ALGORITHMIC COMPUTATION OF DE RHAM
COHOMOLOGY OF COMPLEMENTS OF COMPLEX
AFFINE VARIETIES

ULI WALThER
UNIVERSITY OF MINNESOTA

Abstract. Let $X = \mathbb{C}^n$. In this paper we present an algorithm
that computes the de Rham cohomology groups $H^i_{dR}(U, \mathbb{C})$ where
$U$ is the complement of an arbitrary Zariski-closed set $Y$ in $X$.

Our algorithm is a merger of the algorithm given by T. Oaku
and N. Takayama ([7]), who considered the case where $Y$ is a hy-
persurface, and our methods from [9] for the computation of lo-
cal cohomology. We further extend the algorithm to compute de
Rham cohomology groups with support $H^i_{dR,Z}(U, \mathbb{C})$ where again
$U$ is an arbitrary Zariski-open subset of $X$ and $Z$ is an arbitrary
Zariski-closed subset of $U$.

Our main tool is the generalization of the restriction process
from [8] to complexes of modules over the Weyl algebra.

All presented algorithms are based on Gröbner basis computa-
tions in the Weyl algebra.

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1. Introduction

In his famous paper [4] R. Hartshorne introduced the concept of alge-
braic de Rham cohomology of algebraic varieties as an analog to classi-
cal (singular) cohomology and proved, using results of A. Grothendieck
and P. Deligne, that it agrees with classical cohomology if the base field
is $\mathbb{C}$. Moreover, he also defined the notion of algebraic de Rham cohomology with supports and proved that it fits into certain natural long exact sequences related to inclusion maps (5.1).

In [7], the authors give an algorithm that computes (by Gröbner basis computations) the algebraic de Rham cohomology of the complement $U$ of any given hypersurface $Y$ of $X = \mathbb{C}^n$. Their method is based on the initial definition of Hartshorne, as the hypercohomology of the de Rham complex on $U$. They show that this complex is in the derived category the same as the tensor product over $\mathcal{O}_X$ of the sheaf of differential $n$-forms on $X$ with a resolution of $\mathcal{O}_U$, $\mathcal{O}_U$ considered as a module over the sheaf of differential operators on $X$. The computation of the hypercohomology of the latter complex reduces to computation of usual cohomology of the global sections since $U$ is affine and the sheaves involved are quasi-coherent. An algorithm to compute the cohomology of complexes of the type one gets after taking global sections was given in [8]. The strategy is use the method of restriction of a $D$-module to a linear subvariety ([8], section 5).

In this note we shall prove

**Theorem 4.3.** The de Rham cohomology groups of the complement of an affine complex variety are effectively computable by means of Gröbner basis computations in rings of differential operators.

In fact, we shall first generalize the restriction process to the restriction of a complex to a linear subvariety. Then, as applications, we obtain an algorithm that computes de Rham cohomology of arbitrary Zariski-open $U$, and an algorithm that computes de Rham cohomology of Zariski open sets with support in a Zariski closed subset $Z$ of $U$.

Now we shall give an overview of the structure of this paper. Let $D_n = \mathbb{C}[x_1, \ldots, x_n] \langle \partial_1, \ldots, \partial_n \rangle$, the $n$th Weyl algebra over $\mathbb{C}$.

First of all, in section 2, we show that if $U$ is the complement of any Zariski closed set $Y$ defined by $f_1, \ldots, f_r$ in $X$ then computation of the de Rham cohomology of $U$ can be performed by computing the cohomology of the tensor product over $D_n$ of a $D_n$-free resolution of $D_n/(\partial_1, \ldots, \partial_n)D_n$ with the Mayer-Vietoris complex $MV^\bullet(F)$ associated to $f_1, \ldots, f_r$ (cf. subsection 2.4).

In the following section we compute a certain $D_n$-free complex that is quasi-isomorphic to $MV^\bullet(F)$. In fact, we present a method that computes for an arbitrary complex of finitely generated $D_n$-modules $C^\bullet$ with cohomology that is specializable to the origin a $D_n$-free complex $A^\bullet$ that is quasi-isomorphic to $C^\bullet$ and has certain properties related to the $V_n$-filtration (for facts about the $V_n$-filtration, see also [8]).
Section 4 is devoted to the explicit computation of the derived tensor product $D_n / (\partial_1, \ldots, \partial_n) \cdot D_n \otimes_{D_n} C^\bullet$ where again $C^\bullet$ is required to have specializable cohomology, but otherwise is arbitrary. A corollary of this computation will be an algorithm that computes $H_{dR}^i(U, \mathbb{C})$, (4.3).

In section 5 we review the definition of algebraic de Rham cohomology with supports and exhibit an algorithm that computes $H_{dR,Z}^i(X \setminus Y, C)$ for arbitrary subvarieties $Y, Z$ of $X$. The idea here is similar to the original argument, twisted with the Čech complex associated to $Z$.

2. Algebraic de Rham Cohomology

2.1. Notation. Throughout this article, we shall use the following notation. $\mathbb{C}$ will stand for the field of complex numbers, $X$ denotes affine $n$-dimensional $\mathbb{C}$-space $\mathbb{C}^n$ and $Y$ will be a subvariety of $X$ cut out by polynomials $\{f_1, \ldots, f_r\} \subseteq \mathbb{R}$ where $\mathbb{R}$ is $\mathbb{C}[x_1, \ldots, x_n]$. Let $U = X \setminus Y$.

$D_n$ will be the ring of differential operators on $X$ (also called the Weyl algebra) generated by the multiplications by the $x_i$ (which we will call also $x_i$) and the partial derivatives $\partial_i = \partial / \partial x_i$. Set $\mathcal{O}_X$ to be the structure sheaf on $X$. $D_X$ will be the sheaf version of $D_n$, $D_X = \mathcal{O}_X \otimes_{\mathbb{R}} D_n$. We set $\Omega = D_n / (\partial_1, \ldots, \partial_n) D_n$ and $\Omega(D) = \Omega \otimes_{D_n} D$.

If $\mathcal{M}$ is a $\mathcal{D}$- or $D_n$-module, $\Omega^\bullet(\mathcal{M})$ will throughout stand for the de Rham complex of $\mathcal{M}$. In other words, $\Omega^k(\mathcal{M}) = \mathcal{M} \otimes_{\mathbb{C}} \wedge^k (\mathcal{O}^n)$ and the differential $d$ is defined in the usual way: $d(u \otimes dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = \sum_{j=1}^n \partial_j \cdot u \otimes dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$. If $V$ is a variety, $\Omega^\bullet(V)$ will denote the de Rham complex on $V$.

Furthermore, set $\widehat{\Omega}^\bullet = \Gamma(X, \Omega^\bullet(X))$.

2.2. Definition of de Rham cohomology. Recall the idea of completion $\hat{\mathcal{F}}$ of a quasi-coherent sheaf $\mathcal{F}$ on $A$ with respect to the closed subset $B$: if $V$ is open in $B$, $\hat{\mathcal{F}}(V) = \lim_{\leftarrow k} (\mathcal{F}(V) / \mathcal{I}^k(\mathcal{V}) \cdot \mathcal{F}(V))$ where $\mathcal{I}$ is the sheaf of ideals defining $B$.

Algebraic de Rham cohomology of an arbitrary closed subset $B$ of an arbitrary smooth scheme $A$ over any field $K$ is defined as the hypercohomology of the complex $\hat{\Omega}^\bullet(A)$ where the hat denotes completion of $\Omega^\bullet(A)$ with respect to the system of ideals $\mathcal{I}$ which defines $B$ in $A$. (For a precise definition of the maps in $\hat{\Omega}^\bullet(A)$ see [4], page 22.) It is shown in [4] that this definition does not depend on the embedding of $B$ in $A$ nor in fact on $A$ itself.

In the special case where $B$ is smooth, we may take $B = A$ and then the sheaf of ideals $\mathcal{I}$ is the zero sheaf. In particular, for open subsets of $X$, $H_{dR}^i(U, \mathbb{C})$ is the hypercohomology of the complex $\Omega^\bullet(U)$.
2.3. The Idea of Oaku and Takayama. For this subsection, assume that $Y$ is a hypersurface.

The basic observation is the following

**Lemma 2.1.** The complex $\Omega^\bullet(D_n) = \Omega^\bullet$ is (a complex in the category of right $D_n$-modules and in that category) quasi-isomorphic to the complex that is zero except in position $n$ and whose $n$-th entry is the right $D_n$-module $D_n/(\partial_1, \ldots, \partial_n) \cdot D_n = \Omega$. A corresponding statement holds for $D$.

Notice that $\Omega^\bullet(O_U)$ is the complex $\Omega^\bullet(D_n) \otimes D_OU$. It follows from the lemma that since $\Omega^\bullet(D_n)$ is a complex of free $D_n$-modules, $\Omega^\bullet(O_U)$ is the complex that computes $\text{Tor}^D_n(\Omega, O_U)$.

The hypercohomology of this complex will simply be the cohomology of the global section of this complex, because all sheaves in $\Omega^\bullet(D_n) \otimes D_OU$ are quasi-coherent and $U$ is affine.

So the de Rham cohomology of $U$ is $\text{Tor}^D_n(\Omega, R_f)$ where $Y = \text{Var}(f)$.

Let $A^\bullet$ be a resolution of $R_f$ by finitely generated free $D_n$-modules in the category of left $D_n$-modules of length greater than $n$. That is possible follows for example from the fact that $D_n$ is left-Noetherian and that $R_f$ is $D_n$-cyclic ([1]).

Then the cohomology of $\Omega \otimes D_n A^\bullet$ is the de Rham cohomology of $U$ with coefficients in $\mathbb{C}$ shifted by $n$, since $H^i(\Omega \otimes D_n A^\bullet) = \text{Tor}^D_n(\Omega, R_f)$ and $\text{Tor}^D_n(\Omega, R_f) = 0$ for $i < 0$ and $i > n$.

T. Oaku and N. Takayama gave an algorithm in [8] for the computation of the cohomology groups of this kind of complex. It is in fact explained how one can find the cohomology groups of the complex $D_n/(x_1, \ldots, x_n) \cdot D_n \otimes L_{D_n} M$ where $M$ is an arbitrary holonomic $D_n$-module and the tensor product is considered as an element in the derived category. The present problem can be reduced to that case by applying the Fourier automorphism to $D_n$ which sends $x_i$ to $\partial_i$ and $\partial_i$ to $-x_i$.

So computation of $H^i_{dR}(U, \mathbb{C})$ can be summarized as follows ([8], algorithm 2.1):

- Find a suitable finite free resolution $A^\bullet$ of the $D_n$-module $F(R_f)$, $F(R_f)$ positioned in degree $n$ ($F$ denotes the Fourier automorphism).
- Replace each $D_n$ by the right $D_n$-module $\Omega \cong \mathbb{C}[\partial_1, \ldots, \partial_n]$ in that resolution.
- Truncate the resolution using the method of [8] to a complex of finite dimensional $\mathbb{C}$-vectorspaces.
- Take the $i$th cohomology.
2.4. Computing de Rham cohomology for arbitrary $Y$. Let $Y$ now be cut out by the $r$ polynomials $f_1, \ldots, f_r$. The problem arises from the fact that computation of the hypercohomology of $\Omega^\bullet(\mathcal{D}) \otimes_{\mathcal{D}} \mathcal{O}(U)$ is not just Tor$^D_{\bullet}\mathcal{O}(\Gamma(U, \mathcal{O}(U)))$ anymore, due to the existence of higher cohomology of quasi-coherent sheaves on $U$. The strategy is to find an open covering of $U$ such that each of the open sets in the covering is acyclic for cohomology of quasi-coherent sheaves.

**Definition 2.2.** Set $\mathcal{R} :=$ the ordered nonempty subsets of $\{1, \ldots, r\}$. Define $U_I = X \setminus \text{Var}(f_I)$ and more generally for $I \in \mathcal{R}$, we define $U_I = \bigcap_{i \in I} U_i$.

Similarly, set $f_I = \prod_{i \in I} f_i$ with the special cases $f_I = f_i$ if $I = \{i\}$. Write $\mathcal{O}_{U_I}$ as $\mathcal{O}_I$.

To get started, notice that $U_I = X \setminus \text{Var}(f_I)$. This means in particular, that by Oaku–Takayama the de Rham cohomology groups of $U_I$ with coefficients in $\mathbb{C}$ are computable as the cohomology of $\Omega^\bullet(D_n) \otimes_{D_n} R_{f_I}$.

Notice also that $U = X \setminus Y$ is just the union of all the $U_I$.

In [4], page 28, Hartshorne defines how de Rham cohomology of schemes may be recovered from the de Rham complexes on the open sets in a finite covering. For our $U$ that works as follows.

For each $I$ let $X_I = \prod_{i \in I} U_i$. Then $U_I$ embeds in $X_I$ as the diagonal. As $X_I$ is smooth, $\hat{\Omega}^\bullet(X_I)$ computes de Rham cohomology of $U_I$, the hat denoting completion at the closed subscheme $U_I \subseteq X_I$.

Consider the direct image $\mathcal{M}^\bullet_I$ of $\hat{\Omega}^\bullet(X_I)$ in $U$, induced by the inclusion $j_I : U_I \hookrightarrow U$.

Since $U_I$ is smooth, $\hat{\Omega}^\bullet(X_I)$ is naturally quasi-isomorphic to $\Omega^\bullet(\mathcal{O}_I)$, cf. [4], Proposition II.1.1. So $\mathcal{M}^\bullet_I$ is naturally quasi-isomorphic to $j_{I*}(\Omega^\bullet(\mathcal{O}_I))$, the direct image of $\Omega^\bullet(\mathcal{O}_I) = \Omega^\bullet(D_n) \otimes_{D_n} \mathcal{O}_I$.

For $j \notin I$, the natural maps $X_{I U_J} \hookrightarrow X_I$ and $U_{I U_J} \hookrightarrow U_I$ give a natural map $\hat{\Omega}^\bullet(X_I) \to \hat{\Omega}^\bullet(X_{I U_J})$. Similarly, we get chain maps $\phi_{I j} : \Omega^\bullet(\mathcal{O}_I) \to \Omega^\bullet(\mathcal{O}_{I U_J})$ induced from the inclusion $U_{I U_J} \hookrightarrow U_I$. It is easy to check that the natural quasi-isomorphisms from $\hat{\Omega}^\bullet(X_I)$ to $\Omega^\bullet(D_n) \otimes_{D_n} \mathcal{O}_I$ transform the map $\hat{\Omega}^\bullet(X_I) \to \hat{\Omega}^\bullet(X_{I U_J})$ into $\phi_{I j}$. So the same is true for the direct images in $U$.

Multiply $\phi_{I j}$ by $(-1)^{\text{sgn}(I, j)}$, $\text{sgn}(I, j)$ being the number of elementary permutations that are needed to make $(I, j)$ an actual element of $\mathcal{R}$.

Let us write $\mathcal{J}^\bullet_I := j_{I*}(\Omega^\bullet(D_n) \otimes_{D_n} \mathcal{O}_I)$, a complex of sheaves on $U$. We will now construct a double complex $\mathcal{M}\mathcal{V}(\mathcal{J})$ out of all the $\mathcal{J}^\bullet_I$. Let $\mathcal{M}\mathcal{V}(\mathcal{J})^{k, l} = \bigoplus_{|I| = l} \mathcal{J}^k_I$. The maps in horizontal $(k-)$ direction are simply the directs sums of the differentials of the $\mathcal{J}^\bullet_I$ involved, while
Lemma 2.3. The complex $j_{!*}(\Omega^i \otimes_{D_n} \mathcal{O}_I)$ consists entirely of sheaves that have no higher cohomology on $U$.

Proof. In order to see this observe that it is sufficient to show that $j_{!*}(\mathcal{O}_I)$ has this property, because $\Omega^i$ is $D_n$-free. If $\mathcal{E}^i_*$ is a flasque resolution of $\mathcal{O}_I$ on $U_I$, then $j_{!*}(\mathcal{E}^i_*)$ is a complex of flasque sheaves on $U$ as direct images of flasque sheaves are flasque. Moreover, as $U_I$ is affine, $j_{!*}$ is an exact functor on quasi-coherent sheaves (and $\mathcal{O}_{U_I}$-morphisms), because $R^n j_{!*}(\mathcal{E}^i_*)$ is the sheaf associated to the presheaf $V \mapsto H^i(V \cap U_I, \mathcal{E}^i_*)$ for open subsets $V$ of $U$. Hence we actually get a flasque resolution of $j_{!*}(\mathcal{O}_I)$. Taking global sections we see that $j_{!*}(\mathcal{O}_I)$ has no higher cohomology on $U$, as $\Gamma(U, j_{!*}(\mathcal{E}^i)) = \Gamma(U_I, \mathcal{E}^i)$. □

Remark 2.4. We note in passing that the proof actually shows that $H^i(j_{!*}(\mathcal{F}), U) = 0$ for positive $i$ and all quasi-coherent $\mathcal{F}$ on $U_I$.

So the complex $\text{Tot}^\bullet(\mathcal{M}\mathcal{V}(\mathcal{J}))$ consists of $\Gamma(U, -)$-acyclic sheaves. Thus, in order to compute its hypercohomology it suffices to compute the cohomology of the global sections of that complex. We arrive at

Proposition 2.5. The de Rham cohomology of of $U$ with coefficients in $\mathbb{C}$, which may be computed as the hypercohomology of the complex $\text{Tot}^\bullet(\mathcal{M}\mathcal{V}(\mathcal{J}))$, agrees with the cohomology of the global sections of $\text{Tot}^\bullet(\mathcal{M}\mathcal{V}(\mathcal{J}))$ and can be computed as $H^\bullet(\Omega^\bullet \otimes_{D_n} MV^\bullet)$, where

\begin{equation}
MV^\bullet : 0 \to \bigoplus_{|I|=1} R_{f_1} \to \cdots \to \bigoplus_{|I|=r} R_{f_1} \to 0.
\end{equation}

Proof. This follows from the discussion before the proposition, noting that the global sections on $U$ of $j_{!*}(\Omega^\bullet(\mathcal{O}_I))$ are exactly $\Omega(D_n) \otimes_{D_n} R_{f_1}$ and hence $\Gamma(U, \mathcal{M}\mathcal{V}^\bullet(\mathcal{J})) = \Omega^\bullet(D_n) \otimes_{D_n} MV^\bullet$. □

For any set of polynomials $\{p_i\}_{i=1}^m$, the Čech complex $C^\bullet(R; p_1, \ldots, p_m) := \bigotimes_{1}^m C^\bullet(p_i)$ is defined by $C^\bullet(p_i) = (0 \to R \xrightarrow{\delta_{p_i}} R_{p_i} \to 0)$. 

the vertical (l-) maps are defined to be the sums of all maps which are composed as follows: $\bigoplus_{|I|=l} J^k_I \xrightarrow{\text{nat}} J^k_I \xrightarrow{\phi_{i,j}} J^k_{I \cup J} \leftarrow \bigoplus_{|I|=l+1} J^k_I$.

Notice that this is in fact a double complex (and in particular anti-commutative) due to the sign rule that applies to the $\phi_{i,j}$.

Then, according to Hartshorne, the de Rham cohomology of $U$ is the hypercohomology of the associated total complex $\text{Tot}^\bullet(\mathcal{M}\mathcal{V}(\mathcal{J}))$. Of course, $\mathcal{M}\mathcal{V}(\mathcal{J})$ is just the origin of the usual Mayer-Vietoris spectral sequence of de Rham cohomology and sometimes called the Čech-de Rham complex.

In the hypersurface case $U$ is affine, so hypercohomology on $U$ was cohomology of the global sections. Now we claim

Lemma 2.3. The complex $j_{!*}(\Omega^i \otimes_{D_n} \mathcal{O}_I)$ consists entirely of sheaves that have no higher cohomology on $U$.

Proof. In order to see this observe that it is sufficient to show that $j_{!*}(\mathcal{O}_I)$ has this property, because $\Omega^i$ is $D_n$-free. If $\mathcal{E}^i_*$ is a injective resolution of $\mathcal{O}_I$ on $U_I$, then $j_{!*}(\mathcal{E}^i_*)$ is a complex of flasque sheaves on $U$ as direct images of flasque sheaves are flasque. Moreover, as $U_I$ is affine, $j_{!*}$ is an exact functor on quasi-coherent sheaves (and $\mathcal{O}_{U_I}$-morphisms), because $R^n j_{!*}(\mathcal{E}^i_*)$ is the sheaf associated to the presheaf $V \mapsto H^i(V \cap U_I, \mathcal{E}^i_*)$ for open subsets $V$ of $U$. Hence we actually get a flasque resolution of $j_{!*}(\mathcal{O}_I)$. Taking global sections we see that $j_{!*}(\mathcal{O}_I)$ has no higher cohomology on $U$, as $\Gamma(U, j_{!*}(\mathcal{E}^i)) = \Gamma(U_I, \mathcal{E}^i)$. □

Remark 2.4. We note in passing that the proof actually shows that $H^i(j_{!*}(\mathcal{F}), U) = 0$ for positive $i$ and all quasi-coherent $\mathcal{F}$ on $U_I$.

So the complex $\text{Tot}^\bullet(\mathcal{M}\mathcal{V}(\mathcal{J}))$ consists of $\Gamma(U, -)$-acyclic sheaves. Thus, in order to compute its hypercohomology it suffices to compute the cohomology of the global sections of that complex. We arrive at

Proposition 2.5. The de Rham cohomology of of $U$ with coefficients in $\mathbb{C}$, which may be computed as the hypercohomology of the complex $\text{Tot}^\bullet(\mathcal{M}\mathcal{V}(\mathcal{J}))$, agrees with the cohomology of the global sections of $\text{Tot}^\bullet(\mathcal{M}\mathcal{V}(\mathcal{J}))$ and can be computed as $H^\bullet(\Omega^\bullet \otimes_{D_n} MV^\bullet)$, where

\begin{equation}
MV^\bullet : 0 \to \bigoplus_{|I|=1} R_{f_1} \to \cdots \to \bigoplus_{|I|=r} R_{f_1} \to 0.
\end{equation}

Proof. This follows from the discussion before the proposition, noting that the global sections on $U$ of $j_{!*}(\Omega^\bullet(\mathcal{O}_I))$ are exactly $\Omega(D_n) \otimes_{D_n} R_{f_1}$ and hence $\Gamma(U, \mathcal{M}\mathcal{V}^\bullet(\mathcal{J})) = \Omega^\bullet(D_n) \otimes_{D_n} MV^\bullet$. □

For any set of polynomials $\{p_i\}_{i=1}^m$, the Čech complex $C^\bullet(R; p_1, \ldots, p_m) := \bigotimes_{1}^m C^\bullet(p_i)$ is defined by $C^\bullet(p_i) = (0 \to R \xrightarrow{\delta_{p_i}} R_{p_i} \to 0)$. 

Notice that $MV^i$ is the $i+1$st entry of the Čech complex to $f_1, \ldots, f_r$ if $i \geq 0$ and zero otherwise.

**Remark 2.6.** In the special case where $r = 1$, so $I = (f)$, one sees that the complex $MV^\bullet$ degenerates to $(0 \to R_1 \to 0)$ reducing to the case from [7].

In [9], algorithm 5.1 we gave an algorithm that explicitly computes the Čech complex to a finite set of polynomials as a complex of finitely generated left $D_n$-modules by means of Gröbner basis computations.

It will now be our task to develop an algorithm that computes the cohomology of $\Omega^\bullet \otimes_{D_n} MV^\bullet = \Omega \otimes_{D_n} MV^\bullet$.

3. Computing a certain $D_n$-free complex

In the next two sections we shall be concerned with finding the cohomology of the complex $\Omega^\bullet \otimes_{D_n} MV^\bullet$ where $MV^\bullet$ is the Mayer-Vietoris complex (2.1) to $f_1, \ldots, f_r$, or more generally the cohomology of $\Omega \otimes_{D_n} C^\bullet$ where $C^\bullet$ is an arbitrary complex of finitely generated $D_n$-modules with specializable cohomology (cf. Definition 3.1). In particular, in this section we find a free $D_n$-complex quasi-isomorphic to a given complex $C^\bullet$ with special properties related to the so-called $V$-filtration.

We need to introduce some terminology from [8] related to the $V$-filtration.

**Definition 3.1.** Fix an integer $d$ with $0 \leq d \leq n$ and set $H = \text{Var}(x_1, \ldots, x_d)$. For $\alpha \in \mathbb{Z}^n$, we set $\alpha_H = (\alpha_1, \ldots, \alpha_d, 0, \ldots, 0)$.

On the ring $D_n$ we define the $V_d$-filtration $F^j_H(D_n)$ which consists of all operators $cx^\alpha \partial^\beta$ for which $|\alpha_H| + j \geq |\beta_H|$. More generally, on a free $D_n$-module $A = \oplus_{i=1}^m D_n \cdot e_i$ we define $F^j_H(A)[\mathfrak{m}]$, where $\mathfrak{m}$ is an element of $\mathbb{Z}^m$, as $\sum F^j_H - \text{m}(D_n) \cdot e_i$. We shall call $\mathfrak{m}$ the shift vector.

If $A$ is a free $D_n$-module, the phrase *let a shift vector for $A$ be given* will mean the following. First of all we assume that once and forever a minimal set of generators $\{a_i\}_{i=1}^m$ for $A$ has been chosen, and secondly that there is given $\mathfrak{m} \in \mathbb{Z}^m$ defining the $V_d$-degree on $A$ by the formula of the previous paragraph.

If $M$ is a quotient of the free $D_n$-module $A = \oplus_{i=1}^m D_n \cdot e_i$, so $M = A/I$, we define the filtration on $M$ by $F^j_H[M](M) = F^j_H[A](A) + I$. For submodules $N$ of $A$ we define the $V_d$-filtration by intersection: $F^j_H[N](N) = F^j_H[M](A) \cap N$.

If $A^*$ is a free $D_n$-resolution of the module $M$, $M$ being positioned in degree zero, we say it is $V_d$-strict if there exist shift vectors $\mathfrak{m}_i$ such that $F^j_H[M_i](A^i) \to F^j_H[M_{i+1}](A^{i+1}) \to F^j_H[M_{i+2}](A^{i+2})$ is exact for all...
$i < -1$ and all $j$, and $F^j_H[m_{-1}](A^{-1}) \to F^j_H[m_0](A^0) \to F^j_H[m_0](M) \to 0$ is exact for all $j$.

We define the $V_d$-degree of an operator, $V_d\deg(P)[m]$, to be the smallest $i$ such that $P \in F^i_H[m](A)$.

It has been shown by T. Oaku and N. Takayama ([8]) how to compute for any $D_n$-module $M$ a free $V_d$-strict resolution $(A^*[m_*], \phi^*)$ of $M$, $A^i = \oplus r_iD, r_i = 0$ if $i > 0$. The construction given in [8] allows for arbitrary $m_0$.

The method employed is to construct the free resolution with the usual technique of finding a Gröbner basis for $\ker(A^i \to A^{i+1})$ and calculating the syzygies on this basis. The trick is to impose an order that refines the partial ordering given by $V_d$-degree, together with a homogenization technique.

The vectors $m_i$ are obtained for each $A^i$ with falling $i$: if $A^i$ maps its generators on a Gröbner basis of $\ker(A^{i+1} \to A^{i+2})$ then the shift component $m_i(j)$ of the $j$th generator $e_j$ of $A^i$ is defined as $V_d\deg(\phi^i(e_j)[m_{i+1}])$.

We need to generalize the definitions of [8] to the case where the complex $A^*$ is not a resolution.

**Definition 3.2.** A complex of free $D_n$-modules $\ldots \to A^{k-1} \to A^k \to A^{k+1} \to \ldots$ is said to be $V_d$-adapted at $A^k$ with respect to certain shift vectors $m_{k-1}, m_k, m_{k+1}$ if $\phi^kF^j_H[m_k]A^k \subseteq F^j_H[m_{k+1}]A^{k+1}$ and also $\phi^{k-1}F^j_H[m_{k-1}]A^{k-1} \subseteq F^j_H[m_k]A^k$.

We shall say that the complex is $V_d$-strict at $A^k$ if it is $V_d$-adapted at $A^k$ and moreover $\text{im} \phi^{k-1} \cap F^j_H[m_k]A^k = \text{im}(\phi^{k-1}F^j_H[m_{k-1}]A^{k-1})$ for all $j$.

Suppose $P^0$ and $P^1 = \oplus_1^m D_n \cdot e_i$ are free $D_n$-modules, $\phi: P^1 \to P^0$ a $D_n$-linear map and assume that on $P^0$ a shift vector $m_0$ is given. We define the obvious shift on $P^1$ by setting $m_1(i) = V_d\deg(\phi(e_i))$.

For $1 \leq d \leq n$ we set $\theta_d = x_1\partial_1 + \ldots + x_d\partial_d$. Recall that a $D_n$-module $M = A[m]/I$ is called specializable to $H$ if there is a polynomial in a single variable $b(s)$ such that $b(\theta_d + j)F^j_H[m]M \subseteq F^j_H[m]M$ for all $j$ (cf. [7]). Introducing $\text{gr}^j_H[m]M = (F^j_H[m]M)/(F^{j-1}_H[m]M)$, this can be written as $b(\theta_d + j)\text{gr}^j_H[m]M = 0$. The polynomial $b$ may depend on $m$, while its existence does not ([6]).

Notice that independently of $d$, $\text{gr}^*H(D_n) \cong D_n$, as ring.

The main purpose of this section is to construct for a given finite complex $0 \to C^1 \to \ldots \to C^{r-1} \to 0$ with cohomology specializable to $H$ a quasi-isomorphic free $V_d$-strict complex $A^*[m_*]$.

**Remark 3.3.** Notice that if $A^*$ is a free resolution of $M$ and $V_d$-strict in our sense it is also $V_d$-strict in the sense of Oaku/Takayama. In
fact our definition is a natural generalization to complexes that are not resolutions.

Moreover, let \( \cdots \to A^{k-1}[m_{k-1}] \xrightarrow{\phi^{k-1}} A^k[m_k] \xrightarrow{\phi^k} A^{k+1}[m_{k+1}] \to \cdots \) be \( V_d \)-strict. Then the \( V_d \)-filtration on \( A^k \) induces a filtration on the \( k \)-cycles \( Z^k = \ker \phi^k \) and since the complex is \( V_d \)-strict this gives a natural filtration on the cohomology module \( H^k \),

\[
F^j_H[m_k]H^k = F^j_H[m_k]Z^k / \text{im} F^j_H[m_{k-1}]A^{k-1}.
\]

Let \( B^k \) be im \( \phi^{k-1} \), the \( k \)-boundaries. The short exact sequences

\[ 0 \to Z^k \to A^k \to B^{k+1} \to 0 \]

give rise to short exact sequences of groups

\[ 0 \to F^j_H[m_k]Z^k \to F^j_H[m_k]A^k \to F^j_H[m_{k+1}]B^{k+1} \to 0 \]

since the complex is \( V_d \)-strict. Similarly, the short exact sequences

\[ 0 \to B^k \to Z^k \to H^k \to 0 \]

induce short exact sequences

\[ 0 \to F^j_H[m_k]B^k \to F^j_H[m_k]Z^k \to F^j_H[m_k]H^k \to 0. \]

This in turn induces short exact sequences of the graded objects,

\[ 0 \to \text{gr}^j_H[m_k](B^k) \to \text{gr}^j_H[m_k](Z^k) \to \text{gr}^j_H[m_k](H^k) \to 0 \]

and

\[ 0 \to \text{gr}^j_H[m_k](Z^k) \to \text{gr}^j_H[m_k](C^k) \to \text{gr}^j_H[m_{k+1}](B^{k+1}) \to 0. \]

These sequences are the main feature of \( V_d \)-strict complexes.

We shall break the construction of \( A^\bullet[m_\bullet] \) into several steps.

**Lemma 3.4.** Let \( 0 \to P_A/I_A \to P_B/I_B \to P_C/I_C \to 0 \) be exact and assume that on \( P_C \) there is given a shift vector \( m_C \).

Then \( P_B/I_B \) can be replaced by a certain other quotient of a free module isomorphic to \( P_B/I_B \), such that there exist shift vectors \( m_A, m_B \) making the sequence \( V_d \)-strict.

**Proof.** We remark that making the sequence \( V_d \)-adapted is trivial (but not good enough).

Set \( Q_B = P_A \oplus P_C \) and define \( Q_B \to P_B/I_B \) as \( Q_B \xrightarrow{\phi} P_A \to P_A/I_A \xrightarrow{\phi} P_B/I_B \) for the \( P_A \)-part, while for the \( P_C \)-part we pick some map from \( P_C \) to \( P_B/I_B \) that lifts \( P_B/I_B \to P_C/I_C \) and then map \( Q_B \to P_C \to P_B/I_B \).

Then \( Q_B \) projects onto \( P_B/I_B \). Let’s say the kernel is \( I_{A,C} \). Notice that \( I_{A,C} \) contains \( I_A \oplus 0 \), which corresponds to the injection \( P_A/I_A \hookrightarrow Q_B/I_{A,C} \).

Now define the shift on \( Q_B \) by taking the given shift from \( P_C \) on the second component, and for the generators of \( P_A \) take any shift of your choice.

It is clear that the resulting short exact sequence \( 0 \to P_A/I_A \to Q_B/I_{A,C} \to P_C \to 0 \) is \( V_d \)-adapted. It is just as clear that it is strict at.
Let \( b = \sum \alpha_i a_i + \sum \gamma_i c_i \) be an element of \( Q_B \) that is sent to zero in \( P_C/I_C \). That means that \( \sum \gamma_i c_i \in I_C \). Since \( I_{A,C} \) contains for all elements \( c \in I_C \) an element \((a_c, c)\) (after all, modulo the first component of \( Q_B \), \( 0 \oplus I_C \)-elements must be zero!) we can transform our element \( b \) into an element of \( I_A \oplus 0 \). We would like this element to have \( V_d \)-degree at most \( V_d \deg(b) \).

Return to \( b = \sum \alpha_i a_i + \sum \gamma_i c_i \). The \( V_d \)-degree of \( b \) is the maximum of the degrees of the two sums. Since \( \sum \gamma_i c_i \) is in \( I_C \), we can write it as a sum \( \sum \delta_i n_i \), where the \( V_d \)-degree of the sum, let’s call it \( e \), is the \( V_d \)-degree of the largest summand in the sum, because the \( n_i \) form a \( G \)-basis.

Modulo \( I_{A,C} \), this is the same as the sum \(- \sum \delta_i m_i \), which has lower or equal \( V_d \)-degree, by construction of the shift on \( P_A \).

Then \( \sum \alpha_i a_i - \sum \delta_i m_i \in P_A \) is an expression that maps onto \( b \), modulo \( I_{A,C} \), and has degree at most equal to \( e \). Strictness follows.

We are done. \( \square \)

Notice that this creates a commutative diagram with exact and \( V_d \)-strict rows and columns.

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & P_A/I_A & P_B/I_B & P_C/I_C & 0 \\
0 & P_A & Q_B & P_C & 0 \\
0 & I_A & I_{A,C} & I_C & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

We will have need of a slight improvement of lemma 3.4:

**Lemma 3.5.** Suppose we have 2 short exact sequences

\[
0 \to P_A/I_A \to P_B/I_B \to P_C/I_C \to 0
\]
and

$$0 \to P_D/I_D \to P_A/I_A \to P_F/I_F \to 0$$

and assume that on $P_C$ is given a shift vector $m_C$. Then one can rewrite $P_A/I_A$ as $Q_A/I_{D,F}$ and $P_B/I_B$ as $Q_B/I_{D,F,C}$ and find shift vectors $m_A, m_B, m_D, m_F$ such that the resulting 2 sequences are exact and $V_d$-strict.

**Proof.** First use the first half of the proof of the previous lemma to write $P_A/I_A$ as $Q_A/I_{D,F}$ and then with that representation of $P_A/I_A$ rewrite $P_B/I_B$ as $Q_B/I_{D,F,C}$. So in particular, $Q_B = Q_A \oplus P_C = P_D \oplus P_F \oplus P_C$.

In order to find the proper shift vectors, proceed as follows:

1. Take a G-basis \( \{ c_i \} \) for $I_C$ with respect to an order refining the $V_d$-filtration on $P_C$ relative to the given shift $m_C$. For all $i$ find $a_i = (d'_i, f'_i)$ such that $(d'_i, f'_i, c_i) \in I_{D,F,C}$.
2. Pick a shift on $F$ such that $V_d\deg(f'_i) \leq V_d\deg(c_i)$ for all $i$.
3. Compute a G-basis \( \{ f_i \} \) of $I_F$ using an order that refines $V_d$-degree on $F$, using the shift we just found. For all $i$ find $d_i$ with $(d_i, f_i) \in I_{D,F}$.
4. Pick a shift on $P_D$ such that $V_d\deg(d_i) \leq V_d\deg(f_i)$ for all $i$ and $V_d\deg(d'_i) \leq V_d\deg(c_i)$ for all $i$.

By arguments similar to those that prove lemma 3.4, the sequences are $V_d$-strict. \qed

Lemma 3.4 and 3.5 providing the basis for the construction, the following result is the inductive step:

**Lemma 3.6.** Let $A, B, C$ be three submodules of free modules $F_A, F_B, F_C$. Assume that $0 \to A \to B \to C \to 0$ is exact and $V_d$-strict, relative to some shift vectors on $F_A, F_B, F_C$.

Then one can construct a diagram:

\[
\begin{array}{ccc}
0 & \to & 0 \\
& \uparrow \phi_A & \uparrow \phi_B \\
0 & \to & A \to B \to C \to 0 \\
& \uparrow & \uparrow & \uparrow \\
0 & \to & P_A \to P_B \to P_C \to 0 \\
& \uparrow & \uparrow & \uparrow & \uparrow \\
0 & \to & K_A \to K_B \to K_C \to 0 \\
& \uparrow & \uparrow & \uparrow & \uparrow \\
0 & \to & 0 & \to & 0
\end{array}
\]
such that

• all $P_X$ are free,
• all rows and columns are exact,
• there are shift vectors $\mathbf{m}_A, \mathbf{m}_B, \mathbf{m}_C$ such that if $P_A, P_B, P_C$ are shifted accordingly, all rows and columns become $V_d$-strict.

Proof. Let $\{a_i\}, \{b_i\}$ and $\{c_i\} \cup \phi_B(\{b_i\})$ be G-bases for $A, B, C$ with respect to an order on $F_A, F_B, F_C$ that refines $V_d$-degree.

Let $P_A$ be a free module on the symbols $\{e_{a_i}\}$, define a projection from $P_A$ to $A$ in the obvious way. Let $P_C$ be a free module on the symbols $\{e_{b_i}\} \cup \{e_{c_i}\}$. Define the projection $P_C \to C$ by $e_{c_i} \to c_i, e_{b_i} \to \phi_B(b_i)$. Define degree shifts on $P_A, P_C$ in the obvious way.

Set $P_B = P_A \oplus P_C$. Define $P_B \to B$ on the $P_A$-part in the obvious way, and similarly on the $P_C$-part that corresponds to $\{b_i\}$. For the generators corresponding to $\{c_i\}$, use a lift $\psi : P_C \to B$ for $\phi_B$ (which exists as $P_C$ is free) that satisfies: $\psi(e_{c_i})$ is of no higher (shifted) $V_d$-degree than $V_d\deg(c_i)$ (which exists because $0 \to A \to B \to C \to 0$ is $V_d$-strict). Set $K_A, K_B, K_C$ to be the corresponding kernels.

It is clear that $0 \to P_A \to P_B$ and $P_B \to P_C \to 0$ are $V_d$-strict. If an element of $V_d$-degree $e$ is in the kernel of $P_B \to P_C$, then its second component (the one in $P_C$) is zero, so the $V_d$-degree came from the $P_A$-component. Hence the second row is $V_d$-strict. Then automatically the third row is too.

By [7], the remarks after proposition 3.11, the outer columns are $V_d$-strict.

Let $b \in B$ be of $V_d$-degree $d$. Then $b = \sum \alpha_i b_i$ where $V_d\deg(\alpha_i) + V_d\deg(b_i) \leq V_d\deg(b)$ for all $i$, since $\{b_i\}$ is a G-basis and the image of the element $\sum \alpha_i e_{b_i} \in P_B$ is $b$. Moreover, by our definitions, the $V_d$-degree of this sum in $P_B$ is at most the degree of the image in $B$, which is $e$. It follows that $P_B \to B \to 0$ is $V_d$-strict, and hence the whole column.

We need a version of lemma 3.6 of the type of 3.5.

Lemma 3.7. Assume we have 2 exact $V_d$-strict sequences $0 \to A \to B \to C \to 0$ and $0 \to D \to A \xrightarrow{\phi} F \to 0$ of submodules of free modules. Suppose moreover that for $D$ we are given a short exact $V_d$-strict sequence $0 \to K_D \to P_D[\mathbf{m}_D] \to D \to 0$ where the free module $P_D$ maps its generators on a Gröbner basis of $D$ with respect to the given shift vector. Then for $X \in \{A, B, C, F\}$ one can find free module $P_X$ and submodules $K_X$ creating commutative diagrams with $V_d$-strict and exact rows and columns of the type (3.1) over both given short exact sequences.
Proof. Let \( \{a_i\} \) be a \( G \)-basis for \( A \) and assume \( \{\phi(a_i)\} \cup \{f_i\} \) is one for \( F \). Then set \( P_F \) to be the free \( D_n \)-module on the symbols \( \{\phi(a_i)\} \cup \{f_i\} \) and \( P_A = P_D \oplus P_F \). Define maps \( P_F \to F \) in the obvious way and \( P_A \to A \) as \((P_A \to P_D \to D \to A) + (P_A \to P_F \to A)\) where the map from \( P_F \to A \) is defined by \( e_{\phi(a_i)} \to a_i \) and \( e_{f_i} \to g_i \) where \( g_i \) is a preimage of \( f_i \) in \( A \) of lesser or equal \( V_d \)-degree as \( f_i \).

Let \( K_A, K_F \) be the kernels.

This gives as in lemma 3.6 a \( V_d \)-strict exact commutative diagram over \( 0 \to A \to D \to F \to 0 \).

Now use the sequence \( 0 \to K_A \to P_A \to A \to 0 \) to build a diagram over \( 0 \to A \to B \to C \to 0 \).

\[ \begin{array}{c}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array} \]

\[ \begin{array}{c}
Z^{r-1} \\
P_{Z,r-1}[m_{Z,r-1}] \\
K_{Z,r-1}[m_{Z,r-1}]
\end{array} \]

\[ \begin{array}{c}
C^{r-1} \\
P_{C,r-1}[m_{C,r-1}] \\
K_{C,r-1}[m_{C,r-1}]
\end{array} \]

\[ \begin{array}{c}
B^r = 0 \\
0 \\
0
\end{array} \]

\[ \begin{array}{c}
0 \\
0 \\
0
\end{array} \]

We have now assembled enough machinery to to find for all complexes of \( D_n \)-modules the cohomology of which is specializable to \( H = \text{Var}(x_1, \ldots, x_d) \) a quasi-isomorphic \( V_r \)-strict complex.

So suppose we are given such a complex \( 0 \to C^1 \to \ldots \to C^{r-1} \to 0 \). As a first step, write down all the short exact sequences \( 0 \to B^k \to Z^k \to H^k \to 0 \) and \( 0 \to Z^k \to C^k \to B^{k+1} \to 0 \). That is to say, find representations of these modules and maps in terms of finitely generated free modules modulo a finite number of relations. Observe that all \( B^{i+1}, C^{i+1}, Z^{i+1}, H^{i+1} \) are zero for \( i \geq r - 1 \).

Invoke lemma 3.5 to find a presentation for \( Z^{r-1} \) and for \( C^{r-1} \) together with shift vectors \( m_{Z,r}, m_{C,r}, m_{H,r} \) and \( m_{B,r} \) such that there are commutative diagrams
and

\[
\begin{array}{ccc}
0 & \rightarrow & B^{r-1} \\
\uparrow & & \uparrow \\
0 & \rightarrow & Z^{r-1} \\
\uparrow & & \uparrow \\
0 & \rightarrow & H^{r-1} \\
\uparrow & & \uparrow \\
0 & \rightarrow & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & P_{B,r-1}[m_{B,r-1}]
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & P_{Z,r-1}[m_{Z,r-1}]
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & P_{H,r-1}[m_{H,r-1}]
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & K_{B,r-1}[m_{B,r-1}]
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & K_{Z,r-1}[m_{Z,r-1}]
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & K_{H,r-1}[m_{H,r-1}]
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\end{array}
\]

with all \(P_X\) free, that has exact and \(V_d\)-strict rows and columns.

Then invoke the lemma again, this time starting with the shift just obtained on \(B^{r-1}\) and constructing representations for \(Z^{r-2}, C^{r-2}\), and shifts on \(P_{Z,r-2}, P_{C,r-2}, P_{H,r-2}, P_{B,r-2}\).

And so on. Repetition leads to \(V_d\)-strict commutative diagrams with exact rows and columns

\[
\begin{array}{ccc}
0 & \rightarrow & Z^i \\
\uparrow & & \uparrow \\
0 & \rightarrow & C^i \\
\uparrow & & \uparrow \\
0 & \rightarrow & B^{i+1} \\
\uparrow & & \uparrow \\
0 & \rightarrow & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & P_{Z,i}[m_{Z,i}]
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & P_{C,i}[m_{C,i}]
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & P_{B,i+1}[m_{B,i+1}]
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & K_{Z,i}[m_{Z,i}]
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & K_{C,i}[m_{C,i}]
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & K_{B,i+1}[m_{B,i+1}]
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\end{array}
\]
and

\[
\begin{array}{cccc}
  0 & 0 & 0 \\
  0 & P_{B,i} \to m_{B,i} & P_{Z,i} \to m_{Z,i} & P_{H,i} \to m_{H,i} \\
  0 & K_{B,i} \to m_{B,i} & K_{Z,i} \to m_{Z,i} & K_{H,i} \to m_{H,i}
\end{array}
\]

for \(0 \leq i < r\). The point of this procedure is the creation of a presentation of \(C^i\) as \(P_{C,i}/K_{C,i}\) with \(V_d\)-strict maps between these modules.

Now we assemble a resolution for \(C^\bullet\) as follows. First find a \(V_d\)-strict resolution using the method of [7] for \(B^0\). With lemma 3.7 find a resolution for \(Z^0, H^0, C^0, B^1\). Then feed the obtained resolution for \(B^1\) into lemma 3.7, resulting in resolutions for \(Z^1, H^1, C^1, B^2\). Et cetera.

Let us denote the \(k\)-th module of the resolution for \(X^i\) (with \(X\) being \(Z, C, B\) or \(H\)) by \(P_{X,i}^k\). We define a map \(\delta_{C,i}^k\) from \(P_{C,i}^k\) to \(P_{C,i+1}^k\) as the combined maps \(P_{C,i}^k \to P_{B,i+1}^k \to P_{Z,i+1}^k \to P_{C,i+1}^k\), multiplied by \((-1)^k\). So up to sign \(\delta_{C,i}^k\) is \(P_{C,i}^k = P_{B,i}^k \oplus P_{H,i}^k \oplus P_{B,i-1}^k \to P_{B,i+1}^k \to P_{H,i+1}^k \oplus P_{B,i+1}^k\). Clearly \(P_{C,i}^k \to P_{C,i+1}^k \to P_{C,i+2}^k\) is the zero map and the horizontal maps (in \(i\)-direction) are \(V_d\)-strict.

We have created a double complex \(P_{C,\bullet}^\bullet\) of free \(D_n\)-modules. Moreover, the associated total complex is quasi-isomorphic to \(C^\bullet\) and clearly \(V_d\)-adapted.

**Proposition 3.8.** \(\text{T}^\bullet (P_{C,\bullet}^\bullet)\) is in fact \(V_d\)-strict.

**Proof.** To that end assume that the element \(p = p_i^0 \oplus p_{i+1}^1 \oplus \cdots \oplus p_{r-i}^{r-i}\) \(\in \text{T}^i (P_{C,\bullet}^\bullet) = P_{C,i}^0 \oplus P_{C,i+1}^1 \oplus \cdots \oplus P_{C,r-1}^{r-i}\) is in the image of the total differential \(\delta_T\), and that the \(V_d\)-degree of \(p\) under the shift vectors is \(e\). We need to take a closer look at the maps and modules in front of us.

\(P_{C,i}^k\) is by construction \(P_{B,i}^k \oplus P_{H,i}^k \oplus P_{B,i+1}^k\). The map \(P_{C,i}^{k+1} = P_{B,i}^k \oplus P_{H,i}^k \oplus P_{B,i+1}^k \to P_{C,i}^k = P_{B,i}^k \oplus P_{H,i}^k \oplus P_{B,i+1}^k\) is on the first component the differential from the resolution \(P_{C,\bullet}^\bullet\) while the map from \(P_{H,i}^k \oplus P_{B,i+1}^k \to P_{C,i}^k\) is defined using certain lifts, obtained while using lemma 3.6. Inspection shows that the matrix which represents \(P_{C,i}^{k+1} \to P_{C,i}^k\)
where the $\delta$ are the differentials of the various resolutions for $B$ and $H$ and $\phi^{k+1}_1, \psi^{k+1}_1, \psi^{k+1}_2$ are the maps that are used to produce the mentioned lifts. Note that $\psi_1, \psi_2, \phi$ are all $V_d$-adapted by construction.

We shall argue by falling induction on the variable $s$, starting with $s = r - 1$, that the components $p^{s-i}_s$ of $p$ may be assumed to be zero modulo images of degree no greater than $e$ under the total differential. We will at the same time show that we may assume that the third component of $p^{s-i}_s$ is zero. For $s < r - 1$ this will follow from the induction. For $s = r - 1$ it follows from the fact that $B^r = 0$.

So assume that $0 \leq s \leq r - 1$, that $p$ has only zero components beyond the $s$-th component and that the third piece (to $P^{s-i}_{B,s+1}$) of the $s$-th component of $p$ is zero.

The following lemma will essentially show that our $p$ is then in fact image of an element in $Tot^{s-i}(P^\bullet_{C,s})$ with zero component in $P^{s-i}_{B,s+1}$ and only zeros in all columns beyond the $s$th.

**Lemma 3.9.** Let $(a, b, 0) \in P^{s-i}_{C,s}$ and assume $(a, b, 0) = \delta_C(\alpha, \beta, \gamma)$ with $(\alpha, \beta, \gamma) \in P^{s-i+1}_{C,s}$. Then $(\alpha, \beta, 0) = \delta_C(\alpha', \beta', 0)$ for some $(\alpha', \beta', 0) \in P^{s-i+1}_{C,s}$ where $V_d\deg(\alpha', \beta') \leq V_d\deg(\alpha, \beta, \gamma)$.

**Proof.** By construction $(\psi_1(\gamma), \psi_2(\gamma))$ is in $\ker(P^{s-i}_{B,s} \oplus P^{s-i}_{H,s} \rightarrow P^{s-i-1}_{B,s} \oplus P^{s-i-1}_{H,s})$. Since this kernel is exactly $\delta_Z(P^{s-i+1}_{B,s} \oplus P^{s-i+1}_{H,s}), (\psi_1(\gamma), \psi_2(\gamma)) = \delta_Z(\alpha', \beta')$ where we can pick $\alpha'$ and $\beta'$ to be of $V_d$-degree at most $V_d\deg(\gamma)$. Thus $\delta_C(\alpha, \beta, \gamma) = (\delta_B \alpha + \psi_1 \gamma + \phi \beta, \delta_H \beta + \psi_2 \gamma, 0) = \delta_C(\alpha, \beta, 0) + \delta_C(\alpha', \beta', 0)$. Set $(\alpha', \beta') = (\alpha'' + \alpha, \beta'' + \beta)$. \hfill $\square$

Now write $p^{s-i}_s = (a, b, 0)$. The lemma tells us that $b$ equals $\delta_H(b_1)$ for some $b_1 \in P^{s-i+1}_{H,s}$, of $V_d$-degree at most $e$ because $P^{*}_{B,s}$ is $V_d$-strict. What can we say about $p - \delta_T(b_1)$, which we call $p$ from now on?

Certainly the $V_d$-degree is at most $e$, it is in the image of $\delta_T$, only the first component of $p^{s-i}_s$ is nonzero and all components beyond the $s$th one are zero.

**Lemma 3.10.** Suppose $(a, 0, 0)$ is the image under $\delta_C$ of $(\alpha, \beta, \gamma) \in P^{s-i+1}_{C,s}$. Then there is $\alpha' \in P^{s-i+1}_{B,s}$ with $\delta_C(\alpha', 0, 0) = (a, 0, 0)$ and $\alpha'$ can be chosen to be of $V_d$-degree no bigger than $V_d\deg(a)$.

**Proof.** By the previous lemma, we can assume that $(a, 0, 0)$ is the image of $(\alpha, \beta, 0)$. By construction, $\phi(b) \in \ker(P^{s-i}_{B,s} \rightarrow P^{s-i-1}_{B,s}) =$
\[ \text{im}(P_{B,s}^{s-i+1} \to P_{B,s}^{s-i}). \] As \( P_{B,s}^* \) is a \( V_d \)-strict resolution, \( \phi(b) = \delta_B(\alpha'') \) for some \( \alpha'' \in P_{B,s}^{s-i+1} \). Hence \( \delta_C(\alpha, \beta, 0) = (\delta_B \alpha + \phi \beta, 0, 0) = \delta_C(\alpha, 0, 0) + \delta_C(\alpha'', 0, 0). \)

Since \( \delta_B \) is \( V_d \)-strict and \( \phi \) is \( V_d \)-adapted, we can choose \( \alpha' = \alpha + \alpha'' \) to be of \( V_d \)-degree at most \( V_d \deg(a) \).

The component of \( p \) in \( P_{C,s}^{s-i} \) looks like \( (a, 0, 0) \). Lemma 3.10 tells us that since \( p \) is an image under \( \delta_T \), \( a = \delta_B(\alpha) + (-1)^{s-1} \alpha' \) where \( \alpha \) lives in \( P_{B,s}^{s-i+1} \) (a component of \( P_{C,s}^{s-i+1} \)) and \( \alpha' \in P_{B,s}^{s-i} \) (a component of \( P_{C,s}^{s-i-1} \)).

Then the third component of \( p \) in \( P_{C,s}^{s-i-1} \) must be exactly \( \delta_B(\alpha') = (-1)^{s-1} \delta_B(a) \). Replace \( p \) by \( p - \delta_T((-1)^{s}a) \), this \( -a \) positioned in the \( P_{B,s}^{s-i-1} \)-component of \( P_{C,s}^{s-i-1} \).

The result is a \( p \) with zero component for \( P_{C,s}^{s-i} \) and zero component in the third component for \( P_{C,s}^{s-i-1} \) which differs from the original \( p \) by the \( \delta_T \)-boundary of an element of \( V_d \)-degree at most \( e \).

Now repeat the argument for \( p_{r-2}^{r-i-2}, p_{r-3}^{r-i-3}, \ldots \).

We have proved

**Theorem 3.11.** If \( C^* \) is a complex of \( D_n \)-modules that is bounded below and above and presentations of all \( C^i \) in terms of generators and relations are given, then one can produce a \( V_d \)-strict complex of free \( D_n \)-modules that is quasi-isomorphic to \( C^* \).

**Remark 3.12.** It is not true that the total complex associated to any double complex with \( V_d \)-strict rows and columns is \( V_d \)-strict. This would be equivalent to saying that all finite subsets of a free \( D_n \)-module form a \( G \)-basis for any order refining \( V_d \)-degree. Consider for example the diagram

\[
D_1[1] \xrightarrow{\partial_1} D_1[0] \quad \text{with} \quad 0 \xrightarrow{-(\partial_1 - 1)} D_1[1]
\]

\[ \text{im}(F_H^1[1,1](\text{Tot}^1)) \] but it is not in \( \text{im}(F_H^0[1,1](\text{Tot}^1) \to F_H^0([0](\text{Tot}^0))) \), although it is of \( V_1 \)-degree 0.

4. **Computing Cohomology of \( \Omega^*(D_n) \otimes_{D_n} MV^* \)**

For this section let \( A^*[m_a] \) be a given \( V_d \)-strict free complex. We shall assume further that the cohomology modules of \( A^*[m_a] \) are specializable to \( H = \text{Var}(x_1, \ldots, x_d) \). The main purpose of this section
is to determine a suitable truncation of $A^\bullet[m_\bullet]$ such that the cohomology of $\Omega \otimes_{D_n} A^\bullet$ is captured by the “tensor product” of $\Omega$ with that truncation.

Recall that in [8] $A^\bullet$ is a $V_n$-strict resolution of a specializable module $M$ and the truncation is determined by considering roots of the $b$-function $b(M)$ corresponding to restriction to the origin ([8], Algorithm 5.4).

4.1. Let $H$ be the subspace defined by $x_1 = \ldots = x_d = 0$, and $A^\bullet$ a $V_d$-strict complex of free $D_n$-modules. As a first step, pick generators $\kappa_{i,l}$ for the kernel modules $Z^i = Z^i(A^\bullet)$ of $A^\bullet$. To each of them is associated a degree in the shifted $V_d$-filtration from $A^i$ which we shall call $\lambda_{i,l}$.

Let a bar denote cosets of elements of $Z^i$ in $H^i = H^i(A^\bullet)$. Recall that we agreed to write $\theta_j = x_1 \partial_1 + \ldots + x_j \partial_j$ for $1 \leq j \leq n$. Since $D_n \cdot \bar{\kappa}_{i,l}$ is a specializable $D_n$-module, there is a $b$-function $b_{i,l}(\theta_d)$ associated to it which corresponds to the restriction of $D_n \cdot \bar{\kappa}_{i,l}$ to $x_1 = \ldots = x_d = 0$. Therefore, $b_{i,l}(\theta_d)\kappa_{i,l} \in F_H^{-1}(D_n) \cdot \kappa_{i,l} + \text{im}(A^{i-1} \to A^i)$.

Let $b(\theta_d)$ be the least common multiple of all $b_{i,l}(\theta_d - \lambda_{i,l})$.

4.2. Then consider the associated complex of graded $\text{gr}^H_\bullet(D_n)$-modules $\oplus F_H^j A^\bullet / F_H^{j+1} A^\bullet$.

Now assume that the $\kappa_{i,l}$ form a $G$-basis for $Z^i$ under the order on $A^i$. Then for all $\zeta \in Z^i$, $\zeta = \sum \alpha_{i,l}(\zeta) \kappa_{i,l}$ with $V_{\text{deg}}(\alpha_{i,l}(\zeta) \kappa_{i,l}) \leq V_{\text{deg}}(\zeta)$. Hence the $\kappa_{i,l}$ are generators for $\text{gr}^H_\bullet(H^i)$ and moreover $\text{gr}^H_\bullet(Z^i) = \sum \text{gr}^{j-\lambda_{i,l}}_H(D_n) \kappa_{i,l}$. Since $\text{im}(F_H^{-1}(D_n) \cdot \kappa_{i,l}) + \text{im}(A^{i-1} \to A^i)$ we have $\text{gr}^j_\bullet(H^i A^\bullet) = \sum \text{gr}^{j-\lambda_{i,l}}_H(D_n) \kappa_{i,l}$.

Then observe the following:

\begin{align*}
(4.1) \quad b(\theta_d + j) \text{gr}^j_\bullet(H^i A^\bullet) &= b(\theta_d + j) \sum \text{gr}^{j-\lambda_{i,l}}_H(D_n) \kappa_{i,l} \\
(4.2) &= \sum \text{gr}^{j-\lambda_{i,l}}_H(D_n) b(\theta_d + \lambda_{i,l}) \kappa_{i,l} \\
(4.3) &= 0
\end{align*}

because $b(\theta_d + \lambda_{i,l})$ sends $\kappa_{i,l}$ into $F_H^{-1}(D_n) \cdot \kappa_{i,l} + \text{im}(A^{i-1} \to A^i)$, which is zero in $\text{gr}^{\lambda_{i,l}}_H[m_i](H^i(A^\bullet))$.

**Remark 4.1.** $b(\theta_d)$ is then a multiple of the $b$-function of $H^i(A^\bullet)$ with respect to the given shift vectors.

This paves the way for a result related to [8], Proposition 5.2. The proof is very similar to the one given there.

We need to introduce a number of Koszul complexes. Let $L$ be a $\text{gr}^H_\bullet(D_n)$-module and let $L_j, j \in \mathbb{Z}$ be subgroups of $L$ such that $x_i L_j \subseteq \text{gr}^H_\bullet(D_n)$ for all $i, j$.

Let $\mathcal{L}$ be a $\text{gr}^H_\bullet(D_n)$-module and let $\mathcal{L}_j, j \in \mathbb{Z}$ be subgroups of $\mathcal{L}$ such that $x_i \mathcal{L}_j \subseteq \text{gr}^H_\bullet(D_n)$ for all $i, j$. Then for all $i, j$ we have $\mathcal{L}_j \subseteq \text{gr}_\bullet(D_n)$.

We can then consider the Koszul complex $K^n_{i,j}(\mathcal{L}_j)$ associated to $\mathcal{L}_j$ with respect to the given shift vectors. Let $K^n_{i,j}(\mathcal{L}_j)^\bullet$ be the Koszul complex associated to $\mathcal{L}_j$ with respect to the given shift vectors. Let $K^n_{i,j}(\mathcal{L}_j)^\bullet$ be the Koszul complex associated to $\mathcal{L}_j$ with respect to the given shift vectors.
\( \mathcal{L}_{j-1} \) for \( 1 \leq i \leq d \). In that case we will say that the \( \mathcal{L}_i \) give an \( H \)-filtration for \( \mathcal{L} \). For any integer \( k \) let \( \mathcal{K}^\bullet(\mathcal{L}, x_1, \ldots, x_d)[k] \) be the Koszul complex

\[
0 \to \mathcal{L}_{k+d} \otimes \mathbb{Z} \bigwedge^0 \mathbb{Z}^d \to \mathcal{L}_{k+d-1} \otimes \mathbb{Z} \bigwedge^1 \mathbb{Z}^d \to \cdots \to \mathcal{L}_k \otimes \mathbb{Z} \bigwedge^d \mathbb{Z}^d \to 0
\]
equipped with the usual Koszul maps \( \delta(u \otimes e_{i_1} \wedge \cdots \wedge e_{i_j}) = \sum_i x_i u \otimes e_i \wedge e_{i_1} \wedge \cdots \wedge e_{i_j} \).

Unifying all the graded pieces, let \( \mathcal{K}^\bullet(\mathcal{L}_*, x_1, \ldots, x_d) \) be the usual Koszul complex of \( \mathcal{L} \) relative to \( x_1, \ldots, x_d \).

More generally, for a complex of \( H \)-filtered \( \text{gr}_H^{\bullet}(D_n) \)-modules \( (\mathcal{L}^i, \delta^i) \) with morphisms that respect the \( H \)-filtration we define inductively \( \mathcal{K}^\bullet(\mathcal{L}^i[k], x_1, \ldots, x_d) \) as the total complex of the double complex

\[
\mathcal{K}^\bullet(\mathcal{L}^i; x_1, \ldots, x_d-1)[k]
\begin{array}{c}
\downarrow (-1)^{i}x_d \ \ \ \\
\mathcal{K}^\bullet(\mathcal{L}^i_{i+1}; x_1, \ldots, x_d-1)[k]
\end{array}
\]

where \( \mathcal{K}^\bullet(\mathcal{L}^i_*; \theta)[k] = \mathcal{L}^i[k] = \cdots \to \mathcal{L}^i_k \to \mathcal{L}^i_{k+1} \to \cdots \), the \( k \)-th piece of the original complex.

Notice that \( \mathcal{K}^\bullet(\mathcal{L}^i_*; x_1, \ldots, x_d)[k] \) is the graded component of the usual Koszul complex \( \mathcal{K}^\bullet(\mathcal{L}^i, x_1, \ldots, x_d) \) associated to \( \mathcal{L}^i \) and \( x_1, \ldots, x_d \) that “ends” in \( V_d \)-degree \( k \).

The following theorem explains which graded pieces \( \mathcal{K}^\bullet(\mathcal{L}^i_*; x_1, \ldots, x_d)[k] \) of \( \mathcal{K}^\bullet(\mathcal{L}^i_*; x_1, \ldots, x_d) \) are responsible for nontrivial cohomology pieces of \( \mathcal{K}^\bullet(\mathcal{L}^i_*; x_1, \ldots, x_d) \).

**Theorem 4.2.** Suppose given is a complex of graded \( \text{gr}_H^{\bullet}(D_n) \)-modules \( \mathcal{L}^i \) (i.e., \( \text{gr}_H^{\bullet}(D_n) \mathcal{L}^i_j \subseteq \mathcal{L}^i_{j+k} \)) where the maps between the \( \mathcal{L}^i \) preserve the grading. Assume that there is a polynomial \( b(\theta) \) in \( \mathbb{C}[\theta] \) that satisfies \( b(\theta_d + j) \ker(\mathcal{L}_{i_j}^i \to \mathcal{L}_{i+1}^i) \subseteq \im(\mathcal{L}_{i-1}^i \to \mathcal{L}_i^i) \) for all \( j \) and all \( i \). Let \( k \) be an integer for which \( b(k) \neq 0 \). Then \( \mathcal{K}^\bullet(\mathcal{L}^i, x_1, \ldots, x_d)[k] \) is exact.

**Proof.** The complex \( \mathcal{K}^\bullet(\mathcal{L}^i, x_d) \) is \( V_{d-1} \)-graded in the obvious way. Here is the essential idea of the argument:

Induction claim. If \( b(\theta_d + j) \) kills cohomology in \( \mathcal{L}^i \) of \( V_d \)-degree \( j \), then \( b^2(\theta_{d-1} + j) \) kills cohomology of \( V_{d-1} \)-degree \( j \) in \( \mathcal{K}^\bullet(\mathcal{L}^i, x_d) \). In other words, cohomology of \( \mathcal{K}^\bullet(\mathcal{L}^i, x_d)[j] \).

We may assume that \( b(\theta_d) \) is not a constant since otherwise \( \mathcal{L}^i \) is exact and a spectral sequence argument shows that then \( \mathcal{K}^\bullet(\mathcal{L}^i, x_d) \) is exact as well.
So assume that \((u_{j+1}^{i+1}, u_j^i)\) is in \(K^\bullet(L^\bullet, x_d)[j]\) (whose \(i\)-th piece is \(L_j^{i+1} \oplus L_j^i\)) and suppose this element is in the kernel of the differential in \(K^\bullet(L^\bullet, x_d)\). Then we must have

\[
\delta^{i+1}u_{j+1}^{i+1} = 0,
\]

\[
x_du_{j+1}^{i+1} + \delta^i u_j^i = 0,
\]

\(\delta\) denoting the boundary map in \(L^\bullet\). By hypothesis on \(b\), \(b(\theta_d + j + 1)u_{j+1}^{i+1} = \delta^i u_j^i + 1\) for some \(u_{j+1}^{i+1} \in L_{j+1}^i\). So \(b(\theta_d + j + \partial_d x_d)u_{j+1}^{i+1} = \delta^i u_j^i + 1\) and therefore \(b(\theta_d + j)u_{j+1}^{i+1} + \partial_d x_d u_{j+1}^{i+1} = \delta^i u_j^i + 1\) for some \(V_d\)-homogeneous \(P \in F_H^0(D_n) \setminus F_H^{-1}(D_n)\). Hence \(b(\theta_d + j)u_{j+1}^{i+1} = \delta^i(u_{j+1} - \partial_d Pu_j)\) using relation (4.5). Let us write this as

\[
b(\theta_d + j)u_{j+1}^{i+1} = \delta^i(a_{j+1}^i),
\]

\(a_{j+1}^i \in (L_{j+1}^i \oplus 0) \subset K^{i-1}(L^\bullet, x_d)[j + 1]\).

This implies that if \((u_{j+1}^{i+1}, u_j^i)\) is in the kernel of \(\delta_T\), the differential on \(K^\bullet(L^\bullet, x_d)\), then \(b(\theta_d + j)(u_{j+1}^{i+1}, u_j^i)\) is, modulo the image of \(\delta_T\), congruent to an element \((0, v_j^i)\), which of course is also in the kernel of \(\delta_T\). So it suffices to show that any such kernel element \((0, v_j^i)\) satisfies \(b(\theta_d + j)(0, v_j^i) \in \text{im} \delta_T\).

Since \(\delta_T(0, v_j^i) = 0\), we must have \(\delta^i v_j^i = 0\). Hence \(b(\theta_d + j)v_j^i = \delta^i a_{j+1}^{i-1}\) for some \(a_{j+1}^{i-1} \in L_{j+1}^{i-1}\). Now \(b(\theta_d + j)v_j^i = b(\theta_d - 1 + j) v_j^i + x_d Q \partial_d v_j^i\) for some \(V_d\)-homogeneous \(Q \in F_H^0(D_n) \setminus F_H^{-1}(D_n)\). Therefore \(b(\theta_d + j)v_j^i = \delta^{i-1}a_{j}^{i-1} - x_d Q \partial_d v_j^i\).

Since \(\delta^i(Q \partial_d v_j^i) = -Q \partial_d \delta^i(v_j^i) = 0\), it follows that \(b(\theta_d + j)(0, v_j^i) = \delta_T^{-1}(Q \partial_d v_j^i, a_{j+1}^{i-1})\) and the induction claim is proved.

Now recall the inductive definition of \(K^\bullet(L^\bullet, x_1, \ldots, x_d)\), which together with the induction claim shows that if the cohomology of \(V_d\) degree \(j\) in \(L^\bullet\) is killed by \(b(\theta_d + j)\) then the cohomology of the complex \(K^\bullet(L^\bullet, x_1, \ldots, x_d)[j]\) is killed by \(b^{(2d)}(j)\).

The theorem now follows easily from the fact that \(\mathbb{C}\) is a domain. \(\square\)

Theorem 4.2 leads to the computation of de Rham cohomology as follows.

Recall that we need to compute \(\Omega \otimes_{D^n}^L MV^\bullet\), and notice that \(\Omega \otimes_{D^n}^L (-)\) is for any given complex quasi-isomorphic to \(K^\bullet(-, \partial_1, \ldots, \partial_n)\). In order to cope with the problem that we’d like to compute the complex \(K^\bullet(MV^\bullet, \partial_1, \ldots, \partial_n)\) and not \(K^\bullet(MV^\bullet, x_1, \ldots, x_n)\), we shall make use of the Fourier transform. Namely, \(H^i(K^\bullet(MV^\bullet, \partial_1, \ldots, \partial_n))\) is isomorphic to \(H^i(K^\bullet(MV^\bullet, x_1, \ldots, x_n))\) where \(\tilde{MV}^\bullet\) is the image of the complex \(MV^\bullet\) under the Fourier transform \(x \rightarrow \partial, \partial \rightarrow -x\).
Then $\tilde{MV}^\bullet$ may be replaced by a free $V_n$-strict complex as $A^\bullet$ constructed in section 3. In particular, $A^i$ is then a $\text{gr}_H^r(D_n)$-module and the cohomology of $A^\bullet$ is holonomic ([5]) and therefore specializable to the origin.

Let $b(s) \in \mathbb{C}[s]$ be the polynomial found in subsection 4.1 for $d = n$. Then $b(\theta_d + j) \text{gr}_H^j(H^iA^\bullet[\mathfrak{m}])$ is zero according to (4.1). Therefore by theorem 4.2, $b(j)$ kills the degree $j$ pieces of the cohomology of $K^\bullet(A^\bullet, x_1, \ldots, x_n)$. In other words, if $k_0, k_1$ are integers with $b(s) = 0$ only if $s \in [k_0, k_1] \cap \mathbb{Z}$, then $\text{gr}_H^j K^\bullet(A^\bullet[\mathfrak{m}], x_1, \ldots, x_n)$ is exact if $j \not\in [k_0, k_1] \cap \mathbb{Z}$. Let $\Omega = F(\Omega) = D_n / (x_1, \ldots, x_n) \cdot D_n$.

We need to make a convention about the $V_n$-filtration on tensor products with $\tilde{\Omega}$. If $A[\mathfrak{m}]$ is a free $H$-graded $D_n$-module with shift vector $\mathfrak{m}$ then $\tilde{\Omega} \otimes_{D_n} A[\mathfrak{m}]$ is filtered by $F_H^i[\tilde{\Omega} \otimes_{D_n} A] := \{ \tilde{\mathcal{P}} \otimes_{D_n} \mathcal{Q} | V_n \text{deg}(\mathcal{P}) + V_n \text{deg}[\mathfrak{m}](\mathcal{Q}) \leq i \}$. Note that as $\tilde{\Omega}$ equals $\mathbb{C}[\partial_1, \ldots, \partial_n]$ as right $D_n$-module, $F_H^i[\tilde{\Omega} \otimes_{D_n} A]$ equals $\{(P_1, \ldots, P_{\dim D_n} A) | P_i \in \mathbb{C}[\partial_1, \ldots, \partial_n], \deg_0(P_i) \leq \mathfrak{m}(i) \forall i \}$.

**Theorem 4.3.** The cohomology of $\tilde{\Omega} \otimes_{D_n} MV^\bullet$ can be computed as follows:

1. Compute $MV^\bullet$ as in [9], algorithm 5.1 as complex of finitely generated $D_n$-modules.
2. Compute a $V_n$-strict free complex
   $$\cdots \rightarrow A^{r-2}[\mathfrak{m}_{r-2}] \rightarrow A^{r-1}[\mathfrak{m}_{r-1}] \rightarrow 0$$
   quasi-isomorphic to $\tilde{MV}^\bullet$, $\tilde{MV}^\bullet$ denoting the image of $MV^\bullet$ under the Fourier automorphism.
3. Relative to the induced filtration on $H^i(A^\bullet[\mathfrak{m}])$ compute the $b$-functions $b_i(s)$ for the restriction of $H^i(A^\bullet[\mathfrak{m}])$ to the origin $x_1 = \ldots = x_n = 0$.
4. Let $b(s)$ be the least common multiple of all the $b_i$. Find integers $k_0, k_1$ with $(b(k) = 0, k \in \mathbb{Z}) \Rightarrow (k_0 \leq k \leq k_1)$.
5. $\Omega \otimes_{D_n} MV^\bullet$ is quasi-isomorphic to the complex
   
   $$(4.7) \quad \cdots \rightarrow F_H^{k_i}[\tilde{\Omega} \otimes_{D_n} A^i] \rightarrow F_H^{k_{i+1}}[\tilde{\Omega} \otimes_{D_n} A^{i+1}] \rightarrow \cdots$$

   shifted $n$ spots to the right.
   
   **Proof.** We already remarked that $H^i(K^\bullet(MV^\bullet, \partial_1, \ldots, \partial_n))$ is isomorphic to $H^i(K^\bullet(\tilde{MV}^\bullet, x_1, \ldots, x_n))$, and so we only need to show that the latter can be computed from (4.7).

   Almost tautologically,
   $$\cdots \rightarrow \text{gr}_H^k[m_i]\tilde{MV}^i \rightarrow \cdots \rightarrow \text{gr}_H^k[m_{r-2}]\tilde{MV}^{r-2} \rightarrow \text{gr}_H^k[m_{r-1}]\tilde{MV}^{r-1} \rightarrow 0$$
is quasi-isomorphic to
\[ \cdots \to \operatorname{gr}_H^k[\mathfrak{m}_i]A^i \to \cdots \to \operatorname{gr}_H^k[\mathfrak{m}_{r-2}]A^{r-2} \to \operatorname{gr}_H^k[\mathfrak{m}_{r-1}]A^{r-1} \to 0 \]
and therefore the same is true after application of \( K^\bullet(-,x_1,\ldots,x_n) \).
Moreover, \( K^\bullet(\operatorname{gr}_H^kA'[\mathfrak{m}_i],x_1,\ldots,x_n)[k] \) is quasi-isomorphic to \( \operatorname{gr}_H^k(\bar{\Omega} \otimes_{D_n} A^i)[\mathfrak{m}_i] \) shifted \( n \) places to the right. Hence \( K^\bullet(\operatorname{gr}_H^k MV^\bullet[\mathfrak{m}_i],x_1,\ldots,x_n)[k] \) is quasi-isomorphic to
\[
(4.8) \quad \cdots \to \operatorname{gr}_H^k[\mathfrak{m}_{r-2}](\bar{\Omega} \otimes_{D_n} A^{r-2}) \to \operatorname{gr}_H^k[\mathfrak{m}_{r-1}](\bar{\Omega} \otimes_{D_n} A^{r-1}) \to 0
\]
shifted \( n \) places to the right. By theorem 4.2 this latter complex is exact for all \( k \in (k_0,k_1] \cap \mathbb{Z} \).

Observe that \( F^k_H[0](\bar{\Omega} \otimes_{D_n} D_n) \) is zero if \( k < 0 \) for obvious reasons. Therefore \( F^k_H[\mathfrak{m}_i](\bar{\Omega} \otimes_{D_n} A^i) = 0 \) for \( k < \min_j \{ m_i(j) \} \). This together with the exactness of \( (4.8) \) for \( j < k_0 \) forces \( F^i_H[\mathfrak{m}_i]H^i(\bar{\Omega} \otimes_{D_n} A^\bullet) \) to vanish for \( j < k_0 \). Similarly, \( F^i_H[\mathfrak{m}_i]H^i(\bar{\Omega} \otimes_{D_n} A^\bullet) = F^{i+1}_H[\mathfrak{m}_i]H^i(\bar{\Omega} \otimes_{D_n} A^\bullet) \) for \( j \geq k_1 \).

Thus the cohomology of \( \bar{\Omega} \otimes_{D_n} A^\bullet \) is captured by the quotient of \( F^{k_1}_H[\mathfrak{m}_i]H^i(\bar{\Omega} \otimes_{D_n} A^\bullet) \) modulo \( F^{k_0-1}_H[\mathfrak{m}_i]H^i(\bar{\Omega} \otimes_{D_n} A^\bullet) \). The theorem follows. \( \square \)

**Remark 4.4.**

4.4.1. The quotient \( \frac{F^{k_1}_H[\mathfrak{m}_i](\bar{\Omega} \otimes_{D_n} A^i)}{F^{k_0-1}_H[\mathfrak{m}_i](\bar{\Omega} \otimes_{D_n} A^i)} \) should be thought of as a set of polynomials in \( \partial_1,\ldots,\partial_n \) of degrees bounded between \( k_1 - m_i(j) \) and \( k_0 - m_i(j) \).

4.4.2. Since \( K^{r-i}(\operatorname{gr}_H^kA^\bullet[\mathfrak{m}_i],x_1,\ldots,x_n) \) involves only terms from \( A^{r-1},\ldots,A^{r-i} \), the following statement can be made: if \( MV^\bullet \) is exact at \( r-i \) and beyond, then \( \bar{\Omega} \otimes_{D_n} MV^\bullet \) is exact at \( j \geq r-i \). That follows by considering \( b(s) \cong 1 \) which kills the last \( i-1 \) cohomology terms in \( MV^\bullet \), and inspecting the proof of theorem 4.2 one sees that then 1 also kills the last \( i-1 \) cohomology terms in \( K^\bullet(A^\bullet[\mathfrak{m}_i],x_1,\ldots,x_n) \).

It follows the well-known

**Corollary 4.5.** \( H^*_{dR}(U,\mathbb{C}) = 0 \) if \( i \geq n + \text{cd}(f_1,\ldots,f_r) \).

5. De Rham cohomology with support

Let \( Y,Z \) be two Zariski-closed subsets of \( X \). In this section we are concerned with finding an algorithm that computes the de Rham cohomology groups \( H^*_{dR,Z}(U,\mathbb{C}) \) of \( U = X \setminus Y \) with coefficients in \( \mathbb{C} \) and support in \( Z \).

\( H^*_{dR,Z}(U,\mathbb{C}) \) is defined as follows. Recall the de Rham complex \( \Omega^\bullet(U) \) on \( U \). Usual de Rham cohomology is defined as the hypercohomology of \( \Omega^\bullet(U) \) and not surprisingly de Rham cohomology with supports is
defined as the hypercohomology with supports in $Z$ of $\Omega^*(U)$. In other words, $H^i_{dR,Z}(U, \mathbb{C}) = H^i_*(R\Gamma_U(\Omega^*(U)))$.

As was pointed out by Hartshorne, there is a natural exact sequence
\[ (5.1) \quad \cdots \to H^i_{dR,Z}(U, \mathbb{C}) \to H^i_{dR}(U, \mathbb{C}) \to H^i_{dR}(U \setminus Z, \mathbb{C}) \to H^{i+1}_{dR,Z}(U, \mathbb{C}) \to \cdots \]
which indicates that $H^i_{dR,Z}(U, \mathbb{C})$ measures the change in cohomology due to the removal of $Z \cap U$ from $U$.

For the entire section let us assume that $Y = \text{Var}(F), F = (f_1, \ldots, f_r)$ and $Z = \text{Var}(G), G = (g_1, \ldots, g_s)$. Write $F \cdot G = \{f_i \cdot g_j\}$. As before we will write $F_I$ for $\prod_{i \in I} f_i$ and so on. There is a natural map of Mayer-Vietoris complexes
\[ MV^*(F \cdot G, F) \to MV^*(F \cdot G) \]
given by the natural projection
\[ \bigoplus_{|I|+|J|+|K|=l} R_{F_I G_J} \otimes_R R_{F_K} \to \bigoplus_{|I|+|J|=l} R_{F_I G_J} \]
sending each summand with $|K| > 0$ to zero. This map corresponds to the embedding $X \setminus (Y \cup Z) \hookrightarrow X \setminus Y$. Clearly the map is surjective and the kernel is the subcomplex of $MV^*(F \cdot G, F)$ consisting of those pieces which contain at least one factor from $F$. It is not hard to check that this kernel is exactly $MV^*(F) \otimes_R C^*(F \cdot G), C^*(F \cdot G)$ being the Čech complex to $F \cdot G$ given by $\bigotimes_{i,j}(0 \to R \xrightarrow{\text{nat}} R_{f_i g_j} \to 0)$.

Notice that the sequences
\[ 0 \to (MV^*(F) \otimes_R C^*(F \cdot G))^i \to MV^i(F \cdot G, F) \to MV^i(F \cdot G) \to 0 \]
are all split exact. Let $A^*$ be a resolution of $\Omega$ as right $D_n$-module, for example $A^*$ could be the global sections of the de Rham complex on $X$. Then there is a sequence of complexes
\[ (5.2) \quad A^* \otimes_{D_n} (MV^*(F) \otimes_R C^*(F \cdot G)) \to A^* \otimes_{D_n} MV^*(F \cdot G, F) \to A^* \otimes_{D_n} MV^*(F \cdot G) \]
with split exact rows. In other words, we have a short exact sequence of complexes.

As was explained in previous sections, the cohomology of $A^* \otimes_{D_n} MV^*(F \cdot G, F)$ is $H^i_{dR}(X \setminus Y, \mathbb{C})$ while the cohomology of $A^* \otimes_{D_n} MV^*(F \cdot G)$ is $H^i_{dR}(X \setminus (Y \cup Z), \mathbb{C})$ and the map on cohomology is induced by the natural inclusion.

Comparison of the long exact sequence (5.1) with the long exact sequence that results from the short exact sequence of complexes (5.2) shows that the cohomology of $A^* \otimes_{D_n} (MV^*(F) \otimes_R C^*(F \cdot G))$ is exactly $H^i_{dR,Z}(X \setminus Y, \mathbb{C})$. 


Computationally this is of course horrible: de Rham cohomology of $X \setminus Y$ and $X \setminus Z$ comes from the Mayer-Vietoris complex of $F$ and $G$ while here we have (approximately) the Mayer-Vietoris complex of $F \cup F \cdot G$. We shall try to improve this situation now. As a first step in that direction we point out that the long exact sequence (5.1) shows that for complements of affine closed varieties $H^i_{dR,Z}(X \setminus Y, C)$ is in fact nothing but the relative cohomology group $H^i(X \setminus Y, X \setminus (Y \cup Z); C)$.

Consider the space $X \setminus (Y \cap Z)$ and its open covering by the two sets $X \setminus Y$ and $X \setminus Z$. It follows from [3], Example 17.1, that this is an exact triad for homology with integer coefficients, and from [2], Theorem 11.4, that the same holds for cohomology with coefficients in $\mathbb{C}$. This means that the natural inclusion of pairs

$$(X \setminus Y, X \setminus (Y \cup Z)) \hookrightarrow (X \setminus (Y \cap Z), X \setminus Z)$$

induces an isomorphism between $H^i(X \setminus Y, X \setminus (Y \cup Z); C)$ and $H^i(X \setminus (Y \cap Z), X \setminus Z; C)$. This in turn implies that instead of $H^i_{dR,Z}(X \setminus Y, C)$ we may calculate $H^i_{dR,Z}(X \setminus (Y \cap Z), C)$ since the groups are isomorphic.

Now consider the natural projection of complexes

$$MV^\bullet(F,G) \rightarrow MV^\bullet(G)$$

given by $\bigoplus_{|I|+|J|=l} R_{F_I} \otimes_R R_{G_J} \rightarrow \bigoplus_{|I|=l} R_{F_I}$ induced by the inclusion $X \setminus Z \hookrightarrow X \setminus (Y \cap Z)$. As before, this induces a short exact sequence of complexes

$$0 \rightarrow MV^\bullet(F) \otimes_R C^\bullet(G) \rightarrow MV^\bullet(F,G) \rightarrow MV^\bullet(G) \rightarrow 0.$$ 

Tensoring over $D_n$ with the resolution $A^\bullet$ from above we discover that the cohomology of $A^\bullet \otimes_D (MV^\bullet(F) \otimes_R C^\bullet(G))$ is $H^i_{dR,Z}(X \setminus (Y \cap Z)) \cong H^i_{dR,Z}(X \setminus Y)$. Now the complexity is down to the level of computing $H^i_{dR}(X \setminus (Y \cap Z))$. It follows

**Algorithm 5.1.** Input: polynomials $F = \{f_1, \ldots, f_r\}$ defining $Y$ and $G = \{g_1, \ldots, g_s\}$ defining $Z$; $i \in \mathbb{N}$.

Output: The de Rham cohomology groups of $U = X \setminus Y$ with support in $Z$, $H^i_{dR,Z}(X \setminus Y, C)$, which equal the relative cohomology groups $H^i(X \setminus Y, X \setminus (Y \cup Z); C)$.

Begin

1. Compute the complex $MV^\bullet(F) \otimes_R C^\bullet(G)$ as a complex of left $D_n$-modules as in [9], algorithm 5.1.
2. Compute a free $V_n$-strict complex $A^\bullet$ quasi-isomorphic to the image of $MV^\bullet(F) \otimes_R C^\bullet(G)$ under the Fourier automorphism as in section 4.
3. Find the cohomology groups of $A^\bullet$ and compute the $b$-functions of the cohomology groups under the given shifts. Let $k_0$ and $k_1$ be lower and upper bounds of the roots of the $b$-functions.

4. Replace each $D_n$ in $A^\bullet$ by $k[\partial_1, \ldots, \partial_n] = \Omega$ and restrict the complex to the components between $V_n$-degree $k_0 - 1$ and $k_1$.

5. Take $n-i$-th cohomology of the resulting complex of $\mathbb{C}$-vectorspaces and return it.

End.

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School of Mathematics, University of Minnesota, Minneapolis, MN 55455

E-mail address: walther@math.umn.edu