Atomic decompositions of Besov spaces related to symmetric cones

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Abstract. In this paper we extend the atomic decompositions previously obtained for Besov spaces related to the forward light cone to general symmetric cones. We do so via wavelet theory adapted to the cone. The wavelet transforms sets up an isomorphism between the Besov spaces and certain reproducing kernel function spaces on the group, and sampling of the transformed data will provide the atomic decompositions and frames for the Besov spaces.

1. Introduction

Besov spaces related to symmetric cones were introduced by Bekolle, Bonami, Garrigos and Ricci in a series of papers [1, 3] and [2]. The purpose was to use Fourier-Laplace extensions for the Besov spaces in order to investigate the continuity of Bergman projections and boundary values for Bergman spaces on tube type domains.

Classical homogeneous Besov spaces were introduced via local differences and modulus of continuity. Through work of Peetre [11], Triebel [15] and Feichtinger and Gröchenig [10] these spaces were given a characterization via wavelet theory. The theory of Feichtinger and Gröchenig [10, 13] further provided atomic decompositions and frames for the homogeneous Besov spaces.

In the papers [6] and [4] we gave a wavelet characterization and several atomic decompositions for the Besov spaces related to the special case of the forward light cone. In this paper we will show that the machinery carries over to Besov spaces related to any symmetric cone. Our approach contains some representation theoretic simplifications compared with the work of Feichtinger and Gröchenig, and we in particular exploit smooth representations of Lie groups. The results presented here are also interesting in the context of recent results by Führ [12] dealing with coorbits for wavelets with general dilation groups.
2. Wavelets, sampling and atomic decompositions

In this section we use representation theory to set up a correspondence between a Banach space of distributions and a reproducing kernel Banach space on a group. For details we refer to [4][5][6] which generalizes work in [10].

2.1. Wavelets and coorbit theory. Let $S$ be a Fréchet space and let $S^*$ be the conjugate linear dual equipped with the weak* topology (any reference to weak convergence in $S^*$ will always refer to the weak* topology). We assume that $S$ is continuously embedded and weakly dense in $S^*$. The conjugate dual pairing of elements $\phi \in S$ and $f \in S^*$ will be denoted by $\langle f, \phi \rangle$. Let $G$ be a locally compact group with a fixed left Haar measure $dg$, and assume that $(\pi, S)$ is a continuous representation of $G$, i.e. $g \mapsto \pi(g)\phi$ is continuous for all $\phi \in S$. A vector $\phi \in S$ is called cyclic if $\langle f, \pi(g)\phi \rangle = 0$ for all $g \in G$ means that $f = 0$ in $S^*$. As usual, define the contragradient representation $(\pi^*, S^*)$ by

$$\langle \pi^*(g)f, \phi \rangle = \langle f, \pi(g)\phi \rangle \text{ for } f \in S^*.$$  

Then $\pi^*$ is a continuous representation of $G$ on $S^*$. For a fixed vector $\psi \in S$ define the linear map $W_\psi : S^* \rightarrow C(G)$ by

$$W_\psi(f)(g) = \langle f, \pi(g)\psi \rangle = \langle \pi^*(g^{-1})f, \psi \rangle.$$  

The map $W_\psi$ is called the voice transform or the wavelet transform. If $F$ is a function on $G$ then define the left translation of $F$ by an element $g \in G$ as

$$\ell_gF(h) = F(g^{-1}h).$$

A Banach space of functions $Y$ is called left invariant if $F \in Y$ implies that $\ell_gF \in Y$ for all $g \in G$ and there is a constant $C_g$ such that $\|\ell_gF\|_Y \leq C_g\|F\|_Y$ for all $F \in Y$.

In the following we will always assume that the space $Y$ of functions on $G$ is a left invariant Banach space for which convergence implies convergence (locally) in Haar measure on $G$. Examples of such spaces are $L^p(G)$ for $1 \leq p \leq \infty$ and any space continuously included in an $L^p(G)$.

A non-zero cyclic vector $\psi$ is called an analyzing vector for $S$ if for all $f \in S^*$ the following convolution reproducing formula holds

$$W_\psi(f) * W_\psi(\psi) = W_\psi(f).$$

Here convolution between two functions $F$ and $G$ on $G$ is defined by

$$F * G(h) = \int F(y)G(g^{-1}h) \, dg.$$  

For an analyzing vector $\psi$ define the subspace $Y_\psi$ of $Y$ by

$$Y_\psi = \{ F \in Y \mid F = F * W_\psi(\psi) \},$$

and let

$$\text{Co}_S^\psi Y = \{ f \in S^* \mid W_\psi(f) \in Y \}$$

equipped with the norm $\|f\| = \|W_\psi(f)\|_Y$.

A priori we do not know if the spaces $Y_\psi$ and $\text{Co}_S^\psi Y$ are trivial, but the following theorem lists conditions that ensure they are isometrically isomorphic Banach spaces. The main requirements are the existence of a reproducing formula and a duality condition involving $Y$. 

Theorem 2.1. Let $\pi$ be a representation of a group $\mathcal{G}$ on a Fréchet space $S$ with conjugate dual $S^*$ and let $Y$ be a left invariant Banach function space on $\mathcal{G}$. Assume $\psi$ is an analyzing vector for $S$ and that the mapping
\[ Y \times S \ni (F, \phi) \mapsto \int_{\mathcal{G}} F(g) \langle \pi^*(g)\psi, \phi \rangle \, dg \in \mathbb{C} \]
is continuous. Then
(1) $Y_\psi$ is a closed reproducing kernel subspace of $Y$ with reproducing kernel $K(g, h) = W_\psi(\psi)(g^{-1}h)$.
(2) The space $Co^\psi_S Y$ is a $\pi^*$-invariant Banach space.
(3) $W_\psi : Co^\psi_S Y \to Y$ intertwines $\pi^*$ and left translation.
(4) If left translation is continuous on $Y$, then $\pi^*$ acts continuously on $Co^\psi_S Y$.
(5) $Co^\psi_S Y = \{\pi^*(F)\psi \mid F \in Y_\psi\}$.
(6) $W_\psi : Co^\psi_S Y \to Y_\psi$ is an isometric isomorphism.

Note that (5) states that each member of $Co^\psi_S Y$ can be written weakly as
\[ f = \int_{\mathcal{G}} W_\psi(f)(g)\pi^*(g)\psi \, dg. \]
In the following section we will explain when this reproducing formula can be discretized and how coefficients \{\(c_i(f)\)\} can be determined in order to obtain an expression
\[ f = \sum_i c_i(f)\pi^*(g_i)\psi \]
for any $f \in Co^\psi_S Y$.

2.2. Frames and atomic decompositions through sampling on Lie groups. In this section we will decompose the coorbit spaces constructed in the previous section. For this we need sequence spaces arising from Banach function spaces on $\mathcal{G}$. The decomposition of coorbit spaces is aided by smooth representations of Lie groups.

We assume that $\mathcal{G}$ is a Lie group with Lie algebra denoted $\mathfrak{g}$. A vector $\psi \in S$ is called $\pi$-weakly differentiable in the direction $X \in \mathfrak{g}$ if there is a vector denoted $\pi(X)\psi \in S$ such that for all $f \in S^*$
\[ \langle f, \pi(X)\psi \rangle = \frac{d}{dt} \big|_{t=0} \langle f, \pi(e^{tX})\psi \rangle. \]
Fix a basis \(\{X_i\}_{i=1}^{\dim \mathfrak{g}}\) for $\mathfrak{g}$, then for a multi-index $\alpha$ we define $\pi(D^\alpha)\psi$ (when it makes sense) by
\[ \langle f, \pi(D^\alpha)\psi \rangle = \langle f, \pi(X_{\alpha(k)})\pi(X_{\alpha(k-1)}) \cdots \pi(X_{\alpha(1)})\psi \rangle. \]
A vector $f \in S^*$ is called $\pi^*$-weakly differentiable in the direction $X \in \mathfrak{g}$ if there is a vector denoted $\pi^*(X)f \in S^*$ such that for all $\phi \in S$
\[ \langle \pi^*(X)f, \phi \rangle = \frac{d}{dt} \big|_{t=0} \langle \pi^*(e^{tX})f, \phi \rangle. \]
For a multi-index $\alpha$ define $\pi^*(D^\alpha)\psi$ (when it makes sense) by
\[ \pi^*(D^\alpha)\psi = \pi^*(X_{\alpha(k)})\pi^*(X_{\alpha(k-1)}) \cdots \pi^*(X_{\alpha(1)})\psi \]
Let $U$ be a relatively compact set in $\mathcal{G}$ and let $I$ be a countable set. A sequence \(\{g_i\}_{i \in I} \subseteq \mathcal{G}\) is called $U$-dense if $\{g_iU\}$ cover $\mathcal{G}$, and $V$-separated if for some relatively
compact set $V \subseteq U$ the $g_i V$ are pairwise disjoint. Finally we say that $\{g_i\}_{i \in I} \subseteq G$ is well-spread if it is $U$-dense and a finite union of $V$-separated sequences. For properties of such sequences we refer to [10]. A Banach space $Y$ of measurable functions is called solid, if $|f| \leq |g|$, $f$ measurable and $g \in Y$ imply that $f \in Y$. For a $U$-relatively separated sequence of points $\{g_i\}_{i \in I}$ in $G$, and a solid Banach function space $Y$ on $G$, define the space $Y^#(I)$ of sequences $\{\lambda_i\}_{i \in I}$ for which

$$\|\{\lambda_i\}\|_{Y^#} := \left\| \sum_{i \in I} |\lambda_i| 1_{g_i, U} \right\|_Y < \infty.$$  

These sequence spaces were introduced in [9] (see also [10]), and we remark that they are independent on the choice of $U$ (for a fixed well spread sequence). For a well-spread set $\{g_i\}$ a $U$-bounded uniform partition of unity ($U$-BUPU) is a collection of functions $\psi_i$ on $G$ such that $0 \leq \psi_i \leq 1_{g_i, U}$ and $\sum_i \psi_i = 1$.

In the sequel we will only investigate sequences which are well-spread with respect to compact neighbourhoods of the type

$$U_\epsilon = \{e^{t_1 x_1} \ldots e^{t_n x_n} | t_1, \ldots, t_n \in [-\epsilon, \epsilon]\},$$

where $\{X_i\}_{i=1}^n$ is the fixed basis for $g$.

**Theorem 2.2.** Let $Y$ be a solid and left and right invariant Banach function space for which right translations are continuous. Assume there is a cyclic vector $\psi \in S$ satisfying the properties of Theorem 2.1 and that $\psi$ is both $\pi$-weakly and $\pi^*$-weakly differentiable up to order $\dim(G)$. If the mappings $Y \ni F \mapsto F \ast |W_{\pi(D^\alpha)}(\psi)| \in Y$ are continuous for all $|\alpha| \leq \dim(G)$, then we can choose $\epsilon$ and positive constants $A_1, A_2$ such that for any $U_\epsilon$-relatively separated set $\{g_i\}$

$$A_1 \|f\|_{C_0^\psi Y} \leq \|\{\langle f, \pi(g_i)\psi \rangle\}\|_{Y^#} \leq A_2 \|f\|_{C_0^\psi Y}.$$ 

Furthermore, there is an operator $T_1$ such that

$$f = W_{\psi}^{-1} T_1^{-1} \left( \sum_i W_{\psi}(f) (g_i) \psi_i \ast W_{\psi}(\psi) \right),$$

where $\{\psi_i\}$ is any $U_\epsilon$-BUPU for which $\text{supp}(\psi_i) \subseteq g_i U_\epsilon$.

The operator $T_1 : Y_\psi \mapsto Y_\psi$ (first introduced in [13]) is defined by

$$T_1 F = \sum_i F(g_i) \psi_i \ast W_{\psi}(\psi).$$

**Theorem 2.3.** Let $\psi \in S$ be $\pi^*$-weakly differentiable up to order $\dim(G)$ satisfying the assumptions in Theorem 2.2 and let $Y$ be a solid left and right invariant Banach function space for which right translation is continuous. Assume that $Y \ni F \mapsto F \ast |W_{\pi^*(D^\alpha)}(\psi)| \in Y$ is continuous for $|\alpha| \leq \dim(G)$. We can choose $\epsilon$ small enough that for any $U_\epsilon$-relatively separated set $\{g_i\}$ there is an invertible operator $T_2$ and functionals $\lambda_i$ (defined below) such that for any $f \in C_0^\psi Y$

$$f = \sum_i \lambda_i (T_2^{-1} W_{\psi}(f)) \pi(g_i) \psi$$

with convergence in $S^*$. The convergence is in $C_0^\psi Y$ if $C_c(G)$ are dense in $Y$. 


The operator $T_2 : Y_\psi \mapsto Y_\psi$ (also introduced in [13]) is defined by

$$T_2 F = \sum_i \lambda_i(F) \ell_{g_i} W_\psi(\psi),$$

where $\lambda_i(F) = \int F(g) \psi_i(g) \, dg$.

### 3. Besov spaces on symmetric cones

#### 3.1. Symmetric cones.

For an introduction to symmetric cones we refer to the book [8]. Let $V$ be a Euclidean vector space over the real numbers of finite dimension $n$. A subset $\Omega$ of $V$ is a cone if $\lambda \Omega \subseteq \Omega$ for all $\lambda > 0$. Assume $\Omega$ is open and convex, and define the open dual cone $\Omega^*$ by

$$\Omega^* = \{ y \in V \mid (x, y) > 0 \text{ for all non-zero } x \in \overline{\Omega} \}.$$  

The cone $\Omega$ is called symmetric if $\Omega = \Omega^*$ and the automorphism group $G(\Omega) = \{ g \in \text{GL}(V) \mid g \Omega = \Omega \}$ acts transitively on $\Omega$. In this case the set of adjoints of elements in $G(\Omega)$ is $G(\Omega)$ itself, i.e. $G(\Omega)^* = G(\Omega)$. Define the characteristic function of $\Omega$ by

$$\varphi(x) = \int_{\Omega^*} e^{-(x,y)} \, dy,$$

then

$$\varphi(gx) = |\det(g)|^{-1} \varphi(x).$$

Also,

$$f \mapsto \int_{\Omega} f(x) \varphi(x) \, dx$$

defines a $G(\Omega)$-invariant measure on $\Omega$. The connected component $G_0(\Omega)$ of $G(\Omega)$ has Iwasawa decomposition

$$G_0(\Omega) = KAN$$

where $K = G_0(\Omega) \cap O(V)$ is compact, $A$ is abelian and $N$ is nilpotent. The unique fixed point in $\Omega$ for the mapping $x \mapsto \nabla \log \varphi(x)$ will be denoted $e$, and we note that $K$ fixes $e$. The connected solvable subgroup $H = AN$ of $G_0(\Omega)$ acts simply transitively on $\Omega$ and the integral $\mu_H$ thus also defines the left-Haar measure on $H$.

Throughout this paper we will identify functions on $H$ and $\Omega$ by right-$K$-invariant functions on $G_0(\Omega)$. If $F$ is a right-$K$-invariant function on $G$ and we denote by $f$ the corresponding function on the cone $\Omega$, then

$$F \mapsto \int_{H} F(h) \, dh := \int_{\Omega} f(x) \varphi(x) \, dx$$

gives an integral formula for the left-Haar measure on $H$ which we will denote by $dh$ or $\mu_H$.

**Lemma 3.1.** If $F$ is an $\mu_H$-integrable right-$K$-invariant function on $G_0(\Omega)$, then there is a constant $C$ such that

$$\int F(h) \, dh = C \int F((h^*)^{-1}) \, dh.$$  

Here $h^*$ denotes the adjoint element of $h$ with respect to the inner product on $V$. 

Proof. Without loss of generality we will assume that $F$ is compactly supported. Note first that the function $h \mapsto F((h^*)^{-1})$ is right-$K$-invariant and therefore can be identified with a function on $\Omega$. Since the measure $\varphi(x)\,dx$ on $\Omega$ is $G_0(\Omega)$-invariant, the measure on $H$ is also $G_0(\Omega)$-invariant. For $g \in G_0(\Omega)$ we have that $\ell_g F((h^*)^{-1}) = F((g^*h^*)^{-1})$, and therefore the mapping

$$F \mapsto \int_H F((h^*)^{-1})\,dh$$

defines a left-invariant measure on $H$. By uniqueness of Haar measure we conclude that

$$\int_H F((h^*)^{-1})\,dh = C \int_H F(h)\,dh.$$ 

□

For $f \in L^1(V)$ the Fourier transform is defined by

$$\hat{f}(w) = \frac{1}{(2\pi)^{n/2}} \int_V f(x)e^{-i(x,w)}\,dx \text{ for } w \in V,$n/2

and it extends to an unitary operator on $L^2(V)$ in the usual way. Denote by $S(V)$ the space of rapidly decreasing smooth functions with topology induced by the semi-norms

$$\|f\|_k = \sup_{|\alpha| \leq k} \sup_{x \in V} |\partial^{\alpha} f(x)| (1 + |x|)^k.$$

Here $\alpha$ is a multi-index, $\partial^{\alpha}$ denotes usual partial derivatives of functions, and $k \geq 0$ is an integer. The convolution

$$f \ast g(x) = \int_V f(y)g(x-y)\,dy$$

of functions $f, g \in S(V)$ satisfies

$$\hat{f} \ast \hat{g}(w) = \hat{f}(w)\hat{g}(w).$$

The space $S'(V)$ of tempered distributions is the linear dual of $S(V)$. For functions on $V$ define $\tau_x f(y) = f(y-x)$, $f^\lor(y) = f(-y)$ and $f^\ast(y) = \overline{f(-y)}$. Convolution of $f \in S'(V)$ and $\phi \in S(V)$ is defined by

$$f \ast \phi(x) = f(\tau_x \phi^\lor).$$

The space of rapidly decreasing smooth functions with Fourier transform vanishing on $\Omega$ is denoted $S_\Omega$. It is a closed subspace of $S(V)$ and will be equipped with the subspace topology.

The space $V$ can be equipped with a Jordan algebra structure such that $\Omega$ is identified with the set of all squares. This gives rise to the notion of a determinant $\Delta(x)$. We only need the fact that the determinant is related to the characteristic function $\varphi$ by

$$\varphi(x) = \varphi(e)\Delta(x)^{-n/r},$$

where $r$ denotes the rank of the cone. If $x = ge$ we have

$$\Delta(x) = \Delta(ge) = |\text{Det}(g)|^{1/n}.$$ 

The following growth estimates hold for functions in $S_\Omega$ (see Lemma 3.11 in [2]):
Lemma 3.2. If $\phi \in S_\Omega$ and $k,l$ are non-negative integers, then there is an $N = N(k,l)$ and a constant $C_N$ such that

$$|\hat{\phi}(w)| \leq C_N ||\phi||_N \frac{\Delta(w^j)}{(1 + |w|)^k}.$$ 

3.2. Besov spaces on symmetric cones. The cone $\Omega$ can be identified as a Riemannian manifold $\Omega = G_0(\Omega) / K$ where $K$ is the compact group fixing $e$. The Riemannian metric in this case is defined by $g = \rho := \rho(x) dx$ for $x \in \Omega$. Then $\rho = \rho(x)$ for $x \in \Omega$. Before we prove the theorem, let us note that $\rho = \rho(x)$ for $x \in \Omega$. Let $\rho$ be a function in $S_\Omega$ for which $1 \leq p,q < \infty$ and $s \in \mathbb{R}$ is defined in [2] by

$$\|f\|_{B^{p,q}_s} = \left( \sum_j \Delta(x_j)^{-s} \|f \ast \psi_j\|_p^q \right)^{1/q}.$$ 

The Besov space $\dot{B}^{p,q}_s$ consists of the equivalence classes of tempered distributions $f$ in $S'_\Omega \simeq \{ f \in S'(V) \mid \text{supp}(\hat{f}) \subseteq \overline{\Omega} \}/S'_{\partial \Omega}$ for which $\|f\|_{B^{p,q}_s} < \infty$.

Theorem 3.3. Let $\psi$ be a function in $S_\Omega$ for which $1 \leq \psi \leq 1$. Defining $\hat{\psi}$ by

$$\hat{\psi}_h(w) = \psi(h^{-1}w),$$

then

$$\|f\|_{B^{p,q}_s} \simeq \left( \int_\mathcal{H} \|f \ast \psi_h\|_p^q \det(h)^{-sr/n} dh \right)^{1/q}$$ 

for $f \in S'_\Omega$.

Proof. Before we prove the theorem, let us note that

$$(\psi_h)_g = \psi_{hg}.$$ 

The cover of $\Omega$ corresponds to a cover of $\mathcal{H}$: if $h_j \in \mathcal{H}$ is such that $x_j = h_j e$ then $h_j U$ covers $\mathcal{H}$ with $U = \{ h \in \mathcal{H} \mid he \in B_1(e) \}$.

$$\left( \int_\mathcal{H} \|f \ast \psi_h\|_p^q \det(h)^{-sr/n} dh \right)^{1/q} \leq \left( \sum_j \int_{h_j U} \|f \ast \psi_h\|_p^q \det(h)^{-sr/n} dh \right)^{1/q} \leq C \left( \sum_j \int_{h_j U} \|f \ast \psi_h\|_p^q \det(h_j)^{-sr/n} dh \right)^{1/q}.$$ 

In the last inequality we have used that, if $h \in h_j U$ then $\det(h) \sim \det(h_j)$ uniformly in $j$. This follows since for $h \in h_j U$, $\det(h) = \det(h_j) \det(u)$ for some $u \in U$, and since $U$ is bounded (compact) there is a $\gamma$ such that $1/\gamma \leq \det(u) \leq \gamma$ uniformly in
j. For $h \in h_j U$ all the functions $\widehat{\psi}_{h^{-1} h}$ have compact support contained in a larger compact set. Therefore there is an finite set $I$ such that

$$
\widehat{\psi}_{h^{-1} h} = \sum_{i \in I} \widehat{\psi}_i.
$$

Then, for $h \in h_j U$ we get

$$
\widehat{\psi}_h = \sum_{i \in I} (\widehat{\psi}_i)_{h_j},
$$

and, since $\psi_h \in L^1(V)$,

$$
\|f \ast \psi_h\|_p \leq \sum_{i \in I} \|f \ast (\psi_i)_{h_j}\|_p.
$$

So we get

$$
\left( \int_{\mathcal{H}} \|f \ast \psi_h\|_p^q \det(h)^{-sr/n} \, dh \right)^{1/q} \leq C \sum_{i \in I} \left( \sum_j \|f \ast (\psi_i)_{h_j}\|_p^q \det(h)^{-sr/n} \right)^{1/q} \leq C \|f\|_{\dot{B}^s_{p,q}}.
$$

In the last inequality we used that each of the collections \{$(\psi_i)_{h_j}$\} partitions the frequency plane, and the expression can thus be estimated by a Besov norm (see Lemma 3.8 in [2]).

The opposite inequality can be obtained in a similar fashion.

\[\square\]

4. A Wavelet Characterization of Besov spaces on symmetric cones

We will now show that the Besov spaces can be characterized as coorbits for the group $G = H \ltimes V$, with isomorphism given by the mapping $f \mapsto \tilde{f}$ from $\mathcal{S}'_\Omega$ to $\mathcal{S}'_\Omega$ defined via $\langle \tilde{f}, \phi \rangle = f(\overline{\phi})$. Notice that convolution $f \ast \phi$ can be expressed via the conjugate linear dual pairing as

$$
f \ast \phi(x) = \langle \tilde{f}, \tau_x \phi^* \rangle.
$$

4.1. Wavelets and coorbits on symmetric cones. The group of interest to us is the semidirect product $G = H \ltimes V$ with group composition

$$(h, x)(h_1, x_1) = (hh_1, hx_1 + x).$$

Here $H = AN$ is the connected solvable subgroup of the connected component of the automorphism group on $\Omega \subseteq V$. If $dh$ denotes the left Haar measure on $H$ and $dx$ the Lebesgue measure on $V$, then the left Haar measure on $G$ is given by $\frac{dx \, dh}{\det(h)}$.

The quasi regular representation of this group on $L^2(V)$ is given by

$$
\pi(h, x)f(t) = \frac{1}{\sqrt{\det(h)}} f(h^{-1}(t - x)),
$$

and it is irreducible and square integrable on $L^2_\Omega = \{ f \in L^2(V) \mid \text{supp}(\hat{f}) \subseteq \Omega \}$ (see [11, 7]). In frequency domain the representation becomes

$$
\hat{\pi}(h, x)\hat{f}(w) = \sqrt{\det(h)} e^{-i(x, w)} \hat{f}(h^* w).
$$

By $\pi$ we will also denote the restriction of $\pi$ to $\mathcal{S}'_\Omega$. 

Taking one partial derivative we see that
\[ \rho(h, x) f(t) = \sqrt{\text{Det}(h^*) f(h^*(t - x))} \]
and not the representation \( \pi \). However, Lemma \ref{lemma1} allows us to make a change of variable \( h \mapsto (h^*)^{-1} \) in order to relate the norm equivalence to \( \pi \).

**Remark 4.1.** The norm equivalence we have shown in Theorem \ref{thm1} is related to the unitary representation
\[ \rho(h, x) f(t) = \sqrt{\text{Det}(h^*) f(h^*(t - x))} \]
and the representation \( \pi \). However, Lemma \ref{lemma1} allows us to make a change of variable \( h \mapsto (h^*)^{-1} \) in order to relate the norm equivalence to \( \pi \).

**Lemma 4.2.** The representation \( \pi \) of \( G \) on \( S_\Omega \) is continuous, and if \( \psi \) is the function from from Theorem \ref{thm1}, then the function \( \phi = \psi^* \) is a cyclic vector for \( \pi \).

**Proof.** The Fourier transform ensures that this is equivalent to showing that \( \hat{\pi} \) is a continuous representation. The determinant is continuous, so we will investigate the \( L^\infty \)-normalized representation instead. For \( f \in S(V) \) with support in \( \Omega \) define
\[ f_{h, x}(w) = f(h^*w)e^{-i(x, w)}, \]
for \( h \in H \) and \( x \in V \). Since \( h^*w \in \Omega \) if \( w \in \Omega \) we see that \( f_{h, x} \) is a Schwartz function supported in \( \Omega \), so \( S_\Omega \) is \( \pi \)-invariant.

We now check that \( f_{h, x} \to f \) in the Schwartz semi-norms as \( h \to I \) and \( x \to 0 \). Taking one partial derivative we see that
\[ \frac{\partial f_{h, x}}{\partial w_k}(w) - \frac{\partial f}{\partial w_k}(w) = \sum_l h_{lk} \frac{\partial f}{\partial w_l}(h^*w)e^{-i(x, w)} - i w_k f(h^*w)e^{-i(x, w)} - \frac{\partial f}{\partial w_k}(w) = (h_{kk} - 1) \frac{\partial f}{\partial w_k}(h^*w)e^{-i(x, w)} + \sum_{l \neq k} h_{lk} \frac{\partial f}{\partial w_l}(h^*w)e^{-i(x, w)} - i w_k f(h^*w)e^{-i(x, w)} + \frac{\partial f}{\partial w_k}(h^*w)e^{-i(x, w)} - \frac{\partial f}{\partial w_k}(w) = \sum_{|\beta| \leq |\alpha|} c_\beta(h, x) \partial^\beta f(h^*w)e^{-i(x, w)} + (\partial^\alpha f(h^*w)e^{-i(x, w)} - \partial^\alpha f(w)), \]
where \( \alpha = e_k \) and \( \beta = (h, x) \to 0 \) as \( (h, x) \to (I, 0) \). By repeating the argument we get
\[ \partial^\alpha f_{h, x}(w) - \partial^\alpha f(w) = \sum_{|\beta| \leq |\alpha|} c_\beta(h, x) \partial^\beta f(h^*w)e^{-i(x, w)} + (\partial^\alpha f(h^*w)e^{-i(x, w)} - \partial^\alpha f(w)), \]
where \( c_\beta(h, x) \to 0 \) as \( (h, x) \to (I, 0) \). Using the fact that \( |w| = |(h^*)^{-1}h^*w| \leq ||(h^*)^{-1}||h^*w|| \leq C_N(h) \), where \( C_N(h) \) depends continuously on \( h \). For \( |\alpha| \leq N \) we thus get
\[ (1 + |w|)^N |\partial^\alpha f_{h, x}(w) - \partial^\alpha f(w)| \]
\[ \leq C_N(h) \sum_{|\beta| \leq |\alpha|} c_\beta(h, x)(1 + |h^*w|)^N |\partial^\beta f(h^*w)| + (1 + |w|)^N |\partial^\alpha f(h^*w)e^{-i(x, w)} - \partial^\alpha f(w)| \]
\[ \leq C_N(h) \sum_{|\beta| \leq |\alpha|} c_\beta(h, x)||f||_N + (1 + |w|)^N |\partial^\alpha f(h^*w)e^{-i(x, w)} - \partial^\alpha f(w)|, \]
Since \( c_\beta(h, x) \) tend to 0 as \((h, x) \to (I, 0)\), we investigate the remaining term
\[
|\partial^\alpha f(h^* w) e^{-i(x, w)} - \partial^\alpha f(w)| \leq |\partial^\alpha f(h^* w) - \partial^\alpha f(w)| + |\partial^\alpha f(w) (e^{-i(x, w)} - 1)|
\]
First, let \( \gamma(t) = w + t(h^* w - w) \). For \(|\alpha| = N\) we get
\[
|\partial^\alpha f(h^* w) - \partial^\alpha f(w)|(1 + |w|^2)^N
\]
\[
\leq \int_0^1 |\nabla \partial^\alpha f(\gamma_{h, w}(t))| |\gamma_{h, w}'(t)|(1 + |w|^2)^N dt
\]
\[
\leq \|h^* - I\| \int_0^1 |\nabla \partial^\alpha f(\gamma_{h, w}(t))|(1 + |\gamma(t)|^2)^N+1 \left( \frac{1 + |w|^2}{1 + |\gamma(t)|^2} \right)^{N+1} dt
\]
\[
\leq C \|h^* - I\| \|f\|_{N+1},
\]
where the constant \( C \) is uniformly bounded in \( h \). Next let \( \gamma(t) = tx \), then
\[
(1 + |w|)^N |\partial^\alpha f(w)(e^{-i(x, w)} - 1)| \leq (1 + |w|)^N |\partial^\alpha f(w)| \int_0^1 \langle -iw \gamma'(t) e^{-it(x, w)} \rangle dt
\]
\[
\leq (1 + |w|)^N |\partial^\alpha f(w)| |w| \|x\|
\]
\[
\leq \|f\|_{N+1} |x|.
\]
This shows that the representation \( \pi \) is continuous on \( S_\Omega \).

To show cyclicity, assume that \( \tilde{f} \) is in \( S_{\Omega}^* \) and \( \langle \tilde{f}, \pi(a, x) \phi \rangle = 0 \). Notice that \( \langle \tilde{f}, \pi(a, x) \phi \rangle = f \ast \psi_{(h-1), (x)} \), where \( f \) is the tempered distribution in \( S_{\Omega}^* \) corresponding to \( \tilde{f} \). By the norm equivalence of Theorem 3.3 and Lemma 3.1, we see that \( f = 0 \) in all Besov spaces \( B^p_q \) and thus also in \( S_{\Omega}^* \) (see [2] Lemma 3.11 and 3.22 and note that \( S_{\Omega}^* \) is equipped with the weak* topology). This proves that \( \tilde{f} = 0 \) and \( \phi \) is cyclic. \( \square \)

For \( \psi \in S_\Omega \) define the wavelet transform of \( f \in S_{\Omega}^* \) by
\[
W_\psi(f)(h, x) = \langle f, \pi(h, x) \psi \rangle.
\]
Under certain assumptions on \( \psi \) we get a reproducing formula.

**Lemma 4.3.** If \( \psi \in S_\Omega \) is such that \( \hat{\psi} \) has compact support and
\[
\int_\mathcal{H} |\hat{\psi}(h^* e)|^2 dh = 1,
\]
then
\[
W_\psi(f) \ast W_\psi(\psi) = W_\psi(f)
\]
for all \( f \in S_{\Omega}^* \). Here the convolution is the group convolution on \( \mathcal{G} = \mathcal{H} \ltimes V \).

**Proof.** For \( \phi \) we denote by \( \phi^h \) the function defined by
\[
\hat{\phi^h}(w) = \hat{\phi}(h^* w).
\]
Then \( \phi_1^h \ast \phi_2^h = |\det(h)|(\phi_1 \ast \phi_2)^h, \) and
\[
W_\psi(f) \ast W_\psi(\psi)(h_1, x_1) = \frac{1}{\sqrt{|\det(h_1)|}} \int_\mathcal{H} \langle f, \tau_{x_1} \psi^{h_1} \ast (\psi^*)^h \ast \psi^h \rangle dh \frac{dh}{|\det(h)|^2}
\]
\[
= \frac{1}{\sqrt{|\det(h_1)|}} \int_\mathcal{H} \langle f, \tau_{x_1} \psi^{h_1} \ast (\psi^* \ast \psi)^h \rangle dh \frac{1}{|\det(h)|}.
\]
The function inside the last integral is continuous, so it is enough to show that for \( \phi \in \mathcal{S}_\Omega \) the net
\[
g_C(x) = \int_C \phi * (\psi^* \ast \psi)^h \frac{dh}{|\det(h)|},
\]
converges to \( \phi \) in \( \mathcal{S}_\Omega \) for growing compact sets \( C \to \mathcal{H} \). By the assumption on \( \psi \) we get that \( \hat{g}_C \to \hat{\phi} \) pointwise. Thus we only need to show that \( g_C \) converges, which will happen if the integral
\[
\int_{\mathcal{H}} \sup_x (1 + |x|^2)^N |\partial^\alpha \phi * (\psi^* \ast \psi)^h(x)| \frac{dh}{|\det(h)|} < \infty
\]
is finite for all \( N \) and \( \alpha \). Since both \( \partial^\alpha \phi \) and \( \psi^* \ast \psi \) are in \( \mathcal{S}_\Omega \), we need only focus on showing that
\[
\int_{\mathcal{H}} \sup_x (1 + |x|^2)^N |\phi_1 * \phi_2^h(x)| \frac{dh}{|\det(h)|} < \infty
\]
for all \( N \) and any \( \phi_1, \phi_2 \in \mathcal{S}_\Omega \). We can further assume that \( \hat{\phi}_2 \) has compact support. Note that the Parseval identity, integration by parts and the fact that \( \hat{\phi}_1, \hat{\phi}_2 \) vanish on the boundary of \( \Omega \), give
\[
|\phi_1 * \phi_2^h(x)| = \left| \int_\Omega \hat{\phi}_1(w)\hat{\phi}_2(h^*w)e^{i(x,w)} dw \right| \\
\leq \frac{1}{|x^*|} \int_\Omega \sum_{|\beta| \leq |\alpha|} |p_\beta(h^*)||\partial^\beta \hat{\phi}_1(w)||\partial^{\alpha - \beta} \hat{\phi}_2(h^*w)| dw.
\]
Here \( p_\beta(h^*) \) is a polynomial in the entries of \( h^* \). Choosing \( |\alpha| \) large enough takes care of the terms \( (1 + |x|^2)^N \) for large \( |x| \) (and for small \( |x| \) we use \( \alpha = 0 \)), so
\[
\sup_x |\phi_1 * \phi_2^h(x)|(1 + |x|^2)^N \leq \sum_{|\beta| \leq |\alpha|} |p_\beta(h^*)| \int_{\Omega} |\partial^\beta \hat{\phi}_1(w)||\partial^{\alpha - \beta} \hat{\phi}_2(h^*w)| dw.
\]
Each partial derivative is again in \( \mathcal{S}(V) \) and with support in \( \Omega \), so we investigate terms of the general form \( \int |\hat{\phi}_1(w)\hat{\phi}_2(h^*w)| dw \). Denote by \( h_w \) the unique element in \( \mathcal{H} \) for which \( w = h_w e \), then
\[
\int_{\mathcal{H}} \int_{\Omega} p(||h||)||\hat{\phi}_1(w)||\hat{\phi}_2(h^*w)| dw |dh = \int_{\mathcal{H}} \int_{\Omega} p(||(h_w^*h)^{-1}h)||\hat{\phi}_1(w)||\hat{\phi}_2(h^*e)| dw |dh.
\]
Now \( \hat{\phi}_2 \) is assumed to have compact support and thus the integral over \( \mathcal{H} \) is finite, so we get
\[
\leq C \int_{\mathcal{H}} p(1/||h_w||)|\hat{\phi}_1(w)| dw.
\]
This will be finite, because the estimate \( |\hat{\phi}_1(w)| \leq C \frac{\Delta(w)^l}{(1 + |w|)^l} \). When \( ||h_w|| \sim |w| \) is close to zero we use \( l \) sufficiently large and for large \( ||h_w|| \sim |w| \) the integral is finite for \( k \) large enough. This finishes the proof. \( \square \)
For \(1 \leq p, q < \infty\) and \(s \in \mathbb{R}\) define the mixed norm Banach space \(L^p_{\ast, q}(\mathcal{G})\) on the group \(\mathcal{G}\) to be the measurable functions for which

\[
\|F\|_{L^p_{\ast, q}} := \left( \int_{\mathcal{H}} \left( \int_{V} |F(h, x)|^p \, dx \right)^{q/p} |\text{Det}(h)|^s \, dh \right)^{1/q} < \infty.
\]

**Lemma 4.4.** For \(\psi, \phi \in \mathcal{S}_{\Omega}\) the wavelet transform \(W_{\psi}(\phi)\) is in \(L^p_{\ast, q}(\mathcal{G})\) for \(1 \leq p, q < \infty\) and any real \(s\).

**Proof.** Since \(W_{\psi}(\phi)(h, x) = \phi * \psi^*_{{\phi}(h^{-1})}(x)\), this follows from the norm equivalence of Theorem 3.3 coupled with Lemma 3.1, as well as the fact that a function in \(\mathcal{S}_{\Omega}\) is in any Besov space (see Proposition 3.9 in [2]). \(\square\)

This verifies that the requirements of Theorem 2.1 are satisfied. It also shows that the representation involved has integrable matrix coefficients, which is the basis for the investigation in [10]. We thus complete our wavelet characterization of the Besov spaces by the following result. Remember that \(\tilde{f} \in \mathcal{S}_{\Omega}\) corresponds to \(f \in \mathcal{S}_{\Omega}\) via \((\tilde{f}, \phi) = (f, \phi)\).

**Theorem 4.5.** Given \(1 \leq p, q < \infty\) and \(s \in \mathbb{R}\) let \(s' = sr/n - q/2\). If \(\phi\) is the cyclic vector from Lemma 4.2 normalized to also satisfy Lemma 3.1 then the mapping \(f \mapsto \tilde{f}\) (restricted to \(B^p_{\ast, q}\)) is a Banach space isomorphism from the Besov space \(B^p_{\ast, q}\) to the coorbit \(C_{\mathcal{S}_{\Omega}}(L^p_{\ast, q}(\mathcal{G}))\) for the representation \(\pi\).

**Proof.** We will use Theorem 3.3 to determine \(s'\). Let \(\phi = cv^s\), and notice that

\[
\langle \tilde{f}, \pi(h, x)\phi \rangle = c\sqrt{|\text{Det}(h)|}f * \psi^*_{{\phi}(h^{-1})}(x).
\]

Then by Lemma 3.1 and Theorem 3.3 we get that

\[
\|W_{\psi}(\tilde{f})\|_{L^p_{\ast, q}} = C \left( \int_{\mathcal{H}} \left\|f * \psi_h\right\|_p^q |\text{Det}(h)|^{-q/2 - s'} \, dh \right)^{1/q},
\]

which is equivalent to \(\|f\|_{B^p_{\ast, s}}\) if \(-q/2 - s' = -sr/n\). \(\square\)

### 4.2. Atomic decompositions.

In order to obtain atomic decompositions and frames from Theorems 2.2 and 2.3 we need to show that \(\mathcal{S}_{\Omega}\) are smooth vectors for \(\pi\). A vector \(\psi \in \mathcal{S}_{\Omega}\) is called smooth if \(g \mapsto \pi(g)\psi\) is smooth \(\mathcal{G} \rightarrow \mathcal{S}_{\Omega}\). For smooth vectors \(\psi\) define a representation of \(\mathfrak{g}\) by

\[
\pi^\infty(X)\psi = \frac{d}{dt}
\]

at \(t = 0\) \(\pi(\text{exp}(tX))\psi\).

This also induces a representation of the universal enveloping algebra \(U(\mathfrak{g})\) which we also denote \(\pi^\infty\).

**Theorem 4.6.** The space \(\mathcal{S}_{\Omega}\) is the space of smooth vectors for the representation \((\pi, \mathcal{S}_{\Omega})\), and \((\pi^\infty, \mathcal{S}_{\Omega})\) is a representation of both \(\mathfrak{g}\) and \(U(\mathfrak{g})\).

**Proof.** Again, the determinant does not change the smoothness of vectors so we work with the \(L^\infty\)-normalized representation. Let \(\gamma(t) = (h(t), x(t))\) be a smooth curve in \(\mathcal{G}\) with \(\gamma(0) = (I, 0)\) and \(\gamma'(t) = (H, X)\), then the pointwise derivative (on the frequency side) of functions \(f_{h,x}(w) = f(h^*w)e^{-i(x,w)}\) is

\[
\frac{d}{dt}
\]

at \(t = 0\) \(f(h(t))^*we^{-i(x(t), w)} = (H^*w) \cdot \nabla f(w) - iX \cdot w f(w)\).
This is another Schwartz function and we will show it is also the limit of the derivative in $S_\Omega$.

\[
\frac{1}{t}(f_{h(t),x(t)}(w) - f(w)) - (H^*w) \cdot \nabla f(w) + iX \cdot w f(w),
\]

\[
= \frac{1}{t} \int_0^t (h'(s)^*w) \cdot \nabla f(h(s)^*w) e^{-i(x(s),w)} - (H^*w) \cdot \nabla f(w) ds
\]

\[
+ \frac{1}{t} \int_0^T iX \cdot w f(w) - ix'(t) \cdot w f(h(s)^*w) ds
\]

From the proof of Lemma 4.2 it is evident that each term inside the integral approaches 0 in the Schwartz topology, and the proof is complete. □

This result proves that any vector in $S_\Omega$ is both $\pi$-weakly and $\pi^*$-weakly differentiable of all orders, and this ensures that $W_{\pi(D^*\psi)}(\psi)$ and $W_\psi(\pi(D^*\psi))$ are in $L^s_\psi(G)$ for all $s$ by Lemma 4.3. Thus the continuousities required by Theorems 2.2 and 2.3 are satisfied and we conclude with the promised atomic decompositions.

**Corollary 4.7.** Let $s \in \mathbb{R}$ and $1 \leq p,q < \infty$ be given. There exists an index set $I$, and a well-spread sequence of points $\{(h_i,x_i)\}_{i \in I} \subseteq G$, such that the collection $\pi(h_i,x_i)\psi$ forms both a Banach frame and an atomic decomposition for $B^{p,q}_s$ with sequence space $(L^p_\psi(G))^\#$ when $s' = sr/n - q/2$.

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