Mechanics of Apparent Horizon in Two Dimensional Dilaton Gravity

Rong-Gen Cai and Li-Ming Cao

Abstract In this article, we give a definition of apparent horizon in a two dimensional general dilaton gravity theory. With this definition, we construct the mechanics of the apparent horizon by introducing a quasi-local energy of the theory. Our discussion generalizes the apparent horizons mechanics in general spherically symmetric spacetimes in four or higher dimensions to the two dimensional dilaton gravity case.

1 Introduction

Quantum theory together with general relativity predicts that black hole behaves like a black body, emitting thermal radiation, with a temperature proportional to the surface gravity of the black hole and with an entropy proportional to the area of the cross section of the event horizon [1, 2]. The Hawking temperature and Bekenstein-Hawking entropy together with the black hole mass obey the first law of thermodynamics [3]. The first law of thermodynamics of black hole has two different versions—phase space version or passive version and physical process or active version [4]. In these two versions of discussion, the stationary of the black hole is essential, and the discussion is focused on the event horizon of these stationary spacetimes. However, this kind of horizon strongly depends on the global structure of the spacetime and there exist some practical issues which can not be easily solved [5]. It is very interesting to note that gravitational field equation on the black hole horizon can be expressed into a first law form of thermodynamics [6, 7, 8]. The
usual approach to black hole thermodynamics is to start with the dynamics of gravity and ends with the thermodynamic of black hole spacetimes. Can we turn the logic around and get the dynamics of gravity from some thermodynamic considerations? This can not be fulfilled in this traditional approach because of the dependence of the global spacetime information of the event horizon. Apparent horizon defined by Hawking does not rely on the causal structure of the spacetime. However, it still depend on some global information of the spacetime—one has to select a slicing of the spacetime in advance. Furthermore, it is not clear how to establish thermodynamics on general apparent horizons. To reveal the relation between the spacetime dynamics and thermodynamics, probably local or quasilocal defined horizons are necessary.

A local or quasi-local definition of horizon is based on the local geometry of the spacetime. So it has a potential possibility to provide us more hints to study the relation between some fundamental thermodynamics and the gravitational equations. Along this way, fruitful results have been obtained. In fact, based on local Rindler horizon, Jacobson et al. [9, 10] was able to derive gravity field equation from the fundamental Clausius relation. With the assumption of FRW spacetime, Cai and Kim have obtained the Friedmann equations from the fundamental thermodynamical relation \( dE = T dS \) on the apparent horizon of the spacetimes [11]. A simple summary on the relation between the spacetime dynamics and thermodynamical first law can be found in Ref. [12, 13], while the further understandings of gravitational dynamics from thermodynamical aspects can be seen in Ref. [14].

On the other hand, for general dynamical black holes, Hayward has proposed a new horizon, trapping horizon, to study associated thermodynamics in 4-dimensional Einstein theory without the stationary assumption [15]. In this theory, for general spherically symmetric spacetimes, Einstein equations can be rewritten in a form called “unified first law”. Projecting this unified first law along trapping horizon, one gets the first law of thermodynamics for dynamical black holes. This trapping horizon can be null, spacelike, and timelike, and has no direct relation to the causal structure of the spacetime. Inspired by this quasilocal definition of horizon, Ashtekar et al have proposed two types of horizons, i.e., isolated horizon and dynamical horizon. The former is null, while the later is spacelike [5]. The mechanics of these horizons also has been constructed. In some sense, the trapping horizon is a generalization of the Hawking’s apparent horizon. However, the slicing of the spacetimes is not necessary to define this horizon. Nevertheless, in this paper, we still use the terminology of apparent horizon. Of course, it has the same meanings as the trapping horizon, and can be understood as a generalized apparent horizon.

Two dimensional dilation gravity theory has been widely studied over the past twenty years. One can get this kind of gravity from the spherically symmetric reduction of Einstein gravity theory in higher dimensions. To eliminate Weyl anomaly on string world sheet, one also has such a kind of gravity theory, for example, the famous CGHS model and others, for a nice review see [16]. In these theories, a lot of black hole solutions and cosmological solutions have been found. When some matter fields are included, in general the situations become complicated, and usually we have to study general dynamical solutions. In these two dimensional dilaton
Mechanics of Apparent Horizon in Two Dimensional Dilaton Gravity

Gravity theories, the apparent horizon has been used for a long time. However, what is the meaning of the apparent horizon in a two dimensional theory? Obviously, apparent horizon can not be defined in the usual way because we can not define any expansion scalar of a null congruence in two dimensional spacetimes. In other words, the codimension-2 surface shrinks to a point in a two dimensional spacetime, and intuitively, the size of the point does not change along a light-like geodesic, so the expansion does not make any sense. In this paper, we will propose a definition of apparent horizon in these two dimensional general dilaton gravity theories. With this definition in hand, we can construct the mechanics of the apparent horizon by introducing a quasilocal energy. This energy is similar to the Misner-Sharp energy in four or higher dimensional Einstein gravity theory. Actually, it can be found that this energy reduces to the usual Misner-Sharp energy for a special kind of dilaton gravity theory coming from the spherical reduction of higher dimensional Einstein gravity theory.

2 General dilaton gravity theory in two dimensions

For a general two dimensional dilaton gravity theory, its action can be written into a following form [16]

$$I = \int d^2x \sqrt{-h} \left[ \Phi R + U(\Phi) D^a \Phi D_a \Phi + V(\Phi) + \mathcal{L}_m \right],$$

(1)

where $\Phi$ is the so-called dilaton field, and $R$ is the two dimensional Ricci scalar. The matter Lagrangian is represented by $\mathcal{L}_m$ which may contain tachyon (and others). The matter is denoted by $\psi$ for simplicity. So, in general, the matter Lagrangian can be expressed as

$$\mathcal{L}_m = \mathcal{L}_m(\psi, D_a \psi, \ldots, \Phi, D_a \Phi, \ldots).$$

The equation of motion for the dilaton field $\Phi$ can be written as

$$R - U'(\Phi) D^a \Phi D_a \Phi + V'(\Phi) - 2U(\Phi) \Box \Phi + \mathcal{T}_m = 0,$$

(2)

where $\Box = D_a D^a$ and the prime stands for the derivative with respect to $\Phi$: $d/d\Phi$, while the scalar $\mathcal{T}_m$ is defined as

$$\mathcal{T}_m = \frac{\partial \mathcal{L}_m}{\partial \Phi} - D_a \frac{\partial \mathcal{L}_m}{\partial D_a \Phi} + \ldots.$$

Of course, if the dilation field does not couple to the matter field, this term vanishes. The Euler-Lagrangian equation for the matter field $\psi$ can be obtained in a similar way. The equations of motion for the metric $h_{ab}$ can be put into a form...
\[ U(\Phi)D_a\Phi D_b\Phi - \frac{1}{2} U'(\Phi)D_a\Phi D_b\Phi h_{ab} - D_aD_b\Phi \]
\[ + \Box h_{ab} - \frac{1}{2} V(\Phi) h_{ab} = T_{ab}, \]  
(3)

where \( T_{ab} \) is the energy-momentum tensor of the matter field. Straightforward calculation shows the covariant divergence of this energy-momentum tensor is given by

\[ D^aT_{ab} = -\frac{1}{2} \left[ R - U'(\Phi)D\Phi D_a\Phi + V'(\Phi) - 2U(\Phi)\Box \right] D_b\Phi, \]
(4)

and this suggests that

\[ D^aT_{ab} = \frac{1}{2} T_m D_b\Phi. \]
(5)

Thus we see that the dilation provides an external force to the matter field when the coupling between the matter field and dilaton is present.

### 3 Some solutions of the theory

When the matter field is absent, we have \( T_{ab} = 0 \) and \( T_m = 0 \). In Eddington-Finkelstein gauge, the general solution of Eqs. (3) has a simple form \[ h = e^Q \left[ 2dv \Phi + (w - 2m)dv^2 \right]. \]
(6)

Here, two functions \( Q(\Phi) \) and \( w(\Phi) \) have been introduced, and they are defined by

\[ U = -Q', \quad V = e^{-Q}w' \]
(7)

up to some constants. Note that \( e^Q dv \Phi \) is a closed form, from Poincaré lemma, there exists a function \( r \) satisfying \( dr = e^Q dv \Phi \). This means the general solution of the metric can be transformed into a familiar form

\[ h = \left[ 2dvdr - e^Q(w - 2m)dv^2 \right]. \]
(8)

One can replace the constant \( m \) by a function of \( v \), i.e., \( m(v) \), and then construct a typical dynamical spacetime, i.e., Vaidya-like spacetime. In that case, the energy-momentum tensor of matter field will no longer vanish, and has a nontrivial component \( T_{vv} \) satisfying

\[ \frac{dm(v)}{dv} = T_{vv}. \]
(9)

Naively one can read off the apparent horizon of this spacetime in this Eddington-Finkelstein gauge—it is given by equation

\[ w - 2m(v) = 0 = e^Q D_a\Phi D^a\Phi. \]
(10)
However, we may ask a question here — What is the apparent horizon in this two dimensional spacetime? In the above discussion, we have read off naively the location of apparent horizon from the metric in Eddington-Finkelstein gauge. But in what sense it is an apparent horizon? Usually, the definition of apparent horizon depends on the extrinsic geometry of codimension-2 spacelike surface, i.e., the expansion scalars of the surface. Now we are considering two dimensional spacetime, the codimension-2 surface shrinks to a point, and the expansion scalars cannot be defined. In the next section, we will focus on this question, and explain why the location of $D_a\Phi D^a\Phi = 0$ can be viewed as the apparent horizon in the two dimensional case.

4 Apparent horizon

In this section, we define the apparent horizons in the two dimensional spacetime of the dilaton gravity theory. Assume $\{\ell^a, n^a\}$ is a null frame in the spacetime, and the metric can be expressed as

$$h_{ab} = -\ell_a n_b - n_a \ell_b,$$

(11)

where $\ell^a$ and $n^a$ are two null vector fields which are globally defined on the spacetime and satisfy $\ell_a n^a = -1$. We assume $\ell^a$ and $n^a$ are both future pointing, and furthermore, $\ell^a$ and $n^a$ are outer pointing and inner pointing respectively. On the spacetime, there is a natural vector field $\phi^a = D^a\Phi$. Obviously, the causality of the vector field is determined by the signature of $\phi^a \phi_a$. According to the causality of this vector, the spacetime can be divided into several parts, and in each part the vector field $\phi^a$ either spacelike or timelike. $\phi^a$ is null on the boundary of any part, and this boundary can be defined as a kind of horizon. It is easy to find that on this horizon we have

$$L_\ell \Phi = 0 \text{ or } L_n \Phi = 0.$$ 

So on these horizons we have $L_\ell \Phi = 0$ or $L_n \Phi = 0$. We can further classify the horizons as follows. The horizon is called future if $L_\ell \Phi = 0$ and $L_n \Phi < 0$. In this case, if $L_n L_\ell \Phi < 0$, we call the horizon is outer. The future horizon with $L_n L_\ell \Phi > 0$ is called inner. The past horizon is defined by $L_n \Phi = 0$, and $L_\ell \Phi > 0$. Similarly, the past horizon with $L_\ell L_n \Phi > 0$ is called outer, and the case with $L_\ell L_n \Phi < 0$ is called inner. Mimicking the cases in higher dimensions, the region with $L_\ell \Phi < 0$ and $L_n \Phi < 0$ (or $\phi^a \phi_a < 0$) can be called trapped region of the spacetime [15].

At the first sight, these definitions have nothing to do with the geometry of the spacetime. How do these definitions realize the description of the spacetime region where light can not escape? To answer this question, we have to investigate the detailed structure of the definition and the equations of motion of the dilaton theory. Some calculation shows
where we have used the equation (13) and introduced two scalars $\kappa_\ell = -n_a e^b D_b e^a$, $\kappa_n = -\ell_a n^b D_b n^a$. From these equations, it is easy to find that on the future outer horizon, we have

$$\Box \Phi > 0.$$  

Similarly, on the past outer horizon, we have $\Box \Phi < 0$. In the following discussion, we will focus on the future outer horizon.

Here, we give some explanation why we can define the horizon in this way. From Eqs. (12), we have

$$L_k (\phi \phi^a) = \alpha \left( \Box \Phi (L_\ell \Phi) + 2 (T_{ab} e^a e^b) (L_n \Phi) - 2 U (\Phi) (L_n \Phi) (L_\ell \Phi)^2 \right),$$  

where $k^a = \alpha \ell^a$ is some null vector field and $\alpha$ is a positive function such that the parameter of $k^a$ is affine. We consider the region very near the future outer horizon where $L_\ell \Phi = 0$ and $L_n \Phi < 0$. In this small neighbourhood, from continuity, $\Box \Phi$ should be positive and $L_\ell \Phi$ is very small. Now let us consider the part of the neighbourhood inside the trapped region of the spacetime (where $L_\ell \Phi$ is a small negative quantity and $L_n \Phi$ is a finite negative quantity). In this case, we have

$$L_k \| \phi \| < 0, \quad \| \phi \| = \sqrt{\| \phi \|^2},$$  

only when the null energy condition is broken, i.e., $T_{ab} e^a e^b < 0$. This mathematical relation suggests the light with wave vector $k^a$ can approach the line with $\| \phi \| = 0$ (inside the trapped region) only when the null energy condition of the matter field is broken. This can not happen for usual classical matter field. So the outward propagating light do not exist near the future outer horizon. Similarly, we have

$$L_k (\phi \phi^a) = \alpha \left( \Box \Phi (L_\ell \Phi) + 2 (T_{ab} n^a n^b) (L_n \Phi) - 2 U (\Phi) (L_n \Phi) (L_\ell \Phi)^2 \right),$$  

where $\alpha$ is still a positive function. Obviously, inside the trapped region and near the future outer horizon, we have $L_k \| \phi \| > 0$. This means that inward propagating light is always allowed whatever the energy condition is satisfied or not.

Now, let us consider the neighborhood of the horizon inside the region where $\phi \phi^a > 0$. In the region, $L_\ell \Phi$ is a small positive quantity. From Eq. (14), it is easy to find that it is possible to get $L_k \| \phi \| > 0$ in this case especially when matter field is absent. This means the light has possibility to escape from this region to the region with large value of $\| \phi \|$. For the inner pointing light, Eq. (16) suggests it can cross the horizon and can reach the trapped region.
In a word, light cannot escape from the trapped region we have defined. So the horizon we have defined has the same properties as the apparent horizon in higher dimensions which, roughly speaking, can be viewed as the boundary of trapped region. So in this paper we still use the terminology of apparent horizon to describe this kind of horizon. In the next section, by introducing a quasiloclal energy in this dilaton gravity theory, the mechanics of the apparent horizon will be established.

5 The mechanics of the apparent horizon

To study the mechanics of the apparent horizon, we have to define the quasiloclal energy inside the horizon. Generally, this is not an easy task. However, in the dilaton gravity we are considering, there is a well defined quasiloclal energy. This can be found as follows. From the energy-momentum tensor of the matter field, we can define two useful quantities, i.e., a scalar called generalized pressure

$$P = -\frac{1}{2} T^a_a,$$

and a vector called energy-supply,

$$\Psi_a = T_a^b e^Q D_b \Phi + Pe^Q D_a \Phi.$$

It is easy to find

$$\Psi_a = e^Q \left[ \frac{1}{2} F \Phi D_c \Phi D_a \Phi - D_a D^b \Phi D_b \Phi + \frac{1}{2} D_c D_c \Phi D_a \Phi \right].$$

Thus we have

$$\Psi_a + Pe^Q D_a \Phi = \frac{1}{2} e^Q \left[ U \left( \Phi \right) D_c \Phi D_a \Phi - D_a \left( D^c \Phi D_c \Phi \right) + V \left( \Phi \right) D_a \Phi \right].$$

It is not hard to find that the right hand side of the equation (20) can be written as

$$\frac{1}{2} D_a \left[ w \left( 1 - \frac{e^Q}{w} D_c \Phi D_c \Phi \right) \right].$$

In the above equation, $Q$ and $w$ are the same as those given in Eq.(7). Comparing with the unified first law in higher dimensional spherical symmetric spacetime (15), we can define a similar quasi-local energy

$$E = \frac{1}{2} \left[ w \left( 1 - \frac{e^Q}{w} D_c \Phi D_c \Phi \right) \right].$$
Then the equations of motion for the metric, i.e., equation (3) can be put into the form

\[ D_a E = \Psi_a + Pe^Q D_a \Phi. \] (23)

Since \( e^Q D_a \Phi \) is a closed one form in the spacetime, at least locally, it can be expressed as \( e^Q D_a \Phi = D_a \Psi' \) for some function \( \Psi' \). This suggests the above relation can be transformed into a simple form, i.e.,

\[ dE = \Psi + Pd\Psi'. \] (24)

This energy \( E \) generalizes the Misner-Sharp energy in higher dimensional theory to the two dimensional dilaton gravity theory, and at the same time, the above equation establishes the unified first law in this two dimensional gravity theory.

Now let us consider the special case where the matter field is absent, i.e., \( T_{ab} \) is vanishing, so do \( P \) and \( \Psi \). From the above unified first law (24), we have \( dE = 0 \). This means that \( E \) is a constant which can be denoted by \( m \). By this consideration, from the energy form (22), we have Eq. (10) with \( m(\nu) \). When the energy-momentum tensor is given by some radiation matter, the general solution is just the Vaidya-like spacetime mentioned in the previous section. In this case, it is easy to find that \( E \) is nothing but \( m(\nu) \) and the first law (24) reduces to the Bondi’s energy balance equation (9).

Assume on the apparent horizon, i.e., on the spacetime points which satisfy \( D_a D_c \Phi D_c \Phi = 0 \), that the dilaton field \( \Phi \) takes value \( \Phi_A \), then, on the apparent horizon, the quasi-local energy becomes

\[ E = \frac{1}{2} w(\Phi_A). \] (25)

In general, this energy is not constant because \( \Phi_A \) may depend on the coordinates. For example, for the Vaidya-like spacetime, the total energy inside the apparent horizon is \( \frac{1}{2} w(\Phi_A) = m(\nu) \). For the static case without matter, the apparent horizon coincides with the event horizon (if it can be defined), this energy becomes \( \frac{1}{2} w(\Phi_A) = \frac{1}{2} w(\Phi_e) = m \), where \( \Phi_e \) is the value of dilation on the event horizon.

On the apparent horizon, the energy-supply becomes

\[ \Psi_a = \frac{1}{2} e^Q \left[ -D_a (D^c \Phi D_c \Phi) + \Box \Phi D_a \Phi \right]. \] (26)

Let \( \xi \) be the vector tangent to the apparent horizon. Since \( D^c \Phi D_c \Phi \) is a constant on the apparent horizon, \( \xi^a D_a (D^c \Phi D_c \Phi) = 0 \), then we find

\[ \xi^a \Psi_a = \frac{1}{2} e^Q (\Box \Phi) \delta \xi \Phi, \] (27)

where \( \delta \xi \) is Lie derivative along the vector \( \xi \). In higher dimensions, the surface gravity of an apparent horizon is defined by the Kodama vector field \( K^a D_b K_a = \kappa K_a \) [7]. Here we can also introduce a Kodama-like vector field as \( K_a = -e^Q e^{ab} D_b \Phi \).
Some calculation shows

\[ K^b D_b K^a = \frac{1}{2} \left[ e^Q \Box \Phi K_a - U (e^Q D_b \Phi D^b \Phi) K_a \right]. \]  

(28)

From the definition of the surface gravity, it is easy to find the surface gravity of the apparent horizon can be expressed as

\[ \kappa = \frac{1}{2} e^Q (\Box \Phi). \]

On the future outer apparent horizon, from Eq. (13), we see this surface gravity is always positive. Therefore, we find that the energy-supply projecting onto the apparent horizon gives

\[ \xi^a \Psi_a = \kappa \mathcal{L}_\xi \Phi. \]

(29)

As a result, on the apparent horizon, we have a relation

\[ \mathcal{L}_\xi E = \frac{\kappa}{2\pi} \mathcal{L}_\xi S + P \mathcal{L}_\xi V, \]

(30)

where \( S = 2\pi \Phi \). This relation is the same as the first law of thermodynamics if we identify \( T = \kappa / 2\pi \) and regard \( S \) as entropy. Actually, the “entropy” of the future outer apparent horizon can also be written as

\[ S = 2\pi \Phi_A. \]

(31)

In general, this entropy is not a constant, and it might change with some coordinate. In the static case, the future outer apparent horizon coincides with the event horizon, this entropy becomes \( S = 2\pi \Phi_+ \), this result has been found in many static black holes in the two dimensional dilaton gravity theories [16].

The function \( V \) can be viewed as a kind of “volume” of the system. It comes from the divergence free of the Kodama vector \( K^a \), i.e., \( D_a K^a = 0 \), and can be viewed as a conserved quantity of the theory. Besides the function \( V \), the energy \( E \) can also be viewed as a conserved quantity. Actually, from Eqs. (3) and (5), one can prove that \( J^\mu = T^a_{\mu} K^a \) is conserved, i.e., we have \( D_a J^\mu = 0 \). This suggests that the Hodge dual of \( J_a \) is a closed one form, and locally it is the exterior derivative of a function. This function is nothing but the energy \( E \). Such kind of discussion can also be found in [18] where some special matter Lagrangian has been considered.

For the case of the energy-momentum tensor given by conformal matter, for example, tachyon, the trace of the energy-momentum tensor vanishes. In this case, from the trace part of Eq. (3), the surface gravity becomes \( \kappa = -\frac{1}{2} w^\prime (\Phi_A) \). This is very similar to the static case where the surface gravity vanishes just given by \(-\frac{1}{2} w^\prime (\Phi_+) \).

The work term vanishes due to the traceless of the energy-momentum tensor, so the first law on the apparent horizon becomes

\[ \mathcal{L}_\xi E = \frac{\kappa}{2\pi} \mathcal{L}_\xi S. \]

(32)
This means for the system with conformal matter, there is no external work term. The above result shows that, in the case with conformal matter, all the thermodynamic quantities of the apparent horizon can be obtained from the static case through the replacement of $\Phi$ by $\Phi_A$.

For the two dimensional dilatonic gravity coming from the spherically symmetric reduction of $n$-dimensional Einstein gravity, the potential $U$ and $V$ are given respectively by

$$U(\Phi) = \frac{n-3}{n-2} \Phi^{-1}, \quad V(\Phi) = (n-2)(n-3)\lambda^2 \Phi^{\frac{n-4}{n-2}}, \quad (33)$$

where $\lambda$ is a constant with dimension of mass square, and $\Phi = (\lambda r)^{n-2}$. If the function $U$ and $V$ have the forms (33), the quasi-local energy (22) is just the $n$-dimensional Misner-Sharp energy

$$E_{MS} = \frac{1}{2} (n-2) \rho^{n-3} (1 - D^a D_a r). \quad (34)$$

It is straightforward to see that the entropy of apparent horizon in the two dimensional dilatonic gravity is given by the area of the horizon sphere in $n$-dimensions. This can be obtained by replacing $\Phi_A$ in equation (31) by $(\lambda r_A)^{n-2}$. It should be noted here that our discussion is not restricted to the future outer apparent horizon. Actually, the same discussions can be applied to other types of apparent horizons. For example, in an FRW universe, the entropy of the apparent horizon is given by the one quarter of the area of the apparent horizon, and the radius of the apparent horizon can be expressed by Hubble parameter as

$$\frac{1}{\tilde{r}_A^2} = H^2 + \frac{k}{a^2},$$

where $\tilde{r}_A = ar_A$, and $a$ is the scale factor in the FRW universe, see for example [19]. Thus the apparent horizon associated with the FRW universe also applies here.

Acknowledgements This work is dedicated to celebrate the 60th birthday of Prof. T. Padmanabhan. The work was supported in part by the National Natural Science Foundation of China with grants No.11205148, No.11235010, No.11375247 and No.11435006.

References

1. S. W. Hawking, Commun. Math. Phys. 43, 199 (1975).
2. J. D. Bekenstein, Phys. Rev. D 7, 2333 (1973).
3. J. M. Bardeen, B. Carter and S. W. Hawking, Commun. Math. Phys. 31, 161 (1973).
4. R. M. Wald, Living Rev. Rel. 4:6, 2001.
5. A. Ashtekar, Living Rev. Rel. 7:10, 2004.
6. T. Padmanabhan, Class. Quant. Grav. 19, 5387 (2002) [gr-qc/0204019].
7. T. Padmanabhan, arXiv:0910.0839 [gr-qc].
8. R. G. Cai and N. Ohta, Phys. Rev. D 81, 084061 (2010) [arXiv:0910.2307 [hep-th]].
9. T. Jacobson, Phys. Rev. Lett. 75, 1260 (1995) [arXiv:gr-qc/9504004].
10. C. Eling, R. Guedens, and T. Jacobson, Phys. Rev. Lett. 96, 121301 (2006).
11. R. G. Cai and S. P. Kim, JHEP 0502, 050 (2005) [hep-th/0501055].
12. R.-G. Cai, J. Phys. Conf. Ser. 484, 012003 (2014).
13. R. G. Cai, Prog. Theor. Phys. Suppl. 172, 100 (2008) [arXiv:0712.2142 [hep-th]].
14. T. Padmanabhan, Rept. Prog. Phys. 73, 046901 (2010) [arXiv:0911.5004 [gr-qc]].
15. S. A. Hayward, Class. Quant. Grav. 15, 3147 (1998) [arXiv:gr-qc/9710089].
16. D. Grumiller, W. Kummer and D. V. Vassilevich, Phys. Rept. 369, 327 (2002) [hep-th/0204253].
17. H. Maeda and M. Nozawa, Phys. Rev. D 77, 064031 (2008) [arXiv:0709.1199 [hep-th]].
18. R. B. Mann, Phys. Rev. D 47, 4438 (1993) [hep-th/9206044].
19. R. G. Cai and L. M. Cao, Phys. Rev. D 75, 064008 (2007) [gr-qc/0611071].