Fast Derandomized Low-rank Approximation and Extensions *

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Abstract

• Low-rank approximation of a matrix by means of structured random sampling has been consistently efficient in its extensive empirical studies around the globe, but adequate formal support for this empirical phenomenon has been missing so far.
• Based on our novel insight into the subject, we provide such an elusive formal support and de-randomize and simplify the known numerical algorithms for low-rank approximation and related computations.
• Our techniques can be applied to some other areas of fundamental matrix computations, in particular to the Least Squares Regression, Gaussian elimination with no pivoting and block Gaussian elimination.
• Our formal results and our numerical tests are in good accordance with each other.

Key Words: Low-rank approximation, Random sampling, Derandomization

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1 Introduction

1.1 The problem of low-rank approximation and our progress briefly

Low-rank approximation of a matrix has a variety of applications to the most fundamental matrix computations [HMT11] and numerous problems of data mining and analysis, “ranging from term document data to DNA SNP data” [M11]. Classical solution algorithms use SVD or rank-revealing factorizations, but the alternative solution by means of random sampling is numerically reliable, robust, and computationally and conceptually simple and has become highly and increasingly popular in the last decade (see [HMT11], [M11], and [GL13, Section 10.4.5] for surveys and ample bibliography).

In particular the paper [HMT11] proves that random sampling algorithms applied with Gaussian multipliers produce low-rank approximation with a probability close to 1, but empirically the algorithms work as efficiently with various random structured multipliers.

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Adequate formal support for this empirical evidence has been elusive so far, but based on our new insight we obtain such a support and furthermore derandomize these algorithms and simplify them by applying them with some sparse and structured multipliers. The known links enable immediate extensions of our results to various important computations in numerical linear algebra and data mining and analysis, but we also extend them to other fundamental computational problems solved by using random multipliers. We outline our results in this section. They are in good accordance with our numerical tests of Section 5.

1.2 Some definitions

- Typically we use the concepts “large”, “small”, “near”, “close”, “approximate”, “ill-conditioned” and “well-conditioned” quantified in the context, but we specify them quantitatively as needed.
- Hereafter “≪” means “much less than”; “flop” stands for “floating point arithmetic operation”.
- $I_s$ is the $s \times s$ identity matrix. $O_{k,l}$ is a $k \times l$ matrix filled with zeros. $\mathbf{0}$ is a vector filled with zeros.
- $(B_1 \mid B_2 \mid \ldots \mid B_h)$ denotes a $1 \times h$ block matrix with the blocks $B_1, B_2, \ldots, B_h$.
- $\text{diag}(B_1, B_2, \ldots, B_h)$ denotes a $h \times h$ block diagonal matrix with diagonal blocks $B_1, B_2, \ldots, B_h$.
- rank($W$), $\text{rank}(W)$, and $||W||$ denote the rank, numerical rank, and the spectral norm of a matrix $W$, respectively.
- $W^T$ and $W^H$, and denote its transpose and Hermitian transpose, respectively.
- An $m \times n$ matrix $W$ is called unitary if $W^H W = I_n$ or if $WW^H = I_m$. If this matrix is known to be real, then it is also and preferably called orthogonal.
- $\text{rank}_1(W) = ||W||$ and $\text{rank}_2(W) = ||U||$ if the matrix $U$ is unitary.
- $W = S_{W,\rho} \Sigma_{W,\rho} T_{W,\rho}^T$ is compact SVD of a matrix $W$ of rank $\rho$ with $S_{W,\rho}$ and $T_{W,\rho}$ denoting the unitary matrices of its singular vectors and $\Sigma_{W,\rho} = \text{diag}(\sigma_j(W))_{j=1}^\rho$ the diagonal matrix of its singular values in non-increasing order, $\sigma_1(W) \geq \sigma_2(W) \geq \cdots \geq \sigma_\rho(W) > 0$. ($\sigma_1(W) = ||W||$.)
- $\kappa(W) = \sigma_1(W)/\sigma_\rho(W) \geq 1$ denotes the condition number of a matrix $W$. A matrix is called ill-conditioned if its condition number is large in context and is called well-conditioned if this number $\kappa(W)$ is reasonably bounded.
- “Likely” means “with a probability close to 1”, the acronym “i.i.d.” stands for “independent identically distributed”, and we refer to “standard Gaussian random” variables just as “Gaussian”.
- We call an $m \times n$ matrix Gaussian and denote it $G_{m,n}$ if all its entries are i.i.d. Gaussian variables.
- $\mathcal{G}^{m \times n}$, $\mathbb{R}^{m \times n}$, and $\mathbb{C}^{m \times n}$ denote the classes of $m \times n$ Gaussian, real, or complex matrices, respectively.
- $\mathcal{G}_{m,n,r}$, $\mathbb{R}_{m,n,r}$, and $\mathbb{C}_{m,n,r}$, for $1 \leq r \leq \min\{m,n\}$, denote the classes of $m \times n$ matrices $M = UV$ (of rank at most $r$) where both $m \times r$ matrix $U$ and $r \times n$ matrix $V$ are Gaussian, real, and complex, respectively.
- If $U \in \mathcal{G}^{m \times r}$ and $V \in \mathcal{G}^{r \times n}$, then we call $M = UV$ an $m \times n$ factor-Gaussian matrix of expected rank $r$. (In this case the matrices $U$, $V$ and $M$ have rank $r$ with probability 1 by virtue of Theorem A.1.)
1.3 The basic algorithm

A matrix $M$ can be represented (respectively, approximated) by a product $UV$ of two matrices $U \in \mathbb{C}^{m \times r}$ and $V \in \mathbb{C}^{r \times n}$ if and only if $r \geq \text{rank}(M)$ (respectively, $r \geq \text{nrk}(M)$), and our main goal is the computation of such a representation or approximation.

We begin with the following basic algorithm for the fixed rank problem, where the integer $r = \text{nrk}(M)$ or $r = \text{rank}(M)$ is known. Otherwise we can compute it by means of binary search based on recursive application of the algorithm or proceed, e.g., as in our Algorithm 1.2 of Section 1.5.

**Algorithm 1.1.** Range Finder (See Figure 1 and compare [HMT11, Algorithms 4.1 and 4.2]).

**Input:** An $m \times n$ matrix $M$, a nonnegative tolerance $\tau$, and an integer $r$ such that $0 < r \ll \min\{m, n\}$.

**Initialization:** Fix an integer $l$ such that $r \leq l \ll \min\{m, n\}$. Generate an $n \times l$ matrix $B$.

**Computations:**
1. Compute the $m \times l$ matrix $MB$. Remove its columns that have small norms.
2. Orthogonalize its remaining columns (cf. [GL13, Theorem 5.2.3]), compute and output the resulting $m \times \tilde{l}$ matrix $U = U(MB)$ where $\tilde{l} \leq l$.
3. Estimate the error norm $\Delta = \|\tilde{M} - M\|$ for $\tilde{M} = UU^T M$.
   - If $\Delta \leq \tau$, output SUCCESS; otherwise FAILURE.

![Figure 1: Matrices of Algorithm 1.1](image)

At Stage 3 probabilistic estimate for the norm $\Delta$ can be given by the norm $\|\tilde{M}H - MH\|$ for a random $n \times k$ matrix $H$ and a reasonably small positive integer $k$ (this would extend the Frievalds’ probabilistic test of [F77] (cf. [MR95], [AS00]). [HMT11, Algorithm 4.2] chooses random matrix $B$ at Stage 1 and then proceeds with $k = l$ and $H = B$.

If Stage 3 outputs SUCCESS, then a rank-\(\tilde{l}\) approximation to the matrix $M$ is given by the matrix $\tilde{M} = UU^T M$, but the papers [HMT11, Section 5] and [CW13] avoid costly multiplication of $U^T$ by $M$. Their alternative solution relies on the extension of Stage 1 to randomized approximation of the leading part of the compact SVD of the matrix $M$, associated with its $r$ largest singular values.

The complexity of these algorithms is dominated at Stage 1, which uses $(2n - 1)ml$ flops in the case of generic matrices $M$ and $B$. With intricate application of sparse embedding multipliers $B$, the paper [CW13] computes a low-rank approximation (with failure probability at most $1/5$) by using about $2mn$ flops for generic input $M$, but alternative computations using $O(mnl)$ flops can be still of interest (see Section 1.8).

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1. By applying rank-revealing orthogonalization at Stage 2 we can remove some extraneous columns and obtain rank-$r$ approximation of the matrix $M$. 

1.4 The choice of multipliers: basic observations

We readily verify the following theorem (see Section 4.3):

**Theorem 1.1.** Given an \( m \times n \) matrix \( M \) with \( \text{rank}(M) = r \) and a reasonably small positive tolerance \( \tau \), Algorithm 1.1 outputs SUCCESS if and only if \( \text{rank}(MB) = r \).

**Definition 1.1.** For two integers \( l \) and \( n \), \( 0 < l \leq n \), and any fixed \( n \times l \) multiplier \( B \), partition the set of \( m \times n \) matrices \( M \) with \( \text{rank}(M) = r \) into the set \( M_B = M_{B,\text{good}} \) of “\( B \)-good” matrices such that \( \text{rank}(MB) = r \) and the set \( M_B,\text{bad} \) of “\( B \)-bad” matrices such that \( \text{rank}(MB) < r \).

The following simple observations should be instructive.

**Theorem 1.2.** (Cf. Remark 1.2.) Consider a vector \( v \) of dimension \( n \), an \( n \times n \) unitary matrix \( U \), and an \( n \times l \) unitary matrix \( B \), so that \( n \times l \) matrix \( UB \) is unitary. Then

(i) \( IM_B = (UB)M \), that is, the map \( B \to UB \) multiplies the class \( M_B \) of \( B \)-good \( m \times n \) matrices by the unitary matrix \( U \),

(ii) \( M_B \subseteq M(B \mid v) \), that is, appending a column to a multiplier \( B \) can only expand the class \( M_B \), and

(iii) this class fills the whole space \( C_{m,n,r} \) or \( \mathbb{R}_{m,n,r} \) if \( l = n \).

**Proof.** Part (i) follows because \( (MU)B = M(UB) \). Part (ii) follows because \( \text{rank}(MB) \leq \text{rank}(M(B \mid v)) \). Part (iii) follows because \( \text{rank}(MB) = \text{rank}(M) \) if \( B \) is an \( n \times n \) unitary matrix. \( \qed \)

1.5 A recursive algorithm

Based on Theorem 1.2 we devise the following algorithm where \( \text{rank}(M) \) is not known.

**Algorithm 1.2.** Recursive low-rank representation/approximation of a matrix. See Figure 2 and cf. [HMT11, Algorithm 4.2].

**Input:** An \( m \times n \) matrix \( M \) and a nonnegative tolerance \( \tau \).

**Computations:**
1. Generate an \( n \times n \) unitary matrix \( \hat{B} \).
2. Fix positive integers \( l_1, \ldots, l_h \) such that \( l_1 + \cdots + l_h = n \) (in particular \( l_j = 1 \) for all \( j \) if \( h = n \)) and represent the matrix \( \hat{B} \) as a block vector \( (B_1 \mid B_2 \mid \ldots \mid B_h) \) where the block \( B_i \) has size \( n \times l_i \) for \( i = 1, \ldots, h \).
3. Recursively, for \( i = 1, 2, \ldots, \) apply Algorithm 1.1 to the matrix \( M \) by substituting \( l^{(i)} = \sum_{j=1}^{i} l_j \) for \( l \) and \( B^{(i)} = (B_1 \mid B_2 \mid \ldots \mid B_i) \) for \( B \). Stop when the algorithm outputs SUCCESS.

![Figure 2: Matrices of Algorithm 1.2](image)

By virtue of part (iii) of Theorem 1.2 the algorithm stops and outputs SUCCESS either at the \( h \)th Stage (when \( l^{(h)} = n \)) or earlier, and we are surely interested in yielding SUCCESS already for \( l^{(i)} < n \) and in saving flops for matrix multiplications at Stage 3.
Remark 1.1. We are likely to save some flops if we compute approximate matrix products by using leverage scores \([W14]\) (a.k.a. sampling probabilities \([M11, \text{Sections 3 and 5}]\)).

Remark 1.2. Clearly, the blocks \(B_j\) and \(B^{(i)}\) of the unitary matrix \(\hat{B}\) are unitary as well, but we can readily extend both Theorem 1.2 and Algorithm 1.2 to the case where we apply them to a nonsingular and well-conditioned (rather than unitary) \(n \times n\) matrix \(\hat{B}\). In that case all multipliers \(B^{(i)}\) and all their blocks \(B_j\) are also well-conditioned matrices of full rank, and moreover \(\kappa(B^{(i)}) \leq \kappa(B)\) and \(\kappa(B_j) \leq \kappa(B)\) for all \(i\) and \(j\) (cf. \([GL13, \text{Corollary 8.6.3}]\)).

1.6 Benefits of using Gaussian and random structured multipliers

Theorem 1.3. Let Algorithm 1.1 be applied with a Gaussian multiplier \(B \in \mathcal{G}^{n \times l}\). Then

(i) \(M = M\) with probability 1 if \(l \geq r = \text{rank}(M)\) (cf. Theorem 4.1) and
(ii) it is likely that \(M \approx M\) if \(\text{rank}(M) = r \leq l\), and the probability that \(\hat{M} \approx M\) approaches 1 fast as \(l\) increases from \(r + 1\) (cf. Theorem 4.3).

The theorem implies that Algorithm 1.2 is likely to output SUCCESS at Stage \(h\) for the smallest \(h\) such that \(l^h \geq r\) in the case where \(B\) denotes a Gaussian (rather than unitary) matrix.

An \(n \times l\) matrix of subsample random Fourier or Hadamard transform\(^2\) is defined by \(n + l\) random variables (see Remark 3.3), and we can pre-multiply it by a vector by using \(O(n \log(l))\) flops, for \(l\) of order \(r \log(r)\), (see \([HMT11, \text{Sections 4.6 and 11}],[M11, \text{Section 3.1}],\text{and }[T11])\). For comparison, an \(n \times l\) Gaussian matrix is defined by its \(nl\) random entries, and we need \(l(2n - 1)\) flops in order to pre-multiply it by a vector.

SRFT and SRHT multipliers \(B\) are universal, like Gaussian ones: Algorithm 1.1 applied with such a multiplier is likely to approximate closely a matrix \(M\) having numerical rank at most \(r\), although the estimated failure probability \(3 \exp(-p)\), for \(p = l - r \geq 4\) with Gaussian multipliers increases to order of \(1/l\) in the case of SRFT and SRHT multipliers (cf. \([HMT11, \text{Theorems 10.9 and 11.1}],[M11, \text{Section 5.3.2}],\text{and }[T11])\).

Empirically Algorithm 1.1 with SRFT multipliers fails very rarely even for \(l = r + 20\), although for some special input matrices \(M\) it is likely to fail if \(l = o(r \log(r))\) (cf. \([HMT11, \text{Remark 11.2}]\) or \([M11, \text{Section 5.3.2}]\)). Researchers have consistently observed similar empirical behavior of the algorithm applied with SRHT and various other multipliers (see \([HMT11],[M11],[W14],[PQY15],\) and the references therein),\(^3\) but so far no adequate formal support for that empirical observation has appeared in the huge bibliography on this highly popular subject.

1.7 Our goals, our dual theorem, and its implications

In this paper we are going to

(i) fill the void in the bibliography by supplying a missing formal support for the cited observation, with far reaching implications (see parts (ii) and (iv) below),
(ii) define new more efficient policies of generation and application of multipliers for low-rank approximation,
(iii) test our policies numerically, and
(iv) extend our progress to other important areas of matrix computations.

Our Dual Theorem 1.4 below reverses the assumptions of our Primal Theorem 1.3 that a multiplier \(B\) is Gaussian, while a matrix \(M\) is fixed.

Theorem 1.4. Let \(M - E \in \mathcal{G}_{m,n,r}\) and \(\|E\|_2 \approx 0\) (in which case \(\text{urank}(M) \leq r\) and, with a probability close to 1, \(\text{urank}(M) = r\)). Furthermore let \(B \in \mathbb{R}^{n \times l}\) and \(\text{rank}(B) = l\). Then

(i) Algorithm 1.1 outputs a rank-\(r\) representation of a matrix \(M\) with probability 1 if \(E = 0\) and if \(l \geq r\), and
(ii) it outputs a rank-l approximation of that matrix with a probability close to 1 if \(l \geq r\) and approaching 1 fast as the integer \(l\) increases from \(r + 1\).

\(^2\)Hereafter we use the acronyms SRFT and SRHT.

\(^3\)In view of part (i) of Theorem 1.2, the results cited for the classes of SRFT and SRHT matrices also hold for the products of these classes with any unitary matrix, in particular for the class of \(n \times l\) submatrices of an \(n \times n\) circulant matrix, each made up of \(l\) randomly chosen columns (see Remark 3.3).
Proof. See Theorem 4.4.

Corollary 1.1. Under the assumptions of Theorem 1.4, Algorithm 1.2 is likely to produce a rank-1 approximation to the matrix $M$ at its first Stage $i$ at which $l(i) \geq r$, and the probability that this occurs approaches 1 fast as $l(i)$ increases from $r + 1$.

Part (ii) of Theorem 1.4 implies that Algorithm 1.1 succeeds for the average input matrix $M$ that has a small numerical rank $r \leq l$ (and thus in a sense to most of such matrices) if the multiplier $B$ is any unitary matrix (or even any well-conditioned matrix of full rank) and if the average matrix is defined under the Gaussian probability distribution. The former provision, that $\text{nrank}(B) = l$, is natural for otherwise we could have replaced the multiplier $B$ by an $n \times l_-$ matrix for some integer $l_- < l$. The latter customary provision is natural in view of the Central Limit Theorem.

For an immediate implication of Theorem 1.4, on the average input $M$ having numerical rank at most $r$, Algorithm 1.2 applied with any unitary or even any nonsingular and well-conditioned $n \times n$ multiplier $B$ outputs SUCCESS at its earliest recursive Stage $i$ at which the dimension $l(i) = \sum_{j=1}^{i} l_j$ exceeds $r - 1$. This can be viewed as derandomization of Algorithms 1.1 and 1.2 versus their application with Gaussian sampling.

1.8 Related work, our novelties, and extension of our progress

Part (ii) of our Theorem 1.3 is implied by [HMT11, Theorem 10.8], but our specific supporting estimates are more compact, cover the case of any $l \geq r$ (whereas [HMT11] assumes that $l \geq r + 4$), and we deduce them by using a shorter proof (see Remark 4.3). Our approach and our results of Section 1.7 are new, and so are our families of sparse and structured multipliers and the policies of their generation, combination, and application in Sections 2 and 3 as well.

By applying the well-known links, we can extend our results for low-rank approximation to various fundamental problems of matrix computations and data mining and analysis, but our duality techniques can be extended to other important computational problems as well. In Section 6 (Conclusions) we show such an extension to the Least Squares Regression. Another extension in [PZa] supports numerically safe performance of Gaussian elimination with no pivoting and block Gaussian elimination. The extensions provide new insights and new opportunities and should motivate further effort and further progress.

Extensive decade-long work of a number of authors on an alternative approach to Low-Rank Approximation and Least Squares Regression has culminated in the paper [CW13]. Its algorithms succeed for these problems with a probability at least 4/5, whereas we only reach solution for the average input. Our study, however, leads to some benefits, which should compensate for this deficiency.

1. We show the power of a very large class of multipliers, including various sparse and structured ones. This can be interesting, e.g., for some special structured inputs (see Remark 3.2), but not only for them. Indeed, see our Remark 3.4 and compare the following excerpt from [BCDHKS14]: “The traditional metric for the efficiency of a numerical algorithm has been the number of arithmetic operations it performs. Technological trends have long been reducing the time to perform an arithmetic operation, so it is no longer the bottleneck in many algorithms; rather, communication, or moving data, is the bottleneck”.  

2. In order to make the probability of failure less than $\delta$, the complexity bound of [CW13] involve overhead of order $\log(1/\delta)$, which greatly exceeds the overhead in the case of our average case estimates.

3. [CW13] studies the fixed rank problem; in the case where the input numerical rank is not known, our Algorithm 1.2 substantially decreases the computational overhead versus binary search.

4. Unlike [CW13] we cover the case where the ratio $n/r$ is not very large, which can still be interesting in some applications.

5. Our analysis is quite simple and conceptually distinct and should be of independent interest because it provides elusive explanation of a well-known empirical phenomenon (cf. Section 1.1).

4Hereafter we use the acronym “LSR”.

5Pivoting, that is, row or column interchange of an input matrix for avoiding numerical problems in Gaussian elimination, is communication intensive and has become the bottleneck of Gaussian elimination in the present day computer environment. Preprocessing with randomized and derandomized multipliers is a natural means for overcoming this problem (cf. [BBBDD14], [PQY15], [PZa]).
1.9 Organization of the paper

We organize our presentation as follows:

- In Section 2 we describe our policies for management of the rare failures of Algorithm 1.1 and amend Algorithm 1.2.
- In Section 3 we present some efficient sparse and structured multipliers for low-rank approximation.
- In Section 4 we prove Theorems 1.1, 1.3, and 1.4, extending their claims with more detailed estimates.
- Section 5 (the contribution of the second and the third authors) covers our numerical tests.
- In Section 6 we extend our approach to the LSR computations.
- The Appendix covers some auxiliary results for computations with random matrices.

2 Preventing and managing the unlikely failure of Algorithm 1.1

Our conflicting goals and simple recipes.

We try to decrease:
(i) the cost of the generation of a multiplier $B$ and of the computation of the matrix $MB$,
(ii) the chances for the failure of Algorithm 1.1, and
(iii) the rank of the computed approximation of a matrix $M$.

Towards goal (i) we propose using sparse and structured $n \times l$ multipliers in the next section. They are pre-multiplied by a vector and by a matrix $M$ at a low cost even for $l = n$.

Towards goal (ii) we can expect to succeed whenever integer parameter $l$ exceeds $r + 1$, but our chances for success grow fast as $l$ increases. Such an increase is in conflict with our goal (iii), but we can alleviate the problem by using the following simple technique.

**Randomized Compression Algorithm** (see Figure 3).

1. Fix a sufficiently large dimension $l$, which is still much smaller than $\min\{m, n\}$, generate a sparse and structured $n \times l$ multiplier $B$, and compute the $m \times l$ product $MB$ (by using $(2n - 1)ml$ flops).

2. Fix a smaller integer $l_-$ such that $r \leq l_- < l$, generate a Gaussian $l_\times l_-$ multiplier $G$, and compute and output the $m \times l_-$ matrix $MBG$ (by using $(2l - 1)ml_-$ flops, dominated at Stage 1 if $l \leq \min\{m, n\}$).

![Figure 3: Matrices of Algorithm 1.2](image)

By virtue of Theorem 1.3 this algorithm is likely to succeed, but in our extensive tests in Section 5 even the following simple heuristic recipe has always worked.

**Heuristic Compression Algorithm** (linear combination of failed multipliers): if the first $h$ recursive steps of Algorithm 1.2 have failed for $h > 1$, then apply Algorithm 1.1 with a multiplier $B = \sum_{j=1}^{h} c_j B_j$.
where \( c_j = \pm 1 \) for all \( j \) and for a fixed or random choice of the signs \( \pm \). (More generally, one can choose complex values \( c_j \) on the unit circle, letting \( |c_j| = 1 \) for all \( j \).)

**Remark 2.1.** In our study above we rely on the results of Theorem 1.4, which cover the average input matrices. Real computations can deal with “rare special” input matrices \( M \), not covered by Theorem 1.4, but in our tests in Section 5 with a variety of inputs, our small collection of sparse and structured multipliers of the next section turned out to be powerful enough for handling various important classes of special matrices as well.

**Remark 2.2.** Reusing multipliers. Recall from [HMT11, Sections 8 and 9] that for all \( j \) the matrices \( \tilde{M}^{(j)} = U^{(j)}U^{(j)T}M \) and \( \tilde{M}_j = U_jU_j^T M \), for \( U^{(j)} = U(MB^{(j)}) \), \( U_j = U(MB_j) \), and \( B^{(j)} = (B_1 \mid B_2 \mid \ldots \mid B_j) \), are the orthogonal projections of the matrices \( MB^{(j)} \) and \( MB_j \), respectively, onto the range of the matrix \( M \). Hence \( M - \tilde{M}^{(h)} = M - \tilde{M}^{(h-1)} - \tilde{M}_h \), and so at the \( h \)-th stage of Algorithm 1.2, for \( h > 1 \), we can reuse such projections computed at its Stage \( h - 1 \) rather than recompute them.

### 3 Generation of multipliers. Counting flops and random variables

In our tests we have consistently succeeded by using multipliers from a **limited family** of very sparse and highly structured orthogonal matrices of classes 13–17 of Section 5.3, but in this section we also cover a greater variety of other sparse and structured matrices, which form an **extended family** of multipliers.

We proceed in the following order. Given two integers \( l \) and \( n, l < n \), we first generate four classes of very sparse \( n \times n \) unitary matrices, then combine them into some basic families of \( n \times n \) matrices (we denote them \( B \) in this section), and finally define multipliers \( B \) as \( n \times l \) submatrices made up of \( l \) columns, which can be fixed (e.g., leftmost) or chosen at random. The matrix \( B \) is unitary if so is the matrix \( \tilde{B} \), and more generally \( \kappa(B) \leq \kappa(\tilde{B}) \) (cf. [GL13, Theorem 8.6.3]).

#### 3.1 \( n \times n \) matrices of four primitive types

1. A fixed or random **permutation matrix** \( P \). Their block submatrices form the important class of CountS-ketch matrices from the data stream literature (cf. [W14, Section 2.1], [CCF04], [TZ12]).

2. A **diagonal matrix** \( D = \text{diag}(d_i)_{i=0}^{n-1} \), with fixed or random diagonal entries \( d_i \) such that \( |d_i| = 1 \) for all \( i \) (and so all \( n \) entries \( d_i \) lie on the unit circle \( \{ x : |x| = 1 \} \), being either nonreal or \( \pm 1 \)).

3. An **\( f \)-circular shift matrix**

\[
Z_f = \begin{pmatrix}
0 & \ldots & \ldots & 0 & f \\
1 & \ddots & & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & 0 \\
0 & \ldots & \ldots & 1 & 0
\end{pmatrix}
\]

and its transpose \( Z_f^T \) for a scalar \( f \) such that either \( f = 0 \) or \( |f| = 1 \). We write \( Z = Z_0 \), call \( Z \) **unit down-shift matrix**, and call the special permutation matrix \( Z_1 \) the **unit circulant matrix**.

4. A **\( 2s \times 2s \) Hadamard primitive matrix** \( H^{(2s)} = (I_s \mid -I_s) \) for a positive integer \( s \) (cf. [M11], [W14]).

The latter primitive \( n \times n \) matrices are very sparse, have nonzero entries evenly distributed throughout, and can be pre-multiplied by a vector by using from 0 to 2\( n \) flops.

All our primitive matrices, except for the matrix \( Z \), are unitary or real orthogonal. Hence, for the average input matrix \( M \), Algorithm 1.1 succeeds with any of their \( n \times l \) submatrix \( B \) by virtue of Theorem 1.4, and similarly with any \( n \times l \) submatrix of the matrix \( Z \) of full rank \( l \).

For specific input matrices the algorithm can fail with some of our \( n \times l \) primitive multipliers \( B \) (e.g., this is frequently the case where both input matrix \( M \) and multiplier \( B \) are sparse), but in the next subsections...
we readily combine primitives 1–4 into families of \( n \times n \) sparse and/or structured matrices, and in Section 5 we consistently and successfully test their \( n \times l \) submatrices \( B \) as multipliers.

### 3.2 Family (i): multipliers based on the Hadamard and Fourier processes

At first we recall the following recursive definition of dense and orthogonal (up to scaling by constants) \( n \times n \) matrices \( H_n \) of *Walsh-Hadamard transform* for \( n = 2^k \) (cf. [M11, Section 3.1] and our Remark 3.1):

\[
H_{2q} = \begin{pmatrix} H_q & H_q \\ H_q & -H_q \end{pmatrix} \tag{3.1}
\]

for \( q = 2^h, \ h = 0, 1, \ldots, k-1 \), and the Hadamard primitive matrix \( H_2 = H^{(2)} = \left( \begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix} \right) \) of type 4 for \( s = 1 \).

For demonstration, here are the matrices \( H_4 \) and \( H_8 \) shown with their entries,

\[
H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad H_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \end{pmatrix},
\]

but for larger dimensions \( n \), recursive representation (3.1) enables much faster pre-multiplication of a matrix \( H_n \) by a vector, namely it is sufficient to use \( nk \) additions and subtractions for \( n = 2^k \).

Next we sparsify this matrix by defining it by a shorter recursive process, that is, by fixing a *recursion depth* \( d \), \( 1 \leq d < k \), and applying equation (3.1) where \( q = 2^h, \ h = k - d, k - d + 1, \ldots, k-1 \), and \( H_s I_s \) for \( n = 2^d s \). For two positive integers \( d \) and \( s \), we denote the resulting \( n \times n \) matrix \( H_{n,d} \) and for \( 1 \leq d < k \) call it \( d \)-*Abridged Hadamard (AH) matrix*. In particular,

\[
H_{n,1} = \begin{pmatrix} I_s & I_s \\ I_s & -I_s \end{pmatrix}, \text{ for } n = 2s; \quad H_{n,2} = \begin{pmatrix} I_s & I_s & I_s & I_s \\ I_s & -I_s & -I_s & I_s \\ I_s & I_s & -I_s & -I_s \\ I_s & I_s & I_s & -I_s \end{pmatrix}, \text{ for } n = 4s, \quad \text{and}
\]

\[
H_{n,3} = \begin{pmatrix} I_s & I_s & I_s & I_s & I_s & I_s & I_s & I_s \\ I_s & -I_s & I_s & I_s & -I_s & -I_s & I_s & I_s \\ I_s & I_s & -I_s & -I_s & I_s & I_s & I_s & I_s \\ I_s & I_s & I_s & -I_s & -I_s & I_s & -I_s & I_s \\ I_s & I_s & I_s & I_s & -I_s & I_s & -I_s & -I_s \\ I_s & I_s & I_s & I_s & I_s & -I_s & -I_s & I_s \\ I_s & I_s & I_s & I_s & I_s & I_s & -I_s & -I_s \\ I_s & I_s & I_s & I_s & -I_s & I_s & I_s & -I_s \end{pmatrix}, \text{ for } n = 8s.
\]

For a fixed \( d \), the matrix \( H_{n,d} \) is still orthogonal up to scaling, has \( q = 2^d \) nonzero entries in every row and column, and hence is sparse unless \( k - d \) is a small integer.

Then again, for larger dimensions \( n \), we can pre-multiply such a matrix by a vector much faster if, instead of the representation by its entries, we apply recursive process (3.1), which involves just \( dn \) additions and subtractions and allows highly efficient parallel implementation (cf. Remark 3.4).

We similarly obtain sparse matrices by shortening a recursive process of the generation of the \( n \times n \) matrix \( \Omega_n \) of *discrete Fourier transform (DFT)* at \( n \) points, for \( n = 2^k \):

\[
\Omega_n = (\omega_n^{ij})_{i,j=0}^{n-1}, \text{ for } n = 2^k \text{ and a primitive } n \text{th root of unity } \omega_n = \exp(2\pi i/n), \ i = \sqrt{-1}. \tag{3.2}
\]
In particular $\Omega_2 = H^{(2)} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

\[
\Omega_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -i & 1 & -i \\ 1 & -i & i & i \end{pmatrix}, \quad \text{and} \quad \Omega_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega_8 & i & \omega_8 & -1 & -\omega_8 & -i & -i \omega_8 \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & i \omega_8 & -i & \omega_8 & -1 & -i \omega_8 & i & -\omega_8 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\omega_8 & i & -i \omega_8 & -1 & \omega_8 & i & \omega_8 \\ 1 & -i & 1 & -i & 1 & -i & 1 & i \\ 1 & -i \omega_8 & -i & -i \omega_8 & -1 & -i \omega_8 & i & \omega_8 \end{pmatrix}.
\]

The matrix $\Omega_n$ is unitary up to scaling by $\frac{1}{\sqrt{n}}$. We can pre-multiply it by a vector by using $1.5nk$ flops, and we can efficiently parallelize this computation if, instead of representation by entries, we apply following recursive representation (cf. [P01, Section 2.3] and our Remark 3.1):

\[
\Omega_{2q} = \hat{P}_{2q} \begin{pmatrix} \Omega_q & \Omega_q D_q \\ \Omega_q D_q & -\Omega_q \end{pmatrix} \cdot \hat{D}_q = \text{diag}(\omega_{nq}^{i})_{i=0}^{n-1}.
\] (3.3)

Here $\hat{P}_{2q}$ is the matrix of odd/even permutations such that $\hat{P}_{2q}(u) = v$, $u = (u_i)_{i=0}^{q-1}$, $v = (v_i)_{i=0}^{q-1}$, $v_j = u_{2j}$, $v_{j+2^{h-1}} = u_{2j+1}$, $j = 0, 1, \ldots, 2^{h-1} - 1$; $q = 2^h$, $h = 0, 1, \ldots, k$, and $\Omega_1 = (1)$ is the scalar 1.

We sparsify this matrix by defining it by a shorter recursive process, that is, by fixing a recursion depth $d$, $1 \leq d < k$, replacing $\Omega_s$ for $s = n/2^d$ by the identity matrix $I_s$, and then applying equation (3.3) for $q = 2^h$, $h = k - d, k - d + 1, \ldots, k - 1$.

For $1 \leq d < k$ and $n = 2^d s$, we denote the resulting $n \times n$ matrix $\Omega_{n,d}$ and call it $d$-Abridged Fourier (AF) matrix. It is also unitary (up to scaling), has $q = 2^d$ nonzero entries in every row and column, and thus is sparse unless $k - d$ is a small integer. We can represent such a matrix by its entries, but then again its pre-multiplication by a vector involves just 1.5dn flops and allows highly efficient parallel implementation if we rely on recursive representation (3.3).

By applying fixed or random permutation and scaling to AH matrices $H_{n,d}$ and AF matrices $\Omega_{n,d}$, we obtain the families of $d$-Abridged Scaled and Permutated Hadamard (ASP) matrices, $PDH_{n,d}$, and $d$-Abridged Scaled and Permutated Fourier (ASPF) $n \times n$ matrices, $PD\Omega_{n,d}$ where $P$ and $D$ are two matrices of permutation and diagonal scaling of primitive classes 1 and 2, respectively. Likewise we define the families of ASH, ASF, APH, and APF matrices, $D\Omega_{n,d}$, $D\Omega_{n,d}$, $PD\Omega_{n,d}$, and $PD\Omega_{n,d}$, respectively. Each random permutation or scaling contributes up to $n$ random parameters.

**Remark 3.1.** The following equations are equivalent to (3.1) and (3.3):

\[H_{2q} = \text{diag}(H_q, H_q)H_{(2q)} \quad \text{and} \quad \Omega_{2q} = \hat{P}_{2q} \text{diag}(\Omega_q, \Omega_q \hat{D}_q)H_{(2q)}\]

where $H_{(2q)}$ denotes a $2q \times 2q$ Hadamard’s primitive matrix of type 4. By extending the latter recursive representation we can define matrices that involve more random parameters. Namely we can recursively incorporate random permutations and diagonal scaling as follows:

\[\hat{H}_{2q} = P_{2q} D_{2q} \text{diag}(\hat{H}_q, \hat{H}_q)H_{(2q)} \quad \text{and} \quad \hat{\Omega}_{2q} = P_{2q} D_{2q} \text{diag}(\Omega_q, \Omega_q \hat{D}_q)H_{(2q)}\].

Here $P_{2q}$ are $2q \times 2q$ random permutation matrices of primitive class 1 and $D_{2q}$ are $2q \times 2q$ random matrices of diagonal scaling of primitive class 2, for all $q$. Then again we define $d$-aboridged matrices $\hat{H}_{n,d}$ and $\hat{\Omega}_{n,d}$ by applying only $d$ recursive steps (3.4) initiated at the primitive matrix $I_s$, for $s = n/2^d$.

With these recursive steps we can pre-multiply matrices $\hat{H}_{n,d}$ and $\hat{\Omega}_{n,d}$ by a vector by using at most $2dn$ additions and subtractions and at most $2.5dn$ flops, respectively, provided that $2^d$ divides $n$.

---

6This is a representation of FFT, called decimation in frequency (DIF) radix-2 representation. Transposition turns it into an alternative representation of FFT, called decimation in time (DIT) radix-2 representation.
3.3 $f$-circulant, sparse $f$-circulant, and uniformly sparse matrices

An $f$-circulant matrix

$$Z_f(v) = \begin{pmatrix} v_0 & f v_{n-1} & \cdots & f v_1 \\ v_1 & v_0 & \cdots & f v_0 \\ \vdots & \vdots & \ddots & \vdots \\ v_{n-1} & \cdots & \cdots & v_0 \end{pmatrix} = \sum_{i=0}^{n-1} v_i Z_f^i$$

for the matrix $Z_f$ of $f$-circulant shift, is defined by a scalar $f \neq 0$ and by the first column $v = (v_i)^{n-1}_{i=0}$ and is called *circulant* if $f = 1$ and *skew-circulant* if $f = -1$. Such a matrix is nonsingular with probability 1 (see Theorem A.1) and is likely to be well-conditioned [PSZ15] if $|f| = 1$ and if the vector $v$ is Gaussian.

**Remark 3.2.** One can compute the product of an $n \times n$ circulant matrix with an $n \times n$ Toeplitz or Toeplitz-like matrix by using $O(n \log(n))$ flops (see [P01, Theorem 2.6.4 and Example 4.4.1]).

**FAMILY** (ii) of sparse $f$-circulant matrices $\hat{B} = Z_f(v)$ is defined by a fixed or random scalar $f$, $|f| = 1$, and by the first column having exactly $q$ nonzero entries, for $q \ll n$. The positions and the values of nonzeros can be randomized (and then the matrix would depend on up to $2n + 1$ random values).

Such a matrix can be pre-multiplied by a vector by using at most $(2q - 1)n$ flops or, in the real case where $f = \pm 1$ and $v_i = \pm 1$ for all $i$, by using at most $qn$ additions and subtractions.

The same cost estimates apply in the case of the generalization of $Z_f(v)$ to a uniformly sparse matrix with exactly $q$ nonzeros entries, $\pm 1$, in every row and in every column for $1 \leq q \ll n$. Such a matrix is the sum $\hat{B} = \sum_{i=1}^{q} \hat{D}_iP_i$ for fixed or random matrices $P_i$ and $\hat{D}_i$ of primitive types 1 and 2, respectively.

3.4 Abridged $f$-circulant matrices

First recall the following well-known expression for a $g$-circulant matrix:

$$Z_g(v) = \sum_{i=0}^{n-1} v_i Z_g^i = D_f^{-1} \Omega_n^H D \Omega_n D_f$$

where $g = f^n$, $D_f = \text{diag}(f^i)_{i=0}^{n-1}$, $v = (v_i)^{n-1}_{i=0} = (\Omega_n D_f)^{-1} u$, $u = (u_i)^{n-1}_{i=0}$, and $D = \text{diag}(u_i)^{n-1}_{i=0}$ (cf. [P01, Theorem 2.6.4]). For $f = 1$, the expression is simplified: $g = 1$, $D_f = I_n$, and $Z_g(v) = \sum_{i=0}^{n-1} v_i Z_1^i$ is a circulant matrix:

$$Z_1(v) = \Omega_n^H D \Omega_n, \ D = \text{diag}(u_i)_{i=0}^{n-1}, \text{ for } u = (u_i)^{n-1}_{i=0} = \Omega_n v.$$

(3.5)

Pre-multiplication of an $f$-circulant matrix by a vector is reduced to pre-multiplication of each of the matrices $\Omega$ and $\Omega^H$ by a vector and in addition to performing $4n$ flops (or $2n$ flops in case of a circulant matrix). This involves $O(n \log(n))$ flops overall and then again allows highly efficient parallel implementation.

For a fixed scalar $f$ and $g = f^n$, we can define the matrix $Z_g(v)$ by any of the two vectors $u$ or $v$. The matrix is unitary (up to scaling) if $|f| = 1$ and if $|u_i| = 1$ for all $i$ and is defined by $n + 1$ real parameters (or by $n$ such parameters for a fixed $f$), which we can fix or choose at random.

Now suppose that $n = 2^d s$, $1 \leq d < k$, $d$ and $k$ are integers, and substitute a pair of AF matrices of recursion length $d$ for two factors $\Omega_n$ in the above expressions. Then the resulting abridged $f$-circulant matrix $Z_{g,d}(v)$ of recursion depth $d$ is still unitary (up to scaling), defined by $n+1$ or $n$ parameters $u_i$ and $f$, is sparse unless the positive integer $k - d$ is small, and can be pre-multiplied by a vector by using $(3d + 3)n$ flops. Instead of AF matrices, we can substitute a pair of ASF, APF, ASF, AH, ASPH, APH, or ASF matrices for the factors $\Omega_n$. All such matrices form **FAMILY** (iii) of $d$-abridged $f$-circulant matrices.

**Remark 3.3.** Recall that an $n \times l$ SRFT and SRHT matrices are the products $\sqrt{n/l} \ D \Omega_n R$ and $\sqrt{n/l} \ D \ H_n R$, respectively, where $H_n$ and $\Omega_n$ are the matrices of (3.1) and (3.2), $D = \text{diag}(u_i)_{i=0}^{n-1}$, $u_i$ are i.i.d. variables uniformly distributed on the circle $\{u : |u| = \sqrt{n/l}\}$, and $R$ is the $n \times l$ submatrix formed by $l$ columns of the identity matrix $I_n$, chosen uniformly at random. Equation (3.5) shows that we can obtain a SRFT matrix by pre-multiplying a circulant matrix by the matrix $\Omega_n$ and post-multiplying it by the above matrix $R$. 

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3.5 Inverses of bidiagonal matrices

**FAMILY (iv)** is formed by the inverses of \( n \times n \) bidiagonal matrices

\[
\hat{B} = (I_n + DZ)^{-1} \quad \text{or} \quad \hat{B} = (I_n + Z^T D)^{-1}
\]

for a matrix \( D \) of primitive type 2 and the down-shift matrix \( Z \). In particular,

\[
\hat{B} = (I_n + DZ)^{-1} = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
b_2b_3 & 1 & 0 & \cdots & 0 \\
-b_2b_3b_4 & b_3b_4 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \vdots \\
\pm b_2 \cdots b_n & \cdots & \cdots & -b_{n-2}b_{n-1}b_n & b_{n-1}b_n \end{pmatrix}
\]

if

\[
I_n + DZ = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
-b_2 & 1 & \ddots & 0 & 0 \\
0 & -b_3 & \ddots & \vdots & \vdots \\
\vdots & \ddots & 1 & 0 & \cdots \\
0 & \cdots & \cdots & -b_n & 1 \end{pmatrix}
\]

In order to pre-multiply a matrix \( \hat{B} = (I_n + DZ)^{-1} \) by a vector \( v \), however, we do not compute its entries, but solve the linear system of equations \((I_n + DZ)x = v\) by using \( 2n-1 \) flops or, in the real case, just \( n-1 \) additions and subtractions.

We can randomize the matrix \( \hat{B} \) by choosing up to \( n-1 \) random diagonal entries of the matrix \( D \) (whose leading entry makes no impact on \( \hat{B} \)).

Finally, \( \|\hat{B}\| \leq \sqrt{n} \) because nonzero entries of the lower triangular matrix \( \hat{B} = (I_n + DZ)^{-1} \) have absolute values 1, and clearly \( \|\hat{B}^{-1}\| = \|I_n + DZ\| \leq \sqrt{2} \). Hence \( \kappa(\hat{B}) = \|\hat{B}\| \|\hat{B}^{-1}\| \) (the spectral condition number of \( \hat{B} \)) cannot exceed \( \sqrt{2n} \) for \( \hat{B} = (I_n + DZ)^{-1} \), and the same bound holds for \( \hat{B} = (I_n + Z^T D)^{-1} \).

3.6 Summary of estimated numbers of flops and random variables involved

Table 3.1 shows upper bounds on

(a) the numbers of random variables involved into the \( n \times n \) matrices \( \hat{B} \) of the four families (i)–(iv) and
(b) the numbers of flops for pre-multiplication of such a matrix by a vector.\(^7\)

For comparison, using a Gaussian \( n \times n \) multiplier involves \( n^2 \) random variables and \((2n-1)n\) flops.

One can readily extend the estimates to \( n \times l \) submatrices \( B \) of the matrices \( \hat{B} \).

| family | (i) AH | (i) ASPH | (i) AF | (i) ASPF | (ii) | (iii) | (iv) |
|--------|--------|----------|--------|----------|------|-------|------|
| random variables | 0 | 2n | 0 | 2q + 1 | n | n - 1 |
| flops complex | \(dn\) | \((d + 1)n\) | 1.5\(dn\) | \((1.5d + 1)n\) | \((2q - 1)n\) | \((3d + 2)n\) | \(2n - 1\) |
| flops in real case | \(dn\) | \((d + 1)n\) | * | * | \(qn\) | * | \(n - 1\) |

\(^7\)The asterisks in the table show that the matrices of families (i) AF, (i) ASPF, and (iii) involve nonreal roots of unity.
Remark 3.4. Other observations besides flop estimates can be decisive. E.g., a special recursive structure of an ARSPH matrix $H_{2k,d}$ and an ARSPF matrix $\Omega_{2k,d}$ allows highly efficient parallel implementation of their pre-multiplication by a vector based on Application Specific Integrated Circuits (ASICs) and Field-Programmable Gate Arrays (FPGAs), incorporating Butterfly Circuits [DE].

3.7 Other basic families

There is a number of other interesting basic matrix families. According to [HMT11, Remark 4.6], “among the structured random matrices ..., one of the strongest candidates involves sequences of random Givens rotations”. They are dense unitary matrices

$$\frac{1}{\sqrt{n}}D_1G_1D_2G_2D_3\Omega_n,$$

for the DFT matrix $\Omega_n$, three random diagonal matrices $D_1, D_2$ and $D_3$ of primitive type 2, and two chains of Givens rotations $G_1$ and $G_2$, each of the form

$$G(\theta_1, \ldots, \theta_{n-1}) = P \prod_{i=1}^{n-1} G(i, i + 1, \theta_i)$$

for a random permutation matrix $P$,

$$G(i, i + 1, \theta_i) = \text{diag}(I_{i-1}, (\cos \theta_i, \sin \theta_i), I_{n-i-1})$$

Here $\theta_1, \ldots, \theta_{n-1}$ denote $n - 1$ random angles of rotation uniformly distributed in the range $0 \leq \phi < 2\pi$.

The DFT factor $\Omega_n$ makes the resulting matrices dense, but we can sparsify them by replacing that factor by an AF, ASF, APF, or ASPF matrix having recursion depth $d < \log_2(n)$. This would also decrease the number of flops involved in pre-multiplication of such a multiplier by a vector from order $n \log_2(n)$ to $1.5dn + O(n)$.

We can turn Givens sequences into distinct candidate families of efficient multipliers by replacing either or both of the Givens products with sparse matrices of Householder reflections matrices of the form $I_n - \frac{2\mathbf{v}^T\mathbf{u}}{\mathbf{u}^T\mathbf{u}}$ for fixed or random sparse vectors $\mathbf{h}$ (cf. [GL13, Section 5.1]).

We can obtain a variety of efficient multiplier families by properly combining the matrices of basic families (i)–(iv) and the above matrices. We can use just linear combinations, but can also apply block representation as in the following real $2 \times 2$ block matrix

$$\frac{1}{\sqrt{n}} \begin{pmatrix} Z_1(\mathbf{u}) & Z_1(\mathbf{v}) \\ Z_1(\mathbf{v}) & -Z_1(\mathbf{u}) \end{pmatrix}$$

$D$ for two vectors $\mathbf{u}$ and $\mathbf{v}$ and a matrix $D$ of primitive class 2.

We can define new matrix families by intertwining the Hadamard and Fourier recursive steps.

The reader can find other useful families of multipliers in our Section 5. E.g., according to our tests in Section 5, it turned out to be efficient to use nonsingular well-conditioned (rather than unitary) diagonal factors in the definition of some of our basic matrix families.

4 Proof of Theorems 1.1, 1.3, and 1.4

4.1 Low-rank representation: proof

Hereafter $\mathcal{R}(W)$ denotes the range (column span) of a matrix $W$.

Theorem 4.1. (i) For an $m \times n$ input matrix $M$ of rank $r \leq n \leq m$, its rank-$r$ representation is given by the products $R \bigl(R^T R\bigr)^{-1} R^T M = U(R)U(R)^T M$ provided that $R$ is an $n \times r$ matrix such that $\mathcal{R}(R) = \mathcal{R}(M)$ and that $U(R)$ is a matrix obtained by means of column orthogonalization of $R$.

(ii) $\mathcal{R}(R) = \mathcal{R}(M)$, for $R = MB$ and an $n \times r$ matrix $B$, with probability 1 if $B$ is Gaussian, and

(iii) with a probability at least $1 - r/|S|$ if an $n \times r$ matrix $B$ has i.i.d. random entries sampled uniformly from a finite set $S$ of cardinality $|S|$.

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Algorithm 1.1

Proof. Readily verify part (i) (cf. [S98, pages 60–61]). Then note that \( \mathcal{R}(MB) \subseteq \mathcal{R}(M) \), for an \( n \times r \) multiplier \( B \). Hence \( \mathcal{R}(MB) = \mathcal{R}(M) \) if and only if \( \text{rank}(MB) = r \), and therefore if and only if a multiplier \( B \) has full rank \( r \).

Now parts (ii) and (iii) follow from Theorem A.1. □

Parts (i) and (ii) of Theorem 4.1 imply parts (i) of Theorems 1.3 and 1.4.

4.2 Low-rank approximation: a basic step

Hereafter \( ||W||_F = \left( \sum_{j=1}^{\rho} \sigma_j^2(W) \right)^{1/2} \leq \sqrt{n} ||W|| \) denotes the Frobenius norm of a matrix \( W \) and \( W^+ = T_{W,p}^{\sum_{j=1}^{\rho} S_{W,p}^T} \) denotes the Moore–Penrose pseudo inverse of a matrix \( W \) of rank \( \rho \) having compact SVD \( S_{W,p} \sum_{j=1}^{\rho} S_{W,p}^T \). (Note that \( ||W^+|| = \frac{1}{\sigma_{\min}(W)} \).

In our proofs of Theorems 1.1, 1.3, and 1.4 we rely on the following lemma and theorem.

Lemma 4.1. (Cf. [GL13, Theorem 2.4.8].) For an integer \( r \) and an \( m \times n \) matrix \( M \) where \( m \geq n > r > 0 \), set to 0 the singular values \( \sigma_j(M) \), for \( j > r \), let \( M_r \) denote the resulting matrix, which is a closest rank-\( r \) approximation of \( M \), and write \( M = M_r + E \). Then

\[
||E|| = \sigma_{r+1}(M) \quad \text{and} \quad ||E||_F^2 = \sum_{j=r+1}^{n} \sigma_j^2 \leq \sigma_{r+1}(M)^2(n-r).
\]

Theorem 4.2. The error norm in terms of \( ||(M_rB)^+|| \). Assume dealing with the matrices \( M \) and \( \tilde{M} \) of Algorithm 1.1, \( M_r \) of Lemma 4.1, and \( B \in \mathbb{C}^{n \times l} \) of rank \( l \). Let \( \text{rank}(M_rB) = r \) and write \( E' = EB \) and \( \Delta = ||M - M|| \). Then

\[
||E'||_F \leq ||B||_F ||E||_F \leq ||B||_F \sigma_{r+1}(M) \sqrt{n-r} \quad (4.1)
\]

and

\[
|\Delta - \sigma_{r+1}(M)| \leq \sqrt{S} \quad ||(M_rB)^+|| \quad ||E'||_F + O(||E'||_F^2). \quad (4.2)
\]

Proof. Lemma 4.1 implies bound (4.1).

Next apply part (i) of Theorem 4.1 for matrix \( M_r \), replacing \( M \), recall that \( \text{rank}(M_rB) = l \), and obtain

\[
U(M_r B)U(M_r B)^T M_r = M_r, \quad \mathcal{R}(U(M_r B)) = \mathcal{R}(M_r B) = \mathcal{R}(M_r).
\]

Furthermore \( U(M_r B)^T (M - M_r) = O_{n,n} \). Therefore

\[
U(M_r B)U(M_r B)^T M = U(M_r B)U(M_r B)^T M_r = M_r.
\]

Consequently, \( M - U(M_r B)U(M_r B)^T M = M - M_r = E \), and so (cf. Lemma 4.1)

\[
||M - U(M_r B)U(M_r B)^T M|| = \sigma_{r+1}(M). \quad (4.3)
\]

Apply [PQY15, Corollary C.1], for \( A = M_r B \) and \( E \) replaced by \( E' = (M - M_r)B \), and obtain

\[
||U(M B)U(M B)^T - U(M_r B)U(M_r B)^T|| \leq \sqrt{S} ||(M_r B)^+|| \quad ||E'||_F + O(||E'||_F^2). \quad (4.4)
\]

Combine this bound with (4.3) and obtain (4.2). □

By combining bounds (4.1) and (4.2) obtain

\[
|\Delta - \sigma_{r+1}(M)| \leq \sqrt{S(n-r)} \sigma_{r+1}(M) \quad ||B||_F ||(M_r B)^+|| + O(\sigma_{r+1}^2(M)). \quad (4.4)
\]

In our applications the value \( \sqrt{S(n-r)} \sigma_{r+1}(M)||B||_F \) is small, and so the value \( |\Delta - \sigma_{r+1}(M)| \) is small unless the norm \( ||(M_r B)^+|| \) is large.
4.3 Proof of Theorem 1.1

If rank(MB) = r < r, then rank(\(M\)) \(\leq r < r\), \(\Delta \geq \sigma_{r-1}(M)\). In this case \(\Delta\) is not small because \(\text{rank}(M) = r > r\), and so Algorithm 1.1 applied to \(M\) with the multiplier \(B\) outputs FAILURE.

If rank(MB) = r > rank(MB) = r, then rank(MB - E) = r < r for a small-norm perturbation matrix \(E\). Hence \(\Delta \geq \sigma_{r-1}(M) - O(||E||)\), and then again Algorithm 1.1 applied to \(M\) with the multiplier \(B\) outputs FAILURE. This proves the “only if” part of the claim of Theorem 1.1.

Now let rank(MB) = r and assume that we scale the matrix \(B\) so that \(||B||_F = 1\). Then rank(MB) = r (and so we can apply bound (4.4)), and furthermore rank(MB) = rank(MB) = r. Equation (4.4) implies that \(\Delta \approx \sqrt{8(r - n)\sigma_{r-1}(M)B^+||}\). Therefore \(\Delta\) is a small positive value because rank(M) = r. Thus the value \(|\sigma_{r-1}|\) is small, and part “if” of Theorem 1.1 follows.

4.4 Detailed estimates for primal and dual low-rank approximation

The following theorem, proven in the next subsection, bounds the approximation errors and the probability of success of Algorithm 1.1 for \(B \in \mathbb{G}^{n \times l}\). Together these bounds imply part (ii) of Theorem 1.3.

**Theorem 4.3.** Suppose that Algorithm 1.1 has been applied to an \(m \times n\) matrix \(M\) having numerical rank \(r\) and that the multiplier \(B\) is an \(n \times l\) Gaussian matrix.

(i) Then the algorithm outputs an approximation \(\tilde{M}\) of a matrix \(M\) by a rank-\(l\) matrix within the error norm bound \(\Delta\) such that \(|\Delta - \sigma_{r-1}(M)| < f \sigma_{r-1}(M)\sigma_r(M) + O(\sigma^2_{r-1}(M))\) where \(f = \sqrt{8(n - r)}\nu_{F,n,l}^r\nu^r_{r,l}\) and \(\nu_{F,n,l}\) and \(\nu^r_{r,l}\) are random variables of Definition A.1.

(ii) \(\mathbb{E}(f) < (1 + \sqrt{n + \sqrt{l}})\frac{e}{p}\sqrt{8(n - r)rl}\), for \(p = l - r > 0\) and \(e = 2.71828\ldots\).

**Remark 4.1.** \(\sigma_{r+1}(M)\) is the optimal upper bound on the norm \(\Delta\), and the expected value \(\mathbb{E}(f)\) is reasonably small even for \(p = 1\). If \(p = 0\), then \(\mathbb{E}(f)\) is not defined, but the random variable \(\Delta\) estimated in Theorem 4.3 is still likely to be reasonably close to \(\sigma_{r+1}(M)\) (cf. part (ii) of Theorem A.3).

In Section 4.6 we prove the following elaboration upon dual Theorem 1.4.

**Theorem 4.4.** Suppose that Algorithm 1.1, applied to a small-norm perturbation of an \(m \times n\) factor-Gaussian matrix with expected rank \(r < m\), uses an \(n \times l\) multiplier \(B\) such that rank(B) = \(l\) and \(l \geq r\).

(i) Then the algorithm outputs a rank-\(l\) matrix \(\tilde{M}\) that approximates the matrix \(M\) within the error norm bound \(\Delta\) such that \(|\Delta - \sigma_{r+1}(M)| < f \sigma_{r+1}(M)\sigma_r(M) + O(\sigma^2_{r+1}(M))\), where \(f_d = \sqrt{8(n - r)}|l\nu_{r,l}^+\nu_{m,r}^+\kappa(B)\), \(\kappa(B) = ||B|| \cdot ||B^+||\), and \(\nu_{m,r}^+\) and \(\nu_{r,l}^+\) are random variables of Definition A.1.

(ii) \(\mathbb{E}(f_d) < e^2\sqrt{8(n - r)l}\kappa(B)\frac{1}{(m - r)p}\), for \(p = l - r > 0\) and \(e = 2.71828\ldots\).

**Remark 4.2.** The expected value \(\mathbb{E}(\nu_{m,r}^+) = \frac{e\sigma_r^2}{m - r}\) converges to 0 as \(m \to \infty\) provided that \(r \ll m\). Consequently the expected value \(\mathbb{E}(\Delta) = \sigma_{r+1}(M)\mathbb{E}(f_d)\) converges to the optimal value \(\sigma_{r+1}(M)\) as \(r \mapsto m\) \(\to \infty\) provided that \(B\) is a well-conditioned matrix of full rank and that \(1 < r \ll m\).

**Remark 4.3.** [HMT11, Theorem 10.8] also estimates the norm \(\Delta\), but our estimate in Theorem 4.3, in terms of random variables \(\nu_{F,n,l}\) and \(\nu^r_{r,l}\), is more compact, and our proof is distinct and shorter than one in [HMT11], which involves the proofs of [HMT11, Theorems 9.1, 10.4 and 10.6].

**Remark 4.4.** By virtue of Theorems A.1, rank(M,B) = r with probability 1 if the matrix B or M is Gaussian, which is the case of Theorems 4.3 and 4.4, and under the equation rank(M,B) = r we have proven bound (4.4).

**Remark 4.5.** The Power Scheme of increasing the output accuracy of Algorithm 1.1. See [RST09], [HMST11]. Define the Power Iterations \(M_i = (M^T M)^i\) for \(i = 1, 2, \ldots\). Then \(\sigma_i(M_i) = (\sigma_i(M))^{2i+1}\) for all \(i\) and \(j\) [HMT11, equation (4.5)]. Therefore, at a reasonable computational cost, one can dramatically decrease the ratio \(\frac{\sigma_{r+1}(M)}{\sigma_r(M)}\) and thus decrease the bounds of Theorems 4.3 and 4.4 accordingly.

In the next two subsections we deduce reasonable bounds on the norm \(||(M,B)^+||\) in both cases where \(M\) is a fixed matrix and \(B\) is a Gaussian matrix and where \(B\) is fixed matrix and \(M\) is a factor Gaussian matrix (cf. Theorems 4.5 and 4.6). The bounds imply Theorems 4.3 and 4.4.
4.5 Primal theorem: completion of the proof

Theorem 4.5. For \( M \in \mathbb{R}^{m \times n} \), \( B \in \mathcal{G}^{m \times l} \), and \( \nu_{r,l}^+ \) of Definition A.1, it holds that

\[
|| (M_r B)^+ || \leq \nu_{r,l}^+ / \sigma_r(M). \tag{4.5}
\]

Proof. Let \( M_r = S_r \Sigma_r T_r^T \) be compact SVD.

By applying Lemma A.1, deduce that \( T_r^T B \) is a \( r \times l \) Gaussian matrix.

Denote it \( G_{r,l} \) and obtain \( M_r B = S_r \Sigma_r T_r^T B = S_r \Sigma_r G_{r,l}. \)

Write \( H = \Sigma_r G_{r,l} \) and let \( H = S_H \Sigma_H T_H^T \) be compact SVD where \( S_H \) is a \( r \times r \) unitary matrix.

It follows that \( S = S_r S_H \) is an \( m \times r \) unitary matrix.

Hence \( M_r B = S \Sigma_H T_H^T \) and \( (M_r B)^+ = T_H (\Sigma_H)^+ S^T \) are compact SVDs of the matrices \( M_r B \) and \( (M_r B)^+ \), respectively.

Therefore \( || (M_r B)^+ || = || (\Sigma_H)^+ || = || (\Sigma_r G_{r,l})^+ || \leq || G_{r,l}^+ || || \Sigma_r^- || \).

Substitute \( || G_{r,l}^+ || = \nu_{r,l}^+ \) and \( || \Sigma_r^- || = 1/\sigma_r(M) \) and obtain the theorem. \( \square \)

Combine bounds (4.4), (4.5), and equation \( || B ||_F = \nu_{F,n,l} \) and obtain part (i) of Theorem 4.3. Combine that part with parts (ii) of Theorem A.2 and (iii) of Theorem A.3 and obtain part (ii) of Theorem 4.3.

4.6 Dual theorem: completion of the proof

Theorem 4.6. Suppose that \( U \in \mathbb{R}^{m \times r} \), \( V \in \mathcal{G}^{r \times n} \), \( \text{rank}(U) = r \leq \min\{m, n\} \), \( M = UV \), and \( B \) is a well-conditioned \( n \times l \) matrix of full rank \( l \) such that \( m \geq n > l \geq r \) and \( || B ||_F = 1 \). Then

\[
|| (MB)^+ || \leq || B^+ || \nu_{r,l}^+ ||U^+||. \tag{4.6}
\]

If in addition \( U \in \mathcal{G}^{m \times r} \), that is, if \( M \) is an \( m \times n \) factor-Gaussian matrix with expected rank \( r \), then

\[
|| (MB)^+ || \leq || B^+ || \nu_{m,r}^+ \nu_{r,l}^+. \tag{4.7}
\]

Proof. Combine compact SVDs \( U = S_U \Sigma_U T_U^T \) and \( B = S_B \Sigma_B T_B^T \) and obtain \( UVB = S_U \Sigma_U T_U^T V S_B \Sigma_B T_B^T \).

Here \( U, V, B, S_U, \Sigma_U, T_U, S_B, \Sigma_B, \) and \( T_B \) are matrices of the sizes \( m \times r, r \times n, m \times l, m \times r, r \times r, r \times r, \)

\( n \times l, l \times l \), and \( l \times l \), respectively.

Now observe that \( G_{r,l} = T_U^T V S_B \) is a \( r \times l \) Gaussian matrix, by virtue of Lemma A.1 (since \( V \) is a Gaussian matrix). Therefore \( UVB = S_U FT_B^T \), for \( F = S_U G_{r,l} \Sigma_B \).

Let \( F = S_F \Sigma_F T_F^T \) denote compact SVD where \( \Sigma_F = \text{diag}(\sigma_j(F))_{j=1}^r \) and \( S_F \) and \( T_F^T \) are unitary matrices of sizes \( r \times r \) and \( r \times l \), respectively.

Both products \( S_U S_F \in \mathbb{R}^{m \times r} \) and \( T_F^T T_B^T \in \mathbb{R}^{r \times l} \) are unitary matrices, and we obtain compact SVD \( MB = UVB = S_M B \Sigma_M T_M^T \) where \( S_M = S_U S_F, \Sigma_M = \Sigma_F, \) and \( T_M^T = T_F^T T_B^T \). Therefore

\[
|| (MB)^+ || = || \Sigma_M^+ || = || \Sigma_F^+ || = || F^+ ||.
\]

Note that \( F^+ = \Sigma_B^{-1} G_{r,l}^+ \Sigma_U^{-1} \) because \( \Sigma_B \) and \( \Sigma_U \) are square nonsingular diagonal matrices. Consequently

\[
|| (MB)^+ || = || F^+ || \leq || \Sigma_B^{-1} || || G_{r,l}^+ || || \Sigma_U^{-1} || = || B^+ || \nu_{r,l}^+ ||U^+||,
\]

and (4.6) follows. \( \square \)

We also need the following result implied by [S98, Corollary 1.4.19] for \( P = -C^{-1} E \):

Theorem 4.7. Suppose \( C \) and \( C + E \) are two nonsingular matrices of the same size and

\[
|| C^{-1} E || = \theta < 1.
\]

Then

\[
|| (C + E)^{-1} - C^{-1} || \leq \frac{\theta}{1-\theta} || C^{-1} ||;
\]

e.g., \( || (C + E)^{-1} - C^{-1} || \leq 0.5 || C^{-1} || \) if \( \theta \leq 1/3. \)
Combine (4.4), (4.6) and \( ||B||_F \leq ||B|| \sqrt{l} \) and obtain Theorem 4.4 provided that \( M \) is a factor-Gaussian matrix \( UV \) with expected rank \( r \). Apply Theorem 4.7 to extend the results to the case where \( M = UV + E \) and the norm \( ||E|| \) is small, completing the proof of Theorem 4.4.

**Remark 4.6.** If \( U \in \mathcal{G}^{m \times r} \), for \( m - r \geq 4 \), then it is likely that \( \text{rank}(U) = r \) by virtue of Theorem A.3, and our proof of bound (4.6) applies even if we assume that \( \text{rank}(U) = r \) rather than \( U \in \mathcal{G}^{m \times r} \).

## 5 Numerical Tests

Numerical experiments have been performed by Xiaodong Yan for Tables 5.2–5.4 and by John Svadlenka and Liang Zhao for the other tables. The tests have been run by using MATLAB in the Graduate Center of the City University of New York on a Dell computer with the Intel Core 2 2.50 GHz processor and 4G memory running Windows 7; in particular the standard normal distribution function randn of MATLAB has been applied in order to generate Gaussian matrices.

We calculated the \( \xi \)-rank, i.e., the number of singular values exceeding \( \xi \), by applying the MATLAB function "svd()". We have set \( \xi = 10^{-5} \) in Sections 5.1 and 5.2 and \( \xi = 10^{-6} \) in Section 5.3.

### 5.1 Tests for inputs generated via SVD

In the tests of this subsection we generated \( n \times n \) input matrices \( M \) by extending the customary recipes of [H02, Section 28.3]. Namely, we first generated matrices \( S_M \) and \( T_M \) by means of the orthogonalization of \( n \times n \) Gaussian matrices. Then we defined \( n \times n \) matrices \( M \) by their compact SVDs, \( M = S_M \Sigma_M T_M^T \), for \( \Sigma_M = \text{diag}(\sigma_j)_{j=1}^n; \sigma_j = 1/j, j = 1, \ldots, r, \sigma_j = 10^{-10}, j = r + 1, \ldots, n \), and \( n = 256, 512, 1024 \). (Hence \( ||M|| = 1 \) and \( \kappa(M) = ||M|| ||M^{-1}|| = 10^{10} \.)

Table 5.1 shows the average output error norms \( \Delta \) over 1000 tests of Algorithm 1.1 applied to these matrices \( M \) for each pair of \( n \) and \( r \), \( n = 256, 512, 1024, r = 8, 32 \), and each of the following three groups of multipliers: 3-AH multipliers, 3-ASPH multipliers, both defined by Hadamard recursion (3.3), for \( d = 3 \), and dense multipliers \( B = B(\pm1, 0) \) having i.i.d. entries \( \pm1 \) and 0, each value chosen with probability 1/3.

| \( n \) | \( r \) | 3-AH | 3-ASPH | \( B(\pm1, 0) \) |
|-------|------|------|--------|----------------|
| 256   | 8    | 2.25e-08 | 2.70e-08 | 2.52e-08 |
| 256   | 32   | 5.95e-08 | 1.47e-07 | 3.19e-08 |
| 512   | 8    | 4.80e-08 | 2.22e-07 | 4.76e-08 |
| 512   | 32   | 6.22e-08 | 8.91e-08 | 6.39e-08 |
| 1024  | 8    | 5.65e-08 | 2.86e-08 | 1.25e-08 |
| 1024  | 32   | 1.94e-07 | 5.33e-08 | 4.72e-08 |

Tables 5.2–5.4 show the mean and maximal values of such an error norm in the case of (a) real Gaussian multipliers \( B \) and dense real Gaussian subcirculant multipliers \( B \), for \( q = n \), each defined by its first column filled with either (b) i.i.d. Gaussian variables or (c) random variables \( \pm1 \). Here and hereafter in this section we assigned each random signs + or − with probability 0.5.

Table 5.5 displays the average error norms in the case of multipliers \( B \) of eight kinds defined below, all generated from the following Basic Sets 1, 2 and 3 of \( n \times n \) multipliers:

**Basic Set 1:** 3-APF multipliers defined by three Fourier recursive steps of equation (3.3), for \( d = 3 \), with no scaling, but with a random column permutation.

**Basic Set 2:** Sparse real circulant matrices \( Z_d(v) \) of family (ii) of Section 3.3 (for \( q = 10 \)) having the first column vectors \( v \) filled with zeros, except for ten random coordinates filled with random integers \( \pm1 \).

**Basic Set 3:** Sum of two scaled inverse bidiagonal matrices. We first filled the main diagonals of both matrices with the integer 101 and their first subdiagonals with \( \pm1 \). Then we multiplied each matrix by a diagonal matrix \( \text{diag}(\pm2^b_i) \), where \( b_i \) were random integers uniformly chosen from 0 to 3.

For multipliers \( B \) we used the \( n \times r \) western (leftmost) blocks of \( n \times n \) matrices of the following classes:
Table 5.2: Error norms for SVD-generated inputs and Gaussian multipliers

| r  | n    | mean       | max        |
|----|------|------------|------------|
| 8  | 256  | 7.54 × 10^{-8} | 1.75 × 10^{-9} |
| 8  | 512  | 4.57 × 10^{-8} | 5.88 × 10^{-9} |
| 8  | 1024 | 1.03 × 10^{-7} | 3.93 × 10^{-9} |
| 32 | 256  | 5.41 × 10^{-8} | 3.52 × 10^{-9} |
| 32 | 512  | 1.75 × 10^{-7} | 5.57 × 10^{-9} |
| 32 | 1024 | 1.79 × 10^{-7} | 3.36 × 10^{-9} |

Table 5.3: Error norms for SVD-generated inputs and Gaussian subcirculant multipliers

| r  | n    | mean       | max        |
|----|------|------------|------------|
| 8  | 256  | 3.24 × 10^{-8} | 2.66 × 10^{-9} |
| 8  | 512  | 5.58 × 10^{-8} | 1.14 × 10^{-5} |
| 8  | 1024 | 1.03 × 10^{-7} | 1.22 × 10^{-5} |
| 32 | 256  | 1.12 × 10^{-7} | 3.42 × 10^{-5} |
| 32 | 512  | 1.38 × 10^{-7} | 3.87 × 10^{-5} |
| 32 | 1024 | 1.18 × 10^{-7} | 1.84 × 10^{-5} |

1. a matrix from Basic Set 1;
2. a matrix from Basic Set 2;
3. a matrix from Basic Set 3;
4. the product of two matrices of Basic Set 1;
5. the product of two matrices of Basic Set 2;
6. the product of two matrices of Basic Set 3;
7. the sum of two matrices of Basic Sets 1 and 3, and
8. the sum of two matrices of Basic Sets 2 and 3.

The tests produced the results similar to the ones of Tables 5.1–5.4.

In sum, for all classes of input pairs $M$ and $B$ and all pairs of integers $n$ and $r$, Algorithm 1.1 with our preprocessing has consistently output approximations to rank-$r$ input matrices with the average error norms ranged from $10^{-7}$ or $10^{-8}$ to about $10^{-9}$ in all our tests.

We summarize the results of the tests of this subsection for $n = 1024$ and $r = 8, 32$ in Figure 4.

5.2 Tests for inputs generated via the discretization of a Laplacian operator and via the approximation of an inverse finite-difference operator

Next we present the test results for Algorithm 1.1 applied to input matrices for computational problems of two kinds, both replicated from [HMT11], namely, the matrices of

(i) the discretized single-layer Laplacian operator and
(ii) the approximation of the inverse of a finite-difference operator.

Input matrices (i). We considered the Laplacian operator $[S\sigma](x) = c \int_{\Gamma_1} \log |x - y|\sigma(y)dy, x \in \Gamma_2$, from [HMT11, Section 7.1], for two contours $\Gamma_1 = C(0, 1)$ and $\Gamma_2 = C(0, 2)$ on the complex plane. Its discretization defines an $n \times n$ matrix $M = (m_{ij})_{i,j=1}^n$ where $m_{i,j} = c \int_{\Gamma_1,j} \log |2\omega^i - y|dy$ for a constant $c$ such that $\|M\| = 1$ and for the arc $\Gamma_{1,j}$ of the contour $\Gamma_1$ defined by the angles in the range $[\frac{2j\pi}{n}, \frac{2(j+1)\pi}{n}]$. 

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Table 5.4: Error norms for SVD-generated inputs and random subcirculant multipliers filled with $\pm 1$

| $r$ | $n$ | mean  | max       |
|-----|-----|-------|-----------|
| 8   | 256 | $7.70 \times 10^{-3}$ | $2.21 \times 10^{-7}$ |
| 8   | 512 | $1.10 \times 10^{-8}$ | $2.21 \times 10^{-7}$ |
| 8   | 1024| $1.69 \times 10^{-8}$ | $4.15 \times 10^{-7}$ |
| 32  | 256 | $1.51 \times 10^{-8}$ | $3.05 \times 10^{-7}$ |
| 32  | 512 | $2.11 \times 10^{-8}$ | $3.60 \times 10^{-7}$ |
| 32  | 1024| $3.21 \times 10^{-8}$ | $5.61 \times 10^{-7}$ |

Table 5.5: Error norms for SVD-generated inputs and multipliers of eight classes

| $n$ | $r$ | class 1 | class 2 | class 3 | class 4 | class 5 | class 6 | class 7 | class 8 |
|-----|-----|---------|---------|---------|---------|---------|---------|---------|---------|
| 256 | 8   | 5.94e-09 | 4.35e-08 | 2.64e-08 | 2.20e-08 | 7.73e-07 | 5.15e-09 | 4.08e-09 | 2.10e-09 |
| 256 | 32  | 2.40e-08 | 2.55e-09 | 8.23e-08 | 1.58e-08 | 4.58e-09 | 1.36e-08 | 2.26e-09 | 8.83e-09 |
| 512 | 8   | 1.11e-08 | 8.01e-09 | 2.36e-09 | 7.48e-09 | 1.53e-08 | 8.15e-09 | 1.39e-08 | 3.86e-09 |
| 512 | 32  | 1.61e-08 | 4.81e-09 | 1.61e-08 | 2.83e-09 | 2.35e-08 | 3.48e-08 | 2.25e-08 | 1.67e-08 |
| 1024| 8   | 5.40e-09 | 3.44e-09 | 6.82e-08 | 4.39e-08 | 1.20e-08 | 4.44e-09 | 2.68e-09 | 4.30e-09 |
| 1024| 32  | 2.18e-08 | 2.03e-08 | 8.72e-08 | 2.77e-08 | 3.15e-08 | 7.99e-09 | 9.64e-09 | 1.49e-08 |

We applied Algorithm 1.1 supported by three iterations of the Power Scheme of Remark 4.5 and used with multipliers $B$ being the $n \times r$ leftmost submatrices of $n \times n$ matrices of the following five classes:

- Gaussian multipliers,
- Gaussian Toeplitz multipliers $T = (t_{i-j})_{i=0}^{n-1}$ for i.i.d. Gaussian variables $t_1, \ldots, t_n$.
- Gaussian circulant multipliers $\sum_{i=0}^{n-1} v_i Z_1^i$, for i.i.d. Gaussian variables $v_0, \ldots, v_{n-1}$ and the unit circular matrix $Z_1$ of Section 3.1.
- Abridged permuted Fourier (3-APF) multipliers, and
- Abridged permuted Hadamard (3-APH) multipliers.

As in the previous subsection, we defined each 3-APF and 3-APH matrix by applying three recursive steps of equation (3.3) followed by a single random column permutation.

We applied Algorithm 1.1 with multipliers of all five listed classes. For each setting we repeated the test 1000 times and calculated the mean and standard deviation of the error norm $||\tilde{M} - M||$.

Input matrices (ii). We similarly applied Algorithm 1.1 to the input matrix $M$ being the inverse of a large sparse matrix obtained from a finite-difference operator of [HMT11, Section 7.2] and observed similar results with all structured and Gaussian multipliers.

We performed 1000 tests for every class of pairs of $n \times n$ or $m \times n$ matrices of classes (i) or (ii), respectively, and $n \times r$ multipliers for every fixed triple of $m$, $n$, and $r$ or pair of $n$ and $r$.

Tables 5.6 and 5.7 display the resulting data for the mean values and standard deviation of the error norms, and we summarize the results of the tests of this subsection in Figure 5.

5.3 Tests with additional classes of multipliers

In this subsection we display the mean values and standard deviations of the error norms observed when we repeated the tests of the two previous subsections for the same three classes of input matrices (that is, SVD-generated, Laplacian, and matrices obtained by discretization of finite difference operators), but now we applied Algorithm 1.1 with seventeen additional classes of multipliers (besides its control application with Gaussian multipliers).
We tested Algorithm 1.1 applied to $1024 \times 1024$ SVD-generated input matrices having numerical nullity $r = 32$, to $400 \times 400$ Laplacian input matrices having numerical nullity $r = 3$, and to $408 \times 800$ matrices having numerical nullity $r = 64$ and representing finite-difference inputs.

Then again we repeated the tests 1000 times for each class of input matrices and each size of an input and a multiplier, and we display the resulting average error norms in Table 5.3 and Figures 6–8.

We used multipliers defined as the seventeen sums of $n \times r$ matrices of the following basic families:

- 3-ASPH matrices
- 3-APH matrices
- Inverses of bidiagonal matrices
- Random permutation matrices

Here every 3-APH matrix has been defined by three Hadamard’s recursive steps (3.1) followed by random permutation. Every 3-ASPH matrix has been defined similarly, but also random scaling has been applied.
Table 5.6: Low-rank approximation of Laplacian matrices

| n  | multiplier | r   | mean         | std           |
|----|------------|-----|--------------|---------------|
| 200| Gaussian   | 3.00| 1.58e-05     | 1.24e-05      |
| 200| Toeplitz   | 3.00| 1.83e-05     | 7.05e-06      |
| 200| Circulant  | 3.00| 3.14e-05     | 2.30e-05      |
| 200| 3-APF      | 3.00| 8.50e-06     | 5.15e-15      |
| 200| 3-APH      | 3.00| 2.18e-05     | 6.48e-14      |
| 400| Gaussian   | 3.00| 1.53e-05     | 1.37e-06      |
| 400| Toeplitz   | 3.00| 1.82e-05     | 1.59e-05      |
| 400| Circulant  | 3.00| 4.37e-05     | 3.94e-05      |
| 400| 3-APF      | 3.00| 8.33e-06     | 1.02e-14      |
| 400| 3-APH      | 3.00| 2.18e-05     | 9.08e-14      |
| 2000| Gaussian | 3.00 | 2.10e-05 | 2.28e-05 |
| 2000| Toeplitz | 3.00 | 2.02e-05 | 1.42e-05 |
| 2000| Circulant | 3.00 | 6.23e-05 | 7.62e-05 |
| 2000| 3-APF    | 3.00 | 2.11e-05 | 4.49e-12 |
| 2000| 3-APH    | 3.00 | 2.18e-05 | 3.17e-05 |
| 4000| Gaussian | 3.00 | 2.52e-05 | 3.64e-05 |
| 4000| Toeplitz | 3.00 | 2.52e-05 | 8.27e-05 |
| 4000| Circulant | 3.00 | 5.69e-05 | 1.28e-13 |
| 4000| 3-APF    | 3.00 | 3.17e-05 | 8.64e-12 |

with a diagonal matrix \( D = \text{diag}(d_i)_{i=1}^n \) having the values of random i.i.d. variables \( d_i \) uniformly chosen from the set \( \{1/4, 1/2, 1, 2, 4\} \).

We permuted all inverses of bidiagonal matrices except for Class 5 of multipliers.

Describing our multipliers we use the following acronyms and abbreviations: “IBD” for “the inverse of a bidiagonal”, “MD” for “the main diagonal”, “SB” for “subdiagonal”, and “SP” for “superdiagonal”. We write “MD\(i\)”, “\(k\)th SB\(i\)” and “\(k\)th SP\(i\)” in order to denote that the main diagonal, the \(k\)th subdiagonal, or the \(k\)th superdiagonal of a bidiagonal matrix, respectively, was filled with the integer \(i\).

- **Class 0**: Gaussian
- **Class 1**: Sum of a 3-ASPH and two IBD matrices:
  B1 with MD−1 and 2nd SB−1 and B2 with MD+1 and 1st SP+1
- **Class 2**: Sum of a 3-ASPH and two IBD matrices:
  B1 with MD+1 and 2nd SB−1 and B2 with MD+1 and 1st SP−1
- **Class 3**: Sum of a 3-ASPH and two IBD matrices:
  B1 with MD+1 and 1st SB−1 and B2 with MD+1 and 1st SP−1
- **Class 4**: Sum of a 3-ASPH and two IBD matrices:
  B1 with MD+1 and 1st SB+1 and B2 with MD+1 and 1st SP−1
- **Class 5**: Sum of a 3-ASPH and two IBD matrices:
  B1 with MD+1 and 1st SB+1 and B2 with MD+1 and 1st SP−1
- **Class 6**: Sum of a 3-ASPH and three IBD matrices:
  B1 with MD−1 and 2nd SB−1, B2 with MD+1 and 1st SP+1 and B3 with MD+1 and 9th SB+1
- **Class 7**: Sum of a 3-ASPH and three IBD matrices:
  B1 with MD+1 and 2nd SB−1, B2 with MD+1 and 1st SP−1, and B3 with MD+1 and 8th SP+1
Table 5.7: Low-rank approximation of the matrices of discretized finite-difference operator

| \( m \) | \( n \) | multiplier | \( r \) | mean   | std    |
|------|------|-----------|------|--------|--------|
| 88   | 160  | Gaussian  | 5.00 | 1.53e-05 | 1.03e-05 |
| 88   | 160  | Toeplitz  | 5.00 | 1.37e-05 | 1.17e-05 |
| 88   | 160  | Circulant | 5.00 | 2.79e-05 | 2.33e-05 |
| 88   | 160  | 3-APF     | 5.00 | 4.84e-04 | 2.94e-14 |
| 88   | 160  | 3-APH     | 5.00 | 4.84e-04 | 5.76e-14 |
| 208  | 400  | Gaussian  | 43.00| 4.02e-05 | 1.05e-05 |
| 208  | 400  | Toeplitz  | 43.00| 8.19e-05 | 1.63e-05 |
| 208  | 400  | Circulant | 43.00| 8.72e-05 | 2.09e-05 |
| 208  | 400  | 3-APF     | 43.00| 1.24e-04 | 2.40e-13 |
| 208  | 400  | 3-APH     | 43.00| 1.29e-04 | 4.62e-13 |
| 408  | 800  | Gaussian  | 64.00| 6.09e-05 | 1.75e-05 |
| 408  | 800  | Toeplitz  | 64.00| 1.07e-04 | 2.67e-05 |
| 408  | 800  | Circulant | 64.00| 1.04e-04 | 2.67e-05 |
| 408  | 800  | 3-APF     | 64.00| 1.38e-04 | 6.42e-12 |
| 408  | 800  | 3-APH     | 64.00| 1.38e-04 | 8.65e-12 |

- Class 8: Sum of a 3-ASPH and three IBD matrices:
  B1 with MD+1 and 1st SB−1, B2 with MD+1 and 1st SP−1, and B3 with MD+1 and 4th SB+1

- Class 9: Sum of a 3-ASPH and three IBD matrices:
  B1 with MD+1 and 1st SB−1, B2 with MD+1 and 1st SP−1, and B3 with MD−1 and 3rd SP+1

- Class 10: Sum of three IBD matrices:
  B1 with MD+1 and 1st SB+1, B2 with MD+1 and 1st SP−1, and B3 with MD−1 and 3rd SP+1

- Class 11: Sum of a 3-APH and three IBD matrices:
  B1 with MD+1 and 2nd SB−1, B2 with MD+1 and 1st SP−1, and B3 with MD+1 and 8th SP+1

- Class 12: Sum of a 3-APH and two IBD matrices:
  B1 with MD+1 and 1st SB−1 and B2 with MD+1 and 1st SP−1

- Class 13: Sum of a 3-ASPH and a permutation matrix

- Class 14: Sum of a 3-ASPH and two permutation matrices

- Class 15: Sum of a 3-ASPH and three permutation matrices

- Class 16: Sum of a 3-APH and three permutation matrices

- Class 17: Sum of a 3-APH and two permutation matrices

The tests show quite accurate outputs even where we applied Algorithm 1.1 with very sparse multipliers of classes 13–17.
Figure 6: Relative Error Norm for SVD-generated Input Matrices

Figure 7: Relative Error Norm for Laplacian Input Matrices
Figure 8: Relative Error Norm for Finite-Difference Input Matrices

| Class No. | SVD-generated Matrices | Laplacian Matrices | Finite Difference Matrices |
|-----------|------------------------|-------------------|---------------------------|
|           | Mean       | Std     | Mean     | Std     | Mean     | Std     |
| Class 0   | 3.54E-09  | 3.28E-09| 4.10E-14 | 2.43E-13| 1.61E-06 | 1.35E-06|
| Class 1   | 1.07E-08  | 3.82E-09| 2.05E-13 | 1.62E-13| 4.58E-06 | 9.93E-07|
| Class 2   | 1.16E-08  | 6.62E-09| 6.07E-13 | 5.20E-13| 4.67E-06 | 1.04E-06|
| Class 3   | 1.23E-08  | 5.84E-09| 1.69E-13 | 1.34E-13| 4.52E-06 | 1.01E-06|
| Class 4   | 1.25E-08  | 1.07E-08| 2.46E-13 | 3.44E-13| 4.72E-06 | 9.52E-07|
| Class 5   | 1.13E-08  | 6.09E-09| 1.93E-13 | 1.48E-13| 4.38E-06 | 8.64E-07|
| Class 6   | 1.12E-08  | 8.79E-09| 9.25E-13 | 2.64E-12| 5.12E-06 | 1.29E-06|
| Class 7   | 1.16E-08  | 7.42E-09| 5.51E-13 | 5.35E-13| 4.79E-06 | 1.12E-06|
| Class 8   | 1.33E-08  | 1.00E-08| 1.98E-13 | 1.30E-13| 4.60E-06 | 9.52E-07|
| Class 9   | 1.08E-08  | 4.81E-09| 2.09E-13 | 3.60E-13| 4.47E-06 | 8.57E-07|
| Class 10  | 1.18E-08  | 5.51E-09| 1.87E-13 | 1.77E-13| 4.63E-06 | 9.28E-07|
| Class 11  | 1.18E-08  | 6.23E-09| 1.78E-13 | 1.42E-13| 4.55E-06 | 9.08E-07|
| Class 12  | 1.28E-08  | 1.40E-08| 2.33E-13 | 3.44E-13| 4.49E-06 | 9.67E-07|
| Class 13  | 1.43E-08  | 1.87E-08| 1.78E-13 | 1.61E-13| 4.74E-06 | 1.19E-06|
| Class 14  | 1.22E-08  | 1.26E-08| 2.21E-13 | 2.83E-13| 4.75E-06 | 1.14E-06|
| Class 15  | 1.51E-08  | 1.18E-08| 3.57E-13 | 9.27E-13| 4.61E-06 | 1.08E-06|
| Class 16  | 1.19E-08  | 6.93E-09| 2.24E-13 | 1.76E-13| 4.74E-06 | 1.09E-06|
| Class 17  | 1.26E-08  | 1.16E-08| 2.15E-13 | 1.70E-13| 4.59E-06 | 1.12E-06|

Table 5.8: Relative Error Norm in Tests with Multipliers of Additional Classes
6 Conclusions: Sample Extension to the Computations for Least Squares Regression (LSR)

Our duality techniques for the average inputs can be extended to the acceleration of various matrix computations involving random multipliers. In this concluding section we describe such a sample extension to the following fundamental problem of matrix computations (cf. [GL13]).

Problem 6.1. Least Squares Solution of an Overdetermined Linear System of Equations. Given two integers m and d such that $1 \leq d < m$, a matrix $A \in \mathbb{R}^{m \times d}$, and a vector $b \in \mathbb{R}^m$, compute a vector $x$ that minimizes the norm $\|Ax - b\|$.

If a matrix $A$ has full rank $n$, then unique solution is given by the vector $x = (A^T A)^{-1} A^T b$, satisfying the linear system of normal equations $A^T A x = A^T b$. Otherwise solution is not unique, and a solution $x$ having the minimum norm is given by the vector $A^+ b$. In the important case where $m \gg d$ and an approximate solution is acceptable, Sarlós in [S06] proposed to simplify the computations as follows:

Algorithm 6.1. Least Squares Regression (LSR).

Initialization: Fix an integer $k$ such that $1 \leq k \ll m$.

Computations: 1. Generate a scaled $k \times m$ Gaussian matrix $F$.

2. Compute the matrix $FA$ and the vector $Fb$.

3. Output a solution $\hat{x}$ to the compressed Problem 6.1 where the matrix $A$ and the vector $b$ are replaced by the matrix $FA$ and the vector $Fb$, respectively.

Now write $M = (A | b)$ and $y = \begin{pmatrix} x \\ -1 \end{pmatrix}$ and compare the error norms $\|FM\hat{y}\| = \|FA\hat{x} - Fb\|$, (of the output $\hat{x}$ of the latter algorithm) and $\|My\| = \|Ax - b\|$, (of the solution $x$ of the original Problem 6.1).

Theorem 6.1. [W14, Theorem 2.3]. Suppose that we are given two tolerance values $\delta$ and $\xi$, $0 < \delta < 1$ and $0 < \xi < 1$, three integers $k$, $m$ and $d$ such that $1 \leq d < m$ and

$$k = (d + \log(1/\delta)\xi^{-2})\theta,$$

for a certain constant $\theta$, and a matrix $G_{k,m} \in \mathcal{G}^{k \times m}$. Then, with a probability at least $1 - \delta$, it holds that

$$(1 - \xi)\|My\| \leq \frac{1}{\sqrt{k}}\|G_{k,m}My\| \leq (1 + \xi)\|My\|$$

for all matrices $M \in \mathbb{R}^{m \times (d+1)}$ and all vectors $y = (y_i)_{i=0}^d \in \mathbb{R}^{d+1}$ normalized so that $y_d = -1$.

The theorem implies that with a probability at least $1 - \delta$, Algorithm 6.1 outputs an approximate solution to Problem 6.1 within the error norm bound $\xi$ provided that $k = (d + \log(1/\delta)\xi^{-2})\theta$ and $F = \frac{1}{\sqrt{k}}G_{k,m}$.

For $m \gg k$, the computational cost of performing the algorithm for approximate solution dramatically decreases versus the cost of computing exact solution, but can still be prohibitively high at the stage of computing the matrix product $FM$ for $F = G_{k,m}$. In a number of papers the former cost estimate has been dramatically decreased by means of replacing a multiplier $F = \frac{1}{\sqrt{k}}G_{k,m}$ with various random sparse and structured matrices (see [W14, Section 2.1]), for which the bound of Theorem 6.1 still holds for all matrices $M \in \mathbb{R}^{m \times (d+1)}$, although at the expense of increasing significantly the dimension $k$.

Can we achieve similar progress without such an increase? [CW13] provides positive probabilistic answer based on the Johnson–Lindenstrauss Theorem, while the following theorem does this by using our duality approach in the case where $M$ is the average matrix in $\mathbb{R}^{m \times (d+1)}$ under the Gaussian probability distribution:

Theorem 6.2. Dual LSR. The bound of Theorem 6.1 holds with a probability at least $1 - \delta$ where $\sqrt{k} M \in \mathcal{G}^{m \times (d+1)}$ and $F \in \mathbb{R}^{k \times m}$ is an orthogonal matrix.

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8Such approximate solutions serve as preprocessors for practical implementation of numerical linear algebra algorithms for Problem 6.1 of least squares computation [M11, Section 4.5], [RT08], [AMT10].
Proof. Theorem 6.1 has been proven in [W14, Section 2] in the case where \( \sqrt{k} FM \in G^{k \times (d+1)} \). This is the case where \( \sqrt{k} F \in G^{k \times m} \) and \( M \) is an orthogonal \( m \times (d+1) \) matrix, but is also the case under the assumptions of Theorem 6.2, by virtue of Lemma 6.1.

Theorem 6.2 supports the computation of an approximate randomized solution of LSR Problem 6.1 for any orthogonal multiplier \( F \) (e.g., an abridged scaled Hadamard’s multiplier or a count sketch multiplier) and for an input matrix \( M \in \mathbb{R}^{m \times (d+1)} \) average under the Gaussian probability distribution.

It follows that in this case Algorithm 6.1 can fail only for a narrow class of pairs \( F \) and \( M \) where \( F \) denotes orthogonal matrices in \( \mathbb{R}^{k \times m} \) and \( M \) denotes matrices in \( \mathbb{R}^{m \times (d+1)} \), and even in the unlikely case of failure we can still have good chances for success if we apply heuristic recipes of our Section 2.

Appendix

A Randomized Matrix Computations

Theorem A.1. Suppose that \( A \) is an \( m \times n \) matrix of full rank \( k = \min\{m,n\} \), \( F \) and \( H \) are \( r \times m \) and \( n \times r \) matrices, respectively, for \( r \leq k \), and the entries of these two matrices are nonconstant linear combinations of finitely many i.i.d. random variables \( v_1, \ldots, v_h \).

Then the matrices \( F, FA, H, \) and \( AH \) have full rank \( r \)

(i) with probability \( 1 \) if \( v_1, \ldots, v_h \) are Gaussian variables and

(ii) with a probability at least \( 1 - r/|S| \) if they are random variables sampled under the uniform probability distribution from a finite set \( S \) having cardinality \(|S|\).

Proof. The determinant, \( \det(B) \), of any \( r \times r \) block \( B \) of a matrix \( F, FA, H, \) or \( AH \) is a polynomial of degree \( r \) in the variables \( v_1, \ldots, v_h \), and so the equation \( \det(B) = 0 \) defines an algebraic variety of a lower dimension in the linear space of these variables (cf. [BV88, Proposition 1]). Clearly, such a variety has Lebesgue and Gaussian measures 0, both being absolutely continuous with respect to one another. This implies part (i) of the theorem. Derivation of part (ii) from a celebrated lemma of [DL78], also known from [Z79] and [S80], is a well-known pattern, specified in some detail in [PW08].

Lemma A.1. (Rotational invariance of a Gaussian matrix.) Suppose that \( k, \ m, \) and \( n \) are three positive integers, \( G \) is an \( m \times n \) Gaussian matrix, and \( S \) and \( T \) are \( k \times m \) and \( n \times k \) orthogonal matrices, respectively. Then \( SG \) and \( GT \) are Gaussian matrices.

We state the following estimates for real matrices, but similar estimates in the case of complex matrices can be found in [D88], [E88], [CD05], and [ES05]:

Definition A.1. Norms of random matrices and expected value of a random variable. Write \( \nu_{m,n} = ||G|| \), \( \nu_{m,n}^+ = ||G^+|| \), and \( \nu_{m,n,F}^+ = ||G^+||_F \), for a Gaussian \( m \times n \) matrix \( G \), and write \( E(v) \) for the expected value of a random variable \( v \). (\( \nu_{m,n} = \nu_{n,m} \), \( \nu_{m,n}^+ = \nu_{n,m}^+ \), and \( \nu_{F, m,n} = \nu_{F, n,m} \), for all pairs of \( m \) and \( n \)).

Theorem A.2. (Cf. [DS01, Theorem II.7].) Suppose that \( m \) and \( n \) are positive integers, \( h = \max\{m,n\} \), \( t \geq 0 \). Then

(i) Probability \( \{\nu_{m,n} > t + \sqrt{m} + \sqrt{n}\} \leq \exp(-t^2/2) \) and

(ii) \( E(\nu_{m,n}) < 1 + \sqrt{m} + \sqrt{n} \).

Theorem A.3. Let \( \Gamma(x) = \int_0^x \exp(-t)t^{x-1}dt \) denote the Gamma function and let \( x > 0 \). Then

(i) Probability \( \{\nu_{m,n}^+ > m/x^2\} < \frac{1}{\Gamma(m-n+1)} \) for \( m \geq n \geq 2 \),

(ii) Probability \( \{\nu_{m,n}^+ > x\} \leq 2.35\sqrt{m}/x \) for \( n \geq 2 \),

(iii) \( E(\nu_{m,n}^+) \leq e\sqrt{m}/m - n \), provided that \( m \neq n \) and \( e = 2.71828 \ldots \).

Proof. See [CD05, Proof of Lemma 4.1] for part (i), [SST06, Theorem 3.3] for part (ii), and [HMT11, Proposition 10.2] for part (iii).
The probabilistic upper bounds of Theorem A.3 on $\nu_{m,n}^+$ are reasonable already in the case of square matrices, that is, where $m = n$, but are strengthened very fast as the difference $|m - n|$ grows from 1.

Theorems A.2 and A.3 combined imply that an $m \times n$ Gaussian matrix is well-conditioned unless the integer $m + n$ is large or the integer $|m - n|$ is close to 0. With some grain of salt we can still consider such a matrix well-conditioned even where the integer $|m - n|$ is small or vanishes provided that the integer $m$ is not large. Clearly, these properties can be extended immediately to all submatrices.

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References

[AMT10] H. Avron, P. Maymounkov, S. Toledo, Blendenpik: Supercharging LAPACKs Least-squares Solver, SIAM Journal on Scientific Computing, 32, 1217–1236, 2010.

[AS00] N. Alon, J. Spencer, The Probabilistic Method, Wiley, 2000 (second edition).

[B10] S. Börm, Efficient Numerical Methods for Non-local Operators: $H^2$-Matrix Compression, Algorithms and Analysis, European Math. Society, 2010.

[BBBDD14] M. Baboulin, D. Becker, G. Bosilca, A. Danalis, J. Dongarra, An Efficient Distributed Randomized Algorithm for Solving Large Dense Symmetric Indefinite Linear Systems, Parallel Computing (7th Workshop on Parallel Matrix Algorithms and Applications), 40, 7, 213–223, 2014.

[BCDHKS14] G. Ballard, E. Carson, J. Demmel, M. Hoemmen, N. Knight, O. Schwartz, Communication Lower Bounds and Optimal Algorithms for Numerical Linear Algebra. Acta Numerica, 23, 1–155, 2014.

[BGH03] S. Börm, L. Grasedyck, W. Hackbusch, Introduction to Hierarchical Matrices with Applications, Engineering Analysis with Boundary Elements, 27, 5, 405–422, 2003.

[BGP05] A. Bini, L. Gemignani, V. Y. Pan, Fast and Stable QR Eigenvalue Algorithms for Generalized Semiseparable Matrices and Secular Equation, Numerische Mathematik, 100, 3, 373–408, 2005.

[BV88] W. Bruns, U. Vetter, Determinantal Rings, Lecture Notes in Math., 1327, Springer, 1988.

[C00] B. A. Cipra, The Best of the 20th Century: Editors Name Top 10 Algorithms, SIAM News (Society for Industrial and Applied Mathematics), 33, 4, 2, May 16, 2000.

[CFDF04] M. Charikar, K. Chen, M. Farach-Colton, Finding Frequent Items in Data Streams, Theoretical Computer Science, 312, 1, 3–15, 2004.

[CD05] Z. Chen, J. J. Dongarra, Condition Numbers of Gaussian Random Matrices, SIAM J. on Matrix Analysis and Applications, 27, 603–620, 2005.

[CGR88] J. Carrier, L. Greengard, V. Rokhlin, A Fast Adaptive Algorithm for Particle Simulation, SIAM Journal on Scientific Computing, 9, 669–686, 1988.

[CW13] K. L. Clarkson, D. P. Woodruff, Low Rank Approximation and Regression in Input Sparsity Time, Proceedings of the Forty-fifth Annual ACM Symposium on Theory of Computing (STOC ’13), 81–90, ACM New York, NY, USA, 2013. Full Version in arXiv:1207.6365.

[D88] J. Demmel, The Probability That a Numerical Analysis Problem Is Difficult, Math. of Computation, 50, 449–480, 1988.

[DE] Dillon Engineering, http://www.dilloneng.com/fft_ip/parallel-fft.

[DL78] R. A. Demillo, R. J. Lipton, A Probabilistic Remark on Algebraic Program Testing, Information Processing Letters, 7, 4, 193–195, 1978.
K. R. Davidson, S. J. Szarek, Local Operator Theory, Random Matrices, and Banach Spaces, in Handbook on the Geometry of Banach Spaces (W. B. Johnson and J. Lindenstrauss editors), pages 317–368, North Holland, Amsterdam, 2001.

A. Edelman, Eigenvalues and Condition Numbers of Random Matrices, SIAM J. on Matrix Analysis and Applications, 9, 4, 543–560, 1988.

Y. Eidelman, I. Gohberg, I. Haimovici, Separable Type Representations of Matrices and Fast Algorithms, volumes 1 and 2, Birkhäuser, 2013.

A. Edelman, B. D. Sutton, Tails of Condition Number Distributions, SIAM J. on Matrix Analysis and Applications, 27, 2, 547–560, 2005.

R. Freivalds, Probabilistic Machines Can Use Less Running Time, IFIP Congress 1977, 839–842, 1977.

L. Grasedyck, W. Hackbusch, Construction and Arithmetics of H-Matrices, Computing, 70, 4, 295–334, 2003.

G. H. Golub, C. F. Van Loan, Matrix Computations, The Johns Hopkins University Press, Baltimore, Maryland, 2013 (fourth edition).

L. Greengard, V. Rokhlin, A Fast Algorithm for Particle Simulation, Journal of Computational Physics, 73, 325–348, 1987.

N. J. Higham, Accuracy and Stability in Numerical Analysis, SIAM, Philadelphia, 2002.

N. Halko, P. G. Martinsson, Y. Shkolnisky, M. Tygert, An Algorithm for the Principal Component Analysis of Large Data Sets, SIAM J. Scientific Computation, 33, 5, 2580–2594, 2011.

N. Halko, P. G. Martinsson, J. A. Tropp, Finding Structure with Randomness: Probabilistic Algorithms for Approximate Matrix Decompositions, SIAM Review, 53, 2, 217–288, 2011.

M. W. Mahoney, Randomized Algorithms for Matrices and Data, Foundations and Trends in Machine Learning, NOW Publishers, 3, 2, 2011. Preprint: arXiv:1104.5557 (2011) (Abridged version in: Advances in Machine Learning and Data Mining for Astronomy, edited by M. J. Way et al., pp. 647–672, 2012.)

R. Motwani, P. Raghavan, Randomized Algorithms, Cambridge UNiversity Press, 1995,

P. G. Martinsson, V. Rokhlin, M. Tygert, A Fast Algorithm for the Inversion of General Toeplitz Matrices, Comput. Math. Appl., 50, 741–752, 2005.

V. Y. Pan, Structured Matrices and Polynomials: Unified Superfast Algorithms, Birkhäuser/Springer, Boston/New York, 2001.

V. Y. Pan, Transformations of Matrix Structures Work Again, Linear Algebra and Its Applications, 465, 1–32, 2015.

V. Y. Pan, G. Qian, X. Yan, Random Multipliers Numerically Stabilize Gaussian and Block Gaussian Elimination: Proofs and an Extension to Low-rank Approximation, Linear Algebra and Its Applications, 481, 202–234, 2015.

V. Y. Pan, J. Svadlenka, L. Zhao, Estimating the Norms of Circulant and Toeplitz Random Matrices and Their Inverses, Linear Algebra and Its Applications, 468, 197–210, 2015.

V. Y. Pan, X. Wang, Degeneration of Integer Matrices Modulo an Integer, Linear Algebra and Its Applications, 429, 2113–2130, 2008.
[PZ16] V. Y. Pan, L. Zhao, Low-rank Approximation of a Matrix: Novel Insights, New Progress, and Extensions, Proc. of the Eleventh International Computer Science Symposium in Russia (CSR’2016), (Alexander Kulikov and Gerhard Woeginger, editors), St. Petersburg, Russia, June 2016, Lecture Notes in Computer Science (LNCS), 9691, 352–366, Springer International Publishing, Switzerland (2016).

[PZa] V. Y. Pan, L. Zhao, Numerically Safe Gaussian Elimination with No Pivoting, arxiv 1501.05385 CS, submitted on January 22, 2015, revised in June 2016.

[RST09] V. Rokhlin, A. Szlam, M. Tygert, A Randomized Algorithm for Principal Component Analysis, SIAM Journal on Matrix Analysis and Applications, 31, 3, 1100–1124, 2009.

[RT08] V. Rokhlin, M. Tygert, A Fast Randomized Algorithm for Overdetermined Linear Least-squares Regression, Proc. Natl. Acad. Sci. USA, 105, 36, 13212–13217, 2008.

[S80] J. T. Schwartz, Fast Probabilistic Algorithms for Verification of Polynomial Identities, Journal of ACM, 27, 4, 701–717, 1980.

[S98] G. W. Stewart, Matrix Algorithms, Vol I: Basic Decompositions, SIAM, Philadelphia, 1998.

[S06] T. Sarlós, Improved Approximation Algorithms for Large Matrices via Random Projections, Proceedings of IEEE Symposium on Foundations of Computer Science (FOCS), 143–152, 2006.

[SST06] A. Sankar, D. Spielman, S.-H. Teng, Smoothed Analysis of the Condition Numbers and Growth Factors of Matrices, SIAM J. on Matrix Analysis and Applications, 28, 2, 446–476, 2006.

[T00] E.E. Tyrtyshnikov,Incomplete Cross-Approximation in the Mosaic-Skeleton Method, Computing, 64, 367–380, 2000.

[T11] J. A. Tropp, Improved Analysis of the Subsampled Randomized Hadamard Transform, Adv. Adapt. Data Anal., 3, 1–2 (Special issue ”Sparse Representation of Data and Images”), 115–126, 2011. Also arXiv math.NA 1011.1595.

[TZ12] M. Thorup, Y. Zhang, Tabulation-based 5-independent Hashing with Applications to Linear Probing and Second Moment Estimation, SIAM J. on Computing, 41, 2, 293–331, 2012.

[VVGM05] R. Vandebril, M. Van Barel, G. Golub, N. Mastronardi, A Bibliography on Semiseparable Matrices, Calcolo, 42, 3–4, 249–270, 2005.

[VVM07/08] R. Vandebril, M. Van Barel, N. Mastronardi, Matrix Computations and Semiseparable Matrices (Volumes 1 and 2), The Johns Hopkins University Press, Baltimore, Maryland, 2007.

[VVF10] M. Van Barel, R. Vandebril, P. Van Dooren, K. Frederix, Implicit Double Shift QR-algorithm for Companion Matrices, Numerische Mathematik, 116, 177–212, 2010.

[W14] D.P. Woodruff, Sketching As a Tool for Numerical Linear Algebra, Foundations and Trends in Theoretical Computer Science, 10, 1–2, 1–157, 2014.

[X12] J. Xia, On the Complexity of Some Hierarchical Structured Matrix Algorithms, SIAM J. Matrix Anal. Appl., 33, 388–410, 2012.

[X13] J. Xia, Randomized Sparse Direct Solvers, SIAM J. Matrix Anal. Appl., 34, 197–227, 2013.

[XXCB14] J. Xia, Y. Xi, S. Cauley, V. Balakrishnan, Superfast and Stable Structured Solvers for Toeplitz Least Squares via Randomized Sampling, SIAM J. Matrix Anal. Appl., 35, 44–72, 2014.

[XXG12] J. Xia, Y. Xi, M. Gu, A Superfast Structured Solver for Toeplitz Linear Systems via Randomized Sampling, SIAM J. Matrix Anal. Appl., 33, 837–858, 2012.

[Z79] R. E. Zippel, Probabilistic Algorithms for Sparse Polynomials, Proceedings of EUROSAM’79, Lecture Notes in Computer Science, 72, 216–226, Springer, Berlin, 1979.