Algebraic Bethe ansatz for the six vertex model with upper triangular $K$-matrices

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Abstract

We consider a formulation of the algebraic Bethe ansatz for the six vertex model with non-diagonal open boundaries. Specifically, we study the case where both left and right $K$-matrices have an upper triangular form. We show that the main difficulty entailed by those forms of the $K$-matrices is the construction of the excited states. However, it is possible to treat this problem with the aid of an auxiliary transfer matrix and by means of a generalized creation operator.

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1. Introduction

The introduction of non-periodic boundary conditions into the framework of the quantum inverse scattering method was performed by Sklyanin [1] who resorted to a new object called the $K$-matrix, which satisfies the reflection equations, according to Cherednik’s work [2]. Together with the $R$-matrix, the solution of the Yang–Baxter equation, the $K$-matrix can be used to construct families of commuting transfer matrices.

Although the reflection equations admit general solutions, the algebraic Bethe ansatz (ABA) has been applied directly only for diagonal $K$-matrices. In fact, even multi-state vertex models with diagonal boundaries have since been solved by the ABA [1]. See, for instance, the papers [3–7] and references therein.

In the cases with non-diagonal boundaries, one of the impediments to addressing the ABA is the absence of a simple reference state. For this reason, alternative methods which do not rely on the existence of a reference state have been used frequently. For instance, in [8], exploring functional relations satisfied by the transfer matrix, Nepomechie was able to derive the eigenvalues of the open XXZ chain for special values of the bulk anisotropy, though with a non-standard form of the respective Bethe equations. In subsequent works [9–11], conventional Bethe equations were restored and many situations with constrained and general values of the boundary parameters were studied. See also the recent developments to this approach in [12, 13]. Other methods include the separation of variables [14, 15], the direct use of the Yang–Baxter algebra to derive the eigenvalues [16] and the representation theory of the
so-called $q$-Onsager algebra [17]. One disadvantage of the functional methods is the lack of information in general regarding the eigenvectors of the transfer matrix.

Moreover, results obtained from the ABA method have also been reported in the literature [18–20]. In [18] and its generalization to the spin-$s$ case [19], convenient local gauge transformations were used in order to find a reference state as well as to transform the left and right $K$-matrices into diagonal and upper triangular matrices, respectively. In [20], transformations in both auxiliary and quantum spaces also map the original XXX-$s$ spin chain with two full $K$-matrices into one with one diagonal and one triangular $K$-matrix. A common feature present in these works is the requirement of constraints in the boundary parameters.

Recently, the rational six vertex model with two upper triangular boundaries was solved by Belliard et al [21]. In this case, the excited states do not have a fixed number of magnons and thus the usual ABA does not apply, although the usual reference state is still an eigenvector. Similar settings were first considered in the coordinate Bethe ansatz setup [22] and also in the vertex operator approach [23].

The purpose of this work is to present a constructive approach to obtain generalized excited states, independent of coordinate Bethe ansatz outcomes. The form of the generalized vertex operator approach [23].

This paper is organized as follows. In section 2 we summarize the definitions and relations needed for the ABA method. Next, in section 3, we apply the ABA presenting detailed calculations for the first, second and third generalized excited states, as well as the $n$th generalized Bethe vector. We discuss our results in section 4. We reserve the appendices for useful relations used in the main text.

2. Transfer matrix

The Sklyanin monodromy and transfer matrix for an open vertex model are defined by

$$T_a(u) = K_a^{+}(u)T_a(u)K_a^{-}(u)T_a^{-1}(-u),$$

$$t(u) = \text{Tr}_a[T_a(u)],$$

where $K_a^{\pm}(u)$ are the reflection matrices and $T_a(u)$ and $T_a^{-1}(-u)$ are the monodromy matrices associated with a chain of length $L$ given by an ordered product of $R$-matrices

$$T(u) = R_{a1}(u) \ldots R_{aL}(u), \quad T^{-1}(-u) = R_{aL}(u) \ldots R_{a1}(u).$$

The products in (2.1) and (2.2) are performed in an auxiliary space denoted by $a$, and $n = 1, 2, \ldots, L$ refers to a quantum vector space at the site $n$.

The Yang–Baxter equation

$$R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v),$$

and the reflection equations

$$R_{12}(u - v)K_1^{+}(u)R_{12}(u + v)K_1^{-}(v) = K_2^{+}(v)R_{12}(u + v)K_1^{-}(u)R_{12}(u - v),$$

$$R_{12}(v - u)K_2^{+}(u)vR_{12}(-u - v - 2\eta)K_2^{+}(v)f^2 = K_2^{+}(v)f^2R_{12}(-u - v - 2\eta)K_1^{+}(u)vR_{12}(v - u)$$

guarantee that (2.1) commutes for arbitrary spectral parameters, i.e., $[t(u), t(v)] = 0$.

At least two global relations also follow from (2.3) to (2.5) for the monodromy matrices, namely

$$\hat{R}(u - v)T(u) \otimes T(v) \equiv T(v) \otimes T(u)\hat{R}(u - v)$$

(2.6)
and

\[ R_{12}(u - v)U_1(u)R_{12}(u + v)U_2(v) = U_2(v)R_{12}(u + v)U_1(u)R_{12}(u - v), \]  

(2.7) where \( R(u) = PR(u) \) and \( U_\alpha(u) = T_\alpha(u)K_\alpha(u)T_\alpha^{-1}(-u). \)

For the six vertex model the \( R \)-matrix has the form

\[
R(u) = \begin{pmatrix}
1 & b(u) & c(u) \\
& c(u) & b(u) \\
& & 1
\end{pmatrix},
\]

(2.8) while the upper triangular \( K \)-matrices \([24, 25]\) can be written as

\[
K^-(u) = \begin{pmatrix}
k_{11}^-(u) & k_{12}^-(u) \\
0 & k_{22}^-(u)
\end{pmatrix}, \quad K^+(u) = \begin{pmatrix}
k_{11}^+(u) & k_{12}^+(u) \\
0 & k_{22}^+(u)
\end{pmatrix}
\]

(2.9) where

\[
b(u) = \frac{\sinh(u)}{\sinh(u + \eta)}, \quad c(u) = \frac{\sinh(\eta)}{\sinh(u + \eta)},
\]

(2.10) and

\[
k_{11}^-(u) = \sinh(u + \xi_-), \quad k_{12}^-(u) = \beta_- \sinh(2u),
\]

\[
k_{22}^-(u) = \sinh(\xi_- - u),
\]

\[
k_{11}^+(u) = \sinh(-u - \eta + \xi_+), \quad k_{12}^+(u) = \beta_+ \sinh(-2u - 2\eta),
\]

\[
k_{22}^+(u) = \sinh(u + \eta + \xi_+).
\]

(2.11) In the rational limit, these functions reduce to

\[
b(u) = \frac{u}{u + \eta}, \quad c(u) = \frac{\eta}{u + \eta},
\]

(2.12) and

\[
k_{11}^-(u) = u + \xi_-, \quad k_{12}^-(u) = 2\beta_- u,
\]

\[
k_{22}^-(u) = \xi_- - u,
\]

\[
k_{11}^+(u) = -u - \eta + \xi_+, \quad k_{12}^+(u) = -2\beta_+ (u + \eta),
\]

\[
k_{22}^+(u) = u + \eta + \xi_+.
\]

(2.13) In addition to the spectral parameter \( u \) we have \( \eta \) which parametrizes the anisotropy, and \( \xi_\pm \) and \( \beta_\pm \) are the four free parameters characterizing the boundaries. By taking the first derivative of the transfer matrix \((3.1)\) we can obtain the corresponding XXZ Hamiltonian with non-diagonal boundary terms \([1, 24]\),

\[
H = \sum_{n=1}^{L-1} \left[ \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \cosh(\eta) \sigma_n^z \sigma_{n+1}^z \right] - \frac{\sinh(\eta)}{\sinh(\xi_+)} \left[ \beta_+ (\sigma_1^x + i\sigma_1^y) + \cosh(\xi_+) \sigma_1^z \right]
\]

\[
+ \frac{\sinh(\eta)}{\sinh(\xi_-)} \left[ \beta_- (\sigma_L^x + i\sigma_L^y) + \cosh(\xi_-) \sigma_L^z \right],
\]

(2.14) where \( \sigma_n^{x,y,z} \) are the standard Pauli matrices acting on the site \( n \).
The monodromy matrix \( U_a(u) = T_a(u)K_0^-(u)T_a^-(u) \) can be represented by a \( 2 \times 2 \) matrix

\[
U_a(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},
\]

(2.15)

where \( A(u), B(u), C(u) \) and \( D(u) \) are operators on the Hilbert space \( \bigotimes_{i=1}^L C^2 \). These operators satisfy commutation relations thanks to (2.7). The four relevant relations for this work are

\[
B(u)B(v) = B(v)B(u),
\]

(2.16)

\[
A(u)B(v) = a_1(u, v)B(v)A(u) + a_2(u, v)B(u)A(v) + a_3(u, v)B(u)\tilde{D}(v),
\]

(2.17)

\[
\tilde{D}(u)B(v) = b_1(u, v)B(v)\tilde{D}(u) + b_2(u, v)B(u)\tilde{D}(v) + b_3(u, v)B(u)A(v),
\]

(2.18)

\[
C(u)B(v) = c_1(u, v)B(v)C(u) + c_2(u, v)A(v)A(u) + c_3(u, v)A(u)A(v) + c_4(u, v)A(v)\tilde{D}(u) + c_5(u, v)A(u)\tilde{D}(v) + c_6(u, v)\tilde{D}(u)A(v) + c_7(u, v)\tilde{D}(u)\tilde{D}(v),
\]

(2.19)

where we have used (2.6) to define \( \tilde{D}(u) = D(u) - f(u)A(u) \) with \( f(u) = c(2u) \). The explicit expressions of the coefficients \( a_j(u, v), b_j(u, v) \) and \( c_j(u, v) \) are given in appendix A.

Next, we use these relations in order to diagonalize the transfer matrix (2.1).

3. Bethe ansatz analysis

Taking into account the representation (2.15) as well as the upper triangular form of the left boundary matrix (2.9), the transfer matrix has the form

\[
t(u) = k_{11}^+(u)A(u) + k_{22}^+(u)D(u) + k_{12}^+(u)C(u)
\]

\[
= \omega_1(u)A(u) + \omega_2(u)\tilde{D}(u) + k_{12}^+(u)C(u),
\]

(3.1)

where

\[
\omega_1(u) = k_{11}^+(u) + f(u)k_{22}^+(u),
\]

\[
\omega_2(u) = k_{22}^+(u).
\]

(3.2)

The presence of the annihilation operator \( C(u) \) in (3.1) hinders the usual task of finding its eigenvectors. In fact, the excited states of the periodic or diagonal boundary models are usually constructed by applying the \( B \)-operators to the reference state as a consequence of the commutativity of the transfer matrix and the total spin operator \( \sum_{j=1}^L S_z^j \). For the upper triangular \( K \)-matrices this is no longer true and the excited states have to be constructed in another way.

3.1. The reference state

An important step in the ABA technique is the choice of a reference state from which the excited states are constructed. It turns out that the state

\[
\Psi_0 = \left( \frac{1}{(s_1 \rangle} \right) \otimes \left( \frac{1}{(s_2 \rangle} \right) \otimes \cdots \otimes \left( \frac{1}{(s_L \rangle} \right)
\]

(3.3)
is an eigenstate of \((3.1)\). This is a consequence of the structure of the \(U_0(u)\) matrix elements when the \(K^{-}\) matrix has the form \((2.9)\). In fact, with the help of \((2.6)\), it is not difficult to calculate

\[
\mathcal{A}(u) \Psi_0 = \Delta_1(u) \Psi_0, \quad \tilde{\mathcal{D}}(u) \Psi_0 = \Delta_2(u) \Psi_0, \quad \mathcal{C}(u) \Psi_0 = 0, \quad \mathcal{B}(u) \Psi_0 = *,
\]

where \(*\) denotes a state different from 0 and \(\Psi_0\),

\[
\Delta_1(u) = k_{11}(u),
\]

\[
\Delta_2(u) = [k_{22}(u) - f(u)k_{11}(u)]b(u)^{2L}.
\]

Therefore, we have an eigenvalue problem,

\[
t(u) \Psi_0 = \Lambda_0(u) \Psi_0
\]

where

\[
\Lambda_0(u) = \omega_1(u) \Delta_1(u) + \omega_2(u) \Delta_2(u)
\]

is the corresponding eigenvalue.

### 3.2. The first excited state

In order to construct the first excited state, we introduce an auxiliary transfer matrix given by

\[
\tilde{t}(u) = \omega_1(u) \mathcal{A}(u) + \omega_2(u) \tilde{\mathcal{D}}(u).
\]

We observe that \(t(u)\) and \(\tilde{t}(u)\) share the same reference state, i.e., \(t(u) \Psi_0 = \tilde{t}(u) \Psi_0 = \Lambda_0(u) \Psi_0\).

The one-particle state of the auxiliary transfer matrix \(\tilde{t}(u)\) can be obtained as usual

\[
\Psi_1(u_1) = \mathcal{B}(u_1) \Psi_0,
\]

and the action of \(\tilde{t}(u)\) on the state \((3.9)\), using the relations \((B.1)\) and \((B.2)\) of appendix B, is given by

\[
\tilde{t}(u) \Psi_1(u_1) = \Lambda_1(u, u_1) \Psi_1(u_1) + [\omega_1(u) \tilde{F}_1(u, u_1) + \omega_2(u) \tilde{G}_1(u, u_1)] \mathcal{B}(u) \Psi_0,
\]

where

\[
\Lambda_1(u, u_1) = \omega_1(u) \Delta_1(u) a_1(u, u_1) + \omega_2(u) \Delta_2(u) b_1(u, u_1).
\]

The form of the upper element of \(T_0(u)\), namely \(k_{11}^{+}(u) \mathcal{B}(u) + k_{12}^{+}(u) \tilde{\mathcal{D}}(u)\), suggests that the first excited state should contain two terms: one is the usual, \(\mathcal{B}(u_1) \Psi_0\), while the other one comes from a diagonal operator acting on \(\Psi_0\). We propose accordingly the following first excited state\(^1\) for \(t(u)\)

\[
\Phi_1(u_1) = \Psi_1(u_1) + g(u_1) \Psi_0
\]

where the function \(g(u_1)\) is to be fixed.

By acting the transfer matrix on \(\Phi_1(u_1)\), we find

\[
t(u) \Phi_1(u_1) = \tilde{t}(u) \Psi_1(u_1) + g(u_1) \tilde{t}(u) \Psi_0 + k_{12}^{+}(u) \mathcal{C}(u) \Psi_1(u_1).
\]

Now, we use the expression \((3.10)\) and also \((B.3)\) to obtain

\[
t(u) \Phi_1(u_1) = \Lambda_1(u, u_1) \Phi_1(u_1) + [\omega_1(u) \tilde{F}_1(u, u_1) + \omega_2(u) \tilde{G}_1(u, u_1)] \mathcal{B}(u) \Psi_0
\]

\[
+ [g(u_1)] \Lambda_0(u) \Phi_1(u_1) + k_{12}^{+}(u) \tilde{H}_1(u, u_1) \Psi_0.
\]

\(^1\) We remark that we use the nomenclature ‘excited states’ for the eigenvectors of \(t(u)\) to distinguish them from the ‘particle states’ of \(\tilde{t}(u)\), which have a fixed number of magnons.
Besides the usual unwanted term found for the auxiliary transfer matrix (3.10) there is an additional state in (3.14). Setting the coefficient of $B(u)\Psi_0$ equal to zero we obtain
\begin{equation}
\frac{\Delta_1(u_1)}{\Delta_2(u_1)} = -a_3(u, u_1)\omega_1(u) + b_2(u, u_1)\omega_2(u) - a_2(u, u_1)\omega_1(u) + b_3(u, u_1)\omega_2(u). \tag{3.15}
\end{equation}

One can verify that the right-hand side of (3.15) depends only on $u_1$ \cite{1}. Thus, we find the Bethe equation for the first excited state in the form
\begin{equation}
\frac{\Delta_1(u_1)}{\Delta_2(u_1)} = -\Theta(u_1), \tag{3.16}
\end{equation}

where
\begin{equation}
\Theta(u_1) = \frac{\sinh(2u_1 + \eta)\sinh(u_1 + \eta + \xi_+)}{\sinh(2u_1)\sinh(u_1 - \xi_+)}. \tag{3.17}
\end{equation}

On the other hand, the unwanted term proportional to the reference state is used to extract the expression for $g(u_1)$
\begin{equation}
g(u_1) = \frac{k_{12}^+(u_1)H_1(u, u_1)}{\Lambda_1(u, u_1) - \Lambda_0(u)}. \tag{3.18}
\end{equation}

At a first glance, the right-hand side of (3.18) is also dependent on the spectral variable $u$. However, if we take into account the Bethe equation (3.16), the identities between the coefficients (A.1)–(A.3) and the following relation for the $K$-matrix elements,
\begin{equation}
\frac{1}{k_{12}^{-1}(u_1)[a_2(u, u_1)\omega_1(u) + b_3(u, u_1)\omega_2(u)]} = \frac{1 - a_1(u, u_1)}{a_3(u, u_1)[c_2(u, u_1) + c_3(u, u_1)] - a_2(u, u_1)c_3(u, u_1)} \tag{3.19}
\end{equation}

the expression (3.18) acquires a simple form
\begin{equation}
g(u_1) = \Delta_2(u_1) \left[ k_{11}^+(u_1) \right] \left[ k_{12}^+(u_1) + f(u_1)k_{22}^+(u_1) \right] \tag{3.20}
\end{equation}

which depends only on $u_1$, as expected. The equations (3.16)–(3.20) ensure that $\Phi_1(u_1)$ is an eigenstate of $\bar{t}(u)$ with eigenvalue (3.11).

### 3.3. The second excited state

We proceed in a similar way for the second excited state. First, we consider the two-particle state of the auxiliary transfer matrix
\begin{equation}
\Psi_2(u_1, u_2) = B(u_1)B(u_2)\Psi_0, \tag{3.21}
\end{equation}
in order to get
\begin{equation}
\bar{t}(u)\Psi_2(u_1, u_2) = \Lambda_2(u, u_1, u_2)\Psi_2(u_1, u_2) + [\omega_1(u)F_2(u, u_1, u_2) + \omega_2(u)G_2(u, u_1, u_2)]B(u)\Psi_0 + [\omega_1(u)F_1(u, u_1, u_2) + \omega_2(u)G_1(u, u_1, u_2)]B(u)B(u_2)\Psi_0 \tag{3.22}
\end{equation}

where
\begin{equation}
\Lambda_2(u, u_1, u_2) = \omega_1(u)\Delta_1(u) \prod_{j=1}^{2} a_1(u, u_j) + \omega_2(u)\Delta_2(u) \prod_{j=1}^{2} b_1(u, u_j). \tag{3.23}
\end{equation}

The ansatz for the full second excited state is guessed from the action of $k_{11}^+(u)B(u) + k_{12}^+(u)D(u)$ on $\Psi_0$ twice
\[ \Phi_2(u_1, u_2) = \Psi_2(u_1, u_2) + g_2^{(1)}(u_1, u_2)\Psi_1(u_1) + g_1^{(1)}(u_1, u_2)\Psi_1(u_2) + g_{12}^{(0)}(u_1, u_2)\Psi_0 \]  

(3.24)

with the coefficients \( g_1^{(1)}(u_1, u_2) \) and \( g_{12}^{(0)}(u_1, u_2) \) to be determined \textit{a posteriori}.

The action of \( t(u) \) on the state (3.24), gathering the previous results (3.22), generates many unwanted terms

\[
\begin{align*}
t(u)\Phi_2(u_1, u_2) &= \Lambda_2(u_1, u_2)\Phi_2(u_1, u_2) \\
&+ \left[\omega_1(u)F_2(u_1, u_2) + \omega_2(u)G_2(u_1, u_2)\right]B(u)\Psi_1(u_1) \\
&+ \left[\omega_1(u)F_1(u_1, u_2) + \omega_2(u)G_1(u_1, u_2)\right]B(u)\Psi_1(u_2) \\
&+ \left[g_2^{(1)}(u_1, u_2)\omega_1(u)\right]F_1(u_1, u_2) + \omega_2(u)G_1(u_1, u_2)\right]B(u)\Psi_0 \\
&+ \left[g_2^{(1)}(u_1, u_2)\right]B(u)\Psi_0 \\
&+ \left[g_1^{(1)}(u_1, u_2)\Lambda_1(u) - \Lambda_2(u_1, u_2)\right] + k_{12}^{(1)}(u)H_2(u, u_1, u_2)\Psi_1(u_1) \\
&+ \left[g_1^{(1)}(u_1, u_2)\Lambda_0(u) - \Lambda_2(u_1, u_2)\right] + k_{12}^{(0)}(u)H_1(u, u_2)\Psi_0 \\
&+ k_{12}^{(0)}(u)\left[g_1^{(1)}(u_1, u_2)H_1(u, u_1) + s_{12}^{(1)}(u_1, u_2)H_1(u, u_2)\right]\Psi_0. \\
\end{align*}
\]

(3.25)

The coefficients of \( B(u)\Psi_1(u_1) \) and \( B(u)\Psi_1(u_2) \) lead to the Bethe equations,

\[
\frac{\Delta_0(u_1, u_2)}{\Delta_2(u_1)} = -\Theta(u_1) \frac{b_1(u_1, u_2)}{a_1(u_1, u_2)}, \quad \frac{\Delta_1(u_2)}{\Delta_2(u_2)} = -\Theta(u_2) \frac{b_1(u_2, u_1)}{a_1(u_2, u_1)},
\]

(3.26)

while the coefficients of \( \Psi_1(u_1) \), \( \Psi_1(u_2) \) and \( \Psi_0 \) give us the expressions for \( g_{12}^{(1)}(u_1, u_2) \) and \( g_{12}^{(0)}(u_1, u_2) \). Taking into account the Bethe equations (3.26) we get

\[
\begin{align*}
g_1^{(1)}(u_1, u_2) &= g(u_1)p(u_2, u_1), \\
g_2^{(1)}(u_1, u_2) &= g(u_2)p(u_1, u_2), \\
g_{12}^{(0)}(u_1, u_2) &= g(u_1)g(u_2)q(u_1, u_2)
\end{align*}
\]

(3.27)

where \( g(u_1) \) is given by (3.20) and we have introduced two new functions, namely

\[
p(u, v) = b_1(u, v)\frac{a_1(u, v)}{a_1(v, u)}, \quad q(u, v) = \frac{b_1(u, v)}{a_1(u, v)}.
\]

(3.28)

We can check by direct computation that the other unwanted terms in (3.22) are automatically null if we take into account the Bethe equations (3.26) and the expressions (3.27). Thus, the state

\[
\Phi_2(u_1, u_2) = \Psi_2(u_1, u_2) + g(u_2)p(u_1, u_2)\Psi_1(u_1) + g(u_1)p(u_2, u_1)\Psi_1(u_2) + g(u_1)g(u_2)q(u_1, u_2)\Psi_0
\]

(3.29)

is an eigenstate of the transfer matrix with energy (3.23).

### 3.4. The third excited state

It is expected from the integrability of the model that the second excited state structure should allow the generalization to find the \( n \)th excited structure. Nevertheless, it was not sufficient to guess the \( n \)th excited state from (3.29). Thus, we also proceed to the third excited state.
Following the previous discussions we propose the following structure for the third excited state
\[
\Phi_3(u_1, u_2, u_3) = \Psi_3(u_1, u_2, u_3) + g_3^{(2)}(u_1, u_2, u_3)\Psi_2(u_1, u_2) + g_3^{(1)}(u_1, u_2, u_3)\Psi_1(u_1) + g_1^{(0)}(u_1, u_2, u_3)\Psi_0
\]
where the coefficients \(g^{(k)}(u_1, u_2, u_3)\) will be determined in what follows.

As before we first apply the auxiliary transfer matrix \(\tilde{t}(u)\) to the three-particle state
\[
\mathcal{B}(u_3)\mathcal{B}(u_2)\mathcal{B}(u_1)\Psi_0,
\]
and, as a result, we obtain
\[
\tilde{t}(u)\Psi_3(u_1, u_2, u_3) = \Lambda_3(u_1, u_2, u_3)\Psi_3(u_1, u_2, u_3)
\]
where \(\Lambda_3(u_1, u_2, u_3)\) is given by
\[
\Lambda_3(u_1, u_2, u_3) = \omega_3(u_1, u_2, u_3) = \frac{\Delta_1(u_3)}{\Delta_2(u_3)} = -\Theta(u_3) \prod_{j=1}^{3} \frac{b_j(u_k, u_j)}{a_j(u_k, u_j)},
\]
with
\[
\Delta_1(u_3) = -\Theta(u_3) \prod_{j=1}^{3} \frac{b_j(u_k, u_j)}{a_j(u_k, u_j)},
\]
and
\[
\Delta_2(u_3) = \prod_{j=1}^{3} \frac{b_j(u_k, u_j)}{a_j(u_k, u_j)}.
\]

The next step consists in the determination of \(\tilde{t}(u)\Phi_3(u_1, u_2, u_3)\). We have a proliferation of cumbersome unwanted terms in this case and for this reason we omit their expressions. Following the last subsections, we impose the vanishing of the unwanted terms to fix relations for the unknowns \(g^{(k)}(u_1, u_2, u_3)\) as well as to obtain the Bethe equations. Three of the unwanted terms in \(\tilde{t}(u)\Phi_3(u_1, u_2, u_3)\) coincide with those in (3.32) and lead to the Bethe equations
\[
\frac{\Delta_1(u_k)}{\Delta_2(u_k)} = -\Theta(u_k) \prod_{j=1, j\neq k}^{3} \frac{b_j(u_k, u_j)}{a_j(u_k, u_j)},
\]
and
\[
\Delta_1(u_3) = -\Theta(u_3) \prod_{j=1}^{3} \frac{b_j(u_k, u_j)}{a_j(u_k, u_j)}.
\]

The \(g^{(k)}(u_1, u_2, u_3)\) are obtained from the coefficients of \(\Psi_2(u_i, u_j)\) \((i < j)\), \(\Psi_1(u_i)\) \((i = 1, 2, 3)\) and \(\Psi_0\). Once again, taking into account the Bethe equations (3.34), we get
\[
g_3^{(2)}(u_1, u_2, u_3) = g(u_3)p(u_1, u_3)p(u_2, u_3)
\]
\[
g_3^{(1)}(u_1, u_2, u_3) = g(u_2)p(u_1, u_2)p(u_3, u_2)
\]
\[
g_3^{(0)}(u_1, u_2, u_3) = g(u_1)p(u_2, u_1)p(u_3, u_1)
\]

as the factors of the three \(\Psi_2\),
\[
g_3^{(1)}(u_1, u_2, u_3) = g(u_3)p(u_1, u_3)p(u_2, u_3)q(u_2, u_3)
\]
\[
g_3^{(1)}(u_1, u_2, u_3) = g(u_1)p(u_2, u_1)p(u_3, u_1)q(u_1, u_3)
\]
\[
g_3^{(1)}(u_1, u_2, u_3) = g(u_2)p(u_1, u_2)p(u_3, u_2)q(u_1, u_2)
\]

as the factors of the three \(\Psi_1\), and
\[
g_3^{(0)}(u_1, u_2, u_3) = g(u_1)g(u_2)g(u_3)q(u_1, u_2)q(u_1, u_3)q(u_2, u_3)
\]

as the factor of \(\Psi_0\). Here we notice that the expressions for \(g^{(k)}(u_1, u_2, u_3)\) are factorized in terms of the previously defined functions \(g(u), p(u, v)\) and \(q(u, v)\).

Therefore, (3.30) is an eigenstate of \(\tilde{t}(u)\) with eigenvalue (3.33) provided equations (3.34) are valid.
3.5. The general excited state

The previous results allow us to write general expressions for the eigenvalue problem of the transfer matrix (3.1): the $n$th excited eigenstate is given by

$$\Phi_n(u_1, \ldots, u_n) = \Psi_n(u_1, \ldots, u_n) + \sum_{k=0}^{n-1} \sum_{\ell_1 < \cdots < \ell_{n-k} = 1} g^{(k)}_{\ell_1, \ldots, \ell_{n-k}}(u_1, \ldots, u_n) \times \Psi_k(u_1, \ldots, \hat{u}_{\ell_1}, \ldots, \hat{u}_{\ell_{n-k}}, \ldots, u_n),$$

(3.38)

where the functions $g^{(k)}_{\ell_1, \ldots, \ell_{n-k}}(u_1, \ldots, u_n)$ have the following expression

$$g^{(k)}_{\ell_1, \ldots, \ell_{n-k}}(u_1, \ldots, u_n) = \prod_{m \in \hat{\ell}} g(u_m) \prod_{m' \in \ell, m' < m} q(u_{m'}, u_m) \prod_{m'=1, m' \notin \hat{\ell}} p(u_{m'}, u_m)$$

(3.39)

with $\hat{\ell} = \{\ell_1, \ldots, \ell_{n-k}\}$ and the notation $\hat{u}_j$ indicates the absence of the rapidity $u_j$ in the function. The corresponding eigenvalue is given by

$$\Lambda_n(u, u_1, \ldots, u_n) = \omega_1(u) \Delta_1(u) \prod_{j=1}^{n} a_1(u, u_j) + \omega_2(u) \Delta_2(u) \prod_{j=1}^{n} b_1(u, u_j)$$

(3.40)

while the Bethe rapidities are constrained by

$$\frac{\Delta_1(u_k)}{\Delta_2(u_k)} = -\Theta(u_k) \prod_{j=1, j \neq k}^{n} \frac{b_1(u_k, u_j)}{a_1(u_k, u_j)},$$

(3.41)

where $k = 1, \ldots, n$.

4. Conclusion

We have solved the six vertex model for upper triangular reflection $K$-matrices by means of the algebraic Bethe ansatz. The eigenvalues and the Bethe equations are found to be independent of the upper boundary constants. However, the Bethe states are essentially different. In fact, the wavefunctions of the transfer matrix are a superposition of $2^n$ Bethe states of an auxiliary diagonal transfer matrix. This fact may indicate, for example, the existence of generalized solutions of the Knizhnik–Zamolodchikov equations, inspired by the semiclassical limit of our solution \[26, 27\].

Finally, we remark that our strategy to deal with transfer matrices possessing annihilation operators in their expression, rather than a particular boundary configuration of the six vertex model, may allow the management of the generic boundary case, for instance, attempting to extend the works \[18–20\]. Further directions of investigation include vertex models based on higher hank algebras, e.g., 15- or 19-vertex models.

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Appendix A. Coefficients of the commutation relations

The coefficients of the commutation relation (2.17) are given by

\[ a_1(u, v) = \frac{\sinh(u + v) \sinh(u - v - \eta)}{\sinh(u - v) \sinh(u + v + \eta)}, \quad a_2(u, v) = \frac{\sinh(2v) \sinh(\eta)}{\sinh(u - v) \sinh(2v + \eta)}, \quad a_3(u, v) = -\frac{\sinh(\eta)}{\sinh(u + v + \eta)}, \]

(A.1)

and the coefficients of (2.18) are

\[ b_1(u, v) = \frac{\sinh(u + v + \eta) \sinh(u + v + 2\eta)}{\sinh(u - v) \sinh(u + v + \eta)}, \quad b_2(u, v) = \frac{\sinh(\eta) \sinh(2(u + \eta))}{\sinh(v - u) \sinh(2u + \eta)}, \quad b_3(u, v) = \frac{\sinh(2v) \sinh(\eta) \sinh[2(u + \eta)]}{\sinh(2u + \eta) \sinh(2v + \eta) \sinh(u + v + \eta)}, \]

(A.2)

while the coefficients of (2.19) are given by

\[ c_1(u, v) = 1, \quad c_2(u, v) = \frac{\sinh(2u) \sinh(\eta) \sinh(u - v + \eta)}{\sinh(u - v) \sinh(2u + \eta) \sinh(u + v + \eta)}, \quad c_3(u, v) = \frac{\sinh(2u) \sinh(\eta)}{\sinh(v - u) \sinh(2u + \eta)}, \]

\[ c_4(u, v) = \frac{\sinh(u + v) \sinh(\eta)}{\sinh(u - v) \sinh(u + v + \eta)}, \quad c_5(u, v) = \frac{\sinh(2u) \sinh(\eta)}{\sinh(2u + \eta)}, \quad c_6(u, v) = -\frac{\sinh(\eta)}{\sinh(u + v + \eta)}, \quad c_7(u, v) = -\frac{\sinh(\eta)}{\sinh(u + v + \eta)}. \]

(A.3)

The respective rational coefficients are obtained by the substitution \( \sinh(x) \to x \) in the above expressions.

Appendix B. Reordered operators

An important point in the ABA analysis is to move the operators \( A(u), \tilde{D}(u) \) and \( C(u) \) over the product \( \prod_{j=1}^{n} B(u_j)\Psi_0 \) and then use (3.4). The repeated use of the commutation relations (2.16) to (2.19) allow us to write

\[
A(u) \prod_{j=1}^{n} B(u_j)\Psi_0 = \left[ \Delta_1(u) \prod_{j=1}^{n} a_1(u, u_j) \right] \prod_{j=1}^{n} B(u_j)\Psi_0 + \sum_{k=1}^{n} F_k(u, u_1, \ldots, u_n)B(u) \prod_{j=1,j\neq k}^{n} B(u_j)\Psi_0, \quad (B.1)
\]

\[
\tilde{D}(u) \prod_{j=1}^{n} B(u_j)\Psi_0 = \left[ \Delta_2(u) \prod_{j=1}^{n} b_1(u, u_j) \right] \prod_{j=1}^{n} B(u_j)\Psi_0 + \sum_{k=1}^{n} G_k(u, u_1, \ldots, u_n)B(u) \prod_{j=1,j\neq k}^{n} B(u_j)\Psi_0, \quad (B.2)
\]
\[ C(u) \prod_{j=1}^{n} B(u_j) \Psi_0 = \sum_{k=1}^{n} H_k(u, u_1, \ldots, u_n) \prod_{j=1, j \neq k}^{n} B(u_j) \Psi_0 \]

\[ + \sum_{\ell = 1}^{n} H_{\ell k}(u, u_1, \ldots, u_n) B(u) \prod_{j=1, j \neq \ell, k}^{n} B(u_j) \Psi_0, \]  

(B.3)

where

\[ F_k(u, u_1, \ldots, u_n) = \Delta_1(u_k) a_2(u, u_k) \prod_{\ell = 1, \ell \neq k}^{n} a_1(u, u_\ell) + \Delta_2(u_k) a_3(u, u_k) \prod_{\ell = 1, \ell \neq k}^{n} b_1(u_\ell, u_k), \]  

(B.4)

\[ G_k(u, u_1, \ldots, u_n) = \Delta_1(u_k) b_3(u, u_k) \prod_{\ell = 1, \ell \neq k}^{n} a_1(u_\ell, u_k) + \Delta_2(u_k) b_2(u, u_k) \prod_{\ell = 1, \ell \neq k}^{n} b_1(u_\ell, u_k), \]  

(B.5)

\[ H_k(u, u_1, \ldots, u_n) = \Delta_1(u_k) \Delta_1(u_k) [c_2(u, u_k) + c_3(u, u_k)] \prod_{\ell = 1, \ell \neq k}^{n} a_1(u, u_\ell) a_1(u_\ell, u_k) \]

\[ + \Delta_2(u_k) \Delta_1(u_k) [c_2(u, u_k) + c_3(u, u_k)] \prod_{\ell = 1, \ell \neq k}^{n} b_1(u_\ell, a_1(u_\ell, u_k)) \]

\[ + \Delta_1(u_k) \Delta_2(u_k) c_5(u, u_k) \prod_{\ell = 1, \ell \neq k}^{n} a_1(u, u_\ell) b_1(u_k, u_\ell) \]

\[ + \Delta_2(u_k) \Delta_2(u_k) c_7(u, u_k) \prod_{\ell = 1, \ell \neq k}^{n} b_1(u, u_\ell) b_1(u_k, u_\ell), \]  

(B.6)

\[ H_{\ell k}(u, u_1, \ldots, u_n) = \Delta_1(u_k) \Delta_1(u_\ell) \alpha_{11}(u, u_k, u_\ell) \prod_{m=1, m \neq \ell, k}^{n} a_1(u_k, u_m) a_1(u_\ell, u_m) \]

\[ + \Delta_1(u_k) \Delta_2(u_\ell) \alpha_{12}(u, u_k, u_\ell) \prod_{m=1, m \neq \ell, k}^{n} a_1(u_k, u_m) b_1(u_\ell, u_m) \]

\[ + \Delta_1(u_\ell) \Delta_2(u_k) \alpha_{21}(u, u_k, u_\ell) \prod_{m=1, m \neq \ell, k}^{n} a_1(u_\ell, u_m) b_1(u_k, u_m) \]

\[ + \Delta_2(u_k) \Delta_2(u_\ell) \alpha_{22}(u, u_k, u_\ell) \prod_{m=1, m \neq \ell, k}^{n} b_1(u_k, u_m) b_1(u_\ell, u_m), \]  

(B.7)

with

\[ \alpha_{11}(u, u_k, u_\ell) = a_2(u, u_\ell)[a_1(u_k, u_k) + c_3(u_k, u_k)] \]

\[ + b_3(u, u_\ell)[a_1(u_k, u_\ell) + c_3(u_k, u_k)] \]

\[ + a_2(u_k) [c_3(u_k, u_k) + c_5(u_k, u_k)] \]

\[ + b_3(u_k) [c_3(u_k, u_k) + c_7(u_k, u_k)], \]  

(B.8)

\[ \alpha_{12}(u, u_k, u_\ell) = a_3(u, u_\ell)[a_1(u_k, u_k) + c_3(u_k, u_k)] \]

\[ + b_2(u, u_\ell)[a_1(u_k, u_\ell) + c_3(u_k, u_k)] \]

\[ + a_2(u_k) [c_3(u_k, u_k) + c_5(u_k, u_k)] \]

\[ + b_3(u_k) [c_3(u_k, u_k) + c_7(u_k, u_k)], \]  

(B.9)
\[ \alpha_{21}(u, u_k, u_t) = c_5(u, u_k)[a_2(u, u_t)b_1(u_k, u_t) + a_3(u, u_k)b_5(u_k, u_t)] \\
+ c_7(u, u_k)[b_3(u, u_k)b_1(u_k, u_t) + b_2(u, u_k)b_3(u_k, u_t)] \\
+ a_2(u_k, u_t)[a_3(u, u_k)c_3(u, u_k) + 2(b(u, u_k)c_6(u, u_k))]. \tag{B.10} \]

\[ \alpha_{22}(u, u_k, u_t) = c_5(u, u_k)[a_3(u, u_t)b_1(u_k, u_t) + a_3(u, u_k)b_3(u_k, u_t)] \\
+ c_7(u, u_k)[b_2(u, u_k)b_1(u_k, u_t) + b_2(u, u_k)b_3(u_k, u_t)] \\
+ a_3(u_k, u_t)[a_3(u, u_k)c_3(u, u_k) + 2(b(u, u_k)c_6(u, u_k))]. \tag{B.11} \]

We note that for diagonal boundaries only the expressions (B.1) and (B.2) are necessary [1], while for the upper triangular \( K \)—matrices case we also need the more involved relation (B.3).

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