On the Elliptic Genus and Mirror Symmetry

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ABSTRACT

The elliptic genus for arbitrary two dimensional $N = 2$ Landau-Ginzburg orbifolds is computed. This is used to search for possible mirror pairs of such models. An important aspect of this work is that there is no restriction to theories for which the conformal anomaly is $\hat{c} \in \mathbb{Z}$, nor are the results only valid at the conformal fixed point.

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1. Introduction

$N = 2$ superconformal field theories have attracted much attention during the last couple of years in particular because of their importance in understanding the compactification of the heterotic string. Although $(0, 2)$ models are sufficient for $N = 1$ space-time supersymmetry in the effective four dimensional low-energy theory most effort has been concentrated on the left-right symmetric $(2, 2)$ theories \[1\]. One example of such theories, which is the focus of the present article, is the class of $N = 2$ Landau-Ginzburg vacua. Though not conformally invariant as they stand, they are believed to flow to a fixed point in the infrared, characterized by the superpotential $W$, a quasi homogeneous function of the complex chiral superfields \[2\]. Assuming that the original theory is invariant under a symmetry group $H$, one can consider a new set of models through the process of orbifolding \[3\]. Field configurations in the orbifold model are then identified modulo the action of the group $H$. The process of orbifolding will turn out to be an integral part of the construction of mirror pairs.

Previous studies of $(2, 2)$ compactifications in general and $N = 2$ Landau-Ginzburg models (including orbifold constructions) in particular \[4,5,6,7,8,9,10\] have indicated that there exists a symmetry relating seemingly different models; mirror symmetry \[1\]. At the level of conformal field theory the symmetry can be formulated as an isomorphism between two theories, the only difference being a change of the sign of the left-moving $U(1)$-generator. As trivial as it may seem it has far far reaching consequences. For example, in terms of Calabi-Yau sigma models the effect of mirror symmetry is to equate the space-time physics of target spaces of different topology.

Unfortunately, the original construction by Greene and Plesser \[4\], who considered orbifolds of tensor products of $N = 2$ minimal models, is still the only one for which mirror symmetry has been rigorously proven at the level of conformal field theory. The problem is that in order to compare the theories we have to compute the partition function $Z(q, \gamma_L, \gamma_R) = \text{Tr}(-1)^F q^{L_0} \bar{q}^{\bar{L}_0} \exp(i\gamma_L J_0 + i\gamma_R \bar{J}_0)$ where $L_0$ ($\bar{L}_0$) and $J_0$ ($\bar{J}_0$) are the energy and $U(1)$ charge operator respectively of the left-moving (right-moving) $N = 2$ algebra. Apart from the minimal models and their orbifolds this is in general not a feasible task for an $N = 2$ theory. The situation is much better for the elliptic genus \[12\], which is simply the restriction of the partition function to $\gamma_R = 0$, i.e. $Z(q, \gamma, 0)$. This restricted partition function has an interpretation as an index of the $N = 1$ right-moving supercharge. This property was recently used by Witten \[14\] to calculate the elliptic genera of certain Landau-Ginzburg models which are believed to flow to the minimal models. The results were compared to the elliptic genera of the minimal models, calculated from the known characters of the $N = 2$ discrete series representations, in \[14,15\]. The affirmative outcome of these computations lends further support to the conjectured isomorphism between Landau-Ginzburg theories (at the conformal fixed point) and the minimal series \[16\]. For other applications of the elliptic genus, see \[17,18,19\].

In the following we will use the elliptic genus as a tool for studying mirror symmetry in the context of Landau-Ginzburg orbifolds. Since the spectrum of an $N = 2$ theory is

\[1\] For a review of mirror symmetry, see other articles in this volume as well as \[11\].
symmetric under charge conjugation, the elliptic genus is invariant under $J_0 \to -J_0$, i.e. $Z(q, \gamma, 0) = Z(q, -\gamma, 0)$. Thus, the elliptic genera of two models that constitute a mirror pair must therefore be equal (up to a sign, which could be thought of as arising from different normalizations of the path integral measures of the two theories). Although the equality of the elliptic genera is only an indication of mirror symmetry, and by no means a proof, we will for brevity refer to two models with the same elliptic genus as constituting a mirror pair.

Finally, we want to stress a point which is not of direct relevance for our work but which may have its own merits. Mirror symmetry is mostly discussed in the context of models with $\hat{c} = 3$ (or at least with integer $\hat{c}$). The theory may then in many cases be interpreted as a sigma model where the target space is a (complex) $\hat{c}$ dimensional Calabi-Yau. (For a discussion of theories which may have a geometric interpretation in a generalized sense, see for example [20].) However, mirror symmetry seems to be an inherent two dimensional feature and the value of $\hat{c}$ is of less importance. In addition, the conformal aspect could also turn out to be of subordinate value since our computations indicate that mirror symmetry may be a property of any two dimensional $N = 2$ quantum field theory. In all this is telling us to study mirror symmetry in a broader context and hopefully this will teach us about phenomena relevant to string vacua as well.

This article is organized as follows: In section 2, we calculate the elliptic genus for an arbitrary Landau-Ginzburg orbifold as well as the Poincaré polynomial of the theory by taking the $q \to 0$ limit of the elliptic genus. In section 3, we discuss a plausible scenario and find a sufficient condition for mirror symmetry between Landau-Ginzburg orbifolds. Indeed, it can be shown [21] that the models proposed in [6] satisfy this condition. We conjecture that all pairs of conjugate Landau-Ginzburg models may be obtained by taking products of these models.

2. Landau-Ginzburg orbifolds and the elliptic genus

2.1. The elliptic genus

Let us start by reviewing some important facts about $N = 2$ Landau-Ginzburg models and their orbifolds. Consider the action of a $(2,2)$ Landau-Ginzburg model written in superspace as

$$S = \int d^2 z d^4 \theta K(X_i, \bar{X}_i) + \epsilon \int d^2 z d^2 \theta W(X_i) + c.c.$$  \hspace{1cm} (2.1)

The $X_i$ for $i \in N$, with the total number of fields denoted by $|N|$, are complex chiral superfields with component expansions

$$X_i = x_i + \theta_+ \psi_i^+ - \theta_+ \psi_i^- + \theta_+ \theta_- F_i.$$  \hspace{1cm} (2.2)

The superpotential $W$ is a holomorphic and quasi-homogeneous function of the $X_i$, i.e. it should be possible to assign some weights $k_i \in \mathbb{Z}$ to the fields $X_i$ for $i \in N$ and a degree of homogeneity $D \in \mathbb{Z}$ to $W$ such that

$$W(\lambda^{k_i} X_i) = \lambda^D W(X_i).$$  \hspace{1cm} (2.3)
The central charge for the theory is given by \( \hat{c} = \sum_i (1 - 2q_i) \) with \( q_i = k_i / d \) the left- (and right-) moving \( U(1) \)-charge \(^2\). The model defined by (2.1) is believed to flow to a conformally invariant model in the infrared. Under this renormalization group flow, the Kähler potential \( K(X_i, \bar{X}_i) \) will get renormalized in some complicated way, but there are strong reasons to believe that the superpotential \( W(X_i) \) is an invariant of the flow \(^3\).

In general, \( W \) will be invariant under a discrete, abelian group \( G \) of phase symmetries. The fields \( X_i \) transform in some representations \( R_i \) under \( G \). In the following, we denote the set of representations \( \{R_i\} \) for \( i \in N \) collectively as \( R \). A representation \( R_i \) of \( G \) is specified by a function \( R_i(g) = \exp(\imath 2\pi \theta_i(g)) \) defined for \( g \in G \), which fulfills \( R_i(g_1 g_2) = R_i(g_1) R_i(g_2) \) for all \( g_1, g_2 \in G \). The invariance of \( W \) means that

\[
W(R_i(g)X_i) = W(X_i) \quad \text{for} \quad g \in G. \tag{2.4}
\]

By taking \( \lambda = \exp \imath 2\pi / D \) in (2.3), we see that \( G \) will always contain an element \( q \) such that \( R_i(q) = \exp \imath 2\pi q_i \) for \( i \in N \). From a theory invariant under some symmetry group \( H \), we may construct a new theory by taking the \( H \) orbifold of the original theory, i.e. by modding out by the action of \( H \). In our case, \( H \) could be any subgroup of the group \( G \) of phase symmetries of \( W \). We will denote the theory thus obtained as \( W/H \) \(^4\).

Our object is to calculate the elliptic genus of \( W/H \). This calculation is feasible because of the invariance of the elliptic genus under deformations of the theory which preserve the right-moving supersymmetry \(^4\). We may therefore smoothly turn off the superpotential interactions by letting \( \epsilon \to 0 \), which turns the model into a free field theory. (We take \( K \) to be the Kähler potential of \( \Phi^{\left|N\right|} \) with the flat metric.) The elliptic genus of the model is determined by the set of representations \( R \) and the group \( H \) and may be denoted as \( Z[R/H] \), where we have suppressed the dependence on \( \gamma \) and \( q \). Being essentially a genus one correlation function, it may be written as

\[
Z[R/H] = \frac{1}{\left| H \right|} \sum_{h_a, h_b \in H} Z[R](h_a, h_b). \tag{2.5}
\]

Here \( Z[R](h_a, h_b) \) denotes the contribution from field configurations twisted by \( h_a \) and \( h_b \) around the \( a \) and \( b \) cycles of the torus respectively, and \( \left| H \right| \) is the number of elements of \( H \). In the free field limit, the contribution from each twist sector is a product of contributions from each of the fields \( X_i \) in the theory, i.e.

\[
Z[R](h_a, h_b) = \prod_{i \in N} Z[R_i](h_a, h_b). \tag{2.6}
\]

A by now standard calculation gives \(^4\)

\[
Z[R_i](h_a, h_b) = e^{-\imath 2\pi \theta_i(h_a)} \frac{\Theta_1((1 - q_i) \gamma - \theta_i(h_b) - \tau \theta_i(h_a) |\tau)}{\Theta_1(q_i \gamma + \theta_i(h_b) + \tau \theta_i(h_a) |\tau)}. \tag{2.7}
\]

\(^2\) In general \( q \not\in H \) and hence \( W/H \) is not a valid superstring vacuum, even if \( \hat{c} = 3 \).
where $\Theta_1$ is the Jacobi $\Theta_1$ function \[^{22}\]. The prefactor is chosen such that the elliptic genus is well-defined, \textit{i.e.} it should only depend on $R_i(h_a) = \exp(i2\pi\theta_i(h_a))$ and $R_i(h_b)$. Furthermore, with this choice the complete elliptic genus, given by \(^{2.5}\) with \(^{2.6}\), has the correct double quasi periodicity in $\gamma$ and modular transformation properties \[^{21}\].

2.2. The Poincaré polynomial

A necessary condition for the elliptic genera of two $N = 2$ models to be equal is that they coincide in the $\tau \to i\infty$ limit, \textit{i.e.} that the two models have the same Poincaré polynomial. Following Francesco and Yankielowicz \[^{15}\], this condition can be shown to be sufficient in the case of orbifolds of Landau-Ginzburg models with isomorphic groups of phase symmetries \[^{21}\].

The Poincaré polynomial is a polynomial in $t^{1/D}$, where $t = \exp(i2\pi\gamma)$ and $D$ is some integer (defined by \(^{2.3}\) for the case of Landau-Ginzburg orbifolds). It can be seen as the generating function for the charges of the Ramond sector ground states under the $U(1)$ symmetry of the left-moving $N = 2$ algebra. From our results in the last section, the Poincaré polynomial for a Landau-Ginzburg orbifold may be written as a sum over contributions from different twist-sectors;

\begin{equation}
P[R/H] = \frac{1}{|H|} \sum_{h_a, h_b \in H} P[R](h_a, h_b). \tag{2.8}
\end{equation}

Furthermore, the contribution from each twist-sector is a product of the contributions from each of the fields;

\begin{equation}
P[R](h_a, h_b) = \prod_{i \in N} P[R_i](h_a, h_b). \tag{2.9}
\end{equation}

Finally, by taking the $\tau \to i\infty$ limit of \(^{2.7}\), we find that

\begin{equation}
P[R_i](h_a, h_b) = \begin{cases} 
-t^{1/2} - [\theta_i(h_a)] & \text{for } R_i(h_a) \neq 1 \\
-t^{-1/2}(t^{q_i}R_i(h_b) - t)(1-t^{q_i}R_i(h_b))^{-1} & \text{for } R_i(h_a) = 1
\end{cases}. \tag{2.10}
\end{equation}

Here, $[x]$ denotes the fractional part of $x$, \textit{i.e.} $[x] = x \mod \mathbb{Z}$ and $0 \leq [x] < 1$. Note that if $R_i(h_a) = 1$, or equivalently $[\theta_i(h_a)] = 0$, then the field $X_i$ is left untwisted by the transformation $h_a$. It will prove convenient to introduce

\begin{equation}
P^{tw}[R](g) = \prod_{i \in N} (-t^{1/2})^{tw[R_i](g)} t^{-\sum_{i \in N} [\theta_i(g)]} \tag{2.11}
\end{equation}

\[^3\text{Although the canonical choice, nontrivial phase factors can be introduced, so called discrete torsion}\] \[^{23}\].

\[^4\text{In a more general case it may not be sufficient to merely consider the } q \to 0 \text{ limit}\] \[^{13}\].
and
\[ P^{\text{inv}}[R_i](g) = t^{-1/2}(t^{q_i} R_i(g) - t)(1 - t^{q_i} R_i(g))^{-1}, \] (2.12)
where \( \text{tw}[R](g) \) denotes the number of fields that are twisted by \( g \). We may then write
\[ P[R](h_a, h_b) = P^{\text{tw}}[R](h_a) \prod_{R_i(h_a) = 1} P^{\text{inv}}[R_i](h_b). \] (2.13)

Incidentally, there is a generalized Poincaré polynomial, which is sensitive to the charges of the Ramond sector ground states under both the left- and right-moving \( U(1) \) symmetry. Since mirror symmetry acts by reversing the sign of one of the \( U(1) \) charges and leaving the other unaffected, this generalized Poincaré polynomial is useful to check whether two models might be each others mirror partners rather than being completely equivalent theories. It may be defined as
\[ P(t, \bar{t}) = \text{Tr}(t^{J_0} \bar{t}^{\bar{J}_0}), \] (2.14)
where the trace is over the ground states in the Ramond sector. For a Landau-Ginzburg orbifold, this generalized Poincaré polynomial has been calculated by Intrilligator and Vafa [24]. The only difference with respect to the Poincaré polynomial that we have discussed is that the arguments of \( P^{\text{tw}}[R](g_a) \) and \( P^{\text{inv}}[R_i](g_b) \) in (2.13) are \( t/\bar{t} \) and \( t\bar{t} \) respectively instead of \( t \). We may therefore continue to work with the restricted Poincaré polynomial which only depends on \( t \). A criterion for mirror symmetry is then that the Poincaré polynomials of the two models must be equal and that contributions from twisted (untwisted) fields in one model should correspond to contributions from untwisted (twisted) fields in the other.

3. Mirror symmetry for Landau-Ginzburg orbifolds

3.1. General results

Suppose that we have two Landau-Ginzburg models, each with \( |N| \) superfields, the phase symmetry groups of which are both isomorphic to the same abelian group \( G \). We denote the sets of representations as \( R \) and \( \tilde{R} \) respectively for the two models. In general, we distinguish all quantities pertaining to the second model with a tilde. We are interested in the situation in which the \( H \) orbifold of the first model is the mirror partner of the \( \tilde{H} \) orbifold of the second model for some subgroups \( H \) and \( \tilde{H} \) of \( G \). This means that their elliptic genera should be equal up to a sign:
\[ Z[R/H] = \pm Z[\tilde{R}/\tilde{H}]. \] (3.1)
We will propose a natural way for two models to be each others mirror partners in this sense, but to do so, we first need to discuss some aspects of abelian groups.
All irreducible representations of the abelian group $G$ are one-dimensional. Given two irreducible representations we may construct a new irreducible representation by taking their tensor product. Clearly, the set of irreducible representations of $G$ form a group $G^*$ under the tensor product $\otimes$. This group is in fact isomorphic to $G$ itself. Given a subgroup $H$ of $G$ we define its dual as the subgroup $\tilde{H}$ of $G^*$ of representations on which $H$ is trivially represented, i.e. $\tilde{H}$ is the set of $R \in G^*$ such that $R(g) = 1$ for $g \in H$. In particular, the dual of $G$ itself is the trivial subgroup of $G^*$ which only consists of the identity element, and the dual of the trivial subgroup of $G$ is $G^*$. Clearly, if $H_1 \subset H_2$ then $\tilde{H}_2 \subset \tilde{H}_1$. Similarly, we note that a representation of the group $G^*$ has a natural interpretation as an element of $G$. Therefore, given a subgroup $\tilde{H}$ of $G^*$ we may define its dual $\tilde{\tilde{H}}$ as the subgroup of elements of $G$ which are trivially represented by all $R \in \tilde{H}$. We see that $\tilde{\tilde{H}} = H$. The situation is thus completely symmetric under interchange of $G$ and $G^*$. As an alternative definition of the duality between $H$ and $\tilde{H}$ we have the following relation

$$\frac{1}{|H|} \sum_{g \in H} \tilde{g}(g) = \begin{cases} 1 & \text{for } \tilde{g} \in \tilde{H} \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

and its partner obtained by changing the roles of $G$ and $G^*$. The summand $\tilde{g}(g)$ is the function, defined on $G$, which specifies the $G$ representation $\tilde{g} \in G^*$.

We now interpret our candidate mirror pair of Landau-Ginzburg models so that the fields of the first model transform in the representations $R_i$ for $i \in \mathbb{N}$ under the symmetry group $G$, whereas the fields of the second model transform in the representations $\tilde{R}_i$ for $i \in \tilde{\mathbb{N}}$ under the group $G^*$. As our notation suggests, we will take the $H$ orbifold of the first model and the $\tilde{H}$ orbifold of the second model, where $H$ and $\tilde{H}$ are dual subgroups.

For $g \in G$ and $\tilde{g} \in G^*$ we now define the (partial) Fourier transform of the elliptic genus contributions as

$$\hat{Z}[R](g, \tilde{g}) = \frac{1}{|G|} \sum_{g' \in G} \tilde{g}(g') Z[R](g, g'), \tag{3.3}$$

which may be inverted by means of (3.2) to yield

$$Z[R](g, g') = \sum_{\tilde{g} \in G^*} \tilde{g}^{-1}(g') \hat{Z}[R](g, \tilde{g}). \tag{3.4}$$

Inserting this in (2.5) and using (3.2) we get

$$Z[R/H] = \sum_{h \in H} \sum_{\tilde{h} \in \tilde{H}} \hat{Z}[R](h, \tilde{h}). \tag{3.5}$$

Analogously, we calculate

$$Z[\tilde{R}/\tilde{H}] = \sum_{\tilde{h} \in \tilde{H}} \sum_{h \in H} \hat{Z}[\tilde{R}](\tilde{h}, h). \tag{3.6}$$
A very natural way to fulfill (3.1) is then to require that

$$\hat{Z}[R](g, \bar{g}) = \pm \hat{Z}[\hat{R}](\bar{g}, g)$$

(3.7)

for $g \in G$ and $\bar{g} \in G^*$. We will say that sets of representations $R$ and $\hat{R}$ which fulfill (3.7) are conjugates of each other. Note that this condition implies (5.1) for any $H$ and its dual $\hat{H}$. Furthermore, given two Landau-Ginzburg models $R_1$ and $R_2$ with symmetry groups isomorphic to $G_1$ and $G_2$ respectively, we may construct the product model $R = R_1 \times R_2$ with symmetry group $G \simeq G_1 \times G_2$. If now $R_1$ and $R_2$ are conjugates to $\hat{R}_1$ and $\hat{R}_2$ respectively so that each pair satisfies (3.7), then the pair of product models $R = R_1 \times R_2$ and $\hat{R} = \hat{R}_1 \times \hat{R}_2$ also satisfies (3.7). This means that any $H$ orbifold of $R$, for $H$ a subgroup of $G \simeq G_1 \times G_2$, will be the mirror partner of the corresponding $\hat{H}$ orbifold of $\hat{R}$, even if $H$ is not of the form $H_1 \times H_2$ for any subgroups $H_1$ of $G_1$ and $H_2$ of $G_2$.

As we have seen in the previous section, in the case of Landau-Ginzburg orbifolds it is sufficient to compare the $\tau \rightarrow i \infty$ limits of elliptic genera, i.e. the Poincaré polynomials, to establish their equality for all $\tau$. With the $\tau \rightarrow i \infty$ limit of $Z[R](g, g')$ given by (2.13), the corresponding limit of $\hat{Z}[R](g, \bar{g})$ is

$$\hat{P}[R](g, \bar{g}) = P^{tw}[R](g) \frac{1}{|G|} \sum_{g' \in G} \bar{g}(g') \prod_{R_i(g) = 1} P^{inv}[R_i](g').$$

(3.8)

Similarly, we have

$$\hat{P}[\hat{R}](\bar{g}, g) = P^{tw}[\hat{R}](\bar{g}) \frac{1}{|G|} \sum_{g' \in G^*} \bar{g}'(g) \prod_{\hat{R}_i(\bar{g}) = 1} P^{inv}[\hat{R}_i](\bar{g}').$$

(3.9)

Our condition for mirror symmetry now reads

$$\hat{P}[R](g, \bar{g}) = \pm \hat{P}[\hat{R}](\bar{g}, g).$$

(3.10)

If this condition is fulfilled, then obviously $P[R/H] = \pm P[\hat{R}/\hat{H}]$ for any subgroup $H$. Our results from the last section then imply that also (5.1) is satisfied. The natural way for this to come about is that also (5.7) holds. Although we have no proof, we strongly believe that this is indeed always the case.

The obvious question is now how we may find a pair of Landau-Ginzburg orbifolds such that (3.10) is obeyed. To answer this, we must first introduce some more notation. Let $s$ be a subset of the set $N$ that indexes the fields. We will only be interested in $s$ such that there is at least one element of $G$ which leaves untwisted the $X_i$ for $i \in s$ and twists the remaining fields. We denote the set of such $s$ as $S$, and henceforth we will always assume that $s \in S$. For $g \in G$ we define $\sigma(g) \in S$ by the condition that $i \in \sigma(g)$ if and only if $X_i$ is left untwisted by $g$. Next, we introduce the subgroups $G_s$ of elements of $G$ that leave untwisted the fields $X_i$ for $i \in s$. The remaining fields may be twisted or untwisted depending on which element of $G_s$ we choose. The corresponding objects in the conjugate model are denoted as $\bar{s} \check{,} S, \bar{\sigma}(\bar{g})$ and $G_s^\check{e}$ respectively.
To find a pair of conjugate models, we assume that there is a one-to-one map $\rho$ from $S$ to $\tilde{S}$ such that $(-1)^{|s|} = (-1)^{|N|-|\rho(s)|}$ and $G_s \simeq G_{\rho(s)}^*$ for $s \in S$. Here, $|s|$ denotes the number of elements in $s$, and as usual the tilde over $G_{\rho(s)}^*$ denotes the dual group. Furthermore, we demand that

$$\prod_{i \in s} P^{\text{inv}}[R_i](g) = \sum_{G_s^* \subseteq \tilde{G}_s} (-1)^{|N|-|\tilde{s}|} \sum_{\tilde{g} \in G_\tilde{s}^*} \tilde{g}(g^{-1}) P^{\text{tw}}[\tilde{R}](\tilde{g}).$$

Our notation means that the first sum runs over all $\tilde{s} \in \tilde{S}$ such that $G_\tilde{s}^* \subseteq \tilde{G}_s$. We also postulate the corresponding relation with the roles of the two models interchanged. At this point, the conditions that we have imposed may seem rather ad hoc. Our main justification is that they cover all cases of mirror symmetry between Landau-Ginzburg orbifolds that we know of. We have performed some limited computer searches, which support the hypothesis that this is indeed the general mechanism for mirror symmetry between Landau-Ginzburg orbifolds.

A short calculation \[21\] shows that when these conditions are fulfilled

$$\hat{P}[R](g, \tilde{g}) = P^{\text{tw}}[R](g) P^{\text{tw}}[\tilde{R}](\tilde{g}) \sum_{\tilde{g} \in G_\tilde{s}^* \subseteq \tilde{G}_\sigma(\tilde{g})} (-1)^{|N|-|\tilde{s}|},$$

$$\hat{P}[\tilde{R}](\tilde{g}, g) = P^{\text{tw}}[\tilde{R}](\tilde{g}) P^{\text{tw}}[R](g) \sum_{g \in G_s \subseteq G_{\tilde{\sigma}(\tilde{g})}} (-1)^{|N|-|s|},$$

where the sum in, for example, the first equation runs $\tilde{s} \in \tilde{S}$ such that $G_\tilde{s}^* \subseteq \tilde{G}_s$. We also postulate the corresponding relation with the roles of the two models interchanged. It follows then that

$$\hat{P}[R](g, \tilde{g}) = (-1)^{|N|} \hat{P}[\tilde{R}](\tilde{g}, g),$$

which proves (3.10). We see that in this way of implementing mirror symmetry, contributions from twisted fields in one model correspond to contributions from untwisted fields in the other model and vice versa, just as it should be for a mirror pair.

3.2. Examples

Finally, we will briefly discuss two classes of solutions, first proposed in \[6\], to the conditions (3.11). Both of these can be seen as generalizations of Landau-Ginzburg analogs of the $(2, 2)$ minimal models for which mirror symmetry was first discovered \[4\]. Let us here once again stress that the $N = 2$ minimal models (including products and/or quotients thereof) are the only theories for which mirror symmetry has been rigorously proven. In terms of the Landau-Ginzburg models of Fermat type it has been conjectured that orbifolds of a Fermat potential come in mirror pairs. This conjecture is based on studies of the spectra of Landau-Ginzburg vacua \[3\] and their orbifolds \[8\] as well as more detailed investigations of the moduli space of particular $c = 3$ theories \[7, 10\]. In particular the work in \[3, 8\] as
well as recent advances in terms of toric geometry \cite{9} indicate that mirror symmetry must hold for a much larger class of theories than the minimal models. As was noted in the previous section, we may construct new solutions by taking the product of old ones. We conjecture that by taking products of the models we will describe in this section, one may in fact construct all solutions to the conditions (3.11).

The first class of models is given by a potential of the form

\[ W_1 = X_1^{\alpha_1} + X_1 X_2^{\alpha_2} + \ldots + X_N^{-1} X_N^{\alpha_N}. \]  

(3.14)

The group of phase symmetries is isomorphic to \( G \simeq \mathbb{Z}_D \), where \( D = \alpha_1 \ldots \alpha_N \). Note the particular case \( N = 1 \) for which the model is of Fermat type which is the conjectured equivalent of the \( A_{\alpha_1-2} \) minimal model. Another special case is \( N = 2 \) and \( \alpha = 2 \) which corresponds to the \( D \)-series.

The conjugate partner of the potential \( W_1 \) is a potential of the same type but with the order of the exponents reversed \cite{6}, i.e.

\[ \tilde{W}_1 = \tilde{X}_1^{\tilde{\alpha}_1} + \tilde{X}_1 \tilde{X}_2^{\tilde{\alpha}_2} + \ldots + \tilde{X}_N^{-1} \tilde{X}_N^{\tilde{\alpha}_N}, \]  

(3.15)

where the exponents are given by

\[ \tilde{\alpha}_i = \alpha_{N+1-i} \text{ for } 1 \leq i \leq N. \]  

(3.16)

The group of phase symmetries is obviously isomorphic to \( G^* \simeq \mathbb{Z}_D \).

Our second example is in many respects similar to the first one. This class of models is given by a potential of the form

\[ W_2 = X_N X_1^{\alpha_1} + X_1 X_2^{\alpha_2} + \ldots + X_N^{-1} X_N^{\alpha_N}. \]  

(3.17)

The cyclical nature of this potential makes it natural to take the variable \( i \), which indexes the fields, to be defined modulo \( N \), i.e. \( i \in \mathbb{Z}_N \). The group of phase symmetries is isomorphic to \( G \simeq \mathbb{Z}_D \), where \( D = \alpha_1 \ldots \alpha_N + (-1)^{N-1} \). The conjugate partner of this potential is a potential of the same type but with the order of the exponents \( \alpha_i \) reversed \cite{3}, i.e.

\[ \tilde{W}_2 = \tilde{X}_N \tilde{X}_1^{\tilde{\alpha}_1} + \tilde{X}_1 \tilde{X}_2^{\tilde{\alpha}_2} + \ldots + \tilde{X}_N^{-1} \tilde{X}_N^{\tilde{\alpha}_N}, \]  

(3.18)

where the exponents are given by

\[ \tilde{\alpha}_i = \alpha_{N+1-i} \text{ for } i \in \mathbb{Z}_N. \]  

(3.19)

The group of phase symmetries is isomorphic to \( G^* \simeq \mathbb{Z}_D \).

Without going into the details, which are tedious but straightforward \cite{21}, one can show that (3.11) is fulfilled for both of the above examples. From the general arguments in the previous section it now follows that the \( H \simeq \mathbb{Z}_m \) orbifold and the \( \tilde{H} \simeq \mathbb{Z}_{\tilde{m}} \) orbifold, with \( \tilde{m} = D/m \), of the Landau-Ginzburg models with potentials \( W_{1,2} \) and \( \tilde{W}_{1,2} \) are mirror partners.

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