Some invariant properties of quasi-Möbius maps

Lorenzo Heer

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Abstract

We investigate properties which remain invariant under the action of quasi-Möbius maps of quasi-metric spaces. A metric space is called doubling with constant $D$ if every ball of finite radius can be covered by at most $D$ balls of half the radius. It is shown that the doubling property is an invariant property for (quasi-)Möbius maps. Additionally it is shown that the property of uniform disconnectedness is an invariant for (quasi-)Möbius maps as well.

Keywords Möbius structures, doubling property, quasi-Möbius maps, uniform disconnectedness

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1 Introduction

Let $(X, d)$ be a metric space. $X$ is doubling if there exists a constant $D > 0$, such that every ball of finite radius can be covered by at most $D$ balls of half the radius. $X$ is uniformly disconnected if there exists a constant $\theta < 1$, such that $X$ contains no $\theta$-chain, i.e. a sequence of (at least 3 distinct) points $(x_0, x_1, \ldots, x_n)$ such that

$$d(x_i, x_{i+1}) \leq \theta d(x_0, x_n).$$

A map $f : (X, d) \to (Y, d')$ is quasi-Möbius if it is a homeomorphism and there exists a homeomorphism $\nu : [0, \infty) \to [0, \infty]$, such that for all quadruples $Q = (x_1, x_2, x_3, x_4)$ of distinct points of $X$ and $Q' := (f(x_1), f(x_2), f(x_3), f(x_4))$,

$$\text{cr}(Q', d') \leq \nu(\text{cr}(Q, d))$$

holds. Here the cross-ratio $\text{cr}$ is given by

$$\text{cr}(Q, d) := \frac{d(x_1, x_3)d(x_2, x_4)}{d(x_1, x_4)d(x_2, x_3)}.$$

The aim of this paper is to prove the following two theorems:
Theorem 1 (Invariance of doubling under quasi-Möbius maps). Let \((X, d)\) be a doubling space. Let \(f : (X, d) \to (Y, d')\) be a quasi-Möbius homeomorphism. Then \((Y, d')\) is doubling.

Theorem 2 (Invariance of uniform disconnectedness under quasi-Möbius maps). Let \((X, d)\) be a metric uniformly disconnected space and let \(f : (X, d) \to (Y, d')\) be a quasi-Möbius homeomorphism. Then \((Y, d')\) is uniformly disconnected.

The results are related to results of Lang-Schlichenmaier [5] and Xie [11] who proved that quasi-symmetric maps respectively quasi-Möbius maps preserve the Nagata dimension of metric spaces. The present work has been inspired by the article of Xie [11] and the work of Väisälä [10]. We note that a space is doubling if and only if it has finite Assouad dimension [7]. However the Assouad dimension is not a quasi-symmetric (and therefore also not a quasi-Möbius) invariant [9].

We would like to note that we have been informed that Theorem 1 is also a consequence of a published result of Li-Shanmugalingam [6].

It is well known that uniform disconnectedness is invariant under quasi-symmetric maps [7, 4]. However its behaviour under quasi-Möbius maps has not been studied before.

The related property of uniform perfectness has been shown to be invariant under the metric inversion in [8]. It is therefore also invariant under quasi-Möbius maps.

In Appendix A we prove a slight generalization of Theorem 1 and Theorem 2 for \(K\)-quasi-metric spaces.

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2 Basic Definitions and Preparations

We introduce the necessary definitions which we will require later.

2.1 Extended Metrics

Let \(X\) be a set with cardinality at least 3. We call a map \(d : X \times X \to [0, \infty]\) an extended metric on \(X\) if there exists a set \(\Omega(d) \subset X\) with cardinality 0 or 1 and furthermore all of the following requirements are satisfied:

1. \(d_{|X\setminus\Omega(d) \times X\setminus\Omega(d)} : X\setminus\Omega(d) \times X\setminus\Omega(d) \to [0, \infty]\) is a metric;

2. \(d(x, \omega) = d(\omega, x) = \infty\) for all \(x \in X\setminus\Omega(d)\) and \(\omega \in \Omega(d)\);
3. \( d(\omega, \omega) = 0 \) for \( \omega \in \Omega(d) \).

If \( \Omega(d) \) is non empty we call \( \omega \in \Omega(d) \) the infinitely remote point of \( X \). By abuse of notation we may write \( \infty \) for the point \( \omega \).

### 2.2 Doubling Property

We call a metric space doubling with constant \( D \) if every ball of finite radius can be covered by at most \( D \) balls of half the radius.

### 2.3 Uniform Disconnectedness

For \( \theta < 1 \) we call a sequence of (at least 3 distinct) points \((x_0, x_1, \ldots, x_n)\) in a metric space \((X, d)\) a \( \theta \)-chain if

\[
d(x_i, x_{i+1}) \leq \theta d(x_0, x_n)
\]

holds for all \( i \in \{0, 1, \ldots, n - 1\} \). A metric space is called uniformly disconnected with constant \( \theta \) if it contains no \( \theta \)-chains.

### 2.4 Quasi-Möbius and Quasi-Symmetric Maps

We call a homeomorphism \( f : (X, d) \to (Y, d') \) \( \nu \)-quasi-symmetric if for all pairwise distinct \( x_1, x_2, x_3 \in X \) we have

\[
\frac{d'(f(x_1), f(x_2))}{d'(f(x_1), f(x_3))} \leq \nu \left( \frac{d(x_1, x_2)}{d(x_1, x_3)} \right).
\]

A homeomorphism \( f : (X, d) \to (Y, d') \) is called quasi-symmetric if it is \( \nu \)-quasi-symmetric for some homeomorphism \( \nu : [0, \infty[ \to [0, \infty[ \). It is called symmetric if for all pairwise distinct \( x_1, x_2, x_3 \in X \) we have

\[
\frac{d'(f(x_1), f(x_2))}{d'(f(x_1), f(x_3))} = \frac{d(x_1, x_2)}{d(x_1, x_3)}.
\]

### 3 Invariance of Doubling Property

#### 3.1 Preparations for the Proof

For the proof we need the following proposition of Xie and a result of Väisälä which we cite verbatim

**Proposition 1** (Proposition 3.6 in [11]). Let \( f : (X_1, d_1) \to (X_2, d_2) \) be a quasi-Möbius homeomorphism. Then \( f \) can be written as \( f = f_2^{-1} \circ f' \circ f_1 \), where \( f' \) is a quasi-symmetric map, and \( f_i \) for \( i \in \{1, 2\} \) is either a metric inversion or the identity map on the metric space \((X_i, d_i)\).

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\[1\] And therefore also no \( \theta' \)-chains for any \( \theta' \leq \theta \).
Proposition 2 (Theorem 3.10 in [10]). Let \((X, d)\) be an unbounded metric space and let \(f : X \to Y\) be a quasi-Möbius map. Then \(f\) is quasi-symmetric if and only if \(f(x) \to \infty\) as \(x \to \infty\). If \(X\) is any metric space and if \(f : X \cup \{\infty\} \to Y \cup \{\infty\}\) is quasi-Möbius with \(f(\infty) = \infty\), then \(f|_X\) is quasi-symmetric.

Remark 1. Let \((X, d)\) be an unbounded space. Then we can build the completed space with respect to the infinitely remote point \(\bar{X} := X \cup \{\infty\}\) together with an extended metric \(\bar{d}\). Let \(d(x, y) := d(x, y)\) and \(d(\infty, x) := \bar{d}(x, \infty) = \infty\) for all \(x, y \in X\). Furthermore let \(\bar{d}(\infty, \infty) = 0\). Then clearly \((X, d)\) is doubling if and only if \((\bar{X}, \bar{d})\) is doubling.

Theorem 3. Let \((X, d)\) be an metric doubling space with doubling constant \(D\), where \(d\) is an extended metric [3] and denote by \(\infty \in X\) the infinitely remote point in \((X, d)\). Furthermore let \(p \in X\) with \(p \neq \infty\) and let \(i_p\) be given by \(i_p(x, y) := \frac{d(x, y)}{d(p, x)d(p, y)}\) for all \(x, y \in X \setminus \{\infty\}\) and \(i_p(\infty, x) := i_p(x, \infty) := \frac{1}{d(p, x)}\). Define \(d_p(x, y) := \inf\{\sum_{i=1}^{k} i_p(x_i, x_{i-1}) \mid x = x_0, \ldots, x_k = y \in X \setminus \{p\}\}. \(\bar{X}, d_p\) is doubling with constant at most \(D^{10} + 1\).

Proof. If \((X, d)\) is bounded, consider the space \((\bar{X}, \bar{d})\), with \(\bar{X} := X \cup \{\infty\}\) and \(\bar{d}(x, y) := d(x, y)\) for all \(x, y \in X\) and \(d(\infty, \infty) := \infty\). \((\bar{X}, \bar{d})\) is doubling. Furthermore if \((\bar{X}, \bar{d})\) is doubling, then so is \((X, d_p)\). We therefore only need to show the theorem for unbounded \(X\).

We have the following relation for all \(x, y \in X \setminus \{p\}\) [2]:

\[
\frac{1}{4} i_p(x, y) \leq d_p(x, y) \leq i_p(x, y) \leq \frac{1}{d(x, p)} + \frac{1}{d(y, p)}.
\]

Let \(x_0 \in X \setminus \{p\}\) and \(r > 0\). Let \(B' := B'_r(x_0) := \{x \in X \mid d_p(x_0, x) \leq r\}\) be the ball of radius \(r\) in the space \((X, d_p)\). We consider the following two cases

1. If \(B' \cap B'_r(\infty) \neq \emptyset\), then \(A' := B'_r(x_0) \setminus B_{\frac{r}{2}}(\infty)\). Take \(y_0 \in A'\).

For any two points \(x, y \in A'\) we have by definition of the metric \(d_p\) and the above relation that

\[
i_p(x, y) = \frac{d(x, y)}{d(p, x)d(p, y)} \leq 4d_p(x, y) \leq 8r,
\]

and \(\frac{1}{d(y, p)} = i_p(\infty, y) \geq d_p(\infty, y) > \frac{1}{2}r\). From this it follows that

\[
d(x, y) \leq 8rd(p, x)d(p, y) \leq \frac{32}{r}.
\]

In particular we know that \(A' \subseteq B_{\frac{32}{r}}(y_0) := \{x \in X \mid d(y_0, x) \leq \frac{32}{r}\}\).

By the assumption we furthermore have for all \(x \in B'\) that

\[
d_p(x, \infty) \leq 2r + \frac{1}{2}r = \frac{5}{2}r.
\]
and therefore also
\[ \frac{1}{d(p, x)} \leq \frac{5}{2} r, \]
from which it follows that
\[ d(p, x) \geq \frac{2}{5r}. \]

The space \((X, d)\) is doubling and we can find \(D^N\) balls \(b_i\) of radius \(\frac{4}{5} 2^{-N}\) with centerpoints \(x_i\) covering \(B_{\frac{4}{5}}(y_0)\). Let \(\tilde{b}_i := b_i \cap A'\) then we have for all \(x, y \in \tilde{b}_i:\)
\[ d_p(x, y) \leq i_p(x, y) = \frac{d(x, y)}{d(p, x) d(p, y)} \leq \frac{\frac{64}{2} 2^{-N}}{\frac{2}{5r} \frac{2}{5r}} = \frac{64 \cdot 5^2 \cdot r^2}{2^2 2^N r} = \frac{400}{2^N} r. \]

In particular for \(N := 10\) we know that we have constructed a cover of \(B' \subseteq A' \cup B_\frac{1}{2r}(\infty)\) by \(D^{10} + 1\) balls of radius \(\frac{1}{2r}\).

2. In case that \(B' \cap B_\frac{1}{2r}(\infty) = \emptyset\), we know that \(d_p(x_0, \infty) > r\) and also \(d_p(B', \infty) := \inf_{x \in B'} d_p(x, \infty) \geq \frac{1}{2} r\). For all \(y \in B'\) we have
\[ i_p(x_0, y) = \frac{d(x_0, y)}{d(p, x_0) d(p, y)} \leq 4d_p(x_0, y) \leq 4r, \]
from which it follows that
\[ d(x_0, y) \leq 4rd(p(x_0) d(p, y) \leq \frac{4r}{d_p(\infty, x_0) d_p(\infty, y)} \leq \frac{4r}{d_p(\infty, B')^2}. \]

We therefore have \(B' \subseteq B_{\frac{4r}{d_p(\infty, B')^2}}(x_0)\) and by the doubling property of \((X, d)\) we can cover by \(D^N\) balls \(b_i\) of radius \(\frac{4r}{d_p(\infty, B')^2} 2^{-N}\) with center points \(x_i\). Let \(\tilde{b}_i := b_i \cap B'\), then we have for any two \(x, y \in \tilde{b}_i:\)
\[ d_p(x, y) \leq i_p(x, y) = \frac{d(x, y)}{d(p, x) d(p, y)} \leq \frac{\frac{8r}{d_p(\infty, B')^2} 2^{-N}}{d_p(\infty, B')^2 d_p(\infty, x) d_p(\infty, y)} = \frac{2^{-N+4} d_p(\infty, x) d_p(\infty, y)}{d_p(\infty, B')^2 r}. \]

Furthermore we have
\[ d_p(x, \infty) \leq d_p(x_0, x) + d_p(x_0, \infty) \leq r + d_p(B', \infty) + r \leq 5d_p(B', \infty). \]

In conclusion we get that
\[
2^{-N+4} \frac{d_p(\infty, x) d_p(\infty, y)}{d_p(\infty, B')^2} \leq 2^{-N+4} \frac{5^2 d_p(\infty, B')^2}{d_p(\infty, B')^2} = \frac{8 \cdot 5^2}{2^N}.
\]

It therefore follows that if we take \(N := 9\), then we have a covering of \(B'\) by \(D^9\) balls of radius \(\frac{1}{2r}\).

\[ \square \]

**Remark 2.** Note that if in addition \(d \in \mathcal{M}\) where \((X, \mathcal{M})\) is Ptolemy Möbius, then \(i_p = d_p\) and in particular \((X, d_p)\) is doubling with constant at most \(D^8 + 1\).
3.2 Proof of Theorem 1

Proof of Theorem 1. It remains to show the theorem for \((X, d)\) being a doubling metric space, \(f : (X, d) \to (X, d')\) a metric inversion and we have the following cases to check:

1. \((X, d)\) unbounded, \((X, d')\) bounded;

2. \((X, d)\) and \((X, d)\) both unbounded but with different points at infinity.

Case 2 follows directly from Theorem 3. In the situation of 1, \(d'\) is a metric inversion \(d_p\) where \(p\) is an isolated point in \(X\). That is there exists a \(\epsilon > 0\) such that \(d(p, x) > \epsilon\) for all \(x \in X \setminus \{p\}\). The proof of Theorem 3 still holds.

4 Invariance of Uniform Disconnectedness

The proof of Theorem 2 will again make use of some of the propositions from the previous sections. In the following let \((X, d)\) be a metric space, \(p \in X\) and \(\theta \leq \frac{1}{32}\). We assume that \((X, d_p)\) is not \(\theta\)-uniformly disconnected, in particular there is some \(\theta\)-chain \((x_0, x_1, \ldots, x_n)\) in \((X \setminus \{p\}, d_p)\). We keep this notation for the rest of this section. In addition we introduce the following notation for convenience: Let \(r_i := d(p, x_i), \ l := d(x_0, x_n)\) and \(l_i := d(x_i, x_{i+1})\). This is illustrated in Figure 1. Without loss of generality we can assume \(r_n \geq r_0\).
Remark 3. The condition for \((x_0, x_1, \ldots, x_n)\) being a \(\theta\)-chain in \((X, d_p)\) implies that
\[
\frac{l_i}{r_i r_{i+1}} \leq \frac{4\theta l}{r_n r_0} \quad \forall i \in \{0, \ldots, n-1\}.
\]

On the other hand if
\[
\frac{l_i}{r_i r_{i+1}} \leq \frac{\theta l}{4 r_n r_0} \quad \forall i \in \{0, \ldots, n-1\}
\]
holds, then \((x_0, x_1, \ldots, x_n)\) is a \(\theta\)-chain in \((X, d_p)\).

Lemma 1. Assume that \((X, d)\) contains no \(\sqrt[3]{4\theta}\)-chains. Then there is an index \(s \in \{0, \ldots, n-1\}\) such that
\[
l_s > l \sqrt[3]{4\theta}
\]
and
\[
\max\{r_s, r_{s+1}\} \sqrt[3]{4\theta} \geq r_0.
\]

Proof. Assume for a contradiction that \(r_s \sqrt[3]{4\theta} < r_0\) and \(r_{s+1} \sqrt[3]{4\theta} < r_0\). Then from the condition in the remark above it follows
\[
\frac{l_s}{r_s r_{s+1}} < \frac{4\theta l}{r_n r_0} < \frac{4\theta l_s}{\sqrt[3]{4\theta} r_n r_0 r_{s+1}} = \frac{l_s}{r_s r_{s+1}}
\]
which is a contradiction.

Proposition 3. \((X, d)\) contains a \(\sqrt[3]{4\theta}\)-chain.

Proof. By the previous lemma we know that there must be some index \(q\) such that \(r_q \sqrt[3]{4\theta} \geq r_0\) and for all \(i \in \{0, \ldots, q-1\}\) we have that \(r_i \sqrt[3]{4\theta} < r_0\).
We claim that \((x_q, x_{q-1}, \ldots, x_1, x_0, p)\) is a \(\sqrt[3]{4\theta}\)-chain in \((X, d)\). If this were not so, there would be some \(i \in \{0, \ldots, q-1\}\) for which \(r_q \sqrt[3]{4\theta} < l_i\). But then
\[
\frac{r_q \sqrt[3]{4\theta}^2}{r_0 r_q} < \frac{r_q \sqrt[3]{4\theta}}{r_i r_q} \leq \frac{r_q \sqrt[3]{4\theta}}{r_i r_{i+1}} < \frac{l_i}{r_i r_{i+1}} \leq \frac{4\theta l}{r_n r_0}
\]
implies
\[
r_n < \sqrt[3]{4\theta} l \leq \frac{1}{2} l
\]
which is a contradiction to the triangle inequality of the metric space \((X, d)\).

Proof of Theorem 2. The proof of the theorem now follows directly from Proposition 1.
5 Applications of the Theorems

For the following we need a short definition [4]: Let $F$ be a finite set with $k \geq 2$ elements and let $F^\infty$ denote the set of sequences $\{x_i\}_{i=1}^\infty$ with $x_i \in F$. For two elements $x = \{x_i\}, y = \{y_i\} \in F^\infty$ let

$$L(x, y) = \sup \{I \in \mathbb{N} | \forall 1 \leq i \leq I : x_i = y_i\}.$$ 

In particular we have $L(x, x) = \infty$ and $L(x, y) = 0$ if $x_1 \neq y_1$. Given $0 < a < 1$ set $\rho_a(x, y) = a^{L(x, y)}$. This defines an ultrametric on $F^\infty$. We call $(F^\infty, \rho_a)$ the symbolic $k$-Cantor set with parameter $a$.

As an application of the theorems we provide a generalization of the following result by David and Semmes:

**Proposition 4** (Proposition 15.11 (Uniformization) in [4]). Suppose that $(M, d)$ is a compact metric space which is bounded, complete, doubling, uniformly disconnected, and uniformly perfect. Then $M$ is quasi-symmetrically equivalent to the symbolic Cantor set $F^\infty$, where we take $F = \{0, 1\}$ and we use the metric $\rho_a$ on $F^\infty$ with parameter $a = \frac{1}{2}$.

We can generalize this result as follows:

**Theorem 4.** Suppose that $(M, d)$ is a complete, doubling, uniformly perfect and uniformly disconnected metric space. Then $M$ is quasi-Möbius equivalent to the symbolic Cantor set as given above.

**Proof.** Let $p \in M$ be some point and let $s_p(x, y) = \frac{d(x, y)}{(d(x, p) + 1)(d(y, p) + 1)}$. Let
$$\hat{d}_p(x,y) = \inf \{ \sum_{i=1}^{k} s_p(x_i, x_{i-1}) : x = x_0, \ldots, x_k = y \in X \}. \text{ We have } [2]$$

$$\frac{1}{4} s_p(x, y) \leq \hat{d}_p(x, y) \leq s_p(x, y) \leq \frac{1}{1+d(x, p)} + \frac{1}{1+d(y, p)}.$$ 

Then the space \((M, \hat{d}_p)\) is bounded and satisfies all the properties of the above proposition: The map \(f : (X, d) \to (X, \hat{d}_p)\) given by \(d \mapsto \hat{d}_p\) is Möbius. By [Theorem 2] and [Theorem 1] doubling and uniformly disconnectedness are invariant under Möbius maps. The invariance of uniformly perfectness follows from [8], and the invariance of completeness follows from [1]. Totally boundedness follows from the doubling property and therefore the space \((X, \hat{d}_p)\) is compact. 

We can apply the same idea to Proposition 16.9 in [4] and we get:

**Corollary 1.** Let \((M,d)\) be a complete Ahlfors regular metric space of dimension \(\gamma\) which is uniformly disconnected. Then there exists a doubling measure \(\mu\) on \(F^\infty\), and \((M, d)\) is quasi-Möbius equivalent to \((F^\infty, D)\), where \(D\) is given by

$$D(x, y) = \left( \mu(\bar{B}(x, d_a(x, y))) + \mu(\bar{B}(y, d_a(x, y))) \right)^{\frac{1}{\gamma}},$$

and \(0 < a < 1\).

This follows from the above remarks and the invariance of Ahlfors regularity under \(d \mapsto \hat{d}_p\) as shown in [6].
A Appendix

Proposition 5. Let \((X, d)\) be a \(K\)-quasi-metric space \([3]\). Let \(X_\infty\) denote the infinite remote set and let \(\infty \in X_\infty\), i.e. the space satisfies the relations

1. \(d(x, y) = 0 \iff x = y\),
2. \(d(x, y) = d(y, x)\),
3. \(d(x, y) \leq K \max\{d(x, z), d(z, y)\}\) for all \(x, y, z \in X\) for which all distances are defined,
4. \(d(x, y) < \infty \iff x, y \in X \setminus X_\infty\).

Let \(\lambda : X \to [0, \infty], L > 0\) and \(K' \geq K\) be such that \(X_\infty = \lambda^{-1}(\infty)\) and

1. \(d(x, y) \leq K' \max\{L\lambda(x), L\lambda(y)\}\),
2. \(L\lambda(x) \leq K' \max\{d(x, y), L\lambda(y)\}\).

Denote by \(X'_\infty := \{\lambda^{-1}(0)\}\). Define a new metric \(d_\lambda : (X \times X) \setminus (X'_\infty \times X'_\infty) \to [0, \infty]\) by

1. \(d_\lambda(x, y) := \frac{d(x, y)}{\lambda(x)\lambda(y)}\) for \(x, y \in X \setminus X'_\infty\),
2. \(d_\lambda(x, \infty) := d_\lambda(\infty, x) := \frac{L}{\lambda(x)}\) for \(\infty \in X_\infty\),
3. \(d_\lambda(\infty, \infty) = 0\) for \(\infty \in X_\infty\),
4. \(d_\lambda(x, p) := d_\lambda(p, x) := \infty\) for \(p \in X'_\infty\).

If \((X, d)\) is doubling with constant \(D\) then \((X, d_\lambda)\) is doubling with constant at most \(D^{\log_2(8K'^2 K) + 1}\).

Proof. By Prop 5.3.6 in \([3]\), \(d_\lambda\) is a \(K'^2\)-quasi-metric. In particular we have for all \(x, y, z \in X\) for which all distances are defined, that:

\[ d_\lambda(x, y) \leq K'^2 \max\{d(d, z), d(z, y)\}. \]

Let \(x_0 \in X, x_0 \neq p \in X'_\infty\) and \(r > 0\) and let \(B' := B'_r(x_0) := \{x \in X \mid d_\lambda(x_0, x) \leq r\}\). Consider the following cases

1. If \(B' \cap B'_r(\infty) \neq \emptyset\), then let \(A' := B' \setminus B'_r(\infty)\). For all \(x, y \in B'\) we have

\[ d_\lambda(x, y) = \frac{d(x, y)}{\lambda(x)\lambda(y)} \leq K'^2 r, \]

from which it follows that

\[ d(x, y) \leq K'^2 r \lambda(x)\lambda(y). \]
Furthermore we have for all \( x \in A' \) that \( d_\lambda(\infty, x) = \frac{L}{\lambda(x)} > \frac{1}{2}r \) and therefore also \( \lambda(x) < \frac{2r}{L} \). Combining both equations we get that for all \( x, y \in A' \) we have

\[
d(x, y) \leq K^{\ell_2} \frac{2L}{r} 2L \frac{2L}{r} = K^{\ell_4} L^2 / r.
\]

Without loss of generality assume \( x_0 \in A' \). By the doubling property of \((X, d)\) we can cover \( B_{K^{\ell_4} L^2}(x_0) \) by at most \( D^N \) balls \( b_i \) of radius \( \frac{K^{\ell_4} L^2}{r} 2^{-N} \). Let \( \tilde{b}_i := b_i \cap A' \) then we have for all \( x, y \in \tilde{b}_i \):

\[
d_\lambda(x, y) \leq \frac{K^{\ell_4} L^2}{\lambda(x) \lambda(y)}.
\]

By the assumption there is a \( \tilde{x} \in B' \cap B'_{\frac{r}{2}}(\infty) \) and we have for \( x \in B' \) that \( d_\lambda(x, \tilde{x}) \leq K^{\ell_2} r \), therefore we also have \( \frac{L}{\lambda(x)} = d_\lambda(x, \infty) \leq K^{\ell_4} r \) and \( \lambda(x) \geq \frac{r}{K^{\ell_4} r} \). In conclusion we get for all \( x, y \in \tilde{b}_i \):

\[
d_\lambda(x, y) \leq \frac{K^{\ell_4} L^2}{\lambda(x) \lambda(y)} \leq \frac{K^{\ell_4} L^2}{K^{\ell_4} r} \frac{K^{\ell_4} L^2}{K^{\ell_4} r} = \frac{K^{\ell_4} L^2}{K^{\ell_4} r} 2^{-N}.
\]

In particular for \( N := \lceil \log_2(8K^{\ell_4} K) \rceil \) we get a cover of \( B' \) by at most \( D^N + 1 \) balls of half the radius.

2. If \( B' \cap B'_{\frac{r}{2}}(\infty) = \emptyset \), then we have \( d_\lambda(x_0, \infty) > r \) and \( d_\lambda(B', \infty) > \frac{1}{2}r \).

For all \( y \in B' \) we have \( d_\lambda(x_0, y) = \frac{d(x_0, y)}{\lambda(x_0) \lambda(y)} \leq r \) and therefore also

\[
d(x_0, y) \leq r \lambda(x_0) \lambda(y) \leq \frac{rL^2}{d_\lambda(\infty, x_0) d_\lambda(\infty, y)} = \frac{rL^2}{d_\lambda(B', \infty)^2}.
\]

By the doubling property of \((X, d)\) we can find \( D^N \) balls \( b_i \) of radius \( \frac{rL^2}{d_\lambda(B', \infty)^2} 2^{\ell_4} \) covering \( B' \). Let \( \tilde{b}_i := b_i \cap B' \), then we have for any \( x, y \in \tilde{b}_i \):

\[
d_\lambda(x, y) = \frac{d(x, y)}{\lambda(x) \lambda(y)} \leq \frac{K^{\ell_4} L^2 2^{-N}}{d_\lambda(B', \infty)^2} = \frac{K^{\ell_4} L^2 2^{-N} d_\lambda(\infty, x) d_\lambda(\infty, y)}{d_\lambda(B', \infty)^2}.
\]

Furthermore for any \( x \in B' \) we have

\[
d_\lambda(x, \infty) \leq K^{\ell_2} \max\{d_\lambda(x_0, x), d_\lambda(x_0, \infty)\} \leq K^{\ell_2} r \leq K^{\ell_2} d_\lambda(B', \infty).
\]

We can combine the estimates to get

\[
d_\lambda(x, y) \leq \frac{K^2 r 2^{-N} d_\lambda(B', \infty)^2}{d_\lambda(B', \infty)^2} = K^2 r 2^{-N} K^{\ell_4} 4.
\]

In particular for \( N := \lceil \log_2(8K^{\ell_4}) \rceil \) we have constructed a covering by \( D^N \) balls of radius at most \( \frac{1}{2}r \).
Proposition 6. Let \((X, d)\) be a \(K\)-quasi-metric space \([3]\). Let \(X_{\infty}\) denote the infinite remote set and let \(\infty \in X_{\infty}\), i.e. the space satisfies the relations

1. \(d(x, y) = 0 \iff x = y\),
2. \(d(x, y) = d(y, x)\),
3. \(d(x, y) \leq K \max\{d(x, z), d(z, y)\}\) for all \(x, y, z \in X\) for which all distances are defined,
4. \(d(x, y) < \infty \iff x, y \in X \setminus X_{\infty}\).

Let \(\lambda : X \to [0, \infty]\), \(L > 0\) and \(K' \geq K\) be such that \(X_{\infty} = \lambda^{-1}(\infty)\) and

1. \(d(x, y) \leq K' \max\{L\lambda(x), L\lambda(y)\}\),
2. \(L\lambda(x) \leq K' \max\{d(x, y), L\lambda(y)\}\).

Denote by \(X'_{\infty} := \{\lambda^{-1}(0)\}\). Define a new metric \(d_{\lambda} : (X \times X) \setminus (X'_{\infty} \times X'_{\infty}) \to [0, \infty]\) by

1. \(d_{\lambda}(x, y) := \frac{d(x, y)}{\lambda(x)\lambda(y)}\) for \(x, y \in X \setminus X'_{\infty}\),
2. \(d_{\lambda}(x, \infty) := d_{\lambda}(\infty, x) := \frac{L}{\lambda(x)}\) for \(\infty \in X_{\infty}\),
3. \(d_{\lambda}(\infty, \infty) = 0\) for \(\infty \in X_{\infty}\),
4. \(d_{\lambda}(x, p) := d_{\lambda}(p, x) := \infty\) for \(p \in X'_{\infty}\).

Let \(\theta \leq \frac{1}{K'4}\). If \((X, d_{\lambda})\) has a \(\theta\)-chain, then \((X, d)\) has a \(\sqrt[\theta]{K'^4}\)-chain.

Proof. Using the same notation as before in section 4 we note that for all \(i \in \{0, \ldots, n - 1\}\) the following relation holds:

\[
\frac{l_i}{K'^2r_{i}r_{i+1}} \leq \frac{l_i}{\lambda(x_i)\lambda(x_{i+1})} \leq \frac{l\theta}{\lambda(x_0)\lambda(x_n)} \leq \frac{l\theta}{K'Lr_0r_n}.
\]

We can apply a similar argument as in Lemma 1 to get an index \(q\) for which

\[r_0 \leq \sqrt[\theta]{K'^4r_q},\]

and such that for all \(i \in \{0, \ldots, q - 1\}\) we have

\[r_0 > \sqrt[\theta]{K'^4r_i}.\]
Assume again for a contradiction that \((x_q, x_{q-1}, \ldots, x_0, p)\) is not a \(\sqrt{\theta K^2}l\)-chain. Then for some \(i \in \{0, \ldots, q-1\}:

\[
\frac{\sqrt[3]{\theta K^2 r_q}}{L^2 r_0 r_q} \leq \frac{\sqrt[3]{\theta K^2 r_q}}{L^2 r_i r_{i+1}} \leq \frac{\sqrt[3]{\theta K^2 r_q}}{L^2 r_i r_{i+1}} \leq \frac{\sqrt[3]{\theta K^2 r_q}}{L^2 r_i r_{i+1}} \leq \frac{l_i}{\lambda(x_i) \lambda(x_{i+1})} \leq \frac{l_i}{\lambda(x_i) \lambda(x_{i+1})} \leq \frac{l_i}{\lambda(x_i) \lambda(x_{i+1})} \leq \frac{l_i}{\lambda(x_i) \lambda(x_{i+1})} (4)
\]

From this it follows that

\[ r_n < \sqrt[3]{\theta K^2 K^2 l} \leq K^{-1} l. \]

\[ \square \]

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Loreno Heer
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
CH-8057 Zürich
loreno.heer@math.uzh.ch