REFLECTION POSITIVE AFFINE ACTIONS AND STOCHASTIC PROCESSES

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Abstract. In this note we continue our investigations of the representation theoretic aspects of reflection positivity, also called Osterwalder–Schrader positivity. We explain how this concept relates to affine isometric actions on real Hilbert spaces and how this is connected with Gaussian processes with stationary increments.

1. Introduction

In this note we continue our investigations of the mathematical foundations of reflection positivity, also called Osterwalder–Schrader positivity, a basic concept in constructive quantum field theory [1, 2, 3, 4, 5]. It arises as a requirement on the euclidean side to establish a duality between euclidean and relativistic quantum field theories [6]. It is closely related to “Wick rotations” or “analytic continuation” in the time variable from the real to the imaginary axis.

The underlying concept is that of a reflection positive Hilbert space, introduced in [7]. This is a triple \((\mathcal{E}, \mathcal{E}^+, \theta)\), where \(\mathcal{E}\) is a Hilbert space, \(\theta : \mathcal{E} \to \mathcal{E}\) is a unitary involution and \(\mathcal{E}^+\) is a closed subspace of \(\mathcal{E}\) which is \(\theta\)-positive in the sense that the hermitian form \(\langle \theta u, v \rangle\) is positive semidefinite on \(\mathcal{E}^+\). We write \(\hat{\mathcal{E}}\) for the corresponding Hilbert space and \(q : \mathcal{E}^+ \to \hat{\mathcal{E}}\) for the canonical map.

To see how this relates to group representations, let us call a triple \((G, S, \tau)\) a symmetric Lie semigroup if \(G\) is a Lie group, \(\tau\) is an involutive automorphism of \(G\) and \(S \subseteq G\) a unital subsemigroup (with dense interior) invariant under the involution \(s \mapsto s^\tau := \tau(s)^{-1}\). The Lie algebra \(g\) of \(G\) decomposes into \(\tau\)-eigenspaces \(g = h \oplus q\) and we obtain the Cartan dual Lie algebra \(g^c = h \oplus i q\). \(G^c\) then stands for a Lie group with Lie algebra \(g^c\), often taken simply connected to make it unique. The prototypical pair \((G, G^c)\) consists of the euclidean motion group \(E(d) = \mathbb{R}^d \times O_d(\mathbb{R})\) and the simply connected covering of the orthochronous Poincaré group \(P(d)^\uparrow = \mathbb{R}^d \times O_{1,d-1}(\mathbb{R})^\uparrow\).

If \((G, H, \tau)\) is a symmetric Lie group and \((\mathcal{E}, \mathcal{E}^+, \theta)\) a reflection positive Hilbert space, then we say that a unitary representation \(\pi : G \to U(\mathcal{E})\) is reflection positive with respect to \((G, S, \tau)\) if

\[
\pi(\tau(g)) = \theta \pi(g) \theta \quad \text{for } g \in G \quad \text{and} \quad \pi(S) \mathcal{E}^+ \subseteq \mathcal{E}^+.
\]

Key words and phrases. reflection positivity, reflection positive function, reflection positive representation, reflection positive affine action reflection negative function, Bernstein function.

1As customary in physics, we follow the convenient that the inner product of a complex Hilbert space is linear in the second argument.
If \((\pi, \mathcal{E})\) is a reflection positive representation of \(G\) on \((\mathcal{E}, \mathcal{E}_+, \theta)\), then \(\tilde{\pi}(s)q(v) := q(\pi(s)v)\) defines a representation \((\tilde{\pi}, \hat{\mathcal{E}})\) of the involutive semigroup \((S, \hat{\theta})\) by contractions (Lemma 1.4 in Ref. [7] or Ref. [4]). However, we would like to have a unitary representation \(\pi^c\) of the simply connected Lie group \(G^c\) with Lie algebra \(\mathfrak{g}^c\) on \(\hat{\mathcal{E}}\) whose derived representation is compatible with the representation of \(S\). If such a representation exists, then we call \((\pi, \mathcal{E})\) a euclidean realization of the representation \((\pi^c, \hat{\mathcal{E}})\) of \(G^c\). Sufficient conditions for the existence of \(\pi^c\) have been developed in [8] (see also [3, 9]).

The main new aspects introduced in this short note is a notion of reflection positivity for affine actions of \(G\) on a real Hilbert space. Here \(\mathcal{E}_+\) is generated by the \(S\)-orbit of the origin. On the level of positive definite functions, this leads to the notion of a reflection negative function. For \((G, S, \tau) = (\mathbb{R}, \mathbb{R}^+, -\text{id}_\mathbb{R})\), reflection negative functions \(\psi\) are easily determined because reflection negativity is equivalent to \(\psi|_{[0, \infty)}\) being a Bernstein function. We conclude this note with a brief discussion of connections with some stochastic processes on the real line. Proofs, more details, and more complete results will appear in [10].

2. FROM KERNELS TO REFLECTION POSITIVE REPRESENTATIONS

Since our discussion is based on positive definite kernels, the corresponding Gaussian processes and the associated Hilbert spaces, we first recall the pertinent definitions.

**Definition 2.1.** (a) Let \(X\) be a set. A kernel \(Q: X \times X \to \mathbb{C}\) is called hermitian if \(Q(x, y) = \overline{Q(y, x)}\). A hermitian kernel \(Q\) is called positive, respectively negative, definite, if for \(x_1, \ldots, x_n \in X, c_1, \ldots, c_n \in \mathbb{C}\), we have \(\sum_{j,k=1}^n c_j \overline{c_k} Q(x_j, x_k) \geq 0\), respectively if in addition \(\sum c_j = 0\), we have \(\sum_{j,k=1}^n c_j \overline{c_k} Q(x_j, x_k) \leq 0\) \([11]\).

(b) If \(G\) is a group, then a function \(\varphi: G \to \mathbb{C}\) is called positive (negative) definite if the kernel \((\varphi(gh^{-1}))\) is positive (negative) definite. More generally, if \((S, \ast)\) is an involutive semigroup, then \(\varphi: S \to \mathbb{C}\) is called positive (negative) definite if the kernel \((\varphi(st^\ast))\) is positive (negative) definite.

**Remark 2.2.** Let \(X\) be a set, \(K: X \times X \to \mathbb{C}\) be a positive definite kernel and \(\mathcal{E} = H_K \subseteq \mathbb{C}^X\) the corresponding reproducing kernel Hilbert space. This is the unique Hilbert subspace of \(\mathbb{C}^X\) on which all point evaluations \(f \mapsto f(x)\) are continuous and given by \(f(x) = \langle K_x, f \rangle\) for \(K_x(y) = K(x, y)\). Then the map \(\gamma: X \to H_K, \gamma(x) = K_x\) has total range and satisfies \(K(x, y) = \langle \gamma(x), \gamma(y) \rangle\). The latter property determines the pair \((\gamma, H_K)\) up to unitary equivalence.

**Example 2.3.** (a) Suppose that \(K: X \times X \to \mathbb{C}\) is a positive definite kernel and \(\tau: X \to X\) is an involution leaving \(K\) invariant and that \(X_+ \subseteq X\) is a subset with the property that the kernel \(K^+(x, y) := K(\tau y, x)\) is also positive definite on \(X_+\). Then the closed subspace \(\mathcal{E}_+ \subseteq \mathcal{E} = H_K\) generated by \((K_x)_{x \in X_+}\) is \(\theta\)-positive for \((\theta f)(x) := f(\tau x)\). We thus obtain a reflection positive Hilbert space \((\mathcal{E}, \mathcal{E}_+, \theta)\). We call such kernels \(K\) reflection positive with respect to \((X, X_+, \tau)\).

(b) If \((\mathcal{E}, \mathcal{E}_+, \theta)\) is a reflection positive Hilbert space, then the scalar product defines a reflection positive kernel \(K(v, w) := \langle v, w \rangle\) with respect to \((\mathcal{E}, \mathcal{E}_+, \theta)\). In this sense all reflection positive Hilbert spaces can be obtained in the context of (a), which provides a “non-linear” setting for reflection positive Hilbert spaces.
For symmetric Lie semigroups \((G, S, \tau)\), we obtain natural examples of reflection positive kernels: A function \(\varphi: G \to \mathbb{C}\) is called reflection positive if the kernel \(K(x, y) := \varphi(xy^{-1})\) is reflection positive with respect to \((G, S, \tau)\) in the sense of Example 2.2. These are two simultaneous positivity conditions, namely that the kernel \(\varphi(gh^{-1}))_{g, h \in G}\) is positive definite on \(G\) and that the kernel \(\varphi(st^2))_{s, t \in S}\) is positive definite on \(S\). Here \(X = G\) and \(X_+ = S\).

Prototypical examples are the functions \(\varphi(t) = e^{-\lambda|t|}, \lambda \geq 0\), for \((\mathbb{R}, \mathbb{R}_+, -id_\mathbb{R})\) with \(\hat{E}\) one dimensional. For this triple every continuous reflection positive function has an integral representation \(\varphi(t) = \int_0^\infty e^{-\lambda|t|} d\mu(\lambda)\) for a positive measure \(\mu\) on \([0, \infty)\) (see Cor. 3.3 in Ref. [7]). In Ref. [7] we also discuss generalizations of this concept to distributions and obtain integral representations for the case where \(G\) is abelian.

3. Second quantization and Gaussian processes

**Definition 3.1.** [12] Let \(T\) be a set and \(\mathbb{K} = \mathbb{R}\) or \(\mathbb{C}\). A \(\mathbb{K}\)-valued stochastic process \((X_t)_{t \in T}\) is said to be Gaussian if, for all finite subsets \(F \subseteq T\), the corresponding distribution of the random vector \(X_F = (X_t)_{t \in F}\) with values in \(\mathbb{K}^F\) is Gaussian.

**Definition 3.2.** (Second quantization) (a) For a real Hilbert space \(\mathcal{H}\), we write \(\mathcal{H}^*\) for its algebraic dual, i.e., the set of all (not necessarily continuous) linear functionals \(\mathcal{H} \to \mathbb{R}\), continuous or not. Let \(\Gamma(\mathcal{H}) := L^2(\mathcal{H}^*, \gamma, \mathbb{C})\), where \((\mathcal{H}^*, \Sigma, \gamma)\) is the canonical Gaussian measure space on \(\mathcal{H}^*\) (Ex. 4.3 in Ref. [13]; see also Ref. [14]). Here \(\Sigma\) is the smallest \(\sigma\)-algebra for which all evaluations \(\phi(v): \mathcal{H}^* \to \mathbb{R}, \alpha \mapsto \alpha(v)\), \(v \in \mathcal{H}\), are measurable and the probability measure \(\gamma\) is determined uniquely by

\[
\mathbb{E}(e^{i\phi(v)}) = e^{-\|v\|^2/2} \quad \text{for} \quad v \in \mathcal{H}. \tag{3.1}
\]

Considering the \(\phi(v)\) as random variables, we thus obtain the canonical Gaussian process \((\phi(v))_{v \in \mathcal{H}}\) over \(\mathcal{H}\). It satisfies

\[
\mathbb{E}(\phi(v)) = 0 \quad \text{and} \quad \mathbb{E}(\phi(v)\phi(w)) = \langle v, w \rangle \quad \text{for} \quad v, w \in \mathcal{H}.
\]

**Remark 3.3.** There are many realizations of Gaussian measure spaces over real Hilbert spaces. The one chosen above has the advantage that it directly leads to a natural unitary representation of the motion group \(\text{Mot}(\mathcal{H}) \cong \mathcal{H} \times O(\mathcal{H})\) of \(\mathcal{H}\) by

\[
(\rho(b, g)F)(\alpha) = e^{i\phi(b)\alpha}F(g^*\alpha)
\]

for which the map \(v \mapsto e^{i\phi(v)} = \rho(v, 1)1\) is equivariant (cf. Remark 3.7). Its range is total, and the corresponding positive definite function on \(\text{Mot}(\mathcal{H})\) is given by

\[
\varphi(b, g) = \langle 1, \rho(b, g)1 \rangle = \mathbb{E}(e^{i\phi(b)}) = e^{-\|b\|^2/2}. \tag{3.2}
\]

**Remark 3.4.** Suppose that \(\mathcal{H}\) is real or complex Hilbert space, \(T\) a set, \(\gamma: T \to \mathcal{H}\) a map, and \((\phi(v))_{v \in \mathcal{H}}\) the canonical Gaussian process indexed by \(\mathcal{H}\) (Definition 3.1). Then \((\phi(\gamma(t)))_{t \in T}\) is a centered Gaussian process indexed by \(T\) with covariance kernel

\[
C(s, t) = \mathbb{E}(\phi(\gamma(s))\phi(\gamma(t))) = \langle \gamma(s), \gamma(t) \rangle.
\]

In view of Remark 3.2, a kernel on \(T\) is the covariance kernel of a Gaussian process if and only if it is positive definite. If \(\gamma(T)\) is total in \(\mathcal{H}\), then the corresponding Gaussian process is full in the sense that, up to sets of measure 0, \(\Sigma\) is the smallest \(\sigma\)-algebra for which every \(X_t\) is measurable. Conversely, for every full and centered Gaussian process \((X_t)_{t \in T}\) with covariance kernel \(C\) on the
probability space \((Q, \Sigma, \mu)\), there exists a uniquely determined unitary operator \(U: \Gamma(\mathcal{H}) \to L^2(Q, \Sigma, \mu)\) with \(U(\phi(\gamma(t))) = X_t\) for \(t \in T\) (Thm. 1.10 in Ref. [12]).

**Definition 3.5.** (Normalization) If \((X_t)_{t \in T}\) is a centered Gaussian process with \(E(|X_t|^2) > 0\) for every \(t \in T\) and covariance kernel \(C\), then \(\tilde{X}_t := X_t / \sqrt{E(|X_t|^2)}\) is called the associated normalized process. Its covariance kernel has the form

\[
\tilde{C}(t, s) = \frac{C(t, s)}{\sqrt{C(t, t)C(s, s)}} \quad \text{for} \quad t, s \in T.
\]

**Definition 3.6.** Let \((X_t)_{t \in T}\) be a centered real-valued Gaussian process with covariance kernel \(C : T \times T \to \mathbb{R}\) and \(\sigma : G \times T \to T, (g, t) \mapsto g.t\) a group action.

(a) The process \((X_t)_{t \in T}\) is called stationary if

\[
C(g.t, g.s) = C(t, s) \quad \text{for} \quad g \in G, t, s \in T.
\]

For any realization \(\gamma : T \to \mathcal{H}\) with total range as in Remark 3.4 we thus obtain a uniquely determined orthogonal representation \(U : G \to O(\mathcal{H})\) satisfying \(U(\gamma(t)) = \gamma(g.t)\) for \(g \in G, t \in T\).

(b) The process \((X_t)_{t \in T}\) is said to have stationary increments if the kernel

\[
D(t, s) := E((X_t - X_s)^2) = C(t, t) + C(s, s) - 2C(t, s), \quad (3.3)
\]

is \(G\)-invariant. For any realization \(\gamma : T \to \mathcal{H}\) with total range as in Remark 3.4 we thus obtain a unique affine action \(\alpha : G \to \text{Mot}(\mathcal{H})\) satisfying \(\alpha_g \gamma(t) = \gamma(g.t)\) for \(g \in G, t \in T\) [15,16].

**Remark 3.7.** (a) The canonical Gaussian process \((\phi(v))_{v \in \mathcal{H}}\) over the real Hilbert space \(\mathcal{H}\) is stationary for the orthogonal group \(O(\mathcal{H})\) and has stationary increments for the motion group \(\text{Mot}(\mathcal{H})\) because the kernel

\[
\mathcal{D}(v, w) = E((\phi(v) - \phi(w))^2) = \|v\|^2 + \|w\|^2 - 2\langle v, w \rangle = \|v - w\|^2
\]

is invariant under all isometries.

(b) We also observe that

\[
K(v, w) := E(e^{i\phi(v)}e^{i\phi(w)}) = E(e^{i\phi(w-v)}) = e^{-\|v-w\|^2/2}.
\]

The functions \((e^{i\phi(v)})_{v \in \mathcal{H}}\) form a total subset of \(\Gamma(\mathcal{H})\) and the map \(\eta : \mathcal{H} \to \Gamma(\mathcal{H}), \eta(v) = e^{i\phi(v)}\) is \(\text{Mot}(\mathcal{H})\)-equivariant with respect to the representation \(\rho\). This reflects the \(\text{Mot}(\mathcal{H})\)-invariance of the kernel \(K\).

(b) The random field \((e^{i\phi(v)})_{v \in \mathcal{H}}\) arises by normalization of the process given by

\[
\Gamma(v) := e^{\|v\|^2/2}e^{i\phi(v)} \quad \text{with} \quad E(\Gamma(v)\Gamma(w)) = e^{\|v\|^2 + \|w\|^2 - \|v-w\|^2} = e^{\langle v, w \rangle}.
\]

4. **Reflection positive affine actions**

Let \((G, S, \tau)\) be a symmetric Lie group and \(\mathcal{E}\) be a real Hilbert space, endowed with an isometric involution \(\theta\). We consider an affine isometric action

\[
\alpha_g v = U_g v + \beta_g \quad \text{for} \quad g \in G, v \in \mathcal{E},
\]

where \(U : G \to O(\mathcal{E})\) is an orthogonal representation and \(\beta : G \to \mathcal{E}\) a 1-cocycle, i.e., \(\beta_{gh} = \beta_g + U_g \beta_h\) for \(g, h \in G\). We further assume that \(\theta \alpha_g \theta = \alpha_{\tau(g)}\), which is equivalent to

\[
\theta U_g \theta = U_{\tau(g)} \quad \text{and} \quad \theta \beta_g = \beta_{\tau(g)} \quad \text{for} \quad g \in G.
\]
Then the positive definite kernel $C(s, t) := \langle \beta_s, \beta_t \rangle$ and the negative definite function $\psi(g) := \|\beta_g\|^2$ on $G$ are related by
\[
C(s, t) = \frac{1}{2} (\psi(s) + \psi(t) - \psi(s^{-1}t)), \quad \psi(s^{-1}t) = C(s, s) + C(t, t) - 2C(t, s) = \|\beta_s - \beta_t\|^2
\]
(cf. (3.3)). In particular, the affine action $\alpha$ can be recovered completely from the function $\psi$ (cf. Definition 3.6 [15, 16]). We also note that $\theta \beta_g = \beta_{\tau(g)}$ implies that $\psi \circ \tau = \psi$. The action of $G$ on $E$ preserves the positive definite $\theta$-invariant kernel $Q(x, y) := e^{-\|x-y\|^2/2}$ (Remark 3.7).

**Definition 4.1.** (Reflection positive affine actions) We now consider the closed subspace $E$ of natural Gaussian processes on the real line, such as Brownian motion and its relatives.

**Remark 4.3.** According to Schoenberg’s Theorem for kernels (Thm. 3.2.2 in Ref. [7]), this leads us to the following concept:

**Theorem 4.4.** A symmetric continuous function $\psi: \mathbb{R} \to \mathbb{R}$ is reflection negative if and only if $\psi_{|\mathbb{R}_+}$ is a Bernstein function. In particular, this is equivalent to the existence of $a, b \geq 0$ and a positive measure $\mu$ on $(0, \infty)$ with $\int_0^\infty (1 + \lambda) \, d\mu(\lambda) < \infty$ such that
\[
\psi(t) = a + b|t| + \int_0^\infty (1 - e^{-\lambda t}) \, d\mu(\lambda)
\]
(Lévy–Khintchine representation). Here $a, b$ and $\mu$ are uniquely determined by $\psi$.

**Example 4.5.** (Cor. 3.2.10 in Ref. [11]) For $\alpha \geq 0$, the function $\psi(t) := |t|^\alpha$ is reflection negative on $\mathbb{R}$ if and only if $0 \leq \alpha \leq 1$. For $0 < \alpha < 1$, this follows from the integral representation $t^\alpha = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-\lambda t}) \lambda^{-1-\alpha} \, d\lambda$ for $t > 0$.

5. **RELATIONS TO STOCHASTIC PROCESSES**

In this last section, we discuss the abstract concepts from above in the context of natural Gaussian processes on the real line, such as Brownian motion and its relatives.
Example 5.1. Two sided Brownian motion is a real-valued centered Gaussian process $(B_t)_{t \in \mathbb{R}}$ with covariance kernel

$$C(s,t) = \mathbb{E}(B_s B_t) = \frac{1}{2}(|s| + |t| - |s-t|) \quad \text{for } s, t \in \mathbb{R}$$

and

$$D(s,t) = \mathbb{E}((B_s - B_t)^2) = C(s,s) + C(t,t) - 2C(s,t) = |s - t|.$$  

In particular, it has stationary increments. A natural realization in $\Gamma(L^2(\mathbb{R}))$ is given by $B_t = \phi(b_t)$ for $b_t := \text{sgn}(t) \chi_{[t,0,t\vee 0]}$. Then the corresponding affine isometric action of the additive group $\mathbb{R}$ on $L^2(\mathbb{R})$ is given by

$$\alpha f = S_t f + b_t, \quad \text{where } (S_t f)(x) = f(x - t).$$

This action is reflection positive because the function $\psi(t) := \|b_t\|^2 = |t|$ is reflection negative for $(\mathbb{R}, \mathbb{R}_+, -\text{id}_{\mathbb{R}})$. Since $C(s,-t) = 0$ for $t, s \geq 0$, we get $\mathcal{E} = \{0\}$ if $\mathcal{E} = L^2(\mathbb{R})$ and $\mathcal{E}_+ = L^2(\mathbb{R}_+)$. More generally, for a reflection positive affine action $(\alpha, \mathcal{E})$ of $\mathbb{R}$ one can show that $\mathcal{E}$ is trivial if and only if $(\beta_\alpha, \beta_\mathcal{E}) = 0$ for $ts < 0$ (see Ref. [10]). This property characterizes Brownian motion (up to positive multiples) among Gaussian processes with stationary increments.

Example 5.2. For $0 < H < 1$, fractional Brownian motion is a centered Gaussian process $(X_t)_{t \in \mathbb{R}}$ with stationary increments, covariance kernel

$$C(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}) \quad \text{and} \quad D(t,s) = C(t,t) + C(s,s) - 2C(t,s) = |t-s|^{2H}.$$  

Therefore it corresponds to the negative definite function $\psi(t) := |t|^{2H}$, which is reflection negative if and only if $0 \leq H \leq \frac{1}{2}$ (Example 4.5). Note that $H = 1/2$ corresponds to Brownian motion.

Example 5.3. One sided Brownian motion $(B_t)_{t \geq 0}$ has the covariance kernel $C(s,t) = s \wedge t$ and the corresponding normalized process on $(0, \infty)$ has the kernel $\widetilde{C}(s,t) = \frac{s \wedge t}{\sqrt{s t}} = \frac{s t}{\sqrt{s t}}$. This kernel is invariant under the dilation group $\mathbb{R}_+^\times$, so that the process $(\widetilde{B}_t)_{t \geq 0}$ is stationary with respect to dilations.

Accordingly, the natural realization by $B_t = \phi(b_t) \in \Gamma(L^2(\mathbb{R}_+))$ with $b_t = \chi_{[0,t]}$ and its normalization $\widetilde{b}_t = t^{-1/2}b_t$ are compatible with the unitary representation of the dilation group on $L^2(\mathbb{R}_+)$ by $(\tau_t f)(x) = e^{t/2}f(e^t x)$ in the sense that $\tilde{\tau}_{e^{-t}} = \tau_{\chi_{[0,1]}} = t \tau_{\tilde{b}_1}$.

The invariance of the kernel $\widetilde{C}$ under the involution $\tau(t) = t^{-1}$ of $\mathbb{R}_+$ implies on $L^2(\mathbb{R}_+)$ the existence of a unique unitary involution $\theta$ with $\theta(\tilde{b}_t) = \tilde{b}_{1/t}$. On $(0,1)$, the reflected kernel $\widetilde{C}(s,t^{-1}) = \sqrt{\frac{s t}{s + t}} = \sqrt{\frac{s t}{s + t}}$ is positive definite, so that we obtain with $\mathcal{E} = L^2(\mathbb{R}_+)$ and $\mathcal{E}_+ = L^2([0,1])$ a reflection positive Hilbert space on which the dilation representation of $\mathbb{R}$ is reflection positive. The corresponding reflection positive function on $\mathbb{R}$ is

$$\varphi(t) = \langle \chi_{[0,1]}, \tau\chi_{[0,1]} \rangle = \langle \tilde{b}_1, \tilde{b}_{e^{-t}} \rangle = \tilde{C}(1, e^{-t}) = e^{-|t|/2}$$

and $\mathcal{E}$ is one dimensional.
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