1 \leq M_2 \) by Lemma 6.1 such that \( M_2 > M_1 + d(o,o') \) and \( \omega(\mathcal{O}_{M_1,M_2}) < \omega(G) \). Define \( M'_1 = M_1 + d(o,o') \) and \( M'_2 = M_2 - d(o,o') \). We claim that \( \mathcal{O}'_{M'_1,M'_2} \subset \mathcal{O}_{M_1,M_2} \), where \( \mathcal{O}'_{M'_1,M'_2} \) is the concave region defined using the basepoint \( o' \).

Indeed, let \( g \in \mathcal{O}'_{M'_1,M'_2} \) be some geodesic \([x',y']\) between \( B(o',M'_2) \) and \( B(go',M'_2) \) has the interior disjoint with \( N_{M'_1}(Go') \). Since \( M'_2 = M_1 + d(o,o') \), we have \( N_{M'_1}(Go') \subset N_{M'_1}(Go) \) so the interior of \([x',y']\) lies outside \( N_{M'_1}(Go) \) as well. By the choice of \( M'_2 = M_2 - d(o,o') \), the geodesic \([x',y']\) lies between \( B(o,M_2) \) and \( B(go,M_2) \). Thus, \( g \in \mathcal{O}_{M_1,M_2} \) so the claim thus follows. Thus, \( \omega(\mathcal{O}_{M'}) < \omega(G) \) by Lemma 6.1, concluding the proof of lemma.

6.2. Free product combination. With the critical gap criterion \([2.23]\), the following combination result is not surprising.

Proposition 6.3. Assume that \( G \) acts properly on a geodesic metric space \((Y,d)\). Consider two subgroups \( H,K \) such that \( H \) is a residually finite, contracting subgroup and \( d_{H,H}(Ko) < \infty \) for a basepoint \( o \in Y \).

If either \( K \) is residually finite, or \( \{kh \cdot o : k \in K\} \) has bounded intersection, then there exist finite index subgroups \( \hat{H}, \hat{K} \) of \( H,K \) respectively such that the subgroup \( \Gamma \) generated by \( \hat{H}, \hat{K} \) is isomorphic to \( \hat{H} \ast \hat{K} \).

In addition, if \( \omega(H) \geq \omega(K) \) or \( \omega(H) = \omega(K) = 0 \), then \( \Gamma \) admits a SCC action on \( Y \).

Proof. By hypothesis, \( \text{diam}(\pi_{H,o}(Ko)) \leq \tau \) for some constant \( \tau > 0 \). Let \( C \) be the contraction constant of the subset \( Ho \). Choose a large constant \( D > \max\{D(\tau),c(\tau)\} \) where \( D(\tau),c(\tau) \) are constants given by Proposition \( [2.7] \). Since \( H \) is residually finite and the action is proper, there exists a finite index subgroup \( \hat{H} \) such that \( \hat{H} \cap K = \{1\} \) and \( d(o,ho) > D \) for any \( 1 \neq h \in \hat{H} \). In the second case of the assumptions on \( K \), we just let \( \hat{K} = K \), otherwise, since \( K \) is residually finite, we choose a finite index subgroup \( \hat{K} \) such that \( d(o,k_o) > D \) for any \( 1 \neq k \in \hat{K} \). We are now going to prove that \( (\hat{H},\hat{K}) = \hat{H} \ast \hat{K} \).

Indeed, it suffices to establish that each non-trivial word \( W \) with letters alternating in \( \hat{H} \) and \( \hat{K} \) labels a \((D,\tau)\)-admissible path so a \( c(\tau)\)-quasi-geodesic. To be precise, write \( W = h_1h_1\cdots h_i\cdots h_nk_n \) where \( h_i \in \hat{H}, k_i \in \hat{K} \) and \( h_1, k_n \) may be trivial. Let \( \gamma \) be the path labeled by \( W \) for which the system of contracting sets is given by \( Ho,k_1Ho,\cdots,h_1\cdots k_{n-1}Ho \). Note first that the conditions \( [LL1] \) and \( [BP] \) are clear by the choice of \( \hat{H} \) as above. Two possibilities in the condition \( [LL2] \) correspond to the two facts that either \( \mathcal{K} = \{kh \cdot o : k \in K\} \) has bounded intersection, or \( d(o,k_o) > D \) for \( k \in \hat{K} \). So \( \gamma \) is a \((D,\tau)\)-admissible path of length at least \( D > c(\tau) \) and, since it is a \( c(\tau)\)-quasi-geodesic, we obtain by computation that \( h_1h_1\cdots h_i\cdots h_nk_n \neq 1 \in (\hat{H},\hat{K}) \). This proves that \( (\hat{H},\hat{K}) = \hat{H} \ast \hat{K} \).
For sufficiently large $D$, the goal is to prove that the concave region $\mathcal{O}_C$ belongs to $\hat{K}$. Consider an element $g = h_1 h_2 \cdots h_n h_{n+1} \in \Gamma \setminus K$, which labels an admissible path $\gamma$. Since $g \in \Gamma \setminus K$, there must exist a contracting set $X$ from $\mathcal{X}$ associated to a subpath $p$ labeling an element $h_i$ such that, for a constant $B = B(\tau)$ given by Proposition 2.7, we have $d_X^\gamma(\gamma_1 \cup \gamma_2) \leq 2B$ where $\gamma_1 = [\gamma, -\gamma_2], \gamma_2 = [p, \gamma_2], \gamma_2$ are subpaths of $\gamma$ before and after $X$.

Let $\alpha$ be a geodesic between $B(o, C)$ and $B(g, C)$, so $d(\gamma, \alpha) \leq C$. By Proposition 2.4.4, we have $d_X^\gamma(\gamma_1 \cup \gamma_2) \leq 2B$. Choose further $D > 7C + 2B$, we see that $N_C(X) \cap \alpha \neq \emptyset$. Indeed, if $N_C(X) \cap \alpha = \emptyset$ so $d_X^\gamma(\alpha) \leq C$, we obtain by a projection argument that

$$
\ell(p) \leq d(p, X) + d_X^\gamma(\gamma_1 \cup \gamma_2) + d(p, X) + d_X^\gamma(\gamma, \alpha) + d_X^\gamma(\gamma, \alpha) \\
\leq C + 2B + C + 2C + C + 2C \leq 7C + 2B,
$$

which is a contradiction to the choice of $D > 7C + 2B$ for $\ell(p) \geq D$. Thus, $N_C(X) \cap \alpha \neq \emptyset$ is proved. Since $X$ is a translate of $Ho$ so $X \subset Go$, it follows that $\alpha \cap N_C(Go) \neq \emptyset : g$ does not belong to $\mathcal{O}_C$. Hence, $\mathcal{O}_C \subset \hat{K}$ is proved.

We next show that the action is SCC, provided that $\omega(H) > \omega(K)$ or $\omega(H) = \omega(K) = 0$. For a contracting subgroup, any orbit is quasi-convex by Proposition 2.4.2. So $\hat{H}$ acts by a SCC action: its Poincaré series $\mathcal{P}_p(s)$ diverges at $s = \omega(\hat{H})$ by Corollary 5.4. Consider the case $\omega(H) \geq \omega(K)$. Since the growth rate remains the same for finite index subgroups so $\omega(H) = \omega(\hat{H})$ and $\omega(H) = \omega(\hat{K})$, it follows by Lemma 2.23 that $\omega(\Gamma) \geq \omega(H) > \omega(K)$. Hence, $\Gamma$ admits a SCC action. For the case $\omega(H) = \omega(K) = 0$, since $\Gamma$ contains non-abelian free subgroups so $\omega(\Gamma) > 0$, the action of $\Gamma$ is SCC as well. The proof is thus complete.

6.3. Mapping class groups. In mapping class groups, we use a boundary separation argument to fulfill the criterion of Proposition 6.3. This idea is well-known in the setting of (relatively) hyperbolic groups. In what follows, we explain how to implement it in $\hat{\mathcal{M}}\mathcal{F}$ using Thurston boundary. The references are [36], [34] and in particular, the theory of limit sets in mapping class groups developed in [55].

Recall that Teichmüller space $(Y, d)$ of a closed orientable surface $\Sigma$ with negative Euler characteristic is compactified by the space $\mathcal{P}\mathcal{M}\mathcal{F}$ of projective measured laminations on $\Sigma$, which is the quotient of measured laminations $\mathcal{M}\mathcal{F}$ by positive reals. The proper action on $Y$ of the mapping class group $G$ extends to $\mathcal{P}\mathcal{M}\mathcal{F}$ by homeomorphisms, which is called Thurston boundary of $Y$.

There exists non-zero, symmetric, $G$-invariant, bi-homogeneous intersection function

$$
i : \mathcal{M}\mathcal{F} \times \mathcal{M}\mathcal{F} \to \mathbb{R}_{\geq 0}.
$$

Let $\Lambda$ be a subset in $\mathcal{P}\mathcal{M}\mathcal{F}$. The intersection completion $Z(\Lambda)$ consists of projective measured laminations $[G] \in \mathcal{P}\mathcal{M}\mathcal{F}$ such that $i(F, G) = 0$ for some $[F] \in \Lambda$. A uniquely ergodic point $[F]$ in $\mathcal{P}\mathcal{M}\mathcal{F}$ has the property that if $i(G, F) = 0$ then $[G] = [F]$.

According to [55], the limit set of a group action on a topological space is the set of accumulation points of all orbits. In this regard, the enlarged limit set $\Lambda^e(\Gamma)$ (resp. limit set $\Lambda(\Gamma)$) of a subgroup $\Gamma$ is the set of accumulation points of all $\Gamma$-orbits in $\mathcal{P}\mathcal{M}\mathcal{F} \cup Y$ (resp. $\mathcal{P}\mathcal{M}\mathcal{F}$). By Proposition 8.1 in [55], it follows that $\Lambda^e(\Gamma) \subset \Lambda(\Gamma)$.

An infinite reducible subgroup $H$ preserves a finite family of disjoint simple closed curves called a reduction system on $\Sigma$. The following elementary fact will be useful later on.

Lemma 6.4. The enlarged limit set of an infinite reducible subgroup $H$ is disjoint with that of the subgroup $P$ generated by a pseudo-Anosov element $p$.

Proof. By [55] Section 7.1, the limit set $\Lambda(H)$ of $H$ is the union of an essential reduction system $A$ with projective measured laminations on pseudo-Anosov components obtained by cutting $A$ on $\Sigma$. As this description, all points in $\Lambda(H)$ are non-filling. On the other
hand, the limit set $\Lambda(P)$ of $P$ consists of two filling uniquely ergodic points so $Z(\Lambda(P)) = \Lambda(P)$. Therefore, $\Lambda(H) \cap \Lambda(P) = \emptyset$ and thus $Z(\Lambda(H)) \cap Z(\Lambda(P)) = \emptyset$. 

The proof of the next result makes use of the following fact in [19] Lemma 1.4.2. Suppose that $x_n \in Y$ is a sequence of points converging to a uniquely ergodic point $\xi \in \mathcal{PMF}$. If 

$$d(o, y_n) - d(x_n, y_n) \to +\infty$$

for a sequence $y_n \in Y$, then $y_n \to \xi$.

Lemma 6.5. For any basepoint $o \in Y$, there exists a constant $D > 0$ such that $Ko$ have a $D$-bounded projection to $Ho$.

Proof. Suppose to the contrary that there exists $k_n \in K$ such that $d_{Ho}([o, k_n o]) \to \infty$. Let $x_n \in Ko$ be a projection point of $y_n := k_n o$ to $Po$. Since $d(o, x_n) \to \infty$, up to passage to subsequence, it converges to a limit point in $Z(\Lambda(P)) = \Lambda(P)$.

On the other hand, the contracting property shows that $d(o, y_n) - d(x_n, y_n)$ differs from $d(o, x_n)$ up to a uniform bounded amount. Hence, by Lemma 1.4.2 in [19], one concludes that $y_n$ and $x_n$ converge to the same limit point, giving a contradiction to Lemma 6.4. The proof is thus complete. 

We are ready to prove the following main result of this section.

Proposition 6.6. Let $p$ be a pseudo-Anosov element and $K$ be an infinite torsion-free reducible subgroup in a mapping class group. There exists $n > 0$ such that $\langle p^n, K \rangle$ is a free product of $(p^n)$ and $K$ acting on the Teichmüller space via a SCC action.

Proof. We apply Proposition 6.3 to $H = E(p)$ and $K$, where $E(p)$ defined in [7] is an elementary contracting subgroup with bounded intersection by Lemma 2.11. The only reducible elements in $E(p)$ are torsions so $E(p) \cap K = \{1\}$. Hence, the collection $\{kE(p) \cdot o : k \in K\}$ has bounded intersection. By Proposition 6.3 the conclusion thus follows from Lemma 6.5. 

Two examples deriving from Proposition 6.6 are as follows:

1. Let $p$ be pseudo-Anosov and $k$ an infinite reducible element in $\mathbb{Mod}$. Then $\langle p^n, k \rangle$ admits a SCC action for $n \gg 0$.

2. Let $p$ be pseudo-Anosov and $K$ be an abelian group generated by Dehn twists around disjoint essential simple closed curves. Then $\langle p^n, K \rangle$ admits a SCC action for $n \gg 0$.

References

1. Y. Antolin and L. Ciobanu, Formal conjugacy growth in acylindrically hyperbolic groups, arXiv:1508.06229, 2015.
2. G. Arzhantseva, C. Cashen, and J. Tao, Growth tight actions, Pacific Journal of Mathematics 278 (2015), 1–49.
3. G. Arzhantseva and I. Lysenok, Growth tightness for word hyperbolic groups, Math. Z. 241 (2002), no. 3, 597–611.
4. N. Arzhantseva, C. Cashen, D. Gruber, and D. Hume, Contracting geodesics in infinitely presented graphical small cancellation groups, arXiv:1602.03767.
5. J. Athreya, A. Bufetov, A. Eskin, and M. Mirzakhani, Lattice point asymptotics and volume growth on Teichmüller space, Duke Math. J. 161 (2012), no. 6, 1055–1111.
6. W. Ballmann, Lectures on spaces of nonpositive curvature, Birkhäuser, Basel, 1995.
7. J. Behrstock, Asymptotic geometry of the mapping class group and Teichmüller space, Geom. Topol. 10 (2006), 1523–1578.
8. J. Behrstock and R. Charney, Divergence and quasimorphisms of right-angled artin groups, Mathematische Annalen 352 (2012), 339–356.
9. J. Behrstock, C. Drutu, and L. Mosher, Thick metric spaces, relative hyperbolicity, and quasi-isometric rigidity, Math. Annalen 344 (2009), no. 3, 543–595.
10. J. Behrstock, M. Hagen, and A. Sisto, Thickness, relative hyperbolicity, and randomness in coxeter groups, To appear in Algebr. Geom. Topol.
47. Random walk in random groups, GAFA 13 (2003), no. 1, 73–146.
48. U. Hamenstadt, Bounded cohomology and isometry groups of hyperbolic spaces, J. Eur. Math. Soc. 10 (2008), no. 2, 315–349.
49. V. Kaimanovich and H. Masur, The poisson boundary of the mapping class group, Invent. math. 125 (1996), 221–264.
50. A. Karlsson, Free subgroups of groups with non-trivial Floyd boundary, Comm. Algebra. 31 (2003), 5361–5376.
51. G. Kneiper, On the asymptotic geometry of nonpositively curved manifolds, GAFA 7 (1997), 755–782.
52. T. Koberda, Right-angled artin groups and their subgroups, Course notes at Yale University.
53. H. Masur and Y. Misnky, Geometry of the complex of curves i: Hyperbolicity, Invent. math. 138 (1999), 103–149.
54. Geometry of the complex of curves II: Hierarchical structure, Geometric and Functional Analysis 10 (2000), 902–974.
55. J. McCarthy and A. Papadopoulos, Dynamics on Thurston’s sphere of projective measured foliations, Comment. Math. Helvetici 66 (1991), 133–166.
56. P. Mercat, Entropy of isometries semi-groups of hyperbolic space, arXiv:1602.07809.
57. Y. Minsky, Quasi-projections in Tschmüller space, J. Reine Angew. Math. 473 (1996), 121–136.
58. B. Neumann, Groups covered by finitely many cosets, Publ. Math. Debrecen 3 (1954), 227–242.
59. A. Olshanskii, Subnormal subgroups in free groups, their growth and cogrowth, arXiv:1312.0129, 2015.
60. J. Osborne and W. Yang, Statistical hyperbolicity of relatively hyperbolic groups, arXiv:1504.01011, accepted to AGT, 2015.
61. D. Osin, Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems, vol. 179, Mem. Amer. Math. Soc., 2006.
62. Acylindrically hyperbolic groups, Trans. Amer. Math. Soc. 368 (2016), no. 2, 851–888.
63. S. Patterson, The limit set of a Fuchsian group, Acta Mathematica (1976), no. 1, 241–273.
64. F. Paulin, Un groupe hyperbolique est déterminée par son bord, J. London Math. Soc. (1996), 50–74.
65. M. Peigné, Autour de l’exposant critique d’un groupe klenien, arXiv:1010.6022v1.
66. On some exotic Schottky groups, Discrete and Continuous Dynamical Systems 118 (2011), no. 31, 559 – 579.
67. R. Phillips and P. Sarnak, The Laplacian for domains in hyperbolic space and limit sets of Kleinian groups, Acta Math. 155 (1985), 173–241.
68. L. Potyagailo and W. Yang, Hausdorff dimension of boundaries of relatively hyperbolic groups, Preprint, 2016.
69. T. Roblin, Sur la fonction orbitale des groupes discrets en courbure négative, Ann. Inst. Fourier 52 (2002), 145–151.
70. Ergodicité et équidistribution en courbure négative, no. 95, Mémoires de la SMF, 2003.
71. A. Sambusetti, Growth tightness of free and amalgamated products, Ann. Sci. École Norm. Sup. série 35 (2002), no. 4, 477 – 488.
72. Asymptotic properties of coverings in negative curvature, Geometry & Topology (2008), no. 1, 617–637.
73. S. Schleimer, The end of the curve complex, Groups, Geometry, and Dynamics 5 (2011), no. 1, 169–176.
74. A. Sisto, Contracting elements and random walks, arXiv:1112.2666, 2011.
75. D. Sullivan, The density at infinity of a discrete group of hyperbolic motions, Publ. Math. IHES (1979), 171–202.
76. W. Yang, Genericity of contracting elements in groups.
77. Patterson-Sullivan measures and growth of relatively hyperbolic groups, Preprint, arXiv:1308.6326, 2013.
78. Growth tightness for groups with contracting elements, Math. Proc. Cambridge Philos. Soc 157 (2014), 297 – 319.
79. Purely exponential growth of cusp-uniform actions, Preprint, Arxiv: 1602.07897, to appear in Ergodic theory and dynamical systems, 2016.