The automorphism group and the non-self-duality of $p$-cones

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Abstract
In this paper, we determine the automorphism group of the $p$-cones ($p \neq 2$) in dimension greater than two. In particular, we show that the automorphism group of those $p$-cones are the positive scalar multiples of the generalized permutation matrices that fix the main axis of the cone. Next, we take a look at a problem related to the duality theory of the $p$-cones. Under the Euclidean inner product it is well-known that a $p$-cone is self-dual only when $p = 2$. However, it was not known whether it is possible to construct an inner product depending on $p$ which makes the $p$-cone self-dual. Our results shows that no matter which inner product is considered, a $p$-cone will never become self-dual unless $p = 2$ or the dimension is less than three.

1 Introduction
In this work, we prove two results on the structure of the $p$-cones $L_{n+1}^p = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n \mid t \geq \|x\|_p\}$.

First, we describe the automorphism group of the $p$-cones $L_{n+1}^p$ for $n \geq 2$ and $p \neq 2, 1 < p < \infty$. We show that every automorphism of $L_{n+1}^p$ must have the format

$$\alpha \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix},$$

where $\alpha > 0$ and $P$ is an $n \times n$ generalized permutation matrix. The second result is that, for $n \geq 2$ and $p \neq 2$, it is not possible to construct an inner product on $\mathbb{R}^{n+1}$ for which $L_{n+1}^p$ becomes self-dual. In fact, the second result is derived as a corollary of a stronger result that $L_{n+1}^p$ and $L_{n+1}^q$ cannot be linearly isomorphic if $p < q$ and $n \geq 2$, except when $(p, q, n) = (1, \infty, 2)$.

The motivation for this research is partly due to the work by Gowda and Trott [5], where they determined the automorphism group of $L_{1+1}^1$ and $L_{\infty+1}^\infty$. However, they left open the problem of determining the automorphisms of the other $p$-cones, for $p \neq 2$. Here, we recall that the case $p = 2$ correspond to the second order cones and they are symmetric, i.e., self-dual and homogeneous. The structure of second-order cones and their automorphisms follow from the more general theory of Jordan Algebras [4], see also [8].

In [5], Gowda and Trott also proved that $L_{1+1}^1$ and $L_{\infty+1}^\infty$ are not homogenous cones and they posed the problem of proving/disproving that $L_{p+1}^p$ is not homogenous for $p \neq 2, n \geq 2$. Recall that a cone is said to be homogeneous if its group of automorphisms acts transitively on the interior of the cone. In [6], using the theory of $T$-algebras [11], we gave a proof that $L_{p+1}^p$ is not homogenous for $p \neq 2, n \geq 2$. However, there are two unsatisfactory aspects of our previous result. The first is that we were

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not able to compute the automorphism group of $L_p^{n+1}$. The second is that although we showed that $L_p^{n+1}$ is not homogeneous, we were unable to obtain two elements $x, y$ in interior of $L_p^{n+1}$ such that no automorphism of $L_p^{n+1}$ maps $x$ to $y$. That is, we were unable to show concretely how homogeneity breaks down on $L_p^{n+1}$. The results discussed here remedy those flaws and provide an alternative proof that $L_p^{n+1}$ is not homogeneous.

Another motivation for this work is the general problem of determining when a closed convex cone $K \subseteq \mathbb{R}^n$ is self-dual. If $\mathbb{R}^n$ is equipped with some inner product $\langle \cdot, \cdot \rangle$, the dual cone of $K$ is defined as

$$K^* = \{ y \in \mathbb{R}^n | \langle x, y \rangle \geq 0, \forall x \in K \}. $$

As discussed in Section 1 of [6], an often overlooked point is that $K^*$ depends on $\langle \cdot, \cdot \rangle$. Accordingly, it is entirely plausible that a cone that is not self-dual under the Euclidean inner product might become self-dual if the inner product is chosen appropriately.

This detail is quite important because sometimes we see articles claiming that a certain cone is not a symmetric cone because it is not self-dual under the Euclidean inner product. This is, of course, not enough. As long as a cone is homogeneous and there exists some inner product that makes it self-dual, the cone can be investigated under the theory of Jordan algebras.

This state of affairs brings us to the case of the $p$-cones. Up until the recent articles [5, 6], there was no rigorous proof that the $p$-cones $L_p^{n+1}$ were not symmetric when $p \neq 2$ and $n \geq 2$. Now, although we know that $L_p^{n+1}$ is not homogeneous for $p \neq 2$ and $n \geq 2$, it still remains to investigate whether $L_p^{n+1}$ could become self-dual under an appropriate inner product. This question was partly discussed by Miao, Lin and Chen in [9], where they showed that a $p$-cone (again, $p \neq 2$, $n \geq 2$) is not self-dual under an inner product induced by a diagonal matrix. The results described here show, in particular, that no inner product can make $L_p^{n+1}$ self-dual, for $p \neq 2$, $n \geq 2$.

We now explain some of the intuition behind our proof techniques. Let $n \geq 2$ and let $f_p : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be the function that maps $x$ to $\|x\|_p$. When $p \in (1, 2)$, we have that $f_p$ is twice differentiable only at points $x$ for which $x_i \neq 0$, for all $i$. In contrast, if $p \in (2, \infty)$, $f_p$ is twice differentiable throughout $\mathbb{R}^n \setminus \{0\}$. Now, we let $M_p$ be the boundary without the zero of the cone $L_p^{n+1}$. With that, $M_p$ is exactly the graph of the function $f_p$. Furthermore, $M_p$ is a $C^1$-embedded smooth manifold if $p \in (1, 2)$. If $p \in (2, \infty)$, $M_p$ is a $C^2$-embedded smooth manifold.

Now any linear bijection between $L_p^{n+1}$ and $L_q^{n+1}$ must map the boundary of $L_p^{n+1}$ to the boundary of $L_q^{n+1}$, thus producing a map between $M_p$ and $M_q$. Then, if $p \in (1, 2)$ and $q \in (2, \infty)$, there can be no linear bijection between $L_p^{n+1}$ and $L_q^{n+1}$ because this would establish a diffeomorphism between submanifolds that are embedded with different levels of smoothness.

Now suppose that $p, q$ are both in $(1, 2)$ and that there exists some linear bijection $A$ between $L_p^{n+1}$ and $L_q^{n+1}$ if $(f_p(x), x) \in M_p$ is such that $f_p$ is not twice differentiable at $x$, then $A$ must map $(f_p(x), x)$ to a point $(f_q(y), y)$ for which $f_q$ is not twice differentiable at $y$. This idea is made precise in Proposition 4. In particular, this fact imposes severe restrictions on how Aut($L_p^{n+1}$) acts on $L_p^{n+1}$ and this is the key observation necessary for showing that the matrices in Aut($L_p^{n+1}$) can be written as in (1).

This work is divided as follows. In Section 2 we present the notation used in this paper and review some facts about cones, self-duality and $p$-cones. In Section 3, we discuss the tools from manifold theory necessary for our discussion. Finally, in Section 4 we prove our main results.

## 2 Preliminaries

A convex cone is a subset $K$ of some real vector space $\mathbb{R}^n$ such that $\alpha x + \beta y \in K$ holds whenever $x, y \in K$ and $\alpha, \beta \geq 0$. A cone $K$ is said to be pointed if $K \cap -K = \{0\}$. For a subset $S$ of $\mathbb{R}^n$, the (closed) conical hull of $S$, denoted by cone($S$), is the smallest closed convex cone in $\mathbb{R}^n$ containing $S$. 

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If \( v \in \mathbb{R}^n \), we write \( \mathbb{R}_+(v) \) for the half-line generated by \( v \) and \( \mathbb{R}^+ \) for \( \mathbb{R}_+(v) \setminus \{0\} \), i.e.,

\[
\mathbb{R}_+(v) = \{ \alpha v \mid \alpha \geq 0\}, \\
\mathbb{R}^+ = \{ \alpha v \mid \alpha > 0\}.
\]

A convex subset \( \mathcal{F} \) of \( \mathcal{K} \) is said to be a face of \( \mathcal{K} \) if the following condition hold: If \( x, y \in \mathcal{K} \) satisfies \( \alpha x + (1 - \alpha)y \in \mathcal{F} \) for some \( \alpha \in (0, 1) \) then \( x, y \in \mathcal{F} \). A one dimensional face is called an extreme ray. A polyhedral convex cone is a convex cone that can be expressed as the solution set of finitely many linear inequalities.

If \( \langle \cdot, \cdot \rangle \) is an inner product on \( \mathbb{R}^n \), we can define the dual cone of \( \mathcal{K} \) with respect to the inner product \( \langle \cdot, \cdot \rangle \) by

\[
\mathcal{K}^* = \{ x \in \mathbb{R}^n \mid \langle x, y \rangle \geq 0, \forall y \in \mathcal{K} \}.
\]

A convex cone \( \mathcal{K} \) is self-dual if there exists an inner product on \( \mathbb{R}^n \) for which the dual cone coincides with \( \mathcal{K} \) itself.

Two convex cones \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) in \( \mathbb{R}^n \) are said to be isomorphic if there exists a linear bijection \( A \in GL_n(\mathbb{R}) \), called an isomorphism, such that \( \mathcal{K}_1 = A \mathcal{K}_2 \). An automorphism of a convex cone \( \mathcal{K} \) in \( \mathbb{R}^n \) is a map \( A \in GL_n(\mathbb{R}) \) such that \( A \mathcal{K} = \mathcal{K} \). The group of all automorphisms of \( \mathcal{K} \) is written by \( \text{Aut}(\mathcal{K}) \) and called the automorphism group of \( \mathcal{K} \).

A convex cone \( \mathcal{K} \) is said to be homogeneous if \( \text{Aut}(\mathcal{K}) \) acts transitively on the interior of \( \mathcal{K} \), that is, for every elements \( x \) and \( y \) of the interior of \( \mathcal{K} \), there exists \( A \in \text{Aut}(\mathcal{K}) \) such that \( y = Ax \).

### 2.1 On self-duality

Let \( \mathcal{K} \subseteq \mathbb{R}^n \) be a closed convex cone. As we emphasized in Section 1, self-duality is a relative concept and depends on what inner product we are considering. Let \( \langle \cdot, \cdot \rangle_E \) denote the Euclidean inner product and consider the dual of \( \mathcal{K} \) with respect \( \langle \cdot, \cdot \rangle_E \).

\[
\mathcal{K}^* = \{ y \in \mathbb{R}^n \mid \langle x, y \rangle_E \geq 0, \forall x \in \mathcal{K} \}.
\]

We have the following proposition.

**Proposition 1.** Let \( \mathcal{K} \subseteq \mathbb{R}^n \) be a closed convex cone and let \( \mathcal{K}^* \) be the dual of \( \mathcal{K} \) with respect to the Euclidean inner product \( \langle \cdot, \cdot \rangle_E \). Then, there exists an inner product on \( \mathbb{R}^n \) that turns \( \mathcal{K} \) into a self-dual cone if and only if there exists a symmetric positive definite matrix \( A \) such that \( A \mathcal{K} = \mathcal{K}^* \).

**Proof.** First, suppose that there exist some inner product \( \langle \cdot, \cdot \rangle_K \) for which \( \mathcal{K} \) becomes self-dual. Then, there is a symmetric positive definite matrix \( A \) such that

\[
\langle x, y \rangle_K = \langle x, Ay \rangle_E,
\]

for all \( x, y \in \mathbb{R}^n \). In fact, \( A_{ij} = \langle e_i, e_j \rangle_K \), where \( e_i \) is the \( i \)-th standard unit vector in \( \mathbb{R}^n \). By assumption, we have

\[
\mathcal{K} = \{ x \in \mathbb{R}^n \mid \langle x, Ay \rangle_E \geq 0, \forall y \in \mathcal{K} \} \\
\quad= \{ x \in \mathbb{R}^n \mid \langle Ax, y \rangle_E \geq 0, \forall y \in \mathcal{K} \} \\
\quad= A^{-1} \{ z \in \mathbb{R}^n \mid \langle z, y \rangle_E \geq 0, \forall y \in \mathcal{K} \} \\
\quad= A^{-1} \mathcal{K}^*.
\]

This shows that \( A \mathcal{K} = \mathcal{K}^* \).

Reciprocally, if \( A \mathcal{K} = \mathcal{K}^* \), we define the inner product \( \langle \cdot, \cdot \rangle_K \) such that

\[
\langle x, y \rangle_K := \langle x, Ay \rangle_E,
\]

for all \( x, y \in \mathbb{R}^n \). Then, a straightforward calculation shows that the dual of \( \mathcal{K} \) with respect \( \langle \cdot, \cdot \rangle_K \) is indeed \( \mathcal{K} \). \( \square \)
Therefore, determining whether $\mathcal{K}$ is self-dual for some inner product boils down to determining the existence of a positive definite linear isomorphism between cones, which is a difficult problem in general.

2.2 $p$-cones

Here we present some basic facts on $p$-cones. The $p$-cone is the closed convex cone in $\mathbb{R}^{n+1}$ defined by

$$L_p^{n+1} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n \mid t \geq \|x\|_p\}$$

where $\|x\|_p$ is the $p$-norm on $\mathbb{R}^n$:

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p} \text{ for } p \in [1, \infty) \text{ and } \|x\|_\infty = \max(|x_1|, \ldots, |x_n|).$$

The dual cone of the $p$-cone with respect to the Euclidean inner product is given by $(L_p^{n+1})^* = L_q^{n+1}$ where $q$ is the conjugate of $p$, that is, $\frac{1}{p} + \frac{1}{q} = 1$. The cones $L_1^{n+1}$ and $L_\infty^{n+1}$ are polyhedral. In fact, $L_1^{n+1}$ has $2n$ extreme rays

$$\mathbb{R}_+(1, \sigma e_i^n), \quad i = 1, \ldots, n, \quad \sigma \in \{-1, 1\},$$

where $e_i^n$ denotes the $i$-th standard unit vector in $\mathbb{R}^n$. Moreover, $L_\infty^{n+1}$ has $2^n$ extreme rays

$$\mathbb{R}_+(1, \sigma_1, \ldots, \sigma_n), \quad \sigma_1, \ldots, \sigma_n \in \{-1, 1\}.$$  

The difference in the number of extreme rays shows that $L_1^{n+1}$ and $L_\infty^{n+1}$ are not isomorphic if $n \geq 3$. However, for $n = 2$, they are indeed isomorphic as

$$AL_1^3 = L_\infty^3, \quad A = \begin{pmatrix}
1 & 0 \\
0 & \sqrt{2} \cos(\pi/4) & 0 \\
0 & \sqrt{2} \sin(\pi/4) \\
0 & \sqrt{2} \cos(\pi/4)
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -1 \\
0 & 1 & 1
\end{pmatrix}. \quad (2)$$

The second order cone $L_2^{n+1}$ is known to be a symmetric cone, that is, it is both self-dual and homogeneous, admitting a Jordan algebraic structure [4]. The automorphism group of the second order cone can be identified by the result of Loewy and Schneider [8]: $AL_2^{n+1} = L_2^{n+1}$ or $AL_2^{n+1} = -L_2^{n+1}$ holds if and only if $A^T J_{n+1} A = \mu J_{n+1}$ for some $\mu > 0$ where $J_{n+1} = \text{diag}(1, -1, \ldots, -1)$.

Gowda and Trott determined the structure of the automorphism group of the $p$-cones in the case $p = 1, \infty$:

**Proposition 2** (Gowda and Trott, Theorem 7 in [5]). For $n \geq 2$, $A$ belongs to $\text{Aut}(L_1^{n+1})$ if and only if $A$ has the form

$$A = \alpha \begin{pmatrix}
1 & 0 \\
0 & P
\end{pmatrix},$$

where $\alpha > 0$ and $P$ is an $n \times n$ generalized permutation matrix, that is, a permutation matrix multiplied by a diagonal matrix whose diagonal elements are $\pm 1$. Moreover, $\text{Aut}(L_\infty^{n+1}) = \text{Aut}(L_1^{n+1})$ holds.

In particular, Proposition 2 yields the following consequences.

- $L_1^{n+1}$ and $L_\infty^{n+1}$ are not homogeneous for $n \geq 2$ because any $A \in \text{Aut}(L_1^{n+1}) = \text{Aut}(L_\infty^{n+1})$ fixes the “main axis” $\mathbb{R}_+(1, 0, \ldots, 0)$ of these cones.

- $L_1^{n+1}$ and $L_\infty^{n+1}$ are never self-dual for $n \geq 2$. This is a known fact, but we will also obtain this result as a consequence of Corollary 14 where Proposition 2 will be helpful to prove the case $n = 2$. At this point, we should remark that Barker and Foran proved in Theorem 3 of [1] that a self-dual polyhedral cone in $\mathbb{R}^3$ must have an odd number of extreme rays. Since $L_1^3$ and $L_\infty^3$ have four extreme rays, Barker and Foran’s result implies that they are never self-dual.
3 Manifolds, tangent spaces and the Gauss map

In this subsection, we will provide a brief overview of the tools we will use from manifold theory, more details can be seen in Lee’s book [7] or the initial chapters of do Carmo’s book [3]. First, we recall that a \( n \)-dimensional smooth manifold \( M \) is a second countable Hausdorff topological space equipped with a collection \( \mathcal{A} \) of maps \( \varphi : U \rightarrow \mathbb{R}^n \) with the following properties.

(i) each map \( \varphi \in \mathcal{A} \) is such that \( \varphi(U) \) is an open set of \( \mathbb{R}^n \). Furthermore, \( \varphi \) is an homeomorphism between \( U \) and \( \varphi(U) \), i.e., \( \varphi \) is a continuous bijection with continuous inverse.

(ii) if \( \varphi : U \rightarrow \mathbb{R}^n, \psi : V \rightarrow \mathbb{R}^n \) both belong to \( \mathcal{A} \) and \( U \cap V \neq \emptyset \), then \( \psi \circ \varphi^{-1} : \varphi^{-1}(U \cap V) \rightarrow \psi(U \cap V) \) is a \( C^\infty \) diffeomorphism, i.e., \( \psi \circ \varphi^{-1} \) is a bijective function such that \( \psi \circ \varphi^{-1} \) and \( \varphi \circ \psi^{-1} \) have continuous derivatives of all orders.

(iii) for every \( x \in M \), we can find a map \( \varphi \in \mathcal{A} \) for which \( x \) belongs to the domain of \( \varphi \).

(iv) if \( \psi \) is another map defined on a subset of \( M \) satisfying (i) and (ii), then \( \psi \in \mathcal{A} \). That is, \( \mathcal{A} \) is maximal.

The set \( \mathcal{A} \) is called a maximal smooth atlas and the maps in \( \mathcal{A} \) are called charts. If \( \varphi : U \rightarrow \mathbb{R}^n \) is a chart and \( x \in U \), we say that \( \varphi \) is a chart around \( x \).

Let \( M_1, M_2 \) be smooth manifolds and \( f : M_1 \rightarrow M_2 \) be a function. The function \( f \) is said to be differentiable at \( x \in M_1 \) if there is a chart \( \varphi \) of \( M_1 \) around \( x \) and a chart \( \psi \) of \( M_2 \) around \( f(x) \) such that

\[
\psi \circ f \circ \varphi^{-1}
\]

is differentiable at \( \varphi(x) \). Then, \( f \) is said to be differentiable, if it is differentiable throughout \( M_1 \). Similarly, we say that \( f \) is differentiable of class \( C^k \) if \( \psi \circ f \circ \varphi^{-1} \) is of class \( C^k \), for every pair of charts of \( M_1 \) and \( M_2 \) such that the image of \( \varphi^{-1} \) and the domain of \( \psi \) intersect. Whether a function is differentiable at some point or is of class \( C^k \) does not depend on the particular choice of charts. The function \( \psi \circ f \circ \varphi^{-1} \) is also said to be a local representation of \( f \). If \( f \) is a bijection such that it is \( C^k \) everywhere and whose inverse \( f^{-1} \) is also \( C^k \) everywhere, then \( f \) is said to be a \( C^k \) diffeomorphism.

Let \( M \) be a \( n \)-dimensional smooth manifold. Let \( C^\infty(M) \) denote the ring of \( C^\infty \) real functions \( g : M \rightarrow \mathbb{R} \). A derivation of \( M \) at \( x \) is a function \( v : C^\infty(M) \rightarrow \mathbb{R} \) such that for every \( g, h \in C^\infty(M) \), we have

\[
v(gh) = (v(g))h(x) + g(x)v(h).
\]

Given a \( n \)-dimensional smooth manifold \( M \) and \( x \in M \), we write \( T_x M \) for the tangent space of \( M \) at \( x \), which is the subspace of derivations of \( M \) at \( x \). It is a basic fact that the dimension of \( T_x M \) as a vector space coincides with the dimension of \( M \) as a smooth manifold.

Let \( f : M_1 \rightarrow M_2 \) be a \( C^1 \) map between smooth manifolds. Then, at each \( x \in M_1 \), \( f \) induces a linear map between \( df_x : T_x M_1 \rightarrow T_{f(x)} M_2 \) such that given \( v \in T_x M_1, df_x(v) \) is the derivation of \( M_2 \) at \( f(x) \) satisfying

\[
(df_x(v))(g) = v(g \circ f),
\]

for every \( g \in C^\infty(N) \). The map \( df_x \) is the differential map of \( f \) at \( x \). If the linear map \( df_x \) is injective everywhere, then \( f \) is said to be an immersion. Furthermore, if \( f \) is a \( C^k \) diffeomorphism with \( k \geq 1 \), then \( df_x \) is a linear bijection for every \( x \). Recall that in order to check whether \( f \) is immersion, it is enough to check that the local representations of \( f \) are immersions.

Now, suppose that \( \alpha : (-\epsilon, \epsilon) \rightarrow M \) is a \( C^\infty \) curve with \( \alpha(0) = x \). Then \( d\alpha_0(0) \in T_x M \). Furthermore, \( T_x M \) coincides with the set of velocity vectors of smooth curves passing through \( x \). With a slight abuse of notation, let us write \( \alpha'(t) = d\alpha_0(t) \). With that, we have

\[
T_x M = \{ \alpha'(0) \mid \alpha : (-\epsilon, \epsilon) \rightarrow M, \alpha(0) = x, \alpha \text{ is } C^1 \}, \tag{3}
\]
see more details in Proposition 3.23 and pages 68-71 in [7]. With this, we can compute a differential $df_x(v)$ by first selecting a $C^1$ curve $\alpha$ contained in $M$ with $\alpha(0) = x$, $\alpha'(0) = v$. Then, we have $df_x(v) = (f \circ \alpha)'(0)$, see Proposition 3.24 in [7].

A map $\iota : M_1 \to M_2$ is said to be a $C^k$-embedding if it is a $C^k$ immersion and a homeomorphism on its image (here, $\iota(M_1)$ has the subspace topology induced from $M_2$). Now, suppose that, in fact, $M_1 \subseteq M_2$ and let $\iota : M_1 \to M_2$ denote the inclusion map, i.e., $\iota(x) = x$, for all $x \in M_1$. If $\iota$ is a $C^k$ embedding, we say that $M_1$ is a $C^k$-embedded submanifold of $N$.

We remark that when $M$ is a $m$-dimensional $C^k$-embedded submanifold of $\mathbb{R}^n$, the requirement that $\iota$ be an $C^k$ embedding has the following consequences. First, the topology of $M$ has to be the subspace topology of $\mathbb{R}^n$, i.e., the open sets of $M$ are open sets of $\mathbb{R}^n$ intersected with $M$. Now, let $\varphi : U \to \mathbb{R}^m$ be a chart of $M$. Then, $\iota \circ \varphi^{-1} : \varphi(U) \to U$ is a $C^k$ diffeomorphism. That is, although $\varphi^{-1}$ is $C^\infty$ when saw as a map between $\varphi(U)$ and $M$, its class of differentiability might decrease\(^1\) when seen as a map between $U$ and $\mathbb{R}^m$. For embedded manifolds of $\mathbb{R}^n$, as a matter of convention, we will always see the inverse of a chart $\varphi$ as a function whose codomain is $\mathbb{R}^n$ and we will omit the embedding $\iota$.

Furthermore, whenever $M$ is a $C^k$-embedded submanifold of $\mathbb{R}^n$, we will define tangent spaces in a more geometric way. Given $x \in M$, we will define $T_x M$ as the space of tangent vectors of $C^1$ curves that pass through $x$:

$$T_x M = \{ \alpha'(0) \mid \alpha : (-\epsilon, \epsilon) \to \mathbb{R}^n, \alpha(0) = x, \alpha \subseteq M, \alpha \text{ is } C^1 \},$$

(4)

where $\alpha \subseteq M$ means that $\alpha(t) \in M$, for every $t \in (-\epsilon, \epsilon)$. Here, since we have an ambient space, $\alpha'(0)$ is the derivative of $\alpha$ at 0 in the usual sense.

Both definitions of tangent spaces presented so far are equivalent in the following sense. Let $\tilde{T}_x M$ denote the space of derivations of $M$ at $x$ and let $\iota : M \to \mathbb{R}^n$ denote the inclusion map. Then, $d\iota_x$ is a map between $\tilde{T}_x M$ and $T_x \mathbb{R}^n$. Then, identifying $T_x \mathbb{R}^n$ with $\mathbb{R}^n$, it holds that $d\iota_x(\tilde{T}_x M) = T_x M$. In particular, $\tilde{T}_x M$ and $T_x M$ have the same dimension.

Finally, we recall that for smooth manifolds, the topological notion of connectedness is equivalent to the notion of path-connectedness, see Proposition 1.11 in [7]. Therefore, a manifold $M$ is connected if and only if for every $x, y \in M$ there is a continuous curve $\alpha : [0, 1] \to M$ such that $\alpha(0) = x$ and $\alpha(1) = y$.

### 3.1 Graphs of differentiable maps

For a real valued function $f : U \to \mathbb{R}$ defined on $U \subseteq \mathbb{R}^n$, the graph of $f$ is defined by

$$\text{graph } f := \{ (y, x) \in \mathbb{R} \times U \mid y = f(x) \} \subseteq \mathbb{R}^{n+1}.$$ 

In item (i) of the next proposition, for the sake of completeness, we give a proof of the well-known fact that if $f$ is a $C^k$ function, then $\text{graph } f$ must be a $C^k$-embedded manifold. In item (ii) we observe the fact, also known but perhaps less well-known, that the converse also holds. This is important for us because if we know that $f$ is $C^1$ but not $C^2$, then this creates an obstruction to the existence of certain maps between graph $f$ and $C^2$ manifolds.

**Proposition 3.** For $k \geq 1$, let $f : U \to \mathbb{R}$ be a $C^1$ function defined on an open subset $U$ of $\mathbb{R}^n$.

(i) If $f$ is $C^k$ on an open subset $V$ of $U$, then $\text{graph } f|_V$ is an $n$-dimensional $C^k$-embedded submanifold of $\mathbb{R}^{n+1}$.

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\(^1\)Here is an example of what can happen. Let $M$ be graph of the function $f(x) = |x|$. $M$ is a differentiable manifold and to create a maximal smooth atlas for $M$ we first start with a set $A$ containing only the map $\varphi : M \to \mathbb{R}$ that takes $((|x|, x)$ to $x$. At this point, conditions (i), (ii), (iii) of the definition of atlas are satisfied. Then, we add to $A$ every map $\psi$ such that $A \cup \{ \psi \}$ still satisfies (i), (ii), (iii). The resulting set must be a maximal atlas. Following the definition of differentiability between manifolds, the map $\varphi^{-1}$ is $C^\infty$ if we see it as a map between $\mathbb{R} \to M$, since $\varphi \circ \varphi^{-1}(x) = x$. However, $\varphi \circ \varphi^{-1}$ is not even a $C^1$ map, because $|x|$ is not differentiable at 0.
(ii) Suppose that a subset $M$ of graph $f$ is an $n$-dimensional $C^k$-embedded submanifold of $\mathbb{R}^{n+1}$, with $k \geq 1$. Then $f$ is $C^k$ on the open set $\pi_U(M)$, where $\pi_U : \mathbb{R} \times U \to U$ is the projection onto $U$.

Proof. (i) The proof here is essentially the one contained Example 1.30 and Proposition 5.4 of [7], except that here we take into account the level of smoothness of the embedding.

First, let $M = \text{graph}(f_V)$ and consider the subspace topology inherited from $\mathbb{R}^{n+1}$ (again, see Examples 1.3 and 1.30 in [7] for more details). With the subspace topology, the map $\varphi : V \to M$, given by

$$\varphi(x) = (f(x), x)$$

is a homeomorphism between $V$ and $M$, whose inverse is the projection restricted to $M$, that is $\varphi^{-1}(f(x), x) = x$. Furthermore, $\varphi^{-1}$ induces a maximal smooth atlas of $M$ making $\varphi^{-1} : M \to V$ a chart. We now check that the inclusion $\iota : M \to \mathbb{R}^{n+1}$ is a $C^k$ embedding. A local representation for $\iota$ is obtained by considering $\iota \circ \varphi : V \to \mathbb{R}^{n+1}$, which shows that $\iota$ is a $C^k$ differentiable map. The inverse $\iota^{-1} : \iota(M) \to M$ is given by restricting the identity map in $\mathbb{R}^{n+1}$ to $M$. Since the topology on $M$ is the subspace topology, this establishes that $\iota$ is an homeomorphism.

Furthermore, since the $(n + 1) \times n$ Jacobian matrix $J_{\iota\varphi}$ of the representation of $\iota$ has rank $n$, we see that $\iota$ is an immersion. Hence, $M$ is a $C^k$-embedded submanifold of $\mathbb{R}^{n+1}$.

(ii) Take $x_0 \in \pi_U(M)$. Let $\Phi : V \to \mathbb{R}^n$ be a chart of $M$ around $(f(x_0), x_0)$. We can write the map $\Phi^{-1}$ as

$$\Phi^{-1}(z) = (\psi(z), \varphi(z)) \in \mathbb{R} \times U \text{ for } z \in \Phi(V),$$

for functions $\psi : \Phi(V) \to \mathbb{R}$, $\varphi : \Phi(V) \to U$. Since $\text{Im} \Phi^{-1} \subseteq M \subseteq \text{graph } f$, we have $\psi(z) = f(\varphi(z))$ for all $z \in \Phi(V)$. Then we obtain a local representation $\iota : \Phi(V) \subseteq \mathbb{R}^n \to \mathbb{R}^{n+1}$ of the inclusion map $\iota : M \to \mathbb{R}^{n+1}$ as follows:

$$\iota(z) := \iota \circ \Phi^{-1} = (\psi(z), \varphi(z)) = (f \circ \varphi(z), \varphi(z)).$$

Since $M$ is $C^k$-embedded, $\varphi$ and $\psi$ are $C^k$ when seen as maps $\Phi(V) \to \mathbb{R}$ and $\Phi(V) \to \mathbb{R}^n$, respectively. Let $z_0 = \Phi((f(x_0), x_0))$. Then $\varphi(z_0) = x_0$ since

$$(f(x_0), x_0) = \Phi^{-1}(z_0) = (\psi(z_0), \varphi(z_0)).$$

Note that $\text{rank}(J_{\iota}(z_0)) = n$ holds because $\iota$ is an immersion. On the other hand, since $f$ is $C^1$ by the assumption, it follows by the chain rule for the function $\psi = f \circ \varphi$ that

$$J_{\psi}(z_0) = J_f(\varphi(z_0))J_{\varphi}(z_0) = J_f(x_0)J_{\varphi}(z_0).$$

This means that each row of $J_{\varphi}(z_0)$ is a linear combination of rows of $J_{\varphi}(z_0)$. Therefore, we conclude that

$$n = \text{rank } J_{\iota}(z_0) = \text{rank } (J_{\psi}(z_0)^T, J_{\varphi}(z_0)^T)^T = \text{rank } J_{\varphi}(z_0).$$

Namely, the $n \times n$ matrix $J_{\varphi}(z_0)$ is nonsingular. Since $\varphi$ is $C^k$, the inverse function theorem states that there exists a $C^k$ inverse $\varphi^{-1} : W \to \mathbb{R}^n$ defined on a neighborhood $W$ of $\varphi(z_0) = x_0$. Then, we conclude that the function

$$\psi \circ \varphi^{-1} = f \circ \varphi \circ \varphi^{-1} = f$$

is $C^k$ on $W$.

To conclude, we will show that $\pi_U(M)$ is open. Since $\varphi^{-1}(W)$ is contained in the domain $\Phi(V)$ of the map $\varphi$, it follows that $W = \varphi \circ \varphi^{-1}(W) \subseteq \varphi(\Phi(V))$. Now, let $z \in \Phi(V)$. By definition, we have

$$(\psi(z), \varphi(z)) = \Phi^{-1}(z) \in V,$$

which shows that $\varphi(z) \in \pi_U(V)$. Therefore, $\varphi(\Phi(V)) \subseteq \pi_U(V) \subseteq \pi_U(M)$. Hence, we have $W \subseteq \pi_U(M)$ and so $\pi_U(M)$ is open in $\mathbb{R}^n$, since $x_0$ was arbitrary. 

\footnote{The idea is the same as in Footnote 1, we start with $A = \{\varphi^{-1}\}$ and add every map $\psi$ for which $A \cup \{\psi\}$ still satisfies properties (i), (ii), (iii) of the definition of atlas.}
Given a diffeomorphism $A$ between two graphs of $C^1$ maps $f,g : U \to \mathbb{R}$, the next proposition shows a relation of the categories of differentiability of $f$ and $g$ through the diffeomorphism $B : U \to U$ defined by

$$B(x) = \pi_U(A(f(x),x))$$

where $\pi_U : \mathbb{R} \times U \to U$ is the projection onto $U$. The map $B$ will play a key role in the proof of our main result applied with $U = \mathbb{R}^n \setminus \{0\}$, $f(x) = \|x\|_p$ and $g(x) = \|x\|_q$. We give an illustration of the map $B$ in Figure 1.

**Proposition 4.** Let $f,g : U \to \mathbb{R}$ be $C^1$ maps defined on an open subset $U$ of $\mathbb{R}^n$. Suppose that $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a $C^\infty$ diffeomorphism such that $A(\text{graph } f) = \text{graph } g$.

(i) The map $B : U \to U$, $B(x) := \pi_U(A(f(x),x))$ is a $C^1$ diffeomorphism, where $\pi_U : \mathbb{R} \times U \to U$ satisfies $\pi_U(y,x) = x$.

(ii) For $k \geq 1$, $f$ is $C^k$ on a neighborhood of $x$ if and only if $g$ is $C^k$ on a neighborhood of $B(x)$.

**Proof.** (i) Since $f$ is $C^1$ while $\pi_U$ and $A$ are $C^\infty$ maps, it is must be the case that $B(x) = \pi_U(A(f(x),x))$ is $C^1$.

Let us check that the inverse of $B$ is the map $B^{-1}(y) = \pi_U(A^{-1}(g(y),y))$. Denote

$$B'(y) = \pi_U(A^{-1}(g(y),y)).$$

For any $x \in U$, the relation $A(\text{graph } f) = \text{graph } g$ implies the existence of $y \in U$ such that $A(f(x),x) = (g(y),y)$. Then we have

$$B(x) = \pi_U(A(f(x),x)) = \pi_U(g(y),y) = y.$$

and, therefore,

$$B'(B(x)) = B'(y) = \pi_U(A^{-1}(g(y),y)) = \pi_U(f(x),x) = x.$$

Similarly, we obtain $B(B'(y)) = y$. Hence, $B^{-1}(y) = B'(y)$ holds.

Since $B^{-1}(y) = \pi_U(A^{-1}(g(y),y))$ is also $C^1$, we conclude that $B$ is a $C^1$ diffeomorphism.

(ii) If $f$ is $C^k$ on a neighborhood $V$ of $x$, then $\text{graph}(f|_V)$ is an $n$-dimensional $C^k$-embedded submanifold of $\mathbb{R}^{n+1}$ by Proposition 3 (i). Then, by the assumption on $A$, the set $M := A(\text{graph } f|_V)$ is also an $n$-dimensional $C^k$-embedded submanifold of $\mathbb{R}^{n+1}$ which satisfies $M \subseteq \text{graph } g$. Therefore Proposition 3 (ii) implies that $g$ is $C^k$ on the open set $\pi_U(M) = \pi_U(A(\text{graph } f|_V))$ which contains the point $\pi_U(A(f(x),x))$.

The converse of the assertion follows by applying the same argument to the diffeomorphism $A^{-1}$ because $A^{-1}(\text{graph } g) = \text{graph } f$ and $\pi_U(A^{-1}(g(y),y)) = x$ holds for $y = B(x) = \pi_U(A(f(x),x)).$
3.2 The Gauss map

In this subsection, let $M$ be a $C^k$-embedded submanifold of $\mathbb{R}^n$ with dimension $n - 1$ and $k \geq 1$. In this case, $M$ is sometimes called a hypersurface and when $n = 3$, $M$ is called a surface. The differential geometry of surfaces is, of course, a classical subject discussed in many books, e.g., [2].

In the theory of surfaces, a Gauss map is a continuous function that associates to $x \in M$ a unit vector which is orthogonal to $T_x M$. Unless $M$ is an orientable surface, it is not possible to construct a Gauss map that is defined globally over $M$. However, given any $x \in M$, it is always possible to construct a Gauss map in a neighborhood of $x$. For the sake of self-containment, we will give a brief account of the construction of the Gauss map for hypersurfaces.

For what follows, we suppose that $\mathbb{R}^n$ is equipped with some inner product $\langle \cdot, \cdot \rangle$ and the norm is given by $\|x\| = \sqrt{\langle x, x \rangle}$, for all $x \in \mathbb{R}^n$. Recalling (4), $T_x M$ is seen as a subspace of $\mathbb{R}^n$ and we will equip $T_x M$ with the same inner product $\langle \cdot, \cdot \rangle$.

Definition 5. Let $M$ be a $C^k$-embedded submanifold of $\mathbb{R}^n$ and let $x \in M$. A $C^r$ Gauss map around $x$ is a $C^r$ function $N : U \to \mathbb{R}^n$ such that $U \subseteq M$ is a neighborhood of $x$ in $M$ and

$$N(x) \in (T_x M)^\perp \quad \text{and} \quad \|N(x)\| = 1,$$

for all $x \in U$, where $(T_x M)^\perp$ is the orthogonal complement to $T_x M$.

For what follows, let $x^1, \ldots, x^n \in \mathbb{R}^n$ and let $\det(x^1, \ldots, x^n)$ denote the determinant of the matrix such that its $i$-th column is given by $x^i$. Since the determinant is a multilinear function, if we fix the first $n - 1$ elements, we obtain a linear functional $f$ such that

$$f(x) = \det(x^1, \ldots, x^{n-1}, x).$$

Since $f$ is a linear functional, there is a unique vector $\Lambda(x^1, \ldots, x^{n-1}) \in \mathbb{R}^n$ satisfying

$$\langle \Lambda(x^1, \ldots, x^{n-1}), x \rangle = f(x),$$

for all $x \in \mathbb{R}^n$. Furthermore, $\Lambda(x^1, \ldots, x^{n-1}) = 0$ is zero if and only if the $x^i$ are linearly dependent.

Proposition 6. Let $M \subseteq \mathbb{R}^n$ be an $(n - 1)$-dimensional $C^k$-embedded manifold, with $k \geq 1$. Then, for every chart $\varphi : U \to \mathbb{R}^{n-1}$, there exists a $C^{k-1}$ local Gauss map of $M$ defined over $U$.

Proof. Let $\varphi : U \to \mathbb{R}^{n-1}$ be a chart of $M$. Then, $\varphi^{-1}$ is a function with domain $\varphi(U)$ (which is an open set of $\mathbb{R}^{n-1}$) and codomain $\mathbb{R}^n$. Let $u \in U$. It is well-known that the partial derivatives of $\varphi^{-1}$ at $\varphi(u)$ are a basis for $T_u M$, e.g., page 60 and Proposition 3.15 in [7]. Let $v^i(u)$ be the partial derivative of $\varphi^{-1}$ at $\varphi(u)$ with respect the $i$-th variable. We define a Gauss map $N$ over $U$ by letting

$$N(x) = \frac{\Lambda(v^1(u), \ldots, v^{n-1}(u))}{\|\Lambda(v^1(u), \ldots, v^{n-1}(u))\|}.$$ 

Since the $v^i(u)$ are a basis for $T_u M$, $\Lambda(v^1(u), \ldots, v^{n-1}(u))$ is never zero. In addition, because $\varphi^{-1}$ is of class $C^k$, $N$ must be of class $C^{k-1}$.

\[
\]

3.3 A lemma on hyperplanes and embedded submanifolds

Let $M$ be a connected $C^1$-embedded $n - 1$ dimensional submanifold of $\mathbb{R}^n$ (i.e., a hypersurface) that is contained in a finite union of distinct hyperplanes $H_1, \ldots, H_r$. The goal of this section is to prove that $M$ must be entirely contained in one of the hyperplanes. The intuition comes from the case $n = 3$: a surface in $\mathbb{R}^3$ cannot, say, be contained in $H_1 \cup H_2$ and also intersect both $H_1$ and $H_2$ because it would generate a “corner” at the intersection $M \cap H_1 \cap H_2$, thus destroying smoothness. This is illustrated in Figure 2.
Figure 2: A surface $M$ cannot be smooth if it is connected, contained in $H_1 \cup H_2$, but not entirely contained in neither $H_1$ nor $H_2$.

This is probably a well-known differential geometric fact but we could not find a precise reference, so we give a proof here. Nevertheless, our discussion is related to the following classical fact: a point in a surface for which the derivative of the Gauss map vanishes is called a planar point and a connected surface in $\mathbb{R}^3$ such that all its points are planar must be a piece of a plane, see Definitions 7, 8 and the proof of Proposition 4 of Chapter 3 of [2].

In our case, the fact that $M$ is contained in a finite number of hyperplanes hints that the image of any Gauss map of $M$ should be confined to the directions that are orthogonal to those hyperplanes. This, by its turn, suggests that the derivative of $N$ should vanish everywhere, i.e., all points must be planar. In fact, our proof is inspired by the proof of Proposition 4 of Chapter 3 of [2] and we will use the same compactness argument at the end.

To start, we observe that the tangent of a curve contained in $H_1, \ldots, H_r$ must also be contained in those hyperplanes.

**Proposition 7.** Let $H_i = \{a_i\}^\perp$ be hyperplanes in $\mathbb{R}^n$ for $i = 1, \ldots, r$. Suppose that a $C^1$ curve $\alpha: (-\epsilon, \epsilon) \to \mathbb{R}^n$ is contained in $X = \bigcup_{i=1}^r H_i$. Then, $\alpha'(0) \in X$.

**Proof.** Changing the order of the hyperplanes if necessary, we may assume that

\[
\alpha(0) \in H_1 \cap \cdots \cap H_s \\
\alpha(0) \notin H_{s+1}, \ldots, H_r.
\]

Since $\alpha$ is contained in $X$, we have $s \geq 1$. Furthermore, because $\alpha$ is continuous, there is $\hat{\epsilon} > 0$ such that

\[
\alpha(\epsilon) \notin H_{s+1}, \ldots, H_r, \tag{5}
\]

for $-\hat{\epsilon} < \epsilon < \hat{\epsilon}$.

Now, suppose for the sake of obtaining a contradiction that $\alpha'(0)$ does not belong to any of these hyperplanes $H_1, \ldots, H_s$. Therefore, for all $i \in \{1, \ldots, s\}$, we have

\[
\langle \alpha(0), a_i \rangle = 0, \quad \langle \alpha'(0), a_i \rangle \neq 0.
\]

Since $\alpha'(\cdot)$ is continuous, we can select $0 < \check{\epsilon} < \hat{\epsilon}$ such that for all $i \in \{1, \ldots, s\}$ and $\epsilon \in (-\check{\epsilon}, \check{\epsilon})$, we have

\[
\langle \alpha'(\epsilon), a_i \rangle \neq 0.
\]

By the mean value theorem applied to $\langle \alpha(\cdot), a_i \rangle$ on the interval $[0, \check{\epsilon}/2]$, we obtain that $\langle \alpha(\check{\epsilon}/2), a_i \rangle \neq 0$, for all $i \in 1, \ldots, s$. Since $\check{\epsilon}/2 \in (-\check{\epsilon}, \check{\epsilon})$, (5) implies that

\[
\langle \alpha(\check{\epsilon}/2), a_i \rangle \neq 0,
\]

for $i \in \{s+1, \ldots, r\}$ too. This shows that $\alpha(\check{\epsilon}/2) \notin X$, which is a contradiction. \qed
Before we prove the main lemma of this subsection, we need the following observation on finite dimensional vector spaces.

**Proposition 8.** A finite dimensional real vector space $V$ is not a countable union of subspaces of dimension strictly smaller than $\dim V$.

**Proof.** Suppose that $V$ is a countable union $\bigcup W_i$ of subspaces of dimension smaller than $\dim V$. Take the unit ball $B \subseteq V$. Then, $B = \bigcup W_i \cap B$. However, this is not possible since each $W_i \cap B$ has measure zero, while $B$ has nonzero measure.

We now have all the necessary pieces to prove the main lemma.

**Lemma 9.** Let $X \subseteq \mathbb{R}^n$ be a union of finitely many hyperplanes $H_i = \{a_i\}^\perp$, $a_i \neq 0$, $i = 1, \ldots, r$. Let $M$ be an $(n - 1)$ dimensional differentiable manifold that is connected, $C^1$-embedded in $\mathbb{R}^n$ and contained in $X$. Then, $M$ must be entirely contained in one of the $H_i$.

**Proof.** We proceed by induction in $r$. The case $r = 1$ is clear, so suppose that $r > 1$.

Consider a chart $\phi : U \rightarrow \mathbb{R}^{n-1}$ such that $U \subseteq M$ is connected and construct a $C^0$ (i.e., continuous) Gauss map $N$ in $U$, as in Proposition 6. Let $u \in U$ and let us examine the tangent space $T_u M$. We have

$$T_u M = \{ \alpha'(0) \mid \alpha : (-\epsilon, \epsilon) \rightarrow M, \alpha(0) = u, \alpha \text{ is } C^1 \}.$$

By Proposition 7,

$$T_u M \subseteq X.$$

Therefore,

$$T_u M = \bigcup_{i=1}^r H_i \cap T_u M.$$

Each $H_i \cap T_u M$ is a subspace of $T_u M$ (an intersection of subspaces is also a subspace!). By Proposition 8, $T_u M$ cannot be a union of subspaces of dimension less than $\dim T_u M = n - 1$. Therefore, there exists some index $j$ such that $H_j \cap T_u M = T_u M$. Since both $T_u M$ and $H_j$ have dimension $n - 1$, we conclude that $H_j = T_u M$.

In particular, the Gauss map $N$ satisfies $N(u) = a_j/\|a_j\|$ or $N(u) = -a_j/\|a_j\|$. Therefore, for all $u \in U$, we have

$$N(u) \in \left\{ \pm \frac{a_i}{\|a_i\|} \mid i = 1, \ldots, r \right\}.$$

Since $U$ is connected and $N$ is continuous, we conclude that the Gauss map $N$ is constant. Denote this constant vector by $v$.

Let $\psi = \langle \varphi^{-1}(\cdot), v \rangle$. Since $\varphi$ is a chart, given any $w \in \varphi(U)$, the differential

$$d\varphi^{-1}_w : \mathbb{R}^{n-1} \rightarrow T_{\varphi^{-1}(w)} M$$

is a linear bijection. Since $T_{\varphi^{-1}(w)} M$ is orthogonal to $v$, we conclude that $\psi' = 0$. Therefore $\psi$ must be constant and there is $\kappa_0$ such that $\langle \varphi^{-1}(w), v \rangle = \kappa_0$, for all $w \in \varphi(U)$. That is, $\langle u, v \rangle = \kappa_0$, for all $u \in U$.

Recall that, given $x \in M$, we can always obtain a chart $\varphi : U \rightarrow M$ around $x$ such that $U$ is connected. Therefore, the discussion so far shows that every $x \in M$ has a neighborhood $U$ such that $U$ is entirely contained in a hyperplane

$$\{ z \mid \langle z, v_x \rangle = \kappa_x \},$$

where $v_x$ has the same direction as one of the $a_1, \ldots, a_r$. Now, fix some $x \in M$ and let $y \in M$, $y \neq x$. Since $M$ is connected, there is a continuous path $\alpha : [0,1] \rightarrow M$ such that $\alpha(0) = x$ and $\alpha(1) = y$. 

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Similarly, for every \( t \in [0, 1] \), we can find a neighborhood \( U_t \subseteq M \) of \( \alpha(t) \) such that \( U_t \) is contained in a hyperplane \( \{ z \mid \langle z, v_t \rangle = \kappa_t \} \) where \( v_t \) is parallel to one of \( a_1, \ldots, a_r \). In particular

\[
[0, 1] \subseteq \bigcup_{t \in [0, 1]} \alpha^{-1}(U_t).
\]

Since the \( U_t \) are open in \( M \) and \( \alpha \) is continuous, the \( \alpha^{-1}(U_t) \) form an open cover for the compact set \( [0, 1] \). Therefore, the Heine-Borel theorem implies that a finite number of the \( \alpha^{-1}(U_t) \) are enough to cover \( [0, 1] \). As a consequence, \( \alpha \) itself is contained in finitely many neighborhoods \( U_{t_1}, \ldots, U_{t_\ell} \). Now, we note the following:

- If \( U_{t_i} \cap U_{t_j} \neq \emptyset \) then \( U_{t_i} \cap U_{t_j} \) is a nonempty open set in \( M \) and therefore, an embedded submanifold of dimension \( n - 1 \), see Proposition 5.1 in [7]. Furthermore \( U_{t_i} \cap U_{t_j} \) is contained in the set

\[
H = \{ z \in \mathbb{R}^n \mid \langle z, v_{t_i} \rangle = \kappa_{t_i}, \langle z, v_{t_j} \rangle = \kappa_{t_j} \}.
\]

Therefore, the smooth manifold \( H \) must have at least dimension \( n - 1 \). We conclude that “\( \langle z, v_{t_i} \rangle = \kappa_{t_i} \)” and “\( \langle z, v_{t_j} \rangle = \kappa_{t_j} \)” define the same hyperplane. So, \( U_{t_i} \) and \( U_{t_j} \) are in fact, contained in the same hyperplane.

- \( U_{t_i} \) must intersect some of the \( U_{t_2}, \ldots, U_{t_\ell} \) because if it does not, then \( \alpha^{-1}(U_{t_i}) \) and \( \alpha^{-1}(\bigcup_{i=2}^{\ell} U_{t_i}) \) disconnect the connected set \( [0, 1] \). Changing the order of the sets if necessary, we may therefore assume that \( U_{t_1} \) and \( U_{t_2} \) intersect and, therefore, lie in the same hyperplane. Similarly, the union \( U_{t_1} \cup U_{t_2} \) must intersect one of the remaining neighborhoods \( U_{t_3}, \ldots, U_{t_\ell} \), lest we disconnect the interval \( [0, 1] \). By induction, we conclude that all neighborhoods lie in the same hyperplane.

In particular, \( x \) and \( y \) lie in the same hyperplane and, therefore, \( M \) is entirely contained in some hyperplane whose normal direction has the same direction as one of the \( a_1, \ldots, a_r \).

So far, we have shown that \( M \) is entirely contained in a hyperplane of the form

\[
\{ z \in \mathbb{R}^n \mid \langle z, v \rangle = \kappa_0 \}.
\]

Without loss of generality, we may assume that \( v \) has the same direction as \( a_1 \). If \( \kappa_0 = 0 \), we are done. Otherwise, since \( v \) has the same direction as \( a_1 \), it follows that \( M \) does not intersect \( H_1 \) and

\[
M \subseteq \bigcup_{i=2}^{r} H_i.
\]

By the induction hypothesis, \( M \) must be contained in one of the \( H_2, \ldots, H_r \). \( \square \)

4 Main results

In this section, we show the main results on \( p \)-cones. We begin by observing a basic fact on the differentiability of \( p \)-norms.

**Lemma 10.** Let \( n \geq 2 \) and \( p \in (1, \infty) \).

(i) \(|.|_p \) is \( C^1 \) on \( \mathbb{R}^n \setminus \{0\} \).

(ii) If \( p \in (1, 2) \) then \(|.|_p \) is \( C^2 \) on a neighborhood of \( x \) if and only if \( x_i \neq 0 \) for all \( i \).

(iii) If \( p \in [2, \infty) \) then \(|.|_p \) is \( C^2 \) on \( \mathbb{R}^n \setminus \{0\} \).
Proof. (i) $\|\cdot\|_p$ is $C^1$ on $\mathbb{R}^n \setminus \{0\}$ because

$$\frac{\partial \|\cdot\|_p}{\partial x_i}(x) = \frac{\partial \|\|_p^1-p\|x_i\|^{p-1}}{\partial x_i} \text{sign}(x_i).$$

(ii) If $x_i \neq 0$ for all $i$, it is easy to see that $\|\cdot\|_p$ is $C^2$ on a neighborhood of $x$. For the converse, consider a point $x \neq 0$ with $x_i = 0$ for some $i$. Then, $\frac{\partial \|\cdot\|_p}{\partial x_i}(x) = 0$ holds and so

$$\lim_{h \to 0} \frac{1}{h} \left( \frac{\partial \|\cdot\|_p}{\partial x_i}(x + h e_j) - \frac{\partial \|\cdot\|_p}{\partial x_i}(x) \right) = \lim_{h \to 0} h^{-1} \frac{\partial \|\cdot\|_p}{\partial x_i}(x + he_j)$$

$$= \lim_{h \to 0} h^{-1} \|x + he_i\|_p^{1-p} ||h|^{p-2}h$$

$$= \lim_{h \to 0} \|x + he_j\|_p^{1-p} ||h|^{p-2}$$

$$= \begin{cases} +\infty & (p < 2) \\ 0 & (p > 2). \end{cases}$$

Hence, when $p \in (1,2)$, the derivative $\frac{\partial \|\cdot\|_p}{\partial x_i}(x)$ exists if and only if $x_i \neq 0$.

(iii) For $p > 2$ (the assertion in the case $p = 2$ is clear),

$$\frac{\partial \|\cdot\|_p}{\partial x_i}(x) = (1 - p) \|x\|_p^{1-2p} ||x_i x_j\|^{p-1} \text{sign}(x_i) \text{sign}(x_j)$$

holds if $i \neq j$, otherwise we have

$$\frac{\partial \|\cdot\|_p}{\partial x_i}(x) = (1 - p) \|x\|_p^{1-2p} x_i^{2(p-1)} + (p - 1) \|x\|_p^{1-p} x_i^{p-2}.$$ 

We now move on to the main result of this paper.

**Theorem 11.** Let $p,q \in [1,\infty], p \leq q$, $n \geq 2$ and $(p,q,n) \neq (1,\infty,2)$. Suppose that $\mathcal{L}^n_{p+1}$ and $\mathcal{L}^n_{q+1}$ are isomorphic, that is,

$$ A \mathcal{L}^n_{p+1} = \mathcal{L}^n_{q+1} $$

holds for some $A \in GL_{n+1}(\mathbb{R})$. Then $p = q$ must hold. Moreover, if $p \neq 2$, then we have $A \in Aut(\mathcal{L}^n_1)$. 

**Proof.** The proof consists of three parts I, II, and III.

**I** First we consider the case $p \in \{1,\infty\}$ corresponding to the case when $\mathcal{L}^n_{p+1}$ is polyhedral. Since $A$ preserves polyhedrality, $q$ must be 1 or $\infty$ too. Note that $\mathcal{L}^n_1$ and $\mathcal{L}^n_{q+1}$ cannot be isomorphic if $n \geq 3$ because they have different numbers of extreme rays, see Section 2.2. Therefore, $p = q = 1$ or $p = q = \infty$ must hold. Since $Aut(\mathcal{L}^n_{\infty}) = Aut(\mathcal{L}^n_{1+1})$ holds (Proposition 2), the assertion is verified in the case $p \in \{1,\infty\}$. 

**II** Now let $p,q \in (1,\infty)$. Then the set

$$ M_p := \{(t,x) \in \mathbb{R} \times \mathbb{R}^n \setminus \{0\} \mid t = \|x\|_p\} $$

becomes a $C^1$-embedded submanifold of $\mathbb{R}^{n+1}$ by Lemma 10 (i) and Proposition 3 (i). Note that $A M_{p+1} = M_{q+1}$ implies $AM_q = M_q$ since $A$ maps the boundary of $\mathcal{L}^n_{p+1}$ onto the boundary of $\mathcal{L}^n_{q+1}$.

It suffices to consider the case $p,q \in (1,2)$ by the following observation.

(a) The case $1 < p < 2 \leq q < \infty$ does not happen in view of Proposition 4 and Lemma 10. In fact, since $\|\cdot\|_q$ is $C^2$ on $\mathbb{R}^n \setminus \{0\}$ and $A^{-1} M_q = M_p$ holds, Proposition 4 implies that $\|\cdot\|_p$ is $C^2$ on $\mathbb{R}^n \setminus \{0\}$ but this is a contradiction.
(b) If \(2 \leq p \leq q < \infty\) holds, then taking the dual of the relation \(A L_p^{n+1} = L_q^{n+1}\) with respect to the Euclidean inner product, it follows that
\[
A^{-T} L_p^{n+1} = L_q^{n+1}
\]
where \(p^*\) and \(q^*\) in \((1, 2]\) are the conjugates of \(p\) and \(q\), respectively. Either \(p^* = q^* = 2\) or \(p^*, q^* \in (1, 2)\) must hold by (a). If \(p^* = q^* = 2\), then we are done since this implies that \(p = q = 2\). Now, suppose that \(p^*, q^* \in (1, 2)\). If we prove that \(p^* = q^*\) and \(A^{-T} \in \text{Aut}(L_1^{n+1})\), then we conclude that \(p = q\) and \(A \in \text{Aut}(L_1^{n+1})\). However, by Proposition 2, \(\text{Aut}(L_1^{n+1})^{-T} = \text{Aut}(L_1^{n+1})\) (Note that, if \(P\) is a generalized permutation matrix, then so is \(P^{-T}\)).

From cases (a), (b) we conclude that it is enough to consider the case \(p, q \in (1, 2)\), which we will do next.

III. Let \(p, q \in (1, 2)\). We show by induction on \(n\) that every \(A \in GL_{n+1}(\mathbb{R})\) with \(A L_p^{n+1} = L_q^{n+1}\) is a bijection on the set
\[
E = \bigcup_{i=1}^{n} \bigcup_{\sigma \in \{-1, 1\}} \mathbb{R}^{+}(1, \sigma e_i^n),
\]
where \(e_i^n\) is the \(i\)-th standard unit vector in \(\mathbb{R}^n\). First, let us check that this claim implies \(A \in \text{Aut}(L_1^{n+1})\) and \(p = q\). Taking the conical hull of the relation \(AE = E\), we conclude that
\[
A L_1^{n+1} = A(\text{cone}(E)) = \text{cone}(AE) = \text{cone}(E) = L_q^{n+1},
\]
where the relation \(\text{cone}(E) = L_q^{n+1}\) holds because a pointed closed convex cone is the conical hull of its extreme rays (see Theorem 18.5 in [10]) and \(E\) is precisely the union of all the extreme rays of \(L_1^{n+1}\) with the origin removed, see Section 2.2. Therefore, we have
\[
A \in \text{Aut}(L_1^{n+1}) \subseteq \text{Aut}(L_p^{n+1}),
\]
where the last inclusion follows by Proposition 2 because \(\|P x\|_p = \|x\|_p\) for any generalized permutation matrix \(P\). Then \(L_p^{n+1} = A L_q^{n+1} = L_q^{n+1}\) and so \(p = q\) must hold.

Now, let us show the claim that \(A\) is a bijection on \(E\). Consider the map \(\xi_p : \mathbb{R}^n \setminus \{0\} \to M_p\) defined by \(\xi_p(x) = (\|x\|_p, x)\) whose inverse \(\xi_p^{-1} : M_p \to \mathbb{R}^n \setminus \{0\}\) is the projection \(\xi_p^{-1}(t, x) = x\). By Proposition 4, the map \(B : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}\) defined by
\[
B(x) = \xi_p^{-1} \circ A|_{M_p} \circ \xi_p(x)
\]
is a \(C^1\) diffeomorphism. Moreover, \(\|\cdot\|_p\) is \(C^2\) on a neighborhood of \(x\) if and only if \(\|\cdot\|_q\) is \(C^2\) on a neighborhood of \(B(x)\). Since \(p, q \in (1, 2)\), each of the functions \(\|\cdot\|_p\) and \(\|\cdot\|_q\) is \(C^2\) on a neighborhood of \(x\) if and only if \(x_i \neq 0\) for all \(i\) (Lemma 10). This implies that the set
\[
X = \{x \in \mathbb{R}^n \setminus \{0\} \mid x_i = 0 \text{ for some } i\}
\]
satisfies
\[
B(X) = X
\]
because \(x\) belongs to \(X\) if and only if \(\|\cdot\|_p\) and \(\|\cdot\|_q\) are never \(C^2\) on any neighborhood of \(x\).

III.a. Consider the case \(n = 2\). Then the set \(X\) can be written as
\[
X = \{x \in \mathbb{R}^2 \setminus \{0\} \mid x_1 = 0 \text{ or } x_2 = 0\}
\]
\[
= \mathbb{R}^{++}(0, 1) \cup \mathbb{R}^{++}(0, -1) \cup \mathbb{R}^{++}(1, 0) \cup \mathbb{R}^{++}(-1, 0)
\]
\[
= \bigcup_{i=1}^{2} \bigcup_{\sigma \in \{-1, 1\}} \mathbb{R}^{++}(\sigma e_i^2).
\]
Then \( \xi_p(X) \) and \( \xi_q(X) \) coincide with \( E \):

\[
\xi_p(X) = \xi_q(X) = \bigcup_{i=1}^{2} \bigcup_{\sigma \in \{-1,1\}} \mathbb{R}_+ \{(1, \sigma e_i^3) \} = E.
\]

Moreover, \( A \) is bijective on \( E \) because

\[
A(\xi_q(X)) = \xi_q \circ \xi_q^{-1} \circ A|_{M_p} \circ \xi_p(X) = \xi_q \circ B(X) = \xi_q(X).
\]

Thus, the claim \( AE = E \) holds in the case \( n = 2 \).

Now let \( n \geq 3 \) and suppose that the claim is valid for \( n - 1 \). Denote

\[
X_i := \{ x \in \mathbb{R}^n \setminus \{0\} \mid x_i = 0 \}, \quad M_p^i := \xi_p(X_i) = \{ (t, x) \in \mathbb{R} \times \mathbb{R}^n \setminus \{0\} : t = \|x\|_p, \ x_i = 0 \}.
\]

With that, we have

\[
X = \bigcup_{i=1}^{n} X_i.
\]

We show that for any \( i \in \{1, \ldots, n\} \) there exists \( j \in \{1, \ldots, n\} \) such that

\[
B(X_i) = X_j.
\]

For any \( i \), the set \( X_i \) is a connected \( (n - 1) \) dimensional \( C^1 \)-embedded submanifold of \( \mathbb{R}^n \) contained in \( X \). Since \( B : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\} \) is a \( C^1 \) diffeomorphism satisfying \( B(X) = X \), the set \( B(X_i) \) is also a connected \( (n - 1) \) dimensional \( C^1 \)-embedded submanifold of \( \mathbb{R}^n \) contained in \( X \). Then, since \( X \cup \{0\} \) is the union of the hyperplanes \( X_i \cup \{0\}, \ i = 1, \ldots, n \), it follows from Proposition 9 that \( B(X_i) \) is entirely contained in some hyperplane \( X_j \cup \{0\} \). Then we have

\[
B(X_i) \subseteq X_j.
\]

By the same argument, the set \( B^{-1}(X_i) \) is contained in some hyperplane \( X_k \cup \{0\} \), that is, \( B^{-1}(X_i) \subseteq X_k \) holds. This shows that

\[
X_i = B^{-1}(B(X_i)) \subseteq B^{-1}(X_j) \subseteq X_k.
\]

Since \( X_i \) cannot be a subset of \( X_k \) if \( i \neq k \), it follows that \( i = k \). Then, we obtain \( X_i = B^{-1}(X_j) \), i.e., \( B(X_i) = X_j \).

Since \( B \) is a bijection, the above argument shows that there exists a permutation \( \tau \) on \( \{1, \ldots, n\} \) such that

\[
B(X_i) = X_{\tau(i)}.
\]

Then we have

\[
A(M_p^i) = \xi_q \circ \xi_q^{-1} \circ A|_{M_p} \circ \xi_p(X_i) = \xi_q \circ B(X_i) = \xi_q(X_{\tau(i)}) = M_q^{\tau(i)}.
\]

Taking the linear span both sides, we also have

\[
A(V_i) = V_{\tau(i)} \quad \text{where} \quad V_i := \{ (t, x) \in \mathbb{R} \times \mathbb{R}^n \mid x_i = 0 \}.
\]

Now we apply the induction hypothesis to the isomorphism \( A|_{V_i} \) as follows. Define the isomorphism \( \varphi_i : V_i \to \mathbb{R}^n \) by

\[
\varphi_i(t, x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = (t, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)
\]

and consider the isomorphism \( A_i := \varphi_{\tau(i)} \circ A|_{V_i} \circ \varphi_i^{-1} : \mathbb{R}^n \to \mathbb{R}^n \). By the above argument, we see that \( A_i(\mathcal{L}_p^m) = \mathcal{L}_q^n \):

\[
A_i(\mathcal{L}_p^m) = \varphi_{\tau(i)} \circ A|_{V_i} \circ \varphi_i^{-1}(\mathcal{L}_p^m) = \varphi_{\tau(i)} \circ A(\text{cone } M_p^i) = \varphi_{\tau(i)}(\text{cone } M_q^{\tau(i)}) = \mathcal{L}_q^n.
\]
So the induction hypothesis implies that \( A_i \) is bijective on
\[
\bigcup_{j=1}^{n-1} \bigcup_{\sigma \in \{-1,1\}} \mathbb{R}_{++}(1, \sigma e_j^{n-1}).
\]

Therefore, \( A|_{V_i} = \varphi_{\tau(i)}^{-1} \circ A_i^{-1} \circ \varphi_i \) is a bijection from
\[
\bigcup_{j \in \{1, \ldots, n\} \setminus \{i\}} \bigcup_{\sigma \in \{-1,1\}} \mathbb{R}_{++}(1, \sigma e_j^n)
\]
on to
\[
\bigcup_{j \in \{1, \ldots, n\} \setminus \{\tau(i)\}} \bigcup_{\sigma \in \{-1,1\}} \mathbb{R}_{++}(1, \sigma e_j^n).
\]
Combining this result for each \( i = 1, \ldots, n \), it turns out that \( A \) is bijective on
\[
E = \bigcup_{i=1}^{n} \bigcup_{\sigma \in \{-1,1\}} \mathbb{R}_{++}(1, \sigma e_i^n).
\]

Combining the latter assertion of Theorem 11 and Proposition 2, we obtain the description of the automorphism group of the \( p \)-cones.

**Corollary 12.** For \( p \in [1, \infty] \), \( p \neq 2 \) and \( n \geq 2 \), we have \( \text{Aut}(L_p^{n+1}) = \text{Aut}(L_1^{n+1}) \). In particular, any \( A \in \text{Aut}(L_p^{n+1}) \) can be written as
\[
A = \alpha \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix},
\]
where \( \alpha > 0 \) and \( P \) is an \( n \times n \) generalized permutation matrix.

We can also recover our previous result on the non-homogeneity of \( p \)-cones with \( p \neq 2 \). In contrast to [6], here we do not require the theory of \( T \)-algebras.

**Corollary 13.** For \( p \in [1, \infty] \), \( p \neq 2 \) and \( n \geq 2 \), the \( p \)-cone \( L_p^{n+1} \) is not homogeneous.

**Proof.** By Corollary 12, for any \( A \in \text{Aut}(L_p^{n+1}) = \text{Aut}(L_1^{n+1}) \), we have that the vector \((1, 0, \ldots, 0)\) is an eigenvector of \( A \). So, there is no automorphism of \( L_p^{n+1} \) that maps \((1, 0, \ldots, 0)\) to an interior point of \( L_p^{n+1} \) that does not belong to
\[
\{(\beta, 0, \ldots, 0) \mid \beta > 0\}.
\]
Hence, \( L_p^{n+1} \) cannot be homogeneous.

Now the non-self-duality of \( p \)-cones \( L_p^{n+1} \) for \( p \neq 2 \) and \( n \geq 2 \) is an immediate consequence of Theorem 11 in view of Proposition 1, while we need an extra argument for the case \((p, q, n) = (1, \infty, 2)\).

**Corollary 14.** For \( p \in [1, \infty] \), \( p \neq 2 \) and \( n \geq 2 \), the \( p \)-cone \( L_p^{n+1} \) is not self-dual under any inner product.

**Proof.** Suppose that \( L_p^{n+1} \) is self-dual under some inner product. Then, by Proposition 1, there exists a symmetric positive definite matrix \( A \) such that
\[
AL_p^{n+1} = L_q^{n+1} \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.
\]
If \((p,q,n) \neq (1,\infty,2), (\infty,1,2)\), then \(p = q = 2\) must hold by Theorem 11. Now let us consider the case \((p,q,n) = (1,\infty,2)\), i.e., \(A L^3_1 = L^3_\infty\). Recalling (2), we have \(B L^3_1 = L^3_\infty\) with
\[
B = \begin{pmatrix}
1 & 0 & 0 \\
0 & \sqrt{2} \cos(\pi/4) & -\sqrt{2} \sin(\pi/4) \\
0 & \sqrt{2} \sin(\pi/4) & \sqrt{2} \cos(\pi/4)
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -1 \\
0 & 1 & 1
\end{pmatrix}.
\]
Therefore, \(B^{-1} A \in \text{Aut}(L^3_1)\) holds. Then, by Proposition 2, the matrix \(A\) can be written as \(A = BC\) where \(C\) is of the form
\[
C = \alpha \begin{pmatrix}
1 & 0 & 0 \\
0 & \pm 1 & 0 \\
0 & 0 & \pm 1
\end{pmatrix} \quad \text{or} \quad \alpha \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & \pm 1 \\
0 & \pm 1 & 0
\end{pmatrix}, \quad \alpha > 0.
\]
Since \(A\) is symmetric, it has one of the following forms:
\[
\alpha \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & -1 \\
0 & -1 & 1
\end{pmatrix}, \quad \alpha \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}, \quad \alpha \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1 & -1
\end{pmatrix}, \quad \alpha \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}.
\]
None of them is positive definite. Therefore, we obtain a contradiction. \(\square\)

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