Seiberg-Witten Theory, Integrable Systems and D-branes

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Abstract

In this note it is demonstrated how the Seiberg-Witten solutions and related integrable systems may arise from certain brane configurations in M-theory. Some subtleties of the formulation of the Seiberg-Witten theory via integrable systems are discussed and interpreted along the lines of general picture of string/M-theory dualities.

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1 Introduction

A lot of ideas has appeared recently in theoretical physics due to developments in what can be called nonperturbative string theory or $M$-theory\(^1\). The basic concept is that the physically interesting quantum field theories (QFT’s) could be considered as various vacua of $M$-theory and the stringy symmetries or dualities may relate spectra and correlation functions in one QFT with those in another QFT; then typical string duality $R \leftrightarrow \frac{\alpha’}{R}$ allows in principle to relate perturbative regime in one model with the nonperturbative in another.

This general idea at the moment was put to more solid ground only for the case of complex backgrounds in string theory ($\equiv$ supersymmetric (SUSY) quantum field theories). In such theories physical data of the model (masses, couplings, etc) can be considered as functions on moduli spaces of complex manifolds and the duality symmetry can be regarded as action of a modular group. Useful information can be extracted by powerful machinery of complex geometry. Despite some progress achieved along these lines in the popular scheme of string compactifications on the Calabi-Yau manifolds (see, for example [1] and references therein), rigorous statements about the net result of nonperturbative string theory can be made till now only when nontrivial complex geometry can be effectively reduced to the geometry of one-dimensional complex ($\equiv$ two-dimensional real) manifolds – complex curves $\Sigma$. One of the most interesting examples of exact statements in this field is the Seiberg-Witten anzatz for the Coulomb phase of $\mathcal{N} = 2$ SUSY Yang-Mills theories in four dimensions\(^2\).

In this note I will explain once more how the Seiberg-Witten (SW) anzatz arises from the brane configurations in $M$-theory along the lines of [3, 4]. Following the approach of [5, 6, 8], the language of integrable systems will be used for the formulations of the exact results in $\mathcal{N} = 2$ SUSY four-dimensional gauge theories. I will try to pay attention to the subtleties of the exact formulation of the results in these terms and demonstrate how some of them are governed by the Diaconescu-Hanany-Witten-Witten (DHWW) construction.

2 Seiberg-Witten Anzatz: Integrable Systems

For the $\mathcal{N} = 2$ SUSY gauge theory the SW anzatz can be formulated in the following way. The $\mathcal{N} = 2$ SUSY vector multiplet has necessarily (complex) scalars with the potential $V(\phi) = \text{Tr}[\phi, \phi^\dagger]^2$ whose minima (after factorization over the gauge group) correspond to the diagonal ($[\phi, \phi^\dagger] = 0$), constant and (in the theory with $SU(N_c)$ gauge group) traceless matrices. Their invariants

$$\det(\lambda - \phi) \equiv P_{N_c}(\lambda) = \sum_{k=0}^{N_c} s_{N_c-k} \lambda^k$$

\(^1\)There is no fixed terminology yet in this field – sometimes the term $M$-theory is applied in more ”narrow” sense – to the theory of membranes, M(atrix) models etc. In this note we will use the term $M$-theory in wide sense – identify it with the hypothetical (11-dimensional) nonperturbative ”string” theory.

\(^2\)
(the total number of algebraically independent ones is rank $SU(N_c) = N_c - 1$) parameterize the moduli space of the theory. Due to the Higgs effect the off-diagonal part of the gauge field $A_\mu$ becomes massive, since

$$[\phi_i, A_\mu]_{ij} = (\phi_i - \phi_j) A_{ij}^\mu$$  \hspace{1cm} (2)

while the diagonal part, as it follows from (2) remains massless, i.e. the gauge group $G = SU(N_c)$ breaks down to $U(1)^{\text{rank} G} = U(1)^{N_c - 1}$.

The effective abelian theory is formulated in terms of a finite-dimensional integrable system: the spectral curve $\Sigma$ defined over the genus-dimensional subspace of the full moduli space, e.g.

$$\Lambda^{N_c} \left( w + \frac{1}{w} \right) = 2P_{N_c}(\lambda)$$  \hspace{1cm} (3)

for the pure $SU(N_c)$ gauge theory and the generating differential

$$dS = \lambda \frac{dw}{w}$$  \hspace{1cm} (4)

whose basic property is that its derivatives over $N_c - 1$ moduli give rise to holomorphic differentials. The data $(\Sigma, dS)$ with such properties are exactly the definition of the integrable system in the sense of [7] (see [8] and references therein for details). The period matrix of $\Sigma T_{ij}(a)$ as a function of the action variables

$$a = \oint_A dS \quad a^D = \oint_B dS$$

$$T_{ij} = \frac{\partial a^D_i}{\partial a_j}$$  \hspace{1cm} (5)

gives the set of coupling constants in the effective abelian $U(1)^{N_c - 1}$ theory while action variables themselves are identified with the masses of the BPS states $M^2 \sim |na + ma_D|^2$ with the $(n, m)$ "electric" and "magnetic" charges.

### 3 Spectral Curve as Topologically Nontrivial Part of M-theory 5-brane World Volume

Let us show now that the spectral curve and generating differential can naturally arise from brane configurations.

- First step is to obtain a gauge group $SU(N_c)$ broken down to $U(1)^{N_c - 1}$. The most elegant way of doing this in string theory is to introduce D-branes into type II string theory – the submanifolds in target space where strings can have their ends. $N_c$ parallel D-branes would correspond exactly to what we need now since string stretched between $i$-th and $j$-th brane $(i, j = 1, \ldots, N_c)$ (see Fig.1) will have

\footnote{The genus of the curve is $N_c - 1$, i.e. exactly equal to the number of independent parameters of the polynomial $P_{N_c}(\lambda)$.}
Figure 1: Open strings, stretched between $D$-branes induce the interaction via non-Abelian gauge fields $A^{ij}$. 
Figure 2: The 4-branes restricted by 5-branes to the finite volume (in horizontal $x^6$-direction) give rise to macroscopically 4-dimensional theory.

A vector field $A^{ij}$ in its spectrum such that mass of this vector field is proportional to the length of the string, i.e. to the distance between $i$-th and $j$-th branes. This the $U(1)^{N_c-1}$ massless factor will come from strings having both ends on the same D-brane while the $A^{ij}$ fields with $i \neq j$ will acquire "Higgs" masses \(^{(2)}\) where scalars vev’s are us usual proportional to the "transverse" co-ordinate of the D-brane $\phi \sim \sqrt{\frac{x^2}{\alpha'}}$. 

Next step is that from 10-dimensional type II string theory ($A = \|A^{ij}\|$ is 10-dimensional gauge field in string picture) one wants to get 4-dimensional one. A natural way to reduce the number of space-time dimensions is to restrict ourselves to the effective theory on D-brane world volume. The world volume of the Dirichlet $p$-brane is $p+1$-dimensional, so naively in order to get 4-dimensional theory one should consider D3-branes. This scenario is quite possible and realized in another context; however to get the SW anzatz it is better to use another option, the DHWW brane configuration with the $N_c$ parallel D4-branes stretched between two vertical walls (see Fig.2), so that the naive 5-dimensional D4-world-volume theory is macroscopically (in the light sector) 4-dimensional by conventional Kaluza-Klein argument for a system compactified on a circle or put into a box.

The role of vertical walls should be played by 5-branes \(^{[3]}\), this follows from the $\beta$-function consid-
Figure 3: The brane configuration, represented as a result of "blowing up" Fig.2 – the ladder turns into hyperelliptic Riemann surface being at the same time $N_c$-fold covering of the horizontal cylinder.

operations: the logarithmic behavior of the macroscopic coupling constant can be ensured in the first approximation if corresponding co-ordinate ($x^9$) has logarithmic behavior as a function of "transverse" direction, i.e. satisfy two-dimensional Laplace equation. The effective space is two-dimensional if parallel D-branes are stretched between the 5-branes.

• The obtained picture of 4 and 5 branes in 10 dimensions is of course very rough and true in (semi)-classical approximation. In particular it is naively singular at the points where 4-branes meet 5-branes. These singularities were resolved by Witten [3] who suggested to put the whole picture into 11-dimensional target space of M-theory with compact 11-th dimension and to consider D4-branes as M-theory 5-branes compactified to 11-th dimension. Thus the picture in Fig.2 becomes similar rather to the surface of "swedish ladder" [3] and apart of macroscopic directions $x^0, \ldots, x^3$ looks like (non-compact) Riemann surface with rather special properties (see Fig.3).

[3] I am grateful to V.Kazakov for this not quite exact but illuminating comparison.
In other words, one gets a 5-brane parameterized by \((x^0, x^1, x^2, x^3, x^6, x^9)\), which leaving aside four flat dimensions \((x^0, x^1, x^2, x^3)\) along these lines ends up to \(N_c\) cylinders \(R \times S^1\) embedded into the target space along, say, \((x^6, x^9)\) dimensions (using notation \(z = x^6 + ix^9\) for the corresponding complex co-ordinate). Different cylinders have different positions in the space \(V^\perp = (x^4, x^5, x^7, x^8)\). Moreover the cylinders are all glued together (see Fig.3). The ”effective” two-dimensional subspace of \(V^\perp\) we will describe it in terms of the complex coordinate \(\lambda = x^4 + ix^5\).

Introducing coordinate \(w = e^z\) to describe a cylinder, we see that the system of non-interacting branes (Fig.1) is given by \(z\)-independent equation

\[
P_{N_c}(\lambda) = \prod_{\alpha=1}^{N_c} (\lambda - \lambda_\alpha) = 0, \tag{6}
\]

while their bound state (Fig.3) is described rather by the complex curve \(\Sigma_{N_c}\) or:

\[
\Lambda^{N_c} \cosh z = P_{N_c}(\lambda) \tag{7}
\]

In the weak-coupling limit \(\Lambda \to 0\) (i.e. \(1/g^2 \sim \log \Lambda \to \infty\)) one comes back to disjoint branes \([6]\) \([7]\). Thus we finally got a 5-brane of topology \(R^4 \times \Sigma_{N_c}\) embedded into a subspace \(R^6 \times S^1\) (spanned by \(x^1, ..., x^6, x^9\)) of the full target space. The periodic coordinate is

\[
x^9 = \arg P_{N_c}(\lambda) = \Im \log P_{N_c}(\lambda) = \sum_{\alpha=1}^{N_c} \arg(\lambda - \lambda_\alpha) \tag{9}
\]

### 4 Integrable equations from brane picture

The arguments of the previous section show that the nontrivial part of the 5-brane world-volume looks rather similar to the spectral curves arising in the exact formulas of the SW anzatz. The way to justify this proposed in \([6]\) was based on parallels with the theory of integrable systems.

The integrable equations in this context arise as reductions on ”invisible” dimensions of the equations of motion (better the ”square-root” of the: the BPS-like conditions) of the world-volume theory. In \([4]\) Diaconescu using this idea obtained the Nahm equations. In \([6]\) it was demonstrated that the simplest way for getting algebraic equation for the topologically nontrivial part of brane configurations may be searched among the Hitchin systems \([10]\).

The Hitchin system on elliptic curve

\[
y^2 = (x - e_1)(x - e_2)(x - e_3)
\]

\[
x = \phi(z) \quad y = 2\phi'(z) \quad dz \sim 2\frac{dx}{y} \tag{10}
\]

\(4\) Eqs.(3), (7) and Fig.3 describe a hyperelliptic curve – a double covering of a punctured Riemann sphere,

\[
y^2 = \frac{\Lambda^{2N_c}}{4} \left( w - \frac{1}{w} \right)^2 = P_{N_c}^2(\lambda) - \Lambda^{2N_c} \tag{8}
\]
with $p$ marked points $z_1, \ldots, z_p$ can be defined by \[ \bar{\partial} \Phi_{ij} + (a_i - a_j) \Phi_{ij} = \sum_{\alpha=1}^{p} J_{ij}^{(\alpha)} \delta(z - z_\alpha) \] (11)

so that the solution has the form $(a_{ij} \equiv a_i - a_j)$

\[
\Phi_{ij}(z) = \delta_{ij} \left( p_i + \sum_{\alpha} J_{ii}^{(\alpha)} \partial \log \theta(z - z_\alpha|\tau) \right) + (1 - \delta_{ij}) e^{a_{ij}(z - z)} \sum_{\alpha} J_{ij}^{(\alpha)} \frac{\theta(z - z_\alpha + \frac{\text{Im} a_{ij}}{\pi})}{\theta(z - z_\alpha)} (12)
\]

The exponential (nonholomorphic) part can be removed by a gauge transformation

\[
\Phi_{ij}(z) \rightarrow (U^{-1} \Phi)_{ij}(z)
\] (13)

with $U_{ij} = e^{a_{ij} \bar{z}}$.

The additional conditions to the matrices $J_{ij}^{(\alpha)}$ are

\[
\sum_{\alpha=1}^{p} J_{ii}^{(\alpha)} = 0
\] (14)

having the clear meaning that the sum of all residues of a function $\Phi_{ii}$ is equal to zero, and

\[
\text{Tr} J^{(\alpha)} = m_\alpha
\] (15)

with $m_\alpha = \text{const}$ being some parameters ("masses") of a theory. The spectral curve equation becomes

\[
P(\lambda; z) \equiv \det_{N \times N} (\lambda - \Phi(z)) = \lambda^N + \sum_{k=1}^{N} \lambda^{N-k} f_k(z) = 0
\] (16)

where $f_k(z) \equiv f_k(x, y)$ are some functions (in general with $k$ poles) on the elliptic curve (10). If, however, $J^{(\alpha)}$ are restricted by

\[
\text{rank} J^{(\alpha)} \leq l \quad l < N
\] (17)

the functions $f_k(z)$ will have poles at $z_1, \ldots, z_p$ of the order not bigger than $l$. The generating differential, as usual, should be

\[
dS = \lambda dz
\] (18)

and its residues in the marked points $(z_\alpha, \lambda^{(i)}(z_\alpha))$ (different $i$ correspond to the choice of different sheets of the covering surface) are related with the mass parameters (17) by

\[
m_\alpha = \text{res}_{z_\alpha} \lambda dz \equiv \sum_{i=1}^{N} \text{res}_{z_\alpha} \frac{\lambda^{(i)}(z)}{z_\alpha} dz = \text{res}_{z_\alpha} \text{Tr} \Phi dz
\] (19)

It is easy to see that the general form of the curve (16) coincides with the general curves proposed in (3) at least for $l = 1$, i.e. sources of rank 1 (for the "rational" case the torus should be degenerated into a cylinder).

In (3) the Toda-chain spectral curve (3), (7) has been derived from the $SU(N_c)$ Hitchin system on torus with one marked point $p = 1$ in the double-scaling limit. Of course it is possible to write down the Hitchin
equations directly on the bare cylinder with trivial gauge connection

\[
\tilde{\partial}_v \tilde{L}^\text{TC} (v) = - \left( e^{-\alpha_0 \phi} E_{\alpha_0} + \sum_{\text{simple } \alpha} e^{\alpha_0 \phi} E_{-\alpha} \right) \delta (P_{\infty}) + \left( e^{-\alpha_0 \phi} E_{\alpha_0} + \sum_{\text{simple } \alpha} e^{\alpha_0 \phi} E_{\alpha} \right) \delta (P_0) \tag{20}
\]

where they can be easily solved giving rise to

\[
\tilde{L}^\text{TC} (v) = U^{-1} \tilde{L}^\text{TC} (w) U =
\]

\[
= \text{pH} + v^{-1} \left( e^{-\alpha_0 \phi} E_{-\alpha_0} + \sum_{\text{simple } \alpha} e^{\alpha_0 \phi} E_{\alpha} \right) + v \left( e^{-\alpha_0 \phi} E_{\alpha_0} + \sum_{\text{simple } \alpha} e^{\alpha_0 \phi} E_{-\alpha} \right) \tag{21}
\]

5 Generating Differential

Let us now turn to more subtle point and discuss how the auxiliary spectral Riemann surface is embedded into 11-dimensional target space. Partially that has been illustrated already above when the explicit formulas relating 11-dimensional co-ordinates \( x^I \) with the internal co-ordinates on the surface \( \lambda \) or \( z \) by \( \lambda = x^4 + ix^5 \) and \( z = x^6 + ix^9 \) were presented. In this section I will demonstrate that this embedding is in fact governed by the generating differential and its variations.

Already looking at Fig.3 it is clear that the corresponding Riemann surface is not compact; it means that the metric should have singularities at “infinities”. Indeed, the metric in the target-space is flat (in case of absense of matter) \( ds^2 \sim \sum_I (dx^I)^2 \) and it means that the area of the surface is measured by

\[
\Omega_\Sigma \sim \delta z \wedge \delta \bar{z} + \delta \lambda \wedge \delta \bar{\lambda} \tag{22}
\]

Since the BPS massive spectrum in the theory is determined by the states corresponding to the 2-branes wrapped over the nontrivial cycles of ”internal” complex manifold – in our case the Seiberg-Witten curve – the BPS masses should be proportional to the area of this surface [13]. This area is measured by another (holomorphic) two form

\[
\Omega \sim \delta \lambda \wedge \delta z \tag{23}
\]

which is directly related to the variation of generating differential or to the symplectic form of the corresponding integrable system.

The variation of generating differential over distinguished subfamily of moduli gives rise to holomorphic differentials

\[
\delta_{a_i} dS \sim d\omega_i \tag{24}
\]

The derivative over moduli (24) is taken after some connection is chosen – for example under condition that some function is covariantly constant. For the differential (4), (18) the canonical procedure implies that the covariantly constant function is \( z = \log w \) so that

\[
\delta dS |_{z = \text{const}} = \delta \lambda dz = - \sum_i \delta a_i \frac{p_{a_i}}{p_{\lambda}} dz = \sum_i \delta a_i d\omega_i \tag{25}
\]
From the point of view of $M$-theory one considers an effective theory on 5-brane world-volume with the co-ordinates $(x^0, \ldots, x^3, x^6, x^9)$. It means that when studying the 4-dimensional effective theory on "horizontal cylinders" on Fig.3 one should take the variation of $\lambda$ which has the sense of vev of some (Higgs) field keeping fixed the world-volume coordinates $x^6 = \text{Re} z$ and $x^9 = \text{Im} z$. Physically this corresponds to the fact that we are taking the variation \((24), (25)\) over the vev's of scalar fields only – which play the role of physical moduli in the system.

In principle, this procedure can be correctly defined if one notices that the differential $dS$ possesses double zeroes\(^5\) and it is the action of $g$ singular vector fields $L_{(i}^{\parallel}$ at these points that gives rise to the distinguished subfamily of co-ordinates on the moduli space. More detailed discussion of the properties of generating differential is beyond the scope of this note (see the last ref. of \([8]\) for some details).

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