DEMUSHKIN’S THEOREM IN CODIMENSION ONE

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Abstract. Demushkin’s Theorem says that any two toric structures on an affine variety \( X \) are conjugate in the automorphism group of \( X \). We provide the following extension: Let an \((n-1)\)-dimensional torus \( T \) act effectively on an \( n \)-dimensional affine toric variety \( X \). Then \( T \) is conjugate in the automorphism group of \( X \) to a subtorus of the big torus of \( X \).

Introduction

This paper deals with automorphism groups of toric varieties \( X \) over an algebraically closed field \( K \) of characteristic zero. We consider the following problem: Let \( T \times X \rightarrow X \) be an effective regular torus action. When is this action conjugate in \( \text{Aut}(X) \) to the action of a subtorus of the big torus \( T_X \subset X \)? Some classical results are:

- For complete \( X \), the answer is always positive, because then \( \text{Aut}(X) \) is an affine algebraic group with maximal torus \( T_X \), compare [5] and [4].
- For \( X = \mathbb{K}^m \) and \( \dim(T) \geq m - 1 \), positive answer is due to Białynicki-Birula, see [2] and [3].
- For \( X \) affine and \( \dim(T) = \dim(X) \), positive answer is due to Demushkin [6] and Gubeladze [8].

We focus here on the case \( \dim(T) = \dim(X) - 1 \). As in [6] and [8], we shall assume that \( X \) has no torus factors. We do not insist on \( X \) being affine; we just require that \( X \) has no “small holes” in the sense that there is no open toric embedding \( X \rightarrow X' \) with \( X' \setminus X \) nonempty of codimension at least two. Under these assumptions we prove, see Theorem 3.1:

**Theorem.** Let \( T \times X \rightarrow X \) be an effective regular action of an algebraic torus \( T \) of dimension \( \dim(X) - 1 \). Then \( T \) is conjugate in \( \text{Aut}(X) \) to a subtorus of the big torus \( T_X \subset X \).

In the case of tori \( T \) of dimension strictly less than \( \dim(X) - 1 \) the “toric linearization problem” stated at the beginning is wide open, even for actions of an \((n - 2)\)-dimensional torus on the affine space \( \mathbb{A}^n \). In the latter setting, there is a positive result for the case of a fixed point set of positive dimension, see [11], and a deep theorem saying that \( C^* \)-actions on \( \mathbb{C}^3 \) are linearizable, see [10] and [12].

Let us outline the main ideas of the proof of our theorem. In contrast to [6] and [8], our approach is geometric. Since any two toric structures on \( X \) are conjugate in the automorphism group of \( X \), see [1], it suffices to extend the \( T \)-action to an almost homogeneous torus action on \( X \). This is done in three steps:

First lift the \( T \)-action (up to a finite homomorphism \( T \rightarrow T \)) to Cox’s quotient presentation \( \mathbb{K}^m \rightarrow X \), see Section 1. Next extend the lifted \( T \)-action to a toric structure on \( \mathbb{K}^m \). This involves linearization of a certain diagonalizable group.
action, see Section 3. Finally, push down the new toric structure of $K^m$ to $X$. For this we need that $X$ has no small holes, see Section 3.

1. LIFTING TORUS ACTIONS

We provide here a lifting result for torus actions on a toric variety $X$ to the quotient presentation of $X$ introduced by Cox [3]. First we recall the latter construction. For notation and the basic facts on toric varieties, we refer to Fulton’s book [3].

We shall assume that the toric variety $X$ is nondegenerate, that is $X$ admits no toric decomposition $X \cong Y \times \mathbb{K}^*$. Note that this is equivalent to requiring that every invertible $f \in O(X)$ is constant.

Let $X$ arise from a fan $\Delta$ in a lattice $N$. Denote the rays of $\Delta$ by $\varrho_1, \ldots, \varrho_m$. Let $Q: \mathbb{Z}^m \to N$ be the map sending the canonical base vector $e_i$ to the primitive generator of $\varrho_i$. For a maximal cone $\tau \in \Delta$, set

$$\sigma(\tau) := \text{cone}(e_i; \varrho_i \subset \tau).$$

Then these cones $\sigma(\tau)$ are the maximal cones of a fan $\Sigma$ consisting of faces of the positive orthant in $\mathbb{Q}^m$. Moreover, $Q: \mathbb{Z}^m \to N$ is a map of the fans $\Sigma$ and $\Delta$.

The following properties of this construction are well known:

**Proposition 1.1.** Let $Z \subset K^m$ be the toric variety defined by $\Sigma$, let $q: Z \to X$ be the toric morphism corresponding to $Q: \mathbb{Z}^m \to N$, and let $H \subset T_Z$ be the kernel of the homomorphism $T_Z \to T_X$ of the big tori obtained by restricting $q: Z \to X$.

(i) The complement $K^m \setminus Z$ is of dimension at most $m - 2$.
(ii) The map $q: Z \to X$ is a good quotient for the action of $H$ on $Z$.
(iii) $X$ is smooth if and only if the group $H$ acts freely.

In general, the diagonalizable group $H \subset T_Z$ may be disconnected. Hence we can at most expect liftings of a given action $T \times X \to X$ in the sense that $q: Z \to X$ becomes $T$-equivariant up to a (finite) epimorphism $T \to T$. But such liftings exist:

**Proposition 1.2.** Notation as in [1, 4]. Let $T \times X \to X$ be an effective algebraic torus action. Then there exist an effective regular action $T \times Z \to Z$ and an epimorphism $\kappa: T \to T$ such that

(i) $t \cdot (h \cdot z) = h \cdot (t \cdot z)$ holds for all $(t, h, z) \in T \times H \times Z$,
(ii) $q(t \cdot z) = \kappa(t) \cdot q(z)$ holds for all $(t, z) \in T \times X$.

**Proof.** First we reduce to the case that $X$ is smooth. So, suppose for the moment that the assertion is proven in the smooth case. Then we can lift the $T$-action over the set $U \subset X$ of smooth points. The task then is to extend the lifted action from $U' := q^{-1}(U)$ to $Z$.

By Sumihiro’s Theorem [14, Cor. 2], $X$ is covered by $T$-invariant affine open subsets $V \subset X$. The inverse images $V' := q^{-1}(V)$ are affine and $V' \setminus U'$ is of codimension at least 2 in $V'$. This allows to extend uniquely the lifted $T$-action from $V' \cap U'$ to $V'$ and hence from $U'$ to $Z$.

Therefore we may assume in the remainder of this proof that the toric variety $X$ is smooth. As noted in Proposition 1.1, this means that the group $H \subset T_Z$ acts freely on $Z$.

The most convenient way to lift the $T$-action is to split the procedure into simple steps. For this, write $H$ as a direct product of a torus $H_0$ with finite cyclic groups.
This gives rise to a decomposition of the quotient presentation \( Z \rightarrow X \):

\[
\begin{array}{cccccccc}
Z & \frac{H_k}{H_{k-1}} & \frac{H_{k-1}}{H_{k-2}} & \cdots & \frac{H_1}{H_0} & Z_0 & \cdots & X
\end{array}
\]

This decomposition allows us to lift the \( T \)-action step by step with respect to the geometric quotients by the free actions of the factors \( H_i \). We shall write again \( Z, H \) and \( X \) instead of \( Z_i, H_i \) and \( Z_{i-1} \).

The action of \( H \) on \( Z \) defines a grading of the \( \mathcal{O}_X \)-algebra \( \mathcal{A} := q_*(\mathcal{O}_Z) \). Namely, denoting by \( \Gamma \) the character group of \( H \), we have for every open \( V \subset X \) the decomposition into homogeneous functions:

\[
\mathcal{O}(q^{-1}(V)) = \mathcal{A}(V) = \bigoplus_{\chi \in \Gamma} \mathcal{A}_\chi(V).
\]

Since \( H \) acts freely on \( Z \), all homogeneous components \( \mathcal{A}_\chi \) are locally free \( \mathcal{O}_X \)-modules of rank one. We shall use this fact to make the \( \mathcal{O}_X \)-algebra \( \mathcal{A} \) into a \( T \)-sheaf over the \( T \)-variety \( X \). Then it is canonical to extract the desired lifting from this \( T \)-sheaf structure.

If the group \( H \) is connected, then we can prescribe \( T \)-linearizations on the \( \mathcal{O}_X \)-modules \( \mathcal{A}_i \) corresponding to the members \( \chi_i \) of some lattice basis of \( \Gamma \). Tensoring these linearizations gives the desired \( T \)-sheaf structure on the \( \mathcal{O}_X \)-algebra \( \mathcal{A} \), compare also [3, Section 3].

Since \( X \) is covered by \( T \)-invariant affine open subsets, we can easily check that this \( T \)-sheaf structure of \( \mathcal{A} \) arises from a regular \( T \)-action on \( Z \) that commutes with the action of \( H \) and makes the quotient map \( q: Z \rightarrow X \) even equivariant. This settles the case of a connected \( H \).

Assume that \( H \) is finite cyclic of order \( d \). Let \( \chi \) be a generator of \( \Gamma \). Again, we choose a \( T \)-linearization of \( \mathcal{A}_\chi \). But now it may happen that the induced \( T \)-linearization on \( \mathcal{A}_{d\chi} = \mathcal{O}_X \) is not the canonical one. However, since \( \mathcal{O}^*(X) = \mathbb{K}^* \) holds, these two linearizations only differ by a character \( \xi \) of \( T \).

Let \( \kappa: T \rightarrow T \) be an epimorphism such that \( \xi \circ \kappa = \xi \circ \xi_0^d \) holds for some character \( \xi_0 \) of \( T \). Consider the action \( t \cdot x := \kappa(t) \cdot x \) on \( X \). Then \( \mathcal{A}_\chi \) is also linearized with respect to this action by setting \( t \cdot f := \kappa(t) \cdot f \). Twisting with \( \xi_0^{-1} \), we achieve that the induced linearization on \( \mathcal{A}_{d\chi} = \mathcal{O}_X \) is the canonical one:

\[
(t \cdot f)(x) = \xi_0^{-d}(t)(\xi(\kappa(t)) f(t^{-1} \cdot x)) = f(t^{-1} \cdot x).
\]

The rest is similar to the preceding step: The \( T \)-sheaf structure of \( \mathcal{A} \) defines a \( T \)-action \( (t, z) \mapsto t \cdot z \) on \( Z \) commuting with the action of \( H \) and making the quotient map \( q: Z \rightarrow X \) equivariant with respect to \( (t, x) \mapsto t \cdot x \). Dividing by the kernel of ineffectivity, we can make the action on \( Z \) effective and obtain the desired lifting. \( \square \)

2. Diagonalizable group actions

In this section we show that any effective regular action of an \((m-1)\)-dimensional diagonalizable group \( G \) on \( \mathbb{K}^m \) can be brought into diagonal form by means of an algebraic coordinate change. The result extends a well known analogous statement on torus actions due to Białynicki-Birula, see [3].

We would like to thank the referee for his valuable proposals in order to make our first proof more transparent.
Proposition 2.1. Let $G \times \mathbb{K}^m \to \mathbb{K}^m$ be an effective algebraic action of an $(m - 1)$-dimensional diagonalizable group $G$. Then there exist $\alpha \in \text{Aut}(\mathbb{K}^m)$ and characters $\chi_i: G \to \mathbb{K}^*$ such that for every $g \in G$ and every $(z_1, \ldots, z_m) \in \mathbb{K}^m$ we have
\[
\alpha(g \cdot \alpha^{-1}(z_1, \ldots, z_m)) = (\chi_1(g)z_1, \ldots, \chi_m(g)z_m).
\]

Proof. If the quotient space $\mathbb{K}^m//G$ is a point, then an application of Luna’s slice theorem shows that the action of $G$ is linearizable; this works even more generally for reductive groups, see [4, Proposition 5.1]. Since any linear $G$-action is diagonalizable, we obtain the assertion in the case of $\mathbb{K}^m//G$ being a point.

So we are left with the case that $\mathbb{K}^m//G$ is of dimension one. We write $G = G_0 \times G_1$ with an algebraic torus $G_0$ and a finite abelian group $G_1$. According to the main result of [4], we may assume that the action of $G_0$ is already diagonal. In the sequel, we view $G_0$ as a subtorus of the torus $(\mathbb{K}^*)^m$.

We shall show that the group $G_1$ permutes the coordinate hyperplanes $V(z_i)$ of $\mathbb{K}^m$. Indeed, this is all we need, because then $G_1$ acts by linear automorphisms and hence the action of $G$ is diagonalizable.

Let $p: \mathbb{K}^m \to \mathbb{K}^m//G_0$ be the quotient map. This is a toric morphism. In particular, since the quotient space $\mathbb{K}^m//G_0$ is of dimension one, it is isomorphic to $\mathbb{K}$. Thus we can write down the quotient map explicitly: There are relatively prime nonnegative integers $a_i$ such that
\[
p(z_1, \ldots, z_m) = z_1^{a_1} \cdots z_m^{a_m}.
\]

Let us renumber the coordinates such that in the above presentation of the quotient map we have $a_i > 0$ for all $i \leq k$ and $a_i = 0$ for all $i > k$ with a suitable integer $k \leq m$.

Consider an $i > k$, say $i = m$. Then the points $(1, \ldots, 1)$ and $(1, \ldots, 1, 0)$ lie in the regular fiber $p^{-1}(1)$ of the quotient map $p: \mathbb{K}^m \to \mathbb{K}$. Thus $(1, \ldots, 1, 0)$ lies in the closure of $G_0$ and hence it lies in the closure of some onedimensional subtorus $T_m \subset G_0$. But this subtorus is necessarily of the form
\[
T_m = \{(1, \ldots, 1, t); t \in \mathbb{K}^*\}.
\]

Since the actions of $G_1$ and $T_m$ commute, the group $G_1$ leaves the fixed point set of $T_m$ invariant. But the latter is just the coordinate hyperplane $V(z_m)$. Analogously, we conclude that the remaining $V(z_i)$, where $i > k$, are invariant under the group $G_1$.

We discuss now what happens to the coordinate hyperplanes $V(z_i)$, where $i \leq k$. Note that these are precisely the irreducible components of the fiber $p^{-1}(0)$ of the quotient map. We shall distinguish the cases $k > 1$ and $k = 1$.

For $k > 1$, the fiber $p^{-1}(0)$ is the unique reducible fiber of the quotient map. On the other hand the action of $G_1$ commutes with the action of $G_0$ and hence $G_1$ permutes the fibres of $p$. Thus $G_1$ has to leave $p^{-1}(0)$ invariant. Hence $G_1$ permutes the coordinate hyperplanes $V(z_1), \ldots, V(z_k)$ provided $k > 1$.

Finally, we treat the case $k = 1$. Then, as seen before, $G_0$ contains the one dimensional tori $T_2, \ldots, T_m$. In other words, we have
\[
G_0 = \{(1, t_2, \ldots, t_m); t_i \in \mathbb{K}^*\}.
\]

So the fixed point set of $G_0$ is just the $z_1$-axis. Hence $G_1$ leaves the $z_1$-axis invariant. Now, $G_1$ acts on the $z_1$-axis with a fixed point, say $b$. By conjugating the $G$-action with the translation by $b$, we achieve that $b = 0$ holds. But then $G_1$ leaves $V(z_1) = p^{-1}(0)$ invariant. \qed
3. Proof of the main result

We say that a toric variety $X$ has no small holes, if it does not admit an open toric embedding $X \subset X'$ such that $X' \setminus X$ is nonempty of codimension at least 2 in $X'$. Examples are the toric varieties arising from a fan with convex support. This comprises in particular the affine ones.

**Theorem 3.1.** Let $X$ be a nondegenerate toric variety without small holes, and let $T \times X \to X$ be an effective regular action of an algebraic torus $T$ of dimension $\dim(X) - 1$. Then $T$ is conjugate in $\text{Aut}(X)$ to a subtorus of the big torus $T_X \subset X$.

**Proof.** According to [1, Theorem 4.1], any two toric structures of $X$ are conjugate in $\text{Aut}(X)$. Consequently, it suffices to show that the action of $T$ on $X$ extends to an effective regular action of a torus of dimension $\dim(X)$ on $X$.

Consider Cox’s construction $q: Z \to X$ and its kernel $H := \ker(q)$ as defined in [1]. Choose a lifting of the $T$-action to $Z$ as provided by Proposition 1.2. This gives us an action of the $(m-1)$-dimensional diagonalizable group $G := T \times H$ on the open set $Z \subset \mathbb{K}^m$.

Since the complement $\mathbb{K}^m \setminus Z$ is of dimension at most $m-2$, the action of $G$ extends regularly to $\mathbb{K}^m$. Let $G_0$ be the (finite) kernel of ineffectivity. Applying Proposition 2.1 to the action of $G/G_0$, we can extend the $G$-action to an almost homogeneous action of a torus $S$ on $\mathbb{K}^m$.

We show that $Z$ is invariant with respect to the action of $S$. According to [15, Corollary 2.3], we obtain this if we can prove that the set $Z$ is $H$-maximal in the following sense: If $Z' \subset \mathbb{K}^m$ is an $H$-invariant open subset admitting a good quotient $q': Z' \to X'$ by the action of $H$ such that $Z$ is a $q'$-saturated open subset of $Z'$, then we already have $Z' = Z$.

To verify $H$-maximality of $Z$, consider $Z' \subset \mathbb{K}^m$ and $q': Z' \to X'$ as above. We may assume that $Z'$ is $H$-maximal in $\mathbb{K}^m$. Applying [15, Corollary 2.3] to the actions of $H$ and the standard torus $k^*$ on $\mathbb{K}^m$, we obtain that $Z'$ is invariant with respect to the action of $k^*$. Hence we obtain a commutative diagram of toric morphisms:

$$
\begin{array}{ccc}
Z & \longrightarrow & Z' \\
\| H \downarrow & & \| H \downarrow \\
X & \longrightarrow & X'
\end{array}
$$

By the choice of $Z'$, the horizontal arrows are open toric embeddings. Moreover, the complement $X' \setminus X$ is of codimension at least two in $X'$, because its inverse image $Z' \setminus Z$ under $q'$ is a subset of the small set $\mathbb{K}^m \setminus Z$ and $q'$ is surjective. By the assumption on $X$, we obtain $X' = X$. This verifies $H$-maximality of $Z$. Hence our claim is proved.

The rest is easy: The torus $S/H$ acts with a dense orbit on $X$. Dividing $S/H$ by the kernel of ineffectivity of this action, we obtain the desired extension of the action of $T$ on $X$. □

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