Semiclassical limit of the FZZT Liouville theory

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Abstract

The semiclassical limit of the FZZT Liouville theory on the upper half plane with bulk operators of arbitrary type and with elliptic boundary operators is analyzed. We prove the Polyakov conjecture for an appropriate classical Liouville action. This action is calculated in a number of cases: one bulk operator of arbitrary type, one bulk and one boundary, and two boundary elliptic operators. The results are in agreement with the classical limits of the corresponding quantum correlators.

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1 Introduction

In the last few years the quantum Liouville theory attracted again a considerable attention, mainly for its application in describing D-brane dynamics in non-compact curved or time dependent backgrounds (for review and references see [1]). This renewed interest was supported by a recent progress in the quantum Liouville theory on bordered surfaces, where exact analytic forms of all basic correlators were derived [2–7] by powerful conformal bootstrap methods. These methods were previously successfully applied in rederiving [8] the DOZZ bulk three point function [9, 10] and proving its consistency [11–13].

In the case of the DOZZ three point function on the sphere the exact result agrees with the perturbative calculations [14] based on the hamiltonian approach [15–17]. A perturbative check of the exact formula for the one point function on the pseudosphere was given in [4, 18]. This suggests that the bootstrap solution should have a well defined path integral representation. In all the cases in which a classical solution exists one then could expect that the quantum expressions arise as a result of integrating fluctuations of classical background geometries.

In the case of “heavy” elliptic weights on compact surfaces a systematic functional formulation of Liouville theory was first proposed long time ago by Takhtajan [19, 20] (see [21] for recent generalizations). This so called geometric approach has been recently compared with the exact bootstrap solution on the pseudosphere [22]. It was shown in particular that the choice of regularization is crucial for the agreement with the bootstrap approach. Recently a functional representation of Liouville correlators with heavy elliptic charges on the sphere [23], the pseudosphere [24], and the disc [25] has been developed. This representation agrees with the bootstrap solution at least up to one loop calculations.

In the present paper we address the problem of calculating the classical limit of the FZZT Liouville theory for heavy charges. Our motivation is twofold. In spite of the recent progress in constructing the path integral representation of Liouville correlators for elliptic weights there is still an open and interesting problem of factorization in functional approach. Due to the global character of the Liouville action related to its specific dependence on the background metric it is not simply described by the standard cutting-open procedure. It seems that in order to handle this problem one should first construct a functional representations for Liouville correlators with hyperbolic weights. The FZZT theory provides simple cases where exact solutions are known and both problems can be relatively easily analyzed. The classical limit is just the first step of such calculations.

The second, probably more important motivation is the semiclassical limit itself. It turned out that analyzing the quantum Liouville theory in this limit one gets essentially new insight into the classical hyperbolic geometry. One of the first results of this type was the so called Polyakov conjecture, originally obtained as a classical limit of the Ward identity and proved latter as an exact theorem [26–31]. It says that the classical Liouville action is a generating function for the accessory parameters of the Fuchsian uniformization of the punctured sphere.
More intriguing results are related with the classical conformal block \( f_\delta^{[\delta_3, \delta_2]_{\delta_4, \delta_1}}(x) \), defined by the classical limit of the BPZ quantum conformal block [32] with heavy weights \( \Delta = Q^2 \delta \), \( \Delta_i = Q^2 \delta_i \), [10, 33, 34]:

\[
\mathcal{F}_{1+6Q^2, \Delta}
\left[
\begin{array}{c}
\Delta_1 \\
\Delta_1 \\
\Delta_1 \\
\end{array}
\right]
(x)
\sim
\exp\left\{Q^2 f_\delta^{[\delta_3, \delta_2]_{\delta_4, \delta_1}}(x)\right\}.
\]

(1.1)

It was argued in [10] that the 4-point Liouville classical action can be expressed in terms of 3-point actions and the classical conformal block in a given channel calculated for a saddle point value of the intermediate weight. Since there are three different decompositions of the 4-point action one gets consistency conditions called the classical bootstrap equations [35]. It was also conjectured [35] that the saddle point weight is closely related to the length of the closed geodesic in the corresponding channel. These statements are far from being proved in a rigorous way, but there are many nontrivial numerical checks instead [35]. Let us finally stress that once the 4-point Liouville action is known one can in principle calculate the uniformization of the 4-punctured sphere [36], which is a long standing open mathematical problem. Analyzing various classical limits one may hope to gain some new information on the classical conformal block, which up to now is only available through term by term symbolic calculations from the limit (1.1).

The organization of the paper is as follows. In Sect. 2, following the ideas of [37], we formulate the \( SL(2, \mathbb{R}) \)-monodromy problem with boundary. In Sect. 3 we introduce an appropriate classical Liouville action and derive the formulae for its partial derivatives with respect to bulk and boundary conformal weights, and bulk and boundary cosmological constants. A novel technical result is the formula for the partial derivative of the action with respect to the ratio \( \omega_j = m_j / \sqrt{m} \) in terms of special values of some conformal map. Details of its derivation are explained in the Appendix.

In Sect. 4 we present a proof of the Polyakov conjecture for elliptic boundary weights and arbitrary bulk weights. This extends the previous results [26–31] to the FZZT Liouville theory and is one of the main results of this paper.

The last section contains four examples of explicit calculations of the classical actions both from classical solutions and from the classical limit of exact quantum expressions. The simplest one is the case of one bulk elliptic singularity considered in Subsect. 5.1. The case of one bulk hyperbolic singularity is considered in Subsect. 5.2. This calculations provide an additional support for the construction of the classical action for hyperbolic singularities proposed and analyzed in [31, 38]. The cases of two boundary, and one bulk one boundary elliptic weights are calculated in Subsect. 5.3 and Subsect. 5.4, respectively. Both ways of calculation the classical Liouville action are rather complicated in these cases. The full agreement of the results provides an additional strong evidence that the classical limit of the DOZZ bootstrap solution exists and is properly described by a classical action satisfying the equations derived in Sect. 3 and 4.
2 The monodromy problem with boundary

Let us consider the Fuchs equation

\[ \partial^2 \psi(z) + T(z)\psi(z) = 0, \tag{2.1} \]

with the energy–momentum tensor of the form

\[ T(z) = \sum_{i=1}^{m} \left[ \frac{\delta_i}{(z - z_i)^2} + \frac{\delta_i}{z - \bar{z}_i} \right] + \sum_{j=1}^{n} \left[ \frac{\delta^B_j}{(z - x_j)^2} + \frac{c^B_j}{z - \bar{x}_j} \right], \tag{2.2} \]

where \( \delta_i \) are bulk conformal weights located at the points \( z_1, \ldots, z_m \) of the upper half-plane, \( \delta^B_j \) are boundary conformal weights located at the points \( x_1 < \ldots < x_n \) of the real axis, and \( c_i, c^B_j \) are accessory parameters. We assume that \( \delta_i, \delta^B_j, c^B_j \) are real so that \( T(z) = T(\bar{z}) \).

The requirement that \( T(z) \) is regular at the infinity implies the relations

\[ 2 \sum_{i=1}^{m} \Re c_i + \sum_{j=1}^{n} c^B_j = 0, \]
\[ 2 \sum_{i=1}^{m} \Re (z_i c_i) + \sum_{j=1}^{n} x_j c^B_j = -2 \sum_{i=1}^{m} \delta_i - \sum_{j=1}^{n} \delta^B_j, \]
\[ 2 \sum_{i=1}^{m} \Re (z^2_i c_i) + \sum_{j=1}^{n} x^2_j c^B_j = -4 \sum_{i=1}^{m} \Re z_i \delta_i - 2 \sum_{j=1}^{n} x_j \delta^B_j. \]

A fundamental systems of solutions \( \Psi = \begin{bmatrix} \psi^- \\ \psi^+ \end{bmatrix} \) to the Fuchs equation (2.1) is normalized if

\[ \psi^- \partial \psi^+ - \psi^+ \partial \psi^- = 1. \]

Let \( L_j \) denote the part of the real axis between \( x_j \) and \( x_{j+1} \) (with the exception of \( L_n \) denoting the set of points on the real axis to the right of \( x_n \) and to the left of \( x_1 \)). It follows from (2.3) that for each boundary segment \( L_j \) there exist normalized solutions \( \Psi_j \) to the Fuchs equation (2.1) regular and real along \( L_j \). For any other normalized solution \( \Psi \) we define the matrices

\[ M_j = \Sigma^{-1} B^T_j \sum \Sigma B_j, \quad j = 1, \ldots, n, \]

where \( \Sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) and the matrix \( B_j \) is determined by the relation \( \Psi = B_j \Psi_j \).

We are interested in the following version of the Riemann-Hilbert problem [37]. For given sets of positive weights \( \{\delta_i\}_{i=1}^{m}, \{\delta^B_j\}_{j=1}^{n} \) and real numbers \( \{\omega_j\}_{j=1}^{n} \) one has to adjust the accessory parameters in such a way that the Fuchs equation (2.1) admits a normalized fundamental system \( \Psi \) of solutions, such that:
1. monodromies around all singularities $z_i$ of the upper half plane belong to $SL(2, \mathbb{R})$;

2. the function $-i\Psi^T \cdot \Sigma \cdot \overline{\Psi}$ is strictly positive or strictly negative on the upper half plane except the points $z_i$, $i = 1, \ldots, m$ and $x_j$, $j = 1, \ldots, n$;

3. for each boundary segment $L_j$ the boundary condition
   $$\text{Tr} M_j = \begin{cases} +\omega_j & \text{if } -i\Psi^T \cdot \Sigma \cdot \overline{\Psi} > 0, \\ -\omega_j & \text{if } -i\Psi^T \cdot \Sigma \cdot \overline{\Psi} < 0, \end{cases}$$
   is satisfied.

If $\Psi$ is a solution to the monodromy problem above then the relation
   $$e^{-\varphi} = \left( \frac{\sqrt{m}}{2t} \Psi^T(z) \cdot \Sigma \cdot \overline{\Psi}(z) \right)^2$$
   determines a conformal factor $\varphi$ on the upper half plane satisfying the Liouville equation
   $$\partial \bar{\partial} \varphi = \frac{m}{2} e^\varphi$$
   and the boundary conditions
   $$\left. \partial_y e^{-\frac{\varphi}{2}} \right|_{L_j} = -\frac{\sqrt{m}}{2} \omega_j, \quad j = 1, \ldots, n. \quad (2.6)$$

They are usually written in the form
   $$\left. \partial_y \varphi \right|_{L_j} = m_j e^{\frac{\varphi}{2}}, \quad m_j = \omega_j \sqrt{m}, \quad (2.7)$$

where $m_j$ are so called boundary cosmological constants. Note that $T(z)$ of (2.2) is the classical energy momentum tensor of the solution $\varphi$:
   $$T(z) = T_{\text{cl}}(z) = -\frac{1}{4} \left( \partial \varphi \right)^2 + \frac{1}{2} \partial^2 \varphi = -e^{\frac{\varphi}{2}} \partial^2 e^{-\frac{\varphi}{2}}. \quad (2.8)$$

The conformal factor $\varphi$ is a regular single valued function on the upper half plane $\Im z \geq 0$ except the singular points $z_i, x_j$. It defines the hyperbolic metric $e^\varphi dz d\bar{z}$ with the constant negative scalar curvature
   $$R = -e^{-\varphi} \Delta \varphi = -4e^{-\varphi} \partial \bar{\partial} \varphi = -2m,$$

and with the constant geodesic curvature of each boundary sector $L_j$:
   $$\kappa_j = \frac{1}{2} e^{-\frac{\varphi}{2}} n^a \partial_a \varphi \bigg|_{L_j} = -\frac{m_j}{2} = -\frac{\sqrt{m}}{2} \omega_j.$$
3 The classical Liouville action

In order to simplify our considerations we assume that all weights are elliptic:

\[ \delta_i = \frac{1 - \xi_i^2}{4}, \quad j = 1, \ldots, m, \quad \delta_j^a = \frac{1 - \nu_j^2}{4}, \quad j = 1, \ldots, n, \quad 0 < \xi_i, \nu_j < 1. \]

It follows from (2.2), (2.3), and (2.8) that the most singular terms in the expansions of the Liouville field around the locations of elliptic weights read

\[ \varphi(z, \bar{z}) \sim -2(1 - \xi_i) \log |z - z_i|, \quad \varphi(z, \bar{z}) \sim -2(1 - \nu_j) \log |z - x_j|. \]  

(3.1)

Let \( X \) be the upper half plane with the discs of radii \( \epsilon \) around the points \( z_i, \quad i = 1, \ldots, m \) and the semi-discs of radii \( \epsilon \) around the points \( x_j, \quad j = 1, \ldots, n \) removed. Denote by \( S_i \) the boundary of the disc around \( z_i \) and by \( s_j \) the semicircle forming a boundary of the semi-discs around \( x_i \). Finally, let \( l_i \) denote the part of the real axis between \( x_i + \epsilon \) and \( x_i + 1 - \epsilon \) (with the exception of \( l_n \) denoting the set \([-R, x_1 - \epsilon] \cup [x_n + \epsilon, R]\)). The regularized action functional is defined by

\[
S[\phi] = \lim_{\epsilon \to 0} S_\epsilon[\phi], \\
S_\epsilon[\phi] = \lim_{R \to \infty} S_\epsilon^R[\phi], \\
S_\epsilon^R[\phi] = \frac{1}{4\pi} \int_X d^2 z \left[ \partial \phi \partial \bar{\phi} + m e^\phi \right] \\
+ \sum_{i=1}^m \left[ \frac{1 - \xi_i}{4\pi} \int_{S_i} \kappa_z |dz| \phi - \frac{(1 - \xi_i)^2}{2} \log \epsilon \right] \\
+ \sum_{j=1}^n \left[ \frac{1 - \nu_j}{4\pi} \int_{s_j} \kappa_z |dz| \phi - \frac{(1 - \nu_j)^2}{4} \log \epsilon \right] \\
+ \frac{1}{2\pi R} \int_{s_R} |dz| \phi + \log R,
\]

(3.2)

where \( s_R \) is the semicircle \(|z| = R\) on the upper half plane. The action is constructed in such a way that for fields satisfying

\[
\phi \sim -2(1 - \xi_i) \log |z - z_i| \quad \text{for} \quad z \to z_i, \\
\phi \sim -2(1 - \nu_j) \log |z - x_j| \quad \text{for} \quad z \to x_j, \\
\phi \sim -4 \log |z| \quad \text{for} \quad z \to \infty,
\]

the limit in (3.2) exists and the equation \( \delta S[\phi] = 0 \) gives (2.5) and (2.7).

Let \( \varphi(z, \bar{z}) \) denote a solution of (2.5) and (2.7) with some specified values of \( m, \xi_i, z_i \) and \( m_j, \nu_j, x_j \). We define the classical Liouville action \( S_{cl} \) as the value of the action functional (3.2) calculated on this solution. Using the equations of motion and the boundary condition
satisfied by \( \varphi(z, \bar{z}) \) one immediately gets

\[
\frac{\partial S_{\text{cl}}}{\partial \xi_i} = \lim_{\epsilon \to 0} \left[ -\frac{1}{4\pi} \int_{s_i} \kappa_z |dz| \varphi + (1 - \xi_i) \log \epsilon \right], \quad (3.3)
\]

\[
\frac{\partial S_{\text{cl}}}{\partial \nu_j} = \lim_{\epsilon \to 0} \left[ -\frac{1}{4\pi} \int_{s_j} \kappa_z |dz| \varphi + \frac{1 - \nu_j}{2} \log \epsilon \right]. \quad (3.4)
\]

Shifting the classical solution \( \varphi = \tilde{\varphi} - \log m \) one obtains

\[
S_{\text{cl}}(m, \xi_i, z_i, m_j, \nu_j, x_j) = S_{\text{cl}}(1, \xi_i, z_i, m_j, \nu_j, x_j) \quad (3.5)
\]

\[+ \left( \sum_{i=1}^m \frac{1 - \xi_i}{2} + \sum_{j=1}^n \frac{1 - \nu_j}{4} - \frac{1}{2} \right) \log m.
\]

It is convenient to regard the classical action as a function of the variables \( m \) and \( \omega_j \) (instead of \( m \) and \( m_j \)). One then has

\[
\frac{\partial S_{\text{cl}}}{\partial \omega_j} = \frac{1}{4\pi} \int_{L_j} dx e^{\tilde{\varphi}}, \quad j = 1, \ldots, n. \quad (3.6)
\]

Equations (3.3) and (3.4) express the derivatives of the classical action in terms of the classical metric in the vicinity of the singularities. It is useful to work out a similar “local” expression for the integral in the r.h.s. of (3.6). To this end note that the solution \( \Psi \) to the monodromy problem described in the previous section determines a multivalued analytic function from

\[
\mathcal{M} = \{z \in \mathbb{C} : \Im z \geq 0\} \cup \{\infty\} \setminus \{z_1, \ldots, z_n, x_1, \ldots, x_j\}
\]

to the upper half plane \( \mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\} \),

\[
\mathcal{M} \ni z \mapsto \rho(z) = \frac{\psi^-(z)}{\psi^+(z)} \in \mathbb{H}. \quad (3.7)
\]

The pull-back of the standard hyperbolic metric on \( \mathbb{H} \) by any branch of \( \rho \) yields a regular hyperbolic metric on \( \mathcal{M} \):

\[
e^{\varphi(z, \bar{z})} dzd\bar{z} = \frac{1}{m(\Im \rho)^2} d\rho d\bar{\rho}. \quad (3.8)
\]

One can always choose a branch of \( \rho \) such that the image \( \rho(L_j) \) of the boundary segment \( L_j \) is a connected open curve in \( \mathbb{H} \) joining the points

\[
\rho_j = \lim_{x \to x_j^+} \rho(x), \quad \rho_{j+1} = \lim_{x \to x_{j+1}^-} \rho(x).
\]

This curve has the constant geodesic curvature \( -\frac{\omega_j}{\rho} \) with respect to the Poincaré metric on \( \mathbb{H} \) (the sign being determined with respect to \( \rho(\mathcal{M}) \)). It admits a simple description on the Lobachevsky plane as the arc of the Euclidean circle containing the points \( \rho_j, \rho_{j+1} \) and with
its radius $R$ and its center $O$ determined by the condition that the Euclidean distance of $O$ from the real axis is equal to $|\omega_j| R$. For $|\omega_j| \leq 2$ there are two and for $|\omega_j| > 2$ four different arcs with this property. It is shown in the Appendix that in both cases the hyperbolic length depends only on the location of the endpoints $\rho_j, \rho_{j+1}$ and $|\omega_j|$. It follows from (3.8) that the length of the boundary component $L_j$ with respect to the metric $e^{\varphi(z,\bar{z})}$ is equal to the length of its image $\rho(L_j)$ in the Lobachevsky plane. Using the explicit expressions (A.1) and (A.2) one gets

$$\frac{\partial S_{cl}}{\partial \omega_j} = \frac{1}{\pi \sqrt{4 - \omega_j^2}} \arcsin \left[ \sqrt{1 - \left(\frac{\omega_j}{2}\right)^2} \beta(\rho_j, \rho_{j+1}) \right]$$

(3.9)

for $|\omega_j| < 2$ and

$$\frac{\partial S_{cl}}{\partial \omega_j} = \frac{1}{\pi \sqrt{\omega_j^2 - 4}} \arcsin \left[ \sqrt{\left(\frac{\omega_j}{2}\right)^2 - 1} \beta(\rho_j, \rho_{j+1}) \right]$$

(3.10)

for $|\omega_j| > 2$, where

$$\beta(z, w) \overset{\text{def}}{=} \frac{|z - w|}{2\sqrt{3} \Im z \Im w}.$$

### 4 Polyakov conjecture

The Polyakov conjecture states that

$$\frac{\partial S_{cl}}{\partial z_i} = -c_i, \quad i = 1, \ldots, m;$$

(4.1)

$$\frac{\partial S_{cl}}{\partial x_j} = -c_j^B, \quad j = 1, \ldots, n.$$  

(4.2)

The equations (4.1) can be derived by essentially the same methods one uses in proving the Polyakov conjecture for the Riemann sphere and we shall skip here the derivation. Let us only mention that the equations (4.1) are valid in the case of parabolic and hyperbolic bulk singularities as well, although the classical Liouville action is different in those cases.

We shall prove the equations (4.2). Using the Liouville equation (2.5) and the boundary conditions (2.7) one gets

$$\frac{\partial S_{cl}}{\partial x_j} = \lim_{\epsilon \to 0} D_\epsilon[\varphi],$$

$$D_\epsilon[\varphi] = \frac{i}{8\pi} \int_{s_j} (\bar{\partial} \varphi d\bar{z} - \partial \varphi dz) \frac{\partial \varphi}{\partial x_j} + \frac{1 - \nu_j}{4\pi} \int_{s_j} \kappa_z |dz| \frac{\partial \varphi}{\partial x_j}$$

$$+ \frac{i}{8\pi} \int_{s_j} (dz - d\bar{z}) (\partial \varphi \bar{\partial} \varphi + m e^\varphi) + \frac{1 - \nu_j}{4\pi} \int_{s_j} \kappa_z |dz| \partial_x \varphi$$

(4.3)

$$+ \frac{m_k - 1}{4\pi} e^{\frac{\varphi}{2}} \bigg|_{z=x_j-\epsilon} - \frac{m_k}{4\pi} e^{\frac{\varphi}{2}} \bigg|_{z=x_j+\epsilon}.$$
Applying the identity
\[ \frac{\partial \varphi}{\partial x_j} = -\partial_x \varphi + h_j, \]
where
\[ h_j = - \sum_{k \neq j} \frac{\partial \varphi}{\partial x_k} - \sum_{i=1}^m \left( \frac{\partial \varphi}{\partial z_i} + \frac{\partial \varphi}{\partial \bar{z}_i} \right), \]
one has
\[ \frac{1 - \nu_j}{4\pi} \int_{s_j} \kappa_z |dz| \frac{\partial \varphi}{\partial x_j} + \frac{1 - \nu_j}{4\pi} \int_{s_j} \kappa_z |dz| \partial_x \varphi = \frac{1 - \nu_j}{4\pi} \int_{s_j} \kappa_z |dz| h_j, \]
and
\[ \frac{i}{8\pi} \int_{s_j} \left( \bar{\partial} \varphi \frac{d\bar{z}}{dz} - \partial \varphi dz \right) \frac{\partial \varphi}{\partial x_j} = \]
\[ = - \frac{i}{8\pi} \int_{s_j} \left( \bar{\partial} \varphi \frac{d\bar{z}}{dz} - \partial \varphi dz \right) \left( \partial \varphi + \bar{\partial} \varphi \right) + \frac{i}{8\pi} \int_{s_j} \left( \bar{\partial} \varphi \frac{d\bar{z}}{dz} - \partial \varphi dz \right) h_j. \]
Note that the function \( h_j \) is regular for \( z \to x_j \). Taking into account the asymptotic behavior of \( \varphi \) for \( z \to x_j \) one thus gets in the limit \( \epsilon \to 0 \):
\[ \frac{1 - \nu_j}{4\pi} \int_{s_j} \kappa_z |dz| h_j + \frac{i}{8\pi} \int_{s_j} \left( \bar{\partial} \varphi \frac{d\bar{z}}{dz} - \partial \varphi dz \right) h_j = o(1). \]
It follows that up to terms vanishing in the limit \( \epsilon \to 0 \) the first two lines on the r.h.s. of (4.3) yield:
\[ - \frac{i}{8\pi} \int_{s_j} \left( \bar{\partial} \varphi \frac{d\bar{z}}{dz} - \partial \varphi dz \right) \left( \partial \varphi + \bar{\partial} \varphi \right) + \frac{i}{8\pi} \int_{s_j} \left( d\bar{z} - dz \right) \left( \partial \varphi \partial_x \varphi + m \omega^\varphi \right) \]
\[ = \frac{i}{8\pi} \int_{s_j} dz \left( (\partial \varphi)^2 - m \omega^\varphi \right) - \frac{i}{8\pi} \int_{s_j} dz \left( (\bar{\partial} \varphi)^2 - m \omega^\varphi \right) \]
\[ = \frac{i}{4\pi} \int_{s_j} dz \left( \partial^2 \varphi - \partial \bar{\partial} \varphi - 2T_{cl}(z) \right) - \frac{i}{4\pi} \int_{s_j} d\bar{z} \left( \bar{\partial}^2 \varphi - \partial \bar{\partial} \varphi - 2\bar{T}_{cl}(\bar{z}) \right), \]
where we have used the expression (2.8) for the classical energy-momentum tensor and its complex conjugate \( \bar{T}_{cl}(\bar{z}) \) along with the Liouville equation (2.5). Since \( \bar{T}_{cl}(\bar{z}) = T_{cl}(\bar{z}) \) we have
\[ - \frac{i}{2\pi} \int_{s_j} dz \ T_{cl}(z) + \frac{i}{2\pi} \int_{s_j} d\bar{z} \ \bar{T}_{cl}(\bar{z}) = \frac{i}{2\pi} \int_{|z-x_j|=\epsilon} dz \ T_{cl}(z) = -e_j^3. \]
On the other hand
\[ \frac{i}{4\pi} \int_{s_j} dz \left( \partial^2 \varphi - \partial \bar{\partial} \varphi \right) - \frac{i}{4\pi} \int_{s_j} d\bar{z} \left( \bar{\partial}^2 \varphi - \partial \bar{\partial} \varphi \right) \]
\[ = \frac{i}{4\pi} \int_{s_j} (dz \partial + d\bar{z} \bar{\partial}) \left( \partial \varphi - \bar{\partial} \varphi \right) = \frac{1}{4\pi} \int_{s_j} d\theta \frac{\partial}{\partial \theta} \partial_y \varphi \]
\[ = \frac{1}{4\pi} \left( \partial_y \varphi \big|_{x=x_j+\epsilon} - \partial_y \varphi \big|_{x=x_j-\epsilon} \right) = \frac{m_k}{4\pi} \ e^{\frac{2\pi}{\epsilon}} \bigg|_{z=x_j+\epsilon} - \frac{m_k-1}{4\pi} \ e^{\frac{2\pi}{\epsilon}} \bigg|_{z=x_j-\epsilon}, \]
so that one finally gets
\[ D_e[\varphi] = -c_j^B + o(1), \]
and
\[ \frac{\partial S_{\text{cl}}}{\partial x_j} = -c_j^B. \]
Let us note that the proof presented above applies to the bulk singularities as well.

5 Classical solutions

5.1 One bulk elliptic singularity

In the case of an elliptic singularity located at \( z_1 \) on the upper half-plane \( \Im z_1 > 0 \) the energy-momentum takes the form
\[ T(z) = \frac{1 - \xi^2}{4} \left( \frac{1}{(z - z_1)^2} + \frac{1}{(z - \bar{z}_1)^2} - \frac{2}{(z - z_1)(z - \bar{z}_1)} \right), \quad 0 < \xi < 1. \tag{5.1} \]
Since the singularity is elliptic and there are no singularities on the real axis it is more convenient to work with the normalized solutions to the Fuchs equation (2.1) with a diagonal \( SU(1,1) \) monodromy:
\[
\Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{\xi(z_1 - \bar{z}_1)}} (z - z_1)^{-\xi} (z - \bar{z}_1)^{-1+\xi} \\ \frac{1}{\sqrt{\xi(z_1 - \bar{z}_1)}} (z - z_1)^{1+\xi} (z - \bar{z}_1)^{1-\xi} \end{bmatrix}. \tag{5.2} \]
In this setting a real, single valued conformal factor of the hyperbolic geometry with constant scalar curvature \( -2m \) is given by
\[
e^{-\varphi(z, \bar{z})/2} = \sqrt{m/2} \left( e^t \psi_1(z) \overline{\psi_1(z)} - e^{-t} \psi_2(z) \overline{\psi_2(z)} \right) = \sqrt{m/2} \left( \frac{(z - z_1)(z - \bar{z}_1)}{z_1 - \bar{z}_1} \right) \left( e^t \left| \frac{z - z_1}{z - \bar{z}_1} \right|^{-\xi} - e^{-t} \left| \frac{z - z_1}{z - \bar{z}_1} \right|^{\xi} \right), \tag{5.3} \]
where \( t \) is real and \( t \geq 0 \) so that the r.h.s. is non negative on the upper half plane and the real axis. By direct calculations one gets
\[
\partial_y e^{-\varphi/2} \bigg|_R = \sqrt{m/2} \left( e^t + e^{-t} \right) = \sqrt{m} \cosh t = -\frac{m_B}{2}. \]
Thus a regular metric can be constructed if and only if the bulk and boundary cosmological constants satisfy the conditions
\[ m_B < 0, \quad m_B^2 \geq 4m. \]
This in particular means that the geodesic curvature \( \kappa \) of the boundary is bounded from below, \( \kappa \geq m \).
For the solution (5.3) the r.h.s. of the equations (3.3), (3.6), (4.1), and (3.5) can be easily calculated,

\[
\frac{\partial S_{cl}}{\partial \xi} = \log \xi - \log \sqrt{\frac{m}{z}} - t - \xi \log |z_1 - \bar{z}_1|,
\]

\[
\frac{\partial S_{cl}}{\partial \omega_B} = \frac{\xi}{\sqrt{\omega_B^2 - 4}},
\]

\[
\frac{\partial S_{cl}}{\partial z_1} = \frac{1 - \xi^2}{2} \frac{1}{z_1 - \bar{z}_1},
\]

where \(\omega_B = -2 \cosh t\). The solution reads

\[
S_{cl} = \xi \log \xi - \xi - \frac{\xi}{2} \log m + \xi \log 2 - \frac{1 - \xi^2}{2} \log |z_1 - \bar{z}_1| + \text{const.} \quad (5.4)
\]

Let us note in passing that using the form of the elliptic basis (5.2) one gets for the map (3.7)

\[
\rho(z) = \frac{e^{\frac{\pi}{2} \psi_1(z)} - e^{-\frac{\pi}{2} \psi_2(z)}}{e^{\frac{\pi}{2} \psi_1(z)} - e^{-\frac{\pi}{2} \psi_2(z)}} = i \frac{e^{\xi(z - \bar{z}_1) \xi} - (z - z_1) \xi}{e^{\xi(z - \bar{z}_1) \xi} - (z - z_1) \xi},
\]

so that

\[
\rho_1 = \lim_{z \to -\infty} \rho(z) = i \frac{\sinh t + i \sin 2\pi \xi}{\cosh t + \cos 2\pi \xi}, \quad \rho_2 = \lim_{z \to \infty} \rho(z) = i \frac{\sinh t}{\cosh t + 1}.
\]

This immediately gives

\[
\frac{|\rho_{j+1} - \rho_j|}{2\sqrt{3 \rho_j + 1}} = \frac{\sin \pi \xi}{|\sinh t|} = \frac{\sin \pi \xi}{\sqrt{\left(\frac{\omega_B}{2}\right)^2 - 1}}
\]

and from (3.10) we get as above

\[
\frac{\partial S_{cl}}{\partial \omega_B} = \frac{\xi}{\sqrt{\omega_B^2 - 4}},
\]

which confirms formulae (3.9) and (3.10).

We now shall compare (5.4) with the semi-classical limit of the FZZ 1-point function:

\[
\langle V_\alpha(z_1) \rangle = \frac{U(\alpha, \mu_B)}{|z_1 - \bar{z}_1|^{2\Delta_\alpha}},
\]

\[
U(\alpha, \mu_B) = \frac{2}{b} (\pi \mu_\gamma(b^2))^{2 - \frac{\alpha}{2}} \Gamma(2b\alpha - b^2) \Gamma \left( \frac{2\alpha}{b} - \frac{1}{b^2} - 1 \right) \cosh(2\alpha - Q) \pi s, \quad (5.5)
\]

where \(s\) is defined by the relation

\[
\cosh^2 \pi bs = \frac{\mu_\gamma^2}{\mu} \sin \pi b^2. \quad (5.6)
\]

\[\text{In the case under consideration the classical action could be also calculated by explicit integration in (3.2) (see [25]).}\]
In our notation
\[ \mu = \frac{m}{4\pi b^2}, \quad \mu_B = \frac{m_B}{4\pi b^2}, \quad \alpha = \frac{Q}{2}(1 - \xi), \quad Q = b + \frac{1}{b}, \quad (5.7) \]
and
\[ U(\xi, m_B) = \frac{2}{b^2} (\pi \mu \gamma(b^2))^{\frac{1}{2}(1 + \frac{1}{b^2})} \Gamma(b^2(1 - \xi) + 1 - \xi) \Gamma(-(1 + \frac{1}{b^2})\xi) \cosh(-(b + \frac{1}{b})\xi \pi s). \]

In the limit \( b \to 0 \) the relation (5.6) yields \( \pi bs = t \), hence
\[ \frac{2}{b^2} (\pi \mu \gamma(b^2))^{\frac{1}{2}(1 + \frac{1}{b^2})} \Gamma(b^2(1 - \xi) + 1 - \xi) \cosh(-(b + \frac{1}{b})\xi \pi s) \approx e^{-\frac{1}{b^2} \left[ -\xi + \xi \log \xi - \xi \log \frac{b}{\pi} \right]}. \]

Due to the poles of the gamma function along the negative real axis the \( b \to 0 \) asymptotic of the term \( \Gamma(-1 + \frac{1}{b^2})\xi \) is more subtle. Within the path integral approach the poles in (5.5) arise due to the integration over the zero mode of the Liouville field. If the classical solution exists, one should not expect any pole structure in the quasi-classical limit. In order to show that the poles can be regarded as a sub-leading correction one may use the formula
\[ \Gamma(-x) = -\frac{\pi}{x \Gamma(x) \sin \pi x}, \]
along with the Stirling asymptotic expansion
\[ x \Gamma(x) \approx e^{-x + \left( x + \frac{1}{2} \right) \log x}. \]

In the limit \( b \to 0 \) one obtains:
\[ \Gamma(-(1 + \frac{1}{b^2})\xi) \approx e^{\frac{1}{b^2} \xi - \left( \frac{1}{2} \xi \log \frac{1}{b^2} \right) \log \frac{1}{b^2} \xi - \log \left( \frac{1}{2} \sin \frac{\pi}{b^2} \xi \right)} \]
\[ = e^{-\frac{1}{b^2} \left[ -\xi + \xi \log \xi - \xi \log \frac{b^2 + \frac{1}{b^2} \log \frac{1}{b^2} \xi + b^2 \log \left( \frac{1}{2} \sin \frac{\pi}{b^2} \xi \right) }{\pi} \right]}. \]

Keeping only the leading terms one thus have
\[ U(\xi, m_B) \approx e^{-\frac{1}{b^2} \left[ -\frac{1}{4} \log m + \xi \log 2 - \xi \log \xi - \xi \right]}, \]
in perfect agreement with the classical action (5.4).

### 5.2 One bulk hyperbolic singularity

Hyperbolic weight corresponds to the energy-momentum of the form
\[ T(z) = \frac{1 + \lambda^2}{4} \left( \frac{1}{(z - z_1)^2} + \frac{1}{(z - \bar{z}_1)^2} - \frac{2}{(z - z_1)(z - \bar{z}_1)} \right). \quad (5.8) \]

Repeating (with obvious modifications) the calculations from the previous subsection one gets the metric
\[ e^{-\varphi(z, \bar{z})/2} = \sqrt{m} \left| \frac{(z - z_1)(z - \bar{z}_1)}{z_1 - \bar{z}_1} \right| \sin \left[ \lambda \log \left| \frac{z - z_1}{z - \bar{z}_1} \right| - t \right], \quad (5.9) \]
where

$$\cos t = -\frac{\omega_b}{2} \equiv -\frac{m_B}{2\sqrt{m}}. \tag{5.10}$$

The metric can be constructed only if $|\omega_b| \leq 2$. In the coordinates

$$w = \tau + i\sigma = \log \frac{z - z_1}{z - \bar{z}_1}$$

it takes the form

$$e^{-\varphi(w,\bar{w})/2} = \sqrt{m} \sin(\lambda \tau - t). \tag{5.11}$$

The conformal factor is singular along the lines

$$\lambda \tau = t + \pi k, \quad k \in \mathbb{Z},$$

and the metric $e^{\varphi(w,\bar{w})} |dw|^2$ has closed geodesics located at

$$\lambda \tau = t + \frac{\pi}{2}(2k + 1), \quad k \in \mathbb{Z}.$$

For positive $\omega_b$ there exists a solution of (5.10) satisfying $-\pi < t < -\frac{\pi}{2}$. The metric between the real axis $\tau = 0$ and the geodesic corresponding to $k = 0$,

$$\lambda \tau_g = \frac{\pi}{2} + t < 0,$$

is then regular. As a final step of the construction of the $C^1$ metric on the upper half-plane we shall “fill in” the hole $\tau < \tau_g$ with a flat metric determined by

$$e^{-\varphi_0(w,\bar{w})/2} = \sqrt{m} \lambda$$

or, in the $z$ coordinates,

$$e^{-\varphi_0(z,\bar{z})/2} = \sqrt{m} \lambda \left| \frac{(z - z_1)(z - \bar{z}_1)}{z_1 - \bar{z}_1} \right|.$$

The classical Liouville action in the presence of hyperbolic singularities [31, 38] is constructed as a sum of the standard Liouville action functional calculated on the conformal factor of the hyperbolic metric in the region “between the holes” and the actions for “holes” around each hyperbolic singularity. In our case the first contribution is

$$S_1(m, m_B, \lambda, z_1) = \frac{1}{4\pi} \lim_{R \to \infty} \left[ \int_{\mathcal{M}_R} d^2z \left[ |\partial \varphi|^2 + me^{\varphi} \right] + m_B \int_{-R}^{R} dy e^{\varphi/2} + \int_{\partial \mathcal{M}_R} \kappa_z |dz| \varphi \right], \tag{5.12}$$

where $\mathcal{M}_R$ is a part of the upper half plane outside the hole, $\log \left| \frac{z - z_1}{z - \bar{z}_1} \right| > \frac{1}{\lambda} \left( \frac{\pi}{2} + t \right)$, bounded by the semi-circle of radius $R$. The second one is a regularized Liouville action functional, calculated for the flat metric $\varphi_0$ on the hole around $z = z_1$ with a small disc of radius $\epsilon$ removed,

$$H_\epsilon = \left\{ z \in \mathbb{H} : \log \left| \frac{z - z_1}{z - \bar{z}_1} \right| < \frac{1}{\lambda} \left( \frac{\pi}{2} + t \right) \land |z - z_1| < \epsilon \right\}.$$
It reads
\[ S_2(m, m_B, \lambda, z_1) = \lim_{\epsilon \to 0} S_{2, \epsilon}(m, m_B, \lambda, z_1), \] (5.13)
\[ S_{2, \epsilon}(m, m_B, \lambda, z_1) = \frac{1}{4\pi} \int_{H_c} d^2 z \left[ |\partial \varphi_0|^2 + me^{\varphi_0} \right] + \frac{1}{4\pi} \int_{\partial H_c} \kappa_z |dz| \varphi_0 + (\lambda^2 - 1) \log \epsilon. \]

Shifting
\[ \varphi = \tilde{\varphi} - \log m, \quad \varphi_0 = \tilde{\varphi}_0 - \log m, \]
one checks that the classical action depends on \( m \) only through \( \omega_B \),
\[ S_{cl} = S \left( \frac{m_B}{\sqrt{m}}, \lambda, z_1 \right) \equiv S (\omega_B, \lambda, z_1). \]

From the Polyakov conjecture and eq. (5.8) one gets
\[ \frac{\partial S_{cl}}{\partial z_1} = \frac{1 + \lambda^2}{2} \frac{1}{z_1 - \tilde{z}_1}. \] (5.14)

One also has (using the form of functionals (5.12) and (5.13))
\[ \frac{\partial S_{cl}}{\partial \omega_B} = \frac{1}{4\pi} \int_{\mathbb{R}} dy e^{\tilde{\varphi}/2} = \frac{\lambda}{\sqrt{4 - \omega_B^2}}, \] (5.15)

and finally
\[ \frac{\partial S_{cl}}{\partial \lambda} = \lim_{\epsilon \to 0} \left[ \frac{1}{4\pi} \int_{H_c} d^2 z \frac{\partial \varphi_0}{\partial \lambda} e^{\tilde{\varphi}_0} + \lambda \log \epsilon \right] \]
\[ = \lim_{\epsilon \to 0} \left[ \int_{\tau_0}^{\tau} d\tau \frac{1}{2} \frac{\partial \lambda^2}{\partial \lambda} + \lambda \log \epsilon \right] = \left( t + \frac{\pi}{2} \right) + \lambda \log |z_1 - \tilde{z}_1|, \] (5.16)

where the change of the integration variables from \( z \) to \( w \) and the fact that \( |z - \xi| = \epsilon \) corresponds to \( \tau = \log \epsilon - \log |z_1 - \tilde{z}_1| \) have been used.

Integrating the equations (5.14), (5.15) and (5.16) we get
\[ S(m, m_B, \lambda, z_1) = \frac{1 + \lambda^2}{4} \log |z_1 - \tilde{z}_1|^2 + \lambda \left( t + \frac{\pi}{2} \right) + \text{const}, \] (5.17)

where the constant is independent of \( m, m_B, \lambda, z_1 \).

For hyperbolic weights \( \Delta = \alpha(Q - \alpha) > \frac{Q^2}{4} \), i.e. for \( \alpha \) of the form
\[ \alpha = \frac{Q}{2} (1 + i\lambda), \quad \lambda \in \mathbb{R}, \]
the Liouville one-point coupling constant \( U(\alpha) \) given by (5.5) is complex. Let us write
\[ U(\alpha) = \Phi(\alpha) U_s(\alpha) \]
where $\Phi$ is the phase and $U_s$ the modulus$^4$ of $U$. The phase $\Phi$ coincides with a square root of the Liouville reflection amplitude,

$$\Phi^2(\alpha) = S_L(\alpha),$$

$$S_L(\alpha) = (\pi \mu \gamma (b^2))^{\frac{2\alpha - Q}{b}} \frac{\Gamma(-(2\alpha - Q)/b)\Gamma(1 - b(2\alpha - Q))}{\Gamma((2\alpha - Q)/b)\Gamma(1 + b(2\alpha - Q))},$$

and

$$U_s(\alpha) = 2 \cosh \left[ \frac{(2\alpha - Q)\pi \sigma}{b} \right] \sqrt{\Gamma(1 + (2\alpha - Q)b)\Gamma(1 - (2\alpha - Q)b) \Gamma \left( \frac{2\alpha - Q}{b} \right) \Gamma \left( \frac{Q - 2\alpha}{b} \right)}.$$

Using (5.6), (5.7) and the fact that $t < 0$, we get for $b \to 0$

$$\frac{U_s(\alpha)}{|z_1 - \bar{z}_1|^{2\Delta_{\alpha}}} \sim \exp \left\{ -\frac{1}{b^2} \left( \frac{1 + \lambda^2}{4} \log |z_1 - \bar{z}_1|^2 + \lambda \left( t + \frac{\pi}{2} \right) \right) \right\},$$

in perfect agreement with the classical action (5.17) again.

### 5.3 Two boundary elliptic singularities

In the case of two singularities the conformal weights must be the same, $\nu_1 = \nu_2 = \nu$. Using the $SL(2, \mathbb{R})$ symmetry one can place them at $x_1 = 0$ and $x_2 = \infty$. This corresponds to the following energy-momentum tensor

$$T(z) = \frac{1 - \nu^2}{4} \frac{1}{z^2}.$$

Normalized solutions, regular and real along the positive and the negative semi-axes, are given by

$$\Psi_1 = \begin{bmatrix} \psi_1^- (z) \\ \psi_1^+ (z) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{\nu}} \frac{1 + \nu}{2} \\ \frac{1}{\sqrt{\nu}} \frac{1 - \nu}{2} \end{bmatrix}, \quad \Psi_2 = \begin{bmatrix} \psi_2^- (z) \\ \psi_2^+ (z) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{\nu}} (-z)^{\frac{1 + \nu}{2}} \\ \frac{1}{\sqrt{\nu}} (-z)^{\frac{1 - \nu}{2}} \end{bmatrix}, \quad (5.18)$$

respectively. They are related on the upper half plane by

$$\Psi_1 = \begin{bmatrix} 0 \\ e^{\pi \nu / (2i)} \frac{e^{\pi (1 - \nu) / 2}}{e^{\pi (1 + \nu) / 2}} \end{bmatrix} \Psi_2. \quad (5.19)$$

In terms of $\Psi_1$ the solution to the Liouville equation reads

$$e^{-\varphi / 2} = \frac{\sqrt{m}}{2i} \Psi_1^T (z) \Sigma M \Psi_1 (z), \quad (5.20)$$

where the matrix

$$M = \begin{bmatrix} a & i\beta \\ i\gamma & a \end{bmatrix}, \quad |a|^2 + \beta \gamma = 1, \quad \gamma, \beta \in \mathbb{R}, \quad a \in \mathbb{C},$$

$^4$The subscript $s$ is meant to remind that $U_s(\alpha)$ is symmetric under reflection $\alpha \to Q - \alpha$.\]
can be chosen such that the r.h.s. of (5.20) is positive on the upper half plane. This implies in particular that $\gamma$ and $-\beta$ are positive. The boundary conditions

$$\partial_y \varphi|_{y=0} = m_1 e^{c/2} \text{ for } x > 0, \quad \partial_y \varphi|_{y=0} = m_2 e^{c/2} \text{ for } x < 0,$$

(5.21)

imply

$$a + \bar{a} = \frac{m_1}{\sqrt{m}} = \omega_1, \quad \alpha e^{i\pi \nu} + \bar{\alpha} e^{-i\pi \nu} = -\frac{m_2}{\sqrt{m}} = -\omega_2.$$

Solving for $a$ one gets

$$a = \frac{-\omega_2 - e^{-i\pi \nu} \omega_1}{2i \sin \pi \nu}.$$

From the Gauss-Bonnet theorem one may expect that the geodesic curvature of boundary components should be positive. It can be easily checked that this is really so in the symmetric case $\omega_1 = \omega_2 = \omega$, when the hyperbolic metric exists if the condition

$$-\omega > 2 \sin \frac{1}{2} \pi \nu$$

is satisfied. We assume in the following that the boundary geodesic curvatures are positive and such that the function

$$e^{-\varphi/2} = \frac{\sqrt{m}}{2\nu} |z| \left[ |\gamma| - \nu - \beta |\nu - 2 \Re \alpha e^{i\nu \theta} \right],$$

(5.22)

is positive for $z = |z| e^{i\theta}$ in the upper half plane. Introducing parametrization

$$\omega_1 = -2 \cosh t_1, \quad \omega_2 = -2 \cosh t_2, \quad t_1, t_2 \geq 0,$$

we have for the solution (5.22)

$$\frac{\partial S_{cl}}{\partial \nu} = \lim_{\epsilon \to 0} \left[ -\frac{1}{4\pi} \int_{|z|=\epsilon, 3z>0} \kappa |dz| \varphi + \frac{1-\nu}{2} \log \epsilon + \frac{1}{4\pi} \int_{|z|>\frac{1}{2}, 3z>0} \kappa |dz| \varphi - \frac{1+\nu}{2} \log \epsilon \right]$$

$$= \log \nu - \log \sqrt{m} + \log \sin \pi \nu$$

$$-\frac{1}{2} \log \sin \left( \frac{\pi (1-\nu) + i(t_1 + t_2)}{2} \right) - \frac{1}{2} \log \sin \left( \frac{\pi (1-\nu) - i(t_1 + t_2)}{2} \right),$$

$$\frac{\partial S_{cl}}{\partial \omega_1} = \frac{\sqrt{m}}{4\pi} \int_0^\infty dx e^{\frac{x}{2\pi}} \frac{i}{2\pi} \log \left[ \frac{\sin \left( \frac{\pi (1-\nu) + i(t_1 + t_2)}{2} \right)}{\sin \left( \frac{\pi (1-\nu) - i(t_1 + t_2)}{2} \right)} \right] \frac{\partial t_1}{\partial \omega_1},$$

$$\frac{\partial S_{cl}}{\partial \omega_2} = \frac{\sqrt{m}}{4\pi} \int_{-\infty}^0 dx e^{\frac{x}{2\pi}} \frac{i}{2\pi} \log \left[ \frac{\sin \left( \frac{\pi (1-\nu) + i(t_1 + t_2)}{2} \right)}{\sin \left( \frac{\pi (1-\nu) - i(t_1 + t_2)}{2} \right)} \right] \frac{\partial t_2}{\partial \omega_2}.$$
Integrating these equations one gets (up to a constant)

$$S_{cl} = \nu \left( \log \nu - \frac{1}{2} \log m - 1 \right) + s(\nu) + \sum_{\tau_1 = \pm} \sum_{\tau_2 = \pm} s \left( \frac{\pi(1 - \nu) + i(\tau_1 t_1 + \tau_2 t_2)}{2} \right), \quad (5.23)$$

where

$$s(x) \overset{\text{def}}{=} \frac{1}{\pi} \int_{\pi/2}^{x} dy \log \sin y.$$ 

The quantum boundary two-point coupling constant has the form [3]:

$$d(\beta|s_1, s_2) = \frac{\left[ \pi \mu \gamma \left( b^2 \right)^{(Q-2\beta)/2b} \right] \Gamma_b(2\beta - Q) \Gamma_b^{-1}(Q - 2\beta)}{S_b \left( \beta + i \frac{z_1 + z_2}{2} \right) S_b \left( \beta + i \frac{z_1 - z_2}{2} \right) S_b \left( \beta - i \frac{z_1 + z_2}{2} \right) S_b \left( \beta - i \frac{z_1 - z_2}{2} \right)}, \quad (5.24)$$

with appropriate counterparts of relations (5.6), (5.7) assumed. Taking into account

$$t_i = \pi b s_i, \quad \beta \sim \frac{1}{2b}(1 + \nu),$$

and the asymptotic behavior

$$\log S_b(x) \sim \frac{1}{b^2} s(\pi bx) + \frac{\log 2}{b^2} \left( xb - \frac{1}{2} \right), \quad (5.25)$$

one can check that the semiclassical asymptotic of (5.24) is given by the classical action (5.23) indeed.

### 5.4 One bulk, one boundary elliptic singularities

For $T(z)$ having a single pole at the real axis and a single pole in the interior of the upper half plane one can always choose the “boundary” pole to be located at $z = \infty$. The second pole we shall take at $z = z_1, \ Re z_1 > 0$. The energy-momentum tensor takes the form

$$T(z) = \frac{\delta}{(z - z_1)^2} + \frac{\delta}{(z - \bar{z}_1)^2} + \frac{\delta_b - 2\delta}{(z - z_1)(z - \bar{z}_1)} \quad (5.26)$$

with elliptic weights

$$\delta = \frac{1 - \xi^2}{2}, \quad \delta_b = \frac{1 - \nu^2}{2}.$$ 

A normalized basis in the space of solutions of (2.1), with diagonal SU(1,1) monodromy around $z = z_1$, can be chosen in the form

$$\psi_1(z) = \frac{1}{\sqrt{\xi(z - z_1)}(z - z_1)^{1/2}(z - \bar{z}_1)^{1/2}} F_1 \left( \frac{1-\nu}{2}, \frac{1+\nu}{2}, 1 - \xi, \frac{z_1 - z}{z_1 - \bar{z}_1} \right),$$

$$\psi_2(z) = \frac{1}{\sqrt{\xi(z - z_1)}(z - z_1)^{1/2}(z - \bar{z}_1)^{1/2}} F_1 \left( \frac{1-\nu}{2}, \frac{1+\nu}{2}, 1 + \xi, \frac{z_1 - z}{z_1 - \bar{z}_1} \right). \quad (5.27)$$

The functions $\psi_{1,2}(z)$ are analytic in the vicinity of the real axis (the cuts between the branching points $z = z_1, z = \infty$ and $z = \bar{z}_1$ can be chosen such that they do not intersect
the real axis). A real, single-valued around $z = z_1$ solution to the Liouville equation (2.5) can be expressed through $\psi_1(z), \psi_2(z)$ as

$$e^{-\varphi/2} = \frac{m}{2} (a|\psi_1(z)|^2 - a^{-1}|\psi_2(z)|^2),$$

(5.28)

with a (real) constant $a$ to be determined from (2.6).

To this end it is convenient to express $\psi_i(z)$ in terms of $\psi_i(\bar{z})$. Using the formulae for analytic continuation of the hypergeometric functions one gets:

$$\begin{bmatrix} \psi_1(z) \\ \psi_2(z) \end{bmatrix} = C \begin{bmatrix} \psi_1(\bar{z}) \\ \psi_2(\bar{z}) \end{bmatrix},$$

(5.29)

where

$$C = i \left[ \frac{\Gamma(1-\xi)\Gamma(\xi)}{\Gamma(1+\nu)\Gamma(\frac{1+\nu}{2}+\xi)} \frac{\Gamma(1-\xi)\Gamma(-\xi)}{\Gamma(1+\nu)\Gamma(\frac{1+\nu}{2}-\xi)} \right].$$

(5.30)

Hence

$$e^{-\varphi/2} = \frac{m}{2} \Psi(z) \cdot A \cdot C \cdot \Psi(\bar{z}),$$

(5.31)

where $A = \text{diag}(a, -a^{-1})$. The boundary condition

$$\partial_y e^{-\varphi/2} \bigg|_{y=0} = -\frac{m_B}{2} \equiv -\frac{\sqrt{m}}{2} \omega_B$$

yields

$$\omega_B = -a \frac{\Gamma(1-\xi)\Gamma(-\xi)}{\Gamma(1-\xi)\Gamma(1+\xi)} - a^{-1} \frac{\Gamma(1+\xi)\Gamma(\xi)}{\Gamma(1+\xi)} \frac{\Gamma(1-\xi)\Gamma(-\xi)}{\Gamma(1+\xi)}.$$  

(5.32)

Solving with respect to $a$ one gets

$$a_{\pm} = \frac{\Gamma\left(\frac{1-\nu}{2} - \xi\right) \Gamma\left(\frac{1+\nu}{2} - \xi\right)}{\pi \Gamma(1-\xi)} \Gamma(1+\xi)$$

$$\times \left(\frac{\omega_B}{2} \sin \pi \xi \pm \sqrt{\cos^2 \frac{\pi \nu}{2} + \left(\frac{\omega_B^2}{4} - 1\right) \sin^2 \pi \xi}\right).$$

(5.33)

Let us note that the change of sign in $a$ is equivalent to the change of sign of $\omega_B$ and the change of sign of the r.h.s. of (5.28). It does not lead therefore to any new solutions of the Liouville equation. With no loss of generality we can then work with $a = a_+$. It should be stressed that not for all parameters $\omega_B, \xi, \nu$, for which $a$ is real, the formula (5.28) yields a regular solution for the Liouville equation. Indeed, the r.h.s. of (5.28) may change sign on the upper half plane and the zero lines appearing in this situation give rise to singular lines of the corresponding hyperbolic geometry. Even in the simple situation at hand, the problem of determining the ranges of parameters for which a regular solution exists and its classical action is well defined is rather involved and we are not aware of any compete solution to it. In the following we simply assume that $\omega_B, \xi, \nu$ are such that a regular solution does exist.
Taking into account the asymptotic behavior of the solution $\Psi$ (5.27) for $z \to z_1$ and (5.33) one gets from (3.3):

$$\frac{\partial S_{cl}}{\partial \xi} = -\frac{1}{2} \log m - \xi \log |z_1 - \bar{z}_1| + \log 2$$

$$- \log \left[ \frac{\Gamma(\xi) \Gamma \left( \frac{1+\nu}{2} - \xi \right)}{\Gamma(1-\xi) \Gamma \left( \frac{1+\nu}{2} + \xi \right)} \right] + \log \cos \left( \frac{\nu}{2} + \xi \right)$$

$$- \log \left[ \frac{\omega_\nu}{2} \sin \pi \xi + \sqrt{\cos^2 \frac{\pi \nu}{2} + \left( \frac{\omega_\nu^2}{4} - 1 \right) \sin^2 \pi \xi} \right].$$

Using the asymptotic behavior of the hypergeometric function for large arguments one finds for $z \to \infty$:

$$\psi_1(z) \sim \frac{(z_1 - \bar{z}_1)^{\frac{-1}{2}} \Gamma(\nu) \Gamma(1 - \xi)}{\sqrt{\xi} \Gamma \left( \frac{1+\nu}{2} \right) \Gamma \left( \frac{1+\nu}{2} - \xi \right)} (z - z_1)^{\frac{\nu-1}{2}} (z - \bar{z}_1)^{\frac{\nu+1}{2}},$$

$$\psi_2(z) \sim \frac{(z_1 - \bar{z}_1)^{\frac{-1}{2}} \Gamma(\nu) \Gamma(1 + \xi)}{\sqrt{\xi} \Gamma \left( \frac{1+\nu}{2} \right) \Gamma \left( \frac{1+\nu}{2} + \xi \right)} (z - z_1)^{\frac{\nu+1}{2}} (z - \bar{z}_1)^{\frac{\nu-1}{2}},$$

and

$$\frac{\partial S_{cl}}{\partial \nu} = -\frac{1}{4} \log m + \nu \log |z_1 - \bar{z}_1| + \log \left[ \frac{\Gamma \left( \frac{1+\nu}{2} \right)}{\Gamma(\nu)} \right] - \frac{1}{2} \log \pi$$

$$+ \frac{1}{2} \log \left[ \frac{\Gamma \left( \frac{1+\nu}{2} - \xi \right) \Gamma \left( \frac{1+\nu}{2} + \xi \right)}{\Gamma(1-\xi) \Gamma \left( \frac{1+\nu}{2} + \xi \right)} \right] + \frac{1}{2} \log 2$$

$$- \frac{1}{2} \log \left[ \frac{B_+}{\sin \pi \xi \cos \left( \frac{\nu}{2} + \xi \right)} - \frac{B_-}{\sin \pi \xi \cos \left( \frac{\nu}{2} - \xi \right)} \right],$$

where

$$B_\pm = \pm \frac{\omega_\nu}{2} \sin \pi \xi + \sqrt{\cos^2 \frac{\pi \nu}{2} + \left( \frac{\omega_\nu^2}{4} - 1 \right) \sin^2 \pi \xi}.$$
Calculating the limits along the real axis

\[
\rho_1 = \lim_{x \to -\infty} \rho(x) = i \frac{\sinh r + i \sin 2\pi \xi}{\cosh r + \cos 2\pi \xi}, \quad \rho_2 = \lim_{x \to \infty} \rho(x) = i \frac{\sinh r}{\cosh r + 1},
\]

one obtains

\[
\frac{|\rho_{j+1} - \rho_j|}{2\sqrt{3} \rho_{j+1} \rho_j} = \frac{\sin \pi \xi}{\sinh r}.
\]

Hence for \( \omega_B > 2 \) formula (3.10) implies

\[
\frac{\partial S_{cl}}{\partial \omega_B} = \frac{1}{\pi} \sqrt{\frac{\omega_B^2 - 4}{\omega_B^2}} \arcsin \left[ \frac{\sin \pi \xi}{\sinh r \sqrt{\left(\frac{\omega_B}{2}\right)^2 - 1}} \right],
\]
or

\[
\frac{\partial S_{cl}}{\partial t} = \frac{1}{\pi} \arcsin \left[ \frac{\sinh t \sin \pi \xi}{\sinh r} \right], \quad (5.37)
\]

where \( \omega_B = 2 \cosh t \).

Checking integrability conditions or direct integration of the equations (5.34), (5.36) and (5.37) is rather involved and not especially illuminating. We shall check instead that (5.34), (5.36) and (5.37) coincide with the corresponding derivatives of the classical action obtained from the classical limit of the exact quantum expression.

The bulk-boundary correlation function for the location of the bulk operator at \( z = z_1 \) and the boundary operator at \( z = \infty \) is given by [6]:

\[
\langle V_{\alpha}(z_1) B_{ss}^{\beta}(\infty) \rangle = \frac{P(\alpha, \beta, |s|) I(\alpha, \beta|s)}{|z_1 - z_1|^{2\Delta_\alpha - \Delta_\beta}},
\]

\[
P(\alpha, \beta|s) = -2\pi i \left[ \pi \mu \gamma(b^2) b^{2 - 2|s|} \right] \frac{1}{\Gamma(Q - 2\alpha - \beta)} \times \frac{\Gamma_b(Q - \beta) \Gamma_b(2\alpha - \beta) \Gamma_b(2Q - 2\alpha - \beta)}{\Gamma_b(Q) \Gamma_b(Q - 2\beta) \Gamma_b(Q - 2\alpha) \Gamma_b(2\alpha)},
\]

\[
I(\alpha, \beta|s) = \int_{i\mathbb{R}} du e^{-2\pi i s u} \frac{S_b(u + \beta/2 + \alpha - Q/2) S_b(u + \beta/2 - \alpha + Q/2)}{S_b(u - \beta/2 - \alpha + 3Q/2) S_b(u - \beta/2 + \alpha + Q/2)},
\]

where the relations (5.6) and (5.7) are still assumed.

Using the asymptotic of the Barnes gamma function

\[
\log \Gamma_b \left( \frac{x}{b} \right) \sim - \frac{1}{b^2} \left[ g(x) + \frac{1}{2} \left( x - \frac{1}{2} \right) \log 2\pi + \frac{1}{2} \left( x - \frac{1}{2} \right)^2 \log b \right],
\]

where

\[
g(x) = \int_{\frac{1}{2}}^x dy \log \Gamma(y),
\]

one obtains

\[
\log P(\alpha, \beta|s) \sim \frac{1}{\nu} \left[ -\frac{1}{2} \left( \frac{1}{2} - \frac{\nu}{2} - \xi \right) \log m + \left( \frac{1}{2} - \frac{\nu}{2} - \xi \right) \log 2 + \nu \log 2\pi \\
-3g(1/2) + g(\nu) + g(\xi) + g(1 - \xi) - g(1 + \nu) + \xi \right].
\]
Rescaling the integration variable \( u \rightarrow y = bu \) and using (5.25) one has

\[
I(\alpha, \beta|s) \sim \frac{1}{b} \int_{\mathbb{R}} dy \exp \left\{ \frac{1}{b^2} f(y, t, \xi, \nu) \right\},
\]

where

\[
f(y, t, \xi, \nu) = -(1 + \nu) \log 2 - 2ty + \frac{s}{4} \left( \pi y + \frac{\pi}{4} \right) (1 - \nu - 2\xi) + \frac{s}{4} \left( \pi y + \frac{\pi}{4} \right) (1 - \nu + 2\xi) - s \left( \pi y + \frac{\pi}{4} \right) (1 + \nu + 2\xi) + \frac{\pi}{2}.
\]

This integral can be approximated by its saddle point value. The saddle point equation

\[
e^{2t} = \frac{\sin \left( \pi y + \frac{\pi}{4} (1 - \nu - 2\xi) \right) \sin \left( \pi y + \frac{\pi}{4} (1 - \nu + 2\xi) \right)}{\cos \left( \pi y + \frac{\pi}{4} (1 + \nu - 2\xi) \right) \cos \left( \pi y + \frac{\pi}{4} (1 + \nu + 2\xi) \right)} = \frac{\cos \pi \xi + \sin \left( 2\pi y - \frac{\pi \nu}{2} \right)}{\cos \pi \xi - \sin \left( 2\pi y + \frac{\pi \nu}{2} \right)}
\]

yields two solutions

\[
\cos 2\pi y_{s,\pm} = \frac{1}{\cosh^2 t - \sin^2 \frac{\pi \nu}{2}} \left[ \sinh^2 t \cos \pi \xi \sin \frac{\pi \nu}{2} \right.
\]

\[
\pm \cosh t \cos \frac{\pi \nu}{2} \sqrt{\cos^2 \frac{\pi \nu}{2} + \left( \frac{\omega^2}{4} - 1 \right) \sin^2 \pi \xi} \right].
\]

The appropriate solution could be in principle selected by a careful analysis of the position of the contour with respect to poles located on the real axis. In the present calculations we have chosen \( y_s = y_{s+} \) on the basis of numerical checks of the final result instead.

The asymptotic takes the form

\[
\log I(\alpha, \beta|s) \sim \frac{1}{b^2} f(y_s(t, \xi, \nu), t, \xi, \nu),
\]

where \( y_s(t, \xi, \nu) = y_{s+}(t, \xi, \nu) \). The classical action calculated from the classical limit of the quantum expression

\[
\langle V_\alpha(z_1) B^{ss}_B(\infty) \rangle \xrightarrow{b \to 0} e^{-\frac{1}{b^2} S_{cl}}
\]

reads

\[
S_{cl} = \left( \frac{1 + \nu}{2} - \xi \right) \log m + \left( \frac{1}{4} + \frac{\nu^2}{4} - \frac{\xi^2}{2} \right) \log |z_1 - \bar{z}_1| - \left( \frac{1 + \nu}{2} - \xi \right) \log 2
\]

\[
- \nu \log \pi + 3g \left( \frac{1 + \nu}{2} \right) - g \left( \frac{1 + \nu}{2} \right) - g \left( \nu \right)
\]

\[
+ g \left( \frac{1 + \nu}{2} - \xi \right) + g \left( \frac{1 + \nu}{2} + \xi \right) + 2ty_s
\]

\[
- s \left( \pi y_s + \frac{\pi}{4} (1 - \nu - 2\xi) \right) - s \left( \pi y_s + \frac{\pi}{4} (1 - \nu + 2\xi) \right)
\]

\[
- s \left( \pi y_s + \frac{\pi}{4} (1 + \nu - 2\xi) + \frac{\pi}{2} \right) - s \left( \pi y_s + \frac{\pi}{4} (1 + \nu + 2\xi) + \frac{\pi}{2} \right).
\]
and therefore
\[
\frac{\partial S_{\text{cl}}}{\partial \xi} = \frac{1}{2} \log m - \xi \log |z_1 - \bar{z}_1| + \log 2 - \log \left[ \frac{\Gamma(\xi) \Gamma\left( \frac{1+\nu}{2} - \xi \right)}{\Gamma(1-\xi) \Gamma\left( \frac{1+\nu}{2} + \xi \right)} \right] + \frac{1}{2} \log \left[ \frac{\cos 2\pi y_s - \sin (\pi \xi + \frac{\pi \nu}{2})}{\cos 2\pi y_s + \sin (\pi \xi - \frac{\pi \nu}{2})} \right],
\]
\[
\frac{\partial S_{\text{cl}}^{(1)}}{\partial \nu} = -\frac{1}{4} \log m + \frac{\nu}{2} \log |z_1 - \bar{z}_1| + \log \left[ \frac{\Gamma\left( \frac{1+\nu}{2} \right)}{\Gamma(\nu)} \right] - \frac{1}{2} \log \pi + \frac{1}{4} \log \left[ \frac{\cos 2\pi y_s - \sin (\pi \xi + \frac{\pi \nu}{2})}{\cos 2\pi y_s + \sin (\pi \xi - \frac{\pi \nu}{2})} \right],
\]
\[
\frac{\partial S_{\text{cl}}^{(1)}}{\partial t} = 2y_s = \frac{1}{\pi} \arcsin \sqrt{1 - \cos^2(2\pi y_s)}.
\]

These equations have to be compared with the equations (5.34), (5.36) and (5.37). We have checked the corresponding equalities by Mathematica 5.2 obtaining a perfect agreement in all three cases.

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**Appendix**

We shall calculate the length of the \( j \)-th boundary component
\[
\ell_j = \int_{L_j} dx \ e^{\tilde{\varphi}(x, \bar{z})/2} = \int_{\rho(L_j)} \frac{|d\rho|}{\Im \rho},
\]
in terms of the endpoints \( \rho_j = \lim_{x \to x_j^+} \rho(x), \rho_{j+1} = \lim_{x \to x_{j+1}^-} \rho(x) \). For arbitrary two points \( \rho_j, \rho_{j+1} \in \mathbb{H} \) one can always find an \( SL(2, \mathbb{R}) \) transformation \( w(\rho) \) such that
\[
\Re w(\rho_{j+1}) = -\Re w(\rho_j) > 0, \quad \Im w(\rho_{j+1}) = \Im w(\rho_j).
\]

It is then sufficient to calculate the hyperbolic length of the curve \( w \circ \rho(L_j) \) connecting the points \( q_j = w(\rho_j), \ q_{j+1} = w(\rho_{j+1}) \). Let us note that the sign of the boundary geodesic curvature depends on the location of \( \rho(M) \) with respect to \( \rho(L_j) \), so one can assume \( \omega_j > 0 \) in the following.
For $0 < \omega_j \leq 2$ there are always two curves with the geodesic curvature $\frac{\omega_j}{2}$ connecting points $q_j, q_{j+1}$ (see Fig.1). These are arcs of two circles with their centers on the imaginary axis and their radii determined by the condition that the Euclidean distance of the center from the real axis equals $\frac{\omega_j}{2}R$:

$$R_{\pm} = \frac{2 \Im q_j}{4 - \omega_j^2} \left( \pm \omega_j + \sqrt{(4 - \omega_j^2)\beta^2 + 4} \right),$$

where $\beta = \beta(q_j, q_{j+1})$ and

$$\beta(z, w) = \frac{|z - w|}{2\sqrt{3} |z| |w|}$$

is an $SL(2, \mathbb{R})$-invariant.

![Diagram](image_url)

**Fig. 1** The geometry involved in the determination of the boundary length, $\omega_j < 2$.

The hyperbolic lengths of the corresponding arcs can be easily calculated

$$\ell_{\pm} = \int_{\varphi_{\pm}}^{\pi - \varphi_{\pm}} \frac{2d\varphi}{2\sin \varphi + \omega_j} = \frac{8}{\sqrt{4 - \omega_j^2}} \arctanh \left[ \sqrt{\frac{2 \pm \omega_j}{2 + \omega_j}} \tan \left( \frac{\pi}{4} - \frac{\varphi_{\pm}}{2} \right) \right],$$

where $\varphi_{\pm} = \text{Arg}(q_j \pm i\frac{\omega_j}{2}R_{\pm})$. It follows from Fig.1 that

$$\tan \left( \frac{\pi}{4} - \frac{\varphi_{\pm}}{2} \right) = \frac{|q_{j+1} - q_j|}{\frac{|q_{j+1} - q_j|^2}{2 R_{\pm} + \sqrt{R_{\pm}^2 - \frac{|q_{j+1} - q_j|^2}{4}}}} = \frac{(2 \mp \omega_j)\beta}{2 + \sqrt{(4 - \omega_j^2)\beta^2 + 4}}.$$

Using the identity

$$2 \arctanh \frac{x}{1 + \sqrt{1 + x^2}} = \text{arcsinh } x$$

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one gets

\[ \ell_\pm = \frac{4}{\sqrt{4 - \omega_j^2}} \arcsinh \left[ \sqrt{1 - \left(\frac{\omega_j}{2}\right)^2} \beta(q_j, q_{j+1}) \right]. \]  
(A.1)

Since the lengths of both arcs are the same one obtains \( \ell_j = \ell_\pm \) which yields formula (3.9).

In the case \( \omega_j > 2 \) the points \( q_j, q_{j+1} \) can be connected by curves with the geodesic curvature \( \frac{\omega_j}{2} \) if and only if \( (\omega_j^2 - 4)\beta^2 \leq 4 \) (which is satisfied for \( \beta = \beta(q_j, q_{j+1}) \) by construction).

One has two circles with the radii

\[ R_\pm = \frac{2\sqrt{\omega_j}}{\omega^2 - 4} \left( \omega_j \pm \sqrt{(4 - \omega_j^2)\beta^2 + 4} \right), \]

and four different arcs (see Fig. 2).

Fig. 2 The geometry involved in the determination of the boundary length, \( \omega_j > 2 \).

Let \( \ell_\pm^+ \), \( \ell_\pm^- \) be the lengths of the upper, and lower arcs of the circle with radius \( R_\pm \), respectively. Following essentially the same calculations as above and using

\[
2 \arctan \frac{x}{1 + \sqrt{1 - x^2}} = \arcsin x, \\
2 \arctan \frac{x}{1 - \sqrt{1 - x^2}} = \pi - \arcsin x, \quad (0 < x < 1),
\]

one gets

\[
\ell_\pm^+ = \ell_\pm^- = \frac{4}{\sqrt{\omega_j^2 - 4}} \arcsinh \left[ \sqrt{1 - \left(\frac{\omega_j}{2}\right)^2} \beta(q_j, q_{j+1}) \right],
\]  
(A.2)

\[
\ell_-^+ = \ell_+^- = \frac{4}{\sqrt{\omega_j^2 - 4}} \left( \pi - \frac{4}{\sqrt{\omega_j^2 - 4}} \arcsinh \left[ \sqrt{1 - \left(\frac{\omega_j}{2}\right)^2} \beta(q_j, q_{j+1}) \right] \right).
\]
As the image \( w \circ \rho(L_j) \) of the \( j \)-th boundary component is one of the four arcs, its hyperbolic length is either \( \ell_j^+ \) or \( \ell_j^- \). On the other hand, the length of the \( L_j \) depends analytically on \( \omega_j \), so the formula for \( |\omega_j| > 2 \) should be an analytic continuation of that for \( |\omega_j| < 2 \). Taking this into account one finally gets \( \ell_j = \ell_j^+ = \ell_j^- \), what proves the formula (3.10).

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