Dynamics of a domain wall in a magnetic nano-strip: a toy model

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In this report we demonstrate a simple model for the motion of a vortex domain wall in a ferromagnetic strip of submicron width under the influence of an external magnetic field. The model exhibits three distinct dynamical regimes. In a viscous regime at low fields the wall moves rigidly with a velocity proportional to the field. Above a critical field the motion becomes underdamped as the vortex moves periodically across the strip; these oscillations are accompanied by a slow drift with a decreasing velocity. At still higher fields the drift velocity starts rising linearly with the field again but with a much lower mobility \( dv/dH \) than in the low-field regime. We calculate the relevant quantities and compare them to experimentally observed values.

I. INTRODUCTION

Dynamics of domain walls in ferromagnetic strips and rings with submicron dimensions is a subject of active research. This topic is directly relevant to several proposed schemes of magnetic memory and is also interesting from the standpoint of basic physics. The dynamics of domain walls under an applied magnetic field has distinct regimes: viscous motion with a relatively high mobility at low fields and underdamped oscillations with a slow drift at higher fields.

The nontrivial dynamics is related to the composite nature of a domain wall in a nano-strip: it consists of a few—typically two or three—elementary topological defects in the bulk and at the edge of the strip. As a result, a domain wall has several low-energy degrees of freedom that are relevant to the dynamics. Weak external perturbations engage only the softest (zero) mode—rigid translations along the strip. Larger external forces excite higher modes thereby altering the character of motion.

The general approach to the dynamics of domain walls in thin ferromagnetic strips with a submicron width \( w \) and thickness \( t \ll w \) was described recently by Tretiakov et al. The configuration of a domain wall is parametrized by a few collective coordinates \( \xi = \{\xi_1, \xi_2, \ldots, \xi_N\} \) and the free energy of the system \( U \) is treated as a function of \( \xi \). The resulting equations of motion can be written in the vector notation as

\[
\mathbf{F} - \hat{\Gamma} \mathbf{\xi} + \hat{G} \mathbf{\dot{\xi}} = 0. \tag{1}
\]

Here components of the vector \( \mathbf{F} \) are generalized forces \( F_i = -\partial U/\partial \xi_i \); the symmetric matrix \( \hat{\Gamma} \) and antisymmetric matrix \( \hat{G} \) represent the viscous and gyrotropic tensors, respectively.

The main goal of this paper is to illustrate the collective-coordinate approach on a very simple model of a vortex domain wall that served as a prototype for a more realistic model of Youk et al. Despite its simplicity, the model captures all of the main features of a vortex domain wall and yields simple analytical results for the relevant physical quantities. Quantitatively speaking, the values of the forces computed in this model deviate by no more than 30\% from those obtained in the more realistic model of the vortex wall. Thus one can make meaningful comparisons between the analytical results obtained in this paper and experimental data.

In the main body of the paper we describe the simplified model of the wall and compute the generalized forces and the viscous and gyrotropic tensors. By substituting these quantities into Eq. (1) we obtain simple equations of motion. At low fields the equations describe steady viscous motion of the wall with a velocity proportional to the applied field. The vortex is shifted in the transverse direction by an amount proportional to the velocity of the wall. At a critical velocity the vortex is expelled from the strip and the steady motion breaks down giving way to an oscillatory regime. As the applied field increases further, the drift velocity decreases at first but then again becomes proportional to the applied field; the mobility coefficient \( dv/dH \) is substantially lower than the corresponding value in the viscous regime at low fields. These results are compared to experimental data.

II. MODEL WALL

In our calculations of the wall dynamics, we use a simple model of the vortex domain wall consisting of four domains with uniform magnetization and separated by 90° Neel walls (Fig. 1).

We assume that only two softest modes of the vortex wall are involved in magnetization dynamics, so that the configuration of the wall is fully described by the two coordinates \( (X, Y) \) of the vortex core. In that case the equations of motion reflect the balance of forces acting on a particle moving in two dimensions with a velocity \( \mathbf{V} = (X, Y) \). The forces include a conservative term \( -\partial U/\partial X, -\partial U/\partial Y \), a viscous term \( \hat{\Gamma} \mathbf{V} \), and a gyrotropic term \( \hat{G} \mathbf{V} + p \mathbf{G} \mathbf{\hat{\alpha}} \times \mathbf{V} \). The gyrotropic force depends on the out-of-plane polarization of the vortex core \( p = \mu_0 M/|\mathbf{M}_{\text{core}}| \) and the gyrotropic constant \( G = 2\pi J \), where \( J = \mu_0 M/\gamma \) is the density of angular momentum. The conservative and viscous terms are discussed next.
w 2

that make up the vortex wall. Because the length of these
the exchange cost comes entirely from the four Neel walls
strip. In general, the total exchange energy of the wall
−a force
w

nearly all of this charge is expelled to the edge. In our
unit length \( \rho = 2Q/Mt \).

The magnetostatic energy of this wall \( E(Y) = E(0) + kY^2/2 + O(4) \) has a minimum at \( Y = 0 \). This leads to a
force \(-kY\) that acts to keep the vortex centered on the
strip. In general, the total exchange energy of the wall
may change with the position of the vortex, altering the
restoring force slightly. However, in our simplified model,
the exchange cost comes entirely from the four Neel walls
that make up the vortex wall. Because the length of these
w 2

walls does not change as the vortex moves, we need not
consider the exchange interaction in our analysis of the
wall dynamics.

A line of charges of length \( L \) has the self-energy

\[
E_0(L) = \frac{\mu_0 M^2 t^2}{8\pi} \int_0^L \int_0^L \frac{dx \, dx'}{|x - x'|}.
\]

The divergence at \( x = x' \) requires a regularization. In a
crude way this can be done by introducing a short-range
cutoff in the integral, i.e. by integrating over distances
\(|x - x'| > Ct\), where \( C \) is a numerical constant. We then
obtain a logarithmic dependence on \( Ct \):

\[
E_0(L) = \frac{\mu_0 M^2 t^2}{4\pi} L \log (L/Ct) - 1.
\]

The self-energies of the two lines of charge is

\[
E_{\text{self}}(Y) = E_0(w - 2Y) + E_0(w + 2Y)
= E_{\text{self}}(0) + \frac{\mu_0 M^2 t^2 Y^2}{\pi w} + O(Y^4).
\]

Note that the cutoff \( Ct \) affects only the constant term
\( E_{\text{self}}(0) \); the quadratic term is not sensitive to the exact
value of \( C \).

In a similar way we evaluate the interaction of the two
lines of charges,

\[
E_{\text{int}}(Y) = \frac{\mu_0 M^2 t^2}{4\pi} \int_{-3w/2}^{2Y - w/2} \int_{2Y + w/2}^{3w/2} \frac{dx \, dx'}{\sqrt{w^2 + (x - x')^2}}
= E_{\text{int}}(0) - \frac{\mu_0 M^2 t^2 Y^2}{\pi w \sqrt{5}} + O(Y^4).
\]

The sum of the quadratic terms in Eqs. (1) and (5) yields
the “spring” energy \( kY^2/2 \), from which we determine the
spring constant:

\[
k = \frac{2(1 - 1/\sqrt{5}) \mu_0 M^2 t^2}{\pi w}.
\]

Next we deal with the Zeeman energy of the wall
\(-\mu_0 |\mathbf{H}| \int d^2x \mathbf{H} \cdot \mathbf{M} \) in the presence of an applied magnetic
field \( \mathbf{H} \) parallel to the axis of the strip. A longitudinal
shift of the vortex by \( \Delta X \) results in a pure translation of the
wall. Independently of the wall shape, the rigid
shift changes the Zeeman energy by \(-QH\Delta X \), where
\( Q = 2\mu_0 Mtw \) is the magnetic charge of the wall. Therefore
the longitudinal Zeeman force is \( QH \) in any model.

The Zeeman force also has a transverse component. As can be seen from Fig. [1] transverse motion of the vor-
tex core changes the total magnetization \( M_x \) of the strip
and thus affects its Zeeman energy. As the vortex core
crosses the strip from top to bottom (Fig. [1]), the Zeeman
energy decreases linearly by \( 4\mu_0 H Mt w^2 \). Therefore
the transverse component of the Zeeman force is \(-2QH \).

The total free energy of a wall with the vortex core at
\((X, Y)\) is thus

\[
U(X, Y) = kY^2/2 + 2QHY - QHX.
\]
B. Viscosity tensor $\hat{\Gamma}$ and viscous drag

We next consider the viscosity of the vortex wall. The viscosity that appears in Eq. (1) is a symmetric matrix whose components are given by

$$\Gamma_{ij} = \alpha Jt \int d^2x \frac{\partial \phi}{\partial \xi_i} \frac{\partial \phi}{\partial \xi_j}, \quad (8)$$

where $\phi$ is the azimuthal angle characterizing magnetization.

An infinitesimal shift in the collective coordinates $X$ and $Y$ affects magnetization in the vicinity of the Neel walls only. We begin by considering the contribution of a single Neel wall emanating from the vortex core at $\pm 45^\circ$, $f(x, y, X, Y) = f(x - X \mp y \pm Y)$. For such a wall, derivatives with respect to collective coordinates can be reduced to ordinary gradients: $\partial \phi / \partial X = -\partial \phi / \partial x = -f'$ and $\partial \phi / \partial Y = -\partial \phi / \partial y = \pm f'$. As a result, the tensor components are equal to each other, up to a sign:

$$\Gamma_{XX} = \Gamma_{YY} = \mp \Gamma_{XY} = \alpha Jt \int d^2x f'^2. \quad (9)$$

Note that this represents, up to a trivial constant, the exchange energy of the Neel wall, which has been calculated, e.g., in Ref. [5]. We thus obtain viscosity coefficients for the Neel walls intersecting at the vortex core, $\Gamma_{XX} = \Gamma_{YY} = \mp \Gamma_{XY} = 0.152 \alpha J tw / \lambda$, where the exchange length $\lambda = \sqrt{A / \mu_0 M^2} = 3.8$ nm in permalloy.

Opposite signs of the off-diagonal component $\Gamma_{XY}$ can be understood by noting that, as the vortex moves along $Y$, the two Neel walls shift along $+X$ and $-X$ creating equal and opposite viscous forces in the $X$ direction.

The two peripheral Neel walls have the functional form $\phi(x, y, X, Y) = f(x + y - X + Y \pm w)$, so that their contributions are the same as that of the central Neel wall perpendicular to them. Adding the contributions of all four Neel walls yields a total

$$\Gamma_{XX} = \Gamma_{YY} = -2 \Gamma_{XY} = 0.608 \alpha J tw / \lambda, \quad (9)$$

independently of the vortex position.

It is instructive to compute the ratio of the viscous and gyrotropic forces:

$$\frac{\Gamma_{XX}}{G} = 0.097 \alpha w / \lambda. \quad (10)$$

The small value of Gilbert’s damping in permalloy, $\alpha \approx 0.008$, leads to the dominance of the gyrotropic force in strips with submicron widths. The smallness of $\Gamma_{XX}/G$ can be exploited to organize an expansion in powers of this small parameter.

III. WALL DYNAMICS

Equations of motion (1) for two generalized coordinates $\xi_1 = X$ and $\xi_2 = Y$ read

$$F_i - \Gamma_{ij} \dot{\xi_j} + p G \epsilon_{ij} \dot{\xi_j} = 0 \quad (11)$$

where $\Gamma_{ij} = \Gamma_{ji}$ is a viscosity tensor, $p$ is the polarization of the vortex core, and $\epsilon_{ij}$ is the $2 \times 2$ antisymmetric tensor with $\epsilon_{12} = +1$. The generalized forces $F_i = -\partial U / \partial \xi_i$ are derived from the free energy $U$. We thus arrive at the following equations of motion for the vortex core:

$$\dot{X} = \frac{QH}{\Gamma_{XX}} + \frac{k(\Gamma_{XY} - pG)}{\det \Gamma + G^2} (Y - Y_{eq}), \quad (12)$$

$$\dot{Y} = -\frac{k \Gamma_{XY}}{\det \Gamma + G^2} (Y - Y_{eq}), \quad (13)$$

where the equilibrium $Y$ position of the vortex is given by

$$k Y_{eq} = -pG \frac{QH}{\Gamma_{XX}} (1 + pg), \quad (13)$$

where $g = (2 \Gamma_{XX} + \Gamma_{XY})/G \ll 1$. It is worth noting that the magnitudes of the transverse displacement $|Y_{eq}|$ are slightly different for the two values of the vortex polarization $p$. This effect can be traced to the lack of the reflection symmetry $y \rightarrow -y$ in a vortex wall, which leads to nonzero transverse components of the Zeeman force $-2QH$ and the viscous force $\Gamma_{XY} X$. As a result, trajectories of vortex cores with $p = +1$ and $-1$ are slightly different.

Analysis of the equations of motion yields three distinct regimes (Fig. 2). Below a critical field $H_c$ we find steady viscous motion with a high mobility $\mu = dV/dH$. Immediately above the critical field $H_c$ the motion exhibits an oscillatory component; the drift velocity quickly decreases as the applied field grows. At much higher fields, $H \gg H_0$, the drift velocity rises linearly again but with a much lower mobility $\mu$ than at low fields. The separation of scales $H_c$ and $H_0$ is guaranteed by the smallness of the parameter $\Gamma_{XX}/G$.

A. Low field: $H < H_c$

In a low applied field the wall exhibits simple viscous motion. The transverse coordinate of the vortex will asymptotically approach its equilibrium position $Y_{eq}$, so long as the latter is within the strip. The wall then moves rigidly with a steady longitudinal velocity

$$\dot{X} = QH / \Gamma_{XX}. \quad (14)$$

Experimental data of Beach et al. [6] yield $Q / \Gamma_{XX} \sim 25$ (m/s) Oe$^{-1}$ at low fields for a strip 600 nm wide, which gives $\Gamma_{XX} / G = 0.13$. While our Eq. (10) yields $\Gamma_{XX} / G = 0.12$ if we use the value of $\alpha = 0.008$ measured by Freeman et al. [10].

B. Critical field: $H = H_c$

The low-field regime ends when the equilibrium position of the vortex core is pushed outside the strip edge,
reached when \( Y_{eq} \geq w/2 \), making the steady state unreachable. As pointed out above, in permalloy strips with a width below \( 1 \, \mu\text{m} \) the viscous force is small in comparison with the gyrotropic one. As a result, the equilibrium of a vortex in the transverse direction is set mostly by the balance of the transverse components of the gyrotropic force \( GV \) and the restoring force \( -kY_{eq} \). The critical point is reached when \( Y_{eq} = w/2 \):

\[
GV_c = kw/2.
\]  

(15)

With the aid of Eq. (10) we obtain the critical velocity

\[
V_c = \frac{1 - 1/\sqrt{5}}{2\pi^2} \gamma Mt
\]

(16)

and the critical field

\[
H_c = \Gamma_{XX} V_c/Q = kw\Gamma_{XX}/(2QG).
\]

(17)

For permalloy, \( \gamma = 2.21 \times 10^5 \, \text{m}^{-1} \, \text{s}^{-1} \) and \( M = 8.6 \times 10^5 \, \text{m}^{-1} \, \text{A}^{-1} \). Taking the thickness of \( t = 20 \, \text{nm} \) we obtain \( V_c = 106 \, \text{m/s} \). This is not too far from the critical velocity of \( 80 \, \text{m/s} \) observed by Beach et al.\cite{Beach}

Equation (16) shows that the critical velocity should grow linearly with the film thickness \( t \). It is easy to see that this result is valid beyond the crude model of a vortex wall adopted in this calculation. The two forces balancing each other (15) scale differently with \( t \). While the gyrotropic force is linear in \( t \), the restoring force comes from the magnetostatic energy, which represents Coulomb-like interaction of charges with density \( O(t) \), hence (the dipolar part of) the restoring force is quadratic in \( t \). That gives \( V_c \propto t \).

C. High field: \( H > H_c \), General remarks

Numerical simulations indicate that, after the original vortex with a core polarization \( p \) is expelled from the strip, a new vortex is injected at the same location with the opposite polarization \(-p\). The vortex thus moves between the edges switching its core polarization each time it reaches an edge.

Once the transverse coordinate of the vortex \( Y \) becomes a dynamical variable, the motion acquires an entirely different character. As we already pointed out, the gyrotropic force \( GV \) dwarfs the viscous one, \( V \), in permalloy strips. To zeroth order in \( \Gamma_{XX}/G \), the dynamics is conservative: the vortex core moves along equipotential lines \( U(X,Y) = \text{const.} \). At this order, the wall would oscillate back and forth but would not move on average. Drift requires a nonzero viscosity: as the wall coordinate \( X \) increases on average, the loss of Zeeman energy must be accounted for through viscous friction.

D. Very high field: \( H \gg H_0 \)

We first demonstrate that at a very high field the velocity is again proportional to the field and calculate the high-field mobility. The new field scale \( H_0 \) is set by the requirement that the restoring force \(-kY\) be negligible in comparison with the Zeeman force \( QH \). The characteristic field is

\[
H_0 = kw/(2Q) = H_cG/\Gamma_{XX} \gg H_c.
\]

(18)

When \( H \gg H_0 \), the dynamics is dominated by the Zeeman and gyrotropic forces, so that the vortex moves along an equipotential line \( Y = X/2 + \text{const.} \), or \( X = 2Y \).

As a result of the drift with a velocity \( V_d \), the Zeeman energy goes down on average at the rate \( QHV_d \). It is dissipated through heat generated at the rate

\[
V^T \hat{\Gamma} V = \hat{\gamma}^2 / \hat{\Gamma} \left( \begin{array}{c} 2 \\ 1 \end{array} \right).
\]

The transverse velocity of the vortex core reflects the balance between the longitudinal components of the gyrotropic and Zeeman forces: \( \dot{Y} \approx QH/G \). We thus find the drift velocity

\[
V_d = QH/G^2 \left( \Gamma_{YY} + 4\Gamma_{XX} + 4\Gamma_{XY} \right) = 3\Gamma_{XX}QH/G^2.
\]

(19)

In the last transition we have used the relation between the coefficients of the viscosity tensor specific to this model (17).

The high-field (HF) mobility (19) is suppressed in comparison to the low-field (LF) one (14):

\[
\frac{\mu_{HF}}{\mu_{LF}} = \frac{3\Gamma_{XX}^2}{G^2} \ll 1.
\]

(20)

In the experiment of Beach et al.\cite{Beach} \( \mu_{HF}/\mu_{LF} \approx 0.1 \), while the theoretical result is \( 3(\Gamma_{XX}/G)^2 \approx 0.05 \), i.e. twice as small.
moves beyond the edge of the strip. The critical velocity
breaks down when the equilibrium position of the vortex
steady-state viscous regime at low fields agrees well with
tions of motion. The calculated mobility of the wall in the
domain wall described in this paper yields solvable equa-
tion of the vortex core. A simplified model of the vortex
applied the method of collective coordinates
to expel than an upwardly polarized one. An expansion
wardly polarized vortex requires a slightly higher field
larized vortices must be expelled from the strip for the
the vortex expulsion field
Fig. 2. Note that the critical field is not exactly
The resulting curve is shown for several strip widths in
Fig. 3. Note that the critical field is not exactly $H_c$ and
actually changes slightly with the width. This is because
the equilibrium points for both up- and downwardly po-
larized vortices must be expelled from the strip for the
character of the motion to change. By Eq. (13) a down-
wardly polarized vortex requires a slightly higher field
to expel than an upwardly polarized one. An expansion
of Eq. (22) in powers of $1/H$ yields the high-field result

\[ \Delta X_\pm = \frac{QH \Delta T_\pm}{\Gamma_{XX}} - \frac{G + \Gamma_{XY}}{\Gamma_{XX}} w, \]
\[ \Delta T_\pm = \frac{1}{k \Gamma_{XX}} \ln \left( \frac{1 + H_c/H \pm g}{1 - H_c/H \pm g} \right). \]  
\[ (21) \]

The drift velocity is
\[ V_d = \frac{\Delta X_+ + \Delta X_-}{\Delta T_+ + \Delta T_-} \]
\[ = V_c \left( \frac{H}{H_c} - \frac{4}{1 + \det \Gamma/G^2 \ln \left( \frac{(1+H_c/H)^2-g^2}{(1-H_c/H)^2-g^2} \right)} \right). \]  
\[ (22) \]

IV. DISCUSSION

We have explored the dynamics of a vortex domain
wall in a magnetic strip of a submicron width. We have
applied the method of collective coordinate\textsuperscript{5} to the case
when the wall has two soft modes related to the mo-
tion of the vortex core. A simplified model of the vortex
domain wall described in this paper yields solvable equa-
tions of motion. The calculated mobility of the wall in the
steady-state viscous regime at low fields agrees well with
the value measured by Beach et al.\textsuperscript{2} The steady motion
breaks down when the equilibrium position of the vortex
moves beyond the edge of the strip. The critical velocity
depends just on the magnetization length and the
sample thickness; its calculated value agrees reasonably
well with the data of Beach et al.\textsuperscript{2} The dynamics above
the breakdown changes the character from overdamped
to underdamped: the ratio of the viscous and gyrotropic
forces acting on the wall $\Gamma_{XX}/G = 0.13$ in their exper-
iment. In this regime the velocity sharply declines at
first but later starts to rise again as the field strength in-
creases. The high-field mobility is reduced in comparison
with the low-field value by the factor $3(\Gamma_{XX}/G^2 = 0.05$; the
observed reduction is not as strong: $\mu_{HF}/\mu_{LF} \approx 0.1$\textsuperscript{2}

In addition to simplifying the geometry (but not the
topology) of the domain wall, we have made other as-
sumptions that require further checking. First, we have
assumed that any vortex absorbed by the edge is imme-
diately reemitted. At fields lower than that required for
emission to occur, the wall may simply stay transverse
and continue to move in a viscous fashion. At higher
fields, there may be short delays between absorption and
reemission during which the motion of the wall is again
viscous; the higher mobility of a transverse wall would
tend to increase the drift velocity.

Second, just as the appearance of $Y$ as a new degree
of freedom completely changes the character of the wall
dynamics above the critical field $H_c$, at still higher fields
additional modes of the wall may become important. The
number and dynamical characteristics of soft modes may
also change discontinuously as additional vortices or an-
tivortices are created and annihilated in the bulk of the
strip. We have observed the creation and subsequent an-
nililation of a vortex-antivortex pair near the original
vortex of the wall. Like the process described by Van
Waeyenberge et al.\textsuperscript{11} the pair creation mediates the flip-
ning of the polarization of the wall vortex and results in
the reversal of the gyrotropic force. Thus the dynamics
is similar to that described in this paper: the vortex moves
back and forth, while the domain wall slowly drifts along
the strip. A possible way to detect this new regime is
to measure the frequency of longitudinal oscillations: be-
cause the vortex does not reach the edge, the frequency
should be $\omega > 2 \gamma H$ expected when the vor-
tex moves from edge to edge.\textsuperscript{11} We shall describe the onset
of this type of motion more fully in future work.

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