Weak Solutions to Vlasov-McKean Equations under Lyapunov-Type Conditions

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Abstract

We present a Lyapunov type approach to the problem of existence and uniqueness of general law-dependent stochastic differential equations. In the existing literature most results concerning existence and uniqueness are obtained under regularity assumptions of the coefficients w.r.t the Wasserstein distance. Some existence and uniqueness results for irregular coefficients have been obtained by considering the total variation distance. Here we extend this approach to the control of the solution in some weighted total variation distance, that allows us now to derive a rather general weak uniqueness result, merely assuming measurability and certain integrability on the drift coefficient and some non-degeneracy on the dispersion coefficient. We also present an abstract weak existence result for the solution of law-dependent stochastic differential equations with merely measurable coefficients, based on an approximation with law-dependent stochastic differential equations with regular coefficients under Lyapunov type assumptions.

Keywords: Vlasov-McKean equations; Girsanov theorem; existence and uniqueness of weak solution; Lyapunov method; weighted total variation.

1 Introduction

The purpose of this paper is to provide general existence and uniqueness results for the solution of Vlasov-McKean equations, and more general law-dependent stochastic differential equations, using a Lyapunov approach. The existence and uniqueness of solutions of Vlasov-McKean equations under global Lipschitz conditions is well-known. Surprisingly, uniqueness fails under local Lipschitz assumptions (see [12]). However, in these counterexamples, the noise is degenerate (in fact zero). As the following Example of uniqueness with merely measurable coefficients shows, the situation changes, if the noise becomes non-degenerate.

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Example 1.1. On the complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with real valued $(\mathcal{F}_t)_{t \geq 0}$-Wiener process $(W_t)_{t \geq 0}$ on $\mathbb{R}$, consider the following Vlasov-McKean equation

$$\begin{cases}
  dX_t = \mathbb{E}(h(X_t)) \, dt + dW_t \\
  X_0 = \xi
\end{cases}$$

(1)

with measurable $h$ satisfying the growth condition $|h(x)| \leq Ce^{\frac{x^2}{2T}}$ for some $T > 0$. Let $\mu_0 := \mathbb{P} \circ \xi^{-1}$ be absolutely continuous with continuous differentiable density, and define

$$\phi_h(t, x) := \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} h(x_0 + x + w) e^{-\frac{w^2}{2t}} \, dw \, d\mu_0(dx_0)$$

Then for $t < T$, $x \mapsto \phi_h(t, x)$ is continuous differentiable, hence locally Lipschitz continuous. Let $X_t = \xi + g(t) + W_t$ be a solution of (1), then

$$g'(t) \, dt + dW_t = dX_t = \mathbb{E}(h(X_t)) \, dt + dW_t$$

$$= \mathbb{E}(h(\xi + g(t) + W_t)) \, dt + dW_t$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} h(x_0 + g(t) + w) e^{-\frac{(w-x)^2}{2t}} \, dw \, d\mu_0(dx_0) \, dt + dW_t$$

$$= \phi_h(t, g(t)) \, dt + dW_t.$$

So $g : [0, T) \to \mathbb{R}$ is the unique solution to the equation $g'(t) = \phi_h(t, g(t))$, with initial value $g(0) = 0$. Therefore equation (1) has a unique strong solution on $[0, T)$.

Hence there is a considerable interest in relaxing the assumptions on the coefficients of Vlasov-McKean equations. Strong well-posedness of Vlasov-McKean equation with Hölder drift and Lipschitz dispersion coefficient has been obtained in $[3]$. Strong existence and uniqueness of solutions to the Vlasov-McKean equation under one-sided Lipschitz continuity for the drift and Lipschitz continuous dispersion coefficient have been obtained in $[4]$. The paper $[13]$ considers strong well-posedness of distribution dependent stochastic differential equations with one-sided Lipschitz continuous drift and Lipschitz-continuous dispersion coefficients, $[6]$ generalizes the latter result to path-distribution dependent stochastic differential equations.

Weak existence and strong uniqueness of solutions to the Vlasov-McKean equation with continuous coefficients have been obtained with the help of a Lyapunov method in $[5]$. The recent preprint $[11]$ proves weak and strong well-posedness of the solutions of Vlasov-McKean equations under non-degeneracy assumptions on the noise term with even non-regular drift of at most linear growth.

Existence and uniqueness of weak solutions of Vlasov-McKean equations have been obtained in $[9]$, with regularity of the coefficients w.r.t. the total variation distance. $[1]$ obtains existence and uniqueness of weak and strong solutions of Vlasov-McKean equations with additive noise and drift coefficients that can be decomposed into bounded measurable part and a part that is Lipschitz continuous w.r.t. the Kantorovich distance.
The paper [7] contains an existence result of a weak solution of a distribution-dependent stochastic differential equation with merely measurable coefficients based on an approximation with stochastic differential equations with Lipschitz continuous coefficients. This result requires uniform boundedness of the diffusion term.

In the present paper now, we will extend the result for the existence of weak solutions to Vlasov-McKean equations with measurable coefficients and uniformly non-degenerate and merely integrable diffusion matrix (see the Theorem 3.1). The abstract conditions in this theorem can be verified with the help of a Lyapunov type growth condition on the coefficients in Theorem 3.4. Sufficient conditions, in terms of the coefficients only, are presented in Corollary 3.5.

We also obtain a corresponding uniqueness result for weak solutions of functional law-dependent stochastic differential equations under Lyapunov type growth conditions on the coefficients (see Corollary 2.5), based on an abstract stability result for weak solutions w.r.t. a weighted total variation distance (see Theorem 2.4). Two sets of sufficient conditions in terms of the coefficients are presented in Example 2.6. Our uniqueness results generalize the corresponding result obtained in [11] not only w.r.t. the general law-dependence but also w.r.t. the more general growth conditions. In [11], only linear growth is allowed. Stability results for Vlasov-McKean equations w.r.t. weighted total variation distances have been obtained previously in the references [2, 10], using an analytic approach, that cannot, however, cover general functional law-dependent stochastic differential equations considered in the present work.

## 2 Uniqueness Result

Let $\mathcal{M}$ be the space of signed measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Given a measurable function $\phi : \mathbb{R}^d \to (0, \infty)$, we define the $\phi$-weighted total variation of $\mu \in \mathcal{M}$ by

$$
\|\mu\|_\phi := \int_{\mathbb{R}^d} \phi(y) |\mu|(dy)
$$

Here $|\mu|$ denotes the total variation measure associated with $\mu$. For a continuous function $\phi$, this norm is lower semi-continuous with respect to the weak topology by the following Lemma.

**Lemma 2.1.** Let $\phi : \mathbb{R}^d \to (0, \infty)$ be continuous and assume that the sequence of signed measures $(\mu_n)_{n \in \mathbb{N}}$ in $\mathcal{M}$ converges weakly to the measure $\mu$ and assume that $\phi \in L^1(|\mu|)$. Then

$$
\|\mu\|_\phi \leq \liminf_{n \to \infty} \|\mu_n\|_\phi.
$$

**Proof.** Using the Hahn decomposition theorem we can find a measurable subset $A \in \mathcal{B}(\mathbb{R}^d)$ such that $|\mu| = \mu_A - \mu_{A^c}$, where $\mu_A(B) = \mu(B \cap A)$ and $\mu_{A^c}(B) = \mu(B \cap A^c)$, $B \in \mathcal{B}(\mathbb{R}^d)$, are finite nonnegative Borel measures. Let $\varepsilon > 0$ be arbitrary. Since $\phi \in L^1(|\mu|)$ we can find $R > 0$ such that

$$
\|\mu\|_\phi \leq \int_{\mathbb{R}^d} (\phi(y) \wedge R) (1_A(y) - 1_{A^c}(y)) \mu(dy) + \varepsilon.
$$
Since \((\phi \wedge R) \, d\mu\) is a finite Borel measure we can find a continuous function \(\psi : \mathbb{R}^d \to [-1, 1]\) satisfying
\[
\int_{\mathbb{R}^d} (\phi(y) \wedge R) \left(1_A(y) - 1_{A^c}(y)\right) \mu(dy) \leq \int_{\mathbb{R}^d} (\phi(y) \wedge R \psi)(y) \mu(dy) + \varepsilon.
\]
Consequently,
\[
\|\mu\|_\phi \leq \int_{\mathbb{R}^d} (\phi(y) \wedge R) \psi(y) \mu(dy) + 2\varepsilon \leq \lim_{n \to \infty} \int_{\mathbb{R}^d} (\phi(y) \wedge R) \psi(y) \mu_n(dy) + 2\varepsilon
\leq \liminf_{n \to \infty} \|\mu_n\|_\phi + 2\varepsilon.
\]
Since \(\varepsilon > 0\) is arbitrary, this implies the assertion. \(\square\)

Fix \(T, \tau > 0\). Let \(\mathfrak{M}\) be the Borel \(\sigma\)-algebra induced by the weak topology on \(\mathcal{M}\). Let us define
\[
\mathcal{M}_T := \{\mu : [-\tau, T] \to \mathcal{M} ; \mu \text{ is } \mathcal{B}([-\tau, T]) / \mathfrak{M}\text{-measurable}\}.
\]
Let \((W_t)_{t \geq 0}\) be the standard Brownian motion on \(\mathbb{R}^d\). We consider the nonlinear equation
\[
\begin{aligned}
\text{d}X_t &= b(t, X, \mu) \text{d}t + \sigma(t, X) \text{d}W_t, \quad t \in [0, T], \\
X_t &= \xi_t, \quad t \in [-\tau, 0], \\
\mu \in \mathcal{M}_T, \mu_s = \mathcal{L}(X_s), \quad \text{where } \mathcal{L}(X_s) \text{ denotes the law of } X_s, s \in [-\tau, T]
\end{aligned}
\tag{2}
\]
with initial condition \(\xi \in C([-\tau, 0]; \mathbb{R}^d)\), independent of \((W_t)_{t \geq 0}\), where \(b \equiv \sigma \tilde{b}\) and
\[
\begin{aligned}
\tilde{b} : [0, T] \times C([-\tau, T], \mathbb{R}^d) \times \mathcal{M}_T \to \mathbb{R}^d, \\
\sigma : [0, T] \times C([-\tau, T], \mathbb{R}^d) \to \mathbb{R}^{d \times d}
\end{aligned}
\]
are measurable functions and adapted, i.e. \(\tilde{b}(t, x, \mu)\) and \(\sigma(t, x)\) depend only on the path of \(x\) and \(\mu\) on \([-\tau, t]\). This equation is called a Vlasov-McKean equation.

**Definition 2.2.** We say that equation \((2)\) has a weak solution on \([0, T]\) with initial distribution \(\Xi\) on \(C([-\tau, 0], \mathbb{R}^d)\) if there exist a probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\), an \((\mathcal{F}_t)_{t \geq 0}\) - Wiener process \((W_t)_{t \geq 0}\) on \(\mathbb{R}^d\), an \(\mathcal{F}_0\)-measurable random variable \(\xi \in C([-\tau, 0], \mathbb{R}^d)\) with the law \(\Xi\), and an \((\mathcal{F}_t)_{t \geq 0}\)-adapted stochastic process \(X \in C([-\tau, T], \mathbb{R}^d)\) such that
\[
\begin{aligned}
X_t &= \xi_0 + \int_0^t b(s, X, \mu) \text{d}s + \int_0^t \sigma(s, X) \text{d}W_s, \quad t \in [0, T], \\
X_t &= \xi_t, \quad t \in [-\tau, 0], \\
\mu \in \mathcal{M}_T, \mu_s = \mathcal{L}(X_s),
\end{aligned}
\tag{3}
\]
which requires that the integrals are well defined, i.e.,
\[
\int_0^T |b(s, X, \mu)| + |\sigma(s, X)|^2 \text{d}s < \infty, \quad \mathbb{P}\text{-a.s.}
\tag{4}
\]

\[
\int_{\mathbb{R}^d} (\phi(y) \wedge R) (1_A(y) - 1_{A^c}(y)) \mu(dy) \leq \int_{\mathbb{R}^d} (\phi(y) \wedge R) \psi(y) \mu(dy) + \varepsilon.
\]
Remark 2.3. Note that by Levy’s theorem on characterization of Brownian motion, for any \((\mathcal{F}_t)_{t \geq 0}\)-Wiener process \((W_t)_{t \geq 0}\), \(W_t - W_s\) is independent of \(\mathcal{F}_s\). Specially \((W_t)_{t \geq 0}\) is independent of \(\mathcal{F}_0\), that means in Definition 2.2, \(\xi\) is in fact independent of \((W_t)_{t \geq 0}\).

We will first state an abstract uniqueness result for the weak solution to the Vlasov-McKean equation (2) in the following theorem, that is based on an estimate of the distance of the laws of two weak solutions with different drift and same dispersion coefficient w.r.t. the weighted total variation distance introduced above.

**Theorem 2.4.** Suppose that equation
\[
\begin{align*}
\begin{cases}
    dX_t^0 = \sigma(t, X_t^0) \, dW_t, & t \in [0, T], \\
    X_t^0 = \xi_t, & t \in [-\tau, 0],
\end{cases}
\end{align*}
\]
has a unique strong solution on the probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) for some \(\mathcal{F}_0\)-measurable random variable \(\xi \in C([-\tau, 0], \mathbb{R}^d)\). Let \(\tilde{b}_1, \tilde{b}_2 : [0, T] \times C([-\tau, T], \mathbb{R}^d) \to \mathbb{R}^d\) be such that for \(i = 1, 2\),
\[
\int_0^T |\tilde{b}_i(s, X^0)|^2 \, ds < \infty, \quad \mathbb{P}\text{-a.s.}
\]
and \(\tilde{b}_i(t, x)\) depends only to the path of \(x\) on \([-\tau, t]\). Let \(X^{(i)}, \ i = 1, 2\), defined on the probability spaces \((\Omega^{(i)}, \mathcal{F}^{(i)}, (\mathcal{F}^{(i)}_t)_{t \geq 0}, \mathbb{Q}^{(i)})\) be weak solutions to the equations
\[
\begin{align*}
\begin{cases}
    dX^{(i)}_t = \tilde{b}_i(t, X^{(i)}_t) \, dt + \sigma(t, X^{(i)}_t) \, dW^{(i)}_t, & t \in [0, T] \\
    X^{(i)}_t = \xi^{(i)}_t, & t \in [-\tau, 0],
\end{cases}
\end{align*}
\]
where \(\tilde{b}_i \equiv \tilde{\sigma}_i\), and \(\xi^{(i)}\) is independent of \(W^{(i)}\) and has the same law as \(\xi\). Assume that for \(i = 1, 2\), \(X^{(i)}\) satisfies for \(j = 1, 2\),
\[
\int_0^T |\tilde{b}_j(s, X^{(i)}_s)|^2 \, ds < \infty, \quad \mathbb{Q}^{(i)}\text{-a.s.}
\]
If \(\mu^{(i)}_t, i = 1, 2\) denotes the law of \(X^{(i)}_t\), then for any continuous function \(\phi : \mathbb{R}^d \to (0, \infty)\)
\[
\left\| \mu^{(1)}_t - \mu^{(2)}_t \right\|_\phi \leq \sum_{i=1}^2 \mathbb{E}_{\mathbb{Q}^{(i)}} \left[ \phi \left( X^{(i)}_t \right) \int_0^t \left| \tilde{b}_1(s, X^{(i)}_s) - \tilde{b}_2(s, X^{(i)}_s) \right|^2 \, ds \right]
+ \sum_{i=1}^2 \left( \mathbb{E}_{\mathbb{Q}^{(i)}} \left[ \phi^2 \left( X^{(i)}_t \right) \right] \right)^{1/2} \left( \mathbb{E}_{\mathbb{Q}^{(i)}} \left[ \int_0^t \left| \tilde{b}_1(s, X^{(i)}_s) - \tilde{b}_2(s, X^{(i)}_s) \right|^2 \, ds \right] \right)^{1/2}.
\]
In addition, let \(b_2(t, x) := b(t, x, \mu^{(i)}_t)\) and assume that there exist measurable functions \(\varphi : [0, T] \to C(\mathbb{R}^d, (0, \infty))\) and \(\psi : [0, T] \times C([-\tau, T], \mathbb{R}^d) \to [0, \infty)\) and an increasing positive valued function \(g\) with \(\int_{0^+} \frac{1}{g(u)} \, du = \infty\) such that for every \(\mu, \nu \in \mathcal{M}_T\) with \(\mu|_{[-\tau, 0]} = \nu|_{[-\tau, 0]}\),
\[
\left| \tilde{b}(t, x, \mu) - \tilde{b}(t, x, \nu) \right| \leq \psi(t, x) g^{1/2} \left( \sup_{s \in [0, t]} \|\mu_s - \nu_s\|_{\varphi_s} \right)^2,
\]
Then \( Q^{(1)} \circ (X^{(1)})^{-1} = Q^{(2)} \circ (X^{(2)})^{-1} \) provided that

\[
\int_0^T \sup_{t \in [s,T]} \left\{ \sum_{i=1}^2 \mathbb{E}_{Q^{(i)}} \left[ \varphi_t \left( X^{(i)}_t \right) \int_0^t \left| \tilde{b} \left( u, X^{(i)}_u, \mu^{(i)} \right) - \tilde{b} \left( u, X^{(i)}_u, \mu^{(2)} \right) \right|^2 \, du \right] \right\} \cdot \mathbb{E}_{Q^{(i)}} \left[ \psi^2 \left( s, X^{(i)}_s \right) \right] \, ds < \infty \quad (11)
\]

for \( i = 1, 2 \).

**Proof.** Let \( X^0 \) be the unique strong solution to the following equation

\[
\begin{cases}
    dX^0_t = \sigma(t, X^0) \, dW_t, & t \in [0, T], \\
    X^0_t = \xi_t, & t \in [-\tau, 0].
\end{cases}
\]

Using Girsanov transformation, it turns out that equation (7) has at most one weak solution satisfying (8). Let us define the stopping time \( \tau_n \) as

\[
\tau_n := \inf \left\{ t \geq 0, \min_{i=1,2} \int_0^t \left| \tilde{b}_i(s, X^0) \right|^2 \, ds > n \right\}.
\]

Then the following process for \( i = 1, 2 \) is a martingale

\[
M_{t \wedge \tau_n}^{(i)} := \exp \left( \int_0^{t \wedge \tau_n} \tilde{b}_i(s, X^0) \cdot dW_s - \frac{1}{2} \int_0^{t \wedge \tau_n} \left| \tilde{b}_i(s, X^0) \right|^2 \, ds \right), \quad t \in [0, T].
\]

Let \( \mathbb{P}^{i,n} \) be the probability measure with density

\[
\frac{d\mathbb{P}^{i,n}}{d\mathbb{P}} \bigg|_{\mathcal{F}_T} = M_{t \wedge \tau_n}^{(i)}. \]

By Girsanov theorem, the process

\[
\tilde{W}_{t \wedge \tau_n}^{(i)} = W_{t \wedge \tau_n} - \int_0^{t \wedge \tau_n} \tilde{b}_i(s, X^0)ds, \quad t \in [0, T],
\]

with respect to the probability measure \( \mathbb{P}^{i,n} \) for \( i = 1, 2 \), is a standard Brownian motion on \( \mathbb{R}^d \) until time \( \tau_n \) and we have

\[
X_{t \wedge \tau_n}^0 = \xi_0 + \int_0^{t \wedge \tau_n} b_i(s, X^0) \, ds + \int_0^{t \wedge \tau_n} \sigma(s, X^0) \, d\tilde{W}^{(i)}_s.
\]

Let

\[
\zeta_n^{(i)} := \inf \left\{ t \geq 0, \min_{j=1,2} \int_0^t \left| \tilde{b}_j(s, X^{(i)}) \right|^2 \, ds > n \right\}.
\]

Then if we define

\[
\frac{dQ^{i,n}}{dQ^{(i)}} \bigg|_{\mathcal{F}_{\tau_n}^{(i)}} := \exp \left( -\int_0^{T \wedge \zeta_n^{(i)}} \tilde{b}_i(s, X^{(i)}) \cdot d\tilde{W}^{(i)}_s - \frac{1}{2} \int_0^{T \wedge \zeta_n^{(i)}} \left| \tilde{b}_i(s, X^{(i)}) \right|^2 \, ds \right)
\]

for \( i = 1, 2 \).
then
\[ \bar{W}_{t \wedge \xi_n}^{(i)} = W_{t \wedge \xi_n}^{(i)} + \int_0^{t \wedge \xi_n} \tilde{b}_t(s, X_t^{(i)}) ds, \quad t \in [0, T], \]
with respect to the probability measure \( \mathbb{Q}^{(i, n)} \), for \( i = 1, 2 \), is a standard Brownian motion in \( \mathbb{R}^{d_i} \) until time \( \xi_n^{(i)} \) and we have that
\[ X_t^{(i)} = \xi_0^{(i)} + \int_0^{t \wedge \xi_n^{(i)}} \sigma(s, X_s^{(i)}) d\bar{W}_s^{(i)}, \]
and \( (X_t^{(i)})_{-\tau \leq t \leq 0} \) w.r.t \( \mathbb{Q}^{(i, n)} \) has the same law as \( \xi \). Since equation (5) has a unique strong solution, there exists a measurable function
\[ F : C([-\tau, 0], \mathbb{R}^d) \times C([0, T], \mathbb{R}^{d_i}) \to C([-\tau, T], \mathbb{R}^d) \]
such that \( X_0 = F(\xi, W) \) and similarly \( X_0^{(i)} = F(\xi^{(i)}, \bar{W}^{(i)}) \). Hence for \(-\tau \leq t_0 \leq t_1 \leq \cdots \leq t_m \leq T\),
\[ Q^{(i)} \left[ (X_{\xi_n^{(i)}}, \ldots, X_{t_m \wedge \xi_n^{(i)}}) \in \Gamma \right] \]
\[ = \int_{\Omega^{(i)}} \exp \left( \int_0^{T \wedge \xi_n^{(i)}} \tilde{b}_s(s, X_s^{(i)}) \cdot d\bar{W}_s^{(i)} - \frac{1}{2} \int_0^{T \wedge \xi_n^{(i)}} \left| \tilde{b}_s(s, X_s^{(i)}) \right|^2 ds \right) \cdot \mathbf{1} \left\{ (X_{\xi_n^{(i)}}, \ldots, X_{t_m \wedge \xi_n^{(i)}}) \in \Gamma \right\} dQ^{(i, n)} \]
\[ = \int_{\Omega} \exp \left( \int_0^{T \wedge \tau_n} \tilde{b}_s(s, X_0) \cdot dW_s - \frac{1}{2} \int_0^{T \wedge \tau_n} \left| \tilde{b}_s(s, X_0) \right|^2 ds \right) \cdot \mathbf{1} \left\{ (X_{\tau_n}, \ldots, X_{t_m \wedge \tau_n}) \in \Gamma \right\} d\mathbb{P}^{(i, n)} \]
\[ = \mathbb{P}^{(i, n)} \left[ (X_{\tau_n}, \ldots, X_{t_m \wedge \tau_n}) \in \Gamma \right]. \]
By taking the limit of \( n \to \infty \), we get that the law of \( X_{\tau_n} \) with respect to \( \mathbb{P}^{(i, n)} \) converges weakly to the law of \( X^{(i)} \) with respect to \( \mathbb{Q}^{(i)} \) since \( \mathbb{Q}^{(i)} \left( \sup_{n \geq 1} \xi_n^{(i)} \geq T \right) = 1 \). Let us define the function
\[ \phi_{\varepsilon}(y) := \frac{\phi(y)}{1 + \varepsilon \phi(y)}. \]
Using Lemma 2.1 applied to the bounded function \( \phi_{\varepsilon} \in L^1(|\mu_1 - \mu_2|) \), we obtain that
\[ \left\| \mu_1 - \mu_2 \right\|_\phi = \int_{\mathbb{R}^d} \phi_{\varepsilon}(y) \left| \mu_1 - \mu_2 \right| (dy) \]
\[ = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \phi_{\varepsilon}(y) \left| \mu_1 - \mu_2 \right| (dy) \]
\[ \leq \liminf_{n \to \infty} \liminf_{\varepsilon \to 0} \int_{\mathbb{R}^d} \phi_{\varepsilon}(y) \left| (\mathbb{P}^{1, n}) \circ (X_{\tau_n})^{-1} - (\mathbb{P}^{2, n}) \circ (X_{\tau_n})^{-1} \right| (dy). \]
For $A \in \mathcal{B} (\mathbb{R}^d)$, we have
\[
\left| (\mathbb{P}^{1,n}) \circ (X^0_{t \wedge T_n})^{-1} - (\mathbb{P}^{2,n}) \circ (X^0_{t \wedge T_n})^{-1} \right| (A)
\]
\[
= \sup_{m \geq 1} \sum_{i=1}^{m} \left| \mathbb{P}^{1,n} (X^0_{t \wedge T_n} \in A_i) - \mathbb{P}^{2,n} (X^0_{t \wedge T_n} \in A_i) \right|
\]
\[
= \sup_{m \geq 1} \sum_{i=1}^{m} \left| \int_{\Omega} \left( M^{(1)}_{t \wedge T_n} - M^{(2)}_{t \wedge T_n} \right) \mathbf{1}_{\{X^0_{t \wedge T_n} \in A_i\}} d\mathbb{P} \right|
\]
\[
\leq \sup_{m \geq 1} \sum_{i=1}^{m} \int_{\Omega} \left| M^{(1)}_{t \wedge T_n} - M^{(2)}_{t \wedge T_n} \right| \mathbf{1}_{\{X^0_{t \wedge T_n} \in A_i\}} d\mathbb{P}
\]
\[
= \int_{\Omega} \left| M^{(1)}_{t \wedge T_n} - M^{(2)}_{t \wedge T_n} \right| \mathbf{1}_{\{X^0_{t \wedge T_n} \in A\}} d\mathbb{P},
\]
where $\bigcup_{i=1}^{m} A_i$ means the disjoint union of Borel measurable sets $A_i$, $1 \leq i \leq m$. Therefore
\[
\left\| \mu^{(1)}_t - \mu^{(2)}_t \right\|_{\phi} \leq \liminf_{\epsilon \to 0} \liminf_{n \to \infty} \mathbb{E}_{\mathbb{P}} \left[ \phi_{\epsilon} (X^0_{t \wedge T_n}) \left| M^{(1)}_{t \wedge T_n} - M^{(2)}_{t \wedge T_n} \right| \right].
\]
By using the inequality $|e^x - e^y| \leq |x - y| (e^x + e^y)$, we get
\[
\left| M^{(1)}_{t \wedge T_n} - M^{(2)}_{t \wedge T_n} \right| \leq (M^{(1)}_{t \wedge T_n} + M^{(2)}_{t \wedge T_n}) N_{t \wedge T_n}
\]
where
\[
N_t := \left| \int_0^t \left[ \tilde{b}_1(s, X^0) - \tilde{b}_2(s, X^0) \right] \cdot dW_s - \frac{1}{2} \int_0^t \left[ \tilde{b}_1(s, X^0) \right]^2 - \left| \tilde{b}_2(s, X^0) \right|^2 ds \right|
\]
\[
= \left| \int_0^t \left[ \tilde{b}_1(s, X^0) - \tilde{b}_2(s, X^0) \right] \cdot d\tilde{W}^{(1)}_s - \frac{1}{2} \int_0^t \left[ \tilde{b}_1(s, X^0) \right]^2 - \left| \tilde{b}_2(s, X^0) \right|^2 ds \right|
\]
\[
+ \left| \int_0^t \left[ \tilde{b}_1(s, X^0) - \tilde{b}_2(s, X^0) \right] \cdot \tilde{b}_1(s, X^0) ds \right|
\]
\[
= \left| \int_0^t \left[ \tilde{b}_1(s, X^0) - \tilde{b}_2(s, X^0) \right] \cdot d\tilde{W}^{(2)}_s + \frac{1}{2} \int_0^t \left| \tilde{b}_1(s, X^0) - \tilde{b}_2(s, X^0) \right|^2 ds \right|
\]
and also similarly
\[
N_t := \left| \int_0^t \left[ \tilde{b}_1(s, X^0) - \tilde{b}_2(s, X^0) \right] \cdot d\tilde{W}^{(2)}_s - \frac{1}{2} \int_0^t \left| \tilde{b}_1(s, X^0) - \tilde{b}_2(s, X^0) \right|^2 ds \right|
\]
Therefore by (12), we get inequality (9). Let us now turn to the case where \( \tilde{\phi}_t(s, x) = b(s, x, \mu^{(i)}) \). First we square both sides of (9) with \( \phi = \varphi_t \) and then we substitute inequality
\[ \left\| \mu_t^{(1)} - \mu_t^{(2)} \right\|_{\varphi_t}^2 \leq C \sum_{i=1}^{2} \left( \mathbb{E}_{Q(i)} \left[ \varphi_t \left( X_t^{(i)} \right) \int_0^t \left| \bar{b} \left( s, X_t^{(i)}, \mu_t^{(1)} \right) - \bar{b} \left( s, X_t^{(i)}, \mu_t^{(2)} \right) \right|^2 ds \right) \right]^2 \\
+ C \sum_{i=1}^{2} \mathbb{E}_{Q(i)} \left[ \varphi_t \left( X_t^{(i)} \right) \int_0^t \left| \bar{b} \left( s, X_t^{(i)}, \mu_t^{(1)} \right) - \bar{b} \left( s, X_t^{(i)}, \mu_t^{(2)} \right) \right|^2 ds \right]\].

Then for the function
\[ H(t, s) := C \sum_{i=1}^{2} \mathbb{E}_{Q(i)} \left[ \varphi_t \left( X_t^{(i)} \right) \int_0^t \left| \bar{b} \left( u, X_t^{(i)}, \mu_t^{(1)} \right) - \bar{b} \left( u, X_t^{(i)}, \mu_t^{(2)} \right) \right|^2 du \right] . \]

we have
\[ \left\| \mu_t^{(1)} - \mu_t^{(2)} \right\|_{\varphi_t}^2 \leq \int_0^t H(t, s) g \left( \sup_{u \in [0, s]} \left\| \mu_u^{(1)} - \mu_u^{(2)} \right\|_{\varphi_u}^2 \right) ds. \]

Now define \( h(s) := \sup_{u \in [s, T]} H(u, s) \). The assumption (11) implies that \( h \) is integrable and on the other hand,
\[ \sup_{u \in [0, t]} \left\| \mu_u^{(1)} - \mu_u^{(2)} \right\|_{\varphi_u}^2 \leq \int_0^t h(s) g \left( \sup_{u \in [0, s]} \left\| \mu_u^{(1)} - \mu_u^{(2)} \right\|_{\varphi_u}^2 \right) ds. \]

Now consider the function
\[ F(t) := \int_0^t h(s) g \left( \sup_{u \in [0, s]} \left\| \mu_u^{(1)} - \mu_u^{(2)} \right\|_{\varphi_u}^2 \right) ds. \]

Since \( \sup_{u \in [0, t]} \left\| \mu_u^{(1)} - \mu_u^{(2)} \right\|_{\varphi_u}^2 \leq F(t) \) and \( g \) is increasing, we have
\[ F'(t) = h(t) g \left( \sup_{u \in [0, t]} \left\| \mu_u^{(1)} - \mu_u^{(2)} \right\|_{\varphi_u}^2 \right) \leq h(t) g (F(t)) \]
and therefore
\[ \int_0^{F(t)} \frac{1}{g(u)} \, du = \int_0^t \frac{F'(s)}{g(F(s))} \, ds \leq \int_0^t h(s) \, ds < \infty. \]

Since \( \int_0^1 \frac{1}{g(u)} \, du = \infty \), \( F(t) \) must be zero and hence \( \sup_{u \in [0,1]} \| \mu^{(1)}_u - \mu^{(2)}_u \| \varphi \equiv 0 \). Since \( \varphi \) is positive, this implies \( \mu^{(1)}_t = \mu^{(2)}_t \), for all \( t \in [0,T] \). Therefore \( \mathbb{P}^{1,n} = \mathbb{P}^{2,n} \) and since \( \mathbb{F}^{i,n} \circ (X^0 \wedge T_n) \) converges weakly to \( \mathbb{Q}^{(1)} \circ (X^0)^{-1} \), we get \( \mathbb{Q}^{(1)} \circ (X^0)^{-1} = \mathbb{Q}^{(2)} \circ (X^0)^{-1} \). \( \square \)

**Corollary 2.5.** Let
\[ b(t, x, \mu) := \int_{[-\tau,0]} \int_{\mathbb{R}^d} \beta(t, s, x, y) \mu_{t+s}(dy) \kappa(ds) \]
where \( \beta \equiv \sigma \tilde{\beta} \) and
\[ \tilde{\beta} : [0, \infty) \times [-\tau, 0] \times C([-\tau, T], \mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^{d_1}, \]
\[ \sigma : [0, \infty) \times C([-\tau, T], \mathbb{R}^d) \to \mathbb{R}^{d \times d_1} \]
are measurable functions and \( \kappa \) is a probability measure on \([-\tau, 0]\). Assume that
\[ \begin{align*}
   &dX^0_t = \sigma(t, X^0_t) \, dW_t, \quad t \in [0, T], \\
   &X^0_t = \xi_t, \quad t \in [-\tau, 0],
\end{align*} \]
has a unique strong solution. Suppose there exist a function
\[ V \in C^{1,2}([-\tau, T] \times \mathbb{R}^d, [0, \infty)) \]
and measurable functions \( \varphi : [-\tau, T] \to C(\mathbb{R}^d, (0, \infty)) \) and \( \eta : [-\tau, T] \times \mathbb{R}^d \to [0, \infty) \) such that for all \( x \in C([-\tau, T], \mathbb{R}^d) \) and all \( y \in \mathbb{R}^d \) the following properties hold:

(C1) \( \partial_t V(t, x_t) + \langle \nabla V(t, x_t), \beta(t, s, x, y) \rangle + \frac{1}{2} \text{tr} \left( \sigma^T(t, x) D^2 V(t, x_t) \sigma(t, x) \right) \leq CV(t, x_t), \)
(C2) \( \left| \tilde{\beta}(t, s, x, y) \right| \leq C \varphi(t + s, y) \eta(t + s, x_{t+s}), \)
(C3) \( \eta^4(t, y) + \varphi^2(t, y) \leq CV(t, y), \)
(C4) \( \sup_{s \in [-\tau, 0]} \mathbb{E} V(s, \xi_s) < \infty. \)

Then uniqueness holds for the weak solution to Vlasov-McKean equation (2) in the sense of Definition 2.2 with initial value \( \xi \).

**Proof.** Let \( X^{(1)}_t \) and \( X^{(2)}_t \) be two solutions to the Vlasov-McKean equation (2) with laws \( \mu^{(1)}_t \) and \( \mu^{(2)}_t \). We want to prove that the assumptions of Theorem 2.4 hold with the function \( \varphi \) and
\[ \psi(t, x) := C \int_{-\tau}^0 \eta(t + s, x_{t+s}) \kappa(ds), \quad g(u) = u. \]
We have for \( \mu, \nu \in \mathcal{M}_T \) with \( \mu_{[-\tau,0]} = \nu_{[-\tau,0]} \):

\[
|b(t, x, \mu) - b(t, x, \nu)| \leq \int_{-\tau}^{0} \int_{\mathbb{R}^d} C \varphi(t+s,y) \eta(t+s,x_{t+s}) \, |\mu_{t+s} - \nu_{t+s}| \, (dy)\kappa(ds)
\]

\[
\leq C \int_{-\tau}^{0} \eta(t+s,x_{t+s}) \, \|\mu_{t+s} - \nu_{t+s}\| \varphi_{t+s} \, \kappa(ds)
\]

\[
\leq \psi(t, x) \sup_{u \in [0,t]} \|\mu_u - \nu_u\| \varphi_u
\]

All expectations and integrals in Theorem 2.4 are finite via \([\text{C3}]\) provided that

\[
\sup_{s \in [-\tau,T]} \mathbb{E} V(s, X_s^{(i)}) < \infty.
\]

We have by inequality \([\text{C1}]\):

\[
e^{-Ct} V(t, X_t^{(i)}) = V(0, \xi_0) + \int_0^t e^{-Cs} \left[ -CV(s, X_s^{(i)}) + \partial_t V(s, X_s^{(i)}) + \nabla V(s, X_s^{(i)}) \cdot \tilde{\beta}_s(s, X_s^{(i)}, \tilde{X}_s^{(i)}) \kappa(du) \right]
\]

\[
+ \frac{1}{2} \text{tr} \left( \sigma^T(s, X_s^{(i)}) D^2 V(s, X_s^{(i)}) \sigma(s, X_s^{(i)}) \right) \right] ds + M_t
\]

where, according to \([4]\),

\[
M_t := \int_0^t e^{-Cs} \langle \nabla V(s, X_s^{(i)}) , \sigma(s, X_s^{(i)}) dW_s \rangle , \quad t \geq 0
\]

is a local martingale, i.e. there exist stopping times \(\sigma_n \uparrow \infty\) as \(n \to \infty\) such that \((M_{t \wedge \sigma_n})_{t \geq 0}\) is martingale. By Fatou’s lemma,

\[
\mathbb{E} \left[ e^{-Ct} V(t, X_t^{(i)}) \right] \leq \liminf_{n \to \infty} \mathbb{E} \left[ e^{-C(t \wedge \sigma_n)} V(t \wedge \sigma_n, X_s^{(i)}_{t \wedge \sigma_n}) \right] \leq \mathbb{E} V(0, \xi_0).
\]

This implies \(\sup_{t \in [-\tau,T]} \mathbb{E} V(t, X_t^{(i)}) \leq C_T \sup_{s \in [-\tau,0]} \mathbb{E} V(s, \xi_s) < \infty\). Hence we have by \([\text{C2}]\) and \([\text{C3}]\) and locally boundedness of \(\eta\) that for \(x \in C([-\tau,T], \mathbb{R}^d)\),

\[
\int_0^T \left| \tilde{b}(t, x, \mu^{(i)}) \right|^2 \, dt \leq \int_0^T \int_{-\tau}^{0} \int_{\mathbb{R}^d} \left| \tilde{\beta}(t, s, x, y) \right|^2 \mu_{t+s}^{(i)}(dy) \kappa(ds) \, dt
\]

\[
\leq C^2 \int_0^T \int_{-\tau}^{0} \int_{\mathbb{R}^d} \varphi^2(t+s,y) \eta^2(t+s,x_{t+s}) \mu_{t+s}^{(i)}(dy) \kappa(ds) \, dt
\]

\[
\leq C^3 \int_0^T \int_{-\tau}^{0} \mathbb{E} \left[ V(t+s, X_{t+s}^{(i)}) \right] \eta^2(t+s,x_{t+s}) \kappa(ds) \, dt
\]

\[
\leq C^3 \sup_{t \in [-\tau,T]} \mathbb{E} \left[ V(t, X_t^{(i)}) \right] \int_0^T \int_{-\tau}^{0} \eta^2(t+s,x_{t+s}) \kappa(ds) \, dt < \infty. \quad (13)
\]
So the conditions \((\mathcal{Q})\) and \((\mathcal{S})\) in Theorem 2.4 hold. The right hand side of inequality \([10]\) has the following bound,

\[
\int_0^T \sup_{t \in [s,T]} \left\{ \sum_{i=1}^2 \mathbb{E} \left[ \varphi_t \left( X_t^{(i)} \right) \int_0^t \left| \tilde{b} \left( u, X_t^{(i)}, \mu^{(1)} \right) - \tilde{b} \left( u, X_t^{(i)}, \mu^{(2)} \right) \right|^2 du \right] \cdot \mathbb{E} \left[ \varphi_t \left( X_t^{(i)} \right) \psi^2(s, X_t^{(i)}) \right] + \sum_{i=1}^2 \mathbb{E} \left[ \varphi_t^2 \left( X_t^{(i)} \right) \right] \cdot \mathbb{E} \left[ \psi^2(s, X_t^{(i)}) \right] \right\} \, ds
\]

\[
\leq \int_0^T \sup_{t \in [s,T]} \left\{ \sum_{i=1}^2 \mathbb{E} \left[ \varphi_t^2 \left( X_t^{(i)} \right) \right] \cdot \left( \mathbb{E} \left[ \psi^4(s, X_t^{(i)}) \right] \right)^{1/2} \cdot \left( \mathbb{E} \left[ \int_0^T \left( \left| \tilde{b} \left( u, X_t^{(i)}, \mu^{(1)} \right) \right|^2 + \left| \tilde{b} \left( u, X_t^{(i)}, \mu^{(2)} \right) \right|^2 du \right]^2 \right)^{1/2} \right) \right\} \, ds
\]

Hence by \([13]\) to prove inequality \([11]\), it suffices to show that for \(i = 1, 2,\)

\[
\sup_{t \in [0,T]} \mathbb{E} \left[ \varphi_t^2 \left( X_t^{(i)} \right) + \left( \int_{[-\tau,0]} \eta^2 \left( t + s, X_{t+s}^{(i)} \right) \kappa(ds) \right)^2 \right] < \infty
\]

which is obvious by \(\sup_{t \in [-\tau,T]} \mathbb{E} V(t, X_t^{(i)}) < \infty\) and \((C3)\). \(\square\)

**Example 2.6.** Assume the equation

\[dX_t^0 = \sigma(t, X^0) \, dW_t, \quad X_t^0 = \xi_t, \quad t \in [-\tau, 0]\]

has unique strong solution for a locally bounded measurable function \(\sigma : [0, T] \times C \left( [-\tau, T], \mathbb{R}^d \right) \rightarrow \mathbb{R}^{d \times d_1}\). Let

\[b(t, x, \mu) := \int_{\mathbb{R}^d} \beta(t, x, y) \mu_t(dy)\]

where \(\beta \equiv \sigma \tilde{\beta}\) for a measurable function \(\tilde{\beta} : [0, T] \times C \left( [-\tau, T], \mathbb{R}^d \right) \times \mathbb{R}^d \rightarrow \mathbb{R}^d\). Assume that there exist \(\alpha \geq 0\) and \(p \in [0, 2]\) such that one of the following assumptions holds for all \(x \in C \left( [-\tau, T], \mathbb{R}^d \right)\) and \(y \in \mathbb{R}^d,\)

\[\left| \tilde{\beta}(t, x, y) \right| \leq C \left( 1 + |y|^{\alpha/2} \right) \left( 1 + |x|^{\alpha/4} \right),\]

\[\mathbb{E} \left[ |\xi_0|^\alpha \right] < \infty; \]

\[\left| \tilde{\beta}(t, x, y) \right| \leq C \exp \left( \frac{\alpha}{2} |y|^p + \frac{\alpha}{4} |x|^p \right),\]

\[\mathbb{E} \left[ \exp \left( \alpha |\xi_0|^p \right) \right] < \infty.\]
Then the assumptions of Corollary 2.5 hold with \( \kappa = \delta_0 \) (the Dirac measure at point zero) and
\[
\varphi(y) := 1 + |y|^{\alpha/2}, \\
\eta(y) := 1 + |y|^{\alpha/4}, \\
V \in C^2(\mathbb{R}^d, [0, \infty)) \text{ such that } V(y) = 1 + |y|^\alpha \text{ for } |y| \geq 1,
\]
in the case 1 and
\[
\varphi(y) := \exp\left(\frac{\alpha}{2} |y|^p\right), \\
\eta(y) := \exp\left(\frac{\alpha}{4} |y|^p\right), \\
V \in C^2(\mathbb{R}^d, [0, \infty)) \text{ such that } V(y) = \exp(\alpha |y|^p) \text{ for } |y| \geq 1,
\]
in the case 2. In particular the solution to the Vlasov-McKean equation (2) with initial value \( \xi \) is weakly unique.

3 Existence Result

We first show an abstract theorem on the existence of weak solutions to Vlasov-McKean equations with measurable coefficients by approximating the respective equation with more regular coefficients. We then present explicit Lyapunov type assumptions on the coefficients that imply the assumptions made in the abstract approximation result.

**Theorem 3.1.** Let \( b, \sigma : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, \mathbb{R}^{d \times d_1} \) be measurable and locally bounded. Consider the equation
\[
dX_t = \tilde{E}b(t, X_t, \tilde{X}_t)dt + \tilde{E}\sigma(t, X_t, \tilde{X}_t)dW_t
\]
with initial value \( X_0 = \xi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d) \). Here \( \tilde{X}_t \) is an independent copy of \( X_t \). Assume that there exist sequences of measurable functions
\[
b_n, \sigma_n : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, \mathbb{R}^{d \times d_1}, n \in \mathbb{N}
\]
such that for all \( t \in [0, T] \), the functions \( (x, y) \mapsto b_n(t, x, y), \sigma_n(t, x, y) \) are continuous and equation
\[
dX^n_t = \tilde{E}b_n(t, X^n_t, \tilde{X}^n_t)dt + \tilde{E}\sigma_n(t, X^n_t, \tilde{X}^n_t)dW_t
\]
with initial value \( X^n_0 = \xi \) has a weak solution satisfying
\[
\sup_{t \in [0, T]} \mathbb{E} \mathbb{E} \left[ \left| b_n(t, X^n_t, \tilde{X}^n_t) \right|^q + \left| \sigma_n(t, X^n_t, \tilde{X}^n_t) \right|^q \right] < \infty
\]
for some \( q > 2 \). Assume one of the following hypotheses hold:

**Case A:** For every \( t \in [0, T] \), the mappings \( (x, y) \mapsto b(t, x, y), \sigma(t, x, y) \) are continuous and for every \( R > 0 \), \( b_n(t, \cdot, \cdot) \to b(t, \cdot, \cdot), \sigma_n(t, \cdot, \cdot) \to \sigma(t, \cdot, \cdot) \) as \( n \to \infty \) in \( C(B_R \times B_R) \).
**Case B:** The function \((t, x) \mapsto \sup_{n \in \mathbb{N}} \tilde{E} b_n(t, x, \tilde{X}_n^x)\) is locally bounded and for every \(R > 0\),

\[
\liminf_{n \to \infty} \left[ \inf \left\{ h^T \sigma_n(t, x, y) \sigma_n^T(t, x, y) h : |h| = 1; t \in [0, T]; |x|, |y| \leq R \right\} \right] > 0,
\]

and also \(b_n \to b\) and \(\sigma_n \to \sigma\) as \(n \to \infty\) in \(L^{4d+2}([0, T] \times B_R \times B_R, \lambda)\).

Here \(B_R\) is the ball with radius \(R\) centered at the origin and \(\lambda\) is the Lebesgue measure on \(\mathbb{R}^{2d+1}\). Then equation (14) has a weak solution on \([0, T]\).

We will use the following lemma in the proof of case B, which is a consequence of the Krylov’s estimate (see Theorem 2.2.4 in [K]).

**Lemma 3.2.** Consider the probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) and an \((\mathcal{F}_t)_{t \geq 0}\)-Wiener process \((W_t)_{t \geq 0}\) on \(\mathbb{R}^d\). Let \(Z(t) = \int_0^t f(t, \omega)dt + \int_0^t g(t, \omega)dW_t\) be an Itô process on \(\mathbb{R}^d\) where \(f, g : [0, T] \times \Omega \to \mathbb{R}^d, \mathbb{R}^{d \times d}\) are \(\mathcal{F}_t\)-adapted stochastic processes. Let us denote the exit time of \(Z\) from domain \(D \subset \mathbb{R}^d\) by \(\tau_D\), i.e.,

\[
\tau_D := \inf\{t \geq 0 : Z(t) \notin D\}.
\]

Assume that there exist constants \(K\) and \(\delta\) such that for all \((t, \omega) \in [0, T] \times \Omega\) with the property \(t < \tau_D(\omega)\), the following inequalities hold

\[
|f(t, \omega)| \leq K, \quad \inf_{|h| = 1} h^T g(t, \omega) g^T(t, \omega) h \geq \delta
\]

Then there exists a constant \(N_{K, d, D}\) depending only on \(\delta, K, d\) and the diameter of the region \(D\) such that for any measurable function \(u : [0, T] \times D \to \mathbb{R},\)

\[
\mathbb{E} \left[ \int_0^{T \wedge \tau_D} u(t, Z(t)) dt \right] \leq N_{K, d, D} \left( \int_{[0, T] \times D} |u(t, x)|^{d+1} dt dx \right)^{\frac{1}{d+1}}.
\]

**Proof of Theorem 3.7** First we prove tightness of distributions of \(X^n\) on \(C([0, T], \mathbb{R}^d)\). Using

\[
X^n_t - X^n_s = \int_s^t \tilde{E} b_n \left( u, X^n_u, \tilde{X}_u^n \right) du + \int_s^t \tilde{E} \sigma_n \left( u, X^n_u, \tilde{X}_u^n \right) dW_u
\]

and Burkholder-Davis-Gundy inequality, it follows that

\[
\mathbb{E} \left| X^n_t - X^n_s \right|^q \leq 2^{q-1} \mathbb{E} \left| \int_s^t \tilde{E} b_n \left( u, X^n_u, \tilde{X}_u^n \right) du \right|^q + 2^{q-1} \mathbb{E} \left| \int_s^t \tilde{E} \sigma_n \left( u, X^n_u, \tilde{X}_u^n \right) dW_u \right|^q
\]

\[
\leq 2^{q-1} |t-s|^{q-1} \mathbb{E} \left| \int_s^t \tilde{E} b_n \left( u, X^n_u, \tilde{X}_u^n \right) du \right|^q + 2^{q-1} C \mathbb{E} \left[ \int_s^t \tilde{E} \sigma_n \left( u, X^n_u, \tilde{X}_u^n \right)^2 du \right]^{q/2}
\]

\[
\leq 2^{q-1} |t-s|^{q-1} \mathbb{E} \left| \int_s^t \tilde{E} \left( u, X^n_u, \tilde{X}_u^n \right)^q du + 2^{q-1} C |t-s|^\frac{q}{2} \mathbb{E} \int_s^t \tilde{E} \left( u, X^n_u, \tilde{X}_u^n \right)^q du \leq C |t-s|^\frac{q}{2}.
\]
Since \( q > 2 \), the laws of \( X^n \) in the space of \( C \left( [0, T], \mathbb{R}^d \right) \) are tight and there exist some subsequence \( X^{n_k} \) which converges in law to some law \( \mu \) on \( C \left( [0, T], \mathbb{R}^d \right) \). According to Skorokhod’s theorem, there exist random variables say \( \left( Y^{n_k}, \tilde{Y}^{n_k} \right) \) given on some probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with the same distribution as \( \left( X^{n_k}, \tilde{X}^{n_k} \right) \) converging to some random variable \( \left( Y, \tilde{Y} \right) \) having distribution \( \mu \otimes \mu \). Let us define

\[
M_t^{n_k} := Y_t^{n_k} - \int_0^t \tilde{E}b(s, Y_s^{n_k}, \tilde{Y}_s^{n_k}) \, ds.
\]

\( (M_t^{n_k})_{t \geq 0} \) is a martingale with quadratic variation

\[
N_t^{n_k} := \int_0^t \tilde{E}\sigma_n(s, Y_s^{n_k}, \tilde{Y}_s^{n_k}) \tilde{E}\sigma_n^T(s, Y_s^{n_k}, \tilde{Y}_s^{n_k}) \, ds.
\]

We have by Burkholder-Davis-Gundy inequality and (16) that

\[
\sup_{k \in \mathbb{N}} \mathbb{E} |M_t^{n_k}|^q \leq C \sup_{k \in \mathbb{N}} \mathbb{E} |N_t^{n_k}|^{q/2} \leq C_T.
\]

Let us also define

\[
M_t := Y_t - \int_0^t \tilde{E}b(s, Y_s, \tilde{Y}_s) \, dt,
\]

and

\[
N_t := \int_0^t \tilde{E}\sigma(s, Y_s, \tilde{Y}_s) \tilde{E}\sigma^T(s, Y_s, \tilde{Y}_s) \, ds.
\]

If we can show that \( M_t^{n_k} \to M_t \) and \( N_t^{n_k} \to N_t \) in probability, then we have for bounded continuous function \( F : C([0, s], \mathbb{R}^d) \to \mathbb{R} \) and \( v, u \in \mathbb{R}^d \)

\[
\mathbb{E} \left[ \langle M_t - M_s, v \rangle F(Y|_{[0,s]}) \right] = \lim_{k \to \infty} \mathbb{E} \left[ \langle M_t^{n_k} - M_s^{n_k}, v \rangle F(Y^{n_k}|_{[0,s]}) \right] = 0
\]

and also

\[
\mathbb{E} \left[ \left( \langle M_t - M_s, v \rangle (M_t - M_s, u) - v^T N_t u \right) F(Y|_{[0,s]}) \right] = \lim_{k \to \infty} \mathbb{E} \left[ \left( \langle M_t^{n_k} - M_s^{n_k}, v \rangle (M_t^{n_k} - M_s^{n_k}, u) - v^T N_t^{n_k} u \right) F(Y^{n_k}|_{[0,s]}) \right] = 0
\]

So \( M_t \) is a martingale with quadratic variation \( N_t \) and the proof is completed by using the martingale representation theorem. Now we continue the proof for each set of assumptions separately.

**Case A:** For \( \Theta \in \{ b, \sigma \} \), we have

\[
\left| \Theta_{n_k}(t, Y_t^{n_k}, \tilde{Y}_t^{n_k}) - \Theta(t, Y_t, \tilde{Y}_t) \right| \leq \left| \Theta_{n_k}(t, Y_t^{n_k}, \tilde{Y}_t^{n_k}) - \Theta(t, Y_t^{n_k}, \tilde{Y}_t^{n_k}) \right| + \left| \Theta(t, Y_t^{n_k}, \tilde{Y}_t^{n_k}) - \Theta(t, Y_t, \tilde{Y}_t) \right|.
\]
Since the sequence \((Y_{tnk}, \tilde{Y}_{tnk})\) tends to \((Y_t, \tilde{Y}_t)\) almost surely as \(k \to \infty\), it is a bounded sequence in \(\mathbb{R}^{2d}\) almost surely and the right hand side of inequality above tends to zero as \(k \to \infty\). So by uniform integrability, we get the convergence of \(M_{tnk}^n \to M_t\) and \(N_{tnk}^n \to N_t\) in \(L^1\) as \(k \to \infty\).

**Case B:** Let
\[
\tau_{nk}(R) := \inf \left\{ t \geq 0 : |Y_{tnk}^n| + |\tilde{Y}_{tnk}^n| > R \right\}, \quad \tau(R) := \inf \left\{ t \geq 0 : |Y_t| + |\tilde{Y}_t| > R \right\},
\]
and \(\bar{\tau}(R) := \liminf_{k \to \infty} \tau_{nk}(R)\). Since \((Y_{tnk}, \tilde{Y}_{tnk})\) tends to \((Y, \tilde{Y})\) in \(C([0, T], \mathbb{R}^d)\), \(\bar{\tau}(R) \leq \tau(R)\). We have
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |X_t^n|^2 \right] \leq C\mathbb{E} (|\xi|^2) + C T \mathbb{E} \int_0^T \mathbb{E} \left[ b_n \left( t, X_t^n, \tilde{X}_t^n \right) \right]^2 dt + C \mathbb{E} \left[ \int_0^T \mathbb{E} \left[ \sigma_n \left( t, X_t^n, \tilde{X}_t^n \right) \right]^2 dt \right] \leq C_T.
\]

So the stopping times \(\tau_{nk}(R)\) satisfy
\[
\lim_{R \to \infty} \limsup_{k \to \infty} \mathbb{P} \otimes \bar{\mathbb{P}} (\tau_{nk}(R) < T) = 0. \tag{18}
\]

We have
\[
\mathbb{P} \otimes \bar{\mathbb{P}} \left( \int_0^T \left| \Theta_{nk}(t, Y_{tnk}^n, \tilde{Y}_{tnk}^n) - \Theta(t, Y_t, \tilde{Y}_t) \right|^2 dt > \delta \right)
\leq \mathbb{P} \otimes \bar{\mathbb{P}} \left( T > \tau_{nk}(R) \wedge \bar{\tau}(R) \right)
+ \mathbb{P} \otimes \bar{\mathbb{P}} \left( T \leq \tau_{nk}(R) \wedge \bar{\tau}(R) ; \int_0^T \left| \Theta_{nk}(t, Y_{tnk}^n, \tilde{Y}_{tnk}^n) - \Theta_{nk0}(t, Y_{tnk}^n, \tilde{Y}_{tnk}^n) \right|^2 dt > \delta/9 \right)
+ \mathbb{P} \otimes \bar{\mathbb{P}} \left( T \leq \tau_{nk}(R) \wedge \bar{\tau}(R) ; \int_0^T \left| \Theta_{nk}(t, Y_{tnk}^n, \tilde{Y}_{tnk}^n) - \Theta_{nk0}(t, Y_{tnk}^n, \tilde{Y}_{tnk}^n) \right|^2 dt > \delta/9 \right)
+ \mathbb{P} \otimes \bar{\mathbb{P}} \left( T \leq \tau_{nk}(R) \wedge \bar{\tau}(R) ; \int_0^T \left| \Theta_{nk0}(t, Y_{tnk}^n, \tilde{Y}_{tnk}^n) - \Theta(t, Y_t, \tilde{Y}_t) \right|^2 dt > \delta/9 \right)
= I_1 + I_2 + I_3 + I_4, \text{ say.}
\]

Now observe that
\[
I_1 \leq \mathbb{P} \otimes \bar{\mathbb{P}} (\tau_{nk}(R) < T) + \mathbb{P} \otimes \bar{\mathbb{P}} (\tau(R) < T)
\leq \mathbb{P} \otimes \bar{\mathbb{P}} (\tau_{nk}(R) < T) + \limsup_{l \to \infty} \mathbb{P} \otimes \bar{\mathbb{P}} (\tau_{nl}(R) < T).
\]

From (18) we obtain that \(\lim_{R \to \infty} \limsup_{k \to \infty} I_1 = 0\).

By continuity of \(\Theta_{nk}\), it is clear that for fixed \(k_0\), the second term, i.e. \(I_2\) tends to zero as \(k \to \infty\). To take the limit of \(I_3\) and \(I_4\), we use Lemma 3.2. Since \((t, x) \mapsto \sup_{n \in \mathbb{N}} \mathbb{E} b_n(t, x, \tilde{Y}_t^n)\)
is locally bounded, for \( t \leq T \land \tau_{n_k}(R) \), \( \sup_{k \in \mathbb{N}} \tilde{b}_{n_k} \left( t, Y^{n_k}_t, \bar{Y}^{n_k}_t \right) \) is bounded. Inequality \( (17) \) implies that there exists \( K_R \in \mathbb{N} \) such that for all \( k \geq K_R \),

\[
\inf_{t \in [0, T \land \tau_{n_k}(R)] \atop |h| \leq 1} h^T \sigma_{n_k} \left( t, Y^{n_k}_t, \bar{Y}^{n_k}_t \right) \sigma_{n_k}^T \left( t, Y^{n_k}_t, \bar{Y}^{n_k}_t \right) h \geq \varepsilon(R, T) > 0.
\]

Therefore the conditions of Lemma 5.2 for Itô process \( (Y^{n_k}_t, \bar{Y}^{n_k}_t) \) and the exit time \( \tau_{n_k}(R) \) hold for all \( k \geq K_R \) and there exists a constant \( C(R, T) \) such that,

\[
I_3 \leq \frac{9}{\delta} \mathbb{E} \mathbb{E} \left( \int_0^{T \land \tau_{n_k}(R)} \left| \Theta_{n_k}(t, Y^{n_k}_t, \bar{Y}^{n_k}_t) - \Theta_{n_k}(t, Y^{n_k}_t, \bar{Y}^{n_k}_t) \right|^2 \, dt \right) \leq C(R, T) \left( \int_0^{T \land \tau_{n_k}(R)} \left| \Theta_{n_k}(t, x, y) - \Theta_{n_k}(t, x, y) \right|^{4d+2} \, dtdxdy \right)^{1/4d+2} \to 0,
\]

which tends to zero as \( k, k_0 \to \infty \) since \( \Theta_n \to \Theta \) in \( L_{loc}^{4d+2} \) as \( n \to \infty \). Let \( w \in C \left( \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R} \right) \) be compactly supported with \( 1_{B_R \times B_R} \leq w \leq 1 \). Then

\[
I_4 \leq \frac{9}{\delta} \mathbb{E} \mathbb{E} \int_0^{T \land \tau_{n_k}(R)} w(Y_t, \bar{Y}_t) \left| \Theta_{n_k}(t, Y_t, \bar{Y}_t) - \Theta(t, Y_t, \bar{Y}_t) \right|^2 \, dt.
\]

Since continuous functions are dense in

\[
L^2 \left( [0, T] \times B_R \times B_R, \mu \right) \cap L^{4d+2} \left( [0, T] \times B_R \times B_R, \lambda \right),
\]

where \( \lambda \) is the Lebesgue measure and \( \mu \) is the following finite Borel measure,

\[
\mu(A) := \mathbb{E} \mathbb{E} \int_0^T 1_{\{t, Y_t, \bar{Y}_t \in A\}} w(Y_t, \bar{Y}_t) \, dt,
\]

we can find for every \( \varepsilon > 0 \), a continuous function \( g \) on \( [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \) such that

\[
\left( \mathbb{E} \mathbb{E} \int_0^T w(Y_t, \bar{Y}_t) \left| \Theta_{n_k}(t, Y_t, \bar{Y}_t) - \Theta(t, Y_t, \bar{Y}_t) - g(t, Y_t, \bar{Y}_t) \right|^2 \, dt \right)^{1/2}
\]

\[
+ \left( \int_0^T \int_{B_R} \int_{B_R} \left| \Theta_{n_k}(t, x, y) - \Theta(t, x, y) - g(t, x, y) \right|^{4d+2} \, dxdydt \right)^{1/4d+2} \leq \varepsilon.
\]

So

\[
\frac{(\delta I_4)^{1/2}}{3} \leq \left( \mathbb{E} \mathbb{E} \int_0^{T \land \tau(R)} w(Y_t, \bar{Y}_t) \left| g(t, Y_t, \bar{Y}_t) \right|^2 \, dt \right)^{1/2} + \varepsilon
\]

\[
= \left( \mathbb{E} \mathbb{E} \int_0^T 1_{\{t \leq \tau(R)\}} w(Y_t, \bar{Y}_t) \left| g(t, Y_t, \bar{Y}_t) \right|^2 \, dt \right)^{1/2} + \varepsilon
\]

\[
\leq \liminf_{l \to \infty} \left( \mathbb{E} \mathbb{E} \int_0^T 1_{\{t \leq \tau_{n_l}(R)\}} w(Y_t^{n_l}, \bar{Y}_t^{n_l}) \left| g(t, Y_t^{n_l}, \bar{Y}_t^{n_l}) \right|^2 \, dt \right)^{1/2} + \varepsilon.
\]
Thus, we get for large enough \( l \in \mathbb{N} \),
\[
\frac{(\delta I_4)^{1/2}}{3} \leq \left( \mathbb{E} \mathbb{E} \int_0^{T \wedge \tau_{n_1}(R)} \left| w(Y_{t}^{n_1}, \tilde{Y}_{t}^{n_1}) \right|^2 \text{d}t \right)^{1/2} + 2\varepsilon
\]
Then by Lemma 3.2 we have
\[
\frac{(\delta I_4)^{1/2}}{3} \leq C(R, T) |g|_{L^{4d+2}([0, T] \times B_R \times B_R, \lambda)} + 2\varepsilon
\]
\[
\leq C(R, T) \left( \left| \Theta_{n_k_0} - \Theta \right|_{L^{4d+2}([0, T] \times B_R \times B_R, \lambda)} + \varepsilon \right) + 2\varepsilon
\]
So, \( I_4 \) also tends to zero as \( k_0 \to \infty \). Hence
\[
\int_0^T \left| \Theta_{n_k}(t, Y_{t}^{n_k}, \tilde{Y}_{t}^{n_k}) - \Theta(t, Y_t, \tilde{Y}_t) \right|^2 \text{d}t \to 0 \quad \text{in probability},
\]
as \( k \to \infty \) and therefore \( M_{t}^{n_k} \to M_t \) and \( N_{t}^{n_k} \to N_t \) almost surely. \( \square \)

**Remark 3.3.** The proof of Theorem 3.4 is shorter than the proof of weak existence theorem in [11] because in case \( B \), we estimated \( b, \sigma \) in the smaller space \( L^{4d+2} \) instead of \( L^{2d+1} \) and also we used the representation theorem for martingales. In fact Theorem 3.1 is more general than the weak existence result stated in [11] and to prove that, it is enough to approximate \( b, \sigma \) in the space \( L^{4d+2} \) instead of \( L^{2d+1} \).

**Theorem 3.4.** Let \( b, \sigma : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, \mathbb{R}^{d \times d_1} \) be measurable. Consider the equation
\[
\text{d}X_t = \tilde{\mathbb{E}}b(t, X_t, \tilde{X}_t) \text{d}t + \tilde{\mathbb{E}}\sigma(t, X_t, \tilde{X}_t) \text{d}W_t
\]
with initial value \( X_0 = \xi \). Assume that there exists a convex function \( V \in C^2 \left( \mathbb{R}^d, [0, \infty) \right) \) such that for some \( q > 2 \),
\[
(H1) \ \langle \nabla V(x), b(t, x, y) \rangle + \frac{1}{2} \text{tr} \left( \sigma^T(t, x, y) \text{D}^2 V(x) \sigma(t, x, y) \right) < CV(x),
\]
\[
(H2) \ |b(t, x, y)|^q + |\sigma(t, x, y)|^q < V(x)V(y),
\]
\[
(H3) \ \mathbb{E}V(\xi) + \mathbb{E} \| \xi \|^2 < \infty.
\]
Also assume that \( (x, y) \mapsto b(t, x, y), \sigma(t, x, y) \) are continuous or \( \sigma \), for every \( R > 0 \), satisfies
\[
\inf_{t \in [0, T], |x| < R} \inf_{\mu \in \mathcal{P}} \lambda^T \int_{\mathbb{R}^d} \sigma(t, x, y) \mu(\text{d}y) \int_{\mathbb{R}^d} \sigma^T(t, x, y) \mu(\text{d}y) \lambda > 0.
\]
where \( \mathcal{P} \) is the space of probability measures on \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \). Then (19) has a weak solution, say \( X \), satisfying \( \mathbb{E}V(X_t) \leq e^{Ct} \mathbb{E}V(\xi) \).
Proof. We define for \( n \in \mathbb{N} \) and \( z = (x, y) \) the following globally Lipschitz continuous functions

\[
b_{n,r}(t, z) := \psi^2 \left( \frac{z}{n} \right) \int_{\mathbb{R}^{2d}} b(t, z + \tilde{z}) r^{2d} \phi(r \tilde{z}) d\tilde{z},
\]

\[
\sigma_{n,r}(t, z) := \psi \left( \frac{z}{n} \right) \int_{\mathbb{R}^{2d}} \sigma(t, z + \tilde{z}) r^{2d} \phi(r \tilde{z}) d\tilde{z},
\]

where \( 0 \leq \psi \leq 1 \) and \( 0 \leq \phi \) are compactly supported radial smooth functions with \( \psi|_{B_1} = 1 \) and \( \int_{\mathbb{R}^{2d}} \phi(x) dx = 1 \). Since \( D^2V \) is positive semi-definite, the function

\[
\sigma \mapsto \frac{1}{2} \text{tr} \left( \sigma^T D^2V(x) \sigma \right)
\]

is convex. Since \( V \in C^2(\mathbb{R}^d, [0, \infty)) \), for an arbitrary \( \varepsilon > 0 \), there exists \( r_n > 0 \) large enough such that

\[
\langle \nabla V(x), b_{n,r_n}(t, z) \rangle + \frac{1}{2} \text{tr} \left( \sigma_{n,r_n}^T(t, z) D^2V(x) \sigma_{n,r_n}(t, z) \right)
\]

\[
\leq \psi^2 \left( \frac{z}{n} \right) \left[ \langle \nabla V(x), \int_{\mathbb{R}^{2d}} b(t, z + \tilde{z}) r_n^{2d} \phi(r_n \tilde{z}) d\tilde{z} \rangle 
\right.
\]

\[
+ \frac{1}{2} \text{tr} \left( \int_{\mathbb{R}^{2d}} \sigma^T(t, z + \tilde{z}) r_n^{2d} \phi(r_n \tilde{z}) d\tilde{z} \int_{\mathbb{R}^{2d}} \sigma(t, z + \tilde{z}) r_n^{2d} \phi(r_n \tilde{z}) d\tilde{z} \right) \n\]

\[
\leq \psi^2 \left( \frac{z}{n} \right) \int_{\mathbb{R}^{2d}} \left[ \langle \nabla V(x), b(t, z + \tilde{z}) \rangle + \frac{1}{2} \text{tr} \left( \sigma^T(t, z + \tilde{z}) D^2V(x) \sigma(t, z + \tilde{z}) \right) \right] r_n^{2d} \phi(r_n \tilde{z}) d\tilde{z}
\]

\[
\leq \psi^2 \left( \frac{z}{n} \right) \int_{\mathbb{R}^{2d}} \left[ \langle \nabla V(x + \tilde{x}), b(t, z + \tilde{z}) \rangle 
\right.
\]

\[
+ \frac{1}{2} \text{tr} \left( \sigma^T(t, z + \tilde{z}) D^2V(x + \tilde{x}) \sigma(t, z + \tilde{z}) \right) \n\]

\[
\left. \right] r_n^{2d} \phi(r_n \tilde{z}) d\tilde{z} + \frac{\varepsilon}{2} \leq CV(x) + \varepsilon
\]

The same argument implies

\[
|b_{n,r_n}(t, x, y)|^q + |\sigma_{n,r_n}(t, x, y)|^q < V(x)V(y) + \varepsilon,
\]

Let us take \( r_n > 0 \) large enough as above and \( r_n \to \infty \) as \( n \to \infty \). It is clear that \( b_n = b_{n,r_n} \) and \( \sigma_n = \sigma_{n,r_n} \) are bounded and globally Lipschitz. Therefore there exists a unique solution to

\[
dX^n_t = \bar{E}b(t, X^n_t, \hat{X}^n_t) dt + \bar{E}\sigma(t, X^n_t, \hat{X}^n_t) dW_t
\]

(20)

with any arbitrary \( \mathcal{F}_0 \) measurable initial value \( X^n_0 = \xi \). To show that the assumptions of Theorem 3.1 hold, it is sufficient to prove that \( \sup_{n \in \mathbb{N}} \mathbb{E}V(X^n_t) < C_T \) for all \( t \in [0, T] \). We
have by convexity of $V$ and [H1]
\[
e^{-Ct}V(X^n_t) = V(\xi) + \int_0^t e^{-Cs} \left[ \left\langle \nabla V(X^n_s), \tilde{E}b_n \left( s, X^n_s, \tilde{X}^n_s \right) \right\rangle \right.
\]
\[
+ \frac{1}{2} \text{tr} \left( \tilde{E}\sigma_n^T \left( s, X^n_s, \tilde{X}^n_s \right) D^2V(X^n_s)\tilde{E}\sigma_n \left( s, X^n_s, \tilde{X}^n_s \right) \right)
\]
\[
- CV(X^n_s) \right] ds + M_t
\]
\[
\leq V(\xi) + \tilde{E} \int_0^t e^{-Cs} \left[ \left\langle \nabla V(X^n_s), b_n \left( s, X^n_s, \tilde{X}^n_s \right) \right\rangle \right.
\]
\[
+ \frac{1}{2} \text{tr} \left( \sigma_n^T \left( s, X^n_s, \tilde{X}^n_s \right) D^2V(X^n_s)\sigma_n \left( s, X^n_s, \tilde{X}^n_s \right) \right) - CV(X^n_s) \right] ds + M_t
\]
\[
\leq V(\xi) + \int_0^t \varepsilon e^{-Cs} ds + M_t = V(\xi) + \frac{\varepsilon C}{1 - e^{-C}} (1 - e^{-Ct}) + M_t
\]
where $M_t$ is a local martingale starting from zero. Let $\tau_m \uparrow \infty$ be a corresponding localizing sequence. Then by Fatou’s lemma,
\[
e^{-Ct} \mathbb{E}V(X^n_t) \leq \liminf_{m \to \infty} \mathbb{E} \left[ e^{-C(t \wedge \tau_m)} V(X^n_{t \wedge \tau_m}) \right] = \mathbb{E}V(\xi) + \frac{\varepsilon C}{1 - e^{-C}} (e^{Ct} - 1).
\]
By Theorem 3.4 there exist a weak solution to (19) like $X_t$ and some subsequence $X^{n_k}$ which converges in law to $X$ on $C([0,T], \mathbb{R}^d)$ as $k \to \infty$. Hence $\mathbb{E}V(X_t) \leq e^{Ct} \mathbb{E}V(\xi)$.}

\begin{flushright}
\text{□}
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**Corollary 3.5.** Let $b, \sigma : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, \mathbb{R}^{d \times d_1}$ be measurable. Consider the equation
\[
dX_t = \tilde{E}b(t, X_t, \tilde{X}_t)dt + \tilde{E}\sigma(t, X_t, \tilde{X}_t)dW_t
\]
(21)
with initial value $X_0 = \xi$. Here $\tilde{X}_t$ is an independent copy of $X_t$. Suppose that one of the following assumptions holds for $q > 2$:

1. Assume that there exists $\alpha \geq 1$ such that
\[
|x|^2 \left( 2 \langle x, b(t, x, y) \rangle + |\sigma(t, x, y)|^2 \right) + (\alpha - 2) |\sigma^T(t, x, y)x|^2 \leq C(1 + |x|^4),
\]
\[
|b(t, x, y)|^q + |\sigma(t, x, y)|^q \leq C(1 + |x|^\alpha)(1 + |y|^\alpha),
\]
and $\mathbb{E} \left( |\xi|^{\alpha_1} \right) < \infty$.

2. Assume that there exist $p \in [1, 2]$ and $\alpha > 0$ such that
\[
|x|^2 \left( 2 \langle x, b(t, x, y) \rangle + |\sigma(t, x, y)|^2 \right) + (\alpha p |x|^p + p - 2) |\sigma^T(t, x, y)x|^2 \leq C(1 + |x|^{4-p}),
\]
\[
|b(t, x, y)|^q + |\sigma(t, x, y)|^q \leq C \exp(\alpha |x|^p + \alpha |y|^p),
\]
and $\mathbb{E} \left[ \exp(\alpha |\xi|^p) \right] < \infty$.

Also assume that $(x, y) \mapsto b(t, x, y), \sigma(t, x, y)$ are continuous or $\sigma$ is symmetric and uniformly positive definite, i.e.,
\[
\inf_{s, x, y} \inf_{|\lambda| = 1} \lambda^T \sigma(s, x, y) \lambda > 0.
\]

Then equation (21) has a weak solution.
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