CONTROL PROBLEMS AND INvariant Subspaces For
Sabra shell model of Turbulence

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Abstract. Shell models of turbulence are representation of turbulence equations in Fourier domain. Various shell models and their existence theory along with numerical simulations have been studied earlier. One of the most suitable shell model of turbulence is so called sabra shell model. The existence, uniqueness and regularity property of this model are extensively studied in [11]. We follow the same functional setup given in [11] and study control problems related to it. We associate two cost functionals: one ensures minimizing turbulence in the system and the other addresses the need of taking the flow near a priori known state. We derive optimal controls in terms of the solution of adjoint equations for corresponding linearized problems. Another interesting problem studied in this work is to establish feedback controllers which would preserve prescribed physical constraints. Since fluid equations have certain fundamental invariants, we would like to preserve these quantities via a control in the feedback form. We utilize the theory of nonlinear semi groups and represent the feedback control as a multi-valued feedback term which lies in the normal cone of the convex constraint space, under certain assumptions. Moreover, one of the most interesting result of this work is that we can design a feedback control with only finitely many modes, which is able to preserve the flow in the neighborhood of the constraint set.

1. Introduction. Mathematical modeling of turbulence is very complicated. Various theories and models are proposed in [6], [18], [14], [16]. Turbulent flows show large interactions at local levels/nodes. Hence it is suitable to model them in frequency domain or commonly known as Fourier domain. Shell models of turbulence are simplified caricatures of equations of fluid mechanics in wave-vector representation. They exhibit anomalous scaling and local non-linear interactions in wave number space. We would like to study control problem related to one such widely accepted shell model of turbulence known as sabra shell model.

Shell models are well known as they retain certain features of Navier Stokes Equations. The spectral form of Navier Stokes Equations motivated people to study shell models. But, unlike spectral model of Navier Stokes Equations, shell models contain local interaction between the modes, that is interaction in the short

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range which is important in turbulent phenomena. Several shell models have been proposed in literature. The form of the governing equations is derived by the necessity that the helicity and energy are conserved as in the case of Navier Stokes Equations. The most popular and well studied shell model was proposed by Gledzer and was investigated numerically by Yamada and Okhitani, which is referred as the Gledzer-Okhitani-Yamada or GOY model in short [15], [19]. The numerical experiments performed by them showed that the model exhibits an enstrophy cascade and chaotic dynamics. This garnered lot of interest in the study of shell models and many papers investigating shell models have been published since then. For more details about the shell models, we refer to [13].

1.1. Spectral form of NSE and shell model. The spectral form of Navier Stokes Equations is a starting point of shell models. The Navier Stokes Equation is given by

$$\frac{du}{dt} + (u \cdot \nabla)u = -\nabla p + \nu \Delta u + f$$

with the continuity equation

$$\text{div } u = 0.$$

in the domain $\Omega \subset \mathbb{R}^d$ where $d = 2$ or $3$. Here, $u$ denotes the velocity of the fluid, $p$ is the pressure and $f$ is the forcing term. To rewrite Navier Stokes Equations in spectral form we take Fourier transform of the equation to get,

$$\frac{du_j(n)}{dt} = -i \left(\frac{2\pi}{L}\right) n_j \sum_{n'} \left(\delta_{il} - \frac{n_i n'_l}{n^2} \right) u_i(n') u_l(n - n') - \nu k_n^2 u_j(n) + f_j(n) \quad (1)$$

where $n$ and $n'$ are vectors in $\mathbb{R}^d$,

$$u_j(k) = \frac{1}{(2\pi)^d} \int \exp^{-ikx} u_j(x) dx$$

and the wave vectors $k(n)$ are given by $k(n) = \frac{2\pi n}{L}$ [see [13]].

To describe the shell model, the spectral spaces are divided into concentric spheres of exponentially growing radius,

$$k_n = k_0 \lambda^n \quad (2)$$

with fixed $\lambda > 1$ and $k_0 > 0$. The one dimensional wave numbers are denoted by $k_n$’s such that $k_{n-1} < |k| < k_n$. The set of wave numbers contained in the $n^{th}$ sphere is called $n^{th}$ shell and $\lambda$ is called shell spacing parameter. The spectral velocity $u_n$ is a kind of mean velocity, of the complex Fourier coefficients of the velocity in the $n^{th}$ shell. Various shell models are studied in the literature in which different types of interactions between velocities in adjacent shells are considered. One such choice leads to well studied GOY model given by following equations [see [17] ]:

$$\frac{du_n}{dt} = i(ak_{n+1}u_{n+2}u_{n+1} + bk_nu_{n+1}u_{n-1} + ck_{n-1}u_{n-1}u_{n-2}) - \nu k_n^2 u_n + f_n.$$ 

for $n = 1, 2, 3, \cdots$. The boundary conditions are set to be $u_{-1} = u_0 = 0$. The kinematic viscosity is denoted by $\nu > 0$ and $f_n$’s are the Fourier components of the forcing term. The nonlinear term defines the nonlinear interaction between the nearest nodes. The constants $a, b, c$ are chosen such that the energy and enstrophy is conserved which gives the relation $a + b + c = 0$. Observe that we arrive at model which is a linear sequence of coupled first order ODE’s. Each equation is
nonlinear and quadratic in a set of velocities $u_n$ associated with discrete wave numbers $k_n$, $n = 1, 2, \cdots$.

In this work we consider a model known as sabra shell model, introduced in [17]. The main difference of this model with respect to the GOY model lies in the number of complex conjugation operators used in the nonlinear terms. As shown in [17], this slight change, is responsible for a difference in the phase symmetries of the two models. The sabra shell model exhibits shorter ranged correlations than the GOY model. Apart from this difference all calculations for GOY model remain similar to the calculations for sabra shell model. Moreover, both models also share the same quadratic invariants. Thus the results obtained in this work are equally applicable to GOY model.

In the sabra shell model the nonlinear part of the spectral Navier - Stokes Equation (1) will not only conserve energy and helicity (in the 3D case) globally but also locally that is in each triad. To derive the form of sabra shell model, the usual construction of local interactions in $k$-space, inviscid conservation of energy and fulfillment of Liouville’s theorem are used apart from the demand that the momenta involved in the triad interactions must add up to zero. For more details one can see in Chapter 3 of [13]. The equations of motion for the sabra shell model are given by

$$\frac{du_n}{dt} = i(ak_{n+1}u_{n+2}^*u_{n+1} + bk_n u_{n+1}^*u_{n-1} - ck_{n-1}u_{n-1}^*u_{n-2}) - \nu k_n^2 u_n + f_n$$

for $n = 1, 2, 3, \cdots$, with the convention that $u_{-1} = u_0 = 0$. The kinematic viscosity is represented by $\nu > 0$ and $f_n$’s are the Fourier components of the forcing term. The nonlinear term defines the nonlinear interaction between the nearest nodes. The constants $a, b, c$ are chosen such that $a + b + c = 0$.

In [11], Constantin, Levant and Titi have studied this model analytically and have proved the global regularity of solutions. The given system is infinite set of ODE’s and to rewrite the model in the functional analytic way authors of [11] define various operators on infinite dimensional space $l^2(\mathbb{C})$. Thus they arrive at the abstract formulation of the problem in $l^2$ space and have obtained existence and uniqueness of the strong and weak solutions for the equations in appropriate spaces.

In [12], the same authors have further studied the global existence of weak solutions of the inviscid sabra shell model and have shown that these solutions are unique for some short interval of time. Moreover, they give a Beal-Kato-Majda type criterion for the blowup of solutions of the inviscid sabra shell model and show the global regularity of the solutions in the two-dimensional parameters regime. We are going to study control problem for the equations considered in [11] and hence functional framework and existence uniqueness theorem proved by these authors are used directly with due reference, wherever necessary.

Control problems associated with turbulence equations in general and shell models in particular, have not been studied widely. To our knowledge, there are no known results for the control problems associated with shell models of turbulence. However, various optimal control problems are studied in literature for Navier Stokes Equations and other nonlinear fluid flow problems see [1], [21], [20] and references therein. In the current work our aim is to study optimal control problems associated with the sabra shell model of turbulence. For, we assume the control to be acting as a forcing term in the equation and try to minimize the associated cost functional. We consider two control problems with two different cost functionals: in the first
one aim is to minimize the vorticity in the flow which is equivalent to minimizing the turbulence. Whereas the other one aims to find a control which can steer the flow close to the a priori known desired state.

Another important problem studied in this work is about preserving prescribed physical constraints of the model via feedback controllers. The flow equations have some invariant quantities, which are preserved, e.g., energy, helicity and enstrophy. In engineering applications, the requirement can be of ensuring the enstrophy in a specific spatial region to be kept within a bound. We wish to find a control in a feedback form which can fulfill such a requirement. For Navier Stokes Equations, flow preserving feedback controllers are studied by Barbu and Sritharan in [4]. They have shown that the controllers in the feedback form lie in the appropriate normal cone to the state space. Since the sabra shell model has flow preserving quantities like helicity and enstrophy, we wish to investigate similar questions for this model. The question of finding feedback control can be looked upon as an optimization problem. For operator equations in abstract Hilbert spaces, the problem of finding best interpolation from a convex subset has been studied and shown to be in the normal cone of the convex subset. Various other optimization problems studied in [8], [9], [10], give similar results.

The novelty of this work lies in treating a shell model describing turbulence which displays different properties than Navier Stokes equations, though the techniques for optimal control problem are well known. Moreover, the semi group approach for this model is not studied elsewhere in the literature. We have not only studied the problem using semi group theory of nonlinear operator analysis but have also obtained the feedback controllers which are capable of preserving the flow properties. One of the most interesting results of the paper is last theorem where we can find feedback control with finitely many modes which would retain the flow in the neighborhood of constraint set.

The paper is organized as follows: We discuss the functional setting of the problem, important properties of the operators involved and the existence result in the next section. Since we are using the functional framework used in [11], we use same notations for clarity and hence this section has some resemblance with [11]. We are reiterating few of the properties and important theorems from [11] and [12] and further proving properties needed in our work here. Section 3 is devoted to the study of two control problems where, we characterize the optimal control using the adjoint equation. In section 4 we prove three important theorems about flow preserving feedback controllers. At the end of this section, we demonstrate, how these theorems can be used to determine the feedback controls, with the help of examples. We conclude the paper by summarizing our results and list few interesting problems which can be studied further.

2. Functional setting. In this section we consider the functional framework considered in [11] so that equation (3) can be written in operator form in infinite dimensional Hilbert space. We look upon \( \{u_n\} \) as an element of \( H = l^2(\mathbb{C}) \) and rewrite the equation (3) in the following functional form by appropriately defining operators \( A \) and \( B \),

\[
\frac{du}{dt} + \nu Au + B(u, u) = f \quad u(0) = u^0.
\]

For defining operators \( A \) and \( B \) we introduce certain functional spaces below. For every \( u, v \in H \) the scalar product \( (\cdot, \cdot) \) and the corresponding norm \( |\cdot| \) are defined
as,

\[(u, v) = \sum_{n=1}^{\infty} u_n v_n^*, \quad |u| = \left( \sum_{n=1}^{\infty} |u_n|^2 \right)^{\frac{1}{2}}.
\]

Let \((\phi_j)_{j=1}^{\infty}\) be the standard canonical orthonormal basis of \(H\). The linear operator \(A : D(A) \to H\) is defined through its action on the elements of the canonical basis of \(H\) as

\[A\phi_j = k_j^2 \phi_j
\]

where the eigenvalues \(k_j^2\) satisfy relation (2). The domain of \(A\) contains all those elements of \(H\) for which \(|Au|\) is finite. It is denoted by \(D(A)\) and is a dense subset of \(H\). Moreover, it is a Hilbert space when equipped with graph norm

\[\|u\|_{D(A)} = |Au| \quad \forall u \in D(A).
\]

The bilinear operator \(B(u, v)\) will be defined in the following way. Let \(u, v \in H\) be of the form \(u = \sum_{n=1}^{\infty} u_n \phi_n\) and \(v = \sum_{n=1}^{\infty} v_n \phi_n\). Then,

\[B(u, v) = -i \sum_{n=1}^{\infty} (a_k u_{n+2}^* u_{n+1} + b_k u_{n+1}^* u_n + a_k u_{n-1}^* u_{n-2} + b_k u_{n-1}^* u_{n-2}) \phi_n.
\]

With the assumption \(u^0 = u_{-1} = v_{0} = v_{-1} = 0\) and together with the energy conservation condition \(a + b + c = 0\), we can simplify and rewrite \(B(u, v)\) as

\[B(u, u) = -i \sum_{n=1}^{\infty} (a_k u_{n+2}^* u_{n+1} + b_k u_{n+1}^* u_n - c k_{n-1} u_{n-1}^* u_{n-2}) \phi_n
\]

With above definitions of \(A\) and \(B\), (3) can be written in the form

\[\frac{du}{dt} + \nu Au + B(u, u) = f; \quad u_0 = u^0.
\]

We now give some properties of \(A\) and \(B\).

Clearly, \(A\) is a positive definite, diagonal operator. Since \(A\) is a positive definite operator, the powers of \(A\) can be defined for every \(s \in \mathbb{R}\). For \(u = (u_1, u_2, \ldots) \in H\), define \(A^s u = (k_1^{2s} u_1, k_2^{2s} u_2, \ldots)\).

Furthermore we define the spaces

\[V_s := D(A^{\frac{s}{2}}) = \{ u = (u_1, u_2, \ldots) \mid \sum_{j=1}^{\infty} k_j^{2s} |u_j|^2 < \infty \}
\]

which are Hilbert spaces equipped with the following scalar product and norm,

\[(u, v)_s = (A^{s/2} u, A^{s/2} v) \forall u, v \in V_s, \quad \|u\| = (u, u)_s \quad \forall u \in V_s.
\]

Using above definition of the norm we can show that \(V_s \subset V_0 = H \subset V_{-s} \forall s \geq 0\). Moreover, it can be shown that the dual space of \(V_s\) is given by \(V_{-s}\). Domain of \(A^{1/2}\) is denoted by \(V\) and is equipped with scalar product \((u, v) = (A^{1/2} u, A^{1/2} v) \forall u, v \in D(A^{1/2})\). Thus we get the inclusion

\[V \subset H = H' \subset V',
\]

where \(V'\), the dual space of \(V\) which is identified with \(D(A^{-1/2})\). The norm in \(V\) is denoted by \(\| \cdot \|\). We denote by \(\langle \cdot, \cdot \rangle\) the action of the functionals from \(V'\) on the
elements of \( V \). Hence for every \( u \in V \), the \( H \) scalar product of \( f \in H \) and \( u \in V \) is same as the action of \( f \) on \( u \) as a functional in \( V' \).

\[
V'(f, u)_V = (f, u)_H \quad \forall f \in H, \; \forall u \in V.
\]

So for every \( u \in D(A) \) and for every \( v \in V \), we have \( \langle (u, v) \rangle = \langle Au, v \rangle = \langle u, v \rangle \).

Since \( D(A) \) is dense in \( V \) we can extend the definition of the operator \( A : V \rightarrow V' \) in such a way that \( \langle Au, v \rangle = \langle (u, v) \rangle \; \forall u, v \in V \).

In particular it follows that

\[
\|Au\|_{V'} = \|u\|_V \; \forall u \in V.
\]

**Theorem 2.1. (Properties of bilinear operator \( B \))**

1. \( B : H \times V \rightarrow H \) and \( B : V \times H \rightarrow H \) are bounded, bilinear operators.

   Specifically
   
   \[
   \langle a \rangle \quad |B(u, v)| \leq C_1 |u||v| \quad \forall u \in H, v \in V
   \]
   
   \[
   \langle b \rangle \quad |B(u, v)| \leq C_2 |v||u| \quad \forall u \in V, v \in H
   \]

   where
   \[
   C_1 = (|a|\lambda^{-1} + \lambda) + |b|(\lambda^{-1} + 1)
   \]
   \[
   C_2 = (2|a| + 2\lambda|b|).
   \]

2. \( B : H \times H \rightarrow V' \) is a bounded bilinear operator and

   \[
   \|B(u, v)\|_{V'} \leq C_1 |u||v| \quad \forall u, v \in H.
   \]

3. \( B : H \times D(A) \rightarrow V \) is a bounded bilinear operator and for every \( u \in H \) and \( v \in D(A) \)

   \[
   \|B(u, v)\| \leq C_3 |u| |Av|
   \]

   \[
   C_3 = (|a|\lambda^3 + \lambda^{-3}) + |b|(\lambda + \lambda^{-2}).
   \]

4. For every \( u \in H \) and \( v \in V \), \( \Re(B(u, v), v) = 0 \).

5. Let \( u, v, w \in V \). Denote \( b(u, v, w) = \langle B(u, v), w \rangle \). Then

   \[
   \langle a \rangle \quad b(u, v, w) = -b(v, u, w)
   \]

   \[
   \langle b \rangle \quad b(v, u, w) = -b(v, w, u).
   \]

   \[
   \langle c \rangle \quad b(u, v, v) = 0.
   \]

6. Let us denote \( B(u) = B(u, u) \). Then the map \( B : V \rightarrow V' \) which takes \( u \rightarrow B(u) \) is Gateaux differentiable. Moreover, for each \( u \in V \) the Gateaux derivative of \( B \) in the direction of \( v \in V \) is denoted by \( B'(u)v : V \rightarrow V' \) and is given by

   \[
   B'(u)v = B(u, v) + B(v, u), \quad \forall v \in V.
   \]

   and \( \langle B'(u)v, w \rangle_{(V', V)} = \langle b(u, v, w) + b(v, u, w) \rangle \quad \forall u, v, w \in V. \)

7. Let \( B'(u)^* \) denote the adjoint of \( B'(u) \). Therefore for each \( v \in V \), we have

   \[
   \langle B'(u)v, w \rangle_{(V', V)} = \langle v, B'(u)^*w \rangle_{(V, V')} \quad \forall w \in V.
   \]

   Hence, \( B'(u)^*w : V \rightarrow V' \) is given by

   \[
   B'(u)^*w = -B(u, w) - B(w, u) \quad \forall w \in V.
   \]
Proof. The proofs 1.-4. are given in [11]. We will prove the properties 5., 6., 7.

\[ b(u, v, w) = -i \sum_{n=1}^{\infty} (ak_{n+1}v_{n+2}u_{n+1}^* + bk_nv_{n+1}u_{n-1}^* + ak_{n-1}u_{n-1}v_{n-2} + bk_{n-1}v_{n-1}u_{n-2})w_n^* \]

\[ = -i \sum_{n=1}^{\infty} ak_{n+1}v_{n+2}u_{n+1}^* + \sum_{n=1}^{\infty} bk_nv_{n+1}u_{n-1}w_{n-1}^* + \sum_{n=1}^{\infty} ak_{n-1}u_{n-1}v_{n-2}w_{n}^* + \sum_{n=1}^{\infty} bk_{n-1}v_{n-1}u_{n-2}w_{n}^* \]

\[ = -i \sum_{n=3}^{\infty} ak_{n-1}v_{n-2}u_{n-1}^* + \sum_{n=2}^{\infty} bk_{n-1}v_{n-2}u_{n-1}w_{n-1}^* + \sum_{n=-1}^{\infty} ak_{n+1}u_{n+1}v_{n+2}w_{n+2}^* + \sum_{n=0}^{\infty} bk_{n}v_{n+1}u_{n}w_{n+1}^* \]

Using \( u_{-1} = u_{0} = v_{-1} = v_{0} = 0 \), we get,

\[ = -i \sum_{n=1}^{\infty} ak_{n-1}v_{n-2}u_{n-1}^* + \sum_{n=1}^{\infty} bk_{n-1}v_{n-2}u_{n-1}w_{n-1}^* + \sum_{n=1}^{\infty} ak_{n+1}u_{n+1}v_{n+2}w_{n+2}^* + \sum_{n=1}^{\infty} bk_{n}v_{n+1}u_{n}w_{n+1}^* \]

Hence,

\[ B^*(u, v) = i \sum_{n=1}^{\infty} (ak_{n+1}v_{n+2}u_{n+1}^* + bk_nv_{n+1}u_{n-1} + ak_{n-1}u_{n-1}v_{n-2} + bk_{n-1}v_{n-1}u_{n-2})\phi_n^* \]

\[ \langle B^*(u, w), v^* \rangle = i \sum_{n=1}^{\infty} (ak_{n+1}w_{n+2}u_{n+1}v_n + bk_{n}w_{n+1}u_{n-1}v_n + ak_{n-1}u_{n-1}w_{n-2}v_n + bk_{n-1}w_{n-1}u_{n-2}v_n) \]

Therefore,

\[ \langle B^*(u, w), v^* \rangle = -b(u, v, w) \]

Now using it repeatedly we get,

\[ \langle B(u, v), w \rangle = -\langle B^*(u, w), v^* \rangle \]

\[ = -\langle B(u, w), v \rangle^* \]

\[ = -\langle \langle v, B(u, w) \rangle \rangle^* \]

\[ = -\langle v, B(u, w) \rangle, \]

which proves 5(a). Similarly we can prove 5(b).

To prove \( u \rightarrow B(u) \) is differentiable, it is enough to show that:

\[ \sup_{w \in V, \; w \neq 0} \left( \frac{\|B(u) - B(v) - B'(u)(v - u), w\|}{\|v - u\|\|w\|} \right) \rightarrow 0 \text{ as } \|v - u\| \rightarrow 0 \]

with \( B'(u)(v - u) = B(v - u, u) + B(u, v - u) \).
For all \( u, v, w \in V \) we get,
\[
B(u) - B(v) - B'(u)(v - u) = B(u, u) - B(v, v) - B(v - u, u) - B(u, v - u) \\
= B(-v, u) - B(v, v) - B(u, v - u) \\
= -B(v, u - v) - B(u, v - u) \\
= B(-v, u - v) - B(u, u - v) \\
= B(u - v, u - v).
\] (6)

Now using (6) and 2. of Theorem 2.1 we can estimate
\[
\sup_{w \in V, w \neq 0} \left( \frac{|(B(u) - B(v) - B'(u)(v - u), w)|}{\|v - u\|\|w\|} \right) \leq C \frac{\|u - v\|\|u - v\|}{\|v - u\|} \\
\leq C\|u - v\| \rightarrow 0 \text{ as } \|v - u\| \rightarrow 0.
\]
This proves [6].

Using 5. of Theorem 2.1 we get,
\[
\langle B'(u)v, w \rangle = \langle B(u, w), v \rangle + \langle B(v, w), v \rangle \\
= -\langle v, B(u, w) \rangle - \langle v, B(w, u) \rangle \\
= -\langle v, B(u, w) + B(w, u) \rangle
\]
So, we can denote \( B'(u)^*w = -B(u, w) - B(w, u) \). This proves 7.

The existence and uniqueness for shell model of turbulence (4) are thoroughly studied in [11]. The proof uses mainly Galerkin approximation and Aubin’s Compactness lemma. The existence of weak and strong solution for the problem as obtained in [11] [Theorem 2, Theorem 4] are stated below.

**Theorem 2.2.** Let \( f \in L^2([0,T], V') \) and \( u^0 \in H \). Then there exists a unique weak solution \( u \in L^\infty([0,T], H) \cap L^2([0,T], V) \) to (4). Moreover the weak solution \( u \in C([0,T], H) \).

**Theorem 2.3.** Let \( f \in L^\infty([0,T], H) \) and \( u^0 \in V \). Then there exists a unique strong solution \( u \in C([0,T], V) \cap L^2([0,T], D(A)) \) to (4).

3. **Optimal control problem.** In this section, we study optimal control problem for the shell model of turbulence. We will consider two control problems with two different cost functionals.

3.1. **Vorticity control problem.** For shell model, the vorticity of flow is the velocity derivative as mentioned in [12]. We choose velocity derivative in the cost functional and minimize it so as to reduce the turbulence effect of the flow. The curl \( \nabla u \) gives the vorticity in the flow. The smaller this quantity is, the less agitated the fluid will become. In the functional framework set up for our problem we can show that \( A^{1/2}u \) estimates the vorticity. So we choose our first cost functional as following
\[
J_1(u, g) = \frac{1}{2} \int_0^T |g(t)|^2 dt + \frac{1}{2} \int_0^T |A^{1/2}u(t)|^2 dt,
\]
subject to
\[
du \over dt + \nu Au + B(u, u) = f + g, \, u(0) = u^0,
\] (7)
where \( g \) serves as a control parameter. Let us choose \( u^0 \in V \) and \( f \in L^2([0,T],H) \). If we take \( g \in L^2([0,T],H) \), a unique strong solution \( u \) of (7) exists and \( u \in L^2([0,T],D(A)) \cap C([0,T],V) \). Thus the cost functional is well defined.

We define the optimal control problem in the following way:

\[
\inf_{g \in L^2([0,T],H)} \{J_1(u,g) : u \text{ is the unique strong solution of (7) with control } g\} \quad (8)
\]

From the well known theory for the optimal control problems governed by partial differential equations, we know that the optimal control is derived in terms of the adjoint variable which satisfies a linear system. Hence we need to prove the existence and uniqueness theorem for the linearized system.

Towards this end, let us consider \( \hat{u} \) be the solution of uncontrolled system (4). Let us denote the linearized variable by \( w \) where \( w = u - \hat{u} \). The linearized variable \( w \) satisfies the following system

\[
\frac{dw}{dt} + \nu Aw + B'(\hat{u})w = g, \quad w(0) = 0 \quad (9)
\]

where \((B'(\hat{u}))w\) is as defined in (5). The existence and uniqueness of the above linearized system can be proved by using Galerkin approximation as done in Theorem 2 [11]. In fact the system under consideration is linear and the estimates are simpler than the ones proved in Theorem 2 [11]. Hence we can state following existence and uniqueness theorem for the system (9).

**Theorem 3.1.** Let, \( g \in L^2([0,T],H) \) and \( \hat{u} \) is the unique weak solution of (4). Then there exists a unique weak solution \( w \) to (9) such that

\[
w \in L^\infty([0,T],H) \cap L^2([0,T],V).
\]

Moreover the weak solution \( w \in C([0,T],H) \).

Now, we state and prove the existence of an optimal pair \((\hat{u}, \hat{g})\) for the control problem (8).

**Theorem 3.2.** Let \( u_0 \in V \) be given, then there exists at least one \( \hat{g} \in L^2([0,T],H) \) and \( \hat{u} \in C([0,T],V) \cap L^2([0,T],D(A)) \) such that the functional \( J_1(u,g) \) attains its minimum at \((\hat{u}, \hat{g})\), where \( \hat{u} \) is the unique strong solution of (7) with control \( \hat{g} \).

**Proof.** Let \( p = \inf J_1(u,g) \).

If \( g = 0 \), then by Theorem 2.3 corresponding solution \( u \) exists. Therefore \( \inf J_1(u,g) \) is well defined.

Since \( 0 \leq p = \inf J_1(u,g) < \infty \), there exists a minimizing sequence \( \{g_m\} \in L^2([0,T],H) \) such that \( \lim_{m \to \infty} J_1(u_m,g_m) = p \), where \( u_m \) is the unique strong solution to (7) with control \( g_m \).

w.l.o.g, we assume that \( J_1(u_m,g_m) \leq J_1(u,0) \). This implies that

\[
\frac{1}{2} \int_0^T |g_m|^2 dt + \frac{1}{2} \int_0^T |A^{1/2}u_m|^2 dt \leq \frac{1}{2} \int_0^T |A^{1/2}u|^2 dt.
\]

Since \( u \in C([0,T],V) \), we get,

\[
\frac{1}{2} \int_0^T |g_m|^2 dt \leq \frac{1}{2} \int_0^T |A^{1/2}u|^2 dt < \infty.
\]

Similarly we get,

\[
\frac{1}{2} \int_0^T |A^{1/2}u_m|^2 dt \leq \frac{1}{2} \int_0^T |A^{1/2}u|^2 dt < \infty.
\]
Therefore \( \{g_m\} \) is bounded in \( L^2([0,T], H) \) and \( \{u_m\} \) is bounded in \( C([0,T], V) \cap L^2([0,T], D(A)) \). So we can find a subsequence indexed by let’s say \( \{g_m\} \) and \( \{u_m\} \) such that \( g_m \to \bar{g} \) in \( L^2([0,T], H) \) and \( u_m \to \bar{u} \) in \( C([0,T], V) \cap L^2([0,T], D(A)) \).

Since the cost functional \( J_1 \) is convex and continuous on \( C([0,T], V) \cap L^2([0,T], D(A)) \times L^2([0,T], H) \), so \( J_1 \) is weakly lower semicontinuous. Therefore,

\[
p \leq J_1(\bar{u}, \bar{g}) \leq \liminf_{m \to \infty} J_1(u_m, g_m) \leq \lim_{m \to \infty} J_1(u_m, g_m) = p.
\]

Hence \( J_1 \) attains infimum at \((\bar{u}, \bar{g})\).

Now it remains to show that \( \bar{u} \) is the strong solution of (7) with r.h.s. \( \bar{g} \), which will complete the proof. We will first prove that \((\bar{u}, \bar{g})\) satisfies the weak formulation of (7) together with the initial condition i.e.

\[
\forall v \in V, \quad \left( \frac{d\bar{u}}{dt}, v \right) + \nu(A\bar{u}, v) + b(\bar{u}, \bar{u}, v) = (f, v) + (\bar{g}, v), \quad (\bar{u}(0), v) = (\bar{u}_0, v). \tag{10}
\]

Since \((u_m, g_m)\) is the solution of (7), it satisfies the weak formulation i.e. we show

\[
\forall v \in V, \quad \left( \frac{du_m}{dt}, v \right) + \nu(Au_m, v) + b(u_m, u_m, v) = (f, v) + (g_m, v), \quad (u_m(0), v) = (u_0, v).
\]

Since \( u_m \to \bar{u} \) in \( L^2([0,T], V) \), so passing to the limit we get \( \left( \frac{du_m}{dt}, v \right) \to \left( \frac{d\bar{u}}{dt}, v \right) \) and \( (u_m(0), v) \to (\bar{u}(0), v) \). As \( A \) is a linear operator \( \nu(Au_m, v) \to \nu(A\bar{u}, v) \).

Similarly since \( g_m \to \bar{g} \) in \( L^2([0,T], H) \), so \((g_m, v) \to (\bar{g}, v)\).

We only need to prove that \( B(u_m, u_m) \to B(\bar{u}, \bar{u}) \) in \( L^2([0,T], V') \). For, it is enough to check that \( |(B(u_m, u_m), v) - B(\bar{u}, \bar{u}), v)| \to 0 \) as \( m \to \infty \).

\[
|b(u_m, u_m, v) - b(\bar{u}, u_m)| = |b(u_m - \bar{u}, u_m, v) + b(\bar{u}, u_m, v) - b(u_m - \bar{u}, v)| = |b(u_m - \bar{u}, u_m, v) + b(\bar{u}, u_m - \bar{u}, v) + b(\bar{u}, u_m - \bar{u}, v)| \leq |b(u_m - \bar{u}, u_m, v)| + |b(\bar{u}, u_m - \bar{u}, v)|
\]

Using 2 of Theorem 2.1 we get,

\[
|b(u_m, u_m, v) - b(\bar{u}, u_m)| \leq C|u_m - \bar{u}||u_m||v| + C|\bar{u}||u_m - \bar{u}||v|.
\]

We know \( \{u_m\} \) is bounded. So as \( m \to \infty \), \( |(B(u_m, u_m), v) - B(\bar{u}, \bar{u}), v)| \to 0 \). Hence \((\bar{u}, \bar{g})\) satisfies the weak formulation (10). Therefore \( \bar{u} \) is the weak solution of (7) with control \( \bar{g} \). If \( \bar{u} \) is any other strong solution of (7) with control \( \bar{g} \), it would also be weak solution of (7) with control \( \bar{g} \). And hence by uniqueness of weak solution we will get \( \bar{u} = \bar{u} \) which proves that \( \bar{u} \) is the strong solution of (7) with control \( \bar{g} \). This completes the proof.

In the previous theorem, we have proved the existence of an optimal solution of (8). Next, we want to characterize the optimal control which will be in terms of the adjoint variable. To prove our main theorem of this section namely characterization of optimal control we first prove the following lemma.

**Lemma 3.3.** Let, \( h_1 \in L^2([0,T], H) \) and \( w_{h_1} \) be the solution of

\[
\frac{dw}{dt} + \nu Aw + B'(u)w = h_1, \quad w(0) = 0. \tag{11}
\]

where \( u \) is a solution of non linear controlled equation (7). Then for all \( h_2 \in L^2([0,T], H) \) we have

\[
\int_0^T \langle h_2, w_{h_1} \rangle dt = \int_0^T \langle \tilde{w}_{h_2}, h_1 \rangle dt
\]
where \( \hat{w}_{h_2} \) is the solution of the adjoint linearized problem

\[
- \frac{d\hat{w}}{dt} + \nu A^* \hat{w} + B'(u_g)^* \hat{w} = h_2, \quad \hat{w}(T) = 0.
\]  

(12)

**Proof.**

\[
\int_0^T (h_2, w_{h_1}) dt \\
= \int_0^T \left( -\frac{d\hat{w}_{h_2}}{dt} + \nu A^* \hat{w}_{h_2} + B'(u_g)^* \hat{w}_{h_2}, w_{h_1} \right) dt \\
= \int_0^T \left( [(-\frac{d\hat{w}_{h_2}}{dt}, w_{h_1}) + (\nu A^* \hat{w}_{h_2}, w_{h_1}) + (B'(u_g)^* \hat{w}_{h_2}, w_{h_1})] \right) dt \\
= \int_0^T (-\frac{d\hat{w}_{h_2}}{dt}, w_{h_1}) dt + \int_0^T (\nu A^* \hat{w}_{h_2}, w_{h_1}) dt + \int_0^T (B'(u_g)^* \hat{w}_{h_2}, w_{h_1}) dt \\
= \int_0^T (\hat{w}_{h_2}, \frac{dw_{h_1}}{dt}) dt + (\hat{w}_{h_2}(T), w_{h_1}(T)) - (\hat{w}_{h_2}(0), w_{h_1}(0)) \\
+ \nu \int_0^T (\hat{w}_{h_2}, Aw_{h_1}) dt + \int_0^T (\hat{w}_{h_2}, B'(u_g)w_{h_1}) dt.
\]

Since \( w_{h_1}(0) = 0 \) and \( \hat{w}_{h_2}(T) = 0 \) we get

\[
\int_0^T (h_2, w_{h_1}) dt = \int_0^T [(\hat{w}_{h_2}, \frac{dw_{h_1}}{dt} + Aw_{h_1} + B'(u_g)w_{h_1})] dt
\]

which implies

\[
\int_0^T (h_2, w_{h_1}) dt = \int_0^T (\hat{w}_{h_2}, h_1) dt.
\]

\( \square \)

Now, we will characterize the optimal control of the problem (8) i.e. \( \bar{g} \) in terms of adjoint of the linearized problem. We state and prove the following theorem.

**Theorem 3.4.** Let \((\bar{u}, \bar{g})\) be an optimal pair for the control problem (8), then the optimal control \( \bar{g} \) can be characterized as

\[
\bar{g} = -\hat{w}_{A\bar{u}},
\]

where \( \hat{w}_{A\bar{u}} \) is the solution of the linearized adjoint system

\[
- \frac{d\hat{w}}{dt} + \nu A^* \hat{w} + B'(u_g)^* \hat{w} = A\bar{u}, \quad \hat{w}(T) = 0.
\]

**Proof.** Let \((\bar{u}, \bar{g})\) be the optimal pair for the control problem (8). We denote \( u_g \) as the solution of the controlled system (7) with control \( g \).

Let \( F(g) = J_1(u_g, g) \).

Then we have

\[
F(\bar{g} + \lambda g) - F(\bar{g}) \\
= J_1(u_{\bar{g} + \lambda g}, \bar{g}) + \lambda J_1(u_{\bar{g}}, \bar{g}) \\
= \frac{1}{2} \int_0^T |\bar{g} + \lambda g|^2 dt + \frac{1}{2} \int_0^T |A^{1/2}u_{\bar{g} + \lambda g}|^2 dt \\
- \frac{1}{2} \int_0^T |\bar{g}|^2 dt - \frac{1}{2} \int_0^T |A^{1/2}u_{\bar{g}}|^2 dt
\]
\[
\begin{align*}
&= \frac{1}{2} \int_0^T (\bar{g} + \lambda g, \bar{g} + \lambda g) dt - \frac{1}{2} \int_0^T (\bar{g}, \bar{g}) dt \\
&+ \frac{1}{2} \int_0^T |A^{1/2} u_{\bar{g}} + \lambda g|^2 dt - \frac{1}{2} \int_0^T |A^{1/2} u_{\bar{g}}|^2 dt,
\end{align*}
\]

Similarly, if we take the directional derivative of \( F \) in the direction of \(-g\), we will get,

\[
F'(\bar{g}) \cdot g \leq 0.
\]

Hence,

\[
F'(\bar{g}) \cdot g = \int_0^T (g, \bar{g}) dt + \int_0^T (A^{1/2} w_g, A^{1/2} u_{\bar{g}}) dt = 0.
\]

Further, with the notation \( u_{\bar{g}} = \bar{u} \), we can write

\[
\int_0^T (g, \bar{g}) dt + \int_0^T (w_g, A\bar{u}) dt = 0.
\]

Using lemma 3.3 we get,

\[
\int_0^T (g, \bar{g}) + \int_0^T (g, \bar{w}_{A\bar{u}}) = 0
\]

Since the above equality is true for all \( g \in L^2([0, T], H) \) we get,

\[
(g, \bar{g}) + (g, \bar{w}_{A\bar{u}}) = 0.
\]

Therefore,

\[
\bar{g} = -\bar{w}_{A\bar{u}}.
\]

In the following lemma, we prove the limit used in the theorem namely

\[
\lim_{\lambda \to 0} \frac{u_{\bar{g}} + \lambda g - u_k}{\lambda} = w_g
\]

which would then complete the proof of the theorem.
Lemma 3.5. Let \( u^0 \in V \) be given. Then the mapping \( g \to u_g \) from \( L^2([0,T],H) \) into \( C([0,T],V) \cap L^2([0,T],D(A)) \) is Gateaux differentiable. Furthermore we have, for each \( h \in L^2([0,T],H) \),

\[
\lim_{\lambda \to 0} \frac{u_{g+\lambda h} - u_g}{\lambda} = w_h
\]

where \( w_h \) is the solution of the linearized equation (11) with right hand side \( h \).

Proof. Let us fix \( u^0 \in V \) and \( g, h \in L^2([0,T],H) \).

We have to prove that \( g \to u_g \) is Gateaux differentiable from \( L^2([0,T],H) \) into \( C([0,T],V) \cap L^2([0,T],D(A)) \). So it is enough to prove that

\[
\lim_{|\lambda| \to 0} \left( \frac{|u_{g+\lambda h} - u_g - \lambda w_h|_{C([0,T],V) \cap L^2([0,T],D(A))}}{|\lambda|} \right) = 0.
\]

Set \( z = u_{g+\lambda h} - u_g - \lambda w_h \).

Using (7) and (11) we get,

\[
\frac{dz}{dt} = -\nu A u_{g+\lambda h} - B(u_{g+\lambda h}) + f + g + \lambda h + \nu A u_g + B(u_g) - f - g
\]

\[-\lambda (-\nu A u_h - B'(u_g) w_h + h)
\]

\[-\nu A z - B(u_{g+\lambda h}) + B(u_g) + \lambda B'(u_g) w_h.
\]

So \( z \) satisfies

\[
\frac{dz}{dt} + \nu A z + B(u_{g+\lambda h}) - B(u_g) - \lambda B'(u_g) w_h = 0, \quad z(0) = 0.
\]

Define the operator \( Q : [0,T] \to V' \),

\[
Q(t) = B(u_{g+\lambda h}) - B(u_g) - B'(u_g)(u_{g+\lambda h} - u_g).
\]

From (16) we have \( z \) is the solution of

\[
\frac{dz}{dt} + \nu A z + B'(u_g) z = -Q(t), \quad z(0) = 0.
\]

Using 2. and 6. of Theorem 2.1 we estimate

\[
\|Q(t)\|_{V'} \leq C\|u_{g+\lambda h} - u_g\| \leq C\|u_{g+\lambda h} - u_g\|^2.
\]

Integrating (18) over \([0,T]\) we get,

\[
\|Q\|_{L^2([0,T],V')} \leq C\|u_{g+\lambda h} - u_g\|^2_{L^2([0,T],V')}.
\]

From (17) we get,

\[
\|z\|_{C([0,T],V') \cap L^2([0,T],D(A))} \leq \|Q\|_{L^2([0,T],V')} \leq C\|u_{g+\lambda h} - u_g\|^2_{L^2([0,T],V')}.
\]

If we show that

\[
\|(u_{g+\lambda h} - u_g)\|_{L^2([0,T],V')} \leq C|\lambda|,
\]

then proof is done. For observe that, \( u_{g+\lambda h} - u_g \) solves the equation

\[
\frac{du}{dt} + \nu A u + B(u, u) + B(u, u_g) + B(u) = \lambda h , \quad u(0) = 0.
\]
By Gronwall’s lemma we get,
\[ \|(u_{g^*} + \lambda - u_g)\|_{C([0,T], V) \cap L^2([0,T], D(A))} \leq |\lambda| \|h\|_{L^2([0,T], V')} \leq C|\lambda|. \]
Therefore by (19) we get,
\[ \|z\|_{C([0,T], V) \cap L^2([0,T], D(A))} \leq C|\lambda|^2 \]
Now dividing by \( \lambda \) and sending \( \lambda \to 0 \), (15) is proved. \( \square \)

3.2. Linear quadratic control problem. Next we prove the existence and characterization of optimal control for linear quadratic cost functional. Let,
\[ J_2(u, g) = \frac{1}{2} \int_0^T |u - u_d|^2 + \frac{\beta}{2} \int_0^T |g|^2. \]
where \( u_d \) is the desired state. Let us choose \( u_0 \in H \), \( g \in L^2([0,T], H) \) and \( u_d \in L^\infty([0,T], H) \cap L^2([0,T], V) \). We define the control problem as following
\[ \inf \{ J_2(u, g) : u \text{ is the unique weak solution of (7) with control } g \} \quad (21) \]
As before we will show that an optimal pair exists and then we will characterize the optimal control. However, it is important to note that we can prove the results using weak solution of (7) unlike in Theorem 3.2 and 3.4 where we require the existence of strong solution.

**Theorem 3.6.** Let \( u^0 \in H \) be given. Then there exists at least one \( \bar{g} \in L^2([0,T], H) \) and \( \bar{u} \in L^\infty([0,T], H) \cap L^2([0,T], V) \) such that the functional \( J_2(u, g) \) attains its minimum at \( (\bar{g}, \bar{u}) \), where \( \bar{u} \) is the weak solution of (7) with control \( \bar{g} \).

**Proof.** If \( g = 0 \), then by Theorem 3.1 corresponding solution \( u \) exists in \( L^\infty([0,T], H) \cap L^2([0,T], V) \). So \( J(u, 0) \) exists. Therefore \( \inf J_2(u, g) \) exists.

Let \( \alpha = \inf J_2(u, g) \).

Since, \( 0 \leq \alpha < \infty \), there exists a minimizing sequence \( \{g_m\} \subseteq L^2([0,T], H) \) such that \( J_2(u_m, g_m) \to \alpha \), where \( u_m \in L^\infty([0,T], H) \cap L^2([0,T], V) \) is the unique weak solution of (7) with control \( g_m \).

w.l.o.g we assume that \( J_2(u_m, g_m) \leq J_2(u_0, 0) \). This implies,
\[ \frac{1}{2} \int_0^T |u_m - u_d|^2 + \frac{\beta}{2} \int_0^T |g_m|^2 \leq \frac{1}{2} \int_0^T |u - u_d|^2 dt. \]
Since \( u, u_d \in L^\infty([0,T], H) \), so \( (u - u_d) \in L^\infty([0,T], H) \). Which gives
\[ \frac{\beta}{2} \int_0^T |g_m|^2 dt \leq \frac{1}{2} \int_0^T |u - u_d|^2 dt < \infty. \]
Similarly we get,
\[ \frac{1}{2} \int_0^T |u_m - u_d|^2 \leq \frac{1}{2} \int_0^T |u - u_d|^2 dt < \infty. \]
Therefore, \( \{g_m\} \) is bounded in \( L^2([0,T], H) \). It is easy to see that \( \{u_m\} \) is bounded in \( L^\infty([0,T], H) \cap L^2([0,T], V) \) by using the energy estimate proved in Theorem 2 of [11]. So we can find subsequences indexed by let’s say \( \{g_m\} \) and \( \{u_m\} \) such that \( g_m \to \bar{g} \) in \( L^2([0,T], H) \) and \( u_m \to \bar{u} \) in \( L^\infty([0,T], H) \cap L^2([0,T], V) \). Moreover by Aubin-Lion’s compactness argument we also get \( u_m \to \bar{u} \) in \( C([0,T], H) \). Therefore \( \bar{u} \) is the solution of (7) with control \( \bar{g} \).

Since $J_2$ is continuous and convex on $L^2([0, T], H) \times L^\infty([0, T], H) \cap L^2([0, T], V)$, so $J_2$ is weakly lower semicontinuous on $L^2([0, T], H) \times L^\infty([0, T], H) \cap L^2([0, T], V)$. Therefore,

$$\alpha \leq J_2(\bar{u}, \bar{g}) \leq \lim_{m \to \infty} J_2(u_m, g_m) \leq \lim_{m \to \infty} J_2(u_m, g_m) = \alpha.$$ 

Hence, $J_2(\bar{u}, \bar{g}) = \alpha$. □

The next theorem characterizes the optimal control in terms of solution of the adjoint system.

**Theorem 3.7.** Let $(\bar{u}, \bar{g})$ be an optimal pair for the control problem (21), then the optimal control $\bar{g}$ is given by

$$\bar{g} = -\frac{1}{\beta} \tilde{w}_{\bar{u} - u_d},$$

where $\tilde{w}_{\bar{u} - u_d}$ is the solution of the linearized adjoint system

$$-\frac{d\tilde{w}}{dt} + \nu A^* \tilde{w} + B'(u_g)^* \tilde{w} = \bar{u} - u_d, \quad \tilde{w}(T) = \bar{u}(T) - u_d(T).$$

**Proof.** Let, $(\bar{u}, \bar{g})$ be the optimal pair for the control problem (21).

Let, $F_1(g) = J_2(u_g, g)$.

Then we have,

$$F_1(\bar{g} + \lambda g) - F_1(\bar{g}) = J_2(u_{\bar{g} + \lambda g}, \bar{g} + \lambda g) - J_2(u_{\bar{g}}, \bar{g})$$

$$= \frac{1}{2} \int_0^T |u_{\bar{g} + \lambda g} - u_d|^2 + \frac{\beta}{2} \int_0^T \bar{g} + \lambda g|^2 dt - \frac{1}{2} \int_0^T |u_{\bar{g}} - u_d|^2$$

$$+ \frac{\beta}{2} \int_0^T \bar{g}|^2 dt$$

$$= \frac{1}{2} \int_0^T (u_{\bar{g} + \lambda g} - u_{\bar{g}}, u_{\bar{g} + \lambda g} + u_{\bar{g}} - 2u_d) + \frac{\beta}{2} \int_0^T 2\lambda(g, \bar{g})$$

$$+ \frac{\beta}{2} \int_0^T \lambda^2(g, g)$$

$$= \frac{1}{2} \int_0^T (u_{\bar{g} + \lambda g} - u_{\bar{g}}, u_{\bar{g} + \lambda g} - u_{\bar{g}}) + \frac{\beta}{2} \int_0^T \lambda(g, \bar{g})$$

$$\leq \frac{1}{2} \int_0^T |u_{\bar{g} + \lambda g} - u_{\bar{g}}|^2 + \int_0^T (u_{\bar{g} + \lambda g} - u_{\bar{g}}, u_{\bar{g}} - u_d) + \frac{\beta}{2} \int_0^T \lambda^2(g, g)$$

$$+ \beta \int_0^T \lambda(g, \bar{g}).$$

We can estimate as $|u_{\bar{g} + \lambda g} - u_{\bar{g}}|_{L^2([0, T], H)} \leq \lambda \|g\|_{L^2([0, T], H)}$. Thus after dividing by $\lambda$ and taking $\lambda \to 0$, the first term goes to zero. For the second term we use lemma 3.5. Thus, dividing by $\lambda$ and taking $\lambda \to 0$ we obtain,

$$0 \leq F'_1(\bar{g}) \cdot g = \lim_{\lambda \to 0} \frac{F(\bar{g} + \lambda g) - F(\bar{g})}{\lambda} = \int_0^T (w_g, u_{\bar{g}} - u_d) + \beta \int_0^T (g, \bar{g}).$$
Further using Lemma 3.3 we get,
\[ 0 \leq F_1'(\bar{g}) \cdot g = \int_0^T (g, \bar{w}_u - u_d) + \beta \int_0^T (g, \bar{g}) \]

Similarly if we take the directional derivative of \( F_1 \) in the direction of \(-g\), we will get,
\[ F_1'(g) \cdot g \leq 0 \quad \text{and hence,} \quad F_1'(g) \cdot g = 0 \]

Thus,
\[ \int_0^T (g, \bar{w}_u - u_d) + \beta \int_0^T (g, \bar{g}) = 0 \]

Since the above equality is true for all \( g \in L^2([0,T],H) \), we get,
\[ \bar{g} = \frac{1}{\beta} \bar{w}_u - u_d. \]

This completes the proof. \( \Box \)

4. Invariant subspace. In this section our aim is to find feedback controllers such that certain physical properties associated with the flow are preserved. Let \( K \) be a given closed convex subset of the state space. We wish to find a controller \( g \) such that, whenever an initial value of flow equation lies in \( K \) so does the solution of the controlled equation (7). In many cases the controller \( g \) which preserves flow in \( K \) lies in the normal cone to \( K \). Under some assumptions on \( K \) we show that it is possible to find flow preserving feedback controllers. Later on for specific examples of \( K \) we show how these theorems can be applied. Our proofs use the theory of nonlinear partial differential equation associated to maximal monotone operator.

In the following theorem, we assume that \( K \) is invariant under the operator \((I + \lambda K)^{-1}\). This assumption helps us to get the control which belongs to the normal cone to the set \( K \), so that \( K \) is invariant with respect to initial value.

Before stating the theorem we define necessary terminology.

**Definition 1. Normal Cone.** Let \( K \) be a non-empty convex subset of a Hilbert space \( H \), and \( x \in K \). Then the normal cone to \( K \) at \( x \) is defined as \( N_K(x) = \{ y \in H : \langle y, x - z \rangle \geq 0 \ \forall z \in K \} \).

**Definition 2. Quasi m-accretive.** An operator \( A : H \to H \) is called accretive if for every pair \([x_1, Ax_1], [x_2, Ax_2] \in H \times H\), \((x_1 - x_2, Ax_1 - Ax_2) \geq 0\) holds.

\( A \) is called m-accretive if \( R(\lambda I + A) = H \ \forall \lambda > 0 \).

The operator \( A \) is called quasi m-accretive if \((A + \mu I)\) is m-accretive for some \( \mu > 0 \).

**Theorem 4.1.** Let \( K \) be a closed convex subset of \( H \) such that \( 0 \in K \) and \((I + \lambda A)^{-1}K \subset K \ \forall \lambda > 0 \). Let \( u^0 \in D(A) \cap K \) and \( f \in W^{1,1}([0,T], H) \cap L^\infty([0,T], H) \).

Then, there exists a feedback controller \( g \in L^\infty([0,T] , H) \) and \( g(t) \in N_K(u(t)) \) a.e. \( t \in [0,T] \) such that the corresponding solution to the closed loop system
\[ \frac{du}{dt} + \nu Au + B(u) = f + g, \quad u(0) = u^0 \]  \hspace{0.5cm} (22)

satisfies \( u \in W^{1,\infty}([0,T], H) \cap L^\infty([0,T], K \cap D(A)) \cap C([0,T], V) \). Moreover, \( \frac{d^+u}{dt} + (-f + \nu Au + B(u) + N_K(u))^0 = 0, u(0) = u^0 \) \hspace{0.5cm} (23)

and \( u(t) \in K \ \forall \ t \in [0,T] \). Here \( N_K(u(t)) := \{ w \in H : \langle w, u - z \rangle \geq 0, \forall z \in K \} \) is the normal cone to \( K \) at \( u(t) \) and \( u \to (-f(t) + \nu Au(t) + B(u(t)) + N_K(u(t))) \) is a
multivalued map. If one defines \((-f(t) + \nu A u(t) + B(u(t))) + N_K(u(t))\) to be the projection of origin on \((-f(t) + \nu A u(t) + B(u(t))) + N_K(u(t))\) which is of minimum norm.

So by (23) we can deduce the feedback controller \(g\) is given by,

\[
g(t) = -f(t) + \nu A u(t) + B(u(t)) + N_K(u(t))
\]

\[
-(-f(t) + \nu A u(t) + B(u(t))) + N_K(u(t)) = 0 \quad \forall \, t \in [0, T].
\]

**Proof.** Let us define the modified nonlinear map \(B_N(\cdot) : V \to V^*\),

\[
B_N(\cdot) := \begin{cases} 
B(u) & \text{if } ||u|| \leq N \\
\left(\frac{N}{||u||}\right)^2 B(u) & \text{if } ||u|| > N
\end{cases}
\]

First, we will prove that the operator \(u \to (-f + \nu A u + B_N(u) + N_K(u(t)))\) is quasi m-accretive in \(H\). By definition 2 we have to prove that there exists \(\alpha_N > 0\) such that \(u \to (-f + \nu A u + B_N(u) + N_K(u(t)) + \alpha_N u)\) is m-accretive in \(H\).

From [3][Chapter II, Theorem 2.1] we know that \(u \to N_K(u)\) is m-accretive. We prove \(\nu A + B_N\) is m-accretive as a first step of the proof. For, we define a new operator \(\Gamma_N\) such that \(\Gamma_N : D(\Gamma_N) \to H\) given by,

\[
\Gamma_N = \nu A + B_N.
\]

We claim that \(D(\Gamma_N) = D(A)\). Clearly, \(D(\Gamma_N) \subset D(A)\). So it remains to show the other inclusion.

For, let \(x \in D(A)\). Using 3 of Theorem 2.1 we get,

\[
||\Gamma_N(x)|| = ||\nu A(x) + B_N(x)|| \\
\leq ||\nu A(x)|| + ||B_N(x)|| \\
\leq \nu ||A(x)|| + C_3 ||A(x)|| < \infty.
\]

Which proves \(D(\Gamma_N) = D(A)\).

Now, we will show that, \(u \to \Gamma_N(u)\) is m-accretive. By definition 2, it is enough to show that \(u \to (\Gamma_N(u) + \alpha_N I)\) is accretive, for some \(\alpha_N > 0\).

For \(||u|| \leq N\), \(||z|| \leq N\), using 2. of Theorem 2.1 we get,

\[
||B_N(u) - B_N(z), u - z|| = |b(u, u - z, u - z)| + |b(u - z, z, u - z)| \\
\leq C ||u - z|| ||u - z||.
\]

Since \(||u|| \leq N\), \(||z|| \leq N\) and using Young’s inequality we can estimate,

\[
||(B_N(u) - B_N(z), u - z)|| \leq C_N ||u - z|| ||u - z|| \\
\leq C_N ||u - z||^2 + \frac{\nu}{2} ||u - z||^2. \tag{25}
\]

Similarly, for \(||u|| > N, ||z|| > N\) we get,

\[
||(B_N(u) - B_N(z), u - z)|| \\
= \left|\left(\frac{N}{||u||}\right)^2 b(u, u, u - z) - \left(\frac{N}{||u||}\right)^2 b(z, z, u - z)\right| \\
= \left|\left(\frac{N}{||u||}\right)^2 b(u, u, u - z) - \left(\frac{N}{||u||}\right)^2 b(z, z, u - z)\right|
\]
Therefore, if we choose $\alpha \parallel u = 0$, then similar calculations as above yield (26).

If $\|u\| > N$, $\|z\| \leq N$ or $\|u\| \leq N$, $\|z\| > N$ then similar calculations as above yield estimate as in (25),(26).

Consider $(\Gamma_N + \lambda)u > 0$, 

$$(\Gamma_N + \lambda)u - (\Gamma_N + \lambda)z, u - z) = ((\nu A + B_N + \lambda)u - (\nu A + B_N + \lambda)z, u - z) = \nu (Au - Az, u - z) + (B_N u - B_N z, u - z) + \lambda (u - z, u - z).$$

Since $A$ is m-accretive, $(Au - Az, u - z) \geq \nu \|u - z\|^2$ and using (25),(26) we get,

$$(\Gamma_N + \lambda)u - (\Gamma_N + \lambda)z, u - z) \geq \nu \|u - z\|^2 - C|u - z|^2 - \frac{\nu}{2} \|u - z\|^2 + \lambda|u - z|^2$$

$$= \frac{\nu}{2} |u - z|^2 + (\lambda - C)|u - z|^2$$

$$\geq 0 \text{ if } \lambda > C.$$ 

Therefore, if we choose $\alpha_N > C > 0$, $u \rightarrow (\Gamma_N + \alpha_N I)$ will be accretive.

Now it remains to show that, for fixed $\mu > \alpha_N$

$$R(\mu u + \nu Au + B_N (u) + N_K (u)) = H.$$ 

This would be proved using Hille-Yosida theorem. For, consider the Yosida approximation of $N_K$,

$$F_{\lambda} = \frac{1}{\lambda} (I - (I + \lambda N_K)^{-1}).$$
Let \( f \in H \) be arbitrary but fixed. Then by Minty’s theorem from [3] there exists a unique \( u_\lambda \in D(A) \) such that
\[
\mu u_\lambda + \nu Au_\lambda + B_N(u_\lambda) + F_\lambda(u_\lambda) = f
\]  
(27)
Taking inner product of (27) with \( u_\lambda \) we get,
\[
\mu |u_\lambda|^2 + \nu \|u_\lambda\|^2 + (B_N(u_\lambda), u_\lambda) + (F_\lambda(u_\lambda), u_\lambda) = (f, u_\lambda).
\]
By 2. of Theorem 2.1 and (28) and (29) we get,
\[
\mu |u_\lambda|^2 + \nu \|u_\lambda\|^2 \leq \frac{1}{2\mu} |f|^2 + \frac{\mu}{2} |u_\lambda|^2.
\]
This yields,
\[
\frac{\mu}{2} |u_\lambda|^2 + C_\nu \|u_\lambda\|^2 \leq C.
\]  
(28)
Taking inner product of (27) with \( Au_\lambda \) we get,
\[
\mu \|u_\lambda\|^2 + \nu |Au_\lambda|^2 + (B_N(u_\lambda), Au_\lambda) + (F_\lambda(u_\lambda), Au_\lambda) = (f, Au_\lambda).
\]
Together with \((f + \lambda A)^{-1}K \subset K \forall \lambda > 0 \) and [3][Chapter IV, Proposition 1.1] we get \((F_\lambda(u_\lambda), Au_\lambda) \geq 0\). Therefore, Using 3. of Theorem 2.1,
\[
\mu \|u_\lambda\|^2 + \nu |Au_\lambda|^2 \leq |b(u_\lambda, Au_\lambda, u_\lambda)| + |f||u_\lambda| \\
\leq C |u_\lambda||Au_\lambda\| \|u_\lambda\| + |f||Au_\lambda| \\
\leq C_\nu \|u_\lambda\|^2 \|u_\lambda\|^2 + \frac{\nu}{2} |Au_\lambda|^2 + \frac{1}{\nu} |f|^2
\]
Hence (28) gives,
\[
C_\mu \|u_\lambda\|^2 + \frac{\nu}{2} |Au_\lambda|^2 \leq C.
\]  
(29)
Therefore, we get from (28) and (29),
\[
\|u_\lambda\|^2 + |Au_\lambda|^2 \leq C \forall \lambda > 0.
\]  
(30)
Also, for fixed \( N, B_N \) and \( F_\lambda \) are bounded linear operators. So we get respectively,
\[
|B_N(u_\lambda)| \leq C_N \text{ and } |F_\lambda u_\lambda| \leq C_N.
\]  
(31)
Therefore by (30) we can conclude that there exists a subsequence (let’s denote by \( \lambda \)) such that, \( u_\lambda \to u \) weakly in \( V \) and \( Au_\lambda \to Au \) weakly in \( H \).
Moreover, by (31) we conclude that \( F_\lambda u_\lambda \to \gamma \) weakly in \( H \) and \( B_N(u_\lambda) \to B_N(u) \) weakly in \( H \).
Since, \( F_\lambda \) are the Yosida approximation of \( N_K \), so we get \( \gamma \in N_K(u) \) and \( u \) is the solution of
\[
\mu u + \nu Au + B_N(u) + N_K(u) = f.
\]
Therefore, \( u \to \mu I + \nu Au + B_N(u) + N_K(u) \) is quasi m-accretive. Also \( u^0 \in D(A) \).
So by [5][Chapter 4, Theorem 4.5, Theorem 4.6] we get,
\[
\frac{du}{dt} + \nu Au + B_N(u) + N_K(u) = f, \; u(0) = u^0 \; \text{ a.e. } t \in [0, T],
\]
has a unique solution \( u_N \in W^{1,\infty}([0, T], H) \cap L^\infty([0, T], D(A) \cap K) \cap C([0, T], V) \)
which satisfies,
\[
\frac{d^t u_N}{dt} + (\nu Au + B_N(u) + N_K(u_N) - f)^0 = 0 \; \forall t \in [0, T].
\]
Now, we will show that $\|u_N\| < C$ for some large $N$. For, taking inner product of
\[
\frac{du_N}{dt} + \nu Au_N + B_N(u_N) + N_K(u_N) = f
\]  
with $u_N$ and using the fact that $(N_K(u_N), u_N) \geq 0$ we get,
\[
\frac{d}{dt}|u_N|^2 + \nu \|u_N\|^2 + (N_K(u_N), u_N) \leq +|f, u_N)|
\]
Further we get,
\[
\frac{d}{dt}|u_N|^2 + \nu \|u_N\|^2 \leq \frac{1}{2}|f|^2 + \frac{1}{2}\|u_N\|^2
\]  
(33)

Similarly taking inner product of (32) with $Au_N$ and we get,
\[
\frac{d}{dt}\|u_N\|^2 + \nu |Au_N|^2 + (N_K(u_N), Au_N) \leq -(B_N(u_N), Au_N) + (f, Au_N)
\]
Using $(I + \lambda A)^{-1}K \subset K \forall \lambda > 0$ and [3][Chapter IV, Proposition 1.1] we get
\[
(B_N(u_N), Au_N) \leq C\|u_N\|\|Au_N\|
\]
and we get,
\[
\|u_N\|^2 + \nu |Au_N|^2 \leq C\|u_N\|^2 + \frac{\nu}{4}|Au_N|^2.
\]  
(34)

Further we get,
\[
\frac{d}{dt}\|u_N\|^2 + \frac{\nu}{2}|Au_N|^2 \leq C\|u_N\|^2 + \frac{1}{2}|f|^2.
\]
Using Gronwall’s lemma to (33) we get,
\[
|u_N(t)|^2 \leq \exp \left(\frac{t}{2}(|u_0|^2 + \frac{1}{2}\int_0^t |f(s)|^2 ds)\right) \forall t \in [0, T].
\]
Moreover, integrating (33) over $[0, t]$ and using $f \in L^\infty([0, T], H)$ we get,
\[
|u_N(t)|^2 + \nu \int_0^t \|u_N(s)\|^2 ds \leq C \forall t \in [0, T].
\]
Using Gronwall’s lemma to (34) we get,
\[
\|u_N(t)\|^2 \leq \left(\|u_0\|^2 + \frac{1}{\nu}\int_0^t |f(s)|^2 ds\right) \exp \left(C\nu \int_0^t |u_N(s)|^2 ds\right) \forall t \in [0, T].
\]
Moreover, integrating over $[0, t]$ and using $f \in L^\infty([0, T], H)$ and $u_N \in L^\infty([0, T], H)$ leads to,
\[
\|u_N(t)\|^2 + \frac{\nu}{2}\int_0^t |Au_N(s)|^2 \leq C
\]
So we have,
\[
\|u_N(t)\|^2 \leq C \forall t \in [0, T].
\]
So, for $N$ large enough, (such that $N > C$) ; $\|u_N(t)\| \leq C$ on $t \in [0, T]$. Hence, we will get $B_N(u) = B(u)$ for all $t \in (0, T)$ and $u_N = u$ is the solution to (22).

This proves the theorem. $\square$
In theorem 4.1 we require that \((I + \lambda A)^{-1}K \subset K \forall \lambda > 0\). This is a very strong assumption and may not be satisfied in practical problems. We want to relax this condition. If we do not assume that \(K\) is invariant under \((I + \lambda A)^{-1}\), we still get a result but of weaker form. We assume \(K \subset V\) and show that the feedback controller \(g\) belongs to \(L^2([0,T],V')\), can be found which will ensure that trajectory does not leave \(K\).

**Theorem 4.2.** Let, \(K\) be a closed convex subset of \(V\) such that \(0 \in K\). Let \(u^0 \in D(A) \cap K\) and \(f \in W^{1,2}([0,T],H)\). Then there exists a feedback controller \(g \in L^2([0,T],V')\) and \(g(t) \in -N^*_K(u(t))\ a.e. \ t \in [0,T]\), such that the corresponding solution to the closed loop system (22) i.e.

\[
\frac{du}{dt} + \nu Au + B(u) = f + g, \quad u(0) = u^0
\]

satisfies \(u \in W^{1,\infty}([0,T],H) \cap W^{1,2}([0,T],V)\). Moreover,

\[
\frac{d^+u}{dt} + (-f + \nu Au + B(u) + N^*_K(u))^0 = 0, \quad u(0) = u^0 
\]

and \(u(t) \in K \ \forall t \in [0,T]\).

For each \(t\), \(N^*_K(u(t))\) is defined by \(N^*_K(u(t)) := \{w \in V'; (w,u-z) \geq 0, \forall z \in K\}\), is the \(V'\) valued normal cone to \(K\) at \(u(t)\). Similar to the theorem 4.1, \(u(t) \rightarrow (-f(t) + \nu Au(t) + B(u(t)) + N^*_K(u(t)))\) is a multivalued map and we define for each \(u(t), (-f(t) + \nu Au(t) + B(u(t)) + N^*_K(u(t)))^0\) as the projection of origin onto the closed convex set \((-f(t) + \nu Au(t) + B(u(t)) + N^*_K(u(t)))\).

By (35) we get the feedback controller \(g\) as,

\[
g(t) = -f(t) + \nu Au(t) + B(u(t)) + N^*_K(u(t)) \\
- (-f(t) + \nu Au(t) + B(u(t)) + N^*_K(u(t)))^0 \ \forall t \in [0,T].
\]

**Proof.** We have to show that \(u \rightarrow (-f + \nu Au + B_N(u) + N^*_K(u))\) is quasi m-accretive where \(B_N\) is defined as earlier. By [3][Chapter II, Theorem 2.1] we know \(u \rightarrow N^*_K(u)\) is maximal monotone in \(V \times V'\). Hence \(u \rightarrow (-f + \nu Au + B_N(u) + N^*_K(u) + \alpha_N I)\) is accretive for some \(\alpha_N > 0\). We can prove it as in the proof of Theorem 4.1. Therefore this map is m-accretive by definition 2. Also we can show that \(u \rightarrow (-f + \nu Au + B_N(u) + N^*_K(u) + \alpha_N I)\) is coercive. So by [7][Chapter II, Example 2.3.7] its restriction to \(H\) is maximal monotone in \(H \times H\). This gives, \(u \rightarrow (-f + \nu Au + B_N(u) + N^*_K(u))\) is quasi m-accretive by definition 2. So by [2][Chapter 1, Theorem 1.15, Theorem 1.16] we get,

\[
\frac{du}{dt} + \nu Au + B_N(u) + N^*_K(u) = f, \quad u(0) = u^0 \ a.e. \ t \in [0,T]
\]

has a unique solution \(u_N \in L^\infty([0,T],H) \cap L^2([0,T],V)\). Moreover \(u_N\) satisfies

\[
\frac{d^+u_N}{dt} + (\nu Au_N + B_N(u_N) + N^*_K(u_N) - f)^0 = 0, \ \forall t \in [0,T].
\]

Taking inner product of

\[
\frac{du_N(t)}{dt} + \nu Au_N(t) + B_N(u_N(t)) + N^*_K(u_N(t)) = f(t), \quad u_N(0) = u^0.
\]
with \( u_N \) we get,
\[
\frac{d}{dt} |u_N(t)|^2 + \nu \|u_N(t)\|^2 + (B_N(u_N(t)), u_N(t)) + (N_K^*(u_N(t)), u_N(t)) = (f, u_N(t))\]
\[
\frac{d}{dt} |u_N(t)|^2 + \nu \|u_N(t)\|^2 \leq (f(t), u_N(t))\]
\[
\frac{d}{dt} |u_N(t)|^2 + \nu \|u_N(t)\|^2 \leq \frac{1}{2} |f(t)|^2 + \frac{1}{2} |u_N(t)|^2.
\]
Using Gronwall's lemma and \( f \in W^{1,2}([0, T], H) \) and by integrating we get,
\[
|u_N(t)|^2 + \nu \int_0^t \|u_N\|^2 \leq C. \tag{38}
\]
Now, we formally differentiate (37) with respect to \( t \) to get,
\[
\frac{d}{dt}u_N(t) + \nu(Au_N(t))' + (B_N(u_N(t)))' + (N_K^*(u_N(t)))' = f'(t). \tag{39}
\]
Taking inner product of (39) with \( u_N' \) we get,
\[
\left(\frac{d}{dt}u_N(t), u_N'(t)\right) + \nu((A(u_N(t)))', u_N'(t)) + ((B_N(u_N(t)))', u_N'(t))
\]
\[
+((N_K^*(u_N(t)))', u_N'(t)) = (f'(t), u_N'(t)).
\]
Since, \( A \) is a linear operator \((Au_N(t))' = A(u_N'(t)). \) So we get,
\[
\frac{1}{2} \frac{d}{dt} |u_N(t)|^2 + \nu \|u_N'(t)\|^2 + ((B_N(u_N(t)))', u_N'(t)) = (f'(t), u_N'(t))
\]
\[
-((N_K^*(u_N(t)))', u_N'(t)).
\]
Now we will estimate \(((B_N(u_N(t)))', u_N'(t))\) and \(((N_K^*(u_N(t)))', u_N'(t)).\)
For, \(((N_K^*(u_N(t)))', u_N'(t))\) can be written as
\[
\lim_{h \to 0} \left( \frac{1}{h} (N_K^*(u_N(t+h)) - N_K^*(u_N(t)), \frac{1}{h} (u_N(t+h) - u_N(t)) \right)
\]
which can be written as
\[
((N_K^*(u_N(t)))', u_N'(t)) = \frac{1}{h^2} (N_K^*(u_N(t+h)), u_N(t+h) - u_N(t))
\]
\[
+ \frac{1}{h^2} (N_K^*(u_N(t)), u_N(t) - u_N(t+h))
\]
Since, both the term on the right hand side are non-negative we get,
\[
((N_K^*(u_N(t)))', u_N'(t)) \geq 0.
\]
Therefore we can write an inequality,
\[
\frac{1}{2} \frac{d}{dt} |u_N'(t)|^2 + \nu \|u_N'(t)\|^2 + ((B_N(u_N(t)))', u_N'(t)) \leq (f'(t), u_N'(t)). \tag{40}
\]
Now, for \( \|u_N\| \leq N \) we get,
\[
((B_N(u_N(t)))', u_N'(t)) = \frac{1}{h^2} (B_N(u_N(t+h)) - B_N(t), u_N(t+h) - u_N(t))
\]
\[
= \frac{1}{h^2} \left[ b(u_N(t+h), u_N(t+h) - u_N(t), u_N(t+h) - u_N(t))
\]
\[
- b(u_N(t+h) - u_N(t), u_N(t), u_N(t+h) - u_N(t)) \right].
\]
As the limit \( h \to 0 \) we get,
\[
((B_N(u_N(t)))', u_N'(t)) = b(u_N(t), u_N'(t), u_N'(t)) + b(u_N'(t), u_N(t), u_N'(t)).
\]
Using Theorem 2.1 we get,
\[
|((B_N(u_N(t)))', u_N'(t))| \leq |b(u_N'(t), u_N(t), u_N'(t))|
\leq C|u_N||u_N'||u_N'|
\leq C_\nu|u_N|^2|u_N'|^2 + \frac{\nu'}{2}|u_N'|^2.
\]

And for \(\|u_N\| > N\) we get,
\[
\begin{align*}
((B_N(u_N(t)))', u_N'(t)) &= \frac{1}{h^2}(B_N(u_N(t + h)) - B_N(u_N(t)), u_N(t + h) - u_N(t)) \\
&= \frac{1}{h^2} \left( \frac{N^2}{\|u(t + h)\|^2} B(u_N(t + h)) - \frac{N^2}{\|u(t)\|^2} B(u_N(t)), u_N(t + h) - u_N(t) \right) \\
&= \frac{1}{h^2} \left( \frac{N^2}{\|u(t + h)\|^2} B(u_N(t + h)) - \frac{N^2}{\|u(t)\|^2} B(u_N(t)), u_N(t + h) - u_N(t) \right) \\
&\quad + \frac{1}{h^2} \left( \frac{N^2}{\|u(t + h)\|^2} B(u_N(t + h)) - \frac{N^2}{\|u(t)\|^2} B(u_N(t)), u_N(t + h) - u_N(t) \right) \\
&= \frac{1}{h} \left( B(u_N(t + h)) - B(u_N(t)) \right) \left( \frac{2}{\|u(t)\|^2} b(u_N(t), u_N'(t)) - \frac{2}{\|u(t)\|^2} b(u_N(t), u_N'(t)) \right) \\
&\quad + \frac{2}{\|u(t)\|^2} b(u_N(t), u_N'(t)) + \frac{2}{\|u(t)\|^2} b(u_N(t), u_N'(t)), u_N(t).
\end{align*}
\]

Now, as \(h \to 0\) we get,
\[
\begin{align*}
((B_N(u_N(t)))', u_N'(t)) &= \left( B(u_N(t)) \frac{2N^2\|u_N'(t)\|}{\|u_N(t)\|^3}, u_N'(t) \right) \\
&\quad + \frac{N^2}{\|u(t)\|^2} (B(u_N(t)), u_N(t)) \\
&= \frac{2N^2\|u_N'(t)\|}{\|u_N(t)\|^3} b(u_N(t), u_N'(t)) + \frac{N^2}{\|u(t)\|^2} b(u_N(t), u_N'(t), u_N(t)).
\end{align*}
\]

Since, \(\|u_N(t)\| \neq 0\) we get,
\[
|((B_N(u_N(t)))', u_N'(t))| \leq \frac{2N^2\|u_N'(t)\|}{\|u_N(t)\|^3} |b(u_N(t), u_N(t), u_N'(t))|
\leq \frac{2\|u_N'(t)\|}{\|u_N(t)\|^3} |b(u_N(t), u_N'(t), u_N(t))| + |b(u_N'(t), u_N(t), u_N'(t))|
\leq C\frac{2\|u_N'(t)\|}{\|u_N(t)\|^3} |u_N(t)||u_N'(t)||u_N'(t)| + C\|u_N'(t)||u_N(t)||u_N'(t)|
\leq 2C\|u_N'(t)||u_N(t)||u_N'(t)| + C\|u_N'(t)||u_N(t)||u_N'(t)|.
\]

Using Young’s inequality we get,
\[
|((B_N(u_N(t)))', u_N'(t))| \leq \frac{\nu'}{2}|u_N'(t)|^2 + C_\nu|u_N(t)|^2|u_N'(t)|^2
\]
So, from (40), using above estimates we get,
\[
\frac{d}{dt}|u_N'(t)|^2 + \frac{\nu'}{2}|u_N'(t)|^2 \leq C_\nu|u_N(t)|^2|u_N'(t)|^2 + \frac{1}{2}|f'|^2 + \frac{1}{2}|u_N'(t)|^2.
\]
\[ \frac{d}{dt}|u_N'(t)|^2 + \frac{\nu}{2} \|u_N'(t)\|^2 \leq \left( \frac{1}{2} + C_\nu |u_N(t)|^2 \right) |u_N'(t)|^2 + \frac{1}{2} |f'|^2. \] (41)

Therefore using Gronwall’s inequality we get,

\[ |u_N'(t)|^2 \leq \exp \left\{ \int_0^t \left( \frac{1}{2} + C_\nu |u_N(s)|^2 \right) ds \right\} \left( \frac{1}{2} \int_0^t |f'(s)|^2 \right). \]

And integrating (41) and using \( u_N \in L^\infty([0,T],H) \), \( f \in W^{1,2}([0,T],H) \) and the above estimate we get,

\[ |u_N'(t)|^2 + \frac{\nu}{2} \int_0^t \|u_N'(s)\|^2 ds \leq C, \quad \forall t \in [0,T], \text{ and for all } N. \]

Hence, we conclude that \( \|u_N(t)\| \leq C \quad \forall t \in [0,T]. \)

So, for \( N \) large enough, (such that \( N > C \)), \( \|u_N(t)\| \leq C \) for all \( t \in [0,T] \). Hence, we will get \( B_N(u) = B(u) \quad \forall t \in [0,T] \) and \( u_N = u \) is the solution to (22). This completes the proof. \( \square \)

The next theorem is the most general one where we wish to find a control that has support only in the small subset of the state space. From the view point of application, it would be most interesting if we put the condition that the control be of finite dimension. For the sabra shell model under consideration, \( H \) being the \( \ell^2 \) space, this condition would mean that we are projecting \( u \) on a finite subset of \( \mathbb{N} \) which in turn would mean that we are considering only finitely many modes of \( u \). In the next theorem, we show that we can find a controller with finitely many modes such that corresponding solution to the closed loop system remains close to \( K \), however it may not be in the \( K \) for all \( t \). The control is written in terms of the projection operator on the set \( K \).

**Theorem 4.3.** Let, \( K \) be a closed convex subset of \( H \) and \( 0 \in K \). Let \( P_K : H \to K \) be the projection operator on \( K \) and \( m \) is the characteristic function of a finite set \( \omega \subset \mathbb{N} \) such that \( P_K(mu) = mP_K(u) \quad \forall u \in H \). Also assume \( u^0 \in D(A) \) such that \( mu^0 \in K \) and \( f \in W^{1,2}([0,T],H) \). Then, for each \( \lambda > 0 \) there exists a feedback controller \( g_\lambda = -\frac{1}{\lambda} (mu_\lambda - mP_K(u_\lambda)) \) such that the solution \( u_\lambda \) of the closed loop system

\[ \frac{du(t)}{dt} + \nu Au(t) + B(u(t)) = f(t) + g(t), \quad u(0) = u^0 \] (42)

satisfies, \( u_\lambda \in W^{1,\infty}([0,T],H) \cap L^\infty([0,T],D(A)) \). Moreover there exist a \( C > 0 \) such that

\[ \frac{1}{\lambda} \int_0^T dK_i(mu_\lambda(t)) dt \leq C \quad \forall \lambda > 0. \] (43)

**Proof.** Let the operator \( \chi_N : D(\chi_N) \to H \) be defined by,

\[ \chi_N u = \nu Au + B_N(u) - \frac{1}{\lambda} (mu - P_K(mu)). \]

Clearly, \( D(\chi_N) = D(A) \).

Define the operator \( F u = -\frac{1}{\lambda} (mu - P_K(mu)) \). We can show that \( F \) is non-expansive on \( H \) i.e.

\[ |F u - F v| \leq |u - v|. \]
Then by Proposition 1.4 of [2] we get, $\chi_N + \alpha_N I$ where $\alpha_N > 0$ is m-accretive on $H$. The proof will be similar as in 4.1. Therefore we get, $u \rightarrow \chi_N u$ is a quasi m-accretive by definition 2. So by [2][Chapter 1, Proposition 1.8] the Cauchy problem,

$$\frac{d u(t)}{dt} + \chi_N u(t) = f(t), \quad u(0) = u^0$$

has a unique solution $u^N_\lambda \in W^{1,\infty}([0, T], H) \cap L^\infty([0, T], D(A)) \cap C([0, T], V)$.

Taking inner product of

$$(44)$$

with $u^N_\lambda$ we get,

$$\frac{d u^N_\lambda(t)}{dt} + \chi_N u^N_\lambda(t) = f(t), \quad u^N_\lambda(0) = u^0$$

with $u^N_\lambda$ we get,

$$\frac{d}{dt} |u^N_\lambda|^2 + \nu(Au^N_\lambda, u^N_\lambda) = (f, u^N_\lambda).$$

i.e. $\frac{d}{dt} |u^N_\lambda|^2 + \nu|Au^N_\lambda|^2 \leq C_{\nu}|u^N_\lambda|^2 + \nu \frac{3}{2} |Au^N_\lambda|^2 + \nu \frac{3}{6} |u^N_\lambda|^2$

$$\frac{d}{dt} |u^N_\lambda|^2 + \nu \frac{3}{2} |Au^N_\lambda|^2 \leq C_{\nu}|u^N_\lambda|^2 + \nu \frac{3}{2} |Au^N_\lambda|^2 + \frac{3}{2} |f|^2. \quad (45)$$

Similarly taking inner product of $(44)$ with $Au^N_\lambda$ and using 5. of Theorem 2.1 we get,

$$\frac{d}{dt} |u^N_\lambda|^2 + \nu|Au^N_\lambda|^2 \leq C_{\nu}|u^N_\lambda|^2 + \nu \frac{3}{2} |Au^N_\lambda|^2 + \nu \frac{3}{6} |u^N_\lambda|^2$$

$$\frac{d}{dt} |u^N_\lambda|^2 + \nu \frac{3}{2} |Au^N_\lambda|^2 \leq C_{\nu}|u^N_\lambda|^2 + \nu \frac{3}{2} |Au^N_\lambda|^2 + \frac{3}{2} |f|^2. \quad (46)$$

Using Gronwall’s lemma and using $f \in W^{1,2}([0, T], H)$, then integrating $(45)$ gives,

$$|u^N_\lambda|^2 + \nu \frac{3}{2} \int_0^t \|u^N_\lambda\|^2 \leq C \forall t \in [0, T].$$

Similarly integrating $(46)$ we get,

$$\|u^N_\lambda(t)\|^2 \leq \|u^0\|^2 + C_{\nu} \int_0^t |u^N_\lambda|^2 \|u^N_\lambda\|^2 + \int_0^t |Fu^N_\lambda(s)|^2$$

$$+ \frac{3}{2} \nu \int_0^t |f(s)|^2$$

Since $f \in W^{1,2}([0, T], H)$, $u^N_\lambda \in L^\infty([0, T], D(A))$ and $Fu^N_\lambda$’s are bounded operators, it follows

$$\|u^N_\lambda(t)\|^2 \leq C_{\nu} \int_0^t |u^N_\lambda|^2 \|u^N_\lambda\|^2 + \frac{C}{\lambda^2} + C,$$

for all $N \in \mathbb{N}$, $\lambda > 0$ and $t \in [0, T]$.

Using Gronwall’s inequality we get,

$$\|u^N_\lambda(t)\|^2 \leq C \left(1 + \frac{1}{\lambda^2} \right) \exp \left\{ \int_0^t |u^N_\lambda|^2 \right\}$$
Since, $u^N_\lambda \in L^\infty([0,T],H),\ 
\|u^N_\lambda(t)\|^2 \leq C \left(1 + \frac{1}{\lambda}\right)^2.
$

Therefore, for all $t \in [0,T]$ and $\lambda > 0$ we get,
$$\|u^N_\lambda(t)\| \leq C \left(1 + \frac{1}{\lambda}\right)$$

So, for $N > C \left(1 + \frac{1}{\lambda}\right)$ we get,
$$u^N_\lambda = u_\lambda, \text{ which is the solution of}$$
$$\frac{du_\lambda(t)}{dt} + \nu A u_\lambda(t) + B(u_\lambda(t)) + F(u_\lambda(t)) = f(t), \quad u(0) = u^0, \forall t \in [0,T].$$

This proves (42). We are left to show (43). For we show that,
$$(m u_\lambda - m P_K(u_\lambda), m u_\lambda - m P_K(u_\lambda)) \geq d^2_K(m u_\lambda),$$

which is same as to show that
$$(m u_\lambda, m u_\lambda - m P_K(u_\lambda)) - (m P_K(u_\lambda), m u_\lambda - m P_K(u_\lambda)) \geq d^2_K(m u_\lambda).$$

Clearly,
$$F u_\lambda = \frac{1}{\lambda} (m u_\lambda - m P_K(u_\lambda)), \quad (m u_\lambda, \lambda F u_\lambda) \geq d^2_K(m u_\lambda)$$

i.e
$$(m u_\lambda, F u_\lambda) \geq \frac{1}{\lambda} d^2_K(m u_\lambda).$$

Using Cauchy-Schwartz and Young’s inequality we get,
$$\frac{1}{\lambda} d^2_K(m u_\lambda) \leq \frac{|F u_\lambda|^2}{2} + \frac{|u_\lambda|^2}{2}.$$

Integrating over $[0,T]$ gives,
$$\frac{1}{\lambda} \int_0^T d^2_K(m u_\lambda) \leq \int_0^T \frac{|F u_\lambda|^2}{2} + \int_0^T \frac{|u_\lambda|^2}{2}.$$

Since, $u_\lambda \in L^\infty([0,T],H), \forall \lambda > 0$ there exists $C > 0$ such that
$$\frac{1}{\lambda} \int_0^T d^2_K(m u_\lambda) \leq C.$$

This proves the theorem.

Now we illustrate with two examples how we can find flow preserving controllers using theorems proved above. The quantities like enstrophy and helicity are conserved for fluid flow equations. In [13], it has been shown that shell model of turbulence also preserves these quantities.

The first example is concerned with enstrophy of a system. We choose convex set $K$ to be an enstrophy ball, and show that we can find a control which would preserve the flow if the initial value is assumed to be in $K$. We are going to use the Theorem 4.1 and prove it. In the second example, we consider $K$ to be a subset of $H$ for which helicity is bounded by some bound say $\rho$ and we apply the Theorem 4.3.
Example 4.1. Enstrophy Invariance: Consider the set $K = \{u \in V; |u| = |A^{1/2}u| \leq \rho\}$. We want to find $N_K(u)$ such that if the control belongs to $N_K(u)$, the solution will be in $K$.

Clearly, $K$ is closed, convex subset of $H$. We show that using Theorem 4.1 we can find a feedback controller so that the solution of (7) does not leave $K$. First we show that $(I + \lambda A)^{-1}K \subset K$.

For any $f \in K$ we have to show there exist $y \in K$ such that

\[(I + \lambda A)y = f.\]  \hspace{1cm} (47)

Taking inner product of (47) with $Ay$ we get,

\[(y, Ay) + \lambda(Ay, Ay) = (f, Ay).\]

Using Young’s inequality we get,

\[|A^{1/2}y|^2 + \lambda|Ay|^2 \leq \frac{1}{2}|A^{1/2}f|^2 + \frac{1}{2}|A^{1/2}y|^2.\]

Therefore we get,

\[|A^{1/2}y| \leq |A^{1/2}f| \leq \rho.\]

Thus the solution $y$ to (47) belongs to $K$. Therefore $(I + \lambda A)^{-1}K \subset K$. We have

\[N_K(u) := \left\{ w \in H; \begin{cases} 0 & \text{if } |A^{1/2}u| < \rho \\ \cup_{\lambda>0} \lambda Au & \text{if } |A^{1/2}u| = \rho \end{cases} \right\}

Using the theorem 4.1 we get the feedback controller

\[g(t) := \begin{cases} 0 & \text{if } |A^{1/2}u| < \rho \\ Z(u) & \text{if } |A^{1/2}u| = \rho \end{cases}\]

where $Z(u) = -\frac{A u}{|A u|^2} (f(t) - \nu Au(t) - B(u(t)), Au(t))$ such that the corresponding solution $u(t) \in K \forall t \in [0, T]$.

Example 4.2. Helicity Invariance: Let’s consider $K = \{y \in D(A^{1/4}) : \sum n|y_n|^2 = |A^{1/4}y|^2 \leq \rho^2\}$.

Clearly, $K$ is closed convex subset of $H$. We can show that $K$ is not invariant under $(I + \lambda A)^{-1}$. So we can not use the Theorem 4.1. We are going to apply Theorem 4.3.

Let $L : V_{1/2} \rightarrow H$ be the operator defined by $L(y) = A^{1/4}y$. Then the functional $N(y) = |L(y)|^2$ is continuous on $V_{1/2}$.

So using Theorem 4.3 we get a sequence of feedback controller $g_\lambda \in L^2([0, T], H)$ such that $g_\lambda = \frac{1}{4}(u_\lambda - P_K(u_\lambda))$ and

\[\lim_{\lambda \rightarrow 0} \int_0^T d_K^2(u_\lambda(t))dt = 0.\]

Now to write $g_\lambda$, we need to find projection of $u_\lambda$ on $K$. Let us denote $P_K(u) = z$ for any $u \in V_{1/2}$. We will show that $z$ solves

\[z + 4\lambda L(z)A^{1/2}z = u, \quad z(0) = z^0\]  \hspace{1cm} (48)

where $z^0$ is the projection of $u^0$ on $K$. To prove that $z$ solves (48) we use Lagrange multiplier method. Define the minimization problem,

\[\text{minimize } d(x, u) \quad \text{subject to } N(x) \leq \rho^2.\]
Let the solution of the minimization problem be $z$. Then $z$ will satisfy the following Lagrange multiplier equation.

$$z - u + \lambda N'(z) = 0.$$  

(49)

Now to find $N'(z)$, it is enough to find $L'(z)$ as $N'(z) = 2L(z)L'(z)$.

$$L'(z)h = \lim_{\lambda \to 0} \frac{L(z + \lambda h) - L(z)}{\lambda}$$

$$= \lim_{\lambda \to 0} \frac{\sum k_n(z_n + \lambda h_n)^2 - \sum k_n z_n^2}{\lambda}$$

$$= \lim_{\lambda \to 0} \frac{\sum k_n 2\lambda z_n h_n + \sum k_n \lambda^2 h_n^2}{\lambda}$$

$$= 2 \sum k_n z_n h_n$$

$$= 2(A^{1/4}z, A^{1/4}h)$$

Since, $D(A^{1/2})$ is dense in $D(A^{1/4})$ we can extend the definition of $A^{1/2} : V_{1/2} \to V_{-1/2}$ in such a way that

$$\langle A^{1/2}u, v \rangle = \langle A^{1/4}u, A^{1/4}v \rangle.$$ 

So we get,

$$L'(z) = A^{1/2}z.$$ 

Therefore, by (49) $z$ solves (48) i.e.

$$z + 4\lambda L(z)A^{1/2}z = u.$$ 

5. Conclusion. In this work we have studied two optimal control problems for equations of sabra shell model of turbulence with the control acting as a forcing term. We have studied two cost functionals, one aims to reduce turbulence in the flow and the other one is to find an optimal control which can take the flow to the desired state. In both cases, with the help of the adjoint equation we have shown that if the optimal pair which minimizes the cost functional exists then the optimal control can be characterized by using the solution of appropriate the adjoint equation. However, converse of our theorem does not hold true. That is the control designed by above method need not give the optimal solution for the minimization of cost functional. This is expected because the adjoint equation is written for the linearized equations of sabra shell model. Thus for nonlinear equation the control which is designed via solution of the linearized adjoint equation would not give the optimal solution.

In the second part of our work we have looked at another control problem of finding feedback controller which would preserve certain quantities in the flow. This is mainly useful because the shell models have natural invariants like enstrophy and helicity. We have proved three different theorems about constructing feedback controllers. The first theorem is proved under strong assumption that the constrained set $K$ is invariant under $(I + \lambda A)^{-1}$. In this case we get the control which will be in $H$-valued normal cone to $K$. In the second theorem the assumption is relaxed, but as a result, we get the control in the $V$ valued normal cone to $K$ which is a weaker space than before. In the third theorem, we have proved that we can always find a sequence of controls such that the sequence of corresponding solutions will remain close to $K$. At the end, we have discussed two example where these theorems have been applied.
As noted earlier control problems for shell models of turbulence are not well studied in literature in spite of potential applications. We further plan to study certain controllability problems related to sabra shell model of turbulence. Internal stabilization and H infinity control problem for the shell model also seem promising avenues to explore.

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