A class of explicit solutions disproving the positive energy conjecture in all dimensions

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Abstract

In this article, we construct a class of explicit, smooth and spherically symmetric solutions to the asymptotically flat vacuum constraint equations which have ADM mass of arbitrary sign ($-\infty$, negative, zero, positive). As a direct consequence of the result, there exist asymptotically flat vacuum initial data sets whose metrics are exactly negative mass Schwarzschild outside a given ball. We emphasize that our result does not contradict the positive energy theorem proven by Eichmair [11], instead it shows that the decay rate at infinity of the symmetric $(0,2)$-tensor $k$ stated in the theorem is sharp. The key argument we use in the article is classical, based on the conformal method, in which the conformal equations are equivalently transformed into a single nonlinear equation of functions of one variable.

1 Introduction

An asymptotically flat (AF) initial data set for the Cauchy problem in general relativity is an AF manifold $(M,g)$ of $n$ dimensions, with $n \geq 3$, coupled with a symmetric $(0,2)$-tensor $k$ such that $(M,g,k)$ satisfies the system

\begin{align}
R_g - |k|^2_g + (\text{tr}_g k)^2 &= \mu \quad \text{[Hamiltonian constraint]} \\
\text{div}_g (k - (\text{tr}_g k)g) &= J, \quad \text{[Momentum constraint]}
\end{align}

where $R_g$ is the scalar curvature of $g$ and where $\mu$ is a non-negative scalar field and $J$ is a 1-form on $M$, representing the energy and momentum densities of the matter and non-gravitational fields, respectively. These equations are called the *Einstein constraint equations* and the study of solutions to (1.1) has been a topical issue for many decades.

One of major achievements in the constraint equations is the positive energy theorem (PET) proven by Schoen–Yau [19, 20], Witten [21] and later Chrusciel–Maerten [7], Eichmair [11], which roughly states that every AF initial data $(M,g,k)$ with decay rates at infinity

$$
|g_{ij} - \delta_{ij}| + |x||\partial g_{ij}| + |x||k_{ij}| = O(|x|^{-\frac{n+2}{2}} - \epsilon)
$$

and satisfying the dominant energy $\mu \geq |J|$ has positive ADM mass, unless $(M,g,k)$ is Cauchy initial data for Minkowski space, i.e. $(M,g)$ can be isometrically embedded in Minkowski

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spacetime with the second fundamental form \( k \). This theorem is very important in general relativity since it practically denies existence of negative mass in reality. However, this feature is also a challenge in physics because the repulsion of negative mass has a key role in the accelerated expansion of the universe as well as the dark energy, but finding such a model is clearly difficult due to the theorem. From the modern physical point of view, a natural question to ask is whether there is an AF initial data set that has negative mass without violation of the dominant energy condition. In this article, we will give an answer to this question by considering the simple setting where \( M \equiv \mathbb{R}^n \) and \( \mu \equiv |J| \equiv 0 \), that is the vacuum case on \( \mathbb{R}^n \). Our first main result is the following theorem that affirms existence of smooth solutions with negative mass Schwarzschild metric outside a ball. In what follows, we state only typical results and we refer to Section 4 for our precise statements.

**Main Theorem 1.** Given a constant \( m > 0 \), let \( \varphi \) be a radial and increasing function in \( \mathbb{R}^n \) satisfying
\[
\varphi(x) = 1 - \frac{m}{2|x|^{n-2}}, \quad \text{for all } |x| \geq m^{1/(n-2)}.
\]
Then there exists a symmetric \((0, 2)\)-tensor \( k \) with \( |k| = O(|x|^{-\frac{n}{2}}) \) such that \((\mathbb{R}^n, \varphi^{\frac{2}{n-2}} \delta_{\text{Euc}}, k)\) is a solution to the vacuum constraint equations (1.1).

Clearly, the ADM mass of solutions in Main Theorem 1 is \(-m\). In regard to (1.2), we emphasize that this result does not contradict the PET because the decay rate of \( k \) in our result is lightly reduced to be critical, i.e. \(|k_{ij}| = O(|x|^{-\frac{n}{2}})\) at infinity. This reduction makes \( k \) not be square integrable any longer, and so, as mentioned in the letter of Chrusciel [6], that’s why the arguments of Witten or Eichmair cannot work. To see more clearly how the decay rate of \( k \) impacts on the sign of ADM mass, we prove the following result.

**Main Theorem 2.** Given a constant \( c \geq 0 \) and a decay exponent \( q \in \left( \frac{n+2}{4}, n \right) \), let \( \tau \) be a radial function in \( \mathbb{R}^n \) satisfying \(|\tau| \sim c|x|^{-q}\) at infinity. Then there exists an AF vacuum initial data \((\mathbb{R}^n, g, k)\) such that \( \text{tr}_g k = \tau \) and
\[
|k_{ij}| = O(|x|^{-q}), \quad |g_{ij} - \delta_{ij}| + |x||\partial_k g_{ij}| = O(|x|^{-2q+2}).
\]
Moreover,
- if \( q < \frac{n}{2} \), the ADM mass is \(-\infty\),
- if \( q = \frac{n}{2} \), the ADM mass is negative,
- if \( q > \frac{n}{2} \), the ADM mass is zero, and hence \((\mathbb{R}^n, g, k)\) is an entire spacelike hypersurface of mean curvature \( \tau \) in Minkowski space-time provided that the rigidity part of the PET holds.

It is worth noting that all solutions stated in our results are explicit, spherically symmetric in some sense and quite simple (see. Theorem 4.1). It provides a variety of models in general relativity, and so helps us understand better structure and behavior of the initial data, especially in numerical general relativity.

The outline of this article is as follows. In Section 2, we present the conformal method introduced by Lichnerowicz [15] and later Choquet-Bruhat–York [5] for constructing solutions

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1These are nothing but \( 2q - 2 > \frac{n}{2} \), and hence the mass of \( g \) is a geometric invariant (see. Barnik [2]).
to the constraint equations. In Section 3, we will show that the conformal equations, a couple of nonlinear elliptic equations associated with the conformal method, can be well treated in the space of radial functions. Restricting our consideration to this space, we will transform equivalently the conformal equations into a single nonlinear equation of functions of distance, and then find explicitly solutions. In Section 4, we will study the properties of these solutions and give counterexamples to the positive energy conjecture by proving Main Theorems 1 and 2.

2 Preliminaries

2.1 The conformal equations

In this section, we revisit the conformal method, a traditional way to generate Einstein solutions from scratch. For our purpose, we only focus on the Euclidean space \((\mathbb{R}^n, \delta_{\text{Euc}})\) with \(n \geq 3\). For a treatment of the general case, the interested reader is referred to [10] and [17].

The given data set on \((\mathbb{R}^n, \delta_{\text{Euc}})\) consists of a function \(\tau\) and a TT-tensor \(\sigma\) (i.e. a symmetric, trace-free, divergence-free \((0,2)\)-tensor), and one is required to find a positive function \(\varphi\) tending to 1 at infinity and a 1-form \(W\) satisfying

\[-\frac{4(n-1)}{n-2}\Delta \varphi + \frac{n-1}{n} \frac{\varphi^{N-1}}{n} = |\sigma + LW|^2 \varphi^{-N-1}\]  

[Lichnerowicz equation] (2.1a)

\[\text{div}(LW) = \frac{n-1}{n} \varphi^N d\tau,\]  

[vector equations] (2.1b)

where \(N = \frac{2n}{n-2}\) and \(L\) is the conformal Killing operator defined by

\[(LW)_{ij} = \nabla_i W_j + \nabla_j W_i - \frac{2\delta_{ij}}{n} (\text{div} W).\]  

(2.2)

These equations are called the vacuum Einstein conformal constraint equations, or simply the conformal equations. Once such a solution \((\varphi, W)\) exists, it follows that

\[g = \varphi^{N-2} \delta_{\text{Euc}}\]

\[k = \frac{T}{n} \varphi^{N-2} \delta_{\text{Euc}} + \varphi^{-2} (\sigma + LW),\]  

(2.3)

is a solution to the AF vacuum constraint equations. In this situation, we keep in mind all along the article that \(\text{tr}_g k = \tau\).

2.2 Elliptic operators on weighted Hölder spaces

We next review some standard facts on elliptic operators used for studying the conformal equations. For the proofs, we refer the reader to [9] [10].

Given an integer \(l \geq 0\), a Hölder exponent \(\alpha \in [0,1]\), and a decay exponent \(\beta > 0\), we will use the weighted Hölder spaces \(C^{l,\alpha}_{-\beta}\) to capture asymptotic of functions and tensors near infinity. For \(\alpha = 0\), we will write \(C^{l,0}_{-\beta}\) instead of \(C^{0,0}_{-\beta}\). The weighted norm convention we are using is that the \(C^{s,\alpha}_{-\beta}\) norm is given by

\[
\|f\|_{l,\alpha,-\beta} := \sum_{|s| \leq l} \sup_{\mathbb{R}^n} (\rho^{|s|+\beta} |\partial^s f|) + \sum_{|s| = l} \sup_{\mathbb{R}^n} \rho^{l+\beta+\alpha} \sup_{0 < |y-x| \leq \rho} \left( \frac{|\partial^s f(y) - \partial^s f(x)|}{|y-x|^\alpha} \right).
\]
where in this context \( \rho \) is a positive function which equals \(|x|\) outside the unit ball and \( s \) is a multi-index. It will be clear from the context if the notation refers to a space of functions on \( \mathbb{R}^n \), or a space of sections of some bundle over \( \mathbb{R}^n \).

**Proposition 2.1** (Compact embedding for weighted Hölder spaces). If \( l_1 + \alpha_1 > l_2 + \alpha_2 \) and \( \beta_1 > \beta_2 \) then the inclusion \( C^{l_1,\alpha_1}_{-\beta_1} \subset C^{l_2,\alpha_2}_{-\beta_2} \) is compact.

**Proposition 2.2** (Weighted elliptic regularity for Laplacian). Let \( V \geq 0 \) be a function in \( C^{l_1,\alpha}_{-\beta_1} \) with \( l_1 \geq 2 \), \( \alpha \in (0,1) \) and \( \beta > 0 \).

(a) \( \Delta - V : C^{l_1,\alpha}_{-\beta} \rightarrow C^{l_1,\alpha-\beta}_{-\beta-2} \) is an isomorphism if and only if \( 0 < \beta < n - 2 \).

(b) If \( u \in C^{l_1,\alpha}_{-\beta} \) and \( \Delta u - Vu \in C^{l_1,\alpha}_{-\beta-2} \), then

\[
\|u\|_{l_1,\alpha,-\beta} \leq c (\|u\|_{l_1,\alpha-\beta} + \|\Delta u - Vu\|_{l_1,\alpha-\beta-2})
\]

for some constant \( c > 0 \) independent of \( u \).

Similarly, we also have the following proposition for the operator \( \text{div}L \) appearing in the vector equations (2.1b), where \( L \) is the conformal Killing operator defined in (2.2).

**Proposition 2.3** (Weighted elliptic regularity for vector Laplacian). \( \text{div}L : C^{l_1,\alpha}_{-\beta} \rightarrow C^{l_1,\alpha-\beta}_{-\beta-2} \) is an isomorphism if and only if \( 0 < \beta < n - 2 \).

Finally, we give the theorem of existence and uniqueness of solutions to Lichnerowicz’s equation on the Euclidean space, which is one of two main parts of the conformal equations:

\[
-4\frac{(n-1)}{n-2} \Delta u + \frac{n-1}{n} \tau^2 u^{N-1} = w^2 u^{-N-1}.
\] (2.4)

**Theorem 2.4** (Existence and uniqueness of solution to the Lichnerowicz equation). If \( \tau \) and \( w \) are in \( C^{l_2-2,\alpha}_{-\beta/2} \) with \( l \geq 2 \), \( \alpha \in (0,1) \) and \( \beta \in (0,n-2) \), then the Lichnerowicz equation (2.4) admits a unique positive solution \( u \) satisfying \( u - 1 \in C^{l_2,\alpha}_{-\beta} \).

3 Construction of solutions to the conformal equations

We now seek a class of solutions to the vacuum constraint by solving the conformal equations in a simple setting where \( \sigma \equiv 0 \) and \( \tau \) is a radial function. In this case, the conformal equations (2.1) are rewritten as

\[
-4\frac{(n-1)}{n-2} \Delta \varphi + \frac{n-1}{n} \tau^2 \varphi^{N-1} = |LW|^2 \varphi^{-N-1}
\] (3.1a)

\[
\Delta W_i + \frac{n-2}{n} \partial_i \left( \sum_{j=1}^{n} \partial_j W_j \right) = \frac{n-1}{n} \varphi^{N-1} \tau r x_i
\] (3.1b)

Here and subsequently, \( r \) is the usual Euclidean distance, we denote by \( f' \) the derivative of \( f \) with respect to \( r \) and we call a radial function increasing if it is increasing with respect to \( r \). Clearly, these equations consist of \((\tau, \varphi, W)\) and when we know two of them, we can find the third. Hence, for simplicity of expression, we may say such as \((\varphi, W)\) or \((\varphi, \tau)\) or \(\varphi\)
when \( \tau \) is fixed is a solution to (3.1). The idea behind our use of this data is that in view of Schwarzschild metrics and spherically symmetric solutions obtained by the gluing method [8], we want to look for \( \varphi \) in the space of radial functions, which, by the Lichnerowicz equation, will be guaranteed as long as \( \varphi \) and \(|\sigma + LW|\) are radial. Therefore, a simple way to think of this is restricting ourselves to the data of null \( \sigma \) and radial \( \tau \) and expecting that \(|LW|\) is radial. The following result fulfills our desire and plays a central role in the article.

**Theorem 3.1.** Assume that \( \varphi > 0 \) and \((\varphi-1, \tau) \in C^{3, \alpha}_{-2\beta+2} \times C^{1, \alpha}_1 \) with \( \alpha \in (0, 1) \) and \( \beta > 0 \). Assume furthermore that \( \varphi \) and \( \tau \) are radial functions. Then \((\varphi, \tau)\) is a solution to the conformal equations (3.1) if and only if \( \varphi \) is increasing and

\[
|\tau(r)| = \begin{cases} 
\sqrt{2nN\varphi^{-N+1}\varphi''} & \text{if } \varphi'(r) = 0 \\
2\left(r^{2n-1}\left(\varphi^{(N+2)/2}\right)'ight)' & \text{otherwise.}
\end{cases} \tag{3.2}
\]

Moreover, when \((\varphi, \tau)\) is the solution, the 1-form \( W \) is computed by

\[
W_i = -\frac{x_i}{2r^n} \int_0^r s^{n-1} \left( \int_s^{+\infty} \varphi^N(s_1)\tau'(s_1) ds_1 \right) ds.
\]

**Remark 3.2.** Since \( \varphi' \geq 0 \), by (3.2) we have \( \tau(r) = 0 \) as long as \( \varphi'(r) = 0 \) and \( r > 0 \). Therefore, when \( \tau \) does not change sign, \( \varphi' > 0 \) in \( \mathbb{R}^n \).

**Remark 3.3.** It follows from Theorem 3.1 that for any constant \( c > 0 \), \((c\tau(c), \varphi(c))\) is a radial solution to the conformal equations if and only if \((c\tau(cr), \varphi(cr))\) is a radial solution to the conformal equations.

**Proof of Theorem 3.1.** We will divide the proof into three steps.

**Step 1. Solving the vector equations.** Consider the vector equations

\[
\Delta W_i + \frac{n-2}{n} \partial_i \left( \sum_{j=1}^n \partial_j W_j \right) = \frac{n-1}{n} \varphi^N \tau' \frac{x_i}{r}.
\]

Letting

\[
f(r) = -\int_r^{+\infty} \varphi^N \tau' ds,
\]

the system becomes

\[
\Delta W_i + \frac{n-2}{n} \partial_i \left( \sum_{j=1}^n \partial_j W_j \right) = \frac{n-1}{n} \partial_i f. \tag{3.3}
\]

Differentiating (3.3) with respect to \( i \) and summing all equations of the new system, we obtain

\[
\Delta \left( \sum_{j=1}^n \partial_j W_j \right) = \frac{1}{2} \Delta f.
\]

Therefore, by Proposition 2.2(a) we have

\[
\sum_{j=1}^n \partial_j W_j = \frac{f}{2}
\]
Taking into account (3.3), we get  
\[ \Delta W_i = \frac{1}{2} \partial_i f. \]

Since \( f \) is radial, it follows by computations that  
\[ W_i = \frac{x_i}{2r^n} \int_0^r s^{n-1} f \, ds. \]

Thus, we have by definition  
\[ (LW)_{ij} = -\left( \frac{\delta_{ij}}{n} - \frac{x_i x_j}{r^2} \right) \left( f - \frac{n}{r^n} \int_0^r s^{n-1} f \, ds \right) \]
\[ = -\left( \frac{\delta_{ij}}{n} - \frac{x_i x_j}{r^2} \right) \int_0^r s^n f' \, ds \]
\[ = -\left( \frac{\delta_{ij}}{nr^n} - \frac{x_i x_j}{r^{n+2}} \right) \int_0^r s^n \varphi^N \tau' \, ds \]

and so  
\[ |LW| = \frac{1}{r^n} \sqrt{n-1} \left| \int_0^r s^n \varphi^N \tau' \, ds \right|, \]

which is a radial function as we have expected.

**Step 2. Solving the Lichnerowicz equation.** It simplifies the argument, and causes no loss of generality, to assume \( \varphi' \neq 0 \) almost everywhere. We first take (3.4) into the Lichnerowicz equation, it then follows that \((\varphi, \tau)\) is a solution of the conformal equations (3.1) if and only if they satisfy  
\[ -\frac{4n}{n-2} \left( \varphi'' + \left( \frac{n-1}{r} \right) \varphi' \right) + \tau^2 \varphi^{N-1} = \frac{1}{r^{2n}} \left( \int_0^r s^n \varphi^N \tau' \, ds \right)^2 \varphi^{-N-1}. \]

(3.5)

Integrating by parts (3.5) we have  
\[ -\frac{4n}{n-2} \left( \varphi'' + \left( \frac{n-1}{r} \right) \varphi' \right) + \tau^2 \varphi^{N-1} = \frac{1}{r^{2n}} \left( r^n \varphi^N \tau - \int_0^r \left( s^n \varphi^N \right)' \tau \, ds \right)^2 \varphi^{-N-1}, \]

equivalently,
\[ 2r^n \varphi^N \tau \int_0^r \left( s^n \varphi^N \right)' \tau \, ds - \left( \int_0^r \left( s^n \varphi^N \right)' \tau \, ds \right)^2 = 2Nr^{2n} \varphi^{N+1} \left( \varphi'' + \frac{(n-1)}{r} \varphi' \right). \]

(3.6)

Next, multiplying (3.6) by \( (r^n \varphi^N)' / (r^n \varphi^N)^2 \), we obtain  
\[ \left( \frac{1}{r^n \varphi^N} \left( \int_0^r \left( s^n \varphi^N \right)' \tau \, ds \right)^2 \right)' = 2N \left( \frac{r^n \varphi^N}{\varphi^{N-1}} \right) \left( \varphi'' + \frac{(n-1)}{r} \varphi' \right). \]

(3.7)

We observe here that the equations (3.6) and (3.7) are equivalent as long as \( \varphi' \geq 0 \), which will be proven later, so we can continue our process without undue worry about equivalence among equations. Now, since  
\[ \lim_{r \to 0} \left( \frac{1}{r^n \varphi^N} \left( \int_0^r \left( s^n \varphi^N \right)' \tau \, ds \right)^2 \right) = 0, \]
the equation (3.7) is equivalent to
\[ \left( \int_0^r (s^n \varphi'^N) ds \right)^2 = 2N (r^n \varphi'^N) \left( \int_0^r \frac{\varphi'^N}{\varphi'^N-1} \left( \varphi'' + \frac{(n-1)\varphi'}{s} \right) ds \right). \] (3.8)
Therefore, assuming for the moment that
\[ \int_0^r \frac{\varphi'^N}{\varphi'^N-1} \left( \varphi'' + \frac{(n-1)\varphi'}{s} \right) ds > 0 \quad \text{a.e in } \mathbb{R}^n, \] (3.9)
we obtain
\[ |\tau| = \left| \frac{N \left( (r^n \varphi'^N) \left( \int_0^r \frac{\varphi'^N}{\varphi'^N-1} \left( \varphi'' + \frac{(n-1)\varphi'}{s} \right) ds \right) \right)^{1/2}}{(r^n \varphi'^N) \sqrt{2N (r^n \varphi'^N) \left( \int_0^r \frac{\varphi'^N}{\varphi'^N-1} \left( \varphi'' + \frac{(n-1)\varphi'}{s} \right) ds \right)}} \right| \]
\[ = \left| \frac{N (r \varphi')(r^{n-1} \varphi'') + N \int_0^r \frac{(s^n \varphi'^N)}{\varphi'^N-1} \left( \varphi'' + \frac{(n-1)\varphi'}{s} \right) ds}{\sqrt{(r^n \varphi'^N) \left( \int_0^r \frac{(s^n \varphi'^N)}{\varphi'^N-1} \left( \varphi'' + \frac{(n-1)\varphi'}{s} \right) ds \right)}} \right|. \]
To simplify the formula, we calculate
\[ \int_0^r \frac{(s^n \varphi'^N)}{\varphi'^N-1} \left( \varphi'' + \frac{(n-1)\varphi'}{s} \right) ds = \int_0^r (n \varphi + N s \varphi')(r^{n-1} \varphi'') ds, \]
and so by integration by parts
\[ \int_0^r \frac{(s^n \varphi'^N)}{\varphi'^N-1} \left( \varphi'' + \frac{(n-1)\varphi'}{s} \right) ds = (n \varphi + N s \varphi') r^{n-1} \varphi' - \frac{N}{2} \int_0^r (s^n \varphi'^N)^2 ds \] (3.10)
Taking into account, we have
\[ |\tau| = \left| \frac{2N (r \varphi')(r^{n-1} \varphi'') + N r^{n-1} (\varphi'^N)^2 + N^2 r^n (\varphi'^N)^2}{2 \sqrt{(r^n \varphi'^N) \left( N r^{n-1} (\varphi'^N)^2 + N^2 r^n (\varphi'^N)^2 \right)}} \right| \]
\[ = \left| \frac{2 \left( r^{2n-1} (\varphi'^N)^{2n-1} \right)^{1/2}}{(N + 2) r^{3n/2} \varphi'^N \sqrt{(r^{n-2} \varphi'^N)}} \right|. \] (3.11)
**Step 3.** \( \varphi \) is increasing. We remind the reader that \( \varphi \) was assumed to be different from 0 a.e in \( \mathbb{R}^n \) for simplicity. In view of (3.8)–(3.10), we see that the necessary condition for \((\varphi, \tau)\) to be a solution to (3.11) is
\[ n r^{n-1} (\varphi'^N)^2 + N r^n (\varphi'^N)^2 = r^{n-1} \varphi' (2n \varphi + N r \varphi') > 0 \quad \text{a.e in } \mathbb{R}^n. \] (3.12)
We will show that this condition is equivalent to the fact that \( \varphi \) is increasing. In fact, if \( \varphi' > 0 \) a.e in \( \mathbb{R}^n \), (3.12) is obvious. Conversely, assume that (3.12) holds. Since \( \varphi(0) > 0 \), we have \( 2n \varphi + N r \varphi' > 0 \) near 0. It then follows by (3.12) that \( \varphi' \geq 0 \) near 0. Therefore, if \( \varphi' < 0 \) somewhere, then there exists a convergent sequence \( \{r_m\} \) such that \( \varphi'(r_m) < 0 \) and \( \varphi'(r_m) \to 0 \). This leads to the contradiction that
\[ 0 \leq \varphi'(r_m) (2n \varphi(r_m) + N r m \varphi'(r_m)) < 0. \]
Therefore, we have \( \varphi' > 0 \) a.e, and so the inequality (3.12) holds as we assumed in (3.9). The proof is completed. □
4 Counterexamples to the positive energy conjecture

In this section, we will apply Theorem 3.1 to smoothing out the singular of the negative mass Schwarzschild metric and so disprove the positive energy conjecture in some sense. We will also explain why the decay rate $k = O(r^{-2-\epsilon})$ at infinity is not merely a simple way to guarantee existence of mass, but it also plays an important role in the positive mass inequality. We begin our discussion by recalling the positive mass conjecture on $\mathbb{R}^n$. Let $(\mathbb{R}^n, g)$ be an AF manifold with

$$g - \delta_{\text{Euc}} \in C^{2,\alpha}_{-\frac{n-2}{2}-\epsilon}$$

for some $\epsilon > 0$. The ADM mass of $(\mathbb{R}^n, g)$ is defined by

$$m_{\text{ADM}}(g) := \frac{1}{2(n-2)\omega_{n-1}} \lim_{r \to +\infty} \int_{|x| = r} \sum_{i,j=1}^n (g_{ij,i} - g_{ii,j}) \frac{x_j}{r} dM_{n-1}^0,$$

where $M_{n-1}^0$ is the $(n-1)$-dimensional Euclidean Hausdorff measure and $\omega_{n-1}$ is the volume of the standard unit sphere in $\mathbb{R}^n$. In particular, when $g = \varphi^{N-2} \delta_{\text{Euc}}$ with $\varphi$ radial, the formula becomes

$$m_{\text{ADM}}(g) = -\frac{n-1}{2(n-2)} \lim_{r \to +\infty} (r^{n-1} \varphi').$$

Bartnik in [2] showed that under the decay condition (4.1), the mass is a geometric invariant. A long-standing conjecture in general relativity states that the ADM mass of an AF initial data set satisfying the dominant condition $\mu \geq |J|$ is positive unless it is a Cauchy hypersurface of Minkowski space. This conjecture was proven to be true under suitable decay assumptions of $(g, k)$ at infinity. The most recent progress on such results is the PET proven by Eichmair [11], which is expected to be true for all dimension $n \geq 3$, but so far we have only achieved it in dimensions less than eight. The theorem states for a general AF manifold, but for our purpose, we just recall it on $\mathbb{R}^n$. We also remark that for $n = 3$ the rigidity part in [11] requires additionally the assumption $\text{tr}_{\mu} k = O(r^{-2-\epsilon})$, however in the statement below, since $\mathbb{R}^n$ is spin for all $n \geq 3$, it is removed thanks to the PET version for spin manifolds of dimension 3 shown in [7].

**Positive Energy Theorem** (Chrusciel–Maerten [7] and Eichmair [11]). Let $3 \leq n \leq 7$. Let $(\mathbb{R}^n, g, k)$ be an AF initial data set satisfying the dominant energy condition $\mu \geq |J|$. Assume that $(g, k) \in C^{2,\alpha}_{-\frac{n-2}{2}-\epsilon} \times C^{1,\alpha}_{-\frac{n}{2}-\epsilon}$ and $(\mu, J) \in (C^0_{-n-\epsilon})^2$ for some $\epsilon > 0$. Then the ADM mass is non-negative. Moreover, if $m_{\text{ADM}}(g) = 0$, then $(\mathbb{R}^n, g, k)$ is Cauchy initial data for Minkowski space.

Nowadays, negative mass is no longer viewed as a bizarre situation due to its central role in interpretation on the accelerated expansion of the universe. Physicists do not explicitly exclude the existence of negative, but the dominant energy condition and the PET are clearly difficult to reconcile with a negative ADM mass. From physical point of view, a natural question to ask is whether there exists an AF initial data set of negative ADM mass without violation of the dominant energy condition. In what follows, we will give a positive answer to the question by considering the vacuum case on $\mathbb{R}^n$. In fact, in view of the conformal method presented in Subsection 2.1 we first express Theorem 3.1 in terms of the constraint equations as follows.
Theorem 4.1 (Freely specified conformal factor). Let $\varphi > 0$ be a radial and increasing function in $\mathbb{R}^n$. Assume that $\varphi - 1 \in C^{3,\alpha}_{-2,2}$ with $\alpha \in (0,1)$ and $\beta > 1$. We define $\tau \in C^{1,\alpha}_{-\beta}$ by

$$|	au(r)| = \begin{cases} \sqrt{2nN\varphi^{-N+1}} & \text{if } \varphi'(r) = 0 \\ \frac{N-1}{2} \left(r^{2n-1} \left(\varphi(N+2)/2\right)^{1/2}\right) & \text{otherwise.} \end{cases}$$

(4.3)

Then

$$g_{ij} = \varphi^{N-2} \delta_{ij}$$

$$k_{ij} = \frac{\tau}{n} \varphi^{N-2} \delta_{ij} - \varphi^{-2} \left(\frac{\delta_{ij}}{n} - \frac{x_i x_j}{r^{n+2}}\right) \int_0^r s^n \varphi^N \tau' ds$$

(4.4)

is a solution to the AF vacuum constraint equations (1.1).

As the reader may have noticed, the surprising point of this result is that the increasing property of $\varphi$ makes the ADM mass non-positive. In particular, this property agrees with negative mass Schwarzschild metrics instead of the positive mass ones, therefore, we can construct smooth solutions of negative mass Schwarzschild metric outside a given ball, and so a class of spherically symmetric initial data sets violating the positive energy conjecture is given. We observe that these smooth solutions are no longer time-symmetric like Schwarzschild ones. This is certain since otherwise the PMT will lead to the contradiction that the ADM mass of a negative mass Schwarzschild metrics is non-negative. We will discuss this feature in more detail after stating explicitly what we say above about smoothing out the singular of negative mass Schwarzschild metric. The following result is a direct consequence of Theorem 4.1.

Corollary 4.2 (Smoothing out negative mass Schwarzschild metric). Given a constant $m > 0$, let $\varphi$ be a radial and increasing function in $C^\infty(\mathbb{R}^n)$ satisfying

$$\varphi(r) = 1 - \frac{m}{2r^{n-2}}, \quad \forall r \geq m^{1/(n-2)}.$$

Then $(g,k)$ defined in (4.3)-(4.4) is a smooth solution to the AF vacuum constraint equations (1.1).

Proof. The proof is straightforward by Theorem 4.1.

We now explain why Theorem 4.1 and Corollary 4.2 do not contradict the PET. In fact, for instance when $\varphi = 1 - \frac{m}{2r^{n-2}}$ outside a ball, i.e. $g$ is negative mass Schwarzschild at large distance, we can calculate by (4.3) that

$$\text{tr}_g k = |\tau| \sim cr^{-\frac{n}{2}}$$

as $r \to +\infty$ for some constant $c > 0$. Since $R_g = R_{\text{Sch}} = 0$ near infinity and since $(g,k)$ is an AF vacuum solution, it follows by the Hamiltonian constraint (1.1a) that

$$|k|_g = |\text{tr}_g k| \sim cr^{-\frac{n}{2}}$$

as $r \to +\infty$. This means the decay rate of $k$ at infinity is exactly $r^{-\frac{n}{2}}$, which is critical in the decay assumption of symmetric $(0,2)$—tensors in the PET, and that’s why the theorem fails in this situation.

In order to understand better how the decay rate of $k$ at infinity drives the sign of mass, we would like to give the following result.
**Theorem 4.3** (Freely specified mean curvature). Let $\tau$ be an arbitrary radial function in $C^{1,\alpha}_{\beta}(\mathbb{R}^n)$ with $\alpha \in (0,1)$ and $\beta \in (1,\frac{n}{2})$. There exists a solution $(g,k)$ to the vacuum constraint equations (1.1) such that $(g - \delta_{\text{Euc}},k) \in C^{3,\alpha}_{-2\beta+2} \times C^{1,\alpha}_\beta$ and $\text{tr}gk = \tau$. Moreover, given a constant $c \geq 0$ and a decay exponent $q \in (\frac{n+2}{4},n)$, assume that $|\tau| \sim cr^{-q}$ at infinity. Then we have

$$(g - \delta_{\text{Euc}},k) \in C^{3,\alpha}_{-2q+2} \times C^{1,\alpha}_q$$

and furthermore

(i) if $q < \frac{n}{2}$, then $m_{\text{ADM}}(g) = -\infty$,
(ii) if $q = \frac{n}{2}$, then $-\infty < m_{\text{ADM}}(g) < 0$,
(iii) if $q > \frac{n}{2}$, then $m_{\text{ADM}}(g) = 0$, and hence $(\mathbb{R}^n,g,k)$ is Cauchy initial data for Minkowski space provided that $3 \leq n \leq 7$.

The main tool we use for dealing with the theorem is Leray–Schauder’s fixed point. For convenience and ease of presentation, we would like to recall its statement.

**Leray–Schauder’s Fixed Point** (Gilbarg–Neil [14, Theorem 11.6]). Let $X$ be a Banach space and assume that $T : [0,1] \times X \to X$ is a continuous compact operator satisfying $T(0,x) = 0$ for all $x \in X$. If the set $K := \{(t,x) \in [0,1] \times X \mid T(t,x) = x\}$ is bounded, then $T(1,\cdot)$ has a fixed point.

**Proof of Theorem 4.3** We first define the operator $T : [0,1] \times L^\infty \to L^\infty$ as follows. For any $\phi \in L^\infty$, by Proposition 2.3, there exists a unique $W \in C^{2,\alpha}_{-\beta+1}$ satisfying

$$\text{div}(LW_\phi) = \frac{n-1}{n} |\phi|^N d\tau,$$

and hence, thanks to Theorem 2.4, there exists a unique $\varphi > 0$ such that $\varphi - 1 \in C^{3,\alpha}_{-2\beta+2}$ and

$$-\frac{4(n-1)}{n-2} \Delta \varphi + \frac{n-1}{n} t^{2N} \varphi^{N-1} = |LW|^2 \varphi^{N-1}. \quad (4.7)$$

We define

$$T(t,\phi) := t\varphi.$$ 

It is clear that a fixed point of $T(1,\cdot)$ is a solution to the conformal equations (2.1). In the spirit of the previous section, we will look for a fixed point of $T(1,\cdot)$ in the subspace

$$RL^\infty := \{f \in L^\infty \mid f \text{ is radial}\}.$$ 

The following observations are the key in our arguments:

- Let $\mathcal{V} : L^\infty \to C^{2,\alpha}_{-\beta+1}$ and $\mathcal{L} : [0,1] \times C^{2,\alpha}_{-\beta+1} \to C^{3,\alpha}_{-2\beta+2}$ be defined by

$$\mathcal{V}(\phi) := W, \quad \mathcal{L}(t,W) := \varphi - 1,$$
where \( W \) and \( \varphi \) are determined by (4.6) and (4.7) respectively. Let \( J : C^{3,\alpha}_{-2\beta+2} \to L^\infty \) be the compact weighted Hölder embedding map given by Proposition 2.1. It is clear that

\[
T = t(J \circ (1 + \mathcal{L}) \circ \nu).
\]

We have shown in Section 3 that if \( \phi \) is radial, then so is \( |LV(\phi)| \). On the other hand, since Laplace’s operator \( \Delta \) is invariant under rotations, we deduce from existence and uniqueness of solutions to the Lichnerowicz equation guaranteed by Theorem 2.4 that if the source \((\tau, |LW|)\) is radial, then so is \( \mathcal{L}(t, W) \). Therefore, we can conclude by (4.8) that \( T(t,.) \) maps the subspace \( RL^\infty \) into itself.

- If \( T(t, \phi) = \phi \) with \( (t, \phi) \in (0,1] \times RL^\infty \), then \( \phi/t \) is a solution to the conformal equations (3.1) associated with the seed data \((\delta_{\text{Euc}}, t^N \tau)\). Therefore, it follows from Theorem 3.1 that \( \phi/t \) must be increasing, and so \( \|\phi/t\|_{L^\infty} = 1 \). In particular, the set

\[
K = \{(t, \phi) \in (0,1] \times RL^\infty \mid T(t, \phi) = \phi\}
\]

is bounded.

From these observations, once \( T \) is proven to be continuous and compact in \([0,1] \times RL^\infty\), we can ensure by Leray–Schauder’s fixed point that \( T(1,.) \) has a fixed point in \( RL^\infty \) which is what we have desired. Observing furthermore that in view of (4.8), since \( \nu \) is continuous and since \( J \) is continuous compact, the fact that \( T \) is continuous and compact will follow immediately once we obtain the continuity of \( \mathcal{L} \). Therefore, the task is now to prove that \( \mathcal{L} \) is a continuous operator. The argument we give here is essentially the same as in Maxwell [16], which is the equivalent result for compact manifolds.

In fact, we define \( F(t, W, \psi) : [0,1] \times C^{2,\alpha}_{-\beta+1} \times C^{3,\alpha}_{-2\beta+2} \to C^{1,\alpha}_{-2\beta} \) by

\[
F(t, W, \psi) := -\frac{4(n-1)}{n-2}\Delta(\psi + 1) + \frac{n-1}{n}t^{2N} L^2 (\psi + 1)^{N-1} - |LW|^2 (\psi + 1)^{-N-1}
\]

It is clear that \( F \) is \( C^1 \) map and \( F(t, W, \mathcal{L}(t, W)) = 0 \) for all \((t, W) \in [0,1] \times C^{2,\alpha}_{-\beta+1} \). A standard computation shows that the Fréchet derivative of \( F \) with respect to \( \psi \) is given by

\[
F_\psi(t, W)(u) = -\frac{4(n-1)}{n-2}\Delta u + \frac{(n-1)(N-1)}{n}t^{2N} L^2 (\psi + 1)^{N-2} u + (N+1)|LW|^2 (\psi + 1)^{-N-2} u
\]

It follows that \( F_\psi \in C([0,1] \times C^{2,\alpha}_{-\beta+1}, L(C^{3,\alpha}_{-2\beta+2}, C^{1,\alpha}_{-2\beta})) \), where we denote \((L(C^{3,\alpha}_{-2\beta+2}, C^{1,\alpha}_{-2\beta}))\) the Banach space of all linear continuous maps from \( C^{3,\alpha}_{-2\beta+2} \) into \( C^{1,\alpha}_{-2\beta} \). In particular, setting \( \psi_0 = \mathcal{L}(t, W) \) we have

\[
F_{\psi_0}(t, W)(u) = -\frac{4(n-1)}{n-2}\Delta u + \left(\frac{(n-1)(N-1)}{n}t^{2N} L^2 (\psi_0 + 1)^{N-2} + (N+1)|LW|^2 (\psi_0 + 1)^{-N-2}\right) u
\]

Since

\[
\frac{(n-1)(N-1)}{n}t^{2N} L^2 (\psi_0 + 1)^{N-2} + (N+1)|LW|^2 (\psi_0 + 1)^{-N-2} \geq 0,
\]

it follows by Proposition 2.2(a) that \( F_{\psi_0}(t, W) : C^{3,\alpha}_{-2\beta+2} \to C^{1,\alpha}_{-2\beta} \) is an isomorphism. Therefore, the implicit function theorem implies that \( \mathcal{L} \) is a \( C^1 \)-function in a neighborhood of \((t, W),\)
which deduces \( T \) is continuous compact in \([0, 1] \times L^\infty\), and so \( T(1, \cdot) \) admits a fixed point in \( RL^\infty\).

On account of what we have shown above, let \( \varphi \in RL^\infty \) be a fixed point of \( T(1, \cdot) \). For the convenience of the reader, we summarize the properties of \( \varphi \) as follows:

- As claimed earlier, \( \varphi > 0 \) and \( \varphi - 1 \in C^{3,\alpha}_{-2\beta + 2}\).

- Since \((\tau, \varphi)\) is a radial solution to the conformal equations \((3.1)\), it follows by Theorem \(3.1\) that \( \varphi \) is increasing and \((\tau, \varphi)\) satisfies \((4.3)\).

- Let \((g, k) \in C^{3,\alpha}_{-2\beta + 2} \times C^{1,\alpha}_{-\beta}\) be defined in \((4.4)\). Similarly to Theorem \(4.1\), we deduce from the conformal method that \((g, k)\) is a solution to the vacuum constraint equations \((1.1)\) with \(\text{tr}_g k = \tau\).

Therefore, the rest of the proof is devoted to the decay rate \((4.5)\) and assertions (i–iii). In fact, assume that \(|\tau| \sim cr^{-q}\), then we have by \((4.3)\)

\[
\lim_{r \to +\infty} \left( \frac{2 \left| r^{2n-1} (\varphi^{(N+2)/2} / \varphi)^{(N-2)/2} \right|}{(N + 2) r^{\frac{4n}{2} - q} \varphi^{N-1} \sqrt{\varphi'/r^{n-2}\varphi}} \right) = c. \tag{4.9}
\]

On the other hand, by straightforward calculations we have

\[
\frac{2 \left| r^{2n-1} (\varphi^{(N+2)/2} / \varphi)^{(N-2)/2} \right|}{(N + 2) r^{\frac{4n}{2} - q} \varphi^{N-1} \sqrt{\varphi'/r^{n-2}\varphi}} = \frac{(2n - 1) r^{-1} \varphi^{N/2} / \varphi' + N \varphi^{(N-2)/2} (\varphi')^2 + \varphi^{N/2} \varphi''}{r^{-q} \varphi^{N-1} \sqrt{(\varphi')^2 + (n - 2) r^{-1} \varphi/\varphi'}}. \tag{4.10}
\]

Since \( \varphi - 1 \in C^{3,\alpha}_{-2\beta + 2} \), this gives us

\[
\lim_{r \to +\infty} \left( \frac{2 \left| r^{2n-1} (\varphi^{(N+2)/2} / \varphi)^{(N-2)/2} \right|}{(N + 2) r^{\frac{4n}{2} - q} \varphi^{N-1} \sqrt{\varphi'/r^{n-2}\varphi}} \right) = \frac{1}{\sqrt{n - 2}} \lim_{r \to +\infty} \frac{(2n - 1) r^{-1} \varphi'/\varphi'}{r^{-1} \varphi'-q} \frac{\sqrt{r^{2n-1} \varphi'}}{\sqrt{r^{2n-1} \varphi'}}. \tag{4.11}
\]

Combined with \((4.9)\), we get

\[
\lim_{r \to +\infty} \left( \frac{\sqrt{r^{2n-1} \varphi'}}{r^{n-q} \varphi'} \right) = \frac{c \sqrt{n - 2}}{2(n - q)} \tag{4.11}
\]

Now, by L'Hôpital’s rule it then follows that

\[
\lim_{r \to +\infty} \left( \frac{\varphi'}{r^{-2q+1}} \right) = \left( \lim_{r \to +\infty} \frac{\sqrt{r^{2n-1} \varphi'}}{r^{n-q} \varphi'} \right)^2 \tag{4.11}
\]

\[
= \left( \lim_{r \to +\infty} \left( \frac{\varphi'}{r^{n-q} \varphi'} \right)^2 \right) = \frac{c \sqrt{n - 2}}{2(n - q)}. \tag{4.11}
\]
Therefore, we have
\[ \varphi - 1 = - \int_{r}^{+\infty} \varphi' \, ds \in C^{0, \alpha}_{-2q+2} \]
and hence thanks to the Lichnerowicz equation and Proposition 2.2(b), we deduce from (4.3) that \((g - \delta_{\text{Euc}}, k) \in C^{\beta, \alpha}_{-2q+2} \times C^{\alpha, \alpha}_{-q} \).

Finally, taking (4.11) into the formula (4.2) of the ADM mass yields
\[ m_{\text{ADM}}(g) = - \frac{c \sqrt{n-2}}{4(n-q)} \lim_{r \to +\infty} r^{n-2q}, \]
which implies (i–iii) except the statement about entire spacelike hypersurfaces in the assertion (iii). However, since \((g, k)\) now satisfies assumptions in the PET, this follows by the rigidity part of the theorem. The proof is completed.

**Remark 4.4.** We observe that our argument for entire spacelike hypersurfaces stated in the assertion (iii) of Theorem 4.3 only uses the rigidity part of the PET. Therefore, the restriction on \(n\) can be omitted once the rigidity holds in arbitrary dimensions. In this context, since \(\mathbb{R}^n\) is spin for all \(n \geq 3\), the assumption \(3 \leq n \leq 7\) in this assertion can be removed as long as \(q > n - 2\) due to the version of PET for spin manifolds proven in [7].

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