EXPLICIT WODZICKI EXCISION IN CYCLIC HOMOLOGY

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Abstract. Assuming local one-sided units exist, I give an elementary proof of Wodzicki excision for cyclic homology. The proof is also constructive and provides an explicit inverse excision map. As far as I know, the latter is new.

We work over a field of characteristic zero. For every algebra extension $I \hookrightarrow A \twoheadrightarrow A/I$, where $I$ is a two-sided ideal, there is a long exact sequence in cyclic homology

$$\cdots \to HC_n(A,I) \to HC_n(A) \to HC_{n-1}(A,I) \to \cdots,$$

where $HC_n(A,I)$ is relative cyclic homology. In general this group really depends on $A$, but in favourable situations it agrees with the group $HC_n(I)$. M. Wodzicki proved that this happens iff the bar complex of $I$ is acyclic [Wod88, Thm. 3], [Wod89, §3]. $I$ is then called $H$-unital. In this case the straightforward excision map, i.e.

$$\rho : HC_n(I) \longrightarrow HC_n(A,I),$$

is an isomorphism. In general it does not seem realistic to hope for a closed formula for its inverse, already by the abstract nature of $H$-unitality. Chances should get better if the bar complex comes with an explicit contracting homotopy. A prominent such case is the following: An algebra $I$ has local left units if for every finite set $S \subseteq I$ there exists an element $e$ such that $\forall s \in S : es = s$. Wodzicki shows that such $I$ are $H$-unital [Wod88, Prop. 2], [Wod89, Cor. 4.5].

In the present paper we want to give an elementary proof for this special case of Wodzicki’s theorem:

**Theorem 1.** Suppose $I \hookrightarrow A \twoheadrightarrow A/I$ is an algebra extension such that $I$ and $A$ have local left (or right) units. Then the excision map $\rho : HC_n(I) \to HC_n(A,I)$ is an isomorphism.

The proof circumvents the use of spectral sequences, but on the downside is of course not as general as the original result. In fact, the proof is constructive and leads to an explicit inverse map $\rho^{-1}$.

**Theorem 2.** We keep the assumptions as in Thm. (1). Then for every finite-dimensional subspace of $V \subseteq HC_n(A,I)$ there exists a finite-dimensional subspace $V' \subseteq I \otimes_A A^\otimes n$ and (non-canonical) elements $e_1, \ldots, e_n \in I$ allowing to define a map...
V′ → I⊗n+1 sending f_0 ⊗ f_1 ⊗ ⋯ ⊗ f_n to

\[(0.3) \sum_{s_1, \ldots, s_n \in \{\pm\}} (-1)^{s_1 + \cdots + s_n} e_1 ⊗ f_1 e_2 ⊗ f_2 \cdots e_n ⊗ f_n \]

(whore for each underbrace we take the upper term if s_i = +, the lower if s_i = −) which in homology induces the inverse map ρ⁻¹ : HC_n(A, I) ⊆ V → HC_n(I).

As V can be picked large enough to contain any finite set of elements, this describes ρ⁻¹ entirely. In low-degree cases the formula unwinds as

\[f_0 ⊗ f_1 \mapsto e_1 ⊗ f_1 f_0 − f_1 e_1 ⊗ f_0\]

for n = 1; and for n = 2 one gets

\[f_0 ⊗ f_1 ⊗ f_2 \mapsto e_1 ⊗ f_1 e_2 ⊗ f_2 f_0 − f_1 e_1 ⊗ e_2 ⊗ f_2 f_0\]
\[- e_1 ⊗ f_1 f_2 e_2 ⊗ f_0 + f_1 e_1 ⊗ f_2 e_2 ⊗ f_0.\]

We finish by spelling out the case n = 3, which is already fairly involved: f_0 ⊗ f_1 ⊗ f_2 ⊗ f_3 maps to

\[e_1 ⊗ f_1 e_2 ⊗ f_2 e_3 ⊗ f_3 f_0 − e_1 ⊗ f_1 e_2 ⊗ f_2 f_3 e_3 ⊗ f_0\]
\[- e_1 ⊗ f_1 f_2 e_2 ⊗ e_3 ⊗ f_3 f_0 + e_1 ⊗ f_1 f_2 e_2 ⊗ f_3 e_3 ⊗ f_0\]
\[- f_1 e_1 ⊗ e_2 ⊗ f_2 e_3 ⊗ f_3 f_0 + f_1 e_1 ⊗ e_2 ⊗ f_2 f_3 e_3 ⊗ f_0\]
\[+ f_1 e_1 ⊗ f_2 e_2 ⊗ e_3 ⊗ f_3 f_0 − f_1 e_1 ⊗ f_2 e_2 ⊗ f_3 e_3 ⊗ f_0.\]

The unpleasant restriction to finite-dimensional subspaces in the theorem is of technical nature. As we enlarge such a space, e.g. by taking the union of two such subspaces, it need not be possible to choose the elements e_i so that the maps as in eq. (0.3) prolong compatibly. Only after going to homology, they all describe the same map ρ⁻¹.

This artifact comes from the fact that we can always pick local units for sets S, but have no control how they behave as S enlarges.

**Theorem 3.** We keep the assumptions as in Thm. Then every class [φ] ∈ HC_n(A, I) is represented by a cycle

\[φ = \sum \lambda_j φ_j \quad \text{with} \quad φ_j = f_0 ⊗ f_1 ⊗ ⋯ ⊗ f_n \in I ⊗ A⊗n\]

i.e. a linear combination of pure tensors φ_j with initial slot in I. Take φ_1, ..., φ^d any elements whose classes are spanning the finite-dimensional space V in Thm. and write φ^α_j ∈ I ⊗ A⊗n for the respective pure tensor components. Now pick V′ := span{φ^α_j}_{α, j},

- e_n ∈ I as a local left unit for \(\bigcup_{\{\alpha\}} \{f_0\}\), where the union runs over the f_0-slots of all φ^α_j; and inductively going downward for each 2 \leq i \leq n:
- e_{i−1} ∈ I as a local left unit for \(\{e_i\} ∪ \bigcup_{\{\alpha\}} \{f_i e_i\}\), where the union runs over the f_i-slots of all φ^α_j.

This provides a concrete choice of V′ and the e_i in the statement of Thm.

We also obtain some (weaker) results for Hochschild homology, see Prop.
1. Preparations

Let \( k \) be a field of characteristic zero. In this text, the word \textit{algebra} refers to an associative \( k \)-algebra which need not be commutative and especially not unital. Even if units exist, algebra morphisms are not required to preserve them. Tensor products are always over \( k \).

Let \( A \) be an algebra. Write \( C_1(A) := A^{\otimes i+1} \) for the underlying groups of Hochschild homology; equip them with the usual differential

\[
b(f_0 \otimes \cdots \otimes f_n) := \sum_{i=0}^{n-1} (-1)^i f_0 \otimes \cdots \otimes f_i f_{i+1} \otimes \cdots \otimes f_n + (-1)^n f_n f_0 \otimes f_1 \otimes \cdots \otimes f_n
\]

so that \( HH_i(A) = H_i(\{C_\bullet(A), \mathbf{b}\}) \) is the (naive) Hochschild homology of \( A \). We shall also need the cyclic permutation operator

\[
t(f_0 \otimes \cdots \otimes f_n) := (-1)^n f_n \otimes f_0 \otimes f_1 \otimes \cdots \otimes f_{n-1}.
\]

Define \( CC_i(A) := C_n(A)_{(t)}, \) the co-invariants under \( t \), so that (naive) cyclic homology is given by \( HC_i(A) = H_i(\{CC_\bullet(A), \mathbf{b}\}). \)

\textit{Remark} (this definition suffices). This is the correct definition of Hochschild homology only if \( A \) is unital; in general one defines ‘correct Hochschild homology’ \( HH^\text{cor}_i(A) \) as the homology of the two-row bicomplex

\[
C^\text{cor}_i(A) := [C_i(A) \xrightarrow{1-t} B_{i+1}(A)],
\]

where \( B_\bullet \) is the bar complex and \( t \) the cyclic permutation operator; all details can be found in \cite{Woodhouse} §2, esp. p. 598 l. 5]. Equivalently, one can define \( HH_i^\text{cor}(A) := \ker(\text{coker}(C_i(k) \rightarrow C_i(A_+))) \), where \( A_+ \) denotes the unitalization of \( A \) and the map is induced by \( 1_k \rightarrow 1_{A_+} \) to the formal unit of \( A_+ \) \cite{Woodhouse} paragr. before Thm. 3.1 (this is the definition used in \cite{Woodhouse} and \cite{Lowen} §1.4.1). However, in this text we will only ever make claims about algebras with one-sided local units. Their bar complexes \( B_\bullet \) are acyclic by \cite{Woodhouse} Cor. 4.5] so that the obvious map \( C_\bullet(A) \rightarrow C^\text{cor}_\bullet(A) \) is a quasi-isomorphism. As a result, for the present text it is sufficient to take the naive complex \( C_\bullet(A) \) as the definition, may \( A \) be unital or not – this is also the favourable choice when intending to perform concrete computations.

Given an algebra extension \( I \hookrightarrow A \rightarrow S \), define relative groups \( C_i(A, I) := \ker(C_i(A) \rightarrow C_i(A/I)) \) and denote their homology by \( HH_i(A, I) \), this is relative Hochschild homology. Similarly \( CC_i(A, I) := \ker(CC_i(A) \rightarrow CC_i(A/I)) \) whose homology is relative cyclic homology, denoted by \( HC_i(A, I) \). Then the sequence in eq. 0.1 is exact, trivially by construction. The obvious \textit{excision map}

\[
\rho : CC_i(I) \rightarrow CC_i(A, I)
\]

sending a tensor to itself is clearly well-defined. It induces the homological excision map of eq. 0.2 so this direction of the map in Thm. 2 is easy to describe explicitly. Providing an explicit inverse is less immediate.

2. The proof

2.1. The Guccione-Guccione filtration. Henceforth, we shall assume that \( I \) has local left units. The case of local right units would be entirely analogous. We shall also assume that \( A \) has local left units; this less natural assumption solely serves
the purpose to have the simple description of cyclic homology as in \(\Phi\) available (cf. Rmk. in \([1]\)). Define vector subspaces

\[
F_pC_n(A) = \left\{ \text{k-linear subspace spanned by } f_0 \otimes \cdots \otimes f_n \text{ with } f_0, \ldots, f_{n-p} \in I \right\}
\]

This is a filtration \(F_0C_n(A) \subseteq F_1C_n(A) \subseteq \ldots, F_{[n]+1}C_n(A) = C_n(A)\) so that \(\bigcup_{p\geq 0} F_pC_n(A) = C_n(A)\). This also induces a filtration \(F_pC_n(A, I)\). We have \(F_0C_n(A) = C_n(I)\).

(this filtration is of course inspired by the filtration in the original proof \([\text{Wod}88]\)), but by prescribing the position of the slots with values in \(I\) the computations simplify. This idea originates from \([\text{GG}96]\).

Split \(A\) as a \(k\)-vector space as \(\nu : A \simeq I \oplus (A/I)\). Pick bases \(B_I\) of \(I, B_{A/I}\) of \(A/I\). Then \(B = B_I \cup B_{A/I}\) identifies a basis of \(A\) through \(\nu^{-1}\). Then equip the \(F_pC_n(A)\) with bases \(\{b_0 \otimes \cdots \otimes b_n \mid b_0, \ldots, b_{n-p} \in B_I; b_{n-p+1}, \ldots, b_n \in B\}\). Call this standard tensor basis. All this depends on choices, either of which will be good enough. By direct inspection:

**Lemma 4.** \(F_*\) is a filtration by subcomplexes, i.e. \(bF_pC_{n-1}(A) \subseteq F_pC_{n-1}(A)\).

For a pure tensor \(\varphi = f_0 \otimes \cdots \otimes f_n\) we will use the shorthand notation \(\varphi^{(\ell)} := f_0 \otimes \cdots \otimes f_{n-\ell}\) (the last \(\ell\) slots removed) and write \(\text{in}(\varphi) := f_0\) and \(\text{term}(\varphi) := f_n\) for the initial and terminal slot.

**Proposition 5.** Suppose \(\varphi \in F_pC_n(A)\) with \(p \leq n\) is a cycle (i.e. \(b\varphi = 0\)).

1. Then it is homologous to a representative \(\varphi' \in F_{p-1}C_n(A)\).
2. Write \(\varphi = \sum \lambda_j \varphi_j\) with \(\lambda_j \in k\) and \(\varphi_j\) a pure tensor in the standard tensor basis. Suppose \(e \in I\) is a local left unit for \(\{\text{in}(\varphi_j)\}\). Then for each \(\varphi_j = f_0 \otimes f_1 \otimes \cdots \otimes f_n\) one can define

\[
\varphi'_j := (-1)^{n+1} \left( e \otimes \text{term}(\varphi_j) \varphi_j^{(1)} - \text{term}(\varphi_j) e \otimes \varphi_j^{(1)} \right)
\]

so that \(\varphi' := \sum \lambda_j \varphi'_j\) is an explicit solution.

Note that we could equivalently demand \(\varphi \in F_pC_n(A, I)\) and obtain \(\varphi' \in F_{p-1}C_n(A, I)\) for the output.

**Proof.** Write \(\varphi = \sum \lambda_j \varphi_j\) with \(\lambda_j \in k\) and \(\varphi_j\) a pure tensor in the standard tensor basis. Since \(p \leq n\) the initial slot of each \(\varphi_j\) lies in \(I\). Let \(e \in I\) be a local left unit for the finite set \(\{\text{in}(\varphi_j)\}\); exists by our assumption on \(I\). For each \(\varphi_j = f_0 \otimes f_1 \otimes \cdots \otimes f_n \in I \otimes A^\otimes n\) define \(\varphi'_j\) as in eq. 2.1 more precisely (ignoring the superscripts)

\[
\varphi'_j = (-1)^{n+1} \left( e \otimes f_n f_0 \otimes f_1 \otimes \cdots \otimes f_{n-p} \otimes \cdots \otimes f_{n-1} \right)
\]

Next, define \(\varphi' := \sum \lambda_j \varphi'_j\). Firstly, we observe \(\varphi' \in F_{p-1}C_n(A)\); this is clear from counting indices (which for the comfort of the reader we have spelled out above).

Next, we need to check that \(\varphi'\) is homologous. To this end, define for all \(j\) the element \(G\varphi_j := e \otimes \varphi_j \in F_pC_{n+1}(A)\); this even lies in \(F_{p-1}\). We compute \(b(G\varphi_j)\) straight from the definition, giving

\[
bG\varphi_j = \varphi_j - e \otimes b\varphi_j + (-1)^n \left( e \otimes f_n \varphi_j^{(1)} - f_n e \otimes \varphi_j^{(1)} \right).
\]
To obtain this we have used the crucial fact that $e$ acts as a left unit on the initial slots of all $\varphi_j$. Thus,

$$\varphi = \sum \lambda_j \varphi_j = b(\sum \lambda_j G\varphi_j) + e \otimes b \left( \sum \lambda_j \varphi_j \right) + \sum \lambda_j (-1)^{n+1} \left( e \otimes f_n \varphi_j^{(1)} - f_n e \otimes \varphi_j^{(1)} \right) = b(\ldots) + e \otimes b \varphi + \varphi'$$

However, by assumption $\varphi$ is a cycle, i.e. $b \varphi = 0$, and $b(\ldots)$ is a boundary, so in homology we have $\varphi \equiv \varphi'$.

**Corollary 6.** Every cycle $\varphi \in F_n C_n(A, I)$ is homologous to a cycle $\varphi' \in F_0 C_n(A, I)$.

**Proof.** Just apply Prop. 5 repeatedly $n$ times.

2.2. A refined filtration. There is a better filtration than $F_p$, namely the cyclic symmetrization: Write $tF_p$ for the filtration after applying $t$ (as in eq. 11), i.e. $tF_p C_n(A) = \{ \varphi \mid t \cdot \varphi \in F_p C_n(A) \}$. Now define $\tilde{F}_p C_n(A) := \sum_{j=0}^n (t^j F_n) C_n(A)$.

Explicitly, $\tilde{F}_p C_n(A)$ is the subspace spanned by pure tensors with $n - p + 1$ cyclically successive slots in $I$. As for $F_p$, we find $\tilde{F}_0 C_n(A) = C_n(I)$ and $\tilde{F}_n C_n(A)$ is the subspace spanned from pure tensors with at least one slot in $I$. The true advantage of $\tilde{F}_p$ is that it exhausts the relative homology group:

**Lemma 7.** $C_n(A, I) = \tilde{F}_n C_n(A, I)$.

**Proof.** Clearly in the standard tensor basis $C_i(A, I) = \ker (C_i(A) \rightarrow C_i(A/I))$ is the subspace spanned by those $\varphi := f_0 \otimes \cdots \otimes f_n$ with at least one $f_j \in B_I$.

Now we transport the above considerations to cyclic homology. Almost everything goes through: The filtration $F_p$ does not make sense on $CC^\bullet(A)$ since it is not preserved by $t$, but $\tilde{F}_p$ is clearly well-defined.

**Lemma 8.** We have

1. $CC_n(A, I) = \tilde{F}_n CC_n(A, I)$ and
2. for every cycle $\varphi \in \tilde{F}_n CC_n(A, I)$ there is a representative in $F_n C_n(A, I)$ (under the map $F_n C_n(A, I) \rightarrow \tilde{F}_n CC_n(A, I)$) and we have $\varphi \equiv \varphi'$ with $\varphi' \in \tilde{F}_0 CC_n(A, I)$.

**Proof.** For the first claim pick a lift from $CC_n(A, I)$ to $C_n(A, I)$, then apply Lemma 7. For the second claim, write $[\varphi]$ for an equivalence class under the cyclic permutations $t$. Let $[\varphi] \in \tilde{F}_n CC_n(A, I)$ be given. Then $[\varphi] = \sum \lambda_j [\varphi_j]$ with $\varphi_j$ pure tensors in our standard tensor basis. For each $\varphi_j = [f_0 \otimes \cdots \otimes f_n]$ at least one slot $f_i$ lies in $I$, so we may pick the suitable permutation $t^i \varphi_j$ so that wlog $f_0 \in I$; giving a lift of $\varphi_j$ to $F_n C_n(A, I)$; and then wlog we have a representative $\varphi \in F_n C_n(A, I)$. Corollary 4 applies, giving $\varphi \equiv \varphi'$; this holds invariably since cyclic homology has the same differential $b$.

**Proof of Thm. 3.** By Lemma 8 every $\varphi \in HC_n(A, I)$ has a representative in the filtration step $\tilde{F}_n CC_n(A, I)$ and it satisfies $\varphi \equiv \varphi'$ with $\varphi' \in \tilde{F}_0 CC_n(A, I) = CC_n(I)$. But this just means that $HC_n(A, I) = \tilde{F}_0 HC_n(A, I) = HC_n(I)$.

It is clear that this actually yields a method to produce a concrete representative in $HC_n(I)$ just by evaluating $\varphi'$ in concrete terms. We will do this in the next section.
3. Proof of the explicit formula

In this section we prove Thm. 2 & Thm. 3. We keep the assumptions of the last section.

Proposition 9. Suppose \( \varphi \in F_nC_n(A,I) \) is a cycle, i.e. \( b\varphi = 0 \). Write \( \varphi = \sum \lambda_j \varphi_j \) with each \( \varphi_j \) a pure tensor in our standard tensor basis, say \( \varphi_j = f_0 \otimes f_1 \otimes \cdots \otimes f_n \) with \( f_0 \in I \).

- Let \( e_\alpha \in I \) be a local left unit for \( \bigcup \{ f_0 \} \), where the union runs over the \( f_0 \)-slots of all \( \varphi_j \),
- and for \( i \leq n \) let \( e_{\alpha_{i-1}} \) be a local left unit for \( \{ e_1 \} \cup \{ f_i e_i \} \), where the union runs over the \( f_i \)-slots of all \( \varphi_j \).

For each \( \varphi_j = f_0 \otimes f_1 \otimes \cdots \otimes f_n \) define

\[
\varphi'_j = \sum_{s_1, \ldots, s_n \in \{ \pm \}} (-1)^{s_1 + \cdots + s_n} e_1 \otimes f_1 \otimes e_2 \otimes f_2 \otimes \cdots \otimes e_n \otimes f_n \otimes f_0
\]

(3.1) where for each underbrace we take the upper term if \( s_i = + \), the lower if \( s_i = - \) and then \( \varphi' := \sum \lambda_j \varphi'_j \in C_n(I) \) is an explicit representative of the same homology class as \( \varphi \).

Remark. Instead of a single \( \varphi \in F_nC_n(A,I) \) we can work with finitely many \( \varphi^1, \ldots, \varphi^d \in F_nC_n(A,I) \) and find a uniform choice of the \( e_i \) by taking the finite union of the sets \( \{ \text{in}(\varphi_j) \} \), \( \{ e_i \} \cup \{ f_i e_i \} \) appearing for the individual \( \varphi'^n \) instead.

Remark. The same result holds for cyclic homology, with exactly the same proof. This result does not prove excision for Hochschild homology since it will generally not be true that \( F_nHH_n(A,I) = HH_n(A,I) \).

Proof. We can construct a representative of the homology class \([\varphi]\) in \( F_0C_n(A) = C_n(I) \) by using the procedure \( \varphi \leadsto \varphi' \) of Prop. 5 iteratively \( n \) times. For each iteration the element \( e \) will need to be different; let us write \( e_i \) for the element appearing in the \( (n + 1 - i) \)-th iteration – we start counting with \( i := 1 \); we pick and fix these \( e_n, \ldots, e_1 \in I \). We can now reduce the computation to pure tensors in our standard tensor basis by linearity: Suppose \( \varphi = f_0 \otimes \cdots \otimes f_n \) with \( f_0 \in I \). Recall that we have

\[
\varphi' = (-1)^{n+1} \left( e \otimes \text{term}(\varphi)\varphi^{(1)} - \text{term}(\varphi)e \otimes \varphi^{(1)} \right) \quad \text{(for a suitable } e) \tag{3.2}
\]

Write \( t_0 := \varphi \) and \( t_i := t_{i-1}' \) (with the prime superscript indicating the procedure of Prop. 5). Note that this construction applied to a pure tensor gives a linear combination of two pure tensors. Clearly \( t_i \) will be a linear combination of \( 2^i \) pure tensors, we will write

\[
t_i = \sum_{s_1, \ldots, s_n \in \{ \pm \}} t_i^{s_1 \cdots s_i}, \quad \text{(for } i \geq 1) \tag{3.3}
\]

where the superscripts \( s_j \in \{ +, - \} \) encode whether we have picked the first or second term in eq. (3.2). To be precise: If \( t_{i-1} \) (for \( i \geq 1 \)) comes with a presentation
as in eq. 3.3

\[
(3.4) \quad t_i = t_{i-1}' = \sum_{s_1 \ldots s_i, i \in \{\pm\}} \left( t_{i-1}^{s_1 \ldots s_i} \right)' = (-1)^{n+1} \sum_{s_1 \ldots s_i, i \in \{\pm\}} \sum_{s_{i-1} \in \{\pm\}} e_{n+1-i} \otimes \text{term}(t_{i-1}^{s_1 \ldots s_i}) \cdot (t_{i-1}^{s_1 \ldots s_i})^{(1)} \quad \text{if } s_i = + \\
- \text{term}(t_{i-1}^{s_1 \ldots s_i}) e_{n+1-i} \otimes (t_{i-1}^{s_1 \ldots s_i})^{(1)} \quad \text{if } s_i = -
\]

(3.5) \cdots

where the formula in eq. 3.2 was applicable in eq. 3.5 since the expression was decomposed into pure tensors already; \( e_{n+1-i} \) denotes the local left unit picked in this step. Eq. 3.4 gives us a presentation of \( t_i \) as in eq. 3.3

\[
(3.6) \quad t_i^{s_1 \ldots s_i} = (-1)^{n+1} \left\{ e_{n+1-i} \otimes \text{term}(t_{i-1}^{s_1 \ldots s_i}) \cdot (t_{i-1}^{s_1 \ldots s_i})^{(1)} \quad \text{if } s_i = + \\
- \text{term}(t_{i-1}^{s_1 \ldots s_i}) e_{n+1-i} \otimes (t_{i-1}^{s_1 \ldots s_i})^{(1)} \quad \text{if } s_i = -
\right.
\]

We may now inductively evaluate \( \text{term}(t_{i-1}^{s_1 \ldots s_i}) \). The above equation (just as well as eq. 3.2) shows that irrespective of \( s_i \) each iteration of eq. 3.2 removes the last slot in each pure tensor. Thus, \( \text{term}(t_i^{s_1 \ldots s_i}) = f_{n-i} \) for \( i \geq 0 \). To evaluate the initial term, we follow eq. 3.3 and see that

\[
(3.7) \quad \text{in}(t_i^{s_1 \ldots s_i}) = (-1)^{n+1} \left\{ e_{n-i+1} \quad \text{if } s_i = + \\
- f_{n-i+1} e_{n-i+1} \quad \text{if } s_i = -
\right.
\]

for \( i \geq 1 \). This provides us with an inductive description of the initial terms which we will need later. Eq. 3.6 simplifies in view of our explicit knowledge of the terminal terms and unraveling this inductive formula we get

\[
(3.8) \quad t_i^{s_1 \ldots s_n} = (-1)^{n+1} \left\{ e_1 \otimes f_1 \quad e_2 \otimes f_2 \quad \ldots \quad e_n \otimes f_n \right. \\
\left. \quad f_1 e_1 \otimes f_2 e_2 \otimes \ldots \otimes f_n e_n \otimes t_0^{(n)}
\right.
\]

Using eq. 3.3 and \( t_0^{(n)} = \varphi^{(n)} = f_0 \), we get eq. 3.3. Finally, we need to identify the requirements on how the \( e_i \) can be chosen: From the assumptions of Prop. 5 (and the discussion after eqs. 3.4 & 3.5) we see that \( e_{n+1-i} \) needs to act as a left unit on all the initial slots of the pure tensors appearing in \( t_{i-1} \). By eq. 3.7 this means that \( e_n \) needs to act as a left unit on the \( f_0 \), and for \( i \geq 1 \) the element \( e_{n+1-i} \) needs to act as a left unit on \( \{ e_{n-i+2}, f_{n-i+2}, e_{n-i+2} \} \). We obtain our claim by re-indexing: With \( i' := n - i + 1 \) we get that \( e_{i'} \) needs to act as a left unit on the initial slots of the pure tensors in \( t_{n-i'} \), these are those with initial slots \( f_0 \) or \( \{ e_{i'+1}, f_{i'+1}, e_{i'+1} \} \).

**Proof of Thms. 2 and 3.** For cyclic homology the same argument applies, but is stronger: By Lemma 8 every \( \varphi \in \text{HC}_n(A, I) \) has a representative in \( F_n C_n(A, I) \). As above, we get a representative in \( F_0 C_n(A, I) = CC_n(I) \). As we had already remarked above, the procedure generalizes to finitely many elements \( \varphi^a \) by picking the local units common for them, so we get the result for \( V \) with \( V' \) the span of the choice of representatives \( \varphi^a \).

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