Scaling laws and the rate of convergence in thin magnetic films

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Abstract. We study static 180 degree domain walls in thin infinite magnetic films. We establish the scaling of the minimal energy by \( \Gamma \)-convergence and the energy minimizer profile, which turns out to be the so called transverse wall as predicted in earlier numerical and experimental work. Surprisingly, the minimal energy decays faster than the area of the film cross section at an infinitesimal cross section diameter. We establish a rate of convergence of the rescaled energies as well.

Keywords: Thin magnetic films; Magnetic wires, Magnetization reversal; Domain wall

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1 Introduction

In the theory of micromagnetics the energy of micromagnetics is given by

\[
E(m) = A_{ex} \int_\Omega |\nabla m|^2 + K_d \int_{\mathbb{R}^3} |\nabla u|^2 + Q \int_\Omega \varphi(m) - 2 \int_\Omega H_{ext} \cdot m,
\]

where \( \Omega \subseteq \mathbb{R}^3 \) is a region occupied by a ferromagnetic body, \( m: \Omega \to \mathbb{S}^2 \) with \( m = 0 \) in \( \mathbb{R}^3 \setminus \Omega \) is the magnetization vector, \( A_{ex}, K_d, Q \) are material parameters, \( H_{ext} \) is the
externally applied magnetic field, $\varphi$ is the anisotropy energy density and $u$ is induced field potential, obtained from Maxwell’s equations of magnetostatics,

\[
\begin{align*}
\text{curl} H_{\text{ind}} &= 0 & \text{in} & \mathbb{R}^3 \\
\text{div}(H_{\text{ind}} + m) &= 0 & \text{in} & \mathbb{R}^3,
\end{align*}
\]

where $H_{\text{ind}} = \nabla u$. Namely $u$ is a weak solution of

\[
\triangle u = \text{div} m \quad \text{in} \quad \mathbb{R}^3.
\]

According to the theory of micromagnetics, stable magnetization patterns are described by the minimizers of the micromagnetic energy functional, e.g. $[14,6,7,8]$. In the recent years the study of thin structures in micromagnetics, in particular thin films and wires have been of great interest, see $[12,10,16,17,18,19,20,21]$ for nanowires and $[3,7,8,9,15,17]$. It was suggested in $[1]$ that magnetic nanowires can be used as storage devices. It is known that the magnetization pattern reversal time is closely related to the writing and reading speed of such a device, thus it has been suggested to study the magnetization reversal and switching processes. In $[10]$ the magnetization reversal process has been studied numerically in cobalt nanowires by the Landau-Lifshitz-Gilbert equation. In thin wires the transverse mode has been observed: the magnetization in almost constant on each cross section forming a domain wall that propagates along the wire, while in relatively thick wires the vortex wall has been observed: the magnetization is approximately tangential to the boundary and forms a vortex which propagates along the wire. In $[13]$ similar study has been done for thin nickel wires and the same results have been observed. When a homogenous external field is applied in the axial direction of the wire facing the homogenous magnetization direction, then at a critical strength the reversal of the magnetization typically starts at one end of the wire creating a domain wall, which moves along the wire. The domain wall separates the reversed and the not yet reversed parts of the wire. In $[5]$ Cantero-Alvarez and Otto considered the problem of finding the scaling of critical field in terms of the thin film cross section and material parameters. The authors found four different scaling and corresponding four different regimes. In Figure 1 one can see the transverse and the vortex wall longitudinal and cross section pictures for wires with a rectangular cross section.
A distinctive crossover has been observed between the two different modes, which is expected to occur at a critical diameter of the wire. It has been suggested that the magnetization switching process can be understood by analyzing the micromagnetics energy minimization problem for different diameters of the cross section. In [16] K. Kühn studied 180 degree static domain walls in magnetic wires with circular cross sections. Kühn proved that indeed, the transverse mode must occur in thin magnetic wires as was predicted by experimental and numerical analysis before in [10] and in [13], while in thick wires a vortex wall has the optimal energy scaling. Some of the results proven by K. Kühn for thin wires has been later generalized in [12] to any wires with a bounded, Lipschitz and rotationally symmetric cross sections, see also [11]. Slastikov and Sonnenberg proved the energy $\Gamma$-convergence result in [20] for any $C^1$ cross sections in finite curved wires. It is shown in [16], [12] and [20] that the minimal energy scales like $d^2$, where $d$ is the diameter of the wire, provided the wire cross section has comparable dimensions. It turns out that if the dimensions of the cross section are not comparable, then the minimal energies decay faster than $d^2$ and a logarithmic term occurs. In this paper we study the minimal energy scaling in infinite thin films, as both sides of the cross section go to zero, but one faster that the other. The minimal appears to scale like $d^2(\ln l - \ln d)$, where $0 < d < l$ are the dimensions of the cross section. The paper is organized as follows: In section 2 we make some notations and formulate the main results. In section 3 we prove that for small cross section diameters the magnetostatic energy can be approximated by a quadratic form in the second and the third components of the magnetization $m$. In section 4 we prove when the diameter goes to zero the energy minimization problems $\Gamma$-converge to a one-dimensional problem. In section 5 we prove a rate of convergence on the minimal energies as the diameter of the film goes to zero. Finally, in section 6 we prove to auxiliary lemmas.
2 The main results

Denote $\Omega(l, d) = \mathbb{R} \times R(l, d)$, where $R(l, d) = [-l, l] \times [-d, d]$ and throughout this work it will be assumed that $0 < d \leq l$. Denote the aspect ratio $c = \frac{d}{l}$. Consider the energy of micromagnetics without an external field and anisotropy energy, i.e., the energy of an isotropic ferromagnet with the absence of an external field:

$$E(m) = A_{ex} \int_{\Omega} |\nabla m|^2 + K_d \int_{\mathbb{R}} |\nabla u|^2.$$ 

By scaling of all coordinates one can reach the situation when $A_{ex} = K_d = 1$. Denote $\lambda_n = \frac{1}{c_n |\ln c_n|}, \mu_n = \frac{d_n l_n}{\lambda_n}$ and rescale the magnetization $m$ as follows: $\hat{m}(x, y, z) = m(\lambda_n x, l_n y, d_n z)$. Note that, in contrast to $[16,20,11]$ we rescale $m$ in the $x$ direction as well.

Denote now $\hat{E}(\hat{m}^n) = \frac{E(m^n)}{\mu_n}$ and consider the rescaled minimization problems

$$\inf_{m \in \hat{A}_n} \hat{E}(\hat{m}) \quad (2.2)$$

instead of the original problem

$$\inf_{m \in \hat{A}_n} E(m).$$

The rescaled energy functional will have the form:
\[ \dot{E}(\dot{m}^n) = \frac{1}{\mu_n} \int_{\Omega(1,1)} \left( |\partial_x \dot{m}^n(\xi)|^2 + \frac{\lambda_n^2}{\rho_n} |\partial_y \dot{m}^n(\xi)|^2 + \frac{\lambda_n^2}{d_n^2} |\partial_\xi \dot{m}^n(\xi)|^2 \right) \mathrm{d}\xi + \frac{E_{\text{mag}}(m^n)}{\mu_n}. \]

The limit (reduced) energy functional \( E_0 \) turns out to be

\[
E_0(m) = \begin{cases} 
4 \int_{\mathbb{R}} |\partial_x m|^2 \mathrm{d}x + \frac{4}{\pi} \int_{\mathbb{R}} |m_2|^2 \mathrm{d}x, & \text{if } m_3 \equiv 0 \\
+\infty, & \text{otherwise}
\end{cases}
\]

and the admissible set \( A_0 \) for the reduced variational problem is

\[
A_0 = \{ m: \mathbb{R} \to S^2 \mid m(\pm\infty) = \pm 1 \}.
\]

The reduced or limit variational problem is to minimize the reduced energy functional \( E_0 \) over the admissible set \( A_0 \), i.e.,

\[
\inf_{m \in A_0} E_0(m).
\]

(2.3)

Define furthermore

\[
A_0^3 = \{ m \in A_0 \mid m_3 \equiv 0 \}.
\]

The equality \( \min_{m \in A_0} E_0(m) = \min_{m \in A_0^3} E_0(m) \) suggests considering the minimization problem \( \min_{m \in A_0^3} E_0(m) \) instead of \( \min_{m \in A_0} E_0(m) \). Next define the notion of convergence of the magnetizations like in [16,12].

**Definition 2.1.** The sequence \( \{ m^n \} \subset A(\Omega) \) is said to converge to \( m^0 \in A(\Omega) \) as \( n \) goes to infinity if,

\begin{align*}
(i) \quad \nabla m^n & \rightharpoonup \nabla m^0 \quad \text{weakly in } L^2(\Omega) \\
(ii) \quad m^n & \to m^0 \quad \text{strongly in } L^2_{\text{loc}}(\Omega)
\end{align*}

**Theorem 2.2** (\( \Gamma \)-convergence). The reduced variational problem is the \( \Gamma \)-limit of the full variational problem with respect to the convergence stated in Definition 2.1. This amounts to the following three statements:

- **Lower semicontinuity.** If a sequence of rescaled magnetizations \( \{ \dot{m}^n \} \) with \( m^n \in A_n \) converges to some \( m^0 \in A(\Omega) \) in the sense of Definition 2.1, then

\[
E_0(m^0) \leq \liminf_{n \to \infty} \dot{E}(\dot{m}^n)
\]

- **Recovery sequence.** For every \( m^0 \in A_0 \) and every sequence of pairs \( \{(l_n, d_n)\} \) with \( l_n, d_n \to 0 \), \( c_n \to 0 \), there exists a sequence \( \{ m^n \} \) with \( m^n \in \dot{A}_n \) such that

\[
\dot{m}^n \to m^0 \quad \text{in the sense of Definition 2.1}
\]

\[
E_0(m^0) = \lim_{n \to \infty} \dot{E}(\dot{m}^n)
\]
• **Compactness.** Let \( \{(l_n,d_n)\} \) be such that \( l_n, d_n \to 0 \) and \( c_n \to 0 \). Assume \( m^n \in \tilde{A}_n \) and \( \tilde{E}(\tilde{m}^n) \leq C \) for all \( n \in \mathbb{N} \). Then there exists a subsequence of \( \{m^n\} \) (not relabeled) such that after a translation in the \( x \) direction the sequence \( \tilde{m}^n \) converges to some \( m^0 \in A_3^0 \) in the sense of Definition 2.7.

**Corollary 2.3.** Due to the above theorem we have

\[
\lim_{n \to \infty} \min_{m^n \in \tilde{A}_n} \tilde{E}(\tilde{m}^n) = \min_{m \in A_0} E(m). \tag{2.4}
\]

As will be seen later \( \min_{m \in A_0} E(m) = \frac{16}{\sqrt{\pi}} \).

The next theorem establishes a rate of convergence for (2.4).

**Theorem 2.4** (Rate of convergence). For sufficiently small \( d \) and \( c \) the following bound holds:

\[
\left| \min_{m \in \tilde{A}} \tilde{E}(\tilde{m}) - \min_{m \in A_0} E_0(m) \right| \leq \frac{200}{\sqrt{|\ln c|}} + 20l.
\]

### 3 An approximation of the magnetostatic energy

Recall that the map \( u \) is a weak solution of \( \triangle u = \text{div} m \) if and only if

\[
\int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi = \int_{\Omega} m \cdot \nabla \varphi \quad \text{for all} \quad \varphi \in C^\infty_0(\mathbb{R}^3). \tag{3.1}
\]

The left hand side of the above equality can be written as a sum volume and surface contributions as:

\[
\int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi = -\int_{\Omega} \text{div} m \cdot \varphi + \int_{\partial \Omega} m \cdot n \varphi \quad \text{for all} \quad \varphi \in C^\infty_0(\mathbb{R}^3), \tag{3.2}
\]

where \( n \) is the outward unit normal to \( \partial \Omega \).

Denoting

\[
u_v(\xi) = -\int_{\Omega} \Gamma(\xi - \xi_1)(\text{div} m)(\xi_1) \text{d}\xi \quad \text{and} \quad u_s(\xi) = \int_{\partial \Omega} \Gamma(\xi - \xi_1)(m \cdot n)(\xi) \text{d}\xi,
\]

where \( \Gamma(\xi) = \frac{1}{4\pi|\xi|} \) is the Green function in \( \mathbb{R}^3 \), we obtain

\[
\int_{\mathbb{R}^3} \nabla u_v \cdot \nabla \varphi = \int_{\Omega} v \cdot \varphi, \quad \int_{\mathbb{R}^3} \nabla u_s \cdot \nabla \varphi = \int_{\partial \Omega} s \cdot \varphi \quad \text{for all} \quad \varphi \in C^\infty_0(\mathbb{R}^3). \tag{3.3}
\]

Denote furthermore

\[
E_v = \int_{\mathbb{R}^3} |\nabla u_v|^2, \quad E_s = \int_{\mathbb{R}^3} |\nabla u_s|^2, \quad E_{vs} = \int_{\mathbb{R}^3} \nabla u_v \cdot \nabla u_s.
\]

Following Kohn and Slastikov as in [15] define the average of the magnetization vector over the cross section:

\[
\bar{m}(x, y, z) = \frac{1}{4ld} \int_{R(l,d)} m \text{d}y \text{d}z, \quad (x, y, z) \in \Omega.
\]
Like \( m \) we extend \( \hat{m} \) as 0 outside \( \Omega \). In this section we prove upper and lower bound on the magnetostatic energy for thin films. We start with the \( E_s \) part of the energy. If the parametrization

\[
\begin{aligned}
y &= y(t), \quad t \in [0, 2] \\
z &= z(t), \quad t \in [0, 2]
\end{aligned}
\]

of \( \partial R(l, d) \) is chosen by symmetry so that \( y(t + 1) = -y(t), \ z(t + 1) = -z(t) \) then Theorem 3.3.5 of [12] delivers a formula for \( E_s(m) \) in Fourier space for \( m = m(x) \), namely:

**Theorem 3.1.** For every \( m = m(x) \in A(\Omega) \) there holds

\[
E_s(m) = \frac{1}{4\pi^2} \int_{\mathbb{R}^3} \frac{1}{|k|^2} \left\{ |a|^2 |\hat{m}_2(k)|^2 + |b|^2 |\hat{m}_3(k)|^2 \right. \\
+ \left. ab (\hat{m}_2(k) \hat{m}_3(k) + \hat{m}_2(k) \hat{m}_3(k)) \right\} \, dk,
\]

where

\[
a(k_2, k_3, \omega) = -2i \int_0^1 z'(t) \sin(k_2 y(t) + k_3 z(t)) \, dt,
\]

\[
b(k_2, k_3, \omega) = 2i \int_0^1 y'(t) \sin(k_2 y(t) + k_3 z(t)) \, dt.
\]

Observe, that when the cross section is the rectangle \( R(l, d) \) then the formula for \( E_s \) can be easily simplified in more steps, namely, for any \( m = m(x) \in A(\Omega) \), we have the following representation formula

\[
E_s(m) = \frac{4}{\pi^2} \int_{\mathbb{R}^3} \frac{\sin^2(ly) \sin^2(dz)}{|\xi|^2} \left( \frac{|\hat{m}_2(x)|^2}{z^2} + \frac{|\hat{m}_3(x)|^2}{y^2} \right) \, d\xi.
\]

Set now for convenience

\[
I(l, d, x) = \int_{\mathbb{R}^2} \frac{\sin^2(ly) \sin^2(dz)}{y^2 |\xi|^2} \, dy \, dz,
\]

then

\[
E_s(m) = \frac{4}{\pi^2} \int_{\mathbb{R}} \left( I(l, d, x)|\hat{m}_3(x)|^2 + I(d, l, x)|\hat{m}_2(x)|^2 \right) \, dx.
\]

The following functions will play an important role in this work. Denote for any \( c > 0 \),

\[
a_c = \frac{c}{2} \int_0^\infty \frac{\sin^2 t}{t^2} \cdot \frac{1 - e^{-2t}}{t} \, dt, \quad b_c = a_\frac{1}{c}.
\]

**Lemma 3.2.** For any \( 0 < d \leq l \), we have

(i) Then \( I(d, l, x) \leq 2\pi ld a_c \) and \( I(l, d, x) \leq 2\pi l db_c \) for all \( x \in \mathbb{R} \),

(ii) \( I(d, l, x) \leq \pi ldc(3 - \ln c) \), for all \( x \in \mathbb{R} \),

(iii) \( I(d, l, x) \geq \pi ldc |\ln c| \left( 1 - \frac{5}{\sqrt{|\ln c|}} \right) \), for all \( x \in [-\frac{1}{4}, \frac{1}{4}] \).
Proof. We will use the following two identities, that are well known and be found in most advanced calculus and complex analysis textbooks:

\[
\int_0^\infty \frac{\sin^2 t}{t^2} \, dt = \frac{\pi}{2}, \quad \int_0^\infty \frac{\sin^2(pt)}{t^2 + q^2} \, dt = \frac{\pi}{4q} (1 - e^{-2pq}), \quad p, q > 0. \tag{3.5}
\]

For any \( x \neq 0 \) we have by making a change of variables \( y \rightarrow |x|y, z \rightarrow |x|z \) and putting \( a = l|x|, b = d|x| \),

\[
I(l, d, x) = 4 \int_0^\infty \int_0^\infty \frac{\sin^2(dy) \sin^2(\xi)}{y^2|\xi|^2} \, dy \, d\xi = 4 \int_0^\infty \int_0^\infty \frac{\sin^2(ay) \sin^2(bz)}{y^2(1 + y^2 + z^2)} \, dy \, dz.
\]

Utilizing now the second identity of (3.5) and making a change of variables \( y = \frac{t}{a} \) we obtain

\[
I(l, d, x) = \frac{\pi}{x^2} \int_0^\infty \frac{\sin^2(ay)}{y^2} \cdot \frac{1 - e^{-2b\sqrt{y^2+1}}}{\sqrt{y^2+1}} \, dy = 2\pi ab \int_0^\infty \frac{\sin^2 t}{t^2} \cdot \frac{1 - e^{-2b\sqrt{t^2+a^2}}}{2b\sqrt{t^2+a^2}} \, dt = 2\pi ld \int_0^\infty \frac{\sin^2 t}{t^2} \cdot \frac{1 - e^{-\frac{2d}{l}\sqrt{t^2+l^2x^2}}}{\frac{2d}{l}\sqrt{t^2+l^2x^2}} \, dt.
\]

By the inequality

\[
\frac{2l}{d}\sqrt{t^2 + d^2x^2} \geq \frac{2l}{d} t = \frac{2t}{c}
\]

and the fact that the function \( \frac{1-e^{-t}}{t} \) decreases over \((0, +\infty)\) we get

\[
I(d, l, x) \leq 2\pi ld \int_0^\infty \frac{\sin^2 t}{t^2} \cdot \frac{1 - e^{-\frac{2d}{l}}}{\frac{2d}{l}} \, dt = 2\pi ld a_c. \tag{3.6}
\]

Similarly we have \( I(l, d, x) \leq 2\pi ldb_c. \)

For \((ii)\) we have that \( I(d, l, x) \leq I_1 + I_2 + I_3 \), where

\[
I_1 = \pi ldc \int_0^c \frac{\sin^2 t}{t^2} \cdot \frac{1 - e^{-\frac{2t}{l}}}{t} \, dt,
\]

\[
I_2 = \pi ldc \int_c^1 \frac{\sin^2 t}{t^2} \cdot \frac{1 - e^{-\frac{2t}{l}}}{t} \, dt,
\]

\[
I_3 = \pi ldc \int_1^{+\infty} \frac{\sin^2 t}{t^2} \cdot \frac{1 - e^{-\frac{2t}{l}}}{t} \, dt.
\]

It is clear that
\[ I_1 = 2\pi ld \int_0^c \frac{\sin^2 t}{t^2} \cdot \frac{1 - e^{-\frac{2t}{c}}}{t} \, dt \leq 2\pi ld \int_0^c \, dt = 2\pi ldc, \]

\[ I_2 \leq \pi ldc \int_c^1 \frac{1}{t} \, dt = -\pi ldc \ln c, \]

\[ I_3 \leq \pi ldc \int_1^{+\infty} \frac{\sin^2 t}{t^2} \, dt \leq \pi ldc \int_1^{+\infty} \frac{1}{t^2} \, dt = \pi ldc. \]

Therefore we obtain \( I(d, l, x) \leq \pi ldc(3-\ln c) \) and \((ii)\) is proved.

To get a lower bound on \( I(d, l, x) \) we note that the main contribution to the integral comes from the interval \([c, 1]\). The idea is replacing in the previous argument \([c, 1]\) by \([c^{1-\epsilon}, c^\epsilon]\) where \(\epsilon\) is a small positive number yet to be chosen. Assume \(\epsilon < \frac{1}{3}\) and \(x \in \left[-\frac{1}{l}, \frac{1}{l}\right]\). For any \(t \in [c^{1-\epsilon}, c^\epsilon]\) we have

\[ \frac{2l}{d} \sqrt{t^2 + x^2d^2} \geq \frac{2t}{c} \geq 2c^{-\epsilon}, \]

and

\[ \sqrt{t^2 + x^2d^2} \leq t + |x|d \leq t + \frac{d}{l} = t + c, \]

hence

\[ I(d, l, x) \geq \pi ldc \int_{c^{1-\epsilon}}^{c^\epsilon} \frac{\sin^2 t}{t^2} \cdot \frac{1 - e^{-2c^{-\epsilon}}}{t + c} \, dt. \] (3.7)

If we choose now \(\epsilon = \frac{1}{\sqrt{\ln c}}\) then \(c^\epsilon \to 0\), thus we get,

\[ 1 - e^{-2c^{-\epsilon}} > 1 - \frac{1}{2c^{-\epsilon}} = 1 - \frac{c^\epsilon}{2}, \]

\[ \frac{\sin^2 t}{t^2} \geq \left(\frac{t - \frac{c^\epsilon}{6}}{t^2}\right)^2 \geq 1 - t^2, \quad t \in [0, c^\epsilon]. \]

Thus we obtain by (3.7),

\[ I(d, l, x) \geq \pi ldc \left(1 - c^{2\epsilon}\right) \left(1 - \frac{c^\epsilon}{2}\right) \int_{c^{1-\epsilon}}^{c^\epsilon} \frac{1}{t + c} \, dt \]

\[ \geq \pi ldc \left(1 - 2c^\epsilon\right) \left(\ln(c + c^\epsilon) - \ln(c + c^{1-\epsilon})\right). \]

It is clear that

\[ \ln(c + c^{1-\epsilon}) = \ln c + \ln(1 + c^{-\epsilon}) \]

\[ \leq \ln c + \ln(2c^{-\epsilon}) \]

\[ \leq (1 - 2\epsilon) \ln c, \]

and

\[ \ln(c + c^\epsilon) \geq \ln c^\epsilon = \epsilon \ln c, \]

\[ \therefore I(d, l, x) \geq \pi ldc \left(1 - 2c^\epsilon\right)^2 \left(1 - \frac{c^\epsilon}{2}\right) \left(\ln(c + c^\epsilon) - \ln(c + c^{1-\epsilon})\right) \]

\[ \geq \pi ldc \left(1 - 2c^\epsilon\right)^2 \left(1 - \frac{c^\epsilon}{2}\right) \left(\epsilon \ln c\right) \]

\[ \geq \pi ldc \left(1 - 2c^\epsilon\right)^2 \left(\frac{\epsilon}{2}\right) \ln c \]

\[ \geq \pi ldc \left(1 - 2c^\epsilon\right)^2 \left(\frac{\epsilon}{2}\right) \ln c \]

\[ \geq \pi ldc \left(1 - 2c^\epsilon\right)^2 \left(\frac{\epsilon}{2}\right) \ln c. \]
\[ 1 - 2e^\epsilon = 1 - 2e^{\epsilon \ln c} > 1 - 2e^{-\frac{1}{\epsilon}} > 1 - 2\epsilon. \]

Concluding we obtain
\[
I(d, l, x) \geq \pi(1 - 2\epsilon)(1 - 3\epsilon)ldc \ln c \\
\geq \pi ldc \ln c(1 - 5\epsilon) \\
= \pi ldc \ln c \left(1 - \frac{5}{\sqrt{|\ln c|}} \right).
\]

\[ \square \]

**Corollary 3.3.** We have that
\[
\lim_{c \to 0} \frac{a_c}{c|\ln c|} = \frac{1}{2}
\]

**Proof.** The proof follows from (ii) and (iii) parts of the above lemma. \( \square \)

It is straightforward to see that due to the symmetry of the cross section \( R(l, d) \) one has \( E_{vs}(m) = 0 \) for all \( m = m(x) \in A(\Omega) \). We estimate now the volume contribution \( E_v \) to \( E_{mag} \).

**Lemma 3.4.** For any \( 0 < d \leq l \) and \( m = m(x) \in A \) the following bound holds:
\[
E_v(m) \leq M_m \left( l^2 d^2 + ld^2 \left(1 + \ln \frac{l}{d}\right) \right),
\]
where \( M_m \) is a constant depending on the magnetization \( m \).

**Proof.** By density argument (3.3) holds for \( \varphi = u_v \) thus,
\[
E_v(m) = \int_{\mathbb{R}^3} |\nabla u_v|^2 = - \int_{\Omega} \text{div} m \cdot u_v = \int_{\Omega} \int_{\Omega} \Gamma(\xi - \xi_1) \text{div} m(\xi) \text{div} m(\xi_1) \, d\xi \, d\xi_1.
\]

For any \( m = m(x) \in A \) we have \( \text{div} m = \partial_x m_1(x) \), thus
\[
E_v(m) = \frac{1}{4\pi} \int_{\Omega} \int_{\Omega} \frac{\partial_x m_1(x) \partial_x m_1(x_1)}{|\xi - \xi_1|} \, d\xi \, d\xi_1
\]
where \( \xi = (x, y, z) \) and \( \xi_1 = (x_1, y_1, z_1) \). We have by integration by parts
\[
\int_{\mathbb{R}} \frac{\partial_x m_1(x)}{|\xi - \xi_1|} \, dx = \int_{-\infty}^{0} \frac{dm_1(x)}{|\xi - \xi_1|} + \int_{0}^{+\infty} \frac{dm_1(x)}{|\xi - \xi_1|} \\
= \frac{2}{\sqrt{x_1^2 + (y - y_1)^2 + (z - z_1)^2}} - \int_{\mathbb{R}} \frac{(x - x_1)m_1(x)}{|\xi - \xi_1|^3} \, dx,
\]
where
\[
m^*(x) = \begin{cases} 
m_1(x) + 1 & \text{if } x \leq 0 \\
m_1(x) - 1 & \text{if } x > 0.
\end{cases}
\]

Then it has been shown in [12] that \( m^*(x) \in L^2(\mathbb{R}) \) and we can estimate \( E_v(m) \leq I_1 + I_2 \), where
Recall now Lemma A2 from [11], which asserts that for any point \((y_1, z_1) \in \mathbb{R}^2\) one has

\[
\int_{R(l,d)} \frac{1}{\sqrt{(y - y_1)^2 + (z - z_1)^2}} \, dy \, dz \leq 10d \left(1 + \ln \frac{d}{l}\right). \tag{3.9}
\]

Thus we obtain for \(I_1\),

\[
I_1 \leq \frac{4}{\pi} \|\partial_x m_1\|_{L^2(\mathbb{R})}^2 l^2 d^2 + \frac{1}{4} \int_{R(l,d)} \int_{R(l,d)} \frac{1}{(y - y_1)^2 + (z - z_1)^2} \, dy_1 \, dz_1 \, dy \, dz \\
\leq \frac{4}{\pi} \|\partial_x m_1\|_{L^2(\mathbb{R})}^2 l^2 d^2 + 10ld^2 \left(1 + \ln \frac{l}{d}\right).
\]

By making a change of variables \(\xi_2 = \xi_1 - \xi\) and utilizing again (3.9) we can estimate,

\[
I_2 = \int_{\Omega} \int_{\mathbb{R} \times [-l-y,l-y] \times [-d-z,d-z]} \frac{|m^*(x)\cdot |\partial_x m_1(x_2 + x)||_{\xi_2}^2}{\xi_2^2} \, d\xi_2 \, d\xi \\
\leq \frac{1}{2} \int_{R(l,d)} \int_{R(l,d)} \int_{\mathbb{R} \times [-l-y,l-y] \times [-d-z,d-z]} \frac{|m^*(x)|^2 + |\partial_x m_1(x_2 + x)|^2}{\xi_2^2} \, dx \, d\xi_2 \, dy \, dz \\
= 2ld(|m^*|^2_{L^2(\mathbb{R})} + \|\partial_x m_1\|_{L^2(\mathbb{R})}^2) \int_{R(l,d)} \int_{\mathbb{R} \times [-l-y,l-y] \times [-d-z,d-z]} \frac{d\xi_2}{\xi_2^2} \\
= 2\pi ld(|m^*|^2_{L^2(\mathbb{R})} + \|\partial_x m_1\|_{L^2(\mathbb{R})}^2) \int_{R(l,d)} \frac{1}{(y_1 - y)^2 + (z_1 - z)^2} \, dy_1 \, dz_1 \\
\leq 20\pi ld^2 \left(1 + \ln \frac{l}{d}\right) (|m^*|^2_{L^2(\mathbb{R})} + \|\partial_x m_1\|_{L^2(\mathbb{R})}^2).
\]

The summary of the estimates on \(I_1\) and \(I_2\) completes the proof.
4 The convergence of the energies

Consider a sequence of domain-magnetization-energy triples \( \{ (\Omega_n, m^n, E(m^n)) \} \) where \( \Omega_n = \mathbb{R} \times R(l_n, d_n) \), \( m^n \in \tilde{A}_n = \tilde{A}(\Omega_n) \) and \( l_n, c_n \to 0 \). Lemma 3.2 suggests that for sufficiently big \( n \) one can formally write for any \( m = m(x) \),

\[
E_s(m^n) \approx \frac{8}{\pi} l_n d_n c_n \int_{\mathbb{R}} |m_2^n(x)|^2 \, dx + \frac{8}{\pi} l_n d_n b_n \int_{\mathbb{R}} |m_3^n(x)|^2 \, dx
\]

Next, Lemma A.2 asserts that \( a_{c_n} \) scales like \( c_n \ln c_n \) and \( b_{c_n} \to \frac{2}{\pi} \). Furthermore, by Lemma 3.4 for a fixed \( m^n = m^n(x) \) the summand \( E_s(m^n) \) decays at least like \( l_n d_n^2 \ln^2 \frac{l_n}{d_n} \). Rescaling the magnetizations \( \tilde{m}^n(x, y, z) = m^n(\lambda_n x, l_n y, d_n z) \) we can rewrite the exchange energy for all \( m^n(x) \in A_n \) as

\[
E_{ex}(m^n(x)) = \frac{l_n d_n}{\lambda_n} \int_{\Omega(1,1)} (|\partial_x \tilde{m}^n(x)|^2 + \frac{\lambda_n^2}{4} |\partial_y \tilde{m}^n(x)|^2 + \frac{\lambda_n^2}{d_n^2} |\partial_z \tilde{m}^n(x)|^2) \, dx,
\]

and it is clear that \( \tilde{m}^n : \Omega(1, 1) \to S^2 \).

Thus one would expect that for sufficiently big \( n \) the approximation holds

\[
E_{ex}(m^n(x)) \approx \frac{4 l_n d_n}{\lambda_n} \int_{\mathbb{R}} |\partial_x m^n(x)|^2 \, dx
\]

and

\[
E_s(m^n(x)) \approx \frac{4}{\pi} l_n d_n c_n |\ln c_n| \lambda_n \int_{\mathbb{R}} (|m_2^n(x)|^2 + \frac{\pi}{c_n |\ln c_n|} |m_3^n(x)|^2) \, dx.
\]

This calculation suggests that the coefficients \( \frac{l_n d_n}{\lambda_n} \) and \( l_n d_n c_n |\ln c_n| \lambda_n \) should be taken equal and they will both be the scaling of \( E(m^n) \). This leads to \( \lambda_n = \frac{1}{\sqrt{|\ln c_n|}} \).

Proof of Theorem 2.2.

**Lower semicontinuity.** One can without loss of generality assume that \( \hat{E}(\tilde{m}^n) \leq M \) for some \( M > 0 \) and all \( n \in \mathbb{N} \). Following Kohn and Slastikov [15] let us prove that

\[
\liminf_{n \to \infty} \frac{E_{mag}(m^n)}{\mu_n} = \liminf_{n \to \infty} \frac{E_{mag}(\tilde{m}^n)}{\mu_n}.
\]

By the Poincaré inequality we have

\[
\int_{\Omega} |m - \bar{m}|^2 \leq C(d^2 + l^2) \int_{\Omega} |\nabla m|^2 \leq C(d^2 + l^2) E(m).
\]

Owing now to the third inequality in Lemma A.1 and the above inequality we have

\[
|E_{mag}(m^n) - E_{mag}(\tilde{m}^n)| \leq M_1 \mu_n \sqrt{l_n^2 + d_n^2},
\]

(4.2)
for some $M_1$, which implies (4.1). Let now \( \{q_n\} \) be a sequence with \( 0 < q_n < 1 \) yet to be defined. We have by the Plancherel equality,

\[
q_n \frac{E_{ex}(m^n)}{\mu_n} \geq q_n \frac{1}{\mu_n} \int_{\Omega_n} |\partial_x \hat{m}^n(\xi)|^2 \, d\xi
\]

\[
= 4 q_n l_n d_n \int_{R} |\partial_x \hat{m}^n(x)|^2 \, dx
\]

\[
= 4 q_n l_n d_n \int_{R} |x \cdot \hat{m}^n(x)|^2 \, dx
\]

\[
\geq \frac{4 q_n l_n}{l_n \mu_n} \int_{R \setminus [-\frac{1}{c_{n}}, \frac{1}{c_{n}}]} (|\hat{m}^n_2(x)|^2 + |\hat{m}^n_3(x)|^2) \, dx,
\]

and according to part (iii) of Lemma 3.2 we have for big $n$ as well

\[
\frac{E_s(\hat{m}^n)}{\mu_n} \geq \frac{4}{\pi \mu_n} l_n d_n c_n \ln c_n \left(1 - \frac{5}{\sqrt{\ln c_n}}\right) \int_{\Omega_n} \frac{1}{\ln c_n} (|\hat{m}^n_2(x)|^2 + |\hat{m}^n_3(x)|^2) \, dx.
\]

Now choose $q_n$ so that

\[
\frac{4}{\pi \mu_n} l_n d_n c_n |\ln c_n| \left(1 - \frac{5}{\sqrt{\ln c_n}}\right) = \frac{4 q_n d_n}{l_n \mu_n},
\]

or

\[
q_n = \frac{1}{\pi} l_n d_n c_n |\ln c_n| \left(1 - \frac{5}{\sqrt{\ln c_n}}\right),
\]

and it is clear that $q_n \to 0$.

Applying now the obtained inequalities, (4.1) and the convergence $\nabla \hat{m}^n \to \nabla m^0$ in $L^2(\Omega(1,1))$ we obtain

\[
\liminf_{n \to \infty} \frac{E(m^n)}{\mu_n} \geq \liminf_{n \to \infty} (1 - q_n) \int_{\Omega(1,1)} |\partial_x \hat{m}^n|^2 \, d\xi + \liminf_{n \to \infty} q_n \frac{E_{ex}(m^n)}{\mu_n} + \liminf_{n \to \infty} \frac{E_{mag}(m^n)}{\mu_n}
\]

\[
= \liminf_{n \to \infty} (1 - q_n) \int_{\Omega(1,1)} |\partial_x \hat{m}^n|^2 \, d\xi + \liminf_{n \to \infty} q_n \frac{E_{ex}(m^n)}{\mu_n} + \liminf_{n \to \infty} \frac{E_{mag}(\hat{m}^n)}{\mu_n}
\]

\[
\geq \liminf_{n \to \infty} (1 - q_n) \int_{\Omega(1,1)} |\partial_x \hat{m}^n|^2 \, d\xi + \liminf_{n \to \infty} q_n \frac{E_{ex}(m^n)}{\mu_n} + \liminf_{n \to \infty} \frac{E_s(\hat{m}^n)}{\mu_n}
\]

\[
\geq 4 \int_{R} |\partial_x m^0|^2 + \liminf_{n \to \infty} \frac{4}{\pi \mu_n} l_n d_n c_n |\ln c_n| \left(1 - \frac{5}{\sqrt{\ln c_n}}\right) \int_{R} (|\hat{m}^n_2(x)|^2 + |\ln c_n| |\hat{m}^n_3(x)|^2)
\]

\[
= 4 \int_{R} |\partial_x m^0(x)|^2 \, dx + \liminf_{n \to \infty} \frac{4}{\lambda_n} \int_{R} (|\hat{m}^n_2(x)|^2 + |\ln c_n| |\hat{m}^n_3(x)|^2) \, dx.
\]

It is then standard to prove that the convergence $\hat{m}^n \to m^0$ in $L^2_{loc}(\Omega(1,1))$ implies

\[
\liminf_{n \to \infty} \frac{1}{\lambda_n} \int_{R} |\hat{m}^n_2(x)|^2 \geq \int_{R} |m^0_2(x)|^2 \quad \text{and} \quad \liminf_{n \to \infty} \frac{1}{\lambda_n} \int_{R} |\hat{m}^n_3(x)|^2 \geq \int_{R} |m^0_3(x)|^2.
\]
thus since $|\ln c_n| \to \infty$ we conclude that

$$\liminf_{n \to \infty} \frac{E(m^n)}{\mu_n} \geq E_0(m^0).$$

**Recovery sequence.** Let us prove that the sequence $m^n(x)$, where

$$m^n(\lambda_n x, y, z) = m^0(x) \quad \text{if} \quad \xi \in \Omega(l_n, d_n) \quad \text{and} \quad m^n(\xi) = 0 \quad \text{if} \quad \xi \in \mathbb{R}^3 \setminus \Omega(l_n, d_n)$$

satisfies the required condition. If $m^0_3$ is not identically zero, then $E_0(m^0) = \infty$ and due to the lower semi-continuity part of the foregoing theorem we have that $E_0(m^0) \leq \liminf_{n \to \infty} \hat{E}_n(m^n)$, thus the proof follows. Assume now that $m^0_3 \equiv 0$. It remains to only prove the reverse inequality $\limsup_{n \to \infty} \hat{E}_n(m^n) \leq E_0(m^0)$. It is clear that

$$E(m^n) = 4\mu_n \int_{\mathbb{R}} |\partial_x m^0|^2 \, dx + E_{mag}(m^n).$$

Due to Lemma 3.6 and the Plancherel equality we have,

$$E_s(m^n) \leq \frac{4}{\pi} l_n d_n c_n (|\ln c_n| + 3) \int_{\mathbb{R}} |m^0_2(x)|^2 \, dx,$$

thus

$$\limsup_{n \to \infty} \frac{E_s(m^n)}{\mu_n} \leq \frac{4}{\pi} \int_{\mathbb{R}} |m^0_2(x)|^2 \, dx.$$

We have furthermore by Lemma 3.4 that

$$\limsup_{n \to \infty} \frac{E_v(m^n)}{\mu_n} = 0,$$

thus combining all the obtained inequalities for the energy summands we discover

$$\limsup_{n \to \infty} \hat{E}_n(m^n) \leq E_0(m^0).$$

**Compactness.** The inequality $E(m^n) \leq C \mu_n$ implies

$$\int_{\Omega(1,1)} |\partial_x \hat{m}^n|^2 \leq C, \quad \int_{\Omega(1,1)} |\partial_y \hat{m}^n|^2 \leq C \frac{\mu_n}{\lambda^2_n} \quad \text{and} \quad \int_{\Omega(1,1)} |\partial_z \hat{m}^n|^2 \leq C \frac{\mu_n}{\lambda^2_n},$$

thus a subsequence (not relabeled) of $\{\nabla \hat{m}^n\}$ has a weak limit $f = f(x)$ in $L^2(\Omega(1,1))$. On the other hand $\hat{m}^n$ has a unit length pointwise, thus a subsequence (not relabeled) of $\{\hat{m}^n\}$ has a strong local limit $m^0$ in $L^2(\Omega(1,1))$. It is then straightforward to show that $m^0$ is weakly differentiable with $f = \nabla m^0$, thus $\{\hat{m}^n\}$ converges to $m^0$ in the sense of Definition 2.1. It has been proven in [11], that actually one can translate the subsequence $\{\hat{m}^n\}$ in the $x$ direction so that the limit $m^0$ satisfies $m^0(\pm \infty) = \pm 1$. Finally owing to the lower semi-continuity part of the lemma we discover $E_0(m^0) \leq \liminf \hat{E}(\hat{m}^n) \leq C < \infty$, thus $m^0_3 \equiv 0$, i.e., $m^0 \in A^3_0$. 

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5 The rate of convergence

Recall that for any $\alpha > 0$ one can explicitly determine the minima of the energy functional e.g., [16,12,11],

$$E_\alpha(m) = \int_\mathbb{R} |\partial_x m(x)|^2 \, dx + \alpha \int_\mathbb{R} (|m_y(x)|^2 + |m_z(x)|^2) \, dx$$

in the admissible set

$$A_0 = \{ m : \mathbb{R} \to \mathbb{R}^3 : |m| = 1, m(\pm \infty) = \pm 1 \}.$$ 

The minimizer is given by the formula

$$m = m^{\alpha,\beta} = \left( \frac{e^{2\sqrt{\alpha}x} \cdot \beta^2 - 1}{e^{2\sqrt{\alpha}x} \cdot \beta^2 + 1}, \frac{2\beta e^{2\sqrt{\alpha}x}}{e^{2\sqrt{\alpha}x} \cdot \beta^2 + 1} \cos \theta, \frac{2\beta e^{2\sqrt{\alpha}x}}{e^{2\sqrt{\alpha}x} \cdot \beta^2 + 1} \sin \theta \right),$$

(5.1)

where $\beta \in \mathbb{R}$. Note that the minimal value of the energy does not depend on $\theta$, i.e., it is invariant under rotations in the cross section plane, and for a fixed $\theta$ any minimizer can be obtained from $m^{\alpha} := m^{\alpha,1}$ by translation in the $x$ direction. The minimizer $m^{\alpha}$ satisfies $m^{\alpha}_x(0) = 0$. The minimal value of $E_\alpha$ in $A_0$ will be $4\sqrt{\alpha}$. Therefore, due to the fact $m^0 \in A_0^3$, the minimizers $m^0$ of $E_0$ have the form

$$m^0 = \left( \frac{e^{\sqrt{\pi}x} \cdot \beta^2 - 1}{e^{\sqrt{\pi}x} \cdot \beta^2 + 1}, \frac{2\beta e^{\sqrt{\pi}x}}{e^{\sqrt{\pi}x} \cdot \beta^2 + 1}, 0 \right).$$

(5.2)

The minimal value of $E_0$ is $\frac{16}{\sqrt{\pi}}$.

**Proof of Theorem 2.4** We need to get accurate lower and upper bounds on $E(m)$. For an upper bound we choose the recovery sequence $m(x, y, z) = m^0(\frac{x}{\lambda_n})$, where $m^0 \equiv 0$ and $m^0$ is a minimizer of the energy functional

$$E_0(m) = 4 \int_\mathbb{R} |\partial_x m|^2 \, dx + \frac{4}{\pi} \int_\mathbb{R} (|m_2(x)|^2 + |m_3(x)|^2) \, dx.$$

Due to Lemma 3.2 we have for big $n$

$$E(m^0) \leq \frac{4M_0d_n}{\lambda_n} \int_\mathbb{R} |\partial_x m|^2 \, dx + \frac{4M_0d_n c_n (3 - \ln c_n)}{\pi} \int_\mathbb{R} m^0_2(x)^2 \, dx + E_v(m^0).$$

Next, due to Lemma 3.4 we get for big $n,$

$$\frac{E(m)}{\mu_n} \leq 4E_0(m) + \frac{12}{\pi |\ln c_n|} \int_\mathbb{R} |m^0_2(x)|^2 \, dx + 2M_0d_n \lambda_n (1 - \ln c_n)$$

$$\leq \frac{16}{\sqrt{\pi}} + \frac{10}{|\ln c_n|} + 2\sqrt{M_0d_n |\ln c_n|},$$

thus the minimal energy satisfies the inequality

$$\frac{\min_{m \in \tilde{A}_n} E(m)}{\mu_n} - \frac{16}{\sqrt{\pi}} \leq \frac{10}{|\ln c_n|} + 2\sqrt{M_0d_n |\ln c_n|}.$$  

(5.3)
Assume now $m \in \tilde{A}_n$ is an energy minimizer in $\Omega_n$. We have that $I(l_n, d_n, x) \geq I(d_n, l_n, x)$, thus by Lemma 3.2 we have

$$E_{mag}(\bar{m}) \geq \frac{4}{\pi} l_n d_n c_n |\ln c_n| \left(1 - \frac{5}{\sqrt{\ln c_n}}\right) \int_{\frac{1}{2} \ln c_n}^{\ln c_n} (|\bar{m}_2|^2 + |\bar{m}_3|^2) \, dx.$$  

According to (5.3) we have for big $n$,

$$\frac{\min_{m \in \tilde{A}_n} E(m)}{\mu_n} \leq \frac{16}{\sqrt{\pi}} + 1 < 11.$$

We have furthermore for big $n$ that

$$\int_{\mathbb{R} \setminus [-\frac{1}{n}, \frac{1}{n}]} (|\bar{m}_2|^2 + |\bar{m}_3|^2) \, dx \leq l_n^2 \int_{\mathbb{R}} (|x \cdot \bar{m}_2|^2 + |x \cdot \bar{m}_3|^2) \, dx$$

$$= l_n^2 \int_{\mathbb{R}} (|\partial_x \bar{m}_2|^2 + |\partial_x \bar{m}_3|^2) \, dx$$

$$\leq \frac{l_n}{4 d_n} \int_{\Omega_n} (|\partial_x m_2|^2 + |\partial_x m_3|^2) \, dx$$

$$\leq \frac{l_n E_{ex}(m)}{4 d_n}$$

$$\leq \frac{11 l_n \mu_n}{4 d_n},$$

thus

$$\frac{4}{\pi} l_n d_n c_n |\ln c_n| \int_{\mathbb{R} \setminus [-\frac{1}{n}, \frac{1}{n}]} (|\bar{m}_2|^2 + |\bar{m}_3|^2) \, dx \leq \frac{11}{\pi} l_n^2 c_n |\ln c_n| \mu_n,$$

therefore we obtain

$$E_{mag}(\bar{m}) \geq \frac{4}{\pi} l_n d_n c_n |\ln c_n| \left(1 - \frac{5}{\sqrt{\ln c_n}}\right) \int_{\mathbb{R}} (|\bar{m}_2|^2 + |\bar{m}_3|^2) \, dx - \frac{11}{\pi} l_n^2 c_n |\ln c_n| \mu_n \quad (5.4)$$

It is straightforward to see using the definition of the average that

$$\int_{\Omega_n} (|m_2|^2 + |m_3|^2) - \int_{\Omega_n} (|\bar{m}_2|^2 + |\bar{m}_3|^2) = \int_{\Omega_n} (|m_2 - \bar{m}_2|^2 + |m_3 - \bar{m}_3|^2),$$

thus by the Poincaré inequality we get for big $n$,

$$\int_{\Omega_n} (|m_2|^2 + |m_3|^2) \leq \int_{\Omega_n} (|\bar{m}_2|^2 + |\bar{m}_3|^2) + 11 C \mu_n (l_n^2 + d_n^2). \quad (5.5)$$

Next, due to the estimate (4.1) we have for big $n$ that

$$E_{mag}(m) \geq E_{mag}(\bar{m}) - M_1 \mu_n \sqrt{d_n^2 + l_n^2},$$
where $M_1 = 11C$ and $C$ is the Poincaré constant for $R(l_n, d_n)$. Combining now the last inequality with (5.3) and (5.5), for bin $n$ we discover

$$E_{\text{mag}}(m) \geq \frac{4}{n} l_n d_n c_n |\ln c_n| \left(1 - \frac{5}{\sqrt{|\ln c_n|}}\right) \int_{\mathbb{R}} (|m_2|^2 + |m_3|^2) \, dx - \frac{11}{\pi} l_n^2 c_n |\ln c_n| \mu_n - 12C \mu_n \sqrt{l_n^2 + d_n^2}.$$  

(5.6)

For the whole energy we obtain for big $n$,

$$\frac{E(m)}{\mu_n} \geq 4 \left(1 - \frac{5}{\sqrt{|\ln c_n|}}\right) \left( \int_{\Omega(1,1)} (|\partial_x \hat{m}|^2 \, d\xi + \frac{1}{\pi} \int_{\Omega(1,1)} (|\hat{m}_2|^2 + |\hat{m}_3|^2) \, d\xi \right) - 20C l_n.$$

It has been shown in [11, Lemma 3.3], that if $m \in A_n$ then $\hat{m}(\pm \infty) = \pm 1$, thus we have $\hat{m}(\pm \infty, y, z) = \pm 1$ on a full measure subset $Q$ of $R(1,1)$. Therefore we have for any $(y, z) \in Q$ that

$$\int_{\mathbb{R}} (|\partial_x \hat{m}(x, y, z)|^2 \, dx + \frac{1}{\pi} \int_{\Omega(1,1)} (|\hat{m}_2(x, y, z)|^2 + |\hat{m}_3(x, y, z)|^2) \, dx \geq \frac{4}{\sqrt{\pi}},$$

which gives

$$\int_{\Omega(1,1)} (|\partial_x \hat{m}|^2 \, d\xi + \frac{1}{\pi} \int_{\Omega(1,1)} (|\hat{m}_2|^2 + |\hat{m}_3|^2) \, d\xi \geq \frac{16}{\sqrt{\pi}}.$$

Finally we get for the energies,

$$\frac{E(m)}{\mu_n} - \frac{16}{\sqrt{\pi}} \geq - \frac{200}{\sqrt{|\ln c_n|}} - 20C l_n.$$

A combination of the last inequality and (5.3) completes the proof. In conclusion, let us mention that for sufficiently small $d$ and $l$ the minimizer $m$ must have almost the shape of $m^{\alpha, \beta}$ i.e., must be a transverse wall.

A Appendix

In this section we recall a key inequality and study the function $a_c$.

**Lemma A.1.** For any vector fields $m_1, m_2 \in M(\Omega)$ with finite energies there holds

$$|E_{\text{mag}}(m_1) - E_{\text{mag}}(m_2)| \leq \|m_1 - m_2\|^2_{L^2(\Omega)} + 2\|m_1 - m_2\|_{L^2(\Omega)} \sqrt{E_{\text{mag}}(m_1)}.$$

**Proof.** The proof is trivial and can be found in [10].

Consider now $c \to a_c$ as a map from $(0, +\infty)$ to $(0, +\infty)$.

**Lemma A.2.** The function $a_c$ has the following properties:

(i) $a_c$ increases in $(0, +\infty)$

(ii) $\lim_{c \to 0} \frac{a_c}{c |\ln c|} = \frac{1}{2}$. 

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\( (iii) \lim_{c \to +\infty} a_c = \frac{\pi}{2}. \)

**Proof.** The first property follows from the fact that the function \( \frac{1}{t} e^{-t} \) decreases over \((0, +\infty)\).

The second property is Corollary 3.3. Assume now \( c \geq 4 \). It is clear that

\[
\frac{1 - e^{-\frac{2t}{c}}}{\frac{2t}{c}} \geq 1 - \frac{t}{c} \quad \text{if} \quad t \in \left[0, \frac{c}{2}\right],
\]

thus taking into account the inequality \( \sqrt{c} \leq \frac{c}{2} \), we discover

\[
\frac{1 - e^{-\frac{2t}{c}}}{\frac{2t}{c}} \geq 1 - \frac{t}{c} \geq 1 - \frac{1}{\sqrt{c}} \quad \text{if} \quad t \in [0, \sqrt{c}].
\]

Therefore for \( a_c \) we have on one hand

\[
\liminf_{c \to \infty} a_c \geq \liminf_{c \to \infty} \left(1 - \frac{1}{\sqrt{c}}\right) \int_0^{\sqrt{c}} \frac{\sin^2 t}{t^2} \, dt = \int_0^{+\infty} \frac{\sin^2 t}{t^2} \, dt = \frac{\pi}{2},
\]

but on the other hand

\[
a_c \leq \int_0^{+\infty} \frac{\sin^2 t}{t^2} \, dt = \frac{\pi}{2} \quad \text{for any} \quad c > 0,
\]

which achieves the proof.

\[\square\]

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**References**

[1] D. Atkinson, G. Xiong, C. Faulkner, D. Allwood, D. Petit, and R. Cowburn. Magnetic domain wall logic. *Science*, 309:1688-1692, 2005.

[2] G. Beach, C. Nistor, C. Knuston, M. Tsoi, and J. Erskine. Dynamics of field-driven domain-wall propagation in ferromagnetic nanowires. *Nat. Mater.*, 4:741-744, 2005.

[3] A. Capella, C., Melcher and F. Otto. Wave type dynamics in ferromagnetic thin film and the motion of Néel walls. (Nonlinearity,) 20 (2007) 11, p. 2519-2537

[4] G. Carbou and S. Labbé. Stability for static walls in ferromagnetic nanowires. *Discrete Contin. Dyn. Syst. Ser. B*, 6(2): 273-290(electronic), 2006

[5] R. Cantero-Alvarez and F. Otto. Critical fields in ferromagnetic thin films : identification of four regimes. *Journal of nonlinear science*, 16(4), 351-383 (2006).
[6] Antonio De Simone, Robert V. Kohn, Stefan Müller, Felix Otto: Recent analytical developments in micromagnetics, In G. Bertotti and I. Mayergoyz, editors, The Science of Hysteresis. Academic Press, Inc., 2005

[7] Antonio Desimone, Robert V. Kohn, Stefan Müller, and Felix Otto. A reduced theory for thin-film micromagnetics. Comm. Pure Appl. Math., 55(11):1408-1460, 2002.

[8] Antonio DeSimone, Robert V. Kohn, Stefan Müller, Felix Otto, and Rudolf Schäfer. Two-dimensional modelling of soft ferromagnetic films. R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 457(2016):2983-2991, 2001.

[9] C. J. García-Cervera and Weinan E. Effective dynamics for ferromagnetic thin films. J. Appl. Phys., 90:370-374, 2001.

[10] H. Forster, T. Schrefl, D. Suess, W. Scholz, V. Tsiantos, R. Dittrich, and J. Fidler. Domain wall motion in nanowires using moving grids. J. Appl. Phys., 91:6914-6919, 2002.

[11] D. Harutyunyan. Existence of minimizers and convergence of almost minimizers in ferromagnetic nanowires. An energy barrier for thick wires. submitted 2013, http://arxiv.org/pdf/1207.5195.pdf

[12] D. Harutyunyan. On the $\Gamma$-convergence of the energies and the convergence of almost minimizers in infinite magnetic cylinders, Dissertation, Universität und Landesbibliothek Bonn, Submitted in June 2011, published in 2012, http://hss.ulb.uni-bonn.de/2012/2886/2886.htm

[13] R. Hertel and J. Kirschner. Magnetization reversal dynamics in nickel nanowires. Physica B, 343:206-210, 2004.

[14] A Hubert and R. Schäfer. Magnetic Domains. The analysis of Magnetic Microstructuures. ISBN 3-540-64108-4, Springer-Verlag Berlin-Heidelberg New York, 1998

[15] R. V. Kohn and V. Slastikov. Effective dynamics for ferromagnetic thin films: a rigorous justification. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 461(2053): 143-154, 2005.

[16] K.Kühn. Reversal modes in magnetic nanowires, Dissertation, preprint, Universität Leipzig, 2007 www.mis.mpg.de/preprints/thesis/thesis – 0107.pdf

[17] Y. Nakatani, and A. Thaiville. Domain-wall dynamics in nanowires and nanostripes. In Spin dynamics in confined magnetic structures. 3rd Volume 101 of Topics in applied physics, pages 161-205. Springer Verlag, Berlin, 2006

[18] K. Nielsch, R.B. Wehrspohn, J. Barthel, J. Kirschner, U. Gösele, S.F. Fischer, and H. Kronmüller. Hexagonally ordered 100nm period nickel nanowire arrays. Appl. Phys. Lett., 79(9):1360-1362, 2001.
[19] L. Piraux, J.M. George, J.F. Despres, C. Leroy, E.Ferain, R. Legras, K. Ounadjela, and A. Fert. Giant magnetoresistance in magnetic multilayered nanowires. *Appl. Phys. Lett.*, 65(19):2484-2486, 1994.

[20] V.V. Slastikov and C. Sonnenberg. Reduced models for ferromagnetic nanowires. *IMA J. Appl. Math.*, 77,N2,220-235, 2012.

[21] R. Wieser, U. Nowak, and K. D. Usadel. Domain wall mobility in nanowires: Transverse versus vortex walls. *Phys. Rev. B*, 69:0604401, 2004.