TOTALLY GEODESIC SUBVARIETIES IN THE MODULI SPACE OF CURVES

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Abstract. In this paper we study totally geodesic subvarieties \( Y \subset A_g \) of the moduli space of principally polarized abelian varieties with respect to the Siegel metric, for \( g \geq 4 \). We prove that if \( Y \) is generically contained in the Torelli locus, then \( \dim Y \leq (7g - 2)/3 \).

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1. Introduction

Denote by \( M_g \) the moduli space of smooth complex projective curves of genus \( g \), by \( A_g \) the moduli space of principally polarized abelian varieties and by \( j : M_g \to A_g \) the period map. The Torelli locus \( T_g \) is the closure of \( j(M_g) \) in \( A_g \). It is interesting to relate \( T_g \) to the geometry of \( A_g \) as a locally symmetric variety. We refer to [19, 4, 5, 10] for more information and motivation. In particular we are interested in totally geodesic subvarieties \( Y \) of \( A_g \), i.e. algebraic subvarieties that are images of totally geodesic submanifolds of Siegel space \( \mathcal{S}_g \). Shimura subvarieties are an important subclass of totally geodesic subvarieties, related to Hodge theory and arithmetic \([20]\). One expects that there are very few totally geodesic subvarieties of \( A_g \) that are generically contained in \( T_g \) i.e. such that \( Y \subset T_g \) and \( Y \cap j(M_g) \neq \emptyset \). As for Shimura varieties, following Coleman and Oort, one expects that for large \( g \) there are no such varieties generically contained in \( T_g \), see [19].

An important step in the study of the extrinsic geometry of \( T_g \) inside \( A_g \) was the computation of the second fundamental form of the period map (which is an embedding outside the hyperelliptic locus [21]). This was accomplished in [6] and refined in [5]. Unfortunately this leads only rarely to explicit formulae. But it is enough to get an upper bound for the dimension of a totally geodesic subvariety \( Y \) generically contained in \( T_g \) in terms of the gonality of a point of \( Y \cap j(M_g) \), see [5]. From this one gets...
a bound without gonality assumptions: \( \dim Y \leq \frac{5(g - 1)}{2} \), as soon as \( g \geq 4 \) and \( Y \) is not contained in the hyperelliptic locus.

In this paper we prove the following.

**Theorem 1.1.** Let \( Y \) be a totally geodesic subvariety of \( A_g \) that is generically contained in the Torelli locus. If \( g \geq 4 \), then \( \dim Y \leq \frac{7g - 2}{3} \).

Our proof contains two new ideas. First of all, we use directly the geodesic curves in \( \mathcal{S}_g \). We are able to relate them to Hodge theory (see Lemma 3.3). The Hodge bundle of the (real one-dimensional) family of abelian varieties represented by the geodesic has nice properties with respect to the Fujita decomposition \([9, 2]\)). Such properties do not hold for general families \([12]\).

Secondly, our proof depends heavily on some recent results obtained in \([22, 11]\). In fact starting from the geodesics, we build a complex one-dimensional family of curves (contained in the \( Y \)) and we apply the above-mentioned results to this family. This allows to split the proof in two cases: in the first case one of the main theorems in \([22, 11]\) yields a map of the whole family onto a fixed curve. From this one easily gets a better bound on \( \dim Y \). In the second case one is able to control the Clifford index and the gonality of a point of \( Y \). Then an application of the bound in \([5]\) yields the result.

The plan of the paper is the following: in §2 we recall some definitions and some results from \([22, 11]\). In §3 we study the relation of geodesics to Hodge theory. In §4 we prove Theorem 1.1.

We refer to \([15, 7, 14, 16, 17, 23]\) for related results obtained by different methods.

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2. **Preliminaries on weight 1 variations of the Hodge structure.**

The sheaf of \( k \)-forms (resp. \((p, q)\)-forms) on a complex manifold \( M \) will be denoted by \( \mathcal{A}^k_M \) (resp. \( \mathcal{A}^{p,q}_M \)). The sheaf holomorphic \( k \)-forms is denoted by \( \Omega^k_M \).

2.1. Given an exact sequence \( 0 \to E \to F \to \pi G \to 0 \) of holomorphic vector bundles over a complex manifold \( M \) and a (complex) connection \( \nabla \) on \( F \), the second fundamental form \( \sigma \in \mathcal{A}^{1,0}_M(E^* \otimes G) \) is defined in the following way: given \( u \in E_x \), extend \( u \) to a local section \( \tilde{u} \) of \( E \) and set \( \sigma(u) := \pi((\nabla \tilde{u})(x)) \). When \( \sigma \equiv 0 \), we have \( \nabla s \in \mathcal{A}^1_M(F) \) for any section \( s \in \Gamma(F) \). This means that the connection \( \nabla \) restricts to the bundle \( F \) and defines a connection there.

If \( \nabla \) is flat, \( \sigma \) is in fact holomorphic, i.e. \( \sigma \in H^0(B, \Omega^1_B \otimes E^* \otimes G) \).

2.2. Let \((\mathbb{H}_Z, \mathcal{H}^{1,0}, Q)\) be a polarized variation of the Hodge structure (shortly, PVHS) of weight 1 over a complex manifold \( B \). Here \( \mathbb{H}_Z \) denotes the local system of lattices, \( \mathcal{H}^{1,0} \) the Hodge bundle (that in weight 1 determines the Hodge filtration) and \( Q \) the polarization. We also let \( \mathbb{H}_C = \mathbb{H}_Z \otimes \mathbb{C} \) denote the local system of complex vector spaces and \( \mathcal{H} = \mathbb{H}_C \otimes \mathcal{O}_B \) the associated holomorphic flat bundle with flat connection \( \nabla \). This flat holomorphic connection is in fact defined by setting the kernel equal to \( \mathbb{H}_C \). The Hodge metric is defined by \( h(v, w) := iQ(v, w) \). It is positive definite on \( \mathcal{H}^{1,0} \). Hence the orthogonal projection \( p : \mathcal{H} \to \mathcal{H}^{1,0} \) is well-defined and \( \nabla^{hdg} = p\nabla_{\mathcal{H}^{1,0}} : \mathcal{H} \to \mathcal{H}^{1,0} \otimes \mathcal{A}^1_B \) is the Chern connection of the Hermitian bundle \((\mathcal{H}^{1,0}, h)\).
2.3. We recall the definition of Siegel upper half-space. Let \( \omega = \sum_i dx_i \wedge dy_i \) be the standard symplectic form on \( V := \mathbb{R}^{2g} \). If \( J \in \text{End} V \), \( J^2 = -\text{id}_V \) and \( J^* \omega = \omega \), then \( g_J(x, y) := \omega(x, Jy) \) is a nondegenerate symmetric bilinear form on \( \mathbb{R}^{2g} \). The Siegel upper half-space is defined as \( \mathcal{S}_g := \{ J \in \text{End} V : J^2 = -\text{id}_V, J^* \omega = \omega, g_J \text{ is positive definite} \} \). It is a symmetric space of the non-compact type.

Set \( V_C := V \otimes \mathbb{C} \). For any \( J \in \mathcal{S}_g \) the space \( V \) can be endowed with the complex structure \( J \). We denote by \( V_{1,0} \subset V_C \) the space of its vectors of type \((1, 0)\). We also set \( H_{j,0}^1 := \text{Ann}(V_{1,0}) \subset V_C \). Let \( L \subset V_C^* \) denote the form of sets that are integer-valued on \( \mathbb{Z}^{2g} \subset V \). \( L \) is a lattice in \( V^* \). The symplectic form \( \omega \) induces an isomorphism \( \phi : V \cong V^* \) in the usual way. We denote by \( \mathcal{Q} \) the symplectic form on \( V^* \) obtained by transporting \( \omega \) to \( V^* \) via \( \phi \). With these data we get a polarized variation of the Hodge structure on \( \mathcal{S}_g \); the local system is \( \mathbb{H} := \mathcal{S}_g \times \Lambda \), Hodge bundle is \( H_{1,0} \) and the polarization is \( \mathcal{Q} \).

2.4. If \( J \in \mathcal{S}_g \), then \( T_J \mathcal{S}_g = \{ X \in \text{End} V : XJ + JX = 0, \omega(X, y) + \omega(x, y) = 0, \forall x, y \in V \} \). The Siegel upper half-space has an integrable complex structure that on \( T_J \mathcal{S}_g \) acts by the rule \( X \mapsto JX \). If \( X \in T_J \mathcal{S}_g \), we can complexify \( X \) and its complexification (still denoted by \( X \)) maps \( V_{1,0} \) to \( V_{0,1} \) and vice versa. The transpose of \( X \), denoted by \( X^* \) maps therefore \( H_{j,0}^1 \) to \( H_{j,1}^0 := H_{j,0}^1 \). The map \( X \mapsto X^{|H_{j,1}} \in \text{Hom}(H_{1,0}, H_{0,1}) \) yields an isomorphism

\[
(2.1) \quad T_J \mathcal{S}_g \cong W_J := \{ L \in \text{Hom}(H_{1,0}, H_{0,1}) : \mathcal{Q}(L\alpha, \beta) + \mathcal{Q}(\alpha, T\beta) = 0 \}.
\]

Therefore we can identify the tangent bundle \( T \mathcal{S}_g \) with the subbundle \( W \subset \text{Hom}(H_{1,0}, H_{0,1}) \) defined by \( (2.1) \).

The Hodge bundle \( H_{1,0} \) is provided with the connection \( \nabla^{\text{hdg}} \), as happens for every VHS. Since \( H_{0,1} \cong (H_{1,0})^* \), the connection \( \nabla^{\text{hdg}} \) induces a connection on \( H_{0,1} \) that we denote by \( \nabla^* \). So we get an induced connection \( \nabla^{\text{Hom}} \) on the bundle \( \text{Hom}(H_{1,0}, H_{0,1}) \). The Levi-Civita connection of the symmetric metric coincides with the restriction of \( \nabla^{\text{Hom}} \) to \( W \cong T \mathcal{S}_g \).

2.5. Assume now that \( B \) is a Riemann surface and that \( (\mathbb{H}^g, H_{1,0}, \mathcal{Q}) \) is a PVHS of weight 1 on \( B \). Consider the exact sequence

\[
(2.2) \quad 0 \longrightarrow H_{1,0} \longrightarrow H \longrightarrow \pi_{0,1} \longrightarrow H/H_{1,0} \longrightarrow 0.
\]

Consider on \( H \) the flat connection and let \( \sigma : H_{1,0} \rightarrow H/H_{1,0} \otimes \Omega_B^1 \) be the corresponding second fundamental form, as defined in \( (2.1) \). Using \( \nabla \) and \( \sigma \) we define two vector subbundles of \( H_{1,0} \).

**Definition 2.6.** Let \( \mathcal{U} \) denote the subsheaf of \( H_{1,0} \) spanned by \( \nabla^{\text{hdg}} \)-flat sections (equivalently by \( \nabla \)-flat sections). Set

(i) \( \mathcal{U} := \mathcal{U} \otimes \mathcal{O}_B \),

(ii) \( \mathcal{K} := \ker(\sigma : H_{1,0} \rightarrow H/H_{1,0} \otimes \Omega_B^1) \).

We call \( \mathcal{U} \) the unitary flat bundle and \( \mathcal{K} \) the kernel bundle of the variation, respectively.

By definition \( \mathcal{U} \) is a holomorphic flat bundle. Since \( \nabla^{\text{hdg}} \) is the metric connection, it is unitary. Since \( \nabla \) is flat, \( \sigma \) is holomorphic, so \( \mathcal{K} \) is a coherent subsheaf. If \( \sigma \) is a morphism of constant rank, then \( \mathcal{K} \) is also a vector subbundle of \( H_{1,0} \).
Proposition 2.7. We have $\mathcal{U} \subset \mathcal{K}$.

Proof. If $u \in \mathcal{U}_x$, there is a section $s \in \Gamma(A, \mathcal{U})$ defined on a neighbourhood $A$ of $x$ such that $s(x) = u$. Since $\nabla^{hdg}s = \nabla s \equiv 0$, $\sigma(u) = \pi^{0,1}((\nabla s)(x)) = 0$. 

2.8. The VHS we are interested in come from families of curves. Let $f : C \to B$ be a smooth family of genus $g$ curves. This is a proper and submersive morphism $f$ from a smooth complex surface $C$ to a smooth complex curve $B$ whose fibres are curves of genus $g$. The map $f$ defines a PVHS $(\mathbb{H}^1, \mathcal{H}^{1,0}, \mathcal{Q})$ (called geometric) by taking the local system $\mathbb{H}_Z := R^1f_*\mathbb{Z}$ and the Hodge bundle $\mathcal{H}^{1,0} := f_*\Omega^1_{C/B} = f_*\omega_{C/B}$. The polarization $\mathcal{Q}$ is given by the intersection form. As usual we also have the flat bundle $\mathcal{H} = R^1f_*\mathcal{O} \otimes \mathcal{O}_B$ with the Gauss Manin connection $\nabla$, whose flat sections define the local system $R^1f_*\mathcal{O}$.

We consider the bundles $\mathcal{U}$ and $\mathcal{K}$ of this VHS. In this case both $\mathcal{U}$ and $\mathcal{K}$ can be described in terms of the submersion $f : C \to B$ using holomorphic 1-forms on $C$. We outline shortly this description, referring to [22] and [11] for details. Let

$$\Omega^1_{C,d} = \ker\{d : \Omega^1_{C} \to \Omega^2_{C}\} \subseteq \Omega^1_{C}$$

be the subsheaf of closed holomorphic 1-forms on $C$.

To describe $\mathcal{K}$ consider the exact sequence

$$0 \longrightarrow f^*\omega_B \longrightarrow \Omega^1_{C} \longrightarrow \Omega^1_{C/B} \cong \omega_{C/B} \longrightarrow 0$$

defined by duality using the morphism $df : T_C \to f^*T_B$. Here the cokernel $\Omega^1_{C/B}$ is the sheaf of relative differentials and $\Omega^1_{C/B} \cong \omega_{C/B}$ since $f$ is smooth. Pushing forward (2.3) to $B$, we get the exact sequence

$$0 \longrightarrow f_*f^*\omega_B \cong \omega_B \longrightarrow f_*\Omega^1_{C} \longrightarrow f_*\omega_{C/B} \longrightarrow \partial(R^1f_*\mathcal{O}_C) \otimes \omega_B.$$ 

By a fundamental result of Griffiths (see [13] and [24 Ch. 10]) $\partial$ is the vector bundle morphism that acts on the fibre over $b \in B$ as follows:

$$\partial_b : H^0(C_b, \omega_{C_b}) \longrightarrow H^1(C_b, \mathcal{O}_{C_b}) \otimes T^*_bB,$$

where $\xi_b : T_bB \to H^1(C_b, T_{C_b})$ is the Kodaira-Spencer map. Using the isomorphism $\mathcal{H} \cong H^0(C_b) \cong H^1(C_b, \mathcal{O}_{C_b})$ one can identify $\partial_b$ with the second fundamental form $\sigma_b$ of (2.2). Consequently, we have that

$$\mathcal{K} = \ker \partial.$$

Since $\dim B = 1$, if $v \in T_bB$ is a non-zero vector, then we have

$$\mathcal{K}_b = \ker \sigma_b(v) = \ker \xi_b(v).$$

In particular $\sigma$ has constant rank and $\mathcal{K}$ is a vector subbundle of $f_*\omega_{C/B}$. Moreover the sequence

$$0 \longrightarrow \omega_B \longrightarrow f_*\Omega^1_{C} \longrightarrow \pi \mathcal{K} \longrightarrow 0.$$

is exact.

We recall shortly the definition of Massey products. Fix $b \in B$ and a generator $v \in T_bB$. Using $v$ we get an isomorphism $T_bB \cong \mathbb{C}$. So the exact sequence (2.3) restricted to the fibre $C_b$ reads

$$0 \longrightarrow \mathcal{O}_{C_b} \longrightarrow \Omega^1_{C}|C_b \longrightarrow \omega_{C_b} \longrightarrow 0.$$
We set for simplicity
\[(2.7) \quad E_b := \Omega^1_b|_{C_b}.
\]
The extension class of \((2.6)\) is \(\xi_b(v) \in H^1(C_b, T_{C_b})\). It follows that \(\det E_b \cong \omega_{C_b}\). So we get the adjoint map, first defined in [3]:
\[(2.8) \quad \Phi_b : \bigwedge^2 H^0(E_b) \longrightarrow H^0\left(\bigwedge^2 E_b\right) \cong H^0(\omega_{C_b}).
\]
The long cohomology exact sequence associated to \((2.6)\) starts as follows:
\[
0 \longrightarrow \mathbb{C} \longrightarrow H^0(C_b, E_b) \longrightarrow H^0(\omega_{C_b}) \longrightarrow H^1(C_b, \mathcal{O}_{C_b})
\]
Given \(v_1, v_2\) in \(\mathcal{K}_b = \ker \cup \xi_b\), we lift them, i.e. we take vectors \(\tilde{v}_1, \tilde{v}_2 \in H^0(C_b, E_b)\) such that \(p(\tilde{v}_i) = v_i\).
Let \(\langle v_1, v_2 \rangle \subset H^0(C_b, \omega_{C_b})\) denote the span of \(v_1, v_2\).

**Definition 2.9.** The element
\[
m_b(v_1, v_2) = [\Phi_b(\tilde{v}_1, \tilde{v}_2)] \in H^0(\omega_{C_b})/(v_1, v_2)
\]
is independent of the choice of the liftings and it is called Massey product of \(v_1, v_2 \in \ker \cup \xi_b\).

We notice that \(m_b(v_1, v_2) = [\Phi_b(\tilde{v}_1, \tilde{v}_2)] = 0\) if and only if \(\Phi_b(\tilde{v}_1, \tilde{v}_2) \in (v_1, v_2)\). Since we are assuming that \(f\) is submersive, we have \(\mathcal{K}_b = \ker \cup \xi_b\) by \((2.5)\), so the Massey product is defined for any \(v_1, v_2 \in \mathcal{K}_b\) and for any \(b \in B\).

We now consider the bundle \(\mathcal{U}\) for the VHS coming from the family \(f : C \to B\). On \(C\) there is an exact sequence of sheaves
\[
0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{O}_C \longrightarrow \Omega^1_{\mathcal{C},d} \longrightarrow 0.
\]
We push it forward to \(B\). Since \(f\) is a submersion with compact connected fibres, \(f_* \mathcal{C} \simeq \mathcal{C}_B, f_* \mathcal{O}_C \simeq \mathcal{O}_B\) and \(\omega_B = \ker(d : \mathcal{C} \to \mathcal{O}_b) = \ker(f_* d : f_* \mathcal{C} \to f_* \mathcal{O}_C)\). So we get the exact sequence
\[
0 \longrightarrow \omega_B \longrightarrow f_* \Omega^1_{\mathcal{C},d} \longrightarrow R^1 f_* \mathcal{C} \longrightarrow R^1 f_* \mathcal{O}_C.
\]
By the Splitting Lemma [22] Lemma 3.2] the local system \(U\) underlying \(\mathcal{U}\) fits into the above exact sequence as
\[
0 \longrightarrow \omega_B \longrightarrow f_* \Omega^1_{\mathcal{C},d} \longrightarrow U \longrightarrow 0.
\]
Since \(\mathcal{U} \subset \mathcal{K}\), we can restrict our attention to Massey products on \(\mathcal{U}\).

**Proposition 2.10.** Assume that the Massey products of \(\mathcal{U}\) are all zero, i.e. \(m(v_1, m_2) = 0\) for any \(v_1, v_2 \in \mathcal{U}_b\) and for any \(b \in B\). Then, for any local flat frame \(s_1, \ldots, s_k\) of \(\mathcal{U}\), there are \(\omega_1, \ldots, \omega_m \in H^0(B, f_* \Omega^1_{\mathcal{C},d})\) such that \(\pi(\omega_i) = s_j\) and \(\omega_i \wedge \omega_j = 0\) for any \(i\) and \(j\).

See [22] Prop. 4.3] for the proof. Massey products on \(\mathcal{U}\) contain deep geometric information as is shown by the following version of the classical Castelnuovo de Franchis theorem.

**Theorem 2.11 ([11]).** Let \(f : S \to \Delta\) be a submersive family of smooth projective curves over a disk \(\Delta\). Let \(\omega_1, \ldots, \omega_k \in H^0\left(S, \Omega^1_S\right) (k \geq 2)\) be closed holomorphic 1-forms such that \(\omega_i \wedge \omega_j = 0\) for every \(i, j\) and whose restrictions to a general fibre \(F\) are linearly independent. Then (possibly after shrinking \(\Delta\)) there exist a projective curve \(C\) and a morphism \(\phi : S \to C\) such that \(\omega_i \in \phi^* H^0\left(C, \omega_C\right)\) for every \(i\).
3. A LEMMA ON GEODESICS

3.1. Let $B \subset \mathfrak{G}_g$ be a complex submanifold with $\dim_C B = 1$ and consider the restriction to $B$ of the tautological PVHS on $\mathfrak{G}_g$ introduced in [2.3]. Consider the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{H} \xrightarrow{\pi} \mathcal{H}/\mathcal{K} \longrightarrow 0.$$ 

Using on $\mathcal{H}$ the flat connection $\nabla$ we get a second fundamental form as described in [2.1]

$$\tau_x : T_x B \otimes \mathcal{K}_x \longrightarrow \mathcal{H}_x/\mathcal{K}_x, \quad \tau_x(v \otimes e) := \pi((\nabla_x e)(x)),$$

where $\tilde{e}$ is a local section of $\mathcal{K}$ such that $\tilde{e}(x) = e$. Moreover $\tau$ is a holomorphic section of $\Omega_B^1 \otimes \mathcal{K}^* \otimes \mathcal{H}/\mathcal{K}$.

**Lemma 3.2.** If $\tau \equiv 0$ on $B$, then $\mathcal{K} = \mathcal{U}$.

**Proof.** If $\tau \equiv 0$, the connection $\nabla$ preserves the subbundle $\mathcal{K}$. If $u \in \mathcal{K}_x$, let $s$ be a local section of $\mathcal{H}$ such that $s(x) = u$ and $\nabla s = 0$. Then $s$ lies in $\mathcal{K}$ since $\mathcal{K}$ is preserved by the parallel displacement of $\nabla$. Thus $s$ in fact lies in $\mathcal{U}$ and $u \in \mathcal{U}_x$. Hence $\mathcal{K} \subset \mathcal{U}$. The opposite inclusion is always true. \hfill $\square$

If $\gamma : \mathbb{R} \to \mathfrak{G}_g$ is a non-constant geodesic and $-\infty < a < b < +\infty$, we call $\Gamma := \gamma([a, b])$ a geodesic segment.

**Lemma 3.3.** Let $B \subset \mathfrak{G}_g$ be a complex submanifold with $\dim_C B = 1$. If $B$ contains a geodesic segment, then $\mathcal{K} = \mathcal{U}$.

**Proof.** By the previous lemma it is enough to show that $\tau \equiv 0$ on $B$. Since $\tau$ is holomorphic, if we prove that $\tau$ vanishes on $\Gamma$, then $\tau \equiv 0$ on all $B$ by the identity principle.

To show that $\tau = 0$ on $\Gamma$ fix $x = \gamma(t_0) \in \Gamma$, $v \in T_x B$ and $e \in \mathcal{K}_x$. Since $T_x B$ has complex dimension 1, $v = \lambda \dot{\gamma}(t_0)$ for some $\lambda \in \mathbb{C}$, so it is enough to consider $v = \dot{\gamma}(t_0)$. Let $\tilde{e} = \tilde{e}(t)$ be the section of $\mathcal{K}$ over $\Gamma$ obtained by parallel translation of the vector $e$ with respect to the connection $\nabla^{\text{hdg}}$. A priori $\tilde{e}$ is only a section of $\mathcal{H}^{1,0}$. We claim that in fact $\tilde{e}(t) \in \mathcal{K}_{\gamma(t)}$.

Set $\mathcal{H}^{0,1} := \mathcal{H}/\mathcal{H}^{1,0}$. Since $\mathcal{H}^{0,1} \cong (\mathcal{H}^{1,0})^*$, the connection $\nabla^{\text{hdg}}$ induces a connection on $\mathcal{H}^{0,1}$ that we denote by $\nabla^*$. We get an induced connection on $\text{Hom}(\mathcal{H}^{1,0}, \mathcal{H}^{0,1})$ that we denote by $\nabla^{\text{Hom}}$.

The tangent bundle $T \mathfrak{G}_g$ is a subbundle of $\text{Hom}(\mathcal{H}^{1,0}, \mathcal{H}^{0,1})$ and the Levi-Civita connection for the symmetric metric agrees with the restriction of the connection on $\text{Hom}(\mathcal{H}^{1,0}, \mathcal{H}^{0,1})$. So if $\xi = \xi(t)$ is a section of $T \mathfrak{G}_g$ and $s = s(t)$ is a section of $\mathcal{H}^{1,0}$ we have

$$\nabla^*_\gamma(s) = (\nabla^{\text{Hom}}_\gamma \xi)(s) + \xi(\nabla^{\text{hdg}}_\gamma e).$$

Take $\xi = \dot{\gamma}$ and $s = \tilde{e}$. We have

$$\nabla^{\text{Hom}}_\gamma \dot{\gamma} = \nabla^{\text{Levi-Civita}}_\gamma \dot{\gamma} = 0, \quad \nabla^{\text{hdg}}_\gamma \tilde{e} = 0.$$ 

So $\nabla^*_\gamma(\dot{\gamma}(\tilde{e})) \equiv 0$. Recall now that $e = \tilde{e}(t_0) \in \mathcal{K}_x$. Moreover $\mathcal{K}_{\gamma(t)} = \{ u \in \mathcal{H}^{1,0}_x : \dot{\gamma}(t)(u) = 0 \}$. So $\dot{\gamma}(\tilde{e})(t_0) = 0$. Since we have checked that $\dot{\gamma}(\tilde{e})$ is a parallel section over $\Gamma$, we conclude that $\dot{\gamma}(\tilde{e}) \equiv 0$ on $\Gamma$. This means that $\tilde{e}(t) \in \mathcal{K}_{\gamma(t)}$ for any $t$, as claimed.

To conclude the proof, notice that on $\mathcal{K}$ the connections $\nabla^{\text{hdg}} e$ and $\nabla$ coincide. So

$$\nabla_{\dot{\gamma}(t)} \tilde{e} = \nabla^{\text{hdg}}_\gamma \tilde{e} \equiv 0.$$
Since \( \hat{\varepsilon} \) is a section of \( \mathcal{K} \) we can use it to compute the second fundamental form and we get
\[
\tau_x(v, e) = \pi(\nabla_{\hat{\varepsilon}}(v))(t_0) = 0.
\]

\( \square \)

**Lemma 3.4.** Let \( f : I = [a, b] \rightarrow M \) be a real analytic map in a complex manifold \( M \). Then there is an open subset \( A \subset \mathbb{C} \) containing \( I \) and a holomorphic extension \( h : A \rightarrow M \) of \( f \).

**Proof.** We can easily find a finite family of disks \( \{D_i\}_{i=1}^m \) centred at points \( t_i \in I \) such that (a) \( f(D_i \cap I) \) is contained in the domain of a chart \( U_i = (U_i, \phi_i) = (z^1_i, \ldots, z^n_i) \) of \( M \), (b) on \( D_i \) there are \( n \) holomorphic functions \( h_i^j, j = 1, \ldots, n \), (c) \( z^j_i \circ f = h_i^j \) on \( D_i \cap I \). The function \( f_i := \phi_i^{-1} \circ (h_i^1, \ldots, h_i^n) \) is a holomorphic extension of \( f|_{D_i \cap I} \) to \( D_i \). We claim that these functions glue together. Indeed we start by setting \( h_1 := f_1 \) on \( D_1 \) and we proceed inductively. Assume that a holomorphic extension \( h_{k-1} \) is given on \( D_1 \cup \cdots \cup D_{k-1} \). We claim that \( h_{k-1} = f_k \) on \( D_k \cap (D_1 \cup \cdots \cup D_{k-1}) \). Indeed this set is connected and contains a subinterval of \( I \). \( h_{k-1} = f = f_k \) on this subinterval. Therefore by the identity principle \( h_{k-1} = f_k \) on \( D_k \cap (D_1 \cup \cdots \cup D_{k-1}) \). Thus using \( h_{k-1} \) and \( f_k \) we get a well-defined holomorphic extension \( h_k \) on \( D_1 \cup \cdots \cup D_k \). At the end it is enough to set \( h := h_n \). \( \square \)

4. PROOF OF THE THEOREM

**Proposition 4.1.** Assume that \( g \geq 3 \) and let \( Y \subset M_g \) be a subvariety of of codimension \( c \). For a smooth point \( y \in Y \), there is \( \xi \in T_yY \) such that
\[
\dim(\ker \cup \xi) \geq g - k_0 - 1,
\]
where
\[
k_0 := \left\lfloor \frac{c - 1}{2} \right\rfloor.
\]

**Proof.** Restricting \( Y \) we can assume that it embeds in the Kuranishi family. So \( T_yY \hookrightarrow H^1(C, T_C) \) where \( [C] = y \). Consider the bicanonical image \( X := \text{Bic}(C) := \phi_{2K_C}(C) \subset \mathbb{P}(H^1(C, T_C)) = \mathbb{P}(H^0(C, \omega_C^2)^*) \). Denote by \( S^kX \) the variety of \( k \)-secants of \( X \). Since \( \dim S^kX = 2k + 1 \), we have \( \dim S^{k_0}X \geq c = \text{codim}(\mathbb{P}(T_yY) \subset \mathbb{P}(H^1(C, T_C))) \). Hence \( S^{k_0} \cap \mathbb{P}(T_yY) \) contains at least some point \([\xi]\). By construction there is an effective divisor \( D \) of degree \( k_0 + 1 \) such that
\[
\xi \in \ker(\rho_D : H^1(C, T_C) \rightarrow H^1(C, T_C(D))
\]
where \( \rho_D \) is the map induced by the inclusion \( T_C \hookrightarrow T_C(D) \). Indeed, given an effective divisor \( D \), denote by \( \langle D \rangle \) the intersection of all hyperplanes \( H \subset \mathbb{P}(H^0(C, \omega_C^2)^*) \) such that \( D \leq \phi_{2K_C}H \). Set \( X_{k_0+1} := \{(D, p) \in C^{(k_0+1)} \times \mathbb{P}(H^0(C, \omega_C^2)^* : p \in \langle D \rangle \} \) and denote by \( p_2 \) the second projection. Then \( S^{k_0}X = p_2(X_{k_0+1}) \). So \([\xi] \in \langle D \rangle \) for some \( D \in C^{(k_0+1)} \), which yields (4.3). But
\[
\dim(\ker(\cup \xi : H^0(C, \omega_C^2) \rightarrow H^1(C, O_C))) \geq g - \deg D = g - k_0 - 1.
\]

(See e.g. \[\text{[II]} \text{ Lemma 2.3}.\])

**Proof of Theorem [I].** We argue by contradiction, assuming the existence of a totally geodesic subvariety \( Y \subset A_g \) that is generically contained in \( M_g \) and with \( \dim Y > (7g - 2)/3 \). If \( c \) denotes the codimension of \( Y \cap M_g \) in \( M_g \), this is equivalent to
\[
c < \frac{2g - 7}{3}.
\]
Observe that for \( k_0 \) defined in (4.2), this implies
\[
2k_0 \leq g - 4.
\]
Observe also that by dimension \( Y \) is not contained in the hyperelliptic locus. Fix a smooth point \( y \in Y \) that represents a non-hyperelliptic curve. By Proposition 4.1 there is \( \xi \in T_y Y \) such that \( \xi \neq 0 \) and
\[
\dim(\ker \cup \xi) \geq g - k_0 - 1,
\]
where \( k_0 \) is defined as in (4.2). Let \( \Delta \) be a polydisk and let \( F : \mathcal{X} \to \Delta \) be a Kuranishi family with \( [X_0] = y \). The moduli map \( \pi : \Delta \to M_g, \pi(t) := [X_t] \) is finite, satisfies \( \pi(0) = y \) and its image is a neighbourhood of \( y \). The period mapping \( j \) can be lifted to a map \( \tilde{j} : \Delta \to \mathcal{G}_y \).

Thus \( Y' := \tilde{j}(\pi^{-1}(Y)) \) is a germ of totally geodesic submanifold of \( \mathcal{G}_y \), that contains the point \( y' := \tilde{j}(0) \) and is contained in \( \tilde{j}(\Delta) \). Let \( \gamma : \mathbb{R} \to \mathcal{G}_y \) be the geodesic in \( \mathcal{G}_y \) such that \( \gamma(0) = y' \) and \( \dot{\gamma}(0) = d\tilde{j}(\xi) \). Since \( \xi \neq 0 \) the curve \( \gamma \) is non-constant. Moreover it is real analytic since the Siegel metric is real analytic. Fix \( \varepsilon > 0 \) such that \( \gamma([-\varepsilon, \varepsilon]) \subset Y' \). By Lemma 3.4 there is an open subset \( A \subset \mathbb{C} \) containing \([0, \varepsilon]\) and a holomorphic extension \( h : A \to Y' \) of \( \gamma \). Restricting \( A \) we can assume that \( h \) is an embedding and that it avoids the hyperelliptic locus. Set \( B := h(A) \). Since \( B \subset Y' \subset \tilde{j}(\Delta) \) we can restrict the Kuranishi family to \( \tilde{j}^{-1}(B) \cong B \) and we get a universal family of curves \( f : C \to B \) such that \( b_0 := h(0) = \tilde{j}(y) \). The VHS of this family is simply the restriction to \( B \) of the tautological VHS on \( \mathcal{G}_y \) described in 2.3. We consider the bundles \( \mathcal{U} \) and \( K \) for this VHS on \( B \) (see Definition 2.4). By Lemma 3.3 we have \( \mathcal{U} = K \). But using (2.4) we see that \( K_{b_0} = \ker(\cup \dot{\gamma}(0)) = \ker(\cup \xi) \). At this point we use the bound (4.10). Summing up \( K = \mathcal{U} \) has rank at least \( g - k_0 - 1 \).

Now we consider the Massey products of \( K = \mathcal{U} \) on \( B \) (see definition 2.9).

Assume first that these are identically 0. Take a basis \( u_1, \ldots, u_m \) of \( K_{b_0} \). Up to shrinking \( B \) we can extend these vectors to flat sections \( u_1, \ldots, u_m \) of \( \mathcal{U} \) on \( B \). By Proposition 2.10 there exist unique liftings to sections \( \omega_1, \ldots, \omega_m \) of \( f_*\Omega^1_{C/B} \) (i.e. of closed holomorphic 1-forms on \( C \)) such that \( \omega_i \wedge \omega_j = 0 \), for any \( i, j \). By Theorem 2.11 (i.e. [11, Theorem 1.5]), we get a a morphism \( \phi : C \to C' \) onto a genus \( g' \geq 2 \) smooth compact curve \( C' \), whose restriction to every fibre of \( f \) gives a non-constant degree \( n \) morphism \( \phi : C_0 \to C' \) such that \( \omega_1, \ldots, \omega_m \in \phi^* H^0(\omega_{C'}) \). It follows immediately that \( \text{rk} \mathcal{U} \leq g' \). But also \( g' \leq \text{rk} \mathcal{U} \), since any section given by pull back from \( C' \) is flat and has wedge zero with the others. So we conclude that \( \text{rk} \mathcal{U} = g' \). Recalling the bound on \( \text{rk} K \) established above, we get
\[
g(C') = \text{rk} \mathcal{U} = \text{rk} K \geq g - k_0 - 1.
\]
Since \( f \) is non isotrivial by construction, \( n \geq 2 \), so by the Riemann-Hurwitz formula
\[
2g - 2 \geq 4g - 4k_0 - 4 - 4, \quad 2k_0 \geq g - 3.
\]
But we were assuming (4.4) and hence (4.5). So we would get \( g - 3 \leq 2k_0 \leq g - 4 \), which is clearly absurd. This shows that the Massey products cannot vanish identically.

So there is \( b \in B \) and \( u_1, u_2 \in K_b \), with \( m(u_1, u_2) \neq 0 \). In particular there are \( \tilde{u}_i \in H^0(E_b) \) with \( \tilde{u}_1 \wedge \tilde{u}_2 \neq 0 \). Before using this information, we need to recall a construction already used e.g. in
Consider the sequence (2.6) with extension class $\xi := \xi_b(v)$. From the associated cohomology sequence

$$0 \rightarrow H^0(\mathcal{O}_{C_b}) \cong C \rightarrow H^0(E_b) \rightarrow H^0(\omega_{C_b}) \xrightarrow{\cup \xi} H^1(\mathcal{O}_{C_b})$$

we get $h^0(E_b) = \dim \ker(\cup \xi) + 1$. Recall that $\det E_b \cong \omega_{C_b}$ and the definition of the adjunction map (2.3). We claim that $\ker \Phi_b$ contains a non-zero decomposable element. Indeed we have $h^0(E_b) = \dim \ker(\cup \xi) + 1 \geq g - k_0$, hence $2h^0(E_b) - 4 \geq 2g - 2k_0 - 4$. But recall that we are assuming (4.4). Hence from (4.5) we get

$$(4.7) \quad 2h^0(E_b) - 4 \geq g = h^0(\omega_{C_b}).$$

Since $g \geq \dim \Lambda^2 H^0(E_b) - \dim \ker \phi = \dim \mathbb{P}(\Lambda^2 H^0(E_b)) - \dim \mathbb{P}(\ker \phi) = \dim(\Lambda^2 H^0(E_b)) = 2h^0(E_b) - 4$, it follows from (1.7) that $\mathbb{G}(2, \Lambda^2 H^0(E_b)) \cap \mathbb{P}(\ker \phi) \neq \emptyset$, so there is a decomposable element $s_1 \wedge s_2 \in \ker \phi - \{0\}$, as claimed. In other words there are linearly independent sections $s_1, s_2 \in H^0(E)$ such that $s_1(x)$ and $s_2(x)$ are always proportional. Let $\mathfrak{F}$ be the subsheaf of $\mathcal{O}_{C_b}(E_b)$ generated by $s_1$ and $s_2$. The saturation $\mathcal{L}$ of $\mathfrak{F}$ is the sheaf of sections of a line bundle $L$. By construction $s_1, s_2 \in H^0(L)$ are linearly independent, so $h^0(L) \geq 2$. The quotient sheaf $\mathcal{O}_{C_b}(E_b)/\mathcal{L}$ is also the sheaf of sections of a line bundle $M$. Thus

$$0 \longrightarrow \mathcal{O}_{C_b} \xrightarrow{i} E_b \longrightarrow \omega_{C_b} \longrightarrow 0$$

$s_3 := \beta \circ i$ is a section of $M$. We claim that $s_3 \neq 0$. Indeed if $s_3 \equiv 0$, we would have $i(1) = \alpha(s)$ for a nonvanishing section $\sigma \in H^0(L)$. But then $L$ would be trivial and $s_1$ and $s_2$ would be linearly dependent. Thus $s_3 \neq 0$. Let $D$ be the divisor of zeros of $s_3$. Then $M = \mathcal{O}_{C_b}(D)$. Since $\det E_b = L \otimes M \cong \omega_{C_b}$, $L = \omega_{C_b}(-D)$.

Now we are finally able to use $\tilde{u}_1$ and $\tilde{u}_2$. Since $\tilde{u}_1 \wedge \tilde{u}_2 \neq 0$, the sections $\tilde{u}_i$ do not lie both in $L$. Hence at least one of them has a non-trivial image $s_4$ in $M$. The section $s_3$ and $s_4$ are independent, so $h^0(M) = h^0(\mathcal{O}(D)) \geq 2$. From the diagram one gets

$$\dim \ker(\cup \xi) \leq g - (\deg D - 2h^0(D) + 2) = g - \text{Cliff}(D).$$

(see e.g. [1 Lemma 2.3]). Since $h^0(D) \geq 2$ and $h^0(\omega_C(-D)) \geq 2$, the divisor $D$ contributes to the Clifford index. Therefore $\text{Cliff}(C_b) \leq \text{Cliff}(D)$. It is known that $\text{gon}(C_b) \leq \text{Cliff}(C_b) + 3$, see [8 Thm 2.3]. By the Lemma and [4.1] $\text{Cliff}(D) \leq g - \dim \ker(\cup \xi) \leq k_0 + 1$. Thus

$$\text{gon}(C_b) \leq \text{Cliff}(C_b) + 3 \leq \text{Cliff}(D) + 3 \leq k_0 + 4.$$
Now we can apply [5, Theorem 4.2]: since $Y \subset M_g$ is totally geodesic and $[C_b] \in Y$ is not hyperelliptic, we have
\[
\dim Y \leq 2g + \text{gon}(C_b) - 4.
\]
For $g \geq 4$ we have $g - 2 \geq (2/3)g - 1$. Hence
\[
3g - 3 - c = \dim Y \leq 2g + k_0 + 4 - 4 = 2g + \left\lfloor \frac{c - 1}{2} \right\rfloor \leq 2g + \frac{c - 1}{2} + 1.
\]
But this gives $(2g - 7)/3 \leq c$, which yields the desired contradiction with (4.4). \hfill \Box

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