Broadband noise decoherence in solid-state complex architectures

E Paladino\textsuperscript{1,2}, A D’Arrigo\textsuperscript{1,2}, A Mastellone\textsuperscript{1,2,3} and G Falci\textsuperscript{1,2}

\textsuperscript{1} Dipartimento di Metodologie Fisiche e Chimiche (DMFCI), Universit\`a di Catania. Viale A Doria 6, 95125 Catania, Italy
\textsuperscript{2} MATIS CNR—INFM, Catania, Italy
\textsuperscript{3} CIRA Centro Italiano Ricerche Aerospaziali, Via Maiorise snc - 81043 Capua, CE, Italy

E-mail: epaladino@dmfc.unict.it

Received 16 October 2009
Accepted for publication 23 October 2009
Published 14 December 2009
Online at stacks.iop.org/PhysScr/T137/014017

Abstract
Broadband noise represents a severe limitation toward the implementation of a solid-state quantum information processor. Considering common spectral forms, we propose a classification of noise sources based on the effects produced instead of their microscopic origin. We illustrate a multi-stage approach to broadband noise, which systematically includes only the relevant information on the environment, out of the huge parameterization needed for a microscopic description. We apply this technique to a solid-state two-qubit gate in a fixed coupling implementation scheme.

PACS numbers: 85.25.Cp, 03.65.Yz, 03.67.Lx, 05.40.2a

(Some figures in this article are in colour only in the electronic version.)

1. Introduction

Building scalable multi-qubit systems is presently the main challenge toward the implementation of a solid-state quantum information processor [1]. High-fidelity single qubit gates based both on semi- [2] and super-conducting technologies are nowadays available. In particular, for the three basic types of superconducting qubits (charge, flux and phase) single-qubit operations with high quality factor have been demonstrated in different laboratories [3, 4]. Further improvement has been recently achieved with circuit-quantum electrodynamics (QED) architectures [5, 6]. Multi-qubit systems instead have been proved harder to operate, the main limitation arising from the broadband solid state noise. The requirements for building an elementary quantum processor are in fact quite demanding on the efficiency of the protocols. This includes both severe constraints on readout and a sufficient isolation from fluctuations to reduce decoherence effects.

Based on experiments on the different Josephson junction (JJ) setups, there is presently a general consensus on the most common noise spectral forms and on the main consequences on systems evolutions. Typically noise is broadband and structured, i.e. the noise spectrum extends to several decades, it is non-monotonic, sometimes a few resonances are present.

Noise with 1/f spectrum is common to virtually all nanodevices. Its physical origin varies from device to device and depends on the specific material. Different implementations are in fact more sensitive either to charge, flux or to critical current fluctuations with spectral density scaling as the inverse of the frequency. The presence of slow components in the environment makes the decay of the coherent signal strongly dependent on the experimental protocol being used [4, 7–9]. Measurement protocols requiring numerous repetitions are particularly sensitive to the unstable device calibration due to low-frequency fluctuations. The leading effect is defocusing of the measured signal, analogous to inhomogeneous broadening in nuclear magnetic resonance (NMR) [10]. The intrinsic high-frequency cut-off of 1/f noise is hardly detectable, measurements typically extending to 100 Hz (recently charge noise up to 10 MHz has been detected in a single electron transistor (SET) [11]). Incoherent energy exchanges between system and environment, leading to relaxation and decoherence, occur at typical operating frequencies (about 10 GHz). Indirect measurements of noise spectrum in this frequency range often suggest a white or ohmic behavior [12, 13].
In addition, narrow resonances at selected frequencies (sometimes resonant with the nanodevice relevant energy scales) have been observed [14, 15]. In certain devices, they originate from the circuitry [16] and may eventually be reduced by circuit design. More often, resonances are signatures of the presence of spurious fluctuators which also show up in the time-resolved evolution, unambiguously proving the discrete nature of the noise sources [17]. Such fluctuators may severely limit the reliability of nanodevices [9, 18].

Explanation of this rich physics is beyond phenomenological theories describing the environment as a set of harmonic oscillators. On the other hand, an accurate characterization of the noise sources might be a priori inefficient, since a huge number of parameters would be required for a microscopic description. Therefore one may follow a different route, consisting of classifying noise sources on the basis of their effects instead of on their nature and to understand, case by case, which is the efficient description of the environment. The required information may in fact depend on the specific protocol, especially when the environment is long-time correlated. This program is meaningful for quantum information where relevant timescales are much smaller than the decoherence time. This means that in the generic favorable situation coupling with the environment has simple effects on the system dynamics as different from condensed matter physics where long time behavior is emphasized and interesting problems involve entanglement of a many-body system.

Here we illustrate a road-map to treat broadband noise which allows reasonable approximations to be obtained by systematically including only the relevant information on the environment, out of the huge parameterization needed to specify it. The multi-stage approach for the different classes of broadband noise was originally introduced for single qubit gates [9]. The obtained predictions for the decay of the coherent signal are in agreement with observations in various JJ implementations and in different protocols [8, 9]. We mention the observed decay of Ramsey fringes in charge-phase qubits [4] and recent results on flux/phase qubits [19]. Here we extend the procedure to complex solid-state architectures. As an illustrative case, we perform a systematic analysis of the effects and interplay of low and high frequency noise components in a two-qubit gate in a fixed coupling scheme. Such a systematic analysis points out that efficient operations in the solid state require an accurate preliminary characterization of the noise spectral characteristics and tuning appropriately the device working point.

The paper is organized as follows: in section 2, we introduce the general framework of solid-state multi-qubit gates and illustrate the characteristics of the most common spectral forms. In section 3, we propose a classification of the noise sources and in section 4, we present a multi-stage approach to deal with broadband noise. Section 5 is dedicated to adiabatic noise. In section 6, the multistage approach is applied to a universal two-qubit gate. Section 7 summarizes our main findings.

2. Multi-qubit systems and noise

A multi-qubit gate can be modeled by the following Hamiltonian ($\hbar = 1$):

$$\mathcal{H}_G(t) = \sum_i \mathcal{H}_{Q(i)}(t) + \sum_{i,j} \mathcal{H}_{ij}(t)$$  \hspace{1cm} (1)

where for each qubit, labeled by $i = 1, \ldots, n$, $\mathcal{H}_{Q(i)}(t) = -\frac{1}{2} \bar{\Omega}_i(t) \cdot \vec{\sigma}^{(i)}$ includes set of parameters intrinsic to the device and time-dependent classical control fields. Manipulation of tunable fields allows control of the dynamics and design of arbitrary single qubit operations via unitary transformations. Interactions among qubits and the needed additional parameterization to describe the multi-qubit gate are included in $\sum_{i,j} \mathcal{H}_{ij}(t)$. Depending on the design, the control Hamiltonian may span a reduced subspace of the qubit Liouville space. In other words, there is a limited number of ports available for control, for tuning, for state preparation and measurement. Noise is also coupled via these ports. It is rather usual that only one of the control fields allows the required fast addressing of each qubit. Therefore, we focus on a model where both the control fields, $A(t)$, and the environment are coupled to a single qubit operator, say $\sigma_z^{(i)}$.

$$\mathcal{H}_Q(i) = -\frac{\Omega_i}{2} \cos \theta_i \sigma_x^{(i)} - \frac{\Omega_i}{2} \sin \theta_i \sigma_y^{(i)} - \frac{1}{2} A(t) \sigma_z^{(i)}$$  \hspace{1cm} (2)

$$\mathcal{H}(t) = \mathcal{H}_G(t) + \frac{1}{2} \sum_i \sigma_z^{(i)} \otimes \hat{X}_i + \mathcal{H}_R$$,  \hspace{1cm} (3)

where the polar angles $\theta_i$ define qubit-$i$ working point. The environment Hamiltonian is $\mathcal{H}_R$ and $\hat{X}_i$ is a collective environment variable acting on qubit $i$. Model (3) implies a projection of the device Hamiltonian onto the subspace spanned by the two lowest energy eigenstates for each qubit. This description is valid provided that manipulations with the control fields do not induce leakage to higher energy states of the device [20].

We leave unspecified the nature of the noise sources described by $\mathcal{H}_R$, relevant cases being either discrete or Gaussian fluctuations. In addition, the environment Hamiltonian may include correlations among noise sources affecting each qubit. In the spirit of the present analysis, we assume that the only information at our disposal is the power spectrum of $\hat{X}_i$ fluctuations

$$S_{X_i}(\omega) = \int_0^\infty dt \ e^{i\omega t} \left\{ \frac{1}{2} \langle \hat{X}_i(t) \hat{X}_i(0) + \hat{X}_i(0) \hat{X}_i(t) \rangle - \langle \hat{X}_i \rangle^2 \right\}$$  \hspace{1cm} (4)

where $\langle \cdots \rangle$ denotes the equilibrium average with respect to $\mathcal{H}_R$. We remark that the effect of the environment on the system dynamics is completely characterized by $S_{X_i}(\omega)$ only if the environment is composed of harmonic oscillators or if it is weakly coupled to the system and short time correlated. Of course this is not the case if low-energy excitations determine memory effects. This is the typical situation in the solid state, where, in general, additional statistical information on the environment is required.

We assume that each noise component $\hat{X}_i$ has broadband spectrum $S_{X_i}(\omega) = \frac{\Delta}{\pi}$, $\omega \in [\gamma_m, \gamma_M]$ followed by

---

4 Experiment based on the setup of Poletto et al [19].
a white or ohmic flank at frequencies \( \omega \gg \gamma_M \). Low-
and high-frequency cut-offs depend on the specific setup. Impurities of various origin responsible for random telegraph fluctuations contribute Lorentzian peaks to the power spectrum \( S_{\text{stn}}(\omega) = \frac{\omega^2}{\gamma^2} \frac{1}{\gamma + \gamma_M} \). Such a spectrum originates from classical fluctuations \( \tilde{x} \rightarrow X(t) \), where \( X(t) \) randomly switches between two values \( 0, v_0 \) with rate \( \gamma \). An ensemble of \( N_\text{bi} \) bistable impurities with a distribution of switching rates \( \propto 1/\gamma \), \( \gamma \in [\gamma_M, \gamma_M] \) and average coupling strength \( \overline{\mathbf{r}} \), gives rise to 1/f-spectrum, \( S[1/\mathbf{f}(\omega) \approx \frac{N_\text{bi}}{\gamma_M^2} \ln'(\gamma_M/\gamma_0) \omega^{-1} \) [21]. Selected Lorentzian peaks may be visible in the spectrum if individual impurities are strongly coupled (SC), i.e. when \( v_0/\gamma \gg 1 \). Instead, damped coherent fluctuators at frequencies \( \omega_0 \) in the simplest cases contribute to the total spectrum with additional peaks, \( S_{\text{dc}}(\omega) = \frac{\omega^2}{\gamma^2} \frac{\gamma_0}{\gamma + \gamma_0} \). Spectroscopic evidence of coherent impurities of frequency \( \omega_0 \) close to the qubit Larmor frequency has been reported in [14, 22]. Remarkably, the possibility to exploit spurious quantum two-level systems as qubits [23] or for quantum memory operations has recently been demonstrated [24]. In these cases a quantum description of the impurity is required [25, 26].

3. Three classes of noise

The above description illuminates that in the solid state we have to deal with broadband and structured noise. In other words, the noise spectrum extends to several decades, it is non-monotonic, sometimes a few resonances are present. The various noise sources responsible for the above phenomenology have a qualitatively different influence on the system evolution. This naturally leads to a classification of the noise sources according to the effects produced rather than to their specific nature.

The effects of high-amplitude noise at low frequencies, like 1/f noise, vary from protocol to protocol. This feature is typical of non-Markovian baths. Quantum operations necessarily require repetitions of single detections, each leading to Boolean answer. Therefore, even ‘single shot’ measurements result from numerous repetitions of single runs in an overall process which may last minutes. In the presence of low frequency fluctuations, this leads to unstable device calibration and random clock frequencies in the various repetitions. As a result, a de-focused signal is observed, a phenomenon analogous to inhomogeneous broadening in NMR. Re-focusing protocols, like echo or some dynamical decoupling schemes, allow partial recovery of the signal [7, 13]. Since the environment is long-time correlated, statistical information beyond the power spectrum may be required to describe its effects. This is the case for instance in echo protocols [8, 18]. Environments with long-time memory belong to the class of adiabatic noise, for which the Born–Oppenheimer approximation is applicable. We classify this part of the noise spectrum as ‘adiabatic noise’.

Noise at higher frequency, around the system typical scales (e.g. single qubit Larmor frequencies), results in incoherent energy exchanges between the quantum device and the environment. In particular, high-frequency noise is responsible for spontaneous decay. For systems relevant for quantum information, the system–bath coupling is rather small, in addition high-frequency noise is short-time correlated. Therefore, in simplest cases, effects can be described by a Born–Markov master equation [27]. For single qubit gates, it leads to the relaxation and secular dephasing times, \( T_1 = \sin^2 \theta \frac{S_X(\Omega)}{2} \) and \( T_2 = 2T_1 \) [10]. We classify this part of the noise spectrum as ‘quantum noise’.

Finally, resonances in the spectrum unveil the presence of discrete noise sources which severely affect the system performances, in particular reliability of devices. This is the case when classical impurities are slow enough to induce a visible bistable instability in the system intrinsic frequency. For instance, single qubit gates in the presence of random telegraph noise (switching rate \( \gamma \)) may display two effective frequencies, \( \Omega \) and \( \Omega' \), depending on the impurity state. Their visibility is measured by the ratio \( g = (\Omega' - \Omega)/\gamma \). Beatings can be observed in the ‘strong coupling’ regime \( g > 1 \) [28]. Quantum impurities may also entangle with the device. This additionally leads to a variety of features, like peculiar temperature dependences of decay rates [25]. Under these conditions, knowledge of the power spectrum is absolutely insufficient and in order to describe these effects the relevant system Hilbert space has to be enlarged to include the responsible environmental degrees of freedom. Effects in general depend on the specific protocol and require a microscopic model of the fluctuators. We classify this part of the noise spectrum as ‘SC noise’.

Each noise class requires a specific approximation scheme, which is not appropriate for the other classes. The overall effect results from the interplay of the three classes of noise. In the following section, we will illustrate a multi-scale theory to deal with solid state broadband noise.

4. Multi-scale theory for broadband noise

We are interested in a reduced description of the \( n \)-qubit system, expressed by the reduced density matrix \( \rho^n(t) \). It is formally obtained by tracing out environmental degrees of freedom from the total density matrix \( W^{Q,A,SC}(t) \), which depends on quantum (Q), adiabatic (A) and SC variables. The elimination procedure can be conveniently performed by separating in the interaction Hamiltonian, \( \sum_i \sigma_i^{(z)} \otimes \hat{X}_i \), various noise classes, e.g. by formally writing

\[
\sigma_i^{(z)} \otimes \hat{X}_i = \sigma_i^{(z)} \otimes \hat{X}_i^Q + \sigma_i^{(z)} \otimes \hat{X}_i^A + \sigma_i^{(z)} \otimes \hat{X}_i^{SC}.
\]

Adiabatic noise is typically correlated on a timescale much longer than the inverse of the natural frequencies \( \Omega_i \), then application of the Born–Oppenheimer approximation is equivalent to replace \( \hat{X}_i^A \) with a classical stochastic field \( X_i^A(t) \). This approach is valid when the contribution of adiabatic noise to spontaneous decay is negligible, a necessary condition being \( t \ll T_i^A \propto S_i^A(\Omega_i)^{-1} \). This condition is usually satisfied at short enough times, since \( S_i^A(\omega) \) is substantially different from zero only at frequencies \( \omega \ll \Omega_i \).

This fact already suggests a route to trace-out different noise classes in the appropriate order. The total density matrix parametrically depends on the specific realization of the slow random drives \( \hat{X}(t) \equiv \{X_i^A(t)\} \) and may be written as \( W^{Q,A,SC}(t) = W^{Q,SC}(t) \hat{X}(t) \). The first step is to trace out
quantum noise. In the simplest cases, this requires solving a master equation. In a second stage, the average over all the realizations of the stochastic processes, \( \hat{X}(t) \), is performed. This leads to a reduced density matrix for the \( n \)-qubit system plus the SC degrees of freedom. These have to be traced out in a final stage by solving the Heisenberg equations of motion, or by approaches suitable to the specific microscopic Hamiltonian or interaction. For instance, the dynamics may be solved exactly for some special quantum impurity models at pure dephasing, \( \theta_i = 0 \), when impurities are longitudinally coupled to each qubit [18, 28]. The ordered multi-stage elimination procedure can be formally written as

\[
\rho^0(t) = T_{\text{SC}} \left\{ \int D[\hat{X}(t)] P[\hat{X}(t)] \text{Tr}_n \left[ W^{0,\text{SC}}(t|\hat{X}(t)) \right] \right\}.
\]

(6)

In the following section, we concentrate on the elimination of adiabatic noise. The ordered procedure will be illustrated in section 6 for a two-qubit gate.

5. Adiabatic noise

In general, the adiabatic approximation holds true for times short enough to fulfill the necessary condition, \( t \ll T_n^0 \). In the peculiar, pure dephasing regime, \( \theta_i = 0 \), relaxation processes are forbidden and the adiabatic approximation is exact for any \( S_X(\omega) \). In addition, the adiabatic scheme can be applied also in the presence of correlations between processes \( X_i(t) \) and \( X_j(t) \) [29]. The sake of clarity, here, we consider adiabatic noise affecting each qubit independently. Moreover, we exclude time-dependent drives in \( A(t) \) and in \( H(t) \). The procedure can be straightforwardly extended for instance to Rabi oscillations and other ac drives [30].

Suppose we are able to diagonalize, for each realization \( \tilde{X}(t) \) of the stochastic processes, the system Hamiltonian,

\[
\mathcal{H}(t) = -\sum_i \frac{1}{2} \vec{\sigma}_i \cdot \vec{\sigma}(t) + \sum_{i,j} H_{ij} + \frac{1}{2} \sum_i \sigma_z^{(i)} \otimes X_i(t)
\]

(7)

and denote \( |m(\tilde{X}_i)\rangle \) an instantaneous eigenstate of (7) with eigenvalue \( E_m(t) \). If the system is prepared in a pure state, \( |\psi_0\rangle \), in the adiabatic approximation, each component \( |m(\tilde{X}_i)\rangle \langle m(\tilde{X}_j) |\psi_0\rangle \) evolves in time according to

\[
|m(\tilde{X}_i)\rangle e^{i \Phi_m^{(D)}(t)} |m(\tilde{X}_j)\rangle \langle m(\tilde{X}_j) |\psi_0\rangle,
\]

where \( \Phi_m^{(D)} = -i \int_0^t ds E_m(s) \) is the dynamic phase. The stray fields \( \tilde{X}(t) \) affect the time evolution of the gate fidelity via modifications of the phases \( \Phi_m^{(D)} \) and of the eigenstates \( |m(\tilde{X}_i)\rangle \). In addition, \( |m(\tilde{X}_i)\rangle \) reflects imperfect preparation due to the random field. This occurs, for instance, when the system is prepared by a reset to a pure state vector \( |\psi_0\rangle \) followed by an initialization pulse whose effect depends also on the stray field at \( t = 0 \). The evolution of the system conditioned density matrix can be presented in a compact form

\[
\rho^0(t|\tilde{X}(s)) = \sum_{mp} R_{mp}[\tilde{X}_0, \tilde{X}_1] e^{-i \int_0^t ds \Omega_{mp}(s)},
\]

(8)

where the instantaneous splittings appearing in the phase factor are \( \Omega_{mp}(t) = -\Omega_{pm}(t) = E_m(t) - E_p(t) \), and we have introduced the operator

\[
R_{mp}[\tilde{X}_0, \tilde{X}_1] = |m(\tilde{X}_1)\rangle \langle m(\tilde{X}_0) | \rho(0) |p(\tilde{X}_0)\rangle \langle p(\tilde{X}_1)|, \tag{9}
\]

which contains information about the preparation and the eigenstate errors. Finally, the average of \( \rho^0(t|\tilde{X}(s)) \) over the stochastic process yields the system density matrix, which can be presented as a path integral

\[
\rho^0(t) = \int D[\tilde{X}(s)] P[\tilde{X}(s)] \rho^0(t|\tilde{X}(s)). \tag{10}
\]

Here \( P[\tilde{X}(s)] \) contains informations both on the stochastic processes and on details of the specific protocol. It is convenient to split it as follows:

\[
P[\tilde{X}(s)] = F[\tilde{X}(s)] \rho(0)|p(\tilde{X}(s))|p(\tilde{X}(s)).
\]

where \( p(\tilde{X}(s)) \) is the probability of the realization \( \tilde{X}(s) \). The filter function \( F[\tilde{X}(s)] \) describes the specific operation. For most present day experiments on solid-state qubits \( F(\tilde{X}(s)) = 1 \). For an open-loop feedback protocol, which allows initial control of some collective variable of the environment, say \( \tilde{X}_0 = 0 \), we should put \( F(\tilde{X}(s)) = \Pi(s, \tilde{X}_0) \).

A critical issue is the identification of \( p(\tilde{X}(s)) \) for specific noise sources, such as those displaying 1/f power spectrum. If we sample the stochastic process at times \( t_k = k \Delta t \), with \( \Delta t = 1/m \) and \( k = 0, \ldots, m \), we can identity

\[
p[\tilde{X}(s)] = \lim_{m \to \infty} p_{m+1}(\tilde{X}_m, t_1; \ldots; \tilde{X}_1, t_1; \tilde{X}_0, 0),
\]

(11)

where \( p_{m+1}(s) \) is an \( m + 1 \) joint probability and we have used the shorthand notation \( \tilde{X}_k \equiv \tilde{X}_{t_k} \). In the following, we will propose a systematic method to select only the relevant statistical information on the stochastic process out of the full characterization included in \( p[\tilde{X}(s)] \).

We would like to remark, at this point, that in the adiabatic treatment the word decoherence is perhaps abused. Decoherence is ultimately due to entanglement of the qubit to a quantum environment [31, 32], whereas classical adiabatic noise produces only de-focusing. However, unless the signal can be totally re-focused, and this is impossible in practice, the behavior of \( \rho(t) \) is the same as for true decoherence [33]. In other words, although proper and improper mixed states at the fundamental level are quite distinct concepts [34], they are not distinct in the density matrix description. We also observe that present day ‘applied’ research on solid-state coherent nanodevices for quantum information focuses on the ‘short-time’ dynamics, since a signal that is almost decayed is useless. In this context, methods such as the adiabatic approximation are valuable even if they are not accurate at long timescales and are not valid down to zero temperature.

5.1. Longitudinal approximation

The longitudinal approximation consists of neglecting modifications of the eigenstates \( |m(\tilde{X}_i)\rangle \) and preparation effects. Without loss of generality, we may assume vanishing average of the stochastic processes after the preparation pulse, \( \tilde{X}_0 \). The longitudinal approximation amounts to putting

\[\text{Correlated noise and cross-talk effects have been addressed in [29].}\]
\[ |m(\vec{X}_i)\rangle = |m(\vec{X}_0)\rangle = |m\rangle, \text{ where } |m\rangle \text{ is an eigenstate of the system Hamiltonian } \mathcal{H}. \text{ Then } \rho_{mp}(\vec{X}_0, \vec{X}_1) \approx |m\rangle \rho_{mp}^n(0) \langle p| \text{ is a projected element of the initial density matrix and equation (10) simplifies to} \]

\[ \rho_{mp}^n(t) = \rho_{mp}^n(0) \int D[\vec{X}(s)] P[\vec{X}(s)] e^{-i\int_0^t d\mathcal{H}(s)}. \tag{12} \]

The significance of the longitudinal approximation is easily illustrated for a single qubit gate, when the system Hamiltonian reduces to

\[ \mathcal{H}(t) = -\Omega/2 \cos \theta \sigma_z - \Omega/2 \sin \theta \sigma_x - 1/2 \sigma_z \otimes X(t). \tag{13} \]

In this case, retaining variations of the splitting \( \Omega \) amounts to considering only fluctuations of the length of the vector \( \vec{\Omega} \). This would be the only effect if the noise acted longitudinally, \( \theta = 0 \). Variations of the eigenstates \( \langle m|X_i\rangle \) originate instead from ‘transverse’ variations of \( \vec{\Omega} \).

The longitudinal assumption has been performed in [35] to discuss the effect of Gaussian adiabatic environments. The present approach automatically provides constraints on its validity and shows that whereas errors due to transverse fluctuations are weakly dependent on time, phase errors accumulate. Therefore, transverse noise in the adiabatic approximation has possibly some effect only at very short times, but the phase damping channel eventually prevails. These considerations strongly depend on the amplitude of the noise. We checked analytically and with simulations that they hold true for realistic figures of noise such as those measured in experiments by Zorin et al [36].

In the longitudinal approximation, diagonal elements of the reduced density matrix in the eigenstate basis, \( |m\rangle = |\pm\rangle \), do not decay. Instead, the decay of the off-diagonal elements results from \( \rho_{+-}(t) = \rho_{-+}(0) \exp[-i\Phi(t)] \), where the complex average phase becomes

\[ \Phi(t) = -\Omega t + i \ln \int D[\vec{X}(s)] P[\vec{X}(s)] e^{i\int_0^t d\mathcal{H}(s)} X(s). \tag{14} \]

We notice that the longitudinal approximation may be exact for certain protocols, which therefore are not affected by adiabatic transverse noise. For instance, this is the case if the system is described by (13) and it is prepared in a state \( \rho(0) = 1/2 (|\pm\rangle \langle \pm|) \). If we measure \( \sigma_z \), since \( \langle m(X_i)|\sigma_z|p(X_i)\rangle \) do not depend on \( X_i \), the result is not affected by transverse fluctuations and \( \langle \sigma_z(t)\rangle = \langle \sigma_z(0) \rangle \cos \theta \exp[i\Phi(t)] \). This the case of the decaying oscillations pattern measured with Ramsey interference [4], if the effect of imperfect \( \pi/2 \) pulses is negligible.

5.1.1. Static path approximation (SPA). A standard approximation of the path integrals (10) and (12) consists of neglecting the time dependence in the path, \( \vec{X}(s) = \vec{X}_0 \) and taking \( F[\vec{X}] = 1 \). In this SPA, the problem reduces to ordinary integrations with \( p(\vec{X}_0, 0) \equiv p(\vec{X}_0) \). In the single qubit case, for instance, equation (14) gives average phase shift

\[ \Phi(t) \approx -\Omega t + i \ln \int dX_0 p(X_0) e^{i\Omega X_0 t}, \tag{15} \]

where \( \Omega(X_0) = \sqrt[3]{\Omega^2 \sin \theta + X_0^2 + (\Omega \cos \theta)^2} \). Equation (15) describes the effect of a distribution of stray energy shifts \( \Omega_{mp}(X) - \Omega_{mp}(0) \) and corresponds to the rigid lattice line breadth contribution to inhomogeneous broadening [10]. In experiments with solid state devices, this approximation describes the measurement procedure consisting of signal acquisition and averaging over a large number \( N \) of repetitions of the protocol, for an overall time \( t_m \) (which may also be minutes in actual experiments). Due to slow fluctuations of the solid state environment calibration, the initial value, \( \Omega \sin \theta + X_0 \), fluctuates during the repetitions blurring the average signal, independently of the measurement being single-shot or not.

The probability \( p(X_0) \) describes the distribution of the random variable obtained by sampling the stochastic process \( X(t) \) at the initial time of each repetition, i.e. at times \( t_k = k t_m / N \), \( k = 0, N - 1 \). If \( X_0 \) results from many independent random variables of a multimode environment, the central limit theorem applies and \( p(X_0) \) is a Gaussian distribution with standard deviation \( \sigma \)

\[ \sigma^2 = \langle X^2 \rangle = \int \frac{d\omega}{\pi} S^\Lambda_\omega(\omega), \]

with integration limits \( 1/t_m \) and the high-frequency cut-off of the 1/f spectrum, \( \gamma_M \). In the SPA, the distribution standard deviation, \( \sigma \), is the only adiabatic noise characteristic parameter. If the equilibrium average of the stochastic process vanishes, equation (15) reduces to

\[ \Phi(t) \approx -\Omega t + i \ln \int dX_0 \frac{X_0^2}{2\pi \sigma^2} e^{X_0^2/2\sigma^2} e^{i\Omega X_0 t} d\omega. \tag{16} \]

A convenient approximation is obtained by expanding \( \Omega(X_0) \) to second order in \( X_0 \), which leads to [9]

\[ -i\Phi(t) = i\Omega t - \frac{1}{2} \left( \frac{(\cos \theta \sigma_t)^2}{1 + \sin^2 \sigma \sigma_t / \Omega} - 2 \ln \left( 1 + i \sin^2 \frac{\sigma t}{\Omega} \right) \right). \tag{17} \]

The short-times decay of coherent oscillations qualitatively depends on the working point. In fact, the suppression of the signal, \( \exp[I\omega \Phi(t)] \), turns from a \( \exp(-1/2 \omega^2) \) behavior at \( \theta = 0 \) to a power law, \( [1 + (\sin^2 \sigma_t / \Omega)^2]^{-1/2} \), at \( \theta = \pi/2 \). In these limits, equation (17) reproduces known results for Gaussian 1/f environments. In particular, at \( \theta = 0 \) we obtain the short-times limit, \( t \ll 1/\gamma_M \), of the exact result of [32]. At \( \theta = \pi/2 \), the short and intermediate times result of [35] is reproduced. The fact that results of a diagrammatic approach with a quantum environment, such as those of [35], can be reproduced and generalized already at the simple SPA level makes the semi-classical approach quite promising. It shows that, at least for not too long times (but surely longer than times of interest for quantum state processing), the quantum nature of the environment may not be relevant for the class of problems which can be treated in the Born–Oppenheimer approximation. Note also that the SPA itself has surely a wide validity since it does not require
information about the dynamics of the noise sources, provided they are slow [37]6.

5.1.2. Beyond SPA: first correction. Going beyond the SPA amounts to sampling more accurately the stochastic process \( \tilde{X}(s) \) in (11). The first correction to the SPA is obtained by parameterizing the random process as follows: \( \tilde{X}(s) = \tilde{X}_0 + \tilde{X}_0 - \frac{\tilde{X}_0 - \tilde{X}_0}{s} \). Inserting this expression in (10) and approximating \( p[X(s)] \) in (11) with the conditional probability \( p[X, t; X_0, 0] \), we obtain

\[
\rho(t) = \int d\tilde{X}_t d\tilde{X}_0 p_{\tilde{X}_0}(\tilde{X}_t, t; \tilde{X}_0, 0)\rho \left( t| \tilde{X}_0 + \frac{\tilde{X}_0 - \tilde{X}_0}{s} \right).
\]

(18)

The joint probability depends on the statistics of the noise sources. Therefore the first correction to the SPA distinguishes discrete and Gaussian processes. Considering again a single qubit, the average phase \( \langle 14 \rangle \) at the working point \( \theta = \pi/2 \) for generic Gaussian noise becomes

\[
i\Phi(t) = \frac{1}{2} \ln \left[ 1 + \frac{\gamma^2 (1 - \sigma(t))}{\Omega} \right] + \frac{1}{2} \ln \left[ 1 + \frac{\gamma^2 (\sigma(t))}{3\Omega} \right],
\]

where \( \pi(t) = \frac{1}{\Omega} \int_0^\infty (d\rho/\rho) S(\omega)(1 - e^{-i\omega t}) \) is a transition probability, depending on the stochastic process. For Ornstein–Uhlenbeck processes, it reduces to the result of [38]. The first correction suggests that the SPA, in principle, valid for \( t < 1/\gamma_M \), may have a broader validity. This is illustrated in figure 1, where the adiabatic approximation (numerical evaluation of equation (14)) is compared with the exact (numerical) dynamics in the presence of 1/f noise and with the analytic forms resulting from the SPA and its first correction. For 1/f noise due to a set of bistable impurities the SPA valid also for \( t \gg 1/\gamma_M \), if \( \gamma_M < \Omega \). Of course the adiabatic approximation is tenable if \( t < T_1^A \equiv 2/\gamma(\Omega) \).

By sampling more accurately the adiabatic process \( \tilde{X}(t) \), it is possible to selectively include the statistical information needed for the specific measurement process. For instance, echo protocols are able to partly re-focus the signal, in other words de-focusing described by the SPA is almost canceled by echo pulse sequence. The decay of the echo signal is due to the uncanceled dynamics of the low-frequency fluctuations and to quantum noise. The leading effect of adiabatic noise can be estimated by a proper parameterization of \( \tilde{X}(t) \), similar to the one considered in the present paragraph [30].

6. Two-qubit universal gate: multi-stage approach

In the present section, we apply the multi-stage approach to a universal two-qubit gate based on a fixed coupling scheme. Capacitive and inductive fixed couplings [39] have been used to demonstrate two-qubit logic gates in different JJ implementations [40]. Entanglement is generated by tuning single-qubit energy spacing to achieve mutual resonance. To this end, during the gate operation at least one qubit has to be moved away from the working point of minimal sensitivity to parameter variations, the ‘optimal point’. This has so far represented the main drawback of the fixed-coupling scheme for Josephson implementations, with the exception of phase qubits [41]. More recent proposals have attempted to solve this problem by introducing tunable coupling schemes [42]. Most of them rely on additional circuit elements and gain their tunability from ac-driving [43] or from ‘adiabatic’ couplers [44]. Some of these coupling schemes have been tested in experiments and are potentially scalable [45]. None of these implementations is, however, totally immune from imperfections. In general, introducing additional on-chip circuit elements opens new ports to noise. The possibility to employ ‘minimal’ fixed coupling schemes has been recently reconsidered in [46], pointing out the possibility to single out ‘optimal coupling’ conditions which ensure reasonable protection from 1/f fluctuations.

In order to implement a \( \sqrt{i - \text{SWAP}} \) gate in a fixed coupling scheme, we need two resonant qubits with a transverse coupling, as described by the Hamiltonian

\[
H_0 = -\frac{\Omega}{2} \sigma_z^{(1)} \otimes \sigma_z^{(2)} - \frac{\Omega}{2} \sigma_z^{(1)} \otimes \sigma_z^{(2)} + \frac{\omega_c}{2} \sigma_z^{(1)} \otimes \sigma_z^{(2)},
\]

(19)

where \( \omega_c \) is the coupling strength, and \( \sigma_z^{(i)} \) the pseudo-spin operators whose eigenstates \( |\pm\rangle \) (eigenvalues \( \pm 1 \)) are the computational states of qubit \( i \). Eigenvalues and eigenvectors

---

6 Equation (17) is also valid for Ornstein–Uhlenbeck processes, see [38]. For random telegraph noise the discrete nature of the process modifies this result see [37].
Each noise component $\dot{X}_i$ has broadband spectrum $S_X(\omega) = \frac{\omega}{\pi}$, $\omega \in \{\gamma_m, \gamma_M\}$ followed by a white flank at frequencies $\omega \gg \gamma_m$. Correlated noise sources acting on both qubits have been addressed in [29].

If the system is initialized in the SWAP subspace, for instance in the state $|+−\rangle$, bitwise readout gives the qubit 1 switching probability $P^{(1)}(t)$, i.e. the probability that it will pass to the state $|−\rangle$, and the probability $P^{(2)}(t)$ of finding the qubit 2 in the initial state $|−\rangle$. Cyclic anti-correlation of the probabilities signals the formation of the entangled state during the $\sqrt{1-}\text{SWAP}$ operation. In terms of the two qubit reduced density matrix in the eigenstate basis the switching probabilities read

$$P^{(1)}(t) = \langle−|\text{Tr}_2\rho(t)|−\rangle = \frac{1}{2} [\rho_{11}(t) + \rho_{22}(t)] + \rho_{00}(t) \sin^2 \frac{\varphi}{2} + \text{Re}[\rho_{12}(t)] + \text{Re}[\rho_{03}(t)] \sin \varphi,$$

$$P^{(2)}(t) = \langle−|\text{Tr}_1\rho(t)|−\rangle = \frac{1}{2} [\rho_{11}(t) + \rho_{22}(t)] + \rho_{00}(t) \sin^2 \frac{\varphi}{2} - \text{Re}[\rho_{12}(t)] + \text{Re}[\rho_{13}(t)] \sin \varphi.$$

The matrix elements entering the above probabilities can be evaluated in the multi-stage approach.

### 6.1. Multi-stage approach

We split the interaction as in equation (5), $\dot{X}_i = \dot{X}^{\dagger}_{i\theta} + X_i(t)$. Low-frequency noise is treated in the adiabatic and longitudinal approximation. In addition, we limit the analysis to the SPA. We denote with $\omega_i(X_1, X_2)$ the eigenvalues of $\mathcal{H}_0 + \mathcal{H}_1$ with $\dot{X}_i \rightarrow X_i$.

#### 6.1.1. First stage: elimination of quantum noise.

Quantum noise is traced out by solving the Born–Markov master equation for the reduced density matrix [49] with eigenvalues parametrically dependent on the random fields $\{X_i\}$. In the system eigenstate basis and performing the secular approximation (to be self-consistently checked) it takes the standard form

$$\dot{\rho}_{ij}(t) = -\sum_{m \neq j} \Gamma_{im} \rho_{ij}(t) + \sum_{m \neq i} \Gamma_{mj} \rho_{mi}(t),$$

$$\dot{\rho}_{ij}(t) = -\langle i|\omega_j + \bar{\Gamma}_{ij}\rangle \rho_{ij}(t),$$

where $\omega_j = \omega_j(X_1, X_2) - \omega_j(X_1, X_2)$. The rates $\Gamma_{im}$, $\bar{\Gamma}_{ij}$ depend on the real parts of the lesser and greater Green’s functions, describing emission (absorption) rates to (from) the quantum reservoirs [47]. In addition, for white quantum noise energy shifts are vanishing and do not appear in equation (28).

Because of the symmetry of (19), dissipative transitions inside each—SWAP or Z—subspace are forbidden. The allowed inelastic energy exchange processes are evidenced in figure 2.

---

7 In principle, noise can also couple longitudinally, i.e. via $\sigma^{(1)}_z$ [46]. This is the case of the charge-phase two-port design of Vion et al [4].
The only independent emission rates are $\Gamma_{10} = \Gamma_{32}$, $\Gamma_{20} = \Gamma_{13}$. They read
\[ \Gamma_{10} = \frac{1}{8}(1 + \sin \varphi)[C_X(\omega_{10}) + C_X(\omega_{10})], \]
\[ \Gamma_{20} = \frac{1}{8}(1 - \sin \varphi)[C_X(\omega_{20}) + C_X(\omega_{20})], \] (29)
where the absorption rates, $C_X(\omega) = \frac{2S_X^Q(\omega)}{1 + \exp(-\omega/\kappa_T)}$, are related to the spectrum of quantum noise, $S_X^Q(\omega)$. Emission rates have the same form with $C_X(\omega_{10})$ replaced by $C_X(\omega_{13})$. For the considered initial condition, $| + - \rangle$, the only non-vanishing elements of the reduced density matrix in the eigenbasis of $\hat{H}_0$ are the populations and the SWAP coherence, $\rho_{12}(t)$. For independent quantum noise sources acting on the two qubits, the SWAP decay rate reads $\Gamma_{12} = \frac{1}{4}[(\Gamma_{10} + \Gamma_{30}) + (\Gamma_{20} + \Gamma_{12})] = \frac{1}{2}(\Gamma_{11} + \Gamma_{22})$, where $\Gamma_{i} = \Gamma_{i0} + \Gamma_{i3}$, $i = 1, 2$, are the relaxation rates of the SWAP levels. From equation (28) we obtain
\[ \rho_{21}(t) = -\frac{i}{2}e^{i\omega_{12}(X_1 - X_2)}e^{-\hat{T}_{12}}. \] (30)

The secular approximation is valid provided that $\omega_{21} \approx \omega_c \gg \Gamma_{12}$. This condition is fulfilled, for instance, for white noise levels extrapolated from single charge-phase qubit experiments, as can be evinced from the slow decay due to quantum noise reported in figure 3, gray line.

Equations (27) for the populations do not decouple even in the secular limit. General solutions are quite lengthy. Here we report the approximate forms in the small temperature limit with respect to the uncoupled qubits splittings, $k_B T \ll \Omega$. In this case, level 3 remains populated and
\[ \rho_{00}(t) \approx 1 - \frac{i}{2}[e^{-\Gamma_{30}} + e^{-\Gamma_{03}}], \]
\[ \rho_{11}(t) \approx \frac{i}{2}e^{-\hat{T}_{01}}, \]
\[ \rho_{22}(t) \approx \frac{i}{2}e^{-\hat{T}_{12}}. \] (31)

6.1.2. Second stage: elimination of adiabatic noise. We now consider the effect of low frequencies in the adiabatic, longitudinal and static path approximations. Populations are unaffected by adiabatic noise, whereas the SWAP coherence, $\rho_{12}(t)$, has to be averaged over $X_1$. Here we disregard the negligible $X_1$ dependence of the rates $\Gamma_{ij}$ via $\omega_{ij}$. Under these simplifying assumptions, we are left with the following average:
\[ \rho_{21}(t) \approx -\frac{1}{2}e^{-\hat{T}_{12}} \int dX_1 dX_2 p(X_1) p(X_2) e^{i\omega_{12}(X_1 - X_2)}dt. \] (32)
with $p(X_1) = \frac{1}{\sqrt{2\pi \sigma}} \exp[-X_1^2/2\sigma^2]$. The SWAP splitting can be estimated by treating in perturbation theory $\hat{H}_1$, with $\hat{X}_1 \rightarrow \{X_1\}$, with respect to $H_0$. This leads to
\[ \omega_{21}(X_1, X_2) \approx \omega_c \left( \frac{\omega_c}{2\sigma^2} (X_1^2 + X_2^2) + \frac{\omega_c}{8\sigma^2} \left( 1 + \frac{\omega_c^2}{\Omega^2} \right) \right. \]
\[ \times \left. \left( X_1^2 + 6X_1^2X_2^2 + 2X_2^4 \right) + \frac{1}{8\sigma^2 \Omega^2} (X_1^2 - X_2^2)^2. \right. \] (33)

The average in equation (32) can be evaluated in the analytic form and gives [46]
\[ \rho_{12}(t) = -\frac{1}{2}e^{-\hat{T}_{12}} \left( \int \frac{2\omega_c}{\pi \sigma^2} e^{i\omega_{12}(t+h)(h)} K_0[h(t)] \right) \] (34)
where $h(t) = \omega_c t (1/\Omega^2/\sigma^2 + i\omega_c t)^2/(4\Omega^2)$ and $K_0[h]$ is the K-Bessel function of order zero [48]. We considered equal standard deviations for noise sources acting on both qubits, $\sigma_i = \sigma$. Inserting (31) and (34) in (25) and (26), we obtain the switching probabilities in the multi-stage approach. Out of phase oscillations signals two-qubit states anti-correlations and follows from $P^{(2)}(t) = P(t) \pm \Re[\rho_{12}(t)]$, with $P(t) = -\frac{1}{2} \cos \varphi [\rho_{11}(t) + \rho_{22}(t)] + \cos^2\frac{\varphi}{2} \approx -\frac{1}{2} \cos \varphi [e^{-T_{01}} + e^{-T_{12}}] + g^2/4$.

The efficiency of the gate results from the interplay of quantum and adiabatic noise. High-frequency noise levels expected from single qubit experiments [13] weakly affect the switching probability, whose decay is mainly due to low-frequency noise, figure 3. Remarkably, considerable recovery of short-times oscillation amplitude may be achieved by an optimal choice of the coupling strength, $\omega_c \approx \sigma$ [46]. This regime is illustrated in figure 4.

7. Conclusions
In this article, we have presented a road-map to treat broadband noise typical of solid state nanodevices. The introduced multi-stage approach allows us to obtain reasonable approximations by systematically including only the relevant information on the complex environment, out of the huge parameterization which would be required for a microscopic description. Since the environment is in general long-time correlated, the required information depends on the specific protocol.

The predictions obtained with the present approach are in agreement with observations in various single qubit JJ implementations and in different protocols [8, 9]. We extended the procedure to deal with complex solid-state
architectures. This is a required step in order to predict efficiency and possibly appropriately design nanodevices for quantum information processing. Both because of the complexity of architectures and of the unavoidable broadband nature of solid state noise, theoretical tools allowing systematic and controlled approximations are particularly valuable.

As an illustrative case, we performed a simplified analysis of the effects and interplay of low and high frequency noise components in a two-qubit gate in a fixed coupling scheme. Our results point out that efficient operations in the solid state require an accurate preliminary characterization of the noise spectral characteristics and appropriate tuning of the device working point.

Acknowledgments

We acknowledge support from the EU-EuroSQIP (IST-3-015708-IP).

References

[1] Nielsen M and Chuang I 2005 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[2] Hyashi T et al 2003 Phys. Rev. Lett. 91 226804
Petta J R et al 2004 Phys. Rev. Lett. 93 186802
Petta J R et al 2005 Science 309 2180
Gorman J, Hasko D G and Williams D A 2005 Phys. Rev. Lett. 95 090502
Koppens F H L et al 2006 Nature 442 766
[3] Nakamura Y et al 1999 Nature 398 786
Yu Y et al 2002 Science 296 889
Martinis J M et al 2002 Phys. Rev. Lett. 89 117901
Chiorescu I et al 2003 Science 299 1869
Yamamoto T et al 2003 Nature 425 941
Saito S et al 2004 Phys. Rev. Lett. 93 037001
Johansson J et al 2006 Phys. Rev. Lett. 96 127006
Deppe F et al 2008 Nat. Phys. 4 686
[4] Vion D et al 2002 Science 296 886
[5] Koch J et al 2007 Phys. Rev. A 76 042319
Schreier J A et al 2008 Phys. Rev. B 77 180502
[6] Schuster D I et al 2004 Nature 431 162
Schuster D I et al 2007 Nature 445 515
Göppl M et al 2008 Nature 454 315
Bishop L S et al 2008 Nat. Phys. 5 105
[7] Nakamura Y et al 2002 Phys. Rev. Lett. 88 047901
[8] Falci G, Paladino E and Fazio R 2003 Quantum Phenomena of Mesoscopic Systems Proc. Int. School ‘Enrico Fermi’ (Varenna 2002) ed B L Altschuler and V Tognetti (Amsterdam: IOS Press) (arXiv:cond-mat/0312550)
[9] Falci G et al 2005 Phys. Rev. Lett. 94 167002
[10] Slichter C P 1996 Principles of Magnetic Resonance (Berlin: Springer)
[11] Kafanof S et al 2008 Phys. Rev. B 78 125411
Astařev O et al 2004 Phys. Rev. Lett. 93 267007
[13] Ithier G et al 2005 Phys. Rev. B 72 134519
[14] Simmonds R W et al 2004 Phys. Rev. Lett. 93 077003
Cooper K B et al 2004 Phys. Rev. Lett. 93 180401
[15] Eroms J et al 2006 Appl. Phys. Lett. 89 122516
[16] Van der Wal C H et al 2000 Science 290 773777
[17] Duty T et al 2004 Phys. Rev. B 69 140503
[18] Galperin Y M et al 2006 Phys. Rev. Lett. 96 097009
Bergli J, Galperin Y M and Altschuler B L 2009 New J. Phys. 11 025002
[19] Chiarello F 2008 private communication
Poletto S et al 2008 New J. Phys. 11 013009
Fazio R, Palma G M and Siewert J 1999 Phys. Rev. Lett. 83 5385
Weissman M B 1988 Rev. Mod. Phys. 60 537
Clausn J et al 2007 Phys. Rev. B 76 024508
Tian L and Simmonds RW 2007 Phys. Rev. Lett. 99 137002
Deppe F et al 2007 Phys. Rev. B 76 214503
Kim Z et al 2008 Phys. Rev. B 78 144506
Lupascu A et al 2008 arXiv:0810.0590
Zagoskin A M et al 2006 Phys. Rev. Lett. 97 077001
Neeley M et al 2008 Nat. Phys. 4 523
[26] Paladino E et al 2008 Phys. Rev. B 77 041303
Ashab S et al 2006 New J. Phys. 8 103
Oxtoby N P et al 2009 New J. Phys. 11 063028
[27] Cohen-Tannoudji C, Dupont-Roc J and Grynberg G 1993 Atom–Photon Interactions (New York: Wiley-Interscience)
Paladino E et al 2002 Phys. Rev. Lett. 88 228304
D’Arrigo A et al 2008 New J. Phys. 10 115006
Fazio G et al in preparation
[30] Zurek W 1991 Phys. Today 44 36
Palma G M, Suominen K A and Ekert A K 1996 Proc. R. Soc. A 452 567
Cory D G et al 1998 Phys. Rev. Lett. 81 2152
Isham C 1995 Lectures on Quantum Theory: Mathematical and Structural Foundations (London: Imperial College Press)
Makhlin Y and Shnirman A 2004 Phys. Rev. Lett. 92 178301
Zorin A B et al 1996 Phys. Rev. B 53 13682
Paladino E et al 2003 Adv. Sol. State Phys. 43 747
Rabenstein K, Sverdlov V A and Averin D V 2004 JETP Lett. 79 646
Makhlin Yu et al 1999 Nature 398 305
Youn J Q et al 2002 Phys. Rev. Lett. 89 197902
Pashkin Yu A et al 2003 Nature 421 823
Berkley A J et al 2003 Science 300 1548
Yamamoto T et al 2003 Nature 425 941
McDermott R et al 2005 Science 307 1299
Majer J B et al 2005 Phys. Rev. Lett. 94 090501
Saffman M et al 2006 Science 313 1423
Plantenberg J H et al 2007 Nature 447 836
Sillanpää M A, Park J I and Simmonds R K 2007 Nature 449 438

Figure 4. Effect of adiabatic noise under optimal tuning: switching probability of qubit 1 in the presence of low-frequency noise parameterized by $\alpha/\Omega = 0.08$ for ‘optimal coupling’ $\omega_c/\Omega = 0.08$ (orange) and non-optimal tuning $\omega_c/\Omega = 0.01$ (red).
[42] Averin D V and Bruder C 2003 Phys. Rev. Lett. 91 057003
    Blais A et al 2003 Phys. Rev. Lett. 90 127901
    Plastina F and Falci G 2003 Phys. Rev. B 67 224514
    Plourde B et al 2004 Phys. Rev. B 70 140501(R)
    Niskanen A O, Nakamura Y and Tsai J S 2006 Phys. Rev. B 73 094506
    Bertet P, Harmans C J and Mooij J E 2006 Phys. Rev. B 73 064512
    Wang Y-D, Kemp A and Semba K 2009 Phys. Rev. B 79 024502

[43] Rigetti C, Blais A and Devoret M 2005 Phys. Rev. Lett. 94 240502
    Liu Y-X et al 2006 Phys. Rev. Lett. 96 067003
    Paraoanu G S 2006 Phys. Rev. B 74 140504(R)

[44] Filippov T V et al 2003 IEEE Trans. Appl. Supercond. 13 1005
    Lantz J et al 2004 Phys. Rev. B 70 140507(R)
    Maassen van den Brink A et al 2005 New J. Phys. 7 230

[45] Izmalkov A et al 2004 Phys. Rev. Lett. 93 037003
    Grajcar M et al 2005 Phys. Rev. B 72 02503
    Hime T et al 2006 Science 314 1427
    Niskanen A et al 2006 Science 316 723
    Niskanen A O et al 2007 Nature 447 386
    Majer J et al 2007 Nature 449 443
    van der Ploeg S H W et al 2007 Phys. Rev. Lett. 98 057004
    Fay A et al 2008 Phys. Rev. Lett. 100 187003
    Niskanen A O et al 2008 Phys. Rev. B 77 064505
    Harrabi K et al 2009 Phys. Rev. B 79 020507

[46] Paladino E, Mastellone A, D’Arrigo A and Falci G 2009 arXiv:0906.3115

[47] Weiss U 2008 Quantum Dissipative Systems 3rd edn
    (Singapore: World Scientific)

[48] Abramowitz M and Stegun I A 1965 Handbook of Mathematical Functions
    (New York: Dover)

[49] Paladino E et al 2009 Physica E at press doi:
    10.1016/jphyse.2009.06.0426