Learning Query Inseparable $\mathcal{ELH}$ Ontologies

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Abstract

We investigate the complexity of learning query inseparable $\mathcal{ELH}$ ontologies in a variant of Angluin’s exact learning model. Given a fixed data instance $A_s$ and a query language $Q$, we are interested in computing an ontology $\mathcal{H}$ that entails the same queries as a target ontology $T$ on $A_s$, that is, $\mathcal{H}$ and $T$ are inseparable w.r.t. $A_s$ and $Q$. The learner is allowed to pose two kinds of questions. The first is ‘Does $(T, A_s) \models q$?’, with $A_s$ an arbitrary data instance and $q$ and query in $Q$. An oracle replies this question with ‘yes’ or ‘no’. In the second, the learner asks ‘Are $\mathcal{H}$ and $T$ inseparable w.r.t. $A_s$ and $Q$?’. If so, the learning process finishes, otherwise, the learner receives $(A_s, q)$ with $q \in Q$. $(T, A_s) \models q$ and $(\mathcal{H}, A_s) \not\models q$ (or vice-versa). Then, we analyse conditions in which query inseparability is preserved if $A_s$ changes. Finally, we consider the PAC learning model and a setting where the algorithms learn from a batch of classified data, limiting interactions with the oracles.

Introduction

Ontologies are a formal and popular way of representing knowledge. Taxonomies, categorisation of websites, products and their features, as well as more complex and specialized domain knowledge, can be represented with ontologies. Domain experts use ontologies while sharing and annotating information in their fields because in this way knowledge can be unambiguously understood and easily distributed. Medicine, for example, has produced large, standardised, and scalable ontologies (e.g. Galen and SNOMED CT). Broad and general so called knowledge graphs are emerging such as DBPedia (Bizer et al. 2009), Wikidata (Vrandečić and Krötzsch 2014), YAGO (Suchanek, Kasneci, and Weikum 2008). An ontology enables machines to process relations and definitions and reason about that knowledge. Sharing information, formalising a domain, and making assumptions explicit are some of the main reasons for using an ontology.

Designing ontologies is a hard and error-prone task. The research community has approached the problem by developing editors that help ontology engineers to build ontologies manually (Knublauch et al. 2004) and defined design principles (Stuckenschmidt, Parent, and Spaccapietra 2009). Even with tools, building ontologies is a laborious task that also needs expertise. An expert in designing ontologies is called an ontology engineer. Such an expert is normally familiar with the tools, languages, and techniques necessary to design an ontology through the communication with domain experts. The ontology engineer must communicate and understand the knowledge given by domain experts and then design an ontology that captures what is relevant in the domain.

One of the main challenges in the process of building an ontology is that it often relies on the communication between the ontology engineer (or multiple ontology engineers) and domain experts that, in order to share knowledge, use the ambiguous natural language. Indeed, it can happen that some errors are made while designing it due to the difficulty of sharing knowledge. The ontology engineer can misunderstand the domain experts or they can inadvertently omit precious details. Moreover, knowledge can be implicit and while designing an ontology it is not easy to understand what are the real relationships between concepts and imagine all the possible consequences of designing concepts and their relationships in a particular way. There are several aspects which can influence the difficulty of creating an ontology.

Following the approach by (Konev et al. 2018; Konev, Ozaki, and Wolter 2016), we focus on the problem of finding how concepts should be logically related, assuming that the relevant vocabulary is known by the domain experts, which can share this information with the ontology engineer. In this approach, the problem of building an ontology is treated as a learning problem in which the ontology engineer plays the role of a learner and communicates with the domain experts, who play the role of a teacher (also called an oracle). We assume that (1) the domain experts know the relevant knowledge about the domain and act in a consistent way as a single teacher; (2) the vocabulary that should be used in the ontology is known by the teacher and the learner; (3) the learner can pose queries to the teacher in order to acquire missing knowledge or check if it has learned enough in order to stop learning. The described model can be seen as an instance of Angluin’s exact learning model (Angluin 1988) with membership and equivalence queries. The queries asked by the learner in order to acquire knowledge can be considered as membership queries and the queries that ask if the hypothe-
we investigate a more flexible setting, where the ontology w.r.t. a query language (Lutz and Wolter 2010) and a fixed versioning, modularization, update, and forgetting, depend with polynomially many polynomial size queries. Here, exactly learnability of ontologies formulated in the very popular $\mathcal{ELH}$ (Baader et al. 2007a) ontology language is not possible with polynomially many polynomial size queries. Here, we investigate a more flexible setting, where the ontology does not need to be logically equivalent but only inseparable w.r.t. a query language (Lutz and Wolter 2010) and a fixed data instance\(^1\). Query inseparability is the basic requirement for ontology mediated query answering (OMQA) (Bienvenu 2016). In OMQA, relevant tasks such as ontology versioning, modularisation, update, and forgetting, depend on comparisons between ontologies based on answers given to queries (Botoeva et al. 2019). We study polynomial query and time learnability of the $\mathcal{ELH}$ ontology language in the OMQA setting. The query languages considered are atomic queries (AQ), instance queries (IQ), conjunctive queries (CQ) and a fragment of CQs called rooted CQs (denoted CQ\(_r\)). In particular, we show that in our setting the picture is brighter and $\mathcal{ELH}$ can be polynomially learned from IQs, however, it is still not possible to learn this language from CQs.

Table 1 shows the different results obtained by previous works (Konev et al. 2018; Konev, Ozaki, and Wolter 2016) for logical equivalence and our results (shaded in gray) for query inseparability, taking into account the ontology (upper side) and the query languages (left side). $\checkmark$ means a positive result, i.e. polynomial query learnability; $\xmark$ means that the query language is not expressive enough for exchanging information and $\mathcal{X}$ means that polynomial query learnability cannot be achieved. Polynomial time learnability implies polynomial query learnability but the converse does not hold. All positive results for AQs, IQs, and CQs also hold for polynomial time learnability. Since the learned ontology is not equivalent to the target, it may be the case that after the data is updated the learned ontology and the target are no longer query inseparable. We thus investigate conditions under which query inseparability is preserved when the data changes, which means that no further learning steps are needed after the change. In many application scenarios, interactions with teachers may not be a viable option. We also consider learnability when the learner has only access to a batch of classified examples. Finally, we adapt the Probably Approximately Correct (PAC) model (Valiant 1984) to our OMQA setting, which we separate from the exact and query inseparable problem settings (Theorem 15). Our polynomial time results for query inseparability are transferable to the PAC model extended with membership queries (Theorem 14). Omitted proofs are available in the appendix.

**Related Work.** To cover the vast literature on ontology learning, we point to the collection edited by (Lehmann and Völker 2014) and surveys authored by (Cimiano, Völker, and Buitelaar 2010) and (Wong, Liu, and Bennamoun 2012). The closest works are the already mentioned papers on exact learning of lightweight description logics (DLs) (Konev et al. 2018; Duarte, Konev, and Ozaki 2018; Konev, Ozaki, and Wolter 2016). Exact learning of concepts formulated in the DL CLASSIC has been investigated by (Cohen and Hirsh 1994) and (Frazier and Pitt 1996). Some other works which are more closely related include works on learning $\mathcal{EL}$ concepts (Funk et al. 2019; Lehmann and Haase 2009). Formal Concept Analysis has been applied for learning DL ontologies (Rudolph 2004; Baader et al. 2007b; Borchmann and Distel 2011; Borchmann 2014; Ganter et al. 2016). Learnability of $\mathcal{EL}$ ontologies from finite interpretations has also been investigated (Klarman and Britz 2015). Association rule mining has been used to learn DL ontologies (with concept expressions of limited depth) (Sazonau and Sattler 2017; Völker and Niepert 2011; Fleischhacker, Völker, and Stuckenschmidt 2012; Völker, Fleischhacker, and Stuckenschmidt 2015).

**Basic Definitions**

**Ontologies and Queries.** The $\mathcal{ELH}$ syntax is defined upon mutually disjoint countably infinite sets of concept names $\mathcal{N}_C$, denoted with $A, B$, role names $\mathcal{N}_R$, denoted with $r, s$, and individual names $\mathcal{N}_I$, denoted with $a, b$. $\mathcal{EL}^\text{-concept expressions} \ C$ are defined inductively according to the rule $C ::= A \mid T \mid C \cap C \mid \exists r.C$, where $A \in \mathcal{N}_C$ and $r \in \mathcal{N}_R$. For simplicity, we omit $\mathcal{EL}$- from $\mathcal{EL}$-concept expressions. An $\mathcal{ELH}$ ontology, also called TBox, is a finite set of concept inclusions (CI) $C \subseteq D$, where $C, D$ are concept expressions, and role inclusions (RI) $r \subseteq s$, where $r, s \in \mathcal{N}_R$. We call an $\mathcal{ELH}$ TBox $T$ a terminology if for all $C \subseteq D \in T$ either $C$ or $D$ is a concept name\(^2\) and $T$ has at most one\(^3\) inclusion of

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\(^1\)Here the term ‘query’ refers to queries in the context of databases and query answering. Our data instances are ABoxes.

\(^2\)In the literature, the term terminology commonly refers to sets of concept inclusions $A \subseteq C$ and concept definitions $A \equiv C$, with no concept name occurring more than once on the left. As $A \equiv C$ can be equivalently rewritten as $A \subseteq C$ and $C \subseteq A$, our definition is a natural extension of this one.

\(^3\)If a terminology contains $A \subseteq C$ and $A \subseteq D$ one can always rewrite it into $A \subseteq C \cap D$. 

| Framework | $\mathcal{EL}(\mathcal{H})_{\text{hs}}$ | $\mathcal{EL}(\mathcal{H})_{\text{ns}}$ | $\mathcal{EL}(\mathcal{H})$ |
|-----------|-------------------------------|-------------------------------|------------------|
| AQ        | $\checkmark$                  | $\xmark$                      | $\mathcal{X}$    |
| IQ        | $\checkmark$                  | $\xmark$                      | $\mathcal{X}$    |
| CQ        | $\checkmark$                  | $\checkmark$                 | $\mathcal{X}$    |
| PAC       | $\checkmark$                  | $\checkmark$                 | $\checkmark$     |

Table 1: Polynomial query learnability of learning frameworks.
the form $A \sqsubseteq C$ for every $A \in \mathcal{N}_C$. From now on we assume all $\mathcal{ELH}$ TBoxes we deal with are terminologies. An ABox $A$ is a finite set of expressions of the form $A(a)$ or $r(a, b)$, called assertions, with $A \in \mathcal{N}_C$, $r \in \mathcal{R}_K$ and $a, b \in \mathcal{I}$. We denote by $\text{ind}(A)$ the set of individual names occurring in an ABox $A$. The signature $\Sigma_T$ of a TBox $T$ is the set of concept and role names occurring in it, and similarly for the signature $\Sigma_A$ of an ABox $A$. A knowledge base (KB) is a pair $(T, A)$ where $T$ is a TBox and $A$ is an ABox. We investigate classical query languages considered in the OMG literature. An AQ takes the form of an assertion. An IQ is of the form $C(a)$ or $r(a, b)$, where $C$ is a concept expression, $r \in \mathcal{R}_K$ and $a, b \in \mathcal{I}$. A CQ is a first-order sentence $\exists \vec{x} \varphi(\vec{a}, \vec{x})$, where $\varphi$ is a conjunction of atoms of the form $r(t_1, t_2)$ or $A(t)$, where $t_1, t_2, t$ (called terms) can be individual names from $\vec{a}$ or individual variables from $\vec{x}$. With an abuse of notation, we denote by AQ, IQ and CQ the sets of atomic, instance and conjunctive queries, respectively, and we call a query language a set $Q \subseteq \{\text{AQ, IQ, CQ}\}$. The size of a concept expression $C$ (TBox $T$, ABox $A$, query $q$), denoted by $\|C\|$ (and, respectively, $\|T\|$, $\|A\|$, $\|q\|$) is the length of the string that represents it, where concept, role, and individual names are considered to be of length one.

The semantics of $\mathcal{ELH}$ is given as follows. An interpretation is a pair $I = (\Delta^I, \tau^I)$, where $\Delta^I$ is a non-empty set, called domain, and $\tau^I$ is a function that maps every $a \in \mathcal{N}_I$ to $a^I \in \Delta^I$, every $A \in \mathcal{N}_C$, $A^I \subseteq \Delta^I$ and every $r \in \mathcal{R}_K$ to $r^I \subseteq \Delta^I \times \Delta^I$. The function $\tau^I$ extends to other concept expressions as follows: $T^I = \Delta^I \times \Delta^I$, $(C \cap D)^I = C^I \cap D^I$ and $(\exists r.C)^I := \{d \in \Delta^I \mid \exists e \in \Delta^I. (d, e) \in r^I\}$. An interpretation $I$ satisfies a CI $C \subseteq D$ if $C^I \subseteq D^I$, and an RI $r \subseteq s$ if $r^I \subseteq s^I$. It satisfies an assertion $A(a)$ if $a^I \in A^I$, and an assertion $r(a, b)$ if $(a^I, b^I) \in r^I$. $I$ satisfies an AQ if it satisfies the corresponding assertion, and it satisfies an IQ $C(a)$, or $r(a, b)$, if $a^I \in C^I$, or $(a^I, b^I) \in r^I$. $I$ satisfies a CQ $q$ (or there is a homomorphism from $q$ to $I$) if there is a function $\pi$, mapping terms of $q$ to elements of $\Delta^I$, such that: $\pi(t) = t^I$, if $t \in \mathcal{N}_I$; $\pi(t) = a^I$, for every $A(t)$ of $q$; and $(\pi(t_1), \pi(t_2)) = (r^I, \tau^I)$, for every $r(t_1, t_2)$ of $q$. We write $I \models q$ to state that $I$ satisfies an IQ $q$, $I$ satisfies a TBox $T$, if it is a model of every CI and RI in $T$, and it satisfies an ABox $A$ if it satisfies every assertion in $A$. $I$ satisfies a KB $K = (T, A)$, written $I \models K$, if it satisfies both $T$ and $A$. A KB $K$ entails a CI, an RI, or an assertion, or a query $q$, written $K \models q$, if, for all interpretations $I$, $I \models K$, implies $I \models q$. A KB $K$ entails a KB $K'$, written $K \models K'$, if $I \models K$ implies $I \models K'$; $K$ and $K'$ are equivalent, in symbols $K \equiv K'$, if $K \models K'$ and $K' \models K$. We may also speak of entailments and equivalences of TBoxes and ABoxes, defined as usual (Baader et al. 2007a). For $Q \subseteq \{\text{AQ, IQ, CQ}\}$, the KBs $K$ and $K'$ are $Q$-inseparable, in symbols $K \equiv^Q K'$, if for every query $q \in Q$, we have that $K \models q$ iff $K' \models q$.

Tree Representation and Homomorphisms. We will also represent a concept expression $C$ as a finite directed tree $T_C = (\mathcal{N}_C, \mathcal{E}_C, l_C)$, where $\mathcal{N}_C$ is the set of all vertices, with the root denoted by $r_C$, $\mathcal{E}_C$ is the set of all edges, and $l_C$ is a labelling function that maps every node to a set of concept names and every edge to a role name. This tree representation of $C$ uniquely represents the corresponding concept expression, and it is inductively defined as follows:

- for $C = \top$, $\mathcal{N}_C = \{\rho_C\}$ and $l_C(\rho_C) = \emptyset$;
- for $C = A$, where $A \in \mathcal{N}_C$, $\mathcal{N}_C = \{\rho_C\}$ and $l_C(\rho_C) = A$;
- for $C = \exists r.D$, $T_C$ is obtained from $T_D$ by adding a new root $\rho_C$ and an edge from $\rho_C$ to the root $\rho_D$ of $T_D$ with label $l_C(\rho_C), l_D(\rho_D)$ = (we call $\rho_D$ an $r$-successor of $\rho_C$);
- for $C = D_1 \sqcap D_2$, $T_C$ is obtained by identifying the roots of $T_{D_1}$ and $T_{D_2}$, with $l_C(\rho_{D_1}), l_D(\rho_{D_2})$.

The ABox representation of $C$, $A_C$, is the ABox encoding the tree representation $T_C$ of $C$, defined as follows. For each $v \in \mathcal{N}_C$, we associate $a_v \in \mathcal{I}$. Then, for every $u \in \mathcal{N}_C$ and every $(a, v) \in \mathcal{E}_C$, we put: $A(a_v) \in A_C$ iff $l_C(u, v) = A$; $r(a_u, a_v) \in A_C$ iff $l_C(u, v) = r$. Given $A, A'$ be ABoxes, a function $h: \text{ind}(A) \rightarrow \text{ind}(A')$ is called an ABox homomorphism from $A$ to $A'$ if: for every $C(a) \in A$, $C(h(a)) \in A'$; and, for every $r(a, b) \in A$, $r(h(a), h(b)) \in A'$.

Learning Model. We provide basic notions related to the exact learning model, extending the notations in (Konev et al. 2018). A learning framework $\mathcal{F}$ is a quadruple $(\mathcal{E}, \mathcal{S}, \mathcal{L}, \mu)$, where $\mathcal{E}$ is a set of examples, $\mathcal{S}$ is a subset of $\mathcal{E}$, $\mathcal{L}$ is a set of concept representations (also called hypothesis space), and $\mu$ is a mapping from $\mathcal{L}$ to $[0, 1]$. We omit $\mathcal{S}$ if $\mathcal{E} = \mathcal{L}$. Each element $l \in \mathcal{L}$ is assumed to be represented using a set of symbols $\Sigma_l$ (if $l$ is a TBox $T$, $\Sigma_T$ is the signature of $T$). We say that $e \in \mathcal{E}$ is a positive example for $l \in \mathcal{L}$ if $e \in \mu(l)$ and a negative example for $l$ if $e \notin \mu(l)$.

Given a learning framework $\mathcal{F} = (\mathcal{E}, \mathcal{S}, \mathcal{L}, \mu)$, we are interested in the exact identification of a target concept representation $t \in \mathcal{L}$ w.r.t. the subset $\mathcal{S}$ of examples, by posing queries to oracles. Let $\mathcal{M}_{\mathcal{F}, \mathcal{S}}$ be the oracle that takes as input some $e \in \mathcal{E}$ and returns ‘yes’ if $e \in \mu(t)$ and ‘no’ otherwise. A membership query is a call to the oracle $\mathcal{M}_{\mathcal{F}, \mathcal{S}}$. For every $t \in \mathcal{L}$, we denote by $\mathcal{E}_{\mathcal{F}, \mathcal{S}, t}$ the oracle that takes as input a hypothesis concept representation $h \in \mathcal{L}$ and returns ‘yes’ if $\mu(h) \cap S = \mu(t) \cap S$ and a counterexample $e \in (\mu(h) \cap S) \cap (\mu(t) \cap S)$. We say that $\mathcal{F}$ is a learning algorithm for $\mathcal{F}$ if: for every $l \in \mathcal{L}$, we write ‘with respect to $\mathcal{S}$’. An (exact) learning algorithm for $\mathcal{F} = (\mathcal{E}, \mathcal{S}, \mathcal{L}, \mu)$ is a deterministic algorithm that, for a fixed but arbitrary $t \in \mathcal{L}$, takes $\mathcal{S}$ as input, is allowed to make queries to $\mathcal{M}_{\mathcal{F}, \mathcal{S}}$, and that eventually halts and outputs some $h \in \mathcal{L}$ with $\mu(h) \cap S = \mu(t) \cap S$. We say that the learning algorithm is positive bounded if, in addition, $\mu(h) \subseteq \mu(t)$. We say that $\mathcal{F}$ is exact learnable if there is a learning algorithm for $\mathcal{F}$ and $\mathcal{F}$ is polynomial query learnable if it is exactly learnable by an algorithm $A$ such that at every step the sum of the sizes of the inputs to membership and equivalence queries made by $A$ up to that step is bounded by a polynomial $p(|t|, |\mathcal{S}|)$, where $t$ is the target and $e \in \mathcal{S}$ is the largest counterexample seen so far (Arias 2004). Similarly, $\mathcal{F}$ is polynomial time learnable if it is exactly learnable by an algorithm $A$ such that at every step (we count each call to an oracle as one step of
computation) of computation the time used by A up to that step is bounded by a polynomial \( p(|t|, |e|) \), where \( t \in L \) is the target and \( e \in S \) is the largest counterexample seen so far.

We denote by PQUERY and PTIME\(_L\) the class of learning frameworks which are, respectively, polynomial query and polynomial time learnable. Clearly, PTIME\(_L \subseteq PQUERY\(_L\).

We now introduce the special case of learning frameworks which we focus in this work, called OMQA learning frameworks. Let \( \mathcal{L}, A_s, \) and \( Q \) be, respectively, an ontology language, a fixed but arbitrary ABox, and a query language. An OMQA learning framework \( \mathcal{F}(\mathcal{L}, A_s, Q) \) is a learning framework \( (E, S, L, \mu) \) where \( E \) is the set of all pairs \( (A, q) \) with \( A \) an ABox (may be different from \( A_s) \) and \( q \in Q \); \( L \) is the set of all TBoxes formulated in \( L \) sharing a common finite signature (we write \( \Sigma_T \) to refer to the signature of the target, which is assumed to be as the one used for the hypothesis); \( S \) is the set of elements \( (A, q) \) of \( E \) where \( A = A_s \); and, for all \( T \in L, \mu(T) = \{ (A, q) \in E \mid (T \cdot A) = q \} \).

For all \( T \in L \), \( \mu(T) \) is the set of \( (A, q) \) in \( E \) such that \( (T \cdot A) = q \). We assume that the signature of \( q \), i.e., the set of concept and role names occurring in \( q \), is \( \Sigma_T \cup \Sigma_A \). Moreover, we define the size of an example \( (A, q) \), denoted by \( |(A, q)| \), as the sum of the size of \( A \) and \( q \). Given an OMQA learning framework \( \mathcal{F}(\mathcal{L}, A_s, Q) \) and \( q \in Q \), \( A \in A_s \), and \( T \in L \), if \( \mu(T) \cap \Sigma_A \), \( q \) is \( E \)-inseparable with \( \mu(T) \cap \Sigma_A \).

Since an equivalence query in an OMQA learning framework with \( E = E \) is in fact asking whether a given TBox is \( E \)-inseparable from the target TBox with a fixed ABox and a query language, we may call it an \( E \)-inseparability query.

**Polynomial Learnability**

We investigate whether, for a given fixed ABox and query language, the problem of learning query inseparable \( E \)-boxes is polynomial. We first discuss the relationship between our OMQA setting and the data retrieval one (Konev, Ozaki, and Wolter 2016). The difference between the two settings is that here the oracle can only choose counterexamples of the form \( (A, q) \), with the fixed ABox \( A_s \) given as input, whereas in the mentioned work the ABox in a counterexample can be arbitrary. In both settings the learner can pose membership queries with an arbitrary ABox in the examples.

Formally, given an ontology language \( \mathcal{L} \) and a query language \( Q \), we denote by \( \mathcal{F}(\mathcal{L}, Q) \) the learning framework \( (E, S, L, \mu) \) where \( E \) is the set of all pairs \( (A, q) \) with \( A \) an ABox and \( q \in Q \); \( L \) is the set of all TBoxes formulated in \( L \); \( E = S \); and, for all \( T \in L, \mu(T) = \{ (A, q) \in E \mid (T \cdot A) = q \} \). We denote by \( \mathcal{F}(\mathcal{L}, A_s, Q) \) and \( \mathcal{F}(\mathcal{L}, A_s, \text{IQ}) \) the fragments of \( \mathcal{F}(\mathcal{L}) \) which only allow complex concept expressions on the left-hand side and on the right-hand side of CIs, respectively. It is known that the learning frameworks \( \mathcal{F}(\mathcal{L}, A_s, \text{AQ}) \) and \( \mathcal{F}(\mathcal{L}, A_s, \text{IQ}) \) are in PTIME\(_L\), whereas \( \mathcal{F}(\mathcal{L}, Q) \) and \( \mathcal{F}(\mathcal{L}, \text{IQ}) \) are not in PQUERY\(_L\) (Konev, Ozaki, and Wolter 2016). It follows from our definitions that, for any ontology language \( \mathcal{L} \), query language \( Q \) and ABox \( A_s \), polynomial learnability of \( \mathcal{F}(\mathcal{L}, Q) \) implies polynomial learnability of \( \mathcal{F}(\mathcal{L}, A_s, Q) \). Thus, the positive results for \( \mathcal{F}(\mathcal{L}, A_s, \text{AQ}) \) and \( \mathcal{F}(\mathcal{L}, A_s, \text{IQ}) \) in the OMQA setting. That is, for any ABox \( A_s \), the learning frameworks \( \mathcal{F}(\mathcal{L}, A_s, \text{AQ}) \) and \( \mathcal{F}(\mathcal{L}, A_s, \text{IQ}) \) are in PTIME\(_L\).

**Proposition 1.** For any ontology language \( \mathcal{L} \), query language \( Q \) and ABox \( A_s \), if \( \mathcal{F}(\mathcal{L}, Q) \) is in PTIME\(_L\)/PQUERY\(_L\), then \( \mathcal{F}(\mathcal{L}, A_s, Q) \) is in PTIME\(_L\)/PQUERY\(_L\).

In the following, we extend the positive results for \( \mathcal{F}(\mathcal{L}, A_s, \text{AQ}) \) and \( \mathcal{F}(\mathcal{L}, A_s, \text{IQ}) \) in the OMQA setting based on IQs. This is in contrast with the negative result for the class of \( \mathcal{F}(\mathcal{L}) \) terminologies in the data retrieval setting with IQs, thus, showing that the converse direction of Proposition 1 does not hold. Throughout this section \( A_s \) is a fixed but arbitrary ABox. We also analyse the learning framework \( \mathcal{F}(\mathcal{L}, A_s, \text{AQ}) \), which is also not in PQUERY\(_L\) in the OMQA setting.

**Learning \( \mathcal{F}(\mathcal{L}) \) ontologies with AQ.** We argue that the learning framework \( \mathcal{F}(\mathcal{L}, A_s, \text{AQ}) \) is in PTIME\(_L\). There are only polynomially many counterexamples that the oracle can give since they can only be of the form \( (A, q) \), with \( q \) an atomic query (using symbols from \( \Sigma_T \)). The set of RIs entailed by \( T \) can be learned in polynomial time by adding to the hypothesis \( H \) all RIs \( r \subseteq s \) such that the membership oracle replies ‘yes’ given an example \( \langle r(a, b), s(a, b) \rangle \) as input. Therefore we only need to show that one can compute concept inclusions that together with \( A_s \) entail precisely the same atomic queries as the target ontology \( T \) on \( A_s \). The following lemma establishes that indeed one can compute such concept inclusions in polynomial time.

**Lemma 2.** Let \( T \) and \( H \) be resp. the target and the hypothesis (of size polynomial in \(|T|\)) terminologies. Given a positive counterexample \( (A_s, A(a)) \), one can compute in polynomial time in \(|A_s| + |T| \) a CI \( \mathcal{C} \subseteq B \) such that \( (T, A_s) \models B(b), (H, A_s) \not\models B(b) \), and \( (H \cup \{ C \subseteq B \}, A_s) \models B(b) \), for some \( b \in \text{ind}(A_s) \). Moreover, \( T \models C \subseteq B \).

The main idea for proving Lemma 2 is to transform the structure of \( A_s \) into a tree shaped structure, as in (Konev, Ozaki, and Wolter 2016). However, in the mentioned work only CIs of the form \( C \subseteq A \) are considered, while in our case CIs of the form \( A \subseteq C \) may also be present. With such CIs the computed tree shaped ABox may be smaller than the original concept expression in \( T \) (as in Example 3).

**Example 3.** Assume that \( T = \{ B \subseteq \exists s.B, \exists r.\exists s.B \subseteq A \} \) and \( A = \{ r(a, b), B(b) \} \). We have that \( (T, A) \models A(a) \) and the concept expression \( \exists r.\exists s.B \) encoded in \( A \) is smaller than the original concept expression \( \exists r.\exists s.B \) in \( T \) implying the concept name \( A \). Intuitively, even though there is no morphism from \( A \) to \( B \) to \( A \), we have that \( B \) ‘abbreviates’ \( \exists s.B \), because it is implied by \( B \).

Since there are polynomially many possible counterexamples, our upper bound for AQs follows from Lemma 2.

**Theorem 4.** \( \mathcal{F}(\mathcal{L}, A_s, \text{AQ}) \) is in PTIME\(_L\). Moreover, there is a polynomial bounded learning algorithm for showing such upper bound which only poses membership queries (no inseparability queries are needed).

**Learning \( \mathcal{F}(\mathcal{L}) \) ontologies with IQ.** We build on the result for AQs (Theorem 4) and argue that the learning framework \( \mathcal{F}(\mathcal{L}, A_s, \text{IQ}) \) is in PTIME\(_L\). By Theorem 4, CIs
of the form \( C \subseteq A \) which ensure AQ-inseparability w.r.t. \( A_\ast \) can be learned with membership queries. It remains to show how one can learn CIs of the form \( A \subseteq C \), so that \( \mathcal{H} \) is IQ-inseparable from \( T \) (w.r.t. \( A_\ast \)). By Theorem 4, we can assume that one can construct a hypothesis \( \mathcal{H} \) such that \( T \models \mathcal{H} \), since there is a positive bounded learning algorithm.

The next lemma states that if \( T \models \mathcal{H} \) and \( (\mathcal{H}, A_\ast) \equiv_{AQ} (T, A_\ast) \) then one can transform a counterexample of the form \( (A_\ast, D(a)) \) into a CI such that \( T \models \mathcal{H} \). The number of subpatterns of \( D \) and \( A \in \Sigma_T \) such that \( (\mathcal{H}, A_\ast) \models A(b) \) for some \( b \in \mathbb{N}_0 \). Since \( |\Sigma_T| \) is polynomial in \( |T| \) and the number of subpatterns of \( D \) is also polynomial in \( |D| \). One can find such \( A \subseteq C \) in polynomial time w.r.t. \(|T|\) and \(|(A_\ast, D(a))|\).

**Lemma 5.** Let \( T \) and \( \mathcal{H} \) be \( \mathcal{E} \mathcal{L} \mathcal{H} \) terminologies and let \( A_\ast \) be an ABox. Assume \( (\mathcal{H}, A_\ast) \equiv_{AQ} (T, A_\ast) \). If \( (A_\ast, D(a)) \in \mu(T) \& \mu(\mathcal{H}) \) then there is a \( A(b) \) and a subconcept of \( D \) such that \( (\mathcal{H}, A_\ast) \models A(b), T \models A \subseteq C \), and \( \mathcal{H} \not\models A \subseteq C \).

Given \( A \subseteq C \) such that \( T \models A \subseteq C \), and \( \mathcal{H} \not\models A \subseteq C \), one can compute with polynomially many polynomial size queries another CI \( A' \subseteq C' \) entailed by \( T \) but not by \( \mathcal{H} \) belonging to a class of CIs called \( \mathcal{T}\text{-essential} \) (Konev et al. 2018, Lemma 29). In fact this can be done in polynomial time given the complexity of entailment checking (Baader, Lutz, and Brandt 2008) (no inverse roles). Such \( \mathcal{T}\text{-essential} \) CIs have the property that their size is bounded polynomially in the size of \( T \) (Konev et al. 2018, Lemma 32) and if \( \alpha_1 = A' \sqsubseteq C_1 \) and \( \alpha_2 = A' \sqsubseteq C_2 \) are \( \mathcal{T}\text{-essential} \) and not equivalent then one can compute in polynomial time a \( \mathcal{T}\text{-essential} \) CI \( A' \subseteq C' \) such that it entails \( \alpha_1 \), but not \( \alpha_2 \) (Konev et al. 2018, Lemma 30). All in all, if the learner computes such \( \mathcal{T}\text{-essential} \) counterexamples and adds/refines them in the hypothesis (see (Konev et al. 2018, Algorithm 2)) then, after learning from polynomially many counterexamples, it will terminate and output a hypothesis IQ-inseparable from the target (w.r.t. \( A_\ast \)). The presence of CIs of the form \( C \subseteq A \) does not affect this result (Duarte, Konev, and Ozaki 2018).

**Theorem 6.** \( \mathfrak{g}(\mathcal{E} \mathcal{L} \mathcal{H}, A_\ast) \) is in \( \mathsf{PTIME} \).

In contrast, the learning framework \( \mathfrak{g}(\mathcal{E} \mathcal{L} \mathcal{H}, \mathcal{I} \mathcal{Q}) \) is not in \( \mathsf{PQUERY} \) (Konev, Ozaki, and Wolter 2016). We observe that the counterexamples used in such hardness proof are based on an exponential number of ABoxes encoding concept expressions of the form \( C_b = \bigcap_{i \leq n} C_i \), where \( b = b_1 \ldots b_n \) is a sequence with \( b_i \in \{0, 1\} \), \( C_i = A_i \) if \( b_i = 1 \), and \( C_i = B_i \) if \( b_i = 0 \). In the OMQA setting, the ABox is fixed and, as stated in Theorem 6, this lowers the complexity.

**Learning \( \mathcal{E} \mathcal{L} \mathcal{H} \) ontologies with \( \mathcal{CQ} \).** \( \mathcal{E} \mathcal{L} \mathcal{H} \) ontologies are not polynomial query learnable in the data retrieval setting with CQs as the query language (in fact not even the fragment \( \mathcal{E} \mathcal{L} \mathcal{H}_{\mathcal{I} \mathcal{Q}} \)) (Konev, Ozaki, and Wolter 2016). The counterexamples used by the oracle in the hardness proof are of the form \( \{A(a)\}, q \) (Konev, Ozaki, and Wolter 2016, proof of Lemma 8), so \( \{A(a)\} \) can be considered as the fixed ABox given as part of the input in an OMQA learning framework. Thus, the mentioned hardness result can be transferred to our setting. We formalise this result with the next theorem.
bismutation. Let $I = (\Delta^I, \cdot^I), J = (\Delta^J, \cdot^J)$ be two interpretations. A bimutation is a non-empty relation $Z \subseteq \Delta^I \times \Delta^J$ satisfying the following conditions, for all $(d, e) \in Z$: (1) for all concept names $A \in N_c, d \in A^J \iff e \in A^I$; (2) for all role names $r \in N_r, if (d, d') \in r^J, (d', e') \in Z$, then there exists $e' \in \Delta^J$ such that $(e, e') \in r^J$ and $(d', e') \in Z$; (3) for all role names $r \in N_r, if (e, e') \in r^I, e' \in \Delta^J$, then there exists $d' \in \Delta^I$ such that $(d, d') \in r^I$ and $(d', e') \in Z$. If $(d, e) \in Z$, we write $(I, d) \sim (J, e)$.

**Theorem 10.** Let $T$ and $H$ be $\mathcal{ELH}$ terminologies entailing the same RIs, and let $A_1, \ldots, A_n$ be ABoxes. If, for all $b \in \text{ind}(A_i)$, there is a $a \in \text{ind}(A_j)$ such that $(I_{A_i}, a) \sim (I_{A_j}, b)$, then $(H, A_1) \models_T (T, A_2)$ implies $(H, A) \models_T (T, A)$.

Theorem 10 does not hold if we require the ABoxes $A_1, \ldots, A_n$ to be homomorphically equivalent, i.e., if there are ABox homomorphisms from $A_1$ to $A$ and from $A$ to $A_n$.

**Example 11.** Consider $T = \{\exists r.A_1 \subseteq B\}$ and $A_1 = \{r(a, b), A_1(b), A_2(b)\}$. The hypothesis $H = \{\exists r.(A_1 \sqcap A_2) \subseteq B\}$ is IQ-inseparable. However, if $A = A_1 \cup \{r(a', b')\}$, $A_1(b')\}$, then IQ-inseparability is not preserved, even though $A$ and $A_1$ are homomorphically equivalent.

The problem here is that the left-hand side of CIs in $H$ could be biased and too specific for the individuals in $A_i$. Indeed, in Example 11, if the more general concept expression $\exists A_1$ on the left-side had been learned, then IQ-inseparability would have been preserved after the update. If we allow for modifications to the learned hypothesis $H$, we can extend the class of updated ABoxes $A$ not only to those in which every individual in $A_i$ is bisimilar to an individual in $A_i$ but in which a more relaxed condition is required. It is easy for the learner to make certain kinds of generalisation, for instance, check whether $T \models \exists A_1 \subseteq B$ and add such more general CI to $H$. Therefore, the idea is to suitably ‘generalise’ the left-hand side of CIs in the hypothesis $H$ computed by the learning algorithm.

**Generalisation of $C \subseteq A$ in $H$ for $T$ consists of replacing $C$ by the result $C'$ of (1) replacing a concept name $C$ in $T$ by $B'$ such that $T \models B \subseteq B'$ and $T \not\models B' \subseteq B$ if $T \models C \subseteq A$; or (2) replacing a role name $r$ in $C$ with $s$ such that $T \models r \subseteq s$ and $T \not\models s \subseteq r$ if $T \models C' \subseteq A$. We say that $C \subseteq A \in H$ is generalised for $T$ if generalization of $C \subseteq A \in H$ for $T$ has been exhaustively applied. $H$ is generalised for $T$ if all the CIs in it are generalised. We may omit ‘for $T$’ if this is clear from the context.

With the following definitions, we define a class of ABoxes that are guaranteed to preserve IQ-inseparability if the hypothesis is generalised. Given a TBox $T$ and concept names $A, B \in \Sigma_T \cap N_c$, we say that there is a linear derivation from $A$ to $B$ if $T \models A \subseteq B$ and for all $B' \in N_c$ such that $T \models A \subseteq B'$ we have that $T \models B' \subseteq B$. Similarly, for $r, s \in \Sigma_T \cap N_r$, there is a linear derivation from $r$ to $s$ if $T \models r \subseteq s$ and for all $s' \in N_r$ such that $T \models r \subseteq s'$ we have that $T \models s' \subseteq s$. We write $A \prec A'$ if $A'$ is the result of replacing $A(a) \in A$ by $B(a)$ and there is a linear derivation from $A$ to $B$, or if $A'$ is the result of replacing $r(a, b) \in A$ by $s(a, b)$ and there is a linear derivation from $r$ to $s$. We define $g_T(A_i)$ as the set of all ABoxes $A$ such that there is a sequence $A_1 = \ldots < A_n$ with $A_1 = A_i$ and $A_n = A$.

The following theorem establishes an upper bound for learning frameworks extending $S$ with all the examples of the form $(A, q)$, where $A \in g_T(A_i)$ and $q \in \mathcal{IQ}$.

**Theorem 12.** Let $\mathcal{F}$ be the learning framework that results from adding all pairs of the form $(A, q)$, with $A \in g_T(A_i)$ and $q \in \mathcal{IQ}$, to the set $S$ in $\mathcal{F}(\mathcal{ELH}, A_i, \mathcal{IQ}) = (E, S, L, \mu)$, where $T \in L$. Assume $\Sigma_T \subseteq \Sigma_A$. Then, $\mathcal{F}$ is in $\mathcal{PTIME}$.  

**Learning from Data**

The existence of oracles that correctly answer to all the queries posed by the learner does not naturally fit those settings in which only a direct access to data is available. In this section, we study how the oracle-based approach presented so far can be modified so to allow access to examples retrieved from data, thus reducing the dependency of our learning model on membership and inseparability queries. Firstly, we consider a finite batch of examples (Arias, Khardon, and Maloberti 2007), to be used as a representative of the entire data pool, and study conditions under which it is guaranteed the existence of such a batch that allows us to learn inseparable ontologies. Then, we analyse how a data-driven approach can be used as a basis for a learning model for DL ontologies based on the well-known PAC learning model, possibly extended with membership queries (Valiant 1984).

**Learning from batch.** Given an OMQA learning framework $\mathcal{F}(E, A_i, \mathcal{Q}) = (E, S, L, \mu)$, a batch $B$ is a finite subset of $E$. One could ask under which conditions a batch that allows us to construct an ontology $H \in L$ which is IQ-inseparable from a target $T \in L$ is guaranteed to exist. If no restrictions are imposed on the form of the examples occurring in $B$, the answer is trivial for $\mathcal{EL}$ ontologies and IQs. Indeed, for every $C \subseteq D \in T$, consider the set $B$ of examples of the form $(A_C, D(\rho_C))$, obtained by representing the concept $C$ as a labelled tree with root $\rho_C$ and encoded in the ABox $A_C$. These examples have the property that, for every $T \in L$, $T \models C \subseteq D$ iff $(T, A_C) \models D(\rho_C)$. By setting

$$H = \{C \subseteq D \mid (A_C, D(\rho_C)) \in B\},$$

we obtain that $H$ is equivalent to (and thus IQ-inseparable, w.r.t. any ABox, from) $T$. This construction can be easily extended to $\mathcal{ELH}$ by using examples of the form $\{r(a, b)\}, s(a, b)\}$. However, instead of allowing the examples retrieved from our data to have no restrictions in their size and in the shape of the ABoxes, it would be more natural to require that these ABoxes contain less information than $A_i$, given as a parameter. This intuition can be made precise by imposing that, for each ABox $A$ in the batch, there is an ABox homomorphism from $A$ to $A_i$, and $|A|$ is polynomial in $|T|$. Our next theorem states that, under these assumptions, one can construct a $\mathcal{Q}$-inseparable $\mathcal{ELH}$ termology.

**Theorem 13.** Let $\mathcal{F}(\mathcal{ELH}, A_i, \mathcal{Q}) = (E, S, L, \mu)$ be an OMQA learning framework, with $Q \in \{\mathcal{AQ}, \mathcal{IQ}, \mathcal{CQ}\}$, and let $T \in L$ be such that $\Sigma_T \subseteq \Sigma_A$. Let $X \subseteq E$ be the set of examples $(A, q)$ such that there is an ABox homomorphism from $A$ to $A_i$. Then, there is a batch $B \subseteq X$, polynomial in $|T|$, and an algorithm such that it takes $B$ as input, it eventually halts, and returns $H \in L$ such that $\mu(H) \cap S = \mu(T) \cap S$.  

\[\mu(H) \cap S = \mu(T) \cap S.\]
PAC learning. Let \( \mathcal{F} = (\mathcal{E}, \mathcal{S}, \mathcal{L}, \mu) \) be a learning framework. A probability distribution \( D \) on \( S \) is a function \( D : 2^S \to [0, 1] \subset \mathbb{R} \) such that \( \sum_{x \in S} D(x) = 1 \). Given a target \( t \in \mathcal{L} \), let \( EX^D_{\mathcal{F}, t} \) be the oracle that takes no input, and outputs a classified example \( (e, \ell_t(e)) \), where \( e \in S \) is sampled according to the probability distribution \( D, \ell_t(e) = 1 \) if \( e \in \mu(t) \cap S \) (positive example), and \( \ell_t(e) = 0 \), otherwise (negative example). An example query is a call to the oracle \( EX^D_{\mathcal{F}, t} \). A sample generated by \( EX^D_{\mathcal{F}, t} \) is a (multi-)set of indexed classified examples, independently and identically distributed according to \( D \), sampled by calling \( EX^D_{\mathcal{F}, t} \). A learning framework \( \mathcal{F} \) is \textit{PAC learnable} if there is a function \( f : (0,1)^2 \to \mathbb{N} \) and a deterministic algorithm such that, for every \( \epsilon, \delta \in (0,1) \subset \mathbb{R} \), every probability distribution \( D \) on \( S \), and every target \( t \in \mathcal{L} \), given a sample of size \( m \geq f(\epsilon, \delta) \) generated by \( EX^D_{\mathcal{F}, t} \), the algorithm always halts and outputs \( h \in \mathcal{L} \) such that with probability at least \( (1 - \delta) \) over the choice of \( m \) examples in \( S \), we have that \( D((\mu(h) \cap \mu(t)) \cap S) \leq \epsilon \). If the time used by the algorithm is bounded by a polynomial function \( p(|t|, |e|, 1/\epsilon, 1/\delta) \), where \( e \) is the largest example in the sample, then we say that \( \mathcal{F} \) is \textit{polynomial time PAC learnable}. If, in addition, the algorithm is allowed to make membership queries (where each call to \( MQ_{\mathcal{F}, t} \) counts as one step of computation), we say that \( \mathcal{F} \) is \textit{polynomial time PAC learnable with membership queries}.

Theorem 14 ((Angluin 1988), (Mohri, Rostamizadeh, and Talwalkar 2012)). If \( \mathcal{F} \) is in \texttt{PT1MEL}, then \( \mathcal{F} \) is polynomial time PAC learnable with membership queries.

However, the converse direction of Theorem 14 does not hold (Blum 1994). The argument in the mentioned paper is based on the assumption that one-way functions exist and cannot be easily adapted to serve as a counterexample for OMQA learning frameworks. Our next result, showing that the converse direction of Theorem 14 does not hold in our setting, does \textit{not} rely on cryptographic assumptions, however, it is representation-dependent. Given a sequence \( \sigma = \sigma_1 \sigma_2 \ldots \sigma_n \), with \( \sigma_j \in \{r, s\} \), let the expression \( \exists \sigma.C \) stand for \( \exists \sigma_1 \exists \sigma_2 \ldots \exists \sigma_n.C \) (clearly, there are \( 2^n \) expressions of this form). Consider the OMQA learning framework \( \mathcal{F}(L, A_*, Q) \) where \( A_* = \{ A(a) \} \) and \( Q \) is the query language that extends IQs with a CQ of the form \( \exists r M(x) \); and \( L \) is an ontology language allowing only \( E\mathcal{L}H_{\mathrm{mho}} \) TBoxes of the form \( T_{\sigma} = \{ A \sqsubseteq \exists \sigma.M \} \) or \( T_{\sigma} \) to be expressed, where

\[
T_0 = \{ A \sqsubseteq X_0, M \sqsubseteq \exists r.M \sqsubseteq \exists s.M \} \\
\{ X_i \sqsubseteq \exists r.X_{i+1} \cap \exists s.X_{i+1} | 0 \leq i < n \}
\]

It can be shown, with an argument similar to the one used in (Konev, Ozaki, and Wolter 2016, proof of Lemma 8) (and Theorem 7 above), that such framework is not polynomial query learnable. However, due to the restrictions on the hypothesis space, it is polynomial time PAC learnable, even without membership queries.

Theorem 15. There is a polynomial time PAC learnable OMQA learning framework that is not in \texttt{PQUERYL}.

A learning framework \( \mathcal{F} = (\mathcal{E}, \mathcal{S}, \mathcal{L}, \mu) \) shatters a set of examples \( X \subseteq S \) if \( |\{x \in \mu(h) \cap \mu(t) \cap S | h \in L\}| = 2^{|X|} \). The \textit{VC-dimension} (Vapnik 1995) of \( \mathcal{F} \), denoted \( \text{VC}(\mathcal{F}) \), is the maximal size of a set \( X \subseteq S \) such that \( \mathcal{F} \) shatters \( X \). If \( \mathcal{F} \) can shatter arbitrarily large sets then \( \mathcal{F} \) has infinite VC-dimension.

Example 16. For \( n \in \mathbb{N} \), let

\[
A^n_i = \{ r(a_i, a_{i+1}), s(a_i, a_i) | 1 \leq i < n \} \cup \{ r(a_n, a_1) \}
\]

Each \( a_i \) can be identified by \( C_i = \exists r^{n-1} . \exists s.T, \) since \( a_i \not\equiv C_i^{\exists n} (\exists r^k) \) is a shorthand for \( k \) nestings of the form \( \exists r \). E.g., \( a_1 \) is the only individual not in \( (\exists r . \exists s . T)^{\exists n^2} \) (see Figure 2). For all \( n \in \mathbb{N} \), \( \mathcal{F}(E\mathcal{L}H, A^n, \text{CQ}) \) shatters \( \{ A^n_i, A(a_i) \} | 1 \leq i \leq n \}. \) This does not hold if we add \( s(a_n, a_0) \) to \( A^n_i \). For discrete cases, in particular, for fragments of first-order logic, the lower bounds obtained with the VC-dimension cannot be larger than the size of the learned expressions assuming a reasonable encoding scheme (Arias and Khardon 2006). The authors argue that many VC-dimension bounds in the literature showing exponential or infinite growth are in terms of some parameters (number of clauses, etc.) determining the size of the target, while other parameters (number of literals, etc.) are ignored. Let \( \mathcal{F}^{m} = (\mathcal{E}, \mathcal{S}, \mathcal{L}, \mu) \) be a learning framework where the string size of the elements of \( \mathcal{L} \) is bounded by \( m \). Since the VC-dimension is bounded by a logarithm of \( |L| \) (for \( L \) finite), \( \text{VC}(\mathcal{F}^m) = O(m) \) (Vapnik 1995).

Proposition 17. For all \( m \in \mathbb{N} \), \( \mathcal{F}^m(E\mathcal{L}H, A_*, \text{CQ}) \) is PAC learnable with a polynomial number of example queries.

Since the sample complexity (number of classified examples) is polynomial in the size of the target, polynomial time PAC learnability amounts to showing that one can compute a hypothesis in \( \mathcal{L} \) that is consistent with the classification of the examples in polynomial time. However, even if \( A_* \) is fixed, checking whether \( (A_*, q) \) is a positive example for a hypothesis \( \mathcal{H} \) is NP-hard if the underlying structure of \( A_* \) is non-bipartite (Hell and Nešetřil 1990). So (unless \( P = NP \) is not true), there is no hope for polynomial time learnability, even with membership queries, since in this case one may not be able to convert the CQ into an IQ (as we did in Theorem 9).

Conclusion

We introduced the OMQA learning setting and investigated the complexity of learning \( E\mathcal{L}H \) ontologies in this setting with various query languages. We then considered what happens when the data changes and adaptations to settings where the algorithm learns from classified data, limiting interactions with oracles. Our positive result for IQ-inseparable \( E\mathcal{L}H \) TBoxes paves the way for further studies.
on the complexity of learning ontologies formulated in more expressive languages. We leave the problem of exactly learning $\mathcal{EL}$ TBoxes with $\mathcal{CQ}$'s as an open problem. Learning with a more expressive query language is not easier because the oracle can formulate counterexamples which are not informative. Neither it is more difficult because on the other hand, with a more expressive language, the learner can pose more informative membership queries. It would also be interesting to investigate a similar data model in which the ABox is fixed for all the examples, so that the data pool contains examples in the form of queries alone.

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Section “Polynomial Learnability”

We introduce some basic definitions and lemmas that will be used in our proofs. We are going to use the definition of the tree interpretation of a concept, the notion of a canonical model, of a homomorphism, and of a simulation.

**Definition 18 (Canonical model of an ABox).** The canonical model $\mathcal{I}_A = (\Delta^{A,\mathcal{I}_A})$ of an ABox $A$ is defined as follows:

- $\Delta_A = \text{ind}(A)$;
- $a^T = a$ for all individuals $a \in \text{ind}(A)$;
- $A^A = \{a \mid A(a) \in A\}$;
- $r^A = \{(a,b) \mid r(a,b) \in A\}$.

A path in an $\mathcal{ELH}$ concept expression $C$ is a finite sequence of the form $C_0 \cdot r_0 \cdot C_1 \cdot r_1 \cdots r_n \cdot C_n$ where $C_i$ is a concept expression, $r_i$ is a role name, $C_0 = C$, and, for all $i \in \{0, \ldots, n-1\}$, the concept expression $\exists r_{i+1}, C_{i+1}$ is a top-level conjunct of $C_i$. The set paths$(C)$ contains all paths in $C$. The set tail$(p) = \{A \mid A \in \text{N}_C$ is a top-level conjunct of $C_k$ where $C_k$ is the last concept expression in $p\}$.

**Definition 19 (Tree interpretation).** Let $C$ be an $\mathcal{ELH}$ concept expression. The tree interpretation $\mathcal{I}_C = (\Delta^{C,\mathcal{I}_C})$ of a concept $C$ is defined as follows:

- $\Delta^{C} = \text{paths}(C)$;
- $A^{C} = \{p \in \text{paths}(C) \mid A \in \text{tail}(C)\}$;
- $r^{C} = \{p \in \text{paths}(C) \times \text{paths}(C) \mid \text{there exists } p' = p \cdot r \cdot D$ and $p, p' \in \Delta^{C}\}$ for some concept expression $D$.

We may denote the root path of $\mathcal{I}_C$ with $\rho_C$.

**Definition 20 (Canonical model of a KB).** The canonical model $\mathcal{I}_T (\mathcal{I}_A)$ of an $\mathcal{ELH}$ KB $(T, A)$ is defined inductively. For each $r \in \text{N}_R$, $\mathcal{I}_0$ is defined by extending the canonical model of an ABox $A$ with

$$r^{I_0} = \{(a, b) \mid s(a, b) \in A \text{ and } T \models s \subseteq r\}.$$  

Assume that $\mathcal{I}_r$ has been defined. We define $\mathcal{I}_{r+1}$ as follows. If there exist $C \subseteq D \in T$ such that $p \in \Delta^{I_r}, p \in C^{I_r}, p \notin D^{I_r}$ and $D = \bigcap\{A_j \mid \exists 1 \leq j' \leq \exists s \cdot r_j. D_j\}$, we create $\mathcal{I}_{r+1}$ in the following way:

- $\Delta^{I_{r+1}} = \Delta^{I_r} \cup \{p \cdot q. \cdot \cdot \cdot \cdot q \mid q \in \text{paths}(D_{j'}) \}$;
- $A^{I_{r+1}} = A^{I_r} \cup \{p \mid A_j = A, 1 \leq j \leq j'\}$;
- $r^{I_{r+1}} = r^{I_r} \cup \{(p \cdot q. \cdot \cdot \cdot q. \cdot q') \mid (q, q') \in \Delta^{I_r} \}$.

Finally, the canonical model of a KB $(T, A)$ is $\mathcal{I}_T, A = \bigcup_{r \geq 0} \mathcal{I}_r$.

**Definition 21 (Homomorphism).** Let $\mathcal{I}$ be an interpretation and let $T_C = (\mathcal{V}_C, \mathcal{E}_C, C)$ be the tree representation of concept expression $C$. A homomorphism $h : T_C \to \mathcal{I}$ is a function from $\text{V}_C$ to $\Delta^{C}$ such that

- for all $\nu \in \text{V}_C$ and $A \in \text{N}_C$, if $A \in \text{I}_C(\nu)$ then $h(\nu) \in \Delta^A$;
- for all $\nu_1, \nu_2 \in \text{V}_C$ and $r \in \text{N}_R$, if $l_C(\nu_1, \nu_2) = r$ then $(h(\nu_1), h(\nu_2)) \in r^A$.

**Definition 22 (Simulation).** Let $\mathcal{I} = (\Delta^I, T)$, $\mathcal{J} = (\Delta^J, J)$ be two interpretations. A simulation $S$ is a non-empty relation $S \subseteq \Delta^I \times \Delta^J$ satisfying the following properties:

- for all concept names $A \in \text{N}_C$ and all $(d,e) \in S$ it holds that $d \in A^I$ then $e \in A^J$;
- for all role names $r \in \text{N}_R$, if $(d,e) \in S$ and $(d,d_1) \in r^I, d_1 \in \Delta^I$ then there exists $e_1 \in \Delta^J$ such that $(e, e_1) \in r^J$ and $(d_1, e_1) \in S$.

If $\mathcal{I}$ and $\mathcal{J}$ are tree interpretations, we write $\mathcal{I} \Rightarrow \mathcal{J}$ if there is a simulation $S \subseteq \Delta^I \times \Delta^J$ containing the pair with the roots of $\mathcal{I}$ and $\mathcal{J}$.

The following standard lemmas are going to be used throughout the remainder of this section in order to show properties of the algorithms.

**Lemma 23.** If $(I_1, d_1) \sim (I_2, d_2)$, then the following holds for all $\mathcal{ELH}$ concept expressions $C$: $d_1 \in C^{I_2}$ iff $d_2 \in C^{I_2}$. Moreover, given an arbitrary TBox $T$ and ABoxes $A_1$ and $A_2$, $(I_{A_1}, d_1) \sim (I_{A_2}, d_2)$ implies that, for all $\mathcal{ELH}$ concept expressions $C$: $d_1 \in C^{I_{AC}}$ iff $d_2 \in C^{I_{AC}}$.

**Lemma 24.** Let $C$ be an $\mathcal{ELH}$ concept expression and let $\mathcal{I}$ be an interpretation with $d \in \Delta^I$. Then, $d \in \Delta^C$ if, and only if, there is a homomorphism $h : T_C \to \mathcal{I}$ such that $h(\rho_C) = d$.

**Lemma 25.** Let $C$ be an $\mathcal{ELH}$ concept expression, $T$ a TBox and $A$ an ABox. The following statements are equivalent:

1. $T \models C \subseteq D$;
2. $\rho_C \in D^T \cap r_e$;
3. There is a homomorphism $h : T_D \to T_{C, A}$ such that $h(\rho_D) = \rho_C$.

**Learning $\mathcal{ELH}$ ontologies with AQ**

We now present in full detail a learning algorithm for $\mathcal{ELH}, A_*, \text{AQ}$). Our algorithm is based on the approach used to learn a fragment of $\mathcal{ELH}$ where complex concept expressions are only allowed on the left side of inclusions (Konev, Ozaki, and Wolter 2016). Algorithm 1 shows the steps that the learner should do to learn an AQ-inseparable TBox. It first computes an initial part of the hypothesis $\mathcal{H}$ which consists of all RIs and all CIs with concept names on both sides entailed by the target ontology $T$. This initial $\mathcal{H}$ is constructed by asking $O(|S_T|^2)$ membership queries. More precisely, in this phase the learner calls $M\mathcal{Q}_{\mathcal{H}}(\mathcal{ELH}, A_*, \text{AQ})$ with $\{(A(a))\}$, $B(a))$ as input, for all $A, B \in \Sigma_T \cap \text{N}_C$. If the answer is positive, the learner adds $A \subseteq B$ to $\mathcal{H}$. In this way the learner adds to $\mathcal{H}$ all CIs of the form $A \subseteq B$ entailed by $T$. After that, similarly, for each combination $r, s \in \Sigma_T \cap \text{N}_R$, the learner calls $M\mathcal{Q}_{\mathcal{H}}(\mathcal{ELH}, A_*, \text{AQ}) \mid T$ with $\{(r_{\text{a}}, b)\}$ as input and, if the answer is positive, adds $r \subseteq A$ to $\mathcal{H}$. As a consequence, the learner learns all RIs and the next counterexamples, if any, will only be of the form $(A_*, A(a))$.

After this initial step, the learner enters in a while loop and asks inseparability queries in order to check if it has already found an AQ-inseparable hypothesis. If this is true, it stops and returns an AQ-inseparable TBox, otherwise it gets a positive counterexample $(A_*, A(a))$ from the oracle, which is used to add missing knowledge to the hypothesis $\mathcal{H}$. The assumption
in Line 3 that counterexamples are positive is justified by the fact that the algorithm maintains the invariant that $T \models H$. Indeed, the algorithm adds only CIs that are logical consequences of the target $T$. This property holds before the first inseparability query is asked and it remains true when the next (positive) counterexamples CIs are received. When a new counterexample $(A, A(a))$ is received, the learner should find a tree shaped ABox $A$ rooted in $\rho$ obtained from $A$, such that there is $B \in \Sigma_T \cap N_C$ with $(T, A) \models B(\rho)$ and $(H, A) \not\models B(\rho)$, in order to add an informative CI $C_A \subseteq B$ to $H$. We would like $|C_A|$ to be polynomial in $|T|$ so that $|H|$ is polynomial in $|T|$.

**Algorithm 2: TreeShape**

**Input:** Signature $\Sigma_T$, ABox $A$, TBox $H$

**Output:** ABox $A$

1. $A \leftarrow$ Minimize($\Sigma_T, A, H$)
2. while there is a cycle $c \in A$
   3. $A \leftarrow$ Unfold($\Sigma_T, c, A$)
   4. $A \leftarrow$ Minimize($\Sigma_T, A, H$)
3. return $A$

**Algorithm 3: Minimize**

**Input:** Signature $\Sigma_T$, ABox $A$, TBox $H$

**Output:** ABox $A$

1. Saturate $A$ with $H$
2. foreach $A \in \Sigma_T \cap N_C$ and $a \in \text{ind}(A)$ such that $(T, A) \models A(a)$ and $(H, A) \not\models A(a)$
   3. Domain Minimize $A$ with $A(a)$
   4. Role Minimize $A$ with $A(a)$
3. return $A$

In Line 4 of Algorithm 1, Algorithm 2 is called in order to find a tree shaped ABox. “Unfold” (defined next), doubles cycles in $A$ and “Minimize” removes redundant assertions (Konev, Ozaki, and Wolter 2016).

**Unfold.** We say that $A$ has a (undirected) cycle if there is a finite sequence $a_0 \cdot r_1 \cdot a_1 \cdot \ldots \cdot r_k \cdot a_k$ such that (i) $a_0 = a_k$ and (ii) there are mutually distinct assertions of the form $r_{i+1}(a_i, a_{i+1})$ or $r_{i+1}(a_{i+1}, a_i)$ in $A$, for $0 \leq i < k$. For a cycle $c = a_0 \cdot r_1 \cdot a_1 \cdot \ldots \cdot r_k \cdot a_k$, denote as nodes($c$) = $\{a_0, a_1, \ldots, a_k\}$ the set of individuals that occur in $c$. We denote by $\hat{a}$ the copy of an element $a$ created by the unfolding cycle operation described below. The set of copies of individuals that occur in $c$ is denoted by nodes($\hat{c}$) = $\{\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_{k-1}\}$. Let $\mathcal{I}_A$ be the canonical interpretation of an ABox $A$. An element $a \in \Delta^T_A$ is folded if there is a cycle $c = a_0 \cdot r_1 \cdot a_1 \cdot \ldots \cdot r_k \cdot a_k$ with $a = a_0 = a_k$. Without loss of generality we assume that $r_1(a_0, a_1) \in A$. The cycle unfolding of $c$ is described below.

1. We first open the cycle by removing $r_1(a_0, a_1)$ from $A$. So $r_1^A := r_1^A \setminus \{(a_0, a_1)\}$.
2. Then we create copies of the nodes in the cycle:
   - $\Delta^T_A := \Delta^T_A \cup \{\hat{b} \mid b \in \text{nodes}(c)\}$
   - $A^T_A := A^T_A \cup \{\hat{b} \mid b \in A^T_A\}$
   - $r^T_A := r^T_A \cup \{(\hat{b}, \hat{d}) \mid (b, d) \in r^T_A\} \cup \{(\hat{b}, e) \mid (b, e) \in r^T_A, e \notin \text{nodes}(c)\}$
3. As a third step we close again the cycle, now with double size. So we update $r_1^T_A := r_1^T_A \cup \{(a_0, \hat{a}_1), (\hat{a}_0, a_1)\}$.

**Minimize.** Given an ABox $A$, we denote by $A^{-\alpha}$ the result of removing assertion $\alpha$ from $A$. Also, given $a \in \text{ind}(A)$, we denote by $A^{-a}$ the result of removing from $A$ all ABox assertions in which $a$ occurs. The ABox transformation rules (exhaustively applied) in Algorithm 3 are defined as follows:

- (Saturate $A$ with $H$) Let $A$ be an ABox, let $H$ be a TBox, and let $\alpha$ be an assertion built from symbols in $\Sigma_T$ and $\text{ind}(A)$. If $a \notin A$ and $(H, A) \not\models \alpha$ then replace $A$ with $A \cup \{\alpha\}$;
- (Domain minimize $A$ with $A(a)$) If $(A, A(a))$ is a positive example and $(T, A^{-b}) \models A(a)$ then replace $A$ by $A^{-b}$ where $b \in \text{ind}(A)$;
- (Role minimize $A$ with $A(a)$) If $(A, A(a))$ is a positive example and $(T, A^{-r(b_1, b_2)}) \models A(a)$ then replace $A$ by $A^{-r(b_1, b_2)}$.

We say that an ABox is minimal if there is $A \in \Sigma_T \cap N_C$ and $a \in \text{ind}(A)$ such that $(T, A) \models A(a)$ and $\not\models B(b)$ and the rules ‘Domain minimize $A$ with $A(a)$’ and ‘Role minimize $A$ with $B(b)$’ have been exhaustively applied with all $B \in \Sigma_T \cap N_C$ and all $b \in \text{ind}(A)$.

**Lemma 26.** (Konev, Ozaki, and Wolter 2016, Lemma 49) Let $A'$ be the result of unfolding a cycle $c$ in $A$. Then the following relation $S \subseteq \Delta^T_A \times \Delta^{T'}$ is a simulation $\mathcal{I}_A \Rightarrow \mathcal{I}_A'$:

- for $a \in \Delta^T_A \setminus \text{nodes}(c)$, $(a^T_A, a^T_A') \in S$;
- for $a \in \text{nodes}(c)$, $(a^T_A, a^T_A') \in S$ and $(a^{\mathcal{I}_A}, a^{\mathcal{I}_A'}) \in S$.

**Lemma 27.** (Konev, Ozaki, and Wolter 2016, Lemma 50) Let $A'$ be the result of unfolding a cycle $c$ in $A$. Let $h_* : \mathcal{I}_A \to \mathcal{I}_A$ be the following mapping:

- for $a \in \Delta^T_A \setminus \text{nodes}(c)$, $h_*(a^{\mathcal{I}_A'}) = a^{\mathcal{I}_A}$;
- for $a \in \text{nodes}(c)$, $h_*(a^{\mathcal{I}_A'}) = a^{\mathcal{I}_A}$ and $h_*(\hat{a}^{\mathcal{I}_A'}) = a^{\mathcal{I}_A}$.
Then, \( h_A : \mathcal{I}_A \to \mathcal{I}_A \) is a homomorphism.

**Remark 1.** In the domain and role minimization steps, we assume that the algorithm always attempts to remove the ’clones’ (that is, elements in nodes(\( \widehat{c} \)) generated in the unfold step first (that is, before attempting to remove elements of \( \text{ind}(A_*)) \). By Lemmas 26 and 27 this assumption is w.l.o.g. This means that we can assume that there is a positive example \((A, A(a))\) with \( a \in \text{ind}(A_*)) \) whenever an ABox \( A \) is returned by Algorithm 2.

**Lemma 28.** If \((T, A) \models A(a) \) and \((H, A) \not\models A(a) \) where \( A \) is the output of Algorithm 3, then \((A_*, A(a)) \) is a positive counterexample.

**Proof.** Assume \((T, A) \models A(a) \) and \((H, A) \not\models A(a) \) where \( A \) is the output of Algorithm 3. By Lemma 27 and the fact that minimization only removes nodes and role assertions from \( A \), there is an ABox homomorphism from \( A \) to \( A_* \), and so, \((T, A_* \backslash A) \models A(a) \) (by Remark 1, we can assume that \( a \in \text{ind}(A_*)) \). We need to show that \((H, A_* \backslash A) \not\models A(a) \). In Line 1 the algorithm saturates \( A_* \) with \( H \). This means that if \((H, A_* \backslash A) \models A(a) \) then \( A(a) \in A_* \) (after saturating \( A_* \) with \( H \)). By construction of \( A_* \), if \( A(a) \in A_* \) then \( A(a) \in A \), which contradicts the fact that \((H, A) \not\models A(a) \). So \((H, A_* \backslash A) \not\models A(a) \).

By Lemma 28 and the fact that \( A \) is tree-shaped, we can see that one can compute a CI \( C_A \subseteq B \) such that \((T, A_* \backslash A) \models B(b), (H, A) \not\models B(b) \), and \((H \cup \{ C_A \subseteq B \}, A_* \backslash A) \models B(b) \), for some \( b \in \text{ind}(A_*)) \). We now show that one can compute such CI in polynomial time in \( |A_*| \) and \(|T| \).

**Lemma 29.** For any \( \mathcal{ELH} \) target \( T \) and any \( \mathcal{ELH} \) hypothesis \( H \) with size polynomial in \(|T| \) given a positive counterexample \((A_*, A(a)) \), Algorithm 3 terminates in polynomial time in \(|A_*| \) and \(|T| \) .

**Proof.** Entailment in \( \mathcal{ELH} \) is in PTIME (Baader et al. 2007a) and saturating an ABox with \( H \) does not require any query to the oracle, thus all steps in saturation with \( H \) can be computed in polynomial time. For each concept name \( A \in \Sigma_T \), domain minimization asks at most \(|\text{ind}(A_*)) \) membership queries and role minimization asks at most \(|A_*| \) membership queries (each query to the oracle counts as one computation step)

We now show that Algorithm 2 also runs in polynomial time in \(|A_*| \) and \(|T| \). In each iteration, unfold adds new individuals to the ABox and minimize removes individuals. To show that the algorithm terminates (in polynomial time), we show that on one hand the size of an ABox after minimization is polynomial in \(|T| \). On the other hand, we show that the time the algorithm unfolds the number of individuals after minimization is larger than before the unfolding step (see (Konev, Ozaki, and Wolter 2016)).

**Lemma 30.** Let \( T \) be an \( \mathcal{ELH} \) ontology, \( C' \) a concept expression and \( A \) a concept name such that \( \emptyset \not\models C' \subseteq A \). If \( T \models C' \subseteq A \) then there is \( C \subseteq A \in T \) such that \( T \models C' \subseteq C \).

**Proof.** Let \( \mathcal{I}_{T, C'} \) be the canonical model of \( C' \) with respect to \( T \). We have that \( \rho_{C'} \in \mathcal{I}_{T, C'} \) because \( T \models C' \subseteq A \). Since \( \emptyset \not\models C' \subseteq A \), by the construction of the canonical model, there is \( C \subseteq A \in T \) such that there is a homomorphism \( h : T_C \to \mathcal{I}_{T, C'} \) with \( h(\rho_{C'}) = \rho_{C'} \). Then, by Lemma 25, \( T \models C' \subseteq C \). Given a concept expression \( C \), we write \( C' \prec C \) if \( C' \) is the concept expression corresponding to the tree that results from replacing a subtree \( T_D \) of \( T_C \) by a concept name \( A \) such that \( T_D \models A \subseteq D \). We write \( C' \prec^* C \) if there is a sequence \( C_1 \prec C_2 \prec \ldots \prec C_n \) with \( C_1 = C' \), \( C_n = C \) and \( n > 1 \). By this definition, the following lemma is straightforward.

**Lemma 31.** For all concept expressions \( C', C \), we have that \( C' \prec^* C \) if \( T \models C' \subseteq C \). Moreover, \( |C'| \leq |C| \).

**Lemma 32.** Let \( A \) be a minimal ABox. Then \(|\text{ind}(A)| \leq |T| \).

**Proof.** Let \( A \) be the ABox returned by Algorithm 3. Then there is \( A(a) \) such that \((T, A) \models A(a) \) and \((H, A) \not\models A(a) \). This means that there is a CI \( C' \subseteq A \) such that \((T, A_* \backslash A) \models C' \subseteq A \), \( H \not\models C' \subseteq A \) and \( a \in (C' \setminus A)^{T_A} \). By Lemma 30, there is \( C \subseteq A \in T \) such that \((T, A_* \backslash A) \models C' \subseteq A \). By Lemma 31, \( C' \prec^* C \) and \(|C'| \leq |C| \). If \((T, A_{c_{\leq}}) \models A(a) \) then by Lemma 25 there is a homomorphism \( h : \mathcal{I}_{C'} \rightarrow \mathcal{I}_A \) mapping \( \rho_{C'} \) to \( a \), where \( \rho_{C'} \) is the root of \( T_C \). Since \(|C'| \leq |C| \), we only need to show that \( h \) is surjective. Suppose this is not the case. Then, there is \( b \in \Delta^{T_A} \) such that \( b \not\models \text{Im}_h \), where \( \text{Im}_h = \{ e \in \Delta^{T_A} | e = h(p) \text{ for some } p \in \Delta^{T_{C'}} \} \). Denote as \( A_{b_{\leq}} \) the result of removing \( b \not\models \text{Im}_h \) from \( A \). Since \( A_{b_{\leq}} \) is a subinterpretation of \( A \), if \( a \not\in A^{T_A} \) then \( a \not\in A^{T_A+b_{\leq}} \). So \( a \in (C' \setminus A)^{T_A+b_{\leq}} \), which means that \((T, A_{b_{\leq}}) \models A(a) \) while we still have \((H, A_{b_{\leq}}) \not\models A(a) \). This contradicts the fact that \( A \) is minimal. Thus, \(|\text{ind}(A)| \leq |\Delta^{T_{C'}}| \leq |\Delta^{T_{C}}| \leq |T| \).

**Lemma 33.** Let \( A_n \) be the minimal ABox computed in the \( n \)-th iteration in Line 4 of Algorithm 2. Assume \( A_n \) has a cycle. For all \( n \geq 0 \), \(|\text{ind}(A_{n+1})| > |\text{ind}(A_n)| \).

**Proof.** The argument is similar to the one presented in (Konev, Ozaki, and Wolter 2016, Lemma 51). There is an ABox homomorphism from \( A_{n+1} \) to \( A_n \), which is surjective because otherwise \( A_n \) would not be minimal. This homomorphism is not injective because at least one element of a cycle in \( A_n \) has been unfolded and remained in \( A_{n+1} \) after minimization.

Lemmas 32 and 33 bound the number of iterations of Algorithm 2. We already argued in Lemma 29 that Algorithm 3 terminates in polynomial time in \(|T| \) and \(|A_*| \) and it is easy to see that unfolding can also be performed in polynomial time (first it doubles a cycle in \( A_n \) and in subsequent iterations it doubles a cycle in an ABox of size polynomial in \(|T| \)). Thus, Algorithm 2 also runs in polynomial time.

**Learning \( \mathcal{ELH} \) ontologies with IQ**

Our proof strategy is based on the proof for the learning framework \( \mathcal{ELH} \) (Konev, Ozaki, and Wolter 2016). We assume w.l.o.g. that the target \( T \) does not entail non-trivial role equivalences (that is, there are no distinct \( r \) and \( s \) such that \( T \models r \subseteq s \) and \( T \models s \subseteq r \)). This simplifies our presentation, in particular, Line 8 of Algorithm 4 relies on it. The assumption can be dropped using the notion of a representative role (see (Konev et al. 2018, Theorem 34)).

The learning algorithm for \( \mathcal{ELH}(A, \text{IQ}) \) is given by Algorithm 4. In Line 1, Algorithm 4 computes a hypothesis \( H \) that is AQ-inseparable from \( T \) (with only membership queries).
Then, the algorithm iterates in a ‘while loop’ posing inseparability queries regarding the IQ language. Upon receiving a positive counterexample \( \{A, C(a)\} \), some operations are made in order to learn a CI of the form \( A \sqsubseteq D \) entailed by \( T \) but not by \( H \). The assumption in Line 3 that counterexamples are positive is justified by the fact that the algorithm maintains the invariant that \( T \models H \). Indeed, in the OMQA setting, a learning algorithm can always ensure this property by keeping in the hypothesis only CIs \( C \sqsubseteq D \) such that a membership query with \( \{A, D(p(c))\} \) as input has returned ‘yes’. In particular, Algorithm 4 only adds to the hypothesis CIs that are entailed by \( T \).

We now explain the notion of a \( T \)-essential CI (Konev et al. 2018) which appears in Algorithm 4. This notion is based on some operations performed on examples of the form \( \{\{A(a)\}, C(a)\} \) representing CIs \( A \sqsubseteq C \), where \( A \in N_C \), \( a \in N_a \), and \( C \) an arbitrary concept expression. Intuitively, a \( T \)-essential CI is a CI that is in a sense informative for the learning algorithm and its size is bounded by a polynomial in \( T \).

For the ontology language \( \mathcal{ELH} \), the operations are: concept saturation for \( T \), role saturation for \( T \), sibling merging for \( T \) and decomposition on the right for \( T \). We may omit ‘for \( T \)’ if this is clear from the context. We now recall these operations (Konev et al. 2018), adapted to the OMQA setting. In the following, we assume we are given an example of the form \( \{\{A(a)\}, C(a)\} \).

- Concept saturation for \( T \) consists of updating \( C \) with the result \( C' \) of choosing a node \( \nu \) in \( T_C \) and adding a new concept name from \( \Sigma_T \) to the label of \( \nu \) if \( \{\{A(a)\}, C'(a)\} \) is still a positive example. A positive example \( \{\{A(a)\}, C(a)\} \) is concept saturated for \( T \) if concept saturation for \( T \) is applied exhaustively. Similarly, role saturation for \( T \) consists of updating \( C \) with the result \( C' \) of choosing an edge \( (\nu, \nu') \) in \( T_C \) and replacing the role name \( r \) in the label of \( (\nu, \nu') \) by a distinct \( s \in \Sigma_T \) such that \( T \models s \sqsubseteq r \) if \( \{\{A(a)\}, C'(a)\} \) is still a positive example. A positive example \( \{\{A(a)\}, C(a)\} \) is role saturated for \( T \) if role saturation for \( T \) is applied exhaustively.

- Example 34. Let the signature be \( \Sigma_T = \{A, B, r, s\} \), the target be \( T = \{B \sqsubseteq A, A \sqsubseteq \exists s.B, s \sqsubseteq r\} \), the fixed ABox be \( A_\alpha = \{A(a)\} \), the hypothesis be \( H = \{B \sqsubseteq A, s \sqsubseteq r\} \) and the received counterexample be \( \{\{A(a)\}, \exists r.A(a)\} \).

- Algorithm 4: \( \mathcal{ELH} \) IQ learning algorithm

```plaintext
Input: \( \Sigma_T \), \( T \), \( A_\alpha \) and \( H \)

Output: Hypothesis \( H \)

1. Compute \( H \) such that \( (T, A_\alpha) \equiv_{AQ} (H, A_\alpha) \)
2. \( (T, A_\alpha) \not\equiv_{IQ} (H, A_\alpha) \) do
3. \( \alpha = \text{ReduceCounterexample}(\{A, \forall A, C(a)\}, H) \)
4. Compute a \( T \)-essential CI \( A \sqsubseteq D \) from \( \alpha \)
5. If there is \( A \sqsubseteq D' \) in then
6. Find a \( T \)-essential \( A \sqsubseteq D^* \) such that \( \emptyset \sqsubseteq D^* \sqsubseteq D \sqcap D' \)
7. Replace \( A \sqsubseteq D \) in \( H \) by \( A \sqsubseteq D^* \)
8. end
9. else
10. Add \( A \sqsubseteq D \) to \( H \)
11. end
12. end
13. return \( H \)
```

To show that Algorithm 4 runs in polynomial time (where each call to an oracle counts as one step of computation), we show that (1) the number of iterations is polynomially bounded; and (2) each iteration can be computed in polynomial time. We start by arguing that each iteration requires polynomial many steps (Point 2). Line 5 of Algorithm 4 can be computed in polynomial time using the same algorithm as the one used to learn DL-Lite\(_{R}^\mathcal{I}\) ontologies (Konev, Ozaki, and Wolter 2016), except that here we do not use parent-child merging because \( \mathcal{ELH} \) does not have inverse roles. Line 4 ensures that one can indeed find a singleton ABox, even though \( \mathcal{ELH} \) allows CIs of the form \( C \sqsubseteq A \). For convenience of the reader we provide here the algorithm (Algorithm 5) for refining counterexamples. Algorithm 5 ‘walks inside’ \( A \) and \( C \) in order to find a singleton ABox which together with \( T \) entails a subconcept of \( C \).

- Example 36. Assume \( T = \{A \sqsubseteq \exists r.D\} \), \( H = \emptyset \), and \( A_\alpha = \{r(a, b), A(b)\} \). Let \( (A_\alpha, \exists r.A.D(a)) \) be the positive counterexample received at Line 3 of Algorithm 3. In the next line it calls the function ‘ReduceCounterexample’ (Algorithm 5), with \( (A_\alpha, \exists r.A.D(a)) \) and \( H \) as input. The function makes a recursive call with the positive counterexample \( \{r(a, b), A(b), \exists r.D(b)\} \) and \( H \) as input. In the new function call, Algorithm 5 finds the singleton ABox \( \{A(b)\} \) which satisfies the condition in Line 6 (i.e., \( (T, A(b)) \models \exists r.D(b) \) and returns \( A \sqsubseteq \exists r.D \).
We first argue that Algorithm 5 is implementable. That is, in Line 1, one can indeed assume the existence of a top-level conjunct \( D = \exists r. C' \) of \( C \) such that \((T, A) \models \exists r. C'(a)\) and \((\mathcal{H}, A) \not\models \exists r. C'(a)\). This follows from Lemma 37.

**Lemma 37.** Let \( T \) be an \( \mathcal{ELH} \) TBox and \( A \) an ABox. If \((T, A) \not\models C(a)\) with \( C = \bigcap_{i=1}^{n} D_i \), then there is a \( D_i \) with \( i \in \{1, \ldots, n\} \) such that \((T, A) \not\models D_i(a)\).

**Proof.** If \((T, A) \models D_i(a)\) for all \( i \in \{1, \ldots, n\} \) means that there is a homomorphism from \( h_i : T_{D_i} \to \mathcal{I}_{T, A} \) mapping \( \rho_{D_i} \) to \( a \in \Delta^\mathcal{I}_{T, A} \). It trivially follows that there is a homomorphism \( h : T_{\mathcal{E}} \to \mathcal{I}_{T, A} \) obtained by merging all \( h_i \) into a unique function. By Lemma 25, we reach a contradiction because this would mean that \((T, A) \models C(a)\). \( \square \)

Also, if there is no \( s(a, b) \in A \) such that \( \mathcal{H} \models a \subseteq r \) and \((T, A) \models C'(b)\) (that is, the condition in Line 2 is not satisfied) then one can indeed find a singleton \( \{A(c)\} \subseteq A \) such that \((T, \{A(c)\}) \models D(c)\) in Line 6. This follows from Lemma 40. To prove Lemma 40, we first prove the following two technical lemmas. Let \( I_{T, C} \) be an interpretation, the one-neighbourhood \( N_{I_{T, C}}(a) \) of \( a \in \Delta^{I_{T, C}} \) is the set of concept names \( A \) with \( a \in A^{I_{T, C}} \).

**Lemma 38.** Let \( T \) be an \( \mathcal{ELH} \) terminology and let \( C \) and \( D = \exists r. D' \) be concept expressions. Assume there is a homomorphism \( h : T_D \to I_{T, C} \) such that \( h(\rho_D) = d \in \Delta^{I_{T, C}} \) and the image of the subtree \( T_{D'} \) of \( T_D \) under \( h \) is included in \( \Delta^{I_{T, C}} \). Then there exists \( A \in N_{I_{T, C}}(d) \) such that \((T, A) \models A \subseteq D\).

**Proof.** Since \( T \) is an \( \mathcal{ELH} \) terminology, it contains only CIs of the form \( A \subseteq F \) or \( F \subseteq A \) with \( F \) an \( \mathcal{ELH} \) concept expression and \( A \) a concept name. In the construction of the canonical model (Definition 20), the definition of \( I_{n+1} \) is the result of either
- adding a new element of \( \Delta^{I_{n}} \) to the extension \( A^{I_{n}} \) of a concept name \( A \) (to satisfy a CI of the form \( F \subseteq A \)); or
- identifying the root of the tree interpretation \( T_F \) of a concept expression \( F \) with some element of \( \Delta^{I_{n}} \) in the extension \( A^{I_{n}} \) of a concept name \( A \) (to satisfy a CI of the form \( A \subseteq F \)).

Thus, if there is a homomorphism \( h : T_D \to I_{T, C} \) such that \( h(\rho_D) = d \in \Delta^{I_{T, C}} \) and the image of the subtree \( T_{D'} \) of \( T_D \) under \( h \) is included in \( \Delta^{I_{T, C}} \), this can only be if there is \( A \in N_{I_{T, C}}(d) \) such that \((T, A) \models A \subseteq D\). \( \square \)

**Lemma 39.** If there is a homomorphism \( h : T_C \to I_{T, A} \) for an arbitrary concept expression \( C \), an \( \mathcal{ELH} \) terminology \( T \), an ABox \( A \), and a node \( v' \) in \( T_C \) is mapped to \( a \in \Delta^A \), then all ancestors of \( v' \) are mapped into \( \Delta^A \).

**Proof.** \( \mathcal{ELH} \) does not allow inverse roles. So, if \( v' \) has a parent \( v \), it must be in \( \Delta^A \) because in the construction of canonical model \( I_{T, A} \) only role successors are connected to the elements of \( \Delta^A \). \( \square \)

**Lemma 40.** Let \( T \) be the target, \( \mathcal{H} \) the hypothesis and \( A \) an ABox. If \((T, A) \models \exists r. C(a)\), there is no \( s(a, b) \in A \) such that \( \mathcal{H} \models a \subseteq r \) and \((T, A) \models C(b)\), then there is a singleton ABox \( A' \) such that \((T, A') \models \exists r. C(a)\).

**Proof.** Let \( C' = \exists r. C \), we know that there is a homomorphism \( h : T_{C'} \to I_{T, A} \) such that \( h(\rho_{C'}) = a \in \Delta^A \). If we show that all nodes in \( T_{C'} \) are mapped into \( \Delta^A \), then we are able to conclude that there is a concept name \( A \) such that \((T, A) \models A \subseteq C' \) by using Lemma 38.

Let \( v \) a descendant of \( \rho_{C'} \). For a proof by contradiction we assume that a node \( v \) is mapped by \( h \) to an element in \( \Delta^A \). By Lemma 39, all its ancestors must be mapped into \( \Delta^A \) by \( h \). Remember that \( h(\rho_{C'}) = a \), this means that exists a \( s(a, h(\rho_{C'})) \in A \) such that \( \mathcal{H} \models s \subseteq r \) and \((T, A) \models C(h(\rho_{C'})) \), which contradicts our assumption. \( \square \)

So we have that Algorithm 5 is implementable. It is easy to see that if the input is a positive counterexample then the output is also a positive counterexample (for \( T \) and \( \mathcal{H} \)). Each line of Algorithm 5 can be computed using polynomially many steps. Regarding the recursive calls, we point out that, each time Algorithm 5 makes a recursive call, the example passed as parameter to the function is strictly smaller, so the number of recursive calls is bounded by the size of the counterexample received in Line 3 of Algorithm 4.

Now we have that in Line 5 of Algorithm 4 we obtain a positive counterexample of the form \( \{A(a)\}, C(a)\) in polynomial time. It follows from the proof of Lemma 2 in (Duarte, Konev, and Ozaki 2018) that Line 6 can also be computed in polynomial time, and moreover, the size of resulting CI is polynomial in \( |T| \). Indeed Lemma 35 is an easy consequence of (Duarte, Konev, and Ozaki 2018, Lemma 2) and the only difference in the setting of Lemma 2 and ours is that \( \mathcal{ELH} \) allows RIs. We observe that our notion of \( T \)-essential includes role saturation (as in (Konev et al. 2018)). That is, a positive example \( \{A(a)\}, C(a)\) is \( T \)-essential if it is concept saturated, role saturated, sibling merged and decomposed on the right for \( T \). This notion is also used for CIs. A CI \( A \subseteq C \) is \( T \)-essential if this is the case for \( \{A(a)\}, C(a)\). Role saturation for \( T \) can be implemented in polynomial time (Konev et al. 2018) and the result of Lemma 2 in (Duarte, Konev, and Ozaki 2018) can be easily extended to our case.

The fact that Line 8 is also implementable in polynomial time follows from Lemma 41 below, which is an adaptation of (Duarte, Konev, and Ozaki 2018, Lemma 7 used to show Theorem 2) (see also (Konev et al. 2018, Lemma 30)).
Lemma 41. Let \( A \subseteq C_1 \) and \( A \subseteq C_2 \) be \( T \)-essential positive examples. One can construct a \( T \)-essential \( A \subseteq C \) such that \( \emptyset \models C \subseteq C_1 \cap C_2 \) in polynomial time in \(|C_1| + |C_2|\).

Sketch. The main difference is that here the ontology language is \( \mathcal{ELH} \), which also includes RIs. Since \( A \subseteq C_1 \) and \( A \subseteq C_2 \) are \( T \)-essential, these examples are role saturated. By assumption, the target ontology does not entail non-trivial RIs (see begin of paragraph "Learning \( \mathcal{ELH} \) ontologies with IQ'" in Section 4). Thus, the presence of RIs does not affect the application of sibling merging.

The rest of this subsection is devoted to show that the number of iterations is polynomial in the size of \( T \). In each iteration either a CI is replaced or it is added to the hypothesis. The number of times it is added to the hypothesis is bounded by \(|\Sigma_T|\). Thus, it remains to show that the number of replacements is also bounded polynomially in \(|T|\).

To show this, we use Lemmas 42 and 43 below. Given the tree representation \( T_C \) of a concept \( C \) and an interpretation \( I \), we say that a homomorphism \( h : T_C \rightarrow I \) is an isomorphic embedding for \( T \) if it is injective, \( A \in I(\nu) \) if \( h(\nu) \in AT \) for all concept names \( A \), and for a \( \nu = (\nu_1, \nu_2) \) it holds that \( T \models r \subseteq s \) for all \((h(\nu_1), h(\nu_2)) \in s \).

Lemma 42 (Isomorphic Embedding). Let \( A \subseteq C \) be a \( T \)-essential CI. If \( T \models A \subseteq D \) and \( T \models D \subseteq C \) then any homomorphism \( h : T_C \rightarrow I_{D,T} \) such that \( h(\rho_C) = \rho_D \) is an isomorphic embedding for \( T \).

Sketch. The argument is as in (Konev et al. 2018, Lemma 31). Non-injectivity would contradict that \( A \subseteq C \) is sibling merged (which follows from the assumption that it is \( T \)-essential). The remaining conditions for the notion of isomorphic embedding for \( T \) follow from the fact that \( A \subseteq C \) is concept and role saturated for \( T \) (which again follow from the assumption that the example is \( T \)-essential).

By the Claim of Lemma 33 in (Konev et al. 2018) the following holds.

Lemma 43. If \( A \subseteq C \) is \( T \)-essential, \( T \models A \subseteq C' \) and \( \emptyset \models C' \subseteq C \) and \( \emptyset \not\models C \subseteq C' \), then \( T_C \) is obtained from \( T_{C'} \) by removing at least one subtree.

We are now ready for Lemma 44, which bounds the number of replacements.

Lemma 44. For every \( A \in \Sigma_T \), the number of replacements of a CI in \( H \) in Algorithm 4 is bounded polynomially in \(|T|\).

Proof. All CIs that are added or replace a CI in \( H \) are \( T \)-essential and, therefore, their size is polynomially bounded by \(|T|\) (Lemma 35). As already argued in (Konev et al. 2018, Lemma 33), when \( A \subseteq C \) is replaced with \( A \subseteq C' \), then \( \emptyset \models C' \subseteq C \) and \( \emptyset \not\models C \subseteq C' \). Otherwise the positive counterexample returned by the oracle would be a consequence of \( H \). Moreover, both \( A \subseteq C \) and \( A \subseteq C' \) are consequences of \( T \). So the conditions of Lemma 43 are satisfied, and so, whenever a CI \( A \subseteq C \) is replaced by some other CI \( A \subseteq C' \), we have that \( \Sigma_{C'} > \Sigma_C \).

The presence of CIs of the form \( C \subseteq A \) in our setting does not affect the argument in (Konev et al. 2018, Lemma 33) because the CIs we mention are \( T \)-essential, in particular, they are concept saturated for \( T \), and so, they always satisfy the inclusions of the form \( C \subseteq A \).

This concludes our proof of Theorem 6.

Learning \( \mathcal{ELH} \) ontologies with CQ

We show that one can convert in polynomial time any \( q \in \text{CQ} \), in a positive counterexample of the form \((A_\ast, q)\) into an IQ \( q' \) such that \((A_\ast, q')\) is a positive counterexample. Then our upper bound follows from Theorem 6. To transform the query in a positive counterexample into an IQ, we use the following three operations.

1. Given a positive counterexample \((A_\ast, q)\), individual saturation for \( T \) consists of updating \( q = \exists \bar{x} \varphi(\bar{a}, \bar{x}) \) with the result \( q' \) of choosing \( x \in \bar{x} \) and \( a \in \text{ind}(A_\ast) \) and replacing \( x \) by \( a \) if \((A_\ast, q')\) is still a positive counterexample.

2. Given a positive counterexample \((A_\ast, q)\), merging for \( T \) consists of updating \( q = \exists \bar{x} \varphi(\bar{a}, \bar{x}) \) with the result \( q' \) of choosing distinct \( x, x' \in \bar{x} \) and replacing all occurrences of \( x \) by \( x' \) if \((A_\ast, q')\) is still a positive counterexample.

3. Given a positive counterexample \((A_\ast, q)\), query role saturation for \( T \) consists of updating \( q = \exists \bar{x} \varphi(\bar{a}, \bar{x}) \) with the result \( q' \) of choosing an atom \( s(t, t') \) and a role \( r \in \Sigma_T \) such that \( T \models r \subseteq s \) (under the assumption of not having equivalent roles, as described above for learning with IQs) and replacing \( s(t, t') \) by \( r(t, t') \) if \((A_\ast, q')\) is still a positive counterexample.

We say that a positive counterexample \((A_\ast, q)\) is individual saturated/merged/query role saturated for \( T \) if individual saturation/merging/query role saturation for \( T \) has been exhaustively applied. Given a query \( q \in \text{CQ} \), we denote by \( G_q^T \) the induced subgraph of \( G_q = (V, E) \) that has as nodes \( x \) where \( x \in V \), and nodes in \( V \) that are reachable from \( x \) via a directed path. We say that \( G_q^T \) is tree shaped if there is a unique element (the root), denoted by \( \rho_G \), such that (i) for every node \( d \) there is a directed path from \( \rho_G \) to \( d \) and (ii) for every distinct directed paths \( p_1, p_2 \) starting from \( \rho_G \) their last elements are distinct.

Lemma 45. Let \( T \) and \( H \) be \( \mathcal{ELH} \) TBoxes and assume \( T \) and \( H \) entail the same RIs. If a positive counterexample \((A_\ast, q)\) (for \( T \) and \( H \)), with \( q \in \text{CQ} \), is individual saturated/merged and query role saturated for \( T \), then for all \( x \) occurring in \( q \), the graph \( G_q^T \) is tree shaped and only one individual name reaches \( x \) with a path that visits only variables.

Sketch. Since \((T, A_\ast) \models q\), there is a homomorphism \( h \) from \( q = \exists \bar{x} \varphi(\bar{a}, \bar{x}) \) to \( I_{T,A_\ast} \), mapping every individual to itself. Since individual saturation has been exhaustively applied to \( q \), every \( x \in \bar{x} \) is mapped by \( h \) into \( \Delta^{2 \tau, A_\ast} \setminus \Delta^{2 \tau, A_\ast} \). If this is not the case a variable is mapped by \( h \) into \( \Delta^{2 \tau, A_\ast} \), and individual saturation is not exhaustively applied, reaching a contradiction.

Suppose that \( G_q^T \) is not tree-shaped. Since \( T \) and \( H \) entail the same RIs and query role saturation for \( T \) has been applied it is not the case that for \( r, s \in \Sigma_T \) there is a variable that is both \( r \)-successor and \( s \)-successor of another variable. Therefore, there is an undirected cycle in \( G_q^T \) with three or more nodes. This means that if \( w, y, z \in \bar{x} \) with \( w \neq y \neq z \), atoms of the form \( r(w, y), s(y, z) \) are in \( q \). We know that \( h(y) \) is mapped into \( \Delta^{2 \tau, A_\ast} \setminus \Delta^{2 \tau, A_\ast} \) and by the construction of the canonical model \( I_{T,A_\ast} \), it follows that \( h(y) \) has only one parent. Thus \( h(w) = h(z) \). But this means that every occurrence of \( w \) can be replaced by \( z \). This contradicts the fact that \((A_\ast, q)\) has been merged for \( T \). Therefore \( G_q^T \) is tree-shaped.
The fact that, for all $x \in \mathcal{X}$, only one individual name reaches $x$ with a path that visits only variables follows from the structure of the canonical model $I_{\mathcal{T}, A}$, and the fact that $q$ is individual saturated for $\mathcal{T}$.

**Lemma 8.** Let $\mathcal{T}$ and $\mathcal{H}$ be $\mathcal{ELH}$ TBoxes and assume $\mathcal{T}$ and $\mathcal{H}$ entail the same RIs. Given a positive counterexample $(A_*, q)$ (for $\mathcal{T}$ and $\mathcal{H}$), with $q \in \mathcal{CQ}_p$, one can construct a positive counterexample $(A_*, q')$ with $q' \in \mathcal{IQ}$ in polynomial time in $|\{A_*, q\}| |\Sigma_{\mathcal{T}}|$.

**Proof.** We use the following claim.

**Claim 1.** Given a positive counterexample $(A_*, q)$ with $q \in \mathcal{CQ}_p$, one can construct a positive counterexample $(A_*, q')$ that is individual saturated/merged/query role saturated for $\mathcal{T}$ in polynomial time with respect to $|\{A_*, q\}|$ and $|\Sigma_{\mathcal{T}}|$.

We prove this claim by analysing the running time of individual saturation, merging and query role saturation for $\mathcal{T}$. There are at most $|q|$ variables in $q$ that can potentially be replaced with individuals. Since the number of individuals is bounded by $|A_*|$, after at most $|A_*| |q|$ membership queries, $(A_*, q)$ is individual saturated for $\mathcal{T}$. There are at most $|q|$ variables in $q$ that can be merged. Therefore, after at most $|q|^2$ membership queries $(A_*, q)$ is merged for $\mathcal{T}$. We assume without loss of generality that role names have a representative, so no equivalent role name is in $q$. There are at most $|q|$ atoms of the form $s(t, t')$ with $t, t' \in a \cup \bar{a}$ in $q$ and at most $|\Sigma_{\mathcal{T}}|$ different role names $r$ such that $\mathcal{T} \models r \sqsubseteq s$. For every $s(t, t')$ in $q$, it is checked in polynomial time if the query that results from replacing $s(t, t')$ by $r(t, t')$ in $q$ is still entailed by $(\mathcal{T}, A_*)$ (that is, after modifying $q$, whether $(A_*, q)$ is still a positive counterexample). After at most $|q| |\Sigma_{\mathcal{T}}|$ membership queries $(A_*, q)$ is query role saturated for $\mathcal{T}$. This finishes the proof of this claim.

Given a positive counterexample $(A_*, q)$ with $q \in \mathcal{CQ}_p$, let $(A_*, q')$ be the result of exhaustively applying individual saturation/merging/query role saturation for $\mathcal{T}$. Then, for all $x$ occurring in $q'$, the graph $G^*_{q'}$ is tree shaped and only one individual name reaches $x$ with a path that visits only variables.

**Lemma 9.** Let $\mathfrak{g}(\mathcal{ELH}, A_*, \mathcal{CQ}_p)$ be in PTIME.

**Proof.** The target $\mathcal{T}$ is a terminology, therefore it contains CIs of the form $C \subseteq A$ or $A \subseteq C$. All needed CIs of the form $C \subseteq A$ can be learned without asking any inseparability query according to Theorem 4. Moreover, all RIs $r \sqsubseteq s$ can be learned in polynomial time using membership queries of the form $(\{r(a, b), s(a, b)\})$. It remains to learn CIs of the form $A \subseteq C$ after the learner receives a positive counterexample $(A_*, q)$ with $q \in \mathcal{CQ}_p$. By Lemma 8 we can find a positive counterexample of the form $(A_*, C(a))$ from a positive counterexample $(A_*, q)$, with $q \in \mathcal{CQ}_p$, in polynomial time. Therefore, by Theorem 6 we can learn an IQ-inseparable hypothesis in PTIME. If $\mathcal{ELH}$ TBoxes entail the same RIs and are IQ-inseparable then, by Lemma 46, they are CQ$_p$-inseparable as well (w.r.t. some ABox $A_*$).

**Proofs for Section “Data Updates”**

**Theorem 10.** Let $\mathcal{T}$ and $\mathcal{H}$ be $\mathcal{ELH}$ terminologies entailing the same RIs, and let $A_*$ and $A$ be ABoxes. If for all $a \in \text{ind}(A)$, there is $a \in \text{ind}(A_*)$ such that $(I_{\mathcal{A}_*}, a) \sim (I_{\mathcal{A}}, b)$, then $(\mathcal{H}, A_*) \equiv_{\mathcal{IQ}} (\mathcal{T}, A_*)$ implies $(\mathcal{H}, A) \equiv_{\mathcal{IQ}} (\mathcal{T}, A)$.

**Proof.** For all concept expressions $C$, it holds by Lemma 23 that $(\mathcal{H}, A) \models C(b)$ iff $(\mathcal{H}, A_*) \models C(a)$. Since we assumed $(\mathcal{T}, A_*) \equiv_{\mathcal{IQ}} (\mathcal{H}, A_*)$, it holds that $(\mathcal{H}, A) \models C(a)$ iff $(\mathcal{T}, A_*) \models C(a)$.

**Theorem 12.** Let $\mathfrak{g}$ be the learning framework that results from adding all pairs of the form $(\mathcal{A}, q)$, with $A \in \text{g}_{\mathcal{T}}(A_*)$ and $q \in \mathcal{IQ}$, to the set $S$ in $\mathfrak{g}(\mathcal{ELH}, A_*, \mathcal{IQ}) = (\mathcal{E}, S, \mathcal{L}, \mu)$, where $T \subseteq \mathcal{E}$. Assume $\Sigma_{\mathcal{T}} \subseteq \Sigma_{\mathcal{A}_*}$. Then, $\mathfrak{g}$ is in PTIME.

**Sketch.** We first focus on CIs of the form $C \subseteq A$, with $A \in \mathcal{N}$. Let $\mathcal{H}$ be a hypothesis with CIs computed with Algorithm 1 (modified to ask only membership queries, cf. Theorem 4) and then generalised. Clearly, $\mathcal{H}$ can be generalised for $\mathcal{T}$ in polynomial time in $|\Sigma_{\mathcal{T}}|$ and $|\mathcal{H}|$.

**Claim 1.** For all $A \in \text{g}_{\mathcal{T}}(A_*)$ and all $a \in A$, $(\mathcal{H}, A) \models a$. Moreover, if there is a homomorphism $h : T_C \rightarrow I_{\mathcal{H}, A}$, mapping $\rho_C$ to some $a \in \text{ind}(A_*)$, where $C \subseteq B \in \mathcal{H}$, then there is a homomorphism $h : T_C \rightarrow I_{\mathcal{H}, A}$ mapping $\rho_C$ to $a$. 

**Theorem 46.** Let $\mathcal{T}$ and $\mathcal{H}$ be $\mathcal{ELH}$ TBoxes which entail the same RIs and let $A_*$ be an ABox. If $(\mathcal{T}, A_*) \equiv_{\mathcal{IQ}} (\mathcal{H}, A_*)$ then $(\mathcal{T}, A_*) \equiv_{\mathcal{IQ}} (\mathcal{H}, A_*)$. 

**Sketch.** Assume to the contrary that there is a query $q \in \mathcal{CQ}_p$ such that $(\mathcal{T}, A_*) \models q$ but $(\mathcal{H}, A_*) \not\models q$ (or vice-versa). By Lemma 8, one can construct a positive counterexample $(A_*, q')$ with $q' \in \mathcal{IQ}$ in polynomial time in $|\{A_*, q\}| |\Sigma_{\mathcal{T}}|$. This contradicts the assumption that $(\mathcal{T}, A_*) \equiv_{\mathcal{IQ}} (\mathcal{H}, A_*)$. Thus, $(\mathcal{T}, A_*) \equiv_{\mathcal{IQ}} (\mathcal{H}, A_*)$. 

**Theorem 9.** $\mathfrak{g}(\mathcal{ELH}, A_*, \mathcal{CQ}_p)$ is in PTIME.
By definition of $g_T(A_\alpha)$, $A$ is a subset of $A_\alpha \cup \{\alpha \in AQ \mid (T, A_\alpha) \models \alpha\}$. Therefore, since $(H, A_\alpha) \equiv_{IQ} (T, A_\alpha)$, for every assertion $\alpha \in A_\alpha$, $(H, A_\alpha) \models \alpha$. The second statement follows from the fact that $\mathcal{H}$ is generalised for $T$ and the definition of $g_T(A_\alpha)$.

Let $A \in g_T(A_\alpha)$. Assume $(T, A) \models A(a)$. By definition of $A$, $(T, A_\alpha) \models A(a)$. As $(H, A_\alpha) \equiv_{IQ} (T, A_\alpha)$, $(H, A_\alpha) \models A(a)$. If $(H, A_\alpha) \models A(a)$, with $A \in NC$, and $A(a) \notin \mathcal{H}$, then there is $C \subseteq A \in \mathcal{H}$ such that $a \in C^{\mathcal{H}, A_\alpha}$.

By Claim 1 and Lemma 24, if $A(a) \notin \mathcal{H}$, then $a \in C^{\mathcal{H}, A_\alpha}$. Thus, by the first statement of Claim 1, $(H, A_\alpha) \models A(a)$.

As $T \models \mathcal{H}$ (see Lemma 2 and the definition of generalisation for $T$), if $(H, A) \models A(a)$, then $(T, A) \models A(a)$.

Thus, after computing a generalised $\mathcal{H}$ for $T$ we obtain a TBox that is not only IQ-inseparable from $T$ w.r.t. $A$ but also w.r.t. all $A \in g_T(A_\alpha)$. CIs of the form $A \subseteq D$, with $A \in NC$, are not affected by the ABox updates since $\Sigma_T \subseteq \Sigma_A$, and we can use Algorithm 4 with counterexamples of the form $(A, C(a))$ with $A \in g_T(A_\alpha)$ in the same way as with $A = A_\alpha$. All RIs can be easily learned with membership queries of the form $\{r(a, b); s(a, b)\}$. 

Proofs for Section “Learning from Data”

Theorem 13. Let $\mathcal{H}(\Sigma \mathcal{L}, A_\alpha, Q) = (\mathcal{E}, \mathcal{S}, \mathcal{L}, \mu)$ be an OMQA learning framework, with $Q \in \{AQ, IQ, CQ\}$, and let $\mathcal{L} \subseteq \mathcal{L}$ be such that $\Sigma_T \subseteq \Sigma_A$. Let $X \subseteq \mathcal{E}$ be the set of examples $(A, q)$ such that there is an ABox homomorphism from $A$ to $\mathcal{A}$. Then, there is a batch $B \subseteq X$, polynomial in $|T|$, and an algorithm such that it takes $B$ as input, it eventually halts, and returns $\mathcal{H} \in \mathcal{L}$ such that $\mu(\mathcal{H}) \cap S = \mu(T) \cap S$.

Proof. Let $Q = AQ$. Algorithm 1, given a positive counterexample $(A_\alpha, A(a))$, calls Algorithm 2 on it, returning an example $(A, B(b))$ (where $A$ is tree shaped ABox encoding a concept $C_A$) such that there exists an ABox homomorphism from $A$ to $A_\alpha$. Let $S$ be the set of all such examples, and consider the batch defined as follows:

$$B = S \cup \bigl\{ ((A(a), B(a)) \mid (T, A) \models A \subseteq B, \text{ with } A, B \in \Sigma_T \bigr\} \cup \bigl\{ (r(a, b), s(a, b)) \mid (r, s) \in \Sigma_T \bigr\}$$

Given $B$ as input, which is polynomial in $|T|$, we can construct $\mathcal{H} \in \mathcal{L}$ by setting:

$$\mathcal{H} := \{C_A \subseteq B \mid (A, B(b)) \in S\} \cup \{A \subseteq B \mid (A(a), B(a)) \in B\} \cup \{r \subseteq s \mid (r(a, b), s(a, b)) \in B\}$$

so that $(\mathcal{H}, A_\alpha) \equiv_{IQ} (T, A_\alpha)$. For $Q = IQ$, let $S'$ be the set of all the polynomially many (in $|T|$) positive counterexamples $(A_\alpha, C(a))$ considered in Algorithm 4. We extend the batch $B$, as defined above, to another batch $B' = B \cup S'$. We can define an ontology $\mathcal{H}' \in \mathcal{L}$ by adding to $\mathcal{H}$ (defined as above) the CIs obtained by applying the procedure in the while-loop of Algorithm 4 on the examples in $S'$. Clearly, $(\mathcal{H}', A_\alpha) \equiv_{IQ} (T, A_\alpha)$. Since by assumption $\Sigma_T \subseteq \Sigma_A$, we can add to the batch examples that allow a learning algorithm to learn all the RIs. Then, by Lemma 46, we also have that $(\mathcal{H}', A_\alpha) \equiv_{IQ} (T, A_\alpha)$.

Theorem 14 ((Angluin 1988), (Mohri, Rostamizadeh, and Talwalkar 2012)). If $\mathcal{H}$ is in PTIME, then $\mathcal{H}$ is polynomial time PAC learnable with membership queries.

Proof. The proof slightly generalises the one presented in (Mohri, Rostamizadeh, and Talwalkar 2012. Th. 13.3), in that we allow the probability distribution on a subset $S \subseteq \mathcal{E}$, and is reported here for the convenience of the reader.

Let $\mathcal{H} = (\mathcal{E}, \mathcal{S}, \mathcal{L}, \mu)$ be a learning framework that is polynomial time exact learnable. In the execution of the algorithm used to learn $\mathcal{H}$, we replace each equivalence query with respect to $S$ by a suitable number of calls to $\text{EX}_{\mathcal{H}}^D$, given a distribution $D$. If all the classified examples returned by $\text{EX}_{\mathcal{H}}^D$ are consistent with the hypothesis $h$, then the algorithm proceeds as if the equivalence query with respect to $S$ had returned ‘yes’. Otherwise, there is a classified example $(e, \ell_e(e))$ such that either $e \in \mu(h)$ and $\ell_e(e) = 0$, or $e \notin \mu(h)$ and $\ell_e(e) = 1$. In this case, the algorithm proceeds as if the equivalence query with respect to $S$ had returned ‘no’, with $e$ as a counterexample. Consider the $i$-th equivalence query as the $i$-th stage of the algorithm, and assume that at stage $i$ the algorithm makes $m_i$ calls to $\text{EX}_{\mathcal{H}}^D$ in place of the equivalence query.

Fix $\epsilon, \delta \in (0, 1)$ and let

$$S_{m_i} = \{(e_1, \ell_{e_1}(e_1)), \ldots, (e_{m_i}, \ell_{e_{m_i}}(e_{m_i}))\}$$

be a sample of size $m_i$ generated by $\text{EX}_{\mathcal{H}}^D$ at stage $i$. Denote by $S_{m_i}^{-\ell_i}$ the set of elements of $S$ occurring in $S_{m_i}$. As the examples in $S$ are assumed to be independently and identically distributed according to $D$, the probability distribution $D$ induces $D'$, with

$$D'(\{S_{m_i}^{-\ell_i}\}) := \prod_{j=1}^{m_i} D(\{e_j\})$$

As described, the algorithm will return a hypothesis such that all classified examples returned by $\text{EX}_{\mathcal{H}}^D$ at some stage $i$ are consistent with the hypothesis. In symbols:

$$(S_{m_i} \cap \mu(h)) = (S_{m_i}^{-\ell_i} \cap \mu(t))$$

We write $S_{m_i} \approx \mu(h)$ as an abbreviation for the above. Fix some $h \in \mathcal{L}$ such that $D(\mu(h) \cup \mu(t)) > \epsilon$. Those hypotheses are called ‘bad’ because the error is larger than $\epsilon$. We want to bound the probability of finding a bad hypothesis. Let $n$ be the total number of equivalence queries needed by the algorithm (which exists by assumption) to exactly learn $\mathcal{H}$ in polynomial time. For all $t, h \in \mathcal{L}$, we know that $D(\mu(h) \cup \mu(t)) \leq (1 - \epsilon)$. Then we have the following:

$$\sum_{i=1}^{n} D'(\{S_{m_i}^{-\ell_i} \mid S_{m_i} \approx \mu(h)\}) \leq \sum_{i=1}^{n} D'(\{S_{m_i}^{-\ell_i} \mid S_{m_i} \approx \mu(h)\}) \leq \sum_{i=1}^{n} \prod_{j=1}^{m_i} D(\{e_j \mid e_j \in S_{m_i}^{-\ell_i}, S_{m_i} \approx \mu(h)\}) \leq \sum_{i=1}^{n} (1 - \epsilon)^{m_i}$$

The latter is bounded by $\delta$ for $m_i \geq \frac{1}{\epsilon} \ln \frac{1}{\delta} + i \ln 2$. Indeed, we have:
\[
\sum_{i=1}^{n} (1 - \epsilon)^{m_i} \leq \sum_{i=1}^{n} e^{-\epsilon m_i} \leq \sum_{i=1}^{n} e^{-\epsilon \left( \ln \left( \frac{1}{\delta} + i \ln 2 \right) \right)} \leq \sum_{i=1}^{n} e^{-\epsilon \left( \ln \frac{1}{\delta} + i \ln 2 \right)} \leq \sum_{i=1}^{n} e^{-\epsilon \left( \ln \frac{1}{\delta} + i \ln 2 \right)} \leq \sum_{i=1}^{n} \frac{\delta}{2^n} \leq \delta.
\]

Since we assumed that the original algorithm learns \( \mathcal{G} \) is polynomial time, \( i \) is polynomial in \( |t| \) and \( |e| \), where \( e \) is the largest example in \( S_{m,n} \). Therefore, \( \mathcal{G} \) is polynomial time PAC learnable with membership queries.

**Theorem 15.** There is a polynomial time PAC learnable OMQA learning framework that is not in PQUERYL.

**Sketch.** Consider the OMQA learning framework \( \mathcal{G}(L, A, Q) = (E, S, L, \mu) \) defined in the main part (Section “Learning from Data”). We have that \( L \) is exponential in \( n \), but finite. Given \( \epsilon, \delta \in (0, 1) \) and \( f: (0, 1)^2 \to \mathbb{N} \) such that \( f(\epsilon, \delta) \leq \left\lceil \frac{1}{\epsilon} \ln \frac{1}{\delta} \right\rceil \), and a target \( T \in L \), let

\[
S_m = \{(e_1, \ell_T(e_1)), \ldots, (e_m, \ell_T(e_m))\}
\]

be a sample generated by \( \text{EX}^D_{G(L, A, Q), \tau} \) of size \( m \geq f(\epsilon, \delta) \). We can compute in polynomial time a hypothesis \( \mathcal{H} \) consistent with the \( m \) examples as follows.

1. Set \( \mathcal{H} = T_0 \).
2. If \( \exists \sigma. M(a) \) appears in a positive example of \( S_m \), add \( A \subseteq \exists \sigma. M \) to \( \mathcal{H} \).

By definition of \( L \), at most one example of the form \( (\{A(a)\}, \exists \sigma. M(a), 1) \) can occur in the sample. One can verify that \( \mathcal{H} \) is consistent with all the examples in \( S_m \). Since \( L \) is of exponential size, a sample of polynomial size suffices (Vapnik 1995), and as we already argued, a hypothesis consistent with any sample (generated by \( \text{EX}^D_{G(L, A, Q), \tau} \) can be constructed in polynomial time. Thus, the learning framework is polynomial time PAC learnable. On the other hand, \( \mathcal{G}(L, A, Q) \) is not in PQUERYL (Konev, Ozaki, and Wolter 2016, proof of Lemma 8).