CLASSIFICATION OF POISSON SURFACES
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Abstract. We study complex projective surfaces admitting a Poisson structure; we prove a classification theorem and count how many independent Poisson structures there are on a given Poisson surface.

1. Introduction

The notion of Poisson manifold naturally arises within the framework of analytical mechanics. We briefly recall that a Poisson structure on a $C^\infty$ manifold $M$ is given by a bilinear skew-symmetric bracket $\{\cdot,\cdot\}$ defined on the sheaf of functions $\mathcal{C}^\infty_M$, such that
\[
\begin{align*}
\{f,g\} &= -\{g,f\}; \\
\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} &= 0 \text{ (Jacobi identity)}; \\
\{f,gh\} &= \{f,g\}h + \{f,h\}g.
\end{align*}
\]

As it was pointed out by Lichnerowicz [8], the assignment of a Poisson bracket is equivalent to the assignment of a skew-symmetric bilinear form on the cotangent bundle $T^*M$, i.e., a global section $\Pi \in \Gamma(M, \Lambda^2 TM)$, satisfying the condition
\[
[[\Pi, \Pi]] = 0,
\]
where $[\cdot, \cdot] : \Gamma(M, \Lambda^2 TM) \otimes \Gamma(M, \Lambda^2 TM) \to \Gamma(M, \Lambda^3 TM)$ is the Schouten-Nijenhuis bracket [12]. Any symplectic manifold $(M, \omega)$ carries a canonical Poisson structure, given by $\Pi(\alpha, \beta) = \omega(X_\alpha, X_\beta)$ where $i(X_\alpha)\omega = \alpha$ and $i(X_\beta)\omega = \beta$; the condition $[[\Pi, \Pi]] = 0$ is ensured by $d\omega = 0$.

The definition of Poisson structure extends in a natural fashion to complex manifolds. In particular, on a complex surface $X$ any (holomorphic) global section $\sigma$ of the anticanonical bundle $\mathcal{O}_X(-K_X) = \Lambda^2 TX$ gives rise to a (holomorphic) Poisson structure, since the condition $[\sigma, \sigma] = 0$ is automatically satisfied.

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Complex Poisson surfaces play a major role in the theory of algebraically completely integrable Hamiltonian systems. Indeed, as was proved in [3, 4, 11], the choice of a Poisson bivector on a surface $X$ determines natural Poisson structures both on the moduli space of stable sheaves on $X$ and on the Hilbert scheme of points of $X$. By using this construction, under suitable hypotheses, it is possible to associate an integrable system to a linear system defined on a Poisson surface, generalizing the results obtained by Beauville for linear systems on K3 surfaces [2]. Important examples, like the Neumann system, the Hitchin system, etc., can be obtained in this way [7, 13].

It is quite immediate to get convinced that a projective Poisson surface can be only an abelian, or a K3 or a ruled surface. (By “ruled surface” we mean any projective surface birationally equivalent to $C \times \mathbb{P}^1$, where $C$ is a smooth curve). However, not every ruled surface admits a Poisson structure. The following classification theorem holds.

**Theorem 1.1.** Let $X$ be a minimal ruled surface over the curve $C$ of genus $g$, determined by a normalized rank two vector bundle $V$ over $C$. Let $e = -\text{deg}V$.

1) If $g = 0$, then $X$ is a Poisson surface.

2) If $g = 1$, then
   • if $e = -1$, $X$ does not admit any Poisson structure;
   • if $e \geq 0$, $X$ is a Poisson surface.

3) If $g \geq 2$, then
   • if $-g \leq e \leq 2g - 3$, $X$ does not admit any Poisson structure;
   • if $e = 2g - 2$ and $V$ is indecomposable, $X$ is a Poisson surface;
   • if $e = 2g - 2$ and $V$ is decomposable, or $2g - 2 < e \leq 3g - 3$, $X$ is a Poisson surface if and only if $-K_C - \Lambda^2V$ is effective;
   • if $e \geq 3g - 2$, $X$ is a Poisson surface.

This theorem can be obtained as a corollary of Sakai’s results about the anti-Kodaira dimension of ruled surfaces [10]. However, Sakai’s proof cannot be adapted to answer an important question: how many independent Poisson structures are there on a given ruled Poisson surface? In this note we provide a new (and completely elementary) proof of Theorem 1.1 in Corollary 2.1 and Corollary 2.2 we compute, whenever it is possible, the dimension of $H^0(X, O_X(-K_X))$.

Related results about $| - mK_X |$, for an integer $m \geq 1$, can be found in [3].
2. Classification theorem

Let $X$ be a smooth projective surface over $\mathbb{C}$ endowed with a (nontrivial) Poisson structure, namely a nonzero section $\beta$ of $\mathcal{O}_X(-K_X)$. Let $D$ be the divisor associated to $\beta$; from the exact sequence

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X \to \mathcal{O}_D \to 0,$$

it follows that the Kodaira dimension of $X$ has to be equal either to 0 or to $-\infty$ (see e.g. [3, Prop. 2.3]). In the first case, $X$ is a K3 or an Abelian surface, and the canonical bundle is trivial: thus, the Poisson structure is induced by a (holomorphic) symplectic structure on $X$. If $\text{kod}X = -\infty$, Enriques' theorem implies that $X$ is a ruled surface. We notice that, since the section $\beta$ does not vanish on the open subset $X \setminus D$, the inverse of the Poisson bivector is a symplectic form on $X \setminus D$. In other words, $X \setminus D$ is the unique symplectic leaf of the foliation determined by $\Pi$ [12]. In particular, it follows that Poisson structures on projective surfaces have no nontrivial Casimir (holomorphic) functions.

**Example 2.1.** Let $C$ be a smooth curve. The cotangent bundle $T^*C$ carries a canonical symplectic form $\Omega = d\theta$, where $\theta$ is the Liouville one-form. Denoting by $Q$ the total space of $T^*C$, it follows that $Q$ is a non-compact symplectic surface. It is easy to show that $Q$ can be embedded as an open set into the ruled surface $X = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(K_C))$. $X$ is a Poisson surface and $Q$ a symplectic leaf.

It is obvious, however, that not every ruled surface carries a Poisson structure. For instance, the anticanonical bundle of the surface $X = C \times \mathbb{P}^1$ has no nonzero sections unless $g(C) \leq 1$.

In order to classify the ruled surfaces admitting Poisson structures, we have first to understand what happens when a Poisson surface $X$ is blown-up at a point $p$.

**Lemma 2.1.** Let $\rho : \tilde{X} \to X$ the blow-up at the point $p \in X$. Then

$$h^0(X, \mathcal{O}_X(-K_X)) - 1 \leq h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-K_{\tilde{X}})) \leq h^0(X, \mathcal{O}_X(-K_X)),$$

and $h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-K_{\tilde{X}})) = h^0(X, \mathcal{O}_X(-K_X))$ if and only if $p$ is a base point of $|-K_X|$.
Proof
Let $E$ be the exceptional divisor. Since $\rho$ is an isomorphism on $\tilde{X} \setminus E$, any Poisson bracket on $\tilde{X}$ induces a Poisson bracket on $X$ by Hartogs’ theorem; this proves the right inequality in (1). Since $-K_{\tilde{X}} \cdot E > 0$, any section $\beta \in H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-K_{\tilde{X}}))$ coming from a section $\tilde{\beta} \in H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-K_{\tilde{X}}))$ passes through the point $p$. Conversely, if $\beta$ passes through the point $p$, then $-K_{\tilde{X}} \sim \rho^* D$, where $D$ is the divisor associated to $\beta$, and therefore $-K_{\tilde{X}}$ is effective. 

We can now restrict our attention to minimal ruled surfaces. We shall freely use the results and notations in [6], Chap. V, §2.

If $q(X) = 0$, then $X$ is $\mathbb{P}^2$ or the rational ruled surface $\mathbb{F}_n$, with $n \neq 1$. In both cases, a straightforward computation shows that $h^0(X, \mathcal{O}_X(-K_X)) \geq 9$.

Proposition 2.1. Any minimal ruled surface $X$ with $q(X) = 0$ is a Poisson surface. 

When $q(X) \geq 1$, then $\pi : X \to C$ is a geometrically ruled surface, with $g(C) = q(X)$. We can assume $X = \mathbb{P}(V)$, where $V$ is rank 2 vector bundle on $C$ such that $H^0(C, V) > 0$ and $H^0(C, V \otimes M) = 0$ for every line bundle $M$ of negative degree. We shall say that such an $V$ is normalized. Under this hypothesis, there exists a section $\tau : C \to X$ such that $\tau^2 = \deg V =: -e$; we have $\mathcal{O}_X(\tau) \cong \mathcal{O}_X(1)$ and so $V = R^0 \pi_* \mathcal{O}_X(\tau)$. There are some restrictions on the possible values of the invariant $e$ [9, 6]:

1) if $V \cong \mathcal{O}_C \oplus L$, then $e \geq 0$;
2) if $V$ is indecomposable, then $-g \leq e \leq 2g - 2$.

Moreover, all these values are admissible.

Any normalized vector bundle $V$ over the curve $C$ fits into an exact sequence

\begin{equation}
0 \to \mathcal{O}_C \to V \to L \to 0,
\end{equation}

where $L$ is a line bundle over $C$; we have $L \cong \Lambda^2 V$. Let $\mathcal{L}$ be the divisor on $C$ corresponding to $L$; it is easy to show that

\begin{equation}
-K_X \sim 2\tau + \pi^*(-K_C - \mathcal{L}).
\end{equation}

By the projection formula, we obtain

\begin{equation}
H^0(X, \mathcal{O}(-K_X)) \cong H^0(C, S^2(V) \otimes L^*(-K_C)).
\end{equation}
Lemma 2.2. There is an exact sequence of vector bundles over $C$:
\begin{equation}
0 \to V \otimes L^\wedge(-K_C) \to S^2(V) \otimes L^\wedge(-K_C) \to L(-K_C) \to 0.
\end{equation}

**Proof** Let us consider the exact sequence
\begin{equation*}
0 \to \mathcal{O}_X(1) \to \mathcal{O}_X(2) \to \mathcal{O}_X(2) \otimes \mathcal{O}_\tau \to 0;
\end{equation*}
since $R^1\pi_*(\mathcal{O}_X(1)) = 0$ (see [6], Chap. V, Lemma 2.4), we get the exact sequence
\begin{equation*}
0 \to V \to S^2(V) \to L \otimes L \to 0.
\end{equation*}
The result follows by tensoring this sequence by the line bundle $L^\wedge(-K_C)$. \qed

**Remark 2.1.** By using the exact sequence (5) it is an easy exercise to compute the number of independent Poisson structures on the surfaces $F_n$ (in this case, the invariant $e$ coincides with $n$):
- if $X = F_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$, then $h^0(X, \mathcal{O}(-K_X)) = 9$;
- if $X = F_2$, then $h^0(X, \mathcal{O}(-K_X)) = 9$;
- if $X = F_n, n \geq 3$, then $h^0(X, \mathcal{O}(-K_X)) = n + 6$.
\qed

We recall a useful criterion of ampleness that we shall exploit in the proof of Proposition 2.3.

**Proposition 2.2.**
1) If $e \geq 0$, a divisor $D \equiv a\tau + bf$ is ample if and only if $a > 0$ and $b > ae$.
2) If $e < 0$, a divisor $D \equiv a\tau + bf$ is ample if and only if $a > 0$ and $b > \frac{1}{2}ae$.
\qed

**Theorem 2.1.** Let $X$ be a minimal ruled surface with $q(X) = 1$.
1) If $e = -1$, $X$ does not admit any Poisson structure;
2) if $e \geq 0$, $X$ is a Poisson surface.

**Proof**
If $e = 0$, we have to distinguish 3 cases: $V$ indecomposable, $V = \mathcal{O}_C \oplus \mathcal{O}_C$ and $V = \mathcal{O}_C \oplus L$ with $L \not\cong \mathcal{O}_C$ (recall that $V$ is normalized). If $V$ is indecomposable,
then, as shown in Theorem V.2.15 of [3], it is uniquely determined by the exact sequence

\[(6) \quad 0 \to \mathcal{O}_C \to V \to \mathcal{O}_C \to 0.\]

Since the sequence (6) does not split, we have \(h^0(C, V) = 1\). The long exact cohomology sequence associated to the exact sequence (6) is

\[(7) \quad 0 \to H^0(C, V) \to H^0(C, S^2(V)) \to \mathbb{C} \to \mathbb{C} \to H^1(C, S^2(V)) \to \mathbb{C} \to 0.\]

Now, \(S^2(V)\) is indecomposable (see [1] Th. 9), so that one has \(h^0(C, S^2(V)) = h^0(X, \mathcal{O}(-K_X)) = 1\). The case \(V = \mathcal{O}_C \oplus \mathcal{O}_C\), corresponding to \(X = C \times \mathbb{P}^1\), is trivial; we have

\[H^0(X, \mathcal{O}(-K_X)) \cong H^0(C, \mathcal{O}_C) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)),\]

hence, \(h^0(X, \mathcal{O}(-K_X)) = 3\). If \(V = \mathcal{O}_C \oplus L\) with \(L \not\cong \mathcal{O}_C\), then \(h^0(C, L) = h^1(C, L^\ast) = 0\). From the exact sequence

\[0 \to L^\ast \to V \otimes L^\ast \to \mathcal{O}_C \to 0,\]

we get \(H^0(C, V \otimes L^\ast) \cong H^0(C, \mathcal{O}_C) \cong \mathbb{C}\). By using again the sequence (6), it follows \(h^0(X, \mathcal{O}(-K_X)) = 1\).

If \(e > 0\), then \(V\) is decomposable: \(V \cong \mathcal{O}_C \oplus L\), with \(\deg L < 0\). By reasoning as in the previous case, we get \(h^0(X, \mathcal{O}(-K_X)) = e + 1\).

Finally, if \(e = -1\), then \(V\) is uniquely determined by the exact sequence

\[(8) \quad 0 \to \mathcal{O}_C \to V \to \mathcal{O}_C(P) \to 0,\]

where \(P\) is a point of \(C\). Dualizing the sequence (6), we get at once that \(V^\ast \cong V \otimes \mathcal{O}_C(-P)\). Now, \(V\) is indecomposable, of rank 2 and degree 1; so, by [1] Lemma 22, one has \(V \otimes V^\ast \cong \oplus_{i=0}^{3} \Xi_i\), where the \(\Xi_i\) are the line bundles on \(C\) of order dividing 2 (in particular, we set \(\Xi_0 \cong \mathcal{O}_C\)). An easy computation shows that \(S^2(V) \otimes \mathcal{O}_C(-P) \cong \oplus_{i=1}^{3} \Xi_i\). One concludes that \(h^0(X, \mathcal{O}(-K_X)) = h^0(C, S^2(V) \otimes \mathcal{O}_C(-P)) = 0\).

\[\square\]

**Corollary 2.1.** Let \(X\) be a minimal ruled surface with \(q(X) = 1\).

1) If \(e = 0\) and \(V = \mathcal{O}_C \oplus \mathcal{O}_C\) (hence \(X \cong \mathbb{P}^1 \times C\)), then \(h^0(X, \mathcal{O}_X(-K_X)) = 3\);
2) if \(e = 0\) and \(V\) is indecomposable, then \(h^0(X, \mathcal{O}_X(-K_X)) = 1\);
3) if \(e = 0\) and \(V = \mathcal{O}_C \oplus L\), with \(L \not\cong \mathcal{O}_C\), then \(h^0(X, \mathcal{O}_X(-K_X)) = 1\);
4) if \(e \geq 1\), then \(h^0(X, \mathcal{O}_X(-K_X)) = e + 1\).
It follows from the proof of Theorem 2.1 that, for a ruled surface over an elliptic curve, it may happen that the divisor 
\(-L\) corresponding to 
\(-\Lambda^2V\) is not effective while 
\(-K_X \sim 2\tau + \pi^*(-L)\) is effective. This is not the case when 
g(C) > 1, as we shall prove in the following Proposition, which can be rephrased as follows: the divisor 
\(-K_X \sim 2\tau + \pi^*(-K_C - L)\) is effective if and only if 
\(-K_C - L\) is effective, where \(L\) is the divisor corresponding to the line bundle \(\Lambda^2V\).

**Proposition 2.3.** Let \(\pi : X \to C\) be a minimal ruled surface with \(q(X) = g(C) > 1\). Then, 
\(h^0(X, O_X(-K_X)) = h^0(C, L^*(-K_C))\).

**Proof.** From the exact sequence (5) we obtain:

\[
0 \to H^0(C, V \otimes L^*(-K_C)) \to H^0(C, S^2(V) \otimes L^*(-K_C)) \to H^0(C, L(-K_C)).
\]

Now, \(\deg L(-K_C) = -e - 2g + 2\); since \(-e \leq g\), one has \(\deg L(-K_C) < 0\) in all cases except when \(g = 2, e = -2\). So, for \(g \geq 3\) and for \(g = 2, e \neq -2\), we get

\(h^0(C, V \otimes L^*(-K_C)) = h^0(C, S^2(V) \otimes L^*(-K_C)).\)

The exact sequence (2) implies \(h^0(C, V \otimes L^*(-K_C)) = h^0(C, L^*(-K_C))\); thus, by (1) we get 
\(h^0(X, O_X(-K_X)) = h^0(C, L^*(-K_C)).\) To deal with the missing case \(g = 2, e = -2\), we use the criterion in Proposition 2.2: the divisor \(\tau\) is ample. But, \(-K_X \cdot \tau = 0\), so \(-K_X\) is not effective. This ends the proof.

We can make the previous statement somewhat more precise. By noticing that if \(e = 2g - 2\) and \(V\) is indecomposable, then \(h^1(C, L^*) \neq 0\), hence 
\(h^0(C, L(K_C)) \neq 0\), it is indeed easy to prove the following result.

**Corollary 2.2.** Let \(X\) be a minimal ruled surface with \(q(X) = g \geq 2\).

1) If \(-g \leq e \leq 2g - 3\), \(X\) does not admit any Poisson structure;
2) if \(e = 2g - 2\) and \(V\) is indecomposable, \(X\) is a Poisson surface;
3) if \(e = 2g - 2\) and \(V\) is decomposable, or \(2g - 2 < e \leq 3g - 3\), \(X\) is a Poisson surface if and only if 
\(-K_C - \Lambda^2V\) is effective;
4) if \(e \geq 3g - 2\), \(X\) is a Poisson surface.

We recall that, whenever \(2g - 2 < e \leq 3g - 3\), the vector bundle \(V\) is decomposable.
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