On the equivariant Gromov-Witten Theory of \(\mathbb{P}^2\)-bundles over curves

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Abstract

We compute section class relative equivariant Gromov-Witten invariants of the total space of \(\mathbb{P}^2\)-bundles of the form

\[ \mathbb{P}(\mathcal{O} \oplus L_1 \oplus L_2) \to C, \]

where \(C\) is a genus \(g\) curve, and \(\mathcal{O}\) is the trivial bundle, and \(L_1\) (resp. \(L_2\)) is an arbitrary line bundle of degree \(k_1\) (resp. \(k_2\)) over \(C\).

We prove a gluing formula for the partition functions of these invariants. The gluing formula allows us to compute the partition function in general case in terms of the basic partition functions for the case of \(g = 0\), relative to one, two or three fibers. We compute these basic partition functions via localization techniques combined with relations arising from the gluing formula. These give rise to explicit \(3 \times 3\) matrices \(G, U_1\) and \(U_2\) with entries in \(\mathbb{Q}(u)(t_0, t_1, t_2)\), where \(u\) is the genus parameter, and \(t_0, t_1, t_2\) are the equivariant parameters. Then we prove that the partition function of the section class, ordinary equivariant Gromov-Witten invariants of \(X\) is given by (Theorem 1.6):

\[ \text{tr} \left( G^{g-1} U_1^{k_1} U_2^{k_2} \right). \]

As an application, we establish a formula for the partition function of the ordinary Gromov-Witten invariants of any \(\mathbb{P}^2\)-bundle \(X\) over a curve of genus \(g\) for any class which is a Calabi-Yau section class, i.e. a curve class \(\beta_{cs}\) such that \(K_X \cdot \beta_{cs} = 0\) and \(F \cdot \beta_{cs} = 1\), where \(K_X\) is the canonical bundle of \(X\) and \(F\) is the class of the fiber. We prove that this partition function is given by (Theorem 1.7):

\[ 3^g \left( 2 \sin \frac{u}{2} \right)^{2g-2}. \]
1 Introduction

Let $X$ be a $\mathbb{P}^2$-bundle over a curve $C$ of genus $g$, and we denote the cohomology class of the fiber by $F$.

Definition 1.1. A class $\beta \in H^4(X, \mathbb{Z})$ is called section class if

$$F \cdot \beta = 1.$$ 

We say $\beta$ is a Calabi-Yau class if

$$K_X \cdot \beta = 0.$$ 

$\beta$ is called Calabi-Yau section class if both conditions above hold.

Remark 1.2. A section class is not necessarily represented by a geometric section of the bundle $X$. It could be for example a section with a number of fiber curves (curves which are included in the fibers of $X$) are attached to it.
If $\beta$ is a Calabi-Yau class then the virtual dimension of the moduli space of degree $\beta$, genus $h$ stable maps to $X$ is zero:
\[
\text{virdim } \overline{M}_h(X, \beta) = 0.
\]

Let $\beta_{cs} \in H^4(X, \mathbb{Z})$ be a Calabi-Yau section class. The partition function of the class $\beta_{cs}$ Gromov-Witten invariants of $X$ by
\[
Z_{\beta_{cs}}(g) = \sum_{h=0}^{\infty} u^{2h-2} \int \left[ \overline{M}_h(X, \beta_{cs}) \right]^{\text{vir}} 1,
\]
where
\[
\left[ \overline{M}_h(X, \beta_{cs}) \right]^{\text{vir}} \in A_0 \left( \overline{M}_h(X, \beta_{cs}) \right)
\]
is in the $0^{\text{th}}$ Chow group.

Now let $X$ be a $\mathbb{P}^2$-bundle of the form
\[
\mathbb{P}(L_0 \oplus L_1 \oplus L_2) \to C,
\]
where $C$ is a curve of genus $g$, and $L_0 \to C$ is the trivial bundle, and $L_1 \to C$ and $L_2 \to C$ are two arbitrary line bundles of degrees $k_1$ and $k_2$, respectively. As in [2], we use the word level to refer to the degree of $L_1$ or $L_2$. Sometimes we use the notation $O$ for the trivial bundle instead of $L_0$.

By Leray-Hirsch theorem we have
\[
H^{\text{even}}(X, \mathbb{Z}) = \mathbb{Z}[H,F]/ \left( H^3 + (k_1 + k_2)F \cdot H^2 \right),
\]
where $H$ is the class of the divisor
\[
\mathbb{P}(L_1 \oplus L_2) \subset \mathbb{P}(L_0 \oplus L_1 \oplus L_2),
\]
and $F$ is the class of the fiber of the bundle $X$. Note also that $H$ is cohomologous to the first Chern class of the anti-tautological bundle over $X$.

In this cohomology ring, we have
\[
H^3 = -(k_1 + k_2),
\]
\[
F \cdot H^2 = 1,
\]
and it can be shown that the canonical class of $X$ is given by
\[
K_X = -3H + (2g - 2 - k_1 - k_2)F.
\]
**Definition 1.3.** There is a distinguished section in $X$ which is by definition the locus of $(1 : 0 : 0)$ in

$$X = \mathbb{P}(O \oplus L_1 \oplus L_2) \to C.$$ 

We denote by $\beta_0$ the cohomology class in $H^4(X, \mathbb{Z})$ which is represented by this locus. We also define

$$f := H \cdot F \in H^4(X, \mathbb{Z}).$$

$\{\beta_0, f\}$ is a set of generators for $H^4(X, \mathbb{Z})$. $\{H^2, f\}$ is another set of generators. It is not hard to see that $H \cdot \beta_0 = 0$ and $F \cdot \beta_0 = 1$, and also

$$\beta_0 = H^2 + (k_1 + k_2) f.$$ 

**Remark 1.4.** One can see easily that for $\mathbb{P}^2$-bundles of this form, $\beta \in H^4(X, \mathbb{Z})$ is a section class (see Definition 1.1) if and only if it is of the form

$$\beta = \beta_0 + nf$$

for an integer $n$ (see also Remark 1.2).

The complex torus $\mathbb{T} = (\mathbb{C}^*)^3$ acts on

$$X = \mathbb{P}(L_0 \oplus L_1 \oplus L_2) \to C$$

by

$$(z_0, z_1, z_2)(x_0 : x_1 : x_2) \mapsto (z_0 x_0 : z_1 x_1 : z_2 x_2).$$

Let $\beta_s \in H^4(X, \mathbb{Z})$ be a section class. The partition function of the class $\beta_s$ equivariant Gromov-Witten invariants of $X$ is given by:

$$Z_{\beta_s}(g | k_1, k_2) = \sum_{h=0}^\infty u^{2h-2-K_X \cdot \beta_s} \int_{[\overline{M}_h(X, \beta_s)]^\text{vir}} 1,$$

where $\overline{M}_h(X, \beta_s)$ is the moduli space of degree $\beta_s$, genus $h$ stable maps\footnote{We assume that all domain curves are connected (see Remark 2.6).} to $X$, and

$$[\overline{M}_h(X, \beta_s)]^\text{vir} \in A^*_D(\overline{M}_h(X, \beta_s))$$

is in the $D^{th}$ equivariant Chow group for

$$D = -K_X \cdot \beta_s = \text{virdim} \overline{M}_h(X, \beta_s).$$

Since we are working equivariantly, our definition makes sense even for negative values of $D$ (c.f. Section 2.2.1 of [2]).
**Remark 1.5.** The equivariant Gromov-Witten partition functions are invariant under equivariant deformations. The space

\[ X = \mathbb{P}(\mathcal{O} \oplus L_1 \oplus L_2) \to C \]

that we work with is determined up to equivariant deformation by \( g \), the genus of \( C \), and the levels \( k_1 \) and \( k_2 \) of \( L_1 \) and \( L_2 \), so in this paper we can refer to \( X \) by specifying only these parameters.

Let \( t_0, t_1, t_2 \) be the generators for the equivariant cohomology of a point:

\[ H^*_T(pt) = H^*((\mathbb{C}P^\infty)^3) \cong \mathbb{Q}[t_0, t_1, t_2]. \]

\( Z_{\beta_s}(g \mid k_1, k_2) \) is a homogeneous polynomial in \( t_0, t_1, t_2 \) of degree \(-D\) with coefficients in \( \mathbb{Q}(u) \). In particular, it is zero if \( D \) is positive, and it is a Laurent series in \( u \), independent of \( t_0, t_1, t_2 \), when \( D = 0 \). In the later case, \( Z_{\beta_s}(g \mid k_1, k_2) \) is equal to the usual Gromov-Witten partition function. (c.f. Section 2.2.1 of \[2\]).

The partition function of the section class equivariant Gromov-Witten invariants of the space \( X \) is given by:

\[ Z(g \mid k_1, k_2) = \sum_{\beta_s \text{ is a section class}} Z_{\beta_s}(g \mid k_1, k_2). \]

Note that we can recover any partition function \( Z_{\beta_s}(g \mid k_1, k_2) \) from \( Z(g \mid k_1, k_2) \) by looking at terms in \( Z(g \mid k_1, k_2) \) homogeneous in \( t_0, t_1, t_2 \) (see Remark 2.3).

In Section 4, we will prove the main result of this paper that gives a formula for

\[ Z(g \mid k_1, k_2) \]

for any given genus \( g \) and levels \( k_1, k_2 \):

**Theorem 1.6.** Let \( X \) be a \( \mathbb{P}^2 \) bundle over a curve \( C \) of genus \( g \) of the form

\[ \mathbb{P}(\mathcal{O} \oplus L_1 \oplus L_2) \to C, \]

where \( L_1 \) and \( L_2 \) are two line bundles of degrees \( k_1 \) and \( k_2 \), respectively, then

\[ Z(g \mid k_1, k_2) = \text{tr} \left( G^{g-1} U_1^{k_1} U_2^{k_2} \right), \]
where the matrices $G$, $U_1$ and $U_2$ with entries in the ring $\mathbb{Q}((u))(t_0, t_1, t_2)$ are given by

$$
G = \begin{bmatrix}
(t_0 - t_1)(t_0 - t_2) & 0 & 0 \\
0 & (t_1 - t_0)(t_1 - t_2) & 0 \\
0 & 0 & (t_2 - t_0)(t_2 - t_1)
\end{bmatrix}
$$

$$
+ \begin{bmatrix}
\frac{2(t_0 - t_1 - t_2)}{t_0+t_1-2t_2} & \frac{t_0+t_1-2t_2}{(t_0-t_1)(t_0-t_2)} & \frac{t_0+t_2-2t_1}{(t_0-t_1)(t_0-t_2)} \\
\frac{(t_1-t_0)(t_1-t_2)}{t_1+t_1-2t_2} & \frac{2(2t_1-t_0-t_2)}{(t_1-t_0)(t_1-t_2)} & \frac{(t_1-t_0)(t_1-t_2)}{t_1+t_2-2t_0} \\
\frac{(t_2-t_0)(t_2-t_1)}{t_2+t_2-2t_0} & \frac{(t_1-t_0)(t_1-t_2)}{2(2t_2+t_0-t_1)} & \frac{(t_2-t_0)(t_2-t_1)}{(t_2-t_0)(t_2-t_1)}
\end{bmatrix} \phi^3,
$$

$$
U_1 = \begin{bmatrix}
\frac{\phi}{t_0-t_1} & \frac{(t_0-t_1)(t_0-t_2)}{t_0-t_0} & 0 \\
\frac{t_0-t_1}{t_1-t_0} & \frac{(t_1-t_0)(t_1-t_2)}{(t_1-t_0)(t_1-t_2)} & 0 \\
0 & \frac{t_1-t_0}{t_2-t_1} & \frac{t_1-t_2}{t_2-t_1}
\end{bmatrix},
$$

$$
U_2 = \begin{bmatrix}
\frac{\phi}{t_0-t_2} & 0 & \frac{(t_0-t_2)(t_0-t_2)}{(t_0-t_0)(t_0-t_2)} \\
0 & \frac{t_0-t_2}{t_1-t_2} & \frac{(t_1-t_0)(t_1-t_2)}{(t_1-t_0)(t_1-t_2)} \\
\frac{t_0-t_2}{t_2-t_1} & \frac{t_0-t_2}{t_2-t_1} & \frac{(t_0-t_0)(t_0-t_2)}{(t_0-t_0)(t_0-t_2)}
\end{bmatrix},
$$

where $\phi = 2 \sin \frac{u}{2}$.

In section 5 as an application of Theorem 1.6 we will prove the following result:

**Theorem 1.7.** Let $X \to C$ be any $\mathbb{P}^2$-bundle over a curve $C$ of genus $g$, and let $\beta_{cs} \in H^4(X, \mathbb{Z})$ be a Calabi-Yau section class, then

$$
Z_{\beta_{cs}}(g) = 3^g \left( 2 \sin \frac{u}{2} \right)^{2g-2}.
$$

**Plan of the paper**

In Section 2 we define the partition function of the section class relative equivariant Gromov-Witten invariants of the space

$$
X = \mathbb{P}(\mathcal{O} \oplus L_1 \oplus L_2) \to C.
$$

Then we express a gluing theorem for these partition functions.

In Section 3 we compute some of the basic partition functions we defined in Section 2 in the case $g = 0$. There are some basic partition functions...
Relative invariants and the gluing theorem

Let \((C, p_1, \ldots, p_r)\) be a nonsingular curve of genus \(g\) with \(r\) marked points. Following the notations of Section 1, we take

\[ X = \mathbb{P}(O \oplus L_1 \oplus L_2) \to (C, p_1, \ldots, p_r). \]

We will review the definition of the section class equivariant Gromov-Witten invariants relative to divisors \(F_1, \ldots, F_r\), where \(F_i\) is the fiber over the point \(p_i\). For a treatment of the foundations of equivariant relative Gromov-Witten theory, see [3].

The complex torus \(\mathbb{T} = (\mathbb{C}^*)^3\) acts on \(X\) as in Section 1. We need to fix a basis, \(\mathcal{B}_p\), for the equivariant cohomology of each fiber, \(F_p\), which is a copy

in this case that we can compute via localization, we compute them in 3.1. We use the gluing theorem of Section 2 to compute those that we cannot compute via localizations. This will be done in Section 3.2.

In Section 4, using the results of Section 3, we construct the matrices \(G, U_1 \text{ and } U_2\) appeared in Theorem 1.6 and then we prove the theorem.

In Section 5, we first prove (Lemma 5.1) that any \(\mathbb{P}^2\)-bundle over a curve \(C\) is deformation equivalent to a \(\mathbb{P}^2\)-bundle over \(C\) of the form

\[ \mathbb{P}(O \oplus O \oplus L). \]

Having this, we use Theorem 1.6 to prove Theorem 1.7.

In Appendix A, we first prove that it is enough in this paper to only consider the moduli space of maps with connected domains (Lemma A.1). After that we give a proof for the gluing theorem expressed in Section 2.

In Appendix B, we prove formulas for the partition function of equivariant Gromov-Witten invariants of the space \(X\) for some special cases of class, \(\beta\), and levels, \(k_1\) and \(k_2\).

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of $\mathbb{P}^2$:  
\[ H^*_T(F_p) \cong H_T^*(\mathbb{P}^2) \cong \mathbb{Z}[H](t_0, t_1, t_2)/\left( \prod_{j=0}^2 (H - t_j) \right). \]

Let $\beta_s \in H^4(X, \mathbb{Z})$ be a section class (defined in Section 1). We take 
\[ Z^h_{\beta_s}(g|k_1, k_2)_{\alpha_1...\alpha_r} \]
to be class $\beta_s$, genus $h$, equivariant Gromov-Witten invariant of $X$ relative to the divisors $F_1, \ldots, F_r$, with restrictions given by $\alpha_p \in \mathcal{B}_p$, one for each divisor. More precisely, we take 
\[ \vec{L} = (l_1, \ldots, l_r) \in (\mathbb{Z}^+)^r, \]
\[ \vec{F} = (F_1, \ldots, F_r). \]

Then following Section 2 of [9], let $X[\vec{L}]$ be the $l_i$-step degeneration of $X$ along each $F_i$, and let \[ \overline{M}_h(X/\vec{F}, \beta_s) \]
be the moduli space of relative stable maps 
\[ \left[ q : C' \to X[\vec{L}] \right] \]
from nodal genus $h$ curves$^2$, $C'$, to $X[\vec{L}]$, for some $\vec{L}$, which are representing the class $\beta_s$. Then $\overline{M}_h(X/\vec{F}, \beta_s)$ is a DM-stack of virtual dimension $-K_X \cdot \beta_s$ (see also [6]).

**Remark 2.1.** Since $F \cdot \beta_s = 1$, $\overline{M}_h(X/\vec{F}, \beta_s)$ does not involve partition vectors, as it does in a more general case in [9]. Our moduli space is more general than the one in [9] in the sense that it parameterizes maps relative to more than or equal to one divisor.

For each $p$, we have an evaluation map which is determined by relative points, and is $\mathbb{T}$-equivariant (see [7]):
\[ \text{ev}_p : \overline{M}_h(X/\vec{F}, \beta_s) \to F_p. \]

$^2$We assume that all domain curves are connected (see Remark 2.6)
Then
\[ Z_{\beta_s}^h(g \mid k_1, k_2)_{\alpha_1, \ldots, \alpha_r} = \int_{[\mathcal{M}_h(X/F, \beta_s)]^{\text{vir}}} \ev^*_1(\alpha_1) \cup \cdots \cup \ev^*_r(\alpha_r), \]
where
\[ [\mathcal{M}_h(X, \beta_s)]^{\text{vir}} \in A^T_D([\mathcal{M}_h(X, \beta_s)]) \]
is in the \( D \)th equivariant Chow group for
\[ D = -K_X \cdot \beta_s = \text{virdim } \mathcal{M}_h(X/F, \beta_s). \]

Note that the invariants can be non-zero even for negative values of \( D \) (c.f. Section 2.2.1 of \[2\], and also see Remark 1.5). Then the partition function of the class \( \beta_s \), relative, equivariant Gromov-Witten invariants of the space \( X \) relative to \( F \) with relative multiplicities \( \alpha_p \in B_p \) is given by:
\[
Z_{\beta_s}(g \mid k_1, k_2)_{\alpha_1, \ldots, \alpha_r} = \sum_{h=0}^{\infty} Z_{\beta_s}^h(g)_{\alpha_1, \ldots, \alpha_r} u^{2h - 2 - K_X \cdot \beta_s}. \quad (1)
\]

We can also write the partition function of the section class, relative, equivariant Gromov-Witten invariants of the space \( X \) relative to \( F \) with relative multiplicities \( \alpha_p \in B_p \) as
\[ Z(g \mid k_1, k_2)_{\alpha_1, \ldots, \alpha_r} = \sum_{\beta_s \text{ is a section class}} Z_{\beta_s}(g \mid k_1, k_2)_{\alpha_1, \ldots, \alpha_r}. \]

It is evident that when \( r = 0 \) we get the partition function for the ordinary invariants, defined in Section \[1\].

**Remark 2.2.** \( Z_{\beta_s}(g \mid k_1, k_2)_{\alpha_1, \ldots, \alpha_r} \) is a homogeneous polynomial in \( t_0, t_1, t_2 \) of degree
\[ N = \sum_{p=1}^{r} \deg(\alpha_p) - D, \]
with coefficients in \( \mathbb{Q}((u)) \).

In particular it is zero if \( N < 0 \), and it is a Laurent series in \( u \), independent of \( t_0, t_1, t_2 \), when \( N = 0 \) (c.f. \[2\], Section 2.2.1).
Remark 2.3. We can reexpress the definition of the partition function for
the section class invariants as follows (see Remark 1.4):

$$Z(g | k_1, k_2)_{\alpha_1...\alpha_r} = \sum_{n \in \mathbb{Z}} Z_{\beta_0 + nf}(g | k_1, k_2)_{\alpha_1...\alpha_r}.$$ 

This sum is finite, because by Remark 2.2, it is clear that the sum is termi-
nated from above, and it is also terminated from below because for the large
negative values of $n$, there is no curve representing the class $\beta_0 + nf$ which
means that

$$\overline{M}_h(X/F, \beta_0 + nf) = \emptyset$$

for $n \ll 0$. To see the last claim, let $E = \mathcal{O} \oplus L_1 \oplus L_2$, and notice that for a
negative $n$ there is a one to one correspondence between geometric sections
representing $\beta_0 + nf$ and degree $-n$ sub-line bundles of $E$. $E$ has no sub-line
bundle of degree greater than $k_1 + k_2$. Therefore for $n \ll 0$, $E$ has no sub-line
bundle of degree $-n$, which proves our claim.

One can also recover $Z_{\beta_0 + nf}(g | k_1, k_2)_{\alpha_1...\alpha_r}$ from $Z(g | k_1, k_2)_{\alpha_1...\alpha_r}$, by
looking at terms in the sum above which are homogeneous in $t_0, t_1, t_2$ of
degree

$$N = \sum_{p=1}^{r} \deg(\alpha_p) - D$$

$$= \sum_{p=1}^{r} \deg(\alpha_p) + 2g - 2 - k_1 - k_2 - 3n.$$ 

We use the following useful lemma in Section 3:

Lemma 2.4. Let $X = \mathbb{P}(\mathcal{O} \oplus L \oplus L')$, where $L$ and $L'$ have levels $n$ and $m$,
respectively. Let $\xi_n$ be the class which is represented by the locus of $(0 : 1 : 0)$
in

$$X \cong \mathbb{P}(L^{-1} \oplus \mathcal{O} \oplus L'L^{-1}).$$

Then we have the following relation in $H^4(X, \mathbb{Z})$:

$$\xi_n = \beta_0 - nf.$$ 

Proof. First notice that there exist $a, b \in \mathbb{Z}$ such that

$$\xi_n = a\beta_0 + bf.$$
By trivial relations
\[ F \cdot \beta_0 = F \cdot \xi_n = 1 \]
and also \( F \cdot F = 0 \), it is clear that \( a = 1 \).
For finding \( b \), notice that the normal bundle of \( \xi_n \) in \( X \) is isomorphic to
\[ L \oplus L'^{-1} \to C, \]
which means that
\[ H \cdot \xi_n = -n \]
(Recall from Section 1 that \( H \) is cohomologous to \( \mathbb{P}(L \oplus L') \subset X \)). Now combining this with \( H \cdot \beta_0 = 0 \) and \( H \cdot f = 1 \), we see that \( b = -n \), which proves the lemma. \( \square \)

Before expressing the gluing theorem, we fix a basis, \( \mathcal{B} \), for the equivariant cohomology of \( \mathbb{P}^2 \). We take
\[
\begin{align*}
x_0 &:= (H - t_1)(H - t_2), \\
x_1 &:= (H - t_0)(H - t_1), \\
x_2 &:= (H - t_0)(H - t_2).
\end{align*}
\]
We have \( x_i \in H^4_T(\mathbb{P}^2) \). \( x_0, x_1, x_2 \) are in fact equivariant cohomology classes represented by three fixed points of the torus action on \( \mathbb{P}^2 \). We define
\[ \mathcal{B} := \{x_0, x_1, x_2\}. \]
It is easy to see that \( \mathcal{B} \) is a basis for \( H^*_T(\mathbb{P}^2) \otimes \mathbb{Q}(t_0, t_1, t_2) \), for example, we can recover the ordinary basis for the cohomology of \( \mathbb{P}^2 \):
\[
\begin{align*}
1 &= \frac{x_0}{(t_0 - t_1)(t_0 - t_2)} + \frac{x_1}{(t_1 - t_0)(t_1 - t_2)} + \frac{x_2}{(t_2 - t_0)(t_2 - t_1)}, \\
H &= \frac{t_0x_0}{(t_0 - t_1)(t_0 - t_2)} + \frac{t_1x_1}{(t_1 - t_0)(t_1 - t_2)} + \frac{t_2x_2}{(t_2 - t_0)(t_2 - t_1)}, \\
H^2 &= \frac{t_0^2x_0}{(t_0 - t_1)(t_0 - t_2)} + \frac{t_1^2x_1}{(t_1 - t_0)(t_1 - t_2)} + \frac{t_2^2x_2}{(t_2 - t_0)(t_2 - t_1)}.
\end{align*}
\]
We also have these relations:
\[
\begin{align*}
x_0^2 &= (t_0 - t_1)(t_0 - t_2)x_0, \\
x_1^2 &= (t_1 - t_0)(t_1 - t_2)x_1, \\
x_2^2 &= (t_2 - t_0)(t_2 - t_1)x_2, \\
x_ix_j &= 0 \quad \text{for } i \neq j. \quad (2)
\end{align*}
\]
Convention. From now on, we assume that each $\alpha_p$ for $p = 1, \ldots, r$, in the definition of relative partition functions belong to this basis set, $\mathcal{B}$, with the identification of each $F_p$ with $P^2$.

We take

$$T(x_0) := (t_0 - t_1)(t_0 - t_2),$$

$$T(x_1) := (t_1 - t_0)(t_1 - t_2),$$

$$T(x_2) := (t_2 - t_0)(t_2 - t_1).$$

Then we raise the indices for the relative partition functions by the following rule:

$$Z(g | k_1, k_2)^{\gamma_1 \ldots \gamma_t}_{\alpha_1 \ldots \alpha_s} := \left( \prod_{p=1}^{t} \frac{1}{T(\gamma_p)} \right) Z(g | k_1, k_2)^{\gamma_1 \ldots \gamma_t}_{\alpha_1 \ldots \alpha_s}.$$ 

Then we have the following gluing rules similar to Theorem 3.1 in [2]:

**Theorem 2.5.** For any choices of elements $\alpha_1, \ldots, \alpha_s$ and $\gamma_1, \ldots, \gamma_t$ from the set $\mathcal{B}$, and integers satisfying $g = g' + g'', k_1 = k'_1 + k''_1$, and $k_2 = k'_2 + k''_2$ we have

$$Z(g | k_1, k_2)^{\gamma_1 \ldots \gamma_t}_{\alpha_1 \ldots \alpha_s} = \sum_{\lambda \in \mathcal{B}} Z(g' | k'_1, k'_2)^{\gamma_1 \ldots \gamma_t}_{\alpha'_1 \ldots \alpha'_s} \lambda Z(g'' | k''_1, k''_2)^{\gamma_1 \ldots \gamma_t}_{\alpha''_1 \ldots \alpha''_s},$$

and

$$Z(g | k_1, k_2)^{\gamma_1 \ldots \gamma_t}_{\alpha_1 \ldots \alpha_s} = \sum_{\lambda \in \mathcal{B}} Z(g | k_1, k_2)^{\lambda}_{\alpha_1 \ldots \alpha_s}.$$ 

The proof of this theorem will be given in Appendix A.

**Remark 2.6.** In most of the contexts in which the relative Gromov-Witten invariants are being used, maps with disconnected domain curves are considered as well as ones with connected domains. In Lemma A.1 we prove that in our case, where we only deal with section classes, we don’t need to consider disconnected domain curves.

**Remark 2.7.** In exactly the same way as in [2], one can prove by using Theorem 2.5 that the partition functions $Z(g | 0, 0)_{\alpha_1 \ldots \alpha_s}$ give rise to a 1 + 1-dimensional TQFT taking values in the ring $R = \mathbb{Q}((u))(t_0, t_1, t_2)$. The Frobenius algebra corresponding to this TQFT (see [3], Theorem 2.1) is

$$H = \bigoplus_{i=0}^{2} \text{Re} x_i.$$
3 Calculations

for $x_i \in B$, with multiplication given by

$$e_{x_i} \otimes e_{x_j} = \sum_{k=0}^{2} Z(g \mid 0, 0)^{x_k}_{x_i x_j} e_k.$$  

We will prove that this Frobenius algebra and hence the corresponding TQFT is semisimple (Corollary 3.2.5). In Section 5, we use this fact for proving Theorem 1.6 and 1.7 and also the results in Appendix B for the case $g = 0$.

We will use the following corollary of Theorem 2.5 in our calculations:

**Corollary 2.8.** With the same notation as in Theorem 2.5 we have

$$Z_{\beta_0 + n_f}(g \mid k_1, k_2)_{\alpha_1 \ldots \alpha_s \gamma_1 \ldots \gamma_t} = \sum_{\lambda \in B} \sum_{n=n'+n''} Z_{\beta_0 + n'f}(g' \mid k'_1, k'_2)_{\alpha_1 \ldots \alpha_s \lambda} Z_{\beta_0 + n''f}(g'' \mid k''_1, k''_2)_{\lambda}^\gamma_{\gamma_1 \ldots \gamma_t}. $$

\[ \square \]

3 Calculations

We work with the space

$$X = \mathbb{P}(O \oplus L_1 \oplus L_2) \rightarrow (C, p_1, \ldots, p_r),$$

throughout this section. In accordance with the notations in 2, we will use the words cap, tube and pants to refer to the case where the base curve, $C$, is a genus zero curve with one, two and three marked points, respectively, and by $(k_1, k_2)$ we mean that the level of the line bundle $L_i$ is $k_i$ for $i = 1, 2$ (see Remark 1.5). We sometimes refer to the partition functions by referring to the space to which they correspond. Finally, for simplicity, we will use the notation

$$\phi := 2 \sin \frac{u}{2}$$

in later calculations.
Similar to Section 4.3 in [2] one can see that the following partition functions determine the theory completely:

- $Z(0 \mid 0, 0)_{\alpha}$: corresponding to the level $(0, 0)$ cap,
- $Z(0 \mid 0, 0)_{\alpha_1\alpha_2}$: corresponding to the level $(0, 0)$ tube,
- $Z(0 \mid 0, 0)_{\alpha_1\alpha_2\alpha_3}$: corresponding to the level $(0, 0)$ pants,
- $Z(0 \mid -1, 0)_{\alpha}$: corresponding to the level $(-1, 0)$ cap,
- $Z(0 \mid 0, -1)_{\alpha}$: corresponding to the level $(0, -1)$ cap,
- $Z(0 \mid 1, 0)_{\alpha}$: corresponding to the level $(1, 0)$ cap,
- $Z(0 \mid 0, 1)_{\alpha}$: correspond to the level $(0, 1)$ cap.

We refer to the partition functions above as the basic partition functions.

By the discussion given in Remark 2.3, one can prove the following lemma:

**Lemma 3.1.** The basic partition functions are given by

\[
\begin{align*}
Z(0 \mid 0, 0)_{\alpha} &= Z_{\beta_0}(0 \mid 0, 0)_{\alpha}, \\
Z(0 \mid 0, 0)_{\alpha_1\alpha_2} &= Z_{\beta_0}(0 \mid 0, 0)_{\alpha_1\alpha_2}, \\
Z(0 \mid 0, 0)_{\alpha_1\alpha_2\alpha_3} &= Z_{\beta_0}(0 \mid 0, 0)_{\alpha_1\alpha_2\alpha_3} + Z_{\beta_0 + f}(0 \mid 0, 0)_{\alpha_1\alpha_2\alpha_3}, \\
Z(0 \mid -1, 0)_{\alpha} &= Z_{\beta_0}(0 \mid -1, 0)_{\alpha}, \\
Z(0 \mid 0, -1)_{\alpha} &= Z_{\beta_0}(0 \mid 0, -1)_{\alpha}, \\
Z(0 \mid 1, 0)_{\alpha} &= Z_{\beta_0 - f}(0 \mid 1, 0)_{\alpha}, \\
Z(0 \mid 0, 1)_{\alpha} &= Z_{\beta_0 - f}(0 \mid 0, 1)_{\alpha}.
\end{align*}
\]

**Proof:** We prove the third equality as follows:

In the right hand side, we do not have any partition function of class $\beta_0 + nf$ for $n < 0$, because $O \oplus L_1 \oplus L_2$ does not have any sub-line bundle of a positive degree, as $L_1$ and $L_2$ are level zero.

We also do not have any partition function of class $\beta_0 + nf$ for $n > 1$, because

\[
N = \sum_{p=1}^{3} \deg(\alpha_p) - D = (2 + 2 + 2) - (3H + 2F) \cdot (\beta_0 + nf) = 4 - 3n
\]
which is negative for \(n > 1\) (see Remark 2.2).
The other equalities are proved similarly.

The rest of this section is devoted to computing the terms appeared in
the right hand sides of the equations in Lemma 3.1.

### 3.1 Calculations via localization

The complex torus acts on \(X\) as before. We define

\[
S_0: \text{ the locus of } (1:0:0) \text{ in } X \cong \mathbb{P}(\mathcal{O} \oplus L_1 L_0^{-1} \oplus L_2 L_0^{-1}),
\]

\[
S_1: \text{ the locus of } (0:1:0) \text{ in } X \cong \mathbb{P}(L_0 L_1^{-1} \oplus \mathcal{O} \oplus L_2 L_1^{-1}),
\]

\[
S_2: \text{ the locus of } (0:0:1) \text{ in } X \cong \mathbb{P}(L_0 L_2^{-1} \oplus L_1 L_2^{-1} \oplus \mathcal{O}).
\]

\(S_0, S_1\) and \(S_2\) are fixed under the torus action, and by Lemma 2.4, they
represent the classes \(\beta_0, \beta_0 - k_1 f\) and \(\beta_0 - k_2 f\), respectively.

As before, let \(\beta_s\) be a section class. The torus action on \(X\) induces
an action on \(\overline{M}_h(X/\vec{F}, \beta_s)\). We denote the fixed locus of this action by
\(\overline{M}_h(X/\vec{F}, \beta_s)^T\).

By notations of Section 2, we let \(S_i[\vec{L}] \subset X[\vec{L}]\) be the \(l_i\)-step degeneration
of \(S_i\) along the intersection point \(\tau_{ip} = S_i \cap F_p\) for \(p = 1, \ldots, r,\) and \(i = 0, 1, 2\).

Then \(\overline{M}_h(X/\vec{F}, \beta_s)^T\) parameterizes maps

\[
[q: C' \to X[\vec{L}]]
\]

for some \(\vec{L}\), whose images are either of

\[
S_i[\vec{L}] \cup_{n=1}^{m_i} f_n
\]

for \(i = 0, 1\) or \(2\), where by the last expression we mean \(S_i[\vec{L}]\) with \(m_i\) \(\mathbb{T}\)-fixed
fiber curves, \(f_n\) (\(f_n\) represents the class \(af\) for some \(a \in \mathbb{Z}^+\)), are attached to
it at some points. Note that the choice of \(i \in \{0, 1, 2\}\), and also the number
of fibers which are attached to \(S_i[\vec{L}]\), \(m_i\), are constrained by the class \(\beta_s\).

In general, the moduli space \(\overline{M}_h(X/\vec{F}, \beta_s)^T\) can be quiet complicated,
because of the existence of the fibers attached to each \(S_i[\vec{L}]\). However, in the
special case where \( m_i = 0 \) for some \( i \), an elementary observation shows that the component of \( \overline{M}_h(X/\bar{F}, \beta_s)^T \), parameterizing maps with images equal to \( S_i[\bar{L}] \) is exactly the moduli space of degree one relative stable maps to curves, which we denote by \( \overline{M}_h(S_i/\bar{\tau}_i, 1) \), where

\[
\bar{\tau}_i = (\tau_{i1}, \ldots, \tau_{ip}).
\]

**Assumption 1.** For the rest of Section 3.1 we assume that

\[
\overline{M}_h(X/\bar{F}, \beta_s)^T = \bigcup_{i \in I} \overline{M}_h(S_i/\bar{\tau}_i, 1),
\]

where \( I \subset \{0, 1, 2\} \), depending on the class \( \beta_s \).

Then one can see that the \( \mathbb{T} \)-fixed part of the perfect obstruction theory of \( \overline{M}_h(X/\bar{F}, \beta_s) \) is exactly the usual obstruction theory of \( \bigcup_{i \in I} \overline{M}_h(S_i/\bar{\tau}_i, 1) \), and therefore

\[
[\overline{M}_h(X/\bar{F}, \beta_s)^T]^\text{vir} \cong \sum_{i \in I} [\overline{M}_h(S_i/\bar{\tau}_i, 1)]^\text{vir}.
\]

In the special case where \( m_i = 0 \) for all possible \( i \), Assumption 1 holds. One can see easily that this is the case for all the partition functions in the right hand sides of equations in Lemma 3.1, except for

\[
Z_{\beta_0 + f(0 | 0, 0)}^{\alpha_1 \alpha_2 \alpha_3}.
\]

These partition functions are calculated in this section via localization. \( Z_{\beta_0 + f(0 | 0, 0)}^{\alpha_1 \alpha_2 \alpha_3} \) will be calculated in Section 3.2 by combining the results of this section with the results of the gluing techniques.

Applying the relative virtual localization formula (see Section 3 of \cite{3} to \cite{11}), we can write

\[
Z_{\beta_s}(g | k_1, k_2)_{\alpha_1 \ldots \alpha_r} =
\sum_{h=0}^{\infty} u^{2h-2-K_{X: \beta_s}} \int_{[\overline{M}_h(X/\bar{F}, \beta_s)^T]^\text{vir}} \frac{\text{ev}^*_1(\alpha_1) \cap \cdots \cap \text{ev}^*_r(\alpha_r)}{e(\text{Norm}^{\text{vir}})},
\]

where \( \text{Norm}^{\text{vir}} \) is the equivariant virtual normal bundle of

\[
\overline{M}_h(X/\bar{F}, \beta_s)^T \subset \overline{M}_h(X/\bar{F}, \beta_s),
\]

\footnote{In \cite{3}, the authors assume for convenience that the relative divisor is in the fixed locus, but it is straightforward to adapt their methods to the case at hand \cite{4}.}
and $e($Norm$^{\text{vir}}$) is its equivariant Euler class.

Let

$$\pi : U \rightarrow \overline{M}_h(X/\vec{F}, \beta_s)$$

and

$$q : U \rightarrow X$$

be the universal curve and universal map, respectively, where

$$X \rightarrow \overline{M}_h(X/\vec{F}, \beta_s)$$

is the universal target space.

Now notice that the normal bundle of each $F_p$ in $X$ is the trivial bundle with the trivial torus action. So the deformations of the singularities of the degenerated target spaces does not contribute in $e($Norm$^{\text{vir}}$) (see Section 3 of [3]). We also have the following short exact sequence

$$0 \rightarrow \Omega_X \rightarrow \Omega_X(\log F_p) \rightarrow N_{F_p|X} \rightarrow 0,$$

where $\Omega_X(\log F_p)$ is the sheaf of Kähler differential with logarithmic poles along $F_p$ (see [3]), and $N_{F_p|X}$ is the normal bundle of $F_p$ in $X$. Again since the torus action on $N_{F_p|X}$ is trivial, from this sequence we can see that the moving part of $H^\bullet(C', q^*T_X(-\log F_p))$ is equal to the moving part of $H^\bullet(C', q^*T_X)$, where $T_X(-\log F_p)$ is the dual sheaf of $\Omega_X(\log F_p)$. So from (2) and (3) in [3] (the third term in (3) has no moving part), we can write

$$e($Norm$^{\text{vir}}$) = e(R^\bullet \pi_\ast q^*(T_X)^{\text{mov}})$$

$$= \sum_{i \in I} e(R^\bullet \pi_\ast q^*(N_{S_i|X})),$$

where $N_{S_i|X}$ is the normal bundle of $S_i$ in $X$. One can see easily that

$$N_{S_0|X} \cong L_1L_0^{-1} \oplus L_2L_0^{-1},$$
$$N_{S_1|X} \cong L_0L_1^{-1} \oplus L_2L_1^{-1},$$
$$N_{S_2|X} \cong L_0L_2^{-1} \oplus L_1L_2^{-1}.$$

(5)

For manipulating the evaluation functions in (4), we use the following cartesian diagram for each $p = 1, \ldots, r$, and $i \in I$:

$$\begin{array}{c}
\overline{M}_h(S_i/\vec{r}_i, 1) \longrightarrow \{\tau_{ip}\} \\
\downarrow j_i \quad \quad \quad \downarrow \\
\overline{M}_h(X/\vec{F}, \beta_s) \longrightarrow \text{ev}_p \quad F_p
\end{array}$$
where two vertical maps are inclusions, and $\tau_{ip}$ is the intersection point of $S_i$ with $F_p$, which is the fixed point of the torus action on $F_p$ representing the class $x_i \in B$. From this diagram it is clear that $ev_p^*(\alpha_p)$, restricted to $\overline{M}_h(S_i/\tau_i, 1)$, is a class of pure weight for each $p$, and can be taken out of the integrals.

We summarize all the discussion above in the following equation:

$$Z_{\beta}(g \mid k_1, k_2)_{\alpha_1 \ldots \alpha_r} = \sum_{h=0}^{\infty} u^{2h-2-K_{X, \beta}} \sum_{i \in I} \left( \prod_{p=1}^{r} (ev_p \circ j_i)^*(\alpha_p) \right) \int_{[\overline{M}_h(S_i/\tau_i, 1)]^{vir}} e \left( -R^* \pi_* q^*(N_{S_i | X}) \right).$$

(6)

We will handle the integrals of this form in our calculations by appealing to the results in [2].

Applying Atiah-Bott localization theorem to $\mathbb{P}^2$, one can see easily that

$$(ev_p \circ j_i)^*(x_k) = \begin{cases} T(x_i) & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

(7)

for $k \in \{0, 1, 2\}$ (see Section 2 for the definition of $T(\cdot)$).

### 3.1.1 Computing class $\beta_0$, level (0, 0) cap, tube and pants:

**Lemma 3.1.1.** Partition functions for the class $\beta_0$, level (0, 0) cap, tube and pants are given by

$$Z_{\beta_0}(0 \mid 0, 0)_{x_a} = 1$$

$$Z_{\beta_0}(0 \mid 0, 0)_{x_a x_b} = \begin{cases} T(x_a) & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

$$Z_{\beta_0}(0 \mid 0, 0)_{x_a x_b x_c} = \begin{cases} T(x_a)^2 & \text{if } a = b = c \\ 0 & \text{otherwise} \end{cases}$$

for $a, b, c \in \{0, 1, 2\}$.

**Proof:** Since $k_1 = k_2 = 0$, by Lemma 2.3 all $S_0, S_1$ and $S_2$ represent the class $\beta_0$, so in (6) we have $I = \{0, 1, 2\}$. 
We use the results of Sections 6.2 and 6.4.2 in [2] to evaluate the integrals in (6), for the cap, tube and pants. We prove the formula for the tube, the other cases are similar. By (6) and Lemma 6.1 in [2] (for \(d = 1\)) and also (5) and (7) we have

\[
Z_{\beta_0}(0 | 0, 0)_{x_a x_b} = \sum_{h=0}^{\infty} u^{2h-2-K_X \cdot \beta} \left( (ev_1 \circ j_0)^*(x_a)(ev_2 \circ j_0)^*(x_b) + (ev_1 \circ j_1)^*(x_a)(ev_2 \circ j_1)^*(x_b) + (ev_1 \circ j_2)^*(x_a)(ev_2 \circ j_2)^*(x_b) \right)
\]

\[
\cdot \int_{\bar{M}_h(S_{0/(\tau_{01}, \tau_{02}), 1})} e \left( -R^* \pi_* q^* (L_1 L_0^{-1} \oplus L_2 L_0^{-1}) \right) e \left( -R^* \pi_* q^* (L_0 L_1^{-1} \oplus L_2 L_1^{-1}) \right)
\]

\[
\cdot \int_{\bar{M}_h(S_{2/(\tau_{21}, \tau_{22}), 1})} e \left( -R^* \pi_* q^* (L_0 L_2^{-1} \oplus L_1 L_2^{-1}) \right)
\]

\[
= (\delta_0^0 T(x_0))(\delta_0^0 T(x_0)) \frac{1}{T(x_0)} + (\delta_1^0 T(x_1))(\delta_1^0 T(x_1)) \frac{1}{T(x_1)} + (\delta_2^0 T(x_2))(\delta_2^0 T(x_2)) \frac{1}{T(x_2)} = \delta_0^0 T(x_a). \quad \square
\]

### 3.1.2 Computing class \(\beta_0\), level \((0, -1)\) and \((-1, 0)\) and class \(\beta_0 - f\), level \((0, 1)\) and \((1, 0)\) caps:

**Lemma 3.1.2.** Partition functions for the class \(\beta_0\), level \((0, -1)\) and \((-1, 0)\) caps are given by

\[
Z_{\beta_0}(0 | 0, -1)_{x_a} = (t_a - t_2) \phi^{-1}
\]

\[
Z_{\beta_0}(0 | -1, 0)_{x_a} = (t_a - t_1) \phi^{-1}
\]

for \(a = 0, 1, 2\).
Proof: We prove the first formula, the second one is proved in a similar way. We have $k_1 = 0$ and $k_2 = -1$, so by Lemma 2.4 $S_0, S_1$ represent the class $\beta_0$, but $S_2$ represents the class $\beta_0 + f$. Therefore in (6) we have $I = \{0, 1\}$.

By Lemma 6.3 in [2] (for $d = 1$) and also (5) and (7) we can rewrite (6) as

$$Z_{\beta_0 - f}(0 \mid 0, 1 \mid a) = \sum_{h=0}^{\infty} u^{2h - 2 - Kx \cdot \beta_0}$$

$$\left( (ev \circ j_0)^*(x_a) \int_{\overline{M}_\delta(S_0 / \tau_{01,1})^{vir}} e(-R^*\pi_*q^*(L_1L_0^{-1} \oplus L_2L_0^{-1})) \right)$$

$$+ (ev \circ j_1)^*(x_a) \int_{\overline{M}_\delta(S_1 / \tau_{11,1})^{vir}} e(-R^*\pi_*q^*(L_0L_1^{-1} \oplus L_2L_1^{-1}))$$

$$= \left( \frac{\delta_0 T(x_0)}{t_0 - t_1} + \frac{\delta_1 T(x_1)}{t_1 - t_0} \right) \phi^{-1}$$

$$= (t_a - t_2)\phi^{-1}. \quad \square$$

Lemma 3.1.3. Partition functions for the class $\beta_0 - f$, level $(0, 1)$ and $(1, 0)$ caps are given by

$$Z_{\beta_0 - f}(0 \mid 0, 1 \mid a) = (t_a - t_0)(t_a - t_1)\phi^{-2}$$

$$Z_{\beta_0 - f}(0 \mid 1, 0 \mid a) = (t_a - t_0)(t_a - t_2)\phi^{-2}$$

for $a = 0, 1, 2$.

Proof: We again prove the first relation. The proof of the second one is similar. In this case only $S_2$ represents the class $\beta_0 - f$, so we have $I = \{2\}$. The integral that we need to know in order to prove the Lemma can be found in Section 8 of [2]. The rest of the proof is quiet similar to the proof of Lemma 3.1.2. \square

3.2 Calculations via gluing techniques

In this section, we use Corollary 2.8 (which is referred to as the gluing formula), and the results of Section 3.1 to find $Z_{\beta_0 + f}(0 \mid 0, 0)_{\alpha_1 \alpha_2 \alpha_3}$. For a treatment of gluing spaces and applying the gluing theorem see Appendix A.
3 CALCULATIONS

We first need to find the following partition functions of tubes:

\[
Z(0 | 0, -1)_{\alpha_1 \alpha_2} = Z_{\beta_0}(0 | 0, -1)_{\alpha_1 \alpha_2} + Z_{\beta_0+f}(0 | 0, -1)_{\alpha_1 \alpha_2},
\]

\[
Z(0 | -1, 0)_{\alpha_1 \alpha_2} = Z_{\beta_0}(0 | -1, 0)_{\alpha_1 \alpha_2} + Z_{\beta_0+f}(0 | -1, 0)_{\alpha_1 \alpha_2},
\]

\[
Z(0 | 0, 1)_{\alpha_1 \alpha_2} = Z_{\beta_0-f}(0 | 0, 1)_{\alpha_1 \alpha_2} + Z_{\beta_0}(0 | 0, 1)_{\alpha_1 \alpha_2},
\]

\[
Z(0 | 1, 0)_{\alpha_1 \alpha_2} = Z_{\beta_0-f}(0 | 1, 0)_{\alpha_1 \alpha_2} + Z_{\beta_0}(0 | 1, 0)_{\alpha_1 \alpha_2}.
\]

These equalities can be proved similar to the proof of Lemma 3.1. Now we are going to find the partition functions in the right hand sides of equations in (8):

3.2.1 Computing class $\beta_0$, level $(0, -1)$ and $(-1, 0)$ and class $\beta_0 - f$, level $(0, 1)$ and $(1, 0)$ tubes:

**Lemma 3.2.1.** Partition functions for the class $\beta_0$, level $(0, -1)$ and $(-1, 0)$ tubes are given by

\[
Z_{\beta_0}(0 | 0, -1)_{x_a x_b} = \begin{cases} 
(t_0 - t_1)(t_0 - t_2)^2 \phi^{-1} & \text{if } a = b = 0, \\
(t_1 - t_0)(t_1 - t_2)^2 \phi^{-1} & \text{if } a = b = 1, \\
0 & \text{otherwise}.
\end{cases}
\]

\[
Z_{\beta_0}(0 | -1, 0)_{x_a x_b} = \begin{cases} 
(t_0 - t_2)(t_0 - t_1)^2 \phi^{-1} & \text{if } a = b = 0, \\
(t_2 - t_0)(t_2 - t_1)^2 \phi^{-1} & \text{if } a = b = 2, \\
0 & \text{otherwise}
\end{cases}
\]

for $a, b \in \{0, 1, 2\}$.

**Proof:** The first relation is simply proved by attaching the level $(0, -1)$ cap to the level $(0, 0)$ pants and applying the gluing formula. This is schematically indicated by the following picture:
The result is now obvious by applying Lemma 3.1.2 and Lemma 3.1.1. The proof of the second relation is similar.

Similar to the proof of Lemma 3.2.1, we can prove this:

**Lemma 3.2.2.** Partition functions for the class $\beta_0 - f$, level $(0,1)$ and $(1,0)$ tubes are given by

\[
Z_{\beta_0 - f}(0 \mid 0,1)_{x_a x_b} = \begin{cases} 
(t_2 - t_0)^2(t_2 - t_1)^2 \phi^{-2} & \text{if } a = b = 2, \\
0 & \text{otherwise.}
\end{cases}
\]

\[
Z_{\beta_0 - f}(0 \mid 1,0)_{x_a x_b} = \begin{cases} 
(t_1 - t_2)^2(t_1 - t_0)^2 \phi^{-2} & \text{if } a = b = 1, \\
0 & \text{otherwise.}
\end{cases}
\]

for $a, b \in \{0, 1, 2\}$.

3.2.2 **Computing class $\beta_0$, level $(0,1)$ and $(1,0)$, and also class $\beta_0 + f$, level $(0,-1)$ and $(-1,0)$ tubes:**

We do the calculations for class $\beta_0$, level $(0,1)$ and class $\beta_0 + f$, level $(0,-1)$ tubes, the cases where levels are on the second line bundle are similar.

We attach two tubes of levels $(0, -1)$ and $(0,1)$ to get a tube of level $(0,0)$ (see the picture). Now applying gluing formula and using Lemma 3.2.1 and Lemma 3.2.2, we get

\[
Z_{\beta_0}(0 \mid 0,0)_{x_a x_a} = Z_{\beta_0}(0 \mid 0, -1)_{x_a x_a} Z_{\beta_0}(0 \mid 0, 1)_{x_a x_a} + Z_{\beta_0 + f}(0 \mid 0, -1)_{x_a x_a} Z_{\beta_0 - f}(0 \mid 0, 1)_{x_a x_a}
\]

for $a = 0, 1, 2$.

Again by Lemmas 3.2.1, 3.2.2 and 3.1.1.
so we can solve the equations above for the other unknowns:

\[ Z_{\beta_0}(0 \mid 0, 1)_{x_0x_0} = (t_0 - t_1)\phi, \]
\[ Z_{\beta_0-f}(0 \mid 0, 1)_{x_1x_1} = (t_1 - t_0)\phi, \]
\[ Z_{\beta_0+f}(0 \mid 0, -1)_{x_2x_2} = \phi^2. \]  

(9)

By changing relative conditions, we can get more relations:

\[ x_0(0,0) - \beta_0 x_1 = x_0(0,-1, \beta_0) x_1 \]
\[ 0 = Z_{\beta_0}(0 \mid 0, 0)_{x_0x_1} = Z_{\beta_0}(0 \mid 0, -1)_{x_0x_1} Z_{\beta_0}(0 \mid 0, 1)_{x_1}, \]

which implies that

\[ Z_{\beta_0}(0 \mid 0, 1)_{x_0x_1} = 0. \]  

(10)

We can also write

\[ x_2(0,0) - \beta_0 x_a = x_2(0,-1, \beta_0) x_a + x_2(0,-1, \beta_0+1) x_a \]
\[ 0 = Z_{\beta_0}(0 \mid 0, 0)_{x_2x_a} = Z_{\beta_0}(0 \mid 0, -1)_{x_2x_a} Z_{\beta_0}(0 \mid 0, 1)_{x_a}
 + Z_{\beta_0+f}(0 \mid 0, -1)_{x_2x_a} Z_{\beta_0-f}(0 \mid 0, 1)_{x_2} \]

for \( a = 0, 1 \). This implies that

\[ Z_{\beta_0}(0 \mid 0, 1)_{x_0x_2} = Z_{\beta_0+f}(0 \mid 0, -1)_{x_0x_2} (t_2 - t_1)\phi^{-1}, \]
\[ Z_{\beta_0}(0 \mid 0, 1)_{x_1x_2} = Z_{\beta_0+f}(0 \mid 0, -1)_{x_1x_2} (t_2 - t_0)\phi^{-1}. \]  

(11)

Attaching the level \( (0, 0) \) cap to the level \( (0, 1) \) tube, we get three relations:
0 = Z_{β_0}(0 | 0, 1)^x_0 + Z_{β_0}(0 | 0, 1)^x_1 + Z_{β_0}(0 | 0, 1)^x_2

for \( a = 0, 1, 2 \). We already know that

\[
Z_{β_0}(0 | 0, 1)^x_0 = Z_{β_0}(0 | 0, 1)^x_1 = 0,
\]

so we get

\[
Z_{β_0}(0 | 0, 1)^x_2 = (t_2 - t_1)φ;
\]

\[
Z_{β_0}(0 | 0, 1)^x_1 = (t_2 - t_0)φ.
\]

Combining with (11) we find

\[
0 = Z_{β_0+f}(0 | 0, -1)^x_0 = Z_{β_0+f}(0 | 0, -1)^x_1 = 0.
\]

Finally, in order to find

\[
Z_{β_0+f}(0 | 0, -1)^x_2 = φ^2,
\]

(13)

we attach the level (0, 1) tube to the level (0, -1) tube to obtain the class \( β_0 + f \), level (0, 0) tube:

\[
0 = Z_{β_0+f}(0 | 0, 0)^x_1 = Z_{β_0+f}(0 | 0, 1)^x_2
\]

for \( a = 0, 1 \). This implies that

\[
Z_{β_0+f}(0 | 0, -1)^x_0 = φ^2,
\]

\[
Z_{β_0+f}(0 | 0, -1)^x_1 = φ^2.
\]

(14)

And similarly,
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\[
\begin{align*}
0 &= Z_{\beta_0 + f}(0 \, | \, 0, 0)_{x_0 x_1} = Z_{\beta_0}(0 \, | \, 0, 1)_{x_0 x_0} Z_{\beta_0 + f}(0 \, | \, 0, -1)_{x_0}^x \\
&\quad + Z_{\beta_0}(0 \, | \, 0, 1)_{x_0 x_2} Z_{\beta_0 + f}(0 \, | \, 0, -1)_{x_1}^x,
\end{align*}
\]

which implies that

\[
Z_{\beta_0 + f}(0 \, | \, 0, -1)_{x_0 x_1} = \phi^2. \tag{15}
\]

We now summarize (9-15) into the following lemma:

**Lemma 3.2.3.** Partition functions for the class \(\beta_0\), level \((0, 1)\) and \((1, 0)\) tubes, and also for the class \(\beta_0 + f\), level \((0, -1)\) and \((-1, 0)\) tubes are given by

\[
\begin{align*}
[Z_{\beta_0}(0 \, | \, 0, 1)_{x_a x_b}] &= \begin{bmatrix}
t_0 - t_1 & 0 & t_2 - t_1 \\
0 & t_1 - t_0 & t_2 - t_0 \\
t_2 - t_1 & t_2 - t_0 & 2t_2 - t_0 - t_1
\end{bmatrix} \phi, \\
[Z_{\beta_0}(0 \, | \, 1, 0)_{x_a x_b}] &= \begin{bmatrix}
t_0 - t_2 & t_1 - t_2 & 0 \\
t_1 - t_2 & 2t_1 - t_0 - t_2 & t_1 - t_0 \\
0 & t_1 - t_0 & t_2 - t_0
\end{bmatrix} \phi, \\
[Z_{\beta_0 + f}(0 \, | \, 0, -1)_{x_a x_b}] &= \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix} \phi^2, \\
[Z_{\beta_0 + f}(0 \, | \, -1, 0)_{x_a x_b}] &= \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix} \phi^2,
\end{align*}
\]

for \(a, b \in \{0, 1, 2\}\), where partition functions with the index \(x_a x_b\) are the \((a + 1, b + 1)\) entry of the matrices above.
3.2.3 computing class $\beta_0 + f$, level $(0,0)$ pants:

Lemma 3.2.4. Partition functions for the class $\beta_0 + f$, level $(0,0)$ pants are given by

$$Z_{\beta_0 + f}(0 | 0, 0)_{x_0x_1x_2} = 0,$$
$$Z_{\beta_0 + f}(0 | 0, 0)_{x_0x_2x_2} = (t_2 - t_1)\phi^3,$$
$$Z_{\beta_0 + f}(0 | 0, 0)_{x_1x_2x_2} = (t_2 - t_0)\phi^3,$$
$$Z_{\beta_0 + f}(0 | 0, 0)_{x_0x_0x_2} = (t_1 - t_0)\phi^3,$$
$$Z_{\beta_0 + f}(0 | 0, 0)_{x_1x_1x_2} = (t_0 - t_1)\phi^3,$$
$$Z_{\beta_0 + f}(0 | 0, 0)_{x_2x_2x_2} = (2t_2 - t_0 - t_1)\phi^3,$$
$$Z_{\beta_0 + f}(0 | 0, 0)_{x_0x_1x_1} = (t_1 - t_2)\phi^3,$$
$$Z_{\beta_0 + f}(0 | 0, 0)_{x_0x_0x_1} = (t_0 - t_2)\phi^3,$$
$$Z_{\beta_0 + f}(0 | 0, 0)_{x_0x_0x_0} = (2t_0 - t_1 - t_2)\phi^3,$$
$$Z_{\beta_0 + f}(0 | 0, 0)_{x_1x_1x_1} = (2t_1 - t_0 - t_2)\phi^3,$$

for $a, b \in \{0,1,2\}$.

Proof: We attach the level $(0,1)$ cap to the level $(0,0)$ pants to obtain the class $\beta_0$, level $(0,1)$ tube. Applying the gluing formula together with Lemma 3.1.3 and 3.2.3 we get the following relation

$$Z_{\beta_0 + f}(0 | 0, 0)_{x_0x_1x_2} = Z_{\beta_0 + f}(0 | 0, 0)_{x_1x_2x_2},$$

From this we can get all $Z_{\beta_0 + f}(0 | 0, 0)_{x_0x_1x_2}$ with at least one of $a, b, c$ is equal to 2.

If we attach the level $(0,-1)$ cap to the level $(0,0)$ pants to obtain the class $\beta + f$, level $(0,-1)$ tube we will get

$$Z_{\beta_0 + f}(0 | 0, 0)_{x_0x_1x_2} = Z_{\beta_0 + f}(0 | 0, 0)_{x_1x_2x_2}.$$
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\[ Z_{\beta_0+f}(0|0,0)_{x_0x_0x_1} - Z_{\beta_0+f}(0|0,0)_{x_0x_1x_0} = (t_0 - t_1)\phi^3, \]
\[ Z_{\beta_0+f}(0|0,0)_{x_0x_0x_0} - Z_{\beta_0+f}(0|0,0)_{x_0x_0x_1} = (t_0 - t_1)\phi^3, \]
\[ Z_{\beta_0+f}(0|0,0)_{x_0x_1x_1} - Z_{\beta_0+f}(0|0,0)_{x_1x_1x_0} = (t_0 - t_1)\phi^3. \] 

(16)

We now write a Frobenius relation as follows

\[ 0 = Z_{\beta_0+f}(0|0,0)_{x_0x_1x_1} Z_{\beta_0}(0|0,0)_{x_0x_0} + Z_{\beta_0}(0|0,0)_{x_1x_1x_1} Z_{\beta_0+f}(0|0,0)_{x_0x_0}, \]
where the left hand side is zero by Lemma 3.1.1.

By Lemma 3.1.1, this simplifies to

\[ (t_0 - t_2)Z_{\beta_0+f}(0|0,0)_{x_0x_1x_1} = (t_1 - t_2)Z_{\beta_0+f}(0|0,0)_{x_0x_0x_0}. \]

Combining this with (16), we will find the rest of the partition functions in the lemma.

We now know all the partition functions of pants, so we are able to prove

the semisimplicity of the TQFT (see Remark 2.7):

**Proposition 3.2.5.** The level \((0,0)\) TQFT, resulted from our setting and Theorem 2.5 is semisimple.

**Proof:** By Lemma 3.1.1 and 3.2.4 we have

\[ Z(0|0,0)_{x_a x_b} |_{u=0} = Z_{\beta_0}(0|0,0)_{x_a x_b} = \begin{cases} T(x_a) & \text{if } a = b = c, \\ 0 & \text{otherwise.} \end{cases} \]

This means that for \(u = 0\), the basis \(\{ \frac{e_{x_0}}{T(x_0)}, \frac{e_{x_1}}{T(x_1)} \} \) of the corresponding Frobenius algebra (see Remark 2.7) is idempotent:

\[ \frac{e_{x_i}}{T(x_i)} \otimes \frac{e_{x_j}}{T(x_j)} = \delta_{ij} \frac{e_{x_i}}{T(x_i)}. \]

This proves the semisimplicity when \(u = 0\) (see [1], Section 2). Now the proposition follows from Proposition 2.2 in [1].
4 Proof of Theorem 1.6

We now know everything we need in order to prove Theorem 1.6. We first find the first and second level creation operators and also genus adding operator which are by definition

\[
U_1 = [Z(0 \mid 1, 0)^{x_b}_{x_a}],
\]

\[
U_2 = [Z(0 \mid 0, 1)^{x_b}_{x_a}],
\]

\[
G = [Z(1 \mid 0, 0)^{x_b}_{x_a}],
\]

respectively. Here partition functions with the lower index \(x_a\) and the upper index \(x_b\) are the \((a + 1, b + 1)\) entry of the matrices above.

We can find \(U_1\) and \(U_2\) by simply raising the indices in Lemma 3.2.2 and

\[3.2.3\]

\[
U_1 = \begin{bmatrix}
0 & 0 & 0 \\
0 & (t_1 - t_0)(t_1 - t_2) & 0 \\
0 & 0 & 0
\end{bmatrix} \phi^{-2} + \begin{bmatrix}
\frac{1}{t_0-t_1} & \frac{1}{t_0-t_2} & 0 \\
\frac{1}{t_1-t_0} & \frac{1}{t_1-t_2} & 0 \\
0 & 0 & 1/t_2-t_1
\end{bmatrix} \phi
\]

\[
U_2 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & (t_2 - t_0)(t_2 - t_1)
\end{bmatrix} \phi^{-2} + \begin{bmatrix}
\frac{1}{t_0-t_2} & \frac{1}{t_0-t_1} & 0 \\
\frac{1}{t_1-t_2} & \frac{1}{t_1-t_0} & 0 \\
\frac{1}{t_2-t_0} & \frac{1}{t_2-t_1} & 1/t_2-t_1
\end{bmatrix} \phi
\]

Equation (17)

By Lemma 3.1.1 \([Z_{\beta_0}(0 \mid 0, 0)^{x_b}_{x_a}]\) is the identity matrix, and by the gluing formula we can write

\[
[Z(0 \mid 0, 0)^{x_b}_{x_a}] = [Z(0 \mid 0, 1)^{x_b}_{x_a}] [Z(0 \mid 0, -1)^{x_b}_{x_a}],
\]

\[
[Z(0 \mid 0, 0)^{x_b}_{x_a}] = [Z(0 \mid 1, 0)^{x_b}_{x_a}] [Z(0 \mid -1, 0)^{x_b}_{x_a}].
\]
Therefore, by Lemma 3.2.1 and 3.2.3 we have
\[
U_1^{-1} = [Z_{\beta_0}(0 \mid 1, 0)^{x_b}_{x_a}] + [Z_{\beta_0+f}(0 \mid 1, 0)^{x_b}_{x_a}]
\]
\[
= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \phi^{-1}
\]
\[
+ \begin{bmatrix} \frac{1}{(t_0-t_1)(t_0-t_2)} & \frac{1}{(t_0-t_1)(t_0-t_2)} & \frac{1}{(t_0-t_1)(t_0-t_2)} \\ \frac{1}{(t_1-t_0)(t_1-t_2)} & \frac{1}{(t_1-t_0)(t_1-t_2)} & \frac{1}{(t_1-t_0)(t_1-t_2)} \\ \frac{1}{(t_0-t_0)(t_2-t_1)} & \frac{1}{(t_0-t_0)(t_2-t_1)} & \frac{1}{(t_0-t_0)(t_2-t_1)} \end{bmatrix} \phi^2
\]
\[
= \begin{bmatrix} (t_0-t_1)(t_0-t_2) & (t_0-t_1)(t_0-t_2) & (t_0-t_1)(t_0-t_2) \\ (t_1-t_0)(t_1-t_2) & (t_1-t_0)(t_1-t_2) & (t_1-t_0)(t_1-t_2) \\ (t_0-t_0)(t_2-t_1) & (t_0-t_0)(t_2-t_1) & (t_0-t_0)(t_2-t_1) \end{bmatrix} \phi^{-1} + \phi^2
\]
\[
U_2^{-1} = [Z_{\beta_0}(0 \mid 0, -1)^{x_b}_{x_a}] + [Z_{\beta_0+f}(0 \mid 0, -1)^{x_b}_{x_a}]
\]
\[
= \begin{bmatrix} 0 & 0 & t_0 \\ 0 & 0 & t_1 \\ 1 & 0 & 0 \end{bmatrix} \phi^{-1}
\]
\[
+ \begin{bmatrix} \frac{1}{(t_0-t_1)(t_0-t_2)} & \frac{1}{(t_0-t_1)(t_0-t_2)} & \frac{1}{(t_0-t_1)(t_0-t_2)} \\ \frac{1}{(t_1-t_0)(t_1-t_2)} & \frac{1}{(t_1-t_0)(t_1-t_2)} & \frac{1}{(t_1-t_0)(t_1-t_2)} \\ \frac{1}{(t_0-t_0)(t_2-t_1)} & \frac{1}{(t_0-t_0)(t_2-t_1)} & \frac{1}{(t_0-t_0)(t_2-t_1)} \end{bmatrix} \phi^2
\]
\[
= \begin{bmatrix} (t_0-t_1)(t_0-t_2)^2 & (t_0-t_1)(t_0-t_2)^2 & (t_0-t_1)(t_0-t_2)^2 \\ (t_1-t_0)(t_1-t_2)^2 & (t_1-t_0)(t_1-t_2)^2 & (t_1-t_0)(t_1-t_2)^2 \\ (t_0-t_0)(t_2-t_1)^2 & (t_0-t_0)(t_2-t_1)^2 & (t_0-t_0)(t_2-t_1)^2 \end{bmatrix} \phi^{-1} + \phi^2 \tag{18}
\]

$U_1^{-1}$ and $U_2^{-1}$ are the first and second level annihilation operators, respectively.

Now we are going to find the matrix $G$. By the same argument as one given for Lemma 3.4.1 one can prove that
\[
Z(1 \mid 0, 0)^{x_a}_{x_b} = Z_{\beta_0}(1 \mid 0, 0)^{x_a}_{x_b} + Z_{\beta_0+f}(1 \mid 0, 0)^{x_a}_{x_b}
\]
for $a, b \in \{0, 1, 2\}$. Thus we have
\[
G = [Z_{\beta_0}(1 \mid 0, 0)^{x_b}_{x_a}] + [Z_{\beta_0+f}(1 \mid 0, 0)^{x_b}_{x_a}].
\]

For calculating the terms in the right hand side of this, we attach two pants at two points (see the picture), and apply the gluing formula:
which implies that

\[
Z_{\beta_0}(1|0,0)_{x_a x_a} = Z_{\beta_0}(0|0,0)_{x_a x_a} Z_{\beta_0}(0|0,0)_{x_a x_a},
\]

\[
Z_{\beta_0+f}(1|0,0)_{x_a x_a} = Z_{\beta_0}(0|0,0)_{x_a x_a} Z_{\beta_0+f}(0|0,0)_{x_a x_a} + Z_{\beta_0+f}(0|0,0)_{x_a x_a} Z_{\beta_0}(0|0,0)_{x_a x_a},
\]

Now to prove the formula in Theorem 1.6 we glue a chain of \( g - 1 \) genus one, level \((0,0)\) cobordisms from a circle to another circle, which are attached subsequently at their ends to a chain of \( k_1 \) level \((1,0)\) and \( k_2 \) level \((0,1)\) tubes, and after all, two ends of the resulting chain are also attached to each other (see the picture below).
In this picture, objects with the same mark at their end are attached to each other. Applying the gluing formula each time we glue along a fiber in the procedure above, we obtain the formula in Theorem 1.6:

\[ Z(g | k_1, k_2) = \text{tr} \left( G^{g-1} U_1^{k_1} U_2^{k_2} \right). \]

Thus we have proven Theorem 1.6 in the case where \( g \geq 1 \). The same formula holds for \( g = 0 \), which follows from the semisimplicity of the level (0,0) TQFT (see Remark 2.7 and Theorem 3.2.5).

5 Proof of Theorem 1.7

We begin by outlining our proof of Theorem 1.7.

We first prove that any \( \mathbb{P}^2 \)-bundle over a curve \( C \) is deformation equivalent to

\[ \mathbb{P}(\mathcal{O} \oplus \mathcal{O} \oplus L) \to C. \]

This is the space for which we obtained a formula for finding the section class equivariant Gromov-Witten invariants (Theorem 1.6). We will expand this formula and look for those terms which are corresponding to the Calabi-Yau section classes. There are lots of such terms, but we will see that all of them except three terms are zero for some reason; all equivariant variable will cancel each other and these three terms will add up to give the formula in the theorem.

We first prove the following lemma:
Lemma 5.1. Any \( \mathbb{P}^2 \)-bundle over a curve \( C \) is deformation equivalent to

\[
\mathbb{P}(\mathcal{O} \oplus \mathcal{O} \oplus L) \to C,
\]

where \( \mathcal{O} \to C \) is the trivial bundle and \( L \to C \) is a line bundle of degree \( k \).

Proof: First, we show that every \( \mathbb{P}^2 \)-bundle over a curve \( C \) is of the form

\[
\mathbb{P}(E) \to C,
\]

where \( E \to C \) is a rank 3 bundle.

A rank three vector bundle (resp. \( \mathbb{P}^2 \)-bundle) over \( C \) is classified by an element in \( \check{H}^1(C, \text{Gl}(3)) \) (resp. \( \check{H}^1(C, \text{PGl}(3)) \)), where \( \text{Gl}(3) \) (resp. \( \text{PGl}(3) \)) is the (non-abelian) sheaf of \( \text{Gl}(3) \) (resp. \( \text{PGl}(3) \)) valued holomorphic functions on \( C \). From the exact sequence of sheaves

\[
0 \to \mathcal{O}^* \to \text{Gl}(3) \to \text{PGl}(3) \to 0,
\]

we get a map

\[
\check{H}^1(C, \text{PGl}(3)) \to \check{H}^2(C, \mathcal{O}^*).
\]

By examining the cocycles, one can see that a \( \mathbb{P}^2 \)-bundle over \( C \) is of the form \( \mathbb{P}(E) \to C \) if and only if the corresponding element in \( \check{H}^1(C, \text{PGl}(3)) \) goes to zero under the above map. This element is represented by the Čech cocycle obtained from the transition functions of the bundle. But

\[
\check{H}^2(C, \mathcal{O}^*) = 0
\]

for a curve \( C \); this completes the first part of the proof of the lemma.

Next, we show that \( \mathbb{P}(E) \to C \) is deformation equivalent to

\[
\mathbb{P}(\mathcal{O} \oplus \mathcal{O} \oplus L) \to C.
\]

It is an standard fact that for a rank 3 bundle \( E \) over a curve we have the following exact sequence of bundles over \( C \) (see [5], Example 5.0.1):

\[
0 \to \mathcal{O} \oplus \mathcal{O} \to E(m) \to L \to 0,
\]

for some line bundle \( L \) and some \( m \gg 0 \) such that \( E(m) \) is globally generated by its sections. Thus \( E(m) \) corresponds to an element

\[
v \in \text{Ext}^1(L, \mathcal{O} \oplus \mathcal{O}).
\]
We can deform $E(m)$ by deforming the extension class $v$ to 0 inside this vector space. But $0 \in \text{Ext}^1(L, \mathcal{O} \oplus \mathcal{O})$ corresponds to

$$\mathcal{O} \oplus \mathcal{O} \oplus L \rightarrow C.$$ 

So we have proved that $E(m)$ is deformation equivalent to $\mathcal{O} \oplus \mathcal{O} \oplus L$. Now we use the isomorphism

$$\mathbb{P}(E) \cong \mathbb{P}(E(m))$$

to complete the proof of the lemma.

By this lemma, we can assume that the space $X$ in the theorem is of the form

$$\mathbb{P}(\mathcal{O} \oplus \mathcal{O} \oplus L) \rightarrow C.$$ 

For simplicity, we use the following notations in this section:

\begin{align*}
A &= \left[ Z_{\beta_0}(1 \mid 0, 0)_{x_0} \right] \text{ given by (19)}, \\
B &= \left[ Z_{\beta_0 + f}(1 \mid 0, 0)_{x_0} \right] \text{ given by (20)}, \\
C &= \left[ Z_{\beta_0 - f}(0 \mid 0, 1)_{x_0} \right] \text{ given by (17)}, \\
E &= \left[ Z_{\beta_0}(0 \mid 0, 1)_{x_0} \right] \text{ given by (17)}, \\
N &= \left[ Z_{\beta_0}(0 \mid 0, -1)_{x_0} \right] \text{ given by (18)}, \\
M &= \left[ Z_{\beta_0 + f}(0 \mid 0, -1)_{x_0} \right] \text{ given by (18)}.
\end{align*}

Then we have

\begin{align*}
G &= A + B, \\
U_2 &= C + E, \\
U_2^{-1} &= N + M,
\end{align*}

and we can write the formula in Theorem 1.6 for $k_1 = 0$ and $k_2 = k$ as follows:

$$Z(g \mid 0, k) = \text{tr} \left( (A + B)^{g-1}(C + E)^{k} \right).$$

Now we are looking for those terms in this formula that correspond to Calabi-Yau section class. If we denote this class by

$$\beta_{cs} = \beta_0 + nf,$$
then \( n \) must satisfy this equation:

\[
K_x \cdot \beta = 0 \quad \Rightarrow
\]

\[
2g - 2 - k - 3n = 0.
\]

If for given \( g \) and \( k \), there is an integral solution for \( n \) in this equation then the Calabi-Yau class exists. We write the above equation in terms of \( n \) instead of \( k \):

\[
Z(g \mid 0, k) = \text{tr} \left( (A + B)^{g-1}(C + E)^{2g-2-3n} \right).
\]

Now by the gluing formula, \( G = A + B \) commutes with \( U_2 = C + E \), so we have

\[
Z(g \mid 0, k) = \text{tr} \left( ((A + B)(C + E)^2)^{g-1}(C + E)^{-3n} \right). \tag{21}
\]

**Notation.** For two matrices \( U \) and \( V \), by \((U^a, V^b)\) for \( a, b \in \mathbb{Z}^+ \), we mean the sum of all products that we can write containing \( a \) copies of \( U \) and \( b \) copies of \( V \). For example

\[
(U^2, V) = U^2V + UVU + VU^2.
\]

We first assume that \( g > 0 \). We distinguish two cases:

(i) \( n < 0 \)

One can see that \( E^3 = 0 \) and \( BE^2 = 0 \), so we have

\[
Z(g \mid 0, k) = \text{tr} \left( ((A + B)(C^2 + E^2 + (E, C)))^{g-1}((E^2, C) + (E, C^2) + C^3)^{-n} \right)
\]

\[
\quad = \text{tr} \left( (AE^2 + B(E, C) + \ldots)^{g-1}((E^2, C) + \ldots)^{-n} \right).
\]

\( A, B, C \) and \( E \) correspond to the class \( \beta_0, \beta_0 + f, \beta_0 - f, \) and \( \beta_0 \), respectively. One can see that the only those terms that have been written in the last equality above contribute to make the class \( \beta_{cs} = \beta_0 + nf \). Thus

\[
Z_{\beta_{cs}}(g \mid 0, k)
\]

\[
= \text{tr} \left( (AE^2 + B(E, C))^{g-1}((E^2, C))^{-n} \right)
\]

\[
= \text{tr} \left( (AE^2 + BEC + BCE)^{g-1}(E^2C + CE^2 + ECE)^{-n} \right)
\]

\[
= \text{tr} \left( (AE^2)^{g-1}(CE^2)^{-n} \right) + \text{tr} \left( (BEC)^{g-1}(E^2C)^{-n} \right)
\]

\[
+ \text{tr} \left( (BCE)^{g-1}(ECE)^{-n} \right). \tag{22}
\]
For the last equality we only used the fact that
\[ E^3 = BE^2 = 0 \]
again and also
\[ \text{tr}(UV) = \text{tr}(VU), \]
for any two matrices \( U \) and \( V \).
Now one can see easily by induction that for any nonnegative integer \( a \)
\[ AE^2 (CE^2)^a = AE^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \phi^2, \]
therefore the first term in (22) is
\[ \text{tr} \left( (AE^2)^{g-1}(CE^2)^{-n} \right) = \text{tr} \left( \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{g-1} \phi^{2g-2} \right) = 3^{g-1} \phi^{2g-2}. \tag{23} \]
Again induction on nonnegative integers \( a, b \) together with simple calculations imply that
\[ (BEC)^b(E^2C)^a = \begin{bmatrix} 0 & 0 & \frac{t_1-t_2}{t_0-t_2} \\ 0 & 0 & \frac{t_0-t_2}{t_1-t_0} \\ 0 & 0 & 1 \end{bmatrix} \phi^b, \]
therefore the the second term in (22) is
\[ \text{tr} \left( (BEC)^{g-1}(E^2C)^{-n} \right) = 3^{g-1} \phi^{2g-2}. \tag{24} \]
Powers of \( BCE \) are more difficult to compute, so for computing the third term in (22) we first notice that
\[ CEB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 3 & 3 \end{bmatrix} \phi^2, \]
and also
\[ (ECE)^a = ECE \]
for any positive integer $a$, so for $b > 1$, we can write

$$(BCE)^b(ECE)^a = (B(CEB)^{b-1}CE)(ECE) = 3^{b-1}(BCE)(ECE).$$

Easy calculation shows that

$$\text{tr}((BCE)(ECE)) = \text{tr}(BCE) = 3\phi^2.$$ 

Putting all together, we can find the third term in (22):

$$\text{tr}((BCE)^{g-1}(ECE)^{-n}) = 3^{g-1}\phi^{2g-2}. \quad (25)$$

By (22)-(25), we find

$$Z_{\beta cs}(g | 0, k) = 3^g\phi^{2g-2},$$

which proves the theorem in this case.

(ii) $n \geq 0$

We have $U_2^{-1} = M + N$, so we can rewrite (21) as

$$Z(g | 0, k) = \text{tr}\left( ((A + B)(C + E)^2)^{g-1}(M + N)^{3n}\right).$$

One can check that $(M^2, N) = M^3 = 0$, so

$$Z(g | 0, k) = \text{tr}\left( (AE^2 + B(E, C) + \ldots)^{g-1}(M, N^2 + N^3)^{n}\right).$$

By the same reason as in the last case we have

$$Z_{\beta cs}(g | 0, k) = \text{tr}\left( (AE^2)^{g-1}(N^2M)^n\right) + \text{tr}\left( (BEC)^{g-1}(NMN)^n\right) + \text{tr}\left( (BCE)^{g-1}(MN^2)^n\right). \quad (26)$$

Easy calculations shows that

$$BM = MB = 0,$$

and for any positive integer $a$

$$(N^2M)^a = N^2M,$$

$$(MN^2)^a = MN^2,$$

$$(NMN)^a = MN,$$

$$(AE^2)^a(N^2M)^a = AE^2,$$
and also
\[ \text{tr}((BCE)(NMN)) = 3\phi^2, \]
\[ \text{tr}((BEC)(MN^2)) = 3\phi^2. \]

Now calculations similar to the last case together with these equalities imply that each term in the right hand side of (26) is equal to
\[ 3^{g-1}\phi^{2g-2}, \]
and this proves the theorem in this case.

For \( g = 0 \), the result is deduced from the semisimplicity of the TQFT (Corollary 3.2.5, see also Remark 2.7). \( \square \)

A Proof of the Gluing theorem

We first prove the assertion of Remark 2.6, which deals with the fact that we don’t need to consider maps with disconnected domains:

**Lemma A.1.** The contribution of maps with disconnected domain curves in the section class equivariant Gromov-Witten invariants of the space
\[ \mathbb{P}(\mathcal{O} \oplus L_1 \oplus L_2) \]
is zero.

**Proof:** A disconnected domain curve whose image represents the class \( \beta_0 + nf \) is a union of some connected components such that at least the image of one of them represents the class \( n'f \) for a positive \( n' \). We have
\[ \text{virdim} \mathcal{M}(X/\overline{F}, n'f) = -(-3H + (2g - 2 - k_1 - k_2)F) \cdot n'f = 3n' > 0, \]
so by a discussion similar to Remark 2.2 one can see that
\[ \int_{[\mathcal{M}_h(X/\overline{F},k')]}^{\text{vir}} 1 = 0. \]
Disconnected invariants can be expressed in terms of the product of connected ones, and so the lemma follows. \( \square \)
Now we return to the proof of Theorem 3.2. We prove the first formula, the proof of the second one is similar. For simplicity we prove the case \( s = 0 \) and \( t = 0 \). Extending the argument to the general case is straightforward.

Let \( C_0 \) be a connected curve of genus \( g \) with two irreducible components, \( C' \) and \( C'' \) of genera \( g' \) and \( g'' \) which are attached together at one point \( p \).

In other words

\[
C_0 = C' \bigcup_{p = p' = p''} C'' ,
\]

where \( p' \in C' \) and \( p'' \in C'' \). Now we consider two \( \mathbb{P}^2 \)-bundles

\[
X' = \mathbb{P}(\mathcal{O} \oplus L'_1 \oplus L'_2) \to C',
\]

and

\[
X'' = \mathbb{P}(\mathcal{O} \oplus L''_1 \oplus L''_2) \to C'',
\]

where \( L'_1, L'_2, L''_1 \) and \( L''_2 \) are line bundles of degrees \( k'_1, k'_2, k''_1 \) and \( k''_2 \), respectively. We attach these two spaces by identifying the fibers, \( F' \) and \( F'' \) over \( p' \) and \( p'' \), such that the resulting space is

\[
W_0 = \mathbb{P}(\mathcal{O} \oplus L_1 \oplus L_2) \to C_0 ,
\]

where \( L_1 \) and \( L_2 \) are line bundles of degrees \( k_1 = k'_1 + k'_2, k_2 = k''_1 + k''_2 \), respectively. In other words

\[
W_0 = X' \bigcup_{F = F' = F''} X'',
\]

where \( F \) is the fiber over \( p \).

Let \( W \to \mathbb{A}^1 \) be a generic, 1-parameter deformation of \( W_0 \) for which the fibers \( W_t \) for \( t \neq 0 \in \mathbb{A}^1 \) are

\[
\mathbb{P}(\mathcal{O} \oplus L_1 \oplus L_2) \to C,
\]

where \( C \) is a smooth curve of genus \( g \), and \( L_1 \) and \( L_2 \) are line bundles of degrees \( k_1, k_2 \).

We follow Jun Li’s proof of the degeneration formula in [7]. Let \( \mathcal{M} \) be the stack of expanded degenerations of \( W \), with central fiber \( \mathcal{M}_0 \), and let \( \overline{M}_h(\mathcal{M}, \beta) \) be the stack of non-degenerate, pre-deformable, genus \( h \), class \( \beta \) maps to \( \mathcal{M} \), where \( \beta \) is a section class (see [6]).
We have the evaluation maps

\[ \text{ev}': \overline{M}_{h'}(X'/F', \beta') \to F' = F, \]

and

\[ \text{ev}''': \overline{M}_{h''}(X'/F'', \beta'') \to F'' = F. \]

Li constructs a map

\[ \Phi_\eta : \overline{M}(X'/F', \beta') \times_F \overline{M}(X''/F'', \beta'') \to \overline{M}(\mathfrak{W}_0, \beta), \]

where \( \eta \) includes a pair of classes \((\beta', \beta'')\), such that \( \beta = \beta' + \beta'' \) and a pair of genera \((h', h'')\), such that \( h = h' + h'' \). Then he gives a virtual cycle formula, which in our case is

\[ [\overline{M}(\mathfrak{W}_0, \beta)]^{\text{vir}} = \sum_\eta (\Phi_\eta)_* \Delta^!([\overline{M}(X'/F', \beta')]^{\text{vir}} \times [\overline{M}(X''/F'', \beta'')]^{\text{vir}}), \quad (27) \]

where \( \Delta : F \to F \times F \) is the diagonal map.

**Remark A.2.** The torus action on the family \( W \to \mathbb{A}^1 \) gives an action on the stack of expanded degeneration, \( \mathfrak{W} \). One can check that pre-deformability condition is invariant under this action, so it induces (canonically) an action on each of the moduli spaces \( \overline{M}_{h'}(X'/F', \beta') \), \( \overline{M}_{h''}(X''/F'', \beta'') \) and \( \overline{M}_h(\mathfrak{W}, \beta) \). Therefore Li’s formula holds in the equivariant Chow groups.

If we work with the basis elements \( x_0, x_1, x_2 \) (introduced in Section 2), for the equivariant cohomology of the fiber \( F \), by using (22), one can see easily that

\[ x_0^\vee = \frac{x_0}{T(x_0)}, \quad x_1^\vee = \frac{x_1}{T(x_1)}, \quad x_2^\vee = \frac{x_2}{T(x_2)}, \]

is its dual basis, so the cohomology class of the diagonal of \( F \times F \) is given by (see [8], Theorem 11.11)

\[ \text{im}(\Delta) = \sum_{i=0}^{2} x_i \times x_i^\vee \]

\[ = x_0 \times \frac{x_0}{T(x_0)} + x_1 \times \frac{x_1}{T(x_1)} + x_0 \times \frac{x_2}{T(x_2)}. \]
Using this, we can rewrite (27) as
\[
[M(W_0, \beta)]^{\text{vir}} = \sum_{\eta} \left( \sum_{i=0}^2 (x_i \cap [M(X'/F', \beta')]^{\text{vir}}) \times \frac{x_i}{T(x_i)} \cap [M(X''/F'', \beta'')]^{\text{vir}} \right).
\]

We now have
\[
Z_{\beta}(g | k_1, k_2) = \int_{[M(W_{\beta}, \beta)]^{\text{vir}}} \frac{1}{[M(W_{\beta}, \beta)]^{\text{vir}}}
= \int \sum_{\eta} \sum_{i=0}^2 \int_{[M(X'/F', \beta')]^{\text{vir}}} x_i \int_{[M(X''/F'', \beta'')]^{\text{vir}}} \frac{x_i}{T(x_i)}
= \sum_{\eta} \sum_{i=0}^2 Z_{\beta'}(g' | k_1', k_2') x_i Z_{\beta''}(g'' | k_1'', k_2'') x_i.
\]

Then we can write
\[
Z(g | k_1, k_2) = \sum_{\beta} \sum_{h} u^{2h-2-K_{X} \cdot \beta} Z_{\beta}(g | k_1, k_2)
= \sum_{\beta} \sum_{h} u^{2h-2-K_{X} \cdot \beta} \sum_{\eta} \sum_{i} Z_{\beta'}(g' | k_1', k_2') x_i Z_{\beta''}(g'' | k_1'', k_2'') x_i
= \sum_{\beta, h, \eta, i} u^{2h'-2-K_{X'} \cdot \beta'} Z_{\beta'}(g' | k_1', k_2') x_i u^{2h''-2-K_{X''} \cdot \beta''} Z_{\beta''}(g'' | k_1'', k_2'') x_i
= \sum_{i} Z(g' | k_1', k_2') x_i Z(g'' | k_1'', k_2'') x_i.
\]

B Some special cases

In this Appendix, we apply Theorem 1.6 and drive formulas for some special partition functions of the equivariant Gromov-Witten invariants of the space
\[
P(O \oplus L_1 \oplus L_2) \to C,
\]
where $C$ is smooth curve of genus $g$. 
Similar to Section 5, we use the following notations for simplicity:

\[ A = \begin{bmatrix} Z_{\beta_0}(1 | 0, 0)_{x_b} \\ \end{bmatrix}, \quad B = \begin{bmatrix} Z_{\beta_0+f}(1 | 0, 0)_{x_b} \\ \end{bmatrix}, \]
\[
C_1 = \begin{bmatrix} Z_{\beta_0-f}(0 | 1, 0)_{x_b} \\ \end{bmatrix}, \quad C_2 = \begin{bmatrix} Z_{\beta_0-f}(0 | 0, 1)_{x_b} \\ \end{bmatrix},
\]
\[
E_1 = \begin{bmatrix} Z_{\beta_0}(0 | 1, 0)_{x_b} \\ \end{bmatrix}, \quad E_2 = \begin{bmatrix} Z_{\beta_0}(0 | 0, 1)_{x_b} \\ \end{bmatrix},
\]
\[
N_1 = \begin{bmatrix} Z_{\beta_0}(0 | -1, 0)_{x_b} \\ \end{bmatrix}, \quad N_2 = \begin{bmatrix} Z_{\beta_0}(0 | 0, -1)_{x_b} \\ \end{bmatrix},
\]
\[
M_1 = \begin{bmatrix} Z_{\beta_0+f}(0 | -1, 0)_{x_b} \\ \end{bmatrix}, \quad M_2 = \begin{bmatrix} Z_{\beta_0+f}(0 | 0, -1)_{x_b} \\ \end{bmatrix}.
\]

These matrices are given by (17), (18), (19) and (20).

For proving the theorems in this section, we first write the formula in Theorem 1.6 in each case, in terms of the matrices above. After expanding it we will get a polynomial in \( t_0, t_1 \) and \( t_2 \) (see Remark 2.2). We then try to compute the specific term of that polynomial, which corresponds to the partition function that we are interested in. All the proofs that we are going to provide is for the case \( g > 0 \). The case \( g = 0 \) in each theorem, follows from the semisimplicity of the level \((0, 0)\) TQFT (Corollary 3.2.5, see also Remark 2.7).

**Theorem B.1.** Assume that \( k_1 > 0 \) and \( k_2 \geq 0 \). Then the class \( \beta_0 - k_1 f \), level \((k_1, -k_2)\) equivariant Gromov-Witten partition function of \( X \) is given by

\[
Z_{\beta_0-k_1 f}(g | k_1, -k_2) = (t_1 - t_0)^{g+k_1-1} (t_1 - t_2)^{g+k_1+k_2-1} \left( 2 \sin \frac{u}{2} \right)^{-2k_1-k_2}.
\]

**Proof:** By Theorem 1.6 we have

\[
Z(g | k_1, -k_2) = \text{tr} \left( (A + B)^{g-1} (C_1 + E_1)^{k_1} (M_2 + N_2)^{k_2} \right).
\]

After expanding this, one can see easily that the only term that corresponds to the class \( \beta_0 - k_1 f \) is \( A^{g-1} C_1^{k_1} N_2^{k_2} \), or in other words

\[
Z_{\beta_0-k_1 f}(g | k_1, -k_2) = \text{tr} \left( A^{g-1} C_1^{k_1} N_2^{k_2} \right).
\]

All three matrices \( A, C_1 \) and \( N_2 \) are diagonal, so one can easily compute the
right hand side of the above equation:

\[ A^{g-1} C_1^{k_1} N_2^{k_2} = \begin{bmatrix}
(t_0 - t_1)(t_0 - t_2) & 0 & 0 \\
0 & (t_1 - t_0)(t_1 - t_2) & 0 \\
0 & 0 & (t_2 - t_0)(t_2 - t_1)
\end{bmatrix}^{g-1}
\]

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & (t_1 - t_0)(t_1 - t_2)\phi^{-2} & 0 \\
0 & 0 & 0
\end{bmatrix}^{k_1}
\]

\[
\begin{bmatrix}
(t_0 - t_2)\phi^{-1} & 0 & 0 \\
0 & (t_1 - t_2)\phi^{-1} & 0 \\
0 & 0 & 0
\end{bmatrix}^{k_2}
\]

\[
= \begin{bmatrix}
0 & 0 & 0 \\
0 & (t_1 - t_0)^{g+k_1-1}(t_1 - t_2)^{g+k_1+k_2-1}\phi^{-2k_1-k_2} & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

so

\[
\text{tr} \left( A^{g-1} C_1^{k_1} N_2^{k_2} \right) = (t_1 - t_0)^{g+k_1-1}(t_1 - t_2)^{g+k_1+k_2-1}\phi^{-2k_1-k_2}.
\]

This prove the theorem. □

Similar to the theorem above one can prove

**Theorem B.2.** Assume that \( k_1 \geq 0 \) and \( k_2 > 0 \). Then the class \( \beta_0 - k_2 f \), level \((-k_1, k_2)\) equivariant Gromov-Witten partition function of \( X \) is given by

\[
Z_{\beta_0 - k_2 f}(g \mid -k_1, k_2) = (t_2 - t_0)^{g+k_2-1}(t_2 - t_1)^{g+k_1+k_2-1} \left( 2 \sin \frac{u}{2} \right)^{-2k_2-k_1}.
\]

□

Also we have the following theorem:

**Theorem B.3.** Assume that \( k > 0 \). Then the class \( \beta_0 - kf \), level \((k, k)\) equivariant Gromov-Witten partition function of \( X \) is given by

\[
Z_{\beta_0 - kf}(g \mid k, k) = \left( (t_1 - t_0)^{g+k-1}(t_1 - t_2)^{g-1} + (t_2 - t_0)^{g+k-1}(t_2 - t_1)^{g-1} \right) \left( 2 \sin \frac{u}{2} \right)^{-k}.
\]

**Proof:** By Theorem 1.6 we have

\[
Z(g \mid k, k) = \text{tr} \left( (A + B)^{g-1}(C_1 + E_1)^k(C_2 + E_2)^k \right)
= \text{tr} \left( (A + B)^{g-1}(E_1E_2 + C_1E_2 + E_1C_2)^k \right),
\]

because $C_1C_2=0$.

One can now check that
\[ Z_{\beta_0-k_f}(g \mid k, k) = \text{tr} \left( A^{g^{-1}}(C_1E_2 + E_1C_2)^k \right). \]

We also have
\[ C_1E_2 + E_1C_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_1-t_0 & 0 \\ 0 & 0 & t_2-t_0 \end{bmatrix} \phi^{-1}. \]

So we can write
\[ A^{g^{-1}}(C_1E_2 + E_1C_2)^k = \begin{bmatrix} (t_0-t_1)(t_0-t_2) & 0 & 0 \\ 0 & (t_1-t_0)(t_1-t_2) & 0 \\ 0 & 0 & (t_2-t_0)(t_2-t_1) \end{bmatrix}^{g^{-1}} \]
\[ \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_1-t_0 & 0 \\ 0 & 0 & t_2-t_0 \end{bmatrix}^k \phi^{-k} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & (t_1-t_0)^{g+k-1}(t_1-t_2)^{g-1} & 0 \\ 0 & 0 & (t_2-t_0)^{g+k-1}(t_2-t_1)^{g-1} \end{bmatrix} \phi^{-k}. \]

Thus
\[ \text{tr} \left( A^{g^{-1}}(C_1E_2 + E_1C_2)^k \right) = ((t_1-t_0)^{g+k-1}(t_1-t_2)^{g-1}
+ (t_2-t_0)^{g+k-1}(t_2-t_1)^{g-1}) \left( 2 \sin \frac{u}{2} \right)^{-k}. \]

This proves the theorem. \(\square\)

With the same method one can prove the following result:

**Theorem B.4.** Assume that $k_1 \geq 0$ and $k_2 \geq 0$. Then the class $\beta_0$, level
\((-k_1, -k_2)\) equivariant Gromov-Witten partition function of \(X\) is given by

\[
Z_{\beta_0}(g | -k_1, -k_2) =
\begin{cases}
(t_0 - t_1)^{g+k_1-1}(t_0 - t_2)^{g+k_2-1} \left( 2 \sin \frac{u}{2} \right)^{k_1+k_2} & k_1 > 0, k_2 > 0, \\
((t_0 - t_1)^{g+k_1-1}(t_0 - t_2)^{g-1} + (t_2 - t_0)^{g-1}(t_2 - t_1)^{g+k_1-1}) \left( 2 \sin \frac{u}{2} \right)^{k_1} & k_1 > 0, k_2 = 0, \\
((t_0 - t_1)^{g-1}(t_0 - t_2)^{g+k_2-1} + (t_1 - t_0)^{g-1}(t_1 - t_2)^{g+k_2-1}) \left( 2 \sin \frac{u}{2} \right)^{k_2} & k_1 = 0, k_2 > 0, \\
(t_0 - t_1)^{g-1}(t_0 - t_2)^{g-1} + (t_1 - t_0)^{g-1}(t_1 - t_2)^{g-1} & k_1 = 0, k_2 = 0.
\end{cases}
\]

Every theorem that we have already proved in this section is about computing the partition function for the section class \(\beta_0 + nf\), where \(n\) is the smallest possible integer such that there is curve in \(X\) that represents this class (see Remark 2.3 and Lemma 2.4). This partition function is the first term in the sum in Remark 2.3. The next theorem is about computing the partition function which is the last term in that sum. We restrict ourselves to the level \((0,0)\) case. As we will see, even in this case the proof is more complicated than the the proof of the previous theorems in this section.

In the following theorem we are interested in the greatest possible value of \(n\) such that the class \(\beta = \beta_0 + nf\), level \((0,0)\) partition function is not trivially zero. By Remark 2.2 \(n\) is the greatest integer that satisfies

\[
d = -K_X \cdot \beta = -(-3H + (2g - 2)F)(\beta_0 + nf) = 3n - (2g - 2) < 0.
\]

For \(g = 3k\), we have \(\beta = \beta_0 + (2k - 1)f\) for which \(Z_\beta(g|0,0)\) must be a polynomial of degree 1. For \(g = 3k + 1\), we have \(\beta = \beta_0 + 2kf\) which is the Calabi-Yau section class. For \(g = 3k + 2\), we have \(\beta = \beta_0 + 2kf\), and \(Z_\beta(g|0,0)\) must be a polynomial of degree 2.

**Theorem B.5.** Let \(n\) be the greatest integer that satisfies

\[
3n \leq 2g - 2,
\]
then the class $\beta_0 + nf$, level $(0,0)$ equivariant Gromov-Witten partition function of $X$ is given by

$$Z_{\beta_0 + nf}(g \mid 0, 0) = \begin{cases} 0 & g = 3k, \\ 3^g \left(2 \sin \frac{\pi}{2}\right)^{2g-2} & g = 3k + 1, \\ 3^{g-2}(g-1)(t_0^2 + t_1^2 + t_2^2) & g = 3k - 1, \\ -t_0 t_1 - t_0 t_2 - t_1 t_2) \left(2 \sin \frac{\pi}{2}\right)^{2g-4} & g = 3k + 2. \end{cases}$$

**Proof:** Applying Theorem 1.6 to this case, we can write

$$Z(g \mid k, k) = \text{tr} \left( (A + B)^g \right).$$

Now we prove each case in this theorem separately.

(i) $g = 3k$

In this case $n = 2k - 1$, and one can see that (by using the notation introduced in Section 5)

$$Z_{\beta_0 + (2k-1)f}(g \mid 0, 0) = \text{tr} \left( (A^k, B^{2k-1}) \right).$$

We have

$$AB^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} 9 \phi^6,$$

so for any positive integer $a$ we have

$$(AB^2)^a = 3^{3a-3} \phi^{6a-6} AB^2. \quad (28)$$

One can prove easily that

$$B^3 = 0, \quad (ABAB^2)^2 = 0. \quad (29)$$

Now we use the fact that for any two square matrices, $U$ and $V$,

$$\text{tr}(UV) = \text{tr}(VU). \quad (30)$$

Applying this to each term of $\text{tr} \left( (A^k, B^{2k-1}) \right)$ for some number of times, and using the other equalities above, we can prove that each of these terms is either zero or is equal to

$$\text{tr} \left( AB(AB^2)^{k-1} \right) = 3^{3k-6} \phi^{6k-12} \text{tr}(ABAB^2).$$
However,

\[ ABAB^2 = \begin{bmatrix} 2t_0 - t_1 - t_2 & 2t_0 - t_1 - t_2 & 2t_0 - t_1 - t_2 \\ 2t_2 - t_0 - t_1 & 2t_1 - t_0 - t_2 & 2t_1 - t_0 - t_2 \\ 2t_2 - t_0 - t_1 & 2t_2 - t_0 - t_1 & 2t_2 - t_0 - t_1 \end{bmatrix} 27\phi^9, \]

so

\[ \text{tr} (ABAB^2) = 0. \]

This proves that each term of \( \text{tr} ((A^k, B^{2k-1})) \) is zero. So

\[ Z_{\beta_0+(2k-1)f}(g|0,0) = 0. \]

(ii) \( g = 3k + 1 \)

In this case \( \beta_0 + nf \) is the Calabi-Yau section class, and we have proved the theorem for this case in more generality in Section 5.

(iii) \( g = 3k + 2 \)

In this case \( n = 2k \), and this time we have

\[ Z_{\beta_0+2kf}(g|0,0) = \text{tr} ((A^{k+1}, B^{2k})). \]

The cases \( k = 0, 1 \) can be proved by easy calculations, so we assume that \( k > 1 \). Applying (30) to each term of this for some number of times, and using (29), we can prove that each term of \( \text{tr} ((A^{k+1}, B^{2k})) \) is either zero or is equal to either of

\[ \text{tr} (A(AB^2)^k), \ \text{tr} ((AB)^2(AB^2)^{k-1}), \ \text{tr} (A^2B^2(AB^2)^{k-1}), \]

where the number of the terms of the first, the second and the third kind is \( 2k + 1, 3k + 1, \) and \( k \), respectively. So we can write

\[ Z_{\beta_0+2kf}(g|0,0) = (2k + 1) \text{tr} (A(AB^2)^k) + (3k + 1) \text{tr} ((AB)^2(AB^2)^{k-1}) + k \text{tr} (A^2B^2(AB^2)^{k-1}). \]
One can see that
\[
A(AB^2) = \\
\begin{bmatrix}
(t_0 - t_1)(t_0 - t_2) & (t_0 - t_1)(t_0 - t_2) & (t_0 - t_1)(t_0 - t_2) \\
(t_1 - t_0)(t_1 - t_2) & (t_1 - t_0)(t_1 - t_2) & (t_1 - t_0)(t_1 - t_2) \\
(t_2 - t_0)(t_2 - t_1) & (t_2 - t_0)(t_2 - t_1) & (t_2 - t_0)(t_2 - t_1)
\end{bmatrix}
9\phi^9,
\]
\[
(AB)^2(AB^2) = \\
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}
162(t_0^2 + t_1^2 + t_2^2 - t_0t_1 - t_0t_2 - t_1t_2)\phi^{12},
\]
\[
A^2B^2(AB^2) = \\
\begin{bmatrix}
(t_0 - t_1)(t_0 - t_2) & (t_0 - t_1)(t_0 - t_2) & (t_0 - t_1)(t_0 - t_2) \\
(t_1 - t_0)(t_1 - t_2) & (t_1 - t_0)(t_1 - t_2) & (t_1 - t_0)(t_1 - t_2) \\
(t_2 - t_0)(t_2 - t_1) & (t_2 - t_0)(t_2 - t_1) & (t_2 - t_0)(t_2 - t_1)
\end{bmatrix}
243\phi^{12}.
\]
Using \((28),\) we can write
\[
\text{tr} \left( A(AB^2)^k \right) = 3^{3k-3}\phi^{6k-6}\text{tr} \left( A(AB^2) \right) \\
= 3^{3k-1}(t_0^2 + t_1^2 + t_2^2 - t_0t_1 - t_0t_2 - t_1t_2)\phi^{6k},
\]
\[
\text{tr} \left( (AB)^2(AB^2)^{k-1} \right) = 3^{3k-6}\phi^{6k-12}\text{tr} \left( (AB)^2(AB^2)^{k-1} \right) \\
= 2\cdot3^{3k-1}(t_0^2 + t_1^2 + t_2^2 - t_0t_1 - t_0t_2 - t_1t_2)\phi^{6k},
\]
\[
\text{tr} \left( A^2B^2(AB^2)^{k-1} \right) = 3^{3k-6}\phi^{6k-12}\text{tr} \left( A^2B^2(AB^2)^{k-1} \right) \\
= 3^{3k-1}(t_0^2 + t_1^2 + t_2^2 - t_0t_1 - t_0t_2 - t_1t_2)\phi^{6k},
\]
We define
\[
Q := t_0^2 + t_1^2 + t_2^2 - t_0t_1 - t_0t_2 - t_1t_2.
\]
Putting these equalities into \((31),\) we get
\[
Z_\beta(g \mid 0, 0) = 3^{3k-1}(2k + 1 + 2(3k + 1) + k)Q\phi^{6k} \\
= 3^{3k}(3k + 1)Q\phi^{6k} \\
= 3^{9-2}(g - 1)Q\phi^{2g-4},
\]
and this proves the theorem in this case.
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