Inflation is the generic feature of phantom field—not the big-rip

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A class of solutions for phantom field corresponding to a generalized k-essence lagrangian has been presented, employing a simple method which provides the scope to explore many such. All the solutions having dynamical state parameter are found to touch the magic line $w = -1$ asymptotically. The solutions with constant equation of state can represent phantom, quintessence or an ordinary scalar field cosmologies depending on the choice of a couple of parameters of the theory. For $w \approx -1$, quintessence and phantom models are indistinguishable through the Hubble parameter. Finally, inflation rather than big-rip has been found to be the generic feature of phantom cosmology.

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I. INTRODUCTION

Current astronomical observations of luminosity distance-redshift relation, by standard candles such as supernova SN Ia, Galactic cluster measurements, particularly by Sloan digital Sky survey, cosmic microwave background radiation with WMAP results and corresponding data analysis [1] suggest that about 73% of the total matter density of the Universe is in the form of a slowly varying dark energy with negative pressure, which is smoothly distributed all over the Universe with an equation of state $-1.45 < w < -0.74$, at 95% confidence level. In this connection Caldwell [2] suggested a cosmological model with super-negative equation of state and dubbed it as phantom energy, which leads to the late time acceleration of the Universe. Such a field corresponds to the action of an ordinary scalar field, with a reverse sign in the kinetic energy term. However, the solution presented in [2], found to suffer from some serious set backs [3].

Other than instability, the main two problems associated with phantom type fields in the cosmological context are, the big-rip, and the problem of crossing the phantom divide line $w = -1$. Big-rip is a future (finite time) singularity that replaces the big-bang singularity of Friedmann model, at which all the cosmological parameters blow up, including curvature invariants. However, soon York [4] has shown that the final state of phantom cosmology may be inflation rather than big rip. Aref’eva et al [5] constructed a string inspired model whose solutions don’t suffer from big-rip and are stable under small fluctuations of the initial conditions and special deviations of the form of the potential. Recently, Guo et al [6] have also demonstrated that the de-Sitter like solution is the late time attractor of phantom cosmology.

The second problem, on the other hand is much more serious. It has recently been argued [7] that the best fit for the currently available data from all the measurements mentioned at the beginning, is provided by a model of a rapidly evolving dark energy from a dust-like $w = 0$ at high redshift value $z = 1$ to a phantom-like $w < -1$, at the present. So, for a viable cosmological model, the dark-energy equation of state must go over dynamically to $w = 0$, starting from below, $w < -1$, or vice versa. However, a dynamical transition from $w < -1$ to $w > -1$, where, the equation of state corresponding to the cosmological constant, $w = -1$, commonly known as phantom divide line, seems to be impossible. This is because, matter with $w < -1$ violates dominant energy condition and so such models are apparently supposed to suffer from instability even in the classical level. Though the crossing of phantom divide line is possible in some complicated multiple field theories, and non-minimally coupled scalar field theories [8], however, it has been proved [9] that such crossing is impossible in a single field theory model, even for generalized k-essence [10] non-canonical Lagrangian,

$$L = g(\phi)F(X) - V(\phi),$$

where, $X = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi$, though Andrianov et al [11] have claimed other way round. In any case, if we believe in an equation of state less than $-1$, we have to deal with phantom models seriously, since nothing other than phantom can live on the other side of the barrier. Following the above discussions, we are thus motivated to explore the cosmological behaviour of the phantom field, through mostly a new class of an exact solutions of a generalized single field phantom model. In the process we shall show that, though it is indeed impossible to cross the phantom divide line by a single field model, however, all the solutions corresponding to dynamical equation of state parameter, asymptotically touch and others can be fit with, the magic line $w = -1$. This proves inflation, rather than big-rip is the generic feature of phantom cosmology. Further, it has been shown that the solutions with constant equation of state represent all the phantom, quintessence and ordinary scalar field cosmological models depending on the choice of a couple of parameters of the theory. For $w \approx -1$, the quintessence and the phantom models are found to be indistinguishable through the asymptote of the Hubble parameter.
In the following section we write the action and the field equations of a generalized k-essence Lagrangian and develop a rather straightforward method to extract a class of solutions. In section 3, we present the set of solutions for different form of the potential.

II. ACTION AND THE FIELD EQUATIONS

We start with a single field generalized k-essence action (10) (see also Vikman in [9]) which can be expressed in its simplest form as,

$$S = \int d^4x \sqrt{-g} \left[ \frac{\kappa^2}{2} R + \frac{g(\phi)}{2} \phi_{\mu} \phi^{\mu} - V(\phi) \right],$$

where, we dub $g(\phi)$ as the k-essence parameter, which has got a Brans-Dicke origin, $g = \frac{\kappa^2}{2}$ too, $\omega(\phi)$ being the Brans-Dicke parameter, though the coupling here is minimal. Note that the sign of the kinetic energy term depends on the k-essence parameter for a real scalar field $\phi$. For a positive potential $V(\phi)$, the action (1) represents that for phantom field if $g(\phi)$ is positive, while it turns to ordinary scalar field action for negative $g(\phi)$. Since phantom field has originated from string field theory, so the dimensionless coupling constant $\kappa^2$ is related to the string and the reduced Planck masses. We have chosen $\kappa^2 = 1$, for all practical purpose corresponding to the standard cosmological units $8\pi G = c = 1$. For spatially flat, homogeneous and isotropic Robertson-Walker spacetime $k = 0$,

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2],$$

the field equations in terms of the Hubble parameter $H = \dot{a}/a$ are

$$2\dot{H} + 3H^2 = \frac{g}{2} \ddot{\phi}^2 + V(\phi) = -p,$$  \hspace{1cm} (2)

$$\dddot{\phi} + 3H \ddot{\phi} + \frac{g'}{2} \dot{\phi}^2 - \frac{V'}{g} = 0,$$  \hspace{1cm} (3)

$$3H^2 = -\frac{1}{2} g\dot{\phi}^2 + V(\phi) = \rho,$$  \hspace{1cm} (4)

where, dash (') stands for derivative with respect to the scalar field $\phi$, while $p$ and $\rho$ are the pressure and the energy density of the phantom field respectively. Now, differentiating equation (4) we get,

$$g\dddot{\phi} = -6H \dot{H} - \frac{1}{2} g' \dot{\phi}^2 + V' \dot{\phi}.$$  \hspace{1cm} (5)

So, eliminating $\ddot{\phi}$ between the above equation and equations (3) we obtain,

$$\ddot{\phi} = \frac{2H'}{g},$$  \hspace{1cm} (6)

Note that in the above equation $H$ has been expressed as a function of $\phi$ instead of time, which is referred to as the superpotential $H(t) = W(\phi(t))$ for a single field. However, we do not use this notation, instead we use, $H(t) = H(\phi(t))$. It is possible to write yet another equation by eliminating $\dot{\phi}$ between equations (4) and (5), viz.,

$$2H'^2 + 3gH^2 = gV(\phi).$$

So, (5) and (6) are the two equations that we need to solve for the Hubble parameter $H$ and correspondingly the scale factor $a$ and also for the phantom field $\phi$. This is achieved only if $g(\phi)$ and the phantom field potential $V(\phi)$ are known. Not being sceptic, one can choose $\phi$ as a function of $\phi$ so that equation (5) gets integrated immediately and the Hubble parameter is expressed in terms of the field $\phi$, rather than choosing the so called super-potential as a polynomial in $\phi$, restricting it to not more than third degree [9]. In this method one can explore variety of forms for the so called super-potential, including inverse power law and the exponential ones. Then for different choice of potential $V(\phi)$, it is possible to extract a class of exact solutions of equation (6).

In the following section we have explored a class of exact solutions choosing in each of the three subsections different functional form of $\phi$ and correspondingly different form of the potential $V(\phi)$.

III. SOME EXACT SOLUTIONS AND THE BEHAVIOUR OF THE PHANTOM FIELD

As already mentioned we shall have to solve equations (5) and (6) for $H, \phi, g(\phi)$ and $V(\phi)$, which is not possible unless two are known. In the following three subsections we have chosen three different forms of $g(\phi)$ in such a way that equation (5) gets integrated immediately. Then for different choice of the potential $V(\phi)$ mostly a new class of exact solutions has been excavated and corresponding behaviour of the phantom field has been studied.

A. Let, $g(\phi) = n - a$ constant.

Under the above choice equation (5) can be integrated to yield,

$$H = \frac{n}{2} \phi,$$  \hspace{1cm} (7)

which is devoid of a constant of integration. Thus, this situation corresponds to the linear functional dependence of the so called superpotential [9] on the field $\phi$. Hence, equation (6) now takes the form,

$$\ddot{\phi} + 3n \dot{\phi}^2 - \frac{2V(\phi)}{n} = 0,$$  \hspace{1cm} (8)

as a result, the k-essence parameter $g(\phi)$, the kinetic energy ($K$), the energy-density ($\rho$), the pressure ($p$) of the
phantom field and hence the equation of state, under consideration take the following form,

\[ g(\phi) = \frac{n}{\phi} = \frac{2n^2}{4V(\phi) - 3n^2\phi^2}; \]

\[ K = -\frac{1}{2}g\phi^2 = -\frac{1}{4}(4V(\phi) - 3n^2\phi^2); \rho = \frac{3n^2}{4}\phi^2; \quad (9) \]

\[ p = \frac{3n^2}{4}\phi^2 - 2V(\phi); \quad w = \frac{p}{\rho} = 1 - \frac{8V(\phi)}{3n^2\phi^2}. \]

Now, under some suitable choice of the potential \( V(\phi) \) one can solve for the scale factor and the field variable in view of the above two equations (7) and (8), and can study the behaviour of all other parameters of the theory from equation (9), which we carry out in the following.

Case 1, Let, \( V = V_0, \) a constant.

For the above choice of a constant potential the action (1) now takes the following form,

\[ S = \int d^4x \sqrt{-g}[R + \frac{n^2}{4V_0 - 3n^2\phi^2}\phi, \phi'' - V_0]. \quad (10) \]

Equation (8) can now be integrated to yield,

\[ \phi = \frac{2}{n}V_0^{3/4}\tanh[\sqrt{3V_0}(t - t_0)]. \quad (11) \]

Hence, equation (7) can be solved for the scale factor as,

\[ a = a_0\sqrt{\cosh[\sqrt{3V_0}(t - t_0)]. \quad (12) \]

The deceleration parameter is

\[ q = -\frac{\ddot{a}/a}{\dot{a}^2} = -3\coth^2[\sqrt{3V_0}(t - t_0)] + 2. \quad (13) \]

The so called k-essence parameter, \( g, \) is found to have the following form,

\[ g = \frac{n^2}{2V_0}\cosh^2[\sqrt{3V_0}(t - t_0)]. \quad (14) \]

Finally, the kinetic energy, the pressure, the energy-density and the equation of state of the phantom field under consideration are evaluated as,

\[ K = -V_0\sech^2[\sqrt{3V_0}(t - t_0)]; \]

\[ p = -V_0[2 - \tanh^2[\sqrt{3V_0}(t - t_0)]]; \]

\[ \rho = V_0\tanh^2[\sqrt{3V_0}(t - t_0)]; \quad w = 1 - 2\coth^2[\sqrt{3V_0}(t - t_0)]. \quad (15) \]

Thus the above solutions depict that the cosmic evolution started from a constant value of the scale factor which grew exponentially. The phantom field which was created as the Universe started evolving, asymptotically settles down to a constant value \( \frac{2}{n}\sqrt{\frac{V_0}{3}}, \) while the Hubble parameter is zero initially and is \( \frac{V_0}{3}\) at the end. The deceleration parameter guarantees cosmic acceleration, which falls off and ends up with a value \(-1\) asymptotically. The k-essence parameter, \( g(\phi) \) starts from a constant value \( \frac{2n^2}{4V_0} \) grows indefinitely without a flip in sign, while the kinetic energy of the phantom field starts from \(-V_0\), increases with the evolution and asymptotically vanishes, demonstrating that such model ends up with a bare cosmological constant and is not viable of crossing the phantom divide line. The equation of state which starts from a large negative value and finally settles down to \(-1\), also depicts the same feature. The energy density of the model increases from zero value and ends up at \( V_0 \). In this simple model, with it’s natural cut off, the energy density never grows large enough to tear off gravitationally bound objects. Thus, this model never encounters big-rip rather it clearly indicates inflation with \( p = -\rho, \) corresponding to a bare cosmological constant \( V_0 \) at the end \[ \frac{3}{2}. \]

Case 2, Let, \( V = V_0\phi^2. \)

Under the above assumption, \( w \) becomes nondynamical, as can be seen in view of equation (9), so there does not arise any question of the crossing. The action in this case reads,

\[ S = \int d^4x \sqrt{-g}[\frac{R}{2} + \frac{n^2}{(4V_0 - 3n^2\phi^2)\phi, \phi'' - V_0\phi^2}]. \quad (16) \]

Following the same above procedure, \( \phi \) can be evaluated as,

\[ \phi = -\frac{2n}{4V_0 - 3n^2}(t - t_0)^{-1}. \]

The action (16) guarantees that the field is phantom only if \( 4V_0 > 3n^2 \), which indicates that big-rip can clearly be avoided for \( n < 0 = -m^2. \) Hence, the complete set of solutions is found as,

\[ \phi = \frac{2m^2}{4V_0 - 3m^4}(t - t_0)^{-1}; \quad a = a_0(t - t_0)^{-\frac{m^4}{4V_0 - 3m^4}}; \]

\[ q = -1 - \frac{4V_0 - 3m^4}{m^4}; \quad g = -\frac{4V_0 - 3m^4}{2}(t - t_0)^2; \]

\[ K = -\frac{m^4}{(4V_0 - 3m^4)(t - t_0)^2}; \quad \rho = \frac{3m^8}{(4V_0 - 3m^4)^2(t - t_0)^2}; \quad (17) \]

\[ p = -m^4 \left( \frac{8V_0 - 3m^4}{(4V_0 - 3m^4)^2} \right) \frac{1}{(t - t_0)^2}; \quad w = -\frac{8V_0 - 3m^4}{3m^4}. \]
Thus the field vanishes asymptotically, while the k-essence parameter grows indefinitely starting from zero. The energy density also vanishes asymptotically, starting from an indefinitely large value, while the kinetic energy and the pressure acquire the same fate starting from a large negative value. Thus at the end there is no dark energy at all and we are left with vacuum Einstein’s equation. However, the problem encountered in this model as mentioned earlier, is that the deceleration parameter and the equation of state do not evolve with time. Nevertheless, $V_0$ can be chosen close enough to $\frac{3\pi^2}{8}$ in order to fit the present observable constraint on the equation of state parameter, which is pretty close to one (see, eg., Melchiorri et al in [1]). It is interesting to note that the same set of solutions represents a model for quintessence field [12] provided, $n > 0$ and $4V_0 < 3n^2 = 3m^4 < 8V_0$, which can be a best fit with the magic line $w = -1$ from above, if $3n^2 \approx 4V_0$. Finally, the solutions represent a model for an ordinary scalar field too, with $w > 0$ provided, $n > 0$ and $3n^2 = 3m^4 > 8V_0$. No one yet knows whether the state parameter is dynamical. If it is not, then this is the “three-in-one” situation that covers the whole picture which are of course mutually exclusive. It can be mentioned at this juncture that a form invariance transformation [14] leads to phantom cosmology, starting from scalar field theories. We have been able to show that such transformation corresponds simply to adjustment of a couple of parameters of the theory. In figure 1, we have shown how the Hubble parameter evolves in time, both in the phantom and quintessence model. It demonstrates that for an equation of state close enough to $-1$, it is impossible to identify between the two. However, since the Hubble parameter vanishes asymptotically for both the models, so either we are living at the early stage of evolution or we have to discard this model.

B. Let, $g\dot{\phi} = n\phi$.

Under the above choice, where, $n$ is a constant, equation (5) can be integrated to yield,
\[ H = \frac{n}{4} \phi^2, \] (18)
 ignoring the constant of integration. Now in view of equation (18), equation (6) takes the following form,
\[ \phi \dot{\phi} + \frac{3n}{8} \phi^4 - \frac{2}{n} V(\phi) = 0. \] (19)
 According to this, the k-essence parameter, the kinetic energy, the pressure, the energy-density of the phantom field and finally the equation of state take the following forms,
\[ p = \frac{3n^2}{16} \phi^4 - 2V(\phi); \rho = \frac{3n^2}{16} \phi^4; w = 1 - \frac{32}{3n^2} \frac{V(\phi)}{\phi^4}. \] (20)

Now, we proceed as before to generate solutions of the field equations (18) and (19) under some suitable choice of the potential $V(\phi)$, which we carry out in the following.

Case 1, Let, $V = V_0$, a constant.

For a constant potential the action (1) now takes the following form,
\[ S = \int d^4x \sqrt{-g} \left[ R + \frac{4n^2 \phi^2}{16V_0 - 3n^2 \phi^4} \phi_{\mu} \phi^{\mu} - V_0 \right]. \] (21)

As before the field variables can be evaluated in view of the equations (18) and (19) as,
\[ \phi = 2 \sqrt{V_0/3n^2} \left[ \sqrt{\text{tanh}[\sqrt{3V_0}(t - t_0)]} \right]; \]
\[ a = a_0 \sqrt{\text{cosh}[\sqrt{3V_0}(t - t_0)]}. \] (22)

All other parameters are now found in view of equation (20) as follows,
\[ q = -3 \coth^2[\sqrt{3V_0}(t - t_0)] + 2; g = \frac{2n \tanh[\sqrt{3V_0}(t - t_0)]}{\sqrt{3V_0 \text{sech}^2[\sqrt{3V_0}(t - t_0)]}}; \]
\[ K = -V_0 \text{sech}^2[\sqrt{3V_0}(t - t_0)]; \rho = V_0 \tanh^2[\sqrt{3V_0}(t - t_0)]; \]
\[ p = -V_0(2 - \tanh^2[\sqrt{3V_0}(t - t_0)]); w = 1 - 2 \coth^2[\sqrt{3V_0}(t - t_0)]. \] (23)

These solutions are almost identical to those obtained in case 1 of subsection A. The only difference is that the
phantom field asymptotically settles at a slightly different value, $2 \sqrt{V_0/3n^2}$. While, the k-essence parameter starts from zero instead of a finite value, but grows indefinitely, as before. So, it appears that, as long as the form of the potential remains the same, the form of the Hubble parameter does not make any appreciable difference in the solutions. Even for $H \propto \dot{\phi}^3$, or $H \propto \dot{\phi}^{-1}$, it has been checked that (not shown here) the solutions do not differ.

**Case 2.** Let, $V = V_0\dot{\phi}^2$.

For such a quadratic form of the potential the action (1) can be expressed as,

$$S = \int d^4 x \sqrt{-g} \left[ \frac{R}{2} + \frac{4n^2}{(16V_0 - 3n^2\dot{\phi}^2)} \phi, \mu \phi^{\mu} - V_0\dot{\phi}^2 \right].$$

(24)

In view of equations (18) and (19) the field variables in this case are found as,

$$\phi = \sqrt{\frac{2V_0}{n}} e^{\frac{4V_0}{n}(t-t_0)} \left[ 1 + \frac{3n}{8} e^{\frac{4V_0}{n}(t-t_0)} \right]^\frac{1}{n};$$

$$a = a_0 \left[ 1 + \frac{3n}{8} e^{\frac{4V_0}{n}(t-t_0)} \right]^\frac{1}{n}.$$  \hspace{1cm}  (25)

All the parameters of the theory are obtained in view of equation (20) as,

$$q = -\left[ 1 + \frac{8}{n} e^{-\frac{4V_0}{n}(t-t_0)} \right];$$

$$g = \frac{n^2}{2V_0} \left[ 1 + \frac{3n}{8} e^{\frac{4V_0}{n}(t-t_0)} \right];$$

$$K = -\frac{2V_0^2 e^{\frac{4V_0}{n}(t-t_0)}}{n\left[ 1 + \frac{3n}{8} e^{\frac{4V_0}{n}(t-t_0)} \right]^2};$$

$$\rho = \frac{3V_0^2 e^{\frac{4V_0}{n}(t-t_0)}}{4\left[ 1 + \frac{3n}{8} e^{\frac{4V_0}{n}(t-t_0)} \right]^2};$$

$$p = -\frac{2V_0^2 e^{\frac{4V_0}{n}(t-t_0)}}{n\left[ 1 + \frac{3n}{8} e^{\frac{4V_0}{n}(t-t_0)} \right]^2} \left[ 2 + \frac{3n}{8} e^{\frac{4V_0}{n}(t-t_0)} \right];$$

$$w = -\left[ 1 + \frac{16}{3n} e^{\frac{4V_0}{n}(t-t_0)} \right].$$  \hspace{1cm}  (26)

The Universe here again undergoes an exponential expansion. The field variable $\phi$ grows, starting from a constant value rather than zero and asymptotically becomes yet another constant. The deceleration parameter starts with a value $< -1$ but finally settles down to $-1$, while the k-essence parameter also starts from a constant value and ultimately becomes infinitely large. The pressure as well as the kinetic energy begin with negative values and while the former ends up with less negative value $-\frac{16V_0^2}{3n^2}$, the latter vanishes at the end. The energy density grows and finally reaches a moderate value $\frac{16V_0^2}{3n^2}$, and so, $p + p = 0$, at the end. The equation of state starts with a value much less than $-1$ and at the end settles down to $-1$, indicating inflation \cite{4}, rather than big-rip. It is noticeable that under no circumstances $n$ can be made negative, and so quintessence or the ordinary scalar field solutions are not realizable here, as was in earlier situation with quadratic potential. Thus the form of the Hubble parameter indeed has got a role towards the nature of the solutions.

**Case 3.** Let, $V = V_0\dot{\phi}^4$.

The above choice of quartic potential leads to the following form of the action (1),

$$S = \int d^4 x \sqrt{-g} \left[ \frac{R}{2} + \frac{4n^2}{(16V_0 - 3n^2\dot{\phi}^2)} \phi, \mu \phi^{\mu} - V_0\dot{\phi}^4 \right].$$

(27)

Further, equations (18) and (19) are used to determine the evolution of the phantom field and the scale factor as,

$$\phi = \sqrt{\frac{-4n}{(16V_0 - 3n^2)(t-t_0)}} = \frac{2m}{\sqrt{(16V_0 - 3m^4)(t-t_0)}};$$

$$a = a_0(t-t_0)^\left(\frac{m^4}{16V_0 - 3m^4}\right).$$  \hspace{1cm}  (28)

where, we have made $n$ negative ($n = -m^2$). Rest of the parameters are found in view of equation (20) as,

$$q = -\frac{16V_0 - 2m^4}{m^4};$$

$$K = -\frac{m^4}{(16V_0 - 3m^4)(t-t_0)^2};$$

$$g = 2m^2(t-t_0);$$

$$\rho = \frac{3m^8}{(16V_0 - 3m^4)^2(t-t_0)^2};$$

$$p = -\frac{32V_0^2 - 3m^4}{(16V_0 - 3m^4)^2(t-t_0)^2};$$

$$w = -\frac{32V_0^2 - 3m^4}{3m^4}.$$  \hspace{1cm}  (29)

So, this solution behaves almost identically to that we obtained with quadratic potential (case 2 of subsection A). The Universe undergoes constant acceleration with constant equation of state for the phantom field which can be tuned close to 1 to meet with the present observable constraint (see, eg., Melchiorri et al in \cite{11}). The phantom field and the energy-density both start from an indefinitely large value but vanish at the end, while the pressure and the kinetic energy encounter the same fate starting from a large negative value. It should be noted that one requires $(16V_0 - 3m^4) > 0$ to ensure action (27) admits phantom field. As observed in case 2 of previous subsection, here again the solutions represent those for a quintessence field \cite{13} for $n > 0$ provided, $3n^2 > 16V_0$. Note that $3n^2 \approx 16V_0$ may be the best fit with the magic line from both ends. The above solutions represent an ordinary scalar field too, provided, $3n^2 > 32V_0$. Thus this situation is identical to that obtained earlier (case 2 of subsection A). Hence the same
Case 1, Let,

\[ g \dot{\phi} = 2n e^{l \phi}. \]

Under such assumption equation (5) is integrated and we find,

\[ H = \frac{ne^{l \phi}}{l}, \quad (30) \]

which indicates that the so called superpotential [12] has exponential functional dependence on \( \phi \), in sharp contrast to Aref'eva et al [5], who restricted it to a polynomial in \( \phi \), allowing not more than third degree. Equation (6) can now be expressed as,

\[ nl^2 e^{l \phi} \phi + 3n^2 e^{2l \phi} - l^2 V(\phi) = 0 \quad (31) \]

as a result the parameters of the theory are

\[ g = \left( \frac{2n^2 l^2 e^{2l \phi}}{l^2 V(\phi) - 3n^2 e^{2l \phi}} \right); \quad K = 3 \frac{n^2}{l^2} e^{2l \phi} - V(\phi); \]

\[ p = -2V(\phi) + \frac{3n^2}{l^2} e^{2l \phi}; \quad \rho = \frac{3n^2}{l^2} e^{2l \phi}; \quad (32) \]

\[ w = 1 - \frac{2l^2}{3n^2} e^{-2l \phi} V(\phi). \]

Now one can solve the set of equations (30), (31) and (32) to study the evolution of the phantom field, the scale factor and all the parameters of the theory under some suitable choice of the field potential \( V(\phi) \), as before.

Case 1, Let, \( V = V_0 \) - a constant.

Considering the potential to be a constant the action (1) can now be cast in the following form,

\[ S = \int d^4x \sqrt{-g} [R + \frac{n^2 l^2 e^{2l \phi}}{V_0} \phi_{\mu} \phi^{\mu} - V_0], \quad (33) \]

Using equations (30) and (31), the field and the scale factor can now be evaluated as

\[ \phi = \frac{1}{l} \log \left[ \frac{3}{V_0} \tanh \left( \sqrt{3V_0(t - t_0)} \right) \right]; \]

\[ a = a_0 \cosh \left( \sqrt{3V_0(t - t_0)} \right). \]

It is quite clear from the above expression that unless both the constants \( n \) and \( l \) are chosen negative, \( \phi \) will not be well behaved. So we choose \( n = -m^2 \) and \( l = -\lambda^2 \) and rewrite the above expressions as

\[ \phi = \frac{1}{\lambda^2} \log \left[ \frac{\lambda^2}{m^2} V_0 \right] \cosh \left( \sqrt{3V_0(t - t_0)} \right); \]

\[ a = a_0 \cosh \left( \sqrt{3V_0(t - t_0)} \right). \]

Rest of the parameters of the theory can as such be evaluated from equation (32) as,

\[ q = -[1 + V_0 \coth^2 \left( \sqrt{3V_0(t - t_0)} \right)] + V_0; \]

\[ g = \frac{6m^8 \lambda^4 \tanh^2 \left( \sqrt{3V_0(t - t_0)} \right)}{\lambda^8 V_0^2 - 9m^8 \tanh^2 \left( \sqrt{3V_0(t - t_0)} \right)}; \]

\[ K = \frac{9m^8}{\lambda^8 V_0} \tanh^2 \left( \sqrt{3V_0(t - t_0)} \right) - V_0; \]

\[ p = \frac{9m^8}{\lambda^8 V_0} \tanh^2 \left( \sqrt{3V_0(t - t_0)} \right) - 2V_0; \]

\[ \rho = \frac{9m^8}{\lambda^8 V_0} \tanh^2 \left( \sqrt{3V_0(t - t_0)} \right); \]

\[ w = 1 - \frac{2\lambda^8 V_0^2}{9m^8 \tanh^2 \left( \sqrt{3V_0(t - t_0)} \right)} . \]

Above set of solutions behave properly, provided, \( V_0 \lambda^4 > 3m^4 \), as a result the field variable \( \phi \) vanishes at the end, starting from an indefinite large value, while the scale factor grows exponentially. The deceleration parameter settles down to \(-1\) starting from an indefinitely large value. The pressure starts from \(-2V_0\), gets reduced at the end while the energy-density starts from zero, increases but remains finite. The kinetic energy remains negative throughout the evolution, thus the dominant energy condition is violated all the time. The state parameter, starts from indefinitely large negative value and asymptotically comes close to, but less than \(-1\), assuming, \( 3m^4 \), though less than \( V_0 \lambda^4 \), is pretty close to it. Note that, this is the only situation where \( g \) could change sign provided \( V_0 \lambda^4 < 3m^4 \), indicating a possible crossing of phantom divide line. However, this is not possible physically, since it makes the field negative. This solution is quite different from those obtained earlier with constant potential. Hence, the Hubble parameter plays crucial role in building solutions.

Case 2, Let, \( V = V_0 e^{l \phi} \).
All other parameters of the theory (32) are:

\[
S = \int d^4x\sqrt{-g}\left[\frac{R}{2} + \frac{n^2l^2}{V_0^2 e^{-\nu(t-t_0)} - 3n^2 \phi \phi^\mu - V_0 e^{\nu\phi}}\right],
\]

and equation (30) and (31) can be solved to obtain

\[
\phi = \frac{1}{1} \log \left[ \frac{V_0 l^2}{3n^2 + e^{-\frac{\nu\phi}{2}(t-t_0)}} \right];
\]

\[
a = a_0 \sqrt{3n^2 e^{\frac{\nu\phi}{2}(t-t_0)} + 1}.
\]

All other parameters of the theory (32) are:

\[
q = -\left[1 + \frac{e^{-\frac{\nu\phi}{2}(t-t_0)}}{n^2}\right];
g = 2n^2 l^2 e^{-\frac{\nu\phi}{2}(t-t_0)};
\]

\[
p = -V_0^2 l^2 \left[ \frac{3n^2 + 2e^{-\frac{\nu\phi}{2}(t-t_0)}}{(3n^2 + e^{-\frac{\nu\phi}{2}(t-t_0)})^2} \right]; \quad (38)
\]

\[
K = \frac{-V_0^2 l^2 e^{-\frac{\nu\phi}{2}(t-t_0)}}{(3n^2 + e^{-\frac{\nu\phi}{2}(t-t_0)})^2};
\]

\[
\rho = \frac{3V_0^2 l^2 n^2}{(3n^2 + e^{-\frac{\nu\phi}{2}(t-t_0)})^2};
\]

\[
w = -\left[1 + \frac{2}{3n^2} e^{-\frac{\nu\phi}{2}(t-t_0)}\right].
\]

The above set of solutions (37) and (38), represents an ever accelerating universe which slows down to \(q = -1\), while the increasing phantom field remains finite and well behaved at both ends, provided \(V_0 > \frac{2n^2 + 1}{l^2}\). The k-essence parameter increases indefinitely starting from a finite value, the pressure becomes less negative at the end, just as much positive as the energy density contributes, and as a result the kinetic energy vanishes, while the state parameter tends to \(-1\), again indicating inflation at the end. Figure 2 demonstrates how both the state and the deceleration parameters, touch the \(-1\) line rather quickly, starting from large negative values. Figure 3 depicts the evolution of the Hubble parameter. The situation does not change if both \(n\) and \(l\) are considered negative, i.e., \(\rho = -2\kappa^2 e^{-l\phi}\) and \(V = V_0 e^{-m^2\phi}\). However, \(n\) can not be negative since it makes the field negative. Interestingly enough, if one considers, \(l < 0\), particularly \(l = -m^2\), the above set of solutions represent a collapsing Universe, whose scale factor reduces by a factor \((3n^2 + 1)^{\frac{1}{2}}\), while the field starting from a finite value provided, \(V_0 < \frac{2n^2 + 1}{l^2}\), grows indefinitely. Under such circumstances, the k-essence parameter, the energy density, the kinetic energy, the pressure all vanish at the end, while the deceleration parameter and the state parameter grow to indefinitely large negative value. So, it’s a vacuum but different. Thus the positive and negative exponential potentials behave irreversibly.

**IV. CONCLUDING REMARKS**

There exists a few exact solutions in the literature, corresponding to phantom fields, particularly in view of such generalized k-essence Lagrangian. Though most of the scientists agree that a single field Lagrangian is not viable to cross the phantom divide line [8], yet there is an exception. It has been claimed other way, by Andriyov et al [11]. The confusion can be removed only by presenting a class of exact solutions. Though we have presented a few, however it is possible to extract numerous solutions following the above method. The solutions with constant equation of state are found to represent all the phantom, quintessence and ordinary scalar field cos-
mological models, upon adjustment of a couple of parameters of the theory. Further, for $w \approx -1$, it is impossible to identify phantom with quintessence models. Crossing of the phantom divide line is indeed found to be impossible in such single field models, but all the solutions with dynamical state parameter are found to touch $-1$ line asymptotically and none of the solutions presented above suffer from instability or cosmological singularity like big rip. Thus we conclude in the spirit of [4] and [6] that phantom leads to inflation rather than big-rip.

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