Novel methods to construct nonlocal sets of orthogonal product states in any bipartite high-dimensional system

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Nonlocal sets of orthogonal product states (OPSs) are widely used in quantum protocols owing to their good property. Thus a lot of attention are paid to how to construct a nonlocal set of orthogonal product states though it is a difficult problem. In this paper, we propose a novel general method to construct a nonlocal set of orthogonal product states in $\mathbb{C}^{d} \otimes \mathbb{C}^{d}$ for $d \geq 3$. We give an ingenious proof for the local indistinguishability of those product states. The set of product states, which are constructed by our method, has a very good structure. Subsequently, we give a construction of nonlocal set of OPSs with smaller members in $\mathbb{C}^{d} \otimes \mathbb{C}^{d}$ for $d \geq 3$. On the other hand, we present two construction methods of nonlocal sets of OPSs in $\mathbb{C}^{m} \otimes \mathbb{C}^{n}$, where $m \geq 3$ and $n \geq 3$. Furthermore, we propose the concept of isomorphism for two nonlocal sets of OPSs. Our work is of great help to understand the structure and classification of locally indistinguishable OPSs.

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I. INTRODUCTION

Local distinguishability of orthogonal product states (OPSs) is an important problem in quantum information theory. Many related works \cite{1,16} are proposed so far since nonlocal sets of orthogonal product states (OPSs) can be used to design quantum protocols, such as quantum voting \cite{17} and quantum cryptography \cite{18,19,20}. Although great progress \cite{21,24} has been made in the research of the local distinguishability of OPSs, there are still some problems that have not been solved effectively. For example, the structure and classification of the nonlocal sets of OPSs.

As we know, a set of orthogonal quantum states can be exactly discriminated by a global positive operator-valued measure. However, this task may not be accomplished if only local operations and classical communication (LOCC) are permitted. Many people intuitively think that quantum entanglement causes some sets of quantum states cannot be exactly discriminated by LOCC. But Bennett et al. \cite{1} firstly shows that a set of product states cannot be reliably discriminated by LOCC in $\mathbb{C}^{d} \otimes \mathbb{C}^{d}$. This counterintuitive phenomenon is called quantum nonlocality without entanglement by Bennett et al. Encouraged by Bennett et al.’s work, many people began to engage in the research of nonlocal sets of OPSs and a lot of results \cite{2,23} were proposed.

Now we introduce the development of the construction methods of locally indistinguishable OPSs in bipartite quantum systems. Zhang et al. \cite{23} gave a method to construct a complete orthogonal product basis that cannot be exactly distinguished by LOCC in $\mathbb{C}^{d} \otimes \mathbb{C}^{d}$ for $d \geq 3$. Based on this result, Wang et al. \cite{26} pointed out that a subset of zhang et al.’s product basis is still indistinguishable by using only LOCC. Meanwhile, they generalized their method to a more general quantum system $\mathbb{C}^{m} \otimes \mathbb{C}^{n}$ for $m \geq 3$ and $n \geq 3$. Zhang et al. \cite{27} gave a new method to construct a nonlocal set of OPSs with $4d - 4$ elements in $\mathbb{C}^{d} \otimes \mathbb{C}^{d}$ for $d \geq 3$. Although the set constructed by this method has fewer states, it cannot be perfectly depicted by a graph. Recently, Zhang et al. \cite{25} gave a general construction method of locally indistinguishable OPSs in $\mathbb{C}^{m} \otimes \mathbb{C}^{n}$ for $3 \leq m \leq n$, which is different from Wang et al.’s result. Zhang et al. \cite{29} proposed a method to construct a nonlocal set of orthogonal product states with $3(n + m) - 8$ members in $\mathbb{C}^{m} \otimes \mathbb{C}^{n}$. Although many achievements have been made in the local discrimination of orthogonal product states, there still exist some problems needed to be solved, as we previously mentioned.

In this paper, we propose a novel method to construct nonlocal sets of OPSs in $\mathbb{C}^{d} \otimes \mathbb{C}^{d}$ quantum system for $d \geq 3$. The sets constructed by our method have very good structures, which can be used to design quantum cryptographic protocols. We give a perfect proof for the local indistinguishability of those sets. Meanwhile, we give a more general construction method of locally indistinguishable OPSs with $2n + 2m - 4$ members in $\mathbb{C}^{m} \otimes \mathbb{C}^{n}$ for $m \geq 3$ and $n \geq 3$. On the other hand, we analyze the structures of the nonlocal sets constructed respectively by our method and zhang et al.’s, then give the concept of isomorphism of two nonlocal sets of OPSs. All the results improve the theory of local indistinguishability of bipartite orthogonal product states.

II. PRELIMINARIES

Some preliminaries, which will be used in what follows, are introduced in this section.
Definition 1. [30, 31] If a set of orthogonal quantum states cannot be exactly discriminated by LOCC, we say it is locally indistinguishable or nonlocal.

Definition 2. [11, 13, 22, 26, 28] A quantum measurement is trivial if all its positive operator-valued measure elements are proportional to identity operator.

Lemma 1. (Vandermonde determinant [32]) The following determinant is called Vandermonde determinant since it is firstly researched by Vandermonde.

$$
\begin{vmatrix}
1 a_1 & a_1^2 & \cdots & a_1^{n-1} \\
1 a_2 & a_2^2 & \cdots & a_2^{n-1} \\
1 a_3 & a_3^2 & \cdots & a_3^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 a_n & a_n^2 & \cdots & a_n^{n-1} \\
\end{vmatrix} = \sum_{1 \leq j < t \leq n} (a_t - a_j),
$$

where \( t, j, n \) are integers.

Lemma 2. (Kramer’s rule [33]) A system of equations

$$
\begin{align*}
& a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
& \vdots \\
& a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n
\end{align*}
$$

have a unique solution if the coefficient determinant

$$
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} \\
\end{vmatrix} \neq 0,
$$

Lemma 3. [34] If \( \omega = e^{\frac{2\pi i}{d}} \), we have

$$
(\omega)^t \neq (\omega)^j
$$

and

$$(\omega^p)^{d-1} = 1$$

for \( 1 \leq t < j \leq d - 1 \) and \( 1 \leq p \leq d - 1 \), where \( i = \sqrt{-1} \); \( t, j, n \) are integers; and \( d \geq 3 \).

III. NONLOCAL SETS OF ORTHOGONAL PRODUCT STATES

In Ref. [1], Bennett et al. constructed nine orthogonal product states, which cannot be exactly discriminated by two separated observers if only LOCC are allowed between them. Feng et al. [10] showed that eight states (see FIG. 1 and Eqs. (1)) of those nine product states are still indistinguishable using only LOCC in \( \mathbb{C}^3 \otimes \mathbb{C}^3 \).

$$
\begin{align*}
|\theta_1\rangle &= |0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B \\
|\theta_2\rangle &= |0\rangle_A |0\rangle_B - |1\rangle_A |1\rangle_B \\
|\theta_3\rangle &= |0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B \\
|\theta_4\rangle &= |0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B \\
|\theta_5\rangle &= |0\rangle_A |2\rangle_B + |1\rangle_A |2\rangle_B \\
|\theta_6\rangle &= |0\rangle_A |2\rangle_B - |1\rangle_A |2\rangle_B \\
|\theta_7\rangle &= |1\rangle_A |0\rangle_B + |2\rangle_A |0\rangle_B \\
|\theta_8\rangle &= |1\rangle_A |0\rangle_B - |2\rangle_A |0\rangle_B
\end{align*}
$$

It should be noted that all the product states in this paper are not normalized for convenience.

The set of these eight states has a very good structure, which is very helpful for people to understand quantum nonlocality of orthogonal product states since these states can be expressed intuitively by a graph. FIG. 2. Though many works about quantum nonlocality of product states are proposed, most of them have complicated structures, which makes it difficult for us to understand quantum nonlocality in bipartite high dimensional systems.

A. Nonlocal set of OPSs in \( \mathbb{C}^4 \otimes \mathbb{C}^4 \)

In Ref. [22], Halder et al. gave a method to construct an unextendible product basis in \( \mathbb{C}^d \otimes \mathbb{C}^d \) with \( d \) odd and \( d \geq 5 \). Inspired by their work, we give a novel method to construct a nonlocal set of OPSs in bipartite high dimensional systems. In \( \mathbb{C}^4 \otimes \mathbb{C}^4 \), the states in Eqs. (2) construct a set of domino states. FIG. 2 shows a clear
and intuitive structure of domino states.
\[
|\phi_1\rangle = |0\rangle_A|(0) + |1\rangle + |2\rangle\rangle_B, \\
|\phi_2\rangle = |0\rangle_A|(0) + \omega|1\rangle + \omega^2|2\rangle\rangle_B, \\
|\phi_3\rangle = |0\rangle_A|(0) + \omega^2|1\rangle + (\omega^2)^2|2\rangle\rangle_B, \\
|\phi_4\rangle = (|0\rangle + |1\rangle + |2\rangle)_A|3\rangle_B, \\
|\phi_5\rangle = (|0\rangle + \omega|1\rangle + \omega^2|2\rangle)_A|3\rangle_B, \\
|\phi_6\rangle = (|0\rangle + \omega^2|1\rangle + (\omega^2)^2|2\rangle)_A|3\rangle_B, \\
|\phi_7\rangle = |3\rangle_A(|1\rangle + |2\rangle + |3\rangle)_B, \\
|\phi_8\rangle = |3\rangle_A(|1\rangle + \omega|2\rangle + \omega^2|3\rangle)_B, \\
|\phi_9\rangle = |3\rangle_A(|1\rangle + \omega^2|2\rangle + (\omega^2)^2|3\rangle)_B, \\
|\phi_{10}\rangle = (|1\rangle + |2\rangle + |3\rangle)_A|0\rangle_B, \\
|\phi_{11}\rangle = (|1\rangle + \omega|2\rangle + \omega^2|3\rangle)_A|0\rangle_B, \\
|\phi_{12}\rangle = (|1\rangle + \omega^2|2\rangle + (\omega^2)^2|3\rangle)_A|0\rangle_B,
\]

(2)

where \(\omega = e^{2\pi i/3}\) and \(i = \sqrt{-1}\).

**Theorem 1.** The set of domino states in Eqs. (2) cannot be exactly discriminated by using only LOCC.

**Proof.** It should be noted that the proof method is originally given in Refs. [2, 11]. To discriminate one of the twelve states, one party has to start with an orthogonal preserving and nontrivial measurement \(\{M_k^\dagger M_k\}\), i.e., the post-measurement states should be mutually orthogonal and not all \(M_k^\dagger M_k\) are proportional to identity operator.

Suppose that Alice firstly starts with a set of general 4×4 positive operator-valued measure (POVM) elements \(\{M_k^\dagger M_k : k = 1, 2, \ldots, l\}\), where

\[
M_k^\dagger M_k = \begin{bmatrix}
  m_{00}^k & m_{01}^k & m_{02}^k & m_{03}^k \\
  m_{10}^k & m_{11}^k & m_{12}^k & m_{13}^k \\
  m_{20}^k & m_{21}^k & m_{22}^k & m_{23}^k \\
  m_{30}^k & m_{31}^k & m_{32}^k & m_{33}^k
\end{bmatrix}
\]

under the basis \(\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}\). The post-measurement states \(\{(M_k \otimes I_{4\times4})|\phi_j\rangle : j = 1, 2, \ldots, 12\}\) should preserve their orthogonality.

Because \((M_k \otimes I_{4\times4})|\phi_1\rangle\) is orthogonal to \((M_k \otimes I_{4\times4})|\phi_{10}\rangle\), \((M_k \otimes I_{4\times4})|\phi_{11}\rangle\) and \((M_k \otimes I_{4\times4})|\phi_{12}\rangle\), i.e.,

\[
\begin{align*}
\langle \phi_1|(M_k^\dagger M_k \otimes I_{4\times4})|\phi_{10}\rangle &= 0, \\
\langle \phi_1|(M_k^\dagger M_k \otimes I_{4\times4})|\phi_{11}\rangle &= 0, \\
\langle \phi_1|(M_k^\dagger M_k \otimes I_{4\times4})|\phi_{12}\rangle &= 0,
\end{align*}
\]

and

\[
\begin{align*}
\langle \phi_{10}|(M_k^\dagger M_k \otimes I_{4\times4})|\phi_1\rangle &= 0, \\
\langle \phi_{11}|(M_k^\dagger M_k \otimes I_{4\times4})|\phi_1\rangle &= 0, \\
\langle \phi_{12}|(M_k^\dagger M_k \otimes I_{4\times4})|\phi_1\rangle &= 0
\end{align*}
\]

we get two systems of linear equations

\[
\begin{align*}
m_{01}^k + m_{02}^k + m_{03}^k &= 0, \\
m_{01}^k + \omega m_{02}^k + \omega^2 m_{03}^k &= 0, \\
m_{01}^k + \omega^2 m_{02}^k + (\omega^2)^2 m_{03}^k &= 0
\end{align*}
\]

and

\[
\begin{align*}
m_{10}^k + m_{20}^k + m_{30}^k &= 0, \\
m_{10}^k + \overline{\omega} m_{20}^k + \omega^2 m_{30}^k &= 0, \\
m_{10}^k + \overline{\omega^2} m_{20}^k + (\overline{\omega^2})^2 m_{30}^k &= 0
\end{align*}
\]

where \(\overline{\omega}\) is the complex conjugate of \(\omega\).

By Lemma 1, Lemma 2 and Lemma 3, we get the unique solution of Eqs. (3),

\[
m_{01}^k = m_{02}^k = m_{03}^k = 0
\]

and the unique solution of Eqs. (4),

\[
m_{10}^k = m_{20}^k = m_{30}^k = 0.
\]

Similarly, we can get two systems of linear equations

\[
\begin{align*}
m_{30}^k + m_{31}^k + m_{32}^k &= 0, \\
m_{30}^k + \omega m_{31}^k + \omega^2 m_{32}^k &= 0, \\
m_{30}^k + \omega^2 m_{31}^k + (\omega^2)^2 m_{32}^k &= 0
\end{align*}
\]

and

\[
\begin{align*}
m_{03}^k + m_{13}^k + m_{23}^k &= 0, \\
m_{03}^k + \overline{\omega} m_{13}^k + \omega^2 m_{23}^k &= 0, \\
m_{03}^k + \overline{\omega^2} m_{13}^k + (\overline{\omega^2})^2 m_{23}^k &= 0
\end{align*}
\]

since \((M_k \otimes I_{4\times4})|\phi_7\rangle\) is orthogonal to \((M_k \otimes I_{4\times4})|\phi_4\rangle\), \((M_k \otimes I_{4\times4})|\phi_5\rangle\) and \((M_k \otimes I_{4\times4})|\phi_6\rangle\).

By Lemma 1-3, we get the unique solution of Eqs. (7),

\[
m_{30}^k = m_{31}^k = m_{32}^k = 0
\]

and the unique solution of Eqs. (8),

\[
m_{03}^k = m_{13}^k = m_{23}^k = 0.
\]

Because each of the set \(\{M_k \otimes I_{4\times4})|\phi_4\rangle, (M_k \otimes I_{4\times4})|\phi_5\rangle, (M_k \otimes I_{4\times4})|\phi_6\rangle\} is orthogonal to the other two states, we have

\[
\begin{align*}
\langle \phi_4|(M_k^\dagger M_k \otimes I_{4\times4})|\phi_1\rangle &= 0, \\
\langle \phi_4|(M_k^\dagger M_k \otimes I_{4\times4})|\phi_1\rangle &= 0
\end{align*}
\]
\[
\begin{aligned}
&\{\phi_5| (M_k^1 \otimes I_{4 \times 4}^4) | \phi_4 \} = 0 \\
&\{\phi_5| (M_k^4 \otimes I_{4 \times 4}^4) | \phi_6 \} = 0
\end{aligned}
\]

and
\[
\begin{aligned}
&\{\phi_6| (M_k^1 \otimes I_{4 \times 4}^4) | \phi_4 \} = 0 \\
&\{\phi_6| (M_k^4 \otimes I_{4 \times 4}^4) | \phi_6 \} = 0
\end{aligned}
\]

That is,
\[
\begin{align}
&\sum_{p=1}^{2} (\omega^p \sum_{j=0}^{2} m_j^k) = -\sum_{j=0}^{2} m_j^0, \quad (11) \\
&\sum_{p=1}^{2} (\omega^p \sum_{j=0}^{2} \omega m_j^k) = -\sum_{j=0}^{2} \omega m_j^0
\end{align}
\]

and
\[
\begin{align}
&\sum_{p=1}^{2} (\omega^p \sum_{j=0}^{2} \omega m_j^k) = -\sum_{j=0}^{2} \omega \omega m_j^k, \quad (12) \\
&\sum_{p=1}^{2} (\omega^p \sum_{j=0}^{2} \omega \omega m_j^k) = -\sum_{j=0}^{2} \omega \omega m_j^0
\end{align}
\]

where \(\omega^0 = 1\) and \((\omega^2)^0 = 1\).

For simplicity, we denote the coefficient determinants of Eqs. (11), (12) and (13) as \(D_1\), \(D_2\) and \(D_3\), respectively. Since
\[
\begin{align}
D_1 &= \begin{vmatrix}
\omega & \omega^2 \\
\omega^2 & (\omega^2)^2
\end{vmatrix} = \omega^3 \begin{vmatrix}
1 & \omega \\
1 & \omega^2
\end{vmatrix} = \omega^2 - \omega 
eq 0, \\
D_2 &= \begin{vmatrix}
1 & 1 \\
\omega^2 & (\omega^2)^2
\end{vmatrix} = (\omega^2)^2 - \omega^2 
eq 0, \\
D_3 &= \begin{vmatrix}
1 & 1 \\
\omega & \omega^2
\end{vmatrix} = \omega^2 - \omega 
eq 0,
\end{align}
\]

we obtain the unique solutions of Eqs. (12), (13) and (14), respectively, i.e.,
\[
\begin{align}
&\begin{cases}
\sum_{j=0}^{2} m_j^1 = \sum_{j=0}^{2} m_j^0 \\
\sum_{j=0}^{2} m_j^2 = \sum_{j=0}^{2} m_j^0
\end{cases}, \quad (14) \\
&\begin{cases}
\sum_{j=0}^{2} \omega m_j^1 = \sum_{j=0}^{2} \omega m_j^0 \\
\sum_{j=0}^{2} \omega \omega m_j^2 = \sum_{j=0}^{2} \omega m_j^0
\end{cases}, \quad (15)
\end{align}
\]

By Lemma 1-3, we have the unique solutions of Eqs. (17) and Eqs. (18), respectively, i.e.,
\[
\begin{align}
&\begin{cases}
m_{11} = m_{00} \\
m_{01} = m_{21} = 0
\end{cases}, \quad (19) \\
&\begin{cases}
m_{22} = m_{00} \\
m_{02} = m_{12} = 0
\end{cases}, \quad (20)
\end{align}
\]

Similarly, since each state of the set \(\{ (M_k \otimes I_{4 \times 4}) | \phi_{10} \}, (M_k \otimes I_{4 \times 4}) | \phi_{11} \) and \( (M_k \otimes I_{4 \times 4}) | \phi_{12} \)\) is orthogonal to the other two states, we have
\[
\begin{align}
&\begin{cases}
\sum_{p=0}^{1} [\omega^p \sum_{j=0}^{3} m_j^{k(p+1)}] = -\omega^2 \sum_{j=1}^{3} m_j^{k3}, \\
\sum_{p=0}^{1} (\omega^p)^2 \sum_{j=1}^{3} m_j^{k(p+1)} = -(\omega^2)^2 \sum_{j=1}^{3} m_j^{k3}
\end{cases}, \quad (21)
\end{align}
\]
we obtain the unique solutions of Eqs. (21), (22) and (23) as \( D_4, D_5 \) and \( D_6 \), respectively. Since

\[
D_4 = \begin{vmatrix} 1 & \omega^2 \\ 1 & \omega \end{vmatrix} \neq 0,
\]

\[
D_5 = \begin{vmatrix} 1 & 1 \\ 1 & \omega^2 \end{vmatrix} \neq 0,
\]

\[
D_6 = \begin{vmatrix} 1 & 1 \\ 1 & \omega \end{vmatrix} \neq 0,
\]

we obtain the unique solutions of Eqs. (21), (22) and (23), respectively, i.e.,

\[
\begin{align*}
\sum_{j=1}^{3} m_{j1} &= \sum_{j=1}^{3} m_{j3}, \\
\sum_{j=1}^{3} m_{j2} &= \sum_{j=1}^{3} m_{j3},
\end{align*}
\]

\[
\begin{align*}
\sum_{j=1}^{3} \omega^{j-1} m_{j1} &= \omega^2 \sum_{j=1}^{3} \omega^{j-1} m_{j3}, \\
\sum_{j=1}^{3} \omega^{j-1} m_{j2} &= \omega \sum_{j=1}^{3} \omega^{j-1} m_{j3},
\end{align*}
\]

\[
\begin{align*}
\sum_{j=1}^{3} (\omega^2)^{j-1} m_{j1} &= (\omega^2)^2 \sum_{j=1}^{3} (\omega^2)^{j-1} m_{j3}, \\
\sum_{j=1}^{3} (\omega^2)^{j-1} m_{j2} &= \omega \sum_{j=1}^{3} (\omega^2)^{j-1} m_{j3},
\end{align*}
\]

By Eqs. (10), (24), (25) and (26), we can get

\[
\begin{align*}
\sum_{j=1}^{3} m_{j1} &= m_{j3}, \\
\sum_{j=1}^{3} \omega^{j-1} m_{j1} &= \omega m_{j3}, \\
\sum_{j=1}^{3} (\omega^2)^{j-1} m_{j1} &= \omega^2 m_{j3},
\end{align*}
\]

By Lemma 1-3, we get the unique solutions of Eqs. (27) and Eqs. (28), respectively, i.e.,

\[
\begin{align*}
m_{11} &= m_{33}, \\
m_{21} &= m_{41} = 0,
\end{align*}
\]

and

\[
\begin{align*}
m_{22} &= m_{33}, \\
m_{12} &= m_{32} = 0.
\end{align*}
\]

By Eqs. (5), (6), (9), (10), (19), (20), (29) and (30), we know

\[
M_k^1 M_k = \begin{bmatrix} m_{33} & 0 & 0 & 0 \\
0 & m_{33} & 0 & 0 \\
0 & 0 & m_{33} & 0 \\
0 & 0 & 0 & m_{33} \end{bmatrix}
\]
for $k = 1, 2, \cdots, l$. This means that all the POVM elements are proportional to the identity matrix. Thus Alice cannot start with a nontrivial measurement to keep the post-measurement states orthogonal.

In fact, Bob will face a similar case as Alice does since these twelve states have a symmetrical structure. Therefore, these states cannot be exactly distinguished using only LOCC. This completes the proof.

\section{Nonlocal set of OPSs in $\mathbb{C}^d \otimes \mathbb{C}^d$}

In this section, a general method to construct a nonlocal set of bipartite OPSs, which cannot be exactly distinguished using only LOCC, is given.

\textbf{Theorem 2.} In $\mathbb{C}^d \otimes \mathbb{C}^d$, the $4d - 4$ product states in Eqs. (31) cannot be exactly discriminated using only LOCC.

\begin{equation}
|\phi_{t+1}\rangle = |0\rangle_A (\sum_{j=0}^{d-2} \omega^{ij} |j\rangle)_B, \\
|\phi_{t+d}\rangle = (\sum_{j=0}^{d-2} \omega^{ij} |j\rangle)_A |(d-1)\rangle_B, \\
|\phi_{t+2d-1}\rangle = |(d-1)\rangle_A (\sum_{j=0}^{d-2} \omega^{ij} |(j+1)\rangle)_B, \\
|\phi_{t+3d-2}\rangle = (\sum_{j=0}^{d-2} \omega^{ij} |(j+1)\rangle)_A |0\rangle_B,
\end{equation}

where $\omega = \frac{e^{2\pi i}}{4}$, $t = \sqrt{-1}$, $d \geq 3$ and $t = 0, 1, \cdots, d - 2$.

The structure of our nonlocal set of OPSs is showed in FIG. 3. Here, we give a proof by the similar method as Theorem 1. Thus, in order to make it easier to understand, readers should understand the proof of Theorem 1 before reading this proof of Theorem 2.

\textbf{Proof.} Suppose that Alice firstly starts with a set of general $d \times d$ POVM elements $\{M^k_k : k = 1, 2, \cdots, l\}$, where

\begin{equation*}
M^k_k = \begin{bmatrix}
m^k_{00} & m^k_{01} & \cdots & m^k_{0(d-1)} \\
m^k_{10} & m^k_{11} & \cdots & m^k_{1(d-1)} \\
\vdots & \vdots & \ddots & \vdots \\
m^k_{(d-1)0} & m^k_{(d-1)1} & \cdots & m^k_{(d-1)(d-1)}
\end{bmatrix}
\end{equation*}

under the basis $\{|0\rangle, |1\rangle, \cdots, |(d-1)\rangle\}$. The post-measurement states $\{(M_k \otimes I_{d \times d})|\phi_j\rangle : j = 1, 2, \cdots, 4d - 4\}$ should preserve their orthogonality.

Because $(M_k \otimes I_{d \times d})|\phi_1\rangle$ is orthogonal to each state of the set $\{(M_k \otimes I_{d \times d})|\phi_{t+3d-2}\rangle : t = 0, 1, \cdots, d - 2\}$, we get

\begin{equation}
\begin{aligned}
\sum_{j=1}^{d-1} m^k_{0j} = 0 \\
\sum_{j=1}^{d-1} \omega^{j-1} m^k_{0j} = 0 \\
\vdots \\
\sum_{j=1}^{d-1} (\omega^{d-2})^{j-1} m^k_{0j} = 0
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
\sum_{j=1}^{d-1} m^k_{j0} = 0 \\
\sum_{j=1}^{d-1} \omega^{j-1} m^k_{j0} = 0 \\
\vdots \\
\sum_{j=1}^{d-1} (\omega^{d-2})^{j-1} m^k_{j0} = 0
\end{aligned}
\end{equation}

where $\overline{\omega}$ is the complex conjugate of $\omega$. Since the coefficient determinants of the two systems of equations (32) and (33) are not equal to zero by Lemma 1 and Lemma 3, i.e.,

\begin{equation*}
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
1 & \omega & \cdots & \omega^{d-2} \\
1 & \omega^2 & \cdots & (\omega^2)^{d-2} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \omega^{d-2} & (\omega^{d-2})^2 & \cdots & (\omega^{d-2})^{d-2}
\end{vmatrix} \neq 0,
\end{equation*}

\begin{equation*}
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
1 & \overline{\omega} & \cdots & \overline{\omega}^{d-2} \\
1 & \overline{\omega}^2 & \cdots & (\overline{\omega}^2)^{d-2} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \overline{\omega}^{d-2} & (\overline{\omega}^{d-2})^2 & \cdots & (\overline{\omega}^{d-2})^{d-2}
\end{vmatrix} \neq 0,
\end{equation*}
we have the unique solutions of Eqs. (32) and (33), respectively, \( m_{10}^k = m_{20}^k = \cdots = m_{(d-1)0}^k = 0 \) \( (35) \) and

\[
m_{01}^k = m_{02}^k = \cdots = m_{0(d-1)}^k = 0 \quad (34)
\]

by Lemma 2.

Similarly, because \((M_k \otimes I_{d \times d})|\phi_{2d-1}\) is orthogonal to each state of the set \(\{(M_k \otimes I_{d \times d})|\phi_{t}+d\} : t = 0, 1, \cdots, d-2\)\) we get

\[
\begin{align*}
\sum_{j=0}^{d-2} m_{j(d-1)}^k &= 0 \\
\sum_{j=0}^{d-2} \omega^{j} m_{j(d-1)}^k &= 0 \\
\sum_{j=0}^{d-2} (\omega^2)^{j} m_{j(d-1)}^k &= 0 \\
&\vdots \\
\sum_{j=0}^{d-2} (\omega^{d-2})^{j} m_{j(d-1)}^k &= 0
\end{align*}
\]

and

\[
\begin{align*}
\sum_{j=0}^{d-2} m_{j(d-1)}^k &= 0 \\
\sum_{j=0}^{d-2} \omega^{j} m_{j(d-1)}^k &= 0 \\
\sum_{j=0}^{d-2} (\omega^2)^{j} m_{j(d-1)}^k &= 0 \\
&\vdots \\
\sum_{j=0}^{d-2} (\omega^{d-2})^{j} m_{j(d-1)}^k &= 0
\end{align*}
\]

By Lemma 1-3, we get the unique solutions of Eqs. (36) and (37), respectively, \( m_{(d-1)0}^k = m_{(d-1)1}^k = \cdots = m_{(d-1)(d-2)}^k = 0 \) \( (38) \) and

\[
m_{0(d-1)}^k = m_{1(d-1)}^k = \cdots = m_{(d-2)(d-1)}^k = 0.
\]

Because each state of the set \(\{(M_k \otimes I_{d \times d})|\phi_{t+3d-2}\} : t = 0, 1, \cdots, d-2\) is orthogonal to all the other states, we get the following \(d-2\) systems of equations

\[
\begin{align*}
\sum_{p=0}^{d-3} \sum_{j=1}^{d-1} \omega^p m_{j(p+1)}^k &= -\omega^{d-2} \sum_{j=1}^{d-1} m_{j(d-1)}^k \\
\sum_{p=0}^{d-3} [(\omega^2)^p \sum_{j=1}^{d-1} m_{j(p+1)}^k] &= - (\omega^2)^{d-2} \sum_{j=1}^{d-1} m_{j(d-1)}^k \\
&\vdots \\
\sum_{p=0}^{d-3} [(\omega^{d-2})^p \sum_{j=1}^{d-1} m_{j(p+1)}^k] &= - (\omega^{d-2})^{d-2} \sum_{j=1}^{d-1} m_{j(d-1)}^k
\end{align*}
\]
\[
\begin{align*}
\sum_{p=0}^{d-3} \sum_{j=1}^{d-1} (\omega^j)^{p-1} m_{j(p+1)}^k &= -\sum_{j=1}^{d-1} (\omega^j)^{p-1} m_{j(d-1)}^k \\
\sum_{p=0}^{d-3} (\omega^2)^p \sum_{j=1}^{d-1} (\omega^j)^{p-1} m_{j(p+1)}^k &= -(\omega^2)^{d-2} \sum_{j=1}^{d-1} (\omega^j)^{p-1} m_{j(d-1)}^k \\
\sum_{p=0}^{d-3} (\omega^3)^p \sum_{j=1}^{d-1} (\omega^j)^{p-1} m_{j(p+1)}^k &= -(\omega^3)^{d-2} \sum_{j=1}^{d-1} (\omega^j)^{p-1} m_{j(d-1)}^k \\
\vdots \\
\sum_{p=0}^{d-3} (\omega^{d-2})^p \sum_{j=1}^{d-1} (\omega^j)^{p-1} m_{j(p+1)}^k &= -(\omega^{d-2})^{d-2} \sum_{j=1}^{d-1} (\omega^j)^{p-1} m_{j(d-1)}^k \\
\end{align*}
\]
By Lemma 1-3 and Eqs. (39), we get the unique solutions of the above $d - 2$ systems of equations, i.e.,

$$
\begin{align*}
\sum_{j=1}^{d-2} m_{jq}^k &= m_{(d-1)(d-1)}^k \\
\sum_{j=1}^{d-2} \omega^j m_{jq}^k &= \omega^{-1} m_{(d-1)(d-1)}^k \\
\sum_{j=1}^{d-2} (\omega^2)^j m_{jq}^k &= (\omega^2)^{-1} m_{(d-1)(d-1)}^k, \\
\vdots \\
\sum_{j=1}^{d-2} (\omega^{d-2})^j m_{jq}^k &= (\omega^{d-2})^{-1} m_{(d-1)(d-1)}^k
\end{align*}
$$

(40)

where $q = 1, 2, \ldots, d - 2$.

By Lemma 1-3 and Eqs. (40), we have

$$
\begin{align*}
m_{jq}^k &= 0 \\
m_{qq}^k &= m_{(d-1)(d-1)}^k
\end{align*}
$$

(41)

for $q = 1, 2, \ldots, d - 2; j = 1, 2, \ldots, d - 2$ and $j \neq q$.

Similarly, since each state of the set $\{ (M_k \otimes I_{d \times d}) | \phi_{t+4} \rangle : t = 0, 1, \ldots, d - 2 \}$ is orthogonal to all the other states, we have

$$
\begin{align*}
m_{jq}^k &= 0 \\
m_{qq}^k &= m_{00}^k
\end{align*}
$$

(42)

for $q = 1, 2, \ldots, d - 2; j = 1, 2, \ldots, d - 2$ and $j \neq q$.

By Eqs. (34), (35), (38), (39), (41) and (42), we have

$$
M_k^\dagger M_k = \begin{bmatrix}
    m_{(d-1)(d-1)}^k & 0 & \cdots & 0 \\
    0 & m_{(d-1)(d-1)}^k & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & m_{(d-1)(d-1)}^k
\end{bmatrix}
$$

for $k = 1, 2, \ldots, l$. This means that all the POVM elements are proportional to identity matrix. That is, Alice cannot start with a nontrivial measurement to keep the post-measurement states orthogonal.

In fact, Bob will face the similar case as Alice does since these states have a symmetrical structure. Therefore, these states cannot be exactly distinguished by using only LOCC. This completes the proof.

By Theorem 2, we can get the following corollary when $d = 3$.

**Corollary 1.** In $C^3 \otimes C^3$, the states in Eqs. (43) cannot be perfectly distinguished by LOCC.

$$
\begin{align*}
|\phi_1\rangle &= |0\rangle_A |0\rangle_B, \\
|\phi_2\rangle &= |0\rangle_A |1\rangle_B, \\
|\phi_3\rangle &= |1\rangle_A |0\rangle_B, \\
|\phi_4\rangle &= |1\rangle_A |1\rangle_B, \\
|\phi_5\rangle &= |2\rangle_A |0\rangle_B, \\
|\phi_6\rangle &= |2\rangle_A |1\rangle_B, \\
|\phi_7\rangle &= |2\rangle_A |2\rangle_B.
\end{align*}
$$

(43)

In fact, the states in Eqs. (43) are just the states in Eqs. (1). That is, the set of the states in Eqs. (1) is a special case of Theorem 2.

**C. Nonlocal set of OPSs in $C^m \otimes C^n$**

In this section, we extend our result to a more general situation.

**Theorem 3.** In $C^m \otimes C^n$, the $2(m+n)-4$ product states in Eqs. (44) cannot be exactly discriminated using only LOCC.

$$
\begin{align*}
|\phi_{\sigma + 1}\rangle &= |0\rangle_A \sum_{j=0}^{n-2} (\omega_1)^n |j\rangle_B, \\
|\phi_{\eta + n}\rangle &= \sum_{j=0}^{m-2} (\omega_2)^n |j\rangle_A |(n-1)\rangle_B, \\
|\phi_{\sigma + n + m - 1}\rangle &= |(m-1)\rangle_A \sum_{j=0}^{n-2} (\omega_1)^n |(j + 1)\rangle_B, \\
|\phi_{\eta + 2n + m - 2}\rangle &= \sum_{j=0}^{m-2} (\omega_2)^n |(j + 1)\rangle_A |0\rangle_B.
\end{align*}
$$

(44)
where \( m \geq 3, n \geq 3, i = \sqrt{-1}, \omega_1 = e^{\frac{2\pi i}{m}}, \omega_2 = e^{\frac{2\pi i}{n}}, \sigma = 0, 1, \cdots, n - 2 \) and \( \eta = 0, 1, \cdots, m - 2 \).

In fact, we can prove Theorem 3 with the method that was used in the proof of Theorem 2. It should be noted that a set of \( m \times m \) POVM elements for Alice while a set of \( n \times n \) POVM elements for Bob are needed when we prove each of them cannot start with a nontrivial measurement. Here we omit the proof.

Now, we compare our construction with the existing results. In fact, our nonlocal set of OPs has fewer elements than the sets constructed in \[23, 26, 28, 29\] when the “stoper” state \[35\] is not include. Table 1 gives us the detailed data. Most importantly, the structure of our nonlocal set of OPs is symmetrical and more intuitive.

### TABLE I. Comparison of the numbers of elements between different sets of locally indistinguishable OPs in \( \mathbb{C}^m \otimes \mathbb{C}^n \) for \( 3 \leq m \leq n \)

| Different sets | The number of elements |
|---------------|-----------------------|
| The set of Ref. \[23\] | \( 3n + 3m - 9 \) |
| The set of Ref. \[26\] | \( 3m + 3n - 9 \) |
| The set of Ref. \[28\] | \( 3n + m - 4 \) |
| The set of Ref. \[29\] | \( 3n + 3m - 8 \) |
| our set of Theorem 3 | \( 2n + 2m - 4 \) |

### IV. ISOMORPHISM OF TWO SETS OF ORTHOGONAL PRODUCT STATES

In Ref. \[35\], DiVincenzo et al. gave the concept of orthogonality graph to describe the structure of a set of orthogonal product states in a bipartite Hilbert space.

**Definition 4.** \[35\] Let \( H = H_A \otimes H_B \) be a bipartite Hilbert space with \( \dim H_A = \dim H_B \). Let \( S = \{|\psi_j\rangle = |\varphi_{A_j}\rangle \otimes |\varphi_{B_j}\rangle| j = 1, 2, \cdots, s \rangle \) is a set of orthogonal product states in \( H \). We represent \( S \) as a graph \( G = (V, E_A \cup E_B) \), where \( E_A \) and \( E_B \) are the sets of edges. Each state \(|\psi_j\rangle \rangle \in S \) is represented as a vertex of \( V \). If the states \(|\psi_j\rangle \rangle \) and \(|\psi_k\rangle \rangle \) are orthogonal on \( H_A \) (\( H_B \)), there will be a solid (dotted) line between the vertices \( v_j \) and \( v_k \).

Now we give the definition of isomorphism of two sets, which will be used to describe the relations between different sets of locally indistinguishable OPs.

**Definition 4.** In \( \mathbb{C}^d \otimes \mathbb{C}^d \), we say two sets of orthogonal product states are isomorphic if their orthogonality graphs are same both on Alice’s side and Bob’s side; Otherwise, we say these two sets are not isomorphic.

The orthogonality graph of the set (see Eqs. \(43\)) on Alice’s is showed in FIG. 5. In FIG. 5, the vertex \( j \) denotes the states \(|\phi_j\rangle \rangle \) for \( j = 1, 2, \cdots, 8 \).

In Refs. \[27\], Zhang et al. gave a method to construct a nonlocal set of OPs in \( \mathbb{C}^d \otimes \mathbb{C}^d \). Let \( d = 3 \), we can get the locally indistinguishable OPs in \( \mathbb{C}^3 \otimes \mathbb{C}^3 \), i.e.,

\[
\begin{align*}
|\psi_1\rangle &\rangle = |1\rangle_A (|0\rangle + |1\rangle)_B \\
|\psi_2\rangle &\rangle = |1\rangle_A (|0\rangle - |1\rangle)_B \\
|\psi_3\rangle &\rangle = |2\rangle_A (|0\rangle + |2\rangle)_B \\
|\psi_4\rangle &\rangle = |2\rangle_A (|0\rangle - |2\rangle)_B \\
|\psi_5\rangle &\rangle = (|0\rangle + |1\rangle)_A |2\rangle_B \\
|\psi_6\rangle &\rangle = (|0\rangle - |1\rangle)_A |2\rangle_B \\
|\psi_7\rangle &\rangle = (|0\rangle + |2\rangle)_A |1\rangle_B \\
|\psi_8\rangle &\rangle = (|0\rangle - |2\rangle)_A |1\rangle_B
\end{align*}
\]

(45)

Here we give a mapping between the states of Eqs. (45) and the serial numbers of the vertices, i.e.,

\[
\begin{align*}
|\psi_1\rangle &\rangle \mapsto 1, & |\psi_2\rangle &\rangle \mapsto 2, & |\psi_3\rangle &\rangle \mapsto 3, & |\psi_4\rangle &\rangle \mapsto 4, \\
|\psi_5\rangle &\rangle \mapsto 5, & |\psi_6\rangle &\rangle \mapsto 6, & |\psi_7\rangle &\rangle \mapsto 7, & |\psi_8\rangle &\rangle \mapsto 8
\end{align*}
\]

(46)

By the mapping relation of Eqs. (46), we give the orthogonality graph (see FIG. 6) of the states in Eq. (45) on Alice’s subsystem. It is easy to see that FIG. 5 and FIG. 6 are identical. This means that the two sets of orthogonal product states have the same orthogonal graph.
FIG. 7. The orthogonality graph of the states in Eqs. (43) or Eqs. (45) on Bob’s side.

FIG. 8. The orthogonality graph of Zhang et al.’s states on Alice’s side.

FIG. 9. The orthogonality graph of our states in Eqs. (2) on Alice’s side.

FIG. 10. The orthogonality graph of Zhang et al.’s states on Bob’s side.

FIG. 11. The orthogonality graph of our states on Bob’s side.

Now we consider the orthogonality graphs of Zhang et al’s set and ours in $\mathbb{C}^4 \otimes \mathbb{C}^4$. The orthogonality graphs of our construction, i.e., the states in Eqs. (2), are showed in FIG. 9 and FIG. 11. The states constructed by Zhang et al.’s method are as follows (see Eq. (47)).

$$
\begin{align*}
|\psi_1\rangle &= |1\rangle_A(|0\rangle + |1\rangle)_B, \\
|\psi_2\rangle &= |1\rangle_A(|0\rangle - |1\rangle)_B, \\
|\psi_3\rangle &= |2\rangle_A(|0\rangle + |2\rangle)_B, \\
|\psi_4\rangle &= |2\rangle_A(|0\rangle - |2\rangle)_B, \\
|\psi_5\rangle &= |3\rangle_A(|0\rangle + |3\rangle)_B, \\
|\psi_6\rangle &= |3\rangle_A(|0\rangle - |3\rangle)_B, \\
|\psi_7\rangle &= (|0\rangle + |1\rangle)_A|2\rangle_B, \\
|\psi_8\rangle &= (|0\rangle - |1\rangle)_A|2\rangle_B, \\
|\psi_9\rangle &= (|0\rangle + |2\rangle)_A|3\rangle_B, \\
|\psi_{10}\rangle &= (|0\rangle - |2\rangle)_A|3\rangle_B, \\
|\psi_{11}\rangle &= (|0\rangle + |3\rangle)_A|1\rangle_B, \\
|\psi_{12}\rangle &= (|0\rangle - |3\rangle)_A|1\rangle_B.
\end{align*}
$$

(47)

The orthogonality graphs of the states constructed by Zhang et al.’s method are exhibited in FIG. 8 and FIG. 10. Obviously, FIG. 8 has 39 edges while FIG. 9 has 33 edges. This means that the orthogonality graph of Zhang et al. states in $\mathbb{C}^4 \otimes \mathbb{C}^4$ and ours are not identical on Alice’s side. So do they on Bob’s side. Thus the set of the states constructed by Zhang et al. is not isomorphic with ours. But why does this happen? This is because any two states of our set in $\mathbb{C}^m \otimes \mathbb{C}^n$ are orthogonal only on one side while some states of Zhang et al. are orthogonal both on Alice’s side and on Bob’s side. For example, $|\psi_3\rangle$ and $|\psi_{11}\rangle$, $|\psi_5\rangle$ and $|\psi_7\rangle$ in Eqs. (47).

In fact, any two states constructed by our method in $\mathbb{C}^m \otimes \mathbb{C}^n$ are orthogonal only on one subsystem for $3 \leq m \leq n$. This is a very good property, which will be widely
used to design quantum protocols.

V. CONCLUSION

Quantum nonlocality without entanglement is a peculiar phenomenon in nature. Many people are devoted to this research work since it is a significant thing to perfect and develop quantum information theory. In addition, many results are used to design quantum cryptographic protocols.

In this paper, we present a construction method of a set of locally indistinguishable OPSs in $\mathbb{C}^d \otimes \mathbb{C}^d$, where $d \geq 3$. We give a perfect proof for the local indistinguishability of our set. Any two states of the set are orthogonal only on one subsystem. We analysis the structures of our set and Zhang et al.’s set by orthogonality graph.

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