Gravitational Quantum Cohomology

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**Dedicated to the Memory of Claude Itzykson**

**Abstract**

We discuss how the theory of quantum cohomology may be generalized to “gravitational quantum cohomology” by studying topological $\sigma$-models coupled to two-dimensional gravity. We first consider $\sigma$-models defined on a general Fano manifold $M$ (manifold with a positive first Chern class) and derive new recursion relations for its two point functions. We then derive bi-Hamiltonian structures of the theories and show that they are completely integrable at least at the level of genus 0. We next consider the subspace of the phase space where only a marginal perturbation (with a parameter $t$) is turned on and construct Lax operators (superpotentials) $L$ whose residue integrals reproduce correlation functions. In the case of $M = \mathbb{C}P^N$ the Lax operator is given by $L = Z_1 + Z_2 + \cdots + Z_N + e^t Z_1^{-1} Z_2^{-1} \cdots Z_N^{-1}$ and agrees with the potential of the affine Toda theory of the $A_N$ type. We also obtain Lax operators for various Fano manifolds; Grassmannians, rational surfaces etc. In these examples the number of variables of the Lax operators is the same as the dimension of the original manifold. Our result shows that Fano manifolds exhibit a new type of mirror phenomenon where mirror partner is a non-compact Calabi-Yau manifold of the type of an algebraic torus $\mathbb{C}^*^N$ equipped with a specific superpotential.
1. Introduction

The theory of quantum cohomology introduced in [1, 2] describes how the quantum effects due to instantons modify the classical cohomology of a given manifold \( M \). Relations of the quantum cohomology ring are derived from the study of correlation functions of a topological sigma model on the sphere with \( M \) being its target space. The quantum cohomology has been an important arena for the study of mirror symmetry of Calabi-Yau (CY) manifolds where the A-twisted superconformal field theory on one CY is equivalent with the B-twisted theory on another [3]. When the target space has a positive first Chern class (the case of a Fano manifold), the underlying sigma model is asymptotically free and has a mass gap, and one of the two \( U(1)_R \) symmetries is anomalously broken. Accordingly, only A-twisting is possible and the resulting topological theory has an intrinsically broken scale invariance. Nevertheless, the quantum cohomology of a Fano manifold can also be described [1, 2, 4, 5, 6] by a B-twisted \( N = 2 \) supersymmetric field theory (topological Landau-Ginzburg (LG) model [7]). These phenomena, however, will become far more interesting if the topological sigma model is coupled with topological gravity [1], higher genus contributions are included, and the gravitational descendants are also incorporated. Recall that the gravitational descendants (Mumford-Morita classes) played a prominent role in the theory of two dimensional gravity [1, 8, 9]. Recently, there have been extensive studies on topological sigma models coupled with gravity (or Gromov-Witten invariants) [10, 11, 12, 13, 14, 15] and the rigorous definition is accomplished in [16, 17], but the structure of the theory has not yet been well-understood.

In ref. [18, 19, 20] we have considered the simplest Fano manifold \( \mathbb{C}P^1 \) and used the standard method of topological field theory [1, 8, 21, 22] to analyze the integrable structure of the sigma model coupled with gravity. We have constructed a matrix model which reproduces the sum over all instantons from Riemann surfaces of arbitrary genus onto \( \mathbb{C}P^1 \). The action of the matrix model contains a logarithmic potential which reflects the broken scale invariance of the theory. In ref. [19] it was shown that a Landau-Ginzburg description of the \( \mathbb{C}P^1 \) model is given by a superpotential of the form \( \exp x + \exp -x \) and thus the \( \mathbb{C}P^1 \) model is identified as the \( N = 2 \) sine-Gordon theory.

In this article, at the level of genus 0, we generalize our construction for a wide class of Fano varieties including \( \mathbb{C}P^N \), Grassmannians, Del Pezzo surfaces and the products of such spaces. Based on these constructions we would like to propose a theory of “gravitational quantum cohomology” which is a gravitational version of quantum cohomology theory where the sigma models are coupled with gravity and the gravitational descendants are incorporated. We shall show that gravitational quantum cohomology can be described by a topological LG model based on a non-compact CY manifold having the same dimension as the original Fano manifold \((\mathbb{C}^*)^N \) in the case of \( \mathbb{C}P^N \). Namely, we can represent the correlation functions of the original model by residue integrals of a superpotential on the CY. This indicates the existence of a mirror phenomenon for the case of Fano varieties as well as in the case of Calabi-Yau manifolds.

Contents of this paper are as follows: in section 2 we derive fundamental recursion relations which relate two-point functions \( \langle \sigma_n(O_\alpha)O_\beta \rangle \) to \( \langle \sigma_{n-1}(O_\alpha)O_\gamma \rangle \) using the machinery
of topological field theory \[1, 21, 22\]. Here \(O_\alpha\)'s are the primary fields and \(\sigma_n(O_\alpha)\) is the \(n\)-th gravitational descendant of \(O_\alpha\). These relations hold for arbitrary Fano manifolds and at general values of the coupling constants (in the large phase space). In section 3 we derive the bi-Hamiltonian structure of the topological sigma models making use of these recursions relations. The existence of the bi-Hamiltonian structure shows explicitly the complete integrability of the sigma models on general Fano varieties.

In section 4 we consider the case of the projective spaces \(M = \mathbb{CP}^N\) and discuss the \(\mathbb{CP}^2\) case in detail. We in particular analyze the structure of the theory at a point in the phase space where all the coupling constants vanish except the one \(t\) coupled to the Kähler class (marginal operator). This is the point where the topological sigma models lead to quantum cohomology relations. In our case the recursion relations among two-point functions \(\langle \sigma_n(O_\alpha)O_\beta \rangle\) become simplified when the coupling constants vanish except \(t\) and the two-point functions can be represented by period integrals using a suitable superpotential \(L\) (Lax operator). In the case of \(\mathbb{CP}^N\) \(L\) is given by \(L = Z_1 + Z_2 + \cdots + Z_N + e^t Z_1^{-1}Z_2^{-1}\cdots Z_N^{-1}\). This is the form of the potential of the affine Toda field theory of \(A_N\) type and is a natural generalization of the sine-Gordon potential \(L = Z + e^t Z^{-1}\) of the \(\mathbb{CP}^1\) case. Unlike the case of the quantum cohomology described by one variable \(\mathbb{CP}^1\) we need \(N\) variables \(Z_i, i = 1, \cdots, N\) in order to describe the gravitational quantum cohomology of \(\mathbb{CP}^N\).

Representation of cohomologies by means of period integrals indicate an analogue of the mirror phenomenon. In the case of \(\mathbb{CP}^N\), for instance, the space \((\mathbb{C}^*)^N\) may be interpreted as the mirror manifold of \(\mathbb{CP}^N\). The measure of the period integral \(dZ_1dZ_2\cdots dZ_N/(Z_1Z_2\cdots Z_N)\) gives the analogue of the holomorphic \(N\)-form. In all the examples we have studied in this paper the superpotentials of gravitational cohomologies have the same number of variables as the original manifolds. Thus in the case of Fano varieties we have a mirror phenomenon where the A model coupled with gravity is equivalent with the B model on a non-compact CY manifold of the same dimension (algebraic torus for a toric variety) equipped with a specific superpotential.

Section 5 is devoted to the study of a product space where we obtain a sum of superpotentials for each space, and also some rational surfaces. In the Appendix A we present a proof of an important fact concerning the LG description of descendants and in the Appendix B the Lax operator of some Grassmann manifolds.

After the completion of this work we have noticed a preprint by Givental \[23\] where a mirror phenomenon of a certain class of Fano manifolds is presented.

Throughout this paper we use the following notations;

\(M\): a Kähler manifold with the first Chern class \(c_1(M)\),

\(\{O_\alpha\}\): the base of \(H^*(M; \mathbb{C})\) with \(\dim O_\alpha = 2q_\alpha\),

\(\eta_{\alpha\beta} = \int_M O_\alpha \wedge O_\beta\): intersection pairing or the topological metric.

We lower and raise the indices \(\alpha, \beta, \ldots\) using the metric \(\eta_{\alpha\beta}\) and its inverse \(\eta^{\alpha\beta}\).

2. Fundamental Recursion Relation
First, we summarize what is known about topological string amplitudes

\[ \langle \sigma_{n_1}(O_{\alpha_1}) \cdots \sigma_{n_s}(O_{\alpha_s}) \rangle_{g,d}, \]  

(2.1)

where \( g \) is the genus of surfaces and \( d \) is the degree of maps which is defined as the homology class \( d = f_\ast [\Sigma] \in H_2(M) \).

The general relations among correlation functions in the topological string theory are given as follows.

- Selection rule (ghost number conservation): Non-vanishing of (2.1) requires
  
  \[ c_1(M) \cdot d + (\dim M - 3)(1 - g) = \sum_{i=1}^{s} (n_i + q_{\alpha_i} - 1) \]  

(2.2)

Here, \( c_1(M) \cdot d \in \mathbb{Z} \) is the pairing of \( c_1(M) \in H^2(M) \) and \( d \in H_2(M) \):

\( c_1(M) \cdot d = \int_{f_\ast (\Sigma)} c_1(M) = \int_{\Sigma} f_\ast (c_1(M)). \)

- Puncture equation \cite{21}:
  
  \[ \langle P \sigma_{n_1}(O_1) \cdots \sigma_{n_s}(O_s) \rangle_{g,d} = \sum_{i=1}^{s} n_i \langle \sigma_{n_i-1}(O_i) \prod_{j \neq i} \sigma_{n_j}(O_j) \rangle_{g,d} \]  

(2.3)

- Dilaton equation \cite{8}:
  
  \[ \langle \sigma_1(P) \sigma_{n_1}(O_1) \cdots \sigma_{n_s}(O_s) \rangle_{g,d} = (2g - 2 + s) \langle \sigma_{n_1}(O_1) \cdots \sigma_{n_s}(O_s) \rangle_{g,d} \]  

(2.4)

- Equation associated with \( \omega \in H^2(M; \mathbb{C}) \) \cite{22}:
  
  \[ \langle \sigma_0(\omega) \sigma_{n_1}(O_1) \cdots \sigma_{n_s}(O_s) \rangle_{g,d} \]
  
  \[ = \omega \cdot d \langle \sigma_{n_1}(O_1) \cdots \sigma_{n_s}(O_s) \rangle_{g,d} + \sum_{i=1}^{s} n_i \langle \sigma_{n_i-1}(\omega \wedge O_i) \prod_{j \neq i} \sigma_{n_j}(O_j) \rangle_{g,d} \]  

(2.5)

- Topological recursion relation (TRR) \cite{1}:
  
  This is a relation for \( g = 0 \). We denote the sum over the degrees \( d \) of \( g = 0 \) amplitudes by \( \langle \cdots \rangle = \sum_d \langle \cdots \rangle_{0,d} \).

  \[ \langle \sigma_n(O)_{XY} \rangle = n \langle \sigma_{n-1}(O)_{O\alpha} \rangle \eta^{\alpha\beta} \langle O_{\beta XY} \rangle. \]  

(2.6)

Here, \( X \) and \( Y \) are arbitrary observables. This holds in the large phase space.

In the following we consider only the tree \( (g = 0) \) amplitudes. In the case of minimal models, TRR is powerful enough to express correlators of descendants in terms of correlators of primaries because 0,1, and 2-point functions vanish at the origin of the phase space. In the case of topological \( \sigma \) models, due to the instanton corrections, we have non-vanishing 0,1, and 2-point functions to which TRR cannot be applied. However, if we
use the selection rule (2.2) and the equation (2.3) for the first Chern class \( c_1(M) \), we can convert an \( i \)-point function to \( i + 1 \)-point function. For example, let \( \Omega_M \in H^{2 \dim M}(M) \) be the volume form of \( M \) (whose integral is normalized to 1), and consider the two point function \( \langle \sigma_n(\Omega_M)O_\alpha \rangle \) at the origin of the phase space. Then, we have from (2.3)

\[
\langle \sigma_n(\Omega_M)c_1(M)O_\alpha \rangle = \sum_d \langle \sigma_n(\Omega_M)c_1(M)O_\alpha \rangle_{0,d} = \sum_d c_1(M) \cdot d \langle \sigma_n(\Omega_M)O_\alpha \rangle_{0,d}.
\]  

(2.7)

The selection rule (2.2) says that \( \langle \sigma_n(\Omega_M)O_\alpha \rangle_{0,d} \) is non-vanishing only when \( c_1(M) \cdot d = n + q_\alpha + 1 \). Therefore, (2.7) coincides with \( (n + q_\alpha + 1)\langle \sigma_n(\Omega_M)O_\alpha \rangle \). Namely, the two point function \( \langle \sigma_n(\Omega_M)O_\alpha \rangle \) is proportional to the three point function \( \langle \sigma_n(\Omega_M)c_1(M)O_\alpha \rangle \). If we apply TRR to the latter, we obtain the following relation

\[
\langle \sigma_n(\Omega_M)O_\alpha \rangle = \frac{n}{n + q_\alpha + 1} \langle c_1(M)O_\alpha O_\beta \rangle \langle \sigma_{n-1}(\Omega_M)O_\beta \rangle.
\]  

(2.8)

For a general observable \( \sigma_n(O_\beta) \), similar relation holds but there is a contribution coming from the contact term in (2.3):

\[
\langle \sigma_n(O_\beta)O_\alpha \rangle = \frac{n}{n + \tilde{q}_\alpha + \tilde{q}_\beta} \left( \langle c_1(M)O_\alpha O_\gamma \rangle \langle \sigma_{n-1}(O_\beta)O_\gamma \rangle - \langle \sigma_{n-1}(c_1(M) \wedge O_\beta)O_\alpha \rangle \right),
\]  

(2.9)

where

\[
\tilde{q}_\alpha := q_\alpha + 1 - \frac{\dim M}{2}.
\]

With more care, these formulae can be generalized to the following recursion relations that hold in the large phase space. Let us introduce a matrix \( M_{\alpha\beta} \) by

\[
M_{\alpha\beta} := (q_\alpha + q_\beta + 1 - \dim M) \langle O_\alpha O_\beta \rangle + \int_M c_1(M) \wedge O_\alpha \wedge O_\beta.
\]  

(2.10)

Then, we have for \( n, m \geq 1 \) the

**Fundamental Recursion Relations**

\[
\langle \sigma_n(O_\alpha)O_\beta \rangle = \frac{n}{n + \tilde{q}_\alpha + \tilde{q}_\beta} \left( M_\beta \gamma \langle \sigma_{n-1}(O_\alpha)O_\gamma \rangle - \langle \sigma_{n-1}(c_1(M) \wedge O_\alpha)O_\beta \rangle \right),
\]  

(2.11)

\[
\langle \sigma_n(O_\alpha)\sigma_m(O_\beta) \rangle = \frac{1}{n + m + \tilde{q}_\alpha + \tilde{q}_\beta} \left( nm M^{\rho\sigma} \langle \sigma_{n-1}(O_\alpha)O_\rho \rangle \langle \sigma_{m-1}(O_\beta)O_\sigma \rangle - n \langle \sigma_{n-1}(c_1(M) \wedge O_\alpha)\sigma_m(O_\beta) \rangle - m \langle \sigma_n(O_\alpha)\sigma_{m-1}(c_1(M) \wedge O_\beta) \rangle \right).
\]  

(2.12)

In particular,

\[
\langle \sigma_n(\Omega_M)O_\alpha \rangle = \frac{n}{n + q_\alpha + 1} M_\alpha \beta \langle \sigma_{n-1}(\Omega_M)O_\beta \rangle.
\]  

(2.13)
Note that these are powerful relations among correlation functions which relate two point functions of descendants to those of primaries. As compared with the usual case where one inserts $P$, uses TRR and then integrates with respect to $x = t_0^\rho$, the new relations are purely algebraic and are easier to handle.

**proof.** The equation (2.5) for $\omega = c_1(M)$ together with the selection rule (2.2) yields

$$
\langle c_1(M)\sigma_{n_1}(O_1)\cdots\sigma_{n_s}(O_s)\rangle_{g,d} = (3 - \dim M)(1 - g)\langle \sigma_{n_1}(O_1)\cdots\sigma_{n_s}(O_s)\rangle_{g,d}
+ \sum_{i=1}^s \left\{ (n_i + q_i - 1)\langle \sigma_{n_1}(O_1)\cdots\sigma_{n_s}(O_s)\rangle_{g,d} + n_i\langle \sigma_{n_i-1}(c_1(M)\land O_i)\prod_{j\neq i}\sigma_{n_j}(O_j)\rangle_{g,d} \right\},
$$

which is equivalent at $g = 0$ to the following equation for the free energy

$$
\langle c_1(M) \rangle = (3 - \dim M)F_0 + \sum_{m,\sigma} \left\{ (m + q_\sigma - 1)t^{\sigma}_m \frac{\partial}{\partial t^{\sigma}_m} + mt^{\sigma}_m c_1(M)^\rho \frac{\partial}{\partial t^{\rho}_{m-1}} \right\} F_0
+ \frac{1}{2} t^{\sigma}_0 t^{\rho}_0 \int_M c_1(M) \land O_\sigma \land O_\rho.
$$

(2.16)

The last term comes from the degree zero contribution to three point functions to which the formula (2.5) cannot be applied. Let us introduce the perturbed first Chern class

$$
C_1(M) := c_1(M) - \sum_{m,\sigma} (m + q_\sigma - 1)t^{\sigma}_m \sigma_m(O_\sigma) - \sum_{m,\sigma} mt^{\sigma}_m \sigma_{m-1}(c_1(M) \land O_\sigma).
$$

(2.17)

Then, the above equation is neatly expressed as

$$
\langle C_1(M) \rangle = (3 - \dim M)F_0 + \frac{1}{2} t^{\sigma}_0 t^{\rho}_0 \int_M c_1(M) \land O_\sigma \land O_\rho.
$$

(2.18)

Taking the derivative of this equation with respect to $t^{\sigma}_n$ and $t^{\beta}_0$, we get

$$
\langle C_1(M)\sigma_n(O_\alpha)O_\beta \rangle - (n + q_\alpha + q_\beta - 2)\langle \sigma_n(O_\alpha)O_\beta \rangle - n\langle \sigma_{n-1}(c_1(M) \land O_\alpha)O_\beta \rangle
= (3 - \dim M)\langle \sigma_n(O_\alpha)O_\beta \rangle + \delta_{n,0}\int_M c_1(M) \land O_\alpha \land O_\beta.
$$

(2.19)

For $n = 0$, this gives

$$
\langle C_1(M)O_\alpha O_\beta \rangle = M_{\alpha\beta}.
$$

(2.20)

For $n > 0$, we have

$$
\langle C_1(M)\sigma_n(O_\alpha)O_\beta \rangle = (n + q_\alpha + q_\beta + 1 - \dim M)\langle \sigma_n(O_\alpha)O_\beta \rangle + n\langle \sigma_{n-1}(c_1(M) \land O_\alpha)O_\beta \rangle,
$$

(2.21)

while it follows from the topological recursion relation that

$$
\langle C_1(M)\sigma_n(O_\alpha)O_\beta \rangle = n\langle \sigma_{n-1}(O_\alpha)O_\gamma \rangle \langle O_\gamma C_1(M)O_\beta \rangle.
$$

(2.22)

Combining the above three equations, we obtain the recursion formula (2.12). Proof of the other relation is similar.
Remark. With only the primary marginal perturbation, we have \( \mathcal{C}_1(M) = c_1(M) \), and therefore

\[ M_{\alpha\beta} = \langle c_1(M) O_\alpha O_\beta \rangle \quad \text{for } t_n^\gamma = 0 \text{ unless } n = 0 \text{ and } q_\gamma = 1. \quad (2.23) \]

Then, the relations (2.12) and (2.14) coincides with the ones (2.10) and (2.9) obtained previously.
3. Bi-Hamiltonian Structure

Every topological string theory can be considered, at least at the tree level, as an integrable system where the two point functions constitute the densities for commuting Hamiltonians. Here we study the structure of the integrable system using the fundamental recursion relation (2.12) among the Hamiltonian densities. In particular, we determine the bi-Hamiltonian structure of the topological $\sigma$ models.

3.1 The Bi-Hamiltonian Structure

We take $x = t_0^P$ as the basic (or "spatial") coordinate and regard other coupling constants as infinitely many "time" coordinates. The order parameters of the theory are defined by

$$u_\alpha := \langle PO_\alpha \rangle.$$  \hfill (3.1)

Other two point functions are regarded as functions of these order parameters ("constitutive relations" [21]). The particular two point functions with a puncture insertion $R_{n,\alpha} := \langle \sigma_n(O_\alpha)P \rangle$ generates flows in the phase space in the following sense:

First Hamiltonian Structure

With respect to the first Poisson bracket

$$\{u_\alpha(x), u_\beta(y)\}_1 = \eta_{\alpha\beta}\partial_x \delta(x - y)$$  \hfill (3.3)

$R_{n+1,\alpha}$ acts as the Hamiltonian for the evolution in the variable $t_n^\alpha$

$$\frac{\partial u_\beta}{\partial t_n^\alpha} = \frac{1}{n + 1} \left\{ u_\beta, \int R_{n+1,\alpha} dx \right\}_1$$ \hfill (3.4)

Second Hamiltonian Structure

With respect to the second Poisson bracket

$$\{u_\alpha(x), u_\beta(y)\}_2 = \left( M_{\alpha\beta} \partial_x + \tilde{q}_\beta \langle O_\alpha O_\beta' \rangle' \right) \delta(x - y).$$ \hfill (3.5)

$R_{n,\alpha}$ generates the evolution in the parameter $t_n^\alpha$

$$\frac{\partial u_\beta}{\partial t_n^{\Omega_M}} = \frac{1}{n + \frac{\dim M + 1}{2}} \left\{ u_\beta, \int R_{n,\Omega_M} dx \right\}_2,$$ \hfill (3.6)

$$\frac{\partial u_\beta}{\partial t_n^\alpha} + \frac{n}{n + \tilde{q}_\alpha} c_1(M) \gamma \frac{\partial u_\beta}{\partial t_{n-1}^\alpha} = \frac{1}{n + \tilde{q}_\alpha} \left\{ u_\beta, \int R_{n,\alpha} dx \right\}_2.$$ \hfill (3.7)

Below, we present a derivation and consistency check of these Hamiltonian structures.
The flow equation \((3.4)\) is a consequence of a lemma

**Lemma 1.**

\[
\eta_{\beta\gamma} \frac{\partial}{\partial u_\gamma} R_{n,\alpha} = n \langle \sigma_{n-1}(O_\alpha)O_\beta \rangle \tag{3.8}
\]

**proof.** From the constitutive relation it follows that

\[
\langle \sigma_n(O_\alpha)PO_\rho \rangle = \frac{\partial u_\rho}{\partial t_0} R_{n,\alpha} = \langle O_\rho P O_\beta \rangle \frac{\partial}{\partial u_\gamma} R_{n,\alpha}, \tag{3.9}
\]

while TRR gives

\[
\langle \sigma_n(O_\alpha)PO_\rho \rangle = n \langle \sigma_{n-1}(O_\alpha)O_\beta \rangle \langle O_\beta PO_\rho \rangle. \tag{3.10}
\]

Connecting the above two, and multiplying the inverse matrix of \(\langle O_\beta P O_\rho \rangle\), we see that the equation \((3.8)\) holds.

Let us introduce a convenient notation

\[
P_{\alpha\beta} := \langle PO_\alpha O_\beta \rangle. \tag{3.11}
\]

Then, we have

**Lemma 2.** The matrices \(M\) and \(P\) commute:

\[
P_\alpha \gamma M_\gamma \beta = M_\alpha \gamma P_\gamma \beta. \tag{3.12}
\]

**proof.** Since \(M_{\alpha\beta}\) can be written as a three point function \(\langle C_1(M)O_\alpha O_\beta \rangle\) (see \((2.20)\)), the lemma follows from the associativity relation \(\langle XYO^\gamma \rangle \langle O_\gamma ZW \rangle = \langle XZO^\gamma \rangle \langle O_\gamma YW \rangle\) which holds for arbitrary observables \(X, Y, Z, W\).

Now, we present a derivation of the second Poisson bracket \((3.5)\) and the flow equation \((3.6)\). From the fundamental recursion relation \((2.14)\), we have \((n+q_\gamma)\langle \sigma_{n-1}(\Omega M)O_\gamma \rangle = (n-1)M_\gamma \beta \langle \sigma_{n-2}(\Omega M)O_\beta \rangle\). Multiplying \(P_\alpha \gamma\) we get

\[
\sum_\gamma P_\alpha \gamma (n+q_\gamma)\langle \sigma_{n-1}(\Omega M)O_\gamma \rangle = (n-1)P_\alpha \gamma M_\gamma \beta \langle \sigma_{n-2}(\Omega M)O_\beta \rangle \tag{3.13}
\]

\[
= (n-1)M_\alpha \gamma P_\gamma \beta \langle \sigma_{n-2}(\Omega M)O_\beta \rangle \tag{3.14}
\]

\[
= M_\alpha \gamma \langle \sigma_{n-1}(\Omega M)O_\gamma P \rangle, \tag{3.15}
\]

where we used the commutativity \((3.12)\) in the second step and the TRR in the last step. Using the TRR in the left hand side, we find that

\[
\langle \sigma_n(\Omega M)O_\alpha P \rangle = \sum_\gamma \left(M_\alpha \gamma \partial_x - P_\alpha \gamma q_\gamma \right) \langle \sigma_{n-1}(\Omega M)O_\gamma \rangle \tag{3.16}
\]

Multiplying by a factor \(n + \xi\) (\(\xi\) is a constant to be determined later), we obtain

\[
(n + \xi)\langle \sigma_n(\Omega M)O_\alpha P \rangle = \sum_\gamma \left(M_\alpha \gamma \partial_x - P_\alpha \gamma q_\gamma \right) n \langle \sigma_{n-1}(\Omega M)O_\gamma \rangle + \xi n P_\alpha \gamma \langle \sigma_{n-1}(\Omega M)O_\gamma \rangle
\]

\[
= \sum_\gamma \left(M_\alpha \gamma \partial_x + P_\alpha \gamma (\xi - q_\gamma) \right) \eta_{\gamma\beta} \frac{\partial}{\partial u_\beta} R_{n,\Omega M}
\]

\[
= \sum_\beta \left(M_{\alpha\beta} \partial_x + P_{\alpha\beta}(\xi - \dim M + q_\beta) \right) \frac{\partial}{\partial u_\beta} R_{n,\Omega M}, \tag{3.17}
\]
where we used the TRR in the first step and the Lemma 1 in the second step. In the last step, we have used the fact that \( \eta_{\gamma \beta} \neq 0 \) only if \( q_\gamma + q_\beta = \dim M \). If we can define a Poisson bracket \( \{ \, , \} \) by

\[
\{ u_\alpha(x), u_\beta(y) \} = \left( M_{\alpha \beta} \partial_x + P_{\alpha \beta}(\xi - \dim M + q_\beta) \right) \delta(x - y),
\]

then the above equation will be expressed as

\[
\frac{\partial}{\partial t} \Omega^M_{\alpha n} u_\alpha = \frac{1}{n + \xi} \left\{ u_\alpha, \int R_n \Omega^M dx \right\}.
\] (3.19)

For (3.18) to be a Poisson bracket, it must be anti-symmetric and satisfy the Jacobi identity. Anti-symmetry requires

\[
P_{\alpha \beta}(\xi - \dim M + q_\beta) + P_{\beta \alpha}(\xi - \dim M + q_\alpha) = M'_{\alpha \beta} = (q_\alpha + q_\beta + 1 - \dim M) P_{\alpha \beta},
\] (3.20)

or

\[
\xi = \frac{\dim M + 1}{2}.
\] (3.21)

If we plug this into (3.18) and (3.19), we get (3.5) and (3.6).

Finally, let us check the Jacobi identity. For a test function \( a^\alpha(x) \), we put

\[
u[a] := \int u_\alpha a^\alpha dx.
\] (3.22)

Let \( a, b, \) and \( c \) be test functions.

\[
\{ u[a], \{ u[b], u[c] \} \} = \{ u[a], \int dx b^\beta \left( M_{\beta \gamma} c^{\gamma'} + \langle O_\beta O_\gamma \rangle' \tilde{q}_\gamma c^{\gamma'} \right) \}
\]

\[
= \{ u[a], \int dx \left( M_{\beta \gamma} b^\beta c^{\gamma'} - \langle O_\beta O_\gamma \rangle' \tilde{q}_\gamma (b^\beta c^{\gamma'})' \right) \}
\]

\[
= \int dx \left( -a^\alpha M_{\alpha \rho} - a^\alpha M'_{\alpha \rho} + a^\alpha \langle O_\alpha O_\rho \rangle' \tilde{q}_\rho \right) \left( \frac{\partial M_{\beta \gamma}}{\partial u_\rho} b^\beta c^{\gamma'} - \frac{\partial \langle O_\beta O_\gamma \rangle}{\partial u_\rho} \tilde{q}_\gamma (b^\beta c^{\gamma'})' \right).
\]

Here we note that \( M_{\alpha \beta} = (\tilde{q}_\alpha + \tilde{q}_\beta) \langle O_\alpha O_\beta \rangle + \text{const.} \), So,

\[
-a^\alpha M'_{\alpha \rho} + a^\alpha \langle O_\alpha O_\rho \rangle' \tilde{q}_\rho = -a^\alpha \tilde{q}_\alpha \langle O_\alpha O_\rho \rangle',
\] (3.23)

\[
\frac{\partial M_{\beta \gamma}}{\partial u_\rho} b^\beta c^{\gamma'} - \frac{\partial \langle O_\beta O_\gamma \rangle}{\partial u_\rho} \tilde{q}_\gamma (b^\beta c^{\gamma'})' = \frac{\partial \langle O_\beta O_\gamma \rangle}{\partial u_\rho} (\tilde{q}_\beta b^\beta c^{\gamma'} - \tilde{q}_\gamma b^\beta c^{\gamma'}).
\] (3.24)

Let us put

\[
M_{\alpha \beta \gamma} := M_{\alpha \rho} \frac{\partial \langle O_\beta O_\gamma \rangle}{\partial u_\rho},
\] (3.25)
and note that
\[
\langle O_\alpha O_\beta O_\gamma \rangle = \langle O_\alpha O_\rho \rangle \frac{\partial \langle O_\beta O_\gamma \rangle}{\partial u_\rho}.
\] (3.26)

Then, we have
\[
\{u[a], \{u[b], u[c]\}\} = -\int dx \left( M_{\alpha\beta\gamma}(\bar{q}_\alpha a^{\alpha'} b^{\beta'} c^{\gamma'} - \bar{q}_\gamma a^{\alpha'} b^{\beta'} c^{\gamma'}) + \langle O_\alpha O_\beta O_\gamma \rangle (\bar{q}_\alpha \bar{q}_\beta a^{\alpha'} b^{\beta'} c^{\gamma'}) \right).
\] (3.27)

Lemma 3. \(M_{\alpha\beta\gamma}\) is symmetric.

proof. In the small phase space, \(\frac{\partial \langle O_\beta O_\gamma \rangle}{\partial u_\rho} = \langle O_\rho O_\beta O_\gamma \rangle\), and hence
\[
M_{\alpha\beta\gamma} \frac{\partial \langle O_\beta O_\gamma \rangle}{\partial u_\rho} = \langle O_\alpha C_1(M) O_\rho \rangle \langle O_\rho O_\beta O_\gamma \rangle
\] = \(\langle O_\beta C_1(M) O_\rho \rangle \langle O_\rho O_\alpha O_\gamma \rangle = M_{\beta\rho} \frac{\partial \langle O_\alpha O_\gamma \rangle}{\partial u_\rho}\). (3.28)

Namely, \(M_{\alpha\beta\gamma} = M_{\beta\alpha\gamma}\). As this is a relation of two point functions, it holds also in the large phase space.

By looking at the expression (3.27) we note the symmetry of \(M_{\alpha\beta\gamma}\) and \(\langle O_\alpha O_\beta O_\gamma \rangle\), and find that the Jacobi identity
\[
\{u[a], \{u[b], u[c]\}\} + \{u[b], \{u[c], u[a]\}\} + \{u[c], \{u[a], u[b]\}\} = 0\] (3.29)
holds.

3.2 Examples

The CP\(^1\) Model

The CP\(^1\) model has two primaries \(P\) and \(Q\) corresponding to the identity and the Kähler form (normalized to have a unit volume). The metric is given by \(\eta_{PQ} = 1\) and \(\eta_{PP} = \eta_{QQ} = 0\). Order parameters are denoted as
\[
u = \langle PP \rangle, \quad v = \langle PQ \rangle.
\] (3.30)

The remaining primary two point function \(\langle QQ \rangle\) is expressed in terms of an order parameter by the constitutive relation
\[
\langle QQ \rangle = e^u.
\] (3.31)

The fundamental matrix \((M_\alpha^\beta) = (M_{\alpha\gamma})(\eta^{\gamma\beta})\) is then expressed as
\[
\begin{pmatrix}
v & 2 \\
2e^u & v
\end{pmatrix}
\] (3.32)
(with respect to the order $P,Q$). Using this matrix, we can recursively construct all the two point functions. Applying (2.14), we have for example

$$v_n := \left( \frac{\sigma_n(Q)P}{\sigma_n(Q)Q} \right): \quad v_0 = \left( \frac{v}{e^u} \right), \quad v_1 = \left( \frac{v^2 + e^u}{ve^u} \right), \quad v_2 = \left( \frac{v^3 + 2ve^u}{v^2e^u + e^{2u}} \right), \quad (3.33)$$

$$v_3 = \left( \frac{v^4 + 3ve^u + e^{2u}}{v^3e^u + 3ve^{2u}} \right), \quad v_4 = \left( \frac{v^5 + 4ve^u + 6ve^{2u}}{v^4e^u + 6ve^{2u} + 2e^{3u}} \right), \ldots$$

The second Poisson bracket is expressed as

$$(\{u(x),u(y)\} \{v(x),v(y)\} \{w(x),w(y)\} \{f(x),f(y)\} = \left( \begin{array}{cc} 2\partial_x & v\partial_x + v' \\ v\partial_x & 2e^u\partial_x + e^uu' \end{array} \right) \delta(x - y). (3.34)$$

The CP$^2$ model

The CP$^2$ model has three primaries $P,Q,R$ corresponding to $1, \omega, \omega^2 \in H^*(\text{CP}^2)$, respectively where $\omega \in H^*(\text{CP}^2)$ is the Kähler class such that $\omega^2$ has a unit volume. The metric is given by $\eta_{PR} = \eta_{QQ} = 1$, the others $= 0$. Order parameters are denoted as

$$u = \langle PP \rangle, \quad v = \langle PQ \rangle, \quad w = \langle PR \rangle. (3.35)$$

Other two point functions of primaries are essentially the derivatives of a function $f(u,v)$

$$\langle QQ \rangle = w + f_{vv}, \quad \langle QR \rangle = f_{uv}, \quad \langle RR \rangle = f_{uu}. (3.36)$$

Here, $f$ is the instanton contribution to the free energy:

$$f(u,v) = \sum_{d=1}^{\infty} N_d \frac{u^{3d-1}}{(3d-1)!} e^{dv}, (3.37)$$

which is determined [1, 4, 13] by the associativity equation [1, 8] (or “WDVV equation”)

$$f_{uuu} = (f_{uvv})^2 - f_{uuvf_{vvv}}, (3.38)$$

together with the initial value $N_1 = 1$. For example,

$$N_1 = 1, \quad N_2 = 1, \quad N_3 = 12, \quad N_4 = 620, \quad N_5 = 87304, \ldots (3.39)$$

The fundamental $M$ matrix is then expressed as

$$\left( \begin{array}{ccc} w & 3 & -u \\ 2f_{uv} & w + f_{vv} & 3 \\ 3f_{uu} & 2f_{uw} & w \end{array} \right). (3.40)$$
The 2nd Poisson bracket is given by

\[
\begin{pmatrix}
\{u(x), u(y)\} & \{u(x), v(y)\} & \{u(x), w(y)\} \\
\{v(x), u(y)\} & \{v(x), v(y)\} & \{v(x), w(y)\} \\
\{w(x), u(y)\} & \{w(x), v(y)\} & \{w(x), w(y)\}
\end{pmatrix}
\]

(3.41)

\[
= \begin{pmatrix}
-u\partial_x - \frac{1}{2}u' & \frac{3}{2}\partial_x + \frac{1}{2}v' & w\partial_x + \frac{3}{2}w' \\
3\partial_x - \frac{1}{2}v' & (w + f_{uv})\partial_x + \frac{1}{2}(w + f_{vv})' & 2f_{uv}\partial_x + \frac{3}{2}f_{uv}' \\
w\partial_x - \frac{1}{2}w' & 2f_{uv}\partial_x + \frac{1}{2}f_{uv}' & 3f_{uv}\partial_x + \frac{3}{2}f_{uv}'
\end{pmatrix} \delta(x - y).
\]

(3.42)

**Minimal Models**

The \(k\)th minimal model can formally be considered as a topological string theory with a fictitious “target space” \(M_k\) of dimension \(k/(k + 2)\) and cohomology classes \(O_\alpha\) \((\alpha = 0, 1, \ldots, k)\) of dimensions \(\alpha/(k + 2)\). The first Chern class of \(M_k\) is assumed to vanish \(c_1(M_k) = 0\). Namely, the correlation functions obey the basic equations (the selection rule, topological recursion relation, puncture and dilaton equation) as if the model had such a target space. The fundamental recursion relation (2.12) and the formulae (3.3)-(3.6) for the bi-Hamiltonian structure apply also to this case, since the derivation only needs the selection rule and TRR in a theory with scale invariance.

For example, let us consider the case \(k = 0\) of pure topological gravity. There is only a single primary \(P\) and the order parameter is denoted as \(u = \langle PP\rangle\). The fundamental matrix is given by \(M_0^0 = M_{00} = u\) and hence the recursion relation (2.12) reads as

\[
\langle \sigma_n(P)P \rangle = \frac{n}{n + 1} u \langle \sigma_{n-1}(P)P \rangle.
\]

(3.43)

This yields

\[
\langle \sigma_n(P)P \rangle = \frac{1}{n + 1} u^{n+1},
\]

(3.44)

which is the well-known constitutive relation \([21]\). The second Poisson bracket (3.4) is expressed as

\[
\{u(x), u(y)\}_2 = \left( u(x)\partial_x + \frac{1}{2}u'(x) \right) \delta(x - y).
\]

(3.45)

We see that this coincides with the second Poisson bracket of the full KdV hierarchy

\[
\{u(x), u(y)\} = \left( -\frac{1}{4}\partial_x^3 + u(x)\partial_x + \frac{1}{2}u'(x) \right) \delta(x - y),
\]

(3.46)

in the dispersionless limit \(\partial_x^3 \to 0\).

**Remark.** The fundamental recursion relation (2.12) in the minimal model has never been noted in the previous studies of generalized KdV hierarchy.

### 3.3 Virasoro Algebra

So far, we have been studying the integrable hierarchy by taking \(t_0^P\) as the basic spatial coordinate. At least formally, however, we may regard the variable \(t_0^\alpha\) of an arbitrary
primary field $O_{\alpha}$ as the basic coordinate. (“Democracy” among the primary fields in minimal models is discussed in [24].)

Let us take as the basic spatial coordinate $x$ the coupling constant $t_{0}^{\Omega M}$ for the volume class $\Omega_{M}$. The class $\Omega_{M}$ has the property $c_{1}(M) \wedge \Omega_{M} = 0$ and we expect some restoration of scale invariance as is pointed out in [22] in a different context. The order parameters are defined by $U_{\alpha} = \langle \Omega_{M}O_{\alpha} \rangle$. The formulae (3.3)-(3.6) for the bi-Hamiltonian structure still hold, provided we make the replacement $u_{\alpha} \rightarrow U_{\alpha}$ and $R_{n,\alpha} = \langle \sigma_{n}(O_{\alpha})P \rangle \rightarrow \langle \sigma_{n}(O_{\alpha})\Omega_{M} \rangle$. Here, we note that the second Poisson bracket of the two point function

$$T := \frac{2}{\dim M + 1}U_{\Omega M} = \frac{2}{\dim M + 1} \langle \Omega_{M}\Omega_{M} \rangle$$

(3.47)

gives nothing but the commutation relation of the Virasoro algebra

$$\{T(x), T(y)\}_{2} = (2T(x)\partial_{x} + T'(x))\delta(x - y).$$

(3.48)

Other order parameters $U_{\alpha}$ become the ”primary fields”

$$\{U_{\alpha}(x), T(y)\}_{2} = \left( \frac{2(q_{\alpha} + 1)}{\dim M + 1} U_{\alpha}(x)\partial_{x} + U'_{\alpha}(x) \right) \delta(x - y),$$

(3.49)

of dimensions $2(q_{\alpha} + 1)/(\dim M + 1)$. 

13
4. Lax Operator and Landau-Ginzburg Formulation

The fundamental recursion relation (2.12) completely determines the structure of the integrable hierarchy of the tree-level topological string theory. The integrable system for the full theory including higher genera will be a quantization or central extension of the tree level theory. Conventional way to achieve this is to formulate a Lax pair representation of the tree level system, and then to quantize it or discretize it by finding matrix integral representation. In this paper, we make a modest step toward the formulation of Lax pair representation. In particular, in this section we determine the Lax operator for the \( \mathbb{CP}^N \) model at the tree level in the relevant and marginal (Kähler) perturbations. We shall also see that we can develop a Landau-Ginzburg description of the system by regarding the Lax operator as its superpotential. In the next section, we treat the case of some other target spaces.

4.1 A Review Of The \( \mathbb{CP}^1 \) Model

First, we briefly review the Lax pair representation for the tree level theory of the \( \mathbb{CP}^1 \) model \[18\]. As in §3, the order parameters are denoted as \( u = \langle PP \rangle \) and \( v = \langle PQ \rangle \). The Lax operator is given by

\[
L = p + v + e^u p^{-1}.
\] (4.1)

Here, \( p \) is the momentum variable in the dispersionless Toda lattice hierarchy in which the Poisson bracket is defined by

\[
\{ A, B \} = p \frac{\partial A}{\partial p} \frac{\partial B}{\partial t_0} - p \frac{\partial B}{\partial p} \frac{\partial A}{\partial t_0}.
\] (4.2)

The flow equation (3.4) or (3.6),(3.7) can be written in the Lax form

\[
\frac{\partial L}{\partial t_n^+} = \{(G_{n,\alpha}(L))_+, L\},
\] (4.3)

where \((\cdots)_+\) means to take the non-negative powers of \( p \), and \( G_{n,P} \) and \( G_{n,Q} \) are given by

\[
G_{n,Q} = \frac{1}{n+1} L^{n+1},
\] (4.4)

\[
G_{n,P} = 2 L^n (\log L - c_n).
\] (4.5)

In the second expression, \( c_n \) is given by

\[
c_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n},
\] (4.6)

and the logarithm of \( L \) is defined as a Laurent series in \( p \) by taking the average

\[
\log L = \frac{1}{2} \left( \log \left\{ p(1 + vp^{-1} + e^u p^{-2}) \right\} + \log \left\{ e^{u} p^{-1}(e^{-u} p^2 + ve^{-u} p + 1) \right\} \right)
\] (4.7)

\[
= \frac{1}{2} \left( u + \log(1 + vp^{-1} + e^u p^{-2}) + \log(1 + ve^{-u} p + e^{-u} p^2) \right).
\] (4.8)
From (4.3) we have

\[
\frac{\partial u}{\partial t_n^\alpha} = \frac{\partial}{\partial t_0^\alpha} \left( G_{n,\alpha}(L) \right)_0, \tag{4.9}
\]

\[
\frac{\partial v}{\partial t_n^\alpha} = \frac{\partial}{\partial t_0^\alpha} \left( G_{n,\alpha}(L) \right)_{-1}, \tag{4.10}
\]

where \( \left( G_{n,\alpha} \right)_0 \) and \( \left( G_{n,\alpha} \right)_{-1} \) denote the constant (or \( p \)-independent) term and the coefficient of \( p^{-1} \) of \( G_{n,\alpha} \) respectively. This implies

\[
\langle \sigma_n(O^\alpha) P \rangle = \left( G_{n,\alpha}(L) \right)_0, \tag{4.11}
\]

\[
\langle \sigma_n(O^\alpha) Q \rangle = \left( G_{n,\alpha}(L) \right)_{-1}, \tag{4.12}
\]

or equivalently,

\[
\langle \sigma_n(O^\alpha) O^\beta \rangle = \oint G_{n,\alpha}(L) \hat{O}_\beta \frac{dp}{p}, \tag{4.13}
\]

where \( \hat{P} = 1 \) and \( \hat{Q} = p \). One can check that (4.13) satisfy the fundamental recursion relation (2.12) by using (3.32) for the matrix \( (M^\alpha_\beta) \) and hence provide a correct representation of two point functions.

### 4.2 Generalization

In the case of a general target space, we search for the Lax operator so that the two point functions can be expressed in a way similar to (4.13): Let \( L \) be a polynomial of several variables \( X_1, X_2, \ldots, X_N \) and their inverse powers, whose coefficients are given in terms of the order parameters \( u^\alpha \). We require that it satisfies

\[
\langle \sigma_n(O^\alpha) O^\beta \rangle = \oint G_{n,\alpha}(L) \hat{O}_\beta \Omega, \tag{4.14}
\]

\[
\Omega = \frac{dX_1}{X_1} \wedge \cdots \wedge \frac{dX_N}{X_N}, \tag{4.15}
\]

for a suitable choice of functions \( G_{n,\alpha}(L) \) and \( \hat{O}_\beta(X_i) \). We assume \( \hat{P} = 1 \).

We first see that we can almost determine the function \( G_{n,\alpha}(L) \) under a few assumptions on \( L \). Note that the operator

\[
\mathcal{D} = M_{0\beta} \frac{\partial}{\partial u^\beta} \tag{4.16}
\]

counts the dimension assigned to the parameters \( u^\alpha \) in such a way that

\[
[u^\alpha] = q^\alpha + 1 - \dim M, \tag{4.17}
\]

\[
[\exp(u^j d_j)] = c_1(M) \cdot d. \tag{4.18}
\]
We assume

**Affine Dependence on** \( u^0 = \eta^{0\beta} u_\beta \):\[L = u^0 + (u^0\text{-independent terms}). \tag{4.19}\]

**Homogeneity:**

\[\mathcal{D} L + \sum_i q_i X_i \frac{\partial}{\partial X_i} L = L, \tag{4.20}\]

where \( q_i \) is a dimension assigned to \( X_i \).

Recall that the two point functions satisfy for \( n \geq 1 \) (see lemma 1 and (2.12))

\[
\frac{\partial}{\partial u^0} \langle \sigma_n(O_\alpha)P \rangle = n \langle \sigma_{n-1}(O_\alpha)P \rangle, \quad \mathcal{D} \langle \sigma_n(O_\alpha)P \rangle = (n + 1 + q_\alpha - \dim M) \langle \sigma_n(O_\alpha)P \rangle + n \langle \sigma_{n-1}(c_1(M) \wedge O_\alpha)P \rangle. \tag{4.21}\]

Under the above assumptions, these imply

\[
\frac{d}{dL} G_{n,\alpha} = n G_{n-1,\alpha}, \tag{4.23}\]

\[
L \frac{d}{dL} G_{n,\alpha} = (n + 1 + q_\alpha - \dim M) G_{n,\alpha} + n c_1(M)^\beta G_{n-1,\beta}. \tag{4.24}\]

These equations have enough power to determine the functions \( G_{n,\alpha} \) up to some finite number of arbitrary constants. For instance, in the case of the volume class \( \Omega_M \) the second term of (4.24) is absent and we have (up to an overall constant)

\[
G_{n,\Omega_M} = \frac{1}{n + 1} L^{n+1}. \tag{4.25}\]

In particular, we can use the equation

\[
\langle \sigma_n(\Omega_M)P \rangle = \frac{1}{n + 1} \oint L^{n+1} \Omega, \tag{4.26}\]

to determine \( L \).

In the rest of the paper, we focus our attention on the subspace of the phase space where only the relevant \((t_0,P)\) and the marginal (Kähler) perturbations are turned on. We often turn off also the relevant perturbation, since its dependence is controlled by the puncture equation (2.3) and is easy to recover. On this subspace, the fundamental matrix is greatly simplified and this enables us to find a simple expression for the Lax operator for a wide class of target spaces.

The subspace of moduli space where only the marginal (Kähler) perturbation is turned on has been the area where the quantum cohomology and mirror symmetry played an important role. The objective of this paper is to extend the theory, on the same subspace, to the case of *gravitational* quantum cohomology where the two-dimensional gravity is
coupled to the $\sigma$ model and the Mumford-Morita classes interact with the cohomology classes of $M$. As we are going to discuss in the following, we will observe a mirror symmetry in a generalized sense when the system is coupled to gravity.

4.3 CP$^2$ Model In Detail

For the case of CP$^2$ model, the equations (4.23) and (4.24) for $G_{n,\alpha}$ are

\[
\frac{d}{dL}G_{n,\alpha} = nG_{n-1,\alpha}, \quad \alpha = P, Q, R, \quad (4.27)
\]

\[
L \frac{d}{dL}G_{n,R} = (n+1)G_{n,R}, \quad (4.28)
\]

\[
L \frac{d}{dL}G_{n,Q} = nG_{n,Q} + 3nG_{n-1,R}, \quad (4.29)
\]

\[
L \frac{d}{dL}G_{n,P} = (n-1)G_{n,P} + 3nG_{n-1,Q}. \quad (4.30)
\]

The general solution is

\[
G_{n,R} = \frac{1}{n+1}L^{n+1}, \quad (4.31)
\]

\[
G_{n,Q} = 3L^n(\log L - c_n + c), \quad (4.32)
\]

\[
G_{n,P} = 9nL^{n-1}\left(\frac{1}{2}(\log L)^2 - c_{n-1}\log L + d_{n-1} + c(\log L - c_{n-1}) + d\right), \quad (4.33)
\]

where $c$ and $d$ are arbitrary constants and

\[
d_n = 1 + \frac{c_2}{2} + \cdots + \frac{c_n}{n}. \quad (4.34)
\]

We will find the Lax operator $L$ so that (4.14) holds with these expressions for $G_{n,\alpha}$ (for a suitable choice of $c$ and $d$).

The Two Point Functions

When we turn of all the couplings except $t_0^Q = t$, the fundamental matrix (3.44) is simplified as

\[
\begin{pmatrix}
0 & 3 & 0 \\
0 & 0 & 3 \\
3e^t & 0 & 0
\end{pmatrix}.
\]

Then, the recursion relation (2.12) reads as

\[
\langle \sigma_n(O_\alpha)O_\beta \rangle = \frac{3n}{(n + \alpha + \beta - 1)} \left\{ \begin{array}{ll}
\langle \sigma_{n-1}(O_\alpha)O_{\beta+1} \rangle - \langle \sigma_{n-1}(O_{\alpha+1})O_\beta \rangle & \text{if } O_\beta \neq R \\
3e^t \langle \sigma_{n-1}(O_\alpha)P \rangle - \langle \sigma_{n-1}(O_{\alpha+1})O_\beta \rangle & \text{if } O_\beta = R
\end{array} \right. \quad (4.36)
\]

\[17\]
where $O_0 = P, O_1 = Q, O_2 = R$ and $O_3 = 0$. Together with the boundary conditions $\langle RR \rangle = e^t, \langle PQ \rangle = t$ and $\langle \sigma_1(P)P \rangle = t^2/2, (4.33)$ yields

\[
\langle \sigma_{3m-1}(R)P \rangle = \frac{(3m-1)!}{(m!)^3} e^{mt}, \tag{4.37}
\]

\[
\langle \sigma_{3m}(Q)P \rangle = (t - 3c_m)\frac{(3m)!}{(m!)^3} e^{mt}, \tag{4.38}
\]

\[
\langle \sigma_{3m+1}(P)P \rangle = \left(\frac{t^2}{2} - 3c_mt + 9d_m - 3\tilde{c}_m\right)\frac{(3m+1)!}{(m!)^3} e^{mt}, \tag{4.39}
\]

where $\tilde{c}_m = \sum_{j=1}^m 1/j^2$. On dimensional ground, $\langle \sigma_n(O_{\alpha})P \rangle = 0$ if $n + \alpha - 1 \not\equiv 0 \mod 3$.

The Lax Operator

Let us introduce two variables $X$ and $Y$. The value (4.37) is nothing but the constant term of

\[
\frac{1}{3m}(X + X^{-1}Y + e^t Y^{-1})^{3m}, \tag{4.40}
\]

There are no constant terms in $(X + X^{-1}Y + e^t Y^{-1})^{3m\pm 1}$. Thus, we define the Lax operator by

\[
L = X + X^{-1}Y + e^t Y^{-1}. \tag{4.41}
\]

It is easy to see that the equation (4.14) for $O_\alpha = R$ holds if we take

\[
\Omega = \frac{dX}{X} \wedge \frac{dY}{Y}, \tag{4.42}
\]

and

\[
\hat{P} = 1, \quad \hat{Q} = X, \quad \hat{R} = Y. \tag{4.43}
\]

The logarithm of $L$ is defined by taking the “average” of three kinds of expansions:

\[
\log L = \frac{1}{3} \left( \log(1 + X^{-2}Y + e^t X^{-1}Y^{-1}) + \log(X^2Y^{-1} + 1 + e^t XY^{-2}) + t + \log(e^{-t}XY + e^{-t}X^{-1}Y^{-2} + 1) \right) \tag{4.44}
\]

\[
= \frac{t}{3} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left\{ (X^{-2}Y + e^t X^{-1}Y^{-1})^n + (X^2Y^{-1} + e^t XY^{-2})^n + (e^{-t}XY + e^{-t}X^{-1}Y^{-2})^n \right\}.
\]

The square of log $L$ can also be defined, though we need a care for cross terms between different types of expansions. Then, we can check that (4.32),(4.33) satisfy the equation (4.14) if we take $c = 0$ and $d = -\zeta(2)/3 = -\sum_{n=1}^{\infty} 1/(3n^2)$. (checked for $G_{n,Q}$, $n =
1, . . . , 18 and \( G_{n,p}, n = 2, . . . , 7 \). To summarize, the expression for the descendants reads as

\[
\begin{align*}
G_{n,R} &= \frac{1}{n+1} L^{n+1}, \\
G_{n,Q} &= 3L^n (\log L - c_n), \\
G_{n,P} &= 9nL^{n-1} \left( \frac{1}{2} (\log L)^2 - c_{n-1} \log L + d_{n-1} - \frac{1}{3} \zeta(2) \right),
\end{align*}
\]

where \( c_n \) and \( d_n \) are given in (4.6) and (4.34) respectively.

### 4.4 The CP\(N\) Model

It is straightforward to generalize the above results to the case of CP\(N\) model. The CP\(N\) model has \( N + 1 \) primaries \( O_0 = P, O_1, . . . , O_N \) corresponding to \( 1, \omega, . . . , \omega^N \in H^*(\text{CP}^N) \) respectively, where \( \omega \) is the Kähler class with a unit volume. Here we consider only the marginal perturbation \( t^0_1 = t \). The Lax operator is expressed in terms of \( N \) variables \( X_1, . . . , X_N \) by

\[
L = X_1 + X_1^{-1}X_2 + \cdots + X_{N-1}^{-1}X_N + e^tX_N^{-1}.
\]

This satisfies

\[
\langle \sigma_n(O_\alpha)O_\beta \rangle = \oint G_{n,\alpha}(L)\hat{O}_\beta \Omega,
\]

\[
\Omega = \frac{dX_1}{X_1} \wedge \cdots \wedge \frac{dX_N}{X_N}.
\]

The primary fields \( O_i \) (\( i = 0, 1, . . . , N \)) are represented by

\[
\hat{O}_i = X_i.
\]

The representation \( G_{n,\alpha} \) for descendants are determined (up to some constants of integration) by the equations (4.23) and (4.24). Some of them are given by

\[
\begin{align*}
G_{n,N} &= \frac{1}{n+1} L^{n+1}, \\
G_{n,N-1} &= (N + 1)L^n (\log L - c_n^{(1)}), \\
G_{n,N-2} &= (N + 1)^2 nL^{n-1} \left( \frac{(\log L)^2}{2} - c_{n-1}^{(1)} \log L + c_{n-1}^{(2)} \right), \\
G_{n,N-3} &= (N + 1)^3 n(n-1)L^{n-2} \left( \frac{(\log L)^3}{3!} - c_{n-2}^{(1)} \frac{(\log L)^2}{2} + c_{n-2}^{(2)} \log L - c_{n-2}^{(3)} \right),
\end{align*}
\]

\[
\begin{align*}
\cdots & \cdots \\
\end{align*}
\]

where \( c_n^{(i)} \) are constants satisfying \( c_n^{(i)} = c_{n-1}^{(i)} + c_{n-1}^{(i-1)}/n \) and \( c_n^{(1)} = c_n \).

### 4.5 Landau-Ginzburg Formulation
In the tree-level topological string theory of minimal \[25, 26\] or \(\mathbb{CP}^1\) model \[13\], a Landau-Ginzburg (LG) description has been developed where the superpotential is given by the Lax operator of the corresponding integrable system. Here, we develop a LG formulation for the \(\mathbb{CP}^N\) model.

The superpotential we consider here is the Lax operator (4.48). When the relevant perturbation \((t_0^P)\) is added, it is given by
\[
L = t_0^P + X_1 + X_1^{-1}X_2 + \cdots + X_{N-1}^{-1}X_N + e^tX_N^{-1}.
\] (4.56)

Topological LG model \[7\] is a B-twisted \(N = 2\) sigma model (equipped with the F-term potential \(\int d^2\theta L\)) and upon quantization a nowhere vanishing holomorphic form of middle dimension must be specified \[27\]. In particular, the target space must be a Calabi-Yau manifold. In our case, we take as the target space the algebraic torus \((\mathbb{C}^*)^N\) with the holomorphic \(N\)-form (4.50):
\[
\Omega = \frac{dX_1}{X_1} \wedge \cdots \wedge \frac{dX_N}{X_N}.
\] (4.57)

The vacua are identified with the critical points of \(L, X_i\partial_{X_i}L = 0\):
\[
X_1 = X_1^{-1}X_2 = X_2^{-1}X_3 = \cdots = X_{N-1}^{-1}X_N = e^tX_N^{-1},
\] (4.58)
which consist of the \(N + 1\) points
\[
X_{*k} = e^{\frac{2\pi i}{N+1}}\zeta^k, \quad k = 0, 1, 2 \cdots N,
\] (4.59)
\[
X_{*j} = (X_{*1})^j, \quad j = 2, \cdots, N
\] (4.60)
where \(\zeta = \exp(\frac{2\pi i}{N+1})\). These are all non-degenerate (Hessians are non-vanishing) and hence the number of vacua is \(N + 1\) which is consistent with Tr\((-1)^F = \chi(\mathbb{CP}^N) = N + 1\) of the \(\mathbb{CP}^N\) sigma model.

The three point function is given by
\[
\langle ABC \rangle = \sum_{X_*: \text{critical}} \frac{A(X_*)B(X_*)C(X_*)}{\text{Hess}_{X_*}(L)}
\] (4.61)
\[
= \oint \frac{A(X)B(X)C(X)}{\prod_{j=1}^N X_j \partial_{X_j}L} \Omega,
\] (4.62)
where the last integration is over the small \(N\)-dimensional tori encircling the \(N + 1\) vacua \[28\]. The Hessian \(\text{Hess}_{X_*}(L)\) at the critical point \(X_*\) is given by
\[
\text{Hess}_{X_*}(L) := \det(X_i\partial_{X_i}X_j\partial_{X_j}L)|_{X = X_*}
\] (4.63)
\[
= (X_{*1})^N \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ \vdots & \vdots & \ddots \\ 0 & 0 & 2 \\ -1 & 0 & 2 \end{vmatrix} = (N + 1)(X_{*1})^N.
\] (4.64)
Thus, for example,

\[
\langle P X_i X_j \rangle = \sum_{\text{vacua}} \frac{(X_{*1})^{i+j}}{(N+1)(X_{*1})^N} = \delta_{i+j,N}
\]

and

\[
\langle X_1 X_i X_j \rangle = \sum_{\text{vacua}} \frac{(X_{*1})^{i+j+1}}{(N+1)(X_{*1})^N} = \delta_{i+j+1,N} + e^t \delta_{i,N} \delta_{i,N}.
\]

These coincide with the 3-point functions \( \langle PO_i O_j \rangle \) and \( \langle O_1 O_i O_j \rangle \), respectively, of the \( \text{CP}^N \) model and reproduce the quantum cohomology relation

\[
X_1^{N+1} = e^t.
\]

Therefore, we can identify \( \hat{O}_i = X_i \) as the LG representative for the primary field \( O_i \) of the \( \text{CP}^N \) model.

Next, we provide a LG description for gravitational descendants \( \sigma_n(O_i) \). Here, it is convenient to introduce new LG variables \( Z_1, \ldots, Z_N \) defined by

\[
Z_1 = X_1, \quad Z_2 = X_1^{-1} X_2, \quad Z_3 = X_2^{-1} X_3, \quad \ldots, \quad Z_N = X_{N-1}^{-1} X_N.
\]

The holomorphic form \( \Omega \) is still represented as \( \Omega = \frac{dZ_1}{Z_1} \wedge \cdots \wedge \frac{dZ_N}{Z_N} \) and therefore, the residue formulae (4.61),(4.62) also hold with this choice of variable. The Lax operator is expressed as

\[
L = t_0' + Z_1 + Z_2 + \cdots + Z_N + e^t Z_1^{-1} \cdots Z_N^{-1}.
\]

The three point function \( \langle \sigma_n(O_i) PP \rangle \) is given by

\[
\langle \sigma_n(O_i) PP \rangle = \frac{\partial}{\partial t_{0,P}} \langle \sigma_n(O_i) P \rangle = \oint G'_{n,i}(L) \Omega,
\]

where \( G'_{n,i}(L) = dG_{n,i}/dL \) and the integration is performed along \(|Z_i| = \text{const}\). The contours can be deformed to large ones \(|Z_i| = \text{const} \gg 1\) and then we find

\[
\langle \sigma_n(O_i) PP \rangle = \oint \frac{\tilde{\sigma}_n(O_i)}{\prod_{j=1}^N Z_j \partial Z_j L} \Omega,
\]

where

\[
\tilde{\sigma}_n(O_i) = \left( G'_{n,i}(L) \prod_{j=1}^N Z_j \partial Z_j L \right) .
\]

Here \( (\cdots)_+ \) is the projection to non-negative powers of \( Z_i \):

\[
(Z_1^{n_1} \cdots Z_N^{n_N})_+ := \left\{ \begin{array}{ll} Z_1^{n_1} \cdots Z_N^{n_N} & \text{if } n_1, \ldots, n_N \geq 0 \\ 0 & \text{otherwise} \end{array} \right.
\]

We propose \( [1.71] \) as the LG representative for the descendants. This identification may be justified by the fact that \( [1.71] \) satisfies the “topological recursion relation”:

\[
\tilde{\sigma}_n(O_i) \equiv n \langle \sigma_{n-1}(O_i)O^j \rangle \hat{O}_j.
\]
modulo terms that vanish at the critical points (BRST-exact terms divisible by $Z_j \partial Z_j L = Z_j - e^t Z_1^{-1} \cdots Z_N^{-1}$ for some $j$). Note that the fields $\hat{O}_j$ are given by

$$\hat{O}_j = X_j = Z_1 \cdots Z_j \equiv Z_{i_1} \cdots Z_{i_j}$$

for arbitrary $i_1, \ldots, i_j$, and also

$$n \langle \sigma_{n-1}(O)O \rangle = \oint G'_{n,i}(L) X_j \Omega = \left( G'_{n,i}(L)Z_{i_1} \cdots Z_{i_j} \right)_0$$

for distinct $i_1, \ldots, i_j$, where $(\cdots)_0$ denotes the term independent of $Z_1, \ldots, Z_N$. Therefore, what we need to establish is

$$(f(L) \prod_{j=1}^N Z_j \partial Z_j L)_+ \equiv \left( f(L) \right)_0 Z_1 \cdots Z_N + \left( f(L) Z_1 \right)_0 Z_2 \cdots Z_N + \cdots \equiv \left( f(L) Z_1 \cdots Z_{N-1} \right)_0 Z_N + \left( f(L) Z_1 \cdots Z_N \right)_0,$$

for an arbitrary function $f(L)$ of $L$. This is the multi-variable version of a similar formula for the one-variable case of minimal models [26] and the $\mathbb{C}P^1$ model [19]. In Appendix A, we present a proof of (4.76) in the case of $\mathbb{C}P^2$ model.

The LG representative (4.71) also satisfies an equation

$$\frac{\partial}{\partial t_{0,P}} \tilde{\sigma}_n(O_i) = \left( G''_{n,i}(L) \prod_{j=1}^N Z_j \partial Z_j L \right)_+ = n\tilde{\sigma}_{n-1}(O_i)$$

as a consequence of the $t_{0,P}$-dependence of $L = t_{0,P} + \cdots$ and the identity $G''_{n,i}(L) = nG'_{n-1,i}(L)$. (4.77) leads to the puncture equation (2.3). Note that the shift of the potential by the “cosmological constant” $t_{0,P}$ is irrelevant as far as we only consider primary three-point functions. This is the case for a theory without gravity. However, the shift has a non-trivial effect in the three point functions including the descendants. This is a manifestation of the effect of two-dimensional gravity.

The equation (2.5) for the Kähler class can also be derived in the LG description in the same way as in the $\mathbb{C}P^1$ model [19]. We turn off $t_{0,P}$. Let us introduce new LG variables $\bar{Z}_i = e^{-\frac{t}{\sqrt{\pi}t_{0,P}}} Z_i$. Then, $\bar{L} = e^{-\frac{t}{\sqrt{\pi}t_{0,P}}} L$ has an expression independent of $t$, and the $t$ dependence in

$$\sigma_n(O_i) := e^{-\frac{t}{\sqrt{\pi}t_{0,P}}} \tilde{\sigma}_n(O_i)$$

appears only through logarithms. Then, it is easy to see

$$\frac{\partial}{\partial t} \sigma_n(O_i) = n\sigma_{n-1}(O_{i+1})$$

($O_{N+1} := 0$), which leads to (2.5).

The superpotential (4.56) is nothing but the potential of the affine Toda field theory of $A_N$ type. Thus, this generalizes the correspondence of $\mathbb{C}P^1$ model and sine-Gordon
theory found in the previous work [19]. The relation of the CP^N sigma model and the AN affine Toda field theory has been observed at various stages. (For N = 2 supersymmetric models, see [29] for the comparison of the S-matrices and [30, 31] for the t-t* equation obeyed by the ground state metric. There exist also N = 0 literature [32, 33].) More close to our situation is the work of Batyrev [34]. He observed that the quantum cohomology ring of toric manifolds (including CP^N) is the Jacobian ring of the superpotentials of the affine Toda type. In the case of CP^N, our result shows that the description in terms of the affine Toda potential extends to the gravitational quantum cohomology.

In the next section, we treat several examples of Fano manifolds M. In each of them, we find a Lax operator whose number of variables is equal to the dimension of the original manifold. It turns out that when M is a toric manifold, the LG description is given based on the algebraic torus (C^*)^{\dim M}. When M is not toric (e.g. Grassmannians), the LG description on (C^*)^{\dim M} has a possible trouble due to a run-away behaviour of the potential. In such a case, however, the disease may be cured if we partially compactify (C^*)^{\dim M}. We may refer to the correspondence of the topological string model based on a Fano manifold and the LG model based on a non-compact CY manifold with superpotential of the affine Toda type as a mirror symmetry in a generalized sense. Here deformation of the Kahler class in the sigma model side corresponds to deformation of superpotential in the LG. In the case of ordinary quantum cohomology the LG variables are identified as the generators of the classical cohomology ring H^*(M) and do not possess direct geometrical significance. In the gravitational cohomology, however, we have LG variables as many as \dim M and they are identified as the coordinates of manifolds of the type of algebraic torus (C^*)^{\dim M}.

5. Examples

In this section, we construct Lax operators for more general target spaces. We turn off all the couplings except for the marginal (Kahler) perturbation. We recall from §4.2 (see [1, 20]) that the Lax operator L for a theory with target space M is defined as a polynomial in several variables X_1^{\pm 1}, ..., X_m^{\pm 1} such that

\[ \langle \sigma_n(\Omega_M)O_\alpha \rangle = \frac{1}{n+1} \int L^{n+1} \hat{O}_\alpha \frac{dX_1}{X_1} \wedge \cdots \wedge \frac{dX_m}{X_m}, \] (5.1)

hold for a suitable expression \hat{O}_\alpha, where \Omega_M is the volume class of M. We will consider (i) product spaces, (ii) rational surfaces, and we also treat (iii) Grassmannians in Appendix B.

5.1 Product Spaces

Let M and N be two Kahler manifolds with positive first Chern classes. Suppose that we already have the Lax operators L_M and L_N of the topological string theories on M and N. Then, we will see under a plausible assumption that the Lax operator of the theory for M \times N is given by
\[ L_{M \times N} = L_M + L_N. \] (5.2)

Let \( \{ \Omega_\alpha \} \) and \( \{ \omega_j \} \) be basis of the cohomology groups \( H^*(M) \) and \( H^*(N) \) respectively, where \( \Omega_\alpha \) and \( \omega_j \) have dimensions \( q_\alpha \) and \( q_j \). We denote by \( O_\alpha, O_j \) and \( O_{\alpha,j} \) the primary fields for the classes \( \Omega_\alpha, \omega_j \) and \( \Omega_\alpha \wedge \omega_j \). One of the most important facts in quantum cohomology of a product is that the 3-point functions factorize:

\[ \langle O_{\alpha,i} O_{\beta,j} O_{\gamma,k} \rangle = \langle O_\alpha O_\beta O_\gamma \rangle \langle O_i O_j O_k \rangle. \] (5.3)

This can be seen by noting that the evaluation maps commute with the map

\[ \overline{M}_{0,3}(M \times N, d_M \oplus d_N) \longrightarrow \overline{M}_{0,3}(M, d_M) \times \overline{M}_{0,3}(N, d_N), \] (5.4)

between moduli spaces of stable maps, which is isomorphic among open dense subsets corresponding to smooth curves. (See also §2.5 of [10], and [35].) Since the first Chern class of \( M \times N \) is just a sum of those for \( M \) and \( N \), \( c_1(M \times N) = c_1(M) + c_1(N) \), we see from (5.3) that the fundamental matrix for \( M \times N \) is given by

\[ M_{\alpha,i}^{\beta,j} = M_\alpha^\beta \delta_i^j + \delta_\alpha^\beta N_i^j, \] (5.5)

where \( M_\alpha^\beta \) and \( N_i^j \) are the fundamental matrices for \( M \) and \( N \) respectively. As a consequence, the two point functions of primaries decompose as

\[ \langle O_{\alpha,i} O_{\beta,j} \rangle = \langle O_\alpha O_\beta \rangle \eta_{ij} + \eta_{\alpha\beta} \langle O_i O_j \rangle. \] (5.6)

By our assumption, the Lax operators \( L_M \) and \( L_N \) are defined as polynomials in several variables, say \( Z_1, \ldots, Z_\mu \) for \( L_M \) and \( W_1, \ldots, W_\nu \) for \( L_N \) so that the following formulae hold

\[ \langle \sigma_n(\Omega_M) O_\alpha \rangle = \frac{1}{n+1}(L_M^{n+1} \hat{O}_\alpha)_0, \] (5.7)
\[ \langle \sigma_n(\Omega_N) O_i \rangle = \frac{1}{n+1}(L_N^{n+1} \hat{O}_i)_0. \] (5.8)

Here \( \hat{O}_\alpha \) and \( \hat{O}_i \) depend on \( Z_a \) and \( W_b \), respectively, and \( (\cdots)_0 \) stands for constant terms, i.e. terms independent of \( Z_a \) or \( W_b \). Now we claim that

\[ \langle \sigma_n(\Omega_{M \times N}) O_{\alpha,i} \rangle = \frac{1}{n+1}((L_M + L_N)^{n+1} \hat{O}_{\alpha,i})_{0,0}, \] (5.9)
\[ \hat{O}_{\alpha,i} := \hat{O}_\alpha \hat{O}_i. \] (5.10)

For \( n = 0 \), since \( \Omega_{M \times N} = \Omega_M \wedge \Omega_N \), this follows from (5.6) if

\[ (\hat{O}_\alpha)_0 = \delta_\alpha^0, \quad (\hat{O}_i)_0 = \delta_i^0, \] (5.11)

where \( \alpha = 0 \) (or \( i = 0 \)) stands for the identity class. We also assume that these hold. It is then a straightforward calculation to see that the right hand side of (5.9) satisfies
the fundamental recursion relation (2.14) for the theory of $M \times N$ with the matrix being given by (5.3):

\[
(n + 1 + q_\alpha + q_i) \frac{1}{n + 1} \left( (L_M + L_N)^{n+1} \hat{\omega}_\alpha \hat{\omega}_i \right)_{0,0}
\]

\[
= (n + 1 + q_\alpha + q_i) \frac{1}{n + 1} \sum_{l=0}^{n+1} \binom{n + 1}{l} \left( L_M^l \hat{\omega}_\alpha \right)_0 \left( L_N^{n+1-l} \hat{\omega}_i \right)_0
\]

\[
= \frac{1}{n + 1} \sum_{l=0}^{n+1} \binom{n + 1}{l} \left\{ l \frac{l + q_\alpha}{l} + (n + 1 - l) \frac{n + 1 - l + q_i}{n + 1 - l} \right\} \left( L_M^l \hat{\omega}_\alpha \right)_0 \left( L_N^{n+1-l} \hat{\omega}_i \right)_0
\]

\[
= \frac{1}{n + 1} \sum_{l=0}^{n+1} \binom{n + 1}{l} \{ l M_\alpha^\beta (L_M^{l-1} \hat{\omega}_\beta)_0 (L_N^{n+1-l} \hat{\omega}_i)_0 + (n + 1 - l) N_i^j (L_M^l \hat{\omega}_\alpha)_0 (L_N^{n+1-l} \hat{\omega}_j)_0 \}
\]

\[
= \sum_{l=1}^{n+1} \binom{n}{l-1} M_\alpha^\beta \left( L_M^{l-1} \hat{\omega}_\beta \right)_0 \left( L_N^{n+1-l} \hat{\omega}_i \right)_0 + \sum_{l=0}^{n} \binom{n}{l} N_i^j \left( L_M^l \hat{\omega}_\alpha \right)_0 \left( L_N^{n+1-l} \hat{\omega}_j \right)_0
\]

\[
= M_\alpha^\beta \left( (L_M + L_N)^n \hat{\omega}_\beta \hat{\omega}_i \right)_{0,0} + N_i^j \left( (L_M + L_N)^n \hat{\omega}_\alpha \hat{\omega}_j \right)_{0,0}. \tag{5.12}
\]

In the third step, we have used the fundamental recursion relation for each theory. Since the initial value at $n = 0$ and the recursion relation completely determine the two point functions, the claim (5.9) holds for all $n$. Thus, we can identify $L_M + L_N$ as the Lax operator for the product space $M \times N$. Note that the number of variables of $L_{M+N}$ equals $\text{dim} M + \text{dim} N$ which coincides again with the dimension of $M \times N$. We also note that we recover the relation $\chi(M \times N) = \chi(M) \chi(N)$ for the supersymmetry index, since the number of vacua equals that of the critical points of the superpotential in the LG description: the number of critical points of $L_{M+N}$ is given by the product of those of $L_M$ and $L_N$.

Suppose that the gravitational descendants are representable as LG fields in theories for $M$ and $N$. Namely, we assume for a suitable choice of LG variables $Z_a$ and $W_b$, we have

\[
(f(L_M) \prod_a Z_a \partial_{Z_a} L_M)_+ \equiv \left( f(L_M) \hat{\omega}_\alpha \right)_0 \hat{\omega}_\alpha, \tag{5.13}
\]

\[
(f(L_N) \prod_b W_b \partial_{W_b} L_N)_+ \equiv \left( f(L_N) \hat{\omega}_i \right)_0 \hat{\omega}_i, \tag{5.14}
\]

modulo terms that vanish at the critical points. Here $(\cdots)_+$ are defined as in (4.172). Then, we find

\[
(f(L_M + L_N) \prod_a Z_a \partial_{Z_a} L_M \prod_b W_b \partial_{W_b} L_N)_{+,+} \equiv \left( \left( f(L_M + L_N) \hat{\omega}_\alpha \right)_0 \hat{\omega}_\alpha \prod_b W_b \partial_{W_b} L_N \right)_+
\]

\[
\equiv \left( f(L_M + L_N) \hat{\omega}_\alpha \hat{\omega}_i \right)_{0,0} \hat{\omega}_\alpha \hat{\omega}_i. \tag{5.15}
\]

Thus the descendants of the product theory are also representable as LG fields.

5.2 Rational Surfaces
It is interesting to see how the Lax operator behaves under birational transformations of the target space. The simplest example for such a study is provided by rational surfaces—two-dimensional complex manifolds that can be obtained from \( \mathbb{CP}^2 \) by a series of blow-ups and downs. There are only ten rational surfaces of positive first Chern classes (Del Pezzo surfaces, see \([36]\)): \( \mathbb{CP}^2 \), the quadric \( \mathbb{CP}^1 \times \mathbb{CP}^1 \), and blow up \( M_r \) of \( \mathbb{CP}^2 \) at \( r \)-points in general position \( (r = 1, \ldots, 8) \). Here we consider \( \mathbb{CP}^1 \times \mathbb{CP}^1, M_1, \) and \( M_2 \).

In general, the genus 0 free energy for a complex surface \( M \) has the instanton expansion

\[
F_0 = \sum_d N_d \frac{e^{c_1(M) \cdot d - 1}}{(c_1(M) \cdot d - 1)!} e^{t \cdot d},
\]

where \( t = t_R = t^R_0 \) is the coupling constant of the unique irrelevant operator \( R \) (the volume class of \( M \)). The sum runs over effective classes, i.e., homology classes that are positive on the Kähler cone. For Del Pezzo surfaces, the numbers \( N_d \) are determined \([10, 11]\) by the associativity equation and an initial condition (see also \([37]\)). For the space \( M_r \), the initial condition is stated as \( N_d = 1 \) for every exceptional class \( d \). (A (co)homology class \( E \) is called exceptional when it is represented by a \( \mathbb{CP}^1 \) with a self-intersection number \(-1\), namely, \( c_1(M) \cdot E = 1 \) and \( E \cdot E = -1 \).) For our purpose of determining the fundamental matrix, it is enough to know the free energy \( F \) up to order \( t^2 \).

The Quadric Surface \( \mathbb{CP}^1 \times \mathbb{CP}^1 \)

Since this is a product, the general argument of the previous subsection should apply. There are four primaries; \( P, Q_1, Q_2, R \) where \( Q_1 \) and \( Q_2 \) correspond to the Kähler classes of one \( \mathbb{CP}^1 \) and the other. The first Chern class is \( 2Q_1 + 2Q_2 \) and the free energy is given by

\[
F_0 = \sum_{d_1, d_2 \geq 0} N_{d_1, d_2} \frac{t^R_{d_1 + 2d_2 - 1}}{(2d_1 + 2d_2 - 1)!} e^{d_1 t^1 + d_2 t^2} = t_R e^{t^1} + t_R e^{t^2} + \cdots.
\]

From this expansion, we can read off the fundamental matrix and find that it is indeed a tensor product of two matrices for \( \mathbb{CP}^1 \)'s:

\[
\begin{pmatrix}
0 & 2 & 2 & 0 \\
2e^{t^1} & 0 & 0 & 2 \\
2e^{t^2} & 0 & 0 & 2 \\
0 & 2e^{t^2} & 2e^{t^1} & 0
\end{pmatrix}
= \begin{pmatrix} 0 & 2 \\ 2e^{t^1} & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 2 \\ 2e^{t^2} & 0 \end{pmatrix}.
\]

As we have seen in the previous subsection, the Lax operator is the sum of those for the \( \mathbb{CP}^1 \) model and the fields are products:

\[
L = p_1 + e^{t^1} p_1^{-1} + p_2 + e^{t^2} p_2^{-1},
\]

\[
\hat{P} = 1, \quad \hat{Q}_1 = p_1, \quad \hat{Q}_2 = p_2, \quad \hat{R} = p_1 p_2.
\]

The theory admits the LG description (based on \( \mathbb{C}^* \times \mathbb{C}^* \)) since the \( \mathbb{CP}^1 \) model does.

**Blow Up at One Point**
The space $M_1$ is obtained by blowing up $\mathbb{CP}^2$ at one point $q \in \mathbb{CP}^2$. There are four primaries $P, H, E, R$ where $H$ is the pull back of the Kähler class of $\mathbb{CP}^2$ by the projection $\pi : M_1 \to \mathbb{CP}^2$, and $E$ is the unique exceptional class $[\pi^{-1}(q)]$. The metric is given by

$$\eta_{PR} = \eta_{HH} = -\eta_{EE} = 1.$$  

The first Chern class is $3H - E$ and the free energy is given by $(t^H_0 = t_H, t^E_0 = t_E)$

$$F_0 = \sum_{a \geq b, 0} N_{a,b}(3a - b - 1)! e^{at_H + bt_E} = e^{-t_E} + t_R e^{t_H + t_E} + \frac{t^2_R}{2} e^{t_H} + \cdots, \quad (5.21)$$

which yields the following expression for the fundamental matrix $(\beta := e^{t_H}, \beta_E := e^{t_E})$:

$$\begin{pmatrix}
0 & 3 & -1 & 0 \\
2\beta_H \beta_E & 0 & 0 & 3 \\
2\beta_H \beta_E & 0 & -\beta_E^{-1} & 1 \\
3\beta_H & 2\beta_H \beta_E & -2\beta_H \beta_E & 0 \\
\end{pmatrix}. \quad (5.22)$$

The two point functions are recursively determined by using this expression. Then, we can construct the Lax operator and the expression for the primary fields so that the equation (5.1) holds. The result is (5.1) is checked up to $n = 15$

$$L = X + X^{-1}Y + \beta_H Y^{-1} + \beta_E Y, \quad (5.23)$$

$$\hat{P} = 1 \quad (5.24)$$

$$\hat{H} = X + \beta_E Y \quad (5.25)$$

$$\hat{E} = \beta_E Y \quad (5.26)$$

$$\hat{R} = Y + \beta_E XY. \quad (5.27)$$

As $\beta_E \to 0$ (the limit in which the volume of the exceptional curve $\pi^{-1}(q)$ becomes minus infinity), $\hat{E}$ vanishes, $L$ approaches the Lax operator of $\mathbb{CP}^2$ and $\hat{P}, \hat{H}, \hat{R}$ become $\hat{P}, \hat{Q}, \hat{R}$ of $\mathbb{CP}^2$. We note that $\hat{E}/\beta_E$ turns into $\hat{R}$.

It is easy to see that the vacuum equation $X \partial_X L = Y \partial_Y L = 0$ has four non-zero solutions and there is no vacuum at infinity. This is consistent with the mass gap and the supersymmetry index $\chi(M_1) = 4$ of the sigma model. Thus, we have a sound LG description with superpotential $L$ based on the algebraic torus $C^* \times C^* = \{(X,Y)\}$.

**Blow Up at Two Points**

The space $M_2$ is obtained by blowing up $\mathbb{CP}^2$ at two points $q_1, q_2 \in \mathbb{CP}^2$. There are five primaries $P, H, E_1, E_2, R$ where $H$ is the pull back of the Kähler class of $\mathbb{CP}^2$ by $\pi_2 : M_2 \to \mathbb{CP}^2$, and $E_i$ is the exceptional class $[\pi_2^{-1}(q_i)], i = 1, 2$. There is in addition another exceptional class $E_{12} = H - E_1 - E_2$. The metric is given by $\eta_{PR} = \eta_{HH} = -\eta_{EE_1E_1} = -\eta_{EE_2E_2} = 1$. The first Chern class is $3H - E_1 - E_2$ and the free energy is given by $(t^H_0 = t_H, t^{E_1}_0 = t_{E_1}, t^{E_2}_0 = t_{E_2})$

$$F_0 = \sum_{a \geq b_1, b_2, 0} N_{a,b_1,b_2}(3a - b_1 - b_2 - 1)! e^{at_H + bt_{E_1} + bt_{E_2}} \quad (5.28)$$

$$= e^{t_H + t_{E_1} + t_{E_2}} + e^{-t_{E_1}} + e^{-t_{E_2}} + t_R (e^{t_H + t_{E_1}} + e^{t_H + t_{E_2}}) + \frac{t^2_R}{2} e^{t_H} + \cdots. \quad (5.29)$$
which yields the following expression for the fundamental matrix \((\beta_H := e^{t_H}, \beta_{E_1} := e^{t_{E_1}})\):

\[
\begin{pmatrix}
0 & 3 & -1 & -1 & 0 \\
2(\beta_H\beta_{E_1} + \beta_H\beta_{E_2}) & \beta_H\beta_{E_1}\beta_{E_2} & -\beta_H\beta_{E_1}\beta_{E_2} & -\beta_H\beta_{E_1}\beta_{E_2} & 3 \\
2\beta_H\beta_{E_1} & \beta_H\beta_{E_1}\beta_{E_2} & -\beta_{E_1} - \beta_H\beta_{E_1}\beta_{E_2} & -\beta_H\beta_{E_1}\beta_{E_2} & 1 \\
2\beta_H\beta_{E_2} & \beta_H\beta_{E_1}\beta_{E_2} & -\beta_H\beta_{E_1}\beta_{E_2} & -\beta_{E_1} - \beta_H\beta_{E_1}\beta_{E_2} & 1 \\
3\beta_H & 2(\beta_H\beta_{E_1} + \beta_H\beta_{E_2}) & -2\beta_H\beta_{E_1} & -2\beta_H\beta_{E_2} & 0 \\
\end{pmatrix}
\]

As in the previous example we can determine the Lax operator and the representatives for the fields. The result is (5.31) is checked up to \(n = 15\)

\[
L = X + X^{-1}Y + \beta_HY^{-1} + (\beta_{E_1} + \beta_{E_2})Y + \beta_{E_1}\beta_{E_2}XY,
\]

(5.31)

\[
\hat{P} = 1
\]

(5.32)

\[
\hat{H} = X + (\beta_{E_1} + \beta_{E_2})Y + 2\beta_{E_1}\beta_{E_2}XY
\]

(5.33)

\[
\hat{E}_1 = \beta_{E_1}Y + \beta_{E_1}\beta_{E_2}XY
\]

(5.34)

\[
\hat{E}_2 = \beta_{E_2}Y + \beta_{E_1}\beta_{E_2}XY
\]

(5.35)

\[
\hat{R} = Y + (\beta_{E_1} + \beta_{E_2})XY + \beta_{E_1}\beta_{E_2}(2Y^2 + (\beta_{E_1} + \beta_{E_2})XY^2).
\]

(5.36)

As \(\beta_{E_2} \to 0\), the Lax operator \(L\) and the fields \(\hat{P}, \hat{H}, \hat{E}_1, \hat{R}\) become the Lax operator and the fields \(\tilde{P}, \tilde{H}, \tilde{E}, \tilde{R}\) of \(M_1\), while \(\hat{E}_2/\beta_{E_2} \to \tilde{R}\). Note that the above Lax operator (5.31) is manifestly symmetric under exchange of \(\beta_{E_1}\) and \(\beta_{E_2}\).

Let us discuss whether a consistent LG description is possible by regarding (5.31) as the superpotential. Eliminating the variable \(X\) from the vacuum equation \(X\partial_X L = Y\partial_Y L = 0\), we have

\[
\beta_{E_1}\beta_{E_2}(\beta_{E_1} - \beta_{E_2})^2Y^5 + (\beta_{E_1} - \beta_{E_2})^2Y^4 + (2\beta_H\beta_{E_1}\beta_{E_2}(\beta_{E_1} + \beta_{E_2}) - 1)Y^3
\]

\[
+2\beta_H(\beta_{E_1} + \beta_{E_2})Y^2 + \beta_H^2\beta_{E_1}\beta_{E_2}Y + \beta_H^2 = 0
\]

(5.37)

Generically, there are five non-zero solutions to (5.37) and there is no vacuum at infinity, which is consistent with \(\chi(M_2) = 5\) and the mass gap. However, as the Kähler structure approaches a \(Z_2\) orbifold point \(\beta_{E_1} = \beta_{E_2}\) of the moduli space, two of the vacua run away to infinity and the above LG description breaks down. Thus, at this point we must resort to another description that is insensitive to the singularity of the moduli space. Let us change the variables as \(X \to X\) and \(Y \to Y = (1 + \beta_{E_2}X)Y\). Then, we have a new representation

\[
L = X + X^{-1}Y + \beta_HY^{-1} + \beta_{E_1}Y + \beta_H\beta_{E_2}XY^{-1},
\]

(5.38)

\[
\hat{H} = X + \beta_{E_1}Y + \beta_H\beta_{E_2}XY^{-1}
\]

(5.39)

\[
\hat{E}_1 = \beta_{E_1}Y
\]

(5.40)

\[
\hat{E}_2 = \beta_H\beta_{E_2}XY^{-1}
\]

(5.41)

\[
\hat{R} = Y + \beta_{E_1}X^{-1}Y^2.
\]

(5.42)
At the price of losing the manifest $\mathbb{Z}_2$ symmetry, this description has a good behavior at the orbifold points. The vacuum equation $X \partial_X L = \tilde{Y} \partial_Y L = 0$ has always five non-zero solutions and there is no vacuum at infinity even if $\beta^X \partial^X \gamma$ the orbifold points. The vacuum equation is

$$\pi\colon \tilde{M} \rightarrow M$$

by $\pi\colon \tilde{M} \rightarrow M$ the orbifold points $\gamma^X \partial^X \gamma$. Thus, we expect that (5.38) gives a sound LG description based on $C^* \times C^* = \{(X,Y)\}$ everywhere on the moduli space of Kähler structure. Away from the orbifold points $\gamma^X = \gamma^Y$, this is equivalent to the symmetric description (5.31) based on $C^* \times C^* = \{(X,Y)\}$.

**A General Remark**

From the above examples, we extract the following general features of the Lax operator. Let $M$ be a complex surface and $\tilde{M}$ be the blow up of $M$ at one point $p \in M$. We denote by $\pi\colon \tilde{M} \rightarrow M$ the projection, and by $E$ the exceptional class $[\pi^{-1}(p)]$. The second cohomology group of $\tilde{M}$ is given by

$$H^2(\tilde{M}) = \pi^* H^2(M) \oplus \mathbb{Z} E,$$

(5.43)

which is an orthogonal decomposition with respect to the intersection form. We choose a base $\{\omega_1, \ldots, \omega_s, E\}$ of $H^2(\tilde{M})$ so that $\omega_i \in \pi^* H^2(M)$ and denote parameters of the marginal perturbations as $t_1, \ldots, t_r, t_E$. We put $\beta_E = e^{\xi E}$ and consider the limit $\beta_E \rightarrow 0$ with other parameters $t_i$ being fixed. In the above examples of $M_2 \rightarrow M_1$ and $M_1 \rightarrow \mathbb{C}P^2$, Lax operator $L_{\tilde{M}}$ and the fields of $\tilde{M}$ behave as

$$L_{\tilde{M}} \rightarrow L_M$$

(5.44)

$$\pi^* O \rightarrow \tilde{O}$$

(5.45)

$$\tilde{E}/\beta_E \rightarrow \tilde{R},$$

(5.46)

where $L_M$ and $O$ are the Lax operator and a primary field of $M$. As a consequence, we have

$$\lim_{\beta_E \rightarrow 0} \langle \sigma_n(R) \pi^* (O) \rangle_{\tilde{M}} = \langle \sigma_n(R)O \rangle_M,$$

(5.47)

$$\lim_{\beta_E \rightarrow 0} \frac{1}{\beta_E} \langle \sigma_n(R)E \rangle_{\tilde{M}} = \langle \sigma_n(R)R \rangle_M.$$  

(5.48)

We conjecture that these features are generally true. As a non-trivial check, let us consider the case of $\pi\colon M_2 \rightarrow \mathbb{C}P^1 \times \mathbb{C}P^1$ (See Figure 1).

The blow up $M_2$ of $\mathbb{C}P^2$ at two points $q_1$ and $q_2$ has three exceptional curves; $E_1 = \pi_2^{-1}(q_1)$, $E_2 = \pi_2^{-1}(q_2)$, and $E_{12} = H - E_1 - E_2$ as we noted before. If we blow down along $E_{12}$, we obtain the quadric surface $\mathbb{C}P^1 \times \mathbb{C}P^1$. The subspace of $H^2(M_2)$ orthogonal to $E_{12}$ is spanned by $H - E_1 = \tilde{Q}_2$ and $H - E_2 = \tilde{Q}_1$. Each of them has vanishing self-intersection and can be considered as the pull back of $Q_1$ or $Q_2$. Hence, it is appropriate to take the parametrization

$$t_{Q_1} = t_H + t_{E_1},$$

(5.49)

$$t_{Q_2} = t_H + t_{E_2},$$

(5.50)

$$t_{E_{12}} = -t_H - t_{E_1} - t_{E_2}.$$  

(5.51)
and consider the limit $t_{E_{12}} \to -\infty$, or $\beta_{12} = e^{t_{E_{12}}} \to 0$ with $t_{Q_1}$ and $t_{Q_2}$ being fixed. We work in the asymmetric parametrization (5.38) and make the change of variables

$$X = e^{t_H} p_1^{-1} p_2^{-1}$$

$$\tilde{Y} = e^{t_H} p_1^{-1}.$$  

Then, the Lax operator and the fields in $\pi^* H^*(\mathbb{CP}^1 \times \mathbb{CP}^1)$ are expressed as

$$L = \beta_{Q_1} \beta_{Q_2} \beta_{12} p_1^{-1} p_2^{-1} + p_2 + p_1 + \beta_{Q_1} p_1^{-1} + \beta_{Q_2} p_2^{-1},$$

$$\hat{Q}_1 = \beta_{Q_1} \beta_{Q_2} \beta_{12} p_1^{-1} p_2^{-1} + \beta_{Q_1} p_1^{-1},$$

$$\hat{Q}_2 = \beta_{Q_1} \beta_{Q_2} \beta_{12} p_1^{-1} p_2^{-1} + \beta_{Q_2} p_2^{-1},$$

$$\hat{R} = \beta_{Q_1} \beta_{Q_2} \beta_{12} p_1^{-1} + \beta_{Q_1} p_1^{-1} p_2,$$

which have the well-defined limits as $\beta_{12} \to 0$:

$$L \to p_2 + p_1 + \beta_{Q_1} p_1^{-1} + \beta_{Q_2} p_2^{-1}$$

$$\hat{Q}_1 \to \beta_{Q_1} p_1^{-1} \equiv p_1,$$

$$\hat{Q}_2 \to \beta_{Q_2} p_2^{-1} \equiv p_2,$$

$$\hat{R} \to \beta_{Q_1} p_1^{-1} p_2 \equiv p_1 p_2.$$
Thus, as $\beta_{12} \to 0$ the Lax operator and the fields become those of the $\text{CP}^1 \times \text{CP}^1$ model (5.19),(5.20). Note also that $\hat{E}_{12} = X = \beta Q_1 \beta Q_2 \beta_{12}^{-1} p_1^{-1} p_2^{-1}$ and hence

$$\hat{E}_{12}/\beta_{12} \to \beta Q_1 \beta Q_2 p_1^{-1} p_2^{-1} \equiv p_1 p_2.$$ (5.61)

This shows that the conjecture holds in this case.

We have left many important problems untouched in this paper. (1): Our treatment of the Lax formulation is limited to the case of marginal perturbations. Is it possible to generalize the formalism to a larger phase space? (2): We have restricted ourselves to the genus=0 case throughout this paper (except the results of the $\text{CP}^1$ model). It is a challenging problem to generalize our discussions to higher genera. The known topological recursion relation for $g = 1$ unfortunately seems not strong enough to determine genus=1 amplitudes except in the $\text{CP}^1$ case. Beyond $g = 1$ no general relations are known for topological string amplitudes. Is it possible that a suitable central extension of the Virasoro algebra which we encountered in section 3 may describe the higher genus extension? (3): We need a deeper understanding of the mirror phenomenon in Fano varieties. We hope to discuss these issues in a future publication.

T.E. would like to thank P. Di Francesco for discussions at an early stage of this work. We also wish to thank R. Dijkgraaf, K. Intriligator, M. Jinzenji, S.-K. Yang and especially I. Nakamura for conversations, discussions and educations. This research has been performed while C.S.X. was a JSPS fellow at Univ. of Tokyo. Research of T.E. is supported by Grant-in-Aid for Scientific Research on Priority Area 213 “Infinite Analysis”, Japan Ministry of Education.
Appendix A

In this appendix, we give a proof of the formula \((1.70)\) for the case of two variables. We may put \(t = t_0 = 0\). First, we review the case of the \(\mathbb{CP}^1\) model with \(L = p + p^{-1}\) [19]. Since we have

\[
\frac{f(L)}{p} = \left( \frac{f(L)}{p} \right)_+ + \left( (f(L))_0 p^{-1} + (f(L)p)_0 p^{-2} + O(p^{-3}) \right),
\]

and \(p^2 \partial_p L = p^2 - 1\) has only + components, we obtain

\[
\left( f(L)p \partial_p L \right)_+ = \left( \frac{f(L)}{p} (p^2 - 1) \right)_+ = \left( \frac{f(L)}{p} \right)_+ (p^2 - 1) + \left( (f(L))_0 p + (f(L)p)_0 \right)
\]

\[
\equiv \left( f(L) \right)_0 p + \left( (f(L)p)_0 \right).
\]

The above argument applies also to multi-variable cases, though we need more efforts. We present the derivation in the case of \(\mathbb{CP}^2\) of two variables. Let us rename \(Z_1\) as \(Z\) and \(Z_2\) as \(W\). The Lax operator is then rewritten as \(L = Z + W + Z^{-1}W^{-1}\) and we note that

\[
Z^2 W^2 Z \partial_Z LW \partial_W L = Z^3 W^3 - Z W^2 - Z^2 W + 1
\]

(A.5)

has only + components. The negative part of \(f(L)/Z^2 W^2\) can be expressed as

\[
\left( \frac{f(L)}{Z^2 W^2} \right)_- := \frac{f(L)}{Z^2 W^2} - \left( \frac{f(L)}{Z^2 W^2} \right)_+ = \left( \sum_{a<0} + \sum_{a \geq 0 \ b < 0} + \sum_{a < 0 \ b \geq 0} \right) \left( \frac{f(L)}{Z^{a+2} W^{b+2}} \right)_0 Z^a W^b.
\]

(A.6)

We note that \(Z \equiv W\) and \(Z^3 \equiv Z^2 W \equiv Z W^2 \equiv W^3 \equiv 1\) but we cannot use them before taking the + part \((\cdot \cdot \cdot)_+\). Taking this into account, we have

\[
\left( \frac{f(L)}{Z^2 W^2} \right)_- Z^3 W^3 = \left( \sum_{-a=1,2,3 \ b \geq 0} + \sum_{a \geq 0 \ b=1,2,3} + \sum_{-a=1,2,3 \ b=1,2,3} \right) \left( \frac{f(L)}{Z^{a+2} W^{b+2}} \right)_0 Z^{a+3} W^{b+3}
\]

\[
\equiv \left( \sum_{-a=1,2,3 \ b=1,2,3} + \sum_{a \geq 0 \ b=1,2,3} + \sum_{-a=1,2,3 \ b=1,2,3} \right) \left( \frac{f(L)}{Z^{a+2} W^{b+2}} \right)_0 Z^a W^b,
\]

\[
\left( \frac{f(L)}{Z^2 W^2} \right)_- Z W^2 = \left( \sum_{-a=-1 \ b=0} + \sum_{a \geq 0 \ b=-1,2} + \sum_{a=-1 \ b=-1,2} \right) \left( \frac{f(L)}{Z^{a+2} W^{b+2}} \right)_0 Z^a W^b
\]

\[
\equiv \left( \frac{f(L)}{Z^{a+2} W^{b+2}} \right)_0 Z^2 W
\]

\[
\left( \left( \frac{f(L)}{Z^2 W^2} \right)_- \cdot 1 \right)_+ \equiv 0.
\]
Then, we have

\[
\left( \left( \frac{f(L)}{Z^2W^2} \right)_- Z^2W^2 Z \partial_Z LW \partial_W L \right)_+ \equiv \left( 2 \sum_{a=-3}^{b=0} + \sum_{a=-2}^{b=-3} + 2 \sum_{a=-1}^{b=-1} - 2 \sum_{a=-1}^{b=0} \right) \left( \frac{f(L)}{Z^a+2W^{b+2}} \right)_0 Z^a W^b.
\]  

(A.7)

Here, we note that \( L \) is invariant under the exchange of \( Z \) and \( W \), and also under \( Z \to Z^{-1}W^{-1}, W \to W \). Hence,

\[
\left( f(L)Z^{-a}W^{-b} \right)_0 = \left( f(L)Z^{a+2}W^{a-b} \right)_0.
\]  

(A.8)

In particular, \( \left( f(L)/Z^{a+2}W^{b+2} \right)_0 Z^a W^b \) is invariant (at the critical points) under \( (a, b) = (-3, b) \to (-1, b + 1). \)  

(A.9)

Then, it is easy to see that (A.7) acquires contribution from \( (a, b) = (-2, -2), (-2, -3) \) and \( (-3, -3) \), all with multiplicity 1. Namely,

\[
\left( \left( \frac{f(L)}{Z^2W^2} \right)_- Z^2W^2 Z \partial_Z LW \partial_W L \right)_+ \equiv \left( f(L) \right)_0 Z^{-2}W^{-2} + \left( f(L)W \right)_0 Z^{-2}W^{-3} + \left( f(L)ZW \right)_0 Z^{-3}W^{-3}
\]  

(A.10)

\[
\equiv \left( f(L) \right)_0 ZW + \left( f(L)Z \right)_0 W + \left( f(L)ZW \right)_0.
\]  

(A.11)

Since \( \left( f(L)/Z^2W^2 \right)_+ Z^2W^2 Z \partial_Z LW \partial_W L \) \( \equiv 0 \), we see that (4.76) holds for \( N = 2. \)
Appendix B: Grassmannians

Cohomology group of the complex Grassmann manifold $Gr(k, N)$ (the space of $k$-dimensional subspaces in $\mathbb{C}^N$) is spanned by Schubert classes which are in one to one correspondence with Young diagrams of height $\leq N - k$ and width $\leq k$. We denote by $O_Y$ the primary field corresponding to a Young diagram $Y$. Pairing of $O_Y$ and $O_{Y'}$ is nonvanishing (and is 1) if and only if the union of $Y$ and $Y'$ form the rectangular diagram of size $(N - k) \times k$. The dimension is $k(N - k)$ and the first Chern class is given by $NO_O$.

B.1 Lax Operators

$Gr(2, 4)$

First, we consider the simplest case $Gr(2, 4)$. There are six primaries $P, O_{\square}, O_{\square\square}, O_{\square\square\square}, O_{\square\square\square\square}$. The fundamental matrix in the marginal perturbation $t_0^\square = t$ is given by

$$
\begin{pmatrix}
0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 \\
4e^t & 0 & 0 & 0 & 0 & 4 \\
0 & 4e^t & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

(B.1)

Here we have used the result for the instanton contribution $\langle O_{\square}O_{\square\square}O_{\square\square\square} \rangle = e^t$ which will be derived in the following for a more general situation. (See also \[4, 33, 39, 40\].) We find that the Lax operator and the fields are given by

$$
L = X + X^{-1}(Y + Z) + (Y^{-1} + Z^{-1})W + e^tW^{-1},
$$

(B.2)

$$
\hat{P} = 1
$$

(B.3)

$$
\hat{O}_{\square} = X
$$

(B.4)

$$
\hat{O}_{\square\square} = Y, \quad \hat{O}_{\square\square\square} = Z
$$

(B.5)

$$
\hat{O}_{\square\square\square\square} = (1 + YZ^{-1})W
$$

(B.6)

$$
\hat{O}_{\square\square\square\square\square} = XW
$$

(B.7)

$Gr(2, 5)$

There are ten primaries; $P, O_{\square}, O_{\square\square}, O_{\square\square\square}, O_{\square\square\square\square}, O_{\square\square\square\square\square}, O_{\square\square\square\square\square\square}, O_{\square\square\square\square\square\square\square}, O_{\square\square\square\square\square\square\square\square}$, The
The fundamental matrix in the marginal perturbation is given by

\[
\begin{pmatrix}
0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & 5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\
5e^t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
0 & 0 & 5e^t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

(B.8)

The Lax operator and the fields are given by

\[
L = X + X^{-1}(Y + Z) + Y^{-1}W + (Y^{-1} + Z^{-1})U + (W^{-1} + U^{-1})V + e^tV^{-1},
\]

(B.9)

\[
\hat{P} = 1
\]

(B.10)

\[
\hat{O}_\square = X
\]

(B.11)

\[
\hat{O}_\square = Y, \quad \hat{O}_\bigcirc = Z
\]

(B.12)

\[
\hat{O}_\square = W, \quad \hat{O}_\bigcirc = (1 + YZ^{-1})U
\]

(B.13)

\[
\hat{O}_\square = XU, \quad \hat{O}_\bigcirc = (1 + WU^{-1})V
\]

(B.14)

\[
\hat{O}_\square = (1 + UW^{-1})XV
\]

(B.15)

\[
\hat{O}_\square = YV.
\]

(B.16)

General $Gr(k, N)$

In the above two examples, we observe a certain relation between the fundamental matrix and the Lax operator. We shall call a Young diagram *slim* when it has at most one row and one column. We assign a variable $X_Y$ to each slim diagram $Y$, and we put $X_{\text{empty}} = 1$. Then, in the above two examples we find

\[
L = \frac{1}{N} \sum_{\text{slim diagrams}} X_Y^{-1} M_{Y'} X_{Y'}.
\]

(B.17)

We conjecture that this generally holds. In order to express it more explicitly, we denote by $[a, b]$ the slim diagram of height $a$ and width $b$:

\[
[a, b] = a \begin{array}{c}
\bigcirc \\
\ldots \\
\bigcirc
\end{array}
\]

(B.18)
First, we determine the minor \( M_Y^{Y'} \). By the selection rule,
\[
\langle O_Y O^{Y'} \rangle_d \neq 0 \implies Nd + k(N - k) - 1 = |Y| + k(N - k) - |Y'|,
\]
where \(|Y|\) is the number of boxes of \( Y \). Since \(|Y| \leq N - 1 \) for slim diagrams, we see that the degree must be 0 or 1 and \( d = 1 \) occurs only for \( Y = [N - k, k] \), and \( Y' = [0, 0] \). Let us consider the latter case of degree-1 instanton contribution. The Poincaré dual of \( O^{Y'} \) is a point \( p \) and that of \( O_Y \) is represented by the closure of a Schubert cell \( C_Y \subset Gr(k, N) \). We choose as the point \( p \) the subspace spanned by \( e_{N-k+1}, \ldots, e_N \), where \( e_1, \ldots, e_N \) is the standard base of \( \mathbb{C}^N \). We choose as the cell \( C_Y \) the \( B_+ \)-orbit of the point corresponding to the subspace spanned by \( e_1, e_{N-k+1}, \ldots, e_{N-1} \), where \( B_+ \) is the subgroup of \( GL(N, \mathbb{C}) \) consisting of upper triangular matrices. A degree one map \( \mathbb{CP}^1 \to Gr(k, N) \) is given by a family \((s : t) \mapsto W(s : t)\) of subspaces of the form
\[
W(s : t) = \{ v_0 s + v_1 t, v_2, \ldots, v_k \}_{\mathbb{C}},
\]
where \( v_0, \ldots, v_k \) are linearly independent elements of \( \mathbb{C}^N \). Here we consider \((s, t)\) as the homogeneous coordinates of \( \mathbb{CP}^1 \). Let \((0 : 1)\) and \((1 : 0)\) be the insertion point of \( O^{Y'} \) and \( O_Y \) respectively. We count the number of families (up to the automorphism \( C^* \) of \( \mathbb{CP}^1 - \{(0 : 1), (1 : 0)\}\)) such that \( W(0 : 1) = p \) and \( W(1 : 0) \in \overline{C_Y} \). The requirements are restated as
\[
\begin{align*}
\{v_1, v_2, \ldots, v_k\}_{\mathbb{C}} &= \{e_{N-k+1}, \ldots, e_N\}_{\mathbb{C}}, \\
e_1 \in \{v_0, v_2, \ldots, v_k\}_{\mathbb{C}} &\subset \{e_1, \ldots, e_{N-1}\}_{\mathbb{C}}.
\end{align*}
\]
Then, we find
\[
W(s : t) = \{ e_1 cs + e_N t, e_{N-k+1}, \ldots, e_{N-1} \}_{\mathbb{C}},
\]
where \( c \in C^* \). Thus, the number is one up to the \( C^* \) action:
\[
\langle O_Y O^{Y'} \rangle_1 = 1, \quad Y = [N - k, k], \quad Y' = [0, 0]
\]

The classical part of the cohomology ring is well-known. Finally, we can express the conjectured form \((B.17)\) for the Lax operator:
\[
L = X_{[1,1]} + \sum_{1 \leq a \leq N-k \atop 1 \leq b \leq k} X_{[a,b]}^{-1} (X_{[a+1,b]} + X_{[a,b+1]}) + e^t X_{[N-k,k]}^{-1},
\]
where we put \( X_{[a,b]} = 0 \) if \( a > N - k \) or \( b > k \). Note that the number of variables or the number of non-empty slim diagrams is \((N - k) \times k\), which is the same as the dimension of the original target space \( Gr(k, N) \).

### B.2 Landau-Ginzburg Description

Finally, let us discuss the LG description in the simplest case \( Gr(2, 4) \). The equation determining the vacua \( X \partial_X L = Y \partial_Y L = Z \partial_Z L = W \partial_W L = 0 \) has four solutions
\[
Y = Z = \frac{1}{2} X^2, \quad W = \frac{1}{4} X^3, \quad X^4 = 4e^t,
\]
each having non-vanishing Hessian $8e^t$. Four is less than the index $\chi(Gr(2,4)) = 6$, and there are vacua at infinity: $X \to 0$ keeping $XW = Y^2$ and $Z + Y = 0$. Namely, the LG model based on the algebraic torus $(\mathbb{C}^*)^4 = \{(X,Y,Z,W)\}$ with superpotential $L$ and holomorphic form $\Omega = \frac{dX}{X} \wedge \frac{dY}{Y} \wedge \frac{dZ}{Z} \wedge \frac{dW}{W}$ has a disease and can not be considered as a sound description of the original system. To obtain a good one, we must partially compactify the torus so that $L$ and $\Omega$ are extended and there are six vacua in total. We make the change of variables $X, Y, Z, W \to X, Y, \zeta, \tilde{W}$ in a neighborhood of the “vacua at infinity” $X = Y + Z = W^{-1} = 0$:

$$W = X^{-1}\tilde{W}, \quad Z + Y = -XY\zeta,$$

and include the points with $X = 0$. Then, we obtain a new expression

$$L = X - Y\zeta + Y^{-1}(1 + X\zeta)^{-1}\zeta\tilde{W} + e^tX\tilde{W}^{-1},$$

and

$$\Omega = \frac{dX}{X} \wedge \frac{dY}{Y} \wedge \frac{d\zeta}{1 + X\zeta} \wedge \frac{d\tilde{W}}{\tilde{W}}.$$  

Solving the equation $\partial_X L = Y\partial_Y L = \partial_\zeta L = \tilde{W}\partial_{\tilde{W}} L = 0$, we find two additional vacua

$$X = \zeta = 0, \quad Y^2 = \tilde{W} = -e^t,$$

each having non-vanishing Hessian $-4e^t$. In total there are $4 + 2 = 6$ vacua and there are no more at infinity. Moreover, it is easy to see that the three point functions of the original model coincide with those of this LG model under the identification of fields (B.3)-(B.7).

In summary, we have found a sound LG description of the $Gr(2,4)$ model based on a partial compactification of $(\mathbb{C}^*)^4$. This is the first example in which the mirror partner of a Fano manifold is not just an algebraic torus.
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