On the Ramsey Numbers for Bipartite Multigraphs∗

Ming-Yang Chen†  Hsueh-I Lu‡  Hsu-Chun Yen§

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Abstract

A coloring of a complete bipartite graph is shuffle-preserved if it is the case that assigning a color \( c \) to edges \((u, v)\) and \((u', v')\) enforces the same color assignment for edges \((u, v')\) and \((u', v)\). (In words, the induced subgraph with respect to color \( c \) is complete.) In this paper, we investigate a variant of the Ramsey problem for the class of complete bipartite multigraphs. (By a multigraph we mean a graph in which multiple edges, but no loops, are allowed.) Unlike the conventional \( m \)-coloring scheme in Ramsey theory which imposes a constraint (i.e., \( m \)) on the total number of colors allowed in a graph, we introduce a relaxed version called \( m \)-local coloring which only requires that, for every vertex \( v \), the number of colors associated with \( v \)'s incident edges is bounded by \( m \). Note that the number of colors found in a graph under \( m \)-local coloring may exceed \( m \). We prove that given any \( n \times n \) complete bipartite multigraph \( G \), every shuffle-preserved \( m \)-local coloring displays a monochromatic copy of \( K_{p,p} \) provided that \( 2(p-1)(m-1) < n \). Moreover, the above bound is tight when (i) \( m = 2 \), or (ii) \( n = 2^k \) and \( m = 3 \cdot 2^{k-2} \) for every integer \( k \geq 2 \). As for the lower bound of \( p \), we show that the existence of a monochromatic \( K_{p,p} \) is not guaranteed if \( p > \lceil \frac{n}{m} \rceil \).

Finally, we give a generalization for \( k \)-partite graphs and a method applicable to general graphs. Many conclusions found in \( m \)-local coloring can be inferred to similar results of \( m \)-coloring.

1 Introduction

Ramsey theory, originated in a seminal paper by Ramsey [11] in 1930, has emerged as a fast growing and fascinating research topic in mathematics and theoretical computer science in recent years. Ramsey theory deals with the investigation of the conditions under which a

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†Department of Electrical Engineering, National Taiwan University, Taipei 106, Taiwan, Republic of China. Email: roc3.chen@msa.hinet.net.
‡Corresponding author: Institute of Information Science, Academia Sinica, Taipei 115, Taiwan, Republic of China. Email: hil@iis.sinica.edu.tw. URL: www.iis.sinica.edu.tw/~hil/
§Department of Electrical Engineering, National Taiwan University Taipei 106, Taiwan, Republic of China. Email: yen@cc.ee.ntu.edu.tw. URL: www.ee.ntu.edu.tw/~yen/
sufficiently large complete graph always includes a certain substructure \[7\]. Among various graphs for which Ramsey-type problems have been investigated, bipartite graphs constitute a class for which several deep results have been obtained for a variety of Ramsey numbers (see, e.g., \[2, 3, 5, 6\]). For example, in \[6\], Erdős and Rousseau proved that in every 2-coloring of \(K_{n,n}\), there is a monochromatic copy of \(K_{p,p}\) if \(n > (p - 1)\left(\frac{n}{p}\right)\).

In a more recent article \[3\], Carnielli and Carmelo showed that in any 2-coloring of a bipartite complete graph \(K_{n,n}\), one can always find a monochromatic subgraph isomorphic to \(K_{p,q}\) if \(n \geq 2p(q - 1) + 2^{p-1} + 1\). As in asymptotic versions \[4, 9\], Caro and Rousseau achieved that there are constants \(c_1\) and \(c_2\) such that
\[
c_1 \left(\frac{p}{\log p}\right)^{(q+1)/2} < b(p, q) < c_2 \left(\frac{p}{\log p}\right)^q,
\]
holds as \(p\) tends to infinity, where \(b(p, q)\) is the smallest integer \(n\) such that every 2-coloring (say red and green) of the edges of \(K_{n,n}\) contains either a red \(K_{p,p}\) or a green \(K_{q,q}\).

Some recent research focuses turn to seek other types of Ramsey theory which give tighter bounds or stronger relations with other well-known combinatorial numbers, but usually require additional constraints on the graph. For example, Alon, Erdős, Gunderson, and Molloy \[1\] proved that the corresponding largest integer \(m\) is asymptotically equal to the Turán number \(t(n, \lfloor \frac{n^2}{2k}\rfloor)\) if the smallest Ramsey number \(n\) satisfies that, for any \(k\)-coloring of complete graph \(K_{n,n}\), there exists a copy of \(K_m\) whose edges receive at most \(k - 1\) colors. Similarly, we address how to find a monochromatic copy of biclique in a bipartite multigraph with a shuffle-preserved coloring, i.e., a coloring \(\eta\) such that for all \(u, u' \in U\) and \(v, v' \in V\), the condition \((u, v)_c \land (u', v')_c\) implies the condition \((u, v')_c \land (u', v)_c\), where \((u, v)_c\) denotes the existence of an edge colored \(c\) between vertices \(u\) and \(v\). (See Figure \[1\].) In the present paper, we prove that given any \(n \times n\) complete bipartite multigraph \(G\), every shuffle-preserved \(m\)-local coloring displays a monochromatic copy of \(K_{p,p}\) provided that \(2(p - 1)(m - 1) < n\). Moreover, the above bound is tight when (i) \(m = 2\), or (ii) \(n = 2^k\) and \(m = 3 \cdot 2^{k-2}\) for every integer \(k \geq 2\). As for the lower bound of \(p\), we show that the existence of a monochromatic \(K_{p,p}\) is not guaranteed if \(p > \left\lceil \frac{m}{n} \right\rceil\). Finally, we give a generalization for \(k\)-partite graphs and a method applicable to general graphs.

It is worthy of pointing out that the constraint of shuffle-preserving, to a certain extent, resembles the requirement found in the notion of induced Ramsey numbers (see, e.g., \[8\]) which deals with conditions under which the existence of a monochromatic induced subgraph is guaranteed. Note that what shuffle-preserved means is that for every color \(c\), the subgraph induced by \(c\) is complete. (A subgraph \((V', E')\) of \((V, E)\) (where \(V' \subseteq V\) and \(E' \subseteq E\)) is said to be induced if \(E' = \{(u, v) : u, v \in V', (u, v) \in E\}\).) As we shall see later, the underlying coloring being shuffle-preserved plays a critical role in the existence of a much smaller Ramsey number, in comparison with those reported in the literature for bipartite
Figure 1: A monochromatic $K_{3,3}$ in a coloring of a $4 \times 4$ bipartite multigraph. Notice that here $m$ is bounded by 3, although the total number of colors used in the graph is 5.

graphs. For more about Ramsey numbers for a variety of graphs, the interested reader is referred to [10].

The rest of the paper is organized as follows. Section 2 presents the key theorem of this paper, whose tightness is addressed in Section 3. Section 4 shows our results for $k$-partite graphs and a generalization to general graphs. Section 5 concludes the paper with some open questions.

2 Ramsey theory for bipartite multigraphs

An $n \times n$ bipartite multigraph $G$ is said to be complete if every two vertices of different bipartitions are adjacent. Given a vertex $u$, we write $C(u)$ to denote the set $\{c \in C :$ an incident edge of $u$ is colored $c\}$. For an $n \times n$ bipartite multigraph $G = (V, E)$ and a color set $C$, a coloring $\eta$ is said to be an $m$-local coloring if for every vertex $u$, $|C(u)| \leq m$ (i.e., the number of colors assigned to $u$’s incident edges is at most $m$). Notice that for any $m$-local coloring, the total number of colors used in the entire graph may exceed $m$. Now the following main theorem gives Ramsey theory for bipartite multigraphs.

**Theorem 2.1** Let $G$ be an $n \times n$ complete bipartite multigraph. If $p$ and $m$ are positive integers such that $2(p-1)(m-1) < n$, then any shuffle-preserved $m$-local coloring has a monochromatic copy of $K_{p,p}$.

**Proof.** Suppose that the vertex set of $G$ is partitioned as $U \cup V$. Let $\eta$ be an arbitrary shuffle-preserved $m$-local coloring on $G$ using color set $C$. Without loss of generality, we may assume that all the multiple edges between each pair of vertices have distinct colors under
We define \( U(c) \) (respectively, \( V(c) \)) to be the set of vertices in \( U \) (respectively, \( V \)) each of which has at least one incident edge colored \( c \). To prove our theorem, it suffices to show that there exists a \( c \in C \) such that \(|U(c)| \geq p\) and \(|V(c)| \geq p\). (In this case, the existence of a monochromatic \( K_p,p \) follows immediately from \( \eta \) being shuffle-preserved.)

We prove the theorem by contradiction. Suppose, on the contrary, that there were no \( c \in C \) satisfying \(|U(c)| \geq p\) and \(|V(c)| \geq p\). For convenience, we write the ordered triple \((u, v, c)\) to represent the relation \( c \in C(u) \cap C(v) \), and let \( T \) be the set of all such triples. By \( K_{n,n} \subseteq G \), it is reasonably easy to observe

\[
|T| = \sum_{u \in U} \sum_{v \in V} |C(u) \cap C(v)| \geq |U| \cdot |V| = n^2.
\]

Furthermore, by changing order in double summation, the following also hold.

\[
\sum_{c \in C} |U(c)| = \sum_{u \in U} |C(u)| \leq m |U| \quad (2)
\]

\[
\sum_{c \in C} |V(c)| = \sum_{v \in V} |C(v)| \leq m |V|. \quad (3)
\]

Let

\[
C_1 = \{ c \in C : |U(c)| \geq p \};
\]

\[
C_2 = \{ c \in C : |U(c)| < p \}.
\]

For a vertex \( u \) and a color \( c \), we write \( u \xrightarrow{c} \) to represent the set of \( u \)'s adjacent vertices each of which is connected to \( u \) through some edge of color \( c \). For every \( u \in U \), by the Pigeonhole Principle there must be a color \( c \in C \) such that \(|u \xrightarrow{c}| \geq \lceil n/m \rceil\) (i.e., at least \( \lceil n/m \rceil \) incident edges of \( u \) are colored \( c \)). Note that \( p - 1 < \frac{n}{2(m-1)} \) implies

\[
\left\lfloor \frac{n}{2(m-1)} \right\rfloor \geq p.
\]

By \( m \geq 2 \) we have

\[
|u \xrightarrow{c}| \geq \left\lceil \frac{n}{m} \right\rceil \geq p.
\]

Hence, \(|V(c)| \geq |u \xrightarrow{c}| \geq p\) (at least \( p \) of vertex \( u \)'s neighbors are in \( V(c) \)). Due to our assumption that no \( c \in C \) satisfies \(|U(c)| \geq p\) and \(|V(c)| \geq p\) simultaneously, \( c \) must be in \( C_2 \). For every vertex \( u \in U \), there exists a \( c \in C(u) \) such that \( c \in C_2 \). Hence,

\[
\sum_{c \in C_2} |U(c)| \geq |U|
\]

and from Equations (2) and (3) we have

\[
\sum_{c \in C_1} |U(c)| \leq (m - 1) |U|.
\]
Similarly for each $v \in V$, there must be a color $c$ such that $|v \xrightarrow{c}| \geq p$ (i.e., at least $p$ vertices in $U$ are adjacent to $v$ through edges colored $c$). Thus $c$ will be in $C_1$ and we have

$$
\sum_{c \in C_1} |V(c)| \geq |V|;
$$

$$
\sum_{c \in C_2} |V(c)| \leq (m - 1)|V|.
$$

Therefore, using the above inequalities, we find

$$
|T| = \sum_{c \in C} |U(c)| \cdot |V(c)|
= \sum_{c \in C_1} |U(c)| \cdot |V(c)| + \sum_{c \in C_2} |U(c)| \cdot |V(c)|
\leq (p - 1)\sum_{c \in C_1} |U(c)| + (p - 1)\sum_{c \in C_2} |V(c)|
\leq (p - 1)(m - 1)|U| + (p - 1)(m - 1)|V|
= 2(p - 1)(m - 1)n
< n^2,
$$

which contradicts Equation (1). Our theorem follows. 

Since every $m$-coloring (i.e., coloring a graph with at most $m$ distinct colors) is clearly an $m$-local coloring, the following is straightforward.

**Corollary 2.2** In every shuffle-preserved $m$-coloring of an $n \times n$ complete bipartite multi-graph $G$, there is a monochromatic copy of $K_{p,p}$ if

$$
2(p - 1)(m - 1) < n.
$$

For $m = 2$, Corollary 2.2 suggests a sufficient condition of $n > 2(p - 1)$ for the existence of a monochromatic $K_{p,p}$. The interested reader should contrast this result with a much larger bound (i.e., $n > 2^p$) in [6] in which the shuffle-preserved constraint is lifted.

## 3 Necessary and sufficient conditions

In this section we provide necessary and sufficient conditions for the existence of a monochromatic copy of $K_{p,p}$ for some special cases.

**Lemma 3.1** Let $G$ be a complete $n \times n$ bipartite graph without multiple edges.

1. If $p > \left\lceil \frac{n}{m} \right\rceil$, then there exists a shuffle-preserved $m$-coloring (and thus a shuffle-preserved $m$-local coloring) of $G$ such that $G$ does not contain any monochromatic copy of $K_{p,p}$. 

Figure 2: An example which does not contain any monochromatic copy of $K_{3,3}$ for $(n, m, p) = (5, 3, 3)$ in Lemma 3.1(1).

2. If $n = 2^k$, $m = 3 \times 2^{k-2}$, and $p = 2$ for some $k \geq 2$, then there exists a shuffle-preserved $m$-local coloring of $G$ such that $G$ does not contain any monochromatic copy of $K_{p,p}$.

Proof. Let $u_i$ (respectively, $v_i$) be the $i$-th node of $U$ (respectively, $V$).

Statement 1. We color the edge between $u_i$ and $v_j$ by color $(i \mod m)$. One can easily verify that such a coloring is indeed a shuffle-preserved $m$-coloring. By $p > \left\lceil \frac{n}{m} \right\rceil$, each $v_i$ has at most $p - 1$ incident edges with the same color. Therefore, $G$ contains no monochromatic copy of $K_{p,p}$. (See Figure 2)

Statement 2. For each $\ell = 2, 3, \ldots, k$, let $M_\ell$ denote the $2^\ell \times 2^\ell$ color matrix whose the $(i, j)$-entry specifies the color of the edge between $u_i$ and $v_j$. We construct $M_k$ recursively by letting

$$M_2 = \begin{pmatrix} 1 & 5 & 2 & 2 \\ 1 & 4 & 3 & 4 \\ 8 & 5 & 8 & 7 \\ 6 & 6 & 3 & 7 \end{pmatrix}$$

(see Figure 3) and

$$M_{\ell+1} = \begin{pmatrix} M_\ell & \mu_\ell + M_\ell \\ 2\mu_\ell + M_\ell & 3\mu_\ell + M_\ell \end{pmatrix}$$

for each $\ell = 3, 4, \ldots, k$, where $t\mu_\ell + M_\ell$ denotes the matrix obtained by adding $t$ times of the maximum of $M_\ell$ to each entry of $M_\ell$. Clearly, $M_2$ is 3-local coloring since each row
Figure 3: The edge coloring of the matrix $M_2$ in Lemma 3.1(2) shows that no monochromatic subgraph of $K_{2,2}$ in the case $(n, m) = (4, 3)$.

(and column) consists of three colors. The lack of a monochromatic $K_{2,2}$ in $M_2$ is also straightforward. Likewise, it is reasonably easy to verify that the constructed $M_k$ gives a shuffle-preserved $3 \times 2^{k-2}$-local coloring of $G$ such that no monochromatic copy of $K_{p,p}$ can be found in $G$. □

Combining Theorem 2.1 and Lemma 3.1, we have the following results.

**Theorem 3.2** Every shuffle-preserved 2-local coloring of an $n \times n$ complete bipartite graph $G$ gives a monochromatic copy of $K_{p,p}$ if and only if

$$2(p - 1) < n.$$ 

**Proof.** Since $(p - 1) \geq \frac{n}{2}$ and $p > \left\lfloor \frac{n}{2} \right\rfloor$ are equivalent, the theorem follows from Theorem 2.1 and Lemma 3.1(1) with $m = 2$.† □

**Theorem 3.3** Let $n = 2^k$ and $m = 3 \times 2^{k-2}$ for some integer $k \geq 2$. Any shuffle-preserved $m$-local coloring of an $n \times n$ complete bipartite graph gives a monochromatic copy of $K_{p,p}$ if and only if

$$2(p - 1)(m - 1) < n.$$ 

**Proof.** By Theorem 2.1 and Lemma 3.1(1), the theorem holds when $2(p - 1)(m - 1) < n$ or $p > \left\lceil \frac{n}{m} \right\rceil$. When $2(p - 1)(m - 1) \geq n$ and $p \leq \left\lfloor \frac{n}{m} \right\rfloor$, by $n = 2^k$, $m = 3 \times 2^{k-2}$, and $k \geq 2$, one can obtain $p = 2$. Thus, the theorem follows from Lemma 3.1(2). □

†Observe that Theorem 3.2 also holds even if changing 2-local coloring to 2-coloring due to the nature of Lemma 3.1(1).
4 Generalization

In this section, we give a generalization of the results in Section 2 for $k$-partite graphs as well as for general graphs. For complete $k$-partite graphs, the following results generalize Corollary 2.2 and Lemma 3.1(1).

**Corollary 4.1** In every shuffle-preserved 2-coloring of an $n \times n \times \cdots \times n$ complete $k$-partite multigraph, there is a monochromatic copy of a complete $p \times p \times \cdots \times p$ $k$-partite graph $G$ if

$$2(p - 1) < n.$$  

**Proof.** Let the $k$-partitions of the vertex set be

$$S_1 = \{v_{1,1}, v_{1,2}, \ldots, v_{1,n}\};$$

$$S_2 = \{v_{2,1}, v_{2,2}, \ldots, v_{2,n}\};$$

$$\vdots$$

$$S_k = \{v_{k,1}, v_{k,2}, \ldots, v_{k,n}\}.$$ 

The case $k = 2$ follows from Corollary 2.2. When $k = 3$, we can find one monochromatic $K_{p,p}$ in each bipartite subgraph of $G$ induced by $S_i$ and $S_j$ with $i \neq j$. Since the graph is two-colored, from the Pigeonhole Principle at least two monochromatic copies of $K_{p,p}$ have the same color, say $c$. This tells us that for every set $S_i$, there are at least $p$ vertices, each of which has some incident edge colored $c$, and thus from the coloring being shuffle-preserved, we have a monochromatic $p \times p$ tripartite graph.

Suppose now $k > 3$. Let $G'_i$ denote the $p \times p \times \cdots \times p$ ($k - 1$)-partite subgraph of $G$ induced by all but the nodes in $S_i$. By the inductive hypothesis, there are $k$ monochromatic $p \times p \times \cdots \times p$ ($k - 1$)-partite graphs $G'_1, G'_2, \ldots, G'_k$. By the Pigeonhole Principle, at least two of them have the same color. By the same argument, a monochromatic $p \times p \times \cdots \times p$ $k$-partite graph is obtained.  

**Corollary 4.2** If $p > \left\lceil \frac{2n}{m} \right\rceil$, then there exists an $m$-colored complete $n \times n \times \cdots \times n$ $k$-partite multigraph $G$ that does not contain any monochromatic complete $p \times p \times \cdots \times p$ $k$-partite subgraph.

**Proof.** The proof is a natural generalization of that for Lemma 3.1(1). Let $S_1, S_2, \ldots, S_k$ be the $k$-partitions of the vertex set of $G$. The edges of $G$ are constructed and colored as follows. Let the subgraph of $G$ induced by $S_i$ and $S_j$ for any $j$ with $2 \leq j \leq k$ contain no multiple edges. We color all the incident edges of the $i$-th node of $S_1$ by color $(i \mod m)$. For any indices $i$ and $j$ with $2 \leq i < j \leq k$, let the subgraph of $G$ induced by $S_i$ and $S_j$ be $m$ superimposed copies of $K_{n,n}$, each with a distinct color. It is not difficult to verify that the resulting coloring is shuffle-preserved. By $p > \left\lceil \frac{2n}{m} \right\rceil$, any node in $S_2 \cup S_3 \cup \cdots \cup S_k$ is adjacent to at most $p - 1$ nodes in $S_1$ through edges with the same color. Therefore, $G$ does not contain any complete $p \times p \times \cdots \times p$ $k$-partite subgraph.  

Hence, we get the following theorem immediately.
**Theorem 4.3** In every shuffle-preserved 2-coloring of an $n \times n \times \cdots \times n$ complete $k$-partite multigraph, there is a monochromatic copy of a complete $p \times p \times \cdots \times p$ $k$-partite graph $G$ if and only if

$$2(p - 1) < n.$$ 

Now we consider general graphs. Clearly, we have to generalize the definition of shuffle-preserved coloring as follows: a coloring $\eta$ is shuffle-preserved if the induced subgraph with respect to every color $c$ is complete. Recall that the classical Ramsey theory focuses on the situation that the particular subgraph is required to be monochromatic. In the next theorem we give a sufficient condition for a more general situation that the shuffle-preserved coloring guarantees the existence of a complete subgraph whose edges receive at most some number of colors. (See [1] for a similar theorem on complete graphs.)

**Theorem 4.4** Let $G$ be a graph with $n$ vertices and shuffle-preserved $m$-coloring. Suppose that $d_i$ represents the number of vertices, each of which has exactly $i$ different colors among its incident edges. Then $G$ has a $t$-superimposed copy of $K_p$ if

$$p \leq \left[ \frac{\sum_{i=t}^{m} d_i \binom{i}{t}}{\binom{m}{t}} \right].$$

**Proof.** Let $P_t(S)$ be the set of all $t$-element subsets of $S$ and $V_i$ consist of the vertices of the complete subgraph induced by the $i$-th color. Then by two-way counting we have

$$\sum_{\{i_1, i_2, \ldots, i_t\} \in P_t(\{1, 2, \ldots, m\})} |V_{i_1} \cap V_{i_2} \cap \cdots \cap V_{i_t}| = \sum_{i=t}^{m} d_i \binom{i}{t} \text{ since both sides give the number of all vertices in any } t\text{-element combination among } \{V_1, V_2, \ldots, V_m\}. \text{ Let } S_t = \max_{\{i_1, i_2, \ldots, i_t\} \in P_t(\{1, 2, \ldots, m\})} |V_{i_1} \cap V_{i_2} \cap \cdots \cap V_{i_t}|.$$

Combining the above expressions we have

$$\sum_{i=t}^{m} d_i \binom{i}{t} \leq S_t \binom{m}{t},$$

proving the theorem. \[ \square \]

**5 Concluding remarks**

When $m(p - 1) < n \leq 2(m - 1)(p - 1)$, the necessary and sufficient conditions for the existence of monochromatic $K_{p,p}'s$ in bipartite graphs remain open. It would be interesting to see tighter results for $k$-partite graphs.

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