STRUCTURE FUNCTIONS AND MINIMAL PATH SETS

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Abstract. We give and discuss a general multilinear expression of the structure function of an arbitrary semicoherent system in terms of its minimal path and cut sets. We also examine the link between the number of minimal path and cut sets consisting of 1 or 2 components and the concept of structural signature of the system.

1. Introduction

Consider an $n$-component system $(C, \phi)$, where $C$ is the set $[n] = \{1, \ldots, n\}$ of its components and $\phi: \{0, 1\}^n \to \{0, 1\}$ is its structure function which expresses the state of the system in terms of the states of its components. We assume that the system is semicoherent, which means that the structure function is nondecreasing in each variable and satisfies the conditions $\phi(0, \ldots, 0) = 0$ and $\phi(1, \ldots, 1) = 1$.

Throughout we identify Boolean $n$-vectors $x \in \{0, 1\}^n$ and subsets $A \subseteq [n]$ in the usual way, that is, by setting $x_i = 1$ if and only if $i \in A$. This identification enables us to use the same symbol to denote both a function $f: \{0, 1\}^n \to \mathbb{R}$ and the corresponding set function $f: 2^{[n]} \to \mathbb{R}$ interchangeably. For instance, we write $\phi(0, \ldots, 0) = \phi(\emptyset)$ and $\phi(1, \ldots, 1) = \phi(C)$.

As a Boolean function, the structure function can always be written in the multilinear form

$$\phi(x) = \phi(x_1, \ldots, x_n) = \sum_{A \subseteq C} \phi(A) \prod_{i \in A} x_i \prod_{i \in C \setminus A} (1 - x_i).$$

Since the coefficients in this form are exactly the structure function values, we will refer to this form as the self-descriptive form of the structure function. By expanding the second product in Eq. (1) and then collecting terms, we obtain the simple form of the structure function, namely

$$\phi(x) = \sum_{A \subseteq C} d(A) \prod_{i \in A} x_i,$$

where the link between the new coefficients $d(A)$ and the values $\phi(A)$, which can be obtained from the Möbius inversion theorem, is given through the following linear conversion formulas (see, e.g., [9, p. 31])

$$\phi(A) = \sum_{B \subseteq A} d(B) \quad \text{and} \quad d(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \phi(B).$$

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Recall that a path set of the system is a component subset $P \subseteq C$ such that $\phi(P) = 1$. A path set $P$ of the system is said to be minimal if $\phi(P') = 0$ for every $P' \subset P$. It is well known [3, Ch. 2] that if $P_1, \ldots, P_r$ denote the minimal path sets of the system, then

$$\phi(x) = \bigotimes_{j \in [r]} \prod_{i \in P_j} x_i = 1 - \prod_{j \in [r]} \left( 1 - \prod_{i \in P_j} x_i \right),$$

where $[r] = \{1, \ldots, r\}$ and $\bigotimes$ is the coproduct operation (i.e., the dual of the product operation) defined by $\bigotimes_i x_i = 1 - \prod_i (1 - x_i)$.

**Example 1.** Consider the bridge structure as indicated in Figure 1. This structure is characterized by four minimal path sets, namely $P_1 = \{1, 4\}$, $P_2 = \{2, 5\}$, $P_3 = \{1, 3, 5\}$, and $P_4 = \{2, 3, 4\}$. Equation (3) then shows that the structure function is given by

$$\phi(x_1, \ldots, x_5) = x_1 x_4 \bigotimes x_2 x_5 \bigotimes x_1 x_3 x_5 \bigotimes x_2 x_3 x_4 .$$

The simple form of the structure function can be easily computed by expanding the coproducts in Eq. (3) and simplifying the resulting algebraic expression using $x_i^2 = x_i$. We then obtain

$$\phi(x_1, \ldots, x_5) = x_1 x_4 + x_2 x_5 + x_1 x_3 x_5 + x_2 x_3 x_4$$

$$- x_1 x_2 x_3 x_4 - x_1 x_2 x_3 x_5 - x_1 x_2 x_4 x_5 - x_1 x_3 x_4 x_5 - x_2 x_3 x_4 x_5$$

$$+ 2 x_1 x_2 x_3 x_4 x_5 ,$$

which reveals the coefficients $d(A)$ of the simple form of the structure function. □

![Figure 1. Bridge structure](image-url)

Example 1 illustrates the important fact that the simple form (2) of any structure function can be expressed in terms of the minimal path sets of the system simply by expanding the coproduct in Eq. (3) and then simplifying the resulting polynomial expression (using $x_i^2 = x_i$) until it becomes multilinear. It seems, however, that such a general expression for the structure function is unknown in the literature.

In Section 2 of this note we yield an expression of the simple form of the structure function in terms of the minimal path sets. The derivation of this expression is inspired from the exact computation of the reliability function of the system by means of the inclusion-exclusion principle. We also provide the dual version of this expression in terms of the minimal cut sets and discuss some interesting consequences of these expressions. In Section 3 we show that the number of minimal path and cut sets consisting of 1 or 2 components can be computed easily from the concept of structural signature of the system.
2. Structure functions and minimal path and cut sets

By extending formally the structure function to the hypercube $[0, 1]^n$ by linear interpolation, we define its multilinear extension (a concept introduced in game theory by Owen [8]) as the multilinear polynomial function $\hat{\phi} : [0, 1]^n \to [0, 1]$ defined by

$$\hat{\phi}(x) = \sum_{A \subseteq C} \phi(A) \prod_{i \in A} x_i \prod_{i \in C \setminus A} (1 - x_i).$$

Let $\phi^D : \{0, 1\}^n \to \{0, 1\}$ be the dual structure function defined as $\phi^D(x) = 1 - \phi(1 - x)$, where $1$ stands for the $n$-vector $(1, \ldots, 1)$, and let $d^D(A) (A \subseteq C)$ be the coefficients of the simple form of $\phi^D$. By using the dual structure function we can easily derive various useful forms of the structure function and its multilinear extension (see also Grabisch et al. [5]). Table 1 summarizes the best known of these forms (in addition to the minimal path set representation given in Eq. (3)).

| Name                        | $\phi(x)$ and $\phi(x)$ |
|------------------------------|--------------------------|
| Self-descriptive form        | $\sum_{A \subseteq C} \phi(A) \prod_{i \in A} x_i \prod_{i \in C \setminus A} (1 - x_i)$ |
| Dual self-descriptive form   | $1 - \sum_{A \subseteq C} \phi^D(A) \prod_{i \in C \setminus A} x_i \prod_{i \in C \setminus A} (1 - x_i)$ |
| Simple form                  | $\sum_{A \subseteq C} d(A) \prod_{i \in A} x_i$ |
| Dual simple form             | $\sum_{A \subseteq C} d^D(A) \prod_{i \in A} x_i$ |
| Disjunctive normal form      | $\prod_{A \subseteq C} \phi(A) \prod_{i \in A} x_i$ |
| Conjunctive normal form      | $\prod_{A \subseteq C} \phi^D(A) \prod_{i \in A} x_i$ |

Table 1. Various forms of the structure function and its multilinear extension

The concept of multilinear extension of the structure function has the following important interpretation in reliability theory. When the components are statistically independent, the function $\hat{\phi}$ is nothing other than the reliability function $h_\phi : [0, 1]^n \to [0, 1]$, which gives the reliability

$$h_\phi(p) = h_\phi(p_1, \ldots, p_n) = \sum_{A \subseteq C} \phi(A) \prod_{i \in A} p_i \prod_{i \in C \setminus A} (1 - p_i)$$

of the system in terms of the reliabilities $p_1, \ldots, p_n$ of the components (see, e.g., [4], Ch. 2)).
The exact computation of system reliability $h_{\phi}(p)$ in terms of minimal path sets $P_1, \ldots, P_r$ is usually done by means of the inclusion-exclusion method (see, e.g., [1 Sect. 6.2] and [3 Ch. 2]). In this section we recall this method and show how we can adapt it to derive an expression of the simple form of the structure function in terms of the minimal path sets of the system.

For every $j \in [r]$, let $E_j$ be the event that all components in minimal path set $P_j$ work. Then, using the inclusion-exclusion formula for probabilities, we obtain

$$h_{\phi}(p) = \Pr\left( \bigcup_{j \in [r]} E_j \right) = \sum_{\emptyset \neq B \subseteq [r]} (-1)^{|B|-1} \Pr\left( \bigcap_{j \in B} E_j \right).$$

Let $D_i$ denote the event that component $i$ works. Then, we have $p_i = \Pr(D_i)$ and, using the independence assumption, we have

$$\Pr(E_j) = \Pr\left( \bigcap_{i \in P_j} D_i \right) = \prod_{i \in P_j} p_i$$

and, more generally,

$$\Pr\left( \bigcap_{j \in B} E_j \right) = \Pr\left( \bigcap_{j \in B} \bigcap_{i \in P_j} D_i \right) = \Pr\left( \bigcap_{i \in \bigcup_{j \in B} P_j} D_i \right) = \prod_{i \in \bigcup_{j \in B} P_j} p_i.$$

Substituting Eq. (6) in Eq. (5), we obtain the following multilinear expression of $h_{\phi}(p)$ in terms of the minimal path sets of the system

$$h_{\phi}(p) = \sum_{\emptyset \neq B \subseteq [r]} (-1)^{|B|-1} \prod_{i \in \bigcup_{j \in B} P_j} p_i,$$

or equivalently,

$$h_{\phi}(p) = \sum_{j \in [r]} \prod_{i \in P_j} p_i - \sum_{\{j,k\} \subseteq [r]} \prod_{i \in P_j \cup P_k} p_i + \sum_{\{j,k,l\} \subseteq [r]} \prod_{i \in P_j \cup P_k \cup P_l} p_i - \cdots$$

We now show that the corresponding formula for the structure function can be obtained similarly without an appeal to the independence assumption on the system components. Actually, our result and its proof are purely combinatorial and does not need probability theory.

**Theorem 1.** If $P_1, \ldots, P_r$ denote the minimal path sets of the system, then

$$\phi(x) = \sum_{\emptyset \neq B \subseteq [r]} (-1)^{|B|-1} \prod_{i \in \bigcup_{j \in B} P_j} x_i.$$

**Proof.** The proof relies on the classical polynomial inclusion-exclusion formula

$$1 - \prod_{j \in [r]} (1 - z_j) = \sum_{\emptyset \neq B \subseteq [r]} (-1)^{|B|-1} \prod_{j \in B} z_j,$$

$z_1, \ldots, z_n \in \mathbb{R}$.

Combining this formula with the right-hand expression in Eq. (8), we immediately obtain

$$\phi(x) = \sum_{\emptyset \neq B \subseteq [r]} (-1)^{|B|-1} \prod_{j \in B} \prod_{i \in P_j} x_i.$$  

Formula (8) then follows by simplifying the latter expression using $x_i^2 = x_i$. □

Clearly, Eq. (8) still holds on $[0,1]^n$ if we replace the structure function with its multilinear extension. In particular, Eq. (7) immediately follows from Eq. (8).
Example 2. Consider a 4-component system defined by the three minimal path sets $P_1 = \{1, 2\}$, $P_2 = \{2, 3\}$, and $P_3 = \{3, 4\}$. The constituting elements of the

| $B$   | $(-1)^{|B|} - 1$ | $\bigcup_{j \in B} P_j$ |
|-------|------------------|--------------------------|
| $\{1\}$ | 1                | $\{1, 2\}$              |
| $\{2\}$ | 1                | $\{2, 3\}$              |
| $\{3\}$ | 1                | $\{3, 4\}$              |
| $\{1, 2\}$ | $-1$           | $\{1, 2, 3\}$          |
| $\{1, 3\}$ | $-1$           | $\{1, 2, 3, 4\}$       |
| $\{2, 3\}$ | $-1$           | $\{2, 3, 4\}$          |
| $\{1, 2, 3\}$ | 1              | $\{1, 2, 3, 4\}$       |

Table 2. Example 2

sum in Eq. (8) are gathered in Table 2. Summing up the monomials defined by the subsets given in the third column, each multiplied by the corresponding number (+1 or $-1$) from the second column, by Eq. (8) we obtain

$$
\phi(x_1, x_2, x_3, x_4) = x_1 x_2 + x_2 x_3 + x_3 x_4 - x_1 x_2 x_3 - x_2 x_3 x_4,
$$

which is the simple form of the structure function. □

Interestingly, Theorem 1 enables us to identify the minimal path sets of the system from the simple form of the structure function by quick inspection. We state this result in the following immediate corollary.

Corollary 2. The minimal path sets $P_1, \ldots, P_r$ are exactly the minimal elements (with respect to inclusion) of the family of subsets defined by the monomials (or equivalently, the monomials with coefficient +1) in the simple form of the structure function.

Corollary 2 enables us to reconstruct the minimal path set representation (3) of the structure function from its simple form. Considering for instance the simple form of the bridge structure function as described in Example 1 by Corollary 2 we see that the corresponding minimal path sets are $P_1 = \{1, 4\}$, $P_2 = \{2, 5\}$, $P_3 = \{1, 3, 5\}$, and $P_4 = \{2, 3, 4\}$. We then immediately retrieve Eq. (4).

Theorem 1 has the following additional consequence. Recall that a formation of a subset $A$ of $C$ is a collection of minimal path sets whose union is $A$. A formation of $A$ is said to be odd (resp. even) if it is the union of an odd (resp. even) number of minimal path sets. Note that a particular formation can be both odd and even. By equating the corresponding terms in Eqs. (2) and (8), we obtain the following identity

$$
d(A) = \sum_{\emptyset \neq B \subseteq |r| \bigcup_{j \in B} P_j = A} (-1)^{|B| - 1}.
$$

From this identity, we immediately retrieve the fact (see, e.g., Barlow and Iyer [2]) that the coefficient $d(A)$ is exactly the number of odd formations of $A$ minus the number of even formations of $A$.

A dual argument enables us to yield an expression of the simple form of the structure function in terms of the minimal cut sets of the system. Recall that a subset $K$ of $C$ is a cut set of the system if $\phi(C \setminus K) = 0$. It is minimal if
\( \phi(C \setminus K') = 1 \) for every \( K' \subsetneq K \). If \( K_1, \ldots, K_s \) denote the minimal cut sets of the system, then

\[
\phi(x) = \prod_{j \in [s]} \prod_{i \in K_i} x_i = \prod_{j \in [s]} \left(1 - \prod_{i \in K_i} (1 - x_i)\right).
\]

Starting from the well-known fact that the minimal cut sets of the system are the minimal path sets of the dual, and vice versa, we immediately derive from Theorem 1 and Corollary 2 the following dual versions.

**Theorem 3.** If \( K_1, \ldots, K_s \) denote the minimal cut sets of the system, then

\[
(9) \quad \phi^D(x) = \sum_{\emptyset \neq B \subseteq [r]} (-1)^{|B|-1} \prod_{i \in \bigcup_{j \in B} K_j} x_i.
\]

**Corollary 4.** The minimal cut sets \( K_1, \ldots, K_s \) are exactly the minimal elements (with respect to inclusion) of the family of subsets defined by the monomials (or equivalently, the monomials with coefficient +1) in the simple form of the dual structure function.

Locks [6] described a method for generating all minimal cut sets from the set of minimal path sets (and vice versa) using Boolean algebra. Interestingly, an alternative method consists in applying Corollary 4 to the dual structure function \( \phi^D(x) = 1 - \phi(1 - x) = \prod_{j \in [r]} \prod_{i \in P_j} x_i \).

Consider for instance the structure function defined in Example 2. The dual structure function is given by

\[
\phi^D(x) = (x_1 \Pi x_2)(x_2 \Pi x_3)(x_3 \Pi x_4) = x_1x_3 + x_2x_3 + x_4x_4 - x_1x_2x_3 - x_2x_3x_4.
\]

Corollary 4 then immediately yields the minimal cut sets of the system, namely \( K_1 = \{1, 3\} \), \( K_2 = \{2, 3\} \), and \( K_3 = \{2, 4\} \).

### 3. Minimal Path and Cut Sets of Small Sizes

By identifying the variables \( x_1, \ldots, x_n \) in the multilinear extension \( \hat{\phi}(x) \) of the structure function, we define its diagonal section \( \hat{\phi}(x, \ldots, x) \), which will be simply denoted by \( \hat{\phi}(x) \). From the simple form (2) of the structure function, we immediately obtain the polynomial function

\[
\hat{\phi}(x) = \sum_{k=1}^{n} d_k x^k, \quad \text{where} \quad d_k = \sum_{A \subseteq C} d(A).
\]

For instance, considering the bridge structure function defined in Example 1, we obtain \( \hat{\phi}(x) = 2x^2 + 2x^3 - 5x^4 + 2x^5 \).

By definition, the diagonal section of the multilinear extension of the structure function is also the one-variable reliability function \( h_\phi : [0, 1] \to [0, 1] \), which gives the system reliability \( h_\phi(p) = h_\phi(p, \ldots, p) \) of the system whenever the components are statistically independent and have the same reliability \( p \).
Using Theorem 1 we can easily express function $\hat{\phi}(x)$ in terms of the minimal path sets. We simply have

$$\hat{\phi}(x) = \sum_{\emptyset \neq B \subseteq [r]} (-1)^{|B|-1} x^{|\bigcup_{j \in B} P_j|}$$

and the coefficient $d_k$ of $x^k$ in $\hat{\phi}(x)$ is then given by

$$d_k = \sum_{B \subseteq [r]} (-1)^{|B|-1} x^{|\bigcup_{j \in B} P_j| = k}$$

(10)

Dually, the coefficient $d^D_k$ of $x^k$ in $\hat{\phi}^D(x)$ is given by

$$d^D_k = \sum_{B \subseteq [s]} (-1)^{|B|-1} (|\bigcup_{j \in B} K_j| = k)$$

(11)

Let $\alpha_k$ (resp. $\beta_k$) denote the number of minimal path (resp. cut) sets of size $k$ of the system. The knowledge of these numbers for small $k$ may be relevant when analyzing the reliability of the system. For instance, if the system has no minimal cut set of size 1, it may be informative to count the number $\beta_2$ of minimal cut sets of size 2 and so forth.

The following proposition shows that $\alpha_1$ and $\alpha_2$ (resp. $\beta_1$ and $\beta_2$) can be computed directly from the coefficients $d_1$ and $d_2$ (resp. $d^D_1$ and $d^D_2$), and vice versa.

**Proposition 5.** We have $\alpha_1 = d_1$, $\beta_1 = d^D_1$, $\alpha_2 = \binom{d_1}{2} + d_2$, and $\beta_2 = \binom{d^D_1}{2} + d^D_2$.

**Proof.** On the one hand, setting $k = 1$ in Eqs. (10) and (11) shows that $d_1 = \alpha_1$ and $d^D_1 = \beta_1$. On the other hand, setting $k = 2$ in Eq. (11), we obtain

$$d_2 = \left| \{i \in [r] : |P_i| = 2\} \right| - \left| \{\{i, j\} \subseteq [r] : |P_i \cup P_j| = 2\} \right|,$$

that is $d_2 = \alpha_2 - \binom{\alpha_1}{2}$. Dually, we obtain $d^D_2 = \beta_2 - \binom{\beta_1}{2}$.

We note that, in general, for $k \geq 3$ neither $\alpha_k$ nor $\beta_k$ can be determined from the coefficients of $\hat{\phi}(x)$ and $\hat{\phi}^D(x)$. For instance, consider the structure functions

$$\phi_1(x) = x_1 x_2 \varprod x_3 x_4 = x_1 x_2 + x_3 x_4 - x_1 x_2 x_3 x_4$$

and

$$\phi_2(x) = x_1 x_2 \varprod x_1 x_3 \varprod x_2 x_3 x_4 = x_1 x_2 + x_1 x_3 + x_2 x_3 x_4 - x_1 x_2 x_3 - x_1 x_2 x_3 x_4.$$

We have $\hat{\phi}_1(x) = \hat{\phi}_2(x) = 2x^2 - x^4$ and hence $\hat{\phi}^D_1(x) = 1 - \hat{\phi}_1(1-x) = 1 - \hat{\phi}_2(1-x) = \hat{\phi}^D_2(x)$. However, we clearly have $\alpha_3 = 0$ for function $\phi_1$ and $\alpha_3 = 1$ for function $\phi_2$.

We end this section by giving expressions for the numbers $\alpha_1$, $\alpha_2$, $\beta_1$, and $\beta_2$ in terms of the structural signature of the system.

Recall that the structural signature of the system is the $n$-vector $s = (s_1, \ldots, s_n)$ whose $k$-th coordinate is defined as

$$s_k = \sum_{A \subseteq C, |A| = n-k+1} \frac{1}{\binom{n}{|A|}} \phi(A) - \sum_{A \subseteq C, |A| = n-k} \frac{1}{\binom{n}{|A|}} \phi(A).$$

(13)
This concept was introduced in 1985 by Samaniego [10] for systems whose components have continuous and i.i.d. lifetimes. He originally defined $s_k$ as the probability that the $k$-th component failure causes the system to fail (hence the property $\sum_{k=1}^n s_k = 1$). More recently, Boland [4] showed that this probability can be explicitly given by Eq. (13). Thus defined, the structural signature depends only on the structure function and can actually be considered for any system, without any assumption on the distribution of the component lifetimes.

The following equation gives an expression of $s_k$ in terms of the coefficients $d_1, \ldots, d_n$ (see, e.g., [7, Cor. 12])

$$s_k = \sum_{j=1}^{n-k+1} \frac{(n-k)}{(j-1)} d_j.$$  

Interestingly, combining Eq. (10) with Eq. (14) gives a way to express the structural signature from the minimal path sets. From Eq. (14) we also derive the conversion formulas $d_1 = ns_n$ and $d_2 = \binom{n}{2}(s_{n-1} - s_n)$.

If $s^D = (s_1^D, \ldots, s_n^D)$ denotes the structural signature associated with the dual structure function $\phi^D$, then we have $s_k^D = s_{n+1-k}$ for $k = 1, \ldots, n$. Combining this observation with Eq. (14), we obtain

$$s_k = \sum_{j=1}^{k} \frac{(k-1)}{(j-1)} d_j^D.$$  

In particular, we have $d_1^D = ns_1$ and $d_2^D = \binom{n}{2}(s_2 - s_1)$.

Combining these conversion formulas with Proposition 5 shows that $\alpha_1$ and $\alpha_2$ (resp. $\beta_1$ and $\beta_2$) can be computed directly from $s_n$ and $s_{n-1}$ (resp. $s_1$ and $s_2$), and vice versa. We have

$$\alpha_1 = d_1 = ns_n,$$

$$\beta_1 = d_1^D = ns_1,$$

$$\alpha_2 = \frac{d_1}{2} + d_2 = \frac{ns_n}{2} + \frac{n}{2}(s_{n-1} - s_n),$$

$$\beta_2 = \frac{d_1^D}{2} + d_2^D = \frac{ns_1}{2} + \frac{n}{2}(s_2 - s_1).$$

For instance, consider again the structure function $\phi_1$ given in Eq. (12). We have $\phi_1(x) = 2x^2 - x^4$, $\phi_1^D(x) = 4x^2 - 4x^3 + x^4$, and therefore $d_1 = d_1^D = 0$, $d_2 = 2$, and $d_2^D = 4$. Using Eqs. (15)–(18), we finally obtain $\alpha_1 = \beta_1 = 0$, $\alpha_2 = 2$, $\beta_2 = 4$, and $s = (0, \frac{2}{3}, \frac{1}{3}, 0)$.

**Example 3.** Consider an $n$-component system having $\beta_2$ minimal cut sets of size 2 and no cut set of size 1. By Eqs. (19) and (13) we necessarily have $s_1 = 0$ and $s_2 = \beta_2/\binom{n}{2}$. This result was expected since $s_k$ is the probability that, assuming that the component lifetimes are continuous and i.i.d. (and hence exchangeable), the system fails exactly at the $k$-th component failure. Thus, $s_1$ is clearly zero and $s_2$ is the ratio of the number $\beta_2$ of minimal cut sets of size 2 (favorable cases) over the number $\binom{n}{2}$ of pairs of components (possible cases).
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