NONSTANDARD SOLUTIONS OF THE YANG-BAXTER EQUATION

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Abstract. Explicit solutions of the quantum Yang-Baxter equation are given corresponding to the non-unitary solutions of the classical Yang-Baxter equation for \( \mathfrak{sl}(5) \).

1. Introduction

Etingof and Kazhdan recently proved that any finite dimensional Lie bialgebra \( g \) may be quantized [3]. That is, there exists a topological Hopf algebra structure on \( U(\mathfrak{g})[[\hbar]] \) such that the Lie bialgebra structure on \( g \) is the one induced on \( g \) by passing to the “semi-classical limit”. From this they deduced a general procedure for quantizing solutions of the classical Yang-Baxter equation (CYBE). Thus, at least in theory, one can construct solutions of the quantum Yang-Baxter equation from given solutions of the classical Yang-Baxter equation. Unfortunately, their procedure is not easy to implement explicitly, even in small dimensional situations.

In this note we exhibit an explicit answer to this problem for a particularly interesting family of Lie bialgebra structures on \( \mathfrak{sl}(5) \). These are the bialgebra structures associated to non-unitary solutions of the CYBE (or equivalently of the modified classical Yang-Baxter equation (MCYBE)) as classified by Belavin and Drinfeld [1]. For each such solution of the CYBE we construct an \( R \)-matrix using the Gerstenhaber-Giaquinto-Schack (GGS) conjecture [4]. The YBE was verified in each case using Mathematica.

The GGS conjecture concerns the form of the quantization of such solutions of the CYBE in the case of \( \mathfrak{sl}(n) \). The case of \( \mathfrak{sl}(5) \) is to some extent the first interesting case. For \( \mathfrak{sl}(2) \) there are no solutions of the MCYBE except the standard one. For \( \mathfrak{sl}(3) \) the only non-standard solution is that associated to the well-known Cremmer-Gervais quantization and for \( \mathfrak{sl}(4) \) the nonstandard solutions are essentially of three types, the Cremmer-Gervais solution and two other fairly simple examples. The corresponding \( R \)-matrices for the latter two types can be constructed using other techniques [5]. On the other hand for \( \mathfrak{sl}(5) \) there are 13 different types of solutions to the MCYBE and for many of these the corresponding \( R \)-matrix was hitherto unknown. The validity of the GGS conjecture for \( \mathfrak{sl}(5) \) gives strong evidence that the conjecture should be true for all \( n \).

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2. Solutions to the CYBE and quantization

2.1. The Belavin-Drinfeld description of solutions to the CYBE. Let \( g \) be a complex simple Lie algebra and let \( h \) be a Cartan subalgebra. Let \( \Delta \) be the associated root system and \( \Gamma \) a set of simple roots. A classical \( r \)-matrix over \( g \) is an element \( r \in g \otimes g \) satisfying the classical Yang-Baxter equation

\[
[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0
\]

Take an invariant bilinear form on \( g \) and let \( t \in g \otimes g \) be the associated Casimir element. In \( [1] \) Belavin and Drinfeld gave the following description of solutions of the CYBE which satisfy \( r_{12} + r_{21} = t \). These are the “non-unitary” solutions.

Let \( \Gamma_1, \Gamma_2 \) be two subsets of \( \Gamma \) and let \( \tau : \Gamma_1 \to \Gamma_2 \) be a bijection satisfying

1. \( (\tau \alpha, \tau \beta) = (\alpha, \beta) \) for all \( \alpha, \beta \in \Gamma \);
2. For every \( \alpha \in \Gamma_1 \), there is a \( k \geq 0 \) with \( \tau^k \alpha \in \Gamma_1 \) but \( \tau^{k+1} \alpha \notin \Gamma_1 \).

The data \( (\tau, \Gamma_1, \Gamma_2) \) (or more concisely just \( \tau \)) is often called a Belavin-Drinfeld triple. Given such a triple \( \tau \), an element \( r^0 \in h \otimes h \) is called \( \tau \)-admissible if

1. \( r^0_{12} + r^0_{21} = t^0 \)
2. \( (\tau \alpha \otimes 1)r^0 + (1 \otimes \tau \alpha)r^0 = r^0 \)

where \( t^0 \) is the component of \( t \) in \( h \otimes h \). A \( \tau \)-admissible \( r^0 \) is necessarily of the form \( t^0/2 + \bar{r}^0 \) where \( \bar{r}^0 \in h \wedge h \). The set of all \( \bar{r}^0 \) forms a linear subvariety of \( h \wedge h \) of dimension \( \binom{\Delta}{2} \) where \( d = \#(\Gamma - \Gamma_1) \).

Now \( \tau \) can be extended to an isomorphism of Lie subalgebras \( \tau: g_1 \to g_2 \) where \( g_i \) is the Lie subalgebra of \( g \) associated to \( \Gamma_i \). Choose \( e_\alpha \in g_\alpha \) such that \( (e_\alpha, e_{-\alpha}) = 1 \) and \( \tau(e_\alpha) = e_{\tau \alpha} \) and define an ordering on \( \Delta \) by \( \alpha \prec \beta \) if \( \tau^k \alpha = \beta \) for some positive integer \( k \). View \( g \otimes g \) as a subset of \( g \otimes g \) via the identification \( x \wedge y = 1/2(x \otimes y - y \otimes x) \). Then Belavin and Drinfeld showed \( [1] \)

\[
r = r^0 + \sum_{\alpha > 0} e_{-\alpha} \otimes e_\alpha + \sum_{\alpha, \beta > 0, \alpha < \beta} e_{-\alpha} \wedge e_\beta
\]

is a solution of the Yang-Baxter equation satisfying \( r_{12} + r_{21} = t \) and that every such solution is of this form for some choice of \( h \), \( \Gamma \), \( \tau \) and \( r^0 \).

For any \( g \) there is the “trivial” triple which has \( \Gamma_1 = \Gamma_2 = \emptyset \) and \( \bar{r}^0 \in h \wedge h \) arbitrary. A particularly interesting triple for \( sl(n) \) is the “Cremmer-Gervais” triple which has \( \Gamma_1 = \{2\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\} \), \( \Gamma_2 = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-2}\} \), and \( \tau(\alpha_i) = \alpha_{i-1} \). In contrast to the trivial triple, there is a unique admissible \( \bar{r}^0 \) for the Cremmer-Gervais triple.

2.2. The Gerstenhaber-Giaquinto-Schack conjecture. The Gerstenhaber-Giaquinto-Schack conjecture is a conjectured form for the quantization of the above classical \( r \)-matrices in the case where \( g = sl(n) \), considered as a subset of \( M_n(\mathbb{C}) \). In this setting, a quantization of a classical \( r \)-matrix is an \( R \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \) which has semi-classical limit \( r \) and satisfies the quantum Yang-Baxter equation

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.
\]

Take the form to be the trace form \( (x, y) = \text{Tr}(xy) \) and let \( h \) be the Cartan subalgebra consisting of diagonal matrices of trace zero. The standard Cartan-Weyl basis is then \( e_{\alpha_i} = e_{i, i+1} \), \( e_{-\alpha_i} = e_{i+1, i} \) and \( h_{\alpha_i} = [e_{\alpha_i}, e_{-\alpha_i}] = e_{ii} - e_{i+1, i+1} \). Let
\( \tau \) be a Belavin-Drinfeld triple as described above and let \( r^0 \in h \otimes h \) be \( \tau \)-admissible. Set
\[
a = \sum_{\alpha, \beta > 0} e_{-\alpha} \wedge e_\beta
\]
and
\[
c_+ = \sum_{\alpha > 0} e_{-\alpha} \otimes e_\alpha, \quad c = \sum_{\alpha > 0} e_{-\alpha} \wedge e_\alpha.
\]
Set \( \epsilon = -(ac + ca + a^2) \). Now define \( \tilde{a} \) by
\[
\tilde{a} = \sum_{i,j,k,l} a_{ij}^{ik} q^{a_{ij}^{ik}} e_{ij} \otimes e_{kl}
\]
where \( a = \sum a_{ij}^{kl} e_{ij} \otimes e_{kl} \) and similarly for \( \epsilon \). Set \( \tilde{q} = q - q^{-1} \). The standard \( R \)-matrix is then
\[
R_s = q^{\tilde{q}^2 + 1/n} + \tilde{q} c_+ = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + \tilde{q} \sum_{i > j} e_{ij} \otimes e_{ji}.
\]
It is easy to check that \( R_s \) satisfies the quantum Yang-Baxter equation and that \( PR_s \) satisfies the Hecke relation \( (PR_s - q)(PR_s + q^{-1}) = 0 \) where \( P \) is the permutation matrix.

**Gerstenhaber-Giaquinto-Schack Conjecture.** Let \( \tau \) be a Belavin-Drinfeld triple for \( \mathfrak{sl}(n) \) and suppose \( r^0 = t^0/2 + r^0 \) is \( \tau \)-admissible. Then the matrix
\[
R = q^{\tilde{q}^2} (R_s + \tilde{q} \tilde{a}) q^{\tilde{q}^2}
\]
satisfies the quantum Yang-Baxter equation and \( PR \) satisfies the Hecke relation.

Taking \( \tau \) to be the trivial triple yields the standard \( R \)-matrix when \( r^0 = t^0/2 \) and the standard multiparameter \( R \)-matrices when \( r^0 \) is arbitrary. For use later, let \( R(r^0) = q^{\tilde{q}^2} (R_s) q^{\tilde{q}^2} \) denote the standard multiparameter \( R \)-matrices. As is well known, if \( r^0 = t^0/2 + \sum_{i < j} e_{ii} \otimes e_{jj} \) then
\[
R(r^0) = q^{\tilde{q}^2} (R_s) q^{\tilde{q}^2} = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i < j} (q^{e_{ii}} e_{ii} \otimes e_{jj} + q^{-e_{ii}} e_{jj} \otimes e_{ii}) + \tilde{q} \sum_{i > j} e_{ij} \otimes e_{ji}.
\]
For the Cremmer-Gervais triples described above the formula gives the Cremmer-Gervais \( R \)-matrices.

### 2.3. The GGS conjecture for \( \mathfrak{sl}(5) \)

We now consider the explicit form of the \( R \)-matrices associated to the Belavin-Drinfeld triples on \( \mathfrak{sl}(5) \). According to the GGS Conjecture, each \( R \) is of the form \( R(r^0) + \tilde{q} q^{\tilde{q}^2} a q^{\tilde{q}^2} \) for an admissible \( r^0 \). The specific form of \( R(r^0) \) has already been exhibited. The other summand, \( q^{\tilde{q}^2} a q^{\tilde{q}^2} \), is always a sum of "quantized" wedge products. Specifically, for positive roots \( \alpha \) and \( \beta \) and any constant \( c \), set \( e_{-\alpha} \wedge c e_\beta = q^{-c} e_{-\alpha} \otimes e_\beta - q^c e_\beta \otimes e_{-\alpha} \). For all triples, the term \( q^{\tilde{q}^2} a q^{\tilde{q}^2} \) is always of the form \( \sum_{\alpha, \beta > 0} e_{-\alpha} \wedge c(\alpha, \beta) e_\beta \) where the constants \( c(\alpha, \beta) \) are determined by \( r^0 \) and \( \epsilon \).

Denote by \( \mathcal{T} \) the set of triples on \( \mathfrak{sl}(5) \). Notice that if \( (\tau, \Gamma_1, \Gamma_2) \) is a triple, then \( (\tau^{-1}, \Gamma_2, \Gamma_1) \) is also a triple. Also the graph automorphism of \( A_4 \) induces a bijection on the set of triples. Since these two involutions of \( \mathcal{T} \) commute, this gives an action of the group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) on \( \mathcal{T} \).
Proposition 2.1. The Gerstenhaber-Giaquinto-Schack conjecture is true for \( n = 5 \). The triples below comprise a complete set of representatives from the 13 orbits under the action of \( C_2 \times C_2 \) on \( T \). For each triple the generic admissible \( r^0 \) and the Hecke \( R \)-matrix produced by the GGS conjecture are also explicitly given.

1. \(|\Gamma_1| = 3\)

   (a) The “Cremmer-Gervais” triple: \( \Gamma_1 = \{\alpha_2, \alpha_3, \alpha_4\} \), \( \Gamma_2 = \{\alpha_1, \alpha_2, \alpha_3\} \), \( \tau(\alpha_i) = \alpha_{i-1} \).

   \[
   r^0 = t^0/2 + \frac{1}{5}(-3h_{\alpha_1} \wedge h_{\alpha_2} - 4h_{\alpha_1} \wedge h_{\alpha_3} - 3h_{\alpha_1} \wedge h_{\alpha_4} \\
   - 4h_{\alpha_2} \wedge h_{\alpha_3} - 4h_{\alpha_2} \wedge h_{\alpha_4} - 3h_{\alpha_3} \wedge h_{\alpha_4})
   \]

   \[
   R = R(r^0) + \hat{q}(e_{54} \wedge_2 e_{34} + e_{54} \wedge_4 e_{23} + e_{54} \wedge_6 e_{12} + e_{43} \wedge_2 e_{23} \\
   + e_{43} \wedge_4 e_{12} + e_{32} \wedge_2 e_{12} + e_{53} \wedge_2 e_{24} + e_{53} \wedge_4 e_{13} \\
   + e_{42} \wedge_2 e_{13} + e_{52} \wedge_2 e_{14})
   \]

   (b) The “generalized Cremmer-Gervais” triple: \( \Gamma_1 = \{\alpha_1, \alpha_3, \alpha_4\} \), \( \Gamma_2 = \{\alpha_1, \alpha_2, \alpha_4\} \), \( \tau(\alpha_i) = \alpha_j \), where \( j \equiv i + 3 \pmod{5} \).

   \[
   r^0 = t^0/2 + \frac{1}{5}(h_{\alpha_1} \wedge h_{\alpha_2} - 2h_{\alpha_1} \wedge h_{\alpha_3} + h_{\alpha_1} \wedge h_{\alpha_4} \\
   - 2h_{\alpha_2} \wedge h_{\alpha_3} - 2h_{\alpha_2} \wedge h_{\alpha_4} + h_{\alpha_3} \wedge h_{\alpha_4})
   \]

   \[
   R = R(r^0) + \hat{q}(e_{54} \wedge_2 e_{23} + e_{21} \wedge_4 e_{23} + e_{43} \wedge_6 e_{23} + e_{21} \wedge_2 e_{45} \\
   + e_{43} \wedge_4 e_{45} + e_{43} \wedge_2 e_{12} + e_{53} \wedge_2 e_{13})
   \]

2. \(|\Gamma_1| = 2\)

   (a) \( \Gamma_1 = \{\alpha_3, \alpha_4\} \), \( \Gamma_2 = \{\alpha_1, \alpha_2\} \), \( \tau(\alpha_i) = \alpha_{i-2} \).

   \[
   r^0 = t^0/2 + ch_{\alpha_1} \wedge h_{\alpha_2} + ((c - 1)/2)h_{\alpha_1} \wedge h_{\alpha_3} + ch_{\alpha_1} \wedge h_{\alpha_4} \\
   - ((1 + 3c)/4)h_{\alpha_2} \wedge h_{\alpha_3} + ((c - 1)/2)h_{\alpha_2} \wedge h_{\alpha_4} + ch_{\alpha_3} \wedge h_{\alpha_4}
   \]

   \[
   R = R(r^0) + \hat{q}(e_{43} \wedge (3+c)/8 e_{12} + e_{54} \wedge (3+c)/8 e_{23} + e_{53} \wedge (1-c)/2 e_{13})
   \]

   (b) \( \Gamma_1 = \{\alpha_3, \alpha_4\} \), \( \Gamma_2 = \{\alpha_1, \alpha_2\} \), \( \tau(\alpha_i) = \alpha_{5-i} \).

   \[
   r^0 = t^0/2 + \frac{1}{5}(-c h_{\alpha_1} \wedge h_{\alpha_2} - (1 + c)h_{\alpha_1} \wedge h_{\alpha_3} - 3h_{\alpha_1} \wedge h_{\alpha_4} \\
   - 2h_{\alpha_2} \wedge h_{\alpha_3} + (c - 1)h_{\alpha_2} \wedge h_{\alpha_4} + ch_{\alpha_3} \wedge h_{\alpha_4})
   \]
\[ R = R(r^0) + \hat{q}(e_{54} \wedge_{3/5} e_{12} + e_{43} \wedge_{4/5} e_{23} + e_{53} \wedge_{-9/5} (-e_{13})) \]

(c) \( \Gamma_1 = \{\alpha_2, \alpha_4\}, \quad \Gamma_2 = \{\alpha_1, \alpha_3\}, \quad \tau(\alpha_i) = \alpha_{i-1} \)

\[ r^0 = \frac{t^0}{2} + c h_{\alpha_1} \wedge h_{\alpha_2} + (1 + 3c) h_{\alpha_1} \wedge h_{\alpha_3} + (8c/3 + 1) h_{\alpha_1} \wedge h_{\alpha_4} \]
\[ + (1 + 3c) h_{\alpha_2} \wedge h_{\alpha_3} + (1 + 3c) h_{\alpha_2} \wedge h_{\alpha_4} + c h_{\alpha_3} \wedge h_{\alpha_4} \]

\[ R = R(r^0) + \hat{q}(e_{32} \wedge_{1+c} e_{12} + e_{54} \wedge_{1+c} e_{34}) \]

(d) \( \Gamma_1 = \{\alpha_2, \alpha_4\}, \quad \Gamma_2 = \{\alpha_1, \alpha_3\}, \quad \tau(\alpha_4) = \alpha_1, \quad \tau(\alpha_2) = \alpha_3 \)

\[ r^0 = \frac{t^0}{2} + \frac{1}{5}(2 - c) h_{\alpha_1} \wedge h_{\alpha_2} + (1 - c) h_{\alpha_1} \wedge h_{\alpha_3} - 3 h_{\alpha_1} \wedge h_{\alpha_4} \]
\[ + 2 h_{\alpha_2} \wedge h_{\alpha_3} + (c - 1) h_{\alpha_2} \wedge h_{\alpha_4} + c h_{\alpha_3} \wedge h_{\alpha_4} \]

\[ R = R(r^0) + \hat{q}(e_{32} \wedge_{2/5} e_{12} + e_{32} \wedge_{2/5} e_{34}) \]

(e) \( \Gamma_1 = \{\alpha_1, \alpha_3\}, \quad \Gamma_2 = \{\alpha_1, \alpha_4\}, \quad \tau(\alpha_i) = \alpha_j, \text{ where } j \equiv i + 3 \pmod{5}. \)

\[ r^0 = \frac{t^0}{2} + ((1 - 3c)/2) h_{\alpha_1} \wedge h_{\alpha_2} + ((c - 1)/2) h_{\alpha_1} \wedge h_{\alpha_3} + c h_{\alpha_1} \wedge h_{\alpha_4} \]
\[ + (3c - 1) h_{\alpha_2} \wedge h_{\alpha_3} + (3c - 1) h_{\alpha_2} \wedge h_{\alpha_4} + c h_{\alpha_3} \wedge h_{\alpha_4} \]

\[ R = R(r^0) + \hat{q}(e_{43} \wedge_{1+3c}/4 e_{12} + e_{21} \wedge_{(1+3c)/4} e_{45} + e_{43} \wedge_{1-e} e_{45}) \]

(f) \( \Gamma_1 = \{\alpha_3, \alpha_4\}, \quad \Gamma_2 = \{\alpha_1, \alpha_2\}, \quad \tau(\alpha_i) = \alpha_{i-1} \)

\[ r^0 = \frac{t^0}{2} + ((c - 3)/6) h_{\alpha_1} \wedge h_{\alpha_2} + ((c - 1)/2) h_{\alpha_1} \wedge h_{\alpha_3} + c h_{\alpha_1} \wedge h_{\alpha_4} \]
\[ + ((c - 1)/2) h_{\alpha_2} \wedge h_{\alpha_3} + (4c/3) h_{\alpha_2} \wedge h_{\alpha_4} + c h_{\alpha_3} \wedge h_{\alpha_4} \]

\[ R = R(r^0) + \hat{q}(e_{32} \wedge_{1/2+c/6} e_{12} + e_{43} \wedge_{1/2+c/6} e_{23} + e_{43} \wedge_{1+c/3} e_{12}) \]

3. \( |\Gamma_1| = 1 \)

(a) \( \Gamma_1 = \{\alpha_1\}, \quad \Gamma_2 = \{\alpha_2\}, \quad \tau(\alpha_1) = \alpha_2 \)

\[ r^0 = \frac{t^0}{2} + ((1 + y)/3) h_{\alpha_1} \wedge h_{\alpha_2} + y h_{\alpha_1} \wedge h_{\alpha_3} + ((3z - x)/3) h_{\alpha_1} \wedge h_{\alpha_4} \]
\[ + y h_{\alpha_2} \wedge h_{\alpha_3} + z h_{\alpha_2} \wedge h_{\alpha_4} + x h_{\alpha_3} \wedge h_{\alpha_4} \]
\[ R = R(r^0) + \hat{q} (\varepsilon_{21} \wedge (2-y)/3 \varepsilon_{23}) \]

(b) \( \Gamma_1 = \{\alpha_1\}, \quad \Gamma_2 = \{\alpha_3\}, \quad \tau(\alpha_1) = \alpha_3 \)

\[
r^0 = t^0/2 + ((z-2y)/2) h_{\alpha_1} \wedge h_{\alpha_2} + ((1+x)/2) h_{\alpha_1} \wedge h_{\alpha_3} + x h_{\alpha_1} \wedge h_{\alpha_4} \\
+ y h_{\alpha_2} \wedge h_{\alpha_3} + z h_{\alpha_2} \wedge h_{\alpha_4} + x h_{\alpha_3} \wedge h_{\alpha_4} 
\]

\[ R = R(r^0) + \hat{q} (\varepsilon_{21} \wedge (z-2)/4 \varepsilon_{34}) \]

(c) \( \Gamma_1 = \{\alpha_1\}, \quad \Gamma_2 = \{\alpha_4\}, \quad \tau(\alpha_1) = \alpha_4 \)

\[
r^0 = t^0/2 + ((y-2z)/2) h_{\alpha_1} \wedge h_{\alpha_2} + ((-2x)/2) h_{\alpha_1} \wedge h_{\alpha_3} + ((1-x-z)/2) h_{\alpha_1} \wedge h_{\alpha_4} \\
+ y h_{\alpha_2} \wedge h_{\alpha_3} + z h_{\alpha_2} \wedge h_{\alpha_4} + x h_{\alpha_3} \wedge h_{\alpha_4} 
\]

\[ R = R(r^0) + \hat{q} (\varepsilon_{21} \wedge (y+2)/4 \varepsilon_{45}) \]

(d) \( \Gamma_1 = \{\alpha_2\}, \quad \Gamma_2 = \{\alpha_3\}, \quad \tau(\alpha_2) = \alpha_3 \)

\[
r^0 = t^0/2 + (-1+3y-z) h_{\alpha_1} \wedge h_{\alpha_2} + (-1-x+3y) h_{\alpha_1} \wedge h_{\alpha_3} + 3(z-x) h_{\alpha_1} \wedge h_{\alpha_4} \\
+ y h_{\alpha_2} \wedge h_{\alpha_3} + z h_{\alpha_2} \wedge h_{\alpha_4} + x h_{\alpha_3} \wedge h_{\alpha_4} 
\]

\[ R = R(r^0) + \hat{q} (\varepsilon_{32} \wedge (1-x-y+z) \varepsilon_{34}) \]

4. \(|\Gamma_1| = 0\) The “trivial triple:” \( \Gamma_1 = \Gamma_2 = \emptyset \)

\[
r^0 = t^0/2 + \tilde{r}^0 \quad \text{with} \quad \tilde{r}^0 \in \mathfrak{h} \wedge \mathfrak{h} \quad \text{arbitrary.} 
\]

\[ R = R(r^0) \quad \text{is the standard multiparameter } R\text{-matrix.} \]

Perhaps the most interesting new \( R\)-matrix is that associated to type 1 (b), the generalized Cremmer-Gervais triple. Like the Cremmer-Gervais triple, its \( \Gamma_1 \), which must omit at least one root, omits precisely one and thus its \( r^0 \) is uniquely determined. Setting \( \tilde{p} = -\hat{q} \), the matrix form of the generalized Cremmer-Gervais \( R\)-matrix is
produce genuine nonstandard quantizations of $C$ to the category of comodules over $A$ as in the commutative case. Thus Poincaré series of the associated quantum space and exterior algebra are the same. 

We have constructed here quantizations of each type of non-unitary solution of the classical Yang-Baxter equation for $\mathfrak{sl}(5)$. In so doing we verified in this case the conjecture of Gerstenhaber, Giaquinto and Schack. This gives further evidence that the GGS conjecture should be true for all Belavin-Drinfeld triples on $\mathfrak{sl}(n)$.

One can proceed in the usual way to construct for each of these $R$, a quantization of $\mathbb{C}[SL(5)]$, the algebra of algebraic functions on $SL(5)$. First one constructs the associated bialgebra $A(R)$. Using a case-by-case analysis one can see that the Poincaré series of the associated quantum space and exterior algebra are the same as in the commutative case. Thus $A(R)$ contains a group-like $q$-determinant element $D$ which turns out to be central. Hence one may define a Hopf algebra structure on $\mathbb{C}_R[SL(5)] = A(R)/(D-1)$. Since $R$ is a Hecke symmetry in the sense of Gurevich, it is possible to exploit some Hecke algebra techniques to show that the category of comodules over these Hopf algebras is equivalent as a rigid monoidal category to the category of comodules over $\mathbb{C}_q[SL(5)]$. Hence these $R$-matrices do produce genuine nonstandard quantizations of $\mathbb{C}[SL(5)]$.

3. Conclusion

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