Variations on the Sensitivity Conjecture

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Abstract

We present a selection of known as well as new variants of the Sensitivity Conjecture and point out some weaker versions that are also open.

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1 The Sensitivity Conjecture

Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function. Let $e_i \in \{0, 1\}^n$ denote an $n$-bit Boolean string whose $i$th bit is 1 and the rest of the bits are 0. On an input $x \in \{0, 1\}^n$, the $i$th bit is said to be sensitive for $f$ if $f(x \oplus e_i) \neq f(x)$, i.e., flipping the $i$th bit results in flipping the output of $f$. The sensitivity of $f$ on input $x$, denoted by $s(f, x)$, is the number of bits that are sensitive for $f$ on input $x$.

Definition 1.1. The sensitivity of a Boolean function $f$, denoted by $s(f)$, is the maximum value of $s(f, x)$ over all choices of $x$.

Study of sensitivity of Boolean functions originated from Cook and Dwork [10] and Reischuk [24]. They showed an $\Omega(\log s(f))$ lower bound on the number of steps required to compute a Boolean function $f$ on a CREW PRAM. A CREW PRAM, abbreviated from Consecutive Read Exclusive Write Parallel RAM, is a collection of synchronized processors computing in parallel with access to a shared memory with no write conflicts. The minimum number of steps required to compute a function $f$ on a CREW PRAM is denoted by CREW$(f)$. After Cook, Dwork and Reischuk introduced sensitivity, Nisan [20] found a way to modify the definition of sensitivity to characterize CREW$(f)$ exactly. He introduced a related notion called block sensitivity. A block $B$ is a subset of $[n] = \{1, 2, \ldots, n\}$. Let $e_B \in \{0, 1\}^n$ denote the characteristic vector of $B$, i.e., the $i$th bit of $e_B$ is 1 if $i \in B$ and 0 otherwise. We say that a block $B$ is sensitive for $f$ on $x$ if $f(x \oplus e_B) \neq f(x)$. The block sensitivity of $f$ on $x$, denoted by $bs(f, x)$, is the maximum number of pairwise disjoint sensitive blocks of $f$ on $x$. 
Definition 1.2. The block sensitivity of a Boolean function $f$, denoted by $bs(f)$, is the maximum possible value of $bs(f, x)$ over all choices of $x$.

Obviously, for every Boolean function $f$,
\[ s(f) \leq bs(f). \]

Nisan’s influential result [20] states that $CREW(f) = \Theta(\log bs(f))$ for every Boolean function $f$. Block sensitivity turned out to be polynomially related to a number of other complexity measures (see Section 2); however, to this day it remains unknown whether block sensitivity is bounded above by a polynomial in sensitivity. The following conjecture, known as the Sensitivity Conjecture, is due to Nisan and Szegedy [21].

Conjecture 1.3 (Sensitivity Conjecture, Nisan and Szegedy). For every Boolean function $f$,
\[ bs(f) \leq \text{poly}(s(f)). \]

The rest of the paper is organized as follows. In Section 2 we describe complexity measures of Boolean functions polynomially related to block sensitivity. In Section 3 we review progress on the Sensitivity Conjecture. In Section 4 we present alternative formulations of the Sensitivity Conjecture and point out weaker versions that are also open. Along the way we encounter important examples of Boolean functions. We present these functions in Section 5.

2 Measures Related to Block Sensitivity

Block sensitivity is polynomially related to several other complexity measures of Boolean functions, which we describe in this section.

A deterministic decision tree on $n$ variables $x_1, \ldots, x_n$ is a rooted binary tree, whose internal nodes are labeled with variables, and the leaves are labeled 0 or 1. Edges are also labeled 0 or 1. To evaluate such a tree on input $x$, start at the root and query the corresponding variable, then move to the next node along the edge labeled with the outcome of the query. Repeat until a leaf is reached, at which point the label of the leaf is declared to be the output of the evaluation. A decision tree computes a Boolean function $f$ if it agrees with $f$ on all inputs.

Definition 2.1. The deterministic decision tree complexity of a Boolean function $f$, denoted by $D(f)$, is the depth of a minimum-depth decision tree that computes $f$.

One way to extend the deterministic decision tree model is to add randomness to the computation. In this extended model, each node $v$ has an associated bias $p_v \in [0, 1]$. Evaluation proceeds as before, except when deciding which edge to follow after querying $x_i$ at node $v$, we follow an edge corresponding to the outcome of the query with probability $p_v$ and the other edge with probability $1 - p_v$. 
Definition 2.2. The \textit{bounded-error randomized decision tree complexity} of a Boolean function \(f\), denoted by \(R_2(f)\), is the depth of a minimum-depth randomized decision tree computing \(f\) with probability at least \(2/3\) for all \(x \in \{0, 1\}^n\).

A \textit{certificate} of a Boolean function \(f\) on input \(x\), is a subset \(S \subset [n]\), such that \((\forall y \in \{0, 1\}^n)(x|_S = y|_S \Rightarrow f(x) = f(y))\). The \textit{certificate complexity} of a Boolean function \(f\) on input \(x\), denoted by \(C(f, x)\), is the minimum size of a certificate of \(f\) on \(x\).

Definition 2.3. The \textit{certificate complexity} of a Boolean function \(f\), also known as \textit{non-deterministic decision tree complexity} and denoted by \(C(f)\), is the maximum of \(C(f, x)\) over all choices of \(x\).

Definition 2.4. A polynomial \(p : \mathbb{R}^n \rightarrow \mathbb{R}\) \textit{represents} \(f\) if 
\[(\forall x \in \{0, 1\}^n)(p(x) = f(x)).\]
The \textit{degree} of a Boolean function \(f\), denoted by \(\text{deg}(f)\), is the degree of the unique multilinear polynomial that represents \(f\).

Definition 2.5. A polynomial \(p : \mathbb{R}^n \rightarrow \mathbb{R}\) \textit{approximately represents} \(f\) if 
\[(\forall x \in \{0, 1\}^n)(|p(x) - f(x)| < 1/3).\]
The \textit{approximate degree} of a Boolean function \(f\), denoted by \(\widetilde{\text{deg}}(f)\), is the minimum degree of a polynomial that approximately represents \(f\).

We denote the \textit{quantum decision tree complexity with bounded error} of a Boolean function \(f\) by \(Q_2(f)\). Discussion of quantum complexity is outside the scope of this note. For an introduction to quantum complexity see a survey by de Wolf [11].

Definition 2.6. Complexity measures \(A\) and \(B\) are \textit{polynomially related} if
\[(\forall f)[A(f) \leq \text{poly}(B(f))] \quad \text{and} \quad B(f) \leq \text{poly}(A(f))].\]

Theorem 2.7. The following complexity measures of Boolean functions are all polynomially related:
\[bs(f), \ D(f), \ R_2(f), \ C(f), \ \text{deg}(f), \ \widetilde{\text{deg}}(f), \ Q_2(f).\]

Table I presents a quick summary of the known polynomial relations between complexity measures that play a prominent role in this note. An entry from the table shows the smallest known exponent of a polynomial in the corresponding measure from the column that gives an upper bound on the corresponding measure from the row, as well as the exponent of the biggest known gap between two measures. An entry also contains references to papers, where the result can be found. References of the form [*] indicate that the result is immediate from the definitions of complexity measures. For example, entry 3 [19] \((\log_3 6 [22])\) in the second row and third column means that \(D(f) = O(\text{deg}(f)^3)\) (see [19]) and
there is a Boolean function $f$, for which $D(f) = \Omega(\deg(f) \log 3)$ (see [22]). For a thorough treatment of polynomial relation between various complexity measures of Boolean functions (including variants of quantum query complexity) see a survey by Buhrman and de Wolf [6].

Using Theorem 2.7, one immediately obtains many equivalent formulations of the “sensitivity versus block sensitivity” conjecture. The purpose of this note is to point out some nontrivial variations on this conjecture that, to our knowledge, have not been stated explicitly in the literature. We also propose several weaker versions of the Sensitivity Conjecture, which might provide starting points.

We introduce the following pictorial notation to indicate relations between the statements appearing in this note and the Sensitivity Conjecture.

- a consequence of the Sensitivity Conjecture.
- implies the Sensitivity Conjecture, but the reverse implication is not known. These might be good candidates for refutation.
- equivalent to the Sensitivity Conjecture.
- conditionally equivalent to the Sensitivity Conjecture.

### Table 1: Known polynomial relations between various complexity measures

|       | $bs(f)$ | $D(f)$ | $\deg(f)$ | $C(f)$ |
|-------|---------|--------|-----------|--------|
| $bs(f)$ | 1(1)    | 1 [*](1 [*]) | 2 $[21](\log_3 6 [22])$ | 1 [*](1 [*]) |
| $D(f)$  | 3 $[3, 20](2 [1])$ | 1(1)    | 3 $[19](\log_3 6 [22])$ | 2 $[3](2 [1])$ |
| $\deg(f)$ | 3 $[3, 20](2 [1])$ | 1 [*](1 [*]) | 1(1)    | 2 $[3](2 [1])$ |
| $C(f)$  | 2 $[20](\log_4 5 [11])$ | 1 [*](1 [*]) | 3 $[19](\log_3 6 [22])$ | 1(1)    |

Table 1: Known polynomial relations between various complexity measures. An entry in the table shows the polynomial upper bound on the measure from a row in terms of a measure from a column and the biggest known gap between two measures. The references to the papers, where the corresponding results can be found, are given in square brackets.

### 3 Progress on the Sensitivity Conjecture

The progress on the Sensitivity Conjecture has been limited. Simon [29] proved that for any Boolean function that depends on all $n$ variables, sensitivity is at least $\frac{1}{2} \log n - \frac{1}{2} \log \log n + \frac{1}{2}$. An immediate corollary is that for any Boolean function $f$, $bs(f) = O(s(f)4s(f))$. Kenyon and Kutin [16] proved that sen-

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1The construction appeared in [5] before the notion of block sensitivity was introduced. The analysis of $C(f)$ and $bs(f)$ of the example appears in [11].
2The result is due to [4, 15, 30].
3The example is due to Kushilevitz and appears in footnote 1 on p. 560 of the Nisan-Wigderson paper [22]. See Example 5.3 in Section 5 of this paper.
4These gaps are demonstrated by a commonly known AND-of-ORs function, see Example 5.2 in Section 5 of this paper.
tivity is polynomially related to $\ell$-block sensitivity for any constant $\ell$ ($\ell$-block sensitivity considers only the sensitive blocks of size at most $\ell$). The best known upper bound on block sensitivity in terms of sensitivity is exponential and appears in the work of Kenyon and Kutin [16] on $\ell$-block sensitivity:
\[
bs(f) \leq \left(\frac{2}{\sqrt{2\pi}}\right) e^{s(f)} \sqrt{s(f)}.
\]

In the 80s, Rubinstein [26] exhibited a function with sensitivity $\Theta(\sqrt{n})$ and block sensitivity $\Theta(n)$ (see Example 5.1 in Section 5). Gaps between sensitivity and some other complexity measures are surveyed by Buhrman and de Wolf [6].

In the light of Rubinstein’s example, the best possible upper bound on block sensitivity in terms of sensitivity could be quadratic. Nisan and Szegedy [21] asked the following question:

**Question 3.1.** Is $bs(f) = O(s(f)^2)$ for every Boolean function $f$?

A Boolean function $f : \{0,1\}^n \to \{0,1\}$ is invariant under a permutation $\sigma : [n] \to [n]$, if for any string $x$, $f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. The set of all permutations, under which $f$ is invariant, forms a group, called the invariance group of $f$. A Boolean function is said to be transitive if its invariance group $\Gamma$ is transitive, i.e., for each $i, j \in [n]$ there is a permutation $\sigma \in \Gamma$ such that $\sigma(i) = j$.

Turán [31] proved that any property of $n$-vertex graphs (viewed as a Boolean function on $\binom{n}{2}$ variables) has sensitivity $\Omega(n)$. Turán asked if every transitive function on $n$ variables has sensitivity at least $\Omega(\sqrt{n})$. Chakraborty [7] answered this question in the negative by constructing a transitive function with sensitivity $\Theta(n^{1/3})$ and block sensitivity $\Theta(n^{2/3})$ (see Example 5.5 in Section 5). We propose the following modification of Turán’s question:

**Question 3.2.** If $f : \{0,1\}^n \to \{0,1\}$ is transitive and $f(0) \neq f(1)$, is then $s(f) = \Omega(\sqrt{n})$?

An example of a different behavior of transitive Boolean functions with the property $f(0) \neq f(1)$ is due to Rivest and Vuillemin [25]. They proved that if $n$ is a prime power, and $f(0) \neq f(1)$, then $D(f) = n$. From their proof it can be immediately inferred that in fact $\deg(f) = n$. A conjecture appearing in the Gotsman-Linial paper [13] states that $\deg(f) = O(s(f)^2)$, which, if true, would answer Question 3.2 positively for $n$ that are prime powers.

### 4 Sensitivity vs Other Complexity Measures

Unlike the complexity measures mentioned in Section 2, the complexity measures in this section are not polynomially related to block sensitivity and yet proving a polynomial relation of these measures to sensitivity turns out to be equivalent to proving a polynomial relation between block sensitivity and sensitivity, i.e., the Sensitivity Conjecture itself.
Let $F(x,y)$ be a Boolean function. Consider a setting in which Alice has a Boolean string $x$ and Bob has a Boolean string $y$, and their goal is to compute the value of $F(x,y)$ by communicating as few bits as possible. Alice and Bob agree on a communication protocol beforehand. Having received inputs, they communicate in accordance with the protocol. At the end of the communication one of the parties declares the value of the function $F$. The cost of the protocol is the number of bits exchanged on the worst-case input.

**Definition 4.1.** The deterministic communication complexity of $F$, denoted by $DC(F)$, is the cost of an optimal communication protocol computing $F$.

For more information on communication complexity see [17]. Given a Boolean function on $n$ variables, we will typically consider $F(x,y) = f(x \circ y)$ where $\circ$ is bitwise $\wedge, \lor$ or $\oplus$.

### 4.1 Log-rank vs Sensitivity

For a Boolean function $F$ of two arguments, $\text{rank}(F)$ denotes the rank of the corresponding matrix $M_{x,y} = F(x,y)$ over $\mathbb{R}$.

In this section we present some implications of a recent result by Sherstov. It appears as Theorem 6.4 in [27].

**Theorem 4.1.1 (Sherstov).** For every Boolean function $f$,

$$\max\{\log \text{rank}(f(x \wedge y)), \log \text{rank}(f(x \lor y))\} = \Omega(\deg(f)).$$

☐ **Conjecture 4.1.2.** For every Boolean function $f$,

$$\log \text{rank}[f(x \wedge y)] \leq \text{poly}(s(f)).$$

**Theorem 4.1.3.** Sensitivity Conjecture $\iff$ Conjecture 4.1.2.

**Proof.**

$\Rightarrow$ Beals et al. [3] showed that $D(f) \leq C(f)bs(f)$ and Nisan [20] proved that $C(f) \leq bs(f)^2$. Therefore, we get $D(f) \leq bs(f)^3$ as a simple corollary. It is easy to see that $DC(f(x \wedge y)) \leq 2D(f)$. Finally, by a classical result in communication complexity due to Mehlhorn and Schmidt [18],

$$(\forall F)(\log \text{rank}(F(x,y)) \leq DC(F)).$$

$\Leftarrow$ For a Boolean function $f$, define $g(x) = f(\neg x)$. Clearly, $s(g) = s(f)$ and $\log \text{rank}(g(x \wedge y)) = \log \text{rank}(f(x \lor y))$. Applying the hypothesis to both $g$ and $f$, we get that

$$\max\{\log \text{rank}(f(x \lor y)), \log \text{rank}(f(x \wedge y))\} \leq \text{poly}(s(f)).$$

It follows that $\deg(f) \leq \text{poly}(s(f))$ by Theorem 4.1.1. This completes the proof, since $bs(f) \leq 2\deg(f)^2$ [21].

☐ **Conjecture 4.1.4.** For every Boolean function $f$,

$$\log \text{rank}[f(x \oplus y)] \leq \text{poly}(s(f)).$$
Corollary 4.1.5. Sensitivity Conjecture $\iff$ Conjecture 4.1.4.

Proof.

$\Rightarrow$ Similar to the same direction in Theorem 4.1.3.

$\Leftarrow$ For a Boolean function $f$, define $F(x, y) = f(x \land y)$. It is easy to check that $s(f) \leq s(F) \leq 2s(f)$, and also that

$$\text{rank}(f(x \land y)) \leq \text{rank}(F(x \oplus x', y \oplus y')).$$

The result now follows from Theorem 4.1.3. \hfill $\square$

Definition 4.1.6. The sign-rank of Boolean function $F$ of two arguments, denoted by $\text{rank}_\pm$, is defined as

$$\text{rank}_\pm(F) = \min_M \{ \text{rank}(M) \mid (\forall x, y) \left( (-1)^{F(x, y)} M_{x,y} > 0 \right) \}.$$

The notion of sign-rank was introduced by Paturi and Simon [23] to give a characterization of the unbounded error probabilistic communication complexity.

Since $\text{rank}_\pm(F) \leq \text{rank}(F)$ for every $F$, we propose a possibly weaker version of Conjecture 4.1.4 stated for the sign-rank.

Conjecture 4.1.7. For every Boolean function $f$,

$$\log \text{rank}_\pm(f(x \oplus y)) \leq \text{poly}(s(f)).$$

Question 4.1.8. Does Conjecture 4.1.7 imply Conjecture 4.1.4? I.e., is Conjecture 4.1.7 equivalent to the Sensitivity Conjecture?

4.2 Parity Decision Trees

Parity decision trees are similar to decision trees; the difference is that instead of querying only one variable at a time, one may query the sum modulo 2 of an arbitrary subset of variables (see [32] for a brief introduction to parity decision trees). The parity decision tree complexity of a Boolean function $f$ is denoted by $D_\oplus(f)$. Obviously, $D_\oplus(f) \leq D(f)$. In this section, we explore the relationship between $D_\oplus$ and sensitivity. Note that parity decision trees are strictly more powerful than decision trees. For instance, parity of $n$ bits requires a decision tree of depth $n$ whereas a parity decision tree of depth 1 suffices.

Conjecture 4.2.1. For every Boolean function $f$, $D_\oplus(f) \leq \text{poly}(s(f))$.

This seemingly weaker conjecture is actually equivalent to the Sensitivity Conjecture.

Theorem 4.2.2. Sensitivity Conjecture $\iff$ Conjecture 4.2.1.
Proof.

⇒ $D_\oplus(f) \leq D(f) \leq bs(f)^3$.

⇐ Since $\log \text{rank}(f(x \oplus y)) \leq DC(f(x \oplus y)) \leq 2D_\oplus(f)$, the proof follows from Corollary 4.1.5.

Next we present a quadratic gap between $D_\oplus$ and sensitivity. Consider the Boolean function $h(x) = \bigwedge_{i=1}^{2^k} \bigvee_{j=1}^{2^k} x_{ij}$, on $n = 2^{2^k}$ variables. Clearly, $s(h) = 2^k = \sqrt{n}$. To see that $D_\oplus(h) = n$, consider the mod 2 degree of $h$ defined as:

**Definition 4.2.3.** The mod 2 degree of a Boolean function $f$, denoted by $\deg_\oplus(f)$, is the degree of the unique multilinear polynomial over $\mathbb{F}_2$ (the field of two elements) that represents $f$.

Observe that the OR function (and consequently the AND function) has full mod 2 degree. It follows that $h$ has full mod 2 degree, which shows that $D_\oplus(h) = n$, since for any Boolean function $f$, $\deg_\oplus(f) \leq D_\oplus(f)$.

Similar to the question stated in the survey by Buhrman and de Wolf [6] whether $D(f) \leq O(bs(f)^2)$, we ask the following:

**Question 4.2.4.** Is $D_\oplus(f) = O(bs(f)^2)$?

**Remark 4.2.5.** A positive answer to the above question would imply that $\deg(f) \leq bs(f)^2$, improving the current best known bound $\deg(f) \leq bs(f)^3$ (see [3]).

### 4.3 Analytic Setting

In the previous sections, we considered Boolean functions from $\{0,1\}^n$ to $\{0,1\}$. For the purpose of studying the Fourier spectrum of Boolean functions, it is convenient to use range $\{+1,-1\}$, replacing 0 with +1 and 1 with −1. This operation preserves the complexity measures up to an additive constant. For a brief introduction to Fourier Analysis on the Boolean cube, see, for instance, the survey by de Wolf [12].

**Definition 4.3.1.** For $S \subseteq [n]$, the character $\chi_S$ is defined as

$$\chi_S(x) = (-1)^{\sum_{i \in S} x_i}.$$ 

**Definition 4.3.2.** The Fourier coefficient of $f$ corresponding to $S$ is defined as

$$\hat{f}(S) := \mathbb{E}_{x \in \{0,1\}^n} [f(x)\chi_S(x)].$$

**Σ Conjecture 4.3.3.** For every Boolean function $f$,

$$\min_{S: \hat{f}(S) \neq 0} |\hat{f}(S)| \geq 2^{-\text{poly}(s(f))}.$$ 

**Theorem 4.3.4.** Sensitivity Conjecture $\iff$ Conjecture 4.3.3
Proof.
⇒ It is easy to see that if \( f \) has a decision tree of depth \( d \), then all non-zero Fourier coefficients are integer multiples of \( 2^{-d} \). The result follows from \( D(f) \leq bs(f)^3 \), as stated in the proof of Theorem 4.1.3.

\( \leq \) Let \( \alpha = \min_{S, \hat{f}(S) \neq 0} |\hat{f}(S)| \). Since \( \sum_{S} \hat{f}(S)^2 = 1 \) (Parseval’s Identity), the number of non-zero Fourier coefficients is at most \( \alpha^{-2} \). Consider matrix \( M \) with entries \( M_{x,y} = f(x \oplus y) \). It is easy to check that for each \( S \subseteq [n] \), the vector \( (\chi_S(y))_{y \in \{0,1\}^n} \) is an eigenvector to \( M \) with a corresponding eigenvalue \( 2^n \hat{f}(S) \). Since the \( \chi_S \) form an orthogonal set of vectors, the \( 2^n \hat{f}(S) \) are all the eigenvalues of \( M \).

Hence, \( \alpha \geq 2^{-\text{poly}(s(f))} \) implies that rank of \( f(x \oplus y) \) is at most \( 2^{\text{poly}(s(f))} \).

The proof is complete by Corollary 1.1.5.

The following consequence of the Sensitivity Conjecture appears to be open.

\[ \text{Conjecture 4.3.5.} \quad \text{For every Boolean function } f, \]
\[ \sum_{S} |\hat{f}(S)| \leq 2^{\text{poly}(s(f))}. \]

**Definition 4.3.6.** Let \( F(x,y) \) be a Boolean function. Suppose Alice has a Boolean string \( x \) and Bob has a Boolean string \( y \). The bounded-error randomized communication complexity with shared randomness of \( F \), denoted by \( RC_2(F) \), is the least cost of a randomized protocol that computes \( F \) correctly with probability at least 2/3 on every input, when Alice and Bob are given the same random bits.

Next we prove that Conjecture 4.3.5 is equivalent to the Sensitivity Conjecture under the following variant of the Log-rank Conjecture due to Grolmusz.

**Conjecture 4.3.7 (Grolmusz).** Let \( F : \{0,1\}^{m+n} \rightarrow \{-1,1\} \). Suppose Alice has \( x \in \{0,1\}^m \) and Bob has \( y \in \{0,1\}^n \), then:

\[ RC_2(F(x,y)) \leq \text{poly}(\log \sum_{S \subseteq [m+n]} |\hat{F}(S)|). \]

To prove the equivalence we will need the following result by Sherstov (see [27], Theorem 5.1).

**Theorem 4.3.8 (Sherstov).** Let \( F_1(x,y) := f(x \land y) \) and \( F_2(x,y) := f(x \lor y) \), then
\[ \max\{RC_2(F_1), RC_2(F_2)\} = \Omega(bs(f)^{1/4}). \]

**Theorem 4.3.9.** Conjecture 4.3.7 \( \Rightarrow \) (Sensitivity Conjecture \( \iff \) Conjecture 4.3.5).
Proof. Assume Conjecture 4.3.7. Now, we want to prove that Sensitivity Conjecture $\iff$ Conjecture 4.3.5.

$\Rightarrow$ 1. $= \sum_{S} \hat{f}(S)^2$ by Parseval’s Identity

$\geq (\min \{\hat{f}(S)\}) (\#\{S \mid \hat{f}(S) \neq 0\})$ by Theorem 4.3.4

$\geq 2^{-\text{poly}(s(f))} (\sum_{S} |f(S)|)$ by Theorem 4.3.4

$\Leftarrow$ Consider two Boolean functions $F_1$ and $F_2$ on $2n$ variables defined as in Theorem 4.3.8. It is easy to check that $s(f) \leq s(F_1)$ and $s(F_2) \leq 2s(f)$. Applying Conjecture 4.3.5 to both $F_1$ and $F_2$, we get: $\log \sum |\hat{F}_1(S)| \leq \text{poly}(s(f))$ and $\log \sum |\hat{F}_2(S)| \leq \text{poly}(s(f))$. Now $bs(f) \leq \text{poly}(s(f))$ follows from Theorem 4.3.8 assuming Conjecture 4.3.7.

4.4 Shi’s Characterization of Sensitivity

In this section we present some applications of Shi’s work [28], which contains an interesting characterization of the sensitivity of Boolean functions.

A polynomial representing a Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ (see Section 2) provides a multilinear extension of $f$ from $\mathbb{R}^n$ to $\mathbb{R}$, which (abusing notation) we denote by the same letter $f$.

Let $\ell = (a, b)$ denote the line segment in $[0, 1]^n$ that starts at point $a$ and ends at point $b$.

Definition 4.4.1. The linear restriction of $f$ on $\ell = (a, b)$, $f_\ell : [0, 1] \rightarrow \mathbb{R}$, is defined as

$f_\ell(t) := f((1-t)a + tb), \ \forall t \in [0, 1]$. 

Denote the supremum norm of a function $g : [0, 1] \rightarrow \mathbb{R}$ by $||g||_\infty = \sup_{t \in [0, 1]} |g(t)|$.

Let $g’$ denote the first derivative of $g$.

Theorem 4.4.2 (Shi [28]). For every Boolean function $f$, $s(f) = \sup_\ell ||f_\ell’||_\infty$.

Proof. It is easy to check that it suffices to consider the lines that join two points of the Boolean cube.

For $x \in [0, 1]^n$, let $x^{(i,1)} (x^{(i,0)})$ denote a vector whose $i^{th}$ coordinate is 1 (0) and the other coordinates match with those of $x$. Let $a, b \in \{0, 1\}^n$ and $\ell = (a, b)$ be the line joining $a$ and $b$.

$$f_\ell’(t) = \sum_{i=1}^{n} (b_i - a_i) \cdot \frac{\partial f}{\partial x_i}((1-t)a + tb).$$

Since $f$ is multilinear, we have:

$$\frac{\partial f}{\partial x_i}(x) = f(x^{(i,1)}) - f(x^{(i,0)}).$$

Thus we have:

$$|f_\ell’(t)| \leq \mathbb{E}_{p \in D_t} \left[ \sum_{i=1}^{n} |f(p^{(i,1)}) - f(p^{(i,0)})| \right], \quad (1)$$

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where $D_t$ denotes the following probability distribution on Boolean cube: for each $k$, $\Pr(p_k = 1) = (1-t)a_k + b_k$. Notice that the right hand side of $H$ is at most $s(f)$.

For the other direction let $a \in \{0, 1\}^n$ and $b$ be obtained from $a$ by flipping each bit. It is easy to check that:

$$f'_I(0) = \sum |f(a \oplus e_i) - f(a)| = s(f,a).$$

Choosing a vector $a$ with maximum sensitivity completes the proof. \qed

Combining Theorem 4.4.2 with Conjecture 4.3.3 puts the original “sensitivity versus block sensitivity” problem into an analytic setting.

**Definition 4.4.3.** The approximate degree of linear restrictions of a Boolean function $f$ is defined as follows:

$$\overline{\deg}(f) = \max \left\{ \min \{ \deg(g) \mid g \in \mathbb{R}^t, \| f_t - g \|_\infty \leq 1/6 \} \right\}.$$

**Theorem 4.4.4** (Shi [28]). The complexity measures $\overline{\deg}(f)$ and $s(f)$ are polynomially related.

Observe that, unlike all previous equivalence results, Theorem 4.4.4 gives a complexity measure polynomially related to $s(f)$ rather than $bs(f)$. It follows that the following conjecture is equivalent to the Sensitivity Conjecture.

**Conjecture 4.4.5** (Shi [28]). For every Boolean function $f$, $$\widetilde{\deg}(f) \leq \text{poly} \left( \overline{\deg}(f) \right).$$

### 4.5 Subgraphs of the $n$-cube

Let $Q_n$ denote the $n$-cube graph, i.e., $V(Q_n) = \{0, 1\}^n$ and two vertices are adjacent if the corresponding vectors differ in exactly one position. Denote the maximum degree of graph $G$ by $\Delta(G)$. For a subgraph $H$ of a graph $G$ define $\Gamma(H) = \max\{ \Delta(H), \Delta(G - H) \}$. Gotsman and Linial [13] proved the following remarkable equivalence.

**Theorem 4.5.1** (Gotsman and Linial [13]). The following are equivalent for any monotone function $h : \mathbb{N} \rightarrow \mathbb{R}$:

**A** For any induced subgraph $G$ of $Q_n$ with $|V(G)| \neq 2^{n-1}$ we have $\Gamma(G) \geq h(n)$.

**B** For any Boolean function $f$ we have $s(f) \geq h(\deg(f))$.

**Proof.**

Statement **B** is equivalent to the following:

**B’** For any Boolean function $f$ with $\deg(f) = n$ we have $s(f) \geq h(n)$.
Clearly, $B$ implies $B'$. To prove the reverse implication, let $f$ be a Boolean function of degree $d$. Fix a monomial of degree $d$ of the representing polynomial of $f$. Without loss of generality we may assume the monomial is $x_1 \cdots x_d$. Define $g(x_1, \ldots, x_d) := f(x_1, \ldots, x_d, 0, \ldots, 0)$. Then, $s(f) \geq s(g) \geq h(d)$, as desired.

$A \Rightarrow B'$ We prove the contrapositive. Given a Boolean function $f$ with $s(f) < h(n)$, consider an induced subgraph $G$ of $Q_n$ with

$$V(G) = \{x \in \{0,1\}^n \mid f(x)p(x) = +1\},$$

where $p(x) = (-1)\sum x_i$ is the parity function (as in Section 4.3 we take the range of Boolean functions to be $\{+1,-1\}$). Observe that $f(I) = \hat{f}(n-I)$ for any subset $I \subseteq [n]$. Hence, $\hat{f}(\emptyset) = \hat{f}([n]) \neq 0$, since $\deg(f) = n$. Straight from the definition of Fourier coefficients, $\hat{f}(\emptyset) = \mathbb{E}_x[f(x)]$, so $|V(G)| \neq 2^{n-1}$. Furthermore, $s(fp, x) = n - s(f, x)$ and $deg_G(x) = n - s(fp, x) = s(f, x)$. Thus, $\Gamma(G) < h(n)$.

$A \Leftrightarrow B'$ Observe that the steps in the proof of $A \Rightarrow B'$ are reversible. \qed

The proof of Theorem 4.5.1 translates a Boolean function with a polynomial gap between degree and sensitivity into a graph with the same polynomial gap between $\Gamma$ and $n$, and vice versa. For example, observe that Rubinstein’s function (see Example 5.1 in Section 5) has sensitivity $\Theta(n)$ and full degree, which can be easily verified by a direct computation of $\hat{f}([n])$. Therefore, Rubinstein’s function can be used to obtain a graph $G$ with the surprising property $\Gamma(G) = \Theta(\sqrt{n})$. Chung et al. [9] independently constructed a graph $G$ with $\Gamma(G) < \sqrt{n} + 1$. Their example can be also obtained from Theorem 4.5.1 by applying the reduction in the proof of $A \Rightarrow B'$ to the AND-of-ORs function (see Example 5.2 in Section 5), but note that the Gotsman-Linial theorem was not available at the time when Chung et. al. gave their construction.

It immediately follows that the following conjecture is equivalent to the Sensitivity Conjecture, by taking $h$ to be an inverse polynomial in Theorem 4.5.1.

**Conjecture 4.5.2.** For any induced subgraph $G$ of $Q_n$ such that $|V(G)| \neq 2^{n-1}$ we have $\Gamma(G) \geq \text{poly}^{-1}(n)$.

4.6 Two-colorings of Integer Lattices

Two points $a, b \in \mathbb{Z}^d$ are called neighbors if $||a - b||_2 = 1$. A two-coloring $C$ of $\mathbb{Z}^d$ with colors red and blue is non-trivial if the origin is colored red, and there is a point colored blue on each of the coordinate axes. Sensitivity of a point $a \in \mathbb{Z}^d$ under coloring $C$, denoted by $S(a, C)$, is the number of neighbors of $a$ that are colored differently from $a$.

**Definition 4.6.1.** The sensitivity of a coloring is defined by $S(C) = \max_a S(a, C)$.

Aaronson [2] stated the following question, a positive answer to which would imply the Sensitivity Conjecture. For completeness, we also present a reduction.
Question 4.6.2 (Aaronson). Does every non-trivial coloring of \( \mathbb{Z}^d \) have sensitivity at least \( d^{Ω(1)} \)?

Claim 4.6.3 (Aaronson). A positive answer to Question 4.6.2 implies that Sensitivity Conjecture.

Proof. Given a Boolean function \( f \) on \( n \) variables, let \( x \) be an input, on which \( f \) achieves the highest block sensitivity \( b = bs(f) \). Let \( S_1, \ldots, S_b \) be pairwise disjoint sensitive blocks of \( f \) on \( x \), and let \( R = [n] - (\bigcup_i S_i) \). Let \( γ_i : \mathbb{Z} \to \{0, 1\}^{|S_i|} \) represent a Gray code with \( γ_i(0) = x|_{S_i} \). Consider the following mapping \( φ : \mathbb{Z}^b \to \{0, 1\}^n \): a point \( a \in \mathbb{Z}^b \) is mapped to a Boolean string \( y \in \{0, 1\}^n \) with \( y|_{S_i} = γ_i(a_i) \) and \( y|_R = x|_R \). Finally, obtain coloring \( C \) of \( \mathbb{Z}^b \) by composing \( f \) with \( φ \). Clearly, \( C \) is non-trivial and \( s(C) \leq 2s(f) \), hence

\[
bs(f) = b \leq \text{poly}(s(C)) \leq \text{poly}(s(f)).
\]

5 Some Boolean Functions

In this section we present some interesting examples of Boolean functions. They provide lower or upper bounds for various complexity measures, and some of them appear in more than one context.

The following function was exhibited by Rubinstein [26]. It was discussed in Section 3 and Section 4.5 of this note.

Example 5.1 (Rubinstein’s function). Rubinstein’s function is defined on \( n = k^2 \) variables, which are divided into \( k \) blocks with \( k \) variables each. The value of the function is 1 if there is at least one block with exactly two consecutive 1s in it, and it is 0 otherwise.

Block sensitivity of Rubinstein’s function on \( k^2 \) variables is \( Θ(k^2) \) (hence, certificate complexity and decision tree complexity is also \( Θ(k^2) \)) and sensitivity is \( Θ(k) \). It has full degree as can be verified by a direct computation of Fourier coefficient of the set \([k^2]\).

The following folklore example was discussed in Section 2 and Section 4.5 of this paper.

Example 5.2 (AND-of-ORs function). AND-of-ORs function is defined on \( k \) blocks of \( k \) variables each:

\[
f(x_{11}, \ldots, x_{kk}) = \bigwedge_{i=1}^{k} \bigvee_{j=1}^{k} x_{ij}.
\]

The block sensitivity and sensitivity of AND-of-ORs function on \( k^2 \) variables is \( k \). AND-of-ORs has full degree and hence its decision tree complexity is also \( k^2 \). The certificate complexity of AND-of-ORs function is \( k \).

Definition 5.3. For a Boolean function \( f : \{0, 1\}^m \to \{0, 1\} \) and a Boolean function \( g : \{0, 1\}^n \to \{0, 1\} \), we define the composition function \( f \circ g \) on \( mn \) variables as follows:

\[
f \circ g(x_{11}, \ldots, x_{mn}) = f(g(x_{11}, \ldots, x_{1n}), \ldots, g(x_{m1}, \ldots, x_{mn})).
\]
Kushilevitz exhibited a function $f$ that provides the largest gap in the exponent of a polynomial in $\deg(f)$ that gives an upper bound on $bs(f)$. Never published by Kushilevitz, the function appears in footnote 1 of the Nisan-Wigderson paper [22]. It was discussed in Section 2 of this paper.

**Example 5.4** (Kushilevitz’s function). Define an auxiliary function $h$ on 6 variables:

$$h(z_1, \ldots, z_6) = \sum_i z_i - \sum_{ij} z_i z_j + z_1 z_3 z_4 + z_1 z_2 z_5 + z_1 z_4 z_5 + z_2 z_3 z_4 +$$

$$z_2 z_5 z_6 + z_1 z_2 z_6 + z_1 z_3 z_6 + z_2 z_4 z_6 + z_3 z_5 z_6 + z_4 z_5 z_6.$$

Kushilevitz’s function is defined as $h \circ h \circ \ldots \circ h$.

Observe that the auxiliary function $h$ on 6 variables in Kushilevitz’s example has degree 3 and full sensitivity on the 0 input. Let the function $f$ be obtained by composing $h$ with itself $k$ times. It is defined on $n = 6^k$ variables and has full sensitivity, block sensitivity, decision tree complexity and certificate complexity.

Degree of $f$ is $3^k = n^{\log_6 3}$.

The following function was constructed by Chakraborty [7]. It was discussed in Section 3 of this note.

**Example 5.5** (Chakraborty’s function). Define an auxiliary function $h$ on $k^2$ variables by a regular expression:

$$h(z_{11}, \ldots, z_{kk}) = 1 \iff z \in 11^{k-2}(11111(0 + 1)^{k-5})^k 11111(0 + 1)^{k-8}111.$$

Chakraborty’s function $f$ on $n \geq k^2$ variables is defined as follows:

$$f(x_0, \ldots, x_{n-1}) = 1 \iff (\exists i \in [n]) (g(x_i, x_{(i+1)}), \ldots, x_{(i+k^2)}) = 1),$$

where indices in the arguments of function $g$ are taken modulo $n$.

Chakraborty shows that for $n = k^3$ his function has sensitivity $\Theta(n^{1/3})$ [7], block sensitivity $\Theta(n^{2/3})$ and certificate complexity $\Theta(n^{2/3})$ [8].

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**References**

[1] Aaronson, S. Quantum certificate complexity. In *CCC 2003: Proc. of the 18th IEEE CCC* (2003), pp. 171–178.

[2] Aaronson, S. The “sensitivity” of 2-colorings of the $d$-dimensional integer lattice. [http://mathoverflow.net/questions/314827](http://mathoverflow.net/questions/314827), july 2010.
[3] Beals, R., Buhrman, H., Cleve, R., Mosca, M., and de Wolf, R. Quantum lower bounds by polynomials. *J. of the ACM* 48, 4 (2001), 778–797.

[4] Blum, M., and Impagliazzo, R. Generic oracles and oracle classes. In *FOCS '87: Proc. of the 28th FOCS* (1987), pp. 118–126.

[5] Bublitz, S., Schürfeld, U., Wegener, I., and Voigt, B. Properties of complexity measures for PRAMs and WRAMs. *Theoret. Comput. Sci.* 48, 1 (1986), 53–73.

[6] Buhrman, H., and de Wolf, R. Complexity measures and decision tree complexity: a survey. *Theoret. Comput. Sci.* 288, 1 (2002), 21–43.

[7] Chakraborty, S. On the sensitivity of cyclically-invariant Boolean functions. *CCC 2005: Proc. of the 20th annual IEEE CCC* (2005), 163–167.

[8] Chakraborty, S. Sensitivity, block sensitivity and certificate complexity of Boolean functions. Master’s thesis, University of Chicago, USA, 2005.

[9] Chung, F. R. K., Füredi, Z., Graham, R. L., and Seymour, P. On induced subgraphs of the cube. *J. Comb. Theory, Ser. A* 49, 1 (1988), 180–187.

[10] Cook, S., and Dwork, C. Bounds on the time for parallel RAM’s to compute simple functions. In *STOC '82: Proc. of the 14th annual ACM STOC* (New York, NY, USA, 1982), ACM, pp. 231–233.

[11] de Wolf, R. Quantum communication and complexity. *Theoret. Comput. Sci.* 287, 1 (2002), 337–353.

[12] de Wolf, R. Brief introduction to Fourier analysis on the Boolean cube. *Theory of Computing (Library, Graduate Surveys)* 1 (2008). doi: 10.4086/toc.gs.2008.001.

[13] Gotsman, C., and Linial, N. The equivalence of two problems on the cube. *J. Comb. Theory, Ser. A* 61, 1 (1992), 142–146.

[14] Grolmusz, V. On the power of circuits with gates of low L^1 norms. *Theoret. Comput. Sci.* 188, 1-2 (1997), 117–128.

[15] Hartmanis, J., and Hemachandra, L. A. One-way functions, robustness and the non-isomorphism of NP-complete sets. In *Proc. of the 2nd IEEE Struct. in Complexity Theory Conf.* (1987), pp. 160–174.

[16] Kenyon, C., and Kutin, S. Sensitivity, block sensitivity, and l-block sensitivity of Boolean functions. *Inf. Comput.* 189, 1 (2004), 43–43.

[17] Kushilevitz, E., and Nisan, N. *Communication Complexity*. Cambridge University Press, 1997.
[18] MEHLHORN, K., AND SCHMIDT, E. M. Las Vegas is better than determinism in VLSI and distributed computing (extended abstract). In STOC ’82: Proc. of the 14th annual ACM STOC (New York, NY, USA, 1982), pp. 330–337.

[19] MIDRIJANIS, G. Exact quantum query complexity for total Boolean functions. http://arxiv.org/abs/quant-ph/0403168, 2004.

[20] NISAN, N. CREW PRAMs and decision trees. In STOC ’89: Proc. of the 21st annual ACM STOC (New York, NY, USA, 1989), pp. 327–335.

[21] NISAN, N., AND SZEGEDY, M. On the degree of Boolean functions as real polynomials. Computational Complexity 4 (1992), 462–467.

[22] NISAN, N., AND WIGDERSON, A. On rank vs. communication complexity. Combinatorica 15 (1995), 557–565. 10.1007/BF01192527.

[23] PATURI, R., AND SIMON, J. Probabilistic communication complexity. J. of Comput. and System Sci. 3, 1 (1986), 106 – 123.

[24] REISCHUK, R. A lower time-bound for parallel random access machines without simultaneous writes. Tech. Rep. RJ3431, IBM, New York, 1982.

[25] RIVEST, R., AND VUILLEMIN, J. On recognizing graph properties from adjacency matrices. Theoret. Comput. Sci. 3, 3 (1976), 371–384.

[26] RUBINSTEIN, D. Sensitivity vs. block sensitivity of Boolean functions. Combinatorica 15, 2 (1995), 297–299.

[27] SHERSTOV, A. On quantum-classical equivalence for composed communication problems. Quantum Inf. and Comput. 10, 5&6 (2010), 435–455.

[28] SHI, Y. Approximating linear restrictions of Boolean functions. Tech. rep., Manuscript, 2002.

[29] SIMON, H. U. A tight Ω(log log n)-bound on the time for parallel RAMs to compute non-degenerate Boolean functions. FCT ’83: Proc. of the 1983 Int. FCT 158 (1983), 439–444.

[30] TARDOS, G. Query complexity, or why is it difficult to separate $NP^A \cap coNP^A$ from $P^A$ by random oracles $A$? Combinatorica 9 (1989), 385–392.

[31] TURÁN, G. The critical complexity of graph properties. Inform. Process. Lett. 18 (1984), 151–153.

[32] ZHANG, Z., AND SHI, Y. On the parity complexity measures of Boolean functions. Theoret. Comput. Sci. 411, 26-28 (2010), 2612–2618.