Estimation of Riemannian distances between covariance operators and Gaussian processes

Hà Quang Minh

RIKEN Center for Advanced Intelligence Project, Tokyo, JAPAN

In this work we study two Riemannian distances between infinite-dimensional positive definite Hilbert-Schmidt operators, namely affine-invariant Riemannian and Log-Hilbert-Schmidt distances, in the context of covariance operators associated with functional stochastic processes, in particular Gaussian processes. Our first main results show that both distances converge in the Hilbert-Schmidt norm. Using concentration results for Hilbert space-valued random variables, we then show that both distances can be consistently and efficiently estimated from (i) sample covariance operators, (ii) infinite, normalized covariance matrices, and (iii) infinite samples generated by the given processes, all with dimension-independent convergence. Our theoretical analysis exploits extensively the methodology of reproducing kernel Hilbert space (RKHS) covariance and cross-covariance operators. The theoretical formulation is illustrated with numerical experiments on covariance operators of Gaussian processes.

KEYWORDS
Riemannian distance, Gaussian process, Gaussian measure, covariance operator, reproducing kernel Hilbert space
1 | INTRODUCTION

This work studies two Riemannian distances, namely the affine-invariant Riemannian distance [1] and Log-Hilbert-Schmidt distance [2] between centered Gaussian processes, and more generally, between covariance operators associated with functional stochastic processes. Our main focus is on the estimation of these distances from finite samples generated by the given stochastic processes. In both cases, we show that the distances can be consistently and efficiently estimated from finite samples, with \textit{dimension-independent} convergence rates.

The study of functional data has received increasing interests recently in statistics and machine learning, see e.g. [3, 4, 5]. One particular approach for analyzing functional data has been via the analysis of covariance operators and the distance/divergence functions between them. Recent work along this direction includes [6, 7], which utilize the Hilbert-Schmidt distance between covariance operators and [8, 9], which utilize non-Euclidean distances, in particular the Procrustes distance, also known as Bures-Wasserstein distance. The latter distance corresponds to the $L^2$-Wasserstein distance between two centered Gaussian measures on Hilbert space in the context of optimal transport [10] and can better capture the intrinsic geometry of the set of covariance operators. In the context of covariance operators and Gaussian processes, the $L^2$-Wasserstein distance and its entropic regularization, the Sinkhorn divergence, have been analyzed in [11, 12]. In [13, 14], the Kullback-Leibler divergence between stochastic processes was studied, the latter in the context of functional Bayesian neural networks. In this work, we study the non-Euclidean distances between covariance operators that arise from the Riemannian geometric viewpoint of positive definite Hilbert-Schmidt operators, including in particular the affine-invariant Riemannian and Log-Hilbert-Schmidt distances.

Contributions of this work\footnote{An extended abstract summarizing several preliminary results of the current work, without proofs, in particular Theorems 6 and 7, was presented in the Proceedings of the International Workshop on Functional and Operatorial Statistics (IWFOS 2020) [15].} The following are the main novel contributions of the current work

1. We show that both the affine-invariant Riemannian distance [1] and Log-Hilbert-Schmidt distance [2] between positive definite Hilbert-Schmidt operators converge in the Hilbert-Schmidt norm.
2. From the Hilbert-Schmidt norm convergence, we show that both the affine-invariant/Log-Hilbert-Schmidt distances between centered Gaussian processes/covariance operators can be consistently estimated using sample covariance operators. The convergence rate is \textit{dimension-independent}.
3. By representing the affine-invariant/Log-Hilbert-Schmidt distances between centered Gaussian processes/covariance operators via reproducing kernel Hilbert space (RKHS) covariance and cross-covariance operators, we show that they can be consistently estimated using (i) finite, normalized covariance matrices and (ii) finite samples generated by the corresponding random processes. The convergence rates in all cases are \textit{dimension-independent}.
4. We show the theoretical consistency of the empirical distances between Gaussian measures defined on an RKHS, which are induced by a positive definite kernel [2], as employed in computer vision applications.

2 | DISTANCES BETWEEN GAUSSIAN PROCESSES

We first review the correspondence between Gaussian processes and Gaussian measures/covariance operators on Hilbert spaces, followed by the formal distance formulation. Throughout the paper, we assume the following

1. A1 $T$ is a $\sigma$-compact metric space, that is $T = \bigcup_{i=1}^{\infty} T_i$, where $T_1 \subset T_2 \subset \cdots$, with each $T_i$ being compact.
2. A2 $\nu$ is a non-degenerate Borel probability measure on $T$, that is $\nu(B) > 0$ for each open set $B \subset T$. 

1 An extended abstract summarizing several preliminary results of the current work, without proofs, in particular Theorems 6 and 7, was presented in the Proceedings of the International Workshop on Functional and Operatorial Statistics (IWFOS 2020) [15].
3. **A3** $K, K^1, K^2 : T \times T \rightarrow \mathbb{R}$ are continuous, symmetric, positive definite kernels and $\exists \kappa > 0, \kappa_1 > 0, \kappa_2 > 0$ with

$$
\int_T K(x, x) \, d\nu(x) \leq \kappa^2, \quad \int_T K'(x, x) \, d\nu(x) \leq \kappa_1^2.
$$

(1)

4. **A4** $\xi \sim \text{GP}(0, K)$, $\xi^i \sim \text{GP}(0, K^i)$, $i = 1, 2$, are centered Gaussian processes with covariance functions $K, K^i$, respectively, satisfying assumptions A1-A3.

For $K$ satisfying assumption A3, let $\mathcal{H}_K$ denote the corresponding reproducing kernel Hilbert space (RKHS). Let $K_x : T \rightarrow \mathbb{R}$ be defined by $K_x(t) = K(x, t)$. Assumption A3 implies in particular that

$$
\int_T K(x, t)^2 \, d\nu(t) < \infty \forall x \in T, \quad \int_{T \times T} K(x, t)^2 \, d\nu(t) \, d\nu(x) < \infty.
$$

(2)

It follows that $K_x \in L^2(T, \nu) \forall x \in T$, hence $\mathcal{H}_K \subset L^2(T, \nu)$ [16]. Define the following linear operator

$$
R_K = R_{K, v} : L^2(T, \nu) \rightarrow \mathcal{H}_K, \quad R_K f = \int_T K_x(t) f(t) \, d\nu(t), \quad (R_K f)(x) = \int_T K(x, t) f(t) \, d\nu(t).
$$

(3)

The operator $R_K$ is bounded, with $||R_K : L^2(T, \nu) \rightarrow \mathcal{H}_K|| \leq \sqrt{\int_T K(t, t) \, d\nu(t)} \leq \kappa$. Its adjoint is $R^*_K : \mathcal{H}_K \rightarrow L^2(T, \nu) = J : \mathcal{H}_K \hookrightarrow L^2(T, \nu)$, the inclusion operator from $\mathcal{H}_K$ into $L^2(T, \nu)$ [17]. $R_K$ and $R^*_K$ together induce the following self-adjoint, positive, trace class operator (e.g. [18, 16, 17])

$$
C_K = C_{K, v} = R^*_K R_K : L^2(T, \nu) \rightarrow L^2(T, \nu), \quad (C_K f)(x) = \int_T K(x, t) f(t) \, d\nu(t), \quad \forall f \in L^2(T, \nu),
$$

(4)

$$
\text{tr}(C_K) = \int_T K(x, x) \, d\nu(x) \leq \kappa^2, \quad ||C_K||^2_{\text{HS}(L^2(T, \nu))} = \int_{T \times T} K(x, t)^2 \, d\nu(x) \, d\nu(t) \leq \kappa^4.
$$

(5)

Let $\{\lambda_k\}_{k \in \mathbb{N}}$ be the eigenvalues of $C_K$, with normalized eigenfunctions $\{\phi_k\}_{k \in \mathbb{N}}$ forming an orthonormal basis in $L^2(T, \nu)$. Mercer’s Theorem (see version in [16]) states that

$$
K(x, y) = \sum_{k=1}^{\infty} \lambda_k \phi_k(x) \phi_k(y) \quad \forall (x, y) \in T \times T,
$$

(6)

where the series converges absolutely for each pair $(x, y) \in T \times T$ and uniformly on any compact subset of $T$. By Mercer’s Theorem, $K$ is completely determined by $C_K$ and vice versa.

Consider now the correspondence between the trace class operator $C_K$ as defined in Eq.(4) and Gaussian processes with paths in $L^2(T, \nu)$, as established in [19]. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\xi = (\xi(t))_{t \in T} = (\xi(\omega, t))_{t \in T}$ be a real Gaussian process on $(\Omega, \mathcal{F}, P)$, with mean $m$ and covariance function $K$, denoted by $\xi \sim \text{GP}(m, K)$, where

$$
m(t) = E(\xi(t)), \quad K(s, t) = E[(\xi(s) - m(s))(\xi(t) - m(t))], \quad s, t \in T.
$$

(7)

The sample paths $\xi(\omega, \cdot) \in \mathcal{H} = L^2(T, \nu)$ almost $P$-surely, i.e. $\int_T \xi^2(\omega, t) \, d\nu(t) < \infty$ almost $P$-surely, if and only if ([19], Theorem 2 and Corollary 1)

$$
\int_T m^2(t) \, d\nu(t) < \infty, \quad \int_T K(t, t) \, d\nu(t) < \infty.
$$

(8)
The condition for $K$ in Eq.(8) is precisely assumption A3. In this case, $\xi$ induces the following Gaussian measure $P_\xi$ on $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$: $P_\xi(B) = P(\omega \in \Omega : \xi(\omega, \cdot) \in B)$, $B \in \mathcal{B}(\mathcal{H})$, with mean $m \in \mathcal{H}$ and covariance operator $C_K : \mathcal{H} \to \mathcal{H}$, defined by Eq.(4). Conversely, let $\mu$ be a Gaussian measure on $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$, then there is a Gaussian process $\xi = (\xi(t))_{t \in T}$ with sample paths in $\mathcal{H}$, with induced probability measure $P_\xi = \mu$.

**Divergence between Gaussian processes.** Since Gaussian processes are fully determined by their means and covariance functions, the latter being fully determined by their covariance operators, we can define distance/divergence functions, the latter being fully determined by their covariance operators, we can define distance/divergence functions between two Gaussian processes as follows, see also e.g. [6, 7, 8, 20]. Assume Assumptions A1-A4. Let $\mathcal{H} = L^2(T, \nu)$. Let Gauss$(\mathcal{H})$ denote the set of Gaussian measures on $\mathcal{H}$. Let $\xi^i \sim \text{GP}(m_i, K_i)$, $i = 1, 2$, be two Gaussian processes with mean $m_i \in \mathcal{H}$ and covariance function $K_i$. Let $D$ be a divergence function on Gauss$(\mathcal{H})$.

The corresponding divergence $D_{\text{GP}}$ between $\xi^1$ and $\xi^2$ is defined to be
\[
D_{\text{GP}}(\xi^1 || \xi^2) = D(N(m_1, C_{K^1}) || N(m_2, C_{K^2})).
\] (9)

It is clear then that $D_{\text{GP}}(\xi^1 || \xi^2) \geq 0$ and by Mercer's Theorem
\[
D_{\text{GP}}(\xi^1 || \xi^2) = 0 \iff m_1 = m_2, C_{K^1} = C_{K^2} \iff m_1 = m_2, K^1 = K^2.
\] (10)

Subsequently, we assume $m_1 = m_2 = 0$ and focus on $D(N(0, C_{K^1}) || N(0, C_{K^2}))$ with $D$ being a Riemannian distance.

### 2.1 Background: Finite-dimensional distances

In the finite-dimensional setting, many different distance and distance-like functions between covariance matrices and Gaussian measures have been studied. Specifically, let $A, B$ be two covariance matrices corresponding to two Borel probability measures in $\mathbb{R}^n$, then $A, B \in \text{Sym}^+(n)$, the set of $n \times n$ real, symmetric, positive semi-definite matrices. Examples of distance functions that have been studied on $\text{Sym}^+(n)$ include

1. **Euclidean (Frobenius) distance** $d_E(A, B) = ||A - B||_F$, where $|| \cdot ||_F$ denotes the Frobenius norm.
2. **Square root distance** $d_{1/2}(A, B) = ||A^{1/2} - B^{1/2}||_F = (\text{tr}([A + B - 2(A^{1/2}B^{1/2})]^{1/2}))^{1/2}$.
3. **Bures-Wasserstein distance** (see e.g. [22, 23, 24, 25, 26]) $d_{BW}(A, B) = (\text{tr}([A + B - 2(B^{1/2}AB^{1/2})^{1/2}]^{1/2}))^{1/2} = W_2(N(0, A), N(0, B))$, the $L^2$-Wasserstein distance between two zero-mean Gaussian probability measures in $\mathbb{R}^n$ with covariance matrices $A, B$. It coincides with the square root distance if and only if $A$ and $B$ commute.

Consider now the set $\text{Sym}^{++}(n)$ of $n \times n$ real, symmetric, positive definite (SPD) matrices. Elements of this set include, for example, covariance matrices corresponding to Gaussian probability densities on $\mathbb{R}^n$. The set $\text{Sym}^{++}(n)$ is rich in intrinsic geometrical structures and one common approach is to view it as a Riemannian manifold. Examples of Riemannian metrics that have been studied on $\text{Sym}^{++}(n)$ include

1. **Affine-invariant Riemannian metric** (see e.g. [27, 28]), with the corresponding Riemannian distance $d_{\text{aIE}}(A, B) = ||\log(B^{1/2}AB^{1/2})||_F$, where log denotes the principal logarithm of $A$. The distance $d_{\text{aIE}}(A, B)$ corresponds to the Fisher-Rao distance between two zero-mean Gaussian densities with covariance matrices $A, B$ in $\mathbb{R}^n$.
2. **Log-Euclidean metric** [29], with the corresponding Riemannian distance given by $d_{\text{logE}}(A, B) = ||\log(A) - \log(B)||_F$.
3. When restricted on $\text{Sym}^{++}(n)$, the Bures-Wasserstein distance is also the Riemannian distance corresponding to a Riemannian metric [30].
We first discuss the concept of positive definite (unitized) Hilbert-Schmidt operators on a Hilbert space [1]. Specifically, let $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2$ be infinite-dimensional separable real Hilbert spaces. Let $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ denote the set of bounded linear operators between $\mathcal{H}_1$ and $\mathcal{H}_2$. For $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, we write $\mathcal{L}(\mathcal{H})$. The set of trace class operators on $\mathcal{H}$ is defined to be $\text{Tr}(\mathcal{H}) = \{A \in \mathcal{L}(\mathcal{H}) : ||A||_{\text{tr}} = \sum_{k=1}^{\infty} (\lambda_k, (A^* A)^{1/2} e_k) < \infty\}$, where $\{e_k\}_{k=1}^{\infty}$ is any orthonormal basis in $\mathcal{H}$ and the trace norm $||A||_{\text{tr}}$ is independent of the choice of such basis. For $A \in \text{Tr}(\mathcal{H})$, the trace of $A$ is $\text{tr}(A) = \sum_{k=1}^{\infty} (\lambda_k, A e_k) = \sum_{k=1}^{\infty} \lambda_k$, where $\{\lambda_k\}_{k=1}^{\infty}$ denote the eigenvalues of $A$. For two separable Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, the set of Hilbert-Schmidt operators between $\mathcal{H}_1$ and $\mathcal{H}_2$ is defined to be, see e.g. [33], $\text{HS}(\mathcal{H}_1, \mathcal{H}_2) = \{A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) : ||A||_{\text{HS}}^2 = \text{tr}(A^* A) = \sum_{k=1}^{\infty} ||A e_k||_{\mathcal{H}_2}^2 < \infty\}$, the Hilbert-Schmidt norm $||A||_{\text{HS}}$ being independent of the choice of orthonormal basis $\{e_k\}_{k=1}^{\infty}$ in $\mathcal{H}_1$. For $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, we write $\text{HS}(\mathcal{H})$. The set $\text{HS}(\mathcal{H}_1, \mathcal{H}_2)$ is itself a Hilbert space with the Hilbert-Schmidt inner product $(A, B)_{\text{HS}} = \text{tr}(A^* B) = \sum_{k=1}^{\infty} (A e_k, B e_k)$. The set of extended (or unitized) Hilbert-Schmidt operators on $\mathcal{H}$ is defined in [1] to be

$$\text{HS}_X(\mathcal{H}) = \{A + \gamma I : A \in \text{HS}(\mathcal{H}), \gamma \in \mathbb{R}\}. \tag{11}$$

This is a Hilbert space under the extended Hilbert-Schmidt inner product and extended Hilbert-Schmidt norm

$$(A + \gamma I, B + \nu I)_{\text{HS}_X} = (A, B)_{\text{HS}} + \gamma \nu, \quad ||A + \gamma I||_{\text{HS}_X}^2 = ||A||_{\text{HS}}^2 + \gamma^2. \tag{12}$$
Under $(\cdot, \cdot)_{HS_X}$, the scalar operators $\gamma I, \gamma \in \mathbb{R},$ are orthogonal to the Hilbert-Schmidt operators. With the norm $||| \cdot |||_{HS_X},$ $|||I|||_{HS_X} = 1,$ in contrast to the Hilbert-Schmidt norm, where $|||I|||_{HS} = \infty.$

We recall that an operator $A \in \mathcal{L}(\mathcal{H})$ is said to be positive definite [34] if there exists a constant $M_A > 0$ such that $\langle x, Ax \rangle \geq M_A |||x|||^2 \forall x \in \mathcal{H}.$ This condition is equivalent to requiring that $A$ be both strictly positive, that is $\langle x, Ax \rangle > 0 \forall x \neq 0,$ and invertible, with $A^{-1} \in \mathcal{L}(\mathcal{H}).$ Let $\mathcal{P}(\mathcal{H})$ be the set of self-adjoint positive definite bounded operators on $\mathcal{H}.$

**Positive definite (unitized) Hilbert-Schmidt operators.** With the extended Hilbert-Schmidt operators, we define the set of positive definite (unitized) Hilbert-Schmidt operators on $\mathcal{H}$ to be

$$\mathcal{P}^{+}_2(\mathcal{H}) = \mathcal{P}(\mathcal{H}) \cap HS_X(\mathcal{H}) = \{A + \gamma I > 0 : A^* = A, \gamma \in \mathbb{R}\}. \quad (13)$$

This is a Hilbert manifold, being an open subset of the Hilbert space $HS_X(\mathcal{H}).$ On $\mathcal{P}^{+}_2(\mathcal{H}),$ both $\log(A + \gamma I)$ and $(A + \gamma I)\alpha, \alpha \in \mathbb{R},$ are well-defined and bounded.

**Affine-invariant Riemannian distance.** The generalization of the affine-invariant metric on $\text{Sym}^{++}(n)$ to the Hilbert manifold $\mathcal{P}^{+}_2(\mathcal{H})$ was defined in [1], with the corresponding Riemannian distance given by

$$d_{\text{aiHS}}[(A + \gamma I), (B + \nu I)] = |||\log((B + \nu I)^{-1/2}(A + \gamma I)(B + \nu I)^{-1/2})|||_{HS_X}. \quad (14)$$

**Log-Hilbert-Schmidt distance.** Similarly, the generalization of the Log-Euclidean metric on $\text{Sym}^{++}(n)$ to $\mathcal{P}^{+}_2(\mathcal{H})$ was defined in [2], with the corresponding Log-Hilbert-Schmidt distance given by

$$d_{\log HS}[(A + \gamma I), (B + \nu I)] = |||\log(A + \gamma I) - \log(B + \nu I)|||_{HS_X}. \quad (15)$$

The definition of the extended Hilbert-Schmidt norm guarantees that both $d_{\text{aiHS}}[(A + \gamma I), (B + \nu I)]$ and $d_{\log HS}[(A + \gamma I), (B + \nu I)]$ are always well-defined and finite for any pair $(A + \gamma I), (B + \nu I) \in \mathcal{P}^{+}_2(\mathcal{H}).$ In the setting of reproducing kernel Hilbert space (RKHS) covariance operators, both the affine-invariant Riemannian and Log-Hilbert-Schmidt distances admit closed form formulas in terms of the corresponding kernel Gram matrices [2], [35].

**Distances between positive Hilbert-Schmidt operators.** In the case $\gamma = \nu > 0$ is fixed, both $d_{\text{aiHS}}[(A + \gamma I), (B + \gamma I)]$ and $d_{\log HS}[(A + \gamma I), (B + \gamma I)]$ become distances on the set of self-adjoint, positive Hilbert-Schmidt operators on $\mathcal{H}.$ In the following, let $\text{Sym}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ denote the set of self-adjoint, bounded operators and $\text{Sym}^{++}(n) \subset \text{Sym}(\mathcal{H})$ the set of self-adjoint, positive, bounded operators on $\mathcal{H}.$ We immediately have the following result.

**Theorem 1** Let $\gamma \in \mathbb{R}, \gamma > 0$ be fixed. The distances $d_{\text{aiHS}}[(A + \gamma I), (B + \gamma I)], d_{\log HS}[(A + \gamma I), (B + \gamma I)]$ are metrics on the set $\text{Sym}^{++}(\mathcal{H}) \cap HS(\mathcal{H})$ of positive Hilbert-Schmidt operators on $\mathcal{H}.$

**Related and further generalizations.** Similar to the extended Hilbert-Schmidt operators, we can define the extended trace class operators [36] to be $\text{Tr}_X(\mathcal{H}) = \{A + \gamma I : A \in \text{Tr}(\mathcal{H}), \gamma \in \mathbb{R}\}$ along with the extended Fredholm determinant $d_{\text{det}}(A + \gamma I)$ and subsequently the extended Hilbert-Carleman determinant [37]. With these concepts, we obtained the infinite-dimensional Alpha Log-Det divergences [36] and Alpha–Beta Log-Det divergences [38] between positive definite (unitized) trace class operators and subsequently on the entire Hilbert manifold $\mathcal{P}^{+}_2(\mathcal{H})$ [37]. The Alpha-Beta Log-Det divergences form a highly general family of divergences on $\mathcal{P}^{+}_2(\mathcal{H})$ and include the affine-invariant Riemannian distance $d_{\text{aiHS}}$ as a special case. Closely related to the $L^2$-Wasserstein distance is its entropic
regularization, the Sinkhorn divergence. For two Gaussian measures \( \mu_i \sim N(m_i, C_i) \) on \( \mathcal{H} \), \( i = 0, 1 \), it is given by \([39]\)

\[
S^\alpha_\text{Sinkhorn}(\mu_0, \mu_1) = ||m_0 - m_1||^2 + \frac{\epsilon}{4} \text{tr} \left[ M^\alpha_{00} - 2M^\alpha_{01} + M^\alpha_{11} \right] + \frac{\epsilon}{4} \log \det \left[ \frac{(I + \frac{1}{\epsilon} M^\alpha_{00})^2}{(I + \frac{1}{\epsilon} M^\alpha_{11})} \right], \quad \epsilon > 0.
\]  

(16)

with \( \lim_{\epsilon \to 0} S^\alpha_\text{Sinkhorn}(\mu_0, \mu_1) = W_2(\mu_0, \mu_1) \). Here \( M^\alpha_{ij} = -I + \left( I + \frac{15}{2 \epsilon} C_i C_j \right)^{1/2} \), \( i, j = 0, 1 \), and \( \text{det} \) denotes the Fredholm determinant. The \( \alpha \)-Procrustes distances can also be generalized to the infinite-dimensional setting of \( \mathcal{H} \) and include both the Bures-Wasserstein and Log-Hilbert-Schmidt distances as special cases \([31]\) \([32]\).

\[
d^{\alpha}_\text{proHS}((A + \gamma I), (B + \gamma I)) = \frac{\text{tr}[(A + \gamma I)^{2\alpha} + (B + \gamma I)^{2\alpha} - 2((A + \gamma I)^{\alpha}(B + \gamma I)^{\alpha})^{1/2})]}{|\alpha|}, \quad \alpha \in \mathbb{R}, \alpha \neq 0.
\]

(17)

In particular, for \( A, B \in \text{Sym}^+(\mathcal{H}) \cap \text{Tr} \mathcal{H} \) and \( \alpha = 1/2 \), \( \lim_{\gamma \to 0} d^{1/2}_\text{proHS}((A + \gamma I), (B + \gamma I)) = 2(\text{tr}[A + B - 2(B^{1/2}AB^{1/2})])^{1/2} \), which is twice the Bures-Wasserstein distance.

### 3.1 Finite-rank and finite-dimensional approximations

In practice, it is typically necessary to deal with finite-rank and/or finite-dimensional approximations of infinite-dimensional distances. In the cases of \( d_{\text{HS}} \) and \( d_{\text{logHS}} \), finite-rank and finite-dimensional approximations are consequences of the following general convergence results, which are subsequently employed in the Gaussian process setting. We first note the decomposition \( ||\log(A + \gamma I) - \log(B + \nu I)||_\text{HS}^2 = ||\log \frac{A}{\nu} + I - \log \frac{B}{\nu} + I||_\text{HS}^2 + (\log \frac{\nu}{\gamma})^2 \) and similarly \( ||\log[(A + \gamma I)^{-1/2}(B + \nu I)(A + \gamma I)^{-1/2}]||_\text{HS}^2 = ||\log \frac{A}{\nu} + I - \log \frac{B}{\nu} + I||_\text{HS}^2 + (\log \frac{\nu}{\gamma})^2 \), thus for \( \gamma = \nu \)

\[
||\log(A + \gamma I) - \log(B + \gamma I)||_\text{HS} = ||\log(A + \gamma I) - \log(B + \nu I)||_\text{HS} = \left|\left| \log \left( \frac{A}{\nu} + I \right) - \log \left( \frac{B}{\nu} + I \right) \right|\right|_\text{HS}.
\]

(18)

\[
||\log[(A + \gamma I)^{-1/2}(B + \nu I)(A + \gamma I)^{-1/2}]||_\text{HS} = ||\log \frac{A}{\nu} + I - \log \frac{B}{\nu} + I||_\text{HS}.
\]

(19)

**Theorem 2 (Convergence in Log-Hilbert-Schmidt distance)** Let \( \gamma \in \mathbb{R}, \gamma > 0 \) be fixed. Let \( A_n \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H}) \). Assume that \( (\gamma I + A)^0, 0 < \gamma I + A_n > 0 \forall n \in \mathbb{N} \). Then \( \log(\gamma I + A_n), \log(\gamma I + A) \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H}) \).

(i) If \( A, A_n \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H}) \forall n \in \mathbb{N} \), then

\[
||\log(\gamma I + A_n) - \log(\gamma I + A)||_\text{HS} \leq \frac{1}{\gamma} ||A_n - A||_\text{HS} \forall n \in \mathbb{N}.
\]

(20)

(ii) In general, if \( \lim_{n \to \infty} ||A_n - A|| = 0 \), let \( M_A > 0 \) be such that \( \langle x, (\gamma I + A)x \rangle \geq M_A||x||^2 \forall x \in \mathcal{H} \). For \( 0 < \epsilon < M_A \) fixed, let \( N(\epsilon) \in \mathbb{N} \) such that \( ||A_n - A|| < \epsilon \forall n \geq N(\epsilon) \), then

\[
||\log(\gamma I + A_n) - \log(\gamma I + A)||_\text{HS} \leq \frac{1}{M_A - \epsilon} ||A_n - A||_\text{HS} \forall n \geq N(\epsilon).
\]

(21)

In both cases, \( \lim_{n \to \infty} ||A_n - A||_\text{HS} = 0 \) implies \( \lim_{n \to \infty} ||\log(\gamma I + A_n) - \log(\gamma I + A)||_\text{HS} = 0 \).
Remark Scenario (i) in Theorem 2 applies immediately to the case $A, A_n$ are covariance operators on $\mathcal{H}$. In general, the setting in (ii) is needed since we can have $I + A > 0$ without $A$ being positive. For example, for $I + A > 0, I + B > 0$, as in Theorem 4 for the affine-invariant Riemannian distance, we have $(I + B)^{-1/2}(I + A)(I + B)^{-1/2} = I + C > 0$, where the operator $C = (I + B)^{-1/2}(A - B)(I + B)^{-1/2}$ can be positive or negative or indefinite.

Theorem 3 (Approximation of Log-Hilbert-Schmidt distance) Let $\gamma_i \in \mathbb{R}, \gamma_i > 0, i = 1, 2$ be fixed. Let $A, B, \{A_n\}_{n \in \mathbb{N}}$, $(B_n)_{n \in \mathbb{N}} \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$ be such that $\gamma_i I + A > 0, \gamma_i I + B > 0, \gamma_i I + A_n > 0, \gamma_i I + B_n > 0 \forall n \in \mathbb{N}$.

(i) If $A_n, B_n, A, B \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H})$, then

$$\left| \left| \log(\gamma_i I + A_n) - \log(\gamma_i I + B_n) \right| \right|_{\text{HS}(X)} - \left| \left| \log(\gamma_i I + A) - \log(\gamma_i I + B) \right| \right|_{\text{HS}(X)} \leq \frac{1}{\gamma_i} |A_n - A|_{\text{HS}} + \frac{1}{\gamma_i} |B_n - B|_{\text{HS}}.$$ (22)

(ii) In general case, assume that $\lim_{n \to \infty} |A_n - A|_{\text{HS}} = 0, \lim_{n \to \infty} |B_n - B|_{\text{HS}} = 0$. Let $M_A, M_B > 0$ be such that $(x, (\gamma_i I + A)x) \geq M_A |x|^2, (x, (\gamma_i I + B)x) \geq M_B |x|^2 \forall x \in \mathcal{H}$. Then $\forall 0 < e < \min \{M_A, M_B\}, \exists N(e) \in \mathbb{N}$ such that $\forall n \geq N(e)$, $|A_n - A| < e, |B_n - B| < e$, and

$$\left| \left| \log(\gamma_i I + A_n) - \log(\gamma_i I + B_n) \right| \right|_{\text{HS}(X)} - \left| \left| \log(\gamma_i I + A) - \log(\gamma_i I + B) \right| \right|_{\text{HS}(X)} \leq \frac{1}{M_A - e} |A_n - A|_{\text{HS}} + \frac{1}{M_B - e} |B_n - B|_{\text{HS}}.$$ (23)

Theorem 4 (Convergence in affine-invariant Riemannian distance) Let $\gamma \in \mathbb{R}, \gamma > 0$ be fixed. Let $A, \{A_n\}_{n \in \mathbb{N}} \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$ be such that $\gamma I + A > 0, \gamma I + A_n > 0 \forall n \in \mathbb{N}$, and $\lim_{n \to \infty} |A_n - A|_{\text{HS}} = 0$. Let $M_A > 0$ be such that $(x, (\gamma I + A)x) \geq M_A |x|^2 \forall x \in \mathcal{H}$. Then $\forall 0 < e < M_A, \exists N(e) \in \mathbb{N}$ such that $\forall n \geq N(e)$, $|A_n - A| < e$ and

$$\left| \left| \log((\gamma I + A_n)^{-1/2}(\gamma I + A_n)(\gamma I + A)^{-1/2}) \right| \right|_{\text{HS}} \leq \frac{1}{M_A - e} |A_n - A|_{\text{HS}}.$$ (24)

In particular, if $A \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H})$, then we can set $M_A = \gamma$ and consequently, $\forall 0 < e < \gamma$,

$$\left| \left| \log((\gamma I + A_n)^{-1/2}(\gamma I + A_n)(\gamma I + A)^{-1/2}) \right| \right|_{\text{HS}} \leq \frac{1}{\gamma - e} |A_n - A|_{\text{HS}} \forall n \geq N(e).$$ (25)

Theorem 5 (Approximation of affine-invariant Riemannian distance) Let $\gamma_i \in \mathbb{R}, \gamma_i > 0, i = 1, 2$ be fixed. Let $A, B, \{A_n\}_{n \in \mathbb{N}}$, $(B_n)_{n \in \mathbb{N}} \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$ be such that $\gamma_i I + A > 0, \gamma_i I + B > 0, \gamma_i I + A_n > 0, \gamma_i I + B_n > 0 \forall n \in \mathbb{N}$, and $\lim_{n \to \infty} |A_n - A|_{\text{HS}} = 0, \lim_{n \to \infty} |B_n - B|_{\text{HS}} = 0$. Let $M_A, M_B > 0$ be such that $(x, (\gamma_i I + A)x) \geq M_A |x|^2, (x, (\gamma_i I + B)x) \geq M_B |x|^2 \forall x \in \mathcal{H}$. Then $\forall 0 < e < \min \{M_A, M_B\}, \exists N(e) \in \mathbb{N}$ such that $\forall n \geq N(e)$, $|A_n - A| < e, |B_n - B| < e$, and

$$\left| \left| \log((\gamma_i I + A_n)^{-1/2}(\gamma_i I + B_n)(\gamma_i I + A_n)^{-1/2}) \right| \right|_{\text{HS}(X)} - \left| \left| \log((\gamma_i I + A)^{-1/2}(\gamma_i I + B)(\gamma_i I + A)^{-1/2}) \right| \right|_{\text{HS}(X)} \leq \frac{1}{M_A - e} |A_n - A|_{\text{HS}} + \frac{1}{M_B - e} |B_n - B|_{\text{HS}}.$$ (26)
In particular, if $A, B \in \text{Sym}^+ (\mathcal{H}) \cap \text{HS}(\mathcal{H})$, then we can set $M_A = \gamma_1, M_B = \gamma_2$ and consequently, $\forall 0 < \epsilon < \min \{\gamma_1, \gamma_2\}$,

$$\left| | | \log((\gamma_1 I + A_n)^{-1/2}(\gamma_2 I + B_n)(\gamma_1 I + A_n)^{-1/2}) |||_{\text{HS}} - | | | | \log(\gamma_1 I + A)^{-1/2}(\gamma_2 I + B)(\gamma_1 I + A)^{-1/2}) |||_{\text{HS}} \right| \leq \frac{1}{\gamma_1 - \epsilon} ||A_n - A||_{\text{HS}} + \frac{1}{\gamma_2 - \epsilon} ||B_n - B||_{\text{HS}}. \quad (27)$$

**Finite-dimensional approximations via orthogonal projections.** We now consider the finite-dimensional approximations of $d_{\text{dilHS}}$ and $d_{\text{logHS}}$ via orthogonal projections. Let $A \in \text{HS}(\mathcal{H})$. Let $(e_k)_{k=1}^\infty$ be any orthonormal basis for $\mathcal{H}$. For any $f \in \mathcal{H}$, we have $f = \sum_{k=1}^{\infty} \langle f, e_k \rangle e_k$. Let $N \in \mathbb{N}$ be fixed and consider the finite-dimensional subspace $\mathcal{H}_N = \text{span} \{e_k\}_{k=1}^N$. Consider next the projection operator $P_N = \sum_{k=1}^N e_k \otimes e_k : \mathcal{H} \rightarrow \mathcal{H}_N$. For any $f \in \mathcal{H}$, $P_N f = \sum_{k=1}^N \langle f, e_k \rangle e_k$ and for the operator $P_N A P_N : \mathcal{H} \rightarrow \mathcal{H}$,

$$P_N A P_N f = P_N \sum_{k=1}^N \langle f, e_k \rangle A e_k = \sum_{j=1}^N \sum_{k=1}^N \langle f, e_k \rangle \langle A e_k, e_j \rangle e_j \in \mathcal{H}_N. \quad (28)$$

Hence $P_N A P_N$ is a finite rank operator, with rank at most $N$, and $\text{range}(P_N A P_N) \subset \mathcal{H}_N$. In particular, $P_N A P_N |_{\mathcal{H}_N} : \mathcal{H}_N \rightarrow \mathcal{H}_N$ and for $f, g \in \mathcal{H}_N$, we have

$$\langle g, P_N A P_N f \rangle = \sum_{j,k=1}^N \langle f, e_k \rangle \langle g, e_j \rangle \langle A e_k, e_j \rangle = \langle g, A f \rangle_{\mathcal{H}_N}. \quad (29)$$

where $f = (\langle f, e_k \rangle)_{k=1}^N$, $g = (\langle g, e_k \rangle)_{k=1}^N \in \mathbb{R}^N$ and $A_N$ is the $N \times N$ matrix with $(A_N)_{kj} = \langle A e_k, e_j \rangle$. Thus on $\mathcal{H}_N$ with basis $(e_k)_{k=1}^N$, the operator $P_N A P_N |_{\mathcal{H}_N}$ is represented by the matrix $A_N$. Furthermore, $A \in \text{Sym}(\mathcal{H}) \Rightarrow P_N A P_N |_{\mathcal{H}_N} \in \text{Sym}(\mathcal{H}_N) \Rightarrow A \in \text{Sym}(\mathcal{H}_N)$ and $A \in \text{Sym}^*(\mathcal{H}) \Rightarrow P_N A P_N |_{\mathcal{H}_N} \in \text{Sym}^*(\mathcal{H}_N) \Rightarrow A \in \text{Sym}^*(\mathcal{N})$.

Combining Theorems 3 and 5 with the finite-dimensional projection $P_N$, we obtain the following results.

**Theorem 6 (Finite-dimensional approximation of Log-Hilbert-Schmidt distance)** Assume that $(A + I), (B + I) \in \mathcal{P}_2(\mathcal{H})$. Let $A_N = P_N A P_N |_{\mathcal{H}_N}$ and $B = P_N B P_N |_{\mathcal{H}_N}$, with matrix representation $A_N$ and $B_N$, in the basis $(e_k)_{k=1}^N$, respectively. Then

$$\lim_{N \rightarrow \infty} ||\log(A_N + I) - \log(B_N + I)||_F = \lim_{N \rightarrow \infty} ||\log(A_N + I) - \log(B_N + I)||_{\text{HS}} = ||\log(A + I) - \log(B + I)||_{\text{HS}}. \quad (30)$$

Assume that $(A + \gamma I), (B + \gamma I) \in \mathcal{P}_2(\mathcal{H}), \gamma \in \mathbb{R}, \gamma > 0$. Then

$$\lim_{N \rightarrow \infty} ||\log(A_N + \gamma I) - \log(B_N + \gamma I)||_F = \lim_{N \rightarrow \infty} ||\log(A_N + \gamma I) - \log(B_N + \gamma I)||_{\text{HS}} \quad (31)$$

$$= ||\log(A + \gamma I) - \log(B + \gamma I)||_{\text{HS}}. \quad (32)$$

**Theorem 7 (Finite-dimensional approximation of Affine-invariant Riemannian distance)** Assume that $(A + I), (B + I) \in \mathcal{P}_2(\mathcal{H})$. Let $A_N = P_N A P_N |_{\mathcal{H}_N}$ and $B = P_N B P_N |_{\mathcal{H}_N}$, with matrix representation $A_N$ and $B_N$, in the basis $(e_k)_{k=1}^N$, respectively. Then

$$\lim_{N \rightarrow \infty} ||(B_N + I)^{-1/2}(A_N + I)(B_N + I)^{-1/2}||_F = \lim_{N \rightarrow \infty} ||(B_N + I)^{-1/2}(A_N + I)(B_N + I)^{-1/2}||_{\text{HS}}$$

$$= ||(B + I)^{-1/2}(A + I)(B + I)^{-1/2}||_{\text{HS}}. \quad (33)$$
Assume that \((A + \gamma I), (B + \gamma I) \in \mathcal{P}_2(\mathcal{H}), \gamma \in \mathbb{R}, \gamma > 0\). Then
\[
\lim_{N \to \infty} \| \log[(B_N + \gamma I)^{-1/2}(A_N + \gamma I)(B_N + \gamma I)^{-1/2}] \|_F = \lim_{N \to \infty} \| \log[(B_N + \gamma I)^{-1/2}(A_N + \gamma I)(B_N + \gamma I)^{-1/2}] \|_{\text{HS}} = \| \log[(B + \gamma I)^{-1/2}(A + \gamma I)(B + \gamma I)^{-1/2}] \|_{\text{HS}}.
\]

4 \hspace{1cm} \text{ESTIMATION OF DISTANCES BETWEEN GAUSSIAN PROCESSES}

Let \(\xi^i \sim \text{GP}(0, K^i), i = 1, 2\), be two Gaussian processes satisfying Assumptions A1-A4, with paths in \(L^2(T, \nu)\). The Log-Hilbert-Schmidt and affine-invariant Riemannian distances between \(\xi^1\) and \(\xi^2\) are defined via their corresponding centered Gaussian measures with covariance operators \(C_{K^i}\), as follows
\[
D_{\text{logHS}}^T(\xi^1 || \xi^2) = D_{\text{logHS}}^T[N(0, C_{K^1}), N(0, C_{K^2})] = \| \log(y I + C_{K^1}) - \log(y I + C_{K^2}) \|_{\text{HS}(L^2(T, \nu))},
\]
\[
D_{\text{ahlHS}}^T(\xi^1 || \xi^2) = D_{\text{ahlHS}}^T[N(0, C_{K^1}), N(0, C_{K^2})] = \| \log[(y I + C_{K^1})^{-1/2}(y I + C_{K^2})(y I + C_{K^1})^{-1/2}] \|_{\text{HS}(L^2(T, \nu))}.
\]

In the following, we aim to estimate \(D_{\text{logHS}}^T[N(0, C_{K^1}), N(0, C_{K^2})] \) and \(D_{\text{ahlHS}}^T[N(0, C_{K^1}), N(0, C_{K^2})] \) given finite samples \(\{\xi_i^1(x_j)\}_{i=1}^{N_1}, \{\xi_i^2(x_j)\}_{i=1}^{N_2}\) from \(\xi^1, \xi^2\) on a set of points \(X = (x_j)_{j=1}^m\) in \(T\). These correspond to \(N_t\) realizations of process \(\xi^i, i = 1, 2\), sampled at the \(m\) points in \(T\) given by \(X\). For simplicity and without loss of generality, in the theoretical analysis we let \(N_1 = N_2 = N\). We carry out the analysis in three scenarios: (i) using finite-rank sample covariance operators, (ii) using finite covariance matrices, (iii) using finite samples, with the last being the most practical.

4.1 \hspace{1cm} \text{Estimation of distances from sample covariance operators}

Consider the first scenario, where we have access to samples \(\{\xi_i^1(t) = \xi^i(\omega, t)\}_{j=1}^{N}, i = 1, 2\), and the corresponding sample covariance operators. For \(\xi \sim \text{GP}(0, K)\) on the probability space \((\Omega, \mathcal{F}, P)\), define the rank-one operator \(\xi(\omega, .) \otimes \xi(\omega, .) \in \mathcal{L}(L^2(T, \nu))\) by \(\xi(\omega, .) \otimes \xi(\omega, .) f(x) = \xi(\omega, x) \int_T \xi(\omega, t) f(t) d\nu(t), \omega \in \Omega, f \in L^2(T, \nu)\). Then \(\xi(\omega, .) \otimes \xi(\omega, .) \in \text{HS}(L^2(T, \nu))\) \(P\)-almost surely, with
\[
\| \xi(\omega, .) \otimes \xi(\omega, .) \|_{\text{HS}(L^2(T, \nu))} = \int_T \xi(\omega, t)^2 d\nu(t) < \infty \hspace{0.5cm} P\text{-almost surely}.
\]

The covariance operator \(C_K\) as defined in Eq. (4) can be expressed as
\[
C_K = E[\xi \otimes \xi], \hspace{1cm} C_K f(x) = E \int_T \xi(\omega, x) \xi(\omega, t) f(t) d\nu(t) = \int_T K(x, t) f(t) d\nu(t).
\]

Let \(W = (\omega_j)_{j=1}^N\) be independently sampled from \((\Omega, P)\), corresponding to the samples \(\{\xi^j_i(t) = \xi^i(\omega_j, t)\}_{j=1}^{N}\) from \(\xi\). It defines the sample covariance function \(K_W\) and corresponding sample covariance operator \(C_{K, W}\) by
\[
K_W(x, y) = \frac{1}{N} \sum_{j=1}^{N} \xi(\omega_j, x) \xi(\omega_j, y), \hspace{1cm} C_{K, W} = \frac{1}{N} \sum_{j=1}^{N} \xi(\omega_j, .) \otimes \xi(\omega_j, .),
\]
\[
C_{K, W} f(x) = \frac{1}{N} \sum_{j=1}^{N} \int_T \xi(\omega_j, x) \xi(\omega_j, t) f(t) d\nu(t) = \int_T K_W(x, t) f(t) d\nu(t).
\]
For each fixed $W, K_W$ is a positive definite kernel on $T \times T$. It is continuous if the sample paths $\xi(\omega_i)$ are continuous $P$-almost surely. For $\mathcal{H}_W = \text{span}\{\xi(\omega_j)\}_{j=1}^N \subset L^2(T, \nu), \dim(\mathcal{H}_W) \leq N$ and thus $C_{K, W}$ has rank at most $N$. The convergence of $C_{K, W}$ to $C_K$ is obtained given the following additional assumption, which implies A3

$$\int_T |K(x, x)|^2 d\nu(x) \leq \kappa^4, \int_T |K'(x, x)|^2 d\nu(x) \leq \kappa_i^4, \quad i = 1, 2. \quad (41)$$

**Proposition 8 ([12])** Assume Assumptions A1-A5. Let $W = (\omega_j)_{j=1}^N$ be independently sampled from $(\Omega, P)$. For any $0 < \delta < 1$, with probability at least $1 - \delta$,

$$||C_{K, W}||_{\text{HS}(L^2(T, \nu))} \leq \frac{2\kappa^2}{\delta}, \quad ||C_{K, W} - C_K||_{\text{HS}(L^2(T, \nu))} \leq \frac{2\sqrt{3\kappa^2}}{\sqrt{N}\delta}. \quad (42)$$

Combining Proposition 8 with Theorem 3, we obtain the following estimate for $D^r_{\log\text{HS}}[N(0, C_{K_1}), N(0, C_{K_2})]$.

**Theorem 9 (Estimation of Log-Hilbert-Schmidt distance from sample covariance operators)** Under Assumptions A1-A5, let $W_i = (\omega_j^i)_{j=1}^N$, $i = 1, 2$, be independently sampled from $(\Omega_i, P_i)$. For any $0 < \delta < 1$, with probability at least $1 - \delta$,

$$||D^r_{\log\text{HS}}[N(0, C_{K_1}), N(0, C_{K_2})] - D^r_{\log\text{HS}}[N(0, C_{K_1}), N(0, C_{K_2})]| \leq \frac{4\sqrt{3}(\kappa_1^2 + \kappa_2^2)}{\gamma \sqrt{N}\delta}. \quad (43)$$

Combining Propositions 8 with Theorem 5, we obtain the following estimate for $D^r_{\text{alHS}}[N(0, C_{K_1}), N(0, C_{K_2})]$.

**Theorem 10 (Estimation of affine-invariant Riemannian distance from sample covariance operators)** Assume Assumptions A1-A5. Let $y \in \mathbb{R}, y > 0$ be fixed. Let $W_i = (\omega_j^i)_{j=1}^N$, $i = 1, 2$, be independently sampled from $(\Omega_i, P_i)$. For any $0 < \epsilon < y, 0 < \delta < 1$, let $N(\epsilon) \in \mathbb{N}, N(\epsilon) \geq 1 + \max\left\{\frac{4\kappa_1^4}{\epsilon^2\delta^2}, \frac{4\kappa_2^4}{\epsilon^2\delta^2}\right\}$, then $\forall N \geq N(\epsilon)$, with probability at least $1 - \delta$,

$$||D^r_{\text{alHS}}[N(0, C_{K_1}), N(0, C_{K_2})] - D^r_{\text{alHS}}[N(0, C_{K_1}), N(0, C_{K_2})]| \leq \frac{4\sqrt{3}(\kappa_1^2 + \kappa_2^2)}{(y - \epsilon) \sqrt{N}\delta}. \quad (44)$$

If $\kappa_1, \kappa_2$ are absolute constants, then the convergence in both Theorems 9 and 10 is dimension-independent.

### 4.2 Estimation of distances from finite covariance matrices

Consider the second scenario, where we have access to finite covariance matrices associated with the covariance functions $K_1, K_2$. Let $X = (x_i)_{i=1}^m$ be independently sampled from $(T, \nu)$. The Gaussian process assumption $\xi_i \sim \text{GP}(0, K_1)$ means that $(\xi_i(\omega, x_j))_{i, j=1}^m$ are m-dimensional Gaussian random variables, with $(\xi_i(\omega, x_j))_{i, j=1}^m \sim N(0, K_1(X))$, where $(K_1(X))_{j,k} = K_1(x_j, x_k), 1 \leq j, k \leq m$. Assuming that the covariance matrices $K_1(X)$ are known. Let $y \in \mathbb{R}, y > 0$ be fixed, $D^r_{\log\text{E}}[N(0, A), N(0, B)] = d_{\log\text{E}}(A + yI, B + yI), D^r_{\text{alE}}[N(0, A), N(0, B)] = d_{\text{alE}}(A + yI, B + yI)$, we show that

$$D^r_{\log\text{E}}[N\left(0, \frac{1}{m} K_1[X]\right), N\left(0, \frac{1}{m} K_2[X]\right)] \text{ consistently estimates } D^r_{\log\text{HS}}[N(0, C_{K_1}), N(0, C_{K_2})]. \quad (45)$$

$$D^r_{\text{alE}}[N\left(0, \frac{1}{m} K_1[X]\right), N\left(0, \frac{1}{m} K_2[X]\right)] \text{ consistently estimates } D^r_{\text{alHS}}[N(0, C_{K_1}), N(0, C_{K_2})]. \quad (46)$$
Since $\frac{1}{m}K'[X]: \mathbb{R}^m \to \mathbb{R}^m$ and $C_k' : \mathcal{L}^2(T,v) \to \mathcal{L}^2(T,v)$ operate on two different Hilbert spaces, namely $\mathbb{R}^m$ and $\mathcal{L}^2(T,v)$, we express the quantities in Eqs.(45),(46) via RKHS covariance and cross-covariance operators on the same RKHS induced by the kernels $K'$'s. The convergence analysis is then carried out entirely via RKHS methodology.

**RKHS covariance and cross-covariance operators.** Let $K^1, K^2$ be two kernels satisfying Assumptions A1-A4, and $\mathcal{H}_{K^1}, \mathcal{H}_{K^2}$ the corresponding RKHS. Let $R_{K^i} : \mathcal{L}^2(T,v) \to \mathcal{H}_{K^i}, i = 1, 2$ be as defined in Eq.(3). Together, they define the following RKHS cross-covariance operators

$$R_{ij} = R_{K^i}R_{K^j}^*: \mathcal{H}_{K^j} \to \mathcal{H}_{K^i}, \quad i,j = 1, 2, \quad R_{ij} = R_{K^j}R_{K^i}^*: \mathcal{H}_{K^i} \to \mathcal{H}_{K^j} = R_{ij}^*.$$  \hspace{1cm} (47)

$$R_{ij} = \int_T (K_i^j \otimes K_j^i)dv(t), \quad R_{ij}f = \int_T K_i^j(f, K_j^i)\mathcal{H}_{K^j}dv(t).$$  \hspace{1cm} (48)

$$R_{ij}f(x) = \int_T K_i^j(x)(t)dv(t) = \int_T K_i^j(x,t)f(t)dv(t), \quad f \in \mathcal{H}_{K^j}. \hspace{1cm} (49)$$

In particular, $R_{ij} = L_{K^i}$, with the RKHS covariance operator $L_K$ defined by

$$L_K = R_{K^i}R_{K^i}^*: \mathcal{H}_K \to \mathcal{H}_K, \quad L_K = \int_T (K_i \otimes K_i)dv(t), \hspace{1cm} (50)$$

$$L_Kf(x) = \int_T K_i(x)(t)dv(t) = \int_T K_i(x,t)f(t)dv(t), \quad f \in \mathcal{H}_K. \hspace{1cm} (51)$$

$L_K$ has the same nonzero eigenvalues as $C_k$ and thus $L_K \in \text{Sym}^+ (\mathcal{H}_K) \cap \text{Tr}(\mathcal{H}_K)$, with

$$\text{tr}(L_K) = \text{tr}(C_k) \leq \kappa^2, \quad \|L_K\|_{\mathcal{HS}(\mathcal{H}_K)} = \|C_k\|_{\mathcal{HS}(L^2(T,v))} \leq \kappa^2. \hspace{1cm} (52)$$

**Lemma 11 ([12])** Under Assumptions A1-A3, $R_{ij} \in \mathcal{HS}(\mathcal{H}_{K^i}, \mathcal{H}_{K^j})$, with $\|R_{ij}\|_{\mathcal{HS}(\mathcal{H}_{K^i}, \mathcal{H}_{K^j})} \leq \kappa_i \kappa_j, i,j = 1, 2$.

**Empirical RKHS covariance and cross-covariance operators.** Let $X = (x_i)_{i=1}^m$ be independently sampled from $T$ according to $v$. It defines the following sampling operator (see e.g. [40])

$$S_X : \mathcal{H}_K \to \mathbb{R}^m, \quad S_Xf = (f(x_i))_{i=1}^m = ((f, K_{x_i}))_{i=1}^m \quad \text{with adjoint} \quad S_X^* : \mathbb{R}^m \to \mathcal{H}_K, \quad S_X^*b = \sum_{i=1}^m b_i K_{x_i}. \hspace{1cm} (53)$$

The sampling operators $S_i : \mathcal{H}_{K^i} \to \mathbb{R}^m, i = 1, 2$, together define the following empirical version of $R_{ij}$

$$R_{ijX} = \frac{1}{m} S^*_X S_{iX} = \frac{1}{m} \sum_{k=1}^m (K_{k_{x_i}}^i \otimes K_{k_{x_j}}^j) : \mathcal{H}_{K^j} \to \mathcal{H}_{K^i}, \hspace{1cm} (54)$$

$$R_{ijX}f = \frac{1}{m} \sum_{k=1}^m K_{k_{x_i}}^i(f, K_{k_{x_j}}^j)_{\mathcal{H}_{K^j}} = \frac{1}{m} \sum_{k=1}^m f(x_k)K_{k_{x_j}}^j, \quad f \in \mathcal{H}_{K^j}. \hspace{1cm} (55)$$

In particular, $R_{ijX} = L_{K^iX}$, with the empirical RKHS covariance operator $L_K : \mathcal{H}_K \to \mathcal{H}_K$ defined by

$$L_{KX} = \frac{1}{m} S_X^* S_X = \frac{1}{m} \sum_{i=1}^m (K_{x_i} \otimes K_{x_i}) : \mathcal{H}_K \to \mathcal{H}_K, \hspace{1cm} (56)$$

$$L_{KX}f = \frac{1}{m} S_X^*(f(x_i))_{i=1}^m = \frac{1}{m} \sum_{i=1}^m f(x_i)K_{x_i} = \frac{1}{m} \sum_{i=1}^m (f, K_{x_i})_{\mathcal{H}_K} K_{x_i}. \hspace{1cm} (57)$$
Furthermore, the operator $S_X S_X^* : \mathbb{R}^m \to \mathbb{R}^m$ is given by

$$S_X S_X^* : \mathbb{R}^m \to \mathbb{R}^m, \quad S_X S_X^* b = S_X \sum_{i=1}^m b_i K_{x_i} = \left( \sum_{i=1}^m b_i K(x_i, x_i), \ldots, \sum_{i=1}^m b_i K(x_i, x_m) \right) = K[X]b.$$  \hspace{1cm} (58)

In particular, the nonzero eigenvalues of $L_{K,X}$ are precisely those of $\frac{1}{m} K[X]$, corresponding to eigenvectors that must lie in $\mathcal{H}_{K,X} = \text{span} \{ K_{x_i} \}_{i=1}^m$. Thus, the nonzero eigenvalues of $C_K : \mathcal{L}^2(T,v) \to \mathcal{L}^2(T,v)$, $\text{tr}(C_K)$, $\|C_K\|_{\text{HS}}$, which are the same as those of $L_K : \mathcal{H}_K \to \mathcal{H}_K$, can be empirically estimated from those of the $m \times m$ matrix $\frac{1}{m} K[X]$ (see [17]).

The representations of $\| \log(y + C_{K_1}) - \log(y + C_{K_2}) \|_{\text{HS}(\mathcal{L}^2(T,v))}^2$ and $\| \log \left( y I + \frac{1}{m} K^1[X] \right) - \log \left( y I + \frac{1}{m} K^2[X] \right) \|_F^2$ in terms of RKHS covariance and cross-covariance operators and their empirical versions, respectively, are as follows.

**Proposition 12 (Log-Hilbert-Schmidt distance via RKHS operators)** Let $\gamma \in \mathbb{R}, \gamma > 0$ be fixed. Assume A1-A4, then

$$\| \log(y I + C_{K^1}) - \log(y I + C_{K^2}) \|_{\text{HS}(\mathcal{L}^2(T,v))}^2 = \left\| \log \left( I + \frac{1}{\gamma} L_{K^1} \right) \right\|_{\text{HS}(\mathcal{H}_{K^1})}^2 + \left\| \log \left( I + \frac{1}{\gamma} L_{K^2} \right) \right\|_{\text{HS}(\mathcal{H}_{K^2})}^2 - \frac{2}{\gamma^2} \text{tr} \left[ R_{12}^* h \left( \frac{1}{\gamma} L_{K^1} \right) R_{12} h \left( \frac{1}{\gamma} L_{K^2} \right) \right].$$  \hspace{1cm} (59)

Here $h(A) = A^{-1} \log(I + A)$ for $A$ compact, positive, as in Lemma 27, with $h(0) = I$. The corresponding empirical version is

$$\left\| \log \left( \gamma I + \frac{1}{m} K^1[X] \right) - \log \left( \gamma I + \frac{1}{m} K^2[X] \right) \right\|_F^2 = \left\| \log \left( I + \frac{1}{\gamma} L_{K^1} X \right) \right\|_{\text{HS}(\mathcal{H}_{K^1})}^2 + \left\| \log \left( I + \frac{1}{\gamma} L_{K^2} X \right) \right\|_{\text{HS}(\mathcal{H}_{K^2})}^2 - \frac{2}{\gamma^2} \text{tr} \left[ R_{12}^* X h \left( \frac{1}{\gamma} L_{K^1} X \right) R_{12} X h \left( \frac{1}{\gamma} L_{K^2} X \right) \right].$$  \hspace{1cm} (60)

Similarly, $\| (y I + C_{K^1})^{-1/2} (y I + C_{K^2}) (y I + C_{K^1})^{-1/2} \|_{\text{HS}(\mathcal{L}^2(T,v))}, \| \log \left( \gamma I + \frac{1}{m} K^1[X] \right)^{-1/2} \left( \gamma I + \frac{1}{m} K^2[X] \right) \left( \gamma I + \frac{1}{m} K^1[X] \right)^{-1/2} \|_F$ in terms of RKHS covariance and cross-covariance operators and their empirical versions, respectively, are as follows.

**Proposition 13 (Affine-invariant distance via RKHS operators)** Let $\gamma \in \mathbb{R}, \gamma > 0$ be fixed. Under Assumptions A1-A4,

$$\| (y I + C_{K^1})^{-1/2} (y I + C_{K^2}) (y I + C_{K^1})^{-1/2} \|_{\text{HS}(\mathcal{L}^2(T,v))}^2 = \text{tr} \left[ \log \left( I + \left( \frac{1}{\gamma} L_{K^1} \right)^{-1} - \frac{1}{\gamma} \right) \right. \left. \frac{1}{\gamma} \left( I + \frac{1}{\gamma} L_{K^1} \right)^{-1} R_{12} \right]^2.$$  \hspace{1cm} (61)

The corresponding empirical version is

$$\left\| \log \left( \gamma I + \frac{1}{m} K^1[X] \right)^{-1/2} \left( \gamma I + \frac{1}{m} K^2[X] \right) \left( \gamma I + \frac{1}{m} K^1[X] \right)^{-1/2} \right\|_F^2 = \text{tr} \left[ \log \left( I + \left( \frac{1}{\gamma} L_{K^1} X \right)^{-1} - \frac{1}{\gamma} \right) \right. \left. \frac{1}{\gamma} \left( I + \frac{1}{\gamma} L_{K^1} X \right)^{-1} R_{12} X \right]^2.$$  \hspace{1cm} (62)

For simplicity, to estimate the convergence of $R_{ij,X}$ towards $R_{ij}$, in the following we assume that $K$, $K^1$, $K^2$ are bounded.
The convergence in Theorems 15 and 16 is thus both dimension-independent, see [12]. Thus, assume \( \exists \kappa_1, \kappa_2 > 0 \) such that
\[
(A6) \quad \sup_{x \in T} K(x, x) \leq \kappa^2, \quad \sup_{x \in T} K^i(x, x) \leq \kappa_i^2, i = 1, 2.
\]

Proposition 14 (Convergence of RKHS empirical covariance and cross-covariance operators [12]) Under Assumptions A1-A6, \( ||R_{ij}X||_{\text{HS}(\mathcal{H}_{k_i}; \mathcal{H}_{k_j})} \leq \kappa_i \kappa_j, \ i, j = 1, 2, \forall X \in T^m. \) Let \( X = (x_i)_i \) be independently sampled from \((T, \nu)\). Let \( \forall 0 < \delta < 1, \) with probability at least \( 1 - \delta, \)
\[
||R_{ij}X - R_{ij}||_{\text{HS}(\mathcal{H}_{k_i}; \mathcal{H}_{k_j})} \leq \kappa_i \kappa_j \left[ \frac{2 \log \frac{2}{\delta}}{m} + \sqrt{\frac{2 \log \frac{2}{\delta}}{m}} \right].
\]
In particular, \( ||L_{k_i}X||_{\text{HS}(\mathcal{H}_{k_i})} \leq \kappa_i^2 \) and with probability at least \( 1 - \delta, \)
\[
||L_{k_i}X - L_{k_i}||_{\text{HS}(\mathcal{H}_{k_i})} \leq \kappa_i^2 \left[ \frac{2 \log \frac{2}{\delta}}{m} + \sqrt{\frac{2 \log \frac{2}{\delta}}{m}} \right].
\]

Combining Propositions 12 and 14, we obtain the following estimate of \( D_{\text{logHS}}^T [N(0, C_{k_1}), N(0, C_{k_2})]. \)

Theorem 15 (Estimation of Log-Hilbert-Schmidt distance from finite covariance matrices) Let \( \gamma \in \mathbb{R}, \gamma > 0 \) be fixed. Under Assumptions A1-A6, let \( X = (x_i)_i \) be independently sampled from \((T, \nu)\). For any \( 0 < \delta < 1, \) with probability at least \( 1 - \delta, \)
\[
\left\| \log \left( \gamma I + \frac{1}{m} K^1[X] \right) - \log \left( \gamma I + \frac{1}{m} K^2[X] \right) \right\|^2_F - \left\| \log (\gamma I + C_{k_1}) - \log (\gamma I + C_{k_2}) \right\|^2_{\text{HS}(L^2(T, \nu))} \leq \frac{2 (\kappa_1^4 + \kappa_2^4)}{\gamma^2} \left( \frac{2 \log \frac{6}{\delta}}{m} + \sqrt{\frac{2 \log \frac{6}{\delta}}{m}} \right) \left( \frac{2 \log \frac{6}{\delta}}{m} + \sqrt{\frac{2 \log \frac{6}{\delta}}{m}} \right).
\]

Combining Propositions 13 and 14, we obtain the following estimate of \( D_{\text{ahHS}}^T [N(0, C_{k_1}), N(0, C_{k_2})]. \)

Theorem 16 (Estimation of affine-invariant Riemannian distance from finite covariance matrices) Let \( \gamma \in \mathbb{R}, \gamma > 0 \) be fixed. Under Assumptions 1-5, let \( X = (x_i)_i \) be independently sampled from \((T, \nu)\). For any \( 0 < \delta < 1, \) with probability at least \( 1 - \delta, \)
\[
\left\| \log \left( \gamma I + \frac{1}{m} K^1[X] \right)^{-1/2} (\gamma I + \frac{1}{m} K^2[X]) (\gamma I + \frac{1}{m} K^1[X])^{-1/2} \right\|^2_F - \left\| \log (\gamma I + C_{k_1})^{-1/2} (\gamma I + C_{k_2}) (\gamma I + C_{k_1})^{-1/2} \right\|^2_{\text{HS}(L^2(T, \nu))} \leq \frac{1}{\gamma^2} \left( 1 + \frac{\kappa_1^2 + \kappa_2^2}{\gamma} \right)^3 \left( \frac{2 \log \frac{6}{\delta}}{m} + \sqrt{\frac{2 \log \frac{6}{\delta}}{m}} \right).
\]
The convergence in Theorems 15 and 16 is thus both dimension-independent if \( \kappa_1, \kappa_2 \) are absolute constants.

### 4.3 Estimation of distances from finite samples

Consider now the most practical scenario, where we only have access to samples of the Gaussian processes \( \xi^1, \xi^2 \) on a finite set of points \( X = (x_i)_i \) on \( T. \) We can first estimate the covariance matrices \( K^1[X], K^2[X], \) compute their
The empirical version of Theorem 18
Assume Assumptions A1-A6. Let \( X = (x_i)_{i=1}^m \in T^m \) be fixed. Consider the following empirical estimate of \( D \)
Here the probability is with respect to the product space \( (\Omega, \mathcal{F}, P) \),
\[
K[X] = \mathbb{E}[z(\omega)z(\omega)^T] = \int_{\Omega} z(\omega)z(\omega)^T dP(\omega). \tag{68}
\]
The empirical version of \( K[X] \), using the random sample \( W = (\omega_i)_{i=1}^N \), is then
\[
\hat{K}_W[X] = \frac{1}{N} \sum_{i=1}^N z(\omega_i)z(\omega_i)^T = \frac{1}{N}ZZ^T. \tag{69}
\]
The convergence of \( \hat{K}_W[X] \) to \( K[X] \) is given by the following.

Proposition 17 ([12]) Assume Assumptions A1-A6. Let \( \xi \sim \text{GP}(0, K) \) on \( (\Omega, \mathcal{F}, P) \). Let \( X = (x_i)_{i=1}^m \in T^m \) be fixed. Then
\[
||K[X]||_F \leq m \kappa^2. \tag{67}
\]
for any \( 0 < \delta < 1 \), with probability at least \( 1 - \delta \),
\[
||\hat{K}_W[X] - K[X]||_F \leq \frac{2\sqrt{3}m \kappa^2}{\sqrt{N} \delta}, \quad ||\hat{K}_W[X]||_F \leq \frac{2m \kappa^2}{\delta}. \tag{70}
\]
Let now \( \xi^i \sim \text{GP}(0, K^i), i = 1, 2 \), on the probability spaces \( (\Omega_i, \mathcal{F}_i, P_i) \), respectively. Let \( W^i = (\omega_j^i)_{j=1}^N \), independently sampled from \( (\Omega_i, P_i) \), corresponding to the sample paths \( \xi^i(t) = \xi^i(\omega_j^i, t) \) \( j = 1, 2 \). Combining Proposition 17 and Theorem 3, we obtain the following empirical estimate of \( D_{\text{logE}}^Y \)
from two finite samples of \( \xi^1 \sim \text{GP}(0, K^1) \) and \( \xi^2 \sim \text{GP}(0, K^2) \) given by \( W^1, W^2 \).

Theorem 18 Assume Assumptions A1-A6. Let \( X = (x_i)_{i=1}^m \in T^m \), \( m \in \mathbb{N} \) be fixed. Let \( W^1 = (\omega_j^1)_{j=1}^N \), \( W^2 = (\omega_j^2)_{j=1}^N \) be independently sampled from \( (\Omega_1, P_1) \) and \( (\Omega_2, P_2) \), respectively. For any \( 0 < \delta < 1 \), with probability at least \( 1 - \delta \),
\[
\left| D_{\text{logE}}^Y \left( \mathcal{N}\left(0, \frac{1}{m}K^1[X]\right), \mathcal{N}\left(0, \frac{1}{m}K^2[X]\right) \right) - D_{\text{logE}}^Y \left( \mathcal{N}\left(0, \frac{1}{m}K^1[X]\right), \mathcal{N}\left(0, \frac{1}{m}K^2[X]\right) \right) \right| \leq \frac{4\sqrt{3}(\kappa_1^2 + \kappa_2^2)}{\gamma \sqrt{N} \delta}. \tag{71}
\]
Here the probability is with respect to the product space \( (\Omega_1, P_1)^N \times (\Omega_2, P_2)^N \).

Combining Theorems 18 and 15, we are finally led to the following empirical estimate of the theoretical Log-Hilbert-Schmidt distance \( D_{\text{logHS}}^Y \) \( \mathcal{N}(0, C_{K^1}), \mathcal{N}(0, C_{K^2}) \) from two finite samples \( Z^1, Z^2 \) of \( \xi^1 \sim \text{GP}(0, K^1) \) and \( \xi^2 \sim \text{GP}(0, K^2) \).

Theorem 19 (Estimation of Log-Hilbert-Schmidt distance between Gaussian processes from finite samples) Assume Assumptions A1-A6. Let \( X = (x_i)_{i=1}^m \) be independently sampled from \( (T, \nu) \). Let \( W^1 = (\omega_j^1)_{j=1}^N \), \( W^2 = (\omega_j^2)_{j=1}^N \) be indepen-
dently sampled from \((\Omega_1, P_1)\) and \((\Omega_2, P_2)\), respectively. For any \(0 < \delta < 1\), with probability at least \(1 - \delta\),
\[
\left| D_{\log} \left[ N \left( 0, \frac{1}{m} K^{1_\mathcal{W}} [X] \right) \right] - D_{\log} \left[ N(0, C_{K^1}) \right] \right|
\leq \frac{8 \sqrt{3} (\zeta^2 + \theta^2)}{\gamma \sqrt{N}} + \frac{1}{\gamma} \left[ \frac{2 \log \frac{12}{3}}{m} + \frac{2 \log \frac{12}{3}}{m} \right] + \frac{2 \kappa_1^2 + \zeta^2}{2} \left( 1 + \kappa_1^2 + \kappa_2^2 \right) \left( \frac{2 \log \frac{48}{3}}{m} + \frac{2 \log \frac{48}{3}}{m} \right). \tag{72}
\]

Here the probability is with respect to the space \((T, \nu)^m \times (\Omega_1, P_1)^N \times (\Omega_2, P_2)^N\).

Entirely similar results can be obtained for the affine-invariant Riemannian distance.

5 | ESTIMATION OF DISTANCES BETWEEN RKHS GAUSSIAN MEASURES

We now consider the estimation of the distances between two RKHS Gaussian measures induced by two Borel probability measures \textit{via one positive definite kernel} on the same metric space. This setting has been applied practically, see e.g. [2, 41] for computer vision applications.

Throughout this section, let \(X\) be a complete separable metric space. Let \(K\) be a continuous positive definite kernel on \(X \times X\). Then the reproducing kernel Hilbert space (RKHS) \(\mathcal{H}_K\) induced by \(K\) is separable ([42], Lemma 4.33). Let \(\Phi : X \to \mathcal{H}_K\) be the corresponding canonical feature map, so that \(K(x, y) = \langle \Phi(x), \Phi(y) \rangle_{\mathcal{H}_K}\) \(\forall (x, y) \in X \times X\). Let \(\rho\) be a Borel probability measure on \(X\) such that
\[
\int_X ||\Phi(x)||_{\mathcal{H}_K}^2 \, d\rho(x) = \int_X K(x, x) \, d\rho(x) < \infty. \tag{73}
\]

Then the following RKHS mean vector \(\mu_\Phi \in \mathcal{H}_K\) and RKHS covariance operator \(C_\Phi : \mathcal{H}_K \to \mathcal{H}_K\) induced by the feature map \(\Phi\) are both well-defined and are given by
\[
\mu_\Phi = \mu_{\Phi, \rho} = \int_X \Phi(x) \, d\rho(x) \in \mathcal{H}_K, \quad C_\Phi = C_{\Phi, \rho} = \int_X (\Phi(x) - \mu_\Phi) \otimes (\Phi(x) - \mu_\Phi) \, d\rho(x). \tag{74}
\]

Let \(X = [x_1, \ldots, x_m], m \in \mathbb{N}\), be a data matrix randomly sampled from \(X\) according to \(\rho\), where \(m \in \mathbb{N}\) is the number of observations. The feature map \(\Phi\) on \(X\) defines the bounded linear operator \(\Phi(X) : \mathbb{R}^m \to \mathcal{H}_K, \Phi(X)b = \sum_{j=1}^m b_j \Phi(x_j), b \in \mathbb{R}^m\). The corresponding empirical mean vector and covariance operator for \(\Phi(X)\) are defined to be
\[
\mu_{\Phi(X)} = \frac{1}{m} \sum_{j=1}^m \Phi(x_j) = \frac{1}{m} \Phi(X) 1_{m}, \quad C_{\Phi(X)} = \frac{1}{m} \Phi(X) J_m \Phi(X)^*: \mathcal{H}_K \to \mathcal{H}_K, \tag{75}
\]
where \(J_m = I_m - \frac{1}{m} 1_{m}^T 1_{m}, 1_{m} = (1, \ldots, 1)^T \in \mathbb{R}^m\), is the centering matrix with \(J_m^2 = J_m\). The convergence of \(\mu_{\Phi(X_i)}\) and \(C_{\Phi(X_i)}\) towards \(\mu_\Phi\) and \(C_\Phi\), respectively, is quantified by the following

**Theorem 20 (Convergence of RKHS mean and covariance operators - bounded kernels [11])** Assume that \(\sup_{x \in X} K(x, x) \leq \kappa^2\). Let \(X = (x_i)_{i=1}^m, m \in \mathbb{N}\), be independently sampled from \((X, \rho)\). Then \(||\mu_\Phi||_{\mathcal{H}_K} \leq \kappa, ||\mu_{\Phi(X)}||_{\mathcal{H}_K} \leq \kappa \forall X \in X^m\),...
The following shows theoretical consistency for the empirical quantities in Eqs. (81), (82), which are used practically.

Here the probability is with respect to the space measure \( \mu \).

Combining Theorem 20 with Theorems 2 and 4, we first obtain the following convergence of the empirical Gaussian measure \( N(0, C_{\Phi}(X)) \) towards the Gaussian measure \( N(0, C_{\Phi}) \), which are defined on the RKHS \( \mathcal{H}_K \).

**Theorem 21.** Let \( \gamma \in \mathbb{R}, \gamma > 0 \) be fixed. Assume that \( \sup_{x \in X} K(x, x) \leq \kappa^2 \). For any \( 0 < \delta < 1 \), with probability at least \( 1 - \delta \),

\[
D^\gamma_{\text{logHS}}[N(0, C_{\Phi}(X)), N(0, C_{\Phi})] \leq \frac{3\kappa^2}{\gamma} \left( \frac{2 \log \frac{\delta}{\gamma}}{m} + \sqrt{\frac{2 \log \frac{\delta}{\gamma}}{m}} \right).
\]

For \( 0 < \epsilon < \gamma \), let \( N(\epsilon) \in \mathbb{N} \) be such that \( 3\kappa^2 \left( \frac{2 \log \frac{\delta}{\gamma}}{N(\epsilon)} + \sqrt{\frac{2 \log \frac{\delta}{\gamma}}{N(\epsilon)}} \right) < \epsilon \), then \( \forall \epsilon \geq N(\epsilon), \) with probability at least \( 1 - \delta \),

\[
D^\gamma_{\text{logHS}}[N(0, C_{\Phi}(X)), N(0, C_{\Phi})] \leq \frac{3\kappa^2}{\gamma - \epsilon} \left( \frac{2 \log \frac{\delta}{\gamma}}{m} + \sqrt{\frac{2 \log \frac{\delta}{\gamma}}{m}} \right).
\]

Let now \( X^1 = [x^1_i]_{i=1}^m, X^2 = [x^2_i]_{i=1}^m \), be two random data matrices sampled from \( X \) according to two Borel probability distributions \( \rho_1 \) and \( \rho_2 \) on \( X \). Let \( \mu_{\Phi(X^1)}, \mu_{\Phi(X^2)} \) and \( C_{\Phi(X^1)}, C_{\Phi(X^2)} \) be the corresponding mean vectors and covariance operators induced by the kernel \( K \), respectively. Define the following \( m \times m \) Gram matrices

\[
K[X^1] = \Phi(X^1)^* \Phi(X^1), \quad K[X^2] = \Phi(X^2)^* \Phi(X^2), \quad K[X^1, X^2] = \Phi(X^1)^* \Phi(X^2), \quad (K[X^1])_{jk} = K(x^1_j, x^1_k), \quad (K[X^2])_{jk} = K(x^2_j, x^2_k), \quad (K[X^1, X^2])_{jk} = K(x^1_j, x^2_k), \quad 1 \leq j, k \leq m.
\]

For fixed \( \gamma_i \in \mathbb{R}, \gamma_i > 0, i = 1, 2 \), the following distances are expressed explicitly in terms of the Gram matrices ([2, 43])

\[
||\log(C_{\Phi(X^1)} + \gamma_1 I_{\mathcal{H}_K}) - \log(C_{\Phi(X^2)} + \gamma_2 I_{\mathcal{H}_K})||_{\text{HS}_X},
\]

\[
||\log((C_{\Phi(X^1)} + \gamma_1 I_{\mathcal{H}_K})^{-1/2}(C_{\Phi(X^2)} + \gamma_2 I_{\mathcal{H}_K})^{-1/2}(C_{\Phi(X^1)} + \gamma_1 I_{\mathcal{H}_K})^{-1/2})||_{\text{HS}_X}.
\]

The following shows theoretical consistency for the empirical quantities in Eqs. (81), (82), which are used practically.

**Theorem 22.** Let \( \gamma_i \in \mathbb{R}, \gamma_i > 0, i = 1, 2 \), be fixed. Assume that \( \sup_{x \in X} K(x, x) \leq \kappa^2 \). For any \( 0 < \delta < 1 \), with probability at least \( 1 - \delta \),

\[
||\log(C_{\Phi(X^1)} + \gamma_1 I) - \log(C_{\Phi(X^2)} + \gamma_2 I)||_{\text{HS}_X} - ||\log(C_{\Phi, \rho_1} + \gamma_1 I) - \log(C_{\Phi, \rho_2} + \gamma_2 I)||_{\text{HS}_X}
\]

\[
\leq 3\kappa^2 \left( \frac{1}{\gamma_1} + \frac{1}{\gamma_2} \right) \left( \frac{2 \log \frac{\delta}{\gamma_1}}{m} + \sqrt{\frac{2 \log \frac{\delta}{\gamma_1}}{m}} \right).
\]

Here the probability is with respect to the space \((\rho_1 \times \rho_2)^m\). For \( 0 < \epsilon < \min(\gamma_1, \gamma_2) \), let \( N(\epsilon) \in \mathbb{N} \) be such that...
we show the estimation of the Log-Hilbert-Schmidt and affine-invariant Riemannian distances between covariance operators on a set of finite samples as in Eq. (69), using differently. The Log-Hilbert-Schmidt and affine-invariant Riemannian distances perform consistently across distribution on Riemannian and Log-Hilbert-Schmidt distances, we fixed using the uniform distribution, with \( m \), number of sample points is \( m = 500 \), and regularization parameter is \( \gamma = 10^{-9} \)

\[
3 \kappa^2 \left( \frac{2 \log \frac{8}{N(e)}}{m} + \sqrt{\frac{2 \log \frac{8}{N(e)}}{m}} \right) < \epsilon, \text{ then } \forall m \geq N(e), \text{ with probability at least } 1 - \delta,
\]

\[
\left| \left| \log \left( (C_{\Phi(X')} + \gamma_1 I_{H_K})^{-1/2} (C_{\Phi(X')} + \gamma_2 I_{H_K})^{-1/2} (C_{\Phi(X')} + \gamma_1 I_{H_K})^{-1/2} \right) \right| \right|_{HS_X} \\
- \left| \left| \log \left( (C_{\Phi,\rho_1} + \gamma_1 I_{H_K})^{-1/2} (C_{\Phi,\rho_2} + \gamma_2 I_{H_K})^{-1/2} (C_{\Phi,\rho_1} + \gamma_1 I_{H_K})^{-1/2} \right) \right| \right|_{HS_X} \\
\leq 3 \kappa^2 \left( \frac{1}{\gamma_1 - \epsilon} + \frac{1}{\gamma_2 - \epsilon} \right) \left( \frac{2 \log \frac{8}{N(e)}}{m} + \sqrt{\frac{2 \log \frac{8}{N(e)}}{m}} \right). \tag{84}
\]

6 | NUMERICAL EXPERIMENTS ON GAUSSIAN PROCESSES

In this section, we illustrate the theoretical results above with several experiments on Gaussian processes.

**Estimation of distances.** In Figures 1 and 2, we illustrate the convergence behavior studied in Section 4.3. Here we show the estimation of the Log-Hilbert-Schmidt and affine-invariant Riemannian distances between covariance operators of two Gaussian processes \( GP(0, K^1) \) and \( GP(0, K^2) \), on \( T = [0, 1] \), with increasing number of sample paths \( N = 10, 20, \ldots, 1000 \). In Figure 1, \( K^1 = \exp(-a ||x - y||), \ k^2 = \exp \left(-\frac{||x - y||}{\sigma^2} \right) \), where \( a = 1 \) and \( \sigma = 0.1 \). In Figure 2, \( K^1(x, y) = \exp(-a_i ||x - y||) \), for \( a_1 = 1 \), \( a_2 = 1.2 \). In both cases, the set \( X = (x_j)_{j=1}^{m} \) is chosen randomly from \( T \) using the uniform distribution, with \( m = 500 \). The regularization parameter is fixed at \( \gamma = 10^{-9} \).

**Classification of covariance operators.** We carry out the following binary classification of covariance operators corresponding to two centered Gauss-Markov processes \( GP(0, K^1) \) with \( K^1(x, y) = \exp(-a_i ||x - y||), \), \( \sigma_i > 0, \ i = 1, 2 \), on \( T = [0, 1]^d \) for \( d = 1, 5 \). For each process, we generated a set of empirical covariance matrices, each defined by a set of finite samples as in Eq.(69), using \( N = 500 \) sample points on \( m = 200 \) points randomly chosen by the uniform distribution on \( T \). The training and testing sets contain 10 and 100 empirical covariance matrices, respectively, split equally between the two classes. For classification, we utilized the nearest neighbor approach. For the affine-invariant Riemannian and Log-Hilbert-Schmidt distances, we fixed \( \gamma = 10^{-9} \). The experiments are repeated 5 times.

We report the average classification errors on the test set, along with standard deviations, in four different scenarios in Table 1, with examples of confusion matrices in Figure 3. For the setting \( (\sigma_1 = 1, \sigma_2 = 1.3) \), the two Gaussian processes are easily distinguishable and perfect classification is achieved in almost all cases. For the case \( (\sigma_1 = 1, \sigma_2 = 1.1) \), the two Gaussian processes are clearly much closer to each other and the distances performed differently. The Log-Hilbert-Schmidt and affine-invariant Riemannian distances perform consistently across differ-

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**FIGURE 1** Samples of the centered Gaussian processes \( GP(0, K^1) \), \( GP(0, K^2) \) on \( T = [0, 1] \) and approximations of squared distances between them. Left: \( K^1(x, y) = \exp(-a||x - y||), \ a = 1 \). Middle: \( K^2(x, y) = \exp(-||x - y||^2/\sigma^2), \ \sigma = 0.1 \). Here the number of sample paths is \( N = 10, 20, \ldots, 1000 \), number of sample points is \( m = 500 \), and regularization parameter is \( \gamma = 10^{-9} \).
FIGURE 2  Samples of the centered Gaussian processes $\text{GP}(0, K^1)$, $\text{GP}(0, K^2)$ on $T = [0, 1]$ and approximations of squared distances between them. Left: $K^1(x, y) = \exp(-a||x - y||)$, $a = 1$. Middle: $K^2(x, y) = \exp(-a||x - y||)$, $a = 1.2$. Here the number of sample paths is $N = 10, 20, \ldots, 1000$, number of sample points is $m = 500$, and regularization parameter is $\gamma = 10^{-9}$.

TABLE 1  Classification errors on the test set ($\sigma_1 = 1$ in all cases)

| Distance                  | $\sigma_2 = 1.1, d = 1$ | $\sigma_2 = 1.3, d = 1$ | $\sigma_2 = 1.1, d = 5$ | $\sigma_2 = 1.3, d = 5$ |
|---------------------------|-------------------------|-------------------------|-------------------------|-------------------------|
| Hilbert-Schmidt           | 33%(7.97%)              | 4.60%(2.51%)            | 15.00%(5.52%)           | 1.00%(0.71%)            |
| Bures-Wasserstein         | 8.80%(5.54%)            | 0%                      | 29.20%(6.14%)           | 0%                      |
| Sinkhorn ($\epsilon = 0.1$) | 17.40%(7.83%)          | 0%                      | 13.80%(8.4%)            | 0.40%(0.55%)            |
| Log-Hilbert-Schmidt       | 0%                      | 0%                      | 0%                      | 0%                      |
| Affine-invariant          | 0%                      | 0%                      | 0.2%(0.45%)             | 0%                      |

ent scenarios, with almost perfect classification in all settings. The Hilbert-Schmidt distance, which does not take into account the geometrical structures of covariance matrices/operators, incurs considerable error in this case. The Bures-Wasserstein distance, which does not possess dimension-independent convergence, performs much worse on average in the case $d = 5$ compared to $d = 1$.

7 | PROOFS OF MAIN RESULTS

7.1 | Proofs for the convergence of the Log-Hilbert-Schmidt distance

We first prove Theorems 2 and 3. In the following, let $\mathcal{C}_p(\mathcal{H})$ denote the set of $p$-th Schatten class operators on $\mathcal{H}$, under the norm $|| \cdot ||_p$, where $||A||_p = (\text{Tr}(|A|^p))^{1/p}$, $1 \leq p \leq \infty$, with $\mathcal{C}_1(\mathcal{H}) = \text{Tr}(\mathcal{H})$, $\mathcal{C}_2(\mathcal{H}) = \text{HS}(\mathcal{H})$, and $\mathcal{C}_\infty(\mathcal{H})$ being the set of compact operators under the operator norm $|| \cdot ||$.

Lemma 23 (Corollary 3.2 in [44]) For any two positive operators $A, B$ on $\mathcal{H}$ such that $A \geq cI > 0$, $B \geq cI > 0$, for any
bounded operator $X$ on $\mathcal{H}$,

$$||A^rX - XB^r||_p \leq r c^{r-1} ||AX - XB||_p, 0 < r \leq 1, 1 \leq p \leq \infty.$$  \hfill (85)

**Corollary 24** For two operators $A, B \in \text{Sym}^+(\mathcal{H}) \cap \mathcal{Y}_p(\mathcal{H}), 1 \leq p \leq \infty$,

$$||(I + A)^r - (I + B)^r||_p \leq |r| ||A - B||_p, |r| \leq 1.$$  \hfill (86)

In particular, for $|r| \leq 1, r \neq 0$, we have

$$\frac{||(I + A)^r - (I + B)^r||_p}{|r|} \leq ||A - B||_p.$$

**Proof** By assumption, $I + A \geq I, I + B \geq I$, so for $0 < r \leq 1$, the inequality follows immediately from Lemma 23 with $c = 1$. Consider now $r = -s, 0 < s \leq 1$, then

$$||(I + A)^{-r} - (I + B)^{-r}||_p = ||(I + A)^{-r}[(I + A)^r - (I + B)^r](I + B)^{-r}||_p \leq ||(I + A)^{-r}||_p||(I + A)^r - (I + B)^r||_p||(I + B)^{-r}||_p \leq s||A - B||_p.$$  

using result from the first case. 

**Corollary 25** Let $1 \leq p \leq \infty$. Let $r \in \mathbb{R}, 0 < r \leq 1$, be fixed. Let $\{A_n\}_{n \in \mathbb{N}}, A \in \text{Sym}(\mathcal{H}) \cap \mathcal{Y}_p(\mathcal{H})$ be such that $I + A > 0, I + A_n > 0 \forall n \in \mathbb{N}$. Let $M_A > 0$ be such that $(x, (I + A)x) \geq M_A ||x||^2 \forall x \in \mathcal{H}$. Assume that $\lim_{n \to \infty} ||A_n - A||_p = 0$. Then $\forall \varepsilon, 0 < \varepsilon < M_A, \exists N(\varepsilon) \in \mathbb{N}$ such that

$$\frac{||(I + A_n)^r - (I + A)^r||_p}{r} \leq (M_A - \varepsilon)^{r-1} ||A_n - A||_p, \forall n \geq N(\varepsilon).$$  \hfill (87)

$$\frac{||(I + A_n)^{-r} - (I + A)^{-r}||_p}{r} \leq \frac{1}{(M_A - \varepsilon)M_A} ||A_n - A||_p, \forall n \geq N(\varepsilon).$$  \hfill (88)
Proof For any $\varepsilon$ satisfying $0 < \varepsilon < M_A$, there exists $N(\varepsilon) \in \mathbb{N}$ such that $||A_n - A|| < \varepsilon \ \forall n \geq N(\varepsilon)$. By assumption, we have $I + A \geq M_A I$, so that

\[
I + A_n = I + A + (A_n - A) \geq (M_A - \varepsilon)I \ \forall n \geq N(\varepsilon), \quad (I + A_n)^{-1} \leq (M_A - \varepsilon)^{-1}I \ \forall n \geq N(\varepsilon).
\]

For $0 < r \leq 1$, applying Lemma 23 gives

\[
||(I + A_n)^r - (I + A)^r||_p \leq r(M_A - \varepsilon)^{-1}||A_n - A||_p \ \forall n \geq N(\varepsilon).
\]

For $r = -s, 0 < s \leq 1$, we have

\[
||(I + A_n)^{-s} - (I + A)^{-s}||_p = ||(I + A_n)^{-s}[(I + A_n)^s - (I + A)^s](I + A)^{-s}||_p
\]

\[
\leq ||(I + A_n)^s||_p ||(I + A_n)^s - (I + A)^s||_p ||(I + A)^{-s}||_p \leq (M_A - \varepsilon)^{-s}M_A^{-s}||A_n - A||_p = s(M_A - \varepsilon)^{-s}M_A^{-s}||A_n - A||_p,
\]

using the result from the previous case.

---

**Lemma 26 (Lemma 9 in [45])** Let $A \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$ with $I + A > 0$. Then

\[
\lim_{\alpha \to 0} \frac{||(I + A)^\alpha - I - \log(I + A)||_{HS}}{\alpha} = 0.
\]  

(89)

**Proof of Theorem 2**

(a) Consider first the case $\gamma = 1$. Let $n \in \mathbb{N}$ be fixed. By Lemma 26,

\[
\lim_{\alpha \to 0} \frac{||(I + A_n)^\alpha - I - \log(I + A_n)||_{HS}}{\alpha} = 0, \quad \lim_{\alpha \to 0} \frac{||(I + A)^\alpha - I - \log(I + A)||_{HS}}{\alpha} = 0.
\]

(i) If $A, A_n \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H})$, by Corollary 24, for any $0 < \alpha \leq 1$, $\frac{||(I + A_n)^\alpha - (I + A)^\alpha||_{HS}}{\alpha} \leq ||A_n - A||_{HS}$. Thus for any fixed $0 < \alpha \leq 1$, for any fixed $n \in \mathbb{N}$,

\[
||\log(I + A_n) - \log(I + A)||_{HS} \leq \left|\left| \frac{(I + A_n)^\alpha - I - \log(I + A_n)}{\alpha} \right|\right|_{HS} + \left|\left| \frac{(I + A_n)^\alpha - (I + A)^\alpha}{\alpha} \right|\right|_{HS} + \left|\left| \frac{(I + A)^\alpha - I - \log(I + A)}{\alpha} \right|\right|_{HS}.
\]

Letting $\alpha \to 0$ gives $||\log(I + A_n) - \log(I + A)||_{HS} \leq ||A_n - A||_{HS} \forall n \in \mathbb{N}$.

(ii) Consider now the general assumption $I + A > 0, I + A_n > 0 \ \forall n \in \mathbb{N}$. By Corollary 25, for a fixed $0 < \alpha \leq 1, \forall \varepsilon, 0 < \varepsilon < M_A, \exists N(\varepsilon) \in \mathbb{N}$ such that

\[
\left|\left| \frac{(I + A_n)^\alpha - (I + A)^\alpha}{\alpha} \right|\right|_{HS} \leq (M_A - \varepsilon)^{-\alpha - 1}||A_n - A||_{HS} \ \forall n \geq N(\varepsilon).
\]  

(90)
Thus for any fixed \( \alpha \in \mathbb{R} \), \( 0 < \alpha \leq 1 \), and any fixed \( n \in \mathbb{N} \), \( n \geq N(e) \),

\[
\| \log(I + A_n) - \log(I + A) \|_{HS} \leq \left( \frac{I + A_n}{\alpha} - \log(I + A_n) \right)_{HS} + \left( \frac{I + A}{\alpha} - \log(I + A) \right)_{HS}.
\]

Fixing \( n \) and letting \( \alpha \to 0 \) on the right hand side gives

\[
\| \log(I + A_n) - \log(I + A) \|_{HS} \leq (M_A - e)^{-1} || A_n - A ||_{HS}, \quad \forall n \geq N(e).
\]

It thus follows that \( \lim_{n \to \infty} \| \log(I + A_n) - \log(I + A) \|_{HS} = 0 \).

(b) Consider now the general case \( \gamma > 0 \). Part (i) then follows from (a) and the identity \( \| \log(yI + A_n) - \log(yI + A) \|_{HS} = \left( \| \log(I + A_n) - \log(I + A) \|_{HS} \right) \). For part (ii), we note that \( yI + A \geq M_A \iff I + \frac{A}{y} \geq \frac{M_A}{y} \). Furthermore,

\[
\| \frac{A_n - A}{\gamma} \| \leq \frac{\epsilon}{\gamma} \quad \forall n \geq N(e). \]

The result then follows similarly from (a).

**Proof of Theorem 3** Since \( d_{\log HS}(\gamma_1 I + A, \gamma_2 I + B) = \| \log(\gamma_1 I + A) - \log(\gamma_2 I + B) \|_{HS} \) is a metric, by the triangle inequality and Theorem 2, we have in the case \( A_n, B_n, A, B \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H}) \),

\[
d_{\log HS}(\gamma_1 I + A_n, \gamma_2 I + B_n) - d_{\log HS}(\gamma_1 I + A_n, \gamma_2 I + B) \leq d_{\log HS}(\gamma_1 I + A_n, \gamma_1 I + A) + d_{\log HS}(\gamma_2 I + B_n, \gamma_2 I + B).
\]

The general case follows similarly by Theorem 2.

**7.2 | Proofs for the Log-Hilbert-Schmidt distance between Gaussian processes**

We now prove Theorems 9, 15, 18, and 19.

**Proof of Theorem 9** By Theorem 3,

\[
\Delta = \left| D^r_{\log HS}(N(0, C_{k1}^1, \mathcal{W}1), N(0, C_{k2}^2, \mathcal{W}2)) - D^r_{\log HS}(N(0, C_{k1}^1), N(0, C_{k2}^2)) \right|
\]

\[
= \left| \| \log(\gamma I + C_{k1}^1, \mathcal{W}1) - \log(\gamma I + C_{k2}^2, \mathcal{W}2) \|_{HS} - \| \log(\gamma I + C_{k1}^1) - \log(\gamma I + C_{k2}^2) \|_{HS} \right|
\]

\[
\leq \frac{1}{\gamma} || C_{k1}^1, \mathcal{W}1 - C_{k1}^1 ||_{HS} + \frac{1}{\gamma} || C_{k2}^2, \mathcal{W}2 - C_{k2}^2 ||_{HS}.
\]

By Proposition 8, for any \( 0 < \delta < 1 \), with probability at least \( 1 - \delta \),

\[
|| C_{k1}^1, \mathcal{W}1 - C_{k1}^1 ||_{HS} \leq \frac{4\sqrt{3}k_1^2}{\sqrt{N\delta}} \quad \text{and} \quad || C_{k2}^2, \mathcal{W}2 - C_{k2}^2 ||_{HS} \leq \frac{4\sqrt{3}k_2^2}{\sqrt{N\delta}}.
\]

Consequently, combing all previous expressions gives \( \Delta \leq \frac{4\sqrt{3}(k_1^2 + k_2^2)}{\gamma\sqrt{N\delta}} \).

**Lemma 27** Let \( A \in \text{Sym}^+(\mathcal{H}) \) be compact, with eigenvalues \( \{ \lambda_k \}_{k \in \mathbb{N}}, \lambda_k \geq 0 \forall k \in \mathbb{N} \), and corresponding orthonormal
eigenvectors \( \{\phi_k\}_{k \in \mathbb{N}} \). The following operator \( h(A) \) is well-defined, with \( h(A) \in \text{Sym}^+(\mathcal{H}) \),

\[
h(A) = A^{-1} \log(I + A) = \sum_{k=1}^{\infty} \frac{\log(1 + \lambda_k)}{\lambda_k} \phi_k \otimes \phi_k, \quad \text{with } ||h(A)|| \leq 1.
\] (91)

Here we use \( \lim_{x \to 0} \frac{\log(1 + x)}{x} = 1 \) and set \( h(0) = I \).

**Proof** By the spectral decomposition \( A = \sum_{k=1}^{\infty} \lambda_k \phi_k \otimes \phi_k \), we have \( \log(I + A) = \sum_{k=1}^{\infty} \frac{\log(1 + \lambda_k)}{\lambda_k} \phi_k \otimes \phi_k \). Since \( \lambda_k \geq 0 \forall k \in \mathbb{N} \), by the inequality \( \log(1 + x) \leq x \forall x \geq 0 \) and the limit \( \lim_{x \to 0} \frac{\log(1 + x)}{x} = 1 \) by L'Hopital's rule, we have \( 0 \leq \log(1 + \lambda_k) \leq \lambda_k \forall k \in \mathbb{N} \). Thus \( h(A) \in \text{Sym}^+(\mathcal{H}) \), with \( ||h(A)|| \leq 1 \). ■

**Lemma 28** Let \( \mathcal{H}_1, \mathcal{H}_2 \) be two separable Hilbert spaces. Let \( A : \mathcal{H}_1 \to \mathcal{H}_2 \) be compact. Let \( \{\lambda_k (A^*A)\}_{k \in \mathbb{N}} \) be the eigenvalues of \( A^*A \in \text{Sym}^+(\mathcal{H}_1) \), with corresponding orthonormal eigenvectors \( \{\phi_k (A^*A)\}_{k \in \mathbb{N}} \). Then \( A^*A \in \text{Sym}^+(\mathcal{H}_2) \) and

\[
\log(I_{\mathcal{H}_2} + A^*A) = \sum_{k=1}^{\infty} \frac{\log(1 + \lambda_k)}{\lambda_k} (A\phi_k (A^*A)) \otimes (A\phi_k (A^*A)) = Ah(A^*A)A^*.
\] (92)

where \( h \) is as defined in Lemma 27. If \( A^*A \in \text{HS}(\mathcal{H}_1) \), then \( \log(I_{\mathcal{H}_1} + A^*A) \in \text{HS}(\mathcal{H}_1) \) and \( \log(I_{\mathcal{H}_2} + A^*A) \in \text{HS}(\mathcal{H}_2) \).

**Proof** Let \( N_A \) be the number of strictly positive eigenvalues of \( A^*A \). Then [2]

\[
\log(I_{\mathcal{H}_2} + A^*A) = \sum_{k=1}^{N_A} \frac{\log(1 + \lambda_k)}{\lambda_k} (A\phi_k (A^*A)) \otimes (A\phi_k (A^*A)).
\]

If \( \lambda_k (A^*A) = 0 \), i.e. \( A^*A\phi_k (A^*A) = 0 \), then \( ||A\phi_k (A^*A)||^2 = \langle A\phi_k (A^*A), A\phi_k (A^*A) \rangle = \langle \phi_k (A^*A), A^*A\phi_k (A^*A) \rangle = 0 \) \( \iff \) \( A\phi_k (A^*A) = 0 \). Furthermore, \( (A\phi_k (A^*A)) \otimes (A\phi_k (A^*A)) = A[\phi_k (A^*A) \otimes \phi_k (A^*A)]A^* \). Thus

\[
\log(I_{\mathcal{H}_2} + A^*A) = \sum_{k=1}^{\infty} \frac{\log(1 + \lambda_k)}{\lambda_k} (A\phi_k (A^*A)) \otimes (A\phi_k (A^*A)) = Ah(A^*A)A^*.
\]

If \( A^*A \in \text{HS}(\mathcal{H}_1) \), then it follows that \( ||\log(I_{\mathcal{H}_1} + A^*A)||^2_{\text{HS}(\mathcal{H}_1)} = ||\log(I_{\mathcal{H}_2} + A^*A)||^2_{\text{HS}(\mathcal{H}_2)} = \text{tr}[Ah(A^*A)A^*]^2 = \text{tr}[A^*Ah(A^*A)]^2 < \infty \). ■

**Corollary 29** Let \( \mathcal{H}_1, \mathcal{H}_2, \mathcal{H} \) be separable Hilbert spaces. Let \( A : \mathcal{H}_1 \to \mathcal{H}, B : \mathcal{H}_2 \to \mathcal{H} \) be compact operators such that \( A^*A \in \text{Sym}^+(\mathcal{H}_1) \cap \text{HS}(\mathcal{H}_1), B^*B \in \text{Sym}^+(\mathcal{H}_2) \cap \text{HS}(\mathcal{H}_2) \). Then \( AA^*, BB^* \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H}) \), and

\[
\text{tr}[\log(I_{\mathcal{H}_1} + AA^*) \log(I_{\mathcal{H}_2} + BB^*)] = \text{tr}[B^*Ah(A^*A)A^*Bh(B^*B)] = ||[\sqrt{h(A^*A)}A^*B\sqrt{h(B^*B)}]||^2_{\text{HS}(\mathcal{H}_2;\mathcal{H}_1)},
\]

where \( h \) is as defined in Lemma 27.

**Proof** By Lemma 28, \( \log(I_{\mathcal{H}_1} + AA^*) = Ah(A^*A)A^* \in \text{HS}(\mathcal{H}), \log(I_{\mathcal{H}_2} + BB^*) = Bh(B^*B)B^* \in \text{HS}(\mathcal{H}) \), and

\[
\text{tr}[\log(I_{\mathcal{H}_1} + AA^*) \log(I_{\mathcal{H}_2} + BB^*)] = \text{tr}[Ah(A^*A)A^*Bh(B^*B)B^*]
\]

\[
= \text{tr}[B^*Ah(A^*A)A^*Bh(B^*B)] = ||[\sqrt{h(A^*A)}A^*B\sqrt{h(B^*B)}]||^2_{\text{HS}(\mathcal{H}_2;\mathcal{H}_1)}.
\]
Lemma 30 Assume Assumptions A1-A6. Let $\nu \in \mathbb{R}$, $\nu > 0$ be fixed. Let $X$ be independently sampled from $(T, \nu)$. For $h$ as defined in Lemma 27, for any $0 < \delta < 1$, with probability at least $1 - \delta$,

$$\left\| h\left(\frac{1}{Y}L_{K}X\right) - h\left(\frac{1}{Y}L_{K}\right) \right\|_{\text{HS}(H_{K})} \leq \kappa_{1}^{2} \left( \frac{2 \log \frac{2}{\delta}}{m} + \sqrt{\frac{2 \log \frac{2}{\delta}}{m}} \right).$$

(93)

Proof Since $L_{K}\in\text{Sym}^{+}(H_{K}) \cap \text{HS}(H_{K})$, $h\left(\frac{1}{Y}L_{K}X\right) - h\left(\frac{1}{Y}L_{K}\right) \in \text{Sym}(H_{K}) \cap \text{HS}(H_{K})$ by Lemma 42, with $\left\| h\left(\frac{1}{Y}L_{K}X\right) - h\left(\frac{1}{Y}L_{K}\right) \right\|_{\text{HS}(H_{K})} \leq \frac{1}{2} \left\| L_{K}X - L_{K} \right\|_{\text{HS}(H_{K})}$. The result then follows from Proposition 14.

Lemma 31 Assume Assumptions A1-A6. Let $\nu \in \mathbb{R}$, $\nu > 0$ be fixed. For all $X = (x_{i})_{i=1}^{m} \in T^{m}$,

$$\left\| R_{12}h\left(\frac{1}{Y}L_{K}^{2}X\right) \right\|_{\text{HS}(H_{K2}, H_{K1})} \leq \kappa_{1}\kappa_{2}.$$  

(94)

Let $X = (x_{i})_{i=1}^{m}$ be independently sampled from $(T, \nu)$. For any $0 < \delta < 1$, with probability at least $1 - \delta$,

$$\left\| R_{12}h\left(\frac{1}{Y}L_{K}^{2}X\right) - R_{12}h\left(\frac{1}{Y}L_{K}^{2}\right) \right\|_{\text{HS}(H_{K2}^{2}, H_{K1})} \leq \kappa_{1}\kappa_{2} \left( 1 + \frac{1}{2Y} \kappa_{2}^{2} \right) \left( \frac{2 \log \frac{4}{\delta}}{m} + \sqrt{\frac{2 \log \frac{4}{\delta}}{m}} \right).$$

(95)

$$\left\| R_{12}h\left(\frac{1}{Y}L_{K}^{2}X\right) - R_{12}^{*}h\left(\frac{1}{Y}L_{K}^{2}\right) \right\|_{\text{HS}(H_{K2}^{2}, H_{K1})} \leq \kappa_{1}\kappa_{2} \left( 1 + \frac{1}{2Y} \kappa_{2}^{2} \right) \left( \frac{2 \log \frac{4}{\delta}}{m} + \sqrt{\frac{2 \log \frac{4}{\delta}}{m}} \right).$$

(96)

Proof The first inequality follows from $\left\| h\left(\frac{1}{Y}L_{K}^{2}X\right) \right\| \leq 1$ by Lemma 27 and $\left\| R_{12}h\left(\frac{1}{Y}L_{K}^{2}, H_{K1}\right) \right\| \leq \kappa_{1}\kappa_{2}$ by Proposition 14. Similarly, since $\left\| R_{12}h\left(\frac{1}{Y}L_{K}^{2}, H_{K1}\right) \right\| \leq \kappa_{1}\kappa_{2}$.

$$\Delta = \left\| R_{12}h\left(\frac{1}{Y}L_{K}^{2}X\right) - R_{12}h\left(\frac{1}{Y}L_{K}^{2}\right) \right\|_{\text{HS}(H_{K2}^{2}, H_{K1})}$$

$$\leq \left\| R_{12}X - R_{12}||h\left(\frac{1}{Y}L_{K}^{2}X\right)|| + \left\| R_{12}||h\left(\frac{1}{Y}L_{K}^{2}, H_{K1}\right)\right\| h\left(\frac{1}{Y}L_{K}^{2}, X\right) - h\left(\frac{1}{Y}L_{K}^{2}\right) || \right\|$$

$$\leq \left\| R_{12}X - R_{12}||h\left(\frac{1}{Y}L_{K}^{2}, H_{K1}\right)|| + \kappa_{1}\kappa_{2}||h\left(\frac{1}{Y}L_{K}^{2}, X\right) - h\left(\frac{1}{Y}L_{K}^{2}\right)|| \right\|$$

By Proposition 14 and Lemma 30, the following sets satisfy $\nu^{m}(U_{i}) \geq 1 - \frac{\delta}{2}$, $i = 1, 2$,

$$U_{1} = \left\{ X \in (T, \nu)^{m} : \left\| R_{12}X - R_{12}||h\left(\frac{1}{Y}L_{K}^{2}, H_{K1}\right)|| \right\| \leq \kappa_{1}\kappa_{2} \left( \frac{2 \log \frac{4}{\delta}}{m} + \sqrt{\frac{2 \log \frac{4}{\delta}}{m}} \right) \right\}.$$  

$$U_{2} = \left\{ X \in (T, \nu)^{m} : \left\| h\left(\frac{1}{Y}L_{K}^{2}, X\right) - h\left(\frac{1}{Y}L_{K}^{2}\right) || \right\| \leq \frac{1}{2Y} \kappa_{2}^{2} \left( \frac{2 \log \frac{4}{\delta}}{m} + \sqrt{\frac{2 \log \frac{4}{\delta}}{m}} \right) \right\}.$$
Thus on $U = U_1 \cap U_2$, with $\nu^m(U) \geq 1 - \delta$,

$$\Delta \leq \kappa_1 \kappa_2 \left( 1 + \frac{1}{2Y} \kappa_2^2 \right) \left( \frac{2 \log \frac{5}{\delta}}{m} + \sqrt{\frac{2 \log \frac{5}{\delta}}{m}} \right).$$

The last inequality is obtained similarly.

**Proposition 32** Assume Assumptions A1-A6. Let $X = (x_i)_{i=1}^n$ be independently sampled from $(T, \nu)$. For any $0 < \delta < 1$, with probability at least $1 - \delta$,

$$\left\| R_{12}^* x^h \left( \frac{1}{Y} L_{K_1} X \right) R_{12} x^h \left( \frac{1}{Y} L_{K_2} X \right) - R_{12}^* h \left( \frac{1}{Y} L_{K_1} \right) R_{12} h \left( \frac{1}{Y} L_{K_2} \right) \right\| \leq \kappa_1 \kappa_2 \left( 1 + \frac{1}{2Y} \kappa_2^2 \right) \left( \frac{2 \log \frac{5}{\delta}}{m} + \sqrt{\frac{2 \log \frac{5}{\delta}}{m}} \right).$$

Consequently, with probability at least $1 - \delta$,

$$\left| \text{tr} \left[ R_{12}^* x^h \left( \frac{1}{Y} L_{K_1} X \right) R_{12} x^h \left( \frac{1}{Y} L_{K_2} X \right) - \text{tr} \left[ R_{12}^* h \left( \frac{1}{Y} L_{K_1} \right) R_{12} h \left( \frac{1}{Y} L_{K_2} \right) \right] \right] \leq \kappa_1 \kappa_2 \left( 1 + \frac{1}{2Y} \kappa_2^2 \right) \left( \frac{2 \log \frac{5}{\delta}}{m} + \sqrt{\frac{2 \log \frac{5}{\delta}}{m}} \right).$$

**Proof** By Lemma 31, $||R_{12} x^h \left( \frac{1}{Y} L_{K_2} X \right)||_{HS(H_{K_2}, H_{K_1})} \leq \kappa_1 \kappa_2$, $||R_{12}^* h \left( \frac{1}{Y} L_{K_1} \right)||_{HS(H_{K_1}, H_{K_2})} \leq \kappa_1 \kappa_2$, thus

$$\Delta = ||R_{12}^* x^h \left( \frac{1}{Y} L_{K_1} X \right) R_{12} x^h \left( \frac{1}{Y} L_{K_2} X \right) - R_{12}^* h \left( \frac{1}{Y} L_{K_1} \right) R_{12} h \left( \frac{1}{Y} L_{K_2} \right) ||_{\text{tr}(H_{K_2})}$$

$$\leq \left| ||R_{12}^* x^h \left( \frac{1}{Y} L_{K_1} X \right) - R_{12}^* h \left( \frac{1}{Y} L_{K_1} \right) ||_{HS(H_{K_1}, H_{K_2})} \right| ||R_{12} x^h \left( \frac{1}{Y} L_{K_2} X \right) ||_{\text{tr}(H_{K_2})}$$

$$+ ||R_{12}^* h \left( \frac{1}{Y} L_{K_1} \right) ||_{HS(H_{K_1}, H_{K_2})} \left| ||R_{12} x^h \left( \frac{1}{Y} L_{K_2} X \right) - R_{12} h \left( \frac{1}{Y} L_{K_2} \right) ||_{HS(H_{K_2}, H_{K_1})} \right|$$

$$\leq ||R_{12}^* x^h \left( \frac{1}{Y} L_{K_1} X \right) - R_{12}^* h \left( \frac{1}{Y} L_{K_1} \right) ||_{HS(H_{K_1}, H_{K_2})} ||R_{12} x^h \left( \frac{1}{Y} L_{K_2} X \right) ||_{HS(H_{K_1}, H_{K_2})}$$

$$+ ||R_{12}^* h \left( \frac{1}{Y} L_{K_1} \right) ||_{HS(H_{K_1}, H_{K_2})} ||R_{12} x^h \left( \frac{1}{Y} L_{K_2} X \right) - R_{12} h \left( \frac{1}{Y} L_{K_2} \right) ||_{HS(H_{K_2}, H_{K_1})}$$

$$\leq \kappa_1 \kappa_2 ||R_{12}^* x^h \left( \frac{1}{Y} L_{K_1} X \right) - R_{12}^* h \left( \frac{1}{Y} L_{K_1} \right) ||_{HS(H_{K_1}, H_{K_2})} + \kappa_1 \kappa_2 ||R_{12} x^h \left( \frac{1}{Y} L_{K_2} X \right) - R_{12} h \left( \frac{1}{Y} L_{K_2} \right) ||_{HS(H_{K_2}, H_{K_1})}.$$
Thus on $U = U_1 \cup U_2$, with $\nu^m(U) \geq 1 - \delta$, we have $\Delta \leq \kappa_1^2 \kappa_2^2 \left( 1 + \frac{\kappa_1^2 + \kappa_2^2}{2\nu} \right) \left( 2 \log \frac{2}{m} + \sqrt{2 \log \frac{2}{m}} \right)$.

**Proposition 33** Let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}$ be separable Hilbert spaces. Let $A : \mathcal{H}_1 \to \mathcal{H}$, $B : \mathcal{H}_2 \to \mathcal{H}$ be compact operators, such that $A^*A \in \text{Sym}^+(\mathcal{H}_1) \cap \text{HS}(\mathcal{H}_1)$, $B^*B \in \text{Sym}^+(\mathcal{H}_2) \cap \text{HS}(\mathcal{H}_2)$. Then $AA^*, BB^* \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H})$ and

$$||\log(I_H + AA^*) - (I_H + BB^*)||^2_{HS(H)} = ||\log(I_{H_1} + A^*A)||^2_{HS(H_1)} + ||\log(I_{H_2} + B^*B)||^2_{HS(H_2)} - 2tr[B^*Ah(A^*A)A^*Bh(B^*B)].$$

Here $h$ is as defined in Lemma 27.

**Proof** Since $AA^* : \mathcal{H} \to \mathcal{H}$ and $A^*A : \mathcal{H}_1 \to \mathcal{H}_1$ have the same nonzero eigenvalues, we have $||\log(I_{H_1} + A^*A)||^2_{HS(H_1)} = ||\log(I_{H_2} + B^*B)||^2_{HS(H_2)}$. Similarly, $||\log(I_H + BB^*)||^2_{HS(H)} = ||\log(I_{H_2} + B^*B)||^2_{HS(H_2)}$. Thus

$$||\log(I_H + AA^*) - (I_H + BB^*)||^2_{HS(H)} = ||\log(I_{H_1} + A^*A)||^2_{HS(H_1)} + ||\log(I_{H_2} + B^*B)||^2_{HS(H_2)} - 2\text{tr}[\log(I_H + AA^*)\log(I_H + BB^*)]$$

and the last equality follows from Corollary 29.

**Proof of Proposition 12** (i) For the first identity, let $A = \frac{1}{\sqrt{m}}R^*_{K_1} : \mathcal{H}_1 \to \mathcal{L}^2(T, \nu)$, $B = \frac{1}{\sqrt{m}}R^*_{K_2} : \mathcal{H}_2 \to \mathcal{L}^2(T, \nu)$, then $AA^* = \frac{1}{\sqrt{m}}R^*_{K_1}R_{K_1} = \frac{1}{m}C_{K_1} : \mathcal{L}^2(T, \nu) \to \mathcal{L}^2(T, \nu)$, $BB^* = \frac{1}{m}C_{K_2}$, $A^*A = \frac{1}{m}R^*_{K_1}R_{K_1} = \frac{1}{m}L_{K_1} : \mathcal{H}_1 \to \mathcal{H}_1$, $B^*B = \frac{1}{m}L_{K_2}$, $A^*B = \frac{1}{m}R_{K_1}R_{K_2} = \frac{1}{m}R_{12} : \mathcal{H}_2 \to \mathcal{H}_1$, $B^*A = \frac{1}{m}R_{12}$. By Proposition 33,

$$||\log(yI + C_{K_1}) - \log(yI + C_{K_2})||^2_{HS(L^2(T, \nu))} = ||\log(1 + \frac{1}{\sqrt{m}}C_{K_1}) - \log(1 + \frac{1}{\sqrt{m}}C_{K_2})||^2_{HS(L^2(T, \nu))}$$

$$= ||\log(I + AA^*) - (I + BB^*)||^2_{HS(L^2(T, \nu))}$$

$$= ||\log(I_{H_1} + A^*A)||^2_{HS(H_1)} + ||\log(I_{H_2} + B^*B)||^2_{HS(H_2)} - 2\text{tr}[B^*Ah(A^*A)A^*Bh(B^*B)]$$

$$= ||\log(I + \frac{1}{\sqrt{m}}L_{K_1})||^2_{HS(H_1)} + ||\log(I + \frac{1}{\sqrt{m}}L_{K_2})||^2_{HS(H_2)} - 2\frac{\text{tr}[R_{12}^*h(\frac{1}{\sqrt{m}}L_{K_1})R_{12}h(\frac{1}{\sqrt{m}}L_{K_2})]}{\gamma^2}.$$
Lemma 34  Assume Assumptions A1-A6. Let $\gamma \in \mathbb{R}$, $\gamma > 0$ be fixed. Let $\mathbf{X} = (x_1, \ldots, x_m)$ be independently sampled from $(T, \nu)$. For any $0 < \delta < 1$, with probability at least $1 - \delta$,

$$\left\| \log \left( \frac{1}{\gamma} \mathbf{L}_K \mathbf{X} \right) \right\|_{\text{HS}(\mathcal{H}_K)}^2 - \left\| \log \left( \frac{1}{\gamma} \mathbf{L}_K \mathbf{X} \right) \right\|_{\text{HS}(\mathcal{H}_K)}^2 \leq \frac{2\kappa^4}{\gamma^2} \left( \frac{2 \log \frac{2}{\delta}}{m} + \sqrt{\frac{2 \log \frac{2}{\delta}}{m}} \right).$$

(100)

Equivalently, with probability at least $1 - \delta$,

$$\left\| \log \left( \frac{1}{\gamma} \mathbf{K} \mathbf{X} \right) \right\|_F^2 - \left\| \log \left( \frac{1}{\gamma} \mathbf{C}_K \right) \right\|_{\text{HS}(\mathcal{L}^2(T, \nu))}^2 \leq \frac{2\kappa^4}{\gamma^2} \left( \frac{2 \log \frac{2}{\delta}}{m} + \sqrt{\frac{2 \log \frac{2}{\delta}}{m}} \right).$$

(101)

Proof  By Lemma 40 and Proposition 14,

$$\left\| \log \left( \frac{1}{\gamma} \mathbf{L}_K \mathbf{X} \right) \right\|_{\text{HS}(\mathcal{H}_K)}^2 - \left\| \log \left( \frac{1}{\gamma} \mathbf{L}_K \mathbf{X} \right) \right\|_{\text{HS}(\mathcal{H}_K)}^2 = \left\| \log \left( \frac{1}{\gamma} \mathbf{L}_K \mathbf{X} \right) \right\|_{\text{HS}(\mathcal{H}_K)}^2 - \left\| \log \left( \frac{1}{\gamma} \mathbf{L}_K \mathbf{X} \right) \right\|_{\text{HS}(\mathcal{H}_K)}^2 \leq \frac{1}{\gamma^2} \left\| \mathbf{L}_K \mathbf{X} - \mathbf{L}_K \right\|_{\text{HS}(\mathcal{H}_K)} \left\| \mathbf{L}_K \mathbf{X} \right\|_{\text{HS}(\mathcal{H}_K)} + \left\| \mathbf{L}_K \right\|_{\text{HS}(\mathcal{H}_K)} \right] \leq \frac{2\kappa^4}{\gamma^2} \left( \frac{2 \log \frac{2}{\delta}}{m} + \sqrt{\frac{2 \log \frac{2}{\delta}}{m}} \right).$$

Proof of Theorem 15  By Proposition 12,

$$\Delta = \left\| \log \left( \frac{1}{m} \mathbf{K} \mathbf{X} \right) \right\|_F^2 - \left\| \log \left( \frac{1}{m} \mathbf{K}^2 \mathbf{X} \right) \right\|_F^2 - \left\| \log \left( \frac{1}{m} \mathbf{K} \mathbf{X} \right) \right\|_{\text{HS}(\mathcal{H}_K)}^2 + \log \left( \frac{1}{m} \mathbf{K}^2 \mathbf{X} \right) \right\|_{\text{HS}(\mathcal{H}_K)}^2 \leq \frac{2}{\gamma^2} \left( \frac{2 \log \frac{2}{\delta}}{m} + \sqrt{\frac{2 \log \frac{2}{\delta}}{m}} \right).$$
For each $0 < \delta < 1$, the following sets satisfy $\nu^m(U_i) \geq 1 - \frac{\delta}{3}, i = 1, 2, 3,$

$$U_1 = \left\{ X \in (T, \nu)^m : \left\| \log \left( I + \frac{1}{y} L_{K^1} X \right) \right\|_{\text{HS}(\mathcal{H}_K)}^2 - \left\| \log \left( I + \frac{1}{y} L_{K_1} \right) \right\|_{\text{HS}(\mathcal{H}_K)}^2 \leq \frac{2k^4}{y^2} \left( \frac{2 \log \frac{6}{m}}{m} + \frac{2 \log \frac{6}{m}}{m} \right) \right\}.$$  

$$U_2 = \left\{ X \in (T, \nu)^m : \left\| \log \left( I + \frac{1}{y} L_{K^2} X \right) \right\|_{\text{HS}(\mathcal{H}_K)}^2 - \left\| \log \left( I + \frac{1}{y} L_{K_2} \right) \right\|_{\text{HS}(\mathcal{H}_K)}^2 \leq \frac{2k^4}{y^2} \left( \frac{2 \log \frac{6}{m}}{m} + \frac{2 \log \frac{6}{m}}{m} \right) \right\}.$$  

$$U_3 = \left\{ X \in (T, \nu)^m : \left| \text{tr} \left[ R^*_{12} h \left( \frac{1}{y} L_{K^1} X \right) \right] R^*_{12} h \left( \frac{1}{y} L_{K^2} X \right) - \text{tr} \left[ R^*_{12} h \left( \frac{1}{y} L_{K_1} \right) \right] R^*_{12} h \left( \frac{1}{y} L_{K_2} \right) \right| \leq \kappa^2 \left( 1 + \frac{\kappa^2}{2y} \right) \left( \frac{2 \log \frac{24}{m}}{m} + \frac{2 \log \frac{24}{m}}{m} \right) \right\}.$$  

Thus on the set $U = U_1 \cap U_2 \cap U_3$, with $\nu^m(U) \geq 1 - \delta$,

$$\Delta \leq \frac{2(\kappa^4 + \kappa^2)}{y^2} \left( \frac{2 \log \frac{6}{m}}{m} + \frac{2 \log \frac{6}{m}}{m} \right) + \frac{2 \kappa \kappa^2}{y^2} \left( 1 + \frac{\kappa^2}{2y} \right) \left( \frac{2 \log \frac{24}{m}}{m} + \frac{2 \log \frac{24}{m}}{m} \right).$$

**Proof of Theorem 18** By Theorem 3,

$$\Delta = \left\| \log \left( y I + \frac{1}{m} \hat{K}^1_{W^1} [X] \right) - \log \left( y I + \frac{1}{m} \hat{K}^2_{W^2} [X] \right) \right\|_F - \left\| \log \left( y I + \frac{1}{m} K^1 [X] \right) - \log \left( y I + \frac{1}{m} K^2 [X] \right) \right\|_F \leq \left\| \log \left( y I + \frac{1}{m} \hat{K}^1_{W^1} [X] \right) - \log \left( y I + \frac{1}{m} K^1 [X] \right) \right\|_F + \left\| \log \left( y I + \frac{1}{m} \hat{K}^2_{W^2} [X] \right) - \log \left( y I + \frac{1}{m} K^2 [X] \right) \right\|_F \leq \frac{1}{m y} \left\| \hat{K}^1_{W^1} [X] - K^1 [X] \right\|_F + \frac{1}{m y} \left\| \hat{K}^2_{W^2} [X] - K^2 [X] \right\|_F.$$  

By Proposition 17, for any $0 < \delta < 1$, with probability at least $1 - \delta$,

$$\left\| \hat{K}^1_{W^1} [X] - K^1 [X] \right\|_F \leq \frac{4 \sqrt{3} m K^2}{\sqrt{N} \delta} \quad \text{and} \quad \left\| \hat{K}^2_{W^2} [X] - K^2 [X] \right\|_F \leq \frac{4 \sqrt{3} m K^2}{\sqrt{N} \delta}.$$  

It follows that $\Delta \leq \frac{4 \sqrt{3} (\kappa^4 + \kappa^2)}{y \sqrt{N} \delta}$ with probability at least $1 - \delta$.

**Proof of Theorem 19** We combine the results from Theorems 15 and 18. We have

$$\Delta = \left| D_{\log E}^\nu \left[ \mathcal{N} \left( \frac{1}{m} \hat{K}^1_{W^1} [X] \right) \cdot \mathcal{N} \left( \frac{1}{m} \hat{K}^2_{W^2} [X] \right) \right] - D_{\log E}^\nu \left[ \mathcal{N} \left( 0, C_{K^1} \right) \cdot \mathcal{N} \left( 0, C_{K^2} \right) \right] \right| \leq \left| D_{\log E}^\nu \left[ \mathcal{N} \left( \frac{1}{m} \hat{K}^1_{W^1} [X] \right) \cdot \mathcal{N} \left( \frac{1}{m} \hat{K}^2_{W^2} [X] \right) \right] - D_{\log E}^\nu \left[ \mathcal{N} \left( 0, K^1 [X] \right) \cdot \mathcal{N} \left( 0, K^2 [X] \right) \right] \right| \quad \text{and} \quad \left| D_{\log E}^\nu \left[ \mathcal{N} \left( \frac{1}{m} K^1 [X] \right) \cdot \mathcal{N} \left( \frac{1}{m} K^2 [X] \right) \right] - D_{\log E}^\nu \left[ \mathcal{N} \left( 0, C_{K^1} \right) \cdot \mathcal{N} \left( 0, C_{K^2} \right) \right] \right| = \Delta_1 + \Delta_2.$$
By Theorem 18, the following set \( U_1 \subset (T, v)^m \) satisfies \( v^m(U) \geq 1 - \delta \).

\[
U_1 = \left\{ X \in (T, v)^m : \Delta_1 \leq \frac{8 \sqrt{3} (\kappa_2^2 + \kappa_3^2)}{\gamma \sqrt{N} \delta} \right\}.
\]

By Theorem 15, using the inequality \( (a - b)^2 \leq |a^2 - b^2| \) for \( a \geq 0, b \geq 0 \), for a fixed \( X \in (T, v)^m \), the following set \( U_2 \subset (\Omega_1, P_1)^N \times (\Omega_2, P_2)^N \) satisfies \( \left( P_1 \otimes P_2 \right)^N(U_2) \geq 1 - \frac{\delta}{2} \),

\[
U_2 = \left\{ (W^1, W^2) : \Delta_2 \leq \frac{2 (\kappa_4^2 + \kappa_5^2)}{\gamma^2} \left( \frac{2 \log \frac{1}{m}}{\sqrt{m}} + \frac{2 \log \frac{2}{\delta}}{\sqrt{m}} \right) + \frac{2 \kappa_6^2 \kappa_7^2}{\gamma^2} \left( 1 + \frac{\kappa_8^2 + \kappa_9^2}{2 \gamma} \right) \left( \frac{2 \log \frac{1}{m}}{\sqrt{m}} + \frac{2 \log \frac{2}{\delta}}{\sqrt{m}} \right) \right\}.
\]

Let \( U = (U_1 \times (\Omega_1, P_1)^N \times (\Omega_2, P_2)^N) \cap ((T, v)^m \times U_2) \), then \( (v^m \otimes P_1^N \otimes P_2^N)(U) \geq 1 - \delta \) and

\[
\Delta \leq \frac{8 \sqrt{3} \kappa_2^2 + \kappa_3^2}{\gamma \sqrt{N} \delta} + \frac{1}{\gamma} \left( \frac{2 \kappa_4^2 + \kappa_5^2}{\gamma^2} \left( \frac{2 \log \frac{1}{m}}{\sqrt{m}} + \frac{2 \log \frac{2}{\delta}}{\sqrt{m}} \right) + \frac{2 \kappa_6^2 \kappa_7^2}{\gamma^2} \left( 1 + \frac{\kappa_8^2 + \kappa_9^2}{2 \gamma} \right) \left( \frac{2 \log \frac{1}{m}}{\sqrt{m}} + \frac{2 \log \frac{2}{\delta}}{\sqrt{m}} \right) \right)
\]

\( \forall (X, W^1, W^2) \in U. \]

### 7.3 Proofs for the convergence of the affine-invariant Riemannian distance

We now prove Theorems 4 and 5.

**Lemma 35** Let \( A \in \mathcal{C}_p(\mathcal{H}) \) with \( I + A > 0 \) and \( ||A|| < 1 \). Then \( \log(I + A) \in \mathcal{C}_p(\mathcal{H}) \), with

\[
||\log(I + A)||_p \leq \frac{||A||_p}{1 - ||A||}. \quad (102)
\]

**Proof** For \( ||A|| < 1 \), the following series is absolutely convergent,

\[
\log(I + A) = A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \cdots
\]

It follows that, since \( \mathcal{C}_p(\mathcal{H}) \) is a Banach algebra and a two-sided ideal in \( \mathcal{L}(\mathcal{H}) \)

\[
||\log(I + A)||_p \leq ||A||_p \left[ 1 + \frac{||A||}{2} + \frac{||A||^2}{3} + \cdots \right] \leq ||A||_p [1 + ||A|| + ||A||^2 + \cdots] = \frac{||A||_p}{1 - ||A||}. \quad \]

**Proof of Theorem 4** (i) Consider first the case \( \gamma = 1 \). Write \( (I + A)^{-1/2}(I + A)(I + A)^{-1/2} = (I + A)^{-1} + (I + A)^{-1/2}A_n(I + A)^{-1/2} = I + (I + A)^{-1/2}(A_n - A)(I + A)^{-1/2} \). Since \( \lim_{n \to \infty} ||A_n - A|| = 0, \forall 0 < \epsilon < M_A, \exists N(\epsilon) \in \mathbb{N} \) such that \( \forall n \geq N(\epsilon), ||A_n - A|| < \epsilon \) and

\[
||((I + A)^{-1/2}(A_n - A)(I + A)^{-1/2})|| \leq ||(I + A)^{-1/2}|| \cdot ||A_n - A|| \leq \frac{1}{M_A} ||A_n - A|| < \frac{\epsilon}{M_A} < 1.
\]
By Lemma 35, $\forall n \geq N(\epsilon)$,
\[
||\log((I + A)^{-1/2}(I + A_n)(I + A)^{-1/2})||_{HS} \leq \frac{||(I + A)^{-1/2}(A_n - A)(I + A)^{-1/2}||_{HS}}{1 - ||(I + A)^{-1/2}(A_n - A)(I + A)^{-1/2)||} \leq \frac{||(I + A)^{-1}||}{1 - (\epsilon/M_A)} ||A_n - A||_{HS} \leq \frac{1}{M_A - \epsilon} ||A_n - A||_{HS}.
\]

If $A \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H})$, we can set $M_A = 1$ which gives the second bound.

(ii) Consider now the general case $\gamma > 0$. We have $\gamma I + A \geq M_A \iff I + \frac{A}{\gamma} \geq \frac{M_A}{\gamma}$. Let $N(\epsilon) \in \mathbb{N}$ be such that $||A_n - A|| < \epsilon \forall n \geq N(\epsilon)$, then $||\frac{A_n}{\gamma} - \frac{A}{\gamma}|| < \frac{\epsilon}{\gamma} \forall n \geq N(\epsilon)$. Since $||(\gamma I + A)^{-1/2}(\gamma I + A_n)(\gamma I + A)^{-1/2}||_{HS} = ||(I + \frac{A}{\gamma})^{-1/2}(I + \frac{A_n}{\gamma})(I + \frac{A}{\gamma})||_{HS}$, applying part (i) gives
\[
||((I + A)^{-1/2}(\gamma I + A_n)(\gamma I + A)^{-1/2})||_{HS} \leq \frac{1}{(M_A/\gamma) - (\epsilon/\gamma)} \frac{||A_n - A||_{HS}}{\gamma} = \frac{1}{M_A - \epsilon} ||A_n - A||_{HS}.
\]

If $A \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H})$, setting $M_A = \gamma$ gives the last bound.

Proof of Theorem 5 Since $d_{\text{diHS}}(\gamma_1 I + A, \gamma_2 I + B) = ||\log((\gamma_1 I + A)^{-1/2}(\gamma_2 I + B)(\gamma_1 I + A)^{-1/2})||_{HS_X}$ is a metric, by the triangle inequality and Theorem 4, $\forall 0 < \epsilon < \min\{M_A, M_B\}$, $\exists N(\epsilon) \in \mathbb{N}$ such that $\forall n \geq N(\epsilon), ||A_n - A|| < \epsilon, ||B_n - B|| < \epsilon$ and
\[
|d_{\text{diHS}}(\gamma_1 I + A_n, \gamma_2 I + B_n) - d_{\text{diHS}}(\gamma_1 I + A, \gamma_2 I + B)| \leq d_{\text{diHS}}(\gamma_1 I + A_n, \gamma_1 I + A) + d_{\text{diHS}}(\gamma_2 I + B_n, \gamma_2 I + B) \leq \frac{1}{M_A - \epsilon} ||A_n - A||_{HS} + \frac{1}{M_B - \epsilon} ||B_n - B||_{HS}.
\]

If $A, B \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H})$, setting $M_A = \gamma_1, M_B = \gamma_2$ gives the last bound.

7.4 Proofs for the affine-invariant Riemannian distance between Gaussian processes

We now prove Theorems 10 and 16.

Proof of Theorem 10 By Proposition 8, for any $0 < \delta < 1$, with probability at least $1 - \delta$,
\[
\Delta_1 = ||C_{K^1 \cdot W_1} - C_{K^1}||_{HS} \leq \frac{4\sqrt{3}k^2}{\sqrt{N\delta}} \quad \text{and} \quad \Delta_2 = ||C_{K^2 \cdot W_2} - C_{K^2}||_{HS} \leq \frac{4\sqrt{3}k^2}{\sqrt{N\delta}}.
\]

For $0 < \epsilon < \gamma$, let $N(\epsilon) \in \mathbb{N}, N(\epsilon) \geq 1 + \max\left\{\frac{48\sqrt{3}k^4}{\epsilon^2\delta^2}, \frac{48\sqrt{3}k^4}{\gamma^2\delta^2}\right\}$, then $\Delta_1(N) < \epsilon, \Delta_2(N) < \epsilon \forall N \geq N(\epsilon)$. By Theorem 5, $\forall N \geq N(\epsilon)$, with probability at least $1 - \delta$, $\Delta_3 = ||d_{\text{diHS}}(N(0, C_{K^1 \cdot W_1}), N(0, C_{K^2 \cdot W_2})) - d_{\text{diHS}}(N(0, C_{K^1}), N(0, C_{K^2}))||_{HS}$,
\[
= ||\log((\gamma I + C_{K^1 \cdot W_1})^{-1/2}(\gamma I + C_{K^2 \cdot W_2})(\gamma I + C_{K^1 \cdot W_1})^{-1/2})||_{HS}
- ||\log((\gamma I + C_{K^1})^{-1/2}(\gamma I + C_{K^2})(\gamma I + C_{K^1})^{-1/2})||_{HS}||
\leq \frac{1}{\gamma - \epsilon} \left(||C_{K^1 \cdot W_1} - C_{K^1}||_{HS} + ||C_{K^2 \cdot W_2} - C_{K^2}||_{HS}\right) \leq \frac{4\sqrt{3}(k_1^2 + k_2^2)}{(\gamma - \epsilon)\sqrt{N\delta}}.
\]
Proposition 36 Let $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2$ be separable Hilbert spaces. Let $A : \mathcal{H}_1 \rightarrow \mathcal{H}, B : \mathcal{H}_2 \rightarrow \mathcal{H}$ be compact operators such that $A^*A \in \text{HS}(\mathcal{H}_1), B^*B \in \text{HS}(\mathcal{H}_2)$. Then $A^*A, BB^* \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H})$ and

$$||\log((I + B^*B)(I + AA^*))||_{HS(\mathcal{H})}^2 = \text{tr} \left[ \log \left( I + \begin{pmatrix} I_{\mathcal{H}_1} + A^*A & (I_{\mathcal{H}_1} + A^*A)^{-1} - I & (I_{\mathcal{H}_1} + A^*A)^{-1} A^*B \\ -B^*A(I_{\mathcal{H}_1} + A^*A)^{-1} & B^*B - B^*A(I_{\mathcal{H}_1} + A^*A)^{-1} A^*B \\ I_{\mathcal{H}_2} & B^*(I_{\mathcal{H}_2} + AA^*)^{-1} \end{pmatrix} \right) \right] = \text{tr}((I + D)^2). \quad (103)$$

The operator $(I + tD) : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2$ is positive definite for all $t \in [0, 1]$, with $||(I + tD)^{-1}|| \leq \frac{1 + \lambda_1(A^*A)}{1 + (1-t)\lambda_1(A^*A)}$ and $\sup_{t \in [0, 1]} ||(I + tD)^{-1}|| \leq 1 + \lambda_1(A^*A)$, where $\lambda_1(A^*A)$ is the largest eigenvalue of $A^*A$.

We note that the operator $D$ in Eq.(103) has the form $D = \begin{pmatrix} D_{11} & D_{12} \\ -D_{12} & D_{22} \end{pmatrix}$ and is not self-adjoint.

Proof Expanding $(I + AA^*)^{-1/2}(I + BB^*)(I + AA^*)^{-1/2}$ as

$$(I + AA^*)^{-1/2}(I + BB^*)(I + AA^*)^{-1/2} = (I + AA^*)^{-1} + (I + AA^*)^{-1/2}BB^*(I + AA^*)^{-1/2}$$

$$= I - A(I + A^*A)^{-1}A^* + (I + AA^*)^{-1/2}BB^*(I + AA^*)^{-1/2}$$

$$= I + (-A(I + A^*A)^{-1/2}A^*)(I + AA^*)^{-1/2}B^*(I + AA^*)^{-1/2} = I + C.$$ 

Consider the block operators $\begin{pmatrix} -A(I + A^*A)^{-1/2} & (I + AA^*)^{-1/2}B^*(I + AA^*)^{-1/2} \\ B^*(I + AA^*)^{-1/2} & (I + AA^*)^{-1/2} \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}$ and $\begin{pmatrix} (I + A^*A)^{-1/2}A^* \\ B^*(I + AA^*)^{-1/2} \end{pmatrix} : \mathcal{H} \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2$. The nonzero eigenvalues of the operator $C : \begin{pmatrix} -A(I + A^*A)^{-1/2} & (I + AA^*)^{-1/2}B^*(I + AA^*)^{-1/2} \\ B^*(I + AA^*)^{-1/2} & (I + AA^*)^{-1/2} \end{pmatrix} : \mathcal{H} \rightarrow \mathcal{H}$ are the same as those of the operator $D : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2$, where

$$D = \begin{pmatrix} (I + A^*A)^{-1/2}A^* \\ B^*(I + AA^*)^{-1/2} \end{pmatrix} \begin{pmatrix} -A(I + A^*A)^{-1/2} & (I + AA^*)^{-1/2}B^* \\ I_{\mathcal{H}_2} & B^*(I + AA^*)^{-1/2} \end{pmatrix}$$

$$= \begin{pmatrix} (I + A^*A)^{-1/2}A^* - I & (I + A^*A)^{-1}A^*B \\ -B^*A(I + A^*A)^{-1} & B^*B - B^*A(I + A^*A)^{-1}A^*B \end{pmatrix}.$$ 

Here we have used the following identities (which are special cases of Corollary 2 in [45])

$$(I + A^*A)^{-1/2}A^* = A^*(I + AA^*)^{-1/2}, \quad \text{equivalently} \quad A^*(I + AA^*)^{-1/2} = (I + A^*A)^{-1/2}A^*.$$ 

$$(I + A^*A)^{-1}A^*B = (I + AA^*)^{-1/2}A^* = A(I + AA^*)^{-1/2}, \quad \text{equivalently} \quad (I + AA^*)^{-1/2}A = A(I + AA^*)^{-1/2}.$$ 

Thus the nonzero eigenvalues of $\log(I + C)$ are the same as those of $\log(I + D)$. Therefore

$$||\log(I + C)||_{HS}^2 = \text{tr}[(I + C)^2] = \text{tr}((I + D)^2).$$
The operator $D : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$ has the form $D = \begin{pmatrix} D_{11} & D_{12} \\ -D_{12}^* & D_{22} \end{pmatrix}$ and is thus not self-adjoint. For any $t \in [0, 1]$ and $\forall x = (x_1, x_2) \in \mathcal{H}_1 \oplus \mathcal{H}_2$,

$$
\langle x, (I + tD)x \rangle = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^* \begin{pmatrix} I + tD_{11} & -D_{12}^* \\ -D_{12} & I + tD_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \langle x_1, (I + tD_{11})x_1 \rangle + \langle x_2, (I + tD_{22})x_2 \rangle
$$

$$
= \langle x_1, (1-t)I + t(I + A^*A)^{-1}B \rangle x_1 + \langle x_2, (I + tB^*(I + AA^*)^{-1}B) x_2 \rangle \geq \left( 1 - t \right) \frac{t}{1 + \gamma \lambda_1(A^*A)} \|x_1\|^2 + \|x_2\|^2
$$

$$
\geq \frac{1 + (1 - t)\lambda_1(A^*A)}{1 + \gamma \lambda_1(A^*A)} \|x\|^2.
$$

Thus $(I + tD)$ is positive definite $\forall t \in [0, 1]$. By the Cauchy-Schwarz Inequality, $\| (I + tD)x \| \geq \frac{|1 + (1 - t)\lambda_1(A^*A)|}{1 + \gamma \lambda_1(A^*A)} \|x\|$ $\forall x \in \mathcal{H}_1 \oplus \mathcal{H}_2$, from which it follows that $(I + tD)$ is invertible, with $\| (I + tD)^{-1} \| \leq \frac{1 + \gamma \lambda_1(A^*A)}{|1 + (1 - t)\lambda_1(A^*A)|}$. It is clear then that $\sup_{t \in [0, 1]} \| (I + tD)^{-1} \| \leq 1 + \lambda_1(A^*A)$.

Proof of Proposition 13 The first expression follows from $\| \log \left( I + \frac{1}{\gamma} C_{K^1} \right)^{-1/2} (\gamma I + C_{K^2}) (\gamma I + C_{K^1})^{-1/2} \|_{H_2(L^2(T, \nu))} = \| \log \left( I + \frac{1}{\gamma} C_{K^1} \right)^{-1/2} (\gamma I + C_{K^1}) (\gamma I + C_{K^1})^{-1/2} \|_{H_2(L^2(T, \nu))}$ and Proposition 36, with $A = \frac{1}{\sqrt{\gamma}} R_{K^1}, \ B = \frac{1}{\sqrt{\gamma}} R_{K^2}$.

Similarly, the second expression follows from $\| \log \left( I + \frac{1}{\gamma} K^1 \right)^{-1/2} (\gamma I + \frac{1}{\gamma} K^2) (\gamma I + \frac{1}{\gamma} K^1) \|_{H_2(L^2(T, \nu))} = \| \log \left( I + \frac{1}{\gamma} K^1 \right)^{-1/2} (\gamma I + \frac{1}{\gamma} K^1) \|_{H_2(L^2(T, \nu))}$ and Proposition 36, with $A = \frac{1}{\sqrt{\gamma}} S_{1,1}, \ B = \frac{1}{\sqrt{\gamma}} S_{2,1}$.

Lemma 37 Let $A_{11} \in H_2(\mathcal{H}_1), A_{12} \in H_2(\mathcal{H}_2, \mathcal{H}_1), A_{21} \in H_2(\mathcal{H}_1, \mathcal{H}_2), A_{22} \in H_2(\mathcal{H}_2)$. Consider the operator $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$. Then $\| A \|_{\mathcal{H}_2}^2 = \sum_{i,j=1}^2 \| A_{ij} \|_{\mathcal{H}_2}^2$ and $\| A \|_{\mathcal{H}_2} \leq \sum_{i,j=1}^2 \| A_{ij} \|_{\mathcal{H}_2}$.

Proof Let $\{ e_{i,k} \}_{k \in \mathbb{N}}$ be an orthonormal basis for $\mathcal{H}_i, i = 1, 2$, then $\left( \begin{pmatrix} e_{1,k} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_{2,k} \end{pmatrix} \right)_{k \in \mathbb{N}}$ is an orthonormal basis for $\mathcal{H}_1 \oplus \mathcal{H}_2$. By definition of the Hilbert-Schmidt norm,

$$
\| A \|_{\mathcal{H}_2}^2 = \sum_{k=1}^{\infty} \left\| A \left( \begin{pmatrix} e_{1,k} \\ 0 \end{pmatrix} \right) \right\|^2 + \left\| A \left( \begin{pmatrix} 0 \\ e_{2,k} \end{pmatrix} \right) \right\|^2 = \sum_{k=1}^{\infty} \left( \| A_{11} e_{1,k} \|^2 + \| A_{21} e_{1,k} \|^2 + \| A_{12} e_{2,k} \|^2 + \| A_{22} e_{2,k} \|^2 \right)
$$

$$
= \| A_{11} \|_{\mathcal{H}_2}^2 + \| A_{12} \|_{\mathcal{H}_2}^2 + \| A_{21} \|_{\mathcal{H}_2}^2 + \| A_{22} \|_{\mathcal{H}_2}^2.
$$

From this it follows that $\| A \|_{\mathcal{H}_2} = \sqrt{\sum_{i,j=1}^2 \| A_{ij} \|_{\mathcal{H}_2}^2} \leq \sum_{i,j=1}^2 \| A_{ij} \|_{\mathcal{H}_2}$.

Proof of Theorem 16 Define $D = \begin{pmatrix} D_{11} & D_{12} \\ -D_{12}^* & D_{22} \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$.

$H_{K^2}$ and $D_X = \begin{pmatrix} (I + \frac{1}{\gamma} L_{K^1})^{-1} - I & \frac{1}{\gamma} (I + \frac{1}{\gamma} L_{K^1})^{-1} R_{12} \\ -\frac{1}{\gamma} R_{12} (I + \frac{1}{\gamma} L_{K^1})^{-1} & 1/\gamma L_{K^2} - \frac{1}{\gamma} R_{12} (I + \frac{1}{\gamma} L_{K^1})^{-1} R_{12} \end{pmatrix}$ : $\mathcal{H}_1 \oplus \mathcal{H}_2$.
By Lemma 41,

$$
\Delta = \left\| \log \left( yI + \frac{1}{m} K^1 |X| \right) \right\|_F - \left\| \log \left( yI + C_{K^1} \right) \left( yI + \frac{1}{m} K^2 |X| \right) \right\|_F - \left\| \log \left( yI + \frac{1}{m} K^1 |X| \right) \right\|_F - \left\| \log \left( yI + C_{K^1} \right) \left( yI + \frac{1}{m} K^2 |X| \right) \right\|_F \leq c_D |D| ||D_X D - D||_{HS}.$$

Here the constants $c_{DX}, c_D$ are given by Proposition 33 by

$$
c_{DX} = \sup \left\{ t \in [0,1] : ||(I + tD_X)^{-1}|| \leq 1 + \frac{\lambda_1 (L_{K^1}, X)}{y} + \frac{\kappa_1^2}{y^2} \right\},
$$

$$
c_D = \sup \left\{ t \in [0,1] : ||(I + tD)^{-1}|| \leq 1 + \frac{\lambda_1 (L_{K^1})}{y} + \frac{\kappa_1^2}{y^2} \right\}.
$$

By Lemma 37,

$$
||D||_{HS} \leq \left\| \left( I + \frac{1}{y} L_{K^1} \right)^{-1} - I \right\|_{HS} + \frac{2}{\gamma} \left\| \left( I + \frac{1}{y} L_{K^2} \right)^{-1} R_{12} \right\|_{HS} + \frac{1}{\gamma} ||L_{K^2}||_{HS} + \frac{1}{\gamma^2} \left\| R_{12} \right\|^2_{HS} \leq \frac{1}{\gamma} \left\| L_{K^1} \right\|_{HS} + \frac{2}{\gamma} \left\| R_{12} \right\|_{HS} + \frac{1}{\gamma} ||L_{K^2}||_{HS} + \frac{1}{\gamma^2} \left\| R_{12} \right\|^2_{HS} \leq \frac{1}{\gamma} \frac{\kappa_1^2}{y^2} + \frac{2}{\gamma} \kappa_2 + \frac{1}{\gamma} \frac{\kappa_1^2 \kappa_2^2}{y^2} = \frac{1}{\gamma} (\kappa_1 + \kappa_2)^2 + \frac{\kappa_1^2 \kappa_2^2}{y^2}.
$$

Similarly, $||D_X||_{HS} \leq \frac{1}{\gamma} (\kappa_1 + \kappa_2)^2 + \frac{\kappa_1^2 \kappa_2^2}{y^2}$. It thus follows that

$$
\Delta \leq \frac{1}{\gamma} \left( 1 + \frac{\kappa_1^2}{y^2} \right)^3 (\kappa_1 + \kappa_2)^2 + \frac{\kappa_1^2 \kappa_2^2}{y^2} ||D_X - D||_{HS}. \tag{106}
$$

By Lemma 37, $||D_X - D||_{HS} \leq \Delta_1 + 2 \Delta_2 + \Delta_3 + \Delta_4$, with the $\Delta_j$’s given in the following. The first term is

$$
\Delta_1 = \left\| \left( I + \frac{1}{y} L_{K^1} \right)^{-1} - \left( I + \frac{1}{y} L_{K^1} \right)^{-1} \right\|_{HS} = \left\| \left( I + \frac{1}{y} L_{K^1} \right)^{-1} \left[ \left( I + \frac{1}{y} L_{K^1} \right) - \left( I + \frac{1}{y} L_{K^1} \right) \right] \left( I + \frac{1}{y} L_{K^1} \right)^{-1} \right\|_{HS} = \frac{1}{\gamma} \left\| I + \frac{1}{y} L_{K^1} \right\| \left\| L_{K^1} X - L_{K^1} \right\|_{HS} \left\| I + \frac{1}{y} L_{K^1} \right\|^{-1} \left\| L_{K^1} X - L_{K^1} \right\|_{HS} \leq \frac{1}{\gamma} ||L_{K^1} X - L_{K^1}||_{HS}.
$$

The second term is

$$
\Delta_2 = \frac{1}{\gamma} \left\| \left( I + \frac{1}{y} L_{K^1} \right)^{-1} R_{12} X - \left( I + \frac{1}{y} L_{K^1} \right)^{-1} R_{12} \right\|_{HS} \leq \frac{1}{\gamma} \left\| \left( I + \frac{1}{y} L_{K^1} \right)^{-1} R_{12} X - R_{12} \right\|_{HS} + \frac{1}{\gamma} \left\| \left( I + \frac{1}{y} L_{K^1} \right)^{-1} - \left( I + \frac{1}{y} L_{K^1} \right)^{-1} \right\| R_{12} \left\|_{HS} \leq \frac{1}{\gamma} ||R_{12} X - R_{12}||_{HS} + \frac{1}{\gamma^2} ||L_{K^1} X - L_{K^1}||_{HS} \left\| R_{12} \right\| \leq \frac{1}{\gamma} ||R_{12} X - R_{12}||_{HS} + \frac{\kappa_1^2 \kappa_2^2}{y^2} ||L_{K^1} X - L_{K^1}||_{HS}.
$$
The third term is $\Delta_3 = \frac{1}{y} ||L_{K^2X} - L_{K^2}||$. The fourth term is

$$
\Delta_4 = \frac{1}{y^2} \left\| R_{12X}^* \left( I + \frac{1}{y} L_{K^1X} \right)^{-1} R_{12X} - R_{12}^* \left( I + \frac{1}{y} L_{K^1} \right)^{-1} R_{12} \right\|_{HS}
\leq \frac{1}{y^2} \left\| R_{12X}^* - R_{12}^* \right\|_{HS} \left\| \left( I + \frac{1}{y} L_{K^1X} \right)^{-1} \left( I + \frac{1}{y} L_{K^1} \right)^{-1} \right\| \left\| R_{12X} \right\|_{HS} + \frac{1}{y} ||R_{12}|| \left\| R_{12X} \right\|_{HS} + \frac{1}{y} ||R_{12}|| \left\| R_{12}^* \right\|_{HS}
\leq \frac{k_1k_2}{y^2} ||R_{12X} - R_{12}||_{HS} + \frac{k_1k_2}{y^2} \left( \left\| R_{12X} - R_{12} \right\|_{HS} + \frac{k_1k_2}{y} ||L_{K^1} - L_{K^1}||_{HS} \right)
= \frac{2k_1k_2}{y^2} ||R_{12X} - R_{12}||_{HS} + \frac{k_1k_2^2}{y^2} ||L_{K^1X} - L_{K^1}||_{HS}.
$$

Combining the expressions for all the $\Delta_j$'s, we obtain

$$
||D_X - D||_{HS} \leq \frac{2}{y} + \frac{2k_1k_2}{y^2} ||R_{12X} - R_{12}||_{HS} + \left( \frac{1}{y} + \frac{2k_1k_2}{y^2} + \frac{k_1^2k_2^2}{y^3} \right) ||L_{K^1X} - L_{K^1}||_{HS} + \frac{2}{y} ||L_{K^2} - L_{K^2}||_{HS}
= \frac{1}{y} \left\{ 2 \left( 1 + \frac{k_1k_2}{y} \right) ||R_{12X} - R_{12}||_{HS} + \left( \frac{1}{y} + \frac{k_1k_2}{y^2} \right)^2 ||L_{K^1X} - L_{K^1}||_{HS} + \frac{2}{y} ||L_{K^2} - L_{K^2}||_{HS} \right\}.
$$

By Proposition 14, for any $0 < \delta < 1$, with probability at least $1 - \delta$, the following there inequalities hold simultaneously,

$$
||R_{12X} - R_{12}||_{HS(\mathcal{H}_{K^2} ; \mathcal{H}_{K^1})} \leq k_1k_2 \left[ \frac{2 \log \frac{6}{m} + \sqrt{2 \log \frac{6}{m}}}{m} \right],
||L_{K^1X} - L_{K^1}||_{HS(\mathcal{H}_{K^1})} \leq k_1^2 \left[ \frac{2 \log \frac{6}{m} + \sqrt{2 \log \frac{6}{m}}}{m} \right],
||L_{K^2X} - L_{K^2}||_{HS(\mathcal{H}_{K^2})} \leq k_2^2 \left[ \frac{2 \log \frac{6}{m} + \sqrt{2 \log \frac{6}{m}}}{m} \right].
$$

It follows that with probability at least $1 - \delta$,

$$
||D_X - D||_{HS} \leq \frac{1}{y} \left\{ 2 \left( 1 + \frac{k_1k_2}{y} \right) k_1k_2 + \left( \frac{1}{y} + \frac{k_1k_2}{y^2} \right)^2 k_1^2 + k_2^2 \right\} \left[ \frac{2 \log \frac{6}{m} + \sqrt{2 \log \frac{6}{m}}}{m} \right]
= \frac{1}{y} \left( k_1 + k_2 \right)^2 \left[ \frac{2 \log \frac{6}{m} + \sqrt{2 \log \frac{6}{m}}}{m} \right].
$$

Combining this with Eq.(106), we obtain

$$
\Delta \leq \frac{1}{y^2} \left( \frac{k_1^2}{y} \right)^3 \left( k_1 + k_2 \right)^2 \left( \frac{k_1^2}{y} \right)^2 \left( k_1 + k_2 \right)^2 \left[ \frac{2 \log \frac{6}{m} + \sqrt{2 \log \frac{6}{m}}}{m} \right].
$$

with probability at least $1 - \delta$.  

**Lemma 38** Assume that $A \in \mathcal{L}(\mathcal{H})$ with $I + A > 0$. Let $M_A > 0$ be such that $(x, (I + A)x) \geq M_A ||x||^2 \forall x \in \mathcal{H}$. Then for $t \in [0, 1], I + tA > 0, ||I + tA|| \geq (1 - t) + tM_A, and ||(I + tA)^{-1}|| \leq \frac{1}{(1 - t) + tM_A}$. 

**Proof** By the assumption on $M_A, (x, (I + A)x) = ||x||^2 + (x, Ax) \geq M_A ||x||^2 \forall x \in \mathcal{H}$. Then $(x, Ax) \geq (M_A - 1) ||x||^2 \Rightarrow$
\[ t\langle x, Ax \rangle \geq t(M_A - 1) ||x||^2 \text{ for } t \geq 0. \] Thus \( \forall x \in \mathcal{H}, \)

\[ \langle x, (I + tA)x \rangle = ||x||^2 + t\langle x, Ax \rangle \geq [(1 - t) + tM_A]||x||^2. \]

It follows that \( \forall t \in [0,1], (I + tA) > 0. \) By the Cauchy-Schwarz Inequality, \( \forall x \in \mathcal{H}, \)

\[ ||x|| ||(I + tA)x|| \geq \langle x, (I + tA)x \rangle \geq [(1 - t) + tM_A]||x||^2 \Rightarrow ||(I + tA)x|| \geq [(1 - t) + tM_A]||x||. \]

Thus \( ||I + tA|| \geq (1 - t) + tM_A \) and \( I + tA \) is invertible, with a bounded inverse, and \( ||(I + tA)^{-1}|| \leq \frac{1}{(1 - t) + tM_A}. \]

**Lemma 39** Let \( A \in \mathcal{L}(\mathcal{H}) \) be a compact operator, with \( I + A > 0. \) Then the principal logarithm \( \log(I + A) \) is well-defined, compact, and admits the following integral representation

\[ \log(I + A) = A \int_0^1 (I + tA)^{-1} \, dt. \] (107)

**Proof** By Lemma 38, \( \int_0^1 (I + tA)^{-1} \, dt \) is bounded, thus if Eq.(107) holds, then \( \log(I + A) \) is compact.

(i) Consider first the case \( A \in \text{Sym}(\mathcal{H}). \) Let \( \{\lambda_k\}_{k \in \mathbb{N}} \) be the eigenvalues of \( A, \) with corresponding normalized eigenvectors \( \{u_k\}_{k \in \mathbb{N}}. \) Then \( 1 + \lambda_k > 0 \forall k \in \mathbb{N} \) and \( A \) admits the spectral decomposition \( A = \sum_{k=1}^{\infty} \lambda_k u_k \otimes u_k. \) From the identity \( \log(1 + \lambda) = \int_0^1 \frac{\lambda}{1 + \lambda t} \, dt, \) we have

\[ \log(I + A) = \sum_{k=1}^{\infty} \log(1 + \lambda_k) u_k \otimes u_k = \sum_{k=1}^{\infty} \left( \int_0^1 \frac{\lambda_k}{1 + \lambda_k t} \, dt \right) u_k \otimes u_k = \int_0^1 A(I + tA)^{-1} \, dt. \]

(ii) Consider now the case \( A = UBU^{-1}, \) with \( B \) compact, \( I + B > 0 \) and \( U \in \mathcal{L}(\mathcal{H}) \) invertible. By part (i), \( \log(I + B) \) admits the representation \( \log(I + B) = \int_0^1 B(I + tB)^{-1} \, dt. \) Thus \( \log(I + A) \) is well-defined by

\[ \log(I + A) = \log(I + UB) = \log[U(I + B)U^{-1}] = U \log(I + B)U^{-1} = U \left[ \int_0^1 B(I + tB)^{-1} \, dt \right] U^{-1} = \int_0^1 A(I + tA)^{-1} \, dt. \]

(iii) Consider the general case. We first briefly recall the Dunford-Riesz functional calculus ([46], VII.3) for bounded linear operators in the Hilbert space setting. Let \( U \subset \mathbb{C} \) be an open set, with boundary \( \Gamma \) being a piecewise rectifiable curve, positively oriented. Let \( A \in \mathcal{L}(\mathcal{H}) \) with spectrum \( \sigma(A). \) Assume that \( U \supseteq \sigma(A). \) Let \( f : \mathbb{C} \to \mathbb{C} \) be a function analytic in a domain containing \( U \cup \Gamma. \) Then the function \( f(A) \) is well-defined by the following Bochner integral

\[ f(A) = \frac{1}{2\pi i} \oint_{\Gamma} f(z)(zI - A)^{-1} \, dz. \] (108)

Let \( \log(z) \) denote the principal logarithm of \( z \in \mathbb{C}, \) then \( \log(z) \) is analytic on \( \mathbb{C} - (-\infty, 0] \) and thus \( \log(1 + z) \) is analytic on \( \mathbb{C} - (-\infty, -1]. \) For \( A \in \mathcal{L}(\mathcal{H}) \) a compact operator, with \( I + A > 0, \) the principal logarithm of \( I + A \) is then well-defined by the following integral

\[ \log(I + A) = \frac{1}{2\pi i} \oint_{\Gamma} (zI - A)^{-1} \log(1 + z) \, dz. \] (109)

with \( \Gamma \) enclosing \( \sigma(A) \) and not crossing \( (-\infty, -1]). \)
For each fixed \( t \in [0, 1] \), for \( z \neq (-\infty, -1) \), we always have \( 1 + tz \neq 0 \) and the function \( g(z) = z(1 + tz)^{-1} \) is analytic on \( \mathbb{C} - (-\infty, -1] \). Thus for \( \Gamma \) enclosing \( \sigma(A) \), not crossing \( (-\infty, -1] \),

\[
A(I + tA)^{-1} = \frac{1}{2\pi i} \oint_{\Gamma} (zI - A)^{-1}z(1 + tz)^{-1}dz.
\]

Since \( \log(1 + z) = \int_0^1 z(1 + tz)^{-1} dt \), we have by Fubini’s Theorem for Bochner integral (e.g. [46], Theorem III.11.9)

\[
\log(I + A) = \frac{1}{2\pi i} \oint_{\Gamma} (zI - A)^{-1} \log(1 + z)dz = \frac{1}{2\pi i} \oint_{\Gamma} (zI - A)^{-1} \left( \int_0^1 z(1 + tz)^{-1} dt \right) dz
\]

\[
= \frac{1}{2\pi i} \int_0^1 \left\{ \oint_{\Gamma} (zI - A)^{-1} z(1 + tz)^{-1}dz \right\} dt = \int_0^1 A(I + tA)^{-1} dt.
\]

**Lemma 40** Let \( 1 \leq p \leq \infty \) be fixed. (i) For \( A, B \in \text{Sym}^+(\mathcal{H}) \cap \mathcal{C}_p(\mathcal{H}) \),

\[
||\log(I + A)||_p \leq ||A||_p, \quad ||\log(I + A) - \log(I + B)||_p \leq ||A - B||_p.
\]

(ii) More generally, for \( A, B \in \mathcal{C}_p(\mathcal{H}) \), with \( I + A > 0, I + B > 0 \),

\[
||\log(I + A)||_p \leq ||A||_p \sup_{t \in [0,1]} ||(I + tA)^{-1}||,
\]

\[
||\log(I + A) - \log(I + B)||_p \leq ||A - B||_p \sup_{t \in [0,1]} [||((I + tA)^{-1})|| ||(I + tB)^{-1})||].
\]

**Proof** By Lemma 39, \( \log(I + A) = \int_0^1 A(I + tA)^{-1} dt, \log(I + B) = \int_0^1 B(I + tB)^{-1} dt \). Thus

\[
\log(I + A) - \log(I + B) = \int_0^1 [A(I + tA)^{-1} - B(I + tB)^{-1}]dt
\]

\[
= \int_0^1 (I + tA)^{-1} [A(I + tB) - (I + tA)B](I + tB)^{-1}dt = \int_0^1 (I + tA)^{-1}(A - B)(I + tB)^{-1}dt.
\]

It thus follows that

\[
||\log(I + A) - \log(I + B)||_p \leq \int_0^1 ((I + tA)^{-1}) ||(I + tB)^{-1})dt.
\]

(i) For \( A, B \in \text{Sym}^+(\mathcal{H}) \cap \mathcal{C}_p(\mathcal{H}), I + tA \geq I, I + tB \geq I \ \forall t \in [0,1] \) and thus

\[
||\log(I + A) - \log(I + B)||_p \leq ||A - B||_p.
\]

(ii) More generally, since \( I + A > 0, I + B > 0 \), by Lemma 38, \( I + tA, I + tB \) are invertible \( \forall t \in [0,1] \). The results then follow from the integral representations of \( \log(I + A) \) and \( \log(I + B) \).

**Lemma 41** (i) For \( A, B \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H}) \),

\[
|\text{tr}[(\log(I + A))^2 - \text{tr}(\log(I + B))^2] | \leq ||A - B||_{\text{HS}} [||A||_{\text{HS}} + ||B||_{\text{HS}}].
\]
(ii) For $A, B \in \text{HS}(\mathcal{H})$ with $I + A > 0, I + B > 0,$

$$\left| \text{tr} \left[ \log(I + A)^2 \right] - \text{tr} \left[ \log(I + B)^2 \right] \right| \leq c_A c_B \| A - B \|_{\text{HS}} \| c_A A \|_{\text{HS}} + c_B B \|_{\text{HS}}.$$

(114)

where $c_A = \sup_{t \in [0, 1]} \|(I + tA)^{-1}\|$, $c_B = \sup_{t \in [0, 1]} \|(I + tB)^{-1}\|$

Proof Expanding $|\text{tr} \left[ \log(I + A)^2 \right] - \text{tr} \left[ \log(I + B)^2 \right]|$ and using the Cauchy-Schwarz Inequality

$$\left| \text{tr} \left[ \log(I + A)^2 \right] - \text{tr} \left[ \log(I + B)^2 \right] \right| = |\text{tr} \left[ (\log(I + A) - \log(I + B))(\log(I + A) + \log(I + B)) \right]|$$

$$\leq \| \log(I + A) - \log(I + B) \|_{\text{HS}} \| \log(I + A) \|_{\text{HS}} + \| \log(I + B) \|_{\text{HS}}$$

(i) For $A, B \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H})$, by Lemma 40, $\| \log(I + A) \|_{\text{HS}} \leq \| A \|_{\text{HS}}, \| \log(I + B) \|_{\text{HS}} \leq \| B \|_{\text{HS}}$, $\| \log(I + A) - \log(I + B) \|_{\text{HS}} \leq \| A - B \|_{\text{HS}}$, giving the desired bound.

(ii) For $A, B \in \text{HS}(\mathcal{H})$ with $I + A > 0, I + B > 0$, by Lemma 40, $\| \log(I + A) \|_{\text{HS}} \leq \| A \|_{\text{HS}} \sup_{t \in [0, 1]} \|(I + tA)^{-1}\|$, $\| \log(I + A) - \log(I + B) \|_{\text{HS}} \leq \| A - B \|_{\text{HS}} \sup_{t \in [0, 1]} \|(I + tA)^{-1}\| \sup_{t \in [0, 1]} \|(I + tB)^{-1}\|$, giving the desired bound.

Lemma 42 (i) For $A \in \text{Sym}^+(\mathcal{H})$ compact, $h(A) = A^{-1} \log(I + A)$ is bounded and admits the integral representation

$$h(A) = A^{-1} \log(I + A) = \int_0^1 (I + tA)^{-1} dt.$$  

(115)

(ii) Let $1 \leq p \leq \infty$ be fixed. For $A, B \in \text{Sym}^+(\mathcal{H}) \cap \mathcal{C}_p(\mathcal{H}), h(A) - h(B) \in \text{Sym}^+(\mathcal{H}) \cap \mathcal{C}_p(\mathcal{H}),$ with

$$\| h(A) - h(B) \|_p = \| A^{-1} \log(I + A) - B^{-1} \log(I + B) \|_p \leq \frac{1}{2} \| A - B \|_p.$$  

(116)

Proof (i) Since $\log(I + A) = \int_0^1 (I + tA)^{-1} dt$ by Lemma 39, $h(A) = A^{-1} \log(I + A) = \int_0^1 (I + tA)^{-1} dt$ is bounded.

(ii) For $A, B \in \text{Sym}^+(\mathcal{H})$ compact,

$$h(A) - h(B) = A^{-1} \log(I + A) - B^{-1} \log(I + B) = \int_0^1 [(I + tA)^{-1} - (I + tB)^{-1}] dt$$

$$= \int_0^1 [(I + tA) - (I + tB)](I + tB)^{-1} dt$$

It follows that for $A, B \in \text{Sym}^+(\mathcal{H}) \cap \mathcal{C}_p(\mathcal{H}), h(A) - h(B) \in \text{Sym}^+(\mathcal{H}) \cap \mathcal{C}_p(\mathcal{H}),$ with

$$\| A^{-1} \log(I + A) - B^{-1} \log(I + B) \|_p \leq \int_0^1 \|(I + tA)^{-1} - (I + tB)^{-1} \| \| A - B \|_p \| (I + tB)^{-1} \| dt \leq \frac{1}{2} \| A - B \|_p.$$
7.5 | Proofs for the RKHS Gaussian measures

Proof of Theorem 21 By Theorem 2 and Theorem 20, for any $0 < \delta < 1$, with probability at least $1 - \delta$,

$$D_{\log HS}^T \{ N(0, C_{\Phi(X)}) \}, N(0, C_{\Phi}) \} = \| \log (C_{\Phi(X)} - \gamma I) - \log (C_{\Phi} + \gamma I) \|_{HS} \leq \frac{1}{\gamma} \| C_{\Phi(X)} - C_{\Phi} \|_{HS} \leq \frac{3 \kappa^2}{\gamma^2} \left( \frac{2 log \frac{4}{\delta}}{m} + \sqrt{\frac{2 log \frac{4}{\delta}}{m}} \right) + \frac{3 \kappa^2}{\gamma^2} \left( \frac{2 log \frac{4}{\delta}}{m} + \sqrt{\frac{2 log \frac{4}{\delta}}{m}} \right).$$

By Theorem 4 and Theorem 20, for $0 < \epsilon < \gamma$, let $N(\epsilon) \in \mathbb{N}$ be such that $3 \kappa^2 \left( \frac{2 log \frac{4}{\delta}}{m} + \sqrt{\frac{2 log \frac{4}{\delta}}{m}} \right) < \epsilon$, then $\forall m \geq N(\epsilon)$, with probability at least $1 - \delta$,

$$D_{\log HS}^T \{ N(0, C_{\Phi(X)}) \}, N(0, C_{\Phi}) \} = \| \log (C_{\Phi(X)} - \gamma I) - \log (C_{\Phi} + \gamma I) \|_{HS} \leq \frac{1}{\gamma - \epsilon} \| C_{\Phi(X)} - C_{\Phi} \|_{HS} \leq \frac{3 \kappa^2}{\gamma - \epsilon} \left( \frac{2 log \frac{4}{\delta}}{m} + \sqrt{\frac{2 log \frac{4}{\delta}}{m}} \right).$$

Proof of Theorem 22 By the triangle inequality,

$$\Delta = \| \log (C_{\Phi(X)} + \gamma I) - \log (C_{\Phi} + \gamma I) \|_{HS} \leq \| \log (C_{\Phi} + \gamma I) - \log (C_{\Phi} + \gamma I) \|_{HS} \leq \| \log (C_{\Phi} + \gamma I) - \log (C_{\Phi} + \gamma I) \|_{HS} = \Delta_1 + \Delta_2.$$
Proof We have $P_N e_k = e_k$ for $1 \leq k \leq N$ and $P_N e_k = 0$ for $k \geq N + 1$. Since $||A||_{HS}^2 = \sum_{k=1}^{\infty} ||A e_k||^2 < \infty$,

$$||AP_N - A||_{HS}^2 = \sum_{k=1}^{\infty} ||(AP_N - A)e_k||^2 = \sum_{k=N+1}^{\infty} ||A e_k||^2 \to 0$$

as $N \to \infty$. Similarly, since $||A^*||_{HS} = ||A||_{HS}$,

$$||P_N A - A||_{HS}^2 = ||A^* P_N - A^*||_{HS}^2 = \sum_{k=N+1}^{\infty} ||A^* e_k||^2 \to 0$$

as $N \to \infty$. It follows that

$$||P_N AP_N - A||_{HS} \leq ||(P_N A - A)||_{HS} ||P_N|| + ||AP_N - A||_{HS} \leq ||P_N A - A||_{HS} + ||AP_N - A||_{HS} \to 0$$

as $N \to \infty$. 

Lemma 44 Let $A \in \text{Tr}(\mathcal{H})$. Let $\{e_k\}_{k=1}^{\infty}$ be any orthonormal basis in $\mathcal{H}$. For $N \in \mathbb{N}$ fixed, consider the orthogonal projection operator $P_N = \sum_{k=1}^{N} e_k \otimes e_k$. Then

$$\lim_{N \to \infty} ||P_N AP_N - A||_{tr} = \lim_{N \to \infty} ||AP_N - A||_{tr} = \lim_{N \to \infty} ||P_N A - A||_{tr} = 0. \quad (118)$$

Proof We recall the following properties relating the Banach space $(\mathcal{C}_1(\mathcal{H}), ||||)$ = $(\text{Tr}(\mathcal{H}), ||||_{tr})$ of trace class operators and the Hilbert space $(\mathcal{C}_2(\mathcal{H}), ||||_2)$ = $(\text{HS}(\mathcal{H}), ||||_{HS})$ of Hilbert Schmidt operators on $\mathcal{H}$ (see e.g. [47])

1. $A \in \mathcal{C}_1(\mathcal{H})$ if and only if $A = BC$, for some operators $B, C \in \mathcal{C}_2(\mathcal{H})$.
2. $||BC||_1 \leq ||B||_2 ||C||_2$.

Given that $A \in \text{Tr}(\mathcal{H}) = \mathcal{C}_1(\mathcal{H})$, we then write $A = BC$, for some operators $B, C \in \text{HS}(\mathcal{H}) = \mathcal{C}_2(\mathcal{H})$. Then

$$||P_N AP_N - A||_{tr} = ||P_N BCP_N - BC||_{tr} = ||P_N BCP_N - P_N BC + P_N BC - BC||_{tr} = ||P_N B (CP_N - C) + (P_N B - B) C||_{tr} \leq ||P_N B||_{HS} ||CP_N - C||_{HS} + ||P_N B - B||_{HS} ||C||_{HS}.$$ 

By Lemma 43, $\lim_{N \to \infty} ||P_N B - B||_{HS} = 0$, $\lim_{N \to \infty} ||CP_N - C||_{HS} = 0$. It thus follows that $\lim_{N \to \infty} ||P_N AP_N - A||_{tr} = 0$. For the second limit,

$$||P_N A - A||_{tr} = ||P_N BC - BC||_{tr} = ||(P_N B - B) C||_{tr} \leq ||P_N B - B||_{HS} ||C||_{HS} \to 0 \text{ as } N \to \infty.$$ 

The third limit is proved similarly.

Lemma 45 Let $A \in \text{Sym}(\mathcal{H})$ be compact, with $I + A > 0$. Let $\{e_k\}$ be any orthonormal basis in $\mathcal{H}$. Let $P_N = \sum_{k=1}^{N} e_k \otimes e_k$, $N \in \mathbb{N}$ fixed. Let $A_N$ be the matrix representation of $P_N AP_N |_{\mathcal{H}_N}$ in the basis $\{e_k\}_{k=1}^{N}$, where $\mathcal{H}_N = \text{span} \{e_k\}_{k=1}^{N}$. Then

1. The matrix representation of $\log(I + P_N AP_N |_{\mathcal{H}_N})$ in the basis $\{e_k\}_{k=1}^{N}$ is $\log(I + A_N)$.
2. The matrix representation of $(I + P_N AP_N |_{\mathcal{H}_N})^\alpha$ in the basis $\{e_k\}_{k=1}^{N}$ is $(I + A_N)^\alpha \forall \alpha \in \mathbb{R}$. 

Proof Since $I + A > 0$ and $||P_NAP_N|| \leq ||A||$, we have $I + P_NAP_N > 0$, so that $\log(I + P_NAP_N)$ and $(I + P_NAP_N)^\alpha$ are well-defined $\forall \alpha \in \mathbb{R}$. Since $P_NAP_N : \mathcal{H} \to \mathcal{H}_N$, it has rank at most $N$. Let $(\lambda_k^{A,N}, \phi_k^{A,N})_{k=1}^\infty$ be the corresponding spectrum, with eigenvalues $\lambda_k^{A,N} = 0$, $k \geq N + 1$, and normalized eigenvectors $(\phi_k^{A,N})_{k=1}^\infty$ forming an orthonormal basis of $\mathcal{H}$, with $(\phi_k^{A,N})_{k=1}^N$ forming an orthonormal basis in the subspace $\mathcal{H}_N$. Then

$$P_NAP_N = \sum_{k=1}^N \lambda_k^{A,N} \phi_k^{A,N} \otimes \phi_k^{A,N},$$

$$\log(I + P_NAP_N) = \sum_{k=1}^N (1 + \lambda_k^{A,N}) \phi_k^{A,N} \otimes \phi_k^{A,N} : \mathcal{H} \to \mathcal{H}_N.$$ 

The operator $\log(I + P_NAP_N)$ also has rank at most $N$, with all non-zero eigenvalues corresponding to eigenvectors lying in $\mathcal{H}_N$. Since $A_N$ is the matrix representation of $P_NAP_N$ in the basis $(e_k)_{k=1}^N$, the eigenvalues of $A_N$ are precisely $(\lambda_k^{A,N})_k$, and for $1 \leq j, k \leq N$,

$$(A_N)_{kj} = \langle Ae_k, e_j \rangle = \langle e_k, P_NAP_N e_j \rangle = \sum_{i=1}^N \lambda_i^{A,N} \langle \phi_i^{A,N}, e_k \rangle \langle \phi_i^{A,N}, e_j \rangle, \quad A_N = U^* \text{diag}(\lambda_1^{A,N}, \ldots, \lambda_N^{A,N}) U,$$

where $U$ is the $N \times N$ matrix with $U_{ij} = (\phi_i^{A,N}, e_j)$. $U$ is orthonormal, since

$$\sum_{j=1}^N U_{ij} U_{lj} = \sum_{j=1}^N (\phi_i^{A,N}, e_j) \langle \phi_j^{A,N}, e_j \rangle = \langle e_i, e_j \rangle = \delta_{ij}, \quad 1 \leq i, j \leq N.$$

Thus $\log(I + A_N)$ is the $N \times N$ symmetric matrix given by

$$\log(I + A_N) = U^* \text{diag}(\log(1 + \lambda_1^{A,N}), \ldots, \log(1 + \lambda_N^{A,N})) U.$$ 

On the other hand, from the spectral representation of $P_NAP_N$,

$$\langle e_k, \log(I + P_NAP_N) e_j \rangle = \sum_{i=1}^N \log(1 + \lambda_i^{A,N}) \langle \phi_i^{A,N}, e_k \rangle \langle \phi_i^{A,N}, e_j \rangle = ||\log(I + A_N)||_{kj}.$$

It follows that in the basis $(e_k)_{k=1}^N$ of $\mathcal{H}_N$, the matrix representation of $\log(I + P_NAP_N|_{\mathcal{H}_N})$ is $\log(I + A_N)$. Similarly, the matrix representation of $(I + P_NAP_N|_{\mathcal{H}_N})^\alpha$ is $(I + A_N)^\alpha \forall \alpha \in \mathbb{R}$.

Proof of Theorem 6 Let $N \in \mathbb{N}$ be fixed. By Lemma 43,

$$\lim_{N \to \infty} ||P_NAP_N - A||_{\text{HS}} = 0, \lim_{N \to \infty} ||P_NBP_N - B||_{\text{HS}} = 0.$$ 

With $I + A > 0, I + B > 0$ we have $I + P_NAP_N > 0, I + P_NBP_N > 0 \forall N \in \mathbb{N}$. By Theorem 2,

$$\lim_{N \to \infty} ||\log(I + P_NAP_N) - \log(I + A)||_{\text{HS}} = 0, \lim_{N \to \infty} ||\log(I + P_NBP_N) - \log(I + B)||_{\text{HS}} = 0.$$
By the triangle inequality,

\[ ||| \log(I + P_N A P_N) - \log(I + P_N B P_N)|||_{HS} - ||| \log(I + A) - \log(I + B)|||_{HS} \]
\[ \leq ||| \log(I + P_N A P_N) - \log(I + P_N B P_N)|||_{HS} + ||| \log(I + P_N B P_N) - \log(I + B)|||_{HS} \to 0 \text{ as } N \to \infty. \]

By Lemma 45, in the basis \( \{e_k\}_{k=1}^N \) of the subspace \( \mathcal{H}_N \), the matrix representations of \( \log(I + P_N A P_N|_{\mathcal{H}_N}) \) and \( \log(I + P_N B P_N|_{\mathcal{H}_N}) \) are \( \log(I + A_N) \) and \( \log(I + B_N) \), respectively. From the integral representation \( \log(I + A) = A \int_0^1 (I + tA)^{-1} dt \) in Lemma 39, we have \( P_N A P_N e_k = 0 \Rightarrow \log(I + P_N A P_N) e_k = 0 \forall k \geq N + 1. \) Thus

\[ ||| \log(I + P_N A P_N) - \log(I + P_N B P_N)|||_{HS} = \sum_{k=1}^{\infty} ||| \log(I + P_N A P_N) - \log(I + P_N B P_N) |||_{HS} \]
\[ = \sum_{k=1}^{N} ||| \log(I + P_N A P_N) - \log(I + P_N B P_N) |||_{HS} + ||| \log(I + P_N B P_N) - \log(I + B)|||_{HS} \]
\[ = ||| \log(I + A_N) - \log(I + B_N)|||_{HS} \]

It follows that \( \lim_{N \to \infty} ||| \log(I + A_N) - \log(I + B_N)|||_{HS} = ||| \log(I + A) - \log(I + B)|||_{HS} \). For \( \gamma \in \mathbb{R}, \gamma > 0 \), we note that

\[ ||| \log(A + \gamma I) - \log(B + \gamma I)|||_{HS} = \left\| \log\left( \frac{A}{\gamma} \right) - \log\left( \frac{B}{\gamma} \right) \right\|_{HS}. \]

Thus this case reduces to the previous case.

**Proof of Theorem 7** Let \( N \in \mathbb{N} \) fixed. By Lemma 43,

\[ \lim_{N \to \infty} ||| P_N A P_N - A |||_{HS} = 0, \lim_{N \to \infty} ||| P_N B P_N - B |||_{HS} = 0. \]

With \( I + A > 0, I + B > 0 \), we have \( I + P_N A P_N > 0, I + P_N B P_N > 0 \forall N \in \mathbb{N} \). Thus by Theorem 4,

\[ \lim_{N \to \infty} ||| \log\left( (I + P_N B P_N)^{-1/2} (I + P_N A P_N) (I + P_N B P_N)^{-1/2} \right)|||_{HS} = ||| \log\left( (I + B)^{-1/2} (I + A)(I + B)^{-1/2} \right)|||_{HS}. \]

By Lemma 45, in the basis \( \{e_k\}_{k=1}^N \) of the subspace \( \mathcal{H}_N = \text{span}(e_k)_{k=1}^N \), the matrix representation of the operator \( \log\left( (I + P_N B P_N)^{-1/2} (I + P_N A P_N) (I + P_N B P_N)^{-1/2} \right) : \mathcal{H}_N \to \mathcal{H}_N \) is \( \log\left( (I + B_N)^{-1/2} (I + A_N)(I + B_N)^{-1/2} \right) \).

Write \( (I + P_N B P_N)^{-1/2} (I + P_N A P_N) (I + P_N B P_N)^{-1/2} = I + (I + P_N B P_N)^{-1/2} (P_N A P_N - P_N B P_N)(I + P_N B P_N)^{-1/2} \). Since \( P_N A P_N e_k = P_N B P_N e_k = 0 \forall k \geq N + 1 \), we have \( \forall k \geq N + 1, (I + P_N B P_N) e_k = e_k \Rightarrow (I + P_N B P_N)^{-1/2} e_k = e_k \Rightarrow (P_N A P_N - P_N B P_N)(I + P_N B P_N)^{-1/2} e_k = 0. \) Thus \( \forall k \geq N + 1, (I + P_N B P_N)^{-1/2} (P_N A P_N - P_N B P_N)(I + P_N B P_N)^{-1/2} e_k = 0. \)

From the integral representation \( \log(I + A) = A \int_0^1 (I + tA)^{-1} dt \) in Lemma 39, we then have \( \log\left( (I + P_N B P_N)^{-1/2} (I + P_N A P_N) (I + P_N B P_N)^{-1/2} \right) \text{.} \)
Thus this case reduces to the previous case.

\[ P_N A_P (I + P_N B P_N)^{-1/2}) e_k = 0 \forall k \geq N + 1. \text{ Thus} \\
= \sum_{k=1}^{\infty} \| \log((I + P_N B P_N)^{-1/2}(I + P_N A_P (I + P_N B P_N)^{-1/2})) e_k \|^2 \\
= \sum_{k=1}^{N} \| \log((I + P_N B P_N)^{-1/2}(I + P_N A_P (I + P_N B P_N)^{-1/2})) e_k \|^2 \\
= \| \log((I + P_N B P_N |_{H_N})^{-1/2}(I + P_N A_P |_{H_N}) (I + P_N B P_N |_{H_N})^{-1/2}) \|_{HS} \\
= \| \log((I + B_N)^{-1/2}(I + A_N) (I + B_N)^{-1/2}) \|_F^2.

Combining this with the previous limit gives

\[
\lim_{N \to \infty} \| \log((I + B_N)^{-1/2}(I + A_N) (I + B_N)^{-1/2}) \|_F = \| \log((I + B)^{-1/2}(I + A) (I + B)^{-1/2}) \|_{HS}.
\]

For \( y \in \mathbb{R}, y > 0 \), we note that

\[
\| \log((B + y I)^{-1/2} (A + y I) (B + y I)^{-1/2}) \|_{HS} = \left\| \log \left( \frac{B}{y} + I \right)^{-1/2} \left( \frac{A}{y} + I \right) \left( \frac{B}{y} + I \right)^{-1/2} \right\|_{HS}.
\]

Thus this case reduces to the previous case.

7.7 Further technical results

Theorem 46 Let \( A, B, \{A_n\}_{n \in \mathbb{N}}, \{B_n\}_{n \in \mathbb{N}} \in \text{Sym}(H) \cap \text{HS}(H) \) be such that \( \lim_{n \to \infty} \| A_n - A \|_{HS} = 0, \lim_{n \to \infty} \| B_n - B \|_{HS} = 0 \). Assume that \( \langle I + A \rangle > 0, \langle I + B \rangle > 0, I + A_n > 0, I + B_n > 0 \forall n \in \mathbb{N} \). Then \( (I + B_n)^{-1/2}(I + A_n)(I + B_n)^{-1/2} - I \) are in \( \text{Sym}(H) \cap \text{HS}(H) \).

(i) If \( A, B, A_n, B_n \in \text{Sym}^+(H) \cap \text{HS}(H) \forall n \in \mathbb{N} \), let \( M_{AB} > 0 \) be such that \( \langle x, (I + B)^{-1/2}(I + A)(I + B)^{-1/2} x \rangle \geq M_{AB} \| x \|^2 \forall x \in H \). Then \( \forall \epsilon, 0 < \epsilon < M_{AB}, \exists N_{AB}(\epsilon) \in \mathbb{N} \) such that \( \forall n \geq N_{AB}, \)

\[
\| \log((I + B_n)^{-1/2}(I + A_n)(I + B_n)^{-1/2}) - \log((I + B)^{-1/2}(I + A)(I + B)^{-1/2}) \|_{HS} \leq \frac{1}{M_{AB} - \epsilon} \left( \| A_n - A \|_{HS} + 1 + \| A \|_{HS} \right) \| B_n - B \|_{HS}.
\]

(ii) In general, let \( M_B > 0 \) be such that \( \langle x, (I + B) x \rangle \geq M_B \| x \|^2 \forall x \in H \), then \( \forall \epsilon, 0 < \epsilon < \min(M_B, M_{AB}), \exists N(\epsilon) \) such that \( \forall n \geq N(\epsilon), \)

\[
\| \log((I + B_n)^{-1/2}(I + A_n)(I + B_n)^{-1/2}) - \log((I + B)^{-1/2}(I + A)(I + B)^{-1/2}) \|_{HS} \leq \frac{1}{(M_{AB} - \epsilon)(M_B - \epsilon)} \left( \| A_n - A \|_{HS} + \frac{1}{M_B - \epsilon} \left( 1 + \| A \|_{HS} \right) \| B_n - B \|_{HS} \right).
\]

In both cases, the following convergence holds

\[
\lim_{n \to \infty} \| \log((I + B_n)^{-1/2}(I + A_n)(I + B_n)^{-1/2}) - \log((I + B)^{-1/2}(I + A)(I + B)^{-1/2}) \|_{HS} = 0.
\]
Corollary 47 Let $A, B, \{A_n\}_{n \in \mathbb{N}}, \{B_n\}_{n \in \mathbb{N}} \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H})$ be such that $\lim_{n \to \infty} ||A_n - A||_{\text{HS}} = 0, \lim_{n \to \infty} ||B_n - B||_{\text{HS}} = 0$. Let $\gamma \in \mathbb{R}, \gamma > 0$ be fixed. Then $(\gamma I + B_n)^{-1/2}(\gamma I + A_n)(\gamma I + B_n)^{-1/2} - I, (\gamma I + B)^{-1/2}(\gamma I + A)(\gamma I + B)^{-1/2} - I$ are in $\text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$. Let $M_{AB} > 0$ be such that $(x, (\gamma I + B)^{-1/2}(\gamma I + A)(\gamma I + B)^{-1/2}x) \geq M_{AB}||x||^2 \forall x \in \mathcal{H}$. Then $\forall \epsilon, 0 < \epsilon < M_{AB}, \exists N_{AB}(\epsilon) \in \mathbb{N}$ such that $\forall n \geq N_{AB}$,

\[
||\log[(\gamma I + B_n)^{-1/2}(\gamma I + A_n)(\gamma I + B_n)^{-1/2}] - \log[(\gamma I + B)^{-1/2}(\gamma I + A)(\gamma I + B)^{-1/2}]||_{\text{HS}} \leq \frac{1}{(M_{AB} - \epsilon)\gamma} \left(||A_n - A||_{\text{HS}} + \left(1 + \frac{1}{\gamma}||A||_{\text{HS}}\right)||B_n - B||_{\text{HS}}\right).
\] (122)

Lemma 48 Let $\{A_n\}_{n \in \mathbb{N}}, A \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H})$ such that $(I + A) > 0, (I + A_n) > 0 \forall n \in \mathbb{N}$. Then $A_n(I + A_n)^{-1}$ and $A(I + A)^{-1}$ are in $\text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$.

(i) If $A_n, A \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H}) \forall n \in \mathbb{N}$, then

\[
||A_n(I + A_n)^{-1} - A(I + A)^{-1}||_{\text{HS}} \leq ||A_n - A||_{\text{HS}} \forall n \in \mathbb{N}.
\] (123)

(ii) Assume that $\lim_{n \to \infty} ||A_n - A||_{\text{HS}} = 0$. Let $M_A > 0$ be such that $(x, (I + A)x) \geq M_A||x||^2 \forall x \in \mathcal{H}$, then $\forall \epsilon, 0 < \epsilon < M_A, \exists N(\epsilon) \in \mathbb{N}$ such that

\[
||A_n(I + A_n)^{-1} - A(I + A)^{-1}||_{\text{HS}} \leq \frac{1}{M_A(M_A - \epsilon)} ||A_n - A||_{\text{HS}} \forall n \geq N(\epsilon).
\] (124)

In both cases, $\lim_{n \to \infty} ||A_n - A||_{\text{HS}} = 0$ implies $\lim_{n \to \infty} ||A_n(I + A_n)^{-1} - A(I + A)^{-1}||_{\text{HS}} = 0$.

Proof Since $A_n, A \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$, both $A_n(I + A_n)^{-1}$ and $A(I + A)^{-1}$ are in $\text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H})$. We have

\[
||A_n(I + A_n)^{-1} - A(I + A)^{-1}||_{\text{HS}} = ||(I + A_n)^{-1}A_n - A(I + A)^{-1}||_{\text{HS}}
\]

\[
= ||(I + A_n)^{-1}[A_n(I + A) - (I + A_n)A](I + A)^{-1}||_{\text{HS}}
\]

\[
= ||(I + A_n)^{-1}[A_n - A]A(I + A)^{-1}||_{\text{HS}} \leq ||(I + A_n)^{-1}|| ||A_n - A||_{\text{HS}} ||(I + A)^{-1}||.
\]

(i) If $A_n, A \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H})$, then $||(I + A_n)^{-1}|| \leq 1, ||(I + A)^{-1}|| \leq 1$, so that

\[
||A_n(I + A_n)^{-1} - A(I + A)^{-1}||_{\text{HS}} \leq ||A_n - A||_{\text{HS}}.
\]

(ii) In the general case, by the assumption $\lim_{n \to \infty} ||A_n - A|| = 0$, for any $\epsilon$ satisfying $0 < \epsilon < M_A$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that $||A_n - A|| < \epsilon \forall n \geq N$. Therefore, $I + A_n = I + A + A_n - A \geq (M_A - \epsilon)I \forall n \geq N$. Thus we have $I + A \geq M_A > 0, I + A_n \geq M_A - \epsilon > 0 \forall n \geq N = N(\epsilon)$, from which it follows that

\[
||(I + A_n)^{-1}|| \leq \frac{1}{M_A - \epsilon} \forall N \geq N(\epsilon), \quad \|(I + A)^{-1}|| \leq \frac{1}{M_A}.
\]

Combining this with the first inequality, we have

\[
||A_n(I + A_n)^{-1} - A(I + A)^{-1}||_{\text{HS}} \leq \frac{1}{M_A(M_A - \epsilon)} ||A_n - A||_{\text{HS}} \forall n \geq N,
\]

which implies that $\lim_{n \to \infty} ||A_n(I + A_n)^{-1} - A(I + A)^{-1}||_{\text{HS}} = 0$. 

Proposition 49 Let \( \{A_n\}_{n \in \mathbb{N}}, A, \{B_n\}_{n \in \mathbb{N}} \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H}) \). Assume that \( I + A > 0, I + B > 0, I + A_n > 0, I + B_n > 0 \) \( \forall n \in \mathbb{N} \). Then \( (I + B_n)^{-1/2}(I + A_n)(I + B_n)^{-1/2} = I + (I + A)(I + B)^{-1/2} - I \) and \( (I + B)^{-1/2}(I + A)(I + B)^{-1/2} - I \) are in \( \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H}) \).

(i) If \( A, B, A_n, B_n \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H}) \) \( \forall n \in \mathbb{N} \), then

\[
\| (I + B_n)^{-1/2}(I + A_n)(I + B_n)^{-1/2} - (I + B)^{-1/2}(I + A)(I + B)^{-1/2} \|_{\text{HS}} 
\leq \| A_n - A \|_{\text{HS}} + (1 + \| A \|_{\text{HS}})\| B_n - B \|_{\text{HS}}.
\]

(ii) If \( \lim_{n \to \infty} \| B_n - B \| = 0 \), let \( M_B > 0 \) be such that \( \langle x, (I + B)x \rangle \geq M_B \| x \|^2 \) \( \forall x \in \mathcal{H} \). Then \( \forall \varepsilon > 0, 0 < \varepsilon < M_B \), \( \exists N_B(\varepsilon) \in \mathbb{N} \) such that \( \forall n \geq N_B \),

\[
\| (I + B_n)^{-1/2}(I + A_n)(I + B_n)^{-1/2} - (I + B)^{-1/2}(I + A)(I + B)^{-1/2} \|_{\text{HS}} 
\leq \frac{1}{M_B - \varepsilon} \| A_n - A \|_{\text{HS}} + \frac{1}{M_B - \varepsilon} (1 + \| A \|_{\text{HS}})\| B_n - B \|_{\text{HS}}.
\]

In both cases, \( \lim_{n \to 0} \| A_n - A \|_{\text{HS}} = \lim_{n \to 0} \| B_n - B \|_{\text{HS}} = 0 \) implies

\[
\lim_{n \to 0} \| (I + B_n)^{-1/2}(I + A_n)(I + B_n)^{-1/2} - (I + B)^{-1/2}(I + A)(I + B)^{-1/2} \|_{\text{HS}} = 0.
\]

Proof Let \( C_n = -B_n(I + B_n)^{-1} + (I + B_n)^{-1/2}A_n(I + B_n)^{-1/2}, C = -B(I + B)^{-1} + (I + B)^{-1/2}A(I + B)^{-1/2} \), which are in \( \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H}) \), then

\[
(I + B_n)^{-1/2}(I + A_n)(I + B_n)^{-1/2} = I + C_n, (I + B)^{-1/2}(I + A)(I + B)^{-1/2} = I + C.
\]

(i) If \( A_n, A, B_n, B \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H}) \), then we have by Lemma 48

\[
\| B_n(I + B_n)^{-1} - (I + B)^{-1} \|_{\text{HS}} \leq \| B_n - B \|_{\text{HS}}.
\]

Consider the difference between the second terms of \( C_n \) and \( C \), i.e.

\[
\| (I + B_n)^{-1/2}A_n(I + B_n)^{-1/2} - (I + B)^{-1/2}A(I + B)^{-1/2} \|_{\text{HS}} \leq \| (I + B_n)^{-1/2}A_n(I + B_n)^{-1/2} - (I + B_n)^{-1/2}A(I + B_n)^{-1/2} \|_{\text{HS}}
\]

\[
+ \| (I + B_n)^{-1/2}A(I + B_n)^{-1/2} - (I + B)^{-1/2}A(I + B)^{-1/2} \|_{\text{HS}}
\]

\[
+ \| (I + B_n)^{-1/2}A(I + B)^{-1/2} - (I + B)^{-1/2}A(I + B)^{-1/2} \|_{\text{HS}}.
\]

The first term on the right hand side of the inequality in Eq. (128) is

\[
\| (I + B_n)^{-1/2}(A_n - A)(I + B_n)^{-1/2} \|_{\text{HS}} \leq \| A_n - A \|_{\text{HS}}\| (I + B_n)^{-1/2} \|^2 \leq \| A_n - A \|_{\text{HS}}.
\]

By Corollary 24, since \( \| \cdot \| \leq \| \| \cdot \|_{\text{HS}} \),

\[
\| (I + B_n)^{-1/2} - (I + B)^{-1/2} \| \leq \| (I + B_n)^{-1/2} - (I + B)^{-1/2} \|_{\text{HS}} \leq \frac{1}{2} \| B_n - B \|_{\text{HS}}.
\]
Thus the second term in Eq. (128) satisfies
\[ \| (I + B_n)^{-1/2} A [(I + B_n)^{-1/2} - (I + B)^{-1/2}] \|_{HS} \]
\[ \leq \| (I + B_n)^{-1/2} \|_{HS} \| A \|_{HS} \| (I + B_n)^{-1/2} - (I + B)^{-1/2} \| \leq \frac{1}{2} \| A \|_{HS} \| B_n - B \|_{HS}. \]

Similarly, the third term in Eq. (128) satisfies
\[ \| (I + B_n)^{-1/2} A (I + B)^{-1/2} - (I + B)^{-1/2} A (I + B)^{-1/2} \|_{HS} \]
\[ \leq \| A (I + B)^{-1/2} \|_{HS} \| (I + B_n)^{-1/2} - (I + B)^{-1/2} \| \leq \frac{1}{2} \| A \|_{HS} \| B_n - B \|_{HS}. \]

The inequality in Eq.(125) is obtained by combining all the above inequalities.

(ii) In general, by the assumption \( I + B > 0 \), as in the proof of Lemma 48, since \( \lim_{n \to \infty} \| B_n - B \| = 0 \), for any \( 0 < \varepsilon < M_B \), \( \exists N_B = N_B(\varepsilon) \in \mathbb{N} \), such that \( \| B_n - B \| < \varepsilon \) \( \forall n \geq N_B \) and
\[ I + B_n = I + B + B_n - B \geq (M_B - \varepsilon)I \quad \forall n \geq N_B. \]

By Lemma 48, \( \forall n \geq N_B \),
\[ \| B_n (I + B_n)^{-1} - B (I + B)^{-1} \|_{HS} \leq \frac{1}{M_B (M_B - \varepsilon)} \| B_n - B \|_{HS}. \]

The first term on the right hand side of the inequality in Eq. (128) is
\[ \| (I + B_n)^{-1/2} (A_n - A) (I + B_n)^{-1/2} \|_{HS} \leq \| A_n - A \|_{HS} \| (I + B_n)^{-1/2} \| \leq \frac{\| A_n - A \|_{HS}}{M_B - \varepsilon}. \]

for all \( n \geq N_B \). By Corollary 25, \( \forall n \geq N_B \),
\[ \| (I + B_n)^{-1/2} - (I + B)^{-1/2} \| \leq \frac{1}{2 (M_B - \varepsilon) M_B} \| B_n - B \|. \]

Thus the second term in Eq. (128) satisfies
\[ \| (I + B_n)^{-1/2} A [(I + B_n)^{-1/2} - (I + B)^{-1/2}] \|_{HS} \]
\[ \leq \frac{1}{2 (M_B - \varepsilon)^{3/2} M_B^{1/2}} \| A \|_{HS} \| B_n - B \|_{HS} \quad \forall n \geq N_B. \]

Similarly, for the third term in Eq. (128), we have
\[ \| (I + B_n)^{-1/2} A (I + B)^{-1/2} - (I + B)^{-1/2} A (I + B)^{-1/2} \|_{HS} \]
\[ \leq \| A (I + B)^{-1/2} \|_{HS} \| (I + B_n)^{-1/2} - (I + B)^{-1/2} \| \leq \frac{1}{2 (M_B - \varepsilon) M_B} \| A \|_{HS} \| B_n - B \|_{HS} \quad \forall n \geq N_B. \]
Combining all of the above inequalities, we obtain

\[
\|C_n - C\|_{\text{HS}} \leq \frac{1}{M_B - \epsilon} \|A_n - A\|_{\text{HS}} + \frac{1}{M_B - \epsilon} \left( \frac{1}{M_B} + \frac{\|A\|_{\text{HS}}}{2(M_B - \epsilon)^{1/2} M_B^{1/2}} + \frac{\|A\|_{\text{HS}}}{2M_B} \right) \|B_n - B\|_{\text{HS}}
\]

\[
\leq \frac{1}{M_B - \epsilon} \|A_n - A\|_{\text{HS}} + \frac{1}{M_B - \epsilon} \left( 1 + \|A\|_{\text{HS}} \right) \|B_n - B\|_{\text{HS}}.
\]

\(\forall n \geq N_B\), which is Eq.(126).

**Proof of Theorem 46** As in Proposition 49, we write

\[
(I + B_n)^{-1/2} (I + A_n) (I + B_n)^{-1/2} = I + C_n, \quad (I + B)^{-1/2} (I + A) (I + B)^{-1/2} = I + C,
\]

where \(C_n = -B_n (I + B_n)^{-1} + (I + B_n)^{-1/2} A_n (I + B_n)^{-1/2}, \quad C = -B (I + B)^{-1} + (I + B)^{-1/2} A (I + B)^{-1/2} \in \text{Sym}(\mathcal{H}) \cap \text{HS}(\mathcal{H}).\)

By Proposition 49, since \(\lim_{n \to \infty} \|A_n - A\|_{\text{HS}} = \lim_{n \to \infty} \|B_n - B\|_{\text{HS}} = 0\), we have \(\lim_{n \to \infty} \|C_n - C\|_{\text{HS}} = 0\). Let \(M_{AB} > 0\) be such that \(\langle x, (I + C)x \rangle \geq M_{AB}\|x\|^2\), then by Theorem 2, \(\forall \epsilon, 0 < \epsilon < M_{AB}, \exists \gamma_{AB}(\epsilon) \in \mathbb{N}\) such that \(\|C_n - C\| < \epsilon\) \(\forall n \geq N_{AB}\) and

\[
\|\log(I + C_n) - \log(I + C)\|_{\text{HS}} \leq \frac{1}{M_{AB} - \epsilon} \|C_n - C\|_{\text{HS}} \quad \forall n \geq N_{AB}(\epsilon).
\]

(i) If \(A, B, A_n, B_n \in \text{Sym}^+(\mathcal{H}) \cap \text{HS}(\mathcal{H})\), then by Proposition 49,

\[
\|\log(I + C_n) - \log(I + C)\|_{\text{HS}} \leq \frac{1}{M_{AB} - \epsilon} \left( \|A_n - A\|_{\text{HS}} + (1 + \|A\|_{\text{HS}}) \|B_n - B\|_{\text{HS}} \right) \quad \forall n \geq N_{AB}(\epsilon).
\]

(ii) In the general setting, let \(M_B > 0\) be such that \(\langle x, (I + B)x \rangle \geq M_B\|x\|^2\ \forall x \in \mathcal{H}\). By Proposition 49, \(\forall \epsilon, 0 < \epsilon < M_B, \exists N_B(\epsilon) \in \mathbb{N}\) such that \(\forall n \geq N_B\),

\[
\|C_n - C\|_{\text{HS}} \leq \frac{1}{M_B - \epsilon} \left( \|A_n - A\|_{\text{HS}} + \frac{1}{M_B - \epsilon} \left( 1 + \|A\|_{\text{HS}} \right) \|B_n - B\|_{\text{HS}} \right).
\]

Thus by Theorem 2, \(\forall \epsilon, 0 < \epsilon < \min(M_{AB}, M_B), \forall n \geq \max(N_{AB}, N_B)\),

\[
\|\log(I + C_n) - \log(I + C)\|_{\text{HS}} \leq \frac{1}{(M_{AB} - \epsilon)(M_B - \epsilon)} \left( \|A_n - A\|_{\text{HS}} + \frac{1}{M_B - \epsilon} \left( 1 + \|A\|_{\text{HS}} \right) \|B_n - B\|_{\text{HS}} \right),
\]

In both cases, \(\lim_{n \to \infty} \|\log(I + C_n) - \log(I + C)\|_{\text{HS}} = 0\).

**Proof of Corollary 47** Since \((I + A)^{-1/2}(I + \frac{\epsilon}{\gamma})(I + A)^{-1/2} = (yI + A)^{-1/2}(yI + B)(yI + A)^{-1/2} \geq M_{AB},\) applying
Theorem 46 gives, \( \forall n \geq N_{AB}(\epsilon) \),

\[
\| \log \left[ (I + B_n)^{-1/2} (I + A_n) (I + B_n)^{-1/2} \right] - \log \left[ (I + I + A)^{-1/2} (I + I + B)^{-1/2} \right] \|_{HS} \\
= \left\| \log \left( I + \frac{B_n}{\gamma} \right)^{-1/2} \left( I + \frac{A_n}{\gamma} \right) \left( I + \frac{B_n}{\gamma} \right)^{-1/2} \right\|_{HS} \\
\leq \frac{1}{M_{AB} - \epsilon} \left[ \frac{1}{\gamma} \| A_n - A \|_{HS} + \left( 1 + \frac{1}{\gamma} \| A \|_{HS} \right) \frac{1}{\gamma} \| B_n - B \|_{HS} \right] \\
= \frac{1}{(M_{AB} - \epsilon) \gamma} \left[ \| A_n - A \|_{HS} + \left( 1 + \frac{1}{\gamma} \| A \|_{HS} \right) \| B_n - B \|_{HS} \right].
\]

references

[1] Larotonda G. Nonpositive curvature: A geometrical approach to Hilbert-Schmidt operators. Differential Geometry and its Applications 2007;25:679–700.

[2] Minh HQ, Biagio MS, Murino V. Log-Hilbert-Schmidt metric between positive definite operators on Hilbert spaces. In: Advances in Neural Information Processing Systems 27 (NIPS 2014); 2014. p. 388–396.

[3] Ramsay J, Silverman B. Functional data analysis. Springer; 2005.

[4] Ferraty F, Vieu P. Nonparametric functional data analysis: theory and practice. Springer; 2006.

[5] Horváth L, Kokoszka P. Inference for Functional Data with Applications. Springer; 2012.

[6] Panaretos V, Krauss D, Maddocks J. Second-order comparison of Gaussian random functions and the geometry of DNA minicircles. Journal of the American Statistical Association 2010;105(490):670–682.

[7] Fremdt S, Steinebach J, Horváth L, Kokoszka P. Testing the equality of covariance operators in functional samples. Scandinavian Journal of Statistics 2013;40(1):138–152.

[8] Pigoli D, Aston J, Dryden IL, Secchi P. Distances and inference for covariance operators. Biometrika 2014;101(2):409–422.

[9] Masarotto V, Panaretos VM, Zemel Y. Procrustes Metrics on Covariance Operators and Optimal Transportation of Gaussian Processes. Sankhya A 2018;p. 1–42.

[10] Villani C. Optimal transport: old and new, vol. 338. Springer Science & Business Media; 2008.

[11] Minh HQ. Convergence and finite sample approximations of entropic regularized Wasserstein distances in Gaussian and RKHS settings. arXiv preprint arXiv:210101429 2021.

[12] Minh HQ. Finite sample approximations of exact and entropic Wasserstein distances between covariance operators and Gaussian processes. arXiv preprint arXiv:210412368 2021.

[13] Matthews A, Hensman J, Turner R, Ghahramani Z. On sparse variational methods and the Kullback-Leibler divergence between stochastic processes. In: Artificial Intelligence and Statistics PMLR; 2016. p. 231–239.

[14] Sun S, Zhang G, Shi J, Grosse R. Functional variational Bayesian neural networks. International Conference on Learning Representation 2019.

[15] Hà Quang M. Riemannian Distances between Covariance Operators and Gaussian Processes. In: International Workshop on Functional and Operatorial Statistics Springer; 2020. p. 177–185.

[16] Sun H. Mercer theorem for RKHS on noncompact sets. Journal of Complexity 2005;21(3):337–349.
Rosasco L, Belkin M, Vito ED. On Learning with Integral Operators. Journal of Machine Learning Research 2010;11(30):905–934.

Cucker F, Smale S. On the Mathematical Foundations of Learning. Bulletin of the American Mathematical Society 2002 January;39(1):1–49.

Rajput B, Cambanis S. Gaussian processes and Gaussian measures. The Annals of Mathematical Statistics 1972;p. 1944–1952.

Masarotto V, Panaretos V, Zemel Y. Procrustes metrics on covariance operators and optimal transportation of Gaussian processes. Sankhya A 2019;81(1):172–213.

Dryden IL, Koloydenko A, Zhou D. Non-Euclidean Statistics for Covariance Matrices, with Applications to Diffusion Tensor Imaging. Annals of Applied Statistics 2009;3:1102–1123.

Dowson DC, Landau BV. The Fréchet distance between multivariate normal distributions. Journal of Multivariate Analysis 1982;12(3):450 – 455.

Givens C, Shortt R. A class of Wasserstein metrics for probability distributions. Michigan Math J 1984;31(2):231–240.

Gelbrich M. On a formula for the L2 Wasserstein metric between measures on Euclidean and Hilbert spaces. Mathematische Nachrichten 1990;147(1):185–203.

Bhatia R, Jain T, Lim Y. On the Bures–Wasserstein distance between positive definite matrices. Expositiones Mathematicae 2018.;

Malagò L, Montrucchio L, Pistone G. Wasserstein Riemannian geometry of Gaussian densities. Information Geometry 2018 Dec;1(2):137–179.

Pennec X, Fillard P, Ayache N. A Riemannian Framework for Tensor Computing. International Journal of Computer Vision 2006;66(1):41–66.

Bhatia R. Positive Definite Matrices. Princeton University Press; 2007.

Arsigny V, Fillard P, Pennec X, Ayache N. Geometric means in a novel vector space structure on symmetric positive-definite matrices. SIAM J on Matrix An and App 2007;29(1):328–347.

Takatsu A. Wasserstein geometry of Gaussian measures. Osaka Journal of Mathematics 2011;48(4):1005–1026.

Minh HQ. A unified formulation for the Bures-Wasserstein and Log-Euclidean/Log-Hilbert-Schmidt distances between positive definite operators. In: International Conference on Geometric Science of Information Springer; 2019.;

Minh HQ. Alpha Procrustes metrics between positive definite operators: a unifying formulation for the Bures-Wasserstein and Log-Euclidean/Log-Hilbert-Schmidt metrics. arXiv preprint arXiv:190809275 2019.;

Kadison RV, Ringrose JR. Fundamentals of the theory of operator algebras. Volume I: Elementary Theory. Academic Press; 1983.

Petryshyn WV. Direct and iterative methods for the solution of linear operator equations in Hilbert spaces. Transactions of the American Mathematical Society 1962;105:136–175.

Minh HQ. Affine-invariant Riemannian distance between infinite-dimensional covariance operators. In: International Conference on Geometric Science of Information; 2015.;

Minh H. Infinite-dimensional Log-Determinant divergences between positive definite trace class operators. Linear Algebra and Its Applications 2017;528:331–383.
[37] Minh HQ. Infinite-dimensional Log-Determinant divergences between positive definite Hilbert-Schmidt operators. Positivity 2020;24:631–662.

[38] Minh HQ. Alpha-Beta Log-Determinant Divergences Between Positive Definite Trace Class Operators. Information Geometry 2019 December;2(2):101–176.

[39] Minh HQ. Entropic regularization of Wasserstein distance between infinite-dimensional Gaussian measures and Gaussian processes. preprint arXiv:201107489 2020;.

[40] Smale S, Zhou DX. Learning Theory Estimates via Integral Operators and Their Approximations. Constructive Approximation 2007;26:153–172.

[41] Harandi M, Salzmann M, Porikli F. Bregman Divergences for Infinite Dimensional Covariance Matrices. In: IEEE Conference on Computer Vision and Pattern Recognition (CVPR); 2014. .

[42] Steinwart I, Christmann A. Support vector machines. Springer Science & Business Media; 2008.

[43] Minh HQ. Alpha-Beta Log-Determinant divergences between positive definite trace class operators. Information Geometry 2019;2(2):101–176.

[44] Kittaneh F, Kosaki H. Inequalities for the Schatten p-norm V. Publications of the Research Institute for Mathematical Sciences 1987;23(2):433–443.

[45] Minh HQ. Infinite-Dimensional Log-Determinant Divergences III: Log-Euclidean and Log-Hilbert–Schmidt Divergences. In: Information Geometry and its Applications IV Springer; 2018. p. 209–243.

[46] Dunford N, Schwartz JT. Linear operators, part 1: general theory, vol. 10. John Wiley & Sons; 1988.

[47] Reed M, Simon B. Methods of Modern Mathematical Physics: Functional analysis. Academic Press; 1975.