\textbf{PGL}_2\text{-EQUIVARIANT STRATA OF POINT CONFIGURATIONS IN $\mathbb{P}^1$}

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\textbf{Abstract.} We compute the integral Chow ring of the quotient stack $[(\mathbb{P}^1)^n/PGL_2]$, which contains $\mathcal{M}_{0,n}$ as a dense open, and determine a natural $\mathbb{Z}$-basis for the Chow ring in terms of certain ordered incidence strata. We further show that all $\mathbb{Z}$-linear relations between the classes of ordered incidence strata arise from an analogue of the WDVV relations in $A^*(\mathcal{M}_{0,n})$.

Next we compute the classes of unordered incidence strata in the integral Chow ring of the quotient stack $[\text{Sym}^n\mathbb{P}^1/PGL_2]$ and classify all $\mathbb{Z}$-linear relations between the strata via these analogues of WDVV relations.

Finally, we compute the rational Chow rings of the complement of a union of unordered incidence strata.

1. \textbf{Introduction}

We consider $PGL_2$-equivariant Chow classes of incidence strata corresponding to point configurations in $\mathbb{P}^1$. Our results concern both ordered point configurations, parametrized by $((\mathbb{P}^1)^n$, and unordered point configurations, parametrized by $\text{Sym}^n\mathbb{P}^1 \cong \mathbb{P}^n$, previously considered $GL_2$-equivariantly by Fehér, Némethi, and Rimányi [13]. The equivariant Chow rings $A^*_{PGL_2}((\mathbb{P}^1)^n)$ and $A^*_{PGL_2}(\mathbb{P}^n)$ can be defined as the integral Chow rings of the quotient stacks $[(\mathbb{P}^1)^n/PGL_2]$ and $[\mathbb{P}^n/PGL_2]$ respectively, so the $PGL_2$-equivariant Chow classes of incidence strata are of interest because they specialize to relative classes in $A^*(\mathbb{P}^n)$ and $A^*(\text{Sym}^n\mathbb{P})$ respectively for any $\mathbb{P}^1$-bundle $\mathcal{P} \to B$.

The equivariant Chow ring $A^*_{PGL_2}(\text{pt}) \cong \mathbb{Z}[c_2, c_3]/(2c_3)$ was computed by Pandharipande [27], and the 2-torsion is reflected in the fact that $PGL_2$ is not special. Consequently, restriction to a maximal torus is only injective rationally [9, Proposition 6], which is the main obstacle to computing integral Chow classes. This is in contrast with the analogous $GL_2$-equivariant Chow rings, which are substantially easier to work with as $GL_2$ is special. Also, classes in the $GL_2$-equivariant Chow rings of $((\mathbb{P}^1)^n$ and $\mathbb{P}^n$ only specialize to classes in $A^*(\mathbb{P}^n)$ and $A^*(\text{Sym}^n\mathbb{P})$ when $\mathcal{P} \to B$ is the projectivization of a rank 2 vector bundle.

The reader may refer to Section 2.1 for an exposition on how equivariant Chow classes yield universal relations between relative Chow classes in bundles, and Example 1.1 for example applications.

1.1. \textbf{Ordered strata in $[(\mathbb{P}^1)^n/PGL_2]$}. The moduli space $\mathcal{M}_{0,n}$ of $n$ distinct points on $\mathbb{P}^1$ is the quotient of $(\mathbb{P}^1)^n \setminus \bigcup_{i<j} \Delta_{i,j}$ by the free action of $PGL_2$, where $\Delta_{i,j}$ is the locus in $(\mathbb{P}^1)^n$ where the $i$th and $j$th coordinates are equal. This is classically compactified by the variety $\overline{\mathcal{M}}_{0,n}$ of stable genus zero $n$-pointed curves.

We study the quotient stack $[(\mathbb{P}^1)^n/PGL_2]$ containing $\mathcal{M}_{0,n}$ as a dense open and in particular its integral Chow ring $A^*_{PGL_2}((\mathbb{P}^1)^n)$ as defined in [9, Section 5].
This stack is stratified by certain incidence strata $\Delta_P \subset (\mathbb{P}^1)^n$ for $P$ a partition of $[n] := \{1, \ldots, n\}$, the loci where the $i$th and $j$th coordinates are equal if $i$ and $j$ are in the same part of $P$.

We compute a ring presentation in Theorem 1.1 for $A^\bullet_{\text{PGL}_2}((\mathbb{P}^1)^n)$ similar to that of $A^\bullet_{\text{M}_{0,n}}$ computed by Keel [22]. The incidence strata $\Delta_P$ play a fundamental role in the equivariant Chow ring: in Theorem 1.3 we compute a $\mathbb{Z}$-basis for $A^\bullet_{\text{PGL}_2}((\mathbb{P}^1)^n)$, which consists in degree $\leq n - 2$ of certain incidence strata, and in Theorem 1.5 we show all relations between incidence strata arise from an analogue of the WDVV relations on $A^\bullet_{\text{M}_{0,4}}$ (see Section 1.1.1).

**Theorem 1.1.** The following are true.

1. (Theorem 5.10) For $n \geq 3$, the ring $A^\bullet_{\text{PGL}_2}((\mathbb{P}^1)^n)$ is a free $\mathbb{Z}$-module of rank
   \[
   \sum_{i \leq k \atop i + k \equiv 0 \mod 2} \binom{n}{i},
   \]
   where the relations are (noting $\Delta_{j,i} := \Delta_{i,j}$ for $j > i$)
   \begin{enumerate}
   
   
   
   
   \end{enumerate}

2. (Lemma 5.3) For $n \geq 1$, the group $A^n_{\text{PGL}_2}((\mathbb{P}^1)^n)$ is a free $\mathbb{Z}$-module with the subring of

3. (Theorem 3.3) For $n \geq 1$, the natural map from $A^\bullet_{\text{PGL}_2}((\mathbb{P}^1)^n)$ to

   $A^\bullet_{\text{GL}_2}((\mathbb{P}^1)^n) \cong \mathbb{Z}[u, v]^{S_2}[H_1, \ldots, H_n]/(F(H_1), \ldots, F(H_n))$,

   is injective, where $u + v$ and $uv$ are the first and second chern classes of the standard representation of $GL_2$, $F(z) = (z + u)(z + v)$, and $H_i$ is $c_1(\mathcal{O}(1)) \in A^\bullet_{\text{PGL}_2}((\mathbb{P}^1))$ pulled back via projection to the $i$th factor.

   This identifies $A^\bullet_{\text{PGL}_2}((\mathbb{P}^1)^n)$ with the subring of $A^\bullet_{\text{GL}_2}((\mathbb{P}^1)^n)$ generated by $H_i + H_j + u + v$ for distinct $i, j$ and $2H_i + u + v$ for all $i$, and $H_i \mapsto H_i + H_j + u + v$.

   (Remark 3.5) If the base field is $\mathbb{C}$, then for all $n \geq 1$ the map $A^\bullet_{\text{PGL}_2}((\mathbb{P}^1)^n) \to H^\bullet_{\text{PGL}_2}((\mathbb{P}^1)^n)$ to equivariant cohomology is an isomorphism.

   The square relations $\Delta_{i,j} + \Delta_{i,l} = \Delta_{i,k} + \Delta_{j,l}$ for distinct $i, j, k, l$ are analogous to the WDVV relations on $A^\bullet_{\text{M}_{0,4}}$ pulled back from $A^\bullet_{\text{M}_{0,4}} \cong A^\bullet(\mathbb{P}^1)$ (see Section 1.1.1).

   The diagonal relations $\Delta_{i,j} \Delta_{i,k} = \Delta_{i,j} \Delta_{j,k}$ are geometrically obvious as $\Delta_{i,j} \cap \Delta_{i,k}$ and $\Delta_{i,j} \cap \Delta_{j,k}$ both give the locus where the $i$th, $j$th, and $k$th coordinates are all equal. In particular, repeated intersections in this fashion allow us to reconstruct all $\Delta_P$.

   We will in fact show that the classes of the $\Delta_P$ for $P$ a partition of $\{1, \ldots, n\}$ into $d \geq 2$ parts generate $A^\bullet_{\text{PGL}_2}((\mathbb{P}^1)^n)$ $\mathbb{Z}$-linearly. Surprisingly, we can produce a $\mathbb{Z}$-basis for $A^\bullet_{\text{PGL}_2}((\mathbb{P}^1)^n)$ represented by certain $\Delta_P$ (at least in degrees $\leq n - 2$).

**Definition 1.2.** Call a partition $P$ of $\{1, \ldots, n\}$ good if it can be written as $P = \{A_1, \ldots, A_d\}$ with $A_1 \sqcup A_2$ an initial segment of $\{1, \ldots, n\}$, and $A_3, \ldots, A_d$ intervals.

**Theorem 1.3** (Theorem 5.16). For $n \geq 3$, the additive group $A^\bullet_{\text{PGL}_2}((\mathbb{P}^1)^n)$ has a $\mathbb{Z}$-basis consisting of the following.

1. If $k \leq n - 2$, the classes $\Delta_P$ for $P$ a good partition into $n - k$ parts.
(2) If \( k > n - 2 \), the classes \( \Delta_{k,n+2}^k \Delta_P \) for \( P \) a partition of \( \{1, \ldots, n\} \) into two parts and \( \Delta_{i[n],j[n]}^{k} \Delta_{\{n\}} \), where for each \( P \) the pair \( i_P, j_P \) are chosen to lie in the same part of \( P \).

In Section \[5.1\] we describe a simple algorithm to write arbitrary classes in this \( \mathbb{Z} \)-basis, along with a worked example.

In addition, we show that all relations between the \( \Delta_P \) are generated by push-forwards of square relations. The method of proof will in fact supply an algorithm to write every \( \Delta_Q \) as a \( \mathbb{Z} \)-linear combination of \( \Delta_P \) for \( P \) a good partition using only these relations.

**Definition 1.4.** Denote by \( \text{Part}(d,n) \) the set of partitions of \( n \) into \( d \) parts. Let \( \text{Sq}(d,n) \) be the subgroup of the free abelian group \( \mathbb{Z}^{\text{Part}(d,n)} \) generated by formal square relations \( P_{i,j} - P_{j,i} \) for \( P \in \text{Part}(d+1,n) \) and \( i,j \in \{1, \ldots, n\} \) indices in different parts of \( P \), where \( P_{x,y} \) denotes the partition formed by merging the parts of \( P \) containing \( x \) and \( y \).

**Theorem 1.5** (Corollary \[5.4\]). For \( d \geq 2 \), the map
\[
\mathbb{Z}^{\text{Part}(d,n)}/\text{Sq}(d,n) \to A_{\text{PGL}_2}^{n-d}(\mathbb{P}^1)^n
\]
sending \( P \mapsto \Delta_P \) is an isomorphism.

In particular, since every square relation between the \( \Delta_P \) classes comes from an explicit \( \text{PGL}_2 \)-invariant degeneration in \( (\mathbb{P}^1)^n \) (see Section \[1.1.1\]), Theorem \[1.5\] implies that all linear relations between the \( \Delta_P \) classes can be realized by a sequence of \( \text{PGL}_2 \)-invariant degenerations in \( (\mathbb{P}^1)^n \).

Non-equivariantly, there are relations between the classes \( \Delta_P \in A^*(\mathbb{P}^1)^n \) not generated by these square relations. For example, if \( n = 4 \) we have
\[
\Delta_{\{1,2,3\},\{4\}} + \Delta_{\{1,2,4\},\{3\}} + \Delta_{\{1,3,4\},\{2\}} + \Delta_{\{2,3,4\},\{1\}}
= \Delta_{\{1,2\},\{3,4\}} + \Delta_{\{1,3\},\{2,4\}} + \Delta_{\{1,4\},\{2,3\}}
\]
in \( A^2(\mathbb{P}^1)^4 \).

**Remark 1.6.** All of our theorems can be extended to \( n = 1, 2 \) if we include the classes \( \psi_i = \pi_i^*c_1(T^\vee \mathbb{P}^1) \in A^*(\mathbb{P}^1)^n \) pulled back from the \( i \)th projection \( \pi_i \), which for \( n \geq 3 \) can be written in terms of the \( \Delta_{j,k} \)-classes via \( \psi_i = \Delta_{j,k} - \Delta_{i,j} - \Delta_{i,k} \) for any \( j, k \neq i \). They correspond to \( -(2H_i + u + v) \) under the map from item (3) of Theorem \[1.1\] (see Proposition \[1.3\]) and their definition is analogous to the \( \psi \)-classes on \( \overline{M}_{0,4} \) \[26\] Section 2.

1.1.1. **Relation of the square relation to the WDVV relation.** The WDVV relation in \( A^*(\overline{M}_{0,4}) \) says two points in \( \overline{M}_{0,4} \cong \mathbb{P}^1 \) corresponding to reducible curves have the same class \[23\] Section 0.1. It was shown by Keel \[22\] that \( A^*(\overline{M}_{0,n}) \) is generated as a ring by its boundary divisors, and the only nontrivial relations come from pulling back the WDVV relation under forgetful maps \( \overline{M}_{0,n} \to \overline{M}_{0,4} \). The square relations relate to the WDVV relations as follows. Consider the diagram
\[
\begin{array}{ccc}
\overline{M}_{0,4}(\mathbb{P}^1,1) & \xrightarrow{\text{ev}} & (\mathbb{P}^1)^4 \\
\downarrow \quad \psi & & \\
\overline{M}_{0,4} & \xrightarrow{\psi} & (\mathbb{P}^1)^4
\end{array}
\]
where \( \text{ev} \) is the \((\text{PGL}_2\text{-equivariant})\) total evaluation map from the Kontsevich mapping space \([16]\) Section 1] and \( \pi \) remembers only the source of the stable map and stabilizes. The square relation is \( \text{ev} \ast \pi^* \) applied to the \( WDVV \) relation.

Equivalently, for any closed point \( a \in \mathbb{P}^1 \cong \overline{\mathcal{M}}_{0,4} \), we can consider the locus \( A_a \subset (\mathbb{P}^1)^n \) consisting of the quadruples of points with cross ratio \( a \). The square relation comes from equating the classes of \( A_0 \) and \( A_\infty \).

1.1.2. Relation to other moduli spaces. If we pick a linearization of the \( \text{PGL}_2 \)-action on \((\mathbb{P}^1)^n\) and there are no strictly semistable points, then excising the unstable locus and applying \([9]\) Theorem 3] gives the rational Chow ring of the GIT quotient. In this case, the ideal given by excision is generated by the classes of the excised strata. See \([13]\) for an approach via quiver representations. These GIT quotients are Hassett spaces with total weight \( 2 + \epsilon \) \([18]\) Section 8] and receive maps from \( \overline{\mathcal{M}}_{0,n} \) via reduction morphisms \([9, \text{Theorem 3}]\), as induced maps between GIT quotients \([19, \text{Theorem 3.4}]\), or by viewing \( \overline{\mathcal{M}}_{0,n} \) as a Chow quotient \([20]\).

1.2. Unordered strata in \([\text{Sym}^n\mathbb{P}^1]/\text{PGL}_2\). The \( \text{PGL}_2\)-action on \( \mathbb{P}^1 \) induces an action on the symmetric power \( \text{Sym}^n \mathbb{P}^1 \cong \mathbb{P}^n \), which parameterizes degree \( n \) divisors on \( \mathbb{P}^1 \). For each partition \( \lambda = \{\lambda_1, \ldots, \lambda_d\} \) of \( n \), we have the \( \text{PGL}_2\)-invariant subvariety \( Z_\lambda \subset \mathbb{P}^n \) consisting of divisors that can be written in the form \( \sum_{i=1}^d \lambda_i p_i \) where \( p_i \in \mathbb{P}^1 \). For convenience we often write \( \lambda = a_1^{e_1} \cdots a_k^{e_k} \) to be the partition of \( n \) where \( a_i \) appears \( e_i \) times.

1.2.1. Integral classes of strata. We compute the class of \([Z_\lambda]\) in \( A^*_{\text{PGL}_2}(\mathbb{P}^n) \). The class of \([Z_\lambda]\) in \( A^*_{\text{PGL}_2}(\mathbb{P}^n) \) was given in \([13]\), and we will give a quick independent proof and more compact form in Theorem 1.5. If \( n \) is odd, the map \( A^*_{\text{PGL}_2}(\mathbb{P}^n) \to A^*_{\text{GL}_2}(\mathbb{P}^n) \) induced by the projection \( GL_2 \to PGL_2 \) is injective (see Proposition 3.7). Therefore, all of the difficulty lies in computing \([Z_\lambda]\) in \( A^*_{\text{PGL}_2}(\mathbb{P}^n) \) for \( n \) even. It turns out (see Section 7) that it suffices to compute the class in \( A^*_{\text{PGL}_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z} \), which takes on a particularly simple form.

**Theorem 1.7.** Let \( n \) be even and \( \lambda = a_1^{e_1} \cdots a_k^{e_k} \) be a partition of \( n \) into \( d = e_1 + \cdots + e_k \) parts. The class of \([Z_\lambda]\) \( \in A^*_{\text{PGL}_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z} \cong F_2[c_2, c_3, H]/(q_n(H)) \) where

\[
q_n(t) = \begin{cases} 
\frac{t^{(n+4)/4}(t^3 + c_2 t + c_3)^{n/4}}{4} & n \equiv 0 \mod 4, \\
\frac{t^{(n-2)/2}(t^3 + c_2 t + c_3)^{(n+2)/4}}{4} & n \equiv 2 \mod 4
\end{cases}
\]

is non-zero precisely when all \( a_i \) and \( \frac{dt}{e_1! \cdots e_k!} \) are odd and all \( e_i \) are even, in which case it is equal to \( (\frac{q_n}{q_n'})(H) \).

1.2.2. Relations between strata. If \( \lambda = \{\lambda_1, \ldots, \lambda_d\} = a_1^{e_1} \cdots a_k^{e_k} \) is a partition of \( n \), then taking \( \Phi : (\mathbb{P}^1)^n \to \text{Sym}^n \mathbb{P}^1 \) to be the multiplication map, if \( P = \{A_1, \ldots, A_d\} \) is any partition of \([n]\) with \(|A_i| = \lambda_i \), we have

\[
\Phi_* \Delta_P = (\prod e_i!)[Z_\lambda].
\]

In particular, every square relation between the classes of ordered strata induces a relation between \([Z_\lambda]\) classes by pushing forward along \( \Phi \).

**Theorem 1.8.** (Section 7) Fix \( n \) and choose \( a_\lambda \in \mathbb{Z} \) for each partition of \( n \). The following are equivalent:

1. \( \sum a_\lambda [Z_\lambda] = 0 \) in \( A^*_{\text{PGL}_2}(\mathbb{P}^n) \)
Example 1.11. Suppose $n = 6$, then Corollary 1.10 implies

$$[Z_{4,1,1}] + 3[Z_{2,2,2}] = [Z_{3,2,1}].$$

Consider the following two instances:

1. Let $C_t \subset \mathbb{P}^2$ be a general pencil of degree 6 plane curves. Then, as we vary $C_t$ over $t \in \mathbb{P}^1$, the number of hyperflex lines plus thrice the number of tritangent lines is equal to the number of lines that are both flex and bitangent.
2. Let $X \subset \mathbb{P}^3$ be a general degree 6 surface. Then in $\mathbb{G}(1,3)$, the class of the curve of lines that meet $X$ to order 4 at a point plus three times the class of the curve of tritangent lines to $X$ is equal to the class of the curve of lines that meet $X$ at three points with multiplicities 1, 2, 3.
Note that in both examples, in the absence of a transversality argument, the equalities need to be taken with appropriate multiplicities.

**Remark 1.12.** Lines with prescribed orders of contact with a hypersurface were also studied in [30 Section 5]. Counts of these lines are also related to counting line sections of a hypersurface with fixed moduli [3, 23]. For the surface \( X \subset \mathbb{P}^3 \) in Example 1.11, the points \( p \in X \) for which a line meets \( X \) at \( p \) to order 4 is the *flecnodal curve*, which is always of expected dimension 1 if \( X \) is not ruled by lines by the Cayley-Salmon theorem [21, Theorem 6], which is a primary tool for bounding the number of lines on a smooth surface in \( \mathbb{P}^3 \) (see [29] and [25 Appendix]).

Also, there is no reason not to consider a general variety \( X \subset \mathbb{P}^N \) other than the difficulty of finding a projective variety of higher codimension that has at least a 3-dimensional family of 6-secant lines.

### 1.3. Excision

As an application of our results, we compute the rational equivariant Chow ring of the complement of a union of unordered strata \( A_{PGGL}(\mathbb{P}^n \setminus \cup_{\lambda} Z_\lambda) \otimes \mathbb{Q} = (A^*_{PGGL}(\mathbb{P}^n) \otimes \mathbb{Q})/(\sum I_\lambda \otimes \mathbb{Q}) \), where \( I_\lambda \) is the ideal of excision for \( Z_\lambda \).

We show that \( I_\lambda \otimes \mathbb{Q} \) is generated by the classes of strata contained in \( Z_\lambda \).

**Theorem 1.13 (Lemma S.3).** Given a partition \( \lambda \) of \( n \), \( I_\lambda \otimes \mathbb{Q} \) is generated by \( [Z_\lambda] \) for all \( \lambda' \) that can be obtained from \( \lambda \) by merging parts.

**Remark 1.14.** Theorem 1.13 is false if we replace \( I_\lambda \otimes \mathbb{Q} \) with \( I_\lambda \). This already fails nonequivalently in the case \( n = 4 \) and \( \lambda = \{2, 1, 1\} \). Indeed, \( \Phi : \mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^4 \) maps birationally onto \( Z_\lambda \). Let \( H_1 \) and \( H_2 \) be the hyperplane classes in the factors of \( \mathbb{P}^1 \times \mathbb{P}^2 \) and \( H \) be the hyperplane class of \( \mathbb{P}^4 \). Then \( \Phi_* H_1 = H^2 \), while \( [Z_{\{2,2\}}] = 8H^2 \), \([Z_{\{3,1\}}] = 6H^2 \), and \([Z_{\{2,1,1\}}] = 6H \).

We typically don’t need to use every merged partition \( \lambda' \) for dimension reasons by Corollary 1.10. When \( \lambda = \{a, 1^{n-a}\} \) is a partition with only one part of size greater than 1, we in fact show that \( I_\lambda \otimes \mathbb{Q} \) is generated by just two generators.

**Theorem 1.15 (Theorem S.2).** Given the partition \( \lambda = \{a, 1^{n-a}\} \) of \( n \), \( I_\lambda \otimes \mathbb{Q} \) is generated by \( [Z_\lambda] \) and \( [Z_{\lambda'}] \), where

\[
\lambda' = \begin{cases} 
  \{a+1, 1^{n-a-1}\} & \text{if } a \neq \frac{n}{2} \\
  \{a, 2, 1^{n-a-2}\} & \text{if } a = \frac{n}{2}.
\end{cases}
\]

In fact we will also show the analogous results with \( A^*_{GL_2}(\mathbb{P}^n \setminus \cup_{\lambda} Z_\lambda) \otimes \mathbb{Q} \), and if we further replace \( \mathbb{P}^n \setminus \cup_{\lambda} Z_\lambda \) with its affine cone \( \mathbb{A}^{n+1} \setminus \cup_{\lambda} \tilde{Z}_\lambda \) and consider \( A^*_{GL_2}(\mathbb{A}^{n+1} \setminus \cup_{\lambda} \tilde{Z}_\lambda) \) (see Theorem 3.2).

In the special case \( \lambda = \{2, 1^{n-2}\} \), computing \( I_\lambda \) is the technical heart of the computation of Edidan and Fulghesu of the Chow ring of the stack of hyperelliptic curves of even genus 8.

For \( n \) odd and \( Z_\lambda \) the unstable locus, i.e with \( \lambda = \{\frac{n+1}{2}, 1^{\frac{n-1}{2}}\} \), the rational Chow ring \( A^*_{GL_2}(\mathbb{P}^n \setminus Z_\lambda) \otimes \mathbb{Q} \) equals \( A^*(\mathbb{P}^n/\mathbb{G}L_2) \otimes \mathbb{Q} \), the rational Chow ring of the GIT quotient [9 Theorem 3]. For all \( n \) and \( Z_\lambda \subset \mathbb{P}^n \) the locus of unstable and strictly semistable points, Fehér, Némethi, and Rimányi computed \( A^*_{GL_2}(\mathbb{P}^n \setminus Z_\lambda) \otimes \mathbb{Q} \) using a spectral sequence and used the result to compute the rational Chow ring of the GIT quotient [13 Theorems 4.3 and 4.10]. They actually work with the affine space \( Sym^n K^2 \) instead of \( \mathbb{P}^n \), but the two settings are essentially the same (see Lemma 9.5).
Remark 1.16. The affine analogue of Theorem 1.15 as given in Theorem 9.2 in the special case $a = \begin{bmatrix} 4 \\ \end{bmatrix}$ recovers the $GL_2$-equivariant Chow rings of the stable locus computed in 13 Theorems 4.3 and 4.10 as described above. The Chow ring of the semistable locus required a separate argument.

1.3.1. Multiplicative relations of affine analogues. We conclude in Appendix A by describing a combinatorial branching rule for multiplying the affine analogue of the class of a strata $[Z_{\lambda}] \in A_{GL_2}(Sym^n K^2) \cong \mathbb{Z}[u, v]^{S_2}$ by a generator $u + v$ or $uv$. This generalizes 13 Remark 3.9 (1).

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2. Background and conventions

Conventions:

1. The base field $K$ is algebraically closed of arbitrary characteristic
2. $GL_2$ acts linearly on $\mathbb{P}^1$ and hence on all products $(\mathbb{P}^1)^n$, symmetric powers $\text{Sym}^n \mathbb{P}^1 \cong \mathbb{P}^n$, and their duals
3. $T \subset GL_2$ is the standard maximal torus with standard characters $u$ and $v$
4. $[n]$ denotes the set $\{1, \ldots, n\}$
5. $\Phi : (\mathbb{P}^1)^n \to \text{Sym}^n \mathbb{P}^1 \cong \mathbb{P}^n$ denotes the multiplication map, where $n$ will be clear from context.

2.1. Universal relations and equivariant intersection theory. Equivariant intersection theory was formalized in [9] and will be used to help us analyze the following situation. See also [11] for an exposition.

Suppose we have a group $G$ (typically $G = T, GL_2, PGL_2$) acting on a variety $X$ (typically $(\mathbb{P}^1)^n, \text{Sym}^n \mathbb{P}^1 = \mathbb{P}^n$), and $G$-invariant subvarieties $Y_i$ (typically incidence strata in $(\mathbb{P}^1)^n$ or $\mathbb{P}^n$). Given a principal $G$-bundle $\mathcal{P} \to B$, we have the $X$-bundle $X_{\mathcal{P}} \to B$, where $X_{\mathcal{P}} := \mathcal{P} \times^G X$. Inside $X_{\mathcal{P}}$, we have the cycles

$$(Y_i)_{X_{\mathcal{P}}} := (Y_i)_{\mathcal{P}} \subset X_{\mathcal{P}}$$

restricting to $Y_i$ in each fiber $X$, inducing classes $[Y_i]_{X_{\mathcal{P}}} \in A_*(X_{\mathcal{P}})$. We would like to understand what “universal” linear relations exist between these classes (i.e. which don’t depend on $B$ or $\mathcal{P}$).

For example, if we take $G = PGL_2$, then we are seeking universal relations between classes $[Z_{\mathcal{P}}]_{X_{\mathcal{P}}}$ and between classes $[Z_{\lambda}]_{\text{Sym}^n X}$ for $\mathcal{F} \to B$ a $\mathbb{P}^1$-bundle. If we use $G = GL_2$ instead the relations hold a priori only for $\mathcal{F}$ the projectivization of a rank 2 vector bundle on $B$.

As we will see in Section 2.2 there is a universal group $A^G_*(X)$ approximated by certain $A_*(X_{\mathcal{P}})$ which is equipped with maps $A^G_*(X) \to A_*(X_{\mathcal{P}})$ for all $\mathcal{P}$ and there are classes $[Y_i] \in A^G_*(X)$ such that $[Y_i] \mapsto [Y_i]_{\mathcal{P}}$, so any relations in $A^G_*(X)$ between the $[Y_i]$ descend to relations between the $[Y_i]_{\mathcal{P}}$. Conversely, we will see by construction that any relation between the $[Y_i]_{\mathcal{P}}$ for all $\mathcal{P}$ induces a relation between the $[Y_i]$. 


2.2. Equivariant intersection theory. The equivariant Chow group $A^*_G(X)$ is defined as follows. Suppose $G$ acts linearly on a vector space $V$ with an open subset $U$ of codimension $c$ on which it acts freely. Then for any $k < c$, we define $A^*_{\dim(X) - k}(X) := A_{\dim(X \times \mathbb{G}_U) - k}(X \times^G U)$. Note that $X \times^G U = X \mathbb{P}$ where $\mathbb{P}$ is the principal $G$-bundle $U \to U/G$. This does not depend on the choice of $V$ [9 Definition-Proposition 1].

For $\mathbb{P} \to B$ a principal $G$-bundle over an equidimensional base $B$, we have a map

$$A^*_G(X) \to A_{\dim(B) + \bullet}(P \times^G X)$$

via the composition

$$A^*_{>\dim(X) - c}(X) \cong A_{>\dim((P \times X) \times^G U) - c}((P \times X) \times^G U) \to A_{>\dim((P \times X) \times^G U) - c}((P \times X) \times^G V) \cong A_{>\dim((P \times X) \times^G U) - c}((P \times X) \times^G X)$$

where the second map is induced by flat pullback from the projection, the third map follows from excising $(P \times X) \times^G (V \setminus U)$, and the last map follows from the Chow groups of a vector bundle [17 Theorem 3.3(a)].

Now, we define $A^*_G(X)$ to be the ring of operational $G$-equivariant Chow classes on $X$, i.e. $A^*_G(X)$ is all assignments

$$(Y \to X) \mapsto (A^*_G(Y) \to A^*_G(Y))$$

for every $G$-equivariant map $Y \to X$, compatible with the standard operations on Chow groups [9 Section 2.6]. In our case $X$ is always smooth, and we have the Poincaré duality isomorphism $A^*_G(X) = A^*_{\dim(X) - \bullet}(X)$ [9 Proposition 4], and the identification

$$A^*([X/G]) \cong A^*_G(X),$$

where $[X/G]$ is the quotient stack [9 Section 5.3].

2.3. $GL_2$ and $T$-equivariant Chow rings of $(\mathbb{P}^1)^n$ and $\mathbb{P}^n$. We will postpone discussing $PGL_2$-equivariant intersection rings to Section 3. The equivariant Chow rings $A^*_T((\mathbb{P}^1)^n)$, $A^*_T(\mathbb{P}^n)$, (respectively $A^*_{GL_2}((\mathbb{P}^1)^n)$, $A^*_{GL_2}(\mathbb{P}^n)$) can be approximated by the ordinary Chow rings of $(\mathbb{P}^1)^n$ and $\mathbb{P}^n$ bundles over $\mathbb{P}^N \times \mathbb{P}^N$ (respectively the Grassmannian of lines $\mathbb{G}(1, N)$) for $N >> 0$.

Let $u, v$ be the standard characters of $T$. If $p$ is a point with trivial $GL_2$ action, then

$$A^*_T(pt) = \mathbb{Z}[u, v], \quad A^*_{GL_2}(pt) = \mathbb{Z}[u, v]^{S_2}$$

where $S_2$ acts on $\mathbb{Z}[u, v]$ by swapping $u, v$. By the Chow ring of a vector bundle [17 Theorem 3.3(a)], the $T$ (respectively $GL_2$) equivariant Chow ring of an affine space is isomorphic to the equivariant Chow ring of a point. By the projective bundle theorem [11 Theorem 9.6], we have

$$A^*_T((\mathbb{P}^1)^n) = \mathbb{Z}[u, v][H_1, \ldots, H_n]/(F(H_i)), \quad A^*_T(\mathbb{P}^n) = \mathbb{Z}[u, v][H]/(G(H)),$$

$$A^*_{GL_2}((\mathbb{P}^1)^n) = \mathbb{Z}[u, v]^{S_2}[H_1, \ldots, H_n]/(F(H_i)), \quad A^*_{GL_2}(\mathbb{P}^n) = \mathbb{Z}[u, v]^{S_2}[H]/(G(H))$$
where \( H_i \) is \( c_1(\mathcal{O}_{\mathbb{P}^1}(1)) \) pulled back to \((\mathbb{P}^1)^n\) under the \(i\)th projection and \( H \) is \( c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \), and we define

\[
F(z) = (z + u)(z + v), \quad G(z) = \prod_{k=0}^{n}(z + ku + (n-k)v)
\]

for the rest of the document. Even though one might want to use \(GL_2\)-equivariant Chow rings for applications, \(GL_2\)-equivariant Chow rings inject into \(T\)-equivariant Chow rings, so it suffices to only consider \(T\)-equivariant Chow rings.

The formula for the class of the projectivization of a subbundle \[2.4\] Proposition 9.13] shows the \(i\)th coordinate hyperplane in \(\mathbb{P}^n\) has class \(H + iu + (n-i)v\). This gives the formula for any torus fixed linear space (for example the torus-fixed points) in \((\mathbb{P}^1)^n\) or \(\mathbb{P}^n\) by multiplying a subset of these classes.

**2.4. Ordered and unordered strata of \(n\) points on \(\mathbb{P}^1\).**

**Definition 2.1.** Given a collection \(P = \{A_1, \ldots, A_d\}\) of disjoint subsets of \([n]\), let \(\Delta_P \subset (\mathbb{P}^1)^n\) denote the \(d\)-dimensional locus of points \((p_1, \ldots, p_n)\) where \(p_i = p_j\) whenever \(i, j\) are in the same set \(A_k\) of \(P\).

**Example 2.2.** If \(P = \{(1, 2), (3, 6)\}\) and \(A = \{6\}\), then \(Z_P \subset (\mathbb{P}^1)^6\) consists of points \((p_1, \ldots, p_6)\) such that \(p_1 = p_2 = p_4\) and \(p_3 = p_6\).

**Definition 2.3.** Given a partition \(\lambda = \{\lambda_1, \ldots, \lambda_d\}\) of a positive integer \(n\), we define the \(d\)-dimensional subvariety \(Z_{\lambda} \subset \text{Sym}^n \mathbb{P}^1 \cong \mathbb{P}^n\) to be the image of \(\Delta_P\) under the multiplication map \(\Phi : (\mathbb{P}^1)^n \to \mathbb{P}^n\), where \(P = \{A_1, \ldots, A_d\}\) is any partition of \([n]\) with \(|A_i| = \lambda_i\).

**Remark 2.4.** If we view \(\text{Sym}^n \mathbb{P}^1 \cong \mathbb{P}^n\) as binary degree \(n\) forms on the dual of \(\mathbb{P}^1\), then \(Z_{\lambda}\) is the closure of the degree \(n\) forms with multiplicity sequence given by \(\lambda\), whose equivariant Chow classes were studied by Fehér, Némethi, and Rimányi \[13\].

In order to compactify notation, we make the following definitions.

**Definition 2.5.** Given \(P\) a partition of \([n]\) and \(\lambda\) a partition of \(n\), we let

\[
\Delta_P := [\Delta_P] \in H^*_{G}(\mathbb{P}^1)^n \quad [\lambda] := (\prod_{i=1}^{n}e_i^{\lambda_i})[Z_{\lambda}] \in H^*_{G}(\mathbb{P}^n),
\]

where \(G\) is \(T\), \(GL_2\) or \(PGL_2\), depending on the context and \(e_i^{\lambda_i} = \#\{j \mid \lambda_j = i\}\). For \(\lambda = \{a_1, \ldots, a_d\}\), we will often write \([a_1, \ldots, a_d]\) or \([1^{e_1}, \ldots, n^{e_n}]\) for \([\lambda]\).

**Remark 2.6.** For any such partition \(P\) and \(\lambda\) as in Definition 2.3 then \(\Phi\) maps \(\Delta_P\) onto \(Z_{\lambda}\) with degree \(\prod_{i=1}^{n}e_i^{\lambda_i}\), so \(\Phi_* \Delta_P = [\lambda]\).

**2.5. Affine and projective Thom polynomials.**

**Definition 2.7.** Given a \(T\)-invariant subvariety \(V \subset \mathbb{P}^n\), let \(\tilde{V} \subset A(\text{Sym}^n K^2)\) denote the cone of \(V \subset \mathbb{P}^n\) in \(A(\text{Sym}^n K^2) \cong \mathbb{A}^{n+1}\).

Given a \(T\)-invariant subvariety \(V \subset \mathbb{P}^n\), its class \([V] \in A^*_T(\mathbb{P}^n)\) is a polynomial \(p(H, u, v)\) of degree at most \(n\). The degree 0 term in \(H\), \(p_0(u, v)\), is \([\tilde{V}] \in A^*_T(\mathbb{A}^{n+1}) \cong \mathbb{Z}[u, v]\). This is seen by considering the diagram

\[
A^*_T(\mathbb{P}^n) \xrightarrow{c} A^*_T \times \mathbb{G}_m(\mathbb{A}^{n+1} \setminus \{0\}) \to A^*_T(\mathbb{A}^{n+1} \setminus \{0\})
\]
and noting that $A^*_T(\mathbb{A}^{n+1}\setminus \{0\}) \cong A^*_T(\mathbb{A}^n)$ for $k \leq n$.

It turns out $p_0(u, v)$ determines all of $p$.

**Lemma 2.8** ([12, Theorem 6.1]). We have $p(u, v) = p_0(u + \frac{u}{2}, v + \frac{v}{2})$.

**Proof sketch.** As $(\mathbb{A}^{n+1}\setminus \{0\})/\mathbb{G}_m \cong \mathbb{P}^n$, $p$ can be computed from $[\tilde{V}] \in A^*_{T \times \mathbb{G}_m}(\mathbb{A}^{n+1})$ by mapping to $A^*_{T}(\mathbb{P}^n)$ via

$$A^*_{T \times \mathbb{G}_m}(\mathbb{A}^{n+1}) \to A^*_{T \times \mathbb{G}_m}(\mathbb{A}^{n+1}\setminus \{0\}) \cong A^*_{T}(\mathbb{P}^n).$$

However, the diagonal action of $\mathbb{G}_m$ on $\mathbb{A}^{n+1}$ actually factors through the action of $T$ on $\mathbb{A}^{n+1}$, so $A^*_{T \times \mathbb{G}_m}(\mathbb{A}^{n+1})$ contains no more information than $A^*_T(\mathbb{A}^{n+1})$.

Taking the class $p_0$ and following it from $A^*_{T}(\mathbb{A}^{n+1})$ to $A^*_{T \times \mathbb{G}_m}(\mathbb{A}^{n+1})$ and finally to $A^*_{T}(\mathbb{P}^n)$ yields Lemma 2.8. This argument is written down precisely and in its natural generality in [12, Theorem 6.1].

### 3. $\text{PGL}_2$ and $\text{GL}_2$-equivariant Chow rings

In this section we compare certain $\text{PGL}_2$-equivariant Chow rings to their $\text{GL}_2$-equivariant counterparts, which are easier to work with because $\text{GL}_2$ is special, so restricting to the maximal torus is an injection on equivariant Chow rings [9, Proposition 6].

In particular, we show in Theorem 3.3 that $A^*_{\text{PGL}_2}(\mathbb{P}^1)^n \to A^*_{\text{GL}_2}(\mathbb{P}^1)^n$ is injective and identify its image. For the unordered case, we show in Proposition 3.4 that $A^*_{\text{PGL}_2}(\mathbb{P}^n) \to A^*_{\text{GL}_2}(\mathbb{P}^n)$ is injective for $n$ odd and injective up to 2-torsion when $n$ is even.

To start, we recall a lemma.

**Lemma 3.1** ([25, Lemma 2.1]). Given a linear algebraic group $G$ acting on a smooth variety $X$, let $H$ be a normal subgroup of $G$ that acts freely on $X$ with quotient $X/H$. Then, there is a canonical isomorphism of graded rings

$$A^*_G(X) \cong A^*_G(X/H).$$

**Remark 3.2.** Lemma 3.1 was proven in [25, Lemma 2.1] directly from the definitions, but it can also be seen as a consequence of the fact that the ring $A^*_G(X)$ depends only on the quotient stack $[X/G]$ [9, Proposition 16] and $[[X/H]/(G/H)] \cong [X/G]$ (see [25, Remark 2.4] or [3, Lemma 4.3]).

**Theorem 3.3.** For $n \geq 1$, the ring homomorphism

$$A^*_{\text{PGL}_2}(\mathbb{P}^1)^n \to A^*_{\text{GL}_2}(\mathbb{P}^1)^n$$

induced by the quotient map $\text{GL}_2 \to \text{PGL}_2$ is an injection, and the image is generated by the classes $-(2H_i + u + v)$ and $\Delta_{i,j} = H_i + H_j + u + v$.

**Remark 3.4.** We will show in Proposition 4.3 that $\psi_1 := \pi^*_1 c_1(T^*\mathbb{P}^1) = -(2H_i + u + v)$, as mentioned in Remark 1.6. For $n \geq 3$ this class is redundant as

$$-(2H_i + u + v) = \Delta_{j,k} - \Delta_{i,j} - \Delta_{i,k}.$$

**Proof.** We show the injectivity of $A^*_{\text{PGL}_2}(\mathbb{P}^1)^n \to A^*_{\text{GL}_2}(\mathbb{P}^1)^n$ using the commutativity of the diagram.
$A_{\text{GL}_2}^{\bullet}(\mathbb{P}^1)^n) \xrightarrow{\sim} A_{\text{GL}_2}^{\bullet}((\mathbb{A}^2\setminus\{0\}) \times (\mathbb{P}^1)^{n-1})$

$\downarrow f$

$A_{\text{GL}_2}^{\bullet}(\mathbb{P}^1)^n) \xrightarrow{\sim} A_{\text{GL}_2 \times \mathbb{G}_m}^{\bullet}((\mathbb{A}^2\setminus\{0\}) \times (\mathbb{P}^1)^{n-1})$

with $f$ induced by the multiplication map $\text{GL}_2 \times \mathbb{G}_m \to \text{GL}_2$.

We have the isomorphisms $q_1$ and $q_2$ by Lemma 3.1.

To prove commutativity of the diagram, we can identify each of the rings $A_{\mathcal{O}}^{\bullet}(X)$ with $A_{\mathcal{O}}^{\bullet}([X/G])$ as in Section 2.2 so it suffices to show the following diagram of stacks is commutative.

$\begin{array}{c}
((\mathbb{P}^1)^n/\text{PGL}_2) \\
\downarrow \\
((\mathbb{P}^1)^n/\text{GL}_2)
\end{array}
\xleftarrow{\sim}
\begin{array}{c}
((\mathbb{A}^2\setminus\{0\}) \times (\mathbb{P}^1)^{n-1}/\text{GL}_2) \\
\uparrow \\
((\mathbb{A}^2\setminus\{0\}) \times (\mathbb{P}^1)^{n-1}/\text{GL}_2 \times \mathbb{G}_m)
\end{array}$

Suppose we start with a principal $\text{GL}_2 \times \mathbb{G}_m$-bundle $P \to S$ together with a $\text{GL}_2 \times \mathbb{G}_m$-equivariant map $P \to ([\mathbb{A}^2\setminus\{0\}] \times (\mathbb{P}^1)^{n-1}$, giving a map $S \to ([\mathbb{A}^2\setminus\{0\}] \times (\mathbb{P}^1)^{n-1}/([\mathbb{A}^2\times\mathbb{G}_m])$. Following the diagram around clockwise or counterclockwise, we get a map $S \to [(\mathbb{P}^1)^n/\text{PGL}_2]$ given by a $\text{PGL}_2$-equivariant morphism $P \times^{\text{GL}_2 \times \mathbb{G}_m} \mathbb{G}_m \times^{\text{GL}_2 \times \mathbb{G}_m} \text{PGL}_2 \cong P \times^{\text{GL}_2 \times \mathbb{G}_m} \text{PGL}_2 \to (\mathbb{P}^1)^n$.

When going counterclockwise, the product $P \times^{\text{GL}_2 \times \mathbb{G}_m} \text{GL}_2$ is taken with respect to the multiplication map $\text{GL}_2 \times \mathbb{G}_m \to \text{GL}_2$, while when going clockwise, the product is taken with respect to the projection map $\text{GL}_2 \times \mathbb{G}_m \to \text{GL}_2$. However, the resulting principal $\text{PGL}_2$-bundle is the same as the compositions with the quotient $\text{GL}_2 \to \text{PGL}_2$ are identical.

Now, we will find the induced map $A_{\text{GL}_2}^{\bullet}((\mathbb{A}^2\setminus\{0\}) \times (\mathbb{P}^1)^{n-1}) \to A_{\text{GL}_2}^{\bullet}((\mathbb{P}^1)^n)$ in terms of generators and show it is injective. Consider the diagram

$\begin{array}{c}
A_{\text{GL}_2}^{\bullet}((\mathbb{A}^2\setminus\{0\}) \times (\mathbb{P}^1)^{n-1}) \\
\downarrow \sim \\
A_{\text{GL}_2 \times \mathbb{G}_m}^{\bullet}((\mathbb{A}^2\setminus\{0\}) \times (\mathbb{P}^1)^{n-1})
\end{array}
\xrightarrow{f} 
\begin{array}{c}
A_{\text{GL}_2}^{\bullet}((\mathbb{P}^1)^n) \\
\downarrow \sim \\
A_{\text{GL}_2 \times \mathbb{G}_m}^{\bullet}((\mathbb{A}^2\setminus\{0\}) \times (\mathbb{P}^1)^{n-1})
\end{array}$

where $\text{GL}_2$ acts in the standard way in all cases. In the middle term of the top row, $\mathbb{G}_m$ acts by scaling $\mathbb{A}^2 \setminus \{0\}$. In the last term of the second row, $(\mathbb{G}_m)^n$ acts by having the $i$th copy of $\mathbb{G}_m$ scale the $i$th copy of $\mathbb{A}^2 \setminus \{0\}$. In the middle term of the second row, $(\mathbb{G}_m)^n$ acts by having the first copy of $\mathbb{G}_m$ act by scaling all copies of $\mathbb{A}^2 \setminus \{0\}$ and the $i$th copy of $\mathbb{G}_m$ with $2 \leq i \leq n$ acting by scaling the $i$th copy of $\mathbb{A}^2 \setminus \{0\}$. In the first term of the second row, the $i$th copy of $\mathbb{G}_m^{n-1}$ scales the $i+1$st copy of $\mathbb{A}^2 \setminus \{0\}$.

To compute $f'$, we let $H_1$ be the standard character on the first factor of $\mathbb{G}_m$ in $\text{GL}_2 \times (\mathbb{G}_m)^n$ and let $H_2, \ldots, H_n$ be the standard characters on the remaining
$u - 1$ factors and the $n - 1$ factors of $G_m$ in $GL_2 \times (G_m)^{n-1}$. The induced map $T \times (G_m)^n \to T \times (G_m)^{n-1}$ of tori induces $u \mapsto u + H_1$ and $v \mapsto v + H_1$. Therefore,

$$f' : \frac{Z[u,v]S_z[H_2, \ldots, H_n]}{(uv, F(H_2), \ldots, F(H_n))} \to \frac{Z[u,v]S_z[H_1][H_2, \ldots, H_n]}{(uv, F(H_2 + H_1), \ldots, F(H_n + H_1))},$$

where $u \mapsto u + H_1, v \mapsto v + H_1$, and $H_i \mapsto H_i$.

For $q'_2$, the induced map $T \times (G_m)^n \to T \times (G_m)^{n-1}$ of tori induces $H_1 \mapsto H_1$, $H_i \mapsto H_i - H_1$ for $2 \leq i \leq n$ and $u \mapsto u, v \mapsto v$, and gives the map

$$q'_2 : \frac{Z[u,v]S_z[H_1][H_2, \ldots, H_n]}{(uv, F(H_2 + H_1), \ldots, F(H_n + H_1))} \to \frac{Z[u,v]S_z[H_1, \ldots, H_n]}{(F(H_1), \ldots, F(H_n))}.$$  

The composite

$$q'_2 \circ f' : \frac{Z[u,v]S_z[H_2, \ldots, H_n]}{(uv, F(H_2), \ldots, F(H_n))} \to \frac{Z[u,v]S_z[H_1, \ldots, H_n]}{(F(H_1), \ldots, F(H_n))}$$

is given by $u \mapsto u + H_1, v \mapsto v + H_1, H_i \mapsto H_i - H_1$ for $2 \leq i \leq n$. The image is therefore generated by $2H_1 + u + v$ and $H_i - H_1$ for $2 \leq i \leq n$. If $n \geq 3$, then this is generated by the collection

$$\{ H_1 + H_j + u + v \mid 1 \leq i < j \leq n \} = \{ \Delta_{i,j} \mid 1 \leq i < j \leq n \}$$

(see Proposition 3.3).

\[ \square \]

**Remark 3.5.** Suppose our base field is $\mathbb{C}$. We have a commutative diagram

$$\begin{array}{ccc}
A^*_PGL_2((\mathbb{P}^1)^n) & \xrightarrow{q_1} & A^*_GL_2((\mathbb{P}^1)^n) \\
\downarrow & & \downarrow \\
H^*_PGL_2((\mathbb{P}^1)^n) & \xrightarrow{q_1^H} & H^*_GL_2((\mathbb{P}^1)^n)
\end{array}$$

The map $A^*_GL_2((\mathbb{P}^1)^n) \to H^*_GL_2((\mathbb{P}^1)^n)$ is an isomorphism by the Leray-Hirsch theorem applied to $\mathbb{P}^1$-bundles. Running the proof of Theorem 3.3 for the map $q_1^H$ shows $q_1^H$ is injective. Here we replace the projective bundle theorem in algebraic geometry by the Leray-Hirsch theorem applied to $\mathbb{P}^1$-bundles and the application of Lemma 3.1 with the fact that if $G$ acts on $X$ and $H$ is a normal subgroup which acts freely, then $(X \times EG)/G \cong (X \times EG)/(G/H)$, and $(X \times EG)/H$ is homotopy equivalent to $X/H$ and has a free action by $G/H$.

This implies $A^*_PGL_2((\mathbb{P}^1)^n) \to H^*_PGL_2((\mathbb{P}^1)^n)$ is an isomorphism.

By [27] Theorem 1, the injection $SO(3) \to GL_3$ induces a surjection $A^*_SO(3)(pt) \to A^*_{SO(3)}(pt)$ expressing $A^*_{SO(3)}(pt) \cong \mathbb{Z}[c_1, c_2, c_3]/(c_1^2, 2c_3).$ where $c_1, c_2, c_3$ are the generators of $A^*_{SO(3)}(pt)$. Lemma 3.6 expresses the map $A^*_PGL_2(pt) \to A^*_GL_2(pt)$ in terms of this presentation.

**Lemma 3.6.** Under the composition,

$$A^*_{GL_3}(pt) \to A^*_{SO(3)}(pt) \cong A^*_PGL_2(pt) \to A^*_GL_2(pt) \to A^*_T(pt),$$

we have $c_1 \mapsto 0, c_2 \mapsto -(u-v)^2, c_3 \mapsto 0$.

**Proof.** Lemma 3.6 amounts to finding the map

$$T \to GL_2 \to PGL_2 \cong SO(3) \to GL_3$$

inducing the maps of rings.
To describe the isomorphism $SO(3) \cong \text{PGL}_2$, recall that $GL_2$ acts on the space $K^{2 \times 2}$ of 2 by 2 matrices by conjugation. There is a pairing $(\cdot, \cdot)$ on $K^{2 \times 2}$ given by $\langle A, B \rangle = \text{Tr}(AB)$ that restricts to a nondegenerate form on the three-dimensional vector space of trace zero matrices $V \subset K^{2 \times 2}$. Since the action of $GL_2$ preserves $(\cdot, \cdot)$ and the scalar matrices inside $GL_2$ act trivially, we have an injection $\text{PGL}_2 \to SO(3)$, which is an isomorphism for dimension reasons.

Under this isomorphism, $(u \ 0 \ 0 \ v) \in T$ maps into diagonal matrices in $GL_3$ and acts on $V$ with characters $u - v$, $v - u$ and 0 (written additively). Therefore,

$$
\begin{align*}
&c_1 \mapsto (u - v) + (v - u) = 0 \\
&c_2 \mapsto (u - v)(v - u) = -(u - v)^2 \\
&c_3 \mapsto 0(u - v)(v - u) = 0.
\end{align*}
$$

\[ \square \]

**Proposition 3.7.** We have

$$
A^\bullet_{\text{PGL}_2}(P^n) \cong \begin{cases}
\mathbb{Z}[u, v]^{S_2}/(\prod_{i=0}^4 ((\frac{n+1}{2} - i)u + (\frac{n+1}{2} + i)v)) & \text{if } n \text{ is odd} \\
\mathbb{Z}[c_2, c_3, H]/(2c_3, p_n(H)) & \text{if } n \text{ is even}
\end{cases}
$$

where $p_n(t) \in A^\bullet_{\text{PGL}_2}(pt)[t]$ is defined as

$$
p_n(t) = \begin{cases}
t \prod_{k=1}^2 (t^2 + k^2c_2) + t \prod_{k=1}^2 \sum_{i=1}^{\frac{n+2}{2}} \left(\frac{n}{k}\right)(t^3 + c_2t)^{\frac{n-2}{2}} \prod_{j=1}^{\frac{n-2}{2}} c_j^{k} & n \equiv 0 \pmod{4} \\
t \prod_{k=1}^2 (t^2 + k^2c_2) + t \prod_{k=1}^2 \sum_{i=1}^{\frac{n+2}{2}} \left(\frac{n}{k}\right)(t^3 + c_2t)^{\frac{n-2}{2}} \prod_{j=1}^{\frac{n-2}{2}} c_j^{k} & n \equiv 2 \pmod{4}.
\end{cases}
$$

The map

$$
A_{\text{PGL}_2}^\bullet(P^n) \to A_{\text{GL}_2}^\bullet(P^n)
$$

induced by $GL_2 \to \text{PGL}_2$ is given by

$$
u \mapsto H + \frac{n+1}{2}u + \frac{n-1}{2}v \quad v \mapsto H + \frac{n-1}{2}u + \frac{n+1}{2}v \quad \text{if } n \text{ is odd, and}
$$

$$
c_2 \mapsto -(u - v)^2 \quad c_3 \mapsto 0 \quad H \mapsto H + \frac{n}{2}(u + v) \quad \text{if } n \text{ is even}.
$$

Finally, $A_{\text{PGL}_2}^\bullet(P^n) \to A_{\text{GL}_2}^\bullet(P^n)$ is injective for $n$ odd and injective up to 2-torsion when $n$ is even.

**Proof.** The injectivity statements immediately follow from the explicit descriptions of all of the rings maps in the statement of Proposition 3.7. We omit the verification.

We do the cases $n$ is odd and even separately. First suppose $n$ is odd. Consider the commutative diagram

$$
\begin{array}{ccc}
A^\bullet_{\text{PGL}_2}(P^n) & \longrightarrow & A^\bullet_{\text{GL}_2}(P^n) \\
\downarrow \sim & & \downarrow \sim \\
A^\bullet_{\text{GL}_2}(A^{n+1} \setminus \{0\}) & \xrightarrow{\sim \phi_1} & A^\bullet_{\text{GL}_2/\mu_n}(A^{n+1} \setminus \{0\}) \xrightarrow{\phi_2} A^\bullet_{\text{GL}_2 \times \mathbb{G}_m}(A^{n+1} \setminus \{0\})
\end{array}
$$

The map $\phi_1$ is induced by the isomorphism $GL_2/\mu_n \to GL_2$ given by $A \mapsto (\det A)\mathbb{A}$. To determine $A^\bullet_{\text{GL}_2}(A^{n+1} \setminus \{0\})$ it suffices to check how the maximal torus $T \subset GL_2$ acts on $A^{n+1}$. Since the inverse of $GL_2 \to GL_2/\mu_n$
is given by $A \mapsto (\det A)^{1/2} A \left( \begin{array}{c} \lambda_1 \\ \lambda_2 \end{array} \right)$ maps to \( \left( \lambda_1^{\frac{n+1}{2}} \lambda_2^{\frac{n-1}{2}} \right) \) in $GL_2/\mu_n$ and acts on $\mathbb{A}^{n+1}$ with characters \( \{ (\frac{n+1}{2} - i)u + (\frac{n-1}{2} + i)v \mid 0 \leq i \leq n \} \).

This shows

\[
A^*_GL_2(\mathbb{P}^n) = \mathbb{Z}[u,v]^S/\prod_{i=0}^n((\frac{n+1}{2} - i)u + (\frac{n-1}{2} + i)v)
\]

in this case.

To find the map $A^*_GL_2(\mathbb{A}^{n+1} \setminus \{0\}) \to A^*_GL_2 \times G_m(\mathbb{A}^{n+1} \setminus \{0\})$, we consider the map $GL_2 \times G_m \to GL_2$ and find it maps the pair $\left( \lambda_1 \frac{1}{\lambda_2} \right)$, $\lambda$ to $\lambda^\pm \left( \lambda_1 \frac{1}{\lambda_2} \right)$ in $GL_2/\mu_n$ and $\left( \lambda_1^{\frac{n+1}{2}} \lambda_2^{\frac{n-1}{2}} \right)$ in $GL_2$. This shows the map

\[
\mathbb{Z}[u,v]^S/\prod_{i=0}^n((\frac{n+1}{2} - i)u + (\frac{n-1}{2} + i)v) \to \mathbb{Z}[u,v]^S[H]/\prod_{i=0}^n(H + iu + (n-i)v)
\]

giving $A^*_GL_2(\mathbb{A}^{n+1} \setminus \{0\}) \to A^*_GL_2 \times G_m(\mathbb{A}^{n+1} \setminus \{0\})$ is given by

\[
u \mapsto H + \frac{n+1}{2}u + \frac{n-1}{2}v, \quad v \mapsto H + \frac{n-1}{2}u + \frac{n+1}{2}v.
\]

Now, we do the case $n$ is even. Let $V \cong \mathbb{K}^2$ be a 2-dimensional vector space with the standard representation of $GL_2$. Let $D \cong \mathbb{K}$ be a 1-dimensional vector space where $GL_2$ acts by multiplication by the determinant. Then, $(\text{Sym}^n V) \otimes (D^\vee)^{\otimes n}$ is a $GL_2$ representation that descends to a $PGL_2$ representation.

To determine

\[
A^*_PGL_2(\mathbb{P}^n) \cong A^*_PGL_2(\mathbb{P}((\text{Sym}^n V) \otimes (D^\vee)^{\otimes n/2}))
\]

it suffices to find the chern classes of the $PGL_2$ representation $(\text{Sym}^n V) \otimes (D^\vee)^{\otimes n/2}$ regarded as a $PGL_2$-equivariant vector bundle over a point. These chern classes are given in [13 Corollary 6.3]. The reader should also note that [13] contains mistakes elsewhere in the document (see [17 Introduction]). As a result, we have $A^*_PGL_2(\mathbb{P}^n)$ is $\mathbb{Z}[c_2, c_3, H]/(2c_2, p_n(H))$, where $p_n(t) \in A_{PGL_2}(\text{pt})[t]$ is given as in the statement of the proposition.

Therefore, we have

\[
A^*_PGL_2(\mathbb{P}((\text{Sym}^n V) \otimes (D^\vee)^{\otimes n/2})) \to A^*_GL_2(\mathbb{P}((\text{Sym}^n V) \otimes (D^\vee)^{\otimes n/2}))
\]

given by $c_2 \mapsto -(u-v)^2$ and $c_3 \mapsto 0$ by Lemma [10]. Also, the $\mathcal{O}_{\mathbb{P}((\text{Sym}^n V) \otimes (D^\vee)^{\otimes n/2})}(1)$ class in $A^*_PGL_2(\mathbb{P}((\text{Sym}^n V) \otimes (D^\vee)^{\otimes n/2}))$ maps to the $\mathcal{O}_{\mathbb{P}((\text{Sym}^n V) \otimes (D^\vee)^{\otimes n/2})}(1)$ class in $A^*_GL_2(\mathbb{P}((\text{Sym}^n V) \otimes (D^\vee)^{\otimes n/2}))$ by the projective bundle formula.

Finally, since $(\text{Sym}^n V) \otimes (D^\vee)^{\otimes n/2}$ is a twist of $\text{Sym}^n V$ by a $GL_2$-equivariant line bundle, the $\mathcal{O}_{\mathbb{P}((\text{Sym}^n V) \otimes (D^\vee)^{\otimes n/2})}(1)$ class in $A^*_GL_2(\mathbb{P}((\text{Sym}^n V) \otimes (D^\vee)^{\otimes n/2}))$ maps to $\mathcal{O}_{\mathbb{P}(\text{Sym}^n V)}(1) + c_1GL_2(D^\otimes \frac{n}{2})$ in $A^*_GL_2(\mathbb{P}(\text{Sym}^n V)^{\otimes n/2})$. Since $c_1GL_2(D^{\otimes n/2}) = \frac{n}{2}(u+v)$, we find the composite map

\[
A^*_PGL_2(\mathbb{P}((\text{Sym}^n V) \otimes (D^\vee)^{\otimes n/2})) \to A^*_GL_2(\mathbb{P}(\text{Sym}^n V))
\]

given by

\[
c_2 \mapsto -(u-v)^2, \quad c_3 \mapsto 0, \quad H \mapsto H + \frac{n}{2}(u+v).
\]
4. Formulas and initial reductions

In this section we express the $\Delta_P$ and $[\lambda]$ classes in terms of our equivariant Chow ring presentations.

After this, we compute formulas for $\Delta_P \in A^*_T((\mathbb{P}^1)^n)$ and give a quick, alternative computation of the classes $[Z_\lambda] \in A^*_T(\mathbb{P}^n)$ given in \cite[Theorem 3.4]{13}. The simple presentation for the class of the diagonal in $(\mathbb{P}^1)^n$ works especially well with the formula for the pushforward $\Phi_*: A^*_T((\mathbb{P}^1)^n) \to A^*_T(\mathbb{P}^n)$ via the classes of torus fixed points, and appears not to have been previously exploited in this fashion.

4.1. Class of the diagonal in $(\mathbb{P}^1)^n$. We now compute the $T$-equivariant class of the diagonal $\Delta_{\{[n]\}} \subset (\mathbb{P}^1)^n$. This formula would also follow from localization to the torus fixed points, but the derivation below is simpler.

**Proposition 4.1.** The class of $\Delta_{\{[n]\}}$ in $A^*_T((\mathbb{P}^1)^n)$ is given by

$$\Delta_{\{[n]\}} = \frac{1}{u - v} \left( \prod_{i=1}^{n} (H_i + u) - \prod_{i=1}^{n} (H_i + v) \right).$$

**Proof.** This is a result of the fact that $\Delta_{\{[n]\}}$ intersected with $\{[0 : 1]\} \times (\mathbb{P}^1)^{n-1}$ and $\{[1 : 0]\} \times (\mathbb{P}^1)^{n-1}$ are the torus-fixed points $[0 : 1]^n$ and $[1 : 0]^n$ respectively, so

$$(H_1 + u) - (H_1 + v))\Delta_{\{[n]\}} = \prod_{i=1}^{n} (H_i + u) - \prod_{i=1}^{n} (H_i + v).$$

\qed

4.2. Formula for $\Delta_P$. When two strata $\Delta_P$ and $\Delta_P'$ intersect transversely in $(\mathbb{P}^1)^n$, it is easy to describe their intersection as another stratum.

**Proposition 4.2.** The class $\Delta_P \in A^{n-d}_{PGL_2}((\mathbb{P}^1)^n)$ for $P$ a partition of $[n]$ into $d$ parts is given by the product $\prod_{\{i,j\} \in \text{Edge}(T)} \Delta_{i,j}$, where $T$ is any forest with vertex set $[n]$ consisting of one spanning tree for each part of $P$. In particular,

1. If $i, j$ are in distinct parts of $P$, then if $P_{ij}$ is the partition merging the parts containing $i$ and $j$, we have $\Delta_{i,j} \Delta_P = \Delta_{P_{ij}}$.

2. If $i, j, i', j'$ are in the same part of $P$, we have $\Delta_{i,j} \Delta_P = \Delta_{i',j'} \Delta_P$.

**Proof.** Item (1) follows from the transversality of the intersection $\Delta_{i,j} \cap \Delta_P$, from which we can deduce the first part of the proposition, and item (2) then follows from the first part and repeated applications of the diagonal relation $\Delta_{i,j} \Delta_{i',k} = \Delta_{i,j} \Delta_{i',k}$.

\qed

**Proposition 4.3.** Let $P = \{V_1, \ldots, V_d\}$ be a partition of $[n]$, then

$$\Delta_P = \frac{1}{(u - v)^d} \prod_{i=1}^{d} \left( \prod_{j \in V_i} (H_j + u) - \prod_{j \in V_i} (H_j + v) \right).$$

**Proof.** From Proposition 4.2 $\Delta_P = \prod_{i=1}^{d} \Delta_{\{V_i\}}$. Now apply Proposition 4.1. \qed
4.3. The $\psi_i$ and $\Delta_{i,j}$ classes. At this point, we can prove the formula for $\Delta_{i,j}$ in item (3) of Theorem 1.1 and for $\psi_i$ as mentioned in Remark 1.6.

**Proposition 4.4.** We have

$$\Delta_{i,j} = H_i + H_j + u + v$$

$$\psi_i = -(2H_i + u + v).$$

**Proof.** The formula for $\Delta_{i,j}$ is an immediate consequence of Proposition 4.3. To compute $\psi_i$, it suffices to show that $c_1(T^i) \in A^*_{GL_2}(\mathbb{P}^1)$ is $2H + u + v$, where $H = c_1(O(1)).$ We note that $c_{top}(T_X)$ for any smooth $X$ is the pullback of the diagonal under the diagonal map $X \to X \times X$. The pullback $A^*_{GL_2}(\mathbb{P}^1)^2 \to A^*_{GL_2}(\mathbb{P}^1)$ under the inclusion $\mathbb{P}^1 \cong \Delta_{1,2} \mapsto \mathbb{P}^1 \times \mathbb{P}^1$ is given by $H_1, H_2 \mapsto H$. Under this map, $\Delta_{1,2}$ pulls back to $2H + u + v$ as desired. □

4.4. Pullback and Pushforward under $\Phi$. The pullback map $\Phi^* : A^*(\mathbb{P}^n) \to A^*(\mathbb{P}^1)^n)$ is induced by

$$\Phi^*(H) = \sum_{i=1}^{n} H_i.$$ 

We now consider $\Phi_*$. By considering the classes of the torus-fixed loci, we have for any $A \subset [n]$,

$$\Phi_*(\prod_A (H_i + u) \prod_{[n]\setminus A} (H_j + v)) = \prod_{k \in [n]\setminus \{|A|\}} (H + kv + (n - k)u).$$

This in fact uniquely characterizes $\Phi_*$, which can be seen either from localization [10, Theorem 2] or because

$$\frac{\prod_A (H_i + u) \prod_{[n]\setminus A} (H_j + v)}{\prod_A (-v + u) \prod_{[n]\setminus A} (-u + v)}$$

is a Lagrange interpolation basis for polynomials in $H_1, \ldots, H_n$ modulo $F(H_i)$ for each $i$.

4.5. Formula for $[\lambda]$. Fehér, Némethi, and Rimányi computed the class of $[\lambda]$ for $\lambda$ a partition of $n$ [13, Theorem 3.4]. We can give a quick self-contained computation from Section 4.4 and Proposition 4.1 as follows.

**Theorem 4.5** ([13, Theorem 3.4]). The class $[a_1, \ldots, a_d]$ is the result of first expanding the polynomial

$$\prod_{i=1}^{d} (z^{a_i} - 1) = \sum_{k \geq 0} c_k z^k \quad (c_k \in \mathbb{Z}),$$

and then replacing each monomial

$$z^k \mapsto \frac{1}{(u - v)^d} \prod_{j \in [n]\setminus \{k\}} (H + jv + (n - j)u).$$
Proof. Let $P = \{V_1, \ldots, V_d\}$ be a partition of $[n]$ with $|V_i| = a_i$. We expand the formula from Proposition 4.3

$$\Delta_p = \frac{1}{(u-v)^d} \prod_{i=1}^d \left( \prod_{j \in V_i} (H_i + u) - \prod_{j \in V_i} (H_i + v) \right)$$

to a sum of terms of the form $\prod_{i \in A} (H_i + u) \prod_{j \in [n]\setminus A} (H_j + v)$. Then, Section 4.3 implies that each such term pushes forward to $\prod_{j \in [n] \setminus \{i\}} (H + jv + (n-j)u)$. The result follows immediately.

5. STRATA IN $\text{(P}^1\text{)}^n/PGL_2$

In this section we prove all of our results on ordered point configurations in $\text{P}^1$. Up to Section 5.1, the only result that we use is Theorem 3.3, and in particular the identification of $\Delta_{i,j}$ in $A^*_{PGL_2}(\text{(P}^1\text{)})^n$ as $H_i + H_j + u + v$.

**Remark 5.1.** Whenever we write $\Delta_{i,j}$ in any context, we will always treat $\{i,j\}$ as an unordered tuple, so that implicitly

$$\Delta_{i,j} := \Delta_{j,i}$$

for $i > j$.

Recall from Theorem 3.3 and Section 2.3 we have the inclusions

$$A^*_{PGL_2}(\text{(P}^1\text{)})^n \subset A^*_{GL_2}(\text{(P}^1\text{)})^n \subset A^*_{\text{PGL}_2}(\text{(P}^1\text{)})^n.$$

We first consider the square relation in $(\text{P}^1)^4$.

**Proposition 5.2.** In $A^*_{PGL_2}(\text{(P}^1\text{)})^4$, we have the square relation

$$\Delta_{1,2} + \Delta_{3,4} = \Delta_{2,3} + \Delta_{4,1}.$$

**Proof.** Both sides are equal to $H_1 + H_2 + H_3 + H_4 + 2(u + v)$ by Proposition 4.4. This can also be shown using the fact that the diagonal $\Delta \subset \text{P}^1 \times \text{P}^1$ has a torus-equivariant deformation to $\{0\} \times \text{P}^1 \cup \text{P}^1 \times \{\infty\}$. It also holds by Section 1.1.1. □

**Definition 5.3.** Let $R(n)$ be the ring

$$R(n) = \mathbb{Z}[\{\Delta_{i,j} \mid 1 \leq i < j \leq n\}]/\text{relations},$$

generated by the symbols $\Delta_{i,j}$ together with the relations

1. $\Delta_{i,j} + \Delta_{k,l} = \Delta_{i,k} + \Delta_{j,l}$ for distinct $i,j,k,l$ (square relations)
2. $\Delta_{i,j} \Delta_{i,k} = \Delta_{i,j} \Delta_{j,k}$ for distinct $i,j,k$. (diagonal relations)

given in Theorem 1.1(1). If $n$ is clear from context or irrelevant, we will let $R := R(n)$. If we let each $\Delta_{i,j}$ have degree 1, then the ideal of relations is homogeneous, so $R$ is a graded ring, and we will denote by $R_k$ the $k$th graded part of $R$.

By Theorem 3.3 we can identify $A^*_{PGL_2}(\text{(P}^1\text{)})^n$ as a subring of $A^*_{GL_2}(\text{(P}^1\text{)})^n$, where the image

$$A^*_{PGL_2}(\text{(P}^1\text{)})^n \rightarrow A^*_{GL_2}(\text{(P}^1\text{)})^n = \mathbb{Z}[u,v][H_1, \ldots, H_n]/(F(H_1), \ldots, F(H_n))$$

is generated by $\Delta_{i,j} = H_i + H_j + u + v$ for $n \geq 3$. If $n \leq 2$, we also have to add the classes $\psi_i = -(2H_i + u + v)$ (see Proposition 4.4). Therefore for $n \geq 3$ by Proposition 5.2 we have a surjective map

$$R \twoheadrightarrow A^*_{PGL_2}(\text{(P}^1\text{)})^n.$$
Lemma 5.4. For every \( n \geq 1 \), the free abelian group \( A^k_{PGL_2}(\mathbb{P}^1)^n \) has rank
\[
\sum_{i \leq k \mod 2} \binom{n}{i}.
\]

Proof. We compute the rank of \( A^k_{PGL_2}(\mathbb{P}^1)^n \) by working instead with the rational subring
\[
A^k_{PGL_2}(\mathbb{P}^1)^n \otimes \mathbb{Q} \subset A^k_{GL_2}(\mathbb{P}^1)^n) \otimes \mathbb{Q},
\]
which is generated by the elements \( H'_i := H_i + \frac{1}{2}(u + v) \) by Theorem 5.3. Noting that \( H'^2 = \frac{1}{4}(u - v)^2 \), we see the \( \mathbb{Q} \)-vector space \( A^k_{PGL_2}(\mathbb{P}^1)^n) \otimes \mathbb{Q} \) is spanned by the elements
\[
B = \left\{ \left( \frac{u - v}{2} \right)^{n-d-|B|} \prod_{i \in B} H'_i \mid B \subset [n], |B| \leq n - d, |B| \equiv n - d \pmod{2} \right\},
\]
which has size
\[
|B| = \sum_{i \leq k \mod 2} \binom{n}{i}.
\]
To finish, it suffices to show that the elements of \( B \) are linearly independent. Indeed, the elements of \( B \) become distinct monomials in the \( H'_i \) after setting \( u = 1 \) and \( v = -1 \) (after which the defining relations \( F(H_i) = 0 \) become \( H'^2 = 1 \) for each \( i \)).

Definition 5.5. Let \( \text{Part}(d, n) \) denote the set of partitions of \( [n] \) into \( d \) parts. For \( P \in \text{Part}(d, n) \), for any forest \( T \) with vertex set \( [n] \) consisting of one spanning tree for each part of \( P \), we define
\[
\Delta_P = \prod_{(i, j) \in \text{Edge}(T)} \Delta_{i, j} \in R
\]
Note that by the diagonal relations this is independent of the choice of \( T \), and \( \Delta_P \mapsto \Delta_P \) under the map \( R \to A^k_{PGL_2}(\mathbb{P}^1)^n \) by Proposition 1.2.

Remark 5.6. The two items (1), (2) in Proposition 1.2 are also true for the elements \( \Delta_P \in R \) as the proof only uses the diagonal relations in \( A^k_{PGL_2}(\mathbb{P}^1)^n \).

Lemma 5.7. For \( k \leq n - 2 \), \( R_k \) is generated by \( \{ \Delta_P \mid P \in \text{Part}(n - k, n) \} \).

Proof. Given a product \( \prod_{i=1}^k \Delta_{i, j_i} \), we will produce an algorithm for rewriting this product in terms of \( \Delta_P \) with \( P \) a partition of \( [n] \) into \( n - k \) parts.

By induction, we can write \( \prod_{i=1}^k \Delta_{i, j_i} \) as \( \sum_{P' \in \text{Part}(n-k+1, n)} a_{P'} \Delta_{P'} \), so it suffices to show that \( \Delta_{i, j_i} \Delta_{P'} \) for \( P' \in \text{Part}(n-k+1, n) \) can be written as \( \mathbb{Z} \)-linear combination \( \sum_{P \in \text{Part}(n-k, n)} a_{P} \Delta_{P} \).

If \( i_k, j_k \) are in different parts of \( P' \), then \( \Delta_{i_k, j_k} \Delta_{P'} = \Delta_P \) where \( P \) merges the parts containing \( i_k \) and \( j_k \), and we are done. Otherwise, if they are in the same part
A1, let A2, A3 be two parts of P′ distinct from A1 (which exist as n − k + 1 ≥ 3), with elements x2 ∈ A2 and x3 ∈ A3. By applying a square relation, we have
\[ \Delta_{i_k,j_k} \Delta_P = (\Delta_{i_k,x_2} - \Delta_{x_2,x_3} + \Delta_{x_3,j_k}) \Delta_P, \]
and each of the three terms on the right is some \( \Delta_P \) with \( P \in \text{Part}(n-k,n) \).

**Definition 5.8.** Given a partition P of [n] and i, j ∈ [n] in distinct parts of P, let \( P_{i,j} \) be the partition of [n] obtained by merging the parts in P containing i and j.

From Remark 5.6 the following relations hold in R(n) (and hence also in \( A^\bullet_{pG\mathfrak{L}_2}((\mathbb{P}^1)^n) \)).

**Definition 5.9.** For \( i_1, i_2, i_3, i_4 \) in distinct parts of a partition P of [n], define the square relation for P associated to \( i_1, i_2, i_3, i_4 \) to be the relation
\[ \Delta_{P_{i_1,i_2}} - \Delta_{P_{i_2,i_3}} + \Delta_{P_{i_3,i_4}} - \Delta_{P_{i_4,i_1}} = 0. \]

**Definition 5.10.** Inside the free abelian group \( \mathbb{Z}^{\text{Part}(d,n)} \), denote by Sq(d, n) the subgroup generated by formal square relations \( P_{i,j} - P_{j,k} + P_{k,l} - P_{l,i} \) for \( P \in \text{Part}(d+1,n) \) and \( i, j, k, l \) distinct. Then we define
\[ A(d, n) := \mathbb{Z}^{\text{Part}(d,n)} / \text{Sq}(d, n). \]

Lemma 5.7 shows for \( d \geq 2 \) we have a surjection
\[ A(d, n) \rightarrow R_{d-1} \]
that sends \( P \mapsto \Delta_P \). We will in fact show this is an isomorphism.

**Definition 5.11.** Say a partition \( P \in \text{Part}(d,n) \) for \( d \geq 2 \) is good if P can be written as \( P = \{A_1, \ldots, A_n\} \) with \( A_1 \sqcup A_2 \) a partition of an initial segment of [n], and \( A_3, \ldots, A_n \) all contiguous intervals. Denote
\[ \text{Good}(d, n) := \{P \in \text{Part}(d,n) \mid P \text{ good}\}. \]

**Lemma 5.12.** For \( d \leq n - 2 \), \( A(d, n) \) is generated by the set of \( P \in \text{Good}(d,n) \).

**Proof.** We use induction on \( n \) and d. For \( d = 2 \) every partition is good, and for \( n = 2 \) the result is trivial. Suppose now we have \( n, d > 2 \). Take \( Q \in \text{Part}(d,n) \).

If \( n-1 \) and \( n \) are in the same part, then \( Q' := Q \setminus \{n\} \in \text{Part}(d,n-1) \), and by the induction hypothesis applied to \( A(d,n-1) \) we can write \( Q' = \sum_{P' \in \text{Good}(d,n-1)} q_{P'} P' \).

There is a map
\[ A(d,n-1) \rightarrow A(d,n) \]
mapping each \( P' \) for \( P' \in \text{Part}(d,n-1) \) to \( P \), where \( P \) is obtained by adding \( n \) to the same part as \( n-1 \) in \( P' \). Furthermore, under this map \( P \in \text{Good}(d,n) \) if \( P' \in \text{Good}(d,n-1) \), so we get \( Q \) as a \( \mathbb{Z} \)-linear combination of \( P \) for \( P \in \text{Good}(d,n) \).

If \( n \) is isolated in \( Q \), then let \( Q' = Q \setminus \{n\} \in \text{Part}(d-1,n-1) \). By the induction hypothesis applied to \( A(d-1,n-1) \), we can write \( Q = \sum_{P' \in \text{Good}(d-1,n-1)} q_{P'} P' \).

There is a map
\[ A(d-1,n-1) \rightarrow A(d,n) \]
mapping each \( P' \) for \( P' \in \text{Part}(d-1,n-1) \) to \( P \), where \( P \) is obtained by adding \( n \) as an isolated part. Furthermore, under this map \( P \in \text{Good}(d,n) \) if \( P' \in \text{Good}(d-1,n-1) \), so we get \( Q \) as a \( \mathbb{Z} \)-linear combination of \( P \) for \( P \in \text{Good}(d,n) \).

If neither of the above two cases hold, then \( n-1 \) and \( n \) are not in the same part and \( n \) is not isolated in \( Q \). Let \( x \in [n] \) be another element in the same part as \( n \), and let \( y \in [n] \) be in a different part as \( n-1 \) and \( n \) (which exists as \( d > 2 \)). Then
If we let $\bar{Q} \in \text{Part}(d + 1, n)$ be the result of taking $Q$ and isolating $n$ into its own part, the square relation for $\bar{Q}$ associated to $n - 1, n, x, y$ yields $Q$ as a combination of 3 terms, each of which either has $n$ isolated or $n - 1, n$ in the same group. □

**Lemma 5.13.** For $d \geq 2$,

$$\# \text{Good}(d, n) = \sum_{i=\lfloor n/d \rfloor}^{n-d} \binom{n}{i}.$$  

**Proof.** From the definition of Good($d, n$),

$$\# \text{Good}(d, n) = \sum_{k=1}^{n-d+2} (2^{k-1} - 1) \binom{n-k-1}{n-k-d+2}.$$  

Let

$$G_{d,n} = \sum_{k=1}^{n-d+2} (2^{k-1} - 1) \binom{n-k-1}{n-k-d+2},$$

$$G'_{d,n} = \sum_{i=\lfloor n/d \rfloor}^{n-d} \binom{n}{i}.$$  

We will show $G_{d,n} = G'_{d,n}$ for all $n \geq 2$ and $d \geq 2$ by induction on $n$. For the base case if $n = 2$ and $d \geq 2$ arbitrary, we have two cases: if $d = 2$, $|G(2, 2)| = G_{2,2} = 1$ and if $d > 2$, $|G(d, 2)| = G_{d,2} = 0$. If $d = 2$ and $n \geq 2$ arbitrary, then $G'_{d,n} = 2^{n-1} - 1$ by the binomial theorem, and $G_{d,n} = 2^{n-1} - 1$ because only the $k = n$ term $(2^{n-1} - 1)\binom{-1}{0}$ contributes.

Now, assume we know $G_{d,n} = G'_{d,n}$ for some $n$ and all $d \geq 2$. For the induction step,

$$G_{d,n} + G_{d+1,n} = \sum_{k=1}^{n-d+2} (2^{k-1} - 1) \left( \binom{n-k-1}{n-k-d+2} + \binom{n-k-1}{n-k-d+1} \right)$$

$$= \sum_{k=1}^{n-d+2} (2^{k-1} - 1) \binom{n-k}{n-k-d+2} = G'_{d+1,n+1},$$

and similarly applying Pascal’s identity, $G'_{d,n} + G'_{d+1,n} = G'_{d+1,n+1}$. □

**Corollary 5.14.** For $d \geq 2$, and $n \geq 3$ we have the isomorphisms

$$Z^{\text{Good}(d, n)} \to \mathcal{A}(d, n) \to R_{n-d} \to A^{n-d}_{PGL_2}((\mathbb{P}^1)^d).$$  

**Proof.** By Lemmas 5.7 and 5.12 and (1), we have

$$Z^{\text{Good}(d, n)} \to \mathcal{A}(d, n) \to R_{n-d} \to A^{n-d}_{PGL_2}((\mathbb{P}^1)^d).$$

Since $A^{n-d}_{PGL_2}((\mathbb{P}^1)^d)$ is a finitely generated, free $Z$-module of rank equal to the rank of $Z^{\text{Good}(d, n)}$ by Lemmas 5.3 and 5.13 the composite $Z^{\text{Good}(d, n)} \to A^{n-d}_{PGL_2}((\mathbb{P}^1)^d)$ is an isomorphism. □

We now find an explicit basis for $R_k$ for $k > n - 2$ of size $2^{n-1}$.
Lemma 5.15. For each partition $P \in \text{Part}(d, n)$ for $d \leq 2$, arbitrarily choose $i_P, j_P$ that lie in the same part. Then for $k > n - 2$, $R_k$ is generated by the $2^{n-1}$ elements

$$S_k := \{ \Delta([n]) \Delta_{i,j}^{k-n+1} \} \cup \{ \Delta_P \Delta_{i_P, j_P}^{k-n+2} \mid P \in \text{Part}(2, n) \}.$$ 

Proof. Let $P = \{ A, B \} \in \text{Part}(2, n)$. By Lemma 5.7 it suffices to show $\Delta_P \prod_{\alpha=1}^{k-n+2} \Delta_{i_\alpha, j_\alpha}$ is generated by $S_k$ for any choices of $i_\alpha \neq j_\alpha$. We proceed by induction on $k > n - 2$. For the base case $k = n - 1$, it suffices to show $\Delta_{i,j} \Delta_P$ is generated by $S_k$ for any $i \neq j$. If $k > n - 1$, then by the induction hypothesis, it suffices to show $\Delta_{i,j} \Delta_P \Delta_{i_P, j_P}$ and $\Delta_{i,j} \Delta([n]) \Delta_{i,j}$ are generated by $S_k$. Both the base case and the induction step will work in the same way.

First, $\Delta_{i,j} \Delta([n]) \Delta_{i,j}$ is projection by forgetting the last factor, then by definition

$$\Delta_{i,j} \Delta_P = \Delta([n]) \Delta_{i,j} \Delta_{i,j}$$

by Remark 5.6 (2). To deal with $\Delta_{i,j} \Delta_P \Delta_{i_P, j_P}$, we have two cases.

1. If $\{ i, j \}$ is not contained in $A$ or $B$, then $\Delta_{i,j} \Delta_P$ is the diagonal $\Delta([n])$ by Remark 5.6 (1). Then, by Remark 5.6 (2), $\Delta_{i,j} \Delta_P \Delta_{k-n+1} = \Delta([n]) \Delta_{i,j} \Delta([n])$.

2. Suppose now each $\{ i, j \}$ is in $A$ or $B$, and that without loss of generality, $i, j \in A$. If $i, j \in A$, then using Remark 5.6 (2) we may replace $\Delta_{i,j}$ with $\Delta_{i_P,j_P}$. If $i, j \in B$, we can use a square relation to replace it with $\Delta_{i,j} \Delta_P - \Delta_{i_P,j_P} + \Delta_{i_P,j}$. We then have $\Delta_{i,j} \Delta_P = \Delta([n]) \Delta_{i,j} \Delta_P$, so

$$\Delta_{i,j} \Delta_P \Delta_{i_P,j_P} = 2 \Delta([n]) \Delta_{i,j} \Delta([n]) = \Delta_P \Delta_{i_P,j_P}$$

by Remark 5.6 (2).

Theorem 5.16. For $n \geq 3$, the natural surjection $R \to A_{\text{PGL}_2}(\mathbb{P}^1)^n$ is an isomorphism. Furthermore, $R_k$ has $\mathbb{Z}$-basis given by

1. $\{ \Delta_P \mid P \in \text{Good}(n - k, n) \}$ for $k \leq n - 2$
2. $S_k = \{ \Delta([n]) \Delta_{i,j}^{k-n+1} \} \cup \{ \Delta_P \Delta_{i_P,j_P}^{k-n+2} \mid P \in \text{Part}(2, n) \}$, where for each partition $P \in \text{Part}(d, n)$ for $d \leq 2$, arbitrarily choose $i_P, j_P$ that lie in the same part.

Proof. If $k \leq n - 2$, we have $R_k \to A_{\text{PGL}_2}(\mathbb{P}^1)^n$ is an isomorphism with $\mathbb{Z}$-basis given by $\{ \Delta_P \mid P \in \text{Good}(n - k, n) \}$ by Corollary 5.14. Now, consider the case $k > n - 2$.

The $S_k$ span $R_k$ by Lemma 5.15 so applying (1) yields

$$\mathbb{Z}^{S_k} \to R_k \to A_{\text{PGL}_2}(\mathbb{P}^1)^n,$$

whose composite is a surjection of free $\mathbb{Z}$-modules of the same rank $2^{n-1}$ by Lemmas 5.4 and 5.15 so it is an isomorphism. This proves $R_k \to A_{\text{PGL}_2}(\mathbb{P}^1)^n$ is an isomorphism and identifies $S_k$ as a basis.

5.1. Algorithm and Example. We can describe an algorithm for writing arbitrary classes in $A^*_{\text{PGL}_2}(\mathbb{P}^1)^d$ in terms of our $\mathbb{Z}$-basis. The key fact is that if $pr^n : (\mathbb{P}^1)^n \to (\mathbb{P}^1)^{n-1}$ is projection by forgetting the last factor, then by definition of the pushforward of a cycle

$$pr^n \Delta_P = \begin{cases} \Delta_P | n \text{ is not isolated, and} \\ P & 0 \text{ if } n \text{ is isolated.} \end{cases}$$
At the level of formulae, if we write our class as a polynomial in the $H_i, u, v$ with each $H_i$ appearing to degree at most 1, then $\text{pr}^n_\alpha$ extracts the $H_i$-coefficient. Also, if we have a $\Delta P$ and we know that either $n$ is isolated or $n-1, n$ are in the same part, then as $(H_n - H_{n-1}) \cap \Delta_{n-1,n} = 0$ we also have

$$\text{pr}^n_\alpha(\Delta P \cap (H_n - H_{n-1})) = \begin{cases} 0 & \text{if } n-1, n \text{ are in the same group, and} \\ \Delta P \setminus n & \text{if } n \text{ is isolated.} \end{cases}$$

Suppose we have a class

$$\alpha = \sum_{P \in \text{Good}(d,n)} a_P \Delta P = \sum_{P \in \text{Good}(d,n)} a_P \Delta P + \sum_{P \in \text{Good}(d,n)} a_P \Delta P$$

and we want to find the coefficients $a_P$.

We first show how to reduce down to the case $d = 2$. By the above, we have

$$\text{pr}^n_\alpha = \sum_{P \in \text{Good}(d,n)} a_P \Delta P \setminus n, \quad \text{pr}^n_\alpha(\alpha \cap (H_n - H_{n-1})) = \sum_{P \in \text{Good}(d,n)} a_P \Delta P \setminus n.$$ 

In the first case each $P \setminus n \in \text{Good}(d-1,n)$, and in the second case each $P \setminus n \in \text{Good}(d-1,n-1)$ so we can apply induction to determine all of these coefficients.

Once we have reduced down to the case $d = 2$, we can now identify each $a_P$ separately for $P = \{A, B\}$ a partition of $[n]$ into two parts by evaluating at $H_i = -u$ for $i \in A$ and $H_j = -v$ for $i \in B$ (which is localization at a torus-fixed point). By Proposition 4.3, this evaluates to $a_{\{A, B\}}(u - v)^{n-2}(-1)^{|A|}$. The same method for $d = 2$ works for elements $\alpha \in A^k((\mathbb{P}^1)^n)$ with $k > n - 2$. Applying the same substitution to

$$\alpha = \sum a_P \Delta_{i, j_P}^{k-n+2} \Delta P + a([n]) \Delta_{i([n]), j([n])}^{n-k+1} \Delta ([n])$$

extracts the $a_P$-coefficient for $P = \{A, B\}$ a partition of $[n]$ into two parts as this is the only term that does not vanish under this substitution. Then, we subtract off all of these terms to recover $a_{[n]}$.

**Example 5.17.** As a simple example, consider the $\text{PGL}_2$-orbit closure of a generic point in $(\mathbb{P}^1)^5$. The formula computed in [23 Corollary 4.8] shows that the class of this orbit is

$$\alpha = e_2(H_1, H_2, H_3, H_4, H_5) + 2(u + v)(H_1 + H_2 + H_3 + H_4 + H_5) + (3u^2 + 4uv + 3v^2),$$

where $e_2$ is the second elementary symmetric polynomial. We have
We remark that the decomposition into good incidence strata as
\( \Delta \) gives a nonzero result is
\( A \) which can be geometrically explained as follows. Consider the diagram
\[ \begin{array}{c}
\text{pr}_5^3(\alpha) \\
\text{pr}_5^4(\alpha) \\
\text{pr}_5^4(\alpha) \\
\text{pr}_5^4(\alpha) \\
\text{pr}_5^5(\alpha)
\end{array} \]
\( \Delta_{\{1,2,3,\},\{4\}} \)

\( \Delta_{\{1,2,3,\},\{4\}} \)

\( \Delta_{\{1,2,3,\},\{4\}} \)

\( \Delta_{\{1,2,3,\},\{4\}} \)

\( \alpha \)

\( \text{pr}_5^3(\alpha \cap (H_5 - H_4)) = \Delta_{\{1,2,3,\},\{4\}} \)

\( \text{pr}_5^4(\alpha \cap (H_4 - H_3)) = \Delta_{\{1,2,3,\},\{4\}} \)

\( \text{pr}_5^5(\alpha \cap (H_3 - H_2)) = \Delta_{\{1,2,\},\{3\}} \)

\( \text{pr}_5^5(\alpha \cap (H_2 - H_3)) = \Delta_{\{1\},\{2\}} \)

\( \text{pr}_5^5(\alpha) = 0 \)

\( \text{pr}_5^5(\alpha) = (H_1 + H_2 + H_3 + H_4) + 2(u + v) \)

\( \text{pr}_5^4(\alpha \cap (H_5 - H_4)) = e_2(H_1, H_2, H_3) + (u + v)(H_1 + H_2 + H_3) + (u^2 + uv + v^2) \)

\( \text{pr}_5^4(\alpha) = 1 \)

\( \text{pr}_5^4(\alpha \cap (H_4 - H_3)) = H_1 + H_2 + u + v \)

\( \text{pr}_5^3(\alpha) = 0 \)

\( \text{pr}_5^3(\alpha \cap (H_3 - H_2)) = 1 \).

The only non-trivial identification was
\( \text{pr}_5^5(\alpha \cap (H_5 - H_4)) = \Delta_{\{1,2,3,\},\{4\}} \), which we can identify as follows. Substitute \(-u\)'s and \(-v\)'s for the \( H_i \) corresponding to all nontrivial partitions \( \{A, B\} \) of \( [4] \) into two parts. We find the only choice that gives a nonzero result is
\( A = \{1, 2, 3\}, B = \{4\}, \) yielding \((u - v)^2\), which is the same as for \( \Delta_{\{1,2,3,\},\{4\}} \) by Proposition 4.3. Putting this together yields

\( \alpha = \Delta_{\{1\},\{2\},\{3,4,5\}} + \Delta_{\{1,\},\{2,3\},\{4,5\}} + \Delta_{\{1,2,3,\},\{4\},\{5\}} \).

We remark that the \( PGL_2 \)-orbit closure \( X_n \subset (\mathbb{P}^1)^n \) of a general point in \( (\mathbb{P}^1)^n \) decomposes into good incidence strata as

\[ (2) \quad [X_n] = \sum_{a=1}^{n-2} \Delta_{\{1,\ldots,a\},\{a+1,\ldots,n\}} \]

which can be geometrically explained as follows. Consider the diagram

\[ \begin{array}{ccc}
\overline{M}_{0,n}(\mathbb{P}^1, 1) & \xrightarrow{ev} & (\mathbb{P}^1)^n \\
\downarrow \pi & & \\
\overline{M}_{0,n} & & \\
\end{array} \]

(see Section 1.1.1 for notation). The left and right hand side of [2] can both be described as \( ev_* \pi^*(pt) \) for \( pt \in \overline{M}_{0,n} \) being a general point and the point corresponding to a chain of \( n-2 \) rational curves (respectively), and the result follows from the flatness of \( \pi \). See Proposition 4 for a generalization of this degeneration to \( PGL_{r+1} \) orbits closures of general points in \( (\mathbb{P}^r)^n \).
6. \(GL_2\)-equivariant classes of strata in \(\text{Sym}^n\mathbb{P}^1\)

Recall from Definition 2.5 that \(|\lambda| \in A_{GL_2}(\mathbb{P}^n)\) for \(\lambda\) a partition of \(n\) is the pushforward of \(\Delta_P \in A_{GL_2}([\mathbb{P}^1]^n)\) under the multiplication map \((\mathbb{P}^1)^n \to \mathbb{P}^n\) for \(P\) a partition of \(n\) into subsets with cardinalities given by \(\lambda\). Up to a constant factor given in Definition 2.5, this is the class of the closure \(Z_\lambda\) given in Definition 2.3 of degree \(n\) forms on \((\mathbb{P}^1)^\vee\) whose roots have multiplicities given by \(\lambda\) as studied by Fehér, Némethi, and Rimányi [13].

**Definition 6.1.** Denote by \([a, b, 1^c] := \{(a, b, 1, \ldots, 1)\}\) where there are \(c\) 1’s.

From writing the expressions for \(|\lambda|\) in Theorem 4.3 using generating functions, we find the following new Corollary.

**Corollary 6.2.** For \(d \geq 2\), consider the polynomial

\[
-\frac{1}{(z-1)^{d-2}} \prod_{i=1}^{d} (z^{a_i} - 1) = \sum_{0 \leq k_1 \leq k_2, k_1 + k_2 = n - d + 2} \alpha_{k_1}(z^{k_1} + z^{k_2}).
\]

Then \(\alpha_i \in \mathbb{Z}\) and

\[
[a_1, \ldots, a_d] = \sum_{1 \leq k_1 \leq k_2} \alpha_{k_1}[k_1, k_2, 1^{d-2}]
\]

**Proof.** Clearly all \(\alpha_i \in \mathbb{Z}\) except possibly \(\alpha_{n-d+2}\), which a priori only lies in \(\mathbb{Z}[\frac{1}{2}]\).

But plugging in \(z = 1\) to both sides shows the integrality.

By Theorem 4.3 it suffices to show

\[
\prod_{i=1}^{d} (z^{a_i} - 1) = \sum_{k_1 + k_2 = n - d + 2} \alpha_{k_1}(z^{k_1} - 1)(z^{k_2} - 1)(z - 1)^{d-2}.
\]

or equivalently

\[
\frac{1}{(z-1)^{d-2}} \prod_{i=1}^{d} (z^{a_i} - 1) = \sum_{k_1 + k_2 = n - d + 2} \alpha_{k_1}(z^{k_1} - 1)(z^{k_2} - 1).
\]

By definition of \(\alpha_{k}\), the coefficients of both sides agree except possibly the \(z^0\) and \(z^{n-(d-2)}\)-coefficient. Also, the coefficients of \(z^0\) and \(z^{n-(d-2)}\) are equal to each other on the left hand side, and the same is true on the right side. To see they agree between the left and right sides, we note both sides are 0 after substituting \(z = 1\). \(\square\)

**Lemma 6.3.** The rational \(GL_2\)-equivariant classes in \(\mathbb{P}^n\) of the torus fixed points

\[
\prod_{j \in [n] \setminus \{k\}} (H + jv + (n-j)u) \in A_T^\bullet(\mathbb{P}^n) \otimes \mathbb{Q}
\]

are linearly independent.

**Proof.** For fixed \(k\), \(H \mapsto -kv - (n-k)v\) maps \(\prod_{j \in [n] \setminus \{k'\}} (H + jv + (n-j)u)\) to 0 if and only if \(k' \neq k\). \(\square\)

**Theorem 6.4.** For fixed \(c \geq 0\), the classes \([a, b, 1^c]\) with \(a + b = n - c\) and \(a \geq b\) form a \(\mathbb{Q}\)-basis for \(A_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Q} \subset A_{GL_2}^\bullet(\mathbb{P}^n) \otimes \mathbb{Q}\).
Proof. To show the linear independence, first note that \( \prod_{j \in [n] \setminus \{k\}} (H + jv + (n-j)u) \) are linearly independent in \( A^*_G \langle \mathbb{P}^n \rangle \otimes \mathbb{Q} \) by Lemma 3.4. Therefore, it suffices to show for fixed \( c \) that the polynomials \((z^a-1)(z^b-1)(z-1)^c\) with \( a \geq b \) and \( a+b = n-c \) are linearly independent. Indeed, dividing out by \((z-1)^c\), we note that \((z^a-1)(z^b-1)\) is the only such polynomial which contains either of the monomials \( z^a \) or \( z^b \).

To see that the \( \mathbb{Q} \)-linear span of the classes \([a, b, 1]\) is precisely \( A^*_{\text{PGL}_2} \langle \mathbb{P}^n \rangle \otimes \mathbb{Q} \), we note that we have just shown that the dimension of the \( \mathbb{Q} \)-linear span of the \([a, b, 1]\) is precisely \( \frac{n(n+1)}{2} \) by linear independence, which we can check is the same as the dimension of \( A^*_{\text{PGL}_2} \langle \mathbb{P}^n \rangle \otimes \mathbb{Q} \) by Proposition 3.7.

\[ \square \]

7. Integral classes of unordered strata in \([\text{Sym}^n \mathbb{P}^1/\text{PGL}_2]\)

In this section, we compute the integral classes of \([Z_\lambda] \in A^*_{\text{PGL}_2} \langle \mathbb{P}^n \rangle \). By Proposition 3.7, if \( n \) is odd, then \( A^*_{\text{PGL}_2} \langle \mathbb{P}^n \rangle \rightarrow A^*_G \langle \mathbb{P}^n \rangle \) is injective and we know the image of the \([Z_\lambda]\) in \( A^*_G \langle \mathbb{P}^n \rangle \) by Theorem 4.5, so it suffices to consider the case \( n \) is even, which we assume for the remainder of this section.

Recall the polynomials \( p_n(t) \in A^*_{\text{PGL}_2} \langle \mathbb{P}^n \rangle \) defined in Proposition 3.7 for even \( n \) and let \( q_n \) be the image of \( p_n \) in \( A^*_{\text{PGL}_2} \langle \mathbb{P}^n \rangle/(2) \equiv \mathbb{F}_2[c_2, c_3, t] \). It is easy to see by the binomial theorem or directly from [15, Lemma 6.1] that

\[
q_n(t) = \begin{cases} 
t^{(n+4)/4}t^3 + c_2t + c_3)^{n/4} & \text{if } n \equiv 0 \mod 4, \text{ and} \\
t^{(n-2)/4}(t^3 + c_2t + c_3)^{n+2)/4} & \text{if } n \equiv 2 \mod 4, 
\end{cases}
\]

and \( q_n(t) | q_{n+k}(t) \) for \( k = 0 \) or \( k \geq 4 \) for any \( n \).

By Proposition 3.7 for \( n \) even,

\[
A^*_{\text{PGL}_2} \langle \mathbb{P}^n \rangle \cong \mathbb{Z}[c_2, c_3, H]/(2c_3, p_n(H)),
\]

which is isomorphic to

\[
\bigoplus_{i=0}^n \mathbb{Z}[c_2]H^i \oplus \bigoplus_{i=0}^n c_3F_2[c_2, c_3]H^i
\]

as abelian groups. So to determine the class \([Z_\lambda] \in A^*_{\text{PGL}_2} \langle \mathbb{P}^n \rangle \), it suffices to find its image in \( \bigoplus_{i=0}^n \mathbb{Z}[c_2]H^i \) and \( \bigoplus_{i=0}^n c_3F_2[c_2, c_3]H^i \). Equivalently, if we write the class of \([Z_\lambda]\) as a polynomial in \( c_2, c_3, \) and \( H \) with degree at most \( n \) in \( H \), then it suffices to consider the terms not containing \( c_3 \) and the terms containing \( c_3 \) separately. Under the map \( A^*_{\text{PGL}_2} \langle \mathbb{P}^n \rangle \rightarrow A^*_G \langle \mathbb{P}^n \rangle \), Proposition 3.7 shows that the first factor maps injectively and the second factor maps to zero.

We can determine the image of \([Z_\lambda]\) in the first factor using Theorem 4.5 so it suffices to determine the image of \([Z_\lambda]\) in the second factor to identify its class. To do this, we will work modulo 2 and determine \([Z_\lambda] \in A^*_{\text{PGL}_2} \langle \mathbb{P}^n \rangle \otimes \mathbb{Z}/2\mathbb{Z} \).

Discarding those monomials not containing \( c_3 \) then yields the image of \([Z_\lambda]\) in the second factor.

Definition 7.1. We say a partition \( \lambda = a_1^{e_1} \ldots a_k^{e_k} \) of \( n \) into \( d = \sum_{i=1}^k e_i \) parts is special if all \( a_i \) and \( \frac{d}{e_{i_1} \ldots e_{k_1}} \) are odd, and all \( e_i \) are even.
\textbf{Theorem 7.2.} Let $d$ and $n$ be integers with $n$ even. The class of $[Z_\lambda] \in A^*_{\mathbb{P}GL_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z}$ for $\lambda$ a partition of $n$ into $d$ parts is given by
\[
\begin{cases} 
\frac{2q_\lambda}{q_\lambda}(H) & \text{if } \lambda \text{ is special, and} \\
0 & \text{otherwise.}
\end{cases}
\]

\textbf{Remark 7.3.} If $[Z_\lambda] \in A^*_{\mathbb{P}GL_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z}$ is zero, then the component in $\bigoplus_{i=0}^n \mathbb{Z}[c_2]H^i$ is a multiple of 2, and the component in $\bigoplus_{i=0}^n c_3\mathbb{F}_2[c_2, c_3]H^i$ is zero.

Furthermore, given the statement of the theorem, if $[Z_\lambda] \in A^*_{\mathbb{P}GL_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z}$ is non-zero, then the component in $\bigoplus_{i=0}^n c_3\mathbb{F}_2[c_2, c_3]H^i$ is non-zero and is given by discarding anything with a $c_3^i$-coefficient in $\frac{2q_\lambda}{q_\lambda}(H)$.

\textbf{Lemma 7.4.} Given a ring $R[H]/(P(H))$ for $P$ a monic polynomial of degree $n+1$, define the $R$-linear map $\tilde{f} : R[H]/(P(H)) \to R$ given by taking a polynomial $f(H)$, and outputting the $H^n$-coefficient of the reduction $\tilde{f}(H)$ of $f(H)$ (mod $P(H)$) to a polynomial of degree $\leq n$. Then letting $t$ be an indeterminate, we have
\[
\int \frac{P(H) - P(t)}{H - t} f(H) = \tilde{f}(t).
\]

\textbf{Proof.} We have
\[
\int \frac{P(H) - P(t)}{H - t} f(H) = \int \frac{P(H) - P(t)}{H - t} \tilde{f}(H)
\]
\[
= \int P(H) \tilde{f}(H) - \tilde{f}(t) \frac{H - t}{H - t} - \int P(t) \tilde{f}(H) - \tilde{f}(t) \frac{H - t}{H - t} + \int P(H) - P(t) \tilde{f}(t) \frac{H - t}{H - t}
\]
\[
= 0 + 0 + \tilde{f}(t) = \tilde{f}(t).
\]

Where in the second last equality, the first term is zero because the integrand is a multiple of $P(H)$, the second term is zero because $\frac{\tilde{f}(H) - \tilde{f}(t)}{H - t}$ is a polynomial of degree at most $n - 1$, and the last term is $\tilde{f}(t)$ because $\frac{P(H) - P(t)}{H - t}$ is monic of degree $n$. \hfill \Box

\textbf{Remark 7.5.} Let $G$ be a linear algebraic group and $V$ be a representation. Then,
\[
A^*_G(\mathbb{P}(V)) \cong A^*_G(\text{pt})[H]/(P(H))
\]
\[
A^*_G(\mathbb{P}(V) \times \mathbb{P}(V)) \cong A^*_G(\text{pt})[H_1, H_2]/(P(H_1, P(H_2)),$
\]
where $P \in A^*_G[T]$ is $T^{\dim(V)} + c_1^G(V)T^{\dim(V)-1} + \cdots + c_{\dim(V)}^G(V)$ by the projective bundle theorem and the class of the diagonal in $\mathbb{P}(V) \times \mathbb{P}(V)$ is $(P(H_1) - P(H_2))/(H_1 - H_2)$, giving a geometric interpretation of Lemma 7.4. This can be proven, for example, by first noting that it suffices to consider the case $G = GL(V)$. Then, we can restrict to a maximal torus \cite{9} Proposition 6 and use the fact that the diagonal in $\mathbb{P}(V) \times \mathbb{P}(V)$ admits a torus-equivariant deformation into a union of products of coordinate linear spaces \cite{4} Theorem 3.1.2].

\textbf{Proof of Theorem 7.2.} Note that when all $a_i$ are odd and all $e_i$ are even then $n = \sum a_i e_i$ is either equal to $\sum e_i$, or exceeds it by at least 4, so $q_{e_1+\ldots+e_k} | q_n$ and the claimed expression for $[Z_\lambda]$ is well-defined.
We resolve $Z_\lambda$ birationally with the map

$$\Psi : \prod_{i=1}^k \mathbb{P}^{e_i} \to \mathbb{P}^n$$

taking $(D_1, \ldots, D_k) \mapsto a_1D_1 + \ldots + a_kD_k$ (treating $P^r = \text{Sym}^r \mathbb{P}^1$ for all $r$).

If at least one $e_i$ is odd, then we claim $c_3[Z_\lambda] = 0$. Indeed,

$$c_3[Z_\lambda] = \Psi_*c_3,$$

and $c_3 \in A^\bullet_{PGL_2}(\text{pt})$ maps to $0$ in $A^\bullet_{PGL_2}(\prod_{i=1}^k \mathbb{P}^{e_i})$ as the projection $\prod_{i=1}^k \mathbb{P}^{e_i} \to \text{pt}$ can be factored as the composite $\prod_{i=1}^k \mathbb{P}^{e_i} \to \mathbb{P}^{e_i} \to \text{pt}$, and if $e_i$ is odd then $c_3$ pulls back to zero in $A^\bullet_{PGL_2}(\mathbb{P}^{e_i})$ by Proposition 3.7.

Hence, as $c_3[Z_\lambda] = 0$, we must have $[Z_\lambda]$ is zero in $A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z}$.

Now, suppose that all $e_i$ are even. For the remainder of the proof all integrals are in Chow rings after tensoring with $\mathbb{Z}/2\mathbb{Z}$. By Lemma 7.4 it suffices to show

$$\int_{\mathbb{P}^n} \frac{q_n(t) - q_a(H)}{t - H} \cap \Psi_*1 = \begin{cases} \frac{d!}{e_1! \cdots e_k!} & \text{if all } a_i \text{ and } e_i \text{ are odd and } d \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

By the projection formula applied to $\Psi$, we have

$$\int_{\mathbb{P}^n} \frac{q_n(t) - q_a(H)}{t - H} \cap \Psi_*1 = \int_{\prod_{i=1}^k \mathbb{P}^{e_i}} \frac{q_n(t) - q_a(\sum a_iH_i)}{t - \sum a_iH_i}.$$

Now, if any $a_i$ is even, then as we are working modulo 2, $\frac{q_n(t) - q_a(\sum a_iH_i)}{t - \sum a_iH_i}$ will not contain $H_i$, so the integral is clearly zero. Hence we may assume from now on that all $a_i$ are odd, so that $\sum a_iH_i = \sum H_i \mod 2$.

We claim that $q_d(\sum H_i) = 0$ and that

$$\int_{\prod_{i=1}^k \mathbb{P}^{e_i}} \frac{q_d(t) - q_d(\sum H_i)}{t - \sum H_i} = \frac{d!}{e_1! \cdots e_k!},$$

The first of these follows from pulling back $q_d(H)$ under the multiplication map $\prod_{i=1}^k \mathbb{P}^{e_i} \to \mathbb{P}^d$, and the second of these follows from applying Lemma 7.4 to $1 \in A^\bullet_{PGL_2}(\mathbb{P}^d)$ together with the projection formula as the multiplication map has degree $\frac{d!}{e_1! \cdots e_k!}$.

From the vanishing of $q_d(\sum H_i)$, we have

$$\frac{q_n(t) - q_a(\sum H_i)}{t - \sum H_i} = \frac{q_n(t)q_d(t) - q_d(\sum H_i)}{q_d(t) - q_d(\sum H_i)} + \frac{q_d(\sum H_i)}{t - \sum H_i} \frac{q_n(t) - q_n(\sum H_i)}{q_n(t) - q_d(\sum H_i)}$$

$$= \frac{q_n(t)q_d(t) - q_d(\sum H_i)}{t - \sum H_i},$$

and the result now follows from the second claim after applying $\int_{\prod_{i=1}^k \mathbb{P}^{e_i}}$ to both sides. \qed

We now prove surprisingly that despite the presence of occasional 2-torsion, integral relations between $[Z_\lambda]$ classes in $A^\bullet_{GL_2}(\mathbb{P}^n)$ are equivalent to integral relations between $[Z_\lambda]$-classes in $A^\bullet_{PGL_2}(\mathbb{P}^n)$. 

Theorem 7.6. Let \( n, d \) be integers. A linear combination \( \sum a_\lambda [Z_\lambda] \) with \( a_\lambda \in \mathbb{Z} \) and each \( \lambda \) a partition of \( n \) into \( d \) parts is zero in \( A^*_\text{GL}_2(\mathbb{P}^n) \) if and only if it is zero in \( A^*_\text{PGL}_2(\mathbb{P}^n) \). In particular, \( \sum a_\lambda [Z_\lambda] = 0 \) if and only if

\[
\sum_{\lambda=a_1^{\alpha_1} \cdots a_k^{\alpha_k}} a_\lambda \prod_{i=1}^k \frac{(z^{a_i} - 1)^{e_i}}{e_i!} = 0.
\]

Proof. One direction is trivial, as we have the map \( A^*_\text{PGL}_2(\mathbb{P}^n) \to A^*_\text{GL}_2(\mathbb{P}^n) \) induced by \( GL_2 \to PGL_2 \), so if a linear relation holds in \( A^*_\text{PGL}_2(\mathbb{P}^n) \), then it also holds in \( A^*_\text{GL}_2(\mathbb{P}^n) \). Conversely, suppose that we have \( \sum a_\lambda [Z_\lambda] = 0 \) in \( A^*_\text{GL}_2(\mathbb{P}^n) \). We only have to care about the case that \( n \) is even, because when \( n \) is odd, \( A^*_\text{PGL}_2(\mathbb{P}^n) \to A^*_\text{GL}_2(\mathbb{P}^n) \) is an injection by Proposition 3.7.

For \( n \) even, suppose we have a sum \( \sum a_\lambda [Z_\lambda] \), which is 0 in \( A^*_\text{GL}_2(\mathbb{P}^n) \). Then since the kernel of \( A^*_\text{PGL}_2(\mathbb{P}^n) \to A^*_\text{GL}_2(\mathbb{P}^n) \) is 2-torsion by Proposition 3.7, we know \( \sum a_\lambda [Z_\lambda] \) is 2-torsion in \( A^*_\text{PGL}_2(\mathbb{P}^n) \). By Theorem 7.6, the class \([Z_\lambda]\) in \( A^*_\text{PGL}_2(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z} \) is either 0 or \( \frac{PGL_2(H)}{2} \), and the second possibility occurs precisely when \( \lambda \) is special. Hence to prove Theorem 7.6 by Theorem 4.5 and Lemma 6.3 it suffices to show that if

\[
\sum_{\lambda=a_1^{\alpha_1} \cdots a_k^{\alpha_k}} a_\lambda \prod_{i=1}^k \frac{(z^{a_i} - 1)^{e_i}}{e_i!} = 0,
\]

then

\[
\sum_{\lambda \text{ special}} a_\lambda \equiv 0 \pmod{2}.
\]

Note first that if no special \( \lambda \) appears we are done, so we may assume that at least one special \( \lambda \) appears. As \( d = \sum_{i=1}^k e_i \) for any partition \( \lambda = a_1^{\alpha_1} \cdots a_k^{\alpha_k} \) appearing, we must have \( d \) is even if a special \( \lambda \) appears. Multiplying (3) by \( \frac{d!}{(z-1)^d} \) and plugging in \( z = 1 \), we have

\[
\sum_{\lambda=a_1^{\alpha_1} \cdots a_k^{\alpha_k}} a_\lambda \frac{d!}{e_1! \cdots e_k!} \prod_{i=1}^k a_i^{e_i} = 0.
\]

Now we claim that \( \frac{d!}{e_1! \cdots e_k!} \) is even if any \( e_i \) is odd. Indeed, as \( d \) is even, if not all \( e_i \) are even, then at least two of the \( e_i \) are odd. If \( e_i, e_j \) are both odd, then replacing \( e_i!e_j! \) in \( \frac{d!}{e_1! \cdots e_k!} \) with \((e_i-1)!(e_j-1)! \) yields an integer with a smaller power of 2 dividing it.

Hence, \( \frac{d!}{e_1! \cdots e_k!} \prod_{i=1}^k a_i^{e_i} \) is odd precisely when \( \lambda \) is special. Taking the equality \( \pmod{2} \) then yields the desired result. \( \square \)

We complete the proof of Theorem 1.8

Proof of Theorem 1.8. We have (1), (2) and (4) are equivalent by Theorem 7.6. Also (3) implies (2) is clear as \( A^*_\text{GL}_2(\mathbb{P}^n) \) is free as an abelian group, so \( A^*_\text{GL}_2(\mathbb{P}^n) \to A^*_\text{PGL}_2(\mathbb{P}^n) \otimes \mathbb{Q} \).

To finish, it suffices to show (2) implies (3). Let \( \lambda = (\lambda_1, \ldots, \lambda_d) \) for \( \lambda_1 \geq \cdots \geq \lambda_d \).
Claim. Suppose $\lambda_3 > 1$. Then using pushforwards of square relations in $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$, we can express $[\lambda] \in A^\bullet_{PGL_2}(\mathbb{P}^n)$ in terms of classes $[\lambda']$ where $\lambda' = (\lambda_1', \ldots, \lambda_d')$ where $\lambda_1 + \lambda_2 > \lambda_1' + \lambda_2'$.

Proof of Claim. Pick a partition $P = \{A_1, \ldots, A_d\}$ of $[n]$ with $|A_i| = \lambda_i$. Since $|A_3| > 1$, we can partition it as $A_3 = A_3' \cup A_3''$ into nonempty parts. Now, applying the square relation associated to $P' = \{A_1, A_2, A_3', A_3'', \ldots, A_d\}$ of $[n]$ into $d + 1$ parts and the parts $A_1, A_2, A_3, A_3''$ shows

$$[\lambda] = [\lambda_1] + [\lambda_2] - [\lambda_3],$$

where $\lambda_3' = |A_3'|$ and $\lambda_3'' = |A_3''|$ and

$$\begin{align*}
\lambda_1 &= \{\lambda_1 + \lambda_3', \lambda_2, \lambda_3', \ldots, \lambda_d\} \\
\lambda_2 &= \{\lambda_1, \lambda_2 + \lambda_3', \lambda_3', \ldots, \lambda_d\} \\
\lambda_3 &= \{\lambda_1 + \lambda_2, \lambda_3', \lambda_3', \ldots, \lambda_d\}.
\end{align*}$$

\[\square\]

Returning to the proof of Theorem 8.1, iterating the claim shows that the pushforward of square relations allow us to rewrite any $[\lambda]$ in terms of the $\mathbb{Q}$-basis found in Theorem 8.1 which shows (2) implies (3).

8. Excision of unordered strata in $[\text{Sym}^n \mathbb{P}^1 / PGL_2]$

As an application of our results in the ordered case, we will prove the following result on the $PGL_2$-equivariant Chow ring of $\mathbb{P}^n$ with strata excised, which we will adapt in the next section to the case of $GL_2$-equivariant Chow rings with strata in both $\mathbb{P}^n$ and in $\mathbb{A}^{n+1}$.

Theorem 8.1. Given a partition $\lambda = \{\lambda_1, \ldots, \lambda_d\}$ of $n$,

$$A^\bullet_{PGL_2}(\mathbb{P}^n \setminus Z_\lambda) = A^\bullet_{PGL_2}(\mathbb{P}^n)/I,$$

where the ideal $I \otimes \mathbb{Q} \subset A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Q}$ is generated by all $[\lambda']$ for $\lambda'$ a partition formed by merging some of the parts of $\lambda$.

Even though Theorem 8.1 requires many generators for $I$, in some cases fewer generators suffice.

Theorem 8.2. Given the partition $\lambda = \{a, 1^{n-a}\}$ of $n$, the ideal $I \otimes \mathbb{Q}$ in Theorem 8.1 is generated by $[\lambda]$ and $[\lambda']$, where

$$\lambda' = \begin{cases}
\{a + 1, 1^{n-a-1}\} & \text{if } a \neq \frac{n}{2} \\
\{a, 2, 1^{n-a-2}\} & \text{if } a = \frac{n}{2}.
\end{cases}$$

See Remark 8.3 for the connection to similar results proved in 13.

By the excision exact sequence [17, Proposition 1.8], the ideal $I$ is the same as the pushforward ideal $I_\lambda$ which we define in Definition 8.3.

Definition 8.3. Given a partition $\lambda$ of $n$ and for $G = PGL_2$ or $GL_2$, let $I_\lambda^G$ be the ideal of $A^\bullet_G(\mathbb{P}^n)$ given by the pushforward via the inclusion $i_\lambda : Z_\lambda \hookrightarrow \mathbb{P}^n$

$$I_\lambda^G = (i_\lambda)_* A^G_\bullet(Z_\lambda) \subset A^\bullet_G(\mathbb{P}^n)$$

and the identification $A^\bullet_G(\mathbb{P}^n) \cong A^\bullet_{G\bullet}(\mathbb{P}^n)$ via Poincaré duality [15, Proposition 4]. When $G$ is clear from context we will simply write $I_\lambda$. 

Then, when we have a collection of partitions \( G \), according to the partition \( \lambda \), we get a map

\[
i_\lambda : Y_\lambda \to \mathbb{P}^n
\]

that is birational onto its image \( Z_\lambda \) given by the composition

\[
Y_\lambda \leftarrow \prod_{i=1}^{n} \mathbb{P}^{e_i^{\lambda}} 
\]

of the \( i \)th power map on each factor \( \mathbb{P}^{e_i} \) together with the multiplication map. Equivalently, if we view projective space \( \mathbb{P}^n \) as parameterizing degree \( n \) divisors on \( \mathbb{P}^1 \), then the map is given by \( (D_1, \ldots, D_n) \mapsto \sum_{i=1}^{n} iD_i \).

In particular, \( I_\lambda \) is also given by the image of \((i_\lambda)_*\). Since we are working rationally, we can take a finite cover of \( Y_\lambda \).

**Definition 8.5.** Given a partition \( \lambda = \{\lambda_1, \ldots, \lambda_d\} \) of \( n \), define the finite map \( \Phi_\lambda : (\mathbb{P}^1)^d \to Y_\lambda \) to be

\[
\Phi_\lambda : (\mathbb{P}^1)^d = \prod_{i=1}^{n} ((\mathbb{P}^1)^{e_i^{\lambda}}) \to \prod_{i=1}^{n} \mathbb{P}^{e_i^{\lambda}} = Y_\lambda
\]

given by the multiplication map \((\mathbb{P}^1)^{e_i} \to \mathbb{P}^{e_i}\) on each factor. Since \( \Phi_\lambda \) is finite,

\[
(\Phi_\lambda)_* : A^\bullet_{PGL_2}((\mathbb{P}^1)^d) \otimes \mathbb{Q} \to A^\bullet_{PGL_2}(Y_\lambda) \otimes \mathbb{Q}
\]

is surjective, so \( I_\lambda \otimes \mathbb{Q} \) is the image of

\[
(i_\lambda \circ \Phi_\lambda)_* : A^\bullet_{PGL_2}((\mathbb{P}^1)^d) \otimes \mathbb{Q} \to A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \otimes \mathbb{Q}.
\]

The map \( \Phi_\lambda \) has the nice property that given a partition \( P \) of \( [d] \), the pushforward of the strata \((i_\lambda \circ \Phi_\lambda)_* \Delta_P \) is \([\lambda']\), where \( \lambda' \) is the partition of \( n \) given by merging the parts of \( \lambda \) according to the partition \( P \). From this, we will be able to deduce certain symmetrized strata generate \( I_\lambda \otimes \mathbb{Q} \) based on the generation properties of strata in \((\mathbb{P}^1)^d\).

**Definition 8.6.** Given a set of partitions \( \mathcal{P} \) of \([d]\) and \( G = PGL_2 \) or \( GL_2 \), let \( \Lambda^G_P \subset A^\bullet_G((\mathbb{P}^1)^d) \otimes \mathbb{Q} \) be the submodule over \( A^\bullet_G(\mathbb{P}^n) \otimes \mathbb{Q} \) generated by the classes \( \Delta_P \). Explicitly,

\[
\Lambda^G_P = \sum_{P \in \mathcal{P}} \Delta_P \cap \Phi_\lambda^{\bullet}(A^\bullet_G(\mathbb{P}^n) \otimes \mathbb{Q}).
\]

When \( G \) is clear from context we will notate \( \Lambda^G_P \) simply by \( \Lambda_P \).

**Lemma 8.7.** Let \( \lambda = \{\lambda_1, \ldots, \lambda_d\} \) be a partition of \( n \), and let \( G = PGL_2 \) or \( GL_2 \). Suppose we have a collection of partitions \( \mathcal{P} \) of \([d]\) such that in \( A^\bullet_G((\mathbb{P}^1)^d) \otimes \mathbb{Q} \)

\[
A^\bullet_G((\mathbb{P}^1)^d) \prod_{i=1}^{n} S^{e_i^{\lambda}} \otimes \mathbb{Q} \subset \Lambda^G_P.
\]

Then \( \{ (i_\lambda \circ \Phi_\lambda)_* \Delta_P \mid P \in \mathcal{P} \} \) generates \( I^G_\lambda \otimes \mathbb{Q} \subset A^\bullet_G(\mathbb{P}^n) \otimes \mathbb{Q} \).
Proof. Since
\[ \Phi_\lambda^*(A_G^* (Y_\lambda) \otimes \mathbb{Q}) \subset A_G^* ((\mathbb{P}^1)^d)_{\prod_{i=1}^n S_i} \otimes \mathbb{Q} \subset \Lambda^G_P, \]
we have
\[ (\Phi_\lambda)_* \Lambda^G_P \supset (\Phi_\lambda)_* (\Phi_\lambda^*(A_G^* (Y_\lambda)) \otimes \mathbb{Q}) = A_G^* (Y_\lambda) \otimes \mathbb{Q} \]
and by the projection formula, \((\Phi_\lambda)_* \Lambda^G_P\) is
\[ (\Phi_\lambda)_* \sum_{P \in \mathcal{P}} \Delta_P \cap \Phi_\lambda^*(A_G^* (\mathbb{P}^n) \otimes \mathbb{Q}) = \sum_{P \in \mathcal{P}} (\Phi_\lambda)_* \Delta_P \cap \Phi_\lambda^*(A_G^* (\mathbb{P}^n) \otimes \mathbb{Q}). \]
By the projection formula again, we thus have
\[ \Lambda^G_P \otimes \mathbb{Q} = (\Phi_\lambda)_* (A_G^* (Y_\lambda) \otimes \mathbb{Q}) = \sum_{P \in \mathcal{P}} (\Phi_\lambda)_* \Delta_P \cap A_G^* (\mathbb{P}^n) \otimes \mathbb{Q} \]
as desired. \(\square\)

Lemma 8.8. Let \( \lambda = \{\lambda_1, \ldots, \lambda_d\} \) be a partition of \([n]\) and \( \mathcal{P} \) be all partitions of \([d]\). Then
\[ \Lambda^{PGL_2}_{\mathcal{P}} = \begin{cases} A_{PGL_2}^* ((\mathbb{P}^1)^2) \otimes \mathbb{Q} & \text{if } d = 2 \text{ and } \lambda_1 = \lambda_2, \text{ and} \\ A_{PGL_2}^* ((\mathbb{P}^1)^d) \otimes \mathbb{Q} & \text{otherwise.} \end{cases} \]
In particular, given a partition \( \lambda = \{\lambda_1, \ldots, \lambda_d\} \) of \( n \), \( \Lambda^{PGL_2}_{\mathcal{P}} \otimes \mathbb{Q} \) is generated by all \( [\lambda'] \) with \( \lambda' \) formed by merging parts of \( \lambda \).

Proof. Given the description of \( \Lambda^{PGL_2}_{\mathcal{P}} \), the result about \( \Lambda^{PGL_2}_{\mathcal{P}} \otimes \mathbb{Q} \) follows directly from Lemma 8.7. We will now show the description of \( \Lambda^{PGL_2}_{\mathcal{P}} \).

We may identify \( A_{PGL_2}^* ((\mathbb{P}^n) \otimes \mathbb{Q} \subset A_{PGL_2}^* (\mathbb{P}^n) \otimes \mathbb{Q} \) as the subring generated by \( H + \frac{n}{2}(u + v) \) and \( (u - v)^2 \) by Proposition 8.7. Define
\[ H' = H + \frac{n}{2}(u + v) \quad \text{and} \quad H' = H + \frac{n}{2}(u + v). \]
Note that with these definitions, we have
\[ \Phi_\lambda^* \Delta_\lambda = \sum \lambda_i H'_i, \quad H'^2 = \frac{1}{4}(u - v)^2. \]
We have the \( \mathbb{Q} \)-linear span
\[ \Lambda_\mathcal{P} = \text{Span}_\mathbb{Q}(\{\Delta_P (u - v)^{2k} (\sum \lambda_i H'_i)^l \mid k, l \geq 0, P \in \mathcal{P}\}). \]
The trivial partition is in \( \mathcal{P} \), so 1 is automatically in \( \Lambda_\mathcal{P} \).
Recall by Proposition 8.3 that
\[ \Delta_{\iota j} = H'_i + H'_j, \]
and that \( A_{PGL_2}^* ((\mathbb{P}^1)^d) \otimes \mathbb{Q} \) is generated by \( H'_i \) and \((u - v)^2 \). As \( H'^2 = \frac{1}{4}(u - v)^2 \),
to show \( \Lambda_\mathcal{P} = A_{PGL_2}^* ((\mathbb{P}^1)^d) \otimes \mathbb{Q} \) it suffices to show that every monomial \( \prod_{i \in C} H'_i \)
is in \( \Lambda_\mathcal{P} \) for \( C \subset [n] \).
For \( d = 1, \lambda = \{[n]\} \), we are done as \( H'_1 = \frac{1}{\lambda_1} \Phi_\lambda^* \lambda_1 H' \).
For $d = 2$ and $\lambda_1 \neq \lambda_2$, 
\[
H' = \frac{1}{\lambda_1 - \lambda_2} \left( \Phi_1^{\lambda_2} (H') - \lambda_1 \Delta_{1,2} \right)
\]
\[
H'_2 = \frac{1}{\lambda_2 - \lambda_1} \left( \Phi_1^{\lambda_1} (H') - \lambda_2 \Delta_{1,2} \right)
\]
\[
H'_1 H'_2 = \frac{1}{2\lambda_1 \lambda_2} \left( \Phi_1^{\lambda_1} (H')^2 - \frac{1}{4} (\lambda_1^2 + \lambda_2^2) (u-v)^2 \right)
\]

For $d = 2$ and $\lambda_1 = \lambda_2 = a$, we have to show $\Lambda_P = A_{PGL_2}^* ((\mathbb{P}^1)^2)^{S_2} \otimes \mathbb{Q}$. As $H'^2 = \frac{1}{4} (u-v)^2$, it suffices to show $\Lambda_P = A_{PGL_2}^* ((\mathbb{P}^1)^2)^{S_2} \otimes \mathbb{Q}$ when $d \geq 3$.

Up to degree $d - 2$, we can take $k, \ell = 0$ as the classes $\Delta_P$ for $P \in \mathcal{P}$ generate $A_{PGL_2}^{\leq d-2} ((\mathbb{P}^1)^d)$ by Lemma 5.12. Hence to conclude the proof of Lemma 8.8, it suffices to show that $\prod_{k \neq i} H'_k$ for all $i$ and $\prod H'_k$ are in $\Lambda_P$.

For $\prod_{k \neq i} H'_k$, without loss of generality suppose $i = 1$. We have each of
\[
\frac{1}{a_1 a_2} (\prod_{k \neq 1, 2} H'_k) \cap \Phi_1^{\lambda_1} H' = \frac{1}{a_1} \prod_{k \neq 1} H'_k + \frac{1}{a_2} \prod_{k \neq 2} H'_k + \frac{1}{4 a_1 a_2} (u-v)^2 \sum_{j \neq 1, 2} a_j \prod_{k \neq 1, 2} H'_k
\]
\[
\frac{1}{a_1 a_3} (\prod_{k \neq 1, 3} H'_k) \cap \Phi_1^{\lambda_2} H = \frac{1}{a_1} \prod_{k \neq 1} H'_k + \frac{1}{a_3} \prod_{k \neq 3} H'_k + \frac{1}{4 a_1 a_3} (u-v)^2 \sum_{j \neq 1, 3} a_j \prod_{k \neq 1, 3} H'_k
\]
\[
\frac{1}{a_2 a_3} (\prod_{k \neq 2, 3} H'_k) \cap \Phi_1^{\lambda_3} H = \frac{1}{a_2} \prod_{k \neq 2} H'_k + \frac{1}{a_3} \prod_{k \neq 3} H'_k + \frac{1}{4 a_2 a_3} (u-v)^2 \sum_{j \neq 1, 3} a_j \prod_{k \neq 1, 3} H'_k
\]

lie in $\Lambda_P$ as we have already shown each $\prod_{k \neq i,j} H'_k$ lies in $\Lambda$. Also, the last term on each right hand side lies in $\Lambda_P$ as the number of terms in the $H'_k$ monomial is $d - 3$. Hence taking a linear combination we get $\prod H'_k \in \Lambda_P$.

To show $\prod_{i=1}^{n} H'_i \in \Lambda_P$, we can proceed similarly to above, or expand
\[
\frac{1}{a_1 \ldots a_n} \Phi_1^{\lambda_1} (H')^d = \prod H'_i + (u-v)^2 \text{ (lower order terms in the } H'_i),
\]
using $H'^2 = \frac{1}{4} (u-v)^2$. □

**Proof of Theorem 8.7** This follows from the excision exact sequence [17 Proposition 1.8] and Lemma 8.8. □

**Lemma 8.9.** Let $\lambda = \{a, 1^b\}$ be a partition of $n$. Define $\mathcal{P}_\lambda$ to be the set of partitions
\[
\mathcal{P}_\lambda = \{T\} \cup \left\{ \begin{array}{ll}
\{T_{1,i}\}_{i \geq 2} & a \neq b \\
\{T_{i,j}\}_{2 \leq i < j \leq n} & a = b 
\end{array} \right\}
\]
where $T$ is the trivial partition and $T_{i,j}$ is the partition with $n-1$ parts and $i, j$ in the same part. Then
\[
A_{PGL}^{\mathcal{P}_\lambda} = A_{PGL_2}^* ((\mathbb{P}^1)^{b+1})^{S_i \times S_b} \otimes \mathbb{Q}.
\]
Proof. Define

\[ H' = H + \frac{n}{2}(u + v) \quad \text{and} \quad H_i' = H_i + \frac{1}{2}(u + v). \]

Then in particular,

\[ \Delta_{i,j} = H_i' + H_j' \]

\[ \Phi_{\lambda i,\lambda}(H') = aH_i' + H_2' + \ldots + H_{b+1}' , \]

so \( \Lambda_{P,\lambda} \) is the \( \mathbb{Q} \)-linear span

\[ \Lambda_{P,\lambda} = \text{Span}_\mathbb{Q}\{ \Delta_P(u - v)^{2k}(aH_1' + H_2' + \ldots + H_{b+1}')^\ell \mid k, \ell \geq 0, \ P \in P_{\lambda} \}. \]

We first show that \( H_i' \in \Lambda_{P,\lambda} \). Consider the case \( b \neq a \). Then

\[ H_i' = \frac{1}{a - b} \left( \Phi_{\lambda i,\lambda}(H') - \sum_{i \geq 2} \Delta_{1,i} \right) \in \Lambda_{P,\lambda}. \]

Now consider the case \( b = a \). Then

\[ H_i' = \frac{1}{a} \left( \Phi_{\lambda i,\lambda}(H') - \frac{1}{a - 1} \sum_{2 \leq i < j \leq a + 1} \Delta_{i,j} \right) \in \Lambda_{P,\lambda}. \]

Now that we have shown that \( H_i' \in \Lambda \), it therefore suffices to show that the invariant subring \( A^*_{\text{PGL}_2}(\mathbb{P}^{1+b+1})_{S_i \times S_k} \) is given by

\[ \text{Span}_\mathbb{Q}\{(u - v)^{2k}(aH_1' + H_2' + \ldots + H_{b+1}')^\ell, \ H_i'(u - v)^{2k}(aH_1' + H_2' + \ldots + H_{b+1}')^\ell \mid k, \ell \geq 0 \}. \]

Note that by using the relation \( H_i'^2 = \frac{1}{4}(u - v)^2 \), we see this is the same as

\[ \text{Span}_\mathbb{Q}\{H_i'^{2k}(aH_1' + H_2' + \ldots + H_{b+1}')^\ell \mid k, \ell \geq 0 \} = \text{Span}_\mathbb{Q}\{H_i'^{2k}(H_2' + \ldots + H_{b+1}')^\ell \mid k, \ell \geq 0 \} \]

\[ = \text{Span}_\mathbb{Q}\{(u - v)^{2k}(H_i' + H_2' + \ldots + H_{b+1}')^\ell, \ H_i'(u - v)^{2k}(H_i' + H_2' + \ldots + H_{b+1}')^\ell \mid k, \ell \geq 0 \}. \]

By using the relations \( H_i'^2 = \frac{1}{4}(u - v)^2 \) whenever possible, we see that an element of the invariant subring is a sum of terms of the form \((u - v)^{2k}e_j(H_2', \ldots, H_{b+1}') \) and \((u - v)^{2k}H_i'e_j(H_2', \ldots, H_{b+1}') \) where \( e_j \) is the \( j \)th elementary symmetric polynomial, hence it suffices to show that

\[ \text{Span}_\mathbb{Q}\{(u - v)^{2k}e_j(H_2', \ldots, H_{b+1}') \mid j, k \geq 0 \} \subset \text{Span}_\mathbb{Q}\{(u - v)^{2k}(H_2' + \ldots + H_{b+1}')^\ell \mid k, \ell \geq 0 \}. \]

This follows by induction on \( j \) and the relation

\[ e_j(H_2', \ldots, H_{b+1}')(H_2' + \ldots + H_{b+1}') = (j + 1)e_{j+1}(H_2' + \ldots + H_{b+1}') + \frac{1}{4}(u - v)^2(n - j + 1)e_{j-1}(H_2', \ldots, H_{b+1}'). \]

Proof of Theorem 8.2. This follows from the excision exact sequence [17] [Proposition 1.8], Lemma 8.7 and Lemma 8.9.
9. Excision of unordered strata in \([\text{Sym}^n\mathbb{P}^1/GL_2]\) and \([\text{Sym}^nK^2/GL_2]\)

In this section, we show how our results about excision of unordered strata in \([\text{Sym}^n\mathbb{P}^1/PGL_2]\) imply similar results in \([\text{Sym}^n\mathbb{P}^1/GL_2]\) and \([\text{Sym}^nK^2/GL_2]\), recovering and extending some results of [13] (see Remark 9.3).

**Definition 9.1.** Given a partition \(\lambda\) of \(n\), let \(\overline{T}_\lambda\) be the ideal of \(A^*_{GL_2}(\mathbb{A}^{n+1})\) given by the image of the pushforward \(A^*_{GL_2}(\overline{Z}_\lambda) \to A^*_{GL_2}(\mathbb{A}^{n+1})\) and the identification \(A^*_{GL_2}(\mathbb{A}^{n+1}) \cong A^*_{GL_2}(\mathbb{A}^{n+1})\) via Poincaré duality [9, Proposition 4].

**Theorem 9.2.** \(I^{GL_2}_\lambda \otimes \mathbb{Q}\) (respectively \(\overline{T}_\lambda \otimes \mathbb{Q}\)) is generated by all \([Z_\lambda]\) (respectively \([\overline{Z}_\lambda]\)) with \(\lambda\) formed by merging parts of \(\lambda\) for \(\lambda = \{a, 1^{n-a}\}\) only two generators are required, namely \([Z_\lambda]\) (respectively \([\overline{Z}_\lambda]\)) and \([Z_{\lambda'}]\) (respectively \([\overline{Z}_{\lambda'}]\)) where

\[\lambda' = \begin{cases} \{a+1, 1^{n-a-1}\} & \text{if } a \neq \frac{n}{2} \\ \{a, 1^{n-a-2}\} & \text{if } a = \frac{n}{2}. \end{cases}\]

**Remark 9.3.** In the affine case, when \(n\) is odd and \(a = \frac{n}{2}\) this recovers [13, Theorem 4.3], and when \(n\) is even and \(a = \frac{n}{2}\) this recovers the rational Chow ring of the stable locus in [13, Theorem 4.10].

**Lemma 9.4.** We have

\[Q[u,v]^{S_2} (A^*_{PGL_2}(\mathbb{P}^1 \otimes \mathbb{Q})^{S_4\times \cdots \times S_k}) = (A^*_{GL_2}(\mathbb{P}^1 \otimes \mathbb{Q})^{S_4\times \cdots \times S_k}) \quad \text{and} \quad Q[u,v]^{S_2} (A^*_{PGL_2}(\mathbb{P}^n \otimes \mathbb{Q}) = A^*_{GL_2}(\mathbb{P}^n \otimes \mathbb{Q})\]

In particular, if a set of partitions \(\mathcal{P}\) satisfies the hypotheses of Lemma 9.1 for \(G = PGL_2\), then they also satisfy the hypotheses of Lemma 9.7 for \(G = GL_2\).

**Proof.** We identify \(A^*_{PGL_2}(\mathbb{P}^1 \otimes \mathbb{Q})\) as the subring of \(A^*_{GL_2}(\mathbb{P}^1 \otimes \mathbb{Q})\) via Proposition 3.7 generated by \(H' := H + \frac{u+v}{2}(u+v)\) and \((u-v)^2\). Since \(A^*_{GL_2}(\mathbb{P}^1 \otimes \mathbb{Q})\) is generated by \(H'\) over \(Q[u,v]^{S_2}\), and \((u-v)^2\) and \(u+v\) generate \(Q[u,v]^{S_2}\), \(Q[u,v]^{S_2} (A^*_{PGL_2}(\mathbb{P}^n \otimes \mathbb{Q}) = A^*_{GL_2}(\mathbb{P}^n \otimes \mathbb{Q})\).

For the other equality, we use Theorem 3.3 to identify \(A^*_{PGL_2}(\mathbb{P}^n \otimes \mathbb{Q})\) as the subring of \(A^*_{GL_2}(\mathbb{P}^n \otimes \mathbb{Q})\) generated by \(H'_i := H_i + \frac{u+v}{2}\). Then, \(A^*_{PGL_2}(\mathbb{P}^1 \otimes \mathbb{Q})\) is generated \(\mathbb{Z}\)-linearly by all \(p(H'_1, \ldots, H'_n)\), where \(p\) is a polynomial invariant under the action of \(S_4\times \cdots \times S_k\). Similarly, \(A^*_{GL_2}(\mathbb{P}^1 \otimes \mathbb{Q})\) is generated by all such \(p(H'_1, \ldots, H'_n)\), together with \(u+v\) and \(uv\). Therefore, \(Q[u,v]^{S_2} (A^*_{PGL_2}(\mathbb{P}^1 \otimes \mathbb{Q}) = A^*_{GL_2}(\mathbb{P}^1 \otimes \mathbb{Q})\)

As we will now see, the cones over generators of \(I^{GL_2}_\lambda \otimes \mathbb{Q}\) also generate \(\overline{T}_\lambda \otimes \mathbb{Q}\). We will use a certain property about the classes of unordered strata to prove this, which as we will see is that \(Z_\lambda\) contains a cycle whose class divides the class of the origin in \(A^*_{GL_2}(\mathbb{A}^{n+1}) \otimes \mathbb{Q}\).

**Lemma 9.5.** Given a partition \(\lambda\) of \(n\) and a set of generators \(S\) of \(I^{GL_2}_\lambda \otimes \mathbb{Q}\) of degree at most \(n\), \(\overline{T}_\lambda \otimes \mathbb{Q}\) is generated by

\[\{\alpha_0 | \alpha \in S\},\]

where \(\alpha_0\) is the constant term of \(\alpha \in A^*_{GL_2}(\mathbb{P}^n) \otimes \mathbb{Q}\), after writing \(\alpha\) as a polynomial in \(H, u, v\) that is degree at most \(n\) in \(H\) using the relation \(G(H) = 0\) (see Section 2.3).
Proof. Let $I_\lambda \subset A^*_\text{GL}_2(\mathbb{A}^{n+1}) \otimes \mathbb{Q}$ be the ideal generated by $\{\alpha_0 \mid \alpha \in S\}$, so we want to show $I_\lambda = \overline{I}_\lambda \otimes \mathbb{Q}$. Consider the diagram of rational Chow rings (we omit $\otimes \mathbb{Q}$ for brevity)

\[
\begin{array}{cccc}
A^*_\text{GL}_2(\mathbb{P}^n) & \sim & A^*_\text{GL}_2(\mathbb{A}^{n+1}\backslash \{0\}) & \sim & A^*_\text{GL}_2(\mathbb{A}^{n+1}\backslash \{0\}) & \sim & A^*_\text{GL}_2(\mathbb{A}^{n+1}) \\
\downarrow \pi_1 & & \downarrow \pi_2 & & \downarrow \pi_3 & & \downarrow \pi_4 \\
A^*_\text{GL}_2(\mathbb{P}^n \backslash Z_\lambda) & \sim & A^*_\text{GL}_2(\mathbb{A}^{n+1}\backslash \tilde{Z}_\lambda) & \sim & A^*_\text{GL}_2(\mathbb{A}^{n+1}\backslash \tilde{Z}_\lambda) \\
\end{array}
\]

where $\mathbb{G}_m$ acts by Proposition 4.1 and Section 4.4, which divides $I_\lambda$ by $I_\lambda \otimes \mathbb{Q}$. We know $I_\lambda \otimes \mathbb{Q}$ is the kernel of $\pi_1$, so it maps surjectively to the kernel of $\pi_3$ in $A^*_\text{GL}_2(\mathbb{A}^{n+1}\backslash \{0\})$. Each generator $\alpha \in S$ maps to the image of $\alpha_0$ in $A^*_\text{GL}_2(\mathbb{A}^{n+1}\backslash \{0\}) \otimes \mathbb{Q}$. Since the kernel of $A^*_\text{GL}_2(\mathbb{A}^{n+1}) \otimes \mathbb{Q} \to A^*_\text{GL}_2(\mathbb{A}^{n+1}\backslash \{0\}) \otimes \mathbb{Q}$ is generated by $\prod_{i=0}^n (iu + (n-i)v)$, we have $\overline{I}_\lambda + \langle \prod_{i=0}^n (iu + (n-i)v) \rangle = \overline{I}_\lambda \otimes \mathbb{Q}$. To finish, it suffices to see $\prod_{i=0}^n (iu + (n-i)v) \in \overline{I}_\lambda$.

As $Z([n])$ is a cycle in $Z_\lambda$, $\langle [n] \rangle$ can be expressed as an $A^*_\text{GL}_2(\mathbb{P}^n) \otimes \mathbb{Q}$-linear combination of the elements of $S$, and taking the constant terms yields

$$\langle [n] \rangle_0 = n \prod_{i=1}^{n-1} (iu + (n-i)v) \in \overline{I}_\lambda$$

by Proposition 4.1 and Section 4.4, which divides $\prod_{i=0}^n (iu + (n-i)v)$. □

Proof of Theorem 9.2. Apply Lemma 9.4 to Lemmas 8.8 and 8.9 to get the statements on $I^\text{GL}_2 \otimes \mathbb{Q}$. Then, apply Lemma 9.5 to get the statements on $I_\lambda$. □

Appendix A. Multiplicative relations between symmetrized strata

In this section, we investigate certain multiplicative relations between the classes $[Z_\lambda] \in A^*_\text{GL}_2(\text{Sym}^n K^2)$. These are equivalent to certain relations between the degree 0 terms of the expressions for $[\lambda] \in A^*_\text{GL}_2(\mathbb{P}^n)$ by Section 2.3. For this, it suffices to restrict ourselves to the $\mathbb{Q}$-basis given by the $[a, b, 1^c]$-classes from Theorem 6.4.

Definition A.1. Denote by $[a, b, 1^c]_0 \in \mathbb{Z}[u, v]^{S_2}$ be the term of $[a, b, 1^c] \in H^*_\text{GL}_2(\mathbb{P}^n)$ that is degree zero in $H$.

We show how to write $(u + v)[a, b, 1^c]_0$ and $uv[a, b, 1^c]_0$ as a $\mathbb{Q}$-linear combination of strata. A few of these multiplicative relations have been explicitly written down [13, Remark 3.9] and shown to exist abstractly [13, Theorems 4.3 and 4.10] using the degeneration of a spectral sequence of a filtered CW-complex. We give a combinatorial method to do this in general in Theorems A.2 and A.3.

Theorem A.2. For $c \geq 1$ and $a + b + c = n$,

\[
\begin{align*}
(nu + v)[a, b, 1^c]_0 &= (c + a - b)[a + 1, b, 1^{c-1}]_0 \\
&
\quad + (b + c - a)[a, b + 1, 1^{c-1}]_0 \\
&
\quad + (a + b - c)[a + b, 1, 1^{c-1}]_0.
\end{align*}
\]
Proof. We will prove Theorem A.2 by pulling back to \((\mathbb{P}^1)^n\). By Lemma 2.8 we want to show

\[
(2H + nu + nv)[a, b, 1^c] = (c + a - b)[a + 1, b, 1^{c-1}] + (b + c - a)[a, b + 1, 1^{c-1}] + (a + b - c)[a + b, 1, 1^{c-1}].
\]

Let \(A = \{1, \ldots, a\}\), \(B = \{b + 1, \ldots, a + b\}\). By the projection formula, the right hand side is

\[
\Phi_*(\Delta_{(A,B)} \cap \Phi^*(2H + nu + nv)).
\]

The pullback of \(2H + nu + nv\) along \(\Phi\) is

\[
(H_1 + H_2 + u + v) + (H_2 + H_3 + u + v) + \ldots + (H_n + H_3 + u + v)
= \Delta_{1,2} + \Delta_{2,3} + \ldots + \Delta_{n,1}
\]
by Proposition 4.3. In this way, we now only have to intersect strata using Proposition 4.2 and the square relation as in Proposition 5.2.

There are 6 cases: \(1 \leq i \leq a - 1\), \(i = a\), \(a + 1 \leq i \leq a + b - 1\), \(i = a + b\), \(a + b + 1 \leq i \leq n - 1\), and \(i = n\). We will deal with each of these cases in the same way outlined above.

To calculate \(\Phi_*(\Delta_{i,i+1}\Delta_{(A,B)})\) for \(1 \leq i \leq a - 1\), we use the square relation to replace \(\Delta_{i,i+1}\) with \(\Delta_{i,n} - \Delta_{n,a+1} + \Delta_{a+1,i+1}\). Using Proposition 4.2 each of the products is itself a strata, and the pushforward is

\[
[a + 1, b, 1^{c-1}] - [a, b + 1, 1^{c-1}] + [a + b, 1, 1^{c-1}].
\]

For \(i = a\), Proposition 4.2 implies \(\Delta_{a,a+1}\Delta_{A,B} = \Delta_{(A\cup B)}\), which pushes forward to

\[
[a + b, 1, 1^{c-1}].
\]

Similarly to before, for \(a + 1 \leq i \leq a + b - 1\), the pushforward is

\[
[a, b + 1, 1^{c-1}] - [a + 1, b, 1^{c-1}] + [a + b, 1, 1^{c-1}].
\]

For \(i = a + b\), Proposition 4.2 implies \(\Delta_{a+b,a+b+1}\Delta_{(A,B)} = \Delta_{(A\cup B)\cup(a+b+1)}\), which pushes forward to

\[
[a, b + 1, 1^{c-1}].
\]

For \(a + b + 1 \leq i \leq n - 1\), replace \(\Delta_{i,i+1}\) with \(\Delta_{i,a} - \Delta_{a,a+1} + \Delta_{a+1,i+1}\), and similarly to before we get the pushforward is

\[
[a + 1, b, 1^{c-1}] - [a + b, 1, 1^{c-1}] + [a, b + 1, 1^{c-1}].
\]

Finally, for \(i = n\), using Proposition 4.2 \(\Delta_{n,1}\Delta_{(A,B)} = \Delta_{(A\cup (n), B)}\), so this will pushforward to

\[
[a + 1, b, 1^{c-1}].
\]

Combining these yields the desired result. \(\square\)
Remark A.3. Given a partition $\lambda$ of $n$ with at least three nontrivial parts, the argument of Theorem A.2 is a combinatorial algorithm that can non-canonically express $n(u + v)[\lambda]$ in terms of other classes $[\lambda']$ with one fewer part. The number of square relations can be drastically reduced in practice by an appropriate choice of the partition pushing forward to $[a_1, \ldots, a_d]$.

Theorem A.4. For $c \geq 2$, and $a + b + c = n$

$$n^2uv[a, b, 1]_0 = (2ab + ac + bc + c(c - 1))[a + 1, b + 1, 1^{c-2}]_0$$

$$+ (-ab - bc)[a + 2, b, 1^{c-2}]_0$$

$$+ (-ab - ac)[a, b + 2, 1^{c-2}]_0$$

$$+ (-ac - bc - c(c - 1))[a + b + 1, 1, 1^{c-2}]_0$$

$$+ (ac + bc)[a + b, 2, 1^{c-2}]_0.$$ 

Proof. As in the previous theorem letting $A = \{1, \ldots, a\}$, $B = \{a + 1, \ldots, a + b\}$ the statement is equivalent to

$$\Phi_*((\sum H_i + nu)(\sum H_i + nv)\Delta_{(A, B)})$$

$$= (2ab + ac + bc + c(c - 1))[a + 1, b + 1, 1^{c-2}]$$

$$+ (-ab - bc)[a + 2, b, 1^{c-2}]$$

$$+ (-ab - ac)[a, b + 2, 1^{c-2}]$$

$$+ (-ac - bc - c(c - 1))[a + b + 1, 1, 1^{c-2}]$$

$$+ (ac + bc)[a + b, 2, 1^{c-2}].$$

We have $(H_i + u)(H_i + v) = 0$, so

$$\sum H_i + nu)(\sum H_i + nv) = \sum_{1 \leq i < j \leq n} (H_i + u)(H_j + v) + (H_j + u)(H_i + v)$$

$$= \sum_{1 \leq i < j \leq n} -(H_i - H_j)^2$$

$$= \sum_{1 \leq i < j \leq n} -(\Delta_i, k_{i,j} - \Delta_j, k_{i,j})^2$$

where $k_{i,j} \in [n] \setminus \{i, j\}$ is arbitrary. There are 6 cases depending on which of $A, B, [n] \setminus \{A, B\}$ each of $i, j$ lie in, and for each of these cases an appropriate choice of $k_{i,j}$ can be made so that the strata combine via Proposition 4.2 as in the proof of Theorem A.2 and push forward to $[a', b', 1^{c-2}]$-classes. \hfill $\square$

Remark A.5. Similarly to Theorem A.2, the argument of Theorem A.4 is a combinatorial algorithm that can express $n^2uv[\lambda]$ in terms of other classes $[\lambda']$ with two fewer parts for any partition $\lambda$ of $n$ with at least four parts.

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