Integrals containing the logarithm of the Airy Function $Ai'(x)$

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Abstract

Integrals occurring in Thomas-Fermi theory which contain the logarithm of the Airy function $Ai'(x)$ have been obtained in terms of analytical expressions.

1 An integral containing $Ai'(x) \ln(Ai'(x))$

Integrals of the kind

$$J^{(\alpha)} = \int_{0}^{\infty} \left( \frac{Ai'(x)}{Ai'(0)} \right)^{\alpha} \ln \left( \frac{Ai'(x)}{Ai'(0)} \right) \, dx,$$

which contain the logarithm of the Airy function $Ai'(x)$ [1] and which occur in Thomas-Fermi theory, a semiclassical quantum mechanical theory for the electronic structure of many-body systems, [2] have long resisted expression in closed form or accurate values in terms of classical mathematical constants or analytic special functions. Approximate analytic expressions for $J^{(1)}$ accurate to five decimal places have been given [3]. Here we give analytic expressions for $J^{(\alpha)}$ for orders $\alpha = 1, 2$ in the form of infinite series.

We begin by noting that integration by parts of $J^{(1)}$ yields the expression

$$J^{(1)} = -\frac{1}{Ai'(0)} \int_{0}^{\infty} Ai'(x) \frac{d}{dx} \left\{ \ln \left( \frac{Ai'(x)}{Ai'(0)} \right) \right\} \, dx.$$ 

Progress can be made in the calculation of $J^{(1)}$ using the Weierstrass infinite product representation [4] for the quantity $Ai'(x)/Ai'(0)$ i.e.

$$\frac{Ai'(x)}{Ai'(0)} = \prod_{n=1}^{\infty} \left\{ 1 + \frac{x}{|a'_n|} \right\} \exp \left\{ -\frac{x}{|a'_n|} \right\},$$

(1)

where $a'_n$ are the roots [5] of the Airy function $Ai'(x)$. The first few of those roots are contained in the Table I below.
Values of \( a'_n \) for large \( n \) are given by

\[
a'_n = -t^{2/3} \left\{ 1 - \frac{7}{18} t^{-2} + \frac{35}{288} t^{-4} - \frac{181228}{207360} t^{-6} + \cdots \right\},
\]

where

\[
t = \frac{3}{8} \pi (4n - 3).
\]

Using the expressions above we get for the integral in (1)

\[
\mathcal{J}^{(1)} = \frac{1}{Ai''(0)} \sum_{n=1}^{\infty} \frac{1}{|a'_n|} \int_0^\infty \frac{x \, Ai(x)}{(x + |a'_n|)} \, dx.
\]

Employing the differential equation which defines the Airy function i.e.

\[
Ai''(x) - x \, Ai(x) = 0,
\]

(2)

\( \mathcal{J}^{(1)} \) becomes

\[
\mathcal{J}^{(1)} = \frac{1}{Ai''(0)} \sum_{n=1}^{\infty} \frac{1}{|a'_n|} \int_0^\infty \frac{Ai''(x)}{(x + |a'_n|)} \, dx.
\]

A two-fold integration by parts of the latter expression together with the use of the first two Airy zeta sums \( \mathcal{Z}_k \) i.e. (cf. Appendix I)

\[
\mathcal{Z}_2 = \sum_{n=1}^{\infty} \frac{1}{|a'_n|^2} = -Ai(0)/Ai'(0),
\]

\[
\mathcal{Z}_3 = \sum_{n=1}^{\infty} \frac{1}{|a'_n|^3} = 1,
\]

yields

\[
\mathcal{J}^{(1)} = \frac{2}{Ai''(0)} \sum_{n=1}^{\infty} \frac{1}{|a'_n|} \int_0^\infty \frac{Ai(x)}{(x + |a'_n|)^3} \, dx.
\]
To complete this calculation it is essential to obtain accurate values for the integrals occurring in (3) especially in the case of integrals containing the higher roots.

1.1 The Generalized Stieltjes transforms of $Ai(x)$

The integrals $I_k(a)$, $(a > 0)$

$$I_k(a) = \int_0^\infty \frac{Ai(x)}{(x+a)^k} \, dx,$$

which can be identified as the generalized Stieltjes transforms of the Airy function $Ai(x)$, have magnitudes which decrease rapidly for large $k$, $I_k(a)$ varying roughly as $\frac{1}{a^k}$ (cf. Appendix II).

As will be seen below, the $I_k(a)$ integrals are interrelated by recurrence relations. If we rewrite $I_k(a)$ as

$$I_k(a) = a \int_0^\infty \frac{Ai(x)}{(x+a)^k+1} \, dx + \int_0^\infty \frac{x Ai(x)}{(x+a)^k+1} \, dx,$$

we get

$$I_k(a) = a I_{k+1}(a) + \int_0^\infty \frac{Ai''(x)}{(x+a)^k+1} \, dx.$$

Integrating by parts twice, the integral above produces the recurrence relation

$$I_k(a) - a I_{k+1}(a) - (k+1)(k+2)I_{k+3}(a) = -Ai'(0)/a^{k+1} - (k+1)Ai(0)/a^{k+2}.$$

Using (5) together with the differential relations

$$\frac{d}{da} I_{k+1}(a) = -(k+1)I_{k+2}(a),$$

$$\frac{d^2}{da^2} I_{k+1}(a) = (k+1)(k+2)I_{k+3}(a),$$

the recurrence relation (5) is transformed into the second-order differential equation

$$\frac{d^2}{da^2} I_{k+1}(a) + a I_{k+1}(a) = I_k(a) + \frac{Ai'(0)}{a^{k+1}} + \frac{(k+1)Ai(0)}{a^{k+2}}.$$

In the case where $k = 0$

$$I_0(a) = \frac{1}{3}.$$

The integral $I_1(a)$ which will be shown to be important in the sequel satisfies the differential equation

$$\frac{d^2}{da^2} I_1(a) + a I_1(a) = \frac{1}{3} + \frac{Ai'(0)}{a} + \frac{Ai(0)}{a^2},$$

(7)
In the second instances, the integrals

\[ I_1(a) \big|_{a=a_0} = I_1(a_0), \quad \left[ \frac{dI_1(a)}{da} \right]_{a=a_0} = - I_2(a_0), \]

where \( a_0 \) is any positive number. The general solution to (7) is then given by

\[ I_1(a) = \pi I_1(a_0) [Ai(-a) Bi'(a_0) - Bi(-a) Ai'(a_0)] \\
- \pi I_2(a_0) [Ai(-a) Bi(-a_0) - Bi(-a) Ai(-a_0)] \\
+ \frac{\pi}{3} \{Ai(-a) \int_{a_0}^{a} Bi(-z)dz - Bi(-a) \int_{a_0}^{a} Ai(-z)dz\} \\
+ Ai(-a) \int_{a_0}^{a} \left[ -\frac{3^{1/6} \Gamma(2/3)}{2} z + \frac{\pi}{3^{2/3} \Gamma(2/3) z^2} \right] Bi(-z)dz \\
- Bi(-a) \int_{a_0}^{a} \left[ -\frac{3^{1/6} \Gamma(2/3)}{2} z + \frac{\pi}{3^{2/3} \Gamma(2/3) z^2} \right] Ai(-z)dz. \]

The integrals appearing above in the expression for \( I_1(a) \) in the first and second instances are given by Mathematica as

\[ \int Bi(-z)dz = \frac{z}{3^{1/3} \Gamma(2/3)} \text{}_2F_1 \left( \frac{1}{3}; -\frac{2}{3}, \frac{4}{3} \right), \]

\[ \int Ai(-z)dz = \frac{z}{3^{1/3} \Gamma(2/3)} \text{}_2F_1 \left( \frac{1}{3}; -\frac{2}{3}, \frac{4}{3} \right) + \frac{3^{1/6} \Gamma(2/3) \frac{\pi}{6}}{\frac{4}{3}, \frac{5}{3}} \text{}_2F_1 \left( \frac{2}{3}; -\frac{2}{3}, \frac{4}{3} \right), \]

where \( _2F_1 \) are the generalized hypergeometric functions. Gathering terms we have

\[ \frac{\pi}{3} \left\{ Ai(-a) \int_{a_0}^{a} Bi(-z)dz - Bi(-a) \int_{a_0}^{a} Ai(-z)dz \right\} \]

\[ = \frac{\Gamma(1/3)}{3^{1/5} \sqrt{\pi}} \tilde{3}_-(a) \left[ \left( \frac{a}{2} \right) ^2 \text{}_2F_1 \left( \frac{1}{3}; -\frac{2}{3}, \frac{4}{3} \right) \right]_{a_0} \]

\[ + \frac{\Gamma(2/3)}{3^{2/5} \sqrt{\pi}} \tilde{3}_+(a) \left[ \left( \frac{a}{2} \right) ^2 \text{}_2F_1 \left( \frac{2}{3}; -\frac{2}{3}, \frac{4}{3} \right) \right]_{a_0}, \]

where we have defined the functions \( \tilde{3}_\pm(a) \) and \( \tilde{3}_\pm'(a) \) as

\[ \tilde{3}_\pm(a) = \sqrt{3} Ai(-a) \pm Bi(-a), \quad \tilde{3}_\pm'(a) = \sqrt{3} Ai'(-a) \pm Bi'(-a). \]

In the second instances, the integrals

\[ V_1(z) = \int \left[ -\frac{3^{1/6} \Gamma(2/3)}{2} z + \frac{\pi}{3^{2/3} \Gamma(2/3) z^2} \right] Bi(-z)dz, \]

\[ V_2(z) = \int \left[ -\frac{3^{1/6} \Gamma(2/3)}{2} z + \frac{\pi}{3^{2/3} \Gamma(2/3) z^2} \right] Ai(-z)dz. \]
are given by

\[ V_1(z) = -\ln(z) - \frac{\pi}{3^{5/6} \Gamma(2/3)^2} z^2 F_2(-1/3; -3/9) \]
\[ + \frac{3^{5/6} \Gamma(2/3)^2}{4\pi} z F_2(1/3; -3/9) \]
\[ + \frac{1}{72} z^3 \left[ 2 \cdot F_3\left(1, 1; -3/9 \right) + 2 F_3\left(1, 2, 7/3 \right) \right], \]

\[ V_2(z) = -\frac{\pi}{3^{4/3} \Gamma(2/3)^2} z^2 F_2\left(-1/3; -3/9 \right) \]
\[ - \frac{3^{1/3} \Gamma(2/3)^2}{4\pi} z F_2(1/3; -3/9) \]
\[ + \frac{\sqrt{3}}{216} z^3 \left[ 2 \cdot F_3\left(1, 1; -3/9 \right) - 2 F_3\left(1, 2, 7/3 \right) \right]. \]

Collecting all terms containing hypergeometric functions we have for \( I_1(a) \)

\[ I_1(a) = \frac{\pi}{2\sqrt{3}} I_1(a_0) [\Im_-(a_0) \Im_+ (a_0) - \Im_+(a_0) \Im_- (a_0)] \]
\[ - \frac{\pi}{2\sqrt{3}} I_2(a_0) [\Im_-(a_0) \Im_+ (a_0) - \Im_+(a_0) \Im_- (a_0)] \]
\[ + \Im_+(a) \Delta H_+(a) + \Im_-(a) \Delta H_-(a) - \text{Ai}(-a) \Delta \{\ln(a)\}, \]

where

\[ H_+(a) = \frac{3^{1/3} \Gamma(2/3)^2}{4\pi} a F_2(1/3; -3/9) - \frac{9 \Gamma(2/3)^2}{3^{7/6} \pi} a^2 F_2(1/3; -3/9) \]
\[ + \frac{\sqrt{3}}{216} a^3 F_3\left(1, 2, 5/3 \right), \]

\[ H_-(a) = -\frac{3^{1/3} \Gamma(2/3)^2}{4\pi} a F_2\left(-1/3; -3/9 \right) + \frac{9 \Gamma(2/3)^2}{3^{7/6} \pi} a^2 F_2\left(1/3; -3/9 \right) \]
\[ + \frac{\sqrt{3}}{216} a^3 F_3\left(1, 1; -3/9 \right), \]
\[ \Delta f(a) = f(a) - f(a_0). \]

We have obtained a closed form albeit complicated expression for the Stieltjes transform \( I_1(a) \). Using the expression above with \( a_0 \) chosen to be the magnitude of one of the roots of \( \text{Ai}'(a) \) e.g. \( a'_n \), terms containing \( \text{Ai}'(a'_n) \) vanish. Taking \( a_0 = |a'_1| = 1.08792997 \), and the numerically evaluated values for \( I_1(|a'_1|) = 0.2109508346 \) and \( I_2(|a'_1|) = 0.1425319307 \) then allows a calculation of the integral \( \mathcal{J}^{(1)} \) as will be seen below.

The integral \( \mathcal{J}^{(1)} \) as represented by (3) i.e.

\[ \mathcal{J}^{(1)} = -2 \frac{\text{Ai}'(0)}{\text{Ai}'(a'_n)} \sum_{n=1}^{\infty} \frac{1}{|a'_n|} I_1(|a'_n|), \]

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when rewritten using the relation

\[ I_3(a) = \frac{1}{6} + \frac{Ai'(0)}{2a} + \frac{Ai(0)}{2a^2} - \frac{a}{2} I_1(a), \]

gives

\[ \mathcal{J}^{(1)} = \frac{1}{Ai'(0)} \sum_{n=1}^{\infty} \frac{1}{|a'_n|} \left( \frac{1}{3} + \frac{Ai'(0)}{|a'_n|} + \frac{Ai(0)}{|a'_n|^2} - |a'_n| I_1(|a'_n|) \right). \]

The second and third terms in the equation above can be summed using the Airy zeta functions \( Z_k \) (cf. Appendix I) to give the expression

\[ \mathcal{J}^{(1)} = \frac{1}{Ai'(0)} \sum_{n=1}^{\infty} \frac{1}{3|a'_n|} - I_1(|a'_n|). \] (8)

We note that for large \( a \) (cf. Appendix II) the terms within the curly brackets in (8) are on the order of \( |a'_n|^{-2} \sim (4/3\pi n)^{4/3} \) which gives some assurance that the sum converges.

However, the value of the integral \( \mathcal{J}^{(1)} \) as expressed by (8) is slowly convergent. Taking one hundred terms in the sum gives a value of \(-0.73273890\). The integral \( \mathcal{J}^{(1)} \) computed numerically by Maple has value \(-0.81400778\). In contrast, the value of \( \mathcal{J}^{(1)} \) using the sum in (3) which contains one hundred terms produces \(-0.81399655\), a value with error in the 6th decimal place. We see that neither of the sums given by (3) or (8) yield useful analytic representations for \( I^{(1)} \).

An approach aimed at accelerating the rate of convergence of the representation of \( I^{(1)} \) as given by equation (3) begins by examining in finer detail the behavior of \( I_3(a) \) for large \( a \). We write

\[ I_3(a) = \int_0^a \frac{Ai(x)}{(x+a)^3} \, dx + \int_a^\infty \frac{Ai(x)}{(x+a)^3} \, dx. \]

In the first integral where \( x \leq a \) the denominator can be written as a power series in \( x \) and integrated term by term. The resulting value for the first integral is

\[ \frac{1}{a} I_{3,x \leq a} (a) = \frac{1}{a} \int_0^a \frac{Ai(x)}{(x+a)^3} \, dx \]
\[ \approx \frac{1}{6a^2} \sum_{k=0}^{n} \frac{(-1)^k(k+2)!}{\Gamma(k/3+1)} \left( \frac{1}{3^{1/3}a} \right)^k \ldots \]

where we note the implied truncated series at \( k = n \).

In the case of the second integral i.e. where \( x \geq a \) and \( a \) is large, we have

\[ \frac{1}{a} I_{3,x \geq a} (a) = \frac{1}{a} \int_a^\infty \frac{Ai(x)}{(x+a)^3} \, dx \approx \frac{2}{3\sqrt{\pi}a^{9/4}} \int_1^\infty \frac{\exp(-\frac{2}{3}a^{3/2}x^2)}{(1+x^{4/3})^3} \, dx, \]
where the asymptotic form of $Ai(x)$ has been used. The largest contribution to the latter integral occurs in the interval $1 \leq x \leq 2$ so that

$$
\int_a^\infty \frac{Ai(x)}{(x + a)^3} dx \simeq \frac{2}{3\sqrt{\pi}a^{3/4}} \int_1^2 \exp\left(-\frac{2}{3}a^{3/2}x^2\right) \frac{1}{(1 + x^4/3)^3} dx.
$$

Expanding the denominator in the latter integral in powers of $(x - 1)$ and integrating the resulting series we get

$$
\int_a^\infty \frac{Ai(x)}{(x + a)^3} dx \simeq \frac{\exp(-\frac{2}{3}a^{3/2})}{\sqrt{\pi}a^{19/4}} \left(\frac{1345}{1728} + \frac{2659}{2304a^{3/2}} + \frac{13}{38 a^{3/2}}\right) + \exp(-\frac{2}{3}a^{3/2}) \left(\frac{47}{198} + \frac{851}{1152a^{3/2}}\right) + \frac{\sqrt{\pi}Erf\left(\frac{2\sqrt{6}a^{3/4}}{3}\right)}{a^{3/2}} - Erf\left(\frac{\sqrt{6}a^{3/4}}{3}\right)
\right)
\right).
$$

Using the asymptotic expression for the error function $Erf(z)$ we have upon combining the terms in $I_{3, x \geq a}(a)$

$$
\frac{1}{a}I_{3, x \geq a}(a) \sim \frac{\exp(-\frac{2}{3}a^{3/2})}{\sqrt{\pi}a^{19/4}}.
$$

The exponential terms in the expression for $I_{3, x \geq a}(a)$ are small compared to $I_{3, x \leq a}(a)$ and the integration over the interval $x \geq a$ will be neglected in computing the sums in the equation below.

With this we can write the expression for $\mathcal{J}^{(1)}$ using (3) approximately as

$$
\mathcal{J}^{(1)} \simeq \frac{2}{\text{Ai}'(0)} \sum_{n=1}^N I_3(|a_n'|) + \frac{2}{\text{Ai}'(0)} \sum_{n=N+1}^\infty \frac{I_{3, x \leq a}(|a_n'|)}{|a_n'|},
$$

or introducing the infinite sums which contains all of the roots $a_n'$ we have

$$
\mathcal{J}^{(1)} \simeq \frac{2}{\text{Ai}'(0)} \sum_{n=1}^N \frac{1}{|a_n'|} I_3(|a_n'|) - I_{3, x \leq a}(|a_n'|) + \frac{2}{\text{Ai}'(0)} \sum_{n=1}^\infty \frac{I_{3, x \leq a}(|a_n'|)}{|a_n'|}.
$$

In the sums in this expression, the power of $a$ in the expansion representing $I_{3, x \leq a}(|a_n'|)$ given above whose upper limit was denoted $n$, we note that this limit has not been restricted and is a matter of choice as is the case for $N$. Summing those expressions in terms of the Airy zeta function $Z_k(N)$ (cf. Appendix II) we get

$$
\mathcal{J}^{(1)} \simeq \frac{2}{\text{Ai}'(0)} \sum_{n=1}^N \frac{1}{|a_n'|} I_3(|a_n'|) + \frac{1}{3\text{Ai}'(0)} \sum_{k=0}^{N-1} \frac{(-1)^k(k+2)!}{3^{k/3}k^{(k+2)/3+1}} \left\{Z_{k+4} - Z_{k+4}(N)\right\}.
$$

In the case where $N = 10$ and $n = 3$, $\mathcal{J}^{(1)} = -0.8140073597$ a value which is accurate to seven decimal places. No effort has been made to vary $N$ and/or $n$ in an attempt to increase the accuracy of the expression for $\mathcal{J}^{(1)}$. 

7
1.2 Values of $I_k(a)$ for small $a$

Accurate values of $I_k(a)$ for small $a$ are useful in providing analytic expressions for the integration constants $I_1(a_0)$ and $I_2(a_0)$ i.e. the quantities which are needed to make the solution of the differential equation (7) for $I_1(a)$ unique and analytic. In an effort to do that we write the integral $I_n(a)$ as

$$I_n(a) = \int_{\alpha}^{\infty} \frac{Ai(z-a)}{z^n} \, dz,$$

and expand the Airy function $Ai(z-a)$ in a power series in $a$, with the hope that the resulting integrated series would be capable of yielding accurate values of $I_n(a)$ for $a \leq 1$. The power series for $Ai(z-a)$ i.e.

$$Ai(z-a) = \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} \frac{d^k Ai(z)}{dz^k},$$

is seen to require analytic expressions for the higher derivatives of the Airy function. These have been studied [7] and are given by

$$\frac{d^k Ai(z)}{dz^k} = P_k(z) Ai(z) + Q_k(z) Ai'(z), \quad (9)$$

where $P_k(z)$ and $Q_k(z)$ are polynomials. Recursion relations for these polynomial have the forms

$$P_{k+2}(z) = z P_k(z) + k P_{k-1}(z), \quad (10a)$$
$$Q_{k+2}(z) = z Q_k(z) + k Q_{k-1}(z), \quad (10b)$$
$$P_{k+1}(z) = \frac{d P_k(z)}{dz} + z Q_k(z), \quad (10c)$$
$$Q_{k+1}(z) = \frac{d Q_k(z)}{dz} + P_k(z). \quad (10d)$$

Initial values of $P_k(z)$ and $Q_k(z)$ i.e. $P_0(z) = 1, Q_0(z) = 0$ are sufficient to generate the higher polynomials. The first few of these are given in Table 2

| $k$ | $P_k(z)$     | $Q_k(z)$     |
|-----|--------------|--------------|
| 1   | 0            | 1            |
| 2   | $z$          | 0            |
| 3   | 1            | $z$          |
| 4   | $z^2$        | 2            |
| 5   | $4z$         | $z^2$        |
| 6   | $4 + z^3$    | $6z$         |
| 7   | $9z^2$       | $10 + z^4$   |
| 8   | $28z + z^4$  | $12z^2$      |
| 9   | $28 + 16z^3$ | $52z + z^4$  |
| 10  | $100z^2 + z^5$ | $80 + 20z^3$ |
The generating functions $\xi(t, z)$ and $\lambda(t, z)$ for the polynomials $P_k(z)$ and $Q_k(z)$ are

$$
\xi(t, z) = \sum_{k=0}^{\infty} \frac{t^k}{k!} P_k(z) = \pi \left[ Bi'(z) Ai(z + t) - Ai'(z) Bi(z + t) \right],
$$

$$
\lambda(t, z) = \sum_{k=0}^{\infty} \frac{t^k}{k!} Q_k(z) = \pi \left[ Ai(z) Bi(z + t) - Bi(z) Ai(z + t) \right].
$$

The expression for $Ai(z - a)$ is then given by

$$
Ai(z - a) = \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} \{ P_k(z) Ai(z) + Q_k(z) Ai'(z) \}.
$$

As a result, we get for the integral $I_n(a)$

$$
I_n(a) = \int_{a}^{\infty} \left\{ \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} P_k(z) \right\} \frac{Ai(z)}{z^n} \, dz + \int_{a}^{\infty} \left\{ \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} Q_k(z) \right\} \frac{Ai'(z)}{z^n} \, dz.
$$

Or in terms of the generating functions

$$
I_n(a) = \int_{a}^{\infty} \{ (\xi(-a, z) Ai(z) + \lambda(-a, z) Ai'(z)) \} \frac{dz}{z^n}.
$$

Initially we choose to use the former expression (11) as a means of computing $I_3(a)$ i.e.

$$
I_3(a) = \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} \left\{ \int_{a}^{\infty} \frac{P_k(z) Ai(z)}{z^3} \, dz + \int_{a}^{\infty} \frac{Q_k(z) Ai'(z)}{z^3} \, dz \right\}.
$$

The integral containing $Ai'(z)$ in (13) can be integrated by parts with the result

$$
\int_{a}^{\infty} \frac{Q_k(z) Ai'(z)}{z^3} \, dz = -\frac{Q_k(a) Ai(a)}{a^3} + \int_{a}^{\infty} \left[ 3 \frac{Q_k(z)}{z^3} + \frac{P_k(z)}{z^3} - \frac{Q_{k+1}(z)}{z^3} \right] Ai(z) \, dz.
$$

Combining the integrals in (13) we get

$$
I_3(a) = \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} \int_{a}^{\infty} F_k(z) Ai(z) \, dz - \frac{Ai(a)}{a^3} \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} Q_k(a),
$$

where

$$
z^4 F_k(z) = z \{ 2P_k(z) - Q_{k+1}(z) \} + 3Q_k(z).
$$
The sum in (14) which contains the polynomials \( Q_k(a) \) can be related to the Airy functions using its generating function. We get the closed form expression
\[
\sum_{k=0}^{\infty} \frac{(-a)^k}{k!} Q_k(a) = \pi Ai(0)[ \sqrt{3} Ai(a) - Bi(a)] = \pi Ai(0) \mathcal{J}_\pi (-a).
\]

The degrees of the polynomials in \( z^4 F_k(z) \) as seen below in the matrix \( \mathfrak{A} \) are a complicated function of the index \( k \). As a result, rearranging the integrations in (14) in terms of increasing powers of \( z \) is in general also complicated. We write the integrand \( F_k(z) \) as
\[
F_k(z) = \frac{1}{z^4} \sum_{i=0}^{N_k} \mathfrak{A}_{k,i} z^i,
\]
and note that in the case of even and odd values of \( k \) the upper limits \( N_k \) in the sums are just
\[
F_{2k}(z) = \frac{1}{z^4} \sum_{i=0}^{k+1} \mathfrak{A}_{2k,i} z^i,
\]
\[
F_{2k+1}(z) = \frac{1}{z^4} \sum_{i=0}^{k} \mathfrak{A}_{2k+1,i} z^i,
\]
as can be seen by an inspection of the elements \( \mathfrak{A}_{k,i}(a) \) which are displayed in the matrix \( \mathfrak{A} \)

\[
\mathfrak{A} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 16 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 40 & 0 & 0 & 1 & 0 & 0 \\
0 & 132 & 0 & 0 & 15 & 0 & 0 & 0 \\
240 & 0 & 0 & 100 & 0 & 0 & 1 & 0 \\
0 & 0 & 440 & 0 & 0 & 23 & 0 & 0 \\
0 & 1480 & 0 & 0 & 230 & 0 & 0 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

The expression for the integral \( I_3(a) \) is then
\[
I_3(a) = \pi Ai(0) \mathcal{J}_\pi (-a) - 3\pi Ai(0) \mathcal{J}_\pi (a) \int_a^{\infty} \frac{Ai(z)}{z^4} \, dz \\
+ \sum_{i=1}^{\infty} a^{2i} \left( \sum_{k=0}^{\infty} \frac{(-a^2)^k}{(2k+2i)!} \mathfrak{A}_{2k+2i,i} - \left( \frac{a}{2k+2i+1} \right) \mathfrak{A}_{2k+2i+1,i} \right) \int_a^{\infty} \frac{Ai(z)}{z^4} \, dz
\]
where we have used the sum
\[
\sum_{k=0}^{\infty} \frac{(-a)^k}{k!} A_{k,0} = \frac{\sqrt{3}}{2 Ai'(0)} [\sqrt{3} Ai(-a) - Bi(-a)] \\
= -3\pi Ai(0) J_-(a).
\]
The latter sum follows from an inspection of the elements of \(A\) i.e. those occurring in its first column i.e.
\[
A_{3k+1,0} = 3^{k+1} \Gamma(k + 2/3)/\Gamma(2/3), \\
A_{3k,0} = 0, \\
A_{3k+2,0} = 0.
\]
Calculation of the integrals occurring in the expression for \(I_3(a)\) appearing above in (14) require analytical expressions for the integrals \(\int_{a}^{\infty} z^n Ai(z) \, dz\) and \(\int_{a}^{\infty} z^n Ai'(z) \, dz\). These are given below.

1.3 The incomplete Mellin transforms of \(Ai(z)\) and \(Ai'(z)\)

We define the integrals (\(a > 0\))
\[
I_n(a) = \int_{a}^{\infty} z^n Ai(z) \, dz, \\
I'_n(a) = \int_{a}^{\infty} z^n Ai'(z) \, dz,
\]
and note that integration by parts and use of (2) yields the third-order recurrence relation for all \(n\)
\[
I_n(a) = (n-1)(n-2) I_{n-3}(a) - a^{n-1} Ai'(a) + (n-1) a^n - 2 Ai(a). \tag{15}
\]
Initial values of \(I_n(a)\) i.e. \(I_0(a), I_{-1}(a),\) and \(I_{-2}(a)\) are irreducible and are required to obtain the general solution to this difference equation. For \(I_0(a)\) we have immediately
\[
I_0(a) = \pi [Ai(a) Gi'(a) - Ai'(a) Gi(a)],
\]
where \(Gi\) and \(Gi'\) are the Scorer functions [8]. The integral \(I_0(a)\) has also been denoted by \(Ai_1(a)\) and was studied by Aspnes [9]. An expression for \(I_0(a)\) in terms of the more accessible hypergeometric functions being
\[
I_0(a) = \frac{1}{3} - \frac{a}{2\pi i \Gamma(2/3)} \frac{\Gamma(2/3)}{1F_2\left(\frac{1}{3}; \frac{a^3}{9}, \frac{4}{3}\right)} + \frac{\Gamma(2/3)}{4\pi} \frac{\Gamma(2/3)}{1F_2\left(\frac{4}{3}; \frac{2}{3}, \frac{5}{3}\right)}.
\]
The case for \(I_{-1}(a)\) is more complicated and Maple gives (\(\gamma\) is Euler’s constant)
\[
I_{-1}(a) = \frac{3^{1/6} a^{3/2}}{2\pi} \frac{\Gamma(2/3)}{1F_2\left(\frac{4}{3}, \frac{5}{3}\right)} - \frac{3^{1/3} a^3}{54 \Gamma(2/3)} \frac{\Gamma(2/3)}{1F_2\left(\frac{1}{3}, \frac{a^3}{9}\right)} - \frac{3^{1/3}}{18 \Gamma(2/3)} \left\{\ln(a^6/3) + 4\gamma + \sqrt{3}\pi/3\right\}.
\]
Similarly, the value of the integral $I_{-2}(a)$ is given by Maple as

$$I_{-2}(a) = \frac{1}{3^{2/3}a \Gamma(2/3)} \binom{1/3}{a^3/9} + \frac{3^{1/6} \Gamma(2/3) a^3}{72 \pi} 2 F_3 \left( \frac{1, 1; a^3/9}{2, 2, 7/3} \right)$$

$$+ \frac{3^{1/6} \Gamma(2/3)}{12 \pi} \left\{ \ln(a^6/3) + 4 \gamma - 6 - \sqrt{3} \pi/3 \right\}.$$ 

The values of the first few of the required $I_n(a)$ integrals obtained using the recurrence relations (15) are

- $I_{-6}(a) = \frac{1}{36} \left[ I_0(a) + (1/a + 2/a^3) Ai'(a) + (1/a^2 + 8/a^5) Ai(a) \right]$
- $I_{-5}(a) = \frac{1}{12} \left[ I_{-2}(a) + Ai'(a)/a^3 + 3Ai(a)/a^4 \right]$
- $I_{-4}(a) = \frac{1}{12} \left[ I_{-1}(a) + Ai'(a)/a^2 + 2Ai(a)/a^3 \right]$
- $I_{-3}(a) = \frac{1}{2} \left[ I_0(a) + Ai'(a)/a + Ai(a)/a^2 \right]$
- $I_1(a) = -Ai'(a)$
- $I_2(a) = -aAi'(a) + Ai(a)$
- $I_3(a) = 2I_0(a) - a^2Ai'(a) + 2a Ai(a)$
- $I_4(a) = -(6 + a^3)Ai'(a) + 3a^2 Ai(a)$
- $I_5(a) = -(12a + a^4)Ai'(a) + (12 + 4a^3)Ai(a)$
- $I_6(a) = 40I_0(a) - (20a^2 + a^3)Ai'(a) + (40a + 5a^4)Ai(a)$

In general the $I_n(a)$ integrals for $n \geq 0$ are given by (cf. Appendix III)

- $I_{3k}(a) = \frac{(3k)!}{3^k k!} \left\{ I_0(a) + a Ai(a) \sum_{l=0}^{k-1} \frac{(3a)^l}{(3l+1)!} \right\}$
- $a^2 Ai'(a) \sum_{l=0}^{k-1} \frac{(3a)^l}{(3l+2)!}$

- $I_{3k+1}(a) = \frac{3^{2k}k! \Gamma(k+2/3)}{3^k k!} \left\{ \frac{\pi^2}{3} Ai(a) \sum_{l=0}^{k} \frac{(a^3/9)^l}{l! \Gamma(l+5/3)} \right\}$

- $-Ai'(a) \sum_{l=0}^{k} \frac{(a^3/9)^l}{l! \Gamma(l+2/3)}$

- $I_{3k+2}(a) = \frac{3^{2k}k! \Gamma(k+4/3)}{3^k k!} \left\{ 3Ai(a) \sum_{l=0}^{k} \frac{(a^3/9)^l}{l! \Gamma(l+1/3)} \right\}$

- $-aAi'(a) \sum_{l=0}^{k} \frac{(a^3/9)^l}{l! \Gamma(l+1/3)}$

and the $I_{-n}(a)$ for $n \geq 0$ are given by

- $I_{-3k}(a) = \frac{3^{k+1}}{(3k)!} \left\{ I_0(a) + \frac{\Delta(a)}{a} \sum_{l=0}^{k-1} \frac{(3l+1)!}{(3l+2)!} \right\}$

- $+ \frac{\Delta(a)}{a} \sum_{l=0}^{k-1} \frac{(3l+2)!}{(3a)^l} \right\}$

- $I_{-(3k+1)}(a) = \frac{1}{3^{k+1}k! \Gamma(k+2/3)} \left\{ \frac{\Gamma(2/3) I_{-1}(a)}{a^3/9} \sum_{l=0}^{k-1} \frac{(a^3/9)^l}{l! \Gamma(l+5/3)} \right\}$

- $+ \frac{\Delta(a)}{a} Ai'(a) \sum_{l=0}^{k-1} \frac{(a^3/9)^l}{l! \Gamma(l+1/3)}$

- $I_{-(3k+2)}(a) = \frac{1}{3^{k+1}k! \Gamma(k+4/3)} \left\{ \frac{\Gamma(4/3) I_{-2}(a)}{a^3/9} \sum_{l=0}^{k-1} \frac{(a^3/9)^l}{l! \Gamma(l+1/3)} \right\}$

- $+ \frac{\Delta(a)}{a} Ai'(a) \sum_{l=0}^{k-1} \frac{(a^3/9)^l}{l! \Gamma(l+1/3)}$.
In cases where the upper limits of these sum contain negative values the sums above are empty.

The primed integrals $I'_n(a)$ when integrated by parts gives for $n \geq 0$

$$I'_n(a) = -[a^n Ai(a) + n I_{n-1}(a)].$$

In addition we also have the useful relation ($n \geq 2$)

$$I'_{n-2}(a) = [Ai'(a)/a^{n-1} + I_{n+2}(a)]/(n - 1),$$

General expressions for the integrals $I'_n(a)$ are given by

$$I'_{3k}(a) = 9^k k! \Gamma(k + 1/3) \{-Ai(a) \sum_{l=0}^{k} \frac{(a^3/9)^l}{l! (l+1/3)}$$

$$+ \frac{1}{2} aAi'(a) \sum_{l=0}^{k-1} \frac{(a^3/9)^l}{l! (l+4/3)}\},$$

$$I'_{3k+1}(a) = \frac{(3k+1)!}{3^k} \{-I_0(a) - aAi(a) \sum_{l=0}^{k} \frac{(a^3/9)^l}{l! (3k+1)}$$

$$+ a^2 Ai'(a) \sum_{l=0}^{k-1} \frac{(a^3/9)^l}{l! (3k+2)}\},$$

$$I'_{3k+2}(a) = 9^k k! \Gamma(k + 5/3) \{-a^2 Ai(a) \sum_{l=0}^{k} \frac{(a^3/9)^l}{l! (l+5/3)}$$

$$+ 3 Ai'(a) \sum_{l=0}^{k-1} \frac{(a^3/9)^l}{l! (l+2/3)}\}.$$

Some of the $I'_n(a)$ integrals appearing in $I_3(a)$ are explicitly given by

$$I'_{-4}(a) = I_{-2}(a)/3 - Ai'(a)/3a^3,$$

$$I'_{-3}(a) = I_{-1}(a)/2 - Ai'(a)/2a^2,$$

$$I'_{-2}(a) = I_0(a) + Ai'(a)/a,$$

$$I'_{-1}(a) = I_{-2}(a) - Ai(a)/a,$$

$$I_0'(a) = -Ai(a),$$

$$I_1'(a) = -I_0(a) - aAi(a),$$

$$I_2'(a) = -a^2Ai(a) + 2Ai'(a),$$

$$I_3'(a) = -(3 + a^3)Ai(a) + 3aAi'(a),$$

$$I_4'(a) = -8I_0(a) - (8a + a^4)Ai(a) + 4a^4Ai'(a),$$

$$I_5'(a) = -(15a^2 + a^5)Ai(a) + 5(6 + a^3)Ai'(a),$$

$$I_6'(a) = -(72 + 24a^3 + a^6)Ai(a) + 6a(12 + a^3)Ai'(a),$$

$$\ldots$$
(See Appendix III for an alternate means of computing the integrals $I_n(a)$ and $I_n'(a)$.)

In terms of these integrals we have for $I_3(a)$

$$I_3(a) = \pi Ai(0)[\sqrt{3}Ai(a) - Bi(a)] + \frac{3}{2\pi Ai(0)}[\sqrt{3}Ai(-a) - Bi(-a)]I_{-4}(a) + \sum_{i=1}^{\infty} a^{2i} \left( \sum_{k=0}^{\infty} \frac{(-a^2)^k}{(2k+2)!} \left[ A_{2k+2i+1} - \frac{a^{2k+2i+1}}{(2k+2i+1)} A_{2k+2i+1} \right] \right) I_{-4-i}(a).$$

Using this expression with infinite sums truncated to include powers of $a$ up to $a^{10}$ we obtain a value of $I_3(|a'|)$ i.e.

$$I_3(|a'|) = 0.1045962658,$$

whereas the numerical value of $I_3(|a'|) = 0.1045955174$. Using the relation connecting $I_3(a)$ and $I_1(a)$ we have

$$I_1(|a'|) = 0.2082343827,$$

as compared with the numerical value of $I_1(|a'|)$ i.e. 0.2082347508 a value accurate to six decimal places. Since it is possible to increase the accuracy of these calculations as will be seen below we postpone the calculation of $I_2(|a'|)$.

In obtaining the results above we have used closed form expression for the leading terms of some of the infinite sums appearing in equation (11). Now we consider the possibility of continuing the process of including higher order contributions due to $a$ in a more systematic way. Expanding the generating functions of equation (12) in power series in $z$ we obtain an expression which can be readily integrated i.e.

$$I_n(a) = \sum_{i=0}^{\infty} \frac{\xi((n+1)-a,0)}{n!} \int_a^\infty z^{i-n} Ai(z) \, dz + \sum_{i=0}^{\infty} \frac{\lambda((n+1)-a,0)}{n!} \int_a^\infty z^{i-n} A_{i}'(z) \, dz,$$

or in terms of the incomplete Mellin transforms $I_n(a)$ and $I_n'(a)$ we have

$$I_n(a) = \sum_{i=0}^{\infty} \frac{\xi((n+1)-a,0)}{n!} I_{i-n}(a) + \sum_{i=0}^{\infty} \frac{\lambda((n+1)-a,0)}{n!} I_{i-n}'(a).$$

Since the integrals $I_n(a)$ and $I_n'(a)$ are related we have

$$I_n(a) = \sum_{i=1}^{\infty} \frac{\xi((n+1)-a,0)I_i(a) - \lambda((n+1)-a,0)[a^i Ai(a) + iI_{i-1}(a)]}{(n+i)!} + \sum_{i=0}^{n} \frac{\xi((n-i)-a,0)I_{-i}(a) - \lambda((n-i)-a,0)[Ai(a)/a^i - iI_{i-1}(a)]}{(n-i)!},$$

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where the sums containing integral transforms with positive and negative indexes have been written as separate sums. The first few coefficients $\xi^{(i)}(-a, 0)$ and $\lambda^{(i)}(-a, 0)$ (hereafter denoted by $\xi^{(i)}$ and $\lambda^{(i)}$) are

$$\begin{align*}
\xi^{(0)} &= -\pi Ai'(0)\mathfrak{J}_+(a), \\
\xi^{(1)} &= -\pi Ai'(0)\mathfrak{J}'_+(a), \\
\xi^{(2)} &= \pi Ai(0)\mathfrak{J}_-(a) + a \pi Ai'(0)\mathfrak{J}_+(a), \\
\xi^{(3)} &= 3\pi Ai(0)\mathfrak{J}'_-(a) - \pi Ai'(0)[3\mathfrak{J}_+(a) - a \mathfrak{J}'_+(a)].
\end{align*}$$

(16)

$$\begin{align*}
\lambda^{(0)} &= -\pi Ai(0)\mathfrak{J}_-(a), \\
\lambda^{(1)} &= -\pi Ai(0)\mathfrak{J}'_-(a) + \pi Ai'(0)\mathfrak{J}_+(a), \\
\lambda^{(2)} &= a \pi Ai(0)\mathfrak{J}_-(a) + 2\pi Ai'(0)\mathfrak{J}'_+(a), \\
\lambda^{(3)} &= -\pi Ai(0)[2\mathfrak{J}_-(a) - a \mathfrak{J}'_-(a)] - 3a \pi Ai'(0)\mathfrak{J}_+(a).
\end{align*}$$

(17)

We get for $I_n(a)$

$$I_n(a) = -\frac{Ai(a)}{a^n} \sum_{i=0}^\infty \frac{a^i \lambda^{(i)}}{i!} + \sum_{i=0}^\infty b_i I_i(a) + \sum_{i=1}^{n+1} b_{-i} I_{-i}(a).$$

where the coefficients $b_i$ are given by

$$b_i = \frac{1}{(n+1+i)!} \left\{ (n+1+i)\xi^{(n+i)} - (1+i)\lambda^{(n+1+i)} \right\}.$$

A general expression for $I_n(a)$ for all $n$ is quite complicated and is given in Appendix IV below. The corresponding expression for $I_3(a)$ being

$$I_3(a) = I_0(a) \left\{ \frac{2\pi}{\sqrt{3}} \omega_{1,1} + \sum_{k=0}^\infty \Omega_{k,0}(3, a) \right\}$$

$$+ I_{-1}(a) \Gamma(2/3) \{ \omega_{0,2} + \omega_{1,2} \} + I_{-2}(a) \frac{1}{3} \Gamma(1/3) \omega_{0,3}$$

$$+ Ai(a) \left\{ -\frac{a}{\sqrt{a}} \pi Ai(0)\mathfrak{J}_-(a) + \omega_{1,1} \mathfrak{d}_{1,1} + \omega_{1,2} \mathfrak{d}_{1,2} \right\}$$

$$+ Ai'(a) \left\{ \omega_{1,1} \mathfrak{d}'_{1,1} + \omega_{1,2} \mathfrak{d}'_{1,2} - \sum_{\mu=0}^2 \sum_{k=0}^\infty \Omega_{k,\mu}(3, a) \sigma_{k,\mu} \right\}.$$

where the various terms appearing above are defined in Appendix IV. Taking ten terms in the sums over $k$ together with $a = |a'_1| = 1.0187929716$ the terms involving $Ai'(a'_1)$ vanish and we obtain $I_3(a'_1) = 0.1045955174$ a value accurate to ten decimal places. In a similar calculation for $I_4(1.0187929716)$ a value of 0.08085800094 was obtained which is also accurate to ten decimals.
From these expressions the required expressions for $I_1(|a'_1|)$, and $I_2(|a'_1|)$ can be obtained using the relations

\[
I_1(a) = \frac{1}{3a} + \frac{Ai'(0)}{a^2} + \frac{Ai(0)}{a^3} - \frac{2}{a} I_3(a),
\]

\[
I_2(a) = \frac{1}{3a^2} + \frac{2Ai'(0)}{a^3} + \frac{3Ai(0)}{a^4} - \frac{2}{a^3} I_3(a) - \frac{6}{a} I_4(a).
\]

2 An integral containing $Ai'(x)^2 \ln(Ai'(x))$

We next consider the integral

\[
\mathcal{J}^{(2)} = \int_0^\infty \left( \frac{Ai'(x)}{Ai''(0)} \right)^2 \ln \left( \frac{Ai'(x)}{Ai''(0)} \right) dx,
\]

with a value of $-0.2636317105$ when computed numerically. We have

\[
\mathcal{J}^{(2)} = \frac{1}{3Ai'(0)^2} \sum_{k=1}^\infty J(|a'_k|),
\]

with

\[
J(a) = \frac{1}{a} \int_0^\infty \frac{x}{(x+a)} \left( 2Ai(x) Ai'(x) + x Ai'(x)^2 - x^2 Ai(x)^2 \right) dx,
\]

where we have once again used the Weierstrass infinite product representation for the derivative of the $\ln(Ai'(x)/Ai''(0))$ and have integrated by parts. The infinite series expression for $\mathcal{J}^{(2)}$ like $\mathcal{J}^{(1)}$ is slowly convergent. Taking 50 terms in the series above yields a value of $-0.2343590038$ a poor estimate of the integral in question. As in the case of $\mathcal{J}^{(1)}$, the rate of convergence of the sum representing $\mathcal{J}^{(2)}$ can be accelerated as will be seen below.

2.1 Generalized Stieltjes transforms of $Ai(x)^2$ and $Ai'(x)^2$

In order to proceed further with the analysis of $\mathcal{J}^{(2)}$, it is useful to define the integrals

\[
J_n(a) = \int_0^\infty \frac{Ai(x)^2}{(x+a)^n} dx, \quad \text{and} \quad J'_n(a) = \int_0^\infty \frac{Ai'(x)^2}{(x+a)^n} dx,
\]

and to examine their properties. Integration of $J_n(a)$ by parts gives

\[
(n-1) J_n(a) = -\frac{Ai'(0)^2}{a^n} + a n J_{n+1}(a) + n J'_{n+1}(a), \quad (18)
\]

and integrating

\[
\int_0^\infty \frac{x Ai(x)^2}{(x+a)^n} dx,
\]

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by parts gives

\[ J'_n(a) = -\frac{1}{a} Ai(0) Ai'(0) - \frac{n}{2a} A_i(0)^2 - J_{n-1}(a) + a J_n(a) + \frac{n(n+1)}{2} J_{n+2}(a), \]

(19)

In terms of these quantities the expression for \( \mathcal{J}^{(2)} \) becomes

\[ \mathcal{J}^{(2)} = \frac{1}{3 A_i'(0)^2} \sum_{k=1}^{\infty} \left\{ \frac{1}{10|a_k'|} A_i(0)^2 + \frac{1}{3} A_i(0) A_i'(0) - |a_k'| A_i'(0)^2 \right\} \]

Using (18, 19) we obtain a third-order difference equation for the integrals \( J_n(a) \)

\[
(2n - 1) J_n(a) - 2a n J_{n+1}(a) - \frac{1}{2} n(n + 1)(n + 2) J_{n+3}(a) = -\frac{1}{a^n} A_i'(0)^2 - \frac{n}{a^{n+1}} A_i(0) A_i'(0) - \frac{n(n+1)}{2 a^{n+2}} A_i(0)^2,
\]

which can be rewritten as the differential equation i.e.

\[
\frac{1}{2} \frac{d^3 J_n(a)}{a^3} + 2a \frac{d J_n(a)}{a} + (2n - 1) J_n(a) = -\frac{1}{a^n} A_i'(0)^2 - \frac{n}{a^{n+1}} A_i(0) A_i'(0) - \frac{n(n+1)}{2 a^{n+2}} A_i(0)^2,
\]

(20)

having used the first and third of the differential relations

\[
\frac{d J_n(a)}{a} = -n J_{n+1}(a),
\]

\[
\frac{d^2 J_n(a)}{a^2} = n(n + 1) J_{n+2}(a),
\]

\[
\frac{d^3 J_n(a)}{a^3} = -n(n + 1)(n + 2) J_{n+3}(a).
\]

Here we note that a solution of the differential equation for \( J_1(a) \) i.e.

\[
\frac{1}{2} \frac{d^3 J_1(a)}{a^3} + 2a \frac{d J_1(a)}{a} + J_1(a) = -\frac{A_i'(0)^2}{a} - \frac{A_i(0) A_i'(0)}{a^2} - \frac{A_i(0)^2}{a^3}
\]

(21)

is sufficient to obtain \( \mathcal{J}^{(2)} \) since \( J(a) \) can be written in terms of \( J_1(a) \) and its derivatives and the relation

\[ J_1'(a) = -\frac{1}{a} Ai(0) Ai'(0) - \frac{1}{2a} Ai(0)^2 - J_0(a) + a J_1(a) + J_3(a). \]

We get

\[
\mathcal{J}(|a_k'|) = -\frac{2 A_i(0)^2}{5 |a_k'|} + 2 |a_k'|^2 [J_1(|a_k'|)] - \frac{A_i(0) A_i'(0)}{3 |a_k'|^2} - \frac{A_i'(0)^2}{|a_k'|^2}
\]
Before proceeding to the solution of (21), we examine the behavior of $J(|a_k'|)$ for large values of $|a_k'|$. This is necessary given the presence of positive powers of $|a_k'|$ in the expression for $J(|a_k'|)$ shown above. Repeated integration by parts of the integrals $J_1(a)$, $J_1'(a)$, and $J_2(a)$ give

\[
\begin{align*}
J_1(a) & \sim \frac{Ai'(0)^2}{a} + \frac{Ai(0)Ai'(0)}{3a^2} + \frac{Ai(0)^2}{5a^3} + \cdots, \\
J_1'(a) & \sim - \frac{2Ai(0)Ai'(0)}{3a} - \frac{3Ai(0)^2}{10a^2} + \frac{4Ai'(0)^2}{7a^3} + \cdots, \\
J_2(a) & \sim \frac{Ai'(0)^2}{a^2} + \frac{2Ai(0)Ai'(0)}{3a^3} + \frac{3Ai(0)^2}{5a^4} + \cdots
\end{align*}
\]

Using these expressions we find the value of the summand $J(|a_k'|)$ as $k \to \infty$ to be

\[
J(|a_k'|) \sim - \frac{3}{7|a_k'|^2} Ai'(0)^2 - \frac{2}{3|a_k'|^3} Ai(0) Ai'(0) + \cdots
\]

and note that a cancellation of terms containing positive powers of $|a_k'|$ has taken place and the expression for the sum $J^{(2)}$ is seen to be finite for large $k$.

The solution of the differential equation for $J_1(a)$ is then by

\[
\frac{J_1(a)}{\pi^2} = Ai(-a)^2 [c_1 - \Delta u_1(a)] + Bi(-a)^2 [c_2 - \Delta u_2(a)] + Ai(-a)Bi(-a) [c_3 + 2 \Delta u_3(a)],
\]

where the $c$'s are constants of integration and

\[
\begin{align*}
\Delta u_1(a) & = \int_{a_0}^a [Ai'(0)^2/z + Ai(0)Ai'(0)/z^2 + Ai(0)^2/z^3] Bi(-z)^2 dz, \\
\Delta u_2(a) & = \int_{a_0}^a [Ai'(0)^2/z + Ai(0)Ai'(0)/z^2 + Ai(0)^2/z^3] Ai(-z)^2 dz, \\
\Delta u_3(a) & = \int_{a_0}^a [Ai'(0)^2/z + Ai(0)Ai'(0)/z^2 + Ai(0)^2/z^3] Ai(-z) Bi(-z) dz.
\end{align*}
\]

The initial conditions

\[
\begin{align*}
J_1(a_0) & = J_1(a)|_{a=a_0}, \\
J_2(a_0) & = - \frac{dJ_1(a)}{da}|_{a=a_0}, \\
J_3(a_0) & = \frac{1}{2} \frac{d^2 J_1(a)}{da^2}|_{a=a_0},
\end{align*}
\]

where $a_0$ is any non-zero value of $a$ are sufficient to determine the constants of
integration $c_1, c_2, c_3$. The latter are given by

$$c_1(a_0) = J_1(a_0)[a_0 Bi(-a_0)^2 + Bi'(-a_0)^2] - J_2(a_0)Bi(-a_0)Bi'(-a_0) + J_3(a_0)Bi(-a_0)^2,$$

$$c_2(a_0) = J_1(a_0)[a_0 Ai(-a_0)^2 + Ai'(-a_0)^2] - J_2(a_0)Ai(-a_0)Ai'(-a_0) + J_3(a_0)Ai(-a_0)^2,$$

$$c_3(a_0) = -2J_1(a_0)[a_0 Ai(-a_0)Bi(-a_0) + Ai'(-a_0)Bi'(-a_0)] + J_2(a_0)[Ai(-a_0)Bi'(-a_0) + Ai'(-a_0)Bi(-a_0)] - 2J_3(a_0) Ai(-a_0)Bi(-a_0).$$

Using the relations $a_0 = |a_1'|, Ai'(a_1') = 0$ where $a_1'$ is the first root of $Ai'(a)$ and the Wronskian of $Ai(z)$ and $Bi(a)$, we have $Ai(a_1')Bi'(a_1') = 1/\pi$ and the expressions for $c_1, c_2, c_3$ simplify to

$$c_1(|a_1'|) = J_{1,3}(|a_1'|) Bi(a_1')^2 + [J_1(|a_1'|) Bi'(a_1') - J_2(|a_1'|) Bi(a_1')] Bi'(a_1'),$$

$$c_2(|a_1'|) = J_{1,3}(|a_1'|) Ai(a_1')^2,$$

$$c_3(|a_1'|) = -2J_{1,3}(|a_1'|) Ai(a_1')Bi(a_1') + \frac{1}{\pi} J_2(|a_1'|),$$

where

$$J_{1,3}(|a_1'|) = |a_1'| J_1(|a_1'|) + J_3(|a_1'|).$$

2.2 Evaluation of the $u_i(a)$ integrals

The integrals $u_i(a)$ can be computed by expanding the functions $Ai(-a)^2, Bi(-a)^2$ and $Ai(-a)Bi(-a)$ in power series in $a$. The three functions $Ai(z)^2, Bi(z)^2$ and $Ai(z)Bi(z)$ denoted in common by $w(z)$, are solutions of the third-order differential equation

$$w^{(3)}(z) + 4zw^{(1)}(z) + 2w(z) = 0.$$ The $n$th derivative of $w(z)$ evaluated at $z = 0$ is then given by the difference equation

$$(4n + 2)w^{(n)}(0) + w^{(n+3)}(0) = 0.$$ The solution of which is (where $w^{(\square)}, w^{(I)}, w^{(II)}$ are constants)

$$w^{(3n)}(0) = w^{(\square)}\frac{(-12)^n}{2 \pi} \Gamma(5/6)\Gamma(n + 1/6),$$

$$w^{(3n+1)}(0) = w^{(I)}\frac{(-12)^n}{\sqrt{\pi}} \Gamma(n + 1/2),$$

$$w^{(3n+2)}(0) = w^{(II)}\frac{(-12)^n \Gamma(n + 5/6)}{\Gamma(5/6)}.$$
The infinite series expansions for the three Airy products \( Bi(-z)^2 \), \( Ai(-z)^2 \) and \( Ai(-z) Bi(-z) \) are then

\[
Bi(-z)^2 = w_1^{(□)} S_1(z) + w_1^{(I)} S_2(z) + w_1^{(II)} S_3(z),
\]
\[
Ai(-z)^2 = w_2^{(□)} S_1(z) + w_2^{(I)} S_2(z) + w_2^{(II)} S_3(z),
\]
\[
Ai(-z) Bi(-z) = w_3^{(□)} S_1(z) + w_3^{(I)} S_2(z) + w_3^{(II)} S_3(z),
\]

where

\[
S_1(z) = \frac{\Gamma(5/6) \sum_{n=0}^{\infty} \frac{12^n \Gamma(n + 1/6)}{(3n)!}}{2\pi} (-z)^{3n},
\]
\[
S_2(z) = -\frac{z}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{12^n \Gamma(n + 1/2)}{(3n + 1)!} (-z)^{3n},
\]
\[
S_3(z) = \frac{z^2}{\Gamma(5/6)} \sum_{n=0}^{\infty} \frac{12^n \Gamma(n + 5/6)}{(3n + 2)!} (-z)^{3n},
\]

and \( w_1^{(□)}, w_1^{(I)}, w_1^{(II)} \) are given in the Table 3 below.

| \( w_i(z) \)       | \( w_i^{(□)} \) | \( w_i^{(I)} \) | \( w_i^{(II)} \) |
|---------------------|-----------------|-----------------|-----------------|
| 1 \( Bi(z)^2 \)    | 3\( Ai(0)^2 \)  | -6\( Ai(0)Ai'(0) \) | 6\( Ai'(0)^2 \) |
| 2 \( Ai(z)^2 \)    | \( Ai(0)^2 \)   | 2\( Ai(0)Ai'(0) \) | 2\( Ai'(0)^2 \) |
| 3 \( Ai(z)Bi(z) \) | \( 3\( Ai(0)^2 \) \) | 0               | \(-2\sqrt{3}\( Ai'(0)^2 \) |
\[
\int \frac{w_1(-z)}{z^2} \, dz = - \frac{w_1^{(\Box)}}{z} \, {}_2F_3\left(-\frac{1}{3}, \frac{1}{6} ; \frac{-1}{4}z^3 \right) \\
+ w_1^{(I)} \left[ \ln(z) - \frac{z^3}{12} \, {}_3F_4\left(\frac{1}{3, 3/2, 2, 5/3} ; \frac{-1}{4}z^3 \right) \right] \\
+ z \frac{w_1^{(II)}}{2} \, {}_2F_3\left(\frac{1/3, 5/6, 1/2}{4/3, 3/3, 2} ; \frac{-1}{4}z^3 \right),
\]

\[
\int \frac{w_1(-z)}{z^3} \, dz = - \frac{w_1^{(\Box)}}{2z^2} \, {}_2F_3\left(-\frac{2}{3, 1/6} ; -\frac{1}{4}z^3 \right) - \frac{w_1^{(I)}}{z} \, {}_2F_3\left(-\frac{1}{3, 1/2} ; -\frac{1}{4}z^3 \right) \\
+ \frac{w_1^{(II)}}{2} \left[ \ln(z) - \frac{z^3}{18} \, {}_3F_4\left(\frac{1, 11/6, 4}{2, 7/3, 3, 3/3} \right) \right].
\]

Alternately, Maple gives the same results. The resulting explicit expressions
for the $u_i(z)$ are

\[
\begin{align*}
    u_1(z) &= \frac{\pi^2}{2 \cdot 3^{3/4} \Gamma(2/3)^4 z^2} \, 2 F_3 \left( \begin{array}{c} -2/3, 1/6; \ -\frac{4}{3} \frac{z^3}{3} \\ 1/3, 1/3, 2/3 \end{array} \right) \\
    &\quad - \frac{3^{2/3} \Gamma(2/3)^4 z^2}{32 \pi^2} \, 2 F_3 \left( \begin{array}{c} 2/3, 5/6; \ -\frac{4}{3} \frac{z^3}{3} \\ 4/3, 5/3, 5/3 \end{array} \right) \\
    &\quad + \frac{\pi}{3^{1/6} \pi (2/3)^2 z} \left[ 2 F_3 (1/3, 1/2; -\frac{4}{3} \frac{z^3}{3}) - \frac{1}{2} \, 2 F_3 (1/3, 1/2; -\frac{4}{3} \frac{z^3}{3}) \right] \\
    &\quad + \frac{z^3}{216} \left[ 3 F_4 (1, 1, 1, 1/6; -\frac{4}{3} \frac{z^3}{3}) + 2 F_4 (1, 1, 1, 7/6; \frac{4}{3} \frac{z^3}{3}) \right] \\
    &\quad - 3 F_4 (1, 1, 1, 7/6; \frac{4}{3} \frac{z^3}{3}) \right], \\

    u_2(z) &= -\ln(z) + \frac{\pi^2}{2 \cdot 3^{5/3} \Gamma(2/3)^4 z^2} \, 2 F_3 \left( \begin{array}{c} -2/3, 1/6; \ -\frac{4}{3} \frac{z^3}{3} \\ 1/3, 1/3, 2/3 \end{array} \right) \\
    &\quad - \frac{3^{5/3} \Gamma(2/3)^4 z^2}{32 \pi^2} \, 2 F_3 \left( \begin{array}{c} 2/3, 5/6; \ -\frac{4}{3} \frac{z^3}{3} \\ 4/3, 5/3, 5/3 \end{array} \right) \\
    &\quad - \frac{\pi}{3^{6/6} \pi (2/3)^2 z} \left[ 2 F_3 (1/3, 1/2; -\frac{4}{3} \frac{z^3}{3}) + \frac{1}{2} \, 2 F_3 (1/3, 1/2; -\frac{4}{3} \frac{z^3}{3}) \right] \\
    &\quad + \frac{\pi}{8 \pi} \left[ 2 F_3 (1/3, 1/2; -\frac{4}{3} \frac{z^3}{3}) + 2 F_3 (1/3, 1/2; -\frac{4}{3} \frac{z^3}{3}) \right] \\
    &\quad + \frac{z^3}{72} \left[ 3 F_4 (1, 1, 1/6; -\frac{4}{3} \frac{z^3}{3}) + 2 F_4 (1, 1, 1, 7/6; \frac{4}{3} \frac{z^3}{3}) \right] \\
    &\quad + 3 F_4 (1, 1, 1, 7/6; \frac{4}{3} \frac{z^3}{3}) \right], \\

    u_3(z) &= -\ln(z) - \frac{\pi^2}{3^{1/3} \pi (2/3)^4 z^2} \, 2 F_3 \left( \begin{array}{c} -2/3, 1/6; \ -\frac{4}{3} \frac{z^3}{3} \\ 1/3, 1/3, 2/3 \end{array} \right) \right] \\
    &\quad + \frac{3^{7/6} \Gamma(2/3)^4 z^2}{16 \pi^2} \, 2 F_3 \left( \begin{array}{c} 2/3, 5/6; \ -\frac{4}{3} \frac{z^3}{3} \\ 4/3, 5/3, 5/3 \end{array} \right) \\
    &\quad + \frac{\pi}{2^{1/3} \pi (2/3)^2 z} \left[ 2 F_3 (1/3, 1/2; -\frac{4}{3} \frac{z^3}{3}) + \frac{1}{2} \, 2 F_3 (1/3, 1/2; -\frac{4}{3} \frac{z^3}{3}) \right] \\
    &\quad + \frac{3^{1/3} \Gamma(2/3)^2 z}{4 \pi} \, 2 F_3 \left( \begin{array}{c} 2/3, 5/6; \ -\frac{4}{3} \frac{z^3}{3} \\ 4/3, 5/3, 5/3 \end{array} \right) \\
    &\quad + \frac{\sqrt{3} z^3}{108} \left[ 3 F_4 (1, 1, 1, 7/6; \frac{4}{3} \frac{z^3}{3}) - 2 F_4 (1, 1, 1, 7/6; \frac{4}{3} \frac{z^3}{3}) \right].
\end{align*}
\]
2.3 The $J(a)$ terms

The first and second derivatives of $J_1(a)$ are given by

$$\frac{1}{\pi^2} \frac{d J_1(a)}{d a} = -2 Ai(-a)Ai'(-a)[c_1 - \Delta u_1(a)]$$
$$-2 Bi(-a)Bi'(-a)[c_2 - \Delta u_2(a)]$$
$$-[Ai'(-a)Bi(-a) + Ai(-a)Bi'(-a)][c_3 + 2 \Delta u_3(a)],$$

$$\frac{1}{\pi^2} \frac{d^2 J_1(a)}{d a^2} = 2 [Ai'(-a)^2 - a Ai(-a)^2][c_1 - \Delta u_1(a)]$$
$$+2 [Bi'(-a)^2 - a Bi(-a)^2][c_2 - \Delta u_2(a)]$$
$$-2 [a Ai(-a)Bi(-a) - Ai'(-a)Bi'(-a)][c_3 + 2 \Delta u_3(a)].$$

Using these expressions, the terms $J(|a_k'|)$ are

$$J(|a_k'|) = -\frac{2 Ai(0)^2}{5 |a_k'|} - \frac{2}{3} Ai(0)Ai'(0) - 2 |a_k'| Ai'(0)^2 + j(a)|_{a=|a_k'|},$$

where

$$j(a) = \frac{1}{\pi^2} [2a^2 J_1(a) + \frac{d J_1(a)}{d a} + \frac{a}{2} \frac{d^2 J_1(a)}{d a^2}],$$

and $j(a)|_{a=|a_k'|}$ hereafter denoted by $j(|a_k'|)$ is explicitly given by

$$j(|a_k'|) = [a_k'|^2 Ai(a_k')^2 - 2Ai(a_k')Ai'(a_k') + |a_k'| Ai'(a_k')^2]$$
$$+ [a_k'|^2 Bi(a_k')^2 - 2Bi(a_k')Bi'(a_k') + |a_k'| Bi'(a_k')^2]$$
$$+ [c_1 - \Delta u_1(|a_k'|)]$$
$$+ [c_2 - \Delta u_2(|a_k'|)]$$
$$+ [c_3 + 2 \Delta u_3(|a_k'|)]$$
$$- Ai(a_k')Bi'(a_k') - Ai'(a_k')Bi(a_k'),$$

which simplifies to

$$j(|a_k'|) = [a_k'|^2 Ai(a_k')^2 + |a_k'| Bi'(a_k')^2]$$
$$+ [a_k'|^2 Bi(a_k')^2 - 2Bi(a_k')Bi'(a_k') + |a_k'| Ai'(a_k')^2]$$
$$+ [c_1 - \Delta u_1(|a_k'|)]$$
$$+ [c_2 - \Delta u_2(|a_k'|)]$$
$$- [c_3 + 2 \Delta u_3(|a_k'|)] Ai(a_k')Bi'(a_k').$$
Upon insertion of the expressions for $\Delta u_i(|a_k'|)$ and collecting terms we get the complicated form for $j(|a_k'|)$

$$j(|a_k'|) = c_1 \left[ |a_k'|^2 Ai(a_k')^2 - 2Ai(a_k')Ai'(a_k') + |a_k'| Ai'(a_k')^2 \right]$$

$$+ c_2 + \ln(|a_k'/a_k'|)$$

$$+ \left[ |a_k'|^2 Bi(a_k')^2 - 2Bi(a_k')Bi'(a_k') + |a_k'| Bi'(a_k')^2 \right]$$

$$- Ai(a_k')Bi'(a_k' - Ai'(a_k')Bi(a_k'))$$

$$- d_1(|a_k'|)$$

$$- d_2(|a_k'|)$$

$$+ d_3(|a_k'|)$$

where

$$d_1(a) = a^2 [Ai(-a)^2 + 4\sqrt{3} Ai(-a) Bi(-a) + Bi(-a)^2]$$

$$+ a[Ai'(-a)^2 - 4\sqrt{3} Ai'(-a) Bi'(-a) + Bi'(-a)^2]$$

$$- 4\sqrt{3} [Ai(-a)Bi'(-a) + Ai'(-a)Bi(-a)]$$

$$- 2 [Ai(-a)Ai'(-a) + 3Bi(-a)Bi'(-a)],$$

$$d_2(a) = a^2 [Ai(-a)^2 - 4\sqrt{3} Ai(-a) Bi(-a) + Bi(-a)^2]$$

$$+ a[Ai'(-a)^2 - 4\sqrt{3} Ai'(-a) Bi'(-a) + Bi'(-a)^2]$$

$$+ 4\sqrt{3} [Ai(-a)Bi'(-a) + Ai'(-a)Bi(-a)]$$

$$- 2 [Ai(-a)Ai'(-a) + 3Bi(-a)Bi'(-a)],$$

$$d_3(a) = a^2 [Ai(-a)^2 - 3Bi(-a)^2] + a[Ai'(-a)^2 - 3Bi'(-a)^2]$$

$$- 2 [Ai(-a)Ai'(-a) - 3Bi(-a)Bi'(-a)].$$
In the case where $a = |a'_k|$ we have $Ai(a'_k)Bi'(a'_k) = 1/\pi$ and the expressions for $j(|a'_k|)$ and $d_i(a)$ become

$$j(|a'_k|) = c_1 \left[ |a'_k|^2 Ai(a'_k)^2 \right] - c_3/\pi$$

$$+ [c_2 + \ln(a'_k/a'_0)] \left[ |a'_k|^2 Bi(a'_k)^2 - 2Bi(a'_k)Bi'(a'_k) + |a'_k| Bi'(a'_k)^2 \right]$$

$$- d_1(|a'_k|) \Delta H_1 - d_2(|a'_k|) \Delta H_2 + d_3(|a'_k|) \Delta H_3,$$

and where

$$H_1(a) = -\frac{\pi^2}{2 \cdot 3^{2/3} \cdot 4^2} \ 2F_3 \left( -2/3, 1/6; -\frac{a}{a_0^3} \right) - \frac{1}{108} \ a^3 \ 3F_4 \left( 1, 1, 1/3, 2; -\frac{a}{a_0^3} \right)$$

$$+ \frac{\pi}{2 \cdot 3^{2/3} \cdot 4^2} \ a^2 \ 2F_3 \left( -1/3, 1/6; -\frac{a}{a_0^3} \right),$$

$$H_2(a) = \frac{3^{2/3} \Gamma(2/3)^2}{3 \cdot 2^2} a^2 \ 2F_3 \left( -1/3, 1/2; -\frac{a}{a_0^3} \right) - \frac{\Gamma(2/3)^2}{8 \pi \ 3^{1/3}} a \ 2F_3 \left( 1, 1, 1/3, 2; -\frac{a}{a_0^3} \right)$$

$$- \frac{a^3}{16 \ 3 \ 4} \ 3F_4 \left( 1, 1, 1/3, 2; -\frac{a}{a_0^3} \right),$$

and

$$H_3(a) = \frac{\pi}{3 \cdot 2^2 \cdot 3^{1/3}} a^3 \ 2F_3 \left( -1/3, 1/2; -\frac{a}{a_0^3} \right) - \frac{\Gamma(2/3)^2}{4 \pi \ 3^{1/3}} a \ 2F_3 \left( 1, 1, 1/3, 2; -\frac{a}{a_0^3} \right)$$

$$- \frac{a^3}{\pi \ 3} \ 3F_4 \left( 1, 1, 1/3, 2; -\frac{a}{a_0^3} \right),$$

and

$$d_1(|a'_k|) = |a'_k|^2 \left[ Ai(a'_k)^2 + 4\sqrt{3} Ai(a'_k) Bi(a'_k) + Bi(a'_k)^2 \right]$$

$$+ |a'_k| Bi'(a'_k)^2 - 6Bi(a'_k)Bi'(a'_k) - 3|a'_k| Bi'(a'_k)^2,$$

$$d_2(|a'_k|) = |a'_k|^2 \left[ Ai(a'_k)^2 - 4\sqrt{3} Ai(a'_k) Bi(a'_k) + Bi(a'_k)^2 \right]$$

$$+ |a'_k| Bi'(a'_k)^2 - 6Bi(a'_k)Bi'(a'_k) + 3|a'_k| Bi'(a'_k)^2,$$

$$d_3(|a'_k|) = |a'_k|^2 \left[ Ai(a'_k)^2 - 3Bi(a'_k)^2 \right] - 3|a'_k| Bi'(a'_k)^2 + 6Bi(a'_k)Bi'(a'_k).$$

As in the case of $\mathcal{J}^{(1)}$ where the rate of convergence of the sum over $I_1$ could be accelerated if the sum containing $|a'_k|$ with large $k$ was subtracted, a similar procedure can be performed in the case of $\mathcal{J}^{(2)}$. We write for $\mathcal{J}^{(2)}$

$$\mathcal{J}^{(2)} = \frac{1}{3Ai(0)} \sum_{k=1}^{N} \left[ J(|a'_k|) - J(|a'_k| |a'| > x) \right] + \frac{1}{3Ai(0)} \sum_{i=N+1}^{\infty} J(|a'_k| |a'| > x),$$

and defining

$$J(a)_{a \geq x} = \frac{1}{a^2} \int_0^a \frac{x}{(1 + x/a)} [2Ai(x)Ai'(x) + xAi'(x)^2 - x^2 Ai(x)^2],$$

$$J(a)_{a \geq x} \approx \frac{1}{a^2} \int_0^\infty \frac{x}{(1 + x/a)} [2Ai(x)Ai'(x) + xAi'(x)^2 - x^2 Ai(x)^2],$$

where in the last expression, terms of order $\exp(-4a^{3/2})$ have been omitted. Expanding the denominator in the expression above in a power series in $x$ followed by integration (cf. Appendix III) gives the approximate expression for
\[ \mathcal{J}^{(2)} \approx \frac{1}{3\text{Ai}^{(0)}(0)} \sum_{k=1}^{N} J(|a_k|) \]

\[ -\frac{2}{12^{13/6} \sqrt{\text{Ai}^{(0)}(0)^2}} \sum_{k=0}^{n} \left( \frac{-1}{12} \right)^k \frac{(k+4)(k+1)!}{\Gamma(k/3+13/6)} \left\{ \mathcal{Z}_{k+2} - \mathcal{Z}_{k+2}(N) \right\}. \]

In the case where \( N = 10 \) and \( n = 6 \) the expression above produces \( \mathcal{J}^{(2)} = -0.2636317121 \) a value which is accurate to 8 decimal places.

### 2.4 Values of \( J_n(a) \) for small \( a \)

The solution of the third-order differential equation (21) for \( J_1(a) \) requires values of \( J_1(a_0), J_2(a_0), \) and \( J_3(a_0) \). If analytic expressions for those quantities are also sought, then the integrals \( J_n(a) \) involving \( \text{Ai}(x)^2 \) can be obtained in a way similar to that used for the integrals \( I_n(a) \) which contained \( \text{Ai}(x) \) and were treated above. That is to say we write

\[ J_n(a) = \int_{a}^{\infty} \frac{\text{Ai}(x-a)^2}{x^n} dx, \tag{22} \]

and expand \( \text{Ai}(x-a)^2 \) in a power series in \( a \). We get

\[ \text{Ai}(x-a)^2 = \sum_{j=0}^{\infty} \frac{(-a)^j}{j!} \frac{d^j \text{Ai}(x)^2}{dx^j}. \]

To begin with, we compute the \( j \)th derivatives of \( \text{Ai}(x)^2 \). It is easy to see that these derivatives are given by expressions with the following form

\[ \frac{d^j \text{Ai}(x)^2}{dx^j} = \mathfrak{P}_j(x) \text{Ai}(x)^2 + \mathfrak{Q}_j(x) \text{Ai}'(x)^2 + \mathfrak{R}_j(x) \text{Ai}(x) \text{Ai}'(x), \]

where the quantities \( \mathfrak{P}_j(x), \mathfrak{Q}_j(x), \mathfrak{R}_j(x) \) are polynomials. Using these relations we have

\[ \text{Ai}(x-a)^2 = \sum_{j=0}^{\infty} \frac{(-a)^j}{j!} \left[ \mathfrak{P}_j(x) \text{Ai}(x)^2 + \mathfrak{Q}_j(x) \text{Ai}'(x)^2 + \mathfrak{R}_j(x) \text{Ai}(x) \text{Ai}'(x) \right]. \]

Rearranging the latter expression we get

\[ \text{Ai}(x-a)^2 = \text{Ai}(x)^2 \sum_{j=0}^{\infty} \frac{(-a)^j}{j!} \mathfrak{P}_j(x) + \text{Ai}'(x)^2 \sum_{j=0}^{\infty} \frac{(-a)^j}{j!} \mathfrak{Q}_j(x) \]

\[ + \text{Ai}(x) \text{Ai}'(x) \sum_{j=0}^{\infty} \frac{(-a)^j}{j!} \mathfrak{R}_j(x). \]
Expansions of $Ai(x - a)^2$ in a Taylor series which includes powers of $a$ up to $a^{15}$, allows values of $J_1(|a'|)$, $J_2(|a'|)$ and $J_3(|a'|)$ to be obtained which are accurate to 8 decimal places with values of 0.04826441, 0.03654795 and 0.02879280 respectively. As was seen above this method is not easily expressed in general terms and a more systematic way of obtaining an analytical expression for the $J_n(a)$ integrals is given below.

The quantities in the brackets above have the form of generating functions and are here denoted by $\Xi(-a,x)$, $\Lambda(-a,x)$, and $\varrho(-a,x)$ for the polynomials $\Psi_j(x)$, $\Omega_j(x)$, and $\Re_j(x)$ respectively. Written more compactly we have

$$J_n(a) = \int_0^\infty [\Xi(-a,x) Ai(x)^2 + \Lambda(-a,x) Ai'(x)^2 + \varrho(-a,x) Ai(x) Ai'(x)] \frac{dx}{x^n}. \quad (23)$$

From the expression corresponding to the $j+1$ derivative of $Ai(x)^2$ we get the differential recurrence equations for the $\Psi_j(x)$, $\Omega_j(x)$ and $\Re_j(x)$ polynomials i.e.

$$\Psi_{j+1}(x) = \frac{d\Psi_j(x)}{dx} + x \Re_j(x),$$
$$\Omega_{j+1}(x) = \frac{d\Omega_j(x)}{dx} + \Re_j(x),$$
$$\Re_{j+1}(x) = \frac{d\Re_j(x)}{dx} + 2\Psi_j(x) + 2x \Omega_j(x),$$

with initial values given by

$$\Psi_0(x) = 1,$$
$$\Omega_0(x) = 0,$$
$$\Re_0(x) = 0.$$

The first few of these polynomials are given in the Table 4 below.

| $i$ | $\Psi_i$       | $\Omega_i$ | $\Re_i$ |
|-----|----------------|------------|---------|
| 0   | 1              | 0          | 0       |
| 1   | 0              | 0          | 2       |
| 2   | $2x$           | 2          | 0       |
| 3   | 2              | 0          | 8$x$    |
| 4   | $8x^2$         | $8x$       | 12      |
| 5   | $28x$          | 20         | $32x^2$ |
| 6   | $28 + 32x^3$   | $32x^2$    | 160$x$  |
| 7   | $256x^2$       | $224x$     | $216 + 128x^3$ |
| 8   | $728x + 128x^4$ | $440 + 128x^3$ | $1344x^2$ |
The quantity $Ai(x)^2$ satisfies the differential equation
\[
\frac{d^3 Ai(x)^2}{dx^3} - 4x \frac{d Ai(x)^2}{dx} - 2 Ai(x)^2 = 0,
\]
from which it follows that
\[
\frac{d^{j+3} Ai(x)^2}{dx^{j+3}} - 4x \frac{d^{j+1} Ai(x)^2}{dx^{j+1}} - (4j + 2) \frac{d^j Ai(x)^2}{dx^j} = 0.
\]
From this relation we get the recurrence equations
\[
\Psi_{j+3}(x) - 4x \Psi_{j+1}(x) - (4j + 2) \Psi_j(x) = 0, \\
\Omega_{j+3}(x) - 4x \Omega_{j+1}(x) - (4j + 2) \Omega_j(x) = 0, \\
\Re_{j+3}(x) - 4x \Re_{j+1}(x) - (4j + 2) \Re_j(x) = 0. \quad (24)
\]
The generating functions $\Xi(t, x), \Lambda(t, x), \text{and } \wp(t, x)$
\[
\Xi(t, x) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \Psi_j(x), \\
\Lambda(t, x) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \Omega_j(x), \\
\wp(t, x) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \Re_j(x),
\]
can be obtained in closed form by computing their first three derivatives and using the recurrence relations (24) to obtain the differential equations defining them. For example in the case of $\Xi(t, x)$ we get
\[
\frac{d^3 \Xi(t, x)}{dt^3} - 4(x + t) \frac{d \Xi(t, x)}{dt} - 2 \Xi(t, x) = 0,
\]
the solution of which is
\[
\Xi(t, x) = g_1(x) Ai(t + x)^2 + g_2(x) Bi(t + x)^2 + g_3(x) Ai(t + x) Bi(t + x).
\]
The boundary conditions in this case are
\[
[\Xi(t, x)]_{t=0} = \Psi_0(x) = 1, \\
[d \Xi(t, x)/dt]_{t=0} = \Psi_1(x) = 0, \\
[d^2 \Xi(t, x)/dt^2]_{t=0} = \Psi_2(x) = 2x,
\]
with the resulting closed form expression for $\Xi(t, x)$ being
\[
\Xi(t, x) = \pi^2 [Bi'(x) Ai(x + t) - Ai'(x) Bi(x + t)]^2 = \xi(t, x)^2.
\]
In a similar way, the closed form expressions for the generating functions $\Lambda(t, x)$ and $\varrho(t, x)$ are

$$
\Lambda(t, x) = \pi^2 [Bi(x)Ai(x + t) - Ai(x)Bi(x + t)]^2 = \lambda(t, x)^2,
$$

and

$$
\varrho(t, x) = -2\pi^2 [Bi(x)Bi'(x) Ai(x + t)^2 + Ai(x)Ai'(x) Bi(x + t)^2]
+ 2\pi^2 \{Ai(x)Bi'(x) + Bi(x)Ai'(x)\} Ai(x + t) Bi(x + t),
$$

which can we written as

$$
\varrho(t, x) = 2\pi^2 [Bi'(z) Ai(z + t) - Ai'(z) Bi(z + t)]
\cdot [Ai(z) Bi(z + t) - Bi(z) Ai(z + t)]
= 2 \xi(t, x) \lambda(t, x).
$$

As expected we see that $\Xi(t, x)$, $\Lambda(t, x)$ and $\varrho(t, x)$ are related to the generating functions $\xi(t, x)$ and $\lambda(t, x)$ encountered above. We also note that the polynomials $\mathfrak{P}_j(x)$, $\mathfrak{Q}_j(x)$ and $\mathfrak{R}_j(x)$ are related to the polynomials $\mathfrak{P}_k(z)$ and $\mathfrak{Q}_k(z)$ by simple bilinear relations.

Expanding the generating functions $\Xi(-a, x)$, $\Lambda(-a, x)$, and $\varrho(-a, x)$ in power series in $x$ we get an expression which contains the incomplete Mellin transforms of $Ai(x)^2$, $Ai'(x)^2$ and $Ai(x)Ai'(x)$

$$
J_n(a) = \sum_{i=0}^{\infty} \frac{\Xi^{(i)}(-a, 0)}{i!} \int_a^\infty x^{i-n} Ai(x)^2 dx
+ \sum_{i=0}^{\infty} \frac{\Lambda^{(i)}(-a, 0)}{n!} \int_a^\infty x^{i-n} Ai'(x)^2 dx
+ \sum_{i=0}^{\infty} \frac{\varrho^{(i)}(-a, 0)}{i!} \int_a^\infty x^{i-n} Ai(x) Ai'(x) dx. \tag{25}
$$

The first few expansion coefficients of those appearing above are given by

$$
\Xi(-a, 0) = \pi^2 Ai'(0)^2 \mathfrak{J}_+(a)^2,
$$

$$
\Xi^{(1)}(-a, 0) = 2\pi^2 Ai'(0)^2 \mathfrak{J}_+(a) \mathfrak{J}_+'(a),
$$

$$
\Xi^{(2)}(-a, 0) = 2\pi^2 Ai'(0)^2 [\mathfrak{J}_+'(a)^2 - a \mathfrak{J}_+(a)^2]
- 2\pi^2 Ai(0) Ai'(0) \mathfrak{J}_+(a) \mathfrak{J}_-(a),
$$

$$
\Xi^{(3)}(-a, 0) = 2\pi^2 Ai'(0)^2 [3\mathfrak{J}_+(a)^2 - 4a \mathfrak{J}_+(a) \mathfrak{J}_+'(a)]
- 6\pi^2 Ai(0) Ai'(0) [\mathfrak{J}_+(a) \mathfrak{J}_-'(a) + \mathfrak{J}_+'(a) \mathfrak{J}_-(a)].
$$

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\[ \Lambda(-a,0) = \pi^2 \text{Ai}(0)^2 \text{J}_-(a)^2, \]
\[ \Lambda^{(1)}(-a,0) = 2\pi^2 \text{Ai}(0)^2 \text{J}_-(a) \text{J}_+(a) - 2\pi^2 \text{Ai}(0) \text{Ai}'(0) \text{J}_+(a) \text{J}_-(a), \]
\[ \Lambda^{(2)}(-a,0) = 2\pi^2 \text{Ai}'(0)^2 \text{J}_+(a)^2 + 2\pi^2 \text{Ai}(0)^2 [\text{J}_+(a)^2 - \text{J}_-(a)^2] - 4\pi^2 \text{Ai}(0) \text{Ai}'(0) [\text{J}_+(a) \text{J}_+(a) + \text{J}_+(a) \text{J}_-(a)], \]
\[ \Lambda^{(3)}(-a,0) = 12\pi^2 \text{Ai}'(0)^2 \text{J}_+(a) \text{J}_+(a) + 4\pi^2 \text{Ai}(0)^2 [\text{J}_-(a)^2 - 2a \text{J}_-(a) \text{J}_+(a)] + 12\pi^2 \text{Ai}(0) \text{Ai}'(0) [a \text{J}_+(a) \text{J}_-(a) - \text{J}_+(a) \text{J}_+(a)], \]
where the \text{J} terms have been defined above.

2.5 The incomplete Mellin transforms of \( \text{Ai}(x)^2, \text{Ai}'(x)^2, \text{Ai}(x) \text{Ai}'(x) \)

The incomplete Mellin transforms i.e.
\[ \mathcal{I}_n(a) = \int_a^\infty x^n \text{Ai}(x) \text{Ai}'(x) \, dx, \]
and
\[ i_n(a) = \int_a^\infty x^n \text{Ai}(x)^2 \, dx, \quad i_n'(a) = \int_a^\infty x^n \text{Ai}'(x)^2 \, dx, \]
are analogues of the Mellin transforms \( I_n(a), I_n'(a) \) which have appeared above. These transforms can also be written in closed forms. Recurrence equations for these integrals have been given by Vallee et al [4] as
\[ 2(2n-1) \mathcal{I}_n(a) - n(n-1)(n-2) \mathcal{I}_{n-3}(a) = - (n-1)a^n \text{Ai}(a)^2 - na^{n-1} \text{Ai}'(a)^2 + n(n-1)a^{n-2} \text{Ai}(a) \text{Ai}'(a), \]
with
\[ \mathcal{I}_0(a) = -\frac{1}{2} \text{Ai}(a)^2, \]
\[ \mathcal{I}_1(a) = -\frac{1}{2} \text{Ai}'(a)^2, \]
\[ \mathcal{I}_2(a) = -\frac{a^2}{6} \text{Ai}(a)^2 - \frac{a}{3} \text{Ai}'(a)^2 + \frac{1}{3} \text{Ai}(a) \text{Ai}'(a). \]
Solution of the difference equation (26) yields (cf. Appendix V)

\[
\frac{12^{k+1} \Gamma(k+5/6)}{(3k)!} I_{3k}(a) = Ai(a)^2 B_{k,0}^{(0)} + Ai'(a)^2 B_{k,0}^{(1)} + Ai(a) Ai'(a) B_{k,0}^{(2)},
\]

\[
\frac{12^{k+1} \Gamma(k+7/6)}{(3k+1)!} I_{3k+1}(a) = Ai(a)^2 B_{k,1}^{(0)} + Ai'(a)^2 B_{k,1}^{(1)} + Ai(a) Ai'(a) B_{k,1}^{(2)},
\]

\[
\frac{12^{k+1} \Gamma(k+3/2)}{(3k+2)!} I_{3k+2}(a) = Ai(a)^2 B_{k,2}^{(0)} + Ai'(a)^2 B_{k,2}^{(1)} + Ai(a) Ai'(a) B_{k,2}^{(2)},
\]

where the \( B \) polynomials are given by

\[
B_{k,0}^{(0)} = -\sum_{l=0}^{k} \frac{(3l-1) \Gamma(l-1/6) (12a^3)^l}{(3l)!},
\]

\[
B_{k,0}^{(1)} = -\frac{1}{a} \sum_{l=1}^{k} \frac{\Gamma(l-1/6) (12a^3)^l}{(3l-1)!},
\]

\[
B_{k,0}^{(2)} = \frac{1}{a^2} \sum_{l=1}^{k} \frac{\Gamma(l-1/6) (12a^3)^l}{(3l-2)!},
\]

\[
B_{k,1}^{(0)} = -a \sum_{l=1}^{k} \frac{(3l) \Gamma(l+1/6) (12a^3)^l}{(3l+1)!},
\]

\[
B_{k,1}^{(1)} = -\sum_{l=0}^{k} \frac{\Gamma(l+1/6) (12a^3)^l}{(3l)!},
\]

\[
B_{k,1}^{(2)} = \frac{1}{a} \sum_{l=1}^{k} \frac{\Gamma(l+1/6) (12a^3)^l}{(3l-1)!},
\]

\[
B_{k,2}^{(0)} = -a^2 \sum_{l=0}^{k} \frac{(3l+1) \Gamma(l+1/2) (12a^3)^l}{(3l+2)!},
\]

\[
B_{k,2}^{(1)} = -a \sum_{l=0}^{k} \frac{\Gamma(l+1/2) (12a^3)^l}{(3l+1)!},
\]

\[
B_{k,2}^{(2)} = \sum_{l=0}^{k} \frac{\Gamma(l+1/2) (12a^3)^l}{(3l)!},
\]

The remaining integrals \( i_n(a) \) and \( i'_n(a) \) are then given in terms of the \( I_n(a) \) integrals by means of the Vallee relations [4] as

\[
(2n + 1) i_n(a) = -n(n-1) I_{n-2}(a) - a^{n+1} Ai(a)^2 + a^n Ai'(a)^2 - na^{n-1} Ai(a) Ai'(a),
\]

\[
(2n + 3) i'_n(a) = -n(n+2) I_{n-1}(a) + a^{n+2} Ai(a)^2 - a^{n+1} Ai'(a)^2 - (n+2) a^n Ai(a) Ai'(a).
\]

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Integration by parts of $i_n(a)$ and $i'_n(a)$ gives the simple forms ($n \neq -1$)
\[
i_n(a) = -\frac{1}{(n+1)}[2I_{n+1}(a) + a^{n+1}Ai(a)^2],
\]
\[
i'_n(a) = -\frac{1}{(n+1)}[2I_{n+2}(a) + a^{n+1}Ai'(a)^2].
\]
Elimination of $I$ from these two equations then gives
\[
i'_n(a) = \frac{1}{(n+1)}[(n+2)i_{n+1}(a) + a^{n+2}Ai(a)^2 - a^{n+2}Ai'(a)^2].
\]

### 2.6 The integrals $i_{-n}(a)$, $i'_{-n}(a)$, and $I_{-n}(a)$ ($n > 1$)

The integrals $i_{-n}(a)$, $i'_{-n}(a)$, and $I_{-n}(a)$ are similarly interrelated and integration by parts of the first two of these integrals gives
\[
i_{-n}(a) = \frac{1}{(n-1)}[2I_{-n+1}(a) + a^{-n+1}Ai(a)^2],
\]
\[
i'_{-n}(a) = \frac{1}{(n-1)}[2I_{-n+2}(a) + a^{-n+1}Ai'(a)^2].
\]
The recurrence relation for $I_{-n}(a)$ is given by
\[
I_{-n}(a) = \begin{cases} 
(n(n+1)(n+2)I_{-n-3}(a) \\
-(n+1)Ai(a)^2/a^n - nAi'(a)^2/a^{n+1} \\
-n(n+1)Ai(a)Ai'(a)/a^{n+2}/(4n+2).
\end{cases}
\]
The solutions of this difference equation are given by (30)
\[
\frac{\Gamma(3k)}{12^{k-1}\Gamma(k+1/6)}I_{-3k}(a) = \frac{12}{\Gamma(1/6)}I_{-3}(a) + Ai(a)^2 \beta_{k,0}^{(0)}
\]
\[
+Ai'(a)^2 \beta_{k,0}^{(1)} + Ai(a)Ai'(a)\beta_{k,0}^{(2)},
\]
\[
\frac{\Gamma(3k+1)}{12^{k-1}\Gamma(k+1/2)}I_{-3k-1}(a) = \frac{12}{\Gamma(1/2)}I_{-1}(a) + Ai(a)^2 \beta_{k,1}^{(0)}
\]
\[
+Ai'(a)^2 \beta_{k,1}^{(1)} + Ai(a)Ai'(a)\beta_{k,1}^{(2)},
\]
\[
\frac{\Gamma(3k+2)}{12^{k-1}\Gamma(k+5/6)}I_{-3k-2}(a) = \frac{12}{\Gamma(5/6)}I_{-2}(a) + Ai(a)^2 \beta_{k,2}^{(0)}
\]
\[
+Ai'(a)^2 \beta_{k,2}^{(1)} + Ai(a)Ai'(a)\beta_{k,2}^{(2)},
\]
respectively where the $\beta$ polynomials are
\[
\beta_{k,0}^{(0)} = \sum_{l=0}^{k-1} \frac{(3l+1)\Gamma(3l)}{(l+7/6)(12a^3)^l},
\]
\[
\beta_{k,0}^{(1)} = \frac{1}{a} \sum_{l=0}^{k-1} \frac{\Gamma(3l)}{(l+7/6)(12a^3)^l},
\]
\[
\beta_{k,0}^{(2)} = \frac{1}{a^2} \sum_{l=0}^{k-1} \frac{\Gamma(3l+2)}{(l+7/6)(12a^3)^l},
\]
\[ \beta_{k,1}^{(0)} = \frac{1}{a} \sum_{l=0}^{k-1} \frac{(3l+2)\Gamma(3l+1)}{l^{(l+3/2)(12a)^{l}}}, \]

\[ \beta_{k,1}^{(1)} = \frac{1}{a^2} \sum_{l=0}^{k-1} \frac{\Gamma(3l+1)}{l^{(l+3/2)(12a)^{l}}}, \]

\[ \beta_{k,1}^{(2)} = \frac{1}{a^3} \sum_{l=0}^{k-1} \frac{\Gamma(3l+3)}{l^{(l+3/2)(12a)^{l}}}, \]

\[ \beta_{k,2}^{(0)} = \frac{1}{a^2} \sum_{l=0}^{k-1} \frac{(3l+3)\Gamma(3l+2)}{l^{(l+11/6)(12a)^{l}}}, \]

\[ \beta_{k,2}^{(1)} = \frac{1}{a^3} \sum_{l=0}^{k-1} \frac{\Gamma(3l+2)}{l^{(l+11/6)(12a)^{l}}}, \]

\[ \beta_{k,2}^{(2)} = \frac{1}{a^4} \sum_{l=0}^{k-1} \frac{\Gamma(3l+4)}{l^{(l+11/6)(12a)^{l}}}. \]

Written in these terms the expression for \( J_n(a) \) is

\[
J_n(a) = \sum_{k=0}^{\infty} \left\{ \frac{\varepsilon(k+n)}{k+n)!} i_k(a) + \frac{\varepsilon'(k+n)}{(k+n)!} i_k'(a) + \frac{\varepsilon(k+n)}{(k+n)!} I_k(a) \right\} + \sum_{k=1}^{n} \left\{ \frac{\varepsilon(n-k)}{(n-k)!} i_{-k}(a) + \frac{\varepsilon'(n-k)}{(n-k)!} i_{-k}'(a) + \frac{\varepsilon(n-k)}{(n-k)!} I_{-k}(a) \right\}. \tag{31}
\]

A general expression for \( J_n(a) \) in its most reduced form for all \( n \) is quite complicated and is given in Appendix VI.

Integrals in (31) which contain negative powers of \( x \) such as

\[
i_{-1}(a) = \int_{a}^{\infty} \frac{Ai(x)^2}{x} dx,
\]

\[
i_{-1}'(a) = \int_{a}^{\infty} \frac{Ai'(x)^2}{x} dx,
\]

\[I_{-1}(a) = \int_{a}^{\infty} \frac{Ai(x)Ai'(x)}{x} dx,
\]

are irreducible and require special treatment (integrals with larger negative powers of \( x \) being expressible in terms of those integrals). We have

\[
i_{-1}(a) = \frac{-1}{4} a^2 [\ln(a) + \frac{1}{4} a^3 3F_2(1,1,7/6 ; 4a^3/9)]
\]

\[
-2 a Ai(0) Ai'(0) \frac{3}{2} F_3(1,3,1/2 ; 4a^3/9)
\]

\[
-\frac{1}{2} a^2 Ai'(0)^2 \frac{2}{4} F_3(2,3,5/6 ; 4a^3/9) ,
\]

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\[ i_{-2}(a) = \frac{1}{a} Ai(a)^2 + 2 I_{-1}(a), \]
\[ i_{-3}(a) = Ai(a)^2 \left( \frac{1}{2a} - a \right) + Ai'(a)^2 + \frac{1}{a} Ai(x)Ai'(a) + i_{-1}'(a), \]
\[ i_{-2}'(a) = -Ai(a)^2 + \frac{1}{a} Ai'(a)^2, \]
\[ i_{-3}'(a) = \frac{1}{2a} Ai'(a)^2 + I_{-1}(a), \]
\[ I_{-2}(a) = -a Ai(a)^2 + Ai'(a)^2 + \frac{1}{a} Ai(a)Ai'(a) + i_{-1}(a), \]
\[ I_{-3}(a) = -\frac{1}{2} Ai(a)^2 + \frac{1}{2a} Ai'(a)^2 + \frac{1}{2a} Ai(x)Ai'(a) + \frac{1}{2} i_{-1}(a). \]

We note in passing that in the case of large \( a \), the former integrals have the asymptotic forms

\[ i_{-1}(a) = \int_{a}^{\infty} \frac{Ai(x)^2}{x} dx \sim \frac{e^{-\frac{4}{3}a^{3/2}}}{8\pi a^2}, \]
\[ i_{-1}'(a) = \int_{a}^{\infty} \frac{Ai'(x)^2}{x} dx \sim \frac{e^{-\frac{4}{3}a^{3/2}}}{8\pi a^2}, \]
\[ I_{-1}(a) = \int_{a}^{\infty} \frac{Ai(x)Ai'(x)}{x} dx \sim -\frac{e^{-\frac{4}{3}a^{3/2}}}{8\pi a^{3/2}}. \]

where the asymptotic expressions for the Airy functions have been used above.

Integrations based upon the expressions given in Appendix VI, for \( J_1(|a'|) \), \( J_2(|a'|) \), and \( J_3(|a'|) \) yield values of 0.04826441, 0.03654795, and 0.02879281 respectively. Values accurate to eleven decimal places when powers of \( x \) up to \( x^8 \) are included.

2.7 Conclusions

Although the integrals \( J^{(1)} \), and \( J^{(2)} \) have been given in terms of analytic functions, it is unfortunate that these expressions are extremely complicated in form.
and thus of limited usefulness. This is disappointing since the Thomas-Fermi formulation for atomic theory is in essence an analytic i.e. non-numeric approach such as Hartree-Fock methods used in the study of atomic species and analytic evaluation of its integrals would be in keeping with the spirit of the method.
Appendix I

Using the Weirstrass representation for the quantity \(Ai'(z)/Ai'(0)\) we have

\[
\frac{d \ln(Ai'(z)/Ai'(0))}{dz} = \frac{zAi(z)}{Ai'(z)} = \sum_{n=1}^{\infty} \frac{1}{(z+|a_n|)} - \frac{1}{|a_n|}.
\]

Then for \(k > 1\) we can obtain from the equation above the Airy zeta function \(Z_k\) which is defined as

\[
Z_k = \sum_{n=1}^{\infty} \frac{1}{|a_n'|^k},
\]

by means of the expression

\[
Z_k = (-1)^{k-1} \frac{d^{k-1}}{(k-1)!} \left[ dz^{k-1} \left\{ \frac{zAi(z)}{Ai'(z)} \right\} \right]_{z=0}.
\]

The incomplete Airy zeta function \(Z_k(N)\) is defined as the finite sum

\[
Z_k(N) = \sum_{n=1}^{N} \frac{1}{|a_n'|^k}.
\]

The first few of the \(Z_k\) sums are given in Table 5 (\(\eta = Ai(0)/Ai'(0)\)).

Table 5: The Airy zeta function \(Z_k\)

| \(k\) | \(Z_k\)       |
|------|--------------|
| 2    | \(\eta\)     |
| 3    | 1            |
| 4    | \(\frac{1}{2} \eta^2\) |
| 5    | \(-\frac{2}{3} \eta\) |
| 6    | \(\frac{1}{4} - \frac{1}{3} \eta^3\) |
| 7    | \(\frac{7}{15} \eta^2\) |
| 8    | \(-\frac{11}{36} \eta + \frac{1}{8} \eta^4\) |
Appendix II

It is possible to obtain the leading terms in an asymptotic expression for the integral $I_1(a)$ as follows. Rewriting the integral as

$$I_1(a) = \frac{1}{a} \int_0^\infty \frac{Ai(z)}{1 + z/a} dz,$$

expansion of the denominator for large $a$ yields

$$I_1(a) \sim \frac{1}{a} \sum_{k=0}^\infty \left[ -\frac{1}{a} \right]^k \int_0^\infty z^k Ai(z) dz,$$

$$I_1(a) \sim \frac{1}{3a} \sum_{k=0}^\infty \left[ -\frac{1}{3^{1/3}a} \right]^k \frac{k!}{\Gamma(k/3 + 1)},$$

$$I_1(a) \sim \frac{1}{3a} - \frac{1}{3^{1/3} \Gamma(1/3) a^2} + \cdots$$

Repeated differentiation of the expression for $I_1(a)$ then produces large $a$ expressions for $I_k(a)$.

Appendix III

We note that the integrals $I_n(a)$ and $I_n'(a)$ have for $n > 0$ the forms

$$I_n(a) = c_n(a) Ai(a) + d_n(a) Ai'(a) + e_n I_0(a),$$

$$I_n'(a) = c_n'(a) Ai(a) + d_n'(a) Ai'(a) + e_n' I_0(a).$$

Using the recurrence relation (15) we find that $c_n(a), d_n(a)$, are polynomials which satisfy the relations

$$c_n(a) = (n-1)(n-2)c_{n-3}(a) + (n-1)a^{n-2},$$

$$d_n(a) = (n-1)(n-2)d_{n-3}(a) - a^{n-1},$$

and the constants $e_n$ are given by

$$e_n = (n-1)(n-2)e_{n-3}.$$

Initial values of these quantities are given in the first three rows of Table 6 along with their first few values.

| $n$ | $c_n(a)$  | $d_n(a)$   | $e_n$  |
|-----|------------|------------|--------|
| 1   | 0          | -1         | 0      |
| 2   | 1          | -a         | 0      |
| 3   | 2a         | -a^2       | 2      |
| 4   | 3a^2       | -(6 + a^3) | 0      |
| 5   | 12 + 4a^3  | -(12a + a^4) | 0  |
| 6   | 40 + 5a^4  | -(20a^2 + a^5) | 40     |

Table 6: The polynomials $c, d$ and $e$
The solution for $e_n$ follows immediately and we get

$$e_n = [1 + 2\cos(2\pi n/3)] \frac{(n - 1)!}{3^{n/3-2}\Gamma(n/3)},$$

which has non-zero value only for the case $e_{3k}(a)$ i.e.

$$e_{3k} = \frac{(3k)!}{3^k k!}, \quad e_{3k+1} = 0, \quad e_{3k+2} = 0.$$

In addition we have

\[
\begin{align*}
c_{3k}(a) &= \left(\frac{a^2}{3}\right) g^k \Gamma(k + 1/3) \Gamma(k + 2/3) \sum_{l=0}^{k-1} \frac{(a^3/9)^l}{l! \Gamma(l + 2/3) \Gamma(l + 4/3)}, \\
c_{3k+1}(a) &= \left(\frac{a^2}{3}\right) g^k k! \Gamma(k + 2/3) \sum_{l=0}^{k-1} \frac{(a^3/9)^l}{l! \Gamma(l + 5/3)}, \\
c_{3k+2}(a) &= 3 \cdot g^k k! \Gamma(k + 4/3) \sum_{l=0}^{k} \frac{(a^3/9)^l}{l! \Gamma(l + 1/3)}, \\
d_{3k}(a) &= -\left(\frac{a^2}{9}\right) g^k \Gamma(k + 1/3) \Gamma(k + 2/3) \sum_{l=0}^{k-1} \frac{(a^3/9)^l}{l! \Gamma(l + 4/3) \Gamma(l + 5/3)}, \\
d_{3k+1}(a) &= -g^k k! \Gamma(k + 2/3) \sum_{l=0}^{k} \frac{(a^3/9)^l}{l! \Gamma(l + 2/3)}, \\
d_{3k+2}(a) &= -a g^k k! \Gamma(k + 4/3) \sum_{l=0}^{k} \frac{(a^3/9)^l}{l! \Gamma(l + 4/3)}. 
\end{align*}
\]

The integrals $I_n'(a)$ in terms of these polynomials are

$$I_n'(a) = -[a^n + n c_{n-1}(a)] Ai(a) - n d_{n-1}(a) Ai'(a) - n e_{n-1} I_0(a).$$

### 1 The Mellin transforms for the Airy products

Mellin transforms of the functions $Ai(x)^2$, $Ai'(x)^2$, $Ai(x) Ai'(x)$ may be obtained from well known recurrence relations. For example, the moment $\hat{i}_n$ has the recurrence relation

$$\hat{i}_j(0) = \int_0^\infty x^j Ai(x) Ai'(x) dx = \frac{j (j+1)(j+2)}{2(2j-1)} \int_0^\infty x^{j-3} Ai(x) Ai'(x) dx,$$

or viewed as a difference equation i.e.

$$\hat{i}_j = \frac{j (j-1)(j-2)}{2(2j-1)} \hat{i}_{j-3}.$$
with initial values
\[ \hat{\gamma}_0(0) = -\frac{1}{2} A_i(0)^2, \]
\[ \hat{\gamma}_1(0) = -\frac{1}{2} A_i'(0)^2, \]
\[ \hat{\gamma}_2(0) = \frac{1}{3} A_i(0) A_i'(0), \]
the \( \hat{\gamma}_j \) integrals have values given by
\[ \hat{\gamma}_{3j}(0) = \frac{(3j)! \Gamma(5/6)}{2(12)^j \Gamma(j + 5/6)} A_i(0)^2, \]
\[ \hat{\gamma}_{3j+1}(0) = -\frac{(3j + 1)! \pi}{6(12)^j \Gamma(5/6) \Gamma(j + 7/6)} A_i'(0)^2, \]
\[ \hat{\gamma}_{3j+2}(0) = \frac{(3j + 2)! \sqrt{\pi}}{(12)^{j+1} \Gamma(j + 3/2)} A_i(0) A_i'(0). \]

The moments of \( A_i(x)^2 \) and \( A_i'(x)^2 \) are then seen to be multiples of the \( \hat{\gamma}_j(0) \) integrals. We have
\[ \int_0^\infty x^j A_i(x)^2 \, dx = -\frac{j(j - 1)}{2j + 1} \hat{\gamma}_{j-2}(0), \]
\[ \int_0^\infty x^j A_i'(x)^2 \, dx = -\frac{j(j + 2)}{2j + 3} \hat{\gamma}_{j-1}(0). \]

More explicitly
\[ \int_0^\infty x^{3j+2} A_i(x)^2 \, dx = \frac{(3j + 2)! \Gamma(5/6)}{(12)^j \Gamma(j + 11/6)} A_i(0)^2, \]
\[ \int_0^\infty x^{3j+3} A_i(x)^2 \, dx = \frac{\pi (3j + 3)!}{3(12)^j \Gamma(5/6) \Gamma(j + 13/6)} A_i'(0)^2, \]
\[ \int_0^\infty x^{3j+4} A_i(x)^2 \, dx = \frac{2 \sqrt{\pi} (3j + 4)!}{(12)^{j+2} \Gamma(j + 5/2)} A_i(0) A_i'(0), \]
and
\[ \int_0^\infty x^{3j+1} A_i'(x)^2 \, dx = \frac{(3j + 3)(3j + 1)! \Gamma(11/6)}{10(12)^j \Gamma(j + 11/6)} A_i(0)^2, \]
\[ \int_0^\infty x^{3j+2} A_i'(x)^2 \, dx = \frac{5 \pi (3j + 4)(3j + 2)!}{18(12)^j \Gamma(11/6) \Gamma(j + 13/6)} A_i'(0)^2, \]
\[ \int_0^\infty x^{3j+3} A_i'(x)^2 \, dx = \frac{-2 \sqrt{\pi} (3j + 5)(3j + 3)!}{(12)^{j+2} \Gamma(j + 5/2)} A_i(0) A_i'(0). \]
The integrals above can be written in more compact form using relations obtained by Reid [10] i.e.

\[
\int_0^\infty x^{\alpha-1} \text{Ai}(x)^2 \, dx = \frac{2 \Gamma(\alpha)}{\sqrt{\pi} 12^{(\alpha/3 + 5/6)} \Gamma(\alpha/3 + 5/6)},
\]

\[
\int_0^\infty x^{\alpha-1} \text{Ai}'(x)^2 \, dx = \frac{2(\alpha + 1) \Gamma(\alpha)}{\sqrt{\pi} 12^{(\alpha/3 + 7/6)} \Gamma(\alpha/3 + 7/6)},
\]

\[
\int_0^\infty x^{\alpha-1} \text{Ai}(x) \text{Ai}'(x) \, dx = - \frac{2(2\alpha + 3) \Gamma(\alpha)}{\sqrt{\pi} 12^{(\alpha/3 + 3/2)} \Gamma(\alpha/3 + 3/2)}.
\]
Appendix IV

A general expression for $I_n(a)$

$$I_n(a) = -\frac{Ai(a)}{a^n} \sum_{i=0}^{\infty} \frac{a^i \lambda(i)}{i!} + \sum_{i=0}^{\infty} b_i I_i(a) + \sum_{i=1}^{n+1} b_{-i} I_{-i}(a).$$

can be written in terms of only the $I_{\pm i}(a)$ integrals as follows. The first sum above can be written in closed form as

$$\sum_{i=0}^{\infty} \frac{a^i \lambda(i)}{i!} = \lambda(-a, a) = \pi A_i(0) \Im(-a),$$

and the coefficients $b_i$ are given by

$$b_i = \frac{1}{(n+1+i)!} \left\{ (n+1+i) \xi(n+i) - (1+i)\lambda(n+1+i) \right\}.$$  

The last two sums in $I_n(a)$ can be written as

$$\sum_{i=0}^{\infty} b_i I_i(a) = \sum_{i=0}^{\infty} b_{3i} I_{3i}(a) + \sum_{i=0}^{\infty} b_{3i+1} I_{3i+1}(a)$$

$$+ \sum_{i=0}^{\infty} b_{3i+2} I_{3i+2}(a),$$

$$\sum_{i=1}^{n+1} b_{-i} I_{-i}(a) = \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} b_{-3i} I_{-3i}(a) + \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} b_{-3i-1} I_{-3i-1}(a)$$

$$+ \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} b_{-3i-2} I_{-3i-2}(a),$$

respectively, where $\lfloor z \rfloor$ is the floor function. Gathering together the terms common to $I_0(a)$, $Ai(a)$ and $Ai'(a)$ we have

$$\sum_{i=0}^{\infty} b_i I_i(a) = I_0(a) \sum_{k=0}^{\infty} \Omega_{k,0} (n|a) + Ai(a) \sum_{\mu=0}^{\infty} \sum_{k=0}^{\infty} \Omega_{k,\mu} (n|a) \sigma_{k,\mu}(a)$$

$$-Ai'(a) \sum_{\mu=0}^{\infty} \sum_{k=0}^{\infty} \Omega_{k,\mu} (n|a) \sigma_{k,\mu}'(a),$$

where

$$\Omega_{k,\mu} (n|a) = \frac{1}{\Gamma(3k+n+\mu+1)} \left[ \xi(3k+n+\mu) - \lambda(3k+n+\mu+1) \right].$$
and \((z)_n\) is the Pochhammer function. The polynomials \(\sigma_{k,\mu}\) and \(\sigma'_{k,\mu}\) appearing above are given by

\[
\sigma_{k,0}(a) = a \sum_{l=0}^{k-1} \frac{(3a^3)^l l!}{(3l+1)!},
\]
\[
\sigma_{k,1}(a) = a^2 \frac{2\pi}{3\sqrt{3}} \sum_{l=0}^{k-1} \frac{(a^{3/2})^l}{l! \Gamma(l+5/3)},
\]
\[
\sigma_{k,2}(a) = 2\pi \sqrt{3} \sum_{l=0}^{k} \frac{(a^{3/2})^l}{l! \Gamma(l+1/3)},
\]

and

\[
\sigma'_{k,0}(a) = a^2 \sum_{l=0}^{k-1} \frac{(3a^3)^l l!}{(3l+2)!},
\]
\[
\sigma'_{k,1}(a) = \frac{2\pi}{\sqrt{3}} \sum_{l=0}^{k} \frac{(a^{3/2})^l}{l! \Gamma(l+2/3)},
\]
\[
\sigma'_{k,2}(a) = a \frac{2\pi}{\sqrt{3}} \sum_{l=0}^{k} \frac{(a^{3/2})^l}{l! \Gamma(l+4/3)}.
\]

The finite sum can be rewritten in a similar way as

\[
\sum_{i=1}^{n+1} b_{-i} I_{-i}(a) = I_0(a) \left\{ \frac{2\pi}{3\sqrt{3}} \sum_{i=1}^{\left\lfloor \frac{n}{3} \right\rfloor} \omega_{i,0}(n|a) \right\} + I_{-1}(a) \Gamma(2/3) \sum_{i=0}^{\left\lfloor \frac{n-1}{3} \right\rfloor} \omega_{i,1}(n|a) + I_{-2}(a) \frac{1}{3} \Gamma(1/3) \sum_{i=0}^{\left\lfloor \frac{n-2}{3} \right\rfloor} \omega_{i,2}(n|a) + Ai(a) \sum_{\mu=0}^{2} \sum_{i=1}^{\left\lfloor \frac{n+1-\mu}{3} \right\rfloor} \omega_{i,\mu}(n|a) \tilde{\sigma}_{i,\mu}(a) + Ai'(a) \sum_{\mu=0}^{2} \sum_{i=1}^{\left\lfloor \frac{n+1-\mu}{3} \right\rfloor} \omega_{i,\mu}(n|a) \tilde{\sigma}'_{i,\mu}(a),
\]

where

\[
\omega_{i,\mu}(n|a) = \frac{1}{3^{n-1} (n-\mu-3)! \Gamma(i+\frac{1}{3}) \Gamma(i+\frac{2}{3})} \left\{ \frac{(n-\mu-3)^i}{(3i+1)!} + \frac{(n+1-\mu-3)^i}{(n+i-\mu-3)!} \right\}.
\]

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(note the cases $\omega_{0,2} = \ell^{(2)}_{(2/3)}$, $\omega_{1,2} = \ell^{(0)}_{(2/3)}$). The polynomials $\hat{\sigma}_{k,\mu}, \hat{\sigma}'_{k,\mu}$ are

\[
\hat{\sigma}_{k,0}(a) = \frac{2\pi}{a^2\sqrt{3}} \sum_{l=0}^{k-1} \frac{(3l)!}{l! (3a^3)^l}, \\
\hat{\sigma}_{k,1}(a) = \frac{3}{a^3} \sum_{l=0}^{k-1} \frac{l! \Gamma(l+5/3)}{(a^3/9)^l}, \\
\hat{\sigma}_{k,2}(a) = \frac{3}{a^4} \sum_{l=0}^{k-1} \frac{(l+1)! \Gamma(l+4/3)}{(a^3/9)^l},
\]

and
\[
\hat{\sigma}'_{k,0}(a) = \frac{2\pi}{a\sqrt{3}} \sum_{l=0}^{k-1} \frac{(3l)!}{l! (3a^3)^l}, \\
\hat{\sigma}'_{k,1}(a) = \frac{1}{a^2} \sum_{l=0}^{k-1} \frac{l! \Gamma(l+2/3)}{(a^3/9)^l}, \\
\hat{\sigma}'_{k,2}(a) = \frac{1}{a^3} \sum_{l=0}^{k-1} \frac{l! \Gamma(l+4/3)}{(a^3/9)^l}.
\]

Finally we get for $I_n(a)$
\[
I_n(a) = -\frac{\pi \text{Ai}(0) \text{Ai}(a)}{a^n} \text{Ai}(-a) + \\
I_0(a)\sum_{\mu=0}^{\infty} \Omega_{k,\mu}(n|a) + \frac{2\pi}{\sqrt{3}} \sum_{k=1}^{\left\lfloor \frac{n+1}{\mu} \right\rfloor} \omega_{i,0}(n|a)} \\
+ I_{-1}(a) \Gamma(2/3) \sum_{k=1}^{\left\lfloor \frac{n+1}{\mu} \right\rfloor} \omega_{i,1}(n|a) + I_{-2}(a) \Gamma(4/3) \sum_{k=1}^{\left\lfloor \frac{n+1}{\mu} \right\rfloor} \omega_{i,2}(n|a) \\
+ \text{Ai}(a) \sum_{\mu=0}^{\infty} \Omega_{k,\mu}(n|a) \sigma_{k,\mu}(a) + \sum_{k=1}^{\left\lfloor \frac{n+1}{\mu} \right\rfloor} \omega_{k,\mu}(n|a) \hat{\sigma}_{k,\mu}(a) \\
- \text{Ai}'(a) \sum_{\mu=0}^{\infty} \Omega_{k,\mu}(n|a) \sigma_{k,\mu}(a) + \sum_{k=1}^{\left\lfloor \frac{n+1}{\mu} \right\rfloor} \omega_{k,\mu}(n|a) \hat{\sigma}'_{k,\mu}(a)
\]

Appendix V

Another method of computing the $I_n, i_n, i'_n$ integrals follows if we note that $I_n(a)$ has the form
\[
I_n(a) = p_n(a) \text{Ai}(a)^2 + q_n(a) \text{Ai}'(a)^2 + r_n(a) \text{Ai}(a) \text{Ai}'(a),
\]
where \( p_n(a), q_n(a), \) and \( r_n(a) \) are polynomials. As a result we can find recurrence relations for these polynomials using equation (26). We get

\[
2(2n - 1)p_n(a) - n(n - 1)(n - 2)p_{n-3}(a) = -(n - 1)a^n,
\]

\[
2(2n - 1)q_n(a) - n(n - 1)(n - 2)q_{n-3}(a) = -na^{n-1},
\]

\[
2(2n - 1)r_n(a) - n(n - 1)(n - 2)r_{n-3}(a) = n(n - 1)a^{n-2}.
\]

It also follows that \( i_n(a) \) and \( i_n'(a) \) have the same form as \( \mathcal{I}_n(a) \) and we have in summary

\[
\mathcal{I}_n(a) = p_n(a)Ai(a)^2 + q_n(a)Ai'(a)^2 + r_n(a)Ai(a)Ai'(a),
\]

\[
(n + 1)i_n(a) = \left[2p_{n+1}(a) + a^{n+1}\right]Ai(a)^2
\]

\[
-2q_{n+1}(a)Ai'(a)^2 - 2r_{n+1}(a)Ai(a)Ai'(a),
\]

\[
(n + 1)i_n'(a) = -2p_{n+2}(a)Ai(a)^2 - [2q_{n+2}(a) + a^{n+1}]Ai'(a)^2
\]

\[
-2r_{n+2}(a)Ai(a)Ai'(a).
\]

The first few of these polynomials are given in the Table 7.

| \( n \) | \( p_n(a) \) | \( q_n(a) \) | \( r_n(a) \) |
|---|---|---|---|
| 0 | \(-1/2\) | 0 | 0 |
| 1 | 0 | \(-1/2\) | 0 |
| 2 | \(-a^2/6\) | \(-a/3\) | 1/3 |
| 3 | \(-3 + 2a^4)/10\) | \(-3a^2/10\) | 3a/5 |
| 4 | \(-3a^4/14\) | \(-2(3 + a^4)/7\) | 6a^2/7 |
| 5 | \(-5a^5 + 2a^2)/9\) | \(-20a + 5a^3)/18\) | 10(1 + a^3)/9 |

It should be noted that the \( p, q, r \) polynomials are perhaps more fundamental than the integrals \( i_n(a), i_n'(a), \) and \( \mathcal{I}_n(a) \) themselves since they form their common basis.

Appendix VI

Expressing the \( i_k(a) \) and \( i_k'(a) \) integrals in terms of \( \mathcal{I}_k(a) \) we have \( (n \geq 1) \) where \( \Xi^{(n)}(-1, 0), \Lambda^{(n)}(-1, 0), \) and \( g^{(n)}(-1, 0) \) are hereafter denoted \( \Xi^{(n)}, \Lambda^{(n)} \) and \( g^{(n)} \) we have

\[
J_n(a) = \sum_{k=0}^{\infty} \left\{ \frac{\theta^{(k+n)}\mathcal{I}_k(a) - \Xi^{(k+n)}(k+1)\mathcal{I}_{k+1}(a) + a^{k+1}Ai(a)^2]}{\Lambda^{(k+n)}(k+1)\mathcal{I}_{k+1}(a) + a^{k+1}Ai'(a)^2} \right\}/(k + n)!
\]

\[
+ \left\{ \frac{\theta^{(n-1)}\mathcal{I}_{-1}(a) + \Xi^{(n-1)}(n-1)i_{-1}(a) + \Lambda^{(n-1)}i'_{-1}(a)}{\theta^{(n-1)}\mathcal{I}_{-1}(a) + \Xi^{(n-1)}(n-1)i_{-1}(a) + \Lambda^{(n-1)}i'_{-1}(a)} \right\}/(n - 1)!
\]

\[
+ \sum_{k=2}^{n} \frac{\theta^{(n-k)}\mathcal{I}_{-k}(a)}{\theta^{(n-k)}\mathcal{I}_{-k}(a) + \Xi^{(n-k)}(n-k)i_{-k}(a) + \Lambda^{(n-k)}i'_{-k}(a)}/(n - k)!
\]

\[
+ \sum_{k=2}^{n} \frac{\theta^{(n-k)}\mathcal{I}_{-k}(a)}{\theta^{(n-k)}\mathcal{I}_{-k}(a) + \Xi^{(n-k)}(n-k)i_{-k}(a) + \Lambda^{(n-k)}i'_{-k}(a)}/(n - k)!
\]

\[
+ \frac{\Xi^{(n-k)}(n-k)i_{-k}(a) + \Lambda^{(n-k)}i'_{-k}(a)}{\theta^{(n-k)}\mathcal{I}_{-k}(a) + \Xi^{(n-k)}(n-k)i_{-k}(a) + \Lambda^{(n-k)}i'_{-k}(a)}/(n - k)!
\]

\[
+ \frac{\Xi^{(n-k)}(n-k)i_{-k}(a) + \Lambda^{(n-k)}i'_{-k}(a)}{\theta^{(n-k)}\mathcal{I}_{-k}(a) + \Xi^{(n-k)}(n-k)i_{-k}(a) + \Lambda^{(n-k)}i'_{-k}(a)}/(n - k)!
\]

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Rearranging this expression and gathering common $\mathcal{I}_k(a)$ terms we have

$$J_n(a) = -Ai(a)^2 \sum_{k=-n}^{\infty} \frac{\Xi^{(k+n)} a^{k+1}}{(k+1)(n+k)} \Delta_k, -1 - Ai'(a)^2 \sum_{k=-n}^{\infty} \frac{\Xi^{(k+n)} a^{k+1}}{(k+1)(n+k)} \Delta_k, -1$$

$$+ \left\{ \frac{g(n)}{n!} \right\} \mathcal{I}_0(a) + \left\{ \frac{g(n+1)}{(n+1)!} - 2 \Xi(n)/n! \right\} \mathcal{I}_1(a)$$

$$+ \sum_{k=2}^{\infty} \left[ \frac{g(k+n)}{(k+n)!} - 2 \Xi(k+n-1)/(k+n-1)! - 2 \frac{\Xi^{(k+n-2)}}{(k-1)(n+k-2)!} \right] \mathcal{I}_k(a)$$

$$+ 2 \left\{ \frac{\Xi(n-2)}{(n-2)!} \right\} \mathcal{I}_0(a) + \left\{ \Xi(n-1)/(n-1)! \right\} i_{-1}(a)$$

$$+ \left\{ \frac{\Xi(n-1)}{(n-1)!} \right\} i'_{-1}(a)$$

$$+ \sum_{k=1}^{n-2} \left[ \frac{g(n-k)}{(n-k)!} + 2 \frac{\Xi(n-k-1)}{(n-k-1)!} + 2 \frac{\Xi^{(n-k-2)}}{(k-1)(n-k-2)!} \right] \mathcal{I}_{-k}(a)$$

$$+ \left\{ \frac{\Xi(0)}{(n-1)!} \right\} \mathcal{I}_{-n}(a) + \left\{ \frac{\Xi(1)}{(n-1)!} + 2 \Delta_{n,1} \frac{\Xi(0)}{(n-1)!} \right\} \mathcal{I}_{-n+1}(a),$$

with $\Delta_{k, j} \equiv 1 - \delta_{k, j}$ where $\delta_{k, j}$ is the Kronecker delta function. Since the integrals $\mathcal{I}_k(a)$ and $\mathcal{I}_{-k}(a)$ are related to the Airy functions by (27) and (30) we have for $J_n(a)$ in the cases ($n \geq 4$)

$$J_n(a) = Ai(a)^2 \left\{ -\Xi_n(a) + \sum_{\mu=0}^{\infty} \sum_{k=0}^{\infty} \Psi_{k, \mu}(n|a) B_{k, \mu}^{(0)} \right\}$$

$$+ Ai'(a)^2 \left\{ -\Lambda_n(a) + \sum_{\mu=0}^{\infty} \sum_{k=0}^{\infty} \Psi_{k, \mu}(n|a) B_{k, \mu}^{(1)} \right\}$$

$$+ \sum_{\mu=0}^{\infty} \sum_{k=0}^{\infty} \Psi_{k, \mu}(n|a) \beta_{k, \mu}^{(1)}$$

$$+ Ai(a) Ai'(a) \left\{ -\sum_{\mu=0}^{\infty} \sum_{k=0}^{\infty} \Psi_{k, \mu}(n|a) B_{k, \mu}^{(2)} \right\}$$

$$+ \sum_{\mu=0}^{\infty} \sum_{k=0}^{\infty} \Psi_{k, \mu}(n|a) \beta_{k, \mu}^{(2)}$$

$$+ \mathcal{I}_1 \left\{ \frac{\Xi(n+1)}{(n+1)!} - 2 \frac{\Xi(n)}{n!} \right\} + \mathcal{I}_0 \left\{ \frac{\Xi(n)}{n!} + 2 \frac{\Xi(n-2)}{(n-2)!} \right\}$$

$$+ \frac{12}{\Gamma(3/2)} \mathcal{I}_{-1} \left\{ \sum_{k=0}^{\frac{n-2}{2}} \Psi_{k, 1}(n|a) \right\}$$

$$+ \frac{12}{\Gamma(3/2)} \mathcal{I}_{-2} \left\{ \sum_{k=0}^{\frac{n-2}{2}} \Psi_{k, 2}(n|a) \right\} + \frac{12}{\Gamma(1/2)} \mathcal{I}_{-3} \left\{ \sum_{k=0}^{\frac{n-2}{2}} \Psi_{k, 3}(n|a) \right\}$$

$$+ \frac{\Xi(n-1)}{(n-1)!} i_{-1} + \frac{\Xi(n-1)}{(n-1)!} i'_{-1} + \frac{\Xi(n)}{n!} \mathcal{I}_{-n} + \left\{ \frac{\Xi(n)}{n!} + 2 \Delta_{n, 1} \frac{\Xi(n)}{n!} \right\} \mathcal{I}_{-n+1}$$

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where

\[ \Xi_n(a) = \sum_{k=-n}^{\infty} \frac{\Xi(n+k+1)}{(k+1)!} \Delta k, -1, \]

\[ \Lambda_n(a) = \sum_{k=-n}^{\infty} \frac{\Lambda(n+k+1)}{(k+1)!} \Delta k, -1, \]

and the terms \( \Psi_{k,\mu} \) and \( \psi_{k,\mu} \) are given by

\[ \Psi_{k,\mu}(n|a) = \frac{\Gamma(3k+\mu+1)}{12^{k+1} \Gamma(k+\mu/3+5/6)} \left\{ \frac{\phi(n+3k+\mu)}{(n+3k+\mu)!} - 2 \frac{\Xi(n+3k+\mu-1)}{(3k+\mu)!} (n+3k+\mu-1)! - 2 \frac{\Lambda(n+3k+\mu-2)}{(3k+\mu-1)!(n+3k+\mu-2)!} \right\}, \]

\[ \psi_{k,\mu}(n|a) = \frac{\Gamma(3k+\mu+1)}{12^{k-1} \Gamma(k+\mu/3+1/6)} \left\{ \frac{\phi(n-3k-\mu)}{(n-3k-\mu)!} + 2 \frac{\Xi(n-3k-\mu-1)}{(3k+\mu)!} (n-3k-\mu-1)! + 2 \frac{\Lambda(n-3k-\mu-2)}{(3k+\mu+1)!(n-3k-\mu-2)!} \right\}. \]

References

[1] M. Abramowitz and I. A. Stegun, editors. *Handbook of Mathematical Functions With Formulas, Graphs, And Mathematical Tables*, New York: Dover, p. 446 (1992).

[2] B. J. Laurenzi, *An Analytic Solution to the Thomas-Fermi Equations*, J. Math. Phys. 31, p. 2536 (1990).

[3] B. J. Laurenzi, A preliminary account of the present work is found in *Logarithmic Integrals of Airy Functions*, arXiv:1211.0705v1.pdf.

[4] O. Vallee and M. Soares, *Airy Functions and Applications to Physics*, second edition, Imperial College Press, p. 19, (2010).

[5] Ref. 1, p. 478. Table 10.13. Ref. 8, http://dlmf.nist.gov/11.10

[6] F. W. J. Olver, *Airy and Related Functions*, [DLMF]NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/9.9E8

[7] B. J. Laurenzi, *Polynomials Associated with the Higher Derivatives of the Airy Functions \( Ai(z) \) and \( Ai'(z) \)*, arXiv:1110.2025

[8] NIST Digital Library of Mathematical Functions, *Airy and Related Functions*, http://dlmf.nist.gov/9.12

[9] D. E. Aspnes, *Electric-fields Effects on Optical Absorption Near Thresholds in Solids*, Phys. Rev. 147, (1966) pp. 554-566.

[10] W. H. Reid, *Integral Representation for Products of Airy Functions*, Z. Angew. Math. Phys. 46, (1995) pp. 159-170.