Strong Homotopy Lie Algebras, Generalized Nahm Equations and Multiple M2-branes

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Abstract

We review various generalizations of the notion of Lie algebras, in particular those appearing in the recently proposed Bagger-Lambert-Gustavsson model, and study their interrelations. We find that Filippov’s $n$-Lie algebras are a special case of strong homotopy Lie algebras. Furthermore, we define a class of homotopy Maurer-Cartan equations, which contains both the Nahm and the Basu-Harvey equations as special cases. Finally, we show how the super Yang-Mills equations describing a D$p$-brane and the Bagger-Lambert-Gustavsson equations supposedly describing M2-branes can be rewritten as homotopy Maurer-Cartan equations, as well.
1. Introduction and results

Since Witten’s introduction of M-theory, our understanding of this proposal for a non-perturbative unification of the various superstring models has grown quite slowly. Little more than a year ago, however, a description of the effective dynamics of stacks of M2-branes was proposed by Bagger, Lambert and independently by Gustavsson [1, 2]. The construction of the Bagger-Lambert-Gustavsson (BLG) model relies crucially on mathematical structures known as 3-Lie algebras [3]. It was soon found that the BLG model together with the special 3-Lie algebra $A_4$ should be interpreted as the effective description of a stack of two M2-branes [4]. Unfortunately, subsequent analysis showed that this is the only 3-Lie algebra which can be used in the construction of the BLG theory [5]; all other 3-Lie algebras suitable for that purpose are direct sums of $A_4$.

Therefore, a generalization of the BLG model is clearly needed and over the last year, three such generalizations were pursued. First, positive definiteness of the bilinear pairing of the 3-Lie algebra was relaxed. This leads to ghosts in the theory, which have to be removed. At least one of these procedures [6] yields $d = 3$, $N = 8$ super Yang-Mills (SYM) theory, which is not the desired outcome. Second, it was noticed that the BLG model can be rewritten as a gauge theory [7] having a Lie algebra as its gauge algebra. Thus, the original incarnation of the BLG model in terms of 3-Lie algebras might have been misleading and 3-Lie algebras could have entered the theory only accidentally for the case of two M2-branes. Third, one can use an extended version of 3-Lie algebras, which in general leads to theories with a smaller amount of supersymmetry than the original $N = 8$ of the BLG model [8, 9, 10, 11].

In this note, we present evidence that the BLG model based on 3-Lie algebras should be reconsidered within the framework of strong homotopy Lie algebras, also known as $L_\infty$ algebras [12, 13]. These are natural generalizations of Lie algebras arising in algebraic homotopy theory and deformation theory. Just as one can consider the Maurer-Cartan equation describing deformation problems governed by a Lie algebra, an $L_\infty$ algebra defines a ‘homotopy Maurer-Cartan (hMC) equation’ related to an associated deformation problem [14]. In a physics context, $L_\infty$ algebras and homotopy Maurer-Cartan equations have appeared naturally in string field theory [15], BV quantization [16], topological open string theory and topological field theory [17, 18, 19, 20, 21, 22, 23] as well as gauge theory [24].

Certain classical field equations can be written in the form of homotopy Maurer-Cartan equations associated with appropriate $L_\infty$ algebras. For example, Yang-Mills theory and its maximally supersymmetric extension were discussed in [25] and [26], the Einstein equations were rewritten in hMC form up to second order in [27]. All this, together with the results of this paper, supports the conjecture of Andrei S. Losev that for all classical field equations, an interpretation in terms of hMC equations exists.

As a side remark, we note that the twistor description provides an alternate route to reformulating field equations in hMC form. Over twistor space, solutions to classical field equations correspond to topologically trivial holomorphic vector bundles over a twistor space $\mathcal{P}$. Such vector bundles can be described in terms of holomorphic connections given locally by gauge potentials $A^{0,1} \in \Omega^{0,1}(\mathcal{P}, g)$ which are holomorphically flat, i.e. they satisfy the Maurer Cartan equation

$$\mathcal{F}^{0,2} := \bar{\partial}A^{0,1} + A^{0,1} \wedge A^{0,1} = 0 .$$ (1.1)
Here, \( \mathfrak{g} \) is an ordinary Lie algebra, and thus the twistor description yields an equivalence between classical field equations and Maurer-Cartan equations for connection one-forms taking values in a Lie algebra. For more details and examples of field theories allowing for a twistorial description, see e.g. [28] and references therein. This approach, however, requires an auxiliary twistor space, while the above mentioned reformulations of Yang-Mills and Einstein equations in terms of hMC equations live on the same space as the original equations.

In this paper, we place the BLG model into the context of \( L_\infty \) algebras and hMC equations. First, we review the relations between Filippov’s \( n \)-Lie algebras, Lie \( n \)-algebras and \( L_\infty \) algebras. We find that \( n \)-Lie algebras are a special class of \textit{ungraded} \( L_\infty \) algebras. We then construct a family of models given by hMC equations and special \( L_\infty \) algebras, which correspond to Nahm-type equations. In particular, both the Nahm and the Basu-Harvey equations [29] are contained in this family as special cases. After this, we discuss \( L_\infty \) algebras whose hMC equations yield both the SYM equations\(^1\) as well as the full supersymmetric BLG equations. We conclude with a few remarks on the reduction process from the BLG equations to the SYM equations.

2. Higher Lie structures

We shall need certain successive generalizations of the notion of a Lie algebra, all of which are based on promoting the Lie bracket to operations of higher arity, and on successive weakenings of the classical Jacobi identity. There are two types of generalizations, which can be classified into \textit{graded} and \textit{ungraded}, depending on whether the underlying vector space and operations are required or not to be graded or homogeneous\(^2\). The mathematical details can be found in the Appendix.

2.1. Ungraded generalizations of the notion of a Lie algebra

The first ungraded generalization useful for our purpose is provided by the notion of a \( n \)-Lie algebra, which was introduced by Filippov [3]. To define such algebras, one replaces the Lie bracket by an \( n \)-ary operation which is required to be totally antisymmetric and to obey a partial weakening of the Jacobi identity known as Filippov’s fundamental identity.

A wider generalization is provided by the notion of Lie \( n \)-algebra in the sense of Hanlon and Wachs [30], which again uses a single totally antisymmetric \( n \)-operation of higher arity, this time required to obey a further weakening of the Jacobi identity which we shall call the homotopy Jacobi identity. It was shown in [31] that \( n \)-Lie algebras are Lie \( n \)-algebras, because Filippov’s fundamental identity implies the homotopy Jacobi identity.

The widest generalization we shall need is the notion of a (ungraded) \textit{strong homotopy Lie} (or \( L_\infty \)) algebra, which is well-known in algebraic homotopy theory, where it originated. This is obtained by allowing for a countable family of multilinear antisymmetric operations

\(^1\)Our rewriting of this theory differs from that found in [28].

\(^2\)More generally, each higher Lie structure can be defined as an algebra object in a certain monoidal category \( \mathcal{V} \). The ungraded extensions arise by taking \( \mathcal{V} \) to be the category of vector spaces over some field \( k \) of characteristic zero, while the graded extensions arise when \( \mathcal{V} \) is a category of graded vector spaces over \( k \).
of all arities \( n \geq 1 \), constrained by a countable series of generalizations of the Jacobi identity known as the \( L_\infty \) identities. This notion admits specializations indexed by subsets\(^3\) \( S \subset \mathbb{N}^* \) of arities and which are defined by requiring vanishing of all products of arities not belonging to \( S \). This leads to the notion of \( L_S \) algebra. When \( S = \{n_1, \ldots, n_p\} \) is a finite set (with \( n_1 < \ldots < n_p \)), the result of this specialization is called an \( L_{n_1,\ldots,n_p} \) algebra, also known as an \( L_{(N)} \) algebra when \( S = \{1, \ldots, N\} \). The case \( S = \{n\} \), when only a single product of arity \( n \) is non-vanishing, recovers the notion of a Lie \( n \)-algebra, i.e. all \( n \)-Lie algebras are \( L_n \) algebras. All in all, we have a series of successive generalizations:

\[
\text{Lie algebras} \subset \text{\( n \)-Lie algebras} \subset \text{Lie \( n \)-algebras}
\]

\[
\overset{\|}{\text{\( L_n \) algebras} \subset \text{(ungraded) \( L_\infty \) algebras}}
\]

We refer the reader to the Appendix for more details.

2.2. Graded versions

An obvious variant of the algebraic structures above is to consider their graded versions. In this case the underlying vector space is graded, and each operation is graded-antisymmetric as well as homogeneous of a certain degree. The various extensions of the Jacobi identity are replaced by graded variants, which now contain appropriate sign factors. In general, one can consider a grading through elements of an Abelian group \( G \), and define the signs appearing in various equations (such as the graded antisymmetry and graded Filippov/homotopy Jacobi/\( L_\infty \) identities) by using a group morphism \( \phi \) from \( G \) to \( \mathbb{Z}_2 \). This leads to the notions of \((G,\phi)\)-graded \( n \)-Lie algebras, Lie \( n \)-algebras and \( L_\infty \) algebras/\( L_S \) algebras:

\[
\text{graded Lie algebras} \subset \text{graded \( n \)-Lie algebras} \subset \text{graded Lie \( n \)-algebras}
\]

\[
\overset{\|}{\text{graded \( L_n \) algebras} \subset \text{graded \( L_\infty \) algebras}}
\]

The choice \( G = \{0\} \) (the trivial Abelian group) and \( \phi = 0 \) recovers the ungraded theories, whose multilinear operations are antisymmetric in the usual sense rather than graded antisymmetric. In the \( L_\infty \) case, the choices \( G = \mathbb{Z}, \phi(n) = n(\text{mod} \ 2) \) and \( G = \mathbb{Z}_2, \phi = \text{id}_{\mathbb{Z}_2} \) recover the notions of \( \mathbb{Z} \)- and \( \mathbb{Z}_2 \)-graded \( L_\infty \) algebras as they appear in algebraic homotopy theory. More details can be found in the Appendix.

Remark. For other generalizations of 3-Lie algebras appearing in the Bagger-Lambert model as well as their relations to Lie triple systems, see [3, 9, 10]. There, however, the multilinear operations are not totally antisymmetric. As noted in the conclusions of [11], they might find an interpretation within a further extension of Lie algebras, the strong homotopy pre-Lie algebras.

\(^3\)Here \( \mathbb{N}^* = \{1, 2, 3, \ldots\} \) is the set of non-vanishing natural numbers.
2.3. Homotopy Maurer-Cartan equations

Given a $\mathbb{Z}$-graded $L_\infty$ algebra $L$ (see Appendix), the homotopy Maurer-Cartan equations (hMC) are the following equations for an element $\phi$ of $L$ which is weakly-homogeneous of odd parity:

$$\sum_{\ell \geq 1} \frac{(-1)^{\ell(\ell+1)/2}}{\ell!} \mu_\ell(\phi, \ldots, \phi) = 0 .$$

(2.1)

Here $\mu_\ell$ are the multiplications in $L$. These equations are invariant under the infinitesimal gauge transformations:

$$\delta \phi = -\sum_{\ell \geq 1} \frac{(-1)^{\ell(\ell-1)/2}}{(\ell-1)!} \mu_\ell(\lambda, \phi, \ldots, \phi) ,$$

(2.2)

where $\lambda$ is a weakly-homogeneous element of $L$ of even parity, playing the role of a gauge generator. We illustrate this in some two particular cases:

1. In a differential $\mathbb{Z}$-graded Lie algebra with $\mu_1 = d$ and $\mu_2 = [ , ]$, the hMC equations reduce to the classical Maurer-Cartan equations:

$$d\phi + \frac{1}{2} [\phi, \phi] = 0 .$$

(2.3)

2. In an $L_{1,2,n}$ algebra with $\mu_1 = d$ and $\mu_2(a,b) = [a,b]$, they take the form:

$$d\phi + \frac{1}{2} [\phi, \phi] + \frac{(-1)^{n(n+1)/2}}{n!} \mu_n(\phi, \ldots, \phi) = 0 .$$

(2.4)

3. A unifying framework for Nahm-type equations

3.1. A family of $L_\infty$ algebras

Let $V$ be an $(n + 1)$-dimensional complex vector space endowed with a nondegenerate $\mathbb{C}$-bilinear symmetric form $( , )$. Below, we consider the Clifford algebra $C(V,Q)$, where $Q$ is the quadratic form defined by $( , )$; if there is a canonical choice for $Q$, we also write $C(V)$ as a shorthand. Picking an orthonormal basis $e_1 \ldots e_{n+1}$ of $V$, this can be presented as the unital associative algebra generated by $e_i$ with the relations:

$$e_i e_j + e_j e_i = 2 \delta_{ij} .$$

(3.1)

Let $A_m : V^{\times r} \to C(V,Q)$ be the antisymmetrization operator:

$$A_m(v_1, \ldots, v_m) := \frac{1}{m!} \sum_{\sigma \in S_m} \epsilon(\sigma)v_{\sigma(1)} \ldots v_{\sigma(m)} ,$$

(3.2)

where juxtaposition in the right hand side stands for the multiplication in $C(V,Q)$. Since each map $A_m$ is multilinear and alternating, it factors through a map $\tilde{A}_m : \wedge^m V \to C(V,Q)$. Setting $\wedge^* V := \bigoplus_{m=0}^{n+1} \wedge^m V$, it is well-known that the map $\tilde{A} := \sum_{m=0}^{n+1} \tilde{A}_m : \wedge^* V \to C(V,Q)$ is a linear isomorphism. We will use the standard notation $e_{[i_1 \ldots i_m]}$ for the element $A_m(e_{i_1}, \ldots, e_{i_m})$ of $C(V,Q)$. Via the isomorphism $\tilde{A}$, the grading on $\wedge^* V$ induces a grading of the underlying vector space of $C(V,Q)$, whose homogeneous subspaces we denote by
Besides the differential $\mu$, other powers allows us to also describe gauge transformations. Consider the element $e := e_1 \ldots e_{n+1}$ of $C_{n+1}(V, Q)$, which corresponds to the non-vanishing element $e_1 \wedge \ldots \wedge e_{n+1}$ of the determinant line $\wedge^{n+1} V$ via the isomorphism $\tilde{A}_{n+1} : \wedge^{n+1} V \to C_{n+1}(V, Q)$. From (3.1), we have the following identities:

$$e^2 = 1 \quad \text{and} \quad ee_{i_1} \ldots e_{i_n} = n! e_{i_1} \ldots i_n e_i . \quad (3.3)$$

Now let $(L, [\ , \ , \ ])$ be an ungraded $n$-Lie algebra over the complex numbers. If $n > 2$, then we consider the vector space:

$$F_L = g_L \oplus L \quad (3.4)$$

where $(g_L, [\ , \ , \ ])$ is the basic Lie algebra of $L$, whose natural action on $L$ we denote by the symbol $\triangleright$ (see Appendix). If $n = 2$, we set $F_L = L$. An element $a$ of $\Lambda^i L$ will be said to be of order $i$ and we introduce the notation $\tilde{a} = i$. To describe both the Nahm and the Basu-Harvey equations themselves, $L$ is sufficient; however, it turns out that including the other powers allows us to also describe gauge transformations.

Consider the infinite-dimensional complex vector space $L := L^0 \oplus L^1 \oplus L^2$ where $L^0 := \Omega^0(\mathbb{R}) \otimes \mathbb{C} g_L$, $L^1 := \Omega^1(\mathbb{R}) \otimes \mathbb{C} g_L \oplus \Omega^0(\mathbb{R}) \otimes \mathbb{C} V \otimes \mathbb{C} L$ and $L^2 := \Omega^1(\mathbb{R}) \otimes \mathbb{C} V \otimes \mathbb{C} L$. We view $L$ as a $\mathbb{Z}$-graded vector space whose grading is concentrated in degrees 0, 1 and 2. Also, we endow $L$ with its natural differential $\mu_1(\omega) := -(d \otimes \text{id})\omega$, where $d$ is the de Rham differential on the real line $\mathbb{R}$ and $\omega \in \Omega(\mathbb{R}, L)$. A degree zero element of $L$ is simply a $g_L$-valued function $\lambda \in \Omega^0(\mathbb{R}) \otimes \mathbb{C} g_L$. A degree one element $\phi$ of $L$ decomposes as:

$$\phi = X + A \quad (3.5)$$

where $A$ is a $g_L$-valued one-form on the real line and $X$ is a function on the real line taking values in the vector space $V_{n+1} \otimes \mathbb{C} L$. We view $A$ as a connection one-form for the trivial principal bundle $\mathcal{G}$ over the real line whose fiber is the Lie group obtained by exponentiating $g_L$, and treat $X$ as a section of the trivial vector bundle over $\mathbb{R}$ associated to $\mathcal{G}$ via the action (induced by $\triangleright$) of $g_L$ on the vector space $V_{n+1} \otimes \mathbb{C} L$. Accordingly, we let $\nabla_s$ be the covariant derivation operator defined by the connection one-form $A$, i.e. $dX + A \triangleright X = \nabla_s(X)ds$ for any function $X$ on the real line taking values in $V_{n+1} \otimes \mathbb{C} L$. A degree two element $\rho$ of $L$ is a $V_{n+1} \otimes \mathbb{C} L$-valued one-form on the real line. Thus, a general element $x \in L$ decomposes as:

$$x = \lambda + \phi + \rho = \lambda + A + X + \rho . \quad (3.6)$$

Besides the differential $\mu_1$, we introduce the following products on $L$:

$$\mu_2(\lambda_1 + A_1 + X_1 + \rho_1, \lambda_2 + A_2 + X_2 + \rho_2)$$

$$:= \llbracket \lambda_1, \lambda_2 \rrbracket - 2\llbracket \lambda_1, A_2 \rrbracket + 2\llbracket \lambda_2, A_1 \rrbracket - 2\lambda_1 \triangleright X_2 + 2\lambda_2 \triangleright X_1$$

$$- 2A_1 \triangleright X_2 - 2A_2 \triangleright X_1 + \lambda_1 \triangleright \rho_2 - \lambda_2 \triangleright \rho_1$$

\[4\text{Here and in the following, we include factors in the products } \mu_\ell \text{ such that the normalizations in the equations of motion arising from the hMC equations match the canonical choices.}\]
and:
\[
\mu_n(X, \ldots, X) := - (-1)^{n(n-1)/2} e[X, \ldots, X] ds = (-1)^{n(n-1)/2} e e_i \ldots e_i [X^i \ldots X^i] ds 
\]
\[
= - (-1)^{n(n-1)/2} n! \epsilon_{ii_1 \ldots i_n} e_i [X^i \ldots X^i] ds 
\]
where we decomposed \( X = X^i e_i \). In fact, the form of these products is fixed by the condition that the degree of \( \mu_\ell \) equals \( 2 - \ell \). This is in particular the reason for the factor \( ds \) appearing in \( \mu_n \). We set \( \mu_\ell = 0 \) for all \( \ell \in \mathbb{Z} \setminus \{1, 2, n\} \).

It is easy to check that the products introduced above satisfy the homotopy Jacobi identities \( (B.37) \), hence \((\mathcal{L}, \mu_1, \mu_2, \mu_n)\) is a \( \mathbb{Z} \)-graded \( L_{1,2,n} \) algebra, i.e. a \( \mathbb{Z} \)-graded \( L_\infty \) algebra whose only non-vanishing products are \( \mu_1, \mu_2 \) and \( \mu_n \) (see Appendix). The identity involving two copies of \( \mu_n \) holds because \( ds \wedge ds = 0 \) on the real line. The other non-trivial homotopy Jacobi identity involves products \( \mu_2(\lambda, \cdot) \) and \( \mu_n(X, \ldots, X) \), and holds due to relation \( (A.5) \), which in turn is a consequence of Filipov’s fundamental identity \( (A.2) \) satisfied by the bracket of \( L \) (see Appendix).

The homotopy Maurer-Cartan equation \( (2.1) \) for \( \phi = A + X \):
\[
\mu_1(\phi) + \frac{1}{2} \mu_2(\phi, \phi) - \frac{(-1)^{n(n+1)/2}}{n!} \mu_n(\phi, \ldots, \phi) = 0 
\]
reduces to:
\[
\nabla_s X^i + e_{ii_1 \ldots i_n} [X^{i_1}, \ldots, X^{i_n}] = 0 
\]
and identity which we shall call the generalized Nahm equation. Since \( A \) is flat (we have \( dA = 0 \) for dimension reasons) and \( \mathbb{R} \) is contractible, the connection \( A \) can be gauged away provided that its behavior at infinity is tame enough.

The infinitesimal gauge transformations \( (2.2) \) take the form:
\[
\delta \phi = - \mu_1(\lambda) - \frac{1}{2} \mu_2(\lambda, \phi) - \frac{(-1)^{n(n-1)/2}}{(n-1)!} \mu_n(\lambda, \phi, \ldots, \phi) 
\]
Since the last term vanishes in our case, this is the same as:
\[
\delta \phi = \mu_1(\lambda) + \mu_2(\lambda, \phi) 
\]
or, more explicitly:
\[
\delta A = d \lambda + [\lambda, A] 
\]
\[
\delta X = \lambda \triangleright X 
\]
where \( \lambda \in \Omega^0(\mathbb{R}) \otimes g_L \) as above.

In the following, we discuss the two special cases \( n = 2 \) and \( n = 3 \), which correspond to the Nahm and Basu-Harvey equations, respectively.

3.2. The \( L_\infty \) algebra of the Nahm equation

The Nahm equation was originally introduced to extend the ADHM construction of instantons to the case of monopoles \( [32, 33] \). Later, this construction found an interpretation in terms of D-branes in superstring theory \( [34] \). In this interpretation, one considers a stack of
N D1-brane ending on a D3-brane in type IIB string theory, or, in type IIA, a stack of N D2-brane ending on a D4-brane. The D-branes extend into spacetime as follows:

\[
\begin{array}{ccccccc}
\text{dim} & 0 & 1 & 2 & 3 & 4 & 5 \\
D2 & \times & \times & \times & & & \\
D4 & \times & \times & \times & \times & \times & \times
\end{array}
\quad \text{or} \quad
\begin{array}{ccccccc}
\text{dim} & 0 & 1 & 2 & 3 & 4 & 5 \\
D1 & \times & \times & & & & \\
D3 & \times & \times & \times & \times & \times & \times
\end{array}
\]  \tag{3.12}

For simplicity, let us restrict to the D1/D3 case in the following. Furthermore, we will parameterize the real line extending in the 2-direction by the variable \( s \) and the indices \( i, j, k = 1, 2, 3 \) correspond to the three directions \( \mathbb{R}^3_{345} \). The D1-brane is located at \( x_1 = x_2 = x_3 = 0 \) and the configuration is assumed to be time independent.

This D-brane configuration is a BPS configuration, and it is thus effectively described by a dimensional reduction of the self-dual Yang-Mills equations. Adopting the perspective of the D3-brane, one obtains the Bogomolny monopole equations in three dimensions. The D1-brane appears as a monopole in the effective description of the D3-brane, with a scalar \( \phi \) field with profile \( \phi \sim \frac{1}{s} \), where \( r = \sqrt{x_1^2 + x_2^2 + x_3^2} \) \cite{35, 36}. Inversely, the stack of D1-branes is described by the dimensional reduction of the self-dual Yang-Mills equations to the one dimension parameterized by \( s \), which yields the Nahm equation:

\[
\nabla_s X^i + \varepsilon^{ijk}[X^j, X^k] = 0.
\]

Here, \( \nabla_s X \) is defined through \( d_A X = \nabla_s (X) ds \), where \( d_A = d + A \) is the covariant differential of a flat connection \( A \) on \( \mathbb{R} \), which can be gauged away provided that its behavior at infinity is tame enough. The components \( X^i \) of \( X \) are valued in the Lie algebra \( L := \mathfrak{su}(N) \), and the prefactor \( \varepsilon^{ijk} \) comes from the quaternionic structure underlying the Nahm equation. The D3-brane appears in this picture as the so-called fuzzy funnel solution \cite{37}: Making the separation ansatz \( X^i(s) = f(s) G^i \), we obtain the solution

\[
f(s) = \frac{1}{s} \quad \text{and} \quad G^i = \varepsilon^{ijk} [G^j, G^k].
\]  \tag{3.14}

The interpretation of this solution is as follows: For every \( s \), the cross-section of the stack of D1-branes is blown up into a fuzzy sphere described by the representation of \( \mathfrak{su}(2) \) formed by the \( G^i \). The point at \( s = 0 \) is thus increased to that of a D3-brane (with non-commutative world volume for \( N < \infty \)), and at \( s = 0 \), this D3-brane becomes flat as the radius \( R \sim f(s) \) of the fuzzy sphere tends to infinity. Note that the profiles of both interpretations are compatible:

\[
\phi(r) = s(r) = \frac{1}{r} = \frac{1}{f(s)}.
\]  \tag{3.15}

The Nahm equation \cite{33} is now a special case of the homotopy Maurer-Cartan equations we considered above. We choose \( n = 2 \) and work with the 2-Lie algebra \( L = \mathfrak{su}(N) \), from which we construct the ungraded \( L_\infty \) algebra \( \mathcal{L} := \Omega^\bullet(\mathbb{R}) \otimes C(\mathbb{R}^3) \otimes L \). The Clifford algebra \( C(\mathbb{R}^3) \) is generated by \( e_1, e_2, e_3 \) satisfying \( e_i e_j + e_j e_i = 2 \delta_{ij} \); moreover, we have \( e = e_1 e_2 e_3 = 1 \). The only non-trivial products in \( \mathcal{L} \) are \( \mu_1 \) and \( \mu_2 \), where again \( \mu_1(x) = (d \otimes \text{id}) x \) for \( x \in \mathcal{L} \) and \( \mu_2 \) is defined through:

\[
\begin{align*}
\mu_2(\lambda_1, \lambda_2) & := [\lambda_1, \lambda_2], & \mu_2(\lambda, A) := -2[\lambda, A], & \mu_2(\lambda, X) := -2[\lambda, X], \\
\mu_2(A, X) & := -[A, X], & \mu_2(X, X) & := -2[X, X] ds,
\end{align*}
\]  \tag{3.16}
where the bracket \([ , ]\) is the Lie bracket of \(L\). The homotopy Maurer-Cartan equation takes the form:

\[- \mu_1(X) - \mu_2(A, X) - \frac{1}{2} \mu_2(X, X) = 0 ,\]

which is equivalent to the Nahm equation (3.13). The gauge transformations preserving this equation take the form:

\[
\delta A = - \mu_1(\lambda) - \frac{1}{2} \mu_2(\lambda, A) = d\lambda + [\lambda, A] , \quad \delta X = - \frac{1}{2} \mu_2(\lambda, X) = [\lambda, X] ,
\]

where \(\lambda \in \Omega^0(\mathbb{R}) \otimes L\). These, of course, coincide with the homotopy gauge transformations (2.2) as they apply to our case.

### 3.3. The \(L_\infty\) algebra of the Basu-Harvey equation

In their paper [29], Basu and Harvey suggested a generalized Nahm equation as a description of a stack of \(N\) M2-branes ending on an M5-brane. The configuration in flat spacetime is here given by:

\[
\begin{array}{ccccccc}
\text{dim} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
M2 & \times & \times & \times & \times & \times & \times & \times \\
M5 & \times & \times & \times & \times & \times & \times & \times
\end{array}
\]

(3.19)

where we parameterize again the real line extending in the 2-direction by \(s\). The coordinates \(x_i, i = 1, \ldots, 4\) describe here the space \(\mathbb{R}^4_{3456}\) in the M5-brane, which is perpendicular to the M2-branes. From the analysis of the Abelian theory [36], we know that such a stack of M2-branes should appear in the worldvolume theory on the M5-brane as a scalar field with profile \(\frac{1}{r^2}\), where \(r = \sqrt{x_1^2 + \ldots + x_7^2}\). The M5-brane, in turn, should therefore appear as the scalar field in a solution to the worldvolume theory on the M2-brane with profile \(\frac{1}{\sqrt{s}}\).

Assuming a splitting ansatz for the scalar field \(X^i(s) = f(s)G^i, i = 1, \ldots, 4\), one concludes that the equations should take the following form [29]:

\[
\frac{d}{ds}X^i + \varepsilon^{ijkl}[X^j, X^k, X^l] = 0 ,
\]

(3.20)

where the \(X^i\) take values in a 3-Lie algebra \(L\). Again, these equations can be recast in terms of our Maurer-Cartan equations: they correspond to the case \(n = 3\). Given a 3-Lie algebra \(L\), we have an associated 2-Lie algebra \(\mathfrak{g}_L\). As before, we define \(\mathcal{L} := \Omega^* (\mathbb{R}) \otimes C(\mathbb{R}^4) \otimes F_L\), where \(F_L = L \oplus \mathfrak{g}_L\). As generators for the Clifford algebra \(C(\mathbb{R}^4)\) we take an orthonormal basis \(e_1 \ldots e_4\) of \(\mathbb{R}^4\). In this case one has \(e := e_1 \ldots e_4\).

We endow \(\mathcal{L}\) with the non-trivial products \(\mu_1(x) = -(d \otimes \text{id}) x\) as well as \(\mu_2\) and \(\mu_3\) defined through:

\[
\mu_2(\lambda_1, \lambda_2) := [\lambda_1, \lambda_2] , \quad \mu_2(\lambda, A) := -2[\lambda, A] , \quad \mu_2(\lambda, X) := -2\lambda \triangleright X ,
\]

\[
\mu_2(A, X) := -A \triangleright X , \quad \mu_3(X, X, X) := 3! e[X, X, X]ds .
\]

(3.21)

The homotopy Maurer-Cartan equation equivalent to the Basu-Harvey equation (3.20) reads:

\[- \mu_1(X) - \mu_2(A, X) + \frac{1}{3!} \mu_3(X, X, X) = 0 .
\]

(3.22)
The gauge transformations (2.2) preserving this equation take the form:

$$\delta A = -\mu_1(\lambda) - \frac{1}{2}\mu_2(\lambda, A) = d\lambda + [\lambda, A], \quad \delta X = -\frac{1}{2}\mu_2(\lambda, X) = \lambda \triangleright X,$$

where $\lambda \in \Omega^0(\mathbb{R}) \otimes_{\mathbb{C}} g_L$.

4. The full theories

The Nahm equation and supposedly the Basu-Harvey equation are BPS equations in the effective description of D1- and M2-branes respectively. Hence, they partly determine the supersymmetry transformations of these effective theories, and from closure of the supersymmetry algebra, the field equations and ultimately the Lagrangian can be derived. This, in fact, is how Bagger and Lambert [1] and independently Gustavsson [2] obtained the BLG model in the first place. In this section, we show that graded homotopy Maurer-Cartan equations appear, albeit the grading in the underlying $L_\infty$ algebra is slightly less natural than it was in the case of BPS equations.

4.1. The description of D1- and D2-branes dynamics by homotopy Maurer-Cartan equations

It is well-known that the effective dynamics of D-branes is described by maximally supersymmetric Yang-Mills (SYM) theory dimensionally reduced to their worldvolume, see [38] for details. For a stack of $n$ flat D$p$-branes, one starts with the Lagrangian of $\mathcal{N} = 1$, $U(n)$ SYM theory in ten dimensions,

$$S = \int d^{10}x \text{tr}(-\frac{1}{4}F_{MN}F^{MN} + \frac{i}{2}\bar{\psi}\Gamma^M \nabla_M \psi).$$

(4.1)

Here, $F_{MN}$ are the components of the field strength of the $U(n)$ connection, whose covariant derivative we denote by $\nabla$. The fermions of the theory are described by ten-dimensional Majorana-Weyl spinors $\psi$ transforming in the adjoint representation of the gauge group, while $\Gamma^M$ are the Gamma matrices in the corresponding representation of the Clifford algebra $C(\mathbb{R}^{1,9})$. The dimensional reduction of the bosonic part of the action is obtained as usual by splitting the ten-dimensional gauge potential $A_M$ into a $p + 1$-dimensional gauge potential $A_\mu$, $\mu = 0, ..., p$ and $9 - p$ scalar fields $X^i$, $i = 1, ..., 9 - p$. Setting all derivatives along directions transverse to the worldvolume of the D$p$-brane to zero gives:

$$\text{tr}(F_{MN}F^{MN}) \rightarrow \text{tr}(F_{\mu\nu}F^{\mu\nu} + \nabla_\mu X^i \nabla^\mu X^i + [X^i, X^j][X^i, X^j]).$$

(4.2)

The resulting equations of motion are:

$$\nabla_\mu F^{\mu\nu} = [X^i, \nabla^\nu X^i],$$

(4.3a)

$$\nabla_\mu \nabla^\mu X^i = [[X^i, X^j], X^j].$$

(4.3b)

Gauge transformations are generated by $\mathfrak{su}(n)$-valued functions $\lambda$ on $\mathbb{R}^{1,p}$ and on infinitesimal level, the fields transform according to

$$\delta_\lambda A := d\lambda + [\lambda, A], \quad \delta_\lambda X^i := [\lambda, X^i].$$

(4.4)
It has been shown in [26] that the pure Yang-Mills equations can be interpreted as
homotopy Maurer-Cartan equations using a BRST complex. Earlier, in [25], the maximally
supersymmetric Yang-Mills were recast into hMC form using a pure spinor formulation.
Here, we will present a simple reinterpretation of the $\mathcal{N} = 8$ SYM equations in two or three
dimensions as hMC equations without invoking more than classical structures. To be concise,
we first discuss the purely bosonic part of the action in detail; the necessary modifications
for the supersymmetric extension will be listed in the next section.

The main difference to the Nahm and the Basu-Harvey equations is the fact that the
differential equations (4.3) are non-linear. We will therefore have to define $\mu_1$ as a non-linear
differential operator. It should be stressed, that our rewriting of the two-dimensional $\mathcal{N} = 8$
SYM equations as homotopy Maurer-Cartan equations is by no means unique.

On $\mathbb{R}^{1,p}$, we consider the de Rham differential $d$, the Hodge operator $*$ with respect
to the Minkowski metric $\eta_{\mu
u}$ and the top form $\omega = dx^0 \wedge \ldots \wedge dx^p$. Furthermore, let $q = 9 - p$
as a shorthand and, as before, $C(\mathbb{R}^q) = C_0(\mathbb{R}^q) \oplus \ldots \oplus C_q(\mathbb{R}^q)$ is the Clifford algebra with
generators $e_1, \ldots, e_q$ and $e := e_1 e_2 \ldots e_q$. The vector space underlying our $L_\infty$ structure $\mathcal{L}$ for
SYM theory will be the same as for the Nahm equation extended to the two-dimensional
world-volume of the D$p$-branes, and we employ again the 2-Lie algebra $L = su(N)$. The $L_\infty$
agebra will be supported on the infinite-dimensional vector space:

$$\mathcal{L} := \mathcal{L}^0 \oplus \mathcal{L}^1 \oplus \mathcal{L}^2, \quad (4.5)$$

where:

$$\mathcal{L}^0 := \Omega^0(\mathbb{R}^{1,p}) \otimes C \ L, \quad \mathcal{L}^1 := \left[ \Omega^1(\mathbb{R}^{1,p}) \otimes C \ L \right] \oplus \left[ \Omega^0(\mathbb{R}^{1,p}) \otimes C_1(\mathbb{R}^q) \otimes L \right], \quad (4.6)$$

$$\mathcal{L}^2 := \left[ \Omega^1(\mathbb{R}^{1,p}) \otimes C_2(\mathbb{R}^q) \otimes L \right] \oplus \left[ \Omega^2(\mathbb{R}^{1,p}) \otimes C_1(\mathbb{R}^q) \otimes L \right].$$

We view $\mathcal{L}$ as a $\mathbb{Z}$-graded vector space whose grading is concentrated in degrees $0, 1$ and $2$, with $\mathcal{L}^0, \mathcal{L}^1$ and $\mathcal{L}^2$ playing the role of homogeneous subspaces of the corresponding degree.

For what follows, we consider objects:

$$A \in \Omega^1(\mathbb{R}^{1,p}) \otimes C \ g_L, \quad X \in \Omega^0(\mathbb{R}^{1,p}) \otimes C_1(\mathbb{R}^q) \otimes L, \quad \lambda, \lambda_{1,2} \in \Omega^0(\mathbb{R}^{1,p}) \otimes C \ g_L. \quad (4.7)$$

We will treat $A$ as a connection one-form on the worldvolume valued in the trivial principal
bundle $\mathcal{G}$ defined by the Lie group with Lie algebra $g_L$, $X$ as a section of the
trivial vector bundle with fiber $C_1(\mathbb{R}^q) \otimes L$ associated with $\mathcal{G}$ via the representation induced
by $\triangleright$ and $\lambda$ as a generator of the group of gauge transformations of $\mathcal{G}$. For these fields,
we now define various higher products $\mu_k$. The modified Koszul rule of the $L_\infty$ products
gives the relations $\mu_2(X, A) = \mu_2(A, X)$, $\mu_3(X, X, A) = \mu_3(X, A, X) = \mu_3(A, X, X)$ and
$\mu_3(A, A, X) = \mu_3(A, A, A) = \mu_3(X, A, A)$. Thus the homotopy Maurer-Cartan equation for
$\mathcal{L}$ with field $\phi = X + A$ is

$$- \mu_1(X) - \mu_1(A) - \frac{1}{2} \mu_2(X, X) - \frac{1}{6} \mu_2(A, A) - \mu_2(X, A) +$$

$$+ \frac{1}{6} \mu_3(X, X, X) + \frac{1}{6} \mu_3(A, A, A) + \frac{1}{6} \mu_3(X, A, X) + \frac{1}{6} \mu_3(A, A, X) = 0. \quad (4.8)$$

Of course, our approach can be easily extended to arbitrary SYM theories.
So we define the following products $\mu_k$ mapping into $\Omega^1(\mathbb{R}^{1,1}) \otimes C_\infty(\mathbb{R}^8) \otimes \mathfrak{g}_L$

$$\begin{align*}
\mu_1(A) & := -(*d*dA)\gamma, & \mu_2(A, A) & := -2(*[A,*dA]+*d*[A,A])\gamma, \\
\mu_3(A, A, A) & := 6(*[A,*[A,A]])\gamma, & \mu_2(X, X) & := \frac{1}{4}\text{tr}_C([X, dX])\gamma, \\
\mu_3(X, A, X) & := \frac{1}{4}\text{tr}_C([X, [X, A, X]])\gamma.
\end{align*}$$

These are the expressions appearing in equation (4.3a). The products mapping to $\Omega^2(\mathbb{R}^{1,1}) \otimes C_1(\mathbb{R}^8) \otimes \mathfrak{g}_L$ are defined according to

$$\begin{align*}
\mu_1(X) & = -\Delta X \omega, & \mu_3(X, X, X) & = -6(\gamma[X, \gamma[X, X]])\omega, \\
\mu_2(A, X) & = -([A_\mu, \partial^\mu X] \omega + \partial_\mu [A^\mu, X])\omega, & \mu_3(A, A, X) & = 2[A_\mu, [A^\mu, X]]\omega,
\end{align*}$$

and they contain all the terms appearing in equation (4.3b). Every bracket maps elements of degree one into elements of degree two. As all brackets containing degree two elements vanish, the homotopy Jacobi identities are all satisfied trivially. Therefore, $\mathcal{L}$ with the brackets above forms an $L_\infty$ algebra.

To include gauge transformations, however, we evidently need to introduce the following additional products:

$$\begin{align*}
\mu_1(\lambda) & = -d\lambda, & \mu_2(\lambda, A) & = -2[\lambda, A], & \mu_2(\lambda_1, \lambda_2) & := [\lambda_1, \lambda_2], & \mu_2(\lambda, X) & = -2[\lambda, X].
\end{align*}$$

These follow from the homotopy Maurer-Cartan gauge transformation equation,

$$\delta_\lambda(\phi) = -\mu_1(\lambda) - \frac{1}{2}\mu_2(\lambda, \phi).$$

The range of these maps are degree-zero and -one elements of $\mathcal{L}$ and thus the homotopy Jacobi identities impose nontrivial relations. Most of these relations are trivially satisfied, as e.g. the one for the argument $(x_1) = (\lambda)$: $\mu_1(\mu_1(\lambda)) = 0$ follows from $d^2 = 0$. The relation with arguments $(x_1, x_2) = (\lambda_1, \lambda_2)$ imposes the usual Leibniz rule:

$$\mu_1(\mu_2(\lambda_1, \lambda_2)) = \mu_2(\mu_1(\lambda_1), \lambda_2) - \mu_2(\mu_1(\lambda_2), \lambda_1).$$

The remaining relations specify how gauge transformations act on higher products $\mu_k$. For fields $(x_1, x_2) = (\lambda, X)$, we have the relation

$$\mu_1(\mu_2(\lambda, X)) = \mu_2(\mu_1(\lambda), X) - \mu_2(\mu_1(X), \lambda),$$

which defines $\mu_2(\mu_1(X), \lambda)$. Analogously, we have the homotopy Jacobi identity for the fields $(x_1, x_2) = (\lambda, A)$:

$$\mu_1(\mu_2(\lambda, A)) = \mu_2(\mu_1(\lambda), A) - \mu_2(\mu_1(X), \lambda),$$

which defines $\mu_2(\mu_1(A), \lambda)$. As it is easily seen, the remaining identities define the following additional products:

$$\begin{align*}
\mu_2(\lambda, \mu_2(A, A)), & \quad \mu_2(\lambda, \mu_2(A, X)), & \quad \mu_2(\lambda, \mu_2(X, X)), \\
\mu_2(\lambda, \mu_3(A, A)), & \quad \mu_2(\lambda, \mu_3(X, X)), & \quad \mu_2(\lambda, \mu_3(A, A)), \\
\mu_2(\lambda, \mu_3(X, A, X)). & \quad \mu_2(\lambda, \mu_3(X, X)), & \quad \mu_2(\lambda, \mu_3(A, A)),
\end{align*}$$

Plugging $\phi := (A, X)$ into the homotopy Maurer-Cartan equations of this $L_\infty$ algebra, we recover the bosonic part of the SYM equations (4.3). The gauge transformations (4.4) are equivalent with the gauge transformations (2.22) for $\phi$. 

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4.2. Supersymmetric extension

For brevity, we do not rewrite Majorana-Weyl spinors $\psi$ of $SO(1,9)$ in terms of multiple spinors of $SO(1,p)$, but work with the restriction $S = \sqrt{K}$ of the spinor bundle over $\mathbb{R}^{1,p}$. Thus, we split the generators $\Gamma^M$ of $C(\mathbb{R}^{1,9})$ into $\Gamma^\mu$, $\mu = 0, \ldots, p$ and $e^I$, $I = 1, \ldots, q$. In this form, the $\mathcal{N} = 8$ SYM equations in two or three dimensions read as

$$
\nabla_\mu F^{\mu\nu} = [X^I, \nabla^\nu X^I] - \frac{1}{2} \Gamma^\nu_{\alpha\beta} \{ \bar{\psi}^\alpha, \psi^\beta \}, \quad (4.17a)
$$

$$
\Gamma^\mu_{\alpha\beta} \nabla_\mu \psi^\beta = -\Gamma^I_{\alpha\beta} [X^I, \psi^\beta], \quad (4.17b)
$$

$$
\nabla_\mu \Gamma^\mu_{\alpha\beta} X^I = \{ [X^I, X^J], X^J \} - \frac{1}{2} \Gamma^I_{\alpha\beta} \{ \bar{\psi}^\alpha, \psi^\beta \}. \quad (4.17c)
$$

In the homotopy Maurer-Cartan equations, we take $\phi = (A, \psi, \bar{\psi}, X)$, where $\psi \in \Gamma(S)$ and $\bar{\psi} \in \Gamma(\bar{S})$. We therefore extend $\mathcal{L}$ to

$$
\hat{\mathcal{L}} := \Omega^*(\mathbb{R}^{1,p}) \otimes \mathbb{C} C(\mathbb{R}^3) \otimes \mathfrak{g}_L \oplus (\Gamma(S) \oplus \Gamma(\bar{S}) \oplus \Gamma(S^*) \oplus \Gamma(\bar{S}^*)) \otimes \mathfrak{g}_L, \quad (4.18)
$$

which we associate with the following grading:

$$
\deg(\Gamma(S) \otimes \mathfrak{g}_L) = \deg(\Gamma(\bar{S}) \otimes \mathfrak{g}_L) = 1, \quad \deg(\Gamma(S^*) \otimes \mathfrak{g}_L) = \deg(\Gamma(\bar{S}^*) \otimes \mathfrak{g}_L) = 2. \quad (4.19)
$$

The additional products we have to introduce are straightforwardly read off the equations of motion (4.17):

$$
\mu_1(\psi) = -\Gamma^\mu_{\alpha\beta} \partial_\mu \psi^\beta, \quad \mu_2(A, \psi) = -\Gamma^\mu_{\alpha\beta} A_\mu \psi^\beta,
$$

$$
\mu_2(X, \psi) = -\Gamma^I_{\alpha\beta} [X^I, \psi^\beta], \quad \mu_2(\bar{\psi}, \psi) = -\frac{1}{2} \eta^{\mu\nu} \Gamma^\mu_{\alpha\beta} \{ \bar{\psi}^\alpha, \psi^\beta \} dx^\nu - \frac{1}{2} \Gamma_I \Gamma^I_{\alpha\beta} \{ \bar{\psi}^\alpha, \psi^\beta \} \omega. \quad (4.20)
$$

The homotopy Jacobi identities are again trivially satisfied, as a product $\mu_k \bar{\psi}$ amongst its arguments is defined to be vanishing.

To capture gauge transformations, we additionally introduce

$$
\mu_2(\lambda, \psi) := -2[\lambda, \psi]. \quad (4.21)
$$

The homotopy Jacobi identities then extend the action of gauge transformations to products $\mu_k$ containing $\psi$ in the arguments.

The homotopy Maurer-Cartan equations together with the $L_\infty$ algebra $\mathcal{L}$ endowed with all of these brackets reproduce both the two-dimensional $\mathcal{N} = 8$ SYM equations and the associated gauge transformations of the fields.

Note that the association of degrees to the various subspaces of $\mathcal{L}$ seems to be rather ad-hoc. A more enlightening approach to the grading of the $L_\infty$ algebra is obtained from considering ghost number grading of a BRST complex, as done for pure Yang-Mills theory in [26].

4.3. Review of the BLG theory

We start from a metric 3-Lie algebra $(L, (\ , \ ))$ with associated Lie algebra $\mathfrak{g}_L$. The bracket and the metric on $L$ can be used to construct an invariant bilinear form $(\ , \ )$ on $\mathfrak{g}_L$, which is not identical with the Killing form on $\mathfrak{g}_L$, and which is not positive definite [10]:

$$
(x_1 \wedge x_2, b_1 \wedge b_2) := ([x_1, x_2, b_1], b_2), \quad x_1, x_2, b_1, b_2 \in L. \quad (4.22)
$$
One easily verifies symmetry of \( (\, , \, ) \).

The matter field content of the BLG theory is given by the Goldstone fields arising from the spacetime symmetries broken by the presence of the M2-branes. We have thus 8 scalar fields \( X^I, I = 1, \ldots, 8 \) and a Majorana spinor \( \Psi \) of Spin(1,10) as their superpartner. The matter fields take values in the 3-Lie algebra \( L \) with invariant form \( (\, , \, ) \). In addition, we have a gauge potential \( A_\mu \) taking values in \( \Omega^1(\mathbb{R}^{1,2}) \otimes_C g_L \). As gamma matrices, we use the 11-dimensional ones, which are split according to \( \Gamma_M, M = 0, \ldots, 10 \rightarrow (\Gamma_\mu, e_I), \) \( (\mu = 0, \ldots, 2, I = 1, \ldots, 8) \). For simplicity, we introduce \( X = e_I X^I \) and the invariant form \( (\, , \, ) \) is assumed to include a trace where necessary. The Lagrangian density can then be written in the form:

\[
\mathcal{L}_{BLG} = -\frac{1}{2}(\nabla_\mu X, \nabla^\mu X)_{A \otimes C} + \frac{1}{2}(\bar{\Psi}, \Gamma^\mu \nabla\mu \Psi)_{A} + \frac{i}{4}(\bar{\Psi}, [X, X, \Psi])_{A} - \frac{1}{12}([X, X, X], [X, X, X])_{A \otimes C} + \frac{2}{3} e^{\mu\nu\kappa}(\{A_\mu, \partial_\nu A_\kappa\} + \frac{1}{3}(A_\mu, [A_\nu, A_\kappa])) .
\]

(4.23)

where \([\, , \, ]\) denotes the Lie bracket in \( g_L \), \( \rhd \) denotes the natural action of elements of \( g_L \) on \( L \) and \([\, , \, , \, ]\) is the triple bracket in \( L \) extended linearly for \( C(\mathbb{R}^8) \)-valued objects. This action is invariant under the supersymmetry transformations

\[
\delta X = i\Gamma_I \varepsilon \Gamma^I \Psi , \quad \delta \Psi = \nabla_\mu X^I \varepsilon - \frac{i}{6} [X, X, X] \varepsilon , \quad \delta A_\mu = i\varepsilon \Gamma_\mu (X \wedge \Psi)
\]

(4.24)

up to translations, equations of motion and gauge transformations. On infinitesimal level, the latter are generated by \( \lambda \in \Omega^0(\mathbb{R}^{1,2}) \otimes_C g_L \) according to

\[
\delta X = \lambda \rhd X , \quad \delta \Psi = \lambda \rhd \Psi , \quad \delta A_\mu = \partial_\mu \lambda + \{\lambda, A_\mu\} .
\]

(4.25)

4.4. **BLG equations as homotopy Maurer-Cartan equations**

The BLG equations can also be rewritten as homotopy Maurer-Cartan equations; as the equations of motion for the gauge field are of Chern-Simons type, the grading is slightly more natural than in the case of SYM theory. As before, let us first discuss the bosonic part in detail and add the supersymmetric extension later. The equations of motion take the form:

\[
\nabla_\mu \nabla^\mu X + \frac{1}{2} e[X, X, e[X, X, X]] = 0 , \quad [\nabla_\mu, \nabla_\nu] + e_{\mu\nu\kappa}(X^J \wedge (\nabla^\kappa X^J)) = 0 .
\]

(4.26)

The appropriate extension for the algebra underlying the homotopy Maurer-Cartan interpretation of the Basu-Harvey equation turns out to be

\[
\mathcal{L} := \Omega^*(\mathbb{R}^{1,2}) \otimes_C C(\mathbb{R}^8) \otimes_F L ,
\]

(4.27)

and we associate the following degrees to its subspaces

\[
\deg(\Omega^0(\mathbb{R}^{1,2}) \otimes g_L) = 0 , \quad \deg(\Omega^1(\mathbb{R}^{1,2}) \otimes g_L) = 1 , \quad \deg(\Omega^2(\mathbb{R}^{1,2}) \otimes g_L) = 2 .
\]

(4.28)

Our notation for elements of the various subspaces will be

\[
A \in \Omega^1(\mathbb{R}^{1,2}) \otimes_C g_L , \quad X \in \Omega^0(\mathbb{R}^{1,2}) \otimes_C C(\mathbb{R}^8) \otimes_C L , \quad \lambda, \lambda_{1,2} \in \Omega^0(\mathbb{R}^{1,2}) \otimes_C g_L .
\]

(4.29)
The non-trivial products are very similar to those which we used for the Basu-Harvey equation. The homotopy Maurer-Cartan equation, including the \( \mu_5(X,X,X,X,X) \) product term, is

\[
- \mu_1(X) - \mu_1(A) - \frac{1}{2} \mu_2(X,X) - \frac{1}{2} \mu_2(A,A) - \mu_2(X,A) + \frac{1}{6} \mu_3(X,X,X) + \\
\frac{1}{6} \mu_3(A,A,A) + \frac{1}{2} \mu_3(A,X) + \frac{1}{2} \mu_3(A,A) - \frac{1}{12} \mu_5(X,X,X,X,X) = 0 .
\]

(4.30)

The product components mapping into \( \Omega^2(\mathbb{R}^{1,2}) \otimes g_L \) are defined through

\[
\mu_1(A) := -dA , \quad \mu_2(A,A) := -2[A \wedge A] , \\
\mu_2(X,X) := -\frac{1}{4} * \text{tr}_C(X \wedge dX) , \quad \mu_3(A,X,X) := \frac{1}{4} * \text{tr}_C(X \wedge [A,X]) , \\
\text{while the product components mapping into } \Omega^2(\mathbb{R}^{1,2}) \otimes C_1(\mathbb{R}^8) \otimes L \text{ are given by}
\]

\[
\mu_1(A) := -\Delta X \omega , \quad \mu_2(A,X) := -\partial_\mu \llbracket A^\mu, X \rrbracket \omega - \llbracket A_\mu, \partial^\mu X \rrbracket \omega , \\
\mu_3(A,A,X) := 2 \llbracket A_\mu, \llbracket A^\mu, X \rrbracket \rrbracket \omega , \quad \mu_5(X,\ldots,X) := -\frac{2}{5!} \Gamma[X,X,\Gamma[X,X,X,X]] \omega .
\]

(4.32)

Here, \( \omega = dx^0 \wedge dx^1 \wedge dx^2 \) and \(*\) are the top form and the Hodge star on \( \mathbb{R}^{1,2} \), respectively; \( \tau \) denotes the trace over the representation of the Clifford algebra \( C(\mathbb{R}^8) \). For the time being, all other products are put to zero. These products capture all the terms appearing in the equations (4.26). Moreover, they map elements of \( L \) of degree 1 into elements of degree 2, and as in the SYM case, the homotopy Jacobi identities are therefore trivially satisfied. To allow for gauge transformations as well, we furthermore introduce the products

\[
\mu_1(\lambda) := -d\lambda , \quad \mu_2(A,\lambda) := -2[A,\lambda] , \quad \mu_2(\lambda_1,\lambda_2) := \llbracket \lambda_1, \lambda_2 \rrbracket , \quad \mu_2(\lambda,X) := -2\lambda \triangleright X .
\]

(4.33)

The non-trivial Jacobi identities arising from this definition yield the usual Leibniz rule

\[
\mu_1(\mu_2(\lambda_1,\lambda_2)) = \mu_2(\mu_1(\lambda_1),\lambda_2) - \mu_2(\mu_1(\lambda_2),\lambda_1) .
\]

(4.34)

and define the following products:

\[
\mu_2(\lambda,\mu_2(A,A)) , \quad \mu_2(\lambda,\mu_2(X,X)) , \quad \mu_2(\lambda,\mu_3(A,X,X)) , \\
\mu_2(\lambda,\mu_3(A,A,X)) , \quad \mu_2(\lambda,\mu_5(X,X,X,X)) .
\]

(4.35)

Using these definitions and the field vector \( \phi := (A,X) \) in the homotopy Maurer-Cartan equations and the prescription for infinitesimal gauge transformations reproduces the BLG equations together with the appropriate gauge transformations.

4.5. Supersymmetric extension

For the supersymmetric extension, we use the field content presented in section 4.3, that is, we work with the spinor bundle \( S \) over \( \mathbb{R}^{1,10} \) restricted to \( \mathbb{R}^{1,2} \). The fully supersymmetric equations of motion read as

\[
\nabla_\mu \nabla^\mu X + \frac{1}{2} e_K [\bar{\Psi}, X, e^K \Psi] + \frac{1}{2} e [X, X, e[X, X, X]] = 0 , \\
\Gamma^\mu \nabla_\mu \Psi + \frac{1}{2} [X, X, \Psi] = 0 , \\
\llbracket \nabla_\mu, \nabla_\nu \rrbracket + \varepsilon_{\mu \nu \kappa} (X^J \wedge (\nabla^K X^J) + \frac{1}{2} \bar{\Psi} \wedge (\Gamma^K \Psi)) = 0 .
\]

\text{We do not write down separate equations for fields living in } S \text{ and } \bar{S} .

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To incorporate the fermions, we need to extend $\mathcal{L}$ as follows

$$\hat{\mathcal{L}} := \Omega^*(\mathbb{R}^{1,2}) \otimes \mathbb{C} \otimes F_L \oplus (\Gamma(S) \oplus \Gamma(S^*) \oplus \Gamma(S^*)) \otimes L ,$$

and introduce the further products

$$\mu_2(\bar{\Psi}, \Psi) := \frac{i}{2} \varepsilon_{\mu \nu \kappa} \bar{\Psi} \wedge (\Gamma^\kappa \Psi) dx^\mu \wedge dx^\nu , \quad \mu_3(\bar{X}, X, \Psi) := \frac{i}{2} [\bar{X}, X, \Psi] , \quad \mu_3(\bar{X}, X, \Psi) := \frac{i}{2} [\bar{X}, X, \varepsilon^K \Psi] .$$

These are enough to fully capture the equations of motion, and to allow for gauge transformations, we extend the set of defined products by

$$\mu_2(\lambda, \Psi) := -2[\lambda, \Psi] .$$

The homotopy Jacobi identities then define for us the action of $\mu_2(\lambda, )$ on any of the products, and thus the description of the BLG equations together with their gauge symmetry in terms of homotopy Maurer-Cartan equations is complete.

4.6. From M2-branes to D2-branes

In [4], a nice reduction mechanism was proposed to descend from the BLG model with 3-Lie algebra $A_4$ describing two M2-branes to $d=3, \mathcal{N}=8$ SYM theory with gauge group $U(2)$ describing two D2-branes. The idea behind this is to compactify one direction, say $x^{10}$ on a circle and to assume that this compactification associates an expectation value with the corresponding scalar $X^{48} = X^{48} \tau_4$:

$$\langle X^{48} \rangle = \frac{R}{\ell_s^3} = \frac{g_s}{\ell_s} = g_{YM} ,$$

where $\tau_a, a = 1, \ldots, 4$ denote the generators of $A_4$. This induces a splitting $X^I \rightarrow (X^I, g_{YM})$ as well as a splitting

$$f^{\hat{a} \hat{b} \hat{c}} = \varepsilon^{\hat{a} \hat{b} \hat{c}} \delta^{\hat{d} 4} - \varepsilon^{\hat{a} \hat{b} \hat{d}} \delta^{\hat{c} 4} + \varepsilon^{\hat{a} \hat{c} \hat{d}} \delta^{\hat{b} 4} - \varepsilon^{\hat{b} \hat{c} \hat{d}} \delta^{\hat{a} 4} , \quad a, b, c, d = 1, \ldots, 3 .$$

Because of the last decomposition, we can split the gauge potential according to $A_\mu := A^{ab}_\mu \tau_a \wedge \tau_b$

$$A^a_\mu := A^a_\mu \quad \text{and} \quad B^a_\mu := \frac{1}{2} \varepsilon^{a \hat{b} \hat{c}} A^b_\mu .$$

Rewriting the hMC equations of the BLG model using the expectation value for $X^{\hat{8}}$ as well as the decomposition of the gauge potential, the equations of motion for $B^a_\mu$ become algebraic, and we can eliminate this field. The result are the hMC equations of $d=3 \mathcal{N}=8$ SYM theory plus corrections in lower powers of $g_{YM}$. In a strong coupling expansion, in which the subleading terms are neglected, the corrections vanish, and the effective description of a stack of two D2-branes is reproduced [4].

Acknowledgements

DM and AZ are supported by IRCSET (Irish Research Council for Science, Engineering and Technology) postgraduate research scholarships. CS is supported by an IRCSET postdoctoral fellowship.
Appendix: Various extensions of the notion of a Lie algebra

In this appendix, we give a brief account of the various higher Lie structures used in this paper, which in particular fixes our conventions.

Notations. Let $S_n$ be the group of permutations on $n$ elements. Given a permutation $\sigma \in S_n$, we let $\epsilon(\sigma)$ denote its signature. Recall that $\sigma$ is called an $(j, n-j)$-unshuffle (where $1 \leq j \leq n$) if $\sigma(1) < \ldots < \sigma(j)$ and $\sigma(j+1) < \ldots < \sigma(n)$. We let $\text{Sh}(i, n-i)$ denote the set of all $(i, n-i)$-unshuffles.

A. Ungraded extensions of the notion of Lie algebra

Let $k$ be a field of characteristic zero, for example $k = \mathbb{R}$ or $k = \mathbb{C}$ and $R$ be an associative and commutative unital $k$-algebra, for example $R = \mathbb{R}$ or $R = \mathbb{C}$ again.

$n$-Lie algebras in the sense of Filippov

Definition. An $n$-Lie algebra over $R$ is a (unital) $R$-module $L$ endowed with an $R$-multilinear map $[\ , \ldots , \ ] : L^\times n \to L$ such that:

(a) $[\ , \ldots , \ ]$ is totally antisymmetric, i.e.

$$[x_1, \ldots , x_n] = \epsilon(\sigma)[x_{\sigma(1)}, \ldots , x_{\sigma(n)}], \quad \text{for all } \sigma \in S_n \text{ and all } x_1 \ldots x_n \in L \quad (A.1)$$

(b) $[\ , \ldots , \ ]$ satisfies the following fundamental identity for all $x_i, y_j \in L, i = 1, \ldots , n - 1, j = 1, \ldots , n$:

$$[x_1, \ldots , x_{n-1}, [y_1, \ldots , y_n]] = \sum_{i=1}^n [y_1, \ldots , y_{i-1}, [x_1, \ldots , x_{n-1}, y_i], y_{i+1}, \ldots , y_n]. \quad (A.2)$$

Notice that 2-Lie algebras are ordinary Lie algebras, for which the fundamental identity $(A.2)$ corresponds to the usual Jacobi identity. A basic example is the $n$-Lie algebra $A_{n+1}$ over $\mathbb{R}$ whose underlying vector space is an oriented Euclidean vector space of dimension $n + 1$ and whose $n$-bracket is given by the cross product of an ordered system of $n$ vectors:

$$[x_1, \ldots , x_n] := x_1 \times \ldots \times x_n := \begin{vmatrix} x_1^1 & \cdots & x_1^n & e_1 \\ \vdots & \vdots & \vdots & \vdots \\ x_{n+1}^1 & \cdots & x_{n+1}^n & e_{n+1} \end{vmatrix}, \quad (A.3)$$

In the last equality, $e_1, \ldots , e_{n+1}$ is any orthonormal basis of $A_{n+1}$ and $x_i = \sum_{j=1}^{n+1} x_i^j e_j$ with $x_i^j \in \mathbb{R}$. The 3-Lie algebra $A_4$ played an important role in the BLG model, see e.g. [2].

Let us fix an $n$-Lie algebra $L$. Any such algebra defines an ordinary Lie bracket $[\ , \ ]$ (called the associated basic bracket) on the $R$-module $\mathfrak{g}_L := \wedge_{R}^{n-1} L$ via:

$$[[x_1 \wedge \ldots \wedge x_{n-1}, y_1 \wedge \ldots \wedge y_{n-1}]] := \sum_{i=1}^{n-1} (y_1 \wedge \ldots \wedge y_{i-1} \wedge [x_1 \ldots x_{n-1}, y_i] \wedge y_{i+1} \wedge \ldots \wedge y_{n-1}). \quad (A.4)$$

The ordinary Lie $R$-algebra $(\mathfrak{g}_L, [\ , \ ])$ is called the basic Lie algebra of $L$. 

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**Definition.** An $R$-linear map $D: L \to L$ is called a *derivation* of $L$ if:

$$D([x_1, \ldots, x_n]) = \sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, D(x_i), x_{i+1}, \ldots, x_n]$$  \hspace{1cm} (A.5)

for any $x_1, \ldots, x_n \in L$.

The space $\text{Der}_R(L)$ of all $R$-linear derivations of $L$ is an ordinary Lie $R$-algebra with Lie bracket given by the commutator of derivations. For any elements $x_1 \ldots x_{n-1}$ of $L$, we define a linear map $D_{x_1 \ldots x_{n-1}} : L \to L$ through:

$$D_{x_1 \ldots x_{n-1}}(x) = [x_1, \ldots, x_{n-1}, x] \ (x \in L) \ .$$  \hspace{1cm} (A.6)

It is easy to check that $D_{x_1 \ldots x_{n-1}}$ is a derivation of $L$, called the *inner derivation defined by the sequence of elements* $x_1 \ldots x_{n-1}$. Indeed, Filippov’s fundamental (A.2) identity is equivalent with the derivation property:

$$D_{x_1 \ldots x_{n-1}}([y_1, \ldots, y_n]) = \sum_{i=1}^{n} [y_1, \ldots, y_{i-1}, D_{x_1, \ldots, x_{n-1}}(y_i), y_{i+1}, \ldots, y_n] \ .$$  \hspace{1cm} (A.7)

Relation (A.7) implies:

$$[D_{x_1 \ldots x_{n-1}}, D_{y_1 \ldots y_{n-1}}] = \sum_{i=1}^{n-1} D_{y_1, \ldots, y_{i-1}, D_{x_1, \ldots, x_{n-1}}(y_i), y_{i+1}, \ldots, y_{n-1}} \ ,$$  \hspace{1cm} (A.8)

which shows that the subspace $\text{Inn}_R(L)$ of inner $R$-linear derivations is a Lie subalgebra of $\text{Der}_R(L)$.

Since the map $(x_1 \ldots x_{n-1}) \in L^{\times(n-1)} \xrightarrow{D} D_{x_1 \ldots x_{n-1}} \in \text{Der}_R(L)$ is $R$-multilinear and alternating, it factors through an $R$-linear map $\mathfrak{g}_L \xrightarrow{\tilde{D}} \text{Der}_R(L)$, whose value on an element $u \in G$ we denote by $\tilde{D}_u := \tilde{D}(u) \in \text{Der}_R(L)$. Consequently, we can write (A.7) as:

$$[\tilde{D}_{x_1 \wedge \ldots \wedge x_{n-1}}, \tilde{D}_{y_1 \wedge \ldots \wedge y_{n-1}}] = \tilde{D}[[y_1 \wedge \ldots \wedge y_{n-1}, x_1 \wedge \ldots \wedge x_{n-1}]] \ ,$$  \hspace{1cm} (A.9)

which shows that $\tilde{D}$ is a linear representation (called the *canonical representation*) of the basic algebra on the underlying vector space of $L$. The Lie algebra of inner derivations is the image of this representation, i.e. we have $\text{Inn}_R(L) = \text{Im} \tilde{D}$. The action of $u \in \mathfrak{g}_L$ on an element $x \in L$ will be denoted by:

$$u \triangleright x := \tilde{D}_u(x) \ .$$  \hspace{1cm} (A.10)

The fact that the representation $\tilde{D}$ of $\mathfrak{g}_L$ on $L$ acts through derivations of $(L, [\ , \ldots , \ ])$ is equivalent with the statement that the $n$-bracket $[\ , \ldots , \ ]$ satisfies Filippov’s fundamental identity (A.2).
The algebra $F_L$. When $L$ is an ordinary Lie algebra with bracket $[ , ]$, then we have $[[ , ]] = [ , ]$ and $\mathfrak{gl}_L$ coincides with $L$ as a Lie algebra. In this case, the representation $\bar{D}$ of $\mathfrak{g}_L$ in $L$ is the adjoint representation $\text{ad}$ of $L$ in itself. The fact that $\text{ad}$ acts in $L$ through derivations of $(L, [ , ])$ is the well-known reformulation of the classical Jacobi identity. Filippov’s notion of $n$-Lie algebra is inspired by generalizing this take on the classical Jacobi identity.

Let $L$ be an $n$-Lie algebra with $n > 2$. Since we have an action of $\mathfrak{g}_L$ on $L$, the $R$-module $F_L := \mathfrak{g}_L \oplus L$ carries an algebraic structure consisting of two $R$-bilinear and alternating maps, namely a 2-bracket $[ , ] : F_L \times F_L \to F_L$ and an $n$-bracket $[ , ..., ] : F_L^{\times n} \to F_L$, defined as follows for all $u, v, u_1 \ldots u_n \in \mathfrak{g}_L$ and all $x, y, x_1 \ldots x_n \in L$:

$$[u + x, v + y] := [u, v] + u \triangleright y - v \triangleright x,$$

$$[u_1 + x_1 \ldots u_n + x_n] := [x_1, \ldots, x_n]$$

$$+ \sum_{i=1}^{n} (-1)^{i-1} [u_i, x_1 \wedge_R \ldots \wedge_R x_{i-1} \wedge_R x_{i+1} \wedge_R \ldots \wedge_R x_n] .$$

We will see later that the triple $(F_L, [ , ], [ , ..., ])$ is an (ungraded) $L_{2,n}$ algebra over $R$.

**Definition.** A map $\eta : L \times L \to R$ on $L$ is called invariant if:

$$\eta(D(x), y) + \eta(x, D(y)) = 0 \quad \text{for all} \quad x, y \in L \quad \text{and all} \quad D \in \text{Inn}_R(L) . \quad (A.11)$$

Notice that condition (A.11) amounts to:

$$\eta([z_1, \ldots, z_{n-1}, x], y) + \eta(x, [z_1, \ldots, z_{n-1}, y]) = 0 \quad \forall x, y, z_1 \ldots z_{n-1} \in L . \quad (A.12)$$

**Definition.** When $L$ is an $n$-Lie algebra over $\mathbb{C}$, then a Hermitian metric on $L$ is an invariant Hermitian form $( , ) : L \times L \to \mathbb{C}$.

**Lie $n$-algebras**

**Definition.** A Lie $n$-algebra over $R$ is an $R$-module $L$ endowed with an $R$-multilinear map $\mu_n := [ , ..., ] : \mathcal{L}^{\times n} \to \mathcal{L}$ which is totally antisymmetric:

$$[x_1, \ldots, x_n] = \epsilon(\sigma)[x_{\sigma(1)}, \ldots, x_{\sigma(n)}] , \quad (x_i \in L) \quad (A.13)$$

and satisfies the homotopy Jacobi identity:

$$\sum_{\sigma \in \text{Sh}(n,n-1)} \epsilon(\sigma)[x_{\sigma(1)}, \ldots, x_{\sigma(n-1)}, x_{\sigma(n)}], x_{\sigma(n+1)} \ldots, x_{\sigma(2n-1)} = 0 \quad (x_i, y_j \in L) . \quad (A.14)$$

The definition given above coincides with that used by Hanlon and Wachs [30] but differs from that used e.g. in [39]. The following result was proved by Dzhumadil’diev.

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7In fact, Dzhumadil’diev also shows that $n$-Lie algebras as so-called “symmetric algebras”, which in turn are Lie $n$-algebras.
Proposition. Filippov’s fundamental identity (A.2) implies the homotopy Jacobi identity, i.e. any $n$-Lie algebra $(L, [\ldots,])$ is also a Lie $n$-algebra. Dzhumadil’daev also shows that the converse statement is untrue, a counterexample being provided by Jacobian algebras, which are Lie $n$-algebras but not $n$-Lie.

As we shall see below, Lie $n$-algebras are the same as ungraded $L_n$ algebras, a particular case of ungraded $L_\infty$ algebras. Hence Dzhumadil’daev’s result above shows that the theory of $n$-Lie algebras is a special case of the theory of $L_\infty$ algebras.

$L_\infty$ algebras

Definition. An (ungraded) $L_\infty$ (or strong homotopy Lie) algebra over $R$ is an $R$-module $L$ endowed with a family of $R$-multilinear maps $\mu_n : L^\otimes n \to L$ ($n \geq 1$) such that:

1. Each $\mu_n$ is alternating, i.e.:
   
   $$\mu_n(x_{\sigma(1)} \ldots x_{\sigma(n)}) = \epsilon(\sigma)\mu_n(x_1 \ldots x_n) \text{ for all } x_1 \ldots x_n \in L$$  \hspace{1cm} (A.15)

2. Each of the following countable tower of ungraded $L_\infty$ identities is satisfied:

   $$\sum^n_{i=1} \sum_{\sigma \in \text{Sh}(i, n-i)} (-1)^{i(n+1)}\epsilon(\sigma)\mu_{n-i+1}(\mu_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0$$  \hspace{1cm} (A.16)

for all $n \geq 1$ and all $x_1, \ldots, x_n \in L$.

Definition. Let $S \subset \mathbb{N}^*$ be a proper subset of the set $\mathbb{N}^*$ of all non-vanishing natural numbers. An (ungraded) $L_S$ algebra over $R$ is an $L_\infty$ algebra over $R$ whose products $\mu_i$ vanish for $i \notin S$.

Thus an $L_S$ algebra has only the products $\mu_n$ with $n \in S$, and some of the identities (A.16) might be trivially satisfied. Some particular cases which are common in practice are the following:

1. $S = \{1 \ldots n\}$ for some $n \geq 1$. In this case, and $L_S$ algebra is also called an $L_n$ algebra. Such an $L_\infty$ algebra has only the products $\mu_1 \ldots \mu_n$, and one need only consider the first $2n - 1$ identities in (A.16) (those with $k = 1 \ldots 2n - 1$) since all identities with $k \geq 2n$ are satisfied trivially.

2. $S = \{n\}$ for some $n \in \mathbb{N}^*$. Such an $L_S$ algebra is also called an $L_n$ algebra. An $L_n$ algebra has only the product $\mu_n$ and the only nontrivial identity in (A.16) is the one with $k = 2n - 1$:

   $$\sum_{\sigma \in \text{Sh}(n, n-1)} \epsilon(\sigma)\mu_n(\mu_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), x_{\sigma(n+1)}, \ldots, x_{\sigma(2n-1)}) = 0 \ \forall x_1, \ldots, x_{2n-1} \in L$$  \hspace{1cm} (A.17)

This is called the homotopy Jacobi identity (it is also the last nontrivial identity of an $L_n$ algebra). Clearly any $L_n$ algebra is also an $L_n$ algebra.

Remark. It is obvious from the definitions above that a Lie $n$-algebra is the same as an (ungraded) $L_n$ algebra.
Examples. (1) An $L_{(1)}$ algebra $L$ has just one linear map $\mu_1$ and \[\text{(A.16)}\] reduce to the single condition $\mu_1 \circ \mu_1 = 0$, i.e. $(L, \mu_1)$ is a differential space. Clearly an $L_1$ algebra is the same thing as an $L_{(1)}$ algebra.

(2) An $L_{(2)}$ algebra has a linear unary product $\mu_1$ of weak degree and bilinear binary product $\mu_2$, the second being alternating. Conditions \[\text{(A.16)}\] say that $\mu_1$ squares to zero, that is a derivation of $\mu_2$:

$$
\mu_1(\mu_2(x_1, x_2)) = \mu_2(\mu_1(x_1), x_2) + \mu_2(x_1, \mu_1(x_2)) \quad \text{(A.18)}
$$

and that $\mu_2$ satisfies the Jacobi identity:

$$
\mu_2(\mu_2(x_1, x_2), x_3) + \mu_2(\mu_2(x_2, x_3), x_1) + \mu_2(\mu_2(x_3, x_1), x_2) = 0 \quad \text{(A.19)}
$$

These constraints state that $(L, \mu_1, \mu_2)$ is a differential Lie algebra. An $L_2$ algebra is an $L_{(2)}$ algebra with trivial differential, i.e. an ordinary Lie algebra.

(3) An $L_{2,n}$ algebra has an alternating 2-bracket $\mu_2 = [\ , \ ]$ and an alternating $n$-bracket $\mu_n = [\ , \ , \ , \ ]$. Conditions \[\text{(A.16)}\] amount to the following constraints:

(a) $[\ , \ ]$ is a Lie bracket

(b) $[\ , \ , \ , \ ]$ satisfies homotopy Jacobi

(c) The following identity is satisfied for all elements of the $L_{2,n}$ algebra

$$
\sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1}[[x_i, x_j], x_1, \ldots, \hat{x}_j, \ldots, \hat{x}_i, \ldots, x_{n+1}] + 
\sum_{i=1}^{n+1} (-1)^{n+i-1}[x_1, \ldots, \hat{x}_i, \ldots, x_n, x_i] = 0 \quad \text{(A.20)}
$$

B. The graded versions

Preparations. Let $G$ be an Abelian group and fix a group morphism $\phi : G \to \mathbb{Z}_2$. Recall that a $G$-graded $R$-modules is an $R$-module $V$ endowed with a submodule decomposition:

$$
V = \bigoplus_{g \in G} V_g .
$$

Given such a module, its pure (or homogeneous) elements are those elements $x \in V$ for which there exists some $g = g_x$ such that $x \in V_g$. In this case, we let $\deg x := g$. We also let $\hat{x} := \phi(\deg x) = \phi(g) \in \mathbb{Z}_2$, which we will call the parity of $x$. Thus $x$ is even if its parity equals $\hat{0} \in \mathbb{Z}_2$ and odd if its parity equals $\hat{1} \in \mathbb{Z}_2$. Notice that the sign factor $(-1)^{\hat{g}}$ is unambiguously defined. The reduced grading of $V$ is the $\mathbb{Z}_2$-grading $V = V^\text{red}_0 \oplus V^\text{red}_1$ defined through:

$$
V^\text{red}_0 := \bigoplus_{g \in G | \phi(g) = \hat{0}} V_g \ , \ V^\text{red}_1 := \bigoplus_{g \in G | \phi(g) = \hat{1}} V_g .
$$

The tensor powers $\otimes^n_R V$ of $V$ are defined as usual, while the $(G, \phi)$-graded symmetric and graded exterior powers $\circ_R^n V$ and $\wedge^n_R V$ are defined with sign factors induced by the parity of homogeneous elements. All these $R$-modules are again $G$-graded.

Given a permutation $\sigma \in S_n$, we define the Koszul sign $\epsilon(\sigma, x_1 \ldots x_n)$ of $n$ pure elements $x_1 \ldots x_n \in V$ via the identity:

$$
\sigma(1) \circ_R \ldots \circ_R x_{\sigma(n)} = \epsilon(\sigma, x_1 \ldots x_n) x_1 \circ_R \ldots \circ_R x_n ,
$$
where \( x_1 \odot \ldots \odot x_n \) is the pure element of \( \otimes^n_R V \) defined by \( x_1 \ldots x_n \). The modified Koszul sign \( \chi(\sigma, x_1 \ldots x_n) \) is defined through:

\[
\chi(\sigma, x_1 \ldots x_n) := \epsilon(\sigma) \epsilon(\sigma, x_1 \ldots x_n)
\]

and we have:

\[
x_{\sigma(1)} \wedge_R \ldots \wedge_R x_{\sigma(n)} = \chi(\sigma, x_1 \ldots x_n) x_1 \wedge_R \ldots \wedge_R x_n
\]

Given another group morphism \( \psi : G \to \mathbb{Z}_2 \) and a \((G, \psi)\)-graded \( R \)-module \( U \), an \( R \)-linear map \( f : V \to U \) is called homogeneous of degree \( h \in G \) if \( f(V_g) \subset U_{g+h} \) for all \( g \in G \). It is called weakly homogeneous of weak homogeneity degree \( \alpha \in \mathbb{Z}_2 \) if \( f(V_{\beta}^{\text{red}}) \subset U_{\beta + \alpha}^{\text{red}} \) for all \( \beta \in \mathbb{Z}_2 \).

As in the \( \mathbb{Z}_2 \)-graded case, we can define the notions of \((G, \phi)\)-graded symmetric and graded antisymmetric \( R \)-multilinear maps \( \eta : V^{\times n} \to V \), and find that these factor through \( R \)-linear maps \( \check{\eta} \) from \( \otimes^n_R V \) and \( \wedge^n_R V \) to \( V \), respectively.

**Graded Lie algebras**

**Definition.** A \((G, \phi)\)-graded Lie algebra over \( R \) is a (unital) \( G \)-graded \( R \)-module \( L \) endowed with an \( R \)-bilinear map \( [\ , \ ] : L^{\times 2} \to L \) such that:

(a) \([\ , \ ]\) is graded antisymmetric, i.e.

\[
[x, y] = (-1)^{\bar{x}\bar{y}}[y, x] \quad \text{for all homogeneous elements } x, y \in L , \quad (B.21)
\]

(b) \([\ , \ ]\) satisfies the \((G, \phi)\)-graded Jacobi identity for all homogeneous elements \( x, y, z \in L \):

\[
[x, [y, z]] + (-1)^{\bar{x}(\bar{y} + \bar{z})} [y, [z, x]] + (-1)^{\bar{z}(\bar{x} + \bar{y})} [z, [x, y]] = 0 . \quad (B.22)
\]

As for usual Lie algebras, \((L, [\ , \ ])\) has an adjoint representation \( \text{ad} : L \to \text{End}_R(L) \) given by (here \( \text{ad}_x := \text{ad}(x) \)):

\[
\text{ad}_x(y) := [x, y] . \quad (B.23)
\]

A weakly homogeneous graded derivation of parity \( \alpha \in \mathbb{Z}_2 \) is a homogeneous \( R \)-linear map \( D : L \to L \) such that:

\[
D([x, y]) = [D(x), y] + (-1)^{\alpha \bar{x}} [x, D(y)] \quad (B.24)
\]

for all homogeneous \( x, y \in L \). We let \( \check{D} := \alpha \) and say that \( D \) is even if \( \alpha = 1 \) and that \( D \) is odd if \( \alpha = 0 \). The set of weakly homogeneous derivations of \( L \) is denoted by \( \text{Der}_t(L) \). It is a graded Lie algebra when endowed with the usual graded commutator of homogeneous linear operators in \( L \), which makes it into a graded Lie subalgebra of \( \text{End}_R(L) \). For any homogeneous element \( x \in L \), the associated adjoint action \( \text{ad}_x \) is weakly homogeneous of parity \( \bar{x} \), and \( \text{ad}_x \) gives an \( R \)-linear representation of the graded Lie algebra \((L, [\ , \ ])\) on \( L \), i.e. \( \text{ad} : L \to \text{End}_R(L) \) is a morphism of graded Lie algebras. Given this property, the graded Jacobi identity \((B.22)\) is equivalent to the statement that \( \text{ad}_x \) is a weakly homogeneous derivation for every homogeneous element \( x \in L \).
Graded $n$-Lie algebras

**Definition.** A $(G, \phi)$-graded $n$-Lie algebra over $R$ is a (unital) $G$-graded $R$-module $L$ endowed with an $R$-multilinear map $[\ , \ , \ldots , \ ] : L^\times n \to L$ such that

(a) $[\ , \ , \ldots , \ ]$ is totally graded antisymmetric, i.e.

$$[x_1, \ldots , x_n] = \chi(\sigma, x_1 \ldots x_n)[x_{\sigma(1)}, \ldots , x_{\sigma(n)}], \text{ for all } \sigma \in S_n \text{ and all } x_1 \ldots x_n \in L \ (B.25)$$

(b) $[\ , \ , \ldots , \ ]$ satisfies the following graded fundamental identity for all $x_i, y_j \in L$, $i = 1, \ldots , n - 1$, $j = 1, \ldots , n$:

$$[x_1, \ldots , x_{n-1}, [y_1, \ldots , y_n]] = \sum_{i=1}^{n} (-1)^{i} (x_1 + \ldots + x_{n-1})(y_1 + \ldots + y_i-1) [y_1, \ldots , y_i, [x_1, \ldots , x_{n-1}, y_i], y_{i+1}, \ldots , y_n] . \quad (B.26)$$

Notice that 2-Lie algebras are ordinary $(G, \phi)$-graded Lie algebras, for which the graded fundamental identity $\ (B.26)$ becomes the usual $(G, \phi)$-graded Jacobi identity.

Let us fix a $(G, \phi)$-graded $n$-Lie algebra $L$. Such an algebra defines an ordinary $(G, \phi)$-graded Lie bracket $[\ , \ ]$ (called the associated basic bracket) on the graded $R$-module $\mathfrak{g}_L : = \wedge_{R}^{n-1} L$ via:

$$[[x_1 \wedge_R \ldots \wedge_R x_{n-1}, y_1 \wedge_R \ldots \wedge_R y_n]] := \sum_{i=1}^{n-1} (-1)^{(x_1 + \ldots + x_{n-1})(y_1 + \ldots + y_{i-1})} [x_1 \ldots x_{n-1} \wedge y_i \wedge y_{i+1} \wedge \ldots \wedge y_{n-1}] . \quad (B.27)$$

The $(G, \phi)$-graded Lie $R$-algebra $(\mathfrak{g}_L, [\ , \ ] )$ is called the basic graded Lie algebra of $L$.

**Definition.** An $R$-linear map $D : L \to L$ is called a weakly homogeneous derivation of $L$ of parity $\alpha \in \mathbb{Z}_2$ if:

$$D([x_1, \ldots , x_n]) = \sum_{i=1}^{n} (-1)^{\alpha(x_1 + \ldots + x_{i-1})} [x_1, \ldots , x_{i-1}, D(x_i), x_{i+1}, \ldots , x_n] \quad (B.28)$$

for any $x_1, \ldots , x_n \in L$. We set $\tilde{D} := \alpha$.

The space $\text{Der}_R(L)$ of all $R$-linear and weakly homogeneous derivations of $L$ is a $(G, \phi)$-graded Lie $R$-algebra with graded Lie bracket inherited from $\text{End}_R(L)$. For any elements $x_1 \ldots x_{n-1}$ of $L$, we define a linear map $D_{x_1 \ldots x_{n-1}} : L \to L$ through:

$$D_{x_1 \ldots x_{n-1}}(x) = [x_1, \ldots , x_{n-1}, x] \quad (x \in L) . \quad (B.29)$$

When $x_1 \ldots x_{n-1}$ are homogeneous, then it is easy to check that $D_{x_1 \ldots x_{n-1}}$ is a weakly homogeneous derivation of $L$ of parity $\tilde{x}_1 + \ldots + \tilde{x}_{n-1}$, called the inner derivation defined by

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*Here (and in the following), the given conditions hold for homogeneous elements; for other elements, they are linearly extended.*
the sequence of elements \( x_1 \ldots x_{n-1} \). Indeed, the graded version of Filippov’s fundamental identity \((A.2)\) is equivalent with the graded derivation property:

\[
D_{x_1 \ldots x_{n-1}}([y_1, \ldots, y_n]) = \\
\sum_{i=1}^{n} (-1)^i (x_1 + \ldots + x_{n-1}) (y_1 + \ldots + y_{i-1}) [y_1, \ldots, y_{i-1}, D_{x_1 \ldots x_{n-1}}(y_i), y_{i+1}, \ldots, y_n].
\]

(B.30)

Relation \((B.30)\) implies:

\[
[D_{x_1 \ldots x_{n-1}}, D_{y_1 \ldots y_{n-1}}] = \\
\sum_{i=1}^{n-1} (-1)^{(x_1 + \ldots + x_{n-1}) (y_1 + \ldots + y_{i-1})} D_{y_1 \ldots y_{i-1}, D_{x_1 \ldots x_{n-1}}(y_i), y_{i+1} \ldots y_{n-1}}.
\]

(B.31)

which shows that the subspace \( \text{Im}_R(L) \) of inner \( R \)-linear weakly-homogeneous derivations is a Lie subalgebra of \( \text{Der}_R(L) \).

Since the map \((x_1 \ldots x_{n-1}) \in L^{(n-1)} \overset{D}{\rightarrow} D_{x_1 \ldots x_{n-1}} \in \text{Der}_R(L)\) is \( R \)-multilinear and graded antisymmetric, it factors through an \( R \)-linear map \( g_L \overset{\tilde{D}}{\rightarrow} \text{Der}_R(L)\), whose value on an element \( u \in G \) we denote by \( \tilde{D}(u) \in \text{Der}_R(L)\). Consequently, we can write \([B.29]\) as:

\[
[\tilde{D}_{x_1 \ldots x_{n-1}}, \tilde{D}_{y_1 \ldots y_{n-1}, D_{x_1 \ldots x_{n-1}}(y_i), y_{i+1} \ldots y_{n-1}}] = \tilde{D}_{[x_1 \ldots x_{n-1}, y_1 \ldots y_{n-1}, x_1 \ldots x_{n-1}]}.
\]

(B.32)

which shows that \( \tilde{D} \) is a linear representation (called the \emph{canonical representation}) of the basic algebra on the underlying graded vector space of \( L \); this means that \( \tilde{D} \) is a morphism of graded Lie algebras from \( g_L \) to \( \text{End}_R(L) \). The graded Lie algebra of inner derivations is the image of this representation, i.e. we have \( \text{Im}_R(L) = \text{Im} \tilde{D} \). The action of \( u \in g_L \) on an element \( x \in L \) will be denoted by:

\[
u \triangleright x := \tilde{D}(u)(x).
\]

(B.33)

The fact that the representation \( \tilde{D} \) of \( g_L \) on \( L \) acts through graded derivations of \((L, [\ldots, \ldots])\) is equivalent with the statement that the \( n \)-bracket \([\ldots, \ldots]\) satisfies the graded version of Filippov’s fundamental identity \((A.2)\).

#### Graded Lie \( n \)-algebras

**Definition.** A \((G, \phi)\)-graded Lie \( n \)-\emph{algebra} over \( R \) is a \( G \)-graded \( R \)-module \( L \) endowed with an \( R \)-multilinear map \( \mu_n := [\ldots, \ldots] : L^n \rightarrow L \) which is totally graded antisymmetric:

\[
[x_1, \ldots, x_n] = \chi(\sigma, x_1 \ldots x_n)[x_{\sigma(1)}, \ldots, x_{\sigma(n)}], \quad (x_i \in L)
\]

(B.34)

and satisfies the \emph{graded homotopy Jacobi identity}:

\[
\sum_{\sigma \in \text{Sh}(n,n-1)} \chi(\sigma, x_1 \ldots x_n)[[x_{\sigma(1)}, \ldots, x_{\sigma(n-1)}, x_{\sigma(n)}], x_{\sigma(n+1)} \ldots, x_{\sigma(2n-1)}] = 0 \quad (x_i, y_j \in L).
\]

(B.35)

The definition given above coincides with that used by Hanlon and Wachs \([\ldots]\) but differs from that used by Baez and Schreiber \([\ldots]\). The following result follows by a trivial extension of the proof given by Dzhumadil’daev in the ungraded case:
**Proposition.** The graded version of Filippov’s fundamental identity implies the homotopy Jacobi identity, i.e. any \((G, \phi)\)-graded \(n\)-Lie algebra \((L, [\cdot, \ldots, \cdot])\) is also a \((G, \phi)\)-graded Lie \(n\)-algebra.

As we shall see below, graded Lie \(n\)-algebras are the same as graded \(L\)\(_n\) algebras, a particular case of \(L\)\(_\infty\) algebras. Hence the result above shows that the theory of graded \(n\)-Lie algebras is a special case of the theory of graded \(L\)\(_\infty\) algebras.

**Graded \(L\)\(_\infty\) algebras**

**Definition.** A \((G, \phi)\)-graded \(L\)\(_\infty\) (or strong homotopy Lie) algebra over \(R\) is a \(G\)-graded \(R\)-module \(L\) endowed with a family of \(R\)-multilinear maps \(\mu_n : L^{\times n} \rightarrow L\) \((n \geq 1)\) such that:

- (0) Each \(\mu_n\) is weakly homogeneous of degree \(2 - n \pmod{2}\),
- (1) Each \(\mu_n\) is graded antisymmetric, i.e.:
  \[\mu_n(x_{\sigma(1)} \cdots x_{\sigma(n)}) = \chi(\sigma, x_1 \cdots x_n) \mu_n(x_1 \cdots x_n)\]  
  for all homogeneous elements \(x_1 \cdots x_n \in L\).
- (2) Each of the following countable tower of graded \(L\)\(_\infty\) identities is satisfied:
  \[\sum_{i=1}^{n} \sum_{\sigma \in \text{Sh}(i, n-i)} (-1)^{\binom{n+1}{i}} \chi(\sigma, x_1 \cdots x_n) \mu_{n-i+1}(\mu_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0\]  
  for all \(n \geq 1\) and all homogeneous elements \(x_1, \ldots, x_n \in L\).

We say that \(L\) is strictly homogeneous if each \(\mu_n\) is a \(G\)-homogeneous map.

**Observation.** The classical choices for \((G, \phi)\) when studying \(L\)\(_\infty\) algebras are as follows:

- (A) \(G = \mathbb{Z}\), with \(\phi : G \rightarrow \mathbb{Z}_2\) being the mod 2 reduction morphism, i.e. \(\phi(m) = m(\mod 2)\) for all \(m \in \mathbb{Z}\). The strictly homogeneous case with the supplementary condition \(\deg \mu_n = 2 - n\) leads to the usual theory of \(\mathbb{Z}\)-graded \(L\)\(_\infty\) algebras.

- (B) \(G = \mathbb{Z}_2\) with \(\phi = \text{id}_{\mathbb{Z}_2} : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2\) being the identity morphism. This leads to the usual theory of \(\mathbb{Z}_2\)-graded \(L\)\(_\infty\) algebras.

In the present paper, we have also considered the choice:

- (C) \(G = \{0\}\) (the trivial group), in which case the only possibility is to take \(\phi : G \rightarrow \mathbb{Z}_2\) to be the trivial (zero) morphism \((\phi(0) = 0)\), which we denote by \(\phi = 0\). This corresponds to taking everything to be concentrated in degree zero while forgetting any homogeneity constraints on the products \(\mu_n\).

Our approach naturally unifies the various cases listed above which differ only at the level of grading, while being very similar otherwise.

As in the ungraded case, we can define graded \(L_S\) algebras for any non-empty finite subset \(S \subset \mathbb{N}^*\).

**Definition.** A \((G, \phi)\)-graded \(L_S\) algebra is a \((G, \phi)\)-graded \(L\)\(_\infty\) algebra such that \(\mu_n = 0\) for \(n \in \mathbb{N}^* \setminus S\).
When \( S = \{n\} \), the corresponding \( L_S \) algebras are also called graded \( L_n \) algebras. When \( S = \{1, \ldots, n\} \), they are called graded \( L_{(n)} \) algebras.

Thus a \((G, \phi)\)-graded \( L_{(n)} \) algebra has only the products \( \mu_1 \ldots \mu_n \), and one needs only consider the first \( 2n - 1 \) identities in (B.37) (those with \( k = 1 \ldots 2n - 1 \)) since all identities with \( k \geq 2n \) are satisfied trivially. An \( L_n \) algebra has only the product \( \mu_n \) and the only nontrivial identity in (B.37) is the one with \( k = 2n - 1 \):

\[
\sum_{\sigma \in Sh(n,n-1)} \chi(\sigma, x_1 \ldots x_{2n-1}) \mu_n(\mu_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), x_{\sigma(n+1)}, \ldots, x_{\sigma(2n-1)}) = 0 \tag{B.38}
\]

for all \( x_1, \ldots, x_{2n-1} \in L \), which is called the graded homotopy Jacobi identity (this is also the last nontrivial identity of an \( L_{(n)} \) algebra). Clearly any \( L_n \) algebra is also an \( L_{(n)} \) algebra.

**Examples.** (1) Consider a \((G, \phi)\)-graded \( L_{(1)} \) algebra structure on a \( G \)-graded vector space \( L \). There is just one weakly homogeneous linear map \( \mu_1 \) of weak degree \( \hat{1} \), and the conditions (B.37) reduce to the single condition \( \mu_1 \circ \mu_1 = 0 \), which says that \((L, \mu_1)\) is a \((G, \phi)\)-graded complex. Clearly a \((G, \phi)\)-graded \( L_1 \) algebra is the same thing as an \( L_{(1)} \) algebra. The choice (A), (B) above for \((G, \phi)\) lead to the usual notions of \( \mathbb{Z} \)-graded and \( \mathbb{Z}_2 \)-graded complexes respectively.

(2) A graded \( L_{(2)} \) algebra has a linear unary product \( \mu_1 \) of weak degree \( \hat{1} \) and a bilinear binary product \( \mu_2 \), the second being graded antisymmetric and of weak degree \( \hat{0} \). Conditions (B.37) state that \( \mu_1 \) is a differential, that it is a graded derivation of \( \mu_2 \):

\[
\mu_1(\mu_2(x_1, x_2)) = \mu_2(\mu_1(x_1), x_2) + (-1)^{\hat{x}_1 \hat{x}_2} \mu_2(x_1, \mu_1(x_2)) \tag{B.39}
\]

and that \( \mu_2 \) satisfies the graded Jacobi identity:

\[
(-1)^{\hat{x}_1 \hat{x}_3} \mu_2(\mu_2(x_1, x_2), x_3) + (-1)^{\hat{x}_1 \hat{x}_2} \mu_2(\mu_2(x_2, x_3), x_1) + (-1)^{\hat{x}_2 \hat{x}_3} \mu_2(\mu_2(x_3, x_1), x_2) = 0 \tag{B.40}
\]

These three constraints mean that \( L \) is a differential \((G, \phi)\)-graded Lie algebra with differential \( d = \mu_1 \) and graded Lie bracket \([ \cdot, \cdot ] = \mu_2(\cdot, \cdot)\). An \( L_2 \) algebra is an \( L_{(2)} \) algebra with trivial differential, i.e. a \((G, \phi)\)-graded Lie algebra. The choices (A),(B) above for \((G, \phi)\) lead to the usual notions of \( \mathbb{Z} \)-graded respectively \( \mathbb{Z}_2 \)-graded (differential) Lie algebras.

**C. Lie \( n \)-algebras are \( L_\infty \) algebras**

**Proposition.** A Lie \( n \)-algebra over \( k \) is the same as a \((G, \phi)\)-graded \( L_n \) algebra \( L \) over \( k \) based on choice (C) for \((G, \phi)\), i.e. \( G = \{0\} \) and \( \phi(0) = 0 \).

**Proof.** Indeed, substituting this choice of \((G, \phi)\) into the definition of an \( L_n \) algebra obviously reproduces the definition of a Lie \( n \)-algebra. It follows that Lie \( n \)-algebras are simply the ‘ungraded version’ of \( L_n \) algebras. Recall that any \( n \)-Lie algebra is a symmetric and therefore an Lie \( n \)-algebra [31]. It follows that any \( n \)-Lie algebra is an \( L_n \) algebra. Therefore the theory of \( n \)-Lie algebras is a special case of the theory of \( L_\infty \) algebras. Thus, quite a few extensions of the notion of a Lie algebra which have been considered in the literature are particular cases of \( L_\infty \) algebras, which, as expected from the work of Stasheff, play a unifying role.
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