On the Identity Problem for the Special Linear Group and the Heisenberg Group

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Abstract

We study the Identity problem for matrix semigroups. The Identity problem is to decide whether there exists the identity matrix in the given matrix semigroup. It has been recently shown that the Identity problem is \( \text{NP} \)-complete for a matrix semigroup generated by matrices from the Special Linear Group \( \text{SL}(2, Z) \) and it is undecidable for semigroups generated by 48 matrices from \( \text{SL}(4, Z) \).

Many computational problems for matrix semigroups and groups are computationally hard starting from dimension two and very often become undecidable from dimension three and four even in the case of integer matrices. All undecidability proofs for matrix problems require injective semigroup morphism from a pair of words over any finite alphabet into matrices. In this paper we significantly expand the horizon of known decidability area for matrix semigroups by showing that there is no embedding from a set of pairs of words over a semigroup alphabet to any matrix semigroup in \( \text{SL}(3, Z) \) and consequently show that there is no embedding from a set of pairs of group words into \( Z^3 \times Z^3 \).

As the decidability status of the Identity problem in dimension three is still a long standing open problem, we look for an important subgroup of \( \text{SL}(3, Z) \), the Heisenberg group \( \text{H}(3, Z) \), for which the Identity problem could be decidable following our result on non-existence of embedding. We show that the Identity problem for a matrix semigroup generated by matrices from \( \text{H}(3, Z) \) and even \( \text{H}(3, Q) \) is decidable in polynomial time. Furthermore, we extend the decidability result for \( \text{H}(n, Q) \) in any dimension \( n \). As the Identity problem is computationally equivalent to the Group problem all above results hold for the Group problem as well.

Moreover we are tightening the gap between decidability and undecidability results by improving the first undecidability result for the Identity Problem substantially reducing the bound on the size of the generator set from 48 to 9 for \( 4 \times 4 \) matrix semigroups over integers by developing a novel reduction technique which exploits the properties of anti-diagonal entries in contrast to previous repeated lock technique.

1 Introduction

There are many systems and models which are represented by matrices and matrix products. Their analysis and prediction are the challenging problems that appear in verification, control theory questions, biological systems, etc. [7, 8, 13, 14, 17, 23, 28, 30, 31, 32]. Many nontrivial algorithms for solving decision problems on matrix semigroups are developed, when considering matrices under different constraints like the dimension of matrices, number of matrices in the generator set, or considering specific subclasses of matrices: e.g., the general class of commutative matrices [2], non-commutative case of row-monomial matrices [25] or various subclasses of \( 2 \times 2 \) matrix semigroups generated by non-singular integer matrices [35], upper-triangular integer matrices [22], matrices from the special linear group [3, 12], etc.

However we still see a significant lack of algorithms and complexity results for answering decision problems in matrix semigroups. Many computational problems for matrix semigroups and groups are computationally hard starting from dimension two and very often become undecidable from
dimension three and four even in the case of integer matrices. The central decision problem in matrix semigroups is the membership problem, which was originally considered by A. Markov in 1947 [27]. Let \( S = \langle G \rangle \) be a matrix semigroup finitely generated by a generating set of square matrices \( G \). The membership problem is to decide whether or not a given matrix \( M \) belongs to the matrix semigroup \( S \). By restricting \( M \) to be the identity matrix, we call the problem, the Identity problem.

The decidability status of the Identity problem (i.e., when \( M = I \), the identity matrix) was unknown for a long time, see Problem 10.3 in “Unsolved Problems in Mathematical Systems and Control Theory” [8], but it was recently shown to be undecidable for 48 matrices form \( \mathbb{Z}^{4 \times 4} \) [5] and it is still open in dimension three. The Identity problem is computationally equivalent to another fundamental problem – the Group problem (i.e., to decide whether a semigroup is a group, in other words whether every element has the inverse), which means that any hardness and complexity results hold for the Group problem as well [12]. During the last few decades various encodings of the Post correspondence problem and Turing machines into matrices and matrix products have been used to prove the undecidability of many problems in matrix semigroups, however all of them require some smart techniques of finding injective semigroup morphism from a pair of words over any finite alphabet into matrices, which is essential main component of any encoding process [5,10,20,33].

In 1999, Cassaigne, Harju and Karhumäki significantly boosted the research on finding algorithmic solutions for \( 2 \times 2 \) matrix semigroups by showing that there is no injective semigroup morphism from pairs of words over any finite alphabet (with at least two elements) into complex \( 2 \times 2 \) matrices [11]. This result led to substantial interests in finding algorithmic solutions for such problems as the Identity problem, mortality, membership, vector reachability, freeness etc. for \( 2 \times 2 \) matrices.

For example, in 2007 Gurevich and Schupp [19] showed that the membership problem is decidable in polynomial time for the finitely generated subgroups of the modular group and later in 2017 Bell, Hirvensalo and Potapov proved that the Identity problem for a semigroup generated by matrices from \( \text{SL}(2, \mathbb{Z}) \) is NP-complete by developing a new effective technique to operate with compressed word representations of matrices and closing the gap on complexity improving the original EXPSPACE solution proposed in 2005 [12]. The first algorithm for the membership problem which covers the cases beyond \( \text{SL}(2, \mathbb{Z}) \) and \( \text{GL}(2, \mathbb{Z}) \) has been proposed in [35] and provides the solution for a semigroup generated by non-singular \( 2 \times 2 \) integer matrices. Later these techniques have been applied to build another algorithm to solve the membership problem in \( \text{GL}(2, \mathbb{Z}) \) extended by singular matrices [36]. The current limit of decidability is standing for \( 2 \times 2 \) matrices which are defined over hypercomplex numbers (quaternions) for which most of the problems have been shown to be undecidable in [4] and corresponds to reachability problems for 3-sphere rotation.

In analogy to 1999 result from [11] on non-existence of embedding into \( 2 \times 2 \) matrix semigroups, in this paper, we significantly expand a horizon of decidability area for matrix semigroups and show that there is no embedding from a set of pairs of words over a semigroup alphabet to any matrix semigroup in \( \text{SL}(3, \mathbb{Z}) \) and to \( \text{H}(3, \mathbb{C}) \) and consequently show that there is no embedding from a set of pairs of group words into \( \mathbb{Z}^{3 \times 3} \). The matrix semigroup in \( \text{SL}(3, \mathbb{Z}) \) has attracted a lot of attention now as it can be represented by a set of generators and relations [15,16] similar to \( \text{SL}(2, \mathbb{Z}) \) where it was possible to convert numerical problems into symbolic problems and solve them with novel computational techniques, see [3,12,35,36]. Comparing to the relatively simple representation of \( \text{SL}(2, \mathbb{Z}) \), the case of \( \text{SL}(3, \mathbb{Z}) \) looks more challenging as it contains many types of

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1The idea that such result may hold was motivated by analogy from combinatorial topology, where the identity problem is decidable for the braid group \( B_3 \) which is the universal central extension of the modular group \( \text{PSL}(2, \mathbb{Z}) \) [34], the embedding for a set of pairs of words into the braid group \( B_5 \) exists, see [37] and non-existence of embeddings were proved for \( B_4 \) in [1]. So \( \text{SL}(3, \mathbb{Z}) \) was somewhere in the goldilocks zone between \( B_3 \) and \( B_5 \).
non-commutative and partially commutative elements.

As the decidability status of the Identity problem in dimension three is still a long standing open problem, we look for a subclass of \( \text{SL}(3,\mathbb{Z}) \) for which the identity problem could be decidable following our result on existence of embeddings. The Heisenberg group is an important subgroup of \( \text{SL}(3,\mathbb{Z}) \) which is useful in the description of one-dimensional quantum mechanical systems [9, 13, 24]. We show that the Identity problem for a matrix semigroup generated by matrices from \( \text{H}(3,\mathbb{Q}) \) is decidable in polynomial time. Furthermore, we extend the decidability result to \( \text{H}(n,\mathbb{Q}) \) and show that the problem is still solvable in any dimension. As the Identity problem is computationally equivalent to the Group problem all above results hold for the Group problem as well. Moreover we are filling the gap between decidability and undecidability results by improving the first undecidability result for Identity Problem substantially reducing the bound on the size of the generator set from 48 to 9 for \( 4 \times 4 \) matrix semigroups over integers by developing a novel reduction technique which exploits the properties of anti-diagonal coordinates in contrast to previous repeated lock technique introduced in [3].

2 Preliminaries

A \textit{semigroup} is a set equipped with an associative binary operation. We say that a semigroup \( S \) is \textit{generated} by a subset \( X \) of \( S \) if each element of \( S \) can be expressed as a composition of elements of \( X \). Then, we call \( X \) the \textit{generating set} of \( S \). Let \( \Sigma = \{1, 2, \ldots, m\} \) be any alphabet with at least two letters and \( w \in \Sigma^* \) be a word over \( \Sigma \). For a letter \( a \in \Sigma \), we denote by \( \overline{a} \) the inverse letter of \( a \) such that \( a\overline{a} = \varepsilon \) where \( \varepsilon \) is the empty word.

The special linear group is \( \text{SL}(n,\mathbb{K}) = \{M \in \mathbb{K}^{n \times n} \mid \det(M) = 1\} \), where \( \mathbb{K} = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots \). The Heisenberg group \( \text{H}(3,\mathbb{K}) \) is formed by the \( 3 \times 3 \) matrices of the form

\[
M = \begin{pmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix}, \text{ where } a, b, c \in \mathbb{K}.
\]

It is easy to see that the Heisenberg group is a non-commutative subgroup of \( \text{SL}(3,\mathbb{K}) \). We can consider the Heisenberg group as a set of all triples with the following group law:

\[
(a_1, b_1, c_1) \otimes (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2 + a_1b_2).
\]

By \( \psi(M) \) we denote the triple \((a, b, c) \in \mathbb{K}^3\) which corresponds to the upper-triangular coordinates of \( M \). Let \( M \) be a matrix in \( \text{H}(3,\mathbb{K}) \) such that \( \psi(M) = (a, b, c) \). We define the \textit{superdiagonal vector} of \( M \) to be \( \vec{v}(M) = (a, b) \). Given two vectors \( \vec{u} = (u_1, u_2) \) and \( \vec{v} = (v_1, v_2) \), the \textit{cross product} of \( \vec{u} \) and \( \vec{v} \) is defined as \( \vec{u} \times \vec{v} = u_1v_2 - u_2v_1 \). Any two vectors are said to be \textit{parallel} if the cross product is zero.

The Heisenberg group can be also defined in higher dimensions. The Heisenberg group of dimension \( n \) over \( \mathbb{K} \) is denoted by \( \text{H}(n,\mathbb{K}) \) and is the group of square matrices in \( \mathbb{K}^{n \times n} \) of the following form:

\[
\begin{pmatrix}
1 & a^T & c \\
0 & I_{n-2} & b \\
0 & 0 & 1
\end{pmatrix}, \text{ where } a, b \in \mathbb{K}^{n-2}, c \in \mathbb{K} \text{ and } I_{n-2} \text{ is the identity matrix in } \mathbb{K}^{(n-2)\times(n-2)}.
\]

As we have considered for the Heisenberg group in dimension three, we can also consider the Heisenberg group in dimension \( n \) for any integer \( n \geq 3 \) as a set of all triples with the following
group law: \((a_1, b_1, c_1) \otimes (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2 + a_1 \cdot b_2)\), where \(a_1, a_2, b_1, b_2 \in \mathbb{K}^{n-2}\) and \(a_1 \cdot b_2\) is the dot product.

We extend the function \(\psi\) to \(n\)-dimensional Heisenberg group: For a matrix \(M\), \(\psi(M)\) is the triple \((\vec{a}, \vec{b}, c) \in (\mathbb{K}^{n-2})^2 \times \mathbb{K}\) which corresponds to the upper-triangular coordinates of \(M\).

**Lemma 1.** Let \(M_1\) and \(M_2\) be two matrices from the Heisenberg group \(H(n, \mathbb{K})\) and \(\psi(M_i) = (\vec{a}_i, \vec{b}_i, c_i)\) for \(i = 1, 2\). Then, \(M_1 M_2 = M_2 M_1\) holds if and only if \(a_1 \cdot b_2 = a_2 \cdot b_1\).

**Proof.** The product \(M_1 M_2\) has \(c_1 + c_2 + a_1 \cdot b_2\) in the upper-right corner whereas \(M_2 M_1\) has \(c_1 + c_2 + a_2 \cdot b_1\). The other coordinates are identical as we essentially add numbers in the same coordinate. It is easy to see that two products are equivalent if and only if \(a_1 \cdot b_2 = a_2 \cdot b_1\). \(\square\)

Note that, in the Heisenberg group of dimension 3, the condition of Lemma 1 can be stated as superdiagonal vectors of \(M_1\) and \(M_2\) being parallel.

## 3 Non-existence of Embedding in Dimension Three

In this section, we show that there is no embedding from a set of pairs of words over a semi-group alphabet to the special linear group SL(3, \(Z\)). The monoid \(\Sigma^* \times \Sigma^*\) has a generating set \(\{(0, \varepsilon), (1, \varepsilon), (\varepsilon, 0), (\varepsilon, 1)\}\), where \(\varepsilon\) is the empty word. We simplify the notation by setting \(a = (0, \varepsilon), b = (1, \varepsilon), c = (\varepsilon, 0)\) and \(d = (\varepsilon, 1)\). It is easy to see that we have the following relations:

\[
a c = c a, \quad b c = c b, \quad a d = d a, \quad b d = d b. \tag{1}
\]

In other words, \(a\) and \(b\) commute with \(c\) and \(d\). Note that \(a\) and \(b\) should not commute with each other, and neither should \(c\) and \(d\). Let \(\phi: \Sigma^* \times \Sigma^* \to SL(3, \mathbb{Z})\) be an injective morphism and denote \(A = \phi(a), B = \phi(b), C = \phi(c), D = \phi(d)\). Our goal is to show that \(\phi\) does not exist. Unfortunately the technique developed in [11], where the contradiction was derived from simple relations, resulting from matrix multiplication, cannot be used for a case of SL(3, \(Z\)) as it creates a lot of equations which are not limiting the possibility of embeddings. In contrast to [11], we found new techniques to show non-existence of \(\phi\) by analysis of eigenvalues and the Jordan normal forms.

**Lemma 2.** If there is an injective morphism \(\phi: \Sigma^* \times \Sigma^* \to SL(3, \mathbb{Z})\) and the matrices \(A, B, C\) and \(D\) correspond to \((0, \varepsilon), (1, \varepsilon), (\varepsilon, 0)\) and \((\varepsilon, 1)\) respectively, then the matrices \(A, B, C\) and \(D\) have a single eigenvalue and additionally the Jordan normal form is \(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\).

**Proof.** Let \(\phi\) be an injective morphism from \(S\) into \(SL(3, \mathbb{C})\). Let \(M, M', N_1, N_2 \in \{A, B, C, D\}\), such that \(M M' \neq M' M, M N_1 = N_1 M, M N_2 = N_2 M\) and \(N_1 \neq N_2\). For example, if \(M = A\), then \(M' = B, N_1 = C\) and \(N_2 = D\) or \(N_1 = D\) and \(N_2 = C\).

Since the conjugation by an invertible matrix does not influence the injectivity, we suppose that \(M\) is in the Jordan normal form. For a 3 \(\times\) 3 matrix, there are six different types of matrices in the Jordan normal form. If \(M\) has three different eigenvalues, then

\[
M = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}. \tag{2}
\]

If \(M\) has two eigenvalues, then

\[
M = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \quad \text{or} \quad M = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{pmatrix}. \tag{3}
\]
Finally, if $M$ has only one eigenvalue, then
\[ M = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \text{ or } M = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \text{ or } M = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & 0 & \lambda \\ 0 & 0 & \lambda \end{pmatrix}. \tag{4} \]

The first case (2) can be easily ruled out since $M$ only commutes with diagonal matrices. Then, $N_1$ and $N_2$ should be commuting with $M$ by the suggested relations and as a result, $N_1$ and $N_2$ commute with each other.

Now let us consider the second case (3), where $M$ has two eigenvalues $\lambda$ and $\mu$. Note that the determinant of $M$ is 1 since $M \in \text{SL}(3, \mathbb{Z})$. Namely, $\det(M) = \lambda \mu^2 = 1$. That is, the eigenvalues are $\lambda = 1$ and $\mu = \pm 1$. If $\mu = 1$, then there is only one eigenvalue, which is reduced to the case (1).

If $\mu = -1$, then $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ and let $N_1 = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & \ell \end{pmatrix}$. Now
\[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & \ell \end{pmatrix} = \begin{pmatrix} a & b & c \\ g-d & h-e & \ell-f \\ -g & -h & -\ell \end{pmatrix} = \begin{pmatrix} a & -b & b-c \\ d & -e & e-f \\ g & -h & -\ell \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & \ell \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}. \]

Since these matrices are equal, we have that $b = c = d = g = h = 0$ and $e = \ell$. Similar calculation gives us $N_2 = \begin{pmatrix} a' & 0 & 0 \\ 0 & e' & f' \\ 0 & 0 & e' \end{pmatrix}$. Now, matrices $N_1$ and $N_2$ commute as follows:
\[ \begin{pmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & 0 & e \end{pmatrix} \begin{pmatrix} a' & 0 & 0 \\ 0 & e' & f' \\ 0 & 0 & e' \end{pmatrix} = \begin{pmatrix} aa' & 0 & 0 \\ 0 & ee' & ef'+fe' \\ 0 & 0 & ee' \end{pmatrix} = \begin{pmatrix} a' & 0 & 0 \\ 0 & e' & f' \\ 0 & 0 & e \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & 0 & e \end{pmatrix}. \]

Finally, we consider the case (4) where $M$ has only one eigenvalue. If the matrix $M$ is diagonal, it is easy to see that it is not the case since otherwise $M$ commutes with all matrices including $M'$.

If the matrix $M$ is in the following form $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ and let $N_1 = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & \ell \end{pmatrix}$. Now
\[ \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & \ell \end{pmatrix} = \begin{pmatrix} d + a\lambda & e + b\lambda & f + c\lambda \\ g + d\lambda & h + e\lambda & \ell + f\lambda \\ g\lambda + h\ell & h + \ell\lambda \end{pmatrix} = \begin{pmatrix} a\lambda & a + b\lambda & b + c\lambda \\ d + e\lambda & e + f\lambda \\ g + h\ell + \ell\lambda \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & \ell \end{pmatrix} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}. \]

Since these matrices are equal, we have that $d = g = h = 0$, $a = e = \ell$ and $b = f$. Similar calculation gives us $N_2 = \begin{pmatrix} a' & 0 & 0 \\ 0 & a' & c' \\ 0 & 0 & a' \end{pmatrix}$ and now matrices $N_1$ and $N_2$ commute as follows:
\[ \begin{pmatrix} a & b & c \\ 0 & a & b' \\ 0 & 0 & a' \end{pmatrix} \begin{pmatrix} a' & b' & c' \\ 0 & a' & b' \\ 0 & 0 & a' \end{pmatrix} = \begin{pmatrix} aa' & ab' + ba' & ac' + bb' + ca' \\ 0 & aa' & ab' + ba' \\ 0 & 0 & aa' \end{pmatrix} = \begin{pmatrix} a & b & c \\ a & b' & c' \\ a & b & c' \end{pmatrix}. \]

Now the only possibility for $M$ is the following form: $M = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$, where $\lambda$ is the single eigenvalue of $M$. Since $\det(M) = \lambda^3 = 1$, $\lambda$ can be one of cube roots of unity: $1$, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$.
Among these numbers, the latter two cannot be chosen to be \( \lambda \) since \( \text{tr}(M) = 3\lambda \) cannot be an integer.

Based on the restriction on the Jordan normal form of matrices, we prove that there is no injective morphism from the set of pairs of words over an alphabet \( \Sigma \) into \( \text{SL}(3, \mathbb{Z}) \).

**Theorem 3.** There is no injective morphism \( \phi : \Sigma^* \times \Sigma^* \to \text{SL}(3, \mathbb{Z}) \) for any alphabet \( \Sigma \).

**Proof.** Assume to the contrary that there is an injective morphism \( \phi \) from \( \Sigma^* \times \Sigma^* \) into \( \text{SL}(3, \mathbb{Z}) \). Since the conjugation by an invertible matrix does not influence the injectivity, we suppose that the matrix \( A \), which corresponds to the generator \( a \), is in the Jordan normal form as proven in Lemma 2. Then, we have the following matrices corresponding to the generators \( a, b, c \) and \( d \) as follows:

\[
A = \phi(a) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \phi(b) = \begin{pmatrix} a_B & b_B & c_B \\ d_B & e_B & f_B \\ g_B & h_B & \ell_B \end{pmatrix}, \quad C = \phi(c) = \begin{pmatrix} a_C & b_C & c_C \\ d_C & e_C & f_C \\ g_C & h_C & \ell_C \end{pmatrix}, \quad D = \phi(d) = \begin{pmatrix} a_D & b_D & c_D \\ d_D & e_D & f_D \\ g_D & h_D & \ell_D \end{pmatrix}.
\]

Since \( A \) and \( C \) commute with each other by one of the given relations in \( \mathfrak{I} \), we have

\[
AC = \begin{pmatrix} a_C + d_C & b_C + e_C & c_C + f_C \\ d_C & e_C & f_C \\ g_C & h_C & \ell_C \end{pmatrix} = \begin{pmatrix} a_C & a_C + b_C & c_C \\ d_C & d_C + e_C & f_C \\ g_C & g_C + h_C & \ell_C \end{pmatrix} = CA.
\]

It is easy to see that \( d_C = g_C = f_C = 0 \) and \( a_C = e_C \). Therefore,

\[
C = \begin{pmatrix} a_C & b_C & c_C \\ 0 & a_C & 0 \\ 0 & h_C & \ell_C \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} a_D & b_D & c_D \\ 0 & a_D & 0 \\ 0 & h_D & \ell_D \end{pmatrix}.
\]

Since \( C \) and \( D \) are in \( \text{SL}(3, \mathbb{Z}) \), the determinants should be \( 1 \). Now, the determinant of \( C \) is \( a_C^2 \ell_C \) and hence, by Lemma 2, \( a_C = \ell_C = 1 \). Analogously, we can also see that \( a_D = \ell_D = 1 \). Next we observe that the matrices \( C \) and \( D \) commute if and only if \( C_C h_D = C_D h_C \). By relations \( \mathfrak{I} \), \( C \) and \( D \) do not commute and hence there are three cases to be considered:

1) \( C_C = 0 \) and \( h_C \neq 0 \); 2) \( C_C \neq 0 \) and \( h_C \neq 0 \); and 3) \( C_C \neq 0 \) and \( h_C = 0 \).

We prove that each case leads to a contradiction, i.e., that \( C \) and \( D \) commute. Let us examine the three cases as follows:

**Case 1 (\( C_C = 0 \) and \( h_C \neq 0 \)).** We know that \( c_D \) is also non-zero because otherwise \( C \) and \( D \) commute with each other since \( c_C h_D = c_D h_C = 0 \). We have the following calculations:

\[
BC = \begin{pmatrix} a_B & b_B & c_B \\ d_B & e_B & f_B \\ g_B & h_B & \ell_B \end{pmatrix} \begin{pmatrix} 1 & b_C & 0 \\ 0 & 1 & 0 \\ 0 & h_C & 1 \end{pmatrix} = \begin{pmatrix} a_B & a_B b_C + b_B + c_B h_C & c_B \\ d_B & d_B b_C + e_B + f_B h_C & f_B \\ g_B & g_B b_C + h_B + \ell_B h_C & \ell_B \end{pmatrix}
\]

and

\[
CB = \begin{pmatrix} 1 & b_C & 0 \\ 0 & 1 & 0 \\ 0 & h_C & 1 \end{pmatrix} \begin{pmatrix} a_B & b_B & c_B \\ d_B & e_B & f_B \\ g_B & h_B & \ell_B \end{pmatrix} = \begin{pmatrix} a_B + d_B b_C & b_B + e_B b_C & c_B + f_B b_C \\ d_B & e_B & f_B \\ d_B h_C + g_B & e_B h_C + h_B & f_B h_C + \ell_B \end{pmatrix}.
\]
Since $BC = CB$, we have $d_B c_C = 0$, $d_B h_C = 0$, $f_B b_C = 0$, and $f_B h_C = 0$. By the supposition $h_C \neq 0$, we further deduce that $d_B = f_B = 0$. Then, $B$ should be $B = \begin{pmatrix} a_B & b_B & c_B \\ d_B & e_B & f_B \\ g_B & h_B & \ell_B \end{pmatrix}$. Note that we also have
\[
abla B b_C + c_B h_C = e_B b_C \quad \text{and} \quad g_B b_C + \ell B h_C = e_B h_C
\]
by the equality $BC = CB$.

The characteristic polynomial of $B$ is $P(x) = -x^3 + \text{tr}(B)x^2 - (a_B e_B + a_B \ell_B + e_B \ell_B - c_B g_B)x + \det(B)$ which has roots $\lambda = e_B$ and $\lambda = \frac{1}{2}(a_B + \ell_B \pm \sqrt{(a_B - \ell_B)^2 + 4c_B g_B})$. We know that the only eigenvalue of $B$ is 1 by Lemma 2 and therefore, we have $a_B = e_B = \ell_B = 1$ and $c_B B B = 0$.

Moreover, it follows from Equation (5) that $c_B = 0$ and $g_B b_C = 0$. Note that $g_B \neq 0$ because otherwise the matrix $B$ commutes with $A$. Finally, we consider
\[
BD = \begin{pmatrix} 1 & b_B & 0 \\ 0 & h_B & 0 \\ g_B & h_B & 1 \end{pmatrix} = DB.
\]

It is easy to see that $d_B = c_D = 0$ and then $D$ commutes with $C$. Therefore, we have a contradiction.

**Case 2 ($c_C \neq 0$ and $h_C = 0$).** Consider the matrix $B$ which commutes with $C$ as follows:
\[
BC = \begin{pmatrix} a_B & b_B & c_B \\ d_B & e_B & f_B \\ g_B & h_B & \ell_B \end{pmatrix} \begin{pmatrix} 1 & b_C & c_C \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_B a_B b_C + c_B & a_B b_C + b_B & a_B C + c_B \\ d_B d_B b_C + e_B & d_B b_C + e_B & d_B C + f_B \\ g_B g_B b_C + h_B & g_B b_C + h_B & g_B C + \ell_B \end{pmatrix}
\]
\[
= \begin{pmatrix} a_B + d_B b_C + g_B c_C & b_B + e_B b_C + h_B c_C & c_B + f_B b_C + \ell_B c_C \\ d_B & e_B & f_B \\ g_B & h_B & \ell_B \end{pmatrix}
\]
\[
= \begin{pmatrix} 1 & b_C & c_C \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d_B & e_B & f_B \\ g_B & h_B & \ell_B \end{pmatrix} = CB.
\]

By the equivalence, we have $d_B b_C = 0$, $g_B b_C = 0$, $g_B C = 0$, and $d_B C = 0$. By the supposition $c_C \neq 0$, we further deduce that $d_B = g_B = 0$. Then, $B$ should be of the following form: $B = \begin{pmatrix} a_B & b_B & c_B \\ 0 & e_B & f_B \\ 0 & h_B & \ell_B \end{pmatrix}$. Note that we also have
\[
abla B b_C = e_B b_C + h_B c_C \quad \text{and} \quad a_B c_C = f_B b_C + \ell_B c_C
\]
by the equality $BC = CB$.

The characteristic polynomial of $B$ is $P(x) = -x^3 + \text{tr}(B)x^2 - (a_B e_B + a_B \ell_B + e_B \ell_B - f_B h_B)x + \det(B)$ which has roots $\lambda = e_B$ and $\lambda = \frac{1}{2}(e_B + \ell_B \pm \sqrt{(e_B - \ell_B)^2 + 4f_B h_B})$. We know that the only eigenvalue of $B$ is 1 by Lemma 2 and therefore, we have $a_B = e_B = \ell_B = 1$ and $f_B h_B = 0$.

We can further deduce from Equation (6) that $h_B = 0$ and $f_B b_C = 0$. By a similar argument for the matrices $B$ and $D$ that should commute with each other as in the first case, we have a contradiction.

**Case 3 ($c_C \neq 0$ and $h_C \neq 0$).** It is obvious that $c_D$ and $h_D$ are also non-zero because otherwise $C$ and $D$ should commute. Now consider the matrix $B$ which is commuting with $C$ and $D$. We
can deduce from the relation $BC = CB$ that $d_B = g_B = f_B = 0$ and $a_B = e_B = \ell_B = 1$ since they are eigenvalues of $B$. Hence, $B = \begin{pmatrix} 1 & b_B & c_B \\ 0 & 1 & 0 \\ 0 & h_B & 1 \end{pmatrix}$.

Now we have $c_C h_B = c_B h_C$ since $B$ and $C$ commute with each other. Note that $h_B$ and $c_B$ are both non-zero since $A$ and $B$ commute if $h_B = c_B = 0$. Let us denote $\frac{c_C}{h_B} = \frac{c_B}{h_B} = x$. We also have $c_D h_B = c_B h_D$ from the relation $BD = DB$ and have $\frac{c_D}{h_B} = \frac{c_B}{h_B} = x$. From $x = \frac{c_C}{h_B} = \frac{c_B}{h_B}$, we have $c_C h_D = c_D h_C$ which results in the relation $CD = DC$. Therefore, we also have a contradiction.

Since we have examined all possible cases and found contradictions for every case, we can conclude that there is no injective morphism from $\Sigma^* \times \Sigma^*$ into the special linear group $SL(3, \mathbb{Z})$.

**Corollary 4.** There is no injective morphism $\phi : FG(\Sigma) \times FG(\Sigma) \rightarrow \mathbb{Z}^{3 \times 3}$ for any alphabet $\Sigma$.

**Proof.** We proceed by contradiction. Assume that there exists such an injective morphism $\phi$ from the set of pairs of words over a group alphabet to the set of matrices in $\mathbb{Z}^{3 \times 3}$. Suppose that $A = \phi(a, \varepsilon)$ where $a \in \Sigma$. Then, the inverse matrix $A^{-1}$ corresponding to $(a, \varepsilon)$ must be in $\mathbb{Z}_{3 \times 3}$. This implies that the determinant of $A$ is 1 because otherwise the determinant of $A^{-1}$ becomes a non-integer. By Theorem 3 such injective morphism $\phi$ does not exist.

Next, we show that there does not exist an embedding from pairs of words over a semigroup alphabet into matrices from $H(3, \mathbb{C})$.

**Theorem 5.** There is no injective morphism $\phi : \Sigma^* \times \Sigma^* \rightarrow H(3, \mathbb{C})$ for any alphabet $\Sigma$.

**Proof.** Assume to the contrary that there is an injective morphism $\phi$ from $\Sigma^* \times \Sigma^*$ into $H(3, \mathbb{C})$. Using the notations and relations of (1), we set $\phi(a) = A$, $\phi(b) = B$, $\phi(c) = C$, $\phi(d) = D$ for some matrices $A, B, C, D \in H(3, \mathbb{C})$. By Lemma 1 two matrices $M, N \in H(3, \mathbb{C})$ commute if and only if $\bar{v}(M) \times \bar{v}(N) = 0$. Denote $\bar{v}(A) = (a_1, a_2)$ and $\bar{v}(B), \bar{v}(C), \bar{v}(D), \bar{v}(E)$ are denoted analogously. From the relations (1), it follows that

$$a_1 c_2 = c_1 a_2, \quad a_1 d_2 = d_1 a_2, \quad b_1 c_2 = c_1 b_2, \quad b_1 d_2 = d_1 b_2, \quad a_1 b_2 \neq b_1 a_2, \quad c_1 d_2 \neq d_1 c_2.$$  

Observe first, that $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \neq 0$. If, say, $a_1 = 0$, then from the first two equations it follows that either $a_2 = 0$ or $b_1 = d_1$. If $a_2 = 0$, then the first inequality does not hold, since $a_1 b_2 = a_2 b_1$, and if $c_1 = d_1 = 0$, then the second inequality does not hold, since $c_1 d_2 = 0 = d_1 c_2$.

Now, we can solve $a_1$ from the first two equalities, $\frac{a_1 c_2}{c_1} = a_1 = \frac{a_1 d_2}{d_2}$. That is, $c_1 d_2 = d_1 c_2$, which contradicts the last relation and proves our claim.

## 4 Decidability of the Identity Problem in the Heisenberg group

The decidability of the identity problem in dimension three is a long standing open problem. Following our finding on non-existence of embedding into $SL(3, \mathbb{Z})$, in this section we consider the decidability of the important subgroup of $SL(3, \mathbb{Z})$, the Heisenberg group, which is well known in the context of quantum mechanical systems [9, 13, 24]. Recently a few decidability results have been obtained for a knapsack variant of the membership problem in dimension three (i.e $H(3, \mathbb{Z})$), where the goal was to solve a single matrix equation with a specific order of matrices [23].

In this paper we prove that the Identity problem is decidable for the Heisenberg group over rational numbers. First we provide more intuitive solution for dimension three, i.e $H(3, \mathbb{Q})$, which still requires a number of techniques to estimate possible values of elements under permutations in matrix products. In the end of the section we generalize the result for $H(n, \mathbb{Q})$ case using analogies in the solution for dimension three.
Here we prove that the Identity problem for matrix semigroups in the Heisenberg group over rationals is decidable by analyzing the behaviour of multiplications especially in the upper-right coordinate of matrices. From Lemma 11 it follows that the matrix multiplication is commutative in the Heisenberg group if and only if matrices have pairwise parallel superdiagonal vectors. So we analyse two cases of products for matrices with pairwise parallel and none pairwise parallel superdiagonal vectors and then provide the algorithms that solves the problem in polynomial time. The most difficult part is to show that only limited number of conditions should be checked to guarantee the existence of a product that will gives the identity.

**Lemma 6.** Let \( G = \{M_1, M_2, \ldots, M_r\} \subseteq H(3, \mathbb{R}) \) be a set of matrices from the Heisenberg group such that superdiagonal vectors of matrices are pairwise parallel. If there exists a sequence of matrices \( M = M_i M_j \cdots M_k \), where \( i_j \in [1, r] \) for all \( 1 \leq j \leq k \), such that \( \psi(M) = (0, 0, c) \) for some \( c \in \mathbb{R} \), then,

\[
c = \sum_{j=1}^{k} (c_{i_j} - \frac{x}{2} a_{i_j}^2)
\]

for some \( x \in \mathbb{R} \).

**Proof.** Consider the sequence \( M_i M_j \cdots M_k \) and let \( M_i = \left( \begin{array}{ccc} a_i & b_i & c_i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \) for each \( i \in [1, r] \). Since the superdiagonal vectors are parallel, we have \( x = \frac{b_i}{a_i} \in \mathbb{R} \) and thus \( a_i x = b_i \) for all \( i \in [1, r] \). Let us consider the product of the matrices. Then, the value \( c \) is equal to

\[
c = \sum_{j=1}^{k} c_{i_j} + \sum_{\ell=1}^{k-1} \left( \sum_{j=1}^{\ell} a_{i_j} \right) a_{i_{\ell+1}} x = \sum_{j=1}^{k} c_{i_j} + \frac{1}{2} \left( \sum_{\ell=1}^{k} \sum_{j=1}^{k} a_{i_j}^2 x - \sum_{j=1}^{k} a_{i_j}^2 x \right) = \sum_{j=1}^{k} \left( c_{i_j} - \frac{x}{2} a_{i_j}^2 \right).
\]

The value \( c \) would be preserved in case of reordering of matrices due to their commutativity. \( \square \)

It is worth mentioning that the Identity problem in the Heisenberg group is decidable if any two matrices have pairwise parallel superdiagonal vectors since now the problem reduces to solving a system of two Diophantine equations. Hence, it remains to consider the case when there exist two matrices with non-parallel superdiagonal vectors in the sequence generating the identity matrix. In the following, we prove that the identity matrix is always constructible if we can construct any matrix with the zero superdiagonal vector by using matrices with non-parallel superdiagonal vectors.

**Lemma 7.** Let \( S = \langle M_1, M_2, \ldots, M_r \rangle \subseteq H(3, \mathbb{Q}) \) be a finitely generated matrix semigroup. Then, the identity matrix exists in \( S \) if there exists a sequence of matrices \( M_{i_1} M_{i_2} \cdots M_{i_k} \), where \( i_j \in [1, r] \) for all \( 1 \leq j \leq k \), satisfying the following properties:

1. \( \psi(M_{i_1} M_{i_2} \cdots M_{i_k}) = (0, 0, c) \) for some \( c \in \mathbb{Q} \), and
2. \( \overline{v}(M_{i_{j_1}}) \) and \( \overline{v}(M_{i_{j_2}}) \) are not parallel for some \( j_1, j_2 \in [1, k] \).

**Proof.** Let \( M = M_{i_1} M_{i_2} \cdots M_{i_k} \) and \( \psi(M) = (0, 0, c) \) for some \( c \in \mathbb{Q} \). If \( c = 0 \), then \( M \) is the identity matrix, hence we assume that \( c > 0 \) as the case of \( c < 0 \) is symmetric.

Given that \( M_i \) is the \( i \)th generator and \( \psi(M_i) = (a_i, b_i, c_i) \), we have \( \sum_{j=1}^{i} a_{i_j} = 0 \) and \( \sum_{j=1}^{k} b_{i_j} = 0 \). Since \( c > 0 \), the following also holds:

\[
c = \sum_{\ell=1}^{k-1} \sum_{j=1}^{\ell} a_{i_j} b_{i_{\ell+1}} + \sum_{j=1}^{k} c_{i_j} > 0.
\]
If the matrix semigroup $S \subseteq \text{H}(3, \mathbb{Q})$ has two different matrices $M_1$ and $M_2$ such that $\psi(M_1) = (0, 0, c_1)$ and $\psi(M_2) = (0, 0, c_2)$ and $c_1c_2 < 0$, then the identity matrix should exist in $S$. Let $\psi(M_1) = (0, 0, \frac{p_1}{q_1})$ and $\psi(M_2) = (0, 0, \frac{p_2}{q_2})$ where $p_1, q_1, q_2 \in \mathbb{Z}$ are positive and $p_2 \in \mathbb{Z}$ is negative. Then, it is easy to see that the matrix $M_1^{-q_1p_2}M_2^{q_2p_1}$ exists in $S$ such that $\psi(M_1^{-q_1p_2}M_2^{q_2p_1}) = (0, 0, 0)$.

Now we will prove that if $S$ contains a matrix $M$ such that $\psi(M) = (0, 0, c)$ where $c > 0$, then there also exists a matrix $M'$ such that $\psi(M') = (0, 0, c')$ where $c' < 0$.

First, we classify the matrices into four types as follows. A matrix with a superdiagonal vector $(a, b)$ is classified as 1) the $(+, +)$-type if $a, b > 0$, 2) the $(+, -)$-type if $a \geq 0$ and $b \leq 0$, 3) the $(-, -)$-type if $a, b < 0$, and 4) the $(+, +)$-type if $a < 0$ and $b > 0$. Let $G = \{M_1, M_2, \ldots, M_r\}$ be the generating set of the matrix semigroup $S$. Then, $G = G_{(+, +)} \cup G_{(+, -)} \cup G_{(-, -)} \cup G_{(-, +)}$ such that $G_{(\xi_1, \xi_2)}$ is the set of matrices of the $(\xi_1, \xi_2)$-type where $\xi_1, \xi_2 \in \{+, -\}$.

Recall that we assume $M = M_{i_1} \cdots M_{i_k}$ and $\psi(M) = (0, 0, c)$ for some $c > 0$. The main idea of the proof is to generate a matrix $M'$ such that $\psi(M') = (0, 0, c')$ for some $c' < 0$ by duplicating the matrices in the sequence $M = M_{i_1} \cdots M_{i_k}$ multiple times and reshuffling. Note that any permutation of the sequence generating the matrix $M$ such that $\psi(M) = (0, 0, c)$ still generates matrices $M'$ such that $\psi(M') = (0, 0, c')$ since the multiplication of matrices changes the front two coordinates in a commutative way. Moreover, we can still obtain matrices $M''$ such that $\psi(M'') = (0, 0, c'')$ for some $c'' \in \mathbb{Q}$ if we shuffle two different permutations of the sequence $M_{i_1} \cdots M_{i_k}$ by the same reason.

Let us illustrate the idea with the following example. See Fig.1 and Fig. 2 for pictorial descriptions of the idea. Let $\{M_i \mid 1 \leq i \leq 4\} \subseteq G_{(+, +)}$, $\{M_i \mid 5 \leq i \leq 7\} \subseteq G_{(+, -)}$, $\{M_i \mid 8 \leq i \leq 9\} \subseteq G_{(-, +)}$, and $\{M_i \mid 10 \leq i \leq 13\} \subseteq G_{(-, -)}$. Then, assume that $M_1M_2 \cdots M_{13} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where $x$ is computed by (7). As we mentioned above, $x$ changes if we change the order of multiplications. In this example, we first multiply $(+, +)$-type matrices and accumulate the values in the superdiagonal coordinates since these matrices have positive values in the coordinates. Indeed, the blue dotted area implies the value we add to the upper-right corner by multiplying such matrices. Then, we multiply $(+, -)$-type matrices and still increase the ‘$a$’-value. The ‘$b$’-values in $(+, -)$-type matrices are negative thus, the red lined area is subtracted from the upper-right corner. We still subtract...
where $z$.

Then, the upper-triangular coordinates of the $m$ value. 

by multiplying $(-,-)$-type matrices since the accumulated ‘a’-value is still positive and ‘b’-values are negative. Then, we finish the multiplication by adding exactly the last blue dotted area to the upper-right corner. It is easy to see that the total subtracted value is larger than the total added value.

However, we cannot guarantee that $x$ is negative since $\sum_{i=1}^{13} c_i$ could be larger than the contribution from the superdiagonal coordinates. This is why we need to copy the sequence of matrices generating the matrix corresponding to the triple $(0,0,c)$ for some $c \in \mathbb{Q}$. In figure on the right, we describe an example where we duplicate the sequence eight times and shuffle and permute them in order to minimize the value in the upper-right corner. Now the lengths of both axes are $m$ ($m = 8$ in this example) times larger than before and it follows that the area also grows quadratically in $m$. Since the summation $m \cdot \sum_{i=1}^{13} c_i$ grows linearly in $m$, we have $x < 0$ when $m$ is large enough.

From the sequence $M_{i_1} \cdots M_{i_k}$, we obtain four multisets $S(\xi_1,\xi_2)$, where $\xi_1,\xi_2 \in \{+, -\}$, such that each multiset $S(\xi_1,\xi_2)$ contains the matrices that appear in the sequence and belong to the set $G(\xi_1,\xi_2)$. That is, $S(\xi_1,\xi_2)$ has as many elements as matrices of $(\xi_1,\xi_2)$-type in the product.

For each $\xi_1,\xi_2 \in \{+,-\}$, let us define $a(\xi_1,\xi_2), b(\xi_1,\xi_2), c(\xi_1,\xi_2)$ such that $(a(\xi_1,\xi_2), b(\xi_1,\xi_2), c(\xi_1,\xi_2)) = \sum_{M \in S(\xi_1,\xi_2)} \psi(M)$. In other words, $a(\xi_1,\xi_2), b(\xi_1,\xi_2)$ and $c(\xi_1,\xi_2)$, respectively, is the sum of the values in the ‘a’ (‘b’ and ‘c’, respectively) coordinate from the matrices in the multiset $S(\xi_1,\xi_2)$.

Now consider a permutation of the sequence $M_{i_1} \cdots M_{i_k}$, where the first part of the sequence only consists of the $(+, +)$-type matrices, the second part only consists of the $(+, -)$-type matrices, the third part only consists of the $(-, +)$-type, and finally the last part only consists of the $(-, -)$-type.

Let us denote by $M(+)\psi$ the matrix which results from the multiplication of the first part, namely, $M(+) = \prod_{M \in S(+, +)} M$. Then, $\psi(M(+) = (a(+, +), b(+, +), x(+, +))$ should hold where $x(+) < c(+) + a(+)b(+)$. Let us define $M(+)\psi$, $M(-)\psi$ and $M(+)\psi$ analogously.

Now we claim that there exists an integer $m > 0$ such that $M(+)\psi M(-)\psi M(+)\psi M(-)\psi$ corresponds to the triple $(0,0,c)$ for some $c' < 0$. Let $M$ be a matrix in $\mathbb{H}(3, \mathbb{Q})$ and $\psi(M) = (a, b, c)$. Then, the upper-triangular coordinates of the $m$th power of $M$ are calculated as follows:

$$\psi(M^m) = (am, bm, cm + ab \cdot \frac{1}{2} m(m - 1)).$$

Let us consider the first part $M(+)\psi$ and from (8), $\psi(M(+)\psi = (a(+)m, b(+)m, x(+)m + z_1)$, where $z_1 = |a(+)||b(+)| \cdot \frac{1}{2} m(m - 1)$. Now we multiply $M(+)\psi$ by the second part $M(-)\psi$. Then,
the resulting matrix corresponds to \((a_{(+,+)}+a_{(+,-)})m, (b_{(+,+)}+b_{(+,-)})m, (x_{(+,+)}+x_{(+,-)})m + z_1 - z_2\), where \(z_2 = m^2 |a_{(+,+)}| |b_{(+,+)}| + |a_{(+,-)}| |b_{(+,-)}| \cdot \frac{1}{2}m(m-1)\). Similarly, we compute \(z_3\) and \(z_4\) that will be added to the upper-right corner as a result of multiplying \(M^m_{(+,-)}\) and \(M^m_{(-,+)}\) as follows:

\[
z_3 = |a_{(-,+)}| |b_{(-,+)}| m^2 + |a_{(-,-)}| |b_{(-,-)}| \cdot \frac{1}{2}m(m-1) \text{ and } z_4 = |a_{(-,+)}| |b_{(-,+)}| \cdot \frac{1}{2}m(m-1).
\]

After the calculations, we have \(\psi(M^m_{(+,+)}M^m_{(+,-)}M^m_{(-,-)}M^m_{(-,+)} = (0, 0, (x_{(+,+)} + x_{(+,-)} + x_{(-,-)})m + z_1 - z_2 - z_3 + z_4)\). Let \(z = z_1 - z_2 - z_3 + z_4\). Then, we can see that \(z\) can be represented as a quadratic equation of \(m\) using the above results and then the coefficient of \(m^2\) is always negative if \(S_{(\xi_1, \xi_2)} \neq \emptyset\) for all \(\xi_1, \xi_2 \in \{+, -\}\). The coefficient of \(m^2\) is

\[
\frac{1}{2}(|a_{(+,+)}| |b_{(+,+)}| + |a_{(+,-)}| |b_{(+,-)}|) - \frac{1}{2}(|a_{(+,+)}| |b_{(+,-)}| + |a_{(+,-)}| |b_{(+,+)}|) + |a_{(-,+)}| |b_{(-,+)}| + |a_{(-,-)}| |b_{(-,-)}|).
\]

Let \(|a_{(+,+)}| + |a_{(+,-)}| = |a_{(-,+)}| + |a_{(-,-)}| = a'\) and \(|b_{(+,+)}| + |b_{(+,-)}| = |b_{(-,+)}| + |b_{(-,-)}| = b'\). Then, \(a'b' = a'(|b_{(+,+)}| + |b_{(+,-)}|) = a'|b_{(+,+)}| + a'|b_{(+,-)}| = (|a_{(+,+)}| + |a_{(+,-)}|) |b_{(+,+)}| + (|a_{(-,+)}| + |a_{(-,-)}|) |b_{(-,-)}|\). Now the coefficient of \(m^2\) in \(z\) can be written as

\[
-a'b' + \frac{1}{2}(|a_{(+,+)}| |b_{(+,+)}| + |a_{(+,-)}| |b_{(+,-)}| + |a_{(-,+)}| |b_{(-,+)}| + |a_{(-,-)}| |b_{(-,-)}|) = 0.
\]

Without loss of generality, suppose that \(|a_{(+,+)}| \geq |a_{(+,-)}|\). Then, we have

\[
|a_{(+,+)}| |b_{(+,+)}| + |a_{(+,-)}| |b_{(+,-)}| \leq |a_{(+,+)}| b' \text{ and } |a_{(-,+)}| |b_{(-,+)}| + |a_{(-,-)}| |b_{(-,-)}| \leq |a_{(-,-)}| b'.
\]

From \((|a_{(+,+)}| + |a_{(+,-)}|) b' \leq 2a'b'\), we can see that the coefficient of the highest power of the variable is negative in \(z\) if \(|a_{(+,+)}| + |a_{(+,-)}| < 2a'\). By comparing two terms in (9), we can see that the coefficient is negative if all subsets \(S_{(+,-)}, S_{(-,+)}, S_{(+,+)}\) and \(S_{(-,-)}\) are not empty. Since the coefficient of the highest power of the variable is negative, \(z\) becomes negative when \(m\) is large enough. Therefore, we have a matrix corresponding to the triple \((0, 0, c')\) for some \(c' < 0\) as a product of multiplying matrices in the generating set and the identity matrix is also reachable.

It should be noted that there are some subcases where some of subsets from \(S_{(+,+)}\), \(S_{(+,-)}\), \(S_{(-,+)}\), and \(S_{(-,-)}\) are empty. We examine all possible cases and prove that the coefficient of \(m^2\) should be negative in every case and the matrix with a negative integer in the corner is constructible. First we prove that the coefficient of \(m^2\) in \(z\) should be negative when only one of the subsets from \(S_{(+,+)}\), \(S_{(+,-)}\), \(S_{(-,+)}\), and \(S_{(-,-)}\) is empty as follows:

- **Case of \(S_{(+,+)} = \emptyset\):** Note that \(|a_{(+,-)}| = a'\) and \(|b_{(+,-)}| = b'\) since \(|a_{(+,+)}| = |b_{(+,+)}| = 0\) by \(S_{(+,+)} = \emptyset\) being empty. Then, the coefficient of \(m^2\) becomes \(-a'b' + |a_{(+,-)}| b' + |a_{(-,-)}| |b_{(-,-)}| + a'b_{(+,-)}|\) can be maximized to \(a'b'\). If we maximize \(|a_{(+,-)}| b'\) by setting \(|a_{(+,-)}| = a'\), then \(|a_{(-,-)}| = 0\) should be 0 since \(|a_{(+,+)}| = |a_{(+,-)}| = a'\). Then, \(|a_{(-,-)}| |b_{(-,-)}| + a'|b_{(+,-)}|\) can be \(a'b'\) only when \(|b_{(+,-)}| = b'\). This leads to the set \(S_{(-,-)}\) being empty since we have \(|a_{(-,-)}| = 0\) and \(|b_{(-,-)}| = 0\) and therefore, we have a contradiction.

- **Case of \(S_{(+,-)} = \emptyset, S_{(-,+)} = \emptyset, \text{ or } S_{(+,+)} = \emptyset\):** We can prove the remaining cases by the similar argument as above.

Fig. 4 shows the cases when one of subsets from \(S_{(+,+)}\), \(S_{(+,-)}\), \(S_{(-,+)}\), and \(S_{(-,-)}\) is empty. Lastly, it remains to consider the cases where two of the subsets are empty. Note that we do not consider the cases where three of the subsets are empty because the sum of \(a'\)'s and \(b'\)'s cannot be both zero in such cases. Here we assume one of \(S_{(+,+)}\) and \(S_{(-,-)}\) should contain two matrices whose superdiagonal vectors are not parallel by the statement of this lemma. Then, we can always make
the negative contribution larger by using matrices with different superdiagonal vectors. See Fig. 4 for an example. More formally, we consider the two cases as follows:

- **Case of** $S_{(+,+)} = \emptyset$ **and** $S_{(-,-)} = \emptyset$: Without loss of generality, assume that $S_{(-,+)}$ contains two matrices $M_1$ and $M_2$ with non-parallel superdiagonal vectors. Let $\vec{v}(M_1) = (a_1, b_1)$ and $\vec{v}(M_2) = (a_2, b_2)$ be superdiagonal vectors for $M_1$ and $M_2$, respectively, such that $|\frac{a_1}{b_1}| > |\frac{a_2}{b_2}|$. To simplify the proof, we assume the set $S_{(-,+)}$ only uses one matrix $M_3$, where $\vec{v}(M_3) = (a_3, b_3)$. Let $\vec{v}(M_1) = (a_1, b_1)$ and $\vec{v}(M_2) = (a_2, b_2)$ be superdiagonal vectors for $M_1$ and $M_2$, respectively, such that $|\frac{a_1}{b_1}| > |\frac{a_2}{b_2}|$. To simplify the proof, we assume the set $S_{(-,+)}$ only uses one matrix $M_3$, where $\vec{v}(M_3) = (a_3, b_3)$, to generate a matrix with a zero superdiagonal vector. This implies that $a_1 x + a_2 y + a_3 = 0$ and $b_1 x + b_2 y + b_3 = 0$ for some $x, y \in \mathbb{Q}$. Here the idea is that we first multiply the matrix $M_1$ and then multiply $M_2$ later. For instance, we first multiply $M_{m_1}$ and then $M_{m_2}$. Then, the coefficient of the highest power in $z$ becomes $a' = |a_1| + |a_2|$ and $b' = |b_1| + |b_2|$, the coefficient of $m^2$ is now $\frac{|a_2||b_1|-|a_1||b_2|}{2}$. By the supposition $|\frac{a_1}{b_1}| > |\frac{a_2}{b_2}|$, we prove that the coefficient of the highest power in $z$ is always negative.

- **Case of** $S_{(+,+)} = \emptyset$ **and** $S_{(-,-)} = \emptyset$: We can prove this case by the similar argument as above.

![Figure 3: Subcases where one of the subsets from $S_{(+,+)}$, $S_{(-,+)}$, $S_{(+,-)}$, and $S_{(-,-)}$ is empty](image)

![Figure 4: Subcases where two of the subsets from $S_{(+,+)}$, $S_{(-,+)}$, $S_{(+,-)}$, and $S_{(-,-)}$ are empty](image)

As we have proven that it is always possible to construct a matrix $M'$ such that $\psi(M') = (0, 0, c')$ for some $c' < 0$, we complete the proof. □
Example 8. We illustrate Lemma \(\overline{\Gamma}\) Consider a semigroup \(S\) generated by matrices

\[
\begin{pmatrix}
1 & -4 & 20 \\
0 & 1 & -6 \\
0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 3 & 20 \\
0 & 1 & -2 \\
0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & -1 & 20 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 2 & 20 \\
0 & 1 & 7 \\
0 & 0 & 1
\end{pmatrix}.
\]

A simple calculation shows that a product of the four matrices (in any order) is a matrix \(M\) such that \(\psi(M) = (0,0,80 + x)\) for some \(x \in \mathbb{Z}\). Our goal, is to minimize \(x\) by multiplying the matrices in a different order. Denote the given matrices by \(M_{(+,+)}\), \(M_{(\psi)}\), \(M_{(-,-)}\), and \(M_{(-,+)}\). The Identity problem for finitely generated matrix semigroups in the Heisenberg group \(\psi\) Theorem 9. There are two possible cases of having the identity matrix in the matrix semigroup in \(H(3, \mathbb{Q})\) vectors or there are at least two matrices with non-parallel superdiagonal vectors. Since superdiagonal vectors being parallel is a transitive and symmetric property, each matrix needs to be compared to a representative of each subset. If there are no matrices with parallel superdiagonal vectors, then there are \(r\) subsets \(G_i\) containing exactly one matrix and \(O(r^2)\) tests were done. Let us consider \(G_i = \{M_{k_1}, \ldots, M_{k_{s_i}}\}\) be one of the subsets containing \(s_i\) matrices and \(\psi(M_{k_j}) = (a_{kj}, b_{kj}, c_{kj})\) for all \(k \in [1, s]\) and \(j \in [1, s_i]\). By Lemma \(\overline{\Gamma}\) we transform each matrix \(M_{k_j}\) into the following form:

\[
\psi(M_{k_j}) = (a_{kj}, a_{kj}x, c_{kj} - \frac{x}{2}a_{kj}^2), \text{ for a fixed } x \in \mathbb{Q}.
\]

We solve the system of two linear Diophantine equations \(A\overline{\gamma} = \overline{0}\), where

\[
A = \begin{pmatrix}
a_{k_1} & a_{k_2} & \cdots & a_{k_{s_i}} \\
2c_{k_1} - xa_{k_1}^2 & 2c_{k_2} - xa_{k_2}^2 & \cdots & 2c_{k_{s_i}} - xa_{k_{s_i}}^2
\end{pmatrix}
\]

and \(\overline{\gamma} \geq 0\). The first row corresponds to element \(a\) being zero, and thus also the superdiagonal vector being zero, and the second row to the upper corner being zero.
It is obvious that the identity matrix is in the semigroup if we have a solution in the system of homogeneous linear Diophantine equations for any subset \(G_i\). That is, we need to solve at most \(r\) systems of homogeneous linear Diophantine equations.

Next, we consider the second case, where by Lemma 7 it is enough to check whether there exists a sequence of matrices generating a matrix with zero superdiagonal vector and containing two matrices with non-parallel superdiagonal vectors. Let us say that \(M_{i_1}, M_{i_2} \in G\) where \(1 \leq i_1, i_2 \leq r\) are the two matrices. Recall that \(G = \{M_1, M_2, \ldots, M_r\}\) is a generating set of the matrix semigroup and let \(\psi(M_i) = (a_i, b_i, c_i)\) for all \(1 \leq i \leq r\). We can see that there exists such a product containing the two matrices by solving a system of two homogeneous linear Diophantine equations of the form \(B\bar{y} = \vec{0}\), where

\[
B = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},
\]

with an additional constraint that the numbers in the solution \(\bar{y}\) that correspond to \(M_{i_1}\) and \(M_{i_2}\) should be non-zero since we must use two matrices in the product. We repeat this process at most \(r(r-1)\) times until we find a solution. Therefore, the problem reduces again to solving at most \(O(r^2)\) systems of linear Diophantine equations.

Finally, we conclude the proof by mentioning that the Identity problem for matrix semigroups in the Heisenberg group over rationals can be even decided in polynomial time as a system of homogeneous linear Diophantine equations can be solved in polynomial time when the solution is restricted to non-negative integers [21]. \(\square\)

**Decidability in higher dimensions.** Now we generalize our algorithm for the Identity problem in the Heisenberg group \(H(3, \mathbb{Q})\) to the domain of the Heisenberg groups for any dimension over the rational numbers. Similarly to the case of dimension 3, we establish the following result for the case of matrices where multiplication is commutative.

**Lemma 10.** Let \(G = \{M_1, M_2, \ldots, M_r\} \subseteq H(n, \mathbb{R})\) be a set of matrices from the Heisenberg group such that \(\psi(M_i) = (\vec{a}_i, \vec{b}_i, c)\) and \(\psi(M_j) = (\vec{a}_j, \vec{b}_j, c)\) and \(\vec{a}_i \cdot \vec{b}_j = \vec{a}_j \cdot \vec{b}_i\) for any \(1 \leq i \neq j \leq r\). If there exists a sequence of matrices \(M = M_{i_1}M_{i_2} \cdots M_{i_k}\), where \(i_j \in [1, r]\) for all \(1 \leq j \leq k\), such that \(\psi(M) = (0, 0, c)\) for some \(c \in \mathbb{R}\), then,

\[
c = \sum_{j=1}^{k} (c_{ij} - \frac{1}{2} \vec{a}_{ij} \cdot \vec{b}_{ij}).
\]

**Proof.** Consider the sequence \(M_{i_1}M_{i_2} \cdots M_{i_k}\) and let \(\psi(M_i) = (\vec{a}_i, \vec{b}_i, c_i)\) for each \(i \in [1, r]\). From the multiplication of matrices, we have the following equation:

\[
c = \sum_{j=1}^{k} c_{ij} + \sum_{l=1}^{k-1} \left( \sum_{j=1}^{l} \vec{a}_{ij} \right) \cdot \vec{b}_{i_{l+1}} = \sum_{j=1}^{k} c_{ij} + \frac{1}{2} \left( \sum_{l=1}^{k} \sum_{j=1}^{l} \vec{a}_{ij} \cdot \vec{b}_{ij} - \sum_{j=1}^{k} \vec{a}_{ij} \cdot \vec{b}_{ij} \right) = \sum_{j=1}^{k} (c_{ij} - \frac{1}{2} \vec{a}_{ij} \cdot \vec{b}_{ij}).
\]

From the above equation, we prove the statement claimed in the lemma. Moreover, due to the commutativity of multiplication, the value \(c\) does not change even if we change the order of multiplicands. \(\square\)

Lemma 7 does not generalize to \(H(n, \mathbb{Q})\) in the same way as we cannot classify matrices according to types to control the value in upper-right corner, so we prove that the value in the upper corner will be diverging to both positive and negative infinity quadratically as we repeat the same sequence generating any matrix \(M\) such that \(\psi(M) = (0, 0, c)\).
Lemma 11. Let $S = \langle M_1, M_2, \ldots, M_r \rangle \subseteq H(n, \mathbb{Q})$ be a finitely generated matrix semigroup. Then, the identity matrix exists in $S$ if there exists a sequence of matrices $M_{i_1}M_{i_2}\cdots M_{i_k}$, where $i_j \in [1, r]$ for all $1 \leq j \leq k$, satisfying the following properties:

1. $\psi(M_{i_1}M_{i_2}\cdots M_{i_k}) = (0, 0, c)$ for some $c \in \mathbb{Q}$, and
2. $a_{i_1j_1}^- b_{i_2j_2}^- \neq a_{i_1j_2}^+ b_{i_2j_1}^-$ for some $j_1, j_2 \in [1, k]$ where $\psi(M_i) = (a_i^- b_i^+, c_i)$ for $1 \leq i \leq r$.

Proof. It is easy to see that if there exists a sequence of matrices $M_{i_1}M_{i_2}\cdots M_{i_k}$, where $i_j \in [1, r]$ for all $1 \leq j \leq k$ satisfying the conditions claimed in this lemma, we can always construct the identity matrix in a similar way to the proof of Lemma [Lemma]

From the first property claimed in the lemma, we know that any permutation of the sequence of matrix multiplications of $M_{i_1}M_{i_2}\cdots M_{i_k}$ will lead to matrices $M'$ such that $\psi(M') = (0, 0, y)$ for some $y \in \mathbb{Q}$ since the multiplication of matrices in the Heisenberg group performs additions of vectors which is commutative in the top row and the rightmost column excluding the upper-right corner. From the commutative behavior in the horizontal and vertical vectors of matrices in the Heisenberg group, we also know that if we duplicate the matrices in the sequence $M_{i_1}M_{i_2}\cdots M_{i_k}$ and multiply the matrices in any order, then the resulting matrix can have a non-zero coordinate in the upper triangular coordinates only in the upper right corner.

Now let $j_1, j_2 \in [1, k]$ be two indices such that $a_{i_1j_1}^- b_{i_2j_2}^- \neq a_{i_1j_2}^+ b_{i_2j_1}^-$. Then, consider the following matrix $M_d$ that can be obtained by duplicating the sequence $M_{i_1}M_{i_2}\cdots M_{i_k}$ of matrices into $\ell$ copies and shuffle the order as follows: $M_d = M_{i_1}^{\ell_1} M_{i_2}^{\ell_2} M_{i_k}^{\ell_k}$, where $M_x$ is a matrix that is obtained by multiplying the matrices in $M_{i_1}M_{i_2}\cdots M_{i_k}$ except the two matrices $M_{j_1}$ and $M_{j_2}$. Then, it is clear that $\psi(M_d) = (0, 0, z)$ for some $z$. Let us say that $\psi(M_x) = (a_x^-, b_x^+, c_x)$. Then, it is easy to see that $a_{i_1j_1}^- + a_{i_2j_2}^- + a_x^- = 0$ and $b_{i_1j_1}^- + b_{i_2j_2}^- + b_x^- = 0$. Now we show that we can always construct two matrices that have only one non-zero rational number in the upper right corner with different signs. After which, we can always construct the identity matrix as in the proof of Lemma [Lemma]

First, let us consider the $\ell$th power of the matrix $M_{i_1j_1}$ as follows:

$$
\psi(M_{i_1j_1}^\ell) = (a_{i_1j_1}^- \ell, b_{i_1j_1}^- \ell, c_{i_1j_1} \ell + \sum_{h=1}^{\ell-1} h(a_{i_1j_1}^- b_{i_1j_1}^-)) = (a_{i_1j_1}^- \ell, b_{i_1j_1}^- \ell, c_{i_1j_1} \ell + a_{i_1j_1}^- b_{i_1j_1}^- (\ell - 1)\frac{\ell}{2}).
$$

It follows that the matrix $M_d$ satisfies the equation $\psi(M_d) = (0, 0, z)$ such that

$$
z = y\ell + (a_{i_1j_1}^- b_{i_1j_1}^- + a_{i_2j_2}^- b_{i_2j_2}^- + a_x^- b_x^-)(\ell - 1)\frac{\ell}{2} + (a_{i_1j_1}^- b_{i_2j_2}^- + a_{i_1j_2}^- b_{i_2j_1}^-)\ell^2
\quad + 1/2(2y - (a_{i_1j_1}^- b_{i_1j_1}^- + a_{i_2j_2}^- b_{i_2j_2}^- + a_x^- b_x^-))\ell.
$$

Now the coefficient of the highest term $\ell^2$ in $z$ can be simplified as follows:

$$
\frac{1}{2}((a_{i_1j_1}^- b_{i_1j_1}^- + a_{i_2j_2}^- b_{i_2j_2}^- + a_x^- b_x^-) + 2(a_{i_1j_1}^- b_{i_1j_2}^- + (a_{i_1j_1}^- + a_{i_1j_2}^-) b_x^-))
= \frac{1}{2}((a_{i_1j_1}^- + a_{i_2j_2}^-) (b_{i_1j_1}^- + b_{i_2j_2}^-) + a_{i_1j_1}^- b_{i_2j_2}^- - a_{i_1j_2}^- b_{i_1j_1}^- + (a_{i_1j_1}^- + a_{i_1j_2}^-) b_x^-)
= \frac{1}{2}((-a_x^-) (-b_x^-) + a_{i_1j_1}^- b_{i_2j_2}^- - a_{i_1j_2}^- b_{i_1j_1}^- + (-a_x^-) b_x^-)
= \frac{1}{2}(a_{i_1j_1}^- b_{i_1j_2}^- - a_{i_1j_2}^- b_{i_1j_1}^-).
$$
By the second property claimed in the lemma, we know that the coefficient of the highest term $\ell^2$ in $z$ cannot be zero. Moreover, the value of $z$ will be diverging to negative or positive infinity depending on the sign of $a_{i_1}^* \cdot b_{i_2} - a_{i_2}^* \cdot b_{i_1}$. Now we consider a different matrix $M_e$ which is defined to be the following product $M_{i_2}^e M_{i_1}^e M_{i_2}^e$ and say that $\psi(M_e) = (0, 0, e)$ for some $e \in \mathbb{Q}$. Since we have changed the role of two matrices $M_{i_1}$ and $M_{i_2}$, the value of $e$ can be represented by a quadratic equation where the coefficient of the highest term is $a_{i_2}^* \cdot b_{i_2} - a_{i_1}^* \cdot b_{i_1}$. Therefore, we have proved that it is always possible to construct two matrices that have only one non-zero rational number in the upper right corner with different signs and further obtain the identity matrix if there exists a product of matrices that satisfies the two conditions claimed in the lemma.

**Theorem 12.** The Identity problem for finitely generated matrix semigroups in the Heisenberg group $H(n, \mathbb{Q})$ is decidable.

**Proof.** Similarly to the proof of Theorem 9 there are two cases to ways the identity matrix can be generated. Either all the matrices commute or there is at least two matrices that do not commute. In contrast to Theorem 9 we do not claim that the problem is decidable in polynomial time since partitioning matrices according to dot products takes an exponential time in the number of matrices in the generating set.

Let $S$ be the matrix semigroup in $H(n, \mathbb{Q})$ generated by the set $G = \{M_1, M_2, \ldots, M_r\}$. Indeed, for matrices $N_1, N_2$ and $N_3$, such that $\psi(N_1) = (a_1^*, b_1^*, c_1)$, $\psi(N_2) = (a_2^*, b_2^*, c_2)$ and $\psi(N_3) = (a_3^*, b_3^*, c_3)$, if $a_1^* \cdot b_2 = a_2^* \cdot b_1$ and $a_2^* \cdot b_3 = a_3^* \cdot b_2$, it does not imply that $a_1^* \cdot b_3 = a_3^* \cdot b_1$. Therefore, the number of subsets of $G$, where each subset contains matrices that commute with other matrices in the same subset, is exponential in $r$ as two different subsets are not necessarily disjoint.

Now we examine whether it is possible to generate the identity matrix by multiplying matrices in each subset by Lemma 10. If it is not possible, we need to consider the case of having two matrices that do not commute with each other in the product with zero values in the upper-triangular coordinates except the corner. Let us say that $M_{i_1}, M_{i_2} \in G$ where $1 \leq i_1, i_2 \leq r$ are the two matrices. Recall that $G = \{M_1, M_2, \ldots, M_r\}$ is a generating set of the matrix semigroup and let $\psi(M_i) = (a_i^*, b_i^*, c_i)$ for all $1 \leq i \leq r$. We also denote the $m$th element of the vector $a_i^*$ (respectively, $b_i^*$) by $\tilde{a}_i(m)$ (respectively, $\tilde{b}_i(m)$) for $1 \leq m \leq n - 2$.

Then, we can see that there exists such a product by solving a system of $2(n - 2)$ homogeneous linear Diophantine equations of the form $B\tilde{y} = \tilde{0}$, where

$$B = \begin{pmatrix}
     a_1^*(1) & a_2^*(1) & \cdots & a_r^*(1) \\
     a_1^*(2) & a_2^*(2) & \cdots & a_r^*(2) \\
     \vdots & \vdots & \ddots & \vdots \\
     a_1^*(n-2) & a_2^*(n-2) & \cdots & a_r^*(n-2) \\
     b_1^*(1) & b_2^*(1) & \cdots & b_r^*(1) \\
     b_1^*(2) & b_2^*(2) & \cdots & b_r^*(2) \\
     \vdots & \vdots & \ddots & \vdots \\
     b_1^*(n-2) & b_2^*(n-2) & \cdots & b_r^*(n-2)
\end{pmatrix},$$

with an additional constraint that the numbers in the solution $\tilde{y}$ that correspond to $M_{i_1}$ and $M_{i_2}$ should be non-zero since we must use two matrices in the product. We repeat this process at most $r(r-1)$ times until we find a solution.

Hence, we can view the Identity problem in $H(n, \mathbb{Q})$ for $n \geq 3$ as the problem of solving systems of $2(n - 2)$ homogeneous linear Diophantine equations with some constraints on the solution. As we can solve systems of linear Diophantine equations, we conclude that the Identity problem in $H(n, \mathbb{Q})$ is also decidable. 

\[\square\]
5 The Identity Problem in Matrix Semigroups in Dimension Four

In this section, we prove that the Identity problem is undecidable for $4 \times 4$ matrices, when the generating set has 9 matrices, by introducing a new technique exploiting the anti-diagonal entries.

**Theorem 13.** Given a semigroup $S$ generated by nine $4 \times 4$ integer matrices, determining whether the identity matrix belongs to $S$ is undecidable.

**Proof.** We prove the claim by reducing from a famous undecidable problem called the Post correspondence problem (PCP). Let $\Sigma = \{a, b\}$ be a binary alphabet and $P = \{(u_1, v_1), \ldots, (u_n, v_n)\} \subseteq \Sigma^* \times \Sigma^*$ be a set of pairs of words where $n \geq 2$. Then, the PCP is to determine if there exists a finite sequence of indices $\ell_1, \ldots, \ell_k$ with each $1 \leq \ell_i \leq n$ such that: $u_{\ell_1} \cdots u_{\ell_k} = v_{\ell_1} \cdots v_{\ell_k}$.

We shall use an encoding to embed an instance of the PCP into a set of $4 \times 4$ integer matrices. It is well-known that $(a^i b \bar{a}^j | i \geq 1)$ freely generates a free subgroup of the free group $\langle a, b \rangle$ [4] and that the matrices $(\frac{1}{2} 1 \ 1 0)$ and $(\frac{1}{2} 0)$ freely generate a free subgroup of $\text{SL}(2, \mathbb{Z})$ [26].

Let $FG(\Sigma) = \{z_1, z_2, \ldots, z_\ell\}$ be a group alphabet and $FG(\Sigma_2) = \{a, b, \bar{a}, \bar{b}\}$ be a binary group alphabet. Define the mapping $\alpha : FG(\Sigma) \to FG(\Sigma_2)$ by: $\alpha(z_i) = a^i b \bar{a}, \alpha(\bar{a}) = a^i b \bar{a}$, where $1 \leq i \leq \ell$. It is easy to see that $\alpha$ is a monomorphism. We also define a monomorphism $f : (\Sigma_2) \to \mathbb{Z}^{2 \times 2}$ as $f(a) = (\frac{1}{2} 1), f(\bar{a}) = (\frac{1}{2} 1)$, $f(b) = (\frac{1}{2} 0)$ and $f(\bar{b}) = (\frac{1}{2} 0)$. The composition of two monomorphisms $\alpha$ and $f$ gives us the embedding from an arbitrary group alphabet into the special linear group $\text{SL}(2, \mathbb{Z})$. We use the composition of two monomorphisms $\alpha$ and $f$ to encode a set of pairs of words over an arbitrary group alphabet into a set of $4 \times 4$ integer matrices in $\text{SL}(4, \mathbb{Z})$.

Let $P = \{(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\} \subseteq \Sigma^* \times \Sigma^*$ be an instance of the PCP. Without loss of generality, we can assume that the first pair of words of the solution is $(u_1, v_1)$. We define the alphabet $\Gamma = FG(\Sigma_2) \cup FG(\Sigma_B)$, where $\Sigma_2 = \{a, b\}$ is the alphabet used in the instance of the PCP and $\Sigma_B = \{q_0, q_1, p_0, p_1\}$ is the alphabet for the border letters that enforce the form of a solution.

Then, we define the following sets of words $W_1 \cup W_2 \subseteq \Gamma \times \Gamma$, where

- $W_1 = \left\{ \begin{array}{c} \frac{q_0}{p_0} \cdot \frac{\bar{q}_0}{\bar{p}_0} \ 0 \ b \ rac{q_0}{p_0} \ b \ rac{\bar{q}_0}{\bar{p}_0} \end{array} \right\} \text{ and } a, b \in \Sigma, q_0, p_0 \in \Sigma_B$

- $W_2 = \left\{ \begin{array}{c} \frac{q_0}{p_0} \cdot \frac{q_1}{p_1} \cdot \frac{\bar{q}_0}{\bar{p}_0} \cdot \frac{q_1}{p_1} \cdot \frac{\bar{q}_0}{\bar{p}_0} \end{array} \right\} \ 1 \leq i \leq n, (u_i, v_i) \in P, q_0, q_1, p_0, p_1 \in \Sigma_B$

First we prove that $(q_0 q_1, p_0 p_1) \in (W_1 \cup W_2)^*$ if and only if the PCP has a solution. It is easy to see that any pair of words in $W_1^*$ is of the form $(q_0 w \bar{q}_0, p_0 w \bar{p}_0)$ for $w \in \Sigma_2^*$. Then, there exists a pair of words in $W_2^*$ of the form $(q_0 \bar{w} q_1, p_0 \bar{w} p_1)$ for some word $w \in \Sigma^*$ if and only if the PCP has a solution. Therefore, the pair of words $(q_0 q_1, p_0 p_1)$ can be constructed by concatenating pairs of words in $W_1$ and $W_2$ if and only if the PCP has a solution.

For each pair of words $(u, v) \in W_1 \cup W_2$, we define a matrix $A_{u,v}$ to be $\begin{pmatrix} f(a(u)) & 0_2 \\ 0_2 & f(a(v)) \end{pmatrix} \in \text{SL}(4, \mathbb{Z})$, where $0_2$ denotes the zero matrix in $\mathbb{Z}^{2 \times 2}$. Moreover, we define the following matrix

$$B_{q_0 q_1, p_0 p_1} = \begin{pmatrix} 0_2 & f(a(q_1 \bar{q_0})) \\ f(a(p_1 \bar{p_0})) & 0_2 \end{pmatrix} \in \text{SL}(4, \mathbb{Z})$$

Let $S$ be a matrix semigroup generated by the set $\{A_{u,v}, B_{q_0 q_1, p_0 p_1} | (u, v) \in W_1 \cup W_2\}$. We already know that the pair $(q_0 q_1, p_0 p_1)$ of words can be generated by concatenating words in $W_1$ and $W_2$ if and only if the PCP has a solution. The matrix semigroup $S$ has the corresponding matrix $A_{q_0 q_1, p_0 p_1}$ and thus,

$$\begin{pmatrix} f(a(q_0 \bar{q_1})) & 0_2 \\ 0_2 & f(a(p_0 \bar{p_1})) \end{pmatrix} \begin{pmatrix} 0_2 & f(a(q_1 \bar{q_0})) \\ f(a(p_1 \bar{p_0})) & 0_2 \end{pmatrix} = \begin{pmatrix} 0_2 & f(a(\varepsilon)) \\ f(a(\varepsilon)) & 0_2 \end{pmatrix} \in S.$$
Then, we see that the identity matrix $I_4$ exists in the semigroup $S$ as follows:

\[
\begin{pmatrix}
0 & f(\alpha(\varepsilon)) \\
f(\alpha(\varepsilon)) & 0
\end{pmatrix}
\begin{pmatrix}
0 & f(\alpha(\varepsilon)) \\
f(\alpha(\varepsilon)) & 0
\end{pmatrix}
= 
\begin{pmatrix}
0 & f(\alpha(\varepsilon)) \\
0 & f(\alpha(\varepsilon))
\end{pmatrix}
= 
\begin{pmatrix}
I_2 & 0 \\
0 & I_2
\end{pmatrix}
= I_4 \in S.
\]

Now we prove that the identity matrix does not exist in $S$ if the PCP has no solution. It is easy to see that we cannot obtain the identity matrix only by multiplying ‘A’ matrices since there is no possibility of cancelling every border letter. We need to multiply the matrix $B_{q_1\bar{q}_0,p_1\bar{p}_0}$ with a product of ‘A’ matrices at some point to reach the identity matrix. Note that the matrix $B_{q_1\bar{q}_0,p_1\bar{p}_0}$ cannot be the first matrix of the product, followed by the ‘A’ matrices, because the upper right block of $B_{q_1\bar{q}_0,p_1\bar{p}_0}$, which corresponds to the first word of the pair, should be multiplied with the lower right block of ‘A’ matrices, which corresponds to the second word of the pair. Suppose that the ‘A’ matrix is of the following form: 

\[
\begin{pmatrix}
0 \\
f(\alpha(q_0 u \bar{q}_1))
\end{pmatrix}
\begin{pmatrix}
of(\alpha(p_0 v \bar{p}_1)) \\
2
\end{pmatrix}.
\]

Since the PCP instance has no solution, either $u$ or $v$ is not the empty word. We multiply $B_{q_1\bar{q}_0,p_1\bar{p}_0}$ to the matrix and then obtain the following matrix:

\[
\begin{pmatrix}
0 \\
f(\alpha(q_0 u \bar{q}_1))
\end{pmatrix}
\begin{pmatrix}
of(\alpha(p_0 v \bar{p}_1)) \\
2
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix}
= \begin{pmatrix}
I_2 & 0 \\
0 & I_2
\end{pmatrix}
= I_4 \in S.
\]

We can see that either the upper right part or the lower left part cannot be $f(\alpha(\varepsilon))$, which actually corresponds to the identity matrix in $\mathbb{Z}^{2 \times 2}$. Now the only possibility of reaching the identity matrix is to multiply matrices which have $\text{SL}(2, \mathbb{Z})$ matrices in the anti-diagonal coordinates like $B_{q_1\bar{q}_0,p_1\bar{p}_0}$. However, we cannot cancel the parts because the upper right block (the lower left block) of the left matrix is multiplied with the lower left block (the upper right block) of the right matrix as follows:

\[
\begin{pmatrix}
0 & A \\
B & 0
\end{pmatrix}
\begin{pmatrix}
0 & C \\
D & 0
\end{pmatrix}
= 
\begin{pmatrix}
AD & 0 \\
0 & BC
\end{pmatrix},
\]

where $A, B, C$ and $D$ are matrices in $\mathbb{Z}^{2 \times 2}$. As the first word of the pair is encoded in the upper right block of the matrix and the second word is encoded in the lower left block, it is not difficult to see that we cannot cancel the remaining blocks.

Currently, the undecidability bound for the PCP is five [29] and thus $|W_1 \cup W_2| = 9$. \,$\square$

Consider the membership problem called the special diagonal membership problem, where the task is to determine whether a scalar multiple of the identity matrix exists in a given matrix semigroup. The most recent undecidability bound is shown to be 14 by Halava et al. [20]. We improve the bound to nine, as the identity matrix is the only diagonal matrix of the semigroup $S$ in the proof of Theorem 13. We also prove that the Identity problem is undecidable in $\mathbb{H}(\mathbb{Q})^{2 \times 2}$ as well by replacing the composition $f \circ \alpha$ of mappings with a mapping from a group alphabet to the set of rational quaternions [4].

**Corollary 14.** Given a semigroup $S$ generated by nine $4 \times 4$ integer matrices, determining whether there exists any diagonal matrix in $S$ is undecidable.

**Corollary 15.** Given a semigroup $S$ generated by nine $2 \times 2$ rational quaternion matrices, determining whether there exists the identity matrix $S$ is undecidable.

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