Localization of Vortex Partition Functions in $\mathcal{N} = (2, 2)$ Super Yang-Mills theory

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Abstract

In this article, we study the localization of the partition function of BPS vortices in $\mathcal{N} = (2, 2)$ $U(N)$ super Yang-Mills theory with $N$-flavor on $\mathbb{R}^2$. The vortex partition function for $\mathcal{N} = (2, 2)$ super Yang-Mills theory is obtained from the one in $\mathcal{N} = (4, 4)$ super Yang-Mills theory by mass deformation. We show that the partition function can be written as $Q$-exact form and integration in the partition functions is localized to the fixed points which are related to $N$-tuple one dimensional partitions of positive integers.
1 Introduction

The vortices in two dimensional field theory are instanton like objects and have finite energy and action. We can consider the partition functions of vortices. In supersymmetric theories with eight or four supercharges, our interest is the vortices described by Bogomol’nyi equation which preserve half of the supersymmetries. The vortex moduli spaces in $p$-dimensional theories with eight supercharges are constructed in $D(p-2)-Dp$ system in [1], as the Kähler quotient spaces. Especially, the vortex partition functions for $\mathcal{N}=(4,4)$ super Yang-Mills theories with vortex number $k$ are $U(k)$ gauged matrix model with four supercharges [1, 2]. But, it is difficult to perform multi-variable integrations directly in vortex partition functions.

Another road to evaluate the vortex partitions is the reduction from instanton partition functions in four dimensional $\mathcal{N}=2$ super Yang-Mills theory with surface operators. The instanton partition functions with surface operators are expanded in double series with respect to vortex number and instanton number. When surface operators are half-BPS in $U(N)\mathcal{N}=2$ super Yang-Mills theory, the theory on the plane where the surface operators define becomes $\mathcal{N}=(2,2) U(1)$ gauge theory with matter fields. The insertion of surface operators in four dimensional gauge theory correspond to introduction of open string in topological A-model amplitudes in toric Calabi-Yau 3-folds [3] which can be calculated by refined topological vertex [4]. The authors of [5] calculated abelian vortex partition functions in $\mathcal{N}=(2,2)$ SQED by taking decoupling limit of instaton parts.

In this article, we consider the evaluation of the vortex partition functions for $\mathcal{N}=(2,2) U(N)$ gauge theory in two dimensions with $N_f=N$ fundamental chiral multiplets and with generic twisted masses. This article is organized as follow: In section two, we first explain the Bogomol’nyi equation in 2d $\mathcal{N}=(4,4)$ supersymmetric gauge theory and present vortex partition functions. In section three, deforming the $\mathcal{N}=(4,4)$ theory by adding a superpotential term and taking large mass limit, we obtain the vortex partition functions in $\mathcal{N}=(2,2)$ theory. In section four, we evaluate the vortex partitions by equivariant localization method. In section five,
we also calculate K-theoretic vortex partition function from the equivariant character of the Kähler quotient space. In section six, we discuss our results.

2 Vortex in $\mathcal{N} = (4, 4)$ $U(N)$ super Yang-Mills theories

In this section, we review $\mathcal{N} = (4, 4)$ $U(N_c)$ super Yang-Mills theory in two dimensions \cite{[3]} with the number of hypermultiplets $N_f = N_c = N$. We can construct $\mathcal{N} = (4, 4)$ multiplets by combining a pair of $\mathcal{N} = (2, 2)$ multiplets. The vector multiplet in $\mathcal{N} = (4, 4)$ consists of a pair of $\mathcal{N} = (2, 2)$ superfields $(\Sigma, \Phi)$.

\[
\begin{align*}
\Sigma &= \sigma + i\sqrt{2}\theta^+ \lambda + i\sqrt{2}\theta^- \bar{\lambda} + \frac{1}{\sqrt{2}}\theta^+ \theta^-(D^3 - iF_{12}) + \cdots, \\
\Phi &= \tilde{\sigma} + \sqrt{2}\theta^+ \tilde{\lambda} + i\sqrt{2}\theta^- \tilde{\bar{\lambda}} + \frac{1}{\sqrt{2}}\theta^+ \theta^-(D^1 - iD^2) + \cdots, 
\end{align*}
\]

(2.1)

where $\Sigma$ is the twisted chiral multiplet and $\Phi$ is the chiral multiplet in adjoint representation in $U(N_c)$. Here $\sigma$ and $\tilde{\sigma}$ are scalars, $\lambda$ and $\tilde{\lambda}$ are fermions. $D^i (i = 1, 2, 3)$ are auxiliary fields. The hypermultiplet in $\mathcal{N} = (4, 4)$ consists of a pair of $\mathcal{N} = (2, 2)$ superfields $(Q_i, \tilde{Q}_i), (i = 1, \ldots, N_f)$.

\[
\begin{align*}
Q_i &= q_i + \sqrt{2}\theta^+ \psi_i + \sqrt{2}\theta^- \bar{\psi}_i + \theta^+ \theta^- F_i + \cdots, \\
\tilde{Q}_i &= \tilde{q}_i + \sqrt{2}\theta^+ \tilde{\psi}_i + \sqrt{2}\theta^- \tilde{\bar{\psi}}_i + \theta^+ \theta^- \tilde{F}_i + \cdots, 
\end{align*}
\]

(2.2)

where $Q$ are chiral multiplets in fundamental representation for $U(N_c)$ and $N_f$ flavor. $\tilde{Q}$ are chiral multiplets in anti-fundamental representation for $U(N_c)$ and $N_f$ flavor.

The Lagrangian for 2d $\mathcal{N} = (4, 4)$ super Yang-Mills theory with gauge group $U(N)$ and $N$ flavor consists of

\[
\begin{align*}
S_K &= \int d^2 x d^4 \theta (\text{Tr} \Sigma \Sigma + \text{Tr} \Phi e^V \Phi e^{-V} + Q_i e^{-2V} Q_i + \tilde{Q}_i e^{2V} \tilde{Q}_i), \\
S_V &= \int d^2 x d^2 \theta \bar{Q}_i \Phi Q_i.
\end{align*}
\]

(2.3)

We can also include the Fayet-Iliopoulos terms and the bosonic part in the Lagrangian becomes

\[
L_{\text{boson}} = -\left[\text{Tr} \left( \frac{1}{2e^2} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2e^2} D_\mu \sigma D^\mu \bar{\sigma} + \frac{1}{2e^2} D_\mu \bar{\sigma} D^\mu \sigma + e^2 \left(q_i q_i^\dagger - \tilde{q}_i \tilde{q}_i^\dagger - r \bar{N}_N \right)^2 \right) \right]
\]

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$$+D_\mu q_i^\dagger D^\mu q_i + D_\mu \tilde{q}_i^\dagger D^\mu \tilde{q}_i + \tilde{q}_i^\dagger \{\bar{\sigma}, \sigma\} \tilde{q}_i + \cdots.$$ \hfill (2.4)

When the Fayet-Iliopoulos parameter $r$ is positive, the vacuum is unique up to Weyl symmetry,

$$q_i^a = \sqrt{r} \delta_i^a, \quad \tilde{q}_i^a = 0, \quad \sigma = 0, \quad \tilde{\sigma} = 0,$$ \hfill (2.5)

where we write color indices $a (a = 1, \cdots, N)$ and flavor indices $i (i = 1, \cdots, N)$ explicitly. In this phase, gauge and flavor symmetry group breaks to little symmetry group

$$U(N)_G \times SU(N)_F \rightarrow SU(N)_{\text{diag}}.$$ \hfill (2.6)

In this case, half-BPS equation exist which minimize the energy. This half-BPS solutions of equation(Bogomoln’yi equation) are defined by

$$F_{12b}^a = \frac{e^2}{2} (q_i^a q_i^{\dagger b} - r \delta_i^b),$$

$$(D_z q)^a_i = 0,$$ \hfill (2.7)

where we define covariant derivative $D_z = D_1 - iD_2$.

The vortex instanton number is given by the first Chern number $k$;

$$k := \frac{1}{2\pi} \int \text{Tr} F.$$ \hfill (2.8)

The bosonic solution with vortex number-$k$ has action

$$S_k = 2\pi (r + i\theta)k.$$ \hfill (2.9)

Next we consider the $k$-vortex partition function for $\mathcal{N} = (4, 4)$ super Yang-Mills theories. In two dimensions, the vortex partition function in $\mathcal{N} = (4, 4)$ Yang-Mills theories is constructed by $k$ D0-branes and $N$ D2-branes system. This is called the vortex matrix model \cite{2}.

This model is obtained by dimensional reduction of $\mathcal{N} = 1$ supersymmetric action in four dimensions to zero dimension or equivalently dimensional reduction of $\mathcal{N} = (2, 2)$ supersymmetric theory in two dimensions to zero dimension.
This matrix model is given by
\[ Z_{k,N}^{(4,4)} = \frac{1}{\text{Vol}(U(k))} \int \mathcal{D}X \exp(-\frac{1}{g^2} S_G - S_m - S_A), \] (2.10)
where
\[ S_G = \text{Tr} \left( -\frac{1}{2} \left[ \varphi, \bar{\varphi} \right]^2 - \frac{1}{2} \left[ \phi, \bar{\phi} \right]^2 + \left[ \varphi, \bar{\phi} \right] \left[ \varphi, \phi \right] + \left[ \varphi, \bar{\phi} \right] \left[ \varphi, \phi \right] + D^2 + g^2 \zeta D 
+ \sqrt{2} (\lambda_+ \left[ \lambda_+, \lambda_+ \right] + \lambda_- \left[ \lambda_+, \lambda_- \right] - \bar{\lambda}_+ \left[ \lambda_+, \lambda_- \right] - \bar{\lambda}_- \left[ \lambda_+, \lambda_- \right] ) \right), \]
\[ S_m = \sum_{i=1}^{N} \left( -I_i^\dagger \{ \varphi, \bar{\varphi} \} I_i - I_i^\dagger \{ \phi, \bar{\phi} \} I_i + I_i^\dagger DI 
+ \sqrt{2} (\mu_{i+}^\dagger \bar{\varphi} \mu_{i+} + \mu_{i-}^\dagger \varphi \mu_{i-} - \mu_{i+}^\dagger \bar{\phi} \mu_{i-} - \mu_{i-}^\dagger \phi \mu_{i+} ) 
+ i \sqrt{2} \left[ I_i^\dagger \left( \lambda_+ \mu_{i+} - \lambda_- \mu_{i-} \right) + \left( \mu_{i+}^\dagger \bar{\lambda}_- - \mu_{i-}^\dagger \bar{\lambda}_+ \right) I_i \right] \right), \]
\[ S_A = \text{Tr} \left( \left[ \left[ \varphi, B \right]^\dagger \right]^2 + \left[ \left[ \phi, B \right]^\dagger \right]^2 + D \left[ B, B \right]^\dagger \right) 
+ \sqrt{2} \left( -\left[ \varphi, \rho_- \right]^\dagger \rho_- - \left[ \bar{\varphi}, \rho_+ \right]^\dagger \rho_+ \right) \left[ \phi, \rho_+^\dagger \right] \rho_+ - \left[ \bar{\phi}, \rho_+^\dagger \right] \rho_+ 
+ i \sqrt{2} \left[ B \left( \left[ \rho_+^\dagger, \bar{\lambda}_+ \right] + \left[ \rho_-^\dagger, \bar{\lambda}_- \right] \right) + B^\dagger \left( \left[ \lambda_+, \rho_- \right] + \left[ \lambda_-, \rho_+ \right] \right) \right) \right). \] (2.11)

\((\varphi, \bar{\varphi}, \phi, \bar{\phi}, \lambda, \bar{\lambda}, D)\) come from the 2d \(N = (2, 2)\) vector multiplet in the reduction. \((I_i, \mu_{\pm i}), (i = 1, \cdots, N)\) appear through dimensional reduction of the 2d chiral multiplets to zero dimension, which belong to fundamental representation of \(U(k)\). \((B, \rho_{\pm})\) is the dimensional reduction of 2d chiral multiplet to zero dimension, adjoint representation of \(U(k)\). \(\mathcal{D}X\) is the integration measure. The Fayet-Illioporous parameter in the vortex matrix model is related to 2d gauge coupling
\[ \zeta = \frac{2\pi}{e^2}. \] (2.12)

In order to decouple gravity from the gauge theories, the gauge coupling constant \(g\) in vortex matrix model goes to infinity. Then, the fields \(\lambda\) and \(D\) become the Lagrangian multipliers and produce the constraint;
\[ [B, B]^\dagger + \sum_{i=1}^{N} I_i I_i^\dagger = \zeta \mathbb{I}_k. \] (2.13)

Solutions of this equation characterize the moduli space for \(k\)-vortex
\[ \mathcal{M}_{k,N} = \{(B, I_i)[[B, B]^\dagger + \sum_{i=1}^{N} I_i I_i^\dagger = \zeta \mathbb{I}_k]\}/U(k). \] (2.14)
We can define the vortex partition function by using (2.10)

\[ Z^{N=(4,4)} = 1 + \sum_{k=1}^{\infty} Z_{k,N}^{N=(4,4)} e^{2\pi(r+i\theta)k}. \] (2.15)

So far we have considered only \( N_c = N_f \) case, but vortex partition functions for general \( N_c = N < N_f \) is already given in [1], [2]. In general \( N_f \) flavor case, the vortex partition functions are described by introducing additional \( N_f - N_c \) dimensional reduced 2d chiral multiplets to zero dimension. \( (\tilde{I}_j, \tilde{\mu}_j), (j = 1, \cdots, N_f - N) \). The moduli space for \( k \)-vortex is modified to

\[ \mathcal{M}_{k,N_c,N_f} = \{(B, I_i, \tilde{I}_j)| [B, B^\dagger] + \sum_{i=1}^{N_c} I_i I_i^\dagger - \sum_{j=1}^{N_f-N_c} \tilde{I}_j \tilde{I}_j^\dagger = \zeta \mathbb{I}_k\}/U(k). \] (2.16)

But, for simplicity, we restrict our attention to \( N_c = N_f \) type vortices.

### 3 \( \mathcal{N} = (2,2) \) vortex partition function

In order to obtain \( \mathcal{N} = (2,2) \) vortex partition functions, we add a superpotential \( \hat{W}(\Phi) \) to the 2d theory in (2.3).

\[ \int d^2x d^2\theta \text{Tr} \hat{W}(\Phi). \] (3.1)

This breaks the \( \mathcal{N} = (4,4) \) supersymmetry to \( \mathcal{N} = (2,2) \) in two dimensions. The superpotential contains the quadratic term

\[ \hat{W}(\Phi) = M\Phi^2, \] (3.2)

In the heavy mass limit, the bosonic part of Lagrangian in two dimensions is obtained by setting \( \Phi = 0 \) in (2.3). The vacuum in Higgs branch is still given by

\[ q^a_i = \sqrt{r} \delta^a_i, \quad \tilde{q}^a_i = 0, \quad \sigma = 0. \] (3.3)

Apparently, the \( \tilde{Q} \) fields are trivial, the vortex equation is equivalently to vortex for the following action

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This is bosonic parts of the Lagrangian for $\mathcal{N} = (2, 2)$ $U(N)$ super Yang-Mills with $N$ fundamental chiral multiplets.

Let us consider how the superpotential deformation (3.2) affects the vortex matrix model (2.11). The vortices preserve the half of supersymmetry, so the vortex matrix model is deformed to preserve half of the $\mathcal{N} = (2, 2)$ supersymmetry. As discussed in [7], in the presence of superpotential, vortex matrix model is deformed by adding

$$\text{Tr} \int d\theta^+ \Xi \left. \frac{\partial^2 \hat{W}(\Sigma^{(0,2)})}{\partial \Sigma^{(0,2)}} \right|_{\theta^+=0} + (\text{h.c.})$$

(3.5)

Here $\Sigma^{(0,2)}$ comes from the dimensional reduction of 2d $\mathcal{N} = (0, 2)$ chiral multiplet,

$$\Sigma^{(0,2)} = \phi - i\sqrt{2}\theta^+ \lambda_+ + \cdots.$$  

(3.6)

$\Xi$ appears in the dimensional reduction of 2d $\mathcal{N} = (0, 2)$ fermi multiplet,

$$\Xi = \rho_- - \sqrt{2}\theta^+ G + \cdots,$$  

(3.7)

where $G$ is the auxiliary field in the fermi multiplet.

The left moving fermions $\mu_-, \mu_-^\dagger, \rho_-$ and $\rho_-^\dagger$ are absent in the mass deformation. Moreover in the heavy mass limit, $\Xi$ and $\Sigma^{(0,2)}$ multiplets are decoupled from the vortex matrix model. Thus the vortex partition functions for $\mathcal{N} = (2, 2)$ theory consist of three pieces:

$$S'_G = \text{Tr} \left( \frac{1}{2} [\varphi, \bar{\varphi}]^2 + D^2 + g^2 \zeta D + 2\bar{\lambda}_-[\varphi, \lambda_-] \right),$$

$$S'_m = \sum_{i=1}^{N} \left( -I_i^\dagger \{\varphi, \bar{\varphi}\} I_i + I_i^\dagger D I_i \right)$$
\[-\sqrt{2} \mu^\dagger_+ \bar{\varphi} \mu_+ + i \sqrt{2} [I^1_+ \lambda_- \mu_+ - \mu^\dagger_+ \bar{\lambda}_- I_+] \),
\[ S'_{A} = \text{Tr} \left( [\varphi, B^\dagger] [\bar{\varphi}, \rho_+] + i \sqrt{2} \left( B [\rho^\dagger_+, \bar{\lambda}_-] + B^\dagger [\lambda_-, \rho_+] \right) \right) \].

The k-vortex partition function for \( \mathcal{N} = (2, 2) \) super Yang-Mills theory is
\[
Z_{k, \mathcal{N}} = \frac{1}{\text{Vol}(U(k))} \int \mathcal{D} \varphi \mathcal{D} \bar{\varphi} \mathcal{D} B \mathcal{D} \bar{\rho} \mathcal{D} \eta \mathcal{D} \mu_+ \mathcal{D} \rho_+ \exp \left( -\frac{1}{g^2} S'_G - S'_m - S'_{A} \right).
\]

4 Localization

In this section, we compute the vortex partition function in \( \mathcal{N} = (2, 2) \) super Yang-Mills theories by equivariant localization formula for supersymmetric system [8], [9], [10]. The vortex partition function given by (3.8) is invariant under the following supersymmetric transformation,
\[
Q_\epsilon \varphi = 0,
Q_\epsilon \bar{\varphi} = \eta, \quad Q_\epsilon \eta = [\varphi, \bar{\varphi}],
Q_\epsilon D = [\varphi, \chi], \quad Q_\epsilon \chi = D,
Q_\epsilon I_+ = \mu_+, \quad Q_\epsilon \mu_+ = \varphi I_+,
Q_\epsilon I^\dagger_+ = -\mu^\dagger_+, \quad Q_\epsilon \mu^\dagger_+ = I^\dagger_+ \varphi,
Q_\epsilon B = \rho_+, \quad Q_\epsilon \rho_+ = [\varphi, B] - \epsilon B,
Q_\epsilon B^\dagger = -\rho^\dagger_+, \quad Q_\epsilon \rho^\dagger_+ = [\varphi, B^\dagger] - \epsilon B^\dagger.
\]

Here we defined \( \eta = -i(\lambda_- + \bar{\lambda}_-)/\sqrt{2} \), \( \chi = -i(\lambda_- - \bar{\lambda}_-)/\sqrt{2} \) and rescaled \( \varphi \to -\sqrt{2} \varphi \). The vortex partition function (3.9) can be written in \( Q_\epsilon \)-exact form. For the abelian vortex case, we can apply the localization formula straightforwardly. But, \( Q_\epsilon \) does not generate appropriate fixed points for the nonabelian case. We recall that vev of adjoint scalars in the vector multiplet play a crucial role to localization formula work well and generate appropriate fixed points in Nekrasov partition functions in \( \mathcal{N} = 2 \) super Yang-Mills theory. The twisted masses in chiral multiplets play a role of
vev of adjoint scalars in the vortex partition functions at Higgs branch. We introduce the generic twisted masses $m_i, (i = 1, \cdots, N)$, then the Lagrangian of 2d $\mathcal{N} = (2, 2)$ super Yang-Mills theory is modified as

$$L_{\text{boson}} = - \left[ \text{Tr} \left( \frac{1}{2\epsilon^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2\epsilon^2} D_\mu \sigma D^\mu \bar{\sigma} + \frac{1}{2\epsilon^2} [\sigma, \bar{\sigma}]^2 + \frac{e^2}{2} (qq^\dagger - r^{\| N})^2 \right) + D_\mu q_i^\dagger D^\mu q_i + q_i^\dagger \{ \sigma - m_i, \bar{\sigma} - m_i^* \} q_i \right]. \quad (4.2)$$

The vacuum is labeled by

$$q_i^a_a = \sqrt{\lambda} \delta_i^a, \quad \sigma^a_b = m_i^a \delta_i^b. \quad (4.3)$$

The introduction of twisted mass terms also introduce twisted masses in vortex partitions. The supersymmetry transformation is modified as

$$Q_\epsilon \mu_{+i} = \varphi I_i \rightarrow Q_\epsilon \mu_{+i} = (\varphi - m_i) I_i,$$

$$Q_\epsilon \mu_{+i}^\dagger = I_i^\dagger \varphi \rightarrow Q_\epsilon \mu_{+i}^\dagger = I_i^\dagger (-\varphi - m_i). \quad (4.4)$$

The vortex partition function (3.8) can be written in $Q_\epsilon$-exact form

$$S'_G = Q_\epsilon \text{Tr} \left( \frac{1}{4} [\varphi, \bar{\varphi}] \eta + D\chi + g^2 \zeta \chi \right),$$

$$S'_m = \sum_{i=1}^{N_c} Q_i \left[ \mu_+^i (\bar{\varphi} - m_i^*) I_i + I_i^\dagger (\bar{\varphi} - m_i^*) \mu_+^i + I_i^\dagger \chi I_i \right],$$

$$S'_A = Q_\epsilon \text{Tr} \left[ -B^\dagger [\varphi, \rho_+] + \rho_+^\dagger [\varphi, B] + \chi [B, B^\dagger] \right]. \quad (4.5)$$

$Q_\epsilon$ is nilpotent up to infinitesimal gauge transformation and with flavor rotation.

Let us consider the localization method. First, we introduce the vector field $Q^*$ which acts on the fields and generates the supersymmetry transformation

$$Q^* = [\varphi, \chi] \frac{\partial}{\partial D} + \eta \frac{\partial}{\partial \bar{\varphi}} + \mu_{+i} \frac{\partial}{\partial I_i} + \rho_+ \frac{\partial}{\partial B} + D \frac{\partial}{\partial \chi} + [\varphi, \bar{\varphi}] \frac{\partial}{\partial \eta} + (\varphi - m_i) I_i \frac{\partial}{\partial \mu_+} + ([\varphi, B] - \epsilon B) \frac{\partial}{\partial \rho_+}$$

$$= Q_{\varphi}^i \frac{\partial}{\partial B_i} + Q_{\bar{\varphi}}^i \frac{\partial}{\partial F_i}. \quad (4.6)$$
The critical points $Q^* = 0$ is given by

$$
\begin{align*}
(\varphi_I - m_i)I_i &= 0, \\
(\varphi_I - \varphi_J - \epsilon)B_{IJ} &= 0.
\end{align*}
\quad (4.7)
$$

All the other fields are zero. The $B$ and $I$ are all zero except for $B_{i,i-1}$ and $I_{1,1}, I_{k_1+1,2}, \ldots, I_{k_N+1,N}$. Here $k_i$'s are the partitions of $N$ integers ($i = 1, \ldots, N$) and satisfy a relation $k = \sum i k_i$.

The localization formula [10] is expressed as contour integral

$$
Z_k = \oint \prod_{I=1}^k d\varphi_I \prod_{I \neq J} \frac{(\varphi_I - \varphi_J)}{\text{Sdet} L},
\quad (4.8)
$$

The superdeterminant of $L$ is defined by

$$
\text{Sdet} L = \text{Sdet} \left( \begin{array}{cc}
\frac{\partial Q^*_1}{\partial B_1} & \frac{\partial Q^*_1}{\partial F_1} \\
\frac{\partial Q^*_2}{\partial B_1} & \frac{\partial Q^*_2}{\partial F_2}
\end{array} \right),
\quad (4.9)
$$

and

$$
\text{Sdet} \left( \begin{array}{cc}
A & B \\
C & D
\end{array} \right) = \det(A - BD^{-1}C) \det(D)^{-1}.
\quad (4.10)
$$

Thus, the vortex partition function $Z_k$ for $\mathcal{N} = (2,2)$ super Yang-Mills with twisted mass terms becomes

$$
Z_k = \frac{1}{k!(2\pi i)^k} \oint \prod_{I=1}^k d\varphi_I \left( \prod_{I=1}^k \prod_{i=1}^N \frac{1}{\varphi_I - m_i} \right) \prod_{I \neq J} \frac{\varphi_I - \varphi_J}{\varphi_I - \varphi_J - \epsilon}
\quad (4.11)
$$

For $N_c = N_f = 1$ and $m_1 = 0$, the vortex partition function becomes

$$
Z_k = \frac{1}{k!(2\pi i)^k} \oint \prod_{I=1}^k d\varphi_I \left( \prod_{I=1}^k \frac{1}{\varphi_I} \right) \prod_{I \neq J} \frac{\varphi_I - \varphi_J}{\varphi_I - \varphi_J - \epsilon}
\quad (4.12)
$$

This reproduces the vortex partition function of abelian $k$-vortex in refined topological A-model amplitude of resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}P^1$ in [11]. We do not directly evaluate the contour integral of (4.11). Instead, in the next section, we calculate the equivariant character which reproduce the residues of (4.11) [11].
5 Vortex partition functions and equivariant character

In this section, we calculate equivariant character of the vortex moduli spaces with twisted masses. This is similar to five dimensional instanton counting or K-theoretic instanton counting in $\mathcal{N} = 2$ super Yang-Mills theory. We can recover the results in previous section by taking two dimensional limit.

First, we recall the vortex moduli space $\mathcal{M}_{k,N}$ with $k$-vortex number is represented as \[12\]

\[\mathcal{M}_{k,N} = \{(B, I)|[B, B^\dagger] + II^\dagger = \zeta I_k\}/U(k)\]
\[\simeq \{(B_C, I_C)\}/GL(k; \mathbb{C}).\] \hspace{1cm} (5.1)

where we define the $k \times N$ matrix $I = (I_1, \cdots, I_N)$. We define the spaces $V$ and $W$ on which $B_C$ and $I_C$ act, namely $B_C \in \text{Hom}_\mathbb{C}(V, V)$, $I_C \in \text{Hom}_\mathbb{C}(W, V)$ with $\text{dim}_\mathbb{C}V = k$ and $\text{dim}_\mathbb{C}W = N$.

We define torus action $U_\epsilon(1) \times U(1)^N$ on $\mathcal{M}_{k,N}$. First, we introduce $U_\epsilon(1)$ action on $(B, I)$ by

\[U_\epsilon(1) : (B_C, I_C) \rightarrow (q^{-1}B_C, I_C).\] \hspace{1cm} (5.2)

According to the action of $U_\epsilon(1)$, we modify the $B_C \in \text{Hom}_\mathbb{C}(V, V) \otimes Q$. $Q$ is the one dimensional space on which $U_\epsilon(1)$ acts. where $q = e^{-\beta \epsilon} (\epsilon \in \mathbb{R})$. Next, the action $U(1)^N$ on the $(B_C, I_C)$ is defined by

\[U(1)^N : (B_C, I_C) \rightarrow (B_C, I_C Q_m),\] \hspace{1cm} (5.3)

where $Q_m = \text{diag}(e^{-\beta m_1}, e^{-\beta m_2}, \cdots, e^{-\beta m_N})$, $Q_m = e^{-\beta m_i}$.

The $g \in \text{Hom}(U_\epsilon(1) \times U(1)^N, U(k))$ acts $(B, I)$ by

\[g : (B_C, I_C) \rightarrow (g(t)B_C g(t)^{-1}, g(t)I_C) \quad (t = (q, Q_m) \in U_\epsilon(1) \times U(1)^N).\] \hspace{1cm} (5.4)

The fixed point conditions are written by (5.2), (5.3) and (5.4)

\[(q^{-1}B_C, I_C Q_m) = (g(t)B_C g(t)^{-1}, g(t)I_C).\] \hspace{1cm} (5.5)
We set the $U(k)$ gauge transformation

$$g(t) = \text{diag}(e^{\beta \varphi_1}, e^{\beta \varphi_2}, \ldots, e^{\beta \varphi_k}).$$

(5.6)

Then, the fixed point conditions (5.5) become

$$(\varphi_I - \varphi_J - \epsilon)B_{CIJ} = 0,$$

$$(\varphi_I - m_i)I_{CII} = 0, \quad (I = 1, \ldots, k \quad i = 1, \ldots, N).$$

(5.7)

The solution of the fixed point conditions are

$$\varphi_i = m_i + (I_i - 1)\epsilon$$

(5.8)

which satisfies the fixed points conditions (4.7).

We can decompose the representation space as follows

$$W = \bigoplus_{i=1}^{N} W_i,$$

$$V = \bigoplus_{i=1}^{N} \bigoplus_{n} V_i(n),$$

(5.9)

with

$$W_i := \{ w \in W | Q^{-1}_m w = Q^{-1}_m w \},$$

$$V_i(n) := \{ v \in V | g(t)v = q^n Q_m v \}.$$  (5.10)

The dimensions of $V_i(n)$ are $0 \leq \dim V_i(n) \leq 1$. $(B_C, I_C)$ which satisfy the fixed point conditions (5.5) define the map

$$B_C : V_i(n) \to V_i(n - 1),$$

$$I_C : W_i \to V_i(0).$$

(5.11)

This means that the space $V$ is decomposed into $N$-tuple one dimensional partitions of $k_i$. In the instanton case, recall Nekrasov partition for $U(N)$ gauge theory is expressed by $N$-tuple two dimensional partitions; namely, Young diagrams.
Next, we consider the tangent space of $\mathcal{M}_{k,N}$. Infinitesimal $\mathfrak{gl}(k; \mathbb{C})$ gauge transformation act, $b : \mathfrak{gl}(k; \mathbb{C}) \to \text{Hom}(V, V \otimes Q) \oplus \text{Hom}(W, V)$

$$b : \xi \to ([\xi, B], I\xi), \quad (\xi \in \mathfrak{gl}(k; \mathbb{C})). \quad (5.12)$$

Then, the tangent space $T_x \mathcal{M}_{k,N}$ of $\mathcal{M}_{k,N}$ is

$$T_x \mathcal{M}_{k,N} = \text{Hom}(V, V \otimes Q) \oplus \text{Hom}(W, V)/\text{Im} b$$

$$= \text{Hom}(V, V \otimes Q) \oplus \text{Hom}(W, V)/\text{Hom}(V, V). \quad (5.13)$$

We will use same symbols for the characters and representation spaces. The characters are

$$V = \sum_{i=1}^{N} \sum_{l_i=1}^{k_i} q^{l_i-1} Q_{m_i},$$

$$W = \sum_{i=1}^{N} q_{m_i},$$

$$Q = q^{-1}. \quad (5.14)$$

where we defined $\dim (\oplus_n V_i(n)) = k_i$. The character of the moduli space becomes

$$ch(T_x \mathcal{M}_{k,N}) = (V^* \times Q - V^* + W^*) \times V$$

$$= \sum_{i, i=1}^{N} Q_{\tilde{a}} \sum_{l_i=1}^{k_i} q^{l_i-k_i-1}$$

$$= \sum_{i, i=1}^{N} Q_{\tilde{a}} \left( \sum_{l_i=1}^{a_i} q^{l_i-k_i-1} + \sum_{j_i=1}^{b_i} q^{j_i-1} \right), \quad (5.15)$$

where we have defined $Q_{\tilde{a}} = Q_{m_i} q_{m_i}^{-1}$; $a_i = \min(k_i, k_i)$ and $b_i = \max(k_i - k_i, 0)$. Finally, we obtain the K-theoretic vortex partition functions for $U(N)$ $\mathcal{N} = (2, 2)$ super Yang-Mills theory is

$$Z_{K-\text{theoretic}}^{N = (2, 2)} = \sum_{k=0}^{\infty} e^{2\pi i (r+i\theta)k} \sum_{k_1, \ldots, k_N = k} \prod_{i, i=1}^{N} a_i \prod_{l_i=1}^{b_i} \prod_{j_i=1}^{1} \frac{1}{1 - Q_{\tilde{a}} q^{k_i-l_i} (1 - Q_{\tilde{a}} q^{1-j_i})}. \quad (5.16)$$
When we take the two dimensional limit, $\beta \rightarrow 0$ with $Q_{m_i} = e^{-\beta m_i}$ and $q = e^{-\beta \epsilon}$ and rescale $Z_{k,N}$ by $\beta$. Then, we obtain the residues of (4.11)

$$Z_{N}^{(2,2)} = \sum_{k=0}^{\infty} e^{2\pi(r+\theta)k} \sum_{k_1+\cdots+k_N = k \atop \sum_{i=1}^{N} a_i = 0} \prod_{i=1}^{N} \prod_{l_1 = 1}^{b_{\alpha}} \frac{1}{((k_i + 1 - l_i)\epsilon - m_{\bar{\alpha}})((1-j_i)\epsilon - m_{\bar{\alpha}})},$$

(5.17)

where we have defined $m_{\bar{\alpha}} = m_i - m_j$. For example, in abelian vortex case, the vortex partition function becomes

$$Z_{N}^{(2,2)} = \sum_{k=0}^{\infty} e^{2\pi(r+\theta)k} \prod_{i=1}^{k} \frac{1}{(k+1-i)\epsilon} = \exp\left(\frac{e^{2\pi(r+\theta)}}{\epsilon}\right).$$

(5.18)

6 Discussion

We have calculated the vortex partition of $\mathcal{N} = (2, 2)$ $U(N)$ super Yang-Mills theory with $N$ fundamental chiral multiplets. The introduction of twisted mass terms break the non-abelian symmetry to collection of $U(1)^N$ symmetry. Thus, our results is the collection of $N$-tuple abelian vortex rather than purely non-abelian vortex. Moreover, the localization with twisted mass terms works well in the case of $N_c = N_f$. How the vortex partitions become in the general flavor cases $N_c < N_f$? When the gauge group is $U(1)$, [5] obtain general $N_f$ flavor vortex partition functions by refined BPS state counting and they also reveal the vortex partition function for $U(1)$ gauge group with $N_f$ fundamental chiral multiplets is related to $J$-function of $\mathbb{C}P^{N_f-1}$; the generating function of genus zero Gromov-Witten invariants for $\mathbb{C}P^{N_f-1}$ up to appropriate parameters identification. At strong coupling limit $e^2 \rightarrow \infty$, the vortex moduli space for $U(N_c)$ super Yang-Mills with $N_f$ flavor becomes the moduli space of holomorphic maps from the complex plane $\mathbb{C}$ to the grassmann manifold $\text{Gr}(N_c, N_f)$ [12]. Roughly speaking, we can regard the vortex partitions functions is the geometric index(or volume) of the moduli space of holomorphic maps. So we expect to the purely non-abelian vortex
partition functions for $U(N_c)$ gauge group with $N_f$ fundamental flavor is related to the $J$-function for the grassmann manifold $\text{Gr}(N_c, N_f)$, [13], [14], [15].

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