AN INTEGRAL INEQUALITY FOR THE INVARIANT MEASURE OF SOME FINITE DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATION

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Dedicated to Björn Schmalfuß on occasion of his Sixtieth Birthday

Abstract. We prove an integral inequality for the invariant measure \( \nu \) of a stochastic differential equation with additive noise in a finite dimensional space \( H = \mathbb{R}^d \). As a consequence, we show that there exists the Fomin derivative of \( \nu \) in any direction \( z \in H \) and that it is given by \( v_z = \langle D \log \rho, z \rangle \), where \( \rho \) is the density of \( \nu \) with respect to the Lebesgue measure. Moreover, we prove that \( v_z \in L^p(H, \nu) \) for any \( p \in [1, \infty) \). Also we study some properties of the gradient operator in \( L^p(H, \nu) \) and of his adjoint.

1. Introduction and preliminaries. In the recent paper [5] the following inequality involving the invariant measure \( \nu \) of the Burgers equation was proved

\[
\left| \int_H \langle RD \phi, z \rangle \, d\nu \right| \leq C_p \| \phi \|_{L^p(H, \nu)} |z|,
\]

for all \( \phi \in C^1_b(H) \), all \( z \in H \) and all \( p > 1 \), \( R \) being a suitable negative power of the Laplace operator equipped with Dirichlet boundary conditions.

As noted in [5], by estimate (1.1) it follows that \( RD \) is closable in \( L^p(H, \nu) \) for all \( p > 1 \). Moreover, for each \( z \in H \) there exists \( v_z \in L^p(H, \nu) \) such that

\[
\int_H \langle RD \phi, z \rangle \, d\nu = \int_H v_z \phi \, d\nu, \quad \forall \phi \in C^1_b(H).
\]

Identity (1.2) implies that \( \nu \) is Fomin differentiable in all directions of the range of \( R(H) \) of \( R \). We recall that if \( \nu = N_Q \) (the Gaussian measure of mean 0 and covariance \( Q \)) identity (1.2) is well known in Malliavin Calculus. In this case the adjoint \( (Q^{1/2}D)^* \) of \( Q^{1/2}D \) is called the Skorohod operator.

The aim of the present paper is to show that the inequality (1.1), with \( R \) replaced by the identity operator, can also be proved for the invariant measures of some stochastic differential equations in \( H = \mathbb{R}^d \) of the form

\[
\begin{cases}
\, dX(t) = b(X(t)) \, dt + dW(t), \\
\, X(0) = x \in H,
\end{cases}
\]

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where $W$ is an $\mathbb{R}^d$--valued standard Brownian motion and $b$ fulfills the following assumptions.

**Hypothesis 1.1.** (i) There exist $\omega > 0$, $a \geq 0$ such that
\[
\langle b(x), x \rangle \leq -\omega |x|^2 + a, \quad \forall \ x \in \mathbb{R}^d, \tag{1.4}
\]
(ii) $b : H \to H$ is continuously differentiable and there exists $K > 0$, $N \in \mathbb{N}$ such that
\[
|b(x)| + \|b'(x)\| \leq K(1 + |x|^{2N}), \quad \forall \ x \in \mathbb{R}^d. \tag{1.5}
\]

By (ii) it follows that $b$ is Lipschitz continuous on bounded sets of $H$, whereas (i) allows to estimate $|X(t, x)|^2$ by Itô’s formula; therefore existence and uniqueness of a strong solution $X(\cdot, x)$ of (1.3) is classical, see e.g. the monograph [9]. We shall denote by $P_t$ the transition semigroup
\[
P_t \varphi(x) = \mathbb{E}[\varphi(X(t, x))], \quad t \geq 0, \ x \in H, \ \varphi \in B_b(H) \tag{1.6}
\]

For proving (1.1) we argue as in [5] starting from the elementary identity, see (3.2)
\[
P_t(\langle D\varphi, h \rangle) = \langle DP_t\varphi, h \rangle - \int_0^t P_{t-s}(\langle Db, h, DP_s\varphi \rangle)ds.
\]

Then we prove suitable estimates for $DP_t\varphi$ and their integrals with respect to $\nu$. These estimates require some work because, due to the polynomial growth of the derivative of $b$, see (1.5), we cannot exploit the classical Bismut–Elworthy–Li formula, see [8]. To overcome this problem we shall argue as in [3], [4] and [5], introducing a suitable potential (in the present case $V(x) = K(1 + |x|^{2N})$) and the Feynman–Kac semigroup
\[
S_t \varphi(x) = \mathbb{E}[\varphi(X(t, x)) e^{-\int_0^t V(X(s, x)) \, ds}]. \tag{1.7}
\]

We shall first estimate $\langle DS_t \varphi(x), h \rangle$ then $\langle DP_t \varphi(x), h \rangle$, by taking advantage of the identity
\[
P_t \varphi = S_t \varphi + \int_0^t S_{t-s}(VP_s \varphi) \, ds, \tag{1.8}
\]

which follows from the variation of constants formula, see Section 2 below.

In Section 3 we prove that inequality (1.1) and identity (1.2) hold with $R = I$. Moreover, for any $z \in H$ we show that the Fomin derivative $v_z$ in the direction $z \in H$ is given by $\langle D \log \rho, z \rangle$, where $\rho$ is the density of $\nu$ with respect to the Lebesgue measure. Moreover $v_z \in L^p(H, \nu)$ for all $p \in [1, \infty)$. Finally, we prove a formula for the adjoint $D^*$ of $D$ and also for the elliptic operator $-\frac{1}{2} D^* D$ which can be seen as a generalisation of the Ornstein–Uhlenbeck operator.

We end this section with some notations. We set $H = \mathbb{R}^d$, $d \geq 1$ (norm $| \cdot |$, inner product $(\cdot, \cdot)$) and denote by $L(H)$ the space of all linear bounded operators from $H$ into $H$. Moreover, $C_b(H)$ is the space of all real continuous and bounded mappings $\varphi : H \to \mathbb{R}$ endowed with the sup norm
\[
\| \varphi \|_\infty = \sup_{x \in H} | \varphi(x) |
\]

whereas $C^k_b(H)$, $k > 1$, is the space of all real functions which are continuous and bounded together with their derivatives of order lesser than $k$. Finally, $B_b(H)$ will represent the space of all real, bounded and Borel mappings on $H$. 
2. Estimates of the derivative of the transition semigroup. Let us start by giving an estimate of \( \mathbb{E}([X(t,x)]^{2m}), m \in \mathbb{N} \). The following lemma is standard, we shall give some details of the proof for the reader’s convenience.

**Lemma 2.1.** Assume Hypothesis 1.1(i). Then for any \( m \in \mathbb{N} \) there exists \( a_m > 0 \) such that

\[
\mathbb{E}[|X(t,x)|^{2m}] \leq e^{-2m\omega t}|x|^{2m} + a_m, \quad \forall \ x \in H, \ t \geq 0.
\]

**Proof.** Let first consider the case \( m = 1 \). Then by Itô’s formula, taking into account (1.4) we find

\[
\frac{d}{dt} \mathbb{E}[|X(t,x)|^2] = 2 \mathbb{E}[(X(t,x), b(X(t,x)))] + d \leq -2\omega \mathbb{E}[|X(t,x)|^2] + 2a + d.
\]

We deduce that

\[
\frac{d}{dt} \mathbb{E}[|X(t,x)|^2] \leq -2\omega \mathbb{E}[|X(t,x)|^2] + 2a + d.
\]

By a standard comparison result it follows that

\[
\mathbb{E}[|X(t,x)|^2] \leq e^{-2\omega t}|x|^2 + a_2, \quad \forall \ x \in H, \ t \geq 0,
\]

where

\[
a_2 = \frac{1}{\omega}(2a + d).
\]

Now let \( m > 1 \) and \( \varphi_m(x) = |x|^{2m} \). Then we have

\[
D\varphi_m(x) = 2m|x|^{2m-2} x
\]

and

\[
D^2\varphi_m(x) = 4m(m-1)|x|^{2m-4} x \otimes x + 2m|x|^{2m-2} I,
\]

where \( I \) represents identity in \( H \). Consequently

\[
\frac{1}{2} \text{Tr} [D^2\varphi_m(x)] = m(2m - 2 + d)|x|^{2m-2}
\]

Then again by Itô’s formula we have

\[
\frac{d}{dt} \mathbb{E}[|X(t,x)|^{2m}] = 2m \mathbb{E}[|X(t,x)|^{2m-2}] (X(t,x), b(X(t,x)))
\]

\[
+ m(2m - 2 + d) \mathbb{E}[|X(t,x)|^{2m-2}]
\]

\[
\leq -2m\omega \mathbb{E}[|X(t,x)|^{2m}] + m(2a + 2m - 2 + d) \mathbb{E}[|X(t,x)|^{2m-2}].
\]

It follows that

\[
\mathbb{E}[|X(t,x)|^{2m}] \leq e^{-2m\omega t}|x|^{2m}
\]

\[
+ m(2a + 2m - 2 + d) \int_0^t e^{-2m\omega (t-s)} \mathbb{E}[|X(s,x)|^{2m-2}] ds.
\]

The conclusion follows easily by recurrence. \( \square \)

Now we are going to prove an estimate for the derivative \( D_s X(t,x)h \), which we denote by \( \eta^h(t,x), h \in H \). As well known \( \eta^h(t,x) \) is a solution to the random equation

\[
\begin{aligned}
\frac{d}{dt} \eta^h(t,x) &= b'(X(t,x)) \cdot \eta^h(t,x), \\
\eta^h(0,x) &= h
\end{aligned}
\]

**Lemma 2.2.** Assume Hypothesis 1.1. Then the following estimate holds

\[
|\eta^h(t,x)| \leq e^{K \int_0^t (1 + |X(s,x)|^{2N}) ds} |h|, \quad t \geq 0, \ x, h \in H.
\]
Proof. By (2.3) we deduce, taking into account (1.5), that
\[ \frac{1}{2} \frac{d}{dt} |\eta^h(t,x)|^2 = \langle b'(X(t,x) \cdot \eta^h(t,x), \eta^h(t,x)) \rangle \leq K(1 + |X(t,x)|^{2N}) |\eta^h(t,x)|^2. \]

So, the conclusion follows from Gronwall’s lemma.

Now we are going to estimate \( D_x P_t \varphi \).

2.1. **Pointwise estimate.** As we said in the introduction, we cannot estimate \( D_x P_t \varphi \) for \( \varphi \in C_b(H) \) using the Bismut–Elworthy–Li formula see [8], because we do not know whether the expectation on the right hand side of (2.4) does exist. For this reason, we introduce the potential
\[ V(x) = K(1 + |x|^{2N}), \quad x \in H \]
and the Feynman–Kac semigroup
\[ S_t \varphi(x) = E[\varphi(X(t,x)) e^{-\int_0^t V(X(s,x)) ds}]. \]

We recall that the Bismut–Elworthy–Li formula generalises to \( S_t \), see [7]. In fact for all \( \varphi \in C_b(H) \), setting
\[ \beta(t) = \int_0^t V(X(s,x)) ds, \]
the following identity holds
\begin{align*}
\langle DS_t \varphi(x), h \rangle &= \frac{1}{t} \mathbb{E} \left[ \varphi(X(t,x)) e^{-\beta(t)} \int_0^t \langle \eta^h(s,x), dW(s) \rangle \right] \\
&= - \mathbb{E} \left[ \varphi(X(t,x)) e^{-\beta(t)} \int_0^t \left( 1 - \frac{s}{t} \right) \langle V'(X(s,x), \eta^h(s,x)) \rangle ds \right] \\
&= I_1(\varphi, x, h, t) + I_2(\varphi, x, h, t) = I_1 + I_2. \tag{2.5}
\end{align*}

We shall first estimate \( \langle DS_t \varphi(x), h \rangle \), then \( \langle DP_t \varphi(x), h \rangle \). In the latter case, we take advantage of the identity
\[ P_t \varphi = S_t \varphi + \int_0^t S_{t-s}(VP_s \varphi) ds, \]
which follows from the variation of constants formula; in fact, denoting by \( \mathcal{L} \) and \( \mathcal{K} \) the infinitesimal generators of \( P_t \) and \( S_t \) respectively, it holds
\[ \mathcal{L} = \mathcal{K} + V. \]

Lemma 2.3. Let \( \varphi \in C_b(H) \), \( t \geq 0 \), \( x \in H \). Then for \( p > 1 \), there exists a constant \( C_p > 0 \) such that
\[ |D_x S_t \varphi(x)| \leq C_p (1 + t^{-1/2})(1 + |x|^{2N-1}) \mathbb{E} (\varphi^p(X(t,x)))^{1/p}. \tag{2.6} \]

Proof. We start by estimating \( I_1 \). By Hölder’s inequality with exponents \( p, q = \frac{p}{p-1} \), we have
\begin{align*}
|I_1| &\leq \frac{1}{t} \mathbb{E} \left[ \varphi^p(X(t,x)) \right]^{1/p} \left[ \mathbb{E} \left( e^{-q\beta(t)} \int_0^t \langle \eta^h(s,x), dW(s) \rangle \right)^q \right]^{1/q} \tag{2.7} \\
&= \frac{1}{t} \mathbb{E} \left[ \varphi^p(X(t,x)) \right]^{1/p} \mathbb{E} \left[ (|z(t)|^q)^{1/q} \right],
\end{align*}
where
\[ z(t) = e^{-\beta(t)} \int_0^t \langle \eta^h(s,x), dW(s) \rangle, \quad t \geq 0. \tag{2.8} \]
We now apply Itô’s formula to \( g(z(t)) \) where \( g(r) = |r|^q, \ r \in \mathbb{R} \). Since
\[
g'(r) = q|r|^{q-2}r, \quad g''(r) = q(q-1)|r|^{q-2},
\]
and
\[
dz(t) = -\beta'(t)e^{-\beta(t)}\int_0^t \langle \eta^h(s, x), dW(s) \rangle ds + e^{-\beta(t)} \langle \eta^h(t, x), dW(t) \rangle
\]
we find
\[
d|z(t)|^q = q|z(t)|^{q-2}z(t)(-\beta'(t)z(t) + e^{-\beta(t)} \langle \eta^h(t, x), dW(t) \rangle) + \frac{1}{2} q(q-1)|z(t)|^{q-2}e^{-2\beta(t)}|\eta^h(t, x)|^2 dt.
\]
Integrating from 0 to \( t \), yields
\[
|z(t)|^q = -q \int_0^t |z(s)|^q \beta'(s) ds + q \int_0^t |z(s)|^{q-2}z(s)e^{-\beta(s)} \langle \eta^h(s, x), dW(s) \rangle
\]
\[
+ \frac{1}{2} q(q-1) \int_0^t e^{-2\beta(s)}|z(s)|^{q-2}|\eta^h(s, x)|^2 ds.
\]
Neglecting the negative first term in the previous identity and taking expectation, we find
\[
\mathbb{E} \left( \sup_{r \in [0,t]} |z(r)|^q \right) 
\leq q \mathbb{E} \left( \sup_{r \in [0,t]} \left| \int_0^r e^{-\beta(s)} |z(s)|^{q-2}z(s) \langle \eta^h(s, x), dW(s) \rangle \right| \right)
\]
\[
+ \frac{1}{2} q(q-1) \mathbb{E} \left( \int_0^t e^{-2\beta(s)} |z(s)|^{q-2} |\eta^h(s, x)|^2 ds \right)
\]
\[
=: A_1 + A_2.
\]
By the Burkholder inequality we have, taking into account Lemma 2.1
\[
A_1 \leq 3q \mathbb{E} \left[ \left| \int_0^t e^{-2\beta(s)} |z(s)|^{2(q-1)} |\eta^h(s, x)|^2 ds \right|^{1/2} \right]
\]
\[
\leq 3q \mathbb{E} \sup_{r \in [0,t]} |z(r)|^{q-1} \left( \int_0^t e^{-2\beta(s)} |\eta^h(s, x)|^2 ds \right)^{1/2}
\]
\[
\leq 3qt^{1/2} \mathbb{E} \left[ \sup_{r \in [0,t]} |z(r)|^{q-1} \right] \mathcal{H}.
\]
By Hölder’s inequality with exponents \( q, \frac{2}{q-1} \), it follows that
\[
A_1 \leq 3qt^{1/2} |\mathcal{H}| \left[ \mathbb{E} \left( \sup_{r \in [0,t]} |z(r)|^q \right) \right]^{\frac{q-1}{q}}.
\]
Now by the Young inequality
\[
ab \leq \frac{1}{u} a^u + \frac{1}{v} a^v, \quad a > 0, \ b > 0, \ \frac{1}{u} + \frac{1}{v} = 1
\]
(2.13)
with \( u = q, \ v = \frac{q-1}{q} \), there exists \( c_1 > 0 \) such that
\[
A_1 \leq \frac{1}{4} E \left( \sup_{r \in [0,t]} |z(r)|^{q} \right) + c_1 t^{q/2} |h|^q. \tag{2.14}
\]

Concerning \( A_2 \), using again Lemma 2.1, we find
\[
A_2 = \frac{1}{2} q(q-1) E \left( \int_0^t e^{-2\beta(s)} |z(s)|^{q-2} |\eta^h(s,x)|^2 ds \right)
\leq \frac{1}{2} q(q-1) E \left[ \sup_{r \in [0,t]} |z(r)|^{q-2} \left( \int_0^t e^{-2\beta(s)} |\eta^h(s,x)|^2 ds \right) \right]
\leq \frac{1}{2} q(q-1) E \left[ \sup_{r \in [0,t]} |z(r)|^{q-2} \right] |h|^2 t.
\]

By Hölder’s inequality with exponents \( \frac{q}{2}, \frac{q}{q-2} \) we have
\[
A_2 \leq \frac{1}{2} q(q-1) |h|^2 \left[ E \left( \sup_{r \in [0,t]} |z(r)|^{q} \right) \right]^{\frac{q-2}{q}}.
\]

By the Young inequality (2.13) with \( u = \frac{q}{2} \) and \( v = \frac{q}{q-2} \), it follows that there exists \( c_2 > 0 \) such that
\[
A_2 \leq \frac{1}{4} E \left( \sup_{r \in [0,t]} |z(r)|^{q} \right) + c_2 |h|^q t^{q/2}. \tag{2.15}
\]

Taking into account (2.10), (2.14) and (2.15) we conclude that
\[
E \left( \sup_{r \in [0,t]} |z(r)|^{q} \right) \leq \frac{1}{2} E \left( \sup_{r \in [0,t]} |z(r)|^{q} \right) + (c_1 + c_2) |h|^q t^{q/2}.
\]

Therefore
\[
E \left( \sup_{r \in [0,t]} |z(r)|^{q} \right) \leq (c_1 + c_2) |h|^q t^{q/2}. \tag{2.16}
\]

Finally, by (2.7) it follows that
\[
I_1 \leq (c_1 + c_2) t^{-1/2} |h| \left| E \left( \varphi^p(X(t,x)) \right) \right|^{1/p}.
\]

Now let us consider \( I_2 \), and write
\[
I_2 \leq 2KN \left| E \left( \varphi^p(X(t,x)) \right) \right|^{1/p} \Lambda(t)^{1/q}, \tag{2.17}
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \) and
\[
\Lambda(t) = E \left[ e^{-q\beta(t)} \left( \int_0^t |X(s,x)|^{2N-1} |\eta^h(s,x)| ds \right)^q \right]
\leq E \left[ \left( \int_0^t e^{-\beta(s)} |X(s,x)|^{2N-1} |\eta^h(s,x)| ds \right)^q \right]
\leq E \left[ \sup_{r \in [0,t]} \left( |X(r,x)|^{(2N-1)q} \right) \left( \int_0^t e^{-\beta(s)} |\eta^h(s,x)| ds \right)^q \right]
\leq E \left[ \sup_{r \in [0,t]} \left( |X(r,x)|^{(2N-1)q} \right) \right] |h|^q t^q. \tag{2.19}
\]
So

\[ I_2 \leq 2KN \left( \mathbb{E} \left[ \varphi^p(X(t,x)) \right] \right)^{1/p} \left( \mathbb{E} \left[ \sup_{r \in [0,t]} \left( |X(r,x)|^{(2N-1)q} \right) \right] \right)^{1/q} |h| t. \tag{2.20} \]

Recalling finally (2.1) we see that there exists \( c_3 > 0 \) such that

\[ \left( \mathbb{E} \left[ \varphi^p(X(t,x)) \right] \right)^{1/p} \left( \mathbb{E} \left[ \sup_{r \in [0,t]} \left( |X(r,x)|^{2N-1} \right) \right] \right)^q \leq c_3 (1 + |x|^{2N-1}), \]

so that

\[ I_2 \leq 2KN c_3 (1 + |x|^{2N-1}) \left[ \mathbb{E} \left[ \varphi^p(X(t,x)) \right] \right]^{1/p} |h| t. \tag{2.21} \]

Finally, by (2.5), (2.17) and (2.21), the conclusion follows easily.

2.2. The invariant measure \( \nu \). We shall denote by \( \pi_{t,x} \) the law of \( X(t,x) \) so that for each \( \varphi \in \mathcal{B}_b(H) \) we have

\[ P_t \varphi(x) = \int_H \varphi(y) \pi_{t,x}(dy), \quad x \in H, \ t > 0. \tag{2.22} \]

**Lemma 2.4.** Assume Hypothesis 1.1(i). Then there is an invariant measure \( \nu \) of \( P_t \), moreover for all \( m \in \mathbb{N} \) we have

\[ \int_H |x|^{2m} \nu(dx) \leq a_m, \tag{2.23} \]

where \( a_m \) is the constant in (2.1).

**Proof.** Let \( r > 0 \) and fix \( x \in H \). Set \( B_r^c = \{ y \in H : |y| \geq r \} \). Then, taking into account (2.2) it follows that

\[
\pi_{t,x}(B_r^c) = \int_{|y| \geq r} \pi_{t,x}(dy) \leq \frac{1}{r^2} \int_H |y|^2 \pi_{t,x}(dy) = \frac{1}{r^2} \mathbb{E} \left[ |X(t,x)|^2 \right] \leq \frac{|x|^2 + a_2}{r^2}.
\]  

(2.24)

Therefore by the Krylov–Bogoliubov theorem, see e.g [6], there exists a sequence \( T_n \uparrow +\infty \) such that

\[ \lim_{n \to +\infty} \frac{1}{T_n} \int_0^{T_n} \pi_{t,x} dt = \nu \quad \text{weakly,} \tag{2.25} \]

where \( \nu \) is an invariant measure of \( P_t \).

Now we can prove (2.23). By (2.1) we deduce

\[ \int_H |y|^{2m} \pi_{t,x}(dy) \leq e^{-m\omega t}|x|^{2m} + a_m, \quad \forall \ x \in H, \ t \geq 0. \tag{2.26} \]

It follows that for any \( \epsilon > 0 \)

\[ \int_H \frac{|y|^{2m}}{1 + \epsilon |y|^{2m}} \pi_{t,x}(dy) \leq e^{-m\omega t}|x|^{2m} + a_m, \quad \forall \ x \in H, \ t \geq 0. \tag{2.27} \]

Consequently integrating both sides with respect to \( t \) over \([0,T_n]\) and dividing by \( T_n \), yields

\[ \frac{1}{T_n} \int_0^{T_n} dt \int_H \frac{|y|^{2m}}{1 + \epsilon |y|^{2m}} \pi_{t,x}(dy) \leq \frac{1}{m\omega T_n} (1 - e^{-m\omega T_n}) |x|^{2m} + a_m, \tag{2.28} \]
for all $x \in H$, $t \geq 0$. Finally, letting $n \to +\infty$ and taking into account (2.25), we find
\[
\int_H \frac{|y|^{2m}}{1 + \epsilon |y|^{2m}} \nu(dy) \leq a_m
\]
and the conclusion follows letting $\epsilon$ tend to 0.

2.3. Integral estimates. Let us start with an estimate of $\int_H \langle D_x S_t \varphi(x), h(x) \rangle \nu(dx)$.

**Lemma 2.5.** Let $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} < 1$. Then there is $C_1 > 0$ such that
\[
\left| \int_H \langle D_x S_t \varphi(x), h(x) \rangle \nu(dx) \right| \leq C_1 (1 + t^{-1/2}) \|\varphi\|_{L^p(H,\nu)} \|h\|_{L^q(H,\nu)}. \tag{2.29}
\]

**Proof.** Taking into account (2.6) we have
\[
\left| \int_H \langle D_x S_t \varphi(x), h(x) \rangle \nu(dx) \right| \leq C_p (1 + t^{-1/2}) \times \int_H (1 + |x|^{2N-1}) \left[ \|P_t^p(x)\| \|h(x)\| \nu(dx) \right]. \tag{2.30}
\]
Let
\[
\frac{1}{r} = 1 - \frac{1}{p} - \frac{1}{q},
\]
then by the triple Hölder inequality with exponents $r, p, q$ we have, taking into account the invariance of $\nu$,
\[
\left| \int_H \langle D_x S_t \varphi(x), h(x) \rangle \nu(dx) \right| \leq c (1 + t^{-1/2}) \times \left[ \int_H (1 + |x|^{N-1})^r \nu(dx) \right]^{1/r} \left( \int_H \|P_t^p(x)\| \nu(dx) \right)^{1/p} \|h\|_{L^q(H,\nu)} \|\varphi\|_{L^p(H,\nu)}. \tag{2.31}
\]
The conclusion follows from (2.23). \qed

Now we are ready to estimate $\int_H \langle D_x P_t \varphi(x), h(x) \rangle \nu(dx)$. We start from the identity
\[
P_t \varphi(x) = S_t \varphi(x) + K \int_0^t S_{t-s} (1 + |x|^{2N}) P_s \varphi(x) ds, \tag{2.32}
\]
from which
\[
\langle D_x P_t \varphi(x), h(x) \rangle
\]
\[
= \langle D_x S_t \varphi(x), h(x) \rangle + K \int_0^t \langle D_x S_{t-s} ((1 + |x|^{2N}) P_s \varphi)(x), h(x) \rangle ds. \tag{2.33}
\]

**Proposition 2.6.** Let $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} < 1$. Then there is $C_p^1$ such that
\[
\left| \int_H \langle D_x P_t \varphi(x), h(x) \rangle \nu(dx) \right| \leq C_p^1 (1 + t^{-1/2}) \|\varphi\|_{L^p(H,\nu)} \|h\|_{L^q(H,\nu)}. \tag{2.34}
\]

**Proof.** The first term of (2.33) is bounded by (2.29). Let us estimate the second one. Again by (2.29) we have
Now let us choose $\epsilon > 0$ such that

$$\frac{1}{p+\epsilon} + \frac{1}{q} < 1.$$ 

Then by Hölder’s inequality with exponents $\frac{p+\epsilon}{p}$ and $\frac{p+\epsilon}{q}$ it follows that

$$\| (1 + |x|^{2N}) P_s \phi \|_{L^p(H, \nu)}^p = \int_H (1 + |x|^{2N})^p (P_s \phi)^p d\nu$$

$$\leq \left( \int_H (1 + |x|^{2N})^{\frac{p(p+\epsilon)}{p}} d\nu \right)^{\frac{p}{p+\epsilon}} \left( \int_H (P_s \phi)^{p+\epsilon} d\nu \right)^{\frac{p}{p+\epsilon}}$$

$$\leq \left( \int_H (1 + |x|^{2N})^{\frac{p(p+\epsilon)}{p}} d\nu \right)^{\frac{p}{p+\epsilon}} \| \phi \|_{L^{p+\epsilon}(H, \nu)}^p,$$

by the invariance of $\nu$. Now by (2.2) there exists a constant $C'$ such that

$$\int_H (1 + |x|^{2N})^{\frac{p(p+\epsilon)}{p}} d\nu \leq C'.$$

Therefore

$$\| (1 + |x|^{2N}) P_s \phi \|_{L^p(H, \nu)}^p \leq (C')^{\frac{p}{p+\epsilon}} \| \phi \|_{L^{p+\epsilon}(H, \nu)}^p.$$ 

Substituting in (2.35), yields

$$\left| \int_0^t \langle D_x S_t \cdot (|x|^N P_s \phi), h(x) \rangle ds \right|$$

$$\leq C_1' \left( \int_0^1 (1 + (t-s)^{-\frac{1}{2}}) \| \phi \|_{L^{p+\epsilon}(H, \nu)} \| h \|_{L^p(H, \nu)} ds \right)$$

Non the conclusion follows by the arbitrariness of $\epsilon, p, q$. \qed

3. The main inequality and its consequences.

**Theorem 3.1.** For all $p > 1$ there exists a constant $C_p > 0$ such that for all $\phi \in L^p(H, \nu)$ and all $h \in H$ we have

$$\left| \int_H \langle D_x \phi(x), h \rangle \nu(dx) \right| \leq c \| \phi \|_{L^p(H, \nu)} |h|.$$ 

**Proof.** Step 1. For any $\phi \in C^1_b(H)$ and any $h \in H$ the following identity holds.

$$P_t(\langle D\varphi, h \rangle) = \langle DP_t \varphi, h \rangle - \int_0^t P_{t-s}(\langle \mathbb{D} h, DP_s \varphi \rangle) ds, \quad t > 0.$$ 

To prove (3.2) we consider a sequence $(b_n)$ of mappings $H \to H$ of class $C^\infty$ such that

(i) $\lim_{n \to \infty} b_n(x) = b(x)$, uniformly on bounded sets of $H$.

(ii) $\langle b_n(x), x \rangle \leq -\omega |x|^2 + a, \quad \forall x \in H$.

To construct $(b_n)$ we first set

$$f_n(x) = \frac{b(x) + \omega x}{1 + n^{-1} |x|^{2N+2}} - \omega x,$$
so that
\[ \langle f_n(x), x \rangle \leq -\omega |x|^2 + a, \quad \forall x \in H, \]
and \( f_n \) is sub-linear, then we regularise \( f_n \) using mollifiers.

Now we prove the identity
\[ P^n_t(\langle D\varphi, h \rangle) = \langle DP^n_t \varphi, h \rangle - \int_0^t P^n_{t-s}(\langle Db \cdot h, DP^n_s \varphi \rangle) ds, \quad (3.3) \]
where \( P^n_t \) is the transition semigroup corresponding to \( b_n \).

It is enough to show (3.3) for each \( \varphi \in C_0^b(H) \). In such a case set \( u_n(t, x) = P^n_t \varphi(x) \) and write
\[
\begin{cases}
D_t u_n(t, x) = \frac{1}{2} \Delta u_n(t, x) + \langle Du_n(t, x), b_n(x) \rangle, \\
u_n(0, x) = \varphi(x).
\end{cases} \tag{3.4}
\]

Now, taking \( h \in H \) and setting \( v_n(t, x) = \langle Du_n(t, x), h \rangle \) we see, by a simple computation, that
\[
\begin{cases}
D_t v_n(t, x) = \frac{1}{2} \Delta v_n(t, x) + \langle Dv_n(t, x), b_n(x) \rangle \\
v_n(0, x) = \langle D\varphi(x), h \rangle.
\end{cases} \tag{3.5}
\]

By the variation of constants formula it follows that
\[ v_n(t, x) = P^n_t(\langle D\varphi(x), h \rangle) + \int_0^t P^n_{t-s}(\langle Du_n(s, x), Ah + b_n'(x)h \rangle) ds, \quad (3.6) \]
which coincides with (3.3). Letting \( n \to \infty \), yields (3.2).

**Step 2.** Conclusion.

Integrating (3.2) with respect to \( \nu \) over \( H \) and taking into account the invariance of \( \nu \), yields
\[
\int_H \langle D\varphi(x), h \rangle \nu(dx) = \int_H \langle DP_t \varphi(x), h \rangle \nu(dx)
- \int_H \int_0^t \langle b'(x)h, DP_s \varphi(x) \rangle ds \nu(dx) =: J_1 + J_2 \quad (3.7)
\]
Setting and \( t = 1 \) we deduce
\[ |J_1| \leq \left| \int_H \langle D_2 P_t \varphi(x), h \rangle \nu(dx) \right| \leq 2C_p^1 \| \varphi \|_{L^p(H, \nu)} \| h \|. \quad (3.8) \]
Concerning \( J_2 \) we have by (2.34) and taking into account (1.5)
\[
|J_2| \leq \int_0^t \int_H C_p^1 (1 + (t - s)^{-1/2}) \| \varphi \|_{L^p(H, \nu)} \| b'(\cdot)h \|_{L^q(H, \nu)} ds
\leq K \int_0^t \int_H C_p^1 (1 + (t - s)^{-1/2}) \| \varphi \|_{L^p(H, \nu)} \| (1 + |x|^{2N}) \|_{L^q(H, \nu)} ds |h|. \quad (3.9)
\]
Finally, recalling (2.23) and setting \( t = 1 \) the conclusion follows. \qed
3.1. Consequences of the integral inequality (3.1). The following result can be proved exactly as in [5], replacing \( R \) by \( I \) so, we omit the proof.

**Proposition 3.2.** Assume Hypothesis 1.1 and let \( \nu \) be the invariant measure of problem (1.3). Then for any \( p > 1 \) the gradient

\[
D : C^1_b(H) \subset L^p(H, \nu) \to L^p(H, \nu; H), \quad \varphi \to D\varphi,
\]

is closable.

For any \( p > 1 \) we shall denote by \( D_p \) the closure of \( D \) and by \( D^*_p \) the adjoint operator of \( D_p \). \( D_p \) is a mapping

\[
D_p : D(D_p) \subset L^p(H, \nu) \to L^p(H, \nu; H)
\]

and \( D^*_p \) is a mapping

\[
D^*_p : D(D^*_p) \subset L^q(H, \nu; H) \to L^q(H, \nu),
\]

where \( q = \frac{1}{1-p} \). We have obviously

\[
\int_H \langle D\varphi, F \rangle \, d\nu = \int_H \varphi D^*_p(F) \, d\nu,
\]

(3.10)

for any \( \varphi \in D(D_p) \) and any \( F \in D(D^*_p) \). We recall that \( F \in D(D^*_p) \) if and only if there exists a positive constant \( K_F \) such that

\[
\left| \int_H \langle D\varphi, F \rangle \, d\nu \right| \leq K_F ||\varphi||_{L^p(H, \nu)}, \quad \forall \varphi \in C^1_b(H).
\]

(3.11)

In this case we have

\[
\|D^*_p(F)\|_{L^q(H, \nu)} \leq K_F.
\]

(3.12)

If no confusion may arise we shall omit sub–indices \( p \) in \( D_p \) and \( D^*_p \).

**Proposition 3.3.** For any \( z \in H \) there is \( v_z \in L^q(H, \nu) \) for all \( q \in [1, +\infty) \) such that

\[
\int_H \langle D\varphi, z \rangle \, d\nu = \int_H v_z \varphi \, d\nu.
\]

(3.13)

**Proof.** Let \( z \in H \) and set \( F_z(x) = z, \quad \forall x \in H \). Then by (3.1) it follows that

\[
\left| \int_H \langle D\varphi, F_z \rangle \, d\nu \right| \leq C_{1,p} \|\varphi\|_{L^p(H, \nu)} |z|
\]

(3.14)

This implies \( F_z \in D(D^*_p) \) and \( \|D^*_p(F_z)\|_{L^q(H, \nu)} \leq C_{1,p} |z| \). Setting \( D^*_p(F_z) = v_z \), identity (3.13) follows.

**Remark 3.4.** By Proposition 3.3 \( \nu \) possesses the Fomin derivative of \( \nu \) at the direction \( z \) which is given precisely by \( v_z \) and so, it belongs to \( L^q(H, \nu) \) for all \( q \in [1, \infty) \),

Now we are going to identify \( v_z \).

**Proposition 3.5.** Assume Hypothesis 1.1. Then for any \( z \in H \) we have \( v_z = \langle D\log \rho, z \rangle \), where \( \rho \) is the density of \( \nu \) with respect to the Lebesgue measure on \( \mathbb{R}^d \). Therefore \( \langle D\log \rho, z \rangle \) belong to \( L^p(H, \nu) \) for any \( p \in [1, +\infty) \).
Proof. First notice that by (3.13) it follows in particular that
\[ \left| \int_H \langle D\varphi, z \rangle d\nu \right| \leq \|v_z\|_{L^1(H,\nu)} \|\varphi\|_\infty. \] (3.15)
Therefore, by an argument due to Malliavin, \( \nu \) has a density \( \rho \) with respect to the Lebesgue measure on \( \mathbb{R}^d \) with \( \rho \in L^{d-1}(\mathbb{R}^d) \), see [12].

To prove the last statement, we write (3.13) as
\[ \int_H \langle D\varphi, z \rangle \rho dx = \int_H \varphi v_z \rho dx. \]
This implies in the sense of distributions that
\[ v_z = \langle D\log \rho, z \rangle. \]
Now the conclusion follows from Proposition 3.3. \( \square \)

Remark 3.6. The fact that \( \nu \) has a density \( \rho \) with respect to the Lebesgue measure, together with several properties of \( \rho \) have already been proved in [11], [1] and [2].

Let us finally study some properties of operators \( D^* \) and \( D^*D \).

Proposition 3.7. Let
\[ F(x) = \sum_{h=1}^d f_h(x)e_h, \quad x \in H, \] (3.16)
where \((e_1, \ldots, e_d)\) is an orthonormal basis in \( H \) and \( f_h \in C^1_b(H) \), \( h = 1, \ldots, d \). Then \( F \) belongs to the domain of \( D^* \) and it results
\[ D^*(F) = -\text{div} F + \sum_{h=1}^d v_{e_h} f_h. \] (3.17)
Moreover, if \( \varphi \in C^2_b(H) \) we have
\[ -\frac{1}{2} D^*D(\varphi) = \frac{1}{2} \Delta \varphi - \frac{1}{2} \sum_{h=1}^d v_{e_h} D_h \varphi. \] (3.18)

Proof. Write
\[ \int_H \langle D\varphi, F \rangle d\nu = \sum_{h=1}^d \int_H D_h \varphi f_h d\nu \]
\[ = \sum_{h=1}^d \int_H D_h(\varphi f_h) d\nu - \sum_{h=1}^d \int_H \varphi D_h f_h d\nu \] (3.19)
Since, in view of (3.13)
\[ \int_H D_h(\varphi f_h) d\nu = \int_H \varphi v_{e_h} f_h d\nu, \]
(3.17) follows. Now (3.18) follows as well setting \( F = D\varphi \) in (3.17). \( \square \)

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