More Results on Polygonal Sum Labeling of Graphs

Rajakishore Samal\textsuperscript{1*} and Debdas Mishra\textsuperscript{2}

\textsuperscript{1}Department of Mathematics, O.P.S.Mohavidyalaya, Hindol Road, Dhenkanal – 759019, Odisha, India; rksamalmath@gmail.com
\textsuperscript{2}Department of Mathematics, C.V.Raman College of Engineering and Technology, Bhubaneswar – 752054, Odisha, India; debdasmishra@gmail.com

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Abstract

Objectives: To explore and identify some new classes of graphs which exhibit polygonal sum labeling. Methods: In this article we use a methodology which fundamentally involves formulation and subsequent mathematical validation. Findings: Here we establish that the graphs – Star (\(K_{1,n}\)), Coconut Tree, Bistar (\(S_{m,n}\)), the Graph \(mnk\) Comb (\(P_n\ K_k\)) and Subdivision graph \(S(K_{1,n})\) admit pentagonal, hexagonal, heptagonal, octagonal, nonagonal and decagonal sum labeling. Applications: One can explore to generalize these results and extend to give n-gonal labeling to some classes graphs. Sum labeling has already been used in the problems involving relational database management and hence one can try out to use polygonal sum labeling as well in these problems.

1. Introduction

The graphs considered here are finite, connected, undirected and simple. The notations and terminologies involving graph theory may be found in\textsuperscript{3} and the same involving number theory may be found in\textsuperscript{4}. The study undertaken in this paper involves Polygonal sum labeling of graphs. A \((p, q)\) graph \(G\) is said to admit a polygonal sum labeling if its vertices are labeled by non-negative integers such that the induced edge labels obtained by the sum of the labels of end vertices are the first \(q\) polygonal numbers. A graph possessing a polygonal sum labeling is called a polygonal sum graph. Here we show that some classes of graph can be embedded as induced sub graphs of a Polygonal sum graph. We recapitulate some important definitions useful for the present investigation.

1.1 Definition\textsuperscript{4,5}

The numbers which generate a \(k\)-gon are known as \(k\)-gonal numbers. The \(n^{th}\) \(k\)-gonal (i.e. \(k\) - sided polygonal) number is given by \(P_k(n) = \frac{n((k-2)n-k+4)}{2}\) where \(k \geq 3\). For Example, The \(n^{th}\) pentagonal number is denoted by \(A_n\) and is given by the formula \(A_n = \frac{1}{2}n(3n-1)\). The few pentagonal numbers are 1, 5, 12, 22, 35, 51, 70, 92, 117, 145, 176,...... Figure 1 illustrates pentagonal numbers.

![Figure 1. Pentagonal numbers 1, 5, 12, 22, 35.](image)

1.2 Definition\textsuperscript{3,5}

A \(k\)-gonal sum labeling of a graph \(G\) is a one to one function \(f: V(G) \rightarrow N\) that induces a bijection

*Author for correspondence
of the edges of $G$ defined by 
$$f^+(uv) \rightarrow f(u) + f(v)$$
for every $e = uv \in E(G)$. The graph which admits such labeling is called a $k$-gonal sum graph.

1.3 Example
Figure 2 illustrates a pentagonal sum labeling of $P_5$.

![Pentagonal sum labeling of $P_5$.](image)

1.4 Example
Figure 3 illustrates a decagonal sum labeling of $P_{10}$.

![Decagonal sum labeling of $P_{10}$.](image)

In$^2$ give pentagonal, hexagonal, heptagonal, octagonal, nonagonal and decagonal sum labeling to paths. Amuthavalli and Dineshkumar$^3$ have given pentagonal sum labeling to bistars $S_{m,m}$. In this paper an attempt has been made to prove that the Star $K_{1,n}$, Coconut Tree, Bistar $S_{m,n}$, the Graph $S_{m,n,k}$, Comb $P_n \diamond K_1$ and Subdivision graph $S(K_{1,n})$ admit pentagonal, hexagonal, heptagonal, octagonal, nonagonal and decagonal sum labeling. The graphs have been discussed in brief below.

1.5 Definition
Centre $c$ with $n$ pendant edges incident with $c$ is called a Star graph and is denoted by $K_{1,n}$ or $S_n$. Hence it has $n + 1$ vertices and $n$ edges.

1.6 Definition
Coconut tree is a tree with central path $u_1, u_2, ..., u_n$ having length $n - 1$ and $w_1, w_2, ..., w_k$ be the pendant vertices being adjacent with $u_i$. Hence it has $n + k$ vertices and $n + k - 1$ edges.

1.7 Definition
The graph obtained from $K_{1,m}$ and $K_{1,n}$ by joining their centers with an edge is called Bistar or Double star and is denoted by $S_{m,n}$. Let 
$$V(S_{m,n}) = \{u, v, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$$
and 
$$E(S_{m,n}) = \{uv, uu_i, vv_j : 1 \leq i \leq m, 1 \leq j \leq n\}.$$ Hence it has $m + n + 2$ vertices and $m + n + 1$ edges.

1.8 Definition
The graph $S_{m,n,k}$ is a graph obtained from a path of length $k$ by attaching the stars $K_{1,m}$ and $K_{1,n}$ with its pendant vertices. Hence it has $m + n + k + 1$ vertices and $m + n + k$ edges.

1.9 Definition
A graph obtained by attaching a single pendant edge to each vertex of a path $P_n = u_1u_2...u_n$ is called a comb. A comb graph is obtained from the path by joining a vertex $u_i$ to $w_i$, $1 \leq i \leq n$. It is denoted by $P_n \diamond K_1$. The edges are labeled as $e_{2i-1} = u_iw_i$ and $e_{2i} = u_{i+1}u_i$ for $1 \leq i \leq n$. Hence it has $2n$ vertices and $(2n - 1)$ edges.

1.10 Definition
The Subdivision of the star $K_{1,n}$ is a graph $S(K_{1,n})$ with vertex set $V(S(K_{1,n})) = \{v, v_i, u_i : 1 \leq i \leq n\}$ and edge set $E(S(K_{1,n})) = \{vv_i, uv_i : 1 \leq i \leq n\}$. Hence it has $2n + 1$ vertices and $2n$ edges.

2. Results

2.1 Pentagonal Sum Labeling of Graphs
In this section, we prove that stars $S_n$, coconut trees, bistars or double stars $S_{m,n}$, the graphs $S_{m,n,k}$, combs $P_n \diamond K_1$, subdivision graphs $S(K_{1,n})$ of the star $K_{1,n}$
admit pentagonal sum labeling.

2.1.1 Theorem
The star graph $K_{1,n}$ or $S_n$ possesses a pentagonal sum labeling.

Proof
Let $u$ be the apex vertex and let $u_1, u_2, \ldots, u_n$ be the pendant vertices of the star $S_n$. Define the labeling $f$ by

$$f(u) = 0, \quad f(v) = 1, \quad f(u_i) = \frac{1}{2}i(3i-1), 1 \leq i \leq m,$$

and

$$f(v_j) = \frac{1}{2}(m+j)(3m+3j-1)-1, 1 \leq j \leq n.$$

We see that the induced edge labels are the first $m+n+1$ pentagonal numbers. Hence the star graph $K_{1,n}$ or $S_n$ possesses a pentagonal sum labeling.

2.1.2 Theorem
The coconut trees have pentagonal sum labeling.

Proof
Let $u_1, u_2, \ldots, u_n$ be the vertices of a path having length $n-1$ and let $w_1, w_2, \ldots, w_k$ be the pendant vertices being adjacent with $u_1$.

Define the labeling $f$ by

$$f(u) = 0, \quad f(v) = 1, \quad f(u_i) = \frac{1}{2}i(3i-1), 1 \leq i \leq n,$$

and

$$f(w) = \frac{1}{2}(n+j-1)(3n+3j-4), \quad \text{for } 1 \leq j \leq k.$$

We see that the induced edge labels are the first $n+k-1$ pentagonal numbers. Hence the coconut trees have pentagonal sum labeling.

2.1.3 Theorem
The bistar $S_{m,n}$ admits pentagonal sum labeling.

Proof
Let $V(S_{m,n}) = \{u,v,u_1,v_1 : 1 \leq i \leq m, 1 \leq j \leq n\}$ and

$E(S_{m,n}) = \{uv,u_i,v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$

Define the labeling $f$ by

$$f(u) = 0, \quad f(v) = 1, \quad f(u_i) = \frac{1}{2}i(3i-1), 1 \leq i \leq m,$$

and

$$f(v_j) = \frac{1}{2}(m+j)(3m+3j-1)-1, 1 \leq j \leq n.$$
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\[
f(u) = \begin{cases}
\frac{i}{2}(3i-1), & \text{if } i \text{ is odd} \\
\frac{1}{2}(3i-4), & \text{if } i \text{ is even}
\end{cases} \quad \text{and} \quad f(w) = \frac{1}{2}(n+3i-1) - f(u).
\]

Thus the induced edge labels are the first \(2n-1\) pentagonal numbers. Hence comb \(P_n \square K_1\) possesses a pentagonal sum labeling.

### 2.1.6 Theorem

\(S(K_{1,n})\) the subdivision of the star graphs \(K_{1,n}\) possesses a pentagonal sum labeling.

**Proof**

Let \(V(S(K_{1,n})) = \{v, v_i, u_i : 1 \leq i \leq n\}\) and \(E(S(K_{1,n})) = \{v_j, v_j u_i : 1 \leq i \leq n\}\).

Define the labeling \(f\) by \(f(v) = 0, f(v_i) = \frac{1}{2}i(3i-1), 1 \leq i \leq n, f(u_i) = \frac{1}{2}(n+i)(3n+3i-1) - f(v_i), 1 \leq i \leq n\).

We see that the induced edge labels are the first \(2n\) pentagonal numbers and as such \(S(K_{1,n})\) has a pentagonal sum labeling.

### 2.2 Hexagonal Sum Labeling of Graphs

In this section, we prove that star graphs \(S_n\), coconut trees, bistars or double stars \(S_{m,n}\), the graphs \(S_{m,n,k}\), combs \(P_n \square K_1\), subdivision graphs \(S(K_{1,n})\) of the star \(K_{1,n}\) compatible with hexagonal sum labeling.

#### 2.2.1 Theorem

The star graph \(K_{1,n}\) or \(S_n\) has a hexagonal sum labeling.

**Proof**

Let \(u\) be the apex vertex and let \(u_1, u_2, \ldots, u_n\) be the pendant vertices of the star \(S_n\).

Define the labeling \(f\) by \(f(u) = 0\) and \(f(u_i) = i(2i-1), 1 \leq i \leq n\).

We see that the induced edge labels are the first \(n\) hexagonal numbers. Hence the star graph \(S_n\) has a hexagonal sum labeling.

#### 2.2.2 Theorem

The coconut trees compatible with hexagonal sum labeling.

**Proof**

Let \(u_1, u_2, \ldots, u_n\) be the vertices of a path having length \(n-1\) and let \(w_1, w_2, \ldots, w_k\) be the pendant vertices being adjacent with \(u_i\).

Define the labeling \(f\) by \(f(u_i) = \begin{cases}
\frac{i}{2}(i-1)(2i-1), & \text{if } i \text{ is odd} \\
\frac{1}{2}(2i-3), & \text{if } i \text{ is even}
\end{cases}
\quad \text{for } 1 \leq i \leq n
\quad \text{and} \quad f(w_j) = (n+j-1)(2n+2j-3), \quad \text{for } 1 \leq j \leq k.
\]

We see that the induced edge labels are the first \(n+k-1\) hexagonal numbers and as such the coconut trees compatible with hexagonal sum labeling.

#### 2.2.3 Theorem

The bistar \(S_{m,n}\) admits hexagonal sum labeling.

**Proof**

Let \(V(S_{m,n}) = \{u, v, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}\) and \(E(S_{m,n}) = \{uv_i, uu_i, vv_j : 1 \leq i \leq m, 1 \leq j \leq n\}\).

Define the labeling \(f\) by \(f(u) = 0, f(v) = 1, f(u_i) = i(2i-1), 1 \leq i \leq m, f(v_j) = (m+j)(2m+2j-1)-1, 1 \leq j \leq n\).

We see that the induced edge labels are the first \(m+n+1\) hexagonal numbers. Hence the bistar \(S_{m,n}\) admits hexagonal sum labeling.

#### 2.2.4 Theorem

The graph \(S_{m,n,k}\) admits hexagonal sum labeling.

**Proof**

Let \(P_k : v_1, v_2, \ldots, v_{k+1}\) be a path of length \(k\) with initial vertex \(v_1\) and terminal vertex \(v_{k+1}\). Let \(u_1, u_2, \ldots, u_m\) be the adjacent vertices to \(v_1\) and \(w_1, w_2, \ldots, w_n\) be the adjacent vertices to \(v_{k+1}\).
Define the labeling \( f \) by
\[

def \(f(v) = \begin{cases} 
\frac{1}{2}(i-1)(2i-1), & \text{if } i \text{ is odd} \\
\frac{1}{2}(2i-3), & \text{if } i \text{ is even} 
\end{cases} \)
\text{ for } 1 \leq i \leq k+1,
\]
and \( f(w) = (k+i)(2k+2i-1) - f(v), \) for \( 1 \leq i \leq n. \)

We see that the induced edge labels are the first \( m+n+k \) hexagonal numbers. Hence the graph \( S_{m,n,k} \) admits hexagonal sum labeling.

### 2.2.5 Theorem

The comb \( P_n \square K_1 \) admits hexagonal sum labeling.

**Proof**

Let \( P_n := u_1, u_2, \ldots, u_n \) be a path of length \( n-1 \) and let \( w_1, w_2, \ldots, w_n \) be the pendant vertices adjacent to \( u_1, u_2, \ldots, u_n \) respectively.

For \( i = 1, 2, \ldots, n \), define the labeling \( f \) by
\[

def \( f(u_i) = \begin{cases} 
\frac{1}{2}(i-1)(2i-1), & \text{if } i \text{ is odd} \\
\frac{1}{2}(2i-3), & \text{if } i \text{ is even} 
\end{cases} \) and \( f(w_i) = (n+i)(2n+2i-1) - f(u_i) \).

Thus the induced edge labels are the first \( 2n-1 \) hexagonal numbers and as such the comb \( P_n \square K_1 \) admits hexagonal sum labeling.

### 2.2.6 Theorem

\( S\left( K_{1,n} \right) \) the subdivision of the star \( K_{1,n} \) admits hexagonal sum labeling.

**Proof**

Let \( V\left( S\left( K_{1,n} \right) \right) = \{v, v_j, u_i : 1 \leq i \leq n\} \)
and \( E\left( S\left( K_{1,n} \right) \right) = \{v, v_j, u_i : 1 \leq i \leq n\} \).

Define the labeling \( f \) by
\[

def \( f(v) = 0 \), \( f(v_j) = i(2i-1), 1 \leq i \leq n \) and
\[
def \( f(u_i) = (n+i)(2n+2i-1) - f(v), 1 \leq i \leq n \).

As the induced edge labels are the first \( 2n \) hexagonal numbers, \( S\left( K_{1,n} \right) \) admits hexagonal sum labeling.

### 2.3 Heptagonal Sum Labeling of Graphs

Here we prove that stars \( S_n \), coconut trees, bistars or double stars \( S_{m,n,k} \), the graphs \( S_{m,n,k} \), combs \( P_n \square K_1 \), subdivision graphs \( S\left( K_{1,n} \right) \) of the star \( K_{1,n} \) admit heptagonal sum labeling.

### 2.3.1 Theorem

The star graph \( K_{1,n} \) or \( S_n \) admits heptagonal sum labeling.

**Proof**

Let \( u \) be the apex vertex and let \( u_1, u_2, \ldots, u_n \) be the pendant vertices of the star \( S_n \).

Define the labeling \( f \) by
\[
f(u) = 0 \quad \text{and} \quad f(u_i) = \frac{1}{2}(5i-3), \quad 1 \leq i \leq n.
\]

We see that the induced edge labels obtained by the sum of the labels of the vertices are the first \( n \) heptagonal numbers. Hence star graph \( S_n \) admits heptagonal sum labeling.

### 2.3.2 Theorem

The coconut trees admit heptagonal sum labeling.

**Proof**

Let \( u_1, u_2, \ldots, u_n \) be the vertices of a path having length \( n-1 \) and let \( w_1, w_2, \ldots, w_k \) be the pendant vertices being adjacent with \( u_1 \). Define the labeling \( f \) by
\[

def \( f(w) = \begin{cases} 
\frac{1}{2}(i-1)(2i-1), & \text{if } i \text{ is odd} \\
\frac{1}{2}(2i-3), & \text{if } i \text{ is even} 
\end{cases} \) \quad \text{for } 1 \leq i \leq n \quad \text{and} \quad f(w_j) = \frac{1}{2}(n+j-1)(5n+5j-8), \quad 1 \leq j \leq k.
\]

We see that the induced edge labels are the first \( n+k-1 \) heptagonal numbers. Hence the coconut trees admit heptagonal sum labeling.

### 2.3.3 Theorem

The bistar \( S_{m,n} \) admit heptagonal sum labeling.

**Proof**

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Let \( V(S_{m,n}) = \{u,v,u_i,v_j : 1 \leq i \leq m, 1 \leq j \leq n\} \) and 
\[ E(S_{m,n}) = \{uv, uv_i, vv_j : 1 \leq i \leq m, 1 \leq j \leq n\}. \]

Define the labeling \( f \) by 
\[ f(u) = 0, f(v) = 1, f(u_i) = \frac{1}{2}i(5i - 3), 1 \leq i \leq m, \]
and 
\[ f(v_j) = \frac{1}{2}(m + j)(5m + 5j - 3) - 1, 1 \leq j \leq n. \]

We see that the induced edge labels are the first \( m + n + 1 \) heptagonal numbers. Hence bistar \( S_{m,n} \) admit heptagonal sum labeling.

**2.3.4 Theorem**
The graph \( S_{m,n,k} \) admits heptagonal sum labeling.

**Proof**
Let \( P_k : v_1, v_2, ..., v_{k+1} \) be a path of length \( k \) with initial vertex \( v_1 \) and terminal vertex \( v_{k+1} \). Let \( u_1, u_2, ..., u_m \) be the adjacent vertices to \( v_1 \) and \( w_1, w_2, ..., w_n \) be the adjacent vertices to \( v_{k+1} \).

Define the labeling \( f \) by 
\[ f(v_i) = \begin{cases} 
\frac{1}{4}(i-1)(5i-3), & \text{if } i \text{ is odd} \\
\frac{1}{4}(5i-8), & \text{if } i \text{ is even}
\end{cases} \quad \text{for } 1 \leq i \leq k + 1, \]
\[ f(u_i) = \frac{1}{2}(k + j)(5k + 5j - 3), \quad \text{for } 1 \leq j \leq m; \]
and 
\[ f(w_l) = \frac{1}{2}(k + m + l)(5k + 5m + 5l - 3) - f(v_{i+1}), \quad \text{for } 1 \leq l \leq n. \]

We see that the induced edge labels are the first \( m + n + k \) heptagonal numbers. Hence the graph \( S_{m,n,k} \) admits heptagonal sum labeling.

**2.3.5 Theorem**
The comb \( P_n \sqcup K_1 \) admits heptagonal sum labeling.

**Proof**
Let \( P_n : u_1, u_2, ..., u_n \) be a path of length \( n - 1 \) and let \( w_1, w_2, ..., w_n \) be the pendant vertices adjacent to \( u_1, u_2, ..., u_n \) respectively.

For \( i = 1, 2, ..., n \) : define 
\[ f(u_i) = \begin{cases} 
\frac{1}{4}(i-1)(5i-3), & \text{if } i \text{ is odd} \\
\frac{1}{4}(5i-8), & \text{if } i \text{ is even}
\end{cases} \quad \text{and } f(v_i) = \frac{1}{2}(n + i - 1)(5n + 5i - 8) - f(u_i). \]

Thus the induced edge labels are the first \( 2n - 1 \) heptagonal numbers. Hence comb \( P_n \sqcup K_1 \) admits heptagonal sum labeling.

**2.3.6 Theorem**
\( S(K_{1,n}) \) the subdivision of the star \( K_{1,n} \) admits heptagonal sum labeling.

**Proof**
Let \( V(S(K_{1,n})) = \{v, v_i, u_i : 1 \leq i \leq n\} \) and 
\[ E(S(K_{1,n})) = \{vv_i, vu_i : 1 \leq i \leq n\}. \]

Define the labeling \( f \) by 
\[ f(v) = 0, f(v_i) = \frac{1}{2}(5i - 3), 1 \leq i \leq n \\
\text{and } f(u_i) = \frac{1}{2}(n + i)(5n + 5i - 3) - f(v_i), 1 \leq i \leq n. \]

We see that the induced edge labels are the first \( 2n \) heptagonal numbers. Hence \( S(K_{1,n}) \) admits heptagonal sum labeling.

**2.4 Octagonal Sum Labeling of Graphs**
In this section, we prove that stars \( S_n \), coconut trees, bistars or double stars \( S_{m,n} \), the graphs \( S_{m,n,k} \), combs \( P_n \sqcup K_1 \), subdivision graphs \( S(K_{1,n}) \) of the star \( K_{1,n} \) admit octagonal sum labeling.

**2.4.1 Theorem**
The star graph \( K_{1,n} \) or \( S_n \) admits octagonal sum labeling.

**Proof**
Let \( u \) be the apex vertex and let \( u_1, u_2, ..., u_n \) be the pendant vertices of the star \( S_n \).

Define the labeling \( f \) by 
\[ f(u_i) = i(3i - 2), 1 \leq i \leq n. \]
We see that the induced edge labels obtained by the sum of the labels of the vertices are the first $n$ octagonal numbers. Hence star graph $S_n$ admits octagonal sum labeling.

### 2.4.2 Theorem
Coconut trees admit octagonal sum labeling.

**Proof**

Let $u_1, u_2, \ldots, u_n$ be the vertices of a path having length $n - 1$ and let $w_1, w_2, \ldots, w_k$ be the pendant vertices being adjacent with $u_1$.

Define the labeling $f$ by

$$ f(u_i) = \begin{cases} \frac{1}{2}(i-1)(3i-2), & \text{if } i \text{ is odd} \\ \frac{1}{2}(3i-5), & \text{if } i \text{ is even} \end{cases} \quad \text{for } 1 \leq i \leq n. $$

and $f(w_j) = (n + j - 1)(3n + 3j - 5), \quad \text{for } 1 \leq j \leq k.$

We see that the induced edge labels are the first octagonal numbers. Hence the coconut trees admit octagonal sum labeling.

### 2.4.3 Theorem
The bistar $S_{m,n}$ admits octagonal sum labeling.

**Proof**

Let $V(S_{m,n}) = \{u, v, u_1, v_1, \ldots, u_m, v_m\}$ and $E(S_{m,n}) = \{uv, uu_i, vv_j : 1 \leq i \leq m, 1 \leq j \leq n\}$.

Define the labeling $f$ by

$$ f(u) = 0, \quad f(v) = 1, \quad f(u_i) = (3i - 2), \quad 1 \leq i \leq m, \quad \text{and} \quad f(v_j) = (3m + 3j - 2) - 1, \quad 1 \leq j \leq n. $$

We see that the induced edge labels are the first $m + n + 1$ octagonal numbers. Hence the bistar $S_{m,n}$ admits octagonal sum labeling.

### 2.4.4 Theorem
The graph $S_{m,n,k}$ admits octagonal sum labeling.

**Proof**

Let $P_k : v_1, v_2, \ldots, v_{k+1}$ be a path of length $k$ with initial vertex $v_1$ and terminal vertex $v_{k+1}$.

Let $u_1, u_2, \ldots, u_m$ be the adjacent vertices to $v_1$ and $w_1, w_2, \ldots, w_n$ be the adjacent vertices to $v_{k+1}$.

Define the labeling $f$ by

$$ f(v_i) = \begin{cases} \frac{1}{2}(i-1)(4i-3), & \text{if } i \text{ is odd} \\ \frac{1}{2}i(4i-7), & \text{if } i \text{ is even} \end{cases} \quad \text{for } 1 \leq i \leq k+1, $$

and $f(u_i) = (k + j)(4k + 4j - 3), \quad \text{for } 1 \leq j \leq m,$ and $f(w_i) = (k + m + l)(4k + 4m + 4l - 3) - f(v_{k+1}), \quad \text{for } 1 \leq l \leq n.$

We see that the induced edge labels are the first $2n$ octagonal numbers. Hence the graph $S_{m,n,k}$ admits octagonal sum labeling.

### 2.4.5 Theorem
The comb $P_n \square K_1$ admits octagonal sum labeling.

**Proof**

Let $P_n : u_1, u_2, \ldots, u_n$ be a path of length $n - 1$ and let $w_1, w_2, \ldots, w_n$ be the pendant vertices adjacent to $u_1, u_2, \ldots, u_n$ respectively. For $i = 1, 2, \ldots, n$ define the labeling $f$ by

$$ f(u) = \begin{cases} \frac{1}{2}(i-1)(3i-2), & \text{if } i \text{ is odd} \\ \frac{1}{2}i(3i-5), & \text{if } i \text{ is even} \end{cases} \quad \text{and} \quad f(w_i) = (n + i - 1)(3n + 3i - 5) - f(u). $$

Thus the induced edge labels are the first $2n - 1$ octagonal numbers. Hence comb $P_n \square K_1$ admits octagonal sum labeling.

### 2.4.6 Theorem
The subdivision of the star $K_{1,n}$ admits octagonal sum labeling.

**Proof**

Let $V(S(K_{1,n})) = \{v, v_1, u_1 : 1 \leq i \leq n\}$ and $E(S(K_{1,n})) = \{vv_i, vv_{i+1} : 1 \leq i \leq n\}$.

Define the labeling $f$ by

$$ f(v) = \begin{cases} \frac{1}{2}(i-1)(4i-3), & \text{if } i \text{ is odd} \\ \frac{1}{2}i(4i-7), & \text{if } i \text{ is even} \end{cases} \quad \text{for } 1 \leq i \leq k+1, $$

and $f(u_i) = (k + j)(4k + 4j - 3), \quad \text{for } 1 \leq j \leq m,$ and $f(w_i) = (k + m + l)(4k + 4m + 4l - 3) - f(v_{k+1}), \quad \text{for } 1 \leq l \leq n.$

We see that the induced edge labels are the first $2n$ octagonal numbers. Hence $S(K_{1,n})$ admits octagonal sum labeling.
2.5 Nonagonal Sum Labeling of Graphs

In this section, we prove that stars $S_n$, coconut trees, bistars or double stars $S_{m,n}$, the graphs $S_{m,n,k}$, combs $P_n \sqcap K_1$, subdivision graphs $S(K_{1,n})$ of the star $K_{1,n}$ admit nonagonal sum labeling.

2.5.1 Theorem

The star graph $K_{1,n}$ or $S_n$ admits nonagonal sum labeling.

**Proof**

Let $u$ be the apex vertex and let $u_1, u_2, \ldots, u_n$ be the pendant vertices of the star $S_n$.

Define $f$ by $f(u) = 0$ and $f(u_i) = \frac{1}{2}i(7i-5)$, $1 \leq i \leq n$.

We see that the induced edge labels obtained by the sum of the labels of the vertices are the first $n$ nonagonal numbers. Hence star graph $S_n$ admits nonagonal sum labeling.

2.5.2 Theorem

The coconut trees admit nonagonal sum labeling.

**Proof**

Let $u_1, u_2, \ldots, u_n$ be the vertices of a path having length $n-1$ and let $w_1, w_2, \ldots, w_k$ be the pendant vertices being adjacent with $u_1$.

Define the labeling $f$ by

$$f(u_1) = \begin{cases} \frac{1}{2} i(7i-5) & \text{if } i \text{ is odd} \\ \frac{1}{2} i(7i-12) & \text{if } i \text{ is even} \end{cases}, \text{ for } 1 \leq i \leq n$$

and $f(u_i) = \frac{1}{2}(i+j)(7i+7j-5)$, $1 \leq j \leq m$.

We see that the induced edge labels are the first $n+k-1$ nonagonal numbers. Hence Coconut trees admit nonagonal sum labeling.

2.5.3 Theorem

The bistar $S_{m,n}$ admit nonagonal sum labeling.

**Proof**

Let $V(S_{m,n}) = \{u, v, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and

$E(S_{m,n}) = \{uv, uu_i, vv_j : 1 \leq i \leq m, 1 \leq j \leq n\}$.

Define $f$ by $f(u) = 0$, $f(v) = 1$, $f(u_i) = \frac{1}{2}i(7i-5)$, $1 \leq i \leq m$, and $f(v_j) = \frac{1}{2}(m+j)(7m+7j-5)-1$, $1 \leq j \leq n$.

We see that the induced edge labels are the first $m + n + 1$ nonagonal numbers. Hence bistar $S_{m,n}$ admit nonagonal sum labeling.

2.5.4 Theorem

The graph $S_{m,n,k}$ admits nonagonal sum labeling.

**Proof**

Let $P_k : v_1, v_2, \ldots, v_k$ be a path of length $k$ with initial vertex $v_1$ and terminal vertex $v_{k+1}$.

Let $u_1, u_2, \ldots, u_m$ be the adjacent vertices to $v_1$ and $w_1, w_2, \ldots, w_n$ be the adjacent vertices to $v_{k+1}$.

Define $f$ by $f(v_1) = \begin{cases} \frac{1}{4}(i-1)(7i-5), & \text{if } i \text{ is odd} \\ \frac{1}{4}(7i-12), & \text{if } i \text{ is even} \end{cases}$ for $1 \leq i \leq k+1$,

and $f(v_j) = \frac{1}{2}(k+j)(7k+7j-5)$, $1 \leq j \leq m$, and $f(v_{k+1}) = \frac{1}{2}(k+m+1)(7k+7m+7l-5) - f(v_{k+1})$, for $1 \leq l \leq n$.

We see that the induced edge labels are the first $m + n + k$ nonagonal numbers. Hence the graph $S_{m,n,k}$ admits nonagonal sum labeling.

2.5.5 Theorem

The comb $P_n \sqcap K_1$ admits nonagonal sum labeling.

**Proof**

Let $P_n : u_1, u_2, \ldots, u_n$ be a path of length $n-1$ and let $w_1, w_2, \ldots, w_n$ be the pendant vertices adjacent to $u_1, u_2, \ldots, u_n$ respectively.

For $i = 1, 2, \ldots, n$ : define the labeling $f$ by

$$f(u_i) = \begin{cases} \frac{1}{4}(i-1)(7i-5), & \text{if } i \text{ is odd} \\ \frac{1}{4}(7i-12), & \text{if } i \text{ is even} \end{cases}$$

and $f(v_i) = \frac{1}{2}(n+i-1)(7n+7i-12) - f(u_i)$. 

Thus the induced edge labels are the first $2n - 1$ nonagonal numbers. Hence comb $P_n \square K_1$ admits nonagonal sum labeling.

### 2.5.6 Theorem 2.5.6

$S(K_{1,n})$ the subdivision of the star $K_{1,n}$ admits nonagonal sum labeling.

**Proof**

Let $V(S(K_{1,n})) = \{v, v_i, u_i : 1 \leq i \leq n\}$ and

$$E(S(K_{1,n})) = \{vv_i, v_ju_i : 1 \leq i \leq n\}.$$

Define the labeling $f$ by $f(v) = 0$, $f(v_i) = \frac{1}{2}i(7i - 5), 1 \leq i \leq n,$ and $f(u_i) = \frac{1}{2}(n+i)(7n+7i-5)-f(v_i), 1 \leq i \leq n.$

We see that the induced edge labels are the first $2n - 1$ nonagonal numbers. Hence $S(K_{1,n})$ admits nonagonal sum labeling.

### 2.6 Decagonal Sum Labeling of Graphs

In this section, we prove that stars $S_n$, coconut trees, bistars or double stars $S_{m,n}$, the graphs $S_{m,n,k}$, combs $P_n \square K_1$, subdivision graphs $S(K_{1,n})$ of the star $K_{1,n}$ admit decagonal sum labeling.

#### 2.6.1 Theorem

The star graph $K_{1,n}$ or $S_n$ admits decagonal sum labeling.

**Proof**

Let $u$ be the apex vertex and let $u_1, u_2, \ldots, u_n$ be the pendant vertices of the star $S_n$.

Define $f$ by $f(u) = 0$ and

$$f(u_i) = i(4i - 3), 1 \leq i \leq n.$$

We see that the induced edge labels are the first $n$ decagonal numbers. Hence star graph $S_n$ admits decagonal sum labeling.

#### 2.6.2 Theorem

The coconut trees admit decagonal sum labeling.

**Proof**

Let $u_1, u_2, \ldots, u_n$ be the vertices of a path having length $n - 1$ and let $w_1, w_2, \ldots, w_k$ be the pendant vertices being adjacent with $u_1$. Define the labeling $f$ by

$$f(u_i) = \begin{cases} \frac{1}{2}(i-1)(4i-3), & \text{if } i \text{ is odd} \\ \frac{1}{2}(4i-7), & \text{if } i \text{ is even} \end{cases}$$

for $1 \leq i \leq n$ and $f(w_j) = (a+j-1)(4a+4j-7), 1 \leq j \leq k.$

We see that the induced edge labels are the first $n + k - 1$ decagonal numbers. Hence Coconut trees admit decagonal sum labeling.

#### 2.6.3 Theorem

The bistar $S_{m,n}$ admit decagonal sum labeling.

**Proof**

Let $V(S_{m,n}) = \{u, v_u, v_v : 1 \leq i \leq m, 1 \leq j \leq n\}$ and

$$E(S_{m,n}) = \{uv, uu_i, vv_j : 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Define the labeling $f$ by

$$f(u_i) = \begin{cases} \frac{1}{2}(i-1)(4i-3), & \text{if } i \text{ is odd} \\ \frac{1}{2}(4i-7), & \text{if } i \text{ is even} \end{cases}$$

for $1 \leq i \leq m$ and $f(v_j) = (m+j)(4m+4j-3)-1, 1 \leq j \leq n.$

We see that the induced edge labels are the first $m + n + 1$ decagonal numbers. Hence bistar $S_{m,n}$ admit decagonal sum labeling.

#### 2.6.4 Theorem

The graph $S_{m,n,k}$ admits decagonal sum labeling.

**Proof**

Let $P_k : v_1, v_2, \ldots, v_{k+1}$ be a path of length $k$ with initial vertex $v_1$ and terminal vertex $v_{k+1}$.

Let $u_1, u_2, \ldots, u_m$ be the adjacent vertices to $v_1$ and $w_1, w_2, \ldots, w_n$ be the adjacent vertices to $v_{k+1}$.

Define the labeling $f$ by

$$f(v_i) = \begin{cases} \frac{1}{2}(i-1)(4i-3), & \text{if } i \text{ is odd} \\ \frac{1}{2}(4i-7), & \text{if } i \text{ is even} \end{cases}$$

for $1 \leq i \leq k + 1$, and $f(v_j) = (k+j)(4k+4j-3)-f(v_{i+1}), 1 \leq l \leq n.$
We see that the induced edge labels are the first $m + n + k$ decagonal numbers. Hence the graph $S_{m,n,k}$ admits decagonal sum labeling.

### 2.6.5 Theorem 2.6.5

The comb $P_n \square K_1$ admits decagonal sum labeling.

**Proof**

Let $P_n : u_1, u_2, \ldots, u_n$ be a path of length $n - 1$ and let $w_1, w_2, \ldots, w_n$ be the pendant vertices adjacent to $u_1, u_2, \ldots, u_n$ respectively.

For $i = 1, 2, \ldots, n$ : define the labeling $f$ by

$$f(u_i) = \begin{cases} \frac{1}{2}(i-1)(4i-3), & \text{if } i \text{ is odd} \\ \frac{1}{2}(4i-7), & \text{if } i \text{ is even} \end{cases}$$

and $f(w_i) = (n+i-1)(4n+4i-3) - f(u_i)$.

Thus the induced edge labels are the first $2n - 1$ decagonal numbers. Hence comb $P_n \square K_1$ admits decagonal sum labeling.

### 2.6.6 Theorem

$S(K_{1,n})$ the subdivision of the star $K_{1,n}$ admits decagonal sum labeling.

**Proof**

Let $V(S(K_{1,n})) = \{v, v_i, u_i : 1 \leq i \leq n\}$ and $E(S(K_{1,n})) = \{v v_i, v u_i : 1 \leq i \leq n\}$.

Define the labeling $f$ by $f(v) = 0$, $f(v_i) = i(4i-3)$, $1 \leq i \leq n$, and $f(u_i) = (n+i)(4n+4i-3) - f(v_i)$, $1 \leq i \leq n$.

We see that the induced edge labels are the first $2n$ decagonal numbers. Hence $S(K_{1,n})$ admits decagonal sum labeling.

### 3. References

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