A quasi-spectral method for Cauchy problem of 2/D Laplace equation on an annulus

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Abstract. Real numbers are usually represented in the computer as a finite number of digits hexa-decimal floating point numbers. Accordingly the numerical analysis is often suffered from rounding errors. The rounding errors particularly deteriorate the precision of numerical solution in inverse and ill-posed problems. We attempt to use a multi-precision arithmetic for reducing the rounding error evil. The use of the multi-precision arithmetic system is by the courtesy of Dr. Fujiwara of Kyoto University. In this paper we try to show effectiveness of the multi-precision arithmetic by taking two typical examples; the Cauchy problem of the Laplace equation in two dimensions and the shape identification problem by inverse scattering in three dimensions. It is concluded from a few numerical examples that the multi-precision arithmetic works well on the resolution of those numerical solutions, as it is combined with the high order finite difference method for the Cauchy problem and with the eigenfunction expansion method for the inverse scattering problem.

1. Cauchy Problem of the 2/D Laplace Equation

We start our discussion with rather simple configuration: Given the Cauchy data \( u(x) = \bar{u}(x) \) and \( \frac{\partial u}{\partial n}(x) = \bar{q}(x) \) at the point \( x = (x_1, x_2) \in \mathbb{R}^2 \) with unit outward normal \( n(x) = (n_1, n_2) \) on the external circle \( \Gamma_1 \) with the radius \( r_1 = 1 \) of an annulus \( \Omega \), we consider the Cauchy problem of the Laplace equation \( \Delta u(x) = 0 \) in two dimensions to find unknown function \( u(x) \) on the internal circle \( \Gamma_2 \) with the radius \( r_2 = 0.5 \). This problem setting is a primitive model of the inverse problem in human electro-cardiography \([1]\).

1.1. High order finite difference method

To solve this problem numerically, we apply a quasi-spectral method \([2]\) consisting of a high order finite difference method \([3]\), \([4]\) to its approximation. The conventional finite difference method approximates unknown derivatives, based on some polynomials \([5]\), \([6]\). Some other recent methods \([7]\), \([8]\), \([9]\) are worthy of note in this connection on polynomials. On the other hand, our high order finite difference method approximates unknown derivatives, based on exponential functions. This novel finite difference method uses a linear combination of exponential functions at all sampling points, that can be distributed arbitrarily in and even out of the domain \( \Omega \). Thereby the approximation of derivatives at a point can refer all sampling points in its arbitrary
neighborhood. Therefore a highly precise solution can be achieved and the method is furnished with a mesh-free property.

We shortly explain the idea of the high order finite difference method by taking perhaps the simplest example. Let us consider a one-dimensional initial value problem to find a function \( u(x) \) for \( x > 0 \) satisfying the differential equation \( \frac{du}{dx} = 2 \) with the initial condition \( u(0) = 0.5 \). The exact solution is \( u(x) = 0.5 + 2x \).

We take an arbitrary spacing \( h > 0 \), and pick up an arbitrary number of points (sampling points, in this case three points) \( x(1) = 0 \), \( x(2) = h \), \( x(3) = 2h \) in order to approximate the derivative appearing in the differential equation in the form

\[
\frac{du}{dx}(x) = w_1(x)u(x(1)) + w_2(x)u(x(2)) + w_3(x)u(x(3)) + \varepsilon \left( \frac{d}{dx}; x \right).
\]

The last term is the error committed in the approximation. We take the linearly independent functions (polynomial basis) \( 1, x, x^2 \) respectively as for \( u(x) \) in the approximation to determine the weighting functions \( w_j(x) \) by

\[
\begin{align*}
0 &= x_1(x) + w_2(x) + w_3(x), \\
1 &= w_2(x)h + w_3(x) \cdot 2h, \\
2x &= w_2(x)h^2 + w_3(x) \cdot (2h)^2.
\end{align*}
\]

The solution is \( w_1(x) = \frac{1}{2h^3}(-3h + 2x) \), \( w_2(x) = \frac{2}{h^2}(h - x) \), and \( w_3(x) = \frac{1}{2h}(-h + 2x) \).

We apply this finite difference approximation to our initial value problem to obtain the linear system of equations

\[
\begin{align*}
\text{at } x &= x_1(= 0): & 0.5 &= u_1, \\
\text{at } x &= x_2(= h): & 2 &= -\frac{1}{2h}u_1 + \frac{1}{2h}u_3, \\
\text{at } x &= x_3(= 2h): & 2 &= \frac{1}{2h}u_1 - \frac{2}{h}u_2 + \frac{3}{2h}u_3
\end{align*}
\]

with unknown \( u_j \), which is the approximate value to the exact \( u(x_j) \). The solution of the linear system is

\[
\begin{align*}
u_1 &= 0.5, \\
u_2 &= 0.5 + 2h, \\
u_3 &= 0.5 + 4h,
\end{align*}
\]

which coincides with the exact solution.

The idea can be readily extended to our Cauchy problem. We take \( N \) sampling points \( x^{(j)} \in \mathbb{R}^2 \) for \( j = 1, 2, \ldots, N \) on the domain \( \Omega \) with its boundary \( \partial \Omega \) included. The Cauchy problem of the Laplace equation is approximated at these sampling points as follows.

\[
\begin{align*}
\Delta u(x) &= \sum_{j=1}^{N} w_j(x)u(x^{(j)}) + \varepsilon(\Delta; x), & x &\in \Omega, \\
u(x) &= \sum_{j=1}^{N} w_j(x)u(x^{(j)}) + \varepsilon(1; x), & x &\in \Gamma_1, \\
\frac{\partial u}{\partial n}(x) &= \sum_{j=1}^{N} w_j(x)u(x^{(j)}) + \varepsilon \left( \frac{\partial}{\partial n}; x \right), & x &\in \Gamma_1
\end{align*}
\]

with weighting functions \( w_j(x) \). The last term \( \varepsilon(\cdot; x) \) in each equation denotes the discretization error depending on the operators \( \Delta, \ 1, \text{ and } \partial/\partial n \) to be approximated.
In practice we take a set $S_k$ of $M_k$ sampling points $x^{(j)}$ neighboring on each $x^{(k)}$ ($k = 1, 2, \ldots, N$). If $M_k$ is constant for the index $k$, we use a symbol $M$ instead of $M_k$. We assume that $\Omega$ includes $N_\Omega$ sampling points $x^{(k)}, k = 1, 2, \ldots, N_\Omega$ and $\Gamma_1$ includes $N_\Gamma$ sampling points $x^{(k)}, k = N_\Omega + 1, N_\Omega + 2, \ldots, N_\Omega + N_\Gamma$. We discretize the Cauchy problem at $x^{(k)}$ rather locally in the form

$$\Delta u(x^{(k)}) = \sum_{x^{(j)} \in S_k} w_{jk} u(x^{(j)}), \quad x^{(k)} \in \Omega,$$

$$u(x^{(k)}) = \sum_{x^{(j)} \in S_k} w_{jk} u(x^{(j)}), \quad x^{(k)} \in \Gamma_1,$$

$$\frac{\partial u}{\partial n}(x^{(k)}) = \sum_{x^{(j)} \in S_k} w_{jk+N_\Gamma} u(x^{(j)}), \quad x^{(k)} \in \Gamma_1$$

with the discretization error terms discarded. Since each $x^{(k)} \in \Omega$ is a single point and each $x^{(k)} \in \Gamma_1$ is a double point, the number of data is $N_\Omega + 2N_\Gamma$. Here we must set $N = N_\Omega + 2N_\Gamma$ in order to construct the square matrix $(w_{jk})$. The points $x^{(k)}, k = N_\Omega + N_\Gamma + 1, N_\Omega + N_\Gamma + 2, \ldots, N_\Omega + 2N_\Gamma = N$ are so chosen according to $x^{(k)} = x^{(k-N_\Gamma)} + \frac{1}{\sqrt{N}} n(x^{(k-N_\Gamma)})$ that the matrix may not become singular.

The unknown weighting values $w_{jk}$ are determined as follows. We take linearly independent exponential functions $e^{\xi(i)(x^{(k)})} x$ with a vector $\xi(i)(x^{(k)}) = \rho(x^{(i)} - x^{(k)})$ for all $x^{(i)} \in S_k$ and the inner product $\cdot$. We substitute them into $u(x^{(k)})$ in the expressions above to obtain

$$|\xi(i)(x^{(k)})|^2 e^{\xi(i)(x^{(k)})} x^{(k)} = \sum_{x^{(j)} \in S_k} w_{jk} e^{\xi(i)(x^{(k)})} x^{(j)},$$

$$e^{\xi(i)(x^{(k)})} x^{(k)} = \sum_{x^{(j)} \in S_k} w_{jk} e^{\xi(i)(x^{(k)})} x^{(j)},$$

$$n(x^{(k)}) \cdot \xi(i)(x^{(k)}) e^{\xi(i)(x^{(k)})} x^{(k)} = \sum_{x^{(j)} \in S_k} w_{jk+N_\Gamma} e^{\xi(i)(x^{(k)})} x^{(j)}.$$

The constant $\rho$ is a parameter. This linear system of equations, if the coefficient matrix $(e^{\xi(i)(x^{(k)})} x^{(j)})$ is non-singular, uniquely determines the weighting values $w_{jk}$ for $k = 1, 2, \ldots, N$. We require a condition such as all the points $x^{(k)}$ are distinct for non-singularity of the matrix. The weighting values $w_{jk}$ always become $w_{kk} = 1$ and $w_{jk} = 0, j \neq k$ for $k = N_\Omega + 1, N_\Omega + 2, \ldots, N_\Omega + N_\Gamma$. Once the weighting values $w_{jk}$ are determined, we can obtain the linear system of equations

$$\sum_{x^{(j)} \in S_k} w_{jk} u_j = 0, \quad x^{(k)} \in \Omega,$$

$$u_k = \bar{u}(x^{(k)}), \quad x^{(k)} \in \Gamma_1,$$

$$\sum_{x^{(j)} \in S_k} w_{jk+N_\Gamma} u_j = \bar{q}(x^{(k)}), \quad x^{(k)} \in \Gamma_1$$

for unknown $u_j$.

Moreover, the idea can be extended to a more general partial differential equations. To this end, we consider the following $\mu$-th order partial differential operator $P(\partial) = \sum_{\alpha \in \mathbb{Z}_n^\mu} a_\alpha(x) \partial^\alpha$ with the multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, the smooth coefficients $a_\alpha(x)$ for which $a_\alpha(x) = 0$ for
\(|\alpha| > \mu\), and \(\partial = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right)\). One may refer to Iijima [3] for the case that \(P(\partial)\) is the heat operator \(\partial/\partial t - \Delta\). Corresponding to the operator \(P(\partial)\), we consider the \(\mu\)-th order polynomial \(P(\xi) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha(x)\xi^\alpha\) with \(\xi = (\xi_1, \xi_2, \ldots, \xi_n)\).

We take \(N\) sampling points \(x^{(j)} \in \mathbb{R}^n\) for \(j = 1, 2, \ldots, N\). The derivative of a function \(u(x)\) is approximated in the form

\[
P(\partial)u(x) = \sum_{j=1}^{N} w_j(x)u(x^{(j)}) + \varepsilon(P(\partial); x)
\]

with an error term \(\varepsilon(P(\partial); x)\) in the discretization. The weighting functions \(w_j(x)\) are determined by \(N\) linearly independent exponential functions \(u(x) = e^{\xi(y)x}\) with \(\xi^{(i)}(y) = \rho(x^{(i)} - y)\) for \(i = 1, 2, \ldots, N\), if we substitute them into the approximated form. Namely

\[
P(\xi^{(i)}(y))e^{\xi^{(i)}(y)x} = \sum_{j=1}^{N} w_j(x)e^{\xi^{(i)}(y)x^{(j)}}
\]

with the error term discarded. The constant \(\rho\) is a parameter, and \(y\) is the location parameter.

When we determine weighting values \(w_{jk} = w_j(x^{(k)})\), some of them can be set equal to 0 by taking only \(M_k\) neighboring points to \(x^{(k)}\). Let \(S_k\) denote the set of all neighboring points on \(x^{(k)}\). At \(y = x^{(k)}\) the weighting values \(w_{jk}\) for which \(x^{(j)} \in S_k\) are determined by the system of linear equations

\[
P(\xi^{(i)}(x^{(k)}))e^{\xi^{(i)}(x^{(k)})x^{(k)}} = \sum_{x^{(j)} \in S_k} w_{jk}e^{\xi^{(i)}(x^{(k)})x^{(j)}} \quad , \quad x^{(i)} \in S_k.
\]

Otherwise, namely when \(x^{(j)} \notin S_k\), we set \(w_{jk} = 0\).

We now apply the high order finite difference approximation to rather more general Cauchy problem of the Laplace equation as follows. Let \(\Omega\) is the domain of interest, in which the Laplace equation is considered. On a part of the boundary \(\Gamma_d\) of the domain, Cauchy data \(u(x) = \bar{u}(x)\) and \(\frac{\partial u}{\partial n}(x) = \bar{q}(x)\) are prescribed. Then the Cauchy problem can be written compactly in the form

\[
P_k(\partial)u(x^{(k)}) = f_k \quad \text{at} \quad x^{(k)}
\]

for \(k = 1, 2, \ldots, N\), where

\[
P_k(\partial) = \begin{cases} 
\Delta \\
\frac{\partial}{\partial n}
\end{cases}, \quad f_k = \begin{cases} 
0 \\
\bar{u}(x^{(k)}) \quad \text{on} \quad \Gamma_d \\
\bar{q}(x^{(k)}) \quad \text{on} \quad \Gamma_d
\end{cases}
\]

1.2. Numerical examples

As a preliminary example, we consider the conventional Dirichlet problem to find the function inside the annulus, in which boundary values of the harmonic function \(u(x_1, x_2) = x_1^2 - x_2^2\) are prescribed on both \(\Gamma_1\) and \(\Gamma_2\). The annulus is divided 50 times 50 respectively in both directions of the radius and the central angle. The sampling points are distributed regularly as shown in Figure 1(a). In the high precision computation, 50 decimal digits are used in the multi-precision arithmetic [10], [11]. One hundred (100) sampling points are referred in each neighborhood of the internal sampling point in \(\Omega\) in the high order finite difference method. The parameter \(\rho\) is
set equal to 1. Calculated surface is shown in Figure 1(b). The maximum absolute error in the calculated solution among all the sampling points is $7.83 \times 10^{-13}$.

We now consider the Cauchy problem. The annulus is divided into 50 times 50 in both directions of the radius and the central angle as shown in Figure 2(a). Cauchy data $u(x_1, x_2) = x_1^2 - x_2^2$ and \( \frac{\partial u}{\partial n}(x_1, x_2) = 2x_1n_1 - 2x_2n_2 \) are prescribed at 50 points on the external circle $\Gamma_1$. The parameter $\rho = 1$ is set in the high order finite difference method. Neighboring 50 points around each internal sampling point in $\Omega$ and around each boundary point on $\Gamma_1$ where $\frac{\partial u}{\partial n}(x_1, x_2)$ are specified are used.

In the multi-precision arithmetic, 50 decimal digits are used. Figure 2(b) shows the calculated surface on the whole domain $\Omega$. The maximum absolute error among all the sampling points is $9.00 \times 10^{-6}$, which occurred at the point $(0.5 \cos 26\pi/50, 0.5 \sin 26\pi/50)$ on the internal circle $\Gamma_2$.

Figure 3(a) shows the distribution of the boundary values on $\Gamma_1$, so called an initial curve, and Figure 3(b) the distribution on $\Gamma_2$. The calculated results are plotted in dots, and the exact
solution is drawn in solid lines. The calculated results are in good agreement with the exact solution.

![Graphs showing calculated and exact solutions](image)

**Figure 3.** Calculated (●) and exact (—) solutions.

The synthetic function used in the previous example is a harmonic function, which has no singularities at finite distance. We now consider the Cauchy problem using the harmonic function $u(x_1, x_2) = \ln r$ (with $r = \sqrt{x_1^2 + x_2^2}$), which has a singularity at the origin. The number of 2567 sampling points are distributed in the annulus $\Omega$ as shown in Figure 4(a). Cauchy data are prescribed at 151 points on $\Gamma_1$. The parameter $\rho = 1$ is set in the high order finite difference method. Neighboring 50 points are used around each internal sampling point and around each boundary point on $\Gamma_1$ where $\frac{\partial u}{\partial n}(x_1, x_2)$ are specified.

In the multi-precision arithmetic, 50 decimal digits are used. Figure 4(b) shows the calculated surface on the whole domain $\Omega$. The maximum absolute error is 0.0017 which occurs at the point $(0.5 \cos (220\pi/151), 0.5 \sin(220\pi/151))$.

![Figure 4 with allocation of sampling points and calculated surface](image)

**Figure 4.** Cauchy problem in an annulus.
Figure 5(a) shows the distribution of the boundary values on $\Gamma_2$. Calculated solutions plotted in dots oscillate above and below the exact solution $\ln 0.5 = -0.6931$ within the absolute error of order 0.001. As the number of sampling points $N$ is increased, the maximum absolute error is decreased as shown in Figure 5(b). However the order of accuracy is not high as compared with the high order of accuracy gained in the previous example. The present example suggests us that the singularity of the solution at finite distance, even if it does not belong to the domain $\Omega$ of interest, deteriorates the order of accuracy.

![Figure 5](https://example.com/figure5.png)

(a) Calculated (•) and exact (—) solutions 
(b) Errors in numerical solutions

**Figure 5.** Errors on the internal circle $\Gamma_2$.

We are now interested in extending the harmonic function $u(x_1, x_2) = \ln r$ toward the exterior of the initial curve $\Gamma_2$ of the annulus. Cauchy data $u(x_1, x_2)$ and $\frac{\partial u}{\partial n}(x_1, x_2)$ are prescribed on the internal circle $\Gamma_2$. The annulus is divided into 13 in direction of the radius and 121 in direction of the central angle. The parameter $\rho = 1$ is set in the high order finite difference method. Neighboring 50 points around each internal sampling point in $\Omega$ and around each boundary point on $\Gamma_2$ where $\frac{\partial u}{\partial n}(x_1, x_2)$ are specified are used.

In the multi-precision arithmetic, 50 decimal digits are used. Calculated surface together with the corresponding contours are shown in Figure 6. The maximum absolute error is 0.0014 occurred at the point $(\cos 118\pi/121, \sin 118\pi/121)$. The magnitude of this error is smaller than the magnitude of the error occurred in the previous example in which the harmonic function is extended toward the interior singular point.

Figure 7(a) shows the distribution of the boundary values on $\Gamma_1$. As the number of sampling points $N$ is increased, the maximum absolute error is decreased as shown in Figure 7(b). The order of accuracy remains low.

Figure 8(a) shows contours of approximate solution and the corresponding exact solution $u(x) = \ln |x| - \ln |x - (0.5, -0.5)|$ having singularities when the exact Cauchy data are given on the whole external elliptic boundary $\Gamma_1$. Figure 8(b) shows contours of approximate solution corresponding to the exact solution $u(x) = \ln |x|$ when the Cauchy data are given only on the first quadrant part $\Gamma_d$ of the external circular boundary $\Gamma_1$. We can see in Figure 8(b) that the harmonic function is reasonably extended toward the sunny side of the initial curve $\Gamma_d$, but the extension is poor aside from $\Gamma_d$. This difficulty seems due to the singularity of the harmonic function $\ln |x|$ at the origin, as the next example reveals the situation.

We consider the harmonic function $u(x_1, x_2) = x_1^2 - x_2^2$ with no singularity at finite distance in the annulus $\Omega = \{(x_1, x_2)| 0.5 < \sqrt{x_1^2 + x_2^2} < 1\}$ again. Cauchy data are prescribed only on the first quadrant part $\Gamma_d$ of the external circular boundary $\Gamma_1$. As we can see in Figure 9(a),
contours of the calculated solution are in good agreement with contours of the exact solution. The function is continued harmonically to the other side of the initial curve beyond a central circular hole. Figure 9(b) shows the maximum absolute errors against numbers $N$ of the sampling points when 100 decimal digits are used. The number of sampling points $M_k$ contained in the neighborhood $S_k$ is varied as $M = 50, 100, \text{ and } 200$ independently on the point number $k$. When the conventional double point arithmetic is used, the accuracy of the calculated solution ceases increasing at the error level $10^{-6}$.

2. Inverse Scattering of Sound in Space
We identify an unknown shape using the series of eigenfunction expansion in the inverse scattering problem of sound. In the high precision computation, we can make the truncation error as small as possible by taking sufficiently many terms in the series.

Let $\Omega$ be an obstacle in the space $\mathbb{R}^3$, which is assumed to be a sound-soft convex scatterer.
An incident plane wave of the form \( u^I(x) = \exp(ik \cdot d) \) with the wave number \( k \) is emitted to the scatterer in the direction \( d \), where \( x = (x_1, x_2, x_3) \) and \( d = (d_1, d_2, d_3) \) with \( |d| = 1 \). We assume that the origin of the spherical coordinates in \( \mathbb{R}^3 \) is located inside the scatterer. The spherical coordinates of the point \( x \) is denoted by \((r, \theta, \varphi)\).

The scattering field is described by the total wave \( u(x) \) as the sum of the incident wave \( u^I(x) \) and the scattered wave \( u^S(x) \). We take a sphere \( S_b \) of radius \( b > 0 \) inside the scatterer \( \Omega \). Then scattered wave can be represented by using the single layer sound potential

\[
u^S(x) = \int_{S_b} \Phi(x - y)\sigma(y)ds(y)
\]

with the density \( \sigma(y) \) distributed on \( S_b \), where \( \Phi(x-y) = \frac{e^{ik|x-y|}}{4\pi|x-y|} \) is the fundamental solution...
to the three-dimensional Helmholtz equation $\Delta u + k^2 u = 0$ satisfying the radiation condition
\[ \frac{\partial u^S}{\partial r} - iku^S = o \left( \frac{1}{r} \right) \] as $r \to +\infty$.

Suppose that the corresponding far field pattern $f(\vartheta, \varphi)$ is exactly available. With the coefficients
\[ a_m^n = \frac{\int_S f(\vartheta, \varphi) Y_m^n(\vartheta, \varphi) ds}{ib^2 \exp \left[ i \left( -\frac{1}{2} (n+1)\pi \right) \right] j_n(kb)}, \] we can expand the scattered wave into the series
\[ u^S(x) = ikb^2 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_m^n h_n^{(1)}(kr)j_n(kb)Y_m^n(\vartheta, \varphi). \] (2)

Here $S$ is the unit sphere, $Y_n^m(\vartheta, \varphi)$ is the spherical harmonic function, $h_n^{(1)}$ is the spherical Hankel function of the first kind, and $j_n$ is the spherical Bessel function.

Once the scattered wave has been computed using this series, we can identify a possible shape of the obstacle by tracing the zero-level set \( \{ x \in \mathbb{R}^3 \mid u(x) = 0 \} \) of the total wave \( u(x) = u^I(x) + u^S(x) \), because our obstacle is assumed to be sound soft.

As a preliminary example, we consider the case that our obstacle is a unit sphere (the radius $R = 1$), to which an incident wave with the wave number $k = 3$ is incident in the direction $d = (1, 0, 0)$. Then the scattered wave can be represented by the series [12]
\[ u^S(x) = -\sum_{n=0}^{\infty} i^n(2n + 1) j_n(kR) \frac{h_n^{(1)}(k|x|)}{h_n^{(1)}(kR)} P_n \left( \frac{x}{|x|} \cdot d, \right), \]
where $P_n(z)$ is the Legendre polynomial. The first $n \leq N$ terms of the series are taken with the higher order terms truncated in order to evaluate the total wave, and so the truncated wave is denoted by $u_N(x)$. One hundred (100) decimal digits are used in the multi-precision arithmetic [10]. Figure 10(a) shows calculated contours on $x_1x_3$-plane at $x_2 = 0$ for $u_{100}(x)$, while Figure 10(b) for $u_{200}(x)$. The differences among the contours plotted amounts to $\max_x |u_{200}(x) - u_{100}(x)| < 10^{-11}$.

![Figure 10](image-url)
Figure 11 shows calculated contours of the total wave $u_{100}(x)$ on $x_1x_2$, $x_1x_3$, and $x_2x_3$-plane, respectively. The surface of the unit sphere can be seen as a contour at the level of very small numbers.

![Figure 11. Contours of the total wave for $u_{100}(x)$ on each plane.](image)

We notice that the corresponding far-field pattern $f(\theta, \varphi)$ is given by [12]

$$f(\theta, \varphi) = \frac{i}{k} \sum_{n=0}^{\infty} (2n + 1) \frac{j_n(kR)}{l_n^{(1)}(kR)} P_n\left(\frac{x}{|x|}\right) \cdot d.$$

Now we consider the shape identification problem, based on the exact knowledge of the pattern $f(\theta, \varphi)$. Suppose that the radius $R$ of the obstacle is unknown. One hundred (100) decimal digits are used in the multi-precision arithmetic. The integral on the unit surface in Eq.(1) is evaluated numerically by the trapezoidal rule. The domain of integral $0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi$ is divided into $240 \times 240$ segments. Figure 12 shows the far-field pattern for $R = 1$ with leading 50 terms.

The $n \leq N$ number of coefficients $a_n^m$ are used for the reconstruction of the scattered wave in Eq.(2). The locations with the level $|u(x)| = 0.007$ is detected for $N = 6$ as shown in Figure 13. The sphere with the radius $R = 1$ is well identified. The maximum absolute error in the identification is 0.01 in terms of the radius.

3. Concluding Remarks
A numerical approach is presented to some familiar examples in two typical inverse problems, namely the Cauchy problem of the Laplace equation and the shape identification problem in inverse scattering. In the numerical solutions of the examples, expected accuracy is achieved. The key to the success is the combination of the multi-precision arithmetic system *exflib* developed by Dr. Fujiwara with the high order finite difference scheme devised by Iijima. The multi-precision arithmetic system is shown to be effective also for the computation of the eigenfunction expansion. The numerical demonstration in this paper is confined to the case in which the exact input data are available. The authors are grateful to Professor Imai at Tokushima University and to Professor Iso at Kyoto University for their valuable advice.
Figure 12. Far-field pattern $|f(\vartheta, \varphi)|$ with $k = 3$

Figure 13. Identified surface of the obstacle.

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