Schwarzschild $1/r$-singularity is not permissible in ghost free quadratic curvature
infinite derivative gravity

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(Dated: September 13, 2018)

In this paper we will study the complete equations of motion for a ghost free quadratic curvature
infinite derivative gravity. We will argue that within the scale of non-locality, Schwarzschild-type
singular metric solution is not permissible. Therefore, Schwarzschild-type vacuum solution which
is a prediction in Einstein-Hilbert gravity may not persist within the region of non-locality. We
will also show that just quadratic curvature gravity, without infinite derivatives, always allows
Schwarzschild-type singular metric solution.

I. INTRODUCTION

Arguably Einstein’s theory of general relativity is one of the most successful description of spacetime which has seen
numerous confirmations of observational tests at different length scales, predominantly in the infrared (IR) (far away
from the source and at late time scales) $^1$, including the fascinating detection of gravitational waves $^2$. In spite of this
success, at short distances and at small time scales, i.e. in the ultraviolet (UV), the Einstein-Hilbert action leads to
well-known singular solutions, in terms of blackhole solutions in the vacuum, and the cosmological singularity in a
time dependent background $^3$. The nature of latter singularity is indeed very different from the former, which brings
uncertainty in the cosmological models at the level of initial conditions for inflation and the big bang cosmology. In
reality, one would expect that nature would avoid any kind of classical singularities, whether they are covered by an
event horizon or they are naked - a stronger version of the cosmic censorship hypothesis $^4, 5$. In this respect, it
can be argued that the singularities present in the Einstein-Hilbert action are mere artefacts of the action, and there
must be a way to ameliorate the singularities in nature. Indeed, removing the singularities is one of the foremost
fundamental questions of gravitational physics.

Recently, Biswas, Gerwick, Koivisto and Mazumdar (BGKM) have shown that the quadratic curvature infinite
derivative theory of gravity in 4 spacetime dimensions can be made ghost free and avoid both cosmological and black-
hole singularities at the linearised level around the Minkowski background $^6$, $^7$, while the cosmological singularity
can be resolved even at the full non-linear level $^8, 9$. At the linear level (around asymptotically Minkowski background),
resolution of blackhole singularities has been studied both in the context of static $^7, 10, 11$, $^5$ and in a rotating
case $^6$ by various groups. Furthermore, lack of formation of singularity at the linear level has also been studied in
a dynamical context by Frolov and his collaborators $^22, 23$.

In Ref. $^6$, the authors have shown that for a ghost free quadratic curvature gravitational form factors, at short
distances the gravitational metric-potential tends to be a constant, while at large distances from the source the metric
takes the usual form of $1/r$ behavior in the IR. Furthermore, the gravitational force quadratically vanishes
towards the center in the UV. Such a system behaves very much like a compact object, but by construction there is no
curvature singularity, nor there is an event horizon. The gravitational entropy calculated by the Wald’s formalism $^24$
leads to the $\text{Area-law}$ $^25$. Since, all the interactions are purely derivative in nature, the gravitational form factors
give rise to non-local interactions for such a spacetime $^26, 27$. The non-locality is indeed confined within the scale
$M_\alpha$, which has very a interesting behavior.

The aim of this short note is to show that the full non-linear metric solution of the BGKM gravity will not permit
$1/r$-type metric potential, i.e. Schwarzschild type solution, for the static background. Note that what is relevant for
us is indeed $1/r$ part of the metric potential, be it in isotropic coordinates or Schwarzschild’s coordinates. Near the
vicinity of singularity, at $r = 0$, what dominates is indeed the $1/r$ part of the metric potential in Einstein’s theory of

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$^1$ See previous to this work other relevant references $^2, 3$, where the authors have argued absence of singularity in infinite derivative
gravity motivated from the string theory, however, the full quadratic curvature action including the Weyl term with two gravitational
metric potentials were first presented in $^4$. 

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gravity. Also, such a singular solution exists in quadratic curvature gravity as well [31], see for instance [32], therefore it is a pertinent question to ask whether 1/r-kind of metric potential would survive infinite derivative theory of gravity or not?

II. THE INFINITE COVARIANT DERIVATIVE ACTION

The most general quadratic curvature action (parity invariant and free from torsion) has been derived around constant curvature backgrounds in Refs. [6, 14, 15], given by

\[ S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R + \alpha c \left[ R F_1(\Box_s) R + \tilde{R}^\mu F_2(\Box_s) R_{\mu \nu} + W^{\mu \nu \lambda \sigma} F_3(\Box_s) W_{\mu \nu \lambda \sigma} \right] \right) , \]  

(2)

where \( G = 1/M_p^2 \) is the Newton’s gravitational constant, \( \alpha c \sim 1/M_p^2 \) is a dimensionful coupling, \( \Box_s \equiv \Box/M_p^2 \), where \( M_p \) signifies the scale of non-locality at which new gravitational interaction becomes important. In the limit \( M_p \to \infty \), the action reduces to the Einstein-Hilbert term. The d’Alembertian term is \( \Box = g^\mu \nu \nabla_\mu \nabla_\nu \), where \( \mu, \nu = 0, 1, 2, 3 \), and we work with a metric convention which is mostly positive (-, +, +, +). The \( F_i \)'s are three gravitational form-factors,

\[ F_i(\Box_s) = \sum_n f_{i,n} \Box_s^n , \]  

(3)

reminiscence to any massless theory possessing only derivative interactions. In this theory, the graviton remains massless with transverse and traceless degrees of freedom. However, the gravitational interactions are non-local due to the presence of the form factors \( F_i \)'s, see [6]. These form factors contain infinite covariant derivatives, which shows that the interaction vertex in this class of theory becomes non-local. In fact, the gravitational interaction in this class of theory leads to smearing out the point source by modifying the gravitational potential, as shown in [6]. The non-local gravitational interactions are also helpful to ameliorate the quantum aspects of the theory, which is believed to be UV finite [26–29]. The scale of non-locality is governed by \( M_p^{-1} \).

Around the Minkowski background the three form factors obey a constraint equation, in order to maintain only the transverse-and traceless graviton degrees of freedom, i.e. the perturbative tree-level unitarity [6, 14, 15]

\[ 6F_1(\Box_s) + 3F_2(\Box_s) + 2F_3(\Box_s) = 0 . \]  

(4)

Let us first discuss very briefly the linear properties of this theory around an asymptotically Minkowski background before addressing the non-linear equations of motion. The linear solutions are indeed insightful and provides a lot of

\[ m M_p \leq M_p^2 , \]  

(5)

with the gravitational metric potential in static and isotropic coordinates is given by [6]:

\[ \phi(r) = \frac{G \delta^3(r)}{r} , \]  

(6)

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2 The original action was first written in terms of the Riemann, but it is useful to write the action in terms of the Weyl term which is related to the Riemann as:

\[ W^{\mu \nu \alpha \beta} = R^{\mu \nu \alpha \beta} - \frac{1}{2} (\delta^\mu_\nu R_{\alpha \beta} - \delta^\nu_\beta R_{\alpha \mu} + R_{\mu \alpha \beta} - R_{\beta \mu \alpha} + \frac{R}{6} (\delta^\mu_\nu g_{\alpha \beta} - \delta^\nu_\beta g_{\alpha \mu}) + \frac{R}{6} (\delta^\mu_\nu g_{\alpha \beta} - \delta^\nu_\beta g_{\alpha \mu}) ) . \]  

(1)

3 In order to make sure that the full action Eq. (2) contains the same original dynamical degrees of freedom as that of the massless graviton in 4 dimensions. This is to make sure that the action is ghost free, there are no other dynamical degrees of freedom in spite of the fact that there are infinite derivatives. The graviton propagator for the above action gives rise to

\[ \Pi(k^2) = \frac{1}{a(k^2) \Pi(k^2)_{GR}} = \frac{1}{a(k^2) \left[ \frac{P^{(2)}}{k^2} - \frac{P^{(0)}}{2k^2} \right] } , \]

where \( P^{(2)} \) and \( P^{(0)} \) are spin-2 and 0 projection operators, and \( a(k^2) = e^{\gamma k^2/M_p^2} \), is exponential of an entire function - γ, which does not contain any poles in the complex plane, therefore no new degrees of freedom other than the transverse and traceless graviton, see for details [6, 33]. The gravitational form factors \( F_i(\Box_s) \) cannot be determined simultaneously in terms of \( a(\Box_s) \), if we switch one of the \( F_i = 0 \), then we can express the other form factors in terms of \( a(\Box_s) \), for instance for \( F_2 = 0 \) yields, \( F_1 = -[a(\Box_s) - 1)/12\Box_s] \) and \( F_3 = [(a(\Box_s) - 1)/4\Box_s] \), see [12].
approaches to be constant with a magnitude less than 1 for \( r < 2/M_s \). Since the error function goes linearly in \( r \) for \( r < 2/M_s \), the metric potential becomes finite in this ultraviolet region. For \( r > 2/M_s \), the metric potential follows as \( \sim Gm/r \), in the infrared region. However, in our case the typical scale of non-locality is actually larger than the Schwarzschild’s radius as shown in \([19, 20]\):

\[
\frac{r_{NL}}{M_s} \sim \frac{2}{M_s^2},
\]

thus avoiding the event horizon as well \[4\] Now, for the rest of the discussion, let us focus on the full non-linear equations for the above action Eq. \([2]\).

### III. TOWARDS IMPOSSIBILITY OF THE SCHWARZSCHILD METRIC SOLUTION

The complete equations of motion have been derived from action Eq. \([2]\), and they are given by \([14]\),

\[
P^{\alpha\beta} = -\frac{G^{\alpha\beta}}{8\pi G} + \frac{\alpha_c}{8\pi G} \left(4G^{\alpha\beta}F_1(\square)\right) + \frac{g^{\alpha\beta}R F_1(\square) R - 4(\nabla^{\alpha}\nabla^{\beta} - g^{\alpha\beta}\square) F_1(\square) R}{4}\]

\[\begin{align*}
&- 2\Omega_1^{\alpha\beta} + g^{\alpha\beta} (\Omega_1^{\sigma\tau} + \tilde{\Omega}_1) + 4R^{\alpha\beta}_{\mu\nu} F_2(\square) R^{\mu\nu} \\
&- g^{\alpha\beta} R^{\mu\nu}_{\mu\nu} F_2(\square) R^{\mu\nu} - 4\nabla^{\mu}\nabla^{\nu} (F_2(\square) R^{\mu\nu}) + 2\square(\square) R^{\mu\nu} \\
&+ 2g^{\alpha\beta} (\nabla^{\mu}\nabla^{\nu} (F_2(\square) R^{\mu\nu})) - 2\Omega_2^{\alpha\beta} + g^{\alpha\beta} (\Omega_3^{\tau} + \tilde{\Omega}_2) - 4\Delta_2^{\alpha\beta} \\
&- g^{\alpha\beta} W^{\mu\nu\lambda} F_3(\square) W^{\mu\nu\lambda} + 4W^{\alpha\beta}_{\mu\nu\lambda} F_3(\square) W^{\mu\nu\lambda} \\
&- 4(R_{\mu\nu} + 2\nabla_{\mu}\nabla_{\nu})(F_3(\square) W^{\mu\nu\lambda}) - 2\Omega_3^{\alpha\beta} + g^{\alpha\beta} (\Omega_3^{\tau} + \tilde{\Omega}_3) - 8\Delta_3^{\alpha\beta} \\
&= - T^{\alpha\beta},
\end{align*}\]

where \( T^{\alpha\beta} \) is the stress energy tensor for the matter components, and we have defined the following symmetric tensors, for the detailed derivation, see \([14]\):

\[
\Omega_1^{\alpha\beta} = \sum_{n=1}^{\infty} f_1 \sum_{l=0}^{n-1} R^{\alpha\beta}_{\mu\nu} R^{(\mu)(l)} R^{(\nu)(n-l-1)}, \quad \Omega_1 = \sum_{n=1}^{\infty} f_1 \sum_{l=0}^{n-1} R^{(l)} R^{(n-l-1)},
\]

\[
\Omega_2^{\alpha\beta} = \sum_{n=1}^{\infty} f_2 \sum_{l=0}^{n-1} R^{\alpha\beta}_{\mu\nu} R^{(\mu)(l)} R^{(\nu)(n-l-1)}, \quad \Omega_2 = \sum_{n=1}^{\infty} f_2 \sum_{l=0}^{n-1} R^{(l)} R^{(n-l-1)},
\]

\[
\Delta_2^{\alpha\beta} = \sum_{n=1}^{\infty} f_2 \sum_{l=0}^{n-1} [R^{\alpha\beta}_{\mu\nu} R^{(\mu)(l)} R^{(\nu)(n-l-1)} - R^{\mu\nu}_{\nu\lambda} R^{(\mu)(l)} R^{(\nu)(n-l-1)}],
\]

\[
\Omega_3^{\alpha\beta} = \sum_{n=1}^{\infty} f_3 \sum_{l=0}^{n-1} W^{\alpha\beta}_{\mu\nu\lambda} W^{\mu\nu\lambda} R^{(\mu)(l)} R^{(\nu)(n-l-1)}, \quad \Omega_3 = \sum_{n=1}^{\infty} f_3 \sum_{l=0}^{n-1} W^{(l)} W^{(n-l-1)},
\]

\[
\Delta_3^{\alpha\beta} = \sum_{n=1}^{\infty} f_3 \sum_{l=0}^{n-1} [W^{\alpha\beta}_{\mu\nu\lambda} W^{\mu\nu\lambda} R^{(\mu)(l)} R^{(\nu)(n-l-1)} - W^{\mu\nu}_{\mu\nu\lambda} W^{(\mu)(l)} W^{(\nu)(n-l-1)}],
\]

The notation \( R^{(l)} \equiv \square^l R \) has been used for the curvature tensors and their covariant derivatives. The trace equation is much more simple, and just for the purpose of illustration, we write it below \([14]\):

\[
P = \frac{R}{8\pi G} + \frac{\alpha_c}{8\pi G} \left(12\square F_1(\square) + 2\square (F_2(\square) R) + 4\nabla_{\mu}\nabla_{\nu} (F_2(\square) R^{\mu\nu})
\]

\[\begin{align*}
&+ 2(\Omega_1^{\sigma\tau} + 2\tilde{\Omega}_1) + 2(\Omega_2^{\sigma\tau} + 2\tilde{\Omega}_2) + 2(\Omega_3^{\sigma\tau} + 2\tilde{\Omega}_3) - 4\Delta_2^{\sigma\tau} - 8\Delta_3^{\sigma\tau} \\
&= - T \equiv -g^{\alpha\beta} T^{\alpha\beta}.
\end{align*}\]

\[4\] This could potentially resolve the information-loss paradox, since there is no event horizon and the graviton interactions for \( r_{NL} \sim 2/M_s \) becomes nonlocal, therefore for interacting gravitons the spacetime ceases to hold any meaning in the Minkowski sense.
The Bianchi identity has been verified explicitly in Ref. [14]. Here we briefly sketch the Weyl part, since this will be the most important part of our discussion. To accomplish this, note that the computations are simplified if one uses the following tricks by rewriting the equations of motion with one upper and one lower index, express Ricci tensor through the Einstein tensor (who’s divergence is zero due to the Bianchi identities), and recalling the fact that the divergence of the Weyl tensor is the third rank Cotton tensor, which can be expressed through the Schouten tensor:

$$\nabla^{\gamma}W_{\alpha\mu\nu\gamma} = -\nabla_{\alpha}S_{\mu\nu} + \nabla_{\mu}S_{\alpha\nu},$$

where the Schouten tensor in four dimensions is given by:

$$S_{\mu\nu} = \frac{1}{2} \left( R_{\mu\nu} - \frac{1}{6} g_{\mu\nu} R \right).$$

With this in mind the rest of the computations amount to careful accounting of the symmetry properties of the Weyl tensor (which are identical to those of the Riemann tensor). We should also observe that the Bianchi identities should hold irrespectively of the precise form of functions $f_{i,n}$, because these are mere numerical coefficients, which are required to make the theory ghost free [14]. Technically, this means that we should not bother about the summation over $n$, but rather concentrating on the inner summation over $l$ in Eqs. (9-13). Finally, the symmetry with respect to $\alpha \leftrightarrow \beta$ permutation in the equations of motion can be accounted by re-arranging the summation over $l$ in the inverse order from $n - 1$ to 0. With all the above precautions in mind we can perform a direct substitution and check term by term that all the contributions vanish upon computing the divergence of the equations of motion. As stated above, it is not a surprise that the Bianchi identities hold. However, it is a very good check for the equations of motion, mostly for the mutual coefficients in front of the different terms, details can be found in Ref. [14].

Let us note that in GR, we have a vacuum solution, around an asymptotically Minkowski background for a static case,

$$R = 0, \quad R_{\mu\nu} = 0.$$  \hspace{1cm} (15)

In this case the energy momentum tensor vanishes in all the region except at $r = 0$, where the source $m\delta^{3}(r)$ is localized. One of the properties of such a vacuum solution is the presence of 1/r- static and spherically symmetric metric solution, similar to the Schwarzschild metric, given by:

$$ds^{2} = -b(r)dt^{2} + b^{-1}(r)dr^{2} + r^{2}(d\theta^{2} + \sin^{2}(\theta)d\phi^{2})$$, \hspace{1cm} (16)

where $b(r) = 1 - 2Gm/r$ with the presence of a central singularity at $r = 0$, and also the presence of an event horizon. As we have already discussed, for $r < 2Gm$, what dominates is the 1/r part of $b(r)$, which dictates the rise in the gravitational potential all the way to $r = 0$. Note that, although the vacuum solution permits $R = 0$, $R_{\mu\nu} = 0$, the Weyl-tensor is non-vanishing in the case of a Schwarzschild metric, where

$$W_{\mu\nu\lambda\sigma} W^{\mu\nu\lambda\sigma} \rightarrow \infty,$$

as $r \rightarrow 0$. Now in our case, indeed the full equations of motion are quite complicated, nevertheless, we might be able to test this hypothesis of setting $R = 0$, $R_{\mu\nu} = 0$, and study whether the Schwarzschild metric, or $1/r$ type metric potential is a viable metric solution of our theory of gravity or not?

Let us then demand that the above action, Eq. (2), along with the equations of motion Eq. (8), permits a solution which is Schwarzschild metric with $P_{\alpha\beta} = 0$, and $R = 0$ and $R_{\mu\nu} = 0$. In fact, in the region of non-locality where higher derivative terms in the action are dominant, it suffices to demand that $R = \text{constant}$ and $R_{\mu\nu} = \text{constant}$. Let us now concentrate on the full equations of motion (8) with the Weyl part of the full equations of motion:

$$P^{\alpha\beta} = 0 = P^{\alpha\beta}_{3} = \frac{\alpha_{c}}{8\pi G} \left( -g^{\alpha\beta}W_{\mu\nu\lambda\sigma}F_{3}(\square_{s})W_{\mu\nu\lambda\sigma} + 4W_{\mu\nu\lambda\sigma}F_{3}(\square_{s})W^{\mu\nu\lambda\sigma} \right. \hspace{1cm}$$

$$\left. - 4(R_{\mu\nu} + 2\nabla_{\mu}\nabla_{\nu})(F_{3}(\square_{s})W^{\beta\mu\nu}) - 2\Omega_{i}^{\beta\gamma} + g^{\alpha\beta}(\Omega_{i}^{\gamma} + \bar{\Omega}_{i}) - 8\Delta_{3}^{\alpha\beta} \right). \hspace{1cm} (17)$$

Indeed, we would expect that in order to fulfill the necessary condition (but not sufficient) for the Schwarzschild metric to be a solution of Eq. (8), we would have both the left and the right hand side of the above equation vanishes.

\[5\] We have checked that the Bianchi identity holds true at each and every order of $\square_{s}$. 


identically. The failure of this test will imply that the Schwarzschild metric cannot be the permissible solution of the equation of motion for Eq. (8).

There are couple of important observations to note, which we summarize below:

1. $\mathcal{F}_s(\square_s)$ contain an infinite series of $\square_s$.

2. The Bianchi identity holds for each and every order in $\square_s$, as we have already discussed.

3. The right hand side of Eq. (17) should vanish at each and every order in $\square_s$. This is due to the fact that when we compare the terms, assigned to coefficients $f_{i,n}$ (where the box operator has been applied $n$ times, i.e. $\square^n_s$) with terms where the box operator has been applied $n + 1$ times ($\square^{n+1}_s$, assigned to coefficient $f_{i,n+1}$), then the $1/r^n$ dependence would at least be changed to $1/r^{n+2}$ in this process. Note that the box operator has roughly two covariant derivatives in $r$. Therefore, if we are not seeking for any miraculous cancellation, between different orders in $\square_s$, it is paramount that each and every order in $\square_s$, the right hand side must vanish to yield the Schwarzschild-like metric solution.

4. In fact, we could repeat the same argument for higher order singular metric ansatze, such as $1/r^\alpha$, for $\alpha > 0$ at short distances, near the ultraviolet.

In order to obtain some insight into this problem, let us first consider the right hand side of $P_3^{\alpha\beta}$ with one $\square_s$ only, such that

$$\mathcal{F}_3(\square_s) = (f_{30} + f_{31}\square_s).$$

Therefore, Eq. (17) becomes

$$P_3^{\alpha\beta} = \frac{\alpha_c}{8\pi G} \left( -g^{\alpha\beta} W^{\mu\nu\lambda\sigma}(f_{30} + f_{31}\square_s)W_{\mu\nu\lambda\sigma} + 4W^{\alpha}_{\mu\nu\sigma}(f_{30} + f_{31}\square_s)W^{\beta\mu\nu\sigma} 
- 4(R_{\mu\nu} + 2\nabla_\mu \nabla_\nu)(f_{30} + f_{31}\square_s)W^{\beta\mu\nu\sigma} - 2f_{31}\nabla^\alpha W^{\mu\nu\rho\gamma} \nabla^\beta W_{\mu\nu\rho\gamma} 
+ 4f_{31} W^{\gamma\delta\mu\nu} \nabla^\alpha W^\beta_{\gamma\delta\mu\nu} - 8f_{31} W^{\gamma\delta\mu\nu} \nabla^\alpha W^\beta_{\gamma\delta\mu\nu} \right).$$

In the static limit, after some computations, we can infer the following:

1. All the terms combining $f_{30}$ terms cancel each other from the above expression in Eq. (18). This is indeed reminiscence, and agrees to the earlier computations performed in this regard in Ref. [32], where the action corresponds to just the quadratic in curvature, but with local quadratic curvature action:

$$S = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} \left( R + \alpha_c[R^2 + R_{\mu\nu}R_{\mu\nu} + W^{\mu\nu\lambda\sigma} W_{\mu\nu\lambda\sigma}] \right).$$

Such an action indeed provides singular solutions with metric coefficients $b(r) \sim 1/r$ for $r << r_{sch}$, as the leading order contribution, in spite of the fact that the above action has been shown to be renormalizable, but with an unstable vacuum, due to spin-2 ghost [31]. The BGKM action indeed attempts to address the ghost problem of quadratic curvature gravity.

2. The first non-trivial result comes from the fact that the only terms which do not cancel, and survive from the right hand side of Eq. (18) are those proportional to $f_{31}$, and one can show explicitly that they go as $1/r^8$,

in the UV ( $r \ll 1/M_s$), for details see the Appendix. This means, that indeed $1/r$ as a metric solution does not pass through our test, since the right hand side of the above equation of motion is non-vanishing, but the left hand side ought to vanish in lieu of the vacuum condition, $P^{\alpha\beta} = 0$. 


3. In fact, we may be able to generalize our results to any orders in \( \Box_s \) by noting that the higher orders beyond one box would contribute, at least, two more covariant derivatives in \( r \) in going from \( \Box_s^\alpha \) to \( \Box_s^{\alpha+1} \) terms (assuming that \( \Box_s \sim \frac{1}{M_s^2} \partial_r^2 \)). This means that the full computation for the right hand side of Eq. (13) would yield:

\[
P_3^{\alpha \beta} \sim g^{\alpha \beta} \left( f_{31} \mathcal{O}(\frac{1}{r^6}) + f_{32} \mathcal{O}(\frac{1}{r^{10}}) + \ldots + f_{3n} \mathcal{O}(\frac{1}{r^{6+2n}}) \right),
\]

\((g^{\alpha \beta} \) is defined from the metric \([10]\), see the exact definition of \( P_3^{\alpha \beta} \) in the appendix\) which would require too much fine tuning to cancel each and every term, while keeping in mind that \( f_{3n} \) are mere constant coefficients. Barring such unjustified cancellation, it is fair to say that indeed \( 1/r \) for \( r \ll 1/M_s \) as a metric potential for the BGKM gravity is not a valid solution, if \( F_3 \) has a nontrivial dependence on \( \Box_s \).

Similar conclusions have already been drawn in Ref. [20], with a complementary arguments. In Ref. [20], the argument was based on taking a smooth limit from non-linear solution of Eq. (2) to the linear solution. For any physical solution to be valid, the non-linear solution must pave to the linear solution smoothly.

At the linear level (where the metric potential is bounded below 1), it was shown that the Weyl term vanishes quadratically in \( r \) [20], for a non-singular metric solution given by a metric potential Eq. (4). Therefore, at the full non-linear level \( 1/r \)-type metric potential cannot be promoted as a full solution for the non-linear equations of motion for the BGKM action, since there is no way it can be made to vanish quadratically at the linear level. Similar conclusions can be made for any metric potential which goes as \( 1/r^\alpha \) for \( \alpha > 1 \).

Indeed, this intriguing and potentially very powerful conclusion leads to the fact that the BGKM action with quadratic curvature, infinite covariant derivative gravitational action not only ameliorates the curvature singularity at \( r = 0 \), but also gives rise to a metric potential which is bounded below one in the entire spacetime regime. The notion about the physical mechanism which avoids forming a trapped surface, also yields a static metric solution of gravity, which has no horizon, see [54]. The only viable solution of Eq. (2) remains that of the linear solution, already described by Eq. (6). Indeed, this last step has to be shown more rigorously, which we leave for future investigation.

Another important conclusion arises due to the non-local interactions in the gravitational sector, which yields a non-vacuum solution, such that \( R \neq 0 \) and \( R_{\mu \nu} \neq 0 \), within length scale \( \sim 2/M_s \) [24]. This is due to the fact that the BGKM gravity smears out the Dirac-delta source, and therefore a vacuum solution does not exist any more like in the case of the Einstein’s gravity, or any \( f(R) \) gravity, or even in the context of local quadratic curvature gravity, see Eq. (19).

\[\Box \text{IV. CONCLUSION}\]

The conclusion of this paper is very powerful. We have argued that Schwarzschild metric or \( 1/r \)-type metric potentials cannot be the solution of the full BGKM action given by Eq. (2), and the full non-linear equations of motion Eq. (3). By \( 1/r \)-type metric potential, we mean the non-linear part of the Schwarzschild metric, for \( r < 2Gm \), where \( m \) is the Dirac delta source. The presence or absence of singularity is judged by the Weyl contribution. In the pure Einstein-Hilbert action, indeed the Weyl term in the Schwarzschild metric is non-vanishing, and contributes towards the Kretschmann singularity at \( r = 0 \). In the case of infinite derivatives in 4 dimensions, we have shown here that this is not the case, and the infinite derivative Weyl contribution contradicts with \( 1/r \) being the metric solution for a vacuum configuration, for which the energy momentum tensor vanishes, for a static and spherically symmetric solution for the BGKM action. By itself the result does not prove or disprove a non-singular metric potential, but it provides a strong hint that the full equations of motion cannot support Schwarzschild type of \( 1/r \) type metric potential. We have also argued that on a similar basis even \( 1/r^\alpha \) for \( \alpha > 0 \) will not serve as a full solution to the BGKM gravity. It would be very interesting to explore that if the BGKM gravity may allow other static/non-static singular metric solutions or not [35].

\[\Box\]

6 For the Schwarzschild metric this takes the form \( \Box_s = \frac{1}{M_s^2} g^{\mu \nu} \partial_\mu \partial_\nu = \frac{1}{M_s^2} \left[ \left( 1 - \frac{2m}{r} \right) \partial^2 - 2 \left( -\frac{m}{r^2} + \left( 1 - \frac{2m}{r} \right) \frac{1}{r} \right) \partial_r \right] \).
V. APPENDIX

A. Explicit non-vanishing contributions from the Weyl term

Here we show the relevant terms, present in Eq. (18), assuming \( b(r) = 1 - 2Gm/r \) in the metric (16). The explicit enumeration of each term is important to understand how the coefficient \( f_{30} \) and \( f_{31} \) appear and how they might cancel. Let us define

\[
P^\alpha_3 = \frac{\alpha_c}{8\pi G} \sum_{i=1}^{6} F^\alpha_i
\]

1. For the first term, \( F^\alpha_1 = -g^{\alpha\beta}W^{\mu\nu\lambda\sigma}(f_{30} + f_{31}\Box_s)W_{\mu\nu\lambda\sigma} \), the calculation yields

\[
F^\alpha_1 = g^{\alpha\beta} \frac{48G^2 m^2}{r^8} \left( \frac{1}{M^2} \begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 
\end{pmatrix} \right)
\]

2. The second term, \( F^\alpha_2 = +4W^{\alpha\beta\mu\nu}(f_{30} + f_{31}\Box_s)W^{\beta\mu\nu} \), is given by

\[
F^\alpha_2 = g^{\alpha\beta} \frac{48G^2 m^2}{r^8} \left( \frac{1}{M^2} \begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 
\end{pmatrix} \right)
\]

3. The third term, \( F^\alpha_3 = -4(2R_{\mu\nu} + \nabla_\mu \nabla_\nu)(f_{30} + f_{31}\Box_s)W^{\beta\mu\nu\lambda} \), is given by

\[
F^\alpha_3 = g^{\alpha\beta} \frac{28G^2 m^2}{r^8} f_{31} \left( \frac{1}{M^2} \begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 
\end{pmatrix} \right)
\]

which only depends on the \( f_{31} \) coefficient.

4. The fourth term, \( F^\alpha_4 = -2f_{31}\nabla^\alpha W^\lambda_{\mu\nu\sigma}\nabla^\beta W^\lambda_{\mu\nu\sigma} \), is given by

\[
F^\alpha_4 = g^{\alpha\beta} \frac{28G^2 m^2}{r^8} f_{31} \left( \frac{1}{M^2} \begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 
\end{pmatrix} \right)
\]

5. The fifth term, \( F^\alpha_5 = +g^{\alpha\beta}f_{31}(\nabla^\alpha W^{\mu\nu\rho\sigma}\nabla^\beta W_{\mu\nu\rho\sigma} + W^{\mu\nu\rho\sigma}\Box_s W_{\mu\nu\rho\sigma}) \), is given by

\[
F^\alpha_5 = g^{\alpha\beta} \frac{144G^2 m^2}{r^8} f_{31} \left( \frac{1}{M^2} \begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 
\end{pmatrix} \right)
\]

6. The sixth term, \( F^\alpha_6 = -8f_{31}(W^{\gamma\nu}_{\mu\rho}\nabla^\alpha W_{\gamma\beta\rho\mu} - W_{\gamma\beta\rho\mu}\nabla^\alpha W^{\gamma\nu}_{\mu\rho}) \mu, \nu \), is given by

\[
F^\alpha_6 = g^{\alpha\beta} \frac{576G^2 m^2}{r^8} f_{31} \left( \frac{1}{M^2} \begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 
\end{pmatrix} \right)
\]
Having computed each term of $P_{\alpha \beta}^{\gamma \delta}$, we can conclude that the stress energy momentum tensor dependence on the $f_{30}$ coefficient is vanishing, and only the one box, $\Box_s$, contributions survive. Finally, we have the non-vanishing contribution,

$$P_{\alpha \beta}^{\gamma \delta} = g^{\alpha \beta} \frac{144 G m^2}{8 \pi r^3 M_s} f_{31} \left( \begin{array}{cccc}
-\frac{5(r-2Gm)}{r} & 0 & 0 & 0 \\
0 & \frac{7(r-2Gm)}{r} & 0 & 0 \\
0 & 0 & \frac{21(r-50Gm)}{r} & 0 \\
0 & 0 & 0 & \frac{21(r-50Gm)}{r} \\
\end{array} \right).$$  (27)

**Second order in $\Box_s$ contributions**: In order to strengthen our arguments, we present below the additional contribution for the second order in box, i.e., $\Box_s^2$:

$$P_{\alpha \beta}^{\gamma \delta} (\Box_s^2) = -g^{\alpha \beta} \frac{576 G m^2}{8 \pi r^4 M_s^2} f_{32} \left( \begin{array}{cccc}
a_{00} & 0 & 0 & 0 \\
0 & a_{11} & 0 & 0 \\
0 & 0 & a_{22} & 0 \\
0 & 0 & 0 & a_{33} \\
\end{array} \right).$$  (28)

with the dimensionless matrix elements, defined as

$$
\begin{align*}
a_{00} &= \frac{(939G^2m^2 - 744Gmr + 140r^2)}{r^2}, \\
a_{11} &= \frac{(195G^2m^2 - 132Gmr + 20r^2)}{r^2}, \\
a_{22} &= \frac{-(789G^2m^2 - 534Gmr + 80r^2)}{r^2}, \\
a_{33} &= \frac{-(789G^2m^2 - 534Gmr + 80r^2)}{r^2}.
\end{align*}
$$

Let us now consider, for example, the $P_{32}^{22}$ element at $\Box_s^2$, namely:

$$P_{32}^{22} = \frac{\alpha_c}{8\pi G} f_{31} \left( \frac{3024 G^2 m^2}{r^{10} M_s^2} - \frac{7200 G^3 m^3}{r^{11} M_s^2} \right) + f_{32} \left( \frac{46080 G^2 m^2}{r^{12} M_s^4} - \frac{307584 G^3 m^3}{r^{13} M_s^4} + \frac{454464 G^4 m^4}{r^{14} M_s^4} \right) + \cdots.$$  (29)

Demanding that $P_{32}^{22} = 0$, implies that $f_{31} = f_{32} = 0$. We can now ask what would happen for higher orders in $\Box_s$. Since $\Box_s \sim \frac{\partial^2}{M_s^2 \partial r^2}$, we have at the lowest third order contribution in box, in powers of $r$, is proportional to

$$f_{33} \frac{G^2 m^2}{r^{14} M_s^6}. \quad (30)$$

Therefore, since we already concluded that $f_{31} = f_{32} = 0$, we now have to demand that the contribution of $f_{33} \frac{G^2 m^2}{r^{14} M_s^6}$ vanishes identically. The lowest fourth order contribution, in powers of $r$, will go as

$$f_{34} \frac{G^2 m^2}{r^{16} M_s^8}, \quad (31)$$

we are left with the option that $f_{33} = f_{34} = 0$. Obviously, we do not claim that this is a rigorous mathematical demonstration, however we can hint, by dimensional analysis, that the lowest $n^{th}$ order contribution will be always proportional to

$$f_{3n} \frac{G^2 m^2}{r^{8+2n} M_s^{2n}}. \quad (32)$$

Indeed, the aforementioned analysis will make it hard to make all the expansion coefficient set to vanish, i.e. $f_{3n} = 0$.

**ACKNOWLEDGMENTS**

We would like to thank Luca Buoninfante, Tirthabir Biswas, Valeri Frolov, Tomi Koivisto, and Robert Brandenberger for numerous discussions at various stages of this project. AK and JM are supported by the grant
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