On a solution to display non-filled-in quaternionic Julia sets

Alessandro Rosa

February 1, 2008

Keywords: Quaternionic Julia sets, iterates, dynamical systems, graphics, escape time method, cut-off method, illumination.

Abstract

During early 1980s, the so-called ‘escape time’ method, developed to display the Julia sets for complex dynamical systems, was exported to quaternions in order to draw analogous pictures in this wider numerical field. Despite of the fine results in the complex plane, where all topological configurations of Julia sets have been successfully displayed, the ‘escape time’ method fails to render properly the non-filled-in variety of quaternionic Julia sets. So their digital visualisation remained an open problem for several years. Both the solution for extending this old method to non-filled-in quaternionic Julia sets and its implementation into a program are explained here.

1 Introduction

The worldwide hype for holomorphic dynamics during the turn from 1970s to 1980s claimed the development of graphical methods to display these systems on the computer screen, with emphasis to complex Julia sets \( J \) \[18\]. On the same wave, thanks to the joint work by Mandelbrot and Norton, the first pictures of quaternionic Julia sets came also out by applying the same method as used in \( \mathbb{C} \) and known under the two definitions of ‘escape time’ or ‘level sets’. Few works, either in graphics terms, were published on this subject since early 1980s: namely, the book \[13\] and the articles by Norton \[16\] \[17\], Holbrook \[9\] \[10\], by Hart, Kauffman and Sandin \[7\] \[8\]. For analogous studies in higher dimensional dynamics, see \[5\]. Quaternions are multi-dimensional values consisting of 4 vectors in the form

\[ h = r + mi + nj + pk, \quad r, m, n, p \in \mathbb{R}, \quad i^2 = j^2 = k^2 = -1 \]  \hspace{1cm} (1.1)

and generate a non-abelian group. The quaternion space is named \( \mathbb{H} \) for convention, in honor of William R. Hamilton (1805–1865), inventor of such numerical values for solving some issues arising from rotation in 3-D Euclidean spaces.
The ‘escape time’ method enjoys one major limitation: as one tries to display, for example, the non-filled-in Julia sets of basins $B$ of attraction to a finite limit fixed point $\delta$ and extending to $\infty$ (often retrieved by the iterates of rational maps). This article explains how to modify the ‘standard’ method so to fill this leak and extend it to the ‘non-filled-in’ case. After resuming the technical details of the standard method in section 2, we discuss the novelties in section 3; finally, the implementation into the program QHD is explained in section 4.

2 The background

2.1 Julia sets topology

Up to complex numbers and given a rational map

$$f = \frac{P}{Q}$$

where $P, Q$ are co-prime polynomials, i.e. with no common roots, it was observed that Julia sets $J$ (in any number field) may:

1. be (pointwisely) totally disconnected;
2. have $n - 1$ dimensions;
3. have $n$ dimensions.

**QUESTION 2.1.1.** Could this classification hold for any $n$-dimensional number field?

In $\mathbb{R}$, where $n = 1$, points 1) and 2) are the same, so that real Julia sets $R$ are pointwisely totally disconnected (in fact, for point 2), we have: $n - 1 = 0$ or the whole real line (for point 3), $R$ is $n = 1$ dimensional).

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We liked to use this naming convention in order to differ the ‘escape time’ method from our new version.
In \( \mathbb{C} \), where \( n = 2 \), \( J \) could be either a dust of points (0-dimensional), also a line (\( n - 1 = 1 \) dimensional) or an area (\( n = 2 \) dimensional).

As far as the author knows, it is not known whether point 1) holds for quaternionic \( J \) too, or they are just disconnected sets with (hyper-)volumes. We will implicitly assume here the first two cases only, discarding \textit{a priori} the third one, which leads to the trivial case where \( J \) spans over the whole 4-dimensional space.

### 2.2 The open problem

For complex numbers, any Julia set \( J \) can be successfully displayed with the already available methods; new and customised methods are anyway required to explore local vector fields\(^2\) about the fixed points \( \delta \). But quaternionic \( J \) did not share the same luck. The method employed in \( \mathbb{H} \) was radically borrowed from \( \mathbb{C} \). However, as the related literature of those times indicates – together with the book \[3\], the road-map was to display the iterations of polynomials and to develop the details enhancement: in fact, those graphical examples always concerned of disconnected or of the filled-in varieties \( J \) only, mostly coming from the family of general quadratic polynomials

\[
ph^2 + q, \quad h, p, q \in \mathbb{H}.
\]  

Figure 2.2: Non-filled-in example. A case arising for the map \[2.2\]: this view is taken on the Riemann sphere, in the neighborhood of the point at infinity.

Julia sets are mostly a mathematical question. This flaw in graphical terms was just the side effect from the interests of those pioneering researchers, who mainly wanted to start from the easier properties of polynomials\(^3\). Quaternions are non-commutative and the study of rational maps in \( \mathbb{H} \) may get to high level of complication, in mathematical terms: so the investigation on these maps was not pursued. So it came that the graphics development was not considered

\(^{2}\)Like for the so-called 'hedgehogs', invariant sets about irrationally indifferent fixed points not enjoying the diophantine condition or rational approximation; the digital display was also an open problem here. A related graphical method arranging a good solution is going to be explained in \[2\].

\(^{3}\)In the course of these early mathematical explorations, they proved that quaternionic \( J \) display in 3 dimensions as rotations of the related \( J \) in \( \mathbb{C} \) around the X-axis.
along that direction: there was no need to quest on the central question of this article throughout the older works, because all topological cases of Julia sets for polynomials were already fulfilled by the standard method. (The complete related literature was listed in the bibliography.) Thus we can reasonably state that this problem was still open. A filled-in Julia set \( J \) includes all seed points whose forward iterated orbits do not tend\(^4\) to \( \infty \). Topologically speaking, the boundary of a filled-in \( J \) is homeomorphic to a closed and bounded curve (even with fixed points). But exceptions are thrown when \( J \) also extends to \( \infty \): for example, in the classic example of the cubic rational map, resulting from Newton’s method applied to the complex function \( f(z) \): \( z^3 - 1 \),

\[ f(z) = \frac{2z^3 + 1}{3z^2}. \]  

This variety of Julia sets \( J \) will be said ‘non-filled-in’: this case is out of the range of possibilities for the ‘escape time’ method when applied to quaternions.

Being \( J \) displayed as a three dimensional model, the latest improvements, dating back to early 1990s (see references list), mainly concerned of the application of efficient illumination techniques or of improving the boundary refinement. Successful results in the former direction were achieved by means of Phong illumination model \([6]\); whereas the reader may consult \([7, 8]\) about the second issue. Today software performs higher quality pictures of \( J \) than ever before, but these programs\(^5\) still rely on the standard method: thus only the filled-in \( J \) can be properly displayed. After deeply scanning the Web and consultations with experts in this topic, programs displaying the non-filled-in family have been not obviously coded yet. Author’s solution wants to overcome this limitation; the related code has been already implemented and tested to work into one PC program ‘QHD’, freely downloadable from author’s site\(^6\) and applied to render some among the further pictures.

### 2.3 The standard method

From \([1, 1]\), one understands that Julia sets \( J \) exist in four dimensions. So one is not able to figure them in mind or to display somewhere. One is just able to embed them into two \((\mathbb{R}^2)\) or three \((\mathbb{R}^3)\) real Euclidean spaces onto the computer screen. In this sense, different two-staged methods are available but in general they all root to a first stage where values/points are computed and to a second one, when the screen points are colored. In \( \mathbb{C} \) colors palettes may be arbitrarily set (even to render purely aesthetic effects), but in \( \mathbb{H} \) rigor shall be followed because \( J \) shall give their solid model look.

In practice, this translates into this three steps approach:

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\(^4\)This is not the same as stating that these orbits should converge to a constant limit fixed point \( \delta \), due to the complicate dynamics – where one cannot properly speak of convergence – in the neighborhood of indifferent fixed points, both for the rational and irrational case.

\(^5\)In fig. ??, the reader can watch the earliest results from the application of our method by one of such high quality programs.

\(^6\)http://malilla.supereva.it
1. to scan a given region, provided a discretization of it in order to have a $n-$dimensional grid of values $p$;

2. to iterate each $p$ for a number of times $k$;

3. to test the resulting iterated value $p_k$ in order to color them differently or choose which ones have to be plot.

The ‘standard’ method was applied by Mandelbrot and Norton and by Holbrook in their first visualizations and displays the 3-dimensional models of $J$. We will follow the same line here, focusing on Julia sets $J$ possibly showing up as totally disconnected or 3-D surfaces. The care of illumination is fundamental for any 3-D model, in order to prevent the flat looking. Phong approach guarantees high performance, for example, with regard to details, enhanced as the shining effect is rendered very accurately and very close to real light. In general the standard method, applied to $J$, is son of what usually denoted ‘escape method’ in the early literature devoted to the digital visualization of $J$. We pointed out that malfunctions come up for non-filled-in $J$ from iterates of rational maps, for example from the application of the Newton-Raphson’s method. In this case, one might find out those basins of attraction to a finite limit point and extending to $\infty$. See p. 4 including a picture from the cubic quaternionic function $f(h) : h^3 - 1$, which turns into:

$$T_f = h - \frac{f(h)}{f'(h)} = \frac{2h^3 + 1}{3h^2}.$$  \hspace{1cm} (2.4)

The Julia set curve $J$ includes infinitely many fixed points $\delta$ and also extends to infinity. As the standard method applies to $\{2.4\}$, one sees fig. $\{5.6\}$. One
also notices how roughly it works in this situation: the shape of $J$ is not so clear and just a part of it is displayed, but imprecisely. The grey region is the evidence of the failure of the standard method, which cannot adequately check the sequences of iterated values here.

### 2.4 Tuning the parameters

Before entering the heart chapter, one should be assured that a modification really urges. So we tried to tune the two parameters involved in this kind of experiments: the bailout disc radius and the number of iterations. Finally, we reported the resulting pictures in table 2.1. This experiment is relevant because, for sake of completeness, one shall either show the reasons why something is required and why it is not: these pictures want to be the empirical evidence of what will be further discussed in more details at section 3.2. The table 2.1 at p. [6] includes figures of the same Julia set $J$, the large figure at the bottom
is correctly drawn by the ‘cut-off rate’ approach and shall be compared with the small ones, rendered by the ‘escape time’: the number of iterates and the bailout circle radius (centered at the origin in the classical method) were tuned to distinct values. In some cases, one finds out a solid piece (the radius at 1.0 was even too large and gave a great approximated shape); for larger radii, the grey region indicates that the location of \( J \) cannot be distinguished from the basin locus, because the detection rule failed\(^7\). Finally, the black pictures denote that the chosen conditions were rather insufficient.

3 Opening quaternionic Julia sets to rational maps

3.1 Disclaiming the Analysis

After implementing the currently available methods to display quaternionic Julia sets \( J \) for polynomials, the author wanted to achieve the same even for iterated rational maps. In mathematical terms, the study of the algebraic properties related to them may be harder than in \( \mathbb{R} \) or in \( \mathbb{C} \), owing to the non-commutativity property enjoyed by quaternion numbers. The next sections have been deliberately compiled to solve a practical problem exclusively, not regarding the mathematics behind the nature of the functions involved. In fact, the question on quaternionic Julia sets connectivity still deserves a theoretical systematisation, because the arising topological configurations are not quite the same as in \( \mathbb{C} \), for example disconnected Julia sets in \( \mathbb{H} \) do show connected sub-regions, whereas they are totally disconnected (Cantor dust) in \( \mathbb{C} \). This also invalidates the same definition of Mandelbrot set as it holds in \( \mathbb{C} \). But this goal is out of scope for this article and it goes beyond author’s knowledge.

Figure 3.5: Julia set for (2.4) in \( \mathbb{C} \).

Figure 3.6: Malfunctions. The quaternionic Julia set generated by (2.4) and processed by the standard method.

\(^7\)See section 3.3 for technical details.
3.2 Starting considerations

The standard method scans the given area (in \( \mathbb{C} \)) or volume (in \( \mathbb{R}^3 \), the reduced vector space from \( \mathbb{H} \)), associating a numerical value per each element of this region; the value is iterated for a number of times under a given function. If the resulting value is still inside a sphere of given radius, then the orbit is said to be ‘trapped’ and then it belongs to a basin of attraction bounded by a Julia set \( J \). But this only works well when \( J \) is a closed curve which the test disc is able to catch completely up; on the contrary, the disc does not fit any more for the whole \( J \) extending to \( \infty \). Historically speaking, the standard method inherits the obvious limitation from the approach devoted to display Julia sets \( J \) on a computer and whose location and topology inside \( \mathbb{C} \) were \textit{a priori} known, either if filled-in or disconnected. In fact, the iterated maps, whose \( J \) were going to be shown onto a computer screen for the first times ever during early 1980s, were quadratic polynomials and namely in the form

\[
z^2 + c, \quad z, c \in \mathbb{C};
\]  

(3.5)

here all cases of \( J \), their topology and the attracting fixed points are known to approximately lie inside a bailout disc with radius 2 at most. The standard method was invented with these ‘bounded’ dynamics in mind and it fits the range of cases offered by (3.5) only; thus, in this sense, it is a customised method which unfits the cases not covered by its possibilities.

The standard method can be reset by changing the radius of the trapping disc in \( \mathbb{C} \) but limitations come up as one deals with quaternions: such method strictly depends on the same functionality of the bounded ball (as the concept of trapping disc turns into for the embedding of \( \mathbb{H} \) into \( \mathbb{R}^3 \)): i.e. when orbits are tested to stay bounded or to escape to \( \infty \); in fact the same method implicitly assumes that there is only \textit{one} finite and attracting fixed point and that its location, together with the related basin, \textit{is a priori} known to lie inside the test ball itself. So, for more general purposes – i.e. \textit{regardless of fixed points locations and quantity} – the standard method does not work well.

This will be the starting point upon which the enhancement will be based.

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**Figure 3.7:** First step. Quaternionic Julia set boundary detection.

**Figure 3.8:** Second step. The recurrent process for approximating the Julia set point location.
3.3 Crash course reasons

Therefore why does the standard approach fail here?

First we say why it does not in $\mathbb{C}$: actually because all points of a given region are iterated and tested, regardless of their location and even visibility is not a relevant condition; in fact one draws both the basin and the Julia set in $\mathbb{C}$. But in 3-D, one needs to differ them because the basins shall be not drawn or one would just watch a cubic block instead! The points of quaternionic $J$ do need to be scanned and sorted if they are visible to be plot or not. This is the main direction which methods for quaternionic Julia sets must follow straight.

The exclusion of the basins during the plot is a delicate feature.

‘Filled-in’ or disconnected Julia sets configurations keep an element making the difference: the basin $B_\infty$ to infinity. Its role is clearer in the ‘filled-in’ case (fig. 1.1/B): one plays a boolean situation where points of $B_\infty$ exit the test sphere, others do not. The rule is that $J$ is met when, given two close seeds to $J$, the related orbits are checked by the ‘escape time’ and one finds that an orbit runs to infinity, the other to the finite attracting point (fig. 3.7). So the spatial interval, wherein a point of $J$ should be approximately located, shrinks over and over again up to lower bound, while the fates of the two orbits keep different (fig. 3.8).

Figure 3.9: The above figures resume three topological configurations for complex Julia sets: (A) totally disconnected and with the only basin to infinity, (B) connected and splitting $\mathbb{C}$ into two basins only, a basin to infinity and one to the finite fixed point. Then (C) a Julia set and all basins (two or more) of attraction to finite points.

The standard method crashes because, for such maps as (2.4), infinity is no longer attracting and $J$ splits $\mathbb{R}$ into basins of attraction to finite $\delta$: as $J$ is even crossed, it is not detected because neighboring orbits never escape the test sphere, and the above rule blows away (see fig. 3.6).
3.4 The ‘cut-off rate’ method

The modification mainly relies upon the same region scanning process like in the standard method, but it adds the previous considerations, together with the inspiration from the application of normal families to holomorphic dynamics.

Let \( f \) be an holomorphic function, then

\[
f^0(z) = z, f^1 = f(z), \ldots f^n(z) = f^{n-1}(f(z))
\]

are said to be the iterates of rank 0, 1, \ldots, \( n \). From iteration theory, one knows that, for a same attracting fixed point \( \gamma_k \), the total basin \( B_{\gamma_k} \) of attraction is the union set of all the points whose orbits converge to \( \gamma_k \):

\[
\lim_{n \to \infty} f^n(z) = \gamma, \quad \forall z \in B_{\gamma_k}.
\]

Let \( D_1, D_2, \ldots, D_n \), be Jordan curves bounding simply connected domains \( D_1, D_2, \ldots, D_n \) around \( \gamma_k \) respectively (see fig. 3.10), so that \( D_{n-1} \supset D_n \); thus, in metric terms, the distance \( d_n \), of any pair of points \( p_{n-1} \in D_{n-1}, p_n \in D_n \) (points of an orbit), shrinks to 0 as the orbit itself gets closer to \( \gamma_k \):

\[
d_n = |p_n - p_{n-1}| < \epsilon, \quad \epsilon > 0, \lim \epsilon = 0.
\]   (3.6)

So, generally speaking, given a seed point \( w \in B_{\gamma_k} \) and close to \( J \), the distance \( d_n \to 0 \) as the iterative index \( n \to \infty \). Conversely – very important to understand how the modification works –, one may also state that if \( d^n \) is set to a constant \( s \), then one can re-elaborate \( |f_n(w) - f_{n+1}(w)| > s \) as the seed \( w \) gets closer and closer to \( J \). This offers the chance to part all possible orbits generated throughout one basin \( B_{\gamma_k} \) into two groups:

1. the orbits of the seed points \( p \) which converge, to the fixed point \( \gamma_k \), under a given distance \( d_n \) and after a smaller number of iterations;
2. other orbits of seeds $p$ which need a larger number of iterates to converge under $d_n$.

The points being closer to $\mathcal{J}$ belong to the second group. Hence the goal of revisiting the standard method is the ‘isolation’ of the second group of points and then plot them exclusively, with regard to other basins distributions where even the test ball of the standard method, with fixed location, does not fit any longer to return clear and good graphical results. If the test ball condition is:

$$|p_n - p_{n-1}| < r. \quad (3.7)$$

This isolation can be simply performed by assuming the reverse inequality of (3.6), that is:

$$|p_n - p_{n-1}| \geq r. \quad (3.8)$$

The operator ‘\(\geq\)’ achieves a reverse task: rather than trapping, external orbits to the ball are assumed or in other terms, the successive iterated domains $D_n$ exclusively, whose distances $d_n \geq r$. That is, the domains of points which are farther, in terms of iterates, from $\gamma_k$: conversely speaking, the points which are closer and closer to Julia sets. Moreover one drops off the test condition of the escape method, about the upper bound of the iterative index, which prevent infinite loops; we preferred to adopt the first member of inequality (3.8), because it showed to work more finely: the major enhancement is to offer, by means of just one condition of convergence, the work on $n$ test balls, as many as the number of attracting fixed point $\gamma_k$ of the iterated map. Therefore the reader has no longer to deal with the issue of knowing a priori the number and the location of the attracting fixed points $\delta$ or how long one orbit takes to converge in some sufficiently close neighborhood of the fixed point. One sees such test balls, which do not depend on the location anymore, rising up from (3.8) and they work at the same time without their centers to be preset $ad$ hoc in the algorithm code. This condition grants any arbitrary input map.

The black regions in fig. (??) indicate that those points have not been deliberately plot, as result from the isolation task achieved by the ‘cut-off rate’ method.

## 4 Related software

In this section we will give a short overview on some related software which is currently implementing the ‘cut-off rate’ method.

### 4.1 QHD : Quaternionic Holomorphic Dynamics

First we show one author’s program, coded to make the early experiments on the ‘cut-off rate’ method: QHD is a branch of another application, coded years before and named ‘Inwards to Chaos’, which is devoted to iterations in real, complex and quaternion fields. In order to focus on the visualisation techniques, QHD came up and became the natural environment to display the graphical results related to the ‘cut-off rate’ method.
Figure 3.11: **Cut-off rate method.** The quaternionic Julia set for
\[
\frac{2h^3 - 2h^2 - 3}{3h^2 + 4h + 1}
\]
The set was sectioned to notice either the rotational symmetry with the Julia set in \( \mathbb{C} \) and the existence of interweaving basins which would make the classic method crash.

4.2 The visual interface

This application opens with a user-friendly interface to choose the set to draw (‘Julia’ or ‘Mandelbrot’), the drawing algorithm (‘Standard’ or ‘Extended’), a default set of formulas and the illumination method (‘Simple Lambertian’, ‘Lambertian’ and ‘Phong’).

When properties are displayed, the same window expands and shows additional parameters. First the details level yielding as finer pictures as it is incremented (as well as computation times). Then the number of iterates, the degree of predefined maps, the region coordinates and the value of parameter \( q \), like in formula 2.2).

Input features also list the command to load/save the parameters and the metrics related to Julia set configuration, such as the bailout value playing as the radius of the disc entrapping the orbits in the standard method or as the value \( r \) in the inequality of the extended approach.

The main interface offers a very little preview shot by a quick plot, but one can display magnifications into a larger window; a set of four arrows allows finally to change the input region coordinates.

Additional features include a player to stop the drawing process manually, a window to rotate the given object along the three X-Y-Z axes, together with one panel to manage all parameters related to the illumination models (location of light points, intensity, ambient diffusion). Then, there is the possibility of setting any triplet of the four quaternionic components into the X-Y-Z coordinates system in order to watch the possible shapes assumed by quaternionic Julia sets into a reduced 3-D vector space (see examples in table 4.12 at p. 13). The output allows to save the current picture into one graphics format file or to be copied into the computer clipboard for exchanging it with other applications.
4.3 Looking around

My goal was to test if that method really worked in other applications outside mine and verify if my results were achieved in particular situations or due to possible mistakes in the code. In the recent period (February 2006), after testing the method on QHD, I wanted to let the ‘cut-off’ method circulate among groups interested in computer graphics devoted to quaternionic Julia sets, which are usually frequented by ‘mathematized’ programmers, that is, non-mathematicians, strictly meaning, but skillful people liking to deal with Mathematics. One is Terry W. Gintz who welcomed the new method and implemented it into a program, retrieving high-quality renderings of such fractals; for example see the pictures in the table ?? at p. ??.

5 Conclusions

The program ‘QHD’ can be freely downloaded from the web-site address at http://www.malilla.supereva.it; Opinions and suggestions are welcomed to enhance it. The C++ code can be asked via author’s e-mail.

Alessandro Rosa
Brindisi, Italy
zandor_raz@yahoo.it

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