Universal Codes from Switching Strategies

Wouter M. Koolen and Steven de Rooij

Abstract—We discuss algorithms for combining sequential prediction strategies, a task which can be viewed as a natural generalisation of the concept of universal coding. We describe a graphical language based on Hidden Markov Models for defining prediction strategies, and we provide both existing and new models as examples. The models include efficient, parameterless models for switching between the input strategies over time, including a model for the case where switches tend to occur in clusters, and finally a new model for the scenario where the prediction strategies have a known relationship, and where jumps are typically between strongly related ones. This last model is relevant for coding time series data where parameter drift is expected. As theoretical contributions we introduce an interpolation construction that is useful in the development and analysis of new algorithms, and we establish a new sophisticated lemma for analysing the individual sequence regret of parameterised models.

Index Terms—Universal Coding, Regret, Individual Sequence, Hidden Markov Models, Prediction with Expert Advice, Expert Tracking

I. INTRODUCTION

For the most delectable universal codes, fill a cooking pot with water, add a bunch of experts, a.k.a. codes, and put it on a slow fire, stirring constantly. The resulting mix is guaranteed to delight, achieving a codelength close to that of the best among the ingredient codes.

In this paper we investigate such cookery in detail, deviating from the usual recipe in two ways. First, following Shtarkov [4] and Rissanen [5], universality is expressed in terms of the individual sequence regret: the difference between the length of the considered code and the shortest codelength among any of the ingredient codes, for the data that were actually observed. As such, there are no distributional assumptions. Second, the setting is generalised somewhat: rather than always comparing our performance to that of the best code, we will also consider combinations of the ingredient codes as baselines for the regret. Among other things, this allows us to compete with the best possible way to split the data sequence into a small number of blocks of consecutive outcomes, and encode each block with the best original code for that block, a problem known as expert tracking in online learning [6], [7], which is also a core focus of this work.

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We identify sequential coding with sequential prediction as follows. In each round $t = 1, 2, \ldots$, a sequential prediction strategy issues a probability distribution $Q_t$ on the outcome space $\mathcal{X}$, which for simplicity we assume to be countable. Subsequently, a new outcome $x_t$ is observed, and the prediction is evaluated using logarithmic loss $-\log_2 Q_t(x_t)$. The Kraft inequality states that there exists a prefix code such that the accumulated logarithmic loss of the prediction strategy, rounded up to the nearest integer, corresponds exactly to the codelength for the data; in fact there are many practical algorithms for implementing such a code, such as arithmetic coding [8]. For this reason, we will use the words "prediction strategy" and "code" interchangeably; similarly we use "codelength" as a synonym for "logarithmic loss", and forget about the rounding. Also, for convenience we will use natural logarithms such that codelengths are expressed in nats. See [9], [10], [11] for more information about the connection between sequential prediction and data compression.

Given a set of codes, our aim is to build universal codes with efficient implementations, and evaluate their performance. Borrowing terminology from learning theory, we henceforth call the ingredient codes "experts" to emphasize that they are black boxes that can be interpreted as prediction strategies. Borrowing terminology from learning theory, we henceforth call the ingredient codes “experts” to emphasize that they are black boxes that can be interpreted as prediction strategies. The overall protocol is as follows (see Figure 1). Let $\Xi$ be a set of experts, that we fix throughout this paper. Each round $t$, each expert $\xi \in \Xi$ issues a prediction $P_{\xi,t}$ of the next outcome in the form of a probability distribution. Our universal prediction strategy collects all these predictions and uses them to form a prediction $Q_t$ of its own. We then wait until the new data item $x_t$ is observed, and incur a logarithmic loss of (encode it using) $-\log_2 Q_t(x_t)$ nats. By the end of the game we compare our accumulated loss (total codelength) to that of the best among a set of reference strategies.

Algorithms for combining prediction strategies can be found in the literature under various headings. On the one hand there are results in (Bayesian) statistics and source coding. Most relevant is Wolf and Willems’ algorithm for combining two sequential data compression algorithms [12], called the “Switching Method”, which we discuss in Section IV-D. Important precursors of this work include [13], [14]: the algorithms described there do not combine expert predictions...
but can be used for that purpose (see Section IV-B for details). The tradeoff between time complexity and regret has received substantial further analysis, see [15], [16], [17], [18], [19], but such work is outside the scope of this introduction. On the other hand, the learning theory community has produced a lot of work on universal prediction under the heading “prediction with expert advice” [7], [20], [21], [22], [6], [23]. In this case the experts’ predictions are not necessarily probabilistic, and scored using an arbitrary loss function. In this paper we focus on results for logarithmic loss, although our results apply to any mixable loss function, as discussed in Section V-D.

In order to give a clear, consistent and comprehensive introduction/overview of the topic, we use the Bayesian framework to describe the considered codes, and show how algorithms for prediction with expert advice can often be described using a prior distribution on sequences of experts. This approach allows unified description of many of the mentioned results as well as some new models that represent interesting trade-offs between time complexity and modelling power. Following in the footsteps of, e.g., [13], [12], [15], [24], we use hidden Markov models (HMMs) as an intuitive graphical language to describe such priors, and obtain a computationally efficient implementation using standard algorithms such as the forward algorithm. Our main focus will be on algorithms for expert tracking.

Let us emphasize that, although technically the algorithms we consider are Bayesian, we use a worst-case individual sequence analysis. Thus, we do not adopt the usual subjective Bayesian interpretation of the prior distribution as an expression of belief; in fact, we do not make any statements or assumptions about the data-generating machinery whatsoever.

A. Our Contribution

The aim of this work is to provide a readily useful and accessible introduction to prediction with expert advice for an information theoretic audience. To this end we present models in terms of HMM state transition diagrams which, although widely used, have not been applied as consistently as to the problem of prediction with expert advice. This graphical language allows us to conveniently describe many existing models and design new ones. The resulting diagrams can be understood and compared with ease; moreover, computational efficiency can be gleaned directly from their structure.

Beyond its tutorial nature, the paper also has two theoretical contributions: first, we describe an interpolation construction that is useful in the development and analysis of new algorithms (Section IV-B). Second, Lemma 3 provides a substantial generalisation of earlier methods from [25], [24] for analysing the regret of expert models; the lemma is later used in the proof of Theorem 16 which cannot be proven using earlier results.

We describe a number of straightforward applications of the theory developed in the paper, which can be interpreted both as examples and as models of practical utility. A number of these models are new, in particular the Quickly Decreasing Switching Rate model in Section IV-C and the Ordered Experts model in Section IV-E.

B. Overview

In Section II we describe Bayesian prediction strategies based on a prior distribution on sequences of experts, and discuss how the resulting prediction strategies can be cast in the form of a hidden Markov model.

The performance of the algorithms we are interested in is expressed in terms of guarantees about their individual sequence regret. Section III provides the main theoretical tools for analysing the regret.

In Section IV we illustrate our approach by rendering various models for tracking the best expert in HMM form, thus showing the relationships between them. The seminal Fixed Share algorithm [20], [6] serves as a starting point. Fixed Share has two drawbacks: first, one has to specify a fixed switching rate in advance; choosing a suboptimal value here produces a linear penalty in the regret. Second, the incurred regret depends on the number of observations t, even when the optimal number of switches is bounded. To address these problems, the switching probabilities need to be modelled differently. Section IV-B explains how the part of the model that describes switching probabilities can be isolated from the rest; we then proceed to describe several alternative models for the switching probabilities, and discuss how these modifications affect the regret bound. In particular, Section IV-C describes a new, simple and effective approach to solve both problems associated with Fixed Share, Section IV-D describes several methods for learning the switching rate from data, and Section IV-E describes another model that is especially well suited to the scenario where changes in predictive performance of experts are expected to appear in clusters.

So far, none of the considered models for expert tracking made any assumptions as to the inner workings of, or the relationships between, the various experts – they are black boxes. However, as an interesting special case we consider the scenario where the experts are ordered. For example, if the experts are prediction strategies associated with a parametric model, instantiated with various parameter values, then switches between two experts seem intuitively more likely if they represent parameter values that are close. This scenario is explored in Section IV-F, the notion is taken to its extreme in Section IV-G where the regret is no longer analysed in terms of all-or-nothing “switches”, but rather in terms of a more smooth characterisation of the amount of “parameter drift”.

A number of loose ends are discussed in Section V. We specifically discuss a number of useful results from the literature that would distract from the exposition but are nevertheless too important to skip over altogether (V-A), as well as a variant evaluation criterion called adaptive regret (V-B). We then consider how one might estimate which expert made the best prediction at a certain time (V-C). Finally we describe how our results can be generalised to any mixable loss function (V-D) and to online investment (V-E).
II. Expert Sequence Priors and Hidden Markov Models

Since we do not know who among our set $\Xi$ of experts will issue the best predictions and achieve minimal codelength, the straightforward Bayesian response is to define a prior $w$ on $\Xi$, and then construct a distribution $B_w$ for the joint space $X^\infty \times \Xi$ with

$$B_w(x^t, \xi) := w(\xi)p_\xi(x^t), \quad \text{where} \quad p_\xi(x^t) := \prod_{i=1}^t p_{\xi,i}(x_i). \quad (1)$$

The Bayesian prediction is obtained by conditioning on past observations and marginalising over $\Xi$ as follows:

$$Q_{t+1}(x_{t+1}) := B_w(x_{t+1} \mid x^t).$$

(Random variables are denoted in bold face.) Note that this prediction depends on the expert predictions from times 1 through $t + 1$ but not beyond. This strategy is simple and effective. Compared to the single best expert, the regret of this strategy is

$$- \ln B_w(x^t) - \left( - \ln p_\xi(x^t) \right), \quad (2)$$

where $\hat{\xi} = \text{argmax}_\xi p_\xi(x^t)$, breaking ties arbitrarily. To bound the regret, note that

$$p_\xi(x^t) \geq \sum_{\xi \in \Xi} w(\xi)p_\xi(x^t) \geq w(\hat{\xi})p_\hat{\xi}(x^t); \quad (3)$$

substitution in (2) reveals that the regret must be in the interval $[0, - \ln w(\hat{\xi})]$. Thus, the good news is that this first strategy guarantees a codelength that is within a constant of the performance of the best available expert $\hat{\xi}$. On the other hand, with this strategy we never do any better than $\hat{\xi}$ either! Consider that the nature of the data generating process may evolve over time; consequently different experts may be better during different periods of time. It is also possible that not the data generating process, but the experts themselves change as more and more outcomes are being observed: they may learn from past mistakes, possibly at different rates, or they may have occasional bad days, etc. By generalising the Bayesian modelling a little bit, we can compete with prediction strategies that perform far better than the best individual expert.

A. Including Transient Behaviour

Rather than using a prior distribution to represent our uncertainty about which single expert is best, we generalise the setup by considering which expert is best in each round. Let $\pi$ be a prior on infinite sequences of experts, called an expert sequence prior (ES-prior). We subsequently define the Bayesian joint distribution $B_\pi$ by

$$B_\pi(x^t, \xi^t) := \pi(\xi^t)p_\xi(x^t), \quad \text{where} \quad p_\xi(x^t) := \prod_{i=1}^t p_{\xi,i}(x_i). \quad (4)$$

We can recover the previous prediction strategy by defining $\pi$ such that it assigns probability zero to any expert sequence that lists more than one expert.

In the simplest case, the prior models the sequence of experts as a Markov chain, but it is often desirable to carry along some additional state information besides the identity of the previous expert in the definition of the prior. Therefore, we will construct the Bayesian prediction strategy as a Hidden Markov Model of the form depicted in Figure 2 where the state of the prior process is captured by the $q_i$ variables.

Being a black box, an expert $\xi$ may use any strategy to form the prediction $P_{\xi,t}$ for $x_t$. The framework could even incorporate psychic experts who have some metaphysical access to future data! Or, as is the standard assumption for proving regret bounds, we may consider experts that conspire to maximally frustrate the prediction task. As such, an application may involve complicated dependencies between the $x_t$ and the predictions of the experts in general. However, as not only the data but also the predictions of the experts are observed, there is no need to include these dependencies in the model.

B. Graphical Specification of Expert HMMs

We now outline a graphical language for describing HMMs of the form shown in Figure 2 which allows us to cleanly and intuitively display model structure. The resulting prediction strategy can be read off directly from these diagrams, and diagrammatic simplicity implies computational efficiency.

An Expert Hidden Markov Model (EHMM) is a joint distribution on sequences of states, experts and outcomes. It is defined by the following ingredients. We start by choosing a set of states $Q$ and a designated start state $q_0 \in Q$. We then specify the transition probability between states using a Markov kernel $P$. (We now have a regular Markov chain on the sequence of states.) A subset $Q^p \subseteq Q$ of the states are called productive. Experts are deterministically assigned to the productive states by $\Lambda : Q^p \rightarrow \Xi$. The role of the non-productive silent states is to provide fine-grained independencies in the Markov chain, allowing us to "spill out" the transitions between productive states conceptually and computationally efficiently.

Given expert predictions $P_{\xi,t}$ for all $\xi \in \Xi$ and $i = 1, 2, \ldots$, the joint probability is given as follows. Let $q^t$ be a sequence of states, and let $q^t_1, \ldots, q^t_i$ be the subsequence of its productive
states. Then
\[
Q(q^t, ξ^t, x^t) :=
\begin{cases}
P_C(x^t) \prod_{i=0}^{λ-1} P(q_i \rightarrow q_{i+1}) & \text{if } \forall i : Λ(q_i) = ξ_i, \\
0 & \text{otherwise.}
\end{cases}
\] (5)

For convenience, we identify the EHMM distribution with its defining 5-tuple, i.e. \(Q \equiv \langle Q, Q', 0, P, Λ\rangle\).

The advantage of our setup is that we can specify EHMMs using intuitive state transition diagrams, as done for example in Figures 3 and 4. We first draw a node \(N_q\) for each state \(q\). We use an open dot • for the start state, black dots • for the silent states, and we display each productive state \(q\) as an open circle labelled by the expert \(Λ(q)\) who is assigned to make the prediction, like this: \(A\). We draw an arrow from \(N_q\) to \(N_{q'}\) if the transition probability \(P(q \rightarrow q')\) is nonzero. The transition probabilities themselves could be written along such arrows, but this quickly becomes messy. Instead we write them below our graphs.

The forward algorithm, see e.g. [27], can be used to compute the predictive distribution on experts given past data, i.e.

\[Q(ξ^t+1|x^t)\].

Intuitively, this is done by maintaining weights on the productive states. These weights are then alternately used for prediction and subsequently conditioned on observations, and percolated forward through the network of silent states according to the transition probabilities \(P\).

The total running time of the forward algorithm for all time steps is proportional to the number of edges in the graph.

C. Examples

We give the ES-priors and EHMMs that correspond to the simplest models: Bayesian mixtures and elementwise mixtures with fixed parameters.

Example 1 (Bayesian Mixtures): The EHMM \(B_w\) for the Bayesian mixture is shown in Figure 3. The figure illustrates that in the standard Bayesian mixture there is no provision for a-priori redistribution of probability mass between experts: the posterior distribution will concentrate very quickly on the state corresponding to the expert that assigns the overall highest likelihood to the data. The vector-style definition of \(P\) is a shorthand; each row specifies one or more transition probabilities. The reader may check that this EHMM formally corresponds to the Bayesian mixture in the sense that the marginal likelihoods on data of (5) and (1) coincide. It is well-known that the Bayesian prediction \(P_{\text{EM}}(x_{t+1}|x^t)\) can be computed in \(O(k)\) time per trial by maintaining the posterior. The forward algorithm on \(B_w\) has the same efficiency. □

Example 2 (Elementwise Mixtures): The elementwise mixture \(\text{EM}_w\) with mixture weights \(w\) on experts \(Ξ\) predicts as follows:

\[Q_t(x_t) := \sum_{ξ ∈ Ξ} P_ξ(x_t)w(ξ).\]

This prediction strategy for elementwise mixtures can be implemented by the EHMM \(\text{EM}_w\) defined in Figure 4. The EHMM has a single silent state per outcome, whose transition probabilities are the mixture weights \(w\). Intuitively, funneling all weight through a single silent state all differentiation between the experts based on past performance is forgotten. As such no learning occurs, the strength of this model instead lies in the fact that the model always assigns reasonable probability to any outcome that is likely according to any of the experts. The forward algorithm calculates the prediction of \(\text{EM}_w\) in \(O(k)\) time per trial. □

III. REGRET BOUNDS

Here we provide some handles for analysing the predictive performance of EHMMs. In each case, the idea is to compare the loss incurred by some model \(Q\) to the loss incurred by another prediction strategy from a set \(M\) of reference strategies. For example, \(M\) might be the set of all prediction strategies based on a fixed expert sequence that contains \(m\) blocks, or it might be the set of all prediction strategies that

\[\frac{1}{m} \sum_{m} \text{loss}(Q)\]
can be obtained by mixing over the predictions of the experts with fixed weights. The goal is now to provide an upper bound on the excess codelength of the model Q compared to the best of these reference strategies. Note that for many models we provide simultaneous regret guarantees with respect to several distinct reference classes; e.g. the regret bounds for expert tracking algorithms hold for all numbers of blocks \( m \).

Throughout this paper, we use three types of regret bounds, which are given in order of increasing sophistication. The first bound is appropriate if only a few expert sequences contribute significantly to the probability of the data. In that case it is sufficient to simply drop some terms from the Bayesian mixture.

**Lemma 1 (Regret w.r.t. Expert Sequence \( \xi^t \))**: Let Q denote an EHMM, and let \( \xi^t \) denote a particular reference expert sequence. Then, for all data \( x^t \),

\[
\ln \frac{P_{\xi^t}(x^t)}{Q(x^t)} \leq -\ln Q(\xi^t).
\]

**Proof**: The bound is obtained by dropping all terms in the mixture except for the one corresponding to \( \xi^t \). \( \square \)

We obtain an expression for the regret w.r.t. some reference set \( \mathcal{M} \subseteq \Xi^t \) by maximising \( \xi^t \) over its elements. We have already seen an example application: the regret bound \( 3 \) for \( Q = B_w \) is derived in this way.

In the second kind of bound, another EHMM \( R \) plays the role of reference prediction strategy. It can be useful even if the number of different expert sequences with significant contribution to the probability is very large. The following lemma forms the basis for such bounds.

**Lemma 2 (Regret w.r.t. another EHMM)**: Fix data \( x^t \) and EHMMs \( Q \) and \( R \). We have

\[
\ln \frac{R(x^t)}{Q(x^t)} \leq -\ln E_{\nu} \left[ \frac{Q(\xi^t)}{R(\xi^t)} \right] \leq E_{\nu} \left[ \ln \frac{R(\xi^t)}{Q(\xi^t)} \right],
\]

where \( \nu(\xi^t) = R(\xi^t|x^t) \).

**Proof**: Rewrite

\[
\frac{Q(x^t)}{R(x^t)} \geq \sum_{\xi^t:R(x^t,\xi^t) > 0} \frac{R(x^t,\xi^t)}{Q(x^t,\xi^t)} \frac{Q(x^t,\xi^t)}{R(x^t,\xi^t)} = E_{\nu} \left[ \frac{Q(x^t,\xi^t)}{R(x^t,\xi^t)} \right]
\]

\[
= E_{\nu} \left[ \frac{Q(\xi^t)}{R(\xi^t)} \right],
\]

take the \(-\ln\) and subsequently apply Jensen’s inequality. \( \square \)

Although this bound still involves the actual data through the distribution \( \nu \), sometimes the expectation can be replaced by a maximum over \( \xi^t \). This may be sufficiently sharp for the job at hand if a good uniform bound is available for the ES-priors. However, it is possible to say more about the regret if \( Q \) and \( R \) share a certain structure.

The next and final lemma applies to EHMMs in which some of the transition probabilities are a function of a parameter vector \( \beta \). Intuitively, models with fewer parameters have more constrained transition dynamics and are hence less expressive, making it easier to compete with the maximum likelihood parameter values.

The lemma is a generalisation of Theorems 1 and 3 in \( \textcircled{23} \). Theorem 1 is concerned with the special case of Fixed Share, where the transition matrix is parameterised by the switching rate \( \alpha \). Theorem 3 applies to unconstrained Markov transition dynamics (with \( | \Xi |^2 \) parameters). The lemma below yields sharper results for restricted transition dynamics that can be expressed in exponential family form. We will apply this result to reobtain Monteleoni & Jaakkola’s sophisticated regret bound \( \textcircled{24} \) of Fixed Share (in Theorem 5) and then use it for the kernel models of Sections IV-F and IV-G (Theorem 16), for which the added generality is essential.

The parameterisation of the EHMM should be of the following form. Let \( T_\beta(j) = e^{\beta^T \phi(j)/h(j)} / Z(\beta) \) be an exponential family of distributions with parameter vector \( \beta \), some sufficient statistic \( \phi \), carrier \( h \) and normalisation \( Z(\beta) = \sum_j e^{\beta^T \phi(j)} / h(j), \) where \( j \) takes values in a finite set \( \mathcal{J} \). Let \( \mathcal{Q} \subseteq \mathcal{Q} \) be a subset of the state space for which the transition probabilities are governed by this exponential family model \( T_\beta \) in the sense that there is an injective function \( S : \mathcal{Q}^1 \times \mathcal{J} \rightarrow \mathcal{Q} \) that indicates for each such state how the symbols in \( \mathcal{J} \) map to the successor states:

\[
P(q \rightarrow S(q,j)) = T_\beta(j).
\]

Intuitively, the more often such parameterised transitions are traversed, the larger the difference between two distinct values for \( \beta \). Therefore we need to count the number of transitions from states in \( \mathcal{Q}^1 \). Define the random variable \( q(t) : q(\infty) := q^\lambda \) where \( \lambda \) is chosen such that \( q_i \) is the \( i \)th productive state of \( q^\infty \). We assume throughout that \( q(t) \) is well-defined almost surely for all \( t \). Further define \( n_j(q(t)) := |\{ i | 0 \leq i < \lambda, q_i \in \mathcal{Q}^1, S(q_i,j) = q_{i+1}\}|, \) where \( \lambda \) is the length of \( q(t) \), so that \( n_j(q(t)) \) is the number of transitions in \( q(t) \) between states in \( \mathcal{Q}^1 \) and their \( j \)-successors, and let \( n(q(t)) = \sum_j n_j(q(t)) \) be the total parameterised transition count. We can now state our result:

**Lemma 3 (Regret w.r.t. ML Parameter \( \hat{\beta} \))**: Let \( Q_\beta \) be parameterised as defined above. Fix outcomes \( x^t \) and let \( \hat{\beta} = \arg \max_{\beta} Q_\beta(x^t) \) with the assumption that \( Q_\beta(x^t) > 0 \). Furthermore let \( \mathcal{W}(q(t)) = Q_\beta(q(t)|x^t) \) denote the posterior distribution of \( q(t) \) under prior \( Q_\beta \). We then have

\[
\ln \frac{Q_\beta(x^t)}{Q_\beta(x^t)} \leq -\ln E_w \left[ \frac{Q_\beta(q(t))}{Q_\beta(q(t))} \right] \leq E_w \left[ \ln \frac{Q_\beta(q(t))}{Q_\beta(q(t))} \right]
\]

\[
= E_w [n(q(t)) D(T_\beta || T_\beta)],
\]

where \( D \) is the Kullback-Leibler divergence.

As before, the goal of this lemma is to say something about the overhead incurred by using a particular strategy \( Q_\beta \) instead of the reference strategy \( Q_{\hat{\beta}} \). The result still depends on the data via the distribution \( w \), but in applications of the lemma the idea will be to replace \( E_w [n(q(t)) \] by a bound on the number of states in \( \mathcal{Q}^1 \) that may be traversed by relevant state sequences.

\( \textcircled{2} \) Theorem 1 also appears in \( \textcircled{23} \). Although both theorems are valid, Theorem 3 does not generalise Theorem 1; there is a problem with Lemma 3.3.1 in the cited work.
As an important special case, the distribution on $\mathcal{J}$ may be fully specified by a multinomial distribution with parameter vector $w$; we can then apply the lemma above by setting $\phi(j) = (0, \ldots, 1, 0, \ldots)$, with the 1 appearing at the $j^{th}$ position, and $h(j) = 1$. Instantiated in this way the Lemma expresses a bound with respect to all possible distributions on the successor state, much like Theorem 3 of [25]. However, if we choose a more constrained exponential family, the reference strategy $T_\beta$ has fewer degrees of freedom and the divergence $D(T_\beta \| T_\beta)$ that appears in the bound is reduced. This can be used to obtain Theorem 1 of [25] (which also appears here as Theorem 3, we also use it to prove Theorem 16 below.

Proof of Lemma 3. The first two inequalities are Lemma 2 on the level of state sequences. The contribution of this lemma lies in the last equality. First expand

$$E_w \left[ \ln \frac{Q_\beta(q^{(t)})}{Q_\beta(q^{(t-1)})} \right] = E_w \left[ \sum_{j \in \mathcal{J}} n_j(q^{(t)}) \ln \frac{T_\beta(j)}{T_\beta(j)} \right]$$

$$= \sum_j E_w[n_j(q^{(t)})] \left( (\hat{\beta} - \beta)^T \phi(j) + \ln \frac{Z(\beta)}{Z(\beta)} \right)$$

$$= (\hat{\beta} - \beta)^T \sum_j \phi(j) E_w[n_j(q^{(t)})] + E_w[n(q^{(t)})] \ln \frac{Z(\beta)}{Z(\beta)}. \quad (7)$$

Since $\hat{\beta}$ maximises the probability of $Q_\beta$ and since $\nabla_\beta \ln Q_\beta(x^t, q^{(t)}) = \nabla_\beta \ln (Q_\beta(q^{(t)})/Q_\beta(q^{(t)}))$ we obtain

$$\hat{\beta} = \arg\max_{\beta} \frac{Q_\beta(x^t)}{Q_\beta(x^t)} = \arg\max_{\beta} \left. \frac{-\sum_{q^{(t)}} \nabla_\beta Q_\beta(x^t, q^{(t)})}{Q_\beta(x^t)} \right|_{\beta = \hat{\beta}}$$

$$= \arg\max_{\beta} \left. \frac{\nabla_\beta \ln Q_\beta(x^t, q^{(t)})}{Q_\beta(x^t)} \right|_{\beta = \hat{\beta}}$$

$$= \nabla_\beta E_w \left[ \ln \frac{Q_\beta(q^{(t)})}{Q_\beta(q^{(t-1)})} \right]_{\beta = \hat{\beta}}. \quad (8)$$

This shows that the vector differential of (7) must be zero at $\hat{\beta}$. Reordering terms we obtain

$$\sum_j \phi(j) E_w[n_j(q^{(t)})] = E_w[n(q^{(t)})] \nabla_\beta \ln \frac{Z(\beta)}{Z(\beta)} = E_w[n(q^{(t)})] \sum_{j \sim T_\beta} \phi(j) \quad (8)$$

where the last step follows from

$$\nabla_\beta \ln Z(\beta) = \frac{\nabla_\beta Z(\beta)}{Z(\beta)} = \sum_{j \in \mathcal{J}} \left. e^{\beta^T \phi(j)} h(j)/Z(\beta) \right|_{\beta = \hat{\beta}} \phi(j) = E_{j \sim T_\beta} \phi(j).$$

Using (8) we may now simplify (7) to

$$E_w \left[ \ln \frac{Q_\beta(q^{(t)})}{Q_\beta(q^{(t-1)})} \right]$$

$$= E_w \left[ n(q^{(t)}) \left( (\hat{\beta} - \beta)^T \sum_{j \sim T_\beta} \phi(j) + \ln \frac{Z(\beta)}{Z(\beta)} \right) \right]$$

$$= E_w \left[ n(q^{(t)}) \left( D(T_\beta \| T_\beta) \right) \right],$$

completing the proof.

We now apply the lemma to our two running examples. Note that in both cases, the model is parameterised such that the total number of parameterised transitions is known.

Example 3 (Bayesian Mixture Regret): We have already obtained the bound (3) for $E_w$ using Lemma 1 but it is instructive to do the same using Lemma 3. Let $Q^t$ contain just the initial silent state and identify the experts $\Xi$ with $\mathcal{J}$. We now have $E_w[n(q^{(t)})] = 1$, so the lemma tells us that our regret is $D(\hat{w} \| w)$, where $\hat{w}$ is the hindsight optimal prior weight vector that maximises the probability of the available data, and $w$ is the prior we actually use. Now observe that in order to maximise probability, $\hat{w}$ must assign all mass to a single expert $\xi$, so $D(\hat{w} \| w) = -\ln w(\xi)$ as before.

Example 4 (Elementwise Mixture Regret): We now compute the regret of $E_m(w)$. Let $Q^t$ contain the silent states and again identify the experts $\Xi$ with $\mathcal{J}$. For elementwise mixtures, $E_w[n(q^{(t)})] = t$. So by Lemma 3, the regret of predicting the outcomes $x^t$ with an elementwise mixture with weights $w$ instead of the hindsight optimal mixture weights $\hat{w}$ is bounded by $t D(\hat{w} \| w)$.

IV. Switching Strategies

A. Fixed Share

Mark Herbster and Manfred Warmuth’s paper on tracking the best expert [26], [6] is the first to consider the scenario where the best predicting expert may change over time. They compare the loss of their algorithm to the smallest loss that can be achieved by splitting the data of size $t$ into $m$ segments, and within each segment, copying the predictions of the expert who in hindsight turns out to be best for that particular segment. They give two algorithms called Fixed Share and Variable Share, but the motivation for the second algorithm applies only to loss functions other than log-loss (see Section V-A), so we focus on Fixed Share, which matches the EHMM $F S_{w, \alpha}$ defined in Figure 5. Note that all arcs into the silent states have fixed probability $\alpha \in [0, 1]$ and all arcs from the silent states have some fixed distribution $w$ on $\Xi$ (the original algorithm uses a uniform $w(\xi) = 1/k$). The same algorithm is also described as an instance of the Aggregating Algorithm in [22].

Fixed Share reduces to fixed elementwise mixtures by setting $\alpha = 1$ and to Bayesian mixtures by setting $\alpha = 0$. Each productive state represents that a particular expert is used at a certain sample size. Once a transition to a silent state is made, all expert history is forgotten and a new expert is chosen according to $w$.

We now bound the regret of Fixed Share with respect to a given partition, i.e. sequence of experts.
Theorem 4 (Herbster and Warmuth [6]): Fix experts $\Xi$ and data $x^t$, and let $\xi^t$ be a sequence of experts with $m$ blocks (i.e., $|\{1 \leq i < t \mid \xi^i \neq \xi^i_{i+1}\}| = m-1$), $k = |\Xi|$; and $w(\xi) = 1/k$. Let $\alpha^* = (m-1)/(t-1)$ denote the switching frequency in $\xi^t$. Write $H(\alpha^*, \alpha) = -\alpha^* \ln \alpha - (1 - \alpha^*) \ln(1 - \alpha)$ for the cross entropy. Then

$$\ln \frac{P_{\xi t}(x^t)}{P_{\xi t}(x^t)} \leq m \ln k + (t-1) H(\alpha^*, \alpha). \quad (9)$$

Proof: Let $q^{(t)}$ be the state sequence that produces $\xi^t$ and that passes through silent state $(i)$ iff $\xi_i \neq \xi_{i+1}$. Then

$$\frac{FS_{w, \alpha}(x^t)}{P_{\xi t}(x^t)} \geq FS_{w, \alpha}(q^{(t)}) = k^{-m} (1 - \alpha)^{t-m-1}. \quad (10)$$

Taking logarithms and substituting $\alpha^*$ completes the proof. ■

Note that in Herbster and Warmuth’s algorithm, switches to the same expert are disallowed, allowing our models to keep the exposition clean and simple, but this can be changed in the same way that this is done in the Fixed Share paper.

While $\alpha^*$ optimises the bound, it does not necessarily maximise the probability of the data. We therefore also calculate the regret of $FS_{w, \alpha}$ with respect to the set of Fixed Share algorithms for all $\alpha \in [0, 1]$.

Theorem 5 (Monteleoni and Jaakkola [24]): For all data $x^t$ and switching rate $\alpha \in [0, 1]$,

$$\ln \frac{FS_{w, \alpha}(x^t)}{FS_{w, \alpha}(x^t)} \leq (t-1) D(\hat{\alpha} || \alpha),$$

where $\hat{\alpha}$ maximises the Fixed Share likelihood.

Proof: Apply Lemma 3 setting $Q^\dagger$ to $Q^\dagger$, the set of all productive states, whose outgoing transitions are parameterised by the switching rate $\alpha$. ■

Judging from this theorem and from [9], the regret appears to grow linearly with time, but if we substitute the switching rate $\alpha = \alpha^*$ that optimises the bound, cross entropy reduces to ordinary entropy, and we find that if $m$ is kept fixed, the regret only has a logarithmic dependence on $t$: we have

$$(m-1) \ln \frac{t-1}{m-1} \leq (t-1) H(\alpha^*) \leq (m-1) \ln \frac{t-1}{m-1} + m \quad (10)$$

where $H(\alpha^*) = H(\alpha^*, \alpha^*)$ is the binary entropy. The problem is that such asymptotics can only be achieved if we are somehow able to guess the optimal switching rate before observing the data. This issue is addressed in the following sections. We will evaluate the performance of the other models for expert tracking using the loss of Fixed Share with $\alpha = \alpha^*$ as a baseline.

B. Intermezzo: Interpolation

Note how Fixed Share (Figure 5) interpolates between the Bayesian Mixture (Figure 3) and the Elementwise Mixture (Figure 4). The parameter $\alpha$ determines when switches occur. If no switch occurs then the Bayesian Mixture’s transitions are used: all experts’ weights remain unchanged. On the other hand, if a switch occurs then the Elementwise Mixture’s transitions are used: all experts’ weights are gathered and redistributed. Fixed Share can thus be interpreted as an algorithm that interpolates between the Bayesian and Elementwise mixtures.

In this section we first give an intuitive high-level definition of interpolation, and then follow it up with a detailed definition that carefully manipulates silent states for the sake of efficiency.

Interpolations are natural to the switching domain. In [18], Bernoulli HMMs are used to produce switching rates; these are really just interpolators as defined below. In similar fashion, interpolation allows us to lift pretty much any algorithm for predicting binary data, such as [13], [14], to the context of prediction with expert advice.

Interpolations are useful as a tool to build models in a modular fashion. For example, the Bayesian Mixture model (Figure 3) can be interpreted as a method for learning which expert is best; the Fixed Share algorithm augments this model by introducing the possibility to reset the weights, so that the Bayesian learning process starts anew. By reinterpreting Fixed Share as an interpolation between the Bayesian Mixture and the Elementwise Mixture, as we do below, it becomes possible to replace the Bayesian component of Fixed Share with any other EHMM. Useful examples include EHMMs that are designed to learn elementwise mixture coefficients. This endows the model with the option to reset its state, effectively restarting the learning process. Such possibilities are explored in detail in [28].

Interpolation works as follows. We start with two EHMMs $Q_n$ and $Q_s$ on state space $Q$. We interpret $Q_n(q_{i+1} | q_i)$ as a specification of the state evolution under normal ($n$) behaviour, while we regard $Q_s(q_{i+1} | q_i)$ as a codification of what happens upon a switch ($s$). The decision whether switches are taken is left to a third EHMM $C$, now on state space $C$, with node labels $\Sigma$ in $\{n, s\}$ that we use to select which evolution is desired at each time step. The resulting interpolation $R = Q_n \otimes_C Q_s$ is displayed in Figure 6 as a Bayesian network on variables $c_i$, $\sigma_i$, $q_i$, $\xi_i$ and $x_i$ for each time $i = 1, 2, \ldots$ (Note
that $C$ differs from regular EHMMs in that it does not have a data layer (the $x_i$ variables), and the produced symbols are from $\{n,s\}$ rather than the fixed expert set $\Xi$.) We will now define the interpolation distribution on these variables. Most conditional distributions are copied from the input EHMMs: as before $\xi_i = \Lambda(q_i), \sigma_i = \Sigma(c_i)$, the state evolution for the selection process $R(c_{i+1}|c_i)$ is copied from $C$, and $R(x_i|\xi_i)$ denotes the prediction of expert $\xi_i$ for the $i$th outcome. As indicated, the switch decisions $\sigma_i = \Sigma(c_i)$ are made between rounds, with $\sigma_i = s$ indicating that a switch occurs between time $i$ and $i+1$. In the interpolation the probability of a state $q_{i+1}$ now depends not only on the previous state $q_i$, but also on the switch decision $\sigma_i$, which determines which of the original two dynamics is selected:

$$R(q_{i+1}|q_i, \sigma_i) := Q_{\sigma_i}(q_{i+1}|q_i).$$

In addition, we have to define which of the two dynamics is used initially. We arbitrarily choose to define the model to start with the switching dynamics, and set $R(q_1) := Q_s(q_1)$. We note that the interpolation $R$ is again a prediction strategy of the form shown in Figure 2 now with joint state space $Q \times C$.

Interpolation separates concerns; $C$, on the highest level, determines when to switch. Below, $Q_n$ and $Q_s$ determine the normal and switching behaviour. This separation is reflected in the following modular loss bound:

**Lemma 6 (Interpolation Decomposition):** Fix interpolation $R = Q_n \otimes_C Q_s$. For each sequence $\sigma^{t-1} \in \{n,s\}^{t-1}$ of switch decisions (on the $C$ level) and each sequence $q^t \in Q^t$ of productive states (on the $Q$ level)

$$R(q^t) \geq C(\sigma^{t-1}) Q_s(q_1) \prod_{i=1}^{t-1} Q_{\sigma_i}(q_{i+1}|q_i).$$

**Proof:** By dropping terms from the marginal, we find $R(q^t) \geq R(\sigma^{t-1}) R(q^t|\sigma^{t-1})$. By the definition of interpolation, we have $R(\sigma^{t-1}) = C(\sigma^{t-1})$ and $R(q^t|\sigma^{t-1}) = Q_s(q_1) \prod_{i=1}^{t-1} Q_{\sigma_i}(q_{i+1}|q_i)$.

**Corollary 7 (Default Interpolation Regret):** We apply this lemma to our Q-level EHMMs of interest, $Q_n = B_w$ and $Q_s = EM_w$ where $w$ is uniform on $k$ experts. Fix $\xi^t$. Set $\sigma_i = s$ iff $\xi_{i+1} \neq \xi_i$, and let $m$ be the number of blocks in $\xi^t$, i.e. the number of $s$ in $\sigma^{t-1}$ plus one. Then for all data $x^t$

$$\ln \frac{P_{\xi^t}(x^t)}{R(x^t)} \leq - \ln C(\sigma^{t-1}) + m \ln k.$$
of the state, as indicated by the bit. First it evolves the first state component \((q \in Q)\) to the next productive state in \(Q\) using either \(P^n\) or \(P^s\) as determined by the produced label \(\Sigma(c) \in \{n, s\}\). Then it forwards the second state component \((c \in C)\) to the next productive state using \(P\). That is
\[
P^\otimes (q, c, 0) \rightarrow (q, c', 0)
\]
Finally, the node label is that of the first component
\[
\Lambda^\otimes (q, c, 0) := \Lambda(q).
\]
Figure 7b shows the state transition diagram of an interpolation, with the interpolator shown in Figure 7a.

As mentioned before, the predictions of an EHMM can be computed with constant work per edge, where an edge is defined as a pair of states \(q, q'\) with non-zero transition probability \(P(q \rightarrow q') > 0\). We now bound the number of edges of an interpolation.

**Lemma 9:** Let \(e^c, e^s, e^n\) and \(e^r\) be the numbers of edges in EHMMs \(C, Q_n, Q_s\) and the interpolation \(R = Q_n \otimes C \otimes Q_s\). Then
\[
e^r \leq |C| \max \{e^c, e^s\} + |Q|^n \cdot e^n
\]
**Proof:** From the definition of \(P^\otimes\).

This theorem provides an upper bound, for not all these edges may be reachable from the start state. A careful counting for Fixed Share yields that between the reachable productive states at time \(t\) and \(t + 1\) sit \(c|\Sigma\) edges, for some constant \(c\) independent of \(t\). The running time of the interpolation version is hence no worse than that of the classical version.

This concludes the intermezzo. In the remainder of this section, we discuss the benefits and costs of several choices for \(c\), both in terms of loss bound and in terms of running time. We also briefly discuss alternatives for \(Q_n\) and \(Q_s\).

### C. Decreasing Switching Rate

Fixed Share uses a fixed switching rate \(\alpha\). However it is possible to get good bounds without having to choose \(\alpha\), by letting the switching probability decrease as a function of time. This trick was employed in the source coding setting in [15].

This trick was employed in the source coding setting in [15].

**Theorem 10:** Let \(\alpha_i = 1 - \exp(-c/t_i)\) for some \(c > 0\). Let \(w\) be the uniform distribution on the set \(\Xi\) of \(k\) experts. For any data \(x^t\) and expert sequence \(\xi^t\) with \(m\) blocks
\[
\ln \frac{P(x^t)}{\text{DSR}_{w,\alpha^\otimes}(x^t)} \geq m \ln k + c - (m - 1) \ln c + (m - 1 + c) \ln(t - 1).
\]
**Proof:** By (11), using \(\sum_{i=1}^t \frac{1}{t} < \ln t + 1\) and \(e^r \geq x^t + 1\),
\[
- \ln \text{DSR}_{w,\alpha^\otimes}(x^{t-1}) = c \leq \sum_{i=1}^{t-1} \frac{1}{t} - \sum_{j=2}^m \ln(e^c/t_j) - 1
\]
\[
\leq c \ln(t - 1) + c - (m - 1) \ln c + \sum_{j=2}^m \ln t_j.
\]
The sum is bounded by substituting each \(t_j\) by \(t - 1\), and the result follows by Corollary 7.

Note that while we succeeded in eliminating the parameter \(\alpha\), we have in fact introduced a new parameter \(c\), so it would appear that matters have not improved much. But unlike \(\alpha\), a suboptimal value for \(c\) only yields a regret penalty of order \(\ln t\), so it may safely be set to some convenient constant like \(c = 1\). The optimising value is \(c^* = (m - 1)/(1 + \ln(t - 1))\), which yields slightly better asymptotics, but this defeats the purpose as it would require a priori knowledge of \(m\) and \(t\) again.

We now compare the regret bound (12) to the bound (9) for Fixed Share. To maximise the difference, we use the optimising parameter \(\alpha^*\) for Fixed Share, and we lower bound the entropy using (10). The difference is
\[
c - (m - 1) \ln c + c \ln(t - 1) + (m - 1) \ln(m - 1),
\]
where the last two terms dictate asymptotic behaviour. Which of these terms is dominant depends on how quickly \(m\) grows as
a function of $t$. If there are relatively few switches, $m \ln m = o(\ln t)$, then the $c \ln(t - 1)$ term dominates, so it pays to use a small value for $c$ to get good asymptotics in that case. If, on the other hand, the number of switches is large, then the last term is larger, and it may be substantial; careful judgement is then required to decide whether or not this is an acceptable price to pay or that a more sophisticated method for learning the switching rate (Section V-D) is preferable.

2) Switching with More Quickly Decreasing Probability:
In some settings the optimal number of switches between experts may remain bounded. A natural example is Bayes factors model selection, where the considered experts are Bayesian prediction strategies associated with model classes of varying complexity; at small sample sizes, simple model classes typically yield the best predictions (as their parameter estimates are quicker to converge to their optimal values), but if one of the more complex model classes contains the data generating distribution, then that model class eventually produces the best predictions. From that point in time onwards, no more switches away from that model class are required.

In such a scenario, a simple Bayesian mixture of the experts with uniform prior yields a regret bound of $\ln k$ w.r.t. the ultimately best expert (see (3), which depends on the number of experts but not on the sample size. Asymptotically, this is therefore a better solution than the one presented in the previous section, where even if there are no switches at all ($m = 1$), the incurred regret bound of $\ln k + c \ln(t - 1)$ grows without bound due to the first term of (13). This happens because the ES-prior $\text{DSR}_{w,\omega}$ assigns zero probability to the event that no more switches occur from some time $t$ onwards. The problem with the Bayesian mixture, as apparent in (3), is that it cannot take advantage of the superior performance of the simpler models at small sample sizes. As shown in [19], this results in a suboptimal rate of convergence in the nonparametric case for the Bayesian mixture: its overhead compared to a switching model (such as the one from the previous section) can be arbitrarily large!

To achieve the best of both worlds, we must tweak the model from the previous section somewhat: while we still assign positive prior probability to the occurrence of switches, we also ensure that the probability that no more switches occur from any given time onwards is strictly positive. This section describes a simplification of the “Switch Distribution”[19] proposed in [19] for which the results of that paper still hold. In brief, this model achieves the optimal rate of risk convergence when used for sequential prediction, but at the same time, it defines a consistent model selection criterion (it selects the model containing the true distribution with probability 1 as sufficient data become available).

**Theorem 11:** Let $\alpha_i = 1 - e^{-\tau(t)}$ for some $c > 0$ and a decreasing probability mass function $\tau$ on the positive integers. Let $w$ be the uniform distribution on the set of $k$ experts. For any data $x^t$ and expert sequence $\xi^t$ with $m$ blocks

$$\ln \frac{p_{\xi^t}(x^t)}{\text{DSR}_{w,\omega}(x^t)} \leq m \ln k + c - (m - 1) \ln c - (m - 1) \ln \tau(t_m).$$

**Proof:** Using (11), $\sum_i \tau(i) = 1$ and $e^x \geq x + 1$,

$$- \ln \text{DSR}_{w,\omega}(\sigma^{t-1}) = c \sum_{i=1}^{t-1} \tau(i) - \sum_{j=2}^{m} \ln(e^{c \tau(t_j)} - 1) \leq c - (m - 1) \ln c - \sum_{j=2}^{m} \ln \tau(t_j).$$

For decreasing $\tau$, we obtain an upper bound by substituting $t_i = t_{m}$ for $1 \leq i < t_m$, and the theorem follows from Corollary [19].

A desirable feature of this bound is that it is expressed in terms of the index $t_m$ of the last switch rather than in terms

Incidentally, the switch distribution is a tracking model whose interpolator has the structure depicted in Figure 7. The idea is that with every switch, there is a certain fixed probability of “stabilisation”, meaning that the interpolator enters a special “band” of states where further switching is impossible.
of the time $t$, as we obtained in Section [V-C1]. The role of $c$ is even weaker than before, as there is no $c \ln t$ penalty term; its optimal value is now $c^* = m - 1$, but in practice $c = 1$ would be a sensible value. On the flip side, a $\ln t$ cost per switch as in [12] can no longer be guaranteed. A convenient fat-tailed prior that comes very close is

$$\tau(t) = \frac{1}{\ln(t + e - 1)} = \frac{1}{\ln(t + e)},$$

(14)

which satisfies $-\ln \tau(t) \leq \ln(t) + 2 \ln \ln(t + e) + e/t$.

D. Learning the Switching Rate

1) The Switching Method: In a very early publication [12], Volf and Willems describe an algorithm called the switching method, which is very similar to Herbster and Warmuth’s Fixed Share, except that it is able to learn the optimal switching rate $\alpha$ on-line. A similar method was developed independently in [22]. Here we describe the Switching Method as an interpolation and bound its regret. Whereas Fixed Share interpolates using a fixed Bernoulli[\alpha] distribution, the switching method “integrates out” the parameter using Jeffreys’ prior (which is Beta[$\frac{1}{2}$, $\frac{1}{2}$]).

The switching method EHHM is defined as the interpolation

$$\text{SM}_w \ := \ \text{BH}_w \otimes_{\text{SM}} \text{EM}_w,$$

with the interpolator $\text{SM}$ defined in Figure 8. Each productive state $\langle n_t, n_s, \sigma \rangle$ represents the fact that after observation $n_t + n_s + 1$ a switch occurs ($\sigma = 8$) or not ($\sigma = n$), while there have been $n_s$ switches in the past.

We now bound the regret of the switching method with respect to Fixed Share with any switching rate $\alpha$ (in particular the maximum likelihood rate $\hat{\alpha}$), and thereby show that it is universal for the Fixed Share model class $\{\text{FS}_{w,\alpha} \mid \alpha \in [0, 1]\}$.

**Theorem 12 (The Switching Method Regret):** For any switching rate $\alpha$ and data $x^t$

$$\ln \frac{\text{FS}_{w,\alpha}(x^t)}{\text{SM}_w(x^t)} \leq \ln 2 + \frac{1}{2} \ln t.$$

**Proof:** Fixed Share and the switching method interpolate the same EHHMs, so we have the following information processing inequalities (c.f. Lemma 2)

$$\max_{\xi^t} \frac{\text{FS}_{w,\alpha}(\xi^t)}{\text{SM}_w(\xi^t)} \leq \max_{\xi^t} \frac{\text{FS}_{w,\alpha}(\xi^t)}{\text{SM}_w(\xi^t)} \leq \max_{\sigma^t} \frac{\text{FS}_{w,\alpha}(\sigma^t)}{\text{SM}_w(\sigma^t)} \leq \max_{\sigma^t} \frac{\text{FS}_{w,\alpha}(\sigma^t)}{\text{SM}_w(\sigma^t)}.$$

Thus we may transfer regret bounds from the interpolator level via the expert-sequence level to the data level. The rightmost term is the worst-case regret for the Bernoulli model with Jeffreys prior, which can be bounded (see e.g. [13]) by $2 + \frac{1}{2} \ln t$ for all $\alpha$.

By the previous theorem and the Fixed Share regret bound Theorem 4 we obtain for all $\xi^t$ with switching frequency $\alpha^*$

$$\ln \frac{P(\xi^t)}{\text{SM}_w(\xi^t)} \leq m \ln k + (t - 1) H(\alpha^*) + \ln 2 + \frac{1}{2} \ln t.$$

The switching method was independently derived in [29], where this last inequality is also proved. Our theorem is slightly sharper, as it bounds the regret w.r.t. the actual maximum-likelihood Fixed Share performance instead of its regret bound.

2) Improving Time Efficiency for Learning the Switching Rate: The new ingredient of the switching method compared to Fixed Share is that the EHHM includes a switch count in each state. This allows us to adapt the switching probability to the data, but it also renders the number of states quadratic. The quadratic running time $O(k^2 t^2)$ restricts its use to moderately sized data sets. The approach taken by Monteleoni and Jaakkola [24] is to place a discrete prior on the switching rate $\alpha$: the prior mass is distributed over $\sqrt{t}$ well-chosen points, where the ultimate sample size $t$ is assumed known. This way they still achieve the bound of Theorem 12 up to a constant, while reducing the running time to $O(k^2 \sqrt{t})$.

This approach has two disadvantages of its own: first, the ultimate sample size $t$ has to be known in advance, which means that the presented algorithm is only quasi-online. Second, the discretisation of the prior is obtained by a numeric optimisation procedure, which means that both the number and the locations of the discretisation points are not known in closed form. As a consequence, the resulting regret bound can only be determined up to $O(1)$. In [18] a simple explicit discretisation scheme is presented which allows the regret bound to be calculated exactly. Furthermore, it is shown how, at the cost of a somewhat worse regret bound, this discretisation scheme can be refined online such that $t$ no longer has to be known in advance.

E. The Run-length Model for Clustered Switching

Run-length codes have been used extensively in the context of data compression, see e.g. [30]. The corresponding probability distributions are known in the statistical literature as renewal processes, see [31].
Rather than applying run-length codes directly to the observations, we use them as interpolators, as they constitute good models for the distances between consecutive switches.

The run-length model is especially useful if the switches are clustered, in the sense that some parts of the expert sequence contain relatively few switches, while other parts contain many. The Fixed Share algorithm remains oblivious to such properties, as its interpolator is a Bernoulli model: the probability of switching remains the same, regardless of the index of the previous switch. Essentially the same limitation also applies to the switching method, whose switching probability normally converges as the sample size increases. The decreasing \( \alpha \) models perform well while the switches are clustered toward the beginning of the sample, but depending on the application this may be unrealistic and may introduce a new unnecessary loss overhead.

The run-length model, which is related to Willems’ “linear complexity coding method” from [13] and its subsequent refinement in [15], models the intervals between successive switches as independently distributed according to some distribution \( \tau \). After the switching method and decreasing \( \alpha \) models, this is a third generalisation of the Fixed Share algorithm, which is recovered by taking a geometric distribution for \( \tau \): the interpolation then becomes memoryless and reduces to the interpolator of Fixed Share.

Let \( \tau \) be a distribution on \( \mathbb{Z}_+ \cup \{ \infty \} \), which is used to model the lengths of the blocks. We assume \( \tau(\infty) > 0 \); this allows our regret constant when the reference number of switches is bounded while the number of samples goes to infinity. The run-length interpolator \( RL_\tau \) is defined in Figure 9. Intuitively, the state \( (t, \delta) \) means that we are at time \( t \), and that sample \( t + 1 \) will be the \( \delta \)th sample since the last switch. The EHMM for the run-length model is given by the interpolation

\[
RL_{w,\tau} := \text{ER}_{w,\tau}^\dagger \text{EM}_{w}.
\]

As may be read from the diagram of the interpolator, we require quadratic running time \( O(k t^2) \) to evaluate the run-length model in general.

**Theorem 13 (Run-length Model Regret):** Let \( w \) be the uniform distribution on \( k \) experts. Assume there is a log-convex function \( \vartheta \) on \( [1, \infty) \) that agrees with \( \tau \) on \( \mathbb{Z}_+ \). With abuse of notation, we identify \( \tau \) with \( \vartheta \). Then, for all data \( x^t \) and expert sequences \( \xi^t \) with \( m \) blocks, we have

\[
\ln \frac{P_{\xi^t}(x^t)}{RL_{w,\tau}(x^t)} \leq m \ln k - \ln \tau(\infty) - (m-1) \ln \tau\left(\frac{t_m}{m-1}\right).
\]

**Proof:** Fix a switch sequence \( \sigma^{t-1} \) with \( m-1 \) occurrences of \( s \) at positions \( t_2, \ldots, t_m \), and let \( t_1 = 0 \). For \( j = 1, \ldots, m-1 \), let \( \delta_j = t_{j+1} - t_j \) denote the length of block \( j \). From the definition of the interpolator above, we obtain

\[
-\ln RL_{\vartheta}(\sigma^{t-1}) = -\ln \tau(z \geq t - t_m) - \sum_{j=1}^{m-1} \ln \tau(\delta_j)
\]

\[
\leq -\ln \tau(\infty) - \sum_{j=1}^{m-1} \ln \tau(\delta_j).
\]

Since \( -\ln \tau \) is concave, by Jensen’s inequality we have

\[
\sum_{j=1}^{m-1} \ln \tau(\delta_j) \leq -\ln \left( \frac{\sum_{j=1}^{m-1} \delta_j}{m-1} \right) = -\ln \left( \frac{t_m}{m-1} \right).
\]

In other words, the block lengths \( \delta_j \) are all equal in the worst case. Combining this with Corollary 7 we obtain the result.

We have seen that the run-length model reduces to Fixed Share if the prior on switch distances \( \tau \) is geometric, so it can be evaluated in linear time in that case. The geometric prior is not the only one for which the complexity can be reduced; for example, the negative binomial distribution can also be implemented efficiently, as well as any \( \tau \) with finite support. However, such priors must have exponentially small tails; for priors \( \tau \) with thick tails, which are desirable in our worst-case analysis, one may use the fully general EHMM as depicted in Figure 9.

To compare the performance of the run-length model to the bound (9) for Fixed Share, assume \( t_m = t - 1 \) and define \( \tau \) as in (14). The bound (15) becomes

\[
m \ln k - \ln \tau(\infty) + (m-1) \ln \left( \frac{t - 1}{m-1} \right) + 2(m-1) \ln \left( \frac{t - 1}{m-1} + e \right) + (m-1)e.
\]

To maximise the difference, we use the optimising parameter \( \alpha^\ast \) for Fixed Share, and we bound the entropy from below using (10). The gap between the bounds is then given by

\[
-\ln \tau(\infty) + 2(m-1) \ln \left( \frac{t - 1}{m-1} + e \right) + (m-1)e.
\]

At this modest price, the run-length model does not require tuning any parameters, its regret depends on \( t_m \) instead of \( t \).
F. Ordered Experts

In the models discussed so far, once a switch occurs, it is equally easy to switch to any of the available experts, as \( Q_3 \) prescribes uniform redistribution of the probability mass. This approach is reasonable if we do not know anything about the relationship between the experts; furthermore it has the advantage that percolating probabilities through \( Q_4 \) requires only \( O(k) \) operations, while we would need \( O(k^2) \) operations to support arbitrary transition probabilities between the experts. In this section we consider an interesting alternative that both makes intuitive sense and allows for efficient computation.

Assume that the experts can be sensibly organised using a line or ring topology, with the interpretation that switches between two experts are more likely if they are close together on this structure than if they are far apart. An example is given by Vovk [22] who considers a polynomial regression problem with one expert to represent the polynomials of each degree. In this case it is clear that, typically, the optimal degree increases gradually as more observations are gathered, so switching from degree 10 to degree 11 is more likely than, say, switching instead to degree 1,000.

We will simplify matters further by postulating that the probability of a switch between any pair of experts who are \( \delta \) apart is the same. Furthermore, for simplicity of exposition we identify the experts with the integers, \( \Xi = \mathbb{Z} \). (In practice it is of course not possible to work with an infinite set of experts, but this can be resolved by simply changing the forward algorithm to drop all probability mass that at any time becomes negligible, while the regret bound deteriorates, also by a logarithmic factor. This approach was later refined to obtain more general time complexity / regret tradeoffs in [15, 16, 52, 53].

\[
F \text{ Ordered Experts}
\]

\[
Q = Q^0 \cup Q^r \quad Q^r = \mathbb{Z} \times \mathbb{Z}_+ \quad \text{mod } 2
\]

Theorem 13 shows how the complexity of the algorithm can be improved at a cost of a slightly worse regret bound. They force a switch at carefully chosen states in the run-length EHMM (Figure 9); all subsequent states in the same row become unreachable and hence can be pruned from the model. Using this scheme the number of states reachable in the model at time \( t \) (and therefore the complexity per time step) can be reduced to \( \log_2 t \), while the regret bound deteriorates, also by a logarithmic factor. This approach was later refined to obtain more general time complexity / regret tradeoffs in [15, 16, 52, 53].
discretisation of a continuous parameter space). However, in this section we will bound the regret in terms of the total amount of drift in $\xi^t$:  
$$ d = \sum_{i=1}^{t} |\delta_i|, \text{ where } \delta_1 = \xi_1 \text{ and } \delta_i = \xi_i - \xi_{i-1} \text{ for } 1 < i \leq t, $$
which can be viewed as the length of the path described by $\xi^t$.

As an example, one may consider the switching model proposed by Monteleoni and Jaakkola (see Section IV-D2). They essentially instantiate a number of Fixed Share models, for various values of the switching rate $\alpha$. These Fixed Share instances are prediction strategies, and can therefore be interpreted as experts themselves. However, it seems reasonable to assume that in many cases the optimal switching rate $\alpha$ might be subject to drift: it might vary somewhat as time progresses. Therefore it may be beneficial to combine these “Fixed Share experts” using a model that can represent parameter drift. The resulting loss can be bounded in terms of the amount of drift that occurs in the reference sequence of switching parameters. For parameter drift we no longer use an interpolation, as in previous sections, because switches no longer have special status. Instead, shifts between experts are possible at each time step, through convolution with the following kernel, parameterised by $0 < \alpha < 1$:

$$ \kappa_\alpha(\delta) := \alpha^{|\delta|} \frac{1-\alpha}{1+\alpha}. \tag{17} $$

This kernel can be implemented with the EHMM $\text{KERNEL}^\prime_{\kappa_\alpha}$, from the previous section, but as it turns out it is possible to represent the same kernel using a different EHMM $\text{PD}_\alpha$, defined in Figure 10, that uses silent states to reduce the number of edges, allowing the convolution to be carried out in time proportional to the number of experts considered. Linear time convolution is possible because the kernel is a memoryless distribution conditional on the sign of the drift. This sign information is represented by the distinction between two columns of silent states for each time step.

**Theorem 15 (Parameter Drift Regret):** Fix any data $x^t$ and reference sequence $\xi^t$ with total drift $d$. Let $H(P,Q) = -\sum_x P(x) \ln Q(x)$ denote the cross entropy. Then

$$ \frac{\ln P_{\xi^t}(x^t)}{\text{PD}_\alpha(x^t)} \leq t H(\kappa_\alpha^*, \kappa_\alpha) = -t \ln \frac{1-\alpha}{1+\alpha} - d \ln \alpha, $$

where $\alpha^* = \arg\max_{\alpha} \text{PD}_\alpha(\xi^t) = \sqrt{1 + (t/d)^2} - (t/d)$.

**Proof:** By Lemma 1 the left-hand side is bounded above by $-\ln \text{PD}_\alpha(\xi^t)$. Since $\{\kappa_\alpha\}$ is an exponential family with unit carrier (see e.g. Proposition 19.11),

$$ -\ln \text{PD}_\alpha(\xi^t) = -\ln \prod_{i=1}^{t} \kappa_\alpha(\delta_i) = t E_{\kappa_\alpha^*} [-\ln \kappa_\alpha(\delta)] = t H(\kappa_\alpha^*, \kappa_\alpha). $$

The equality of the Theorem further follows from

$$ \text{PD}_\alpha(\xi^t) = \prod_{i=1}^{t} \kappa_\alpha(\delta_i) = \alpha^d \left( \frac{1-\alpha}{1+\alpha} \right)^t. $$

![Figure 10 Parameter drift: PD_α](image)

The parameter $\alpha^*$ that maximises the likelihood of $\xi^t$ is found by equating the derivative to zero.

We can be somewhat more precise about how much it can hurt performance to use a suboptimal parameter $\alpha$. The following theorem, which bounds the regret with respect to the optimal parameter-drift model, is an analogue of Theorem 5 for Fixed Share. The theorem applies to a wide class of kernel EHMMs, but in particular it holds for the parameter-drift model $\text{PD}_\alpha$, for which the transition dynamics are governed by the one-dimensional exponential family (17). It is a strong result that uses the full generality of Lemma 3.

**Theorem 16 (Kernel ML Regret):** Fix $x^t$ and let $\tilde{\beta} = \arg\max_{\beta} \text{KERNEL}_{\kappa_\beta}(x^t)$ for some exponential family $\{\kappa_\beta\}$. We have

$$ \frac{\ln \text{KERNEL}_{\kappa_{\tilde{\beta}}}(x^t)}{\ln \text{KERNEL}_{\kappa_\beta}(x^t)} \leq d(\kappa_{\tilde{\beta}} \parallel \kappa_\beta). $$

**Proof:** Since the transition probabilities associated with each productive state (i.e. the kernel $\kappa_\beta$) are an exponential family distribution, we can apply Lemma 3 with $Q^t$ equal to the set of all productive states.
Instantiating this theorem for parameter drift we obtain:

**Corollary 17 (Parameter Drift ML Regret):** Fix $x^t$ and let $\hat{\alpha} = \arg\max_{\alpha} P_{\alpha}(x^t)$. We have

\[
\ln \frac{P_{\alpha}(x^t)}{\mathcal{D}_\alpha(x^t)} \leq t \left( \frac{2\hat{\alpha} \ln \left( \frac{\hat{\alpha}}{\alpha} \right)}{(1-\hat{\alpha})(1+\alpha)} + \ln \left( \frac{1+\alpha}{(1-\hat{\alpha})(1+\hat{\alpha})} \right) \right).
\]

The parameter drift model as discussed so far shares both the elegance of the Fixed Share algorithm and its awkward dependence on a parameter $\alpha$. However, most of the techniques to avoid specifying $\alpha$ that were discussed in previous sections can be adapted to the parameter drift model. In particular, we can compete with the maximum likelihood drift parameter $\hat{\alpha}$ using a discretisation scheme akin to [24], [18], such that the discretisation point $\alpha$ closest to $\hat{\alpha}$ reduces the right hand side of Corollary 17 to a uniformly bounded constant. As before, this is possible using $O(\sqrt{t})$ discretisation points, leading to a $O(t\sqrt{t})$ total running time. We omit the details.

Moreover we can adapt the trick we used in Section IV-C and let the kernel parameter $\alpha$ decrease with time. This yields a linear run time, at the cost of deteriorating the bound for large drifts.

**Theorem 18 (Decreasing Drift Regret):** Let $\mathcal{D}$ denote the ES-joint based on the parameter drift model with time-dependent kernel $\kappa_{\alpha_t}$ with $\alpha_t = 1/(i+1)$. For any data $x^t$ and reference sequence $\xi^t$ with total drift $d$, we have

\[
\ln \frac{P_{\xi^t}(x^t)}{\mathcal{D}(x^t)} \leq (d+2) \ln(t+1).
\]

**Proof:** We first expand

\[
\mathcal{D}(\xi^t) = \prod_{i=1}^{t} \kappa_{\alpha_t}(\delta_i) = \prod_{i=1}^{t} \alpha_t^{\delta_i} \frac{1-\alpha_t}{1+\alpha_t}
\]

\[
= \prod_{i=1}^{t} (i+1)^{\delta_i} \frac{2}{(t+1)(t+2)} \prod_{i=1}^{t} (i+1)^{-\delta_i}.
\]

For fixed total drift $d$, it is clear that this probability is minimised by $|\delta_i| = 0$ for $1 \leq i < t$ and $|\delta_t| = d$. Therefore

\[
\mathcal{D}(\xi^t) \geq \frac{2}{(t+1)(t+2)}(t+1)^{-d} \geq (t+1)^{-d-2}.
\]

We now take the $-\ln$ and apply Lemma 1 to complete the proof.

V. EXTENSIONS

In this section we describe a number of extensions to the framework described above. In Section V-A we discuss a number of tracking algorithms for which different, potentially useful performance guarantees can be given. Then in Section V-B we discuss adaptive regret, a criterion for evaluating performance more locally. In Section V-C we try to find out which expert was best at a particular time step, and finally in Sections V-D and V-E and we indicate how our approach can be generalised to work with any mixable loss function and how it can be applied to online investment.

A. OTHER APPROACHES TO SWITCHING

The models described in the present paper can be described in the Bayesian framework using prior distributions on sequences of experts. The priors we presented so far did not depend on any contextual information, such as the outcomes, or any other external information. Below, we will list three other important models for expert tracking whose EHMM transition probabilities depend on the past losses of the experts. We are not aware of any models in which the transition probabilities depend on properties of the observed data other than the losses; this is an interesting area for future research.

In all three cases it is straightforward to find counter-examples that show that these algorithms cannot be represented as EHMMs with fixed transition probabilities.

The first such algorithm, Variable Share, was introduced together with Fixed Share in [6]. It is useful for loss functions like square loss, that are not only mixable, but also bounded (also see Section V-D). For this setting, which is outside the scope of this paper, a regret of $O(m \ln k + m \ln \ln(L/m))$ can be established, where $m$ and $k$ are the number of blocks and the number of experts as usual, but $L$ is the loss of the best reference strategy using $m$ blocks. Thus, if the data can be predicted well by partitioning it into blocks and using a fixed expert within each block, then the overhead of the algorithm is small. In the log loss setting however, the algorithm incurs infinite regret in the worst case, so no useful guarantees can be provided.

The other two approaches do in fact work in the log loss setting. The first is known as “Mixing Past Posteriors” [37]. Like Fixed Share, this algorithm allows for efficient tracking of the best predicting expert; but unlike Fixed Share, it is especially efficient for sparse problems, where the predictions of only a few out of the full set of experts ever need to be used. For Mixing Past Posteriors, a regret of $O(u \ln k + m \ln t)$ can be achieved, where $u$ is the number of blocks that occur in the comparator sequence, and $t$, $k$, and $m$ are as usual. This is beneficial if $k$ is large and $u$ is small. Whereas the original algorithm is rather eclectic, a proper Bayesian interpretation (making use of specialists) and a slight improvement of the bound can be found in [38].

The last result combines two experts in such a way that the regret is controlled in terms of the fluctuations in the cumulative loss difference of the two experts as a function of time. The idea is that if the fluctuations are large, the regret is relatively high, but in that case you also gain a lot from switching between the experts in the first place. The paper [39] is phrased in terms of investment policies, but the setting is equivalent to ours. In financial terms, the bound expresses that you have a large overhead only when you are making a lot of money anyway!

B. ADAPTIVE REGRET

Also of interest is the notion of adaptive regret proposed by Hazan and Seshadhri in [32]. The adaptive regret of an algorithm on a given interval is the difference between the loss of the algorithm on that interval and that of the best expert for that interval. The new goal is then to design algorithms with
low worst-case adaptive regret on all intervals. An algorithm with low adaptive regret will automatically have low tracking regret, the tracking bound is obtained simply by summing the adaptive bound over all blocks of a segmentation of the data; the converse is not always possible. It is proved in \cite{40} that Fixed Share, and its generalisations with time-varying switching rates (as e.g. in Section IV-C) are optimal algorithms for adaptive regret: no other algorithm can guarantee lower adaptive regret on all intervals. Moreover in that paper’s forthcoming journal version \cite{41} it is shown that the worst-case adaptive regret of any algorithm is dominated by that of such a generalised Fixed Share. As such, whereas the local perspective taken by adaptive regret allows giving stronger performance guarantees for Fixed Share, it cannot capture the global benefit of modelling the switching dynamics, as expressed e.g. by the bounds for run-length and the Switching Method.

C. Expert Estimation

We focused on EHMM models for sequential prediction. However, EHMMs may also be used for batch data analysis. Below we indicate how to obtain the best regularised expert at hand. This turns out to be possible for Fixed Share, and obtained efficiently by exploiting the structure of the EHMM. Running time explodes. Still, the MAP $\xi$ is clear from the motivating definition (4).

The Viterbi algorithm \cite{27} is used to compute the most likely sequence of states $q(t)$ that contributes most to it:

$$\arg \max_{q(t)} Q(x^t, q(t)) = \arg \max_{q(t)} Q(x^t | q(t)) Q(q(t)).$$

The Viterbi algorithm \cite{27} is used to compute the most likely sequence of states for HMMs. It can be easily adapted to handle silent states. However, we may also write

$$Q(x^t) = \sum_{\xi^t} Q(x^t, \xi^t),$$

and wonder about the sequence of experts $\xi^t$ that contributes most. This problem is harder because several states can produce the same expert simultaneously; in other words, a single sequence of experts can be generated by many different sequences of states. So we cannot use the Viterbi algorithm as it is. The Viterbi algorithm can be extended to compute the MAP expert sequence for general EHMMs, but the resulting running time explodes. Still, the MAP $\xi$ can be sometimes be obtained efficiently by exploiting the structure of the EHMM at hand. This turns out to be possible for Fixed Share, and also for the more sophisticated Switch Distribution mentioned in Section IV-C. The algorithm for the latter is given in \cite{11}.

As an alternative way to gain insight, one may run the forward and backward algorithms to compute $Q(x^t, \xi^t)$ and $Q(x^t | q^t, x^t)$. Recall that $q^t$ is the productive state that is used at time $i$. From these we can compute the a posteriori probability $Q(q^t | x^t)$ of each productive state $q^t$. That is, the posterior probability taking all the available data into account (including observations that were made later than time $i$). This is a standard way to analyse data in the HMM literature, see e.g. \cite{27}. We can then project the posterior on $\text{states}$ down to obtain the posterior probability $Q(\xi^t | x^t)$ of each expert $\xi^t \in \Xi$ at each time $i = 1, \ldots, t$. This gives us a sequence of mixture weights over the experts that we can, for example, plot on a $\Xi \times t$ grid. On the one hand this gives us a mixture over experts for each time instance, obviously a richer representation than just single experts. On the other hand we lose the temporal correlations that can be important in MAP calculation, as each time instance is treated separately.

D. Mixable Loss Functions

We presented log-loss regret bounds for experts that sequentially produce probability distributions on the next outcome. Not all prediction tasks are in this form, for example, we may be asked to make a point prediction based on real-valued expert advice and be scored using quadratic loss. Fortunately, several loss functions are mixable \cite{7, 23}, in that for each mixture of predictions, there is a single prediction whose loss is always less than the exponentiated average loss. Mixable losses include log loss, quadratic loss, Hellinger loss and entropic loss. 0/1 loss and absolute loss are not mixable.

Prediction strategies that are obtained by running the forward algorithm on any EHMM can be adapted to mixable losses straightforwardly, by preprocessing the input to and post-processing the output of the forward algorithm for sequential prediction. On the input side, expert predictions are transformed into probabilities. On the output side, the posterior distribution on the next expert is transformed (using the mixability condition) into a single prediction. The resulting prediction strategy has the same mixable-loss regret bound as the original prediction strategy (although possibly expressed in different units). The details of the general reduction can be found in \cite{28}. In the special case of online investment an even tighter correspondence holds, as outlined in the next section.

E. Online Investment

As it happens, all algorithms for prediction with expert advice discussed in this paper can also be used as strategies for online investment. The key observation is that the weights on the experts issued by any considered prediction algorithm and the resulting codelength only depend on the losses incurred by the experts, not on any other aspect of their behaviour. This is clear from the motivating definition (4).

To predict on the stock market, we start again from (4), but replace expert $\xi$’s data likelihood $P_{\xi,s}(x_s)$ with the multiplication factor $r_{\xi,s}$ incurred by stock $\xi$ in trading round $s$. Each round, the investment algorithm uses the same expression to compute the posterior distribution on the next stock. It then divides its capital among the stock according to this distribution. In the prediction algorithm this posterior was used instead to mix the predictions of the experts to form its own predictive distribution; for the investment strategy this last step has been abstracted away.
Note that while the expert data likelihoods are proper probability distributions, the multiplication factors \( r_{\xi,t} \) may be larger than one; this does not cause problems since the posterior weights are renormalised by Bayes’ rule.

The returns obtained by the investment strategy are given by the formula for the data likelihood with the above substitution. As such, all regret guarantees derived in this paper carry over unmodified when using these algorithms as investment strategies.

For further discussion of the relation between investment and prediction algorithms see [44, Chapter 10].

**Cover’s Portfolio Selection:** Perhaps the best known link between information theory and finance is provided by Cover’s seminal results on portfolio selection [42, 43]. These algorithms fit exactly in the formalism described in this paper: they can be obtained by applying the reduction described above to specific EHMM models. The simplest such model was introduced in Example 2, where each round, the predictions of the experts are mixed using a fixed weight vector. Applying the reduction to finance, we recover the constant rebalanced portfolio strategy. It is also possible to obtain Cover’s universal portfolios by using a more sophisticated EHMM that learns the optimal weight mixtures. For the case of two experts, this EHMM has already been defined, albeit for a different purpose: the EHMM depicted in Figure 8 was previously used as an interpolator, for learning the switching rate in an expert tracking strategy. However, when applied in its own right it learns the optimal elementwise mixture weights for combining the predictions of two experts labelled “s” and “n”.

The construction of Figure 8 can be augmented to more than two experts, but the state space quickly grows large: for \( k \) experts, the number of states in round \( t \) is \( 4^{k-1} \). As such, the algorithm will process \( t \) outcomes in \( O(t^k) \) time, matching the complexity of Cover’s algorithm. Interestingly, the methods discussed in Section IV-D2 for reducing the time complexity of the Switching Method carry over to learning mixtures, allowing an easy speedup to \( O(t^{k+1/2}) \).

Substantial advances have been made in making Cover’s universal portfolio selection practical for large numbers of stocks (by imposing some assumptions); these fall outside the scope of this paper. For more information see [44].

**VI. Conclusion**

We generalise the concept of universal coding for some model class \( \mathcal{M} \), by comparing the performance of the universal code not just to the performance of the codes in \( \mathcal{M} \), but also to other reference classes.

We evaluate performance in terms of individual sequence regret, and make no distributional assumptions. We summarise and unify existing algorithms from two domains: information-theoretic literature about universal coding on the one hand and universal prediction (also known as “prediction with expert advice”) from learning theory. Thanks to the well-known equivalence between prefix coding and probability theory the algorithms and techniques of this paper can immediately be applied in both settings.

We present all models in Bayesian form using prior distributions on expert sequences (ES-priors). The (infinitely long) expert sequence defines which expert is used at which time. Prediction then amounts to “integrating out” those experts in the sequence that are used at other time steps than the one predicted. The challenge is to identify those models that provide good tradeoffs between predictive performance and time complexity.

Throughout the paper, hidden Markov models (HMMs) are used to specify ES-priors, since their explicit representation of the current state and state-to-state evolution naturally fit the temporal correlations we seek to model. For reasons of efficiency we use HMMs with silent states. The standard algorithms for HMMs (Forward, Backward, Viterbi and Baum-Welch) can be used to answer questions about the ES-prior as well as the induced distribution on data. The running time of the forward algorithm can be read off directly from the graphical representation of the HMM.

This approach allows a unified presentation of many existing expert models. We focus on models for tracking the best expert, where the loss incurred by a prediction strategy is compared to the loss incurred if the data are optimally divided into \( m \) blocks, and the best expert is used within each block. The discrepancy (“regret”) is then bounded in terms of variables such as the current time \( t \), the number of experts \( k \), and the number of blocks \( m \). In each case, we recover both the regret bound and the running time known from the literature.

We not only succinctly summarise and contrast many key algorithms from the literature, but also describe a number of new models. In particular the model with quickly decreasing probability of switching (Section IV-C), and the models that assume the experts are ordered (Section IV-F) are new and computationally efficient, and have competitive regret bounds.

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