Generalized Möbius Ladder and Its Metric Dimension

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Abstract. In this paper we introduce generalized Möbius ladder $M_{m,n}$ and give its metric dimension. Moreover, it is observed that, depending on even and odd values of $m$ and $n$, it has two subfamilies with constant metric dimensions.

Keywords. Metric dimension, Resolving set, Generalized Möbius ladder

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1. Introduction

The concepts of metric dimension and resolving set were introduced by Slater in [16, 17] and studied independently by Harary and Melter in [5]. Since then the resolving sets have been widely investigated, as you can see in [2, 3, 4, 5, 15, 18, 19]. Applications of metric dimension to the navigation of robots in networks are discussed in [11], to chemistry in [4], and to image processing in [13].

A graph $G$ is a pair $(V(G), E(G))$, where $V$ is the set of vertices and $E$ is the set of edges. A path from a vertex $v$ to a vertex $w$ is a sequence of vertices and edges that starts from $v$ and stops at $w$. The number of edges in a path is the length of that path. A graph is said to be connected if there is a path between any two of its vertices. The distance $d(u, v)$ between two vertices $u, v$ of a connected graph $G$ is the length of a shortest path between them.
Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of $G$ and let $v$ be a vertex of $G$. The representation $r(v|W)$ of $v$ with respect to $W$ is the $k$-tuple $(d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. If distinct vertices of $G$ have distinct representations with respect to $W$, then $W$ is called a resolving set for $G$ [1]. A resolving set of minimum cardinality is called a basis of $G$; the number of elements in this basis is the metric dimension of $G$, $\dim(G)$. A family $\mathcal{G}$ of connected graphs is said to have constant metric dimension if it is independent of any choice of member of that family.

The metric dimension of wheel $W_n$ is determined by Buczkowski et al. [1], of fan $f_n$ by Caceres et al. [2], and of Jahangir graph $J_{2n}$ by Tomescu et al. [3]. Chartrand et al. [4] proved that the family of path $P_n$ has the constant metric dimension 1. Javaid et al. proved in [5] that the plane graph antiprism $A_n, n \geq 5$ constitutes a family of regular graphs with constant metric dimension 3. The metric dimensions of some classes of plane graphs and convex polytopes have been studied in [6], of generalized Petersen graphs $P(n, 3)$ in [7], of some rotationally-symmetric graphs in [8]. The part of the metric dimension of the Möbius ladder $M_n$ is determined in [9], while the remaining part is determined in [10].

This paper is organized as follows: The generalized Möbius Ladder is introduced in Section 2, its metric dimension is given in Section 3, and the examples are given in Section 4. The conclusive remarks are given in the last section.

2. Generalized Möbius Ladder

Consider the Cartesian product $P_m \times P_n$ of paths $P_m$ and $P_n$ with vertices $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$, respectively. Take a $180^\circ$ twist and identify the vertices $(u_1, v_1), (u_1, v_2), \ldots, (u_1, v_n)$ with the vertices $(u_m, v_1), (u_m, v_{n-1}), \ldots, (u_m, v_1)$, respectively, and identify the edge $((u_1, i), (u_1, i + 1))$ with the edge $((u_m, v_{n+1-i}), (u_m, v_{n-i}))$, where $1 \leq i \leq n - 1$. What we receive is the generalized Möbius ladder $M_{m,n}$. You may observe that we receive the usual Möbius ladder for $n = 2$ and for any odd integer $m \geq 4$. You can see $M_{7,3}$ in the following figure.
For brevity we shall use the symbol $v_{ij}$ (or simply $ij$) to represent the vertex $(u_i, v_j)$ of $M_{m,n}$, as you can see in the figure:

The generalized Möbius ladder obtained from $P_7 \times P_3$ is:

3. The Main Results

This section contains the metric dimension of the generalized Möbius ladder $M_{m,n}$. The results confirm that $M_{m,n}$ has two subfamilies with constant metric dimensions.

**Theorem 3.1.** The metric dimension of $M_{m,n}$, $m - n \geq 3$, is 3 when one of $m$ and $n$ is even and other is odd.
Proof. We claim that the resolving set in this case is \( W = \{v_{1,1}, v_{1,n}, v_{m-1,1}\} \). It means the distance vectors corresponding to the vertices of \( W \) are distinct, and no set with less than 3 vertices serves as a resolving set. In order to find distance vectors we involve two parameters, \( q \) and \( i \), and depending on their different values we divide the entries of distance vectors into three steps:

**Step I.** [Distances of \( v_{1,1} \) with all vertices of \( M_{m,n} \)] In this case for each value of \( q \in \{1, \ldots, n\} \) the parameter \( i \) varies from 1 to \( m - 1 \). The entries of distance vectors are

\[
d(v_{1,1}, v_{i,q}) = \begin{cases} i + q - 2 & 1 \leq i \leq \frac{1}{2}(m + n - 2q + 1) \\ m + n - q - i & \frac{1}{2}(m + n - 2q + 3) \leq i \leq m - 1 \end{cases}
\]

**Step II.** [Distances of \( v_{1,n} \) with all vertices of \( M_{m,n} \)] Here, again, for each value of \( q \in \{1, \ldots, n\} \) the parameter \( i \) varies from 1 to \( m - 1 \), and we get

\[
d(v_{1,n}, v_{i,q}) = d(v_{1,1}, v_{i,n+1-q}).
\]

**Step III.** [Distances of \( v_{m-1,1} \) with all vertices of \( M_{m,n} \)] In the following we have two parts:

a) For \( q = 1 \) we have

\[
d(v_{m-1,1}, v_{i,q}) = \begin{cases} i + n - 1 & 1 \leq i \leq \frac{1}{2}(m - n - 1) \\ m - 1 - i & \frac{1}{2}(m - n + 1), 1 \leq i \leq m - 1 \end{cases}
\]

b) For each value of \( q \in \{2, \ldots, n\} \) the parameter \( i \) varies from 1 to \( m - 1 \), and we get

\[
d(v_{m-1,1}, v_{i,q}) = d(v_{1,1}, v_{i,n+2-q}).
\]

Now we show that no set with less than three vertices is a resolving set. For this it is enough to show that the removal of a single vertex from any resolving set of three vertices does resolve the graph anymore. For easy understanding let us take \( W = \{v_{1,1}, v_{1,n}, v_{m-1,1}\} \) as the resolving set; in this case three different possibilities arise:

**Possibility I.** If we take \( W_1 = \{v_{1,1}, v_{1,n}\} \), then \( d(v_{i,j}|W_1) = d(v_{m-i+1,n-j+1}|W_1) \); here for each value of \( i \in \{2, \ldots, m - 1\} \) the value of \( j \) varies from 1 to \( n \).

**Possibility II.** If we take \( W_2 = \{v_{1,1}, v_{m-1,1}\} \), then \( d(v_{i,j}|W_2) = d(v_{m-i,n-j+2}|W_2) \); here for each value of \( i \in \{1, \ldots, m - 1\} \) the value of \( j \) varies from 2 to \( n \).

**Possibility III.** If we take \( W_3 = \{v_{1,n}, v_{m-1,1}\} \), then \( d(v_{i,j}|W_3) = d(v_{i+1,j+1}|W_3) \); here \( i = j \) with \( 1 \leq i, j \leq n - 1 \).

\[ \square \]

**Theorem 3.2.** The metric dimension of \( M_{m,n}, m - n \geq 4 \), is 4 when \( m \) and \( n \) are both even or odd.

Proof. Here the resolving set is \( W = \{v_{1,1}, v_{1,n}, v_{m-1,1}, v_{m-1,n}\} \). It means the distance vectors corresponding to these vertices are distinct, and no set with less than 4 vertices serves as a resolving set. Here we involve two parameters, \( q \) and \( i \), and depending on their different values we divide the entries of distance vectors into four steps:
Step I. [Distances of $v_{1,1}$ with all vertices of $M_{m,n}$] In this case for each value of $q \in \{1, \ldots, n\}$ the parameter $i$ varies from 1 to $m - 1$. The entries of distance vectors are:

$$d(v_{1,1}, v_{i,q}) = \begin{cases} i + q - 2 & 1 \leq i \leq \frac{1}{2}(m + n - 2q + 2) \\ m + n - q - i & \frac{1}{2}(m + n - 2q + 4) \leq i \leq m - 1 \end{cases}$$

Step II. [Distances of $v_{1,n}$ with all vertices of $M_{m,n}$] Here, again, for each value of $q \in \{1, \ldots, n\}$ the parameter $i$ varies from 1 to $m - 1$, and we get $d(v_{1,n}, v_{i,q}) = d(v_{1,1}, v_{i,n+1-q})$.

Step III. [Distances of $v_{m-1,1}$ with all vertices of $M_{m,n}$] In the following we have two parts:

a) For $q = 1$ we have

$$d(v_{m-1,1}, v_{i,q}) = \begin{cases} i + n - 1 & 1 \leq i \leq \frac{1}{2}(m - n) \\ m - 1 - i & \frac{1}{2}(m - n + 2), 1 \leq i \leq m - 1 \end{cases}$$

b) For each value of $q \in \{2, \ldots, n\}$ the parameter $i$ varies from 1 to $m - 1$, and we get $d(v_{m-1,1}, v_{i,q}) = d(v_{1,1}, v_{i,n+2-q})$.

Step IV. Here for each value of $q \in \{1, \ldots, n\}$ the parameter $i$ varies from 1 to $m - 1$, and we get $d(v_{m-1,n}, v_{i,q}) = d(v_{m-1,1}, v_{i,n+1-q})$.

Now we show that no set with less than four vertices is a resolving set. For this it is enough to show that the removal of a single vertex from any resolving set of four vertices does resolve the graph anymore. For easy understanding let us take $W = \{v_{1,1}, v_{1,n}, v_{m-1,1}, v_{m-1,n}\}$ as the resolving set. In this case four different possibilities arise:

**Possibility I.** If we take $W_1 = \{v_{1,1}, v_{1,n}, v_{m-1,1}\}$, then $d(v_{m-n+2,1}|W_1) = d(v_{m-n+2,1}|W_1)$.

**Possibility II.** If we take $W_2 = \{v_{1,n}, v_{m-1,1}, v_{m-1,n}\}$, then $d(v_{m-n,1}|W_2) = d(v_{m-n,1}|W_2)$.

**Possibility III.** If we take $W_3 = \{v_{1,1}, v_{m-1,1}, v_{m-1,n}\}$, then $d(v_{m-n,2}|W_3) = d(v_{m-n,2}|W_3)$.

**Possibility IV.** If we take $W_4 = \{v_{1,1}, v_{1,n}, v_{m-1,n}\}$, then $d(v_{m-n+2,2}|W_4) = d(v_{m-n+2,2}|W_4)$.

□

4. Examples

In this section two examples are presented, one related to Theorem 3.1 and second related to Theorem 3.2.

**Example.** Consider $M_{7,4}$. The resolving set in this case is $W = \{v_{1,1}, v_{1,4}, v_{6,1}\}$.
The full table of size $24 \times 24$ representing the distances among the vertices of $M_{7,4}$ is split up into two sub-tables, each having size $24 \times 12$, and the distances corresponding to resolving vertices are given in bold form:

\[
\begin{array}{cccccccccccc}
  & v_{11} & v_{12} & v_{13} & v_{14} & v_{21} & v_{22} & v_{23} & v_{24} & v_{31} & v_{32} & v_{33} & v_{34} \\
v_{11} & 0 & 1 & 2 & 3 & 1 & 2 & 3 & 4 & 2 & 3 & 4 & 4 \\
v_{12} & 1 & 0 & 1 & 2 & 2 & 1 & 2 & 3 & 3 & 2 & 3 & 4 \\
v_{13} & 2 & 1 & 0 & 1 & 3 & 2 & 1 & 2 & 4 & 3 & 2 & 3 \\
v_{14} & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 5 & 4 & 3 & 2 \\
v_{21} & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 1 & 2 & 3 & 4 \\
v_{22} & 2 & 1 & 2 & 3 & 1 & 0 & 1 & 2 & 2 & 1 & 2 & 3 \\
v_{23} & 3 & 2 & 1 & 2 & 2 & 1 & 0 & 1 & 3 & 2 & 1 & 2 \\
v_{24} & 4 & 3 & 2 & 1 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 \\
v_{31} & 2 & 3 & 4 & 4 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 \\
v_{32} & 3 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 0 & 1 & 2 \\
v_{33} & 4 & 3 & 2 & 3 & 2 & 1 & 2 & 2 & 1 & 0 & 1 &  \\
v_{34} & 4 & 4 & 3 & 2 & 4 & 3 & 2 & 1 & 3 & 2 & 1 & 0 \\
v_{41} & 3 & 4 & 4 & 3 & 2 & 3 & 4 & 4 & 1 & 2 & 3 & 4 \\
v_{42} & 4 & 3 & 4 & 4 & 3 & 2 & 3 & 4 & 2 & 1 & 2 & 3 \\
v_{43} & 4 & 3 & 3 & 4 & 4 & 3 & 2 & 3 & 3 & 2 & 1 & 2 \\
v_{44} & 3 & 4 & 4 & 3 & 4 & 4 & 3 & 2 & 4 & 3 & 2 & 1 \\
v_{51} & 4 & 4 & 3 & 2 & 3 & 4 & 4 & 3 & 2 & 3 & 4 & 4 \\
v_{52} & 4 & 3 & 2 & 3 & 4 & 3 & 3 & 4 & 3 & 2 & 3 & 4 \\
v_{53} & 3 & 2 & 3 & 4 & 4 & 3 & 3 & 4 & 4 & 3 & 2 & 3 \\
v_{54} & 2 & 3 & 4 & 4 & 3 & 4 & 4 & 3 & 4 & 3 & 2 &  \\
v_{61} & 4 & 3 & 2 & 1 & 4 & 5 & 4 & 3 & 3 & 4 & 5 & 4 \\
v_{62} & 3 & 2 & 1 & 2 & 4 & 3 & 2 & 3 & 4 & 3 & 3 & 4 \\
v_{63} & 2 & 1 & 2 & 3 & 3 & 2 & 3 & 4 & 4 & 3 & 3 & 4 \\
v_{64} & 1 & 2 & 3 & 4 & 2 & 3 & 4 & 4 & 3 & 4 & 4 & 3 \\
\end{array}
\]
**Example.** The resolving set of $M_{10,2}$ is $W = \{v_{1,1}, v_{1,2}, v_{9,1}, v_{9,2}\}$. Here, again, the full table of size $18 \times 18$ representing the distances among the vertices of $M_{7,4}$ is split up into two sub-tables, each having size $18 \times 9$:  

|   | $v_{41}$ | $v_{42}$ | $v_{43}$ | $v_{44}$ | $v_{51}$ | $v_{52}$ | $v_{53}$ | $v_{54}$ | $v_{61}$ | $v_{62}$ | $v_{63}$ | $v_{64}$ |
|---|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $v_{11}$ | 3 | 4 | 4 | 3 | 4 | 4 | 3 | 2 | 4 | 3 | 2 | 1 |
| $v_{12}$ | 4 | 3 | 3 | 4 | 3 | 2 | 3 | 3 | 3 | 2 | 1 | 2 |
| $v_{13}$ | 4 | 3 | 3 | 4 | 3 | 2 | 4 | 2 | 1 | 2 | 2 | 3 |
| $v_{14}$ | 3 | 4 | 4 | 3 | 2 | 3 | 4 | 4 | 1 | 2 | 3 | 4 |
| $v_{21}$ | 2 | 3 | 4 | 4 | 3 | 4 | 4 | 3 | 4 | 4 | 3 | 2 |
| $v_{22}$ | 3 | 2 | 3 | 4 | 4 | 3 | 3 | 4 | 4 | 3 | 2 | 3 |
| $v_{23}$ | 4 | 3 | 2 | 3 | 4 | 3 | 3 | 4 | 3 | 2 | 3 | 3 |
| $v_{24}$ | 4 | 4 | 3 | 2 | 3 | 4 | 4 | 3 | 2 | 3 | 4 | 4 |
| $v_{31}$ | 1 | 2 | 3 | 4 | 2 | 3 | 4 | 4 | 3 | 4 | 4 | 3 |
| $v_{32}$ | 2 | 1 | 2 | 3 | 3 | 2 | 3 | 4 | 4 | 3 | 4 | 4 |
| $v_{33}$ | 3 | 2 | 1 | 2 | 4 | 3 | 2 | 3 | 4 | 4 | 3 | 4 |
| $v_{34}$ | 4 | 3 | 2 | 1 | 4 | 4 | 3 | 2 | 3 | 4 | 4 | 3 |
| $v_{41}$ | 0 | 1 | 2 | 3 | 1 | 2 | 3 | 4 | 2 | 3 | 4 | 4 |
| $v_{42}$ | 1 | 0 | 1 | 2 | 2 | 1 | 2 | 3 | 3 | 2 | 3 | 4 |
| $v_{43}$ | 2 | 1 | 0 | 1 | 3 | 2 | 1 | 2 | 4 | 3 | 2 | 3 |
| $v_{44}$ | 3 | 2 | 1 | 0 | 4 | 3 | 2 | 1 | 4 | 4 | 3 | 2 |
| $v_{51}$ | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 1 | 2 | 3 | 4 |
| $v_{52}$ | 2 | 1 | 2 | 3 | 1 | 0 | 1 | 2 | 2 | 1 | 2 | 3 |
| $v_{53}$ | 3 | 2 | 1 | 2 | 2 | 1 | 0 | 1 | 3 | 2 | 1 | 2 |
| $v_{54}$ | 4 | 3 | 2 | 1 | 3 | 2 | 1 | 0 | 4 | 3 | 2 | 1 |
| $v_{61}$ | 2 | 3 | 4 | 5 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 |
| $v_{62}$ | 3 | 2 | 3 | 4 | 2 | 1 | 2 | 3 | 1 | 0 | 1 | 2 |
| $v_{63}$ | 4 | 3 | 2 | 3 | 3 | 2 | 1 | 2 | 2 | 1 | 0 | 1 |
| $v_{64}$ | 4 | 4 | 3 | 2 | 4 | 3 | 2 | 1 | 3 | 2 | 1 | 0 |
5. Conclusion

In this article we introduced generalized Möbius Ladder $M_{m,n}$ and proved that it has two subfamilies, each having a constant metric dimension. The metric dimension of $M_{m,n}$ is 3 when either $m$ is odd and $n$ is even or when $m$ is even and $n$ is odd; the metric dimension of $M_{m,n}$ is 4 when $m$ and $n$
are both odd or when are both even. It is remarkable that the present results cover the results about the M"{o}bius Ladder $M_m$ already presented in [14] and [20] as subcases.

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