Symbol-Pair Distance of Repeated-Root Constacyclic Codes of Prime Power Lengths over \( \mathbb{F}_{p^m}[u]/\langle u^3 \rangle \)

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Abstract: Let \( p \) be a prime, \( s, m \) be positive integers, \( \gamma \) be a nonzero element of the finite field \( \mathbb{F}_{p^m} \), and let \( R = \mathbb{F}_{p^m}[u]/\langle u^3 \rangle \) be the finite commutative chain ring. In this paper, the symbol-pair distances of all \( \gamma \)-constacyclic codes of length \( p^s \) over \( R \) are completely determined.

Keywords: repeated-root codes; constacyclic codes; symbol-pair distance; finite chain ring

1. Introduction

Initially, in information theory, the message communicated in a noisy channel was divided into information units, which were called individual symbols. The research on the process of writing and reading is often presumed to be performed on individual symbols. With the development of high-density data storage technologies, symbol-pair codes are proposed to protect efficiently against a certain number of pair-errors. In [1,2], Cassuto and Blaum established a new coding framework for channels whose outputs are overlapping pairs of symbols. In 2011, by using algebraic methods, Cassuto and Litsyn [3] constructed cyclic symbol-pair codes. Applying Discrete Fourier Transform for coefficients of codeword polynomials \( c(x) \in \mathbb{F}_q[x]/\langle x^n - 1 \rangle \) and BCH bounds, Cassuto and Litsyn proved that for a cyclic code with dimensions greater than 1 and Hamming distance \( d_H \), the corresponding symbol-pair distance is at least \( d_H + 2 \) [3] [Th. 10]. In particular, Kai et al. [4] extended the result of Cassuto and Litsyn [3] [Th. 10] for the case of simple-root constacyclic codes. Many researchers have scrutinized symbol-pair distances over constacyclic codes since then in [5-9] over many years.

Constacyclic codes are the pivotal and profound part of linear codes. It includes as a subclass the important class of cyclic codes, which form the most important and well studied class of error-correcting codes. This family of codes is thus interesting for both theoretical and practical reasons. Repeated-root constacyclic codes were first initiated in the most generality by Castagnoli in [10] and Van Lint in [11]. They established that the repeated-root constacyclic codes have a sequential structure, which motivated the researchers to further study these codes.

For any \( a \geq 2 \), let \( R \) be the ring \( \mathbb{F}_{p^m}[u]/\langle u^a \rangle = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + \cdots + u^{a-1}\mathbb{F}_{p^m} \) \( (u^a = 0) \). The ring \( R \) has been widely used as alphabets in certain constacyclic codes (see, for instance ([12-14])).

When \( a = 2 \), there is significant literature on constacyclic codes over rings \( \mathbb{F}_{p^m}[u]/\langle u^2 \rangle = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} \) for various prime \( p \) and positive integers \( m \) (see, e.g., [15-23]). In particular, the structure of and symbol-pair distance distribution of all constacyclic codes of length \( p^s \) over \( \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} \) were completely determined in [7,8,17].

When \( a = 3 \), in 2015, the authors of [24] determined the structure of \( (\delta + au^2) \)-constacyclic codes of length \( p^s \) over \( \mathbb{F}_{p^m}[u]/\langle u^3 \rangle = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + u^2\mathbb{F}_{p^m} \). DNA cyclic codes...
over $\mathbb{F}_2 + u\mathbb{F}_2 + u^2\mathbb{F}_2$ were studied in [25]. In [26], Laaouine et al. obtained the structure of all $\gamma$-constacyclic codes of length $p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m} + u^2\mathbb{F}_{p^m}$ by classifying them into eight types, where $\gamma$ is a nonzero element of $\mathbb{F}_{p^m}$. The Hamming distances of $\gamma$-constacyclic codes of length $p^s$ over $R$ have been computed by Dinh et al. [27]. Symbol-pair distances of $\gamma$-constacyclic codes have remained open. Motivated by that, we solved this problem in this paper.

The organization of this paper is as follows. Some preliminary results are discussed in Section 2. In Section 3, the symbol-pair minimum distances of $\gamma$-constacyclic codes of length $p^s$ are established over the ring $R$. Section 4 contains some examples for different values of $p$ and $s$. We conclude the paper in Section 5.

2. Some Preliminaries

For a finite ring $R$, consider the set $R^n$ of $n$-tuples of elements from $R$ as a module over $R$ in the usual way. A code $C$ of length $n$ over $R$ is an $R$-submodule of $R^n$. Some Preliminaries

Let $\lambda$ be an invertible element of $R$. The $\lambda$-constacyclic shift $\tau_\lambda$ on $R^n$ is defined as

$$\tau_\lambda((x_0, x_1, \ldots, x_{n-1})) = (\lambda x_{n-1}, x_0, x_1, \ldots, x_{n-2}),$$

and a code $C$ is said to be $\lambda$-constacyclic if $\tau_\lambda(C) = C$, i.e., if $C$ is closed under the $\lambda$-constacyclic shift $\tau_\lambda$. In case $\lambda = 1$, those $\lambda$-constacyclic codes are called cyclic codes, and when $\lambda = -1$, such $\lambda$-constacyclic codes are called negacyclic codes.

Each codeword $c = (c_0, c_1, \ldots, c_{n-1}) \in C$ is customarily identified with its polynomial representation $c(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1}$, and the code $C$ is in turn identified with the set of all polynomial representations of its codewords. Then in the ring $R[x]/(x^n - \lambda)$, $xc(x)$ corresponds to a $\lambda$-constacyclic shift of $c(x)$. From that, the following fact follows at once (cf. [28,29]).

Proposition 1. A linear code $C$ of length $n$ is $\lambda$-constacyclic over $R$ if and only if $C$ is an ideal of $R[x]/(x^n - \lambda)$.

Let $\Sigma$ be an alphabet of size $q$, whose elements are called symbols. Suppose that $x = (x_0, x_1, \ldots, x_{n-1})$ is a vector in $\Sigma^n$. The symbol-pair vector of $x$ is defined as

$$\pi(x) = ((x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_0)).$$

In 2010, Cassuto and Blaum [1] gave the definition of the symbol-pair distance as the Hamming distance over the alphabet $(\Sigma, \Sigma)$. Given $x = (x_0, x_1, \ldots, x_{n-1})$, $y = (y_0, y_1, \ldots, y_{n-1})$, the symbol-pair distance between $x$ and $y$ is defined as

$$d_{sp}(x, y) = d_H(\pi(x), \pi(y)) = \left|\left\{i \mid (x_i, x_{i+1}) \neq (y_i, y_{i+1})\right\}\right|.$$

The symbol-pair distance of a symbol-pair code $C$ is defined as

$$d_{sp}(C) = \min\{d_{sp}(x, y) \mid x, y \in C, x \neq y\}.$$

The symbol-pair weight of a vector $x$ is defined as the Hamming weight of its symbol-pair vector $\pi(x)$:

$$wt_{sp}(x) = \left|\left\{i \mid (x_i, x_{i+1}) \neq (0, 0), 0 \leq i \leq n-1, x_n = x_0\right\}\right|.$$

If the code $C$ is linear, its symbol-pair distance is equal to the minimum symbol-pair weight of nonzero codewords of $C$:

$$d_{sp}(C) = \min\{wt_{sp}(x) \mid 0 \neq x \in C\}.$$
Throughout this paper, let \( p \) be a prime, \( s, m \) be positive integers, \( \mathbb{F}_{p^m} \) be the finite field of order \( p^m \), and let \( R = \mathbb{F}_{p^m}[x]/(x^s) \) be the finite commutative chain ring with unity.

By applying Proposition 1, all \( \gamma \)-constacyclic codes of length \( p^s \) over \( R \) are precisely the ideals in the ring
\[
\mathcal{R}_\gamma = \mathcal{R}[x]/(x^{p^s} - \gamma),
\]
where \( \gamma \) is a nonzero element of \( \mathbb{F}_{p^m} \).

In [26], Laaouine et al. classified all \( \gamma \)-constacyclic codes of length \( p^s \) over \( R \), that is, ideals of the ring \( \mathcal{R}_\gamma \), are

**Theorem 1** (cf. [26]). The ring \( \mathcal{R}_\gamma \) is a local finite non chain ring with maximal ideal \( (u, x - \gamma_0) \), where \( \gamma_0 \in \mathbb{F}_{p^m} \) such that \( \gamma_0^{p^s} = \gamma \). The \( \gamma \)-constacyclic codes of length \( p^s \) over \( \mathcal{R} \), that is, ideals of the ring \( \mathcal{R}_\gamma \), are

**Type 1** (\( C_1 \)):
\[
\langle 0 \rangle, \; \langle 1 \rangle.
\]

**Type 2** (\( C_2 \)):
\[
C_2 = \langle u^2(x - \gamma_0)^\tau \rangle, \quad 0 \leq \tau \leq p^s - 1.
\]

**Type 3** (\( C_3 \)):
\[
C_3 = \langle u(x - \gamma_0)^d + u^2(x - \gamma_0)^4 h(x) \rangle,
\]
where \( 0 \leq L \leq \delta \leq p^s - 1, 0 \leq t < L \), either \( h(x) \) is 0 or \( h(x) \) is a unit in \( \mathcal{R}_\gamma \). Here \( L \) is the smallest integer such that \( u^2(x - \gamma_0)^L \in \mathcal{C}_3 \).

**Type 4** (\( C_4 \)):
\[
C_4 = \langle u(x - \gamma_0)^d + u^2(x - \gamma_0)^4 h(x), u^2(x - \gamma_0)^\omega \rangle,
\]
where \( 0 \leq \omega < L \leq \delta \leq p^s - 1, 0 \leq t < \omega \), either \( h(x) \) is a unit in \( \mathcal{R}_\gamma \) or 0, and \( L \) is same as in Type 3.

**Type 5** (\( C_5 \)):
\[
C_5 = \langle (x - \gamma_0)^a + u(x - \gamma_0)^4 h_1(x) + u^2(x - \gamma_0)^5 h_2(x) \rangle,
\]
where \( 0 < V \leq U \leq a \leq p^s - 1, 0 \leq t_1 < U, 0 \leq t_2 < V \), either \( h_1(x), h_2(x) \) are 0 or are units in \( \mathcal{R}_\gamma \). Here \( U \) and \( V \) are the smallest integer satisfying \( u^2(x - \gamma_0)^V \in \mathcal{C}_5 \).

**Type 6** (\( C_6 \)):
\[
C_6 = \langle (x - \gamma_0)^a + u(x - \gamma_0)^4 h_1(x) + u^2(x - \gamma_0)^5 h_2(x), u^2(x - \gamma_0)^c \rangle,
\]
where \( 0 \leq c < V \leq U \leq a \leq p^s - 1, 0 \leq t_1 < U, 0 \leq t_2 < c \), either \( h_1(x), h_2(x) \) are 0 or are units in \( \mathcal{R}_\gamma \). Here \( U, V \) are same as in Type 5.

**Type 7** (\( C_7 \)):
\[
C_7 = \langle (x - \gamma_0)^a + u(x - \gamma_0)^4 h_1(x) + u^2(x - \gamma_0)^5 h_2(x), u(x - \gamma_0)^b + u^2(x - \gamma_0)^5 h_3(x) \rangle,
\]
where \( 0 \leq W \leq b < U \leq a \leq p^s - 1, 0 \leq t_1 < b, 0 \leq t_2 < W, 0 \leq t_3 < W \), either \( h_1(x), h_2(x), h_3(x) \) are 0 or are units in \( \mathcal{R}_\gamma \). Here \( W \) and \( U \) are the smallest integer satisfying \( u^2(x - \gamma_0)^W \in \mathcal{C}_7 \).

**Type 8** (\( C_8 \)):
\[
C_8 = \langle (x - \gamma_0)^a + u(x - \gamma_0)^4 h_1(x) + u^2(x - \gamma_0)^5 h_2(x), u(x - \gamma_0)^b + u^2(x - \gamma_0)^5 h_3(x), u^2(x - \gamma_0)^c \rangle,
\]
where \( 0 \leq c < W \leq L_1 \leq b < U \leq a \leq p^s - 1, 0 \leq t_1 < b, 0 \leq t_2 < c, 0 \leq t_3 < c \), either \( h_1(x), h_2(x), h_3(x) \) are 0 or are units in \( \mathcal{R}_\gamma \). Here \( L_1 \) is the smallest integer satisfying \( u^2(x - \gamma_0)^{L_1} \in \langle u(x - \gamma_0)^b + u^2(x - \gamma_0)^5 h_3(x) \rangle \) as in Type 5 and \( W \) is same as in Type 7.
Proposition 2 (cf. [26]). We have

\[ L = \begin{cases} \delta, & \text{if } h(x) = 0, \\ \min \{ \delta, p^s - \delta + t \}, & \text{if } h(x) \neq 0. \end{cases} \]

\[ L_1 = \begin{cases} b, & \text{if } h_3(x) = 0, \\ \min \{ b, p^s - b + t_3 \}, & \text{if } h_3(x) \neq 0. \end{cases} \]

\[ U = \begin{cases} a, & \text{if } h_1(x) = 0, \\ \min \{ a, p^s - a + t_1 \}, & \text{if } h_1(x) \neq 0. \end{cases} \]

\[ V = \begin{cases} a, & \text{if } h_1(x) = h_2(x) = 0, \\ \min \{ a, p^s - a + t_2 \}, & \text{if } h_1(x) = 0 \text{ and } h_2(x) \neq 0, \\ \min \{ a, p^s - a + t_1 \}, & \text{if } h_1(x) \neq 0. \end{cases} \]

\[ W = \begin{cases} b, & \text{if } h_1(x) = h_2(x) = h_3(x) = 0 \\ \text{or } h_1(x) \neq 0 \text{ and } h_3(x) = 0, \\ \min \{ b, p^s - a + t_2 \}, & \text{if } h_1(x) = h_3(x) = 0, h_2(x) \neq 0, \\ \min \{ b, p^s - b + t_3 \}, & \text{if } h_1(x) = h_2(x) = 0, h_3(x) \neq 0, \\ \text{or } h_1(x) \neq 0 \text{ and } h_3(x) \neq 0, \\ \min \{ b, p^s - a + t_2, p^s - b + t_3 \}, & \text{if } h_1(x) = 0, h_2(x) \neq 0, h_3(x) \neq 0. \end{cases} \]

Theorem 2 (cf. [26]). Let \( C \) be a \( \gamma \)-constacyclic codes of length \( p^s \) over \( \mathbb{R} \). Then following the same notations as in Theorem 1, we have the following results:

- If \( C = \langle 0 \rangle \), then \( |C| = 1 \).
- If \( C = \langle 1 \rangle \), then \( |C| = p^{3mp^s} \).
- If \( C = \langle u^2(x - \gamma_0)^{\tau} \rangle \) with \( 0 \leq \tau \leq p^s - 1 \), then
  \[ |C| = p^{m(p^s - \tau)}. \]
- If \( C = \langle u(x - \gamma_0)^{\delta} + u^2(x - \gamma_0)^{4h(x)} \rangle \) is of the Type 3, then
  \[ |C| = p^{m(2p^s - \delta - 3)}. \]
- If \( C = \langle u(x - \gamma_0)^{\delta} + u^2(x - \gamma_0)^{4h(x)}, u^2(x - \gamma_0)^{\omega} \rangle \) is of the Type 4, then
  \[ |C| = p^{m(2p^s - \delta - \omega)}. \]
- If \( C = \langle (x - \gamma_0)^{a} + u(x - \gamma_0)^{t_1}h_1(x) + u^2(x - \gamma_0)^{t_2}h_2(x) \rangle \) is of the Type 5, then
  \[ |C| = p^{m(3p^s - a - u - v)}. \]
- If \( C = \langle (x - \gamma_0)^{a} + u(x - \gamma_0)^{t_1}h_1(x) + u^2(x - \gamma_0)^{t_2}h_2(x), u^2(x - \gamma_0)^{c} \rangle \) is of the Type 6, then
  \[ |C| = p^{m(3p^s - a - u - c)}. \]
- If \( C = \langle (x - \gamma_0)^{a} + u(x - \gamma_0)^{t_1}h_1(x) + u^2(x - \gamma_0)^{t_2}h_2(x), u(x - \gamma_0)^{b} + u^2(x - \gamma_0)^{t_3}h_3(x) \rangle \) is of the Type 7, then
  \[ |C| = p^{m(3p^s - a - b - c)}. \]
- If \( C = \langle (x - \gamma_0)^{a} + u(x - \gamma_0)^{t_1}h_1(x) + u^2(x - \gamma_0)^{t_2}h_2(x), u(x - \gamma_0)^{b} + u^2(x - \gamma_0)^{t_3}h_3(x), u^2(x - \gamma_0)^{c} \rangle \) is of the Type 8, then
  \[ |C| = p^{m(3p^s - a - b - c)}. \]
3. Symbol-Pair Distance

In this section, we shall determine symbol-pair distances of all $\gamma$-constacyclic codes of length $p^s$ over $R$. To do this, we need the following theorem.

Theorem 3 (cf. [6]). Let $C$ be a $\gamma$-constacyclic code of length $p^s$ over $\mathbb{F}_{p^m}$. Then $C = \langle (x - \gamma_0)^r \rangle$, for $\kappa \in \{0, 1, \ldots, p^s\}$, and its symbol-pair distance $d_{sp}(C)$ is completely determined by:

$$d_{sp}(C) = \begin{cases} 
2, & \text{if } \kappa = 0, \\
3p^v, & \text{if } \kappa = p^s - p^{\delta-v} + 1, \text{ where } 0 \leq v \leq s - 2, \\
4p^v, & \text{if } p^s - p^{\delta-v} + 2 \leq \kappa \leq p^s - p^{\delta-v} + p^{\delta-1}, \\
& \text{where } 0 \leq v \leq s - 2, \\
2(\zeta + 2)p^v, & \text{if } p^s - p\eta + \zeta \eta + 1 \leq \kappa \leq p^s - p\eta + (\zeta + 1)\eta, \\
& \text{where } \eta = p^{\delta-1}, 0 \leq v \leq s - 2 \text{ and } 1 \leq \zeta \leq p - 2, \\
(\zeta + 2)p^{s-1}, & \text{if } \kappa = p^s - p + \zeta, \text{ where } 0 \leq \zeta \leq p - 2, \\
p^s, & \text{if } \kappa = p^s - 1, \\
0, & \text{if } \kappa = p^s.
\end{cases}$$

Note that $\mathbb{F}_{p^m}$ is a subring of $R$, for a code $C$ over $R$, we denote $d_{sp}(C_{\mathbb{F}})$ as the symbol-pair distance of $C|_{\mathbb{F}_{p^m}}$.

Now, we compute the symbol-pair distance for each type of $\gamma$-constacyclic codes of length $p^s$ over $R$ one by one.

Type 1 consists of the trivial ideals $\langle 0 \rangle$, $\langle 1 \rangle$. Hence, they have symbol-pair distances 0 and 2, respectively.

For a code $C_2 = \langle u^2(x - \gamma_0)^\tau \rangle$ of Type 2, $0 \leq \tau \leq p^s - 1$, the codewords of $C_2$ are exactly same as the codewords of the $\gamma$-constacyclic codes $\langle (x - \gamma_0)^\tau \rangle$ in $\mathbb{F}_{p^m}[x]/\langle x^p - \gamma \rangle$ multiplied by $u^2$. Thus, we obtain $d_{sp}(C_2) = d_{sp}(\langle (x - \gamma_0)^\tau \rangle_{\mathbb{F}})$, which are given in Theorem 3.

Theorem 4. Let $C_2 = \langle u^2(x - \gamma_0)^\tau \rangle$, $0 \leq \tau \leq p^s - 1$, be a $\gamma$-constacyclic codes of length $p^s$ over $R$ of Type 2 (as classified in Theorem 1). Then the symbol-pair distance $d_{sp}(C_2)$ of the code $C_2$ is given by

$$d_{sp}(C_2) = d_{sp}(\langle (x - \gamma_0)^\tau \rangle_{\mathbb{F}}) = \begin{cases} 
2, & \text{if } \tau = 0, \\
3p^v, & \text{if } \tau = p^s - p^{\delta-v} + 1, \text{ where } 0 \leq v \leq s - 2, \\
4p^v, & \text{if } p^s - p^{\delta-v} + 2 \leq \tau \leq p^s - p^{\delta-v} + p^{\delta-1}, \\
& \text{where } 0 \leq v \leq s - 2, \\
2(\zeta + 2)p^v, & \text{if } p^s - p\eta + \zeta \eta + 1 \leq \tau \leq p^s - p\eta + (\zeta + 1)\eta, \\
& \text{where } \eta = p^{\delta-1}, 0 \leq v \leq s - 2 \text{ and } 1 \leq \zeta \leq p - 2, \\
(\zeta + 2)p^{s-1}, & \text{if } \tau = p^s - p + \zeta, \text{ where } 0 \leq \zeta \leq p - 2, \\
p^s, & \text{if } \tau = p^s - 1.
\end{cases}$$

Now, we are going to determine the symbol-pair distances of those codes for the remaining cases (Type 3, 4, 5, 6, 7 and 8). To do this, we first observe that

$$\text{wt}_{sp}(a(x)) \geq \text{wt}_{sp}(ua(x)), \quad (1)$$

where $a(x) \in R[x]$.

The symbol-pair distance of Type 3 $\gamma$-constacyclic codes can be calculated as follows:
Theorem 5. Let \( C_3 = \langle u(x - \gamma_0)\delta + u^2(x - \gamma_0)^{\delta}h(x) \rangle \) be a \( \gamma \)-constacyclic codes of length \( p^s \) over \( R \) of Type 3 (as classified in Theorem 1). Then the symbol-pair distance \( d_{sp}(C_3) \) of \( C_3 \) is given by

\[
d_{sp}(C_3) = d_{sp}((u(x - \gamma_0)^{\delta_})_F)
\]

\[
= \begin{cases} 
2, & \text{if } L = 0, \\
3p^r, & \text{if } L = p^s - p^{s-v} + 1, \text{ where } 0 \leq v \leq s - 2, \\
4p^r, & \text{if } p^s - p^{s-v} + 2 \leq L \leq p^s - p^{s-v} + p^{s-v-1}, \text{ where } 0 \leq v \leq s - 2, \\
2(\zeta + 2)p^r, & \text{if } p^s - p\eta + \zeta\eta + 1 \leq L \leq p^s - p\eta + (\zeta + 1)\eta, \text{ where } \eta = p^{s-v-1}, 0 \leq v \leq s - 2 \text{ and } 1 \leq \zeta \leq p - 2, \\
(\zeta + 2)p^{s-1}, & \text{if } L = p^s - p + \zeta, \text{ where } 0 \leq \zeta \leq p - 2, \\
p^s, & \text{if } L = p^s - 1.
\end{cases}
\]

Proof. Let \( C_3 = \langle u(x - \gamma_0)^{\delta} + u^2(x - \gamma_0)^{\delta}h(x) \rangle \) be of Type 3. Let \( c(x) \) be an arbitrary nonzero element of \( C_3 \). That means there exist \( f_0(x), f_u(x), f_{u^2}(x) \in F_{p^s}[x] \) such that

\[
c(x) = [f_0(x) + uf_u(x) + u^2f_{u^2}(x)][u(x - \gamma_0)^{\delta} + u^2(x - \gamma_0)^{\delta}h(x)].
\]

Thus,

\[
uc(x) = u^2f_0(x)(x - \gamma_0)^{\delta}.
\]

By (1), we obtain that

\[
w_{tsp}(c(x)) \geq w_{tsp}(uc(x))
= w_{tsp}(u^2f_0(x)(x - \gamma_0)^{\delta})
\geq d_{sp}(u^2(x - \gamma_0)^{\delta})
= d_{sp}((u(x - \gamma_0)^{\delta})_F).
\]

Since, \((u(x - \gamma_0)^{\delta}) \subseteq ((x - \gamma_0)^{L})_F\), we have

\[
d_{sp}((u(x - \gamma_0)^{\delta})_F) \geq d_{sp}((x - \gamma_0)^{L})_F).
\]

From this, we obtain \( w_{tsp}(c(x)) \geq d_{sp}(((x - \gamma_0)^{L})_F) \) for each \( c(x) \) nonzero element of \( C_3 \). This implies that

\[
d_{sp}(C_3) \geq d_{sp}((x - \gamma_0)^{L})_F). \tag{2}
\]

On the other hand, we have that

\[
\langle u^2(x - \gamma_0)^{L} \rangle \subseteq C_3,
\]

which implies that

\[
d_{sp}((x - \gamma_0)^{L})_F) = d_{sp}((u^2(x - \gamma_0)^{L})) \geq d_{sp}(C_3). \tag{3}
\]

Now by (2) and (3), we obtain

\[
d_{sp}(C_3) = d_{sp}((x - \gamma_0)^{L})_F).
\]

Now by applying Theorem 3, we obtain the desired result. \( \square \)

Now, we determine the symbol-pair distance of Type 4 \( \gamma \)-constacyclic codes.

Theorem 6. Let \( C_4 = \langle u(x - \gamma_0)^{\delta} + u^2(x - \gamma_0)^{\delta}h(x), u^2(x - \gamma_0)^{\omega} \rangle \) be a \( \gamma \)-constacyclic code of length \( p^s \) over \( R \) of Type 4 (as classified in Theorem 1). Then the symbol-pair distance \( d_{sp}(C_4) \) of \( C_4 \) is given by
\[ d_{sp}(C_4) = d_{sp}(((x - \gamma_0)^{\omega})_F) \]

\[
= \begin{cases} 
2, & \text{if } \omega = 0, \\
3p^v, & \text{if } \omega = p^s - p^{s-v} + 1, \text{ where } 0 \leq v \leq s - 2, \\
4p^v, & \text{if } p^s - p^{s-v} + 2 \leq \omega \leq p^s - p^{s-v} + p^{s-v-1}, \text{ where } 0 \leq v \leq s - 2, \\
2(\zeta + 2)p^v, & \text{if } p^s - p\eta + \zeta\eta + 1 \leq \omega \leq p^s - p\eta + (\zeta + 1)\eta, \text{ where } \eta = p^{s-v-1}, 0 \leq v \leq s - 2 \text{ and } 1 \leq \zeta \leq p - 2, \\
(\zeta + 2)p^{v-1}, & \text{if } \omega = p^s - p + \zeta, \text{ where } 0 \leq \zeta \leq p - 2. 
\end{cases}
\]

**Proof.** First of all, since \( u^2(x - \gamma_0)^{\omega} \in C_4 \), it follows that 
\[ d_{sp}(C_4) \leq d_{sp}((u^2(x - \gamma_0)^{\omega})_F) = d_{sp}(((x - \gamma_0)^{\omega})_F). \]

To prove that \( d_{sp}(((x - \gamma_0)^{\omega})_F) \leq d_{sp}(C_4) \), we assume an arbitrary polynomial \( c(x) \in C_4 \setminus \langle u^2(x - \gamma_0)^{\omega} \rangle \) and move on to show that \( wt_{sp}(c(x)) \geq d_{sp}(((x - \gamma_0)^{\omega})_F). \)

By (1), we obtain that 
\[
wt_{sp}(c(x)) \geq wt_{sp}(uc(x)) \geq d_{sp}((u^2(x - \gamma_0)^{\delta})) = d_{sp}(((x - \gamma_0)^{\delta})_F) \geq d_{sp}(((x - \gamma_0)^{\omega})_F) \text{ (because } (x - \gamma_0)^{\delta} \subseteq (x - \gamma_0)^{\omega}).
\]

Hence, \( d_{sp}(((x - \gamma_0)^{\omega})_F) \leq d_{sp}(C_4) \), forcing 
\[ d_{sp}(C_4) = d_{sp}(((x - \gamma_0)^{\omega})_F). \]

Now by applying Theorem 3, we obtain the desired result. \( \square \)

Next, we calculate the symbol-pair distance of Type 5 \( \gamma \)-constacyclic codes as follows:

**Theorem 7.** Let \( C_5 = ((x - \gamma_0)^a + u(x - \gamma_0)^{h_1}(x) + u^2(x - \gamma_0)^{h_2}(x)) \) be a \( \gamma \)-constacyclic codes of length \( p^s \) over \( R \) of Type 5 (as classified in Theorem 1). Then the symbol-pair distance \( d_{sp}(C_5) \) of \( C_5 \) is given by

\[ d_{sp}(C_5) = d_{sp}(((x - \gamma_0)^V)_F) \]

\[
= \begin{cases} 
3p^v, & \text{if } V = p^s - p^{s-v} + 1, \text{ where } 0 \leq v \leq s - 2, \\
4p^v, & \text{if } p^s - p^{s-v} + 2 \leq V \leq p^s - p^{s-v} + p^{s-v-1}, \text{ where } 0 \leq v \leq s - 2, \\
2(\zeta + 2)p^v, & \text{if } p^s - p\eta + \zeta\eta + 1 \leq V \leq p^s - p\eta + (\zeta + 1)\eta, \text{ where } \eta = p^{s-v-1}, 0 \leq v \leq s - 2 \text{ and } 1 \leq \zeta \leq p - 2, \\
(\zeta + 2)p^{v-1}, & \text{if } V = p^s - p + \zeta, \text{ where } 0 \leq \zeta \leq p - 2, \\
p^v, & \text{if } \omega = p^s - 1.
\end{cases}
\]

**Proof.** Let \( C_5 = ((x - \gamma_0)^a + u(x - \gamma_0)^{h_1}(x) + u^2(x - \gamma_0)^{h_2}(x)) \) be of Type 5. Now for each nonzero \( c(x) \in C_5 \), there exist \( g_0(x), g_u(x), g_{u2}(x) \in \mathbb{F}_{p^m}[x] \) such that

\[
c(x) = [g_0(x) + u g_u(x) + u^2 g_{u2}(x)]((x - \gamma_0)^a + u(x - \gamma_0)^{h_1}(x) + u^2(x - \gamma_0)^{h_2}(x)).
\]

Thus, \( u^2 c(x) = u^2 g_0(x) (x - \gamma_0)^a \).

By (1), we see that
Let Theorem 8.

Let obtain that distance \(d\) γ-constacyclic codes of length \(p\) \(x\) − \(V\) \(sp\) \(x\), we have

\[
\langle \gamma \rangle \subseteq \langle \gamma \rangle_{\mathbb{F}}.
\]

This implies that \(d_{sp}(\langle \gamma \rangle_{\mathbb{F}}) \leq d_{sp}(C_5)\).

On the other hand we have that

\[
u^2(x - \gamma_0) \in C_3,
\]

then \(d_{sp}(C_3) \leq d_{sp}(\langle u^2(x - \gamma_0)^c\rangle) = d_{sp}(\langle \gamma \rangle_{\mathbb{F}})\) and we obtain \(d_{sp}(C_3) = d_{sp}(\langle \gamma \rangle_{\mathbb{F}})\). Now by applying Theorem 3, we obtain the desired result. \(\square\)

The symbol-pair distance of Type 6 γ-constacyclic codes can be established as follows:

**Theorem 8.** Let \(C_6 = \langle (x - \gamma_0)^a + u(x - \gamma_0)^b h_1(x) + u^2(x - \gamma_0)^c h_2(x), u(x - \gamma_0)^k + u^2(x - \gamma_0)^d h_3(x) \rangle\) be a γ-constacyclic codes of length \(p^v\) over \(\mathbb{R}\) of Type 6 (as classified in Theorem 1). Then the symbol-pair distance \(d_{sp}(C_6)\) of \(C_6\) is given by

\[
d_{sp}(C_6) = d_{sp}(\langle (x - \gamma_0)^c\rangle_{\mathbb{F}})
\]

\[
= \begin{cases}
  2, & \text{if } c = 0, \\
  3p^v, & \text{if } c = p^v - p^{s - v} + 1, \text{ where } 0 \leq v \leq s - 2, \\
  4p^v, & \text{if } p^v - p^{s - v} + 2 \leq c \leq p^v - p^{s - v} + p^v - 1, \text{ where } 0 \leq v \leq s - 2, \\
  2(\xi + 2)p^v, & \text{if } p^v - p^v + \xi + 1 \leq c \leq p^v - p^v + (\xi + 1), \text{ where } \eta = p^v - 1, 0 \leq v \leq s - 2 \text{ and } 1 \leq \xi \leq p - 2, \\
  (\xi + 2)p^v, & \text{if } c = p^v - p + \xi, \text{ where } 0 \leq \xi \leq p - 2.
\end{cases}
\]

**Proof.** First of all, since \(u^2(x - \gamma_0)^c \in C_6\), it follows that

\[
d_{sp}(C_6) \leq d_{sp}(\langle u^2(x - \gamma_0)^c\rangle) = d_{sp}(\langle (x - \gamma_0)^c\rangle_{\mathbb{F}}).
\]

Now, consider an arbitrary polynomial \(c(x) \in C_6 \setminus \langle u^2(x - \gamma_0)^c\rangle\). Thus, by (1), we obtain that

\[
wt_{sp}(c(x)) \geq wt_{sp}(u^2c(x))
\]

\[
= wt_{sp}(u^2g_0(x)(x - \gamma_0)^a)
\]

\[
\geq d_{sp}(\langle u^2(x - \gamma_0)^a\rangle)
\]

\[
= d_{sp}(\langle (x - \gamma_0)^a\rangle_{\mathbb{F}})
\]

\[
\geq d_{sp}(\langle (x - \gamma_0)^c\rangle_{\mathbb{F}}) \text{ (because } \langle (x - \gamma_0)^a\rangle \subseteq \langle (x - \gamma_0)^c\rangle)\).
\]

Hence, \(d_{sp}(\langle (x - \gamma_0)^c\rangle_{\mathbb{F}}) \leq d_{sp}(C_6)\), forcing

\[
d_{sp}(C_6) = d_{sp}(\langle (x - \gamma_0)^c\rangle_{\mathbb{F}}).
\]

Now by applying Theorem 3, we obtain the desired result. \(\square\)

Now, we determine Theorem 3, we obtain the desired result.

**Theorem 9.** Let \(C_7 = \langle (x - \gamma_0)^a + u(x - \gamma_0)^b h_1(x) + u^2(x - \gamma_0)^c h_2(x), u(x - \gamma_0)^b + u^2(x - \gamma_0)^d h_3(x) \rangle\) be a γ-constacyclic codes of length \(p^v\) over \(\mathbb{R}\) of Type 7 (as classified in Theorem 1). Then the symbol-pair distance \(d_{sp}(C_7)\) of \(C_7\) is given by
Let \( \gamma \) be of Type 8. Consider an arbitrary polynomial \( c(x) \in C_7 \). We consider two cases.

* Case 1: \( c(x) \in \langle u \rangle \). In this case, by (1). We have

\[
\wtsp(c(x)) \geq \wtsp(u^2(x-\gamma_0)^{b}) \\
= \spd(\langle (x-\gamma_0)^{b} \rangle).
\]

* Case 2: \( c(x) \notin \langle u \rangle \). In this case, by (1). We have

\[
\wtsp(c(x)) \geq \wtsp(u^2c(x)) \\
= \spd(\langle (x-\gamma_0)^a \rangle). \]

Now, consider an arbitrary polynomial \( c(x) \in C_7 \), it follows that

\[
d_{sp}(C_7) = d_{sp}(\langle (x-\gamma_0)^{W} \rangle).
\]

Since, \( \langle (x-\gamma_0)^a \rangle \subseteq \langle (x-\gamma_0)^b \rangle \subseteq \langle (x-\gamma_0)^{W} \rangle \), we have

\[
d_{sp}(\langle (x-\gamma_0)^{W} \rangle) \geq d_{sp}(\langle (x-\gamma_0)^{b} \rangle) \geq d_{sp}(\langle (x-\gamma_0)^{a} \rangle).
\]

Hence, \( d_{sp}(\langle (x-\gamma_0)^{W} \rangle) \leq d_{sp}(C_7) \), forcing

\[
d_{sp}(C_7) = d_{sp}(\langle (x-\gamma_0)^{W} \rangle).
\]

Now by applying Theorem 3, we obtain the desired result. \(\square\)

Finally, we determine the symbol-pair distance of Type 8 \( \gamma \)-constacyclic codes.

**Theorem 10.** Let \( C_8 = \langle (x-\gamma_0)^a + u(x-\gamma_0)^{b}h_1(x) + u^2(x-\gamma_0)^{b}h_2(x), u(x-\gamma_0)^b + u^2(x-\gamma_0)^b h_3(x), u^2(x-\gamma_0)^c \rangle \) be a \( \gamma \)-constacyclic codes of length \( p^s \) over \( R \) of Type 8 (as classified in Theorem 1). Then the symbol-pair distance \( d_{sp}(C_8) \) of \( C_8 \) is given by

\[
d_{sp}(C_8) = d_{sp}(\langle (x-\gamma_0)^c \rangle).
\]

\[
= \begin{cases} 
2, & \text{if } c = 0, \\
3p^v, & \text{if } c = p^v - p^{s-v} + 1, \text{ where } 0 \leq v \leq s - 2, \\
4p^v, & \text{if } p^v - p^{s-v} + 2 \leq c \leq p^v - p^{s-v} + p^{s-v-1}, \text{ where } 0 \leq v \leq s - 2, \\
2(\zeta + 2)p^v, & \text{if } p^v - p\eta + \zeta \eta + 1 \leq c \leq p^v - p\eta + (\zeta + 1)\eta, \text{ where } \eta = p^{s-v-1}, 0 \leq v \leq s - 2 \text{ and } 1 \leq \zeta \leq p - 2, \\
(\zeta + 2)p^{s-1}, & \text{if } c = p^v - p + \zeta, \text{ where } 0 \leq \zeta \leq p - 2.
\end{cases}
\]

**Proof.** Let \( C_8 = \langle (x-\gamma_0)^a + u(x-\gamma_0)^{b}h_1(x) + u^2(x-\gamma_0)^{b}h_2(x), u(x-\gamma_0)^b + u^2(x-\gamma_0)^b h_3(x), u^2(x-\gamma_0)^c \rangle \) be of Type 8. Consider an arbitrary polynomial \( c(x) \in C_8 \setminus \langle u^2(x-\gamma_0)^c \rangle \). Now, we consider two cases as follows:
* Case 1: \( c(x) \in \langle u \rangle \). In this case, by (1), we have
\[
wt_{sp}(c(x)) \geq wt_{sp}(uc(x))
\]
\[
\geq d_{sp}(\langle u^2(x - \gamma \theta)^3 \rangle)
\]
\[
= d_{sp}(\langle (x - \gamma \theta)^3 \rangle)
\]
\[
\geq d_{sp}(\langle (x - \gamma \theta)^3 \rangle) \text{ (because } \langle (x - \gamma \theta)^3 \rangle \subseteq \langle (x - \gamma \theta)^5 \rangle \text{).}
\]

* Case 2: \( c(x) \notin \langle u \rangle \). In this case, by (1), we have
\[
wt_{sp}(c(x)) \geq wt_{sp}(u^2c(x))
\]
\[
\geq d_{sp}(\langle u^2(x - \gamma \theta)^3 \rangle)
\]
\[
= d_{sp}(\langle (x - \gamma \theta)^3 \rangle)
\]
\[
\geq d_{sp}(\langle (x - \gamma \theta)^3 \rangle) \text{ (because } \langle (x - \gamma \theta)^3 \rangle \subseteq \langle (x - \gamma \theta)^5 \rangle \text{).}
\]

This implies that \( d_{sp}(\langle (x - \gamma \theta)^3 \rangle) \leq d_{sp}(C_8) \).

On the other hand, we have that
\[
\langle u^2(x - \gamma \theta)^3 \rangle \subseteq C_8,
\]
then \( d_{sp}(C_8) \leq d_{sp}(\langle u^2(x - \gamma \theta)^3 \rangle) = d_{sp}(\langle (x - \gamma \theta)^3 \rangle) \) and we obtain \( d_{sp}(C_8) = d_{sp}(\langle (x - \gamma \theta)^3 \rangle) \). Now by applying Theorem 3, we obtain the desired result. \( \square \)

4. Examples

In this section, we present some examples of symbol-pair distances of constacyclic codes of length \( p^s \) over \( \mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p \) (\( u^3 = 0 \)).

**Example 1.** Consider the ring \( \mathcal{R} = \mathbb{F}_7 + u\mathbb{F}_7 + u^2\mathbb{F}_7 \), where \( p = 7 \), \( m = 1 \) and \( \gamma \in \mathbb{F}_7^\times \), \( \gamma \)-constacyclic codes of length 49 over \( \mathcal{R} \) has eight types of generator. The symbol-pair distances corresponding to different generators are given as follows:

- **Type 1** (\( C_1 \)): \( \langle 0 \rangle, \langle 1 \rangle \),
  \( d_{sp}(\langle 0 \rangle) = 0 \), \( d_{sp}(\langle 1 \rangle) = 2 \).

- **Type 2**: \( C_2 = \langle u^2(x - \gamma)^3 \rangle \), where \( 0 \leq t \leq 48 \), then the symbol-pair distance, \( d_{sp}(C_2) \), is determined in Table 1.

- **Type 3**: \( C_3 = \langle u(x - \gamma)^3 + u^2(x - \gamma)^3h(x) \rangle \), where \( 0 \leq \delta \leq 48 \), \( 0 < t \leq \delta \), either \( h(x) \) is 0 or a unit in \( \mathcal{R}_\gamma \), and then symbol-pair distance, \( d_{sp}(C_3) \), is given in Table 2.

- **Type 4**: \( C_4 = \langle u(x - \gamma)^3 + u^2(x - \gamma)^3h(x), u^2(x - \gamma)^3 \rangle \), where \( 0 \leq \omega < \delta \leq 48 \), \( 0 \leq t < \omega \), either \( h(x) \) is 0 or a unit in \( \mathcal{R}_\gamma \), then the symbol-pair distance, \( d_{sp}(C_4) \), is determined in Table 3.

- **Type 5**: \( C_5 = \langle (x - \gamma)^a + u(x - \gamma)^b h_1(x) + u^2(x - \gamma)^bh_2(x) \rangle \), where \( 1 \leq a \leq 48 \), \( 0 \leq t_1 < a \), \( 0 \leq t_2 < a \), either \( h_1(x), h_2(x) \) are 0 or are units in \( \mathcal{R}_\gamma \). Then the symbol-pair distance, \( d_{sp}(C_5) \), is given in Table 4.

- **Type 6**: \( C_6 = \langle (x - \gamma)^a + u(x - \gamma)^b h_1(x) + u^2(x - \gamma)^bh_2(x), u^2(x - \gamma)^c \rangle \), where \( 0 \leq c < a \leq 48 \), \( 0 \leq t_1 < a \), \( 0 \leq t_2 < c \), either \( h_1(x), h_2(x) \) are 0 or are units in \( \mathcal{R}_\gamma \). Then the symbol-pair distance, \( d_{sp}(C_6) \), is determined in Table 5.

- **Type 7**: \( C_7 = \langle (x - \gamma)^a + u(x - \gamma)^b h_1(x) + u^2(x - \gamma)^bh_2(x), u(x - \gamma)^b + u^2(x - \gamma)^b h_3(x) \rangle \), where \( 0 \leq b < a \leq 48 \), \( 0 \leq t_1 < b \), \( 0 \leq t_2 < b \), \( 0 \leq t_3 < b \), either \( h_1(x), h_2(x), h_3(x) \) are 0 or are units in \( \mathcal{R}_\gamma \). Then the symbol-pair distance, \( d_{sp}(C_7) \), is given in Table 6.

- **Type 8**: \( C_8 = \langle (x - \gamma)^a + u(x - \gamma)^b h_1(x) + u^2(x - \gamma)^bh_2(x), u(x - \gamma)^b + u^2(x - \gamma)^b h_3(x), u^2(x - \gamma)^c \rangle \), where \( 0 \leq c < b < a \leq 48 \), \( 0 \leq t_1 < b \), \( 0 \leq t_2 < c \), \( 0 \leq t_3 < c \), either \( h_1(x), h_2(x), h_3(x) \) are 0 or are units in \( \mathcal{R}_\gamma \), then the symbol-pair distance, \( d_{sp}(C_8) \), is determined in Table 7.
Table 1. Symbol-pair distance of $\gamma$-constacyclic codes of Type 2 over $F_7 + uF_7 + u^2F_7$.

| Range of $\tau$ | $d_{sp}(C_2)$ |
|-----------------|----------------|
| $\tau = 0$      | 2              |
| $\tau = 1$      | 3              |
| $2 \leq \tau \leq 7$ | 4          |
| $8 \leq \tau \leq 14$ | 6          |
| $15 \leq \tau \leq 21$ | 8          |
| $22 \leq \tau \leq 28$ | 10         |
| $29 \leq \tau \leq 35$ | 12         |
| $36 \leq \tau \leq 42$ | 14         |
| $\tau = 43$    | 21             |
| $\tau = 44$    | 28             |
| $\tau = 45$    | 35             |
| $\tau = 46$    | 42             |
| $\tau = 47$    | 49             |
| $\tau = 48$    | 49             |

Table 2. Symbol-pair distance of $\gamma$-constacyclic codes of Type 3 over $F_7 + uF_7 + u^2F_7$.

| Range of $L$ | $d_{sp}(C_3)$ |
|--------------|----------------|
| $L = 0$      | 2              |
| $L = 1$      | 3              |
| $2 \leq L \leq 7$ | 4          |
| $8 \leq L \leq 14$ | 6          |
| $15 \leq L \leq 21$ | 8          |
| $22 \leq L \leq 28$ | 10         |
| $29 \leq L \leq 35$ | 12         |
| $36 \leq L \leq 42$ | 14         |
| $L = 43$    | 21             |
| $L = 44$    | 28             |
| $L = 45$    | 35             |
| $L = 46$    | 42             |
| $L = 47$    | 49             |
| $L = 48$    | 49             |

Table 3. Symbol-pair distance of $\gamma$-constacyclic codes of Type 4 over $F_7 + uF_7 + u^2F_7$.

| Range of $\omega$ | $d_{sp}(C_4)$ |
|-------------------|----------------|
| $\omega = 0$      | 2              |
| $\omega = 1$      | 3              |
| $2 \leq \omega \leq 7$ | 4          |
| $8 \leq \omega \leq 14$ | 6          |
| $15 \leq \omega \leq 21$ | 8          |
| $22 \leq \omega \leq 28$ | 10         |
| $29 \leq \omega \leq 35$ | 12         |
| $36 \leq \omega \leq 42$ | 14         |
| $\omega = 43$    | 21             |
| $\omega = 44$    | 28             |
| $\omega = 45$    | 35             |
| $\omega = 46$    | 42             |
| $\omega = 47$    | 49             |
Table 4. Symbol-pair distance of $\gamma$-constacyclic codes of Type 5 over $\mathbb{F}_7 + u\mathbb{F}_7 + u^2\mathbb{F}_7$.

| Range of $V$ | $d_{sp}(C_5)$ |
|--------------|---------------|
| $V = 1$      | 3             |
| $2 \leq V \leq 7$ | 4           |
| $8 \leq V \leq 14$ | 6           |
| $15 \leq V \leq 21$ | 8           |
| $22 \leq V \leq 28$ | 10          |
| $29 \leq V \leq 35$ | 12          |
| $36 \leq V \leq 42$ | 14          |
| $V = 43$     | 21            |
| $V = 44$     | 28            |
| $V = 45$     | 35            |
| $V = 46$     | 42            |
| $V = 47$     | 49            |
| $V = 48$     | 49            |

Table 5. Symbol-pair distance of $\gamma$-constacyclic codes of Type 6 over $\mathbb{F}_7 + u\mathbb{F}_7 + u^2\mathbb{F}_7$.

| Range of $c$ | $d_{sp}(C_6)$ |
|--------------|---------------|
| $c = 0$      | 2             |
| $c = 1$      | 3             |
| $2 \leq c \leq 7$ | 4           |
| $8 \leq c \leq 14$ | 6           |
| $15 \leq c \leq 21$ | 8           |
| $22 \leq c \leq 28$ | 10          |
| $29 \leq c \leq 35$ | 12          |
| $36 \leq c \leq 42$ | 14          |
| $c = 43$     | 21            |
| $c = 44$     | 28            |
| $c = 45$     | 35            |
| $c = 46$     | 42            |
| $c = 47$     | 49            |

Table 6. Symbol-pair distance of $\gamma$-constacyclic codes of Type 7 over $\mathbb{F}_7 + u\mathbb{F}_7 + u^2\mathbb{F}_7$.

| Range of $W$ | $d_{sp}(C_7)$ |
|--------------|---------------|
| $W = 0$      | 2             |
| $W = 1$      | 3             |
| $2 \leq W \leq 7$ | 4           |
| $8 \leq W \leq 14$ | 6           |
| $15 \leq W \leq 21$ | 8           |
| $22 \leq W \leq 28$ | 10          |
| $29 \leq W \leq 35$ | 12          |
| $36 \leq W \leq 42$ | 14          |
| $W = 43$     | 21            |
| $W = 44$     | 28            |
| $W = 45$     | 35            |
| $W = 46$     | 42            |
| $W = 47$     | 49            |
Table 7. Symbol-pair distance of \( \gamma \)-constacyclic codes of Type 8 over \( \mathbb{F}_7 + u\mathbb{F}_7 + u^2\mathbb{F}_7 \).

| Range of \( c \) | \( d_{sp}(C_{S}) \) |
|------------------|------------------|
| \( c = 0 \)     | 2                |
| \( c = 1 \)     | 3                |
| \( 2 \leq c \leq 7 \) | 4                |
| \( 8 \leq c \leq 14 \) | 6                |
| \( 15 \leq c \leq 21 \) | 8                |
| \( 22 \leq c \leq 28 \) | 10               |
| \( 29 \leq c \leq 35 \) | 12               |
| \( 36 \leq c \leq 42 \) | 14               |
| \( c = 43 \)    | 21               |
| \( c = 44 \)    | 28               |
| \( c = 45 \)    | 35               |
| \( c = 46 \)    | 42               |

Example 2. Table 8 shows the representation of all \( \gamma \)-constacyclic codes of length 3 over the chain ring \( \mathbb{F}_3 + u\mathbb{F}_3 + u^2\mathbb{F}_3 \), where \( \gamma \in \{1, 2\} \), together with the symbol-pair distances \( d_{sp} \) of such codes and the number of codewords \( |C| \) in each of those constacyclic codes. In all codes we have \( h_0, a_0, b_0, c_0 \in \{1, 2\} \) and \( b_1 \in \{0, 1, 3\} \).

Table 8. \( \gamma \)-constacyclic codes of length 3 over the chain ring \( \mathbb{F}_3 + u\mathbb{F}_3 + u^2\mathbb{F}_3 \).

| Ideal \( (C) \) | \( d_{sp} \) | \( C \) |
|------------------|--------------|--------|
| \( \langle 0 \rangle \) | 0            | \( 3^1 \) |
| \( \langle 1 \rangle \) | 2            | \( 3^2 \) |
| \( \langle u^2 \rangle \) | 2            | \( 3^2 \) |
| \( \langle u^2(x - \gamma) \rangle \) | 3            | \( 3^3 \) |
| \( \langle u^2(x - \gamma)^2 \rangle \) | 3            | \( 3^3 \) |
| \( \langle u(x - \gamma) \rangle \) | 3            | \( 3^4 \) |
| \( \langle u(x - \gamma)^2 \rangle \) | 3            | \( 3^4 \) |
| \( \langle u(x - \gamma) + h_0 u \rangle \) | 3            | \( 3^4 \) |
| \( \langle u(x - \gamma)^2 + h_0 u \rangle \) | 3            | \( 3^4 \) |
| \( \langle u(x - \gamma)^3 + h_0 u \rangle \) | 3            | \( 3^4 \) |
| \( \langle u(x - \gamma)^3 \rangle \) | 3            | \( 3^4 \) |

| \( \langle u(x - \gamma), u^2 \rangle \) | 2            | \( 3^5 \) |
| \( \langle u(x - \gamma)^2, u^2 \rangle \) | 2            | \( 3^4 \) |
| \( \langle u(x - \gamma)^3, u^2(x - \gamma) \rangle \) | 3            | \( 3^5 \) |
| \( \langle (x - \gamma), (x - \gamma)^2 \rangle \) | 3            | \( 3^6 \) |
| \( \langle (x - \gamma)^2, (x - \gamma)^3 \rangle \) | 3            | \( 3^5 \) |
| \( \langle (x - \gamma)^3, (x - \gamma)^4 \rangle \) | 3            | \( 3^6 \) |
| \( \langle (x - \gamma)^3 + a_0 u \rangle \) | 3            | \( 3^6 \) |
| \( \langle (x - \gamma)^3 + a_0 u(x - \gamma) \rangle \) | 3            | \( 3^6 \) |
| \( \langle (x - \gamma)^3 + a_0 u + b_0 u \rangle \) | 3            | \( 3^6 \) |
| \( \langle (x - \gamma)^3 + a_0 u(x - \gamma) + b_0 u \rangle \) | 3            | \( 3^6 \) |
| \( \langle (x - \gamma)^3 + a_0 u(x - \gamma) + b_0 u^2 \rangle \) | 3            | \( 3^6 \) |
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Table 8. Cont.

| Ideal (C) | d^\text{op} | C |
|-----------|-------------|---|
| → Type 6: |             |   |
| ⃗((x−γ),u^2) | 2 | 3^7 |
| ⃗((x−γ)^2,u^2) | 2 | 3^5 |
| ⃗((x−γ)^2,u^2(x−γ)) | 3 | 3^4 |
| ⃗((x−γ) + αu,u^2) | 2 | 3^7 |
| ⃗((x−γ)^2 + αu,u^2) | 2 | 3^6 |
| ⃗((x−γ)^2 + αu(x−γ),u^2) | 2 | 3^5 |
| ⃗((x−γ)^2 + αu(x−γ),u^2(x−γ)) | 3 | 3^4 |
| ⃗((x−γ)^2 + αu(x−γ) + βu^2,u^2(x−γ)) | 3 | 3^4 |

→ Type 7:
| ⃗((x−γ),u) | 2 | 3^8 |
| ⃗((x−γ)^2,u) | 2 | 3^7 |
| ⃗((x−γ)^2,u(x−γ)) | 3 | 3^5 |
| ⃗((x−γ)^2,u(x−γ) + co^2) | 3 | 3^5 |
| ⃗((x−γ)^2 + bu^2,u(x−γ)) | 3 | 3^5 |
| ⃗((x−γ)^2 + bu^2,u(x−γ) + co^2) | 3 | 3^5 |

→ Type 8:
| ⃗((x−γ)^2,u(x−γ),u^2) | 2 | 3^6 |

5. Conclusions

In this research article, all symbol-pair distances of repeated-root γ-constacyclic codes having length q^n have been determined over \( R = \mathbb{F}_{q^n} + u\mathbb{F}_{q^n} + u^2\mathbb{F}_{q^n} \), where \( γ \in \mathbb{F}_{q^n} \) and \( u^3 = 0 \). MDS symbol-pair codes have the highest possible error-detecting and error-correcting capability among codes of given length and size. It would be interesting to identify MDS symbol-pair codes within this class of constacyclic codes.

In [30], Ding et al. generalized the symbol-pair codes to b-symbol codes. Thus, it would also be interesting to determine b-symbol distances of γ-constacyclic codes of length q^n over R.

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