ENTROPY THEORY FOR THE PARAMETRIZATION OF THE EQUILIBRIUM STATES OF PIMSNER ALGEBRAS

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ABSTRACT. We consider Pimsner algebras that arise from C*-correspondences of finite rank, as dynamical systems with their rotational action. We revisit the Laca-Neshveyev classification of their equilibrium states at positive inverse temperature along with the parametrizations of the finite and the infinite parts simplices by tracial states on the diagonal. The finite rank entails an entropy theory that shapes the KMS-structure. We prove that the infimum of the tracial entropies dictates the critical inverse temperature, below which there are no equilibrium states for all Pimsner algebras. We view the latter as the entropy of the ambient C*-correspondence. This may differ from what we call strong entropy, above which there are no equilibrium states of infinite type. In particular, when the diagonal is abelian then the strong entropy is a maximum critical temperature for those. In this sense we complete the parametrization method of Laca-Raeburn and unify a number of examples in the literature.

1. INTRODUCTION

The Fock space construction gives a concrete quantization of systems in terms of Hilbertian operators. Originating from Quantum Mechanics, it has seen an important generalization to Hilbert bimodules over C*-algebras, better known as C*-correspondences. The key element is the existence of a C*-algebra \( A \) acting “externally” on \( X \) and of an \( A \)-valued inner product. Rieffel [34] originally envisioned C*-correspondences as a tool to identify C*-algebras in terms of their representation theory. Pimsner [32] much later extended the theory to accommodate a range of examples of Operator Algebras arising from C*-dynamics and graphs. The Pimsner algebras generalize the well known Toeplitz- and Cuntz-algebras and they have been under considerable study since their introduction. By now they form a topic in its own respect with several interactions with graph theory and ring theory. The C*-correspondence machinery is now viewed as an effective way for quantizing geometric structures that evolve in discrete time.

Nevertheless, the interplay of C*-algebras with Quantum Statistical Mechanics goes well beyond that point. Taking motivation from ideal gases, there is an analogue of a Kubo-Martin-Schwinger condition for states of C*-algebras that admit an \( \mathbb{R} \)-action, even when moving beyond the trace class operators. See for example the seminal monographs of Bratelli-Robinson [5, 6]. The parametrization of equilibrium states has been an essential task in the past 30 years, as they can serve as an invariant for T-equivariant isomorphisms. To give only but a fragment of a very long list we mention the Cuntz-algebra [12, 30], C*-algebras of different types of dynamical systems [4, 13, 15, 16, 20, 22, 26, 27, 33, 36, 37], graph C*-algebras [1, 2, 18, 35], C*-algebras related to number systems [3, 9, 24, 25] and to subshifts [11, 28], and Pimsner algebras [17, 23].

The major steps for classifying the equilibrium states of Pimsner algebras were established in the seminal paper of Laca and Neshveyev [23]. Their arguments were further refined by Laca and Raeburn [25] in their study of C*-algebras arising from number systems. The approach of Laca-Raeburn has been very influential, and effectively applicable in a big variety of examples, e.g. [1, 2, 20, 26, 27]. However each occasion ad-hoc data is used to trigger the algorithm. The aim of this paper is to show how these ideas combine with the notion of entropy of Pinzari, Watatani and Yotetani [33] that is induced when the ambient C*-correspondences have finite rank; an assumption that holds in the aforementioned cases. The KMS-structure of the Pimsner algebras in [1, 20, 26, 27] follows as an application of this analysis.

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1.1. Motivation. The Toeplitz-Pimsner algebra $\mathcal{T}_X$ is the $C^*$-algebra generated by the left creation operators of $X$ and $A$ acting on the Fock space $F_X$. In addition, there is a range of Pimsner algebras that encodes desirable redundancies. Every quotient of $\mathcal{T}_X$ by $T$-equivariant relations gives rise to a $J$-relative Cuntz-Pimsner algebras $O(J,X)$, where $J \subseteq \phi_X^{-1}(KX)$ for the left action $\phi_X$ of $A$ and the compact operators $KX$. Among those the Cuntz-Pimsner algebra $O_X$ is of central importance and arises when $J$ is Katsura’s ideal [21]. It is the smallest $T$-equivariant quotient of $\mathcal{T}_X$ that admits a faithful copy of $A$ and $X$ [19]. In general $O_X \neq O(\phi_X^{-1}(KX), X)$ but they coincide with $O(A, X)$ when $X$ is injective and $\phi_X(A) \subseteq KX$.

Laca and Neshveyev [23] studied actions implemented by one-parameter unitaries for injective $C^*$-correspondences. Their main tool was the use of induced traces from [8, 10, 31]. In this way they were able to classify the equilibrium states of $\mathcal{T}_X$ in terms of their restrictions on the diagonal by using iterations of the inducing map at each level of the Fock space. Following Exel-Laca [15], they proved a Wold decomposition into a finite part (given by a series of iterations of a tracial state on the diagonal) and an infinite part (where iterations are stable). They showed that $\mathcal{T}_X$ admits a rich KMS-structure from which they could derive that of $O_X$ (when $X$ is injective) through the infinite part. A characterization was also given for ground states.

Later Laca-Raeburn [25] refined the main tools of [23] for a specific class of Pimsner algebras coming from number systems. From then on the interest was restricted to dynamics implemented by the rotational action. Most notably they use the statistical approximations of [23] to parameterize the finite part by tracial states on $T$. As we shall explain later there is a difference between the parametrizations in [23] and in [25]. Likewise, weak*-homeomorphic parametrizations were given for both ground states and KMS$_\infty$-states in [25].

In turn, a number of subsequent works, e.g. [1, 2, 20, 26, 27], were greatly influenced by the parametrization of [23, 25] and applied their method to other examples of Pimsner algebras. A re-appearing theme is the existence of two critical temperatures $\beta_c \geq \beta'_c$ for which:

(a) for $\beta > \beta_c$ the algorithm of [25] gives all equilibrium states for $\mathcal{T}_X$;

(b) for $\beta = \beta'_c$ there is an association with averaging states; and

(c) there are no equilibrium states below $\beta'_c$.

At the other extreme $O_X$ is not amenable to the construction of (a) but it provides the states for (b). Such an example is the averaging state on the Cuntz-algebra $O_d$ which is the only possible equilibrium state (and it appears at $\beta = \log d$).

The critical temperatures often coincide and can be associated to structural data of the original construct. For example, an Huef-Laca-Raeburn-Sims [1] show that $\beta_c = \beta'_c$ is the logarithm of the Perron-Frobenius eigenvalue when the graph is irreducible. In a continuation [2] the authors also show that a more rich structure appears for general graphs. That was also verified by Kajiwara-Watatani [18] who studied the KMS-structure of Cuntz-Krieger $C^*$-algebras. In the process they achieve also a parametrization of the infinite part of $O(J,X)$ for $J$ inside Katsura’s ideal under some assumptions on the $C^*$-correspondence. However this does not cover the infinite part in the non-injective case, i.e., it does not cover the case $J = \phi_X^{-1}(KX)$. These works motivate the following question:

**Q.** How $A$ and $X$ dictate the critical temperature(s) beyond which we don’t have equilibrium states of Pimsner algebras?

In the current paper we show how this is done under the assumption that $X$ attains a finite set $\{x_1, \ldots, x_d\}$ of vectors in its unit ball such that $1_X = \sum_{i \in [d]} \theta_{x_i} x_i$. Equivalently, when the adjointable operators of $X$ are compact. This is satisfied in the aforementioned examples, and sometimes on the much stronger side of the vectors being orthogonal. We are not assuming orthogonality here.

Also, we mention that we consider just the dynamics coming from the rotational $T$-action for which there is a physical interpretation. Recall that the starting point for Gibbs states is the action implemented by $r \mapsto e^{\iota r(H-KN)}$, where $H$ is the Hamiltonian, $N$ is the number operator and $\kappa$ is the chemical potential. When $H$ is the Hamiltonian of a Quantum Harmonic Oscillator then it admits the solution $H = \hbar \omega(1/2 + N)$ for the energy dimension $\hbar \omega/2$ of the ground state, and the
action is implemented by $e^{ir\hbar\omega/2}e^{ir(\hbar\omega - \kappa)N}$. Since $N$ is unbounded some effort is required to make $\sigma$ precise. This can be seen for example in Proposition 2.2 where it is shown that the rotational action $r \mapsto \gamma_{e^{ir}}$ realizes any action implemented by $r \mapsto e^{ir(c+sN)}$ for $c \in \mathbb{C}$ and $s \in \mathbb{R}$. In what follows we make the normalization $\hbar\omega - \kappa = 1$. Recall that $\kappa < \hbar\omega/2$ for any Quantum Harmonic Oscillator and thus substituting $\beta$ by $(\hbar\omega - \kappa)\beta$ covers all cases.

1.2. Decomposition and parametrization. We write $E_\beta(\mathcal{O}(J,X))$ for the $(\sigma, \beta)$-KMS states of the $J$-relative Cuntz-Pimsner algebra $\mathcal{O}(J,X)$ with respect to the action $r \mapsto \sigma_r := \gamma_{e^{ir}}$. Every $\mathcal{O}(J,X)$ is the quotient of $\mathcal{T}_X$ by a $T$-equivariant ideal and hence in order to understand $E_\beta(\mathcal{O}(J,X))$ it suffices to do so for $E_\beta(\mathcal{T}_X)$. We need to revisit in detail the main points of [18, 23] and in particular see how the method of [25] extends to unify [1, 20, 26, 27].

In what follows fix $\{x_1, \ldots, x_d\}$ be a finite unit decomposition. Then $\{x_\mu | |\mu| = n\}$ yields a unit decomposition for $X^\otimes n$, where we write $x_{\mu_{n-1}\cdots \mu_1} = x_{\mu_n} \otimes \cdots \otimes x_{\mu_1}$ for a word $\mu = \mu_n \cdots \mu_1$ on the $d$ symbols. Consequently the projections $p_\mu : FX \to X^{\otimes n}$ and the compacts $\mathcal{K}(FX)$ are in $\mathcal{T}_X$. The finite and the infinite parts of the Wold decomposition from [23] form respectively the convex sets:

\begin{equation}
E_\beta^\infty(\mathcal{T}_X) := \{ \varphi \in E_\beta(\mathcal{T}_X) | \varphi(p_k) = 1 \} \quad \text{and} \quad E_\beta^\infty(\mathcal{T}_X) := \{ \varphi \in E_\beta(\mathcal{T}_X) | \varphi(p_0) = 0 \}.
\end{equation}

In particular $E_\beta^\infty(\mathcal{T}_X)$ corresponds to the states annihilating $\mathcal{K}(FX)$ (and thus to those that factor through $\mathcal{O}(A,X)$), and $E_\beta^\infty(\mathcal{T}_X)$ corresponds to those that restrict to states on $\mathcal{K}(FX)$ (see Theorem 4.6). We then construct the parametrization of each convex set by a specific convex set in the tracial states $T(A)$ of $A$. This is linked to the formal series

\begin{equation}
c_{\tau,\beta} := \sum_{k=0}^{\infty} e^{-k\beta} \sum_{|\mu|=k} \tau(\langle x_\mu, x_\mu \rangle) \quad \text{for } \tau \in T(A) \text{ and } \beta > 0.
\end{equation}

We thus need to consider the sets that arise from two extreme cases:

\begin{equation}
T_\beta(A) := \{ \tau \in T(A) | c_{\tau,\beta} < \infty \} \quad \text{and} \quad AVT_\beta(A) := \{ \tau \in T(A) | e^{\beta} \tau = \sum_{i \in [d]} \tau((x_i, x_i)) \}.
\end{equation}

Notice that $c_{\tau,\beta} = \sum_{k=0}^{\infty} 1$ for every $\tau \in AVT_\beta(A)$.

The parametrization of $E_\beta^\infty(\mathcal{T}_X)$ is constructive and follows from [23, 25]. In Theorem 6.1 we show that there is a bijection

\begin{equation}
\Phi : T_\beta(A) \to E_\beta^\infty(\mathcal{T}_X) \text{ such that } \Phi(\tau)(p_0) = c_{\tau,\beta}^{-1}.
\end{equation}

In particular $\Phi$ can be reconstructed by

\begin{equation}
\Phi(\tau)(t(\xi^{\otimes n})t(\eta^{\otimes m})) = \delta_{n,m}c_{\tau,\beta} \sum_{k=0}^{\infty} e^{-(k+n)\beta} \sum_{|\mu|=k} \tau(\langle \eta^{\otimes m} \otimes x_\mu, \xi^{\otimes n} \otimes x_\mu \rangle)
\end{equation}

for all $\xi^{\otimes n} \in X^{\otimes n}$ and $\eta^{\otimes m} \in X^{\otimes m}$. When $E_\beta^\infty(\mathcal{T}_X)$ is weak*-closed then $\Phi$ is a weak*-homeomorphism. As a new outcome of this analysis we derive that the map $\Phi$ preserves convex combinations (by weighting over the $c_{\tau,\beta}$), and thus it preserves the extreme points.

Theorem 6.1 uses the crux of the arguments of [23, proof of Theorem 2.1] but as with [1, 2, 20, 25, 26, 27] there are slight differences. First of all the correspondence between $E_\beta(\mathcal{T}_X)$ and a subset of $T(A)$ in [23, Theorem 2.1] is given as a correspondence between $E_\beta(\mathcal{T}_X) \to T(A)$ by restriction $\Phi \mapsto |\Phi|_A$, and it is not linked to $T_\beta(A)$. In the comments preceding [23, Definition 2.3] it is hinted how a $\tau$ might be obtained from $\Phi(\tau)$ but the suggested map requires normalization (by the possibly non-constant $c_{\tau,\beta}$). Secondly, $\Phi$ is obtained through induced representations of Toeplitz-Pimsner algebras rather than the theory of induced traces from [8, 10, 31].

The infinite part is dealt with in Theorem 7.1 where an affine weak*-homeomorphism is constructed:

\begin{equation}
\Psi : \{ \tau \in AVT_\beta(A) | \tau|_I = 0 \} \to E_\beta^\infty(\mathcal{T}_X) \text{ such that } \Psi(\varphi)|_A = \tau,
\end{equation}
for the ideal of $A$

\[(1.7)\quad I := \{a \in A \mid \lim_n \|\phi_X(a) \otimes \text{id}_{X^\otimes n-1}\| = 0\}.
\]

The ideal $I$ is the kernel of the canonical quotient $q : T_X \to \mathcal{O}(A, X)$ and arises from the fact that every $\varphi \in E^\infty_T(T_X)$ factors through $q$. The proof follows the lines of [18, Theorem 3.18] with the additional use of $I$. The main tool is that the fixed point algebra is the inductive limit of the $\mathcal{K} X^\otimes n$ when $X$ is injective. It has been implicitly applied in [28, 20] to obtain equilibrium states at the critical temperature.

The affine weak*-homeomorphism is obtained in [23, Theorem 2.1 and Theorem 2.5], when $X$ is injective and non-degenerate, but with an entirely different line of attack. At the end of [23, proof of Theorem 2.1] it is shown that any equilibrium state can be given as a limit of finite type states on $\sigma^\varepsilon$ perturbed actions so that $\lim_{\varepsilon \to 0} \sigma^\varepsilon = \sigma$. Hence they verify that [23, Formula (2.2)] gives a well defined extension of a state from $A$ to $T_X$. Then [23, Theorem 2.5] asserts that [23, Formula (2.2)] gives a state $\varphi$ of infinite type if and only if $\varphi|_A \in \text{AVT}_\beta(A)$. Theorem 7.1 on the other hand constructs directly the extension within the same action $\sigma$.

By passing to a $T$-equivariant quotient we derive a similar characterization for any $J$-relative Cuntz-Pimsner algebra $\mathcal{O}(J, X)$ through the following scheme:

\[
T_\beta(A) \cap \{\tau \in T(A) \mid \tau|_J = 0\} \xrightarrow{\Phi} E^\text{fin}_\beta(\mathcal{O}(J, X)) \\
\text{AVT}_\beta(A) \cap \{\tau \in T(A) \mid \tau|_I = 0\} \xrightarrow{\Psi} E^\infty_\beta(\mathcal{O}(J, X))
\]

Figure. Parametrization of equilibrium states of $\mathcal{O}(J, X)$.

Of course this has to be taken with care as it may be that $E^\infty_\beta(\mathcal{O}(J, X))$ or $E^\text{fin}_\beta(\mathcal{O}(J, X))$ is empty for some choices of $\beta$ and $J$. This brings us to the main point of the discussion that captured our interest in the first place.

1.3. Entropy. Taking motivation from the classical case, entropy has been used in various guises. See the excellent monograph of Neshveyev-Størmer in this respect [29]. Our approach is closer to that of Pinzari-Watatani-Yotetani [33] who considered imprimitivity bimodules with finite left and right unit decompositions. The starting point is that the statistical approximation (1.5) works only when $c_{r,\beta} < \infty$. The ratio test may not be conclusive for all formal series $c_{r,\beta}$ but it can be used to define the following notions of entropies. The entropy of a tracial state $\tau$ of $A$ is given by

\[(1.8)\quad h_X^\tau := \limsup_k \frac{1}{k} \log \sum_{|\mu|=k} \tau(\langle x_\mu, x_\mu \rangle).
\]

Notice that $h_X^\tau \leq \beta$ if $\tau \in T_\beta(A)$, and that $h_X^\tau = \beta$ if $\tau \in \text{AVT}_\beta(A)$. Moreover $h_X^\tau$ is independent of the choice of the unit decomposition. On the other hand for a fixed unit decomposition $x = \{x_1, \ldots, x_d\}$ we can define

\[(1.9)\quad h_X^x := \limsup_k \frac{1}{k} \log \|\sum_{|\mu|=k} \langle x_\mu, x_\mu \rangle\|_A
\]

where the lim sup is actually a limit. The strong entropy of $X$ is then given by

\[(1.10)\quad h_X := \inf\{h_X^x \mid x = \{x_1, \ldots, x_d\} \text{ is a unit decomposition for } X\}.
\]

If $A$ is abelian then $h_X^x$ is the same for all unit decompositions. Finally we define the entropy of $X$ as the critical temperature below which we do not attain equilibrium states for any Pimsner algebra, i.e.,

\[(1.11)\quad h_X := \inf\{\beta > 0 \mid E_\beta(T_X) \neq \emptyset\} \quad (\text{with } \inf \emptyset := \infty).
\]

By weak*-compactness the infimum is actually a minimum. In Proposition 5.7, Corollary 6.4, Proposition 7.2 and Corollary 7.3 we show that:

(i) $h_X \leq h_X^* \leq \log d$, and $h_X^\tau \leq h_X^*$ for every $\tau \in T(A)$. 

(ii) If $\beta > h_X^s$ then $T_\beta(A) = T(A)$ and thus $E^{\infty}_{\beta}(T_X) = \emptyset$ and $E_\beta(T_X) = E^{\text{fin}}_\beta(T_X)$.

An essential application of [33] gives also that $E^{\infty}_{\beta, A}(T_X) \neq \emptyset$ when $A$ is abelian. In Corollary 7.3 we provide one of the main conclusions of this analysis; namely, that the entropy of $X$ can be recovered from the state entropies in the following way:

$$h_X = \max\{0, \inf \{h_X^s \mid \tau \in T(A)\}\}. \quad (1.12)$$

In fact, if $h_X > 0$ or if $h_\tau \geq 0$ for all $\tau \in T(A)$ then $h_X = \min \{h_X^s \mid \tau \in T(A)\}$. Consequently, if $h_X^s = h_X^c$ for all $\tau \in T(A)$ then $E^{\infty}_{\beta}(T_X) = \emptyset$ whenever $\beta > h_X$. This gives the KMS-states theory of $O_d$ in a nutshell, and reflects what is known for specific cases in the literature. In Section 8 we emphasize by examples that:

(iii) The infimum over all $h_X^s$ is required in the definition of $h_X^s$, as the notion of an orthonormal basis is not well defined for Hilbert modules over non-abelian C*-algebras.

(iv) If $A$ is abelian and $X$ attains an orthonormal basis then $h_X^s = h_X^c = \log d$ for all $\tau \in T(A)$. Moreover $E_\beta(T_X) = E^{\text{fin}}_{\beta, A}(T_X) \neq \emptyset$ for all $\beta > \log d$, and $E_{\log d}(T_X) = E^{\infty}_{\log d}(T_X) \neq \emptyset$.

(v) There may be both finite and infinite parts for $T_X$ when $\beta \in (h_X(h_X^c), h_X(h_X^c)]$.

As a second application we show how the entropy theory fully recovers the KMS-structure of Pimsner algebras of irreducible graphs [1, 18], and that of Pimsner algebras related to dynamical systems or self-similar actions of [20, 26, 27]. For these examples we derive item (v) above, where the value $d$ is specified by the intrinsic data of the related C*-correspondence.

1.4. States at the upper half plane. We follow [25] and make a distinction between states that are bounded on the upper half plane (ground states) and states that arise at the limit of $\beta \uparrow \infty$ (KMS$_\infty$-states). The parametrization in Theorem 9.2 resembles that of [1, 20, 26, 27], which in turn are inspired by [23, Theorem 2.2]. Namely, the mapping $\tau \mapsto \varphi_\tau$ given by

$$\varphi_\tau(f) := \begin{cases} 
\tau(f) & \text{if } f \in qJ(A) \subseteq O(J, X), \\
0 & \text{otherwise,}
\end{cases} \quad (1.13)$$

defines an affine weak*-homeomorphism from the states $S(A)$ of $A$ (resp. from $T(A)$) that vanish on $J$, onto the ground states of $O(J, X)$ (resp. the KMS$_\infty$-states of $O(J, X)$).

2. Preliminaries

2.1. Kubo-Martin-Schwinger states. Let $\sigma : \mathbb{R} \to \text{Aut}(A)$ be an action on a C*-algebra $A$. Then there exists a norm-dense $\sigma$-invariant $*$-subalgebra $A_{\text{an}}$ of $A$ such that for every $f \in A_{\text{an}}$ the function $\mathbb{R} \ni r \mapsto \sigma_r(f) \in A$ is analytically continued to an entire function $\mathbb{C} \ni z \mapsto \sigma_z(f) \in A$ [5, Proposition 2.5.22]. If $\beta > 0$, then a state $\varphi$ of $A$ is called a $(\sigma, \beta)$-KMS state (or equilibrium state at $\beta$) if it satisfies the KMS-condition:

$$(2.1) \quad \varphi(fg) = \varphi(g \sigma_{i\beta}(f)) \text{ for all } f, g \text{ in a norm-dense } \sigma \text{-invariant } *\text{-subalgebra of } A_{\text{an}}.$$ 

If $\beta = 0$ or if the action is trivial then a KMS-state is a tracial state on $A$. The KMS-condition follows as an equivalent for the existence of particular continuous functions [6, Proposition 5.3.7]. More precisely, a state $\varphi$ is an equilibrium state at $\beta > 0$ if and only if for any pair $f, g \in A$ there exists a complex function $F_{f, g}$ that is analytic on $D = \{z \in \mathbb{C} \mid 0 < \text{Im}(z) < \beta\}$ and continuous (hence bounded) on $\overline{D}$ such that

$$F_{f, g}(r) = \varphi(f \sigma_r(g)) \text{ and } F_{f, g}(r + i\beta) = \varphi(\sigma_r(g)f) \text{ for all } t \in \mathbb{R}.\quad (2.1)$$

A state $\varphi$ of $A$ is called a KMS$_\infty$-state if it is the weak*-limit of $(\sigma, \beta)$-KMS states as $\beta \uparrow \infty$. A state $\varphi$ of a C*-algebra $A$ is called a ground state if the function $z \mapsto \varphi(f \sigma_z(g))$ is bounded on $\{z \in \mathbb{C} \mid \text{Im}z > 0\}$ for all $f, g$ inside a dense analytic subset of $A$. The distinction between ground states and KMS$_\infty$-states is not apparent in [6] and is coined in [25].
2.2. C*-correspondences. The reader should be familiar with the theory of C*-correspondences, e.g. [21]. A C*-correspondence X over A is a right Hilbert A-module with a left action given by a *-homomorphism \( \phi_X : A \to \mathcal{L}(X) \). We write \( KX \) for the ideal of compact operators and we denote the rank one operators by \( \theta_{\xi,\eta} : X \to X : \zeta \mapsto \xi \langle \eta, \zeta \rangle \).

For \( n > 1 \) we write \( X^{\otimes n} = X^{\otimes n-1} \otimes X \) for the stabilized \( n \)-tensor product, with the left action given by \( \phi_n = \phi_X \otimes \text{id}_{X^{\otimes n-1}} \). We write \( \xi^{\otimes n} := \xi_1 \otimes \cdots \otimes \xi_n \) for the elementary tensors of \( X^{\otimes n} \).

We fix \((\pi, t)\) be the Fock representation of \( X \). That is, on \( \mathcal{F}X := \bigoplus_{n=1}^{\infty} X^{\otimes n} \) we define the adjointable operators given on the elementary tensors \( \eta^{\otimes n} \in X^{\otimes n} \) by

\[
\pi(a)\eta^{\otimes n} = \phi_n(a)\eta^{\otimes n} \quad \text{for} \quad a \in A \quad \text{and} \quad t(\xi^{\otimes n}) = \xi \otimes \eta^{\otimes n} \quad \text{for} \quad \xi \in X.
\]

In order to reduce the use of superscripts we will abuse notation and write \( t(\xi^{\otimes n}) \) instead of the more appropriate \( t^n(\xi^{\otimes n}) \), and \( t(\xi^{\otimes 0}) = \pi(a) \) for \( a = \xi^{\otimes 0} \in A \). We write \( \mathcal{T}_X \) for the Toeplitz-Pimsner C*-algebra that is generated by \( \pi(A) \) and \( t(X) \). It follows that

\[
\mathcal{T}_X = \overline{\text{span}} \{ t(\xi^{\otimes m})n(x) \mid \xi^{\otimes n} \in X^{\otimes n}, \eta^{\otimes m} \in X^{\otimes m}, n, m \in \mathbb{Z}_+ \}
\]

with the understanding that \( X^{\otimes 0} = A \). It is clear that \( \mathcal{T}_X \) admits a gauge action \( \gamma_z := \text{ad}_{u_z} \) given by the unitaries

\[
u_z(\xi_n) = z^n \xi_n \quad \text{for} \quad \xi_n \in X^{\otimes n}.
\]

The Gauge-Invariant-Uniqueness-Theorem (in the full generality obtained by Katsura [21]) asserts that \( \mathcal{T}_X \) is the universal C*-algebra with respect to pairs \((\rho, v)\) such that

\[
v(\xi)^* v(\eta) = \rho(\xi, \eta) \quad \text{and} \quad \rho(a) t(\xi) = t(\phi_X(a) \xi).
\]

Any such pair induces a map \( \psi_v \) on \( KX \) such that \( \psi_v(\theta_{\xi,\eta}) = (\pi(\xi)^*) v(\eta) \). In fact \((\rho, v)\) induces a faithful representation of \( \mathcal{T}_X \) if and only if it admits a gauge action and \( \rho(A) \cap \psi_v(KX) = (0) \) (hence \( \rho \) is injective). We also fix the projections

\[
p_n : \mathcal{F}X \to X^{\otimes n}.
\]

It is straightforward that the \( p_n \) commute with the diagonal operators of \( \mathcal{L}(\mathcal{F}X) \) and thus with the elements in the fixed point algebra \( \mathcal{T}_X^\mathbb{C} \).

Let \( J \subseteq \phi_X^{-1}(KX) \). The \( J \)-relative Cuntz-Pimsner algebra \( \mathcal{O}(J, X) \) is defined as the quotient of \( \mathcal{T}_X \) by the ideal generated by

\[
\pi(a) - \psi_v(\phi_X(a)) \quad \text{for} \quad a \in J.
\]

As such it inherits the gauge action from \( \mathcal{T}_X \). In particular \( \mathcal{O}(J, X) \) is the universal C*-algebra with respect to pairs \((\rho, v)\) that in addition satisfy the \( J \)-covariance \( \rho(a) = \psi_v(\phi_X(a)) \) for all \( a \in J \). If \( J = J_X \) for Katsura’s ideal

\[
J_X := \ker \phi_X \bigcap \phi_X^{-1}(KX)
\]

then the quotient is the Cuntz-Pimsner algebra \( \mathcal{O}_X [21] \). It is shown in [19] that \( A \) embeds in \( \mathcal{O}(J, X) \) if and only if \( J \subseteq J_X \). In this case the Gauge-Invariant-Uniqueness-Theorem asserts that a pair \((\rho, v)\) defines a faithful representation of \( \mathcal{O}(J, X) \) if and only if it is \( J \)-covariant, it admits a gauge action, \( \rho \) is injective and \( J = \{ a \in A \mid \rho(a) \in \psi_v(KX) \} \).

The Fock space itself admits Hilbert spaces quantizations. For convenience we take Hilbert spaces to be conjugate linear in the first entry (so they are right Hilbert \( \mathbb{C} \)-modules). Suppose that \( \rho_0 : A \to \mathcal{B}(H_0) \) is a *-representation and form the Hilbert module \( \mathcal{F}X \otimes_{\rho_0} H_0 \). It is a Hilbert space with the inner product given by

\[
(\xi^{\otimes n} \otimes x, \eta^{\otimes m} \otimes y) := \langle x, \rho_0((\xi^{\otimes n}, \eta^{\otimes m})_{\mathcal{F}X}) y \rangle_{H_0}.
\]

and the induced pair \((\rho, v) := (\pi \otimes I_{H_0}, t \otimes I_{H_0}) \) defines a representation of \( \mathcal{T}_X \). Now if we consider \((H_{\tau}, x_{\tau}, \rho_{\tau})\) be the GNS-representation of \( A \) and \((H_u, \rho_u)\) be the universal representation of \( A \) then \((\pi \otimes I_{H_u}, t \otimes I_{H_u}) \) defines a faithful representation of \( \mathcal{T}_X \) on

\[
\mathcal{F}X \otimes_{\rho_u} H_u \simeq \bigoplus_{\tau \in \mathcal{S}(A)} \mathcal{F}X \otimes_{\rho_{\tau}} H_{\tau}.
\]
2.3. The KMS-simplex and the number operator. Fix \( s \in \mathbb{R} \). We use the gauge action to define \( \sigma : \mathbb{R} \to \text{Aut}(T_X) \) by \( \sigma_r = \gamma_{e^{ir\tau}} \). It is standard to see then that it extends to an entire function on the analytic elements \( f = t(\xi^{\otimes n})t(\eta^{\otimes m})^* \) of \( T_X \) by setting
\[
\sigma_z(t(\xi^{\otimes n})t(\eta^{\otimes m})^*) = e^{(n-m)iz}t(\xi^{\otimes n})t(\eta^{\otimes m})^*.
\]
We emphasize here that we consider just elementary tensors. The \((\sigma, \beta)\)-KMS condition for a state \( \varphi \) is thus written as
\[
\varphi(t(\xi^{\otimes n})t(\eta^{\otimes m})^*) \cdot t(\zeta^{\otimes k})t(y^{\otimes l})^*) = e^{-(n-m)\beta s}\varphi(t(\zeta^{\otimes k})t(y^{\otimes l})^* \cdot t(\xi^{\otimes n})t(\eta^{\otimes m})^*)
\]
Likewise we get the \((\sigma, \beta)\)-KMS condition for the relative Cuntz-Pimsner algebras \( O(J, X) \).

**Definition 2.1.** Let \( X \) be a \( C^*\)-correspondence and \( J \subseteq \mathcal{O}_X^{-1}(KX) \). For \( \beta > 0 \) we write \( E_\beta(\mathcal{O}(J, X)) \) for the set of the \((\sigma, \beta)\)-KMS states of \( \mathcal{O}(J, X) \) where \( \sigma : \mathbb{R} \to \text{Aut}(\mathcal{O}(J, X)) \) is given by \( r \mapsto \gamma_{e^{ir\tau}} \) for the gauge action \( \gamma \) of \( \mathcal{O}(J, X) \).

The rotational action formalizes the distribution \( e^{-\beta N} \) for the number operator \( N \) given by \( N\xi^{\otimes n} = n\xi^{\otimes n} \). This is similar to what is done in Quantum Mechanics and let us include some details here.

**Proposition 2.2.** Let \( X \) be a \( C^*\)-correspondence over \( A \) and let \( c \in \mathbb{C} \) and \( s \in \mathbb{R} \). If \( N \) is the number operator on \( FX \otimes \rho_r H_\tau \) for the GNS-representation \((H_\tau, x_\tau, \rho_\tau)\) of \( \tau \). Then \( \mathcal{N}_r \otimes I_{H_\tau} = \sum_{k=0}^\infty kp_k \otimes I_{H_\tau} \) is an unbounded selfadjoint operator on \( FX \otimes \rho_r H_\tau \). It suffices to show that
\[
e^{ir(c+sN)} \otimes I_{H_\tau} = e^{ic}u_{c+s} \otimes I_{H_\tau}.
\]
For convenience let us write \( p_{k,\tau} = p_k \otimes I_{H_\tau} \). By the Spectral Theorem for unbounded normal operators we deduce that
\[
e^{ir(c+sN)} \otimes I_{H_\tau} = \text{sot-lim}_m e^{ic} \prod_{k=0}^{m} e^{iskp_{k,\tau}}.
\]
For any \( z \in \mathbb{C} \) we can use the functional calculus to approximate \( e^{z p_{k,\tau}} \) by \( P_t(p_{k,\tau}) \) such that the \( P_t(x) = \sum_{j} \alpha_{j,\tau} x^j \) converge to \( e^{xz} \) for \( x \in \{0, 1\} \). Then we get
\[
P_t(p_{k,\tau})(\xi^{\otimes n} \otimes x_\tau) = \begin{cases} \sum_{j} \alpha_{j,\tau}(\xi^{\otimes n} \otimes x_\tau) & \text{if } n = k, \\ \alpha_{0,\tau}(\xi^{\otimes n} \otimes x_\tau) & \text{if } n \neq k, \end{cases}
\]
and so \( e^{zp_{k,\tau}} = e^z p_{k,\tau} + \sum_{m \neq k} p_{m,\tau} \). Therefore
\[
e^{ir(c+sN)} \otimes I_{H_\tau}(\xi^{\otimes n} \otimes x_\tau) = \lim_{m} e^{ic} \prod_{k=0}^{m} e^{iskp_{k,\tau}}(\xi^{\otimes n} \otimes x_\tau)
\]
\[
= e^{ic} e^{isN}(\xi^{\otimes n} \otimes x_\tau) = e^{ic}(u_{c+s} \otimes I_{H_\tau})(\xi^{\otimes n} \otimes x_\tau),
\]
and the proof is complete. \( \blacksquare \)

Henceforth we focus on the case where \( s = 1 \). Substituting \( \beta \) by \( s\beta \) in what follows yields the results for any \( s \in \mathbb{R}^+ \).
3. CHARACTERIZATION OF EQUILIBRIUM STATES

We start by giving an equivalent characterization of the KMS-condition.

Proposition 3.1. Let $X$ be a $C^*$-correspondence and let $\beta \in \mathbb{R}$. Then $\varphi \in E_\beta(T_X)$ if and only if
\[
\varphi(t(\xi^\otimes n)t(\eta^\otimes m)^*) = \delta_{n,m} e^{-n\beta} \varphi(t(\eta^\otimes m)^* t(\xi^\otimes n))
\]
for all elementary tensor vectors $\xi^\otimes n \in X^\otimes n$, $\eta^\otimes m \in X^\otimes m$, with $n,m \in \mathbb{Z}_+$. Consequently two $(\sigma,\beta)$-KMS states coincide if and only if they agree on $\pi(A)$.

An analogous description holds for the states in $E_\beta(O(J,X))$ for any relative Cuntz-Pimsner algebra $O(J,X)$.

Proof. Suppose that $\varphi \in E_\beta(T_X)$. If $n = m$ then the KMS-condition in (2.2) directly gives that
\[
\varphi(t(\xi^\otimes n)t(\eta^\otimes m)^*) = e^{-n\beta} \varphi(t(\eta^\otimes m)^* t(\xi^\otimes n)).
\]
If $n \neq m$ then we use that $\varphi$ is $\sigma$-invariant and therefore for every $r \in \mathbb{R}$ we get
\[
\varphi(t(\xi^\otimes n)t(\eta^\otimes m)^*) = \varphi_r(t(\xi^\otimes n)t(\eta^\otimes m)^*) = e^{(n-m)r} \varphi(t(\xi^\otimes n)t(\eta^\otimes m)^*)
\]
As $(n-m) \neq 0$ we must have that $\varphi(t(\xi^\otimes n)t(\eta^\otimes m)^*) = 0$.

Conversely suppose that $\varphi$ is a state on $T_X$ satisfying (3.1). It will be convenient to refer to elements of the form $t(\xi^\otimes n)t(\eta^\otimes m)^*$ as $(n,m)$-products. We have to verify equation (2.2), i.e.,
\[
\varphi(t(\xi^\otimes n)t(\eta^\otimes m)^*) = e^{-(n-m)\beta} \varphi(t(\xi^\otimes n)t(\eta^\otimes m)^*)
\]
We will proceed by considering cases on $n,m,k,l$. The left hand side of (2.2) gives either an $(n,m-k+l)$-product or an $(n+k-m,l)$-product, depending on whether $m \geq k$ or $m \leq k$. Similarly the right hand side gives either a $(k,l-n+m)$-product or a $(k+n-l,m)$-product. In each case we get that $\varphi$ is zero on these products, and thus equation (2.2) holds when $n+k \neq l+m$. Now suppose that $n+k = l+m$. Without loss of generality we may assume that $m \geq k$ and so $n \geq l$ (otherwise take adjoints). Let us write
\[
\eta^\otimes m = \eta^\otimes k \otimes \eta^\otimes m-k \quad \text{and} \quad \xi^\otimes n = \xi^\otimes l \otimes \xi^\otimes n-l.
\]
By using (3.1), the left hand side of (2.2) equals to
\[
\varphi(t(\xi^\otimes n)t(\eta^\otimes m)^*) = \varphi(t(\xi^\otimes n)t(\eta^\otimes m)^*)
\]
\[
= e^{-n\beta} \varphi(t(\eta^\otimes l \otimes (\xi^\otimes k)\eta^\otimes m-k)^* t(\xi^\otimes n))
\]
Likewise, the right hand side of equation 2.2 equals to
\[
e^{-(n-m)\beta} \varphi(t(\xi^\otimes n)t(\eta^\otimes m)^*) = e^{-(n-m)\beta} \varphi(t(\eta^\otimes l \otimes (\xi^\otimes k)\eta^\otimes m-k)^* t(\xi^\otimes n))
\]
\[
e^{-(n-m)\beta} \varphi(t(\eta^\otimes l \otimes (\xi^\otimes k)\eta^\otimes m-k)^* t(\xi^\otimes n))
\]
Therefore equation 2.2 is satisfied, and the proof is complete.

The following proposition allows us to consider just unital $C^*$-correspondences from now on. When $\phi_X$ is not unital, we define $X^1$ be the space $X$ which becomes a $C^*$-correspondence over $A^1 = A + \mathbb{C}$ by extending the operators $\phi_X(1)\xi = \xi = 1$. Note here that $A^1 = A \oplus \mathbb{C}$ when $A$ is already unital but $\phi_X(1_A) \neq 1_X$.

Proposition 3.2. Let $X$ be a $C^*$-correspondence over $A$. Then $\varphi$ is a $(\sigma,\beta)$-KMS state for $T_{X^1}$ if and only if it restricts to a $(\sigma,\beta)$-KMS state on $T_X$.

Proof. If $\phi_X : A \to \mathcal{L}X$ is unital then there is nothing to show. Otherwise let $(\pi,t)$ be the Fock representation of $X^1$ and notice that $(\pi|A,t)$ defines a faithful representation of $T_X$ by the Gauge-Invariant-Uniqueness-Theorem. Indeed it admits a gauge action and if $(\pi(a) \in \psi_t(KX))$ then
\[
a = p_0 \pi(a) p_0 \in p_0 \psi_t(KX) p_0 \subseteq p_0 \psi_t(KX^1) p_0 = (0)
\]
Remark 4.2. For any non-trivial word \( p \) we have that \( \varphi(p) = 1 \).

Remark 4.3. By construction, \( K(FX) \) is an ideal in \( T_X \). When \( X \) is of finite rank then we can write every projection \( p_k : FX \to A \) with \( k \geq 1 \) by

\[
p_k = \sum_{|\mu| = k} t(x_{\mu})t(x_{\mu})^* - \sum_{|\nu| = k+1} t(x_{\nu})t(x_{\nu})^*,
\]

and thus \( p_k \in T_X \) for all \( k \geq 1 \). Moreover we see that

\[
p_0 = 1_{FX} - \sum_{i \in |d|} t(x_i)t(x_i)^* \in T_{X^1}.
\]

Hence by using the unitization we have that \( p_k \in T_X \) for all \( k \in \mathbb{Z}_+ \). It is straightforward that the \( p_n \) commute with all elements in \( T_{X^1} \), as the latter are supported on the diagonal of \( FX \). Moreover if \( \varphi \in E_\beta(T_{X^1}) \) then

\[
\varphi(p_k) = \sum_{|\mu| = k} \varphi(t(x_{\mu})p_0t(x_{\mu})^*) = \sum_{|\mu| = k} e^{-k\beta} \varphi(p_0 \pi((x_{\mu}, x_{\mu}))p_0) \leq \sum_{|\mu| = k} e^{-k\beta} \varphi(p_0).
\]

Therefore if \( \varphi(p_0) = 0 \) then \( \varphi(p_k) = 0 \) for all \( k \in \mathbb{Z}_+ \).

This triggers the following definition. We will be using the same symbol for the extension of a state from \( T_X \) to \( T_{X^1} \) from Proposition 3.2.

Definition 4.4. Let \( X \) be a C*-correspondence of finite rank over \( A \). For \( \beta \in \mathbb{R} \) we define

\[
E^\text{fin}_\beta(T_X) := \{ \varphi \in E_\beta(T_X) \mid \sum_{k=0}^\infty \varphi(p_k) = 1 \} \quad \text{and} \quad E^\infty_\beta(T_X) := \{ \varphi \in E_\beta(T_X) \mid \varphi(p_0) = 0 \}.
\]

Likewise we define \( E^\infty(J, X) \) and \( E^\text{fin}(J, X) \) for any \( J \)-relative Cuntz-Pimsner algebra with respect to the projections \( q_J(p_k) \), for the canonical *-epimorphism \( q_J : T_X \to \mathcal{O}(J, X) \).

Remark 4.5. We also note that \( E^\infty_\beta(\cdot) \) and \( E^\text{fin}(\cdot) \) may be trivial in some cases. For if \( q : T_X \to \mathcal{O}(A, X) \) is the canonical *-epimorphism, then its kernel \( K(FX) \) is generated by \( p_0 \). This automatically implies that \( E^\text{fin}_\beta(\mathcal{O}(A, X)) = \emptyset \). As another example, in Proposition 5.7 we will show that \( E^\infty_\beta(T_X) = \emptyset \) for sufficiently large \( \beta \).

Notice that \( K(FX) \subseteq T_X \) and thus it inherits the gauge action by restriction. Therefore we also get equilibrium states for \( K(FX) \). Let us give an alternative proof of [23, Proposition 2.4] of the Wold decomposition into a finite and an infinite part.
Theorem 4.6. Let \( X \) be a \( C^\ast \)-correspondence of finite rank over \( A \) and let \( \beta \in \mathbb{R} \). Then for any \( \varphi \in E_\beta(T_X) \) we have:

(i) \( \varphi \in E^\text{fin}_\beta(T_X) \) if and only if \( \varphi|_{\mathcal{K}(FX)} \in E_\beta(\mathcal{K}(FX)) \);
(ii) \( \varphi \in E^\text{fin}_\beta(T_X) \) if and only if \( \varphi|_{\mathcal{K}(FX)} = 0 \) if and only if \( \varphi \) factors through \( O(A, X) \);
(iii) There are unique \( \varphi_\text{fin} \in E^\text{fin}_\beta(T_X) \) and \( \varphi_\infty \in E^\infty_\beta(T_X) \) such that

\[
\varphi = \lambda \varphi_\text{fin} + (1 - \lambda) \varphi_\infty, \text{ for } \lambda := \sum_{k=0}^\infty \varphi(p_k).
\]

Proof. By positivity we have \( \sum_{k=0}^n \varphi(p_k) \leq 1 \) for every \( n \in \mathbb{N} \) and so \( \sum_{k=0}^\infty \varphi(p_k) < \infty \). Moreover equation (4.2) implies that \( \sum_{k=0}^\infty \varphi(p_k) = 0 \) if and only if \( p_0 = 0 \). Now both items (i) and (ii) follow by using \( \sum_{k=0}^n p_k \) as a contractive approximate identity of \( \mathcal{K}(FX) \). For item (iii) use the Wold decomposition with respect to the quotient map \( T_X \to O(A, X) \) and the KMS-condition on \( \varphi \) to define the positive functional \( \psi_\text{fin} : T_X \to \mathbb{C} \) by

\[
\psi_\text{fin}(f) := \sum_{k=0}^\infty \varphi(p_k f p_k) = \sum_{k=0}^\infty \varphi(p_k f p_k) = \sum_{k=0}^\infty \varphi(p_k f p_k), \text{ for all } f \in T_X,
\]

and let \( \psi_\infty := \varphi - \psi_\text{fin} \). If \( \psi_\text{fin} \neq 0 \) then \( \lambda := \sum_{k=0}^\infty \varphi(p_k) = \|\psi_\text{fin}\| \), and so \( \|\psi_\infty\| = 1 - \lambda \). Hence if \( \lambda \in (0, 1) \) we obtain the states

\[
\varphi_\text{fin} := \lambda^{-1} \psi_\text{fin} \quad \text{and} \quad \varphi_\infty := (1 - \lambda)^{-1} \psi_\infty.
\]

Since there is a unique extension of a state from \( \mathcal{K}(FX) \) to \( T_X \) we get uniqueness of this decomposition. As \( \varphi_\infty(p_0) = 0 \) it remains to show that \( \varphi_\text{fin} \) satisfies the KMS-condition. By definition we have that

\[
\psi_\text{fin}(t(\xi^{\otimes n}) t(\eta^{\otimes m})^*) = 0 \quad \text{when } n \neq m.
\]

Now if \( n = m \) then we get

\[
t(\eta^{\otimes n})^* p_k t(\xi^{\otimes n}) = \begin{cases} p_{k-n} t(\eta^{\otimes n})^* t(\xi^{\otimes n}) p_{k-n} & \text{if } k \geq n, \\ 0 & \text{otherwise.} \end{cases}
\]

Therefore for all \( n, m \in \mathbb{Z}_+ \) we obtain

\[
\psi_\text{fin}(t(\xi^{\otimes n}) t(\eta^{\otimes m})^*) = \delta_{n,m} \sum_{k=0}^\infty \varphi(p_k t(\xi^{\otimes n}) t(\eta^{\otimes m})^*) p_k
\]

\[
= \delta_{n,m} \sum_{k=0}^\infty \varphi(t(\eta^{\otimes m})^* p_k t(\xi^{\otimes n}))
\]

\[
= \delta_{n,m} \sum_{k=n}^\infty \varphi(p_{k-n} t(\eta^{\otimes m})^* t(\xi^{\otimes n}) p_{k-n})
\]

\[
= \delta_{n,m} \sum_{k=0}^\infty \varphi(p_k t(\eta^{\otimes m})^* t(\xi^{\otimes n}) p_k) = \delta_{n,m} \psi_\text{fin}(t(\eta^{\otimes m})^* t(\xi^{\otimes n}))
\]

and thus \( \varphi_\text{fin} \) satisfies equation (3.1).

\[ \blacksquare \]

Remark 4.7. The convex decomposition is not weak*-continuous. For example, for fixed \( \varphi_\infty \in E^\infty_\beta(T_X) \) and \( \varphi_\text{fin} \in E^\text{fin}_\beta(T_X) \), the states \( \varphi_n = n^{-1} \varphi_\text{fin} + (1 - n^{-1}) \varphi_\infty \) weak*-converge to \( \varphi_\infty \). However the infinite and the finite parts of all \( \varphi_n \) stay the same.

5. Entropy

We start with a remark that ensures that the quantities we are to introduce are independent of the choice of the unit decomposition.
Remark 5.1. If $\tau \in T(A)$ then the value $\sum_{|\mu|=k} \tau(\langle x_{\mu}, x_{\mu} \rangle)$ is independent of the unit decomposition $x = \{x_1, \ldots, x_d\}$ that we may use. Indeed if $\{y_1, \ldots, y_d\}$ is a second unit decomposition, then

$$\sum_{|\mu|=k} \tau(\langle x_{\mu}, x_{\mu} \rangle) = \sum_{|\mu|=k} \sum_{|\nu|=k} \tau(\langle x_{\mu}, y_{\nu}\rangle) = \sum_{|\nu|=k} \sum_{|\mu|=k} \tau(\langle y_{\nu}, x_{\mu}\rangle) = \sum_{|\nu|=k} \tau(\langle y_{\nu}, y_{\nu}\rangle).$$

Definition 5.2. Let $X$ be a $C^*$-correspondence of finite rank over $A$ with respect to $\{x_1, \ldots, x_d\}$ and let $\beta \in (0, \infty)$. For any $\tau \in T(A)$ we define the formal series

$$(5.1) \quad c_{\tau, \beta} := \sum_{k=0}^{\infty} e^{-k\beta} \sum_{|\mu|=k} \tau(\langle x_{\mu}, x_{\mu} \rangle).$$

We write $T_{\beta}(A) := \{\tau \in T(A) \mid c_{\tau, \beta} < \infty\}$.

Remark 5.1 implies that $c_{\tau, \beta}$ and $T_{\beta}(A)$ do not depend on the unit decomposition. Since $x_0 = 1_A$ then we see that $c_{\tau, \beta} \geq 1$. Moreover the set $T_{\beta}(A)$ is convex. On the other extreme we have the notion of averages.

Definition 5.3. Let $X$ be a $C^*$-correspondence of finite rank over $A$ with respect to $\{x_1, \ldots, x_d\}$ and let $\beta > 0$. Let $\AVT_{\beta}(A)$ be the set of the tracial states $\tau$ of $A$ that satisfy

$$\tau(a) = e^{-\beta} \sum_{i \in [d]} \tau(\langle x_i, ax_i \rangle) \text{ for all } a \in A.$$

As in Remark 5.1 we have that $\AVT_{\beta}(A)$ does not depend on the decomposition $\{x_1, \ldots, x_d\}$ of the unit. The next proposition marks that $T_{\beta}(A) \cap \AVT_{\beta}(A) = \emptyset$.

Proposition 5.4. Let $X$ be a $C^*$-correspondence of finite rank over $A$ and let $\beta > 0$. If $\tau \in \AVT_{\beta}(A)$ then $c_{\tau, \beta} = \infty$.

Proof. Induction yields an average formula for all words of length $k$, i.e.,

$$(5.2) \quad \tau(a) = e^{-k\beta} \sum_{|\mu|=k} \tau(\langle x_{\mu}, ax_{\mu} \rangle) \text{ for all } a \in A, k \in \mathbb{N}.$$\[\square\]

The root test implies a notion of entropy for $\tau \in T(A)$ that connects with convergence of $c_{\tau, \beta}$. We are going to use also two notions of entropy for $X$. As we use entropy for convergence of $c_{\tau, \beta}$ we set $\limsup_k k^{-1} \log a_k = 0$ if $a_k = 0$ eventually.

Definition 5.5. Let $X$ be a $C^*$-correspondence of finite rank over $A$ with respect to a unit decomposition $x = \{x_1, \ldots, x_d\}$.

1. The entropy of a $\tau \in T(A)$ is given by

$$h^\tau_X := \limsup_k \frac{1}{k} \log \sum_{|\mu|=k} \tau(\langle x_{\mu}, x_{\mu} \rangle).$$

2. The entropy of $x$ is defined by

$$h^x_X := \limsup_k \frac{1}{k} \log \| \sum_{|\mu|=k} \langle x_{\mu}, x_{\mu} \rangle \|_A.$$

3. The strong entropy of $X$ is defined by

$$h^s_X := \inf \{h^\tau_X \mid \text{ $x = \{x_1, \ldots, x_d\}$ is a unit decomposition for $X$} \}.$$

4. The entropy of $X$ is defined by

$$h_X := \inf \{\beta > 0 \mid E_{\beta}(T_X) \neq \emptyset\}.$$
Remark 5.6. Due to remark 5.1, the entropy $h^*_X$ is independent of the unit decomposition. Likewise $h^*_X = h^*_X$ for any unit decomposition when $A$ is abelian. Furthermore the lim sup in $h^*_X$ is actually the limit of a decreasing sequence. Indeed for $k_1, k_2 \in \mathbb{N}$ with $k_1 + k_2 = k$ we get
\[
\sum_{\mu = k} \langle x_\mu, x_\mu \rangle = \sum_{|\nu_1| = k_1, |\nu_2| = k_2} \langle x_{\nu_1} \otimes x_{\nu_2}, x_{\nu_1} \otimes x_{\nu_2} \rangle \\
= \sum_{|\nu_1| = k_1} \langle x_{\nu_1}, \left( \sum_{|\nu_2| = k_2} \langle x_{\nu_2}, x_{\nu_2} \rangle \right) \rangle \leq \| \sum_{|\nu_2| = k_2} \langle x_{\nu_2}, x_{\nu_2} \rangle \| A \sum_{|\nu_1| = k_1} \langle x_{\nu_1}, x_{\nu_1} \rangle.
\]

Therefore the sequence $\| \sum_{|\nu| = k} \langle x_\mu, x_\mu \rangle \|_A$ is submultiplicative.

We close this section with a connection between entropies and $E_\beta(T_X)$. We shall see later that Proposition 5.7(iv) can follow from the complete parametrization of $E^\text{fin}_{\beta}(T_X)$ and $E^\text{inf}_{\beta}(T_X)$. Item (v) below is basically a rewording of [33, Theorem 2.5, Corollary 2.6].

Proposition 5.7. Let $X$ be a C*-correspondence of finite rank over $A$ and let $\beta > 0$.

(i) If $\tau \in T_\beta(A) \cup AVT_\beta(A)$ then $h^*_X \leq \beta$.
(ii) For every $\tau \in T(A)$ we have that $h^*_X \leq h^*_X \leq \log d$.
(iii) $T_\beta(A) = T(A)$ for all $\beta \in (h^*_X, \infty)$.
(iv) $E^\text{fin}_{\beta}(T_X) = \emptyset$ and $E_\beta(T_X) = E^\text{inf}_{\beta}(T_X)$ whenever $\beta > h^*_X$.
(v) If $A$ is abelian then $AVT_{h^*_X}(A) \neq \emptyset$.

Proof. Let $x = \{x_1, \ldots, x_d\}$ be a decomposition of the unit. Item (i) follows directly from the root test when $\tau \in T_\beta(A)$ and from Proposition 5.4 when $\tau \in AVT_\beta(A)$. Moreover it is straightforward to check that if $\tau \in T(A)$ then
\[
\sum_{|\mu| = k} \tau(\langle x_\mu, x_\mu \rangle) \leq \| \sum_{|\mu| = k} \langle x_\mu, x_\mu \rangle \|_A \leq d^k.
\]

As the left hand side does not depend on $x$, taking infimum over all unit decompositions gives that $h^*_X \leq h^*_X \leq \log d$. For item (iii) suppose that $\beta \in (h^*_X, \infty)$ and choose a unit decomposition $x = \{x_1, \ldots, x_d\}$ such that $h^*_X \leq h^*_X < \beta$. Then for any $\tau \in T(A)$ we have that
\[
\lim_{k \to \infty} \left( e^{-k\beta} \sum_{|\mu| = k} \tau(\langle x_\mu, x_\mu \rangle) \right)^{1/k} \leq e^{-\beta h^*_X} < 1
\]
giving that $e_{r,\beta} < \infty$. For item (iv), if $\varphi \in E^\text{inf}_{\beta}(T_X)$ then $\varphi(p_0) = 0$ and thus $\varphi(p_k) = 0$ for all $k \in \mathbb{Z}_+$ by equation (4.2). But then the KMS-condition yields
\[
1 = \sum_{|\mu| = k} \varphi(t(x_\mu)t(x_\mu)^*) = e^{-k\beta} \sum_{|\mu| = k} \varphi(\pi(\langle x_\mu, x_\mu \rangle) \leq e^{-k\beta} \| \sum_{|\mu| = k} \langle x_\mu, x_\mu \rangle \|_A,
\]
as $\varphi \in T(A)$. Hence $\beta \leq k^{-1} \log \| \sum_{|\mu| = k} \langle x_\mu, x_\mu \rangle \|_A$ for all $k \in \mathbb{Z}_+$, which gives that $\beta \leq h^*_X$. Taking the infimum over all unit decompositions yields $\beta \leq h^*_X$. Finally, if $A$ is abelian then the arguments of [33, Theorem 2.5, Corollary 2.6] apply to give that $AVT_{h^*_X}(A) \neq \emptyset$. In short let the map
\[
\psi: A \to A \text{ such that } \psi(a) = \sum_{i \in [d]} \langle x_i, ax_i \rangle.
\]
As $\psi$ is a positive map we have $\| \psi^k \| = \| \psi^k(1) \| = \| \sum_{|\mu| = k} \langle x_\mu, x_\mu \rangle \|_A$ for all $k \in \mathbb{N}$. Therefore
\[
h^*_X = \lim_{k \to \infty} \log \| \psi^k \|^{1/k} = \log \lambda_\psi,
\]
where $\lambda_\psi$ is the spectral radius of $\psi$. Then [33, Theorem 2.5, Corollary 2.6] implies that $\lambda_\psi$ is an eigenvalue of the adjoint of $\psi$ on the states of $A$, i.e. there is $\tau \in \mathcal{S}(A)$ such that $\tau \psi = \lambda_\psi \tau \psi$ (the fullness condition of [33] is not required here). Hence $\tau$ gives a tracial state in $AVT_{h^*_X}(A)$. \hfill \blacksquare
6. The finite part of the equilibrium states

In this section we parametrize the states in $E^\text{fin}_\beta(T_X)$ for $\beta \in (0, \infty)$ and consequently we show how this induces a parametrization for all $E^\text{fin}_\beta(O(J,X))$. Passing from $T_\beta(A)$ to $E^\text{fin}_\beta(T_X)$ uses essentially [23, proof of Theorem 2.1]. Showing that this construction is a bijection generalizes the corresponding arguments from [25].

**Theorem 6.1.** Let $X$ be a $C^*$-correspondence of finite rank over $A$ and let $\beta \in (0, \infty)$. Then there is a bijection

$$
\Phi : T_\beta(A) \to E^\text{fin}_\beta(T_X) \text{ such that } \Phi(\tau)(p_0) = c^{-1}_{\tau,\beta}.
$$

If $x = \{x_1, \ldots, x_d\}$ is a decomposition of the unit then $\Phi$ is given by

$$
(6.1) \quad \Phi(\tau)(t(\xi^n) (\xi^m)^*) = \delta_{n,m} e^{-1}_{\tau,\beta} \sum_{\mu \in \mathbb{F}_+} e^{-(\mu+n)\beta} \tau(\langle \eta^n \otimes x_\mu, \xi^n \otimes x_\mu \rangle)
$$

for all $\xi^n \in X^n$ and $\eta^m \in X^m$. If, in addition, $E^\text{fin}_\beta(T_X)$ is weak*-closed then $\Phi$ is a weak*-homeomorphism between weak*-compact sets.

**Proof.** Equation (6.1) is independent of the unit decomposition for $\tau \in T_\beta(A)$. Indeed let $y = \{y_1, \ldots, y_d\}$ be a second decomposition. If $n \neq m$ then there is nothing to show. For $n = m$ we directly verify that

$$
\sum_{|\mu| = k} \tau(\langle \eta^n \otimes x_\mu, \xi^n \otimes x_\mu \rangle) = \sum_{|\mu| = k} \sum_{|\nu| = k} \tau(\langle \eta^n \otimes x_\mu, \xi^n \otimes \theta_{y_\nu,y_\nu}x_\mu \rangle)
$$

for all $\eta^n \in X^n$ and $\xi^n \in X^n$. If, in addition, $E^\text{fin}_\beta(T_X)$ is weak*-closed then $\Phi$ is a weak*-homeomorphism between weak*-compact sets.

Now we proceed to the construction of $\Phi$. First we show that $\varphi_\tau \equiv \Phi(\tau)$ exists and is in $E^\text{fin}_\beta(T_X)$ when $\tau \in T_\beta(A)$. Let $(H_\tau, x_\tau, \rho_\tau)$ be the GNS-representation associated to $\tau$ and consider the induced pair $(\rho, v) := (\pi \otimes I, t \otimes I)$ for $T_X$ acting on $FX \otimes_{\rho_\tau} H_\tau$. For any word $\mu$ on the $d$ symbols define the positive vector state $\varphi_{\tau,\mu}$ of $T_X$ be given by

$$
\varphi_{\tau,\mu}(f) = \langle x_\mu \otimes x_\tau, (\rho \times v)(f)x_\mu \otimes x_\tau \rangle_H \text{ for } f \in T_X.
$$

We then define

$$
\varphi_\tau := c^{-1}_{\tau,\beta} \sum_{k=0}^\infty e^{-k\beta} \sum_{|\mu| = k} \varphi_{\tau,\mu}.
$$

To see that it is indeed well defined (and a state) on $T_X$ first check that

$$
c^{-1}_{\tau,\beta} \sum_{k=0}^\infty e^{-k\beta} \sum_{|\mu| = k} \varphi_{\tau,\mu}(\pi(1_A)) = c^{-1}_{\tau,\beta} \sum_{k=0}^\infty e^{-k\beta} \tau(\langle x_\mu, x_\mu \rangle) = 1.
$$

Likewise we have $\varphi_{\tau,\mu}(f) \leq \|f\| \varphi_{\tau,\mu}(\pi(1_A))$ for all $0 \leq f \in T_X$, and thus

$$
c^{-1}_{\tau,\beta} \sum_{k=0}^\infty e^{-k\beta} \sum_{|\mu| = k} \varphi_{\tau,\mu}(f) \leq c^{-1}_{\tau,\beta} \sum_{k=0}^\infty e^{-k\beta} \sum_{|\mu| = k} \|f\| \varphi_{\tau,\mu}(\pi(1_A)) = \|f\|.
$$

Next we show that $\varphi_\tau$ satisfies equation (6.1). If $n \neq m$ then for all $\mu$ we get that

$$
\varphi_{\tau,\mu}(t(\xi^n) (\xi^m)^*) = \tau(\langle t(\xi^n)^*x_\mu, t(\xi^m)^*x_\mu \rangle_{FX}) = 0,
$$

and thus $\varphi_\tau(t(\xi^n) (\xi^m)^*) = 0$. If $n = m$ and $k \geq n$, then for all $x_\mu$ with $|\mu| < n$ we get that

$$
\varphi_{\tau,\mu}(t(\xi^n) (\xi^m)^*) = \tau(\langle t(\xi^n)^*x_\mu, t(\xi^m)^*x_\mu \rangle_{FX}) = 0.
$$
On the other hand if $|\mu| = k \geq n$ then recall that $\sum_{|\mu|=k} t(x_\mu) t(x_\mu)^* \text{ acts as a unit on } t(X^{\otimes \ell})$ for all $\ell \geq k$. Thus we get

$$\sum_{|\mu|=k} \varphi_{\tau,\mu}(t(\xi^{\otimes n}) t(\eta^{\otimes n})^*) = \sum_{|\mu|=k} \tau^{\pi^{-1}}(t(x_\mu)^* t(\xi^{\otimes n}) t(\eta^{\otimes n})^* t(x_\mu))$$

$$= \sum_{|\mu|=k} \sum_{|\nu|=k-n} \tau^{\pi^{-1}}(t(x_\mu)^* t(\xi^{\otimes n}) t(x_\nu) t(x_\mu)^* t(\eta^{\otimes n})^* t(x_\nu))$$

$$= \sum_{|\nu|=k-n} \tau^{\pi^{-1}}(t(x_\nu)^* t(\eta^{\otimes n})^* t(\xi^{\otimes n}) t(x_\nu))$$

$$= \sum_{|\nu|=k-n} \varphi_{\tau,\nu}(t(\eta^{\otimes n})^* t(\xi^{\otimes n})).$$

Hence we obtain

$$\varphi_\tau(t(\xi^{\otimes n}) t(\eta^{\otimes n})^*) = c_{\tau,\beta}^{-1} \sum_{k=0}^{\infty} e^{-k\beta} \sum_{|\mu|=k} \varphi_{\tau,\mu}(t(\xi^{\otimes n}) t(\eta^{\otimes n})^*)$$

$$= c_{\tau,\beta}^{-1} \sum_{k=0}^{\infty} e^{-k\beta} \sum_{|\mu|=k-n} \varphi_{\tau,\mu}(t(\eta^{\otimes n})^* t(\xi^{\otimes n}))$$

$$= c_{\tau,\beta}^{-1} \sum_{k=0}^{\infty} e^{-(k+n)\beta} \sum_{|\mu|=k} \varphi_{\tau,\mu}(t(\eta^{\otimes n})^* t(\xi^{\otimes n}))$$

$$= c_{\tau,\beta}^{-1} \sum_{k=0}^{\infty} e^{-(k+n)\beta} \varphi_{\tau,\beta}(t(\eta^{\otimes n})^* t(\xi^{\otimes n})).$$

We verify that $\varphi_\tau \in E_\beta(T_X)$ by using Proposition 3.1. By definition we have that if $n \neq m$ then $\varphi_\tau(t(\xi^{\otimes n}) t(\eta^{\otimes m})^*) = 0$. Now if $n = m$ then we directly compute

$$\varphi_\tau(t(\xi^{\otimes n}) t(\eta^{\otimes n})^*) = c_{\tau,\beta}^{-1} \sum_{k=0}^{\infty} e^{-(k+n+1)\beta} \sum_{|\mu|=k} \varphi_{\tau,\mu}(t(\eta^{\otimes n})^* t(\xi^{\otimes n}))$$

$$= e^{-n\beta} c_{\tau,\beta}^{-1} \sum_{k=0}^{\infty} e^{-k\beta} \sum_{|\mu|=k} \varphi_{\tau,\mu}(t(\eta^{\otimes n})^* t(\xi^{\otimes n}))$$

$$= c_{\tau,\beta}^{-1} \sum_{k=0}^{n} e^{-k\beta} \varphi_{\tau,\beta}(t(\eta^{\otimes n})^* t(\xi^{\otimes n})).$$

In order to show that $\varphi_\tau \in E_{\beta}^{\text{fin}}(T_X)$ we compute

$$\sum_{k=0}^{n} \varphi_\tau(p_k) = 1 - \sum_{|\nu|=n+1} \varphi_\tau(t(x_\nu) t(x_\nu)^*)$$

$$= 1 - c_{\tau,\beta}^{-1} \sum_{k=0}^{\infty} e^{-(n+1+k)\beta} \sum_{|\nu|=n+1} \sum_{|\mu|=k} \tau(\langle x_\nu \otimes x_\mu, x_\nu \otimes x_\mu \rangle)$$

$$= c_{\tau,\beta}^{-1} \sum_{k=0}^{n} e^{-k\beta} \sum_{|\mu|=k} \tau(\langle x_\mu, x_\mu \rangle),$$

Applying for $n = 0$ yields $\varphi_\tau(p_0) = c_{\tau,\beta}^{-1}$. Taking the limit $n \to \infty$ gives $\sum_{k=0}^{\infty} \varphi_\tau(p_k) = c_{\tau,\beta}^{-1} c_{\tau,\beta} = 1,$ and so $\varphi_\tau \in E_{\beta}^{\text{fin}}(T_X)$. 

Secondly we show that this correspondence is surjective. To this end fix \( \varphi \in E_{\beta}^{\text{fin}}(T_X) \). Inequality (4.2) gives that \( \varphi(p_0) \neq 0 \) and thus we can define the state \( \tau_{\varphi} \) on \( A \) by

\[
\tau_{\varphi}(a) := \varphi(p_0)^{-1} \varphi(p_0 \pi(a) p_0) \quad \text{for all } a \in A.
\]

Moreover \( \tau_{\varphi} \) is in \( T(A) \) since

\[
\varphi(p_0) \tau_{\varphi}(ab) = \varphi(p_0 \pi(a) \pi(b) p_0) = \varphi(\pi(b) p_0 \pi(a)) = \varphi(p_0 \pi(b) \pi(a) p_0) = \varphi(p_0) \tau(ba),
\]

where we used that \( p_0 \in \pi(A)' \) and \( \sigma_{\beta}(\pi(a)) = \pi(a) \). In order to show that \( \tau_{\varphi} \in T_{\beta}(A) \) it suffices to show that

\[
\varphi(p_0)^{-1} = \sum_{k=0}^{\infty} e^{-k\beta} \sum_{|\mu|=k} \tau_{\varphi}(x_\mu, x_\mu).
\]

However a direct computation yields

\[
\varphi(p_0) \sum_{|\mu|=k} \tau_{\varphi}(x_\mu, x_\mu) = \sum_{|\mu|=k} \varphi(p_0 t(x_\mu)^* t(x_\mu) p_0) = e^{k\beta} \sum_{|\mu|=k} \varphi(t(x_\mu) p_0 t(x_\mu)^*) = e^{k\beta} \varphi(p_k).
\]

Since \( \varphi \in E_{\beta}^{\text{fin}}(T_X) \) we have

\[
\sum_{k=0}^{\infty} e^{-k\beta} \sum_{|\mu|=k} \tau_{\varphi}(x_\mu, x_\mu) = \varphi(p_0)^{-1} \sum_{k=0}^{\infty} \varphi(p_k) = \varphi(p_0)^{-1}.
\]

Surjectivity now follows by showing that \( \varphi = \Phi(\tau_{\varphi}) \). Since both are \((\sigma, \beta)\)-KMS states, by Proposition 3.1 it suffices to show that they agree on \( \pi(A) \). Since \( \varphi \) is implemented by a state on \( K(FX) \), for every \( a \in A \) we have that

\[
\varphi(\pi(a)) = \lim_m \sum_{k,l=0}^{m} \varphi(p_k \pi(a) p_l) = \lim_m \sum_{k=0}^{m} \varphi(p_k \pi(a))
\]

\[
= \lim_m \sum_{k=0 \mid |\mu|=k} \varphi(t(x_\mu) p_0 t(x_\mu)^* \pi(a))
\]

\[
= \lim_m \sum_{k=0 \mid |\mu|=m} e^{-k\beta} \varphi(p_0 t(x_\mu)^* \pi(a) t(x_\mu) p_0)
\]

\[
= c_{\tau_{\varphi},\beta}^{-1} \sum_{k=0}^{\infty} e^{-k\beta} \tau_{\varphi}(x_\mu, a x_\mu)) = \Phi(\tau_{\varphi})(\pi(a)).
\]

To show injectivity let \( \tau \in T_{\beta}(A) \) and use the vector states \( \varphi_{\tau,\mu} \) to get

\[
\sum_{|\mu|=k} \varphi_{\tau,\mu}(p_0 \pi(a) p_0) = \begin{cases} \tau(a) & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}
\]

Therefore we have

\[
\varphi_{\tau}(p_0 \pi(a) p_0) = c_{\tau,\beta}^{-1} \sum_{k=0}^{\infty} e^{-k\beta} \sum_{|\mu|=k} \varphi_{\tau,\mu}(p_0 \pi(a) p_0) = \varphi_{\tau}(p_0) \tau(a)
\]

showing that \( \tau \) is uniquely identified by \( \varphi_{\tau} \).

Finally we show that \( \Phi^{-1} \) is weak*-continuous when \( E_{\beta}^{\text{fin}}(T_X) \) is weak*-closed. To this end let \( \varphi_j, \varphi \in E_{\beta}^{\text{fin}}(T_X) \) such that \( \varphi_j \rightarrow \varphi \) in the weak*-topology. By assumption \( \varphi_j(p_0) \neq 0 \) and \( \varphi(p_0) \neq 0 \) so that \( \tau_{\varphi_j}(a) \rightarrow \tau_{\varphi}(a) \) for all \( a \in A \). Hence \( \Phi^{-1} \) is a continuous bijection from the compact space \( E_{\beta}^{\text{fin}}(T_X) \) onto the Hausdorff space \( T_{\beta}(A) \), and therefore a homeomorphism. \( \blacksquare \)
Remark 6.2. For $\beta > h_X^*$ we have that $T_\beta(A) = T(A)$. It thus follows that the sequence $(\sum_{k=0}^n e^{-k\beta/2} \sum_{|\mu|=k} \langle x_\mu, ax_\mu \rangle)$ converges for all $a \in A$. An application of the Monotone Convergence Theorem gives then that the bijection $\Phi$ of Theorem 6.1 is always weak*-homeomorphism when $\beta > h_X^*$.

**Corollary 6.3.** Let $\Phi : T_\beta(A) \to E^\text{fin}_\beta(T_X)$ be the map of Theorem 6.1. If $\tau = \lambda, \tau_1 + (1 - \lambda) \tau_2$ for $\tau_1, \tau_2 \in T_\beta(A)$ and $\lambda \in [0, 1]$ then

$$\Phi(\tau) = \frac{\lambda c_{\tau_1, \beta}}{c_{\tau, \beta}} \Phi(\tau_1) + (1 - \lambda) \frac{c_{\tau_2, \beta}}{c_{\tau, \beta}} \Phi(\tau_2).$$

Conversely, if $\varphi = \lambda \varphi_1 + (1 - \lambda) \varphi_2$ for $\varphi_1, \varphi_2 \in E^\text{fin}_\beta(T_X)$ and $\lambda \in [0, 1]$ then

$$\Phi^{-1}(\varphi) = \frac{\lambda \varphi_1(p_0)}{\varphi(p_0)} \Phi^{-1}(\varphi_1) + (1 - \lambda) \frac{\varphi_2}{\varphi(p_0)} \Phi^{-1}(\varphi_2).$$

Consequently, the parametrization $\Phi$ fixes the extreme points.

**Proof.** For the forward direction it is clear that $c_{\tau, \beta} = \lambda c_{\tau_1, \beta} + (1 - \lambda)c_{\tau_2, \beta}$. Therefore the state

$$\varphi' = \frac{\lambda c_{\tau_1, \beta}}{c_{\tau, \beta}} \Phi(\tau_1) + (1 - \lambda) \frac{c_{\tau_2, \beta}}{c_{\tau, \beta}} \Phi(\tau_2)$$

is in $E^\text{fin}_\beta(T_X)$ as a convex combination of states in $E^\text{fin}_\beta(T_X)$. Now for every $a \in A$ we have

$$\varphi_\tau(\pi(a)) = \lambda c_{\tau_1, \beta}^{-1} \sum_{k=0}^{\infty} e^{-k\beta} \sum_{|\mu|=k} \tau_1(\langle x_\mu, ax_\mu \rangle) + (1 - \lambda) c_{\tau_1, \beta}^{-1} \sum_{k=0}^{\infty} e^{-k\beta} \sum_{|\mu|=k} \tau_2(\langle x_\mu, ax_\mu \rangle)$$

$$= \frac{\lambda c_{\tau_1, \beta}}{c_{\tau, \beta}} \Phi(\tau_1)(\pi(a)) + (1 - \lambda) \frac{c_{\tau_2, \beta}}{c_{\tau, \beta}} \Phi(\tau_2)(\pi(a)) = \varphi'(\pi(a)).$$

As both $\varphi'$ and $\varphi_\tau$ are in $E_\beta(T_X)$, Proposition 3.1 implies that they are equal. For the converse set $\tau_1 = \Phi^{-1}(\varphi_1)$, $\tau_2 = \Phi^{-1}(\varphi_2)$ and $\tau = \Phi^{-1}(\varphi)$. Then by construction, for every $a \in A$ we get that

$$\varphi(p_0)\tau(a) = \lambda \varphi_1(p_0\pi(a)p_0) + (1 - \lambda) \varphi_2(p_0\pi(a)p_0) = \lambda \varphi_1(p_0)\tau_1(a) + (1 - \lambda) \varphi_2(p_0)\tau_2(a).$$

Applying for $a = 1_A$ also gives that $\varphi(p_0) = \lambda_1 \varphi_1(p_0) + \lambda_2 \varphi_2(p_0)$. Finally to see that $\Phi$ fixes the extreme points just notice that the $c$-constants are all non-zero and the equations for $\Phi(\tau)$ and $\Phi^{-1}(\varphi)$ are convex combinations of states.

**Corollary 6.4.** If $X$ is a $C^*$-correspondence of finite rank over $A$ then $h_X \leq h_X^*$.

**Proof.** If $\beta > h_X^*$ then Proposition 5.7(iii) implies that $T_\beta(A) = T(A)$. Therefore Theorem 6.1 gives that $E^\text{fin}_\beta(T_X) \neq \emptyset$ and so $h_X \leq \beta$.

The gauge action of $T_X$ is inherited by the $J$-relative Cuntz-Pimsner algebras. Thus we can use the previous parametrization for their equilibrium states. For convenience let us write here $(\rho, v) = (q_J\pi, q_Jt)$ for the faithful representation of $O(J, X)$ where $q_J : T_X \to O(J, X)$ is the canonical quotient map. Hence $ker q_J$ is the ideal generated by $\pi(a) p_0$ for all $a \in J$ when $p_0 \in T_X$. We will write simply $q$ when $J = A$.

**Theorem 6.5.** Let $X$ be a $C^*$-correspondence of finite rank over $A$ and let $\beta \in (0, \infty)$. Suppose that $J \subseteq \phi^*_X(KX)$. Then there is a bijection

$$\Phi : \{ \tau \in T_\beta(A) \mid \tau|_J = 0 \} \to E^\text{fin}_\beta(O(J, X)).$$

If $x = \{x_1, \ldots, x_d\}$ is a decomposition of the unit then $\Phi$ is given by

$$(6.2) \quad \Phi(\tau)(v(\xi^m) v(\xi^m)^*) = \delta_{n,m} c_{\tau, \beta}^{-1} \sum_{k=0}^{\infty} e^{-(k+n)\beta} \sum_{|\mu|=k} \tau((\eta^m \otimes x_\mu, \xi^m \otimes x_\mu)).$$

for all $\xi^m \in X^\otimes m$ and $\eta^m \in X^\otimes m$. Moreover $\Phi$ satisfies the convex combination of Corollary 6.3 and thus it preserves extreme points. If, in addition, $E^\text{fin}_\beta(O(J, X))$ is weak*-closed then $\Phi$ is a weak*-homeomorphism between weak*-compact sets.
Proof. Fix $q_J: T_X \to \mathcal{O}(J, X)$ be the canonical $*$-epimorphism. Then the ker $q_J = \mathcal{K}(\mathcal{F}(X); J)$ is generated by $\pi(a)p_0$ for $a \in J$. Let $\tau \in T_\beta(A)$ and fix $\varphi_\tau$ be the associated state in $E_\beta^\text{fin}(T_X)$ given by Theorem 6.1. Then we get that $\varphi_\tau(\pi(a)p_0) = c_{\tau,\beta}(a)$. Therefore, if $\tau$ vanishes on $J$ then $\varphi_\tau$ vanishes on ker $q_J$ and so it induces a state on $\mathcal{O}(J, X)$. As the unital quotient map intertwines the gauge actions the induced state is in $E_\beta^\text{fin}(\mathcal{O}(J, X))$. Conversely if $\varphi \in E_\beta^\text{fin}(\mathcal{O}(J, X))$ then $\varphi q_J \in E_\beta^\text{fin}(T_X)$ and it defines $\tau_\varphi \in T_\beta(A)$ by Theorem 6.1. By construction $\tau_\varphi$ vanishes on $J$ as $q_J(\rho_0p_0p_0) = q_J(\pi(a)p_0) = 0$ for all $a \in J$.

7. The infinite part of the equilibrium states

Let us now see how we can parametrize $E_\beta^\text{inf}(\mathcal{O}(A, X))$ (and thus all $E_\beta^\infty(\mathcal{O}(J, X))$). The main point here is that these states are given by extending tracial states on $A$ rather than by taking statistical approximations. When $X$ is non-degenerate and injective then the existence of such a $\Psi$ can be derived by combining [23, Theorem 2.1 and Theorem 2.5]. However the attack therein is essentially different, as the well definedness of the extension is verified by using perturbations of the action. Following [18, Theorem 3.18] we can directly construct the extension within the fixed point algebra (and by keeping the same action).

Theorem 7.1. Let $X$ be a $C^*$-correspondence of finite rank over $A$ and let $\beta \in (0, \infty)$. Let

$$I := \{a \in A \mid \lim_n ||\phi_X(a) \otimes \text{id}_X\otimes\cdots|| = 0\}.$$ 

Then there is an affine weak*-homeomorphism

$$\Psi: \{\tau \in \text{AVT}_\beta(A) \mid \tau|_I = 0\} \to E_\beta^\infty(\mathcal{O}(A, X))$$

such that $\Psi(\varphi)|_{\rho(A)} = \tau$. In particular $\Psi$ induces an affine weak*-homeomorphism onto $E_\beta^\infty(T_X)$.

Proof. By Theorem 4.6, $E_\beta^\infty(T_X)$ consists exactly of the $(\sigma, \beta)$-KMS states that factor through $E_\beta(\mathcal{O}(A, X))$. The ideal $I$ is the kernel of $q|_{\pi(A)}$ for the canonical $*$-epimorphism $q: T_X \to \mathcal{O}(A, X)$. Suppose first that $X$ is not injective and fix $(\rho, \nu)$ such that $\mathcal{O}(A, X) = C^*(\rho, \nu)$. Then $I = \ker \rho$ and we claim that $\mathcal{O}_X$ is canonically $*$-isomorphic to $\mathcal{O}_Y$ for $Y = \nu(X)$ and $B = \rho(A)$. To this end first notice that $Y$ is injective and of finite rank so that $J_Y = B$. Indeed the covariance gives that $\rho(a) = \rho(a)\sum_{i \in [d]} v(x_i)v(x_i)^*$ for all $a \in A$.

Hence if $a + I \in \ker \phi_Y$ then $\rho(a) = \rho(a)\sum_{i \in [d]} v(x_i)v(x_i)^* = 0$, so that $a \in \ker I = 0$. Secondly it is clear that $(\text{id}_B, \text{id}_Y)$ defines a $B$-covariant pair for $Y$ since for $b = \rho(a)$ we have

$$\psi_{\text{id}_Y}(\phi_Y(b)) = \rho(a) = \text{id}_B(b).$$

Moreover it inherits a gauge action and trivially $\text{id}_B$ is injective on $B$. Thus the Gauge-Invariant-Uniqueness-Theorem asserts that $\mathcal{O}_Y = C^*(\text{id}_B, \text{id}_Y) = C^*(\rho, \nu)$.

Therefore without loss of generality we may assume that $X$ is injective so that $\mathcal{O}(A, X) = \mathcal{O}_X$ and $I = (0)$. We have to produce a weak*-homeomorphism $\Psi: \text{AVT}_\beta(A) \to E_\beta^\infty(\mathcal{O}_X)$ such that $\Psi^{-1}(\varphi) = \varphi \rho$. Let $\varphi \in E_\beta^\infty(\mathcal{O}_X)$ and set $\tau := \varphi \rho \in T(A)$. Therefore the KMS-condition yields

$$\tau(a) = \varphi(\rho(a)\sum_{i \in [d]} v(x_i)v(x_i)^*) = e^{-\beta}\sum_{i \in [d]} \varphi(v(x_i)^*\rho(a)v(x_i)) = e^{-\beta}\sum_{i \in [d]} \tau(|x_i, ax_i|)$$

and thus $\tau \in \text{AVT}_\beta(A)$. Now fix $\tau \in \text{AVT}_\beta(A)$ and we will construct a $\varphi_\tau \in E_\beta^\infty(\mathcal{O}(A, X))$. To this end we use a well known construction, that goes as back as [32]. Namely, when $X$ is injective and the left action is by compacts then the fixed point algebra $\mathcal{O}_X$ can be identified with the direct limit

$$A \xrightarrow{\phi_X} KX \xrightarrow{\otimes \text{id}_X} KX \otimes \cdots \otimes \text{id}_X,$$

where $[\otimes \text{id}_X](t) = t \otimes \text{id}_X$. In our case this identification is given by

$$\theta_{\xi \otimes n, \eta \otimes n} \otimes \text{id}_X = \sum_{i \in [d]} \theta_{\xi \otimes n \otimes x_i, \eta \otimes n \otimes x_i}.$$
as a direct computation on elementary tensors shows. Therefore $O_X^*$ is the inductive limit of the increasing sequence $\{(O_X^n)_n\}_{n \in \mathbb{N}}$ for

$$(O_X^n)_n := \bigoplus_{n \in \mathbb{N}} \{v(\xi^{\otimes n})v(\eta^{\otimes n})* | \xi^{\otimes n}, \eta^{\otimes n} \in X^{\otimes n}\} = \psi_{v,n}(KX^{\otimes n})$$

by writing

$$v(\xi^{\otimes n})v(\eta^{\otimes n})* = \sum_{i \in [d]} v(\xi^{\otimes n})v(x_i)v(x_i)^*v(\eta^{\otimes n})*.$$ 

We define the functionals $\varphi_n$ on $(O_X^n)_n$ by

$$\varphi_n(\psi_{v,n}(k_n)) := e^{-n\beta} \sum_{|\mu| = n} \tau((x_{\mu}, k_n x_{\mu}))$$

for $k_n \in KX^{\otimes n}$. To see that it is well defined notice that for a positive $k_n$ we have

$$e^{-n\beta} \sum_{|\mu| = n} \tau((x_{\mu}, k_n x_{\mu})) \leq \| k_n \| e^{-n\beta} \sum_{|\mu| = n} \tau((x_{\mu}, x_{\mu})) \leq \| k_n \|.$$ 

In particular we have $\varphi_n(\rho(1_A)) = e^{-n\beta} \sum_{|\mu| = n} \tau((x_{\mu}, x_{\mu})) = 1$ and so each $\varphi_n$ is a state. By construction we have that

$$\varphi_n(\psi_{v,n}(\theta_{\xi^{\otimes n}, \eta^{\otimes n}})) = e^{-n\beta} \sum_{|\mu| = n} \tau((x_{\mu}, \xi^{\otimes n}(\eta^{\otimes n}, x_{\mu}))$$

$$= e^{-n\beta} \sum_{|\mu| = n} \tau((\eta^{\otimes n}, x_{\mu})x_{\mu}, \xi^{\otimes n})) = e^{-n\beta} \tau((\eta^{\otimes n}, \xi^{\otimes n})).$$

We see that the collection $\{\varphi_n | n \in \mathbb{N}\}$ is compatible with the direct limit structure since

$$\varphi_{n+1}(\psi_{v,n+1}(\theta_{\xi^{\otimes n}, \eta^{\otimes n}} \otimes \text{id}_X)) = \sum_{i \in [d]} \varphi_{n+1}(v(\xi^{\otimes n})v(x_i)v(x_i)^*v(\eta^{\otimes n})*)$$

$$= e^{-(n+1)\beta} \sum_{|\mu| = n+1, i \in [d]} \tau((x_{\mu}, \xi^{\otimes n} \otimes x_i)(\eta^{\otimes n} \otimes x_i), x_{\mu}))$$

$$= e^{-(n+1)\beta} \sum_{i \in [d]} \sum_{|\mu| = n+1} \tau((\eta^{\otimes n} \otimes x_i), (x_{\mu}, \xi^{\otimes n} \otimes x_i))$$

$$= e^{-(n+1)\beta} \sum_{i \in [d]} \tau((x_i, (\eta^{\otimes n}, \xi^{\otimes n}))x_i))$$

$$= e^{-n\beta} \tau((\eta^{\otimes n}, \xi^{\otimes n})) = \varphi_n(\psi_{v,n}(\theta_{\xi^{\otimes n}, \eta^{\otimes n}})).$$

Therefore it defines a state $\varphi_\tau$ in the limit which extends $\tau$ such that

$$\varphi_\tau(v(\xi^{\otimes n})v(\eta^{\otimes n})*) = \varphi_n(v(\xi^{\otimes n})v(\eta^{\otimes n})*) = e^{-n\beta} \tau((\eta^{\otimes n}, \xi^{\otimes n})).$$

Let $E : O_X \to O_X^*$ be the conditional expectation coming from the gauge action. Then Proposition 3.1 yields that the induced state $\varphi_\tau E$ is a $(\sigma, \beta)$-KMS state on $O_X$. The same proposition implies that $\varphi_\tau E$ is the unique $(\sigma, \beta)$-KMS state with restriction $\tau$ on $A$. Therefore $\Psi$ is injective.

It is immediate that $\Psi^{-1}$ is weak*-continuous and affine. Since $E^{\infty}_\beta(O(A, X))$ is weak*-compact and $AVT_\beta(A)$ is Hausdorff, it follows that $\Psi$ is a weak*-homeomorphism.

However we cannot have arbitrarily large $\beta > 0$ for $O(A, X)$. Obviously $q(p_0) = 0$ and so $E_\beta(O(A, X)) = E^{\infty}_\beta(O(A, X))$. Therefore Theorem 6.5 is void for $O(A, X)$ at $\beta < h_X^*$; and there is a good reason for this.

**Proposition 7.2.** If $X$ is a $C^*$-correspondence of finite rank over $A$ then $E_\beta(O(A, X)) = 0$ for all $\beta > h_X^*$. If, in addition, $A$ is abelian then $O(A, X)$ attains equilibrium states at $h_X^*$.

**Proof.** Since $\varphi q \in E^{\infty}_\beta(T_X)$ for $\varphi \in E_\beta(O(A, X))$, Proposition 5.7 yields $\beta \leq h_X^*$. The same proposition and Theorem 7.1 gives the second part of the statement.

**Corollary 7.3.** Let $X$ be a $C^*$-correspondence of finite rank over $A$. Then:
(i) \( h_X = \max\{0, \inf\{h_X^\tau \mid \tau \in T(A)\}\} \).
(ii) If \( 0 < h_X \), or if \( 0 \leq h_r \) for all \( \tau \in T(A) \), then \( h_X = \min\{h_X^\tau \mid \tau \in T(A)\} \).
(iii) If \( h_X^\tau = h_X^\mu \) for all \( \tau \in T(A) \) then \( E_{\beta}^{\text{fin}}(T_X) = \emptyset \) for all \( \beta > h_X \).

**Proof.** For item (i) let \( \beta \geq h_X \) so that \( E_{\beta}(T_X) \neq \emptyset \). Due to the decomposition and the parametrization we get that \( T_{\beta}(A) \neq \emptyset \) or \( \text{AVT}_{\beta}(A) \neq \emptyset \). In any case there is a \( \tau \in T(A) \) such that \( h_X^\tau \leq \beta \). Therefore \( \inf\{h_X^\tau \mid \tau \in T(A)\} \leq h_X \). Suppose there were a \( \tau \in T(A) \) such that \( h_X^\tau < h_X \). Then \( h_X^\tau \leq h_X < \tau \). Suppose there were a \( \tau \in T(A) \) such that \( h_X^\tau < h_X \). Then \( h_X^\tau \leq h_X < \tau \). Then \( h_X^\tau \leq \beta \), and because \( h_X^\tau \leq h_X < \tau \), it is clear that \( \tau \notin T(A) \) for all \( \beta > 0 \) in which case \( h_X = 0 \). If \( h_X^\tau > 0 \) then choose \( \beta \in (h_X^\tau, h_X) \). Then the root test gives that \( c_{\tau, \beta} < \infty \) and thus the contradiction \( E_{\beta}^{\text{fin}}(T_X) \neq \emptyset \).

For item (ii), weak*-compactness gives that \( E_{h_X}(T_X) \neq \emptyset \). We consider two cases:

**Case (a).** If \( h_X > 0 \) then item (i) implies that \( h_X = \inf\{h_X^\tau \mid \tau \in T(A)\} \). Now we can decompose \( \varphi \in E_{h_X}(T_X) \) and use the parametrization of each component to get a \( \tau_0 \in T_{h_X}(A) \cup \text{AVT}_{h_X}(A) \) with \( 0 \leq h_X^{\tau_0} \leq h_X \). However by item (i) we have that \( h_X^{\tau_0} \geq h_X \) and thus we have equality, i.e., a minimum at \( h_X^{\tau_0} \).

**Case (b).** If \( h_X = 0 \) but \( h_X^\tau \geq 0 \) for all \( \tau \in T(A) \), then \( T_X \) admits a tracial state \( \varphi \) such that

\[
\sum_{|\mu|=k} \varphi(t(x_{\mu})) = \varphi(\sum_{|\mu|=k} t(x_{\mu})) \leq \varphi(\pi(1_A)) = 1.
\]

As this holds for all \( k \in \mathbb{Z}_+ \) we have that \( 0 \leq h_X^{\tau_0} \leq \log 1 \) and so \( h_X^{\tau_0} = 0 = h_X \) for \( \tau_0 := \varphi \pi \in T(A) \).

The third item follows by Proposition 7.2.

8. Comments and applications

8.1. Unit decompositions. The strong entropy requires taking the infimum over all possible unit decompositions. This is because the notion of basis is not well defined for C*-correspondences over non-commutative C*-algebras. Let us give such an example here.

**Example 8.1.** Let \( A = C(K) \oplus O_2 \) for a compact and Hausdorff space \( K \) and the Cuntz algebra \( O_2 = C^*(s_1, s_2) \). Let \( \alpha \in \text{End}(A) \) be given by \( \alpha(a, b) = (a, s_1 b s_1^* + s_2 b s_2^*) \) and let \( X \) be the induced C*-correspondence \( \alpha A \). That is \( X = A \) as a vector space and

\[
\langle \xi, \eta \rangle = \xi^* \eta \quad \text{and} \quad (a, b) \cdot (c, d) = \alpha(a, b)(c, d).
\]

We chose \( A \) to have a commutative part so that \( T(A) \neq \emptyset \). In [20] it is shown that \( \emptyset \neq E_{\beta}(T_X) = E_{\beta}^{\text{fin}}(T_X) \) for all \( \beta \in (0, \infty) \).

Now \( \alpha A \) admits at least two unit decompositions \( x = \{(1, 1)\} \) and \( y = \{(1, s_1), (1, s_2)\} \). It is clear that \( \sum_{|\mu|=k} (x_{\mu}, x_{\mu}) = 1 \). On the other hand we have that \( y_1 = (1, s_1) \) and \( y_2 = (1, s_2) \) are orthonormal and so \( \langle y_1, y_2 \rangle = (1, 1) \) for all \( \mu \in \mathbb{F}_2^2 \). We then see that they have different entropies as

\[
h_X^\tau = \limsup_k \frac{1}{k} \log \| \sum_{|\mu|=k} \langle x_{\mu}, x_{\mu} \rangle \| = 0 < \log 2 = \limsup_k \frac{1}{k} \log \| \sum_{|\mu|=k} \langle y_{\mu}, y_{\mu} \rangle \|.\]

8.2. Orthogonal bases. In several examples, the C*-correspondence is over an abelian \( A \) and admits a finite orthonormal basis. From our analysis, and in particular from Corollary 7.3, we get directly the KMS-structure in these cases:

(i) \( h_X = \log d \) for all \( \tau \in T(A) \) and \( h_X = h_X^\star = \log d \);
(ii) \( E_{\beta}(T_X) = E_{\beta}^{\text{fin}}(T_X) \neq \emptyset \) for all \( \beta > \log d \), and \( E_{\log d}(T_X) = E_{\log d}^{\infty}(T_X) \neq \emptyset \).

Indeed suppose that \( X \) admits a finite orthonormal basis \( x = \{x_1, \ldots, x_d\} \), i.e., \( \langle x_i, x_j \rangle = \delta_{i,j} \).

Then \( \langle x_{\mu}, x_{\nu} \rangle = \delta_{\mu, \nu} \) for \( |\mu| = |\nu| \), so that

\[
h_X^\tau = h_X^\star = \log d \text{ for all } \tau \in T(A).
\]

Corollary 7.3 yields that \( h_X = \log d \) and thus \( E_{\beta}(T_X) = E_{\beta}^{\text{fin}}(T_X) \) for all \( \beta > \log d \). Moreover we see that \( c_{\tau, \log d} = \sum_{k=0}^\infty 1 \) so that

\[
E_{\log d}(T_X) = E_{\log d}^{\infty}(T_X).
\]
As we noted in Proposition 7.2 we have that $E^\infty_{\log d}(T_X) \neq \emptyset$. As applications we get the full KMS-structure for the Pimsner algebras:

(a) In [20], by applying for $d$ the multiplicity of the dynamical system;
(b) In [26], by applying for $d = | \det A |$ and using [14, Lemma 2.6];
(c) In [27], by applying for $d = |X|$ and using [27, Equation 3.1].

We will see below that Corollary 7.3 gives also the KMS-structure of [1, 18] for $C^*$-algebras of irreducible graphs. Notice that in addition to that we provide a clear parametrization of all the equilibrium states at the critical temperature $\beta = \log d$.

$C^*$-correspondences of finite graphs is the first step away from orthonormal bases. To fix notation, the $C^*$-correspondence $X_G$ of a graph $G = (G^{(0)}, G^{(1)}, s, r)$ is the linear span of $\{ x_e \mid e \in G^{(1)} \}$ over the abelian $C^*$-algebra generated by orthogonal projections $\{ p_v \mid v \in G^{(0)} \}$ such that

$$\langle x_e, x_f \rangle = \delta_{e,f} p_{s(f)} \quad p_v x_e = \delta_{e,r(e)} x_e \quad x_e p_v = \delta_{s(e,v)} x_e.$$

It admits the unit decomposition given by the basis $\{ x_e \mid e \in G^{(1)} \}$. The equilibrium states of this category have been extensively investigated in [1, 2, 18]. Pimsner algebras of irreducible graphs had been considered in [1] were Perron-Fr"obernus Theorem is used in an essential way. Let us see here how the entropy theory we have developed applies and recovers the results therein.

**Theorem 8.2.** [1] Let $X$ be the $C^*$-correspondence associated to a finite irreducible graph $G$. Let $\lambda_G$ be the Perron-Fr"obernus eigenvalue of its adjacency matrix. Then:

(i) $h_X = \log \lambda_G$;
(ii) $E_\beta(T_X) = E^\beta_{\lambda_G}(T_X)$ for all $\beta > \log \lambda_G$;
(iii) $E_\lambda(T_X) = E^\infty_{h_X}(T_X)$ is a singleton implemented by the Perron-Fr"obernus eigenvector.

**Proof.** Let us use the same symbol $G$ for the adjacency matrix of $G$. It is easy to see that the quantity $\| \sum_{|j|=k} |x_{\mu}, x_{\mu}| \|$ counts the number of paths of length $k$. Hence from the Perron-Fr"obernus Theorem we have that $h_X = \log \lambda_G$. We will show that

$$h_X = \log \lambda_G$$

for all $\tau \in T(A)$,

and then Corollary 7.3(iii) gives items (i) and (ii). To this end suppose that $G$ consists of $n$ vertices $\{ v_1, \ldots, v_n \}$ and let us write $p_j$ for the projection corresponding to $v_j$. For $\tau \in T(A)$ set $P = \text{diag}(\tau(p_j) \mid j \in [n])$ so that the diagonal entries sum up to one. On one hand we have that

$$\sum_{|j|=k} \tau(|x_{\mu}, x_{\mu}|) \leq \sum_{i,j \in [n]} (PG^k)_{ij} \leq \sum_{i,j \in [n]} (G^k)_{ij},$$

so that $h_X \leq \log \lambda_G$ by the Perron-Fr"obernus Theorem. Now let $w$ be the strictly positive eigenvector of $G$ at $\lambda_G$ and choose $\omega = \max\{ w_j \mid j \in [n] \}$. For $\tau \in T(A)$ there exists a $v_j$ such that $\tau(p_j) \neq 0$. Without loss of generality assume that this happens at $v_1$. Since the $j$-th co-ordinate of $PG^k w$ is $\lambda_G^j \tau(p_j) w_j$, we have that

$$\omega \sum_{i,j \in [n]} (PG^k)_{ij} \geq \sum_{j \in [n]} (PG^k)_{1j} \omega \geq \sum_{j \in [n]} (PG^k)_{1j} w_j = \lambda_G^k \tau(p_1) w_1 > 0.$$

Therefore we conclude that

$$h_X = \lim_{k \to \infty} \frac{1}{k} \log(\omega \sum_{i,j} (PG^k)_{ij}) \geq \lim_{k \to \infty} \frac{1}{k} \log(\lambda_G^k w_1 \tau(p_1)) = \log \lambda_G \geq h_X,$$

and so $h_X = \log \lambda_G$. We also see that

$$c_{\tau, \log \lambda_G} \geq \frac{\tau(p_1) w_1}{\omega} \sum_{k=0}^\infty e^{-k \log \lambda_G} \lambda_G^k = \infty$$

so that $T_{h_X}(A) = \emptyset$. Hence $E_{h_X}(T_X) = E_{h_X}^0(T_X)$ and for item (iii) it remains to show that $\overline{AVT_{h_X}(A)}$ is a singleton. A direct computation gives that

$$\sum_{e \in G^{(1)}} \tau((x_e, p_i x_e)) = \sum_{e \in \tau^{-1}(v_i)} \tau((x_e, x_e)) = \sum_{e \in \tau^{-1}(v_i)} \tau(p_{s(e)}) = \sum_{j \in [n]} g_{ij} \tau(p_j)$$
Therefore \( \tau \in \text{AVT}_{h_X}(A) \) if and only if \([\tau(p_1), \ldots, \tau(p_n)]\) is a \( \lambda_G \)-eigenvector of \( \ell^1 \)-norm one. By uniqueness of the Perron-Fröbenius eigenvector \( w \) we derive that there is only one \( \tau \in \text{AVT}_{h_X}(A) \) and it satisfies

\[
\tau(p_i) = (w_1 + \cdots + w_n)^{-1} w_i \quad \text{for all } i \in [n].
\]

8.3. **Entropies comparison.** The KMS-structure of graph \( C^* \)-algebras gives a nice mixing of cases. We will use this class to provide an example that distinguishes between entropies, i.e.

(i) It may be the case that \( h_X < h_X^\beta \), and in particular \( E_{\beta}^{\text{fin}}(T_X) \neq \emptyset \) for every \( \beta \in (h_X, h_X^\beta) \).

(ii) It may be the case that \( E_{\beta}^\infty(T_X) \neq \emptyset \) and \( E_{\beta}^\infty(T_X) \neq \emptyset \) for some \( \beta > 0 \).

In fact we will give an example for which \( \text{AVT}_\beta(A) \neq \emptyset \) for a finite number of \( \beta \), whereas \( T_\beta(A) \neq \emptyset \) for all \( \beta > h_X \).

**Example 8.3.** Fix a collection of positive integers \( \{1 = a_1 < \cdots < a_n\} \) and let the graph \( (G) \) be

\[
\begin{array}{c}
\bullet \quad v_0 \\
\circ \quad v_1 \\
\circ \quad \ldots \\
\circ \quad v_n \\
\end{array}
\]

where \( (a_j) \) denotes the number of cycles on the vertex \( v_j \). Hence the singular value of the adjacency matrix is greater than \( a_n \). The paths of length \( k \) ending at \( v_j \) are \( a_j^k \) in number when \( j > 0 \). On the other hand the number of paths of length \( k \) from \( v_j \) to \( v_0 \) equals to \( \sum_{j=1}^{k-1} a_j^k = (a_j^k - 1)/(a_j - 1) \) when \( j \neq 1 \), and equals to \( k \) when \( j = 1 \). As the \( p_v \) are orthogonal we see that

\[
\| \sum_{|\mu|=k} \langle x_\mu, x_\mu \rangle \| = \max_{v \in G(0)} \#(\{ x_\mu \mid |\mu| = k, s(x_\mu) = v \}) = \frac{a_n^k - 1}{a_n - 1}.
\]

Since \( a_n^k - 1 \leq (a_n^k - 1)/(a_n - 1) \leq a_n^k \) we get that \( h_X^\tau = \log a_n \). Now let \( \tau \) be a trace on the vertices and notice that

\[
\sum_{|\mu|=k} \tau(\langle x_\mu, x_\mu \rangle) = \sum_v \sum_{|\mu|=k} \#(\{ x_\mu \mid |\mu| = k, s(x_\mu) = v \}) \cdot \tau(p_v).
\]

If \( r = \max\{i \in [n] \mid \tau(p_i) \neq 0\} \) then

\[
\log a_r \leq \frac{1}{k} \log(a_n^k - 1) \tau(p_r) \leq h_X^\tau \leq \frac{1}{k} \log(n + 1) a_r = \log a_r
\]

and so \( h_X^\tau = \log a_r \). Hence any trace supported on \( \{v_1, \ldots, v_r\} \) defines a state in \( E_{\beta}^{\text{fin}}(T_X) \) as long as \( \beta > \log a_r \). Moreover notice that the cycles on each vertex are orthogonal. Thus for every \( \log a_j \) we get that the Dirac measure \( \tau_j \) on \( p_v \) is in \( \text{AVT}_{\log a_j}(A) \) and so \( E_{\log a_j}^\infty(T_X) \neq \emptyset \). Therefore we get the required conclusions:

(i) Since \( a_1 = 0 \) then \( \tau_1 \) induces a state in \( E_{\beta}^{\text{fin}}(T_X) \) for all \( \beta > 0 \); hence \( h_X = 0 < h_X^\beta \).

(ii) If \( j > 1 \) then any convex combination of \( \{ \tau_{j'} \mid j' < j\} \) induces a state in \( E_{\log a_j}^{\text{fin}}(T_X) \) and \( \tau_j \) induces a state in \( E_{\log a_j}^\infty(T_X) \).

With a small tweak we can produce a variant \( (G') \) for which \( E_{\beta}^\infty(C^*(G')) \neq \emptyset \) for any \( \beta \in (0, \infty) \). Indeed add a source to obtain the graph

\[
\begin{array}{c}
\bullet \quad w \\
\circ \quad v_0 \\
\circ \quad v_1 \\
\circ \quad \ldots \\
\circ \quad v_n \\
\end{array}
\]
and notice that all the entropies remain the same (there is only one path ending at \(v_0\) of length \(k\) that can be added). If \(\tau'\) is the Dirac measure on \(w\) then \(h_X^{\tau'} = 0\) and so \(\tau' \in T_\beta(A)\) for all \(\beta > 0\).

9. Ground states and KMS\(_\infty\)-states

We follow [25] and make a distinction between KMS\(_\infty\)-states and ground states. The following theorems make that difference clear. The form of the ground states has been identified in [23, Theorem 2.2].

**Theorem 9.1.** Let \(X\) be a \(C^*\)-correspondence of finite rank over \(A\). Then there exists an affine weak*-homeomorphism \(\tau \mapsto \varphi_\tau\) between the states \(\tau \in \mathcal{S}(A)\) (resp. the tracial states \(\tau \in \mathcal{T}(A)\)) and the ground states (resp. the KMS\(_\infty\)-states) of \(T_X\) such that

\[
\varphi_\tau(\pi(a)) = \tau(a) \quad \text{for all } a \in A \quad \text{and} \quad \varphi_\tau(t(\xi^n)t(\eta^m)^*) = 0 \quad \text{when } n + m \neq 0.
\]

**Proof.** For a state \(\tau \in \mathcal{S}(A)\) consider the GNS-representation \((H_\tau, \pi_\tau, \rho_\tau)\). Let again \((\rho, v)\) be the induced representation of \(T_X\) on \(H = FX \otimes _{\rho_\tau} H_\tau\) and let \(\varphi_\tau\) be the vector state given by

\[
\varphi_\tau(f) := \langle x_0 \otimes x_\tau, (\rho \times v)(f)x_0 \otimes x_\tau \rangle_H = \tau(p_0fp_0).
\]

It is immediate that \(\varphi_\tau\) satisfies the conditions of the statement. This also shows that the map \(\tau \mapsto \varphi_\tau\) is injective.

Next we show that equation (9.1) characterizes the ground states for \(\tau \in \mathcal{S}(A)\). Then surjectivity follows by noting that if \(\varphi\) is a ground state of \(T_X\) then \(\varphi = \varphi_\tau\) for \(\tau = \varphi_\pi\). Let \(\varphi\) be a ground state and let \(m \neq 0\). Then the function

\[
r + is \mapsto \varphi(t(\xi^n)t(\eta^m)^*) = e^{-imr}e^{ms}\varphi(t(\xi^n)t(\eta^m)^*)
\]

has to be bounded for all \(s > 0\). This can happen only if \(\varphi(t(\xi^n)t(\eta^m)^*) = 0\). Now if \(m = 0\) and \(n \neq 0\) then we get that \(\varphi(t(\xi^n)) = 0\) by taking adjoints. In any case \(\varphi(t(\xi^n)) = 0\) when \(n + m \neq 0\). Since \(\sigma_\pi = \text{id}\) on \(\pi(a)\) we also get that \(\varphi_\pi \in \mathcal{S}(A)\) and so \(\varphi\) satisfies equation (9.1). Conversely suppose that \(\varphi\) satisfies equation (9.1). We have to show that, for any pair

\[
f(t(\xi^n)t(\eta^m)^*) \quad \text{and} \quad g(t(\xi^n)t(\eta^m)^*)
\]

the function \(r + is \mapsto \varphi(f(\sigma_{\tau + is}(g)))\) is bounded when \(t > 0\). Indeed we have that

\[
|\varphi(f(\sigma_{\tau + is}(g))|^2 = e^{-(k-l)2s}|\varphi(fg)|^2 \leq e^{-(k-l)2s}\varphi(f^*f)\varphi(g^*g).
\]

This is clearly bounded when \(k - l > 0\). Now if \(k - l < 0\) then \(l > 0\) and so

\[
\varphi(g^*g) = \varphi(t(y^\otimes(\xi^k,\eta^k))t(y^\otimes l)^*) = 0,
\]

and thus \(\varphi(f(\sigma_{\tau + is}(g))) = 0\), which completes the proof.

Now we pass to the KMS\(_\infty\)-states. Suppose that \(\varphi\) is a KMS\(_\infty\)-state. Due to weak*-compactness (and after passing to subsequences), we may choose a sequence \(\beta_j \uparrow \infty\) such that \(w^*\lim_j \varphi_{\tau,\beta_j}\) converges to a KMS\(_\infty\)-state \(\varphi\). Then \(\varphi_{\pi(A)}\) is tracial and when \(n + m \neq 0\) then

\[
\varphi(t(\xi^n)t(\eta^m)^*) = \lim_{\beta_j \uparrow \infty} e^{-|\beta_j|\delta_{n,m}}\varphi_{\tau,\beta_j}(t(\eta^m)^*) = 0,
\]

so that \(\varphi\) satisfies equation (9.1). For surjectivity let \(\varphi\) be a KMS\(_\infty\)-state and set \(\tau = \varphi_\pi\). Let \(\beta_j \uparrow \infty\) and without loss of generality assume that \(\beta_j > h_X^\tau\) for all \(j\). Then we can form \(\varphi_{\tau,\beta_j} \in \mathcal{E}_{\beta_j}^\text{fin}(T_X)\) arising from Theorem 6.1. After passing to a subsequence let \(\varphi_\tau = w^*\lim_j \varphi_{\tau,\beta_j}\). We will show that \(\varphi = \varphi_\tau\). For \(n + m \neq 0\) we have that

\[
\varphi(t(\xi^n)t(\eta^m)^*) = 0 = \varphi_\tau(t(\xi^n)t(\eta^m)^*).
\]

Hence it suffices to show that \(\varphi_\tau \pi = \tau\). Fix a unit decomposition \(x = \{x_1, \ldots, x_d\}\). Then for \(a \in A\) we have

\[
\varphi_{\tau,\beta_j}(\pi(a)) = c_{\tau,\beta_j}^1\tau(a) + c_{\tau,\beta_j}^{-1}\sum_{k=1}^{\infty} e^{-k\beta_j} \sum_{|\mu|=k} \tau(\langle x_\mu, ax_\mu \rangle).
\]
Take $\varepsilon > 0$ so that $h_X^* + \varepsilon < \beta_1 \leq \beta_j$. Then there exists an $N \in \mathbb{N}$ such that $\sum_{[\mu]=k} \tau(\langle x_\mu, x_\mu \rangle) \leq e^{k(h_X^* + \varepsilon)}$ for all $k \geq N$. Therefore we get that
\[
1 \leq c_{\tau, \beta_j} \leq 1 + \sum_{k=1}^{N-1} e^{-k\beta_j} \sum_{[\mu]=k} \tau(\langle x_\mu, x_\mu \rangle) + \sum_{k=N}^{\infty} e^{-k\beta_j} e^{k(h_X^* + \varepsilon)}.
\]

However we have that
\[
\lim_{\beta_j \to \infty} \left[ e^{N(-\beta_j + h_X^* + \varepsilon)} \frac{1}{1 - e^{-\beta_j + h_X^* + \varepsilon}} + \sum_{k=1}^{N-1} e^{-k\beta_j} \sum_{[\mu]=k} \tau(\langle x_\mu, x_\mu \rangle) \right] = 0
\]
which gives $\lim_{\beta_j \to \infty} c_{\tau, \beta_j} = 1$. Combining with positivity of $\tau$ we also derive that
\[
\left\| \sum_{k=1}^{\infty} e^{-k\beta_j} \sum_{[\mu]=k} \tau(\langle x_\mu, ax_\mu \rangle) \right\| \leq \frac{\|a\|}{\sum_{k=1}^{\infty} e^{-k\beta_j} \sum_{[\mu]=k} \tau(\langle x_\mu, x_\mu \rangle)} \leq \frac{\|a\|}{\sum_{k=1}^{N-1} e^{-k\beta_j} \sum_{[\mu]=k} \tau(\langle x_\mu, x_\mu \rangle) + \sum_{k=N}^{\infty} e^{-k\beta_j} \sum_{[\mu]=k} \tau(\langle x_\mu, x_\mu \rangle)} \rightarrow 0.
\]

Thus taking limits $\beta_j \uparrow \infty$ in equation (9.2) we conclude the required $\varphi(\tau(a)) = \tau(a)$.

Finally we have the analogues for the ground states and the KMS$_\infty$-states for $J$-relative Cuntz-Pimsner algebras.

**Theorem 9.2.** Let $X$ be a $C^*$-correspondence of finite rank over $A$. Suppose that $J \subseteq \phi_X^{-1}(\mathcal{K}X)$. Then the mapping $\tau \mapsto \varphi_\tau$ for
\[
\varphi_\tau(\rho(a)) = \tau(a) \text{ for all } a \in A \text{ and } \varphi_\tau((\xi^{\otimes n})v(\eta^{\otimes m})^*) = 0 \text{ when } n + m \neq 0
\]
defines an affine weak*-homeomorphism from the states on $A$ (resp. from the tracial states on $A$) that vanish on $J$ onto the ground states of $\mathcal{O}(J, X)$ (resp. onto the KMS$_\infty$-states) of $\mathcal{O}(J, X)$.

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