I. INTRODUCTION

The central problem in any dynamical theory is to find the time evolution of a system that was prepared in a given initial state. In quantum mechanics there are only a few of these problems that are readily solved by simple analytical methods [1]. In general, we have to rely on approximations to obtain out of the Schrödinger equation the time evolution operator $\hat{U}_S(t,t_0)$ in a suitable form. With the explicit knowledge of $\hat{U}_S(t,t_0)$, we may calculate the expectation value of any physical observable of the system at any time $t$ once we know the state of system at the time $t_0$. Frequently, we are able to find the time evolution operator $\hat{U}_0(t,t_0)$ associated with $\hat{H}_0$, a part of the total Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$. In this case, it is usually convenient to make a transformation to the “interaction picture” such that

$$\hat{U}_S(t,t_0) = \hat{U}_0(t,t_0)\hat{U}_I(t,t_0)$$

(1)

holds with $\hat{U}_0(t_0,t_0) = \hat{I}$, $\hat{U}_I(t_0,t_0) = \hat{I}$. Consequently, the time evolution operator in the interaction picture, $\hat{U}_I(t,t_0)$, satisfies the Schrödinger equation

$$\frac{\partial \hat{U}_I(t,t_0)}{\partial t} = -i\lambda \hat{H}_I(t)\hat{U}_I(t,t_0),$$

(2)

where we have considered $\hbar = 1$ and have defined $\lambda \hat{H}_I(t) = \hat{U}_0^3(t,t_0)\hat{H}_{\text{int}}(t)\hat{U}_0(t,t_0)$, The parameter $\lambda$ is some dimensionless real parameter chosen to give a measure of the relative magnitude of the most important matrix elements of $\hat{H}_{\text{int}}$ and $\hat{H}_0$ in a given problem.

The most popular perturbation approximation method to deal with the Schrödinger equation in Eq. (2) is the Dyson expansion [2]:

$$\hat{U}_I(t,t_0) = \hat{I} - i\lambda \int_0^t dt_1 \hat{H}_I(t_1,t_0) +$$

$$(-i\lambda)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \hat{H}_I(t_2, t_0) \hat{H}_I(t_1, t_2) + \ldots,$$

(3)

where it is very convenient to estimate the solution by truncating the series to a given order $\lambda^n$. Besides the normal difficulties to calculate high-order terms, the Dyson truncation produces an approximated evolution operator that is not unitary. Other expansions of operator $\hat{U}_I(t,t_0)$ have also been proposed in the literature, as the Magnus expansion [3] the Fer product [4] and, more recently, the Aniello expansion [5]. New bounds for the convergence of the Magnus expansion and the Fer product have been recently studied in Ref. [6]. Other product expansions have also been considered in the literature [7].

In this paper, we present an alternative method to calculate the time evolution operator $\hat{U}_I(t,t_0)$ as a product of a finite number of unitary operators

$$\hat{U}_I(t,t_0) = \hat{U}_1(t,t_0)\hat{U}_2(t,t_0)\ldots\hat{U}_k(t,t_0)\ldots\hat{U}_N(t,t_0),$$

(4)

where each operator $\hat{U}_k(t,t_0), k = 1, 2, \ldots, N - 1$, can be written as the exponential of an anti-Hermitian operator proportional to $(\lambda)^k$, while $\hat{U}_N(t,t_0) - \hat{I}$ is at most of order $(\lambda)^N$. The number $N$ of operators in the expansion can be as large as we want. If we approximate the last operator $\hat{U}_N(t,t_0)$ in the product by the unit operator, we obtain an expansion of the evolution operator $\hat{U}_I(t,t_0)$ which is explicitly unitary to order $(\lambda)^N$. Besides this important advantage, this expansion is well suited to treat a variety of problems. In Section II, we derive the expressions for each term in the expansion; in Section III, we provide pedagogical examples; and in Section IV, we present our conclusions.

II. THE METHOD

We start with the simple case of $N = 2$. Equation (4) can then be written as

$$\hat{U}_I(t,t_0) = \hat{U}_1(t,t_0)\hat{U}_2(t,t_0)\hat{U}_3(t,t_0).$$

(5)

Following the same kind of procedure used in the transformation to the interaction representation, we write

$$\hat{U}_I(t,t_0) = e^{-i\lambda W_1(t,t_0)}\hat{U}_2(t,t_0),$$

(6)

where $W_1(t,t_0)$ is some hermitian operator to be chosen conveniently. From now on, we set $t_0 = 0$ and avoid writing it to simplify the notation. From Eqs. (2), (5) and (6), we have

$$\frac{\partial \hat{U}_2(t)}{\partial t} = -i\lambda \hat{H}_2(t)\hat{U}_2(t),$$

(7)
where
\[
\hat{H}_2(t) = \sum_{m=0}^{\infty} \frac{(i\lambda)^m}{m!} (ad \Hat{W}_1(t))^m \left\{ \Hat{H}_1(t) - \frac{1}{m+1} \frac{\partial \Hat{W}_1(t)}{\partial t} \right\}.
\]
(8)

Here, we have defined \(ad \Hat{A}\{ \} = [\Hat{A}, \cdot] \) and used the following relation
\[
e^{\alpha \Hat{A}} \Hat{B} e^{-\alpha \Hat{A}} = \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} (ad \Hat{A})^m \{ \Hat{B} \}
= \Hat{B} + \alpha [\Hat{A}, \Hat{B}] + \frac{\alpha^2}{2!} [\Hat{A}, [\Hat{A}, \Hat{B}]] + \cdots \]
(9)
Choosing
\[
\Hat{W}_1(t) = \int_0^t \Hat{H}_1(t')dt',
\]
we have
\[
\Hat{H}_2(t) = \sum_{m=1}^{\infty} \frac{(i\lambda)^m}{(m+1)!} (ad \Hat{W}_1(t))^m \{ m\Hat{H}_1(t) \},
\]
(11)
which is of order \(\lambda\). In this case, \(\Hat{U}_2(t)\) is the solution of Eq. (6) and should be written as an exponential of a non-Hermitian operator that is, in general, a series on the variable \(\lambda\), starting with \(\lambda^2\).

In the simple case where \([\Hat{H}_1(t), \Hat{H}_1(t')] = -2i f(t, t')\) is a c-number function, then
\[
\Hat{H}_2(t) = \lambda \int_0^t dt' f(t', t)
\]
(12)
is also a c-number function. Consequently, Eq. (7) can be easily integrated to give \(\Hat{U}_2(t) = e^{-i\lambda^2 \phi(t)}\), where
\[
\phi(t) = \int_0^t dt' \int_0^{t'} dt'' f(t'', t')
\]
(13)
and the time evolution operator is
\[
\Hat{U}_1(t) = e^{-i\gamma \int_0^t dt' \Hat{H}_1(t')} e^{-i\lambda^2 \phi(t)}.
\]
(14)

This result is well known and could also be easily obtained using the Magnus expansion [8]. It can be used, for example, to easily obtain the time evolution operator for the quantum state generated by an external time-dependent force acting on a mechanical oscillator, as we show in the next section.

The procedure described above can be generalized for any value of \(N\) greater than 2 by setting
\[
\Hat{U}_n(t) = e^{-i\lambda \Hat{W}_n(t)} \Hat{U}_{n+1}(t), n = 2, 3, \ldots N - 1,
\]
(15)
so that expansion of the operator \(\Hat{U}_1(t)\) may also read
\[
\Hat{U}_1(t) = e^{-i\lambda \Hat{W}_1(t)} e^{-i\lambda \Hat{W}_2(t)} \cdots e^{-i\lambda \Hat{W}_{N-1}(t)} \Hat{U}_N(t).
\]
(16)
The operators \(\Hat{U}_n(t)\), for \(n = 2, 3, \ldots\), satisfy a Schrödinger-like equation
\[
\frac{\partial \Hat{U}_n(t)}{\partial t} = -i\lambda \Hat{H}_n(t) \Hat{U}_n(t),
\]
(17)
where \(\Hat{H}_n(t)\) is given by
\[
\Hat{H}_n(t) = \sum_{m=0}^{\infty} \frac{(i\lambda)^m}{m!} (ad \Hat{W}_{n-1}(t))^m
\times \left\{ \Hat{H}_{n-1}(t) - \frac{1}{m+1} \frac{\partial \Hat{W}_{n-1}(t)}{\partial t} \right\}.
\]
(18)

By choosing operators \(\Hat{W}_j(t)\)'s for \(j = 1, \ldots, N - 1\), we obtain operators \(\Hat{H}_j(t)\) for \(j = 2, \ldots, N\), and the expansion given by Eq. (18).

We now show that it is possible to choose operators \(\Hat{W}_n(t)\) being proportional to \(\lambda^{n-1}\), and such that the operators \(\Hat{H}_n(t)\) are power series in the variable \(\lambda\) starting with the power \(\lambda^{n-1}\). Then, by substituting \(\Hat{W}_n(t), n = 1, 2, \ldots (N - 1)\) in Eq. (4) and noticing that \(\Hat{I} - \Hat{U}_N(t)\) would be at least of \(\mathcal{O}(\lambda^N)\), we will obtain the desired expansion announced in the Introduction. We make the proof by construction. Writing explicitly the dependence of the operators \(\Hat{W}_n(t)\) and \(\Hat{H}_k(t)\) on \(\alpha = i\lambda\), we have
\[
\Hat{W}_k(t) = \alpha^{k-1} \Hat{W}_k(t),
\]
\[
\Hat{H}_k(t) = \sum_{j=k-1}^{\infty} \Hat{H}_{k,j}(t) \alpha^j.
\]
(19)
By substituting Eq. (19) in Eq. (18) we get, for \(j \geq n \geq 1\),
\[
\Hat{H}_{n+1,j} = \sum_{m=1}^{\infty} \sum_{k=1}^{n} \sum_{i=k-1}^{\infty} \frac{1}{m!} (ad \Hat{W}_k(t))^m \left\{ \Hat{H}_{k,i}\delta(km+i-j) \right\} - \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(m+1)!} (ad \Hat{W}_k(t))^m \left\{ \frac{d\Hat{W}_k(t)}{dt} \delta(km+k-1-j) \right\},
\]
(20)
and for \(0 < j < n\),
\[
\frac{d\Hat{W}_{j+1}(t)}{dt} = \sum_{m=1}^{\infty} \sum_{k=1}^{n} \sum_{i=k-1}^{\infty} \frac{1}{m!} (ad \Hat{W}_k(t))^m \left\{ \Hat{H}_{k,i}\delta(km+i-j) \right\} - \sum_{k=1}^{n} \sum_{m=1}^{\infty} \frac{1}{(m+1)!} (ad \Hat{W}_k(t))^m \left\{ \frac{d\Hat{W}_k(t)}{dt} \delta(km+k-1-j) \right\},
\]
(21)
where \( \delta(m-n) = \delta_{mn} \) is the Kronecker delta. Notice that \( j+1 > k \) in Eq. (21) so that \( \frac{dW_{j+1}(t)}{dt} \) is given recursively in terms of the operators \( \tilde{W}_k(t) \) and \( \tilde{H}_k(t) \), for \( k \leq j \). For example, if we set \( n = 2 \) and \( j = 1 \) in the above equations, we easily get

\[
\frac{d\tilde{W}_2(t)}{dt} = \frac{1}{2} (\text{ad} \tilde{W}_1(t)) \{ \tilde{H}_1(t) \},
\]

(22)

where we used Eq. (21) and the fact that \( \tilde{H}_{1,0}(t) = \tilde{H}_1(t) \). Using the initial condition \( \tilde{W}_2(0) = 0 \), we have

\[
\tilde{W}_2(t) = \frac{1}{2} \int_0^t dt' [\tilde{W}_1(t'), \tilde{H}_1(t')],
\]

(23)

which also can be written as

\[
\tilde{W}_2(t) = \frac{1}{2} \int_0^t dt_1 \int_0^{t_1} dt_2 [\tilde{H}_1(t_2), \tilde{H}_1(t_1)].
\]

(24)

To obtain an approximate expression for \( \tilde{U}_1(t) \) valid to order \( O(\lambda^2) \), we first set \( N = 3 \) in Eq. (16):

\[
\tilde{U}_1(t) = e^{-i\lambda \tilde{W}_1(t)} e^{-i(\lambda^2)\tilde{W}_2(t)} \tilde{U}_3(t).
\]

(25)

\( \tilde{U}_3(t) - \tilde{I} \) is of order \( \lambda^3 \), since it satisfies the Schrödinger equation, Eq. (17), with \( \tilde{H}_3(t) \) of the order \( O(\lambda^2) \). If we approximate \( \tilde{U}_3(t) \) by the identity we get an approximation which is unitary and valid to order \( O(\lambda^2) \). Using the expressions for \( \tilde{W}_1(t) = \tilde{W}_1(t) \) and for \( \tilde{W}_2(t) \) given in Eq. (10) and Eq. (23), we have

\[
\tilde{U}_1(t) \approx \exp\left\{ (-i\lambda) \int_0^t dt_1 \tilde{H}_1(t_1) \right\} \times \exp\left\{ \frac{\lambda^2}{2} \int_0^t dt_1 \int_0^{t_1} dt_2 [\tilde{H}_1(t_2), \tilde{H}_1(t_1)] \right\}.
\]

(26)

The procedure described above can be generalized for obtaining approximations involving a product of \( N \) operators, by calculating \( \tilde{W}_k(t), k = 1, \ldots, N \), through Eqs. (20) and (21). Below we give, as examples, the explicit expressions for \( \tilde{W}_3(t) \), \( \tilde{W}_4(t) \), and \( \tilde{W}_5(t) \)

\[
\tilde{W}_3(t) = \frac{1}{3} \int_0^t dt' [\tilde{W}_1(t'), [\tilde{W}_1(t'), \tilde{H}_1(t')]] ,
\]

\[
\tilde{W}_4(t) = \frac{4}{3!} \int_0^t dt' [\tilde{W}_1(t'), [\tilde{W}_1(t'), [\tilde{W}_1(t'), \tilde{H}_1(t')]] ] + \frac{1}{4} \int_0^t dt' [\tilde{W}_2(t'), [\tilde{W}_1(t'), \tilde{H}_1(t')]] ,
\]

\[
\tilde{W}_5(t) = \frac{4}{5!} \int_0^t dt' [\tilde{W}_1(t'), [\tilde{W}_1(t'), [\tilde{W}_1(t'), \tilde{H}_1(t')]] ] + \frac{2}{3} [\tilde{W}_3(t'), [\tilde{W}_1(t'), [\tilde{W}_1(t'), \tilde{H}_1(t')]]].
\]

(27)

As we show in the next section, the expansion obtained above may be useful in several cases and in particular for obtaining effective time independent Hamiltonians, when the operator \( \tilde{U}_n(t') \) in the expansion can be approximated by the exponential of the product of the time with a constant operator.

Notice that besides the fact that they are Hermitian, no restriction was made on the operators \( \tilde{W}_n(t) \) for \( n = 2, 3, \ldots \) until now. Special choices of \( \tilde{W}_n(t) \), other than the one we have chosen to discuss in this paper, may lead to interesting applications in specific cases.

### III. EXAMPLES OF APPLICATIONS

In this section we describe three examples of applications of the method: i) the problem of a linear harmonic oscillator subjected to a driving force; ii) the Raman resonant transition inside a cavity; iii) the ultrastrong coupling (USC) and deep strong coupling (DSC) regimes of the Jaynes-Cummings (JC) model.

We start with the well known problem of a linear harmonic oscillator subject to a driving force \( \pm g f(t) \). The Hamiltonian is given by (\( \hbar = 1 \))

\[
H = \omega (\hat{a} \hat{a}^\dagger + 1/2) + g f(t) (\hat{a} + \hat{a}^\dagger) \]

(28)

where \( \hat{a} \) and \( \hat{a}^\dagger \) are the usual annihilation and creation operators satisfying the algebra \( [\hat{a}, \hat{a}^\dagger] = 1 \).

We first take \( H_0 = \omega (\hat{a} \hat{a} + 1/2) \) and go to the interaction representation by defining \( \tilde{U}(t) = e^{-i\hat{H}_0 t/\hbar} \tilde{U}_1(t) \), where

\[
\frac{\partial \tilde{U}_1(t)}{\partial t} = -ig \tilde{H}_1(t) \tilde{U}_1(t),
\]

(29)

with

\[
\tilde{H}_1(t) = f(t) (\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}).
\]

(30)

In this case, \( [\tilde{H}_1(t), \tilde{H}_1(t')] = -2if(t') f(t) \sin \omega (t - t') \) is a c-number. Therefore, \( \tilde{W}_n = 0 \) for \( n > 2 \), \( \tilde{U}_3 = \tilde{I} \), and

\[
\tilde{W}_1(t) = \int_0^t dt' f(t') (\hat{a} e^{-i\omega t'} + \hat{a}^\dagger e^{i\omega t'}),
\]

(31)

\[
\tilde{W}_2(t) = i \int_0^t dt_1 \int_0^{t_1} dt_2 f(t_1) f(t_2) \sin \omega (t_1 - t_2).
\]

Then, the time evolution operator in the interaction picture is given by

\[
\tilde{U}_1(t) = e^{i\varphi(t)} \tilde{D}(v(t))
\]

(32)

where \( \varphi(t) = g^2 \tilde{W}_2(t) \) is a time-dependent phase and \( \tilde{D}(v(t)) = e^{v(t)\hat{a}^\dagger - v^*(t)\hat{a}} \) is the displacement operator and

\[
v(t) = -ig \int_0^t dt' f(t') e^{i\omega t'}.
\]

(33)

Another example is the case of resonant Raman scattering inside a cavity. Consider a three-level \( \Lambda \) atom interacting quasi-resonantly with a mode of frequency \( \omega_1 \)
of the cavity field and a classical field of frequency $\omega_2$, as schematized in Fig. 1. The two lower levels, $|g\rangle$ and $|e\rangle$, are closely spaced in energy and can make quasi-resonant dipole transitions to an upper level $|i\rangle$. $\omega_{ig}$ and $\omega_{ie}$ are the energy differences between the upper level and the lower levels $|g\rangle$ and $|e\rangle$, respectively. The Hamiltonian that describes the interaction in the rotating-wave approximation is given by

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$$

$$\hat{H}_0 = \omega_{ig}|i\rangle\langle i| + \omega_{ie}|e\rangle\langle e| + \omega_1\hat{a}^\dagger\hat{a}$$

$$\hat{H}_{\text{int}} = \Omega_{ig}|i\rangle\langle g|\hat{a} + \Omega_{ie}e^{-i\omega t}|i\rangle\langle e| + \Omega_ig|g\rangle\langle i|\hat{a}^\dagger + \Omega_ie|e\rangle\langle i|\hat{a}^\dagger,$$

(34)

where $\Omega_{ig}$ is the vacuum Rabi frequency associated to the cavity field of frequency $\omega_1$, while $\Omega_{ie}$ is the Rabi frequency associated to the external classical field of frequency $\omega_2$. Assume that the initial cavity field state has a photon distribution with low photon average number. In Ref. [9], it has been shown that if the detuning $\delta = \omega_{ig} - \omega_1 \approx \omega_{ig} - \omega_{eg} - \omega_2$ is such that $|\delta| \gg \Omega_{gi} \gg \Omega_{ei}$, it is then possible to show that the Raman transition $|g, n_0 + 1\rangle \leftrightarrow |e, n_0\rangle$ is resonant for a certain $n_0$ depending on the detunings of the driving field. Here we rederive the conditions on the frequencies that make the process resonant and the effective Hamiltonian for the system.

**FIG. 1.** Raman transition of a Λ atom inside a cavity.

Assume that the classical field frequency is tuned to

$$\omega_2 = \omega_1 - \omega_{eg} - (n_0 + 1)\Omega_{gi}^2/\Delta + \Omega_{ei}^2/\Delta,$$

(35)

with $\Delta$ satisfying the equation

$$\Delta = \omega_{ig} - \omega_1 + (\Omega_{ei}^2 + 2\Omega_{gi}^2)/\Delta + 2n_0\Omega_{gi}^2/\Delta,$$

(36)

where $n_0$ is an integer and $|\Delta| \gg \Omega_{gi} \gg \Omega_{ei}$.

We first write the Hamiltonian of Eq. (34) as $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$ with

$$\hat{H}_0 = \hat{H}'_0 + \hat{H}_{SS}$$

$$\hat{H}_{\text{int}} = \hat{H}'_{\text{int}} - \hat{H}_{SS},$$

(37)

where $\hat{H}'_0$ and $\hat{H}'_{\text{int}}$ are given by Eq. (34). Also, $\hat{H}_{SS}$ is given by

$$\hat{H}_{SS} = \left(\frac{\Omega_{gi}^2}{\Delta} \hat{a}^\dagger\hat{a}_t + \frac{\Omega_{ei}^2}{\Delta} |i\rangle\langle i| - \frac{\Omega_{gi}^2}{\Delta} \hat{a}^\dagger\hat{a}_t |g\rangle\langle g| - \frac{\Omega_{ei}^2}{\Delta} |e\rangle\langle e|\right),$$

(38)

where

$$\hat{\Delta}_g = \omega_1 - \omega_{ig} + \Omega_{gi}^2(2n + 1)/\Delta$$

and

$$\hat{\Delta}_e = \omega_2 - \omega_{ie} + \Omega_{ie}^2n/\Delta.$$

(39)

We then write the time evolution operator in the interaction representation with respect to $\hat{H}_0$ and use our unitary perturbative expansion. Neglecting terms that vary very rapidly with time, we obtain

$$\lambda\hat{W}_1 = -\hat{H}_{SS}t$$

$$\lambda\hat{W}_2 = \hat{H}_{SS} + \hat{H}_{\text{eff}}t,$$

(40)

where

$$\hat{H}_{\text{eff}} = -\frac{\Omega_{gi}\Omega_{ei}}{\Delta} (|e\rangle\langle g|\hat{a} + |g\rangle\langle e|\hat{a}^\dagger).$$

(41)

Therefore

$$\hat{U}_I(t) \approx e^{-i\hat{H}_{SS}t}e^{-i(\hat{H}_{SS} + \hat{H}_{\text{eff}})t}.$$

(42)

Using the Baker-Hausdorff formula and neglecting the term depending on the commutators of $\hat{H}_{SS}$ and $\hat{H}_{\text{eff}}$, we may write

$$\hat{U}_I(t, 0) \approx e^{-i\hat{H}_{\text{eff}}t}.$$

(43)

That is, $\hat{H}_{\text{eff}}$ can be considered an effective Hamiltonian of the interaction picture associated to the choice of $\hat{H}_0$ given in the Eq. (37).

**FIG. 2.** Survival probability of $|g; 0\rangle$ vs. $\tau = \omega t$. Solid line: exact solution; dashed line: $\hat{U} \approx \hat{U}_0\hat{U}_1$; dot dashed line, first Born approx.; dotted line $\hat{U} \approx \hat{U}_0$, $\omega_0/\omega = 0.6; g/\omega = 0.5$.

Consider now the situation of the Jaynes-Cummings model in the USC regime between a cavity mode and a qubit, $g/\omega \gtrsim 0.1$. This situation is currently accessible to experiments using superconducting qubits and cavities in circuit quantum electrodynamics [10, 11]. In this case,
the rotating-wave approximation is no longer valid and one should consider the full interaction Hamiltonian
\[ \hat{H} = \omega_0 \hat{a}^\dagger \hat{a} + g(\hat{a}^\dagger + \hat{a})\sigma_x + \omega_0 \sigma_z / 2. \] (44)

In the case where \( \omega_0 = 0 \), it reduces to
\[ \hat{H}' = \omega \hat{a}^\dagger \hat{a} + g(\hat{a}^\dagger + \hat{a})\sigma_x. \] (45)

The eigenstates of \( \hat{H}' \) are the product of displaced number states \([12]\) and the eigenstates \(|\pm\rangle\) of \( \sigma_x \), associated with the eigenvalues \( \pm 1 \)
\[ | \pm n; \pm \rangle = \hat{D}(\mp x)|n\rangle \otimes |\pm\rangle, \] (46)

where \( x = g/\omega \), \( \hat{D}(\nu) = e^{\nu \hat{a}^\dagger - \nu^* \hat{a}} \) is the displacement operator, \(|n\rangle \), \( n = 0, 1, 2, \ldots \) are Fock states, and \( \sigma_z |\pm\rangle = \pm |\pm\rangle \). The eigenstates \( \hat{D}(\mp x)|n\rangle \) of \( \hat{H}' \) are degenerate and associated with the eigenvalue \( (n\omega - g^2/\omega) \).

In basis \(| \pm n; \pm \rangle\), \( n = 0, 1, \ldots \), \( \omega_0 \sigma_z / 2 \) is written as
\[ \omega_0 \sigma_z / 2 = \omega_0 / 2 \sum_{n,m} \langle n| \hat{D}(2x)|m\rangle | + n; + \rangle\langle -m; - | + \text{H.c.} \] (47)

In Ref. \([13]\), it has been proposed an approximation which keeps only the terms with \( n = m \) in the right hand side of Eq. (17), that is,
\[ \hat{H}_0 = \hat{H}' + \omega_0 / 2 \sum_n \langle n| \hat{D}(2x)|n\rangle | + n; + \rangle\langle -n; - | + \text{H.c.}, \] (48)

where \( \langle n| \hat{D}(2x)|n\rangle = e^{-2x^2} L_n(4x^2) \). The eigenstates of \( \hat{H}_0 \) can be easily written as
\[ \frac{1}{\sqrt{2}}(| + n; + \rangle \pm | - n; - \rangle, \] (49)

and are associated to the eigenvalues \( n\omega - g^2/\omega \pm (\omega_0 / 2) \langle n| \hat{D}(2x)|n\rangle \).

Using the approximate Hamiltonian \( \hat{H}_0 \) as our zeroth-order approximation, we have found that our method is well suited for describing transition probabilities for a very large range of \( \omega_0 / \omega \) and \( g/\omega \geq 0.1 \), including both, the USC and DSC regimes of the JC model \([11]\). Let us write \( \hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} \), where
\[ \hat{H}_{\text{int}} = \omega_0 / 2 \sum_{n \neq m} \langle n| \hat{D}(2x)|m\rangle | + n; + \rangle\langle -m; - | + \text{H.c.} \] (50)

From Eqs. (48) and (50) we can easily calculate \( \lambda \hat{H}_1(t) = e^{i\hat{H}_0 t} \hat{H}_{\text{int}} e^{-i\hat{H}_0 t} \) and the operators \( \hat{W}_1 \) and \( \hat{W}_2 \) using the expressions given in Eqs. (10) and (23).

FIG. 3. Probability of \(|g; 0\rangle\) to make a transition to the state \(|e; 1\rangle\) as a function of \( \tau = \omega t \). Solid line: exact solution; dashed line: \( \hat{U} \approx \hat{U}_0 \hat{U}_1 \); dotted line: first Born approximation; dashed line \( \hat{U} \approx \hat{U}_0 \). \( \omega_0 / \omega = 0.6; g/\omega = 0.5 \).

FIG. 4. Survival probability of \(|g; 0\rangle\) as a function of \( \tau = \omega t \). Solid line: exact solution; dashed line: \( \hat{U} \approx e^{-i\hat{H}_0 t} e^{-i\lambda \hat{W}_1(t)} \); dot dashed line: \( \hat{U} \approx e^{-i\hat{H}_0 t} e^{-i\lambda \hat{W}_2(t)} \); dotted line \( \hat{U} \approx e^{-i\hat{H}_0 t} \). \( \omega_0/\omega = 1; g/\omega = 0.8 \).

FIG. 5. Transition probability from \(|g; 0\rangle\) to \(|e; 1\rangle\) vs. \( \tau = \omega t \). Solid line: exact solution; dashed line: \( \hat{U} \approx \hat{U}_0 \hat{U}_1 \); dotted line \( \hat{U} \approx \hat{U}_0 \hat{U}_1 \); dotted line \( \hat{U} \approx \hat{U}_0 \). \( \omega_0 / \omega = 1; g/\omega = 0.8 \).

In Figs. 2 and 3, we present curves corresponding to the exact result calculated numerically, the results calculated using three approximations for the time evolution operator: \( e^{-i\hat{H}_0 t} \), \( e^{-i\hat{H}_0 t} e^{-i\lambda \hat{W}_1(t)} \), and the Dyson approximation to first order in \( \omega_0 \). The results show that both, the approximation \( \hat{H} = \hat{H}_0 \) and the first Born approximation, do not describe the transition probabilities for the
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