A Study of Submanifolds of the Complex Grassmannian Manifold with Parallel Second Fundamental Form

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Abstract. We prove an extension of a theorem of A. Ros on a characterization of seven compact Kähler submanifolds by holomorphic pinching [5] to certain submanifolds of the complex Grassmannian manifolds.

1. Introduction

Let $\mathbb{C}P^n(1)$ be the $n$-dimensional complex projective space with the constant holomorphic sectional curvature 1 and $M^m$ an $m$-dimensional compact Kähler submanifold immersed in $\mathbb{C}P^n(1)$. In [5] Ros has proved that the holomorphic sectional curvature of $M$ is greater than or equal to $\frac{1}{2}$ if and only if $M$ has the parallel second fundamental form. Our goal in the present paper is to extend this result to submanifolds immersed in the complex Grassmannian manifold.

Let $Gr_p(\mathbb{C}^n)$ be the complex Grassmannian manifold of complex $p$-planes in $\mathbb{C}^n$. Since the tautological bundle $S \to Gr_p(\mathbb{C}^n)$ is a subbundle of a trivial bundle $Gr_p(\mathbb{C}^n) \times \mathbb{C}^n \to Gr_p(\mathbb{C}^n)$, we obtain the quotient bundle $Q \to Gr_p(\mathbb{C}^n)$. This is called the universal quotient bundle. We notice the fact that the holomorphic tangent bundle $T_{1,0}Gr$ over $Gr_p(\mathbb{C}^n)$ can be identified with the tensor product of holomorphic vector bundles $S^* \to Gr_p(\mathbb{C}^n)$ and $Q$, where $S^* \to Gr_p(\mathbb{C}^n)$ is the dual bundle of $S \to Gr_p(\mathbb{C}^n)$. If $\mathbb{C}^n$ has a Hermitian inner product, $S$, $Q$ have Hermitian metrics and Hermitian connections and so $Gr_p(\mathbb{C}^n)$ has a Hermitian metric induced by the identification of $T_{1,0}Gr$ and $S^* \otimes Q$, which is called the standard metric on $Gr_p(\mathbb{C}^n)$. In the present paper, we prove the following theorem:

THEOREM 1. Let $Gr_p(\mathbb{C}^n)$ be the complex Grassmannian manifold of complex $p$-planes in $\mathbb{C}^n$ with the standard metric $h_{Gr}$ induced from a Hermitian inner product on $\mathbb{C}^n$ and $f$ a holomorphic isometric immersion of a compact Kähler manifold $(M, h_M)$ with a Hermitian metric $h_M$ into $Gr_p(\mathbb{C}^n)$. We denote by $Q \to Gr_p(\mathbb{C}^n)$ the universal quotient bundle over $Gr_p(\mathbb{C}^n)$ of rank $q := n - p$. We assume that the pull-back bundle of $Q \to Gr_p(\mathbb{C}^n)$
is projectively flat. Then the holomorphic sectional curvature of $M$ is greater than or equal to $\frac{1}{q}$ if and only if $f$ has parallel second fundamental form.

We regard $Gr_{n-1}(\mathbb{C}^n)$ as the complex projective space. When we consider a holomorphic map $f : M \to Gr_{n-1}(\mathbb{C}^n)$ of a compact complex manifold into the complex projective space, then the pull-back bundle of $Q \to Gr_{n-1}(\mathbb{C}^n)$ is projectively flat since the rank of $Q$ is 1. Thus a holomorphic map of a compact complex manifold into the complex Grassmannian manifold which satisfies the condition that the pull-back bundle of the universal quotient bundle is projectively flat is a kind of generalization of a holomorphic map into the complex projective space. In the case that $p < n - 1$, see the latter part of Section 2.

It is why Theorem 1 is an extension of a theorem of Ros in [5]. In the case that $p = n - 1$, the sufficient condition in our theorem is that the holomorphic sectional curvature is greater than or equal to $1$, which is distinct from $\frac{1}{2}$ in a theorem of Ros. This is because we take a metric of Fubini-Study type with constant holomorphic sectional curvature 2.

REMARK 1. We can suppose that $p \geq q$ without loss of generality. In fact we can show that there is no immersion satisfying projectively flatness in the case that $p < q$. (See Remark 4.)

2. Preliminaries

Let $Gr_p(\mathbb{C}^n)$ be the complex Grassmannian manifold of complex $p$-planes in $\mathbb{C}^n$ with a standard metric $h_{Gr_p}$ induced from a Hermitian inner product on $\mathbb{C}^n$. We denote by $S \to Gr_p(\mathbb{C}^n)$ the tautological vector bundle over $Gr_p(\mathbb{C}^n)$. Since $S \to Gr_p(\mathbb{C}^n)$ is a subbundle of a trivial vector bundle $\mathbb{C}^n = Gr_p(\mathbb{C}^n) \times \mathbb{C}^n \to Gr_p(\mathbb{C}^n)$, we obtain a holomorphic vector bundle $Q \to Gr_p(\mathbb{C}^n)$ as a quotient bundle. This is called the universal quotient bundle over $Gr_p(\mathbb{C}^n)$. For simplicity, it is called the quotient bundle. Consequently we have a short exact sequence of vector bundles:

$$0 \to S \to \mathbb{C}^n \to Q \to 0.$$ 

Taking the orthogonal complement of $S$ in $\mathbb{C}^n$ with respect to the Hermitian inner product on $\mathbb{C}^n$, we obtain a complex subbundle $S^\bot \to Gr_p(\mathbb{C}^n)$ of $\mathbb{C}^n$. As $C^\infty$ complex vector bundle, $Q$ is naturally isomorphic to $S^\bot$. Consequently, the vector bundle $S \to Gr_p(\mathbb{C}^n)$ (resp. $Q \to Gr_p(\mathbb{C}^n)$) is equipped with a Hermitian metric $h_S$ (resp. $h_Q$) and so a Hermitian connection $\nabla^S$ (resp. $\nabla^Q$). The holomorphic tangent bundle $T_{1,0}Gr_p(\mathbb{C}^n)$ over $Gr_p(\mathbb{C}^n)$ is identified with $S^* \otimes Q \to Gr_p(\mathbb{C}^n)$ and the Hermitian metric on the holomorphic tangent bundle is induced from the tensor product $h_S^* \otimes h_Q$ of $h_S^*$ and $h_Q$.

Let $w_1, \ldots, w_n$ be a unitary basis of $\mathbb{C}^n$. We denote by $\mathbb{C}^p$ the subspace of $\mathbb{C}^n$ spanned by $w_1, \ldots, w_p$ and by $\mathbb{C}^q$ the orthogonal complement of $\mathbb{C}^p$, where $q = n - p$. The orthogonal projection to $\mathbb{C}^p$, $\mathbb{C}^q$ is denoted by $\pi_p$, $\pi_q$ respectively. Let $G$ be the special unitary group $SU(n)$ and $P$ the subgroup $S(U(p) \times U(q))$ of $SU(n)$ according to the decomposition. Then $Gr_p(\mathbb{C}^n) \cong G/P$. The vector bundles $S$, $Q$ are identified with $G \times_p \mathbb{C}^p$, $G \times_p \mathbb{C}^q$. 

$G \times p \mathbb{C}^q$ respectively. We denote by $\Gamma(S)$, $\Gamma(Q)$ spaces of sections of $S$, $Q$ respectively. Let $\pi_Q : \mathbb{C}^n \to \Gamma(Q)$ be a linear map defined by

$$\pi_Q(w)([g]) := [g, \pi_q(g^{-1}w)] \in G \times p \mathbb{C}^q, \quad w \in \mathbb{C}^n, \quad g \in G. $$

The bundle injection $i_Q : Q \to \mathbb{C}^n$ can be expressed as the following:

$$i_Q([g, v]) = ([g], gv), \quad v \in \mathbb{C}^q, \quad g \in G, \quad [g] \in Gr_p(\mathbb{C}^n) \cong G/P. $$

Let $t$ be a section of $Q \to Gr_p(\mathbb{C}^n)$. Since $i_Q(t)$ can be regarded as a $\mathbb{C}^n$-valued function $t : Gr_p(\mathbb{C}^n) \to \mathbb{C}^n$, $\pi_Qd(i_Q(t))$ defines a connection on $Q$. This is nothing but $\nabla Q$.

Similarly, we can write a bundle injection $i_S : S \to \mathbb{C}^n$ and a linear map $\pi_S : \mathbb{C}^n \to \Gamma(S)$:

$$i_S([g, u]) = ([g], gu), \quad u \in \mathbb{C}^p, \quad g \in G, \quad [g] \in G/P,$$

$$\pi_S(w)([g]) := [g, \pi_p(g^{-1}w)], \quad w \in \mathbb{C}^n, \quad g \in G. $$

The connection $\pi_Sd(i_S(s))$ on $S$ is nothing but $\nabla S$.

We introduce the second fundamental form $H$ in the sense of Kobayashi [1], which is a $(1,0)$-form with values in $\text{Hom}(S, Q) \cong S^* \otimes Q$:

$$di_S(s) = \nabla^S_s + H(s), \quad H(s) = \pi_Qd(i_S(s)), \quad s \in \Gamma(S). \tag{1}$$

Similarly, we introduce the second fundamental form $K$, which is a $(0,1)$-form with values in $\text{Hom}(Q, S) \cong Q^* \otimes S$:

$$di_Q(t) = K(t) + \nabla^Q t, \quad K(t) = \pi_Sd(i_Q(t)), \quad t \in \Gamma(Q). \tag{2}$$

**Lemma 1 ([1]).** The second fundamental forms $H$ and $K$ satisfy

$$h_Q(H_U s, t) = -h_S(s, K_U t), \quad s \in S_x, \quad t \in Q_x, \quad U \in T_{1,0} Gr_p(\mathbb{C}^n),$$

for any $x \in Gr_p(\mathbb{C}^n)$.

For a proof, see [1].

**Lemma 2.** For a vector $w \in \mathbb{C}^n$, set $s = \pi_S(w)$ and $t = \pi_Q(w)$. Then

$$\nabla^S_s = -K_U t, \quad \nabla^Q_U t = -H_U(s), \quad (U \in T_{1,0} Gr_p(\mathbb{C}^n)).$$

**Proof.** Since $i_S(s) + i_Q(t) = ([g], w)$, we have

$$0 = \pi_S(di_S(s) + di_Q(t)) = \nabla^S(s) + K(t).$$

Thus $\nabla^S_s = -K(t)$. Similarly $\nabla^Q_U t = -H(s)$.

Since $H$ is a $(1, 0)$-form with values in $S^* \otimes Q$, then $H$ can be regarded as a section of $T_{1,0} Gr_p(\mathbb{C}^n)^* \otimes T_{1,0} Gr_p(\mathbb{C}^n)$.
PROPOSITION 1 ([3]). The second fundamental form $H$ can be regarded as the identity transformation of $T_{1,0}Gr_p(C^n)$.

The unitary basis $w_1, \ldots, w_n$ of $C^n$ provides us with the corresponding sections

$$s_A = \pi_S(w_A) \in \Gamma(S), \quad t_A = \pi_Q(w_A) \in \Gamma(Q), \quad A = 1, \ldots, n.$$ 

PROPOSITION 2 ([3]). For arbitrary $(1,0)$-vectors $U$ and $V$ on $Gr_p(C^n)$, we have

$$h_{Gr}(U, V) = \sum_{A=1}^{n} h_S(K_{\overline{V}}t_A, K_{\overline{V}}t_A) = \sum_{A=1}^{n} h_Q(H_U s_A, H_V s_A).$$

Proposition 1 and Proposition 2 were proved by the second author in [3].

REMARK 2. Let $U, V$ be $(1,0)$-vectors on $Gr_p(C^n)$ at $x \in Gr_p(C^n)$. From Lemma 1 and Proposition 2, we have

$$h_{Gr}(U, V) = -\text{trace}_Q H_U K_{\overline{V}} = -\text{trace}_S K_{\overline{V}} H_U,$$

where $\text{trace}_Q H_U K_{\overline{V}}$ is the trace of the endomorphism $H_U K_{\overline{V}}$ of $Q_x$ and $\text{trace}_S K_{\overline{V}} H_U$ is the trace of the endomorphism $K_{\overline{V}} H_U$ of $S_x$.

Since any vectors in $S_x$ (resp. $Q_x$) can be expressed by a linear combination of $s_1(x), \ldots, s_n(x)$ (resp. $t_1(x), \ldots, t_n(x)$), it follows from Lemma 2 that the curvature $R^S$ of $\nabla^S$ and $R^Q$ of $\nabla^Q$ are expressed by the following:

$$R^S(U, \overline{V}) s_A = \nabla^S_U (\nabla^S s_A)(\overline{V}) - \nabla^S_{\overline{V}} (\nabla^S s_A)(U) = K_{\overline{V}} H_U s_A,$$

$$R^Q(U, \overline{V}) t_A = \nabla^Q_U (\nabla^Q t_A)(\overline{V}) - \nabla^Q_{\overline{V}} (\nabla^Q t_A)(U) = -H_U K_{\overline{V}} t_A.$$ 

It follows from $h_{Gr} = h_{s^*} \otimes h_Q$ that the curvature $R^{Gr}$ of $Gr_p(C^n)$ can be expressed as $R^S \otimes \text{Id}_Q + \text{Id}_{s^*} \otimes R^Q$. Thus we can compute $R^{Gr}$ as follows:

$$R^{Gr}(U, \overline{V}) Z = -H_Z K_{\overline{V}} H_U - H_U K_{\overline{V}} H_Z,$$

for $(1,0)$-vectors $U, V, Z$.

REMARK 3. Let us compute the holomorphic sectional curvature of $Gr_{n-1}(C^n)$. Since the quotient bundle over $Gr_{n-1}(C^n)$ is of rank 1, then it follows from the equations (3) and (6) that

$$R^{Gr}(U, \overline{V}) Z = -H_Z K_{\overline{V}} H_U - H_U K_{\overline{V}} H_Z = h_{Gr}(Z, V) U + h_{Gr}(U, V) Z,$$

where $U, V$ is $(1,0)$-vectors. Thus for any unit $(1,0)$-vector $U$ we obtain

$$\text{Hol}^{Gr}(U) = h_{Gr}(R^{Gr}(U, \overline{U}) U, U) = h_{Gr}(2U, U) = 2,$$

where $\text{Hol}^{Gr}(U)$ is the holomorphic sectional curvature along $U$ of $Gr_{n-1}(C^n)$. 

From now on, we introduce a relation between holomorphic vector bundles over a compact complex manifold and holomorphic maps into the complex Grassmannian manifold. For a detail, see [3].

Let $M$ be a compact complex manifold and $V \rightarrow M$ a holomorphic vector bundle with Hermitian metric and Hermitian connection $\nabla^V$. We denote by $(W, (\cdot, \cdot)_W)$ the space of holomorphic sections of $V \rightarrow M$ with $L_2$-Hermitian inner product. Assume that the bundle homomorphism, which is called an evaluation map,
$$
ev : M \times W \rightarrow V : (x, t) \mapsto t(x)$$
is surjective. In this case $V \rightarrow M$ is called globally generated by $W$. Then we obtain a holomorphic map $f_0 : M \rightarrow Gr_p(W) : x \mapsto \ker \ev_x$. This is called the standard map induced by $V \rightarrow M$.

Conversely, let $M$ be a compact Kähler manifold and $f : M \rightarrow Gr_N(C^n)$ a holomorphic isometric immersion. It follows from a Borel-Weil Theorem that $C^n$ can be regarded as the space of holomorphic sections of $Q \rightarrow Gr$. By restricting each sections of $Q \rightarrow Gr$ to $M$, we obtain a linear map from $C^n$ to the space of holomorphic sections of $f^*Q \rightarrow M$. Then we obtain an evaluation map
$$
ev_C : M \times C^n \rightarrow f^*Q : (x, t) \mapsto t(x), \quad \text{for } x \in M, t \in C^n.$$The bundle isomorphism $\ev_C$ is surjective and we have $\ker \ev_C = S_{f(x)} = f(x)$. Therefore by using $\ev_C$, $f$ is expressed that $f(x) = \ker \ev_C$.

Here we assume that $f^*Q \rightarrow M$ is projectively flat. It follows from the holonomy theorem and $(\ast)$ in Section 3 that there exists a holomorphic line bundle $L \rightarrow M$ such that $f^*Q \rightarrow M$ is decomposed to orthogonal direct sum of $q$-copies of $L \rightarrow M$, where $q = n - p$. We denote by $\tilde{L} \rightarrow M$ the orthogonal direct sum bundle of $q$-copies of $L \rightarrow M$ and also denote by $W$ and $\tilde{W}$ the space of holomorphic sections of $L \rightarrow M$ and $\tilde{L} \rightarrow M$ respectively. We fix an $L_2$-Hermitian inner product $(\cdot, \cdot)_W$ and $(\cdot, \cdot)_{\tilde{W}}$ of $W$ and $\tilde{W}$ respectively. Then $\tilde{W}$ is regarded as the orthogonal $q$-direct sum of $W$. Let $f_0 : M \rightarrow Gr_{N-1}(W)$ be the standard map induced by $L \rightarrow M$, where $N$ is the dimension of $W$. When we denote by $\tilde{f} : M \rightarrow Gr_{q(N-1)}(\tilde{W})$ the standard map induced by $\tilde{L} \rightarrow M$, $\tilde{f}$ can be expressed as
$$\tilde{f}(x) = f_0(x) \oplus \cdots \oplus f_0(x) \subset W \oplus \cdots \oplus W, \quad \text{for } x \in M.$$

Since $f^*Q \rightarrow M$ is isomorphic to $\tilde{L} \rightarrow M$ with metrics and connections, we have a linear map $\iota : C^n \rightarrow \tilde{W}$. We assume that $\iota$ is injective. Then it follows from Theorem 5.5 in [3] that there exists a semi-positive Hermitian endomorphism $T$ of $\tilde{W}$ such that $f : M \rightarrow$
$Gr_p(C^n)$ can be expressed as

$$f(x) = (t^*T_i)^{-1}(\tilde{f}(x) \cap \iota(C^n)),$$

where $t^* : W \to C^n$ is the adjoint linear map of $\iota$.

Consequently, if $f : M \to Gr_p(C^n)$ is holomorphic isometric immersion with the condition that $f^*Q \to M$ is projectively flat, then $f$ can be expressed by using a holomorphic map into the complex projective space and a semi-positive Hermitian endomorphism.

3. Proof of Theorem 1

Let $M$ be a compact Kähler manifold and $f : M \to Gr_p(C^n)$ a holomorphic isometric immersion, where $Gr_p(C^n)$ has the metric $h_{Gr}$ induced by the Hermitian inner product of $C^n$. We denote by $\nabla^M$ and $\nabla^{Gr}$ the Hermitian connections of $M$ and $Gr_p(C^n)$ respectively. We have a short exact sequence of holomorphic vector bundles:

$$0 \to T_{1,0}M \to T_{1,0}Gr|_M \to N \to 0,$$

where $T_{1,0}Gr|_M$ is a holomorphic vector bundle induced by $f$ from the holomorphic tangent bundle over $Gr_p(C^n)$ and $N$ is a quotient bundle. In the same manner as in the previous section, we obtain second fundamental forms $\sigma$ and $A$ of $TM$ and $N$:

$$\nabla^M_U V = \nabla^M_U V + \sigma(U, V), \quad U \in T_C M, \quad V \in \Gamma(T_{1,0}M),$$

(7)

$$\nabla^{Gr}_U \xi = -A_\xi U + \nabla^N_\xi \xi, \quad U \in T_C M, \quad \xi \in \Gamma(N).$$

(8)

For each point $x \in M$, $\sigma : T_{1,0}M \times T_{1,0}M \to N_x$ is a symmetric bilinear mapping. This is called the second fundamental form of $f$. The second fundamental form $A : N_x \times T_{0,1}M \to T_{1,0}M$ is a bilinear mapping. This is called the shape operator of $f$. We follow a convention of submanifold theory to define the shape operator.

Throughout this section, the symbol $\nabla$ means the suitable connection of covariant tensor fields induced by $\nabla^M$, $\nabla^{Gr}$ and $\nabla^N$. Second fundamental forms $\sigma$ and $A$ satisfy the following formulas.

FORMULAS 1. For any $U, V, Z, W \in T_{1,0}M$, we have

- $\sigma(U, V) = 0, \quad A_\xi U = 0$,
- $h_{Gr}(\sigma(U, V), \xi) = h_{Gr}(V, A_\xi U)$,
- $h_{Gr}(R^M(U, \overline{V})Z, W) = h_{Gr}(R^{Gr}(U, \overline{V})Z, W) - h_{Gr}(\sigma(U, Z), \sigma(V, W))$,
- $h_{Gr}(R^N(U, \overline{V})\xi, \eta) = h_{Gr}(R^{Gr}(U, \overline{V})\xi, \eta) + h_{Gr}(A_\xi \overline{V}, A_\eta U)$,
- $(\nabla_V \sigma)(U, Z) = (\nabla_U \sigma)(V, Z)$,
- $(\nabla_{\overline{V}} \sigma)(U, Z) = - (R^{Gr}(U, \overline{V})Z)_{\perp}$.

Note that the quotient bundle $N$ is isomorphic to the orthogonal complement bundle $T_{1,0}^\perp M$ as a $C^\infty$ complex vector bundle. The third, fourth and fifth formulas are called the
equation of Gauss, the equation of Ricci and the equation of Codazzi respectively. From the equation of Codazzi,
\[ \nabla \sigma : T_{1,0} M \otimes T_{1,0} M \otimes T_{1,0} M \rightarrow N_x \]
is a symmetric tensor for any \( x \in M \).

We assume that \( f^\ast Q \rightarrow M \) is projectively flat. The vector bundle \( f^\ast Q \rightarrow M \) is projectively flat if and only if
\[ R^{f^\ast Q}(U, \overline{V}) = \alpha(U, \overline{V}) \text{Id}_{Q_{f(x)}}, \quad \text{for } U, V \in T_{1,0} M, \]
where \( \alpha \) is a complex 2-form on \( M \). Since \( R^{f^\ast Q} \) is a (1,1)-form, so is \( \alpha \). It follows from the equation (3) that
\[ h_M(U, V) = \text{trace} R^Q(U, \overline{V}) = q \cdot \alpha(U, \overline{V}). \]
Therefore, \( f^\ast Q \rightarrow M \) is projectively flat if and only if
\[ R^{f^\ast Q}(U, \overline{V}) = \frac{1}{q} h_M(U, V) \text{Id}_{Q_{f(x)}}, \quad \text{for } U, V \in T_{1,0} M. \]  
\[ (*) \]

**Remark 4.** It follows from the equation (5) that
\[ R^{f^\ast Q}(U, \overline{V}) = -H_{U \overline{K}_U} : Q_x \rightarrow S_x \rightarrow Q_x. \]
Therefore, if an immersion \( f \) satisfies the equation \( (*) \), the rank of \( S \) is greater than or equal to that of \( Q \).

We denote by Hol the holomorphic sectional curvature of a Kähler manifold. By the equation of Gauss, if \( U \) is a unit \((1,0)\)-vector on \( M \), then
\[ \text{Hol}^M(U) = h_M(R^M(U, \overline{U})U, U) = h_{Gr}(R^{Gr}(U, \overline{U})U, U) - \| \sigma(U, U) \|^2 \]
\[ = \text{Hol}^{Gr}(U) - \| \sigma(U, U) \|^2. \]  
\[ (9) \]

**Lemma 3.** Under the assumption of Theorem 1, for any unit \((1,0)\)-vector \( U \) on \( M \) we have
\[ \text{Hol}^{Gr}(U) = \frac{2}{q}. \]

**Proof.** Let \( U \) be a unit \((1,0)\)-vector at \( x \in M \). By the equation \( (*) \), we have
\[ -H_{U \overline{K}_U} = R^{f^\ast Q}(U, \overline{U}) = \frac{1}{q} \text{Id}_{Q_x}. \]  
\[ (10) \]
It follows from equations (6) and (10) that
\[ \text{Hol}^{Gr}(U) = h_{Gr}(R^{Gr}(U, \overline{U})U, U) = -2h_{S \otimes Q}(H_U K_{\overline{U}} H_U, H_U) \]
\[ = \frac{2}{q} h_{S \otimes Q}(H_U, H_U) = \frac{2}{q}. \]  
\[ \square \]
**Lemma 4.** Under the assumption of Theorem 1, for any \((0, 1)\)-vector \(\nabla\) on \(M\) we have 
\[
\nabla_{\nabla \sigma} = 0.
\]

**Proof.** It follows from equation (6) and (*) that
\[
R^{Gr}(U, \nabla)Z = -H_Z K_{\nabla} H_U - H_U K_{\nabla} H_Z
= \frac{1}{q} h_{Gr}(Z, V) U + \frac{1}{q} h_{Gr}(U, V) Z,
\]
where \(U, V, Z\) are \((1, 0)\)-vectors on \(M\). By the equation of Codazzi, we have
\[
\nabla_{\nabla \sigma}(U, Z) = -(R^{Gr}(U, \nabla)Z)^\perp = 0.
\]

\[\square\]

In [5] A. Ros has proved the following Lemma.

**Lemma 5 (A. Ros [5]).** Let \(T\) be a \(k\)-covariant tensor on a compact Riemannian manifold \(M\). Then
\[
\int_{UM} (\nabla T)(X, \ldots, X) dX = 0,
\]
where \(UM\) is the unit tangent bundle of \(M\) and \(dX\) is the canonical measure of \(UM\) induced by the Riemannian metric on \(M\).

For a proof, see [5].

We use the complexification of the above Lemma.

**Lemma 6.** Let \(T\) be a \((p, q)\)-covariant tensor on an \(m\)-dimensional compact Kähler manifold \((M, h_M)\). We consider \(M\) as an \(2m\)-dimensional real manifold with the almost complex structure \(J\). We denote by \(g_M\) the Riemannian metric induced by \(h_M\). Then we have the canonical measure \(dX\) of \(UM\). We obtain the following equality:
\[
\int_{UM} (\nabla T)(\overline{U}_X, U_X, \ldots, U_X, \overline{U}_X, \ldots, \overline{U}_X) dX = 0,
\]
where \(U_X = \frac{1}{\sqrt{2}}(X - \sqrt{-1} JX)\) and \(\overline{U}_X = \frac{1}{\sqrt{2}}(X + \sqrt{-1} JX)\) and \(X\) is a real tangent vector on \(M\).

**Proof.** We define real valued \(k\)-covariant tensors on Riemannian manifold \((M, g_M)\) by
\[
2K(X_1, \ldots, X_k) = T(U_1, \ldots, U_p, \overline{U}_{p+1}, \ldots, \overline{U}_k) + T(U_1, \ldots, U_p, \overline{U}_{p+1}, \ldots, \overline{U}_k),
\]
\[
2L(X_1, \ldots, X_k) = \sqrt{-1}(T(U_1, \ldots, U_p, \overline{U}_{p+1}, \ldots, \overline{U}_k)
- T(U_1, \ldots, U_p, \overline{U}_{p+1}, \ldots, \overline{U}_k)),
\]
\]
where \( k = p + q, \) \( U_i = U_{X_i} \) for \( i = 1, \ldots, k. \) Then \( T, K \) and \( L \) satisfy the following equation:

\[
T(U_1, \ldots, U_p, U_{p+1}, \ldots, U_k) = K(X_1, \ldots, X_k) - \sqrt{-1}L(X_1, \ldots, X_k).
\]

We get the covariant derivative of both sides of this equation:

\[
(\nabla_{U_X} T)(U_X, \ldots, U_X, \ldots) = \frac{1}{\sqrt{2}}(\nabla_{X+\sqrt{-1}JX} K)(X, \ldots, X) - \frac{\sqrt{-1}}{\sqrt{2}}(\nabla_{X+\sqrt{-1}JX} L)(X, \ldots, X).
\] (12)

Since the covariant derivative is linear, then

\[
(\nabla_{X+\sqrt{-1}JX} K)(X, \ldots, X) = (\nabla_{X} K)(X, \ldots, X) + \sqrt{-1}(\nabla_{JX} K)(X, \ldots, X).
\] (13)

Consequently it follows from Lemma 5 that we obtain

\[
\int_{UM} (\nabla T)(U_X, U_X, \ldots, U_X, \ldots, U_X)dX = \frac{\sqrt{-1}}{\sqrt{2}} \int_{UM} (\nabla_{JX} K)(X, \ldots, X)dX + \frac{1}{\sqrt{2}} \int_{UM} (\nabla_{JX} L)(X, \ldots, X)dX.
\]

For the covariant tensor field \( K, \) we define a new covariant tensor fields \( \tilde{K} \) by

\[
\tilde{K}(X_1, \ldots, X_k) = K(JX_1, \ldots, JK_k), \quad \text{for } X_1, \ldots, X_k \in T_x M \quad (x \in M).
\]

Since the almost complex structure \( J \) is parallel and preserves the inner product and orientation of each tangent space of \( M, \) it follows that

\[
\int_{UM} (\nabla_{JX} K)(X, \ldots, X)dX = (-1)^k \int_{UM} (\nabla_{JX} K)(JX, \ldots, JX)dX
\]

\[
= (-1)^k \int_{UM} (\nabla_{JX} \tilde{K})(JX, \ldots, JX)dX
\]

\[
= (-1)^k \int_{UM} (\nabla_X \tilde{K})(X, \ldots, X)dX
\]

\[
= 0.
\]

The last equation follows from Lemma 5. Similarly we have

\[
\int_{UM} (\nabla_{JX} L)(X, \ldots, X)dX = 0.
\]

Therefore we obtain the equality in Lemma 6. \( \square \)

**PROOF OF THEOREM 1.** We define a (2,2)-covariant tensor \( T \) on \( M \) by

\[
T(U, V, \overline{Z}, \overline{W}) = h_{Gr}(\sigma(U, V), \sigma(Z, W)),
\] (14)
where \( U, V, Z, W \) are \((1,0)\)-vectors on \( M \). Using the equation of Ricci and the equation of Codazzi, we obtain
\[
(\nabla^2 T)(U, U, U, U) = h_M((\nabla^2 \sigma)(U, U, U, U), \sigma(U, U)) + \|\nabla \sigma(U, U, U, U)\|^2.
\]
Using the Ricci identity, we obtain
\[
(\nabla^2 \sigma)(U, U, U, U) = -R^N(U, \overline{U})(\sigma(U, U)) + 2\sigma(R^M(U, \overline{U})U, U).
\]
It follows from Lemma 4 that
\[
(\nabla^2 \sigma)(U, U, U, U) = -R^N(U, \overline{U})(\sigma(U, U)) + 2\sigma(R^M(U, \overline{U})U, U).
\]
Therefore, we obtain
\[
(\nabla^2 T)(U, U, U, U) = -h_G r (R^N(U, \overline{U})(\sigma(U, U)), \sigma(U, U)) + 2h_G r (\sigma(R^M(U, \overline{U})U, U), \sigma(U, U)) + \|\nabla \sigma(U, U, U, U)\|^2. \tag{15}
\]
From the equation of Ricci and (6), we have
\[
h_G r (R^N(U, \overline{U})(\sigma(U, U)), \sigma(U, U)) = h_G r (R^G r (U, \overline{U})(\sigma(U, U)), \sigma(U, U)) + \|A_\sigma(U, U, \overline{U})\|^2.
\]
\[
= h_G r (-H_\sigma(U, U) K_{\overline{U}} H_U, H_\sigma(U, U)) + h_G r (-H_U K_{\overline{U}} H_\sigma(U, U), H_\sigma(U, U)) + \|A_\sigma(U, U, \overline{U})\|^2. \tag{16}
\]
In the following calculation, we extend \((1,0)\)-vectors to local holomorphic vector fields if necessary.

**Lemma 7.** For any \((1,0)\)-vectors \( U, V, Z \) on \( M \), we have
\[
-H_\sigma(U, Z) K_{\overline{V}} = (\nabla_Z R^{f^* Q})(U, V).
\]

**Proof.** We have
\[
(\nabla_Z R^{f^* Q})(U, V) = -\nabla_Z (H_U K_{\overline{V}}) + H_{\nabla_Z U} K_{\overline{V}} = - (\nabla_Z H)(U) K_{\overline{V}}.
\]
Since we can easily show that \( H_\sigma(U, Z) = (\nabla_U H)(Z) \), we obtain
\[
-H_\sigma(U, Z) K_{\overline{V}} = (\nabla_U H)(Z) K_{\overline{V}} = (\nabla_Z R^{f^* Q})(U, V). \quad \Box
\]
It follows from (*) in Section 3 that
\[
(\nabla_Z R^{f^* Q})(U, V) = \nabla_Z^{f^* Q}(R^{f^* Q}(U, \overline{V})) - R^{f^* Q}(\nabla_Z M(U, \overline{V})
\]
\[
= \frac{1}{q} \nabla_Z (h_M(U, V)) I_d Q - \frac{1}{q} h_M(\nabla_Z M(U, V)) I_d Q = 0
\]
where $U$, $V$, $Z$ are $(1,0)$-vectors on $M$. Then it follows from Lemma 7, the equations (10) and (16) that

$$h_{Gr}(R^N(U, U)(\sigma(U, U)), \sigma(U, U)) = h_{Gr}(-H_U K_{\overline{U}} H_{\sigma(U, U)}, H_{\sigma(U, U)}) + \|A_{\sigma(U, U)} U\|^2$$

$$= \frac{1}{q} \|\sigma(U, U)\|^2 + \|A_{\sigma(U, U)} \overline{U}\|^2. \quad (17)$$

Using the equation of Gauss and the equation (11), we have

$$h_{Gr}(\sigma(R^M(U, U)U, U), \sigma(U, U)) = h_{Gr}(R^G(U, U)U, A_{\sigma(U, U)} U)$$

$$= h_{Gr}(R^M(U, U)U, A_{\sigma(U, U)} U) - \|A_{\sigma(U, U)} U\|^2$$

$$= -2h_{Gr}(H_U K_{\overline{U}} H_{\sigma(U, U)} - \|A_{\sigma(U, U)} U\|^2$$

$$= \frac{2}{q} \|\sigma(U, U)\|^2 - \|A_{\sigma(U, U)} U\|^2. \quad (18)$$

Combining the equations (17) and (18) with (15), we obtain

$$(\nabla^2 T)(\overline{U}, U, U, U, U) = -\left(\frac{1}{q} \|\sigma(U, U)\|^2 + \|A_{\sigma(U, U)} U\|^2\right)$$

$$+ 2\left(\frac{2}{q} \|\sigma(U, U)\|^2 - \|A_{\sigma(U, U)} U\|^2\right) + \|(\nabla \sigma)(U, U)\|^2$$

$$= \frac{3}{q} \|\sigma(U, U)\|^2 - q \|A_{\sigma(U, U)} U\|^2 + \|(\nabla \sigma)(U, U)\|^2. \quad (19)$$

By integrating both sides of the equation (19) ($U = U_X$), Lemma 6 yields

$$\frac{3}{q} \int_{UM} \left(\|\sigma(U_X, U_X)\|^2 - q \|A_{\sigma(U_X, U_X)} \overline{U}_X\|^2\right) dX$$

$$+ \int_{UM} \|(\nabla \sigma)(U_X, U_X)\|^2 dX = 0. \quad (20)$$

From now on we assume that the holomorphic sectional curvature of $M$ is greater than or equal to $\frac{1}{q}$. Let us compute the first term of the left hand side of the equation (20). We define $\xi \in N$ as $\sigma(U, U) = \|\sigma(U, U)\| \xi$. Then we have

$$A_{\sigma(U, U)} U = \|\sigma(U, U)\| A_\xi \overline{U}. \quad \text{We denote by } \tau \text{ the involutive anti-holomorphic transformation of the complexification } T_{C} M \text{ of } TM \text{ having } TM \text{ as the fixed point set. Let } B := A_\xi \circ \tau. B \text{ is an anti-linear transformation}$$
and satisfies the following equation:

\[ h_{Gr}(BU, V) = h_{Gr}(BV, U), \quad \text{for } U, V \in T_{1,0}M, \ x \in M. \]

If we regard \( B \) as a real linear transformation on the real vector space with an inner product \( \Re(h_{Gr}(\cdot, \cdot)) \), then \( B \) is a symmetric transformation. Let \( \lambda \) be the eigenvalue of \( B \) whose absolute value is maximum and \( e \) the corresponding unit eigenvector. By Cauchy-Schwarz inequality, we have

\[ \lambda = h_{Gr}(Be, e) = h_{Gr}(A\xi e, e) = h_{Gr}(\xi, \sigma(e, e)) \leq \|\sigma(e, e)\|. \]

It follows from the equation (9), Lemma 3 and the hypothesis that

\[ \|A\xi U\|^2 \leq \lambda^2 \leq \|\sigma(e, e)\| \leq \frac{1}{q}. \]

It follows that

\[ \|\sigma(U, U)\|^2 - q\|A\sigma(U, U)U\|^2 = \|\sigma(U, U)\|^2(1 - q\|A\xi U\|^2) \geq \|\sigma(U, U)\|^2\left(1 - q \cdot \frac{1}{q}\right) = 0. \]

Thus it follows from the equation (20) that

\[ \|\nabla\sigma(U, U)\|^2 = 0. \]

Since \( \nabla\sigma \) is a symmetric tensor, \( \nabla\sigma \) vanishes.

Conversely, we assume that \( M \) has parallel second fundamental form. From the equation (9) and Lemmas 3 and 4, it is enough to prove that \( \|\sigma(U, U)\|^2 \leq \frac{1}{q} \), where \( U \) is an arbitrary unit \((1, 0)\)-vector on \( M \). Let \( T \) be a \((2, 2)\)-covariant tensor on \( M \) defined by the equation (14). Since the second fundamental form \( \sigma \) is parallel, \( T \) is also parallel and so \( \nabla^2 T = 0 \). It follows from the equation (19) that

\[ \|\sigma(U, U)\|^2 - q\|A\sigma(U, U)U\|^2 = 0. \tag{21} \]

The Cauchy-Schwarz inequality and the equation (21) imply that

\[ \|\sigma(U, U)\|^2 = h_{Gr}(\sigma(U, U), \sigma(U, U)) = h_{Gr}(U, A\sigma(U, U)U) \leq \|A\sigma(U, U)U\| = \frac{1}{\sqrt{q}}\|\sigma(U, U)\|. \]

Therefore, \( \|\sigma(U, U)\|^2 \leq \frac{1}{q}. \]

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