A Novel Local Transform Inverse S-Transform Algorithm for Statistical Filter

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Abstract. S-transform (ST) is a useful tool for time-frequency filter. However, the conventional inverse S-transform (IST) algorithm suffers from time or frequency leakage. In this paper, we proposed a novel local transform inverse S-Transform (LTIST) algorithm for statistical filter. First, the matrix S-transform (MST) and MIST are derived. Then the proposed LTIST approach applies to denoising. The statistical property of stochastic noise in the MIST is discussed. The results show that the proposed MIST algorithm has better time-frequency localization in statistical filtering than the conventional methods. Illustrative examples verify the effectiveness of the proposed algorithm.

1. Introduction

Time-frequency analysis is a powerful tool for time-varying signal and many algorithms have been proposed [1]. There are two well know linear ones: short-time Fourier transform (STFT) and continuous wavelet transform (CWT). CWT has a progressive resolution thanks to its time-scale analysis methods, which replaces the frequency shift of the STFT by the dilation of a basis function, also called mother wavelet [2]. These methods are applied to many fields including electrical and mechanical engineering, medical imaging and geophysics [3-6].

The S-transform (ST) proposed by Stockwell et al. [7], which can be viewed as an intermediate step between the STFT and the CWT. This method succeeds the advantages of STFT and it enables the use of the frequency variable as well as the multiresolution strategy of the wavelets. An adaptive Gaussian window was used to keep higher frequency resolution with narrower windows and lower frequency resolution with wider windows. Therefore, it provides a powerful performance for time-frequency analysis. Nowadays the ST has been widely applied in many areas, filtering is one of the hotter applications in the field of data processing [8, 9].

There are two factors that influence the filtering performance: IST algorithms and the pass-regions in the S-transfrom domain [10-13]. To solve the former problem, some IST algorithms have been proposed with different filtering properties in the time-frequency domain. Stockwell et al.’s IST algorithm is efficient but has unsatisfactory time resolution. In order to improve this problem, Schimmel and Gallart proposed another excellent time localization IST algorithm [10]. However, their method contains some reconstruction errors in it. Fortunately the distortion can be corrected by deconvolution [11]-[13]. To second problem, Pei and Wang define the pass-regions using global
threshold value method, but the criteria used to define the threshold value in the wideband and narrowband filters is just from empirical value [12].

In this paper, we propose a new method to determine pass-regions in the S-transform time-frequency domain. This method will provide more higher precise in time-frequency filtering and does not require any noise synchronous acquisition as reference.

2. S-Transform and Its Inverses

2.1 Continuous S-Transform and its inverses

The continuous ST derives from STFT, which also can be viewed as a special CWT [6]. The continuous ST of the signal \( x(t) \) is defined as

\[
S(\tau, f) = \int_{-\infty}^{\infty} x(t)w(\tau-t, f)e^{-j2\pi ft}dt
\]

where \( w(\tau-t, f) \) is a Gaussian window, it can be denoted as

\[
w(\tau-t, f) = \frac{|f|}{k2\pi}e^{-\frac{f^2(\tau-t)^2}{2k^2}}, \quad k > 0
\]

where \( f \) is frequency variables, \( t \) is time variables and \( \tau \) is delay and \( k \) is a scaling factor that controls time-frequency resolution.

2.2 Matrix Formulation of ST and IST

Let \( x[p] = x(pT) \), \( p = 0, 1, \ldots, N-1 \) denote a discrete time series corresponding to \( x(t) \) with a time sampling interval of \( T \). Let \( f_s = 1/T \) be the sampling frequency and \( f_0 \) be the frequency step, and \( M = f_s/f_0 \) and \( n = -M/2, \ldots, -M/2-1 \) is the index of frequency range. The discrete ST of the time series \( x[p] \) can then write as [14]

\[
S[m, n] = \sum_{p=0}^{N-1} x[p] \frac{|p|}{\sqrt{2\pi kN}} e^{-i\frac{m-p}{2k^2N^2}t^2}e^{2\pi p|n|N}, \quad n \neq 0
\]

equation (3) can be rewritten in the following way [15,16]:

\[
S[m, n] = \sum_{p=0}^{N-1} x[p] \frac{|p|}{\sqrt{2\pi kN}} e^{-i\frac{m-p}{2k^2N^2}t^2
}e^{2\pi p|n|N} + x[0]\frac{|0|}{\sqrt{2\pi kN}} e^{-i\frac{m}{2k^2N^2}t^2}e^{2\pi 0|n|N} + \\
\cdots + x[p]\frac{|p|}{\sqrt{2\pi kN}} e^{-i\frac{m-p}{2k^2N^2}t^2}e^{2\pi p|n|N} + \\
\cdots + x[N-1]\frac{|N-1|}{\sqrt{2\pi kN}} e^{-i\frac{m-(N-1)}{2k^2N^2}t^2}e^{2\pi (N-1)|n|N}
\]

where \( T_{(m)0,0}, T_{(m)0,1}, \ldots, T_{(m)0,p}, \ldots + T_{(m)(N-1)p}, x[N-1] \) represent transform coefficients of ST, which is related to \( m, n \) and \( p \). Let \( s[(n-1)\times N + m] = S[m, n] \), (4) can be formulated in the form of matrices

\[
\begin{bmatrix}
s[1] \\
s[2] \\
\vdots \\
s[N] \\
s[N+1] \\
s[M \times N]
\end{bmatrix} = 
\begin{bmatrix}
T_{11} & T_{12} & \cdots & T_{1N} \\
T_{21} & T_{22} & \cdots & T_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
T_{N1} & T_{N2} & \cdots & T_{NN} \\
T_{(N+1)1} & T_{(N+1)2} & \cdots & T_{(N+1)N} \\
\vdots & \vdots & \ddots & \vdots \\
T_{(M+N-1)1} & T_{(M+N-1)2} & \cdots & T_{(M+N-1)N}
\end{bmatrix}
\begin{bmatrix}
s[0] \\
x[1] \\
\vdots \\
x[N-2] \\
x[N-1]
\end{bmatrix}
\]
Therefore, discrete ST in matrix form can be modeled as
\[ s = Tx \]  
where \( s \) and \( x \) are \( N^2 \times 1 \) matrix, \( T \) is a \( N^2 \times N \) matrix respectively. Let \( T^+ \) denotes the Moore-Penrose pseudoinverse, discrete IST in the form of matrix form can be modeled as
\[ x = T^+s \]  
where \( T^+ \) exists only if matrix \( T \) has full rank.

3. Local Ist Time-Frequency Filters

Time-frequency analysis can be applied in many areas. Extracting the instantaneous parameters of a signal and de-noising of data are the most important applications in filtering. One of the interests of the ST lies in filtering the noise in the time-frequency domain and then using inverse transform to extract useful signal.

Let \( C[m,n] \) be an time-frequency equalization filter. S-transform time-frequency domain filters as follows
\[ S[m,n] = S[m,n]*C[m,n]. \]  
The corresponding matrix IST filters can be describe as
\[ x_f = T^+s_f \]  
where \( s_f \) is the filtered time-frequency domain \( S_f \), \( x_f \) represents the output signal after filter.

Conventionally, filtering would multiply the pass-region by one and the others by zeros in the time-frequency domain,

Discrete IST in matrix form be modeled as (9), which can be rewritten in the following way
\[
\begin{bmatrix}
  x_f[0] \\
  x_f[1] \\
  \vdots \\
  x_f[N-2] \\
  x_f[N-1]
\end{bmatrix}
= \begin{bmatrix}
  T_{i1} & T_{i2} & \cdots & T_{iN} \\
  T_{j1} & T_{j2} & \cdots & T_{jN} \\
  \vdots & \vdots & \ddots & \vdots \\
  T_{(N-1)i1} & T_{(N-1)i2} & \cdots & T_{(N-1)jN} \\
  T_{MN1} & T_{MN2} & \cdots & T_{MNjN}
\end{bmatrix}
\begin{bmatrix}
  s_{f11} \\
  s_{f12} \\
  \vdots \\
  s_{f21} \\
  s_{fMN}
\end{bmatrix}
\]  
(10)

Notice that the output discrete time series \( x_f[i] \) depend on \( [T_{i1}, T_{i2}, \cdots, T_{iN}] \) multiplied by \( [s_{11}, s_{12}, \cdots, s_{iN}] \). Each element of the time series is related with the whole time-frequency domain matrix and the corresponding row of the inverse transform matrix. In other words, if we want to extract only interesting time regions signal in whole time-frequency domain, it is necessary to take account of the entire time-frequency domain and corresponding row elements of inverse transform matrix. Therefore, the proposed LT IST can be formulated in matrix form as follow:
\[
\begin{bmatrix}
  x_f[i] \\
  \vdots \\
  x_f[j]
\end{bmatrix}
= \begin{bmatrix}
  T_{i1} & T_{i2} & \cdots & T_{iN} \\
  T_{j1} & T_{j2} & \cdots & T_{jN}
\end{bmatrix}^{-1}
\begin{bmatrix}
  s_{f11} \\
  s_{f12} \\
  \vdots \\
  s_{j21} \\
  s_{jMN}
\end{bmatrix}
\]  
(11)

\[ x_f = T_f^+s \]  
(12)

where \( T_f^+ \) correspond to the interesting time-frequency regions \( s_f \) in a certain period of time. LT IST filtering in the pass-region would multiply by one and other time by zeros. Unlike the conventional filtering method, LT IST has good time localization in exacting the transient signals. We
notice that (12) is only an IST algorithm without filtering. It is necessary to take account of the time-frequency domain.

The spectrum of the ST in the time-frequency domain is defined as the squared modulus of $S[m, n]$, and the mean power spectrum (MPS) of the white noise can be defined as

$$ E = |S(\tau, f)|^2 $$

$$ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{E}[h(t)h^*(u)]e^{-i(\tau-u)f^2/2}e^{-i\phi(\tau-u)}du $$

$$ = \frac{1}{2\pi} \sigma^2 \int_{-\infty}^{\infty} \delta(t-u)e^{-i(\tau-u)f^2/2}e^{-i\phi(\tau-u)}du $$

$$ = \frac{1}{2\pi} \sigma^2 $$

(13)

where $h(t)$ is white noise with zero mean and standard deviation is $\sigma^2$, $E$ denotes the expectation.

From (13), we know that the MPS of white noise in the time-frequency domain has linear relationship with the frequency. If white noise is normally distributed with one degree of freedom (DOF), it seems reasonable that the MPS of the ST in the time-frequency domain is $\chi^2$ distributed with two DOF. The hypothesis can be proved by the Monte Carlo statistical method.

First of all, let $x(t) = s(t) + n(t)$, where $s(t)$ is signal and $n(t)$ is white noise. The null hypothesis $H_0$ and the alternative hypothesis $H_1$ are presented. Assume that

$H_0 : x(t)$ is noise,

$H_1 : x(t)$ included the interesting signal .

Give a significant level at $\mu$ and the confidence level is $1 - \mu$, then calculate power spectrum of the noisy signal. Let $Z(\tau, f) = |S(\tau, f)|^2 / |S(\tau, f)|_{mean}^2$. If the probability $P(Z(\tau, f) \leq \chi^2_\mu)$ is an occurrence, we accept $H_0$ and reject $H_1$. Because of the distribution for ST power spectrum is $\chi^2$ with two DOF, therefore,

$$ P(Z(\tau, f) \leq \chi^2_\mu) = 1 - \mu . $$

(14)

The accepted domain

$$ |S(\tau, f)|^2 \leq \frac{1}{2} |S(\tau, f)|_{mean}^2 \chi^2(\mu) $$

(15)

whereas the rejected domain

$$ |S(\tau, f)|^2 > \frac{1}{2} |S(\tau, f)|_{mean}^2 \chi^2(\mu) . $$

(16)

4. Simulation Experiments

Previous studies demonstrate that time-frequency filter efficiency depends on the inverse transform algorithms and the area in time-frequency domain. In this section, there is an example to evaluate the performance of the proposed method. Some synthetic test signals would be used to analyze and MSE in the time domain to evaluate the time-frequency filter performance.

Removing the white noise: In this example, a signal $x(t) = \cos(10\pi t/128)$ interfered by the white noise with zero mean and standard deviation 0.1 are plotted in Fig.1 (a) and Fig.1 (b). First of all, it is important to distinguishing signal from noise in the time-frequency. Fig.1(c) shows the ST spectrum of the noisy signal and statistic characteristics of the white noise. According to the (15), if the ST amplitude of the noisy signal greater than statistic characteristics of the white noise, it is reasonable to believe that the area contains all useful characteristic signals, otherwise, there are noises. Therefore, conventionally time-frequency filters are set the interesting areas to one and the others to zeros, as
illustrated in Fig.1 (d). The filtered output signals after different IST are plotted in Fig.1 (e), which shows that the four ISTs have similar performance, but the proposed LT IST has advantage over the other ones.

FIGURE 1. (a) The original noiseless signal. (b) The noisy signal in time domains interfered by white noise with zero mean and standard deviation 0.1. (c) Noisy signal in the S-transform time-frequency domain relative to MPS. (d) The filtered signal in the time-frequency domain. (e) The left is filtered signals after four different IST and the right is their errors.

5. Conclusion and Discuss
This paper presented a new strategy for the local matrix inverse S-transform. The LT IST was proposed, which can avoid restrict errors when the data processing involves a manipulation of special time in the time-frequency domain. The MPS of the white noise in ST was discussed. The results indicated that the local spectrum of the white noise follow a $0.5\chi^2$ distribution with two DOF. What’s more, a filtering algorithm based on hypothesis testing was proposed, and the simulation results
showed that the proposed algorithm is effective to distinguish signal from noise. In fact, during the realization process of time-frequency filters, errors will always exist in the result because of time-frequency domain are modified and not the same as noiseless signal. The proposed LT IST algorithm is suit to extract specific signal in local time-frequency domain. If the specific frequency domain signal wants to be extracted, LT IST algorithm in matrix form can be modified. The corresponding column components of inverse transform matrix would multiply by one and others by zero.

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