Kraśkiewicz-Pragacz modules and Ringel duality

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Abstract. In [4, 5] Kraśkiewicz and Pragacz introduced representations of the upper-triangular Lie algebras whose characters are Schubert polynomials. In [11] the author studied the structure of Kraśkiewicz-Pragacz modules using the theory of highest weight categories. From the results there, in particular we obtain a certain highest weight category whose standard modules are KP modules. In this paper we show that this highest weight category is self Ringel-dual, and shows that the tensor product operation on \( b \)-modules is compatible with Ringel duality functor.

1 Introduction

The study of Schubert polynomials is an interesting subject in algebraic combinatorics. One of the possible methods for studying Schubert polynomials is via certain representations, introduced by Kraśkiewicz and Pragacz ([4, 5]), of the Lie algebra \( b \) of all upper-triangular matrices. These representations, which we call \( KP \) modules in this paper, has the property that their character with respect to the subalgebra of diagonal matrices is equal to Schubert polynomials: just like Schur polynomials appear as the characters of irreducible representations of \( gl_n \).

In [11] and [12], the author investigated the structure of KP modules. The main motivation there was the investigation of Schubert positivity: i.e. the positivity of coefficients in the expansion of a polynomial into a linear combination of Schubert polynomials. Because of the property of KP modules described above, if a polynomial \( f \) is the character of some \( b \)-module having a filtration by KP modules then \( f \) is Schubert-positive. In [11] we gave a characterization of \( b \)-modules having such filtrations, and we used the result in [12] to prove some positivity results on Schubert polynomials. For details on these results see these two papers.

The method used in [11] is the theory of highest weight categories. A highest weight category is an abelian category with a specified family of objects called standard objects, together with some axioms. For a highest weight category the notion of costandard objects are naturally defined, and it is then shown that an object have a filtration by standard objects (a standard filtration) iff its extension with any costandard object vanishes. In [11] we used the theory to obtain a characterization of modules having a filtration by KP modules: the work was strongly inspired by some works by Polo, van der Kallen, Joseph, etc.
on Demazure modules (see the bibliography in [11]).

For a highest weight category \( \mathcal{C} \), there is a notion called Ringel dual of \( \mathcal{C} \), denoted \( \mathcal{C}^\vee \). It is another highest weight category, with a contravariant equivalence \( \mathcal{C}^\Delta \to (\mathcal{C}^\vee)^\Delta \) between the subcategories of objects having standard filtrations: i.e. \( \mathcal{C}^\vee \) is obtained by “flipping” the standardly-filtered part of \( \mathcal{C} \).

The main purpose of this paper is to show that a certain highest weight category \( \mathcal{C}_n \) having KP modules \( S_w \) (\( w \in S_n \)) as standard modules is self Ringel-dual (Theorem 4.1). This provides an contravariant equivalence \( \mathcal{C}_n^\Delta \to \mathcal{C}_n^\vee \), and gives an interesting corollary (Corollary 4.2) on the extension groups between KP modules. We also investigate the relation between Ringel duality on \( \mathcal{C}_n \) and the tensor product operation on modules (Theorem 5.1).

The paper is organized as follows. In Section 2 we prepare some definitions and results on Schubert polynomials, KP modules, and highest weight categories. In Section 3 we explicitly relate KP modules and highest weight categories: we give a certain highest weight category \( \mathcal{C}_n \) with KP modules \( S_w \) (\( w \in S_n \)) being standard modules (it is in fact an immediate consequence of the results in [11]). In Section 4 we show that the Ringel dual of \( \mathcal{C}_n \) is equivalent to itself, and determine the correspondence between standard modules. In Section 5 we see that the tensor product operation and the Ringel duality functor are in some sense compatible with each other.

2 Preliminaries

2.1 Schubert polynomials

Let \( \mathbb{Z}_{>0} \) denote the set of all positive integers. A permutation is a bijection from \( \mathbb{Z}_{>0} \) to itself which fixes all but finitely many points. Let \( n \) be a positive integer. Let \( S_n = \{ w : \text{permutation}, w(i) = i \ (i > n) \} \) and \( S_n^{(n)} = \{ w : \text{permutation}, w(n+1) < w(n+2) < \cdots \} \). For \( i < j \), let \( t_{ij} \) denote the permutation which exchanges \( i \) and \( j \) and fixes all other points. Let \( s_i = t_{i,i+1} \). For a permutation \( w \), let \( \ell(w) = \# \{ i < j : w(i) > w(j) \} \). Let \( w_0 \in S_n \) be the longest element of \( S_n \), i.e. \( w_0(i) = n + 1 - i \ (1 \leq i \leq n) \). For \( w \in S_n^{(n)} \) we define \( \text{inv}(w) = \{ \text{inv}(w_1), \ldots, \text{inv}(w_n) \} \in \mathbb{Z}_{>0}^n \) by \( \text{inv}(w_i) = \# \{ j : i < j, w(i) > w(j) \} \).

Note that if \( w \in S_n \) we have \( \text{inv}(w) \in \Lambda_n := \{ (a_1, \ldots, a_n) \in \mathbb{Z}^n : 0 \leq a_i \leq n-i \} \).

For a polynomial \( f = f(x_1, \ldots, x_n) \) and \( 1 \leq i \leq n-1 \), we define \( \partial f = \frac{f(x_1, \ldots, x_{i-1}, x_i - x_{i+1}, \ldots, x_n)}{x_i - x_{i+1}} \). For each \( w \in S_n^{(n)} \) we can assign its Schubert polynomial \( \mathcal{S}_w \in \mathbb{Z}[x_1, \ldots, x_n] \), which is recursively defined by

- \( \mathcal{S}_w = x_1^{w(1)-1} x_2^{w(2)-1} \cdots x_n^{w(n)-1} \) if \( w(1) > \cdots > w(n) \), and
- \( \mathcal{S}_{ws_i} = \partial \mathcal{S}_w \) if \( \ell(ws_i) < \ell(w) \).

It is known that the set \( \{ \mathcal{S}_w : w \in S_n \} \) constitutes a \( \mathbb{Z} \)-basis of the ring \( H_n = \mathbb{Z}[x_1, \ldots, x_n]/I \) where \( I \) is the ideal generated by the homogeneous symmetric polynomials of positive degrees. Note also that the natural map \( \mathbb{Z}[x_1, \ldots, x_n] \to H_n \) restricts to an isomorphism of \( \mathbb{Z} \)-modules \( \bigoplus_{\lambda \in \Lambda_n} \mathbb{Z} x^\lambda \cong H_n \), where \( x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n} \) ([8, Proposition 2.5.3, Corollary 2.5.6]). We also need the following basic facts:

**Proposition 2.1.** Let \( \iota : H_n \to H_n \) be the ring automorphism given by \( x_i \mapsto -x_{i+1} \). Then for \( w \in S_n \), \( \ell(\mathcal{S}_w) = \mathcal{S}_{\iota w} \iota w_0 \).
Remark 2.2. The automorphism $\iota$ corresponds to the map between flag varieties which takes a flag to its dual flag: see eg. [3] §10.6, Exercise 13]

Proof. First note that $\iota\partial_{i}\iota=\partial_{n-i}$. Thus we only have to check the proposition for $w=w_{0}$.

Since the only elements in $H_{n}=\bigoplus_{w\in S_{n}}\mathbb{Z}S_{w}$ with degree $\binom{n}{2}$ are the constant multiples of $\mathcal{S}_{w_{0}}$, we see that $\iota(\mathcal{S}_{w_{0}})$ is a constant multiple of $\mathcal{S}_{w_{0}}$. Let $(i_{1},\ldots,i_{l})$ be a longest word, i.e. $l=\ell(w_{0})$ and $w=s_{i_{1}}\cdots s_{i_{l}}$. Note that $(n-i_{1},\ldots,n-i_{l})$ is also a longest word. We have $\partial_{i_{1}}\cdots\partial_{i_{l}}(\mathcal{S}_{w_{0}})=\mathcal{S}_{id}=1$ and $\partial_{i_{1}}\cdots\partial_{i_{l}}\iota(\mathcal{S}_{w_{0}})=(\iota(\partial_{n-i_{1}})\cdots(\iota(\partial_{n-i_{l}}))\iota(\mathcal{S}_{w_{0}})=\iota(\partial_{n-i_{1}}\cdots\partial_{n-i_{l}}\mathcal{S}_{w_{0}})=1$. Thus $\iota(\mathcal{S}_{w_{0}})=\mathcal{S}_{w_{0}}$. \qed

Proposition 2.3. For $w\in S_{\infty}^{(n)}\setminus S_{n}$ we have $\mathcal{S}_{w}\in I$.

Proof. Since $\partial_{i}I\subset I$ for $1\leq i\leq n-1$, it suffices to show that the proposition holds in the case $w(1)>\cdots>w(n)$. Since in this case $\mathcal{S}_{w}=x_{w(1)-1}^{1}x_{w(2)-1}^{2}\cdots x_{w(n)-1}^{n}$ it is enough to show $x_{i}^{n}\in I$. This is immediate from the equation $\prod_{i=2}^{n}(1-x_{i}u)=\frac{1}{1-x_{1}u}=\sum_{j\geq 0}x_{1}^{j}u^{j}$ in $H_{n}[u]$ since the LHS has no terms above $u^{n}$. \qed

Schubert polynomials satisfy the following Cauchy identity:

Proposition 2.4 ([7] (5.10)). $\sum_{w\in S_{n}}\mathcal{S}_{w}(x)\mathcal{S}_{ww_{0}}(y)=\prod_{i+j\leq n}(x_{i}+y_{j})$.

2.2 Kraśkiewicz-Pragacz modules

Let $K$ be a field of characteristic zero. Let $b=b_{n}$ be the Lie algebra of all upper triangular $K$-matrices, and let $\mathfrak{b}\subset b$ be the subalgebra of all diagonal matrices and the subalgebra of all strictly upper triangular matrices respectively. Let $U(b)$ and $U(n^{+})$ be the universal enveloping algebras of $b$ and $n^{+}$ respectively. For a $U(b)$-module $M$ and $\lambda=(\lambda_{1},\ldots,\lambda_{n})\in\mathbb{Z}^{n}$, let $M_{\lambda}=\{m\in M:hm=(\lambda,h)m:\forall h\in\mathfrak{b}\}$ where $(\lambda,h)=\sum\lambda_{j}h_{j}$. $M_{\lambda}$ is called the $\lambda$-weight space of $M$. If $M_{\lambda}\neq 0$ then $\lambda$ is said to be a weight of $M$. If $M=\bigoplus_{\lambda\in\mathbb{Z}^{n}}M_{\lambda}$ and each $M_{\lambda}$ has a finite dimension, then $M$ is said to be a weight $b$-module and we define $ch(M)=\sum_{\lambda}\dim M_{\lambda}x^{\lambda}$. From here we only consider weight $b$-modules. For $1\leq i\leq j\leq n$, let $e_{ij}\in b$ be the matrix with 1 at the $(i,j)$-position and all other coordinates 0. Let $p=(n-1,n-2,\ldots,0)\in\mathbb{Z}^{n}$ and $1=(1,\ldots,1)\in\mathbb{Z}^{n}$. Also let $a_{ij}=(0,\ldots,0,1,0,\ldots,0,1,0,\ldots,0)$ for $1\leq i<j\leq n$, where 1 and −1 are at $i$-th and $j$-th position respectively.

For $\lambda\in\mathbb{Z}^{n}$, let $K_{\lambda}$ denote the one-dimensional $U(b)$-module where $h\in\mathfrak{b}$ acts by $(\lambda,h)$ and $n^{+}$ acts by 0.

In [3], Kraśkiewicz and Pragacz defined certain $U(b)$-modules which here we call Kraśkiewicz-Pragacz modules or KP modules. Here we use the following definition. Let $w\in S_{\lfloor n\rfloor}$. Let $K^{w}=\bigoplus_{1\leq i\leq n}Ku_{i}$ be the vector representation of $b$. For each $j\geq 1$, let $\{i<j:w(i)>w(j)\} = \{i_{1},\ldots,i_{r}\}$ $(i_{1}<\cdots<i_{r})$, and let $u_{w}^{(j)}=u_{i_{1}}\wedge\cdots\wedge u_{i_{r}}\in\bigwedge^{j}K^{w}$. Let $u_{w}=u_{w}^{(1)}\otimes u_{w}^{(2)}\otimes\cdots\in\bigotimes_{j\geq 1}\bigwedge^{j}K^{w}$. Then the KP module $S_{w}$ associated to $w$ is defined as $S_{w}=U(b)u_{w}=U(n^{+})u_{w}$.

Example 2.5. If $w=s_{i}$, then $u_{w}=u_{i}$ and thus we have $S_{w}=K^{i}:=$ $\bigoplus_{1\leq j\leq i}Ku_{j}$.

KP modules have the following property:
Theorem 2.6 ([5] Remark 1.6, Theorem 4.1]). For any \( w \in S^{(n)}_\infty \), \( S_w \) is a weight \( \mathfrak{b} \)-module and \( \text{ch}(S_w) = \mathcal{S}_w \).

We slightly generalize the notion of KP modules. For \( \lambda \in \mathbb{Z}^n \), take \( k \in \mathbb{Z} \) so that \( \lambda + k1 \in \mathbb{Z}^n_{\geq 0} \), and define \( \mathcal{S}_\lambda = S_w \otimes K_{-k1} \) (\( w \in S^{(n)}_\infty \), \( \text{inv}(w) = \lambda + k1 \)). Note that this definition does not depend on the choice of \( k \). We denote by \( u_\lambda \) the generator of \( \mathcal{S}_\lambda \). We also write \( \mathcal{S}_\mu = \text{ch}(S_\mu) \).

2.3 Highest weight categories

Highest weight categories were first introduced by Cline, Parshall and Scott ([1]). In this paper we use the following definition (cf. [8]):

Definition 2.7. Let \( \mathcal{C} \) be an abelian \( K \)-category with enough projectives and injectives, such that every object has a finite length. Let \( \Lambda = (\Lambda, \leq) \) be a finite poset parametrizing the simple objects \( \{ L(\lambda) : \lambda \in \Lambda \} \) in \( \mathcal{C} \) (called weight poset). Moreover, assume that a family of objects \( \{ \Delta(\lambda) : \lambda \in \Lambda \} \) called standard objects is given. Then \( \mathcal{C} = (\mathcal{C}, \Lambda) \) is called a highest weight category if the following axioms hold:

1. \( \text{Hom}_\mathcal{C}(\Delta(\lambda), \Delta(\mu)) = 0 \) unless \( \lambda \leq \mu \).

2. \( \text{End}_\mathcal{C}(\Delta(\lambda)) \cong K \).

3. Let \( P(\lambda) \) denote the projective cover of \( L(\lambda) \). Then there exists a surjection \( P(\lambda) \twoheadrightarrow \Delta(\lambda) \) such that its kernel admits a filtration whose subquotients are of the form \( \Delta(\nu) \) (\( \nu > \lambda \)).

For highest weight category \( (\mathcal{C}, \Lambda) \) and \( \lambda \in \Lambda \), let \( \mathcal{C}_{\leq \lambda} \) denote the subcategory of objects whose simple constituents are all of the form \( L(\mu) \) (\( \mu \leq \lambda \)). Let \( \nabla(\lambda) \in \mathcal{C} \) be the injective hull of \( L(\lambda) \) in the subcategory \( \mathcal{C}_{\leq \lambda} \). These are called costandard objects. The standard modules can also be characterized in this way: \( \Delta(\lambda) \) is the projective cover of \( L(\lambda) \) in \( \mathcal{C}_{\leq \lambda} \). More generally, for any order ideal \( \Lambda' \) such that \( \mu \leq \lambda \) (resp. \( \mu > \lambda \)) implies \( \mu \in \Lambda' \) (resp. \( \mu \not\in \Lambda' \)), \( \Delta(\lambda) \) is the projective cover of \( L(\lambda) \) in the subcategory \( \mathcal{C}_{\Lambda'} \) of modules whose simple constituents are \( L(\mu) \) (\( \mu \in \Lambda' \)).

A standard (resp. costandard) filtration of an object \( M \in \mathcal{C} \) is a filtration such that each of its subquotients is a standard (resp. costandard) object. For a highest weight category \( \mathcal{C} \) let \( \mathcal{C}^\Delta \) denote the subcategory of all objects having standard filtrations.

We recall some facts from the theory of highest weight categories:

Proposition 2.8 ([1, 2 Appendix, Proposition A2.2(ii)], [6]). \( \text{Hom}_\mathcal{C}(\Delta(\lambda), \nabla(\mu)) \cong K \) if \( \lambda = \mu \) and 0 otherwise, and \( \text{Ext}_\mathcal{C}^i(\Delta(\lambda), \nabla(\mu)) = 0 \) for \( i \geq 1 \). Hence if \( M \in \mathcal{C} \) has a standard (resp. costandard) filtration, then for any \( \lambda \in \Lambda \), the number of times \( \Delta(\lambda) \) (resp. \( \nabla(\lambda) \)) appears in (any) standard (resp. costandard) filtration is \( \dim \text{Hom}_\mathcal{C}(M, \nabla(\lambda)) \) (resp. \( \dim \text{Hom}_\mathcal{C}(\Delta(\lambda), M) \)).

The next proposition may be standard in the theory of highest weight categories, but we here give a proof:

Proposition 2.9. Let \( M \in \mathcal{C}^\Delta \) and let \( \Lambda' \subset \Lambda \) be an order ideal. Let \( M^{\Lambda'} \) be the largest quotient of \( M \) whose each simple constituent is isomorphic to some \( L(\lambda) \)
(λ ∈ Λ'). Note that the existence of such quotient follows from the finiteness of the length of M. Then M′ and Ker(M → M′) have filtrations whose subquotients are of the form Δ(λ) (λ ∈ Λ') and Δ(µ) (µ ∈ Λ ∖ Λ') respectively. If λ ∈ Λ is a maximal element in Λ', then Ker(M′ → M′ ∖ {λ}) is isomorphic to a direct sum of some copies of Δ(λ) (note that the former statement follows from this one).

Proof. We prove the latter statement by the induction on |Λ'|. The only simple module which can appear in the head of N := Ker(M′ → M′ ∖ {λ}) is L(λ); i.e. the head of N is a direct sum of copies of L(λ). Thus, since Δ(λ) is the projective cover of L(λ) in C′, it suffices to show that N is projective in C′.

Let µ ∈ Λ'. We want to show Ext1(N, L(µ)) = 0. We have three exact sequences

Hom(N, ∇(µ)/L(µ)) → Ext1(N, L(µ)) → Ext1(N, ∇(µ)), Ext1(M′, ∇(µ)) → Ext1(N, ∇(µ)) → Ext2(M′ ∖ {λ}, ∇(µ)) and Hom(Ker(M → M′), ∇(µ)) → Ext1(M′, ∇(µ)) → Ext1(M, ∇(µ)). Here Ext1(M, ∇(µ)) = 0 by Proposition 2.3 and Ext2(M′ ∖ {λ}, ∇(µ)) = 0 by Proposition 2.8 and the induction hypothesis. Thus it suffices to show Hom(Ker(M → M′), ∇(µ)) and Hom(N, ∇(µ)/L(µ)) vanish.

For the first claim it suffices to show Hom(Ker(M → M′), L(ν)) = 0 for all ν ≤ µ since ∇(µ)/L(µ) is a maximal element in C′. But it follows from the definition of M′ that Ker(M → M′) has no quotient of the form L(ν) (ν ∈ Λ'). Thus the claim follows.

The second one is similar. Since the head of N is a sum of L(λ), we have Hom(N, L(ν)) = 0 unless ν = λ. But the simple constituents of ∇(µ)/L(µ) are of the form L(ν) (ν < µ) because L(µ) has no self-extension (cf. [1] Lemma 3.2(b)); the definition there is dual to our definition but the proof is essentially the same). Thus Hom(N, ∇(µ)/L(µ)) = 0.

An object M ∈ C is called a tilting if it has both standard filtration and costandard filtration. In [2] Ringel showed the followings.

**Proposition 2.10 ([9], [2] Appendix, Theorem A4.2), ([3]).** For each λ, there exists a unique (up to isomorphism) tilting T(λ) which is indecomposable and has the property that there exists an injective morphism Δ(λ) → T(λ) whose cokernel admits a filtration by the objects of the form Δ(µ) (µ < λ). Moreover, every tilting is a direct sum of the objects T(λ) (λ ∈ Λ).

**Proposition 2.11 ([9], [2] Appendix, Theorems A4.7, A4.8), ([3]).** Let T be a tilting which contains every T(λ) at least once (such T is called a full tilting). Then the category C′ (which in fact does not depend on a choice of T), called the Ringel dual of C, of all finite dimensional left EndC(T)-modules is again a highest weight category with standard objects Δ′(λ) = HomC(Δ(λ), T) and the weight poset Λ′, the opposite poset of Λ. Moreover, the contravariant functor F : C → C′ given by FM = HomC(M, T) restricts to a contravariant equivalence between CΔ and (C′)Δ, and gives an isomorphism ExtC(M, N) ≅ ExtC(M', FN, FM) for any M, N ∈ CΔ and any i ≥ 0.

# 3 KP modules and highest weight categories

In this section we relate the notion of highest weight categories with KP modules, using the results from [11]. Let us introduce two ordering relations on Z≥0.
as follows. For \( \lambda = \text{inv}(w) \) and \( \mu = \text{inv}(v) \) \((\lambda, \mu \in \mathbb{Z}_{>0}^n, w, v \in S_{\lambda}^{(n)})\) with \( \ell(w) = \ell(v) \), we define \( \lambda \preceq \mu \iff w^{-1} > v^{-1} \) and \( \lambda \prec \mu \iff w^{-1} > v^{-1} \) (if \( \ell(w) \neq \ell(v) \) we define \( \lambda \) and \( \mu \) to be incomparable). Here for two permutations \( x \) and \( y, x > y \) (resp. \( x > y \)) if there exists an \( i \) such that \( x(j) = y(j) \) for any \( j \neq i \) (resp. \( j > i \)) and \( x(i) > y(i) \). We write \( \lambda \prec \mu \) if both \( \lambda < \mu \) and \( \lambda \prec \mu \) hold. For general \( \lambda, \mu \in \mathbb{Z}^n \), take \( k \in \mathbb{Z} \) so that \( \lambda + k1, \mu + k1 \in \mathbb{Z}^n_{>0} \), and write \( \lambda < \mu \) (resp. \( \lambda \prec \mu \)) iff \( \lambda + k1 < \mu + k1 \) (resp. \( \lambda + k1 \prec \mu + k1 \)), \( \lambda + k1 < \mu + k1 \). This definition does not depend on \( k \).

It can be seen that \( \Lambda_\lambda \) is an order ideal of \( \mathbb{Z}^n \) with respect to \( \prec \), using \cite{11} Lemma 6.2 and the argument in the proof of \cite{11} Lemma 6.3.

**Proposition 3.1.** Let \( \Lambda \subset \mathbb{Z}^n \) be a finite order ideal with respect to the ordering \( \prec \). Let \( \mathcal{C}_\Lambda \) be the category of all weight \( \mathfrak{g} \)-modules whose weights are in \( \Lambda \). Then \( \mathcal{C}_\Lambda \) is a highest weight category with weight poset \((\Lambda, \prec)\) and standard objects \( \{ S_\lambda : \lambda \in \Lambda \} \). In particular, \( \mathcal{C}_\Lambda := \mathcal{C}_\Lambda^\Lambda \) is a highest weight category.

**Proof.** In \cite{11} Proposition 6.4 we showed that if \( \lambda, \mu \in \mathbb{Z}^n \) and \( (S_\mu)_\lambda \neq 0 \) then \( \lambda \preceq \mu \) (more precisely, \( \lambda \leq \mu \) follows from (1) in the proof of \cite{11} Proposition 6.4), and \( \lambda \preceq \mu \) follows from (2) in the same proof and \cite{11} Lemma 6.2). If \( \text{Hom}(S_\Lambda, S_\mu) \neq 0 \), then \( (S_\mu)_\lambda \neq 0 \) must hold since \( S_\Lambda \) is generated by an element of weight \( \Lambda \), and thus \( \lambda \preceq \mu \). This checks (1) in the definition of highest weight category. (2) also follows since \( (S_\mu)_\lambda = Ku_\lambda \).

Let \( \lambda \in \Lambda \). Let \( P_\lambda \) be the projective cover of \( K_\lambda \) in the category of all weight \( \mathfrak{g} \)-modules, so \( P_\lambda \cong U(b)/(h - (\lambda, h))_{h \in \mathfrak{h}} \). Let the elements of \( \Lambda \) be \( \lambda^1, \ldots, \lambda^\ell \) so that:

- \( \lambda^i \prec \lambda^j \) implies \( i < j \), and
- The all elements which are greater than or equal to \( \lambda \) (with respect to the ordering \( \prec \)) are \( \lambda^k = \lambda, \lambda^{k+1}, \ldots, \lambda^\ell \).

Let \( P^m \) denote the largest quotient of \( P_\lambda \) such that all of its weights are in \( \{ \lambda^1, \ldots, \lambda^\ell \} \). Note that \( P^1 \) is the projective cover of \( K_\lambda \) in \( \mathcal{C}_{\{\lambda^1, \ldots, \lambda^\ell\}} \). In particular, \( P^1 \) is the projective cover of \( K_\lambda \) in \( \mathcal{C}_\Lambda \), and \( P^k \cong S_\lambda \) as we showed in \cite{11} Proposition 6.4. By the same argument as in the proof of \cite{11} Lemma 7.1, we see that, there is a quotient filtration \( P^1 \rightarrow P^{l_1-1} \rightarrow \cdots \rightarrow P^k \cong S_\lambda \rightarrow 0 \) such that \( \text{Ker}(P^m \rightarrow P^{m-1}) \cong S_\lambda \) is isomorphic to the direct sum of some copies of \( S_\lambda \). This verifies (3) in the definition.

From \cite{11} Proposition 6.4) (and \cite{11} Lemma 6.2]) we see that the costandard modules in \( \mathcal{C}_\Lambda \) are \( \{ S^*_{\rho - \lambda} \otimes K_\rho : \lambda \in \Lambda \} \).

## 4 Ringel dual of \( \mathcal{C}_n \)

In this section we show the following:

**Theorem 4.1.** The Ringel dual of the highest weight category \( \mathcal{C}_n \) is equivalent to \( \mathcal{C}_n \) itself. The functor \( F \) in Proposition 2.7 acts on the standard modules by \( F(S_\mu) = S_{\omega_1 \omega_2 \cdots \omega_n} \).

This theorem tells us an interesting corollary about the Ext groups between KP modules:
Corollary 4.2. \Ext^n_{C_n}(S_w, S_u) \cong \Ext^n_{C_n}(S_{w_0uw_0}, S_{w_0uw_0}) \text{ for any } w, v \in \Lambda_n \text{ and } i \geq 0.

Remark 4.3. By the same argument as in [12] Lemma 7.1 it holds that \Ext^n_{C_n}(M, N) \cong \Ext^n(M, N) \text{ (Ext group in the category of all weight b-modules) for any } M, N \in C_n. \text{ Hence Corollary 4.2 in fact shows } \Ext^n(S_w, S_u) \cong \Ext^n(S_{w_0uw_0}, S_{w_0uw_0}).

Remark 4.4. By Theorem 4.1 we have a functor defined as the composition of \( C_n \rightarrow C_n \rightarrow C_{n+1} \rightarrow C_{n+1} \). By the theorem we see that this functor acts on the standard modules by \( S_w \mapsto S_{1 \times w} \), where \( 1 \times w \in S_{n+1} \) is defined by \( (1 \times w)(1) = 1, (1 \times w)(i+1) = w(i+1) \text{ for } 1 \leq i \leq n \). Thus this functor can be seen as an analogue of the "shift operator" for Schubert polynomials ([12] (4.21), (4.22)).

First we prepare some definitions and results. For \( \lambda = \inv(w) \in \Lambda_n \) define \( \overline{\lambda} = \inv(w_0uw_0) \). Note that by definition, for \( \lambda, \mu \in \Lambda_n \), \( \lambda \preceq \mu \) if \( \overline{\lambda} \succeq \overline{\mu} \).

For each \( \lambda \in \Lambda_n \), define \( T(\lambda) = \bigotimes_{1 \leq j \leq n-1} K^{n-j} \). Then as we showed in the proof of [12] Lemma 4.2, \( T(\lambda) \) has a filtration whose subquotients are of the form \( S_{\pi} \) (\( \pi \in \Lambda_n, \mu \preceq \lambda \)). Since \( \rho = -\overline{\lambda} - \overline{\mu} \) we have \( T(\lambda) \cong T(\rho - \lambda) \otimes K_{\rho} \).

Let \( T(\lambda) \) also has a filtration whose subquotients are of the form \( S_{\rho} \). Thus \( T(\lambda) \) is a tilting in \( C_n \). Since the weights of \( S_{\rho} \) are all \( \preceq \mu \) ([11] Proposition 6.4), the weights of \( T(\lambda) \) are all \( \leq \lambda \), and the weight space \( T(\lambda) \) is one-dimensional.

By Proposition 2.9 if \( M \) has a standard filtration, then \( \ker(M^{\leq \lambda} \to M^{< \lambda}) \) is isomorphic to a direct sum of copies of \( S_{\lambda} \), where \( M^{\leq \lambda} \) and \( M^{< \lambda} \) are the largest quotients of \( M \) whose weights are all \( \leq \lambda \) and \( < \lambda \) respectively. In this case we see, from the proof of [11] Theorem 8.1, that the isomorphism can be written as \( S_{\lambda} \otimes (M^{\leq \lambda})_{\lambda} \ni xu \mapsto xv \in \ker(M^{\leq \lambda} \to M^{< \lambda}) \), where on the left-hand side \( b \) acts only on \( S_{\lambda} \).

Proof of Theorem 4.1. Let \( \mathcal{C} = C_n \). Let \( T = \bigoplus_{\lambda \in \Lambda_n} T(\lambda) \cong \bigwedge (K^{n-1} \oplus K^{n-2} \oplus \cdots \oplus K^1) \). Note that \( T \) is a full tilting. Let \( b' = \bigoplus_{\lambda} K e'_{ij} \) be a copy of \( b \).

We define an action of \( b' \) on \( T \) which commutes with the action of \( b \) as follows. Take a basis \( \{ u_{ij} : i, j \geq 1, i + j \leq n \} \) of \( K^{n-1} \oplus \cdots \oplus K^1 \) so that the action of \( b \) is given by \( e_{ij} u_{ij} = \delta_{ij} u_{ij} \). Then we define the action of \( b' \) on \( K^{n-1} \oplus \cdots \oplus K^1 \) by \( e'_{ij} u_{ij} = \delta_{ij} u_{ij} \), and define the action on \( T \) as the one induced from this action. In other words, if \( - : T \to T \) is the involution given by \( u'_{ij} = u_{ij} \), then \( e'_{ij} = - e_{ij} \). Since the actions of \( b \) and \( b' \) commute, we have an algebra homomorphism \( \mathcal{U}(b) \cong \mathcal{U}(b') \to \End(T) \), and thus an \( \End(T) \)-module can be naturally seen as a \( \mathcal{U}(b) \)-module. If \( M \) is an \( \End(T) \)-module, then as a \( \mathcal{U}(b) \)-module it has a weight-space decomposition \( M = \bigoplus_{\lambda \in \Lambda_n} \pi_{\lambda} M \) where \( \pi_{\lambda} \in \End(T) \) is the projection \( T = \bigoplus_{\mu \in \Lambda_n} T(\mu) \to T(\lambda) \); in particular the weights of \( M \) are all in \( \Lambda_n \). So we have a functor \( \mathcal{C}^\vee = \End(T)\mod \to \mathcal{C} \). We want to show that this functor is an equivalence and the composition \( \mathcal{C} \to \mathcal{C} \to \mathcal{C} \) sends \( S_w \) to \( S_{w_0uw_0} \).

First we show the second claim. By definition, \( S_w \) is isomorphic to the \( b \)-submodule of \( T \) generated by \( \bigwedge_{i,j < w(i), w(j)} u_{i,j, n+1-j} \); hereafter we identify \( u_w \) with this element. Since \( u'_w = \pm u_w \), we have an injective homomorphism \( S_{w_0uw_0} \to \Hom(S_w, T) \) given by \( xu_{w_0uw_0} \mapsto (v \mapsto x'v) \) \( (x \in \mathcal{U}(b)) \); it is well-defined since if \( xu_{w_0uw_0} = 0 \) then \( x'v = x'yv_w \) \( (3y \in \mathcal{U}(b)) \).
y(x_{w_i,w_{i'}}) = 0. Since $T$ has a costandard filtration, by Proposition 2.3 the dimension of $\text{Hom}(S_w,T)$ is equal to the number of times the costandard module $S_{w_{i_0}} \otimes K_{\rho}$ appears in (any) costandard filtration of $T$. Since $T \cong T^* \otimes K_{\rho}$, this number is equal to the number of times $S_{w_{i_0}}$ appears in (any) standard filtration of $T$. From Cauchy identity we see that $ch(T) = \prod (x_i + 1)^{n-1} = \sum_{x \in S_n} S_v(x) S_{w_{i_0}}(1)$, and thus we see that $\dim \text{Hom}(S_w,T) = S_{w_{i_0},w_{i'0}}(1) = \dim S_{w_{i_0},w_{i'0}}$. So the injection above is in fact an isomorphism and this shows the second claim.

Now let us show that the functor $C' \to C$ given above is an equivalence. First we note the following thing. Define an algebra $A = U(b)/I$, where $I$ is the two-sided ideal generated by all elements in $S(b) \cong K[h^*]$ which vanish on $\Lambda_n$ (here $\Lambda_n \subset \mathbb{Z}^n$ is identified with a subset of $h^*$ via the pairing $(\lambda,h) = \sum \lambda_i h_i$ introduced before). Then the weight $b$-modules with weights in $\Lambda_n$ are just the finite dimensional $A$-modules (note that $A$-modules automatically have weight decompositions since any element $p_{\lambda} \in K[h^*]$ such that $p_{\lambda}(\mu) = \delta_{\mu \lambda}$ ($\forall \mu \in \Lambda_n$) acts as a projection onto the $\lambda$-weight space). Thus it suffices to show that the map $\varphi : A \ni a \mapsto (\text{action of } a \text{ on } T) \in \text{End}(A)$ is an isomorphism. We note here that $A$ has an algebra anti-automorphism $\iota$ defined by $\iota(h) = (h,h) - h$ ($h \in b$) and $\iota(e_{i+1} - e_{i+1}) = -e_{i+1}$ ($1 \leq i \leq n-1$). For each $\lambda \in \Lambda_n$ take $p_{\lambda} \in A$ as above. Note that $\iota(p_{\lambda}) = p_{\lambda}$.

First we show that both $A$ and $\text{End}(T)$, seen as objects in the highest weight category $C_n$, have standard filtrations, and the number of times the standard constituent $S_w$ appears in these filtrations are both $S_{w_{i_0},w_{i'}}(1)$. The claim that $A$ has a standard filtration follows since $A = \bigoplus_{\lambda \in \Lambda_n} Ap_{\lambda}$ and $Ap_{\lambda}$ is the projective cover of $K_{\lambda}$ in $C_n$, which has a standard filtration. The number of times $S_w$ appears in this filtration can be then calculated, by Proposition 2.3 as $\dim \text{Hom}(A,S_{w_{i_0}} \otimes K_{\rho}) = \dim(S_{w_{i_0}} \otimes K_{\rho}) = S_{w_{i_0},w_{i'}}(1)$. For $\text{End}(T)$, first note that $T$ has a standard filtration. Note that, in general, if $0 \to L \to M \to N \to 0$ is exact, $N$ has a standard filtration and $T$ has a costandard filtration then $0 \to \text{Hom}(N,T) \to \text{Hom}(M,T) \to \text{Hom}(L,T) \to 0$ is exact, since $\text{Ext}^1(N,T) = 0$ by Proposition 2.3. Thus, since $T$ has a standard filtration, the isomorphism $\text{Hom}(S_w,T) \cong S_{w_{i_0},w_{i'}}$ we have checked above shows that $\text{End}(T)$ has a standard filtration. The number of times $S_w$ appears in a standard filtration of $\text{End}(T)$ is equal to the number of times $S_{w_{i_0},w_{i'}}$ appears in a standard filtration of $T$. This number is $S_{w_{i_0},w_{i'}}(1)$ as we have seen above $\text{ch}(T) = \sum_{x \in S_n} S_v(x) S_{w_{i_0}}(1)$.

Let $0 \leq d \leq \binom{n}{2}$. It suffices to show that $\varphi$ induces an isomorphism between $A_d := \sum_{\lambda_1 + \cdots + \lambda_n = d} Ap_{\lambda} A_d$ and $\text{End}(T)_d := \text{End}(\bigwedge^d(K^{n-1} \oplus \cdots \oplus K^1))$, since as algebras $A = \bigoplus_d A_d$ (this follows easily from $hp_{\lambda} = \delta_{h\lambda}$ and $e_{ij}p_{\lambda} = p_{\alpha_{ij} \lambda}$) and $\text{End}(T) = \bigoplus_d \text{End}(T)_d$. Let the elements of $\{ \lambda \in \Lambda_n : \sum \lambda_i = d \}$ be $\lambda^{(1)} > \lambda^{(2)} > \cdots > \lambda^{(r)}$. Note $\lambda^{(\ell)} <' \lambda^{(\ell+1)} <' \cdots <' \lambda^{(r)}$. Define $I_d = \sum_{\mu \leq \lambda^{(\ell)}} Ap_{\mu}$. Also define $J_d = \text{Hom}(T \otimes \bigwedge^d, T) \subset \text{End}(T)$ where $T \leq \bigwedge^d$ is the largest quotient of $T$ whose weights are all $\leq \lambda^{(\ell)}$. Define $I_0 = 0$ and $J_0 = 0$. Note that $I_d \subset \cdots \subset I_r = A_d$ and $J_d \subset \cdots \subset J_r = \text{End}(T)_d$. It suffices to show $\varphi(I_d) \subset J_d$ and $\varphi$ induces an isomorphism $I_d/J_d \to J_d/J_{d-1}$.

Let $\lambda = \lambda^{(k)}$. The first claim $\varphi(I_d) \subset J_d$ follows since for $\mu \geq \lambda$, $p_{\mu}$ acts on $T$ as the projection onto $T(\mu)$, and every weight $\nu$ of $T(\mu)$ satisfies $\nu \leq \mu \leq \lambda$. Let us now show that the induced map $I_d/J_d \to J_d/J_{d-1}$ is an isomorphism. We show that $I_d/J_d$ and $J_d/J_{d-1}$ are both isomorphic to $S_\lambda \otimes S_{\rho-\lambda}$ as vector
spaces and the composition of isomorphisms $I_k/I_{k-1} \cong S_\lambda \otimes S_{\rho-\lambda} \cong J_k/J_{k-1}$ is equal to the map induced from $\varphi$.

By definition, $A/I_k \cong A^{<\lambda}$ and $A/I_{k-1} \cong A^{\leq \lambda}$, and thus $I_k/I_{k-1} \cong \operatorname{Ker}(A^{<\lambda} \to A^{\leq \lambda})$. By Proposition 2.3, this is a direct sum of $m$ copies of $S_\lambda$, where $m$ is the number of times $S_\lambda$ appears in a standard filtration of $A$. We have seen above that $m = S_{\rho-\lambda}(1)$. Thus $S_\lambda \otimes S_{\rho-\lambda}$ and $I_k/I_{k-1}$ have the same dimensions.

The map $x_\lambda \otimes y_{\rho-\lambda} \mapsto xp_\lambda(y) \in I_k/I_{k-1}$ is a well-defined surjection, since the weights of $I_k/I_{k-1}$ (resp. $\nu(I_k/I_{k-1})$) are all $\leq \lambda$ (resp. $\leq \rho-\lambda$). Thus this is in fact an isomorphism.

Since $\operatorname{Ext}^1(T^{<\lambda}, T)$ vanishes, $I_k/I_{k-1} \cong \operatorname{Hom}(\operatorname{Ker}(T^{<\lambda} \to T^{<\lambda}), T)$ via the restriction map. The right-hand side is isomorphic to $\operatorname{Hom}(S_\lambda \otimes (T^{<\lambda})^*, T) \cong ((T^{<\lambda})^*)^* \otimes \operatorname{Hom}(S_\lambda, T)$ by the remark before the proof. We have $\operatorname{Hom}(S_\lambda, T) \cong S_\lambda$. On the other hand, since $T \cong T^* \otimes K_{\rho}$, $(T^{<\lambda})^* \cong (T_{\leq \mu})_{\mu}$ where $\mu = \rho-\lambda$ and $T_{\leq \mu}$ denotes the largest submodule of $T$ whose weights are $\leq \mu$. Since $S_\mu$ is the projective cover of $K_{\mu}$ in $C_{\leq \mu}$ we have $(T_{\leq \mu})_{\mu} \cong \operatorname{Hom}(S_\mu, T) \cong \operatorname{Hom}(S_\lambda, T) \cong S_\lambda$.

Now we show that the composition of these isomorphisms is equal to the map induced from $\varphi$, up to a sign depending only on $\lambda$. Chasing the isomorphisms we see that it suffices to show that $\varphi(xp_\lambda(y)(\tau) = \langle \iota(y)^\tau, x^\tau \rangle$ holds, up to a sign depending only on $\lambda$, for all $\tau \in T_{\lambda}$ and all $x, y \in A$, where $\langle -,- \rangle$ is a natural bilinear form on $T$ defined by $T \otimes T \twoheadrightarrow T \cong \wedge (K^{n-1} \oplus \cdots \oplus K^1)$ via $\wedge^i(K^{n-1} \oplus \cdots \oplus K^1) \cong K$. Note that from the definition we see that $\langle u, x \cdot v \rangle = \langle \iota(x)^u, v \rangle$ holds for any $u, v \in T$ and $x \in A$. First we have $\varphi(xp_\lambda(y))(\tau) = x^\tau p_\lambda(y)^\tau$. Since $p_\lambda(y)^\tau \in T_{\lambda_{\lambda}}$ it is a constant multiple of $u_{\lambda_{\lambda}}$.

Using the pairing defined above we see that this is equal to $\pm (p_\lambda(y)^\tau, u_{\lambda_{\lambda}})_{\lambda_{\lambda}}$ where the sign only depends on $\lambda$. Since $\langle p_\lambda(y)^\tau, u_{\lambda_{\lambda}} \rangle = \langle \iota(y)^\tau, x^\tau \rangle$ we are done.

\[ \square \]

**Remark 4.5.** By the isomorphism $\operatorname{End}(T) \cong A$ above we have $\operatorname{End}(T(\lambda)) \cong p_\lambda A$. But it can be seen, using $p_\lambda h = hp_\lambda$ ($h \in \mathfrak{h}$) and $p_\lambda e_{ij} = e_{ij}p_\lambda - e_{ji}$, that $p_\lambda A \cong K$. So we see that $T(\lambda)$ is in fact an indecomposable tilting.

**Remark 4.6.** The full tilting module $T$ introduced above has a relation with double Schur functor introduced by Sam [10]. Since two actions of $b$ and $b'$ commute, the direct sum $b \otimes b'$, which is isomorphic to the even part of $b(n|n)$, naturally acts on $T$. Then it is possible to define an action of the odd part of $b(n|n)$ on $T$ so that $T$ is isomorphic to the double functor image $\mathcal{F}_{wn}(V^*)$.

## 5 Relation with tensor product

In this section we show that the tensor product operation and the Ringel-duality functor $F = \operatorname{Hom}(-, T)$ ($T = \wedge (K^{n-1} \oplus \cdots \oplus K^1)$ as above) are in some sense compatible with each other. Precisely, we show the following:

**Theorem 5.1.** Let $M, N \in C_n$ have standard filtrations. Then $F((M \otimes N)_{\lambda_{\lambda}}) \cong (FM \otimes FN)_{\lambda_{\lambda}}$, where for a weight $b$-module $L$, $L_{\lambda_{\lambda}} \in C_n$ denotes the largest quotient of $L$ which is in $C_n$. 

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Let $\mathcal{C}_+^t$ be the category of all finite dimensional weight $b$-module whose weights are in $\mathbb{Z}_{\geq 0}^t$. Note that if $M,N \in \mathcal{C}_+^t$ then $M \otimes N \in \mathcal{C}_+^t$. Using the terminology from highest weight categories we say that $M \in \mathcal{C}_+^t$ has a standard filtration if $M$ has a filtration whose subquotients are $S_\lambda$ ($\lambda \in \mathbb{Z}_{\geq 0}^t$). Note that if $M,N \in \mathcal{C}_+^t$ have standard filtrations then $M \otimes N$ also has a standard filtration, by the result of [12].

**Remark 5.2.** If $L \in \mathcal{C}^t$ has a standard filtration, then as we show below, $\text{ch}(L^\lambda) = \text{ch}(L)$ holds in the ring $H_n$. So, together with Theorem 5.1 this theorem can be seen as a module theoretic counterpart of Proposition 2.1; i.e. the claim that $\mathfrak{S}_w \mapsto \mathfrak{S}_{\omega w^\lambda}$ is a ring automorphism.

First we prepare some lemmas.

**Lemma 5.3.** Let $\iota : H_n \rightarrow H_n$ be the ring automorphism in Proposition 5.1. If $M \in \mathcal{C}_+^t$ has a standard filtration, then $\text{ch}(FM) = \iota(\text{ch}(M))$ in $H_n$.

**Proof.** Since the extension of KP modules with $T$ vanish, in general if we have an exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ with $L,M,N \in \mathcal{C}_+^t$ having standard filtrations, then $0 \rightarrow FN \rightarrow FM \rightarrow FL \rightarrow 0$ is exact. Thus we only have to show the lemma for $M = S_w$ $(w \in S^{(n)}_\infty)$. The case $w \in S_n$ follows from Theorem 4.1.

If $w \in S^{(n)}_\infty \setminus S_n$, then we have $FS_w = 0$ since $S_w$ is generated by an element of weight $\text{inv}(w)$ while the weight space $T_{\text{inv}(w)}$ is zero. Thus the lemma follows from this case since $\mathfrak{S}_w = 0$ in $H_n$. $\square$

**Lemma 5.4.** Let $M \in \mathcal{C}_+^t$ has a standard filtration. Then $\text{ch}(M^\Lambda_n) = \text{ch}(M)$ as elements of $H_n$. If $M \in \mathcal{C}_n$ is a quotient of $M$ and $\text{ch}(M) = \text{ch}(M)$ in $H_n$, then $M \cong M^\Lambda_n$.

**Proof.** By Proposition 2.3, Ker($M \rightarrow M^\Lambda_n$) has a filtration whose subquotients are of the form $S_v$ $(v \in S^{(n)}_\infty \setminus S_n)$. Thus $\text{ch}(M) = \text{ch}(M^\Lambda_n) + (\text{a linear combination of } \mathfrak{S}_w (w \in S^{(n)}_\infty \setminus S_n))$, and the second term vanishes in $H_n$ by Proposition 2.3. The second claim follows from the first claim since $\bigoplus_{\lambda \in \Lambda_n} \mathbb{Z} \pi^\lambda \cong H_n$. $\square$

**Lemma 5.5.** Let $M,N \in \mathcal{C}_+^t$ have standard filtrations. Suppose that the morphism $FM \otimes FN \rightarrow F(M \otimes N)$ given by $\varphi \otimes \psi \mapsto (m \otimes n \mapsto \varphi(m) \wedge \psi(n))$ is surjective. Then it induces an isomorphism $(FM \otimes FN)^\Lambda_n \cong F(M \otimes N)$ $(\cong F((M \otimes N)^\Lambda_n))$.

**Proof.** We have, as vector spaces, $F(M \otimes N) = \text{Hom}(M \otimes N,T) = \bigoplus_{\lambda \in \Lambda_n} \text{Hom}(M \otimes N,T(\lambda))$. It can be seen that $\text{Hom}(M \otimes N,T(\lambda))$ is the $\lambda$-weight space of the $b$-module $F(M \otimes N)$. Thus $F(M \otimes N) \in \mathcal{C}_n$, and then the hypothesis implies that $F(M \otimes N)$ is a quotient of $FM \otimes FN$ with weights in $\Lambda_n$. But by Lemma 5.3 we have, in $H_n$, $\text{ch}(F(M \otimes N)) = \iota(ch(M)(ch(N)) = \iota(ch(M))(\iota(ch(N)) = ch(FM \otimes FN)$, where $\iota$ is the ring automorphism on $H_n$ which sends $\mathfrak{S}_w$ to $\mathfrak{S}_{\omega w^\lambda}$. Thus the claim follows from Lemma 5.4. $\square$

For $M,N \in \mathcal{C}_+^t$ having standard filtrations, let $\mathcal{P}(M,N)$ be the claim that the map $FM \otimes FN \rightarrow F(M \otimes N)$ above is surjective (and thus $(FM \otimes FN)^\Lambda_n \cong F((M \otimes N)^\Lambda_n)$).

**Lemma 5.6.** Let $L,M,N,X \in \mathcal{C}_+$ have standard filtrations. Then the following implications hold:
(1) If $L$ is a direct sum component of $M$ then $\mathcal{P}(M, X)$ implies $\mathcal{P}(L, X)$.

(2) Suppose that there exists an exact sequence $0 \to L \to M \to N \to 0$. Then $\mathcal{P}(L, X) \wedge \mathcal{P}(N, X) \implies \mathcal{P}(M, X)$ and $\mathcal{P}(M, X) \implies \mathcal{P}(L, X)$ hold (in fact $\mathcal{P}(M, X)$ also implies $\mathcal{P}(N, X)$, but we do not need it here).

(3) $\mathcal{P}(L, M)$ and $\mathcal{P}(L \otimes M, N)$ implies $\mathcal{P}(L, M \otimes N)$.

Proof. (1) is clear.

(2) We have a commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & FN \otimes FX & \longrightarrow & FM \otimes FX & \longrightarrow & FL \otimes FX & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(N \otimes X) & \longrightarrow & F(M \otimes X) & \longrightarrow & F(L \otimes X) & \longrightarrow & 0.
\end{array}
$$

Here the rows are exact since $\text{Ext}^1(N, T)$ and $\text{Ext}^1(N \otimes X, T)$ vanish. This shows $\mathcal{P}(L, X) \wedge \mathcal{P}(N, X) \implies \mathcal{P}(M, X)$ and $\mathcal{P}(M, X) \implies \mathcal{P}(L, X)$.

(3) This holds since

$$
\begin{array}{ccc}
FL \otimes FM \otimes FN & \longrightarrow & F(L \otimes M) \otimes FN \\
\downarrow & & \downarrow \\
FL \otimes F(M \otimes N) & \longrightarrow & F(L \otimes M \otimes N)
\end{array}
$$

commutes.

We also need the following technical lemma:

**Lemma 5.7.** Let $M \in \mathcal{C}_+$ has a standard filtration. Let $\lambda \in \Lambda_n$. Let $V \subset \text{Hom}(M, T)$ be the submodule consisting of all homomorphisms which vanishes on the $\mu$-weight space for any $\mu > \lambda$ (it is a submodule since the action of $b'$ preserves weights with respect to $\mathfrak{b} \subset \mathfrak{b}$). Then $\text{Hom}(M, T)/V \cong \text{Hom}(M, T)^{<\lambda}$, the largest quotient of $\text{Hom}(M, T)$ whose weights are all $<\lambda$.

**Proof.** First note that $V = \text{Hom}(M^{2\lambda}, T)$ where $M^{2\lambda}$ is the largest quotient of $M$ whose weights are all $\not\prec \lambda$. From Proposition 4.4 we see that $M^{2\lambda}$ has a standard filtration and, if $\text{ch}(M) = \sum_{\mu} c_{\mu} \mathcal{S}_\mu$, then the number of times $\mathcal{S}_\mu$ appears in a standard filtration of $M^{2\lambda}$ is $c_\mu$ if $\mu \not\prec \lambda$ and 0 otherwise. Thus we see from Theorem 5.11 that $\text{ch}(V) = \text{ch}(\text{Hom}(M^{2\lambda}, T)) = \sum_{\mu > \lambda} c_{\mu} \mathcal{S}_\mu$ and $\text{ch}(\text{Hom}(M, T)/V) = \sum_{\mu > \lambda} c_{\mu} \mathcal{S}_\mu$.

On the other hand, $\text{Hom}(M, T)$ has a standard filtration by Theorem 5.11 with $\mathcal{S}_\mu$ appearing $c_{\mu}$ times for any $\mu \in \Lambda_n$. So by Proposition 2.3 we see $\text{ch}(\text{Hom}(M, T)^{<\lambda}) = \sum_{\mu < \lambda} c_{\mu} \mathcal{S}_\mu = \sum_{\mu < \lambda} c_{\mu} \mathcal{S}_\mu = \sum_{\mu > \lambda} c_{\mu} \mathcal{S}_\mu = \text{ch}(\text{Hom}(M, T)/V)$. This shows the claim.

**Proof of Theorem 5.7.** First we show that $\mathcal{P}(S_n, S_\delta)$ holds for any $\delta \in S_n$ and any $1 \leq i \leq n - 1$. We write $\overline{w} = \overline{w}_1 \overline{w}_0$ ($\overline{w}_0 \in S_\delta$) and $\overline{F} = n + 1 - k (1 \leq k \leq n)$. Recall from the proof of Theorem 4.11 that $T$ has an action of $b'$, a copy of $\mathfrak{b}$, defined by $e'_{ij} u_{pq} = \delta_{ij} u_{pq}$, which commutes with the usual action of $\mathfrak{b}$. 

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Recall also that the isomorphism $S_w \to \text{Hom}(S_w, T)$ was given by $xw \mapsto (v \mapsto x'v)$. We want to show that the map $\varphi : S_w \otimes K^{n+1} \to F(S_w \otimes K^i)$ given by $yw \otimes u_p \mapsto xw \otimes u_p \mapsto x'yw \wedge u_p$ is a surjection.

Let $(p_1, q_1), \ldots, (p_r, q_r)$ be all the pairs $(p, q)$ such that $1 \leq p \leq i < q \leq n$ and $\ell(w_{p_q}) = \ell(w) + 1$, ordered by the lexicographic order of $(w(p), w(q))$. Let $w^k = w_{p_q}$. Then $\text{inv}(w^i) \leq \cdots \leq \text{inv}(w^r)$ and $\text{inv}(w^i) >' \cdots >' \text{inv}(w^r)$.

For $x \in S_w$ and $1 \leq p < q \leq n + 1$ such that $\ell(xt_{pq}) = \ell(x) + 1$, let $m_{pq}(x) = \# \{ r > q : x(p) < x(r) < x(q) \}$ and let $v_{pq}(x) = e_{pq}^{m_{pq}(x)}u_x \wedge u_p \in S_x \otimes K^n$. Note that $v_{pq}(x)$ has weight $\text{inv}(xt_{pq})$. As we showed in [12, Lemma 3.3], $\{v_{pq}(x) : 1 \leq p \leq i < q \leq n + 1, \ell(xt_{pq}) = \ell(x) + 1 \}$ generates $S_x \otimes K^i$ as a $b$-module.

From the result of [12, §3], we see that, if $U_k$ is the submodule of $S_w \otimes K^i$ generated by $v_{pq}(w)$ $(l > k)$ together with $v_{j,n+1}(w)$ $(1 \leq j < i, \ell(w_{j,n+1}) = \ell(w) + 1)$, then $U_{k-1}/U_k \cong S_w$. In particular, the weights of $(S_w \otimes K^i)/U_k$ are all $\leq \text{inv}(w^k)$. Moreover, $U_k$ has a filtration by modules $S_{w_{j,n+1}}$, and thus $\text{ch}(U_k) = 0$ in $H_n$. Therefore $(S_w \otimes K^i)/U_k \cong (S_w \otimes K^i)^{\wedge n}$. Let $V_k (k = 1, \ldots, r)$ be the submodule of $F(S_w \otimes K^i) = \text{Hom}(S_w \otimes K^i, T)$ consisting of the homomorphisms which vanishes on the $\mu$-weight space for any $\mu > \text{inv}(w^k)$. By Lemma 5.7, $F(S_w \otimes K^i)/V_k \cong F(S_w \otimes K^i)^{< \text{inv}(w^r)} (1 \leq k \leq r)$. We see $V_k = F(S_w \otimes K^i)$ since by the argument above the weights of $(S_w \otimes K^i)^{\wedge n}$ are all $\leq \text{inv}(w^k)$. We also set $V_0 = 0$. Note that the constituents in a standard filtration of $F(S_w \otimes K^i)$ are $S_w^{\otimes (1 \leq k \leq r)}$ by Theorem 1.1. In particular, the only constituent $S_x$ with $\text{inv}(w^{k-1}) >' \text{inv}(x) >' \text{inv}(w^k)$ is $S_w^{\otimes (1 \leq k \leq r)}$. Thus $V_k/V_{k-1} \cong \text{Ker}(F(S_w \otimes K^i)^{< \text{inv}(w^r)}) \to F(S_w \otimes K^i)^{< \text{inv}(w^r)} \cong S_w^{\otimes (1 \leq k \leq r)}$ by Proposition 2.3. In particular $V_k/V_{k-1}$ is cyclic on the generator of weight $\text{inv}(w^r)$.

We show $\varphi(v_{pq}(w)) \in V_k \setminus V_{k-1}$ for each $k$. Note that the desired surjectivity follows from this claim since it shows that $\varphi(v_{pq}(w)) + V_k$ is a cyclic generator of $V_k/V_{k-1}$.

For $1 \leq k < n$ consider $\varphi(v_{pq}(w))(v_{p_1, q_1}(w)) = (e_{pq}^{m_{pq}(w)})(e_{pq}^{m_{pq}(w)} \wedge u_{pq}) \wedge u_{pq}$. We have the following observations.

- If $p_1 < q_k$ and $w(p_1) > w(q_k)$ then $\varphi(v_{pq}(w))(v_{p_1, q_1}(w)) = 0$, in such case $u_w = u_{pq} \wedge \cdots$ and thus $(e_{pq}^{m_{pq}(w)})(e_{pq}^{m_{pq}(w)} \wedge u_{pq}) = u_{pq} \wedge \cdots$ (note that it does not matter whether $u_w$ contains $u_{pq}$ or not).
- If $q_1 < q_k$ and $w(q_1) > w(q_k)$ then $\varphi(v_{pq}(w))(v_{p_1, q_1}(w)) = 0$, in such case $u_w = u_{pq} \wedge \cdots$ and thus $e_{pq}^{m_{pq}(w)}u_w = u_{pq} \wedge \cdots$.
- If $p_1 < q_k$ and $w(p_1) > w(q_k)$ then $\varphi(v_{pq}(w))(v_{p_1, q_1}(w)) = 0$ since in such case $u_w = u_{pq} \wedge \cdots$.

If $\varphi(v_{pq}(w))(v_{p_1, q_1}(w)) \neq 0$, the observations above, together with $p_k, p_l \leq i < q_k, q_l$ and $\ell(w_{p_1q_1}) = \ell(w) + 1$, show that $w(p_1) \leq w(p_k)$ and $w(q_1) \leq w(q_k)$. In particular, $l \leq k$. Thus $\varphi(v_{pq}(w))$ induces a map $(S_w \otimes K^i)/U_k \to T$. Since the weights of $(S_w \otimes K^i)/U_k$ are all $\leq \text{inv}(w^k)$, this shows $\varphi(v_{pq}(w)) \in V_k$. Moreover, it can be seen that $\varphi(v_{pq}(w))(v_{p_1, q_1}(w)) \neq 0$, and this shows $\varphi(v_{pq}(w)) \notin V_{k-1}$. Therefore we checked the claim and thus $\mathcal{P}(S_w, S_w)$ follows.
Now we can proceed to the general case. From (2) of Lemma 5.6 we see that \( P(M, S_n) \) holds for any \( M \) having a standard filtration. Since as we have shown in [12] if \( M \) has a standard filtration then \( M \otimes S_n \) also has a standard filtration, (3) of Lemma 5.6 shows that \( P(M, S_n \otimes S_n \otimes \cdots) \) holds for any \( i, j, \ldots \) and any \( M \). Then from (1) of Lemma 5.6 we see that \( P(M, T(\lambda)) \) holds for any \( \lambda \) and any \( M \), since \( T(\lambda) \) is a direct sum component of \( \bigotimes_{1 \leq i \leq n-1} S_{n-i}^{\lambda_i} \). Thus again from (2) of Lemma 5.6 we get \( P(M, S_\lambda) \), since as we showed in [12] there is an injection \( S_\lambda \hookrightarrow T(\lambda) \) such that the cokernel has a standard filtration. Thus \( P(M, N) \) for general \( M, N \) follows by (2) of Lemma 5.6.

We obtain the following corollary from this theorem (though more direct proofs may be possible):

**Corollary 5.8.** The projective cover of \( K_\lambda \) in \( C_n \) is isomorphic to \( (S^{\lambda_1}(K^1) \otimes \cdots \otimes S^{\lambda_{n-1}}(K^{n-1}))^{\Lambda_n} \).

**Proof.** \( T(\overline{X}) = \bigwedge^{\lambda_1}(K^{n-1}) \otimes \cdots \otimes \bigwedge^{\lambda_{n-1}}(K^1) \). Since \( FT(\overline{X}) \) is a direct sum component of \( \text{End}(T) \), it is projective in \( C_n \). Since \( F(\overline{N}(K^1)) = F(S_{1,2, \ldots, i-j, \ldots, n-j+1, \ldots}) = S_{1,2, \ldots, i-j, \ldots, n-j+1, \ldots} \) by Theorem 4.3 we see that \( F(T(\overline{X})) \cong (S^{\lambda_1}(K^1) \otimes \cdots \otimes S^{\lambda_{n-1}}(K^{n-1}))^{\Lambda_n} \) by Theorem 5.8. Since the head of this module is isomorphic to \( K_\lambda \) (by the argument in [5] Remark 1.6) the claim follows.

**Remark 5.9.** As we saw, \( (M, N) \mapsto (M \otimes N)^{\Lambda_n} \) is a very fundamental operation in the category \( C_n \) and \( C_n^\Delta \); this in fact defines a structure of symmetric tensor category on \( C_n \) and \( C_n^\Delta \). Experimental results suggest an interesting conjecture relating this “restricted tensor product” operation and our full tilting module \( T \): the dimension of \( (T^{\otimes k})^{\Lambda_n} \) seems to be \( (k+1)^{\binom{n}{2}} \) for any \( k \). Also there is a finer form of this conjecture: the dimension of the degree-\( d \) piece (with respect to the grading induced from the natural grading on \( T = \bigwedge^*(\cdots) \)) of \( (T^{\otimes k})^{\Lambda_n} \) seems to be \( k^d \binom{n}{d} \).

Moreover, it can be seen that the latter version of the conjecture implies that \( \text{Hom}(T^{\otimes k}, T) \) also has a dimension \( (k+1)^{\binom{n}{2}} \). Note that this is true for \( k = 1 \) since \( \text{ch}(\text{End}(T)) = \text{ch}(T) = \sum_{v \in S_n} S_v(1)S_{w(v)}(1) \) and thus \( \dim(\text{End}(T)) = \sum_{v \in S_n} S_v(1)S_{w(v)}(1) = \sum_{v \in S_n} S_v(1)S_{w(v)}(1) = \sum_{v \in S_n} S_v(1)S_{w(v)}(1) = 2^\binom{n}{2} \) by Cauchy formula.

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