Spiral spin textures of bosonic Mott insulator with SU(3) spin-orbit coupling

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We study the Mott phase of three-component bosons, with one particle per site, in an optical lattice by mapping it onto an SU(3) spin model. In the simplest case of full SU(3) symmetry, one obtains a ferromagnetic Heisenberg model. Introducing an SU(3) analog of spin-orbit coupling, additional spin-spin interactions are generated. We first consider the scenario of spin-dependent hopping phases, leading to Dzyaloshinskii-Moriya-type interactions. They result in the formation of spiral spin textures, which in one dimension can be understood by a local unitary transformation.

Applying classical Monte Carlo simulations, we extend our study to two-dimensional systems, and systems with "true" spin-orbit coupling, involving spin-changing hoppings.

I. INTRODUCTION

Quantum many-body physics is to a large extent the physics of interacting spins. Literally, the spin is an intrinsic property of particles, formally described by a representation of the SU(2) group. For example, the fundamental representation of SU(2), expressed in terms of the Pauli matrices, describes the 1/2-spin, as present, for instance, in electrons. It requires only a small amount of abstraction to extend the meaning of spin to an arbitrary internal degree of freedom carried by the particles. Such an extended spin picture is very common in quantum engineering, where it may denote the different states in which cold atoms or trapped ions are prepared. While these states may in principal belong to one hyperfine manifold of the atoms, such that the true atomic spin would distinguish between the states, this is not necessarily the case. Then the different states form a so-called pseudospin manifold, which, in contrast to real spin, is independent from the quantum-statistical properties of the particles\textsuperscript{1}. In an even wider sense, one might also map external degrees of freedom onto a spin system. For instance, empty or occupied sites on a lattice could be seen as a spin-1/2 pointing up or down. This is what occurs in the hard boson limit of the Bose Hubbard model (cf. \textsuperscript{2}). The most common way to derive spin models from Hubbard models is to consider the limit of strong interactions, and derive an effective Hamiltonian, describing super-exchange interactions\textsuperscript{3,4}.

What is common to these different spin pictures, is its formal aspect: the operators describing the degree of freedom under scrutiny belong to some representation of the SU(2) group. In particular, for the case of pseudospin, one typically has a two-level system. In this case, indeed any local unitary operation on the spin degree of freedom can be expressed through a single Pauli matrix. However, if the (pseudo)spin degree of freedom involves more than two states, the spin picture might become cumbersome. As illustrated in Fig.\textsuperscript{1}, for a three-level system, the transition from state $|+\rangle$ to state $|\rangle$ cannot be described in terms of a single SU(2) spin matrix. As SU(2) has only three generators, there can only be one spin-raising operator, say $S^+$, which raises state $|\rangle$ to $|0\rangle$, and $|0\rangle$ to $|+\rangle$, but a direct link from $|\rangle$ to $|+\rangle$ is missing. To connect these two states, one has to apply twice the raising operator $S^+$. It may then be more convenient to turn to a new spin, or better “isospin”, picture, in which all operators acting on the degree of freedom span an SU(3) group. Since this group provides three different raising operators, any unitary operation on the three-level Hilbert space can then be expressed by a single SU(3) spin matrix, see Fig.\textsuperscript{1}. This demonstrates that SU(3) spins may play an important role in systems with three-level constituents ranging from high-energy systems of elementary particles with their intrinsic three-fold degrees of freedom like flavor or color\textsuperscript{5,6}, to condensed matter systems with a three-dimensional pseudospin manifold, or quantum optical systems of three level atoms. In these contexts SU(3) spin models and their ground states have recently been investigated\textsuperscript{7,8}. Quantum-chaotic behavior in a SU(3) spin system built of trapped ions has been studied in Ref.\textsuperscript{9}.

In this paper, we focus on multi-component quantum gases, which have recently attracted a lot of attention in the wide field of quantum simulation and quantum
control [11]. In particular, we study those models which arise from a three-fold pseudospin degree of freedom combined with variants of spin-orbit coupling. The latter is the important mechanism, which connects external and internal degrees of freedom of the particles. We stress that in Nature, spin-orbit coupling exists due to the electric charges of the particles and the magnetic moments associated with the spins. In atomic gases, spin-orbit coupling has been investigated in Ref. [32] in the lattice, one makes the Peierls substitution which can be minimally coupled to the momentum. In Mott type [29, 30], this gauge potential mostly via superexchange terms of the Dzyaloshinskii-Moriya type [24–28], that this leads to a biquadratic SU(2) Heisenberg model [35, 36].

II. HEISENBERG MODEL FOR A MOTT PHASE OF THREE-COMPONENT BOSONS

Three-level bosons have attracted a lot of attention already since the early days of Bose-Einstein condensation [33, 34], as it occurs quite frequently that bosonic atoms carry the spin one. Let us therefore start from this most familiar case of spinor gases, and view it from a generalized SU(3) point of view. This prepares for a convenient description of pseudospin gases. Finally, we will turn our attention to SU(3) models with Dzyaloshinskii-Moriya-type interactions, relevant for systems with (pseudo)spin-dependent hopping.

A. Effective Mott Hamiltonian for spinor bosons

In spinor gases, the interaction is characterized by two parameters $U_0$ and $U_2$. Discretized on a lattice, the interaction Hamiltonian reads:

$$H_{\text{int}}^\text{spinor} = \frac{U_0}{2} \sum_i n_i (n_i - 1) + \frac{U_2}{2} \sum_i \left( S_i^2 - 2n_i \right), \quad (1)$$

where $n_i$ is the occupation number on site $i$, and $S_i$ the (total) spin operator acting on the particle on site $i$.

The kinetic energy of bosons in a standard optical lattice is well described by nearest-neighbor tunneling,

$$H_{\text{standard}}^{\text{kin}} = -t \sum_{\langle i, j \rangle} \sum_{\sigma \in \{+,0,-\}} (a_i^\dagger a_j + H.c), \quad (2)$$

with $t$ the tunneling amplitude. For $t \ll U_0, U_2$, the system is in the Mott phase, and insight in its physics can be gained by taking the kinetic part as a second-order perturbation to the interactions. It has been shown that this leads to a biquadratic SU(2) Heisenberg model [35, 36].

$$H_{\text{SU(2)}} = -J \sum_{\langle i,j \rangle} \left[ \cos \theta (S_i \cdot S_j) + \sin \theta (S_i \cdot S_j)^2 \right], \quad (3)$$

with the parameters $J$ and $\theta$ depending on $t, U_0$, and $U_2$. This effective Hamiltonian exhibits spin-nematic and ferromagnetic phases, a prediction which recently has been confirmed by a full quantum Monte-Carlo study of the problem [37].

As argued in the introduction, the biquadratic SU(2) Hamiltonian [35, 36] can be rewritten in terms of SU(3) spin matrices, which reduces it to a quadratic form. While the most general expression is provided in the appendix, a particularly simple special case is the one of spin-independent interactions, $U_2 = 0$, corresponding to $\theta = \pi/4$. The SU(3) symmetry of the system then leads to an effective SU(3) Heisenberg Hamiltonian

$$H_{\text{SU(3)}} = -\frac{t^2}{2U} \sum_{\langle i,j \rangle} \sum_{\nu,\mu} \lambda^{(\nu)} \cdot \lambda^{(\mu)} = -\frac{J}{2\sqrt{2}} \sum_{\langle i,j \rangle} \lambda^T_i \cdot \lambda_j, \quad (4)$$
where we have inserted \( J = -\sqrt{2}t^2/U \). The \( \lambda^{(\nu)} \) are the eight generators of SU(3), which, for convenience, have been arranged together to a eight-component vector \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(8)})^T \) on the right side of Eq. (1). The form of Eq. (1) holds for any filling factor if the \( \lambda^{(\nu)} \) are chosen a symmetric SU(3) representation, corresponding to the desired particle number. An interesting case is the filling with three bosons per site: The low-energy Hilbert space is then described by the 10 representation of SU(3), which is implemented in Nature as the baryon decuplet.

In the following, however, we restrict ourselves to the simpler case of one boson per site. The low-energy Hilbert space then is three-dimensional, and one works in the fundamental representation of SU(3). Accordingly, we associate the eight Gell-Mann matrices with the \( \lambda^{(\nu)} \), cf. Ref. [3]. Note that six of these matrices are related to the raising/lowering operators of Fig. 1b: \( (\lambda^{(1)} \pm i\lambda^{(2)}) = 2T^\pm, \ (\lambda^{(4)} \pm i\lambda^{(5)}) = 2V^\pm, \) and \( (\lambda^{(6)} \pm i\lambda^{(7)}) = 2U^\pm. \) The other two matrices are diagonal: \( \lambda^{(3)} = \text{diag}(1, -1, 0), \) and \( \lambda^{(8)} = \frac{1}{\sqrt{3}} \text{diag}(1, 1, -2). \)

B. Effective Mott Hamiltonian for bosons with SU(3) pseudospin

The biquadratic form of Eq. (2) holds due to the invariance of \( H = H^\text{kin} + H^\text{int} \) under spin rotations. If, instead of considering Bose gases with an intrinsic spin, one switches to pseudospin-1 systems, that is, a trinary mixture of bosons, this symmetry is likely to be broken. In particular, one might not expect interactions as described by Eq. (1). Neglecting spin-changing collisions, the most general two-body contact interaction would be given in terms of six interaction parameters \( U_{\sigma\sigma'} \):

\[
H^\text{int}_\text{pseudospin} = U_{\sigma\sigma'} \sum_i |\sigma\sigma'_i\rangle \langle \sigma\sigma'_i|,
\]

for pairs of particles in state \( |\sigma\rangle \) and \( |\sigma'\rangle \) interacting on site \( i \). The effective Mott Hamiltonian to \( H = H^\text{kin} + H^\text{int}_\text{pseudospin} \) is of the form

\[
H_{\text{eff}} = -\sum_{\langle i,j \rangle} \sum_{\nu,\lambda} J_{\nu} \lambda^{(\nu)} \lambda^{(\lambda)} + J_{38} \lambda^{(3)} \lambda^{(8)} + \lambda^{(8)} \lambda^{(3)} + \sum_i \left[ h_3 \lambda^{(3)}_i + h_8 \lambda^{(8)}_i \right].
\]

Now the interaction parameters \( J_{\nu} \) are spin-dependent, there is an additional interaction \( J_{38} \), and magnetic-field-like terms \( h_3 \) and \( h_8 \). All parameters are given in appendix A2.

C. Effective Mott Hamiltonian in the case of spin-dependent tunneling phases and generalized Dzyaloshinskii-Moriya interactions

We now turn our attention to systems where the standard hopping is replaced by a spin-orbit coupled kinetic term. Recently, spin-orbit coupling in two-component Bose gases has been shown to give rise to effective Dzyaloshinskii-Moriya interactions in the low energy description of the Mott insulator phase [24,27], that is an antisymmetric spin-spin interaction of the form \( \mathbf{D} \cdot (\mathbf{S}_i \times \mathbf{S}_j) \), characterized by the Dzyaloshinskii-Moriya vector \( \mathbf{D} \). The presence of such interaction leads to a variety of different spin textures in the ground state.

In this section, we derive an effective Hamiltonian for three-component bosons, which exhibits Dzyaloshinskii-Moriya type interactions generalized to SU(3) spins. For this, it is sufficient to consider pseudospin-dependent tunneling phases. If these phases were spatially dependent, this would mimic a magnetic field acting differently on the three components, corresponding to different electric charges of the pseudospin states. Note that such a situation occurs naturally in the quark model, where the up quark has an electric charge of \( 2/3e \), while down quark and strange quark have a charge of \( -e/3 \).

We start our analysis by writing down the general form of the kinetic term in the presence of a gauge potential,

\[
H_{\text{SOC}}^\text{kin} = -t \sum_i \sum_{\nu \in \{\epsilon_x, \epsilon_y, \epsilon_z\}} a_i^\dagger e^{-i\mathbf{A}_i} a_i + H.c.,
\]

where we have introduced a vector notation for the three bosonic components \( a_i^\dagger = (a_{i,\epsilon_x}^\dagger, a_{i,\epsilon_y}^\dagger, a_{i,\epsilon_z}^\dagger) \). Spin-orbit coupling is obtained by choosing the gauge potential to be sensitive to the (pseudo)spin. We have assumed a two-dimensional square lattice in the notation above. A one-dimensional system is obtained by freezing out the hopping between adjacent one-dimensional chains.

Except in Sec. IV, we shall consider the simplest choice for the gauge potential corresponding to diagonal \( A_x \) and \( A_y \). In the generic 2-dimensional case we choose standard gauge-free hopping in one direction, say \( y \), i.e. \( A_y = 1 \). The non-trivial gauge choice is made in the \( x \)-direction (also chosen as the chain direction for 1-d lattices) with \( A_x = \text{diag}(\alpha, \beta, \gamma) \). In the effective Mott Hamiltonian, this will lead to modified spin interactions along the \( x \)-direction, as the superexchange terms gain a phase factor.

Let us, in the first place, assume the important special case of SU(3)-symmetric interactions of strength \( U \). The hopping in \( x \)-direction then yields the effective Hamiltonian:

\[
h_x = -\frac{\gamma^2}{2U} \sum_i \sum_{\nu=1}^8 J_{\nu} \lambda^{(\nu)}_i \lambda^{(\nu)}_{i+1} + \sum_{\nu,\mu} V_{\nu\mu} \left( \lambda^{(\nu)}_i \lambda^{(\mu)}_{i+1} - \lambda^{(\nu)}_i \lambda^{(\mu)}_{i+2} \right),
\]

where we have inserted \( \lambda^{(\nu)}_i \) and \( \lambda^{(\nu)}_{i+1} \) into the effective Hamiltonian for bosons with SU(3) pseudospin.
with
\[ J_1 = J_2 = \cos(\alpha - \beta), \quad J_4 = J_5 = \cos(\alpha - \gamma), \]
\[ J_6 = J_7 = \cos(\beta - \gamma), \quad J_3 = J_8 = 1, \]
\[ V_{12} = \sin(\alpha - \beta), \quad V_{45} = \sin(\alpha - \gamma), \quad V_{67} = \sin(\beta - \gamma). \]

All other \( V_{\mu\nu} \) are zero. Apparently, the second term in Eq. (8) describes an antisymmetric spin-spin interaction. To make the analogy to the Dzyaloshinskii-Moriya interaction as close as possible, we introduce a “vector interaction” as close as possible, we introduce a “vector interaction”

\[ \mathbf{u} \times \mathbf{v} \equiv f^{ijk} u_i v_j \hat{e}_k, \]

with \( f^{ijk} \) the antisymmetric structure constants of SU(3), cf. Eq. (8). If \( \sin(\alpha - \beta) = \sin(\alpha - \gamma) - \sin(\beta - \gamma) \), the effective Hamiltonian \( h_x \) can be written as

\[ h_x = -\frac{t^2}{2U} \left( \sum_{\mu=1}^{N} \left( \sum_{\nu=1}^{N} J_{\mu\nu} \lambda_{\mu}^{(\nu)} \lambda_{\nu+1}^{(\mu)} + D \cdot (\lambda_\mu \times \lambda_{\mu+1}) \right) \right), \]

with the \( J \)'s given in Eq. (8), and the non-zero entries of the Dzyaloshinskii-Moriya vector given by

\[ D_3 = \sin(\alpha - \beta), \quad D_8 = \frac{1}{\sqrt{3}} [\sin(\alpha - \gamma) + \sin(\beta - \gamma)]. \]

In the discussion below, we consider Hamiltonian’s of the type given in Eq. (11), determined by the parameter \( \alpha \), with the choice \( \gamma = \alpha, \beta = 0 \).

As will be discussed in greater detail below, the Dzyaloshinskii-Moriya interaction does not fully lift the huge degeneracy of the ferromagnetic SU(3) Heisenberg model. One way to obtain a unique ground state is to break the SU(3) symmetry of the interactions. For instance, by strengthening interactions between identical particles, \( U \), relative to interactions between particles of different (pseudo)spin, \( U_{\text{inter}} \), one might force the system into a fully unpolarized state, with equal occupation numbers in all spin states, \( N_+ = N_0 = N_- \). The effective spin Hamiltonian resulting from these interactions is given again by Eq. (8), but now with the parameters

\[ J_1 = J_2 = \frac{1}{\mu} \cos \alpha, \quad J_4 = J_5 = \frac{1}{\mu}, \]
\[ J_6 = J_7 = \frac{1}{\mu} \cos \alpha, \quad J_3 = J_8 = \frac{2\mu - 1}{\mu}, \]
\[ V_{12} = \frac{1}{\mu} \sin \alpha, \quad V_{45} = 0, \quad V_{67} = -\frac{1}{\mu} \sin \alpha, \]

where \( \mu \equiv U_{\text{inter}}/U \) is the ratio between the two interaction strengths.

### III. SPIRAL ORDER PHASES

For simplicity, we consider first spin models stemming from SU(3) symmetric interactions, Eq. (8), with parameters from Eq. (9), for discussing the effect of the Dzyaloshinskii-Moriya term. We shall distinguish between open and closed 1D systems, and 2D systems.

#### A. Effect of the Dzyaloshinskii-Moriya term in 1D with open boundary

Freezing the dynamics in the \( y \)-direction, it is possible to solve the problem analytically under open boundary conditions. It is convenient to introduce a spin rotation matrix \( U_\alpha \):

\[ U_\alpha = \begin{pmatrix} e^{-i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\alpha} \end{pmatrix}, \]

acting on site \( i \). For a chain of \( N \) spins, labeled \( 1, \ldots, N \) we introduce

\[ \mathbf{U} \equiv \otimes_{k=1}^{N} U_k [(N - k)\alpha], \]

that is, each pair of neighboring spins is rotated relative to each other by an angle \( \alpha \). The Hamiltonian \( h_x \) from Eq. (8) can then be written as

\[ h_x = -\frac{t^2}{2U} U^\dagger \left( \sum_{i=0}^{N-2} \lambda_i^T \lambda_{i+1} \right) U. \]

This means that \( h_x \) can be mapped onto the ferromagnetic SU(3) Heisenberg model via a unitary transformation, and both Hamiltonians have the same energy spectrum. In particular, the ground state degeneracy of the ferromagnetic Heisenberg Hamiltonian is not lifted by the Dzyaloshinskii-Moriya term.

However this solution allows one to gain insight into the magnetic correlations in the SU(3) system by mapping the ground states of ferromagnetic model to the Dzyaloshinskii-Moriya Hamiltonian. The ground state manifold of the SU(3) symmetric Heisenberg ferromagnet corresponds to the maximal symmetric representation of SU(3) on \( N \) sites and is therefore spanned by the symmetric Dicke states, i.e., the symmetric superposition of states with \( N_+ \) particles in \( |+\rangle \), \( N_0 \) particles in \( |0\rangle \), and \( N_- \) particles in \( |-\rangle \), where \( N_+ + N_0 + N_- = N \). Therefore, the number of degenerate ground states is given by

\[ d = \sum_{N_+}^{N} \sum_{N_0=0}^{N-N_+} \frac{1}{2}(N + 1)(N + 2). \]

In the presence of the Dzyaloshinskii-Moriya interactions, each Fock state contributing to an eigenstate gains a phase through the unitary transformation, Eq. (15). To illustrate this phase acquisition, let us first consider a chain of \( N = 3 \) spins. This yields \( d = 10 \) ground states. Three of them are given in terms of a single Fock state, namely the states \( |++\rangle \equiv |+\rangle_i=1 \otimes |+\rangle_{i=2} \otimes |+\rangle_{i=3}, (000), \) and \( |--\rangle \). These states are, apart from an irrelevant overall phase, invariant under \( \mathbf{U} \). The remaining seven ground states are transformed by the Dzyaloshinskii-Moriya interactions. Let us, as an important example, pick out the fully unpolarized state...
with one + particle, one − particle, and one 0 particle. This is the unique ground state of the isotropic ferromagnetic SU(3) model in a subspace of the Hilbert space with \(N_+ = N_- = N_0\). Note that these numbers are constants of motion.

For \(N = 3\), the fully unpolarized Dicke state reads

\[
\Psi_{+0-} = \frac{1}{\sqrt{6}} (|+0-\rangle + |+ -0\rangle + |0 + -\rangle + |0 -+\rangle + |-+0\rangle + |-0+\rangle).
\]

The ground state of \(h_x\) in the fully unpolarized sector reads

\[
\tilde{\Psi}_{+0-} = U_x^\dagger \Psi_{+0-} = \frac{1}{\sqrt{6}} (e^{2i\alpha} |+0-\rangle + e^{3i\alpha} |+ -0\rangle + e^{i\alpha} |0 + -\rangle + e^{i\alpha} |0 -+\rangle + e^{3i\alpha} |-+0\rangle + e^{2i\alpha} |-0+\rangle).
\]

This is easily generalized to any \(N\) being a multiple of 3. We note here that breaking the SU(3) symmetry of the interactions with \(\mu < 1\), as in Eq. (13) (and \(\alpha = 0\)), yields a unique ground state originating from the unpolarized Dicke state. The overlap of this ground state with the unpolarized Dicke state is large for the system sizes for which we have performed exact diagonalization, e.g. for \(\mu = 0.75, N = 9\); it is approximately 0.95. The effect of the phases \(\alpha \neq 0\) can again be taken into account by the transformation Eq. (15), which at any \(\alpha\) produces the correct ground state with a fidelity \(> 0.99\) for \(\mu = 0.75, N = 9\). Therefore the discussion below holds to a good approximation also for the case when the Dzyaloshinskii-Moriya type Hamiltonian ground state is unique.

It can easily be seen that due to strong entanglement in this state, the single site density matrices are all proportional to the identity and therefore the spin (traceless Gellmann matrices) averages to zero, \(\langle \lambda^{(i)} \rangle = 0\), for arbitrary \(N\). By combinatoric analysis of the Dicke state, the spin-spin correlations can be evaluated and they show an interesting behavior as a function of \(\alpha\):

\[
\langle \lambda^{(1)} \lambda^{(1)} \rangle + \lambda^{(2)} \lambda^{(2)} \rangle = \frac{4N}{9(N-1)} \cos(\pi\text{N}),
\]

\[
\langle \lambda^{(4)} \lambda^{(4)} \rangle + \lambda^{(5)} \lambda^{(5)} \rangle = \frac{4N}{9(N-1)},
\]

\[
\langle \lambda^{(6)} \lambda^{(6)} \rangle + \lambda^{(7)} \lambda^{(7)} \rangle = \frac{4N}{9(N-1)} \cos(\pi\text{N}),
\]

\[
\langle \lambda^{(3)} \lambda^{(3)} \rangle = 0,
\]

\[
\langle \lambda^{(9)} \lambda^{(9)} \rangle = 0.
\]

The spin structure can thus be viewed as spirals in the \(\lambda^{(1)} - \lambda^{(2)}\) and the \(\lambda^{(6)} - \lambda^{(7)}\) plane, and a ferromagnet in the \(\lambda^{(4)} - \lambda^{(5)}\) plane.

We note that spiral order has also been found in SU(2) systems with spin-orbit coupling [24, 28], but the coexistence of ferromagnetic and spiral order is a feature only found in the SU(3) system. Moreover, the phase angle \(\alpha\) can be used to freely tune the periodicity of the spirals. In contrast to the systems in Refs. [24, 28], we have not assumed off-diagonal spin-orbit coupling, easing the experimental realization of the system studied here.

**B. Effect of the Dzyaloshinskii-Moriya term with periodic boundary conditions**

The unitary transformation that gauges away the Dzyaloshinskii-Moriya term in the Hamiltonian in the previous section fails, in general, to achieve this completely in the case of a periodically closed boundary. Indeed, the transformation gauges away the Dzyaloshinskii-Moriya term in the entire bulk of the lattice leaving only an uncompensated, frustrated boundary term. The reason for this is the phase relation between the first and the last spin demanded by the periodic boundary condition. Following the logic of the previous section, fixing the phase transformations on the first and last spins results in a phase mismatch of \(N\alpha\) on the bond connecting these spins.

As such the magnetic ordering is therefore expected to be qualitatively very similar to that described in the last section. This is exactly true when the unitary transformation has no conflict with the periodic boundary as in the special cases when \(0 = \text{mod}[N\alpha, 2\pi]\), *i.e.*, for \(\alpha\) being an integer multiple of \(2\pi/N\). Interestingly, our results from exact diagonalization suggest, the ground state for arbitrary Hamiltonian parameter \(\alpha\) is well described by a unitary transformation, with input parameter chosen from the discrete set \(\tilde{\alpha} = m2\pi/N\). The integer \(m\) is a kind of a “winding number” counting the number of times a spin spiral wraps around the axis throughout the system. It depends on \(\alpha\) through a devil staircase-like function:

\[
m = j \quad \text{for } j \in \mathbb{Z}: \frac{2\pi}{N} j \leq \alpha < \frac{2\pi}{N} (j + 1).
\]

We emphasize here, that this unitary transformation whilst providing a description of the ground state, for arbitrary \(\alpha\), does not unitarily connect between the Heisenberg Hamiltonian and the Dzyaloshinskii-Moriya-type
Hamiltonians. This reflects in an energy spectrum depending on $\alpha$.

Due to the discreteness of the input parameters to the unitary transformation, the system with periodic boundary conditions exhibits clearly separated phases for finite systems. It is worth to notice that for $m = 2\pi/N_J$, that is in the vicinity of $\alpha = \pi/2$, the spiral order becomes antiferromagnetic.

C. Effect of the Dzyaloshinskii-Moriya term in 2D

We now turn to a two-dimensional system, under periodic boundary conditions, with spin-orbit coupling in one direction, and standard hopping in the other direction. Again, we are not able to map the problem onto the Heisenberg model by means of the unitary transformation. In order to achieve this, the 8-dimensional on-site vector $\vec{\lambda} = \lambda_1^x, \lambda_2^x, \lambda_1^y, \lambda_2^y, \lambda_3^x, \lambda_4^x, \lambda_3^y, \lambda_4^y$ can be seen as a generalized Bloch vector, where the prefactor is needed to attain unit normalization. It is important to note a crucial difference w.r.t. the well-known SU(2) case: A pure state of a two-level system is characterized by two angles, a mixing angle, and a relative phase, which perfectly maps onto a Bloch vector, where the prefactor is needed to attain unit normalization. Again, we are not able to map the problem onto the Heisenberg model by means of the unitary transformation. In order to achieve this, the 8-dimensional vector $\vec{\lambda}$ can be seen as a generalized Bloch vector, where the prefactor is needed to attain unit normalization.

We obtain here the classical ground state phase of the 2-D model. In order to achieve this, the 8-dimensional SU(3) spins are treated classically by replacing them with 2-D spins. In order to achieve this, the 8-dimensional SU(3) spins are treated classically by replacing them with 2-D spins. In order to achieve this, the 8-dimensional SU(3) spins are treated classically by replacing them with 2-D spins. In order to achieve this, the 8-dimensional SU(3) spins are treated classically by replacing them with 2-D spins. In order to achieve this, the 8-dimensional SU(3) spins are treated classically by replacing them with 2-D spins. In order to achieve this, the 8-dimensional SU(3) spins are treated classically by replacing them with 2-D spins.

In the classical limit, the spin-spin Hamiltonian provides an energy functional which is numerically minimized by simulated annealing. The latter is implemented by starting from an arbitrary initial configuration, which is locally updated through a Metropolis algorithm. We have repeatedly carried out the simulated annealing for a lattice with $30 \times 30$ spins.

For the case of SU(3)-symmetric interaction, Eq. (15), the simulated annealing always led to a ferromagnet fully polarized within the $\lambda_3^x - \lambda_8^x$ plane. This can be understood by noting that on the classical level the Dzyaloshinskii-Moriya interactions do not affect the part of the system which is polarized in the $\lambda_3^x - \lambda_8^x$ plane, in which the energy per spin may remains minimal by forming a ferromagnet.

To force the system into a less trivial configuration than the ferromagnetic one, we have to make the polarization in the $\lambda_3^x - \lambda_8^x$ plane less favorable. This can again be achieved by tuning pseudospin-dependent interactions away from the SU(3)-symmetry point, as described in Eq. (13). For $\mu < 1$, polarization within the $\lambda_3^x - \lambda_8^x$ plane becomes energetically less favorable, and the effect of the Dzyaloshinskii-Moriya interactions is expected to be enhanced.

Accordingly, our simulated annealing lattice with pseudospin interactions characterized by $\mu = 0.75$ yields interesting spin patterns: As shown in Fig. 2 there is spiral order along the $x$-axis, and ferromagnetic order along the $y$-axis with the standard hopping. The spiral ordering occurs only in the 1-, 2-, 6-, and 7-components of $\vec{\lambda}$, that is, in those components directly affected by the Dzyaloshinskii-Moriya term. As expected, the 3- and 8-components of $\vec{\lambda}$ are vanishing small, and the 4- and 5-components of $\vec{\lambda}$, unaffected by the Dzyaloshinskii-Moriya interaction, show ferromagnetic order.

The periodicity of these textures along the $x$-axis depends on the hopping phase $\alpha$, in exactly the same way as in the 1D system with periodic boundary conditions. Indeed, the spiral order found along $x$ is precisely the classical version of the quantum solution we obtained in 1D. To see this, we evaluate spin-spin correlations in the classical solution. The site-dependent parametrization of $\vec{\lambda}$ that is in agreement with the results of simulated annealing, is given by:

$$\vec{\lambda}_{x,y} = \frac{1}{\sqrt{3}} (\cos(x\bar{\alpha}), \sin(x\bar{\alpha}), 0, 1, 0, \cos(x\bar{\alpha}), -\sin(x\bar{\alpha})),$$

(22)

where $\bar{\alpha}$ is an integer multiple of $2\pi/L_x$, with $L_x$ the size of the lattice along the $x$-direction. The value of $\bar{\alpha}$ depends on $\alpha$ according to the devil staircase function (20) described in the previous section. Using this, the correlations are:

$$\lambda_i^{(1)} \lambda_i^{(1)} + \lambda_i^{(2)} \lambda_i^{(2)} = \frac{1}{3} \cos(r\alpha),$$

(23)

$$\lambda_i^{(4)} \lambda_i^{(4)} + \lambda_i^{(5)} \lambda_i^{(5)} = \frac{1}{3} \cos(r\alpha),$$

$$\lambda_i^{(6)} \lambda_i^{(6)} + \lambda_i^{(7)} \lambda_i^{(7)} = \frac{1}{3} \cos(r\alpha),$$

$$\lambda_i^{(8)} \lambda_i^{(8)} = 0.$$
SU(3) planes:
\[ \lambda_1 - \lambda_2 \]
\[ \lambda_4 - \lambda_5 \]
\[ \lambda_6 - \lambda_7 \]

**FIG. 2:** (Color online) **Spin textures in the presence of spin-dependent tunneling phases:** The plots show, for different \( \alpha \), the projection of the local SU(3) spin vectors \( \vec{\lambda} \) into different spin planes. The calculation was done within a 30 \( \times \) 30 lattice, but due to the periodicity of the spin patterns we only plot a representative part of the system. In (a) and (b), the system has a 3 \( \times \) 1 and a 5 \( \times \) 1 unit cell, framed by the green boxes. In (c), the system is antiferromagnetic in the components affected by the Dzyaloshinskii-Moriya interactions. In (d), we give an example for a large unit cell, extending over 15 sites.

state degeneracy of the Heisenberg model. The latter was achieved by breaking the SU(3)-symmetry in the interactions, e.g. by making atoms in the same pseudospin state more repulsive than atoms in different pseudospin states. Furthermore, even in this case, the spiral structures could be traced back to effects of a gauge transformation acting on the (anisotropic) Heisenberg SU(3) model.

Furthermore, the precise interaction Hamiltonian for spin-orbit coupled quantum gases is a delicate issue. One has to bear in mind that schemes to generate the spin-orbit coupling typically dress the atoms. Such dressing might have significant consequences for the interactions, cf. Refs. [38, 39]. It seems, however, to be a reasonable assumption to consider spin-independent interactions. Nature provides atomic manifolds with (almost) SU(N)-symmetric interactions, and if the dressed states are composed only from states of such manifolds, interactions should remain spin-independent.

In this section, we therefore analyze a more complicated (non-abelian) spin-orbit coupling, which includes spin-changing hopping processes with the assumption of SU(3) symmetric interactions. As we shall see, such terms again lead to non-trivial spin textures, but they also remove the degeneracies of the Heisenberg model.

A truly SU(3) spin-orbit coupling has been discussed by Barnett and Galitski in Ref. [32]. It is described by the vector potential

\[
(A_x, A_y) = \frac{\pi}{3} \left( -\frac{2}{\sqrt{3}} (\lambda^2 - \lambda^5) + \lambda^7, \lambda^3 + \sqrt{3} \lambda^8 \right).
\]  

(24)

With this choice, motivated by a mapping onto the SU(N) Hofstadter model, the non-interacting system has been shown to exhibit a topological non-trivial band-structure.

We now derive the effective Hamiltonian in the strongly interacting limit corresponding to the spin-orbit coupling in Eq. (24). We first note that along the \( y \)-direction the spin-orbit coupling remains diagonal, and our analysis of the previous section applies. The interest-
FIG. 3: (Color online) **Spin textures in the presence of off-diagonal spin-orbit coupling**: The plots show the projection of the local SU(3) spin vectors $\lambda$ into different spin planes, for a Mott system with the spin-orbit coupling of Eq. (24). In (a), we restrict the dynamics to the $x$-direction, while (b) shows the results for the two-dimensional system. The calculation was done for 30 lattice sites in each direction. The green boxes frame the unit cell, which are repeated through the whole lattice, chosen to have a periodic boundary.

The off-diagonal feature of the spin-orbit coupling affects the hopping along the $x$-direction. Explicitly, it reads

$$H_{\text{SOC,x}}^{\text{kin}} = -t \sum_i a_i^{\dagger} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} a_{i+1}. \quad (25)$$

Hopping from the left to the right therefore promotes a particle counterclockwise along the triangle of Fig. 1b: Upon hopping to the right, a $|+\rangle$ particle becomes $|0\rangle$, a $|0\rangle$ becomes $|-\rangle$, and a $|-\rangle$ becomes $|+\rangle$. This makes it straightforward to write down the hopping in terms of raising/lowering operators, and one obtains a relatively compact second-order Hamiltonian:

$$h_x = -\frac{t^2}{U} \left\{ \lambda_i^{(4)} \lambda_{i+1}^{(6)} - \lambda_i^{(5)} \lambda_{i+1}^{(7)} - \lambda_i^{(1)} \lambda_{i+1}^{(4)} + \lambda_i^{(2)} \lambda_{i+1}^{(5)} - \lambda_i^{(6)} \lambda_{i+1}^{(1)} - \lambda_i^{(7)} \lambda_{i+1}^{(2)} \right\} - \frac{1}{2} \left( \lambda_i^{(3)} \lambda_{i+1}^{(3)} + \lambda_i^{(8)} \lambda_{i+1}^{(8)} + \sqrt{3} \left( \lambda_i^{(3)} \lambda_{i+1}^{(8)} - \lambda_i^{(8)} \lambda_{i+1}^{(3)} \right) \right) \}. \quad (26)$$

As in the previous section, the simulated annealing algorithm has been used to grasp the spin textures in the ground state of this Hamiltonian. First, we freeze the hopping in the $y$-direction, and study $h_x$ alone.
spins now avoid the $\lambda^{(3)} - \lambda^{(8)}$ plane due to the spin-orbit coupling. In the remaining planes, a three-periodic pattern is exhibited, similarly to the one of the previous section for $\alpha = 1$, see Fig. 3. Different from the case before, however, in none of the planes does the system exhibit ferromagnetic alignment.

If, additionally to the spin-orbit coupling in $x$-direction, one also assumes the diagonal spin-orbit term along the $y$-direction, according to Eq. (24), the two terms together give rise to periodic spin patterns in both direction. As shown in Fig. 3(b), the unit cell of the spin lattice then becomes two-dimensional, containing $3 \times 3$ sites. No other ground state patterns have been obtained using the simulated annealing technique.

V. SUMMARY AND OUTLOOK

We have derived the effective Hamiltonian of three-component bosons corresponding to the Mott phase in the presence of spin-orbit coupling. We have focused on two simple cases: a purely diagonal hopping with spin-dependent phases, as well as a particular purely off-diagonal spin-orbit coupling. The abelian spin-orbit coupling allows us to introduce an SU(3) analog of the Dzyaloshinskii-Moriya interaction (antisymmetric in the pseudo-spin positions). We studied the one-dimensional chain and the square lattice. Both, the diagonal and the off-diagonal coupling, lead to SU(3) spin spiral textures in the ground state. The periodicities of the spin textures are controlled by the spin-orbit coupling parameters. Asymmetry between the inter- and intra-particle interactions $\mu < 1$ lift the SU(3) degeneracies and, for diagonal spin-orbit coupling in one dimension, lead to spiral phases. On the quantum level, these textures can be traced back to a local gauge transformation over Dicke-like states. The classical ground state phase on the two-dimensional lattice was studied using the simulated annealing technique yielding, for diagonal spin-orbit coupling, phases that can be understood from the corresponding one-dimensional case. On the other hand, a generic unique classical ground state spiral phase was obtained for symmetric interactions with non-abelian spin-orbit coupling.

More generally, the interplay between diagonal and off-diagonal spin-orbit coupling leads to a much richer landscape of spin Hamiltonians than discussed here. The work considered here could be extended to the study of the competition between the asymmetry of interactions $\mu$ and the various types of spin-orbit coupling.

Finally, we comment on the realization of the scenarios discussed in this paper. On the one hand these require the realization of the strongly interacting limit, where the particle fluctuations are strongly suppressed as described theoretically e.g. in Refs. 2, 35, 40–42 and already achieved experimentally 43–48. SU(3) Bose-Hubbard like systems can be realized with ultra-cold atoms with three internal spin degrees of freedom 49 while the strong spin orbit coupling could be realized by employing proposals described, for instance, in Refs. 50–53.

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Appendix A: Effective Hamiltonians

1. SU(3) notation of biquadratic Heisenberg Hamiltonian

The SU(2) spin-1 matrices read

\[ S^x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]  

(A1)

The spin-1 vector is thus written in terms of Gell-Mann matrices as

\[ S = \frac{1}{\sqrt{2}} \left( \lambda^{(1)} + \lambda^{(6)}, \lambda^{(2)} + \lambda^{(7)}, \frac{1}{\sqrt{3}}(\lambda^{(3)} + \sqrt{3}\lambda^{(8)}) \right). \]  

(A2)
Using this equation, it is straightforward to translate any spin-1 Hamiltonian into the SU(3) picture. In particular, for the biquadratic Heisenberg Hamiltonian, Eq. (3), we find the following contributions:

\[
\mathbf{S}_i \cdot \mathbf{S}_j = \frac{1}{2} \left( \lambda_i^{(1)} \cdot \lambda_j^{(1)} + \lambda_i^{(2)} \cdot \lambda_j^{(2)} + \lambda_i^{(6)} \cdot \lambda_j^{(6)} + \lambda_i^{(7)} \cdot \lambda_j^{(7)} \right) \\
+ \frac{1}{2} \left( \lambda_i^{(1)} \cdot \lambda_j^{(6)} + \lambda_i^{(6)} \cdot \lambda_j^{(1)} + \lambda_i^{(2)} \cdot \lambda_j^{(7)} + \lambda_i^{(7)} \cdot \lambda_j^{(2)} \right) \\
+ \frac{1}{4} \left( \lambda_i^{(3)} \cdot \lambda_j^{(3)} + 3 \lambda_i^{(5)} \cdot \lambda_j^{(5)} + \sqrt{3} \lambda_i^{(8)} \cdot \lambda_j^{(8)} + \sqrt{3} \lambda_i^{(6)} \cdot \lambda_j^{(2)} \right),
\]

(A3)

\[
(\mathbf{S}_i \cdot \mathbf{S}_j)^2 = -\frac{1}{2} \left( \lambda_i^{(1)} \cdot \lambda_j^{(6)} + \lambda_i^{(6)} \cdot \lambda_j^{(1)} + \lambda_i^{(2)} \cdot \lambda_j^{(7)} + \lambda_i^{(7)} \cdot \lambda_j^{(2)} \right) \\
+ \frac{1}{4} \left( \lambda_i^{(3)} \cdot \lambda_j^{(3)} - \lambda_i^{(8)} \cdot \lambda_j^{(8)} - \sqrt{3} \lambda_i^{(8)} \cdot \lambda_j^{(8)} - \sqrt{3} \lambda_i^{(6)} \cdot \lambda_j^{(2)} \right) \\
+ \frac{1}{2} \left( \lambda_i^{(4)} \cdot \lambda_j^{(4)} + \lambda_i^{(5)} \cdot \lambda_j^{(5)} \right).
\]

(A4)

Apparently, this sum of both contributions yields the SU(3) Heisenberg Hamiltonian.

2. Pseudospin interactions - XYZ-type Hamiltonian

For pseudospin interactions as given in Eq. (5), one obtains the effective Mott Hamiltonian of Eq. (6), with the parameters:

\[
J_1 = J_2 = \frac{t^2}{2U_{++}},
\]

(A5)

\[
J_4 = J_5 = \frac{t^2}{2U_L},
\]

(A6)

\[
J_6 = J_7 = \frac{t^2}{2U_0}.
\]

(A7)

\[
J_3 = \frac{t^2}{2} \left( \frac{1}{U_{++}} + \frac{1}{U_{00}} - \frac{1}{U_{++}} \right),
\]

(A8)

\[
J_8 = \frac{t^2}{6} \left( \frac{1}{U_{++}} + \frac{1}{U_{00}} + \frac{1}{U_{+0}} - \frac{2}{U_{++}} - \frac{2}{U_{00}} \right).
\]

(A9)

\[
J_{38} = \frac{\sqrt{3}t^2}{6} \left( \frac{1}{U_{++}} - \frac{1}{U_{--}} + \frac{1}{U_{00}} - \frac{1}{U_{++}} \right),
\]

(A10)

\[
h_3 = \frac{1}{3} \left( \frac{1}{U_{++}} - \frac{1}{U_{00}} \right),
\]

(A11)

\[
h_8 = \frac{\sqrt{3}t^2}{18} \left( \frac{2}{U_{++}} + \frac{2}{U_{00}} - \frac{4}{U_{--}} + \frac{2}{U_{+0}} - \frac{1}{U_{++}} - \frac{1}{U_{00}} \right).
\]

(A12)
3. Spinor interaction and tunneling phases

Assuming spinor interactions, Eq. [11], in combination with a tunneling phase $\alpha$ for particles in $|\pm\rangle$, the following second-order Hamiltonian is obtained along the direction of tunneling:

$$
H_{\text{eff}} = -\frac{\Delta^2}{U_0 + U_2} \sum_i \left\{ \cos \alpha \left( \lambda_i^{(1)} \cdot \lambda_{i+1}^{(1)} + \lambda_i^{(2)} \cdot \lambda_{i+1}^{(2)} + \lambda_i^{(6)} \cdot \lambda_{i+1}^{(6)} + \lambda_i^{(7)} \cdot \lambda_{i+1}^{(7)} \right) 
+ \sin \alpha \left( \lambda_i^{(1)} \cdot \lambda_{i+1}^{(2)} - \lambda_i^{(2)} \cdot \lambda_{i+1}^{(1)} + \lambda_i^{(6)} \cdot \lambda_{i+1}^{(7)} - \lambda_i^{(7)} \cdot \lambda_{i+1}^{(6)} \right) + \frac{3}{\sqrt{3}} \lambda_i^{(8)} \cdot \lambda_{i+1}^{(8)} 
+ \frac{2 \sqrt{3}}{3} \left( \lambda_i^{(3)} \cdot \lambda_{i+1}^{(8)} + \lambda_i^{(8)} \cdot \lambda_{i+1}^{(3)} \right) + \frac{2}{3} \lambda_i^{(3)} \cdot \lambda_{i+1}^{(3)} 
- \frac{3}{\sqrt{3}} \left( \lambda_i^{(3)} \cdot \lambda_{i+1}^{(8)} + \lambda_i^{(8)} \cdot \lambda_{i+1}^{(3)} \right) + \frac{2}{3} \lambda_i^{(3)} \cdot \lambda_{i+1}^{(3)} \right) \right\}.
$$

(A13)

For non-zero tunneling phases, a SU(3)-field is acting in the $\lambda^{(3)} - \lambda^{(8)}$ plane, favoring the polarization of the system within this plane.

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