Dynamical Projective Curvature in Gravitation

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Abstract

By using a projective connection over the space of two-dimensional affine connections, we are able to show that the metric interaction of Polyakov 2D gravity with a coadjoint element arises naturally through the projective Ricci tensor. Through the curvature invariants of Thomas and Whitehead, we are able to define an action that could describe dynamics to the projective connection. We discuss implications of the projective connection in higher dimensions as related to gravitation.

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1 Summary of Paper

In one dimension, the smooth circle and line are characterized by the algebra of vector fields, called the Virasoro algebra. It can be further endowed with more structure by adding a gauge group so that points on the circle are mapped into the group. The Kac-Moody algebra locally characterizes this added structure. The symmetries of the circle lead to two dimensional physics through symplectic structures associated with the coadjoint orbits of the Virasoro algebra and Kac-Moody algebras. The coadjoint elements \((\mathcal{D}, A)\), corresponding to a quadratic differential \(\mathcal{D}\) and a gauge field \(A\), are realized as a cosmological term in the effective action of two-dimensional gravity and background gauge field coupled to a WZW model, respectively. The gauge field has meaning in any dimension and can be made dynamical by adding a Yang-Mills action in 2D and higher. In this note we address two questions: 1) Is there a principle that can give dimensional ubiquity for the field \(\mathcal{D}\)? and 2) What would be the corresponding dynamical action? In other words, by “lowering” the dimension to 2D, are new gravitational degrees of freedom that are independent of Einstein now manifest and can they have an interpretation in higher dimensions? We answer these questions by first recognizing that \(\mathcal{D}\) is related to projective geometry, a notion that is well-defined in higher dimensions. Then by using the Thomas-Whitehead projective connection \(\tilde{\nabla}\), we are able to describe a projective curvature invariant and build a dynamical action for \(\mathcal{D}\) modelled on the Gauss-Bonnet action in Riemannian geometry. We discuss the gravitational consequences of this action in two and four dimensions.

2 Review of Coadjoint Orbits

The Virasoro algebra is the unique central extension of the Witt algebra, which is the algebra of vectors on the circle \(S^1\). Let \(\xi\) and \(\eta\) be vectors and \(\alpha\) and \(\beta \in \mathbb{R}\). The algebra is defined by its commutation relations,

\[
[(\mathcal{L}_\xi, \alpha), (\mathcal{L}_\eta, \beta)] = (\mathcal{L}_{\xi \eta}, (\xi, \eta)\alpha),
\]

where

\[
\mathcal{L}_\xi \eta = -\xi \partial \eta + \eta \partial \xi = -\mathcal{L}_{\eta} \xi = (\xi \circ \eta),
\]

implying that,

\[
[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{\xi \eta}.
\]
The two-cocycle, \((\xi, \eta)_0\) is the Gelfand-Fuchs cocycle defined by

\[
(\xi, \eta)_0 = \frac{c}{2\pi} \int (\xi \eta''') dx,
\]

and \(c\) a constant. This is a pairing of the vector \(\xi\) with the one-cocycle of \(\eta\). This one-cocycle arises from a projective transformation that has mapped the vector field \(\eta\) to a quadratic differential,

\[
\eta \partial_\theta \rightarrow \eta''' d\theta^2.
\]

By introducing the coadjoint representation on the dual space of the algebra, coadjoint elements of the Virasoro algebra are represented by \((B, c)\), which is in the direct sum of the dual of the tangent bundle of \(S^1\) and the reals through an invariant pairing, \(< (\xi, \alpha) | (B, c) >\). The invariance of the pairing then requires that the action of a Virasoro element \((\xi, \alpha)\) on a dual element \((B, c)\) yields

\[
ad^*_{(\eta, \alpha)} (B, c) = (\eta B' + 2\eta' B - c\eta''', 0).
\]

The first two summands are the Lie derivative of a quadratic differential. However, the last summand is a one-cocycle used in the Gelfand-Fuchs cocycle, making the action an affine module \([1, 2]\). Kirillov \([3]\) observed that this action is the same as the action of vector fields on the space of Sturm-Liouville operators. Here, \((B, c)\) defines a family of Sturm-Liouville operators, i.e.

\[
(B, c) \leftrightarrow -2c \frac{d^2}{dx^2} + B(x),
\]

where on the left side \((B, c)\) is identified with a centrally extended coadjoint element of the Virasoro algebra and on the right side it is a Sturm-Liouville operator. In this way we are able to make direct contact with one-dimensional projective geometry because the invariants of the Virasoro algebra can be expressed in terms of invariants of Sturm-Liouville, which define a projective structures on \(S^1\). So the Virasoro invariant two-cocycle,

\[
(\xi, \eta)_{(B, c)} = \frac{c}{2\pi} \int (\xi \eta''' - \xi''' \eta) dx + \frac{1}{2\pi} \int (\xi \eta' - \xi' \eta) B dx,
\]

is intimately related to the Sturm-Liouville invariant,

\[
L_{(B, c)} \phi(x) \equiv (-2c \frac{d^2}{dx^2} + B(x)) \phi(x) = 0,
\]
identifying $B$ as a projective connection \cite{3}. It is well-known that in both cases, $(B,0)$ transforms as a quadratic differential along with a Schwarzian derivative at the group level. The Schwarzian derivative is perhaps the most well-known and ubiquitous example of a projective differential geometry invariant \cite{1, 2}. In what follows, we will return to this correspondence by relating the two-cocycle explicitly to the Thomas-Whitehead connection in Section 4.2 below.

One of the ways this is of interest to physicists is with respect to string theory and gravity through the quantization of the Virasoro algebra and the anomalous contributions to the 2D effective quantum gravitational action. By studying coadjoint orbits \cite{4–9} one gets both a geometric understanding of the underlying vacuum structure of gravity in 2D and the anomalous contributions to gravitation. By explicitly studying the field theory associated with the semi-direct product of the Virasoro algebra and an affine Lie algebra that defines a gauge theory, one shows that the WZW \cite{10} model and Polyakov \cite{11} 2D quantum gravity, derived via path integral quantization of two-dimensional chiral fermions, are the geometric actions associated with the affine Kac-Moody algebra and the Virasoro algebra respectively.

Each coadjoint orbit admits a natural symplectic two-form, $\Omega_{(B,c)}(\xi, \eta)$ \cite{3, 12} on centrally extended vector fields $\xi$ and $\eta$. Because of this, the dual of the Virasoro algebra can be foliated into distinct classical and quantum mechanical systems via the equivalence classes of the coadjoint elements, i.e. $[(B, c)]$. This gives rise to a Hilbert space structure which is a direct sum of the equivalence classes \cite{4}

$$
\mathcal{H} = \oplus_i \mathcal{H}_{[(B, c)]}.
$$

(2.10)

From a quantum mechanical viewpoint, the coadjoint elements determine the vacuum structure of their respective Hilbert spaces. By using this natural symplectic two form on coadjoint orbits and a strategy for integrating the two-form \cite{13–15}, one also relates each orbit specified by coadjoint element $(\mathcal{D}, c)$ with a 2D gravitational action \cite{5}

$$
S = \frac{c}{2\pi} \int dx d\tau \left[ \frac{\partial^2 s}{(\partial s)^3} \partial_s \partial_s s - \frac{(\partial^2 s)^2 (\partial s)}{(\partial s)^3} \right] - \int dx d\tau \mathcal{D}(x) \frac{(\partial s/\partial \tau)}{(\partial s/\partial x)},
$$

(2.11)

where $s(\theta; \lambda, \tau)$ corresponds to a two-parameter family of elements of the Virasoro group. Changing the notation $x \to x_-, \tau \to x_+, s \to f$, and $\mathcal{D} \to 0$, this action is identical to Polyakov’s action \cite{11}, viz

$$
S = \frac{c}{2\pi} \int d^2 x \left[ (\partial_-^2 f) \left( \partial_+ \partial_- f \right) \left( \partial_- f \right)^{-2} - \left( \partial_-^2 f \right)^2 \left( \partial_+ f \right) \left( \partial_- f \right)^{-3} \right]
$$

(2.12)
where the gauge fixed metric is

\[ g_{ab} = \begin{pmatrix} 0 & \frac{1}{2} h_{++}(\theta, \tau) \\ \frac{1}{2} h_{++}(\theta, \tau) & 0 \end{pmatrix}, \tag{2.13} \]

and where \( h_{++} = (\partial_+ f / \partial_+ f) \) for a function \( f(x_-, x_+) \) in light-cone coordinates. The interaction term with the coadjoint element \( (\mathcal{D}, 0) \) admits a term

\[ S_{(\mathcal{D}, g)} = \int d^2 x \mathcal{D} (\partial_+ f / \partial_+ f), \tag{2.14} \]

suggesting that \( \mathcal{D} \) is the \( \mathcal{D} = \mathcal{D}_{--} \) component of an external background field \( \mathcal{D}_{\mu \nu} \) in two dimensions. Here it serves as a background cosmological term that influences the energy-momentum tensor.

In Section 4.3 we will show that by using a projective connection over the affine connections in two dimensions, this term appears naturally as a background field in the Einstein-Hilbert in 2D. We will go further and ascribe dynamics to this field in Section 4.5. This is akin to adding the Yang-Mills action to a WZW model that is coupled to a background gauge field \( A_\mu \). Understanding the dynamics of the coadjoint elements in the Virasoro algebra will help to understand the stability of the quantization of orbits such as \( \text{diff}S^1/\text{SL}(n, \mathbb{R}) \) as well as non-trivial cosmological contributions to gravitation in higher dimensions. The partnership with the vector potential \( A_\mu \) (we will take \( SU(M) \) as our gauge group) and the quadratic differential \( \mathcal{D}_{\mu \nu} \) in describing the semi-direct product of the Kac-Moody (affine Lie algebra) and the Virasoro algebra is another impetus for making \( \mathcal{D}(\theta) \) dynamical.

To explicitly show this partnership, let’s review the relevant literature [16, 17]. As discussed above, the coadjoint representation appears as the dual of the adjoint representation through a pairing between the two representations, \( \langle (\xi, \alpha) | (B, c) \rangle \). We will consider the analogous pairing for the algebra of the semi-direct product of the Virasoro algebra with an affine Lie algebra and its dual. We write the mode decomposition of the Virasoro algebra and the affine Lie algebra with structure constants \( f^{\alpha \beta \gamma} \) on the circle by

\[ [L_N, L_M] = (N - M) L_{N+M} + c N^3 \delta_{N+M,0} \tag{2.15} \]

for the Virasoro sector,

\[ [J^\alpha_N, J^\beta_M] = i f^{\alpha \beta \gamma} J^\gamma_{N+M} + N k \delta_{N+M,0} \delta^{\alpha \beta} \tag{2.16} \]
for the affine Kac-Moody algebra, and
\[ [L_N, J_M^\alpha] = -M J_{N+M}^\alpha. \]  (2.17)
for the interacting commutation relations. To be explicit, the algebra may be realized on the circle by
\[ L_N = e_N^a \partial_a = ie^{iN\theta} \partial_\theta, \quad J_N^\alpha = \tau^\alpha e^{iN\theta}, \]  (2.18)
so that a basis for the centrally extended algebra can be written as
\[ \left(L_A, J_B^\beta, \rho\right). \]  (2.19)
Here the last component is in the center of the algebra. The adjoint representation acts on itself through the commutation relations and explicitly is given by:
\[ \left(L_A, J_B^\beta, \rho\right) \ast \left(L_N, J_M^\alpha, \mu\right) = \left(L_{\text{new}}, J_{\text{new}}^\beta, \mu\right) \]  (2.20)
where
\begin{align*}
L_{\text{new}} &= (A - N) L_{A+N} \\
J_{\text{new}} &= -M J_{A+M}^\alpha + B J_B^\beta + i f^\beta \alpha \lambda J_{B+M}^\lambda \\
\lambda &= (cA^3) \delta_{A+N,0} + B k \delta^\alpha \delta_{B+M,0}.
\end{align*}  (2.21)
In a similar way, we can construct a basis for the coadjoint representation. The coadjoint elements can be realized as \(\left(\tilde{L}_N, \tilde{J}_M^\alpha, \tilde{\mu}\right)\):
\[ \tilde{L}_N = e_{ab}^N dx^a dx^b = -ie^{-iN\theta} d\theta^2, \quad \tilde{J}_N^\alpha = A_{a}^N,\alpha dx^a = \tau^\alpha e^{-iN\theta}, \]  (2.22)
and \(\tilde{\mu}\) is a constant. The \(e_{ab}^N\) are the components of a quadratic differential and the \(A_{a}^N,\alpha\) are one-form components. The pairing between the modes of the coadjoint representation and the adjoint representation is explicitly:
\[ \left\langle \left(\tilde{L}_N, \tilde{J}_M^\alpha, \tilde{\mu}\right) \middle| \left(L_A, J_B^\beta, \rho\right) \right\rangle = \frac{1}{2\pi i} \int \left(e_{ab}^N e_{ab}^N + \text{tr}(J_B^\beta A_b^M,\alpha)\right) dx^b + \rho \tilde{\mu} \]
\[ = \frac{1}{2\pi i} \int \left(e^{iA\theta} e^{-iN\theta} + \text{tr}(\tau^\beta e^{iB\theta} \tau^\alpha e^{-iM\theta})\right) d\theta + \rho \tilde{\mu} \]
\[ = \delta_{N,A} + \delta^\alpha \delta_{M,B} + \rho \tilde{\mu}, \]  (2.23)
where \( \text{tr}(\tau^\alpha \tau^\beta) = \delta^{\alpha\beta} \). Invariance of the pairing with respect to the action of the adjoint representation gives the transformation laws for the coadjoint representation [16],

\[
\left( L_A, J^\beta_B, \rho \right) \star \left( \tilde{L}_N, \tilde{J}^\alpha_M, \tilde{\mu} \right) = \left( \tilde{L}_{\text{new}}, \tilde{J}^\alpha_M, 0 \right) \quad \text{with},
\]

\[
\tilde{L}_{\text{new}} = (2A - N)\tilde{L}_{N-A} - B\delta^{\alpha\beta}\tilde{L}_{M-B} - \tilde{\mu}(cA^3)\tilde{L}_{-A} \quad \text{and}
\]

\[
\tilde{J}^\alpha_M = (M - A)\tilde{J}^\alpha_{M-A} - if^{\beta\nu}\tilde{J}^\nu_{M-B} - \tilde{\mu}B k j_{-B}^\beta.
\]

In terms of the mode decomposition, the one-dimensional vector fields \( \xi^j \) and the matrix valued gauge parameter \( \Lambda^I_J \) are given by

\[
\xi^j = \sum_{N=-\infty}^{\infty} e^j_N \xi^N \quad \text{and} \quad \Lambda^I_J = \sum_{\alpha=1}^{q} \sum_{N=-\infty}^{\infty} \Lambda^N_{\alpha} (J^\alpha_N)^I_J,
\]

where \( q \) is the dimension of the affine Lie algebra. Then a generic member of the algebra may be written as the three-tuple

\[
\mathcal{F} = (\xi(\theta), \Lambda(\theta), a)
\]

containing a one-dimensional vector field \( \xi^i \), an \( M \times M \) matrix valued gauge parameter \( \Lambda^I_J \), and a central element \( a \). The pairing in the Virasoro sector is a contraction of a vector field \( \xi^i \) and a quadratic differential \( D_{ij} \),

\[
< \xi, D > = \int \xi^i D_{ij} dx^j,
\]

and the pairing in the Kac-Moody sector of a gauge parameter \( \Lambda^I_J \) and a dual element \( (A_i)^I_J \) is given by

\[
< \Lambda, A > = \int \text{tr}(\Lambda A_j) dx^j.
\]

Therefore we may write

\[
D = \sum_{n=-\infty}^{\infty} D^n \tilde{L}_n \quad \text{and} \quad A = \sum_{\alpha=1}^{M} \sum_{n=-\infty}^{\infty} \Lambda^n_{\alpha} \tilde{J}^\alpha_n.
\]

The coadjoint element is the three-tuple

\[
B = (D(\theta), A(\theta), \mu),
\]

6
which consists of a rank two projective connection $\mathcal{D}_{ab}$, a gauge connection $A_a$ and a corresponding central element $\mu$. In this way the coadjoint action of an adjoint element $\mathcal{F} = (\xi(\theta), \Lambda(\theta), a)$ on a coadjoint element $\mathcal{B} = (\mathcal{D}(\theta), A(\theta), \mu)$ gives the transformation law \[16\]

$$\delta \tilde{\mathcal{B}}_F = (\xi(\theta), \Lambda(\theta), a) \ast (\mathcal{D}(\theta), A(\theta), \mu) = (\delta \mathcal{D}(\theta), \delta A(\theta), 0).$$

(2.33)

The transformation laws now have the interpretation of a one-dimensional projective transformation on the projective connection $\mathcal{D}(\theta)$ and gauge connection $A$, and the accompanying gauge transformations on these fields, i.e

$$\delta \mathcal{D}(\theta) = 2\xi' \mathcal{D} + \mathcal{D}' \xi + \frac{c\mu}{2\pi} \xi''' - \text{coordinate transformation} \quad \text{gauge transformation}$$

(2.34)

and

$$\delta A(\theta) = A' \xi + \xi' A' - \left[\Lambda A - A \Lambda\right] + k \mu \Lambda'.$$

(2.35)

We note that $\mathcal{D}(\theta)$ transforms inhomogeneously under the projective transformation, while $A(\theta)$ transforms inhomogeneously under the gauge transformation. In terms of two-dimensional Yang-Mills, $A(\theta)$ may be regarded as the space component of a vector potential, $A_\mu = (A_\tau, A_\theta)$, where one uses the temporal gauge to fix $A_\tau = 0$. When Eq.(2.35) is set to zero and the coordinate transformations are ignored, one sees that this parallels a Gauss Law constraint from a Yang-Mills theory where the one-dimensional electric field $E_\theta$ takes the place of $\Lambda$. In \[16, 18–23\] the authors argued that there was an analogous field theory related to the dual of the Virasoro algebra that, like Yang-Mills, could be written in any dimension. Then one could lift the one-dimensional identity of $\mathcal{D}$ to a field theory in two-dimensions and higher. In Section 4.3, we will show that by identifying $\mathcal{D}$ with a projective connection $\tilde{\Gamma}_{\alpha\beta\gamma}$ that admits a projective Ricci tensor $K_{\alpha\beta}$, the interaction term 2.14 can be written as

$$S_{\text{int}} = \int d^3x \sqrt{(-G)} K_{\alpha\beta} G^{\alpha\beta}$$

(2.36)

where $G_{\alpha\beta}$ will correspond to a specific three-dimensional metric derived in Section 4.1 that contains the 2D metric $g_{ab}$. This identification adds dimensional ubiquity to the diffeomorphism field $\mathcal{D}_{ab}$. Furthermore, in Section 4.5 we postulate that the dynamical
theory for $\mathcal{D}$ may be written as a projective version of the Gauss-Bonnet action,

$$S_\mathcal{D} = \int d^3x \sqrt{(-G)} \left( K^\alpha_{\beta\gamma\rho} K^\beta_{\alpha\gamma\rho} - \frac{1}{4} K^\alpha_{\alpha\beta} K^\beta_{\alpha\beta} + K^2 \right),$$

(2.37)

using the projective curvature tensor. The three-dimensional metric $G_{\alpha\beta}$ arises from the chiral Dirac $\gamma^3$ matrix, which is used to define the third dimension. The construction is designed to be dimensionally independent and furthermore, the projective Gauss-Bonnet action can be used in 2 to 4 space-time dimensions without introducing higher time derivatives on the underlying metric. In Section 4.5, we will also derive the field equations and energy-momentum tensor for any dimension. This work promotes the diffeomorphism field $D_{ab}$ from a remnant of the dual of the Virasoro algebra to a rank two tensor in any dimension. Its purely geometric origin may have merits in describing gravitational and cosmological phenomena, especially as a candidate for dark energy and dark matter.

3 Review of Projective Curvature

3.1 Projective Structure and the Ricci Tensor

Before discussing the results of this paper, we review the relationship between projective structure and the Ricci tensor. The underlying idea is the equivalence between the family of geodesics on a manifold, and the differential operators that give rise to the same geodesics. In the study of sprays (for a review see [24]), which may be viewed as the space of geodesics on a manifold and their derivatives, geodesic equations, Sturm-Liouville and Laplace equations are all placed on similar footing. We are interested in the structure of geodesic equations under reparameterization. For example in Eq. (2.8) one could ask how the coadjoint element transforms under reparameterization so that the two-cocycle remains invariant. Sturm-Liouville theory investigates the same question about how the Liouville potential transforms under reparameterization of its parameter. These operations have a leading quadratic differential operator. Indeed, Weyl [25] asked the same question regarding geodesics more than ninety years ago. The idea of projective connections originated with Weyl, Thomas, Whitehead and Cartan [25–27]. It is born from the idea that a manifold $M$ can be characterized by its geodesics and geodetics (we will discuss the distinction presently). Consider the geodesic equation,

$$\xi^a \nabla_a \xi^b = f \xi^b,$$

(3.1)
where $f$ is a function. Here, the vector field $\xi$ has an affine parameter $\tau$, so that $\xi^a \nabla_a \tau = 1$. This geodesic equation has an inherent symmetry. For one thing, geodesics and geodetics ($f = 0$) can be related by a suitable rescaling of $g \xi^a = \zeta^a$. Indeed for any differentiable function $g$, $\xi^a$ is also geodesic. In particular, when $g$ satisfies $\frac{d}{d\tau} (\log g(\tau)) = -f(\tau)$, $\xi^a$ is geodetic. We will refer to both geodetics and geodesics as geodesics from now on. Writing $\zeta^a = \frac{d}{d\tau} x^a$, the geodesic equation in terms of the affine connection $\Gamma^a_{bc}$ is

$$
\zeta^b \nabla_b \zeta^a = \frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0.
$$

(3.2)

This demonstrates the symmetry of the affine parameter, $\tau \rightarrow c + b \tau$. The study of sprays through projective connections [24] addresses this symmetry.

From the relationship between the affine connection and the geodesics, one may ask, “if given two connections, when do they have the same family of geodesics?” Weyl [25] had shown that if one considers two connections, say $\hat{\nabla}$ and $\tilde{\nabla}$, then they are projectively equivalent, i.e. admit the same family of geodesics, if and only if there exists a one form $\omega$, such that

$$
\hat{\nabla} \alpha Y^\beta = \tilde{\nabla} \alpha Y^\beta + \omega \alpha Y^\beta + \delta_\alpha Y^\beta \omega \mu Y^\mu.
$$

(3.3)

These two connections are said to belong to the same equivalence class $[\tilde{\nabla}]$ which is called a unique projective structure on $M$. Weyl was able to show the existence of a projective curvature, which is an invariant for each projective structure. Consider the Ricci tensor of two projectively related connections. Explicitly,

$$
(\tilde{\nabla}_\alpha \tilde{\nabla}_\beta - \tilde{\nabla}_\beta \tilde{\nabla}_\alpha) Y^\beta = \tilde{\nabla} Y^\beta = \tilde{R} Y^\beta = \tilde{S} Y^\beta - \tilde{A} Y^\beta
$$

and

$$
(\tilde{\nabla}_\alpha \tilde{\nabla}_\beta - \tilde{\nabla}_\beta \tilde{\nabla}_\alpha) Y^\beta = \hat{\nabla} Y^\beta = \hat{R} Y^\beta = \hat{S} Y^\beta - \hat{A} Y^\beta,
$$

where $S_{\alpha\beta}$ is symmetric and $A_{\alpha\beta}$ is anti-symmetric. Then the two-forms are related by,

$$
\hat{A}_{\alpha\beta} = \tilde{A}_{\alpha\beta} + \tilde{\nabla}_\alpha \omega_\beta - \tilde{\nabla}_\beta \omega_\alpha
$$

(3.5)

so $A_{\alpha\beta}$ has changed by an exact two-form. The Bianchi identity shows that $dA = 0$, where $d$ is the exterior derivative. Therefore $A_{\alpha\beta}$ is a closed. This implies a gauge symmetry when $A_{\alpha\beta} \rightarrow A_{\alpha\beta} + \partial_\alpha \Lambda_\beta$. Since the Ricci tensor is related to the Laplacian up to terms associated with the affine connection, the question asked originally regarding the two-cocycle is also being addressed here. In this note, we will take advantage of the
work on projective connections to elevate the one-dimensional coadjoint element to a field in two dimensions and introduce an action for the dynamics of the diffeomorphism field.

3.2 Projective Structure via Thomas and Whitehead

There are many formulations of projective structures. We will follow the work of Thomas [27] and Whitehead [26]. For a review of projective connections, see [1, 2, 24]. We may examine projective structures by treating the diffeomorphism field, \( D \), as a component of the projective connection \( \tilde{\Gamma}^{\alpha}_{\beta\gamma} \) which is a connection on the space of affine connections, \( \nabla \). This may be formulated concretely by studying the family of affine connections on a manifold \( M \) with dimension \( m \) through another connection, “a connection of connections”, on a manifold \( N \) with dimension \( n = m + 1 \).

Here \( N \) is to be regarded the projective space \( \mathbb{P}^m \) which is the quotient space of \( \mathbb{R}^{m+1} - 0 \) under multiplication by positive non-zero real numbers. Let \( \Upsilon^\alpha \) correspond to a vector field on \( N \) which generates this action. A function \( f \) on \( \mathbb{P}^m \) then enjoys the symmetry generated by \( \Upsilon \) so its Lie Derivative vanishes, i.e. \( \mathcal{L}_\Upsilon f = \Upsilon^\alpha \partial_\alpha f = 0 \). In a Cartesian frame, this is \( x^\alpha \partial_\alpha = \Upsilon \). In the projective connection below, we will realize

\[
\Upsilon^\alpha \partial_\alpha f = \lambda \partial_\lambda f,
\]

where \( \lambda \) is the radial or “volume” parameter. Furthermore we require that under parity, \( P f(x) = f(-x) \). The parity restriction keeps us from making a change in affine parameters, as stated above, in such a way as to change an attractive force into a repulsive force. Functions such as this define the set \( \mathcal{F}_\Upsilon \). Vector fields on \( \mathbb{P}^m \) can be realized through their equivalence class defined by;

\[
\mathcal{L}_\Upsilon X \propto \Upsilon
\]

and \( PX = X \). So \( X \equiv Y \) if \( Y - X \propto \Upsilon \). We will use a construction that picks an element from each equivalence class as a representative of the equivalence class by its projection into a preferred one-form \( \omega \) on \( \mathbb{R}^{m+1} - 0 \) that enjoys \( P \omega = \omega \), and is related to \( \Upsilon \) by the conditions that \( \omega_\alpha \Upsilon^\alpha = 1 \) and \( \mathcal{L}_\Upsilon \omega_\rho = 0 \). Explicitly, let \( \tilde{X} \equiv X_E \) epitomize its equivalence class. Then \( \tilde{X}^\beta = X_E^\beta - X_E^\rho \omega_\rho \Upsilon^\beta \), is equivalent to \( X_E \). Then the Lie algebra of equivalence classes can be defined through \( [X_E, Y_E] = \mathcal{L}_{X_E} Y_E = Z_E \). For functions in \( \mathcal{F}_\Upsilon \), say \( g \), we have \( \mathcal{L}_{Z_E} g \in \mathcal{F}_\Upsilon \). From this we can define a covariant derivative operator on the space of equivalence classes of vectors.
3.3 Explicit Thomas-Whitehead Projective Connection

The Thomas-Whitehead theory may be thought of as “gauging” the projective geometry by providing a connection as an $\mathbb{R}_+$ fibration for projective transformations over the manifold $\mathcal{M}$. To construct the covariant derivative operator explicitly, we will need a parameter corresponding to flows as defined by the principle vector $\Upsilon$. This is achieved through the volume form, $\text{vol}(\lambda) = f(\lambda)\epsilon_{a_1\cdots a_n}dx^{a_1}\cdots dx^{a_n}$. Then $\lambda$ becomes the new direction parameter for $\mathcal{N}$ restricted to $\mathbb{R}_+$. This will be the definition used to describe $\lambda$ as stated in Eq. [3.6]. Following [24, 28], we construct the coefficients of the projective connection $\tilde{\nabla}_\alpha$ by requiring that

$$\tilde{\nabla}_\alpha \Upsilon^\beta = \delta^\beta_\alpha. \quad (3.8)$$

The new $m + 1$ coordinates are $x^\alpha = (\lambda, x^1, x^2, \cdots x^m)$, where $\alpha = 0, \cdots, m$. Here the Greek letters denote the coordinates on $\mathcal{N}$ while the Latin correspond to the coordinates on $\mathcal{M}$ and are labeled $a, b = 1, \cdots, m$. The connection coefficients for $\tilde{\nabla}$ on $\mathcal{N}$ will be denoted by $\tilde{\Gamma}^\beta_\rho_\alpha$ and the connection coefficients on $\mathcal{M}$ by $\Gamma^b_c_a$. Then in the frame where $\Upsilon^\alpha = (\lambda, 0, 0, \cdots, 0)$ and $\omega_\beta = (\frac{1}{\lambda}, 0, \cdots, 0)$ [28], the projective connection coefficients are:

$$\tilde{\Gamma}^0_0_a = \tilde{\Gamma}^0_0_0 = 0, \quad \tilde{\Gamma}^0_0_0 = \Upsilon^0_0D_{ab}, \quad \tilde{\Gamma}^\alpha_0_0 = 0, \quad \tilde{\Gamma}^a_0_0 = \omega_0 \delta^a_b, \quad \text{and} \quad \tilde{\Gamma}^a_0_0 = \Gamma^a_0_c. \quad (3.9)$$

This choice guarantees that

$$\mathbf{P}_\omega \omega_\beta = \omega_\beta, \quad \omega_\alpha \Upsilon^\alpha = 1 \quad \text{and} \quad \mathcal{L}_\Upsilon \omega_\beta = 0. \quad (3.10)$$

One then computes the projective curvature in the usual way, i.e

$$[\tilde{\nabla}_\alpha, \tilde{\nabla}_\beta]^\rho = K^\rho_\alpha_\beta_\gamma. \quad (3.11)$$

One finds that projective invariance of the curvature, Eq. (3.11), induces the projective gauge symmetry on $\mathcal{D}_{ab}$ given by,

$$\mathcal{D}'(q)_{cd} = \frac{\partial p^c}{\partial q^a} \frac{\partial p^d}{\partial q^b} \mathcal{D}(p)_{cd} + \frac{\partial p^l}{\partial q^c} \left( \frac{\partial^2 q^c}{\partial p^l \partial p^m} \frac{\partial^2 p^d}{\partial q^m \partial q^b} \right) + \frac{\partial q^m}{\partial p_e} \frac{\partial^3 p^b}{\partial q^m \partial q^a \partial q^b}, \quad (3.12)$$

which for an infinitesimal transformations in the direction of a vector field $\xi^a$ is given by

$$\mathcal{L}_\xi \mathcal{D}(x)_{cd} = \xi^a \partial_a \mathcal{D}(x)_{cd} + \mathcal{D}(x)_{ad} \partial_c \xi^a + \mathcal{D}(x)_{ca} \partial_d \xi^a + \partial_a \partial_c \partial_d \xi^a. \quad (3.13)$$
Thus there is a coordinate transformation worth of gauge symmetry corresponding to \( m \) degrees of freedom. Since \( D(x)_{ab} \) is a symmetric tensor on \( \mathcal{M} \), it will have a total of \( \frac{m(m+1)}{2} - m = \frac{m(m-1)}{2} \) degrees of freedom for \( m > 1 \) \((m = 1 \) is the special case of the coadjoint elements of the Virasoro algebra where the projective invariant is defined by the two-cocycle). We remark that the transformation properties in complex dimensions can be found in \([29–31]\). Furthermore, Gunning \([31]\) discusses the origin of the Schwarzian derivative in one complex dimension.

### 3.4 Schwarzian Derivative and Geodesics

From the point of view of the Thomas-Whitehead projective connection, the Schwarzian derivative enjoys ubiquity in any dimension through reparameterization of geodesics. To see this, consider a geodetic equation for a vector field \( \xi^\alpha \equiv \frac{dx^\alpha}{d\tau} \) on \( \mathcal{N} \):

\[
\xi^\alpha \tilde{\nabla}_a \xi^\beta = 0, \tag{3.14}
\]

where \( \tilde{\nabla} \) is the Thomas-Whitehead projective connection. For the \( a = 1 \cdots m \) components,

\[
\frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = f \frac{dx^a}{d\tau}, \tag{3.15}
\]

where \( f = -2 \left( \frac{d \log (x^b(\tau))}{d\tau} \right) \), and for the \( \beta = 0 \) component,

\[
\frac{d^2 x^0}{d\tau^2} + x^0 D_{ab} \frac{dx^a}{d\tau} \frac{dx^b}{d\tau} = 0. \tag{3.16}
\]

Now changing the parameter, \( \tau \to \tau'(\tau) \), so that Eq.(3.15) is also geodetic, requires

\[
\frac{d^2 \tau'(\tau)}{d\tau^2} = -2 \frac{d \log (x^0(\tau))}{d\tau} \frac{d\tau'}{d\tau}. \tag{3.17}
\]

Then performing this reparameterization on Eq.(3.16) yields \([24]\)

\[
\frac{d}{d\tau} \left( \frac{d^2 \tau'(\tau)}{d\tau^2} \right) - \frac{1}{2} \left( \frac{d^2 \tau'(\tau)}{d\tau^2} - \frac{d\tau'}{d\tau} \right)^2 = 2 D_{ab} \frac{dx^a}{d\tau} \frac{dx^b}{d\tau}. \tag{3.18}
\]

The quantity

\[
\frac{d}{d\tau} \left( \frac{d^2 \tau'(\tau)}{d\tau^2} \right) - \frac{1}{2} \left( \frac{d^2 \tau'(\tau)}{d\tau^2} - \frac{d\tau'}{d\tau} \right)^2 = S(\tau', \tau) \tag{3.19}
\]

is precisely the Schwarzian derivative and is known to be invariant under Möbius transformations, \( SL(2, R) \), where \( \tau' \to \frac{a \tau + b}{c \tau + d} \).
4 Thomas-Whitehead Projective Gravity

In what follows, the metric $G_{\alpha\beta}$ discussed above will be derived by using the chiral algebra of the Dirac matrices in even dimensions (see Section 4.1). The resulting metric structure will then be used in any dimension. From there we can construct the spin-connection on $\mathcal{N}$ and make contact with 2D Polyakov gravity through fermions coupled to the projective geometry in Section 4.3. Section 4.4 will also discuss the possibility of chiral gravitational fields arising from distinct connections for the left-handed and right-handed fermions. In Section 4.2, we will show that instead of using curvature invariants (which vanish in one-dimension), the projective Laplacian, $G^{\alpha\beta}\nabla_\alpha\nabla_\beta$, over the one-dimensional circle can be used to give rise to the 2-cocycle in Eq. (2.8). Then in Section 4.5, we will describe dynamics for the field $D_{ab}$ in higher dimensions. Since the projective curvature can be defined for any Riemannian manifold, we can define the dynamics through a curvature squared type action with $G_{\alpha\beta}$. The projective Gauss-Bonnet action will be deemed the most suitable of the curvature squared actions, as our interest is mainly in 2, 3 and 4 dimensions where the Riemannian Gauss-Bonnet action vanishes in 2 and 3 dimensions, and is a topological invariant in 4 dimensions. Thus higher derivative terms that might arise from curvature squared terms, such as the Kretschmann invariant density and Ricci squared, will not arise in these dimensions. We explicitly derive the field equations for the diffeomorphism field $D_{ab}$, as well as its contribution to the energy-momentum tensor. We briefly examine the constraint dynamics and end with some additional discussion in Section 6.

4.1 The Metric $G_{\alpha\beta}$ and Spin Connection $\tilde{\omega}_{\mu}^{AB}$

As stated above, the Thomas-Whitehead connection is defined on a manifold $\mathcal{N}$ that has one more dimension associated with its volume. In general, there will not be a metric which is compatible with the connection. Consider even-dimensional manifold $\mathcal{M}$. We define a metric on $\mathcal{N}$ through the Clifford algebra of the Dirac matrices and the chiral Dirac matrix. Let us assume that $\mathcal{M}$ admits spinors. We can exploit the Dirac matrices to “lift” a metric to $\mathcal{N}$. The Dirac matrices are related to a metric $g_{ab}$ on the manifold $\mathcal{M}$ via

$$\{\gamma^a, \gamma^b\} = 2g^{ab}. \quad (4.1)$$

Here the coordinates on $\mathcal{M}$ are $a, b = 1, \cdots, m$. Then by using the volume form on $\mathcal{M}$, we can define the chiral Dirac matrix, $\gamma_{m+1}$ (with new index down) that is related
to the volume parameter $\lambda$ via

$$\gamma(\lambda)_{m+1} = \frac{f(\lambda)}{m!}^{\frac{m-2}{2}} \epsilon_{a_1...a_m} \gamma^{a_1} ... \gamma^{a_m}. \quad (4.2)$$

The intimacy of chirality and the volume form allows us to define a metric on the manifold, $\mathcal{N}$, through

$$\{\gamma_\alpha, \gamma_\beta\} = 2G_{\alpha\beta}, \quad (4.3)$$

where now $\alpha, \beta = 0, \ldots, m$. Then in any dimension, we may write the metric $G_{\alpha\beta}$ as

$$G_{\alpha\beta} = \begin{pmatrix} g_{ab} & 0 \\ 0 & f(\lambda)^2 \end{pmatrix} \quad (4.4)$$

We will use $G_{\alpha\beta}$ to contract with the projective curvature tensor in the interaction Lagrangian and the dynamical action for the diffeomorphism field. The volume of $G_{\alpha\beta}$ is given by

$$\sqrt{-\det(G_{\alpha\beta})} = \sqrt{-\det(g_{ab})f(\lambda)}. \quad (4.5)$$

Following the spirit of [32–34] and to guarantee finite volume when $\lambda$ is integrated from 0 to $\infty$ in the action functionals, we choose $f(\lambda)$ so that

$$f(\lambda) = \exp(-2\frac{\lambda}{\lambda_0}). \quad (4.6)$$

Since $G_{\alpha\beta}$ admits frame fields through

$$G_{\mu\nu} = e_\mu^A e_\nu^B \eta_{AB} \quad \text{and} \quad \eta_{AB} = g_{\mu\nu} E^\mu_A E^\nu_B, \quad (4.7)$$

the projective spin connection may be written as

$$\tilde{\omega}^{AB}_\mu = e_\nu^A (\bar{\nabla}_\alpha E^{\nu B}). \quad (4.8)$$

The projective connection then acts on the gamma matrices as

$$\bar{\nabla}_\mu \gamma^\nu = \partial_\mu \gamma^\nu + [\Omega_\mu, \gamma^\nu] + \tilde{\Gamma}^\nu_{\mu\sigma} \gamma^\sigma, \quad (4.9)$$

and

$$\Omega_\mu = \frac{1}{8} \omega^{AB}_\mu [\gamma_A, \gamma_B] \quad (4.10)$$

is the connection on fermions. As we will remark later, in even dimensions, the fermion representation does not change in going from $m$ to $m+1$ dimensions; there may be distinct projective connections for left and right handed chiral fermions. This introduces a natural notion of chiral symmetry in the gravitational sector.
4.2 The 2-Cocycle and the TW Connection

As discussed in Section 2, Eq.(2.7), Kirillov was able to show a relationship between the Sturm-Liouville operator and the coadjoint elements of the Virasoro algebra. By considering the Thomas-Whitehead over a one-dimensional manifold \( M \) and using the metric \( G_{\alpha\beta} \) we will present another way to see the correspondence. Here the conditions from Eq.(3.10) still hold and the Laplacian is used to construct the projective invariant. We consider a projective 2-cocycle on \( N \) for a circle (this can also be defined on a line) given by

\[
\langle \xi, \eta \rangle_{(\zeta)} = q \int_{C(\zeta)} \xi^\alpha (\tilde{\nabla}_\alpha G^{\mu\nu} \tilde{\nabla}_\rho \tilde{\nabla}_\nu \eta^\beta G_{\beta\mu}) \zeta^\mu d\sigma - (\xi \leftrightarrow \eta),
\]

(4.11)

where \( \sigma \) parameterizes the path. Here the coordinates on \( N \) are \( x^\alpha = (\lambda, x) \) and we choose the vector fields as \( \xi^\alpha = (\xi_0, \xi_1) \) and \( \eta^\alpha = (\eta_0, \eta_1) \). The vector \( \zeta^\mu \equiv \frac{dx^\mu}{d\sigma} \) defines the path \( C \). In accord with Eq.(2.5), the vector field \( \eta \) is mapped into a quadratic differential, say \( \eta_{\alpha\mu} \)

\[
(\tilde{\nabla}_\alpha G^{\rho\nu} \tilde{\nabla}_\rho \tilde{\nabla}_\nu \eta^\beta G_{\beta\mu}).
\]

(4.12)

For the sake of description, we write the one-dimensional metric as \( g(x) \) and its Christoffel symbol as \( \Gamma \). Now consider a path given by a fixed value \( \lambda = \lambda_0 \) along the vector, \( \zeta_{\lambda_0}^\mu = (0, \frac{dx}{d\sigma}) \), and vector fields \( \eta^\alpha = (0, \eta_1) \) and \( \xi^\alpha = (0, \xi_1) \). On such a path one finds that

\[
\langle \xi, \eta \rangle_{(\zeta_{\lambda_0})} = q \int_1 \xi_1 (2D_{11} + g(x) \frac{1}{\lambda_0^2 f(\lambda_0)} + 2\Gamma(x)^2 + \Gamma'(x)) \eta_1'' dx + q \int_1 \xi_1'' dx - (\xi \leftrightarrow \eta).
\]

(4.13)

The term \( 2\Gamma(x)^2 + \Gamma'(x) \) dictates the transformation law for \( D_{11} \) and is algebraically equivalent to Eq.(2.34). Also, the metric is a tensor and has no inhomogeneous transformation law. We take the metric to be the standard metric, \( g(x) = 1 \), so that Eq.(4.13) becomes

\[
\langle \xi, \eta \rangle_{(\zeta_{\lambda_0})} = q \int_1 \xi_1 (2D_{11} + \frac{1}{\lambda_0^2 f(\lambda_0)}) \eta_1'' dx + q \int_1 \xi_1'' dx - (\xi \leftrightarrow \eta).
\]

(4.14)

Since \( \lambda_0 \) is fixed, we make the identification that \( 2qD_{11} + \frac{q}{\lambda_0^2 f(\lambda_0)} = B \), which recovers Eq.(2.8) for \( q = \frac{c}{2\pi} \).
4.3 Polyakov-Diffeomorphism Field Interaction Term

The Polyakov metric will determine the $m = 2$ dimensional metric on $\mathcal{M}$, and we will extend it to a metric on $\mathcal{N}$ via the matrix $\gamma^3$ as mentioned above. Explicitly, the Polyakov metric \(^{35,36}\) is

$$g_{ab} = \begin{pmatrix}
0 & \frac{1}{2} \\
\frac{1}{2} & h_{\tau\tau}(\theta, \tau)
\end{pmatrix}, \quad (4.15)$$

where $h_{\tau\tau}(\theta, \tau) = \frac{\partial_{\theta} f(\theta, \tau)}{\partial_{\theta} f(\theta, \tau)}$. With this choice of metric, the $\gamma^3$ extended metric is

$$G_{\alpha\beta} = \begin{pmatrix}
0 & \frac{1}{2} & 0 \\
\frac{1}{2} & h_{\tau\tau} & 0 \\
0 & 0 & f(\lambda)^2
\end{pmatrix}. \quad (4.16)$$

The projective curvature coefficients $K^\rho_{\alpha\beta\gamma}$ for the Polyakov metric on $\mathcal{M}$ are given below:

$$K^3_{\alpha\beta\gamma} = \begin{cases}
\lambda(-\partial_{\tau} D_{\theta\theta} + \partial_{\theta} D_{\theta\tau} + D_{\theta\theta} \partial_{\theta} h_{\tau\tau}), & a = 1, b = 1, c = 2 \\
-\lambda(-\partial_{\tau} D_{\theta\theta} + \partial_{\theta} D_{\theta\tau} - 2 D_{\theta\tau} \partial_{\theta} h_{\tau\tau} + D_{\theta\theta} (\partial_{\tau} h_{\tau\tau} + 2(\partial_{\theta} h_{\tau\tau} h_{\tau\tau}))), & a = 2, b = 1, c = 2 \\
0, & \text{otherwise}
\end{cases} \quad (4.17)$$

$$K^2_{\alpha\beta\gamma} = \begin{cases}
-D_{\theta\tau} - \partial_{\theta}^2 h_{\tau\tau}, & a = 2, b = 1, c = 2 \\
-D_{\theta\theta}, & a = 1, b = 1, c = 2 \\
0, & \text{otherwise}
\end{cases} \quad (4.18)$$

$$K^1_{\alpha\beta\gamma} = \begin{cases}
D_{\theta\tau} + \partial_{\theta}^2 h_{\tau\tau}, & a = 1, b = 1, c = 2 \\
D_{\tau\tau} + 2 h_{\tau\tau} \partial_{\theta}^2 h_{\tau\tau}, & a = 2, b = 1, c = 2 \\
0, & \text{otherwise}
\end{cases} \quad (4.19)$$

The projective-Ricci tensor then is computed via $K^\rho_{\alpha\beta\rho} = K_{\alpha\beta} = K_{\beta\alpha}$ and has coefficients

$$K_{\alpha\beta} = \begin{cases}
D_{\theta\theta}, & \alpha = 1, \beta = 1 \\
-D_{\theta\tau} - \partial_{\theta}^2 h_{\tau\tau}, & \alpha = 1, \beta = 2 \\
-D_{\tau\tau} - 2 h_{\tau\tau} \partial_{\theta}^2 h_{\tau\tau}, & \alpha = 2, \beta = 2.
\end{cases} \quad (4.20)$$
Using the metric $G_{\alpha\beta}$ on $\mathcal{N}$, we can find the interaction term between the projective connection and the induced Polyakov metric through

\[ S_{\text{Diff Inter}} = \int \sqrt{-G} G^{\mu\nu} K_{\mu\nu} d\lambda \, d\theta \, d\tau \]
\[ = \int d\lambda d\theta d\tau \frac{f(\lambda)}{\sqrt{2}} (D_{\theta\theta} h_{\tau\tau} - D_{\theta\tau} - \partial^2_{\theta} h_{\tau\tau}), \]
\[ = \int d\theta d\tau \left( \int d\lambda \frac{f(\lambda)}{\sqrt{2}} \right) (D_{\theta\theta} h_{\tau\tau} - D_{\theta\tau} - \partial^2_{\theta} h_{\tau\tau}) \]  

(4.21)

where the last term is the scalar curvature and is a total derivative. Similarly, the middle term integrates to a constant as it decouples from the interaction with the metric. Therefore we have the interaction term found in [5],

\[ S_{\text{Diff Inter}} = q \int d\theta d\tau D_{\theta\theta} h_{\tau\tau}, \]  

(4.22)

where $q$ is a constant. Thus, using the projective connection over two-dimensional metrics in the Polyakov gauge, we are able to recover the interaction term, Eq. (2.14) between the Polyakov metric and the coadjoint element.

### 4.4 Chiral Gravity through Projective Geometry

One could have recovered this interaction term by computing the chiral anomaly with a Dirac operator defined by the projective connection. Since the dimension of the irreducible representation of the fermions does not change in going from $n$ to $n + 1$ dimensions via the matrix $\gamma^{n+1} = \gamma^{\text{chiral}}$, one can arrive at this result by computing $\text{Tr}(\gamma^3)$ which written in invariant form on the projective space is

\[ \gamma^3 = \frac{\sqrt{2}}{\sqrt{\Upsilon^\alpha \Upsilon_\alpha}} \text{Tr}(\Upsilon^\alpha \gamma_\alpha). \]  

(4.23)

We couple left handed fermions to a Dirac equation by using the projective Dirac operator $\tilde{\nabla}$ in the heat kernel expansion. There the square of the Dirac operator yields

\[ \tilde{\nabla}^2 \tilde{\nabla} = \gamma^\alpha \tilde{\nabla}_\alpha (\gamma^\beta) \tilde{\nabla}_\beta + (G^{\alpha\beta} - i\sigma^{\alpha\beta}) \tilde{\nabla}_\alpha \tilde{\nabla}_\beta. \]  

(4.24)

The requisite term to computing the trace is

\[ \sigma^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta = \frac{1}{2} \sigma^{\alpha\beta} [\tilde{\nabla}_\alpha, \tilde{\nabla}_\beta] = \frac{1}{2} \sigma^{\alpha\beta} K_{\alpha\beta} = \frac{1}{2} \sigma^{\alpha\beta} \frac{1}{2} \sigma_{\delta\lambda} K_{\alpha\beta} \delta^\lambda \propto K, \]  

(4.25)
where $K_{\alpha\beta}$ is the projective curvature two-form constructed from the projective spin connection Eq. (4.8). Since $\gamma^{m+1}$ has two eigenvalues that correspond to distinct orientation forms, we may write a chiral gravitational theory for fermions in even dimensions by adding a Dirac action on the projective space. Then by introducing two distinct projective connections for the vector and axial vector, we can write a chiral action as

\[
S_{(\psi, D^V, D^A)} = \int d^{m+1}x \sqrt{-G} \left( \bar{\psi} \gamma^\mu \tilde{\nabla}_\mu \psi + \bar{\psi} \gamma^\mu \tilde{\nabla}_\mu (\Psi^\alpha \gamma_\alpha) \psi \right). \tag{4.26}
\]

The advantage of this approach is that we do not have to introduce separate metrics for the chiralities, since $D^V_{ab}$ and $D^A_{ab}$ are independent fields that have distinct isotropy groups. When the two diffeomorphism fields are the same, chiral symmetry is broken.

We are presently investigating how such a coupling affects chiral symmetry and chiral symmetry breaking through gravity.

### 4.5 Projective Curvature Squared and Diffeomorphism Field Lagrangian

#### 4.5.1 Thomas Whitehead Projective Gravitational Action

Now we turn our attention to constructing a Lagrangian for the diffeomorphism field in order to describe it as a dynamical field in its own right. Motivated by the discussion in the previous section, a Lagrangian for the diffeomorphism field could be of the form

\[
S_{\text{trial action}} = \int d^m x d\lambda \sqrt{-G} \left( K^\rho_{\alpha\beta\gamma} K^\delta_{\mu\nu\sigma} G^{\rho\mu} G^{\delta\nu} G^{\alpha\sigma} G^{\rho\delta} \right). \tag{4.27}
\]

However, an expansion of this action includes a Kretschmann scalar density,

\[
\int d^m x \sqrt{-g} \ R^a_{\ bcd} R_a^{\ bcd}.
\]

Such a term will give rise to higher derivative terms in the metric. Furthermore, the action is not explicitly projectively invariant. Since the Gauss-Bonnet action is trivial in two and three dimensions and a topological invariant in four dimensions, we are motivated to write our action as

\[
S_{\text{TWPG}} = \alpha_0 \int d^m x d\lambda \sqrt{-\det (G_{\mu\nu})} K_{\alpha\beta} G^{\alpha\beta} + S_{\text{PGB}}, \quad \text{where} \quad \tag{4.28}
\]

\[
S_{\text{PGB}} = \beta_0 \int d^m x d\lambda \sqrt{-\det (G_{\mu\nu})} \left( K^2 - 4K_{\mu\nu} K_{\mu\nu} + K_{\mu\nu\rho} K_{\mu\nu\rho \alpha} \right), \tag{4.29}
\]

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where $K_{ab} = R_{ab} - (m-1)D_{ab}$ and $K = R - (m-1)g^{ab}D_{ab} = R - (m-1)D$. We will call this the Thomas-Whitehead projective gravitational action. This may be expanded as the $m$-dimensional action

$$S_{TWPG} = \alpha_0 \frac{\lambda_0}{2} \int d^m x \sqrt{-g} g^{ab} R_{ab}$$

$$- (m-1) \alpha_0 \frac{\lambda_0}{2} \int d^m x \sqrt{-g} g^{ab} D_{ab}$$

$$- \frac{\beta_0 \lambda_0}{2} \int d^m x \sqrt{-g} \left( 4R_{ab}D^{ab} - 2(m-1)D_{ab}D^{ab} - \frac{\lambda_0^2}{54} K_{bmn}K^{bmn} \right)$$

$$+ 2(m-1) \beta_0 \lambda_0 \int d^m x \sqrt{-g} D_{ab} \left( 2R_{ab} - (m-1)D_{ab} \right)$$

$$- (m-1) \frac{\beta_0}{2} \lambda_0 \int d^m x \sqrt{-g} D_{ab} \left( 2R - (m-1)D \right) + S_{GB}, \quad (4.30)$$

where the $\lambda$ integration has been performed using Eq.(4.6). Here $\alpha_0 \equiv \frac{\kappa^2}{\lambda_0}$ where $\kappa$ is the gravitational constant. The quadratic Gauss-Bonnet action,

$$S_{GB} = \frac{\beta_0}{2} \lambda_0 \int d^m x \sqrt{-g} \left( R^2 - 4R_{ab}R^{ab} + R_{abc}R^{abc} \right), \quad (4.31)$$

vanishes in two and three dimensions and is a topological term in four dimensions. The dynamics for $D_{ab}$ arise from the term $K_{bmn}K^{bmn}$ where $K_{bmn} = \nabla_m D_{bn} - \nabla_n D_{bm}$. This satisfies the Bianchi identity

$$K_{acm} + K_{cma} + K_{mac} = 0. \quad (4.32)$$

The field equations for $D_{ab}$ that arise from this action are

$$\beta_0 \left( \frac{\lambda_0}{3} \right)^3 \nabla_m (K^{cdm} + K^{dcm}) - \lambda_0 (m-1)(\alpha_0 + 2K)g^{cd} + 4\beta_0 \lambda_0 (2m-3)K^{cd} = 0. \quad (4.33)$$

This can be written in the form $\mathcal{O}_{cd,ab}D_{ab} = J^{cd}$, where $\mathcal{O}_{cd,ab}$ is a second order differential operator and $J^{cd}$ a metric dependent source, i.e.

$$\beta_0 \left( \frac{\lambda_0}{3} \right)^3 \nabla_m (K^{cdm} + K^{dcm}) - 4\beta_0 \lambda_0 (2m-3)(m-1)D^{cd} + 2\lambda_0 (m-1)^2 Dg^{cd}$$

$$= -4\beta_0 \lambda_0 (2m-3)R^{cd} + \lambda_0 (m-1)(\alpha_0 + 2R)g^{cd}. \quad (4.34)$$

In Thomas-Whitehead projective gravity, the field $D_{ab}$ is physical and can backreact with the metric. By varying the action with respect to the metric $g_{ab}$, we can write the Einstein equations as

$$\kappa (R_{lm} - \frac{1}{2} Rg_{lm}) = \Theta_{lm} + \Theta^{lm}_{GB}, \quad (4.35)$$

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where we have separated out the energy-momentum contribution from the Riemannian Gauss-Bonnet term as $\Theta_{GB}^{lm}$. The contribution due to the diffeomorphism field is given by $\Theta^{lm}$. This can be written in four parts corresponding to the contribution arising from the projective Einstein-Hilbert action, which is the first summand in Eq. (4.28), and the three terms in Eq. (4.29) arising from the projective scalar squared term (PSS), the projective curvature squared term (PCS) and the the projective Ricci squared term (PRS). Using these labels we write

$$\Theta^{mn} = \Theta_{PEH}^{mn} + \Theta_{PSS}^{mn} + \Theta_{PCS}^{mn} + \Theta_{PRS}^{mn}$$  \hspace{1cm} (4.36)$$

where

$$\Theta_{PEH}^{lm} = \frac{1}{2} \kappa^2 (n-1) (D^l m - \frac{1}{2} D g^lm)$$  \hspace{1cm} (4.37)$$

$$\Theta_{PSS}^{lm} = 2\beta_0 \lambda_0 (1-m)(g^{lm} g^{cd} - g^{lc} g^{md}) \nabla_c \nabla_d D + \beta_0 \frac{\lambda_0}{2} (2(1-m) R D + (1-m) R^m D + (1-m) D^m D)$$  \hspace{1cm} (4.38)$$

$$\Theta_{PCS}^{lm} = \beta_0 \lambda_0 \left( L + 4(R^l_c D^m a + R^m_e D^{cl}) - 2(n-1)(D^l_e D^m c + D^m_e D^{cl}) \right) - \beta_0 \lambda_0 \frac{\lambda_0}{54} K^{mca} K^{bmn} - \beta_0 \lambda_0 \frac{\lambda_0}{27} K^{cdl} K^{emb} - \beta_0 \lambda_0 \nabla_c \nabla_d X^{clm}$$  \hspace{1cm} (4.39)$$

$$\Theta_{PRS}^{lm} = -2\beta_0 \lambda_0 (1-m) g^{lm} \left( 2 R_{ab} D^{ab} + (1-m) D_{ab} D^{ab} \right) - 4(1-m) (R_c^d D^m e + D^{cl} R^m e + (1-m) D^{df} D^m e) + 2(1-m) \nabla_a \nabla_b \left( D^{ab} g^{lm} + D^{lm} g^{ab} - D^{mb} g^{la} - D^{at} g^{mb} \right)$$  \hspace{1cm} (4.40)$$

Here

$$X^{cab} = J^{cab} - J^{abc} - J^{bac},$$  \hspace{1cm} (4.41)$$

$$J^{cab} = g^{ca} \nabla_m D^{mb} + g^{cb} \nabla_m D^{ma} - 2 g^{cd} \nabla_d D^{ef} - \frac{\lambda_0^2}{54} (D_c^m K^{abm} + D_e^c K^{bam})$$  \hspace{1cm} (4.42)$$

and

$$L = -\frac{\lambda_0^2}{54} (K_{mnk} K_{acd}) g^{ab} g^{nc} g^{nd} - 2(n-1) D^{ab} D_{ab} + 4R^{ab} D_{ab}.$$  \hspace{1cm} (4.43)$$

For the sake of completeness, in dimensions higher than four, the Gauss-Bonnet action contributes to the total energy-momentum tensor through divergent-free Lanczos tensor
\[\frac{1}{\beta_0 \lambda_0} \Theta_{GB}^{lm} = \]
\[\frac{1}{2} g^{lm} (R^2 - 4 R_{ab} R^{ab} + R_{abcd} R^{abcd}) - 2 R R^{lm} - 4 R^{alm} R_{ab} - 2 R^{bcd} R_{bcd}^m + 4 R^a R^m_a.\]  
(4.44)

5 2D TWPG

Here we briefly discuss the relevant degrees of freedom in the “free” Lagrangian in 2D. For this we do not include the projective Einstein-Hilbert action and explicitly use a metric where \(\Gamma^a_{bc} = 0\). From here we can examine salient features of the phase space structure of the action. An in-depth analysis will be carried out through Dirac brackets in a future work. For this simplified case, we have the 2D projective action

\[S_{\text{Diff free}} = -\alpha_1 \int d\theta d\tau \left( (\mathcal{D}_{\theta \theta} + \mathcal{D}_{\tau \tau})^2 - 4 \mathcal{D}_{\theta \tau}^2 \right) \]
\[-\alpha_2 \int d\theta d\tau \left( (\partial_{\tau} \mathcal{D}_{\theta \tau} - \partial_{\theta} \mathcal{D}_{\tau \tau})^2 - (\partial_{\tau} \mathcal{D}_{\theta \theta} - \partial_{\theta} \mathcal{D}_{\theta \tau})^2 \right),\]  
(5.1)

where
\[\alpha_1 = \int \frac{1}{\sqrt{2}} f(\lambda) \quad \text{and} \quad \alpha_2 = \int \sqrt{2} \lambda^2 f(\lambda)^3.\]  
(5.2)

The conjugate momenta are given by
\[\Pi_{\theta \tau} = -2 \alpha_2 (\partial_{\tau} \mathcal{D}_{\theta \theta} - \partial_{\theta} \mathcal{D}_{\theta \tau})\]
\[\Pi_{\theta \theta} = +2 \alpha_2 (\partial_{\tau} \mathcal{D}_{\theta \tau} - \partial_{\theta} \mathcal{D}_{\tau \tau})\]
\[\Pi_{\tau \tau} = 0,\]  
(5.3)

where the vanishing of \(\Pi_{\tau \tau}\) signifies a primary constraint, \(\Phi_1 = \Pi_{\tau \tau}\). The equations of motion for \(\mathcal{D}_{\tau \tau}\) then give a secondary constraint \(\Phi_2\), where
\[\Phi_2 = -\sqrt{2} \alpha_1 (\mathcal{D}_{\theta \theta} + \mathcal{D}_{\tau \tau}) + \partial_{\tau} \Pi_{\theta \tau}.\]  
(5.4)

In terms of the conjugate momenta, the field equations for the three fields \(\mathcal{D}_{\theta \theta}, \mathcal{D}_{\theta \tau},\) and \(\mathcal{D}_{\tau \tau}\) respectively are
\[\sqrt{2} \alpha_1 (\mathcal{D}_{\theta \theta} + \mathcal{D}_{\tau \tau}) - \partial_{\theta} \Pi_{\theta \tau} = 0\]
\[\sqrt{2} \alpha_1 \mathcal{D}_{\theta \tau} + \partial_{\theta} \Pi_{\theta \tau} = \partial_{\tau} \Pi_{\theta \tau},\]  
21
Here one may view the $\alpha_1$ terms as mass terms, demonstrating that masses are derived from the projective geometry. The structure of the constraint algebra for Dirac quantization will depend on whether $D_{\tau\tau}$ vanishes, which will lead to first class constraints via Dirac brackets. Only one physical degree of freedom will remain. We expect that known techniques for handling partially massless gauge theories [34] will also be of value here. This is under investigation.

6 Conclusion

We have shown that projective geometry may be used to give dynamics and a field theoretic interpretation to the coadjoint elements which arise in the study of 2D gravity. The fixed coadjoint orbits are interpreted as external energy-momentum tensors in the absence of dynamics. For orbits such as $\text{Diff}S^1/\text{SL}_n$ with $D_{ab} \propto -n^2$, the coadjoint element acts as a negative cosmological constant in 2D. Furthermore, the projective curvature renders the diffeomorphism field a dynamical field theory, putting the orbit construction in the Virasoro sector on the same footing as the affine Kac-Moody orbits, where the coadjoint elements can be regarded as background Yang-Mills fields. The construction uses the property of coadjoint elements as projective structures on $S^1$ which allows us to “lift” the projective connection to any dimension. By introducing a metric structure associated with the chiral Dirac algebra, we can compute curvature squared invariants and provide dynamics to the projective connection. Using the Gauss-Bonnet action, we can compute an energy-momentum tensor for the diffeomorphism field and avoid higher derivatives on the metric for dimensions less than five. This energy-momentum tensor demonstrates that the diffeomorphism field will couple to matter gravitationally. A fully back-reacted solution will modify geodesics and will appear as a source of dark energy. Whether this is related to phenomenologically observed dark energy and dark matter will depend on the $\beta_0$ coupling constant. This is presently under investigation. Indeed, one could fix the affine connection due to general relativity and use the projective connection to study perturbations which give distinct geodesics. As suggested in [19], gravitation may be equipped with not only a metric but another field so that the pair $(g_{ab}, D_{\alpha\beta})$ describes the dynamics. Dynamical projective connections may open a new avenue to the understanding of cosmology, gravitational radiation, and the gravitational dynamics of matter. However, much work is left to be
done to pin down the phenomenology of this approach.

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