Tachyon with an inverse power-law potential in a braneworld cosmology

Neven Bilić¹, Silvije Domazet¹ and Goran S Djordjevic²

¹ Division of Theoretical Physics, Rudjer Bošković Institute, Zagreb, Croatia
² Department of Physics, University of Niš, Niš, Serbia

E-mail: bilic@irb.hr, sdomazet@irb.hr and gorandj@junis.ni.ac.rs

Received 13 April 2017, revised 30 May 2017
Accepted for publication 6 July 2017
Published 24 July 2017

Abstract
We study a tachyon cosmological model based on the dynamics of a 3-brane in the bulk of the second Randall–Sundrum model extended to more general warp functions. A well known prototype of such a generalization is the bulk with a self-interacting scalar field. As a consequence of a generalized bulk geometry the cosmology on the observer brane is modified by the scale dependent four-dimensional gravitational constant. In particular, we study a power law warp factor which generates an inverse power-law potential \( V \propto \varphi^{-n} \) of the tachyon field \( \varphi \). We find a critical power \( n_{\text{cr}} \) that divides two subclasses with distinct asymptotic behaviors: a dust universe for \( n > n_{\text{cr}} \) and a quasi de Sitter universe for \( 0 < n < n_{\text{cr}} \).

Keywords: tachyon, braneworld cosmology, Randall–Sundrum model

1. Introduction

Braneworld cosmology is based on the scenario in which matter is confined on a brane moving in the higher dimensional bulk with only gravity allowed to propagate in the bulk [1, 2]. Among many braneworld models a particularly important are the two versions of the Randall–Sundrum (RS) model. The first RS model (RSI) [3] was originally proposed as a solution to the hierarchy problem in particle physics whereas the second RS model (RSII) [4] renders a mechanism for localizing gravity on the 3+1 dimensional universe embedded in a 4+1 spacetime without compactification of the extra dimension.

Immediately after the papers [3, 4] appeared, it was realized that the RS model is immersed in a wider framework of the so called AdS/CFT correspondence [5] (for a recent retrospect see appendix of [6]). At the same time it was realized that the RS model, as well as similar braneworld models, may have interesting cosmological implications [7, 8]. In particular, owing to the presence of an extra dimension and the bulk cosmological constant related to the brane...
tension, the usual Friedmann equations are modified [7] so the model can have predictions different from the standard cosmology and is therefore subject to cosmological tests [9].

The RS model, originally proposed with a pure 4+1-dimensional anti-de Sitter (AdS$_5$) bulk, can be extended to include matter in the bulk. A massive scalar field in the bulk was first introduced by Goldberger and Wise [10] to stabilize the brane separation in RSI. The RS model with a minimally coupled scalar field in the bulk, referred to as the thick brane model, has been constructed for maximally symmetric solutions on the brane [11]. It has been demonstrated that the bulk scalar potential and the corresponding brane potential can be derived from a superpotential in which case the solution of the static vacuum geometry reduces to a set of first-order BPS-like equations [12].

A noncanonical scalar field in the bulk with bulk tachyon Lagrangian has been considered in [13] where a thick braneworld has been investigated in the cosmological context. In particular, the stability under tensor perturbations and gravity localization has been demonstrated. In a subsequent paper [14] the stability under scalar perturbation has been studied for a braneworld with maximally symmetric geometry. Models where matter in the bulk contains a non-minimally coupled selfinteracting scalar have also been studied. Some interesting features of these models can be found in [15, 16] and references therein.

In this paper we study an RSII-type braneworld cosmology extended to more general warp factors. As an application, we study in particular a braneworld scenario based on this extended RSII with an effective tachyon field on the brane. What distinguishes the tachyon from the canonical scalar field is the Lagrangian of the Dirac–Born–Infeld (DBI) form [17]:

$$\mathcal{L} = V(\varphi) \sqrt{1 - g^{\mu\nu} \frac{\partial \varphi}{\partial x^\mu} \frac{\partial \varphi}{\partial x^\nu}}.$$  \hspace{1cm} (1.1)

A similar Lagrangian appears in the so called DBI inflation models [18]. In these models the inflation is driven by the motion of a D3-brane in a warped throat region of a compact space and the DBI field corresponds to the position of the D3-brane.

As shown by Abramo and Finelly [19] in a standard cosmological scenario, for the class of tachyon models with inverse power-law potentials $V(\varphi) \propto \varphi^{-n}$, the power $n = 2$ divides two subclasses with distinct behaviors in the asymptotic regimes. For $n < 2$ in the limit $\varphi \to \infty$, the pressure $p \to -1$ and the universe behaves as quasi-de Sitter. For $n > 2$, for large $\varphi$ the pressure $p \to 0$ very quickly yielding asymptotically a cold dark matter (CDM) domination. In the context of tachyon inflation, in both cases after the inflationary epoch the tachyon will remain a dominant component unless at the end of inflation, it decayed into inhomogeneous fluctuations and other particles. This period, known as reheating [20–22], links the inflationary epoch with the subsequent thermalized radiation era.

A simple tachyon model can be realized in the framework of RSII. The original RSII consists of two 3-branes in the AdS$_5$ bulk with line element

$$d^2_{(5)} = G_{ab} dX^a dX^b = e^{-2\nu |y|/\ell} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2,$$  \hspace{1cm} (1.2)

with observer’s brane placed at $y = 0$ and the negative tension brane pushed of to $y = \infty$. It may be easily shown [23] that one additional 3-brane moving in the AdS$_5$ bulk behaves effectively as a tachyon with a potential $V(\varphi) \propto \varphi^{-4}$ and hence drives a dark matter attractor. A more general tachyon potential could be obtained from more general bulk geometry. This can be achieved if one assumes a presence of matter in the bulk, e.g. in the form of a selfinteracting scalar field. The bulk scalar would change the bulk geometry depending on the scalar field potential. In addition, the braneworld cosmology would differ from that of the original RSII model.

3 Note that our tachyon field is located on the observer brane in contrast to the bulk tachyon of [13, 14].
A straightforward approach would be to start from a given bulk field potential and derive the bulk geometry which would, in turn, yield a tachyon potential of the effective tachyon field induced by the dynamical 3-brane. A more empirical approach would be to go the other way round: starting from a given, phenomenologically interesting tachyon potential one would fix the warp factor and one could, in principle, construct the bulk scalar-field selfinteraction potential. In this paper we will start from a warp factor of a general form and use the tachyon model to study the cosmology on the brane. In particular, we will study the effects of the warp factor of the power-law form which may be linked to the exponential superpotential of the bulk field. With this warp the tachyon will have an inverse power-law potential. We will analyze this type of tachyon potentials in four different cosmological scenarios: the standard tachyon cosmology, low and high density regimes of our braneworld model, and high density Gauss–Bonnet braneworld cosmology.

The remainder of the paper is organized as follows. In the next section we introduce the extended RSII and derive the corresponding braneworld cosmology. In section 3 we introduce the tachyon as a dynamical brane and derive the field equations in a covariant Hamiltonian formalism. The asymptotic solutions to the field equations for the inverse power law potential are presented in section 4. In section 5 we give the concluding remarks. Finally, in appendix we outline the derivation of the generalized RSII model with a scalar field in the bulk.

2. Braneworld cosmology

Our curvature conventions are as follows: $R^{abcd} = \partial_c \Gamma^a_{db} - \partial_d \Gamma^a_{cb} + \Gamma^e_{cb} \Gamma^a_{de} - \Gamma^e_{db} \Gamma^a_{ce}$ and $R_{ab} = R^c_{cab}$. So Einstein’s equations are $R_{ab} - \frac{1}{2} RG_{ab} = +8\pi G T_{ab}$.

Here, we derive the braneworld cosmology assuming that the bulk spacetime is given by the metric

$$ds^2(5) = G_{ab} dX^a dX^b = \psi(y)^2 \eta_{\mu\nu} dx^\mu dx^\nu - dy^2,$$

and the cosmology is determined by the motion of the brane. It will be sometimes advantageous to work in conformal coordinates with line element

$$ds^2(5) = \frac{1}{\chi(z)} (g_{\mu\nu} dx^\mu dx^\nu - dz^2).$$

The two metrics are related by the coordinate transformation

$$dz = \frac{dy}{\psi(y)},$$

and

$$\chi(z) = \frac{1}{\psi(y(z))}.$$

The observer brane is placed at $y = y_{br}$ ($z = z_{br}$) and, as in the original RSII model, we assume the $Z_2$ orbifold symmetry $y \leftrightarrow y_{br} - y (z \leftrightarrow z_{br}/z)$, so the region $-\infty < y \leq y_{br}$ ($0 < z \leq z_{br}$) is identified with $y_{br} \leq y < \infty$ ($z_{br} \leq z < \infty$).

Next, we assume that observer’s brane has additional matter represented by the Lagrangian $\mathcal{L}$, i.e. the brane action is

$$S_{br}[h] = \int_{\Sigma} d^4x \sqrt{-h} (-\lambda(y) + \mathcal{L}[h]),$$
with a y-dependent brane tension $\lambda$. Then, we allow the brane to move in the bulk along the fifth coordinate $y$. In other words, the brane hypersurface $\Sigma$ is time dependent and may be defined by

$$\psi(y)^2 - a(t)^2 = 0, \quad (2.6)$$

where $a = a(t)$ is an arbitrary function. The normal to $\Sigma$ is then given by

$$n_\mu \propto \partial_\mu (\psi(y) - a(t)) = (-\dot{a}, 0, 0, 0, \psi') \quad (2.7)$$

and, using the normalization $g^{\mu\nu}n_\mu n_\nu = -1$, one finds the nonvanishing components

$$n_t = \frac{\dot{a}}{\psi'} \left( 1 - \frac{\dot{a}^2}{\psi'^2} \right)^{-1/2} \quad (2.8)$$

$$n_y = -\left( 1 - \frac{\dot{a}^2}{\psi'^2} \right)^{-1/2}. \quad (2.9)$$

Using this, we find the induced line element on the brane

$$ds^2_{ind} = (G_{ab} + n_an_b)dx^adx^b = n(t)^2dt^2 - a(t)^2dx^2, \quad (2.10)$$

where

$$n^2 = a^2 - \frac{\dot{a}^2}{\psi'^2}. \quad (2.11)$$

It is understood that the argument $y$ of $\psi'(y)$ is implicitly time dependent through (2.6). The Friedmann equations on the brane follow directly from the junction conditions [24]

$$[[K^\mu_\nu - \delta^\mu_\nu K^\alpha_\alpha]] = 8\pi G_5 (\lambda(y)\delta^\mu_\nu + T^\mu_\nu), \quad (2.12)$$

where $K^\mu_\nu$ is the pullback of the extrinsic curvature tensor defined in appendix (equation (A.4)). The energy momentum tensor $T^\mu_\nu = \text{diag}(\rho, -p, -p, -p)$ corresponds to the Lagrangian $\mathcal{L}$. From (2.12) together with (2.8)–(2.11) we obtain

$$\frac{(\partial_t a)^2}{n^2a^2} + \frac{\psi'^2}{\psi'^2} = \left( \frac{4\pi G_5}{3} \right)^2 (\lambda + \rho)^2. \quad (2.13)$$

The first term on the left-hand side of (2.13) is the square of the Hubble expansion rate for the metric (2.10) on the brane

$$H^2 = \frac{(\partial_t a)^2}{n^2a^2}. \quad (2.14)$$

Then, the first Friedmann equation takes the form

$$H^2 = \frac{8\pi G_N}{3} \left( \frac{4\pi G_5}{3k} \lambda + \left( \frac{4\pi G_5}{3} \right)^2 \rho^2 + \left( \frac{4\pi G_5}{3} \right)^2 \lambda - \frac{\psi'^2}{\psi'^2} \right), \quad (2.15)$$

where $G_N$ is the four-dimensional Newton constant and $k$ is a mass scale related to $G_5$,

$$k = \frac{G_N}{G_5}. \quad (2.16)$$

Generally, $\lambda, \psi$ and $\psi'$ are implicit functions of $a(t)$ through their dependence on $y$ which in turn is a function of $a$ via (2.6). For a pure AdS$_5$ bulk with curvature radius $\ell = 1/k$ we have
\[ \psi' / \psi = -k, \ k = 1/\ell, \ \lambda \text{ is constant and with the RSII fine tuning condition we recover the usual RSII expressions (see, e.g. [6]). Henceforth we will assume } |\psi' / \psi| \lesssim k \text{ and the average of } |\psi' / \psi| \text{ will basically represent a warp compactification scale.} \]

A modified Friedmann equation similar to (2.15) was derived by Brax et al [25] for a brane-world with a scalar field \( \Phi \) in the bulk with a time dependent geometry. The full expression in our notation reads

\[ H^2 = \frac{4 \pi G_5}{9} W \rho \left( \frac{8 \pi G_5}{3} \rho \right)^2 - \frac{4 \pi G_5}{9 a^4} \int \frac{dW}{d\tau} a^4 \rho - \frac{1}{6 a^4} \int d\tau \frac{da^4}{d\tau} \left( \frac{1}{2} (\partial_\tau \Phi)^2 - U_{\text{eff}} \right), \]

(2.17)

where \( \tau \) is the synchronous time and \( W = W(\Phi) \) is the superpotential. The effective potential \( U_{\text{eff}} = U_{\text{eff}}(\Phi) \) is defined as

\[ U_{\text{eff}} = \frac{W^2}{6} - \frac{1}{8} \left( \frac{dW}{d\Phi} \right)^2 + U \]

(2.18)

and \( U = U(\Phi) \) is the bulk-field potential. It is understood that \( \Phi \) and its derivative \( \partial_\tau \Phi \) are functions of \( \tau \) only. The contribution of the last two terms in (2.17) is referred to as the \textit{retarded effect} [25]. In deriving (2.17) the contribution of dark radiation has not been taken into account.

Equation (2.17) reduces to a simpler equation similar to our (2.15) if one assumes that the bulk scalar field is evolving much slower than the scale factor. On this assumption we have

\[ \frac{1}{W} \frac{dW}{d\tau} \ll \frac{1}{a^4 \rho} \frac{da^4}{d\tau} \]

(2.19)

and

\[ \frac{da^4}{d\tau} \left( \frac{1}{2} (\partial_\tau \Phi)^2 - U_{\text{eff}} \right) \simeq \frac{d}{d\tau} \left( a^4 \frac{1}{2} (\partial_\tau \Phi)^2 - a^4 U_{\text{eff}} \right), \]

(2.20)

so the third term on the righthand side of (2.17) can be neglected compared with the first term and the integration in the last term is trivially performed. The constant of integration can be set to zero as it would only contribute to a dark radiation term of the form \( 1/a^4 \) which has been ignored anyway. Hence, up to a dark radiation term, equation (2.17) reduces to

\[ H^2 = \frac{4 \pi G_5}{9} W \rho + \left( \frac{8 \pi G_5}{3} \rho \right)^2 + \frac{1}{6} \left( U_{\text{eff}} - \frac{1}{2} (\partial_\tau \Phi)^2 \right). \]

(2.21)

Now we show explicitly that our equation (2.15) is equivalent to (2.21). First we identify the superpotential and bulk potential

\[ W(\Phi) \equiv 8 \pi G_5 \Lambda(y), \]

(2.22)

\[ U(\Phi) \equiv \frac{1}{2} \Phi'^2 - 6 \frac{\psi'^2}{\psi^2}, \]

(2.23)

where \( \Phi = \Phi(y) \) is yet unspecified function of \( y \). Next, using equations (2.18), (2.22) and (2.23) as the defining equations for \( U_{\text{eff}}, \) we can express (2.15) in the form

\[ H^2 = \frac{4 \pi G_5}{9} W \rho + \left( \frac{8 \pi G_5}{3} \rho \right)^2 + \frac{1}{6} U_{\text{eff}} - \frac{1}{12} \left( \Phi'^2 - \frac{1}{4} \left( \frac{dW}{d\Phi} \right)^2 \right). \]

(2.24)
Finally, by demanding that the function $\Phi$ satisfies

$$\Phi'^2 - \frac{1}{4} \left( \frac{\partial W}{\partial \Phi} \right)^2 = (\partial_t \Phi)^2,$$

our equation (2.15) takes the form identical to (2.21). Hence, we have demonstrated that equations (2.15) and (2.21) are equivalent.

By manipulating equation (2.25) with the help of (2.7), (2.11), and (2.14) we find that $\Phi$ satisfies another equation equivalent to (2.25):

$$\Phi' = \frac{1}{2} \frac{\partial W}{\partial \Phi} \left( 1 - H^2 \frac{\psi'^2}{\psi^2} \right)^{-1/2}.$$

(2.26)

Given the functions $\lambda(y)$ and $\psi(y)$, equations (2.22), (2.23) and (2.26) together with (2.15) determine parametrically $W$ and $U$ as functions of $\Phi$. Note that equation (2.26) is consistent with the first equation in (A.14) in the static limit $H \to 0$.

In this way, the scale dependence of the brane tension $\lambda(y)$ which has not been specified yet, can be attributed to a scalar field dynamics in the bulk. Due to the retarded effect, equation (2.17), even in its reduced form (2.15), is rather complicated. However, we can simplify our braneworld cosmology by assuming that the contribution of the retarded effects in (2.15) are negligible compared to $H^2$, i.e. we will assume

$$\left( \frac{4\pi G_5}{3} \right)^2 \approx \frac{\psi'^2}{\psi^2}.$$

(2.27)

This assumption is motivated by the junction condition (A.10) (as a generalization of the fine tuning condition of RSII) which is exact in the static case. Then our approximated braneworld cosmology is defined by the Friedmann equation

$$H^2 = \frac{8\pi G}{3} \rho + \left( \frac{4\pi G_N}{3k} \right)^2 \rho.$$

(2.28)

Here $G = G(a)$ is a scale dependent effective gravitational constant defined as

$$G(a) = \frac{G_N}{k} \frac{d\chi}{dz} \bigg|_{z=\chi^{-1}(1/a)},$$

(2.29)

where $\chi^{-1}$ denotes the inverse function of $\chi$. The second Friedmann equation is easily obtained by combining the time derivative of (2.28) with energy-momentum conservation yielding

$$\dot{H} = - \left( 4\pi G + 3 \left( \frac{4\pi G_N}{3k} \right)^2 \rho \right) (p + \rho) + \frac{4\pi}{3} \frac{dG}{da} \dot{\rho}.$$

(2.30)

As a promising future research topic it would be of interest to extend our approach along the lines of [26] where the warp factor was allowed to depend on the radial braneworld coordinate $r$ in addition to the usual $y$ and $t$ dependence. A natural extension worth of investigating would be including $r$ dependence of the bulk scalar field in addition to its $y$ dependence.

In the following we shall abbreviate by BWC the braneworld cosmology described by (2.28) and (2.30). Thus, the Friedmann equations of BWC differ from those of the original RSII in the scale dependence of the effective gravitational constant $G$ and in one additional term in the second equation that depends on the derivative of $G$. This term will be suppressed if the scale dependence of $G$ is weak.
Equation (2.29) imposes certain restrictions on the function $\chi$. First, we need $G(a)$ to be positive which restricts $\chi(z)$ to the class of monotonously increasing functions of $z$ and, as a consequence of (2.3), $\psi$ must be a monotonously decreasing function of $y$. Second, the variation of $G(a)$ is constrained by the big bang nucleosynthesis [27] and other cosmological and astrophysical observations [28]. The observations are roughly consistent with the constraint

$$\left| \frac{\dot{G}}{G} \right|_{\text{today}} \lesssim 10^{-12} \text{yr}^{-1}$$

which, in turn, implies

$$\left| \frac{a}{G} \frac{dG}{da} \right|_{\text{today}} = \frac{1}{H_0} \left| \frac{\dot{G}}{G} \right|_{\text{today}} \lesssim 1.43 \times 10^{-2},$$

where we have used the value $H_0^{-1} = 14.3$ Gyr corresponding to the Planck 2015 estimate of the Hubble constant $H_0 = 68$ km s$^{-1}$ Mpc$^{-1}$ [29]. Using the definition (2.29), we find a relation

$$\frac{a}{G} \frac{dG}{da} = -\frac{\chi \chi'}{\chi^2} \bigg|_{z=\chi^{-1}(1/a)}.$$  

Thus, equation (2.32) imposes a constraint also on $\chi(z)$. For our purpose the metric of the power law form $\chi \propto z^{n/4}$ is of particular interest. In this case equation (2.33) reads

$$\frac{a}{G} \frac{dG}{da} = -\frac{n-4}{n},$$

and equation (2.32) imposes a constraint on the power

$$|n-4| \lesssim 0.057,$$

where the central value $n = 4$ corresponds to the original RSII setup with constant $G$.

As demonstrated in appendix, the power-law warp $\chi \propto z^{n/4}$ with $n \geq 4$ can be attributed to a selfinteracting scalar field $\Phi$ in the bulk with exponential superpotential $W \propto \exp(\gamma \Phi)$, Then, the constraint (2.35) implies a constraint on the parameter $\gamma$ (see also [30])

$$\gamma^2 \lesssim 0.477 \times 10^{-2},$$

which is less stringent than the order of magnitude estimate [31] $\gamma \lesssim 0.01$ based on the solar system bounds on the Edington parameter [32].

The constraint (2.32) yielding (2.35) and (2.36) is obtained from astrophysical and cosmological bounds on variation of $G$ for the period between BBN (corresponding roughly to the scales $a \simeq 10^{-9}$) and today. It is conceivable that the bounds on variations of $G$ with $a$ are less restrictive for the early cosmology prior to BBN.

3. Dynamical brane as a tachyon

In this section we introduce a tachyon via the dynamical brane. In addition to the observer brane at $y = y_{\text{br}}$ we place a non BPS positive tension brane at $y > y_{\text{br}}$ in the bulk with metric (2.1). Our setup is similar to that of Lykken and Randall [33].

The action of the 3+1 dimensional brane in the five dimensional bulk is equivalent to the Dirac–Born–Infeld description of a Nambu-Goto 3-brane. [34, 35]. Consider a 3-brane
moving in the 4+1-dimensional bulk spacetime with coordinates $X^a$, $a = 0, 1, 2, 3, 4$. The points on the brane are parameterized by $X^a(x^\mu), \mu = 0, 1, 2, 3$, where $x^\mu$ are the coordinates on the brane. The brane action is given by

$$S_{br} = -\sigma \int d^4x \sqrt{-\det(g^{(ind)})}, \quad (3.1)$$

where $\sigma$ is the brane tension and $g^{(ind)}_{\mu\nu}$ is the induced metric or the ‘pull back’ of the bulk space-time metric $G_{ab}$ to the brane,

$$g^{(ind)}_{\mu\nu} = G_{ab} \frac{\partial X^a}{\partial x^\mu} \frac{\partial X^b}{\partial x^{\nu}}. \quad (3.2)$$

Taking the Gaussian normal parameterization $X^a(x^\mu) = (x^\mu, z(x^\mu))$, we find

$$g^{(ind)}_{\mu\nu} = \frac{1}{\chi(z)} (g_{\mu\nu} - z_{,\mu} z_{,\nu}). \quad (3.3)$$

A straightforward calculation of the determinant yields the brane action in the form

$$S_{br} = -\sigma \int d^4x \sqrt{-g} \chi^{-4} \sqrt{1 - X^{1/2}}, \quad (3.4)$$

where we have introduced the abbreviation

$$X = g^{\mu\nu} z_{,\mu} z_{,\nu}. \quad (3.5)$$

Hence, we have obtained a tachyon Lagrangian with potential

$$V(z) = \sigma \chi(z)^{-4}, \quad (3.6)$$

where the fifth conformal coordinate $z = z(x)$ has become a dynamical tachyon field. An attempt was made to generalize the action (3.4) by replacing $(1 - X)^{1/2}$ by $(1 - X)^q$ with $q$ being an arbitrary positive power [36]. However, only the action with $q = 1/2$ stems from d-brane dynamics in a $d + 1 + 1$ bulk.

Next, we derive the tachyon field equations from the action (3.4). The tachyon Lagrangian takes the form

$$\mathcal{L} = -\frac{\sigma}{\chi(z)^2} \sqrt{1 - g^{\mu\nu} z_{,\mu} z_{,\nu}},$$

and in the following we assume that the function $\chi(z)$ is known. Note that for a pure AdS$_5$ bulk we have $\chi = k z$ and we reproduce the brane action of [23] if we identify the scale $k$ with the inverse of the AdS$_5$ curvature radius $\ell$.

It is important to stress that the cosmology in section 2 was derived assuming that the observer brane is moving in the bulk with time independent geometry, and equation (2.6) relates the position of the observer brane to the cosmological scale $a$. However, in this section we work in the gauge where the observer brane is at a fixed position and the cosmology will reflect the time dependence of the bulk metric in addition to the time dependence of the dynamical brane position. We consider a spatially flat FRW spacetime on the observer brane with four dimensional line element in the standard form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a(t)^2 (dr^2 + r^2 d\Omega^2), \quad (3.8)$$

where, unlike in section 2, the time $t$ is synchronous. In the cosmological context it is natural to assume that the tachyon condensate is comoving, i.e. the velocity components are $u_{\mu} = (1, 0, 0, 0)$ and $X$ becomes simply
\( X = \dot{z}^2 \).  \hfill (3.9)

The treatment of our system is conveniently performed in the covariant Hamiltonian formalism based on earlier works on symplectic formalism of De Donder [37] and Weyl [38] (for recent reviews see [39, 40]; for details and application in cosmology see [23]). For this purpose we first define the conjugate momentum field as

\[ \pi^\mu_z = \frac{\partial L}{\partial \dot{z}_\mu}. \]  \hfill (3.10)

In the cosmological context \( \pi^\mu_z \) is time-like so we may also define its magnitude as

\[ \pi_z = \sqrt{g_{\mu\nu} \pi^\mu_z \pi^\nu_z}. \]  \hfill (3.11)

The Hamiltonian density may be derived from the stress tensor corresponding to the Lagrangian (3.7) or by the Legendre transformation. Either way one finds

\[ \mathcal{H} = \frac{\sigma}{\chi^4} \sqrt{1 + \pi_z^2 / \sigma^2}. \]  \hfill (3.12)

Then, we can write Hamilton’s equations in the form [23]

\[ \dot{z} = \frac{\partial \mathcal{H}}{\partial \pi_z}, \]  \hfill (3.13)

\[ \dot{\pi}_z + 3H\pi_z = -\frac{\partial \mathcal{H}}{\partial z}. \]  \hfill (3.14)

In the spatially flat cosmology the Hubble expansion rate \( H \) is related to the Hamiltonian via the modified Friedmann equation (2.15). As the cosmological scale is no longer related to the observer’s brane position, the function \( \chi(a) \) need not satisfy the condition (2.6). Nevertheless, the brane cosmology is governed by the same (approximate) Friedmann equation (2.28) in which the scale dependent gravitational constant will have a functional dependence on the the warp factor as dictated by equation (2.27), i.e. \( G \propto \chi_z \). However, the functional dependence on the cosmological scale will be subject to the field equations (3.13) and (3.14) together with equation (2.28) which can be written as

\[ H \equiv \frac{\dot{a}}{a} = \sqrt{\frac{8\pi G_N}{3} \mathcal{H} \left( \frac{\chi_z}{k} + \frac{2\pi G_N}{3k^2} \mathcal{H} \right)}. \]  \hfill (3.15)

To solve the system of equations (3.13)--(3.15) it is convenient to rescale the time as \( t = \tau/k \) and express the system in terms of dimensionless quantities. Besides, by appropriately rescaling the tachyon field \( z \) and its conjugate field \( \pi_z \) we can eliminate the brane tension \( \sigma \) from the equations. To this end we introduce the dimensionless functions

\[ h = H/k, \quad \varphi = kz, \quad \pi_\varphi = \pi_z / \sigma, \]  \hfill (3.16)

and we rescale the Lagrangian and Hamiltonian to obtain the rescaled dimensionless pressure and energy density:

\[ p = \frac{L}{\sigma} = -\frac{1}{\chi^4 \sqrt{1 + \chi_z^2 / \sigma^2}} = -\frac{1}{\chi^4 \sqrt{1 - \varphi^2}}, \]  \hfill (3.17)
\[ \rho = \frac{\mathcal{H}}{\sigma} = \frac{1}{\chi^4} \sqrt{1 + \chi^8 \pi_\phi^2} = \frac{1}{\chi^4} \sqrt{1 - \dot{\varphi}^2}. \] (3.18)

In these equations and from now on the overdot denotes a derivative with respect to \( \tau \). Then we introduce a combined dimensionless coupling

\[ \kappa^2 = \frac{8\pi G_S}{k} \sigma = \frac{8\pi G_N}{k^2} \sigma \] (3.19)

and from (3.13)–(3.15) we obtain the following set of equations

\[ \dot{\phi} = \frac{\chi^4 \pi_\varphi}{\sqrt{1 + \chi^8 \pi_\varphi^2}} = \frac{\pi_\varphi}{\rho}, \] (3.20)

\[ \dot{\pi}_\varphi = -3h \pi_\varphi + \frac{4\chi_\varphi}{\chi^5 \sqrt{1 + \chi^8 \pi_\varphi^2}}. \] (3.21)

Here

\[ h = \sqrt{\frac{\kappa^2}{3} \rho \left( \chi_\varphi + \rho \frac{\kappa^2}{12} \right)}, \] (3.22)

where \( \chi_\varphi \) is an abbreviation for \( \partial \chi / \partial \varphi \). Obviously, the explicit dependence on \( \sigma \) and \( k \) in equations (3.20)–(3.22) is eliminated leaving one dimensionless free parameter \( \kappa \) which could, in principle, be fixed from phenomenology.

### 4. Cosmological solutions to the field equations

Next, we analyze in detail the tachyon with potential

\[ V(\varphi) = \varphi^{-n}, \quad \text{or} \quad \chi(\varphi) = \varphi^{n/4}, \] (4.1)

As shown in Appendix, this inverse power-law dependence for \( n > 4 \) can be derived from the exponential superpotential (A.23) in the braneworld model with a scalar in the bulk. According to (2.29) and (3.6), the potential (4.1) being a monotonously decreasing function of \( \varphi \), is consistent with the positivity requirement for \( G(a) \). We will assume that the bounds on variations of \( G \) with \( a \) discussed in section 2 do not apply to the pre-BBN cosmology, in particular during inflationary epoch, so we will ignore the constraint (2.35).

With (4.1) equations (3.20) and (3.21) become

\[ \dot{\phi} = \frac{\varphi^n \pi_\varphi}{\sqrt{1 + \pi_\varphi^2 \varphi^{2n}}}, \] (4.2)

\[ \dot{\pi}_\varphi = -3h \pi_\varphi + \frac{n}{\varphi^{n+1} \sqrt{1 + \pi_\varphi^2 \varphi^{2n}}}. \] (4.3)

We will try to solve these equations by the ansatz

\[ \pi_\varphi = c_n \varphi^{m-n}, \] (4.4)

where the constants \( c_n \) and \( m \) are to be fixed by the field equations. With this ansatz the density is given by
\[ \rho = \varphi^{-n}(1 + \pi_\varphi^2 \varphi^{2m})^{1/2} = \varphi^{-n}(1 + c_n^2 \varphi^{2m})^{1/2} \]  
(4.5)

and equation (4.2) becomes

\[ \dot{\varphi} = \frac{c_n \varphi^m}{\sqrt{1 + c_n^2 \varphi^{2m}}} . \]  
(4.6)

Furthermore, the time derivative of (4.4) together with (4.6) yields

\[ \pi \dot{\varphi} = c_n^2 (m - n) \varphi^{2m - n - 1} \left(1 + c_n^2 \varphi^{2m}\right)^{-1/2} . \]  
(4.7)

We will look for solutions in the low and high energy density regimes of BWC characterized by the conditions \( \kappa^2 \rho / 12 \ll \chi, \varphi \) and \( \kappa^2 \rho / 12 \gg \chi, \varphi \), respectively. In these regimes the Hubble rate is respectively given by

\[ h = \frac{\kappa}{\sqrt{3}} \rho^{1/2} \chi^{1/2} , \quad h = \frac{\kappa^2}{6} \rho . \]  
(4.8)

It is advantageous to analyze these two particular cosmologies by making use of a more general equation

\[ h = h_0 \rho^\alpha \chi_\varphi^\beta , \]  
(4.9)

where the constants \( h_0 \) and \( \alpha \) are positive and \( \beta < 1 \). In particular, \( h_0, \alpha, \) and \( \beta \) are respectively equal to \( \kappa^2 / 6, 1, \) and 0, at high density and \( \kappa / \sqrt{3}, 1 / 2, \) and 1/2 at low density. Equation (4.9) also includes the standard cosmology analyzed by Abramo and Finelli [19] and the Gauss–Bonnet braneworld (GBB) at high density [41–44]. In the standard cosmology we have \( h_0 = \kappa / \sqrt{3}, \alpha = 1 / 2, \) and \( \beta = 0 \) whereas in the high density limit of GBB we have [41, 44] \( h_0 = \kappa^2 / 3, \alpha = 1 / 3, \) and \( \beta = 0 \). In the latter scenario the coupling \( \kappa^2 \) is defined by the first equation (3.19) where we have identified the mass scale \( 4k^2 \) with the inverse of the GB coupling, i.e. we set \( 1 / k = 4 \sqrt{\alpha_{GB}} \).

Applying (4.4)–(4.7) and (4.9) we obtain an identity

\[ c_n^2 (m - n) \varphi^{2m} + 3h_0cn \left(\frac{n}{4}\right)^\beta \varphi^{m+1-\alpha n+\beta n/4-\beta} \left(1 + c_n^2 \varphi^{2m}\right)^{\alpha/2+1/2} - n = 0 , \]  
(4.10)

which must hold for any \( \varphi \). It is easily seen that this identity will be satisfied if and only if \( m = 0 \) and \( n = n_{cr} \), where the critical power \( n_{cr} \) depends on \( \alpha \) and \( \beta \),

\[ n_{cr} = \frac{1 - \beta}{\alpha - \beta / 4} . \]  
(4.11)

In particular \( n_{cr} \) equals 2 in the standard cosmology [19], 3 in the high density GBB, and, respectively, 4/3 and 1 in the low and high density regimes of BWC. For \( n = n_{cr} \) the derivative \( \dot{\varphi} \) is simply a constant yielding

\[ \varphi = \frac{c_{cr}}{\sqrt{1 + c_{cr}^2}} \tau , \]  
(4.12)

where, for simplicity, we have set the integration constant to zero and \( c_{cr} \) stands for \( c_{n_{cr}} \). As a consequence of (4.10), the constant \( c_{cr} \) satisfies the equation

\[ c_{cr}^2 (1 + c_{cr}^2)^{\alpha - 1} = \left(\frac{4^\beta n_{cr}^{1-\beta}}{3h_0}\right)^2 , \]  
(4.13)
which, in general, cannot be solved for \( c_{cr} \). However, for each of the four cases mentioned above, equation (4.13) becomes relatively simple with solutions

\[
c_{cr} = \begin{cases} 
8\sqrt{2}/(3\kappa)^2 \left( 1 + \sqrt{1 + 4(3\kappa/4)^4} \right)^{1/2} & \text{BWC, low density} \\
2/(3\kappa^2) & \text{BWC, high density} \\
2\sqrt{2}/(3\kappa^2) \left( 1 + \sqrt{1 + 9\kappa^4/4} \right)^{1/2} & \text{standard cosmology} \\
1/(3\kappa^2) \left( 1 + u^{1/3} + u^{-1/3} \right) & \text{GBB, high density,} 
\end{cases}
\] (4.14)

where, in the last line

\[
u = 1 + 27\kappa^4/2 + \sqrt{27\kappa^2/2 - \sqrt{4 + 27\kappa^4}}. \tag{4.15}
\]

From (3.17) and (3.18) it follows that the equation of state is a negative constant

\[w \equiv p/\rho = -\frac{1}{1 + c_{cr}^2}\]

and hence describes a dark energy fluid. Note that in the strong coupling limit, i.e. \( \kappa \to \infty \), corresponding to large brane tensions, we have \( c_{cr} \to 0 \) and the fluid approaches the cosmological constant.

From (4.5) and (4.12) we obtain the density as a function of \( \tau \)

\[\rho(\tau) = \rho_0 \tau^{-n_{cr}}, \tag{4.17}\]

where

\[
\rho_0 = \frac{(1 + c_{cr}^2)^{n_{cr}/2 + 1/2}}{c_{cr}^{n_{cr}}}.
\] (4.18)

Furthermore, from (4.8) we find that the cosmological scale behaves as a power of \( \tau \)

\[a(\tau) = a_0 \tau^q, \tag{4.19}\]

where

\[q = \frac{n_{cr}}{3c_{cr}^2}. \tag{4.20}\]

Finally \( \rho \) can be expressed as a function of the cosmological scale as

\[\rho(a) = \rho_0 \left( \frac{a}{a_0} \right)^{-n_{cr}/q} = \rho_0 \left( \frac{a}{a_0} \right)^{-3c_{cr}^2/(1+c_{cr}^2)}. \tag{4.21}\]

Obviously, in the limit \( \kappa \to \infty \) we have \( c_{cr} \to 0 \), and the universe approaches de Sitter. In contrast, in the weak coupling limit \( c_{cr} \to \infty \), the universe behaves as dust.

For \( n \neq n_{cr} \), equation (4.10) admits no solution. Nevertheless, it can be solved in the asymptotic regimes of high and small \( \varphi \). In these regimes we distinguish two cases: a) \( \varphi \to \infty \) and \( m > 0 \) or \( \varphi \to 0 \) and \( m < 0 \), and b) \( \varphi \to \infty \) and \( m < 0 \) or \( \varphi \to 0 \) and \( m > 0 \).

4.1. (a) \( \varphi \to \infty \) and \( m > 0 \) or \( \varphi \to 0 \) and \( m < 0 \)

Keeping only the dominant terms in (4.10) in the limit \( \varphi \to \infty \) for \( m > 0 \) or \( \varphi \to 0 \) for \( m < 0 \) we find
\[ c_\nu^2 (m - n) + 3h_0 (n/4)^\beta c_\nu^{\alpha+2} \varphi^{m-1} = 0, \]  
(4.22)

where

\[ s = n \left( \alpha - \frac{\beta}{4} \right) + \beta - 1 = \left( \frac{n}{n_{\text{cr}}} - 1 \right) (1 - \beta). \]  
(4.23)

Then, to satisfy (4.22) for any \( \varphi \) we must have

\[ m = \frac{s}{\alpha}. \]  
(4.24)

From this it follows \( n \leq n_{\text{cr}} \) for \( m \leq 0 \). The constant \( c_\nu \) is given by

\[ c_\nu = \left( \frac{1 - \beta + n\beta/4}{3h_0 \alpha} \right)^{1/\alpha} \left( \frac{4}{n} \right)^{\beta/\alpha} \]  
(4.25)

and we have

\[ \rho = \pi_\varphi = c_\nu \varphi^{-\frac{1-\beta+n\beta/4}{\alpha}}. \]  
(4.26)

The equation of state

\[ w = -(1 - \varphi^2) = -\frac{1}{1 + c_\nu^{2\varphi^{4\alpha}}}. \]  
(4.27)

approaches 0 for large \( \varphi \) with \( n > n_{\text{cr}} \) or small \( \varphi \) with \( n < n_{\text{cr}} \). Hence, the system (4.2) and (4.3) has a dark-matter attractor at \( \dot{\varphi}^2 = 1 \) or \( \varphi = \tau \) in both limits. The large and small \( \varphi \) limits correspond to the large and small \( \tau \) limits, respectively. In these limits we obtain the following behavior of the density and cosmological scale as functions of time

\[ \rho(\tau) = c_\nu \tau^{-\frac{1-\beta+n\beta/4}{\alpha}}, \]  
(4.28)

\[ a(\tau) = a_0 \tau^{\frac{1-\beta+n\beta/4}{3\alpha}}, \]  
(4.29)

so that the density as a function of the cosmological scale

\[ \rho(a) = c_\nu a_0^3 a^{3(1-\beta+n\beta/4)/\alpha}. \]  
(4.30)

clearly demonstrates the dust behavior.

4.2. (b) \( \varphi \to \infty \) and \( m < 0 \) or \( \varphi \to 0 \) and \( m > 0 \)

This case is relevant for an inflationary scenario. Namely, as we shall shortly see, the slow roll condition \( \dot{\varphi} \ll 1 \) for the tachyon inflation \([45, 46]\) is met in both \( \varphi \to \infty \) and \( \varphi \to 0 \) limits.

Keeping the dominant terms in (4.10) in the limit \( \varphi \to \infty \) for \( m < 0 \) or \( \varphi \to 0 \) for \( m > 0 \) the equation reduces to

\[ 3h_0 (n/4)^\beta c_\nu^{\alpha+2} \varphi^{m-1} = n \]  
(4.31)

yielding

\[ m = s = \left( \frac{n}{n_{\text{cr}}} - 1 \right) (1 - \beta), \]  
(4.32)

so \( m \leq 0 \) implies \( n \leq n_{\text{cr}} \) as before. The coefficient \( c_\nu \) is now given by
\[ c_n = \frac{4^\beta n^{1-\beta}}{3 h_0}. \]  

(4.33)

Then, using

\[ \rho = \varphi^{-n}, \quad \pi_\varphi = c_n \varphi^{s-n}, \]  

(4.34)

we obtain

\[ \dot{\varphi} = \pi_\varphi / \rho = c_n \varphi^s, \]  

(4.35)

from which it follows \( \dot{\varphi} \to 0 \) in both \( \varphi \to \infty \) (with \( s < 0 \)) and \( \varphi \to 0 \) (with \( s > 0 \)) limits. Hence, the equation of state \( w \to -1 \) in both limits. However, the solution to (4.35) critically depends on whether \( s \) is equal, greater or smaller than 1:

\[ \varphi(\tau) = \begin{cases} 
[c_n(1-s)(\tau - \tilde{\tau})]^{1/(1-s)} & s < 1, \tau > \tilde{\tau} \\
\varphi_0 \exp c_n \tau & s = 1, \tau < \tilde{\tau} \\
\exp \{-b_n e^{-2c_n \tau}\} & s > 1, \tau > \tilde{\tau}
\end{cases} \]  

(4.36)

where \( \tilde{\tau} \) and \( \varphi_0 > 0 \) are arbitrary constants of integration. Obviously, the limits \( \varphi \to \infty \) (with \( s < 0 \)) and \( \varphi \to 0 \) (with \( 0 < s < 1 \)) correspond to the limits \( \tau \to \infty \) and \( \tau \to \tilde{\tau} \), respectively. The limit \( \varphi \to 0 \) (with \( s \geq 1 \)) correspond to \( \tau \to -\infty \). As a consequence of (4.36), the cosmological scale factor evolves as

\[ \frac{a(\tau)}{a_0} = \begin{cases} 
\exp \{(\tau - \tilde{\tau})^r / \tau_0^r\} & s < 0, \tau > \tilde{\tau} \\
\exp \{-|\tau - \tilde{\tau}|^r / \tau_0^r\} & 0 < s < 1, \tau > \tilde{\tau} \\
\exp \{-b_n e^{-2c_n \tau}\} & s > 1, \tau < \tilde{\tau}
\end{cases} \]  

(4.37)

where

\[ r = \frac{2s}{s-1} = \frac{2(n - n_\alpha)(1-\beta)}{(n-n_\alpha)(1-\beta) - n_\alpha}, \]  

(4.38)

\[ b_n = \frac{3 h_0^2}{2 \varphi_0 n} \left( \frac{n}{4} \right)^{2\beta}, \]  

(4.39)

and

\[ \tau_0 = \left( \frac{|r|}{h_0} \right)^{1/r} \left( \frac{4}{n} \right)^{\beta/r} (c_n |s - 1|)^{1/r-1}. \]  

(4.40)

Note that the limits \( \varphi \to \infty \) and \( \varphi \to 0 \) correspond to \( a \to \infty \) and \( a \to 0 \), respectively. From (4.34) together with (4.36) using the inverted relations (4.37) we find the density as a function of the cosmological scale:

\[ \rho(a) = \begin{cases} 
|c_n| s - 1| \tau_0 |^{n/(s-1)} |\ln(a/a_0)|^{n/(2s)} & s < 0, \quad a > a_0 \\
|\varphi_0 b_n|^{-n/2} |\ln(a/a_0)|^{n/2} & s = 1, \quad a < a_0
\end{cases} \]  

(4.41)

Hence, in the asymptotic regimes of small and large \( a \) the density varies logarithmically, thus demonstrating a quasi de Sitter behavior.
5. Conclusions

We have studied a braneworld cosmology (BWC) scenario based on the second RS model extended to more general warp factors. We have shown how our BWC is related to the cosmology of the braneworld in the bulk with a selfinteracting scalar field minimally coupled to gravity. In the high density regime of our BWC the modified Friedmann equation is identical to that of the original RSII cosmology and can be relevant for the early stages of inflation. Within a reasonable approximation in the low density regime the modified Friedmann equation remains of the same form as in the standard cosmology except that the effective gravitational constant $G$ is scale dependent.

As an application we have investigated a class of tachyon models in the framework of BWC. Assuming no restrictions on variations of $G$ in a pre-BBN cosmology we have analyzed a power-law variation corresponding to the inverse power-law tachyon potential $V \propto \varphi^{-n}$. We have demonstrated a universal critical behavior for the cosmologies described by (4.9) for the tachyon field theory with inverse power-law potential: there exist a critical power $n = n_{cr}$ that divides a dust universe for $n > n_{cr}$ and a quasi de Sitter universe for $0 < n < n_{cr}$ in both asymptotic regimes of large and small tachyon field $\varphi$ with $n_{cr}$ depending on the details of the cosmological scenarios. In particular we have analyzed three different scenarios: the standard tachyon cosmology and low and high energy-density regimes of the braneworld cosmology. For these three cosmologies, we have found $n_{cr}$ to be equal to 2, $4/3$, and 1, respectively.

Acknowledgments

This work has been supported by the Croatian Science Foundation under the project IP-2014-09-9582 and partially supported by ICTP—SEENET-MTP project PRJ-09 Cosmology and Strings. The work of N Bilić and S Domazet has been partially supported by the H2020 CSA Twinning project No. 692194, RBI-T-WINNING. G Djordjevic acknowledges support by the Serbian Ministry for Education, Science and Technological Development under the projects No. 176021 and No. 174020.

Appendix. Scalar field in the bulk

Following DeWolfe et al [12] we generalize the original RSII action to include a minimally coupled scalar field $\Phi$ in the bulk so that in the absence of $\Phi$ the bulk is reduced to the pure AdSs. The dynamics of a 3-brane in a 4+1 dimensional bulk is described by the total action as the sum of the bulk and brane actions

$$ S = S_{\text{bulk}} + S_{\text{br}}. $$

The bulk action is given by

$$ S_{\text{bulk}} = \frac{1}{8\pi G_5} \int d^5X \sqrt{G} \left[ -\frac{R(5)}{2} + \frac{1}{2} G^{ab} \Phi_a \Phi_b - U(\Phi) \right] + S_{\text{GH}}[h], $$(A.1)

where the Gibbons-Hawking boundary term is given by an integral over the brane hypersurface $\Sigma$:

$$ S_{\text{GH}}[h] = \frac{1}{8\pi G_5} \int _{\Sigma} d^4x \sqrt{-h} K[h]. $$

(A.2)

The quantity $K$ is the trace of the extrinsic curvature tensor $K_{ab}$ defined as
where \( n^a \) is a unit vector normal to the brane pointing towards increasing fifth coordinate, \( h_{ab} \) is the induced metric

\[
h_{ab} = G_{ab} + n_a n_b,
\]

and \( h = \det h_{ab} \) is its determinant. Observers on the brane with action

\[
S_{br}[h] = - \int_{\Sigma} d^4x \sqrt{-h} \sigma(\Phi),
\]

see the induced metric \( h_{\mu\nu} \). As we will shortly see, the relationship between the brane tension \( \sigma \) and the potential \( U \) is dictated by the field equations and the boundary conditions on the brane.

Next we outline a derivation of the RSII like solution assuming the metric and bulk scalar field depend on the fifth coordinate only [12]. The Einstein equations in the bulk are

\[
6 \frac{\psi^/'2}{\psi''} = \frac{1}{2} \Phi'^2 - U(\Phi),
\]

\[
-3 \frac{\psi''}{\psi'} + 3 \frac{\psi^2}{\psi'^2} = \Phi'^2,
\]

together with the \( \Phi \) field equation away from the brane

\[
\Phi'' + 4 \frac{\psi'}{\psi} \Phi' = \frac{dU}{d\Phi},
\]

where the prime \( ' \) denotes a derivative with respect to \( y \). In addition, the solutions are subject to the junction conditions on the brane

\[
-3 \left[ \frac{\psi'}{\psi} \right] = 8\pi G_5 \sigma(\Phi_0),
\]

\[
\left[ \frac{\Phi'}{\psi} \right] = 8\pi G_5 \frac{\partial \sigma(\Phi_0)}{\partial \Phi_0},
\]

where \([f]\) denotes the discontinuity of a function \( f(y) \) across the brane, i.e.

\[
[f(y)] = \lim_{\epsilon \to 0^+} (f(y_{br} + \epsilon) - f(y_{br} - \epsilon)).
\]

Equation (A.10) is a generalized fine tuning condition of RSII.

The system of equations (A.7)–(A.9) can be reduced to three decoupled first order differential equations [12]. Suppose \( U(\Phi) \) is expressed in terms of a superpotential \( W(\Phi) \)

\[
U(\Phi) = \frac{1}{8} \left( \frac{dW}{d\Phi} \right)^2 - \frac{1}{6} (W(\Phi))^2.
\]

Then, a solution to equations

\[
\frac{\psi'}{\psi} = \mp \frac{1}{6} W(\Phi), \quad \Phi' = \pm \frac{1}{2} \frac{dW}{d\Phi},
\]

will also satisfy equations (A.7)–(A.9).

Owing to the assumed orbifold symmetry, the superpotential \( W \) and its derivative \( dW/d\Phi \) will be continuous across the brane if the upper and lower signs in (A.14) are chosen for
y > y_{br}$ and $y < y_{br}$, respectively. Then, the junction conditions (A.10) and (A.11) will be satisfied provided

$$
\sigma(\Phi_0) = \frac{1}{8\pi G_5} W(\Phi_0), \quad \frac{\partial \sigma(\Phi_0)}{\partial \Phi_0} = \frac{1}{8\pi G_5} \frac{dW(\Phi_0)}{d\Phi_0},
$$

(A.15)

where $\Phi_0$ denotes the value of $\Phi$ on the brane. Hence, generally $\sigma(\Phi)$ is tangent to $W(\Phi)$ on the brane. A stronger condition $\sigma(\Phi) \equiv W(\Phi)/(8\pi G_5)$ is required for a BPS brane [47].

The system of equation (A.14) can be expressed in terms of $\chi$ as a single second order differential equation. In the region $y_{br} \leq y < \infty \ (z_{br} \leq z < \infty)$ one finds

$$
\chi \frac{d^2 \chi}{dz^2} - \frac{1}{12} \left( \frac{dW}{d\Phi} \right)^2 = 0,
$$

(A.16)

where the argument of $dW/d\Phi$ is

$$
\Phi = W^{-1}(6 \chi). \quad (A.17)
$$

Here $W^{-1}$ denotes the inverse function of $W$ and $\chi$ is an abbreviation for $d\chi/dz$. Similarly, in the region $-\infty < y \leq y_{br} \ (0 < z \leq z_{br})$ one finds

$$
\frac{z^2}{z_{br}} \chi \frac{d}{dz} \left( \frac{z^2}{z_{br}} \chi \right) - \frac{1}{12} \left( \frac{dW}{d\Phi} \right)^2 = 0,
$$

(A.18)

where the argument of $dW/d\Phi$ is

$$
\Phi = W^{-1}(-6 \chi_{-}/z_{br}^2). \quad (A.19)
$$

As an example of particular interest, consider a warp function of the power-law form

$$
\chi(z) = \begin{cases} 
\left( \frac{z}{z_{br}} \right)^{n}, & \text{for } z_{br} \leq z < \infty \\
\left( \frac{z_{br}}{z} \right)^{n}, & \text{for } 0 < z \leq z_{br}
\end{cases}
$$

(A.20)

Then, using either (A.16) with (A.17) in the region $z_{br} \leq z < \infty$ or (A.18) with (A.19) in the region $0 < z \leq z_{br}$, we obtain the differential equation

$$
\left( \frac{dW}{d\Phi} \right)^2 - \eta W^2 = 0,
$$

(A.21)

where

$$
\eta = \frac{n-1}{3n}. \quad (A.22)
$$

For $n \geq 1$, equation (A.21) has a real solution

$$
W(\Phi) = 6k e^{\sqrt{\eta} \Phi}. \quad (A.23)
$$

In the limiting case of $n = 1$ ($\eta = 0$) one recovers the RSII model with pure AdS$_5$ bulk with curvature radius $\ell = 1/k = z_{br}$. The superpotentials of the exponential form (A.23) have been extensively studied in the context of braneworlds inspired by supergravity [25, 31, 48, 49].

For $n < 1$, equation (A.21) has no real solution. However, the power-law warp factor of the form

$$
\chi = \left( \frac{z}{z_{br}} \right)^{n} - C,
$$

(A.24)
with \( C > 0 \), yields a modified differential equation

\[
\left( \frac{dW}{d\Phi} \right)^2 - \eta W^2 \left( 1 - C \left( \frac{z_{br} W}{6n} \right)^{-1/(3\eta)} \right) = 0,
\]

(A.25)

which has real solutions for any real \( n \). The solution can be expressed in the form

\[
W = \frac{6nC^{3/2}}{z_{br}} \times \left\{ \begin{array}{ll}
[1 + \tan^2(\Phi/(6\sqrt{\eta}) + C_1)]^{-3\eta}, & \text{for } n < 1 \\
[1 - \tanh^2(\Phi/(6\sqrt{\eta}) + C_2)]^{-3\eta}, & \text{for } n \geq 1,
\end{array} \right.
\]

(A.26)

where \( C_1 \) and \( C_2 \) are arbitrary integration constants which can be conveniently chosen. For example, with the choice

\[
C_2 = \frac{1}{6\eta} \ln \frac{4^{3\eta}}{nC^{3\eta}}
\]

(A.27)

one recovers the solution (A.23) in the limit \( C \to 0 \).

References

[1] Arkani-Hamed N, Dimopoulos S and Dvali G 1998 Phys. Lett. B 429 263
[2] Antoniadis I, Arkani-Hamed N, Dimopoulos S and Dvali G R 1998 Phys. Lett. B 436 257
[3] Randal L and Sundrum R 1999 Phys. Rev. Lett. 83 3370
[4] Randal L and Sundrum R 1999 Phys. Rev. Lett. 83 4690
[5] Gubser S S 2001 Phys. Rev. D 63 084017
Nojiri S, Odintsov S D and Zerbini S 2000 Phys. Rev. D 62 064006
Giddings S B, Katz E and Randall L 2000 J. High Energy Phys. JHEP03(2000)023
Hawking S W, Hertog T and Reall H S 2000 Phys. Rev. D 62 043501
Duff M J and Liu J T 2001 Class. Quantum Grav. 18 3207
Duff M J and Liu J T 2000 Phys. Rev. Lett. 85 2052
Nojiri S and Odintsov S D 2000 Phys. Lett. B 484 119
de Haro S, Skenderis K and Solodukhin S N 2001 Class. Quantum Grav. 18 3171
[6] Bilić N 2016 Phys. Rev. D 93 066010
[7] Binetruy P, Defayet C and Langlois D 2000 Nucl. Phys. B 565 269
[8] Flanagan E E, Henry Tye S H and Wasserman I 2000 Phys. Rev. D 62 044039
[9] Kodama Y, Nomura T, Suyama T and Yamauchi J 2010 Phys. Rev. D 81 123509
[10] Goldberger W D, Linde A D and Starobinsky A A 1999 Phys. Rev. Lett. 83 4922
[11] Kobayashi S, Koyama K and Soda J 2002 Phys. Rev. D 65 064014
[12] DeWolfe O, Freedman D Z, Gubser S S and Karch A 2000 Phys. Rev. D 62 046008
[13] German G, Herrera-Aguilar M, Malagon-Morejon D, Mora-Luna R R and da Rocha R 2013 J. Cosmol. Astropart. Phys. JCAP02(2013)035
[14] Germ G, Herrera-Aguilar A, Kuerten A M, Malagon-Morejon D and da Rocha R 2016 J. Cosmol. Astropart. Phys. JCAP01(2016)047
[15] Bogdanos C, Dimitriadiadis A and Tamvakis K 2007 Class. Quantum Grav. 24 3701
[16] Herrera-Aguilar M, Malagon-Morejon D, Mora-Luna R R and Quiros I 2012 Class. Quantum Grav. 29 035012
[17] Sen A 2011 J. High Energy Phys. JHEP10(1999)008
[18] Shandera S E and Tye S-H H 2006 J. Cosmol. Astropart. Phys. JCAP05(2006)007
[19] Abram L R W and Finelli F 2003 Phys. Lett. B 575 165
[20] Kofman L, Linde A D and Starobinsky A A 1994 Phys. Rev. Lett. 73 3195
[21] Kofman L, Linde A D and Starobinsky A A 1997 Phys. Rev. D 56 3258
[22] Bassett B A, Tsujikawa S and Wands D 2006 Rev. Mod. Phys. 78 537
[23] Bilić N and Tupper G B 2014 Central Eur. J. Phys. 12 147
[24] Israel W 1966 Nuovo Cim. B 44 1–14
Israel W 1967 Nuovo Cim. B 48 463 (erratum)
[25] Brax P and van de Bruck C 2003 Class. Quantum Grav. 20 R201
Brax P, van de Bruck C and Davis A C 2004 Rep. Prog. Phys. 67 2183

[26] Bernardini A E, Cavalcanti R T and da Rocha R 2015 Gen. Relativ. Gravit. 47 1840

[27] Accetta F S, Krauss L M and Romanelli P 1990 Phys. Lett. B 248 146

[28] Uzan J P 2003 Rev. Mod. Phys. 75 403
Uzan J P 2011 Living Rev. Relativ. 14 2

[29] Ade P A R et al 2016 Astron. Astrophys. 594 A13

[30] Amarilla L and Vucetich H 2010 Int. J. Mod. Phys. A 25 3835

[31] Davis A C, Brax P and van de Bruck C 2005 Nucl. Phys. Proc. Suppl. 148 64

[32] Bertotti B, Less L and Tortora P 2003 Nature 425 374

[33] Lykken J D and Randall L 2000 J. High Energy Phys. JHEP06(2000)014

[34] Bordemann M and Hoppe J 1994 Phys. Lett. B 325 359
Ogawa N 2000 Phys. Rev. D 62 085023

[35] Jackiw R 2002 Lectures on Fluid Mechanics (Berlin: Springer)

[36] Choudhury S and Panda S 2016 Eur. Phys. J. C 76 278

[37] De Donder T 1930 Théorie Invariantive Du Calcul des Variations (Paris: Gaultier-Villars)

[38] Weyl H 1935 Ann. Math. 36 607

[39] Struckmeier J and Redelbach A 2008 Int. J. Mod. Phys. E 17 435–91

[40] Cremaschini C and Tesserotto M 2016 Appl. Phys. Res. 8 60

[41] Lidsey J E and Nunes N J 2003 Phys. Rev. D 67 103510

[42] Tsujikawa S, Sami M and Maartens R 2004 Phys. Rev. D 70 063525

[43] Calcagni G and Tsujikawa S 2004 Phys. Rev. D 70 103514

[44] Calcagni G 2005 arXiv:hep-ph/0503044

[45] Steer D A and Vernizzi F 2004 Phys. Rev. D 70 043527

[46] Bilic N, Dimitrijevic D, Djordjevic G and Milosevic M 2017 Int. J. Mod. Phys. A 32 1750039

[47] Kim J E, Tupper G B and Viollier R D 2005 Phys. Lett. B 612 293

[48] Brax P and Davis A C 2001 Phys. Lett. B 497 289

[49] Brax P and Davis A C 2001 Phys. Lett. B 513 156