A UNIQUE CONTINUATION PROPERTY FOR A CLASS OF PARABOLIC DIFFERENTIAL INEQUALITIES IN A BOUNDED DOMAIN

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ABSTRACT. This article is concerned with a strong unique continuation property of a forward differential inequality abstracted from parabolic equations proposed on a convex domain Ω prescribed with some regularity and growth conditions. Our results show that the value of the solutions can be determined uniquely by its value on an arbitrary open subset ω in Ω at any given positive time T. We also derive the quantitative nature of this unique continuation, that is, the estimate of a L²(Ω) norm of the initial data, which is majorized by that of solution on the bounded open subset ω at terminal moment t = T.

1. Introduction. Suppose that Ω is a convex bounded domain in Rⁿ (n ≥ 1) with a smooth boundary ∂Ω. Let T be a given positive constant. We consider a forward differential inequality, which reads

|∂ₜu(x, t) − △u(x, t)| ≤ M(|∇u(x, t)| + |u(x, t)|),

in (x, t) ∈ Q = Ω × (0, T], where M is a positive number. In this paper, we will discuss the unique continuation property for solutions of (1.1) under suitable regularity assumption on u(x, t). Let ω ⊂ Ω be a nonempty and open subset of Ω, and we have obtained data u(x, T) in ω.

The contexts of term unique continuation vary according to different application problems. It is a special case of observability that the states in the entire domain is predicted given the states in a local set. In some circumstances, unique continuation refers to the fact that initial states are same, given the observations at terminal are

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same. This term can be quantitatively referred to weak unique continuation, i.e., if $u(x, t) \equiv 0$ in an open subset $\omega \subseteq \Omega \times (0, T)$, then $u(x, t)$ is identically 0 in $\Omega \times (0, T)$ if $\omega$ possesses certain geometric conditions (see [17, 18]); $\Omega$ can be relaxed to the whole spatial domain under some conditions on solutions. This paper pursues a type of quantitative unique continuation for differential inequalities, that is, we can estimate the initial state and even the evolution history of forward evolution equations or inequalities in the whole spatial domain, according to observation in a small nonempty and open subset $\omega$ at a terminal time.

Unique continuation properties of solutions are of interest related to inverse problems and control theories of PDEs, and it was first found for elliptic equations as it naturally holds for harmonic functions, then for some classes of parabolic equations. The first result about unique continuation of strong solutions of parabolic equations with constant coefficients is in [13], where E. Landis and O. Oleinik used reduction to extend the study to the parabolic equations with time-invariable coefficients. In general, to obtain unique continuation, two methodologies, Carleman inequalities and frequency functions, can be involved. The application of Carleman estimates is described systematically in the seminal works [1, 5, 6, 10, 12] authored by L. Escauriaza, C. E. Kenig, H. Koch, D. Tataru, M. Yamamoto et al, and a concise review paper [19], while frequency function method can be found in [8, 14, 15, 16] where similar strong unique continuation in this paper was pursued.

More recently, G. Camliyurt and I. Kukavica in [3] pursued the unique continuation by checking finite orders of vanishing for forward parabolic PDEs with 1st derivative term, whose coefficients are variable but bounded. Frequency functions and a technique of changing variables are invoked in their work. This situation would be a case of the differential inequalities discussed in this paper. We refer readers to similar discussion on elliptic equations by H. Donnelly and C. Fefferman [4], and I. Kukavica [11] for motivations in early years in light of T. Carleman’s seminal work [2] for elliptic equations in 1930s.

Unique continuation properties for inequalities with less smooth variable coefficients have been considered in plenty of seminal literatures, including those for dispersive PDEs such as linear Schrödinger equations. Among them, $|\partial_t u + \Delta u| \leq |V(x, t)u|$ in $\Omega \times [0, T]$ is discussed in L. Sogge’s [18] and L. Escauriaza and F. Fernández [6]. In [18], an unbounded potential $V(x, t) \in L^{\frac{n+2}{n}}_{loc}(dxdt)$ is assumed, and the author obtained weak unique continuation. Strong uniqueness continuation was obtained by D. Jerison and C. Kenig for Schrödinger operators with $V \in L^2_{loc}(dxdt)$ (see, e.g., [9]), which is proved to be sharp. In [6], $V$ is bounded hence the inequalities are similar to ours but backward. We would also refer readers to L. Escauriaza and L. Vega’s [5] for results about heat operators with other different conditions restricted on potential $V$ and references therein.

Most studies of unique continuation of differential inequalities are carried out owing to Carleman inequalities, while this paper will follow the idea provided by frequency functions to pursue the strong unique continuation.

To facilitate our discussion, we make the following assumptions throughout the paper: **Assumption 1.** The regularity for solutions of (1.1) is

$$u(x, t) \in L^2(0, T; H^1_0(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)).$$

**Assumption 2.** The growth condition for $u$ is

$$\int_\Omega |u(x, T)|^2 dx \leq e^{CM(T-t)} \int_\Omega |u(x, t)|^2 dx,$$
for any $t \in [0, T]$.

**Assumption 3.** Suppose that $\partial_t u(x, t) - \Delta u(x, t) \in L^2(\Omega)$, and

$$\|\partial_t u(x, t) - \Delta u(x, t)\|_{H^{-1}(\Omega)} \leq CM\|u(x, t)\|_{L^2(\Omega)},$$

for $t \in (0, T]$ a.e.

Throughout the rest of the paper, the following notations will be used. We denote $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ the usual norm and the inner product of $L^2(\Omega)$ respectively. Besides, variables $x$ and $t$ for functions of $(x, t)$ and variable $x$ for functions of $x$ will be omitted, provided that it does cause some confusion. Then, we have

**Theorem 1.1.** Let $\Omega$ be a bounded convex domain of $\mathbb{R}^n$, $(n \geq 1)$, and $\omega$ be a nonempty open subset of $\Omega$. Suppose that Assumptions 1-3 hold. Then, there are positive numbers: $\gamma = C(\Omega, \omega, T)$ and $C = C(\Omega, \omega)$ such that, any solution $u(x, t)$ of (1.1) has the following estimate:

$$\int_{\Omega} |u(x, T)|^2 dx \leq C \exp \left( \frac{C}{T} + C(MT + M^2T + M^2T^2) \right) \times \left( \int_{\Omega} |u_0(x)|^2 dx \right)^{1-\gamma} \times \left( \int_{\omega} |u(x, T)|^2 dx \right)\gamma.$$ (1.2)

**Remark 1.** The constant $C$ in (1.2) stands for a positive constant only dependent on spatial domains $\Omega$ and $\omega$. This constant varies in different contexts.

The unique continuation property obtained are stated as follows:

**Theorem 1.2.** Let $\Omega$ be a bounded convex domain of $\mathbb{R}^n$, $(n \geq 1)$, and $\omega$ be a nonempty open subset of $\Omega$. Suppose that Assumptions 1-3 hold. If $u(x, 0) \not\equiv 0$, then, there exists a positive number $C = C(\Omega, \omega)$ such that solution $u(x, t)$ of (1.1) has the following estimate:

$$\int_{\Omega} |u(x, 0)|^2 dx \leq C \exp \left( C \left( \frac{1}{T} + 1 + MT + M^2T + M^2T^2 \right) e^{\frac{CM^2T}{\|u_0\|^2_{H^{-1}(\Omega)}}} \|u_0\|^2_{L^2(\Omega)} \right) \times \int_{\omega} (|u(x, T)|^2 dx).$$ (1.3)

**Remark 2.** This result demonstrates that under the conditions of this theorem, solutions of (1.1) must vanish on $\Omega \times [0, T]$, if it vanishes in a nonempty open subset $\omega$ at time $T$.

We organize the paper as follows. In section 2, some preliminary results are presented. Section 3 is devoted to the unique continuation property for the solutions of (1.1), while a nontrivial example will be given in section 4.

Given a positive number $\lambda$, we define

$$G_\lambda(x, t) = \frac{1}{(T-t+\lambda)^{n/2}} e^{-\frac{|x-x_0|^2}{4(T-t+\lambda)}}, \quad \mathbb{R}^n \times [0, T],$$ (1.4)

where $x_0 \in \Omega$. $G_\lambda$ is referred as a Green function.

Then, for each $t \in [0, T]$, we define functions of time by solutions $u(x, t)$ of (1.1):

$$H_\lambda(t) = \int_{\Omega} |u(x, t)|^2 G_\lambda(x, t) dx, \quad (1.5)$$

$$D_\lambda(t) = \int_{\Omega} |\nabla u(x, t)|^2 G_\lambda(x, t) dx, \quad (1.6)$$
and therefore, frequency function is defined as
\[ N_λ(t) = \frac{2D_λ(t)}{H_λ(t)}. \] (1.7)

The function \( N_λ(t) \) was discussed in [5, 8, 16]. We have \( H_λ(t) \neq 0 \) at any moment throughout the paper.

Next, we will discuss the properties for the functions \( G_λ(x, t), H_λ(t), D_λ(t), \) and \( N_λ(t) \). Since \( ω \) is a nonempty and open subset of \( Ω \), we pick an open ball \( B_r \) centered at \( x_0 \in ω \) with radius \( r \) satisfying that \( B_r \subset ω \). Let \( m = \sup_{x \in Ω} |x - x_0|^2 \), and \( ν \) be the exterior unit normal vector on \( ∂Ω \). The following Lemma 1.3 is directly borrowed from [5, 15].

**Lemma 1.3.** For \( λ > 0 \), the function \( G_λ(x, t) \) holds the following four identities over \( \mathbb{R}^n \times [0, T] \):

\[ \partial_t G_λ(x, t) + \Delta G_λ(x, t) = 0, \] (1.8)

\[ \nabla G_λ(x, t) = \frac{-(x - x_0)}{2(T - t + λ)} G_λ(x, t), \] (1.9)

\[ \partial_t^2 G_λ(x, t) = \frac{-1}{2(T - t + λ)} G_λ(x, t) + \frac{|x_i - x_{0i}|^2}{4(T - t + λ)^2} G_λ(x, t), \] (1.10)

and for \( i \neq j \),

\[ \partial_i \partial_j G_λ(x, t) = \frac{(x_i - x_{0i})(x_j - x_{0j})}{4(T - t + λ)^2} G_λ(x, t), \] (1.11)

where \( x_{0i} \)’s are i-th coordinate component of \( x_0 \).

Via straight calculation combining knowledge of \( G_λ(x, t) \) in Lemma 1.3, we have

**Lemma 1.4.** For each \( λ > 0 \), the following identities holds for \( t \in (0, T) \):

\[ \frac{d}{dt} H_λ(t) = -2D_λ(t) + 2 \int_Ω u(∂_i u - \Delta u)G_λ dx, \] (1.12)

\[ 2H_λ'(t)D_λ(t) = -4 \left[ \int_Ω u \left( \frac{x - x_0}{2(T - t + λ)} \cdot \nabla u + \frac{1}{2} (\Delta u - \partial_t u) \right) G_λ dx \right]^2 + \int_Ω u(\Delta u - \partial_t u)G_λ dx \] \] (1.13)

and

\[ D_λ'(t) := -θ - 2 \int_Ω \left( \frac{x - x_0}{2(T - t + λ)} \cdot \nabla u + \frac{1}{2} (\Delta u - \partial_t u) \right)^2 G_λ dx + \frac{1}{2} \int_Ω (\Delta u - \partial_t u)^2 G_λ dx + \frac{1}{2} \int_Ω \frac{1}{(T - t + λ)} D_λ(t), \] (1.14)

where

\[ θ := \int_{∂Ω} |\nabla u|^2 \partial_ν G_λ dσ - 2 \int_{∂Ω} \partial_ν u(\nabla u \cdot \nabla G_λ) dσ. \]

**Lemma 1.5.** For \( λ > 0 \) and \( t \in (0, T) \), frequency function \( N_λ(t) \) holds

\[ \frac{d}{dt} \left[ (T - t + λ) \exp(-2M^2t) N_λ(t) \right] \leq CM^2(T + λ). \] (1.15)
Proof.

\[ N'_\lambda(t) = \frac{2}{H_\lambda^2(t)} \left[ D'_\lambda(t)H_\lambda(t) - D_\lambda(t)H'_\lambda(t) \right] := I_1 + I_2, \]

representing first term and second term; further,

\[
I_1 = \frac{2}{H_\lambda^2(t)} \left\{ -\theta - 2 \int_\Omega \left( \partial_t u - \frac{x-x_0}{2(T-t+\lambda)} \cdot \nabla u + \frac{1}{2}(\triangle u - \partial_t u) \right)^2 G_\lambda \, dx \right. \\
+ \frac{1}{2} \int_\Omega (\triangle u - \partial_t u)^2 G_\lambda + \frac{1}{(T-t+\lambda)} D_\lambda(t) \Big\} H_\lambda(t) \\
= \frac{2}{H_\lambda(t)} \left( -\theta + \frac{D_\lambda(t)}{T-t+\lambda} \right) \\
- \frac{4}{H_\lambda(t)} \left[ \int_\Omega \left( \partial_t u - \frac{x-x_0}{2(T-t+\lambda)} \cdot \nabla u + \frac{1}{2}(\triangle u - \partial_t u) \right)^2 G_\lambda \, dx \right] \\
+ \frac{1}{H_\lambda(t)} \int_\Omega (\triangle u - \partial_t u)^2 G_\lambda dx.
\]

and

\[
I_2 = \frac{1}{H_\lambda^2(t)} \left\{ 4 \left[ \int_\Omega u \left( \partial_t u - \frac{x-x_0}{2(T-t+\lambda)} \cdot \nabla u + \frac{1}{2}(\triangle u - \partial_t u) \right) G_\lambda \, dx \right]^2 \\
- \left[ \int_\Omega u(\triangle u - \partial_t u)G_\lambda \, dx \right]^2 \right\},
\]

from Lemma 1.4. Therefore,

\[
N'_\lambda(t) = \frac{2}{H_\lambda(t)} \left( -\theta + \frac{1}{T-t+\lambda} D_\lambda(t) \right) + \frac{1}{H_\lambda(t)} \int_\Omega (\triangle u - \partial_t u)^2 G_\lambda dx \\
+ \frac{4}{H_\lambda^2(t)} \left[ \int_\Omega u \left( \partial_t u - \frac{x-x_0}{2(T-t+\lambda)} \cdot \nabla u + \frac{1}{2}(\triangle u - \partial_t u) \right) G_\lambda \, dx \right]^2 \\
- \frac{1}{H_\lambda^2(t)} \left[ \int_\Omega \left( \partial_t u - \frac{x-x_0}{2(T-t+\lambda)} \cdot \nabla u + \frac{1}{2}(\triangle u - \partial_t u) \right)^2 G_\lambda \, dx \right] \\
- \frac{4}{H_\lambda(t)} \int_\Omega (\triangle u - \partial_t u)^2 G_\lambda dx. \quad (1.16)
\]

By Cauchy-Schwarz inequality, we have

\[
\left[ \int_\Omega u \left( \partial_t u - \frac{x-x_0}{2(T-t+\lambda)} \cdot \nabla u + \frac{1}{2}(\triangle u - \partial_t u) \right) G_\lambda \, dx \right]^2 \\
\leq \left[ \int_\Omega \left( \partial_t u - \frac{x-x_0}{2(T-t+\lambda)} \cdot \nabla u + \frac{1}{2}(\triangle u - \partial_t u) \right)^2 G_\lambda \, dx \right] \cdot H_\lambda(t).
\]

This, combining with (1.16), indicates

\[
N'_\lambda(t) - \frac{1}{T-t+\lambda} N_\lambda(t) + \frac{2\theta}{H_\lambda(t)} \lesssim \frac{1}{H_\lambda(t)} \int_\Omega (\triangle u - \partial_t u)^2 G_\lambda dx. \quad (1.17)
\]
On the right hand side, by (1.1), we obtain
\[
\frac{1}{H_\lambda(t)} \int_\Omega (\triangle u - \partial_t u)^2 G_\lambda \, dx \leq \frac{2M^2}{H_\lambda(t)} \int_\Omega |\nabla u|^2 G_\lambda \, dx + \frac{2M^2}{H_\lambda(t)} \int_\Omega |u|^2 G_\lambda \, dx.
\]
Hence,
\[
N'_\lambda(t) - \frac{N_\lambda(t)}{T - t + \lambda} + \frac{2\theta}{H_\lambda(t)} - M^2 N_\lambda(t) \leq 2M^2. 
\tag{1.18}
\]

Now, we will discuss the term \( \theta \). Since \( u = 0 \) on \( \partial \Omega \), it holds that \( \nabla u = (\partial_\nu u) \nu \) on \( \partial \Omega \). By direct computation, we obtain
\[
\theta := \int_{\partial \Omega} |\nabla u|^2 \partial_\nu G_\lambda \, d\sigma - 2 \int_{\partial \Omega} \partial_\nu u (\nabla u \cdot \nabla G_\lambda) \, d\sigma
\]
\[
= - \frac{1}{2(T - t + \lambda)} \int_{\partial \Omega} |\nabla u|^2 ((x - x_0) \cdot \nu) G_\lambda \, d\sigma
+ \frac{1}{(T - t + \lambda)} \int_{\partial \Omega} \partial_\nu u ((x - x_0) \cdot \nabla u) G_\lambda \, d\sigma
\]
\[
= - \frac{1}{2(T - t + \lambda)} \int_{\partial \Omega} |\nabla u|^2 ((x - x_0) \cdot \nu) G_\lambda \, d\sigma
+ \frac{1}{(T - t + \lambda)} \int_{\partial \Omega} (\partial_\nu u)^2 ((x - x_0) \cdot \nu) G_\lambda \, d\sigma
\]
\[
= \frac{1}{2(T - t + \lambda)} \int_{\partial \Omega} (\partial_\nu u)^2 ((x - x_0) \cdot \nu) G_\lambda \, d\sigma.
\]
Since the domain is convex, we have \((x - x_0) \cdot \nu \geq 0\) from the geometric point of view. Thus, \( \theta \geq 0 \). This, combining with (1.18), shows
\[
N'_\lambda(t) - \frac{1}{T - t + \lambda} + M^2 N_\lambda(t) \leq CM^2. 
\tag{1.19}
\]

Therefore, by multiplying integral factor \( \exp \left( \ln(T - t + \lambda) - M^2 t \right) \), for any \( t \in (0, T) \), (1.19) yields that
\[
\frac{d}{dt} \left[ N_\lambda(t)(T - t + \lambda) \exp(-M^2 t) \right] \leq CM^2(T - t + \lambda),
\]
which leads to the conclusion. \( \square \)

Let a constant \( K_T \) be
\[
K_T = 4 \ln \left( \frac{\int_\Omega |u(x, 0)|^2 \, dx}{\int_\Omega |u(x, T)|^2 \, dx} \right) + \frac{2m}{T} + CM^2T^2 + CM^2T + CMT + \frac{n}{2}. 
\tag{1.20}
\]

**Lemma 1.6.** For each \( \lambda > 0 \), it holds that:
\[
\lambda e^{-M^2 T} N_\lambda(T) + \frac{n}{2} \leq \left( \frac{\lambda}{T} + 1 \right) K_T. 
\tag{1.21}
\]

**Proof.** Integrating (1.15) over \((t, T)\), we infer
\[
\lambda e^{-M^2 T} N_\lambda(T) - (T - t + \lambda)e^{-M^2 t} N_\lambda(t) \leq CM^2(T + \lambda)T,
\]
integrating the above on \((0, \frac{T}{2})\), we get
\[
\frac{T}{2} \lambda e^{-M^2 T} N_\lambda(T) \leq (T + \lambda) \int_0^{\frac{T}{2}} N_\lambda(t) \, dt + M^2 T^2 \frac{T}{2} (T + \lambda).
\]
Since Lemma 1.4, we have
\[
\int_{0}^{T} N_{\lambda}(t) dt = -\int_{0}^{T} H_{\lambda}'(t) dt + \int_{0}^{T} \frac{2}{H_{\lambda}(t)} \int_{\Omega} u(\partial_{t} u - \Delta u) G_{\lambda} dx dt
\]
\[
= -\ln\left(\frac{H_{\lambda}(\frac{T}{2})}{H_{\lambda}(0)}\right) + \int_{0}^{T} \frac{2}{H_{\lambda}(t)} \int_{\Omega} u(\partial_{t} u - \Delta u) G_{\lambda} dx dt. \tag{1.22}
\]

On the right hand side, by (1.1), we have
\[
\int_{0}^{T} \frac{M}{H_{\lambda}(t)} \int_{\Omega} |\nabla u| + |u| |G_{\lambda}| dx dt
\]
\[
\leq \int_{0}^{T} \frac{M}{H_{\lambda}(t)} \left( \int_{\Omega} |\nabla u||u| G_{\lambda} dx dt + \int_{\Omega} |u|^2 G_{\lambda} dx dt \right) := I_{3} + I_{4}.
\]

Estimating \( I_{3} \) and \( I_{4} \), it follows that
\[
I_{3} := \int_{0}^{T} \frac{M}{H_{\lambda}(t)} \int_{\Omega} |\nabla u||u| G_{\lambda} dx dt \leq \frac{1}{2} \int_{0}^{T} \frac{2}{H_{\lambda}(t)} \int_{\Omega} |\nabla u|^2 G_{\lambda} dx dt + CM^2 T,
\]
and
\[
I_{4} := \int_{0}^{T} \frac{M}{H_{\lambda}(t)} \int_{\Omega} u^2 G_{\lambda} dx dt \leq CMT.
\]

In \( I_{3} \), the term \( \frac{1}{2} \int_{0}^{T} N_{\lambda}(t) dt \) can be moved to the left in (1.22) for combination. Hence,
\[
\frac{T}{2} \lambda e^{-M^2 T} N_{\lambda}(t) \leq 2(T + \lambda) \left[ \ln \frac{H_{\lambda}(0)}{H_{\lambda}(\frac{T}{2})} + CM^2 T + CMT \right] + CM^2 \frac{T^2}{2}(T + \lambda).
\]

In the term \( \ln \frac{H_{\lambda}(0)}{H_{\lambda}(\frac{T}{2})} \),
\[
\frac{H_{\lambda}(0)}{H_{\lambda}(\frac{T}{2})} = \frac{\int_{\Omega} |u(x, 0)|^2 (T + \lambda)^{-\frac{T}{2}} \cdot e^{-\frac{|x-x_0|^2}{4(T+\lambda)}} dx}{\int_{\Omega} |u(x, \frac{T}{2})|^2 (T + \lambda)^{-\frac{T}{2}} \cdot e^{-\frac{|x-x_0|^2}{4(T+\lambda)}} dx} \leq \frac{\int_{\Omega} |u(x, 0)|^2 dx \cdot (T + \lambda)^{\frac{T}{2}}}{\int_{\Omega} |u(x, \frac{T}{2})|^2 dx \cdot e^{-\frac{|x-x_0|^2}{4(T+\lambda)}} dx \cdot (T + \lambda)^{\frac{T}{2}}}
\]
\[
\leq \frac{\int_{\Omega} |u(x, 0)|^2 dx \cdot (T + \lambda)^{\frac{T}{2}} \cdot e^{\frac{m}{2(T+\lambda)}}}{\int_{\Omega} |u(x, \frac{T}{2})|^2 dx} \leq e^{\frac{m}{2T}} \frac{\int_{\Omega} |u(x, 0)|^2 dx}{\int_{\Omega} |u(x, \frac{T}{2})|^2 dx}.
\]

Therefore,
\[
\frac{T}{2} \lambda e^{-M^2 T} N_{\lambda}(t) \leq 2(T + \lambda) \left( \frac{m}{2T} + \ln \left( \frac{\int_{\Omega} |u(x, 0)|^2 dx}{\int_{\Omega} |u(x, \frac{T}{2})|^2 dx} \right) + CM^2 T + CMT \right) + CM^2 \frac{T^2}{2}(T + \lambda).
\]

Using Assumption 2, we have
\[
\frac{\int_{\Omega} |u(x, T)|^2 dx}{\int_{\Omega} |u(x, \frac{T}{2})|^2 dx} \leq \exp(CMT). \tag{1.24}
\]
From (1.24), we can move the ratio term on the left to the right side, then add the result to (1.23), and obtain
\[
\lambda e^{-M^2T} N(T) \leq \left( \frac{\lambda}{T} + 1\right) \left[ \frac{2m}{T} + 4 \ln \left( \frac{\int_\Omega |u(x,0)|^2 dx}{\int_\Omega |u(x,T)|^2 dx}\right) + CMT + CM^2T + CM^2T^2 \right]
\leq \left( \frac{\lambda}{T} + 1\right) (K_T - \frac{n}{2}) \leq \left( \frac{\lambda}{T} + 1\right) K_T - \frac{n}{2},
\]
and we obtain the result.

\[\square\]

\textbf{Lemma 1.7.} For \( T > 0 \), the following estimate holds
\[
\left[ 1 - \frac{8e^{M^2T} \lambda}{r^2} \left( \frac{\lambda}{T} + 1\right) K_T \right] \int_\Omega |x - x_0|^2 |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4r^2}} dx
\leq 8e^{M^2T} \lambda \left( \frac{\lambda}{T} + 1\right) K_T \int_{B_r} |u(x,T)| e^{-\frac{|x-x_0|^2}{4r^2}} dx.
\]

\textbf{Proof.} We borrow an inequality from [7] that for any \( f \in H_0^1(\Omega) \) and for a \( \lambda > 0 \),
\[
\int_\Omega \frac{|x - x_0|^2}{8\lambda} |f(x)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx
\leq 2\lambda \int_\Omega |\nabla f(x)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx + \frac{n}{2} \int_\Omega |f(x)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx.
\]

(1.25)

From this fact,
\[
\int_\Omega |x - x_0|^2 |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4r^2}} dx
\leq 8\lambda \left( 2\lambda \int_\Omega |\nabla u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx + \frac{n}{2} \int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx \right)
\leq 8\lambda \left( \lambda N_\lambda(T) + \frac{n}{2} \right) \int_\Omega |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx
\leq 8\lambda \left( \lambda N_\lambda(T) + \frac{n}{2} \right) \left[ \int_{B_r} |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx + \frac{1}{r^2} \int_{\Omega \setminus B_r} |x - x_0|^2 |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4r^2}} dx \right],
\]
as \( |x-x_0| \geq 1 \) when \( x \notin B_r \).

With help of (1.21), we can observe
\[
\int_\Omega |x - x_0|^2 |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4r^2}} dx
\leq 8\lambda e^{M^2T} \left( \frac{\lambda}{T} + 1\right) K_T \left[ \int_{B_r} |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} dx + \frac{1}{r^2} \int_{\Omega} |x - x_0|^2 |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4r^2}} dx \right].
\]

We can arrive at the result from the following
\[
\left[ 1 - \frac{8e^{M^2T} \lambda}{r^2} \left( \frac{\lambda}{T} + 1\right) K_T \right] \int_\Omega |x - x_0|^2 |u(x,T)|^2 e^{-\frac{|x-x_0|^2}{4r^2}} dx
\]
Combining (2.2) and (2.3),

\[ \leq 8e^{MT}\lambda \left( \frac{\lambda}{T} + 1 \right) K_T \int_{B_r} |u(x, T)|e^{-\frac{|x-x_0|^2}{4\lambda}} \, dx. \]

This completes the proof. \(\square\)

2. The unique continuation property.

2.1. Proof of Theorem 1.1.

Proof. We take

\[ \lambda = \frac{1}{2} \left( -T + \sqrt{T^2 + \frac{r^2 T^2}{4K_T e^{MT}}} \right) > 0, \]

such that

\[ \frac{8e^{MT}\lambda}{r^2} \left( \frac{\lambda}{T} + 1 \right) K_T = \frac{1}{2}. \] (2.1)

It follows that

\[ \frac{1}{\lambda} = 2 \frac{T + \sqrt{T^2 + \frac{4K_T}{e^{MT}e^{MT}K_T}}} = 8 \left( T + \sqrt{T^2 + \frac{4K_T}{e^{MT}e^{MT}K_T}} \right) \frac{1}{T} e^{MT} K_T \]

\[ \leq 8 \left( 2T + \frac{4K_T}{e^{MT}e^{MT}K_T} \right) \frac{1}{T} e^{MT} K_T \leq (16 + \frac{4r}{\sqrt{m^2}}) \frac{1}{r^2} e^{MT} K_T, \]

as \( \frac{m}{r} \leq K_T \) and \( e^{MT} \geq 1 \). We obtain

\[ e^{\frac{m}{r}} \leq e^{\frac{(4m+r\sqrt{m})}{2r} K_T e^{MT}} \]

\[ \leq e^{\frac{(4m+r\sqrt{m})}{2r} e^{MT}} e^{\frac{(4m+r\sqrt{m})}{2r} e^{MT} (\frac{2r}{T} + CM^2T^2 + CM^2T + CMT)} \]

\[ \times \left( \frac{\int_{\Omega} |u(x, 0)|^2 \, dx}{\int_{\Omega} |u(x, T)|^2 \, dx} \right)^{\frac{1}{2}} (4m+r\sqrt{m}) e^{MT} \] (2.2)

From Lemma 1.7, we have

\[ \int_{\Omega} |x-x_0|^2 |u(x, T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} \, dx \leq r^2 \int_{B_r} |u(x, T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} \, dx. \] (2.3)

Combining (2.2) and (2.3),

\[ \int_{\Omega} |u(x, T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} \, dx \leq \int_{\Omega} |u(x, T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} \, dx \]

\[ \leq \int_{\Omega\setminus B_r} |u(x, T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} \, dx + \int_{B_r} |u(x, T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} \, dx \]

\[ \leq \frac{1}{r^2} \int_{\Omega} |x-x_0|^2 |u(x, T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} \, dx + \int_{B_r} |u(x, T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} \, dx \]

\[ \leq 2 \int_{B_r} |u(x, T)|^2 e^{-\frac{|x-x_0|^2}{4\lambda}} \, dx \leq 2 \int_{B_r} |u(x, T)|^2 \, dx, \]
We first address two energy estimates from equation (1.1). Multiplying

equation (1.1) by \( u \) and integrating over \( \Omega \), we obtain:

\[
\int_\Omega |u(x, T)|^2 \, dx \leq 2e^{\frac{CM}{2}} \int_{B_r} |u(x, T)|^2 \, dx
\]

\[
\leq 2e^{(4m + r \sqrt{m}) \frac{1}{2} e^{M^2T} \left( \frac{2m}{4} + CM^2T^2 + CM^2T + CM \right) e^{M^2T}} \times \left( \int_\Omega |u(x, 0)|^2 \, dx \right) \frac{1}{2} (4m + r \sqrt{m}) e^{M^2T} \times \int_{B_r} |u(x, T)|^2 \, dx.
\]

Thus, we have

\[
\int_\Omega |u(x, T)|^2 \, dx \leq 2e^{C\frac{M^2T}{r}} e^{C\frac{2m}{4} + M^2T^2 + M^2T + MT} e^{M^2T} \left( \int_\Omega |u(x, 0)|^2 \, dx \right) \frac{C' e^{M^2T}}{r \sqrt{m}} \times \int_{B_r} |u(x, T)|^2 \, dx.
\]

This is equivalent to the following inequality:

\[
\int_\Omega |u(x, T)|^2 \, dx \leq C e^{C'\frac{2m}{4} + M^2T^2 + M^2T + MT} \left( \int_\Omega |u(x, 0)|^2 \, dx \right) \frac{C' e^{M^2T}}{r \sqrt{m}} \times \left( \int_{B_r} |u(x, T)|^2 \, dx \right)^{\frac{2}{1 - \gamma}}
\]

where \( C' = 4(4m + r \sqrt{m}) e^{M^2T} \). Let \( \gamma = \frac{r^2}{4m + r \sqrt{m}} \), and the above estimate gives

\[
\int_\Omega |u(x, T)|^2 \, dx \leq C e^{C'\frac{2m}{4} + M^2T^2 + M^2T + MT} \left( \int_\Omega |u(x, 0)|^2 \, dx \right) \frac{C' e^{M^2T}}{r \sqrt{m}} \times \left( \int_{B_r} |u(x, T)|^2 \, dx \right)^{\frac{2}{1 - \gamma}}
\]

This completes the proof.

\[\square\]

2.2. Proof of Theorem 1.2.

Proof. We will first prove a backward uniqueness estimate:

\[
\|u(0)\|_{H^{-1}((\omega))}^2 \leq \exp \left( 2e^{CM^2T} \left( \zeta(0) + CM \sqrt{\zeta(0)} \right) T \right) \|u(T)\|_{H^{-1}((\omega))}^2.
\]

We first address two energy estimates from equation (1.1). Multiplying \( u \) and \((-\Delta)^{-1} u\) with \( \partial_t u - \Delta u \), we have two energy identities

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + \|u(t)\|_{H^{-1}((\omega))}^2 = \langle (\partial_t u - \Delta u), u \rangle,
\]

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2_{H^{-1}((\omega))} + \|u(t)\|^2 = \langle (\partial_t u - \Delta u), (-\Delta)^{-1} u \rangle.
\]

Let \( f = \partial_t u - \Delta u \), and \( \zeta(t) := -\frac{\|u(t)\|^2}{\|u(t)\|_{H^{-1}((\omega))}^2} \), then

\[
\zeta(t) = \frac{2}{\|u(t)\|_{H^{-1}((\omega))}^2} \left( \langle f, u \rangle \|u(t)\|_{H^{-1}((\omega))} - \|u(t)\|_{H^2((\omega))}^2 \right) - \langle f, (-\Delta)^{-1} u \rangle \|u\|^2 + \|u\|^4.
\]

By direct computation, we have

\[
\|u\|^4 - \|u\|^2 (\|f\|_{H^{-1}((\omega))}^2 - \|u\|_{H^{-1}((\omega))}^2) - \langle f, (-\Delta)^{-1} u \rangle \|u\|^2 + \|u\|^4 \\
\leq (\|u\|_{H^2((\omega))}^2 + \|f\|_{H^{-1}((\omega))}^2 - \langle f, u \rangle) \|u(t)\|_{H^{-1}((\omega))}^2 - \langle \frac{f}{2}, (-\Delta)^{-1} u \rangle^2.
\]
Therefore, we can obtain the following estimate:

\[
\frac{d}{dt} \zeta(t) \leq \frac{2}{\|u\|_{H^{-1}(\Omega)}^2} \|f\|_{H^{-1}(\Omega)}^2.
\]

By Assumption 2,

\[
\|f\|_{H^{-1}(\Omega)} \leq CM\|u\|,
\]

thus

\[
\zeta(t) \leq e^{CM^2t}\zeta(0).
\]

Applying this to \(H^{-1}\) energy identity (2.6), we have

\[
0 \leq \frac{1}{2} \frac{d}{dt} \|u(t)\|_{H^{-1}(\Omega)}^2 + \zeta(t) \|u(t)\|_{H^{-1}(\Omega)}^2 + \|f\|_{H^{-1}(\Omega)}^2 u_{H^{-1}(\Omega)}^2 + CM \|u\| u_{H^{-1}(\Omega)}^2
\]

\[
\leq \frac{1}{2} \frac{d}{dt} \|u(t)\|_{H^{-1}(\Omega)}^2 + \zeta(t) \|u(t)\|_{H^{-1}(\Omega)}^2 + \|f\|_{H^{-1}(\Omega)}^2 u_{H^{-1}(\Omega)}^2 + CM \|u\| u_{H^{-1}(\Omega)}^2
\]

\[
\leq \frac{1}{2} \frac{d}{dt} \|u(t)\|_{H^{-1}(\Omega)}^2 + \zeta(t) e^{CM^2t} \|u(t)\|_{H^{-1}(\Omega)}^2 + CM \|u\| u_{H^{-1}(\Omega)}^2
\]

\[
\leq \frac{1}{2} \frac{d}{dt} \|u(t)\|_{H^{-1}(\Omega)}^2 + \zeta(t) e^{CM^2t} \|u(t)\|_{H^{-1}(\Omega)}^2 + CM \sqrt{\zeta(t)} \|u\|_{H^{-1}(\Omega)}^2
\]

\[
\leq \frac{1}{2} \frac{d}{dt} \|u(t)\|_{H^{-1}(\Omega)}^2 + e^{CM^2T} \left(\zeta(0) + CM \sqrt{\zeta(0)}\right) \|u(t)\|_{H^{-1}(\Omega)}^2.
\]

Therefore,

\[
\|u(0)\|_{H^{-1}(\Omega)}^2 \leq \exp(2e^{CM^2T} \left(\zeta(0) + CM \sqrt{\zeta(0)}\right) T) \|u(T)\|_{H^{-1}(\Omega)}^2.
\]

This yields

\[
\frac{\|u(0)\|^2}{\|u(T)\|^2} \leq \frac{\|u(0)\|_{H^{-1}(\Omega)}^2}{\|u(T)\|_{H^{-1}(\Omega)}^2} \zeta(0) \leq \zeta(0) \exp(2e^{CM^2T} \left(\zeta(0) + CM \sqrt{\zeta(0)}\right) T)
\]

\[
\leq \exp(C(1 + MT)e^{CM^2T}\zeta(0)).
\]

This, together with (1.2), deduces (1.3). This completes the proof. \( \Box \)

3. A nontrivial example. In this section, we will give a nontrivial parabolic example when theorems hold. We consider the equation as follows:

\[
\begin{cases}
\partial_t u - \Delta u + \sum_{i=1}^n b_i(x,t)\partial_x u + c(x,t)u = 0, & \text{in } \Omega \times (0,T), \\
u = 0, & \text{on } \partial \Omega \times (0,T), \\
u(x,0) = u_0(x), &
\end{cases}
\]

where \( u \) denote state \( u(x,t) \) at spatial position \( x \in \Omega \) and time \( t \geq 0 \), and the initial data \( u_0(x) \in L^2(\Omega) \).

Now, we suppose that the coefficients \( b_i(x,t), c(x,t), (i = 1, 2, \ldots, n) \) satisfy

\[
b_i(x,t), c(x,t) \in L^\infty(\Omega \times (0,T)), (i = 1, 2, \ldots, n),
\]

and

\[
M = \max \{ \|b_i\|_{L^\infty(\Omega \times (0,T))}, \|c\|_{L^\infty(\Omega \times (0,T))} \mid i = 1, 2, \ldots, n \}.
\]

Thus, the solutions of equation (1.1) \( u(x,t) \in L^2(0,T; H^1_0(\Omega)) \cap H^1(0,T; H^{-1}(\Omega)) \) since the initial data \( u_0 \in L^2(\Omega) \). Assumption 1 in Section 1 is satisfied. By the standard energy estimate, Assumption 2 holds as well. In order to get Assumption 3, we will introduce the following Lemma.
Lemma 3.1. Suppose that $h \in L^\infty(\Omega)$, and $g \in H^1_0(\Omega)$. Then, we have
\[
\|h \cdot \partial_i g\|_{H^{-1}(\Omega)} \leq C \|h\|_{L^\infty(\Omega)} \cdot \|g\|_2, \quad i = 1, 2, \ldots, n.
\]

Proof. We will prove for any fixed $i \in \{1, 2, \ldots, n\}$. Clearly, we can find a function $v \in L^\infty(\Omega)$ such that $\partial_i v = h$, and $\|v\|_{L^\infty(\Omega)} \leq C \|h\|_{L^\infty(\Omega)}$. Then,
\[
h \cdot \partial_i g = \partial_i (v \cdot g) - h \cdot g.
\]
Thus,
\[
\|h \cdot \partial_i g\|_{H^{-1}(\Omega)} \leq \|\partial_i (v \cdot g)\|_{H^{-1}(\Omega)} + \|h \cdot g\|_{H^{-1}(\Omega)}
\leq \|v \cdot g\|_2 + \|h \cdot g\|_2 \leq C \|h\|_{L^\infty(\Omega)} \cdot \|g\|_2.
\]
This completes the proof.

By this Lemma, Assumption 3 also holds in this case.

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