Differential graded Koszul duality: An introductory survey

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Abstract
This is an overview on derived nonhomogeneous Koszul duality over a field, mostly based on the author's memoir L. Positselski, Memoirs of the American Math. Society 212 (2011), no. 996, vi+133. The paper is intended to serve as a pedagogical introduction and a summary of the covariant duality between DG-algebras and curved DG-coalgebras, as well as the triality between DG-modules, CDG-comodules, and CDG-contramodules. Some personal reminiscences are included as a part of historical discussion.

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INTRODUCTION

Koszul duality is a fundamental phenomenon in mathematics, including such fields as algebraic topology, algebraic and differential geometry, and representation theory. This phenomenon is so general that it does not seem to admit a ‘maximal natural’ generality. Whatever formulation one comes up with, it can be likely further extended by building something on top or underneath.

Koszul duality is a synthetic subject. Depending on context, its formulation involves such concepts as differential graded structures, curved differential graded structures, comodules and contramodules, conilpotency, coderived and contraderived categories, operads and properads, and whatnot.

Koszul duality has to be distinguished from the comodule–contramodule correspondence. The co–contra correspondence is a less familiar though no less fundamental, but simpler phenomenon, often accompanying the Koszul duality. The following rule of thumb for distinguishing the two phenomena may be helpful. Viewed as a covariant category equivalence, the co–contra correspondence takes injective objects to projective objects or vice versa. Koszul duality takes projective objects to irreducible objects and/or irreducible objects to injective objects, and so on [75, Prologue].

The aim of this paper is to introduce the reader to the subject of derived Koszul duality in the context of differential graded algebras and modules, as well as coalgebras, comodules, and contramodules; and survey some of its main results. We also discuss curved differential structures, but no $A_\infty$–structures are considered in this paper (the reader can find discussions of these in the Koszul duality context in the dissertation [47] and the memoirs [64, 65]). No relative Koszul duality settings (as in [63, sections 0.4 and 11] or [75]) are discussed in this survey, either; so everything in this paper happens over a field. We also do not consider operads and their generalizations, referring instead to the book [48] and the papers [31, 89].

This introductory survey is intended for an audience versed in homological algebra generally, but largely unfamiliar with the subject of derived Koszul duality. It is almost entirely based on the author’s memoir [64]. The more elementary topic of underived quadratic duality [7, 58, 61, 83] is only briefly touched in this paper. Homogeneous versions of derived Koszul duality [7], [64, appendix A] (involving graded modules over positively or negatively graded algebras, with the differentials preserving the grading) are not elaborated upon in this paper, either. Model structures and compact generators are also only briefly touched.

We start with posing the problem in a basic particular case, formulating the main results in an approximate form in the simplest settings, and then proceed to introduce further ingredients and make more precise and general assertions. Some sketches of proofs are included in this
survey, but most results receive a brief outline of an idea of the proof and a reference to a more detailed treatment.

All the Koszul duality functors, triangulated equivalences, and so on, in this paper are covariant functors or category equivalences. A discussion of contravariant version of homogeneous Koszul duality can be found in [64, section A.2].

One disclaimer is in order. As usual in differential graded homological algebra, sign rules (plus or minus) present a tedious problem. In this survey, we skip many descriptions of sign rules, referring the reader to the original publications such as [64] for the details. So, our formulations may be imprecise in that not all the signs are properly spelled out.

1 \ ALGEBRAS AND MODULES

Throughout this paper, we work over a fixed ground field \( k \). Unless otherwise mentioned, all algebras in this paper are associative and unital, all modules are unital, and all homomorphisms of algebras are presumed to take the unit to the unit.

In this section, we start posing the problem of derived nonhomogeneous Koszul duality in the simplest particular case of complexes of modules over an augmented \( k \)-algebra \( A \).

1.1 \ Augmented algebras

Let \( A \) be an (associative, unital) algebra over a field \( k \). An augmentation \( \alpha \) on \( A \) is a (unital) \( k \)-algebra homomorphism \( \alpha : A \rightarrow k \). So, \( \alpha(1) = 1 \), and the existence of an augmentation implies that \( 1 \neq 0 \) in \( A \); hence \( \alpha \) is a surjective map. We denote by \( A^+ = \ker(\alpha) \subset A \) the augmentation ideal; so \( A^+ \) is a two-sided ideal in \( A \), and \( A = k \oplus A^+ \) as a \( k \)-vector space. The \( k \)-algebra homomorphism \( \alpha : A \rightarrow k \) endows the one-dimensional \( k \)-vector space \( k \) with left and right \( A \)-module structures.

The following definition goes back to the papers [20, chapter II] and [1]. The bar-construction \( \text{Bar}_{\alpha}^*(A) \) of an augmented \( k \)-algebra \( A = (A, \alpha) \) is defined as the complex

\[
\begin{align*}
&k \leftarrow A^+ \overset{\partial}{\leftarrow} A^+ \otimes_k A^+ \overset{\partial}{\leftarrow} A^+ \otimes_k A^+ \otimes_k A^+ \leftarrow \cdots \\
\text{with the differential given by the formulae } &\partial(a \otimes b) = ab, \quad \partial(a \otimes b \otimes c) = ab \otimes c - a \otimes bc, \ldots,
\end{align*}
\]

\[
\partial(a_1 \otimes \cdots \otimes a_n) = a_1 a_2 \otimes a_3 \otimes \cdots \otimes a_n - \cdots
\]

\[
+ (-1)^{i+1} a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n + \cdots
\]

\[
+ (-1)^n a_1 \otimes \cdots \otimes a_{n-2} \otimes a_{n-1} a_n,
\]

and so on, for all \( a, b, c, a_i \in A^+ \), \( n \geq 1 \), and \( 1 \leq i \leq n - 1 \). The leftmost differential \( \partial : A^+ \rightarrow k \) is the zero map.

The complex \( \text{Bar}_{\alpha}^*(A) \) computes the \( k \)-vector spaces \( \text{Tor}^4_n(k, k) \) (where \( k \) is endowed with left and right \( A \)-module structures via \( \alpha \)). In other words, there are natural isomorphisms of \( k \)-vector spaces

\[
H^{-n} \text{Bar}_{\alpha}^*(A) \simeq \text{Tor}^4_n(k, k) \quad \text{for all } n \geq 0.
\]
Let \((A, \alpha)\) and \((B, \beta)\) be two augmented \(k\)-algebras, and let \(f : A \to B\) be a \(k\)-algebra homomorphism compatible with the augmentations (that is, satisfying the equation \(\alpha = \beta f\)). Then there is the induced map of augmentation ideals \(f^+ : A^+ \to B^+\) and the induced map of the bar-constructions \(\text{Bar}^*(f) : \text{Bar}^*_{\alpha}(A) \to \text{Bar}^*_{\beta}(B)\). The latter map is a morphism of complexes of \(k\)-vector spaces.

**Example 1.1.** The morphism of complexes \(\text{Bar}^*(f)\) may well be a quasi-isomorphism even when a morphism of augmented \(k\)-algebras \(f : A \to B\) is not an isomorphism.

For example, the complex \(0 \leftarrow B^+ \leftarrow B^+ \otimes_k B^+ \leftarrow B^+ \otimes_k B^+ \otimes_k B^+ \leftarrow \cdots\) is acyclic whenever the augmentation ideal \(B^+\), viewed as an associative \(k\)-algebra, has a unit of its own. The sequence of \(k\)-linear maps \(h : B^+ \otimes^n \to B^+ \otimes^{n+1}\) given by the rules \(h(b_1 \otimes \cdots \otimes b_n) = e \otimes b_1 \otimes \cdots \otimes b_n\) for all \(b_i \in B^+, 1 \leq i \leq n, n \geq 1\), where \(e\) is the unit in \(B^+\), provides a contracting homotopy.

Choose any nonzero associative, unital \(k\)-algebra, denote it by \(B^+\), and adjoin a new unit to it formally, producing the \(k\)-algebra \(B = k \oplus B^+\). Then the natural homomorphisms of augmented \(k\)-algebras \(k \to B \to k\) induce quasi-isomorphisms of the bar-constructions.

More generally, let \(A\) and \(B\) be two associative, unital \(k\)-algebras. Consider the direct sum \(A \oplus B\), and endow it with a \(k\)-algebra structure as the product of \(A\) and \(B\) in the category of \(k\)-algebras. Choose an augmentation \(\alpha : A \to k\), and let the augmentation of \(A \oplus B\) be constructed as the composition \(A \oplus B \to A \to k\). Then the natural homomorphism of augmented \(k\)-algebras \(A \oplus B \to A\) (the direct summand projection) induces a quasi-isomorphism of the bar-constructions.

### 1.2 Bar-constructions of modules

Let \(A\) be an associative \(k\)-algebra with an augmentation \(\alpha : A \to k\). Let \(M\) be a left \(A\)-module.

The *bar-construction* \(\text{Bar}^*_{\alpha}(A, M)\) of \(A\) with the coefficients in \(M\) is the complex

\[
M \overset{\partial}{\leftarrow} A^+ \otimes_k M \overset{\partial}{\leftarrow} A^+ \otimes_k A^+ \otimes_k M \leftarrow \cdots
\]

with the differential given by the formulae \(\partial(a \otimes m) = a m\), \(\partial(a \otimes b \otimes m) = ab \otimes m - a \otimes bm\), \(\cdots\),

\[
\partial(a_1 \otimes \cdots \otimes a_n \otimes m) = a_1 a_2 \otimes a_3 \otimes \cdots \otimes a_n \otimes m - \cdots + (-1)^{n+1} a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n m,
\]

and so on, for all \(a, b, a_i \in A^+, m \in M\), and \(n \geq 1\).

The complex \(\text{Bar}^*_{\alpha}(A, M)\) computes the \(k\)-vector spaces \(\text{Tor}^A_n(k, M)\) (where \(k\) is endowed with a right \(A\)-module structure via \(\alpha\)). In other words, there are natural isomorphisms of \(k\)-vector spaces

\[
H^{-n} \text{Bar}^*_{\alpha}(A, M) \simeq \text{Tor}^A_n(k, M) \quad \text{for all } n \geq 0.
\]

**Examples 1.2.** The complex \(\text{Bar}^*_{\alpha}(A, M)\) may well be acyclic even when \(M \neq 0\).

(1) For example, the complex \(\text{Bar}^*_{\alpha}(A, M)\) is acyclic whenever the augmentation ideal \(A^+\), viewed as an associative \(k\)-algebra, has a unit of its own. The sequence of \(k\)-linear maps
\[ h : A^+ \otimes_k M \to A^+ \otimes_{k+1} M \]
given by the rules \[ h(a_1 \otimes \cdots \otimes a_n \otimes m) = e \otimes a_1 \otimes \cdots \otimes a_n \otimes m \]
for all \( a_i \in A^+ \), \( m \in M \), \( 1 \leq i \leq n \), \( n \geq 0 \), where \( e \) is the unit in \( A^+ \), provides a contracting homotopy.

More generally, let \( A \) and \( B \) be two associative, unital \( k \)-algebras, and let \( \alpha : A \to k \) be an augmentation of \( A \). Consider the direct sum \( A \oplus B \), and endow it with an augmented \( k \)-algebra structure \((\alpha, 0) : A \oplus B \to k\) as in Example 1.1. Pick a left \( B \)-module \( M \), and endow it with an \((A \oplus B)\)-module structure via the natural \( k \)-algebra homomorphism (the direct summand projection) \( A \oplus B \to B \). Then the complex \( \text{Bar}_{\alpha}^\idot (A \oplus B, M) \) is acyclic.

To give a couple of other examples, consider the algebra of polynomials in one variable \( A = k[x] \) over the field \( k \), endowed with the augmentation \( \alpha : A \to k \) given by the rule \( \alpha(x) = 0 \).

For any element \( a \in k \), denote by \( k_a = A/(x - a) \) the one-dimensional \( A \)-module in which the generator \( x \in A \) acts by the operator of multiplication with \( a \). In particular, in the \( A \)-module \( k = k_0 \) the algebra \( A \) acts via the augmentation \( \alpha \). Put \( M = k_a \), where \( a \neq 0 \). Then \( \text{Tor}_n^A(k, M) = 0 \) for all \( n \geq 0 \), hence the complex \( \text{Bar}_{\alpha}^\idot (A, M) \) is acyclic.

Alternatively, consider the \( A \)-module \( M = k[x, x^{-1}] \) of Laurent polynomials in \( x \) (or the \( A \)-module \( M = k(x) \) of rational functions in \( x \)). Once again, in these cases \( \text{Tor}_n^A(k, M) = 0 \) for all \( n \geq 0 \), and the complex \( \text{Bar}_{\alpha}^\idot (A, M) \) is acyclic.

1.3 Posing the problem

Let \( A = (A, \alpha) \) be an augmented \( k \)-algebra, and let \( M^* \) be a complex of left \( A \)-modules. Then the bar-construction \( \text{Bar}_{\alpha}^\idot (A, M^*) \) is a bicomplex of \( k \)-vector spaces. Let us totalize this bicomplex by taking infinite direct sums along the diagonals.

The problem of derived (nonhomogeneous) Koszul duality can be formulated as follows. We would like to endow the complex \( \text{Bar}_{\alpha}^\idot (A, M^*) \) with some natural structure, and define an equivalence relation on complexes with such structures, in a suitable way so that the assignment

\[ M^* \mapsto \text{Bar}_{\alpha}^\idot (A, M^*) \]

would be a triangulated equivalence between the unbounded derived category \( \mathcal{D}(A\text{-mod}) \) of complexes of \( A \)-modules and the triangulated category of complexes with the said structure up to the said equivalence relation.

What structure should it be, and what should be the equivalence relation? The first question is easier to answer: the bar-construction \( \text{Bar}_{\alpha}^\idot (A) \) has a natural \( DG \)-coalgebra structure, and the complexes \( \text{Bar}^\idot (A, M^*) \) are \( DG \)-comodules over \( \text{Bar}_{\alpha}^\idot (A) \), as will be explained below in Subsections 2.3 and 2.4.

The second question is harder, because the conventional notion of quasi-isomorphism is \textit{not} up to the task, as Examples 1.2 illustrate. The complex \( \text{Bar}_{\alpha}^\idot (A, M) \) can be quasi-isomorphic to zero for a quite nonzero one-term complex of \( A \)-modules \( M = M^* \). The relevant definition of the \textit{coderived category} of \( DG \)-comodules will be spelled out in Subsection 7.7.

2 Coalgebras and Comodules

Unless otherwise mentioned, all \textit{coalgebras} in this paper are coassociative, counital coalgebras over the field \( k \), all \textit{comodules} are counital, and all \textit{homomorphisms of coalgebras} are compatible with the counits.
In this section, we present the simplest initial formulations of derived nonhomogeneous Koszul duality for complexes of modules over an augmented algebra $A$ and for complexes of comodules over a conilpotent coalgebra $C$.

2.1 Coalgebras and comodules

The standard reference sources on coalgebras and comodules over a field are the books [49, 88]. The present author’s overview [71] can be used as an additional source.

A (coassociative, counital) coalgebra $C$ over a field $k$ is a $k$-vector space endowed with $k$-linear maps of comultiplication and counit

$$\mu : C \rightarrow C \otimes_k C$$

and

$$\varepsilon : C \rightarrow k$$

satisfying the following coassociativity and counitality axioms. First, the two compositions

$$C \rightarrow C \otimes_k C \Rightarrow C \otimes_k C \otimes_k C$$

must be equal to each other, $(\mu \otimes \text{id}_C) \circ \mu = (\text{id}_C \otimes \mu) \circ \mu$. Second, both the compositions

$$C \rightarrow C \otimes_k C \Rightarrow C$$

must be equal to the identity map, $(\varepsilon \otimes \text{id}_C) \circ \mu = \text{id}_C = (\text{id}_C \otimes \varepsilon) \circ \mu$.

A left comodule $M$ over a coalgebra $C$ is a $k$-vector space endowed with a $k$-linear left coaction map

$$\nu : M \rightarrow C \otimes_k M$$

satisfying the following coassociativity and counitality axioms. First, the two compositions

$$M \rightarrow C \otimes_k M \Rightarrow C \otimes_k C \otimes_k M$$

must be equal to each other, $(\mu \otimes \text{id}_M) \circ \nu = (\text{id}_C \otimes \nu) \circ \nu$. Second, the composition

$$M \rightarrow C \otimes_k M \rightarrow M$$

must be equal to the identity map, $(\varepsilon \otimes \text{id}_M) \circ \nu = \text{id}_M$.

A right comodule $N$ over $C$ is a $k$-vector space endowed with a right coaction map

$$\nu : N \rightarrow N \otimes_k C$$

satisfying the similar coassociativity and counitality axioms.

Let $V$ be a $k$-vector space. Then the comultiplication map $\mu$ on a coalgebra $C$ induces a left coaction map $C \otimes_k V \rightarrow C \otimes_k C \otimes_k V$ on the $k$-vector space $C \otimes_k V$ and a right coaction map $V \otimes_k C \rightarrow V \otimes_k C \otimes_k C$ on the $k$-vector space $V \otimes_k C$. The left $C$-comodule $C \otimes_k V$ and the right $C$-comodule $V \otimes_k C$ are called the cofree $C$-comodules cogenerated by the vector space $V$. 
For any left $C$-comodule $L$, the $k$-vector space of all left $C$-comodule maps $L \rightarrow C \otimes_k V$ is naturally isomorphic to the $k$-vector space of all $k$-linear maps $L \rightarrow V$,

$$\text{Hom}_C(L, C \otimes_k V) \cong \text{Hom}_k(L, V).$$

Hence, the ‘cofree comodule’ terminology.

### 2.2 DG-coalgebras and DG-comodules

The following definitions, going back at least to the paper [23], can be also found in [64, sections 2.1 and 2.3].

A graded coalgebra over $k$ is a graded $k$-vector space $C = \bigoplus_{i \in \mathbb{Z}} C^i$ endowed with a coalgebra structure such that both the comultiplication map $\mu : C \rightarrow C \otimes_k C$ and the counit map $\varepsilon : C \rightarrow k$ are morphisms of graded vector spaces (that is, homogeneous linear maps of degree 0). Here the standard induced grading is presumed on the tensor product $C \otimes_k C$, while the one-dimensional $k$-vector space $k$ is endowed with the grading where it is placed in the degree $i = 0$.

A graded left comodule over a graded coalgebra $C$ is a graded $k$-vector space $M = \bigoplus_{i \in \mathbb{Z}} M^i$ endowed with a left $C$-comodule structure such that the coaction map $\nu : M \rightarrow C \otimes_k M$ is a morphism of graded vector spaces. Graded right $C$-comodules are defined similarly.

In particular, for any graded $k$-vector space $V$, the graded $k$-vector space $C \otimes_k V$ has a natural structure of a cofree graded left $C$-comodule, while the tensor product $V \otimes_k C$ is a cofree graded right $C$-comodule.

A DG-coalgebra $C^\cdot = (C, d)$ over $k$ is a complex of $k$-vector spaces endowed with a coalgebra structure such that both the comultiplication map $\mu : C^\cdot \rightarrow C^\cdot \otimes_k C^\cdot$ and the counit map $\varepsilon : C^\cdot \rightarrow k$ are morphisms of complexes of vector spaces (that is, homogeneous linear maps of degree 0 commuting with the differentials). Here the standard induced differential $d(c' \otimes c'') = d(c') \otimes c'' + (-1)^{i(c')} c' \otimes d(c'')$ for all $c' \in C^i$ and $c'' \in C^j$ is presumed on the tensor product of complexes $C^\cdot \otimes_k C^\cdot$, while the differential on the $k$-vector space $k$ is zero.

A left DG-comodule $M^\cdot = (M, d_M)$ over a DG-coalgebra $C^\cdot$ is a complex of $k$-vector spaces endowed with a left $C$-comodule structure such that the coaction map $\nu : M^\cdot \rightarrow C^\cdot \otimes_k M^\cdot$ is a morphism of complexes of $k$-vector spaces. Right DG-comodules over $C^\cdot$ are defined similarly.

### 2.3 Coalgebra structure on the bar-construction

Let $V$ be a $k$-vector space. Then the direct sum of the tensor powers of $V$,

$$k \oplus V \oplus (V \otimes_k V) \oplus (V \otimes_k V \otimes_k V) \oplus \cdots \cong \bigoplus_{n=0}^{\infty} V^\otimes_n$$

can be naturally endowed with a structure of graded associative algebra over $k$. The multiplication in this algebra, denoted by $T(V) = \bigoplus_{n=0}^{\infty} V^\otimes_n$, is given by the rule

$$(v_1 \otimes \cdots \otimes v_p)(w_1 \otimes \cdots \otimes w_q) = v_1 \otimes \cdots \otimes v_p \otimes w_1 \otimes \cdots \otimes w_q$$
for all \( v_i, w_j \in V, 1 \leq i \leq p, 1 \leq j \leq q, p, q \geq 0 \). The unit element in \( T(V) \) is \( 1 \in k = V^{\otimes 0} \). The algebra \( T(V) \) is the free associative, unital algebra spanned by the vector space \( V \).

The same graded \( k \)-vector space \( \bigoplus_{n=0}^{\infty} V^{\otimes n} \) also has a natural structure of graded coassociative coalgebra over \( k \). The comultiplication in this coalgebra, denoted by \( \mathcal{J}(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n} \), is given by the rule

\[
\mu(v_1 \otimes \cdots \otimes v_n) = \sum_{p+q=n} (v_1 \otimes \cdots \otimes v_p) \otimes (v_{p+1} \otimes \cdots \otimes v_{n})
\]

for all \( v_i \in V, 1 \leq i \leq n, n \geq 0 \). The counit on \( \mathcal{J}(V) \) is the direct summand projection map \( \mathcal{J}(V) \rightarrow V^{\otimes 0} = k \). We refer to Remark 3.2 for a discussion of a cofreeness property of the tensor coalgebra \( \mathcal{J}(V) \).

Let \((A, \alpha)\) be an augmented algebra over \( k \). We observe that the underlying graded vector space \( \bar{\text{Bar}}_{\alpha}(A) \) of the bar-construction \( \bar{\text{Bar}}^\ast_{\alpha}(A) \), as defined in Subsection 1.1, coincides (up to a grading sign change) with the graded \( k \)-vector space \( \bigoplus_{n=0}^{\infty} A^{+ \otimes n} \). Consequently, the graded vector space \( \bar{\text{Bar}}_{\alpha}(A) \) has natural structures of an associative algebra and a coassociative coalgebra.

It turns out that the bar differential \( \partial \) on \( \bar{\text{Bar}}^\ast_{\alpha}(A) \) does not respect the multiplication on \( \bar{\text{Bar}}_{\alpha}(A) = T(A^+) \), that is, \( \partial \) does not satisfy any kind of Leibniz rule with respect to the multiplication on \( T(A^+) \). However, the differential \( \partial \) is compatible with the comultiplication on \( \bar{\text{Bar}}_{\alpha}(A) = \mathcal{J}(A^+) \). In other words, the complex \( \bar{\text{Bar}}^\ast_{\alpha}(A) \) endowed with the graded coalgebra structure of \( \mathcal{J}(A^+) \) is a \( DG \)-coalgebra in the sense of the definition in Subsection 2.2.

### 2.4 Derived Koszul duality formulated for complexes of modules

Let \((A, \alpha)\) be an augmented \( k \)-algebra, and let \( M^\ast \) be a complex of left \( A \)-modules. We will denote simply by \( M \) the underlying graded \( A \)-module of \( M^\ast \).

Then the underlying graded vector space \( \bar{\text{Bar}}_{\alpha}(A, M) \) of the bar-construction \( \bar{\text{Bar}}^\ast_{\alpha}(A, M^\ast) \), as defined in Subsections 1.2 and 1.3, can be viewed as a cofree graded left comodule \( \bar{\text{Bar}}_{\alpha}(A, M) = \mathcal{J}(A^+) \otimes_k M \) over the tensor coalgebra \( \mathcal{J}(A^+) \) (as explained in Subsections 2.1 and 2.2). It turns out that the total differential on \( \bar{\text{Bar}}^\ast_{\alpha}(A, M^\ast) \) is compatible with the bar differential \( \partial \) on the \( DG \)-coalgebra \( \bar{\text{Bar}}^\ast_{\alpha}(A) \) and the left coaction of \( \bar{\text{Bar}}_{\alpha}(A) \) in \( \bar{\text{Bar}}_{\alpha}(A, M) \). In other words, the complex \( \bar{\text{Bar}}^\ast_{\alpha}(A, M^\ast) \) endowed with the cofree graded comodule structure of \( \mathcal{J}(A^+) \otimes_k M \) is a \( left \ DG \)-comodule over the \( DG \)-coalgebra \( \bar{\text{Bar}}^\ast_{\alpha}(A) \).

The following theorem is our first formulation of the derived Koszul duality. At this point in our exposition, it is more of an advertisement than a precise claim, in that it contains details which have not been defined yet but will be defined below.

**Theorem 2.1.** Let \((A, \alpha)\) be an augmented associative algebra over a field \( k \). Then the assignment \( M^\ast \mapsto \bar{\text{Bar}}^\ast_{\alpha}(A, M^\ast) \) induces a triangulated equivalence between the derived category of left \( A \)-modules \( \mathcal{D}(A-\text{mod}) \) and the coderived category of left \( DG \)-comodules over the \( DG \)-coalgebra \( \bar{\text{Bar}}^\ast_{\alpha}(A) \),

\[
\mathcal{D}(A-\text{mod}) \simeq \mathcal{D}^{co}(\bar{\text{Bar}}^\ast_{\alpha}(A)-\text{comod}).
\]

**Proof.** This is a particular case of [64, Theorem 6.3(a)]; see [47, Theorem 2.2.2.2] and [42, section 4] for an earlier approach. The inverse functor, \( \mathcal{D}^{co}(\bar{\text{Bar}}^\ast_{\alpha}(A)-\text{comod}) \rightarrow \mathcal{D}(A-\text{mod}) \), assigns...
to a left DG-comodule $N^*$ over the DG-coalgebra $C^* = \text{Bar}^*_\alpha(A)$ the graded left $A$-module $A \otimes_k N$ endowed with a natural differential whose construction will be explained in Subsections 5.3 and 5.4. The definition of the coderived category will be given in Subsection 7.7.

The triangulated equivalence of Theorem 2.1 takes the irreducible left $A$-module $k$ (with the $A$-module structure defined in terms of the augmentation $\alpha$) to the cofree left DG-comodule $C^*$ over the DG-coalgebra $C^* = \text{Bar}^*_\alpha(A)$ (cf. the discussion of cofree comodules in Subsection 2.1). The same equivalence takes the left DG-comodule $k$ over $C^*$ (with the $C$-comodule structure on $k$ defined in terms of the unique coaugmentation of $C$; see Subsections 2.6 and 4.4) to the free left $A$-module $A$.

For generalizations of Theorem 2.1, see Theorems 2.4 and 5.4.

2.5 | Nonhomogeneous quadratic dual DG-coalgebra

The DG-coalgebra $\text{Bar}^*_\alpha(A)$ produced by the bar-construction is rather big. In particular, it is essentially never finite-dimensional (unless $A = k$). The concept of nonhomogeneous quadratic duality, going back to [45], [83, section 3], [25, section 2.3], and [61], sometimes allows to produce a smaller DG-coalgebra that can be used in lieu of $\text{Bar}^*_\alpha(A)$ in Koszul duality theorems such as Theorem 2.1.

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a nonnegatively graded $k$-algebra with $A_0 = k$. Then the direct summand inclusion $A_1 \to A$ can be uniquely extended to a morphism of graded algebras $\pi : T(A_1) \to A$. The algebra $A$ is said to be quadratic if the homomorphism $\pi$ is surjective and its kernel is generated by elements of degree 2 (simply speaking, this means that the algebra $A$ is generated by elements of degree 1 with relations in degree 2).

The graded algebra $A$ is called Koszul if $\text{Tor}_i^A(k, k) = 0$ for all $i \neq j$. Here the first grading $i$ on the Tor spaces is the usual homological grading, while the second grading $j$, called the internal grading, is induced by the grading of $A$. This condition for $i = 1$ means precisely that the algebra $A$ is generated by $A_1$; assuming this is the case, the same condition for $i = 2$ means precisely that $A$ is defined by quadratic relations. So, any Koszul graded algebra is quadratic [58, chapter 1, Corollary 5.3].

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a quadratic algebra. Put $V = A_1$, and denote by $I \subset V \otimes_k V$ the kernel of the (surjective) multiplication map $A_1 \otimes_k A_1 \to A_2$. So, $V$ is the space of generators of the quadratic algebra $A$, and $I$ is the space of defining quadratic relations of $A$. The grading components of $A$ can be expressed in terms of $V$ and $I$ by the formula

$$A_n = V^{\otimes n} / \sum_{i=1}^{n-1} V^{\otimes i-1} \otimes_k I \otimes_k V^{\otimes n-i-1}, \quad n \geq 1.$$  

Consider the tensor coalgebra $\mathcal{J}(V)$, and denote by $C = A^\vee$ the graded subcoalgebra in $\mathcal{J}(V)$ with the components $C^0 = k$, $C^1 = V$, $C^2 = I$, and generally

$$C^n = \bigcap_{i=1}^{n-1} V^{\otimes i-1} \otimes_k I \otimes_k V^{\otimes n-i-1} \subset V^{\otimes n}, \quad n \geq 1.$$  

The coalgebra $C$ is called the quadratic dual coalgebra to a quadratic algebra $A$ [82, section 2.1].

Remark 2.2. In most expositions, including the present author’s [61] and [58], the quadratic duality is viewed as a contravariant functor constructed using the passage to the dual vector space.
The contravariant duality assigns to a quadratic algebra $A$ the quadratic algebra $A^!$, which can be obtained as the graded dual $k$-vector space to the coalgebra $C = A^*$, that is, $A^!_n = (C^n)^*$ for all $n \geq 0$. The construction of the quadratic algebra $A^!$ works well for quadratic algebras $A$ with finite-dimensional components (see, for example, [58, chapter 1, section 3]), and it can be made to work without the assumption of locally finite dimension by considering linearly compact topological vector spaces. Our general preference is to use coalgebras instead, and consider the covariant quadratic duality between algebras and coalgebras as constructed above. In particular, the intended applications in [82] were quadratic algebras with infinite-dimensional components; that is one reason why the covariant algebra–coalgebra quadratic duality was introduced in [82]. In the context of derived nonhomogeneous Koszul duality over a field, which is the subject of this survey, the language of covariant duality is more illuminating, in our view.

Let $0 = F_{-1}A \subset k = F_0A \subset F_1A \subset F_2A \subset \cdots$ be a $k$-algebra endowed with an increasing filtration $F$ such that $A = \bigcup_{n \geq 0} F_nA$, $1 \in F_0A$, and $F_nA \cdot F_mA \subset F_{n+m}A$ for all $n, m \geq 0$. Then the associated graded vector space $\text{gr}^FA = \bigoplus_{n=0}^{\infty} F_nA/F_{n-1}A$ carries a naturally induced associative algebra structure. A nonhomogeneous quadratic algebra is a filtered algebra $(A,F)$ such that the graded algebra $\text{gr}^FA$ is quadratic. A nonhomogeneous Koszul algebra is a filtered algebra $(A,F)$ such that the graded algebra $\text{gr}^FA$ is Koszul [83, section 2; 58, chapter 5].

**Example 2.3.**

1. Let $V$ be a $k$-vector space. Consider the graded algebra $A = \bigoplus_{n=0}^{\infty} A_n$ with the components $A_0 = k, A_1 = V$, and $A_n = 0$ for all $n \geq 2$. The multiplication on the graded algebra $A$ is trivial: the multiplication map $A_n \otimes A_m \rightarrow A_{n+m}$ vanishes for all $n, m > 0$ (in the cases when $n = 0$ or $m = 0$ the multiplication map is determined by the condition that $1 \in k = A_0$ is the unit element of $A$). The graded algebra $A$ is well-known to be Koszul (see, for example, [58, chapter 2, Corollary 2.4 or 4.3] for much more general results). In this example, the quadratic dual coalgebra $C = A^*$ is the whole tensor coalgebra, that is, $C = \mathcal{I}(V)$.

2. Any nonzero associative algebra $A$ can be endowed with the trivial filtration, $F_0A = k \cdot 1$ and $F_1A = A$. Then the associated graded algebra $\text{gr}^FA$ has grading components $\text{gr}^F_0A = k$, $\text{gr}^F_1A = A/(k \cdot 1)$, and $\text{gr}^F_nA = 0$ for $n \geq 2$. So, the algebra $\text{gr}^FA$ has the form of example (1) (with $V = A/(k \cdot 1)$), and the multiplication on it is trivial in the sense we have explained. Thus, the passage to the associated graded algebra of an algebra $A$ with respect to the trivial filtration $F$ destroys all the information about the multiplication in $A$.

It is clear from this discussion that any nonzero associative algebra $A$ endowed with the trivial filtration $F$ is a nonhomogeneous Koszul algebra. Some associative algebras $A$ admit nontrivial filtrations making them nonhomogeneous Koszul algebras with a more interesting associated graded algebra $\text{gr}^FA$ and a smaller quadratic dual coalgebra $C = (\text{gr}^FA)^!$ (cf. Example 2.5).

Let $(A,F)$ be a nonhomogeneous quadratic algebra and $C = (\text{gr}^FA)^!$ be the quadratic dual coalgebra to the quadratic algebra $\text{gr}^FA$. Assume additionally that the algebra $A$ is endowed with an augmentation $\alpha : A \rightarrow k$ with the augmentation ideal $A^+ \subset A$. Then there is a natural isomorphism $\text{gr}^FA = F_1A/F_0A \cong A^+ \cap F_1A \subset A$. So, the vector space $V = \text{gr}^FA$ can be viewed as a subspace in $A^+$, and the graded coalgebra $C$ can be viewed as a subcoalgebra in $\mathcal{I}(A^+)$,

$$C \subset \mathcal{I}(V) \subset \mathcal{I}(A^+) = \text{Bar}_\alpha(A).$$
One observes that the subcoalgebra $C \subset \text{Bar}_{\alpha}(A)$ is in fact a DG-subcoalgebra $C^* \subset \text{Bar}^*_{\alpha}(A)$, that is $\partial(C) \subset C$. Essentially, the first reason for that is because the multiplication map $A^+ \otimes_k A^+ \longrightarrow A^+$ restricted to the subspace $I \subset \text{gr}_1^A \otimes_k \text{gr}_1^A \cong (A^+ \cap F_1 A) \otimes_k (A^+ \cap F_1 A) \subset A^+ \otimes_k A^+$ lands within the subspace $A^+ \cap F_1 A \subset A^+$ (by the definition of $I$ as the kernel of the multiplication map $\text{gr}_1^A \otimes_k \text{gr}_1^A \longrightarrow \text{gr}_2^A = F_2 A / F_1 A$ in the graded algebra $\text{gr}^k A$). Further details need to be checked; we refer to [83, section 3], [61, Proposition 2.2] or [58, chapter 5, Proposition 4.1]. A generalization to nonaugmented algebras $A$ will be discussed in Example 6.11.

Let $(A, F, \alpha)$ be an augmented nonhomogeneous quadratic algebra and $M^*$ be a complex of left $A$-modules. Then one can easily see that $C \otimes_k M \subset \text{Bar}_{\alpha}(A, M)$ is a subcomplex in $\text{Bar}^*_{\alpha}(A, M)$, that is $\partial(C \otimes_k M) \subset C \otimes_k M$. In anticipation of the discussion in Section 5, we denote the complex $C \otimes_k M$ with the differential induced by the differential $\partial$ on $\text{Bar}_{\alpha}(A, M)$ by $C^* \otimes^\tau M^*$. This notation is intended to emphasize the fact that the differential on $C^* \otimes^\tau M^*$ is not simply the tensor product differential on the tensor product of two complexes $C^* \otimes_k M^*$, but rather some twisted version of it taking the action of $A$ in $M$ into account.

Let $(A, F, \alpha)$ be a nonhomogeneous Koszul algebra. Then the DG-coalgebra $C^*$ is quasi-isomorphic to the ambient DG-coalgebra $\text{Bar}_{\alpha}^*(A, M)$, and the complex $C^* \otimes^\tau M^*$ is quasi-isomorphic to the ambient bar-complex $\text{Bar}_{\alpha}^*(A, M)$. Accordingly, one has

$$H^{-n}(C^*) \cong \text{Tor}_n(k, k) \quad \text{and} \quad H^{-n}(C^* \otimes^\tau M) \cong \text{Tor}_n(k, M)$$

for any left $A$-module $M$ and all $n \geq 0$.

**Theorem 2.4.** Let $(A, F, \alpha)$ be an augmented nonhomogeneous Koszul algebra over a field $k$. Then the assignment $M^* \longmapsto C^* \otimes^\tau M^*$ induces a triangulated equivalence between the derived category of left $A$-modules $\mathcal{D}(A-\text{mod})$ and the coderived category of left DG-comodules over the DG-coalgebra $C^*$,

$$\mathcal{D}(A-\text{mod}) \cong \mathcal{D}_{\text{co}}(C^*-\text{comod}).$$

**Proof.** This is a particular case of [64, Theorems 6.5(a) and 6.6]. The inverse functor, $\mathcal{D}_{\text{co}}(C^*-\text{comod}) \longrightarrow \mathcal{D}(A-\text{mod})$, assigns to a left DG-comodule $N^*$ over $C^*$ the complex of left $A$-modules $A \otimes^\tau N^*$, which means the graded left $A$-module $A \otimes_k N$ endowed with a twisted differential, as explained below in Subsections 5.3 and 5.4. The definition of the coderived category will be given in Subsection 7.7.\qed

Any associative algebra $A$ can be viewed as a nonhomogeneous Koszul algebra with the trivial filtration, as explained in Example 2.3(2). For an augmented algebra $A$ endowed with the trivial filtration, one has $C^* = \text{Bar}^*_{\alpha}(A)$. So, Theorem 2.1 can be obtained as a particular case of Theorem 2.4 for the trivial filtration $F$.

For a generalization of Theorem 2.4, see Theorem 5.4.

**Example 2.5.** Let $\mathfrak{g}$ be a Lie algebra over a field $k$ and $\Lambda(\mathfrak{g})$ be the exterior algebra spanned by the $k$-vector space $\mathfrak{g}$. Then the Chevalley–Eilenberg differential $\partial$ on $\Lambda(\mathfrak{g})$ makes $\Lambda(\mathfrak{g})$ a complex
computing the homology spaces $H_\ast(\mathfrak{g}, k)$ of the Lie algebra $\mathfrak{g}$ with coefficients in the trivial $\mathfrak{g}$-module $k$ [17, chapter III], [45]. The differential $\partial$ does not respect the exterior multiplication on $\Lambda(\mathfrak{g})$, but it is an odd coderivation of the exterior comultiplication; so $C^\ast = (\Lambda(\mathfrak{g}), \partial)$ is a DG-coalgebra.

The enveloping algebra $U(\mathfrak{g})$ is endowed with the natural (Poincaré–Birkhoff–Witt) filtration $F$, making $U(\mathfrak{g})$ (the thematic example of) a nonhomogeneous Koszul algebra over a field. There is also a natural augmentation $\alpha$ on $U(\mathfrak{g})$, corresponding to the action of $\mathfrak{g}$ in trivial $\mathfrak{g}$-modules. The construction above assigns to the augmented nonhomogeneous quadratic algebra $(U(\mathfrak{g}), F, \alpha)$ the DG-coalgebra $(\Lambda(\mathfrak{g}), \partial)$.

Moreover, for any $\mathfrak{g}$-module $M$, the homological Chevalley–Eilenberg complex $(\Lambda(\mathfrak{g}) \otimes_k M, \partial)$ of the Lie algebra $\mathfrak{g}$ with coefficients in $M$ is a DG-comodule over the $(\Lambda(\mathfrak{g}), \partial)$. The construction above assigns the DG-comodule $C^\ast \otimes^\mathbb{L} M = (\Lambda(\mathfrak{g}) \otimes_k M, \partial)$ to the $U(\mathfrak{g})$-module $M$.

Thus, Theorem 2.4 claims that the construction of the homological Chevalley–Eilenberg complex induces an equivalence between the derived category of $\mathfrak{g}$-modules and the coderived category of DG-comodules over the DG-coalgebra $(\Lambda(\mathfrak{g}), \partial)$. For a finite-dimensional Lie algebra $\mathfrak{g}$, the coalgebra $\Lambda(\mathfrak{g})$ is finite-dimensional; so one can pass to the dual algebra $\Lambda(\mathfrak{g}^\ast)$ and say that the construction of the cohomological Chevalley–Eilenberg complex $(\Lambda(\mathfrak{g}^\ast) \otimes_k M, d)$ induces an equivalence between the derived category of $\mathfrak{g}$-modules and the coderived category of DG-modules over the DG-algebra $\Lambda^\ast(\mathfrak{g}^\ast) = (\Lambda(\mathfrak{g}^\ast), d)$,

$$D(\mathfrak{g} \text{-mod}) \simeq D^c(\Lambda^\ast(\mathfrak{g}^\ast) \text{-mod}).$$

This example can be found in [64, Example 6.6].

Let us emphasize that these assertions are certainly not true for the conventional derived category of DG-(co)modules in place of the coderived category. In fact, the derived category $D(\mathfrak{g} \text{-mod})$ is not equivalent to the derived category $D(\Lambda^\ast(\mathfrak{g}^\ast) \text{-mod})$ already when $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra over a field $k$ of characteristic 0, or indeed, a one-dimensional abelian Lie algebra (cf. Example 1.2(2)). We will continue this discussion in Example 7.3.

A relative version of Theorem 2.4, with the ground field $k$ replaced by an arbitrary associative ring $R$, can be found in the book [75]. See [75, sections 3.8 and 6.6].

### 2.6 Cobar-construction; DG-algebra and DG-module structures

Let $C$ be a (coassociative, counital) coalgebra over a field $k$, as defined in Subsection 2.1. A coaugmentation $\gamma$ on $C$ is a (counital) coalgebra homomorphism $\gamma : k \rightarrow C$. So, the composition $k \xrightarrow{\gamma} C \xrightarrow{\epsilon} k$ is the identity map. We denote by $C^+ = \text{coker}(\gamma)$ the cokernel of the augmentation map. Generally speaking, $C^+$ is a coassociative coalgebra without counit (dually to the augmentation ideal, which is an associative algebra without unit). The coalgebra homomorphism $\gamma : k \rightarrow C$ endows the one-dimensional $k$-vector space $k$ with left and right $C$-comodule structures.

The cobar-construction $\text{Cob}^\gamma_\ast(C)$ of a coaugmented coalgebra $C = (C, \gamma)$ is defined as the complex [20, chapter II], [1]

$$k \xrightarrow{0} C^+ \xrightarrow{\partial} C^+ \otimes_k C^+ \xrightarrow{\partial} C^+ \otimes_k C^+ \otimes_k C^+ \rightarrow \cdots$$
with the differential given by the formulae \( \delta(c) = \mu^+(c) \), \( \delta(c_1 \otimes c_2) = \mu^+(c_1) \otimes c_2 - c_1 \otimes \mu^+(c_2) \), ...

\[
\begin{aligned}
\delta(c_1 \otimes \cdots \otimes c_n) &= \mu^+(c_1) \otimes c_2 \otimes \cdots \otimes c_n - \\
&+ (-1)^{n-1} c_1 \otimes \cdots \otimes c_{i-1} \otimes \mu^+(c_i) \otimes c_{i+1} \otimes \cdots \otimes c_n + \cdots + (-1)^n c_1 \otimes \cdots \otimes c_{n-1} \otimes \mu^+(c_n),
\end{aligned}
\]

and so on, for all \( c, c_i \in C^+, 1 \leq i \leq n \), and \( n \geq 1 \). Here \( \mu^+ : C^+ \to C^+ \otimes_k C^+ \) is the comultiplication map on \( C^+ \) (induced by the comultiplication map \( \mu : C \to C \otimes_k C \)). The leftmost differential \( \delta : k \to C^+ \) is the zero map.

Let \((C, \gamma)\) be a coaugmented coalgebra and \( M \) be a left \( C \)-comodule. The cobar-construction \( \text{Cob}^\gamma(C, M) \) of \( C \) with the coefficients in \( M \) is the complex

\[
M \xrightarrow{\partial} C^+ \otimes_k M \xrightarrow{\partial} C^+ \otimes_k C^+ \otimes_k M \to \cdots
\]

with the differentials given by the formulae \( \partial(m) = \nu^+(m) \), \( \partial(c \otimes m) = \mu^+(c) \otimes m - c \otimes \nu^+(m) \), ...

\[
\begin{aligned}
\partial(c_1 \otimes \cdots \otimes c_n \otimes m) &= \mu^+(c_1) \otimes c_2 \otimes \cdots \otimes c_n \otimes m - \\
&+ (-1)^{n-1} c_1 \otimes \cdots \otimes c_{n-1} \otimes \mu^+(c_n) \otimes m + (-1)^n c_1 \otimes \cdots \otimes c_n \otimes \nu^+(m),
\end{aligned}
\]

and so on, for all \( c_i \in C^+, 1 \leq i \leq n \), and \( n \geq 0 \). Here \( \nu^+ : M \to C^+ \otimes_k M \) is the left \( C^+ \)-coaction map obtained by composing the left \( C \)-coaction map \( \nu : M \to C \otimes_k M \) with the natural surjection \( C \otimes_k M \to C^+ \otimes_k M \).

Dually to the discussion in Subsections 2.3 and 2.4, one observes that the underlying graded vector space \( \text{Cob}^\gamma(C) \) of the cobar-construction \( \text{Cob}^\gamma(C) \) coincides with the graded \( k \)-vector space \( \bigoplus_{n=0}^\infty C^+ \otimes^n \). Consequently, the graded vector space \( \text{Cob}^\gamma(C) \) has natural structures of an associative algebra and a coassociative coalgebra.

It turns out that the cobar differential \( \delta \) on \( \text{Cob}^\gamma(C) \) does not respect the comultiplication on \( \text{Cob}^\gamma(C) = T(C^+) \). However, the differential \( \delta \) is compatible with the multiplication on \( \text{Cob}^\gamma(C) = T(C^+) \), that is, it satisfies the Leibniz rule with signs

\[
\delta(ab) = \delta(a)b + (-1)^{|a|} a\delta(b)
\]

for any elements \( a \) and \( b \in T(C^+) \) of degrees \( |a| \) and \( |b| \). In other words, the complex \( \text{Cob}^\gamma(C) \) endowed with the graded algebra structure of \( T(C^+) \) is a DG-algebra over \( k \).

The graded algebra of cohomology of the DG-algebra \( \text{Cob}^\gamma(C) \) is naturally isomorphic to (the graded algebra with the opposite multiplication to) the Ext algebra of the left \( C \)-comodule \( k \),

\[
H^* \text{Cob}^\gamma(C) \approx \text{Ext}^*_C(k, k),
\]

where \( k \) is endowed with the left \( C \)-comodule structure via \( \gamma \) and the Ext is computed in the abelian category of left \( C \)-comodules. Similarly, for any left \( C \)-comodule \( M \), the graded module of cohomology of the DG-module \( \text{Cob}^\gamma(C, M) \) over the DG-algebra \( \text{Cob}^\gamma(C) \) (see the discussion in the next Subsection 2.7) is naturally isomorphic to the Ext module \( \text{Ext}^*_C(k, M) \) over the algebra...
2.7 Derived Koszul duality formulated for complexes of comodules

Let $N^*$ be a complex of left $C$-comodules. Then the cobar-construction $\text{Cob}_r^*(C, N^*)$ is a bicomplex of $k$-vector spaces. We totalize this bicomplex by taking infinite direct sums along the diagonals.

Denote simply by $N$ the underlying graded $C$-comodule of a complex of left $C$-comodules $N^*$. The underlying graded vector space $\text{Cob}_r(C, N)$ of the cobar-construction $\text{Cob}_r^*(C, N^*)$ can be viewed as a free graded left module $T(C^+) \otimes_k N$ over the tensor algebra $T(C^+)$. It turns out that the total differential on $\text{Cob}_r^*(C, N^*)$ is compatible with the cobar differential $\delta$ on the DG-algebra $\text{Cob}_r^*(C)$ and the left action of $\text{Cob}_r(C)$ on $\text{Cob}_r(C, N)$; that is, a Leibniz rule with signs similar to the one in Subsection 2.6 is satisfied for this action. In other words, the complex $\text{Cob}_r^*(C, N^*)$ endowed with the free graded module structure of $T(C^+) \otimes_k N$ is a left DG-module over the DG-algebra $\text{Cob}_r^*(C)$.

We formulate two versions of the derived Koszul duality for complexes of comodules, the non-conilpotent and the conilpotent one. The definitions of the coderived and the absolute derived categories will be given in Subsections 7.6 and 7.7.

**Theorem 2.6.** Let $(C, \gamma)$ be a coaugmented coassociative coalgebra over a field $k$. Then the assignment $N^* \mapsto \text{Cob}_r^*(C, N^*)$ induces a triangulated equivalence between the coderived category of left $C$-comodules $\mathcal{D}^{\text{co}}(C\text{-comod})$ and the absolute derived category of left DG-modules over the DG-algebra $\text{Cob}_r^*(C)$,

$$\mathcal{D}^{\text{co}}(C\text{-comod}) \simeq \mathcal{D}^{\text{abs}}(\text{Cob}_r^*(C)\text{-mod}).$$

**Proof.** This is a particular case of [64, Theorem 6.7(a)]. The absolute derived category $\mathcal{D}^{\text{abs}}(\text{Cob}_r^*(C)\text{-mod})$ coincides with the coderived category $\mathcal{D}^{\text{co}}(\text{Cob}_r^*(C)\text{-mod})$ by [64, Theorem 3.6(a)]; see Theorem 7.8(a). □

**Theorem 2.7.** Let $C$ be a conilpotent coassociative coalgebra over a field $k$, endowed with its unique coaugmentation $\gamma$. Then the same assignment $N^* \mapsto \text{Cob}_r^*(C, N^*)$ induces a triangulated equivalence between the coderived category of left $C$-comodules $\mathcal{D}^{\text{co}}(C\text{-comod})$ and the conventional derived category of left DG-modules over the DG-algebra $\text{Cob}_r^*(C)$,

$$\mathcal{D}^{\text{co}}(C\text{-comod}) \simeq \mathcal{D}(\text{Cob}_r^*(C)\text{-mod}).$$

**Proof.** This is a particular case of [64, Theorem 6.4(a)]; see [47, Theorem 2.2.2.2] and [42, section 4] for an earlier approach. It will be explained in Section 3 what a ‘conilpotent coalgebra’ is. □

In both the theorems, the inverse functor assigns to a left DG-module $M^*$ over the DG-algebra $A^* = \text{Cob}_r^*(C)$ the complex of left $C$-comodules $C \otimes^+ M^*$, which is a notation for the graded left $C$-comodule $C \otimes_k M$ endowed with a twisted differential, as constructed below in Subsections 5.3 and 5.4.
In both Theorems 2.6 and 2.7, the triangulated equivalence takes the irreducible left $C$-comodule $k$ (with the $C$-comodule structure defined in terms of the coaugmentation $\gamma$) to the free left DG-module $A^*$ over the DG-algebra $A^* = \text{Cob}_\gamma^\gamma(C)$. The same equivalences take the left DG-module $k$ over $A^*$ (with the $A$-module structure on $k$ defined in terms of the natural augmentation of $A^*$; see Subsection 4.4) to the cofree (injective) left $C$-comodule $C$.

For generalizations of Theorems 2.6 and 2.7, see Subsection 5.4.

*Remark 2.8.* One can obtain the definition of a coalgebra from the definition of an algebra over a field by inverting the arrows, but inverting the arrows in the formulation of Theorem 2.1 does not exactly produce the formulation of Theorem 2.6 or 2.7. The roles of algebras and coalgebras in derived nonhomogeneous Koszul duality are not symmetric up to inverting the arrows, as there are subtle differences between the constructions on the algebra and coalgebra side.

Of course, one can say that the associative algebras are the monoid objects in the monoidal (tensor) category of vector spaces, while the coassociative coalgebras are the monoid objects in the monoidal category opposite to vector spaces. Then the point is that the monoidal category of infinite-dimensional vector spaces is not equivalent to its opposite monoidal category, and moreover, the properties of these categories are different in a way important for nonhomogeneous Koszul duality.

To begin with, the tensor product of vector spaces preserves infinite direct sums, but not infinite products. Furthermore, the functor of tensor product of vector spaces (with one of the two objects fixed) has a right adjoint, but not a left adjoint functor. It is also relevant that the (filtered) direct limits of vector spaces are exact functors, but the inverse limits aren’t. These differences cause the lack of symmetry between the Koszul duality theorems formulated above and in Subsections 2.4 and 2.5.

In particular, it is illuminating to observe that the constructions of the DG-comodule $\text{Bar}_\alpha^\alpha(A, M^*)$ and the DG-module $\text{Cob}_\gamma^\gamma(C, N^*)$ are not transformed into one another by inverting the arrows, for the subtle reason that the direct sum (rather than the direct product) totalization of the respective bicomplex is used in both the constructions. The fact that the tensor products preserve the direct sums, but not the direct products of vector spaces explains this choice of the totalization procedures, but it breaks the symmetry between the algebra and coalgebra sides.

### 3 | STRUCTURE THEORY OF COALGEBRAS; CONILPOTENCY

All coassociative coalgebras over a field are unions of their finite-dimensional subcoalgebras. For this reason, the structure theory of coalgebras stands approximately on the same level of complexity as the structure theory of finite-dimensional algebras. So, it is much simpler than the structure theory of infinite-dimensional algebras and rings. Some details of this argument are elaborated upon in this section.

#### 3.1 | Brief remarks about coalgebras

The theory of coalgebras is notoriously counterintuitive. Most algebraists seem to find it difficult to invert the arrows properly in their mind in order to pass from the familiar ring theory to the coalgebra theory. One needs to get used to that.
Here we only mention a couple of obvious points. In ring theory, subrings are aplenty and quotient rings are rare. To pass to the quotient ring, one needs to have an ideal in the ring. All ideals are subalgebras (without unit), but most nonunital subalgebras are not ideals. Furthermore, all two-sided ideals are both left ideals and right ideals, but most left or right ideals are not (two-sided) ideals.

In coalgebras, the situation is reversed. A quotient coalgebra is the dual concept to a subalgebra, and a subcoalgebra is the dual concept to a quotient algebra. Quotient coalgebras are aplenty, but subcoalgebras are rare.

To pass to the quotient coalgebra, it suffices to have a coideal in the coalgebra. If one is satisfied with obtaining a noncounital quotient coalgebra, then the definition of a coideal reduces to this: a vector subspace $J$ in a coalgebra $C$ is said to be a coideal (in the noncounital sense) if $\mu(J) \subset J \otimes_k C + C \otimes_k J$ (where $\mu : C \to C \otimes_k C$ is the comultiplication map). The definition of a coideal in the counital sense includes an additional condition that the counit $\epsilon : C \to k$ must vanish on a coideal $J$.

For comparison, a subcoalgebra $D \subset C$ is a vector subspace such that $\mu(D) \subset D \otimes_k D$. One immediately observes that any subcoalgebra is a coideal (in the noncounital sense), but most coideals are not subcoalgebras. Furthermore, any left coideal (that is, a left subcomodule in the left $C$-comodule $C$) is a two-sided coideal, and any right coideal is a two-sided coideal (in the noncounital sense), but most (two-sided) coideals are neither left nor right coideals.

It is also worth mentioning that the comodule structures induced via homomorphisms of coalgebras are pushed forward rather than pulled back. So, if $f : C \to D$ is a homomorphism of coalgebras, then any (left or right) $C$-comodule acquires the induced structure of a left $D$-comodule. The resulting exact, faithful functor $C \text{-comod} \to D \text{-comod}$ on the abelian categories of comodules is called the corestriction of scalars with respect to $f$.

We refer to [88, section 1.4] for further details.

### 3.2 Local finite-dimensionality

The following classical result can be found in [88, Corollary 2.1.4 and Theorem 2.2.1].

**Lemma 3.1.**

(a) Any coassociative coalgebra $C$ is the (directed) union of its finite-dimensional subcoalgebras. The sum of any two finite-dimensional subcoalgebras in $C$ is a finite-dimensional subcoalgebra in $C$.

(b) Any (left) comodule over a coassociative coalgebra $C$ is the union of its finite-dimensional $C$-subcomodules.

(c) Any finite-dimensional $C$-comodule $M$ is a comodule over some finite-dimensional subcoalgebra $E \subset C$ (so the image of the coaction map $\nu : M \to C \otimes_k M$ is contained in $E \otimes_k M$).

**Proof.** Let us sketch a proof of the first assertion of part (a); the other assertions are similar or simpler. Given a coassociative, counital coalgebra $C$, consider an arbitrary finite-dimensional vector subspace $V \subset C$, and denote by $E \subset C$ the full preimage of $C \otimes_k V \otimes_k C$ under the iterated comultiplication map $\mu^{(2)} = (\mu \otimes \text{id}_C) \circ \mu = (\text{id}_C \otimes \mu) \circ \mu : C \to C \otimes_k C \otimes_k C$. Then $E = E(V)$ is a subcoalgebra in $C$, one has $E \subset V$ (so $E$ is finite-dimensional), and $C = \bigcup_{V \subset C} E(V)$. \qed
Specifying a coalgebra structure on a finite-dimensional $k$-vector space $E$ is equivalent to specifying an algebra structure on the dual vector space $E^*$. So, the dual vector space to a finite-dimensional coalgebra is a finite-dimensional algebra, and the dual vector space to a finite-dimensional algebra is a finite-dimensional coalgebra. For infinite-dimensional vector spaces the situation is more complicated: the dual vector space to a coalgebra has a natural algebra structure, but not vice versa.

### 3.3 Conilpotent coalgebras

A nonunital associative algebra $A$ is said to be nilpotent if there exists an integer $n \geq 1$ such that the product of any $n$ elements in $A$ vanishes, that is, $A^n = 0$. An augmented algebra $A$ is said to be nilpotent if its augmentation ideal $A^+$ is a nilpotent algebra without unit, that is, the product of any $n$ elements in $A^+$ vanishes.

Note that the augmentation of a nilpotent augmented algebra $A$ is unique. Since the product of any $n$ elements in $A^+$ vanishes, there are no nonzero $k$-algebra homomorphisms $A^+ \rightarrow k$; hence only one unital $k$-algebra homomorphism $A \rightarrow k$.

Coalgebras are better suited for nilpotency than algebras, in the sense that one can speak of a coalgebra being conilpotent without assuming existence of a related finite integer $n$. One can say that a (unital or nonunital) finite-dimensional coalgebra $E$ is conilpotent if its dual finite-dimensional algebra is nilpotent. Then an arbitrary coalgebra $C$ is said to be conilpotent if all its finite-dimensional subcoalgebras are conilpotent, that is, $C$ is a union of finite-dimensional conilpotent coalgebras. So, a finite-dimensional conilpotent coalgebra always has a finite conilpotency degree, but an infinite-dimensional conilpotent coalgebra need not have it.

Here is an explicit definition of a conilpotent coalgebra without counit that is easy to work with on the technical level. A noncounital coalgebra $D$ is said to be conilpotent if for every element $x \in D$ there exists an integer $n \geq 1$ such that the image of $x$ under the iterated comultiplication map $\mu^{(n)}: D \rightarrow D \otimes n+1$ vanishes. A coaugmented coalgebra $(C, \gamma)$ is called conilpotent if the noncounital coalgebra $C^+ = C / \gamma(k)$ is conilpotent in the sense of the previous definition. A conilpotent coaugmented coalgebra has a unique coaugmentation; moreover, any counital coalgebra homomorphism between two conilpotent coalgebras preserves the coaugmentations.

A conilpotent coalgebra $C$ comes endowed with a natural increasing filtration $F$. Namely, $F_{-1}C = 0, F_0C = \gamma(k) \cong k$, and for every $n \geq 0$ the subspace $F_nC \subset C$ consists of all elements $c \in C$ whose image in $(C^+)^{\otimes n+1}$ under the composition of maps $C \rightarrow C^{\otimes n+1} \rightarrow (C^+)^{\otimes n+1}$ vanishes. Then one has $\mu(F_nC) \subset \sum_{p+q=n} F_pC \otimes_k F_qC \subset C \otimes_k C$, so the filtration $F$ on $C$ is compatible with the comultiplication. The equation $C = \bigcup_{n \geq 0} F_nC$ expresses the condition that $C$ is conilpotent.

A noncounital conilpotent coalgebra $D$ comes endowed with a similarly defined natural increasing filtration $F$. Namely, one has $F_0D = 0$, and for every $n \geq 0$ the subspace $F_nD \subset D$ is the kernel of the iterated comultiplication map $\mu^{(n)}: D \rightarrow D^{\otimes n+1}$. Then one has $\mu(F_nD) \subset \sum_{p+q=n} F_pD \otimes_k F_qD \subset D \otimes_k D$ and $D = \bigcup_{n \geq 1} F_nD$.

**Remark 3.2.** Let $V$ be a $k$-vector space. In the discussion of the tensor algebra $T(V)$ and tensor coalgebra $J(V)$ in Subsection 2.3, it was mentioned that $T(V)$ is the free associative algebra spanned by $V$, but a discussion of a possible cofreeness property of the tensor coalgebra $J(V)$ was postponed. Now it is the time for such a discussion.
The first important observation is that the coaugmented coalgebra $\mathcal{J}(V)$ is always conilpotent. In fact, one has $F_n\mathcal{J}(V) = \bigoplus_{i=0}^{n} V^\otimes i$ for every $n \geq 0$. Moreover, $\mathcal{J}(V)$ is the cofree conilpotent coalgebra cospanned by $V$, in the following sense. Let $C$ be a conilpotent coalgebra, and let $f: C \rightarrow V$ be a $k$-linear map annihilating the coaugmentation, that is, such that the composition $f \gamma : k \rightarrow V$ vanishes. Then there exists a unique coalgebra homomorphism $t: C \rightarrow \mathcal{J}(V)$ whose composition with the direct summand projection $\mathcal{J}(V) \rightarrow V^\otimes 1 = V$ is equal to $f$.

Let us emphasize that the assertion above is not true for nonconilpotent coalgebras $C$. For such coalgebras, a homomorphism like $t$ does not exist in general.

### 3.4 Cosemisimple coalgebras and the coradical filtration

The classical structure theory of finite-dimensional associative algebras over a field tells that any such algebra $A$ has a nilpotent Jacobson radical $J$, while the quotient algebra $A/J$ decomposes uniquely as a finite product of simple algebras. The Jacobson radical $J \subset A$ is simultaneously the unique maximal nilpotent ideal and the unique minimal ideal in $A$ for which the quotient algebra $A/J$ is semisimple.

Moreover, if $f: A \rightarrow B$ is a surjective homomorphism of finite-dimensional $k$-algebras, then the Jacobson radicals $J(A) \subset A$ and $J(B) \subset B$ are well-behaved with respect to $f$ in the sense that $f(J(A)) = J(B)$. The induced surjective homomorphism of semisimple algebras $A/J(A) \rightarrow B/J(B)$ is a projection onto a direct factor; in fact, it represents the algebra $B/J(B)$ as a direct factor of the algebra $A/J(A)$ in a unique way.

Passing to the dual coalgebra, one obtains what can be called ‘the structure theory of finite-dimensional coassociative coalgebras’ $E$. Any such coalgebra has a unique maximal cosemisimple subcoalgebra $E^{ss}$, which decomposes uniquely as a direct sum of cosimple coalgebras. The (noncounital) quotient coalgebra $E/E^{ss}$ is conilpotent.

Moreover, if $E \subset D$ is a subcoalgebra in a finite-dimensional coalgebra, then $E^{ss} = E \cap D^{ss}$. The cosemisimple coalgebra $D^{ss}$ can be decomposed as a direct sum of its subcoalgebra $E^{ss}$ and a complementary subcoalgebra in a unique way.

Let us formulate the general definition of a cosemisimple coalgebra, in the form of a list of equivalent conditions in the next lemma. Here a nonzero coassociative, counital coalgebra over $k$ is called cosimple if it has no nonzero proper subcoalgebras.

### Lemma 3.3

For any coassociative, counital coalgebra $C$ over a field $k$, the following conditions are equivalent.

1. The abelian category of left $C$-comodules is semisimple.
2. The abelian category of right $C$-comodules is semisimple.
3. $C$ is the sum of its cosimple subcoalgebras.
4. $C$ uniquely decomposes as a direct sum of its (finite-dimensional) cosimple subcoalgebras, with all the cosimple subcoalgebras of $C$ appearing as direct summands in this direct sum decomposition.

Now let $C$ be an arbitrary (infinite-dimensional) coassociative coalgebra. Representing $C$ as the directed union of its finite-dimensional subcoalgebras $E$ and passing to the direct limit, one obtains the following structural result.
Proposition 3.4. Any coassociative, counital coalgebra $C$ over a field $k$ contains a unique maximal cosemisimple subcoalgebra $C^{ss}$, which is can be obtained as the sum of all co(semi)simple subcoalgebras of $C$. Simultaneously, $C^{ss}$ is the unique minimal subcoalgebra among all the subcoalgebras $D \subset C$ for which that the (noncounital) quotient coalgebra $C/D$ is conilpotent.

Following the discussion in Subsection 3.3, the conilpotent noncounital coalgebra $D = C/C^{ss}$ is endowed with a natural increasing filtration $F$. Denoting by $F_nC$ the full preimage of $F_n(C/C^{ss})$ under the natural surjective homomorphism $C \rightarrow C/C^{ss}$, one obtains a natural increasing filtration $F$ on $C$, called the coradical filtration. What we call the ‘maximal cosemisimple subcoalgebra’ $C^{ss}$ of a coalgebra $C$ is otherwise known as the coradical of $C$ [88, chapters VIII–IX], [49, chapter 5].

Remark 3.5. The definitions of the coradical filtration in [49, 88] are formulated in terms of the so-called wedge operation on vector subspaces of a coalgebra. Let $C$ be a coassociative coalgebra over $k$ and $X, Y \subset C$ be two vector subspaces. According to [88, section 9.0], [49, proof of Theorem 5.2.2], the wedge $X \wedge Y \subset C$ is defined as the kernel of the composition $C \rightarrow C \otimes_k C \rightarrow C/X \otimes_k C/Y$ of the comultiplication map $C \rightarrow C \otimes_k C$ with the natural surjection $C \otimes_k C \rightarrow C/X \otimes_k C/Y$. According to [88, Proposition 9.0.0(c)], the wedge operation is associative. By the definition, put $\wedge^1 X = X$ and $\wedge^n X = (\wedge^{n-1} X) \wedge X$ for $n \geq 2$.

Note that the definition of the wedge makes perfect sense for noncounital coalgebras $D$ as well. Arguing by induction on $n$, one easily shows that $\wedge^n X \subset D$ is the kernel of the composition $D \rightarrow D \otimes^n \rightarrow (D/X) \otimes^n$ of the iterated comultiplication map $D \rightarrow D \otimes^n$ with the natural surjection $D \otimes^n \rightarrow (D/X) \otimes^n$ (for every $n \geq 1$). Then, for a noncounital conilpotent coalgebra $D$, the definition of the natural increasing filtration $F$ on $D$ given in Subsection 3.3 can be rephrased as $F_nD = \wedge^{n+1} 0 \subset D$.

Let $p : C \rightarrow D$ be a homomorphism of noncounital coalgebras. Then one can easily see that the wedge operation commutes with taking the full preimages of vector subspaces under $p$: for any subspaces $X, Y \subset D$ one has $p^{-1}(X \wedge Y) = p^{-1}X \wedge p^{-1}Y$, hence $p^{-1}(\wedge^n X) = \wedge^n(p^{-1}X)$.

Now let $C$ be a coassociative, counital coalgebra over $k$. The coradical filtration is defined in [88, section 9.1] by the rule $C_n = \wedge^{n+1} C^{ss} \subset C$, where $C^{ss} \subset C$ is what we call the maximal cosemisimple subcoalgebra (called the coradical in [49, 88]). In the book [49, section 5.2], the coradical filtration is defined by induction using the rules $C_0 = C^{ss}$ and, essentially, $C_n = C^{ss} \wedge C_{n-1} \subset C$; this is clearly equivalent to the definition in [88]. Taking $D = C/C^{ss}$, one concludes that $C_n = \wedge^{n+1} C^{ss} = \wedge^{n+1} p^{-1}(0) = p^{-1}(\wedge^{n+1} 0) = p^{-1}(F_nD) = F_nC$ (where $p : C \rightarrow D$ is the natural surjection). Thus, the definition of the coradical filtration by the rule $C_n = \wedge^{n+1} C^{ss}$ in [49, 88] agrees with our definition of it by the rule $F_nC = p^{-1}(F_nD)$ given above in this section.

A counital coalgebra $C$ is conilpotent (in the sense of the definition in Subsection 3.3) if and only if there is an isomorphism of coalgebras $C^{ss} \simeq k$, or in other words, if and only if the coalgebra $C^{ss}$ is one-dimensional.

4 | DG-ALGEBRAS AND DG-COALGEBRAS

The aim of this section is to formulate a simple version of Koszul duality between algebras and coalgebras (rather than modules and comodules). The notion of a filtered quasi-isomorphism of DG-coalgebras [30] plays a key role.
4.1 Augmented DG-algebras

Let $A^* = (A, d)$ be a DG-algebra over a field $k$. So, $A = \bigoplus_{i \in \mathbb{Z}} A^i$ is a graded $k$-algebra and $d: A \longrightarrow A$ is a homogeneous $k$-linear map of degree 1 satisfying the Leibniz rule with signs with respect to the multiplication (in other words, $d$ is an odd derivation of $A$) such that $d^2 = 0$.

An augmentation $\alpha$ on $A^*$ is a (unital) DG-algebra homomorphism $\alpha: A^* \longrightarrow k$. So, $\alpha(1) = 1$, and $\alpha(d(a)) = 0$ for all $a \in A^{-1}$. We denote by $A^*: = \ker(\alpha) \subseteq A^*$ the augmentation ideal. So, $A^*: = \ker(\alpha)$ is a homogeneous two-sided DG-ideal in $A^*$, and $A^* = k \oplus A^*: = \ker(\alpha)$ as a complex of $k$-vector spaces.

The bar-construction $\text{Bar}^*_{\alpha}(A^*)$ of an augmented DG-algebra $(A^*, \alpha)$ is the bicomplex

$$
k \leftarrow A^*: = \ker(\alpha) \leftarrow A^*: \otimes_k A^*: \leftarrow A^*: \otimes_k A^*: \otimes_k A^*: \leftarrow \cdots$$

with the bar differential $\partial$ given by the formulae similar to those in Subsection 1.1 (except for the $\pm$ signs, which are more complicated in the case of a DG-algebra $A^*$) and the differential $d$ induced by the differential $d$ on $A^*$. The differential $d$ on $\text{Bar}^*_{\alpha}(A^*)$ acts on every tensor power $A^*: \otimes^n$ of the augmentation ideal $A^*: = \ker(\alpha)$ by the tensor product of $n$ copies of the differential $d$ on $A^*: = \ker(\alpha)$ (in the sense of the tensor product of complexes).

The total complex of the bar-construction $\text{Bar}^*_{\alpha}(A^*)$ is produced by taking infinite direct sums along the diagonals. The underlying graded vector space of $\text{Bar}^*_{\alpha}(A^*)$ is endowed with the graded coalgebra structure of the tensor coalgebra $\text{Bar}^*_{\alpha}(A) = T(A^* + 1)$. Here the shift of cohomological grading $1$ in the latter formula is a way to express the passage to the total complex of the bicomplex $\text{Bar}^*_{\alpha}(A^*)$, with its total grading equal to the grading induced by the grading of $A$ minus the grading by the number of the tensor factors. This makes $\text{Bar}^*_{\alpha}(A^*)$ a DG-coalgebra (as defined in Subsection 2.2). We refer to [64, section 6.1] for further details, including the (not so trivial) sign rules for the differentials $\partial$ and $d$.

4.2 Coaugmented DG-coalgebras

Let $C^* = (C, d)$ be a DG-coalgebra over $k$ (see section 2.2). A coaugmentation $\gamma$ on $C^*$ is a (counital) DG-coalgebra homomorphism $\gamma: k \longrightarrow C$. So, $\gamma(k) \subseteq C^0$, $\gamma$ is a homomorphism of coalgebras, the composition $k \longrightarrow C \longrightarrow k$ is the identity map, and $d(\gamma(1)) = 0$. We put $C^+: = \text{coker}(\gamma)$; so $C^+$ is, generally speaking, a DG-coalgebra without counit, and $C^* = k \oplus C^+ = \text{coker}(\gamma)$ as a complex of $k$-vector spaces.

The cobar-construction $\text{Cob}^*_{\gamma}(C^*)$ of a coaugmented DG-coalgebra $(C^*, \gamma)$ is the bicomplex

$$
k \longrightarrow C^+: = \text{coker}(\gamma) \longrightarrow C^+: \otimes_k C^+: \longrightarrow C^+: \otimes_k C^+: \otimes_k C^+: \longrightarrow \cdots$$

with the cobar differential $\partial$ given by the formulae similar to those in Subsection 2.6 (except for the $\pm$ signs) and the differential $d$ induced by the differential $d$ on $C^*$. The differential $d$ on $\text{Cob}^*_{\gamma}(C^*)$ acts on every tensor power $C^+: \otimes^n$ of the complex $C^+: = \text{coker}(\gamma)$ by the tensor product of $n$ copies of the differential $d$ on $C^+: = \text{coker}(\gamma)$.

The total complex of the cobar-construction $\text{Cob}^*_{\gamma}(C^*)$ is produced by taking infinite direct sums along the diagonals. The underlying graded vector space of $\text{Cob}^*_{\gamma}(C^*)$ is endowed with the graded algebra structure of the tensor algebra $\text{Cob}_\gamma(C) = T(C^* + [-1])$. Here the shift of cohomological grading $[-1]$ is a way to express the passage to the total complex of the bicomplex $\text{Cob}^*_{\gamma}(C^*)$, with
its total grading equal to the grading induced by the grading by the number of tensor factors. This makes \( \text{Cob}^\gamma(C^*) \) a DG-algebra. We refer to [64, section 6.1] for further details, including the sign rules for the differentials \( \partial \) and \( d \).

### 4.3 Conilpotent DG-coalgebras

A noncounital graded coalgebra \( D \) is called \textit{conilpotent} if its underlying noncounital ungraded coalgebra is conilpotent (see the definition in Subsection 3.3). A coaugmented graded coalgebra \( (C, \gamma) \) is called \textit{conilpotent} if its underlying coaugmented ungraded coalgebra is conilpotent; equivalently, this means that the noncounital graded coalgebra \( D = C/\gamma(k) \) is conilpotent.

A coaugmented DG-coalgebra \( (C^*, \gamma) \) is called \textit{conilpotent} if its underlying graded coalgebra \( C \) is conilpotent. In other words, a DG-coalgebra \( C^* \) is conilpotent if and only if the graded coalgebra \( C \) is conilpotent \textit{and} its coaugmentation \( \gamma : k \rightarrow C \) is a DG-coalgebra map, that is, \( d(\gamma(1)) = 0 \).

By construction, the canonical increasing filtration \( F \) on a conilpotent (noncounital or coaugmented) graded coalgebra is a filtration by homogeneous vector subspaces. The canonical increasing filtration \( F \) on a conilpotent DG-coalgebra \( C^* \) is a filtration by subcomplexes \( F_n C^* \subseteq C^* \).

### 4.4 Duality between DG-algebras and DG-coalgebras

Denote by \( k\text{-alg}_{\text{dg}} \) the category of DG-algebras over \( k \) (with the usual DG-algebra homomorphisms). Furthermore, let \( k\text{-alg}_{\text{aug}} \) denote the category of augmented DG-algebras (with DG-algebra homomorphisms preserving the augmentations).

Similarly, denote by \( k\text{-coalg}_{\text{dg}} \) the category of DG-coalgebras over \( k \), and by \( k\text{-coalg}_{\text{coaug}} \) the category of coaugmented DG-coalgebras. Let \( k\text{-coalg}_{\text{conilp}} \subseteq k\text{-coalg}_{\text{coaug}} \) denote the full subcategory whose objects are the conilpotent DG-coalgebras.

One easily observes that the cobar-construction \( \text{Cob}^\gamma(C^*) \) of any coaugmented DG-coalgebra \( C^* \) is naturally an augmented DG-algebra (with the direct summand projection \( \text{Cob}^\gamma(C) \rightarrow (C^+)^{\otimes 0} = k \) providing the augmentation). The bar-construction \( \text{Bar}^\gamma(A^*) \) of any augmented DG-algebra \( A^* \) is not only naturally coaugmented, but even a conilpotent DG-coalgebra (essentially, because the tensor coalgebra is conilpotent; see Remark 3.2).

So, the bar-construction is a functor

\[
\text{Bar}^\gamma : k\text{-alg}_{\text{aug}} \longrightarrow k\text{-coalg}_{\text{conilp}}
\]

while the cobar-construction is a functor

\[
\text{Cob}^\gamma : k\text{-coalg}_{\text{coaug}} \longrightarrow k\text{-alg}_{\text{aug}}
\]

\textbf{Lemma 4.1.} \textit{The restriction of the cobar-construction to conilpotent DG-coalgebras,} \( \text{Cob}^\gamma : k\text{-coalg}_{\text{conilp}} \longrightarrow k\text{-alg}_{\text{aug}} \)

\textit{is naturally a left adjoint functor to the bar-construction} \( \text{Bar}^\gamma : k\text{-alg}_{\text{aug}} \longrightarrow k\text{-coalg}_{\text{conilp}} \).
Proof. Essentially, the assertion holds because \( T(V) \) is the free (graded) algebra spanned by a (graded) vector space \( V \), while \( \mathcal{L}(V) \) is the cofree conilpotent (graded) coalgebra cospanned by \( V \). A more detailed explanation, based on the notion of a twisting cochain, will be offered in Subsection 5.2. □

One would like to define equivalence relations (that is, the classes of morphisms to be inverted) in the categories of augmented DG-algebras and conilpotent DG-coalgebras so that, after inverting these classes of morphisms, the adjoint functors \( \text{Bar}_\gamma \) and \( \text{Cob}_\gamma \) become equivalences of categories. One important obstacle is that it is not good enough to just invert the usual quasi-isomorphisms in both the categories \( k\text{-alg}^{\text{aug}}_{\text{dg}} \) and \( k\text{-coalg}^{\text{conilp}}_{\text{dg}} \), for the reason demonstrated by Example 1.1.

The problem is resolved by using the concept of a filtered quasi-isomorphism introduced by Hinich in [30, section 4]. In the terminology of [30], an admissible filtration on a coaugmented DG-coalgebra \( (C^*, \gamma) \) is an exhaustive comultiplicative increasing filtration by subcomplexes \( F_nC^* \) such that \( F_{-1}C^* = 0 \) and \( F_0C^* = \gamma(k) \). This means that \( C^* = \bigcup_{n=0}^{\infty} F_nC^* \) and \( \mu(F_nC^*) \subset \sum_{p+q=n} F_pC^* \otimes F_qC^* \) for every \( n \geq 0 \).

In particular, any coaugmented DG-coalgebra admitting an admissible filtration is conilpotent. Conversely, the canonical increasing filtration on any conilpotent DG-coalgebra is admissible.

Let \( f : C^* \longrightarrow D^* \) be a morphism of conilpotent DG-coalgebras. The morphism \( f \) is said to be a filtered quasi-isomorphism if there exist admissible filtrations \( F \) on both the DG-coalgebras \( C^* \) and \( D^* \) such that \( f(F_nC) \subset F_nD \) and the induced map \( F_nC^*/F_{n-1}C^* \longrightarrow F_nD^*/F_{n-1}D^* \) is a quasi-isomorphism of complexes of \( k \)-vector spaces for every \( n \geq 0 \).

Note that there is no reason to expect that the composition of filtered quasi-isomorphisms should be a filtered quasi-isomorphism. Nevertheless, leaving set-theoretical issues aside, to any class of morphisms \( S \) in a category \( C \) one can assign the category \( C[S^{-1}] \) obtained by formally inverting all morphisms from \( S \); in particular, one can formally invert all filtered quasi-isomorphisms of DG-coalgebras.

By contrast, a morphism of DG-algebras \( f : A \longrightarrow B \) is said to be a quasi-isomorphism if it is a quasi-isomorphism of the underlying complexes of vector spaces. Clearly, for any pair of composable morphisms of DG-algebras \( f \) and \( g \), if two of the morphisms \( f \), \( g \), and \( gf \) are quasi-isomorphisms, then so is the third one.

The following theorem goes back to [30, Theorem 3.2] and [47, Theorem 1.3.1.2]. In the stated form, it can be found in [64, Theorem 6.10(b)].

**Theorem 4.2.** Let \( \text{Quis} \) be the class of all quasi-isomorphisms of augmented DG-algebras and \( \text{FQuis} \) be the class of all filtered quasi-isomorphisms of conilpotent DG-coalgebras over \( k \). Then the adjoint functors \( \text{Bar}_\gamma \) and \( \text{Cob}_\gamma \) induce mutually inverse equivalences of categories

\[
\text{Bar}_\gamma : k\text{-alg}^{\text{aug}}_{\text{dg}}[\text{Quis}^{-1}] \simeq k\text{-coalg}^{\text{conilp}}_{\text{dg}}[\text{FQuis}^{-1}] : \text{Cob}_\gamma.
\]

**Remarks 4.3.** Clearly, any filtered quasi-isomorphism of DG-coalgebras is a quasi-isomorphism of their underlying complexes of vector spaces (that is, a quasi-isomorphism in the usual sense of the word). By the converse is not true.

The functor \( \text{Bar}_\gamma : k\text{-alg}^{\text{aug}}_{\text{dg}} \longrightarrow k\text{-coalg}^{\text{conilp}}_{\text{dg}} \) takes quasi-isomorphisms to filtered quasi-isomorphisms. In particular, it follows that it takes quasi-isomorphisms to quasi-isomorphisms.
The functor \( \text{Cob}_\gamma^* : k\text{-coalg}^{\text{conlp}}_{\text{dg}} \longrightarrow k\text{-alg}^{\text{aug}}_{\text{dg}} \) takes filtered quasi-isomorphisms to quasi-isomorphisms. This important property of filtered quasi-isomorphisms of DG-coalgebras is a part of Theorem 4.2.

However, the functor \( \text{Cob}_\gamma^* \) does not take quasi-isomorphisms to quasi-isomorphisms, generally speaking. For example, let \( f : (A, \alpha) \longrightarrow (B, \beta) \) be a morphism of augmented algebras (or augmented DG-algebras) as in Example 1.1, that is, \( f \) is not a quasi-isomorphism, but \( \text{Bar}^*(f) \) is. Put \( C^* = \text{Bar}^*_\alpha(A^*), D^* = \text{Bar}^*_\beta(B^*), \) and \( g = \text{Bar}^*(f) ; \) so \( g : (C^*, \gamma) \longrightarrow (D^*, \delta) \) is a quasi-isomorphism (but not a filtered quasi-isomorphism!) of conilpotent DG-coalgebras.

It is a part of Theorem 4.2 that the adjunction morphism \( \text{Cob}_\gamma^*(\text{Bar}^*_\alpha(E^*)) \longrightarrow E^* \) is a quasi-isomorphism of DG-algebras for any augmented DG-algebra \((E^*, \varepsilon)\). So, the adjunction morphisms \( \text{Cob}_\gamma^*(\text{Bar}^*_\alpha(A^*)) \longrightarrow A^* \) and \( \text{Cob}_\gamma^*(\text{Bar}^*_\beta(B^*)) \longrightarrow B^* \) are quasi-isomorphisms. It follows that \( \text{Cob}_\gamma^*(g) : \text{Cob}_\gamma^*(C^*) \longrightarrow \text{Cob}_\gamma^*(D^*) \) is not a quasi-isomorphism of DG-algebras.

### 4.5 Quillen equivalence between DG-algebras and DG-coalgebras

The derived Koszul duality and comodule-contramodule correspondence results can be usually expressed in the language of Quillen equivalences between model categories [64, section 8.4]. In particular, the Koszul duality between augmented DG-algebras and conilpotent DG-coalgebras stated in Theorem 4.2 can be expressed as a Quillen equivalence. Let us briefly sketch the related assertions.

We suggest the book [33] as a standard reference source on model categories and related concepts. The category \( k\text{-alg}^{\text{aug}}_{\text{dg}} \) of DG-algebras over \( k \) has a standard model category structure in which the quasi-isomorphisms are the weak equivalences and the surjective DG-algebra maps (that is, the DG-algebra maps that are surjective as maps of graded vector spaces) are the fibrations [29, 37]. It is clear from the explicit descriptions of the classes of cofibrations and trivial cofibrations in \( k\text{-alg}^{\text{aug}}_{\text{dg}} \), which can be found in the formulation of [64, Theorem 9.1(a)], that the model category \( k\text{-alg}^{\text{aug}}_{\text{dg}} \) is cofibrantly generated. The similar assertions hold for the category of augmented DG-algebras \( k\text{-alg}^{\text{aug}}_{\text{dg}} \) [64, Theorem 9.1(b)].

The category \( k\text{-coalg}^{\text{conlp}}_{\text{dg}} \) of conilpotent DG-coalgebras over \( k \) also has a natural model structure [30, Theorem 3.1], [47, Theorem 1.3.1.2(a)], [64, Theorem 9.3(b)]. The weak equivalences can be described as the morphisms which get inverted when one inverts the filtered quasi-isomorphisms. The cofibrations are the injective DG-coalgebra maps (that is, the DG-coalgebra maps that are injective as maps of graded vector spaces). Explicit descriptions of all the classes of morphisms involved can be found in the formulation of [64, Theorem 9.3(b)].

The model category \( k\text{-coalg}^{\text{conlp}}_{\text{dg}} \) is likewise cofibrantly generated. In fact, the injective morphisms of finite-dimensional DG-coalgebras are the generating cofibrations (as one can see from the fact that any DG-coalgebra is the directed union of its finite-dimensional DG-subcoalgebras; cf. Subsection 3.2). The injective filtered quasi-isomorphisms strictly compatible with the filtrations, acting between finite-dimensional DG-coalgebras, are the generating trivial cofibrations (this follows from the description of the trivial cofibrations as the retracts of the injective filtered quasi-isomorphisms strictly compatible with the filtrations, as per [64, Theorem 9.3(b)]).

The pair of adjoint functors in Lemma 4.1 and Theorem 4.2 is a Quillen equivalence between \( k\text{-alg}^{\text{aug}}_{\text{dg}} \) and \( k\text{-coalg}^{\text{conlp}}_{\text{dg}} \) [30, Theorem 3.2], [47, Theorem 1.3.1.2(b)], [64, end of section 9.3].
5 | TWISTING COCHAINS

The concept of a twisting cochain explains the adjunction of Lemma 4.1 and the constructions of inverse equivalences in Theorems 2.1, 2.4, 2.6, and 2.7. More generally, the notion of an acyclic twisting cochain, defined in this section, allows to conveniently formulate the Koszul duality theorems in wide contexts.

5.1 | The Hom DG-algebra and twisting cochains

Let $C$ be a coassociative coalgebra and $A$ be an associative algebra over a field $k$. Then the vector space $\text{Hom}_k(C, A)$ of all $k$-linear maps $C \to A$ acquires an associative $k$-algebra structure. Given two linear maps $f, g \in \text{Hom}_k(C, A)$, their product $fg \in \text{Hom}_k(C, A)$ is constructed as the composition

$$C \xrightarrow{\mu} C \otimes_k C \xrightarrow{f \otimes g} A \otimes_k A \xrightarrow{m} A,$$

where $\mu : C \to C \otimes_k C$ is the comultiplication and $m : A \otimes_k A \to A$ is the multiplication map. Given a counit $\varepsilon : C \to k$ on $C$ and a unit $e : k \to A$ in $A$, the unit element $1 \in \text{Hom}_k(C, A)$ is constructed as the composition $C \xrightarrow{\varepsilon} k \xrightarrow{e} A$.

Given two graded vector spaces $U$ and $V$, the graded Hom space $\text{Hom}_k(U, V)$ is endowed with a differential in the usual way, producing the complex $\text{Hom}_k(U, V)$. For any graded coalgebra $C$ and any graded algebra $A$, the graded Hom space $\text{Hom}_k(C, A)$ is a graded algebra over $k$.

Given two complexes of vector spaces $U \cdot$ and $V \cdot$, the graded Hom space $\text{Hom}_k(U, V)$ is endowed with a differential in the usual way, producing the complex $\text{Hom}_k(U \cdot, V \cdot)$. For any DG-coalgebra $C \cdot$ and a DG-algebra $A \cdot$ over $k$, the complex $\text{Hom}_k(C \cdot, A \cdot)$ is a DG-algebra over $k$.

Let $E \cdot = (E, d)$ be a DG-algebra. A homogeneous element $a \in E^1$ of degree 1 in $E$ is said to be a Maurer–Cartan element (or a Maurer–Cartan cochain) if it satisfies the Maurer–Cartan equation $a^2 + d(a) = 0$ in $E^2$.

Let $C^\cdot$ be a DG-coalgebra and $A^\cdot$ be a DG-algebra over $k$. The bar construction, as in Subsection 1.1, is given by $\text{Bar}_\alpha(A) = \bigoplus_{i \geq 0} \text{Hom}_k(A, A^i)$.

Various conditions of compatibility with (co)augmentations are usually imposed on twisting cochains. In particular, let $(C^\cdot, \gamma)$ be a coaugmented DG-coalgebra and $(A^\cdot, \alpha)$ be an augmented DG-algebra. Twisting cochains $\tau : C^\cdot \to A^\cdot$ such that $\alpha \circ \tau = 0 = \tau \circ \gamma$ will be important for us in this Section 5.

5.2 | Bar-cobar adjunction and acyclic twisting cochains

We start with some examples before passing to the general case.

Examples 5.1.

1. Let $(A, \alpha)$ be an augmented algebra over $k$, and let $C^\cdot = \text{Bar}_\alpha(A)$ be its bar-construction, as in Subsection 1.1. Let $\gamma : k \to C^\cdot$ be the natural coaugmentation of $\text{Bar}_\alpha(A)$ (as in Subsection 4.4). Then the composition $\tau$ of the direct summand projection $\text{Bar}_\alpha(A) \to A^+ \otimes^1 = A^+$...
and the inclusion $A^+ \to A$ is a twisting cochain for the DG-coalgebra $C^*$ and the algebra $A$ (viewed as a DG-algebra in the obvious way). The equations of compatibility with the augmentation and the coaugmentation $\alpha \circ \tau = 0 = \tau \circ \gamma$ are satisfied in this example.

(2) More generally, let $(A^*, \alpha)$ be an augmented DG-algebra, and let $C^* = \text{Bar}^*_\alpha(A^*)$ be its bar-construction, as in Subsection 4.1. Denote by $\gamma: k \to C^*$ the natural coaugmentation of $\text{Bar}^*_\alpha(A^*)$. Then the composition $\tau: C^* \to A^*$ of the direct summand projection $\text{Bar}^*_\alpha(A^*) \to A^* \otimes k^1 = A^{*+}$ and the inclusion $A^{*+} \to A^*$ is a twisting cochain for the DG-coalgebra $C^*$ and the DG-algebra $A^*$. The equations $\alpha \circ \tau = 0 = \tau \circ \gamma$ are satisfied for this twisting cochain.

Examples 5.2.

(1) Let $(C, \gamma)$ be a coaugmented coalgebra over $k$, and let $A^* = \text{Cob}^*_\gamma(C)$ be its cobar-construction, as in Subsection 2.6. Let $\alpha: A^* \to k$ be the natural augmentation of $\text{Cob}^*_\gamma(C)$ (as in Subsection 4.4). Then the composition $\tau: C^* \to A^*$ of the natural surjection $C \to C^{[1]}$ and the direct summand inclusion $C^{[1]} \to \text{Cob}^*_\gamma(C)$ is a twisting cochain for the coalgebra $C$ (viewed as a DG-coalgebra in the obvious way) and the DG-algebra $A^*$. The equations of compatibility with the augmentation and the coaugmentation $\alpha \circ \tau = 0 = \tau \circ \gamma$ are satisfied in this example.

(2) More generally, let $(C^*, \gamma)$ be a coaugmented DG-coalgebra, and let $A^* = \text{Cob}^*_\gamma(C^*)$ be its cobar-construction, as in Subsection 4.2. Denote by $\alpha: A^* \to k$ the natural augmentation of $\text{Cob}^*_\gamma(C^*)$. Then the composition $\tau: C^* \to A^*$ of the natural surjection $C^* \to C^{*[1]}$ and the direct summand inclusion $C^{*[1]} \to \text{Cob}^*_\gamma(C^*)$ is a twisting cochain for the DG-coalgebra $C^*$ and the DG-algebra $A^*$. The equations $\alpha \circ \tau = 0 = \tau \circ \gamma$ are satisfied for this twisting cochain.

Example 5.3. Let $(A, F, \alpha)$ be an augmented nonhomogeneous quadratic algebra, as defined in Subsection 2.5, and let $C^* \subset \text{Bar}^*_\alpha(A)$ be the related (nonhomogeneous quadratic dual) DG-subcoalgebra. Then the composition $C^* \to \text{Bar}^*_\alpha(A) \to A$ of the inclusion map $C^* \to \text{Bar}^*_\alpha(A)$ with twisting cochain $\text{Bar}^*_\alpha(A) \to A$ from Example 5.1 (1) is a twisting cochain $\tau: C^* \to A$. Equivalently, the twisting cochain $\tau$ can be constructed as the composition of the direct summand projection $C \to C^{[1]} = V = F_1A/F_0A$ and the inclusion $F_1A/F_0A \simeq A^+ \cap F_1A \to A$.

The natural coaugmentation $k \to \text{Bar}^*_\alpha(A)$ of the cobar DG-algebra $\text{Bar}^*_\alpha(A)$ factorizes as $k \to C^* \to \text{Bar}^*_\alpha(A)$, making $C^*$ a coaugmented (in fact, conilpotent) DG-coalgebra with the coaugmentation $\gamma: k \to C^*$. The twisting cochain $\tau: C^* \to A$ satisfies the equations $\alpha \circ \tau = 0 = \tau \circ \gamma$.

Now we can return to the proof of Lemma 4.1.

Proof of Lemma 4.1: Further details. Let $(A^*, \alpha)$ be an augmented DG-algebra and $(C^*, \gamma)$ be a conilpotent DG-coalgebra. Then the underlying graded coalgebra of the bar-construction $\text{Bar}^*_\alpha(A^*)$ is $\text{Bar}^*_\alpha(A) = J(A^+[1])$ and the underlying graded algebra of the cobar-construction $\text{Cob}^*_\gamma(C^*)$ is $\text{Cob}^*_\gamma(C) = T(C^+[-1])$.

Since $T(C^+[-1])$ is the free graded algebra spanned by $C^+[-1]$, graded algebra homomorphisms $\text{Cob}^*_\gamma(C) \to A$ correspond bijectively to homogeneous $k$-linear maps $C^+[-1] \to A$ of degree 0. Among these, the graded algebra homomorphisms compatible with the augmentations on $\text{Cob}^*_\gamma(C)$ and $A$ correspond precisely to the linear maps $C^+[-1] \to A^+$. This means elements of the vector space $\text{Hom}^1_k(C^+, A^+)$. 

Since $J(A^+[1])$ is the cofree conilpotent graded coalgebra cospanned by $A^+[1]$ and $C$ is a conilpotent graded coalgebra, graded coalgebra homomorphisms $C \to \text{Bar}_\alpha(A)$ correspond bijectively to homogeneous $k$-linear maps $C^+ \to A^+[1]$ of degree 0. This means elements of the same vector space $\text{Hom}_k^1(C^+, A^+)$. Finally, one has to check that a graded algebra homomorphism $\text{Cob}_\gamma(C) \to A^*$ commutes with the differentials on $\text{Cob}_\gamma(C^*)$ and $A^*$ if and only if the related element of $\text{Hom}_k^1(C^+, A^+) \subset \text{Hom}_k^1(C, A)$ satisfies the Maurer–Cartan equation. Similarly, a graded coalgebra homomorphism $C \to \text{Bar}_\alpha(A)$ commutes with the differentials on $C^*$ and $\text{Bar}_\alpha^*(A^*)$ if and only if the related element of $\text{Hom}_k^1(C^+, A^+) \subset \text{Hom}_k^1(C, A)$ satisfies the same Maurer–Cartan equation.

To sum up, both the augmented DG-algebra homomorphisms $\text{Cob}_\gamma(C^*) \to A^*$ and the (conilpotent) DG-coalgebra homomorphisms $C^* \to \text{Bar}_\alpha^*(A^*)$ correspond bijectively to twisting cochains $\tau : C^* \to A^*$ satisfying the equations of compatibility with the augmentation and the coaugmentation $\alpha \circ \tau = 0 = \tau \circ \gamma$.

Let $(A^*, \alpha)$ be an augmented DG-algebra, $(C^*, \gamma)$ be a conilpotent DG-coalgebra, and $\tau : C^* \to A^*$ be a twisting cochain satisfying the equations $\alpha \circ \tau = 0 = \tau \circ \gamma$. The twisting cochain $\tau$ is said to be acyclic if one of (or equivalently, both) the related DG-algebra homomorphism $\text{Cob}_\gamma(C^*) \to A^*$ and the DG-coalgebra homomorphism $C^* \to \text{Bar}_\alpha^*(A^*)$ become isomorphisms after the quasi-isomorphisms of DG-algebras, or, respectively, the filtered quasi-isomorphisms of conilpotent DG-coalgebras, are inverted, as in Theorem 4.2. Simply put, $\tau$ is called acyclic if the related homomorphism of DG-algebras $\text{Cob}_\gamma(C^*) \to A^*$ is a quasi-isomorphism.

For example, the twisting cochain from Examples 5.2 is acyclic, by the definition, whenever the coaugmented coalgebra $(C, \gamma)$ or the coaugmented DG-coalgebra $(C^*, \gamma)$ is conilpotent. It is a part of Theorem 4.2 that the twisting cochain from Examples 5.1 is acyclic for any augmented algebra $(A, \alpha)$ or an augmented DG-algebra $(A^*, \alpha)$.

Concerning the twisting cochain $\tau$ from Example 5.3, it is acyclic for any (augmented) nonhomogeneous Koszul algebra $(A, F)$. This claim is a corollary of the proof of Poincaré–Birkhoff–Witt theorem for nonhomogeneous quadratic/Koszul algebras in [61, sections 3.2 and 3.3] or [58, chapter 5, Proposition 7.2(ii)].

### 5.3 Twisted differential on the tensor product

Let $(A, \alpha)$ be an augmented algebra and $M^*$ be a complex of left $A$-modules. Let us make some basic observations about the bar-constructions $\text{Bar}_\alpha(A)$ and $\text{Bar}_\alpha(A, M^*)$, as defined in Subsections 1.1 and 1.3.

The underlying graded vector space $\text{Bar}_\alpha(A, M)$ of the complex $\text{Bar}_\alpha(A, M^*)$ is isomorphic to the tensor product $\text{Bar}_\alpha(A) \otimes_k M$ of the underlying graded vector spaces of the complexes $\text{Bar}_\alpha(A)$ and $M^*$. But the differential on $\text{Bar}_\alpha(A, M^*)$ is not the tensor product of the differentials on $\text{Bar}_\alpha(A)$ and $M^*$. Rather, in the formula for the differential of $\text{Bar}_\alpha(A, M^*)$, there are the summands coming from the differential of $\text{Bar}_\alpha(A)$, there is the summand coming from the differential of $M^*$, but there is also one additional summand, defined in terms of the action of $A$ in $M$.

Similarly, let $(C, \gamma)$ be a coaugmented coalgebra and $N^*$ be a complex of left $C$-comodules. Consider the cobar-constructions $\text{Cob}_\gamma(C)$ and $\text{Cob}_\gamma(C, N^*)$, as defined in Subsections 2.6 and 2.7.
The underlying graded vector space $\text{Cob}_{\gamma}(C, N)$ of the complex $\text{Cob}^*_\gamma(C, N^*)$ is isomorphic to the tensor product $\text{Cob}^*_\gamma(C) \otimes_k N$ of the underlying graded vector spaces of the complexes $\text{Cob}^*_\gamma(C)$ and $N^*$. But the differential on $\text{Cob}^*_\gamma(C, N^*)$ is not the tensor product of the differentials on $\text{Cob}^*_\gamma(C)$ and $N^*$. Rather, in the formula for the differential of $\text{Cob}^*_\gamma(C, N^*)$, there are the summands coming from the differential of $\text{Cob}^*_\gamma(C)$, there is the summand coming from the differential on $N^*$, but there is also one additional summand, defined in terms of the coaction of $C$ in $N$.

The construction of the twisted differential on the tensor product of a DG-module and a DG-comodule (twisted by a twisting cochain $\tau$) is abstracted from these observations. Let $A^*$ be a DG-algebra and $C^*$ be a DG-coalgebra over $k$. Let $\tau : C^* \longrightarrow A^*$ be a twisting cochain.

Let $M^* = (M, d_M)$ be a right DG-module over $A^*$, and let $N^* = (N, d_N)$ be a left DG-comodule over $C^*$. Consider the tensor product $M \otimes_k N$ of the underlying graded vector spaces of $M^*$ and $N^*$, and endow it with the differential given by the formula
\[
d(x \otimes y) = d_M(x) \otimes y + (-1)^{|x|} x \otimes d_N(y) + d^\tau(x \otimes y)
\]
for all homogeneous elements $x \in M$ and $y \in N$ of degrees $|x|$ and $|y|$. Here $d^\tau : M \otimes_k N \longrightarrow M \otimes_k N$ is the composition
\[
M \otimes_k N \longrightarrow M \otimes_k C \otimes_k N \longrightarrow M \otimes_k A \otimes_k N \longrightarrow M \otimes_k N
\]
of the map induced by the comultiplication map $N \longrightarrow C \otimes_k N$, the map induced by the twisting cochain $\tau : C \longrightarrow A$, and the map induced by the multiplication map $M \otimes_k A \longrightarrow M$. The reader can consult with [64, section 6.2] for the sign rule.

Then one can check that $d^2(x \otimes y) = 0$. So, the tensor product of graded vector spaces $M \otimes_k N$ endowed with the differential $d$ is a complex. We denote this complex by $M^* \otimes^\tau N^*$.

Similarly, let $M^* = (M, d_M)$ be a left DG-module over $A^*$, and let $N^* = (N, d_N)$ be a right DG-comodule over $C^*$. Consider the tensor product $N \otimes_k M$ of the respective underlying graded vector spaces, and endow it with the differential given by the formula
\[
d(y \otimes x) = d_N(y) \otimes x + (-1)^{|y|} y \otimes d_M(x) + d^\tau(y \otimes x),
\]
where $d^\tau : N \otimes_k M \longrightarrow N \otimes_k M$ is the composition
\[
N \otimes_k M \longrightarrow N \otimes_k C \otimes_k M \xrightarrow{\tau} N \otimes_k A \otimes_k M \longrightarrow N \otimes_k M.
\]

Once again, choosing the sign properly (cf. [64, section 6.2]), one can can check that $d^2(y \otimes x) = 0$. So, the tensor product $N \otimes_k M$ endowed with the differential $d$ is a complex. We denote this complex by $N^* \otimes^\tau M^*$.

### 5.4 Derived Koszul duality on the comodule side in the augmented case

Let $A^*$ be a DG-algebra and $C^*$ be a DG-coalgebra over $k$. Suppose that we are given a twisting cochain $\tau : C^* \longrightarrow A^*$.
Given a left DG-comodule $N^\ast$ over $C^\ast$, we consider the complex of vector spaces $A^\ast \otimes^\tau N^\ast$ constructed in Subsection 5.3. Then the structure of right DG-module over $A^\ast$ on $A^\ast$ has been eaten up in the construction of the twisted differential on the tensor product, but the left DG-module structure of $A^\ast$ over $A^\ast$ is inherited by the twisted tensor product, making $A^\ast \otimes^\tau N^\ast$ a left DG-module over $A^\ast$.

Similarly, given a left DG-module $M^\ast$ over $A^\ast$, we consider the complex of vector spaces $C^\ast \otimes^\tau M^\ast$ constructed in Subsection 5.3. Then the right DG-comodule structure over $C^\ast$ on $C^\ast$ has been consumed in the construction of the twisted differential on the tensor product, but the left DG-comodule structure of $C^\ast$ over $C^\ast$ is inherited by the twisted tensor product, making $C^\ast \otimes^\tau M^\ast$ a left DG-comodule over $C^\ast$.

Denoting by $A^\ast\text{-mod}$ the DG-category of left DG-modules over $A^\ast$ and by $C^\ast\text{-comod}$ the DG-category of left DG-comodules over $C^\ast$, one observes that the DG-functor

$$A^\ast \otimes^\tau - : C^\ast\text{-comod} \longrightarrow A^\ast\text{-mod}$$

is left adjoint to the DG-functor

$$C^\ast \otimes^\tau - : A^\ast\text{-mod} \longrightarrow C^\ast\text{-comod}.$$

Similarly to Subsection 2.7, we formulate two versions of the derived Koszul duality theorem, a conilpotent and a nonconilpotent one. Let $(A^\ast, \alpha)$ be an augmented DG-algebra and $(C^\ast, \gamma)$ be a coaugmented DG-coalgebra. Suppose that we are given a twisting cochain $\tau : C^\ast \longrightarrow A^\ast$ satisfying the equations $\alpha \circ \tau = 0 = \tau \circ \gamma$.

**Theorem 5.4.** In the context of the previous paragraph, assume further that $C^\ast$ is a conilpotent DG-coalgebra (as defined in Subsection 4.3) and $\tau$ is an acyclic twisting cochain (as defined in Subsection 5.2). Then the adjoint functors $M^\ast \mapsto C^\ast \otimes^\tau M^\ast$ and $N^\ast \mapsto A^\ast \otimes^\tau N^\ast$ induce a triangulated equivalence between the conventional derived category of left DG-modules over $A^\ast$ and the coderived category of left DG-comodules over $C^\ast$,

$$\text{D}(A^\ast\text{-mod}) \simeq \text{D}^{co}(C^\ast\text{-comod}).$$

**Proof.** This is a particular case of [64, Theorem 6.5(a)]; see [47, Theorem 2.2.2.2] and [42, section 4] for an earlier approach. The definition of the coderived category will be explained below in Subsection 7.7. \[\square\]

**Theorem 5.5.** Let $(C^\ast, \gamma)$ be a coaugmented DG-coalgebra, and let $\tau : C^\ast \longrightarrow \text{Coh}^\ast(C^\ast) = A^\ast$ be the twisting cochain constructed in Example 5.2(2). Then the adjoint functors $M^\ast \mapsto C^\ast \otimes^\tau M^\ast$ and $N^\ast \mapsto A^\ast \otimes^\tau N^\ast$ induce a triangulated equivalence between the absolute derived category of left DG-modules over $A^\ast$ and the coderived category of left DG-comodules over $C^\ast$,

$$\text{D}^{\text{abs}}(A^\ast\text{-mod}) \simeq \text{D}^{co}(C^\ast\text{-comod}).$$

**Proof.** This is a particular case of [64, Theorem 6.7(a)]. For the definitions of the coderived and absolute derived categories, see Subsections 7.6 and 7.7. The absolute derived category $\text{D}^{\text{abs}}(A^\ast\text{-mod})$ coincides with the coderived category $\text{D}^{\text{co}}(A^\ast\text{-mod})$ by [64, Theorem 3.6(a)]; see Theorem 7.8(a). \[\square\]
In both Theorems 5.4 and 5.5, the triangulated equivalence takes the left DG-module $k$ over $A^*$ (with the $A$-module structure on $k$ defined in terms of the augmentation $\alpha$) to the cofree left DG-comodule $C^*$ over $C^*$. The same equivalences take the free left DG-module $A^*$ over $A^*$ to the left DG-comodule $k$ over $C^*$ (with the $C$-comodule structure on $k$ defined in terms of the coaugmentation $\gamma$).

For generalizations of Theorems 5.4 and 5.5, see Subsection 6.9.

### 6 CDG-RINGS AND CDG-COALGEBRAS

Under Koszul duality, lack of chosen (co)augmentation on one side corresponds to curvature on the other side. For example, to a nonaugmented algebra a curved DG-coalgebra is assigned. Moreover, a change of (co)augmentation corresponds to a change of flat connection, or in other words, to a Maurer–Cartan twist of the DG-structure on the other side. The aim of this section is to extend the Koszul duality theorems of Subsections 4.4 and 5.4 to the nonaugmented context.

#### 6.1 Posing the problem of nonaugmented Koszul duality

Suppose that we are given an associative algebra $A$, and we would like to compute the derived category of modules over it, in terms of some kind of Koszul duality. Following the approach of Section 1 and Subsection 2.4, we have to choose an augmentation of $A$ and produce the bar-construction $C^* = \text{Bar}^\alpha_\alpha(A)$ using an augmentation $\alpha : A \to k$.

But an augmentation does not seem to be relevant to the problem of describing $A$-modules or complexes of $A$-modules. What if $A$ does not admit an augmentation? A $k$-algebra homomorphism $A \to k$ does not always exist. Or if $A$ has many possible augmentations, what is the point of choosing one of them?

To be sure, the bar-complex $\text{Bar}^\alpha_\alpha(A, M)$ computes the vector spaces $\text{Tor}^A_n(k, M)$ for a given $A$-module $M$, as mentioned in Subsection 1.2. But one can have $\text{Tor}^A_n(k, M) = 0$ for all $n \in \mathbb{Z}$, while $M \neq 0$, as per Examples 1.2. This is the reason why the coderived category of DG-comodules, rather than the conventional derived category, appears in Theorem 2.1. What is the point of choosing an $A$-module structure on $k$ and considering the homological functor $\text{Tor}^A_*(k, -)$, only to discover that this functor annihilates a big part of the desired derived category $\mathcal{D}(A\text{-mod})$?

The approach worked out in [64] and going back to [61] can be briefly stated as follows. Instead of looking for an augmentation $\alpha : A \to k$, let us choose an arbitrary $k$-linear map $\nu : A \to k$ for which $\nu(1) = 1$. Surely any nonzero $k$-algebra $A$ admits plenty of such maps $\nu$.

Let us extend the bar-construction $\text{Bar}^\alpha_\alpha(A, M)$ to the context of arbitrary $k$-linear maps $\nu$ as above. Then the resulting object $\text{Bar}^\nu_\nu(A, M)$ is not a DG-algebra, and not even a complex at all. But it is a graded coalgebra $C$ endowed with an odd coderivation $\partial : C^n \to C^{n+1}$ with a nonzero square. The square of the differential $\partial$ is described as the commutator with the curvature linear function $h : C^{-2} \to k$.

The resulting algebraic object $C^* = (C, \partial, h)$ is called a curved DG-coalgebra, or a CDG-coalgebra for brevity. To any $A$-module $M$, we assign a CDG-comodule $\text{Bar}^\nu_\nu(A, M)$ over $C^*$; and similarly to any complex of $A$-modules $M^*$. The differential on $\text{Bar}^\nu_\nu(A, M)$ does not square to zero; so the homology spaces of $\text{Bar}^\nu_\nu(A, M)$ (as well as of $\text{Bar}^\nu_\nu(A)$) are undefined. But we have already decided that we are not too much interested in the Tor spaces $\text{Tor}^A_*(k, M)$ anyway.
Replacing the $k$-linear map $v : A \rightarrow k$ by another such map $v' : A \rightarrow k$, satisfying the same condition $v'(1) = 1$, leads to a CDG-coalgebra $(C, d', h')$ naturally isomorphic to $(C, d, h)$. The DG-categories of CDG-comodules over $(C, d', h')$ and $(C, d, h)$ are isomorphic.

Finally, though one cannot speak of quasi-isomorphisms of CDG-comodules in the conventional sense of the word, and therefore the conventional derived category of CDG-comodules over $(C, d, h)$ is undefined, the definition of the coderived category makes perfect sense for CDG-comodules. The problem of computing the derived category of $A$-modules is solved by constructing a triangulated equivalence between the derived category $D(A-\text{mod})$ and the coderived category $D^{co}(C^* - \text{comod})$ of CDG-comodules over $C^* = (C, d, h)$.

Unrelated to the Koszul duality theory, curved DG-modules also appear in the literature in connection with the popular topic of matrix factorizations (which are rather special particular cases of CDG-modules) [14, 24, 40, 56]. The coderived and absolute derived categories are an important technical tool in the matrix factorization theory [2, 19, 57, 59, 77]. (Weakly) curved $A_\infty$-algebras play a fundamental role in the Fukaya theory [18, 26, 27, 65].

6.2 CDG-rings and CDG-modules

The following definitions go back to the paper [61]. The terminology ‘curvature’ and ‘connection’ comes from an analogy with the respective concepts from differential geometry, based on examples from differential geometry [61, section 4], [75, sections 10.2–10.8]. The latter class of examples, namely, the duality between the rings of differential operators and (curved) DG-rings of differential forms, is an instance of a more complicated relative version of nonhomogeneous Koszul duality [39; 6, section 7.2; 63, section 0.4 and chapter 11; 64, appendix B], [75], which falls outside of the scope of this survey. We refer to the book [75] for a definitive treatment.

A curved DG-ring (CDG-ring) $B^* = (B, d, h)$ is a graded ring $B = \bigoplus_{i \in \mathbb{Z}} B^i$ endowed with the following data.

- $d : B \rightarrow B$ is an odd derivation of degree 1, that is, for every $i \in \mathbb{Z}$ an additive map $d_i : B^i \rightarrow B^{i+1}$ is given such that the Leibniz rule with signs

$$d(bc) = d(b)c + (-1)^{|b|}bd(c)$$

is satisfied for all homogeneous elements $b$ and $c \in B$ of degrees $|b|$ and $|c|$.
- $h \in B^2$ is an element.

The following axioms relating $d$ and $h$ must be satisfied.

(i) The square of the differential $d$ on $B$ is described by the formula $d^2(b) = hb - bh$ for all $b \in B$.
(ii) $d(h) = 0$.

The element $h \in B^2$ is called the curvature element.

A DG-ring is the same thing as a CDG-ring with $h = 0$. The category of DG-rings is a subcategory, but not a full subcategory of the category of CDG-rings; the passage from the uncurved to the curved DG-rings involves not only adding new objects to the category, but also adding new morphisms between previously existing objects. Let us define the morphisms of CDG-rings now.

Let $B^* = (B, d_B, h_B)$ and $A^* = (A, d_A, h_A)$ be two CDG-rings. A morphism of CDG-rings $B^* \rightarrow A^*$ is a pair $(f, a)$, where
• $f : B \longrightarrow A$ is a homomorphism of graded rings;

• $a \in A^1$ is an element

such that

(iii) $f(d_B(z)) = d_A(f(z)) + [a, f(z)]$ for all $z \in B$ (where the graded commutator $[−, −]$ is defined by the usual rule $[x, y] = xy - (-1)^{|x||y|}yx$ for all homogeneous elements $x$ and $y$ of degrees $|x|$ and $|y|$).

(iv) $f(h_B) = h_A + d_A(a) + a^2$.

The element $a \in A^1$ is called the change-of-connection element.

The composition of two morphisms of CDG-rings $(C, d_C, h_C) \longrightarrow (B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is given by the rule $(f, a) \circ (g, b) = (f \circ g, a + f(b))$. The identity morphism $(B, d_B, h_B) \longrightarrow (B, d_B, h_B)$ is the morphism $(\text{id}_B, 0)$.

Morphisms of CDG-rings $(\text{id}_B, a) : (B, d', h') \longrightarrow (B, d, h)$ are called the change-of-connection morphisms. All such morphisms of CDG-rings are isomorphisms. Moreover, for any CDG-ring $(B, d, h)$ and an element $a \in B^1$ there exists a unique CDG-ring structure $(B, d', h')$ on the graded ring $B$ such that $(\text{id}_B, a) : (B, d', h') \longrightarrow (B, d, h)$ is a (change-of-connection iso)morphism of CDG-rings. The twisted differential and curvature element $d' : B \longrightarrow B$ and $h' \in B^2$ are given by the formulae $d'(z) = d(z) + [a, z]$ and $h' = h + d(a) + a^2$.

In particular, one can assume that $h = 0$; so $(B, d)$ is a DG-ring. Then one has $h' = 0$ (that is, the pair $(B, d')$ is a DG-ring again) if and only if the equation $d(a) + a^2 = 0$ is satisfied, that is, $a \in B^1$ is a Maurer–Cartan element (in the sense of Subsection 5.1). So, any DG-ring structure can be twisted by any Maurer–Cartan cochain, producing a new DG-ring structure on the same graded ring. The resulting DG-ring $(B, d')$ is naturally isomorphic to the original DG-ring $(B, d)$ as a CDG-ring, but not as a DG-ring.

The following definition of a CDG-module can be found in [58, chapter 5, section 4] or [64, section 3.1]. Let $B^* = (B, d, h)$ be a CDG-ring. A left CDG-module $M^* = (M, d_M)$ over $(B, d, h)$ is a graded left $B$-module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ endowed with the following datum:

• $d_M : M \longrightarrow M$ is an odd derivation of degree 1 compatible with the odd derivation $d$ on $B$, that is, for every $i \in \mathbb{Z}$ an additive map $d_{M,i} : M_i \longrightarrow M_{i+1}$ is given such that the Leibniz rule with signs

\[ d_M(bx) = d(b)x + (-1)^{|b||x|}bd_M(x) \]

is satisfied for all homogeneous elements $b \in B$ and $x \in M$ of degrees $|b|$ and $|x|$.

The following axiom must be satisfied.

(v) The square of the differential $d_M$ on $M$ is described by the formula $d_M^2(x) = hx$ for all $x \in M$.

Similarly, a right CDG-module $N^* = (N, d_N)$ over $(B, d, h)$ is a graded right $B$-module $N = \bigoplus_{i \in \mathbb{Z}} N_i$ endowed with the following datum:

• $d_N : N \longrightarrow N$ is an odd derivation of degree 1 compatible with the odd derivation $d$ on $B$, that is, $d_N$ acts on the grading components of $N$ as $d_{N,i} : N_i \longrightarrow N_{i+1}$ and the Leibniz rule with signs

\[ d_N(yb) = d_N(y)b + (-1)^{|y||b|}yd(b) \]

is satisfied for all homogeneous elements $b \in B^{|b|}$ and $y \in N^{|y|}$.
The following axiom is imposed.

(vi) The square of the differential $d_N$ on $N$ is given by the formula $d_N^2(y) = -yh$ for all $y \in N$.

Comparing the equation for the square of the differential in (i) with (v) and (vi), one observes that they do not agree. So, there is no natural left (or right) CDG-module structure on the underlying graded ring $B$ of a CDG-ring $(B, d, h)$. However, any CDG-ring $B^*$ is naturally a CDG-bimodule over itself, in the sense of the definition in [64, section 3.10] or [75, section 6.1].

To any two (say, left) CDG-modules $L^* = (L, d_L)$ and $M^* = (M, d_M)$ over a CDG-ring $(B, d, h)$, one can assign the complex of morphisms $\text{Hom}_B(L^*, M^*)$ from $L$ to $M$. The degree $n$ component $\text{Hom}_B(L, M)$ of the complex $\text{Hom}_B(L^*, M^*)$ is the group of all homogeneous $B$-linear maps $L \rightarrow M$ of degree $n$ (see [64, sections 1.1 and 3.1] for the sign rule involved). The differential $d$ on the complex $\text{Hom}_B(L^*, M^*)$ is defined by the usual rule $d(f)(l) = d_M(f(l)) - (-1)^{|f|}f(d_L(l))$ for all $l \in L$.

The rule above is well-known to define a differential with zero square on the graded abelian group of homogeneous morphisms between two DG-modules. It turns out that for two CDG-modules over a CDG-ring, the same formula still defines a differential with zero square, as two curvature-related terms in the computation of $d^2(f)$ cancel each other. Consequently, there is the DG-category of left CDG-modules over $B^* = (B, d, h)$, which we denote simply by $B^* \text{-mod}$.

Given a morphism of CDG-rings $(f, a) : (B, d_B, h_B) \rightarrow (A, d_A, h_A)$ and a left CDG-module $(M, d_M)$ over $(A, d_A, h_A)$, one can endow the graded left $A$-module $M$ with the induced structure of graded left $B$-module and a twisted differential $d'_M$, making $(M, d'_M)$ a left CDG-module over $(B, d_B, h_B)$. The twisted differential $d'_M : M \rightarrow M$ is defined by the formula $d'_M(x) = d_M(x) + ax$ for all $x \in M$. Similarly, for a right CDG-module $(N, d_N)$ over $(A, d_A, h_A)$, the twisted differential $d'_N$ defined by the rule $d'_N(y) = d_N(y) - (-1)^{|y|}ya$ for all $y \in N$ makes $(N, d'_N)$ a right CDG-module over $(B, d_B, h_B)$.

In particular, the DG-categories of CDGmodules over two isomorphic CDG-rings are naturally isomorphic. Specializing to CDG-isomorphisms (or in other words, change-of-connection isomorphisms) of DG-rings, we come to the (perhaps not too widely known) observation that a Maurer–Cartan twist of the differential on a DG-ring does not change the DG-category of DG-modules. The reason is that the differentials on DG-modules $(M, d_M)$ over a DG-ring $(B, d)$ can be twisted (by a Maurer–Cartan cochain $a \in B^1$) alongside with a twist $d \rightsquigarrow d' = d + [a, -]$ of the differential on $B$ (according to the rule above, $d'_M(x) = d_M(x) + ax$).

### 6.3 CDG-coalgebras and CDG-comodules

A CDG-algebra $(B, d, h)$ over a field $k$ is a graded (associative, unital) $k$-algebra endowed with a CDG-ring structure with a $k$-linear differential $d$. The definitions of a CDG-coalgebra $(C, d, h)$ and a CDG-comodule $(M, d_M)$ over it are obtained from those of a CDG-algebra and a CDG-module by inverting the arrows. Let us spell out some details.

Partly following the terminology of [44], let us utilize the term precomplex for a graded vector space $K$ endowed with a $k$-linear differential $d : K \rightarrow K$ of degree 1 with not necessarily zero square. A CDG-coalgebra $(C, d, h)$ and a CDG-comodule $(M, d_M)$ are, first of all, precomplexes: both $d : C \rightarrow C$ and $d_M : M \rightarrow M$ are homogeneous $k$-linear maps of degree 1 with, generally speaking, nonzero squares.

Given two precomplexes $V^* = (V, d_V)$ and $W^* = (W, d_W)$, one defines their tensor product $V^* \otimes_k W^* = (V \otimes_k W, d)$ as follows. The graded vector space $V \otimes_k W$ is simply the tensor prod-
uct of the graded vector spaces $V$ and $W$. The differential $d$ on $V \otimes_k W$ is given by the usual rule $d(v \otimes w) = d_V(v) \otimes w + (-1)^{|v|} v \otimes d_W(w)$ for $v \in V^{|v|}$ and $w \in W^{|w|}$.

Now we can define the notions of coderivations on coalgebras and comodules. Let $C$ be a graded coalgebra over $k$. An odd coderivation (of degree 1) on $C$ is a homogeneous $k$-linear map $d : C \rightarrow C$ of degree 1 such that the comultiplication map $\mu : C \rightarrow C \otimes_k C$ is a morphism of precomplexes (that is, $\mu$ commutes with the differentials). Here the differential $d$ on $C \otimes_k C$ is defined by the rule above. One can check that any odd coderivation $d$ on $C$ is compatible with the counit, in the sense that the counit $\varepsilon : C \rightarrow k$ is also a morphism of precomplexes (where the differential on $k$ is zero).

Let $(C, d)$ be a graded coalgebra endowed with an odd coderivation of degree 1, and let $M$ be a graded left $C$-comodule. Then an odd coderivation on $M$ compatible with the coderivation $d$ on $C$ is a $k$-linear map $d_M : M \rightarrow M$ of degree 1 such that the left coaction map $\nu : M \rightarrow C \otimes_k M$ is a morphism of precomplexes. Here, once again, the differential $d$ on $C \otimes_k M$ is defined by the rule above in terms of the differentials $d$ on $C$ and $d_M$ on $M$.

To spell out the rules for the squares of the differentials $d$ and $d_M$ on a CDG-coalgebra $C$ and a CDG-comodule $M$, we need to have a brief preliminary discussion of algebras dual to coalgebras and their actions in comodules. It was already mentioned in Subsection 3.2 that the dual vector space to any coalgebra is an algebra. Similarly, the graded dual vector space $C^*$ to a graded coalgebra $C$ is a graded algebra. Furthermore, any graded left $C$-comodule $M$ can be endowed with a graded $C^*$-module structure with the action map defined as the composition

$$C^* \otimes_k M \rightarrow C^* \otimes_k C \otimes_k M \rightarrow M$$

of the map induced by the left coaction map $\nu : M \rightarrow C \otimes_k M$ and the map induced by the pairing map $C^* \otimes_k C \rightarrow k$.

Let us pause at this point, however, and observe that for any algebra $A$ there is the opposite algebra $A^{op}$. Which one of the two opposite multiplications on $C^*$ should one choose and use as the canonical choice of the multiplication on the dual vector space to a coalgebra? The traditional way of making this choice [88, section 2.1] results in left $C$-comodules becoming right $C^*$-modules and vice versa. Our usual preference is to choose the sides so that left $C$-comodules become left $C^*$-modules and right $C$-comodules become right $C^*$-modules (cf. [71, beginning of section 1.4]).

With these observations in mind, let us introduce some notation. Given a (graded) coalgebra $C$ and two (homogeneous) $k$-linear functions $a, b : C \rightarrow k$ (of some degrees $|a|, |b| \in \mathbb{Z}$), we denote by $a \cdot b : C \rightarrow k$ the product of $a$ and $b$ in the graded algebra $C^*$. Given a graded left comodule $M$ over $C$, an element $x \in M$, and a linear function $b : C \rightarrow k$, we denote by $b \cdot x \in M$ the result of the left action of the element $b \in C^*$ on the element $x$ in the left $C^*$-module $M$. Similarly, for a graded right comodule $N$ over $C$, a homogeneous element $y \in N$, and a homogeneous linear function $b : C \rightarrow k$, we let $y \cdot b \in N$ denote the result of the right action of $b$ on $y$. We refer to [64, section 4.1] for the sign rules.

Now we can dualize the definitions from Subsection 6.2. A curved DG-coalgebra (CDG-coalgebra) $C^* = (C, d, h)$ over a field $k$ is a graded coalgebra $C = \bigoplus_{i \in \mathbb{Z}} C^i$ endowed with the following data.

- $d : C \rightarrow C$ is an odd coderivation of degree 1 (so the grading components of $d$ are $d_i : C^i \rightarrow C^{i+1}$).
- $h : C \rightarrow k$ is a homogeneous linear function of degree 2 (so the only grading component of $h$ is $h : C^{-2} \rightarrow k$).
The following axioms relating $d$ and $h$ must be satisfied:

(i) the square of the differential $d$ on $C$ is described by the formula $d^2(c) = h \ast c - c \ast h$ for all $c \in C$;
(ii) $h(d(c)) = 0$ for all $c \in C^{-3}$.

The linear function $h : C^{-2} \rightarrow k$ is called the curvature linear function.

A DG-coalgebra is the same thing as a CDG-coalgebra with $h = 0$. The category of DG-coalgebras is a subcategory, but not a full subcategory in the category of CDG-coalgebras, as one can see from the following definition of a morphism of CDG-coalgebras.

Let $C^* = (C, d_C, h_C)$ and $D^* = (D, d_D, h_D)$ be two CDG-coalgebras. A morphism of CDG-coalgebras $C^* \rightarrow D^*$ is a pair $(f, a)$, where

- $f : C \rightarrow D$ is a homomorphism of graded coalgebras;
- $a : C \rightarrow k$ is a homogeneous linear function of degree 1

such that

(iii) $d_D(f(c)) = f(d_C(c)) + f(a \ast c) - (-1)^{|c|} f(c \ast a)$ for all homogeneous elements $c \in C^{[c]}$;
(iv) $h_D(f(c)) = h_C(c) + a(d_C(c)) + a^2(c)$ for all $c \in C$.

The linear function $a : C^{-1} \rightarrow k$ is called the change-of-connection linear function.

The composition of morphisms and the identity morphisms in the category of CDG-coalgebras are defined in the way similar/dual to the one for CDG-rings (see [64, section 4.1] for the details).

Let $C^* = (C, d, h)$ be a CDG-coalgebra. A left CDG-comodule $M^* = (M, d_M)$ over $C^*$ is a graded left $C$-comodule endowed with

- an odd coderivation $d_M : M \rightarrow M$ of degree 1 compatible with the coderivation $d$ on $C$

such that

(v) the square of the differential $d_M$ on $M$ is described by the formula $d_M^2(x) = h \ast x$ for all $x \in M$.

Similarly, a right CDG-comodule $N^* = (N, d_N)$ over $C^*$ is a graded right $C$-comodule endowed with

- an odd coderivation $d_N : N \rightarrow N$ of degree 1 compatible with the coderivation $d$ on $C$

such that

(vi) the square of the differential $d_N$ on $N$ is given by the formula $d_N^2(y) = -y \ast h$ for all $y \in N$.

Similarly to the theory of CDG-modules over CDG-rings discussed in Subsection 6.2, left CDG-comodules over a CDG-coalgebra $C^* = (C, d, h)$ form a DG-category $C^* \text{-comod}$. Any morphism of CDG-coalgebras $(f, a) : (C, d_C, h_C) \rightarrow (D, d_D, h_D)$ induces a DG-functor $C^* \text{-comod} \rightarrow D^* \text{-comod}$ assigning to a CDG-comodule $(M, d_M)$ the CDG-comodule $(M, d'_M)$, with the graded $D$-comodule structure on $M$ obtained from the graded $C$-comodule structure on $M$ by the corestriction of scalars (as mentioned in Subsection 3.1) and the twisted differential $d'_M$, given by the rule $d'_M(x) = d_M(x) + a \ast x$. So, an isomorphism of CDG-coalgebras induces an isomorphism of the DG-categories of CDG-comodules over them.

Similarly to Subsection 6.2, the equations for the square of the differential in (i), (v) and (vi) are all different. So, a CDG-coalgebra $C^*$ is naturally neither a left, nor a right CDG-comodule over itself; but it has a natural structure of a CDG-bicomodule over itself [64, section 4.8].
Let $A$ be a nonzero associative $k$-algebra; so $1 \in A$ is a nonzero element. Consider the $k$-vector space $A_+ = A/(k \cdot 1)$ (cf. Subsection 1.1).

Choose a $k$-linear map $v : A \longrightarrow k$ such that $v(1) = 1$. So, $v$ is a $k$-linear retraction of the $k$-algebra $A$ (viewed as a $k$-vector space) onto its unit line $k = k \cdot 1 \subset A$. Put $V = \ker(v) \subset A$. Then the composition of linear maps $V \longrightarrow A \longrightarrow A_+$ is an isomorphism, so the vector space $A_+$ can be identified with the subspace $V \subset A$, and $A = k \oplus V$ as a $k$-vector space.

The multiplication map $m : A \otimes_k A \longrightarrow A$ decomposes into components according to the direct sum decomposition $A = k \oplus V$. Note that the restrictions of $m$ to $k \cdot 1 \otimes_k A$ and $A \otimes_k k \cdot 1$ are prescribed by the condition that $1 \in A$ is a unit, so we do not need to keep track of these. The restriction of the map $m$ to $V \otimes_k V \subset A \otimes_k A$ provides a linear map $V \otimes_k V \longrightarrow A = k \oplus V$. Denote its components by $m_V : V \otimes_k V \longrightarrow V$ and $m_k : V \otimes_k V \longrightarrow k$.

Consider the tensor coalgebra $\mathcal{J}(A_+) = \bigoplus_{n=0}^{\infty} A_+^\otimes n$, and endow it with the differential $\delta$,

$$k \overset{0}{\longleftarrow} A_+ \overset{\delta}{\longleftarrow} A_+ \otimes_k A_+ \overset{\delta}{\longleftarrow} A_+ \otimes_k A_+ \otimes_k A_+ \overset{\delta}{\longleftarrow} \cdots,$$

given by the formulae $\delta(a \otimes b) = m_V(a \otimes b)$, $\delta(a \otimes b \otimes c) = m_V(a \otimes b) \otimes c - a \otimes m_V(b \otimes c)$, ...

and so on, for all $a, b, c, a_i \in A_+, n \geq 1$, and $1 \leq i \leq n - 1$. The identification $A_+ \cong V$ is presumed here. The leftmost differential $\delta : A_+ \longrightarrow k$ is the zero map.

Note that the map $m_V : A_+ \otimes A_+ \longrightarrow A_+$ is not an associative multiplication. Accordingly, the tensor coalgebra $\mathcal{J}(A_+)$ with the differential $\delta$ is not a complex: one has $\delta^2 \neq 0$ (generally speaking). Still, essentially by construction, $\delta$ is an odd coderivation of the graded coalgebra $\mathcal{J}(A_+)$, in the sense of Subsection 6.3 (to make it an odd coderivation of degree 1, one has to change the sign of the grading on the tensor coalgebra).

Denote by $\text{Bar}(A)$ the tensor coalgebra $\mathcal{J}(A_+)$ with the sign of the grading changed, so $\text{Bar}^{-n}(A) = A_+^\otimes n$. Furthermore, denote by $h : \text{Bar}^{-2}(A) \longrightarrow k$ the linear map $-m_k : A_+ \otimes_k A_+ \longrightarrow k$. Then $\text{Bar}_v(A) = (\text{Bar}(A), \delta, h)$ is a CDG-coalgebra over $k$; the linear function $h$ is its curvature linear function. This is the nonaugmented version of the bar-construction for an associative algebra.

Now let $M$ be a left $A$-module. Consider the cofree left comodule $\mathcal{J}(A_+) \otimes_k M$ over the tensor coalgebra $\mathcal{J}(A_+)$, and endow it with the differential $\delta$,

$$M \overset{\delta}{\longleftarrow} A_+ \otimes_k M \overset{\delta}{\longleftarrow} A_+ \otimes_k A_+ \otimes_k M \overset{\delta}{\longleftarrow} \cdots,$$

given by the formulae $\delta(a \otimes x) = ax$, $\delta(a \otimes b \otimes x) = m_V(a \otimes b) \otimes x - a \otimes bx$, ...

and so on, for all $a, b, c, a_i \in A_+, n \geq 1$, and $1 \leq i \leq n - 1$. The identification $A_+ \cong V$ is presumed here. The leftmost differential $\delta : A_+ \longrightarrow k$ is the zero map.

Note that the map $m_V : A_+ \otimes A_+ \longrightarrow A_+$ is not an associative multiplication. Accordingly, the tensor coalgebra $\mathcal{J}(A_+)$ with the differential $\delta$ is not a complex: one has $\delta^2 \neq 0$ (generally speaking). Still, essentially by construction, $\delta$ is an odd coderivation of the graded coalgebra $\mathcal{J}(A_+)$, in the sense of Subsection 6.3 (to make it an odd coderivation of degree 1, one has to change the sign of the grading on the tensor coalgebra).

Denote by $\text{Bar}(A)$ the tensor coalgebra $\mathcal{J}(A_+)$ with the sign of the grading changed, so $\text{Bar}^{-n}(A) = A_+^\otimes n$. Furthermore, denote by $h : \text{Bar}^{-2}(A) \longrightarrow k$ the linear map $-m_k : A_+ \otimes_k A_+ \longrightarrow k$. Then $\text{Bar}_v(A) = (\text{Bar}(A), \delta, h)$ is a CDG-coalgebra over $k$; the linear function $h$ is its curvature linear function. This is the nonaugmented version of the bar-construction for an associative algebra.
and so on, for all \( a, b, a_j \in A_+, x \in M \), and \( n \geq 1 \). As above, the identification \( A_+ \cong V \) is presumed here (explaining, in particular, the notation \( ax \) for the action of an element \( a \in A_+ \cong V \) on an element \( x \in M \)).

Once again, the graded vector space \( L(A_+) \otimes_k M \) with the differential \( \delta \) is not a complex. Instead, the cofree graded comodule \( L(A_+) \otimes_k M \) over the graded coalgebra \( L(A_+) \), with the differential \( \delta \) on the cofree graded comodule, is a left CDG-comodule over the CDG-coalgebra \( \text{Bar}^*_v(A) = (\text{Bar}(A), \delta, h) \). We denote this CDG-comodule by \( \text{Bar}^*_v(A, M) \). This is the nonaugmented version of the bar-construction for a module over an associative algebra.

Now let \( v' \colon A \rightarrow k \) be another \( k \)-linear map such that \( v'(1) = 1 \). Put \( V' = \ker(v') \subset A \), and note natural isomorphisms \( V' \cong A_+ \cong V \). The difference \( v - v' \colon A \rightarrow k \) is a \( k \)-linear map taking 1 to 0; so it factorizes through the natural surjection \( A \rightarrow A_+ \), defining a \( k \)-linear map \( l \colon A_+ \rightarrow k \). The vector subspace \( V' \subset A \) can then be described as

\[
V' = \{ a' = a + l(a) \mid a \in V \} \subset A.
\]

Following the construction above, there are the linear maps \( m_{V'} : V' \otimes_k V' \rightarrow V' \) and \( m_k' : V' \otimes_k V' \rightarrow k \) corresponding to the choice of the retraction \( v' : A \rightarrow k \) of the \( k \)-algebra \( A \) onto its unit line \( k = k \cdot 1 \). Furthermore, there is the related differential \( \delta' \) on the tensor coalgebra \( L(A_+) \) and the related curvature linear function \( h' \), forming together a CDG-coalgebra \( \text{Bar}^*_v(A) = (\text{Bar}(A), \delta', h') \).

The linear function \( l : A_+ \rightarrow k \) can be viewed as a change-of-connection linear function \( l : \text{Bar}^{-1}(A) \rightarrow k \). Then one observes that the pair \((\text{id}, l)\) is a change-of-connection isomorphism of CDG-coalgebras \( \text{Bar}^*_v(A) \rightarrow \text{Bar}^*_v(A) \).

For any left \( A \)-module \( M \), the differential \( \delta' \) on the bar-construction \( \text{Bar}^*_v(A, M) \) is obtained from the differential \( \delta \) on the bar-construction \( \text{Bar}^*_v(A, M) \) by twisting with the change-of-connection linear function \( l \). So, the equivalence of DG-categories \( \text{Bar}^*_v(A) - \text{comod} \cong \text{Bar}^*_v(A) - \text{comod} \) induced by the isomorphism of CDG-coalgebras \( (\text{id}, l) : \text{Bar}^*_v(A) \rightarrow \text{Bar}^*_v(A) \) takes the CDG-comodule \( \text{Bar}^*_v(A, M) \) over \( \text{Bar}^*_v(A) \) to the CDG-comodule \( \text{Bar}^*_v(A, M) \) over \( \text{Bar}^*_v(A) \).

Furthermore, given a homomorphism of associative algebras \( f : A \rightarrow B \), each of them endowed with a \( k \)-linear retraction onto its unit \( v : A \rightarrow k \) and \( v' : B \rightarrow k \), the difference \( v - v' \circ f : A \rightarrow k \) defines a linear function \( l : A_+ \rightarrow k \). At the same time, the map \( f \) induces a \( k \)-linear map \( f_+ : A_+ \rightarrow B_+ \), which in turn induces a homomorphism of the tensor coalgebras \( g : \text{Bar}(A_+) \rightarrow \text{Bar}(B_+) \). Then the pair \( (g, l) \) defines a morphism of CDG-coalgebras \( (g, l) : \text{Bar}^*_v(A) \rightarrow \text{Bar}^*_v(B) \) assigned to a morphism of associative algebras \( A \rightarrow B \). This makes the nonaugmented bar-construction functorial.

### 6.5 Nonaugmented bar-construction for DG-algebras

Having spelled out the details of the nonaugmented bar-construction in the simplest case of a \( k \)-algebra \( A \), we will now briefly sketch the generalization to DG-algebras \( A^* \). The following construction is a nonaugmented version of Subsection 4.1.

Let \( A^* = (A, d) \) be a DG-algebra over a field \( k \). Assume that \( A^* \neq 0 \), or equivalently, \( 1 \in A^0 \) is a nonzero element. Note that \( k \cdot 1 \subset A^* \) is a subcomplex, since \( d(1) = 0 \). Put \( A^*_+ = A^*/(k \cdot 1) \); so \( A^*_+ \) is a complex of \( k \)-vector spaces, whose underlying graded vector space we denote by \( A_+ \).
Choose a homogeneous $k$-linear map $v : A \to k$ of degree 0 such that $v(1) = 1$. So, $v : A \to k$ is a homogeneous $k$-linear retraction of the graded $k$-vector space $A$ onto the unit line $k = k \cdot 1 \subset A$. Put $V = \ker(v)$; so $V$ is a homogeneous $k$-linear subspace in $A$ such that $A = k \oplus V$. The composition $V \to A \to A_+$ is an isomorphism of graded $k$-vector spaces.

Both the multiplication map $m : A \otimes_k A \to A$ and the differential $d : A \to A$ decompose into components according to the direct sum decomposition $A = k \oplus V$. The restrictions of $m$ to $k \cdot 1 \otimes_k A$ and $A \otimes_k k \cdot 1$ are prescribed by the condition that $1 \in A$ is a unit, while the restriction of $d$ to $k \cdot 1$ vanishes; so we do not need to keep track of these. The restriction of the map $m$ to the subspace $V \otimes_k V \subset A \otimes_k A$ provides a linear map $V \otimes_k V \to A = k \oplus V$, whose components we denote by $m_V : V \otimes_k V \to V$ and $d_V : V \to V$, whose components we denote by $d_V : V \to V$ and $d_k : V \to k$.

Denote by $\text{Bar}(A)$ the graded coalgebra $J(A[1])$. (Here, as usual, the cohomological degree shift [1] is responsible for the construction of the total grading on the bigraded vector space $\bigoplus_{n=0}^{\infty} A \otimes^n$.) One observes that, for any graded $k$-vector space $W$, odd coderivations $d$ of degree 1 on the tensor coalgebra $T(W)$ are uniquely determined by their corestrictions to $W$, that is, the compositions $J(W) \to J(W) \to W$, where $J(W) \to W \otimes^1 = W$ is the direct summand projection. Moreover, an arbitrary homogeneous linear map $J(W) \to W$ of degree 1 corresponds to some odd coderivation $d$ on $J(W)$ in this way.

The CDG-coalgebra $\text{Bar}^*(A^*) = (\text{Bar}(A), d, h)$ is now constructed as follows. The odd coderivation $d : \text{Bar}(A) \to \text{Bar}(A)$ of (total) degree 1 is defined by the condition that its composition with the projection $\text{Bar}(A) \to A_+$ has two possibly nonzero bigrading components, given by the maps $m_V : A_+ \otimes_k A_+ \to A_+$ and $d_V : A_+ \to A_+$. The curvature linear function $h : \text{Bar}^{-2}(A) \to k$ also has two possibly nonzero bigrading components, namely, the maps $m_k : (A_+ \otimes_k A_+)^0 \to k$ and $d_k : A_+^{-1} \to k$. We refer to [64, section 6.1] for the sign rules.

Now let $v' : A \to k$ be another homogeneous $k$-linear map such that $v'(1) = 1$. Then the same construction as above produces a CDG-coalgebra structure $\text{Bar}^*_v(A^*) = (\text{Bar}(A), d', h')$ on the graded coalgebra $\text{Bar}(A)$. Similarly to the discussion in Subsection 6.4, the CDG-coalgebras $\text{Bar}^*_v(A^*)$ and $\text{Bar}^*_{v'}(A^*)$ are connected by a natural change-of-connection isomorphism $(\text{id}, l) : \text{Bar}^*_v(A^*) \to \text{Bar}^*_{v'}(A^*)$. Here $l : \text{Bar}^{-1}(A) \to k$ is a change-of-connection linear function whose only nonzero bigrading component is the linear function $l : A_0^0 \to k$ measuring the difference between the retractions $v$ and $v'$.

Recall the notation $k\text{-alg}_{dg}$ introduced in Subsection 4.4 for the category of DG-algebras over $k$. Denote by $k\text{-alg}_{dg}^+ \subset k\text{-alg}_{dg}$ the full subcategory of nonzero DG-algebras; and let $k\text{-coalg}_{cdg}$ denote the category of CDG-coalgebras over $k$.

**Proposition 6.1.** The assignment of the curved DG-coalgebra $\text{Bar}^*_v(A^*)$ to a DG-algebra $A^*$ can be naturally extended to a functor from the category of nonzero DG-algebras to the category of CDG-coalgebras over $k$,

$$\text{Bar}^*_v : k\text{-alg}_{dg}^+ \to k\text{-coalg}_{cdg}.$$  

**Proof.** To construct the functor $\text{Bar}^*_v$, one has to choose for every nonzero DG-algebra $A^*$ over $k$ a homogeneous $k$-linear retraction onto its unit $v_A : A \to k$. Such retractions can be chosen in an arbitrary way, and do not need agree with each other in any sense. The resulting functor $\text{Bar}^*_v$,
viewed up to a uniquely defined isomorphism of functors, does not depend on the choice of the retractions $v_A$.

Given a homomorphism of DG-algebras $f : A^* \rightarrow B^*$, each of them endowed with a homogeneous $k$-linear retraction onto its unit $v_A : A \rightarrow k$ and $v_B : B \rightarrow k$, one constructs the induced morphism of CDG-coalgebras $(g, l) : \text{Bar}^*_{v_A}(A^*) \rightarrow \text{Bar}^*_{v_B}(B^*)$ in the way similar to the one in Subsection 6.4. The map $f$ induces a homogeneous $k$-linear map $f_+ : A_+ \rightarrow B_+$ of degree 0, which in turn induces a graded coalgebra homomorphism $g : \text{Bar}(A) = J(A_+[1]) \rightarrow J(B_+[1]) = \text{Bar}(B)$. The only nonzero bigrading component of the change-of-connection linear function $l : \text{Bar}^{-1}(A) \rightarrow k$ is the linear function $l : A_0^+ \rightarrow k$ measuring the difference between the retractions $v_A$ and $v_B \circ f : A \rightarrow k$.

\[ 6.6 \quad \text{Curved, noncoaugmented cobar-construction} \]

This section is a generalization of Subsection 4.2 to curved, noncoaugmented DG-coalgebras. The constructions below are dual to those of Subsection 6.5, up to a point. But then there are subtle differences between two dual pictures, which we will explain.

Let $C^* = (C, d_C, h_C)$ be a nonzero CDG-coalgebra over a field $k$. Then the counit $\epsilon : C \rightarrow k$ is a nonzero homogeneous linear map of degree 0. Consider the graded vector subspace $C_+ = \ker(\epsilon) \subset C$.

Choose a homogeneous $k$-linear map $w : k \rightarrow C$ of degree 0 such that the composition $k \xrightarrow{w} C \xrightarrow{\epsilon} k$ is the identity map. So, $w$ is a homogeneous $k$-linear section of the counit map $\epsilon$. Put $W = \text{coker}(w)$; then the composition $C_+ \rightarrow C \rightarrow W$ is an isomorphism of graded $k$-vector spaces and $C = k \oplus W$ as a graded $k$-vector space.

Both the comultiplication map $\mu : C \rightarrow C \otimes_k C$ and the differential $d_C : C \rightarrow C$ decompose into components according to the direct sum decomposition $C = k \oplus W$. The projections of $\mu$ onto $k \cdot 1 \otimes_k C$ and $C \otimes_k k \cdot 1$ are prescribed by the counitality axiom, while the composition of $d_C$ with the counit map vanishes; so we do not need to keep track of these. The composition of the map $\mu$ with the projection $C \otimes_k C \rightarrow W \otimes_k W$ provides a $k$-linear map $k \oplus W = C \rightarrow W \otimes_k W$, whose components we denote by $\mu_W : W \rightarrow W \otimes_k W$ and $\mu_k : k \rightarrow W \otimes_k W$. The composition of the map $d_C$ with the projection $C \rightarrow W$ provides a $k$-linear map $k \oplus W = C \rightarrow W$, whose components we denote by $d_W : W \rightarrow W$ and $d_k : k \rightarrow W$.

Denote by Cob(C) the free graded algebra $T(C_+[-1])$. One observes that, for any graded $k$-vector space $V$, odd derivations $d$ of degree 1 on the tensor algebra $T(V)$ are uniquely determined by their restrictions to the subspace (direct summand) of generators, $V = V^0 \subset T(V)$. Moreover, any homogeneous linear map $V \rightarrow T(V)$ of degree 1 corresponds to some odd derivation $d$ on $T(V)$ in this way.

We are going to construct a CDG-algebra $\text{Cob}^*_w(C^*) = (\text{Cob}(C), d_{\text{Cob}}, h_{\text{Cob}})$, which can be called the curved, noncoaugmented cobar-construction of a CDG-coalgebra $C^*$. Specifically, the odd derivation $d_{\text{Cob}} : \text{Cob}(C) \rightarrow \text{Cob}(C)$ of degree 1 is defined by the condition that its restriction to the subspace of generators $C_+ \subset \text{Cob}(C)$ has three possibly nonzero bigrading components, given by the maps $\mu_W : C_+ \rightarrow C_+ \otimes_k C_+$ and $d_W : C_+ \rightarrow C_+$, and also by the curvature linear function $h_C : C^{-2} = C^{-2} \rightarrow k$. The curvature element $h_{\text{Cob}} \in \text{Cob}^2(C^*)$ has two possibly nonzero bigrading components, namely, the elements $\mu_k(1) \in (C_+ \otimes_k C_+)^0$ and $d_k(1) \in C_+^1 = C^1$. The sign rules can be found in [64, section 6.1].
Remark 6.2. The first subtlety which deserves to be mentioned is that the previous paragraph presumes that degree $-2$ is different from degree 0. This is the case with the usual gradings by the group of integers, which are generally presumed in this paper. But sometimes one may be interested in 2-periodic complexes, DG-algebras, CDG-coalgebras, and so on, graded by the group $\mathbb{Z}/2\mathbb{Z}$ (or by some other grading group, as in [64, Remark 1.1] and [59, section 1.1]). When $2 = 0$ in the grading group $\Gamma$, the equality $C^{-2} = C^{-2}$ no longer holds, and the curvature linear function $h_C$ also decomposes into two components, $h_W : W \to k$ and $h_k : k \to k$. In this case, $h_W$ becomes a part of the differential $d_{\text{Cob}}$, and $h_k$ becomes a third component of the curvature element $h_{\text{Cob}}$ in the cobar-construction (see [64, section 6.1]).

One can go further and consider the grading group $\Gamma = \{0\}$, which makes sense over a ground field $k$ of characteristic 2 (otherwise there are sign issues forcing the complexes to be at least $\mathbb{Z}/2\mathbb{Z}$-graded). In this setting, even change-of-connection elements/linear functions can have nonzero unit/counit components (as $1 = 0$ in $\Gamma$). We refer to [65, section 6.1] for a discussion of the bar- and cobar-constructions including a treatment of these unconventional grading effects special to characteristic 2.

A coaugmentation $\gamma$ on a CDG-coalgebra $(C, d, h)$ is a morphism of (graded, counital) coalgebras $\gamma : k \to C$ such that $(\gamma, 0) : (k, 0, 0) \to (C, d, h)$ is a morphism of CDG-coalgebras. This definition includes the condition that the composition $k \xrightarrow{\gamma} C \xrightarrow{d} C$ vanishes. (When $2 = 0$ in the grading group $\Gamma$, this definition also includes the condition that the composition $k \xrightarrow{\gamma} C \xrightarrow{h} k$ vanishes.)

Given a coaugmented CDG-coalgebra $(C^*, \gamma)$, one can take the section $w = \gamma$ in the construction above and produce a DG-algebra $\text{Cob}^*(C^*) = (\text{Cob}(C), d_{\text{Cob}})$. By the definition, the maps $\mu_k$ and $d_k$ vanish in this case, so $h_{\text{Cob}} = 0$.

Otherwise, when no natural coaugmentation is available for $C^*$, one has to choose an arbitrary section $w : k \to C$ to apply the cobar-construction, and the issue of changing the section arises. Let $w' : k \to C$ be another homogeneous $k$-linear map of degree 0 such that $\varepsilon \circ w' = \text{id}_k$. Then the difference $b = w'(1) - w(1)$ is an element of the vector subspace $C^0_+ \subset C$. The vector space $C^0_+$ is one of the bigrading components of the vector space $\text{Cob}^1(C)$; so $b \in C^0_+ \subset \text{Cob}^1(C)$. The CDG-algebras $\text{Cob}_{w'}^*(C^*)$ and $\text{Cob}_{w}^*(C^*)$ are connected by a natural change-of-connection isomorphism $(\text{id}_{\text{Cob}}, b) : \text{Cob}_{w'}^*(C^*) \to \text{Cob}_{w}^*(C^*)$.

Now suppose that we are given a change-of-connection isomorphism of CDG-coalgebras $(\text{id}, a) : (C, d, h) \to (C', d', h')$, where $a : C^{-1} \to k$ is a change-of-connection linear function on $C$. Let $w : k \to C$ be a homogeneous $k$-linear section of the counit. Then the two cobar-constructions $\text{Cob}_{w'}^*(C, d, h)$ and $\text{Cob}_{w}^*(C, d', h')$ are naturally isomorphic as CDG-algebras.

To construct the latter isomorphism, one only needs to follow the philosophy of nonaugmented Koszul duality, which tells that change-of-connection isomorphisms on one side correspond to changes of the chosen retraction of the unit or section of the counit on the other side. The cobar-construction $\text{Cob}_{w}^*(C, d', h')$ comes endowed with a natural retraction onto the unit $v_{\text{Cob}}$, namely, the direct summand projection $\text{Cob}(C) \to C^0_+ = k$. (The map $v_{\text{Cob}}$ is even an augmentation of the graded algebra $\text{Cob}(C)$, but it is not an augmentation of the CDG-algebra $\text{Cob}_{w'}^*(C, d, h)$ unless $h_C = 0$.) The desired isomorphism $\text{Cob}_{w'}^*(C, d, h) \cong \text{Cob}_{w}^*(C, d', h')$ is supposed not to preserve these retractions onto the unit.

Coming to the point, given two graded vector spaces $V$ and $W$, any homogeneous $k$-linear map $V \to T(W)$ of degree 0 can be uniquely extended to a graded $k$-algebra homomorphism $T(V) \to T(W)$. Consider the graded $k$-algebra automorphism $f_a : \text{Cob}(C) \to \text{Cob}(C)$ defined by the rule
c ↦ c + a(c) for all c ∈ C_+ = V = W. Here c + a(c) is an element of the graded vector space V ⊕ k = V ⊗ 1 ⊕ V ⊗ 0 ⊂ T(V) = Cob(C). Then we have a natural isomorphism of CDG-coalgebras (f_a, 0) : Cob^*_w(C, d', h') → Cob^*_w(C, d, h).

Recall the notation k-coalg_{cdg} introduced in Subsection 6.5 for the category of CDG-coalgebras over k. Denote by k-coalg^+_cdg ⊂ k-coalg_{cdg} the full subcategory of nonzero CDG-coalgebras, and by k-coalg^+_{cdg} the category of coaugmented CDG-coalgebras. Furthermore, let k-alg_{cdg} denote the category of CDG-algebras over k.

**Proposition 6.3.**

(a) The assignment of the DG-algebra Cob^*_γ(C) to a coaugmented CDG-coalgebra (C, γ) can be naturally extended to a functor from the category of coaugmented CDG-coalgebras to the category of DG-algebras over k,

\[ \text{Cob}^*_γ : k\text{-coalg}^{\text{coaug}}_{cdg} → k\text{-alg}_{cdg}. \]

(b) The assignment of the CDG-algebra Cob^*_w(C) to a CDG-coalgebra C can be naturally extended to a functor from the category of nonzero CDG-coalgebras to the category of CDG-algebras over k,

\[ \text{Cob}^*_w : k\text{-coalg}^+_{cdg} → k\text{-alg}_{cdg}. \]

**Proof.** Part (a): Let C = (C, d_C, h_C) and D = (D, d_D, h_D) be two CDG-coalgebras endowed with coaugmentations γ : k → C and δ : k → D. Let (f, a) : (C, γ) → (D, δ) be a morphism of coaugmented CDG-coalgebras, that is, a morphism of CDG-coalgebras preserving coaugmentations, in the sense that f ◦ γ = δ. (When 1 = 0 in the grading group Γ, the condition that a ◦ γ = 0 should be added.)

The morphism of DG-algebras g : Cob^*_γ(C) → Cob^*_δ(D) induced by the coaugmented CDG-coalgebra morphism (f, a) is constructed as follows. The map f induces a homogeneous k-linear map f_+ : C_+ → D_+ of degree 0. The graded k-algebra homomorphism g : Cob(C) = T(C_+ [−1]) → T(D_+ [−1]) = Cob(D) is defined by the rule c ↦ f_+(c) − a(c) for all c ∈ C_+ = V. Here f_+(c) − a(c) is an element of the graded vector space W ⊕ k = W ⊗ 1 ⊕ W ⊗ 0 ⊂ T(W) = Cob(D), where W = D_+.

To construct the functor in part (b), one has to choose for every nonzero CDG-coalgebra C over k a homogeneous k-linear section of its counit w_C : k → C. Such sections are chosen in an arbitrary way, and do not need to agree with each other in any sense. The resulting functor Cob^*_w, viewed up to a uniquely defined isomorphism of functors, does not depend on the choice of the sections w_C.

The construction of the morphism of CDG-algebras (g, b) : Cob^*_w(C) → Cob^*_w(D) assigned to a morphism of CDG-coalgebras (f, a) : C → D by the functor Cob^*_w in part (b) combines the construction from the proof of part (a) with the dual version of the construction from the proof of Proposition 6.1. We skip further details, which are rather straightforward: the section/retraction and connection changes are taken care of as described in the discussion preceding the proposition. □

**Remark 6.4.** One conspicuous difference between the expositions in Subsections 6.5 and 6.6 is that the bar-construction was only applied to uncurved DG-algebras A^* = (A, d) in
Subsection 6.5, while the cobar-construction was defined for \textit{curved} CDG-coalgebras $C^* = (C, d, h)$ in Subsection 6.6. Let us explain the situation.

Actually, one can define the bar-construction in the context of curved DG-algebras and assign a CDG-coalgebra $\text{Bar}^\ast (B^*) = (\text{Bar}(B), d_{\text{Bar}}, h_{\text{Bar}})$ to any CDG-algebra $B^* = (B, d_B, h_B)$ with a chosen retraction onto the unit line $\nu : B \rightarrow k$. When the CDG-algebra $B^*$ is augmented and $\beta : B^* \rightarrow k$ is the augmentation, the construction produces a DG-coalgebra $\text{Bar}^\ast_{\beta}(B^*) = (\text{Bar}(B), d_{\text{Bar}})$ (as $h_{\text{Bar}} = 0$ in this case).

The latter approach was suggested in the paper [54]. Subsequently it was discovered and reported in the paper [44, section 5.1] that the approach does not work as intended.

In fact, there is a massive loss of information involved with the passage from an (augmented or nonaugmented) CDG-algebra $B^* = (B, d_B, h_B)$ over a field $k$ with a nonzero curvature element $h_B \neq 0$ to the bar-construction $\text{Bar}^\ast_{\nu}(B^*)$. In particular, \textit{all} (C)DG-comodules over the (C)DG-coalgebra $\text{Bar}^\ast_{\nu}(B^*)$ are contractible when $h_B \neq 0$. This vanishing phenomenon, originally observed by Kontsevich, was recorded in the present author’s memoir [64, Remark 7.3].

The Kontsevich vanishing can be avoided by working with so-called \textit{weakly curved} DG-algebras, which presumes a more complicated setting with a \textit{topological local ring of coefficients} instead of a ground field $k$. Then the theory becomes well-behaved, as it was established in the memoir [65], where the Koszul duality theory for weakly curved DG-algebras and weakly curved $A_{\infty}$-algebras was developed.

Returning to the construction of the CDG-coalgebra $\text{Bar}^\ast_{\nu}(B^*)$ over a field $k$, the reader can find some details of this (well-defined, but not well-behaved) construction in [64, section 6.1], with a further discussion in [64, Remark 6.1]. One point is worth mentioning: even though one can construct the CDG-coalgebra $\text{Bar}^\ast_{\nu}(B^*)$ or even the DG-coalgebra $\text{Bar}^\ast_{\beta}(B^*)$, one \textit{cannot} assign a (C)DG-coalgebra isomorphism to a change-of-connection isomorphism of CDG-algebras $(\text{id}_B, a) : (B, d', h') \rightarrow (B, d, h)$ with a change-of-connection element $0 \neq a \in B^1$.

The explanation is that the tensor coalgebra $\text{Bar}(B) = J(B_+)$ is conilpotent, hence uniquely coaugmented; and while there are coderivations of the tensor coalgebras $J(W)$ which do not preserve the coaugmentations, there exist no such endomorphisms or homomorphisms of tensor coalgebras. A useful intuition may be provided by the construction of the dual algebra $C^*$ to a coalgebra $C$, as mentioned in Subsections 3.2 and 6.3. Suppose that $W$ is a finite-dimensional (ungraded) $k$-vector space, and consider the tensor coalgebra $C = J(W)$. Then the dual algebra $C^*$ is the algebra of noncommutative \textit{formal Taylor power series} in a finite number of variables over $k$. On the other hand, $T(W)$ is the algebra of noncommutative \textit{polynomials}. Now, variable changes like $z \rightarrow z + 1$ are well-defined as automorphisms of (commutative or noncommutative) polynomial rings, but one cannot make such a variable change act on the ring of formal power series in $z$. Still, $\partial / \partial z_i$ are well-defined derivations of the formal power series ring $k[[z_1, \ldots, z_n]]$ (or its noncommutative version); they just cannot be integrated/exponentiated to automorphisms.

To sum up the discussion in this remark, one can say that coderivations of the tensor coalgebras $J(W)$ which do not preserve the coaugmentation may be well-defined, but they are \textit{not} well-behaved in the context of differential graded homological algebra. This is one difference between the properties of the tensor coalgebras and the tensor coalgebras; this is also an explanation of the Kontsevich vanishing.

The problem of nonfunctoriality of the bar-construction with respect to change-of-connection morphisms of CDG-algebras is resolved by passing from the usual (conilpotent) to the ‘extended’ (nonconilpotent) bar-construction of [28, Definition 2.5]. See the discussion in [28, section 4].
6.7  |  Duality between DG-algebras and CDG-coalgebras

Let \( C^\ast = (C, d, h) \) be a CDG-coalgebra and \( \gamma : k \rightarrow C^\ast \) be a coaugmentation of \( C^\ast \) (as defined in Subsection 6.6). The CDG-coalgebra \( C^\ast \) is said to be conilpotent if the coaugmented graded coalgebra \( (C, \gamma) \) is conilpotent. In other words, a CDG-coalgebra \( C^\ast \) is conilpotent if and only if the graded coalgebra \( C \) is conilpotent with the (unique) coaugmentation \( \gamma \) and \( (\gamma, 0) : k \rightarrow C^\ast \) is a morphism of CDG-coalgebras.

As above, we denote by \( k^-\text{alg}_{\text{dg}} \) the category of DG-algebras over \( k \) and by \( k^-\text{alg}_{\text{cdg}}^+ \subset k^-\text{alg}_{\text{dg}} \) the full subcategory of nonzero DG-algebras. Furthermore, \( k^-\text{coalg}_{\text{cdg}} \) is the category of CDG-coalgebras over \( k \), and \( k^-\text{coalg}_{\text{cdg}}^{\text{coaug}} \) is the category of coaugmented CDG-coalgebras (with the CDG-coalgebra morphisms \( (f, \alpha) : C^\ast \rightarrow D^\ast \) forming a commutative triangle with the coaugmentations \( (\gamma, 0) : k \rightarrow C^\ast \) and \( (\delta, 0) : k \rightarrow D^\ast \)). Denote by \( k^-\text{coalg}_{\text{cdg}}^{\text{conilp}} \subset k^-\text{coalg}_{\text{cdg}}^{\text{coaug}} \) the full subcategory of conilpotent CDG-coalgebras.

Somewhat similarly to the discussion in Subsection 4.4, the bar-construction \( \text{Bar}^\ast (A^\ast) \) of any DG-algebra \( A^\ast \) is naturally a coaugmented CDG-coalgebra (with the direct summand inclusion \( k = A^\ast_0 \rightarrow \text{Bar}(A) \) providing the coaugmentation). Moreover, the coaugmented CDG-coalgebra \( \text{Bar}^\ast (A^\ast) \) is conilpotent.

So, the bar-construction of Proposition 6.1 is actually a functor

\[
\text{Bar}^\ast : k^-\text{alg}_{\text{dg}}^+ \rightarrow k^-\text{coalg}_{\text{cdg}}^{\text{conilp}}.
\]

The curved coaugmented cobar-construction of Proposition 6.3(a) obviously produces nonzero DG-algebras, so is a functor

\[
\text{Cob}^\ast : k^-\text{coalg}_{\text{cdg}}^{\text{coaug}} \rightarrow k^-\text{alg}_{\text{dg}}^+.
\]

Lemma 6.5. The restriction of the cobar-construction to conilpotent CDG-coalgebras,

\[
\text{Cob}^\ast : k^-\text{coalg}_{\text{cdg}}^{\text{conilp}} \rightarrow k^-\text{alg}_{\text{dg}}^+.
\]

is naturally a left adjoint functor to the bar-construction \( \text{Bar}^\ast : k^-\text{alg}_{\text{dg}}^+ \rightarrow k^-\text{coalg}_{\text{cdg}}^{\text{conilp}} \).

Lemma 6.5 is a curved/nonaugmented version of Lemma 4.1. A very brief hint of proof will be given in Subsection 6.8.

One would like to define equivalence relations (that is, the classes of morphisms to be inverted) in the categories of (nonzero) DG-algebras and conilpotent CDG-coalgebras so that the adjoint functors \( \text{Bar}^\ast \) and \( \text{Cob}^\ast \) become equivalences of categories after these classes of morphisms are inverted. In the case of DG-algebras, one can use the conventional quasi-isomorphisms; but the notion of a quasi-isomorphism of CDG-coalgebras is undefined because CDG-coalgebras are not complexes and do not have cohomology spaces.

Similarly to Subsection 4.4, the problem is resolved by considering filtered quasi-isomorphisms, which actually make sense for CDG-coalgebras. An admissible filtration on a coaugmented CDG-coalgebra \( (C^\ast, \gamma) \) is defined as an exhaustive comultiplicative increasing filtration by graded vector subspaces \( F_n C^\ast \subset C^\ast \) such that \( F_{-1} C^\ast = 0 \), \( F_0 C^\ast = \gamma(k) \), and \( d(F_n C^\ast) \subset F_{n-1} C^\ast \) for all \( n \geq 0 \).
In particular, any coaugmented CDG-coalgebra having an admissible filtration is conilpotent. Conversely, the canonical increasing filtration (as in Subsection 3.3) on any conilpotent CDG-coalgebra is admissible.

For any admissible filtration $F$ on a coaugmented CDG-coalgebra $(C^*, \gamma)$ one has $d^2(F_n C^*) \subset F_{n-1} C^*$ for all $n \geq 0$, since $d^2(c) = h \cdot c - c \cdot h$ for all $c \in C$ and the curvature linear function $h : C \rightarrow k$ annihilates $F_0 C^*$. Consequently, the induced differential $d$ on the associated graded vector space (in fact, the associated bigraded coalgebra) $\text{gr}^F C^* = \bigoplus_{n=0}^{\infty} F_n C^*/F_{n-1} C^*$ squares to zero, so it makes $\text{gr}^F C^*$ a complex, and in fact, a (bigraded, uncurved) DG-coalgebra.

Let $(f, a) : C^* \rightarrow D^*$ be a morphism of conilpotent CDG-coalgebras. The morphism $(f, a)$ is said to be a filtered quasi-isomorphism if there exist admissible filtrations $F$ on both the CDG-coalgebras $C^*$ and $D^*$ such that $f(F_n C^*) \subset F_n D^*$ and the induced map $F_n C^*/F_{n-1} C^* \rightarrow F_n D^*/F_{n-1} D^*$ is a quasi-isomorphism of complexes of vector spaces for every $n \geq 0$. Equivalently, this means that $\text{gr}^F f : \text{gr}^F C^* \rightarrow \text{gr}^F D^*$ is a quasi-isomorphism of (bigraded) DG-coalgebras.

**Theorem 6.6.** Let $\text{Quis}$ be the class of all quasi-isomorphisms of (nonzero) DG-algebras and $\text{FQuis}$ be the class of all filtered quasi-isomorphisms of conilpotent CDG-coalgebras. Then the adjoint functors $\text{Bar}^*_{\text{v}}$ and $\text{Cob}^*_{\gamma}$ induce mutually inverse equivalences of categories

$$
\text{Bar}^*_{\text{v}} : k\text{-alg}^+_{\text{dg}} [\text{Quis}^{-1}] \approx k\text{-coalg}_{\text{cdg}}^{\text{conilp}} [\text{FQuis}^{-1}] : \text{Cob}^*_{\gamma}.
$$

**Proof.** This is [64, Theorem 6.10(a)]. □

**Example 6.7.** The following trivial example may be instructive. Note that there are many nonzero DG-algebras quasi-isomorphic to the zero DG-algebra; they are all legitimate objects of the category $k\text{-alg}^+_{\text{dg}} [\text{Quis}^{-1}]$. Typical representatives of this isomorphism class are the DG-algebra $A^* = (k[\epsilon], d)$, with $\epsilon \in A^{-1}$ and $d(\epsilon) = 1$, and its quotient DG-algebra $\overline{A}^* = (k[\epsilon]/(\epsilon^2), d)$. For any (unital) DG-algebra $X^*$ quasi-isomorphic to zero, there exists a morphism of (unital) DG-algebras $A^* \rightarrow X^*$; so $X^*$ is isomorphic to $A^*$ in $k\text{-alg}^+_{\text{dg}} [\text{Quis}^{-1}]$.

One may wonder what the corresponding conilpotent CDG-coalgebras are. Let $B^* = (k[h], 0, h)$ be the CDG-algebra over $k$ whose underlying graded algebra is the algebra of polynomials in the curvature element $h$ (while the differential is $d = 0$). As $B^*$ is a graded algebra with finite-dimensional components, the graded dual vector space to $B^*$ has a natural structure of CDG-coalgebra, which we denote by $C^*$. So, one has $C^n \approx k$ whenever $n$ is an even nonpositive integer, and $C^n = 0$ otherwise; the curvature linear function $h : C^{-2} \rightarrow k$ is an isomorphism, the differential on $C^*$ vanishes, and the comultiplication map $C^{-2n} \rightarrow C^{-2p} \otimes_k C^{-2q}$ is an isomorphism for all $p + q = n, p, q \geq 0$. One can easily see that $C^* = \text{Bar}^*_{\text{v}}(\overline{A}^*)$, where $\nu : \overline{A} \rightarrow k$ is the unique homogeneous retraction onto the unit line. (In particular, $C^*$ is obviously a conilpotent CDG-coalgebra.)

Furthermore, consider the CDG-algebra $\overline{B}^* = (k[h]/(h^2), 0, h)$. The graded dual vector space $\overline{C}^*$ to $\overline{B}^*$ is a CDG-coalgebra with $\overline{C}^n = k$ for $n = 0$ and $n = -2$, and $\overline{C}^n = 0$ for all other $n$. The curvature linear function $h : \overline{C}^{-2} \rightarrow k$ is an isomorphism, while the differential on $\overline{C}^*$ vanishes. It is easy to see that $A^* = \text{Cob}^*_{\gamma}(\overline{C}^*)$, where $\gamma : k \rightarrow \overline{C}^*$ is the unique coaugmentation of the conilpotent CDG-coalgebra $\overline{C}^*$. 
The CDG-algebras $B^*$ and $\widehat{B^*}$ were discussed in [44, section 2.2 and Proposition 3.2] under the name of ‘initial CDG-algebras’. In fact, $B^*$ is the initial object of the category of CDG-algebras over $k$ and strict morphisms between them, that is, morphisms $(f,0)$ with a vanishing change-of-connection element $a = 0$. By the same token, the CDG-coalgebras $C^*$ and $\widehat{C^*}$ might be called ‘terminal CDG-coalgebras’ (and indeed, the zero DG-algebra is the terminal object of the category $k\text{-alg}_{dg}$).

Similarly to the discussion in Subsection 4.5, the equivalence of categories in Theorem 6.6 can be expressed as a Quillen equivalence of model category structures. One additional complication as compared to Subsection 4.5, which arises here, is that the category $k\text{-alg}^+_{dg}$ of nonzero DG-algebras has no terminal object (because the zero DG-algebra, which is the terminal object of $k\text{-alg}_{dg}$, is excluded from $k\text{-alg}^+_{dg}$). The category $k\text{-coalg}_{cdg}^{\text{conlp}}$ of conilpotent CDG-coalgebras has no terminal object, either (as only the category of conilpotent CDG-coalgebras and strict morphisms between them has a terminal object, mentioned in Example 6.7).

Still the definition of a model category (as in [33]) requires existence of all limits and colimits. This problem is resolved by ‘finalizing’ (formally adjoining terminal objects to) both the categories $k\text{-alg}^+_{dg}$ and $k\text{-coalg}^{\text{conlp}}_{cdg}$ before defining model structures on them. We refer to [64, section 9.2, Theorem 9.3(a), and end of section 9.3] for the details.

### 6.8 Twisting cochains in the curved context

Let $V^* = (V, d_V)$ and $W^* = (W, d_W)$ be two precomplexes, that is, graded $k$-vector spaces endowed with homogeneous $k$-linear differentials $d_V : V \to V$ and $d_W : W \to W$ of degree 1 with possibly nonzero squares (as in Subsection 6.3). Then the precomplex $\text{Hom}^*_{k}(V^*, W^*)$ is defined as the graded Hom space $\text{Hom}_k(V, W)$ endowed with the differential $d : \text{Hom}_k(V, W) \to \text{Hom}_k(V, W)$ of degree 1 given by the usual formula $d(f)(x) = d_W(f(x)) - (-1)^{|f|}d_V(x)$ for all $f \in \text{Hom}^{|f|}_k(V, W)$ and $x \in V^{|x|}$.

Let $C^* = (C, d_C, h_C)$ be a CDG-coalgebra and $B^* = (B, d_B, h_B)$ be a CDG-algebra over $k$. Consider the graded Hom space $\text{Hom}_k(C, B)$, and endow it with a graded $k$-algebra structure as constructed in Subsection 5.1 and a differential $d$ given by the usual rule as above. Furthermore, let $h \in \text{Hom}^2_k(C, B)$ be the element given by the formula $h(c) = \varepsilon(c)h_B - h_C(c) \cdot 1$ for all $c \in C$, where $1 \in B$ is the unit element and $\varepsilon : C \to k$ is the counit map. Then $\text{Hom}^*_{k}(C^*, B^*) = (\text{Hom}_k(C, B), d, h)$ is a CDG-algebra over $k$ [64, section 6.2].

Let $E^* = (E, d, h)$ be a CDG-ring over $k$. Then an element $a \in E^1$ is said to be a Maurer–Cartan element if it satisfies the equation $a^2 + d(a) + h = 0$ in $E^2$. Equivalently, $a \in E^1$ is a Maurer–Cartan element if and only if there exists an odd derivation $d' : E \to E$ of degree 1 (specifically, $d'(e) = d(e) + [a, e]$ for all $e \in E$) such that $(id_E, a) : (E, d', 0) \to (E, d, h)$ is an isomorphism of CDG-rings. So, Maurer–Cartan elements in a CDG-ring parameterize its change-of-connection isomorphisms with DG-rings.

By the definition, a twisting cochain $\tau$ for a CDG-coalgebra $C^*$ and a CDG-algebra $B^*$ over $k$ is a Maurer–Cartan element in the CDG-algebra $\text{Hom}^*_{k}(C^*, B^*)$. So, $\tau : C^* \to B^*$ is a homogeneous $k$-linear map of degree 1 satisfying the Maurer–Cartan equation. (There is a slight abuse of terminology involved here, as by ‘cochains’ one usually means elements in grading components of a complex, while $\text{Hom}^*_{k}(C^*, B^*)$ is not a complex.)
This definition of a twisting cochain is a generalization of the one in Subsection 5.1. Similarly to the discussion in Subsection 5.1, various conditions of compatibility with (co)augmentations may be imposed on twisting cochains. We will be particularly interested in the context where \((C^*,\gamma)\) is a coaugmented (better yet, conilpotent) CDG-coalgebra, while \(A^*\) is a DG-algebra. In this case, twisting cochains \(\tau: C^* \longrightarrow A^*\) such that \(\tau \circ \gamma = 0\) play an important role.

**Examples 6.8.**

(1) This is a nonaugmented version of Example 5.1(1). Let \(A \neq 0\) be an algebra over \(k\) and \(C^* = \text{Bar}_v^*(A)\) be its bar-construction, as in Subsection 6.4. Let \(\gamma: k \longrightarrow C^*\) be the natural coaugmentation of \(\text{Bar}_v^*(A)\) (as in Subsection 6.7). Then the composition \(\tau\) of the direct summand projection \(\text{Bar}(A) \longrightarrow A_+^0 = A_+\) and the inclusion \(A_+ \approx \text{ker}(v) \longrightarrow A\) is a twisting cochain for the CDG-coalgebra \(C^*\) and the algebra \(A\) (viewed as a (C)DG-algebra in the obvious way). The equation of compatibility with the coaugmentation \(\tau \circ \gamma = 0\) is satisfied in this example.

(2) This is a nonaugmented version of Example 5.1(2). Let \(A^*\) be a nonzero DG-algebra and \(C^* = \text{Bar}_v^*(A^*)\) be its bar-construction, as in Subsection 6.5. Let \(\gamma: k \longrightarrow C^*\) denote the natural coaugmentation of \(\text{Bar}_v^*(A^*)\). Then the composition \(\tau: C^* \longrightarrow A^*\) of the direct summand projection \(\text{Bar}(A) \longrightarrow A_+^0 = A_+\) and the inclusion \(A_+ \approx \text{ker}(v) \longrightarrow A\) is a twisting cochain for the CDG-coalgebra \(C^*\) and the DG-algebra \(A^*\). The equation \(\tau \circ \gamma = 0\) is satisfied for this twisting cochain.

**Example 6.9.** This is a curved version of Example 5.2(2). Let \((C^*,\gamma)\) be a coaugmented CDG-coalgebra, and let \(A^* = \text{Cob}_v^*(C^*)\) be its cobar-construction, as in Subsection 6.6. Then the composition \(\tau: C^* \longrightarrow A^*\) of the natural surjection \(C^* \longrightarrow C^*+ = C^*/\gamma(k)\) and the direct summand inclusion \(C^*+ \longrightarrow \text{Cob}_v^*(C^*)\) is a twisting cochain for the CDG-coalgebra \(C^*\) and the DG-algebra \(A^*\). The equation \(\tau \circ \gamma = 0\) is satisfied for this twisting cochain.

**Example 6.10.** This is a noncoaugmented version of the previous example. Let \(C^* \neq 0\) be a CDG-coalgebra, and let \(B^* = \text{Cob}_v^*(C^*)\) be its (noncoaugmented) cobar-construction, as in Subsection 6.6. Then the composition \(\tau: C^* \longrightarrow B^*\) of the surjection \(C \longrightarrow \text{coker}(\omega) \approx C_+\) and the direct summand inclusion \(C_+ = C_+^0 \longrightarrow \text{Cob}(C)\) is a twisting cochain for the CDG-coalgebra \(C^*\) and the CDG-algebra \(B^*\).

**Hint of proof of Lemma 6.5.** Let \(A^*\) be a nonzero DG-algebra and \(C^*\) be a conilpotent CDG-coalgebra over \(k\). Then both the DG-algebra homomorphisms \(\text{Cob}_v^*(C^*) \longrightarrow A^*\) and the (coaugmented) CDG-coalgebra morphisms \(C^* \longrightarrow \text{Bar}_v^*(A^*)\) correspond bijectively to twisting cochains \(\tau: C^* \longrightarrow A^*\) satisfying the equation of compatibility with the coaugmentation \(\tau \circ \gamma = 0\).

Let \(A^*\) be a nonzero DG-algebra and \((C^*,\gamma)\) be a conilpotent CDG-coalgebra. Let \(\tau: C^* \longrightarrow A^*\) be a twisting cochain satisfying the equation \(\tau \circ \gamma = 0\). Generalizing the definition from Subsection 5.2, the twisting cochain \(\tau\) is said to be *acyclic* if one of (or equivalently, both) the related DG-algebra homomorphism \(\text{Cob}_v^*(C^*) \longrightarrow A^*\) and the CDG-coalgebra morphism \(C^* \longrightarrow \text{Bar}_v^*(A^*)\) become isomorphisms after the quasi-isomorphisms of DG-algebras, or, respectively, the filtered quasi-isomorphisms of conilpotent CDG-coalgebras, are inverted, as in Theorem 6.6.
Simply put, the twisting cochain $\tau$ is called acyclic if the related homomorphism of DG-algebras $\text{Cob}^*_0(C^*) \to A^*$ is a quasi-isomorphism.

For example, the twisting cochain from Example 6.9 is acyclic, by the definition, whenever the coaugmented CDG-coalgebra $(C^*, \gamma)$ is conilpotent. It is a part of Theorem 6.6 that the twisting cochain from Examples 6.8 is acyclic for any nonzero algebra $A$ or DG-algebra $A^*$.

**Example 6.11.** The following example is a nonaugmented version of Subsection 2.5 and Example 5.3.

Let $(A, F)$ be a nonhomogeneous quadratic algebra, as defined in Subsection 2.5, and let $C = (\text{gr}^F A)^\gamma$ be the quadratic dual coalgebra to the quadratic algebra $\text{gr}^F A$. Let $\nu : A \to k$ be a $k$-linear retraction onto the unit line. Then there is a natural isomorphism of vector spaces $\text{gr}^F A = F_1 A / F_0 A \cong \ker(\nu) \cap F_1 A \subset A$. So, the vector space $V = \text{gr}^F A$ can be viewed as a subspace in $A_+ = A / k \cdot 1 \cong \ker(\nu)$, and the graded coalgebra $C$ can be viewed as a subcoalgebra in $\mathcal{I}(A_+)$,

$$C \subset \mathcal{I}(V) \subset \mathcal{I}(A_+) = \text{Bar}(A).$$

Similarly to Subsection 2.5, one observes that the subcoalgebra $C \subset \text{Bar}(A)$ is in fact a CDG-subcoalgebra in the CDG-coalgebra $\text{Bar}^*_0(A)$ constructed in Subsection 6.4, $C^* \subset \text{Bar}^*_0(A)$. Quite simply, this means that the differential $\delta$ on $\text{Bar}^*_0(A)$ preserves $C$, that is $\delta(C) \subset C$. The curvature linear function $h : C \to k$ is produced as the restriction of the curvature linear function $h_{\text{Bar}} : \text{Bar}(A) \to k$ to the subcoalgebra $C \subset \text{Bar}(A)$. This is another way to spell out the nonhomogeneous quadratic duality construction of [61, Proposition 2.2] or [58, chapter 5, Proposition 4.1].

We have constructed the CDG-coalgebra $C^* = (C, \delta, h)$ nonhomogeneous quadratic dual to a nonhomogeneous quadratic algebra $(A, F)$. Now the composition $C^* \to \text{Bar}^*_0(A) \to A$ of the inclusion map $C^* \to \text{Bar}^*_0(A)$ with the twisting cochain $\text{Bar}^*_0(A) \to A$ from Example 6.8(1) is a twisting cochain $\tau : C^* \to A$. Equivalently, the twisting cochain $\tau$ can be constructed as the composition of the direct summand projection $C \to C^1 = F_1 A / F_0 A$ and the inclusion $F_1 A / F_0 A \cong \ker(\nu) \cap F_1 A \to A$.

The direct summand inclusion $\gamma : k = C_0 \to C$ is a coaugmentation of the CDG-coalgebra $C^*$, so $(C^*, \gamma)$ is a coaugmented, and in fact, conilpotent CDG-coalgebra. The twisting cochain $\tau : C^* \to A$ satisfies the equation $\tau \circ \gamma = 0$.

For any nonhomogeneous *Koszul* algebra $(A, F)$, as defined in Subsection 2.5 (that is, a nonhomogeneous quadratic algebra for which the quadratic graded algebra $\text{gr}^F A$ is Koszul), the twisting cochain $\tau : C^* \to A$ is acyclic. As in Subsection 5.2, this claim is a corollary of the proof of Poincaré–Birkhoff–Witt theorem for nonhomogeneous Koszul algebras in [61, sections 3.2 and 3.3] or [58, chapter 5, Proposition 7.2(ii)].

A generalization of these constructions and the acyclicity claim above to filtered *DG-algebras* $(A^*, F)$ can be found in [64, section 6.6].

## 6.9 Derived Koszul duality on the comodule side

Let $B^*$ be a CDG-algebra and $C^*$ be a CDG-coalgebra over $k$, and let $\tau : C^* \to B^*$ be a twisting cochain.
Given a left CDG-comodule $N^*$ over $C^*$, we consider the tensor product of graded vector spaces $B \otimes_k N$ and endow it with a differential $d$ twisted with the twisting cochain $\tau$ using the same formulae as in Subsections 5.3 and 5.4. Then $B^* \otimes^\tau N^* = (B \otimes_k N, d)$ is a left CDG-module over $B^*$.

Similarly, given a left CDG-module $M^*$ over $B^*$, we consider the tensor product of graded vector spaces $C \otimes_k M$ and endow it with a differential $d$ twisted with the twisting cochain $\tau$ using the same formulae as in Subsections 5.3 and 5.4. Then $C^* \otimes^\tau M^* = (C \otimes_k M, d)$ is a left CDG-comodule over $C^*$.

For example, let $A$ be a nonzero associative algebra and $C^* = \text{Bar}^*_v(A)$ be its bar-construction, as in Subsection 6.4. Let $M$ be a left $A$-module. Then the CDG-module $\text{Bar}^*_v(A, M)$ over $\text{Bar}^*_v(A)$ constructed in Subsection 6.4 can be recovered as $\text{Bar}^*_v(A, M) = \text{Bar}^*_v(A) \otimes^\tau M$, where $\tau : C^* \to A$ is the twisting cochain from Example 6.8(1).

Recall the notation $B^* \text{--mod}$ for the DG-category of left CDG-modules over $B^*$ and $C^* \text{--comod}$ for the DG-category of left CDG-comodules over $C^*$ (see Subsections 6.2 and 6.3). Similarly to Subsection 5.4, one observes that the DG-functor

$$B^* \otimes^\tau - : C^* \text{--comod} \to B^* \text{--mod}$$

is left adjoint to the DG-functor

$$C^* \otimes^\tau - : B^* \text{--mod} \to C^* \text{--comod}.$$

**Theorem 6.12.** Let $A^*$ be a nonzero DG-algebra and $(C^*, \gamma)$ be a conilpotent CDG-coalgebra over $k$ (as defined in Subsection 6.7). Let $\tau : C^* \to A^*$ be an acyclic twisting cochain (as defined in Subsection 6.8); this includes the condition that $\tau \circ \gamma = 0$. Then the adjoint functors $M^* \mapsto C^* \otimes^\tau M^*$ and $N^* \mapsto A^* \otimes^\tau N^*$ induce a triangulated equivalence between the conventional derived category of left DG-modules over $A^*$ and the coderived category of left CDG-comodules over $C^*$,

$$D(A^* \text{--mod}) \simeq D^c(C^* \text{--comod}).$$

**Proof.** This is [64, Theorem 6.5(a)]. The particular case corresponding to the twisting cochain from Examples 6.8 can be found in [64, Theorem 6.3(a)], while the case of the twisting cochain from Example 6.9 (for a conilpotent CDG-coalgebra $C^*$) is considered in [64, Theorem 6.4(a)]. For the definition of the coderived category, see Subsection 7.7. $\square$

The triangulated equivalence of Theorem 6.12 takes the free left DG-module $A^*$ over $A^*$ to the left CDG-comodule $k$ over $C^*$ (with the $C$-comodule structure on $k$ defined in terms of the coaugmentation $\gamma$). Note that, unlike in Theorems 5.4–5.5, there is no natural structure of a DG-module over $A^*$ on the one-dimensional vector space $k$, as the DG-algebra $A^*$ is not augmented. This fact is not unrelated to the fact that there is no natural structure of a left CDG-comodule over $C^*$ on the cofree graded left $C$-comodule $C$ (for the reason explained at the end of Subsection 6.3).

**Theorem 6.13.** Let $C^*$ be a nonzero CDG-coalgebra, and let $\tau : C^* \to \text{Cob}^*_v(C^*) = B^*$ be the twisting cochain from Example 6.10. Then the adjoint functors $M^* \mapsto C^* \otimes^\tau M^*$ and $N^* \mapsto B^* \otimes^\tau N^*$ induce a triangulated equivalence between the absolute derived category of left CDG-modules over
\(B^*\) and the coderived category of left CDG-comodules over \(C^*\),

\[
\text{D}^{\text{abs}}(B^* \text{-mod}) \simeq \text{D}^{\text{co}}(C^* \text{-comod}).
\]

**Proof.** This is [64, Theorem 6.7(a)]. The definitions of the coderived and absolute derived categories will be explained in Subsections 7.6 and 7.7. The absolute derived category \(\text{D}^{\text{abs}}(B^* \text{-mod})\) coincides with the coderived category \(\text{D}^{\text{co}}(B^* \text{-mod})\) by [64, Theorem 3.6(a)]; see Theorem 7.8(a). \(\square\)

For extended versions of Theorems 6.12 and 6.13 (‘Koszul triality’), see Subsection 9.7. Some additional comments on the proofs of the theorems will be offered in Subsection 9.8.

A version of nonconilpotent Koszul duality (Theorem 6.13) with the roles of the algebras and coalgebras switched (starting from a DG-algebra \(A^*\) and considering the cofree nonconilpotent coalgebra spanned by \(A^+_\), or rather, the vector space dual topological algebra) was suggested in the paper [28]. For a version of derived nonhomogeneous Koszul duality for \(DG\)-categories, see [32].

## 7 DERIVED CATEGORIES OF THE SECOND KIND

‘Derived categories of the second kind’ is a common name for the coderived, contraderived, and absolute derived categories, while the conventional derived category is called the ‘derived category of the first kind’. The classical homological algebra can be described as the domain where there is no difference between derived categories of the first and the second kind. Derived categories of the second kind play a crucial role in all formulations of derived Koszul duality outside of the contexts in which assumptions of boundedness of the (internal or cohomological) grading ensure applicability of the classical homological algebra.

### 7.1 Importance of derived categories of the second kind

The earliest versions of derived Koszul duality, establishing the very existence of the phenomenon, were formulated for complexes (DG-algebras, and so on) up to conventional quasi-isomorphism. This includes Quillen’s equivalence between the categories of negatively cohomologically graded Lie DG-algebras and connected, simply connected, negatively cohomologically graded cocommutative DG-coalgebras over a field of characteristic zero [84, Theorem I]. In this result, the boundedness conditions on the cohomological grading allow to avoid the problem demonstrated in Example 1.1 (cf. our Theorems 4.2 and 6.6, where the cohomological gradings are completely unbounded, but the notion of a filtered quasi-isomorphism is used). See, for example, [67, section 3] for noncommutative versions of Quillen’s theory.

The most important area of derived Koszul duality results for which the conventional quasi-isomorphisms are sufficient is the homogeneous Koszul duality. This means the study of complexes of modules, DG-modules, and so on, endowed with an additional internal grading which is assumed to be positive or negative on the rings and bounded below or bounded above on the modules. In some classical contexts, the modules are assumed to be finitely generated, which serves to ensure that their internal gradings are bounded. This theory goes back to the paper [8] (see also [55, appendix A]), where the derived \(S\Lambda\) duality (the equivalence between the bounded derived categories of finitely generated graded modules over the symmetric algebra of a finite-
dimensional vector space and the exterior algebra of the dual vector space) was introduced. A standard reasonably modern source on derived homogeneous Koszul duality is [7, section 2.12]; another exposition can be found in [64, appendix A].

To give a simple explanation of the role of positive or bounded internal grading, note that Tor$_0^A(k, M) \neq 0$ for any bounded below graded module $M \neq 0$ over a positively graded algebra $A = k \oplus A_1 \oplus A_2 \oplus \ldots$. This serves to exclude counterexamples such as in Examples 1.2. Say, one can take $A = k[x]$ to be the polynomial (symmetric) algebra in one variable $x$, internally graded so that the generator $x$ sits in the internal grading 1, and endowed with the augmentation $\alpha : A \rightarrow k$ induced by the internal grading (so $\alpha(x) = 0$). Then all the counterexamples of modules $M$ in Example 1.2(2) are either ungraded, or their grading is unbounded.

When the conventional quasi-isomorphism is not suitable as the equivalence relation on one of the sides (which is usually, but not always, the coalgebra or comodule/contramodule side) of Koszul duality, another equivalence relation on the respective side has to be defined instead. When the equivalence relation on the other (usually, the algebra or module side) of the duality is still the conventional quasi-isomorphism, a possibility of quick hack presents itself: call a morphism on the coalgebra side a ‘weak equivalence’ if the Koszul duality functor transforms it into a quasi-isomorphism on the algebra side. In this approach, the Koszul duality becomes an equivalence of two categories one of which is constructed partially in terms of the other one. Such formulations of Koszul duality can be found in the paper [30, Theorem 3.2], the dissertation [47, Theorem 2.2.2.2], and the note [42, section 4], as well as in the preprint [6, section 7.2].

The latter reference concerns the $D$-$\Omega$ duality, that is, Koszul duality between the rings of differential operators and the de Rham (C)DG-rings of differential forms (which is a thematic example of relative nonhomogeneous Koszul duality [75]). For another early approach to the $D$-$\Omega$ duality, see [39]. Such approaches were tried in the absence of understanding of the existence of and the role of the constructions of the derived categories of the second kind.

### 7.2 History of derived categories of the second kind, I

Let me start with some personal reminiscences. A shorter historical account of the same events can be found in [81, Remark 9.2].

The problem of constructing derived nonhomogeneous Koszul duality (for nonhomogeneous Koszul algebras, as in Subsection 2.5) was suggested to me by Misha Finkelberg sometime around 1991–1992, in a handwritten letter sent to Moscow (where I lived) from Massachusetts (where he was doing his PhD studies at Harvard). Alongside with A. Vaintrob, Misha was my main informal de facto teacher and advisor in the second half of 1980s and early 1990s.

Misha wrote that the problem of derived nonhomogeneous Koszul duality (that is, constructing a derived equivalence between complexes of modules over a nonhomogeneous Koszul algebra and DG-modules over the nonhomogeneous quadratic dual DG-algebra) would be a natural extension of my work on nonhomogeneous quadratic duality [58, 61]. (The paper [61] was only submitted in November 1992, but very early short versions of what became the book [58] circulated since 1991, if not 1990.) The case of a finite-dimensional Lie algebra and its Chevalley–Eilenberg DG-algebra (as in Example 2.5) was perceived as a thematic example, of course; and the idea to proceed further to $D$-$\Omega$ duality was suggested in Misha’s letter.

Soon, that is also around 1992, I made an intensive attempt to attack and conquer the problem, which resulted in a tentative understanding that the derived category $D(\mathfrak{g} \text{-mod})$ was equivalent to the homotopy category of DG-modules over $\Lambda^*(\mathfrak{g}^*)$ with injective (or equivalently,
projective) underlying graded $\Lambda(\mathfrak{g}^*)$-modules. From the contemporary point of view (say, after Becker's paper [5]), that was about the best answer one could hope for. Back in the 1990s, I did not see it that way.

Following my thinking of the early '90s, derived nonhomogeneous Koszul duality was supposed to be an equivalence between Verdier quotient categories of the homotopy categories of DG-modules, not their subcategories. I wanted a derived Koszul duality, not a homotopy one! (in the sense of the homotopy categories of complexes or DG-modules, that is, closed morphisms up to cochain homotopy). Specifically, whatever the construction of a category of DG-modules over $\Lambda^*(\mathfrak{g}^*)$ equivalent to $D(\mathfrak{g} \text{-mod})$ would be, there was supposed to be a way to assign an object of this category to an arbitrary DG-module over $\Lambda^*(\mathfrak{g}^*)$. I did not see a natural way to assign a graded-injective DG-module to an arbitrary one. Once again, Becker's [5, Proposition 1.3.6] did not exist back then.

By late 1990s, I perceived derived nonhomogeneous Koszul duality as an important unsolved, perhaps unsolvable, problem obstructing development of the theory of DG-modules generally. In the meantime, I moved from Moscow to the United States, first as a visitor, then as a graduate student, and finally as a postdoc. My access to the literature improved, and I was able to observe that the theory was developing nevertheless. On top of Spaltenstein's paper [86], which was known to me back in early 1990s, there appeared an important paper of Keller [41]. I had also noted several papers by Neeman (for example, [10, 51]), from which I learned the concepts of homotopy (co)limits and the telescope construction in triangulated categories, as well as compactly generated triangulated categories and other things.

In the 1998–1999 academic year, I was an NSF postdoc at the Institute for Advanced Study. My postdoc advisors there were P. Deligne and V. Voevodsky. In March 1999, my term as a postdoc was coming to an end, and Voevodsky suggested that we should meet for a discussion. At the meeting, Voevodsky returned to the topic of DG-algebra and DG-module theory (which he occasion-ally mentioned to me during our conversations over the years, starting from 1994). His idea was to develop a DG version of Tannaka theory for conservative functors from an (enhanced) triangulated category to the (derived) category of vector spaces, with applications to his derived categories of motives and motivic realization functors in mind.

Discussing the problems involved, we arrived to the topic of a Koszul duality between ungraded modules over the symmetric and exterior algebras; and I mentioned that it was an unsolved problem, as the naïve approach with conventional derived categories on both sides of the would-be equivalence did not work. Upon hearing my arguments, Voevodsky immediately replied that this problem can be solved, and must be solved. I went home and solved it. It was immediately obvious to me that an important discovery was made. By mid-April 1999, I had both the definition of what is now called the coderived category (Definitions 7.5 and 7.11), and the proofs of derived nonhomogeneous Koszul duality results such as Theorems 2.1, 2.4, 2.6, and 2.7.

Having made my discovery, I went to the IAS library to look for relevant prior literature. The most important thing I found was the remarkable series of papers by Eilenberg and Moore [21–23], with the final piece of the series authored by Husemoller, Moore, and Stasheff [34]. In particular, the 1962 paper [21] clarified the issue of ‘divergent spectral sequences’, which was how the difficulties in Koszul duality were spoken of in my Moscow circles in the early 1990s (see Remarks 7.1 and 7.2 for a discussion).

The 1974 paper [34] introduced the distinction between what it called the differential derived functors of the first and the second kind. I immediately realized that it was relevant to my exotic derived categories of DG-modules and DG-comodules; so I seized on this terminology and started
to call my categories ‘derived categories of the second kind’ (as opposed to the conventional
derived category, which was the ‘derived category of the first kind’).

Only after my definitions of derived categories of the second kind were already invented, I dis-
covered Hinich’s papers [29, 30]. My recollection is that I may have also reinvented Hinich’s
definition of a filtered quasi-isomorphism of conilpotent DG-coalgebras before reading it in [30].

That direct sums of injective modules and direct products of projective modules play an important
role in my constructions of derived categories of the second kind for DG-modules, I also well-
realized in Spring 1999. In this connection, I found references such as the papers [16] and [4]
(explaining when the products of projective modules are projective), which were eventually cited
in the memoir [64]. But the more advanced conditions (∗) and (∗∗) from [64, Subsections 3.7
and 3.8] (see Theorem 7.9) were only invented in 2009.

In fact, the condition that the direct sums of injective modules are injective characterized
Noetherian rings, which in my thinking of the time was general enough for an interesting the-
ory (cf. [46]). But the rings over which products of projective modules are projective turned out to
be more rare (as per [4, 16]), essentially only Artinian rings and their immediate generalizations.
This created a misconception that the contraderived categories were rarely well-behaved outside
of the realms of contramodules over coalgebras over fields and modules over Artinian rings, which
haunted me for the years to come (until Spring 2012, when [65, appendix B] was largely written
and the notion of contraherent cosheaves was discovered).

The time period 1998–1999 was a year of Geometric Representation Theory at the IAS. My post-
doc was not a part of that Special Program, but many mathematicians I knew, including Finkelberg
and other people from Moscow, were Members of the IAS for that academic year as a part of that
program. In the Spring 1999, we had an informal seminar where I presented my new results on
Koszul duality in the form of a series of talks. Subsequently I spoke about these results at various
seminars (including H.-J. Baues’ Oberseminar Topologie at MPIM-Bonn in July 2001).

The first preprint versions of what became my memoir [64] only appeared in Spring 2009. There
were several reasons for such a protracted delay, the most important of them being that the task of
typing an original research exposition of the size and complexity of [64] was way above my abilities
in the first half of 2000s. My early research work consisted of very short papers; writing longer
texts required a different technique. An acute problem of insufficient mathematical writing skills,
which tormented me since Fall 1995, was only resolved by about Winter 2006/2007. But there was
also a terminological problem.

Naming things is important. The expression ‘derived category of the second kind’ was both too
long and too imprecise, as there were at least two important ‘derived categories of the second
kind’ known to me in Spring 1999 already. Now they are called the coderived and contraderived
categories. I was unable to invent such a nomenclature myself. What I had in Spring 1999 was not a
terminology for the distinction between the two dual concepts of a coderived and a contraderived
category, but only a notation, and not a very good one at that. The coderived category was denoted
by D′, and the contraderived category by D′′ (one can still see traces of this notation in the old
letters [62] and the recent paper [70]). A work such as [64] could not be written in a readable form
with such clumsy terminological tools.

The terminology ‘coderived category’ first appeared in Keller’s note [42]. One of the mathem-
aticians in Moscow directed my attention to this note sometime in mid-2000s. Upon seeing that
the terminological problem had been solved, I knew that the time had come for me to write up
my results. I started with a treatise on semi-infinite homological algebra [63], which was much
heavier with lots of details that I could not keep all in my mind, but could only work out in writ-
ing; so I had more motivation to type that work. A very elaborate terminological system featuring
the prefixes ‘co-’ and ‘contra-’ (and also ‘semi-’) was developed for the purposes of the book [63] based on the idea I borrowed from Keller’s note.

I typed the bulk of the material in [64] while being a visitor at the IHES in March–April 2009. I contacted Bernhard Keller by e-mail from Bures and went to visit him in his office in Paris, and subsequently gave a talk at the Algebra Seminar in the Institut Henri Poincaré. The title of the talk was ‘Koszul Triality’. This talk is mentioned in the acknowledgement to [44].

7.3 Philosophy of derived categories of the second kind, I

A complex (for example, a complex of modules or a DG-module) $K^*$ can be thought of in two ways. One can view $K^*$ as a deformation of its cohomology module $H^*(K^*)$, or as a deformation of its underlying graded module $K$ (say, endowed with the zero differential). In the notation of [64] (going back to [34, Remark I.5.1]), the important operation of forgetting the differential is denoted by the upper index $#; so one can write $K = K^*#$.

These two ways of looking at a complex are reflected in the classical concept of two spectral sequences of hypercohomology [15, section XVII.3]. The old word ‘hypercohomology’ means the cohomology groups one obtains by applying a derived functor to a complex (rather than just to a module). Let $F$ be a left exact functor and $K^*$ be a complex of modules, bounded below, to which we apply the right derived functor $\mathbb{R}^*F$. Then there are two spectral sequences

\[ I^p_{p,q} = \mathbb{R}^p F(H^q(K^*)) \implies \mathbb{R}^{p+q} F(K^*), \]

\[ II^p_{p,q} = \mathbb{R}^q F(K^p) \implies \mathbb{R}^{p+q} F(K^*), \]

the former of which starts from the derived functor $\mathbb{R}^*F$ applied to the cohomology of the complex $K^*$, while the latter one starts from the derived functor $\mathbb{R}^*F$ applied to the terms of the complex $K^*$. So, the spectral sequence $I^p$ expresses the point of view on $K^*$ as a ‘deformation of its cohomology’, while the spectral sequence $II^p$ is an expression of the vision of $K^*$ as a ‘deformation of its terms’. The condition that $K^*$ is a bounded below complex ensures that both the spectral sequences converge to the same limit (namely, the hypercohomology groups $\mathbb{R}^* F(K^*)$).

Remark 7.1. Let us illustrate by examples the role of convergent and ‘divergent’ spectral sequences in Koszul duality, as mentioned in Subsection 7.2. We start with a convergent example; a ‘divergent’ one will be presented below in Remark 7.2.

Let $A^*$ be a DG-algebra, $(C^*, \gamma)$ be a conilpotent DG-coalgebra, and $\tau : C^* \to A^*$ be twisting cochain satisfying the equation $\tau \circ \gamma = 0$ (cf. Theorem 5.4). Let us show that the functor $M^* \mapsto C^* \otimes \tau M^*$ takes acyclic DG-modules over $A^*$ to acyclic DG-comodules over $C^*$.

Indeed, let $F$ be the canonical increasing filtration on the conilpotent DG-coalgebra $C^*$ (as in Subsections 3.3 and 4.3). Denote also by $F$ the induced increasing filtration on the tensor product $C \otimes_k M^*$; so $F_n(C \otimes_k M) = (F_n C) \otimes_k M$ for all $n \geq 0$. Then $F$ is a filtration of $C^* \otimes \tau M^*$ by subcomplexes (in fact, by DG-subcomodules). Moreover, the differential on the successive quotient complexes $F_n(C^* \otimes \tau M^*)/F_{n-1}(C^* \otimes \tau M^*)$ is simply the tensor product differential on the tensor product of complexes $(F_n C^*/F_{n-1} C^*) \otimes_k M^*$, essentially due to the assumption that $\tau \circ \gamma = 0$. If the complex $M^*$ is acyclic, it follows that the complex $F_n(C^* \otimes \tau M^*)/F_{n-1}(C^* \otimes \tau M^*)$ is acyclic as well for every $n \geq 0$. 
Now the increasing filtration $F$ on the complex $C^* \otimes \cdot M^*$ is exhaustive; so the related spectral sequence converges (this is an easy version of [21, Theorem 7.4]), and it follows that the complex $C^* \otimes \cdot M^*$ is acyclic. (Dually one can show, using a similar complete decreasing filtration, that the functor $M^* \longmapsto \text{Hom}_\tau(C^*, M^*)$ from Subsections 9.5 and 9.6 takes acyclic DG-modules over $A^*$ to acyclic DG-contramodules over $C^*$; cf. Theorem 9.6.) This argument can be found in [64, proof of Theorem 6.4], where it is used to prove coacyclicity rather than acyclicity.

Having dispelled the mystery of ‘divergent spectral sequences’, the authors of [21] embarked upon a program of applying derived functors (such as Ext and Tor) to DG-modules, and particularly of defining the derived functor $\text{Cotor}$ (of the functor of cotensor product of comodules) and applying it to DG-comodules [23]. The program culminated in the paper [34], where it was pointed out that the two hypercohomology spectral sequences for DG-modules (as well as, generally speaking, for unbounded complexes) converge, in one sense or another, to two different limits. These two limits were called the differential derived functors of the first and second kind in [34]. The only difference between their constructions consisted in using two different ways of totalize a given bicomplex: taking either direct sums, or direct products along the diagonals [34, Definition I.4.1]. In retrospective, one can say that classical homological algebra (as exemplified by [15] but not [21]) consisted in studying complexes under suitable boundedness or finite homological dimension assumptions guaranteeing that the two kinds of differential derived functors agree.

### 7.4 Philosophy of derived categories of the second kind, II

No derived categories were mentioned in the papers [21, 23, 34], but in the subsequent decades people became interested in derived categories (including derived categories of DG-modules). An important property of a well-behaved derived category is that it can be defined in two or more equivalent ways: either as the category of complexes up to an equivalence relation (such as the quasi-isomorphism), or as a category of adjusted complexes (such as complexes of projective or injective objects), which can be used as resolutions for constructing derived functors. From the derived category perspective, classical homological algebra is the realm where the equivalence relation on complexes is the conventional quasi-isomorphism, and the adjustedness conditions are the usual termwise conditions: the resolutions are the complexes of injective modules, or the complexes of projective modules, and so on.

In particular, for any abelian category $\mathcal{A}$ with enough injective objects, the bounded below derived category $\mathcal{D}^+(\mathcal{A})$ is equivalent to the homotopy category of bounded below complexes of injective objects in $\mathcal{A}$. Dually, for any abelian category $\mathcal{B}$ with enough projectives objects, the bounded above derived category $\mathcal{D}^-\mathcal{B}$ is equivalent to the homotopy category of bounded above complexes of projective objects in $\mathcal{B}$. These are the classical homological algebra settings.

Here is a thematic counterexample showing that the assertions in the previous paragraph are not true for unbounded complexes. Let $\Lambda = k[\varepsilon]/(\varepsilon^2)$ be the exterior algebra in one variable (or in other words, the ring of dual numbers). Consider the unbounded complex of free $\Lambda$-modules with one generator

$$\cdots \longrightarrow \Lambda \longrightarrow \Lambda \longrightarrow \Lambda \longrightarrow \cdots,$$

where all the differentials are the operators of multiplication with $\varepsilon$. Then (2) is an unbounded complex of projective (also, injective) $\Lambda$-modules which is acyclic, but not contractible. So, (2) is
a zero object in the derived category \(D(\Lambda\text{-mod})\), but a nonzero object in the homotopy category of complexes of projective (or injective) objects in \(\Lambda\text{-mod}\). The point is that (2) is a quite nonzero object in the coderived category \(D^{(\Lambda)}(\Lambda\text{-mod})\).

Denote by \(C = \Lambda^*\) the dual vector space to \(\Lambda\) with its natural coalgebra structure. Then the triangulated equivalence of Theorem 2.7 essentially (up to obvious grading effects) assigns the acyclic complex (2), viewed as a complex of \(C\)-comodules (as per the construction in Subsection 6.3), to the \(k[x]\)-module \(k[x, x^{-1}]\) from Example 1.2(2).

**Remark 7.2.** Now we can present an example of ‘divergent spectral sequence’ in Koszul duality promised in Remark 7.1. Let \((C, \gamma)\) be a (say, conilpotent) coaugmented coalgebra over \(k\). Similarly to Remark 4.3, we observe that the functor \(N^* \mapsto \text{Cob}^\gamma(C, N^*)\) from Theorem 2.7 need not take acyclic complexes of comodules over \(C\) to acyclic DG-modules over the DG-algebra \(\text{Cob}^\gamma(C)\).

Indeed, consider the coalgebra \(C = \Lambda^*\) as above, and the complex of \(C\)-comodules (2). Then the DG-algebra \(A^* = \text{Cob}^\gamma(C)\) is, in fact, the polynomial ring \(A = k[x]\), endowed with the obvious grading and the zero differential. The functor \(M^* \mapsto C \otimes \tau M^*\) takes the graded \(A\)-module \(k[x, x^{-1}]\) (viewed as a DG-module with zero differential) to the acyclic complex of \(C\)-comodules (2).

It is a part of Theorem 2.7 that the adjunction morphism \(A^* \otimes \tau C^* \otimes \tau M^* \rightarrow M^*\) is a quasi-isomorphism of DG-modules for any DG-module \(M^*\) over \(A^*\). It follows that the functor \(\text{Cob}^\gamma(C, -) = A^* \otimes \tau -\) takes the acyclic complex of \(C\)-comodules (2) to a nonacyclic DG-module over \(A^*\) quasi-isomorphic to \(k[x, x^{-1}]\).

How can it happen? Consider the decreasing filtration \(F\) on the graded vector space \(\text{Cob}^\gamma(C) = \bigoplus_{n=0}^{\infty} C^+ \otimes_n C^+\) defined by the obvious rule \(F^n \text{Cob}^\gamma(C) = \bigoplus_{i=n}^{\infty} C^+ \otimes_i C^+\). Given a complex of \(C\)-comodules \(N^*\), let \(F\) be the induced filtration on the graded vector space \(\text{Cob}^\gamma(C, N) = \text{Cob}^\gamma(C) \otimes_k N\); so \(F^n \text{Cob}^\gamma(C, N) = F^n \text{Cob}^\gamma(C) \otimes_k N\). Then \(F^n \text{Cob}^\gamma(C, N^*)\) is a subcomplex in \(\text{Cob}^\gamma(C, N^*)\) for every \(n \geq 0\). Furthermore, the quotient complex \(F^n \text{Cob}^\gamma(C, N^*) / F^{n+1} \text{Cob}^\gamma(C, N^*)\) is the tensor product \(C^+ \otimes_n C^+\) with the differential induced by the differential on \(N^*\). So, all the quotient complexes \(F^n \text{Cob}^\gamma(C, N^*) / F^{n+1} \text{Cob}^\gamma(C, N^*)\) are acyclic whenever a complex of \(C\)-comodules \(N^*\) is acyclic. Still, the complex \(\text{Cob}^\gamma(C, N^*)\) need not be acyclic. How can it happen?

This is what people naïvely call a ‘divergent spectral sequence’. In fact, the spectral sequence associated with the filtration \(F\) on the cobar-complex \(\text{Cob}^\gamma(C, N^*)\) does not diverge at all; it just does not converge to the cohomology of the complex \(\text{Cob}^\gamma(C, N^*)\). It is not supposed to. The point is that the decreasing filtration \(F\) on the complex \(\text{Cob}^\gamma(C, N^*)\) is not complete. Simply put, if one wants the spectral sequence of a decreasing filtration on the totalization of a bicomplex to converge to the cohomology of the totalization, one needs to totalize the bicomplex by taking direct products along the diagonals and not the direct sums. The spectral sequence associated with the filtration \(F\) on the cobar-complex \(\text{Cob}^\gamma(C, N^*)\) converges (at least in the weak sense of [21, Theorem 7.4]) to the cohomology of the completion of the complex \(\text{Cob}^\gamma(C, N^*)\), that is, to the cohomology of the direct product totalization of the bicomplex.

Let us denote by \(\overline{\text{Cob}}^\gamma(C, N^*)\) the completed cobar-complex, that is, the direct product totalization of the cobar-bicomplex of \(C\) with the coefficients in \(N^*\). Then, by [21, Theorem 7.4], the complex \(\overline{\text{Cob}}^\gamma(C, N^*)\) is indeed acyclic for any acyclic complex of \(C\)-comodules \(N^*\) (cf. Remark 2.8). The completed cobar-construction is an interesting functor on its own; a discussion of it can be found, for example, in [58, chapter 5, section 8]. But it is not the adjoint functor to the functor \(M^* \mapsto C \otimes \tau M^*\) in Theorem 2.7 (cf. Subsection 5.4).
As suggested in the first paragraph of the introduction to [64] (see [64, section 0.1]), if one’s spectral sequence diverges, one should either replace the complex with its completion, or choose a different filtration. In the context of Koszul duality, it is better to choose a different filtration. That is what one accomplishes by replacing acyclic complexes or DG-comodules with coacyclic ones.

The reader can find a further introductory discussion of the complex (2) and its place in Koszul duality in the book [75, Prologue], and an introductory discussion of derived categories of the second kind largely centered around the example (2) in the paper [78, section 5]. This material goes back to [64, Example 3.3], where another counterexample can be also found: a totally finite-dimensional (in fact, two-dimensional) DG-module $M^*$ over a (two-dimensional) DG-algebra $B^*$ such that the underlying graded $B$-module $M$ is projective and injective, while the DG-module $M^*$ is acyclic but not contractible.

The triangulated equivalence of Theorem 5.4 assigns the acyclic DG-module $M^*$ (viewed as a DG-comodule over the DG-coalgebra $C^* = B^{**}$) to the $k[x]$-module $k_1 (a = 1)$ from Example 1.2(2). Both the acyclic complex (2) and the acyclic DG-module $M^*$ can be also viewed as assigned to the respective modules over the abelian Lie algebra $\mathfrak{g} = k$ by the triangulated equivalence (1) from Example 2.5.

Example 7.3. To give a further series of examples, let us continue the discussion of Lie algebras from Example 2.5. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over a field $k$ of characteristic 0, and let $\Lambda^*(\mathfrak{g}^*)$ be its cohomological Chevalley–Eilenberg complex. To any complex of modules $M^*$ over $\mathfrak{g}$, the Koszul duality (1) assigns the cohomological Chevalley–Eilenberg complex (or rather, the total complex of the bicomplex) $\Lambda^*(\mathfrak{g}^*) \otimes_k M^*$, with the differential including summands induced by the Lie bracket on $\mathfrak{g}$, the action of $\mathfrak{g}$ in $M$, and the differential on $M$. Here the complex $\Lambda^*(\mathfrak{g}^*) \otimes_k M^*$ is viewed as a DG-module over the DG-algebra $\Lambda^*(\mathfrak{g}^*)$.

In particular, for any $\mathfrak{g}$-module $M$, the Chevalley–Eilenberg complex $(\Lambda^*(\mathfrak{g}^*) \otimes_k M, d)$ computes the Lie algebra cohomology $H^*(\mathfrak{g}, M)$ with the coefficients in $M$. Now one has $H^*(\mathfrak{g}, M) = 0$ for any nontrivial finite-dimensional irreducible $\mathfrak{g}$-module $M$. So, $(\Lambda^*(\mathfrak{g}^*) \otimes_k M, d)$ is another example of a finite-dimensional acyclic, noncontractible DG-module whose underlying graded module is both projective and injective.

Let us explain in yet another way why a triangulated equivalence such as (1) cannot hold with the conventional derived category $\mathcal{D}(\Lambda^*(\mathfrak{g}^*)-\text{mod})$ used instead of the coderived category. Any quasi-isomorphism of DG-algebras $f: A^* \rightarrow B^*$ induces a triangulated equivalence of the derived categories of DG-modules, $\mathcal{D}(A^*-\text{mod}) \simeq \mathcal{D}(B^*-\text{mod})$ [41, Example 6.1], [64, Theorem 1.7]. It is well-known that, for a semisimple Lie algebra $\mathfrak{g}$ in char $k = 0$, the DG-algebra $\Lambda^*(\mathfrak{g}^*)$ is formal (that is, quasi-isomorphic to its cohomology algebra); indeed, the subalgebra of $\mathfrak{g}$-invariant elements in $\Lambda^*(\mathfrak{g}^*)$ is a DG-subalgebra with zero differential mapping quasi-isomorphically onto the cohomology. Moreover, the cohomology algebra $H^*(\mathfrak{g}) = H^*(\mathfrak{g}, k)$ is not very informative: it is an exterior (free graded commutative) algebra with generators in certain odd degrees. There is no hope of recovering the derived category of $\mathfrak{g}$-modules $\mathcal{D}(\mathfrak{g}-\text{mod})$ from the quasi-isomorphism class of the DG-algebra $\Lambda^*(\mathfrak{g}^*)$.

So, phenomena appearing to be specific to unbounded complexes also manifest themselves in totally bounded, finite-dimensional DG-modules. The explanation is that, if one wants to stay within the realm of classical homological algebra, then one has to restrict oneself to suitably bounded DG-modules over DG-rings $A^*$ belonging to one of the following two classes: either $A^*$ is nonpositively cohomologically graded [64, Theorem 3.4.1], or $A^*$ is connected, simply connected,
and positively cohomologically graded [64, Theorem 3.4.2]. The DG-algebra $B^*$ from [64, Example 3.3] (discussed above), as well as, more generally, the DG-algebra $\Lambda^*(g^*)$ from Examples 2.5 and 7.3, belong to neither of the two classes.

### 7.5 Philosophy of derived categories of the second kind, III

Thus, the choice between derived categories of the first and the second kind can be presented as consisting in having to decide what to do with the complex (2). Is it to be considered as a trivial object and not acceptable as a resolution (for computing derived functors with it), or as a nontrivial object and acceptable as a resolution? We can now summarize the philosophy of two kinds of derived categories as follows.

In derived categories of the first kind (the conventional derived categories),

- a complex or a DG-module is viewed as a deformation of its cohomology;
- the conventional quasi-isomorphism (that is, the property of a morphism to induce an isomorphism on the cohomology objects) is the equivalence relation on complexes or DG-modules in the derived category;
- strong and complicated conditions need to be imposed on resolutions: the adjusted complexes or DG-modules are known as the homotopy projectives, homotopy flats, homotopy injectives, and so on; in slightly different terminological systems they are called $K$-projectives, $K$-flats, $K$-injectives, and so on, or $DG$-projectives, $DG$-flats, $DG$-injectives, and so on (see [81, Remark 6.4] for the terminological discussion).

It is important here that one cannot tell whether a complex or DG-module is homotopy adjusted (that is, can be used as a resolution for derived categories or derived functors of the first kind) by looking only on the terms of the complex or on the underlying graded module of a DG-module. The property of a complex/DG-module to be homotopy adjusted depends on the differential. The classical works where the theory of derived categories of the first kind was developed are [29, 41, 86].

In derived categories of the second kind (the coderived, contraderived, and absolute derived categories):

- a complex or a (C)DG-module is viewed as a deformation of its underlying graded module, or of itself endowed with the zero differential;
- strong and complicated equivalence relations are imposed on complexes or DG-modules in the derived categories; the related classes of trivial objects are called the coacyclic, contraacyclic, or absolutely acyclic complexes or (C)DG-modules; for conilpotent (C)DG-coalgebras, the equivalence relation of filtered quasi-isomorphism is used;
- all the conventional termwise injective or termwise projective complexes, or (C)DG-modules with injective/projective underlying graded modules are adjusted (that is, can be used as resolutions); more precisely:
  - in the coderived categories, all complexes of injective objects or graded-injective (C)DG-modules are acceptable as resolutions;
  - in the contraderived categories, all complexes of projective objects or graded-projective (C)DG-modules are acceptable as resolutions.

It is important here that one cannot tell whether a morphism is a weak equivalence for a derived category of the second kind, or whether a complex or DG-module is coacyclic/contraacyclic/and
so on, by looking only on its underlying complex of abelian groups. The property of a complex/DG-module to be co- or contraacyclic depends on the module structure. The classical works where the theory of derived categories of the second kind was developed are \[5, 19, 30, 35, 38, 42, 44, 46, 47, 52, 53, 63, 64, 87\].

The terminology coderived category reflects an understanding that the coderived categories are most suitable for comodules. This was perhaps an intended meaning of the term in Keller’s note [42], where it was first introduced (in fact, in the original definition in [42], the coderived category was defined for DG-comodules only; the definition was not applicable to modules or DG-modules).

Dually, the contraderived category is most suitable for contramodules, while the conventional derived category works best for complexes of modules or DG-modules. This philosophy was developed and played a central role in the present author’s monograph on semi-infinite homological algebra [63]. The technical aspects of the assertion ‘coderived categories are best behaved for comodules, contraderived categories for contramodules, and conventional derived categories for modules’ are discussed in detail in the memoir [64, sections 1–4] (cf. Subsections 7.6, 7.7, and 9.3).

But there are also other considerations. In particular, the conventional derived category makes no sense for curved structures. There is a partial exception to this rule in the (complicated, technical, and special) weakly curved case, as developed in [65]. But generally speaking, CDG-modules (as well as CDG-rings, CDG-comodules, and so on) have no cohomology groups or modules, so the notion of a conventional quasi-isomorphism is undefined for them (as we mentioned already in Subsection 6.1). Thus, for CDG-modules one has to consider derived categories of the second kind.

This presents no problem for Koszul duality. But, say, derived categories of matrix factorizations can be only defined as derived categories of the second kind \[19, 57, 59\]. So, sometimes one is forced to work with the coderived categories of modules, technically complicated as they might be. Conversely, differential derived functors of the first kind for DG-comodules appear in the context of the Eilenberg–Moore spectral sequence [23]; see [63, section 0.2.10] for a discussion.

### 7.6 Coderived and contraderived categories of CDG-modules

To any bicomplex one can assign its total complex. In fact, there are several ways to do it: at least, one has to choose between taking infinite direct sums or infinite products along the diagonals. There are further possibilities, which may be called the Laurent totalizations (as in ‘the Laurent formal power series’): one can take direct sums in one direction and direct products in the other one [34, Definition 1.3.4]. For bicomplexes with only a finite number of nonzero terms on every diagonal, there is only one way of producing the total complex.

Here we are interested in a special class of bicomplexes with three rows: namely, the short exact sequences of complexes. It is straightforward to see that the total complex of any short exact sequence of complexes (say, in an abelian category, or even in an exact category) is acyclic.

The concept of totalization can be extended to DG-categories, where one can speak of the (product or coproduct) totalization of a complex of objects and closed morphisms of degree 0 between them [64, section 1.2], [77, section 1.3]. In particular, one can consider complexes (for example, short exact sequences) of DG-modules or CDG-modules, and their totalizations.

Specifically, let \(B^\ast = (B, d)\) be a DG-ring. A short exact sequence of DG-modules \(0 \rightarrow K^\ast \rightarrow L^\ast \rightarrow M^\ast \rightarrow 0\) over \(B^\ast\) consists of three (say, left) DG-modules \(K^\ast, L^\ast, M^\ast\) and two morphisms \(f \in \text{Hom}_B^0(K, L)\) and \(g \in \text{Hom}_B^0(L, M)\) such that \(d(f) = 0 = d(g)\) in the respective complexes.
Hom∗(K, L) and Hom∗(L, M), and 0 → K \xrightarrow{f} L \xrightarrow{g} M → 0 is a short exact sequence of graded B-modules. The totalization Tot(K* → L* → M*) of 0 → K* → L* → M* → 0 is constructed by passing to the direct sums of the (at most three) grading components along every diagonal, and setting the total differential to be the sum of dK, dL, dM, f, and g with suitable signs. Then Tot(K* → L* → M*) is again a (left) DG-module over B*.

The definitions of a short exact sequence of CDG-modules 0 → K* → L* → M* → 0 over a CDG-ring B* and its total CDG-module Tot(K* → L* → M*) over B* are spelled out literally the same (recall that the Hom of two CDG-modules over B* is a complex, as explained in Subsection 6.2). The totalization can also be constructed as an iterated cone: take the cone \text{cone}(f) of the closed morphism f : K* → L* in the DG-category of CDG-modules B*-mod, and put \text{Tot}(K* → L* → M*) = \text{cone}(\text{cone}(f) → M*). (We are only speaking of the totalization as defined up to some cohomological grading shift, and refrain from specifying any preferred choice of such grading shift, as it is not important for our purposes.)

Let Hot(B*-mod) denote the homotopy category of left CDG-modules over a CDG-ring B*, that is, the category whose objects are the CDG-modules over B* and morphisms are closed morphisms of degree 0 up to cochain homotopy. As the DG-category B*-mod has shifts and cones, its homotopy category Hot(B*-mod) is a triangulated category [11].

**Definition 7.4** [64, section 3.3]. Let B* be a CDG-ring. A left CDG-module over B* is said to be absolutely acyclic if it belongs to the minimal thick subcategory of Hot(B*-mod) containing the totalizations of short exact sequences of left CDG-modules over B*. The full subcategory of absolutely acyclic CDG-modules is denoted by Ac^{abs}(B*-mod) ⊂ Hot(B*-mod). The triangulated Verdier quotient category

\[ D^{\text{abs}}(B*-\text{mod}) = \text{Hot}(B*-\text{mod}) / \text{Ac}^{\text{abs}}(B*-\text{mod}) \]

is called the absolute derived category of left CDG-modules over B*.

**Definition 7.5** [63, section 2.1; 64, section 3.3]. A left CDG-module over B* is said to be coacyclic if it belongs to the minimal triangulated subcategory of Hot(B*-mod) containing the totalizations of short exact sequences of left CDG-modules over B* and closed under infinite direct sums. The thick subcategory of coacyclic CDG-modules is denoted by Ac^{co}(B*-mod) ⊂ Hot(B*-mod). The triangulated Verdier quotient category

\[ D^{\text{co}}(B*-\text{mod}) = \text{Hot}(B*-\text{mod}) / \text{Ac}^{\text{co}}(B*-\text{mod}) \]

is called the coderived category of left CDG-modules over B*.

**Definition 7.6** [63, section 4.1; 64, section 3.3]. A left CDG-module over B* is said to be contraacyclic if it belongs to the minimal triangulated subcategory of Hot(B*-mod) containing the totalizations of short exact sequences of left CDG-modules over B* and closed under infinite products. The thick subcategory of contraacyclic CDG-modules is denoted by Ac^{ctr}(B*-mod) ⊂ Hot(B*-mod). The triangulated Verdier quotient category

\[ D^{\text{ctr}}(B*-\text{mod}) = \text{Hot}(B*-\text{mod}) / \text{Ac}^{\text{ctr}}(B*-\text{mod}) \]

is called the contraderived category of left CDG-modules over B*.
Example 7.7 [64, Examples 3.3]. The doubly unbounded acyclic complex of $\Lambda$-modules in the formula (2) from Subsection 7.4 is neither coacyclic, nor contraacyclic. Its (bounded above) acyclic subcomplex of canonical truncation is contraacyclic, but not coacyclic. Dually, the (bounded below) acyclic quotient complex of canonical truncation of (2) is coacyclic, but not contraacyclic. Consequently, neither one of the three complexes is absolutely acyclic. We refer to [78, beginning of section 5] for a detailed discussion.

The maximal natural generality for Definitions 7.4–7.6 is that of exact DG-categories [77, section 5.1]. This concept, suggested in [64, section 3.2 and Remarks 3.5–3.7], was worked out in the preprint [77].

The following two theorems are the main results of the basic theory of coderived, contraderived, and absolute derived categories of CDG-modules.

Theorem 7.8.

(a) Let $B^* = (B, d, h)$ be a CDG-ring whose underlying graded ring $B$ has finite left global dimension (as a graded ring; that is, the abelian category of graded left $B$-modules has finite homological dimension). Then the three classes of coacyclic, contraacyclic, and absolutely acyclic CDG-modules over $B^*$ coincide,

$$Ac^{co}(B^*\text{-mod}) = Ac^{abs}(B^*\text{-mod}) = Ac^{ctr}(B^*\text{-mod}),$$

and accordingly, the three derived categories of the second kind coincide,

$$D^{co}(B^*\text{-mod}) = D^{abs}(B^*\text{-mod}) = D^{ctr}(B^*\text{-mod}).$$

(b) Let $A^* = (A, d)$ be a DG-ring whose underlying graded ring $A$ has finite left global dimension (as a graded ring). Assume additionally that either $A^n = 0$ for all $n > 0$, or otherwise $A^n = 0$ for all $n < 0$, the ring $A^0$ is (classically) semisimple, and $A^1 = 0$. Then the three classes of coacyclic, contraacyclic, and absolutely acyclic DG-modules over $A^*$ coincide with the class of all acyclic DG-modules,

$$Ac(A^*\text{-mod}) = Ac^{co}(A^*\text{-mod}) = Ac^{abs}(A^*\text{-mod}) = Ac^{ctr}(A^*\text{-mod}).$$

Accordingly, all the four derived categories coincide,

$$D(A^*\text{-mod}) = D^{co}(A^*\text{-mod}) = D^{abs}(A^*\text{-mod}) = D^{ctr}(A^*\text{-mod}).$$

Proof. Part (a) is [64, Theorem 3.6(a)]. For an alternative proof based on the concept of a cotorsion pair, see [76, Corollaries 4.8 and 4.15]. For a generalization to exact DG-categories (covering also Theorems 7.12 and 9.3), see [77, Theorem 5.6], or for a much wider generalization with weaker assumptions, [77, Theorem 8.9].

Part (b) for complexes of modules over a ring of finite global dimension is a particular case of [63, Remark 2.1]. In full generality, part (b) is provable by comparing part (a) with [64, Theorem 3.4.1(d)] (in the first case) or with [64, Theorem 3.4.2(d)] (in the second case).
For a further discussion of assumptions under which a result like Theorem 7.8(b) holds, explore the references in [64, last paragraph of section 3.6]. In particular, by [64, Theorem 9.4], one has $\text{Ac}(A^-\text{-mod}) = \text{Ac}^{co}(A^-\text{-mod}) = \text{Ac}^{ab}(A^-\text{-mod}) = \text{Ac}^{ctr}(A^-\text{-mod})$ for any cofibrant DG-algebra $A^-$ over a commutative ring of finite global dimension.

Let $B$ be a graded ring. The following conditions on $B$ are relevant for the theory of derived categories of the second kind [64, sections 3.7 and 3.8]:

(*) any countable direct sum of injective graded left $B$-modules has finite injective dimension (as a graded $B$-module);

(**) any countable product of projective graded left $B$-modules has finite projective dimension (as a graded $B$-module).

Note that any graded ring $B$ of finite left global dimension (as in Theorem 7.8(a)) satisfies both (*) and (**).

One denotes by $\text{Hot}(B^*\text{-mod}_{inj}) \subset \text{Hot}(B^*\text{-mod})$ the full triangulated subcategory in the homotopy category formed by all the CDG-modules $J^* = (J, d_J)$ over $B^*$ whose underlying graded left $B$-modules $J$ are injective (as graded left $B$-modules). Such CDG-modules $J^*$ over $B^*$ are called graded-injective.

Dually, one denotes by $\text{Hot}(B^*\text{-mod}_{proj}) \subset \text{Hot}(B^*\text{-mod})$ the full triangulated subcategory in the homotopy category formed by all the CDG-modules $P^* = (P, d_P)$ over $B^*$ whose underlying graded left $B$-modules $P$ are projective. Such CDG-modules $P^*$ over $B^*$ are called graded-projective.

Theorem 7.9.

(a) Let $B^* = (B, d, h)$ be a CDG-ring whose underlying graded ring $B$ satisfies condition (*). Then the composition $\text{Hot}(B^*\text{-mod}_{inj}) \rightarrow \text{Hot}(B^*\text{-mod}) \rightarrow \text{D}^{co}(B^*\text{-mod})$ of the triangulated inclusion functor $\text{Hot}(B^*\text{-mod}_{inj}) \rightarrow \text{Hot}(B^*\text{-mod})$ and the Verdier quotient functor $\text{Hot}(B^*\text{-mod}) \rightarrow \text{D}^{co}(B^*\text{-mod})$ is an equivalence of triangulated categories,

$$\text{Hot}(B^*\text{-mod}_{inj}) \simeq \text{D}^{co}(B^*\text{-mod}).$$

(b) Let $B^* = (B, d, h)$ be a CDG-ring whose underlying graded ring $B$ satisfies condition (**). Then the composition $\text{Hot}(B^*\text{-mod}_{proj}) \rightarrow \text{Hot}(B^*\text{-mod}) \rightarrow \text{D}^{ctr}(B^*\text{-mod})$ of the triangulated inclusion functor $\text{Hot}(B^*\text{-mod}_{proj}) \rightarrow \text{Hot}(B^*\text{-mod})$ and the Verdier quotient functor $\text{Hot}(B^*\text{-mod}) \rightarrow \text{D}^{ctr}(B^*\text{-mod})$ is an equivalence of triangulated categories,

$$\text{Hot}(B^*\text{-mod}_{proj}) \simeq \text{D}^{ctr}(B^*\text{-mod}).$$

Proof. Part (a) is [64, Theorem 3.7], and part (b) is [64, Theorem 3.8]. For an alternative proof based on the notion of a cotorsion pair, see [76, Corollary 4.18] for part (a) and [76, Corollary 4.9] for part (b). For a generalization to exact DG-categories (covering also Theorems 7.13 and 9.4), see [77, Theorem 5.10].

7.7 | Coderived category of CDG-comodules

This section is a comodule version of the previous one. In the spirit of the discussion in the end of Subsection 7.5, we will see (in Theorem 7.13) that the coderived categories of comodules are
somewhat better behaved than the coderived categories of modules. On the other hand, it makes no sense to consider ‘contraderived categories of comodules’, as the functors of infinite products are usually not exact in comodule categories.

Let $C^* = (C, d, h)$ be a CDG-coalgebra over $k$. Similarly to Subsection 7.6, one can speak of short exact sequences $0 \to K^* \to L^* \to M^* \to 0$ of left CDG-comodules over $C^*$ and their totalizations (total CDG-comodules) $\text{Tot}(K^* \to L^* \to M^*)$, which are again left CDG-comodules over $C^*$.

Let $\text{Hot}(C^*\text{-comod})$ denote the homotopy category of left CDG-comodules over a CDG-coalgebra $C^*$. Similarly to Subsection 7.6, the DG-category $C^*\text{-comod}$ has shifts and cones, so its homotopy category $\text{Hot}(C^*\text{-comod})$ is triangulated.

**Definition 7.10** [64, section 4.2]. Let $C^*$ be a CDG-coalgebra. A left CDG-comodule over $C^*$ is said to be **absolutely acyclic** if it belongs to the minimal thick subcategory of $\text{Hot}(C^*\text{-comod})$ containing the totalizations of short exact sequences of left CDG-comodules over $C^*$. The full subcategory of absolutely acyclic CDG-comodules is denoted by $\text{Ac}^{\text{abs}}(C^*\text{-comod}) \subset \text{Hot}(C^*\text{-comod})$. The triangulated Verdier quotient category

$$D^{\text{abs}}(C^*\text{-comod}) = \text{Hot}(C^*\text{-comod})/\text{Ac}^{\text{abs}}(C^*\text{-comod})$$

is called the **absolute derived category** of left CDG-comodules over $C^*$.

**Definition 7.11** [63, section 2.1; 64, section 4.2]. A left CDG-comodule over $C^*$ is said to be **coacyclic** if it belongs to the minimal triangulated subcategory of $\text{Hot}(C^*\text{-comod})$ containing the totalizations of short exact sequences of left CDG-comodules over $C^*$ and closed under infinite direct sums. The thick subcategory of coacyclic CDG-comodules is denoted by $\text{Ac}^{\text{co}}(C^*\text{-comod}) \subset \text{Hot}(C^*\text{-comod})$. The triangulated Verdier quotient category

$$D^{\text{co}}(C^*\text{-comod}) = \text{Hot}(C^*\text{-comod})/\text{Ac}^{\text{co}}(C^*\text{-comod})$$

is called the **coderived category** of left CDG-comodules over $C^*$.

In the context of the next theorem, one can keep in mind that, unlike for rings, the left and right global dimensions agree for any (ungraded or graded) coalgebra $C$ over a field $k$ (see [64, beginning of section 4.5]).

**Theorem 7.12.** Let $C^* = (C, d, h)$ be a CDG-coalgebra over $k$ whose underlying graded coalgebra $C$ has finite left global dimension (as a graded coalgebra; that is, the abelian category of graded left $C$-comodules has finite homological dimension). Then the two classes of coacyclic and absolutely acyclic CDG-comodules over $C^*$ coincide,

$$\text{Ac}^{\text{co}}(C^*\text{-comod}) = \text{Ac}^{\text{abs}}(C^*\text{-comod}),$$

and accordingly, the two derived categories of the second kind coincide,

$$D^{\text{co}}(C^*\text{-comod}) = D^{\text{abs}}(C^*\text{-comod}).$$

**Proof.** This is [64, Theorem 4.5(a)]. For a generalization to exact DG-categories, see [77, Theorem 5.6(a) or Theorem 8.9(a)]. □
It would be interesting to obtain a DG-comodule version of Theorem 7.8(b). The nonexistence of a meaningful notion of a ‘contraacyclic DG-comodule’ stands in the way of applying an argument similar to the proof of Theorem 7.8(b) (cf. [64, section 4.3]). The case of complexes of comodules over a coalgebra of finite global dimension is covered by [63, Remark 2.1].

Note that the comodule version of condition (∗) from Subsection 7.6 holds for any (graded) coalgebra $C$ over a field $k$, because the class of all injective $C$-comodules is closed under infinite direct sums. Indeed, the injective $C$-comodules are the direct summands of the cofree ones (cf. Subsection 2.1); and cofree left $C$-comodules have the form $C \otimes_k V$, where $V$ ranges over (graded) $k$-vector spaces; so cofree $C$-comodules obviously form a class closed under infinite direct sums.

Similarly to Subsection 7.6, we denote by $\mathcal{D}(C\text{-comod}_{\text{in}}} \subset \mathcal{D}(C\text{-comod})$ the full triangulated subcategory in the homotopy category formed by all the CDG-comodules whose underlying graded $C$-comodules are injective. Such CDG-comodules are called graded-injective.

**Theorem 7.13.** Let $C^* = (C, d, h)$ be a CDG-coalgebra over $k$. Then the composition $\mathcal{D}(C^*\text{-comod}_{\text{in}}} \rightarrow \mathcal{D}(C^*\text{-comod}) \rightarrow \mathcal{D}(C^*\text{-comod})$ of the triangulated inclusion functor $\mathcal{D}(C^*\text{-comod}_{\text{in}}} \rightarrow \mathcal{D}(C^*\text{-comod})$ and the Verdier quotient functor $\mathcal{D}(C^*\text{-comod}) \rightarrow \mathcal{D}(C^*\text{-comod})$ is an equivalence of triangulated categories,

$$\mathcal{D}(C^*\text{-comod}_{\text{in}}) \simeq \mathcal{D}(C^*\text{-comod}).$$

**Proof.** This is [64, Theorem 4.4(c)]. For a generalization to exact DG-categories, see [77, Theorem 5.10(a)].

**Remark 7.14.** For any DG-ring $A^* = (A, d)$, the derived category of DG-modules $D(A^*\text{-mod})$ is compactly generated (in the sense of [50, 51]). In fact, it follows immediately from the definitions that the free DG-module $A^*$ over $A^*$ is a single compact generator of $D(A^*\text{-mod})$ [43, section 3.5]. On the coalgebra side of Koszul duality, the related observation is that, for any CDG-coalgebra $C^* = (C, d, h)$ over $k$, the coderived category of CDG-comodules $D^c(C^*\text{-comod})$ is compactly generated. The CDG-comodules whose underlying graded $k$-vector spaces are (totally) finite-dimensional form a set of compact generators of $D^c(C^*\text{-comod})$ [64, section 5.5]. The latter result has many analogous versions and far-reaching generalizations, including the case of the coderived category of CDG-modules over a graded Noetherian CDG-ring [64, section 3.11] (see also [19, Proposition 1.5(d)]), the Becker coderived category of a locally coherent abelian category [87, Corollary 6.13], the coderived category of a locally Noetherian abelian DG-category [77, Theorem 9.23], and the coderived category of a locally coherent abelian DG-category under a certain assumption of ‘finite fp-projective dimension’ [77, Theorem 9.39].

### 7.8 History, II, and conclusion

Let me try to explain my original (end of March 1999) motivation for introducing Definitions 7.5 and 7.11. The following remark, purporting to serve as the explanation, is written in the notation of twisting cochains, because it is very convenient; though back in Spring 1999 I was not familiar with twisting cochains. Rather, I was thinking in terms of Koszul duality functors arising in the context of bar- and cobar-constructions, such as in Theorems 2.1, 2.6, and 2.7.
Remark 7.15. The trouble with Koszul duality functors is that they can take acyclic complexes or DG-comodules to nonacyclic ones, as illustrated by the example in Remark 7.2. By contrast, one can note that the Koszul duality functors mentioned above in this survey always take coacyclic objects to contractible ones!

In full generality, let $B^*$ be a CDG-algebra, $C^*$ be a CDG-coalgebra, and $\tau : C^* \to B^*$ be a twisting cochain. Then the CDG-module $B^* \otimes^\mathbb{L} N^*$ over $B^*$ is contractible for any coacyclic CDG-comodule $N^*$ over $C^*$. Similarly, the CDG-comodule $C^* \otimes^\mathbb{L} M^*$ over $C^*$ is contractible for any coacyclic CDG-module $M^*$ over $B^*$.

Indeed, let us consider, for example, a short exact sequence $0 \to K^* \to L^* \to M^* \to 0$ of CDG-comodules over $C^*$. Note that, for any CDG-comodule $N^*$ over $C^*$, the underlying graded $B$-module structure of the CDG-module $B^* \otimes^\mathbb{L} N^*$ does not depend either on the differential or on the graded $C$-comodule structure on $N^*$, but only on the underlying graded vector space of $N^*$. But as a short exact sequence of graded vector spaces, the sequence $0 \to K^* \to L^* \to M^* \to 0$ is split. Consequently, the induced short sequence of CDG-modules $0 \to B^* \otimes^\mathbb{L} K^* \to B^* \otimes^\mathbb{L} L^* \to B^* \otimes^\mathbb{L} M^* \to 0$ over $B^*$ is not only exact, but its underlying short exact sequence of graded $B$-modules is even split exact.

Now one can easily see that, for any short exact sequence of CDG-modules over $B^*$ that is split exact as a short exact sequence of graded $B$-modules, the corresponding total CDG-module over $B^*$ is contractible. Obviously, one has $B^* \otimes^\mathbb{L} \text{Tot}(K^* \to L^* \to M^*) = \text{Tot}(B^* \otimes^\mathbb{L} K^* \to B^* \otimes^\mathbb{L} L^* \to B^* \otimes^\mathbb{L} M^*)$. Thus, the CDG-module $B^* \otimes^\mathbb{L} \text{Tot}(K^* \to L^* \to M^*)$ over $B^*$ is contractible.

While my work on derived categories of the second kind remained unwritten until 2007 or 2009, other people were approaching the same or almost the same concepts from various angles. Filtered quasi-isomorphisms of DG-coalgebras were introduced by Hinich [30] back in 1998, and the same approach was extended to DG-comodules by Lefèvre-Hasegawa [47] in 2003. In the same year, the term coderived category, together with its first definition for DG-comodules, appeared in Keller’s note [42].

The study of the homotopy category of unbounded complexes of projective modules was initiated by Jørgensen [38] in 2003, and for the homotopy category of unbounded complexes of injective objects in a locally Noetherian Grothendieck abelian category this was done by Krause [46] in 2004. Later this line of research was taken up and developed by Neeman in [52, 53] and St’ovíček in [87].

In the meantime, essentially the same constructions as in my Definitions 7.4–7.6, of what they called ’derived categories’ in quotes, were arrived at by Keller, Lowen, and Nicolás [44] in 2006, but their results remained unavailable to the public until after my seminar talk in Paris in April 2009. The first arXiv version of my research monograph [63], where Definitions 7.5–7.6 were spelled out in the context of exact categories, appeared in 2007; and the first arXiv version of the memoir [64], where these definitions were presented and studied for CDG-modules, CDG-comodules, and CDG-contramodules, became available in 2009.

To summarize, the theory of derived categories of the second kind identifies three constructions, showing that they produce one and the same triangulated category. Here $C^* = (C, d, h)$ is a conilpotent CDG-coalgebra over a field $k$:

1. the coderived category of left CDG-comodules over $C^*$ defined as in the note [42, section 4], using the cobar-construction in order to pass to DG-modules over the DG-algebra $\text{Cob}_\gamma^\ast(C^*)$;
(2) the homotopy category of graded-injective left CDG-comodules over $C^*$ (that is, CDG-comodules with injective underlying graded $C$-comodules), which is the CDG-comodule version of the homotopy category of complexes of injective modules as in [46, section 2];

(3) the coderived category of left CDG-comodules over $C^*$ as per Definition 7.11, which is the comodule version of [44, section 3.1 (A1–A2)].

The equivalence of (1) and (3) is a corollary of Theorem 5.4 (for DG-coalgebras) or Theorem 6.12 (in full generality). The equivalence of (2) and (3) does not depend on the conilpotency assumption and holds for any CDG-coalgebra $C^*$ over $k$; this is Theorem 7.13.

### 7.9 Becker’s derived categories of the second kind

From our point of view, the most important development in the theory of derived categories of the second kind after [63, 64] was Becker’s paper [5]. The specific result which is presumed here, [5, Proposition 1.3.6], is stated in the language of abelian model structures; so we will translate it below into the perhaps more familiar language of triangulated categories. This is an extension of the approach of [38, 46, 52, 53] from complexes of modules into the CDG-module realm.

After some hesitation, we dared to put in writing in [81, Remark 9.2] the terminology of derived categories of the second kind in the sense of Positselski versus derived categories of the second kind in the sense of Becker. The definitions in Subsections 7.6 and 7.7 are those of derived categories of the second kind in my sense.

We are following the expositions in [81, sections 7 and 9] (in the case of abelian categories) and [76, sections 4.2–4.3] (in the case case of CDG-modules). Let $B^*$ be a CDG-ring. A left CDG-module $X^*$ over $B^*$ is said to be contraacyclic in the sense of Becker if the complex of abelian groups $\text{Hom}_B^\bullet(P^*, X^*)$ is acyclic for any graded-projective left CDG-module $P^*$ over $B^*$. The thick subcategory of Becker-contraacyclic CDG-modules in the homotopy category is denoted by $\mathcal{A}_{bct}^c(B^* \text{-mod}) \subset \mathcal{H}(B^* \text{-mod})$; and the related triangulated Verdier quotient category

$$D^{btr}(B^* \text{-mod}) = \mathcal{H}(B^* \text{-mod}) / \mathcal{A}_{bct}^c(B^* \text{-mod})$$

is called the contraderived category of left CDG-modules over $B^*$ in the sense of Becker.

Dually, a left CDG-module $Y^*$ over $B^*$ is said to be coacyclic in the sense of Becker if the complex of abelian groups $\text{Hom}_B^\bullet(Y^*, J^*)$ is acyclic for any graded-injective left CDG-module $J^*$ over $B^*$. The thick subcategory of Becker-coacyclic CDG-modules in the homotopy category is denoted by $\mathcal{A}_{bco}^c(B^* \text{-mod}) \subset \mathcal{H}(B^* \text{-mod})$; and the related triangulated Verdier quotient category

$$D^{bco}(B^* \text{-mod}) = \mathcal{H}(B^* \text{-mod}) / \mathcal{A}_{bco}^c(B^* \text{-mod})$$

is called the coderived category of left CDG-modules over $B^*$ in the sense of Becker.

The result of [64, Theorem 3.5] tells that co/contraacyclicity in the sense of Positselski implies co/contraacyclicity in the sense of Becker. Thus, Becker’s co/contraderived categories are ‘nonstrictly smaller’ than mine.

Becker’s result [5, Proposition 1.3.6] tells (or rather, implies) that the compositions of triangulated functors

$$\mathcal{H}(B^* \text{-mod}_{\text{proj}}) \longrightarrow \mathcal{H}(B^* \text{-mod}) \longrightarrow D^{btr}(B^* \text{-mod})$$
and
\[ \text{Hot}(B^\bullet - \text{mod}_{\text{inj}}) \rightarrow \text{Hot}(B^\bullet - \text{mod}) \rightarrow D^{\text{bco}}(B^\bullet - \text{mod}) \]
are triangulated equivalences
\[ D^{\text{bctr}}(B^\bullet - \text{mod}) \simeq \text{Hot}(B^\bullet - \text{mod}_{\text{proj}}) \quad \text{and} \quad D^{\text{bco}}(B^\bullet - \text{mod}) \simeq \text{Hot}(B^\bullet - \text{mod}_{\text{inj}}) \]
for any CDG-ring $B^\bullet$. Thus, one can say that ‘the Becker analogue of Theorem 7.9 holds for any CDG-ring’.

On the other hand, with [5, Proposition 1.3.6] in mind, one can interpret Theorem 7.9 as telling that under the assumption of condition (\(*\)) or (\(\ast\ast\)), Becker’s and Positselski’s derived categories of the second kind agree. It is an open problem whether they agree for an arbitrary CDG-ring; in fact, this is not known even for complexes of modules over a ring (see [70, Examples 2.5(3) and 2.6(3)] for a discussion).

Positselski’s and Becker’s derived categories of the second kind are known to agree for CDG-comodules or CDG-contramodules over a CDG-coalgebra over a field (see Theorem 7.13 and Theorem 9.4). It is in this sense that one says that ‘the coderived categories are (known to be) better behaved for comodules, and the contraderived categories for contramodules’.

The definitions in [19, 44, 59, 63, 64, 70, 75, 77, 78] are those of co/contraderived categories in the sense of Positselski. The definitions in [5, 38, 46, 52, 53, 76, 81, 87] are those of co/contraderived categories in the sense of Becker.

8 | CONTRAMODULES OVER COALGEBRAS

Contramodules are dual analogues of comodules. To any coalgebra (over a field), one can assign the (abelian) categories of left and right contramodules over it, alongside with the comodule categories. The structure theory of contramodules over a coalgebra over a field is more complicated, but not too much more complicated, than the structure theory of comodules. The historical obscurity/neglect of contramodules is responsible for the popular misconception that injective objects are much more common than projective objects in ‘naturally appearing’ abelian categories. We suggest the survey paper [71] as the standard reference source on contramodules.

8.1 | The basics of contramodules

The definitions of coalgebras and comodules (as in Subsection 2.1) are obtained by writing down the definitions of algebras and modules in the tensor notation and inverting the arrows. To arrive to the definition of a contramodule, it remains to note that there are two equivalent ways to express the definition of a module in a tensor (or Hom) notation.

An algebra over a field $k$ is a vector space $A$ endowed with linear maps $m : A \otimes_k A \rightarrow A$ (the multiplication map) and $e : k \rightarrow A$ (the unit map). A left $A$-module $M$ is a $k$-vector space endowed with a linear map $n : A \otimes_k M \rightarrow M$ (the action map). The usual associativity and unitality axioms need to be imposed. But there is a different way to spell out the definition of a module.
A left $A$-module $M$ is the same thing as a $k$-vector space endowed with a $k$-linear map $p : M \to \text{Hom}_k(A, M)$. This is just an expression of the tensor-Hom adjunction: the maps $n$ and $p$ are connected by the rule $p(x)(a) = n(a \otimes x) \in M$ for all $x \in M$ and $a \in A$. We leave it to the reader to write down the associativity and unitality axioms for an $A$-module $M$ in terms of the map $p$.

The definition of a contramodule over a coalgebra $C$ over a field $k$ is obtained by inverting the arrows in the definition of a module given in terms of the map $p$. We refer to [9, section 2.6] for a relevant discussion of monad–comonad and comonad–monad adjoint pairs.

Specifically, let $C$ be a (coassociative and counital) coalgebra over a field $k$. A left $C$-contramodule $P$ is a $k$-vector space endowed with a $k$-linear map of left $C$-contraaction

$$\pi : \text{Hom}_k(C, P) \to P$$

satisfying the following contraassociativity and contraunitality axioms. First, the two compositions

$$\text{Hom}_k(C, \text{Hom}_k(C, P)) \simeq \text{Hom}_k(C \otimes_k C, P) \Rightarrow \text{Hom}_k(C, P) \to P$$

must be equal to each other, $\pi \circ \text{Hom}_k(C, \pi) = \pi \circ \text{Hom}_k(\mu, P)$. Second, the composition

$$P \to \text{Hom}_k(C, P) \to P$$

must be equal to the identity map, $\pi \circ \text{Hom}_k(\epsilon, P) = \text{id}_P$ [71, section 1.1]. Here, as in Subsection 2.1, $\mu : C \to C \otimes_k C$ and $\epsilon : C \to k$ are the comultiplication and counit maps of the coalgebra $C$.

In the definition of a left contramodule, the identification $\text{Hom}_k(C, \text{Hom}_k(C, P)) \simeq \text{Hom}_k(C \otimes_k C, P)$ is obtained as a particular case of the adjunction isomorphism $\text{Hom}_k(U, \text{Hom}_k(V, W)) \simeq \text{Hom}_k(V \otimes_k U, W)$, where $U$, $V$, and $W$ are arbitrary $k$-vector spaces. In the definition of a right contramodule, the isomorphism $\text{Hom}_k(V, \text{Hom}_k(U, W)) \simeq \text{Hom}_k(V \otimes_k U, W)$ is presumed.

For any right $C$-comodule $N$ and any $k$-vector space $V$, the vector space $\text{Hom}_k(N, V)$ has a natural structure of left $C$-contramodule (see [71, section 1.2] for the details). The left $C$-contramodule $\text{Hom}_k(C, V)$ is called the free left $C$-contramodule generated by the vector space $V$.

For any left $C$-contramodule $Q$, the $k$-vector space of all left $C$-contramodule maps $\text{Hom}_k(C, V) \to Q$ is naturally isomorphic to the $k$-vector space of all $k$-linear maps $V \to Q$,

$$\text{Hom}^C(\text{Hom}_k(C, V), Q) \simeq \text{Hom}_k(V, Q).$$

Hence, the ‘free contramodule’ terminology.

### 8.2 Duality-analogy of comodules and contramodules

The duality-analogy (or ‘covariant duality’) between comodules and contramodules is a remarkable yet unfamiliar phenomenon. It can be expressed by saying that the categories of comodules and contramodules look as though they were opposite categories—up to a point. In fact, no category of contramodules is ever the opposite category to a category of comodules; for example, over the coalgebra $C = k$, both the categories of $C$-comodules and $C$-contramodules coincide with the
category of $k$-vector spaces (which is certainly not equivalent to its opposite category). But the analogy is striking.

Let $C$ be a coalgebra over $k$. Then both the category of left $C$-comodules $C\text{-comod}$ and the category of left $C$-contramodules $C\text{-contra}$ are abelian.

The abelian category $C\text{-comod}$ has enough injective objects; in fact, the injective comodules are precisely the direct summands of the cofree comodules (as defined in Subsection 2.1). The abelian category $C\text{-contra}$ has enough projective objects; in fact, the projective contramodules are precisely the direct summands of the free contramodules (as defined in Subsection 8.1) [71, section 1.2].

The forgetful functor $C\text{-comod} \longrightarrow k\text{-vect}$ from the category of $C$-comodules to the category of $k$-vector spaces is exact and preserves infinite coproducts (but not infinite products). Consequently, the coproduct functors in $C\text{-comod}$ are exact (moreover, so are the functors of filtered colimit).

The forgetful functor $C\text{-contra} \longrightarrow k\text{-vect}$ is exact and preserves infinite products (but not infinite coproducts). Consequently, the product functors in $C\text{-contra}$ are exact (but the filtered limits are not exact in $C\text{-contra}$, of course, as they are not exact already in $k\text{-vect}$).

A wide class of abelian categories to which the categories of comodules belong is called the class of Grothendieck categories. A wide class of abelian categories to which the categories of contramodules belong is called the class of locally presentable abelian categories with enough projective objects. The main reference source on the latter class of abelian categories is the preprint [72] (see also [79, section 6]). The duality-analogy (or ‘covariant duality’) between these two classes of abelian categories is emphasized in the paper [81] (cf. [79, 80]).

### 8.3 Comodules and contramodules over power series in one variable

The aim of this section is to introduce the intuition of contramodules as modules with infinite summation operations.

The dual vector space to an infinite-dimensional (discrete) vector space comes endowed with a natural topology; such topological vector spaces are known as linearly compact or pseudocompact (or pro-finite-dimensional). In particular, the dual vector spaces to infinite-dimensional coalgebras are topological algebras. In fact, the category of coalgebras is anti-equivalent to the category of linearly compact topological algebras (see the discussion in [71, first paragraph of section 1.3]).

So, coalgebras can be identified by the names of their dual topological algebras. In particular, let $kz^*$ be a one-dimensional vector space with the basis vector $z^*$. Consider the tensor coalgebra $C = T(kz^*)$, as defined in Subsection 2.3. Then the dual topological algebra of $C$ is the algebra $C^* = k[[z]]$ of formal Taylor power series in the variable $z$ (cf. Remark 6.4).

Let $C$ be a coalgebra over $k$. As explained in Subsection 6.3, any comodule over a coalgebra $C$ is a module over the algebra $C^*$. Similarly, any contramodule over $C$ is also a module over $C^*$. The composition

$$C^* \otimes_k P \longrightarrow \text{Hom}_k(C, P) \longrightarrow P$$

of the natural injective map $C^* \otimes_k P \longrightarrow \text{Hom}_k(C, P)$ with the contraaction map $\pi : \text{Hom}_k(C, P) \longrightarrow P$ defines an action of $C^*$ in $P$.

We would like to describe comodules and contramodules over the coalgebra $C = T(kz^*)$. Namely, a $C$-comodule $M$ is the same thing as a $k[z]$-module with a locally nilpotent action of
the operator \( z \). This means that for every element \( m \in M \) there exists an integer \( n \geq 1 \) such that \( z^n m = 0 \) in \( M \). The coaction map \( M \longrightarrow C \otimes_k M \) is given in terms of the \( k[z] \)-module structure on \( M \) by the formula

\[
m \mapsto \sum_{n=0}^{\infty} z^n \otimes z^n m \quad \text{for all } m \in M,
\]

and the condition of local nilpotence of the action of \( z \) in \( M \) comes from the condition that the sum in the right-hand side must be finite (if it is to define an element of the tensor product \( C \otimes_k M \)).

A \( C \)-contramodule \( P \) is the same thing as a \( k \)-vector space with the following \( z \)-power infinite summation operation. To every sequence of elements \( p_0, p_1, p_2 \ldots \in P \), an element denoted formally by \( \sum_{n=0}^{\infty} z^n p_n \in P \) is assigned. The following axioms must be satisfied:

\[
\sum_{n=0}^{\infty} z^n (a p_n + b q_n) = a \sum_{n=0}^{\infty} z^n p_n + b \sum_{n=0}^{\infty} z^n q_n \quad \text{for all } p_n, q_n \in P, \ a, b \in k
\]
(linearity),

\[
\sum_{n=0}^{\infty} z^n p_n = p_0 \quad \text{if } p_1 = p_2 = \cdots = 0 \text{ in } P
\]
(contraunitality), and

\[
\sum_{i=0}^{\infty} z^i \sum_{j=0}^{\infty} z^j p_{ij} = \sum_{n=0}^{\infty} z^n \sum_{i+j=n}^{i \geq j \geq 0} p_{ij}
\]
for all \( p_{ij} \in P \) (contraassociativity). In the latter equation, the first three summation signs denote the \( z \)-power infinite summation operation, while the forth one means a finite sum of elements in \( P \) [71, section 1.3].

The intuition of contramodules as modules with infinite summation operations is axiomatized in the concepts of contramodules over a commutative ring with a finitely generated ideal [68, sections 3–4], [74, section 1], contramodules over topological rings and topological associative algebras [71, sections 2.1 and 2.3], [79, section 6], contramodules over topological Lie algebras [71, sections 1.7 and 2.4], and so on.

One special property of the coalgebra \( C = J(kz^*) \) is that a \( C \)-contramodule structure can be uniquely recovered from its underlying \( k[[z]] \)-module structure. In fact, the forgetful functor to the category of modules over the polynomial algebra \( C \)-contra \( \longrightarrow k[z] \)-mod is already fully faithful [63, Remark A.1.1], [68, Theorem 3.3], [65, Theorem B.1.1]. This is not true for an arbitrary coalgebra \( C \), of course; still this is true for any finitely cogenerated conilpotent coalgebra \( C \) [69, Theorem 2.1]. See [71, section 3.8] for a discussion of far-reaching generalizations.

### 8.4 Nonseparated contramodules

The duality-analogy between comodules and contramodules has some limitations, though. The examples of nonseparated contramodules demonstrate such limitations.

Let \( f : C \longrightarrow D \) be a homomorphism of coalgebras. Then, as mentioned in Subsection 3.1, every \( C \)-comodule acquires a \( D \)-comodule structure. Similarly, every \( C \)-contramodule acquires a \( D \)-contramodule structure. The resulting functor \( C \)-contra \( \longrightarrow D \)-contra is called the contrarestriction of scalars with respect to \( f \).
In particular, let $E \subset C$ be a subcoalgebra. Then the corestriction and contrarestriction of scalars are fully faithful functors
\[
E\text{-comod} \rightarrow C\text{-comod},
\]
\[
E\text{-contra} \rightarrow C\text{-contra}.
\]

The functor $E\text{-comod} \rightarrow C\text{-comod}$ has a right adjoint functor assigning to every left $C$-comodule $M$ its maximal subcomodule $E M \subset M$ whose $C$-comodule structure comes from an $E$-comodule structure. The $E$-comodule $E M$ can be computed as the kernel of the composition of maps $M \rightarrow C \otimes_k M \rightarrow C/E \otimes_k M$.

The functor $E\text{-contra} \rightarrow C\text{-contra}$ has a left adjoint functor assigning to every left $C$-contramodule $P$ its maximal quotient contramodule $E P$ whose $C$-contramodule structure comes from an $E$-contramodule structure. The $E$-contramodule $E P$ can be computed as the cokernel of the composition of maps $\text{Hom}_k(C/E, P) \rightarrow \text{Hom}_k(C, P) \rightarrow P$.

According to Lemma 3.1, any $C$-comodule $M$ is the union of its finite-dimensional subcomodules, and each of these is a comodule over a finite-dimensional subcoalgebra in $C$. Therefore, one has
\[
M = \bigcup_{E \subset C} E M,
\]
where the union is taken over all the finite-dimensional subcoalgebras $E \subset C$.

The dual assertion for contramodules is not true. Specifically, for any $C$-contramodule $P$ one can consider the natural map
\[
P \rightarrow \lim_{E \subset C} E P,
\]  
where the inverse limit is taken over all the finite-dimensional subcoalgebras $E \subset C$. The map (3) is not injective in general.

In fact, there exists an infinite-dimensional coalgebra $C$ and a two-dimensional $C$-contramodule $P$ for which $\lim_{E \subset C} E P$ is a one-dimensional $C$-contramodule. So, the $C$-contramodule structure on $P$ does not come from a contramodule structure over any finite-dimensional subcoalgebra in $C$, and the intersection of the kernels of the maps $P \rightarrow E P$ is a one-dimensional subcontramodule in the two-dimensional contramodule $P$ [63, section A.1.2].

Let us now consider the example of the coalgebra $C = J(kz^a)$ from Subsection 8.3. Then the map (3) is simply the natural map from $P$ to the $z$-adic completion of $P$,
\[
P \rightarrow \lim_{n \geq 1} P/z^n P.
\]  
The map (4) is known to be surjective for any $C$-contramodule $P$ [63, Lemma A.2.3], [68, Theorems 3.3 and 5.6], [74, section 1]. But there exist $C$-contramodules $P$ for which the map (4) is not injective. In other words, any $C$-contramodule is $z$-adically complete, but it need not be $z$-adically separated. The now-classical counterexample can be found in [85, Example 2.5], [63, section A.1.1], [90, Example 3.20], [68, Example 2.7(1)], [71, section 1.5].

In fact, the mentioned counterexample from [63, 85, 90] is a $C$-contramodule $P$ with the following property. There exists a sequence of elements $p_0, p_1, p_2, \ldots \in P$ such that $z^n p_n = 0$ for every $n \geq 0$, but $\sum_{n=0}^{\infty} z^n p_n \neq 0$ in $P$. Here we use the $z$-power infinite summation operation notation from Subsection 8.3. So, the contramodule infinite summation operations cannot be interpreted as any kind of limits of finite partial sums [63, Preface], [71, section 0.2 of the introduction].
8.5  |  Contramodule Nakayama lemma and irreducible contramodules

As explained in Subsection 8.4, the map (3) need not be injective. However, this map never vanishes (for a nonzero \(C\)-contramodule \(P\)).

Dropping the contraunitality axiom from the definition of a contramodule in Subsection 8.1, one obtains the definition of a contramodule over a noncounital coalgebra \(D\). The following result is called the \textit{contramodule Nakayama lemma} (for contramodules over coalgebras over a field).

\textbf{Lemma 8.1.}

(a) Let \(D\) be a conilpotent noncounital coalgebra (as defined in Subsection 3.3) and \(P\) be a nonzero \(D\)-contramodule. Then the contraaction map \(\pi : \text{Hom}_k(D, P) \rightarrow P\) is not surjective.

(b) Let \(C\) be a (counital) coalgebra and \(P\) be a nonzero left \(C\)-contramodule. Then there exists a finite-dimensional subcoalgebra \(E \subset C\) such that \(EP \neq 0\).

\textbf{Proof.} Part (a) is [63, Lemma A.2.1] (see [65, Lemma 1.3.1], [71, Lemmas 2.1 and 3.22], and the references therein for generalizations). To deduce part (b) from part (a), one has to use the structure theory of coalgebras, or more specifically, the fact that the maximal cosemisimple subcoalgebra \(C^{ss} \subset C\) is a direct sum of finite-dimensional (cosimple) coalgebras and the quotient coalgebra without counit \(D = C/C^{ss}\) is conilpotent (see Subsection 3.4) together with a description of contramodules over an infinite direct sum of coalgebras [63, Lemma A.2.2]. \(\square\)

It follows from Lemma 8.1(b) that any irreducible \(C\)-contramodule is finite-dimensional (moreover, it is a contramodule over a finite-dimensional subcoalgebra in \(C\)).

In fact, while there is, of course, no way to define a contramodule structure on an arbitrary \(C\)-comodule, there does exist a natural way to define a left \(C\)-contramodule structure on any finite-dimensional left \(C\)-comodule [63, section A.1.2]. Indeed, the functor \(N \rightarrow N^*\) is an anti-equivalence between the categories of finite-dimensional right and left \(C\)-comodules, and on the other hand, the dual vector space to a right \(C\)-comodule is a left \(C\)-contramodule (as mentioned in Subsection 8.1). This produces a fully faithful covariant functor from the category of finite-dimensional left \(C\)-comodules to the category of finite-dimensional left \(C\)-contramodules; a finite-dimensional left \(C\)-contramodule \(P\) belongs to the essential image of this functor if and only if its contramodule structure comes from an \(E\)-contramodule structure for some finite-dimensional subcoalgebra \(E \subset C\) (that is, \(P = EP\)).

This covariant fully faithful functor restricts to a bijection between the isomorphism classes of irreducible left \(C\)-comodules and irreducible left \(C\)-contramodules. Both the sets of isomorphism classes of irreducibles are also naturally bijective to the set of all cosimple subcoalgebras in \(C\), of course.

8.6  |  Contratensor product

The formalisms of tensor product and Hom-type operations play a key role in the book [63] and the memoir [65]. In the ring and module theory, the formalism of tensor/Hom operations has only two such operations, namely, the tensor product and Hom of (bi)modules. In the context of coalgebras, contramodules, and so on, the tensor/Hom operations are more numerous.
In particular, the definition of the cotensor product of comodules goes back, at least, to the paper [23]. On the other hand, the construction of the contratensor product of a comodule and a contramodule seems to have first appeared in [62, 63].

Let \( C \) be a coalgebra, \( N \) be a right \( C \)-comodule, and \( P \) be a left \( C \)-contramodule. The contratensor product \( N \otimes_C P \) is a \( k \)-vector space constructed as the cokernel of the difference of two natural maps

\[
N \otimes_k \text{Hom}_k(C, P) \rightrightarrows N \otimes_k P.
\]

Here the first map \( N \otimes_k \text{Hom}_k(C, P) \rightarrow N \otimes_k P \) is induced by the contraction map \( \pi : \text{Hom}_k(C, P) \rightarrow P \), while the second map is the composition \( N \otimes_k \text{Hom}_k(C, P) \rightarrow N \otimes_k C \otimes_k \text{Hom}_k(C, P) \rightarrow N \otimes_k P \) of the map induced by the coaction map \( \nu : N \rightarrow N \otimes_k C \) and the map induced by the natural evaluation map \( C \otimes_k \text{Hom}_k(C, P) \rightarrow P \) [71, section 3.1].

For any right \( C \)-comodule \( N \) and any \( k \)-vector space \( V \), there is a natural isomorphism of vector spaces

\[
N \otimes_C \text{Hom}_k(C, V) \cong N \otimes_k V.
\]

For any right \( C \)-comodule \( N \), any left \( C \)-contramodule \( P \), and any vector space \( V \), there is a natural isomorphism of vector spaces

\[
\text{Hom}_C(P, \text{Hom}_k(N, V)) \cong \text{Hom}_k(N \otimes_C P, V),
\]

where, as in Subsection 8.1, \( \text{Hom}_C \) denotes the space of morphisms in the category of left \( C \)-contramodules.

For a discussion of a generalization of the construction of contratensor product to topological rings, see [79, section 7.2] (some further discussion and references can be found in [71, end of section 3.3]).

### 8.7 Underived co–contra correspondence

Let \( C \) be a (coassociative, counital) coalgebra over a field \( k \). As mentioned in Subsection 8.2, there are enough injective objects in the abelian category of left \( C \)-comodules, and these injectives are precisely the direct summands of the cofree left \( C \)-comodules \( C \otimes_k V \) (where \( V \) ranges over the \( k \)-vector spaces). Similarly, there are enough projective objects in the abelian category of left \( C \)-contramodules, and these projectives are precisely the direct summands of the free left \( C \)-contramodules \( \text{Hom}_k(C, V) \).

It turns out that there is a natural equivalence between the additive categories of injective left \( C \)-comodules and projective left \( C \)-contramodules,

\[
C\text{–comod}_{\text{inj}} \simeq C\text{–contra}_{\text{proj}}.
\]

The simplest way to construct the category equivalence (5) is to define it on cofree comodules and free contramodules by the rule \( C \otimes_k V \leftrightarrow \text{Hom}_k(C, V) \). Then one has to compute that the
groups of morphisms agree,
\[
\text{Hom}_C(C \otimes_k U, C \otimes_k V) \simeq \text{Hom}_k(C \otimes_k U, V) \\
\simeq \text{Hom}_k(U, \text{Hom}_k(C, V)) \simeq \text{Hom}^C(\text{Hom}_k(C, U), \text{Hom}_k(C, V))
\]
for any \( k \)-vector spaces \( U \) and \( V \), in view of the descriptions of morphisms into a cofree comodule and from a free contramodule in Subsections 2.1 and 8.1. A category-theoretic version of this argument (an equivalence of Kleisli categories for adjoint comonad–monad pairs) is discussed in [9, section 2.6(2)].

Another way to obtain the additive category equivalence (5) is to construct a pair of adjoint functors
\[
\text{Hom}_C(C, -) : C \text{–}\mathbb{L}
\mathbb{U}
\mathbb{E}
\mathbb{N}
\mathbb{O}
\mathbb{D}
\mathbb{A}
\mathbb{S}
\mathbb{O}
\mathbb{M}
\mathbb{D}
\text{–}\mathbb{C}
\to C \text{–}\mathbb{C}
\]
\[
C \otimes C - (6)
\]
and check that it restricts to an equivalence between the full subcategories of injective comodules and projective contramodules.

Here the right adjoint functor \( \text{Hom}_C(C, -) \) is the Hom functor in the comodule category \( C \text{–}\mathbb{C} \text{omod} \). Taking the Hom eats up the left \( C \)-comodule structure on \( C \), but the right \( C \)-comodule structure on \( C \) stays and induces a left \( C \)-contramodule structure on the Hom space, essentially as explained in Subsection 8.1. The left adjoint functor \( C \mathcal{O}_C - \) is the functor of contratensor product. Taking the contratensor product consumes the right \( C \)-comodule structure on \( C \), but the left \( C \)-comodule structure on \( C \) remains and induces a left \( C \)-comodule structure on the contratensor product (cf. [71, sections 1.2 and 3.1]).

For much more advanced discussions of the philosophy of comodule-contramodule correspondence, see the introductions to the papers [66, 70]. One important early work related to the co–contra correspondence is [35, section 4]. From the contemporary point of view, the co–contra correspondence can be thought of as a particular case of the tilting-cotilting correspondence [79, 80]. A discussion of underived versions of the co–contra correspondence containing various generalizations of the results above in this section can be found in [71, sections 3.4–3.6].

8.8 History of contramodules

The notion of a contramodule (over a coalgebra over a commutative ring) was invented by Eilenberg and Moore and defined on par with comodules in the 1965 memoir [22, section III.5]. Besides [22], two other papers on contramodules were published in 1965 and 1970, among them the rather remarkable paper [3]. Then contramodules were all but forgotten for 30–40 years.

In the meantime, what came to be known as ‘Ext-\( p \)-complete’ or ‘weakly-\( I \)-complete’ abelian groups [12, 36] or (in a later terminology) ‘cohomologically \( I \)-adically complete’ or ‘derived \( I \)-adically complete modules’ (for a finitely generated ideal \( I \) in a commutative ring) [60] were defined and studied by authors who remained apparently completely unaware of the connection with Eilenberg and Moore’s contramodules. We refer to the presentation [73] for a detailed discussion of this part of the history.

As described in Subsection 7.2, in Spring 1999 the present author went to the IAS library to look for prior literature relevant to my March–April 1999 discovery of derived categories of the second kind and their role in derived nonhomogeneous Koszul duality. Among other things, I found the definition of a contramodule in a paper copy of the memoir [22] which the IAS library had.
In retrospect, I could have realized already then that contramodules form an ideal context for my definition of the contraderived category (which I already had in Spring 1999). But I did not realize that, though I was a bit unhappy about the definition of the contraderived category (called ‘the derived category D′′, at the time) having too little use. The definition of a contramodule felt strange. As many people had before and would later, I remained unimpressed by the definition of a contramodule in 1999, though I kept a recollection of it at the bottom of my memory.

My view of contramodules changed in Summer 2000 when I was working on the first (2000) part of the 'Summer letters on semi-infinite homological algebra' [62]. It was then that I recalled the definition of a contramodule and realized that contramodules are suited for use as the coefficients for semi-infinite cohomology (as opposed to semi-infinite homology) theories. The concept of comodule-contramodule correspondence was also arrived at in 2000–2002 and reflected in [62]. What is now called the contraderived category of contramodules (together with its mixed counterpart, ‘the semicontraderived category of semicontramodules’) was first considered in [62].

The realization that derived nonhomogeneous Koszul duality has a contramodule side, and should be formulated in the form of a ‘triality’ with the comodule side, the contramodule side, and the co–contra correspondence forming three sides of a triangular diagram of triangulated category equivalences, came to me by mid-'00s.

The letters [62] were eventually posted to the internet, where some people could read them, including in particular T. Brzeziński (as transliterated Russian is somewhat similar to Polish). This influenced his thinking, as exemplified by publications such as [9]. My own account of contramodule theory first appeared in the August 2007 preprint version of the book [63] and subsequently in the memoir [64].

The September 2007 presentation of Brzeziński on contramodules [13] featured the following statistics of MathSciNet search hits:

- comodules = 797;
- contramodules = 3.

As I am typing these lines (on July 19, 2022), the current statistics of MathSciNet search hits (‘Anywhere=–(−)’) is

- comodules = 1457;
- contramodules = 30.

From our point of view, contramodules (over coalgebras and corings) should be treated on par and in parallel with comodules in most contexts.

9 | CDG-CONTRAMODULES

The contramodule side of the derived Koszul duality complements the comodule side. Together with the derived comodule–contramodule correspondence, they form a commutative triangle diagram of triangulated category equivalences called the Koszul triality.

9.1 | Graded contramodules

Let $C = \bigoplus_{n \in \mathbb{Z}} C^n$ be a graded coalgebra. The definition of a graded $C$-contramodule is easily formulated by analogy with the definition of an ungraded contramodule over an ungraded coalgebra
in Subsection 8.1. All one needs to do is to replace the usual ungraded notions of the tensor product and Hom spaces with the ones intrinsic to the world of graded vector spaces.

So, a graded left $C$-contramodule is a graded $k$-vector space $P$ endowed with a left contraction map $\pi : \text{Hom}_k(C, P) \to P$, where $\text{Hom}_k(C, P)$ denotes the graded Hom space (of homogeneous $k$-linear maps $C \to P$ of various degrees $n \in \mathbb{Z}$) and $\pi$ is a morphism of graded vector spaces (that is, a homogeneous linear map of degree 0). The same contraassociativity and contraunitality axioms as in Subsection 8.1 are imposed.

However, one can easily get confused trying to define what it means to specify a grading on a given ungraded $C$-contramodule. The problem is that the conventional thinking of a grading as a direct sum decomposition that can be either ignored or taken into account at one’s will is insufficient. Rather, one needs to think of the category of graded vector spaces and the forgetful functor from it to the category of ungraded ones. The point is that there is more than one such forgetful functor. Two of them are important for the contramodule theory.

A suitable formalism was suggested in [63, section 11.1.1] and then again in [75, section 2.1]. To a graded vector space $V$ one can assign two ungraded vector spaces, namely, $\Sigma V = \bigoplus_{n \in \mathbb{Z}} V^n$ and $\Pi V = \prod_{n \in \mathbb{Z}} V^n$. Further possibilities include the Laurent summations $\bigoplus_{n < 0} V^n \oplus \prod_{n \geq 0} V^n$ and $\prod_{n \leq 0} V^n \oplus \bigoplus_{n > 0} V^n$ (cf. the beginning of Subsection 7.6), but for our purposes we do not need these options.

Given a graded module (or even a nonpositively or nonnegatively graded ring), one can choose between producing its underlying ungraded module (respectively, ring) by taking the direct sum or the direct product of the grading components, that is the forgetful functor $\Sigma$ or $\Pi$. For coalgebras, comodules, and contramodules, there is no choice. The underlying ungraded coalgebra or comodule is constructed by applying the direct sum forgetful functor $\Sigma$. The underlying ungraded contramodule is obtained by applying the direct product forgetful functor $\Pi$.

The reason for such preferences in respect to the functors of forgetting the grading lies in the natural isomorphisms

$$\Sigma(U \otimes_k V) \simeq \Sigma U \otimes_k \Sigma V,$$

$$\Pi \text{Hom}_k(V, W) \simeq \text{Hom}_k(\Sigma V, \Pi W),$$

which hold for all graded vector spaces $U$, $V$, and $W$. Here the tensor product/Hom spaces in the left-hand sides of the formulae are taken in the realm of graded vector spaces, while in the right-hand sides of the formulae the tensor product/Hom functors are applied to ungraded vector spaces.

So, if $C$ is a graded coalgebra, $M$ is a graded $C$-comodule, and $P$ is a graded $C$-contramodule, then $\Sigma M$ is an ungraded $\Sigma C$-comodule, while $\Pi P$ is an ungraded $\Sigma C$-contramodule. Thus, specifying a grading on an ungraded $C$-contramodule $Q$ means decomposing $Q$ into a direct product of its grading components [64, Remark 2.2]. We refer to [64, section and Remark 2.2] for a discussion of the related sign rules.

### 9.2 Definition of CDG-contramodules

Given a DG-coalgebra $C^*$, one can spell out the notion of a (left) DG-contramodule $P^*$ over $C^*$ similarly to the definition of a DG-comodule in Subsection 2.2. All one needs to do is to transfer the definition of a contramodule from Subsection 8.1 to the world of complexes of vector spaces.
We skip the details, which can be found in [64, section 2.3], and pass to the more general case of CDG-contramodules.

Continuing the exposition from Subsection 6.3, we use the language of precomplexes of vector spaces. Given two precomplexes $V^* = (V, d_V)$ and $W^* = (W, d_W)$, the Hom precomplex $\text{Hom}_k(V^*, W^*)$ is defined as explained in Subsection 6.8.

Now we can define the notion of a contraderivation of a contramodule. Let $C$ be a graded coalgebra endowed with an odd coderivation $d : C \longrightarrow C$ of degree 1, as defined in Subsection 6.3; and let $P$ be a graded left $C$-contramodule. Then an odd contraderivation on $P$ compatible with the coderivation $d$ on $C$ is a $k$-linear map $d_P : P \longrightarrow P$ of degree 1 such that the left coaction map $\pi : \text{Hom}_k(C, P) \longrightarrow P$ is a morphism of precomplexes. Here the differential $d$ on $\text{Hom}_k(C, P)$ is defined by the rule from Subsection 6.8 in terms of the differentials $d$ on $C$ and $d_P$ on $P$.

Furthermore, as explained in Subsection 8.3, any contramodule $P$ over a coalgebra $C$ is naturally a module over the algebra $C^*$. This construction has an obvious graded version: any graded contramodule $P$ over a graded coalgebra $C$ is naturally endowed with a graded module structure over the graded dual vector space $C^*$ to $C$, with the natural structure of graded algebra on $C^*$. Similarly to the discussion in Subsection 6.3, one has to choose between two opposite ways of defining the multiplication on $C^*$. We prefer to choose the sides so that left $C$-comodules become left $C^*$-modules; then left $C$-contramodules also become left $C^*$-modules.

Given a graded left contramodule $P$ over a graded coalgebra $C$, a homogeneous element $p \in P$, and a homogeneous linear function $b : C \longrightarrow k$ we let $b \ast p \in P$ denote the result of the left action of $b$ on $p$. We refer to [64, section 4.1] for the sign rule.

Now we can present our definition. Let $C^* = (C, d, h)$ be a CDG-coalgebra over $k$. A left CDG-contramodule $P^* = (P, d_P)$ over $C^*$ is a graded left $C$-contramodule endowed with

- an odd contraderivation $d_P : P \longrightarrow P$ of degree 1 compatible with the coderivation $d$ on $C$

such that

\[ (vii) \text{ the square of the differential } d_P \text{ on } P \text{ is described by the formula } d_P^2(p) = h \ast p \text{ for all } p \in P. \]

Similarly to the theories of CDG-modules and CDG-comodules (as in Subsections 6.2 and 6.3), left CDG-contramodules over a CDG-coalgebra $C^* = (C, d, h)$ form a DG-category $C^*$–contra. Any morphism of CDG-coalgebras $(f, a) : (C, d_C, h_C) \longrightarrow (D, d_D, h_D)$ induces a DG-functor $C^*$–contra $\longrightarrow D^*$–contra assigning to a CDG-contramodule $(P, d_P)$ the CDG-contramodule $(P, d'_P)$, with the graded $D$-contramodule structure on $P$ obtained from the graded $C$-contramodule structure on $P$ by the contrarestriction of scalars (as mentioned in Subsection 8.4) and the twisted differential $d'_P$ given by the rule $d'_P(p) = d_P(p) + a \ast p$. So, an isomorphism of CDG-coalgebras induces an isomorphism of the DG-categories of CDG-contramodules over them.

### 9.3 Contraderived category of CDG-contramodules

This section is a contramodule version of Subsections 7.6 and 7.7. In the spirit of the discussion in the end of Subsection 7.5, we will see (in Theorem 9.4) that the contraderived categories of contramodules are somewhat better behaved than the contraderived categories of modules. On the other hand, it makes no sense to consider ‘coderived categories of contramodules’, as the functors of infinite coproducts are usually not exact in contramodule categories (contramodule categories of homological dimension 1 being a remarkable exception; see [65, Remark 1.2.1]).
Let $C^* = (C, d, h)$ be a CDG-coalgebra over $k$. Similarly to Subsections 7.6 and 7.7, one can speak of short exact sequences $0 \longrightarrow P^* \longrightarrow Q^* \longrightarrow R^* \longrightarrow 0$ of left CDG-contramodules over $C^*$ and their totalizations (total CDG-contramodules) $\text{Tot}(P^* \rightarrow Q^* \rightarrow R^*)$, which are again left CDG-contramodules over $C^*$.

Let $\text{Hot}(C^*\text{--contra})$ denote the homotopy category of left CDG-contramodules over a CDG-coalgebra $C^*$. Similarly to Subsections 7.6 and 7.7, the DG-category $C^*\text{--contra}$ has shifts and cones, so its homotopy category $\text{Hot}(C^*\text{--contra})$ is triangulated.

**Definition 9.1** [64, section 4.2]. Let $C^*$ be a CDG-coalgebra. A left CDG-contramodule over $C^*$ is said to be **absolutely acyclic** if it belongs to the minimal thick subcategory of $\text{Hot}(C^*\text{--contra})$ containing the totalizations of short exact sequences of left CDG-contramodules over $C^*$. The full subcategory of absolutely acyclic CDG-contramodules is denoted by $\text{Ac}^{\text{abs}}(C^*\text{--contra}) \subset \text{Hot}(C^*\text{--contra})$. The triangulated Verdier quotient category

$$D^{\text{abs}}(C^*\text{--contra}) = \text{Hot}(C^*\text{--contra})/\text{Ac}^{\text{abs}}(C^*\text{--contra})$$

is called the **absolute derived category** of left CDG-contramodules over $C^*$.

**Definition 9.2** [63, section 4.1], [64, section 4.2]. A left CDG-contramodule over $C^*$ is said to be **contraacyclic** if it belongs to the minimal triangulated subcategory of $\text{Hot}(C^*\text{--contra})$ containing the totalizations of short exact sequences of left CDG-contramodules over $C^*$ and closed under infinite products. The thick subcategory of contraacyclic CDG-contramodules is denoted by $\text{Ac}^{\text{ctr}}(C^*\text{--contra}) \subset \text{Hot}(C^*\text{--contra})$. The triangulated Verdier quotient category

$$D^{\text{ctr}}(C^*\text{--contra}) = \text{Hot}(C^*\text{--contra})/\text{Ac}^{\text{ctr}}(C^*\text{--contra})$$

is called the **contra-derived category** of left CDG-contramodules over $C^*$.

In the context of the next theorem, it is helpful to recall that the homological dimensions of the three abelian categories of (ungraded or graded) left $C$-comodules, right $C$-comodules, and left $C$-contramodules coincide for any (ungraded or graded) coalgebra $C$ over a field $k$ (see [64, beginning of section 4.5]). The common value of these three dimensions is called the **global dimension** of the (ungraded or graded) coalgebra $C$.

**Theorem 9.3.** Let $C^* = (C, d, h)$ be a CDG-coalgebra over $k$ whose underlying graded coalgebra $C$ has finite global dimension (as a graded coalgebra; that is, the abelian category of graded left $C$-contramodules has finite homological dimension). Then the two classes of contraacyclic and absolutely acyclic CDG-contramodules over $C^*$ coincide,

$$\text{Ac}^{\text{ctr}}(C^*\text{--contra}) = \text{Ac}^{\text{abs}}(C^*\text{--contra}),$$

and accordingly, the two derived categories of the second kind coincide,

$$D^{\text{ctr}}(C^*\text{--contra}) = D^{\text{abs}}(C^*\text{--contra}).$$

**Proof.** This is [64, Theorem 4.5(b)]. For a generalization to exact DG-categories, see [77, Theorem 5.6(b) or Theorem 8.9(b)].
Note that the contramodule version of condition \((**)\) from Subsection 7.6 holds for any (graded) coalgebra \(C\) over a field \(k\), because the class of all projective \(C\)-contramodules is closed under infinite products. Indeed, the projective \(C\)-contramodules are the direct summands of the free ones (see Subsections 8.1 and 8.2), and free left \(C\)-contramodules have the form \(\text{Hom}_k(C, V)\), where \(V\) ranges over (graded) \(k\)-vector spaces; so free \(C\)-contramodules obviously form a class closed under infinite products.

Similarly to Subsections 7.6 and 7.7, we denote by \(\mathcal{D}(C, \mathcal{O}) \subset \mathcal{D}(C)\) the full triangulated subcategory in the homotopy category formed by all the CDG-contramodules whose underlying graded \(C\)-contramodules are projective. Such CDG-contramodules are called graded-projective.

**Theorem 9.4.** Let \(C = (C, d, h)\) be a CDG-coalgebra over \(k\). Then the composition \(\mathcal{D}(C, \mathcal{O}) \longrightarrow \mathcal{D}(C)\) of the triangulated inclusion functor \(\mathcal{D}(C, \mathcal{O}) \longrightarrow \mathcal{D}(C)\) and the Verdier quotient functor \(\mathcal{D}(C) \longrightarrow \mathcal{D}(C, \mathcal{O})\) is an equivalence of triangulated categories,

\[
\mathcal{D}(C, \mathcal{O}) \cong \mathcal{D}(C).
\]

**Proof.** This is [64, Theorem 4.4(d)]. For a generalization to exact DG-categories, see [77, Theorem 5.10(b)].

9.4 | Derived co–contra correspondence

Let \(C = (C, d, h)\) be a CDG-coalgebra over \(k\). Then there is a natural pair of adjoint DG-functors between the DG-categories of left CDG-comodules and left CDG-contramodules over \(C\). The right adjoint DG-functor is

\[
\text{Hom}_C(C, -) : C \text{-comod} \longrightarrow C \text{-contra},
\]

while the left adjoint DG-functor is

\[
C \odot_C - : C \text{-contra} \longrightarrow C \text{-comod}.
\]

Let us spell out the constructions of these functors. For any left CDG-comodule \(M\) over \(C\), the underlying graded left \(C\)-comodule \(\text{Hom}_C(C, M)\) of the CDG-contramodule \(\text{Hom}_C(C, M)\) is simply the graded left \(C\)-comodule of homogeneous left \(C\)-comodule morphisms \(C \longrightarrow M\). This is the graded version of the construction of the functor \(\text{Hom}_C(C, -)\) from Subsection 8.7. So, the degree \(n\) component \(\text{Hom}_C^n(C, M)\) of the graded \(C\)-comodule \(\text{Hom}_C(C, M)\) is the vector space of all homogeneous left \(C\)-comodule homomorphisms \(C \longrightarrow M\) of degree \(n\) (for every \(n \in \mathbb{Z}\)). The differential \(d\) on the left CDG-contramodule \(\text{Hom}_C(C, M)\) is given by the usual rule for the differential on the Hom space of two precomplexes, as in Subsections 6.8 and 9.2.

For any left CDG-contramodule \(P\) over \(C\), the underlying graded left \(C\)-comodule \(\text{C} \odot_C P\) of the CDG-comodule \(\text{C} \odot_C P\) is simply the contratensor product of the graded right \(C\)-comodule \(C\) and the graded left \(C\)-contramodule \(P\). Here one has to extend the construction of the contratensor product functor from Subsection 8.6 to the realm of graded coalgebras, graded comodules, and graded contramodules, which is done in the most straightforward way (see [64, section 2.2] for
The graded left $C$-comodule structure on the contratensor product $C \otimes_C P$ is induced by the graded left $C$-comodule structure on $C$, as in Subsection 8.7. The differential $d$ on the left CDG-comodule $C^\ast \otimes_C P^\ast$ is given by the usual rule for the differential of the tensor product of two precomplexes, as in Subsection 6.3.

The standard notation for the comodule–contramodule correspondence functors (as in [63–65]) is $\Psi_{C^\ast} = \text{Hom}_C(C^\ast, -)$ and $\Phi_{C^\ast} = C^\ast \otimes C -$.

**Theorem 9.5.** Let $C^\ast$ be a CDG-coalgebra over $k$. Then the adjoint DG-functors $\text{Hom}_C(C^\ast, -)$ and $C^\ast \otimes_C -$, restricted to the full DG-subcategories of graded-injective CDG-comodules and graded-projective CDG-contramodules, induce a triangulated equivalence

$$\text{Hot}(C^\ast \text{–comod}_{\text{inj}}) \approx \text{Hot}(C^\ast \text{–contra}_{\text{proj}}).$$

Consequently, the right derived functor of $\text{Hom}_C(C^\ast, -)$ and the left derived functor of $C^\ast \otimes_C -$ are mutually inverse triangulated equivalences between the coderived and the contraderived category,

$$\mathbb{R}\text{Hom}_C(C^\ast, -) : \mathcal{D}^{\text{Comp}}(C^\ast \text{–comod}) \approx \mathcal{D}^{\text{Contra}}(C^\ast \text{–contra}) : C^\ast \otimes L_{C^\ast} C -.$$

**Proof.** This is [64, Theorem 5.2]. To prove the first assertion, one observes that the adjunction morphisms $C^\ast \otimes_C \text{Hom}_C(C^\ast, M^\ast) \longrightarrow M^\ast$ and $P^\ast \longrightarrow \text{Hom}_C(C^\ast, C^\ast \otimes_C P^\ast)$ are (closed) isomorphisms of objects in the respective DG-categories for any graded-injective left CDG-comodule $M^\ast$ and graded-projective left CDG-contramodule $P^\ast$ over $C^\ast$. As the property of a closed morphism of CDG-comodules or CDG-contramodules to be an isomorphism depends only on the underlying morphism of graded co/contramodules, the claim essentially follows from the graded version of the underived co–contra correspondence theory of Subsection 8.7. In other words, the DG-functors $\text{Hom}_C(C^\ast, -)$ and $C^\ast \otimes_C -$ are mutually inverse equivalences between the DG-categories of graded-injective CDG-comodules and graded-projective CDG-contramodules, $C^\ast \text{–comod}_{\text{inj}} \approx C^\ast \text{–contra}_{\text{proj}}$; hence they induce an equivalence of the respective homotopy categories.

The second assertion follows from the first one in view of Theorems 7.13 and 9.4. Essentially, the derived functor $\mathbb{R}\text{Hom}_C(C^\ast, -)$ is constructed by applying the functor $\text{Hom}_C(C^\ast, -)$ to graded-injective CDG-comodules, while the derived functor $C^\ast \otimes L_{C^\ast} -$ is constructed by applying the functor $C^\ast \otimes C -$ to graded-projective CDG-contramodules.

For references to further discussions of the philosophy and generalizations of the co–contra correspondence, see the end of Subsection 8.7.

### 9.5 Twisted differential on the graded Hom space

This section is the contramodule version of Subsection 5.3. For the sake of simplicity of the exposition (to avoid a detailed discussion of CDG-bimodules and CDG-bicomodules), we restrict ourselves to the uncurved case.

Let $A^\ast$ be a DG-algebra and $C^\ast$ be a DG-coalgebra over $k$, and let $\tau : C^\ast \longrightarrow A^\ast$ be a twisting cochain (as defined in Subsection 5.1). Let $M^\ast = (M, d_M)$ be a left DG-module over $A^\ast$, and let $Q^\ast = (Q, d_Q)$ be a left DG-contramodule over $C^\ast$. Consider the graded Hom space $\text{Hom}_k(M, Q)$ of
the underlying graded vector spaces of $M^*$ and $Q^*$, and endow it with the differential given by the formula

$$d(f)(x) = d_Q(f(x)) - (-1)^{|f|} f(d_M(x)) \pm d^\tau(f)$$

for all homogeneous $k$-linear maps $f \in \text{Hom}_k^{/[f]}(M, Q)$ and homogeneous elements $x \in M^{[x]}$. Here $d^\tau : \text{Hom}_k(M, Q) \to \text{Hom}_k(M, Q)$ is the homogeneous map of degree 1 given by the rule

$$d^\tau(f)(x) = \pi(c \mapsto f(\tau(c)x)),$$

for all $f \in \text{Hom}_k^{/[f]}(M, Q)$ and $x \in M^{[x]}$. In this formula, $\pi : \text{Hom}_k(C, Q) \to Q$ is the contraction map, while $c \mapsto f(\tau(c)x)$ is the homogeneous $k$-linear map $C \to Q$ assigning to an element $c \in C$ the image of the element $\tau(c)x \in M$ under the map $f$. The reader can consult with [64, section 6.2] for the sign rule.

Then one can check that $d^2(f) = 0$. So, the graded vector space $\text{Hom}_k(M, Q)$ endowed with the differential $d$ is a complex. We denote this complex by $\text{Hom}^\tau(M^*, Q^*)$.

Analogously, let $N^* = (N, d_N)$ be a left DG-comodule over $C^*$, and let $P^* = (P, d_P)$ be a left DG-module over $A^*$. Consider the graded Hom space $\text{Hom}_k(N, P)$ of the underlying graded vector spaces of $N^*$ and $P^*$, and endow it with the differential given by the formula

$$d(f)(y) = d_P(f(y)) - (-1)^{|f|} f(d_N(y)) \pm d^\tau(f)$$

for all homogeneous $k$-linear maps $f \in \text{Hom}_k^{/[f]}(N, P)$ and homogeneous elements $y \in N^{[y]}$. Here $d^\tau : \text{Hom}_k(N, P) \to \text{Hom}_k(N, P)$ is the homogeneous map of degree 1 given by the rule that, for any homogeneous linear map $f \in \text{Hom}_k^{/[f]}(N, P)$, the map $d^\tau(f) \in \text{Hom}_k^{/[f]+1}(N, P)$ is equal to the composition

$$N \xrightarrow{\nu} C \otimes_k N \xrightarrow{\tau \otimes f} A \otimes_k P \xrightarrow{n} P$$

of the comultiplication map $\nu : N \to C \otimes_k N$, the map $\tau \otimes f : C \otimes_k N \to A \otimes_k P$, and the multiplication map $n : A \otimes_k P \to P$.

Once again, choosing the sign properly (cf. [64, section 6.2]), one can check that $d^2(f) = 0$. So, the graded vector space $\text{Hom}_k(N, P)$ endowed with the differential $d$ is a complex. We denote this complex by $\text{Hom}^\tau(N^*, P^*)$.

## 9.6 Derived Koszul duality on the contramodule side

This section is the contramodule version of Subsection 6.9. Let $B^*$ be a CDG-algebra and $C^*$ be a CDG-coalgebra over $k$, and let $\tau : C^* \to B^*$ be a twisting cochain (as defined in Subsection 6.8).

Given a left CDG-contramodule $Q^*$ over $C^*$, we consider the graded Hom space $\text{Hom}_k(B, Q)$ and endow it with the differential $d$ twisted with the twisting cochain $\tau$ using the same formulae as in Subsection 9.5. Then $\text{Hom}^\tau(B^*, Q^*) = (\text{Hom}_k(B, Q), d)$ is a left CDG-module over $B^*$. Here the left $B$-module structure on $B$ has been eaten up in the construction of the twisted differential on the Hom space, but the right $B$-module structure on $B$ remains and induces the underlying graded left $B$-module structure of the CDG-module $\text{Hom}^\tau(B^*, Q^*)$. 
Similarly, given a left CDG-module $P^*$ over $B^*$, we consider the graded Hom space $\text{Hom}_k(C, P)$ and endow it with the differential $d$ twisted with the twisting cochain $\tau$ using the same formulae as in Subsection 9.5. Then $\text{Hom}^\tau(C^*, P^*) = (\text{Hom}_k(C, P), d)$ is a left CDG-contramodule over $C^*$. Here the left $C$-comodule structure on $C$ has been consumed in the construction of the twisted differential on the Hom space, but the right $C$-comodule structure on $C$ stays and induces the underlying graded left $C$-comodule structure of the CDG-contramodule $\text{Hom}^\tau(C^*, P^*)$.

Recall the notation $B^*-\text{mod}$ for the DG-category of left CDG-modules over $B^*$ and $C^*-\text{contra}$ for the DG-category of left CDG-contramodules over $C^*$ (see Subsections 6.2 and 9.2). Dually to Subsection 6.9, one observes that the DG-functor

$$\text{Hom}^\tau(B^*, -): C^*-\text{contra} \longrightarrow B^*-\text{mod}$$

is right adjoint to the DG-functor

$$\text{Hom}^\tau(C^*, -): B^*-\text{mod} \longrightarrow C^*-\text{contra}.$$

**Theorem 9.6.** Let $A^*$ be a nonzero DG-algebra and $(C^*, \gamma)$ be a conilpotent CDG-coalgebra over $k$ (as defined in Subsection 6.7). Let $\tau: C^* \longrightarrow A^*$ be an acyclic twisting cochain (as defined in Subsection 6.8); this includes the condition that $\tau \circ \gamma = 0$. Then the adjoint functors $P^* \longrightarrow \text{Hom}^\tau(C^*, P^*)$ and $Q^* \longmapsto \text{Hom}^\tau(A^*, Q^*)$ induce a triangulated equivalence between the conventional derived category of left DG-modules over $A^*$ and the contraderived category of left CDG-contramodules over $C^*$,

$$\text{D}(A^*-\text{mod}) \simeq \text{D}^{\text{ctr}}(C^*-\text{contra}).$$

**Proof.** This is [64, Theorem 6.5(b)]. The particular case corresponding to the twisting cochain from Examples 6.8 can be found in [64, Theorem 6.3(b)], while the case of the twisting cochain from Example 6.9 (for a conilpotent CDG-coalgebra $C^*$) is considered in [64, Theorem 6.4(b)]. For the definition of the contraderived category, see Subsection 9.3.

The triangulated equivalence of Theorem 9.6 takes the left CDG-contramodule $k$ over $C^*$ (with the $C$-contramodule structure on $k$ defined in terms of the coaugmentation $\gamma$) to the cofree left DG-module $\text{Hom}_k(A^*, k)$ over $A^*$.

Note that there is no natural structure of a DG-module over $A^*$ on the one-dimensional vector space $k$, as the DG-algebra $A^*$ is not augmented. This fact is not unrelated to the fact that there is no natural structure of a left CDG-contramodule over $C^*$ on the free graded left $C$-contramodule $\text{Hom}_k(C, k)$ (because of a mismatch of the equations for the square of the differential involving the curvature, similar to the one explained in the end of Subsection 6.3).

**Theorem 9.7.** Let $C^*$ be a nonzero CDG-coalgebra, and let $\tau: C^* \longrightarrow \text{Cob}^*_w(C^*) = B^*$ be the twisting cochain from Example 6.10. Then the adjoint functors $P^* \longrightarrow \text{Hom}^\tau(C^*, P^*)$ and $Q^* \longmapsto \text{Hom}^\tau(B^*, Q^*)$ induce a triangulated equivalence between the absolute derived category of left CDG-modules over $B^*$ and the contraderived category of left CDG-contramodules over $C^*$,

$$\text{D}^{\text{abs}}(B^*-\text{mod}) \simeq \text{D}^{\text{ctr}}(C^*-\text{contra}).$$

**Proof.** This is [64, Theorem 6.7(b)]. The definitions of the contraderived and absolute derived categories are explained in Subsections 9.3 and 7.6. The absolute derived category
D^{abs}(B^\text{--mod}) coincides with the contraderived category D^{tr}(B^\text{--mod}) by [64, Theorem 3.6(a)]; see Theorem 7.8(a).

\[ \square \]

### 9.7 Koszul triality

In this section, we summarize the results formulated above in this survey in the form of commutative diagrams of Koszul triality. There are two main triality theorems in [64]: the conilpotent and the nonconilpotent one.

**Theorem 9.8.** Let $A^*$ be a DG-algebra and $(C^*, \gamma)$ be a conilpotent CDG-coalgebra over $k$. Let $\tau : C^* \rightarrow A^*$ be an acyclic twisting cochain; this includes the conditions that $A^* \neq 0$ and $\tau \circ \gamma = 0$. Then there is a commutative diagram of triangulated category equivalences

\[
\begin{array}{ccc}
D(A^\text{--mod}) & \cong & D^\text{co}(C^*\text{--comod}) \\
\downarrow & & \downarrow \\
D^\text{tr}(C^*\text{--contra}) & \cong & D(A^*\text{--mod})
\end{array}
\]

where the upper diagonal double line

\[ C^* \otimes \tau - : D(A^\text{--mod}) \cong D^\text{co}(C^*\text{--comod}) : A^* \otimes \tau - \]

is the comodule side conilpotent Koszul duality of Theorem 6.12; the lower diagonal double line

\[ \text{Hom}^\tau(C^*, -) : D(A^\text{--mod}) \cong D^\text{tr}(C^*\text{--contra}) : \text{Hom}^\tau(A^*, -) \]

is the contramodule side conilpotent Koszul duality of Theorem 9.6; and the vertical double line

\[ \mathbb{R} \text{Hom}_C(C^*, -) : D^\text{co}(C^*\text{--comod}) \cong D^\text{tr}(C^*\text{--contra}) : C^* \odot C^- \]

is the derived comodule-contramodule correspondence of Theorem 9.5.

**Proof.** This is [64, Theorem 6.5]. The particular case corresponding to the twisting cochain from Examples 6.8 can be found in [64, Theorem 6.3], while the case of the twisting cochain from Example 6.9 (for a conilpotent CDG-coalgebra $C^*$) is considered in [64, Theorem 6.4]. For the definitions of the coderived and the contraderived category, see Subsections 7.7 and 9.3. For discussions of some elements of the proof of the theorem, see Remarks 7.1 and 7.15 and Subsection 9.8. \[ \square \]

Here the comodule side of the Koszul duality takes the left CDG-comodule $k$ over $C^*$ (with the $C$-comodule structure on $k$ defined in terms of the coaugmentation $\gamma$) to the free left DG-module $A^*$ over $A^*$. The contramodule side of the Koszul duality takes the left CDG-contramodule $k$ over $C^*$ (with the $C$-contramodule structure on $k$ similarly defined in terms of the coaugmentation $\gamma$) to the cofree left DG-module $\text{Hom}_k(A^*, k)$ over $A^*$. 
Now let us assume that \((A^*, \alpha)\) is an augmented DG-algebra and \((C^*, \gamma)\) is a conilpotent DG-coalgebra. Let \(\tau : C^* \rightarrow A^*\) be an acyclic twisting cochain satisfying the equation \(\alpha \circ \tau = 0\) (in addition to the previously assumed \(\tau \circ \gamma = 0\)). For example, the twisting cochains from Examples 5.1, 5.2, and 5.3, as well as generally ‘acyclic twisting cochains’ in the sense of Subsection 5.2, satisfy these additional conditions.

Then the left DG-module \(k\) over \(A^*\) (with the \(A\)-module structure on \(k\) defined in terms of the augmentation \(\alpha\)), the cofree left DG-comodule \(C^*\) over \(C^*\), and the free left DG-contramodule \(\text{Hom}_k(C^*, k)\) over \(C^*\) correspond to each other under the triality (7) of Theorem 9.8.

**Theorem 9.9.** Let \(C^* \neq 0\) be a CDG-coalgebra, and let \(\tau : C^* \rightarrow \text{Cob}^*_w(C^*) = B^*\) be the twisting cochain from Example 6.10. Then there is a commutative diagram of triangulated category equivalences

\[
\begin{array}{ccc}
D^c(C^*\text{-comod}) & \simeq & D^c_\text{co-abs=ctr}(B^*\text{-mod}) \\
\Vert & & \Vert \\
D^c_\text{ctr}(C^*\text{-contra}) & \simeq & D^c_\text{co-abs=ctr}(B^*\text{-mod}) : B^* \otimes \tau -
\end{array}
\]

where the upper diagonal double line

\[C^* \otimes \tau - : D^c_\text{co-abs=ctr}(B^*\text{-mod}) \simeq D^c_\text{co}(C^*\text{-comod}) : B^* \otimes \tau -\]

is the comodule side nonconilpotent Koszul duality of Theorem 6.13; the lower diagonal double line

\[\text{Hom}^\tau(C^*,-) : D^c_\text{abs=ctr}(B^*\text{-mod}) \simeq D^c_\text{ctr}(C^*\text{-contra}) : \text{Hom}^\tau(B^*,-)\]

is the contramodule side nonconilpotent Koszul duality of Theorem 9.7; and the vertical double line

\[\mathbb{R}\text{Hom}_C(C^*,-) : D^c_\text{co}(C^*\text{-comod}) \simeq D^c_\text{ctr}(C^*\text{-contra}) : C^* \otimes_C \mathbb{L}_C -\]

is the derived comodule-contramodule correspondence of Theorem 9.5.

**Proof.** This is [64, Theorem 6.7]. For the definitions of the coderived, contraderived, and absolute derived categories, see Subsections 7.6, 7.7, and 9.3. The notation \(D^c_\text{co-abs=ctr}(B^*\text{-mod})\) is intended to emphasize that the coderived, contraderived, and absolute derived category of left CDG-modules over \(B^*\) coincide by Theorem 7.8(a) (since the free graded algebra \(B\) has finite global dimension 1). For a discussion of some elements of the proof of the theorem, see Remark 7.15 and Subsection 9.8.

□

### 9.8 Some comments on the proof

The aim of this section is to complement the proofs of the Koszul duality theorems [64, Theorems 6.3, 6.4, 6.5, and 6.7] (see Theorems 9.8 and 9.9) with some additional details.
In the context of both the diagrams (7) and (8), there are three points to be explained: that the diagonal (Koszul duality) functors are well-defined, that the diagonal functors acting in the opposite directions are mutually inverse, and that the triangular diagram is commutative. We have already commented upon the vertical (co–contra) equivalence in the proof of Theorem 9.5.

The main thing to observe is that the Koszul duality functors are exact and need not be derived. There is no need to replace a (C)DG-module, CDG-comodule, or CDG-contramodule with any resolution (adjusted version) of it before applying any one of the functors $A^\ast \otimes^\tau \dashv$, $C^\ast \otimes^\tau \dashv$, and so on (defined in Subsections 6.9 and 9.6).

In fact, the Koszul duality functors have even better properties: they take weakly trivial objects to contractible ones. For example, in the context of the conilpotent duality of Theorem 9.8, the functor $A^\ast \otimes^\tau \dashv$ takes coacyclic CDG-comodules to contractible DG-modules, while the functor $C^\ast \otimes^\tau \dashv$ — takes acyclic DG-modules to contractible CDG-comodules. Similarly, in the same theorem the functor $\text{Hom}^\tau(A^\ast, -)$ takes contraacyclic CDG-contramodules to contractible DG-modules, while the functor $\text{Hom}^\tau(C^\ast, -)$ takes acyclic DG-modules to contractible CDG-contramodules.

By adjunction, the same property of the Koszul duality functors can be expressed by saying that they take arbitrary objects to adjusted ones (meaning ‘adjusted’ on the suitable side and for the respective exotic derived category). For example, in the context of Theorem 9.8, the functor $A^\ast \otimes^\tau \dashv$ takes arbitrary CDG-comodules to homotopy projective DG-modules, while the functor $C^\ast \otimes^\tau \dashv$ takes arbitrary DG-modules to graded-injective CDG-comodules. Similarly, in the same theorem the functor $\text{Hom}^\tau(A^\ast, -)$ takes arbitrary CDG-contramodules to homotopy injective DG-modules, while the functor $\text{Hom}^\tau(C^\ast, -)$ takes arbitrary DG-modules to graded-projective CDG-contramodules.

The fact that the Koszul duality functors take co/contraacyclic objects to contractible ones is explained in Remark 7.15. Let us make a comment on the proofs of the claims that the functor $C^\ast \otimes^\tau \dashv$ — takes acyclic DG-modules over $A^\ast$ to contractible CDG-comodules over $C^\ast$, while the functor $\text{Hom}^\tau(A^\ast, -)$ takes contraacyclic CDG-contramodules to contractible DG-modules, while the functor $\text{Hom}^\tau(C^\ast, -)$ takes acyclic DG-modules to contractible CDG-contramodules.

For the sake of accessibility of the exposition, we formulate the following lemma for CDG-modules, while in fact parts (a), (b) are equally applicable to CDG-comodules and parts (a), (c) to CDG-contramodules.

**Lemma 9.10.** Let $B^\ast = (B, d, h)$ be a CDG-ring.

(a) Let $M^\ast$ be a CDG-module over $B^\ast$ endowed with a finite filtration $0 = F_0M^\ast \subset F_1M^\ast \subset \cdots \subset F_nM^\ast = M^\ast$ by CDG-submodules $F_iM^\ast \subset M^\ast$. Assume that the CDG-module $F_iM^\ast / F_{i-1}M^\ast$ over $B^\ast$ is absolutely acyclic for every $1 \leq i \leq n$. Then the CDG-module $M^\ast$ over $B^\ast$ is absolutely acyclic.

(b) Let $N^\ast$ be a CDG-module over $B^\ast$ endowed with an increasing filtration $0 = F_0N^\ast \subset F_1N^\ast \subset \cdots \subset F_iN^\ast \subset F_{i+1}N^\ast \subset \cdots$ by CDG-submodules $F_iN^\ast \subset N^\ast$. Assume that the filtration $F$ is exhaustive, that is $N^\ast = \bigcup_{i \geq 0} F_iN^\ast$, and the CDG-module $F_iN^\ast / F_{i-1}N^\ast$ over $B^\ast$ is coacyclic for every $i \geq 1$. Then the CDG-module $N^\ast$ over $B^\ast$ is coacyclic.

(c) Let $Q^\ast$ be a CDG-module over $B^\ast$ endowed with a decreasing filtration $Q^\ast = F_0Q^\ast \supset F_1Q^\ast \supset \cdots \supset F_iQ^\ast \supset F_{i+1}Q^\ast \supset \cdots$ by CDG-submodules $F_iQ^\ast \subset Q^\ast$. Assume that the filtration $F$ is (separated and) complete, that is $Q^\ast = \varprojlim_{i \geq 1} Q^\ast / F_iQ^\ast$, and the CDG-module $F_iQ^\ast / F_{i+1}Q^\ast$ over $B^\ast$ is contraacyclic for every $i \geq 0$. Then the CDG-module $Q^\ast$ over $B^\ast$ is contraacyclic.
Proof. This lemma is implicit in [63, 64]; cf. [63, proof of Lemma 2.1] or [64, proof of Theorem 3.7]. Part (a): proceeding by induction on \( n \), it suffices to prove that, for any short exact sequence of CDG-modules \( 0 \rightarrow K^* \rightarrow L^* \rightarrow N^* \rightarrow 0 \) over \( B^* \), the CDG-module \( L^* \) is absolutely acyclic whenever both the CDG-modules \( K^* \) and \( N^* \) are. Indeed, the CDG-module \( T^* = \text{Tot}(K^* \rightarrow L^* \rightarrow N^*) \) is absolutely acyclic by the definition, and the CDG-module \( L^* \) is homotopy equivalent to a CDG-module which can be obtained from \( K^*, T^* \), and \( N^* \) using (a small finite number of) passages to shifts and cones.

Part (b): Similarly to part (a), one proves by induction on \( i \) that the CDG-module \( F_i N^* \) is coacyclic for every \( i \geq 1 \). Having that established, consider the telescope short exact sequence
\[
0 \rightarrow \bigoplus_{i=1}^{\infty} F_i N^* \rightarrow \bigoplus_{i=1}^{\infty} F_i N^* \rightarrow N^* \rightarrow 0
\]
of CDG-modules over \( B^* \), and denote by \( T^* \) its total CDG-module. Then the CDG-module \( T^* \) is absolutely acyclic by the definition, while the CDG-module \( \bigoplus_{i=1}^{\infty} F_i N^* \) is coacyclic since the CDG-modules \( F_i N^* \) are. It follows that the CDG-module \( N^* \) is coacyclic.

Part (c): similarly to part (a), one proves by induction on \( i \) that the CDG-module \( Q^*/F_i Q^* \) is contraacyclic for every \( i \geq 1 \). Now one can consider the telescope short exact sequence
\[
0 \rightarrow Q^* \rightarrow \prod_{i=1}^{\infty} Q^*/F_i Q^* \rightarrow \prod_{i=1}^{\infty} Q^*/F_i Q^* \rightarrow 0
\]
of CDG-modules over \( B^* \). This sequence is exact, because the transition maps in the projective system \( Q^*/F_{i+1} Q^* \rightarrow Q^*/F_i Q^* \) are surjective and \( Q^* = \lim_{\leftarrow \geq 1} Q^*/F_i Q^* \). Denote by \( T^* \) its total CDG-module. Then the CDG-module \( T^* \) is absolutely acyclic by the definition, while the CDG-module \( \prod_{i=1}^{\infty} Q^*/F_i Q^* \) is contraacyclic since the CDG-modules \( Q^*/F_i Q^* \) are. It follows that the CDG-module \( Q^* \) is contraacyclic.

Our next comment is on the proofs of the claims that the diagonal functors in the opposite directions are mutually inverse in Theorems 9.8 and 9.9. The following observations are intended to complement the arguments in the proofs of [64, Theorems 4.4, 6.3, and 6.4]. As above, we formulate the assertions for CDG-modules, though those of them not involving contraacyclicity are also applicable to CDG-comodules, while those not involving coacyclicity are applicable to CDG-contramodules.

**Lemma 9.11.** Let \( B^* = (B, d, h) \) be a CDG-ring and \( M^* = (M, d_M) \) be a left CDG-module over \( B^* \). Let \( L \subset M \) be a graded \( B \)-submodule. Assume that the composition \( L \rightarrow M \xrightarrow{d_M} M \rightarrow M/L \) is an isomorphism of graded abelian groups (or of graded left \( B \)-modules) \( L \rightarrow (M/L)[1] \). Then the CDG-module \( M^* \) over \( B^* \) is contractible.

**Proof.** Here the notation \( L \rightarrow (M/L)[1] \) means that the composition of maps in question is a homogeneous map \( L \rightarrow M/L \) of degree 1. In fact, it is always a morphism of graded left \( B \)-modules (with the suitable sign rule). The map \( L \rightarrow (M/L)[1] \) being bijective means that the CDG-module \( M^* \) over \( B^* \) is freely generated by the graded left \( B \)-module \( L \). In the notation of [64, proof of Theorem 3.6], this is expressed by the formula \( M^* = G^+(L) \). All CDG-modules over \( B^* \) freely generated by graded \( B \)-modules are easily seen to be contractible. □
Now let us consider a CDG-module $M^*$ over $B^*$ endowed with a filtration by graded $B$-submodules

$$\cdots \subset F_{-i}M \subset F_{-i+1}M \subset \cdots \subset F_0M \subset F_1M \subset \cdots \subset F_iM \subset F_{i+1}M \subset \cdots$$

The differential $d_M : M \rightarrow M$ is not supposed to preserve this filtration; however, we assume that $d_M(F_iM) \subset F_{i+1}M$. Then for every $i \in \mathbb{Z}$ we have a morphism of graded $B$-modules $F_iM/F_{i-1}M \rightarrow (F_{i+1}M/F_iM)[1]$ induced by $d_M$. The collection of all such maps is a complex of graded left $B$-modules

$$\cdots \rightarrow (F_{i-1}M/F_{i-2}M)[i-1] \rightarrow (F_iM/F_{i-1}M)[i] \rightarrow (F_{i+1}M/F_iM)[i+1] \rightarrow \cdots$$

**Lemma 9.12.**

(a) Let $N^* = (N, d_N)$ be a left CDG-module over $B^*$, and let $0 = F_0N \subset F_1N \subset F_2N \subset \cdots$ be an exhaustive increasing filtration of $N$ by graded $B$-submodules $F_iN \subset N$. Assume that $d_N(F_iN) \subset F_{i+1}N$ for all $i \geq 1$ and the complex of graded left $B$-modules

$$0 \rightarrow F_1N[1] \rightarrow (F_2N/F_1N)[2] \rightarrow (F_3N/F_2N)[3] \rightarrow \cdots$$

with the differential induced by $d_N$ is acyclic. Then the CDG-module $N^*$ over $B^*$ is coacyclic.

(b) Let $Q^* = (Q, d_Q)$ be a left CDG-module over $B^*$, and let $Q = F^0Q \supset F^1Q \supset F^2Q \supset \cdots$ be a complete decreasing filtration of $Q$ by graded $B$-submodules $F^iQ \subset Q$. Assume that $d_Q(F^iQ) \subset F^{i-1}Q$ for all $i \geq 2$ and the complex of graded left $B$-modules

$$\cdots \rightarrow (F^2Q/F^3Q)[-2] \rightarrow (F^1Q/F^2Q)[-1] \rightarrow Q/F^1Q \rightarrow 0$$

with the differential induced by $d_Q$ is acyclic. Then the CDG-module $Q^*$ over $B^*$ is contraacyclic.

**Proof.** Part (a): Define a new increasing filtration $G$ on $N$ by the rule $G_iN = F_iN + d_N(F_iN)$. Then $0 = G_0N^* \subset G_1N^* \subset G_2N^* \subset \cdots$ is an increasing filtration of $N^*$ by CDG-submodules $G_iN^* \subset N^*$. The filtration $G$ is exhaustive if and only if the filtration $F$ is. In view of Lemma 9.11, the condition of exactness of the complex of graded left $B$-modules in part (a) implies that the CDG-modules $M_i^* = G_iN^*/G_{i-1}N^*$ are contractible for all $i \geq 1$. According to Lemma 9.10(b), it follows that the CDG-module $N^*$ is coacyclic. The proof of part (b) is dual and based on Lemmas 9.11 and 9.10(c): the new filtration $G^iQ = F^{i+1}Q + d_Q(F^{i+1}Q)$ is a decreasing filtration by CDG-submodules of $Q^*$, it is complete if and only if the filtration $F$ is, the condition of exactness of the complex implies that the CDG-modules $G^iQ^*/G^{i+1}Q^*$ are contractible, and so on.

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