Uniqueness for Volterra-type stochastic integral equations

Leonid Mytnik ¹ Thomas S. Salisbury ²

Abstract
We study uniqueness for a class of Volterra-type stochastic integral equations. We focus on the case of non-Lipschitz noise coefficients. The connection of these equations to certain degenerate stochastic partial differential equations plays a key role.

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1 Introduction

The aim of this paper is to study uniqueness for a Volterra-type stochastic differential equation. Let $0 < \alpha < 1/2$ and let $\sigma$ be a Hölder continuous function with exponent $\gamma \in (0, 1)$. That is, we assume that there exists $L = L(\gamma)$ such that

$$|\sigma(x) - \sigma(y)| \leq L|x - y|^\gamma, \quad \forall x, y \in \mathbb{R}. \quad (1.1)$$

Consider the stochastic integral equation

$$X_t = x_0 + \int_0^t (t-s)^{-\alpha} g(s) \, ds + \int_0^t (t-s)^{-\alpha} \sigma(X_s) \, dB_s, \quad t \geq 0, \quad (1.2)$$

where $g$ is a bounded continuous function. We will extend the classical Yamada-Watanabe strong uniqueness result [YW71] to the above Volterra-type stochastic integral equation.

Existence and uniqueness for non-singular Lipschitz stochastic Volterra equations was shown in [Pro85]. To the best of our knowledge, there are no known uniqueness results for these equations with non-Lipschitz coefficients. Indeed a major difficulty lies in the absence of any natural semimartingale representation for solutions. However we will show that some of the methodology developed in [MPS05], [MP11] can be applied in this case.

Here is the first main result of this paper.

**Theorem 1.1** Let $\alpha \in (0, 1/2)$, and $\sigma$ satisfy (1.1) for some $\gamma \in \left(\frac{1}{2(1-\alpha)}, 1\right]$. Then, for any $x_0 \in \mathbb{R}$, and bounded continuous $g$, there is a pathwise unique solution to the equation (1.2).

Note that Yamada-Watanabe result states uniqueness for the above equation in the case of $\alpha = 0$ for any $\gamma \geq 1/2$. This gives an indication that our result is close to optimal, but we have not succeeded in constructing a counterexample for the case of $\gamma < \frac{1}{2(1-\alpha)}$. We believe that the methods developed in [BMP10] and [MMP] may be useful to tackle the non-uniqueness problem.

In fact our motivation for studying the above equation came originally from the study of strong uniqueness for SPDEs and of catalytic superprocesses in dimension one. Recall that the density of super-Brownian motion with space-time dependent branching rate, in dimension $d = 1$, can be represented as a solution to the following SPDE

$$\frac{\partial X_t(x)}{\partial t} = \frac{1}{2} \Delta X_t(x) + \sqrt{\lambda_s(x)X_t(x)} \, W, \quad t \geq 0, \quad x \in \mathbb{R}. \quad (1.3)$$

Here $\lambda_s(x)$ maybe interpreted as an instantaneous rate of branching at the point $x$ at time $s$. If one takes $\lambda_s(x) = 1$ and replaces the square root by a more general power, the SPDE

$$\frac{\partial X_t(x)}{\partial t} = \frac{1}{2} \Delta X_t(x) + |X_t(x)|^{\gamma} \, W, \quad t \geq 0, \quad x \in \mathbb{R}. \quad (1.4)$$

has been studied extensively. [MP11] shows strong uniqueness in (1.4) for $\gamma > \frac{2}{7}$, and [MMP] shows that strong uniqueness fails for $\gamma < \frac{3}{7}$. Those results are for
unconstrained (i.e., signed) solutions. If solutions are restricted to be positive, as is the case for the density of super-Brownian motion, then \[\text{[BMP10]}\] adds an immigration term and show that strong uniqueness fails for \(\gamma < \frac{1}{2}\). However altogether there is still no good understanding of uniqueness/non-uniqueness problem in the range \(\gamma \in (0, \frac{3}{4}]\).

One may try to narrow that gap by considering a smoother process, namely catalytic super-Brownian motion. It will turn out that the analogue of the SPDE (1.3) does not make sense, but that closely related SPDEs do, and lead naturally to the stochastic integral equation (1.2).

The study of super-Brownian motion with \(\lambda_s(dx)\) replaced by a singular measure \(\rho_s(dx)\) was initiated in \[\text{[DF90], [DFR91], [DF92]}\]. This pair \((\rho, X)\) serves as a model of a chemical (or biological) reaction of two substances, called the catalyst and reactant. The branching of the particles in the \(X\) population (reactant) occurs only in the presence of catalyst \(\rho\). More specifically, \(X\) is the super-Brownian motion whose branching rate at time \(t\) in the space element \(dx\) is given by \(\rho_t(dx)\). For \(\rho\) the Dirac measure, an elegant approach for studying the catalytic process was introduced in \[\text{[FJ95]}\]. This approach was later extended to a more general catalyst (see \[\text{[MV05]}\]). The relation of catalytic super-Brownian motion to SPDEs was presented in \[\text{[Zah05]}\]. However, in the case where \(\rho\) is the Dirac measure, the catalytic super-Brownian motion cannot be rigorously described as a solution to an SPDE. As we will see, there is a degenerate SPDE that is closely related to (1.2).

Let \(\rho = \delta_0\). The process \(X\) makes non-trivial sense only in dimension \(d = 1\), since only then do the paths of underlying Brownian particles hit the point catalyst. Before describing the corresponding martingale problem, it is necessary to define the local time of a superprocess at point the \(x = 0\).

At a heuristic level the local time \(l^0_t\) of a measure-valued process \(X\) at the point \(x = 0\) is a non-decreasing real-valued process such that

\[
l^0_t = \int_0^t \int_{\mathbb{R}} \delta_0(y)X_s(dy)ds,
\]

where \(\delta_0\) is the Dirac delta function. The precise definition includes using an approximate delta function instead of \(\delta_0\) and passing to the limit. Then we can write the martingale problem for super-Brownian motion with a point catalyst at \(x = 0\) as

\[
\begin{aligned}
X_t(\phi) &= X_0(\phi) + \int_0^t X_s(\Delta \phi/2) ds + M_t(\phi), \forall \phi \in D(\Delta), \\
\end{aligned}
\]

where \(M_t(\phi)\) is a continuous square integrable \(\mathcal{F}_t\)-martingale with \(\langle M(\phi) \rangle_t = \phi(0)^2 t^0_t(t)\) and \(M_0(\phi) = 0\).

If we pretend that the measure \(l^0(ds)\) is absolutely continuous with respect to Lebesgue measure, that is,

\[
l^0_t = \int_0^t X_s(0) ds,
\]

and \(X_t(0)\) is bounded, then it would be easy to derive that \(X_t(\cdot)\) is a solution to the following degenerate SPDE written in a mild form:

\[
X_t(x) = \int_\mathbb{R} p_t(x-y)X_0(dy) + \int_0^t p_{t-s}(x) \sqrt{X_s(0)} dB_s,
\]

\[\text{[1.5]}\]
where $p_t(x)$ is a transition density of Brownian motion (see [Zäh05] for related results). Set $x = 0$ to get the following stochastic integral equation (SIE)

$$X_t(0) = \int_{\mathbb{R}} p_t(y)X_0(dy) + \int_0^t \frac{1}{\sqrt{2\pi(t-s)^{-1}}} \sqrt{X_s(0)} dB_s.$$

(1.6)

However the assumption (1.5) is false – the local time $l^0(ds)$ is singular with respect to Lebesgue measure (see [DF94], [DFLM95]). In fact $X_t(dx)$ does not have a density at the point of catalyst $x = 0$ and hence we do not expect there to be a solution to (1.6) in the ordinary sense. But what we may ask about is the feasible parameters $\alpha$ and $\gamma$ such that there is a solution to the following analogous SIE

$$X_t(0) = x_0 + \int_0^t (t-s)^{-\alpha} \lambda|X_s(0)|^\gamma dB_s,$$

(1.7)

where $\lambda \in \mathbb{R}$. If we replace $|X_s(0)|^\gamma$ inside the stochastic integral, by a general $\gamma$-Hölder continuous function $\sigma(X_s(0))$, we arrive at the equation (1.2), uniqueness for which is the main concern of the current paper.

The connection between our SIE and SPDEs is more than simply an analogy or heuristic. In fact, some of our arguments rely on rewriting the SIE in terms of an SPDE that can be thought of as the $\gamma > \frac{1}{2}$ version of a “catalytic Bessel process”. In the particular case of $\gamma = 1/2$, we can, in fact, show that there is at most one non-negative solution of the equation (1.7) for any $\alpha \in (0, 1/2)$. Moreover, in our second main result, we establish weak uniqueness for non-negative solutions of such equations.

**Theorem 1.2** Assume that $\alpha \in (0, 1/2)$, and $\lambda \in \mathbb{R}$. Then for any $x_0 > 0$ and bounded non-negative continuous $g$, there exists at most one weak non-negative solution to

$$X_t = x_0 + \int_0^t (t-s)^{-\alpha} g(s) ds + \int_0^t (t-s)^{-\alpha} \lambda \sqrt{|X_s|} dB_s, \quad t \geq 0.$$

(1.8)

**Organization of the paper** In Section 2 we first prove existence results for our equations. Then, in Proposition 2.1 we treat the case of $\gamma = 1$ of Theorem 1.1. In the same section, we introduce the SPDE analogues of equations (1.2) and (1.8), and state corresponding uniqueness Theorems 2.5, 2.7. In Section 3 Theorems 2.7 and (1.2) are proved. The rest of the paper, except Section 8 is devoted to the proof of Theorem 2.5 from which Theorem 1.1 is an immediate consequence. In Section 8 the uniqueness for equations with kernels smoother than $(t-s)^{-\alpha}$ is considered.

## 2 Existence and background

Our first goal is to construct the solution to (1.2). In fact, we will prove existence of a solution to the more general equation:

$$X_t = h(t) + \int_0^t (t-s)^{-\alpha} \sigma(X_s) dB_s,$$

(2.1)

where $h$ is a continuous function.
Before we start dealing with the above questions, we introduce some notation, which will be used throughout this work. We write $C(\mathbb{R})$ for the space of continuous functions on $\mathbb{R}$. A superscript $k$ (respectively $\infty$) indicates that functions are in addition $k$ times (respectively infinitely many times) continuously differentiable. A subscript $b$ (respectively $c$) indicates that they are also bounded (respectively have compact support). We also define tempered norms

$$||f||_{\lambda, \infty} := \sup_{x \in \mathbb{R}} |f(x)| e^{-\lambda|x|},$$

set

$$C_{\text{tem}} := \{ f \in C(\mathbb{R}), ||f||_{\lambda, \infty} < \infty \text{ for every } \lambda > 0 \}$$

and endow it with the topology induced by the norms $|| \cdot ||_{\lambda, \infty}$ for $\lambda > 0$. That is, $f_n \to f$ in $C_{\text{tem}}$ iff $\lim_{n \to \infty} ||f - f_n||_{\lambda, \infty} = 0$ for all $\lambda > 0$. Similarly we define

$$C_{\text{rap}} := \{ f \in C(\mathbb{R}), ||f||_{\lambda, \infty} < \infty \text{ for every } \lambda < 0 \}$$

and endow it with the topology induced by the norms $|| \cdot ||_{\lambda, \infty}$ for $\lambda < 0$. $C_{\text{tem}}^k$ (respectively $C_{\text{rap}}^k$) denotes collection of functions in $C_{\text{tem}}$ (respectively in $C_{\text{rap}}$) which are in addition $k$ times continuously differentiable with all the derivatives in $C_{\text{tem}}$ (respectively in $C_{\text{rap}}$). As before $k$ can be equal to $\infty$.

For $I \subset \mathbb{R}_+$, let $C(I, E)$ be the space of all continuous functions on $I$ taking values in a topological space $E$, endowed with the topology of uniform convergence on compact subsets of $I$. In particular, $X \in C(\mathbb{R}_+, C_{\text{tem}})$ denotes a function $X_t(x)$ with $X_t \in C_{\text{tem}}$ varying continuously with $t$. In this context we will use either the notation $X(t, x)$ or $X_t(x)$, depending on which is more convenient. We will also denote by $C_{\text{tem}}^{k+}$ the collection of non-negative functions in $C_{\text{tem}}$.

Let $\mathcal{M}_f = \mathcal{M}_f(\mathbb{R})$ be the space of finite measures on $\mathbb{R}$ endowed with weak topology. Throughout the paper $c_i$ and $c_{i,j}$ will denote fixed positive constants, while $C$ and $c$ will denote positive constants which may change from line to line.

Now we return to the equation (2.1) First, let us treat the case of Lipschitz $\sigma$ (by this, we prove Theorem 1.1 for the case of $\gamma = 1$):

**Proposition 2.1** Let $\sigma$ be a continuous Lipschitz function. Assume $0 < \alpha < \frac{1}{2}$ and $h \in C(\mathbb{R}_+, \mathbb{R})$. Then there exists a unique strong solution $X$ to (2.1) in $C(\mathbb{R}_+, \mathbb{R})$. Moreover, for any $p > 0$, $T > 0$, there exists a constant $\rho = \rho(p, T, \sigma) < \infty$ such that

$$\sup_{0 \leq s \leq T} \mathbb{E} [|X_s|^p] < \rho(p, T, \sigma). \quad (2.2)$$

**PROOF.** We will use the standard Picard scheme. Let

$$X^0_t = h(t)$$

$$X^{n+1}_t = h(t) + \int_0^t (t-s)^{-\alpha} \sigma(X^n_s) \, dB_s, \quad n \geq 0. \quad (2.3)$$

Note that since $\sigma$ is Lipschitz it also satisfies a linear growth bound, that is, there exists a constant $s$ such that

$$|\sigma(x)| \leq s(1 + |x|), \quad \forall x \in \mathbb{R}. \quad (2.4)$$
First, let us prove by induction that $X^n$ is well defined for all $n$. In what follows, we fix an arbitrary $T > 0$. Assume inductively that $X^n$ is a well defined adapted process and

$$\sup_{0 \leq t \leq T} \mathbb{E} \left[ |X^n_t|^2 \right] < \infty. \quad (2.5)$$

Then, by using the growth condition (2.4), one can immediately get that the stochastic integral in (2.3) is well defined and hence $X^{n+1}$ is well defined, and moreover,

$$\sup_{0 \leq t \leq T} \mathbb{E} \left[ |X^{n+1}_t|^2 \right] \leq 2 \sup_{0 \leq t \leq T} h(t)^2 + \frac{4}{1 - 2\alpha} T^{1 - 2\alpha} \sup_{0 \leq t \leq T} \left( 1 + \mathbb{E} \left[ |X^n_t|^2 \right] \right) \quad (2.6)$$

So, by induction, we immediately get that $X^n$ is well defined for all $n$, and (2.5) holds for all $n$. Similarly, by using Burkholder-Gundy-Davis and Hölder inequalities, one can show that, for any $p \geq 2$, there exists a constant $c_{2.7} = c_{2.7}(p, c_{2.4}) < \infty$ such that

$$\sup_{0 \leq s \leq t} \mathbb{E} \left[ |X^{n+1}_s|^p \right] \leq \left( \sup_{0 \leq s \leq t} h(t)^p \right) + \int_0^t (t-s)^{-2\alpha} \left( 1 + \sup_{0 \leq u \leq s} \mathbb{E} \left[ |X^n_u|^p \right] \right) ds \quad (2.7)$$

By this, and by the extension of Gronwall’s lemma (see Lemma 15 in [Dal99]), we get that, in fact, there exists a constant $c_{2.7} = c(p, c_{2.4}) < \infty$ such that

$$\sup_{0 \leq s \leq t} \sup_{n \geq 0} \mathbb{E} \left[ |X^n_s|^p \right] \leq c_{2.7} \quad (2.8)$$

Now in order to show that the sequence $\{X^n_t\}_{n \geq 0}$ converges in $L^p$, define

$$V^n_t = \sup_{0 \leq s \leq t} \mathbb{E} \left[ |X^{n+1}_s - X^n_s|^p \right].$$

Since $\sigma$ is Lipschitz function, we conclude similarly to (2.7) that there exists a constant $c_{2.8} = c_{2.8}(p, c_{2.4}) < \infty$ such that

$$V^{n+1}_t \leq c_{2.8} \int_0^t (t-s)^{-2\alpha} V^n_s \, ds. \quad (2.8)$$

Since, by (2.7), $\sup_{0 \leq t \leq T} V^0_t < \infty$, we again get by the extension of Gronwall’s lemma (see Lemma 15 in [Dal99]) that $\{V^n_t\}_{n \geq 0}$ converges to 0 uniformly on $[0, T]$. This implies that there exists $X_t$ such that $\{X^n_t\}_{n \geq 0}$ converges to $X$ in $L^p$ uniformly on $[0, T]$. It is easy to check that $X$ has a jointly measurable version, and that $X$, in fact, satisfies (2.1) for a.e. $t$. The existence of continuous in time version of the process follows by standard application of Kolmogorv continuity criterion and is left to the reader. As we choose the continuous version of the process we get that $X$ satisfies (2.1) for all $t$. 

6
To prove uniqueness, let \( X^1_t \) and \( X^2_t \) solve (2.1). Suppose \(|\sigma(x) - \sigma(y)| \leq c|x - y|\). For \( K > 0 \), let \( T_K \) be the first time \( t \) that either of \(|X^1_t| > K\). Set
\[
m_K(t) = \sup_{s \leq t} \mathbb{E}(X^1_{s \wedge T_K} - X^2_{s \wedge T_K})^2 \leq 4K^2 < \infty.
\]
Then for \( s \leq t \),
\[
\mathbb{E}[(X^1_{s \wedge T_K} - X^2_{s \wedge T_K})^2] = \mathbb{E}[\int_0^{s \wedge T_K} (s - q)^{-2\alpha} (\sigma(X^1_q) - \sigma(X^2_q))^2 dq] \\
\leq c \mathbb{E}[\int_0^{s} (s - q)^{-2\alpha} (X^1_{q \wedge T_K} - X^2_{q \wedge T_K})^2 dq] \\
\leq cm_K(t) \int_0^{s} (s - q)^{-2\alpha} dq = cm_K(t) \frac{t^{1-2\alpha}}{1 - 2\alpha}.
\]
Therefore \( m_K(t) \leq cm_K(t) \frac{t^{1-2\alpha}}{1 - 2\alpha} \), from which we conclude that \( m_K(t) = 0 \) on some interval \([0, \varepsilon]\). Iterating the argument now shows that \( m_K(t) = 0 \) for all \( t \geq 0 \), and sending \( K \to \infty \) implies the desired result.

As for (2.2), it follows immediately by (2.7). \( \square \)

**Remark 2.2** Note that the constant \( c \) depends on \( \sigma \) only through the constant \( p \).

We now turn to the non-Lipschitz case.

**Lemma 2.3** Let \( \sigma \) be continuous and satisfy the growth bound (2.4). Assume \( 0 < \alpha < \frac{1}{2} \) and that \( h \in C(\mathbb{R}_+, \mathbb{R}) \). Then there exists a weak solution \( X \) to (2.1) in \( C(\mathbb{R}_+, \mathbb{R}) \) and
\[
\sup_{0 \leq s \leq T} \mathbb{E}[|X_s|^p] < \infty.
\] (2.9)

**Proof.** Choose a sequence of Lipschitz functions \( \{\sigma_n\}_{n \geq 1} \) which satisfy the growth condition (2.4) uniformly in \( n \), and such that \( \{\sigma_n\}_{n \geq 1} \) converges to \( \sigma \) uniformly on \( \mathbb{R} \), as \( n \to \infty \). Then by the previous proposition for each \( n \geq 1 \) there exists a process \( X^n \) that solves (2.1) with \( \sigma_n \). Since \( \sigma_n \) satisfy the growth condition (2.4) with the same constant, by (2.7) and Remark 2.2 we get that for any \( T > 0, p \geq 2 \),
\[
\sup_{n \geq 1} \sup_{0 \leq s \leq T} \mathbb{E}[|X^n_s|^p] < \infty.
\] (2.10)

Now, for any \( 0 \leq t < t' \), we have
\[
|X^n_{t'} - X^n_t|^p \leq C_p|h(t') - h(t)| + C_p \int_t^{t'} ((t' - s)^{-\alpha} - (t - s)^{-\alpha}) \sigma_n(X^n_s) dB_s \\
+ C_p \int_t^{t'} (t' - s)^{-\alpha} \sigma_n(X^n_s) dB_s.
\]

To bound the expectations of the three terms on the right hand side, use the Burkholder-Davis-Gundy and Hölder inequalities, (2.4), (2.10) and some simple algebra. This implies
\[
\mathbb{E}[|X^n_{t'} - X^n_t|^p] \leq C|t' - t|^{p(1/2 - \alpha)},
\]
where the constant on the right hand side does not depend on \( n \). By the Kolmogorov criterion we get the tightness of \( \{X^n\}_{n \geq 1} \) in \( C(\mathbb{R}_+, \mathbb{R}) \), and each weak limit point is Hölder continuous with any index less than \( 1/2 - \alpha \).

Let \( \{X^n_k\}_{k \geq 1} \) be some converging subsequence and \( X \) be the corresponding limit point. First, clearly (2.9) follows from (2.10). We will show that \( X \) satisfies (2.1).

Define

\[
Y_t^k = \int_0^t (t-s)^{\alpha-1}X^n_k \, ds, \quad t \geq 0, \quad k \geq 1.
\]

It is easy to check that \( Y^k \) satisfies the following equation

\[
Y_t^k = \int_0^t (t-s)^{\alpha-1}h(s) \, ds + c_\alpha \int_0^t \sigma_n(X^n_k) \, dB^n_k,
\]

where \( c_\alpha = \int_0^1 (1-r)^{\alpha-1}r^{-\alpha} \, dr \). By passing to the limit, due to convergence of \( \{X^n_k\}_{k \geq 1} \) in \( C(\mathbb{R}_+, \mathbb{R}) \) to \( Y_t = \int_0^t (t-s)^{\alpha-1}X \, ds \), \( t \geq 0 \). Moreover

\[
M^k_t = c_\alpha \int_0^t \sigma_n(X^n_k) \, dB^n_k, \quad t \geq 0, \quad k \geq 1,
\]

is a sequence of square integrable martingales with quadratic variations given by

\[
\langle M^k \rangle_t = c_\alpha^2 \int_0^t \sigma_n(X^n_k)^2 \, ds, \quad t \geq 0, \quad k \geq 1.
\]

By the uniform integrability, uniform convergence of \( \{\sigma_n\}_{k \geq 1} \) to \( \sigma \) and again by convergence of \( \{X^n_k\}_{k \geq 1} \) in \( C(\mathbb{R}_+, \mathbb{R}) \), we get that martingales converge to the martingale \( M \) with quadratic variation

\[
\langle M \rangle_t = c_\alpha^2 \int_0^t \sigma(X)^2 \, ds, \quad t \geq 0.
\]

Now it is standard to show that there exists a Brownian motion \( B \) such that \( M_t = c_\alpha \int_0^t \sigma(X) \, dB_s, \quad t \geq 0 \), and hence,

\[
Y_t = \int_0^t (t-s)^{\alpha-1}h(s) \, ds + c_\alpha \int_0^t \sigma(X) \, dB_s, \quad t \geq 0.
\]

By reversing the transformation, that is, by recalling that

\[
X_t = \frac{1}{c_\alpha} \frac{d}{dt} \int_0^t (t-s)^{-\alpha}Y_s \, ds, \quad t \geq 0,
\]

it is easy to verify that \( X \) is a solution to (2.1). \( \square \)

We will now construct an SPDE related (2.1). Fix \( \theta > 0 \). Define

\[
\Delta \theta = \frac{2}{(2+\theta)^2} \frac{\partial}{\partial x} |x|^{-\theta} \frac{\partial}{\partial x} \quad (2.11)
\]

Then, for some constant \( c_\vartheta > 0 \), the function

\[
p^\vartheta_t(x) = \frac{c_\vartheta}{t^{\frac{n+2}{2}}} e^{-\frac{|x|^2 + \vartheta t}{2}} \quad (2.12)
\]
is a classical solution to the following evolution equation

\[
\begin{cases}
\frac{\partial u}{\partial t} = \Delta_\theta u \\
u_0 = \delta_0
\end{cases}
\]

on \( \mathbb{R}_+ \times \mathbb{R} \). By changing variables, we see that \( \int p_t^\theta(x) \, dx \) is independent of \( t \), and we choose \( c_\theta \) to make \( p_t^\theta \) a probability density. Note that \( \Delta_0 = \frac{1}{2} \Delta \), where \( \Delta \) is the classical Laplacian.

Let \( \{ S_t, t \geq 0 \} \) be the semigroup generated by \( \Delta_\theta \). That is,

\[ S_t \phi(x) = \int_{\mathbb{R}} p_t^\theta(x, y) \phi(y) \, dy, \tag{2.13} \]

where \( p_t^\theta(x, y) \) is the transition density for the process with generator \( \Delta_\theta \) (ie the fundamental solution to \( \dot{u} = \Delta_\theta u \)). Define the domain of the operator \( \Delta_\theta \):

\( D(\Delta_\theta) \equiv \{ \phi \in C^2_{\text{rap}} : \Delta_\theta \phi \in C_{\text{rap}} \} \).

(2.14)

In certain cases we will need also domain containing more functions:

\( D_{\text{tem}}(\Delta_\theta) \equiv \{ \phi \in C^2_{\text{tem}} : \Delta_\theta \phi \in C_{\text{tem}} \} \).

(2.15)

The generator is ambiguous at \( x = 0 \), but we choose the semigroup to be symmetric; \( p_t^\theta(0, x) = p_t^\theta(0, -x) \). Because \( \Delta_\theta \) is in divergence form, \( p_t^\theta(x, y) = p_t^\theta(y, x) \), so in particular,

\[ p_t^\theta(x, 0) = p_t^\theta(0, x) = p_t^\theta(x), \]

where the latter is given by (2.12). It is simple to verify that if a process \( \xi_t \) has semigroup \( S_t \) then \( |\xi_t|^2 + \frac{\theta}{2} \text{sign}(\xi_t) \) is a Bessel process of dimension \( \frac{2}{2 - \theta} < 1 \), so in fact, explicit formulas for \( p_t(x, y) \) could be given.

**Lemma 2.4** Let \( X_0 \in C_{\text{tem}} \) and \( g \in C(\mathbb{R}_+, C_{\text{tem}}) \cup C(\mathbb{R}_+, \mathcal{M}_1) \). Assume that \( \theta \in (0, \infty) \) and that \( \sigma \) is continuous and satisfies a linear growth condition (2.14). Then there exists a weak solution \( X \in C(\mathbb{R}_+, C_{\text{tem}}) \) to the following SPDE

\[
X(t, x) = S_t X_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}^\theta(x, y) \sigma(X(s, 0)) \, dB_y + \int_0^t \int_{\mathbb{R}} p_{t-s}^\theta(x, y) g(s, y) \, dy \, ds, \quad t \geq 0. \tag{2.16}
\]

**Proof.** By Lemma 2.3 there exists weak solution \( V \) to the SIE

\[ V_t = h(t) + \int_0^t c_\theta(t - s)^{-\alpha} \sigma(V_s) dB_s, \]

with

\[ h(t) = S_t X_0(0) + \int_0^t \int_{\mathbb{R}} p_{t-s}^\theta(0, y) g(s, y) \, dy \, ds, \]

\( c_\theta \) as in (2.12), and

\[ \alpha = \frac{1}{2 + \theta}, \tag{2.17} \]
Now define

\[ X(t,x) = S_t X_0(x) + \int_0^t \int_\mathbb{R} p^0_{t-s}(x,y) g(s,y) \, dy \, ds + \int_0^t p^0_{t-s}(x) \sigma(V_s) \, dB_s \]

It is trivial to check that \( X \) is indeed solution to (2.10) with \( X(t,0) = V_t, t \geq 0 \), and \( X \) is in \( C(\mathbb{R}_+, C_{tem}) \).

For the rest of the paper we will also assume (2.17).

It is clear from Lemma 2.4 and its proof that there is a correspondence between SPDEs of type (2.10) and SIEs of type (2.1). Consider the particular case with \( X_0 = x_0 = \text{const and } g(s,x) = \frac{1}{c_0} g(s) \delta_0 \) in (2.16). Then \( S_t x_0 = x_0 \), so (2.16) becomes

\[ X(t,x) = x_0 + \int_0^t p^0_{t-s}(x) \frac{g(s)}{c_0} \, ds + \int_0^t p^0_{t-s}(x) \sigma(X(t,0)) \, dB_s. \]

In particular for \( x = 0 \) we have

\[ X(t,0) = x_0 + \int_0^t (t-s)^{-\alpha} g(s) \, ds + \int_0^t c_0 (t-s)^{-\alpha} \sigma(X(s,0)) \, dB_s. \]

Thus we get that \( X_t = X(t,0) \) satisfies the SIE given in (1.2) with \( c_0 \sigma(\cdot) \) instead of \( \sigma(\cdot) \). Conversely, if \( X_t \) is a solution to (1.2) ith \( c_0 \sigma(\cdot) \) instead of \( \sigma(\cdot) \), then as in the proof of Lemma 2.4 we can define

\[ X(t,x) \equiv x_0 + \int_0^t p^0_{t-s}(x) \frac{g(s)}{c_0} \, ds + \int_0^t p^0_{t-s}(x) \sigma(X(s)) \, dB_s. \]

Then \( X(\cdot,\cdot) \) lies in \( C(\mathbb{R}_+, C_{tem}) \) and satisfies (2.10) with \( X(0,\cdot) = x_0 \) and \( g(s,\cdot) = \frac{c_0}{\sigma_0} \delta_0(\cdot) \). Thus Theorem 1.1 will follow if we can show pathwise uniqueness for (2.10). In order to prove the pathwise uniqueness for (1.2) it is enough to prove the pathwise uniqueness for (2.10). In other words, it follows once we prove the following theorem.

**Theorem 2.5** Assume that \( \alpha \in (0,1/2) \) and that \( \sigma : \mathbb{R} \to \mathbb{R} \) satisfies (2.7) and (2.4) for some \( \gamma \in \left( \frac{1}{2}\alpha - \frac{1}{2}, 1 \right] \). Let \( X_0 \in C_{tem} \) and \( g \in C(\mathbb{R}_+, C_{tem}) \cup C(\mathbb{R}_+, \mathcal{M}_t) \). Then pathwise uniqueness holds for solutions of (2.10) in \( C(\mathbb{R}_+, C_{tem}) \).

The proof of our pathwise uniqueness theorems will require some moment bounds and regularity properties for arbitrary continuous \( C_{tem} \)-valued solutions to the equation (2.16). We know that the fractional Brownian motion \( \int_0^t (t-s)^{-\alpha} \, dB_s \) is Hölder continuous with exponent \( \xi \) for any \( \xi < \frac{1}{2} - \alpha \). More generally, if \( X \) is any solution to (2.16) then \( X(t,0) \) is Hölder with exponent \( \xi \) for any \( \xi < \frac{1}{2} - \alpha \). In fact, we have the following result.

**Proposition 2.6** Let \( X_0 \in C_{tem}, g \in C(\mathbb{R}_+, C_{tem}) \cup C(\mathbb{R}_+, \mathcal{M}_t), \alpha \in (0,1/2) \) and let \( \sigma \) be a continuous function satisfying the growth bound (2.7). Then any solution \( X \in C(\mathbb{R}_+, C_{tem}) \) to (2.16) has the following properties.
For any $T, \lambda > 0$ and $p \in (0, \infty)$,
\[ \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} \mathbb{E} \left( |X(t, x)|^p e^{-\lambda |x|} \right) < \infty. \] (2.18)

(b) For any $\xi \in (0, \frac{1}{2} - \alpha)$ the process $X(\cdot, \cdot)$ is a.s. uniformly Hölder continuous on compacts in $(0, \infty) \times [-1, 1]$, and the process
\[ Z(t, x) \equiv X(t, x) - S_t X_0(x) - \int_0^t \int_{\mathbb{R}} p_{t-s}(x, y) g(s, y) \, dy \, ds, \]
is uniformly Hölder continuous on compacts in $[0, \infty) \times [-1, 1]$, with Hölder coefficients $\xi$ in time and space. Moreover, for any $T > 0$, $R > 0$, and $0 \leq t, t' \leq T, x, x' \in \mathbb{R}$ such that $|x|, |x'| \leq 1$ as well as $p \in [2, \infty)$, there exists a constant $c_{2.19} = c_{2.19}(T, p)$ such that
\[ \mathbb{E} \left( |Z(t, x) - Z(t', x')|^p \right) \leq c_{2.19} \left( |t - t'|^{(1/2 - \alpha)p} + |x - x'|^{(1/2 - \alpha)p} \right). \] (2.19)

The proof of Proposition 2.6 is delayed to Section 6.

It is straightforward to show that under the hypotheses of Lemma 2.4, solutions to (2.16) with continuous $C_{\text{tem}}$-valued paths are also solutions to the equation in its distributional form for suitable test functions $\Phi$. More specifically, for $\Phi \in C(\mathbb{R}_+, D(\Delta \theta))$, such that $s \mapsto \frac{\partial \Phi}{\partial s} \in C(\mathbb{R}_+, C_{\text{rap}})$, we have
\[ \int_{\mathbb{R}} X(t, x) \Phi_t(x) \, dx = \int_{\mathbb{R}} X_0(x) \Phi_0(x) \, dx \]
(2.20)
+ $\int_0^t \int_{\mathbb{R}} X(s, x) \left( \Delta \theta \Phi_s(x) + \frac{\partial \Phi_s(x)}{\partial s} \right) \, dx \, ds$

+ $\int_0^t \int_{\mathbb{R}} g(s, x) \Phi_s(x) \, dx \, ds$

+ $\int_0^t \sigma(X(s, 0)) \Phi_s(0) \, dB_s \forall t \geq 0 \text{ a.s.}$

In fact, given an appropriate class of test functions, the two notions of solution (2.16) and (2.20) are equivalent. For the details of a similar proof we refer to Shiga [Shi94] Theorem 2.1 and its proof. There, the setting is a bit different as it works in the setting of a non-degenerate SPDE. However, the arguments do not change as long as the stochastic integral in (2.20) is well defined, which can easily be checked.

Now we say a few words about the proof of Theorem 1.2. Again, by Lemma 2.4 its proof and discussion after it, it is enough to prove the weak uniqueness for corresponding SPDE. That is we are going to prove the following result.

**Theorem 2.7** Assume that $\alpha \in (0, 1/2)$, and $\lambda \in \mathbb{R}$. Let $X_0 \in C_{\text{tem}}$ and $g \in C(\mathbb{R}_+, C_{\text{tem}}) \cup C(\mathbb{R}_+, \mathcal{M}_1)$. Then there exists at most one weak solution
\[ X \in C(\mathbb{R}_+, C^+_\text{tem}) \text{ to the following SPDE} \]
\[ \begin{align*}
X(t, x) &= S_t X_0(x) + \int_0^t p^\theta_{t-s}(x) \lambda \sqrt{X(s, 0)} dB_s \\
&\quad + \int_0^t \int_{\mathbb{R}} p^\theta_{t-s}(x, y) g(s, y) dy ds, \quad t \geq 0.
\end{align*} \tag{2.21} \]

3 Proof of Theorems 2.7 and 1.2

We start with proving Theorem 2.7. By simple scaling, we may and will assume, without loss of generality, that \( \lambda = 1 \).

We need the following lemma.

**Lemma 3.1** Let \( \phi \) be a non-negative function in \( C_c(\mathbb{R}) \). Then there exists a unique, non-negative solution \( u = U\phi \in C(\mathbb{R}_+, C_{\text{rap}}(\mathbb{R})) \) to the following equation
\[ u(t, x) = S_t \phi(x) - \int_0^t p^\theta_{t-s}(x) u(s, 0)^2 ds, \quad t \geq 0, \quad x \in \mathbb{R}. \tag{3.1} \]

**PROOF.** The proof is an easy adaptation of the proof of Proposition 2.3.1 in [DF94] for our situation where the Brownian semigroup and its kernel is replaced by the semigroup and the kernel generated by \( \Delta_\theta \). It is not difficult to see that all the basic estimates hold also in this case. Note that the proof of Proposition 2.3.1 in [DF94] also uses the ideas from the proof of Theorem 3.5 in [DFR91] where a more general set of “catalysts” is considered. The fact that for any \( t \geq 0 \), \( u(t, \cdot) \in C_{\text{rap}}(\mathbb{R}) \) is an easy consequence of the domination
\[ u(t, x) \leq S_t \phi(x), \quad t \geq 0, \quad x \in \mathbb{R}. \]

\[ \square \]

For any two functions \( \phi, \psi \) on \( \mathbb{R} \), denote
\[ \langle \phi, \psi \rangle \equiv \int_{\mathbb{R}} \phi(x)\psi(x) dx \]
whenever integral exists.

Now we need the following lemma.

**Lemma 3.2** Let \( X_0 \in C^+_{\text{tem}} \) and \( g \in C(\mathbb{R}_+, C^+_{\text{tem}}) \cup C(\mathbb{R}_+, M_t) \). Let \( X \) be any solution to (2.21) in \( C(\mathbb{R}_+, C^+_{\text{tem}}) \). Then for any non-negative \( \phi \in C_c(\mathbb{R}) \), we have
\[ \mathbb{E} \left[ e^{-\langle X_t, \phi \rangle} \right] = \mathbb{E} \left[ e^{-\langle X_0, U_t^\phi \rangle - \int_0^t \langle g(s, \cdot), U_{t-s}^\phi \rangle ds} \right], \tag{3.2} \]
for all \( t \geq 0 \).

**PROOF.** First, note that by standard arguments the solution \( U^\phi \) to (3.1) also satisfies the following weak form of the equation
\[ \langle U_t^\phi, \psi \rangle = \langle \phi, \psi \rangle + \int_0^t \langle U_s^\phi, \Delta_\theta \psi \rangle - \frac{1}{2} U_s^\phi(0)^2 \psi(0) ds, \quad t \geq 0, \tag{3.3} \]
forall $\psi \in D_{\text{tem}}(\Delta \theta)$. Moreover $(U^\phi_t, \psi)$ is differentiable in $t$ and

$$\frac{\partial (U^\phi_t, \psi)}{\partial t} = \langle U^\phi_t, \Delta \theta \psi \rangle - \frac{1}{2} U^\phi_t(0)^2 \psi(0), \quad t \geq 0,$$

(3.4)

forall $\psi \in D_{\text{tem}}(\Delta \theta)$.

Fix arbitrary non-negative $\phi \in C^\infty_c(\mathbb{R})$ and $\epsilon > 0$. By properties of $S_t$ and spaces $C_{\text{rap}}, C_{\text{tem}}$, it is easy to check that

$$S_{\epsilon} U^\phi \in C(\mathbb{R}^+, D(\Delta \theta)), \quad \frac{\partial S_{\epsilon} U^\phi}{\partial s} \in C(\mathbb{R}^+, C_{\text{rap}}),$$

(3.5)

and

$$S_{\epsilon} X_t \in C(\mathbb{R}^+, D_{\text{tem}}(\Delta \theta)), \quad \text{a.s.}$$

(3.6)

Fix arbitrary $T > 0$. Use (2.20), (3.5) and (3.6) to get

$$\langle X_t, S_{\epsilon} U^\phi_T - t \rangle = \langle X_0, S_{\epsilon} U^\phi_T \rangle (3.7)$$

$$+ \int_0^t \left\langle X_s, \Delta \theta S_{\epsilon} U^\phi_{T-s} + \frac{\partial S_{\epsilon} U^\phi_{T-s}}{\partial s} \right\rangle ds$$

$$+ \int_0^t \langle g(s, \cdot), S_{\epsilon} U^\phi_{T-s} \rangle ds$$

$$+ \int_0^t \sqrt{X(s, 0)} S_{\epsilon} U^\phi_{T-s}(0) dB_s \quad \forall t \geq 0, \quad \text{a.s.}$$

Now use (3.4) and the fact that

$$\left\langle U^\phi_{T-s}, \Delta \theta p_{\epsilon}(x, \cdot) \right\rangle = \Delta \theta S_{\epsilon} U^\phi_{T-s}(x), \quad \forall x \in \mathbb{R},$$

to get

$$\Delta \theta S_{\epsilon} U^\phi_{T-s}(x) + \frac{\partial S_{\epsilon} U^\phi_{T-s}}{\partial s}(x) \quad \text{(3.8)}$$

$$= \frac{1}{2} U^\phi_{T-s}(0)^2 p_{\epsilon}(x), \quad \forall x \in \mathbb{R}.$$
By the Itô formula we easily get
\[
E\left[e^{-(X_t,S_{t}U_{T-t}^{\phi})-\int_{0}^{T}g(s,\cdot,S_{t}U_{T-t}^{\phi})\,ds}\right] = E\left[e^{-(X_0,S_{0}U_{T}^{\phi})-\int_{0}^{T}g(s,\cdot,S_{t}U_{T-t}^{\phi})\,ds}\right] + \int_{0}^{T}E\left[e^{-(X_0,S_{0}U_{T-t}^{\phi})-\int_{0}^{T}g(s,\cdot,S_{t}U_{T-t}^{\phi})\,ds}\right]\times \frac{1}{2}\left\{X(s,0)(S_{t}U_{T-t}^{\phi}(0))^2 - S_{t}X_s(0)(S_{t}U_{T-t}^{\phi}(0))^2\right\}\,ds. \tag{3.10}
\]

Now let \(\epsilon \to 0\). Use the continuity of \(X_t(\cdot), U_t^{\phi}(\cdot)\) and the dominated convergence theorem to get
\[
E\left[e^{-(X_T,S_{T}U_{T}^{\phi})-\int_{0}^{T}g(s,\cdot,S_{t}U_{T-t}^{\phi})\,ds}\right] = E\left[e^{-(X_0,S_{0}U_{T}^{\phi})-\int_{0}^{T}g(s,\cdot,S_{t}U_{T-t}^{\phi})\,ds}\right],\tag{3.11}
\]
for all \(t \in [0,T]\). By taking \(t = T\), we get
\[
E\left[e^{-(X_T,\phi)}\right] = E\left[e^{-(X_0,U_0^{\phi})-\int_{0}^{T}g(s,\cdot,U_{T-t}^{\phi})\,ds}\right], \quad \forall T > 0, \tag{3.12}
\]
and we are done. \(\square\)

**Proof of Theorem 2.7** By Lemma 3.2 we immediately get that for any solution \(X_t(\cdot)\), and \(U_t^{\phi}(\cdot)\) and the dominated convergence theorem to get
\[
E\left[e^{-(X_t,S_{t}U_{T-t}^{\phi})-\int_{0}^{T}g(s,\cdot,S_{t}U_{T-t}^{\phi})\,ds}\right] = E\left[e^{-(X_0,S_{0}U_{T}^{\phi})-\int_{0}^{T}g(s,\cdot,S_{t}U_{T-t}^{\phi})\,ds}\right],
\]
for all \(t \in [0,T]\). By taking \(t = T\), we get
\[
E\left[e^{-(X_0,\phi)}\right] = E\left[e^{-(X_0,U_0^{\phi})-\int_{0}^{T}g(s,\cdot,U_{T-t}^{\phi})\,ds}\right], \quad \forall T > 0,
\]
and we are done. \(\square\)

**Proof of Theorem 1.2** This follows immediately by correspondance of so-lutions to (2.21) and (1.8) (see Lemma 2.4 and discussion after it). \(\square\)

Note that, in fact, \(U^{\phi}\) is the so-called log-Laplace equation for the catalytic superprocess with single point catalyst at 0, and the motion process generated by \(\Delta_\theta\). So in principal, we could prove Theorem 2.7 by showing that such any such superprocess is in fact a weak solution to (2.21). We gave the more detailed proof just for the sake of completeness.

4 Uniqueness: preliminary estimates

In this section we will develop machinery for proving Theorem 2.5. The proof follows a similar approach to that in [MPS05].

Let \(\rho\) be a strictly increasing function on \(\mathbb{R}_+\), such that
\[
\rho(x) \geq \sqrt{x}. \tag{4.1}
\]
and
\[
\int_{0^+} \rho^{-2}(x) dx = \infty. \tag{4.2}
\]
As in the proof of Yamada and Watanabe [YW71], we may define a sequence of functions \(\phi_n\) in the following way. First, let \(a_n \downarrow 0\) be a strictly decreasing sequence such that \(a_0 = 1\), and
\[
\int_{a_n}^{a_{n-1}} \rho^{-2}(x) dx = n. \tag{4.3}
\]
Second, we define functions \(\psi_n \in C_c^\infty(\mathbb{R})\) such that \(\text{supp}(\psi_n) \subset (a_n, a_{n-1})\), and that
\[
0 \leq \psi_n(x) \leq \frac{2\rho^{-2}(x)}{n} \leq \frac{2}{nx} \quad \text{for all } x \in \mathbb{R} \text{ as well as } \int_{a_n}^{a_{n-1}} \psi_n(x) dx = 1. \tag{4.4}
\]
Finally, set
\[
\phi_n(x) = \int_0^{|x|} \int_0^y \psi_n(z) dz dy. \tag{4.5}
\]
From this it is easy to see that \(\phi_n(x) \uparrow |x|\) uniformly in \(x \geq 0\). Note that each \(\psi_n\) and thus also each \(\phi_n\) is identically zero in a neighborhood of zero. This implies that \(\phi_n \in C_c^\infty(\mathbb{R})\) despite the absolute value in its definition. We have
\[
\phi_n'(x) = \text{sgn}(x) \int_0^{|x|} \psi_n(y) dy,
\]
\[
\phi_n''(x) = \psi_n(|x|).
\]
Thus, \(|\phi_n'(x)| \leq 1\), and \(\int \phi_n''(x) h(x) dx \rightarrow h(0)\) for any function \(h\) which is continuous at zero.

Now let \(X^1\) and \(X^2\) be two solutions of (2.16) with sample paths in \(C(\mathbb{R}_+, C_{tem})\) a.s., with the same initial condition, \(X^1(0) = X^2(0) = X_0 \in C_{tem}\), and the same Brownian motion \(B\) in the setting of Theorem 2.5. Define \(X = X^1 - X^2\). Set \(\Phi_x^m(y) = p_{m,n}^\nu(x,y)\). Note that for any \(x \in \mathbb{R}\), \(\Phi_x^m(\cdot) \in D(\Delta_\theta)\). Use (2.20) to get the semimartingale decomposition of \(\langle X_t, \Phi_x^m \rangle = \langle X_1^1 - X_1^2, \Phi_x^m \rangle\). Then apply Itô’s Formula to the semimartingale \(\langle X_t, \Phi_x^m \rangle\) to get
\[
\phi_n(\langle X_t, \Phi_x^m \rangle)
= \int_0^t \int_{\mathbb{R}} \phi_n'((\tilde{X}_s, \Phi_x^m)) \left( \sigma(X^1(s,0)) - \sigma(X^2(s,0)) \right) \Phi_x^m(0) dB_s
+ \int_0^t \phi_n'((\tilde{X}_s, \Phi_x^m)) \langle \tilde{X}_s, \Delta_\theta \Phi_x^m \rangle ds
+ \frac{1}{2} \int_0^t \psi_n((\tilde{X}_s, \Phi_x^m)) \left( \sigma(X^1(s,0)) - \sigma(X^2(s,0)) \right)^2 \Phi_x^m(0)^2 ds.
\]
We integrate this function of \(x\) against another non-negative test function \(\Psi \in C([0,t], D(\Delta_\theta))\) such that
\[
\Psi_s(0) > 0, \forall s \geq 0 \text{ and } \sup_{s \leq t} \left| \int_{\mathbb{R}} |x|^{-b} \left( \frac{\partial \Psi_s(x)}{\partial x} \right)^2 dx \right| < \infty, \forall t > 0, \tag{4.6}
\]
and \( s \mapsto \frac{\partial \psi_t}{\partial x} \in C(\mathbb{R}, C_{\text{rap}}) \). Also assume \( \Gamma(t) \equiv \{ x : \exists s \leq t, \Psi_s(x) > 0 \} \subset B(0, J(t)) \) for some \( J(t) > 0 \). We then obtain by the classical and stochastic version of Fubini’s Theorem, and arguing as in the proof of Proposition II.5.7 of [Pert02] to handle the time dependence in \( \Psi \), that for any \( t \geq 0 \),

\[
\left\langle \phi_n((\bar{X}_t, \Phi^{m}_t)), \Psi_t \right\rangle
\]

\( \left( \begin{array}{c} t \\ 0 \end{array} \right) = \int_{0}^{t} \left\langle \phi_n'((\bar{X}_s, \Phi^{m}_s)) \Phi^{m}_s(0), \Psi_s \right\rangle (\sigma(X^1(s, 0)) - \sigma(X^2(s, 0))) dB_s \\
+ \int_{0}^{t} \left\langle \phi_n'((\bar{X}_s, \Phi^{m}_s)) \bar{X}_s, \Delta \Phi^{m}_s \right\rangle, \Psi_s ds
\]

\( + \frac{1}{2} \int_{0}^{t} \left\langle \psi_n((\bar{X}_s, \Phi^{m}_s)) |\Phi^{m}_s(0)|^2, \Psi_s \right\rangle (\sigma(X^1(s, 0)) - \sigma(X^2(s, 0)))^2 ds \\
+ \int_{0}^{t} \left\langle \psi_n((\bar{X}_s, \Phi^{m}_s)), \Psi_s \right\rangle ds
\]

\( \equiv I_{1}^{m, n}(t) + I_{2}^{m, n}(t) + I_{3}^{m, n}(t) + I_{4}^{m, n}(t). \)

We need a calculus lemma. For \( f \in C^2(\mathbb{R}) \), let \( \|D^2 f\|_\infty = \|\frac{\partial^2 f}{\partial x^2}\|_\infty. \)

**Lemma 4.1** Let \( f \in C^2(\mathbb{R}) \) be non-negative and not identically zero. Then

\[
\sup \left\{ \left( \frac{\partial f}{\partial x} (x) \right)^2 f(x)^{-1} : f(x) > 0 \right\} \leq 2 \|D^2 f\|_\infty.
\]

**PROOF.** See Lemma 2.1 of [MPS05].

We now consider the expectation of expression \( (4.7) \) stopped at a stopping time \( T \), that we will choose later on. Ultimately we will use the following to show that the contributions of \( I_{1}^{m, n}, I_{2}^{m, n}, \) and \( I_{4}^{m, n} \) to this expectation disappear in the limit. \( I_{3}^{m, n} \) is where the classical Yamada-Watanabe calculation comes into play, to be analyzed in Section [5].

**Lemma 4.2** Assume the hypotheses of Theorem 2.5. For any stopping time \( T \) and constant \( t \geq 0 \) we have:

(a) \( \mathbb{E}(I_{1}^{m, n}(t \wedge T)) = 0 \) for all \( m, n. \)

(b) \( \limsup_{m,n \to \infty} \mathbb{E}(I_{2}^{m, n}(t \wedge T)) \leq \mathbb{E} \left( \int_{0}^{t \wedge T} \int_{\mathbb{R}} |\bar{X}(s, x)| \Delta \Psi_s(x) dx ds \right). \)

(c) \( \lim_{m,n \to \infty} \mathbb{E}(I_{4}^{m, n}(t \wedge T)) = \mathbb{E} \left( \int_{0}^{t \wedge T} |\bar{X}(s, x)| \psi_s(x) ds \right). \)

**PROOF.** (a) Let \( g_{m,n}(s) = \left\langle \phi_n((\bar{X}_s, \Phi^{m}_s)), \Phi^{m}_s(0), \Psi_s \right\rangle. \) Note first that \( I_{1}^{m, n}(t \wedge T) \) is a continuous local martingale with square function

\[
(I_{1}^{m, n})_{t \wedge T} = \int_{0}^{t \wedge T} g_{m,n}(s)^2 (\sigma(X^1(s, 0)) - \sigma(X^2(s, 0)))^2 ds \\
\leq C \int_{0}^{t \wedge T} g_{m,n}(s)^2 (|X^1(s, 0)| + |X^2(s, 0)| + 2)^2 ds.
\]
An easy calculation shows that $|g_{m,n}(s,y)| \leq \|\Psi\|_\infty$, so by (2.15)
\[
\mathbb{E}(I_{m,n}^{m,n}(t \wedge T)) \leq C(t) < \infty \quad \forall t > 0.
\]
This shows $I_{m,n}^{m,n}(t \wedge T)$ is a square integrable martingale and so has mean 0, as required.

(b) We have to rewrite $I_{2,m,n}^{m,n}$. Denote by $\Delta_{x,\theta}$ the $\theta$-Laplacian acting with respect to $x$. We know by symmetry that
\[
\Delta_{y,\theta}\Phi_m^n(y) = \Delta_{x,\theta}\Phi_x^m(y).
\]
Hence, since $\bar{X}_s$ is locally integrable and continuous we have for $|x| \leq J(t)$,
\[
\int_\mathbb{R} \bar{X}(s,y)\Delta_{x,\theta}\Phi_x^m(y)dy = \int_\mathbb{R} \bar{X}(s,y)\Delta_{x,\theta}\Phi_x^m(y)dy = \Delta_{x,\theta}\int_\mathbb{R} \bar{X}(s,y)\Phi_x^m(y)dy,
\]
for all $m$. This implies for any $t \geq 0$,
\[
I_{2,m,n}^{m,n}(t) = \int_0^t \int_\mathbb{R} \phi'_n((\bar{X}_s, \Phi_x^m))\Delta_{x,\theta}((\bar{X}_s, \Phi_x^m)) \Psi_s(x)dxds
\]
\[
= -2\alpha^2 \int_0^t \int_\mathbb{R} \frac{\partial}{\partial x} \left( \phi'_n((\bar{X}_s, \Phi_x^m)) \right) |x|^{-\theta} \frac{\partial}{\partial x} \left( (\bar{X}_s, \Phi_x^m) \right) \Psi_s(x)dxds
\]
\[
= -2\alpha^2 \int_0^t \int_\mathbb{R} \psi_n((\bar{X}_s, \Phi_x^m))|x|^{-\theta} \left( \frac{\partial}{\partial x} (\bar{X}_s, \Phi_x^m) \right)^2 \Psi_s(x)dxds
\]
\[
+ 2\alpha^2 \int_0^t \int_\mathbb{R} \phi'_n((\bar{X}_s, \Phi_x^m))\Delta_{x,\theta}\Psi_s(x)dxds
\]
\[
= \int_0^t I_{2,1,m,n}^{m,n}(s) + I_{2,2,m,n}^{m,n}(s) + I_{2,3,m,n}^{m,n}(s)ds.
\]

Above, we have used that $\phi'_n = \psi_n$ and we have repeatedly used integration by parts, the product rule as well as the chain rule on $\phi'_n((\bar{X}_s, \Phi_x^m))$. In order to deal with the various parts of $I_{2,m,n}^{m,n}$ we will first jointly consider $I_{2,1}^{m,n}$ and $I_{2,2}^{m,n}$.

For fixed $s$ we define a.s.,
\[
A^s = \left\{ x : \left( \frac{\partial}{\partial x} (\bar{X}_s, \Phi_x^m) \right)^2 \Psi_s(x) \leq (\bar{X}_s, \Phi_x^m) \frac{\partial}{\partial x} (\bar{X}_s, \Phi_x^m) \frac{\partial}{\partial x} \Psi_s(x) \right\} \cap \{ x : \Psi_s(x) > 0 \}
\]
\[
= A^{+,s} \cup A^{-,s} \cup A^{0,s},
\]

17
Finally, for any $t$

\[ A^{+,s} = A^s \cap \{ \frac{\partial}{\partial x}(\tilde{X}_s, \Phi_x^m) > 0 \}, \]

\[ A^{-,s} = A^s \cap \{ \frac{\partial}{\partial x}(\tilde{X}_s, \Phi_x^m) < 0 \}, \]

\[ A^{0,s} = A^s \cap \{ \frac{\partial}{\partial x}(\tilde{X}_s, \Phi_x^m) = 0 \}. \]

By (4.6) we can find $\epsilon > 0$ sufficiently small such that

\[ B(0, \epsilon) \subset \Gamma(t), \quad \text{and} \quad \inf_{s \leq t, x \in B(0, \epsilon)} \Psi_s(x) > 0. \quad (4.8) \]

On $A^{+,s}$ we have

\[ 0 < \left( \frac{\partial}{\partial x}(\tilde{X}_s, \Phi_x^m) \right) \Psi_s(x) \leq (\tilde{X}_s, \Phi_x^m) \frac{\partial}{\partial x} \Psi_s(x), \]

and therefore for any $t \geq 0$,

\[
\begin{align*}
& \int_0^t \int_{A^{+,s}} \psi_n(|(\tilde{X}_s, \Phi_x^m)|)|x|^{-\theta}(\tilde{X}_s, \Phi_x^m) \frac{\partial}{\partial x} \Psi_s(x) \frac{\partial}{\partial x} (\tilde{X}_s, \Phi_x^m) dx ds \\
& \quad \leq \int_0^t \int_{A^{+,s}} \psi_n(|(\tilde{X}_s, \Phi_x^m)|)|x|^{-\theta}(\tilde{X}_s, \Phi_x^m)^2 \left( \frac{\partial}{\partial x} \Psi_s(x) \right)^2 dx ds \\
& \quad \leq \int_0^t \int_{A^{+,s}} 2 \frac{1}{n} \frac{1}{1 \leq |(\tilde{X}_s, \Phi_x^m)| \leq a_n} |x|^{-\theta}|(\tilde{X}_s, \Phi_x^m)| \left( \frac{\partial}{\partial x} \Psi_s(x) \right)^2 dx ds \quad \text{by (4.4)} \\
& \quad \leq \frac{2a_n}{n} \int_0^t \int_{B(0, \epsilon)} \frac{(\frac{\partial}{\partial x} \Psi_s(x))^2}{\Psi_s(x)} |x|^{-\theta} dx ds + 2n^2 \Psi_s(x) \int_{\Gamma \cap B(0, \epsilon)} |x|^{-\theta} dx ds \\
& \quad \equiv \frac{2a_n}{n} C(\Psi, t),
\end{align*}
\]

where (4.6), (4.8) and Lemma 4.1 are used in the last two lines. Similarly, on the set $A^{-,s}$,

\[ 0 > \left( \frac{\partial}{\partial x}(\tilde{X}_s, \Phi_x^m) \right) \Psi_s(x) \geq (\tilde{X}_s, \Phi_x^m) \frac{\partial}{\partial x} \Psi_s(x). \]

Hence, with the same calculation

\[
\begin{align*}
& \int_0^t \int_{A^{-,s}} \psi_n(|(\tilde{X}_s, \Phi_x^m)|)|x|^{-\theta}(\tilde{X}_s, \Phi_x^m) \frac{\partial}{\partial x} \Psi_s(x) \frac{\partial}{\partial x} (\tilde{X}_s, \Phi_x^m) dx ds \\
& \quad \leq \frac{2a_n}{n} \int_0^t \int_{\mathbb{R}} \frac{1}{1 \leq |(\tilde{X}_s, \Phi_x^m)| \leq a_n} |x|^{-\theta} \left( \frac{\partial}{\partial x} \Psi_s(x) \right)^2 dx ds \\
& \quad \leq \frac{2a_n}{n} C(\Psi, t).
\end{align*}
\]

Finally, for any $t \geq 0$,

\[
\begin{align*}
& \int_0^t \int_{A^{0,s}} \psi_n(|(\tilde{X}_s, \Phi_x^m)|)|x|^{-\theta}(\tilde{X}_s, \Phi_x^m) \frac{\partial}{\partial x} \Psi_s(x) \frac{\partial}{\partial x} (\tilde{X}_s, \Phi_x^m) dx ds = 0,
\end{align*}
\]
and we conclude that
\[ E(I_{2,3}^{m,n}(t \wedge T) + I_{2,2}^{m,n}(t \wedge T)) \leq 4a^2 C(\Psi, t) \frac{a}{n}, \]
which tends to zero as \( n \to \infty \).

For \( I_{2,3}^{m,n} \) recall that \( \phi_n'(X)X \uparrow |X| \) uniformly in \( X \) as \( n \to \infty \), and that \( \langle \tilde{X}_s, \Phi_x^m \rangle \) tends to \( \bar{X}(s, x) \) as \( m \to \infty \) for all \( s, x \) a.s. by the a.s. continuity of \( \bar{X} \). This implies that \( \phi_n'((\langle \tilde{X}_s, \Phi_x^m \rangle, \langle \bar{X}_s, \Phi_x^m \rangle) \to |\bar{X}(s, x)| \) pointwise a.s. as \( m, n \to \infty \), where it is unimportant how we take the limit. We also have the bound
\[ |\phi_n'((\langle \tilde{X}_s, \Phi_x^m \rangle, \langle \bar{X}_s, \Phi_x^m \rangle)| \leq |\langle \tilde{X}_s, \Phi_x^m \rangle| \leq |\langle \bar{X}_s, \Phi_x^m \rangle|. \]  
(4.9)
The a.s. continuity of \( \tilde{X} \) implies a.s. convergence for all \( s, x \) of \( \langle \tilde{X}_s, \Phi_x^m \rangle \) to \( |\bar{X}(s, x)| \) as \( m \to \infty \). Jensen’s Inequality and (2.18) show that \( \langle \tilde{X}_s, \Phi_x^m \rangle \) is \( L^p \) bounded on \((0, t) \times B(0, J(t)) \times \Omega, ds \times dx \times \mathbb{P}) \) uniformly in \( m \). Therefore
\[ \{\langle \tilde{X}_s, \Phi_x^m \rangle : n, m \} \text{ is uniformly integrable on } (0, t) \times B(0, J(t)) \times \Omega. \]  
(4.10)
This gives uniform integrability of \( \{\phi_n'((\langle \tilde{X}_s, \Phi_x^m \rangle, \langle \bar{X}_s, \Phi_x^m \rangle) : n, m \} \) by our earlier bound (4.11). Since \( \Psi_s = 0 \) off \( B(0, J(t)) \), this implies that
\[ \lim_{m, n \to \infty} E(I_{2,3}^{m,n}(t \wedge T)) = E\left( \int_0^{t \wedge T} |\tilde{X}(s, x)| \Delta \theta \psi_s(x) dx ds \right). \]
Collecting the pieces, we have shown that (b) holds.

(c) As in the above argument we have
\[ \phi_n((\langle \tilde{X}_s, \Phi_x^m \rangle) \to |\bar{X}(s, x)| \text{ as } m, n \to \infty \text{ a.s. for all } x \text{ and all } s \leq t. \]  
(4.11)
The uniform integrability in (4.10) and the bound \( \phi_n((\langle \tilde{X}_s, \Phi_x^m \rangle) \leq \langle |\tilde{X}_s|, \Phi_x^m \rangle \)

imply that
\[ \{\phi_n((\langle \tilde{X}_s, \Phi_x^m \rangle) : n, m \} \text{ is uniformly integrable on } [0, t] \times B(0, J(t)) \times \Omega. \]
Therefore the result now follows from the above convergence and the bound
\[ |\hat{\psi}_s(x)| \leq C1_{\{|x| \leq J(t)\}}. \]

5 Uniqueness: Theorem \[ \[2.5\] \]
Let \( T_K = \inf\{ t \geq 0 : \sup_{x \in [t, 1]}(|X^1(t, x)| + |X^2(t, x)|) > K \} \wedge K \). Note that
\[ T_K \to \infty, \text{ \( \mathbb{P} \)-a.s. as } K \to \infty \]  
(5.1)
since each \( X^i \) is continuous. Also define a metric \( d \) by
\[ d((t, x), (t', x')) = |t - t'|^a + |x - x'|, \text{ } t, t' \in \mathbb{R}, \text{ } x, x' \in \mathbb{R}, \]
and set
\[ Z_{K,N,\xi} \equiv \{ (t, x) \in \mathbb{R}^+ \times \mathbb{R} : t \leq T_K, |x| \leq 2^{-N\alpha}, |t - \hat{t}| \leq 2^{-N}, |x - \hat{x}| \leq 2^{-N\alpha}, \] 
for some \((\hat{t}, \hat{x}) \in [0, T_K] \times \mathbb{R}\) satisfying \(|\hat{X}(\hat{t}, \hat{x})| \leq 2^{-N\xi}\).

We will now use the following key result on improving the Hölder continuity of \(\hat{X}(t, x)\) when \(\hat{X}\) and \(|x|\) are small. We will assume this result in this section, where we will use it to show uniqueness. We will prove Theorem 5.1 in Section 7.

**Theorem 5.1** Assume the hypotheses of Theorem 2.3. \(\hat{X} = X^1 - X^2\), where \(X^i\) is a solution of (2.17) with sample paths in \(C(\mathbb{R}^+; C_{\text{tem}})\) a.s. for \(i = 1, 2\). Let \(\xi \in (0, 1)\) satisfy
\[
\exists N_\xi = N_\xi(K, \omega) \in \mathbb{N} \text{ a.s. such that for any } N \geq N_\xi, \text{ and any } (t, x) \in Z_{K,N,\xi}
\]
\[
|t' - t| \leq 2^{-N}, t, t' \leq T_K, |y - x| \leq 2^{-N\alpha} \implies |\hat{X}(t, x) - \hat{X}(t', y)| \leq 2^{-N\xi}. \tag{5.2}
\]
Let \(\frac{1}{2} - \alpha < \xi < \frac{1}{2} - \alpha \wedge 1\). Then there is an \(N_{\xi,1} = N_{\xi,1}(K, \omega, \xi) \in \mathbb{N}\) a.s. such that for any \(N \geq N_{\xi,1}\), \(\text{in } \mathbb{N}\) and any \((t, x) \in Z_{K,N,\xi}\)
\[
|t' - t| \leq 2^{-N}, t, t' \leq T_K, |y - x| \leq 2^{-N\alpha} \implies |\hat{X}(t, x) - \hat{X}(t', y)| \leq 2^{-N\xi}. \tag{5.3}
\]
Moreover there are strictly positive constants \(R, \delta, \beta, c, \delta_1, \delta_2\) depending only on \((\xi, \xi^1)\) and \(N(K) \in \mathbb{N}\) which also depends on \(K\), such that
\[
\mathbb{P}(N_{\xi,1} \geq N) \leq (5.4), (\mathbb{P}(N_{\xi} \geq N/R) + K \exp(-c(2^{N\delta})) \tag{5.4}
\]
provided that \(N \geq N(K)\).

**Corollary 5.2** Assume the hypotheses of Theorem 2.3. Let \(\hat{X}\) be as in Theorem 2.3 and \(\frac{1}{2} - \alpha < \xi < \frac{1}{2} - \alpha \wedge 1\). There is an a.s. finite positive random variable \(C_{\xi,K}(\omega)\) such that for any \(e \in (0, 1)\), \(t \in [0, T_K]\) and \(|x| \leq e^\alpha\), if \(|\hat{X}(t, x)| \leq e^\xi\) for some \(|\hat{x} - x| \leq e^\alpha\), then \(|\hat{X}(t, y)| \leq C_{\xi,K}(\omega) e^\xi\) whenever \(|x - y| \leq e^\alpha\). Moreover there are strictly positive constants \(\delta, c, \delta_1, \delta_2\), depending on \(\xi\), and \(r_0(K)\), which also depends on \(K\), such that
\[
\mathbb{P}(C_{\xi,K} \geq r) \leq (5.5), \left[\left(\frac{r - 6}{K + 1}\right)^{-\delta} + K \exp\left(-c(2^{N\delta})\right)\right] \tag{5.5}
\]
for all \(r \geq r_0(K) > 6 + (K + 1)\).

**Proof.** Let \(\xi_0 = \frac{3}{4}(\frac{1}{2} - \alpha)\). By Proposition 2.3(b) and the equality \(\hat{X} = Z^1 - Z^2\), where \(Z^i(t, x) = X^i(t, x) - S_i X_0(x) - \int_0^t \int_{\mathbb{R}} P^0_{t-s}(x, y) g(s, y) dy ds\), we have (5.2) with \(\xi = \xi_0\). Indeed, \(\hat{X}\) is uniformly Hölder continuous on compacts in \([0, \infty) \times \mathbb{R}\) in space and in time with any exponent less than \(\frac{1}{2} - \alpha\). This allows to get (5.2) with \(\xi = \xi_0\).

Inductively define \(\xi_{n+1} = \left[\left(\xi_n \gamma + \frac{1}{2} - \alpha\right) \wedge 1\right] \left(1 - \frac{1}{n+1}\right)\). It is easily checked that \(\xi_0 < \xi_1\), from which it follows inductively that \(\xi_n \uparrow \frac{3}{4}(\frac{1}{2} - \alpha) \wedge 1\). Let now \(\xi\)
be as in the statement of the corollary, that is \( \xi \in \left( \frac{1}{2} - \frac{\alpha}{1 - \gamma}, 1 \right) \). Fix \( n_0 \) so that \( \xi_{n_0} \geq \xi > \xi_{n_0 - 1} \). Apply Theorem 5.1 inductively \( n_0 \) times to get (5.2) for \( \xi = \xi_{n_0 - 1} \) and, hence, (5.3) with \( \xi^1 = \xi_{n_0} \).

First consider \( \epsilon \leq 2^{-N_{\xi_{n_0}}} \). Choose \( N \in \mathbb{N} \) so that \( 2^{-N-1} < \epsilon < 2^{-N} \), and so \( N \geq N_{\xi_{n_0}} \). Assume \( t \leq T_K, |x| \leq \epsilon^n \leq 2^{-N_{\alpha}}, \) and \( |\tilde{X}(t, \tilde{x})| \leq \epsilon^\delta \leq 2^{-N_{\xi}} \leq 2^{-N_{\xi_{n_0}} - 1} \) for some \( |\tilde{x} - x| \leq \epsilon^n \leq 2^{-N_{\alpha}} \). Then \( (t, x) \in Z_{K, N_{\xi_{n_0}} - 1} \).

Therefore (5.3) with \( \xi^1 = \xi_{n_0} \) implies that if \( |y - x| \leq \epsilon^n \leq 2^{-N_{\alpha}} \), then

\[
|\tilde{X}(t, y)| \leq |\tilde{X}(t, \tilde{x})| + |\tilde{X}(t, \tilde{x}) - \tilde{X}(t, x)| + |\tilde{X}(t, x) - \tilde{X}(t, y)| \\
\leq 2^{-N_{\xi}} + 2^{-N_{\xi_{n_0}}} \leq 3 \cdot 2^{-N_{\xi}} \leq 3(2\epsilon^\delta) \leq 6\epsilon^\delta.
\]

For \( \epsilon > 2^{-N_{\xi_{n_0}}} \), we have for \( (t, x) \) and \( (t, y) \) as in the corollary,

\[
|\tilde{X}(t, y)| \leq K + 1 \leq (K + 1)2^{-N_{\xi_{n_0}}} \epsilon^\delta.
\]

This gives the conclusion with \( C_{\xi, K} = (K + 1)2^{-N_{\xi_{n_0}}} \epsilon^\delta + 6 \). A short calculation and (5.4) now imply that there are strictly positive constants \( \bar{R}, \bar{\delta}, \), (5.6), depending on \( \xi \) and \( K \), such that

\[
\mathbb{P}(C_{\xi, K} \geq r) \leq \mathbb{P}\left(N_{\frac{1}{2}(\frac{1}{2} - \alpha)} \geq \frac{1}{\bar{R}} \log_2 \left( \frac{r - 6}{K + 1} \right) \right) + K \exp\left(-\mathbb{P}\left(N_{\frac{1}{2}(\frac{1}{2} - \alpha)} \geq \frac{1}{\bar{R}} \log_2 \left( \frac{r - 6}{K + 1} \right) \right) \right)
\]

for all \( r \geq r_0(K) \). The usual Kolmogorov continuity proof applied to (2.19) with \( \tilde{X} = Z^1 - Z^2 \) in place of \( Z \) (and \( \xi = \frac{1}{2}(\frac{1}{2} - \alpha) \)) shows there are \( \bar{c}, \bar{c}_3 > 0 \) such that

\[
\mathbb{P}(N_{\frac{1}{2}(\frac{1}{2} - \alpha)} \geq M) \leq \bar{c}_3 2^{-M\bar{\delta}}
\]

for all \( M \in \mathbb{R} \). Thus, (5.5) follows from (5.6).

Now let us prove a simple lemma that will allow us to choose the “right” \( \xi \) that will satisfy the conditions of the previous corollary and allow us to push through the uniqueness argument. One inequality below is needed to make Corollary 5.2 apply. The other is required for the proof of Lemma 5.3.

**Lemma 5.3** Fix \( \alpha, \gamma \) satisfying the conditions of Theorem 5.5, that is,

\[
1 > \gamma > \frac{1}{2(1 - \alpha)} \geq \frac{1}{2}.
\]

Then we can choose \( \xi \in (0, 1) \) such that

\[
\frac{\alpha}{2\gamma - 1} < \xi < \left( \frac{1}{2} - \frac{\alpha}{1 - \gamma} \wedge 1 \right).
\]

**Proof.** Let us verify that (5.8) is possible. There are two cases, the first being

\[
\frac{1}{2} - \frac{\alpha}{1 - \gamma} < 1.
\]
Recall that \( \alpha \in (0, \frac{1}{2}) \) and \( \gamma \in (\frac{1}{2}, 1) \). Therefore
\[
\frac{1}{1-\gamma} - \frac{\alpha}{2\gamma - 1} = \frac{(\frac{1}{2} - \alpha)(2\gamma - 1) - \alpha(1-\gamma)}{(1-\gamma)(2\gamma - 1)} = \frac{\gamma(1-\alpha) - \frac{\alpha}{2}}{(1-\gamma)(2\gamma - 1)} > 0,
\]
where the last inequality follows by (5.7). Then (5.10) implies that we can fix \( \xi \) satisfying (5.8) in the case of (5.9).

The second case is for \( \frac{1}{2} - \alpha \geq 1 \), that is \( \alpha \leq \gamma - \frac{1}{2} \).
Hence
\[
\frac{\alpha}{2\gamma - 1} \leq \frac{1}{2},
\]
and we can easily fix \( \xi \) satisfying (5.8) in this case as well.

Now fix \( \xi \) as in the previous lemma and define
\[
\eta = \frac{\xi}{\alpha}.
\]
Lemma 5.3 immediately implies that
\[
\eta > \frac{1}{2\gamma - 1}.
\]
(5.11)

We return to the setting and notation of Section 3. In particular \( \Psi \in C_c^\infty([0,t] \times \mathbb{R}) \) with \( \Gamma(t) = \{ x : \exists s \leq t, \Psi_s(x) > 0 \} \subset B(0, J(t)) \).

For \( a_n \) given by (4.3), let \( m(n) := a_n - \frac{1}{2} \). Note that \( m(n) \geq 1 \) for all \( n \). Set \( c_0(K) := r_0(K) \lor K^2 \) (where \( r_0(K) \) is chosen as in Corollary 5.2) and define the stopping time
\[
T_{\xi,K} = \inf \{ t \geq 0 : t > T_K \text{ or } t \leq T_K \text{ and there exist } \epsilon \in (0, 1], \hat{x}, x, y \in \mathbb{R} \text{ with } |x| \leq \epsilon^\alpha, |\hat{X}(t, \hat{x})| \leq \epsilon^\xi, |x - \hat{x}| \leq \epsilon^\alpha, |x - y| \leq \epsilon^\alpha \text{ such that } |\hat{X}(t, y)| > c_0(K)\epsilon^\xi \}.
\]
Assuming our filtration is completed as usual, \( T_{\xi,K} \) is a stopping time by the standard projection argument. Note that for any \( t \geq 0 \), by Corollary 5.2
\[
\mathbb{P}(T_{\xi,K} \leq t) \leq \mathbb{P}(T_K \leq t) + \mathbb{P}(C_{\xi,K} > c_0(K))
\]
\[
\leq \mathbb{P}(T_K \leq t) + \frac{c_5}{c_5.1}\left( \frac{K^2 - 6}{K + 1} \right)^{-\delta}
\]
\[
+ K \exp\left(-\frac{c_5.2}{c_5.1}\left( \frac{K^2 - 6}{K + 1} \right)^{\delta} \right)
\]
(5.12)
which tends to zero as \( K \to \infty \) due to (5.1).

With this set-up we can show the following lemma:
Lemma 5.4 For all $x \in B(0, \frac{1}{m^{|n|}})$ and $s \in [0, T_x, K]$, if $|⟨\tilde{X}_s, \Phi^{m(n)}_x⟩| \leq a_{n-1}$ then

$$\sup_{y \in B(x, \frac{1}{m(n)})} |\tilde{X}(s, y)| \leq c_0(K)a_{n-1}.$$ 

PROOF. Since $|⟨\tilde{X}_s, \Phi^{m(n)}_x⟩| \leq a_{n-1}$ and $\tilde{X}_s(\cdot)$ is continuous there exists an $\hat{x} \in B(x, \frac{1}{m(n)})$ such that $|\tilde{X}(s, \hat{x})| \leq a_{n-1}$. Apply the definition of the stopping time with $\epsilon^0 = 1/m(n) \in [0, 1]$ and so $\epsilon^0 = a_{n-1}$ to obtain the required bound. \hfill \Box

Next, we bound $|I_3^{m(n), n}|$ of (1.17) using the Hölder continuity of $\sigma$, as well as the definition of $\psi_n$.

Lemma 5.5

$$\lim_{n \to \infty} \mathbb{E}\left( |I_3^{m(n), n}(t \wedge T_x, K)| \right) = 0 \tag{5.13}$$

PROOF. By (1.1) we have

$$|I_3^{m(n), n}(t \wedge T_x, K)| \leq \frac{L^2}{n} \int_0^{t \wedge T_x, K} \int_0^{\text{a}_n a_{n-1}} a_{n-1} |\tilde{X}(s, 0)|^{2\gamma} \times \Phi^{m(n)}_x(0)^2 \Psi_s(x) dx ds.$$ 

We obtain from Lemma 5.3

$$|I_3^{m(n), n}(t \wedge T_x, K)| \leq \frac{L^2}{n} \left( \frac{\Psi_n(0)^2}{m(n)} c_0(K)^{2\gamma} \right) \int_0^{t \wedge T_x, K} \left( \int_{\Gamma(t)} \Phi^{m(n)}_x(0)^2 dx \right) ds$$

$$\leq \frac{L^2}{n} \left( \frac{\Psi_n(0)^2}{m(n)} c_0(K)^{2\gamma} \right) t \times \frac{a_{n-1}^{2\gamma}}{a_n} \int_0^{t \wedge T_x, K} \left( \int_{\Gamma(t)} \Phi^{m(n)}_x(0)^2 dx \right) ds$$

$$\leq \frac{L^2}{n} \left( \frac{\Psi_n(0)^2}{m(n)} c_0(K)^{2\gamma} \right) t \times \frac{a_{n-1}^{2\gamma}}{a_n} \int_0^{t \wedge T_x, K} \left( \int_{\Gamma(t)} \Phi^{m(n)}_x(0)^2 dx \right) ds$$

$$\leq \frac{L^2}{n} \left( \frac{\Psi_n(0)^2}{m(n)} c_0(K)^{2\gamma} \right) t \times \frac{a_{n-1}^{2\gamma}}{a_n} \int_0^{t \wedge T_x, K} \left( \int_{\Gamma(t)} \Phi^{m(n)}_x(0)^2 dx \right) ds$$

$$\leq \frac{C(L, \Psi c_0(K)^{2\gamma} t a_{n-1}^{2\gamma} a_n}{n}$$

If we choose $\rho(x) = \sqrt{x}$ then $\int_{a_{n-1}}^{a_n} x^{-1} dx = n$ so that $\frac{a_n}{a_{n-1}} = e^n$ or (using that $a_0 = 1$) $a_n = e^{-n(a_n+1)}$. Thus (5.13) holds if $n(n+1) - (2\gamma + \frac{1}{\eta})(n-1)n < 0$ for $n$ large. This is equivalent to

$$1 - (2\gamma + \frac{1}{\eta}) < 0 \Leftrightarrow \gamma > \frac{1}{2} + \frac{1}{2\eta}$$

which holds by (5.11). A similar argument applies for any $\rho$ satisfying (1.1) and (1.2). \hfill \Box
Use (4.11) and Fatou’s Lemma on the left-hand side of (5.7), and Lemmas 4.2 and 5.5 on the right-hand side, to take limits in this equation and so conclude that

\[
\int_{\mathbb{R}} \mathbb{E}\left( |\tilde{X}(t ∧ T_{\xi,K}, x)| \right) \Psi_t(x) \, dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}} \mathbb{E}\left( \phi_n\left( \langle \tilde{X}_{t ∧ T_{\xi,K}}^m, \Phi_x^{m(n)} \rangle \right) \right) \Psi_t(x) \, dx
\]

\[
\leq \mathbb{E}\left( \int_0^t \int_{\mathbb{R}} |\tilde{X}(s, x)| \left( \Delta_{\phi} \Psi_s(x) + \psi_s(x) \right) \, dx \, ds \right)
\]

\[
\leq \int_0^t \int_{\mathbb{R}} \mathbb{E}\left( |\tilde{X}(s, x)| \right) \left| \Delta_{\phi} \Psi_s(x) + \psi_s(x) \right| \, dx \, ds.
\]

Since \( T_{\xi,K} \) tends in probability to infinity as \( K \to \infty \) according to (5.12), we have that

\[
\tilde{X}(t ∧ T_{\xi,K}, x) \to \tilde{X}(t, x)
\]

and so we finally conclude with another application of Fatou’s Lemma that

\[
\int_{\mathbb{R}} \mathbb{E}\left( |\tilde{X}(t, x)| \right) \Psi_t(x) \, dx \leq \int_0^t \int_{\mathbb{R}} \mathbb{E}\left( |\tilde{X}(s, x)| \right) \left| \Delta_{\phi} \Psi_s(x) + \psi_s(x) \right| \, dx \, ds. \quad (5.14)
\]

Let \( \{g_N\} \) be a sequence of functions in \( C_c^{\infty}(\mathbb{R}) \) such that \( g_N : \mathbb{R} \to [0, 1] \),

\[
B(0, N) \subset \{ x : g_N(x) = 1 \}, \quad B(0, N + 1)^c \subset \{ x : g_N(x) = 0 \},
\]

and

\[
\sup_{N \geq 1} \| |x|^{-\delta} g_N^\prime \|_{\infty} + \| \Delta_{\phi} g_N \|_{\infty} \equiv C < \infty,
\]

where \( g_N^\prime \) denotes the derivative with respect to the spatial variable. Now let \( \phi \in C_c^{\infty}(\mathbb{R}) \), and for \( (s, x) \in [0, t] \times \mathbb{R} \) set \( \Psi_N(s, x) = (S_{t-s}\phi(x))g_N(x) \). It is then easy to check that \( \Psi_N \in C_c^{\infty}([0, t] \times \mathbb{R}) \) and for \( \lambda > 0 \) there is a \( C = C(\lambda, \phi, t) \) such that for all \( N \)

\[
|\Delta_{\phi} \Psi_N(s, x) + \psi_N(s, x)| = \left| 4\alpha^2 |x|^{-\delta} \frac{\partial}{\partial x} S_{t-s}\phi(x) \frac{\partial}{\partial x} g_N(x) + S_{t-s}\phi(x) \Delta_{\phi} g_N(x) \right|
\]

\[
\leq Ce^{-\lambda|x|} 1_{\{|x| > N\}}.
\]

Use this in (5.13) to conclude that

\[
\int_{\mathbb{R}} \mathbb{E}( |\tilde{X}(t, x)| ) \phi(x) \, dx \leq C \int_0^t \int_{\mathbb{R}} \mathbb{E}( |\tilde{X}(s, x)| ) e^{-\lambda|x|} 1_{\{|x| > N\}} \, dx \, ds.
\]

By Proposition 2.6, the right-hand side of the above approaches zero as \( N \to \infty \) and we see that

\[
\mathbb{E}\left( \int_{\mathbb{R}} |\tilde{X}(t, x)| \, dx \right) = 0.
\]

Therefore \( X^1(t) = X^2(t) \) for all \( t \geq 0 \) a.s. by continuity.
6 Hölder continuity: Proposition 2.6

First we will introduce a number of technical lemmas that will be frequently used. The proof of the next lemma is elementary and therefore is omitted.

Lemma 6.1 For any \(x, y \in \mathbb{R}, 0 \leq \beta \leq 1,\)
\[
|p_t(x) - p_t(y)| \leq ct^{-\alpha} \left( \frac{|x-y|}{t} \right)^{\beta} (\max(|x|, |y|))^{(\frac{\beta}{\alpha} - 1)\beta}.
\]

Lemma 6.2 For any \(0 < t < t' \leq T, x \in \mathbb{R},\)
\[
\int_0^t (p_{t-s}(x) - p_{t-s}(x))^2 ds \leq c(T)|t' - t|^{1-2\alpha}.
\]

PROOF. Assume, without loss of generality, that \(t' - t \leq t.\)
\[
\int_0^t (p_{t-s}(x) - p_{t-s}(x))^2 ds \leq c \int_0^t (t-s)^{-2\alpha} ds + \int_0^t (p_{t-s}(x) - p_{t-s}(x))^2 ds \leq c |t' - t|^{1-2\alpha} + c \int_0^t |(t-s)^{-\alpha} - (t'-s)^{-\alpha}|^2 ds.
\]

The second term on the right hand side is trivially bounded by
\[
c \int_0^t |(t-s)^{-\alpha} - (t'-s)^{-\alpha}|^2 ds \leq c \int_0^t (t-s)^{-2\alpha} - (t'-s)^{-2\alpha} ds \leq c |t' - t|^{1-2\alpha}.
\]

The third term on the right hand side of (6.1) is also easy to bound, as
\[
c \int_0^t (t-s)^{-2\alpha} e^{-\frac{|x-s|^{1/\alpha}}{t-s} (t'-s)} (t'-s)^{-2\alpha} ds \leq c(t'-t) \int_0^t (t'-s)^{-2\alpha} ds \leq c |t' - t|^{1-2\alpha}.
\]

This completes the proof. □

Lemma 6.3 For any \(x, y \in [-1, 1], t \leq T, \beta \in (1/2 - \alpha),\)
\[
\int_0^t (p_{t-s}(x) - p_{t-s}(y))^2 ds \leq c(T)(\max(|x|, |y|))^{(\frac{\beta}{\alpha} - 1)\beta} |x - y|^{1-2\alpha}.
\]

PROOF.
\[
\int_0^t (p_{t-s}(x) - p_{t-s}(y))^2 ds \leq \int_0^{t-|x-y|} (p_{t-s}(x) - p_{t-s}(y))^2 ds + \int_{t-|x-y|}^t (p_{t-s}(x) - p_{t-s}(y))^2 ds, \tag{6.2}
\]

so by Lemma 6.1 (by taking \( \beta = 1 \) there) we can bound the first term on the right hand side by

\[
(\max(|x|, |y|))^{\frac{1}{\alpha} - 1} \int_0^{t - |x - y|} (t - s)^{-2\alpha - 2}|x - y|^2 \, ds
\]

\[
\leq C (\max(|x|, |y|))^{\frac{1}{\alpha} - 1} 2^\beta |x - y|^{-2\alpha - 1 + 2}
\]

\[
\leq (\max(|x|, |y|))^{\frac{1}{\alpha} - 1} \beta |x - y|^{-2\alpha + 1}.
\]

By Lemma 6.1 again, with \( \beta \) as given in this lemma, we can bound the second term on the right hand side of (6.2) by

\[
(\max(|x|, |y|))^{\frac{1}{\alpha} - 1} 2^\beta \int_{t - |x - y|}^t (t - s)^{-2\alpha - 2} ds
\]

\[
\leq C (\max(|x|, |y|))^{\frac{1}{\alpha} - 1} 2^\beta |x - y|^{-2\alpha + 1},
\]

and we are done. \( \square \)

**Proof of Proposition 2.6**

(a) follows by the correspondence between the SPDE (2.16) and SIE (2.1) and the moment bound (2.9).

(b) Let \( Z(t, x) = X(t, x) - S_tX_0(x) - \int_0^t \int_{\mathbb{R}} p_{t-s}^\theta(x, y)g(s, y) \, dy \, ds \). Then

\[
|Z(t', x) - Z(t, y)| = \int_0^t (p_{t-s}^\theta(x) - p_{t-s}^\theta(y)) \sigma(X(s, 0)) \, dB_s
\]

\[
+ \int_t^{t'} p_{t-s}^\theta(x) \sigma(X(s, 0)) \, dB_s
\]

\[
= \int_0^t (p_{t-s}^\theta(x) - p_{t-s}^\theta(y)) \sigma(X(s, 0)) \, dB_s
\]

\[
+ \int_0^t (p_{t-s}^\theta(x) - p_{t-s}^\theta(y)) \sigma(X(s, 0)) \, dB_s
\]

By the Burkholder-Gundy-Davis and Hölder inequalities, the moment bound on \( X(s, 0) \), and Lemmas 6.2 and 6.3, the required moment bound

\[
\mathbb{E}[|Z(t', x) - Z(t, y)|^p] \leq C(|t' - t|^{p(1/2 - \alpha)} + |x - y|^{p(1/2 - \alpha)})
\]

now follows. By the Kolmogorov criterion we also get the required Hölder continuity of \( Z \). \( \square \)
Hölder continuity: Theorem 5.1

Proof of Theorem 5.1

We proceed along the lines of the proof of Theorem 4.1 of [MPS05]. Fix arbitrary (deterministic) \((t, x), (t', y)\) such that \(|t - t'| \leq \epsilon \equiv 2^{-N} (N \in \mathbb{N})\), \(|x| \leq 2^{-Na}\), \(|x - y| \leq 2^{-Na}\) and \(t \leq t'\) (the case \(t' \leq t\) works analogously).

In the following we will define small numbers \(\delta, \delta', \delta_1, \delta_2 > 0\) as follows. As \(\xi_1 < (\xi\gamma + \frac{1}{2} - \alpha) \wedge 1\), we may choose \(\delta \in (0, \frac{1}{2} - \alpha)\) such that \(\xi_1 < ((\xi\gamma + 1)2 - \alpha) \wedge 1 - \alpha\delta < 1\).

Fix \(\delta' \in (0, \delta)\). Now choose \(\delta_1 \in (0, \delta')\) sufficiently small such that \(\xi_1 < ((\xi\gamma + \frac{1}{2} - \alpha) \wedge 1) - \alpha\delta + \alpha\delta_1 < 1\). \(\quad (7.1)\)

Moreover define

\[
p \equiv ((\xi\gamma + \frac{1}{2} - \alpha) \wedge 1) - \alpha(1/2 - \alpha) + \alpha\delta_1,
\]

and hence by \(7.1\) we easily get that

\[
p + \alpha(1/2 - \alpha - \delta) = ((\xi\gamma + \frac{1}{2} - \alpha) \wedge 1) - \alpha\delta + \alpha\delta_1 \in (\xi_1, 1). \quad (7.2)
\]

Also define \(\delta_2 > 0\) sufficiently small such that

\[
\delta' - \delta_2 > \delta_1 \quad (7.3)
\]

and define

\[
\tilde{p} \equiv p + \alpha(\delta' - \delta_2 - \delta_1)
= ((\xi\gamma + \frac{1}{2} - \alpha) \wedge 1) - \alpha(1/2 - \alpha) + \alpha(\delta' - \delta_2). \quad (7.4)
\]

By \(7.3\) we get that

\[
\tilde{p} > p.
\]

Now consider for some random \(N_1 = N_1(\omega, \xi, \xi_1)\) (to be chosen below in \(7.19\)),

\[
P \left( |\tilde{X}(t, x) - \tilde{X}(t, y)| \geq |x - y|^{\frac{1}{2} - \alpha - \delta} e^p, (t, x) \in Z_{K, N, \xi}, N \geq N_1 \right)
+ P \left( |\tilde{X}(t', x) - \tilde{X}(t, x)| \geq |t' - t|^{\alpha(\frac{1}{2} - \alpha - \delta)} e^p, (t, x) \in Z_{K, N, \xi}, t' \leq T_K, N \geq N_1 \right).

(7.5)\]

In what follows we are going to obtain the bound on \(\tilde{p}\). Set

\[
D^{x,y,t,t'}(s) = |p_{t-s}(x) - p_{t-s}(y)|^2 |\tilde{X}(s, 0)|^{2\gamma},
D^{x,t}(s) = p_{t-s}(x)^2 |\tilde{X}(s, 0)|^{2\gamma}.
\]

27
With this notation, expression \( \text{(7.6)} \) is bounded by

\[
P\left(|\tilde{X}(t, x) - \tilde{X}(t, y)| \geq |x - y|^{\frac{1}{2} - \alpha - \delta}c^p, (t, x) \in Z_{K,N,\xi}, N \geq N_1 \right) \leq \int_0^t D^{x,y,t,t}(s)ds \leq |x - y|^{1-2\alpha - 2\delta'}c^{2p}
\]

\[
+ P\left(|\tilde{X}(t', x) - \tilde{X}(t, x)| \geq |t' - t|^{\alpha(\frac{1}{2} - \alpha - \delta)}c^p, (t, x) \in Z_{K,N,\xi}, t' \leq T_K, N \geq N_1 \right) \int_0^{t'} D^{x,t}(s)ds + \int_0^t D^{x,x,t,t}(s)ds \leq (t' - t)^{2\alpha(\frac{1}{2} - \alpha - \delta')}c^{2p}
\]

\[
+ P\left( \int_0^{t'} D^{x,y,t,t}(s)ds > |x - y|^{1-2\alpha - 2\delta'}c^{2p}, (t, x) \in Z_{K,N,\xi}, N \geq N_1 \right)
\]

\[
+ P\left( \int_0^t D^{x,y,t}(s)ds + \int_0^t D^{x,x,t,t}(s)ds > (t' - t)^{2\alpha(\frac{1}{2} - \alpha - \delta')}c^{2p}, (t, x) \in Z_{K,N,\xi}, t' \leq T_K, N \geq N_1 \right)
\]

\[
=: P_1 + P_2 + P_3 + P_4.
\]

Notice that the processes

\[
\hat{\iota} \mapsto \int_0^{\hat{\iota}} \left( p_{\iota - \tau}(x) \right) \left( \sigma(X^1(s, 0)) - \sigma(X^2(s, 0)) \right) B(ds)
\]

are continuous local martingales for any fixed \( x, t \) on \( 0 \leq \hat{\iota} \leq t \). We bound the appropriate differences of these integrals by considering the respective quadratic variations of \( \tilde{X}(t, x) - \tilde{X}(t, y) \) and \( \tilde{X}(t', x) - \tilde{X}(t, x) \) (see \( \text{(2.16)} \)). By \( \text{(1.1)} \), we see that the time integrals in the above probabilities differ from the appropriate square functions by a multiplicative factor of \( L^2 \).

If \( \delta'' = \delta - \delta' > 0 \), \( B \) is a standard one-dimensional Brownian motion with \( B(0) = 0 \), and \( B^\ast(t) := \sup_{0 \leq s \leq t} |B(s)| \), then \( P_1 \) of \( \text{(7.6)} \) can be bounded using the Dubins-Schwarz Theorem:

\[
P_1 \leq \mathbb{P}\left( B^\ast(1)L|x - y|^{1-2\alpha - 2\delta'}c^{2p} \right) \geq |x - y|^{\frac{1}{2} - \alpha - \delta}c^p
\]

\[
= \mathbb{P}\left( B^\ast(1)L|x - y|^{\frac{1}{2} - \alpha - \delta'}c^p \geq |x - y|^{\frac{1}{2} - \alpha - \delta}c^p \right)
\]

\[
= \mathbb{P}\left( B^\ast(1) \geq L^{-1}|x - y|^{-\delta''} \right) \leq \exp(-\delta' \dfrac{|x - y|^{-\delta''}}{2 L}), \quad \text{(7.7)}
\]

where we have used the reflection principle in the last line. Similarly,

\[
P_2 \leq \mathbb{P}\left( B^\ast(1)L|t' - t|^{1-2\alpha - \delta'}c^{2p} \right) \geq |t' - t|^{\frac{1}{2} - \alpha - \delta/2}c^p
\]

\[
= \mathbb{P}\left( B^\ast(1)L|t' - t|^{2\alpha(\frac{1}{2} - \alpha - \delta)}c^p \geq |t' - t|^{\frac{1}{2} - \alpha - \delta}c^p \right)
\]

\[
= \mathbb{P}\left( B^\ast(1) \geq L^{-1}|t' - t|^{-\alpha\delta''} \right) \leq \exp(-\delta' \dfrac{|t' - t|^{-\alpha\delta''}}{2 L}), \quad \text{(7.8)}
\]

Here the constants \( \delta', \delta'' \) depend on \( L \).

Before we proceed with bounds on \( P_3, P_4 \), in the next lemma, we will obtain a useful bound on \( \tilde{X}(s, 0) \).

28
Lemma 7.1 Let $N \geq N_\xi$. Then on $\{ \omega : (t, x) \in Z_{K, N, \xi} \}$,
\[
\begin{align*}
|\tilde{X}(s, 0)| &\leq 3\epsilon^\xi \quad \text{for } s \in [t - \epsilon, t'], \\
|\tilde{X}(s, 0)| &\leq (4 + K)2^{N_\xi}(t - s)^\xi \quad \text{for } s \in [0, t - \epsilon].
\end{align*}
\] (7.9) (7.10)

PROOF. Assume $(t, x) \in Z_{K, N, \xi}$, $0 \leq t' \leq T_K$ and choose $(\hat{t}, \hat{x})$ such that
\[
\hat{t} \leq T_K, \ |t - \hat{t}| \leq \epsilon = 2^{-N}, \ |\hat{x} - x| \leq \epsilon^\alpha, \ \text{and} \ |\tilde{X}(\hat{t}, \hat{x})| \leq 2^{-N_\xi} = \epsilon^\xi.
\]
We first observe that for $s \in [t - \epsilon, t']$, we trivially have $|t - s| \leq \epsilon$. Therefore by (5.2) and the definition of $Z_{K, N, \xi}$, for $s \in [t - \epsilon, t']$ we get
\[
|\tilde{X}(s, 0)| \leq |\tilde{X}(\hat{t}, \hat{x})| + |\tilde{X}(\hat{t}, \hat{x}) - \tilde{X}(t, x)| + |\tilde{X}(t, x) - \tilde{X}(s, 0)|
\] 
\[
= 3 \cdot 2^{-N_\xi} 
\]
which proves (7.9).
If $s \in [t - 2^{-N_\xi}, t - \epsilon]$, then there exists $\tilde{N} \geq N_\xi$ such that $2^{-(\tilde{N} + 1)} \leq t - s \leq 2^{-\tilde{N}}$ so that as in (7.11) we can bound
\[
|\tilde{X}(s, 0)| \leq |\tilde{X}(\hat{t}, \hat{x})| + |\tilde{X}(\hat{t}, \hat{x}) - \tilde{X}(t, x)| + |\tilde{X}(t, x) - \tilde{X}(s, 0)|
\] 
\[
\leq 2^{-N_\xi} + 2^{-N_\xi} + 2^\xi \cdot 2^{-(\tilde{N} + 1)\xi}
\] 
\[
\leq 2 \cdot (t - s)^\xi + 2 \cdot (t - s)^\xi
\] 
\[= 4(t - s)^\xi,
\]
which proves (7.10) for $s \in [t - 2^{-N_\xi}, t - \epsilon]$. For $s \in [0, t - 2^{-N_\xi}]$ we bound
\[
|\tilde{X}(s, 0)| \leq K \leq K(t - s)^{-\xi}(t - s)^\xi 
\]
\[
\leq K2^{N_\xi}(t - s)^\xi,
\]
and we are done. □

For the rest of this section $C(K)$ will be a constant depending on $K$ which may change from line to line. The next lemma is crucial for bounding $P_3$.

Lemma 7.2 Let $N \geq N_\xi$. Then on $\{ \omega : (t, x) \in Z_{K, N, \xi} \}$,
\[
\int_0^t D^x_y.t.t(s)ds \leq C(K)2^{2\gamma N_\xi} 2^\gamma |x - y|^{1 - 2\alpha - 2\gamma}.
\]

PROOF. First we split the integral:
\[
\int_0^t D^x_y.t.t(s)ds = \int_{t-\epsilon}^t D^x_y.t.t(s)ds + \int_{t-\epsilon}^0 D^x_y.t.t(s)ds
\]
\[=: D_1(t) + D_2(t).
\]
By Lemma 7.1 we get
\[
D_1(t) \leq \int_{t-\epsilon}^t (p_{t-s}(x) - p_{t-s}(y))^2 \epsilon^{2\xi \gamma} ds.
\]

29
Now apply Lemma \[6.3\] with \( \beta = 1/2 - \alpha - \delta' \) to get
\[
D_1(t) \leq c\epsilon^{2\gamma}|x - y|^{1-2\alpha} \max(|x|, |y|)^{(1/\alpha - 1)2\beta}.
\]

Now recall that
\[
\max(|x|, |y|) \leq c\epsilon^\alpha
\]
and we get
\[
D_1(t) \leq c\epsilon^{2\gamma}\epsilon^{2\beta} |x - y|^{1-2\alpha-2\delta'} \epsilon^{(1/\alpha)2\beta}
= c\epsilon^{2(1/2-\alpha(3/2-\alpha)+\alpha\delta'+\gamma\xi)}|x - y|^{1-2\alpha-2\delta'}
\leq c\epsilon^{2\beta}|x - y|^{1-2\alpha-2\delta'},
\]
where the last line follows by \[7.12\].

Now we will bound \( D_2(t) \). By Lemma \[7.1\] we get
\[
D_2(t) \leq C(K)2^{2\gamma\xi N_{\xi}} \int_0^{t-t_c} (pt-s(x) - pt-s(y))^2 (t-s)^5 ds.
\]
Apply Lemma \[6.1\] with \( \beta = 1 \) and use \[7.12\] to get
\[
D_2(t) \leq C(K)2^{2\gamma\xi N_{\xi}} \int_0^{t-t_c} (t-s)^{-2\alpha + 2\gamma\xi}|x - y|^2 \epsilon^{2(1-\alpha)} ds
= C(K, \delta, \delta_2)2^{2\gamma\xi N_{\xi}}((-2\alpha-1+2\gamma\xi)\epsilon^0-2\alpha\delta_2 \epsilon^{2(1-\alpha)}|x - y|^{1-2\alpha-2\delta'}|x - y|^{1+2\alpha+2\delta'})
\leq C(K, \delta, \delta_2)2^{2\gamma\xi N_{\xi}}\epsilon^0 \alpha (1+2\alpha+2\delta')
= C(K, \delta, \delta_2)2^{2\gamma\xi N_{\xi}}2\beta |x - y|^{1-2\alpha-2\delta'},
\]
where the last equality follows easily by the simple algebra and the definition of \( \hat{\beta} \).

The next lemma is important for bounding \( P_4 \).

**Lemma 7.3** Let \( N \geq N_{\xi} \). Then on \( \{\omega : (t, x) \in Z_{K,N,\xi}\} \),
\[
\int_t^{t'} D^x, t'(s) ds + \int_0^{t-t_c} D^x, x, t'(s) ds \leq C(K)2^{2\gamma\xi N_{\xi}} \epsilon^{2\beta} |t' - t|^{\alpha(1-2\alpha - 2\delta')}.
\]

**PROOF.** By Lemma \[7.1\] we have,
\[
\int_t^{t'} D^x, t'(s) ds = \int_t^{t'} p_{t-s}(x)^2 \{\hat{X}(s, 0)\}^{2\gamma} ds
\leq c \int_t^{t'} p_{t-s}(0)^2 \epsilon^{2\gamma\xi} ds
= c \epsilon^{2\gamma\xi} \int_t^{t'} (t'-s)^{-2\alpha} ds
= c \epsilon^{2\gamma} |t' - t|^{1-2\alpha}
\]
As for the second term at the left hand side of (7.13), we first split it:

\[
\int_0^t D^{x,x,t,t'}(s) ds = \int_{t-\epsilon}^t D^{x,x,t,t'}(s) ds + \int_0^{t-\epsilon} D^{x,x,t,t'}(s) ds
\]

\[
= D_1(t) + D_2(t).
\]

Then by Lemma 6.2 and (7.11) we have

\[
D_1(t) = \int_{t-\epsilon}^t |p_{t-s}(x) - p_{t-s'}(x)|^2 |\tilde{X}(s,0)|^{2\gamma} ds
\]

\[
\leq \alpha c^{2\gamma}\xi|t' - t|^{1-2\alpha}
\]

\[
\leq \alpha c^{2\gamma}\xi \epsilon t^{1-2\alpha-\alpha(1/2-\alpha)} + \alpha \delta' |t' - t|^{2\alpha(1/2-\alpha-\delta')}
\]

\[
\leq \alpha c^{2}\beta|t' - t|^{2\alpha(1/2-\alpha-\delta')}, \quad (7.14)
\]

where the last inequality follows since

\[
\beta < 1/2 - \alpha + \gamma - \alpha(1/2 - \alpha) + \alpha \delta'. \quad (7.15)
\]

As for \(D_2(t)\), we again use Lemma 7.1 and also argue similarly to the proof of Lemma 6.2.

\[
\begin{align*}
D_2(t) &= \int_0^{t-\epsilon} |p_{t-s}(x) - p_{t-s'}(x)|^2 |\tilde{X}(s,0)|^{2\gamma} ds \\
&\leq C(K)^{2\gamma}\xi\epsilon \int_0^{t-\epsilon} \left|((t-s)^{-\alpha} - (t'-s')^{-\alpha})e^{-\frac{1}{\alpha}}\right|^2 (t-s)^{2\gamma} ds \\
&\quad + C(K)^{2\gamma}\xi\epsilon \int_0^{t-\epsilon} \left|(t'-s')^{-\alpha}(e^{-\frac{1}{\alpha}} - e^{-\frac{1}{\alpha}}\right|^2 (t-s)^{2\gamma} ds \\
&= D_{2,1} + D_{2,2}.
\end{align*}
\]

Then we easily have

\[
D_{2,1} \leq C(K)^{2\gamma}\xi\epsilon \int_0^{t-\epsilon} \left|((t-s)^{-2\alpha} - (t'-s')^2) (t-s)^{2\gamma} ds \\
\leq C(K)^{2\gamma}\xi\epsilon \epsilon^{(-2\alpha + 1/2 + 2\gamma)}(t'-s')^2 \\
\leq C(K)^{2\gamma}\xi\epsilon \epsilon^{(-2\alpha + 1/2 + 2\gamma)}(t'-s')^2(1/2 - \alpha - \delta') |t'-t|^{2\alpha(1/2 - \alpha - \delta')} \\
= C(K)^{2\gamma}\xi\epsilon \epsilon^{2\gamma(1/2 - \alpha + \gamma)}(1/2 - \alpha - \delta') |t'-t|^{2\alpha(1/2 - \alpha - \delta')}
\]

and

\[
D_{2,2} \leq C(K)^{2\gamma}\xi\epsilon \int_0^{t-\epsilon} \left|\frac{|x|^{1/2}}{t-s} - \frac{|x|^{1/2}}{t'-s}ight|^2 (t-s)^{2\gamma} ds \\
\leq C(K)^{2\gamma}\xi\epsilon |x|^{2\gamma} \int_0^{t-\epsilon} \left|((t-s)^{-2\alpha} - (t'-s')^{-2\alpha}) (t-s)^{2\gamma} ds \\
\leq C(K)^{2\gamma}\xi\epsilon \epsilon^{(-2\alpha + 2\gamma)}|t'-t|^2 \\
\leq C(K)^{2\gamma}\xi\epsilon \epsilon^{2\gamma(1/2 - \alpha + \gamma - (1/2 - \alpha) + \alpha \delta')} |t'-t|^{2\alpha(1/2 - \alpha - \delta')}
\]

\[
\leq C(K)^{2\gamma}\xi\epsilon \epsilon^{2\gamma(1/2 - \alpha - \delta')} |t'-t|^{2\alpha(1/2 - \alpha - \delta')},
\]

31
where the last inequality follows by (7.15).

Combining the above bounds, we are done.

\[\square\]

We can finally conclude that in (7.6),
\[P_3 = P_4 = 0\]
if
\[\xi > 2^p, 2^p + 2^{-N_K} \eta_K \eta < 2^p\] (7.16)

For (7.16) it is equivalent to show
\[C(K) < 2^{-2(N(p - p)) + 2^{-N_K} \eta_K \eta}\]
and since \(\hat{p} - p = \delta' - \delta_1 - \delta_2 > 0\) we require
\[N > \left\lfloor \frac{2\xi N_K + \log C(K)}{2(\hat{p} - p)} \right\rfloor + 1 = \left\lfloor \frac{2\xi N_K + \log C(K)}{2(\delta' - \delta_1 - \delta_2)} \right\rfloor + 1.\] (7.17)

where \(\lfloor \cdot \rfloor\) is the greatest integer function. Hence by (7.17) we can choose the constant
\[c_{7.18} = c_{7.18}(K, \xi, \delta, \delta_1, \delta_2)\] (7.18)
such that for
\[N \geq \left\lfloor c_{7.18} N \xi \right\rfloor\]
holds. Note that the constant \(c_{7.18}\) depends ultimately on \(\xi, \xi_1\) and \(K\).

Hence (7.6), (7.7), (7.8) imply that if
\[N_1(\omega, \xi, \xi_1, K) = N_1 \lor \left\lfloor c_{7.18} N \xi \right\rfloor\] (7.19)
then for \(d((t, x), (t', y)) \leq 2^{-N_1}, t \leq t'\),
\[\mathbb{P}\left(\left|\tilde{X}(t, x) - \tilde{X}(t, y)\right| \geq |x - y|^{1/2 - \alpha - \delta} 2^{-N_1}, \right) \leq \mathbb{T}_K(\exp\left(-c_{7.19}|x - y|^{-\delta''}\right) + \exp\left(-c_{7.19}|t' - t|^{-\delta''}\right)).\] (7.20)

Now set
\[M_{n, N, K} = \max\{|\tilde{X}(j2^{-n}, (z + 1)2^{-n}) - \tilde{X}(j2^{-n}, z2^{-n})| + |\tilde{X}((j + 1)2^{-n}, z2^{-n}) - \tilde{X}(j2^{-n}, z2^{-n})| : \]
\[|z| \leq 2^m, (j + 1)2^{-n} \leq T_K, j \in \mathbb{Z}, z \in \mathbb{Z},\]
\[(j2^{-n}, z2^{-n}) \in Z_{K, N, \xi}\].

(7.20) implies that if
\[A_N = \{\omega : \text{for some } n \geq N, M_{n, N, K} \geq 2 \cdot 2^{-n(1/2 - \alpha - \delta)} 2^{-N_1}, \text{ } N \geq N_1\},\]
\[N \geq 32\]
then for some fixed constants $C, c_1, c_2 > 0$,
\[
\mathbb{P}(\bigcup_{N' \geq N} A_{N'}) \leq C \sum_{N' = N}^{\infty} \sum_{N = N'}^{\infty} K 2^{(\alpha+1)n} e^{-c_1 2^{\alpha n}} \leq C K \eta_N,
\]
where $\eta_N = e^{-c_2 2^{\alpha n}}$. Therefore $N_2(\omega) = \min\{N \in \mathbb{N} : \omega \in A_{N'}^c \text{ for all } N' \geq N\} < \infty$ a.s. and in fact
\[
\mathbb{P}(N_2 > N) = \mathbb{P}(\bigcup_{N' \geq N} A_{N'}) \leq C K \eta_N. \tag{7.21}
\]
Choose $m \in \mathbb{N}$ with $m > 2/\alpha$ and assume $N \geq (N_2 + m) \lor (N_1 + m)$. Let $(t, x) \in Z_{K,N,\xi}$, $d((t',y),(t,x)) \leq 2^{-N\alpha}$, and $t' \leq T_K$. For $n \geq N$ let $t_n \in 2^{-n} \mathbb{Z}_+$ and $x_n \in 2^{-\alpha n} \mathbb{Z}$ be the unique points so that $t_n \leq t < t_n + 2^{-n}$, $x_n \leq x < x_n + 2^{-\alpha n}$ for $x \geq 0$ and $x_n - 2^{-\alpha n} < x \leq x_n$ if $x < 0$. Similarly define $t'_n$ and $y_n$ with $(t',y)$ in place of $(t,x)$. Choose $(\tilde{t}, \tilde{x})$ as in the definition of $Z_{K,N,\xi}$ (recall $(t,x) \in Z_{K,N,\xi}$). If $n \geq N$, then
\[
d((t'_n,y_n), (\tilde{t}, \tilde{x})) \leq d((t'_n,y_n), (t',y)) + d((t',y), (t,x)) + d((t,x), (\tilde{t}, \tilde{x})) \leq |t'_n - t'|^{\alpha} + |y - y_n| + 2^{-N\alpha} + 2^{-N\alpha} < 4 \cdot 2^{-N\alpha} < 2^{2-N \alpha} < 2^{-\alpha(N-2/\alpha)} \leq 2^{-\alpha(N-m)}.
\]
Therefore $(t'_n,y_n) \in Z_{K,N-m,\xi}$, and similarly (and slightly more simply) $(t_n,x_n) \in Z_{K,N-m,\xi}$. Our definitions imply that $t_N$ and $t'_N$ are equal or adjacent in $2^{-N} \mathbb{Z}_+$ and similarly for the components of $x_N$ and $y_N$ in $2^{-N \alpha} \mathbb{Z}_+$. This, together with the continuity of $\bar{X}$, the triangle inequality, and our lower bound on $N$ (which shows $N - m \geq (N_2 \lor N_1)$), implies
\[
|\bar{X}(t,x) - \bar{X}(t',y)| \leq |\bar{X}(t_N, x_N) - \bar{X}(t'_N, y_N)| + \sum_{n=N}^{\infty} |\bar{X}(t_{n+1}, x_{n+1}) - \bar{X}(t_n, x_n)| + |\bar{X}(t'_{n+1}, y_{n+1}) - \bar{X}(t'_n, y_n)|
\leq M_{N,N-m,K} + \sum_{n=N}^{\infty} 2 M_{n+1,N-m,K}
\leq C \sum_{n=N}^{\infty} 2 \cdot 2^{-n \alpha(1/2 - \alpha - \delta)} 2^{-(N-m)p}
\leq c_0(p) 2^{-N(\alpha(1/2 - \alpha - \delta) + p)}
\leq 2^{-N_{\xi_1}}.
\]
The last line is valid for $N \geq N_3$ because $\alpha(1/2 - \alpha - \delta) + p > \xi_1$ by (7.21). Here $N_3$ is deterministic and may depend on $p, \xi_1, \delta, c_0$ and hence ultimately on $\xi, \xi_1$. This proves the required result with
\[
N_{\xi_1}(\omega) = \max(N_2(\omega) + m, N_{\xi_1}(\omega) + m, [\xi, \delta_1](\xi_{1}\delta_1)N_{\xi_1} + m, N_3).
\]
Now fix $R' = 1 \lor [\xi, \delta_1]$ and $N(K) \equiv N_3$ (deterministic). Then if $N \geq 2m \lor N(K)$, (7.21) implies that
\[
\mathbb{P}(N_{\xi_1} \geq N) \leq \mathbb{P}(N_2 \geq N - m) + 2\mathbb{P}(N_{\xi} \geq N(1 - m/N)/R')
\leq C K \eta_{N-m} + 2\mathbb{P}(N_{\xi} \geq N/R),
\]
for \( R = 2R' \). This gives the required probability bound \([5.4]\).

## 8 Smooth kernels

The strong uniqueness results stated earlier had \( \alpha > 0 \). We did not try to extend those arguments to the cases \( \alpha = 0 \), because in that case a much simpler argument will serve. We present that in this section.

If \( \alpha = 0 \), then the SIE \([1.2]\) is simply the SDE \( dX_t = \sigma(X_t)dt \), and the classical Yamada-Watanabe result gives strong uniqueness for \( \gamma \in [\frac{1}{2}, 1] \). But one can ask about more general SIE’s, with a smooth but non-constant kernel, for which the latter result does not apply directly. That is the content of the following result. Note that this is the only result that in this paper that applies when \( \gamma \) actually equals \( \frac{1}{2} \).

**Proposition 8.1** Suppose that \( \kappa(s, t) \) is a deterministic smooth positive function of variables \( s \leq t \), that is bounded away from 0. Let \( \sigma \) satisfy \([1.1]\) for some \( \gamma \in [\frac{1}{2}, 1] \). Then strong uniqueness holds for the stochastic integral equation

\[
X_t = x_0 + \int_0^t \kappa(s, t) \sigma(X_s) dW_s. \tag{8.1}
\]

**PROOF.** Let \( X^1_t \) and \( X^2_t \) be solutions to \((8.1)\). Set \( Y^1_t = \int_0^t \sigma(X_s) dW_s \), so \( X^1_t = x_0 + \int_0^t \kappa(s, t) dY^1_s \). Therefore \( dX^1_t = \kappa(t, t) dY^1_t + H^1_t dt \), where \( H^1_t = \int_0^t \partial_2 \kappa(s, t) dY^1_s \).

Set \( \tilde{X}_t = X^1_t - X^2_t \), \( \tilde{Y}_t = Y^1_t - Y^2_t \), and \( \tilde{H}_t = H^1_t - H^2_t \), so \( d\tilde{X}_t = \kappa(t, t) d\tilde{Y}_t + \tilde{H}_t dt \). In particular, for \( \phi_n \) as in \((8.3)\),

\[
\phi_n(\tilde{X}_t) = \int_0^t \phi'_n(\tilde{X}_s) \kappa(s, s) d\tilde{Y}_s + \int_0^t \phi''_n(\tilde{X}_s) \tilde{H}_s ds + \frac{1}{2} \int_0^t \phi'''_n(\tilde{X}_s) [\kappa(s, s) - \sigma(X^1_s)]^2 ds.
\]

Let \( K > 0 \) and take \( T_K \) to be the first time either \( X^1_t \) or \( X^2_t \) exceeds \( K \). Recall that \( L \) is the Hölder constant for \( \sigma \). Then the quadratic variation of the first term is bounded, so

\[
E[\phi_n(\tilde{X}_{t \wedge T_K})] \leq E\left[ \int_0^t \phi'_n(\tilde{X}_s) 1_{\{s < T_K\}} \tilde{H}_s ds + \frac{1}{2} \int_0^t \phi''_n(\tilde{X}_s) [\kappa(s, s) - \sigma(X^1_s)]^2 ds \right]
\]

\[
\leq E\left[ \int_0^t |\phi'_n(\tilde{X}_s)\tilde{H}_s| 1_{\{s < T_K\}} ds + \frac{L^2}{2} \int_0^t \phi''_n(\tilde{X}_s) \|\tilde{X}^1_s\|^{2\gamma} ds \right]
\]

\[
\leq \int_0^t E[|\phi'_n(\tilde{X}_{s \wedge T_K})\tilde{H}_{s \wedge T_K}|] ds + \frac{L^2}{2} \int_0^t E[|\phi''_n(\tilde{X}_{s \wedge T_K})\|\tilde{X}_{s \wedge T_K}\|^{2\gamma}] ds \equiv I^n_t + I^2_t.
\]

Then

\[
I^2_t \leq \frac{L^2}{n} \int_0^t E[\|\tilde{X}_{s \wedge T_K}\|^{2\gamma}] ds \leq \frac{L^2t(2K)^{2\gamma-1}}{n} \to 0 \quad \text{as} \quad n \to \infty.
\]
Since
\[ \hat{H}_t = \int_0^t \partial_2 \kappa(s, t) \, d\hat{Y}_s = \partial_2 \kappa(t, t) \hat{Y}_t - \int_0^t \hat{Y}_s \partial_2 \kappa(s, t) \, ds, \]
we have
\[ I^n_t \leq \int_0^t \mathbb{E}[\|\hat{H}_{s \wedge T_K}\|] \, ds \]
\[ \leq \int_0^t \|\partial_2 \kappa(s, s)\| \mathbb{E}[\|\hat{Y}_{s \wedge T_K}\|] \, ds + \int_0^t \int_s^t \|\partial_2 \kappa(q, s)\| \mathbb{E}[\|\hat{Y}_{q \wedge T_K}\|] \, dq \, ds \]
\[ = \int_0^t \mathbb{E}[\|\hat{Y}_{s \wedge T_K}\|] \left( \|\partial_2 \kappa(s, s)\| + \int_s^t \|\partial_2 \kappa(s, q)\| \, dq \right) \, ds. \]

Sending \( n \to \infty \) gives that
\[ \mathbb{E}[\|\hat{X}_{t \wedge T_K}\|] \leq \int_0^t \mathbb{E}[\|\hat{Y}_{s \wedge T_K}\|] \left( \|\partial_2 \kappa(s, s)\| + \int_s^t \|\partial_2 \kappa(s, q)\| \, dq \right) \, ds. \tag{8.2} \]

Let \( m_K(t) = \max_{s \leq t} \mathbb{E}[\|\hat{Y}_{s \wedge T_K}\|] \). Since
\[ \hat{X}_t = \int_0^t \kappa(s, t) \, d\hat{Y}_s = \kappa(t, t) \hat{Y}_t - \int_0^t \partial_1 \kappa(s, t) \hat{Y}_s \, ds, \]
we see that \( |\kappa(t, t)\hat{Y}_t| \leq |\hat{X}_t| + \int_0^t |\partial_1 \kappa(s, t)| \hat{Y}_s \, ds \). In combination with (8.2), this shows that
\[ \mathbb{E}[\|\hat{Y}_{t \wedge T_K}\|] \leq \int_0^t \mathbb{E}[\|\hat{Y}_{s \wedge T_K}\|] \left( \|\partial_2 \kappa(s, s)\| + \int_s^t \|\partial_2 \kappa(s, q)\| \, dq + \|\partial_1 \kappa(s, t)\| \right) \, ds. \]

For any \( t_0 > 0 \), let \( C(t_0) \) be the maximum of the above fraction, over \( 0 \leq s \leq t \leq t_0 \). Therefore
\[ 0 \leq m_K(t) \leq C(t_0) \int_0^t m_K(s) \, ds \tag{8.3} \]
for every \( t \leq t_0 \). This is \( \leq C(t_0) m_K(t) \), from which it follows that \( m_K(t) = 0 \) for \( t \in [0, \frac{1}{C(t_0)}] \). Applying (8.3) a second time now gives this for \( t \in [0, \frac{2}{C(t_0)}] \).

After finitely many iterations we have \( m_K(t) = 0 \) on \( [0, t_0] \), and since \( t_0 \) was arbitrary, in fact this holds for all \( t \geq 0 \). Sending \( K \to \infty \) shows that for every \( t \) we have \( \hat{Y}_t = 0 \) a.s., and therefore also \( \hat{X}_t = 0 \) a.s. \( \square \)

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