SHARP UPPER AND LOWER BOUNDS OF THE ATTRACTOR DIMENSION FOR 3D DAMPED EULER–BARDINA EQUATIONS

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Abstract. The dependence of the fractal dimension of global attractors for the damped 3D Euler–Bardina equations on the regularization parameter $\alpha > 0$ and Ekman damping coefficient $\gamma > 0$ is studied. We present explicit upper bounds for this dimension for the case of the whole space, periodic boundary conditions, and the case of bounded domain with Dirichlet boundary conditions. The sharpness of these estimates when $\alpha \to 0$ and $\gamma \to 0$ (which corresponds in the limit to the classical Euler equations) is demonstrated on the 3D Kolmogorov flows on a torus.

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2000 Mathematics Subject Classification. 35B40, 35B45, 35L70.
Key words and phrases. Regularized Euler equations, Bardina model, unbounded domains, attractors, fractal dimension, Kolmogorov flows.

This work was supported by Moscow Center for Fundamental and Applied Mathematics, Agreement with the Ministry of Science and Higher Education of the Russian Federation, No. 075-15-2019-1623 and by the Russian Science Foundation grant No.19-71-30004 (sections 2-4). The second author was partially supported by the Leverhulme grant No. RPG-2021-072 (United Kingdom).
1. Introduction

Being the central mathematical model in hydrodynamics, the Navier-Stokes and Euler equations permanently remain in the focus of both the analysis of PDEs and the theory of infinite dimensional dynamical systems and their attractors, see [2, 8, 13, 15, 16, 25, 26, 43, 44, 45] and the references therein for more details. Most studied is the 2D case where reasonable results on the global well-posedness and regularity of solutions as well as the results on the existence of global attractors and their dimension are available. However, the global well-posedness in the 3D case remains a mystery and even listed by the Clay institute of mathematics as one of the Millennium problems. This mystery inspires a comprehensive study of various modifications/regularizations of the initial Navier-Stokes/Euler equations (such as Leray-α model, hyperviscous Navier-Stokes equations, regularizations via $p$-Laplacian, etc.), many of which have a strong physical background and are of independent interest, see e.g. [14, 19, 26, 34, 37] and the references therein.

In the present paper we shall be dealing with the following regularized damped Euler system:

\[
\begin{aligned}
\partial_t \bar{u} + (\bar{u}, \nabla_x) \bar{u} + \gamma \bar{u} + \nabla_x p &= g, \\
\text{div } \bar{u} &= 0, \\
\bar{u}(0) &= u_0.
\end{aligned}
\]  

(1.1)

with forcing $g$ and Ekman damping term $\gamma \bar{u}$, $\gamma > 0$. The damping term $\gamma \bar{u}$ makes the system dissipative and is important in various geophysical models [39]. Here and below $\bar{u}$ is a smoothed (filtered) vector field related with the initial velocity field $u$ as the solution of the Stokes problem

\[
\begin{aligned}
\bar{u} &= \bar{u} - \alpha \Delta_x \bar{u} + \nabla_x q, \\
\text{div } \bar{u} &= 0,
\end{aligned}
\]  

(1.2)

where $\alpha > 0$ is a given small parameter. In other words,

\[
\bar{u} = (1 - \alpha A)^{-1} u,
\]

where $A := \Pi \Delta_x$ is the Stokes operator and $\Pi$ is the Helmholtz–Leray projection to divergent free vector fields in the corresponding domain.

System (1.1), (1.2) (at least in the conservative case $\gamma = 0$) is often referred to as the simplified Bardina subgrid scale model of turbulence, see [4, 5, 24] for the derivation of the model and further discussion, so in this paper we shall be calling (1.1) the damped Euler–Bardina equations. We also mention that rewriting (1.1) in terms of the variable $\bar{u}$ gives

\[
\partial_t \bar{u} - \alpha \partial_t \Delta_x \bar{u} + (\bar{u}, \nabla_x) \bar{u} + \gamma \bar{u} + \nabla_x p = \alpha \gamma \Delta_x \bar{u} + g
\]  

(1.3)

which is a damped version of the so-called Navier–Stokes–Voight equations arising in the theory of viscoelastic fluids, see [23, 38] for the details.

Our main interest in the present paper is to study the dimension of global attractors for system (1.1) in 2D and 3D paying main attention to the most complicated 3D case. Note that, unlike the classical Euler equations, Bardina-Euler equations can be interpreted as an ODE with bounded nonlinearity in the proper Hilbert space, so no problems with well-posedness arise, see [5] and also
section §2 below, so the main aim of our study is to get as sharp as possible bounds for the corresponding global attractors. Each case $d = 2$ and $d = 3$ in turn is studied in three different settings as far as the boundary conditions are concerned. More precisely, the system is studied

(1) on the torus $\Omega = \mathbb{T}^d = [0, 2\pi]^d$. In this case the standard zero mean condition is imposed on $u, \bar{u}$ and $g$;

(2) in the whole space $\Omega = \mathbb{R}^d$;

(3) in a bounded domain $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary conditions for $\bar{u}$.

We denote by $W^{s,p}(\Omega)$ the standard Sobolev space of distributions whose derivatives up to order $s$ belong to the Lebesgue space $L^p(\Omega)$. In the Hilbert case $p = 2$ we will write $H^s(\Omega)$ instead of $W^{s,2}(\Omega)$. In order to work with velocity vector fields, we denote by $H^s = H^s(\Omega)$ the subspace of $[H^s(\Omega)]^d$ consisting of divergence free vector fields. In the case of $\Omega \subset \mathbb{R}^d$ we assume in addition that vector fields from $H^s$ satisfy Dirichlet boundary conditions and in the case of periodic boundary conditions $\Omega = \mathbb{T}^d$ we assume that these vector fields have zero mean.

We also recall that equation (1.1) possesses the standard energy identity

$$\frac{1}{2} \frac{d}{dt} \left( \|\bar{u}\|^2_{L^2(\Omega)} + \alpha \|\nabla_x \bar{u}\|^2_{L^2(\Omega)} \right) + \gamma \left( \|\bar{u}\|^2_{L^2(\Omega)} + \alpha \|\nabla_x \bar{u}\|^2_{L^2(\Omega)} \right) = (g, \bar{u}),$$

where $(u, v)$ is the standard inner product in $[L^2(\Omega)]^d$. For this reason it is natural to consider problem (1.1) in the phase space $H^1$ with norm

$$\|\bar{u}\|^2_{H^1} := \|\bar{u}\|^2_{L^2} + \alpha \|\nabla_x \bar{u}\|^2_{L^2}.$$ 

Our first main result is the following theorem which gives an explicit upper bound for the fractal dimension of the attractor in the 3D case.

**Theorem 1.1.** Let $d = 3$, let $\Omega$ be as described above, and let $g \in [L^2(\Omega)]^3$ (in the periodic case we assume also that $g$ has zero mean). Then the solution semigroup $S(t)$ associated with equation (1.1) possesses a global attractor $\mathcal{A} \subset H^1$ with finite fractal dimension satisfying the following inequality:

$$\dim_F \mathcal{A} \leq \frac{1}{12\pi} \frac{\|g\|^2_{L^2}}{\alpha^{5/2} \gamma^4}. \quad (1.4)$$

The analogue of this estimate for the 2D case reads

$$\dim_F \mathcal{A} \leq \frac{1}{16\pi} \frac{\|g\|^2_{L^2}}{\alpha^2 \gamma^4} \quad (1.5)$$

with the following improvement for the case when $\Omega = \mathbb{T}^2$ or $\Omega = \mathbb{R}^2$:

$$\dim_F \mathcal{A} \leq \frac{1}{8\pi} \frac{\|\text{curl } g\|^2_{L^2}}{\alpha \gamma^4} \quad (1.6)$$

due to estimates related with the vorticity equation.
Since the general case $\gamma > 0$ is reduced to the particular one with $\gamma = 1$ by scaling $t \to \gamma^{-1} t$, $u \to \gamma^{-2} u$, $g \to \gamma^{-2} g$, the most interesting in estimates (1.4), (1.5) and (1.6) is the dependence of the RHS on $\alpha$. For the viscous case of equations (1.1)

$$\partial_t u + (\bar{u}, \nabla_x)\bar{u} + \nabla_x q = \nu \Delta_x u + g$$

the following estimate is proved in [5]:

$$\dim_{F} \mathcal{A} \leq C \frac{\|g\|_{L^2}^{6/5}}{\nu^{12/5} \alpha^{18/5}}$$

for the case $\Omega = \mathbb{T}^3$. We see that even in the case $\nu = 1$ this estimate gives essentially worse dependence on $\alpha$ than our estimate (1.4). The upper bounds for 3D Navier-Stokes-Voight equation obtained in [23] give even worse dependence on the parameter $\alpha$ (like $\alpha^{-6}$). Estimates (1.5) and (1.6) have been proved for $\Omega = \mathbb{T}^2$ in a recent paper [22]. The sharpness of these estimates in the limit as $\alpha \to 0$ was also established there for the case of the 2D Kolmogorov flows. However, to the best of our knowledge, no lower bounds for the dimension of the attractor of the Euler–Bardina equations in 3D are available in the literature.

Our second main result covers this gap. Namely, we consider the 3D Kolmogorov flows on the torus $\Omega = \mathbb{T}^3$ for equations (1.1) generated by the family of the right-hand sides parameterized by an integer parameter $s \in \mathbb{N}$:

$$g = g_s = \begin{cases} 
    g_1 = \gamma^2 \lambda(s) \sin(s x_3), \\
    g_2 = 0, \\
    g_3 = 0,
\end{cases}$$

(1.7)

where $s \sim \alpha^{-1/2}$ and $\lambda(s)$ is a specially chosen amplitude, see §5. Then, performing an accurate instability analysis for the linearization of equation (1.1) on the corresponding Kolmogorov flow (in the spirit of [33], see also [20, 21, 32]), we get the following result.

**Theorem 1.2.** Let $\Omega = \mathbb{T}^3$ and let $\gamma > 0$, and $\alpha > 0$. Then in the limit $\alpha \to 0$ the integer parameter $s$ and the amplitude $\lambda(s)$ can be chosen so that the corresponding forcing $g = g_s$ of the form (1.7) produces the global attractor $\mathcal{A} = \mathcal{A}_s$, whose dimension satisfies the following lower bound:

$$\dim_{F} \mathcal{A} \geq c \frac{\|g\|_{L^2}^2}{\alpha^{5/2} \gamma^4},$$

(1.8)

where $c > 0$ is an absolute effectively computable constant.

Estimate (1.8) shows that our upper bound (1.4) is optimal. Again, to the best of our knowledge, this is the first optimal two-sided estimate for the attractor dimension in a 3D hydrodynamical problem.

In this connection we recall the celebrated upper bound in [11] for the attractor dimension of the classical Navier–Stokes system on the 2D torus, which is still
logarithmically larger than the corresponding lower bound in [32]. On the other hand, adding to the system an arbitrary fixed damping makes it possible to obtain the estimate for the attractor dimension that is optimal in the vanishing viscosity limit [21].

We finally observe that the obtained lower estimates for the attractor dimension grow as \( \alpha \to 0 \) in both 2D and 3D cases (and even are optimal for the case of tori), so one may expect that the limit attractor \( A_0 \) (which corresponds to the case of non-modified damped Euler equation) is infinite dimensional. Indeed, the existence of the attractor \( A_0 \) in the proper phase space is well-known in 2D at least if \( g \in W^{1,\infty} \), see [9] and references therein and we expect that some weaker version of the limit attractor \( A_0 \) can be also constructed in 3D using the trajectory approach, see [8], and the concept of dissipative solutions for 3D Euler introduced by P. Lions, see [31]. However, the situation with the dimension is much more delicate since the obtained lower bounds for the instability index on Kolmogorov’s flows are optimal for intermediate values of \( \alpha \) only and do not provide any reasonable bounds for the limit case \( \alpha = 0 \). Thus, the question of finite or infinite-dimensionality of the limit attractor remains completely open even in the 2D case.

The paper is organized as follows. The key estimates for the solutions of problem (1.1) are derived in §2. Global well-posedness and dissipativity are also discussed there. The existence of a global attractor \( \mathcal{A} \) is verified in §4. To make the proof independent of the choice of a (bounded or unbounded) domain \( \Omega \), we use the so called energy method for establishing the asymptotic compactness of the associated semigroup.

The upper bounds for its dimension are obtained in §5 via the volume contraction method [2, 10, 44]. The essential role in getting optimal bounds for the global Lyapunov exponents is played by the collective Sobolev inequalities for \( H^1 \)-orthonormal families proved in Appendix A based on the ideas of [27]. Their role is somewhat similar to the role of the Lieb–Thirring inequalities [28, 29] in the dimension estimates of the attractors of the classical Navier–Stokes equations [2, 44]. The corresponding inequality in the 2D case has also been used in [22]. Finally, the sharp lower bounds of the dimension for the case \( \Omega = \mathbb{T}^3 \) are obtained in §5 by adapting/extending the ideas of [22, 33] to the 3D case.

2. A PRIORI ESTIMATES, WELL-POSEDNESS AND DISSIPATIVITY

We start with the standard energy estimate, which looks the same in the 2D and 3D cases as well as for the three types of boundary conditions.

**Proposition 2.1.** Let \( u \) be a sufficiently regular solution of equation (1.1). Then the following dissipative energy estimate holds:

\[
\| \dot{u}(t) \|_\alpha^2 \leq \| \dot{u}(0) \|_\alpha^2 e^{-\gamma t} + \frac{1}{\gamma^2} \| g \|_{L^2}^2 ,
\]

(2.1)
where
\[ \| \bar{u} \|_{\alpha}^2 := \| \bar{u} \|_{L^2}^2 + \alpha \| \nabla_x \bar{u} \|_{L^2}^2. \] (2.2)

**Proof.** Indeed, multiplying equation (1.1) by \( \bar{u} \), integrating over \( \Omega \) and using the relation between \( u \) and \( \bar{u} \) as well as the standard fact that the inertial term vanishes after the integration, we arrive at
\[
\frac{d}{dt} (\| \bar{u} \|_{L^2}^2 + \alpha \| \nabla_x \bar{u} \|_{L^2}^2) + 2\gamma (\| \bar{u} \|_{L^2}^2 + \alpha \| \nabla_x \bar{u} \|_{L^2}^2) = 2(g, \bar{u}) \leq 2\| g \|_{L^2} \| \bar{u} \|_{L^2} \leq 2\gamma \| \bar{u} \|_{L^2}^2 + \frac{1}{\gamma} \| g \|_{L^2}^2. \] (2.3)

Applying the Gronwall inequality, we get the desired estimate (2.1) and complete the proof. \( \Box \)

The next corollary is crucial for our upper bounds for the attractor dimension.

**Corollary 2.2.** Let \( u \) be a sufficiently smooth solution of problem (1.1). Then the following estimate holds:
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \| \nabla_x u(s) \|_{L^2} ds \leq \frac{1}{\gamma \sqrt{2\alpha}} \| g \|_{L^2}. \] (2.4)

**Proof.** Indeed, integrating estimate (2.3) over \( t \), taking the limit \( t \to \infty \) and using the fact that \( \| u(t) \|_{\alpha}^2 \) remains bounded (due to estimate (2.1), we arrive at
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \| \nabla_x u(s) \|_{L^2}^2 ds \leq \frac{1}{2\alpha \gamma^2 \pi} \| g \|_{L^2}^2. \]

Using after that the H"older inequality
\[
\frac{1}{t} \int_0^t \| \nabla_x u(s) \|_{L^2}^2 ds \leq \left( \frac{1}{t} \int_0^t \| \nabla_x u(s) \|_{L^2}^2 dx \right)^{1/2},
\]
we get the desired result and finish the proof of the corollary. \( \Box \)

We now turn to the two dimensional case without boundary. In this case, more accurate estimates are available due to the possibility to use the vorticity equation. Indeed, applying \( \text{curl} \) to (1.1) and setting \( \omega := \text{curl} u \), we obtain the vorticity equation for \( \omega \):
\[
\partial_t \omega + (\bar{u}, \nabla_x) \omega + \gamma \omega = \text{curl} g, \quad \omega = (1 - \alpha \Delta_x) \bar{\omega}. \] (2.5)

The estimates for the solution on the torus \( \mathbb{T}^2 \) were derived in [22]. Although for \( \mathbb{R}^2 \) they are formally the same, we reproduce them for the sake of completeness.

**Proposition 2.3.** Let \( u \) be a sufficiently smooth solution of (1.1), where \( \Omega = \mathbb{T}^2 \) or \( \mathbb{R}^2 \) and let \( \omega := \text{curl} u \) and \( \bar{\omega} := \text{curl} \bar{u} \). Then, the following dissipative estimate holds:
\[
\| \bar{\omega}(t) \|_{\alpha}^2 \leq \| \bar{\omega}(0) \|_{\alpha}^2 e^{-\gamma t} + \frac{1}{\gamma^2} \| \text{curl} g \|_{L^2}^2. \] (2.6)
Proof. Taking the scalar product of equation (2.5) with \( \bar{\omega} \), we see that the non-linear term vanishes and using that
\[
(\omega, \bar{\omega}) = \|\bar{\omega}\|_{L^2}^2 + \alpha \|\nabla_x \bar{\omega}\|_{L^2}^2,
\]
we obtain
\[
\frac{1}{2} \frac{d}{dt} (\|\bar{\omega}\|_{L^2}^2 + \alpha \|\nabla_x \bar{\omega}\|_{L^2}^2) + \gamma (\|\bar{\omega}\|_{L^2}^2 + \alpha \|\nabla_x \bar{\omega}\|_{L^2}^2) = (\text{curl} \, g, \bar{\omega}) \leq \frac{1}{2\gamma} \|\text{curl} \, g\|_{L^2}^2 + \frac{\gamma}{2} \|\bar{\omega}\|_{L^2}^2.
\]
This gives the desired estimate (2.6) by the Gronwall inequality and finishes the proof of the proposition.

Analogously to Corollary 2.2, we get the following estimate.

**Corollary 2.4.** Let \( \Omega = T^2 \) or \( \mathbb{R}^2 \) and let \( u \) be a sufficiently smooth solution of problem (1.1). Then the following estimate holds:
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \|\nabla_x \bar{u}(s)\|_{L^2} \, ds \leq \frac{1}{\gamma} \min \left\{ \|\text{curl} \, g\|_{L^2}, \frac{\|g\|_{L^2}}{\sqrt{2\alpha}} \right\}.
\]
Indeed, the second inequality was already proved in Corollary 2.2 and the first one is an immediate corollary of (2.6) and the fact that \( \|\nabla \bar{u}\|_{L^2} = \|\bar{\omega}\|_{L^2} \).

Let us conclude this section by discussing the well-posedness of problem (1.1) and justification of the estimates obtained above. We will consider below only the 3D case (the 2D case is analogous and even slightly simpler).

We also note from the very beginning that equation (1.1) can be rewritten in the form of an ODE in a Hilbert space with bounded nonlinearity. Indeed, applying the Helmholtz–Leray projection \( \Pi \) to both sides of (1.1) together with the operator
\[
A_\alpha := (1 - \alpha A)^{-1},
\]
where \( A = \Pi \Delta_x \) is the Stokes operator in \( \Omega \), we arrive at
\[
\partial_t \bar{u} + \gamma \bar{u} + B(\bar{u}, \bar{u}) = A_\alpha \Pi g, \quad \bar{u} \big|_{t=0} = \bar{u}_0,
\]
where \( B(\bar{u}, \bar{v}) := A_\alpha \Pi ((\bar{u}, \nabla_x) \bar{v}) \).

It is natural to consider this system in the phase space \( \bar{u} \in H^1(\Omega) \) with norm (2.2). Then the nonlinear operator \( B \) is bounded from \( H^1 \) to \( H^{3/2} \):
\[
\|B(\bar{u}, \bar{v})\|_{H^{3/2}} \leq C_\alpha \|\bar{u}\|_\alpha \|\bar{v}\|_\alpha,
\]
where \( C_\alpha \) depends only on \( \alpha \). Indeed, if \( \bar{u}, \bar{v} \in H^1 \), then by the Sobolev embedding theorem \( \bar{u}, \bar{v} \in L^6(\Omega) \) and \( (\bar{u}, \nabla_x) \bar{v} \in L^{3/2}(\Omega) \) by Hölder’s inequality. Together with the \( (L^{3/2} \to W^{2,3/2}) \)-boundedness of the operator \((1 - \alpha A)^{-1}\), we get that \( B(\bar{u}, \bar{v}) \in W^{2,3/2}(\Omega) \). Finally, the Sobolev embedding \( W^{2,3/2} \subset H^{3/2} \) proves estimate (2.11).
Thus, $B(\bar{u}, \bar{u})$ is a regularizing operator in $H^1$ and equation (2.10) is an ODE in $H^1$ with bounded nonlinearity. Therefore the local existence and uniqueness of a solution as well as (an infinite) differentiability of the corresponding local solution semigroup are straightforward corollaries of the Banach contraction principle or the implicit function theorem, see e.g. [18] for the details. Thus, to get the global well-posedness and dissipativity we only need to verify the proper a priori estimate. Since this has already been done in Proposition 2.1, we have proved the following theorem.

**Theorem 2.5.** Let $\bar{u}_0 \in H^1(\Omega), g \in [L^2(\Omega)]^d$ (in the case of periodic BC we also assume that $g$ has zero mean). Then there exists a unique global solution $\bar{u} \in C([0, \infty), H^1)$ of problem (2.10) (which is simultaneously the unique solution of (1.1)). Moreover, the function

$$t \to \|\bar{u}(t)\|_{L^2}^2 + \alpha\|\nabla_x \bar{u}(t)\|_{L^2}^2$$

is absolutely continuous and the following energy identity holds:

$$\frac{1}{2} \frac{d}{dt} (\|\bar{u}(t)\|_{L^2}^2 + \alpha\|\nabla_x \bar{u}(t)\|_{L^2}^2) + \gamma (\|\bar{u}(t)\|_{L^2}^2 + \alpha\|\nabla_x \bar{u}(t)\|_{L^2}^2) = (g, \bar{u}). \quad (2.12)$$

In particular, the dissipative estimate (2.1) holds for any solution $u$ of class $u \in C([0, \infty), H^1)$.

**Corollary 2.6.** Let the assumptions of Theorem 2.5 holds. Then equation (2.10) generates a dissipative solution semigroup

$$S(t)\bar{u}_0 := \bar{u}(t), \quad t \geq 0 \quad (2.13)$$

in the phase space $H^1(\Omega)$. Moreover, $S(t)$ is $C^\infty$-differentiable for every fixed $t$.

Indeed, the existence of the semigroup is an immediate corollary of the well-posedness proved in the theorem and the differentiability follows from the ODE structure of (2.10) and the fact that the map $\bar{u} \to B(\bar{u}, \bar{u})$ is $C^\infty$-smooth as a map from $H^1$ to $H^1$.

**3. Asymptotic compactness and attractors**

In this section we construct a global attractor for the solution semigroup $S(t)$ generated by problem (1.1). We start with recalling the definition of a weak and strong global attractor, see [2, 8] for more details. We will mainly consider below the most complicated case $\Omega = \mathbb{R}^3$ since in the case of a bounded domain the asymptotic compactness is an immediate corollary of the fact that $B(\bar{u}, \bar{u}) \in H^{3/2}$ if $u \in H^1$, see Remark 3.5.

**Definition 3.1.** A set $\mathcal{A}_w \subset H^1$ is a weak global attractor of the semigroup $S(t)$ if

1) $\mathcal{A}_w$ is a compact set in $H^1$ with weak topology;
2) $\mathcal{A}_w$ is strictly invariant, i.e., $S(t)\mathcal{A}_w = \mathcal{A}_w$;
3) $\mathcal{A}_w$ attracts the images of all bounded sets in the weak topology of $H^1$, i.e. for every bounded set $B \subset H^1$ and every neighbourhood $O(\mathcal{A}_w)$ of the attractor in the weak topology, there exists $T = T(O, B)$ such that

$$S(t)B \subset O(\mathcal{A}_w) \text{ for all } t \geq T.$$}

Analogously, $\mathcal{A}_s$ is a strong attractor if it is compact in the strong topology of $H^1$, is strictly invariant and attracts the images of bounded sets in the strong topology as well. Obviously

$$\mathcal{A}_w = \mathcal{A}_s$$

if both attractors exist.

We will use the following criterion for verifying the existence of an attractor, see [2, 44] for the details.

**Proposition 3.2.** Let the operators $S(t)$ be continuous in the weak topology for every fixed $t$ and let the semigroup $S(t)$ possess a bounded absorbing set $B$. The latter means that for every bounded $B \subset H^1$ there exists $T = T(B)$ such that

$$S(t)B \subset B \text{ for all } t \geq T.$$}

Then there exists a weak global attractor $\mathcal{A}_w$ of the semigroup $S(t)$ which is generated by all complete (defined for all $t \in \mathbb{R}$) bounded solutions of problem (2.10):

$$\mathcal{A}_w = \mathcal{K}|_{t=0},$$

where $\mathcal{K} := \{\bar{u} \in C_b(\mathbb{R}, H^1), \bar{u} \text{ solves } (2.10)\}$.

Let, in addition, $S(t)$ be asymptotically compact on $B$. The latter means that for every sequence $\bar{u}_n^0 \in B$ and every sequence $t_n \to \infty$, the sequence

$$\{S(t_n)\bar{u}_n^0\}_{n=1}^\infty$$

is precompact in the strong topology of $H^1$. Then $\mathcal{A}_w$ is also a strong global attractor for the semigroup $S(t)$.

We start with verifying the existence of a weak attractor.

**Proposition 3.3.** Let the assumptions of Theorem 2.5 hold. Then the solution semigroup $S(t)$ generated by equation (1.1) possesses a weak global attractor $\mathcal{A}_w$ in the phase space $H^1$.

**Proof.** The existence of a bounded absorbing set $B$ is an immediate corollary of the dissipative estimate (2.1). We may take

$$B := \{\bar{u} \in H^1, \|\bar{u}\|_{L^2}^2 + \alpha\|\nabla_x \bar{u}\|_{L^2}^2 \leq \frac{2}{\gamma^2}\|g\|_{L^2}^2\}.$$}

Thus, we only need to check the weak continuity. Let $\bar{u}_n^0 \in B$ be a sequence of the initial data weakly converging to $\bar{u}_0$: $\bar{u}_n^0 \rightharpoonup \bar{u}_0$ in $H^1$. Denote by $\bar{u}^n(t) := S(t)\bar{u}_n^0$
the corresponding solutions. We need to check that for every fixed $T$, $\bar{u}^n(T) \to \bar{u}(T)$ in $H^1$, where $\bar{u}(t) := S(T)\bar{u}_0$.

To see this we recall that $\bar{u}^n$ is bounded uniformly with respect to $n$ in $L^\infty(0,T;H^1)$ due to estimate \((2.1)\). Moreover, from equation \((2.10)\) we see also that $\partial_t\bar{u}^n$ is uniformly bounded in the same space. Thus, passing to a subsequence, if necessary, we may assume that $\bar{u}^n(t) \to v(t)$ for every $t \in [0,T]$ and $\partial_t\bar{u}_n \to \partial_tv$ in $L^2(0,T;H^1)$ for some function $v(t)$ such that $v,\partial_tv \in L^\infty(0,T;H^1)$. So, it remains to verify that $v(t) = S(t)\bar{u}_0$ by passing to the limit in equations \((2.10)\) for functions $\bar{u}^n$.

This passing to the limit is obvious for linear terms, so we only need to prove the convergence of the nonlinear term $B(\bar{u}_n, \bar{u}_n)$. In turn, this is the same as to prove that, in the sense of distributions,

$$(\bar{u}^n, \nabla_x)\bar{u}^n = \text{div}(\bar{u}^n \otimes \bar{u}^n) \to \text{div}(v \otimes v) = (v, \nabla_x)v.$$ 

The last statement will be proved if we check that

$$\bar{u}^n \otimes \bar{u}^n \to v \otimes v \quad \text{in} \quad L^2((0,T) \times \Omega). \quad \text{(3.2)}$$

To verify \((3.2)\), we recall that the sequence $\bar{u}^n \otimes \bar{u}^n$ is uniformly bounded in $L^2$ due to dissipative estimate \((2.1)\) and the embedding $H^1((0,T) \times \Omega) \subset L^4$. Moreover, since the embedding $H^1((0,T) \times \mathbb{R}^3) \subset L^2((0,T);L^2_{\text{loc}}(\Omega))$ is compact, we have the strong convergence $\bar{u}^n \to v$ in $L^2((0,T);L^2_{\text{loc}}(\Omega))$ and, therefore, the convergence $\bar{u}^n \to v$ almost everywhere. Since the sequence $\bar{u}^n \otimes \bar{u}^n$ is uniformly bounded in $L^2((0,T) \times \Omega)$, we may assume without loss of generality that it is weakly convergent to some $\psi \in L^2((0,T) \times \Omega)$. Along with the established convergence almost everywhere this implies that $\psi = v \otimes v$, see e.g. \([30]\), and proves \((3.2)\).

Thus, we have proved that $v$ solves the equation \((2.10)\) and by the uniqueness $v(t) = \bar{u}(t)$. This finishes the proof of weak continuity of the operators $S(t)$ and the existence of a weak global attractor now follows from Proposition \((3.2)\). The theorem is proved.

We are now ready to verify the existence of a strong global attractor.

**Proposition 3.4.** Let the assumptions of Theorem \((2.5)\) hold. Then the solution semigroup $S(t)$ generated by equation \((2.10)\) possesses a strong global attractor $\mathcal{A} = \mathcal{A}_s$ in the phase space $H^1$.

**Proof.** According to Proposition \((3.2)\) we only need to verify the asymptotic compactness of $S(t)$ on $\mathcal{B}$. We will use the so-called energy method for this purpose, see \([3], [36]\) for more details.

Let $\{\bar{u}_0^n\} \subset \mathcal{B}$, let $t_n \to \infty$ be arbitrary and let $\bar{u}^n(t) := S(t_n)\bar{u}_0^n$. Define also $\bar{v}^n(t) := \bar{u}^n(t + t_n)$. Then these functions are defined on the time intervals $t \in [-t_n, \infty)$ and, due to the existence of a weak global attractor, without loss of generality, we may assume that $\bar{v}(t) \to \bar{u}(t)$ in $H^1$ for all $t \in \mathbb{R}$ to some complete
trajectory $\bar{u} \in K$. In particular,
\begin{equation}
\bar{v}^n(0) = S(t_n)\bar{u}_0^n \rightarrow \bar{u}(0)
\end{equation}
and we only need to check that this convergence is strong.

It is convenient to use the equivalent norm (2.2) in the space $H^1$. Then, the strong convergence in (3.3) will be proved if we verify that
\begin{equation}
\|\bar{v}^n(0)\|^2_\alpha \rightarrow \|\bar{u}(0)\|^2_\alpha.
\end{equation}
To see this we integrate the energy identity (2.12) for $\bar{v}^n(t)$ in time and get
\begin{equation}
\|\bar{v}^n(0)\|^2_\alpha = \|\bar{u}_0^n\|^2_\alpha e^{-2\gamma t_n} + \int_{-t_n}^{0} e^{2\gamma s}(g, \bar{v}^n(s))\,ds.
\end{equation}
Passing to the limit $n \rightarrow \infty$ in this relation and using the weak convergence of $\bar{v}^n$ to $\bar{u}$ and uniform boundedness of $\bar{v}^n$ and the initial data $\bar{u}_0^n$, we conclude that
\begin{equation}
\lim_{n \rightarrow \infty} \|\bar{v}^n(0)\|^2_\alpha = \int_{-\infty}^{0} e^{2\gamma s}(g, \bar{u}(s))\,ds.
\end{equation}
On the other hand, integrating the energy identity for the limit solution $\bar{u}$ in time, we arrive at
\begin{equation}
\|\bar{u}(0)\|^2_\alpha = \int_{-\infty}^{0} e^{2\gamma s}(g, \bar{u}(s))\,ds.
\end{equation}
Equalities (3.6) and (3.7) imply (3.4), therefore the convergence in (3.3) is actually strong. Thus, the desired asymptotic compactness is proved and the proposition is also proved.

**Remark 3.5.** Since the operator $B(\bar{u}, \bar{u})$ is regularizing, one can easily increase the regularity of the global attractor $\mathcal{A}$ using the decomposition of the semigroup into the decaying linear part and the regularizing nonlinear part (see [17]):
\begin{equation}
S(t) := L(t) + K(t),
\end{equation}
where $v(t) = L(t)\bar{u}_0$ solves
\begin{equation}
\partial_t v + \gamma v = 0, \quad v|_{t=0} = \bar{u}_0
\end{equation}
and $w(t) := K(t)\bar{u}_0$ satisfies
\begin{equation}
\partial_t w + \gamma w + B(\bar{u}, \bar{u}) = A_\alpha \Pi g, \quad w|_{t=0} = 0.
\end{equation}
Combining this decomposition with bootstrapping arguments, we may check that the regularity of the attractor $\mathcal{A}$ is restricted by the regularity of $g$ only and it will be $C^\infty$-smooth if $g \in H^\infty(\mathbb{R}^3)$. Moreover, using the proper weighted estimates, see [35], we may get the estimates on the rate of decay for solutions belonging to the attractor as $|x| \rightarrow \infty$ in terms of the decay rate of $g$ which clarify the reason why $\mathcal{A}$ is compact. However, all these estimates do not seem very helpful for estimation of the attractor dimension (since they grow rapidly with respect to $\gamma, \alpha \rightarrow 0$) and therefore we will not go into more details here.
4. Upper bounds for the fractal dimension

In this section we derive upper bounds for the fractal dimension of the attractor \( \mathcal{A} \). As usual for the Navier–Stokes type equations, these bounds will be obtained by means of the volume contraction method, see [2, 10, 44] and the references therein. On the analytical side, the Lieb–Thirring inequalities for \( L^2 \)-orthonormal families [28, 29] which are an indispensable tool for the dimension estimates of the attractors for the Navier–Stokes equations are replaced in our case by the collective Sobolev inequalities for \( H^1 \)-orthonormal families and are proved in the Appendix A.

Furthermore, since system (1.1) in the 2D case has already been studied in [22] (for the case \( \Omega = \mathbb{T}^2 \)), we will concentrate here on the 3D case only.

**Theorem 4.1.** Suppose that \( \Omega \) is either the 3D torus \( \mathbb{T}^3 \), or a bounded domain \( \Omega \subset \mathbb{R}^3 \) (endowed with Dirichlet BC), or the whole space \( \Omega = \mathbb{R}^3 \). Let \( g \in [L^2(\Omega)]^d \) (in the case of \( \mathbb{T}^3 \) we also assume that \( g \) has zero mean). Then the global attractor \( \mathcal{A} \) corresponding to the regularized damped Euler system (1.1) has finite fractal dimension satisfying the following estimate:

\[
\dim_F \mathcal{A} \leq \frac{1}{12\pi} \frac{\|g\|_{L^2}^2}{\alpha^{5/2} \gamma^4}. \tag{4.1}
\]

**Proof.** The solution semigroup \( S(t) : H^1 \to H^1 \) is smooth with respect to the initial data (see Corollary 2.6), so we only need to estimate the \( n \)-traces for the linearization of equation (2.10) over trajectories on the attractor. This linearization of (1.1) reads:

\[
\begin{cases}
\partial_t \tilde{\theta} = -\gamma \tilde{\theta} - B(\bar{u}(t), \tilde{\theta}) - B(\tilde{\theta}, \bar{u}(t)) =: L_{\bar{u}}(t) \tilde{\theta}, \\
\text{div} \tilde{\theta} = 0, \quad \tilde{\theta}|_{t=0} = \tilde{\theta}_0 \in H^1(\Omega),
\end{cases} \tag{4.2}
\]

where \( B(\bar{u}, v) := A_\alpha \Pi((\bar{u}, \nabla_x)\bar{v}) \). In order to utilize the well-known cancelation property

\[
((\bar{u}, \nabla_x)\tilde{\theta}, \tilde{\theta}) \equiv 0
\]

for the inertial term in the Navier-Stokes equations, it is natural to endow the space \( H^1 \) with the scalar product

\[
(\tilde{\theta}, \xi)_\alpha = (\tilde{\theta}, \xi) + \alpha(\nabla_x \tilde{\theta}, \nabla_x \xi) = ((1 - \alpha A)\tilde{\theta}, \xi)
\]

associated with the norm (2.2). Then, using that \( \Pi A_\alpha = A_\alpha \) and \( \Pi \tilde{\theta} = \tilde{\theta} \), we get the cancelation

\[
(B(\bar{u}, \tilde{\theta}), \tilde{\theta})_\alpha = (A_\alpha \Pi(\bar{u}, \nabla_x)\tilde{\theta}, (1 - \alpha \Delta_x)\tilde{\theta}) = (A_\alpha \Pi(\bar{u}, \nabla_x)\tilde{\theta}, (1 - \alpha \Pi \Delta_x)\tilde{\theta}) = (\Pi(\bar{u}, \nabla_x)\tilde{\theta}, \tilde{\theta}) = ((\bar{u}, \nabla_x)\tilde{\theta}, \tilde{\theta}) \equiv 0
\]

of the most singular term \( B(\bar{u}, \tilde{\theta}) \) and, therefore, only the more regular term \( B(\tilde{\theta}, \bar{u}) \) will impact the trace estimates.
Following the general strategy, see e.g. [44], the \( n \)-dimensional volume contraction factors \( \omega_n(\mathcal{A}) \) (=the sums of the first \( n \) global Lyapunov exponents) which control the dimension can be estimated from above by the following numbers:

\[
q(n) := \limsup_{t \to \infty} \sup_{u(t) \in \mathcal{A}} \left( \sup_{\tilde{\theta}_j} \frac{1}{t} \int_0^t \sum_{j=1}^n (L_{u(\tau)} \tilde{\theta}_j, \tilde{\theta}_j)_\alpha d\tau, \right),
\]

where the first (inner) supremum is taken over all orthonormal families \( \{\tilde{\theta}_j\}_{j=1}^n \) with respect to the scalar product \((\cdot, \cdot)_\alpha\) in \( H^1 \):

\[
(\tilde{\theta}_i, \tilde{\theta}_j)_\alpha = \delta_{ij}, \quad \text{div} \theta_j = 0,
\]

and the second (middle) supremum is over all trajectories \( u(t) \) on the attractor \( \mathcal{A} \). Then, using the cancellation mentioned above together with the pointwise estimate (B.1) proved in Appendix B, we get

\[
\sum_{j=1}^n (L_{u(t)} \tilde{\theta}_j, \tilde{\theta}_j)_\alpha \leq -\gamma n + \frac{2}{3} \int_\Omega \rho(x) |\nabla_x \bar{u}(t, x)| dx \leq -\gamma n + \frac{2}{3} \| \nabla_x \bar{u}(t) \|_{L^2} \| \rho \|_{L^2}, \quad (4.5)
\]

where

\[
\rho(x) = \sum_{j=1}^n |\tilde{\theta}_j(x)|^2.
\]

We now use estimate (A.8) from Appendix A

\[
\| \rho \|_{L^2} \leq \frac{1}{2\sqrt{\pi}} \frac{n^{1/2}}{\alpha^{3/4}}
\]

and obtain

\[
\sum_{j=1}^n (L_{u(t)} \tilde{\theta}_j, \tilde{\theta}_j)_\alpha \leq -\gamma n + \frac{1}{\sqrt{6\pi}} \frac{n^{1/2}}{\alpha^{3/4}} \| \nabla_x \bar{u}(t) \|_{L^2}. \quad (4.7)
\]

Finally, using (2.4), we arrive at

\[
q(n) \leq -\gamma n + \frac{1}{2\sqrt{3\pi}} \frac{n^{1/2}}{\alpha^{5/4}} \| g \|_{L^2}.
\]

It only remains to recall that, according to the general theory, \( \omega_n(\mathcal{A}) \leq q(n) \) and any number \( n^* \) for which \( \omega_{n^*}(\mathcal{A}) \leq 0 \) and \( \omega_n(\mathcal{A}) < 0 \) for \( n > n^* \) is an upper bound both for the Hausdorff [2, 44] and the fractal [6, 7] dimension of the global attractor \( \mathcal{A} \). This gives the desired estimate

\[
\dim_F \mathcal{A} \leq \frac{1}{12\pi} \frac{\| g \|_{L^2}^2}{\alpha^{5/2} \gamma^4}
\]

and finishes the proof of the theorem. \( \square \)
Remark 4.2. Estimates (1.5) and (1.6) for $T^2$ (and the fact that it is sharp) were proved in [22]. The upper bound for $\mathbb{R}^2$ is exactly the same once we now know (A.8) for $\mathbb{R}^2$. For a bounded domain we only need to replace in the proof in [22] the estimates of the solutions on the attractor by (2.4). Alternatively, one can go through the proof of Theorem 4.1 and replace the 3D constants by their 2D counterparts accordingly.

5. SHARP LOWER BOUND ON $T^3$

The aim of this section is to show that estimate (4.1) for system (1.1) on $T^3 = [0, 2\pi]^3$ is sharp in the limit as $\alpha \to 0$.

We consider a family of right-hand sides
\[ g = g_s = \begin{cases} g_1 = \gamma \lambda(s) \sin sx_3, \\ g_2 = 0, \\ g_3 = 0, \end{cases} \quad (5.1) \]
depending only on $x_3$ and parameterized by $s \in \mathbb{N}, s \gg 1$. The amplitude function $\lambda(s)$ will be specified in the course of the proof. Corresponding to the family $g_s$ is the family of stationary solutions of (1.1)
\[ \bar{u}_0(x_3) = \begin{cases} u_0(x_3) = \lambda(s) \sin sx_3, \\ 0, \\ 0, \end{cases} \quad (5.2) \]
with $p = 0$. In fact,
\[ \bar{u}_0 = (1 - \alpha \Delta_x)^{-1} \bar{u}_0 = (\bar{u}_0, 0, 0)^T \]
also depends only on $x_3$ and therefore $(\bar{u}_0, \nabla_x)\bar{u}_0 = 0$.

We now consider system (1.1) linearized on the stationary solution (5.2)
\[ \begin{aligned} \partial_t w + \bar{u}_0 \frac{\partial \bar{w}}{\partial x_1} + \bar{w}_3 \frac{\partial \bar{u}_0}{\partial x_3} e_1 + \gamma w + \nabla_x q &= 0, \\ \text{div } w &= 0, \end{aligned} \quad (5.3) \]
where $e_1 = (1, 0, 0)^T$ and $\bar{w} = (1 - \alpha \Delta_x)^{-1} w$. The standing assumption is
\[ \int_{T^3} w(x, t) dx = 0. \quad (5.4) \]
We shall look for the solution of the linear problem (5.3) in the form
\[ w(x, t) = \begin{pmatrix} w_1(x_3) \\ w_2(x_3) \\ w_3(x_3) \end{pmatrix} e^{i(ax_1 + bx_2 - \alpha t)}, \quad q(x, t) = q(x_3) e^{i(ax_1 + bx_2 - \alpha t)}, \quad (5.5) \]
where \( a, b \in \mathbb{Z} \) so that \( w \) and \( q \) are \( 2\pi \)-periodic in each \( x_i \). If such a solution of (5.3) is found, then substituting (5.5) into (5.3) and setting \( t = 0 \) we see that

\[
w(x, 0) = \begin{pmatrix} w_1(x_3) \\ w_2(x_3) \\ w_3(x_3) \end{pmatrix} e^{i(ax_1 + bx_2)}
\]

is a vector-valued eigenfunction of the stationary operator

\[
L_3(\vec{u}_0)w = \bar{u}_0 \frac{\partial \bar{w}}{\partial x_3} + \bar{w}_3 \frac{\partial \bar{u}_0}{\partial x_3} \bar{e}_1 + \gamma w + \nabla x q \tag{5.6}
\]

and \( iac \) is the corresponding eigenvalue. If \( \text{Re}(iac) < 0 \), then the corresponding mode is unstable.

We substitute (5.5) into (5.3) and obtain the system

\[
\begin{align*}
-\gamma w_1 - ia(\bar{u}_0 w_1 - cw_1) &= iaq + \bar{w}_3 u'_0 \\
-\gamma w_2 - ia(\bar{u}_0 w_2 - cw_2) &= ibq, \\
-\gamma w_3 - ia(\bar{u}_0 w_3 - cw_3) &= q', \\
iaw_1 + ibw_2 + w'_3 &= 0,
\end{align*}
\]

(5.7)

Lemma 5.1. There are no unstable solutions of equation (5.3)

\[
\frac{\partial}{\partial t} w = L_3(\bar{u}_0)w
\]

that can be written in the form (5.5) with \( a = 0 \).

Proof. Let \( a = 0 \). Then the solutions of (5.3) are sought in the form

\[
w(x, t) = \begin{pmatrix} w_1(x_3) \\ w_2(x_3) \\ w_3(x_3) \end{pmatrix} e^{i(bx_2 - ct)}, \quad q(x) = q(x_3) e^{i(bx_2 - ct)},
\]

and (5.7) goes over to

\[
\begin{align*}
-\gamma w_1 + icw_1 &= \bar{w}_3 u'_0 \\
-\gamma w_2 + icw_2 &= ibq, \\
-\gamma w_3 + icw_3 &= q', \\
ibw_2 + w'_3 &= 0.
\end{align*}
\]

Let \( b \neq 0 \). Then \( w_2 = -w'_3/(ib) \). Substituting this into the second equation and differentiating the third with respect to \( x_3 \) we obtain

\[
q'' = b^2 q,
\]

which gives that \( q = 0 \), since \( q \) is periodic. Since we are looking for unstable solutions, it follows that \( \text{Re}(ic) < 0 \) and therefore \( -\gamma + ic \neq 0 \). This gives that \( w_2 = w_3 = 0 \), and, finally, \( w_1 = 0 \).

If \( a = b = 0 \), then \( w'_3 = 0 \), and \( w_3 = 0 \) by periodicity and zero mean condition. This gives \( q = 0 \) and \( w_1 = w_2 = 0 \). The proof is complete. \( \square \)
5.1. **Squire’s transformation.** We now reduce the 3D instability analysis to the instability analysis of the transformed 2D problem. The key role is played by the Squire’s transformation (see [41], [12], [33]).

Since we are looking for unstable solutions of (5.3), in view of Lemma 5.1 we may assume that $a \neq 0$ in (5.7). Multiplying the first equation in (5.7) by $a$ and the second by $b$ and adding up the results, we obtain

\[
\begin{align*}
-\hat{\gamma}\hat{w}_1 - ia(\hat{u}_0\hat{w}_1 - \hat{c}\hat{w}_1) &= i\hat{a}\hat{q} + \hat{w}_3u'_0, \\
-\hat{\gamma}\hat{w}_3 - ia(\hat{u}_0\hat{w}_3 - \hat{c}\hat{w}_3) &= \hat{q}', \\
i\hat{a}\hat{w}_1 + \hat{w}_3' &= 0,
\end{align*}
\]

(5.8)

where

\[
\begin{align*}
\hat{a}^2 &= a^2 + b^2, \\
\hat{w}_1 &= aw_1 + bw_2, \\
\hat{w}_3 &= w_3, \\
\hat{\gamma} &= \frac{\gamma}{a}, \\
\hat{q} &= \frac{q}{a}, \\
\hat{c} &= c.
\end{align*}
\]

(5.9)

The solutions of this problem on the 2d torus $T^2_{\hat{a}} = x_1 \in [0, 2\pi/|\hat{a}|], x_3 \in [0, 2\pi]$ are sought in the form

\[
\hat{w}(x_1, x_3, t) = \left(\begin{array}{c}
\hat{w}_1(x_3) \\
\hat{w}_3(x_3)
\end{array}\right) e^{i(\hat{a}x_1 - \hat{a}\hat{c}t)}, \quad \hat{q}(x_1, x_3, t) = q(x_3)e^{i(\hat{a}x_1 - \hat{a}\hat{c}t)},
\]

(5.10)

and if such a solution is found, then the vector function

\[
\hat{w}(x_1, x_3, 0) = \left(\begin{array}{c}
\hat{w}_1(x_3) \\
\hat{w}_3(x_3)
\end{array}\right) e^{i\hat{a}x_1}
\]

(5.11)

is a vector-valued eigenfunction with eigenvalue $i\hat{a}\hat{c}$ of the stationary operator

\[
L_2(\bar{u}_0)\hat{w} = \hat{\gamma}\hat{w} + \bar{u}_0\frac{\partial\hat{w}}{\partial x_1} + \hat{w}_3\frac{\partial\bar{u}_0}{\partial x_3}e_1 + \nabla_x \hat{q}, \quad \text{div}\ \hat{w} = 0,
\]

(5.12)

on $T^2_{|\hat{a}|}$, where the stationary solution and the generating right-hand side are

\[
\bar{u}_0(x_3) = \begin{cases}
u_0(x_3) = \lambda(s)\sin sx_3, \\
0,
\end{cases}
\quad g_s(x_3) = \begin{cases}g_1(x_3) = \lambda(s)\hat{\gamma}\sin sx_3, \\
0,
\end{cases}
\]

(5.13)

and where as before $\bar{u}_0 = \bar{u}_0(x_3) = (1 - \Delta_x)^{-1}u_0$.

To avoid unnecessary complications we assume in what follows that

\[
\sqrt{a^2 + b^2} = \hat{a} > 0, \quad a > 0,
\]

and formulate the main result on the Squire’s reduction of the 3D instability analysis to the 2D case.
Lemma 5.2. Let \( \hat{w} \) in (5.11) be an unstable eigenfunction of the operator (5.12) on the torus \( T^2_\hat{a} = [0, 2\pi/\hat{a}] \times [0, 2\pi] \). Then for any pair of integers \( a, b \in \mathbb{Z} \) with \( a^2 + b^2 = \hat{a}^2 \) there exist an unstable solution of system (5.7) on \( T^3 = [0, 2\pi]^3 \).

Proof. Once the \( \hat{\cdot} \)-variables are known, \( q,w_3,c \) and \( \gamma \) are found from (5.9). It remains to find \( w_1 \) and \( w_2 \). We consider the second equation in (5.7):

\[
A w_2 := (-\gamma + ia)c w_2 - i\bar{u}_0 \bar{w}_2 = ibq.
\]

Since \( \hat{w} \) is unstable, \( \text{Re}(ic) < 0 \) and therefore \( \text{Re}(-\gamma + ic) < 0 \). Suppose that \( A w_2 = 0 \) for some \( w_2 \). Taking the scalar product in (complex) \( L^2([0, 2\pi]) \) with \( w_2 \) and taking into account that the second term is purely imaginary we obtain for the real part

\[
\text{Re}(-\gamma + ic)\|w_2\|_{L^2}^2 = 0,
\]

which gives that \( w_2 = 0 \), and \( A \) has a trivial kernel. In addition, \( A \) is a Fredholm operator, since the second term is compact (smoothing). Hence it has a bounded inverse, and since \( q \) is known, we have found \( w_2 \). Finally,

\[
w_1 = (\hat{a}\hat{w}_1 - bw_2)/a.
\]

\[\square\]

5.2. Instability analysis on \( T^2 \). We now have to recall the instability analysis for the 2D problem that was carried out in detail in our previous work [22]. The problem was studied on the standard torus \( T^2 = [0, 2\pi]^2 \) and we now denote the second coordinate by \( x_3 \), so that \( x_1, x_3 \) are the coordinates on \( T^2 \). The family of the forcing terms and the corresponding stationary solutions are as in (5.13) and the linearized stationary operator is precisely (5.12). Applying \( \text{curl} \) to (5.12) we obtain the equivalent scalar operator in terms of the vorticity whose spectrum was studied in [22]

\[
L_{s}\omega := J\left((\Delta_x - \alpha\Delta_x^2)^{-1}\omega_s, (1 - \alpha\Delta_x)^{-1}\omega\right) + J\left((\Delta_x - \alpha\Delta_x^2)^{-1}\omega_s, (1 - \alpha\Delta_x)^{-1}\omega\right) + \gamma\omega = -\sigma\omega,
\]

where

\[
J(a, b) = \nabla a \cdot \nabla b = \partial_{x_1} a \partial_{x_3} b - \partial_{x_3} a \partial_{x_1} b,
\]

and

\[
\omega_s = \text{curl} \, \overline{u}_0 = -\lambda(s)s \cos sx_2, \quad \omega = \text{curl} \, \hat{\omega}.
\]

The following result was proved in [22] (see Theorem 4.1 and Corollary 4.2.)

Theorem 5.3. Given a large integer \( s > 0 \) let a fixed pair of integers \( t, r \) belong to a bounded region \( A(\delta) \) defined by conditions

\[
t^2 + r^2 < s^2/3, \quad t^2 + (s + r)^2 > s^2, \quad t^2 + (s + r)^2 > s^2, \quad t \geq \delta s,
\]

(5.15)
where \(0 < \delta < 1/\sqrt{3}\). There exists an absolute constant \(c_1\) such that for

\[
\lambda \geq \lambda_2(s, \gamma) = c_1 \gamma \frac{(1 + \alpha s^2)^2}{s}
\]  

(5.16)
in (5.13) the linear operator \(L_s\) on the torus \(\mathbb{T}^2 = [0, 2\pi]^2\) has a real negative (unstable) eigenvalue \(\sigma < 0\) of multiplicity 2. The corresponding eigenfunctions are

\[
\omega_1(x_1, x_3) = \sum_{n=-\infty}^{\infty} a_{t,sn-r} \cos(tx_1 + (sn + r)x_3),
\]

\[
\omega_2(x_1, x_3) = \sum_{n=-\infty}^{\infty} a_{t,sn-r} \sin(tx_1 + (sn + r)x_3).
\]

(5.17)

We now observe that \(\omega_1\) and \(\omega_2\) in (5.17) are the real and imaginary parts of the complex-valued eigenfunction

\[
\omega_1(x_1, x_3) + i\omega_2(x_1, x_3) = \left[ \sum_{n=-\infty}^{\infty} a_{t,sn-r} e^{i(sn+r)x_3} \right] e^{itx_1}.
\]

Recovering the corresponding divergence free vector function, that is, applying the operator \(\nabla_x \Delta_x^{-1}\), we obtain an unstable vector valued eigenfunction of the operator \(L_2(\vec{u}_0)\) written in the required form (5.11):

\[
w(x_1, x_3) = \left( \begin{array}{c} w_1(x_3) \\ w_3(x_3) \end{array} \right) e^{itx_1}. \]

(5.19)

For the 3D instability analysis below we need to repeat the construction of an unstable eigenmode on the torus \(\mathbb{T}_2^2\) with \(x_1 \in [0, 2\pi/\varepsilon], x_2 \in [0, 2\pi]\), where \(\varepsilon > 0\) is arbitrary (not necessarily small).

**Proposition 5.4.** Let \(r\) and \(t' := t\varepsilon\) belong to the region \(A(\delta)\):

\[
t'^2 + r^2 < s^2/3, \quad t'^2 + (-s + r)^2 > s^2, \quad t'^2 + (s + r)^2 > s^2, \quad t' \geq \delta s. \quad (5.18)
\]

Let \(\lambda\) be defined in (5.16) and let \(g_s\) and \(\vec{u}_0\) be the same as before but in two dimensions:

\[
g_s(x_3) = (\gamma \lambda(s) \sin sx_3, 0)^T, \quad \vec{u}_0(x_3) = (\lambda(s) \sin sx_3, 0)^T.
\]

Then there exists an unstable solution

\[
w(x_1, x_3) = \left( \begin{array}{c} w_1(x_3) \\ w_3(x_3) \end{array} \right) e^{it\varepsilon x_1}, \quad x \in \mathbb{T}_2^2.
\]

(5.19)
of the form (5.11) of the operator (5.12) on the torus \(\mathbb{T}_2^2\).

**Proof.** Following the proof of Theorem 4.1 in [22], we see that a word for word repetition of it shows that if \(t' = \varepsilon t, r\) satisfy (5.18), then the corresponding
operator (5.14) has an unstable (real negative) eigenvalue of multiplicity two with eigenfunctions

\[ \omega_1(x_1, x_3) = \sum_{n=-\infty}^{\infty} a_{t,sn+r} \cos(t\varepsilon x_1 + (sn + r)x_3), \]

\[ \omega_2(x_1, x_3) = \sum_{n=-\infty}^{\infty} a_{t,sn+r} \sin(t\varepsilon x_1 + (sn + r)x_3), \]

from which we construct the required vector valued complex eigenfunction (5.19) as before.

It is convenient for us to single out a small rectangle \( D \) in the \((t', r)\)-plane inside the region \( A(\delta) \) defined by (5.18), see Fig. 1:

\[ |r| \leq c_2s, \quad 0 < c_3s \leq t' = t\varepsilon \leq c_4s. \]  

(5.20)

Here \( \delta = \delta^* \in (0, 1/\sqrt{3}) \) is fixed, and all the constants \( c_i \) are absolute constants, whose explicit values can easily be specified.

**Figure 1.** The region \( A(\delta) \) and the rectangle \( D \).
5.3. 3D lower bound. We can now formulate the main results of this section.

**Theorem 5.5.** We consider the linearized system on the 3D torus \( T^3 = [0, 2\pi]^3 \) with right-hand side \( g_s \) and stationary solution \( \vec{u}_0 \) given in (5.1) and (5.2), where

\[
\lambda = \lambda_3(s) = \sqrt{2\lambda_2(s, \gamma)} = \sqrt{2c_1 \gamma \left(1 + \alpha s^2\right)^2/s}.
\]

(5.21)

Then for each triple of integers \( a, b, r \) satisfying

\[
c_3 s \leq \tilde{a} = \sqrt{a^2 + b^2} \leq c_4 s, \quad |r| \leq c_2 s, \quad a \geq |b|,
\]

(5.22)

there exists an unstable solution of the linearized operator (5.13).

**Proof.** We fix \( a, b, r \) satisfying (5.22). Then in view of the first two inequalities in (5.22) the pair \((t', r) \in D \subset A(\delta)\), where \( t' = \tilde{a} \cdot 1 \) (so that we set \( t = 1 \)). Applying Squire’s transformation we obtain a 2D linearized problem on the torus \( T^2_{\tilde{a}} \) of the form (5.12) with \( \hat{\gamma} = \gamma \tilde{a}/a \). Then in view of the third inequality in (5.22) we have

\[
\lambda = \sqrt{2}\lambda_2(s, \gamma) = \lambda_2(s, \sqrt{2}\hat{\gamma} a/\tilde{a}) \geq \lambda_2(s, \gamma).
\]

We now see from Proposition 5.4 that the 2D linearized problem (5.12) has an unstable eigenvalue. Lemma 5.2 says, in turn, that so does the linearized problem (5.6) on the standard torus \( T^3 = [0, 2\pi]^3 \). \(\square\)

The number of integers \((a, b, r)\) satisfying (5.22) is of order \( c_5 s^3 \), where

\[
c_5 = \frac{1}{4} \pi c_2 (c_4^2 - c_3^2).
\]

**Theorem 5.6.** Let the right-hand side in (1.1) be \( g_s \) defined in (5.1) with \( \lambda(s) \) defined in (5.21). Then the dimension of the corresponding attractor \( \mathcal{A} = \mathcal{A}_s \) satisfies for an absolute constant \( c_6 \) the lower bound

\[
\dim_F \mathcal{A} \geq c_6 \frac{\|g_s\|_{L^2}^2}{\alpha^{5/2} \gamma^4}.
\]

(5.23)

**Proof.** We study our system in the limit \( \alpha \to 0 \). Since \( s \) is at our disposal we set

\[
s = \frac{1}{\sqrt{\alpha}}.
\]

Then \( \lambda \) in (5.21) and \( \|g_s\|_{L^2}^2 \) in (5.1) become

\[
\lambda = c_7 \gamma \sqrt{\alpha}, \quad \|g_s\|_{L^2}^2 = c_8 \gamma^4 \alpha.
\]

We can finally write

\[
\dim_F \mathcal{A} \geq c_5 s^3 = c_5 \frac{1}{\alpha^{3/2}} = c_5 \frac{\alpha \|g_s\|_{L^2}^2}{\alpha^{5/2} \|g_s\|_{L^2}^2} = c_5 \frac{\alpha}{\alpha^{5/2} c_8 \alpha^2}.
\]

which gives (5.23) with \( c_6 = c_5/c_8 \). \(\square\)
Remark 5.7. We recall the sharp lower bound in [22] for the attractor dimension for system (1.1) on the 2D torus $\mathbb{T}^2$:

$$\dim F \geq c_{\text{abs}} \frac{\| \text{curl} g_s \|_{L^2}^2}{\alpha \gamma^4}.$$  

(5.24)

Since for $\lambda \sim \gamma \sqrt{\alpha}$ we have $\| \text{curl} g_s \|_{L^2}^2 \sim \gamma^4$ and $\| g_s \|_{L^2}^2 \sim \gamma^4 \alpha$, it follows that estimate (5.24) can equivalently be written in terms of the other dimensionless number in (1.8) as follows

$$\dim F \geq c'_{\text{abs}} \frac{\| g_s \|_{L^2}^2}{\alpha^2 \gamma^4}.$$  

Appendix A. Collective Sobolev inequalities for $H^1$-orthonormal families

We prove here some spectral inequalities for orthonormal families of functions which are the key technical tool in our derivation of sharp upper bounds for the attractor dimension. These inequalities are somehow complementary to the classical Lieb–Thirring inequalities for orthonormal systems in $L^2$ and in our case we estimate the proper norms of the same quantity $\rho(x) = \sum_{j=1}^n |\bar{\theta}_j(x)|^2$, but for families $\{\bar{\theta}_i\}_{i=1}^n$ that are orthonormal in $H^1$ with norm (2.2) depending on $\alpha$. Our exposition utilizes the ideas from [27] as well as extends the results of [22] to 3D case.

We start with the case $\Omega = \mathbb{R}^d$ or $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary conditions (analogously to the classical Lieb–Thirring inequality, the case of Dirichlet boundary conditions follows from the case of whole space by zero extension).

A.1. The case of the whole space and a domain with Dirichlet BC.

Theorem A.1. Let $\Omega \subseteq \mathbb{R}^d$ be an arbitrary domain. Let a family of vector functions $\{\bar{\theta}_i\}_{i=1}^n \in H^1(\Omega)$ with $\text{div} \bar{\theta}_i = 0$ be orthonormal with respect to the scalar product

$$m^2 (\bar{\theta}_i, \bar{\theta}_j)_{L^2} + (\nabla \bar{\theta}_i, \nabla \bar{\theta}_j)_{L^2} = m^2 (\bar{\theta}_i, \bar{\theta}_j)_{L^2} + (\text{curl} \bar{\theta}_i, \text{curl} \bar{\theta}_j)_{L^2} = \delta_{ij},$$  

(A.1)

Then the function $\rho(x) := \sum_{j=1}^n |\bar{\theta}_j(x)|^2$ satisfies

$$\| \rho \|_{L^2} \leq \frac{1}{2 \sqrt{\pi}} \frac{n^{1/2}}{m}, \quad d = 2,$$

$$\| \rho \|_{L^2} \leq \frac{1}{2 \sqrt{\pi}} \frac{n^{1/2}}{m^{1/2}}, \quad d = 3.$$  

(A.2)

Proof. We first let $\Omega = \mathbb{R}^d$ and introduce the operators

$$\mathbb{H} = V^{1/2}(m^2 - \Delta_x)^{-1/2} \Pi, \quad \mathbb{H}^* = \Pi(m^2 - \Delta_x)^{-1/2} V^{1/2}$$

acting in $[L^2(\mathbb{R}^d)]^d$, where $V \in L^2(\mathbb{R}^d)$ is a non-negative scalar function which will be specified below and $\Pi$ is the Helmholtz–Leray projection to divergence.
free vector fields. Then $K = \mathbb{H}^* \mathbb{H}$ is a compact self-adjoint operator acting from $[L^2(\mathbb{R}^d)]^d$ to $[L^2(\mathbb{R}^d)]^d$ and

$$\text{Tr} K^2 \leq \text{Tr} \left( \Pi (m^2 - \Delta_x)^{-1/2} V (m^2 - \Delta_x)^{-1/2} \Pi \right)^2 \leq \text{Tr} \left( \Pi (m^2 - \Delta_x)^{-1} V^2 (m^2 - \Delta_x)^{-1} \Pi \right) = \text{Tr} \left( V^2 (m^2 - \Delta_x)^2 \Pi \right),$$

where we used the Araki–Lieb–Thirring inequality for traces \cite{1, 29, 40}:

$$\text{Tr}(BA^2 B^p) \leq \text{Tr}(B^p A^2 B^p), \quad p \geq 1,$$

and the cyclicity property of the trace together with the facts that $\Pi$ commutes with the Laplacian and that $\Pi$ is a projection: $\Pi^2 = \Pi$.

We want to show that

$$\text{Tr} K^2 \leq \begin{cases} \frac{1}{4\pi} \frac{m}{m^2} ||V||^2_{L^2}, & d = 2; \\ \frac{1}{4\pi} \frac{1}{m} ||V||^2_{L^2}, & d = 3. \end{cases} \quad (A.3)$$

Indeed, the fundamental solution of $(m^2 - \Delta_x)^2 \Pi$ in $\mathbb{R}^d$ is a $d \times d$ matrix

$$F^d_{ij}(x) = G_d(x) \delta_{ij} - \partial_{x_i} \partial_{x_j} \Delta^{-1} G_d(x)$$

with $\mathbb{R}^d$-trace at $x \in \mathbb{R}^d$

$$\text{Tr}_{\mathbb{R}^d} F^d(x) = dG_d(x) - \sum_{i=1}^d \partial_{x,i} \Delta^{-1} G_d(x) = (d-1)G_d(x),$$

where $G_d(x)$ is the fundamental solution of the scalar operator $(m^2 - \Delta_x)^2$ in the whole space $\mathbb{R}^d$:

$$G_d(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{i \xi x} d\xi}{(m^2 + |\xi|^2)^2} = \begin{cases} \frac{1}{8\pi} e^{-|x|m}, & d = 3; \\ \frac{1}{4\pi} \frac{1}{m} |x|mK_1(|x|m), & d = 2. \end{cases} \quad (A.4)$$

The first equality here follows from \cite{12}, while for the second we have (since the function is radial and using formula 13.51(4) in \cite{46})

$$G_2(x) = \frac{1}{2\pi} \mathcal{F}^{-1}((m^2 + |\xi|^2)^2) = \frac{1}{2\pi} \int_0^\infty J_0(|x|r) r dr \frac{1}{(m^2 + r^2)^2} = \frac{1}{4\pi} \frac{1}{m^2} |x|mK_1(|x|m),$$

where $K_1$ is the modified Bessel function of the second kind.

Thus, the operator $V^2 (m^2 - \Delta_x)^2 \Pi$ has the matrix-valued integral kernel

$$V(y)^2 F^d(x - y)$$
and therefore
\[ \text{Tr}(V^2(m^2 - \Delta_x)^2 \Pi) = \int_{\mathbb{R}^d} \text{Tr}_{\mathbb{R}^d} \left( V(y)^2 F^d(0) \right) dy = (d - 1)\|V\|_{L^2}^2 G_d(0) \] (A.5)
which along with (A.4) proves the first inequality in (A.3), and also the second one, since \( (tK_1(t))_{|t=0} = 1 \).

We can now complete the proof as in [27]. Setting
\[ \psi_i := (m^2 - \Delta_x)^{1/2} \bar{\theta}_i, \]
we see from (A.1) that \( \{\psi_j\}_{j=1}^n \) is an orthonormal family in \( L^2 \). We observe that
\[ \int_{\mathbb{R}^d} \rho(x) V(x) dx = \sum_{i=1}^n \| \mathbb{H}\psi_i \|_{L^2}^2. \]
By the orthonormality of the \( \psi_j \)'s in \( L^2 \) and the definition of the trace we obtain
\[ \sum_{i=1}^n \| \mathbb{H}\psi_i \|_{L^2}^2 = \sum_{i=1}^n \langle \mathbb{K}\psi_i, \psi_i \rangle \leq \sum_{i=1}^n \| \mathbb{K}\psi_i \|_{L^2} \leq \left( \sum_{i=1}^n \| \mathbb{K}\psi_i \|_{L^2}^2 \right)^{1/2} = n^{1/2} \left( \sum_{i=1}^n \langle \mathbb{K}\psi_i, \psi_i \rangle \right)^{1/2} \leq n^{1/2} \left( \text{Tr} \mathbb{K} \right)^{1/2}. \]
(A.6)
This gives
\[ \int_{\mathbb{R}^d} \rho(x) V(x) dx \leq n^{1/2} \left( \text{Tr} \mathbb{K} \right)^{1/2}. \]
Setting \( V(x) := \rho(x) \) and using (A.3), we complete the proof of (A.2) for the case of \( \Omega = \mathbb{R}^d, d = 2, 3 \).

Finally, if \( \Omega \) is a proper domain in \( \mathbb{R}^d \), we extend by zero the vector functions \( \vec{\theta}_j \) outside \( \Omega \) and denote the results by \( \tilde{\theta}_j \), so that \( \tilde{\theta}_j \in H^1(\mathbb{R}^d) \) and \( \text{div} \tilde{\theta}_j = 0 \). We further set \( \tilde{\rho}(x) := \sum_{j=1}^n \tilde{\theta}_j(x)^2 \). Then setting \( \tilde{\psi}_i := (m^2 - \Delta_x)^{1/2} \tilde{\theta}_i \), we see that the system \( \{\tilde{\psi}_j\}_{j=1}^n \) is orthonormal in \( L^2(\mathbb{R}^d) \) and \( \text{div} \tilde{\psi}_j = 0 \). Since clearly \( ||\tilde{\rho}||_{L^2(\mathbb{R}^d)} = ||\rho||_{L^2(\Omega)} \), the proof of estimate (A.2) reduces to the case of \( \mathbb{R}^d \) and therefore is complete. \( \square \)

The proved result can be rewritten in terms of orthogonal functions with respect to inner product (4.3) as follows.

**Corollary A.2.** Let the assumptions of Theorem A.1 hold and let \( \{\tilde{\theta}_j\}_{j=1}^n \), \( \text{div} \tilde{\theta}_j = 0 \) be an orthonormal system with respect to
\[ (\tilde{\theta}_i, \tilde{\theta}_j)_{L^2} + \alpha(\nabla \tilde{\theta}_i, \nabla \tilde{\theta}_j)_{L^2} = \delta_{ij}. \]
(A.7)
Then \( \rho(x) = \sum_{j=1}^{n} |\bar{\theta}_j(x)|^2 \) satisfies
\[
\|\rho\|_{L^2} \leq \frac{1}{2\sqrt{\pi}} \alpha^{1/4}, \quad d = 2,
\]
\[
\|\rho\|_{L^2} \leq \frac{1}{2\sqrt{\pi}} \alpha^{3/4}, \quad d = 3.
\] (A.8)

Indeed, this statement follows from (A.2) by the proper scaling.

A.2. The case of periodic BC: Estimates for the lattice sums. We now turn to the case \( \Omega = \mathbb{T}^d \). In this case, we naturally have an extra condition that the considered functions have zero mean. Analogously to the case \( \Omega = \mathbb{R}^d \), the Laplacian commutes with the Helmholtz–Leray projection, so we may define analogously the operator \( K \) and get exactly the same expression (A.5) for its trace. The only difference is that now \( G_d(x) = G_{d,m}(x) \) a fundamental solution of the scalar operator \((m^2 - \Delta_x)^{-2}\) on the torus \( \mathbb{T}^d \) (with zero mean condition), so the integral in (A.4) should be replaced by the corresponding sum over the lattice \( \mathbb{Z}^d_0 = \mathbb{Z}^d \setminus \{0\} \):
\[
G_d(x) = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d_0} e^{ik \cdot x} \left( m^2 + |k|^2 \right)^{-2}.
\] (A.9)

Thus, in order to get estimates (A.2) for the torus \( \mathbb{T}^2 \) arguing as in the proof of Theorem A.1, we only need to check that
\[
G_{d,m}(0) < \begin{cases} 
\frac{1}{8\pi} \frac{1}{m}, & d = 3; \\
\frac{1}{4\pi m^2}, & d = 2.
\end{cases}
\] (A.10)

for all \( m \geq 0 \). Unfortunately, we do not know explicit expressions for the sum (A.9), so we need to do rather accurate estimates of the associated lattice sum in order get (A.10) based on the Poisson summation formula. In the case \( d = 2 \), (A.10) is proved in [22] and the case \( d = 3 \) is considered in Proposition A.4 below. Thus, the following result holds.

**Theorem A.3.** Let a family of divergence free vector functions with zero mean \( \{\bar{\theta}_i\}_{i=1}^{n} \in \dot{H}^1(\mathbb{T}^d) \) be orthonormal with respect to scalar product (A.1). Then estimates (A.2) hold. Analogously, if this family is orthonormal with respect to (A.7), then \( \rho \) satisfies inequalities (A.3).

As explained before, the case \( d = 2 \) is verified in [22] and for proving the result for \( d = 3 \), it is sufficient to prove the following proposition.

**Proposition A.4.** The following inequality holds for all \( m \geq 0 \):
\[
F(m) := m \sum_{k \in \mathbb{Z}^d_0} \frac{1}{(|k|^2 + m^2)^2} < \pi^2
\] (A.11)
Proof. Before we go over to the proof, we first observe that in \(\mathbb{R}^3\) we have the equality
\[
\int_{\mathbb{R}^3} \frac{dx}{(|x|^2 + m^2)^2} = \frac{\pi^2}{m^2}
\]
and, secondly, it is the absence of the term with \(k = 0\) in the sum in (A.11) that makes inequality (A.11) hold at all.

We use the Poisson summation formula (see, e. g., [42])
\[
\sum_{k \in \mathbb{Z}^n} f(k/m) = (2\pi)^{n/2} \sum_{k \in \mathbb{Z}^n} \hat{f}(2\pi km),
\]
where \(F(f)(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)e^{-i\xi x} dx\). For the function
\[
f(x) = \frac{1}{(1 + |x|^2)^2}, \quad x \in \mathbb{R}^3
\]
with \(\hat{f}(\xi) = \frac{\pi^2}{(2\pi)^{3/2}}e^{-|\xi|}\) (A.12), and with \(\int_{\mathbb{R}^3} f(x)dx = \pi^2\) this gives
\[
F(m) = \frac{1}{m^3} \sum_{k \in \mathbb{Z}^3} f(k/m) = \frac{1}{m^3} = \pi^2 \sum_{k \in \mathbb{Z}^3} e^{-2\pi m|k|} - \frac{1}{m^3} = \pi^2 \sum_{k \in \mathbb{Z}^3} e^{-2\pi m|k|} - \frac{1}{m^3}.
\]
(A.13)

In particular, this formula gives a convenient way to compute \(F(m)\) numerically for the case where \(m\) is not very small. We start the proof of inequality (A.11) with the case \(m \leq 1\).

Lemma A.5. Inequality (A.11) holds for all \(m \in [0, 1]\).

Proof. Since
\[
F^\prime(m) = \sum_{k \in \mathbb{Z}^3_0} \left( \frac{1}{(|k|^2 + m^2)^2} - \frac{4m^2}{(|k|^2 + m^2)^3} \right) = \sum_{k \in \mathbb{Z}^3_0} \frac{|k|^2 - 3m^2}{(|k|^2 + m^2)^3},
\]
we see that all terms in the sum for \(F(m)\) with \(|k|^2 \geq 3\) are monotone increasing (after multiplying by \(m\)) with respect to \(m \leq 1\), so we may write
\[
F(m) \leq \frac{6m}{(1 + m^2)^2} + \frac{12m}{(2 + m^2)^2} + \sum_{|k|^2 \geq 3} \frac{1}{(|k|^2 + 1)^2} \leq \\
\leq \max_{m \in [0,1]} \left\{ \frac{6m}{(1 + m^2)^2} \right\} + \max_{m \in [0,1]} \left\{ \frac{12m}{(2 + m^2)^2} \right\} + F(1) - \frac{6}{4} - \frac{12}{9} = \\
= \frac{9\sqrt{3}}{8} + \frac{9\sqrt{6}}{16} - \frac{3}{2} - \frac{4}{3} + \pi^2(1.01306) - 1 = 9.4915 < \pi^2 = 9.8696,
\]
where we have used (A.13) in order to compute \(F(1) = \pi^2(1.01306) - 1\) (the calculations are reliable since the series has an exponential rate of convergence). Thus, the lemma is proved. \(\square\)
We now turn to the case \( m \geq 1 \).

**Lemma A.6.** Inequality (A.11) holds for all \( m \geq 1 \).

**Proof.** It follows from (A.13) that inequality (A.11) goes over to

\[
G(m) := \pi^2 m^3 \sum_{k \in \mathbb{Z}_3^0} e^{2\pi m |k|} - 1 < 0.
\]

We use the inequality

\[
|k| \geq \frac{1}{\sqrt{3}}(|k_1| + |k_2| + |k_3|)
\]

for all terms with \(|k| > 1\) and leave the first 6 terms with \(|k| = 1\) unchanged. This gives

\[
G(m) \leq \pi^2 m^3 \sum_{k \in \mathbb{Z}_3^0} e^{2\pi m (|k_1| + |k_2| + |k_3|)/\sqrt{3}} - 1 + 6\pi^2 m^3 \left( e^{-2\pi m} - e^{-2\pi m/\sqrt{3}} \right)
\]

and we only need to prove that the right-hand side of this inequality is negative. Summing the geometric progression, we get

\[
G(m) \leq G_0(m) := \pi^2 m^3 \left( \left( 1 + \frac{2}{e^{2\pi m/\sqrt{3}} - 1} \right)^3 - 1 \right) - 1 + 6\pi^2 m^3 \left( e^{-2\pi m / \sqrt{3}} - 1 \right) + 12\pi^2 \left( \frac{m^3/2}{e^{2\pi m/\sqrt{3}} - 1} \right)^2 + 8\pi^2 \left( \frac{m}{e^{2\pi m/\sqrt{3}} - 1} \right)^3 + 6\pi^2 m^3 e^{-2\pi m} - 1 = 6\pi^2 \psi_1(m) + 12\pi^2 \psi_2(m)^2 + 8\pi^2 \psi_3(m)^3 + \psi_4(m).
\]

We claim that all functions \( \psi_i(m) \) are monotone decreasing for \( m \geq 1 \). Indeed, the function \( \psi_3(m) \) is obviously decreasing for all \( m \geq 0 \). The function \( \psi_4(m) \) is decreasing for \( m \geq \frac{3}{m} < 1 \). Analogously, as elementary calculations show, the second function is decreasing for \( m \geq m_2 < 1 \) where

\[
m_0 = \frac{\sqrt{3}}{4\pi} \left( 3 + 2W \left( -3e^{-3/2}/2 \right) \right) \approx 0.241,
\]

where \( W \) is a Lambert \( W \)-function. Finally, let us prove the monotonicity of \( \psi_1(m) \). Indeed,

\[
\psi_1'(m) = m^2 \frac{2\pi \sqrt{3} e^{-2\pi m \sqrt{3}/3} - 4\pi m \sqrt{3} - 9e^{-2\pi m \sqrt{3}/3} + 9}{3(e^{2\pi m \sqrt{3}/3} - 1)^2}
\]

and we see that

\[
2\pi \sqrt{3} e^{-2\pi m \sqrt{3}/3} - 4\pi m \sqrt{3} - 9e^{-2\pi m \sqrt{3}/3} + 9 < 2\pi m \sqrt{3} e^{-2\pi m \sqrt{3}/3} - 1 + 9 - 2\sqrt{3} < 0
\]
if \( m \geq 1 \), since \( 9 - 2\pi \sqrt{3} < 0 \). Thus, \( \psi_1'(m) < 0 \) for \( m \geq 1 \) and \( \psi_1(m) \) is also decreasing. Thus, \( G_0(m) \) is decreasing for \( m \geq 1 \) and we only need to note that \( G_0(1) = -0.7562 < 0 \) and the lemma is proved. \( \square \)

Finally, we have verified (A.11) for all \( m \geq 0 \) and the proof is complete. \( \square \)

**Remark A.7.** Of course, the estimates obtained above hold for families of scalar functions \( \{\bar{\theta}_i\}_{i=1}^n \in H^1 \) that are orthonormal with respect to (A.7). In this case, the factor \( (d-1) \) in formula (A.5) is replaced by 1, and we get a \( \sqrt{2} \)-times better constant in the 3D case and the same constant in the 2D case. Namely, the function \( \rho(x) := \sum_{i=1}^n |\bar{\theta}_i(x)| \) satisfies

\[
\|\rho\|_{L^2} \leq \frac{1}{2\sqrt{\frac{\pi}{\alpha}}} n^{1/2}, \quad d = 2, \\
\|\rho\|_{L^2} \leq \frac{1}{\sqrt{8\pi}} n^{3/4\alpha}, \quad d = 3.
\]

These estimates also hold for all three cases \( \Omega = \mathbb{T}^d, \Omega = \mathbb{R}^d, \) and \( \Omega \subset \mathbb{R}^d \) with Dirichlet boundary conditions.

**Appendix B. A pointwise estimate for the nonlinear term**

In this appendix, we prove a pointwise estimate for the inertial term which corresponds to the Navier–Stokes nonlinearity.

**Proposition B.1.** Let for some \( x \in \mathbb{R}^d, u(x) \in \mathbb{R}^d \) and \( \text{div} \ u(x) = 0 \). Then

\[
|((\theta, \nabla_x u, \theta)(x)| \leq \sqrt{\frac{d-1}{d}} |\theta(x)|^2 |\nabla_x u(x)|,
\]

where \( \nabla_x u(x) \) is a \( d \times d \) matrix with entries \( \partial_i u_j \), and

\[
|\nabla_x u|^2 = \sum_{i,j=1}^d (\partial_i u_j)^2.
\]

**Proof.** Basically, this can be extracted from [28]. For the sake of completeness we reproduce the details. We suppose first that \( A \) is a symmetric real \( d \times d \) matrix with entries \( a_{ij} \) and with \( \text{Tr} \ A = 0 \). Then

\[
\|A\|_{\mathbb{R}^d \to \mathbb{R}^d} \leq \frac{d-1}{d} \sum_{i,j=1}^d a_{ij}^2.
\]

In fact, let \( \lambda_1, \ldots, \lambda_d \) be the eigenvalues of \( A \) and let \( \lambda_1 \) be the largest one in absolute value. Then \( \sum_{j=1}^d \lambda_j = 0 \) and therefore

\[
(d - 1) \sum_{j=2}^d \lambda_j^2 \geq \left( \sum_{j=2}^d \lambda_j \right)^2 = \lambda_1^2.
\]
Adding \( (d - 1)\lambda_1^2 \) to both sides we obtain
\[
(d - 1) \sum_{j=1}^{d} \lambda_j^2 \geq d\lambda_1^2,
\]
which gives (B.2) since \( \lambda_1^2 = \|A\|_{\mathbb{R}^d \to \mathbb{R}^d}^2 \) and \( \sum_{j=1}^{d} \lambda_j^2 = \text{Tr} A^2 = \sum_{i,j=1}^{d} a_{ij}^2 \). Now (B.1) follows from (B.2) with \( A := \frac{1}{2}(\nabla_x u + \nabla_x u^T) \) and \( \text{Tr} A = 0 \), since
\[
\sum_{i,j=1}^{d} a_{ij}^2 = \frac{1}{4} \sum_{i,j=1}^{d} \left( \partial_i u_j + \partial_j u_i \right)^2 \leq \sum_{i,j=1}^{d} \left( \partial_i u_j \right)^2.
\]

\[\Box\]

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