Modeling Mild Collisions

I-Sheng Yang
IOP and GRAPPA, Universiteit van Amsterdam,
Science Park 904, 1090 GL Amsterdam, Netherlands
E-mail: I.S.Yang@uva.nl

Abstract. It is a surprisingly common phenomenon that two objects collide with each other and emerge only mildly altered. We motivate a dynamic-independent, analytical framework to study these mild collisions through two specific examples: (1) Head-on collision between two non-integrable solitons, and (2) Gravitational self-interaction for a collapsing shell of radiation.

1. Introduction
Scattering theory connects detail dynamics to observables with clear physical meanings. Such connection can be particularly sharp for a special class of scattering events: when two localized objects collide and end up almost unaffected. These mild collisions support an ex post facto perturbative expansion. As soon as one observes a mild collision, one can immediately write down an abstract expansion \( \phi_0 + \phi_1 \). Here \( \phi_0 \) describes two colliding objects being completely unaltered, and \( \phi_1 \) describes small corrections. The dynamical reason why \( \phi_1 \) is small will later emerge, and sometimes one can answer important physical questions independent of that.

Although we call them special events, mildness of collisions is a signature behavior of solitons which arises in many branches of physics [1, 2]. Thus we will first review the recent progress made by Amin, Lim and myself regarding collisions of solitons [3, 4]. We demonstrated that the most natural reason why soliton collisions are mild is very different from the common impression of (near) integrability.

Mild collisions are also common in gravitational physics. In our second example, we will demonstrate how to map a problem of gravitational self-interaction of a radiation shell into a problem of mild collisions. We exploit the fact that during mild collisions, the small correction \( \phi_1 \) contains all the information describing how are the objects slightly modified. It can be expanded into various components with clear physical meanings, and employing these projections appropriately is the key feature of our approach.

2. Fast colliding solitons
The complete description of a scattering theory for fast (therefore mild) collisions between solitons can be found in [3]. Here we review the key properties by a simple example given in [4]: a (1 + 1) dimensional theory of a scalar field with a periodic potential. \( \partial_t^2 \phi - \partial_x^2 \phi + V'(\phi) = 0 \), \( V(\phi) = V(\phi + \Delta \phi) \). This theory supports a simple soliton which interpolates between two nearby “vacuum”, \( V(N\Delta \phi) = \text{Min}(V) = 0 \), which is usually called a “kink”, \( \phi_N(x) = V[\phi_K(x)] \). This field profile exponentially approaches vacuum values beyond some distance \( \sim (L/2) \) away from its center \( x = 0 \), so we can treat it as a localized object of size \( L \). A reflection takes this
to an “antikink” $\phi_K(-x)$, and Lorentz symmetry allows us to describe moving (anti)kinks, $\phi_{K,v}(x,t) = \phi_K [\gamma(x - vt)]$, $\gamma = (1 - v^2)^{-1/2}$.

A mild collision between two kinks can be easily setup in the center of mass frame, $\phi_0(x,t) = \phi_{K,v}(x,t) + \phi_{K,-v}(x,t)$. This describes two kinks coming toward each other at some $t_i < 0$, colliding at $t = 0$ but nothing happens. They simply pass through each other and continue their merry ways for $t_f > 0$. Assuming that the collision is indeed mild, we can expand true solution as $\phi = \phi_0 + \phi_1$ in which $\phi_1$ is small. Applying this assumption to the equation of motion and keep only the leading order in $\phi_1$, we get

$$\partial_t^2 \phi_1 - \partial_x^2 \phi_1 + V''(\phi_0)\phi_1 = V'(\phi_{K,v} + \phi_{K,-v}) - V'(\phi_{K,v}) - V'(\phi_{K,-v}) \equiv S(x,t) \quad (1)$$

The initial condition of two unaltered kinks is $\phi_1 = \partial_t \phi_1 = 0$ at some $t_i < 0$. Solving for $\phi_1$ then involves an $x - t$ integral of some Green’s function with the source term $S(x,t)$ in Eq. (1), and the smallness of such space-time integral guarantees a small $\phi_1$. This is easily arranged. A moving kink only have a nontrivial field profile within a region $\sim \gamma^{-1}L$, and outside such region it approaches the vacuum value $n\Delta \phi$. That means at any time when the two kinks do not overlap, $V'(\phi_{K,v} + \phi_{K,-v}) = V'(\phi_{K,v}) + V'(\phi_{K,-v})$ so the source term is zero. It is only nonzero in the diamond shape region where the two kinks overlaps, as shown in Fig.1. The space-time area of such region is $L^2/(4\gamma^2v)$, thus a large $\gamma$ always justifies a small $\phi_1$ expansion$^1$, and at leading order we expect $\phi_1 \propto \gamma^{-2}$.

![Figure 1](image1.png)

**Figure 1.** The collision between two fast-moving solitons. They only interact in the center (green) diamond region.

![Figure 2](image2.png)

**Figure 2.** A shell that shrinks and bounces back (shaded region) can be matched to its inverse image and becomes a collision problem.

Before solving Eq. (1) explicitly, it is very helpful to first anticipate what it means by breaking it up into modes. The zero mode is special because it corresponds to the shift symmetry of the kink, $f_0[\gamma(x - vt)] \propto \partial_x \phi_K[\gamma(x - vt)]$. Its coefficient takes the general form of $A_0 + B_0\gamma(t - vx)$. A nonzero $A_0$ means that the kink acquires a constant position shift during this collision, and a nonzero $B_0$ means that the kink is moving at a different velocity after the collision. They can be evaluated into simple close-form expressions as shown in [4], but here we will focus on another striking property. We know $B_0 = 0$ even without evaluating it. In other words, we know that solitons do not slow down at the leading order of $1/\gamma^2$.

In order to see that, take energy conservation and expand it in the power of $\phi_1$, $E[\phi_0 + \phi_1]|_{t=t_f} > 0 - E[\phi_0]|_{t=t_i < 0} = -\delta E_0 \phi_1 + \frac{1}{2} \delta E_0 \phi_1^2 + \ldots$ A few integration tricks show that a nonzero $B_0$

$^1$ The full justification to all order and the expected convergence is described in [3].
3. Self-gravitation of a radiation shell

Consider a thin-shell profile of massless scalar field in a Minkowski space background. When the field amplitude $\epsilon$ is small, we can adopt the post-Minkowski expansion [5].

$$ds^2 = -dt^2 + dr^2 + r^2d\Omega^2; \quad \sim \mathcal{O}(\epsilon^0), \quad \sim \mathcal{O}(\epsilon^1), \quad \sim \mathcal{O}(\epsilon^2) \quad \Delta^2 \phi_0 - \frac{2}{r} \partial_r \phi_0 = 0; \quad \sim \mathcal{O}(\epsilon^3) \quad \partial_r^2 \phi_1 - \frac{2}{r} \partial_r \phi_1 = C \left( \partial_r^2 \phi_0 + \frac{2}{r} \partial_r \phi_0 \right); \quad (2)$$

where $C(r,t) = \left[ V(r,t) - M(r,t)/r \right]$, $V(r,t)$ is the gravitational potential, $M(r,t)$ is the mass of the shell, and $\phi_0$ is the field amplitude. The smallness of $\phi_1$ is guaranteed by staying in the regime of weak gravity.

In fact, we can further improve the accuracy of the approximation by introducing a modified metric that takes into account the effects of the shell. The modified metric is given by

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The divergence only comes from the integral in the first term, and \( g' \) shows that it must be a position shift. A massless field profile is supposed to follow a null geodesic. When the metric is modified like in Eq. (4), \( |dr| = dt \) is no longer a null geodesic, and a shell starting at \( r = r_0 \), \( t_i = -r_0 \) is not coming back to \( r = r_0 \) on time. Tracking the appropriate null geodesic can give us a position shift, \( \Delta r = \int_{r_0}^{r} dt \ C (|r - r_0 + t|, t) \sim \ln r_0 \). The integral here is identical to that in Eq. (8), and it is the only diverging term there. After projecting out an overall position shift, everything else is finite. Thus indeed, gravity is a \( 1/r^2 \) long-range force, but its long range effect here is a boring position shift. After removing that, the next order \( 1/r^3 \) tidal force creates finite “tidal deformations” of this shell.

If we put this shell inside a reflective box, such as a large AdS space, it will be reflected back from the boundary and repeat this self-interaction. One naïve intuition is that since gravity is an attractive force, the shell should becomes narrower from its own gravity, and it will eventually become narrow enough to form a black hole [6]. Using our method, we can show that a shell is equally likely to become wider, since tidal effect tends to pull things apart. Therefore a black hole formation in AdS space is not a completely generic outcome [7].

4. Discussion
We presented two separate cases with very different dynamics, and one method that addresses the outcome of mild collisions in both of them. In the case of soliton collisions, our method provides new insight into the reason why they tend to be mild, and it is very different from the common impression of something related to integrability [3, 4]. The more center of mass kinetic energy enters the collision, the less amount will be lost. In a collision with center of mass energy \( E_{\text{CoM}} \propto \gamma \) where \( \gamma \) is the usual boost factor, the kinetic energy loss is negligible \( E_{\text{loss}} \propto (\gamma^{-4} E_{\text{CoM}}) \propto \gamma^{-3} \). The solitons pass through each other with all of their original energy.

In the case of gravitational self-interaction, we demonstrated a clean way to separate a naïve divergence in the post-Minkowski expansion: a large position shift due to the change of geodesics. Recognizing this divergence allows us to focus on other finite outcomes and answer important questions such as the AdS stability problem [6, 7].

We are quite used to scattering theories formulated in the language of asymptotic plane-waves, which are convenient since they form a complete basis to describe all possible outcomes. In the case of mild collisions, small changes to the colliding objects should effectively form a complete basis to describe all possible outcomes. It is natural to expect that a fully developed scattering theory in this context can be useful in answering many physical questions.

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