The face amplitude of spinfoam quantum gravity

Eugenio Bianchi\textsuperscript{1}, Daniele Regoli\textsuperscript{1,2} and Carlo Rovelli\textsuperscript{1}

\textsuperscript{1}Centre de Physique Théorique de Luminy\textsuperscript{3}, Case 907, F-13288 Marseille, France, European Union
\textsuperscript{2}Dipartimento di Fisica Università di Bologna e INFN sezione di Bologna, via Irnerio 46, 40126 Bologna, Italy

E-mail: rovelli@cpt.univ-mrs.fr

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Abstract
The structure of the boundary Hilbert space and the condition that amplitudes behave appropriately under compositions determine the face amplitude of a spinfoam theory. In quantum gravity the face amplitude turns out to be simpler than originally thought.

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1. Introduction
A spinfoam sum over a given two-complex $\sigma$, formed by faces $f$ joining along edges $e$ in turn meeting at vertices $v$, is defined by the expression

$$Z_{\sigma} = \sum_{j_f, i_e} \prod_f d_f \prod_v A_v(j_f, i_e),$$

where $A_v(j_f, i_e)$ is the ‘vertex amplitude’ and $d_f$ is the ‘face amplitude’. The sum is over an assignment $j_f$ of an irreducible representation of a compact group $G$ to each face $f$ and of an intertwiner $i_e$ to each edge $e$ of the two-complex. Expression (1) is often viewed as a possible foundation for a background-independent quantum theory of gravity [1]. In particular, a vertex amplitude $A_v(j_f, i_e)$ that might define a quantum theory of gravity has been developed in [2–9] and is today under intense investigation (see [10]). But what about the ‘measure factor’ given by the face amplitude $d_f$? What determines it?

The uncertainty in determining the face amplitude has been repeatedly remarked [11–16]. One way of fixing the face amplitude which can be found in the literature, for example, is to derive the sum (1) for general relativity (GR) starting from the analogous sum for a topological BF theory, and then implementing the constraints that reduce BF to GR as suitable constraints on the states summed over. For instance, in the Euclidean context GR is a constrained $SO(4)$ BF theory. The state sum (1) is well understood for $SO(4)$ BF theory: its face amplitude...
is the dimension of the $SO(4)$ representation $(j_+, j_-)$. The simplicity constraint fixes this to be of the form $j_\pm = \gamma \pm j_f$, where $\gamma = \frac{1 + \gamma_2}{2}$ and $\gamma$ is the Barbero–Immirzi parameter, and therefore

$$d_{j_f} = (2j_+ + 1)(2j_- + 1) = (2\gamma_j j_f + 1)(2\gamma_- j_f + 1).$$

(2)

However, doubts can be raised against this argument. For instance, Alexandrov [17] has stressed the fact that the implementation of second class constraints into a Feynman path integral in general requires a modification of the measure, and here the face amplitude plays precisely the role of such measure, since $A_v \sim e^{i \text{Action}}$. Do we have an independent way of fixing the face amplitude?

Here, we argue that the face amplitude is uniquely determined for any spinfoam sum of the form (1) by three inputs: (a) the choice of the boundary Hilbert space; (b) the requirement that the composition law holds when gluing two-complexes; and (c) a particular ‘locality’ requirement, or, more precisely, a requirement for the local composition of group elements.

We argue below that these requirements are implemented if $Z$ is given by the expression

$$Z_{\sigma} = \int dU \prod_v A_v(U_v) \prod_f \delta(U_{1f} \cdots U_{kf}),$$

(3)

where $U_v \in G$, $v_1 \cdots v_2$ are the vertices surrounding the face $f$, and $A_v(U_v)$ is the vertex amplitude $A_v(j_f, i_e)$ expressed in the group element basis [18]. Then we show that this expression leads directly to (1), with an arbitrary vertex amplitude, but a fixed choice of face amplitude, which turns out to be the dimension of the representation $j$ of the group $G$:

$$d_{j} = \text{dim}(j).$$

(4)

In particular, for quantum gravity this implies that the BF face amplitude (2) is ruled out, and should be replaced (both in the Euclidean and in the Lorentzian case) by the $SU(2)$ dimension:

$$d_{j} = 2j + 1.$$  

(5)

Equation (3) is the key expression of this paper; we begin by showing that $SO(4)$ BF theory (the prototypical spinfoam model) can be expressed in this form (section 2). Then we discuss the above three requirements and we show that equation (3) implements these requirements (section 3). Finally we show that (3) gives (1) with the face amplitude (4) (section 4).

The problem of fixing the face amplitude has also been discussed by Bojowald and Perez in [16]. Bojowald and Perez demand that the amplitude be invariant under suitable refinements of the two-complex. This request is strictly related to the composition law that we consider here, and the results we obtain are consistent with those of [16].

2. BF theory

It is well known that the partition function (1) for BF theory can be rewritten in the form (see [1])

$$Z_{\sigma} = \int dU_e \prod_f \delta(U_{1f} \cdots U_{ef}),$$

(6)

where $U_e$ are the group elements associated with the oriented edges of $\sigma$ and $(e_1, \ldots, e_k)$ are the edges that surround the face $f$. Let us introduce group elements $h_{ve}$, labeled by a vertex $v$ and an adjacent edge $e$, such that

$$U_e = h_{ve} h_{v'e}^{-1},$$

(7)
where \( v \) and \( v' \) are the source and target of the edge \( e \) (see figure 1) respectively. Then we can trivially rewrite (6) as

\[
Z_\sigma = \int dh_{ve} \prod_f \delta \left( (h_{ve}^{-1} h_{v'e}) \cdots (h_{v'e}^{-1} h_{ve}) \right).
\]

Now define the group elements

\[
U_{vf}^v = h_{ve}^{-1} h_{v'e}
\]

associated with a single vertex \( v \) and the two edges \( e \) and \( e' \) that emerge from \( v \) and bound the face \( f \) (see figure 1). Using these, we can rewrite (6) as

\[
Z_\sigma = \int dh_{ve} \int dU_{vf}^v \prod_{v,f} \delta \left( U_{vf}^v, h_{ve} h_{v'e}^{-1} \right) \prod_{f} \delta \left( U_{vf}^v \cdots U_{vf}^0 \right),
\]

where the first product is over the faces \( f \) that belong to the vertex \( v \), and then a product over all the vertices of the two-complex.

Note that this expression has precisely the form (3), where the vertex amplitude is

\[
A_v(U_{vf}^v) = \int dh_{ve} \prod_{f} \delta \left( U_{vf}^v, h_{ve} h_{v'e}^{-1} \right),
\]

which is the well-known expression of the 15j Wigner symbol (the vertex amplitude of BF in the spin network basis) in the basis of the group elements.

We have shown that the BF theory spinfoam amplitude can be put into the form (3). We now argue that (3) is the general form of a local spinfoam model that obeys the composition law.

3. Three inputs

(a) Hilbert space structure. Equation (1) is a coded expression to define the amplitudes

\[
W_\sigma (j_i, i_a) = \sum_{j_f, i_c} \prod_f d_{j_f} \prod_v A_v(j_f, i_c; j_i, i_a),
\]
defined for a two-complex \( \sigma \) with boundary, where the boundary graph \( \Gamma = \partial \sigma \) if formed by links \( l \) and nodes \( n \). The spins \( j_l \) are associated with the links \( l \), as well as with the faces \( f \) that are bounded by \( l \); the intertwiners \( i_n \) are associated with the nodes \( n \), as well as with the edges \( e \) that are bounded by \( n \). The amplitude of the vertices that are adjacent to these boundary faces and edges depend also on the external variables \((j_l, i_n)\).

In a quantum theory, the amplitude \( W(j_l, i_n) \) must be interpreted as a (covariant) vector in a space \( H/\Gamma_1 \) of quantum states. We assume that this space has a Hilbert space structure, which we know. In particular, we assume that

\[
H/\Gamma_1 = L^2[G^L, dU_l],
\]

where \( L \) is the number of links in \( \Gamma \) and \( dU_l \) is the Haar measure. Thus, we can interpret (12) as

\[
W_\sigma(j_l, i_n) = \langle j_l, i_n | W \rangle,
\]

where \( |j_l, i_n \rangle \) is the spin network function

\[
\langle U_l | j_l, i_n \rangle = \psi_{j_l, i_n}(U_l) = \bigotimes_l R^j_l(U_l) \cdot \bigotimes_n i_n.
\]

Here, \( R^j_l(U_l) \) are the representation matrices in the representation \( j \) and \( i \) forms an orthonormal basis in the intertwiner space. See for instance [10, 20] for details. Using the scalar product defined by (13), we have

\[
\langle j_l, i_n | j'_l, i'_n \rangle = \int dU_l \overline{\psi_{j_l, i_n}(U_l)} \psi_{j'_l, i'_n}(U_l)
= \prod_l \dim(j_l) \delta_{j_l j'_l} \prod_n \delta_{i_n i'_n},
\]

where \( \dim(j) \) is the dimension of the representation \( j \). Therefore, the spin-network functions \( \psi_{j_l, i_n}(U_l) \) are not normalized. (These \( \dim(j) \) normalization factors are due to the convention chosen: they have nothing to do with the dimension of the representation that appears in (4).) The resolution of the identity in this basis is

\[
1 = \sum_{j_l, i_n} \left( \prod_l \dim(j_l) \right) |j_l, i_n \rangle \langle j_l, i_n |.
\]

(b) Composition law. In non-relativistic quantum mechanics, if \( U(t_1, t_0) \) is the evolution operator from time \( t_0 \) to \( t_1 \), the composition law reads

\[
U(t_2, t_0) = U(t_2, t_1)U(t_1, t_0).
\]

That is, if \( |n \rangle \) is an orthonormal basis, then

\[
\langle f | U(t_2, t_0) | i \rangle = \sum_n \langle f | U(t_2, t_1) | n \rangle \langle n | U(t_1, t_0) | i \rangle.
\]

Let us write an analogous condition of the spinfoam sum. Consider for simplicity a two-complex \( \sigma = \sigma_1 \cup \sigma_2 \) without boundary, obtained by gluing two two-complexes \( \sigma_1 \) and \( \sigma_2 \) along their common boundary \( \Gamma \). Then, we require that \( W \) satisfies the composition law

\[
Z_{\sigma_1 \cup \sigma_2} = \langle \sigma_2 | W_{\sigma_1} \rangle,
\]

as discussed by Atiyah in [21]. Note that to formulate this condition we need the Hilbert space structure in the space of the boundary states.

\[
\text{If } \Gamma \text{ has two disconnected components interpreted as 'in' and 'out' spaces, then } H_\Gamma \text{ can be identified as the tensor product of the 'in' and 'out' spaces of non-relativistic quantum mechanics. In the general case, } H_\Gamma \text{ is the boundary quantum state in the sense of the boundary formulation of quantum theory [19, 20].}
\]
(c) **Locality.** As a vector in $H_r$, the amplitude $W(j_i, i_a)$ can be expressed on the group element basis

$$W(U_i) = \langle U_i | W \rangle = \sum_{j_i, i_a} \left( \prod_l \dim(j_i) \right) \psi_{j_i, i_a}(U_i) W(j_i, i_a).$$

Similarly, the vertex amplitude can be expanded in the group element basis

$$A_v(U_v^f) = \{ U_v^f | A_v \} = \sum_{j_v^f, i_v^f} \left( \prod_{f^*} \dim(j_v^f) \right) \psi_{j_v^f, i_v^f}(U_v^f) A_v(j_v^f, i_v^f).$$

Note that here the group element $U_v^f$ and the spin $j_v^f$ are associated with a vertex $v$ and a face $f$ adjacent to $v$. Similarly, the intertwiner $i_v^f$ is associated with a vertex $v$ and a node $n$ adjacent to $v$. Consider a boundary link $l$ that bounds a face $f$ (see figure 2). Let $v_1 \cdots v_k$ be the vertices that are adjacent to this face. We say that the model is local if the relation between the boundary group element $U_l$ and the vertices group elements $U_v^f$ is given by

$$U_l = U_v^{f_1} \cdots U_v^{f_k}. \quad (23)$$

In other words, the boundary group element is simply the product of the group elements around the face.

Note that a spinfoam model defined by (3) is local and satisfies the composition law in the above sense. In fact, (3) generalizes immediately to

$$W_\sigma(U_l) = \int dU_v^f \prod_v A_v(U_v^f) \prod_{\text{internal } f} \delta(U_v^{f_1} \cdots U_v^{f_{\text{internal } f}}) \prod_{\text{external } f} \delta(U_v^{f_1} \cdots U_v^{f_{\text{external } f}} U_l^{-1}). \quad (24)$$

Here, the first product over $f$ is over the (‘internal’) faces that do not have an external boundary, while the second is over the (‘external’) faces $f$ that are also bounded by the vertices $v_1, \ldots, v_k$.
and by the link \(l\). It is immediate to see that locality is implemented, since the second delta enforces the locality condition (23).

Furthermore, when gluing two amplitudes along a common boundary we have immediately that

\[
\int dU_l W_{\sigma_1} (U_l) W_{\sigma_2} (U_l) = Z_{\sigma_1 \cup \sigma_2}
\]

because the two delta functions containing \(U_l\) collapse into a single delta function associated with the face \(l\), which becomes internal.

Thus, (3) is a general form of the amplitude where these conditions hold.

In [16], Bojowald and Perez have considered the possibility of fixing the face amplitude by requiring the amplitude of a given spin/intertwiner configuration to be equal to the amplitude of the same spin/intertwiner configuration on a finer two-simplex where additional faces carry the trivial representation. This requirement implies essentially that the amplitude does not change by splitting a face into two faces. It is easy to see that (3) satisfies this condition. Therefore, (3) satisfies also the Bojowald–Perez condition.

4. Face amplitude

Finally, let us show that (3) implies (1) and (4). To this purpose, it is sufficient to insert (22) into (3). This gives

\[
Z_{\sigma} = \int dU^\nu_v \prod_v \sum_{j^\nu_v} \left( \prod_{j^\nu} \dim(j^\nu) \right) \psi_{j^\nu_v, i^\nu_v} (U^\nu_v) A_v (j^\nu_v, i^\nu_v) \prod_f \delta(U^\nu_f \cdots U^\nu_1).
\]

Then expand the delta function in a sum over characters

\[
Z_{\sigma} = \int dU^\nu_v \prod_v \sum_{j^\nu_v} \left( \prod_{j^\nu} \dim(j^\nu) \right) \psi_{j^\nu_v, i^\nu_v} (U^\nu_v) A_v (j^\nu_v, i^\nu_v)
\]

\[
\times \prod_f \sum_{j_f} \dim(j_f) \Tr(R^{\nu_f} (U^\nu_1) \cdots R^{\nu_f} (U^\nu_n)).
\]

We can now perform the group integrals. Each \(U^\nu_v\) appears precisely twice in the integral: once in the sum over \(j^\nu_v\) and the other in the sum over \(j^\nu_f\). Each integration gives a delta function \(\delta_{j^\nu_v, j^\nu_f}\), which can be used to kill the sum over \(j^\nu_v\) dropping the \(v\) subscript. Following the contraction path of the indices, it is easy to see that these contract the two intertwiners at the opposite side of each edge. Since intertwiners are orthonormal, this gives a delta function \(\delta_{i^\nu_v, i^\nu_f}\) which reduces the sums over intertwiners to a single sum over \(i^n\).

Bringing everything together, and noting that the \(\dim(j)\) factor from the group integrations cancels the one in the integral, we have

\[
Z_{\sigma} = \sum_{j/i^n} \prod_f \dim(j_f) \prod_v A_v (j^\nu_v, i^\nu_v).
\]

This is precisely equation (1), with the face amplitude given by (4).

Note that the face amplitude is well defined, in the sense that it cannot be absorbed into the vertex amplitude (as any edge amplitude can be). The reason is that any factor in the vertex amplitude depending on the spin of the face contributes to the total amplitude at a power \(k\), where \(k\) is the number of sides of the face. The face amplitude, instead, is a contribution to the total amplitude that does not depend on \(k\). This is also the reason why the normalization
chosen for the spinfoam basis does not affect the present discussion: it affects the expression for the vertex amplitude, not that for the face amplitude.

By an analogous calculation one can show that the same result holds for the amplitudes $W$: equation (12) follows from (24) expanded on a spin network basis.

In conclusion, we have shown that the general form (3) of the partition function, which implements locality and the composition law, implies that the face amplitude of the spinfoam model is given by the dimension of the representation of the group $G$ which appears in the boundary scalar product (13).

In GR, in both the Euclidean and the Lorentzian cases, the boundary space is

$$H_\Gamma = L_2[ SU(2)^L, dU_1];$$

therefore, the face amplitude is $d_\gamma = \dim_{SU(2)}(j) = 2j + 1$, and not the $SO(4)$ dimension (2), as previously supposed.

Note that such $d_\gamma = 2j + 1$ amplitude defines a theory that is far less divergent than the theory defined by (2). In fact, the potential divergence of a bubble is suppressed by a power of $j$ with respect to (2). In [15], it has been shown that the $d_\gamma = 2j + 1$ face amplitude yields a finite main radiative correction to a five-valent vertex if all external legs are set to zero.

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