Difference $L$ operators and a Casorati determinant solution to the $T$-system for twisted quantum affine algebras

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Abstract

We propose factorized difference operators $L(u)$ associated with the twisted quantum affine algebras $U_q(A^{(2)}_{2n}), U_q(A^{(2)}_{2n-1}), U_q(D^{(2)}_{n+1}), U_q(D^{(3)}_4)$. These operators are shown to be annihilated by a screening operator. Based on a basis of the solutions of the difference equation $L(u)w(u) = 0$, we also construct a Casorati determinant solution to the $T$-system for $U_q(A^{(2)}_{2n}), U_q(A^{(2)}_{2n-1})$.

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1 Introduction

In [1], a class of functional relations, a $T$-system, was proposed for commuting transfer matrices of solvable lattice models associated to twisted quantum affine algebras $U_q(X_r^{(r)}_{N}) \ (r > 1)$. For $X_N^{(r)} = A^{(2)}_{N}$, it has the following form:

For $U_q(A^{(2)}_{2n})$ case:

\[
T^{(a)}_m(u-1)T^{(a)}_m(u+1) = T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u) + T^{(a-1)}_m(u)T^{(a+1)}_m(u)
\]

for $1 \leq a \leq n - 1,$ (1.1)

\[
T^{(n)}_m(u-1)T^{(n)}_m(u+1) = T^{(n)}_{m-1}(u)T^{(n)}_{m+1}(u) + T^{(n-1)}_m(u)T^{(n)}_m(u + \frac{\pi i}{2\hbar}).
\]
For $U_q(A_{2n-1}^{(2)})$ case:

$$T_m^{(a)}(u - 1)T_m^{(a)}(u + 1) = T_m^{(a)}(u)T_{m+1}^{(a)}(u) + T_m^{(a-1)}(u)T_{m+1}^{(a+1)}(u)$$

for $1 \leq a \leq n - 1$, \hspace{1cm} (1.2)

$$T_m^{(a)}(u - 1)T_m^{(a)}(u + 1) = T_{m-1}^{(a)}(u)T_m^{(a)}(u) + T_{m-1}^{(a-1)}(u)T_m^{(a+1)}(u)$$

Here $\{T_m^{(a)}(u)\}_{a \in I_a, m \in \mathbb{Z}_+; u \in \mathbb{C}}$ ($I_a = \{1, 2, \ldots, n\}$) are the transfer matrices with the auxiliary space labeled by $a$ and $m$. We shall adopt the boundary condition $T_1^{(a)}(u) = 0, T_0^{(a)}(u) = 1$, which is natural for the transfer matrices. This $T$-system (1.1), (1.2) is a kind of discrete Toda equation, which follows from a reduction of the Hirota-Miwa equation [2],[3]. The original $T$-system [1] contains a scalar function $g_T^{(a)}(u)$ in the second term of rhs of (1.1), (1.2). Throughout this paper, we set $g_T^{(a)}(u) = 1$. This corresponds to the case where the vacuum part is formally trivial. However, structure of the solution of (1.1), (1.2) is essentially independent of the function $g_T^{(a)}(u)$. In this paper, we briefly report on a new expression to the solution of (1.1), (1.2) motivated by the recently found interplay [4] between factorized difference operators and the $q$-characters for non-twisted quantum affine algebras [5, 6].

In section 2, we propose factorized difference operators $L(u)$ for $U_q(A_{2n}^{(2)})$, $U_q(A_{2n-1}^{(2)})$, $U_q(D_{n+1}^{(2)})$, $U_q(D_4^{(3)})$. $L(u)$ generates functions $\{T^{(a)}(u)\}_{a \in \mathbb{Z}, u \in \mathbb{C}}$ which are Laurent polynomials in variables $\{Y^{(a)}(u)\}_{a \in I_a, u \in \mathbb{C}}$. Moreover $Y^{(a)}(u)$ is expressed by a function $Q_a(u)$ which corresponds to the Baxter $Q$-function. When $Q_a(u)$ is suitably chosen in the context of the analytic Bethe ansatz [7, 8, 11, 11], $T^{(a)}(u)$ corresponds to an eigenvalue formula of the transfer matrix in the dressed vacuum form (DVF). In particular for $1 \leq a \leq b$ ($U_q(A_{2n}^{(2)})$, $U_q(A_{2n-1}^{(2)})$): $b = n$; $U_q(D_{n+1}^{(2)})$: $b = n - 1$; $U_q(D_4^{(3)})$: $b = 2$), the auxiliary space for this transfer matrix is expected [11] to be a finite dimensional irreducible module of the quantum affine algebra [10, 11], which is called the Kirillov-Reshetikhin module $W^{(a)}_1(u)$ (see also, section 5 in [12]). One of the intriguing properties of $L(u)$ is that $L(u)$ is annihilated by a screening operator $\{S_a\}_{a \in I_a}$, from which $(S_a \cdot T^{(a)})(u) = 0$ results. In the context of the analytic Bethe ansatz, this corresponds to the pole-freeness of $T^{(a)}(u)$ under the Bethe ansatz equation. For the non-twisted case $U_q(X_N^{(1)})$, one may identify $S_a$ with the Frenkel-Reshetikhin screening operator [5] if $Q_a(u)$ is suitably chosen.

For $U_q(A_N^{(2)})$ case, $L(u)$ becomes order of $N + 1$. By using a basis of the solutions of the difference equation $L(u)w(u) = 0$, in section 3, we give a solution (Theorem 3.5) of the $T$-system for $U_q(A^1_N)$ (1.1), (1.2) as a ratio of two Casorati determinants whose matrix size is constantly $(N + 1) \times (N + 1)$. On solving this $T$-system, a duality relation (Proposition 2.7)
plays an important role. There is another expression of the solution to the $U_q(A^{(2)}_N)$ $T$-system \([14, 12]\) which is described by semi-standard tableaux with rectangular shape \([1]\). This solution follows from a reduction of the Bazhanov and Reshetikhin’s Jacobi-Trudi type formula \([13]\) (see (3.4)). In contrast to the Casorati determinants case, the size of the matrix for this determinant is $m \times m$ and thus increases as $m$ increases. Lemma \([3.3]\) connects these two types of solutions.

In contrast to $U_q(A^{(2)}_N)$ case, $L(u)$ for $U_q(D^{(2)}_{n+1}), U_q(D^{(3)}_4)$ contain factors which have a negative exponent $-1$, thus their order become infinite. Therefore we can not straightforwardly extend the analysis to get the Casorati determinant type solution for $U_q(A^{(2)}_N)$ to this case. However Jacobi-Trudi type formulae are still available in this case as reductions of the solutions in \([14, 15]\). This situation is parallel to the non-twisted case \([4]\).

The deformation parameter $q$ is expressed by a parameter $\hbar$ as $q = e^{\hbar}$. The parameter $\hbar$ often appears as a multiple of $\frac{\pi}{n}. However we note that our argument in this paper is also valid even if one formally set $\frac{\pi}{n} = 0$. In this case, the $T$-system \([11]\) is equivalent to the one for the superalgebra $B^{(1)}(0|n)$ \([16]\).

In this paper, we omit most of the calculations and proofs, which are parallel with those in the non-twisted case \([4]\).

## 2 Difference $L$ Operators

Let $X_N$ be a complex simple Lie algebra of rank $N$, $\sigma$ a Dynkin diagram automorphism of $X_N$ of order $r = 1, 2, 3$. The affine Lie algebras of type $X^{(r)}_N = A^{(1)}_n$ $(n \geq 1)$, $B^{(1)}_n$ $(n \geq 2)$, $C^{(1)}_n$ $(n \geq 2)$, $D^{(1)}_n$ $(n \geq 4)$, $E^{(1)}_n$ $(n = 6, 7, 8)$, $F^{(1)}_4, G^{(1)}_2, A^{(2)}_{n-1}$ $(n \geq 1)$, $A^{(2)}_{n-1}$ $(n \geq 2)$, $D^{(2)}_{n+1}$ $(n \geq 2)$, $E^{(2)}_6$ and $D^{(3)}_4$ are realized as the canonical central extension of the loop algebras based on the pair $(X_N, \sigma)$. We write the set of the nodes of the Dynkin diagram of $X_N$ as $I = \{1, 2, \ldots, N\}$, and let $I_\sigma = \{1, 2, \ldots, n\}$ be the set of $\sigma$-orbits of $I$. In particular, $N = n$ and $I = I_\sigma$ for the non-twisted case $r = 1$. We define numbers $\{r_a\}_{a \in I}$ such that $r_a = r$ if $\sigma(a) = a$, otherwise $r_a = 1$. In our enumeration of the notes of the Dynkin diagram (see, Figure \([1]\)), $r_a$ is 1 except for the case: $r_a = 2$ for $A^{(2)}_{2n-1}$, $r_a = 2$ $(1 \leq a \leq n - 1)$ for $D^{(2)}_{n+1}$, $r_3 = r_4 = 2$ for $E^{(2)}_6$, $r_2 = 3$ for $D^{(3)}_4$. Let $\{\alpha_a\}_{a \in I}$ be the simple roots of $X_N$ with a bilinear form $\langle \cdot | \cdot \rangle$ normalized as $\langle \alpha | \alpha \rangle = 2$ for a long root $\alpha$. Let $I_{ab}$ be an element of the incidence matrix of $X_N$: $I_{ab} = 2\delta_{ab} - 2(\alpha_a | \alpha_b) / (\alpha_a | \alpha_a)$.

Let $U_q(X^{(r)}_N)$ be the quantum affine algebra. We introduce functions $\{Q_a(u)\}_{a \in I_\sigma, u \in \mathbb{C}}$ which correspond to the Baxter $Q$ functions for $U_q(X^{(r)}_N)$,
Figure 1: The Dynkin diagrams of $X_N$ for $r > 1$: The enumeration of the nodes with $I$ is specified under or the right side of the nodes. The filled circles denote the fixed points of the Dynkin diagram automorphism $\sigma$ of order $r$.

| $X_N^{(r)}$ | $X_N$ | automorphism $\sigma$ |
|--------------|--------|-----------------------|
| $A_{2n}^{(2)}$ | ![Diagram](Circles) | $\sigma(2n - a + 1) = a$ for $1 \leq a \leq 2n$ |
| $A_{2n-1}^{(2)}$ | ![Diagram](Circles) | $\sigma(2n - a) = a$ for $1 \leq a \leq 2n - 1$ |
| $D_{n+1}^{(2)}$ | ![Diagram](Circles) | $\sigma(a) = a$ for $1 \leq a \leq n - 1$; $\sigma(n) = n + 1$; $\sigma(n + 1) = n$ |
| $E_6^{(2)}$ | ![Diagram](Circles) | $\sigma(7 - a) = a$ for $a = 1, 2, 5, 6$; $\sigma(3) = 3$; $\sigma(4) = 4$ |
| $D_4^{(3)}$ | ![Diagram](Circles) | $\sigma(1) = 3$; $\sigma(2) = 2$; $\sigma(3) = 4$; $\sigma(4) = 1$ |

and define functions $\{Y_a(u)\}_{a \in I; u \in \mathbb{C}}$ as

$$Y_a(u) = \frac{Q_a(u - \frac{(a_a|a_a|)}{2})}{Q_a(u + \frac{(a_a|a_a|)}{2})}. \quad (2.1)$$

We formally set $Y_0(u) = 1$; $Q_n+1(u) = Q_n(u + \frac{\pi i}{2})$ and $Y_{n+1}(u) = Y_n(u + \frac{\pi i}{2})$ for $X_N^{(r)} = A_{2n}^{(2)}$; $Q_n+1(u) = 1$ and $Y_{n+1}(u) = 1$ for $X_N^{(r)} \neq A_{2n}^{(2)}$. For the twisted case $r > 1$, we assume quasi-periodicity $Q_a(u + \frac{\pi i}{2}) = h_a Q_a(u)$ ($h_a \in \mathbb{C}$), which induces periodicity $Y_a(u + \frac{\pi i}{2}) = Y_a(u)$. For the non-twisted case $r = 1$, one can identify $Y_a(u)$ with the Frenkel-Reshetikhin variable $Y_{a,q}$ denoted as $Y_a(u)$ in [4] if $Q_a(u)$ is suitably chosen. We shall also use notations $Q_a^k(u) = \prod_{j=0}^{k-1} Q_a(u + \frac{\pi i}{2})$ and $Y_a^k(u) = \prod_{j=0}^{k-1} Y_a(u + \frac{\pi i}{2})$.

Next we introduce screening operators $\{S_a\}_{a \in I; u \in \mathbb{C}}$ whose action is given by

$$(S_a \cdot Y_b)(u) = \delta_{ab} Y_a(u) S_a(u). \quad (2.2)$$
Here we assume \( S_a(u) \) satisfies the following relation
\[
S_a(u + (\alpha_a|\alpha_a)) = A_a \left( u + \frac{(\alpha_a|\alpha_a)}{2} \right) S_a(u),
\]
\[
A_a(u) = \prod_{b=1}^{n'} Q_b^{a*}(u - (\alpha_a|\alpha_b)) \frac{Q_b^{a*}(u + (\alpha_a|\alpha_b))}{Q_b^{a*}(u - (\alpha_a|\alpha_b))},
\]
where \( r_{ab} = \max(r_a, r_b) \); \( n' = n+1 \) for \( X_N^{(r)} = A_{2n}^{(2)} \) and \( n' = n \) for \( X_N^{(r)} \neq A_{2n}^{(2)} \). We assume \( S_a \) obeys the Leibniz rule. The origin of (2.4) goes back to the Reshetikhin and Wiegmann’s Bethe ansatz equation [17] (cf. (4.1)). For the non-twisted case \( r = 1 \) case, (2.4) reduces to the corresponding equation in [4]. We have a formal solution of (2.3) (see also, section 5 in [5]):
\[
S_a(u) = \prod_{b=1}^{n'} K_{ab}(u) \frac{Q_b^{a*}(u - \frac{(\alpha_a|\alpha_a)}{2})Q_b^{a*}(u + \frac{(\alpha_a|\alpha_a)}{2})}{Q_b^{a*}(u)},
\]
where
\[
K_{ab}(u) = \begin{cases} 
1 & \text{if } I_{ab} = 0 \\
Q_b^{a*}(u) & \text{if } I_{ab} = 1 \\
Q_b(u - \frac{1}{2})Q_b(u + \frac{1}{2}) & \text{if } I_{ab} = 2 \\
Q_b(u - \frac{1}{2})Q_b(u) & \text{if } I_{ab} = 3.
\end{cases}
\]
Owing to the Leibniz rule, we have
\[
(S_a \cdot Y_b^k)(u) = \delta_{ab} Y_a^k(u) \sum_{j=0}^{k-1} S_a(u + \frac{\pi ij}{\hbar}).
\]
We shall use the following variables for each algebra; the origin of these variables goes back to the analytic Bethe ansatz calculation of DVF [8, 9, 1]. For \( U_q(A_{2n}^{(2)}) \) case:
\[
z_a(u) = \frac{Y_a(u + a)}{Y_{a-1}(u + a + 1)} \quad \text{for} \quad 1 \leq a \leq n,
\]
\[
z_0(u) = \frac{Y_n(u + n + 1 + \frac{\pi i}{2\hbar})}{Y_n(u + n + 2)},
\]
\[
z_{a}(u) = \frac{Y_{a-1}(u + 2n - a + 2 + \frac{\pi i}{2\hbar})}{Y_a(u + 2n - a + 3 + \frac{\pi i}{2\hbar})} \quad \text{for} \quad 1 \leq a \leq n.
\]
We also use the variables: \( x_a(u) = z_a(u) \) and \( x_{2n-a+2}(u) = z_a(u) \) for \( 1 \leq a \leq n; x_{n+1}(u) = z_0(u) \).
For $U_q(A_{2n-1}^{(2)})$ case:

\[
\begin{align*}
  z_a(u) &= \frac{Y_a(u + a)}{Y_{a-1}(u + a + 1)} \quad \text{for} \quad 1 \leq a \leq n - 1, \\
  z_n(u) &= \frac{Y^n_a(u + n)}{Y_{n-1}(u + n + 1)}, \\
  z_{\pi}(u) &= \frac{Y_{n-1}(u + n + 1 + \frac{\pi i}{2\hbar})}{Y^n_a(u + n + 2)}, \\
  z_{\pi}(u) &= \frac{Y_{a-1}(u + 2n - a + 1 + \frac{\pi i}{2\hbar})}{Y^n_a(u + 2n - a + 2 + \frac{\pi i}{2\hbar})} \quad \text{for} \quad 1 \leq a \leq n - 1.
\end{align*}
\]

(2.9)

We also use the variables: $x_a(u) = z_a(u)$ and $x_{2n-a+1}(u) = z_{\pi}(u)$ for $1 \leq a \leq n$.

For $U_q(D_{n+1}^{(2)})$ case:

\[
\begin{align*}
  z_a(u) &= \frac{Y^n_a(u + a)}{Y_{a-1}(u + a + 1)} \quad \text{for} \quad 1 \leq a \leq n, \\
  z_{n+1}(u) &= \frac{Y_n(u + n + \frac{\pi i}{2\hbar})}{Y_n(u + n + 2)}, \\
  z_{n+1}(u) &= \frac{Y_n(u + n)}{Y_n(u + n + 2 + \frac{\pi i}{2\hbar})}, \\
  z_{\pi}(u) &= \frac{Y_{a+1}(u + 2n - a + 1)}{Y^n_a(u + 2n - a + 2)} \quad \text{for} \quad 1 \leq a \leq n.
\end{align*}
\]

(2.10)
For $U_q(D_4^{(3)})$ case:

\[ z_1(u) = Y_1(u + 1), \]
\[ z_2(u) = \frac{Y_3(u + 2)}{Y_1(u + 3)}, \]
\[ z_3(u) = \frac{Y_3(u + 3)}{Y_1(u + 3)Y_2(u + 4)}, \]
\[ z_4(u) = \frac{Y_1(u + 3 - \frac{\pi i}{3\hbar})}{Y_1(u + 5 + \frac{\pi i}{3\hbar})}, \]
\[ z_7(u) = \frac{Y_1(u + 3 + \frac{\pi i}{3\hbar})}{Y_1(u + 5 - \frac{\pi i}{3\hbar})}, \]
\[ z_8(u) = \frac{Y_1(u + 5)Y_2(u + 4)}{Y_3(u + 5)}, \]
\[ z_9(u) = \frac{1}{Y_1(u + 7)}. \]  

(2.11)

Let $D$ be a difference operator such that $Df(u) = f(u + 2)D$ for any function $f(u)$. We shall use notations: $\prod_{k=1}^{m} g_k = g_1g_2 \cdots g_m$ and $\prod_{k=1}^{m} g_k = g_mg_{m-1} \cdots g_1$. By using the variables (2.8)-(2.11), we introduce a factorized difference $L$ operator for each algebra.

For $U_q(A^{(2)}_{2n})$ case:

\[ L(u) = \prod_{a=1}^{n}(1 - z_a(u)D)(1 - z_0(u)D) \prod_{a=1}^{n}(1 - z_a(u)D) \]
\[ = \prod_{a=1}^{2n+1}(1 - x_a(u)D). \]  

(2.12)

For $U_q(A^{(2)}_{2n-1})$ case:

\[ L(u) = \prod_{a=1}^{n}(1 - z_a(u)D) \prod_{a=1}^{n}(1 - z_a(u)D) = \prod_{a=1}^{2n}(1 - x_a(u)D). \]  

(2.13)

For $U_q(D^{(2)}_{n+1})$ case:

\[ L(u) = \prod_{a=1}^{n+1}(1 - z_a(u)D)(1 - z_{n+1}(u)z_{n+1}(u + 2)D^{-1}) \prod_{a=1}^{n+1}(1 - z_a(u)D)(2.14) \]
For $U_q(D_4^{(3)})$ case:

$$L(u) = \prod_{a=1}^{4} (1 - z_a(u)D)(1 - z_4(u)z_4(u + 2)D^2)^{-1} \prod_{a=1}^{4} (1 - z_a(u)D). \quad (2.15)$$

In general, $L(u) \quad (2.12)-(2.15)$ are power series of $D$ whose coefficients lie in $\mathbb{Z}[Y_a(u)^{\pm 1}]_{a \in I_\sigma; u \in \mathbb{C}}$. We assume $S_a$ acts on these coefficients linearly.

**Proposition 2.1.** For $a \in I_\sigma$, we have $(S_a \cdot L)(u) = 0$.

The proof is similar to the non-twisted case [4]. So we just mention the lemmas which are necessary to $U_q(D_4^{(3)})$ case.

**Lemma 2.2.** For $U_q(D_4^{(3)})$ case, let

$$H_1(u) = Y_1(u) + \frac{Y_2^3(u + 1)}{Y_1(u + 2)}, \quad H_2(u) = Y_2^3(u) + \frac{Y_1^3(u + 1)}{Y_2^3(u + 2)},$$

$$K_1(u) = \frac{1}{Y_1(u)} + \frac{Y_1(u - 2)}{Y_2^3(u - 1)}, \quad K_2(u) = \frac{1}{Y_2^3(u)} + \frac{Y_1^3(u - 2)}{Y_1(u - 1)},$$

then $(S_a \cdot H_a)(u) = (S_a \cdot K_a)(u) = 0$ for $a = 1, 2$.

**Lemma 2.3.** For $U_q(D_4^{(3)})$ case, one can rewrite $L(u) \quad (2.15)$ as follows:

$$L(u) = (1 - K_1(u + 7)D + \frac{1}{Y_1^2(u + 8)}D^2)$$

$$\times (1 - \sum_{j=0}^{\infty} A_j(u)D^{2j+1} + \sum_{j=0}^{\infty} B_j(u)D^{2j+2})(1 - H_1(u + 1)D + Y_2^3(u + 2)D^2),$$

where

$$A_j(u) = K_1(u + 4j + 5 + \frac{\pi i}{3\hbar})H_1(u + 3 - \frac{\pi i}{3\hbar})$$

$$+ (1 - \delta_{j0})K_1(u + 4j + 5 - \frac{\pi i}{3\hbar})H_1(u + 3 + \frac{\pi i}{3\hbar}),$$

$$B_j(u) = K_1(u + 4j + 7 + \frac{\pi i}{3\hbar})H_1(u + 3 + \frac{\pi i}{3\hbar})$$

$$+ K_1(u + 4j + 7 - \frac{\pi i}{3\hbar})H_1(u + 3 - \frac{\pi i}{3\hbar}) - \delta_{j0} \frac{Y_2^3(u + 4)}{Y_2^3(u + 6)}.$$
Lemma 2.4. For $U_q(D_4^{(3)})$ case, one can expand the $Y_2$ dependent part in $L(u)$ (2.13):

$$(1 - z_2(u)D)(1 - z_3(u)D) = 1 - Y_1(u + 5)K_2(u + 6)D + \frac{Y_1(u + 5)Y_1(u + 7)}{Y_1^2(u + 7)}D^2,$$

$$(1 - z_3(u)D)(1 - z_2(u)D) = 1 - \frac{H_2(u + 2)}{Y_1(u + 3)}D + \frac{Y_3^2(u + 3)}{Y_1(u + 3)Y_1(u + 5)}D^2.$$ We shall expand $L(u)$ as

$$L(u) = \sum_{a=0}^{\infty} (-1)^a T^a(u + a)D^a. \quad (2.16)$$

In particular, we have $T^0(u) = 1$ and $T^a(u) = 0$ for $a \in \mathbb{Z}_{<0}$. For $U_q(A_N^{(2)})$ case, (2.16) becomes a polynomial in $D$ of order $N + 1$ and $T^a(u) = 0$ for $a \in \mathbb{Z}_{\geq N+2}$.

Remark 2.5. There is a homomorphism $\beta$ analogous to the one in [5].

$$\beta : \mathbb{Z}[Y_a(u)^{\pm 1}]_{a \in I_\alpha} \cap \mathbb{Z} \rightarrow \mathbb{Z}[e^{\pm \frac{\pi i}{2\hbar} \Lambda_a}]_{a \in I_\alpha}; \quad \beta(Y_a(u)^{\pm 1}) = e^{\pm \frac{\pi i}{2\hbar} \Lambda_a},$$

where $\{\Lambda_a\}_{a \in I_\alpha}$ are the fundamental weights of a rank $n$ subalgebra $\mathfrak{g}$ of $X_N^{(r)}$: $(X_N^{(r)}, \mathfrak{g}) = (X_n^{(1)}, X_n), (A_2^{(2)}, C_n), (A_2^{(2)}, C_n), (D_n^{(2)}, B_n), (D_4^{(2)}, G_2), (E_6^{(2)}, F_4)$. Note that the image of $\beta$ is independent of the parameter $\hbar$. In particular, $\beta(T^a(u)) \in \mathbb{Z}[e^{\pm \Lambda_a}]_{a \in I_\alpha}$ is a linear combination of $\mathfrak{g}$ characters (cf. section 6 in [13]). For $1 \leq a \leq b$ (or $U_q(A_2^{(2)}), U_q(A_2^{(2)}): b = n; U_q(D_n^{(2)}): b = n - 1; U_q(D_4^{(2)}): b = 2$), $T^a(u)$ contains a term $Y_a^{\tau_a}(u) = \prod_{k=1}^{a} z_k(u + a - 2k); \beta(Y_a^{\tau_a}(u)) = e^{\Lambda_a}$. In the context of the analytic Bethe ansatz [9] (resp. the theory of $q$-characters [5]), $Y_a^{\tau_a}(u)$ corresponds to the top term of DVF (resp. the highest weight monomial of the $q$-character) for the Kirillov-Reshetikhin module $W^a_{n}(u)$ over $U_q(X_N^{(r)})$.

From the Proposition 2.1 we obtain:

Corollary 2.6. For $a \in I_\alpha$ and $b \in \mathbb{Z}$, we have $(S_a \cdot T^b)(u) = 0$.

For $U_q(A_N^{(2)})$ case, there is a duality among $\{T^a(u)\}_{a \in \mathbb{Z} \cup \{0\}}$.

Proposition 2.7. For $U_q(A_N^{(2)})$ case, we have

$$T^a(u) = T^{N+1-a}(u + \frac{\pi i}{2\hbar}), \quad a \in \mathbb{Z}.$$
This relation is given in [1] as ‘modulo σ relation’. The proof of this proposition is similar to the \(B^{(1)}(0|n)\) case [16], which corresponds to \(N = 2n\) and \(\frac{\pi i}{\hbar} \to 0\).

One can show

\[
L(u)Q_1^1(u) = 0. \tag{2.17}
\]

A \(T - Q\) relation follows from (2.17):

\[
\sum_{a=0}^{\infty} (-1)^a T^a(u + a)Q_1^1(u + 2a) = 0. \tag{2.18}
\]

We shall expand \(L(u)^{-1}\) as

\[
L(u)^{-1} = \sum_{m=0}^{\infty} T_m(u + m)D^m. \tag{2.19}
\]

In particular, we have \(T_0(u) = 1\) and \(T_m(u) = 0\) for \(m \in \mathbb{Z}_{<0}\). From the relation \(L(u)L(u)^{-1} = 1\), we obtain a \(T - T\) relation

\[
\sum_{a=0}^{m} (-1)^a T_{m-a}(u + m + a)T^a(u + a) = \delta_{m0}. \tag{2.20}
\]

From the relation \(L(u)^{-1}L(u) = 1\), we also have

\[
\sum_{a=0}^{m} (-1)^a T_{m-a}(u - m - a)T^a(u - a) = \delta_{m0}. \tag{2.21}
\]

In particular for \(U_q(A_N^{(2)})\) case, the \(T - Q\) relation (2.18) reduces to

\[
\sum_{a=0}^{N+1} (-1)^a T^a(u + a)Q_1(u + 2a) = 0. \tag{2.22}
\]

From the Proposition 2.7, one can rewrite this as follows

\[
\sum_{a=0}^{N+1} (-1)^a T^a(u - a)Q_1(u - 2a + g + \frac{\pi i}{2\hbar}) = 0, \tag{2.23}
\]

where \(g = N + 1\) is the dual Coxeter number of \(A_N^{(2)}\). If one assume \(\lim_{m \to \infty} T_m(u + m)\) (resp. \(\lim_{m \to \infty} T_m(u - m)\)) is proportional to \(Q_1(u)\) (resp. \(Q_1(u + g + \frac{\pi i}{2\hbar})\)), then one can recover the \(T - Q\) relation (2.22) (resp. (2.23)) from the \(T - T\) relation (2.20) (resp. (2.21)).
3 Solution of the $T$-system

The goal of this section is to give a Casorati determinant solution to the $U_q(A^{(2)}_N)$ $T$-system \(^{(1.1), (1.2)}\). Consider the following difference equation

$$L(u)w(u) = 0,$$  \hspace{1cm} (3.1)

where $L(u)$ is the difference $L$ operator \(^{(2.12), (2.13)}\) for $U_q(A^{(2)}_N)$. By using a basis \(\{w_1(u), w_2(u), \ldots, w_{N+1}(u)\}\) of the solutions of (3.1), we define a Casorati determinant:

\[
\begin{vmatrix}
  w_1(u + 2i_1) & w_1(u + 2i_2) & \cdots & w_1(u + 2i_{N+1}) \\
  w_2(u + 2i_1) & w_2(u + 2i_2) & \cdots & w_2(u + 2i_{N+1}) \\
  \vdots & \vdots & \ddots & \vdots \\
  w_{N+1}(u + 2i_1) & w_{N+1}(u + 2i_2) & \cdots & w_{N+1}(u + 2i_{N+1})
\end{vmatrix}
\]

Setting $w = w_1, w_2, \ldots, w_{N+1}$ in (3.1) and noting the relation $T^{N+1}(u) = 1$, we obtain the following relation:

\[ [0, 1, \ldots, N] = [1, 2, \ldots, N + 1]. \]  \hspace{1cm} (3.2)

Owing to the Cramer’s formula, we also have:

**Proposition 3.1.** For $a \in \{0, 1, \ldots, N + 1\}$, we have

$$T^a(u + a) = [0, 1, \ldots, a - 1, a + 1, \ldots, N + 1]_{[0, 1, \ldots, N]}.$$

**Lemma 3.2.** For $U_q(A^{(2)}_N)$ case, one can rewrite $L(u)$ \(^{(2.12), (2.13)}\) as

\[
L(u) = \prod_{a=1}^{N+1} (x_a(u + N + 1 - 2a + \frac{\pi i}{2\hbar}) - D).
\]

Let $\xi_m^{(a)}(u) = [0, 1, \ldots, a - 1, a + m, a + m + 1, \ldots, N + m]$ and $\xi(u) = \xi_0^{(1)}(u) = [0, 1, \ldots, N]$. Note that $\xi_m^{(0)}(u) = \xi(u)$ follows from (3.2). For $1 \leq a \leq N + 1$, we introduce a difference operator

\[
L_a(u) = \prod_{b=N+2-a}^{N+1} (D - x_b(u + N + 1 - 2b + \frac{\pi i}{2\hbar})). \hspace{1cm} (3.3)
\]

In particular we have $L_{N+1}(u) = (-1)^{N+1}L(u)$. We choose a basis of the solutions of (3.1) so that it satisfies $L_a(u)w_b(u) = 0$ for $1 \leq b \leq a \leq N + 1$: $w_a \in \text{Ker}L_a$. For this basis, the following lemma hold.
Lemma 3.3. Let \( \{i_k\} \) be integers such that \( 0 = i_0 < i_1 < \cdots < i_N, \mu = (\mu_k) \) the Young diagram whose \( k \)-th row is \( \mu_k = i_{N+1-k} + k - N - 1 \), and \( \mu' = (\mu'_k) \) the transposition of \( \mu \). We assign coordinates \((j,k)\) \( \in \mathbb{Z}^2 \) on the skew-Young diagram \((\mu_1^{N+1})/\mu\) such that the row index \( j \) increases as we go upwards and the column index \( k \) increases as we go from the left to the right and that \((1,1)\) is on the bottom left corner of \((\mu_1^{N+1})/\mu\).

\[
\frac{[i_0, i_1, \ldots, i_N]}{[0, 1, \ldots, N]} = \sum_b \prod_{(j,k) \in (\mu_1^{N+1})/\mu} x_{b(j,k)}(u + 2j + 2k - 4)
\]

where the summation is taken over the semi-standard tableau \( b \) on the skew-Young diagram \((\mu_1^{N+1})/\mu\) as the set of elements \( b(j,k) \in \{1, 2, \ldots, N+1\} \) labeled by the coordinates \((j,k)\) mentioned above.

The proof is similar to the \( U_q(C_n^{(1)}) \) case \([4]\), where we use a theorem in \([19]\) and Proposition 2.7. Note that Lemma 3.3 reduces to the Proposition 3.1 if we set \( i_b = b \) for \( 0 \leq b \leq a - 1 \) and \( i_b = b + 1 \) for \( a \leq b \leq N \). From Proposition 2.7 and Lemma 3.3 one can show:

Lemma 3.4. For \( a \in \{0, 1, \ldots, N+1\} \), we have \( \frac{\xi^{(a)}(u)}{\xi(u)} = \frac{\xi^{(N-a+1)}(u+2a-N-1+\frac{\pi i}{2\hbar})}{\xi(u+2a-N-1+\frac{\pi i}{2\hbar})} \).

The following relation is a kind of Hirota-Miwa equation \([2, 3]\), which is a Plücker relation and used in a similar context \([20, 21, 22, 4]\).

Lemma 3.5. \( \xi^{(a)}_m(u)\xi^{(a)}_m(u+2) = \xi^{(a)}_{m+1}(u)\xi^{(a)}_{m}(u+2) + \xi^{(a-1)}_m(u)\xi^{(a+1)}_m(u+2) \).

From Lemma 3.4 and Lemma 3.5 we finally obtain:

Theorem 3.6. For \( a \in I_a \) and \( m \in \mathbb{Z}_{\geq 1} \), \( T^{(a)}_m(u) = \frac{\xi^{(a)}(u-a-m+1)}{\xi(u-a-m+1)} \) satisfies the \( T \)-system for \( U_q(A_N^{(2)}) \) \((1.1), (1.2)\).

There is another expression to the solution to the \( U_q(A_N^{(2)}) \) \( T \)-system \((1.1), (1.2)\), which follows from a reduction of the Bazhanov and Reshetikhin’s Jacobi-Trudi type formula \([13]\) (cf. section 5 in \([11]\)).

\[
T^{(a)}_m(u) = \det_{1 \leq j, k \leq m} (T^{a-j+k}(u + j + k - m - 1)) \tag{3.4}
\]

where \( T^a(u) \) obeys the following condition:

\[
T^a(u) = \begin{cases} 
0 & \text{if } a < 0 \text{ or } a > N + 1 \\
1 & \text{if } a = 0 \text{ or } a = N + 1 \\
T^{(a)}_1(u) & \text{if } 1 \leq a \leq n \\
T^{(N-a+1)}_1(u + \frac{\pi i}{2\hbar}) & \text{if } n + 1 \leq a \leq N.
\end{cases} \tag{3.5}
\]
Through the identification $T^a(u) = T^a(u)$ and Lemma 3.3 (3.4) reproduces the solution in Theorem 3.6 and also the tableaux sum expression in [1].

4 Discussion

In this paper, we have dealt with the $T$-system without the vacuum part. On applying our results to realistic problems in solvable lattice models or integrable field theories, we must specify the Baxter $Q$-function, and recover the vacuum part whose shape depends on each model. We can easily recover the vacuum part by multiplying the vacuum function $\psi_a(u)$ by the function $z_a(u)$ so that $\psi_a(u)$ is compatible with the Bethe ansatz equation of the form (cf. [17, 23])

$$\Psi_a(u_j) = \prod_{b=1}^{n'} \frac{Q_{ab}^{(a)}(u_j^{(a)} + (\alpha_a | \alpha_b))}{Q_{ab}^{(a)}(u_j^{(a)} - (\alpha_a | \alpha_b))}, \quad a \in I_\sigma.$$

In the case of the solvable vertex model, it was conjectured [23] that $\Psi_a(u)$ is given as a ratio of Drinfeld polynomials.

A remarkable connection between DVF and the $q$-character was pointed out in [5]. It was also conjectured [4] that $q$-characters of Kirillov-Reshetikhin modules over $U_q(X^{(r)}_N)$ satisfy the $T$-system [24]. It is natural to expect that similar phenomena are also observed for the twisted case $U_q(X^{(r)}_N)$ ($r > 1$).

We can also easily construct difference $L$ operators associated with superalgebras by using the results on the analytic Bethe ansatz [26, 27, 28, 16]. However their orders are infinite as $U_q(B_n^{(1)}), U_q(D_n^{(1)}), U_q(D_{n+1}^{(2)}), U_q(D_4^{(3)})$ case. Thus we will need some new ideas to construct Casorati determinant like solutions to the $T$-system for superalgebras.

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