In this paper scattering equivalences are used to simplify current operators in constituent quark models. The simplicity of the method is illustrated by applying it to a relativistic constituent quark model that fits the meson mass spectrum [1]. This model requires a non-trivial constituent quark current operator to fit the pion form factor data. A model with a different confining interaction, that has the identical spectrum and can reproduce the measured pion form factor using only point-like constituent quark impulse currents is constructed. Both the original and transformed models are relativistic direct-interaction models with a light-front kinematic subgroup [2][3].

1 Introduction

The prominent role played by the choice of representation of the dynamics is a striking feature of relativistic direct-interaction quantum mechanics. Models with different kinematic subgroups [2] (light-front, Euclidean, or Lorentz) that fit the experimental meson and baryon spectra require different representations of quark current [4][5][6][7][8] operators to fit the measured electromagnetic properties of the pion and nucleon.

The representation dependence of operators, like the current, is not surprising. In local quantum field theory all fields in the same Borchers class [9] have the same scattering matrix. In effective field theory, field redefinitions [10] lead to equivalent representations of dynamics on limited energy scales, and in quantum theories of a finite number of degrees of freedom the group of scattering equivalences [11][12] leads to different representations of the dynamics with the same spectrum and scattering matrix elements.

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If the inverse scattering problem has a solution in relativistic or non-relativistic quantum mechanics then it cannot be unique. A large class of equivalent interactions that lead to the same $S$-matrix can be constructed using scattering equivalences \([12]\). Different scattering equivalent models are equivalent representations of the dynamics.

While there is no fundamental preference for a given representation among the class of equivalent representations, there are computational advantages to choosing a representation where one aspect of the dynamics is simple.

We illustrate how the choice of representation may be used to simplify the structure of the quark current operator in a relativistic constituent quark model of mesons. The general construction is discussed in section two. The construction is applied to the model of \([1]\) in section three.

## 2 Construction

Consider a relativistic constituent quark model which fits the meson mass spectrum. Assume that this model is a relativistic direct interaction model with a given kinematic subgroup. Also assume that there is a corresponding constituent quark current operator $J^\mu(x)$ that can be used to compute the electromagnetic properties of the mesons. In general the current operator includes quark form factors and dynamical two-body operators.

The current matrix elements are invariant under simultaneous unitary transformations of the meson eigenstates and current operator

\[
\langle p', \psi | J^\mu(x) | p, \phi \rangle = \langle p', \psi' | J'^\mu(x) | p, \phi' \rangle
\]

where $A$ is any unitary transformation. The freedom to choose $A$ can be used to simplify the structure of the current operator or the meson wave functions.

This work utilizes a restricted class of unitary transformations that differ from the identity by a finite rank operator on the internal Hilbert space. For a suitable choice of the finite rank operator, the original and transformed mass operators differ by a short range interaction. The constituent quark masses and kinetic energy remain unchanged and the original and transformed current operators will differ by a short range operator. If the finite rank operator commutes with the kinematic subgroup, then the original and transformed interactions will be kinematically invariant.

Unitary operators that are finite rank perturbations of the identity are elements of class of scattering equivalences \([12]\). Scattering equivalences are unitary transformations that preserve the scattering matrix without changing the representation of free particles. While general unitary transformations could be used
in models with confining interactions, the scattering equivalences have the desirable feature that they lead to equivalent models that differ from the original model by a short-ranged addition to the original confining interaction.

These unitary transformations have the structure

\[ A = I + \Delta \]  

where \( \Delta \) is the finite rank part of \( A \). The constituent quark-antiquark mass operator and current operator transform as:

\[ M' = M_0 + V_c \rightarrow M' = MAA^\dagger = M_0 + V'_c \]  

where  

\[ V'_c = V_c + \Delta M + M\Delta^\dagger + \Delta M\Delta^\dagger \]  

and  

\[ J'^\mu(x) \rightarrow J'^\mu(x) = AJ^\mu(x)A^\dagger = J^\mu(x) + \Delta J^\mu(x) + J^\mu(x)\Delta^\dagger + \Delta J^\mu(x)\Delta^\dagger. \]

Since \( M \) is unbounded the range of \( \Delta^\dagger \) is required to be in the domain of \( M \). With this choice the differences \( M' - M \) and \( J'^\mu(x) - J^\mu(x) \) are short ranged.

The mass operator \( M \) has the structure:

\[ M = M_0 + V_c \]  

where

\[ M_0 = \sqrt{m^2 + k^2} + \sqrt{\bar{m}^2 + \bar{k}^2} \]  

and

\[ V_c = V_c(\lambda_1, \cdots, \lambda_n) \]

is a phenomenological confining interaction with parameters \( \lambda_1, \cdots, \lambda_n \) chosen to fit the mass spectrum of observed mesons.

The eigenvectors and eigenvalues of \( M \) are denoted by

\[ M|m_n\rangle = m_n|m_n\rangle \]

where the \( n = 0 \) state corresponds to the \( \pi \) meson and \( m_0 = m_\pi \).

To generate the finite rank operator \( \Delta \) let \( |\bar{m}_\pi\rangle \) be the lowest mass eigenstate of a mass operator \( \bar{M} \) obtained from \( M \) changing parameters \( \{\lambda_1, \cdots, \lambda_n\} \) in the confining interaction:

\[ \bar{M} = M_0 + \bar{V}_c \quad \bar{M}|\bar{m}_\pi\rangle = \bar{m}_\pi|\bar{m}_\pi\rangle \]

where

\[ \bar{V}_c = V_c(\bar{\lambda}_1, \cdots, \bar{\lambda}_n). \]  

Assume that it is possible to find a set of parameters \( \{\bar{\lambda}_1, \cdots, \bar{\lambda}_n\} \) with the property that experimental pion form factor can be expressed in terms of a
matrix element of a point-like impulse current, \( J_\mu^p(0) \), in the states \(| p, \bar{m}_\pi, \rangle\), \(| p', \bar{m}_\pi, \rangle\).

The wave function \( \langle k | \bar{m}_\pi \rangle \) is a candidate for the transformed pion wave function. The assumption means that the desired wave function can be found in the class of ground state solutions of mass operators with different choices of the potential parameters. This restricts the possible candidates for the new wave function to a set of functions that depend on a small number of parameters. A larger class of wave functions can be considered if there are no functions in the above class that are able to reproduce the measured form factors. The spectrum of \( \tilde{M} \) contains no physics, however, without loss of generality one of the parameters, \( \lambda_k \), can be taken to be an additive constant which can be adjusted so the eigenvalue \( \bar{m}_\pi = m_\pi \). This choice simplifies the notation, but has no effect on the form factor.

A general feature of relativistic quantum mechanical models is that an impulse current cannot be compatible with current conservation and current covariance. The problem is that the left side of the relations

\[
U(\Lambda, a) J^\mu(x) U^\dagger(\Lambda, a) = J'^\nu(\Lambda x + a) \Lambda_{\nu \mu}
\]

and

\[
[H, J^0(x)] = \sum_i [P^i, J^i(x)]
\]

involves the dynamics. These dynamical constraints, when combined with space reflection and time reversal symmetries, imply that the current matrix elements can be expressed in terms of a smaller set of independent current matrix elements. The number of independent current matrix elements is precisely equal to the number of invariant form factors. For each kinematic subgroup there are natural choices of independent current matrix elements. It is possible to consistently evaluate the independent current matrix elements, and all matrix elements related to them by kinematic transformations, using only impulse currents. In this paper, the impulse approximation means only that the independent matrix elements are calculated in the impulse approximation.

The problem is to find a unitary operator \( A = I + \Delta \) that maps \(| m_\pi \rangle\) to \(| \bar{m}_\pi \rangle\). Since \(| \bar{m}_\pi \rangle\) is not orthogonal to any of the states \(| m_n \rangle\), except where required by selection rules, the transformation \( A \) will effect all of the states.

A minimal solution uses a rank-one \( \Delta \). To construct the desired transformation define two orthogonal bases on the two-dimensional subspace spanned by the vectors \(| m_\pi \rangle\) and \(| \bar{m}_\pi \rangle\). If these vectors are identical then \( A = I \), and there is nothing to do. Otherwise orthogonal bases on this two-dimensional subspace are

\[
|m_\pi\rangle, \quad |m_\perp\rangle := \frac{|\bar{m}_\pi\rangle - |m_\pi\rangle \cos(\theta)}{\sin(\theta)}
\]

and

\[
|\bar{m}_\pi\rangle, \quad |\bar{m}_\perp\rangle := \frac{|m_\pi\rangle - |\bar{m}_\pi\rangle \cos(\theta)}{\sin(\theta)}
\]
where
\[ \cos(\theta) = \cos(\theta)^* := \langle m_\pi | \bar{m}_\pi \rangle \]  \hspace{1cm} (19)
\[ \sin(\theta) > 0. \]  \hspace{1cm} (20)

The overlap \( \langle m_\pi | \bar{m}_\pi \rangle = \cos(\theta) \) can be chosen to be real as a consequence of
the time reversal invariance of \( V_c \) and \( \bar{V}_c \). The \( \sin(\theta) \) is non zero, otherwise
\( |m_\pi\rangle = |\bar{m}_\pi\rangle \).

By construction the states \( \{ |m_\pi\rangle, |\bar{m}_\pi\rangle \} \), \( \{ |m_\pi\rangle, |m_\perp\rangle \} \), and
\( \{ |\bar{m}_\pi\rangle, |\bar{m}_\perp\rangle \} \) all span the same two-dimensional subspace.

The operator \( A \) is chosen to satisfy
\[ A|m_\pi\rangle = |\bar{m}_\pi\rangle \]  \hspace{1cm} (21)
\[ A|m_\perp\rangle = |\bar{m}_\perp\rangle \]  \hspace{1cm} (22)

and
\[ A|\psi\rangle = |\psi\rangle \quad \text{for} \quad \langle m_\pi | \psi \rangle = \langle m_\perp | \psi \rangle = 0. \]  \hspace{1cm} (23)

A unitary \( A \) satisfying (21), (22) and (23) is given by:
\[ A = I + \Delta = I - |m_\pi\rangle \langle m_\pi | - |m_\perp\rangle \langle m_\perp | + |\bar{m}_\pi\rangle \langle \bar{m}_\pi | + |\bar{m}_\perp\rangle \langle \bar{m}_\perp |. \]  \hspace{1cm} (24)

The perturbation \( \Delta \) is the matrix
\[ \Delta = (|\bar{m}_\pi\rangle - |m_\pi\rangle) \langle m_\pi | + (|\bar{m}_\perp\rangle - |m_\perp\rangle) \langle m_\perp |. \]  \hspace{1cm} (25)

It is useful to express \( \Delta \) directly in terms of the non-orthogonal states \( |m_\pi\rangle \) and
\( |\bar{m}_\pi\rangle \):
\[ \Delta = -\rho( |m_\pi\rangle - |\bar{m}_\pi\rangle ) (\langle m_\pi | - \langle \bar{m}_\pi | ) \]  \hspace{1cm} (26)
where
\[ \rho = \frac{\cos(\theta) + 1}{\sin^2(\theta)}. \]  \hspace{1cm} (27)

In this representation \( \Delta \) is easily seen to be a rank-one Hermitian operator.

Using \( \Delta + \Delta^\dagger + \Delta \Delta^\dagger = 0 \) and \( \langle m_\pi | (V_c - \bar{V}_c) |\bar{m}_\pi\rangle = \langle m_\pi | (V_c - \bar{V}_c) |m_\pi\rangle = 0 \) gives the transformed confining interaction:
\[ V_{c}' = V_c + \Delta M + M \Delta^\dagger + \Delta M \Delta^\dagger = V + \rho( |m_\pi\rangle - |\bar{m}_\pi\rangle ) (\langle m_\pi | - \langle \bar{m}_\pi | ) \]
\[ + \rho( V_c - \bar{V}_c)(|\bar{m}_\pi\rangle - |m_\pi\rangle ) (\langle m_\pi | - \langle \bar{m}_\pi | ) \]
\[ + \rho^2( |m_\pi\rangle - |\bar{m}_\pi\rangle ) (\langle m_\pi | - \langle \bar{m}_\pi | ) (\langle m_\pi | - \langle \bar{m}_\pi | ) \]  \hspace{1cm} (28)

By construction the mass operator \( M' = M_0 + V_{c}' \):

a.) Has the same spectrum as \( M \).
b.) Has the same pion wave function as $\tilde{M}$

c.) Differs from $M$ by the short range modification \[(28)\] to the confining interaction.

Even though the transformation $A$ was constructed to transform the ground state wave function, it must also transform the $n > 0$ states to preserve orthogonality. The transformed states have the form

$$|m'_n⟩ = A|m_n⟩ = |m_n⟩ + \rho⟨\tilde{m}_π|m_n⟩ (|m_π⟩ - |\tilde{m}_π⟩). \quad (29)$$

If the current $J^\mu_p(x)$ is identified with the point quark-antiquark impulse current, then the transformation $A$ generates exchange current contributions to the current in the original representation:

$$\langle p', m'_n | J_p^\mu | p, m_l⟩ = \langle p', m_n | [J_p^\mu + \Delta^\dagger J_p^\mu + J_p^\mu \Delta + \Delta^\dagger J_p^\mu \Delta] | p, m_l⟩. \quad (30)$$

Thus, form factors can be calculated using the current \[(30)\] in the original representation or the point quark impulse current in the transformed representation \[(29)\].

This method provides an alternative means for describing electromagnetic properties of constituent quarks which replaces the constituent quark form factors by a different representation of the dynamics.

### 3 Example

The simplicity of the construction is illustrated with an example. The constituent quark-antiquark model of Carlson, Kogut and Pandharipande \cite{1} is taken as the constituent quark model that fits the meson spectrum. The semi-relativistic Hamiltonian in \cite{1} is interpreted as the mass operator in a fully relativistic model with the kinematic subgroup of the light front. This relativistic model is described in the appendix.

The choice of kinematic subgroup does not affect the mass spectrum of the original model. It does affect the value of the impulse current matrix elements.

The mass operator in this model has the form

$$M = \sqrt{m_q^2 + k^2} + \sqrt{m_{\bar{q}}^2 + k^2} + V_c \quad (31)$$

$$V_c(\lambda_1, \cdots, \lambda_4) = \lambda_1 + \frac{\lambda_2}{r} + \lambda_3 r + \delta \bar{s}_q \cdot \bar{s}_{\bar{q}} e^{-\frac{r^2}{2\lambda_4}} \quad (32)$$

$$\delta = -\frac{\lambda_2}{3m_q^2 \lambda_4^3 \sqrt{\pi}}. \quad (33)$$

The model of \cite{1} also includes spin-orbit and tensor interactions that vanish for the pion. The parameters $\lambda_1, \cdots, \lambda_4$ of the original Carlson Kogut and Pandharipande model are listed on the first line of table 1.
Table 1: Interaction Parameters

| Interaction | $m$ (GeV) | $\lambda_1$ (GeV) | $\lambda_2$ | $\lambda_3$ (GeV)$^2$ | $\lambda_4$ (GeV)$^{-1}$ |
|-------------|-----------|-------------------|-------------|----------------------|----------------------|
| $V_e$       | .360      | -.777             | -.5         | .197                 | .66                  |
| $\bar{V}_e$| .360      | -.911             | -.14        | .049                 | .35                  |

The pion form factor can be expressed in terms of the matrix elements of the $+$ component of the current in a frame where the $+$ component of the momentum transfer vanishes\cite{13} \cite{14}. This matrix element is evaluated with a point-like impulse current:

$$F_\pi(Q^2) = \langle \frac{q^2}{2}, m_\pi |(J_{\gamma}^+ p_\gamma(0) + J_{\bar{\gamma}}^+ \bar{p}_\gamma(0)) | -\frac{q^2}{2}, m_\pi \rangle \quad (34)$$

where the states have the normalization

$$\langle p', m_\pi | p, m_\pi \rangle = \delta(p^{+'} - p^+)\delta^2(p'_\perp - \bar{p}_\perp). \quad (35)$$

The calculation of the wave functions are done using the method described in \cite{15}. If the original Carlson, Kogut, Pandharipande wave pion function is used in (34), then the calculated form factor is larger than the experimental form factor for both low \cite{16} and high \cite{17} momentum transfer. The calculated form factor is given by the dotted curves in figure 1 and figure 2.

The parameters in the second line of table 1 define the interaction $\bar{V}_e$. The state $|\bar{m}_\pi\rangle$ is the ground state of the mass operator $\bar{M} = M_0 + \bar{V}_e$. The solid curves in figures 1 and 2 are the result of using $|\bar{m}_\pi\rangle$ in (34). The calculated form factor is consistent with the measured data, but the spectrum of the operator $\bar{M}$ no longer agrees with experiment.

The method described in the previous section constructs a new mass operator, $M'$, that has the same spectrum as $M$ and the same pion wave function as $\bar{M}$. In this example the overlap parameter is

$$\cos(\theta) := \langle m_\pi | \bar{m}_\pi \rangle = .731 \quad (36)$$

which gives $\rho = 3.71$. The states $|\bar{m}_\pi\rangle$ and $|m_\pi\rangle$ and the overlap $\langle \bar{m}_\pi | m_\pi \rangle$ are used to define the unitary operator $A = I + \Delta$ using (26). The transformed mass operator is $M' = M_0 + V'_e$ with $V'_e$ given by (28). The operator $M'$ has the same mass spectrum as the original Carlson, Kogut, Pandharipande mass operator and has a pion eigenstate that can be used with the point impulse quark current to obtain the measured pion form factors. By construction, the pion form factor in this model is given by the solid curves in figures 1 and 2. Eigenstates of $M'$ for mesons other than the pion are obtained by applying $A$ to the corresponding eigenstates of $M$.

Figures 3 and 4 compare the coordinate (fig. 3) and momentum space (fig. 4) wave functions of the pion for the mass operators $M$ (dotted curve) and $M'$ (solid curve). The Carlson, Kogut, and Pandharipande wave functions have a
smaller size in configuration space and more high-momentum components than the wave-functions of the transformed mass operator. The form factor requires a more spread out wave function in configuration space and a wave function with more low momentum components in momentum space. The transformation $A$ has the overall effect of softening the wave functions of the original Carlson, Kogut, and Pandharipande model. This is consistent with the calculations of Cardarelli et. al. [4]. They use a similar quark-antiquark model and need to introduce single quark form factors to obtain measured pion form factors. In [4] the quark form factors also soften the wave function.

The pion decay constant, computed following [14], is 193 MeV in the light-front model with the original Carlson, Kogut, Pandharipande parameters. The transformation $A$ reduces this to 131 MeV, however it is still above the experimental value of 92 MeV. This could be improved by considering more general transformations $A$, or constituent quark models with smaller constituent quark masses [14].

The overlap probability, $P = |\langle m_\pi | \bar{m}_\pi \rangle|^2 = \cos^2(\theta)$, of the dotted and solid wave functions in fig. 3 and fig. 4 is .53. This means that the overlap probability of $|\bar{m}_\pi \rangle$ with the excited states of $M'$ is .47. Figures 5 and 6 show the effect of the transformation $A$ on the point-like impulse form factors for the first radial excitation of the pion. In these figures the dotted curve is the form factor in the original Carlson Kogut and Pandharipande model and the solid curve is the form factor in the transformed model. These plots indicate that the scattering equivalence also has significant effects on the excited mesons states with the same quantum numbers. The form factors in the original and transformed models cross at about 1 (GeV)$^2$. This is because the transformed excited state is not orthogonal to the original Carlson, Kogut, Pandharipande ground state, which has a significant amount of high momentum components. The operator $A$ constructed in this example is the simplest unitary operator that maps $|m_\pi \rangle$ to $|\bar{m}_\pi \rangle$. Different choices of $A$ lead to different predictions for the form factors of the excited states.

The effect of the transformation $A$ is to add an additional short range structure to the original confining interaction. Compared to the original Carlson, Kogut, and Pandharipande interaction, the additional short-ranged part contains non-localities. In a relativistic quantum theory there is no preferred reason to favor a local over a non-local interaction except for mathematical simplicity. Local and non-local interactions have similar strengths and defects. In relativistic models with short-ranged interactions, both the local and non-local interactions are consistent with two-body cluster properties, but both fail to be consistent with microscopic locality. Microscopic derivations of local interactions necessarily make implicit assumptions that lead to local interaction, however these assumptions are not based on physics principles. This shows that there is no reason to consider the transformed interaction to be any more or less fundamental than the original interaction.
4 Conclusions

In this paper the freedom to change representation is used to construct a constituent quark model that fits the meson mass spectrum and reproduces the pion form factor using only point-like constituent quarks.

This was done using scattering equivalences, which are unitary operators of the form $A = I + \Delta$, where $\Delta$ is asymptotically zero [12]. This paper utilized a limited class of these operators where $\Delta$ is a finite rank operator on the internal Hilbert space satisfying a domain restriction. Desirable results were achieved by considering only a small subset of rank-one $\Delta$’s. Improvements to the fit of the pion form factor can be achieved by considering a larger class of $\Delta$’s, such as compact $\Delta$’s. Excellent results can be obtained with constituent quark models having slightly lower constituent quark masses [13]. In this application the quark masses cannot be used as parameters because they appear in both the wave functions and the Clebsch-Gordon coefficients of the Poincaré group.

The example in section three illustrates the simplicity and power of the method. In this example the desired unitary scattering equivalence is an easily constructed rank-one perturbation of the identity. The only required input is the new state vector $|\bar{m}_\pi\rangle$ and the overlap $\langle m_\pi | \bar{m}_\pi \rangle$. The new representation does not require constituent quark/antiquark form factors.

The new model $M'$ is designed to be consistent with the meson spectrum and the pion form factor. For a given choice of $\Delta$ the model makes predictions for elastic and transition form factors involving other mesons which can (in principle) be tested against experiment. In addition, the transformed operators might be useful in formulating many-body constituent quark models of hadrons.

The example started with relativistic constituent quark model with a light-front kinematic symmetry, and produced a new relativistic model with the same light-front kinematic symmetry, same mass spectrum, where the pion form factor can be computed in point-quark impulse approximation. This example illustrates that constraints imposed by the choice of kinematic subgroup and mass spectrum do not determine the form factors.

Scattering equivalences are also known to exist [12] between models with different kinematic subgroups. These relationships can also be exploited to construct equivalent current operators for models with different forms of dynamics. While the ambiguities in representations of the dynamics are sometimes considered a liability, this paper shows that they lead to a flexibility that can lead to a simplification of the dynamics.

5 Appendix

The relativistic interpretation of the Carlson, Kogut, Pandharipande semi-relativistic Hamiltonian is discussed in this appendix.

A relativistic quantum mechanical model is defined by a unitary representation of the Poincaré group acting on a model Hilbert space. The mass operator of this representation is necessarily a dynamical operator.
A representation of the quark-antiquark Hilbert space and a representation of the Poincaré Lie algebra where the mass operator is the semi-relativistic Carlson, Kogut, and Pandharipande Hamiltonian is exhibited in this section. The generators are chosen so the generators of transformations that leave the light front $x^+ = 0$ invariant are kinematic. This defines a relativistic light-front dynamics.

The Hilbert space in this model is the tensor product of the mass $m$ spin $\frac{1}{2}$ irreducible representation of the Poincaré group associated with the quark and the mass $\bar{m}$ spin $\frac{1}{2}$ irreducible representation of the Poincaré group associated with the antiquark. The light-front components of the single particle momenta $p_i := (p_1^+, p_1^-, p_2^\perp)$ and the 3-component of the light front spin $j_3^\perp$ are a maximal set of commuting observables on the single quarks spaces. The model Hilbert space is the tensor product of the space of square integrable functions of $(p_1, \mu_1)$:

$$\psi_q(p_q, \mu_q) \otimes \psi_{\bar{q}}(p_{\bar{q}}, \mu_{\bar{q}})$$

The tensor product of irreducible representations of the Poincaré group is reducible. The Clebsch-Gordon coefficients of the Poincaré group are $\langle p_q, \mu_q, p_{\bar{q}}, \mu_{\bar{q}} | P, \mu | l, s \rangle$.

$$\delta(P - p_1 - p_2) \delta(k - k_1, p_2) \frac{\omega_1(k) \omega_2(k) \omega_{P^+}}{k^2} \frac{\sqrt{\omega_1(k) \omega_2(k) P^+}}{p_1^+ p_2^+ m(k)}$$

$$\times D_{\mu_1 \mu_2}^{1/2} \langle B_f^{-1}(k_1) B_c(k_1) | D_{\mu_2 \mu_3}^{1/2} \langle B_f^{-1}(k_2) B_c(k_2) | \gamma_\mu_{l, k} \rangle \} \rangle$$

where $m = m(k) = \sqrt{k^2 + m_q^2} + \sqrt{k^2 + m_{\bar{q}}^2}$.

The discrete indices $l, s$ label multiple copies of the same irreducible representation that appear in the tensor product. From the structure of the Clebsch-Gordon coefficient above it is obvious that $s \in 0, 1$ and $| j - s | \leq l \leq j + s$.

The Clebsch-Gordon coefficients lead to a representation of the quark-antiquark Hilbert space as the space of square integrable functions of the variables $(P^+, \vec{P}_\perp, \mu, k, j, l, s)$. Vectors in the Hilbert space are square integrable functions of these variables

$$\psi(P, \mu, k, j, l, s) = \langle P^+, \vec{P}_\perp, \mu, k, j, l, s | \psi \rangle.$$
\[ \delta(P - P')\delta_{jj'}\delta_{\mu\mu'}\langle k, l, s | M^j_{\mu \mu} | k', l', s' \rangle \] (42)

This defines \( M \) in terms of the kernel, \( \langle k, l, s | M^j_{\mu \mu} | k', l', s' \rangle \), of the Carlson, Kogut and Pandharipande semi-relativistic Hamiltonian. This kernel is identified with the semi-relativistic Hamiltonian given by (31) and (32).

Given this mass operator the following operators [2]

\[ \vec{E} = -iP^+ \frac{\partial}{\partial \vec{P}_\perp} \quad K_3 = -iP^+ \frac{\partial}{\partial P^+} \] (43)

\[ J_3 = j_3 - \frac{1}{P^+} \hat{z} \cdot (\vec{P} \times \vec{E}) \] (44)

\[ P_- = \frac{\vec{P}_\perp \cdot \vec{P}_\perp - M^2}{P^+} \] (45)

\[ \vec{J}_\perp = \frac{1}{P^+} \left( \frac{P^+ - P^-}{2} (\hat{z} \times \vec{E}) - (\hat{z} \times \vec{P})K_3 + \vec{P}_\perp j_3 + M\vec{J}_\perp \right) \] (46)

along with the multiplication operators \( P^+ \) and \( \vec{P}_\perp \), define a set of self-adjoint operators on the space of square integrable functions of \( \{P, \mu, k, j, l, s\} \) that satisfy the Poincaré Lie algebra with the Carlson Kogut and Pandharipande mass operator (42), (31) and (32).

The interaction in the above expressions appears in both \( P^- \) and \( \vec{J}_\perp \). The remaining generators, which generate transformations that leave the light front invariant, do not involve \( M \) thus are kinematic. It is clear that the same properties hold if \( M \) is replaced by \( M' \). Unlike Fock space motivated models [18], the Poincaré symmetry is exact.

Note that to compute current matrix elements with an impulse current it is necessary to use the Clebsch Gordon coefficients [18] to transform the eigenstates of \( M \) and \( M' \) to single quark variables. Because the Clebsch Gordon coefficients depend on the quark masses, [19], the quark masses were not allowed to vary in determining \( \vec{M} \).

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Figure 1: Pion form factor at low $Q^2$.

Figure 2: Pion form factor at high $Q^2$. 
Figure 3: Comparison of k-space wave functions.

Figure 4: Comparison of r-space wave functions.
Figure 5: $\pi^*$ form factor at low $Q^2$.

Figure 6: $\pi^*$ form factor at high $Q^2$. 