The Congruence Subgroup Problem
for low rank Free and Free Metabelian groups

David El-Chai Ben-Ezra, Alexander Lubotzky

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To Efim Zelmanov, a friend and a leader

Abstract

The congruence subgroup problem for a finitely generated group $\Gamma$ asks whether $\hat{\text{Aut}}(\Gamma) \to \text{Aut}(\hat{\Gamma})$ is injective, or more generally, what is its kernel $C(\Gamma)$? Here $\hat{X}$ denotes the profinite completion of $X$.

In this paper we first give two new short proofs of two known results (for $\Gamma = F_2$ and $\Phi_2$) and a new result for $\Gamma = \Phi_3$:

1. $C(F_2) = \{e\}$ when $F_2$ is the free group on two generators.
2. $C(\Phi_2) = \hat{F}_\omega$ when $\Phi_n$ is the free metabelian group on $n$ generators, and $\hat{F}_\omega$ is the free profinite group on $\aleph_0$ generators.
3. $C(\Phi_3)$ contains $\hat{F}_\omega$.

Results (2) and (3) should be contrasted with an upcoming result of the first author showing that $C(\Phi_n)$ is abelian for $n \geq 4$.

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1 Introduction

The classical congruence subgroup problem (CSP) asks for, say, $G = SL_n(\mathbb{Z})$ or $G = GL_n(\mathbb{Z})$, whether every finite index subgroup of $G$ contains a principal congruence subgroup, i.e., a subgroup of the form $G(m) = \ker(G \to GL_n(\mathbb{Z}/m\mathbb{Z}))$ for some $0 \neq m \in \mathbb{Z}$. Equivalently, it asks whether the natural map $\hat{G} \to GL_n(\hat{\mathbb{Z}})$ is injective, where $\hat{G}$ and $\hat{\mathbb{Z}}$ are the profinite completions of the group $G$ and the ring $\mathbb{Z}$, respectively. More generally, the CSP asks what is the kernel of this map. It is a classical 19th-century result that the answer is negative for $n = 2$. Moreover (but not so classical, cf. [Me], [La]), the kernel, in this case,
is \( \hat{F}_\omega \) - the free profinite group on a countable number of generators. On the other hand, for \( n \geq 3 \), the map is injective and the kernel is therefore trivial.

The CSP can be generalized as follows: Let \( \Gamma \) be a group and \( M \) a finite index characteristic subgroup of it. Denote:

\[
G(M) = \ker(\text{Aut}(\Gamma) \rightarrow \text{Aut}(\Gamma/M)).
\]

Such a finite index normal subgroup of \( G = \text{Aut}(\Gamma) \) will be called a “principal congruence subgroup” and a finite index subgroup of \( G \) which contains such a \( G(M) \) for some \( M \) will be called a “congruence subgroup”. Now, the CSP for \( \Gamma \) asks whether every finite index subgroup of \( G = \text{Aut}(\Gamma) \) is a congruence subgroup. When \( \Gamma \) is finitely generated, \( \text{Aut}(\hat{\Gamma}) \) is profinite and the CSP is equivalent to the question (cf. [BER], §1 and §3): Is the map \( \hat{G} = \hat{\text{Aut}(\Gamma)} \rightarrow \text{Aut}(\hat{\Gamma}) \) injective? More generally, it asks what is the kernel \( C(\Gamma) \) of this map.

As \( GL_n(\mathbb{Z}) = \text{Aut}(\mathbb{Z}^n) \), the classical congruence subgroup results mentioned above can therefore be reformulated as \( C(A_2) = \hat{F}_\omega \) while \( C(A_n) = \{e\} \) for \( n \geq 3 \), when \( A_n = \mathbb{Z}^n \) is the free abelian group on \( n \) generators.

Very few results are known when \( \Gamma \) is non-abelian. A very surprising result was proved in [As] by Asada by methods of algebraic geometry:

**Theorem 1.1.** \( C(F_2) = \{e\} \), i.e., the free group on two generators has the congruence subgroup property, namely \( \text{Aut}(\hat{F}_2) \rightarrow \text{Aut}(\hat{F}_2) \) is injective.

A purely group theoretic proof for this theorem was given by Bux-Ershov-Rapinchuk [BER]. Our first goal in this paper is to give an easier and more direct proof of Theorem 1.1 which also give a better quantitative estimate: we give an explicitly constructed congruence subgroup \( G(M) \) of \( \text{Aut}(F_2) \) which is contained in a given finite index subgroup \( H \) of \( \text{Aut}(F_2) \) of index \( n \). Our estimates on the index of \( M \) in \( F_2 \) as a function of \( n \) are substantially better than those of [BER] - see Theorems 2.7 and 2.9.

We then turn to \( \Gamma = \Phi_2 \), the free metabelian group on two generators. The initial treatment of \( \Phi_2 \) is similar to \( F_2 \), but quite surprisingly, the first named author showed in [Be1] a negative answer, i.e. \( C(\Phi_2) \neq \{e\} \). We also give a shorter proof of this result, deducing that:

**Theorem 1.2.** \( C(\Phi_2) = \hat{F}_\omega \).

We then go ahead from 2 to 3 and prove:

**Theorem 1.3.** \( C(\Phi_3) \) contains a copy of \( \hat{F}_\omega \). In particular, the congruence subgroup property (strongly) fails for \( \Phi_3 \).

This is also surprising, especially if compared with an upcoming paper of the first author [Be2] showing that \( C(\Phi_n) \) is abelian for \( n \geq 4 \). So, while the dichotomy for the abelian case \( A_n = \mathbb{Z}^n \) is between \( n = 2 \) and \( n \geq 3 \), for the metabelian case, it is between \( n = 2, 3 \) and \( n \geq 4 \).

A main ingredient of the proof of Theorem 1.3 is showing that \( \text{Aut}(\Phi_3) \) is large, i.e. it has a finite index subgroup which is mapped onto a non-abelian
free group. For this we use the method developed by Grunewald and the second author in [GL] to produce arithmetic quotients of $\text{Aut}(F_n)$. In particular, it is shown there that $\text{Aut}(F_3)$ is large. Our starting point to prove Theorem 1.3 is the observation that the same proof shows also that $\text{Aut}(\Phi_3)$ is large.

In our proof of Theorem 1.2 the largeness of $\text{Aut}(\Phi_2)$ is also playing a crucial role. But, a word of warning is needed here: largeness of $\text{Aut}(\Gamma)$ by itself is not sufficient to deduce negative answer for the CSP for $\Gamma$. For example, $\text{Aut}(F_2)$ is large but has an affirmative answer for the CSP. At the same time, as mentioned above, $\text{Aut}(F_3)$ is large and we do not know whether $F_3$ has the congruence subgroup property or not. To prove Theorem 1.3 we use the largeness of $\text{Aut}(\Phi_3)$ combined with the fact that every non-abelian finite simple group which is involved in $\text{Aut}(\hat{\Phi}_3)$ is already involved in $GL_3(R)$ for some finite commutative ring $R$, as we will show below.

The paper is organized as follows: In §2 we give a short proof for Theorem 1.1 and in §3 for Theorem 1.2. The 4th section is devoted to the proof of Theorem 1.3. We close in §5 with some remarks and open problems, about free nilpotent and solvable groups.

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2 The CSP for $F_2$

Before we start, let us quote some general propositions which Bux-Ershov-Rapinchuk bring throughout their paper.

**Proposition 2.1.** (cf. [BER], Lemma 2.1) Let:

$$1 \to G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3 \to 1$$

be an exact sequence of groups. Assume that $G_1$ is finitely generated and that the center of its profinite completion $\hat{G}_1$ is trivial. Then, the sequence of the profinite completions

$$1 \to \hat{G}_1 \xrightarrow{\hat{\alpha}} \hat{G}_2 \xrightarrow{\hat{\beta}} \hat{G}_3 \to 1$$

is also exact.

**Proposition 2.2.** (cf. [BER], Corollaries 2.3, 2.4. and 2.7) Let $F$ be the free group on the set $X$, $|X| \geq 2$. Then:

1. The center of $\hat{F}$, the profinite completion of $F$, is trivial.
2. If $x, y \in X$, $x \neq y$, then the centralizer of $[y, x]$ in $\hat{F}$ is $Z_{\hat{F}}([y, x]) = \langle [y, x] \rangle$, the closure of the cyclic group generated by $[y, x]$.

We start now with the following lemma whose easy proof is left to the reader:
Lemma 2.3. Let $H \leq G = \text{Aut}(\Gamma)$ be a congruence subgroup. Then:

$$\text{ker}(\hat{G} \to \text{Aut}(\hat{\Gamma})) = \text{ker}(\hat{H} \to \text{Aut}(\hat{\Gamma}))$$

In particular, the map $\hat{G} \to \text{Aut}(\hat{\Gamma})$ is injective if and only if the map $\hat{H} \to \text{Aut}(\hat{\Gamma})$ is injective.

Denote now $F_2 = \langle x, y \rangle$ the free group on $x$ and $y$. It is a well known theorem of Nielsen (cf. [MKS], 3.5) that the kernel of the natural surjective map:

$$\text{Aut}(F_2) \to \text{Aut}(F_2/F_2') = \text{Aut}(Z^2) = GL_2(Z)$$

is $\text{Inn}(F_2)$, the inner automorphism group of $F_2$. It is also well known that the group $\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$ is free on two generators and of finite index in $GL_2(Z)$ which contains $\text{ker}(GL_2(Z) \to GL_2(Z/4Z))$. Now, if we denote the preimage of $\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$ under the map $\text{Aut}(F_2) \to GL_2(Z)$ by $\text{Aut}'(F_2)$, then $\text{Aut}'(F_2)$ is of finite index in $\text{Aut}(F_2)$ and contains the principal congruence subgroup:

$$\text{ker}(\text{Aut}(F_2) \to GL_2(Z) \to GL_2(Z/4Z) = \text{Aut}(F_2/(F_2')F_2))$$

So, by Lemma 2.3 it is enough to prove that $\hat{\text{Aut}}'(F_2) \to \text{Aut}(\hat{F}_2)$ is injective.

Now, by the description above, we deduce the exact sequence:

$$1 \to \text{Inn}(F_2) \to \text{Aut}'(F_2) \to \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle \to 1.$$ 

As $\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$ is free, this sequence splits by the map:

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \mapsto \alpha = \begin{cases} x \mapsto x \\ y \mapsto x^2 \end{cases}, \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \mapsto \beta = \begin{cases} x \mapsto xy^2 \\ y \mapsto y \end{cases}$$

and thus: $\text{Aut}'(F_2) = \text{Inn}(F_2) \rtimes \langle \alpha, \beta \rangle$. By Propositions 2.1 and 2.2, the exact sequence: $1 \to \text{Inn}(F_2) \to \text{Aut}'(F_2) \to \langle \alpha, \beta \rangle \to 1$ yields the exact sequence:

$$1 \to \hat{\text{Inn}}(F_2) \to \hat{\text{Aut}}'(F_2) \to \langle \hat{\alpha}, \hat{\beta} \rangle \to 1$$

which gives:

$$\text{Aut}'(F_2) = \hat{\text{Inn}}(F_2) \rtimes \langle \hat{\alpha}, \hat{\beta} \rangle$$

Thus, all we need to show is that the following map is injective:

$$\hat{\text{Inn}}(F_2) \rtimes \langle \hat{\alpha}, \hat{\beta} \rangle \to \text{Aut}(\hat{F}_2).$$

We will prove this, in three parts: The first part is that the map $\hat{\text{Inn}}(F_2) \to \text{Aut}(\hat{F}_2)$ is injective, but this is obvious as $\hat{\text{Inn}}(F_2) \cong \hat{F}_2$ is mapped isomorphically to $\hat{\text{Inn}}(\hat{F}_2) \cong \hat{F}_2$. The second part is to show that the map $\rho : \langle \hat{\alpha}, \hat{\beta} \rangle \to$
$Aut(\hat{F}_2)$ is injective, and the last part is to show that the intersection of the images of $Inn(\hat{F}_2)$ and $\langle \alpha, \beta \rangle$ in $Aut(\hat{F}_2)$ is trivial, i.e. $Inn(\hat{F}_2) \cap \text{Im} = \{e\}$.

So it remains to prove the next two lemmas, Lemma 2.4 and Lemma 2.6.

**Lemma 2.4.** The map $\langle \alpha, \beta \rangle \to Aut(\hat{F}_2)$ is injective.

Before proving the lemma, we recall a classical result of Schreier:

**Theorem 2.5.** (cf. [MKS], 2.3 and 2.4) Let $F$ be the free group on the set $X$ where $|X| = n$, and $\Delta$ a subgroup of $F$ of index $m$. Let $T$ be a right Schreier transversal of $\Delta$ (i.e. a system of representatives of right cosets containing the identity, such that the initial segment of any element of $T$ is also in $T$). Then:

1. $\Delta$ is a free group on $m \cdot (n - 1) + 1$ elements.
2. The set $\{tx \mid (tx)^{-1} \neq e, t \in T, x \in X\}$ is a free generating set for $\Delta$, where for every $g \in F$ we denote by $\bar{g}$ the unique element in $T$ satisfying $\Delta g = \Delta \bar{g}$.

**Proof.** (of Lemma 2.4) Define $\Delta = \ker(F_2 \to (\mathbb{Z}/2\mathbb{Z})^2)$. This is a characteristic subgroup of index 4 in $F_2$, that by the first part of Theorem 2.5 is isomorphic to $F_5$. We also have: $\hat{\Delta} = \ker(\hat{F}_2 \to (\mathbb{Z}/2\mathbb{Z})^2)$, and therefore, there is a natural homomorphism: $Aut(\hat{F}_2) \to Aut(\hat{\Delta}) \cong Aut(F_5)$ which induces the composition $\langle \alpha, \beta \rangle \to Aut(\hat{F}_2) \to Aut(\hat{\Delta})$. Thus, it is enough to show that the composition map $\langle \alpha, \beta \rangle \to Aut(\hat{\Delta})$ is injective.

Now, let $X = \{x, y\}$ and $T = \{1, x, y, xy\}$ be a right Schreier transversal of $\Delta$. By applying the second part of Theorem 2.5 for $X$ and $T$, we get the following set of free generators for $\Delta$:

$$e_1 = x^2, \quad e_2 = yxy^{-1}x^{-1}, \quad e_3 = y^2, \quad e_4 = xyxy^{-1}, \quad e_5 = xy^2x^{-1}$$

Hence, the automorphisms $\alpha$ and $\beta$ act on $\Delta$ in the following way:

$$\alpha = \begin{cases} e_1 \mapsto x^2 & = e_1 \\ e_2 \mapsto yxy^{-1}x^{-1} \mapsto yxy^{-1}x^{-1} & = e_2 \\ e_3 \mapsto y^2 \mapsto yx^2yx^2 & = e_2e_4e_3e_1 \\ e_4 \mapsto xyxy^{-1} \mapsto xyxy^{-1} & = e_4 \\ e_5 \mapsto xy^2x^{-1} \mapsto xy^2x^{-1} & = e_4e_2e_5e_1 \end{cases}$$

$$\beta = \begin{cases} e_1 \mapsto xy^2xy^2 & = e_5e_1e_3 \\ e_2 \mapsto yxy^{-1}x^{-1} \mapsto yxy^{-1}x^{-1} & = e_2 \\ e_3 \mapsto y^2 \mapsto y^2 & = e_3 \\ e_4 \mapsto xyxy^{-1} \mapsto xy^3xy & = e_5e_4e_3 \\ e_5 \mapsto xy^2x^{-1} \mapsto xy^2x^{-1} & = e_5 \end{cases}$$
Let us now define the map $\pi : \Delta \to \langle \alpha, \beta \rangle \cong F_2$ (yes! these are the same $\alpha$ and $\beta$) by the following way:

$$
\pi = \begin{cases}
    e_1 &\mapsto \alpha \\
    e_2 &\mapsto 1 \\
    e_3 &\mapsto \beta \\
    e_4 &\mapsto \alpha^{-1} \\
    e_5 &\mapsto \beta^{-1}
\end{cases}
$$

It is easy to see that $N = \ker \pi$ is the normal subgroup of $\Delta$ generated as a normal subgroup by $e_2$, $e_1 e_4$ and $e_3 e_5$, and that $N$ is invariant under the action of the automorphisms $\alpha$ and $\beta$, since:

$$
\begin{align*}
\alpha(e_2) &= e_2 \in N \\
\alpha(e_1 e_4) &= e_1 e_4 \in N \\
\alpha(e_3 e_5) &= e_2 e_4 e_3 e_1 e_2 e_5 e_1 \\
&= e_4 ((e_4^{-1} e_2 e_4) (e_3 ((e_1 e_4) e_2) e_3^{-1}) (e_3 e_5) (e_1 e_4)) e_4^{-1} \in N \\
\beta(e_2) &= e_2 \in N \\
\beta(e_1 e_4) &= e_5 e_1 e_3 e_5 e_4 e_3 = e_5 ((e_1 e_3 e_5 e_1^{-1}) (e_1 e_4) (e_3 e_5)) e_5^{-1} \in N \\
\beta(e_3 e_5) &= e_3 e_5 \in N
\end{align*}
$$

Therefore, the homomorphism $\langle \alpha, \beta \rangle \to \text{Aut}(\hat\Delta)$ induces a homomorphism: $\langle \alpha, \beta \rangle \to \text{Aut}(\hat\Delta)$, and thus it is enough to show that the last map is injective. Now, under this map, $\alpha$ and $\beta$ act on $\langle \alpha, \beta \rangle$ in the following way:

$$
\begin{align*}
\alpha &= \begin{cases}
    \alpha = e_1 N \mapsto \alpha(e_1 N) = \alpha(e_1) N = e_1 N &= \alpha \\
    \beta = e_4 N \mapsto \alpha(e_3 N) = \alpha(e_3) N = e_2 e_4 e_3 e_1 N &= \alpha \beta^{-1} \alpha
\end{cases} \\
\beta &= \begin{cases}
    \alpha = e_1 N \mapsto \beta(e_1 N) = \beta(e_1) N = e_5 e_1 e_3 N &= \beta^{-1} \alpha \beta \\
    \beta = e_4 N \mapsto \beta(e_3 N) = \beta(e_3) N = e_3 N &= \beta
\end{cases}
\end{align*}
$$

Namely, $\alpha$ and $\beta$ act via $\pi$ on $\langle \alpha, \beta \rangle$ by the inner automorphisms $\alpha$ and $\beta$ and hence $\langle \alpha, \beta \rangle$ is mapped isomorphically to $\text{Inn}(\langle \alpha, \beta \rangle)$, yielding that the map $\langle \alpha, \beta \rangle \to \text{Aut}(\hat\Delta)$ is injective and $\langle \alpha, \beta \rangle \to \text{Aut}(\hat F_2)$ is injective as well, as required.

\[\square\]

**Lemma 2.6.** $\text{Inn}(\hat F_2) \cap \text{Im} \rho = \{e\}$, where $\rho : \langle \alpha, \beta \rangle \to \text{Aut}(\hat F_2)$ is the map defined above.

**Proof.** First we observe that $\alpha$ and $\beta$ fix $e_2 = [y, x]$. Thus, by the second part of Proposition 2.2, we have:

$$
\text{Inn}(\hat F_2) \cap \text{Im} \rho \subseteq Z_{\text{Inn}(\hat F_2)}(\text{Inn}([y, x])) = \langle \text{Inn}([y, x]) \rangle = \langle \text{Inn}(e_2) \rangle.
$$
Now, as $e_2 \in \ker \pi$, where $\pi$ is as defined in the proof of Lemma 2.4, the image of $\langle \text{Inn}(e_2) \rangle$ in $\text{Inn}(\langle \alpha, \beta \rangle)$ is trivial. Thus, the image of $\text{Inn}(\hat{F}_2) \cap \text{Im} \rho$ in $\text{Inn}(\langle \alpha, \beta \rangle)$ is trivial, and isomorphic to $\text{Inn}(\hat{F}_2) \cap \text{Im} \rho$ as we saw that $\text{Im} \rho$ is mapped isomorphically to $\text{Inn}(\langle \alpha, \beta \rangle)$. So $\text{Inn}(\hat{F}_2) \cap \text{Im} \rho$ is trivial. \hfill \blacksquare

This finishes the proof of Theorem 1.1. In [BER], the authors give an explicit construction of a congruence subgroup which is contained in a given finite index subgroup of $\text{Aut}(\hat{F}_2)$. They prove the following theorem:

**Theorem 2.7.** (cf. [BER], Theorem 5.1) Let $H$ be a finite index normal subgroup of $G = \text{Aut}(\hat{F}_2)$ such that $\text{Inn}(\hat{F}_2) \leq H \leq \text{Aut}'(\hat{F}_2)$ and let $n = [\text{Aut}'(\hat{F}_2) : H]$. Pick two distinct odd primes $p, q \nmid n$, and set $m = n \cdot p^{n+1}$. Then, there exists an explicitly constructed normal subgroup $M \leq F_2$ of index dividing $144 \cdot m^4 \cdot q^{36 \cdot m^4 + 1}$ such that $G(M) \leq H$, when for a general normal subgroup $M \triangleleft F_2$ we define:

$$G(M) = \{ \sigma \in G \mid \sigma(M) = M, \sigma \text{ acts trivially on } F_2/M \}$$

We end this section with a much simpler explicit construction of a congruence subgroup and with a better bound for the index of $M$. But before, let us recall the ”discrete version” of Proposition 2.2 from [BER]:

**Proposition 2.8.** (cf. [BER], Propositions 2.2. and 2.6.) Let $F$ be the free group on the set $X$, $|X| \geq 2$, and let $F/N$ be a finite quotient of $F$. Pick a prime $p$ not dividing the order of $F/N$ and set $M = N^p N'$. Then:

1. The image of every normal abelian subgroup of $F/M$ through the natural projection $F/M \to F/N$, is trivial.
2. If $N \subseteq F_p F_2$, $x,y \in X$, $x \neq y$, then the image of the centralizer $Z_{F/M}([y,x] \cdot M)$ through the natural projection $F/M \to F/N$, is $|[y,x] \cdot N|$. 

**Theorem 2.9.** Let $H$ be a finite index normal subgroup of $G = \text{Aut}(\hat{F}_2)$ such that $\text{Inn}(\hat{F}_2) \leq H \leq \text{Aut}'(\hat{F}_2) = \text{Inn}(\hat{F}_2) \rtimes \langle \alpha, \beta \rangle$ and let $n = [\text{Aut}'(\hat{F}_2) : H]$. Then for every prime $p \nmid 6n$, there exists an explicitly constructed normal subgroup $M \leq F_2$ of index dividing $144 \cdot n^4 \cdot p^{36 \cdot n^4 + 1}$ such that $G(M) \leq H$.

**Proof.** Recall the map $\pi : F_2 \supseteq \Delta \to \langle \alpha, \beta \rangle$ from the proof of Lemma 2.4, and let $t_1 = 1$, $t_2 = x$, $t_3 = y$, $t_4 = xy$ be the system of representatives of right cosets of $\Delta$ in $F_2$. Denote also $K = H \cap \langle \alpha, \beta \rangle$ and define:

$$N = F_2^6 F_2^6 \cap \bigcap_{g \in F_2} g^{-1} \pi^{-1} (K) g F_2^6 F_2^6 \cap \prod_{i=1}^4 t_i^{-1} \pi^{-1} (K) t_i$$

$$M = F_2^6 F_2^6 \cap N^p N'$$

Then $\pi^{-1}(K)$ is a subgroup of index $n$ in $\Delta$ and $\bigcap_{i=1}^4 t_i^{-1} \pi^{-1}(K) t_i$ is a normal subgroup of $F_2$ of index dividing $n^4$ in $\Delta$, and of index dividing $4n^4$ in $F_2$. So as $F_2^6 F_2^6$ is of index 9 in $\Delta$, $N$ is a normal subgroup of index dividing $36 \cdot n^4$ in $F_2$. 

7
Thus, by the Schreier formula, the index of $N'N^p$ in $F_2$ divides $36 \cdot n^4 \cdot p^{36 \cdot n^4 + 1}$ and the index of $M$ in $F_2$ is dividing $4 \cdot 36 \cdot n^4 \cdot p^{36 \cdot n^4 + 1}$. So it remains to show that $G(M) \leq H$.

Let $\sigma \in G(M)$. As $M \leq F_2^2 F_2^2$ we have:
\[
G(M) \leq \ker(G \to \text{Aut}(F_2/\langle F_2^2 F_2^2 \rangle)) \leq \text{Aut}^f(F_2) \cong \langle \alpha, \beta \rangle
\]
and therefore we can write $\sigma = \text{Inn}(f) \cdot \delta$ for some $f \in F_2$ and $\delta \in \langle \alpha, \beta \rangle$. By assumption, $\sigma$ acts trivially on $F_2/M$ and thus $\delta$ acts on $F_2/M$ as $\text{Inn}(f^{-1})$. Now, as $\alpha$ and $\beta$ fix $[y, x]$, we deduce that so does $\delta$. Thus, $f \cdot M \in Z_{F_2/M}([y, x] \cdot M)$, and by Proposition 2.3, $f \cdot N \in ([y, x] \cdot N)$. Hence, $\delta$ acts on the group $F_2/M$ as $\text{Inn}([y, x]^n \cdot n)$ for some $r \in \mathbb{Z}$ and $n \in N$. Therefore, $\delta$ acts on $\Delta/M$ as $\text{Inn}(e_2 \cdot n)$ for some $r \in \mathbb{Z}$ and $n \in N$. So, $\delta$ acts on $\pi(\Delta)/\pi(M) = \Delta/(M \cdot \ker \pi)$ as $\text{Inn}(\pi(e_2 \cdot n))$ for some $r \in \mathbb{Z}$ and $n \in N$. But $e_2 \in \ker \pi$, so $\delta$ acts on $\pi(\Delta)/\pi(M)$ as $\text{Inn}(\pi(n))$ for some $n \in N$. Now, by the definition of $N$, $\pi(N) \subseteq K$ and also $\pi(M) \subseteq K'K^p$, so $\delta$ acts on $\pi(\Delta)/K'K^p$ as $\text{Inn}(k)$ for some $k \in K$. Moreover, by the definition of $\pi$ we have $\pi(\Delta) = \langle \alpha, \beta \rangle$ and by the computations we made in the proof of Lemma 2.3, $\delta$ acts on $\langle \alpha, \beta \rangle$ as $\text{Inn}(\delta)$. Thus, there exists some $k \in K$ such that $\text{Inn}(\delta) \cdot \text{Inn}(k)^{-1}$ acts trivially on $\langle \alpha, \beta \rangle/K'K^p$, i.e. $\delta \cdot k^{-1} \in Z(\langle \alpha, \beta \rangle/K'K^p)$. Now, by the first part of Proposition 2.3, as $Z(\langle \alpha, \beta \rangle/K'K^p)$ is an abelian normal subgroup of $\langle \alpha, \beta \rangle/K'K^p$, it is mapped trivially to $\langle \alpha, \beta \rangle/K$. I.e. $\delta \cdot k^{-1} \in K$, so also $\delta \in K \subseteq H$. Thus, $\sigma = \text{Inn}(f) \cdot \delta \in H$, as required.

\section{The CSP for $\Phi_2$}

In this section we will prove Theorem 1.2 and will show that the congruence kernel of the free metabelian group on two generators is the free profinite group on a countable number of generators.

Before we start, let us observe that for a group $\Gamma$, one can also ask a parallel congruence subgroup problem for $G = \text{Out}(\Gamma)$. I.e. one can ask whether every finite index subgroup of $G$ contains a principal congruence subgroup of the form:

$$G(M) = \ker(G \to \text{Out}(\Gamma/M))$$

for some finite index characteristic subgroup $M \leq \Gamma$. When $\Gamma$ is finitely generated, this is equivalent to the question whether the congruence map $\hat{G} \to \text{Out}(\hat{\Gamma})$ is injective. Moreover, it is easy to see that Lemma 2.3 has a parallel version for $G$, namely, if $H \leq G$ is a congruence subgroup of $G$, then:

$$\ker(\hat{G} \to \text{Out}(\hat{\Gamma})) = \ker(\hat{H} \to \text{Out}(\hat{\Gamma}))$$

We start now with the next proposition which is slightly more general than Lemma 3.1 in [BER]. Nevertheless, it is proven by the same arguments:

\textbf{Proposition 3.1.} (cf. [BER], Lemma 3.1.) Let $\Gamma$ be a finitely generated residually finite group such that $\hat{\Gamma}$ has a trivial center. Considering the congruence
map $\widehat{Out}(\Gamma) \to Out(\hat{\Gamma})$, we have:

$$C(\Gamma) = \ker(\widehat{Aut}(\Gamma) \to Aut(\hat{\Gamma})) \cong \ker(\widehat{Out}(\Gamma) \to Out(\hat{\Gamma}))$$

It is well known that $\Phi_2$ is a residually finite group (cf. [Be1], Theorem 2.11). It is also proven there that $Z(\hat{\Phi}_2)$ is trivial (proposition 2.10). So by the above proposition:

$$C(\Phi_2) = \ker(\widehat{Aut}(\Phi_2) \to Aut(\hat{\Phi}_2)) \cong \ker(\widehat{Out}(\Phi_2) \to Out(\hat{\Phi}_2))$$

In addition, it is an old result by Bachmuth [Ba] that the kernel of the surjective map:

$$\ker(\widehat{Out}(\Phi_2) \to Out(\Phi_2/\Phi'_2) = Aut(\mathbb{Z}^2) = GL_2(\mathbb{Z}) = Inn(\Phi_2)$$

i.e., $Out(\Phi_2) \cong GL_2(\mathbb{Z})$. Now, the free group $\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \rangle$ is a congruence subgroup of $Out(\Phi_2)$ as it contains:

$$\ker(Out(\Phi_2) \to Out(\Phi_2/\Phi_4^2)) = ker(GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}/4\mathbb{Z})) .$$

So by the appropriate version of Lemma 2.3 and by Proposition 3.1, we obtain that:

$$C(\Phi_2) = \ker(\widehat{Out}(\Phi_2) \to Out(\hat{\Phi}_2))$$
$$= \ker(GL_2(\mathbb{Z}) \to Out(\hat{\Phi}_2))$$
$$= \ker(\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \rangle \to Out(\hat{\Phi}_2))$$

Now, as $\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \rangle$ is a free group, we can also state that:

$$C(\Phi_2) = \ker(\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \rangle \to Out(\hat{\Phi}_2)) \cong \ker(\langle \alpha, \beta \rangle \to Aut(F_2) \to Aut(\hat{\Phi}_2) \to Out(\hat{\Phi}_2))$$

where $\alpha$ and $\beta$ are the automorphisms of $F_2$ that we defined in the previous section, which are the preimages of $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ under the map $Aut(F_2) \to GL_2(\mathbb{Z})$, respectively. So all we need to show is that:

**Lemma 3.2.** $C(\Phi_2) = \ker(\langle \alpha, \beta \rangle \to Out(\hat{\Phi}_2)) = \hat{F}_\omega$. 

9
Proof. As the free group \( \langle \left( \begin{array}{cc} 1 & 2 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 2 & 1 \end{array} \right) \rangle \) is a congruence subgroup of the group \( \text{Aut} (\mathbb{Z}^2) = \text{Out} (\mathbb{Z}^2) = \text{GL}_2 (\mathbb{Z}) \), we have:

\[
C (\mathbb{Z}^2) = \ker \left( \left( \begin{array}{cc} 1 & 2 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 2 & 1 \end{array} \right) \right) \to \text{Out}(\mathbb{Z}^2) \)
\]

\[
= \ker((\alpha, \beta) \to \text{Out}(\hat{\mathbb{Z}}^2))
\]

Thus, if we denote: \( C = \ker((\alpha, \beta) \to \text{Aut}(\hat{\Phi}_2)) \), then using equation (3.1), we have:

\[
C (\mathbb{Z}^2) = \ker((\alpha, \beta) \to \text{Out}(\hat{\Phi}_2) \to \text{Out}(\hat{\Phi}_2) \to \text{Out}(\hat{\mathbb{Z}}^2) = \text{Aut}(\hat{\mathbb{Z}}^2))
\]

Let us now make the following observation: if \( \sigma, \tau \) are two automorphisms of a group \( \Gamma \) which act trivially on \( \Gamma / M \) and on \( M \), where \( M \trianglelefteq \Gamma \) is abelian, then \( \sigma \) and \( \tau \) commute. Indeed, if \( g \in \Gamma \), then \( \sigma (g) = g \cdot m = g \cdot n = \sigma (g \cdot n) = \sigma (\tau (g)) \).

The conclusion from the above observation and from the previous discussion is that \( C (\mathbb{Z}^2) / C \) is abelian, and thus, \( C (\mathbb{Z}^2) / C (\Phi_2) \) is also abelian. Finally, \( C (\mathbb{Z}^2) \) is known to be isomorphic to \( \check{F}_\omega \) (McI, LuII). Moreover, by Proposition 1.10 and Corollary 3.9 of [LV] every normal closed subgroup \( N \) of \( \check{F}_\omega \) such that \( \check{F}_\omega / N \) is abelian, is also isomorphic to \( \check{F}_\omega \). Thus, \( C (\Phi_2) \cong \check{F}_\omega \) as well, as required.

Remark 3.3. Our proof of Theorem 1.2 is much shorter than the one given in [Be1], but the latter gives more information. We show here that \( C (\mathbb{Z}^2) / C (\Phi_2) \) is abelian, while from [Be1] one can deduce that, in fact, \( C (\Phi_2) = C (\mathbb{Z}^2) \). See [E] for more.

4 The CSP for \( \Phi_3 \)

In this section we will prove Theorem 1.3 which claims that \( C (\Phi_3) \) contains a copy of \( \check{F}_\omega \). Let us start by showing that \( \text{Aut} (\Phi_3) \) is large:

**Proposition 4.1.** The group \( \text{Aut} (\Phi_3) \) is large, i.e. it has a finite index subgroup that can be mapped onto a non-abelian free group.
Proof. The proof will follow the method developed in [GL] to produce arithmetic quotients of $\text{Aut}(F_3)$. Denote the free group on 3 generators by $F_3 = \langle x, y, z \rangle$, and the cyclic group of order 2 by $C_2 = \{1, g\}$. Define the map $\pi : F_3 \to C_2$ by: $\pi = \begin{cases} x & \mapsto g \\ y, z & \mapsto 1 \end{cases}$, and its kernel by $R = \ker \pi$. Then, using the right transversal $T = \{1, x\}$, we deduce by Theorem 2.3 that $R$ is freely generated by: $x^2$, $y$, $xyx^{-1}$, $z$, $zxz^{-1}$. Thus, $\overline{R} = R/R' = \mathbb{Z}^2$ is generated as a free abelian group by the images:

$$v_1 = x^2, \ v_2 = y, \ v_3 = xyx^{-1}, \ v_4 = z, \ v_5 = zxz^{-1}$$

Now, the action of $F_3$ on $R$ by conjugation induces an action of $F_3/R = C_2 = \{1, g\}$ on $\overline{R} = R/R'$, sending:

$$g \mapsto \begin{cases} v_1 = x^2 & \mapsto x^2 \\ v_2 = y & \mapsto x^{-2}(xyx^{-1})x^2 = xyx^{-1} \\ v_3 = xyx^{-1} & \mapsto y \\ v_4 = z & \mapsto x^{-2}(zxz^{-1})x^2 = zxz^{-1} \\ v_5 = zxz^{-1} & \mapsto z \end{cases} = B$$

The above matrix has two eigenvalues $\lambda = \pm 1$ and the eigenspaces are:

$$V_1 = \text{Sp} \{v_1, v_2 + v_3, v_4 + v_5\}$$
$$V_{-1} = \text{Sp} \{v_2 - v_3, v_4 - v_5\}$$

Recall, $\Phi_3 = F_3/F_3''$, and as $F_3/R$ is abelian, $F_3/R'$ is metabelian. Thus, we have a surjective homomorphism: $\Phi_3 \to F_3/R'$. Denote now: $S = R/F_3''$, so we can identify: $F_3/R \cong \Phi_3/S$, $F_3/R' \cong \Phi_3/S'$ and $\overline{R} = R/R' \cong S/S' = \overline{S}$. So as before, $\Phi_3/S = C_2$ acts on $\overline{S}$ by the matrix $B$.

Denote also $G(S) = \{\sigma \in \text{Aut}(\Phi_3) \mid \sigma(S) = S\}$. It is clear that $G(S)$ is of finite index in $\text{Aut}(\Phi_3)$ with a natural map: $G(S) \to \text{Aut}(S)$ which induces a map: $\rho : G(S) \to \text{Aut}(\overline{S}) = \text{GL}_5(\mathbb{Z})$. We claim now that if $\sigma \in G(S)$ then $\rho(\sigma)$ commutes with $B$. First observe that there exists some $s \in S$ such that $\sigma(x) = sx$ (x now plays the role of the image of x under the map $F_3 \to \Phi_3$). Now, let $t \in S$, and remember that the action of $B$ on $\overline{S}$ is induced by the action of $x$ on $S$ by conjugation. So:

$$\sigma(x^{-1}tx) = \sigma(x)^{-1}\sigma(t)\sigma(x) = x^{-1}s^{-1}\sigma(t)sx = (x^{-1}sx)^{-1}(x^{-1}\sigma(t)x)(x^{-1}sx)$$

and hence:

$$\left(\rho(\sigma) \cdot B\right)(\overline{t}) = \frac{\sigma(x^{-1}tx)}{(x^{-1}sx)^{-1}(x^{-1}\sigma(t)x)(x^{-1}sx)} = \frac{x^{-1}\sigma(t)x}{x^{-1}s \sigma(t)x} = (B \circ \rho(\sigma))(\overline{t})$$

11
Therefore, $\rho(G(S))$ commutes with $B$. It follows that the eigenspaces of $B$ are invariant under the action of $G(S)$. In particular, we deduce that $V_{-1}$ is invariant under the action of $\rho(G(S))$. Thus, we obtain a homomorphism $\nu: G(S) \to Aut(V_{-1} \cap S) = GL_2(\mathbb{Z})$.

Consider now the following automorphisms of $Aut(\Phi_3)$ ($x, y, z$ play the role of the images of $x, y, z$ under $F_3 \to \Phi_3$):

$$\alpha = \begin{cases} x & \mapsto x \\ y & \mapsto y \\ z & \mapsto zy \end{cases} \quad \beta = \begin{cases} x & \mapsto x \\ y & \mapsto yz \\ z & \mapsto z \end{cases}$$

So $\alpha, \beta \in G(S)$ act on $V_{-1} = Sp\left\{u_1 = v_2 - v_3, u_2 = v_4 - v_5\right\}$ in the following way:

$$\alpha(u_1) = \alpha\left(\bar{y} - xyx^{-1}\right) = \bar{y} - xyx^{-1} = u_1$$
$$\alpha(u_2) = \alpha\left(\bar{z} - xzx^{-1}\right) = \bar{z} + \bar{y} - xzx^{-1} - xyx^{-1} = u_2 + u_1$$

$$\beta(u_1) = \beta\left(\bar{y} - xyx^{-1}\right) = \bar{y} + \bar{z} - xyx^{-1} - xzx^{-1} = u_1 + u_2$$
$$\beta(u_2) = \beta\left(\bar{z} - xzx^{-1}\right) = \bar{z} - xzx^{-1} = u_2$$

Therefore, under the map $\nu: G(S) \to GL_2(\mathbb{Z})$ we have: $\alpha \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\beta \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Thus, the image of $G(S)$ contains $\left< \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right>$, which is free and of finite index in $GL_2(\mathbb{Z})$. Finally, if we denote the preimage $H = \nu^{-1}\left(\left< \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right>\right)$, then $H$ is a finite index subgroup of $Aut(\Phi_3)$ that can be mapped onto a free group, as required.

Let us now continue with the following definition:

**Definition 4.2.** We say that a group $P$ is involved in a group $Q$, if it isomorphic to a quotient group of some subgroup of $Q$.

It is not difficult to see that if a finite group $P$ is involved in a profinite group $Q$, than it is involved in a finite quotient of $Q$. Now, we showed that $Aut(\Phi_3)$ has a finite index subgroup $H$ which can be mapped onto $F_2$. Thus we have a map: $\tilde{H} \to \tilde{F}_2$, but as $\tilde{F}_2$ is free, the map splits, and thus $\tilde{H}$ and hence $Aut(\Phi_3)$, contains a copy of $\tilde{F}_2$. Thus, any finite group is involved in $Aut(\Phi_3)$. On the other hand, we claim:

**Proposition 4.3.** Let $P$ be a non-abelian finite simple group which is involved in $Aut(\Phi_3)$. Then, for some prime $p$ and some $d \in \mathbb{N}$, $P$ is involved in $SL_3(p^d)$, the special linear group over the field of order $p^d$.
Proof. Let $F_n$ be the free group on $x_1, \ldots, x_n$. Then there is a natural injective homomorphism from $F_n$ into the matrix group:

$$\left\{ \begin{pmatrix} g & 0 \\ t & 1 \end{pmatrix} \mid g \in F_n, \ t \in \sum_{i=1}^n \mathbb{Z}[F_n] t_i \right\}$$

defined by the map:

$$x_i \mapsto \begin{pmatrix} x_i & 0 \\ t_i & 1 \end{pmatrix}, \quad 1 \leq i \leq n$$

where $t_i$ is a free basis for a right $\mathbb{Z}[F_n]$-module. This is called the Magnus embedding. Usually, its properties are studied by Fox’s free differential calculus, but we will not need it here explicitly (cf. [Bi], [RS], [Ma]).

One can prove, by induction on its length, that for a word $w \in F_n$, under the Magnus embedding, $w \mapsto \begin{pmatrix} w_0 & 0 \\ \sum_{i=1}^n w_i t_i & 1 \end{pmatrix}$ where:

$$w - 1 = \sum_{i=1}^n (x_i - 1) w_i. \quad (4.1)$$

The identity $(4.1)$ shows that the polynomials $w_i$ determine the word $w$ uniquely.

Thus, we have an injective map (which is not homomorphism) $J : \text{End}(F_n) \to M_n(\mathbb{Z}[F_n])$ defined by:

$$\alpha \mapsto \begin{pmatrix} \alpha(x_1)_1 & \cdots & \alpha(x_n)_1 \\ \vdots & \ddots & \vdots \\ \alpha(x_1)_n & \cdots & \alpha(x_n)_n \end{pmatrix}$$

It is not difficult to check, using the identity $(4.1)$, that the above map satisfies:

$$J(\alpha \circ \beta) = J(\alpha) \cdot J(\beta)$$

where by $\alpha (J(\beta))$ we mean that $\alpha$ acts on every entry of $J(\beta)$ separately.

Now, for $m \in \mathbb{N}$, denote: $K_{n,m} = F_m F_n'$ and $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. Then, the natural maps $F_n \to F_n/K_{n,m} = \mathbb{Z}_m^n$ and $\mathbb{Z} \to \mathbb{Z}_m$ induce a map:

$$\pi_{n,m} : F_n \to \left\{ \begin{pmatrix} g & 0 \\ t & 1 \end{pmatrix} \mid g \in F_n, \ t \in \sum_{i=1}^n \mathbb{Z}[F_n] t_i \right\}$$

$$\to \left\{ \begin{pmatrix} g & 0 \\ t & 1 \end{pmatrix} \mid g \in \mathbb{Z}_m^n, \ t \in \sum_{i=1}^n \mathbb{Z}_m [\mathbb{Z}_m^n] t_i \right\}.$$

It is shown in ([Be1], Proposition 2.6) that $\ker(\pi_{n,m}) = K_{n,m} K_{n,m}'$ and hence $\Phi_{n,m} := \text{Im}(\pi_{n,m}) \cong F_n/K_{n,m} K_{n,m}'$. Moreover, it is proven there (Proposition 2.7) that we have the following equality:

$$\hat{\Phi}_n = \lim_{\leftarrow m} \Phi_{n,m}.$$
Observe now that for every $m_2| m_1$, $\ker (\Phi_{n,m_1} \to \Phi_{n,m_2})$ is characteristic in $\Phi_{n,m_1}$, and for every $m$, $\ker (\Phi_n \to \Phi_{n,m})$ is characteristic in $\Phi_n$. Thus:

$$Aut(\hat{\Phi}_n) = Aut\left(\lim_{\longleftarrow m} \Phi_{n,m}\right) = \lim_{\longleftarrow m} Aut \left(\Phi_{n,m}\right).$$

Now, observe that the identity (4.1) is also valid for the entries of the elements of $\Phi_{n,m}$, and thus, every element of $\Phi_{n,m}$ is determined by its left lower coordinate. Therefore, as every automorphism of $\Phi_{n,m}$ can be lifted to an endomorphism of $F_n$, we have an injective map (which is not a homomorphism) $J_m : Aut (\Phi_{n,m}) \to M_n (\mathbb{Z}_m [\mathbb{Z}_m^n])$ which satisfies the identity:

$$J_m (\alpha \circ \beta) = J_m (\alpha) \cdot (J_m (\beta))$$

where the action of $\alpha$ on $\mathbb{Z}_m [\mathbb{Z}_m^n] = \mathbb{Z}_m [F_n/K_{n,m}]$ is through the natural projection $\Phi_{n,m} \cong F_n/K_{n,m} \to F_n/K_{n,m} \cong \mathbb{Z}_m^n$.

We denote now $KA (\Phi_{n,m}) = \ker (Aut (\Phi_{n,m}) \to Aut (\Phi_{n,m}/K_{n,m}))$. Observe, that as $KA (\Phi_{n,m})$ acts trivially on $\Phi_{n,m}/K_{n,m} = \mathbb{Z}_m^n$, the map $J_m$ gives us a homomorphism, which is also injective, as mentioned above:

$$J_m : KA (\Phi_{n,m}) \to GL_n (\mathbb{Z}_m [\mathbb{Z}_m^n])$$

Now, if $P$ is a non-abelian simple group which is involved in $Aut (\hat{\Phi}_3)$, then it must be involved in $Aut (\hat{\Phi}_{3,m})$ for some $m$. Thus, it must be involved either in $Aut (\Phi_{3,m}/K_{3,m}) = GL_3 (\mathbb{Z}_m)$ or in $KA (\Phi_{3,m}) \leq GL_3 (\mathbb{Z}_m [\mathbb{Z}_m^3])$. So it must be involved in $GL_3 (R)$ for some finite commutative ring $R$. As every finite commutative ring is artinian, it can be decomposed as:

$$R = R_1 \times \ldots \times R_l$$

for some local finite rings $R_1, \ldots, R_l$, so:

$$GL_3 (R) = GL_3 (R_1) \times \ldots \times GL_3 (R_l)$$

and thus $P$ must be involved in $GL_3 (R)$ for some local finite commutative ring $R$. Denote the unique maximal ideal of $R$ by $M_\prime \subset R$. As $R$ is a finite local Noetherian ring, it is well known that $M_\prime = 0$ for some $r \in \mathbb{N}$.

Note now that if $S,T \subset R$ for some commutative ring $R$, and

$$I + A \in \ker (GL_3 (R) \to GL_3 (R/S))$$

$$I + B \in \ker (GL_3 (R) \to GL_3 (R/T))$$

when $I$ denotes the identity element in $GL_3 (R)$, then

$$[I + A, I + B] \in \ker (GL_3 (R) \to GL_3 (R/ST)).$$

Indeed, if $I + C = (I + A)^{-1}$ and $I + D = (I + B)^{-1}$ then, as $AB = CD = AD = CB = 0 \mod (ST)$ we have:

$$[I + A, I + B] = (I + A)(I + B)(I + C)(I + D) =$$

$$= I + AC + A + BD + B + C + D \mod (ST)$$

$$= I + (I + A)(I + C) - I + (I + B)(I + D) - I = I \mod (ST)$$
With the above observation we deduce that for every \( k \geq 1 \), the kernel of the map \( GL_3(R/M^{k+1}) \to GL_3(R/M^k) \) is abelian. So, \( P \) must be involved in \( GL_3(R/M) = GL_3(p^d) \) for some prime \( p \) and \( d \in \mathbb{N} \). Finally, using the fact that \( GL_3(p^d) / SL_3(p^d) \) is abelian, we obtain that \( P \) is involved in \( SL_3(p^d) \), as required.

**Corollary 4.4.** There exists a finite simple group which is not involved in \( Aut(\hat{\Phi}_3) \).

**Proof.** By the proposition above, it is enough to show that there is a finite simple non-abelian group which is not involved in \( SL_3(p^d) \) for any prime \( p \) and \( d \in \mathbb{N} \). Now, by a theorem of Jordan, there exists an integer-valued function \( J(n) \) such that for every field \( F \), \( \text{char}(F) = 0 \), any finite subgroup of \( GL_n(F) \) contains a normal abelian subgroup of index at most \( J(n) \). As a corollary of this theorem, Schur proved that the same holds (with the same function) for any finite subgroup \( Q \leq GL_n(F) \) with \( \text{char}(F) = p > 0 \), provided \( p \nmid |Q| \) (cf. [W e] chapter 9). Clearly, the same holds for any group which is involved in such a finite group \( Q \).

We claim that for \( n \) large enough, \( Alt(n) \) is not involved in \( SL_3(p^d) \) for any \( p \) and \( d \). Indeed, fix two different primes \( q_1 \) and \( q_2 \) larger than \( J(3) \). Then, for \( n \) sufficiently large (e.g. \( n > q_1^3 \)) the \( q_i \)-sylow subgroup \( S_i \) of \( Alt(n) \) is non-abelian (since \( Alt(n) \) contains the non-abelian \( q_i \)-group of order \( q_i^3 \)) and every subgroup of \( S_i \) of index \( \leq J(3) \) is equal to \( S_i \), so also non-abelian. If \( Alt(n) \) were involved in \( SL_3(p^d) \) then for at least one of the \( q_i \), \( q_i \neq p \), a contradiction.

**Corollary 4.5.** The congruence kernel \( C(\Phi_3) \) contains a copy of \( \hat{F}_\omega \).

**Proof.** The immediate conclusion of Corollary 4.4 is that \( Aut(\hat{\Phi}_3) \) does not contain a copy of \( \hat{F}_2 \). Thus, the intersection of \( C(\Phi_3) \) and the copy of \( \hat{F}_2 \) in \( Aut(\Phi_3) \) is not trivial. Thus, \( C(\Phi_3) \) contains a non-trivial normal closed subgroup \( N \) of \( \hat{F}_2 \). By Theorem 3.10 in [LV] it contains a copy of \( \hat{F}_\omega \), as required.

**5 Remarks and open problems**

We end this paper with several remarks and open problems. Denote the free solvable group of derived length \( r \) on 2 generators by \( \Phi_{2,r} \). By combining the results of ([BFM], Theorem 1) and ([KLM], Theorem 1.4) we have:

\[
\ker\left( Aut(\Phi_{2,r}) \to Aut(\mathbb{Z}^2) = GL_2(\mathbb{Z}) \right) = Inn(\Phi_{2,r})
\]

for every \( r \), i.e. \( Out(\Phi_{2,r}) = GL_2(\mathbb{Z}) \). So by the same arguments as in [3] we have:

\[
C(\Phi_{2,r}) = \ker\left( \langle \alpha, \beta \rangle \to Out(\hat{\Phi}_{2,r}) \right)
\]

15
As $\text{Out}(\hat{\Phi}_{2,r+1})$ is mapped onto $\text{Out}(\hat{\Phi}_{2,r})$, we obtain the sequence:

$$C(\mathbb{Z}^2) = C(\Phi_{2,1}) \geq C(\Phi_2) = C(\Phi_{2,2}) \geq C(\Phi_{2,3}) \geq \ldots \geq C(\Phi_{2,r}) \geq \ldots \geq C(F_2) = \{e\}$$

and a natural question is whether the inequalities are strict or not. An equivalent reformulation of this question is the following: the cosets of the kernels

$$\ker(GL_2(\mathbb{Z}) = \text{Out}(\Phi_{2,r}) \to \text{Out}(\Phi_{2,r}/K))$$

for characteristic finite index subgroups $K \leq \Phi_{2,r}$ provide a basis for a topology $\mathcal{C}(r)$ on $GL_2(\mathbb{Z})$, called the congruence topology with respect to $\Phi_{2,r}$, which is weaker (equal) than the profinite topology $\mathcal{F}$ of $GL_2(\mathbb{Z})$, and stronger (equal) than the classical congruence topology of $GL_2(\mathbb{Z})$. The latter is equal to $\mathcal{C}(1)$. So, the question above is equivalent to the question whether these topologies are strictly weaker than $\mathcal{F}$, and whether the topology $\mathcal{C}(r)$, for a given $r$, is strictly weaker than $\mathcal{C}(r+1)$.

For example, Theorem 1.1, which states that $C(F_2) = \{e\}$ is equivalent to the statement that the congruence topology which $\text{Out}(F_2)$ induces on $\text{Out}(F_2) = GL_2(\mathbb{Z})$ is equal to the profinite topology of $GL_2(\mathbb{Z})$.

Considering Theorem 1.2 we deduce that $\mathcal{C}(2) \not\subseteq \mathcal{F}$, but with the proof we gave here one can not decide whether $\mathcal{C}(1) = \mathcal{C}(2)$ or $\mathcal{C}(1) \not\subseteq \mathcal{C}(2)$. Equivalently, we can not decide whether $C(\mathbb{Z}^2) = C(\Phi_2)$ or $C(\mathbb{Z}^2) \not\geq C(\Phi_2)$.

But, in [Be1] it was shown quite surprisingly, that:

**Theorem 5.1.** $\mathcal{C}(1) = \mathcal{C}(2)$, or equivalently $C(\mathbb{Z}^2) = C(\Phi_2)$.

The proof in [Be1] suggested to conjecture that $\mathcal{C}(1) = \mathcal{C}(2) = \mathcal{C}(r)$ for every $r$. But, the explicit construction of a congruence subgroup we gave in [2] gives a counter example:

**Proposition 5.2.** $\mathcal{C}(1) \not\subseteq \mathcal{C}(r)$ for every $r \geq 3$. Equivalently $C(\mathbb{Z}^2) \not\geq C(\Phi_{2,r})$ for every $r \geq 3$.

**Proof.** Denote $G = \left\langle \left( \begin{array}{cc} 1 & 2 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 2 & 1 \end{array} \right) \right\rangle \leq GL_2(\mathbb{Z})$. Then by a theorem of Reiner [Re], for every $p \neq 2$, $G^p$ is not a congruence subgroup of $GL_2(\mathbb{Z})$ in the classical manner, i.e. $G^p \not\in \mathcal{C}(1)$. On the other hand, applying the explicit construction given in Theorem 2.9 we obtain a finite index normal subgroup $M \triangleleft F_2$ such that $F_2/M$ is of solvability length 3 such that

$$\ker(\text{Out}(F_2) = GL_2(\mathbb{Z}) \to \text{Out}(F_2/M)) \leq G^p.$$ 

This shows that $G^p$ is a congruence subgroup of $GL_2(\mathbb{Z})$ with respect to the congruence topology induced by $\text{Out}(\Phi_{2,3})$. Equivalently, $\mathcal{C}(1) \not\subseteq \mathcal{C}(3)$ or $C(\mathbb{Z}^2) \not\geq C(\Phi_{2,3})$, as required.

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1We remark that if one wants $M$ to be characteristic, all we need to do, is to replace $M$ by $\bigcap_{\sigma \in \text{Aut}(F_2)} \sigma(M)$, and this procedure does not change the solvability length of $F_2/M$. 

16
The proposition suggests the following conjecture:

**Conjecture 5.3.** \( C(\Phi_{2,r}) \not\supseteq C(\Phi_{2,r+1}) \) for every \( r \geq 2 \), or equivalently \( \mathcal{C}(r) \subsetneq \mathcal{C}(r+1) \). In particular, \( C(\Phi_{2,r}) \neq \{e\} = C(\Phi_2) \) and \( \mathcal{C}(r) \neq \mathcal{F} \) for every \( r \).

We should remark that we do not even know to decide whether \( C(\Phi_{2,r}) \neq \{e\} \) for \( r \geq 3 \), i.e. we do not know if the congruence subgroup property holds for \( \Phi_{2,r} \) for \( r \geq 3 \) or not. Note that our proofs of Theorems 1.2 and 1.3 claiming that \( \Phi = \Phi_2 = \Phi_2,2 \) and \( \Phi = \Phi_3 \) do not satisfy the CSP were based on two facts:

1. \( Aut(\Phi) \) is large, and hence every finite group is involved in \( \hat{Aut}(\Phi) \), and
2. not every finite group is involved in \( Aut(\hat{\Phi}) \).

Now, for \( \Phi = \Phi_{d,r} \), the free solvable group on \( d \geq 2 \) generators and solvability length \( r \), part 2 is valid for \( 1 \leq r \leq 2 \) and every \( d \) (with the same proof as for \( d = 3 \) in [4]). But, as \( C(\Phi_{d,1}) = \{e\} \) for every \( d \geq 3 \), and \( C(\Phi_{d,2}) \) is abelian for every \( d \geq 4 \) (cf. [5,2]), part 1 is not valid in these cases. On the other hand, for \( \Phi = \Phi_{2,r} \) or \( \Phi = \Phi_{3,r} \), part 1 is still true for every \( r \geq 2 \) but not part 2. Infact, we have:

**Proposition 5.4.** Let \( \Phi_{d,r} \) be the free solvable group on \( d \geq 2 \) generators and solvability length \( r \). Then if \( r \geq 3 \), then every finite group \( H \) is involved in \( \hat{Aut}(\Phi_{d,r}) \).

**Proof.** By the same arguments of ([Lu2], 5.2), it can be deduced from Gaschütz’s Lemma that for every surjective homomorphism \( \pi : \hat{\Phi}_{d,r} \to \Gamma \) where \( \Gamma \) is finite, the homomorphism

\[
\hat{Aut}(\Phi_{d,r}) \geq \left\{ \sigma \in \hat{Aut}(\Phi_{d,r}) \mid \sigma(\ker \pi) = \ker \pi \right\} \to \hat{Aut}(\Gamma)
\]

is surjective. Thus, for proving our proposition it suffices to show that \( \hat{\Phi}_{d,r} \) has a finite quotient \( \Gamma \) such that \( H \) is involved in \( \hat{Aut}(\Gamma) \). Now, by Cayley’s Theorem, \( H \) is a subgroup of \( \text{Sym}(n-1) \) for some \( n \) and the later is a subgroup of \( SL_n(p) \) for every prime \( p \). Thus, the next lemma due to Robert Guralnick, finishes the proof of the proposition.

**Lemma 5.5.** For every \( n \geq 2 \), there exists a prime \( p \) and a finite group \( \Gamma \) generated by two elements and of solvability length three, such that \( SL_n(p) \) is involved in \( \hat{Aut}(\Gamma) \).

**Proof.** Fix a prime \( r \) such that \( r > n + 1 \). Using Dirichlet’s Theorem, pick a prime \( p \) such that \( r \) divides \( p - 1 \). Consider now the general affine group

\[
\Delta = AGL_1(r) = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \mid a \in F_r^*, b \in F_r \right\} = F_r \rtimes F_r^*.
\]
Then $\Delta$ is of order $r(r-1)$. In addition, as $r \mid (p-1)$, $F_p$ contains the unit roots of order $r$, fix one of them $\xi \neq 1$, and consider the diagonal matrix

$$D = \begin{pmatrix} \xi & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \xi^{r-1} \end{pmatrix} \in GL_{r-1}(p).$$

Now, we can embed $\Delta$ in $GL_{r-1}(p)$ by sending an element $b \in \{0, \ldots, r-1\} = F_r$ to the diagonal matrix $D^b$ (giving rise to a subgroup $N = \{D^b \mid b \in F_r\}$) and an element $a \in F_r^*$ to the permutation matrix which normalizes $N$, sending $D^b$ to $D^{ba}$. So $\Delta$ has a module $V$ of dimension $r-1$ over $F_p$. Now, every $\Delta$-submodule of $V$ is also $N$-submodule. The $N$-submodules are direct sums of different one dimensional $N$-modules, the eigen-spaces of $D^1$, on which $F_r^*$ acts transitively. We deduce that $V$ is an irreducible module.

Denote now $W = \oplus_{l=1}^{r-2} V$ and using the obvious action of $\Delta$ on $W$, define: $\Gamma = W \rtimes \Delta$. We claim that $\Gamma$ is generated by two elements. By the description above, it is clear why $\Delta$ is generated by two element, one of them is $D \in F_r$, and we denote the other one by $S \in F_r^*$. Let us now define

$$D' = ((\vec{e}_1, \ldots, \vec{e}_{r-1}), D), S' = ((\vec{0}, \ldots, \vec{0}), S) \in W \rtimes \Delta$$

where $\{\vec{e}_1, \ldots, \vec{e}_{r-1}\}$ is the standard basis of $V$. For a $1 \leq j \leq r-1$ denote $\eta = \xi^j$. Note, that for every $1 \leq k \leq r-2$, $1 + \eta + \ldots + \eta^k = \frac{1-\xi^{r+1}}{1-\xi} \neq 0$.

It follows that $D'^k = ((\alpha_1 \vec{e}_1, \ldots, \alpha_{r-2} \vec{e}_{r-2}), D^k)$ where $0 \neq \alpha_i \in F_p$ for every $1 \leq k \leq r-2$. Now, there is a power $S'^l$ of $S$, $1 \leq l \leq r-2$, which sends $\vec{e}_{r-1}$ to $\vec{e}_1$. We have also $S'^l DS'^{-l} = D'^{-k}$ for some $1 \leq k \leq r-2$. Thus, for some $0 \neq \alpha_i \in F_p$, we can write:

$$w = S'^l D' S'^{-l} D^k$$

$$= ((\vec{0}, \ldots, \vec{0}), S')((\vec{e}_1, \ldots, \vec{e}_{r-2}), D)((\vec{0}, \ldots, \vec{0}), S'^{-l})((\alpha_1 \vec{e}_1, \ldots, \alpha_{r-2} \vec{e}_{r-2}), D^k)$$

$$= ((S'((\vec{e}_1, \ldots, S'^l(\vec{e}_{r-2})), S'^l DS'^{-l})((\alpha_1 \vec{e}_1, \ldots, \alpha_{r-2} \vec{e}_{r-2}), D^k)$$

$$= (S'((\vec{e}_1, + D'^{-k}(\alpha_1 \vec{e}_1), \ldots, S'^l(\vec{e}_{r-2}) + D'^{-k}(\alpha_{r-2} \vec{e}_{r-2}), I) \in W.$$

Now, as $S'$ sends $\vec{e}_{r-1}$ to $\vec{e}_1$, $\vec{e}_1$ does not appear in any entry of $w$ except the first one.

Observe now, that the diagonals of $D^0, \ldots, D^{r-2}$, considered as column vectors of $V = F_r^{r-1}$, form a basis for $V$ as the matrix:

$$\begin{pmatrix} 1 & \xi & \cdots & \xi^{r-2} \\ 1 & \xi^2 & \cdots & \xi^{2(r-2)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \xi^{r-1} & \cdots & \xi^{(r-1)(r-2)} \end{pmatrix}$$

is a Vandermonde matrix, and therefore invertible. Thus, there is a linear
combination

\[ C = \beta_0 D^0 + \ldots + \beta_{r-2} D^{r-2} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \beta_i \in \mathbb{F}_p. \]

Now, observe that \( D' \) acts on \( W \) by conjugation via the action of \( D \) on \( V \). Thus, we obtain an action of \( C \) on \( W \) via its action on \( V \), in which \( C(w) \) has \( \vec{0} \) in every entry except the first one. This shows, as \( V \) is irreducible, that the first copy of \( V \) in \( W \) is inside the group generated by \( D' \) and \( S' \). In a similar way, all the \( r-2 \) copies of \( V \) are generated by \( D' \) and \( S' \), so \( \Gamma \) is generated by two elements.

Now, \( \Delta \times SL_{r-2}(p) \) acts on \( W = \oplus_{i=1}^{r-2} V = V \otimes \mathbb{F}_p^{r-2} \) in a obvious way. Thus \( \Gamma = W \times \Delta \) is normal in \( W \times (\Delta \times SL_{r-2}(p)) \), so \( SL_{r-2}(p) \) is involved in \( Aut(\Gamma) \).

Let us remark that while we do not know the answer to the congruence subgroup problem for free solvable groups on two generators and solvability rank \( r \) (unless \( r = 1 \) or 2), the situation with free nilpotent groups on two generators is easier:

**Proposition 5.6.** For every free nilpotent group on two generators \( \Gamma \), the congruence kernel contains a copy of \( \hat{F}_\omega \) - the free profinite group on countable number of generators.

**Proof.** It is known that if \( \hat{\Gamma} \) is a pro-nilpotent group, then the kernel of the map \( Aut(\Gamma) \to Aut(\hat{\Gamma}/\hat{\Gamma}) \) is pro-nilpotent (cf. [Lu2], 5.3). Thus, if \( \Gamma \) is a free nilpotent group (of arbitrary class) then by the same arguments we brought before, there exists a finite group which is not involved in \( Aut(\hat{\Gamma}) \). On the other hand, if \( \Gamma \) is free nilpotent group on two generators, then \( Aut(\Gamma) \) is large, as it can be mapped onto \( GL_2(\mathbb{Z}) \). Thus, \( \hat{F}_2 \) is a subgroup of \( \hat{Aut}(\hat{\Gamma}) \) and \( C(\Gamma) \cap \hat{F}_2 \) is non-trivial, hence contains a copy of \( \hat{F}_\omega \) (cf. [LV]).

Our last remark is about the CSP for subgroups of automorphism groups. Considering the classical congruence subgroup problem, one can take \( G \) to be a subgroup of \( GL_n(R) \) where \( R \) is a commutative ring, and ask whether every finite index subgroup of \( G \) contains a subgroup of the form \( \ker(G \to GL_n(R/I)) \) for some finite index ideal \( I \triangleleft R \). This direction of generalization of the classical CSP has been studied intensively during the second half of the 20th century (cf. [Rag], [Rap]). One can ask for a parallel generalization for automorphism groups or outer automorphism groups. I.e. let \( G \leq Aut(\Gamma) \) (resp. \( G \leq Out(\Gamma) \)), does every finite index subgroup of \( G \) contain a principal congruence subgroup of the form \( \ker(G \to Aut(\Gamma/M)) \) (resp. \( \ker(G \to Out(\Gamma/M)) \)) for some characteristic finite index subgroup \( M \leq \Gamma \)?

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\( ^2 \)In general, the kernel of the map \( Aut(\Gamma) \to GL_2(\mathbb{Z}) \) strictly contains \( Inn(\Gamma) \) (cf. [BG], [An]).
Now, let $\pi_{g,n}$ be the fundamental group of $S_{g,n}$, the surface of genus $g$ with $n$ punctures, such that $\chi(S_{g,n}) = 2 - 2g - n \leq 0$. Then, there is an injective map of $PMod(S_{g,n})$, the pure mapping class group, into $Out(\pi_{g,n})$ (cf. [FM], chapter 8). Thus, one can ask the CSP for $PMod(S_{g,n})$ as a subgroup of $Out(\pi_{g,n})$. Considering the above problem, it is known that:

**Theorem 5.7.** For $g = 0, 1, 2$ and every $n > 0$, $PMod(S_{g,n})$ has the CSP.

The cases for $g = 0$ were proved by [DDH] and in [Mc], the cases for $g = 1$ were proved by [As], and the cases for $g = 2$ where proved by [Bo]. It can be shown that for every $n > 0$, $\pi_{g,n} \cong F_{2g+n-1}$ = the free group on $2g + n - 1$ generators. Thus, the above cases give an affirmative answer for various subgroups of the outer automorphism group of finitely generated free groups. Though, the CSP for the full $Out(F_d)$ where $d \geq 3$ is still unsettled.

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Institute of Mathematics
The Hebrew University
Jerusalem, ISRAEL 91904

davidel-chai.ben-ezra@mail.huji.ac.il
alex.lubotzky@mail.huji.ac.il