A NOTE ON DERIVED FUNCTORS OF DIFFERENTIAL OPERATORS

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Abstract. In their work on differential operators in positive characteristic, Smith and Van den Bergh define and study the derived functors of differential operators; they arise naturally as obstructions to differential operators reducing to positive characteristic. In this note, we provide formulas for the ring of differential operators as well as these derived functors of differential operators. We apply these descriptions to show that differential operators behave well under reduction to positive characteristic under certain hypotheses. We show that these functors also detect a number of interesting properties of singularities.

1. Introduction

The notion of differential operators on singular varieties, as defined by Grothendieck [15], has attracted interest in algebra for a number of reasons. Questions about the ring structure of differential operators, e.g., finite generation and simplicity, have been well-studied [3, 18, 22, 19]. In Commutative Algebra, differential operators on singularities have found a resurgence of interest due to their interesting connections with $F$-singularities, symbolic powers, and noncommutative resolutions, among other topics [21, 22, 6, 1, 12]. While most commonly one encounters the ring $D_R$ of differential operators from a ring $R$ to itself, differential operators from one module $M$ to another $N$ are defined; the collection of these is denoted $D(M,N)$. As with Hom$(M, -)$, the functor $D(M, -)$ is left-exact, and it admits a sequence of right-derived functors, denoted $R^iD(M, -)$. In particular, $R^0D(R, R) = D_R$. These functors are studied in [22], where it is shown that the $p$-torsion part of $R^1D(R, R)$ is the obstruction to differential operators reducing to positive characteristic; see Proposition 6.1 below for a precise statement. We refer to these as the Smith–Van den Bergh functors.

In this note, we give two descriptions of the differential operators $D(R, \omega_R)$ and the Smith–Van den Bergh functors $R^iD(R, \omega_R)$ for Cohen-Macaulay rings with canonical module $\omega_R$; in particular, when $R$ is Gorenstein, these formulas describe the ring of differential operators on $R$ as well as the aforementioned $R^1D(R, R)$. Our formulas, which we state in the Gorenstein graded $K$-algebra case for convenience, are in terms of local cohomology modules.

Theorem A. Let $K$ be a field, and $R$ be a Gorenstein graded $K$-algebra of dimension $d$.

1. (Theorem 4.1) For all $i \geq 0$, $R^iD(R, R) \cong H^{d+i}_\Delta(R \otimes_K R)$, where $\Delta$ is the ideal of the diagonal.

2. (Theorem 4.5) If $R$ has an isolated singularity, $d \geq 3$, and $1 \leq i \leq d-2$, then $R^iD(R, R) \cong H^{i+1}_R(D_R)$.

The case $i = 0$ of Theorem A(1) generalizes the Grothendieck–Sato formula for differential operators on smooth varieties. This was previously generalized in [2] to Cohen-Macaulay algebras over a field satisfying a certain cohomological assumption; our characterization does

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not require these hypotheses, and interprets them in terms of the Smith–Van den Bergh functors; see Remark 4.4.

Our descriptions of the modules $R^i D(R, R)$ have a number of interesting applications. Apropos to the original motivation, we are able to show that differential operators behave well under reduction to positive characteristic under certain hypotheses. For example:

**Theorem B** (see Theorem 6.3). Let $R$ be a Gorenstein graded $\mathbb{Z}$-algebra. If $R$ is a direct summand of a polynomial ring, then the map $D_R \otimes \mathbb{Z}/p\mathbb{Z} \to D_{R/pR}$ is an isomorphism for all but finitely many primes $p$. Moreover, these isomorphisms preserve the order filtration.

This provides a positive answer to a question of Smith and Van den Bergh [22] in a significant case. Additionally, we apply our description to give an easy conceptual example of a local cohomology module with infinitely many associated primes; see Example 6.6.

While the original motivations for the study of the Smith–Van den Bergh functors pertain to $R^1 D(R, R)$, we find that the vanishing or nonvanishing of $R^i D(R, R)$ for other $i > 0$ has interesting connections to aspects of the singularity of $R$. For example, nonvanishing of these modules yields a lower bound for the dimension of secant varieties; see Proposition 5.2. Furthermore, as an easy consequence of Theorem A(2), if $R$ is an isolated Gorenstein singularity, the (non)vanishing of these modules characterizes the depth of $D_R$ as an $R$-module; see Theorem 5.1. We give an alternative interpretation of Switala’s Matlis duality for $D$-modules, which yields a characterization of when $R$ is a simple $D$-module in terms of these functors; see Proposition 3.2. We also establish a symmetry property for differential operators of Gorenstein rings; see Corollary 4.2.

## 2. Differential operators and their derived functors

**Differential operators.** Let $R$ be a commutative ring, and $A$ be a subring. If $M$ and $N$ are two $R$-modules, the ($A$-linear) differential operators from $M$ to $N$ of order at most $i$, denoted $D^i_{R|A}(M, N)$, are defined inductively as follows:

- $D^0_{R|A}(M, N) = \text{Hom}_R(M, N)$,
- $D^i_{R|A}(M, N) = \{ \delta \in \text{Hom}_A(M, N) \mid \forall f \in R, \delta \circ f_M - f_N \circ \delta \in D^{i-1}_{R|A}(M, N) \}$,

where $f_M$ and $f_N$ denote multiplication by $f$ on $M$ or $N$ respectively. The differential operators from $M$ to $N$ are

$$D_{R|A}(M, N) = \bigcup_{n=0}^{\infty} D^n_{R|A}(M, N).$$

Throughout we denote $D_{R|A}(R, M)$ by $D_{R|A}(M)$ and $D_{R|A}(R, R)$ by $D_{R|A}$, and similarly for differential operators up to a certain order.

Let $L$, $M$, and $N$ be three $R$ modules. It is easily verified that if $\alpha \in D^i_{R|A}(L, M)$ and $\beta \in D^j_{R|A}(M, N)$, then $\beta \circ \alpha \in D^{i+j}_{R|A}(L, N)$. Consequently, $D_{R|A}$ is a ring under composition, and $D_{R|A}(M)$ is a right $D_{R|A}$-module for any $R$-module $M$. Of course, $R$ is a left $D_{R|A}$-module as well. We say that $R$ is $D_{R|A}$-simple if $R$ is a simple left $D_{R|A}$-module.

**Modules of principal parts.** We define the enveloping algebra of $R$ over $A$ to be

$$\mathcal{P}_{R|A} := R \otimes_A R,$$
and the module of n-differentials of \( R \) over \( A \) to be
\[
\mathcal{P}^n_{R|A} := \mathcal{P}^n_{R|A}/\Delta^n_{R|A},
\]
where \( \Delta^n_{R|A} \) is the kernel of the multiplication map \( \mathcal{P}^n_{R|A} \rightarrow R \). We also set
\[
\mathcal{P}^n_{R|A}(M) := R \otimes_A M, \quad \text{and} \quad \mathcal{P}^n_{R|A}(M) := \mathcal{P}^n_{R|A}(M) \otimes_{\mathcal{P}^n_{R|A}} \mathcal{P}^n_{R|A}.
\]
By \([15, 16.7.4]\), if \( R \) is essentially of finite type over \( A \) and \( M \) is a finitely generated \( R \)-module, then \( \mathcal{P}^n_{R|A}(M) \) is a finitely generated \( R \)-module for any \( n \).

There is a natural isomorphism of Hom-tensor adjunction
\[
\text{Hom}_R(\mathcal{P}^n_{R|A}(M), N) \cong \text{Hom}_A(M, N),
\]
given by \( \phi \mapsto \phi \circ d \), where \( d(x) = 1 \otimes x \), and the homomorphisms on the left-hand-side are those linear with respect to the left \( R \)-action on \( \mathcal{P}^n_{R|A}(M) \). Under this isomorphism, the \( \mathcal{P}^n_{R|A} \)-module structure on \( \text{Hom}_R(\mathcal{P}^n_{R|A}(M), N) \) given by its action on the source corresponds to the \( \mathcal{P}^n_{R|A} \)-module structure on \( \text{Hom}_A(M, N) \) where the left factor acts on \( N \) and the right factor acts on \( M \). The adjunction map induces isomorphisms
\[
\text{Hom}_R(\mathcal{P}^n_{R|A}(M), N) \cong \mathcal{D}^n_{R|A}(M, N)
\]
and
\[
\lim_{\to} \text{Hom}_R(\mathcal{P}^n_{R|A}(M), N) \cong \mathcal{D}_{R|A}(M, N).
\]
These are isomorphisms of \( \mathcal{P}^n_{R|A} \)-modules, where the structures are as described above.

**Remark 2.1.** We will often wish to consider \( \mathcal{P}^n_{R|A} \)-modules as \( R \)-modules; in general this can be done in different ways. For a \( \mathcal{P}^n_{R|A} \)-module \( \mathcal{M} \), we will write \( {a} \mathcal{M} \), respectively \( {b} \mathcal{M} \), for the \( R \)-module structure on \( \mathcal{M} \) given by restriction of scalars to the left, respectively right, \( R \)-factor of \( \mathcal{P}^n_{R|A} \). In particular, \( {a}\mathcal{D}_{R|A}(M, N) \) is the structure induced by the action on \( N \), and \( {b}\mathcal{D}_{R|A}(M, N) \) is the structure induced by the action on \( M \).

**The Smith–Van den Bergh functors.** The functor \( M \mapsto \mathcal{D}_{R|A}(M) \) is a left-exact functor from the category of \( R \)-modules to \( \mathcal{P}^n_{R|A} \)-modules. Following \([22]\), we denote its right derived functors by \( \mathcal{R}^i\mathcal{D}_{R|A}(M) \). There is a natural identification \( \mathcal{R}^i\mathcal{D}_{R|A}(M) \cong \lim_{\to} \text{Ext}^i_R(\mathcal{P}^n_{R|A}, M) \).

Under mild hypotheses, there is an alternative description.

**Proposition 2.2** (Smith–Van den Bergh \([22, \text{Proposition 2.2.2}]\)). Suppose that \( R \) is projective over \( A \) and \( \mathcal{P}^n_{R|A} \) is noetherian. Then \( \mathcal{R}^i\mathcal{D}_{R|A}(M) \cong \text{H}^i_{\Delta^n_{R|A}}(\text{Hom}_A(R, M)) \). Consequently, if \( \mathcal{P}^n_{R|A} \) has dimension \( D < \infty \), then \( \mathcal{R}^i\mathcal{D}_{R|A}(-) \equiv 0 \) for \( i > D \).

We note the following consequence.

**Lemma 2.3.** Suppose that \( R \) is projective over \( A \) and \( \mathcal{P}^n_{R|A} \) is noetherian and finite-dimensional. If \( \mathcal{R}^i\mathcal{D}_{R|A}(R) = 0 \) for all \( i > s \), then \( \mathcal{R}^i\mathcal{D}_{R|A}(-) \equiv 0 \) for all \( i > s \). In particular, if \( \mathcal{R}^i\mathcal{D}_{R|A}(R) = 0 \) for all \( i > 0 \), then the functor \( \mathcal{D}_{R|A}(-) \) is exact.

**Proof.** Since the functors \( \mathcal{R}^i\mathcal{D}_{R|A} \) commute with direct sums, \( \mathcal{R}^i\mathcal{D}_{R|A}(F) = 0 \) for any free module \( F \). Given an arbitrary module \( M \), take a short exact sequence
\[
0 \to \text{syz}(M) \to F \to M \to 0.
\]
The long exact sequence in the functors $R^\bullet D_{R|A}$ gives isomorphisms

$$R^i D_{R|A}(M) \cong R^{i+1} D_{R|A}(\text{syz}(M)).$$

Applying this repeatedly gives isomorphisms

$$R^i D_{R|A}(M) \cong R^{i+t} D_{R|A}(\text{syz}^t(M)).$$

But, by Proposition 2.2, for $i+t > \dim(\mathcal{P}_{R|A})$, we have $R^{i+t} D_{R|A}(-) \equiv 0$. Thus, $R^i D_{R|A}(-) \equiv 0$.

For the final statement, we note that the left-exact functor $D_{R|A}(-)$ is exact if and only if all of its right-derived functors vanish. \qed

**Remark 2.4.** The previous lemma is the analogue of a well-known property of local cohomology: for any ideal $I$ in a ring $R$, $H^i_I(R) = 0$ for $i > s$ implies $H^j_I(\cdot) \equiv 0$ for $i > s$.

We will refer to the functors $R^i D_{R|A}(-)$ as the Smith–Van den Bergh functors.

### 3. Duality and $D$-simplicity

Recall that if $K$ is a field, and $(R, m, K)$ is a local ring containing an isomorphic copy of $K$, then the Matlis duality functor $(-)^\vee = \text{Hom}_K(-, E_R(K))$ agrees with the functor $M \mapsto \varinjlim \text{Hom}_K(M/m^n M, K)$ for any finitely generated $R$-module $M$. Similarly, if $K$ is a field, and $R$ is a noetherian $\mathbb{N}$-graded $K$-algebra with $R_0 = K$, then the functor $(-)^\vee_K = \text*{Hom}_K(-, K)$ agrees with the usual Matlis duality functor $(-)^\vee = \text{Hom}_R(M, E_R(K))$ on the category of graded $R$-modules; see [5, Proposition 3.6.16]. Here we use the notation $^\text*{Hom}$ to denote the module of graded homomorphisms. We note that if $M$ is a finitely generated module and $N$ is an arbitrary graded module, $^\text*{Hom}(M, N)$ agrees with $\text{Hom}(M, N)$ after forgetting grading, and likewise for the derived functors $^\text*{Ext}(M, N)$, see [5, Section 1.5].

We will use a version of a result of Switala [23] in the complete local case and Switala–Zhang [24] in the graded case that asserts that the Matlis dual of a left $D_{R|K}$-module is a right $D_{R|K}$-module, and vice versa. We provide here an alternative interpretation of this result, with an auxiliary observation that we employ in the sequel.

Given an $R$-module $M$, let $\text{fg}(M)$ denote the directed system (with respect to containment) of finitely generated submodules of $M$. The *compactly supported differential operators* from $M$ to $N$ are

$$D_{R|A}^{\text{cpt}}(M, N) := \varprojlim_{M_\lambda \in \text{fg}(M)} D_{R|A}(M_\lambda, N).$$

If $M$ is finitely generated, then $D_{R|A}^{\text{cpt}}(M, N) = D_{R|A}(M, N)$ for any module $N$.

The first part of the following proposition is well-known to experts, but we include it here for completeness.

**Proposition 3.1.** Let $(R, m, K)$ be a graded or local ring essentially of finite type over a coefficient field $K$.

1. If $M$ is finitely generated, then $^bD_{R|K}(M, K) \cong M^\vee$.
2. If $M$ is an arbitrary $R$-module, then $^bD_{R|K}^{\text{cpt}}(M, K) \cong M^\vee$.
3. (Switala, Switala–Zhang) If $M$ is a left $D_{R|K}$-module, then $M^\vee$ is a right $D_{R|K}$-module. If $M$ is a right $D_{R|K}$-module, then $M^\vee$ is a left $D_{R|K}$-module.
Proof. (1) We compute

\[ D_{R|K}(M, K) = \lim_{\to} \text{Hom}_R(\mathcal{P}_{R|K}^n(M), K) = \lim_{\to} \text{Hom}_R\left(\frac{R \otimes_K M}{\Delta_{R|K}^n(R \otimes_K M)}, K\right) = \lim_{\to} \text{Hom}_R\left(\frac{R \otimes_K M}{(m \otimes 1 + \Delta_{R|K}^n)(R \otimes_K M)}, K\right) = \lim_{\to} \text{Hom}_R(K \otimes_K M/m^{n+1}M, K), \]

where the third equality follows from the fact that any \( R \)-linear map from a module \( N \) to \( K \) factors uniquely through \( N/\mathfrak{m}N \), and the fourth from the equality \( m \otimes 1 + \Delta_{R|K}^n = m \otimes 1 + 1 \otimes m^{n+1} \). Thus,

\[ bD_{R|K}(M, K) = b\lim_{\to} \text{Hom}_R(K \otimes_K M/m^{n+1}M, K) = \lim_{\to} \text{Hom}_K(M/m^{n+1}M, K) \cong M^\vee. \]

(2) We compute

\[ bD_{R|K}^{\text{cpt}}(M, K) = \lim_{\to} bD_{R|K}(M, K) = \lim_{\to} M^\vee = \left( \lim_{\to} M^\vee \right)^\vee = M^\vee. \]

(3) First, we note that if \( M \) is a left (respectively, right) \( D_{R|K} \)-module, then the action of \( D_{R|K} \) is given by differential operators from \( M \) to \( M \); indeed, this is immediate from the definition. Now, every differential operator \( \delta \) on \( M \) takes finitely generated submodules to finitely generated submodules. Indeed, \( \delta \) factors as \( d : M \to \mathcal{P}_{R|K}^n(M) \) followed by an \( R \)-linear map \( \phi : \mathcal{P}_{R|K}^n(M) \to M \); the restriction of such \( \delta \) to a finitely generated \( M_\lambda \) factors through an \( R \)-linear map \( \phi' : \mathcal{P}_{R|K}^n(M_\lambda) \to M \), and since \( \mathcal{P}_{R|K}^n(M_\lambda) \) is finitely generated, the image is contained in a finitely generated submodule of \( M \). Consequently, \( D_{R|K}(M, M) \) acts on \( D_{R|K}^{\text{cpt}}(M, K) \) by precomposition. Hence, we obtain an action of \( D_{R|K} \) on \( D_{R|K}^{\text{cpt}}(M, K) \) with the order reversed. \( \square \)

We apply the previous proposition to show that \( D_{R|K} \)-simplicity admits a straightforward characterization in terms of the Smith–Van den Bergh functor \( R^1D_{R|K}(-) \).

**Proposition 3.2.** Let \( K \) be a field, and \( R \) be a noetherian \( \mathbb{N} \)-graded \( K \)-algebra, with \( R_0 = K \). Let \( \varpi : D_{R|K}(R) \to D_{R|K}(K) \) be the map obtained by composition with the \( R \)-linear surjection \( \pi : R \to K \). Then \( R \) is a simple left \( D_{R|K} \)-module if and only if \( \varpi \) is surjective. Equivalently, \( R \) is \( D_{R|K} \)-simple if and only if the connecting homomorphism \( D_{R|K}(K) \to R^1D_{R|K}(R_+) \) is the zero map.

**Proof.** By Proposition 3.1 (1), \( bD_{R|K}(K) \) is isomorphic to \( E_R(K) \). Clearly, \( \eta = \pi \circ d \in D_{R|K}(K) \) generates the socle of \( E^\vee \). Now, \( \varpi(\delta) = \eta \circ \delta \); that is, \( \varpi \) agrees with the natural right \( D_{R|K} \)-action on \( E_R(K) \) applied to a socle generator of \( E_R(K) \).
We refer to $E$ submodule. Thus, $d E\Phi H$.

Theorem 4.1. Let $N anian$.

The *dimensions of $R$ are.

The conclusion of the previous corollary may fail if $R$ is not Gorenstein. Let $K$ be a field, and $R = K[x, y]/(x, y)^2$. Then $D_{R/K} = \text{Hom}_K(R, R)$, since $\Delta_{R/K} = 0$ in $\mathcal{P}_{R/K}$.

Thus, $\text{Ext}^i_{R/K}(K, R)^{\oplus 3} \cong R^{\oplus 3}$, whereas $\text{Ext}^i_{R/K}(K, R)^{\oplus 3} \cong (R^\vee)^{\oplus 3}$. Since $R$ is not Gorenstein, these are not isomorphic. The same argument works more generally for any graded artinian $K$-algebra that is not Gorenstein.

Remark 4.4. Our description of $R^i D_{R/A}(R)$ also sheds some light on the hypotheses of [2]. Let $K$ be a field. In ibid., a dimensional Cohen-Macaulay $K$-algebra of dimension $d$ determines a good CM variety if $H^{d+1}_{\Delta_{R/K}}(R \otimes_K \omega_R) = 0$, and a very good CM variety if $H^i_{\Delta_{R/K}}(M) = 0$ for all $\mathcal{P}_{R/K}$-modules $M$ and all $i > d$. If $R$ is Gorenstein and graded,
it follows from Theorem 4.1 and Lemma 2.3 that $R$ is a good CM variety if and only if $R^1D_{R|K}(R) = 0$, and $R$ is a very good CM variety if and only if $D_{R|K}(-)$ is an exact functor. We note also that in ibid., the Grothendieck–Sato formula is obtained for good CM varieties without the Gorenstein assumption.

Using a version of a duality of Horrocks [17] due to Dao and Montaño [7], we can give a different description of the derived functors of the ring of differential operators for isolated singularities.

**Theorem 4.5.** Let $K$ be a field, and $R$ be a graded $K$-algebra with an isolated singularity. Suppose that $d = \dim(R) \geq 3$. Then, $R^iD_{R|K}(\omega_R) \cong H^{i+1}_{R_+}(D_{R|K}(\omega_R))$ for $1 \leq i \leq d - 2$.

**Proof.** By [7], if $R$ is Cohen-Macaulay of dimension at least 3, $M$ is a finitely generated module that is free on the punctured spectrum, and $1 \leq i \leq d - 2$, then there are isomorphisms $\text{Ext}^i_R(M, \omega_R) \cong H^{i+1}_{R_+}(\text{Hom}_R(M, \omega_R))$. Since the formation of $\mathcal{P}^n_{R|K}$ commutes with localization, $\mathcal{P}^n_{R|K}$ is free on the punctured spectrum. Thus, there are isomorphisms

$$\text{Ext}^i_R(\mathcal{P}^n_{R|K}, \omega_R) \cong H^{i+1}_{R_+}(\text{Hom}_R(\mathcal{P}^n_{R|K}, \omega_R)) \cong H^{i+1}_{R_+}(D_{R|K}(\omega_R))$$

for each $n$. The statement follows from passing to the direct limit. \(\square\)

**Remark 4.6.** While Theorems 4.1 and 4.5 are stated in the graded case, the analogous statements hold in the complete case. If $R$ is complete with coefficient field $K$, one replaces $\mathcal{P}_{R|K}$ with its m-adic completion $\hat{\mathcal{P}}_{R|K} \cong R\hat{\otimes}_K R$, and $\mathcal{P}^n_{R|K}$ with $\hat{\mathcal{P}}^n_{R|K}$, and the same proofs hold mutatis mutandis.

5. **Vanishing criteria**

As an easy corollary of Theorem 4.5, we see that these functors determine the depth of the ring of differential operators for isolated Gorenstein singularities.

**Corollary 5.1.** Let $K$ be a field, and $R$ be a Gorenstein graded $K$-algebra with an isolated singularity. Suppose that $d = \dim(R) \geq 3$. Then,

$$\text{depth}_R(D_{R|K}) = \begin{cases} 1 + \min\{i > 0 \mid R^iD_{R|K}(R) \neq 0\} & \text{if this number is } \leq d, \\ d & \text{otherwise}. \end{cases}$$

**Proof.** The hypotheses imply that $R$ is normal. Each $D^n_{R|K}$ is reflexive, so $H^0_{R_+}(D^n_{R|K}) = H^1_{R_+}(D^n_{R|K}) = 0$ for each $n$; one has $H^0_{R_+}(D_{R|K}) = H^1_{R_+}(D_{R|K}) = 0$ by taking direct limits. The formula then follows from Theorem 4.5 and the characterization of depth in terms of local cohomology. \(\square\)

We also apply Proposition 4.1 to give vanishing conditions on $R^iD_{R|K}(\omega_R)$ in terms of meaningful geometric information. Recall that a map of $K$-algebras $R \rightarrow S$ is *radicial* if, for some (equivalently, every) algebraically closed field $L \supseteq K$, the map $\text{Spec}(S \otimes_K L) \rightarrow \text{Spec}(R \otimes_K L)$ is injective [13, Definition 3.5.4]; in particular, if $K$ is algebraically closed, this is equivalent to $\text{Spec}(S) \rightarrow \text{Spec}(R)$ being injective. We note that a radicial map of graded rings or local rings is necessarily module-finite by Nakayama’s lemma, and hence is bijective on spectra.

We define the *radicial rank* of a $K$-algebra $R$ to be

$$\text{rra}_K(R) = \min\{n \mid \exists f_1, \ldots, f_n \in R \text{ such that } K[f_1, \ldots, f_n] \rightarrow R \text{ is radicial}\}.$$
Since the arithmetic rank of an ideal $I \subseteq R$ can be characterized as
\[ \text{ara}(I) = \min \{ n \mid \exists f_1, \ldots, f_n \in R \text{ such that } R/(f_1, \ldots, f_n) \rightarrow R/I \text{ is bijective on Spec} \}, \]
we think of radicial rank as a subalgebra analogue of arithmetic rank.

The following proposition is closely related to [10, §3] and [11, §4].

**Proposition 5.2.** Let $K$ be a field, and $R$ be a Cohen-Macaulay graded $K$-algebra. Then
\[ \max \{ i \mid R^i D_{R[K]}(\omega_R) \neq 0 \} \leq \text{rra}_K(R) - \dim(R). \]
If $R$ is standard graded, $X = \text{Proj}(R)$, and $\text{Sec}(X)$ is the secant variety of $X$, then
\[ \max \{ i \mid R^i D_{R[K]}(\omega_R) \neq 0 \} \leq \dim(\text{Sec}(X)) - \dim(X). \]

**Proof.** Let $d = \dim(R)$. By [14, Lemma 1.8.7.1], $S \rightarrow R$ is radicial if and only if $R \otimes_S R \rightarrow R$ induces a surjection on spectra. We have isomorphisms $R \otimes_S R \cong \mathcal{P}_{R[K]}/(\Delta_{S[K]} \mathcal{P}_{R[K]})$ and $R \cong \mathcal{P}_{R[K]}/\Delta_{R[K]}$, so $S \rightarrow R$ is radicial if and only if $\sqrt{\Delta_{S[K]} \mathcal{P}_{R[K]}} = \Delta_{R[K]}$.

Let $W$ be a canonical module for $\mathcal{P}_{R[K]}$. If $R^i D_{R[K]}(\omega_R) \cong H^d_{\Delta_{R[K]}(W)} \neq 0$ and $\sqrt{\Delta_{S[K]} \mathcal{P}_{R[K]}} = \Delta_{R[K]}$, then $H^d_{\Delta_{R[K]}(W)} \neq 0$. Consequently, $\Delta_{S[K]}$ cannot be generated by fewer than $d + i$ elements. For any $n$-generated $K$-algebra, its diagonal ideal is generated by $n$ elements. Thus, $R^i D_{R[K]}(\omega_R) \neq 0$ implies $\text{rra}_K(R) \geq d + i$, which establishes the first inequality of the statement.

For the second part of the statement, it suffices to note that $\text{rra}_K(R) \leq \dim(\text{Sec}(X)) + 1$. This is well-known; see, e.g., [11, Corollary 4.3].

**Remark 5.3.** We note that the local cohomology modules occurring in Proposition 4.1 are closely related to those that appear in the work [10, 11] of Emilie Dufresne and the present author on invariant theory. A *separating set* for an action of a group $G$ on a polynomial ring $S = \text{Sym}(V)$ consists of a set of invariants $\{f_1, \ldots, f_i\} \subseteq S^G$ such that for any $v, w \in V$ with $h(v) \neq h(w)$ for some $h \in S^G$, there is an $f_i$ with $f_i(v) \neq f_i(w)$; we refer the reader to [8, Section 2.4] for an introduction to separating sets. By [9, Theorem 2.2], if $G$ is reductive and $K = \bar{K}$, then $\{f_1, \ldots, f_i\}$ is a separating set if and only if $K[f_1, \ldots, f_i] \rightarrow S^G$ is radicial. In [10], we show that the smallest cardinality of a separating set for a linear action of a finite group $G$ on a polynomial ring $R$ is bounded below by $\max \{ i \mid H^i_{\Delta_{R[K]}(\mathcal{P}_{S[K]})} \neq 0 \}$, where $R = S^G$. This is applied to give meaningful lower bounds in terms of properties of the representation. A similar approach is executed in [11] to bound separating sets for actions of tori. It follows from Remark 2.4 that the bounds given by Proposition 5.2 are always at least as strong as those mentioned above; they also do not require the ring $R$ to be realized as a subring of a polynomial ring.

**Example 5.4.** Let $K$ be a field of characteristic zero, and $M$ be a $3 \times 3$ matrix of indeterminates. Let $R = K[M]/I_2(M)$, where $I_2(M)$ is the ideal of $2 \times 2$ minors of $M$. The ring $R$ is a 5-dimensional ring, with an isolated Gorenstein singularity at the homogeneous maximal ideal. The secant variety of $X = \text{Proj}(R)$ is $\text{Proj}(K[M]/\det(M))$, which has dimension 7.

By Proposition 5.2, $R^i D_{R[K]}(R) = 0$ for $i > 3$.

We claim that $R^i D_{R[K]}(R) \neq 0$ for some $i > 0$. Equivalently, we show that $H^i_{\Delta_{R[K]}(\mathcal{P}_{R[K]})} \neq 0$ for some $i > 3$. Let $C_i \subset R$ for $i = 1, 2, 3$ be the ideal generated by the images of the variables in the $i$th column of $M$. Consider the ideal
\[ J = (C_2 \otimes 1 + C_3 \otimes 1 + 1 \otimes C_1 + 1 \otimes C_3) \subset \mathcal{P}_{R[K]}. \]
The quotient ring $R_{R/K}/J$ is a polynomial ring in 6 variables, and the image of $\Delta_{R/K}$ generates its maximal ideal, so $H^6_{\Delta_{R/K}}(R_{R/K}/J) \neq 0$. Thus, by Remark 2.4, $H^i_{\Delta_{R/K}}(R_{R/K}) \neq 0$ for some $i \geq 6$.

Consequently,

$$1 \leq \min\{i > 0 \mid R^iD_{R/K}(R) \neq 0\} \leq \max\{i > 0 \mid R^iD_{R/K}(R) \neq 0\} \leq 3.$$ 

By Corollary 5.1, $\text{depth}_R(D_{R/K}) \leq 4$, so $D_{R/K}$ is not Cohen-Macaulay. We note that $R$ is an invariant ring of an action of a torus $T$ on a polynomial ring $S$, and $D_{R/K} = D_{S/K}^T$ for this action [18]. In particular, this gives a counterexample to the analogue of the Hochster–Roberts Theorem [16] for actions of linearly reductive groups on the Weyl algebra.

6. Reduction to positive characteristic

In this section, we apply Theorem 4.1 to determine when differential operators behave well under reduction to positive characteristic. As mentioned in the introduction, this is governed by the first Smith–Van den Bergh functor:

**Proposition 6.1** (Smith–Van den Bergh [22, Subsection 5.1]). Let $A$ be a Dedekind domain, and $R$ be an $A$-algebra such that $\mathcal{P}_{R/A}^n$ is a projective $A$-module for all $n$; in particular, $R$ is $A$-projective. Then, for every $A$-algebra $B$, there is a short exact sequence

$$0 \to D_{R/A}(R) \otimes_A B \to D_{R\otimes_A B}(R \otimes_A B) \to \text{Tor}_1^A(B, R^1D_{R/A}(R)) \to 0.$$

We observe that in the Gorenstein case this question is a particular case of the study of $p$-torsion in local cohomology modules.

**Proposition 6.2.** Let $R$ be a Gorenstein $d$-dimensional noetherian $\mathbb{N}$-graded ring with $R_0 = \mathbb{Z}$. The following are equivalent:

(i) The local cohomology module $H^d_{\Delta_{R/\mathbb{Z}}}(\mathcal{P}_{R/\mathbb{Z}})$ has $p$-torsion for only finitely many primes $p \in \mathbb{Z}$.

(ii) There is an $n \in \mathbb{Z}$ such that, after replacing $R$ by $R[1/n]$ and $\mathbb{Z}$ by $A = \mathbb{Z}[1/n]$, there are isomorphisms

$$D_{R/A} \cong D_{R/A} \otimes_A \mathbb{F}_p,$$

for every prime $p \in \mathbb{Z}$ that does not divide $n$.

**Proof.** First, we note that we can replace $R$ by $R[1/n]$ and $\mathbb{Z}$ by $A = \mathbb{Z}[1/n]$ to make $\mathcal{P}_{R/A}^n$ a projective $A$-module for all $n$ as in [22, Subsection 5.1]. Then, by Proposition 6.1, it suffices to show that, for all $p$ that do not divide $n$, $H^d_{\Delta_{R/A}}(\mathcal{P}_{R/A})$ has nonzero $p$-torsion if and only if $R^1D_{R/A}(R)$ has nonzero $p$-torsion.

Let $p$ be a prime not dividing $n$, and set $A' = \widehat{\mathbb{A}}_{(p)}$ and $R' = R \otimes_A A'$. We claim that $R^1D_{R/A}(R) \otimes_A A' \cong R^1D_{R'/A'}(R')$. Indeed, this is immediate from the fact that Ext commutes with flat base change for finitely generated modules over noetherian rings.

Now, $R'$ as above satisfies the hypotheses of Theorem 4.1, $\text{height}(R_+) = d - 1$, and $\mathcal{P}_{R'/A'}$ is Gorenstein. Thus,

$$R^1D_{R'/A'}(R') \cong H^d_{\Delta_{R'/A'}(\mathcal{P}_{R'/A'})} \cong H^d_{\Delta_{R/A}(\mathcal{P}_{R/A})} \otimes_A A'.$$
where the last isomorphism follows from the behavior of local cohomology under flat base change. Therefore, existence of nonzero $p$-torsion in each of the modules

$$
R^1 \Delta_{R|A}(R), \ R^1 \Delta_{R|A}(R) \otimes_A A', \ \Delta^d_{R|A}(S_{R|A}) \otimes_A A', \ \text{and} \ \Delta^d_{R|A}(S_{R|A})
$$

is equivalent. \hfill \Box

We can apply this observation to show that differential operators behave well under reduction to positive characteristic in certain circumstances. In particular, the next two results provide positive answers to [22, Question 5.1.2] under stronger hypotheses.

**Theorem 6.3.** Let $R$ be a graded direct summand of a polynomial ring $S = \mathbb{Z}[x_1, \ldots, x_m]$. Suppose also that $R$ is Gorenstein. Then there is an $n \in A$ such that, after replacing $R$ by $R[1/n]$ and $\mathbb{Z}$ by $A = \mathbb{Z}[1/n]$, there are isomorphisms

$$
D_{R|\mathbb{F}_p} \cong D_{R|A} \otimes_A \mathbb{F}_p, \quad \text{where} \quad \overline{R} = R/pR,
$$

for every prime $p \in \mathbb{Z}$ that does not divide $n$.

**Proof.** Since $R$ is a graded direct summand of $S$, $S_{R|A}$ is a graded direct summand of $S_{A|A}$, as is $\Delta^i_{R|A}(S_{R|A})$ of $\Delta^i_{R|A}(S_{A|A})$ for each $i$. Since $S_{A|A}$ is a smooth $\mathbb{Z}$-algebra, by [4], $H^i_{\Delta^i_{R|A}}(S_{A|A})$ has $p$-torsion for only finitely many primes $p$ for each $i$. Thus, the same holds for $H^i_{\Delta^i_{R|A}}(S_{R|A})$. We can then apply Proposition 6.2. \hfill \Box

**Corollary 6.4.** Let $R$ be a Gorenstein graded subring of polynomial ring $S = \mathbb{Z}[x_1, \ldots, x_m]$. Suppose that $R \otimes \mathbb{Q} \to S \otimes \mathbb{Q}$ is module-finite and splits as $R \otimes \mathbb{Q}$-modules. Then there is an $n \in \mathbb{Z}$ such that, after replacing $R$ by $R[1/n]$ and $\mathbb{Z}$ by $A = \mathbb{Z}[1/n]$, there are isomorphisms

$$
D_{R|\mathbb{F}_p} \cong D_{R|A} \otimes_A \mathbb{F}_p, \quad \text{where} \quad \overline{R} = R/pR,
$$

for every prime $p \in \mathbb{Z}$ that does not divide $n$.

**Proof.** Any fixed set of module generators $\{f_1, \ldots, f_t\}$ of $S \otimes \mathbb{Q}$ over $R \otimes \mathbb{Q}$ live inside of $S[1/n]$ for some $n \in \mathbb{Z}$; after inverting an element of $\mathbb{Z}$, without loss of generality they live in $\{f_1, \ldots, f_t\} \subset S$. Applying generic freeness, we can localize further on $\mathbb{Z}$ so that $C = S/(\sum RF_i)$ is a free $A = \mathbb{Z}[1/n]$-module. Since $C \otimes \mathbb{Q} = 0$, we must have $C = 0$. Thus, we can assume that $S$ is a finitely generated $R$-module.

Now, let $f_1 \in S$ and $\phi : S \otimes \mathbb{Q} \to R \otimes \mathbb{Q}$ be such that $\phi(f_1 \otimes 1) = 1$. Note that the map $\phi$ is determined by the values $\phi(f_1 \otimes 1)$. Since there are finitely many, we can again invert an element of $A$ to ensure each of these is contained in $R$. Then $\phi$ descends to a $R$-module splitting of the inclusion of $R$ into $S$. The result then follows from Theorem 6.3. \hfill \Box

**Remark 6.5.** The map $D_{R|A}(R) \otimes_A B \to D_{(R \otimes_A B)}(R \otimes_A B)$ in Proposition 6.1 preserves the order filtration. That is, this map occurs as the filtered direct limit of maps $D_{R|A}(R) \otimes_A B \to D_{(R \otimes_A B)}(R \otimes_A B)$. Thus, in Proposition 6.2, Theorem 6.3, and Corollary 6.4, when isomorphisms do occur, they preserve the order filtration.

We can also apply Proposition 6.2 to give a conceptual example of a local cohomology module with infinitely many associated primes; the first example of this phenomenon was given by Singh [20].
Example 6.6. Let $R = \frac{\mathbb{Z}[x, y, z]}{(x^3 + y^3 + z^3)}$. In [21], Smith showed that for every $p \equiv 1 \mod 3$, there is a differential operator in $R \otimes_{\mathbb{Z}} \mathbb{F}_p$ that is not the base change of a differential operator in $R$. The same argument with minor modifications works for all prime characteristics other than three.

By [3], the differential operators on $R \otimes_{\mathbb{Z}} \mathbb{C}$ of degree zero are generated (as an algebra) by the Euler operator, which multiplies any homogeneous element by its degree. Since $D_{R|\mathbb{Z}}$ is torsionfree over $\mathbb{Z}$, we have $D_{R|\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \cong D_{R \otimes \mathbb{C}|\mathbb{C}}$, so any element $\delta \in [D_{R|\mathbb{Z}}]_0$ is given by the rule $\delta(f) = P(\operatorname{deg}(f))f$ for some polynomial $P \in \mathbb{Z}[x]$ for each homogeneous $f$.

Now, fix a prime $p \neq 3$, and set $\bar{R} = R \otimes \mathbb{F}_p$. Both $\bar{R}$ and its ring of $p$th powers $\bar{R}^p$ are Gorenstein with $a$-invariant zero. Thus, there is a degree-preserving isomorphism $\operatorname{Hom}_{\bar{R}^p}(\bar{R}, \bar{R}^p) \cong R$. In particular, there is a nonzero $\bar{R}^p$-linear map $\phi : \bar{R} \to \bar{R}$ of degree zero, the image of which is an ideal of $\bar{R}^p$. By the isomorphism $D_{\bar{R}^p \otimes \mathbb{F}_p} \cong \bigcup_{e \geq 0} \operatorname{Hom}_{\bar{R}^p}(\bar{R}, \bar{R})$ (see, e.g., [22, Lemma 2.2.1]), we see that $\phi$ is a differential operator on $\bar{R}$ of degree zero. However, it follows from the previous paragraph that every degree zero element in the image of $D_{R|\mathbb{Z}} \otimes \mathbb{F}_p$ must have as its own image a direct sum of graded pieces of $\bar{R}$. No ideal of $\bar{R}^p$ is of this form, so such a map $\phi$ cannot be the base change of an operator on $R$.

It follows from Proposition 6.2 that $H^1_{(x-x', y-y', z-z')}(R \otimes_{\mathbb{Z}} R)$ has nonzero $p$-torsion for all but finitely many primes $p$. In particular, this local cohomology module has infinitely many associated primes.

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