THE IDEAL COUNTING FUNCTION IN CUBIC FIELDS

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Abstract

For a cubic algebraic extension $K$ of $\mathbb{Q}$, the behavior of the ideal counting function is considered in this paper. Let $a_K(n)$ be the number of integral ideals of the field $K$ with norm $n$. An asymptotic formula is given for the sum

$$\sum_{n_1^2 + n_2^2 \leq x} a_K(n_1^2 + n_2^2).$$

Keywords: Non-normal extension; Ideal counting function; Rankin-Selberg convolution.

1 Introduction

Let $K$ be an algebraic extension of $\mathbb{Q}$ with degree $d$. The associated Dedekind zeta function is defined by

$$\zeta_K(s) = \sum_a \mathcal{N}(a)^{-s}, \quad \Re s > 1,$$

where the sum runs over all integral ideals in $\mathcal{O}_K$, and $\mathcal{N}(a)$ is the norm of the integral ideal $a$. Since the norm of an integral ideal is a positive rational integer, the Dedekind zeta function can be rewritten as an ordinary Dirichlet series

$$\zeta_K(s) = \sum_{n=1}^{\infty} a_K(n)n^{-s}, \quad \Re s > 1,$$

where $a_K(n)$ counts the number of integral ideals $a$ with norm $n$ in $K$, we call it the ideal counting function. A great number of deep arithmetic properties of a number field are hidden within its Dedekind zeta function.

It is known that the ideal counting function $a_K(n)$ is a multiplicative function, and it has the upper bound

$$a_K(n) \ll \tau^d(n),$$

where $\tau(n)$ is the divisor function, see [1].

Landau [2] in 1927 gave the average behavior of the ideal counting function

$$\sum_{n \leq x} a_K(n) = cx + O(x^{\frac{2}{n+1} + \varepsilon})$$

for arbitrary algebraic number field of degree $d \geq 2$. Nowak [3] then established the best estimation hitherto in any algebraic number field of degree $d \geq 3$. By using the decomposition of

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prime number \( p \) in \( \mathcal{O}_K \) and the analytic properties of \( L \)-functions, in paper [4], Lü considered the average behavior of moments of the ideal function

\[
\sum_{n \leq x} a^l_K(n), \quad l = 1, 2, \ldots
\]

and gave a sharper estimates for \( l = 1 \) in the Galois extension over \( \mathbb{Q} \), while later Lü and the author [5] gave a bound for the sum

\[
\sum_{n \leq x} a^l_K(n^2), \quad l = 1, 2, \ldots
\]

in the Galois extension over \( \mathbb{Q} \).

For a non-normal extension \( K \) of \( \mathbb{Q} \), the decomposition of \( p \) in \( \mathcal{O}_K \) is complex, so we can not use the same method as the normal extension. In 2008, by applying the so-called strong Artin conjecture, Fomenko [6] get the results

\[
\sum_{n \leq x} a^l_K(n), \quad l = 2, 3,
\]

when \( K \) is a non-normal cubic field extension. Later, Lü [8] revised the error term.

In this paper, the author will be interested in the estimation of the following sum

\[
\sum_{n^2 + n_2^2 \leq x} a_K(n_1^2 + n_2^2). \tag{1.1}
\]

where \( K \) is the cubic algebraic extension of \( \mathbb{Q} \).

For the purpose, we first consider the arithmetic function \( r(n) \) which is the number of representation of an integer \( n \) as sums of two square integers. i.e.

\[
r(n) = \# \{ n \in \mathbb{Z} | n = n_1^2 + n_2^2 \}.
\]

Then, we can rewrite the formula (1.1) as

\[
\sum_{n^2 + n_2^2 \leq x} a_K(n_1^2 + n_2^2) = \sum_{n \leq x} a_K(n) \sum_{n=n_1^2+n_2^2} 1 = \sum_{n \leq x} a_K(n)r(n). \tag{1.2}
\]

It is known that \( r(n) \) is the ideal counting function of the Gaussian number field \( \mathbb{Q}(\sqrt{-1}) \) and we have

\[
r(n) = \sum_{d|n} \chi'(d),
\]

where \( \chi' \) is a real primitive Dirichlet character modulo 4.

For general quadratic number field \( L \) with discriminant \( D' \), the ideal counting function of the field \( L \) is

\[
a_L(n) = \sum_{d|n} \chi'(d),
\]

where \( \chi' \) is a real primitive Dirichlet character modulo \( |D'| \). It is an interesting question to consider the sum

\[
\sum_{n \leq x} a_K(n)a_L(n).
\]
Fomenko [7] consider this convolution sum when both $K$ and $L$ are quadratic field. However, we shall discuss a more general case. Assume that $q \geq 1$ is an integer, $\chi$ is a primitive character modulo $q$, define the function

$$f_\chi(n) = \sum_{k|n} \chi(k),$$

then we have the following results

**Theorem 1.1.** Let $K$ be a cubic normal extension of $\mathbb{Q}$ and $q \geq 1$ is an integer, $\chi$ a primitive Dirichlet character modulo $q$, then we have

$$\sum_{n \leq x} a_K(n)f_\chi(n) = xP_5(\log x) + O(x^{5/8+\varepsilon}),$$

(1.3)

where $P_5(t)$ is a polynomial in $t$ with degree 5, and $\varepsilon > 0$ is an arbitrarily small constant.

**Theorem 1.2.** Let $K$ be a cubic non normal extension of $\mathbb{Q}$ and $q \geq 1$ is an integer, $\chi$ a primitive Dirichlet character modulo $q$, then we have

$$\sum_{n \leq x} a_K(n)f_\chi(n) = xP_3(\log x) + O(x^{5/8+\varepsilon}),$$

(1.4)

where $P_3(t)$ is a polynomial in $t$ with degree 3, and $\varepsilon > 0$ is an arbitrarily small constant.

According to the theorems above, we obtain

**Corollary 1.1.** Let $K$ be a cubic normal extension of $\mathbb{Q}$, and $r(n)$ the number of representation of an integer $n$ as sums of two square integers. Then we have

$$\sum_{n \leq x} a_K(n)r(n) = xP_5(\log x) + O(x^{5/8+\varepsilon}),$$

where $P_5(t)$ is a polynomial in $t$ with degree 5.

**Corollary 1.2.** Let $K$ be a cubic non normal extension of $\mathbb{Q}$, and $r(n)$ the number of representation of an integer $n$ as sums of two square integers. Then we have

$$\sum_{n \leq x} a_K(n)r(n) = xP_3(\log x) + O(x^{5/8+\varepsilon}),$$

where $P_3(t)$ is a polynomial in $t$ with degree 3.

Assume that $K$ and $L$ are Galois extensions of $\mathbb{Q}$ with degree $d_1$, $d_2$, respectively. According to the theory of Artin $L$-functions, the ideal counting functions $a_K(n)$ and $a_L(n)$ can be represented by the sum of characters of the representations of $Gal(K/\mathbb{Q})$ and $Gal(L/\mathbb{Q})$, respectively.

## 2 Preliminaries

Let $K$ be a cubic algebraic extension of $\mathbb{Q}$, and $D = df^2$ (d squarefree) its discriminant; the Dedekind zeta function of $K$ is

$$\zeta_K(s) = \sum_{n=1}^{\infty} a_K(n)n^{-s}, \quad \text{for } \Re s > 1.$$
It has the Euler product

\[ \zeta_K(s) = \prod_p \left( 1 + \frac{a_K(p)}{p^s} + \frac{a_K(p^2)}{p^{2s}} + \cdots \right). \]

We will give some results about Dedekind zeta function of cubic field $K$ in the following.

**Lemma 2.1.** $K$ is a normal extension if and only if $D = f^2$. In this case

\[ \zeta_K(s) = \zeta(s)L(s, \varphi)L(s, \overline{\varphi}), \]

where $\zeta(s)$ is the Riemann zeta function and $L(s, \varphi)$ is an ordinary Dirichlet series (over $\mathbb{Q}$) corresponding to a primitive character $\varphi$ modulo $f$.

*Proof.* See the lemma in [9].

By using lemma 2.1, the Euler product of Riemann zeta function $\zeta(s)$ and the Dirichlet $L$-functions, we have

**Lemma 2.2.** Assume that $a_K(n)$ is the ideal counting function of the cubic normal extension $K$ over $\mathbb{Q}$, we get

\[ a_K(n) = \sum_{x,y|n} \varphi(x)\overline{\varphi}(y), \]

Here $x$ and $y$ are integers. In particular, when $n = p$ is a prime, we get

\[ a_K(p) = 1 + \varphi(p) + \overline{\varphi}(p), \quad (2.5) \]

where $\varphi$ is a primitive character modulo $f$.

Assume that $K$ is a non-normal cubic extension over $\mathbb{Q}$, which is given by an irreducible polynomial $f(x) = x^3 + ax^2 + bx + c$. Let $E$ denote the normal closure of $K$ that is normal over $\mathbb{Q}$ with degree 6, and denoted the Galois group $\text{Gal}(E/\mathbb{Q}) = S_3$. Firstly, we will introduce some properties about $S_3$ (see [10], pp. 226-227 for detailed arguments).

The elements of $S_3$ fall into three conjugacy classes

\[ C_1 : (1); \quad C_2 : (1, 2, 3), (3, 2, 1); \quad C_3 : (1, 2), (2, 3), (1, 3). \]

with the following three simple characters: the one dimensional characters $\psi_1$ (the principal character) and $\psi_2$ (the other character determined by the subgroup $C_1 \cup C_2$), and the two dimensional character $\psi_3$.

Let $D$ be the discriminant of $f(x) = x^3 + ax^2 + bx + c$ and $K_2 = \mathbb{Q}(\sqrt{D})$. The fields $K_2$ and $K$ are the intermediate extensions fixed under the subgroups $A_3$ and $\{(1), (1,2)\}$, respectively. The extensions $K_2/\mathbb{Q}$, $E/K_2$ and $E/K$ are abelian. The Dedekind zeta function satisfy the relations

\[ \zeta_E(s) = L_{\psi_1}L_{\psi_2}L_{\psi_3}^2, \]
\[ \zeta_{K_2}(s) = L_{\psi_1}L_{\psi_2}, \]
\[ \zeta_K(s) = L_{\psi_1}L_{\psi_3}, \]
\[ \zeta(s) = L_{\psi_1}, \]

where

\[ L_{\psi_2} = L(s, \psi_2; E/\mathbb{Q}) \quad \text{and} \quad L_{\psi_3} = L(s, \psi_3; E/\mathbb{Q}), \]
and \( L_{\psi_2} = L(s, \psi_2, E/\mathbb{Q}) \) and \( L_{\psi_3} = L(s, \psi_3, E/\mathbb{Q}) \) are Artin L-functions.

Kim in [11] proved that the strong Artin conjecture holds true for the group \( S_3 \). By using the strong Artin conjecture, the function \( L_{\psi_3} \) also can be interpreted in another way [12]. Let \( \rho : S_3 \to GL_2(\mathbb{C}) \) be the irreducible two-dimensional representation. Then \( \rho \) gives rise to a cuspidal representation \( \pi \) of \( GL_2(\mathbb{A}_\mathbb{Q}) \). Let

\[
L(s, \pi) = \sum_{n=1}^{\infty} M(n)n^{-s}.
\]

Below we assume that \( \rho \) is odd, i.e. \( D < 0 \), then \( L(s, \pi) = L(s, f) \), where \( f \) is a holomorphic cusp form of weight 1 with respect to the congruence group \( \Gamma_0(|D|) \),

\[
f(z) = \sum_{n=1}^{\infty} M(n)e^{2\pi inz}.
\]

Here as usual, \( L(s, \pi) \) denotes the L-function of the representation \( \pi \), and \( L(s, f) \) denotes the Hecke L-function of cusp form \( f \). Thus \( L_{\psi_3} = L(s, f) \) and

\[
\zeta_K(s) = \zeta(s)L(s, f).
\]

The formula (2.6) implies that

**Lemma 2.3.** The symbols defined as above, we have

\[
a_K(n) = \sum_{d|n} M(d).
\]

In particular,

\[
a_K(p) = 1 + M(p),
\]

where \( p \) is a prime integer.

To prove the theorem, we also need some well-known estimates of the relative L-functions in the following.

For subconvexity bounds, we have the following well-known estimates.

**Lemma 2.4.** For any \( \varepsilon > 0 \), we have

\[
\zeta(\sigma + it) \ll_{\varepsilon} (1 + |t|)^{(1/3)(1-\sigma)} + \varepsilon
\]

uniformly for \( 1/2 \leq \sigma \leq 1 \), and \( |t| \geq 1 \).

**Proof.** See theorem II 3.6 in the book [13].

For the Dirichlet L-series, By using the Phragmen-Lindelöf principle for a strip [14] and the estimates given by Heath-Brown [15], we have the similar results

\[
L(\sigma + it, \chi) \ll_{\varepsilon} (1 + |t|)^{(1/3)(1-\sigma)} + \varepsilon
\]

uniformly for \( 1/2 \leq \sigma \leq 1 \), and \( |t| \geq 1 \), where \( \chi \) is a Dirichlet character modulo \( q \), and \( q \) is an integer.

For the mean values of the relative L-functions, we have
Lemma 2.5. For any $\varepsilon > 0$, we have
\[
\int_1^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^A \ll T^{1+\varepsilon}
\]
(2.9)
uniformly for $T \geq 1$, where $A = 2, 4$.

Lemma 2.6. For any $\varepsilon > 0$, and $q$ is an integer, $\chi$ is a character modulo $q$. We have
\[
\int_1^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^A \ll T^{1+\varepsilon}
\]
(2.10)
uniformly for $T \geq 1$, where $A = 2, 4$.

For Hecke $L$-functions defined in the formula (2.6), we have

Lemma 2.7. For any $\varepsilon > 0$, we have
\[
\int_1^T \left| L \left(\frac{1}{2} + it, f \right) \right|^2 dt \sim CT \log T
\]
\[
\int_1^T \left| L \left(\frac{1}{2} + it, f \right) \right|^6 dt \ll T^{2+\varepsilon}
\]
(2.11)
uniformly for $T \geq 1$, and the subconvexity bound
\[
L(\sigma + it, f) \ll_t \varepsilon (1 + |t|)^{\max\{(2/3)(1-\sigma), 0\} + \varepsilon}
\]
uniformly for $1/2 \leq \sigma \leq 2$ and $|t| \geq 1$.

Proof. The first and third results due to Good [16], and the second result was proved by Jutila [17].

We also have the convexity bounds for the relative $L$-functions.

Lemma 2.8. Let $L(s, g)$ be a Dirichlet series with Euler product of degree $m \geq 2$, which means
\[
L(s, g) = \sum_{n=1}^{\infty} a_g(n)n^{-s} = \prod_{p<\infty} \prod_{j=1}^{m} \left(1 - \frac{\alpha_g(p, j)}{p^s}\right),
\]
where $\alpha_g(p, j)$, $j = 1, 2, \cdots, m$ are the local parameters of $L(s, g)$ at prime $p$ and $a_g(n) \ll n^{\varepsilon}$.
Assume that this series and its Euler product are absolutely convergent for $\Re(s) > 1$. Also, assume that it is entire except possibly for simple poles at $s = 0, 1$, and satisfies a functional equation of Riemann type. Then for $0 \leq \sigma \leq 1$ and any $\varepsilon > 0$, we have
\[
L(\sigma + it, g) \ll_g \varepsilon (1 + |t|)^{(m/2)(1-\sigma) + \varepsilon},
\]
(2.12)
and for $T \geq 1$, we have
\[
\int_T^{2T} \left| L\left(\frac{1}{2} + \varepsilon + it, g\right) \right|^2 dt \ll_{g, \varepsilon} T^{m/2+\varepsilon}.
\]
(2.13)

Proof. The first result is from the lemma 2.2 in [18], and the second result is from the lemma 2.1 in [18].

□
3 Proof of Theorems

Assume that $K$ is a cubic extension of $\mathbb{Q}$. The Dedekind zeta function of $K$ is

$$\zeta_K(s) = \sum_{n=1}^{\infty} a_K(n)n^{-s}, \quad \Re s > 1.$$ 

Its Euler product is

$$\zeta_K(s) = \prod_p \left(1 + \frac{a_K(p)}{p^s} + \frac{a_K(p^2)}{p^{2s}} + \cdots\right), \quad \Re s > 1.$$

Let $q$ be an integer, and $\chi$ a primitive Dirichlet character modulo $q$. Define the function

$$f_\chi(n) = \sum_{k|n} \chi(k). \quad (3.14)$$

It is easy to check that $f_\chi(mn) = f_\chi(m)f_\chi(n)$, when $(m, n) = 1$.

Since $a_K(n) \ll n^\varepsilon$, so does $a_K(n)f_\chi(n)$. We can define an $L$-function associated to the function $a_K(n)f_\chi(n)$ in the half-plane $\Re s > 1$,

$$L_{K, f_\chi}(s) = \sum_{n=1}^{\infty} a_K(n)f_\chi(n)n^{-s}, \quad (3.15)$$

which is absolutely convergent in this region. Both $a_K(n)$ and $f_\chi(n)$ are multiplicative function, then for $\Re s > 1$, the function $L_{K, f_\chi}(s)$ can be expressed by the Euler product

$$L_{K, f_\chi}(s) = \prod_p \left(1 + \frac{a_K(p)f_\chi(p)}{p^s} + \frac{a_K(p^2)f_\chi(p^2)}{p^{2s}} + \cdots\right),$$

where the product runs over all primes.

**Proof of Theorem 1.1**

When $K$ is cubic normal extension, according to the formula (2.5) and (3.14), we get the formula

$$a_K(p)f_\chi(p) = 1 + \varphi(p) + \overline{\varphi}(p) + \chi(p) + \varphi(p)\chi(p) + \overline{\varphi}(p)\chi(p), \quad (3.16)$$

where $p$ is a nature prime number.

For $\Re s > 1$, we can write

$$M_{K, f_\chi}(s) := \zeta(s)L(s, \varphi)L(s, \overline{\varphi})L(s, \chi)L(s, \varphi \times \chi)L(s, \overline{\varphi} \times \chi)$$

as an Euler product of the form

$$\prod_p \left(1 + \frac{A(p)}{p^s} + \cdots\right),$$

where $A(p) = 1 + \varphi(p) + \overline{\varphi}(p) + \chi(p) + \varphi(p)\chi(p) + \overline{\varphi}(p)\chi(p)$, and the function $L(s, \varphi \times \chi)$ and $L(s, \overline{\varphi} \times \chi)$ are the Rankin-Selberg convolution $L$-function of the Dirichlet $L$-functions $L(s, \varphi)$, $L(s, \overline{\varphi})$ with the Dirichlet $L$-functions $L(s, \chi)$ respectively.
By comparing it with the Euler product of $L_{K,f}(s)$, and using the formula (3.16), we obtain

$$L_{K,f}(s) = M_{K,f}(s) \cdot U_1(s),$$

(3.17)

where $U_1(s)$ denotes a Dirichlet series, which is absolutely convergent for $\Re s > 1/2$, and uniformly convergent for $\Re s > 1/2 + \varepsilon$. Therefore, the function $L_{K,f}(s)$ admits an analytic continuation into the half-plane $\sigma > 1/2$, having as its only singularity a pole of order 6 at $s = 1$, because $\zeta(s)$ and each of the Dirichlet $L$-functions has a simple pole at $s = 1$.

By using the well-known inversion formula for Dirichlet series, we obtain

$$\sum_{n \leq x} a_K(n) f_\chi(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \frac{L_{K,f}(s)}{s} \frac{x^s}{s} ds + O(\frac{x^{1+\varepsilon}}{T}).$$

Where $b = 1 + \varepsilon$ and $1 \leq T \leq x$ is a parameter to be chosen later.

Shifting the path of integration to the line $\sigma = 1/2 + \varepsilon$. By using Cauchy’s residue theorem, we have

$$\sum_{n \leq x} a_K(n) f_\chi(n) = \frac{1}{2\pi i} \left\{ \int_{\frac{1}{2}+\varepsilon-iT}^{b+iT} \frac{x^s}{s} ds + \left( L_{K,f}(s) \frac{x^s}{s} \right)_{s=1} + O(\frac{x^{1+\varepsilon}}{T}) \right\}$$

$$= xP_5(\log x) + J_1 + J_2 + J_3 + O(\frac{x^{1+\varepsilon}}{T}) \quad (3.18)$$

Where $P_5(t)$ is a polynomial in $t$ with degree 5.

Using the lemmas in section 2 about the bound for the Dirichlet series, we will estimate the $J_i, i = 1, 2, 3$ in the following.

For $J_1$, we have

$$J_1 \ll x^{1/2+\varepsilon} + x^{1/2+\varepsilon} \int_1^T |M_{K,f}(1/2 + \varepsilon + it)| t^{-1} dt \quad (3.19)$$

where we have used that $U_1(s)$ is absolutely convergent in the region $\Re s \geq 1/2 + \varepsilon$ and behaves as $O(1)$ there.

By Hölder’s inequality, we have

$$\int_1^T |M_{K,f}(1/2 + \varepsilon + it)| t^{-1} dt \ll \log T \sup_{1 \leq T_1 \leq T} T_1^{-1} \cdot T_1^{1/6+\varepsilon} \cdot T_1^{1/6+\varepsilon} \times$$

$$I_\zeta(T_1)^{1/4} I_\varphi(T_1)^{1/4} I_\varphi(T_1)^{1/4} I_\chi(T_1)^{1/4},$$

where we have used the formula (2.8), and

$$I_\zeta(T_1) := \int_{T_1}^{2T_1} |\zeta(1/2 + \varepsilon + it)|^4 dt,$$

$$I_\varphi(T_1) := \int_{T_1}^{2T_1} |L(1/2 + \varepsilon + it, \varphi)|^4 dt,$$

and
Proof of Theorem 1.2

where $p$ is a nature prime number.

By virtue of (3.23), we have the relation

$$L_{K, f_{\chi}}(s) = \zeta(s)L(s, \chi)L(s, f)L(s, f \times \chi) \cdot U_2(s),$$

where $L(s, f \times \chi)$ is the Rankin-Selberg convolution $L$-function of $L(s, f)$ and $L(s, \chi)$, and $U_2(s)$ denotes a Dirichlet series, which is absolutely convergent for $\sigma > 1/2$. Therefore, the function $L_{K, f_{\chi}}(s)$ admits an analytic continuation into the half-plane $\sigma > 1/2$, having as its
only singularity a pole of order 4 at \( s = 1 \), because \( \zeta(s) \) and each of the relative \( L \)-functions has a simple pole at \( s = 1 \).

The degree of \( L(s, f \times \chi) \) is 2, according to the formula (2.13), we have

\[
\int_{T}^{2T} \left| L(1/2 + \varepsilon + it, f \times \chi) \right|^2 dt \ll_{g, \varepsilon} T^{2/2+\varepsilon}.
\]

Similarly as the proof of Theorem 1.1 using the inversion formula for Dirichlet series and the estimates above, we have the main term of the sum is

\[
\text{Res}_{s=1} L_{K, f\chi}(s)x^s s^{-1} = xP_3(\log x),
\]

and the error term is \( O(x^{5/8+\varepsilon}) \).

The proof is over.

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The ideal function in cubic fields

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