PROOF MINING AND EFFECTIVE BOUNDS IN DIFFERENTIAL POLYNOMIAL RINGS

WILLIAM SIMMONS AND HENRY TOWSNER

Abstract. Using the functional interpretation from proof theory, we analyze nonconstructive proofs of several central theorems about polynomial and differential polynomial rings. We extract effective bounds, some of which are new to the literature, from the resulting proofs. In the process we discuss the constructive content of Noetherian rings and the Nullstellensatz in both the classical and differential settings. Sufficient background is given to understand the proof-theoretic and differential-algebraic framework of the main results.

1. Introduction

This paper is concerned with proofs of finitary statements which pass through an ultraproduct construction as an intermediate step. The basic idea is illustrated by the following theorem:

Theorem ([53], Theorem 2.5). For every $n$ and $d$, there is a bound $b$ so that whenever $K$ is a field and $\Lambda$ is a finite set of generators in $K[\{X\}_1, \ldots, \{X\}_n]$ with total degree bounded by $d$, the following implication holds: if either $f \in (\Lambda)$ or $g \in (\Lambda)$ for all $fg \in (\Lambda)$ such that $fg$ has total degree $\leq b$, then $(\Lambda)$ is prime.

Their proof proceeds as follows: suppose this were false for some $n$ and $d$. That is, for each $b$, there exists some field $k_b$ and some $\Lambda_b$ in $k_b[\{X\}_1, \ldots, \{X\}_n]$ with total degree bounded by $d$ satisfying the assumption but with $(\Lambda_b)$ not prime. They take an ultraproduct $K = \prod U_k$ and $\Lambda = \prod U_{\Lambda}$ and then work in the ring $K[\{X\}_1, \ldots, \{X\}_n]$ to obtain a contradiction. Details on the ultraproduct construction can be found, for instance, in [28], but this will not concern us here, since our interest is on how to eliminate—“unwind”—the use of ultraproducts.

The disadvantage to such a proof is that it appears to be non-constructive—one assumes, towards a contradiction, that the $\Lambda_b$ exist for all $b$, but this does not directly tell us how large the bound $b$ actually is. By eliminating the ultraproduct construction from these proofs, we will obtain explicit calculations of these bounds.

Date: September 27, 2016.

Partially supported by NSF grant DMS-1600263.
1.1. **Unwinding Ultraproduct Proofs.** The essential technique comes from proof theory: one views a proof of a property $\sigma$ in an ultraproduct as a sequence of statements

$$\sigma_1, \ldots, \sigma_n, \sigma$$

where each step follows from the earlier ones. In order to obtain a direct proof, we replace each step “$\sigma_i$ is true in the ultraproduct” with some related fact “$\sigma'_i$ is true in every field”. For most of the results we are interested in, the conclusion $\sigma$ is the same as $\sigma'$, which is, of course, the point. The difficulty is that, for intermediate statements, this may not be true: sometimes $\sigma_i$ is true in an ultraproduct, but *not* true in arbitrary fields. In this case we need to replace $\sigma_i$ with some different formula $\sigma'_i$.

It turns out that the right translation is a tool called the *monotone functional interpretation* \[30\]. The functional (or “Dialectica”) interpretation was introduced by Gödel \[15\]; the monotone variant was developed by Kohlenbach to make it easier to apply to ordinary mathematical proofs. (See also \[2, 14, 29, 51, 52\] for more background on the functional interpretation.)

The functional interpretation tells us to look at the syntactic form of the statement $\sigma$ in (a suitable language of) first-order logic. Our main conclusions, like Theorem 2.5 of \[53\], turn out to be equivalent to $\Pi_2$—that is, $\forall\exists$—statements. (In many cases, including Theorem 2.5 of \[53\], this equivalence is not obvious.) Relatedly, the functional interpretation is essentially the identity on $\Pi_2$ statements, as we would expect.

Intermediate steps, however, may be more complicated. For example, the proofs below will use Hilbert’s Basis Theorem, which may be stated as:

For every $n$ and every increasing sequence of ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ in $K[X_1, \ldots, X_n]$, there is an $m$ so that for all $m' > m$, $I_m = I_{m'}$.

For our purposes, we will take the $I_i$ to be finitely generated. Then this can be seen as a $\Pi_3$ statement:\[1\]:

$$\forall\{I_i\} \forall n \exists d \forall d' \cdots .$$

The functional interpretation tells us to replace this with a function bound \[25\]:

For every $n$, every function $D$, and every function $F$, there is an $M$ so that whenever $(\Lambda_1) \subseteq (\Lambda_2) \subseteq \cdots$ is an increasing sequence of finitely-generated ideals in $K[X_1, \ldots, X_n]$ where each polynomial in $\Lambda_i$ has total degree $\leq D(i)$, there is an $m \leq M$ so that $\Lambda_{F(m)} \subseteq (\Lambda_m)$.

The conclusion is weaker: we no longer ask for an $I_m = (\Lambda_m)$ which is the ultimate union of the sequence of ideals, but instead ask for a long interval

\[1\]Technically, first-order logic cannot include the outer quantifier over the $\{I_i\}$, but a standard trick is to add a symbol to the language which will represent the sequence $\{I_i\}$, and this is equivalent to allowing a single universal quantifier on the outside of the formula, which is precisely what we need here.
on which the sequence seems to have stabilized. In return, we obtain that
the bound is uniform—it does not depend on the field \( K \) or the particular
polynomials \( \Lambda_i \), only on bounds on the total degree of the \( \Lambda_i \)—and can be
computed from \( n, D, \) and \( F \). (We will compute such bounds below in the
simpler but equivalent case where \( F(i) = i + 1 \), which is the only case we
need.)

What the functional interpretation guarantees is that any application
of Hilbert’s Basis Theorem as an intermediate step in the proof of a \( \Pi_2 \)
statement can actually be replaced by the version with a function bound for
the right choice of \( F \): in other words, a proof of a \( \Pi_2 \) statement never needs
to use an \( I_m \) which is truly maximal; it always suffices to choose \( I_m \) so that
\( I_{F(m)} \subseteq I_m \) for some big enough function \( F(m) \).

1.2. Insights from unwound proofs. The functional interpretation is a
formal technique whose properties are established by rigorous theorems (29
presents many of the main technical results). Such theorems establish
critical properties of the functional interpretation such as modularity: rather
than transforming an entire proof at once, one can refine large steps into
sublemmas whose interpretations are more tractable [51].

Importantly, though, the reader need not be familiar with the details in
order to understand the output of the method, which can be expressed as
constructive arguments in standard mathematical language. Even for the
practitioner, the functional interpretation can be used as a heuristic for con-
verting unformalized nonconstructive proofs into algorithmic ones without
having to go through a formal language [14], [51]. Accordingly, throughout
the paper we keep the machinery of the functional interpretation in the
background while retaining the product: explicit procedures for computing
desired objects and quantitative bounds on the complexity of those proce-
dures.

A great virtue of the functional interpretation is that it is systematic
and applies in many situations. Proofs often have a “hidden combinatorial
core” [29] that the functional interpretation can identify. (For instance, the
intricate combinatorics in Szemerédi’s proof of his regularity lemma [49]
automatically emerge from the functional interpretation [16][50].)

In this paper, we aim to use the perspective provided by the functional
interpretation to analyze ultraproduct proofs of the Nullstellensatz and re-
lated results found in [53] and [23]. Our systematic use of tactics suggested
by the functional interpretation gives an alternate route to effective versions
of these important theorems.

1.3. Plan of the paper. We briefly review the Nullstellensatz and differ-
ential Nullstellensatz prior to outlining our path in the rest of the paper.
A standard form of Hilbert’s Nullstellensatz states that for an algebraically
closed field \( K \), given an ideal \( I \subseteq K[X_1, \ldots, X_n] \), the radical ideal \( \sqrt{I} \) con-
sists of all polynomials over \( K \) that vanish on the common zero locus \( V(I) \)
of $I$. This is a nonconstructive statement, owing to the existential nature of the definition of radical ideals, and the usual proofs are also ineffective.

The “effective Nullstellensatz” is the problem of finding uniform bounds on radical ideal membership valid for all fields and only depending on the number of algebraic unknowns and degrees of the generators. Brownawell, Kollár, Dubé, and subsequent authors have employed analytic, algebraic, and combinatorial techniques to show that single exponential bounds suffice \([5, 9, 32]\). In contrast, van den Dries and Schmidt justify their focus on nonstandard methods by observing that “by concentrating on existence proofs for bounds, rather than on their construction, it is possible to gain a lot of efficiency of exposition” \([53]\). We do not explicitly state their results and arguments, but we cite the corresponding nonconstructive analogues. Our unwindings of their proofs show that van den Dries and Schmidt’s ultraproduct strategy is not only elegant, but also (implicitly) preserves much more effective content than one might suppose.

Our other basic source for nonstandard proofs is \([23]\) by Harrison-Trainor, Klys, and Moosa, who adapt the techniques of van den Dries and Schmidt to the more complicated differential case. Differential fields enrich the field structure by adding commuting derivations (additive endomorphisms obeying the usual product rule for derivatives). Ritt \([43]\) and Raudenbush \([41, 42]\) enunciated differential-algebraic versions of the basis theorem and Nullstellensatz, the latter of which Cohn \([7]\) and Seidenberg \([46]\) approached from an algorithmic angle (without giving explicit bounds). The effective differential Nullstellensatz consists of giving bounds on radical differential ideal membership or, equivalently, consistency of systems of polynomial differential equations.

Recently there has been considerable interest in analyzing the effective content of the differential Nullstellensatz \([8, 18, 20, 22, 34]\), with the methods employed coming from algebra and model theory \([11, 37, 40]\). Other constructive problems in differential algebra and differential algebraic geometry have also gained attention \([4, 12, 13, 19, 21, 26, 35]\).

In the rest of the section, we preview the technical part of the paper. We list our main results and indicate how they should be understood. Because our results require stating a series of explicit functions bounding various properties, we include an index to where the definitions of these functions can be found and, where appropriate, where bounds on their rate of growth are proven:
Table 1. Table of Notations and Bounds

| Function | Definition | Calculated Bounds |
|----------|------------|-------------------|
| $d_n$    | Notation 2.14 (p. 8) |                   |
| $c_n$    | Notation 2.9 (p. 9)     |                   |
| $\zeta_n$| Notation 2.17 (p. 12)   |                   |
| $p_n$    | Notation 2.19 (p. 13)   | Lemma 6.7 (p. 37) |
| $m$      | Notation 3.5 (p. 15)    | Lemma 6.10 (p. 39) |
| $m^*$    | Notation 3.5 (p. 15)    | Lemma 6.10 (p. 39) |
| $g$      | Notation 4.14 (p. 21)   |                   |
| $u_F$    | Notation 4.18 (p. 22)   | Lemma 6.12 (p. 39) |
| $u_F^*$  | Notation 4.20 (p. 23)   | Lemma 6.12 (p. 39) |
| $f$      | Notation 4.26 (p. 24)   | Lemma 6.13 (p. 39) |
| $h_{n,m}$| Notation 4.30 (p. 25)   | Lemma 6.18 (p. 41) |
| $i_{n,m}^{\text{sat}}$ | Notation 4.39 (p. 27) | Lemma 6.19 (p. 42) |
| $i_{n,m}^{\text{cohere}}$ | Notation 4.39 (p. 28) | Lemma 6.20 (p. 42) |
| $i_k$    | Notation 4.44 (p. 29)   | Lemma 6.14 (p. 40) |
| $j_{n,m}$| Notation 4.47 (p. 30)   | Lemma 6.21 (p. 43) |
| $\lambda$ | Theorem 5.2 (p. 34)     | Lemma 6.22 (p. 43) |
| $j_{n,m}$| Notation 5.3 (p. 34)    | Lemma 6.27 (p. 45) |

Our goal in Section 2 is to unwind the existence proof of a bound on prime ideals given by Theorem 2.5 of [53]. The main ingredients are finitary counterparts of prime ideals and vector space bases, as well as bounds on flat extensions of polynomial rings (Theorem 2.5 and Lemma 2.6). The result 2.20, which has a form typical of others in the paper, is

**Theorem.** Let $n, d$ be given. If $\Lambda \subseteq K[X_{[n]}]_{\leq d}$ is such that $(\Lambda)$ is prime up to $p_n(d)$ then $(\Lambda)$ is prime.

The subscript denotes a bound on the degree of the polynomials in question (see Definition 2.2). “Primality up to some value” (Definition 2.13) is a “local” - in particular, easily seen to be computable - notion of primality suggested by the functional interpretation. The symbol $p_n$ represents a certain recursively-defined bounding function (Notation 2.19) on the degree of possible counterexamples to primality of an ideal $I \subseteq K[X_1, \ldots, X_n]$, where $K$ is an arbitrary field. Such a bound is implicit in the sense that we must analyze the recursive definition in order to establish the growth rate of $p_n$ in comparison to some well-known benchmark. See Section 6 for such an analysis.

Our actual bounds tend to be rough, and it is not surprising that in many cases carefully optimized arguments (e.g., Theorem 3.4 of [22]) give tighter bounds. The functional interpretation’s output is dependent on its input and so cannot improve on the implicit constructive content of a given ineffective proof. It is nonetheless meaningful to expose that content, especially since general classes of bounds are often of most intrinsic interest. Like the bounds
obtained in [18, 22, 34, 37], our main bounds are non-primitive recursive. This indicates either an actual complexity barrier or the need for fundamentally new ideas that can qualitatively lower the bounds beyond what any existing proofs provide.

Section 3 deals with the underlying complexity of Noetherianity and its consequences. Using a bound on Dickson’s Lemma from the literature (Theorem 3.7) as a shortcut, we unwind proofs of Hilbert’s Basis Theorem and the Nullstellensatz (Theorems 3.9 and 3.10):

**Theorem.** Suppose $(\Lambda_1) \subseteq (\Lambda_2) \subseteq \cdots \subseteq K[X[n]]$ with $\Lambda_i \subseteq K[X[n]] \leq D(i)$. Then there is a $j \leq m^*(D, n)$ such that $(\Lambda_{j+1}) \subseteq (\Lambda_j)$.

(Here $D$ denotes a given nondecreasing function from $\mathbb{N}$ to $\mathbb{N}$ and $m^*$ is the aforementioned bound on Dickson’s Lemma.)

**Theorem** (Based on [53], Cor 2.7(ii)). For any $n, d$ there is $m = m^*(i \mapsto p^i_n(d), n)$ so that if $\Lambda \subseteq K[X[n]] \leq d$ and $f^k \in (\Lambda)$ (for any $k$) then $f = \sum_i c_i r_i$ where each $r_i^{2^m} \in (\Lambda)$.

The use of Noetherianity is the key factor driving our bounds, as well as those found in other papers that examine the complexity of the differential Nullstellensatz. Morally, many of the bounds we extract are non-primitive recursive because the original nonconstructive proof invokes Noetherianity. (Moreno Socías proved non-primitive recursiveness of bounds on Hilbert’s Basis Theorem in [37]. Also, Simpson has shown in the sense of reverse mathematics that proving Hilbert’s Basis Theorem is equivalent to proving that Ackermann’s function–well known to not be primitive recursive–is a total function [47].) The resulting bounds, as shown in Section 6, are far larger than the doubly-exponential bounds from the slick proof of the effective Nullstellensatz in 3.11 which does not appeal to Noetherianity.

Section 4 is the longest of the paper and establishes many of our basic effective results on differential polynomial rings. Our treatment is largely self-contained, but additional details on differential algebra are found in, e.g., [6, 27, 31, 36, 48]. After describing the framework we establish bounds relating differential ideals and algebraic ideals (for example, bounds on the complexity of coherent sets and Rosenfeld’s lemma, Proposition 4.40 and Lemma 4.45). The basic ingredient for these bounds is a quantitative version of the theorem that there are no infinite descending chains of autoreduced sets (Theorem 4.31).

Our efforts in this section culminate in several new bounds. In [12], Freitag, Li, and Scanlon remark that “producing explicit equations for differential Chow varieties in specific cases would require effectivizing Theorem 6.1” of [23], which is precisely the content of Theorem 4.48 and its corollary 4.49.

**Theorem.** Suppose $\Lambda \subseteq K\{X[n]\} \leq b$ and let $P$ be a minimal prime $\Delta$-ideal containing $\Lambda$. Then $P$ has a characteristic set $\Sigma \subseteq K\{X[n]\} \leq \mathrm{char}(b)$.
EFFECTIVE BOUNDS IN DIFFERENTIAL POLYNOMIAL RINGS

In the differential setting, subscripts now indicate bounds on order as well as degree; see Definition 1.2. Moreover, Corollary 4.50 and Lemma 6.21 give an explicit bound, not on primality of differential ideals (which is open and equivalent to the well-known Ritt problem [17]), but on the weaker Theorem 5.4 of [23] that bounds only one factor:

**Theorem.** Let $\Lambda \subseteq K\{X_1\} \leq b$ be given with $1 \not\in [\Lambda]$. If either $f \in [\Lambda]$ or $g \in [\Lambda]$ for all $f, g \in K\{X_1\}$ with $fg \in [\Lambda]$ and $f \in K\{X_1\} \leq \text{char}(b)$, then $[\Lambda]$ is prime.

Section 5 is concerned with bounds on what we call “Ritt-Noetherianity”, the Noetherianity of radical differential ideals. The finitary version 5.4 of the Ritt-Raudenbush basis theorem is new:

**Theorem.** Let $i_0, \Lambda, \Lambda_0 \subseteq \Lambda_1 \subseteq \cdots, D, F, d$ be given such that:

- $\Lambda \subseteq K\{X_1\} \leq d$ is autoreduced, and
- $\Lambda_i \subseteq K\{X_1\} \leq D(i)$ for all $i$.

Then there is an $i \in [i_0, i(i_0, D, F, d, \Lambda)]$ so that $\Lambda_{F(i)} \subseteq \{\Lambda \cup \Lambda_i\}$.

We analyze the corresponding explicit bounds in 6.27. Using our finitary basis theorem, it is possible to unwind the proof of the differential Nullstellensatz found in [23] (Corollary 4.5/Theorem 6.3), but we do not include the details here; see the discussion at the beginning of Section 5.

With the unwinding work behind us, in Section 6 we show how to interpret the bounds produced by the functional interpretation in preceding sections. For most of our results we analyze the functions’ growth rates and find their place in the Grzegorczyk hierarchy of fast-growing functions [38]. For instance, the bound $p_n$ on primality in Section 2 lies in the second stage of the fast-growing hierarchy. To minimize disruption, we place in Appendix A the results on ordinal arithmetic needed to justify the calculations in this section.

2. **Explicit Bounds for Testing Primality**

In [53], van den Dries and Schmidt prove a number of results about ultraproducts of polynomial rings $K[X]$, and derive the existence of uniform bounds independent of $K$. Since these results are used extensively in [23], in this section we obtain effective versions using the methods described in the previous section.

We have two purposes: to demonstrate, in the simpler algebraic setting, the methods we will later use in the differential setting, and to produce actual explicit bounds we will need later. In some cases, effective proofs have been given by other means (often before [53]), and these bounds are often substantially more efficient than those given by unwinding the ultraproduct arguments. When this happens, we will sometimes simply cite the known bounds; at other times, unwinding the ultraproduct proof illustrates a useful technique, so we will also describe the less efficient proof.
Throughout this section we are concerned with an arbitrary field $K$ and its polynomial extension $K[X_1, \ldots, X_n]$.

**Notation 2.1.** We abbreviate $K[X_1, \ldots, X_n]$ by $K[X_n]$. More generally, we abbreviate $K[X_i, X_{i+1}, \ldots, X_j]$ by $K[X_{i,j}]$.

We prefer this to the more common abbreviations $K[\bar{X}]$ or even $K[X]$ because we wish to be explicit about the number of variables.

### 2.1. Internal Flatness and Faithful Flatness.

**Definition 2.2.** We write $K[X_n] \leq d$ for the set of polynomials in $K[X_n]$ of total degree at most $d$.

We say $K[X_n]$ is **internally flat bounded by** $D$ if for every $b$, whenever $f_1, \ldots, f_k \in K[X_n] \leq b$ are coefficients of a homogeneous linear equation $\sum_i f_i y_i = 0$ and $g_1, \ldots, g_k \in K[X_n]$ is a solution, there exist $h_{ij} \in K[X_n] \leq D(b)$ and $c_j \in K[X_n]$ so that $\sum_i f_i h_{ij} = 0$ for each $j$ and $\sum_j c_j h_{ij} = g_i$ for each $i$.

Internal flatness states that every solution to $\sum_i f_i y_i = 0$ is a linear combination of solutions of bounded degree. The name “internal flatness” refers to the fact when $K = \prod K_i$ is an ultraproduct, $K[X_n]_{\text{int}} = \prod K_i[X_n]$ is a flat extension of $K[X_n]$ if and only if there is some $D$ so that most $K_i[X_n]$ are internally flat bounded by $D$.

**Remark 2.3.** Although we will not need this notion, we can define internal flatness for any graded ring $R = \oplus_i R_i$ with $R_{\leq i} = \oplus_{|j| \leq i} R_j$: $R$ is internally flat bounded by $D, S$ if for every $k, b$, whenever $f_1, \ldots, f_k \in R_{\leq b}$ are coefficients of a homogeneous linear equation $\sum_i f_i y_i = 0$ and $g_1, \ldots, g_k \in R$ are a solution, there exist $h_{ij} \in R_{\leq D(k,b)}$ with $1 \leq j \leq S(k, b)$ and $c_j \in R$ so that $\sum_i f_i h_{ij} = 0$ for each $j$ and $\sum_j c_j h_{ij} = g_i$ for each $i$.

The bounds $b, S$ are unnecessary for polynomial rings because $K[X_n] \leq d$ is finite dimensional with dimension bounded in $n, d$.

**Notation 2.4.** We write $d_n(b) = (2b)^n$.

**Theorem 2.5 (H.43).** $K[X_n]$ is internally flat bounded by $d_n$.

More generally, given a system of $m$ homogeneous equations with coefficients in $K[X_n] \leq b$, the space of solutions is generated by solutions in $K[X_n] \leq d_n(m,b)$.

We also expect an analog of faithful flatness. It is standard that a flat extension is faithfully flat exactly when the extension does not create solutions to any unsolvable inhomogeneous linear equations with coefficients from the base ring. Then “internal faithful flatness” just says that if an inhomogeneous equation is solvable, the size of the solution should be bounded in the degrees of the coefficients. This is the same as giving bounds on ideal membership.
Lemma 2.6 ([24]). For any $n$ and any $f_i \in K[X_{[n]}]_{\leq b}$, if $\sum_{i \leq k} f_i g_i = h$ where $h$ has degree $b$ then there are $g_1', \ldots, g'_k \in K[X_{[n]}]_{\leq \vartheta_n(b)}$ such that $\sum_{i \leq k} f_i g_i' = h$.

2.2. Bounds on Primality. Working in the ultraproduct setting with $K = \prod \mathbf{K}_i$ and $K[X_{[n]}]_{\text{int}} = \prod \mathbf{K}_i(K[X_{[n]}])$, van den Dries and Schmidt [53] show

Theorem 2.7. If $I$ is an ideal in $K[X_{[n]}]$ then $I$ is prime iff $IK[X_{[n]}]_{\text{int}}$ is prime in $K[X_{[n]}]_{\text{int}}$.

The standard analog of this is

Theorem 2.8. There is a function $p_n(b)$ so that for any $\Lambda \subseteq K[X_{[n]}]_{\leq b}$ with $(\Lambda)$ not prime, there are $f, g \in K[X_{[n]}]_{\leq p_n(b)}$ so that $fg \in (\Lambda)$ but $f, g \not\in (\Lambda)$.

In [44] Schmidt-Göttsch shows that $p_n(b)$ has the form $b^{\beta(n)}$ for some $\beta$. Here we extract bounds with a worse dependence on $b$ directly from the simpler proof given in [53].

Notation 2.9. $\varepsilon(n, b) = 2(b+\vartheta_n-1(b))^{n-1}+1+1+b+\vartheta_n-1(b)$.

Lemma 2.10. Suppose $\phi : K(X_1) \to L$ is a field extension and $\lambda_1, \ldots, \lambda_k \in K[X_{[n]}]_{\leq b}$. Writing $\phi$ for the map $\phi : K(X_1)[X_{[2,n]}] \to L[X_{[2,n]}]$ as well, any solution in $L$ to $\sum_i \phi (\lambda_i) y_i = 0$ is a linear combination of images under $\phi$ of solutions from $K[X_{[n]}]_{\leq \varepsilon(n, b)}$.

Proof. By internal flatness, solutions to $\sum_i \phi (\lambda_i) y_i$ are linear combinations of solutions from $L[X_{[2,n]}]_{\leq \vartheta_n-1(b)}$. Let $M_1, \ldots, M_j$ list the $\leq (b+\vartheta_n-1(b))^{n-1}$ monomials of degree $\leq b+\vartheta_n-1(b)$ in $X_{[2,n]}$; then we may rewrite $\sum_i \phi (\lambda_i) y_i = 0$ as

$$\sum_i \phi (\sum_j f_{i,j} M_j) (\sum_j y_{i,j} M'_{j}) = 0$$

where $f_{i,j} \in K[X_{[1]}]_{\leq b}$. So we may expand this into a system of $\leq (b+\vartheta_n-1(b))^{n-1}$ equations of the form

$$\sum_i \sum_j \phi (f_{i,j}) y_{i,j} - j = \sum_k \phi (g_{k,j}) x_k = 0.$$

The solutions to a single equation $\sum_i \phi (g_{k,j}) x_k = 0$ are generated by the solutions of the form $(\phi (g_{k,j_0}), \ldots, -\phi (g_{1,j_0}), \ldots)$ (because $L$ is a field); by substituting $x_1 = \sum_k z_k \phi (g_{k,j_0})$ and $x_k = -z_k \phi (g_{k,j_0})$, we obtain a system of equations with one fewer equation and coefficients in $\phi (K[X_{[1]}]_{\leq 2b})$.

Repeating this, we eventually reduce to a single equation whose solutions are generated by the image of solutions from $K[X_{[1]}]_{\leq 2(b+\vartheta_n-1(b))^{n-1}b}$. Undoing the sequence of substitutions, we see that the original $x_i$ are generated by the images of solutions from $K[X_{[1]}]_{\leq 2(b+\vartheta_n-1(b))^{n-1}1b}$, and so the $y_i$ are generated by the images of solutions from $K[X_{[2,n]}]_{\leq 2(b+\vartheta_n-1(b))^{n-1}1b+b+\vartheta_n-1(b)}$. □
Lemma 2.11 (Based on [53], Lemma 2.3). For any \( n, b \) and \( \Lambda \subseteq K[X_{[n]}]_{\leq b} \), if \( f \in K[X_1] \) has degree \( > \varepsilon(n, b) \) and is irreducible then for any \( g \in K[X_{[n]}] \) such that \( fg \in (\Lambda) \), also \( g \in (\Lambda) \).

Proof. Let \( \lambda_1, \ldots \) enumerate \( \Lambda \). Suppose \( fg = \sum_i a_i \lambda_i \). Since \( f \) is irreducible, \( L = K[X_1]/(f) \) is a field; let \( \phi : K \to L \) be the natural embedding. We have a solution \( \sum_i \phi(a_i) \phi(\lambda_i) = 0 \) in \( L[X_2, \ldots, X_n] \). By the previous lemma, the solutions are generated by the images of solutions from \( K[X_{[n]}]_{\leq \varepsilon(n, b)} \).

Since \( f \) has degree \( > \varepsilon(n, b) \), we also have \( a_i = \sum_j c_j a_{ij} + f q_i \). Therefore

\[
fg = \sum_i a_i \lambda_i \\
= \sum_i (\sum_j c_j a_{ij} + f q_i) \lambda_i \\
= \sum_j c_j \sum_i a_{ij} \lambda_i + f \sum_i \lambda_i q_i \\
= f \sum_i \lambda_i q_i
\]

and therefore \( g = \sum_i \lambda_i q_i \in (\Lambda) \). \( \square \)

Lemma 2.12 (Based on [53], Corollary 2.4). For any \( n, b \) and any \( \Lambda \subseteq K[X_{[n]}]_{\leq b} \), one of the following holds:

- there is an \( f \in K[X_1]_{\leq \varepsilon(n-1, k)} \) and a \( g \) so \( \deg(g) \leq \varepsilon(n, b) + \varepsilon_{n-1}(b) + b \), \( fg \in (\Lambda) \), but \( g \notin (\Lambda) \), or

- whenever \( f \in K[X_1] \) and \( fg \in (\Lambda) \), \( g \in (\Lambda) \).

Proof. Suppose there is some \( f \in K[X_1] \) and some \( g \) so that \( fg \in (\Lambda) \) but \( g \notin (\Lambda) \). We may choose \( f \) with minimal degree such that this happens. Then \( f \) is irreducible—if \( f = f_0 f_1 \) then \( f_0(f_1 g) \in (\Lambda) \), so either \( f_1 g \notin (\Lambda) \) (so \( f_0 \) is a witness of smaller degree) or \( f_1 g \in (\Lambda) \) (so \( f_1 \) is a witness of smaller degree). By the previous lemma, \( f \in K[X_1]_{\leq \varepsilon(n, b)} \).

We have \( fg = \sum_i a_i \lambda_i \). Let \( L = K[X_1]/(f) \) and let \( \phi : K[X_1] \to L \) be the natural embedding, so \( \sum_i \phi(a_i) \phi(\lambda_i) = 0 \), and so by internal flatness, \( \phi(a_i) = \sum_j c_j a_{ij} \) where \( a_{ij} \in L[X_{[2, n]}]_{\leq \varepsilon_{n-1}(b)} \). We may assume \( a_{ij} = \phi(a'_{ij}) \) with \( a'_{ij} \in K[X_{[n]}]_{\leq \varepsilon(n, b) + \varepsilon_{n-1}(b)} \). We have \( a_i = \sum_j c_j a_{ij} + f q_i \) and \( \sum_i \lambda_i a_{ij} = f q'_j \).

Since \( \deg(\lambda_i a_{ij}) \leq \varepsilon(n, b) + \varepsilon_{n-1}(b) + b \), also \( \deg(q'_j) \leq \varepsilon(n, b) + \varepsilon_{n-1}(b) + b \).

There must be some \( j \) so \( q'_j \notin (\Lambda) \), and therefore \( f, q'_j \) is the witness to the first case. Otherwise, towards a contradiction, each \( q''_j = \sum_i \lambda_i q'_{ji} \), and therefore

\[
fg = \sum_i a_i \lambda_i \\
= \sum_i (\sum_j c_j a_{ij} + f q_i) \lambda_i \\
= \sum_j c_j \sum_i a_{ij} \lambda_i + \sum_i f q_i \lambda_{k,i}
\]
\[
\sum_j c_j f q_j' + \sum_i f q_i \lambda_i
= f \sum_j c_j \sum_i \lambda_i q_j' + f \sum_i q_i \lambda_i
= f \sum_i \lambda_i (\sum_j c_j q_j' + q_i),
\]
and therefore \(g = \sum_i \lambda_i (\sum_j c_j q_j' + q_i)\) so \(g \in (\Lambda)\), giving the needed contradiction. \(\square\)

**Definition 2.13.** We say an ideal \(I \subseteq K[X_n]\) is prime up to \(b\) if whenever \(fg \in (\Lambda)\) with \(f, g \in K[X_n]_{\leq b}\), either \(f \in (\Lambda)\) or \(g \in (\Lambda)\).

Notation 2.14. We write \(K(X)[Y_n]\) for those elements of \(K(X)[Y_n]\) whose total degree in the \(Y\) variables is at most \(r\). We write \(K(X)[Y_n]_{\leq X,Yr}\) for those elements of \(K(X)[Y_n]\) whose total degree in \(X, 1/X, Y_n\) is at most \(r\).

The proof in the ultraproduct involves using the fact that \(K_{int}(X)\) is freely generated over \(K(X)\). Therefore, given \(fg = \sum \lambda_i\) where the \(\lambda_i\) come from \((K_{int}(X))[Y_n]\), we can view the \(\lambda_i\) as coming from \(K(X, Z[m])[Y_n]\) where the \(Z[m]\) are a basis for some subspace large enough to contain the \(\lambda_i\).

In the finitary world, the analog of the basis \(Z[m]\) is a "local basis": a collection of elements \(Z_1, \ldots, Z_m\) such that, on the one hand, each \(a_i\) is algebraic in \(Z[m]\) using "small" coefficients (in the sense of the grading), but there are no algebraic dependencies among the \(Z(m)\) even using much larger coefficients.

**Definition 2.15.** We write \(K(X)_{\leq d}\) for \([K[X, 1/X]_{\leq d}\). Let \(S\) be a set of elements in \(K(X)\) and let \(F : \mathbb{N} \to \tilde{\mathbb{N}}\). An \(F\)-local basis for \(S\) is a set \(Z\) and a bound \(w\) such that:

- \(S \subseteq K(X, Z)_{\leq w}\),
- if \(z \in Z\) then \(z \notin K(X, Z \setminus \{z\})_{\leq F(w)}\).

**Lemma 2.16.** For any \(S \subseteq K(X, S)\) and any \(F\), letting \(F'(x) = xF(x)\), there is an \(F\)-local basis \(Z, w\) such that \(Z \subseteq S\) and \(w \leq (F')^{|S|}(1)\).

**Proof.** Set \(S_0 = S\) and \(w_0 = 1\). Given \(S_i, w_i\), if this is an \(F\)-local basis for \(S_i\), we are done. Otherwise, define \(S_{i+1}, w_{i+1}\) as follows: chose \(z \in S_i\) so that \(z \in K(X, S_i \setminus \{s\})_{\leq F(w_i)}\) and set \(S_{i+1} = S_i \setminus \{z\}\) and \(w_{i+1} = w_i F(w_i)\). Then for each \(s \in S_i\), since \(s \in K(X, S_i)_{\leq w_i}\), also \(s \in K(X, S_{i+1})_{\leq w_i F(w_i)}\).

Since \(S_{i+1} \subseteq S_i\), this process stops in at most \(|S|\) steps. \(\square\)
Notation 2.17.

- ζ₀(n, d) = \binom{n + b_n(d)}{n}, the number of monomials of degree ≤ d_n(d) in n variables,
- ζ₁(n, d, b) = \binom{b_n(d) + 2}{n}ζ₀(n, d),
- ζ₂(n, d, b) = (ζ₁(n, d, b) + 1)^{2ζ₁(n,d,b)} - 1.

This leads to the following crucial result. We show that sufficient primality in the sense of ≤_{X,Y} implies primality in the sense of ≤_Y. This is a weak form of the result we are attempting to prove: we begin with an ideal which is prime only for f, g with total degree in both X and Y bounded, and we obtain primality for f, g with Y-degree bounded, but arbitrary X-degree.

Lemma 2.18. Let n, b ≤ d be given. Then whenever Λ ⊆ K(X)[Y_{\lfloor n\rfloor}]_{≤_{X,Y}b} so that (Λ) is prime up to ζ₁(n, d, b)ζ₂(n, d, b) in the sense of ≤_{X,Y}, also (Λ) is prime in K(X)[Y_{\lfloor n\rfloor}] up to d in the sense of ≤_Y.

Proof. Let d and Λ ⊆ K(X)[Y_{\lfloor n\rfloor}]_{≤_{X,Y}b} be given so that (Λ) is prime up to ζ₁(n, d, b)ζ₂(n, d, b) in the sense of ≤_{X,Y}. Let f, g ∈ K(X)[Y_{\lfloor n\rfloor}]_{≤_Yd} be given with fg ∈ (Λ). This implies that fg = \sum a_i\lambda_i and, by Lemma 2.16, we may assume the a_i ∈ K(X)[Y_{\lfloor n\rfloor}]_{≤_{Yb_n(d)}}. Note that we may assume |Λ| ≤ (b_n(d) + 1), the dimension of K(X)[Y_{\lfloor n\rfloor}]_{≤_b} as a vector space over K(X).

We enumerate the monomials in Y_{\lfloor n\rfloor} appearing in the a_i as M_0, . . . , M_j, . . . . There are at most ζ₀(n, d) such monomials. We may write a_i = \sum\sum a_{ij}M_j, f = \sum u_jM_j, and g = \sum v_jM_j where a_{ij}, u_j, v_j are elements of K(X).

Let S_0 = \{a_{ij}\}_{i,j} \cup \{u_j, v_j\}. Note that |S_0| ≤ ζ₁(n, d, b). Let F be the function given by F(x) = xζ₁(n, d, b) + 1. By Lemma 2.16 there is an S ⊆ S_0 and a w ≤ ζ₂(n, d, b) so that S, w is an F-local basis for S_0.

We have

\[ (\sum_j u_jM_j)(\sum_j v_jM_j) = fg = \sum_i \sum_j a_{ij}M_j\lambda_i. \]

Writing each u_i, v_i, a_{ij} as an element of K(X, S)_{≤w}—that is, as a rational polynomial involving X, S where the degrees of the top and bottom add to at most w—we may multiply through to clear denominators. So we have

\[ (\sum_j u'_jM_j)(\sum_j v'_jM_j) = \sum_i \sum_j a'_{ij}M_j\lambda_i \]

where the u', v', a'_{ij} are polynomials in X, S with degrees bounded by wζ₁(n, d, b).

We will now rearrange our sums to focus on monomials from S. Write M_0^*, . . . , M_j^*, . . . for the monomials in S arranged so that M_0^* = 1 and M_i^*M_j^* = M_{ij}^* where we have

\[ (\sum_j u_j^*M_j^*)(\sum_j v_j^*M_j^*) = \sum_j (\sum_i a_{ij}^*\lambda_i)M_j^* \]
where the $u''_j, v''_j, a''_{ij}$ are elements of $K[X,Y_{[n]}]$ with $X$ degree bounded by $w\zeta_1(n,d,b)$ and $Y_{[n]}$ degree bounded by $\delta_n(d)$. By our choice of pseudobasis, we can separate this out by monomial: for each $j$,  
\[ \sum_i a''_{ij} \lambda_i = \sum_{k_0 + k_1 = j} u''_{k_0} v''_{k_1}. \]

We follow the standard argument to solve this monomial by monomial, keeping track of bounds along the way. We show by induction on $J$ that there are $k_0, k_1$ with $k_0 + k_1 = J$ so that for each $j < k_0$, $u''_j = \sum_i b_{ij} \lambda_i$ and for each $j < k_1$, $v''_j = \sum_i c_{ij} \lambda_i$.

Suppose we have chosen such $k_0, k_1$. Then  
\[ \sum_i a''_{ij} \lambda_i = \sum_{j \leq J} u''_{j} v''_{j} = u''_{k_0} v''_{k_1} + \sum_{j < k_0} u''_{j} v''_{j} + \sum_{j < k_1} u''_{j} v''_{j}, \]

so  
\[ u''_{k_0} v''_{k_1} = \sum_i (a''_{ij} + \sum_{j < k_0} u''_{j} b_{ij} + \sum_{j < k_1} u''_{j} c_{ij}) \lambda_i. \]

Since $\Lambda$ is prime up to $w\zeta_1(n,d,b)$ in the sense of $\leq_{X,Y}$, we have either $u''_{k_0} = \sum_i b_{k_0} \lambda_i$ (and we replace $k_0$ with $k_0 + 1$) or $v''_{k_1} = \sum_i c_{k_1} \lambda_i$ (and we replace $k_1$ with $k_1 + 1$).

We may continue until either $k_0 = (w\zeta_1(n,d)+w)$ or $k_1 = (w\zeta_1(n,d)+w)$. Suppose the first case happens (the second is symmetric); then we have  
\[ f = \sum_j u''_j M_j^* = \sum_i \sum_j b_{ij} \lambda_i M_j^* = \sum_i (\sum_j b_{ij} M_j^*) \lambda_i, \]

and therefore $f = \sum_i b_i \lambda_i$. \qed

We now arrive at the main result of this section: showing that we can “upgrade” from internal primality up to a certain point to actual primality.

**Notation 2.19.**

- $p_1(d) = d$,
- $v(n,d) = \zeta_1(n-1,p_{n-1}(d),d)\zeta_2(n-1,p_{n-1}(d),d)$,
- $\rho(n,d) = \max\{2^{v(n,d)+n}v(n,d),\epsilon(n-1,d)\}$,
- $p_n(d) = \rho(n,d)$.

**Theorem 2.20** (Based on [53], Theorem 2.5). Let $n,d$ be given. If $\Lambda \subseteq K[X_{[n]}]_{\leq d}$ is such that $(\Lambda)$ is prime up to $p_n(d)$ then $(\Lambda)$ is prime.

**Proof.** By induction on $n$. When $n = 1$ this is straightforward: $(\Lambda)$ is principal iff there is a single element of $\Lambda$ generating the ideal.

So suppose $n > 1$. First, suppose that for each $i$ there is an $h_i \in K[X_{[i]}]_{\leq \rho(n,d)}$ with $h_i \in (\Lambda)$. Then $K[X_{[n]}]/(\Lambda)$ is a field extension of $K$ where each $X_i$ is algebraic of degree $\leq \rho(n,d)$. In particular, any element $f$ of $K[X_{[n]}]$ may be written $f = f_0 + f'$ where $f' \in (\Lambda)$ and $f_0$ has total degree $\leq \rho(n,d)n$. So if $fg \in (\Lambda)$ then we have $fg = (f_0 + f')(g_0 + g') = f_0g_0 + c$ with $c \in (\Lambda)$. Therefore $f_0g_0 \in (\Lambda)$ and since $f_0g_0$ has degree $\leq 2\rho(n,d)n$,
the assumption applies, and either \( f_0 \in (\Lambda) \) or \( g_0 \in (\Lambda) \). Therefore either \( f = f_0 + f' \in (\Lambda) \) or \( g = g_0 + g' \in (\Lambda) \).

So suppose this does not hold: for some \( i \leq n \), \( K[X_i]_{\leq p(n,d)} \cap (\Lambda) = \emptyset \). We will apply the inductive hypothesis to the ring \( K(X_i)[X_{[1,i-1]},X_{[i+1,n]}] \).

By rearranging the variables, it suffices to assume \( i = 1 \).

**Claim 2.20.1:** For every \( u \in K[X_1] \), if \( fu \in (\Lambda) \) then \( f \in (\Lambda) \).

*Proof.* We apply Lemma 2.12. It suffices to rule out the first case: suppose there were an \( f \in K[X_1]_{(n-1,d)} \) and a \( g \in K[X_n]_{\leq \ell(n-1,d)+d} \) so that \( fg \in (\Lambda) \) but \( g \not\in (\Lambda) \). Since, by assumption, \( f \not\in (\Lambda) \), this violates the primality of \( (\Lambda) \) up to \( \ell(n-1,d)+d \).

**Claim 2.20.2:** \( (\Lambda) \) is prime in \( K(X_1)[X_{[2,n]}] \) up to \( v(n,d) \) in \( X_{[1,n]} \)-degree.

*Proof.* Suppose \( fg \in (\Lambda) \) with \( f, g \in K(X_1)[X_{[2,n]}]_{\leq v(n,d)} \), so \( fg = \sum a_i \lambda_i \).

Clearing denominators, \( f'g'h = \sum a_i' \lambda_i \lambda_i \) where \( h \in K[X_1]_{\leq 2(v(n,d)+\ell(n,d))} \) and \( f', g' \in K[X_n]_{\leq v(n,d)} \).

Since \( K[X_1]_{\leq 2(v(n,d)+\ell(n,d))} \cap (\Lambda) = \emptyset \), we have \( h \not\in (\Lambda) \). By primality of \( (\Lambda) \) up to \( 2\ell(n,d) \), \( f', g' \in (\Lambda) \), and so, without loss of generality, \( f' \in (\Lambda) \). Then \( f = f'/h' \) for some \( h' \in K[X_1] \), and since \( f' = \sum b_i \lambda_i \), also \( f = \sum b_i(h'/h') \lambda_i \), and therefore \( f \in (\Lambda) \).

**Claim 2.20.3:** \( (\Lambda) \) is prime in \( K(X_1)[X_{[2,n]}] \).

*Proof.* Since \( (\Lambda) \) is prime up to \( v(n,d) \) in \( K(X_1)[X_2,\ldots,X_n] \) in \( X_{[1,n]} \)-degree, by Theorem 2.13 also \( (\Lambda) \) is prime up to \( p_{n-1}(d) \) in \( K(X_1)[X_{[2,n]}] \) in \( X_{[1,n]} \)-degree. By the inductive hypothesis applied to \( K(X_1)[X_{[2,n]}] \), we have that \( (\Lambda) \) is prime in \( K(X_1)[X_{[2,n]}] \).

We can now complete the proof: suppose \( fg \in (\Lambda) \) in \( K[X_n] \) (with \( \deg(fg) \) arbitrary). Then certainly \( fg \in (\Lambda) \) in \( K(X_1)[X_{[2,n]}] \), so without loss of generality, \( f = \sum b_i \lambda_i \) with the \( b_i \in K(X_1)[X_{[2,n]}] \). Clearing denominators again, \( fu = \sum b_i' \lambda_i \lambda_i \) with the \( b_i' \in K[X_{[n]}] \) and \( u \in K[X_1] \). Applying the first claim above, we must have \( f \in (\Lambda) \), completing the proof. \( \square \)

3. **Hilbert’s Basis Theorem, Noetherianity, and the Nullstellensatz**

For some results we will need an effective version of Hilbert’s Basis Theorem—that is, of the Noetherianity of \( K[X_n] \). Such theorems are given without bounds in several places in the literature, such as Hertz [25] and Perdry and Schuster [39]. Moreno Socías proved that bounds on the length of ascending chains of polynomial ideals are non-primitive recursive in the number of indeterminates (Cor. 7.5, [37]). As a warm-up for the differential case, we use our methods to obtain an effective basis theorem and Nullstellensatz.

To give bounds on Hilbert’s Basis Theorem, we use a function given by Figueira et al [10] to bound witnesses to Dickson’s Lemma. (León Sánchez...
and Ovchinnikov give related bounds in [34].) For the remainder of the discussion we fix an arbitrary monotonically increasing function \( D : \mathbb{N} \to \mathbb{N} \).

**Notation 3.1.** Consider a nonempty finite set \( X \) with elements from \( \mathbb{N}^\omega_1, \ldots, \mathbb{N}^\omega_r \). Let \( \tau_X \) (or simply \( \tau \) when \( X \) is understood) be the multiset containing one copy of \( n_i \) for every element of \( X \) belonging to \( \mathbb{N}^n_i \).

Given any multiset \( \tau \) containing a natural number \( k > 0 \), we denote by \( \tau \langle k, i, D \rangle \) the multiset obtained by removing one copy of \( k \) from \( \tau \) and introducing \( k \cdot (D(i) - 1) \) new copies of \( k - 1 \). This operation introduces 0 into the multiset if \( k = 1 \). If \( \tau \) contains 0, define \( \tau \langle 0, i, D \rangle \) to be the result of removing one copy of 0 from \( \tau \).

**Example 3.2.** Suppose \( X = \{(1, 2, 3), (4, 5, 6), (1, 2)\} \). Then \( \tau \) is the multiset \( \{3, 3, 2\} \) and \( \tau \langle 3, i, D \rangle \) is the multiset \( \{3, 2, \ldots, 2\} \) containing \( 3 \cdot (D(i) - 1) + 1 \) copies of 2. The multiset \( \tau \langle 2, i, D \rangle \) is \( \{3, 3, 1, \ldots, 1\} \) and contains \( 2 \cdot (D(i) - 1) \) copies of 1.

We can compare multisets lexicographically:

**Proposition 3.3.** The collection of finite multisets on \( \mathbb{N} \) is well ordered by the relation \( \leq_{\text{multi}} \) defined as follows:

\[ \sigma \leq_{\text{multi}} \tau \text{ if and only if } \sigma = \tau \text{ or } \tau \text{ contains strictly more copies of } k \text{ than does } \sigma, \text{ where } k \text{ is the greatest value such that } \tau \text{ and } \sigma \text{ contain different numbers of copies of } k. \]

**Example 3.4.**

- \( \{1, 1, 1, 1, 1\} \leq_{\text{multi}} \{2\} \)
- \( \{3, 1, 0\} \leq_{\text{multi}} \{3, 2\} \)

Note that \( \tau \langle k, i, D \rangle \leq_{\text{multi}} \tau \), whence the following recursive definition makes sense:

**Notation 3.5.** We define \( m_{\tau, D}(i) \) by:

- \( m_{\emptyset, D}(i) = 0 \).
- \( m_{\tau, D}(i) = 1 + m_{\tau_{\min \tau, i, D}, D}(i + 1) \), where \( \tau \neq \emptyset \) and \( \min \tau \) is the least element of the multiset \( \tau \).

For future convenience, denote the expression \( m_{\{n\}, D+1}(0) + 1 \) by \( m^*(D, n) \).

**Example 3.6.** Let \( D(i) = i + 2 \).

\[
\begin{align*}
m_{\{2\}, D}(0) &= 1 + m_{\{1, 1\}, D}(1) \\
&= 2 + m_{\{1, 0\}, D}(2) \\
&= 3 + m_{\{1, 0\}, D}(3) \\
&= 4 + m_{\{1\}, D}(4) \\
&= 5 + m_{\{0, 0, 0, 0\}, D}(5) \\
&= 6 + m_{\{0, 0, 0, 0\}, D}(6) \\
&\vdots
\end{align*}
\]
When \( \vec{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n \), we write \(|\vec{a}|\) to represent \( \max_{i \leq n} \{a_i\} \) (the infinity norm). We write \((a_1, \ldots, a_n) \preceq (b_1, \ldots, b_n)\) if for each \( i \leq n, a_i \leq b_i \). The bound we need concerns sequences \( \vec{a}_1, \vec{a}_2, \ldots \) such that for each \( i, |\vec{a}_i| \leq D(i) \).

**Theorem 3.7** (See [10], Lemma V.I). Let \( \vec{a}_1, \vec{a}_2, \ldots \) be a sequence in \( \mathbb{N}^n \) such that for each \( i, |\vec{a}_i| \leq D(i) \). There exist \( i < j \leq m^*(D, n) \) such that \( \vec{a}_i \preceq \vec{a}_j \).

**Remark 3.8.** The existence of bounds follows from Dickson’s Lemma, which implies that there are no infinite bad sequences such that \( \vec{a}_i \preceq \vec{a}_j \) for all \( i < j \) (equivalently, \((\mathbb{N}^n, \preceq)\) is a well-quasiordering) [10].

We now give an effective version of Hilbert’s Basis Theorem.

**Theorem 3.9.** Suppose \((\Lambda_1) \subseteq (\Lambda_2) \subseteq \cdots \subseteq K[X_{[n]}] \) with \( \Lambda_i \subseteq K[X_{[n]}]_{\leq D(i)} \). Then there is a \( j \leq m^*(D, n) \) such that \( (\Lambda_{j+1}) \subseteq (\Lambda_j) \).

**Proof.** We associate the monomial \( X_1^{a_1} \cdots X_n^{a_n} \) with the tuple \( \vec{a} \). We place a linear ordering on \( \mathbb{N}^n \), and so also on monomials, by saying \( \vec{a} < \vec{b} \) if either \( \sum_{1 \leq k \leq n} a_k < \sum_{1 \leq k \leq n} b_k \) or both \( \sum_{1 \leq k \leq n} a_k = \sum_{1 \leq k \leq n} b_k \) and, taking \( l \) least so \( a_l \neq b_l, a_l < b_l \). (The linear ordering < should not be confused with the partial ordering \( \preceq \).)

We define a sequence of elements of \( K[X_{[n]}] \) as follows. Suppose we have defined \( f_i \) for \( i < j \) so that \( f_i \in K[X_{[n]}]_{\leq D(i)} \). We reduce each element of \( \Lambda_j \) by \( f_1, \ldots, f_{j-1} \). That is, if \( f \in \Lambda_j \) and \( f \) contains monomials divided by the leading monomial (i.e., greatest with respect to \( < \)) of \( f_1 \), divide \( f \) by \( f_1 \); the remainder \( r_1 \) is a reduction of \( f \) with respect to \( f_1 \).

The total degree \( \deg(r_1) = \deg(f - \alpha \cdot f_1) \) for some \( \alpha \) such that \( \deg(\alpha \cdot f_1) \leq \deg(f) \), so \( r_1 \in K[X_{[n]}]_{\leq D(j)} \). Reduce \( r_1 \) with respect to \( f_2 \), the resulting remainder with respect to \( f_3 \), and so on. Since \( \Lambda_j \subseteq K[X_{[n]}]_{\leq D(j)} \), it follows that the reductions are contained in \( K[X_{[n]}]_{\leq D(j)} \). If all elements reduce to \( 0 \), we are done. Otherwise, we take \( f_j \) to be the reduction with the greatest leading monomial.

Let \( \vec{a}_1, \ldots, \vec{a}_j, \ldots \) be the leading monomials of \( f_1, \ldots, f_j, \ldots \). If \( i < j \), then \( \vec{a}_i \preceq \vec{a}_j \) because \( f_j \) is reduced with respect to \( f_i \) and hence \( \vec{a}_i \) does not divide \( \vec{a}_j \). Note that if the sum of the entries of \( \vec{a}_j \) is bounded by \( D(j) \), then \( |\vec{a}| \leq D(j) \) so by 3.7 this process must terminate at some \( j \leq m^*(D, n) \). □

The final result from [53] we need is Corollary 2.7(ii). We include the following proof, which is the unwinding of the proof in [53].

**Theorem 3.10** (Based on [53], Cor 2.7(ii)). For any \( n,d \) there is \( m = m^*(i \mapsto p_{[d]}(d), n) \) so that if \( \Lambda \subseteq K[X_{[n]}]_{\leq d} \) and \( f^k \in (\Lambda) \) (for any \( k \)) then \( f = \sum_i c_i r_i \) where each \( r_i \in (\Lambda) \).
Proof. We will produce a tree of finitely generated ideals as follows. When \( \sigma \) is a node of this tree, we write \( \Gamma_\sigma \) for the finite set of generators. We will inductively maintain that:

- \( \Gamma_\sigma \subseteq K[X_{[n]}]_{\leq p_{n}^{|\sigma|}(d)} \) and
- if \( \sigma \subseteq \tau \) then \( \Gamma_\sigma \subseteq \Gamma_\tau \).

We begin by setting \( \Gamma_{[]} = \Lambda \). Given \( \Gamma_\sigma \), we check whether \( f \in (\Gamma_\sigma) \); if so, \( \sigma \) is a leaf. If not, since \( f^k \in (\Gamma_\sigma) \), \( (\Gamma_\sigma) \) is not prime, so we find \( gh \in (\Gamma_\sigma) \) with \( g, h \not\in (\Gamma_\sigma) \) and \( \deg(g), \deg(h) \leq p_{n}(|\sigma|)(d) \). We define \( \Gamma_{\sigma^{-}(0)} = \Gamma_\sigma \cup \{g\} \) and \( \Gamma_{\sigma^{-}(1)} = \Gamma_\sigma \cup \{h\} \). Inductively we see that \( \Gamma_\sigma \subseteq K[X_{[n]}]_{\leq p_{n}^{|\sigma|}(d)} \).

The previous theorem ensures that each branch has length \( \leq m^*(i \mapsto p_{n}^i(d), n) \), so the tree has at most \( 2m^*(i \mapsto p_{n}^i(d), n) \) leaves. Take \( m = m^*(i \mapsto p_{n}^i(d), n) \). Note that the ideal corresponding to each leaf contains \( f \), so for each \( \sigma \) we have \( f = \sum c_{i,\sigma} \gamma_{i,\sigma} \). Fix some leaf \( \sigma_0 \), and consider the system of equations of the form \( \sum_i \gamma_{i,\sigma_0} y_{i,\sigma_0} - \sum_j \gamma_{j,\sigma} y_{j,\sigma} = 0 \). This is a system of at most \( 2^m \) equations whose coefficients have degree at most \( p_{n}^m(d) \). The \( \{c_{i,\sigma}\} \) give a solution, and so by Lemma 2.5 there are solutions \( c'_{i,j,\sigma} \) such that \( c_{i,\sigma} = \sum_j d_{j,i,\sigma} \) and the \( d_{j,i,\sigma} \) have degree at most \( d_{n}(p_{0}^m(d)2^m) \).

Let \( f_j = \sum_i \gamma_{i,\sigma_0} c'_{i,j,\sigma} \). Note that, since for each \( i, j, \sigma, \sum_i c'_{i,j,\sigma} \gamma_{i,\sigma} = \sum_i c'_{i,j,\sigma} \gamma_{i,\sigma} \), also \( f_j = \sum_i \gamma_{i,\sigma} c_{i,j,\sigma} \), so \( f_j \in (\Gamma_\sigma) \) for each leaf \( \sigma \). We now show inductively that if \( |\sigma| = i \) then \( f_{j}^{2^m-i} \in (\Gamma_\sigma) \). For leaves this is immediate. If \( f_{j}^{2^m-i} \in (\Gamma_{\sigma^{-}(0)} \cap (\Gamma_{\sigma^{-}(1)})) \), recall that there are \( g, h \) so \( \Gamma_{\sigma^{-}(0)} = \Gamma_\sigma \cup \{g\} \) and \( \Gamma_{\sigma^{-}(1)} = \Gamma_\sigma \cup \{h\} \), so \( f_{j}^{2^m-i} = \sum_i \gamma_{i,\sigma} u_i + gu = \sum_i \gamma_{i,\sigma} v_i + hv \), so

\[
f_{j}^{2^m-i} = (\sum_i \gamma_{i,\sigma} u_i + gu)(\sum_i \gamma_{i,\sigma} v_i + hv) = \sum_i \gamma_{i,\sigma} u_i' + ghuv,
\]

so \( f_{j}^{2^m-(i-1)} \in (\Gamma_\sigma) \).

In particular, \( f_{j}^{2m} \in (\Gamma) \). Since \( f = \sum_i c_{i,\sigma_0} \gamma_{i,\sigma_0} = \sum_i \sum_j d_{j,i,\sigma_0} \gamma_{i,\sigma_0} = \sum_j d_{j,i,\sigma_0} \), we have shown the claim. \( \square \)

In fact, these bounds are embarrassingly poor compared to those given by a different method.

**Theorem 3.11.** Suppose \( \Lambda \subseteq K[X_{[n]}]_{\leq d} \) and \( f^k \in (\Lambda) \) with \( \deg(f) \leq d \). Then \( f_{n+1}(d+1) \in (\Lambda) \).

**Proof.** We use the Rabinowitsch trick: since \( f^k \in (\Lambda) \), by the Nullstellensatz we have \( 1 = \sum_i g_i \lambda_i + g(1 - Y f) \) for some \( \lambda_i \in \Lambda \) and \( g, g \in K[X_{[n]}, Y] \). By Lemma 2.6 we may assume that the \( g_i \) have degree \( \leq d_{n+1}(d+1) \). Therefore, substituting \( 1/f \) for \( Y \) and multiplying both sides by \( f_{n+1}(d+1) \) to clear denominators, we get \( f_{n+1}(d+1) = \sum_i g_i \lambda_i \). \( \square \)

Note, however, that the proof-mined result is a little more uniform: unlike in 3.11, there is no restriction on the degree of \( f \) in 3.10.
4. Bounds in Differential Polynomial Rings

4.1. Rankings and Faithful Flatness. We now turn to the differential case. Henceforth we fix a field $K$ of characteristic 0 equipped with a set $\Delta = \{\delta_1, \ldots, \delta_m\}$ of commuting partial derivations (i.e., additive homomorphisms satisfying the usual product rule). We write $\Theta = \{\delta_1^{k_1} \cdots \delta_m^{k_m} \mid k_1, \ldots, k_m \geq 0\}$ for the set of $\Delta$-operators; for convenience we will also refer to these as derivations. By the term “derivative” we mean an expression like $\theta X_i$; i.e., a differential indeterminate $X_i$ to which a derivation $\theta \in \Theta$ has been applied. The order of $\theta X_i$ is $\sum_i k_i$.

Definition 4.1. A ranking $<$ on a set of derivatives is a well-ordering such that for all derivatives $u, v$ and nontrivial $\theta \in \Theta$:

1. If $u < v$, then $\theta u < \theta v$ and
2. $u < \theta u$.

Throughout our discussion, we assume a fixed ranking of order-type $\omega$, so we can associate a derivative $\theta X_i$ to a natural number $o(\theta X_i)$. In order to give concrete bounds, we further assume an orderly ranking on derivatives: $\delta_1^{k_1} \cdots \delta_m^{k_m} X_r < \delta_1^{l_1} \cdots \delta_m^{l_m} X_s$ if and only if $\sum_i k_i < \sum_i l_i$ or $\sum_i k_i = \sum_i l_i$ and $k_1 = l_1, k_2 = l_2, \ldots, k_i < l_i$ for some $1 \leq i \leq m$ or all these quantities are the same and $r < s$. This allows us to equate the differential polynomial ring $K\{X_{[n]}\}$ in $n$ differential indeterminates with the algebraic polynomial ring $K[Z_{[b]}]$ in countably many indeterminates, where $Z_{o(\theta X_i)}$ is associated with $\theta X_i$.

There are two natural gradings we might consider: either treating the degree of polynomials and the number of algebraic indeterminates separately, or combining these into a single grading.

Definition 4.2. We write $K[Z_{[b]}] = K[Z_1, \ldots, Z_b]$. 
$K\{X_{[n]}\}_{\leq b}$ is $K[Z_{[b]}]_{\leq b}$.
$K\{X_{[n]}\}_{\leq b,d}$ is $K[Z_{[b]}]_{\leq d}$.

Remark 4.3. Order and ranking are related but not equal. The ranking can be significantly greater because the number of derivatives of a given order grows with the order. For example, using our chosen orderly ranking, if $n = 1, m = 2$, then $\delta_2 X_1$ is the second-least derivative in the ranking but $\delta_1 \delta_2 X_1$ is the fifth-least even though the order only increased by 1. There are $(N+m-1) \cdot n$ derivatives of order $N$ in $K\{X_{[n]}\}, \Delta = \{\delta_1, \ldots, \delta_m\}$.

Definition 4.4. The leader of a differential polynomial $f \in K\{X_{[n]}\} \setminus K$ is the greatest derivative (in the ranking) appearing in $f$. The initial $I_f$ of $f$ is the coefficient of the highest-degree term in the leader of $f$, considering $f$ as a univariate polynomial in the leader. The separant $S_f$ of $f$ is the initial of any proper derivative $\theta f$ of $f$ (equivalently, the formal partial derivative in the usual calculus sense with respect to the leader.) The rank of $f$ is the ordered pair $(\mu_f, \deg(\mu_f))$ consisting of the leader $\mu_f$ of $f$ and the highest degree in which it appears in $f$; ranks are compared lexicographically.
See [48] for further discussion and examples of rankings, leaders, and related notions.

We first show a version of internal flatness for $K \{X_{[n]}\}$.

**Lemma 4.5** (Based on [23], 4.1, flatness). Whenever $f_1, \ldots, f_k \in K \{X_{[n]}\}_{\leq b}$ and $\sum_i g_i f_i = 0$, there exist $h_{ij} \in K \{X_{[n]}\}_{\leq b, d_{0}(b)}$ and $c_j \in K \{X_{[n]}\}$ so that $\sum_i h_{ij} f_i = 0$ for each $j$ and $\sum_j c_j h_{ij} = g_i$ for each $i$.

**Proof.** Consider an equation $\sum_{i \leq l} f_i Y_i = 0$ with the $f_i$ in $K \{X_{[n]}\}_{\leq b}$ and suppose $\sum_i f_i g_i = 0$. We may write the solutions $g_i$ as polynomials in those variables $Z_k$ with $k > b$, with coefficients in $K[Z_{[b]}]$: $g_i = \sum_j g_{ij} W_j$. Since $W_j$ is transcendental over $K[Z_{[b]}]$, $\sum_i f_i g_i = 0$ implies that, for each $j$, $\sum_i f_i g_{ij} = 0$. Internal flatness of $K[Z_{[b]}]$ says that the $g_{ij}$ must be linear combinations of solutions in $K[Z_{[b]}]_{\leq d_{0}(b)}$. The space of such solutions is finite dimensional, say $u_1, \ldots, u_r$ where each $u_k = (u_{k0}, \ldots, u_{kl})$ with $\sum_i f_i u_{ki} = 0$. So for each $j$, the $g_{ij}$ must be a linear combination of such solutions, $g_{ij} = \sum_k c_{jk} u_{ki}$. Then $g_i = \sum_j \sum_k c_{jk} u_{ki} W_j = \sum_k (\sum_j c_{jk} W_j) u_{ki}$. So, setting $c'_k = \sum_j c_{jk} W_j$, we have $g_i = \sum_k c'_k u_{ki}$. \hfill $\square$

**Lemma 4.6** (Based on [23], 4.1, faithful flatness). For any $n$ and any $f_i, h \in K \{X_{[n]}\}_{\leq b}$, if $\sum_{k \leq b} f_i g_i = h$ then there are $g'_i \in K \{X_{[n]}\}_{\leq b, d_{0}(b)}$ such that $\sum_{i \leq b} f_i g'_i = h$.

**Proof.** Suppose $\sum_i f_i g_i = h$ where $f, h \in K \{X_{[n]}\}_{\leq b} = K[Z_{[b]}]_{\leq b}$. Then we may write each $g_i$ as a sum of monomials, and have $g_i = g_i^+ + g_i^-$ where $g_i^+$ consists only of those monomials in $Z_{[b]}$ and $g_i^-$ contains all monomials with at least one term outside of $Z_{[b]}$. Then $\sum_i f_i g_i^- = 0$, so $\sum_i f_i g_i^+ = h$. By Lemma 2.6 there are $g'_i \in K[Z_{[b]}]_{\leq d_{0}(b)} = K \{X_{[n]}\}_{\leq b, d_{0}(b)}$ so that $\sum_i f_i g'_i = h$. \hfill $\square$

### 4.2. Stratified Ideals and Autoreduced Sets.

**Notation 4.7.** If $\Lambda \subseteq K \{X_{[n]}\} \setminus K$ is finite, we write $H_\Lambda$ for $\prod_{\Lambda \subseteq \Lambda} I \Lambda S_\Lambda$.

**Notation 4.8.** Given $\Lambda \subseteq K \{X_{[n]}\}$, we denote by $(\Lambda), [\Lambda]$, and $\{\Lambda\}$, respectively, the ideal, differential ideal (i.e., closed under derivation), and perfect ideal (radical ideal generated by $[\Lambda]$; it is automatically differential in our case) generated in $K \{X_{[n]}\}$ by $\Lambda$.

**Notation 4.9.** We frequently work with the following saturation ideals:

$$(\Lambda): H_{\Lambda}^\infty = \{ g \mid \exists n H_{\Lambda}^n g \in (\Lambda) \}$$

and

$$[\Lambda]: H_{\Lambda}^\infty = \{ g \mid \exists n H_{\Lambda}^n g \in [\Lambda] \}.$$ 

Because the ideals $(\Lambda): H_{\Lambda}^\infty$ and $[\Lambda]: H_{\Lambda}^\infty$ need not be finitely generated, we need a more nuanced way to work with them if we want effective bounds.

**Definition 4.10.** A stratified ideal $(\Lambda_k)_k$ in $K \{X_{[n]}\}$ is an increasing sequence $(\Lambda_1) \subseteq (\Lambda_2) \subseteq \cdots \subseteq K \{X_{[n]}\}$ so that, for each $k$, $\Lambda_k \subseteq K \{X_{[n]}\}_{\leq k}$. 


We identify a stratified ideal \( (\Lambda_k)_k \) with the ideal \((\bigcup_k \Lambda_k)\). Note that there is no assumption that \( K\{X_{[n]}\}_{\leq k} \cap (\bigcup_k \Lambda_k) = (\Lambda_k) \)—new elements of \( K\{X_{[n]}\}_{\leq k} \) might appear in \( \Lambda_k \) with \( k' \) much larger than \( k \).  

We pick canonical stratifications associated with the ideals \((\Lambda) : H^\infty_\Lambda \) and \([\Lambda] : H^\infty_\Lambda \).

**Notation 4.11.** We write
\[
\Lambda^H_{(k)} = \{ g \in K\{X_{[n]}\}_{\leq k} \mid H^k_\Lambda g \in (\Lambda) \},
\]
\[
\Lambda_{[k]} = \{ \theta \lambda \mid \lambda \in \Lambda \text{ and } o(\theta \lambda) \leq o(\lambda) + k \},
\]
and
\[
\Lambda^H_{[k]} = \{ g \in K\{X_{[n]}\}_{\leq k} \mid H^k_\Lambda g \in (\Lambda_{[k]}) \}.
\]

Before discussing primality of stratified ideals, we introduce the important topic of autoreduced sets.

**Definition 4.12.** A differential polynomial \( f \in K\{X_{[n]}\} \) is partially reduced with respect to \( g \in K\{X_{[n]}\} \setminus K \) if \( f \) has no proper derivative of the leader \( \mu_g \) of \( g \). \( f \) is reduced with respect to \( g \) if \( f \) is partially reduced with respect to \( g \) and additionally the degree of \( \mu_g \) in \( f \) is strictly less than the degree of \( \mu_g \) in \( g \). \( f \) is reduced with respect to a subset \( S \subseteq K\{X_{[n]}\} \setminus K \) if \( f \) is reduced with respect to every element of \( S \), and \( S \) is autoreduced if every element of \( S \) is reduced with respect to every other element of \( S \).

If a finite set \( \Lambda \subseteq K\{X_{[n]}\}_{\leq b} = K[Z_{[b]}]_{\leq b} \) is autoreduced, one may check that \((\binom{2b}{b})\) is an upper bound on the number of distinct monomials in \( Z_1, \ldots, Z_b \) and hence on the cardinality of \( \Lambda \). In particular, this implies that \( H_\Lambda \in K\{X_{[n]}\}_{\leq b, 2b(\binom{2b}{b})} \).

Given \( f \in K\{X_{[x]}\} \), we can find a closely related remainder \( \tilde{f} \) that is reduced with respect to \( \Lambda \); the process of obtaining \( \tilde{f} \) is called pseudodivision. The exact remainder obtained from reduction depends on the sequence of elements from \( \Lambda \), but we can still find effective bounds on the complexity regardless of the choices made during pseudodivision.

**Example 4.13.** Choose a ranking in which \( z > \delta_1 x > \delta_2 y \). Let \( g_1 = \delta_2 y (\delta_1 x)^2 + x \delta_1 x, \) \( g_2 = \delta_2 y \delta_1^2 x + x, \) and \( f = z + x \delta_1^2 x + T_f \) (where the trailing terms \( T_f \) have lower rank). We illustrate a single step of pseudodivision of \( f \) with respect to \( g_1 \) and \( g_2 \), respectively:

- \( f \) contains a proper derivative of the leader of \( g_1 \), so we differentiate \( g_1 \) to obtain
  \[ \delta_1 g_1 = (2\delta_2 y \delta_1 x + x) \delta_1^2 x + (\delta_1 \delta_2 y + 1)(\delta_1 x)^2. \]

  Multiply \( f \) by \( S_{g_1} \) and subtract a suitable multiple of \( \delta_1 g_1 \). The remainder \( r_1 \) after one step is
  \[ S_{g_1} f - x \cdot \delta_1 g_1 = S_{g_1}(z + T_f) - x \cdot (\delta_1 \delta_2 y + 1)(\delta_1 x)^2. \]

\footnote{Formally, we are representing membership in these ideals as an existential property.}
• $f$ contains the leader of $g_2$ but no proper derivatives thereof, so to pseudodivide $f$ by $g_2$ we simply multiply $f$ by $I_{g_2}$ and divide to obtain

$$I_{g_2}f - x \cdot g_2 = I_{g_2}(z + T_f) - x^2.$$  

Note that the actual leader $z$ of $f$ never came into play; we only eliminated terms that prevented $f$ from being reduced with respect to $g_1$ or $g_2$. Also, if we were reducing $f$ with respect to the set $\{g_1, g_2\}$, we would instead continue reducing $r_1$ with respect to $g_1$ until the remainder $r$ was reduced with respect to $g_1$. We would then reduce $r$ with respect to $g_2$. See [48] for further explanation of reduction algorithms.

**Notation 4.14.** $g(b, d) = d(1 + b)^d$.

**Lemma 4.15 (Based on [31], I(9), Proposition 1).** Let $\Lambda \subseteq K\{X_{[n]}\}_{\leq b}$ be autoreduced and let $f \in K\{X_{[n]}\}_{\leq d}$. Then there exist $\tilde{f} \in K\{X_{[n]}\}_{\leq d, g(b, d)}$ reduced with respect to $\Lambda$ and $k_\lambda, l_\lambda \leq g(b, d)$ for each $\lambda \in \Lambda$ such that

$$\left(\prod_{\lambda \in \Lambda} I_{\lambda}^{k_\lambda} S_{\lambda}^{l_\lambda}\right) f - \tilde{f} \in (\Lambda_{[d]}).$$

**Proof.** Pseudodivide repeatedly to eliminate the highest ranking term in the current remainder that is not reduced with respect to $\Lambda$. This rank decreases at each step (either by reducing order or degree), so the process terminates with a remainder $\tilde{f}$ that is reduced with respect to $\Lambda$. Repeated multiplication by initials and separatants throughout this process yields the form $\left(\prod_{\lambda \in \Lambda} I_{\lambda}^{k_\lambda} S_{\lambda}^{l_\lambda}\right) f - \tilde{f} \in (\Lambda_{[d]})$ for some $k_\lambda, l_\lambda$.

Recursively define $B(i)$ as follows:

• $B(0) = d$,
• $B(i + 1) = B(i)(1 + b)$.

We claim that $B(i)$ is an upper bound on the degree of the remainder after reducing $i$-many times the highest ranking term not reduced with respect to $\Lambda$.

By hypothesis, $d = B(0)$ bounds the degree of $f$, so we start out correctly. For the inductive step, suppose that the remainder after eliminating $i$-many derivatives belongs to $K\{X\}_{\leq d, B(i)}$. The number of division steps required to eliminate a derivative is at most the current degree in that derivative, hence it is bounded by $B(i)$. Each multiplication by an initial or separant increases the degree by at most $b$. The degree after eliminating the $i + 1$-st derivative is consequently bounded by $(current\ bound) + b \cdot (maximal\ number\ of\ divisions) = B(i) + b \cdot B(i) = B(i)(1 + b) = B(i + 1)$. Pseudodivision cannot increase the order, so the new remainder belongs to $K\{X\}_{\leq d, B(i + 1)}$. Thus the final bound on the degree of $\tilde{f}$ is $B(d) = d(1 + b)^d = g(b, d)$. 
To find a bound on $k_{\lambda}, l_{\lambda}$ we add up the total number of division steps required to eliminate each derivative. Again we see that $g(b, d)$ is an upper bound: $d + d(1 + b) + \cdots + d(1 + b)^{d-1} = d \left(\frac{(1+b)^d-1}{b}\right) \leq g(b, d)$. \hfill \qed

4.3. Local Primality. The right notion of primality for stratified ideals is a “local primality” notion which tells us not only that when $fg \in (\bigcup_k \Lambda_k)$ that either $f \in (\bigcup_k \Lambda_k)$ or $g \in (\bigcup_k \Lambda_k)$, but incorporates a bound $F$, so that $fg \in (\Lambda_k)$ implies that either $f \in (\Lambda_{F(k)})$ or $g \in (\Lambda_{F(k)})$.

**Definition 4.16.** Let $(\Lambda_k)_k$ be a stratified ideal. We say $(\Lambda_k)_k$ is $F$-prime up to $b$ if for each $k \leq b$, whenever $fg \in (\Lambda_k)$, either $f \in (\Lambda_{F(k)})$ or $g \in (\Lambda_{F(k)})$.

We also need the bounded version of primality, analogous to the notion of “prime up to $d$”.

**Definition 4.17.** Let $(\Lambda_k)_k$ be a stratified ideal. We say $(\Lambda_k)_k$ is boundedly $F$-prime up to $b$ if for each $k \leq b$, whenever $fg \in (\Lambda_k) \cap K\{X[n]\}_{n \leq k}$, either $f \in (\Lambda_{F(k)})$ or $g \in (\Lambda_{F(k)})$.

We always assume that the function $F$ is monotone—that is, $a \leq b$ implies $F(a) \leq F(b)$.

**Notation 4.18.** Given $F$, let $F_d(b) = F(p_d(b))$. Set $u_F(x) = F_m^{x^i \mapsto F_i(x)}(x)$.

**Lemma 4.19** (Based on [23, 4.1a].) If $\Lambda \subseteq K\{X[n]\}_{n \leq b}$ is such that $(\Lambda^H_k)$ is boundedly $F$-prime up to $p_d(u_F(d))$, then $(\Lambda^H_k)$ is $u_F$-prime up to $d$.

**Proof.** To show $u_F$-primality up to $d$, we must show the statement for each $d \leq d$, but without loss of generality (because $F$ is monotone) it suffices to show that whenever $fg \in (\Lambda^H_k)$, either $f \in (\Lambda^H_{u_F(d)})$ or $g \in (\Lambda^H_{u_F(d)})$.

Consider $\Lambda^*_{\lambda}(k) = \Lambda^H_{\lambda}(k) \cap K\{X[n]\}_{n \leq d,k}$. Observe that $(\Lambda^*_k)$ is also boundedly $F_d$-prime up to $p_d$($u_F(d)$): if $fg \in (\Lambda^*_k) \cap K\{X[n]\}_{n \leq d,k} \subseteq (\Lambda^H_k)$ for $k \leq p_d(u_F(d))$ then, without loss of generality, $f \in (\Lambda^H_{F(k)}) \cap K\{X[n]\}_{n \leq d,k}$, so $f \in (\Lambda^*_{F(k)})$.

Consider the sequence of ideals $(\Lambda^*_i) \subseteq (\Lambda^*_i_{F_d(d)}) \subseteq (\Lambda^*_i_{F_d(d)}) \subseteq \cdots$. By Theorem 3.9 there is some $i \leq m(i \mapsto F^i_d(d), d)$ so that $(\Lambda^*_i_{F^i_d(d)}) \subseteq (\Lambda^*_i_{F^i_d(d)}(d))$. Let $k = F^i(d) \leq u_F(d)$, so $d \leq k \leq u_F(d)$ and $(\Lambda^*_i_{F_d(d)}) \subseteq (\Lambda^*_i_{F_d(d)}(d))$. In particular, since $p_F(k) \leq u_F(d)$, if $fg \in (\Lambda^*_k) \cap K\{X[n]\}_{n \leq d,k}$ then either $f$ or $g$ belongs to $(\Lambda^*_i_{F_d(d)(k)}) = (\Lambda^*_i_{F_d(d)(k)}) \subseteq (\Lambda^*_i_{F_d(d)}(d))$, so $(\Lambda^*_i)$ is prime up to $p_d(k)$. By Theorem 2.22 $(\Lambda^*_i_{F_d(d)}(d))$ is prime.

Now suppose $fg \in (\Lambda^H_k) \subseteq (\Lambda^H_k)$. Then $fg \in K\{Z[m]\}$ for some $m$. Let $M_0, \ldots, M_j, \ldots$ enumerate the monomials over the variables $Z_{[d+1, m]}$. We may write $fg = \sum_{i,j} u_{i,j} \gamma_i M_j$ with $\gamma_i \in (\Lambda^H_k) \subseteq (\Lambda^*_i)$, $f = \sum_j b_j M_j$ and
\[ g = \sum_j c_j M_j. \] Then for each \( j \),
\[ \sum_{j_0 + j_1 = j} b_{j_0} c_{j_1} = \sum_i u_{i,j} \gamma_i. \]

We resolve this monomial by monomial. We show by induction on \( J \) that there are \( k_0, k_1 \) with \( k_0 + k_1 = J \) so that for each \( j < k_0 \), \( b_j \in (\Lambda^*_k) \) and for each \( j < k_1 \), \( c_j \in (\Lambda^*_k) \). When \( J = 0 \), this is immediate. Suppose the claim holds for \( J \); we have
\[ b_{k_0} c_{k_1} = \sum_i u_{i,j} \gamma_i - \sum_{j_0 < k_0} b_{j_0} c_{j-J-j_0} - \sum_{j_1 < k_1} b_{j-J}, c_{j_1}, \]
and therefore \( b_{k_0} c_{k_1} \in (\Lambda^*_k) \). Since this is a prime ideal, we have either \( b_{k_0} \in (\Lambda^*_k) \), in which case we increment \( k_0 \), or similarly with \( c_{k_1} \).

When \( J \) is large enough, we see that we must have either \( b_j \in (\Lambda^*_k) \) for all \( j \) or \( c_j \in (\Lambda^*_k) \) for all \( j \), so we have either \( f \in (\Lambda^*_k) \subseteq (\Lambda^*_H) \subseteq (\Lambda^*_H(\mathfrak{uf}(d))) \) or \( g \in (\Lambda^*_k) \subseteq (\Lambda^*_H(\mathfrak{uf}(d))) \).

\[ \square \]

**Notation 4.20.** \( \mathfrak{uf}(b) = \max\{\mathfrak{uf}(b), \mathfrak{d}_{b+1}(2b(2b)^b) + 1\} \).

**Lemma 4.21** (Based on [23], 4.2b). If \( \Lambda \subseteq K\{X[n] \leq b\}, |\Lambda| \leq (2b)^b \), and \( \langle \Lambda^H \rangle \) is boundedly \( \mathfrak{F} \)-prime up to \( p_b(\mathfrak{uf}(b)) \), then \( (\Lambda):H^\infty_\Lambda \subseteq (\Lambda^H_{\langle \mathfrak{uf}(b) \rangle}) \).

**Proof.** Suppose \( H^N_\Lambda g \in (\Lambda) \) for some \( N \). Then certainly \( H^N_\Lambda g \in (\Lambda^H_{\langle \mathfrak{uf}(b) \rangle}) \) since \( \Lambda \subseteq (\Lambda^H_{\langle \mathfrak{uf}(b) \rangle}) \). Since \( \langle \Lambda^H \rangle \) is boundedly \( \mathfrak{F} \)-prime up to \( p_b(\mathfrak{uf}(b)) \), by Lemma 4.19 it is \( \mathfrak{uf}(b) \)-prime up to \( b \) and either \( H^N_\Lambda \in (\Lambda^H_{\langle \mathfrak{uf}(b) \rangle}) \) or \( g \in (\Lambda^H_{\langle \mathfrak{uf}(b) \rangle}) \); in the latter case we are done.

Suppose \( H^N_\Lambda \in (\Lambda^H_{\langle \mathfrak{uf}(b) \rangle}) \), so also \( H^{N+\mathfrak{uf}(b)}_\Lambda \subseteq (\Lambda) \). Then by Theorem 3.11 already \( H^b_{\mathfrak{uf}(b)(2b(2b)^b) + 1} \subseteq (\Lambda) \). But this implies that \( 1 \in (\Lambda^H_{\mathfrak{uf}(b)(2b(2b)^b) + 1}) \), so also \( g \in (\Lambda^H_{\mathfrak{uf}(b)(2b(2b)^b) + 1}) \).

\[ \square \]

Later we will need a slight refinement of these notions, where we restrict to the subring of \( K\{X[n] \} \) containing only those indeterminates that are not proper derivatives of the leaders of \( \Lambda \) (that is, those elements partially reduced with respect to \( \Lambda \)).

**Definition 4.22.** \( K\{X[n] \mid \Lambda \} \) is the ring \( K[Z_S] \) where \( S \) is the (possibly infinite) set of indices of indeterminates which are not proper derivatives of any \( u_\lambda \) with \( \lambda \in \Lambda \). We write \( K\{X[n] \mid \Lambda \} \leq b \) for \( K\{X[n] \mid \Lambda \} \cap K\{X[n] \} \leq b \).

We say \( \langle \Lambda \rangle \) is \( \text{pr}(\Lambda) \)-\( \mathfrak{F} \)-prime up to \( b \) if for each \( k \leq b \), whenever \( f g \in (\Lambda_k) \cap K\{X[n] \mid \Lambda \} \), either \( f \in (\Lambda_{\mathfrak{F}(k)}) \) or \( g \in (\Lambda_{(\mathfrak{F}(k)}) \).

We say \( \langle \Lambda \rangle \) is boundedly \( \text{pr}(\Lambda) \)-\( \mathfrak{F} \)-prime up to \( b \) if for each \( k \leq b \), whenever \( f g \in (\Lambda_k) \cap K\{X[n] \mid \Lambda \} \leq k \), either \( f \in (\Lambda_{\mathfrak{F}(k)}) \) or \( g \in (\Lambda_{(\mathfrak{F}(k)}) \).

Inspection of the proofs of the previous two lemmas gives:
Lemma 4.23. If $\Lambda \subseteq K\{X_{[n]}\}_{b}$ is such that $\langle \Lambda^H_{(k)} \rangle$ is boundedly pr($\Lambda$)-$\mathbf{F}$-prime up to $p_d(u_{\mathbf{F}}(d))$, then $\langle \Lambda^H_{(k)} \rangle$ is pr($\Lambda$)-$u_{\mathbf{F}}$-prime up to $d$.

Lemma 4.24. If $\Lambda \subseteq K\{X_{[n]}\}_{b}$, $|\Lambda| \leq \binom{2b}{b}$, and $\langle \Lambda^H_{(k)} \rangle$ is boundedly pr($\Lambda$)-$\mathbf{F}$-prime up to $p_b(u_{\mathbf{F}}(b))$, then $(\Lambda): H^\infty_\Lambda \cap K\{X_{[n]} \uparrow \Lambda\} \subseteq \langle \Lambda^H_{(u_{\mathbf{F}}(b))} \rangle$.

Then we get:

Lemma 4.25. If $\Lambda \subseteq K\{X_{[n]}\}_{b}$ is autoreduced and $\langle \Lambda^H_{(k)} \rangle$ is boundedly pr($\Lambda$)-$\mathbf{F}$-prime up to $p_b(u_{\mathbf{F}}(b))$, then $(\Lambda): H^\infty_\Lambda \subseteq \langle \Lambda^H_{(u_{\mathbf{F}}(b))} \rangle$.

Proof. Given $g \in (\Lambda): H^\infty_\Lambda$, we have $g \in K\{X_{[n]} \uparrow \Lambda\}[Z_T]$ for some finite set of additional indeterminates $Z_T$, each a derivative of some leader in $\Lambda$. Then we may write $g = \sum_i g_i M_i$ with the $g_i \in K\{X_{[n]} \uparrow \Lambda\}$. The $Z_T$ do not appear in $\Lambda$ (by autoreducedness of $\Lambda$), so by Lemma 4.24 and induction on the number of $Z_T$ appearing in $g$ we must have each $g_i \in (\Lambda): H^\infty_\Lambda \cap K\{X_{[n]} \uparrow \Lambda\} \subseteq \langle \Lambda^H_{(u_{\mathbf{F}}(b))} \rangle$. This proves $g \in (\Lambda^H_{(u_{\mathbf{F}}(b))})$. $\square$

Notation 4.26.
- $N = \partial_{u_{\mathbf{F}}^+(b)}(u_{\mathbf{F}}^+(b)) + u_{\mathbf{F}}^+(b)$.
- $f(F, b) = \partial_{u_{\mathbf{F}}^+(b)}((\sum_{i \in b} N_i + b) \cdot u_{\mathbf{F}}^+(b)) + N$.

Lemma 4.27 (Based on [23], 4.2c). Suppose $\Lambda \subseteq K\{X_{[n]}\}_{b}$ is autoreduced and $\langle \Lambda^H_{(k)} \rangle$ is boundedly pr($\Lambda$)-$\mathbf{F}$-prime up to $p_b(u_{\mathbf{F}}(b))$. Suppose there is a $g \in (\Lambda): H^\infty_\Lambda$ which is non-zero and reduced with respect to $\Lambda$. Then there is an $f \in (\Lambda^H_{(u_{\mathbf{F}}(b))}) \cap K\{X_{[n]}\}_{\leq u_{\mathbf{F}}(b), f(F, b)}$ which is non-zero and reduced with respect to $\Lambda$.

Proof. Suppose $g \in (\Lambda): H^\infty_\Lambda$ is non-zero and reduced with respect to $\Lambda$. Then by Lemma 4.25, $g \in (\Lambda^H_{(u_{\mathbf{F}}(b))})$. Let $T[t]$ list the variables appearing as leaders in $\Lambda$ (so $t \leq b$) and let $Y_{[y]}$ list all variables appearing in $g$ which are not derivatives of the variables $T[t]$. Since $g$ is reduced relative to $\Lambda$, $g \in K[T[t], Y_{[y]}]$. Without loss of generality we may assume that $t + y \leq u_{\mathbf{F}}(b)$ and that $\Lambda^H_{(u_{\mathbf{F}}(b))} \subseteq K[T[t], Y_{[y]}]$. We then have $g = \sum_i c_i \gamma_i$ for some $\gamma_i \in \Lambda^H_{(u_{\mathbf{F}}(b))}$ and $c_i \in K[T[t], Y_{[y]}]$.

Consider the field $L = K(Y_{[y]})$. Then $g \in L[T[t]]_{\leq b}$ (since its degree in the leading variables of $\Lambda$ is bounded by the degrees of $\Lambda$). Since $g = \sum_i c_i \gamma_i$, by Lemma 2.6, we have $g = \sum_i c'_i \gamma_i$ with $c'_i \in L[T[t]]_{\leq u_{\mathbf{F}}(b)}$. Clearing denominators, we have $f = hg = \sum_i h_i \gamma_i$ where the $h_i \in K[T[t], Y_{[y]}]$ have the same $T[t]$-degree as the $c'_i$. Note that $f$ is still non-zero and reduced with respect to $\Lambda$.

We expand this into a system of equations with one equation for each monomial from $T[t]$. There are at most $(\sum_i c'_i)$-many equations because there
are \(t\)-many variables and the \(T[y]\)-degree of the system is bounded by \(N\), the sum of the \(T[y]\)-degrees of \(h_i\) and \(\gamma_i\). By Theorem 2.5, the system has solutions \(h_i'\) with \(Y[y]\)-degree at most \(\delta_{u_F^+(b)}((N+t)^i \cdot u_F^+(b))\) such that \(f' = \sum_i h_i'\gamma_i\) is non-zero in the same monomials \(f\) is. Further, since \(f'\) is zero in all monomials \(f\) is, \(f'\) is still reduced with respect to \(\Lambda\). By using the fact that \(t \leq b\) and adding the total degrees of \(h_i'\) and \(\gamma_i\), we obtain the final bound on \(f'\).

\[\square\]

4.4. Chains of Autoreduced Sets and Coherent Sets. The derivatives in \(K\{X_{[n]}\}\) are well-quasiordered: that is, given any infinite sequence \(u_1, u_2, \ldots, u_n, \ldots\), there must be some \(i < j\) and some derivation \(\theta\) so that \(\theta u_i = u_j\).

**Definition 4.28.** A bad leader sequence is a sequence of derivatives \(\langle u_1, \ldots, u_m \rangle\) so that \(u_1 < u_2 < \cdots < u_m\) and if \(i < j\) then there is no \(\theta\) with \(\theta u_i = u_j\).

It is an easy consequence of Dickson’s Lemma that there are no infinite bad leader sequences; in particular, we can speak of the “tree of bad leader sequences”, and carry out proofs by induction on this tree.

There is a standard ordering associated with autoreduced sets:

**Definition 4.29.** Let \(\Lambda \subseteq K\{X_{[n]}\}\) be autoreduced. List \(\Lambda = \{f_1, \ldots, f_r\}\) in order of ascending rank. We write \(\Gamma(\Lambda)\) for the sequence \(\langle (\mu_1, b_1), \ldots, (\mu_r, b_r) \rangle\) where \(\mu_i\) is the leader of \(f_i\) and \(b_i\) is the degree of \(\mu_i\) in \(f_i\)—that is, \((\mu_i, b_i)\) is the rank of \(f_i\).

Given such a sequence \(\gamma = \langle \langle \mu_1, b_1 \rangle, \ldots, \langle \mu_r, b_r \rangle \rangle\), we write \(\gamma_\mu = \langle \mu_1, \ldots, \mu_r \rangle\) and \(\gamma_b = \langle b_1, \ldots, b_r \rangle\).

Given two sequences \(\gamma_1 = \langle \langle \mu_1, b_1 \rangle, \ldots, \langle \mu_r, b_r \rangle \rangle\) and \(\gamma_2 = \langle \langle \mu_1', b_1' \rangle, \ldots, \langle \mu_s', b_s' \rangle \rangle\), we say \(\gamma_1\) has lower rank than \(\gamma_2\) if either:

1. there is an \(i \leq \min(r, s)\) so that for all \(j < i\), \(\langle \mu_j, b_j \rangle = \langle \mu_j', b_j' \rangle\),
   but \(\langle \mu_i, b_i \rangle < \langle \mu_i', b_i' \rangle\), or
2. \(s > r\) and for all \(j \leq r\), \(\langle \mu_j, b_j \rangle = \langle \mu_j', b_j' \rangle\).

By abuse of notation, we often say \(\Lambda_1\) has lower rank than \(\Lambda_2\) when \(\Gamma(\Lambda_1)\) has lower rank than \(\Gamma(\Lambda_2)\).

Rank forms a well-order on the sequences \(\Gamma(\Lambda)\); we now obtain explicit bounds on the length of decreasing sequences in this ordering.

The idea is that we suppose we have a long sequence \(\Gamma(\Lambda_0), \ldots, \Gamma(\Lambda_d)\) all beginning with the same initial sequence \(\gamma\). In the worst case, where this sequence is as long as possible, \(\Gamma(\Lambda_0) = \gamma\) and for all \(i > 0\), \(\Gamma(\Lambda_i)\) must properly extend \(\gamma\)—that is, each \(\Gamma(\Lambda_i)\) must begin with some \(\gamma^{-} \langle \langle u_i, k_i \rangle \rangle\).

We break the interval \([1, d]\) into subsequences based on \(u_i\), and then further subsequences based on \(k_i\).

We first write down hypothetical worst case bounds: we inductively bound each subinterval, and then add these all up to bound the whole interval.
**Notation 4.30.** We define a bound \( h_{n,m}(D, \gamma) \) by recursion on \( \gamma\), taking \( D_{i_0}(i) = D(i_0 + i) \). We usually drop \( n, m \) when they are clear from context. We define:

- when \( \gamma\) is maximal, \( h(D, \gamma) = 1 \),
- when \( \gamma\) is not maximal, we define two helper sequences, \( w_u \) for \( u \in [-1, D(1)] \) and \( v_{u,k} \) for \( u \in [-1, D(1)], k \in [0, D(w_u)] \):
  - \( w_{D(1)} = 1 \),
  - if \( \gamma_{\mu}(u) \) is not a bad leader sequence then \( w_{u-1} = w_u \),
  - if \( \gamma_{\mu}(u) \) is a bad leader sequence then
    - \( v_{u,D(w_u)} = w_u \),
    - \( v_{u,k-1} = v_{u,k} + h(D_v, k, \gamma_{\mu}(k)) \),
    - \( w_{u-1} = v_{u,0} \),
- and set \( h(D, \gamma) = w_{-1} \).

We set \( h(D) = h(D, \emptyset) \).

Roughly speaking, the gap \( v_{u,k-1} - v_{u,k} \) is a bound on how long the subinterval of \( i \) so that \( \Gamma(A_i) \) begins with \( \gamma \) could be.

**Lemma 4.31.** Let a monotonic function \( D : \mathbb{N} \to \mathbb{N} \) be given and let \( \gamma = \langle (\mu_1, b_1), \ldots, (\mu_r, b_r) \rangle \). Suppose that for each \( i, A_i \) is an autoreduced set in \( K\{X_n\}_{\leq D(i)} \) so that \( \Gamma(A_i) \) begins with \( \gamma \). Then there is an \( i \leq h(D, \gamma) \) such that \( \Gamma(\Lambda_{i+1}) \) does not have lower rank than \( \Gamma(A_i) \).

**Proof.** We proceed by induction on \( \mu \). When \( \gamma\) is maximal, 1 suffices: if \( \Gamma(A_0) \) and \( \Gamma(A_1) \) both begin with \( \gamma \) then both must be equal to \( \gamma \) (because \( \gamma\) is a maximal bad leader sequence), so they have the same rank.

Suppose \( \gamma\) is not maximal, so that other than \( A_0 \), we may assume each \( \Gamma(A_i) \) is a proper extension of \( \gamma \). That is, each \( \Gamma(A_i) \) begins \( \gamma \) for some \( u \leq D(1) \). For each \( (u, b) \), take \( \hat{w}_u \) to be least so that \( \Gamma(A_{\hat{w}_u}) \) begins \( \gamma \) for some \( u' \leq u \) and take \( \hat{v}_{u,b} \) to be least so that \( \Gamma(A_{\hat{v}_{u,b}}) \) begins \( \gamma \) for some \( b' \leq b \).

To prove that our bounds work, we compare the actual gaps \( \hat{v}_{u,k-1} - v_{u,k} \) to our bounds \( v_{u,k-1} - v_{u,k} \). The idea is there must be some first interval in which the actual interval is at least as long as our bound, and the inductive hypothesis will guarantee that we find our witness in this interval.

Taking \( \hat{v}_{u-1} = \hat{w}_{u-1} \), we look for the smallest \( (u, b) \) so that \( \hat{v}_{u,b} > v_{u,b} \). Note that we must have \( \hat{v}_{u,b} \leq v_{u,b} \) (otherwise this would have happened for a smaller \( (u, b) \)). Then for every \( i \) such that \( \gamma \) begins \( \gamma \) for every \( i \) in \( [\hat{v}_{u,b}, \hat{v}_{u,b} - 1] \), \( \Gamma(A_i) \) begins \( \gamma \), so we may apply the inductive hypothesis with \( D_{\hat{v}_{u,b}} \) and obtain the desired witness.

In particular, when \( \gamma \) is the empty sequence,

**Corollary 4.32.** Let a monotonic function \( D : \mathbb{N} \to \mathbb{N} \) be given. Suppose that for each \( i, A_i \) is an autoreduced set in \( K\{X_n\}_{\leq D(i)} \). Then there is an \( i < h(D) \) such that \( \Gamma(\Lambda_{i+1}) \) does not have lower rank than \( \Gamma(A_i) \).
This bound is relevant for several important operations. Our first application takes as input a finite set and outputs an autoreduced set that is closely related to the original.

**Notation 4.33.** $D_{b,n,m}^\text{sat}$ is the function defined inductively by:

- $D_{b,n,m}^\text{sat}(0) = b$.
- $D_{b,n,m}^\text{sat}(i + 1) = g(D_{b,n,m}^\text{sat}(i), b)$.

We set $\iota_{n,m}(b) = D_{b,n,m}^\text{sat}(h_{n,m}(D_{b,n,m}^\text{sat}(0)))$.

Here $n$ is the number of indeterminates and $m$ is the number of derivations; when $n, m$ are fixed, we simply write $\iota^\text{sat}(b)$.

**Proposition 4.34.** Let $\Lambda \subseteq K\{X_{[n]}\}^\leq b$ be finite. Then there exists an autoreduced set $\Lambda' \subseteq [\Lambda] \cap K\{X_{[n]}\}^\leq b^\text{sat}(b)$ such that $\Lambda \subseteq [\Lambda'] : H_\Lambda^\infty$. Further, if $\Lambda^*$ is an autoreduced subset of $\Lambda$ then $\Lambda'$ has rank less than or equal to that of $\Lambda^*$.

**Proof.** Consider the following procedure: Let $\Lambda_0$ be an autoreduced subset of $\Lambda$ having minimal rank. We are done if $\Lambda$ is non-zero. Let $\Lambda_1$ be an autoreduced subset of $\Lambda_0 \cup \{\bar{f}\} \subseteq \Lambda$ having minimal rank. Repeat the process to recursively define a sequence $\Lambda_0, \Lambda_1, \Lambda_2, \ldots$ of autoreduced subsets of $\Lambda$. By Corollary 4.32 it suffices to verify that $\Lambda_{i+1} < \Lambda_i$ as autoreduced sets and that $\Lambda_i \subseteq K\{X_{[n]}\}^\leq b^\text{sat}(i)$.

Let $\bar{f}$ be the remainder used in defining $\Lambda_{i+1}$ from $\Lambda_i$. Note that elements of $\Lambda_i$ of lower rank than $\bar{f}$ are reduced with respect to $\bar{f}$. Since $\bar{f}$ is reduced with respect to $\Lambda_i$, the set of elements of $\Lambda_i$ having rank lower than that of $\bar{f}$ is an autoreduced subset of $\Lambda_i \cup \{\bar{f}\}$ having strictly lower rank than that of $\Lambda_i$.

Lastly, note that $\Lambda_0 \subseteq K\{X_{[n]}\}^\leq b = K\{X_{[n]}\}^\leq b^\text{sat}(0)$. Assume $\Lambda_i \subseteq K\{X_{[n]}\}^\leq b^\text{sat}(i)$. By Lemma 4.15 pseudodiving an element of $\Lambda \subseteq K\{X_{[n]}\}^\leq b$ with respect to $\Lambda_i$ gives a remainder belonging to $g(D_b^\text{sat}(i), b) = D_b^\text{sat}(i + 1)$.

**Notation 4.35.** Let $v$ be a derivative and let $\Lambda \subseteq K\{X_{[n]}\}$ be a finite set. We write $\Lambda_{[<v]}$ for the set $\{\theta \lambda \mid \lambda \in \Lambda, \theta \in \Theta, \text{ and } \mu_{\theta\lambda} < v\}$.

**Notation 4.36.** Consider $f, g \in K\{X_{[n]}\} \setminus K$ and $\theta_g, \theta_f$ derivations such that $\theta_g f$ and $\theta_f g$ have the same leader $v$. We define the $\Delta$-$S$-polynomial of $f$ and $g$ with respect to $v$, denoted $\Delta(f, g, v)$, to be $S_c \theta_g f - S_c \theta_f g$. If $v$ is the least such derivative, we simply write $\Delta(f, g)$.

The following property is key:

**Definition 4.37.** An autoreduced set $\Lambda \subseteq K\{X_{[n]}\} \setminus K$ is coherent if for all $f, g \in \Lambda$ with derivatives that share a common leader $v$ we have $\Delta(f, g, v) \in (\Lambda_{[<v]} : H_\Lambda^\infty)$. 

To have coherence it suffices for the least such leader to satisfy this condition. In practice, it is convenient to work with a slightly stronger notion. This costs us nothing, since the standard construction of a coherent set actually gives the stronger property.

**Definition 4.38.** An autoreduced set $\Lambda \subseteq K\{X_{[n]}\}\setminus K$ is reduction-coherent if for all $f, g \in \Lambda$ with derivatives that share a common leader $v$, the $\Delta$-polynomial $\Delta(f, g, v)$ reduces to 0 with respect to $\Lambda$.

It is easy to check that any reduction-coherent set is coherent. See [31, 48] for further details and generalizations.

Using essentially the same strategy as in [43], we give effective bounds on coherent sets. See [33, 5.5.12] for the algorithm (but without the bound).

**Notation 4.39.** $D_b^{\text{cohere}}$ is the function defined inductively by:

1. $D_b^{\text{cohere}}(b, n, m, 0) = b$,
2. $D_b^{\text{cohere}}(b, n, m, i + 1) = g(D_b^{\text{cohere}}(b, n, m, i), (2D_b^{\text{cohere}}(b, n, m, i + 1) + m - 1) \cdot n \cdot D_b^{\text{cohere}}(b, n, m, i + 1))$.

We set $i_{m, n}(b) = D_b^{\text{cohere}}(b, n, m, (D_b^{\text{cohere}}(b, n, m, m)))$.

Again we usually omit $n, m$.

**Proposition 4.40.** Let $\Lambda_0 \subseteq K\{X_{[n]}\}_{\leq b}$ be an autoreduced set. Then there exists a reduction-coherent set $\Lambda \subseteq [\Lambda_0]_{\leq b} \cap K\{X_{[n]}\}_{\leq 1}$.

**Proof.** Given an autoreduced set $\Lambda_0$, form the $\Delta$-polynomials corresponding to pairs of elements of $\Lambda_0$. Pseudodivide each $\Delta$-polynomial by $\Lambda_0$ and let $R_0$ be the set of remainders. If $R_0 = \{0\}$, then $\Lambda_0$ is already reduction-coherent. If not, select an autoreduced subset $\Lambda_1$ of $\Lambda_0 \cup R_0$ having minimal rank and repeat the process.

The output is automatically reduction-coherent if the procedure terminates. Termination follows from Corollary 4.32 because $\Lambda_{i+1} < \Lambda_i$ as autoreduced sets using the same argument as before: given $p \in R_i$, the set of elements of $\Lambda_i$ having rank lower than that of $p$ and $\{p\}$ is an autoreduced subset of $\Lambda_i \cup R_i$ having strictly lower rank than that of $\Lambda_i$. We claim that the intermediate sets are bounded by $D_b^{\text{cohere}}$. Suppose $\Lambda_i \subseteq K\{X_{[n]}\}_{\leq D_b^{\text{cohere}}(i)}$. Then the degree of a $\Delta$-polynomial of two elements of $\Lambda_i$ is at most $2D_b^{\text{cohere}}(i)$. The ranking grows more, though, because there are $(2D_b^{\text{cohere}}(i) + m - 1) \cdot n$ derivatives of order $2D_b^{\text{cohere}}(i)$. Hence doubling the order of a derivative in $K\{X_{[n]}\}_{\leq D_b^{\text{cohere}}(i)}$ cannot place the result beyond $K\{X_{[n]}\}_{\leq (2D_b^{\text{cohere}}(i) + m - 1) \cdot n \cdot D_b^{\text{cohere}}(i) + 1}$.

By Lemma 4.15, pseudodividing with respect to $\Lambda_i$ gives a remainder bounded by $g(D_b^{\text{cohere}}(i), (2D_b^{\text{cohere}}(i) + m - 1) \cdot n \cdot (D_b^{\text{cohere}}(i) + 1)) = D_b^{\text{cohere}}(i + 1)$. Hence the algorithm terminates by step $b(D_b^{\text{cohere}})$. □

An easy extension of this argument gives:
Proposition 4.41. Let \( \Lambda_0 \subseteq K\{X_{[n]}\}_{\leq b} \) be an autoreduced set. Then there exists a reduction-coherent set \( \Lambda \subseteq [\Lambda_0]_{\leq \text{cohere}(b)} \cap K\{X_{[n]}\}_{\leq \text{cohere}(b)} \) such that \( \Lambda_0 \subseteq [\Lambda] : H^\infty_{\Lambda} \).

Proof. As in the previous proof, we form a sequence of autoreduced sets \( \Lambda_0, \Lambda_1, \ldots \) of decreasing rank. If \( \Lambda_i \) is not reduction-coherent then we choose \( \Lambda_{i+1} \) as in the previous lemma.

If \( \Lambda_i \) is reduction-coherent, we ask whether there is any \( f \in \Lambda_0 \) so that the remainder \( \bar{f} \) with respect to \( \Lambda_i \) is non-zero. If so, we take \( \Lambda_{i+1} \) to be a minimal rank autoreduced subset of \( \Lambda_i \cup \{ \bar{f} \} \).

This must still terminate within \( h(D_b^\text{cohere}) \) steps with some \( \Lambda_i \) which is reduction-coherent and whenever \( f \in \Lambda_0 \), the remainder \( \bar{f} \) with respect to \( \Lambda_i \) is 0. Therefore by the definition of the remainder, \( H^k_{\Lambda_i} f \in ((\Lambda_i)[g]) \) for some \( k \), so \( f \in [\Lambda_i] : H^\infty_{\Lambda_i} \).

Lemma 4.42 (Based on \[23\], 4.4(1)). Suppose \( \Lambda \subseteq K\{X_{[n]}\}_{\leq b} \) is coherent and autoreduced and \( \Lambda^H \) is \( F \)-prime up to \( p_0(u_F(b)) \). Suppose \( g \in [\Lambda] : H^\infty_{\Lambda} \cap K\{X_{[n]}\}_{\leq d} \). Then \( g \in (\Lambda^H_{[g][b]+u_F^+(b)}) \).

Proof. Let \( \bar{g} \) be the remainder of \( g \) with respect to \( \Lambda \), so also \( g' \in [\Lambda] : H^\infty_{\Lambda} \).

By \[31\] III(8), Lemma 5, also \( g' \in (\Lambda) : H^\infty_{\Lambda} \), and then by Lemma 4.21
\[ g', g' \in (\Lambda^H_{[g][b]+u_F^+(b)}) \] so \( H^u_{\Lambda} g' \in (\Lambda) \subseteq (\Lambda[d]) \). Since \( H^u_{\Lambda} g' \in (\Lambda[d]) \) as well, we have \( H^u_{\Lambda} g' \in (\Lambda[d]) \), so \( g \in (\Lambda^H_{[b][b]+u_F^+(b)}) \).

Lemma 4.43 (Based on \[23\], 4.4(2)). Suppose \( \Lambda \subseteq K\{X_{[n]}\}_{\leq b} \) is autoreduced and \( [\Lambda] : H^\infty_{\Lambda} \) contains no non-zero elements of degree \( \leq b \) reduced with respect to \( \Lambda \), that \( P \) is any prime \( \Delta \)-ideal, and that there is some \( g \in [\Lambda] : H^\infty_{\Lambda} \) \( \setminus P \). Then there is an \( h \in K\{X_{[n]}\}_{\leq b} \) that \( h \in [\Lambda] : H^\infty_{\Lambda} \setminus \Delta P \).

Proof. If \( \Lambda \not\subseteq P \), this is immediate, so assume \( \Lambda \subseteq P \). Since \( g \in [\Lambda] : H^\infty_{\Lambda} \), also \( H^d g \in P \) for some \( d \). Since \( P \) is prime, either \( g \in P \) or \( I_\lambda \) or \( S_\lambda \) belongs to \( P \) for some \( \lambda \in \Lambda \). Since we have ruled out \( g \in P \), we have \( I_\lambda \in P \) or \( S_\lambda \in P \). But both \( I_\lambda \) and \( S_\lambda \) have degree \( \leq b \) and are reduced with respect to \( \Lambda \), so they do not belong to \( [\Lambda] : H^\infty_{\Lambda} \).

Notation 4.44.

- \( z^0(d, b) = d \),
- \( z^{k+1}(d, b) = 3^k(d_{b+d}((g(b + d - 1, \max\{b + d - 1, 2b\}) + d + 1)(2b)2b + d) + g(b + d - 1, \max\{b + d - 1, 2b\}) + d + 1, b)) \).

Lemma 4.45 (Based on \[31\], III.8, Lemma 5). Suppose \( \Lambda \subseteq K\{X_{[n]}\}_{\leq b} \) is autoreduced and reduction-coherent. Then if \( g \in (\Lambda^H_{[d][b]+u_F^+(b)}) \cap K\{X_{[n]}\}_{\leq d} \) is partially reduced with respect to \( \Lambda \), \( g \in (\Lambda^H_{[b][d]+u_F^+(b)}) \).
Proof. We may write
\[ H^d_{\Lambda} g = \sum_{j \leq r} c_j(\theta_j \lambda_j) + \sum_i d_i \lambda_i \]
where \( \theta_j \lambda_j \in K\{X_{[n]}\}_{\leq b+d} \). Let \( m \) be the least upper bound on indices \( l \)
such that \( Z_l \) is the leader of some proper derivative \( \theta_j \lambda_j \) appearing in the
sum. We will show by induction on \( m \) that \( g \in (A^H_{[\Lambda]}(d,b)) \). When \( m = 0 \)
(that is, there are no \( c_j \theta_j \lambda_j \) terms), this is immediate.

So suppose the claim holds for values less than \( m \) and there is some \( j \)
with \( \mu_{\theta_j \lambda_j} = Z_m \) and \( Z_m \) is the largest leader; we write
\[ H^d_{\Lambda} g = \sum_{j < q} c_j(\theta_j \lambda_j) + \sum_{q < j < r} c_j(\theta_j \lambda_j) + \sum_i d_i \lambda_i \]
where \( \mu_{\theta_j \lambda_j} = Z_{k_j} \) so \( k_j < m \) if \( j < q \) and \( k_j = m \) if \( q \leq j < r \).

We multiply both sides by \( S_{\lambda_q} \), so
\[ S_{\lambda_q} H^d_{\Lambda} g = \sum_{j < q} c_j'(\theta_j \lambda_j) + \sum_i d_i' \lambda_i + \sum_{q < j < r} c_j(S_{\lambda_q} \theta_j \lambda_j - S_{\lambda_j} \theta_q \lambda_q) + \sum_{q < j < r} c_j S_{\lambda_j} \theta_q \lambda_q. \]

By assumption, the remainder of \( S \) with respect to \( \Lambda_{[d-1]} \) is 0. Since
\( S \in K\{X_{[n]}\}_{\leq b+d-1,2b} \) and \( \Lambda_{[d-1]} \subseteq K\{X_{[n]}\}_{\leq b+d-1,b} \), by \[4.15 \] we have
\[ H^d_{\Lambda}^{b(d-1,\max\{b+d-1,2b\})+d+1} S \in (\Lambda_{[d-1]}). \]
It follows that
\[ H^d_{\Lambda}^{b(b+d-1,\max\{b+d-1,2b\})+d+1} g = \sum_{j < q} c_j''(\theta_j \lambda_j) + \sum_i d_i'' \lambda_i + c^*(\theta_q \lambda_q). \]

By internal faithful flatness, we may assume \( c_j'', d_i'', c^* \) have degree at most
\( D = d(b+d-1,\max\{b+d-1,2b\})+d+1 \). Since \( g \) is partially reduced with respect to \( \Lambda \), in particular \( Z_m = \mu_{\theta_q \lambda_q} \) does not
appear on the left side of the equation. We have \( \theta_q \lambda_q = S_{\lambda_q} Z_m + h \) where \( h \)
has lower rank than \( Z_m \), so we may replace \( Z_m \) with \( -\frac{h}{S_{\lambda_q}} \). The degree of
\( Z_m \) is at most \( D \), so we multiply by \( H^D_{\Lambda} \) to clear denominators, giving
\[ H^D_{\Lambda}^{b(b+d-1,\max\{b+d-1,2b\})+d+1} g = \sum_{j < q} c_j'''(\theta_j \lambda_j) + \sum_i d_i''' \lambda_i. \]

The claim now follows by the inductive hypothesis since each \( \mu_{\theta_j \lambda_j} = Z_l \)
with \( l < m \). \qed

4.5. Characteristic Sets.

Definition 4.46. Let \( I \subseteq K\{X_{[n]}\} \) be a \( \Delta \)-ideal. An autoreduced subset \( \Sigma \)
of \( I \) having minimal rank is a characteristic set of \( I \).

We use Rosenfeld’s lemma and effective bounds on reduction-coherent
sets to find effective bounds on characteristic sets.

Notation 4.47. \( D^\text{char}_b \) is the function defined inductively by:
- \( \mathbb{P}^\text{char}_b(k) = j^{k+1}(k, g(c, k)), \)
\( D_{b,n,m}(0) = b, \)

\( D_{b,n,m}(i+1) = \max \{ g(D_{b,n,m}(i)), \left( \frac{2D_{b,n,m}(i) + m - 1}{m - 1} \right) \cdot n \cdot (D_{b,n,m}(i) + 1), \)
\( \mathfrak{p}_{D_{b,n,m}}(i)\left( u_{F_{\text{char}}} \left( D_{b,n,m}(i) \right) \right), \)
\( f(\mathfrak{p}_{D_{b,n,m}}(i), D_{b,n,m}(i)) \} \).

We set \( i_{\text{char}}(b) = D_{b,n,m}(h_{n,m}(D_{b,n,m})) \).

Once again, we usually omit \( n, m \).

**Theorem 4.48** (Based on [23], Lemma 5.6/Theorem 6.1). Let \( \Lambda \subseteq K \{ X_{[n]} \} \leq b \). Let \( P \) be a proper \( \Delta \)-ideal containing \( \Lambda \) such that whenever \( fg \in P \) with \( f \in K \{ X_{[n]} \} \leq \text{char}(b) \), either \( f \in P \) or \( g \in P \). Then \( P \) contains a set \( \Sigma \subseteq K \{ X_{[n]} \} \leq \text{char}(b) \) which is the characteristic set of a prime ideal containing \( \Lambda \) where \( H_\Sigma \not\subseteq P \).

**Proof.** We construct a series of autoreduced sets so that \( \Lambda_{i+1} \) has lower rank than \( \Lambda_i \). Take \( \Lambda_0 \subseteq K \{ X_{[n]} \} \leq b \) to be some minimal rank autoreduced subset of \( \Lambda \). Given \( \Lambda_i \subseteq K \{ X_{[n]} \} \leq D_{b \text{char}(i)} \cap P \), we proceed as follows.

First, if \( \Lambda_i \) is not reduction-coherent, we proceed as in Proposition 4.40 take \( \Lambda_{i+1} \) to be a minimal rank autoreduced subset of \( \Lambda_i \cup R \) where \( R \) consists of the \( \Delta \)-\( S \)-polynomials of pairs from \( \Lambda_i \). As in Proposition 4.40 we have \( \Lambda_{i+1} \subseteq K \{ X_{[n]} \} \leq g(D_{b \text{char}(i)}(A_0^{2D_{b \text{char}(i)}(i)+m-1}) \cdot n \cdot (D_{b \text{char}(i)+1})) \).

If \( \Lambda_i \) is reduction-coherent but \( \Lambda \not\subseteq (\Lambda_i) : H_{\Lambda_i}^{\infty} \), we proceed as in Proposition 4.34, pick \( f \in \Lambda \) so that the remainder \( \tilde{f} \) with respect to \( \Lambda_i \) is non-zero and let \( \Lambda_{i+1} \) be a minimal rank autoreduced subset of \( \Lambda_i \cup \{ f \} \). As in Proposition 4.34 we have \( \Lambda_{i+1} \subseteq K \{ X_{[n]} \} \leq D_{b \text{char}(i)}(g(D_{b \text{char}(i)}, b) \subseteq K \{ X_{[n]} \} \leq g(D_{b \text{char}(i)}(2D_{b \text{char}(i)}(i)+m-1}) \cdot n \cdot (D_{b \text{char}(i)+1})) \).

If neither of the previous cases hold and \( H_{\Lambda_i} \not\subseteq P \), pick some \( u \in \{ I_f, S_f | f \in \Lambda_i \} \cap P \) and take a minimal rank autoreduced subset \( \Lambda_{i+1} \) of \( \Lambda_i \cup \{ u \} \). Then \( \Lambda_{i+1} \subseteq K \{ X_{[n]} \} \leq D_{b \text{char}(i)}(i) \).

If \( \langle (\Lambda_i)_{H_{\Lambda_i}}^d \rangle \) is not boundedly \( \text{pr}(\Lambda_i) \cdot F_{\text{char}}(D_{b \text{char}(i)}(i)) \cdot (D_{b \text{char}(i)}(i)) \) prime up to \( p_{D_{b \text{char}(i)}}(i) \cdot u_{F_{\text{char}}} \left( D_{b \text{char}(i)}(i) \right) \) then there is some \( k \leq p_{D_{b \text{char}(i)}}(i) \cdot u_{F_{\text{char}}} \left( D_{b \text{char}(i)}(i) \right) \) and some \( f, g \in \langle (\Lambda_i)_{H_{\Lambda_i}}^d \rangle \cap K \{ X_{[n]} | \Lambda_i \} \leq k \) so that \( f, g \not\in \langle (\Lambda_i)_{H_{\Lambda_i}}^d \rangle \). Then \( H_{\Lambda_i}^k f, g \in P \). Since \( H_{\Lambda_i} \not\subseteq P \), we may assume \( f \in P \). Let \( \tilde{f} \) be the remainder of \( f \) with respect to \( \Lambda_i \), so there are \( l_0, l_1 \leq g(D_{b \text{char}(i)}, k) \) so that

\[ I_{\Lambda_i}^{l_0} a_{\Lambda_i}^l f - \tilde{f} \in (\Lambda_i) \].
Suppose $\tilde{f} = 0$; then $H^i_{\Lambda_i} f \in (\Lambda_i^{(k)})$, so $f \in (\Lambda_i)^{H^i_{\Lambda_i}}(\mathcal{D}_b^\text{char}(i),k))$. But then by Lemma 4.45 $f \in ((\Lambda_i)^{H^i_{\Lambda_i}}(\mathcal{D}_b^\text{char}(i),k))) = ((\Lambda_i)^{H^i_{\Lambda_i}}(\mathcal{D}_b^\text{char}(i),k)))$. But this contradicts the assumption that $f g$ witnessed the failure of bounded $\mathcal{D}_b^\text{char}(i)$-primality. So $\tilde{f} \neq 0$. Since $\tilde{f} \in P$, let $\Lambda_{i+1}$ be a minimal rank autoreduced subset of $\Lambda_i \cup \{\tilde{f}\}$.

Suppose none of the cases above hold, and there is a $g \in (\Lambda_i) : \Lambda_i^{\infty}$ which is non-zero and reduced with respect to $\Lambda_{i+1}$. Then by Lemma 4.45, there is an $f \in (\Lambda_i^{H^i_{\Lambda_i}}(\mathcal{D}_b^\text{char}(i)),\mathbb{K}[X[n]]) \leq u_{\mathcal{D}_b^\text{char}(i),f}(\mathcal{D}_b^\text{char}(i),\mathbb{K}[X[n]])$ which is non-zero and reduced with respect to $\Lambda_{i+1}$. Then we may take $\Lambda_{i+1}$ to be a minimal rank autoreduced subset of $\Lambda_i \cup \{f\}$.

By Lemma 4.32, there is an $i < b(\mathcal{D}_b^\text{char})$ so that none of these cases occurs: $\Lambda_i$ is reduction-coherent, $H_{\Lambda_i} \notin P$, $(\Lambda_i^{H^i_{\Lambda_i}}) d$ is boundedly prime to $\mathcal{D}_b^\text{char}(i)$ prime up to $\mathcal{D}_b^\text{char}(i)$. But in the former case $\Lambda_i$ is the characteristic set of the prime $\Delta$-ideal $[\Lambda_i] : H_i^{\infty}$. Since $\Lambda \subseteq [\Lambda_i] : H_i^{\infty}$, we are done.

**Corollary 4.49.** Suppose $\Lambda \subseteq K\{X[n]\}_{\leq b}$ and let $P$ be a minimal prime $\Delta$-ideal containing $\Lambda$. Then $P$ has a characteristic set $\Sigma \subseteq K\{X[n]\}_{\leq \text{char}(b)}$.

**Proof.** $P$ contains some $\Sigma$ which is the characteristic set of a prime ideal $[\Sigma] : H_\Sigma^{\infty}$ containing $\Lambda$ with $H_\Sigma \notin P$. Suppose $f \in [\Sigma] : H_\Sigma^{\infty}$, so $H_\Sigma f \in [\Sigma] \subseteq P$ for some $k$. Since $P$ is prime and $H_\Sigma \notin P$, we have $f \in P$, so $[\Sigma] : H_\Sigma^{\infty} \subseteq P$. By minimality of $P$, we must have $[\Sigma] : H_\Sigma^{\infty} = P$. □

**Corollary 4.50** (Based on [23], Proposition 5.3/Theorem 5.4). Let $\Lambda \subseteq K\{X[n]\}_{\leq b}$ be given with $1 \notin [\Lambda]$. If either $f \in [\Lambda]$ or $g \in [\Lambda]$ for all $f, g \in K\{X[n]\}$ with $fg \in [\Lambda]$ and $f \in K\{X[n]\}_{\leq \text{char}(b)}$, then $[\Lambda]$ is prime.

**Proof.** We apply the theorem to obtain a characteristic set $\Sigma \subseteq [\Lambda] \cap K\{X[n]\}_{\leq \text{char}(b)}$ with $H_\Sigma \notin [\Lambda]$. Since $\Lambda \subseteq [\Sigma] : H_\Sigma^{\infty}$, we also have $[\Lambda] \subseteq [\Sigma] : H_\Sigma^{\infty}$. It remains to show the reverse containment. If $g \in [\Sigma] : H_\Sigma^{\infty}$ then for some $N \in \mathbb{N}$ we have $H_\Sigma^N g \in [\Sigma] \subseteq [\Lambda]$. Let $N$ be least so that $H_\Sigma^N g \in [\Lambda]$. If $N > 0$ then either $H_\Sigma \in [\Lambda]$ or $g \in [\Lambda]$ (because the factors of $H_\Sigma$ are bounded by $\text{char}(b)$). But in the former case $H_\Sigma \in [\Lambda] \subseteq [\Sigma] : H_\Sigma^{\infty}$. Therefore $g \in [\Lambda]$.

**Remark 4.51.** Proposition 5.3 of [23] states this for a radical ideal, but their argument similarly applies to a differential ideal with minor modifications.

5. Ritt-Noetherianity

In this section we give an effective version of Ritt-Noetherianity. There are two approaches we might take, depending on the question of whether
we treat membership in a radical differential ideal as decidable—that is, given $h \in K\{X_{[n]}\}_{\leq d}$ and $\Lambda \subseteq K\{X_{[n]}\}_{\leq b}$, whether there is some bound $k$ depending on $n, b, d$ so that $h \in \{\Lambda\}$ iff $h^k \in (\Lambda_{[k]})$.

Of course, there is such a bound, given first in [18] with refinements in [11, 20, 22, 34]. But [23] uses Ritt-Noetherianity to prove the existence of such a bound, so one might choose to avoid existing bounds and use the functional interpretation to obtain an explicit version of the bound from [23]. The resulting version of Ritt-Noetherianity, however, is rather unwieldy (as the functional interpretation of a $\Pi_1$ statement, it requires the use of higher-order functions on functions), and the bounds one gets are much worse than those in the literature.

Therefore in the work below, we use the bounds from [18] on testing membership in radical differential ideals. This makes Ritt-Noetherianity a $\Pi_3$ statement, directly analogous to the effective version of Noetherianity discussed in Section 8. This will have roughly the form

For any functions $D, F$ there is a bound $M$ so that whenever $\Lambda_i \subseteq K\{X_{[n]}\}_{\leq D(i)}$ for all $i$, there is an $m \leq M$ so that $\Lambda_{F(m)} \subseteq \{\Lambda_m\}$. 

Along the way, we will need to inductively prove cases of similar but weaker statements (roughly speaking, we will replace the conclusion with $\Lambda_{F(m)} \subseteq \{\Lambda_m \cup \{u\}\}$ for various choices of $u$). Once we prove several of these, we will need to arrange for their conjunction to hold uniformly—that is, once we can find $m_1$ so that $\Lambda_{F(m_1)} \subseteq \{\Lambda_{m_1} \cup \{u_1\}\}$ and $m_2$ so that $\Lambda_{F(m_2)} \subseteq \{\Lambda_{m_2} \cup \{u_2\}\}$, we will need to find a single $m$ so that $\Lambda_{F(m)} \subseteq \{\Lambda_m \cup \{u_1\}\} \cap \{\Lambda_m \cup \{u_2\}\}$.

**Lemma 5.1 (Knitting Lemma).** Let $J$ be a finite set. Suppose that for each $j \in J$, any function $F$, and any $d$, there is a $k \in [d, \mathcal{G}_j(F, d)]$ so that for all $i \in [k, F(k)]$, $\phi_j(i)$ holds. Then there is a functional $\mathcal{G}_J$ so that for any $d$ and any $F$, there is a $k \in [d, \mathcal{G}_J(F, d)]$ so that, for each $j \in J$ and each $i \in [k, F(k)]$, $\phi_j(i)$ holds.

The existence of such a lemma is guaranteed by the proof of correctness of the functional interpretation. We began with statements $\forall x \exists y \forall z \phi_j(x, y, z)$, and it follows that $\forall x \exists y \forall j \in J \exists y_j \leq Y \forall z \phi_j(x, y_j, z)$. The functional interpretation promises that if the latter follows from the former then bounds on the functional interpretation of the latter must be derivable from bounds on the functional interpretation of the former. The knitting lemma is simply the statement that the functional interpretation works in one specific case. (Compare to Lemma 6.2 of [33], which deals with essentially the same issue.)

**Proof.** By induction on $|J|$. When $|J| = 1$, this is trivial—$\mathcal{G}_{\{j\}}$ is simply $\mathcal{G}_j$. So suppose $|J| > 2$ and pick some $j_0 \in J$. The inductive hypothesis gives us a function $\mathcal{G}_{J \setminus \{j_0\}}$.

For any $d'$, define $F_{d'}(d) = F(\max\{d, d'\})$. Let $G(d') = F(\mathcal{G}_{j_0}(F_{d'}, d'))$. Let $\mathcal{G}_J(F, d) = \mathcal{G}_{J \setminus \{j_0\}}(G, d)$. 


For any \( d \), there is a \( d' \in [d, \mathcal{G}_{J,\{j_0\}}(\mathbf{G}, d)] \) so that for all \( i \in [d', \mathbf{G}(d')] \) and all \( j \in J \setminus \{j_0\} \), \( \phi_j(i) \) holds. There is also a \( k \in [d', \mathcal{G}_{j_0}(\mathbf{F}^{d'}, d')] \) so that, for all \( i \in [k, \mathbf{F}^{d'}(k)] \), \( \phi_{j_0}(i) \) holds. Since \( k \leq \mathcal{G}_{j_0}(\mathbf{F}^{d'}, d') \), also \( \mathbf{F}(k) \leq \mathcal{F} \mathcal{G}_{j_0}(\mathbf{F}^{d'}, d') = \mathbf{G}(d') \), so \( [k, \mathbf{F}(k)] \subseteq [d', \mathbf{G}(d')] \), so \( k \) is the desired witness.

\[ \square \]

**Theorem 5.2 (IS).** There is a function \( \mathfrak{t} \) so that whenever \( \Lambda \subseteq K \{X_{[n]}\}_{\leq b} \) and \( h \in K \{X_{[n]}\}_{\leq b} \cap \{\Lambda\} \), also \( h \in \sqrt{\langle \Lambda |_{[n,b]} \rangle} \).

**Notation 5.3.** We define \( i_{n,m}(i_0, d, \mathbf{F}, d, \Lambda) \) by recursion on the rank of \( \Lambda \).

Suppose \( i_{n,m}(i_0, d, \mathbf{F}, d, \Lambda) \) has been defined for \( \Lambda' \) with rank less than the rank of \( \Lambda \). Let \( J \) consist of all (finitely many) autoreduced sets \( \Lambda^* \subseteq K \{X_{[n]}\}_0 \), for each \( \Lambda^* \in J \) and each \( d \), we have functionals \( \mathcal{G}_{\Lambda^*,d}(\mathbf{F}, i) = i_{n,m}(i, d, \mathbf{F}, i_{s\text{at}}(d), \Lambda^*) \), so by the Knitting Lemma, we have a functional \( \mathcal{G}_{J,d} \).

Then we define \( i_{n,m}(i_0, d, \mathbf{F}, d, \Lambda) \) to be the maximum of:
\[
i_{n,m}(\mathcal{G}_{J,d}(\mathbf{F}, i_0), d, \mathbf{F}, d(\mathcal{G}_{J,d}(\mathbf{F}, i_0))) + \mathfrak{t}(n, d(\mathcal{G}_{J,d}(\mathbf{F}, i_0))), \Lambda_*)
\]
where \( \Lambda_* \) ranges over autoreduced sets in \( K \{X_{[n]}\}_{\leq d} \), with lower rank than \( \Lambda \).

Again, we usually omit \( n, m \).

**Theorem 5.4 (Based on [36], Theorem 1.16, p. 47).** Let \( i_0, \Lambda, \Lambda_0 \subseteq \Lambda_1 \subseteq \cdots, \mathbf{D}, \mathbf{F}, d \) be given such that:

- \( \Lambda \subseteq K \{X_{[n]}\}_{\leq d} \) is autoreduced, and
- \( \Lambda_0 \subseteq K \{X_{[n]}\}_{\leq \mathbf{D}(i)} \) for all \( i \).

Then there is an \( i \in [i_0, i(i_0, \mathbf{D}, d, \Lambda)] \) so that \( \mathbf{A}_{\mathbf{F}(i)} \subseteq \{\Lambda \cup \Lambda_i\} \).

**Proof.** We proceed by induction on the rank of \( \Lambda \). We assume that whenever \( \Lambda' \) is an autoreduced set so that \( \Lambda' \) has lower rank than \( \Lambda \), the claim holds of \( \Lambda' \).

The functional \( \mathcal{G}_{J,d} \) is as above.

**Claim 5.4.1:** There is an \( i \in [i_0, \mathcal{G}_{J,d}(\mathbf{F}, i_0)] \) so that for every \( u \in \{I_x, S_x \mid \lambda \in \Lambda\} \), \( \Lambda_{\mathbf{F}(i)} \subseteq \{\Lambda \cup \{u\} \cup \Lambda_i\} \).

**Proof.** For every \( u \in \{I_x, S_x \mid \lambda \in \Lambda\} \), Proposition 4.34 gives a \( \Lambda_* \subseteq [\Lambda \cup \{u\}] \cap K \{X_{[n]}\}_{\leq s\text{at}(d)} \) so that \( \Lambda \cup \{u\} \subseteq [\Lambda_*]_\Lambda : H^\Lambda_* \) and the rank of \( \Lambda_* \) is less than the rank of \( \Lambda \). Necessarily \( \Lambda_* \subseteq [\Lambda \cup \{u\}]_{\leq s\text{at}(d)} \).

In particular, \( \Lambda_* \in J \), so \( \Lambda_{\mathbf{F}(i)} \subseteq \{\Lambda \cup \Lambda_i\} \subseteq \{\Lambda \cup \{u\} \cup \Lambda_i\} \).

For each \( h \in \Lambda_{\mathbf{F}(i)} \) and each derivation \( \theta \leq \mathfrak{t}(n, \mathbf{D}(i)) \), consider the remainder \( \tilde{h} \) of \( \theta h \) with respect to \( \Lambda_* \). Suppose that for some \( h, \theta \), the remainder \( \tilde{h} \neq 0 \). Let \( \Lambda_* \subseteq [\Lambda \cup \{\tilde{h}\}] \cap K \{X_{[n]}\}_{\leq s\text{at}(\mathfrak{t}(n, \mathbf{D}(i)))} \) be autoreduced so that \( \Lambda \cup \{\tilde{h}\} \subseteq [\Lambda_*]_\Lambda : H^\Lambda_* \) and \( \Lambda_* \) has lower rank than \( \Lambda \). Then the inductive hypothesis applies and we find an \( i' \in [i, j(i, \mathbf{B}, \mathbf{D}, \mathbf{B}(\mathbf{F}(i)), \mathbf{i}_{\text{sat}}(\mathfrak{t}(n, \mathbf{D}(i))), \Lambda_*)] \)
so that $\Lambda_{\mathbf{F}(i')} \subseteq \{\Lambda_u \cup \Lambda_{i'} \} \subseteq \{\Lambda \cup \{\bar{h}\} \cup \Lambda_{i'}\}$. But since $\Lambda_{\mathbf{F}(i')} = \Lambda_{\mathbf{F}(i)}$, we have $\bar{h} \in \{\Lambda_u \cup \Lambda_{i'} \}$. Therefore $i'$ is the desired witness.

In the remaining case, for each $h \in \Lambda_{\mathbf{F}(i)}$ and each derivation $\theta \leq \epsilon(n, \mathbf{D}(i))$, the remainder with respect to $\Lambda_a$ is $0$—that is, for some $M, H_{\lambda}^M \theta h \in (\Lambda_{\mathbf{D}(i)+ord(\theta)}^\theta)$.

**Claim 5.4.2:** For every $h \in \Lambda_{\mathbf{F}(i)}$ and all sets of derivations $\left\{\theta_u \right\}_{u \in \{I_{\lambda}, S_{\lambda} | \lambda \in \Lambda_{a}\}}$, $\theta'$ with $\sum_u ord(\theta_u) + ord(\theta') \leq \epsilon(n, \mathbf{D}(i))$, we have

$$
(\prod_u \theta_u)(\theta'h) \in \{\Lambda \cup \Lambda_i\}.
$$

**Proof.** By induction on $\sum_u ord(\theta_u)$. When $\sum_u ord(\theta_u) = 0$, we have $\Pi_u \theta_u u = \Pi_u H_{\lambda}$. Since the reduction of $\theta'h$ with respect to $\Lambda$ is $0$, we have $H_{\lambda}^M(\theta'h) \in [\Lambda] \subseteq \{\Lambda \cup \Lambda_i\}$. Since this is a radical ideal, also $H_{\lambda}(\theta'h) \in \{\Lambda \cup \Lambda_i\}$.

Otherwise, take some $u_0$ with $ord(\theta_{u_0}) > 0$, so $\theta_{u_0} = \delta\theta_{u_0}'$. For $u \neq u_0$, take $\theta'_u = \theta_u$. Note that

$$
\delta((\prod_u \theta'_u)(\theta'h)) = \sum_u ((\delta\theta'_u) \prod_{u' \neq u} \theta'_u u')(\theta'h) + (\prod_u \theta'_u)(\delta\theta'h).
$$

Since $(\prod_u \theta'_u)(\theta'h)$ and $(\prod_u \theta'_u)(\delta\theta'h)$ are both in $\{\Lambda \cup \Lambda_i\}$, also

$$
\sum_u ((\delta\theta'_u) \prod_{u' \neq u} \theta'_u u')(\theta'h) \in \{\Lambda \cup \Lambda_i\}.
$$

Multiply this by $\prod_u \theta_u$, so $\sum_u ((\delta\theta'_u) \theta_u u) \prod_{u' \neq u} \theta'_u u' u' u'')(\theta'h) \in \{\Lambda \cup \Lambda_i\}$. Any term in this sum where $u \neq u_0$ now has the form $\gamma(\prod_u \theta'_u)(\theta'h)$ and therefore belongs to $\{\Lambda \cup \Lambda_i\}$. Therefore the remaining term,

$$
((\delta\theta'_{u_0}) \theta_{u_0} u_0) \prod_{u' \neq u_0} \theta'_u u' u' u'')(\theta'h) = \prod_u \theta_u u^2(\theta'h) \in \{\Lambda \cup \Lambda_i\}.
$$

Since the ideal is radical, also $(\prod_u \theta_u)(\theta'h) \in \{\Lambda \cup \Lambda_i\}$ as desired.

Consider some $h \in \Lambda_{\mathbf{F}(i)}$. Then for each $u \in \{I_{\lambda}, S_{\lambda} | \lambda \in \Lambda_{a}\}$ we have $h \in \{\Lambda \cup \Lambda_i \cup \{u\}\}$, so for each $u$ there is some $m$ so

$$
h^m = \sum_i \gamma_{i,u} \delta_{i,u}
$$

where each $\delta_{i,u}$ has the form $\theta_{i,u} \mu_{i,u}$ where $\theta_{i,u}$ is a derivative of order $\leq \epsilon(n, \mathbf{D}(i))$ and $\mu_{i,u} \in \Lambda \cup \Lambda_i \cup \{u\}$.

Suppose we multiply these all together, $h^m = \Pi_u \sum_i \gamma_{i,u} \delta_{i,u}$, and so also

$$
h^{m+1} = \Pi_u h \sum_i \gamma_{i,u} \delta_{i,u}.
$$

Expanding the product, each term has the form $h \prod_u \gamma_{i,u} \delta_{i,u}$. Consider some such term. If there is some $u$ with $\mu_{i,u} \in \Lambda \cup \Lambda_i$ then $h \prod_u \gamma_{i,u} \delta_{i,u} \mu_{i,u} = \gamma \theta_{i,u} \mu_{i,u} \in \{\Lambda \cup \Lambda_i\}$. 


For each term $\prod_u \gamma_{i_u, u} \delta_{i_u, u} h$ with every $\mu_{i_u, u} = u$, we have

$$h \prod_u \gamma_{i_u, u} \delta_{i_u, u} = \gamma(\prod_u \theta_{i_u, u} u) h.$$ 

But we have shown that $(\prod_u \theta_{i_u, u}) h \in \{ \Lambda \cup \Lambda_1 \}$.

Therefore $h^{\nu' + 1}$ is a sum of terms belonging to $\{ \Lambda \cup \Lambda_1 \}$, so we have $h \in \{ \Lambda \cup \Lambda_1 \}$.

6. Making Bounds Explicit

6.1. Ordinal Length Iteration of Functions. We need the concept of ordinal length iterations of a function. We first recall some basic theory of ordinals below $\varepsilon_0$ (which more than suffices for our purposes—the largest ordinals we will need are in the vicinity of $\omega^{\omega^\omega}$).

**Definition 6.1.** Any ordinal $\alpha < \varepsilon_0$ has a unique Cantor normal form given by a finite set $I$ of ordinals below $\varepsilon_0$ and, for each $\beta \in I$, a positive natural number $c_\beta$, so that $\alpha = \sum_{\beta \in I} \omega^\beta c_\beta$.

When $\alpha = 0$, we have $I = \emptyset$. We sometimes think of this representation as being given recursively: $\alpha = \sum_{\beta \in I} \omega^\beta c_\beta$, and each $\beta$ can further be expressed in Cantor normal form. Note that each $c_\beta$ must be strictly positive (when $c_\beta = 0$, should omit $\beta$ from $I$).

**Definition 6.2.** When $\alpha = \sum_{\beta \in I} \omega^\beta c_\beta > 0$ (so $I$ is non-empty), we write $\max \alpha$ and $\min \alpha$ for $\max I$ and $\min I$, respectively.

When $\min I > 0$, we call $\alpha$ a limit ordinal. When $\min I = 0$, we call $\alpha$ a successor ordinal.

When $\min I = 0$, we have $\alpha = \alpha' + \omega^0 c_0 = \alpha' + c_0$ where $\alpha'$ is a limit ordinal.

**Definition 6.3.** For any $\alpha = \sum_{\beta \in I} \omega^\beta c_\beta > 0$ and any $x \in \mathbb{N}$, we define $\alpha[x] < \alpha$ recursively by $\alpha - 1$ if $\min \alpha = 0$ and

$$\alpha[x] = \sum_{\beta \in I \setminus \{ \min \alpha \}} \omega^\beta c_\beta + \omega^{\min \alpha}(c_{\min \alpha} - 1) + \omega^{(\min \alpha)[x]} x$$

otherwise.

When $\alpha$ is a successor, $\alpha[x]$ is always $\alpha - 1$. When $\alpha = \gamma + \omega^n$, $\alpha[x] = \gamma + \omega^{n-1} x$. When $\alpha = \gamma + \omega^\omega$, $\alpha[x] = \gamma + \omega^x x$, and so on. $\alpha[x]$ is the canonical sequence of approximations to $\alpha$; in particular, when $\alpha$ is a limit, $\lim_x \alpha[x] = \alpha$.

**Definition 6.4.** If $\alpha = \sum_{\beta} \omega^\beta c_\beta$, we define the coordinate bound $|\alpha| \in \mathbb{N}$ recursively by $|\alpha| = \max \{c_\beta, |\beta|\}$.

$|\alpha|$ is the upper bound on the coefficients that appear anywhere in the Cantor normal form of $\alpha$.

**Definition 6.5.** Let $g$ be a function. We define:
• $g^0(b) = b$,
• $g^\alpha(b) = g^\alpha[b](g(b))$.

In particular, it is helpful to introduce a canonical list of functions to serve as benchmarks for our bounds.

**Definition 6.6.** Let $G(b) = b + 1$.

Then $G^\omega(b) = 2b$, $G^{\omega^2}(b) \geq 2^b b$, $G^{\omega^3}(b)$ is essentially a tower of exponents of height $b$, $G^{\omega^4}$ is roughly the unary Ackermann function.

This is similar to the fast-growing functions [38], sometimes called the Grzegorczyk hierarchy. We have chosen a slower indexing because it matches our applications: roughly speaking, $G^{\omega^\alpha}$ is the $\alpha$-th function in the fast-growing hierarchy.

We will need various properties about the behavior of these iterations which are included in Appendix A.

6.2. Some Bounds on Order of Magnitude.

**Lemma 6.7.** For each $n$ and $d \geq n$, $p_n(d) \leq G^{\omega^{2n^2}}(d)$.

**Proof.** By induction on $n$. Clearly $p_1(d) = d = G^0(d)$.

Suppose the claim holds for $n$. Note that for $d \geq n$,

$$
\epsilon(n, d) = 2^{(d + \delta_{n-1}(d))^{n-1} + 1} d + \delta_{n-1}(d)
$$

$$
= 2^{(d + (2d)^{n-1})^{n-1} + 1} d + (2d)^{2n-1}
$$

$$
\leq G^{\omega^{2 + \omega}}(d).
$$

Then for $d \geq n$,

$$
\zeta_1(n, p_n(d), d) = \left(\frac{d + n}{n}\right) + 2\zeta_0(n, p_n(d))
$$

$$
= \left(\frac{d + n}{n}\right) + 2 \left(\frac{n + \delta_n(p_n(d))}{n}\right)
$$

$$
= \left(\frac{d + n}{n}\right) + 2 \left(\frac{n + (2p_n(d)^{2n})}{n}\right)
$$

$$
< \left(\frac{d + n}{n}\right) (n + (2p_n(d)^{2n})) e^2
$$

$$
\leq ((d + n + 2)(n + (2p_n(d)^{2n})) e^2)^n
$$

$$
\leq ((d + n + 2)(n + (2G^{\omega^{2n^2}}(d)^{2n})) e^2)^n
$$

$$
\leq ((d + n + 2)(n + (2G^{\omega^2}(G^{\omega^{2n^2}}(d))) e^2)^n
$$

$$
\leq ((d + n + 2)G^{\omega^{2 + \omega^4 + n}}(G^{\omega^{2n^2}}(d)))^n
$$

$$
\leq G^{\omega^{2 + \omega^{2n^2}}(d)).
$$
and
\[
\zeta_2(n, p_n(d), d) = (\zeta_1(n, p_n(d), d) + 1)^{2^G(n, p_n(d), d) - 1} \\
\leq G^{2^2 \cdot 3} (G^{\omega^2} (G^{\omega^2} 8n(d))) \\
\leq G^{\omega^2 \cdot 5} (G^{\omega^2} 8n(d)).
\]

We therefore have
\[
\nu(n + 1, d) = \zeta_1(n, p_n(d), d) \zeta_2(n, p_n(d), d) \\
\leq G^{\omega^2 \cdot 2} (G^{\omega^2} 8n(d)) G^{\omega^2 \cdot 5} (G^{\omega^2} 8n(d)) \\
\leq (G^{\omega^2 \cdot 5} (G^{\omega^2} 8n(d)))^2 \\
\leq G^{\omega^2 \cdot 6} (G^{\omega^2} 8n(d))
\]
and
\[
p_{n+1}(d) = \max\left\{ 2^{\left( \nu(n + 1, d) + n + 1 \right)} \nu(n + 1, d), c(n, d) \right\} \\
\leq \max\left\{ 2^{\left( \nu(n + 1, d) + n + 1 \right) e} n + 1 \nu(n + 1, d), G^{\omega^2 + 2 + \omega}(d) \right\} \\
\leq G^{\omega^2 \cdot 7 + \omega^4 + n + 1} (G^{\omega^2} 8n(d)) G^{\omega^2 \cdot 6} (G^{\omega^2} 8n(d)) \\
\leq G^{\omega^2 \cdot 8} (G^{\omega^2} 8n(d)) \\
= G^{\omega^2} 8(n+1)(d).
\]

To get bounds on \( m \), we first need the following observation:

**Lemma 6.8.** Suppose \( \tau = \tau^0 \cup \tau^1 \) and for every \( a \in \tau^0 \) and \( b \in \tau^1 \), \( a \leq b \). Then
\[
m_{\tau, D}(i) = m_{\tau^0, D}(i) + m_{\tau^1, D}(i) + m_{\tau^0, D}(i).
\]

**Proof.** We proceed by induction on \( \tau^0 \). When \( \tau^0 = \emptyset \), this is immediate from the definition. Suppose the claim holds for all \( \tau_0 <_{\text{mult}} \tau^0 \). Then
\[
m_{\tau, D}(i) = 1 + m_{\tau^0, D}(i + 1) \\
= 1 + m_{\tau^0, D}(i + 1) \\
= 1 + m_{\tau^0, D}(i + 1) + m_{\tau^1, D}(i + 1) \\
= m_{\tau^0, D}(i) + m_{\tau^1, D}(i + 1)
\]
as needed. \( \square \)

**Definition 6.9.** If \( \tau \) is a multiset with \( \max \tau = n \) and, for each \( i \leq n \) with \( 0 < i \), \( c_i \) copies of \( i \), then
\[
o(\tau) = \sum_{0 < i \leq n} \omega^{i-1} c_i + 2|\tau|.
\]
Lemma 6.10. For any \( \tau \) and any monotone \( D \) so that \( D(b) \geq 2b \), whenever \( b \geq |\tau| \) we have

\[
m_{\tau, D}(b) \leq D^{\omega}(b).
\]

Proof. We proceed by induction on max \( \tau \).

When max \( \tau = 0 \), \( m_{\tau, D}(b) = r \), so \( m_{\tau, D}(b) \leq D^{0}(b) \) once \( b \geq r = |\text{max } \tau| \).

Suppose the claim holds for values less than \( n \) and proceed by side induction on the number of copies of \( n \) in \( \tau \). Suppose we are given \( \tau \) and \( n \in \tau \) is maximal so \( \tau = \tau_0 \cup \{n\} \). Then, letting \( \sigma_b \) be the multiset with \( D(b) \) copies of \( n - 1 \),

\[
m_{\tau, D}(b) = m_{\tau_0, D}(b) + m_{\{n\}, D}(b + m_{\tau_0, D}(b)) = m_{\tau_0, D}(b) + m_{\sigma_b + m_{\tau_0, D}(b), D}(b + 1 + m_{\tau_0, D}(b)) \leq D^{\omega}(b) + D^{\omega - 2}(b + D^{\omega}(b))(b + 1 + D^{\omega}(b)) \leq D^{\omega}(b) + D^{\omega - 1}(b + D^{\omega}(b)) \leq D^{\omega}(b) + D^{\omega - 1 + \omega}(b) \leq D^{\omega - 1 + \omega + 2}(b) = D^{\omega}(b).
\]

\[ \square \]

Corollary 6.11. If \( D(b) \geq \max\{2b, b + 1\} \) then \( m^*(D, n) \leq D^{\omega - 1 + 1}(0) \).

Recall that \( F_x(b) = F(p_x(b)) \).

Lemma 6.12. If \( F(b) \geq 2b \) for all \( b \) then \( u_F(b) \leq F^{\omega \omega + \omega + 1}(b) \).

Proof. Let \( D_c(i) = F^i(c) \). By induction on \( \alpha \), we claim that \( D^\alpha_c(i) \leq F^{\omega \times \alpha + i}(c) \). When \( \alpha = 0 \), \( D^\alpha_c(0) = 0 \leq F^0(0) \). When \( \alpha > 0 \),

\[
D^\alpha_c(i) = D^\alpha_c[i](D_c(i)) = D_c^a[i](F^i_c(c)) \leq F^{\omega \times (\alpha[i])}(F^i_c(c)) \leq F^{\omega \times \alpha + i}(c).
\]

So we have \( m^*(i) \mapsto F^i_x(x, x) = m^*(D_x, x) \leq D^{\omega^2 - 1 + 1}(0) \leq F^{\omega^2 + 1}(x) \). Therefore

\[
u_F(x) \leq F^{\omega^2 + 1}(x) \leq F^{\omega}(F^{\omega^2 + 1}(x)) \leq F^{\omega^2 + \omega + 1}(x).\]

\[ \square \]

Note that, under the same assumptions, \( F^{\omega \omega + \omega + 1}(b) \) is much larger than \( \omega^b + 2b(2b)^b + 1 \), so the same bound holds for \( u_F^+ \).

Lemma 6.13. If \( F(b) \geq 2b \) for all \( b \) then \( f(F, b) \leq F_b^{\omega \omega + \omega^2 + \omega + 8}(b) \).
Definition 6.15. assignment of ordinals to bad leader sequences.

Bounds on Lemma 4.31 and its Consequences.

Lemma 6.16. \(\mathcal{L}(47)\)

Lemma 6.17. With \(m\) derivatives and \(n\) differential indeterminates, there is an assignment of ordinals \(o(\langle u_1, \ldots, u_k \rangle) \leq \omega^m \cdot n\) to bad leader sequences so that \(o(\langle \rangle) = \omega^n\), \(o(\langle \vec{a}_1, \ldots, \vec{a}_m \rangle) < o(\langle \vec{a}_1, \ldots, \vec{a}_m, \vec{a}_{m+1} \rangle)\), and if each \(u_i \in K\{X_{[n]}\}_{\leq d}\) then \(o(\langle u_1, \ldots, u_k \rangle) \leq nk^m\).
For each $k$, When $u \leq k$ we will show that for each $T$. Taking $o((u_1, \ldots, u_k)) = \sum_{j \leq m} \omega^j \cdot (\sum_i c_{i,j})$ gives the desired bound. \hfill \Box

The quantity $\sum_{j \leq m} \omega^j \cdot (\sum_i c_{i,j})$ is an instance of the “natural” or “commutative” sum for ordinals.

**Lemma 6.18.** Let $g$ be a fixed monotonic function with $g(b) \geq 2b$ for all $b$. For each $c$, let $D_c$ be the function $D_c(i) = g^{c+i}(b)$.

Then for any $\gamma$ with $o(\gamma) = \alpha$ and any $b \geq \max\{|\gamma|, n, m + 2\}$,

$$h_{n,m}(D_c, \gamma) \leq g^{\omega^{\alpha^2+\omega+c+2}}(b).$$

In particular,

$$h_{n,m}(D) \leq g^{\omega^{\omega^m-2n+\omega+2}}(b).$$

**Proof.** By induction on $o(\gamma)$. When $o(\gamma) = 0$, so $\gamma$ is maximal, $h_{n,m}(D_c, \gamma) = 1 \leq b = g^0(b)$.

Suppose $o(\gamma) = \alpha$ and for all $\gamma'$ with $o(\gamma') < \alpha$, the claim holds. Let $d = D_c(1) = g^{c+1}(b)$ and $b' = g^\omega(d) \geq n(|\gamma| + 1)d^m$. Let $\beta = \alpha[b']$. For each $u \in [-1, d]$, we will show that

$$w_u \leq g^{\omega^{\beta^2+1}(d-u)+\omega(d-u)+(d-u)(b')}.$$  

When $u = b'$, this is immediate.

Let $\delta_u = \omega^{\beta^2+1}(d-u) + \omega(d-u) + (d-u)$, and suppose we have shown that $w_u \leq g^{\delta_u}(b')$. Then $D_c(w_u) = g^{u_u+c}(b) \leq g^{u_u}(b')$. Let $c_u = D_c(w_u)$.

We will show that for each $k \in [0, c_u]$,

$$v_{u,k} \leq g^{\omega^{\beta^2}(c_u-k)+\omega(2(c_u-k)+2(c_u-k))(g^{\delta_u}(b')).$$

When $k = c_u$, this is immediate.

Suppose the claim holds for $k$. Then we have

$$v_{u,k-1} = v_{u,k} + h_{n,m}(D_{u,k}, \gamma \sim ((u, k)))$$

$$\leq v_{u,k} + g^{o(\gamma \sim ((u, k))))} + \omega^{\beta^2+1}(u,k+2)(b)$$

$$\leq v_{u,k} + g^{\omega^{\beta^2+1}(u,k+2)+\omega(v_{u,k+2}(b))}$$

$$= v_{u,k} + g^{\omega^{\beta^2+1}(u,k+2)+\omega(g^{\delta_u}(b'))(g^{\delta_u}(b'))}$$

$$\leq v_{u,k} + g^{\omega^{\beta^2+1}(u,k+2)+\omega(2(g^{\delta_u}(b'))(g^{\delta_u}(b'))(g^{\delta_u}(b')))}$$

$$\leq g^{\omega^{\beta^2}(c_u-k)+\omega(2(c_u-k)+2(c_u-k))(g^{\delta_u}(b'))} + g^{\omega^{\beta^2}(c_u-(k+1))+\omega(2(c_u-(k+1))+2(c_u-k))(g^{\delta_u}(b'))}$$

$$\leq 2g^{\omega^{\beta^2}(c_u-(k+1))+\omega(2(c_u-(k+1))+2(c_u-k))(g^{\delta_u}(b'))}$$

$$\leq g^{\omega^{\beta^2}(c_u-(k+1))+\omega(2(c_u-(k+1))+2(c_u-(k+1))(g^{\delta_u}(b')))}. $$
In particular, \( v_{u,0} \leq g^{\omega \beta^2 c_u + \omega 2c_u + 2c_u (g^{\delta_u} (b'))} \). Therefore \( w_{u-1} = v_{u,0} \)
\[
\leq g^{\omega \beta^2 c_u + \omega 2c_u + 2c_u (g^{\delta_u} (b'))} \\
\leq g^{\omega \beta^2 + 1 + 1 (g^{\omega^2 (g^{\delta_u} (b'))})} \\
= g^{\omega \beta^2 + 1 + 1 (g^{\omega (g^{\omega^2 + 1 (d-u) + \omega (d-u) + (d-u) (b'))})} \\
\leq g^{\omega \beta^2 + 1 (d-(u+1) + \omega (d-(u+1) + (d-(u+1) (b')))}
\]

Therefore
\[
h_{n,m}(D, \gamma) = w_{-1} \\
\leq g^{\omega \beta^2 + 1 (d+1) + \omega (d+1) + d+1 (b')} \\
\leq g^{\omega \beta^2 + 1 + (b')} \\
\leq g^{\omega_2 + 1 + (b')} \\
\leq g^{\omega_2 + \omega + c + 2 (b')}
\]

Lemma 6.19.
\[
i^\text{sat}_{n,m}(b) \leq G^{\omega^{mn+2} + \omega^{m^2} + \omega^2} (b).
\]

**Proof.** Let \( g_*(d) = g(d, d) = d(1 + d)^d \leq G^{\omega^2 + 1 (d)}. \) Then the function \( b \mapsto D_{b}^\text{sat}(i) \) is bounded by \( g_*(b) \), and therefore
\[
h_{n,m}(D_{b}^\text{sat}) \leq g_*^{\omega^{mn} + \omega + 2 (b)} \\
\leq (G^{\omega^2 + 1 (\omega^{mn} + \omega + 2 (b)} \\
\leq G^{\omega^{mn+2} + \omega^{m^2} + \omega^2} (b).
\]
Then
\[
i^\text{sat}_{n,m}(b) \leq g_* (G^{\omega^{mn+2} + \omega^{m^2} + \omega^2} (b)) \leq G^{\omega^{2n+2} + \omega^{m^2} + \omega^2} (b).
\]

**Lemma 6.20.** When \( b \geq \max \{m, n\} \),
\[
i^\text{cohere}_{n,m}(b) \leq G^{\omega^{mn+2} + \omega^{m^2} + \omega^2} + \omega^{m^2} + \omega^2 + 3 (b).
\]

**Proof.** We use the same reasoning as in the previous lemma, but with the function
\[
g_* (b) = g(b, \left( \frac{2b + m - 1}{m - 1} \right) \cdot n (b + 1)) \\
\leq G^{\omega^2 + \omega + 1 (b)},
\]
so
\[ i_{n,m}^\text{cohere} (b) \leq \mathcal{G}^{\omega^{m+2n+2}+\omega^{m+2n+1}+\omega^{m+2n}+\omega^26+\omega^2+3} (b). \]

**Lemma 6.21.** When \( b \geq \max\{m, n\} \),
\[ i_{n,m}^\text{char} (b) \leq \mathcal{G}^{\omega^{m+2n+\omega+\omega^m+2n+5+\omega^{m+15}+\omega^2+2+\omega^2+5} (b). \]

**Proof.** Observe that
\[
\mathcal{F}^\text{char}_c (k) = 3^{k+1}(k, g(c, k)) \\
\leq \mathcal{G}^{\omega^24k+\omega6k+3k}(\max\{c, k\}) \leq \mathcal{G}^{\omega^2(4k+1)+\omega6k+3k+1}(\max\{c, k\}).
\]
Also
\[
\mathcal{F}^\text{char}_c (p_c (k)) \leq \mathcal{G}^{\omega^2(4k+1)+\omega6k+3k+1}(\max\{c, \mathcal{G}^\omega 8c (k)\}) \\
\leq \mathcal{G}^{\omega^2(4k+8c+1)+\omega6k+3k+1}(\max\{c, k\}),
\]
\[
\mathcal{u}_{\mathcal{F}^\text{char}_b (j)} (b) \leq (\mathcal{G}^{\omega^2(4i+8b+1)+\omega6i+3i+1}(\max\{b, i\}) \omega^b + \omega^1 \\
\leq \mathcal{G}^{\omega^b+2(4i+8b+1)+\omega^{b+1}6i+\omega^b(3i+1)+\omega^3(4i+8b+1)+\omega^2(10i+8b+1)+\omega7i+3i+1}(\max\{b, i\}),
\]
and
\[
f (\mathcal{F}^\text{char}_b, b) \leq (\mathcal{G}^{\omega^2(12b+1)+\omega6b+3b+1}) \omega^b + \omega^2 \omega^b + \omega^8 (b) \\
\leq \mathcal{G}^{\omega^b+2(12b+1)+\omega^{b+1}6b+\omega^b(3b+1)+\omega^4(72b+6)+\omega^3(48b+1)+\omega^2(102b+14)+\omega(51b+1)+24b+8} (b).
\]
This time we use the function
\[
g_*(b) = \max\{g(b, \frac{2b + m - 1}{m - 1} n(b + 1)), p_b(\mathcal{u}_{\mathcal{F}^\text{char}_b (b)}), f(\mathcal{F}^\text{char}_b, b)\} \\
\leq \mathcal{G}^{\omega^b+2a_0+\omega^{b+1}a_1+\omega^b a_2} (b) \\
\leq \mathcal{G}^{\omega^b+\omega^5} (b)
\]
so
\[ i_{n,m}^\text{char} (b) \leq \mathcal{G}^{\omega^{m+2n+\omega+\omega^m+2n+5+\omega^{m+15}+\omega^2+2+\omega^2+5}+\omega^10} (b). \]

6.4. **Bounds on Ritt-Noetherianity.**

**Lemma 6.22 (IS).** When \( d \geq n \), \( \mathfrak{f} (n, d) \leq \mathcal{G}^{\omega^d+n+8} (d) \).

**Lemma 6.23.** Suppose that \( \mathfrak{B} (F, d) \leq \mathcal{F}^\alpha (d) \). Then \( \mathfrak{B} (F, d) \leq \mathcal{F}^{\alpha + |J| + 1} (d) \).

**Proof.** By induction on \(|J|\), we show that \( \mathfrak{B} (F, d) \leq \mathcal{F}^{(\alpha + 1)^\otimes n} (d) \) where \( (\alpha + 1)^\otimes n \) is iterated commutative multiplication, as described in the appendix. The conclusion follows since \( (\alpha + 1)^\otimes n \leq \alpha^{n+1} \).

When \(|J| = 1\), this is immediate. Observe that \( \mathfrak{G}(d') \leq \mathcal{F}^{\alpha+1}(d') \), so \( \mathfrak{B} (F, d) = \mathfrak{B} (F \setminus \{j_0\}) G, d) \leq \mathcal{G}^{(\alpha + 1)^\otimes (|J|-1)} (d) \leq \mathcal{F}^{(\alpha+1)^\otimes |J|} (d) \).
We assign an explicit ordinal \( o(\Lambda) \) to autoreduced sets so that when \( \Lambda' \) has lower rank than \( \Lambda \), \( o(\Lambda') < o(\Lambda) \).

**Lemma 6.24.** With \( m \) derivatives and \( n \) differential indeterminates, there is an assignment of ordinals \( o(\Lambda) \leq \omega^{\omega m \cdot n} \) to autoreduced sets so that \( o(\Lambda) \leq \omega^{\omega m \cdot n} \) and if \( \Lambda' \) has lower rank than \( \Lambda \) then \( o(\Lambda') < o(\Lambda) \).

**Proof.** Let \( \Gamma(\Lambda) = (\mu_1, b_1), \ldots, (\mu_r, b_r) \). Set
\[
o(\Lambda) = \sum_{i \leq r} \omega^{o(\mu_1, \ldots, \mu_i)} b_i + \omega^{o(\mu_1, \ldots, \mu_r)}.
\]

If \( \Lambda' \) has lower rank than \( \Lambda \), so \( \Gamma(\Lambda') = (\mu'_1, b'_1), \ldots, (\mu'_{r'}, b'_{r'}) \) then either there is some \( i \leq \min\{r, r'\} \) so that \( \sum_{j < i} \omega^{o(\mu_1, \ldots, \mu_j)} b_j = \sum_{j < i} \omega^{o(\mu'_1, \ldots, \mu'_j)} b'_j \) but \( \omega^{o(\mu_1, \ldots, \mu_i)} b_i > \omega^{o(\mu'_1, \ldots, \mu'_i)} b'_i \), and therefore \( \omega^{o(\mu_1, \ldots, \mu_i)} b_i > \sum_{j \geq i} \omega^{o(\mu'_1, \ldots, \mu'_j)} b'_j + \omega^{o(\mu'_1, \ldots, \mu'_i)} \), or \( r' > r \) and we have \( \omega^{o(\mu_1, \ldots, \mu_r)} > \sum_{j > r} \omega^{o(\mu'_1, \ldots, \mu'_j)} b'_j + \omega^{o(\mu'_1, \ldots, \mu'_i)} \). \( \square \)

**Definition 6.25.** Let \( \Lambda \) be autoreduced and let \( \Gamma(\Lambda) = (\mu_1, b_1), \ldots, (\mu_r, b_r) \). Set
\[
\text{Let } \Gamma(\Lambda) = \langle (\mu_1, b_1), \ldots, (\mu_r, b_r) \rangle.
\]

**Lemma 6.26.** Assume \( F(i) \geq i + 1 \) for all \( i \) and \( F, D \) are monotonic. Without loss of generality, let \( F(i) \geq D(i) \) for all \( i \).

Then \( j_{n, m}(i_0, D, F, d, \Lambda) \leq F^{\omega^{o(\Lambda)} + \omega^{m \cdot n + 2}}(\max\{d, n, i_0\}) \).

**Proof.** Write \( o(\Lambda) \) for the ordinal rank of \( \Lambda \). We proceed by induction on \( o(\Lambda) \). When \( o(\Lambda) = 0 \), \( j_{n, m}(i_0, D, F, d, \Lambda) = i_0 \).

Suppose \( o(\Lambda) = \gamma \) and the claim holds for all \( \Lambda \) with \( o(\Lambda) < \gamma \). Let \( b = n(\text{stat}(d) + n)^n \leq n(\text{stat}(d))^n + 1 \) when \( d \geq n \), so when \( \Lambda \subseteq K\{X[n]\} \leq \text{stat}(d) \), \( |o(\Lambda)\| \leq b \). Then
\[
\mathfrak{G}_{j, D}(F, i_0) \leq F^{\omega^{\gamma[b_1] \cdot 2 + \omega^{m \cdot n + 2} + 3}(\max\{d, n, i_0\})}
\]
\[
\leq F^{\omega^{\gamma[b_1] \cdot \omega^{\max\{2(\text{stat}(d) + n)^n, i_0\}}}}
\]
\[
\leq F^{\omega^{\gamma[b_1] + 1 \cdot \omega^2}(\max\{\text{stat}(d) + n, i_0\})}
\]
\[
\leq F^{\omega^{\gamma[b_1] + 1 \cdot \omega^{m \cdot n + 2} + \omega^2 \cdot n^2 + \omega^2 \cdot 7 + 3}(\max\{d, n, i_0\})}
\]

Therefore
\[
D(F(\mathfrak{G}_{j, D}(F, i_0))) \leq F^{\omega^{\gamma[b_1] + 1 + \omega^{m \cdot n + 2} + \omega^2 \cdot n^2 + \omega^2 \cdot 7 + 5}(\max\{d, n, i_0\})}
\]
and
\[
\mathfrak{f}(n, D(\mathfrak{G}_{j, D}(F, i_0))) \leq F^{\omega^{\gamma[b_1] + 1 + \omega^{m \cdot n + 2} + \omega^2 \cdot n^2 + \omega^2 \cdot 8 + \omega^2 \cdot 4}(\max\{d, n, i_0\})}
\]

We may bound the sum of these by \( b' = F^{\omega^{\gamma[b_1] + 1 + \omega^{m \cdot n + 2} + 3}(\max\{d, n, i_0\})} \).
Now let \( b'' = n(b')^{n+1} \), so when \( \Lambda^* \subseteq K\{X_{[n]} \} \leq \text{sat}(\tau(n,D(i))) \), \( |o(\Lambda_*)| \leq b'' \). Then
\[
j_{n,m}(i_0, D, F, d, \Lambda) \leq M^{p(g^n|b'| + \omega^{\gamma|b'|+1} + \omega^m2n+23}(\max\{d, n, i_0\})
\]
\[
\leq M^{\omega^\gamma 2 + \omega^m2n+23}(\max\{d, n, i_0\})
\]
\[\square\]

**Corollary 6.27.** Assume \( F(i) \geq i + 1 \) for all \( i \) and \( F, D \) are monotonic. Without loss of generality, let \( F(i) \geq D(i) \) for all \( i \). Then \( j_{n,m}(i_0, D, F, d, \Lambda) \leq M^{p(\omega^{\omega\omega\omega\omega^m}n^2 + \omega^{\omega^m2n+23}(\max\{d, n, i_0\})} \).

**Appendix A. Ordinal Iterations**

The next several lemmas show identities relating ordinal arithmetic to function iteration. Throughout this section we assume that \( g \) is monotonic and \( g(b) \geq b + 1 \) for all \( b \) and we consider \( b \geq 1 \).

**Lemma A.1.** Let \( \alpha \) and \( \beta \) be ordinals with \( \min \beta \leq \max \alpha \). Then
\[
g^{\alpha + \beta}(b) = g^\alpha(g^\beta(b)).
\]

**Proof.** By induction on \( \beta \). When \( \beta = 0 \), this is trivial.
\[
g^{\alpha + \beta}(b) = g^{(\alpha+\beta)}(g(b)) = g^{\alpha+\beta}(g(b)) = g^{\alpha}(g^\beta(g(b))) = g^{\alpha}(g^\beta(b))
\]
using the inductive hypothesis since \( \beta[b] < \beta \).
\[\square\]

The main difficulty when dealing with ordinal iterations is that they are not strictly monotonic: we do not, in general, have \( \alpha < \beta \) implies \( g^\alpha(b) \leq g^\beta(b) \) (consider the case where \( b \) is much smaller than \( n \): we may have \( g^n(b) > g^\omega(b) \)).

When considering the effect of \( \alpha \) on the size of \( g^\alpha(b) \), both the size of \( \alpha \) and the size of its coefficients matter.

We next establish some lemmas showing some cases when we can obtain monotonicity. First, note that when \( \max \beta \leq \min \alpha \) we have
\[
g^\alpha(b) \leq g^\alpha(g^\beta(b)) = g^{\alpha + \beta}(b).
\]

**Lemma A.2.** \( g^{\omega^\alpha}(b) > g^\alpha(b) \).

**Proof.** By induction on \( \alpha \). When \( \alpha = 0 \),
\[
g^{\omega^0}(b) = g(b) \geq b + 1 > b = g^0(b).
\]

When \( \alpha > 0 \),
\[
g^{\omega^\alpha}(b) = g^{\omega^{\alpha[b]}(g(b))} \geq g^{\omega^{\alpha[b]}(g(b))} > g^{\alpha[b]}(g(b)) = g^\alpha(b).
\]
\[\square\]

**Lemma A.3.** \( g^{\omega^\alpha c}(b) \geq g^c(b) \).
Proof. By induction on $\alpha$. When $\alpha = 0$ the two sides are identical. When $\alpha > 0$ we have

$$g^{\alpha^+}(b) = g^{\alpha^+}(\cdots (g^{\alpha^+}(b)) \cdots)$$

$$> g^\alpha(\cdots (g^\alpha(b)) \cdots)$$

$$\geq g^{\max\alpha}(\cdots (g^{\max\alpha}(b)) \cdots)$$

$$= g^{\max\alpha}(b)$$

$$\geq g^\gamma(b).$$

Lemma A.4. \textit{For any $\epsilon > \delta$ and any $d \geq g^\delta(b)$, $\epsilon[d] \geq \delta$.}

Proof. We proceed by induction on $\delta$. When $\delta = 0$, this is trivial.

Write $\epsilon = \epsilon' + \omega^\gamma$ where $\gamma = \min \epsilon$. If $\gamma \leq \max \delta$ then since $\epsilon > \delta$, we must have $\epsilon' \geq \delta$, so $\epsilon[d] = \epsilon' + \omega^\gamma[d] \geq \epsilon' \geq \delta$.

So suppose $\gamma > \max \delta$. If $\epsilon' \neq 0$ then $\epsilon' \geq \omega^\gamma > \delta$ and we are done, so assume $\epsilon' = 0$ and therefore $\epsilon = \omega^\gamma$. We have $\epsilon[d] = \omega^\gamma[d]d$.

By the inductive hypothesis, since $\gamma > \max \delta$ and $d \geq g^\delta(b) \geq g^{\max\delta}(b)$, we also have $\delta[d] \geq \max \delta$. If $\delta[d] > \max \delta$ then $\epsilon[d] > \delta$.

So suppose $\delta[d] = \max \delta$. Then

$$d \geq g^\delta(b) \geq g^{\max \delta c_{\max \delta}(b)} \geq g^{\max\delta}(b) > c_{\max \delta}.$$

So $\epsilon[d] = \omega^\gamma[d]d \geq \omega^{\max \delta(c_{\max \delta} + 1)} > \delta$. \hfill \Box

Lemma A.5. \textit{Let $\alpha = \sum_{\gamma \in I} \omega^\gamma c_{\gamma}$. Let $\beta, \delta \in I$ with $\delta < \beta$, and let $c'_{\delta} = c_{\beta} - 1$, $c'_{\delta} = c_{\delta} + 1$, and $c'_{\gamma} = c_{\gamma}$ if $\gamma \notin \{\beta, \delta\}$. Let $\alpha' = \sum_{\gamma \in I} \omega^\gamma c'_{\gamma}$. Then $g^{\alpha'}(b) \leq g^{\alpha}(b)$.}

Proof. It suffices to consider the case where $I \cap (\beta, \delta) = \emptyset$, since we get the general case by applying this case several times. So we have

$$\alpha = \alpha^+ + \omega^\gamma + \omega^\delta + \alpha^-$$

where $\min(\alpha^+) \geq \gamma$ and $\delta \geq \max(\alpha^-)$. Since

$$g^{\alpha}(b) = g^{\alpha^+}(g^{\omega^\gamma + \omega^\delta}(g^{\alpha^-}(b)))$$

and

$$g^{\alpha'}(b) = g^{\alpha^+}(g^{\omega^\delta 2}(g^{\alpha^-}(b)))$$

it suffices to show that when $\gamma > \delta$,

$$g^{\omega^\gamma + \omega^\delta}(b) \geq g^{\omega^\delta 2}(b).$$
We show this by induction on $\gamma$:

\[
g^{\omega^\gamma + \omega^\delta} (b) = g^{\omega^\gamma} (g^{\omega^\delta} (b))
\]

\[
= g^{\omega^\gamma [\omega^\delta (b)]} g((g^{\omega^\delta} (b)))
\]

\[
\geq g^{\omega^\gamma [\omega^\delta (b)] + \omega^\delta} (b)
\]

\[
\geq g^{\omega^\delta 2} (b).
\]
So we assume \( \beta = \omega^\delta \) and proceed by induction on \( \alpha \). Writing \( \alpha = \alpha' + \omega^\epsilon \) where \( \epsilon = \min \alpha \); if \( \alpha' \neq 0 \), we have

\[
g^\beta(g^\alpha(b)) = g^\beta(g^{\alpha'}(g^{\omega^\epsilon}(b)))
\leq g^{\alpha'+\beta}(g^{\omega^\epsilon}(b))
= g^{\alpha'}(g^\beta(g^{\omega^\epsilon}(b)))
\leq g^{\alpha'}(g^{\omega^\epsilon+\beta}(b))
= g^{\alpha+\beta}(b)
\]

using the inductive hypothesis twice. So we may reduce to the case where \( \alpha = \omega^\epsilon \).

If \( \epsilon = \delta \) then this follows from work above, so we may assume \( \epsilon > \delta \). Therefore \( \omega^\epsilon[g^\beta(b)] \geq \beta \), which must mean that \( \epsilon[g^\beta(b)] \geq \delta \). Then we have:

\[
g^\beta(g^{\omega^\epsilon}(b)) = g^\beta(g^{\omega^\epsilon}[b](g(b)))
= g^\beta(g^{\omega^\epsilon}[b]g(b))
\leq g^\beta(g^{\omega^\epsilon}[\omega^\beta(b)][b](g(b)))
\leq g^\beta(g^{\omega^\epsilon}[\omega^\beta(b)][b+1](b))
\leq g^{\omega^\epsilon}[\omega^\beta(b)][b+1](g^\beta(b))
\leq g^{\omega^\epsilon}[\omega^\beta(b)][\omega^\beta(b)](g^\beta(b))
= g^{\omega^\epsilon}(g^\beta(b))
= g^{\omega^\epsilon+\beta}(b).
\]

\[\square\]

**Definition A.8.** Let \( \alpha = \sum_{\gamma \in I} \omega^\gamma c_\gamma \) and \( \beta = \sum_{\delta \in J} \omega^\delta d_\delta \). Then \( \alpha \# \beta = \sum_{\gamma \in I, \delta \in J} \omega^{\gamma + \delta}c_\gamma d_\delta \) (where \( c_\gamma = 0 \) for \( \gamma \not\in I \) and \( d_\delta = 0 \) for \( \delta \not\in J \)).

Let \( \alpha = \sum_{\gamma \in I} \omega^\gamma c_\gamma \) and \( \beta = \sum_{\delta \in J} \omega^\delta d_\delta \). Then \( \alpha \otimes \beta = \sum_{\gamma \in I, \delta \in J} \omega^{\gamma \delta}c_\gamma d_\delta \).

These are the “natural” or “commutative” addition and multiplication on ordinals. Our work above shows

**Lemma A.9.** \( g^\alpha(g^\beta(b)) \leq g^{\alpha \# \beta}(b) \).

**Lemma A.10.** \( (g^\alpha)^\beta(b) \leq g^{\alpha \otimes \beta}(b) \).

**Proof.** By induction on \( \beta \). When \( \beta = 0 \) this is immediate. Otherwise,

\[
(g^\alpha)^\beta(b) = (g^\alpha)^{\beta[b]}(g(b))
\leq (g^{\alpha \otimes \beta[b]})(g(b))
\leq (g^{(\alpha \otimes \beta)[b]})(g(b))
= g^{\alpha \otimes \beta}(b).
\]

\[\square\]
Lemma A.11. If $|\beta| < b$ and $\beta < \alpha$ then $\beta \leq \alpha[b]$.

Proof. By main induction on $\beta$ and side induction $\alpha$. When $\alpha = \alpha' + 1$, this is immediate from the inductive hypothesis.

Suppose $\alpha = \alpha' + \omega^\gamma$, so $\alpha[b] = \alpha' + \omega^\gamma[b]$. If $\gamma \leq \max \beta$ then $\alpha' \geq \beta$ so $\alpha[b] \geq \beta$ as well. So suppose $\gamma > \max \beta$. Then by the main inductive hypothesis, $\gamma[b] \geq \max \beta$, and since $|\beta| < b$, $\beta < \omega^{\max \beta}b \leq \omega^\gamma[b] \leq \alpha[b]$. □

Lemma A.12. If $\alpha > \beta$ and $|\beta| < b$ then $g^\beta(b) \leq g^\alpha(b)$.

Proof. Let $\beta, g, b$ be fixed and proceed by induction on $\alpha$. If $\alpha = \beta$ this is trivial, and if $\alpha = \alpha' + 1$ this follows immediately from the inductive hypothesis and the monotonicity of $g$.

If $\alpha$ is a limit ordinal then $g^\alpha(b) = g^{\alpha[b]}(g(b)) \geq g^\beta(b)$ by the inductive hypothesis and the fact that $\alpha[b] \geq \beta$. □

References

[1] Matthias Aschenbrenner. Ideal membership in polynomial rings over the integers. J. Amer. Math. Soc., 17(2):407–441 (electronic), 2004.
[2] Jeremy Avigad and Solomon Feferman. Gödel’s functional (“Dialectica”) interpretation. In Handbook of proof theory, volume 137 of Stud. Logic Found. Math., pages 337–405. North-Holland, Amsterdam, 1998.
[3] Jeremy Avigad and Henry Towsner. Metastability in the Furstenberg-Zimmer tower. Fund. Math., 210(3):243–268, 2010.
[4] François Boulier, Daniel Lazard, François Ollivier, and Michel Petitot. Representation for the radical of a finitely generated differential ideal. In Proceedings of the 1995 international symposium on Symbolic and algebraic computation, pages 158–166. ACM, 1995.
[5] W Dale Brownawell. Bounds for the degrees in the nullstellensatz. Annals of Mathematics, 126(3):577–591, 1987.
[6] Alexandru Buium and Phyllis J Cassidy. Differential algebraic geometry and differential algebraic groups: from algebraic differential equations to diophantine geometry. Bass H, Buium A, Cassidy P (eds) Selected works of Ellis Kolchin. AMS, Providence, 1999.
[7] Richard Cohn. On the analogue for differential equations of the Hilbert-Netto theorem. Bulletin of the American mathematical Society, 47(4):268–270, 1941.
[8] Lisi D’Alfonso, Gabriela Jeronimo, and Pablo Solernó. Effective differential nullstellensatz for ordinary DAE systems with constant coefficients. Journal of Complexity, 30(5):588–603, 2014.
[9] Thomas W Dubé. A combinatorial proof of the effective nullstellensatz. Journal of Symbolic Computation, 15(3):277–296, 1993.
[10] Diego Figueira, Santiago Figueira, Sylvain Schmitz, and Philippe Schnoebelen. Ackermannian and primitive-recursive bounds with Dickson’s lemma. In 26th Annual IEEE Symposium on Logic in Computer Science—LICS 2011, pages 269–278. IEEE Computer Soc., Los Alamitos, CA, 2011.
[11] James Freitag and Omar León Sánchez. Effective uniform bounding in partial differential fields. Advances in Mathematics, 288:308–336, 2016.
[12] James Freitag, Wei Li, and Thomas Scanlon. Differential chow varieties exist. arXiv preprint arXiv:1504.03755, 2015.
[13] James Freitag and Thomas Scanlon. Strong minimality and the $j$-function. arXiv preprint arXiv:1402.4588, 2014.
[14] Philipp Gerhardy. Proof mining in practice. In Logic Colloquium, volume 35, pages 82–91, 2007.
[15] Kurt Gödel. Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. Dialectica, 12:280–287, 1958.
[16] Isaac Goldbring and Henry Towsner. An approximate logic for measures. Israel Journal of Mathematics, 199(2):867–913, 2014.
[17] OD Golubitsky, MV Kondratieva, and AI Ovchinnikov. On the generalized Ritt problem as a computational problem. Journal of Mathematical Sciences, 163(5):515–522, 2009.
[18] Oleg Golubitsky, Marina Kondratieva, Alexey Ovchinnikov, and Agnes Szanto. A bound for orders in differential Nullstellensatz. J. Algebra, 322(11):3852–3877, 2009.
[19] D Yu Grigor’ev. Complexity of quantifier elimination in the theory of ordinary differential equations. In European Conference on Computer Algebra, pages 11–25. Springer, 1987.
[20] Richard Gustavson, Marina Kondratieva, and Alexey Ovchinnikov. New effective differential nullstellensatz. Advances in Mathematics, 290:1138–1158, 2016.
[21] Richard Gustavson, Alexey Ovchinnikov, and Gleb Pogudin. Bounds for orders of derivatives in differential elimination algorithms. arXiv preprint arXiv:1602.00246, 2016.
[22] Richard Gustavson and Omar León Sánchez. Effective bounds for the consistency of differential equations. arXiv preprint arXiv:1601.02995, 2016.
[23] Matthew Harrison-Trainor, Jack Klys, and Rahim Moosa. Nonstandard methods for bounds in differential polynomial rings. J. Algebra, 360:71–86, 2012.
[24] Grete Hermann. Die Frage der endlich vielen Schritte in der Theorie der Polynomideale. Math. Ann., 95(1):736–788, 1926.
[25] Aaron Hertz. A constructive version of the hilbert basis theorem. Master’s thesis, Carnegie Mellon University, 2004.
[26] Ehud Hrushovski and Anand Pillay. Effective bounds for the number of transcendental points on subvarieties of semi-abelian varieties. American Journal of Mathematics, pages 439–450, 2000.
[27] Irving Kaplansky. Introduction to differential algebra. 1957.
[28] H Jerome Keisler. The ultraproduct construction. Ultrafilters Across Mathematics, 530:163–179, 2010.
[29] U. Kohlenbach. Applied proof theory: proof interpretations and their use in mathematics. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2008.
[30] Ulrich Kohlenbach. Analysing proofs in analysis. In Logic: from foundations to applications (Staffordshire, 1993), Oxford Sci. Publ., pages 225–260. Oxford Univ. Press, New York, 1996.
[31] E. R. Kolchin. Differential algebra and algebraic groups. Academic Press, New York-London, 1973. Pure and Applied Mathematics, Vol. 54.
[32] János Kollár. Sharp effective nullstellensatz. Journal of the American Mathematical Society, 1(4):963–975, 1988.
[33] M. V. Kondratieva, A. B. Levin, A. V. Mikhailov, and E. V. Pankratiev. Differential and difference dimension polynomials, volume 461 of Mathematics and its Applications. Kluwer Academic Publishers, Dordrecht, 1999.
[34] Omar León Sánchez and Alexey Ovchinnikov. On bounds for the effective differential Nullstellensatz. J. Algebra, 449:1–21, 2016.
[35] Wei Li and Ying-Hong Li. Computation of differential chow forms for ordinary prime differential ideals. Advances in Applied Mathematics, 72:77–112, 2016.
[36] David Marker, Margit Messmer, and Anand Pillay. Model theory of fields, volume 5 of Lecture Notes in Logic. Association for Symbolic Logic, La Jolla, CA, second edition, 2006.
[37] Guillermo Moreno Socías. Length of polynomial ascending chains and primitive recursiveness. *Math. Scand.*, 71(2):181–205, 1992.
[38] PG Odifreddi. *Classical Recursion Theory, Vol II, Studies in Logic and the Foundations of Mathematics*, Vol. 143. North-Holland Publishing Co., Amsterdam, 1999.
[39] Hervé Perdry and Peter Schuster. Noetherian orders. *Mathematical Structures in Computer Science*, 21(01):111–124, 2011.
[40] David Pierce. Fields with several commuting derivations. *The Journal of Symbolic Logic*, 79(01):1–19, 2014.
[41] HW Raudenbush. Ideal theory and algebraic differential equations. *Transactions of the American Mathematical Society*, 36(2):361–368, 1934.
[42] HW Raudenbush Jr. On the analog for differential equations of the Hilbert-Netto theorem. *Bulletin of the American mathematical Society*, 42(6):371–373, 1936.
[43] Joseph Fels Ritt. *Differential equations from the algebraic standpoint*, volume 14. American Mathematical Soc., 1932.
[44] Karsten Schmidt-Göttsch. Bounds and definablity over fields. *Journal für die reine und angewandte Mathematik*, 377:18–39, 1987.
[45] A. Seidenberg. Constructions in algebra. *Trans. Amer. Math. Soc.*, 197:273–313, 1974.
[46] Abraham Seidenberg. *An elimination theory for differential algebra*. Number 1-4. University of California Press, 1956.
[47] Stephen G. Simpson. Ordinal numbers and the Hilbert basis theorem. *J. Symbolic Logic*, 53(3):961–974, 1988.
[48] William Y. Sit. The Ritt-Kolchin theory for differential polynomials. In *Differential algebra and related topics* (Newark, NJ, 2000), pages 1–70. World Sci. Publ., River Edge, NJ, 2002.
[49] Endre Szemerédi. Regular partitions of graphs. In *Problèmes combinatoires et théorie des graphes* (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), volume 260 of *Colloq. Internat.* CNRS, pages 399–401. CNRS, Paris, 1978.
[50] Terence Tao. Szemerédi’s regularity lemma revisited. *arXiv preprint math/0504472*, 2005.
[51] Henry Towsner. A worked example of the functional interpretation. *arXiv preprint arXiv 1503.05572*, submitted, 2015.
[52] Anne S Troelstra. *Metamathematical investigation of intuitionistic arithmetic and analysis*, volume 344. Springer Science & Business Media, 1973.
[53] L. van den Dries and K. Schmidt. Bounds in the theory of polynomial rings over fields. A nonstandard approach. *Invent. Math.*, 76(1):77–91, 1984.