Investigation of Gear Dynamic Characteristics under Stochastic External Excitations

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Abstract. Engine drive gear systems are widely used in lots of mechanical equipment. In engine drive gear systems, the external excitations, including the driving speed and external load, are generally stochastic due to manufacturing error, energy loss, variation of operating environmental, and so on. The reported works usually considered them as deterministic. In this paper, a stochastic nonlinear gear dynamic model considering the stochastic external excitations is presented. Then, the effects of the stochastic driving speed and external load on gear dynamic characteristics are investigated by Monte Carlo (MC) simulation. By the simulation results, some insightful conclusions, which are important for the gear random dynamics, are obtained.

1. Introduction

Gear systems, especially engine drive gear systems, are widely used in lots of mechanical equipment [1]. For an engine drive gear system, there are generally two kinds of excitations, internal excitations and external excitations. Internal excitations include time-varying mesh stiffness (TVMS), backlash, transmission error, and so on [2]. External excitations contain the external load [3] and driving speed of the pinion [4]. In existing studies, the effects of internal excitations have been well investigated while the research about the effects of external excitations achieve less attention, especially under stochastic domain. However, the dynamic responses of an engine drive gear system are significantly affected by the external excitations [4].

Under deterministic domain, external excitations have been investigated in some existing works. The mathematical models and the solving approaches for the nonlinear dynamics of gear systems under deterministic load was reviewed in [5]. Qiu et al. [6] considered the influence of deterministic input rotating speed on stiffness and introduced a velocity modulated stiffness model. In reality, however, the stochastic feature of external excitations is ubiquitous due to manufacturing error, energy loss, possible change of the engine status, and variation of operating environmental [7]. For example, the stochastic feature of dynamic loads for high speed gears was observed in [8]. Accordingly, to obtain more realistic representation of gear systems, the stochastic anticipated load profile and speed profile should be considered when modelling engine drive gear systems.

In recent works about modelling gear systems, attention of researchers has transferred from the deterministic domain to the stochastic domain. Tobe et al. modelled the randomness in the load as Gaussian white noise and validated their model through experiment results [9]. The random dynamic...
response of a gear pair under constant mesh stiffness, constant damping coefficient, and stochastic external load by adding Gaussian white noise was investigated in [10]. Fang et al. [2] investigated the effects of friction on the dynamic characteristics of a spur gear pair under stochastic external load. However, in existing works, the effects of stochastic driving speed, especially the coupled effects of stochastic driving speed and stochastic external load, are still short of investigation. In this paper, we model an engine drive gear system under stochastic external excitations, i.e., stochastic external load and stochastic driving speed. Then, the dynamic characteristics focusing on the effects caused by the stochastic external excitations are analyzed, and thus, some insightful conclusions are obtained.

The remaining parts of this paper are organized as follows. In Section 2, the proposed stochastic dynamic model for a spur gear pair is described. The effects of the stochastic external excitations are illustrated in Section 3. Section 4 draws the conclusion.

2. Nonlinear stochastic gear dynamic model
In this section, the proposed stochastic dynamic model for a spur gear pair is described. A spur gear pair is given in Fig. 1. Thus, the equations of motion for the system is given as

\[ I_p \ddot{\theta}_p = T_p - (F + G)R_p - \mu(F + G)X_i \]  

\[ I_g \ddot{\theta}_g = T_g + (F + G)R_g + \mu(F + G)(l - X_i) \]  

where \( I_p \), \( T_p \), \( \ddot{\theta}_p \), \( R_p \) are the moments of inertia, driving torque, angular acceleration, and base circle radius of the pinion (similarly the subscript \( g \) denotes the gear), respectively, \( F \) is the total elastic force, \( G \) is the damping force, \( \mu \) is the friction coefficient, \( X_i \) denotes the tangent distance in the action line, \( l = (R_p + R_g)\sin\alpha_0 \) means the length of the action line, and \( \alpha_0 \) denotes the pressure angle. According to Eq. (1) and Eq. (2), we have the following expression

\[ \ddot{\theta}_g = \left[ I_p \ddot{\theta}_p + T + (KG + c\dot{\delta})L \right] / I_g \]  

where \( \delta = R_p\dot{\theta}_p - R_g\dot{\theta}_g \) denotes the relative angular displacement of the gear pair, \( F = Kg(\delta) \), \( L = R_g - R_p + \mu(R_g + R_p)\sin\alpha_0 \), \( T = T_g - T_p \), \( K \) is the effective mesh stiffness, \( g(\dot{\delta}) \) denotes the function of backlash, \( L \) is the equivalent length of the force arm, \( c \) is the damping coefficient, and \( T \) is the equivalent torque (i.e., the external load in this paper) that reflects the combined fluctuation of \( T_g \) and \( T_p \). In this model, we consider the classic backlash function and the mesh stiffness function [2], i.e., \( g(\dot{\delta}) = \begin{cases} \delta - b, & \delta > b \\ 0, & -b \leq \delta \leq b \\ \delta + b, & \delta < -b \end{cases} \) and \( K(\theta) = \begin{cases} k_1, & (n-1)\phi_n \leq \theta < (n-1)\phi_n + \phi_m \\ k_2, & (n-1)\phi_n + \phi_m \leq \theta < n\phi_n \end{cases} \), where \( b \) denotes the backlash, \( \phi_n \) and \( \phi_m \) represent a mesh period of gear teeth and the double pairs of teeth mesh duration in a mesh period, respectively.

Accordingly, the second-order ordinary differential equation (ODE) of the considered gear model is given as

\[ \ddot{\theta}_g = \left[ I_p \ddot{\theta}_p + T + [Kg(\theta_p R_p - \theta_g R_g) + c(\dot{\theta}_p R_p - \dot{\theta}_g R_g)]L \right] / I_g \]  

Note that, the inputs (excitations) in (4) are \( T, \theta_p, \dot{\theta}_p, \ddot{\theta}_p \) while the outputs are \( \theta_g, \dot{\theta}_g, \dddot{\theta}_g \) (or called the responses of the gear dynamic model).
Figure 1. Dynamic model of a gear pair

Then, we need to consider the stochastic external load and driving speed formulation. According to the study in [11], the randomness in the external load and the driving speed is generally set as Gaussian white noise \( \xi_i(t) \),

\[
E(\xi_i(t)) = 0
\]

\[
E[\xi_i(t)\xi_i(t + \tau)] = r_i\Theta(\tau)
\]

where \( \xi_i(t) \) denotes the noise of the driving speed with variance \( r_i \), \( \xi_2(t) \) denotes the noise of the external load with variance \( r_2 \), and \( \Theta \) is the Dirac Delta function. Accordingly, the expression of \( \dot{\theta}_p(t) \) can obtained as

\[
\dot{\theta}_p(t) = \chi_0 + \sum_n \chi_n \sin(\rho_n t + \psi_n) + \xi_1(t)
\]

Similarly, the stochastic external load \( T \) can be expressed as

\[
T(t) = f_0 + \sum_j f_j \cos(\nu_j t + \beta_j) + \xi_2(t)
\]

where \( f_0 \) is a constant, \( j \) denotes the harmonic of the torque, \( \nu_j \), \( f_j \), and \( \beta_j \) are the corresponding angular frequency, amplitude, and initial phase, respectively.

It is known that Gaussian diffusion process \( \Phi(t) \) is the integration of Gaussian noise, and thus, \( \Phi(t) \) can be approximated by Itô stochastic differential equation (SDE) [11][12], which is given as follow.

\[
d\Phi(t) = \lambda dt + \sigma dW(t)
\]
where $\lambda$ is the drift scalar, $\sigma$ is the diffusion scalar, and $W(t)$ is a standard Wiener process. In addition, $dW(t)$ can be approximated as $dW(t) = \sqrt{\Delta t}$ [11]. For Gaussian white noise $\xi(t)$, $\lambda = 0$ and $\sigma = \sqrt{r_i}$. Accordingly, Eq. (7) can be approximated as

$$\theta_p(t) = \theta_p(t_0) + \chi_0 \Delta t + \sum_n \frac{\sin(\rho_n t)}{\rho_n} - \sin(\rho_n t_0) + \sqrt{r_i \Delta t} \tag{10}$$

Then, $\dot{\theta}_p(t)$ can be defined as

$$\dot{\theta}_p(t) = -\sum_n \chi_n \rho_n \sin(\rho_n t) + \frac{\xi(t)}{\Delta t} - \frac{\xi(t_0)}{\Delta t} \tag{11}$$

Note that the proposed stochastic model (i.e., Eq. (4) combined with Eqs. (7), (10), and (11)) is considered in the pinion rotational motion for the first time. Under the proposed model, there are two stochastic factors in the gear dynamic model, and thus, the complexity in solving the model and the analysis is largely increased.

3. Results and discussion

In this section, the numerical simulation results are given, and then, we analyze the simulation results. The gear pair parameters adopted in the simulation are summarized in Table 1.

| Table 1. Gear pair parameters. |
|--------------------------------|
| Inertia moment | $I_p = 2.6 \times 10^{-4}$ Kg $\cdot$ m$^2$ |
| $I_g = 0.0045$ Kg $\cdot$ m$^2$ |
| Base circle | $r_p = 0.034$ m |
| $r_g = 0.052$ m |
| Backlash | $b = 1 \times 10^{-5}$ m |
| Pressure angle | $\alpha_0 = 20^\circ$ |
| Stiffness | $k_1 = 1.6 \times 10^8$ N/m |
| $k_2 = 0.9 \times 10^8$ N/m |
| Friction coefficient | $\mu = 0.04$ |

3.1. Validation of the proposed model

In [4], a model of a spur gear pair under deterministic load and deterministic driving speed was validated with a finite element model. It was concluded that the model in [4] captures the salient behaviour of the actual system. Accordingly, we take that model as a sample to validate our proposed model. The simulation results are given in Fig. 2. Note that Liu’s Model denotes the model in [4].

From Fig. 2, the relative displacement of our proposed model is almost the same with that in Liu’s model. Thus, it validates that our proposed model can capture the characteristics of the actual systems.

3.2. Dynamic responses of the proposed model

In this section, we analyze the dynamic responses of the proposed model. Let $r_i = 25$, and we have the expression of $\dot{\theta}_p(t)$ as follow.

$$\dot{\theta}_p(t) = 800 + 80 \cos(100t + \pi/2) + 20 \cos(80t + \pi/2) + \xi(t) \tag{12}$$
Figure 2. Relative angular displacement with different models

Let \( r_2 = 2500 \), and we have the expression of \( T(t) \) as follow.

\[
T(t) = 500 + 50\cos(100t) + \xi_2(t)
\]  

(13)

Therefore, the gear model is excited by two stochastic excitations, i.e., Eq. (12) and Eq. (13). To demonstrate the effects of the stochastic driving speed, we take the case with deterministic driving speed as a comparison. Fig. 3 presents two instantaneous probability density functions (PDFs) with two profiles (i.e., stochastic driving speed and deterministic speed) of the driving speed at 0.2 s. Fig. 4 presents the standard derivation of \( \delta \) with time variation under two cases. The difference of the PDF and standard derivation of \( \delta \) under two cases is caused by the stochastic term in the driving speed. From Fig. 3, we can see that the responses excited by stochastic driving speed have more dispersion than those excited by deterministic driving speed. The variance of responses under the stochastic driving speed case is larger than that under the deterministic driving speed case. From Fig. 4, it illustrates that the standard deviation of \( \delta \) obtained by stochastic driving speed is always greater (about twice greater) than that obtained by deterministic driving speed during the simulation period. Accordingly, it can further prove that the case with stochastic driving speed has a larger variance than the case with deterministic driving speed.

Figure 3. PDF under two cases

Figure 4. Standard derivation of \( \delta \) with time variation under two cases
3.3. Coupled effects of the stochastic external load and stochastic driving speed

In this section, we analyse the coupled effects of the stochastic external load and stochastic driving speed. First, two sensitivity parameters $\eta_1$ and $\eta_2$ for driving speed and load are defined as follows.

$$\eta_1 = \frac{\sqrt{r_1}}{E(\hat{\theta}_p)}$$  
(14)

$$\eta_2 = \frac{\sqrt{r_2}}{E(T)}$$  
(15)

Keep the mean values $E(\hat{\theta}_p)$ and $E(T)$ invariant and adjust the values of $r_1$ and $r_2$ to change the values of $\eta_1$ and $\eta_2$. Table 2 summarizes the standard deviation of $\delta$ under different $\eta_1$ and $\eta_2$. From Table 2, we can see that the standard deviation of $\delta$ increases with the increase of $\eta_1$ and $\eta_2$. In addition, it also can be concluded that the standard deviation of $\delta$ increases more with the variation in $\eta_1$.

| $\eta_2$ | $\eta_1$ | 0.5%      | 0.65%     | 0.8%      |
|----------|----------|-----------|-----------|-----------|
| 0.5%     | 1.0665\times10^{-5} | 2.0038\times10^{-5} | 3.7009\times10^{-5} |
| 0.65%    | 1.0715\times10^{-5} | 2.1316\times10^{-5} | 4.3268\times10^{-5} |
| 0.8%     | 1.1444\times10^{-5} | 2.4168\times10^{-5} | 4.5109\times10^{-5} |

To further analyse the results, we define the average increase rates of the driving speed and external load as follows.

$$\kappa_1 = \frac{1}{6} \sum_{i=1}^{3} \frac{<i,2>} {<i,1>} + \frac{<i,3>} {<i,2>}$$  
(16)

$$\kappa_2 = \frac{1}{6} \sum_{i=1}^{3} \frac{<2,i>} {<1,i>} + \frac{<3,i>} {<2,i>}$$  
(17)

where $\kappa_1$ and $\kappa_2$ are the average increase rates of the driving speed and external load, respectively. $<i,j>$ represents the element in $i$th row and $j$th column in Table 2 without counting the header. Under the same $\eta_1$, the average increase rate $\kappa_2$ is 1.09. For the same $\eta_2$, the average increase rate $\kappa_1$ is 1.97. Thus, the standard deviation of $\delta$ is more sensitive to the variation of $\eta_1$. In other words, the stochastic feature in the driving speed $\hat{\theta}_p(t)$ is much more dangerous than that in the external load $T(t)$. To some degree, the randomness in the $\hat{\theta}_p(t)$ may cause more vibration and failure. The control of its randomness is very important in real applications.

Then, we consider the phase diagram ($\delta-\hat{\delta}$ relationship) with the stochastic driving speed. The simulation results are given in Fig. 5. Noted that $\delta$ and $\hat{\delta}$ are the mean value of the total samples. Recalling the phase diagram in [2], the orbit of $\delta-\hat{\delta}$ is smooth and periodic. However, in our proposed model, the orbit of $\delta-\hat{\delta}$ in the phase portrait is non-smooth and non-periodic, which is
caused by the stochastic driving speed. This phenomenon could be affected by the coupling of the stochastic driving speed and stochastic external load.

Figure 5. Phase Diagram

3.4. Discussion
According to the simulation results, we can obtain some insightful conclusions.

1. The established model is more realistic by considering the randomness in the driving speed and external load,
2. Under the same stochastic load, a small ratio of randomness in the driving speed will greatly increase the dispersion in responses,
3. The dispersion in responses is more sensitive to the uncertainty in the driving speed than that in the external load.

Therefore, our proposed gear dynamic model and numerical results can be used as a useful tool to investigate the gear random dynamics.

4. Conclusions
In this paper, a gear stochastic dynamic model for a spur gear pair considering TVMS, gear mesh damping, backlash, and friction under stochastic external excitations is established. Then, the stochastic external excitations, i.e., the driving speed and the external load, are modeled in the gear dynamic model. After that, the dynamic responses of this model are investigated using numerical simulation and compared with previous works.

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Acknowledgments
This research is supported by the National Natural Science Foundation of China (Grant No. 51675523), Natural Sciences and Engineering Research Council of Canada (NSERC), and Future Energy Systems under Canada First Research Excellent Fund (CFREF).