A CRITICAL ANALYSIS OF THE PROTON FORM FACTOR
WITH SUDAKOV SUPPRESSION AND INTRINSIC
TRANSVERSE MOMENTUM

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Abstract

The behavior of the proton magnetic form factor is studied within the modified hard scattering picture, which takes into account gluonic radiative corrections in terms of transverse separations. We parallel the analysis given previously by Li and make apparent a number of serious objections. The appropriate cut-off needed to render the form-factor calculation finite is both detailed and analyzed by considering different cut-off prescriptions. The use of the maximum interquark separation as a common infrared cut-off in the Sudakov suppression factor is proposed, since it avoids difficulties with the $\alpha_s$-singularities and yields a proton form factor insensitive to the inclusion of the soft region which therefore can be confidently attributed to perturbative QCD. Results are presented for a variety of proton wave functions including also their intrinsic transverse momentum. It turns out that the perturbative contribution, although theoretically self-consistent for $Q^2$ larger than about 6 GeV$^2$ to 10 GeV$^2$, is too small compared to the data.

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I. INTRODUCTION

The proton magnetic form factor at large momentum transfer has been extensively analy- 
ed within perturbative Quantum Chromodynamics (pQCD) over the last decade [1–10]. 
The theoretical basis of these calculations is the hard scattering formula [11] in which 
the proton form factor is generically expressed as a convolution of a hard-scattering am- 
plitude $T_H$ and proton distribution amplitudes (DA) $\Phi$ which represent valence quark Fock 
state wave functions integrated over quark transverse momenta (defined with respect to the 
momentum of their parent proton):

$$
G_M(Q^2) = \int_0^1 [dx] \int_0^1 [dx'] |f_N(\mu_F)|^2 \Phi^*(x', \mu_F) T_H(x, x', Q, \mu) \Phi(x, \mu_F),
$$

(1.1)

where $Q^2$ is the invariant momentum transfer squared and $[dx] = dx_1 dx_2 dx_3 \delta(1 - \sum x_i)$, 
$x_i$ being the momentum fractions carried by the valence quarks. The renormalization scale 
is denoted by $\mu$ and the factorization scale by $\mu_F$. The latter scale defines the interface 
between soft physics—absorbed in the wave function—and hard physics, treated explicitly 
within pQCD. The dimensionful constant $f_N$ represents the value of the proton wave function 
at the origin of the configuration space and has to be determined nonperturbatively [2,4,8]. 
The residual (mainly perturbative) scale dependence of $f_N$ and that of the proton DA is 
controlled by the evolution equation [1].

To lowest order the hard scattering amplitude is calculated as the sum of all Feynman 
diagrams for which the three quark lines are connected pairwise by two gluon propagators. 
This allows the quarks in the initial and final proton to be viewed as moving collinearly up 
to transverse momenta of order $\mu_F$. It is then easy to show that $T_H \sim \frac{(\alpha_s(\mu))^2}{Q^4}$, wherein $\alpha_s$ 
is the running strong coupling constant in the one-loop approximation.

The Pauli form factor $F_2$ and hence the electric form factor $G_E$ cannot be calculated 
within the hard scattering picture (HSP), since they require helicity-flip transitions which 
are not possible for (almost) massless quarks. These form factors are dominated by sizeable 
higher twist contributions as we know from experiment [12,13]. Eq. (1.1) is obtained by
taking the + component of the electromagnetic vertex and represents the helicity-conserving part of the form factor.

The choice of the renormalization scale in the calculation of the proton form factor is a crucial point. Most authors [2,3,8,14] use a constant $\alpha_s$ outside the integrals over fractional momenta, with an argument rescaled by the characteristic virtualities for each particular model DA. Choosing $\mu$ that way and using DAs calculated by means of QCD sum rules—distributions whose essential characteristic is a strong asymmetry in phase space—results for $G_M$ have been obtained [8,10,14] that compare fairly well with the data [12,15]. On the other hand, the so-called “asymptotic” DA [1] $\Phi_{as} = 120 x_1 x_2 x_3$—into which any DA should evolve with $Q^2 \to \infty$—yields a vanishing result for $G_M^p$. However, for a renormalization scale independent of $x$, large contributions from higher orders are expected in the endpoint region, $x_i \to 0$. Indeed, for the pion form factor this has been shown explicitly, at least for the next-to-leading order [16,17]. Such large higher-order contributions would render the leading-order calculation useless. A more appropriate choice of the renormalization scale would be, e.g., $\sqrt{x_2 x'_2 Q}$, since such a scale would eliminate the large logarithms arising from the higher-order contributions. Unfortunately, this is achieved at the expense that $\alpha_s$ becomes singular in the endpoint regions. It has been conjectured [1] that gluonic radiative corrections (Sudakov factors) will suppress that $\alpha_s$-singularity and, therefore, in practical applications of the HSP one may handle this difficulty by cutting off $\alpha_s$ at a certain value, typically chosen in the range 0.5 to 0.7. Another, semi-phenomenological recipe to avoid the singularity of $\alpha_s$ is to introduce an effective gluon mass [18] which cut-offs the interaction at low $Q^2$ values.

Besides the extreme sensitivity of the form factors on the utilized DA and besides the problem with higher-order contributions and/or the singularity of $\alpha_s$, there is still another—perhaps more fundamental—difficulty with such calculations. Indeed, the applicability of (1.1) at experimentally accessible momentum transfer, typically a few GeV, is not a priori justified. It was argued by Isgur and Llewellyn-Smith [19] and also by Radyushkin [20] that the HSP receives its main contributions from the soft endpoint regions, rendering the pertur-
bative calculation inconsistent. Recently, this criticism has been challenged by Sterman and collaborators [21–23]. Based on previous works by Collins, Soper, and Sterman [24], they have calculated Sudakov corrections to the hard-scattering process taking into account the conventionally neglected transverse momentum, $k_\perp$, of the quarks. The Sudakov corrections damp those contributions from the endpoint regions in which transverse momenta of the quarks are not large enough to keep the exchanged gluons hard. Moreover, as presumed, the Sudakov corrections cancel the $\alpha_s$-singularity without introducing additional *ad hoc* cut-off parameters as for instance a gluon mass. Thus the modified HSP provides a well-defined expression for the form factor which takes into account the perturbative contributions in a self-consistent way, even for momentum transfers as low as a few GeV.

However, an important element has not been considered in the analyses of Refs. [22,23]. This concerns the inclusion of the intrinsic transverse momentum of the hadronic wave function. As it was recently shown by two of us [25] for the case of the pion form factor, the inclusion of the transverse size of the pion extends considerably the self-consistency region of the perturbative contribution down to values of momentum transfer unreachable by the Sudakov corrections alone. On the other hand, the incorporation of the $k_\perp$-dependence leads to a substantial decrease of the magnitude of the (leading-order) pion form factor. Unfortunately, a clear-cut comparison with the available data is not possible because of their low quality and the uncertainty in the determination of the pion-nucleon coupling constant [26]. Nevertheless, it seems reasonable to expect that the pion form factor receives considerable soft contributions in the presently accessible GeV region.

The aim of the present paper is to perform an analysis for the proton form factor within the modified HSP. One of our objectives is to critically examine Li’s approach [23] and to enlarge the theoretical framework by including the intrinsic $k_\perp$-dependence of the proton wave function. At the same time we want to clarify several technical points, which are absent in the pion case and are first encountered in the more complicated calculation of the proton form factor.

The purpose of our analysis is to investigate how reliably the perturbative contribution
to the proton form factor can be calculated and to answer the question whether there is a proton wave function—modeled on the basis of QCD sum rules [4,8]—which is capable of providing, in a theoretically self-consistent way, a good agreement with the data within the modified HSP. It is clear that being able to identify the leading-order perturbative contribution reliably allows us to estimate the size of soft contributions to the proton form factor, contributions which are not accounted for in the modified HSP formalism. [Note that the $k_\perp$-dependent effects taken into account in the modified HSP represent also soft contributions of higher-twist type.]

Sudakov suppression (which can be viewed as the perturbative part of the transverse wave function) and intrinsic $k_\perp$-dependence of the wave function may also have a lot of interesting consequences in other exclusive reactions. Thus, for instance, Sotiropoulos and Sterman [27] have applied these elements to near-forward proton-proton elastic scattering claiming that their interplay drives the transition of the fixed $s$ differential cross section from the $t^{-8}$ behavior at moderate $t$ to the $t^{-10}$ behavior at larger $t$, as predicted by dimensional counting rules [28].

The outline of the paper is as follows. In Sec. II we discuss the proton wave function. The modified HSP is treated in Sec. III. The discussion of the infrared (IR) cut-off prescription in the Sudakov factor and its effect on the $\alpha_s$-singularities is given in Sec. IV. The numerical results are presented in Sec. V and our conclusions are contained in Sec. VI.

**II. THE PROTON WAVE FUNCTION**

Similarly to Sotiropoulos and Sterman [27], we write the valence quark component of the proton state with positive helicity in the form
\[ |P, +> = \frac{1}{\sqrt{N_c}} \int_0^1 [dx] \int [d^2k_\perp] \{ \Psi_{123} M_{a_1a_2a_3}^{a_1a_2a_3} + \Psi_{213} M_{a_1a_2a_3}^{a_1a_2a_3} - (\Psi_{132} + \Psi_{231}) M_{a_1a_2a_3}^{a_1a_2a_3} \} \epsilon_{a_1a_2a_3}, \quad (2.1) \]

where we have assumed the proton to be moving rapidly in the 3-direction. Hence, the ratio of transverse to longitudinal momenta of the quarks is small. The measure over the transverse momentum integration is defined by

\[ [d^2k_\perp] = \frac{1}{(16\pi^3)^2} \delta^{(2)} \left( \sum_{i=1}^3 \vec{k}_{\perp i} \right) d^2k_{\perp 1} d^2k_{\perp 2} d^2k_{\perp 3}. \quad (2.2) \]

In the zero binding energy limit, which is characteristic for the parton picture, one has

\[ x_1 + x_2 + x_3 = 1 \quad \text{and} \quad \vec{k}_{\perp 1} + \vec{k}_{\perp 2} + \vec{k}_{\perp 3} = 0. \quad (2.3) \]

The three quark state with helicities \( \lambda_1, \lambda_2, \lambda_3 \) and colors \( a_1, a_2, a_3 \) is given by

\[ M_{a_1a_2a_3}^{a_1a_2a_3} = \frac{1}{\sqrt{x_1x_2x_3}} |u_{a_1}; x_1, \vec{k}_{\perp 1}, \lambda_1 > |u_{a_2}; x_2, \vec{k}_{\perp 2}, \lambda_2 > |d_{a_3}; x_3, \vec{k}_{\perp 3}, \lambda_3 >. \quad (2.4) \]

Since the orbital angular momentum is assumed to be zero, the proton helicity is the sum of the quark helicities. The quark states are normalized as follows:

\[ <q_{a'_i}; x'_i, \vec{k}'_{\perp i}, \lambda'_i|q_{a_i}; x_i, \vec{k}_{\perp i}, \lambda_i > = 2x_i(2\pi)^3 \delta_{a'_i a_i} \delta_{\lambda'_i \lambda_i} \delta(x'_i - x_i) \delta(\vec{k}'_{\perp i} - \vec{k}_{\perp i}). \quad (2.5) \]

From the permutation symmetry between the two \( u \) quarks and from the requirement that the three quarks have to be coupled to give an isospin 1/2 state it follows that Eq. (2.1) can be expressed in terms of only one independent scalar function \[2\]. In the sequel, \( \Psi \) denotes the momentum space wave function.

The subscripts on \( \Psi \) refer to the order of momentum arguments, for example \( \Psi_{123}(x, \vec{k}_{\perp}) = \Psi(x_1, \vec{k}_{\perp 1}; x_2, \vec{k}_{\perp 2}; x_3, \vec{k}_{\perp 3}) \). Note that, in general, the wave function depends on the factorization scale \( \mu_F \). We make the following convenient ansatz for the wave function:

\[ \Psi_{123}(x, \vec{k}_{\perp}) = \frac{1}{8\sqrt{N_c}} f_N(\mu_F) \Phi(x, \mu_F) \Omega(x, \vec{k}_{\perp}). \quad (2.6) \]

The distribution amplitude \( \Phi = V - A \) (in the notation of \[3\]) is defined in such a way that
\[
\int_0^1 [dx] \Phi_{123}(x, \mu_F) = 1,
\]

where an obvious abbreviated notation has been introduced. The DA can be expressed in terms of the eigenfunctions of the evolution equation \[1\], \(\Phi^n(x_i)\), which are linear combinations of Appell polynomials. Then the proton DA can be cast into the form

\[
\Phi_{123}(x, \mu_F) = \Phi_{123}^{as}(x) \sum_n B_n \left( \frac{\alpha_s(\mu_F)}{\alpha_s(\mu_0)} \right)^{\tilde{\gamma}_n/\beta_0} \tilde{\Phi}_{123}^n(x),
\]

where the notations of \[7\] are adopted. \(\Phi_{123}^{as}(x)\) is the asymptotic DA mentioned in the Introduction. The exponents \(\tilde{\gamma}_n\), driving the evolution behavior of the DA, are related to the anomalous dimensions of trilinear quark operators with isospin \(1/2\) (see \[29\]) and resemble the \(b_n\) in the Brodsky-Lepage notation \[1\]. Because they are positive fractional numbers increasing with \(n\), higher-order terms in \(2.8\) are gradually suppressed. The constants \(\tilde{\gamma}_n\) are given in Table 1; \(\beta_0 = 11 - 2n_f/3 = 9\) for three flavors.

Constraints on the DA are obtained implicitly by restricting their few first moments within intervals determined from QCD sum rules \[2,4,8\], which are evaluated at some self-consistently determined normalization point \(\mu_F\) of order 1 GeV (see, e.g., \[7\]):

\[
\Phi^{(n_1,n_2,n_3)}(\mu_0) = \int_0^1 [dx] x_1^{n_1} x_2^{n_2} x_3^{n_3} \Phi_{123}(x, \mu_0)
\]

In most model calculations, mentioned above, the moment constraints provided by QCD sum rules are used to determine the first five expansion coefficients \(B_n\), where \(B_0 = 1\) due to normalization (2.7). However, since the moments are burdened by errors, these expansion coefficients—although mathematically uniquely determined by the moments of corresponding order \[14\]—in practice their numerical values cannot be fixed precisely giving rise to different options for the proton DA. In our calculation of form factors we employ amplitudes complying with the Chernyak-Ogloblin-Zhitnitsky (COZ) sum-rule moment constraints. It was shown in \[30,31\] that such amplitudes constitute a finite orbit in the \((B_4, R \equiv |G_M|^4/G_M^0)\) plane ranging from COZ-like amplitudes \[8\] with \(R \leq 0.5\) to the recently proposed \[14\] het-
erotic one with \( R \approx 0.1 \). For the convenience of the reader, the QCD sum-rules constraints and the expansion coefficients \( B_n \) of selected model amplitudes are compiled in Table 1.

The \( k_\perp \)-dependence of the wave function is contained in the function \( \Omega \) which is normalized according to

\[
\int [d^2 k_\perp] \Omega_{123}(x, \vec{k}_\perp) = 1.
\]  

Due to (2.7) and (2.10), \( f_N \) is the value of the DA at the origin of the configuration space. Its evolution behavior is given by

\[
f_N(\mu_F) = f_N(\mu_0) \left( \frac{\alpha_s(\mu_F)}{\alpha_s(\mu_0)} \right)^{2/3} \beta_0 \]

and its value has been determined to be \( f_N(\mu_0) = (5.0 \pm 0.3) \times 10^{-3} \text{GeV}^2 \). \( \llbracket \)  

In Eq. (2.6) \( \Psi \) represents the soft part of the proton wave function, which results by removing the perturbative part and absorbing it into the hard-scattering amplitude \( T_H \). The perturbative tail of the full wave function behaves as \( 1/k_\perp^4 \) for large \( k_\perp \), whereas the soft part vanishes as \( 1/k_\perp^6 \) or faster. The nonperturbative or intrinsic \( k_\perp \)-dependence of the soft wave function, being related to confinement, is parametrized as a simple Gaussian according to

\[
\Omega_{123}(x, \vec{k}_\perp) = (16\pi^2)^2 \frac{a^4}{x_1x_2x_3} e^{x_3} \left[ -a^2 \sum_{i=1}^{3} k_{1i}^2 / x_i \right].
\]

This parametrization of the intrinsic \( k_\perp \)-dependence of the wave function, which is due to Brodsky, Huang, Lepage, and Mackenzie \([32]\), seems to be more favorable than the standard form of factorizing \( x \)- and \( k_\perp \)-dependencies. At least for the case of the pion wave function, this has recently been effected by Zhitnitsky \([33]\) on the basis of QCD sum rules. He finds that a factorizing wave function is in conflict with some general theoretical constraints which any reasonable wave function should comply. Zhitnitsky’s QCD sum-rule analysis of the pion wave function seems to indicate that the \( k_\perp \)-distribution may also show a double-hump structure, which means that small and large values of \( k_\perp \) are favored relative to intermediate values. It is likely that the proton wave function may exhibit a similar behavior, though
this kind of analysis has yet to be done. For the purposes of the present work we ignore this
possibility.

In (2.12) the parameter $a$ controls the root mean square transverse momentum (r.m.s.),
$\langle k_{\perp}^2 \rangle^{1/2}$, and the r.m.s. transverse radius of the proton valence Fock state. From the known
charge radius of the proton, we expect the r.m.s. transverse momentum to be larger than
about 250 MeV. The actual value of $\langle k_{\perp}^2 \rangle^{1/2}$ may be much larger than 250 MeV, e.g., 600 MeV
or so. Indeed, Sotiropoulos and Sterman [27] show that application of the modified HSP
to proton-proton elastic scattering leads to an approximate $t^{-8}$-behavior of the differential
cross section at moderate $|t|$. The behavior $d\sigma/dt \sim t^{-10}$, predicted by dimensional counting,
appears only at very large $|t|$. At precisely which value of $|t|$ the transition from the $t^{-8}$
to the $t^{-10}$ behavior occurs, depends on the transverse size of the valence Fock state of the
proton. Since the ISR [34] and the FNAL [35] data are rather compatible with a $t^{-8}$-behavior
of the differential cross section, Sotiropoulos and Sterman conclude that the transverse size
of the proton is small, perhaps $\leq 0.3$ fm. Correspondingly, the r.m.s. transverse momentum
is larger than 600 MeV. It is worth noting that such a value is supported by the findings of
the EMC group [36] in a study of the transverse momentum distribution in semi-inclusive
deep inelastic $\mu p$ scattering. A phenomenologically successful approach to the HSP, in which
baryons are viewed as bound states of a quark and an effective diquark, also uses a value
of this size for $\langle k_{\perp}^2 \rangle^{1/2}$ [37–39]. There is a second constraint on the wave function, viz. the
probability for finding three valence quarks in the proton:

$$P_{3q} = \frac{|f_N|^2}{3} (\pi a)^4 \int_0^1 [dx] \frac{2(\Phi_{123}(x))^2 + \Phi_{132}(x)\Phi_{231}(x)}{x_1 x_2 x_3} \leq 1. \quad (2.13)$$

In our numerical analysis to be presented in Sec. 5, we make use of two different values of
the r.m.s. transverse momentum, namely, one which is obtained by the requirement $P_{3q} = 1$
for a given wave function. [This corresponds to the minimum value of the r.m.s. transverse
momentum.] The other option for the r.m.s. transverse momentum we consider is the rather
large value of 600 MeV. In the latter case, the probability for the valence quark Fock state
depends on the wave function.
III. THE MODIFIED HARD SCATTERING PICTURE

Following Li [23], we write the proton form factor in the form

\[ G_M(Q^2) = \frac{16}{3} \int_0^1 [dx][dx'] \int [d^2 k_{\perp}][d^2 k'_{\perp}] \sum_{j=1}^2 T_{H_j}(x, x', \vec{k}_{\perp}, \vec{k}'_{\perp}, Q, \mu) Y_j(x, x', \vec{k}_{\perp}, \vec{k}'_{\perp}, \mu_F). \]  

(3.1)

Note, however, that our notation is slightly different compared to that of Li. Making use of the symmetry properties of the proton wave function under permutation, the contributions from the 42 diagrams involved in the calculation of the proton form factor in lowest order can be arranged into two reduced hard scattering amplitudes of the form

\[ T_{H_1} = \frac{2}{3} C_F \left( \frac{(4\pi\alpha_s(\mu))^2}{(1 - x_1)(1 - x'_1)Q^2 + (\vec{k}_{\perp 1} - \vec{k}'_{\perp 1})^2} \right) \left[ x_2 x'_2 Q^2 + (\vec{k}_{\perp 2} - \vec{k}'_{\perp 2})^2 \right], \]  

(3.2)

\[ T_{H_2} = \frac{2}{3} C_F \left( \frac{(4\pi\alpha_s(\mu))^2}{x_1 x'_1 Q^2 + (\vec{k}_{\perp 1} - \vec{k}'_{\perp 1})^2} \right) \left[ x_2 x'_2 Q^2 + (\vec{k}_{\perp 2} - \vec{k}'_{\perp 2})^2 \right], \]  

(3.3)

where \( C_F = 4/3 \) is the Casimir operator of the fundamental representation of \( SU(3)_c \). In the hard scattering amplitudes only the \( k_{\perp} \)-dependence of the gluon propagators is included, whereas that of the quark propagators has been neglected. It is expected that this technical simplification introduces only a minor error of about 10% in the final result. For the case of the pion form factor this has been explicitly demonstrated by Li [23].

The functions \( Y_j \) in (3.1) are short-hand notations for linear combinations of products of the initial and final state wave functions \( \Psi_{ijk} \Psi_{i'j'k'} \), weighted by \( x_i \)-dependent factors arising from the fermion propagators, namely:

\[ Y_1 = \frac{1}{(1 - x_1)(1 - x'_1)} \left\{ 4 \Psi'_{123} \Psi_{123} + 4 \Psi'_{132} \Psi_{132} + \Psi'_{231} \Psi_{231} + \Psi'_{321} \Psi_{321} + 2 \Psi'_{231} \Psi_{132} + 2 \Psi'_{321} \Psi_{123} + 2 \Psi'_{123} \Psi_{321} \right\} \]  

(3.4)

\[ Y_2 = \frac{1}{2(1 - x_2)(1 - x'_1)} \left\{ 3 \Psi'_{132} \Psi_{132} - \Psi'_{231} \Psi_{231} - \Psi'_{231} \Psi_{132} - \Psi'_{132} \Psi_{231} \right\} \]  

\[ - \frac{1}{(1 - x_3)(1 - x'_1)} \left\{ 4 \Psi'_{321} \Psi_{321} + \Psi'_{123} \Psi_{123} + 2 \Psi'_{321} \Psi_{123} + 2 \Psi'_{123} \Psi_{321} \right\}. \]  

(3.5)
Ignoring the transverse momenta in the hard scattering amplitudes (3.2) and (3.3), and inserting (2.6) and (2.10), one arrives at the standard HSP result for the magnetic form factor. Although this expression is correct in the asymptotic momentum domain, the transverse degrees of freedom are an essential ingredient of the formalism and neglecting them leads to inconsistencies in the endpoint regions, where one of the fractional momenta \(x_i\) or \(x'_i\) tends to zero. After all, it is precisely this approximation that is responsible for the inconsistencies mentioned in the Introduction. The power of combining the transverse momentum dependence of the hard scattering amplitude and radiative corrections in the form of Sudakov form factors was realized by Sterman and collaborators [21–23]. Ultimately, it leads to a suppression of contributions from the dangerous soft regions, where both the longitudinal and transverse momenta of the quarks are small.

In order to include the Sudakov corrections, it is advantageous to reexpress Eq. (3.1) in terms of the variables \(\vec{b}_i\), which are canonically conjugate to \(\vec{k}_{\perp i}\) and span the transverse configuration space. Then

\[
G_M(Q^2) = \frac{16}{3} \int_0^1 [dx][dx'] \int \frac{d^2b_1}{(4\pi)^2} \frac{d^2b_2}{(4\pi)^2} \sum_j \hat{T}_j(x, x', \vec{b}, Q, \mu) \hat{Y}_j(x, x', \vec{b}, \mu_F) e^{-S_j}, \quad (3.6)
\]

where the Fourier transform of a function \(f(\vec{k}_{\perp}) = f(\vec{k}_{\perp 1}, \vec{k}_{\perp 2})\) is defined by

\[
\hat{f}(\vec{b}) = \frac{1}{(2\pi)^4} \int d^2k_{\perp 1} d^2k_{\perp 2} \exp\{-i\vec{b}_1 \cdot \vec{k}_{\perp 1} - i\vec{b}_2 \cdot \vec{k}_{\perp 2}\} f(\vec{k}). \quad (3.7)
\]

Since the hard scattering amplitudes depend only on the differences of initial and final state transverse momenta, there are only two independent Fourier-conjugate vectors \(\vec{b}_1 (= \vec{b}'_1)\) and \(\vec{b}_2 (= \vec{b}'_2)\). They are, respectively, the transverse separation vectors between quarks 1 and 3 and between quarks 2 and 3. Accordingly, the transverse separation of quark 1 and quark 2 is given by

\[
\vec{b}_3 = \vec{b}_2 - \vec{b}_1. \quad (3.8)
\]

[Note that Sotiropoulos and Sterman [27] define the transverse separations in a cyclic way which results in the interchange \(\vec{b}_1 \leftrightarrow -\vec{b}_2\), as compared to our definition.]
The fact that there are only two independent transverse separation vectors is a consequence of the approximation made in the treatment of the hard scattering amplitudes (3.2) and (3.3) which disregards the $k_\perp$-dependence of the quark propagators. This approximation is justified by the enormous technical simplification it entails, given that the thereby introduced errors are very small. Then by virtue of rotational invariance of the system with respect to the longitudinal axis, the form factor (3.6) can be expressed in terms of a seven-dimensional integral instead of an eleven-dimensional one. Physically, the relations $\vec{b}_1 = \vec{b}'_1$, $\vec{b}_2 = \vec{b}'_2$ mean that the physical probe (i.e., the photon) mediates only such transitions from the initial to the final proton state, which have the same transverse configurations of the quarks.

The Fourier-transformed hard scattering amplitudes appearing in Eq. (3.6) read

$$
\hat{T}_1 = \frac{8}{3} C_F \alpha_s(t_{11}) \alpha_s(t_{12}) K_0 \left( \sqrt{(1-x_1)(1-x'_1)Qb_1} \right) K_0 \left( \sqrt{x_2 x'_2 Qb_2} \right),
$$

$$
\hat{T}_2 = \frac{8}{3} C_F \alpha_s(t_{21}) \alpha_s(t_{22}) K_0 \left( \sqrt{x_1 x'_1 Qb_1} \right) K_0 \left( \sqrt{x_2 x'_2 Qb_2} \right),
$$

where $K_0$ is the modified Bessel function of order 0 and $b_l$ denotes the length of the corresponding vector. We have now chosen the renormalization scale in such a way that each hard gluon carries its own individual momentum scale $t_{ji}$ as the argument of the corresponding $\alpha_s$. The $t_{ji}$ are defined as the maximum scale of either the longitudinal momentum or the inverse transverse separation, associated with each of the gluons:

$$
t_{11} = \max \left[ \sqrt{(1-x_1)(1-x'_1)Q}, 1/b_1 \right],
$$

$$
t_{21} = \max \left[ \sqrt{x_1 x'_1 Q}, 1/b_1 \right],
$$

$$
t_{12} = t_{22} = \max \left[ \sqrt{x_2 x'_2 Q}, 1/b_2 \right],
$$

One may think of other choices. However, they are not expected to lead to very different predictions for the form factor [23].

The quantities $\hat{Y}_j$ contain the same combinations of initial and final state wave functions as those in (3.4) and (3.5), the only difference being that now the products
\[ \Psi_{ij'k'} \Psi_{ijk} \] are replaced by corresponding products of Fourier-transformed wave functions: 
\[ \hat{\Psi}_{ij'k'}(x', \vec{b}, \mu_F) \hat{\Psi}_{ijk}(x, \vec{b}, \mu_F). \] Using (2.6) and (2.12), the Fourier transform of the wave function reads 
\[ \hat{\Psi}_{123}(x, \vec{b}, \mu_F) = \frac{1}{8\sqrt{N_c}} f_N(\mu_F) \Phi_{123}(x, \mu_F) \hat{\Omega}_{123}(x, \vec{b}), \tag{3.12} \]
where the Fourier-transform of the \( k_\perp \)-dependent part is given by 
\[ \hat{\Omega}_{123}(x, \vec{b}) = (4\pi)^2 \exp \left\{ -\frac{1}{4a^2} \left[ x_1 x_3 b_1^2 + x_2 x_3 b_2^2 + x_1 x_2 b_3^2 \right] \right\}. \tag{3.13} \]

The exponentials \( e^{-S_j} \) in (3.6) are the Sudakov factors, which incorporate the effects of gluonic radiative corrections. Because of this, (3.6) is not simply the Fourier transform of (3.1) but an expression comprising an additional physical input. Thus (3.6) may be termed the “modified hard-scattering formula”. On the ground of previous works by Collins and Soper [24], Botts and Sterman [21] have calculated a Sudakov factor using resummation techniques and having recourse to the renormalization group. They find Sudakov exponents of the form 
\[ S_j = \sum_{l=1}^{3} s(x_l, \hat{b}_l, Q) + \int_{1/b_l}^{t_{ij}} \frac{d\hat{\mu}}{\hat{\mu}} \gamma_q(g(\hat{\mu}^2)) \]
\[ + \sum_{l=1}^{3} s(x'_l, \hat{b}_l, Q) + \int_{1/b_l}^{t'_{ij}} \frac{d\hat{\mu}}{\hat{\mu}} \gamma_q(g(\hat{\mu}^2)) \], \tag{3.14} \]
wherein the Sudakov functions \( s(\xi_l, \hat{b}_l, Q) \) are given by 
\[ s(\xi_l, \hat{b}_l, Q) = \frac{A^{(1)}}{2\beta_1} \hat{q}_l \ln \left( \frac{\hat{q}_l}{\hat{b}_l} \right) + \frac{A^{(2)}}{4\beta_1} \left( \frac{\hat{q}_l}{\hat{b}_l} - 1 \right) - \frac{A^{(1)}}{2\beta_1} (\hat{q}_l - \hat{b}_l) \]
\[ - \frac{A^{(1)}}{16\beta_1^2} \hat{q}_l \ln \left( 2 \hat{b}_l + 1 \right) + \frac{A^{(2)}}{4\beta_1} \ln \left( 2 \hat{b}_l + 1 \right) \]
\[ - \left[ \frac{A^{(2)}}{4\beta_1^2} - \frac{A^{(1)}}{4\beta_1} \ln(\hat{q}_l) \right] \ln \left( \frac{\hat{q}_l}{\hat{b}_l} \right) \]
\[ - \frac{A^{(1)}}{32\beta_1^3} \left[ \ln^2(2\hat{q}_l) - \ln^2(2\hat{b}_l) \right]. \tag{3.15} \]

Here \( \xi_l = x_l \) or \( x'_l \) (\( l = 1, 2, 3 \)) and the variables \( \hat{q} \) and \( \hat{b} \) are defined as follows:
\[ \hat{q}_l = \ln[\xi_l Q/(\sqrt{2} \Lambda_{QCD})] \] \tag{3.16}
\[ \hat{b}_l = \ln\left[1/\tilde{b}_l A_{\text{QCD}}\right]. \] (3.17)

The coefficients \(A^{(i)}\) and \(\beta_i\) are

\[
A^{(1)} = \frac{4}{3}, \quad A^{(2)} = \frac{67}{9} - \frac{1}{3} \pi^2 - \frac{10}{27} n_f + \frac{8}{3} \beta_1 \ln\left(\frac{1}{2} e^\gamma\right),
\]

\[
\beta_1 = \frac{33 - 2n_f}{12}, \quad \beta_2 = \frac{153 - 19n_f}{24},
\] (3.18)

where \(n_f\) is the number of quark flavors and \(\gamma = 0.5772\ldots\) is the Euler-Mascheroni constant. In the sequel \(n_f = 3\) is used. \[\gamma_q = -\frac{\alpha_s}{\pi} + O(\alpha_s^2)\] is the anomalous quark dimension in the axial gauge [10].

The Sudakov function, \(s(\xi_l, \tilde{b}_l, Q)\), in (3.15) takes into account leading and next-to-leading gluonic radiative corrections of the form shown in Fig. 1. The quantities \(\tilde{b}_l (l = 1, 2, 3)\) are infrared cut-off parameters, naturally related to, but not uniquely determined by the mutual separations of the three quarks [24]. A physical perspective on the choice of the IR cut-off is provided by the following analogy to ordinary QED. One expects that because of the color neutrality of a hadron, its quark distribution cannot be resolved by gluons with a wave length much larger than a characteristic quark separation scale; meaning that long wave length gluons probe the color singlet proton and hence radiation is damped. Radiative corrections with wave lengths between the IR cut-off and an upper limit (related to the physical momentum \(Q\)) yield to suppression; it is understood that still softer gluonic corrections are already taken care of in the hadron wave function, whereas harder gluons are considered as part of \(T_H\).

Different choices of the IR cut-off have been used in the literature: Thus, Li [23] chooses \(\tilde{b}_l = b_l\) (this choice hereafter is termed the “L” prescription), whereas Hyer [11] in his analysis of the proton-antiproton annihilation into two photons and of the time-like proton form factor as well as Sotiropoulos and Sterman [27] take \(\tilde{b}_1 = b_2, \tilde{b}_2 = b_1, \tilde{b}_3 = b_3\) (this choice is denoted the “H-SS” prescription). Still another possibility, and the one proposed in the present work for reasons that will be explained below is to use as IR cut-off the maximum of the three interquark separations, i.e., to set
\[
\bar{b} \equiv \max\{b_1, b_2, b_3\} = \bar{b}_1 = \bar{b}_2 = \bar{b}_3.
\] (3.19)

This choice, designated by “MAX”, is analogous to that in the meson case, wherein the quark-antiquark distance naturally provides a secure IR cut-off. The specific features of each particular cut-off choice will be discussed in detail in Sec. [IV].

The integrals in (3.14) arise from the application of the renormalization group equation (RGE). The evolution from one scale value to another is governed by the anomalous dimensions of the involved operators. The integrals combine the effects of the application of the RGE on the wave functions and on the hard scattering amplitude. The range of validity of (3.13) for the Sudakov functions is limited to not too small \( \bar{b}_l \) values. Whenever \( 1/\bar{b}_l \) is large relative to the hard (gluon) scale \( \xi_l Q \), the gluonic corrections are to be considered as higher-order corrections to \( T_H \) and hence are not contained in the Sudakov factor but are absorbed in \( T_H \). For that reason, Li [23] sets any Sudakov function \( s(\xi_l, \bar{b}_l, Q) \) equal to zero whenever \( \xi_l \leq \sqrt{2}/(Q\bar{b}_l) \). Moreover, Li holds the Sudakov factor \( e^{-S} \) equal to unity whenever it exceeds this value, which is the case in the small \( \bar{b}_l \)-region. Actually, the full expression (3.14) shows in this region a small enhancement resulting from the interplay of the next-to-leading logarithmic contributions to the Sudakov exponents and the integrals over the anomalous dimensions. We follow the same lines of argument in our analysis.

The IR cut-offs \( 1/\bar{b}_l \) in the Sudakov exponents mark the interface between the nonperturbatively soft momenta, which are implicitly accounted for in the proton wave function, and the contributions from soft gluons, incorporated in a perturbative way in the Sudakov factors. Obviously, the IR cut-off serves at the same time as the gliding factorization scale \( \mu_F \) to be used in the evolution of the wave function. For that reason, Li [23] as well as Sotiropoulos and Sterman [27] take \( \mu_F = \min\{1/\bar{b}_l\} \). The “MAX” prescription (3.19), adopted in the present work, naturally complies with the choice of the evolution scale proposed in [23,27].
IV. DISCUSSION OF THE $\alpha_s$-SINGULARITIES

It is well known that the inclusion of an $x$-dependent renormalization scale in the argument of $\alpha_s$ within the standard HSP of Brodsky-Lepage \cite{1} presents the difficulty that the value of $\alpha_s$ becomes singular in the endpoint regions. To render the form factors (Eq. (1.1)) finite, additional external parameters, like an effective gluon mass \cite{18} or a cut-off prescription have to be introduced. Technically, such parameters play the rôle of IR regulators serving to regularize one of the gluon propagators, which may become soft along the boundaries of phase space (see, e.g., \cite{4}). One of the crucial advantages of the modified HSP, proposed by Sterman and collaborators \cite{21,22,23}, is that there is no need for external regulators because the Sudakov factor may suppress the singularities of the “bare” (one-loop) $\alpha_s$ inherently. Indeed in the pion case, it was shown \cite{22} that the transverse quark-antiquark separation is tantamount to an IR regulator which suffices to cancel all singularities from the soft region.

Concerning the proton form factor, the situation is much more complicated because more scales are involved and hence the choice of the appropriate IR cut-off parameters $\bar{b}_f$ is not obvious, as discussed in Sec. III. As we shall effect in the following, the cancellation of the $\alpha_s$-singularities by the Sudakov factor depends sensitively on that particular choice.

In Fig. 2 we display the exponential of the Sudakov function $\exp[-s(\xi_t, \tilde{b}_f, Q)]$ for $Q = 30 \Lambda_{QCD}$ by imposing Li’s requirement \cite{22}: $s(\xi_t, \tilde{b}_f, Q) = 0$ whenever $\xi_t \leq \sqrt{2}/Q\tilde{b}_f$. Ultimately, the cancellation of the $\alpha_s$-singularities relies on the fact that whenever one of the $\alpha_s$ tends to infinity (owing to the limit $t_{ji} \rightarrow \Lambda_{QCD}$), the Sudakov factor $e^{-S_j}$ rapidly decreases to zero. As it can be observed from Fig. 2 this is not the case in the region determined by $\xi_t \leq \sqrt{2}\Lambda_{QCD}/Q$ and simultaneously $\tilde{b}_f\Lambda_{QCD} \rightarrow 1$, where $\exp[-s(\xi_t, \tilde{b}_f, Q)]$ is fixed to unity. In the pion case this does not matter, since the other $\exp[-s(1 - \xi, \tilde{b}, Q)] \rightarrow 0$ faster than any power of $\ln[1/(\tilde{b}\Lambda_{QCD})]$ and, consequently, the Sudakov factor drops to zero. In contrast, the treatment of the proton form factor is more subtle. In that case, $e^{-S_j}$ does not necessarily vanish fast enough to guarantee the cancellation of the $\alpha_s$-singularities. This
can be illustrated by the following configuration: if, say, $x_1 < \sqrt{2} \Lambda_{\text{QCD}}/Q$ and $\tilde{b}_1 \Lambda_{\text{QCD}} \to 1$ then $x_2 + x_3 \approx 1$ and $x_2$ can have any value between 0 and 1 $- x_1$. Since $\tilde{b}_2 \Lambda_{\text{QCD}}$ is unrestricted within the limits 0 and 1, the corresponding exponentials of the Sudakov functions $\exp[-s(x_2, \tilde{b}_2, Q)]$ and $\exp[-s(x_3, \tilde{b}_3, Q)]$ do not automatically fall off to zero in order to yield sufficient suppression of the $\alpha_s$-singularities, unless all three $\tilde{b}_l$ are coerced to be equal. If the three $\tilde{b}_l$ are allowed to be different, then the Sudakov factor provides suppression only through the contributions of the anomalous dimensions. According to the “L” and “H-SS” prescriptions, which, in general, allow for different $\tilde{b}_l$ in the Sudakov functions, the integrand in (3.6) has singularities behaving as

$$\sim \ln \left( \frac{1}{\tilde{b}_l \Lambda_{\text{QCD}}} \right)^\kappa$$

(4.1)

for $\tilde{b}_l \Lambda_{\text{QCD}} \simeq 1$ and $x_l$ hold fixed. The maximum degree of divergence is given by

$$\kappa = \frac{1}{\beta_0} \left( \frac{4}{3} + 2 \tilde{\gamma}_{\max} - 2 \right) + 1,$$

(4.2)

where the first term $4/3$ comes from the evolution of $f_N$, (2.11) and the constant $\tilde{\gamma}_{\max}$ is related to the anomalous dimension driving the evolution behavior of the proton DA, see (2.8) and Table 1: $\tilde{\gamma}_{\max}$ is the maximum value of the $\{\tilde{\gamma}_n\}$ within a given polynomial order of the expansion of the DA. We reiterate that the $\tilde{\gamma}_n$ are positive fractional numbers increasing with $n$. Thus the singular behavior of the integrand becomes worse as the expansion in terms of Appell polynomials extends to higher and higher orders. The term $-2$ in (4.2) stems from the integrations over the anomalous dimensions in the Sudakov factor $e^{-S_j}$ (see (3.14)). Finally, the term 1 originates from that $\alpha_s(t_{jk})$ which becomes singular in (3.3), c.f., (3.11). Which one of the $\alpha_s$ couplings becomes actually singular, depends on the prescription imposed on the IR cut-off parameters $\tilde{b}_l$. The integral (3.6) does not exist if $\tilde{\gamma}_{\max} \geq 1/3$. As Table 1 reveals, this happens already for proton DAs which include Appell polynomials of order 1, i.e., for all DAs except for the asymptotic one: $\Phi_{\text{as}} = 120 x_1 x_2 x_3$. Thus application of the “L” and “H-SS” prescriptions on the choice of the IR cut-off parameters $\tilde{b}_l$ to the proton form factor entails the modified HSP to be invalid. In view of these results, Li’s analysis of the proton form factor [23] seems to be seriously flawed.
A simple recipe to bypass the singular behavior of the integrand is to ignore completely
the evolution of the DA or to “freeze” it at any (arbitrary) value larger than $\Lambda_{\text{QCD}}$. Hyer [11] suggested to take for the factorization scale $\mu_F = \max (1/b_l)$. In this case, the $\tilde{\gamma}_{\text{max}}$ appears in (4.2) only if all three $\tilde{b}_l$ tend to $1/\Lambda_{\text{QCD}}$ at once. But then at least one of the $\exp[-s(\xi_l, \tilde{b}_l, Q)]$ drops to 0 faster than any power of $\ln \left(1/\tilde{b}_l\Lambda_{\text{QCD}}\right)$. Apparently, Hyer’s choice of the factorization scale avoids singularities of the form (4.1), but seems to us physically implausible. Since he only presents numerical results for the proton form factor in the time-like region, we cannot compare with his results directly.

Another option, and actually the one proposed in this work, is to use a common IR cut-off not only for the evolution of the wave function but also in the Sudakov exponent. For a common cut-off $\tilde{b}$, the Sudakov factors always cancel the $\alpha_s$-singularities; if, for a given $l$, we are in the dangerous region, $\xi_l < \sqrt{2}\Lambda_{\text{QCD}}/Q$, $\tilde{b}\Lambda_{\text{QCD}} \to 1$, at least one of the other two Sudakov functions lies in the region $\xi_{l'} > \sqrt{2}\Lambda_{\text{QCD}}/Q$, $\tilde{b}\Lambda_{\text{QCD}} \to 1$ ($l' \neq l$) and therefore provides sufficient suppression, as outlined above. In particular, we favor $\tilde{b} = \max\{b_l\}$ as the optimum choice (“MAX” prescription), since it does not only lead to a regular integral but also to a non-singular integrand. The Sudakov factor $e^{-S_1}$ subject to the “L” and “MAX” prescriptions is plotted for a specific quark configuration in Fig. 3. This figure makes it apparent that the Sudakov factor in connection with the “MAX” prescription is unencumbered by singularities in the dangerous soft regions. As a consequence of the regularizing power of the “MAX” prescription, the perturbative contribution to the proton form factor (3.6) saturates in the sense that the results become insensitive to the inclusion of the soft regions. A saturation as strong as possible is a prerequisite for the self-consistency of the modified HSP, as will be discussed in Sec. V.

To demonstrate the amount of saturation, we calculate the proton form factor through (3.6), employing a cut-off procedure to the $b_l$-integrations at a maximum value $b_c$. In Fig. 4 the dependence of $G_M$ on $b_c$ for the three choices, labeled: “L”, “H-SS”, and “MAX” is shown using, for reasons of comparison with previous works, the COZ DA and ignoring evolution. [Evolution has been dispensed with to avoid the concomitant singularity in $Q^4 G_M$]
as $b_c \Lambda_{QCD} \rightarrow 1$ when imposing the “L” and “H-SS” prescriptions.] As one sees from the figure, the “MAX” prescription leads indeed to saturation; the soft region $b_c \Lambda_{QCD} > 0.7$ does not contribute to the form factor substantially. In fact, already 50% of the result are obtained from the regions with $b_c \Lambda_{QCD} < 0.48$. Note that $\alpha_s$ increases to a value of 0.95 at $b_c \Lambda_{QCD} \approx 0.48$. This indicates that a sizeable fraction of the contributions to the form factor is accumulated in the perturbative region.

Unfortunately, this saturation is achieved at the expense of a rather strong damping of the perturbative contribution to the proton form factor. Using the two other prescriptions (“L” and “H-SS”) and ignoring evolution, we have found larger results for $G_M$, but no indication for saturation: The additional contributions to the form factor gained this way are accumulated exclusively in the soft regions, i.e., for values of $b_c \Lambda_{QCD}$ near 1. These findings are in evident contradiction to Li’s results (figure 5 in [23]) for which an acceptable saturation has been claimed. On the other hand, we can qualitatively confirm the saturation behavior of the proton form factor calculated by Hyer [41] in the time-like region. Since we regard a saturation behavior as a stringent test for the self-consistent applicability of pQCD, calculations which accumulate large contributions from soft regions (large $b_c$) cannot be considered as theoretically legitimate.

The rôle of the evolution effect subject to the “MAX” prescription is also exhibited in Fig. 4. It shows that the effect of evolution is large, although finite, owing to the strong suppression provided by the Sudakov factor. Note that according to our discussion in Sec. III, the factorization scale is $\mu_F = 1/\tilde{b}$. The significant feature of the evolution effect is that it tends to neutralize the influence of the IR cut-off. Thus one obtains larger values of the proton form factor at the expense of a slightly worse saturation.

V. NUMERICAL ANALYSIS

In this section we give numerical results for the proton form factor. In these calculations we throughout employ the “MAX” prescription with evolution included, using
$\Lambda_{\text{QCD}} = 180$ MeV and $\mu_0 = 1$ GeV. Before proceeding with the presentation of our final results, let us investigate the effect of including the intrinsic transverse momentum in our calculations. The $k_\perp$-dependence of the proton wave function effectively introduces a confinement scale in the formalism, the importance of which may be appreciated by looking at Fig. 4. This figure shows results, obtained for the COZ DA without $k_\perp$-dependence and for two different values of $\langle k_\perp^2 \rangle^{1/2}$. To describe the intrinsic $k_\perp$-dependence, one can use (2.12) or, after transforming to the transverse configuration space, (3.13). Notice that in Li’s approach the Gaussian in (3.13) has been replaced by unity. The oscillator parameter $a$ is determined in such a way that either the normalization of the wave function $P_{3q}$ is unity (resulting into $\langle k_\perp^2 \rangle^{1/2} = 271$ MeV for the COZ DA), or by inputing the value of the r.m.s. transverse momentum. In the second case, we use a value of 600 MeV (see the discussion in Sec. II), which implies $P_{3q} = 0.042$. As can be seen from this figure, the predictions for the form factor are quite different for the three cases. The intrinsic $k_\perp$-dependence of the wave function leads to further suppression of the perturbative contribution, which becomes substantial if the r.m.s. transverse momentum is large. On the other hand, this suppression is accompanied by an increasing amount of saturation, since also the Gaussian (3.13) suppresses predominantly contributions from the soft regions, viz., the large $b$-regions. In contrast to the Sudakov factor, however, this suppression is $Q$-independent. The interplay of the two effects, Sudakov suppression and intrinsic transverse momentum, leads to a different $Q$-behavior of the form factor depending on the value of the r.m.s transverse momentum, as can be seen from Fig. 4. The $Q$-dependence beyond 10 GeV$^2$ is rather weak, being approximately compatible with dimensional counting (modulo logarithmic corrections). For very large values of $Q$ beyond 1000 GeV$^2$ the three curves have approached each other within 10% accuracy. This happens when the Sudakov factor dominates the Gaussian (3.13) and selects those configurations with small interquark separations. In this region, which one may consider as the pure perturbative region, the results for the form factor are independent of the confinement scale introduced by the r.m.s. transverse momentum.

The penalty of the additional suppression of the perturbative contribution caused by the
Gaussian (3.13) is mitigated by the advantage that the perturbative contribution becomes more self-consistent than by the Sudakov factor alone. This is indicated in the enhanced amount of saturation with increasing r.m.s. transverse momentum. Adapting the criterion of self-consistency, originally suggested by Li and Sterman [22] for the pion case, namely that 50% of the results are accumulated at moderate values of the coupling constant, say, $\alpha_s^2 \leq 0.5$, we find self-consistency for $Q^2 \approx 7 \text{ GeV}^2$ (for the COZ DA).

Finally, in Fig. 6, we demonstrate the effect of different proton DAs on the form factor. To this end, we investigate a set of 45 DAs [30,31], which all respect the QCD sum-rule constraints [8]. The results for the various DAs—or more precisely wave functions, since we include their intrinsic transverse momentum dependence—obtained under the “MAX” prescription with evolution included, form the shaded area shown in the figure. All wave functions are normalized to unity and the corresponding r.m.s. transverse momenta vary between 267 MeV and 317 MeV (see Table 1). The theoretical form-factor predictions span a “band” congruent to the “orbit” of solutions found in [30,31]. The upper bound of the “band” corresponds to the DA COZ$_{up}$, which yields the maximum value of the form-factor ratio $|G^n_M|/G^p_M = 0.4881$ in the standard HSP. The lower limit of the “band” is obtained using the DA “low” (sample 8 in [31]) with $|G^n_M|/G^p_M = 0.175$. Explicitly shown are the results for the COZ DA [8], its optimized version (with respect to the sum-rule constraints) and the “heterotic” DA, recently proposed by two of us in [10]. We note that the differences among these curves practically disappear already at about $Q^2 = 80 \text{ GeV}^2$, despite the fact that these amplitudes have distinct geometrical characteristics [14].

Since the true valence Fock state probability is likely much smaller, or invariably the r.m.s. transverse momentum larger than of order of 300 MeV, the “band” describes rather maximal expectations for the (leading-order) perturbative contributions to the form factor; at least for proton wave functions of the type we utilize. Comparison with the experimental data reveals that the theoretical predictions amount, at best, to approximately 50% of the measured values. This is the benchmark against which we have to discern novelties and aberrations. Closing this discussion we note that, since we are calculating only the helicity-
conserving part of the current matrix element it is not obvious whether we should compare the theoretical predictions with the data for the Sachs form factor \( G_M \) or the Dirac form factor \( F_1 \). Therefore we have exhibited in Fig. 6 both sets of data \([15]\) for comparison. Since the two sets of data differ by only 10\%, our conclusions concerning the smallness of the theoretical results remain unaffected.

The various model wave functions led to self-consistency of the perturbative contribution, i.e., 50\% of the results are accumulated in regions where \( \alpha_s^2 \leq 0.5 \), in the range of \( Q^2 \) between 6 and 10 GeV\(^2\).

### VI. SUMMARY AND CONCLUSIONS

The objective of the present work has been to derive the proton magnetic form factor within the modified version (Sec. III) of the standard Brodsky-Lepage HSP \([1]\), a scheme which takes into account gluonic radiative corrections \([24]\) in terms of transverse separations. This is done by incorporating in the formalism the Sudakov factor, calculated by Botts and Sterman \([21]\). There are already some interesting applications of the modified HSP \([22,23,25,27,41–43]\). The significant element of this type of analyses is that the \( \alpha_s \)-singularities, arising from hard-gluon exchange and evolution, can be cancelled without introducing free external parameters. We emphasize that in contrast to pure phenomenological recipes (e.g., the introduction of a gluon mass), the modified HSP provides an explicit scheme how the IR protection of the “bare” \( \alpha_s \) proceeds through gluonic radiation accumulated in the Sudakov factor. Thus, in the modified HSP, one may conceive of the (finite) IR-protected \( \alpha_s \) as being the effective coupling. By this procedure the potentially dangerous soft regions of momenta are suppressed entailing also a reduction of the perturbative contribution to the form factor. While in the pion case \([22]\), it is fortunate that the cancellation of the \( \alpha_s \)-singularities comes out naturally, Li’s approach to the proton form factor \([23]\) leads to a lack of complete cancellation of the \( \alpha_s \)-singularities (see Sec. IV). Without evolution of the proton wave function the emerging singularities in \((3.6)\) are still integrable, but logarithmic
corrections due to evolution yield ultimately to uncompensated singularities.

On the grounds of our discussion, we are reasonably confident that Li’s treatment can be cured within the modified HSP. We suggest to use a a common IR cut-off in the Sudakov exponents (3.14) and Sudakov functions (3.15): viz., the maximum transverse separation. This “MAX” prescription provides sufficient IR protection, since even with evolution, the integrand in (3.6) remains finite. A significant feature of this treatment is that the proton form factor saturates, i.e., it becomes insensitive to the contributions from large transverse separations. The other choices of the IR cut-off (“L”, “H-SS”), we have discussed, do not lead to saturation.

However, this reliable saturation and IR protection of the form factor is achieved at the expense of a strong reduction of the perturbative contribution to the form factor. The damping of the proton form factor becomes even stronger if one takes into account the intrinsic transverse momentum dependence of the proton wave function (see Fig. 4 and Fig. 5). This has been done by assuming a non-factorizing $x$ and $k_\perp$-dependence of the wave function of the Brodsky-Lepage-Huang-Mackenzie type and fixing the value of $\langle k_\perp^2 \rangle^{1/2}$ either via the valence quark probability $P_{3q}$ or by inputing the value 600 MeV by hand. A remarkable finding is that the form factor calculated within the modified HSP, appropriately extended to include the intrinsic transverse momentum of the proton wave function, shows only a mild dependence on the particular model DA.

The perturbative contribution to the form factor becomes self-consistent in all cases for momentum transfers larger than 6 to 10 GeV$^2$. The actual value of the onset of self-consistency depends on the particular wave function and the r.m.s. transverse momentum chosen. Self-consistency is defined such that 50% of the result are accumulated in regions where $\alpha_s^2$ is smaller than 0.5 (Sec. V).

Comparing our theoretical results with the data, it turns out that they fall short by at least 50%. This is true not only for the COZ DA, (which we have exemplarily used to facilitate comparison with previous works) but actually for the whole spectrum of amplitudes determined in [30,31] and found to comply with the COZ sum-rule requirements. Depending
on the actual value of the r.m.s. transverse momentum, the reduction of the perturbative contribution may be even stronger than 50%.

The fact that in all considered cases the self-consistently calculated perturbative contribution to the proton form factor fails to reproduce the existing data, is perhaps a signal that soft contributions (higher twists) not accounted for so far by the modified HSP should be included. Such contributions comprise, e.g., improved and/or more complicated wave functions, orbital angular momentum, higher Fock components, quark-quark correlations (diquarks), radiative corrections to the quark and gluon condensates, quark masses, etc. Also remainders of genuine soft contributions, like vector-meson-dominance terms or the overlap of the soft parts of the wave functions (Feynman contributions), may still be large at accessible momentum transfers. The rather large value of the Pauli form factor $F_2$ around 10 GeV$^2$, as found experimentally [12], indicates that sizeable higher-twist contributions still exist in that region of momentum transfer [13]. One may suspect similar or even larger higher-twist contributions to the helicity non-flip current matrix element controlling $F_1$ and $G_M$. Large (perturbative) higher-order corrections to the hard-scattering amplitude cannot be excluded as well, since their size has not yet been estimated. In analogy to the Drell-Yan process, these corrections may be condensed in a K-factor multiplying the leading-order perturbative result. However, with our choice of the renormalization scale, the K-factor is expected to be close to unity. At least for the case of the pion form factor, calculations of the K-factor to one-loop order exist [16,17], which indicate that choosing the renormalization scale analogously to ours, the value of the K-factor is indeed close to unity.

In conclusion we note that it was not our primary aim to use the modified HSP to obtain best agreement with the data, although from our point of view this scheme represents a decisive step towards a deeper understanding of the electromagnetic form factors. In the present work the focus has been placed on theoretical problems, overlooked previously.
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TABLES

**Table 1.** Expansion coefficients for selected DAs, taken from [8] and [30,31]. Our notation is adopted from [7]. The $\tilde{\gamma}_n$ are related to the anomalous dimensions of trilinear quark operators. The associated r.m.s transverse momentum and oscillator parameter for each model wave function, normalized via $P_{3q} = 1$, are shown.

| $n$ | $\tilde{\gamma}_n$ | $B_n$(COZ$^{up}$) | $B_n$(COZ) | $B_n$(COZ$^{opt}$) | $B_n$(het) | $B_n$(low) |
|-----|------------------|----------------|------------|----------------|------------|------------|
| 1   | 20/9             | 3.2185         | 3.6750     | 3.5268         | 3.4437     | 4.1547     |
| 2   | 24/9             | 1.4562         | 1.4840     | 1.4000         | 1.5710     | 1.4000     |
| 3   | 32/9             | 2.8300         | 2.8980     | 2.8736         | 4.5937     | 3.3756     |
| 4   | 40/9             | -17.3400       | -6.6150    | -4.5227        | 29.3125    | 26.1305    |
| 5   | 42/9             | 0.4700         | 1.0260     | 0.8002         | -0.1250    | -0.5855    |

$\langle k^2_{\perp} \rangle^{1/2}$ [MeV]  271  271  267  317  299

$a$ [GeV$^{-1}$]  0.9893  0.9939  1.0069  0.8537  0.9217
FIGURES

FIG. 1. Illustration of gluonic radiative corrections to the proton magnetic form factor in the axial gauge.

FIG. 2. The exponential of the Sudakov function $s(\xi_l, \vec{b}_l, Q)$ vs. $\xi_l$ and $\vec{b}_l\Lambda_{\text{QCD}}$ for $Q = 30\Lambda_{\text{QCD}}$. In the hatched area the Sudakov function is set equal to zero according to Li’s requirement [23].

FIG. 3. The Sudakov factor $e^{-S_1}$ vs. $b_1\Lambda_{\text{QCD}}$ and $b_2\Lambda_{\text{QCD}}$ evaluated at $Q = 30\Lambda_{\text{QCD}}$, and $x_1 = x'_1 = 0.9$, $x_2 = x_3 = x'_2 = x'_3 = 0.05$ assuming a linear quark configuration ($\vec{b}_1$ and $\vec{b}_2$ are parallel to each other). The upper and lower figures correspond to the “L” and “MAX” prescriptions, respectively.

FIG. 4. The proton magnetic form factor as a function of $b_c\Lambda_{\text{QCD}}$. The curves shown are obtained at $Q = 30\Lambda_{\text{QCD}}$ for the COZ DA. The solid line corresponds to the “MAX” prescription including evolution. The dotted (dashed, dashed-dotted) line represents results using the “H-SS” (“MAX”, “L”) prescription ignoring intrinsic $k_\perp$ and evolution.

FIG. 5. The influence of the intrinsic transverse momentum on the proton magnetic form factor. The curves shown are obtained for the COZ DA by imposing the “MAX” prescription including evolution. The solid line represents the results without $k_\perp$-dependence, whereas the dashed and dotted lines are obtained with $<k_\perp^2>^{1/2} = 271$ MeV and 600 MeV, respectively.

FIG. 6. The proton magnetic form factor vs. $Q^2$. Data are taken from [15]. The $G_M^p$ data are represented by black circles, whereas those for $F_1^p$ are indicated by open circles. The theoretical results are obtained using the “MAX” prescription including evolution and normalizing the wave functions to unity. The shadowed strip indicates the range of predictions derived from the set of DAs determined in [30,31] in the context of QCD sum rules (see text). The solid (dashed, dotted) line corresponds to the COZ (heterotic, optimized COZ) DA.
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\[ \exp[-s(\xi_l, \tilde{b}_l, Q)] \]
Figure 3
Figure 4
\[ Q^4 G_M(Q^2) \text{ [GeV}^4]\]
Figure 6