Energetics of Black Hole-Accretion Disk System with Magnetic Connection: Limit of Low Accretion Rate

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Abstract

We study the energetics of a black hole-accretion disk system with magnetic connection: a Keplerian disk is connected to a Kerr black hole by a large-scale magnetic field going through the transition region. We assume that the magnetic field is locked to the inner boundary of the disk and corotates with the inner boundary, the accretion rate is low but the accretion from the disk can still provide enough amount of cold plasma particles in the transition region so that the magnetohydrodynamics approximation is valid. Then, the magnetic field is dynamically important in the transition region and affects the transportation of energy and angular momentum. Close to the equatorial plane, the motion of particles is governed by a one-dimensional radial momentum equation, which contains a fast critical point as the only intrinsic singularity. By finding solutions that smoothly pass the fast critical point, we find that a system with a fast rotating black hole and that with a slow rotating black hole behave very differently. For a black hole with $a > a_{\text{cr}} \equiv 0.3594M$, where $M$ is the mass, $a$ the specific angular momentum of the black hole, the spinning energy of the black hole is efficiently extracted by the magnetic field and transported to the disk, increasing the radiation efficiency of the disk by many orders of magnitude. For a black hole with $0 \leq a < a_{\text{cr}}$, the inner region of the disk is disrupted by the magnetic field and the inner boundary of the disk moves out to a radius where the Keplerian angular velocity of the disk is equal to the spinning angular velocity of the black hole (which is at infinity if the black hole is nonrotating). As a result, the disk may have an extremely low radiation efficiency if $0 \leq a/M \ll 1$. Although our calculations are restricted in a thin slab close to the equatorial plane, we expect that the solutions are typical for the whole transition region.

Key words: black hole physics—accretion, accretion disks—magnetic fields—magnetohydrodynamics: MHD
1. Introduction

Extraction of energy from a rotating black hole with the aid of magnetic fields has been investigated by many people for several decades. When a magnetic field connects the horizon of a Kerr black hole to remote plasma particles, Blandford and Znajek (1977) showed that the voltage drop on the horizon of the black hole induced by the rotation of the black hole in the magnetic field is large enough to cause a cascade production of electron-positron pairs to form a force-free plasma around the black hole. As a result, a poloidal electric current propagating between the horizon of the black hole (the “battery”) and the remote plasma (the “load”) is produced, which transports energy and angular momentum from the black hole to the remote plasma in the form of Poynting flux. The Blandford-Znajek mechanism was later reformulated and confirmed by Macdonald and Thorne (1982) in terms of the “membrane paradigm” (see also Thorne, Price, Macdonald 1986). Although the original Blandford-Znajek mechanism was formulated with the assumption of a force-free plasma around the black hole, Takahashi et al. (1990) showed that the Blandford-Znajek mechanism also works for an ideal magnetohydrodynamical (MHD) fluid that is not force free. Under some conditions the plasma accreting onto the black hole carries a negative energy flux, and as a result the rotational energy of the black hole is extracted. For a long time the Blandford-Znajek mechanism has been considered as a promising process for powering the radio jets in active galactic nuclei (Blandford, Znajek 1977; Rees et al. 1982; Phinney 1983; Begelman, Blandford, & Rees 1984). Recently, this mechanism has also been invoked in models for the central engines of gamma-ray bursts (Paczyński 1993; Mészáros, & Rees 1997; Paczyński 1998; Lee, Wijers, & Brown 2000; Li 2000b).

Recently, a variant of the Blandford-Znajek mechanism has been discussed in the literature: The magnetic field is assumed to connect the black hole to the accretion disk rotating around it (Blandford 1999; Li 2000a; Li, Paczyński 2000; van Putten, Ostriker 2001; Li 2002a; Blandford 2002; Medvedev, Murray 2002; Wang, Xiao, Lei 2002; Wang et al. 2003; Uzdensky 2004, and references therein). In such a model, the accretion disk replaces the remote plasma (the “load”) in the standard Blandford-Znajek mechanism. Moreover, because of the rotation of the disk, the disk itself is also a “battery”, in addition to the black hole battery. The two batteries have opposite signs and the direction of the transportation of energy and angular momentum is determined by the net effect of the two batteries (Li 2002a). When the black hole rotates faster than the disk, its spin energy is extracted by the magnetic field and transported to the disk region. This energy is eventually dissipated and radiated away by the disk, increasing the power and the radiation efficiency of the disk (Agol, Krolik 2000; Li 2000a; Li 2002a). When the black hole rotates slower than the disk, the magnetic field transports energy and angular momentum from the disk to the black hole, decreasing the power and the radiation efficiency of the disk. Therefore, magnetic fields connecting a black hole to an accretion disk have important effects on the transportation of energy and angular momentum, and the ra-
diation process in the disk. Recent XMM-Newton observations on some Seyfert galaxies and Galactic black hole candidates have revealed some possible evidences for a magnetic connection between a black hole and its disk and extraction of energy from a Kerr black hole (Wilms et al. 2001; Li 2002a; Li 2002b; Miller et al. 2002; Ballantyne, Vaughan, Fabian 2003; Merloni, Fabian 2003; Reynolds, Nowak 2003; Reynolds et al. 2004).

For a standard geometrically thin Keplerian disk, it is usually assumed that the inner boundary of the disk is located at the marginally stable circular orbit and the torque at the inner boundary is zero (Shakura, Sunyaev 1973; Novikov, Thorne 1973; Page, Thorne 1974). This assumption has been justified only for nonmagnetized or weakly magnetized disk flows (Muchotrzeb, Paczyński 1982; Abramowicz, Kato 1989). When the disk is strongly magnetized, the magnetic field may strongly affect the structure and accretion of the disk flow (Camenzind 1990; Nitta, Takahashi, Tomimatsu 1991; Hirotani et al. 1992; Punsly 2001; Li 2003b; Li 2003c). In particular, the assumption of zero torque at the inner boundary of the disk has recently been challenged by Krolik (1999), Gammie (1999), and Agol, Krolik (2000). These authors argued that, when magnetic fields are present, they may be dynamically important in the plunging region between the marginally stable circular orbit and the horizon of the black hole and couple the material in the plunging region to the inner boundary of the disk. As a result, the material in the plunging region exerts a torque to the disk at the inner boundary. The issue is still under debate, however (Paczyński 2000; Armitage, Reynolds, Chiang 2001; Hawley, Krolik 2002; Afshordi, Paczyński 2003; Li 2003a; Li 2003b). For instance, Li (2003b) argued that small-scale and tangled magnetic fields—which are often involved in MHD disks—can never be dynamically important. In the presence of an even small but nonzero resistivity, magnetic reconnection will work efficiently to limit the growth of the magnetic field (see also Blandford 2000), so that in the equilibrium state the energy density of the magnetic field is always much smaller than the kinetic energy density of disk particles. This means that small-scale and chaotic magnetic fields in the plunging region cannot produce a significant torque at the inner boundary of the disk.

However, large-scale and ordered magnetic fields have a completely different story. They are more easily amplified by the shear rotation of the disk. When the resistivity is small, in the equilibrium state the energy density of the magnetic field can be large enough to be equipartitioned with the kinetic energy density of disk particles, then the magnetic field must be dynamically important and affect the structure and the motion of the accretion flow (Li 2003b; Li 2003c).

If in the model of Gammie (1999) the magnetic field is interpreted as a large-scale magnetic field connecting a Kerr black hole to a disk around it and corotating with the disk inner boundary, then Gammie’s results showed that the magnetic field is dynamically important in the transition region, extracts the spin energy of the black hole and transports it to the disk. However, he has obtained a disk radiation efficiency that is only slightly bigger than unity:
$\varepsilon_{\text{total}} \approx 1.04$ for $a/M = 0.95$, where $\varepsilon_{\text{total}}$ is defined by the ratio of the total radiated energy to the mass accretion rate, $M$ is the mass of the black hole, $a$ is the specific spinning angular momentum of the black hole.\footnote{Throughout the paper we use geometrized units $G = c = 1$.} This is caused by the fact that Gammie has assumed that the strength of magnetic field in the transition region is limited by the disk accretion rate: $F_{\theta\phi} \approx r^2 B_r \approx 3\eta r^{1/4} r_{\text{in}}^{3/4}$ \cite{Gammie1999}, where $r_g = M$ is the gravitational radius of the black hole, $r_{\text{in}}$ is the radius of the disk inner boundary, $\eta$ is a factor $\lesssim 1$. However, this relation crucially depends on the validity of the following steady state equation for a standard Keplerian accretion disk (with zero torque at the inner boundary)

$$3\pi \Sigma \nu \approx \dot{M}_D,$$
(1)

where $\Sigma$ is the surface mass density of the disk, $\nu$ is the disk viscosity, and $\dot{M}_D > 0$ is the mass accretion rate of the disk. As we will see, equation (1) is not valid any more when there exists a large-scale magnetic field connecting a black hole to a disk.

A large-scale magnetic field connecting a Kerr black hole to a disk at the inner boundary will produce a nonzero torque at the inner boundary. Then, in the steady state the internal viscous torque in the disk is given by \cite{Li2002a}

$$g = \frac{(E - \Omega_D L)_{\text{in}}}{E - \Omega_D L} g_{\text{in}} + \frac{E - \Omega_D L}{-d\Omega_D/dr} \dot{M}_D f,$$
(2)

where $E$ and $L$ are the specific energy and specific angular momentum of disk particles on Keplerian circular orbits, $\Omega_D$ is the disk angular velocity, the function $f$ is given by equation (15n) of Page, Thorne 1974, and the subscript “in” means evaluation at the inner boundary of the disk (i.e., the marginally stable circular orbit). Apparently, when $g_{\text{in}} \neq 0$, $\dot{M}_D$ does not have to be nonzero to balance the internal viscous torque $g$. In the limiting case $\dot{M}_D = 0$, the viscous torque in the disk just works to transport outward the angular momentum that is pumped into the disk by the magnetic connection.

Using $g \approx 3\pi r^2 \Sigma \Omega_D \nu$, $E - \Omega_D L \approx 1$, $-d\Omega_D/dr \approx 3\Omega_D/(2r)$, and $f \approx (3/2)r\Omega_D^2$, equation (2) then becomes

$$3\pi \Sigma \nu \approx \frac{g_{\text{in}}}{r^2 \Omega_D^2} + \dot{M}_D.$$
(3)

Equation (3) is an extension of the standard equation (1) to the case when the torque at the inner boundary of the disk, $g_{\text{in}}$, is nonzero. Because of equation (3) \cite[or, the equivalent equation (2)]{Gammie1999}, the limit on $F_{\theta\phi}$ derived by Gammie does not apply to the case of a large-scale magnetic field connecting a black hole to a disk. The relaxation of this restriction allows us to consider the limit of low accretion rate and look for solutions with higher total radiation efficiency: $\varepsilon_{\text{total}} \gg 1$.

In this paper, we extend the works of Gammie (1999) and Li (2003a; 2003b). We assume that a large-scale magnetic field connects a Kerr black hole to an accretion disk through the
transition region between the marginally stable circular orbit and the horizon of the black hole. In addition, we assume that the mass accretion rate is low, but the accretion from the disk can still provide sufficient amount of plasma particles in the transition region so that the MHD approximation is valid. The magnetic field is frozen to the perfectly conducting disk at the inner boundary and corotates with the inner boundary. We solve the one-dimensional radial momentum equation, derived by Li (2003b) for the region close to the equatorial plane (but we expect the solutions also apply, at least qualitatively, to the regions well above and below the equatorial plane), for smooth solutions corresponding to accretion flows starting from the inner boundary of the disk and ending on the horizon of the black hole. Based on these solutions, we study the effects of the magnetic field on the energetics of black hole-accretion disk system with magnetic connection, including extraction of energy from a rapidly rotating black hole and the influence on disk radiation efficiency.

2. Outline of the Model and the Mathematical Formalism

The geometry of the model is as follow: A large-scale magnetic field connects a Kerr black hole to a directly rotating Keplerian accretion disk, going through the transition region between the inner boundary of the disk and the horizon of the black hole. The mass accretion rate is assumed to be low, resulting a sharp density contrast between the disk region and the transition region. The high-density disk region is geometrically thin, presumably because the magnetic field in the disk region is not dynamically important (i.e., the energy density of the magnetic field is much smaller than the rotational energy density of disk particles). In the low-density transition region, the magnetic field must be dynamically important due to the low mass density of particles, so both magnetic fields and gases are geometrically thick and they expand significantly in the vertical direction. Possible formation scenarios for this model will be discussed in Sec. 4.

We will focus on a small neighborhood of the equatorial plane in the transition region: \( \cos^2 \theta \ll 1 \). Following Gammie (1999) and Li (2003a; 2003b), we assume that the plasma in the transition region is cold and perfectly conducting, near the equatorial plane the magnetic field and the velocity field of plasma particles have only \( r \)- and \( \phi \)-components. The mathematics for such a model is described in detail by Li (2003a; 2003b) (see also Gammie 1999; Camenzind 1986a; Camenzind 1986b; Takahashi et al. 1990; Hirotani et al. 1992; Punsly 2001; Takahashi 2002), and some results are tested by numerical simulations (Gammie, McKinney, Tóth 2003; De Villiers, Hawley 2003). In summary, in the stationary and axisymmetric state, the motion of

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2 This assumption means that the plasma in the transition region is nearly in a force-free condition (i.e., the electromagnetic forces dominate over the inertial and gravitational forces), except very close to horizon of the black hole.

3 Throughout the paper we adopt the Boyer-Lindquist coordinates \((t, r, \theta, \phi)\). The condition \( \cos^2 \theta \ll 1 \) will not restrict the applicability of our results too much, see the discussions in Sec. 4.
the plasma particles is governed by the one-dimensional radial momentum equation \( F(r, u^r) = 0 \), where the generation function \( F(r, u^r) \) is defined by (Li 2003b)

\[
F(r, u^r) \equiv \left\{ 1 + \frac{c_0^2}{r^2 u^r} \left[ \chi^2 - \frac{A}{r^2} (\omega - \Omega_\Psi)^2 \right] \right\}^2 \left[ \frac{r^2}{\Delta} (u^r)^2 + 1 - \frac{f_E^2}{\chi^2 - \frac{A}{r^2} (\omega - \Omega_\Psi)^2} \right],
\]

where \( \Delta \equiv r^2 - 2Mr + a^2 \), \( A \equiv (r^2 + a^2)^2 - \Delta a^2 \), \( \chi \equiv (r^2 \Delta / A)^{1/2} \) (the lapse function), \( \omega \equiv 2Mar/A \) (the angular velocity of frame dragging), and \( u^r \) is the radial component of the four-velocity of plasma particles.

The definition of \( F(r, u^r) \) contains four integral constants: \( c_0 \equiv C_0/\sqrt{-F_m} \), \( C_0 \) is the radial magnetic flux, \( F_m < 0 \) is the radial mass flux; \( \Omega_\Psi \) corresponds to the “angular velocity of the magnetic field” (see below for a precise definition for the angular velocity of magnetic field lines); \( f_L \equiv -F_L/F_m \), \( F_L \) is the radial angular momentum flux; \( f_E' \equiv f_E - \Omega_\Psi f_L \), where \( f_E \equiv -F_E/F_m \), \( F_E \) is the radial energy flux (for details see Li 2003b). Note, our \( c_0 \) corresponds to Gammie’s \( F_{\theta \phi}/\sqrt{2} \), \( \Omega_\Psi \) corresponds to his \( \Omega_F \). (Since \( F \) depends on \( c_0^2 \), without loss of generality we assume \( c_0 > 0 \).)

We remark that the angular velocity of magnetic field lines—which is often defined for a degenerate electromagnetic field where the electric field is perpendicular to the magnetic field—and the angular velocity of disk particles are two different concepts. The angular velocity of magnetic field lines, \( \Omega_F \), is defined by

\[
F_{ab} \left[ \left( \frac{\partial}{\partial t} \right)^b + \Omega_F \left( \frac{\partial}{\partial \phi} \right)^b \right] = 0,
\]

where \( F_{ab} \) is the antisymmetric electromagnetic field tensor. For the model adopted in this paper, \( F_{ab} \) is given by the equation (13) of Li 2003b. Then from equation (5) we have \( \Omega_F = \Omega_\Psi \). Equation (5) means that the electric field measured by a pseudo-observer who rotates around the black hole with an angular velocity \( \Omega_F \) is vanishing.\(^4\) In terms of the electric field and the magnetic field (who are perpendicular to each other) measured by an observer in the locally nonrotating frame, equation (5) can be written as \( \mathbf{E} = \mathbf{B}_p \times \mathbf{v}_F \) and \( v_F = \chi^{-1} (\Omega_F - \omega) \mathbf{m} \) [see eq. [5.3] of Macdonald, Thorne 1982], where \( m^a = (\partial/\partial \phi)^a \), \( \mathbf{B}_p \) is the poloidal component of the magnetic field. Since the velocity of disk particles have both radial and azimuthal components while \( \mathbf{v}_F \)—the linear velocity associated with the angular velocity \( \Omega_F \)—has only an azimuthal component, generally \( \Omega_F \) is not equal to the disk angular velocity \( \Omega_D \) even though the magnetic field is frozen to the disk. Indeed, since \( |\mathbf{E}| \) does not have to be smaller than or equal to

\(^4\) We remind the reader that the electric field \( E_a \) and the magnetic field \( B_a \) measured by an observer of four-velocity \( u^b \) are defined by the antisymmetric electromagnetic field tensor \( F_{ab} \) through \( E_a = F_{ab} u^b \) and \( B_a = \frac{-1}{2} \epsilon_{abcd} u^b F^{cd} \), where \( \epsilon_{abcd} \) is the totally antisymmetric tensor of the volume element associated with the spacetime metric \( g_{ab} \).
The linear velocity $v_F$ can be greater than the speed of light (Macdonald, Thorne 1982)—hence the name “pseudo-observer” used above. In addition, although the angular velocity of disk particles is usually a function of radius, for a stationary and axisymmetric configuration $\Omega_F$ must be constant along magnetic field lines, otherwise magnetic field lines would wind themselves up in violation of stationarity. This is a relativistic extension of the isorotation law of Ferraro (1937) and a mathematical proof can be found in Carter 1979, Macdonald, Thorne 1982, and Li 2003b.

The corresponding differential equation for $u^r$ can be derived from equation (4) by using

$$\frac{\partial F}{\partial u^r} \frac{du^r}{dr} + \frac{\partial F}{\partial r} = 0.$$  

Equation (6) contains two singularities: one is at the fast critical point ($r_f$), the other is at the Alfvén critical point ($r_A$). The Alfvén critical point, defined by $u_r u^r = c_A^2 / (1 - c_A^2)$ where $c_A = c_A \alpha_c$ is the Alfvén velocity, turns out to be an apparent singularity, since it appears in both $\partial F/\partial u^r$ and $\partial F/\partial r$ so cancels out. However, the fast critical point, defined by $u_r u^r = c_A^2 / (1 - c_A^2)$ where $c_A^2 = c_A \alpha_c$, is a true singularity. For accretion solutions particles must leave the inner boundary of the disk “submagnetosonically” but enter the horizon of the black hole “supermagnetosonically”. Thus, any physical solution must smoothly pass the fast critical point, which requires that $\partial F/\partial u^r = 0$ and $\partial F/\partial r = 0$ simultaneously at $r = r_f$. Because of this restriction, among the four integral constants ($c_0, \Omega_\Psi, f'_E, \text{and } f_L$) only three are independent: the other one has to be determined as a function of them (Li 2003b).

The disk is assumed to be Keplerian (with a tiny inward radial velocity superposed) and has an inner boundary at the marginally stable circular orbit: $r_{in} = r_{ms}$. Although we have argued that the angular velocity of magnetic field lines and the angular velocity of disk particles are two different concepts and in general they are not equal to each other, here following Gammie (1999) we assume that $\Omega_\Psi = \Omega_{in}$, where $\Omega_{in}$ is the angular velocity of the disk at the inner boundary. This boundary condition implies that the magnetic field is locked to and corotates with the inner boundary of the disk, which is a good approximation when the radial velocity is much smaller than the rotational velocity at the inner boundary of the disk. The reasonableness of this assumption is supported by the asymptotic relation $f_E \approx \Omega_\Psi f_L$ as $c_0 / r_g \rightarrow \infty$, as we will see in the end of Sec. 3. Then, following the assumption $\Omega_\Psi = \Omega_{in}$, the constant $f'_E$ is uniquely determined by the specific energy, the specific angular momentum, and the angular velocity of the disk at the inner boundary (Gammie 1999; Li 2003b): $f'_E = -E_{in} + \Omega_{in} L_{in}$, which is always negative.

Then, only one free parameter is left: $c_0$, and $f_L$ will be determined as a function of $c_0$, $\Omega_\Psi$, and $f'_E$. From the definition of $c_0$, we have

$$\alpha_c \equiv \frac{c_0}{r_g} \approx \frac{2 B_H r_H}{\sqrt{M_D}}$$
\[ \approx 5 \left[ 1 + \left( 1 - \frac{a^2}{M^2} \right)^{1/2} \right] \left( \frac{B_H}{10^4 G} \right) \left( \frac{M}{10^9 M_\odot} \right) \left( \frac{\dot{M}_D}{10^{25} g/s} \right)^{-1/2}. \]  

(7)

where \( r_H \) is the radius of the black hole horizon, \( B_H \) is the normal component of the magnetic field on the black hole horizon (Macdonald, Thorne 1982). The square of the dimensionless parameter \( \alpha_c \), roughly speaking, measures the ratio of the “power” of the magnetic connection to the “power” of accretion. As explained in the Introduction, for the model of magnetic connection between a black hole and an accretion disk, \( \alpha_c \) can be \( \gg 1 \) in the limit of low accretion rate, although Gammie (1999) estimated that \( \alpha_c \sim 1 \) for the chaotic magnetic field in the transition region donated by accretion from a magnetized disk. Therefore, in our calculation we will treat \( \alpha_c \) as a free parameter and allow it to vary from \( 10^{-1} \) to \( 10^4 \). [The case of \( \alpha_c \ll 1 \) corresponds to the limit of high accretion rate, or equivalently weak radial magnetic field, as already studied by Li (2003a; 2003b).]

Having specified the values of \( \Omega , \Psi \) and \( f'_E \), and the range of \( c_0 \), we are ready to solve the radial momentum equation \( F(r,u') = 0 \) for solutions that smoothly pass the fast critical point, and determine the values of \( f_L \) and \( f_E = f'_L + \Omega \Psi f_L \) for each solution, following the approach of Li (2003b). The results will be presented in the next section.

3. Results

The constant \( f_L \) can be solved from \( F(r,u') = 0 \). For \( f_L \) to be real, the following condition must be satisfied

\[ \left[ \frac{r^2}{\Delta} (u')^2 + 1 \right] \left[ \chi^2 - \frac{A}{r^2} (\omega - \Omega \Psi)^2 \right] \leq f'_E. \]

(8)

At \( r = r_t \), \( u' \) can be expressed as a function of \( r_t \), then equation (8) becomes

\[ \left[ \chi^2 - \frac{A}{r^2} (\omega - \Omega \Psi)^2 \right] \frac{c_0^4}{r^2} \leq -\Delta \left[ \chi^2 - \frac{A}{r^2} (\omega - \Omega \Psi)^2 - f'_E \right], \quad r = r_t. \]

(9)

Equation (9) restricts the physically allowed region in the \((r_t, c_0)\)-space, while equation (8) restricts the physically allowed region in the \((r,u')\)-phase space.

With \( f_L \) expressed in terms of \( r_t \) and \( u'_t \equiv u'(r_t) \), we can solve for \( r_t \) and \( u'_t \) from the condition \( \partial F/\partial r = 0 = \partial F/\partial u' \) at \( r = r_t \) (Li 2003b). The solutions for \( r_t \) as a function of \( \alpha_c = c_0/r_g \) are shown in Fig. 1. Each panel corresponds to a different spinning state of the black hole, as indicated by the value of \( a \). The vertical dotted line shows the position of the marginally stable circular orbit, i.e., the inner boundary of the disk. The dashed curve represents the boundary for physical solutions, beyond which equation (9) is violated so physical solutions do not exist.

From Fig. 1 we see that, the solutions for \( r_t \) can be divided into two classes according to the different topology they have, one class corresponds to \( a > a_{ct} \equiv 0.3594 M \), the other corresponds to \( 0 \leq a < a_{ct} \). (Throughout the paper we assume that \( 0 \leq a/M < 1 \).) The critical
Fig. 1. Solutions for the radius at the fast critical point, $r_f$ (solid curves). Each panel corresponds to a different spinning state of the black hole, as indicated by the value of $a$. The outer boundary of the flow is at the marginally stable circular orbit, as indicated by the vertical dotted line. The dashed curve represents the boundary for physical solutions: beyond the dashed curve physical solutions do not exist (see the text).

spin $a_{cr}$ corresponds to a state that the Keplerian disk angular velocity at the marginally stable circular orbit is equal to the spinning angular velocity of the black hole, i.e. the disk inner boundary corotates with the black hole horizon (Li 2000a; Li 2002a). When $a > a_{cr}$ (then the black hole rotates faster than the inner boundary of the disk and any point beyond that), there is only one branch of solutions for $r_f$, which is separated from the boundary for physical solutions. As $a \to a_{cr}$ from above, the boundary for physical solutions merges to the solutions for $r_f$, when $c_0/r_g$ is large (see the bottom-left panel). When $0 \leq a < a_{cr}$ (then the black hole rotates slower than the inner boundary of the disk), there are two disconnected branches (A and B) of solutions for $r_f$. Branch A ends at the boundary for physical solutions as $c_0/r_g$ increases (see the dark point in the bottom-right panel). As $a$ increases from 0 to $a_{cr}$, the branch B and the end point of branch A move toward the up-left direction, until they disappear when $a = a_{cr}$ is reached.
For the case of $a = 0$, and generally for any $0 \leq a < a_{\text{cr}}$, the solution for $r_f$ on branch B does not correspond to a physical solution representing a flow that starts from $r_{\text{in}}$ and ends at $r_{\text{H}}$, see Appendix 1. Therefore, for any $0 \leq a < a_{\text{cr}}$, there is a maximum value of $\alpha_c$, beyond which stationary and axisymmetric solutions do not exist. For the case of $a = 0$ (the Schwarzschild case), we have $\alpha_{c,\text{max}} \approx 3.88$, corresponding to $r_f \approx 3r_g$ (the dark dot in the bottom-right panel of Fig. 1).

Note, for all cases, $r_f$ approaches $r_{\text{in}} = r_{\text{ms}}$ as $c_0^2/r_g^2 \to 0$, confirming the earlier results of Li (2003a; 2003b)—although $\Omega_\Psi \neq 0$ here.

With the solutions for $r_f$ found, we can calculate $f_L$, then $f_E$. Then, we can solve for the global solutions that start from the inner boundary of the disk, smoothly pass the fast critical point, and end at the horizon of the black hole. Examples of such solutions are presented in Appendix 1. Those examples confirm the conclusions that global solutions exist, except for parameters on the branch B when $0 \leq a < a_{\text{cr}}$. In this section, we present some integral results, since those integral results are more closely related to observational effects.

First, the total radiation efficiency of the system, defined by the ratio of the total power of the disk to the mass accretion rate,$^5$ can be calculated by (see, e.g., Gammie 1999)

$$\varepsilon_{\text{total}} = 1 + f_E. \tag{10}$$

The results for $\varepsilon_{\text{total}}$ are shown in Fig. 2 as a function of $\alpha_c$. For the cases of $a = 0$ and $a = 0.2M$, the curves end at $\alpha_{c,\text{max}} = 3.88$ and $\alpha_{c,\text{max}} = 4.10$, respectively.

Figure 2 shows that, although as $c_0^2/r_g^2 \to 0$ all efficiencies approach the corresponding values for a standard Keplerian disk, as $c_0^2/r_g^2$ goes up the disk radiation efficiency is increased by the magnetic connection. Furthermore, a disk around a Kerr black hole with $a > a_{\text{cr}}$ behaves very differently from that around a Kerr black hole with $0 \leq a < a_{\text{cr}}$. For $a > a_{\text{cr}}$, as $c_0/r_g$ increases the total efficiency $\varepsilon_{\text{total}}$ keeps increasing without limit. For sufficiently large $c_0/r_g$, $\varepsilon_{\text{total}}$ can be greater than unity, then a part of the energy radiated by the disk must be extracted from the spin energy of the black hole. Note, as $a$ goes from $a_{\text{cr}}$ to $a = 0.99M$, $\varepsilon_{\text{total}}$ keeps going up; but after that, as $a$ goes from $0.99M$ to $a = 0.999M$, $\varepsilon_{\text{total}}$ goes down for large $c_0/r_g$. This is caused by the fact that as $a \to M$, the spinning angular velocity of the black hole approaches the angular velocity at the inner boundary of the disk, so if $a$ is very close to $M$ the power of energy extracted from the black hole decreases with increasing $a/M$ (see, e.g., Li 2000a; 2002a).

For $0 \leq a < a_{\text{cr}}$, as $c_0/r_g$ goes up from small values, $\varepsilon_{\text{total}}$ increases until a maximum of $\varepsilon_{\text{total}}$ is reached at $\alpha_c = \alpha_{c,\text{max}}$. For $c_0/r_g > \alpha_{c,\text{max}}$, stationary and axisymmetric solutions do not exist. We speculate that, for $0 \leq a < a_{\text{cr}}$ and large $c_0/r_g$, the inner part of the disk will be disrupted by the magnetic field, resulting that the inner boundary of the disk moves outward.

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5 The transition region is not likely to be able to radiate much of energy since significant change in the specific energy of particles inside the marginally stable orbit necessarily implies that the non-gravitational force is important then the material in the transition region must be geometrically thick so have low radiative efficiency.
The total radiation efficiency of the system, $\varepsilon_{\text{total}} = 1 + f_E$, as a function of $\alpha_c = c_0/r_g$. Different lines correspond to different spinning states of the black hole, as labeled. For reference, the following typical values for disk radiation efficiencies are indicated: 0.057, corresponding to a standard Keplerian disk around a Schwarzschild black hole; 0.34, a standard Keplerian disk around a Kerr black hole with $a = 0.999M$; and 1, beyond which some of radiated energy must be tapped from the black hole.

(More implications will be discussed near the end of this section.) Even for $c_0/r_g < \alpha_{c,\text{max}}$, the magnetic field can still be dynamically important and leads to an increase in the total radiation efficiency of the disk. For example, for the case of a Schwarzschild black hole ($a = 0$), the total radiation efficiency at $c_0/r_g = \alpha_{c,\text{max}}$ is $\approx 0.19$, 3.4 times higher than the corresponding value ($\approx 0.057$) for a standard Keplerian disk with no magnetic connection.

For the critical case of $a = a_{\text{cr}}$, the radiation efficiency increases as $c_0/r_g$ goes up, but quickly approaches a constant for $c_0/r_g > 10$. This can be shown analytically. From Fig. 1, the bottom-left panel, we see that for $a = a_{\text{cr}}$ and large $c_0/r_g$, the boundary for physical solutions merges to the solutions for $r_t$. Then, from equation (4) we have

$$f_L = -\left[ \frac{4}{\chi^2 - \frac{4}{\chi^2} (\omega - \Omega_{\Psi})^2} \right]_{r=r_t} f'_E. \tag{11}$$

As $c_0/r_g \to \infty$, we have $r_t \to r_H$ and $\omega(r=r_t) \to \Omega_H = \Omega_{\Psi}$ (for $a = a_{\text{cr}}$). Then, both the numerator
and denominator on the right-hand of equation (11) approach zero as \(c_0/r_g \to \infty\), the limiting value in the brackets can be evaluated by L’Hospital’s rule. The results for \(f_L\) and \(f_E\) are then
\[
f_L \approx 1.1687r_g f'_L \approx -0.7488 r_g, \quad f_E \approx -0.7103,
\]
which are independent of \(c_0/r_g\) provided that \(c_0^2/r_g^2 \gg 1\). The corresponding limiting radiation efficiency is \(\varepsilon_{\text{total}}(a = a_{\text{cr}}, c_0^2/r_g^2 \gg 1) \approx 0.29\), four times higher than that for a standard disk around a Kerr black hole of \(a = a_{\text{cr}}\).

The inequality \(\varepsilon_{\text{total}} > 1\) is a sufficient condition for a part of the energy radiated by the disk being extracted from the spin energy of the black hole, since when \(\varepsilon_{\text{total}} > 1\) the net energy flux \(F_E\) must be positive. This can also be explained by the fact that the maximum amount of energy that can be extracted from the disk material is equal to the rest mass of the disk. However, \(\varepsilon_{\text{total}} > 1\) is not a necessary condition, which means that a part of energy can still be extracted from the black hole if \(\varepsilon_{\text{total}} \leq 1\). Indeed, for all models calculated in Fig. 2, the specific energy of disk particles are always finite and positive as they reach the horizon of the black hole, see Fig. 3. For large \(c_0/r_g\), the specific energy of particles on the horizon of the black hole, \(E_H\), is smaller than the specific energy of particles on the marginally stable circular orbit by a significant fraction, implying that the magnetic field in the transition region is dynamically important and keeps removing energy and angular momentum from particles. However, \(E_H\) never gets to zero no matter how large \(c_0/r_g\) is. In fact, Fig. 3 shows that, for \(a \geq a_{\text{cr}}\), \(E_H\) approaches positive constants as \(c_0/r_g \to \infty\). This makes the approach of extracting energy from a Kerr black hole with magnetic fields distinctly different from the Penrose mechanism (Penrose 1969) (c.f. Hirotani et al. 1992; Williams 2003). In the former case negative electromagnetic energy flows into the horizon of the black hole but the energy of particles remains positive, while in the latter the energy of the particle that falls into the black hole has to be negative.

The total radiation efficiency, \(\varepsilon_{\text{total}}\), can be written as a sum of two distinct contributions: the contribution from the mechanical energy (including rest-mass energy) of particles, \(\varepsilon_{\text{mech}} = 1 - E_H\); and the contribution from the electromagnetic energy (i.e., the Poynting flux), \(\varepsilon_{\text{EM}} = \varepsilon_{\text{total}} - \varepsilon_{\text{mech}} = f_E + E_H\). For the model studied in this paper, \(\varepsilon_{\text{mech}}\) is always positive, while \(\varepsilon_{\text{EM}}\) can take either sign. When \(\varepsilon_{\text{EM}} > 0\), electromagnetic energy flows out of the black hole horizon, part of the radiated energy is extracted from the spin energy of the black hole. When \(\varepsilon_{\text{EM}} < 0\), part of the mechanical energy of disk particles is converted into electromagnetic energy and flows into the black hole horizon, the rest is transported to the disk then radiated away, no energy is extracted from the black hole. Thus, when \(\varepsilon_{\text{EM}} \leq 0\), all the energy radiated by the disk eventually comes from the gravitational binding energy of disk particles. The fraction \(\varepsilon_{\text{EM}}/\varepsilon_{\text{total}}\) is calculated and shown in Fig. 4, where different curves correspond to different spinning states of the black hole. We see that, extraction of energy from black holes is possible only if \(a > a_{\text{cr}} = 0.3594M\). For \(a < a_{\text{cr}}\), a fraction of the mechanical energy of particles is
Fig. 3. The specific energy of disk particles as they reach the horizon of the black hole, as a function of $c_0/r_g$. Different lines correspond to different spinning states of the black hole, as labeled. As $c_0^2/r_g^2 \to 0$, $E_H$ approaches the specific energy on the marginally stable circular orbit.

converted to the electromagnetic energy which flows into the horizon of the black hole. In the critical case $a = a_{cr}$, no electromagnetic energy flows into or out of the black hole, the change in the mechanical energy of particles is all eventually radiated away by the disk. We also see that, for $a > a_{cr}$, extraction of energy from black holes happens even when $\varepsilon_{\text{total}} < 1$. For example, when $a = 0.99M$ and $c_0 = 2r_g$, we have $\varepsilon_{\text{total}} \approx 0.59$ and $\varepsilon_{\text{EM}}/\varepsilon_{\text{total}} \approx 0.30$, so about 30% of the total radiation comes from the spinning energy of the black hole.

It is interesting to check the evolution of the black hole spin, $a/M$, under the joint action of magnetic connection and accretion. The evolution direction of the black hole spin is determined by the sign of $f_L - 2af_E$: If $f_L - 2af_E > 0$, the black hole spins down (i.e., $a/M$ decreases with time); if $f_L - 2af_E < 0$, the black hole spins up (i.e., $a/M$ increases with time). If $f_L - 2af_E$ happens to be zero, then the spin of the black hole is in an equilibrium state and does not change with time (Gammie 1999). In Fig. 5 we show the evolution direction of the black hole spin in the $(a,c_0)$-space. We see that, for all models with $a < a_{cr}$, the black hole spins up. While for models with $a > a_{cr}$, the evolution direction of the black hole spin depends on the
Fig. 4. Fraction of the electromagnetic energy (Poynting flux) contribution to the total radiation efficiency. When $\varepsilon_{\text{EM}}/\varepsilon_{\text{total}} > 0$, $\varepsilon_{\text{EM}}/\varepsilon_{\text{total}}$ of the total radiated energy is extracted from the spin energy of the black hole in the form of Poynting flux. When $\varepsilon_{\text{EM}}/\varepsilon_{\text{total}} < 0$, part of the mechanical energy of particles is converted to the energy of electromagnetic fields and falls into the black hole. Different lines correspond to different spinning states of the black hole: $a/M = 0, 0.2, 0.3594, 0.4, 0.6, 0.8,$ and $0.99$ (bottom-up).

value of $c_0/r_g$. For a given $a > a_{\text{cr}}$, the black hole spins down/up if $c_0/r_g$ is greater/smaller than some critical value. As an example, for $a/M = 0.99$, the critical value of $c_0/r_g$ is $\approx 1.12$. For $c_0/r_g \gg 1$, the equilibrium state (corresponding to $f_L - 2af_E = 0$) approaches $a_{\text{cr}}/M = 0.3594$ from above.

It is worth to make some further comments on the implications of our results on black hole-accretion disk systems with $0 \leq a < a_{\text{cr}}$ and large $c_0/r_g$. As we have shown, for sufficiently large $c_0/r_g$ and $0 \leq a < a_{\text{cr}}$, stationary and axisymmetric solutions do not exist. This implies that when $c_0/r_g$ is large, the inner part of a Keplerian disk around a Kerr black hole with $0 \leq a < a_{\text{cr}}$ will be disrupted by the strong magnetic field connecting the black hole to the disk. As a result, the inner boundary of the disk moves out, until a corotation radius is reached where the Keplerian angular velocity of the disk is equal to the spinning angular velocity of the black hole. Then, a new stationary equilibrium state is established, with a Keplerian disk truncated.
Fig. 5. The \((a, c_0)\)-space is divided into three regions: the spin-down region, the spin-up region, and the region where physical solutions do not exist (shaded region). If a black hole-accretion disk system is in the spin-down region, the black hole will be spun down (i.e., its spin \(a/M\) decreases with time) by the joint action of magnetic connection and accretion (as indicated by the leftward arrow). If a black hole-accretion disk system is in the spin-up region, the black hole will be spun up by the magnetic connection and accretion (as indicated by the rightward arrow). The two regions are separated by the equilibrium state (the solid curve), in which the spin of the black hole does not change with time. The curve for the equilibrium state merges to the line of \(a = 0.3594M\), as \(c_0/r_g \to \infty\). (The inner boundary of the disk is at the marginally stable circular orbit.)
inwardly at the corotation radius
\[ r_c = M \left( \frac{a}{M} \right)^{-2/3} \left[ 1 + \sqrt{1 - \left( \frac{a}{M} \right)^2} \right]^{4/3}, \]  \hspace{1cm} (13)
for \( 0 \leq a < a_{\text{cr}}. \) This motivates us to propose the following model for a black hole-accretion disk system with large \( c_0/r_g; \) the inner boundary of the geometrically thin Keplerian disk is given by
\[ r_{\text{in}} = \begin{cases} r_{\text{ms}}, & a > a_{\text{cr}}; \\ r_c, & 0 \leq a < a_{\text{cr}}. \end{cases} \]  \hspace{1cm} (14)
For \( r < r_{\text{in}}, \) the magnetic field is dynamically important so the material is geometrically thick. The magnetic field exerts a torque at the inner boundary of the disk, which pumps angular momentum and energy into the Keplerian disk region. The total radiation efficiency of such a disk is shown in Fig. 6 as a function of the black hole spin.

Figure 6 shows that, the total radiation efficiency of a disk with a strong magnetic connection dramatically differs from that of a standard Keplerian accretion disk. For \( a > a_{\text{cr}}, \) the spin energy of the black hole is efficiently extracted by the magnetic field and fed into the disk, increasing the disk radiation efficiency dramatically. (As \( a/M \) approaches unity, the radiation efficiency drops since when \( a/M = 1 \) the angular velocity at the inner boundary of the disk is equal to the spin angular velocity of the black hole.) While for \( 0 \leq a < a_{\text{cr}}, \) the radiation efficiency is independent of the value of \( c_0/r_g, \) provided that \( c_0/r_g \) is large. For \( 0.03 < a/M < 0.3594, \) the radiation efficiency is still larger than that of a standard Keplerian disk despite the fact the inner boundary of the disk moves out. This is caused by the nonzero torque at the disk inner boundary produced by the magnetic field, which transports energy from the transition region to the disk. For \( 0 \leq a/M < 0.03, \) the effect of the growth in the disk inner boundary radius becomes more important, resulting that the disk radiation efficiency becomes smaller than that predicted by the standard disk model. As \( a \to 0, \) the disk radiation efficiency drops to zero (compare it to the corresponding efficiency \( \approx 0.057 \) for a standard Keplerian disk around a Schwarzschild black hole).

For \( 0 < a/M \ll 1, \) we have \( r_{\text{in}} = r_c \approx M(a/4M)^{-2/3} \gg r_g. \) Correspondingly, we have \( f_E \approx f'_E \approx -1 + 3M/2r_c, \) so we have \( \varepsilon_{\text{total}} = 1 + f_E \approx 3M/2r_c \approx 0.6(a/M)^{2/3}. \) Thus, for a system with an extremely slow rotating black hole, the radiation efficiency of the accretion disk can be extremely low.

Finally, when \( a > a_{\text{cr}} \) and in the limit \( c_0/r_g \to \infty, \) by equation (57) of Li 2003b (and the fact that \( E_H \) and \( L_H \) remain finite, see Fig. 3) we have \( \varepsilon_{\text{total}} \approx f_E \approx \Omega_\Psi f_L, \) and the power of the black hole
\[ P_H \approx -F_m f_E \approx \frac{2M}{r_H} C_0^2 \Omega_{\text{in}} (\Omega_H - \Omega_{\text{in}}), \]  \hspace{1cm} (15)
where we have used \( \Omega_\Psi = \Omega_{\text{in}}. \) From the definition of \( C_0 \) (Li 2003b), we have \( C_0 = 2B_H Mr_H. \)
Fig. 6. The total radiation efficiency of a black hole-accretion disk system with magnetic connection (MC), where the inner boundary radius of the Keplerian disk is given by equation (14). Two models are calculated: one with $c_0/r_\text{g} = 10^2$, the other with $c_0/r_\text{g} = 10^3$. The curves break at $(a/M, \varepsilon_{\text{total}}) = (0.3594, 0.2897)$. For comparison, the radiation efficiency corresponding to a standard Keplerian disk is shown with the dashed line.

Equation (15) can be used to estimate the power of a disk magnetically coupled to a fast rotating Kerr black hole.

4. Discussion and Conclusions

We have studied the energetics of a black hole-accretion disk system with a large-scale magnetic field connecting the disk to the black hole through the transition region. The model that we have adopted is an extension of those used by Gammie (1999) and Li (2003b). In Gammie 1999 (as well as in Krolik 1999), the magnetic field is assumed to have small scales, be advected into the transition region from the disk by accretion, and couple the material in the transition region to the disk. While in our model, we assume that the magnetic field has a large scale, directly connect the disk to the black hole. The role of accretion is only to provide a dilute plasma gas in the transition region so that an electric current can exist there. In
Li 2003b, the large-scale magnetic field is assumed to extend from the black hole horizon to infinity and have a zero angular velocity although the disk has a nonzero angular velocity and the magnetic field is frozen to the disk (see Sec. 2 for the difference between the two angular velocities). While in our model, the large-scale magnetic field is assumed to terminate at the inner boundary of a thin Keplerian disk, beyond which the magnetic field is chaotic, has small scales, and is dynamically unimportant. In addition, the magnetic field is assumed to corotate with the inner boundary of the disk.

We have focused our calculations to a small neighborhood of the equatorial plane, although the material in the transition region must be geometrically thick when the dynamical effects of magnetic fields are important. It is assumed that the system is in a stationary and axisymmetric state, and in the small neighborhood, where \( \cos^2 \theta \ll 1 \), both the magnetic field and the velocity field have only radial and azimuthal components (i.e., \( B^\theta = u^\theta = 0 \)). This model is similar to the cylindrical model that is often used for accretion disks, where the cylindrical coordinates are used and every quantity is independent of the vertical coordinate. Here we use the spherical coordinates and assume that every quantity is independent of \( \theta \)—which is more appropriate for an accretion flow near the central black hole. The self-consistency of this model has been proved by Li (2003b), where it was shown that to the first order of \( \cos \theta \) the dynamical equilibrium in the \( \theta \)-direction is guaranteed. The extension to the regions well above and below the equatorial plane has also been discussed by Li (2003b).

With our simplified model, we find that in the limit of low mass accretion rate, a system with a fast rotating black hole and a system with a slow rotating black hole behave very differently. For a black hole with \( a > a_{\text{cr}} = 0.3594 M \), the spin energy of the black hole is efficiently extracted by the magnetic field and transported to the disk region, increasing the radiation efficiency of the disk dramatically. Indeed, in such a case the total radiation efficiency of the disk is unbounded from above. For a black hole with \( 0 \leq a < a_{\text{cr}} \), stationary solutions do not exist if we assume the inner boundary of the disk is at the marginally stable orbit, which leads us to speculate that in such a case the inner region of the disk is disrupted by the magnetic field and the inner boundary of the disk moves outward until a corotation radius (defined by \( \Omega_D = \Omega_H \)) is reached. When \( 0 \leq a/M \ll 1 \), the disk—which has a very large radius at the inner boundary—has an extremely low radiation efficiency: \( \varepsilon_{\text{total}} \approx 0.6(a/M)^{2/3} \ll 1 \). The above results are in great contrast to those for a standard geometrically thin Keplerian disk, whose radiation efficiency is always in the range \( 0.06 - 0.4 \).

The disruption of the inner region of a disk magnetically coupled to a slow rotating black hole can be interpreted by the following fact: When the black hole rotates slower than the inner boundary of the disk, the black hole exerts a negative torque to the inner boundary of the disk, but a Keplerian disk cannot sustain a negative torque (Li 2000a; Li 2002a). The general consequences of the magnetic interaction between a black hole and an accretion disk have already been explored in the literature (see, e.g., Li 2000a; Li 2002a), while the calculations
presented in this paper confirm those results with mathematics.

Note, the condition for the validity of the solutions is $\cos^2 \theta \ll 1$, not $|\cos \theta| \ll 1$. For $71.6^\circ < \theta < 108.4^\circ$, we have $\cos^2 \theta < 0.1$ where we expect the solutions are not bad. This corresponds to an opening angle of $37^\circ$ around the equatorial plane, which is not a small angle. Therefore, although our calculations are restricted in a thin slab close to the equatorial plane, we expect that the solutions are typical for the whole transition region. This conclusion is reasonable when the following fact is also considered: the accretion flow has highest density on the equatorial plane so the transportation of mass, energy, and angular momentum should dominantly happen in the region near the equatorial plane.

Hence, according to both the general results in the literature and the detailed calculations in this paper, an accretion disk magnetically coupled to a black hole can have a radiation efficiency in a broad range: from an extremely low value $\ll 1$ to an extremely high value $\gg 1$, depending on the spin of the black hole, the strength of the magnetic field, and the mass accretion rate. This is of great interest since observations show that galactic nuclei at high and low redshifts have extremely divergent distribution in luminosities, from the extremely bright active galactic nuclei (AGN) at high redshift to the much less active—sometimes not active at all—nearby galactic nuclei, which is certainly not because of a lack of supermassive black holes in nearby galactic nuclei (Narayan 2002). In order to interpret the extremely low luminosities of nearby galactic nuclei, people were led to consider models of advection- or convection-dominated flow (ADAF/CDAF), and advection-dominated inflow outflow solution (ADIOS) (for reviews see Blandford 1999; Narayan 2002). Our results on the system with a slow rotating black hole suggests a new interpretation for the low luminosities in nearby galactic nuclei: nearby galactic nuclei may have extremely slow rotating supermassive black holes and the inner region of the disks may have been disrupted by a strong magnetic field connecting the disk to the black hole.\(^6\)

Apparently, the model presented in this paper has been highly simplified. In a real situation, the magnetic field should have a more complex structure: The foot points of magnetic field lines should distribute over the surface of the disk rather than all going through the inner boundary of the disk, and some of the magnetic field lines passing through the black hole may extend to infinity directly (see, e.g., Nitta, Takahashi, Tomimatsu 1991; Blandford 2002; Uzdensky 2004). The field lines connecting the black hole to the surface of the disk lead to additional contribution to the transportation of angular momentum and energy between the black hole and the disk (Li 2002a; Li 2002b; Wang et al. 2003). The field lines connecting the black hole directly to the plasma at infinity extract energy and angular momentum from the black hole and transport them to remote plasma through the Blandford-Znajek mechanism, so will affect the evolution of the central black hole (Park, Vishniac 1988; Park, Vishniac 1990; Lu

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\(^6\) An accretion disk around a fast rotating black hole, containing a large-scale magnetic field and having a high accretion rate, may also have an extremely low radiation efficiency and luminosity if the large-scale magnetic field extends to infinity (Li 2003b).
et al. 1996; Wang, Xiao, Lei 2002). Clearly, all these effects will affect some of the calculations in this paper, including the total radiation efficiency of the disk and the evolution of the spin of the black hole. Detailed calculations including the above effects are beyond the scope of the current paper. However, we should emphasize that the main results in this paper are more qualitative rather than quantitative, they are accurate only if the effects mentioned above are weak. The readers should keep this point in their mind to correctly understand and apply these results.

Finally, let us briefly comment on under what conditions a large-scale magnetic field connecting a disk and a black hole through the transition region can be formed. It appears to us that there are three possible ways: (1) Inherited from the progenitor of the black hole and the disk. For example, merger of a magnetized neutron star with a black hole will naturally lead to the formation of a disk with a large-scale magnetic field. The turbulent motion [or, the magneto-rotational instability (Balbus, Hawley 1991; Balbus, Hawley 1998)] in the disk region then disrupts the initially large-scale magnetic field, transform it into highly tangled small-scale magnetic fields. The magnetic field in the transition region, on the other hand, may remain being ordered and having large correlation scales. (2) Large-scale magnetic fields may form from small-scale magnetic fields through reconnection (Tout, Pringle 1996). The chaotic small-scale magnetic fields in the disk region are carried into the transition region by accreting gases, then form a large-scale magnetic field in the transition region through reconnection. (3) The chaotic small-scale magnetic fields in the disk region are carried into the transition region by accreting gases, then stretched by the radial motion to increase the correlation length in the radial direction. Imagine that, in the disk region and near the inner boundary, there is a small loop of magnetic field line with a length \( \sim H \) in the radial direction, where \( H \) is the half-thickness of the disk. As accretion goes on, its inner foot (the foot closer to the inner boundary) leaves the disk first, then moves to the black hole along with the gas. The outer foot is still in the disk region and has a distance \( \sim H \) from the inner boundary of the disk, moves toward the inner boundary with a velocity \( v_r \). Assuming that the time needed by a plasma particle moving from the inner boundary of the disk to the horizon of the black hole is approximately given by the radial free-fall time: \(^7\) \( t_1 \sim r_{\text{in}}(r_{\text{in}}/r_g)^{1/2} \). Then, the correlation length of the magnetic field in the transition region is \( \gtrsim r_{\text{in}} \) if the following condition is satisfied: \( t_1 < t_2 \equiv H/v_r \sim \left(c_s^2/v_{\phi}^2\right)r_{\text{in}}/v_r \), where \( c_s \) is the sound speed in the disk region, and \( v_{\phi} \) is the rotation velocity of the disk. For a Keplerian disk we have \( v_{\phi} \sim (r_g/r)^{1/2} \), then \( t_1 < t_2 \) leads to \( v_r < c_s^2/v_{\phi} \) in the disk. Interestingly, for a standard Keplerian disk we have \( v_r \sim c_s^2/v_{\phi} \) (Balbus, 7).

\(^7\) In a real case a particle leaving from the inner boundary of disk has nonzero angular momentum so will not free-fall radially. However, if the magnetic field is dynamically important in the transition region (as the case in the present paper), the particle may lose its angular momentum quickly so the precise proper time taken by the particle moving from the disk inner boundary to the black hole horizon may not be far from the radial free-fall time.
Hawley 1998). So, if the disk radial velocity is smaller than that predicted by a standard Keplerian disk, then a large-scale magnetic field can be formed in the transition region.

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In the main text we have solved the solutions at the fast critical points and the corresponding integral constants. However, this does not guarantee the existence of global solutions that start from the inner boundary of the disk and end at the horizon of the black hole. In this appendix we show that global solutions exist corresponding to the fast critical point solutions in Fig. 1 except the branch B in the case of $a=0$, by presenting some examples of global solutions. (The inner boundary of the disk is always at the marginally stable circular orbit.)

In Fig. 7, we present the global solution $u^r = u^r(r)$ for $a = 0.99M$ and $c_0 = 10^2r_g$. The corresponding fast critical point is at $r_f = 1.141r_g$ and $u_f^r = -0.1673$, integral constants are $f_L = 1215r_g$ and $f_E = 442.7$ (so the total radiation efficiency is $\varepsilon_{\text{total}} = 443.7$). In the figure, the solid curves correspond to $F(r, u^r) = 0$. The smooth solid curve going from the bottom-right corner to the top-left corner, as labeled with letter “A”, represents the global solution for $u^r = u^r(r)$. The solution describes a flow that starts from the marginally stable circular orbit subsonically (the right end), passes the Alfvén point (the circle) first, then the fast critical point (the filled circle), finally falls into the horizon of the black hole supersonically (the left end).

In Fig. 8 we show the global solution $u^r = u^r(r)$ for $a = a_{\text{cr}}$ and $c_0 = 10^2r_g$. The corresponding fast critical point is at $r_f = 1.934r_g$ and $u_f^r = -0.6627$, integral constants are $f_L = -0.7556r_g$ and $f_E = -0.7110$ (so the total radiation efficiency is $\varepsilon_{\text{total}} = 0.289$). Note, in this example, the fast critical point is almost right on the boundary for physical solutions (see Fig. 1, the panel for $a = 0.3594M$), and the Alfvén critical point merges to the fast critical point. Comparing Fig. 8 to Fig. 7, we can see how the topology of the solutions to $F(r, u^r) = 0$ changes as $a \to a_{\text{cr}}$ from above.

In Fig. 9 we show the global solution $u^r = u^r(r)$ for $a = 0$ and $c_0 = 1r_g$. The corresponding fast critical point is at $r_f = 4.395r_g$ (on the branch A in Fig. 1, the panel for $a = 0$) and $u_f^r = -0.1079$, integral constants are $f_L = -3.078r_g$ and $f_E = -0.9165$ (so the total radiation efficiency is $\varepsilon_{\text{total}} = 0.083$).

In Fig. 10 we show the global solution $u^r = u^r(r)$ for $a = 0$ and $c_0 = 3.88r_g$. The corresponding fast critical point is at $r_f = 3.00r_g$ (the dark point on the branch A in Fig. 1, the panel for $a = 0$) and $u_f^r = -0.488$, integral constants are $f_L = -1.485r_g$ and $f_E = -0.808$ (so the total radiation efficiency is $\varepsilon_{\text{total}} = 0.19$). Similar to Fig. 7, the smooth solid curve labeled with letter “A” is the physical solution that starts from the inner boundary of the disk and ends on the horizon of the black hole. Note, in this example, the fast critical point is on the
Fig. 7. The contour of $F(r, u^r) = 0$ (solid curves), corresponding to $a = 0.99 M$ and $c_0 = 10^2 r_g$. The contours for the fast critical points (the long-dashed curve, defined by $u_r u^r = c_A^2 / (1 - c_A^2)$), the Alfvén point (the short-dashed curve, defined by $u_r u^r = c_A u_r^r$), and the boundary for physical solutions (the dashed-dotted curve, beyond which $f_L$ is complex so physical solutions do not exist) are also shown. The global solution $u^r = u^r(r)$ is represented by the smooth solid curve labeled with letter “A”, which starts from the inner boundary of the disk (the right end) and ends on the horizon of the black hole (the left end), passing the Alfvén critical point (the circle) and the fast critical point (the filled circle) in turn.

Figures 7–10 demonstrate that global solutions exist for $a \geq c_{cr}$ with any $\alpha_c = c_0 / r_g$, and for $0 \leq a < a_{cr}$ with $\alpha_c \leq \alpha_{c,\text{max}}$.

Finally, we show that global solutions do not exist for $0 \leq a < a_{cr}$ and $\alpha_c > \alpha_{c,\text{max}}$, by presenting the contours of $F(r, u^r) = 0$ for $a = 0$ and $c_0 = 10^2 r_g$ in Fig. 11. The corresponding solutions for the fast critical point is at $r_f = 2.00064 r_g$ (on the branch B in Fig. 1, the panel for $a = 0$) and $u_r^r = -0.74377$, integral constants are $f_L = 701.7 r_g$ and $f_E = 47.04$. As we can see from the figure, there is no solid curve representing a solution $u^r = u^r(r)$ that starts from the
Fig. 8. Similar to Fig. 7 but for $a = a_{cr} = 0.3594M$. The Alfvén critical point merges to the fast critical point, and they are right on the boundary for physical solutions (c.f. Fig. 1, the panel for $a = 0.3594M$). (The dashed-dotted curve is moved upward a little bit so that it can be seen. Indeed it almost coincides with the solid curve.)

marginally stable circular orbit ($r_{ms} = 6r_g$) and ends on the horizon of the black hole ($r_H = 2r_g$): the curves $A_1$ and $A_2$ are not connected to each other. This conclusion can also be inferred from the values of $f_E$: it is positive so cannot correspond to physical solutions since no energy can be extracted from a Schwarzschild black hole. (On the branch B in Fig. 1, the panel for $a = 0$, there is another solution for $r_I$ corresponding to $c_0 = 10^2r_g$: $r_I = 2.0315r_g$. It is easy to show that this also does not lead to a physical solution.)

Appendix 2. Some Useful Fitting Formulas

In this Appendix we present some useful fitting formulas. The inner boundary of the disk is always assumed at the marginally stable circular orbit.

For $0.3594 < a/M < 0.999$ and $c_0^2/r_g^2 \gg 1$, the solution for the radius at the fast critical point can be fitted by

$$r_I \approx r_H + f_1(a^*) r_g^3/c_0^2,$$  \hspace{1cm} (A1)
Fig. 9. Similar to Fig. 7 but for $a = 0$ and $c_0 = 1 r_g$. The smooth solid curve labeled with letter “A” is a physical solution for $u^r = u^r(r)$, starting from the inner boundary of the disk (the right end) and ending on the horizon of the black hole (the left end).

where $a^* = a/M$ and

$$f_1(a^*) \equiv 3.637 + 2.319 \sqrt{1 - a^*} + 1.672 a^* - 4.883 a^{*2} - 4.975 a^{*3} + 4.882 a^{*4}. \quad (A2)$$

The relative error in $r_f - r_H$ is $< 6\%$ for $c_0/r_g > 10$, $< 2\%$ for $c_0/r_g > 100$. Then, since $r_f - r_H \ll r_H$, the radial component of the four-velocity at the fast critical point is

$$u_f^r \approx - \left( \frac{M}{r_H} \right)^{4/3} \left\{ 2f_1(a^*) \sqrt{1 - a^{*2}} \left[ f_E^2 + 4M^2 (\Omega_H - \Omega_\Psi)^2 \right] \right\}^{1/3}. \quad (A3)$$

Substituting $r = r_f$ and $u = u_f^r$ into $F(r, u^r) = 0$, we can solve for $f_L$, then $f_E = f_L + \Omega_\Psi f_L$.

For $0 \leq a/M \leq 0.3952$, the maximum value of $c_0$ (beyond which stationary and axisymmetric solutions do not exist, i.e. the $c_0$ on the dashed curve in Fig. 5), can be fitted by

$$\log(c_0/r_g) \approx f_2(a^*),$$

where

$$f_2(a^*) \equiv 0.5718 + 0.01069/\sqrt{0.3594 - a^*} - 0.00006044/\sqrt{0.3594 - a^*} - 0.1928 a^* + 8.255 a^{*2} - 114.5 a^{*3} + 735.4 a^{*4} - 2161 a^{*5} + 2402 a^{*6}. \quad (A4)$$

The relative error in $\log(c_0/r_g)$ is $< 1\%$. 26
Fig. 10. Similar to Fig. 9 but for $c_0 = 3.88 r_g$. In this case the fast critical point is on the boundary for physical solutions (c.f. Fig. 1, the panel for $a = 0$), and the Alfvén critical point merges to the fast critical point.

For $0.3956 \leq a/M \leq 0.999$, the $c_0$ corresponding to the “equilibrium state” in Fig. 5 can be fitted by $\log(c_0/r_g) \approx f_3(a^*)$, where

$$f_3(a^*) = 31.803 + 2.447 \sqrt{1 - a^*} + 0.01371 / \sqrt{a^* - 0.3594} - 297.945 a^* + 1104.82 a^{*2} - 2160.917 a^{*3} + 2357.443 a^{*4} - 1359.334 a^{*5} + 323.972 a^{*6}. \quad (A5)$$

The relative error in $\log(c_0/r_g)$ is $< 2\%$. 

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Fig. 11. The contour of $F(r, u') = 0$ (solid curves), corresponding to $a = 0$ and $c_0 = 10^2 r_g$, and $r_t = 2.00064 r_g$. There is no physical solution corresponding to a flow that starts from the marginally stable circular orbit, passes the fast critical point, then enters the horizon of the black hole. As shown, the curves $A_1$ and $A_2$ are not connected to each other.