Pseudoenergy, Superenergy, and Energy Radiation via Gravitational Waves in General Relativity

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(March 24, 2022)

For slowly spinning matter the rate of energy loss via radiation of gravitational waves is estimated in General Relativity (GR) within a generally covariant superenergy approach. This estimation differs from Einstein’s Quadrupole Formula (EQF) by a suppression factor \( \Pi \ll 1 \). For a symmetric two-body-like distribution of scalar matter \( \Pi \) is estimated to be \( \ll (v/c)^2 (r/R)^2 \), where \( v \) is orbital velocity of the bodies, \( c \) - velocity of light, \( r \) - radius of each body, and \( R \) – the inter-body distance. This contradiction with EQF is briefly discussed.

Until the direct detection of the gravitational waves (GW) produced by binary neutron stars or other astrophysical and cosmological objects (which is expected as highly probable in the nearest decade), the measurements of orbital damping of the binary pulsar PSR1913+16 \([1,2]\), in excellent agreement with Einstein’s Quadrupole Formula (EQF) \([3]\), are usually considered the ultimate indirect test of the energy radiated in General Relativity (GR) \([4]\) by any moving matter.

The generation of gravitational radiation in GR is a long-standing problem that dates back to the first years of the General Theory of Relativity (GR) \([5]\) by any moving matter. The fact that GW are real has been confirmed by coordinate free theorems \([6–8]\) and by short-wave analysis \([9]\).

Quantitative estimations of energy radiation via GW are based on (or agree with) the well known Einstein’s Quadrupole Formula which is derived for pseudoenergy\(^*\) radiation rate \( p\dot{E} \). Then the generally covariant energy radiation rate \( \dot{E} \) is estimated by the noncovariant \( p\dot{E} \) in the transverse-traceless (TT) gauge. The leading order of the post-Newtonian (PN) expansion yields

\[
p\dot{E} = \frac{G}{45} (D_i^k)^2,
\]

where \( G \) is Newton’s gravitational constant\(^1\), and

\[
D_{ik} = \int T^{00} (3x_i x_k - \delta_{ik} r^2)
\]

is the time-dependent, untraced mass quadrupole tensor with mass density \( T^{00} \) given by the energy-momentum tensor of matter \( T^{\mu\nu} \) \([10]\).

In spite of the generally noncovariant relation \( \dot{E} = p\dot{E} \), after the discovery of binary pulsar PSR1913+16 \([1]\), formula (1) with next-to-leading order PN contributions (see \([11–13]\), in particular) is widely accepted for GR estimations of GW radiation, in part because of theoretical works designed to shore up its foundations \([12,13,15–20]\), but mostly because of its excellent agreement with binary pulsar data \([2,21]\).

But an empirical agreement does not imply conceptual adequacy: the covariance in theoretical derivations motivated by many experiments may be more important for the theoretical understanding of the nature of gravitation than a particular agreement with data.

Gauge invariance is an important property of the wave equation and energy-momentum tensor for gravitational waves in GR. The usual gauge transformations used for gravitational waves are related to coordinate transformations and gauge invariance implies invariance under these transformations. In general the wave equation for weak gravitational waves is approximately gauge invariant only for high frequency waves \([9]\). The effective energy-momentum tensor is gauge invariant to leading order only if it is averaged over a region of spacetime whose scale is large compared to the wavelengths of the waves. In other cases, such as low frequency gravitational

\(^*\)In GR the gravitational field (Levi-Civita connection coefficients \( \Gamma^k_{ij} \)) does not possess any energy-momentum tensor but, as a consequence of the Einstein Equivalence Principle (EEP), it only possesses the so-called "energy-momentum pseudotensors" \( T^{\mu\nu} \). The gravitational energy-momentum (and gravitational angular momentum) pseudotensors, as being functions of \( \Gamma^k_{ij} \) (the total gravitational strengths) describe the energy-momentum of the total gravitational field, which is a combination of the real gravitational field (for which the Riemann curvature tensor \( R_{iklm} \neq 0 \)) and the inertial forces field (for which \( R_{iklm} = 0 \)). The inertial forces field is generated by the coordinates used. This is also a consequence of the EEP.

\(^1\)Throughout this paper the units, in which the light velocity \( c = 1 \) and reduced Planck constant \( \hbar = 1 \), are adopted.
waves where the averaging is done over time, either the wave equation, the energy-momentum tensor, or both, are in general not gauge invariant [22].

Another alternative is to define gauge invariant effective energy-momentum tensors for gravitational waves and other gravitational perturbations in almost all situations of interest [22] if only those gauge transformations are used which change the perturbed geometry, but leave the background geometry alone. This point of view allows considering the general covariance for all approaches to energy radiation based on the so-called gauge invariant variables in GR [9,14,23–26]. Attempts to quantify the backreaction effects of GW in GR by defining a gauge invariant effective energy-momentum tensor for the waves were first done in [14] and later in [9] for high frequency gravitational waves in a vacuum. Another effective energy-momentum tensor for high frequency waves, both in a vacuum and in spacetimes containing classical matter, was defined in [23]. An extension of [9] in order to include lower frequency waves and spacetimes containing classical matter has been done in [25]. It should be noted that in each case mentioned above a procedure violating general gauge invariance was used: either some sort of averaging procedure, or a procedure of reduction of the gauge group that gives similar results to an averaging procedure [22].

In many known calculations with GW, the background spacetime manifold is considered non-dynamical and, moreover, flat. This means a partial gauge fixing in order to work only with systems of coordinates such that the background metric takes the Minkowski form. What is left from the general gauge group after this gauge fixing is just the Poincaré group. Thus, the reduction of the gauge group means the transition from a general gauge group after this gauge fixing. This reduction of the gauge group is a possible reason why some formally independent calculations of GW radiation in the TT gauge ([28–30] and [26]1) estimate the rate of energy loss in agreement with each other and with EQF: the energy-momentum pseudotensor transforms like the tensor under transformations of the Poincaré group, as do the gauge invariant variables introduced with the partial fixing of the general gauge group. Namely, because of the partial fixing of the gauge group, all these estimations correspond to the tensor field gravity in Minkowski space-time and therefore they are close to each other. Unfortunately, all these estimations do not correspond exactly to GR.

A classic point of view [31–33] is that the Riemann curvature tensor \( R_{\mu\nu\alpha\beta} \) has to play the main role in the definition of gravitational radiation for the exact solutions to GR: only the covariant expressions dependent on \( R_{\mu\nu\alpha\beta} \) may be used to get real information on gravitational energy-momentum and angular momentum in arbitrary admissible coordinates.

The canonical superenergy [34] and angular supermomentum [35] tensors are exactly this type of quantities. They extract covariant information (hidden in the pseudotensor \( t^{\mu\nu} \) [36]) about the real gravitational field, and can be expressed through \( \partial_{\alpha\beta} t_{\mu\nu} \sim R_{\mu\nu\alpha\beta} \) [37].

Calculations of the superenergy radiation rate \((sE)\) can be made by using conservation of the total superenergy introduced for a massive scalar matter field \( (\phi) \) with metric gravitation [38].

As a contribution to debates in [17], [39], [40], [41,42], [43], [44], this paper is an attempt to consider the problem of energy radiation by scalar matter in GR within a generally covariant superenergy approach [38] with the conservation of total superenergy and a covariant relation between the energy-momentum and superenergy-supermomentum tensors.

Let’s start from an approximate two-body-like solution to the Einstein-Klein-Gordon equation5

\[
R_{\mu\nu} = 8\pi G \left( \phi_{,\mu}\phi_{,\nu} + \frac{1}{2} m^2 \phi^2 g_{\mu\nu} \right),
\] (3)

where \( m \) is the scalar field mass.

For semi-quantitative estimations let’s approximate solutions to (3) by an extended, two-body-like distribution of the scalar matter field \( \phi(t, x, y, z) \) chosen in a form of fuzzy surface ellipsoid (with major semi-axis \( A, B, C \)) which is slowly spinning with angular velocity \( \omega \). In order to study the sensitivity of thus obtained results to deviation of \( \phi \) from the exact solution to (3), let’s consider two different coordinate dependences for \( \phi(t, x, y, z) \) chosen in exponential \( \phi_E(t, x, y, z) \) or Gauss \( \phi_G(t, x, y, z) \) forms**. Neglecting gravitational mass shift, let’s normalize both distributions to the rest mass of the matter.

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1In these numerical calculations the gauge conditions are approximately transverse and traceless in the wave zone.

5Following [37,40], the convention is for Greek indices to run over four time-space values 0, 1, 2, 3, while Latin indices run over three spatial values 1, 2, 3; commas and \( \partial \) denote partial derivatives with respect to a chosen coordinate system, while semicolons and \( \nabla \) denote covariant derivatives; repeated indices are summed over.

**In the \( x-y \) rotation plane \( \phi_E \sim \phi_1, \phi_G \sim \phi_2 \), where

\[
\phi_n = \exp \left( -\frac{\left( x \cos(\omega t) + y \sin(\omega t) \right)}{A} \right) - \left( \frac{x \sin(\omega t) - y \cos(\omega t)}{B} \right)^n - \left( \frac{|z|}{C} \right)^n. \] (4)
(M) at $\omega = 0$. For thus defined $\phi_E$ and $\phi_G$ the energy and superenergy densities of the matter at $A \gg B \sim C$ are two-body-like (Fig.1). Although, $\phi_E$ and $\phi_G$ neglect radial contractions of each body under the gravitational attraction, any radial movement of matter does not radiate the tensor metric waves and is beyond the subject of this paper.

In the TT gauge small deviations ($h_{\alpha\beta}^{\mu} \nu(x)$) of its Minkowski form $(\eta^{\alpha\beta})$ yield [10]

$$h_{\alpha\beta} = \frac{2G}{3R}(D_{\alpha\beta})_{\nu \nu}. \quad (5)$$

Instead of the noncovariant calculations with the energy-momentum pseudotensor and gauge invariant variables, let’s calculate the evolution of the scalar particle Compton wavelength (1/\$m_0$) should be

$$[W_{\alpha\beta\lambda\mu}\xi^{\beta}\xi^{\gamma}]^\alpha = 0. \quad (6)$$

Because the exact conservation of $W$ (6) takes place only for the metric tensor and matter field satisfying (3), all calculations should not be too sensitive to a reasonable variation of the approximate solutions mentioned above.

For the goal of this paper the most reasonable choice of Killing vectors is the 4-velocity $u$ of an observer at rest relative to the radiating matter: $u = (1, 0, 0, 0)$. The reasonably small difference between $E_E$ and $E_G$ roughly approximates possible deviations from the exact result. Because these deviations are not significant for semi-quantitative estimations, only the exponential case is considered below:

$$\dot{E} = \frac{1}{2m^2}\dot{W}. \quad (12)$$

Expressions (8), (9), and (12) together yield the generally covariant quadrupole formula

$$\dot{E} = \frac{G}{45}\Pi\left(D_{\alpha\beta}\right)^2, \quad (13)$$

which differs from (1) by a suppression factor

$$\Pi \approx \frac{2\omega^2}{m^2}, \quad (14)$$

where $m$ (the scalar field mass) is yet an arbitrary parameter.

To be consistent with the classical solutions to the problem of GW radiation, $m$ should be large enough: the scalar particle Compton wavelength (1/$m_0$) should be \( \ll A, B, C \), and correspondingly $m \gg m_0$, where the energy of quantum fluctuations $m_0$ is estimated by the
inverse values of $A$, $B$, $C$. Therefore, in accordance with (14) the upper limit of the energy loss rate can be estimated roughly at $m = m_0$. The following particular choice $m_0 = \sqrt{A^{-2} + B^{-2} + C^{-2}}$ significantly simplifies all analytical expressions.

Taking into account that $m \gg m_0$, $\Pi$ is estimated to be extremely small in the two-body-like case: $\Pi \ll (\omega r)^2 = v^2(r/R)^2 \ll 1$, where $r$ and $R$ are characteristic scales (radial ($R \approx A$) and transverse ($r \approx B \approx C$)) of the matter distribution projected on the rotation plane, and $v (= \omega R \ll 1)$ is the orbital velocity.

The kinematic factor $v^2$ in $\Pi$ results in that the quadrupole gravitational radiation occurs at the order of $v^7$ beyond Newtonian gravity instead of $v^5$ in the other approaches with the partial violations of general covariance. The geometric factor $(r/R)^2$ suppresses the energy radiation to zero for point-like masses ($r \to 0$) in accordance with the equations for geodesics.

Typical values of $\Pi$ are estimated to be $\Pi \ll 10^{-2}$ for the final stage of binary neutron stars ($v \sim 0.3$, $r/R \sim 0.5$), and $\Pi \ll 10^{-16}$ for the current stage of the binary pulsar PSR1913+16. These estimations for $\Pi$ strongly contradict to $\Pi = 1$ in (1), and the binary pulsar data.

A possible reason for the dramatic difference between the covariant (13) and noncovariant (1) estimations of the energy loss rates is the covariance of the superenergy approach which allows extracting only the covariant contributions from the energy-momentum pseudotensor $\mu^\nu$. This pseudotensor contains contributions of inertial forces (with $R_{\alpha\beta\mu\nu} = 0$) which may dominate in $\mu^\nu$ in some particular gauges. Nevertheless, these inertial forces do not contribute to the covariant energy flux of the real gravitational fields with $R_{\alpha\beta\mu\nu} \neq 0$. The strong contradiction between the covariant and noncovariant calculations may mean far-reaching consequences for the structure of any metric theory of gravitation which claims to be realistic.

Acknowledgement: Author is grateful to S. I. Blinnikov, I. B. Khriplovich, M. Yu. Konstantinov, and Yu. S. Vladimirov for their constructive comments. This work was supported in part by research grant # 1885.2003.2 of Russian Ministry of Industry and Science.

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