KÄHLER AND SYMPLECTIC STRUCTURES ON 4-MANIFOLDS
AND HYPERKÄHLER GEOMETRY

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Abstract. A non-linear generalization of the Dirac operator in 4-dimensions, obtained by replacing the spinor representation with a hyperKähler manifold admitting certain symmetries, is considered. We show that the existence of a covariantly constant, generalized spinor defines a Kähler structure on the base 4-dimensional manifold. For a class of hyperKähler manifolds obtained via hyperKähler reduction, we also show that a harmonic spinor, under mild conditions, defines a symplectic structure. Finally, we show that if a covariantly constant, generalized spinor satisfies generalized Seiberg-Witten equations, the metric on the base manifold has a constant scalar curvature.

1. Introduction

Let \((X, g_X)\) be a 4-dimensional, smooth, oriented Riemannian manifold and let \(\pi : Q \to X\) be a Spin\(^c\)-structure on \(X\). The Levi-Civita connection on the frame bundle \(\pi_{SO} : P_{SO(4)} \to X\) and a connection \(A\) on the principal \(U(1)\)-bundle \(\pi_{U(1)} : P_{U(1)} := Q/\text{Spin}(4) \to X\) determine a unique connection \(A\) on \(Q\). Let \(W^\pm \to X\) denote the associated positive and negative spinor bundles respectively and \(\sigma : W^+ \to \Lambda^2_+(X)\) denote the quadratic map. If \(u \in \Gamma(X, W^+)\) is a non-vanishing section, then \(\sigma \circ u\) defines a non-degenerate, self-dual 2-form on \(X\). It was shown in [BLPR00, Sco02] independently that if \(\nabla^A u = 0\) for some \(U(1)\)-connection \(A\), then \(\sigma \circ u\) defines a Kähler structure on \(X\), compatible with a metric \(g'_X = c \cdot g_X\), \(c \in \mathbb{R}\). On the other hand, under mild conditions, it was shown that if \(u\) is a harmonic spinor, then \(\sigma \circ u\) defines a symplectic structure on \(X\). Scorpan [Sco02] gave a characterization of the Kähler and symplectic 2-forms that lie in the image of the quadratic map.

Taubes introduced a non-linear generalization [Tau99] of the Spin-Dirac operator in dimension 3, wherein the spinor representation is replaced by a hyperKähler manifold \((M, g_M, I_1, I_2, I_3)\) - also known as the target hyperKähler manifold - admitting an action of \(Sp(1)\) that permutes the 2-sphere of complex structures. Generalized spinors are defined to be sections of the associated fibre-bundle with a typical fibre \(M\). The Dirac operator is replaced by a first-order, non-linear elliptic differential operator \(D\) for maps taking values in \(M\). For a twisting principal \(G\)-bundle \(P_G\), every connection \(A\) on \(P_G\) defines a twisted, generalized Dirac operator \(D_A\). The idea was extended to dimension 4 by Pidstrygach [Pid04].

The current article investigates the role of certain special generalized spinors and hyperKähler manifolds in defining a Kähler or a symplectic structure on \(X\).

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One way of obtaining hyperKähler manifolds with requisite properties, is via Swann’s construction [Swa91]. Starting with a quaternionic Kähler manifold of positive scalar curvature, Swann’s construction produces a hyperKähler manifold endowed with a permuting $Sp(1)$-action. The manifold is a fibration over the quaternionic Kähler manifold. Additionally an action of a Lie group $G$ that preserves the quaternionic Kähler structure can be lifted to a hyperHamiltonian action on the hyperKähler manifold. For target hyperKähler manifolds obtained via Swann’s construction, covariantly constant spinors define a Kähler structure on $X$.

**Theorem 1.1.** Let $\mathcal{U}(N)$ denote the total space of a Swann bundle over some quaternionic Kähler manifold $N$ with positive scalar curvature and assume that $N$ admits an action of $U(1)$ that preserves the quaternionic Kähler structure. Let $\mu : M \to sp(1)^*$ denote the associated hyperKähler $U(1)$-moment map and $u \in \text{Map}(Q, \mathcal{U}(N))^{Spin^c}$ be a spinor whose range does not contain a fixed point of $U(1)$-action on $\mathcal{U}(N)$. If there exists a connection $A$ on $Q$ such that the covariant derivative $D_Au = 0$, then, under the isomorphism $\Phi : sp(1)^* \to \Lambda^2_\mathbb{C}(\mathbb{R}^4)^*$, $\omega := \Phi(\mu \circ u)$ defines a Kähler structure on $X$.

Another way of constructing such hyperKähler manifolds is via hyperKähler reduction of $\mathbb{H}^n$ by a hyperHamiltonian action of a Lie group $H$. The technique due to Hitchin, Karlhede, Lindström and Rocček [HKLR87], is an analogue of Marsden-Weinstein reduction for symplectic manifolds. It has proven to be quite useful in constructing highly non-trivial hyperKähler manifolds, starting from flat quaternionic spaces. If the usual action of $Sp(1)$ on $\mathbb{H}^n$ preserves the zero set $\mu_H^{-1}(0)$ of the $H$-moment map, then the action descends to a permuting action on the quotient $M_H := \mu_H^{-1}(0)/H$. Moreover, if there exists a hyperHamiltonian $U(1)$-action on $\mathbb{H}^n$ that commutes with $H$-action and preserves $\mu_H^{-1}(0)$, then it descends to a hyperHamiltonian action on $M_H$.

For $M = M_H$, the covariantly constant spinors define a Kähler structure on $X$:

**Theorem 1.2.** Let $\mu : M_H \to sp(1)^*$ denote a hyperKähler $U(1)$-moment map and $u \in \text{Map}(Q, M_H)^{Spin^c}$ be a spinor whose range does not contain a fixed point of $U(1)$-action on $M$. If there exists a connection $A$ on $Q$ such that the covariant derivative $D_Au = 0$, then, $\omega := \Phi(\mu \circ u)$ defines a Kähler structure on $X$.

Moreover, in this case, the harmonic spinors, under mild conditions, define a symplectic structure on $X$:

**Theorem 1.3.** Let $A$, $\mu$ be as in Theorem 1.2. Let $u \in \text{Map}(Q, M_H)^{Spin^c}$ be a spinor whose range does not contain a fixed point of $U(1)$-action on $M_H$. Assume that $D_Au \perp \ker du$. If $D_Au = 0$, then $\omega$ defines a symplectic structure on $X$.

The layout of the article is as follows: Section 2 is divided into two parts. In the first part, sub-section 2.1, we introduce the preliminaries on hyperKähler manifolds and describe the two mentioned constructions; namely hyperKähler reduction and Swann’s construction. In the second part, sub-section 2.2, we introduce the preliminaries needed in order to define the generalized Dirac operator. The details of some of the technical part in this section is left to the Appendix. Section 3, gives the proof of Theorem 1.1, Theorem 1.2 and Theorem 1.3. Finally in Section 5, we prove that the $Spin^c$-structures defined by $\omega$ in both the situations is isomorphic to the one determined by $g_X$. 


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2. Preliminaries and Notations

2.1. HyperKähler manifolds with permuting actions. A 4n-dimensional Riemannian manifold $(M, g_M)$ is said to be hyperKähler if it is endowed with a set of almost-complex structures $I_i \in \text{End}(TM), \quad i = 1, 2, 3, \quad I_i I_j = \delta_{ij} I_k$ that are covariantly constant with respect to the Levi-Civita connection. The quaternionic structure induces a covariantly constant algebra homomorphism

$$I : \mathbb{H} \rightarrow \text{End}(TM), \quad I_\xi := I(\xi) = \xi_0 \text{id}_{TM} + \xi_1 I_1 + \xi_2 I_2 + \xi_3 I_3 \quad \text{for} \quad \xi \in \mathbb{H}$$

Sitting inside the quaternion algebra, is the standard 2-sphere of purely imaginary quaternions $S^2 = \{ \xi = \xi_1 i + \xi_2 j + \xi_3 k \mid |\xi| = 1 \}$

Every $\xi \in S^2$ defines a Kähler structure on $M$. In other words, $M$ has an entire family of Kähler structures parametrized by $S^2 \in \Im(\mathbb{H})$.

Definition 1. An isometric action of a smooth Lie group $G$ on $M$ is said to be tri-holomorphic if it fixes the 2-sphere of complex structures $S^2$; i.e.,

$$T g I_\xi = I_\xi T g, \quad \text{for} \quad g \in G, \quad \xi \in S^2$$

In particular, $G$ preserves the Kähler 2-forms $\omega_i = g_M(I_i(\cdot), \cdot)$, for $i = 1, 2, 3$. We can combine $\omega_i$ to define a single $\text{sp}(1)$-valued 2-form

$$\omega \in \text{sp}(1)^* \otimes \Lambda^2 M, \quad \omega_\xi := \langle \omega, \xi \rangle = g_M(I_\xi(\cdot), \cdot)$$

Additionally, if all the three associated moment maps exist, then the action is said to be tri-Hamiltonian. Again, one can combine the three moment maps into one to define a $\text{sp}(1)$-valued map $\mu : M \rightarrow \text{sp}(1)^* \otimes g^*$, that satisfies

1. $d\mu = t_g \omega$
2. $\mu(gh) = \text{Ad}_g^*(\mu(h))$

The map $\mu$ is called a hyperKähler moment map.

Example 1. Consider $\mathbb{H}^n$ with the hyperKähler structure given by $(R_1, R_2, R_3)$. Let $Sp(n)$ denote the group of $\mathbb{H}$-linear isometries of $\mathbb{H}^n$ and $G \subset Sp(n)$.

The $G$-action on $\mathbb{H}^n$, given by left multiplication $G \times \mathbb{H}^n \ni (g, h) \mapsto gh \in \mathbb{H}^n$ is a tri-holomorphic action with the hyperKähler moment map

$$\langle \mu(x), \xi \otimes \eta \rangle = \frac{1}{2} \xi^x \eta^x$$
HyperKähler reduction. Many non-trivial examples of hyperKähler manifolds can be constructed via hyperKähler reduction, from the quaternionic vector space $\mathbb{H}^n$. HyperKähler reduction is an extension of the well-known Marsden-Weinstein reduction for symplectic manifolds.

Suppose that $M$ is endowed with a hyper-Hamiltonian action of a compact Lie group $H$. Let $\mu_H : M \rightarrow \mathfrak{sp}(1)^* \otimes \mathfrak{h}^*$ be a hyperKähler moment map for the $H$-action and assume that $a \in \mathfrak{h}$ is invariant under the co-adjoint action of $H$. Then $\mu_H^{-1}(a)$ is an $H$-invariant submanifold of $M$.

**Theorem 2.1.** [HKLR87] Let $a \in \mathfrak{h}$ be a central regular value of the moment map $\mu_H$ and assume that $H$ acts freely and properly on $\mu_H^{-1}(a)$. Then the quotient $\frac{M_{\mu_H^{-1}(a)}}{H}$ is again a hyperKähler manifold.

**Example 2.** Many interesting examples of hyperKähler manifolds fall under this category. To name a few

1. Co-tangent bundles of complex Lie groups [Kro88, KS96]
2. Co-adjoint orbits of semi-simple Lie groups [Kro90]
3. Moduli space of framed, charge $k$ instantons on $S^4$ [ADHM78]

Swann bundles. Let $Sp(1)$ denote the group of unit quaternions and $\mathfrak{sp}(1)$ its Lie algebra. Note that $\mathfrak{sp}(1) \cong \mathfrak{im} \mathbb{H}$. A permuting action of $Sp(1)$ or $SO(3)$ on $M$ is an isometric action, such that the induced action on the 2-sphere of complex structures $S^2$ is the standard action of $Sp(1)$ or $SO(3)$ on $S^2$.

\[ Tq I_\xi Tq^{-1} = I_\xi q \tilde{q}, \quad \text{for} \quad q \in Sp(1), \quad \xi \in S^2 \subset \mathfrak{sp}(1) \]

Amongst the hyperKähler manifolds admitting a permuting $Sp(1)$-action, there are those that also admit a hyperKähler potential. A hyperKähler potential is a real-valued function $\rho : M \rightarrow \mathbb{R}$ which is a Kähler potential w.r.t all three complex structures simultaneously; i.e,

\[ -d(I_\xi (d\rho)) = 2\omega_\xi \quad \text{for} \quad \xi \in \mathfrak{sp}(1), \quad ||\xi||^2 = 1. \]

A quaternionic Kähler manifold $N$ is a 4n-dimensional manifold whose holonomy is contained in $Sp(n)Sp(1) := (Sp(n) \times Sp(1))/\pm 1$. Let $F_N$ denote the reduction of the principal frame bundle $P_{SO(4n)}$ to $Sp(n)Sp(1)$-bundle over $N$. Then $\mathcal{S}(N) = F_N/Sp(n)$ is a principal $SO(3)$-bundle, which is a frame bundle of the three dimensional vector subbundle of skew symmetric endomorphisms of $TN$. The $Sp(1)$-action on $\mathbb{H}$ (by left multiplication) descends to an isometric action of $SO(3)$ on $\mathbb{H}^*/\pm 1$. The Swann bundle over $N$ is defined to be the principal $\mathbb{H}^*/\mathbb{Z}_2$-bundle

\[ \mathcal{U}(N) := \mathcal{S}(N) \times_{SO(3)} \mathbb{H}^*/\mathbb{Z}_2 \rightarrow N \]

**Theorem 2.2** ([Swa91]). Let $N$ be a quaternionic Kähler manifold with a positive scalar curvature. Then $\mathcal{U}(N)$ is a hyperKähler manifold with a free, permuting $Sp(1)$ action. Moreover, $\mathcal{U}(N)$ also admits a hyperKähler potential given by

\[ \rho_0 = \frac{1}{2} r^2 \]

where $r$ is the co-ordinate along $\mathbb{H}^*/\mathbb{Z}_2$. If $N$ has an isometric action of a Lie group $G$ that preserves the quaternionic Kähler structure, then the action can be lifted to a hyper-Hamiltonian action of $G$ on $\mathcal{U}(N)$.
The hyperKähler potential on $\mathcal{U}(N)$ is quite special. Namely, if $X_0 := \text{grad}\rho_0$ - also known as Euler vector field - the fundamental vector fields due to permuting $Sp(1)$-action satisfy:

(2) $\iota_{\xi} K^M_{\xi} = -X_0, \quad \xi \in S^2 \subset sp(1)$ and $\rho_0 = \frac{1}{2} g_{\mu}(X_0, X_0)$

The moment map for a hyper-Hamiltonian $G$-action on $\mathcal{U}(N)$ has a simple form [Sch10, Corollary 3.3.1]

(3) $\langle \mu, \xi \otimes \eta \rangle = -\frac{1}{2} g_{\mu}(K^M_{\xi}, K^M_{\eta}), \quad \xi \in sp(1), \quad \eta \in g$

**Example 3.** The flat space $\mathbb{H}^n := \mathbb{H}^n \setminus \{0\}$ is the total space of Swann bundle over $\mathbb{H}P^{n-1}$. Indeed, observe that $\mathbb{H}^n = Sp(n) \times \mathbb{R}^+$. A permuting $Sp(1)$-action on $\mathbb{H}^n$ is given by

$$Sp(1) \times \mathbb{H}^n \ni (q, h) \mapsto h\tilde{q} \in \mathbb{H}^n$$

The Euler vector field $X_0 = \text{id}_{\mathbb{H}^n}$ and therefore, the hyperKähler potential

$$\rho_0(h) = \frac{1}{2} ||h||^2.$$

On the other hand, consider a hyperKähler manifold $M$ endowed with a permuting $Sp(1)$-action.

**Theorem 2.3 ([Swa91]).** If $M$ admits a hyperKähler potential $\rho_0 : M \longrightarrow \mathbb{R}^+$, then for $c \in \mathbb{R}$, $N := \rho_0^{-1}(c)/Sp(1)$ is a quaternionic Kähler manifold of positive scalar curvature. Consequently, $M$ is the total space of a Swann bundle over $N$.

Define $Spin^G_c(4) := Spin(4) \times_{\mathbb{Z}_2} G$, where $\mathbb{Z}_2$ denotes an order 2-subgroup generated by $(-1, \varepsilon)$ with the central element $\varepsilon \in G$. An action of $Spin^G_c(4)$ is said to be permuting if the action of $Sp(1)_+ \hookrightarrow Spin^G_c(4)$ is permuting and the action of $Sp(1)_- \times G \hookrightarrow Spin^G_c(4)$ is tri-holomorphic. For the rest of the article, we will study the case when $G = U(1)$.

**2.2. Generalized Dirac operator.** Fix a $Spin^c$-structure $\pi : Q \longrightarrow X$. The Levi-Civita connection $\phi$ on the frame bundle $F_{SO(4)}$ and a connection $A$ on $F_{U(1)}$ uniquely define a connection $\mathcal{A}$ on $Q$. Let $\mathcal{A} \subset \Lambda^1(Q, spin^c)$ denote the space of all connections which are the lift of the Levi-Civita connection. Let $M$ be a manifold admitting a permuting action of $Spin^c(4)$. We define the space of generalized spinors to be the space of smooth, equivariant maps $S := \text{Map}(Q, M)^{Spin^c} \cong \Gamma(X, Q \times_{Spin^c} M)$.

The covariant derivative of a spinor $u \in S$, w.r.t $\mathcal{A} \in \mathcal{A}$ is defined as

(4) $D_{\mathcal{A}} : C^\infty(Q, M)^{Spin^c} \longrightarrow \text{Hom}(TQ, TM)^{Spin^c}_{\text{hor}} \cong C^\infty(Q, (\mathbb{R}^4)^* \otimes TM)^{Spin^c}$

$D_{\mathcal{A}}u = du + K^M_{\mathcal{A}}|_u$

where $K^M_{\mathcal{A}}|_u : TQ \to u^*TM$ is vector bundle homomorphism

$K^M_{\mathcal{A}}|_u(v) = K^M_{\mathcal{A}(v)}|_u(p), \quad v \in T_pP$

Alternatively, one can view the covariant derivative as

(5) $D_{\mathcal{A}} : C^\infty(Q, M)^{Spin^c} \longrightarrow C^\infty(Q, (\mathbb{R}^4)^* \otimes TM)^{Spin^c}$

(6) $(D_{\mathcal{A}}u(p), w) = du(p)(\tilde{w})$

where, $w \in \mathbb{R}^4$, $\tilde{w}$ denotes the horizontal lift of $\pi_{SO}(p)(w) \in T_{\pi(p)}X$. 
Let $\psi : TTM \to TM$ denote the Levi-Civita connector on $M$ and $\nabla^{A,\psi}$, the linearization of $D_A$ (see Appendix (B), Lemma (B.3))

$$\nabla^{A,\psi} : C^\infty(Q, TM)^{Spin^-} \to \text{Hom}(TQ, TM)^{Spin^-}_{hor}$$

The following Lemma is crucial to our construction in the next section

**Lemma 2.4.** [Sch10, Corollary 4.6.2] Let $M$ be a manifold with a permuting $Sp(1)$-action and a hyperK"ahler potential $\rho_0$ and let $X_0 := \text{grad} \rho_0$. Let $u \in S$. Then $X_0 \circ u \in \Gamma(Q, u^* TM)^{Spin^-}$. For a connection $A$ on $Q$,

$$D_A u = \nabla^{A,\psi}(X_0 \circ u)$$

**Clifford multiplication.** Define $W^+$ to be the $Spin^c$-equivariant bundle $TM \to M$ equipped with an action induced by $\vartheta_+ = [q_+, q_-] \mapsto [q_+, q_-]$. More precisely, for any $w_+ \in W^+$, the action is given by:

$$[q_+, q_-] \cdot w_+ = Tq_+ Tq_- w_+.$$

Define the $W^-$ to be the $Spin^c$-equivariant bundle $TM \to M$ equipped with the following action:

$$[q_+, q_-] \cdot w_- = \|q_+\| \cdot \|q_-\| Tq_+ Tq_- w_-.$$

Now Clifford multiplication is is a map of $Spin(4)$-representations

$$m : \mathbb{R}^4 \to \text{End}(W^+ \oplus W^-)$$

We identify $\mathbb{R}^4$ with $\mathbb{H}$ by mapping the standard, oriented basis $(e_1, e_2, e_3, e_4)$ of $\mathbb{R}^4$, to $(1, i, j, k)$. The hyperK"ahler structure on $\mathbb{H}$ is given by $(R_i, R_j, R_k)$ and the $Spin^c$ action on $\mathbb{H}$ by $[q_+, q_-] \cdot h = q_+ h q_-$. Define the map

$$m : \mathbb{R}^4 \cong \mathbb{Q} \to \text{End}(W^+ \oplus W^-)$$

$$h \mapsto \begin{bmatrix} 0 & -\|h\|^2 \\ \|h\| & 0 \end{bmatrix}$$

Since $m(h)^2 = -g_{\mathbb{H}^*}(h, h) \cdot id_{W^+ \oplus W^-}$, by universality property, extends to a map of algebras $\tilde{m} : Cl_4 \to \text{End}(W^+ \oplus W^-)$. Identifying $\mathbb{H}^4$ with $(\mathbb{H}^4)^*$ we define Clifford multiplication by:

$$\bullet : (\mathbb{H}^4)^* \otimes (W^+ \oplus W^-) \to W^+ \oplus W^-$$

$$g_{\mathbb{H}^*}(h, \cdot)(w_+, w_-) \mapsto m(h)(w_+, w_-).$$

This map is $Spin^c$-equivariant [Sch10].

Composing Clifford multiplication $\mathcal{C}$ with the covariant derivative, we get the non-linear Dirac operator:

$$\mathcal{D}_A u \in C^\infty(Q, u^* W^-)^{Spin^-}$$

More explicitly, from (5), we get

$$\mathcal{D}_A(u) = \sum_{i=0}^3 e_i \bullet D_A u(e_i)$$

Let $\mathcal{C}_u$ be the restriction of the Clifford multiplication $\mathcal{C}$ to $u^* W^+ \oplus u^* W^-$. Consider the first-order differential operator

$$\mathcal{D}^{lin}_{A,u} := \mathcal{C}_u \circ \nabla^{A,\psi} : u^* W^+ \oplus u^* W^- \to \mathcal{D}^{lin}_{A,u}$$

**Lemma 2.5.** [Sch10, Lemma 4.6.1] The linearization of the generalized Dirac operator, at a point $(u, A) \in \mathcal{S} \times \mathcal{A}$, coincides with the linear operator $\mathcal{D}^{lin}_{A,u}$. 
Proof of Theorem 2.6. Generalized Seiberg-Witten equations. Let \( u \) denote hyperKähler moment map for the hyperKähler \( U(1) \)-action on \( M \) and \( F_A \in \text{Map}(Q, \Lambda^2(\mathbb{R}^4)^* \otimes \mathfrak{sp}(1))^{\text{Spin}^c} \) denote the curvature of the connection \( A \). The generalized Seiberg-Witten equations for a pair \( (u, A) \in \mathcal{S} \times \mathcal{A} \), in dimension 4, are

\[
\begin{align*}
D_A u &= 0 \\
F_A^+ - \mu \circ u &= 0
\end{align*}
\]

3. Kähler structure on 4-manifolds and Swann bundles

In this section, we give the proof of Theorem 1.1. Assume that \( U(N) \) is endowed with a hyper-Hamiltonian action of \( U(1) \) and let \( \mu : M \rightarrow \mathfrak{sp}(1)^* \) denote the associated moment map (3). Observe that since the \( \mathfrak{sp}(1) \) action on \( U(N) \) is free, \( \mathfrak{X}_0 \circ u \in \Gamma(Q, u^*W^+) \) is a non-vanishing section.

Proof of Theorem 1.1. Let \( (u, A) \in \mathcal{S} \times \mathcal{A} \) be such that \( D_A u = 0 \) and the image of \( u \) does not contain any fixed points of the \( U(1) \)-action. This has the consequence that \( \langle d\mu \circ u \rangle = \langle d\mu, D_A u \rangle = 0 \), which implies that \( \mu \circ u \) is constant. Under the isomorphism \( \Phi : \mathfrak{sp}(1)^* \rightarrow \Lambda^2(\mathbb{R}^4)^* \), the map \( \mu \circ u : Q \rightarrow \mathfrak{sp}(1) \) defines a non-degenerate, self-dual 2-form \( \omega := \Phi(\mu \circ u) \) on \( X \). We treat \( \omega \) as an element in \( \text{Map}(Q, \Lambda^2(\mathbb{R}^4)^*)^{\text{Spin}^c} \).

\[
\nabla^{A,\psi}(\omega \bullet (\mathfrak{X}_0 \circ u)) = (D_{\omega}) \bullet \mathfrak{X}_0 \circ u + \omega \bullet \nabla^{A,\psi}(\mathfrak{X}_0 \circ u) = (D_{\omega}) \bullet \mathfrak{X}_0 \circ u + \omega \bullet D_A u
\]

Consider the left hand side of (14)

\[
\omega \bullet (\mathfrak{X}_0 \circ u) = (\mu_1 \circ u) \cdot (e_0 \wedge e_1 + e_2 \wedge e_3) \bullet \mathfrak{X}_0 \circ u \\
+ (\mu_2 \circ u) \cdot (e_0 \wedge e_2 - e_1 \wedge e_3) \bullet \mathfrak{X}_0 \circ u \\
+ (\mu_3 \circ u) \cdot (e_0 \wedge e_3 + e_1 \wedge e_2) \bullet \mathfrak{X}_0 \circ u \\
= (\mu_1 \circ u) \cdot ((e_0 \cdot e_1) \bullet \mathfrak{X}_0 \circ u + (e_2 \cdot e_3) \bullet \mathfrak{X}_0 \circ u) \\
+ (\mu_2 \circ u) \cdot ((e_0 \cdot e_2) \bullet \mathfrak{X}_0 \circ u - (e_1 \cdot e_3) \bullet \mathfrak{X}_0 \circ u) \\
+ (\mu_3 \circ u) \cdot ((e_0 \cdot e_3) \bullet \mathfrak{X}_0 \circ u + (e_1 \cdot e_2) \bullet \mathfrak{X}_0 \circ u) \\
= \sum_{i=1}^{3} (\mu_i \circ u) I_i (\mathfrak{X}_0 \circ u)
\]
Since $\mu \circ u$ is constant, this implies that $\mu_l \circ u$ are constant for all $l = 1, 2, 3$. Therefore the left hand side of the eq. (14) reads

$$\nabla^{A,\psi} (\omega \cdot X_0 \circ u) = \nabla^{A,\psi} \left( \sum_{l=1}^{3} (\mu_l \circ u) \ I_l (X_0 \circ u) \right)$$

(15)

$$= \sum_{l=1}^{3} (\mu_l \circ u) \ \nabla^{A,\psi} (I_l (X_0 \circ u))$$

(16)

$$= \sum_{l=1}^{3} (\mu_l \circ u) \ I_l (D_A u)$$

(17)

Note that we have used Lemma 2.4 in the third step. Substituting in (14) we get

(18) $$D_{\phi \omega} \cdot X_0 \circ u = 0$$

We can think of $D_{\phi \omega}$ as a one-form with values in the space of skew-adjoint, traceless endomorphisms of $W^+ \oplus W^-$

$$T_x X \ni v \mapsto D_{\phi \omega} (v)|_x \cdot (\cdot)$$

Since, from (18), one of the eigenvalues of $D_{\phi \omega}|_x$ is zero, it follows that the other one is zero as well. This is true for all $x \in X$. Thus we get $D_{\phi \omega} = 0$ or alternatively, interpreting $\omega \in \Gamma(X, \Lambda^2_+ (X))$, we have $\nabla \omega = 0$.

**Complex structure.** If $\beta$ is some non-degenerate, self-dual 2-from on $X$ such that $D_{\phi \beta} = 0$, then one can suitably modify the metric, from $g_x \rightsquigarrow g'_x$ so that $|\beta| = \sqrt{2}$. Therefore if $\mathfrak{g} \Lambda^2_+(X)$ (a.k.a twistor space) denotes the sphere bundle in $\Lambda^2_+ (X)$, then $\beta \in \Gamma(X, (\mathfrak{g} \Lambda^2_+(X))')$ and determines an almost-complex structure $J_\beta$ on $X$. As the Levi-Civita connection remains unchanged, we get $D_{\phi \beta} = 0$ and therefore $J_\beta$ is integrable. Thus $\omega$ defines a Kähler structure on $X$, compatible with the metric $g'_x$. $\square$

4. **Kähler and symplectic structures for case of hyperKähler reduction**

In this section, we consider the case where $M$ is a hyperKähler reduction of some flat-space $\mathbb{H}^n$. For such hyperKähler manifolds, it is possible to construct both the Kähler and symplectic structures on $X$. The idea here involves lifting the Dirac equation suitably for maps taking values in $\mathbb{H}^n$. Let $H \subset Sp(n)$ be a compact, unitary group, acting hyper-Hamiltonianly on $\mathbb{H}^n$ and $\mu_H : \mathbb{H}^n \rightarrow sp(1)^* \otimes \mathfrak{h}^*$ denote the associated moment map. Assume also that $H$ acts freely and properly on $\mu_H^{-1}(0)$ and let $M_H := \mu_H^{-1}(0)/H$. If $Sp(1)$-preserves $\mu_H^{-1}(0)$, then, the permuting action of $Sp(1)$ on $\mathbb{H}^n$ descends to a permuting action on $M_H$. Given a hyper-Hamiltonian $U(1)$-action on $\mathbb{H}^n$, that commutes with the $H$-action and preserves $\mu_H^{-1}(0)$, it descends to a hyper-Hamiltonian action on $M_H$. 

Consider the following diagram

\[\begin{array}{c}
P_{\hat{H}} \xrightarrow{\hat{u}} P \subset \mathbb{H}^n \\
\pi_1 \downarrow \quad \downarrow \pi_2 \\
Q \xrightarrow{u} M_H \\
\pi \downarrow \\
X
\end{array}\]

Here \(\pi_1\) is a \(\text{Spin}^c\)-equivariant submersion, \(P := \mu_{\hat{H}}^{-1}(0)\) is the principal \(H\)-bundle over \(M_H\). Note that \(P_{\hat{H}} \to X\) which is a principal \(H\)-bundle over \(Q\). Let \(\hat{u}\) be a smooth map. Define \(u : Q \to M_H\) as

\[u(q) = \pi_2(\hat{u}(p)), \quad q \in Q, \quad p \in \pi_1^{-1}(q)\]

Clearly, the diagram commutes. On the other hand, given a smooth spinor \(u : Q \to M_H\), it defines a principal \(H\)-bundle over \(Q\) via pull-back of \(P\). The pull-back of the canonical connection \(a\) on \(P\), defined as

\[K_{a,\hat{H}}^P|_p(v) = -pr_{\text{im}K_{H}^P}(v), \quad v \in T_pP\]

by \(\hat{u}\) along with the connection \(A\) on \(Q\) uniquely define a connection \(A\) on \(P_{\hat{H}}\)

\[A = \pi^*A \oplus A_h \in \Lambda^1\left(P_{\hat{H}}, \text{spin}(4) \oplus \mathfrak{h}\right)^{H \times \text{Spin}^c(4)}\]

where \(A_h = \hat{u}^*a - (\pi^*A, \iota_{\text{spin}(4)}\hat{u}^*a)\).

**Proposition 4.1.** Then, there is a 1-1 correspondence between

\[(\hat{u}, A) \mid D_A\hat{u} = 0, \quad \mu_H \circ \hat{u} = 0 \quad \text{and} \quad (u, A) \mid D_Au = 0\]

where, \(u\) is the projection of \(\hat{u}\) to \(Q\), and also between

\[(\hat{u}, A) \mid D_A\hat{u} = 0, \quad \mu_H \circ \hat{u} = 0 \quad \text{and} \quad (u, A) \mid D_Au = 0\]

**Proof.** To begin with, note that the condition \(\mu_H \circ \hat{u} = 0\) means that the map \(\hat{u}\) is non-vanishing, since the action of \(H\) on \(\mu_{\hat{H}}^{-1}(0)\) is free.

For \(h \in \mathbb{H}^n\), define \(H_h := \ker d\mu_H(h) \cap (\text{im}K_{\hat{H}}^\mathbb{H})^\perp\). It is easy to see that if \(\mu_H(h) = 0\), then \(H_h\) is just the horizontal subspace over \(\hat{h} \in \mu_{\hat{H}}^{-1}(0)\) w.r.t the canonical connection \(a\).

We will prove the proposition in three steps. In what follows, we shall denote the \(H\) and \(\text{Spin}^c\)-components of \(A\) by \(A_h\) and \(\widehat{A}\) respectively.
Step 1: In the first step we will prove that \( \xi D_A \hat{u}(v) \in \mathcal{H}_{\hat{u}} \) for every \( \xi \in \mathfrak{sp}(1) \) and \( v \in \mathcal{H}_A \subset TP_{\hat{u}} \). Indeed, if \( \mu_H \circ \hat{u} = 0 \), then \( d\mu_H(v) \in \ker d\mu_H(\hat{u}(p)) \). Also, \( K^{\mathfrak{sp},H}_A |_{\hat{u}} \in \ker d\mu_H(\hat{u}(p)) \) and \( K^{\mathfrak{sp},Spin^c}_A |_{\hat{u}} \in \ker d\mu_H(\hat{u}(p)) \). Therefore, \( D_A \hat{u}(v) \in \ker d\mu_H(\hat{u}(p)) \). Consequently

\[
0 = \langle d\mu_H(D_A \hat{u}(v)), \xi \otimes \eta \rangle = \langle \xi K^{\mathfrak{sp},H}_A |_{\hat{u}(p)}, D_A \hat{u}(v) \rangle = -\langle K^{\mathfrak{sp},H}_A |_{\hat{u}(p)}, \xi D_A \hat{u}(v) \rangle
\]

for \( \xi \in \mathfrak{sp}(1) \), \( \eta \in \mathfrak{h} \). In other words, \( \xi D_A \hat{u}(v) \in (\text{im } K^{\mathfrak{sp},H})^\perp \) for all \( \xi \in \mathfrak{sp}(1) \). Also, for \( \xi' \in \mathfrak{sp}(1) \),

\[
\langle d\mu_G(\xi \xi', D_A \hat{u}(v)), \xi \otimes \eta \rangle = \langle d\mu_G(D_A \hat{u}(v)), (\xi', \xi) \otimes \eta \rangle = 0
\]

which implies \( \xi D_A \hat{u}(v) \in \ker d\mu_G(\hat{u}(p)) \) for all \( \xi \in \mathfrak{sp}(1) \). Therefore, \( \xi D_A \hat{u}(v) \in \mathcal{H}_{\hat{u}} \).

Step 2: If \( D_A \hat{u} = 0 \), then from (10), we have

\[
0 = D_A \hat{u}(e_0) - \sum_{i=1}^{3} I_i D_A \hat{u}(e_i)
\]

From Step 1, we get \( D_A \hat{u}(e_i) \in \mathcal{H}_{\hat{u}} \). It follows that \( D_A \hat{u}(e_i) \in \mathcal{H}_{\hat{u}} \) for all \( i = 1, 2, 3 \). Consequently, for any \( v \in \mathcal{H}_A \), \( \text{pr}^{\mathfrak{im}} K^{\mathfrak{sp},H}_A \hat{u}(v) = 0 \) and we get \( K^{\mathfrak{sp},H}_A |_{\hat{u}(p)} = -\text{pr}^{\mathfrak{im}} K^{\mathfrak{sp},H}_A d\hat{u}(v) \). In other words, the \( \mathfrak{h} \)-connection component of \( A \) is just the pull-back of the canonical connection on the Riemannian submersion \( \mu^{-1}_H(0) \).

Since the diagram commutes, \( d\pi_2(D_A \hat{u}) = D_A u \). Also, as \( D_A \hat{u}(e_i) \in \mathcal{H}_{\hat{u}} \) for all \( i = 0, 1, 2, 3 \)

\[
0 = d\pi_2(D_A \hat{u}) = d\pi_2 \left( D_A \hat{u}(e_0) - \sum_{i=1}^{3} I_i D_A \hat{u}(e_i) \right)
\]

\[
= d\pi_2(D_A \hat{u}(e_0)) - \sum_{i=1}^{3} \pi_2^* I_i d\pi_2(D_A \hat{u}(e_i))
\]

\[
= D_A u(e_0) - \sum_{i=1}^{3} \tilde{I}_i D_A u(e_i)
\]

\[
= D_A u
\]

Thus, \( D_A \hat{u} = 0 \) implies \( D_A u = 0 \). On the other hand if \( K^{\mathfrak{sp},H}_A |_{\hat{u}(p)} = -\text{pr}^{\mathfrak{im}} K^{\mathfrak{sp},H}_A d\hat{u}(v) \), then \( D_A \hat{u} \in \mathcal{H}_{\hat{u}} \) and so \( d\pi_2(D_A \hat{u}) = D_A u \). Therefore, if \( D_A u = 0 \), it implies that \( D_A \hat{u} \in \text{im } K^{\mathfrak{sp},H} \). But since,

\[
D_A \hat{u} = D_A \hat{u}(e_0) - \sum_{i=1}^{3} \pi_2^* I_i D_A \hat{u}(e_i) \in \mathcal{H}_{\hat{u}}
\]

it follows that \( D_A \hat{u} \in (\text{im } K^{\mathfrak{sp},H})^\perp \) and so \( D_A \hat{u} = 0 \).
Step 3: Using an argument verbatim to the one in Step 2 above, we can prove that there is a 1-1 correspondence between

\[ \{ \nabla_A \hat{u} = 0, \quad \mu_H \circ \hat{u} = 0 \} \quad \text{and} \quad \{ \nabla_A u = 0 \} \]

This proves the statement. □

Kähler and symplectic structures. Proposition 4.1 allows us to use the tools for the usual Spin-Dirac operator.

For the rest of the section, assume that the \( h \)-component of a connection \( A \) is given by \( A_h \) as in (20).

Lemma 4.2. Given a non-vanishing spinor \( \hat{u} : P_H \rightarrow \mathbb{H}^n \), there exists a unique connection \( A \) such that \( \nabla_A \hat{u} = 0 \).

Proof. This is a consequence of the fact that since \( \hat{u} \) is non-vanishing, Clifford multiplication is modelled on quaternionic multiplication. Since the \( h \)-component of the connection is given by the pull-back of the canonical connection on Riemannian submersion \( \mu^{-1}(0) \), the only variable is the \( U(1) \)-connection component.

Let \( A \) be the lift of a connection \( A \) on \( Q \), determined by a \( U(1) \)-connection \( A \).

For a connection \( A' = A + i\alpha \), \( \alpha \in \Omega^2 \), we have \( \nabla_{A'} \hat{u} = \nabla_A \hat{u} + i\alpha \cdot \hat{u} \). To solve \( \nabla_{A'} \hat{u} = 0 \), we need to solve \( \nabla_A \hat{u} = -i\alpha \cdot \hat{u} \).

Consider the map

\[ \mathbb{R}^4 \otimes \mathbb{H}^n^* \rightarrow \mathbb{H}^n^*, \quad a \otimes h \mapsto a \cdot h = h \cdot \hat{a} \]

Fix \( h_0 \in \mathbb{H}^n^* \) and let \( h' \in \mathbb{H}^n^* \). Define \( a := \frac{\bar{h}^t h_0}{|h_0|^2} \in \mathbb{H} \cong \mathbb{R}^4 \). Then,

\[ a \cdot h_0 = h_0 \cdot \frac{\bar{h}^t \cdot h'}{|h_0|^2} = h' \]

Thus, for \( a \) a fixed \( h_0 \) and any \( h' \), there exists an \( a \in \mathbb{R}^4 \) such that \( a \cdot h_0 = h' \). In other words, \( \nabla_A \hat{u} = -i\alpha \cdot \hat{u} \) always has a solution. The uniqueness of the connection \( A \) follows from the unique continuation property of the Dirac operator. □

Observe that if we change the \( U(1) \)-connection, then \( \nabla_A \hat{u} = \nabla_A \hat{u} + i\alpha \cdot \hat{u} \). In other words, the covariant derivative changes along the direction of the \( U(1) \)-orbit of \( \hat{u} \).

Proposition 4.3. [Sco02, Corollary 3.11]

\[ ||\hat{u}||^2 \cdot D_A \hat{u} = i(D\phi \omega) \cdot \hat{u} + (D_A \hat{u}, i\hat{u})_R \cdot i\hat{u} \]

Consequently, if \( D_A \hat{u} = 0 \), then \( \omega \) defines a Kähler structure on the base manifold, compatible with a metric which is a scalar multiple of \( g_x \).

Proof. The hyperKähler \( U(1) \)-moment map is given by

\[ \mu : \mathbb{H}^n \rightarrow \mathfrak{sp}(1)^*, \quad \mu(h) = \frac{1}{2} \bar{h}^t i h \]

Therefore \( \omega \cdot \hat{u} = (\mu \circ \hat{u}) \cdot \hat{u} = -\frac{i}{2} ||\hat{u}||^2 \cdot \hat{u} \). The identity now follows from Corollary 3.11 of [Sco02]. □
Composed with Clifford multiplication, we obtain the splitting formula for Dirac operator.

**Proposition 4.4.** [Sco02, Theorem 3.15] Given a $H \times Spin^c$-structure $P_H$ and a connection $A$ on $P_H$, we have

\[(24) \quad \|\tilde{\mu}\|^2D_{\tilde{\mu}}\tilde{\mu} = i(2d^*\omega + \langle D_{\tilde{\mu}}\tilde{\mu}, i\tilde{\mu}\rangle_R) \cdot \tilde{\mu}\]

Therefore, if $\langle D_{\tilde{\mu}}\tilde{\mu}, i\tilde{\mu}\rangle_R = 0$, then every harmonic spinor $\tilde{\mu}$ defines a symplectic structure on the base manifold, compatible with a metric conformal to $g_X$.

**Note.** Using the existence argument as in Lemma 4.2, one can prove that there exists a unique connection $A'$ such that $\langle D_{\tilde{\mu}}\tilde{\mu}, i\tilde{\mu}\rangle_R = 0$

**Remark 1.** It is worth noting that every non-vanishing, covariantly constant/harmonic spinor defines the same Kähler/symplectic structure. Indeed, one can think of $\mu$ as a quadratic map

\[(25) \quad \mu : S^{4n-1} \rightarrow S^2\]

where $S^{4n-1}$ is the unit sphere in $\mathbb{H}^n$ and $S^2$ is the unit sphere in $\Lambda^2_+^c(\mathbb{R}^4)$. For $m_1 \geq 2m_2$, every polynomial map from $S^{m_1}$ to $S^{m_2}$ is a constant map [Woo68]. Therefore, for $n > 1$, $\mu : S^{4n-1} \rightarrow S^2$, $\mu$ is a constant map.

This has the following consequence: let $A'$ be the unique connection such that $\langle D_{\tilde{\mu}}\tilde{\mu}, i\tilde{\mu}\rangle_R = 0$. Then, from Proposition 4.3 and Theorem 4.4 we conclude

**Lemma 4.5.** If there exists a non-vanishing spinor $\tilde{\mu}$ which is covariantly constant with respect to the unique connection $A'$ such that $\langle D_{\tilde{\mu}}\tilde{\mu}, i\tilde{\mu}\rangle_R = 0$, then every non-vanishing spinor is covariantly constant w.r.t a unique connection and the defines the same complex structure.

Similarly, every harmonic spinor, satisfying $\langle D_{\tilde{\mu}}\tilde{\mu}, i\tilde{\mu}\rangle_R = 0$ defines the same symplectic structure on $X$.

**Remark 2.** One can repeat the entire argument in Theorems 1.1, 1.2 and 1.3 for $G = U(n)$ instead of $U(1)$. One only needs to observe that the $U(n)$-moment map splits into 2 components

$\mu_{u(n)} = \mu_{u(n)} + \mu_{u(1)}$

and $u(1)$-component defines a self-dual 2-form on $X$.

5. **Equivalence of $Spin^c$-structures**

Let $(X, g_X)$ be an oriented, compact, 4-dimensional manifold and let $Q \rightarrow X$ be a fixed $Spin^c$-structure. A non-degenerate, self-dual 2-form $\beta \in \Gamma(X, s\Lambda^2_+^c(X))$ defines an almost-complex structure $J$ on $X$, which is compatible with a metric in the conformal class of $g_X$. Then $(\beta, J)$ define a $Spin^c$-structure $Q_{\beta} \rightarrow X$.

**Proposition 5.1.** Let $M$ be as in Theorem 1.1 or Theorem 1.2 and $u \in S$ be a spinor whose range does not contain a fixed point of the $U(1)$-action. Let $A$ be a connection on $P_{U(1)}$ and $A$ be the induced connection on $Q$. If $D_A u = 0$, then the $Spin^c$-structure $Q_A$ is isomorphic to $Q$.

**Proof.** The condition $D_A u = 0$ implies that $D_A \omega = 0$, or in other words $d(\mu \circ u) = 0$ and thus $\omega$ has a constant length. Modifying the metric $g_X$ to a metric $g_X' := c \cdot g_X$ for some $c \in \mathbb{R}$, we may assume that $|\omega| = \sqrt{2}$. The metric $g_X'$, is thus a Kähler metric defining the $Spin^c$-structure $Q_\omega$. 
Any metric \( g'_X := e^{2f}g_X \) in the conformal class of \( g \) induces an isomorphism between the respective frame bundles

\[
P_{SO(4)} \xrightarrow{e^{-f} \gamma} P'_{SO(4)}
\]

where \( \pi : P_{SO(4)} \rightarrow X \). Any point \( p \in P \) is a linear isomorphism \( p : \mathbb{R}^4 \rightarrow T_{\pi(p)}X \). Consider \( p \in P_{SO(4)} \) and \( p'' \in P'_{SO(4)} \) such that \( \pi(p) = \pi''(p'') = x \). Then, \( p \circ (p'')^{-1} : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \) is an automorphism. But \( p'' = e^{-f(x)}p \). Therefore, \( p \circ (p'')^{-1} \) is an isomorphism of \( \mathbb{R}^4 \) obtained by scalar multiplication by \( e^{-f(x)} \).

If \( e^{-f} \) is constant, say \( c \), then the isomorphism is independent of \( x \in X \). In other words, we get an isomorphism between \((\mathbb{R}^4, g_{\mathbb{R}^4})\) and \((\mathbb{R}^4, c \cdot g_{\mathbb{R}^4})\), where \( g_{\mathbb{R}^4} \) is the standard metric on \( \mathbb{R}^4 \). This induces an isomorphism of the respective complexified Clifford algebras \( \gamma : Cl_4 \otimes \mathbb{C} \rightarrow Cl'_{4} \otimes \mathbb{C} \) which preserves the positive elements and therefore, induces an isomorphism of the respective Spin\(^c\) groups \( \gamma : Spin^c(4) \rightarrow (Spin^c(4))'' \). Therefore, if \( g'_X := c \cdot g_X \) and \( Q \) and \( Q' \) denote the respective Spin\(^c\)-bundles over \( X \), then the map \( \gamma \) induces an isomorphism of the bundles \( Q \) and \( Q' \). In conclusion, the Spin\(^c\)-structure \( Q_u \) is isomorphic to \( Q \).

\[\square\]

6. Constant Scalar Curvature

In this section, we show that if the solution space to equations (13) contains a pair \((u, A)\) such that \( D_A u = 0 \) then the scalar curvature of the base 4-dimensional manifold is necessarily (negative) constant.

Let \( M \) be as in Theorem 1.1 or Theorem 1.2. If a pair \((u, A)\) satisfies \( D_A u = 0 \), we get \( D_A u = 0 \) and the Weitzenböck formula (2.6) gives

\[
0 = \frac{s_X}{4} \lambda_0 \circ u + \mathcal{Y}(F_A^+)|_u
\]

Take the inner product on both the sides, with \( \lambda_0 \circ u \) to get

\[
0 = \frac{s_X}{4} \cdot \rho_0 \circ u + \langle \mu \circ u, F_A^+ \rangle
\]

(27)

The second term in the above expression is computed as follows:

\[
\langle \mathcal{Y}(F_A^+)|_u, \lambda_0 \circ u \rangle = \sum_{l=1}^{3} \left< I_l K M_{(F^+_A, \xi_l)}|_u, I_l K M_{(F^+_A, \xi_l)}|_u \right> = \sum_{l=1}^{3} \left< K^M_{(F^+_A, \xi_l)}|_u, K^M_{(F^+_A, \xi_l)}|_u \right>
\]

\[
= \sum_{l=1}^{3} \mu(\xi_l \otimes (F_A^+, \xi_l)) \circ u = \langle \mu \circ u, F_A^+ \rangle
\]

Since \( D_A u = 0 \), \( d(\rho_0 \circ u) = (d\rho_0, D_A u) = 0 \), \( \rho_0 \circ u = c \), a constant. Therefore,

\[-\frac{s_X}{4} \cdot \rho_0 \circ u = \langle \mu \circ u, F_A^+ \rangle
\]

If the pair \((u, A)\) is a solution to generalized Seiberg-Witten equations (13), then

\[-\frac{s_X}{4} \cdot \rho_0 \circ u = |\mu \circ u|^2
\]

Since \( \rho_0 \circ u \) is a positive constant, this implies \( s_X \) is a negative constant. Therefore, the metric on \( X \) is a metric of constant scalar curvature. Since the metric defined by the Kähler structure is a scalar multiple of \( g_X \), it follows that the Kähler structure defined by \( \Phi(\mu \circ u) \) is of constant scalar curvature.
Appendix A. Vector bundles and connections

Let $\pi_E : E \rightarrow X$ be a vector bundle. Then consider $T\pi_E : TE \rightarrow TX$. Then $V_E \subseteq \ker(T\pi_E) \subseteq TE$ is called the vertical sub-bundle. A connection on $E$ is a choice of a smooth horizontal sub-bundle $H_E$ such that $TE = H_E \oplus V_E$. Denote by $pr_v$ and $pr_{u_E}$ the projections to the vertical and the horizontal sub-bundles respectively. A connection on $E$ is said to be linear if $pr_v$ is linear w.r.t $T\pi_E$

**Vertical lift.** Consider the pull-back bundle $E \times_M E$. Then, the map $vl_E : E \times M \rightarrow V_E, (v,w) \mapsto \frac{d}{dt}(v + tw)|_{t=0}$ gives an isomorphism of vector bundle over $E$ and is called a vertical lift.

**Connector.** A connector is a smooth map $K : TE \rightarrow E$ is a smooth map that satisfies $K \circ vl_E = pr_2 : E \times M \rightarrow E$ and is a vector bundle homomorphism for both the vector bundle structures on $E$; i.e $T\pi_E : TE \rightarrow TX$ and $\pi : TE \rightarrow E$.

Given a linear connection $\Phi : TE \rightarrow TE$, its connector is given by the composition

$TE \xrightarrow{\Phi} V_E \xrightarrow{(vl_E)^{-1}} E \times X \xrightarrow{pr_2} E$

A connector on $E$ induces a covariant derivative $\nabla^K : \Gamma(X, E) \rightarrow \Gamma(X, T^*X \otimes E)$

$\nabla^K_v(s) = K(Ts(v)) \text{ for } v \in TX, s \in \Gamma(X, E)$

Appendix B. Principal bundles and covariant derivatives on associated bundles

Let $\pi_P : P \rightarrow X$ be a principal $G$-bundle and $M$ be a manifold with a smooth action of $G$.

**Theorem B.1.** [Bau09, Satz 3.5] There is a bijection between the space of $G$-equivariant maps from $P \rightarrow M$ and the sections of the fibre bundle $F := P \times_G M \rightarrow X$

$\Gamma^\infty(P, M)^G \rightarrow \Gamma(X, F)$

$u \mapsto s_u$, where $s_u(x) = [p, u(p)]$ and $\pi_P(p) = x$

(28)

**Covariant derivatives on associated fibre-bundles.** Let $A \in \Lambda^1(P, g)^G$ be a connection on $P$. We define the covariant derivative of $u \in \text{Map}(P, M)^G$ w.r.t as

$D_Au = Tu + K^M_A|_u \in \text{Hom}(TP, T^*M)_{\text{hor}}$

where $K^M_A|_u : TP \rightarrow u^*TM$ is vector bundle homomorphism and the subscript “hor” implies that $D_Au$ vanishes for all $v \in V_{TP}$

$K^M_A|_u(v) = K^M_A|_{u(p)}(v), v \in T_pP$

**Proposition B.2.** [KM97, Theorem 42.1] Let $X$ be a compact manifold. Then the space $\text{Map}(P, M)^G$ is a Fréchet manifold modelled on topological vector spaces:

$T_u \text{Map}(P, M)^G = \Gamma(P, u^*TM)^G$

Therefore the covariant derivative can be interpreted as a section of the infinite-dimensional Fréchet vector-bundle:
Note that 
\[ T \text{Hom}(\mathcal{H}_A, TM)^G = \text{Hom}(\mathcal{H}_A, TTM)^G, \quad \forall \text{Hom}(\mathcal{H}_A, TM)^G = \text{Hom}(\mathcal{H}_A, VT^M)^G \]

A connector \( \psi : TTM \rightarrow TM \) induces a connector \( \Psi \) on \( \text{Hom}(\mathcal{H}_A, TM)^G \). Using this, one can compute the linearization of \( \nabla^\Psi D_A \).

**Lemma B.3.** [Sch10, Section 2.4] The linearization \( \nabla^\Psi D_A \) of the covariant derivative coincides with the first-order differential operator

\[ \nabla^{A,\psi} : C^\infty(TP, TM)^G \rightarrow \text{Hom}(TP, TM)^G_{\text{hor}}, \quad v \mapsto \psi \circ T\nu \circ \text{pr}_{\mathcal{H}_A} \]

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