A DYNAMICAL SHAFAREVICH THEOREM FOR ENDOMORPHISMS
OF $\mathbb{P}^N$

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Abstract. We prove a dynamical analogue of the Shafarevich conjecture for morphisms $f : \mathbb{P}^N_K \to \mathbb{P}^N_K$ of degree $d \geq 2$, defined over a number field $K$. This extends previous work of Silverman [8] and others in the case $N = 1$.

1. Introduction

Let $K$ be a number field, let $S$ be a finite set of places of $K$ containing the archimedean places, and let $\mathcal{O}_S$ be the ring of $S$-integers of $K$. An abelian variety $A/K$ is said to have good reduction outside $S$ if there is a proper $\mathcal{O}_S$-group scheme with generic fiber $K$-isomorphic to $A$. The Shafarevich conjecture, proved by Faltings [3], gives a finiteness statement for such abelian varieties.

Theorem 1.1 (Faltings). [3] Up to $K$-isomorphism, there are only finitely many principally polarized abelian varieties $A$ over $K$ of a given dimension which have good reduction outside $S$.

The goal of this paper is to provide a dynamical analogue of Theorem 1.1 for morphisms $f : \mathbb{P}^N_K \to \mathbb{P}^N_K$ of degree $d \geq 2$. Letting $\mathcal{O}_p$ denote the valuation ring of the local field $K_p$ for a prime $p$ of $K$, we say that such a morphism has good reduction outside $S$ if for all $q \notin S$, there is an $\mathcal{O}_q$-morphism $\mathbb{P}^N_{\mathcal{O}_q} \to \mathbb{P}^N_{\mathcal{O}_q}$ with generic fiber $\text{PGL}_{N+1}(K_q)$-conjugate to $f$ (cf. [7, Remark 2.16]). However, it is immediately clear that the naïve dynamical analogue of the Shafarevich conjecture fails; for instance, when $N = 1$, the set of monic degree $d \geq 2$ polynomials $f \in \mathcal{O}_S[x]$ consists of infinitely many $\text{PGL}_{N+1}(K)$-conjugacy classes. It is easy to generalize this construction to higher dimensions, so that the most obvious dynamical counterpart of the Shafarevich conjecture fails for all choices of $d, N, K$, and $S$. In order to recover an appropriate statement, one must impose more structure on the space of maps. As has been done under various guises in [5], [6], [8], [9], and [10], we study pairs $(f, X)$ consisting of a map $f$ having good reduction outside $S$ and an appropriate finite $\text{Gal}(\overline{K}/K)$-invariant
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set \( X \subseteq \mathbb{P}^N(K) \) also having good reduction outside \( S \). Naturally, we would expect \( X \) to be dynamically related to \( f \) — in particular, to have \( X = Y \cup f(Y) \) for some \( Y \subseteq \mathbb{P}^N(K) \) — and for the points of \( X \) to lie in a certain kind of general position modulo reduction by all primes \( p \notin S \).

**Definition 1.2.** Let \( x \in \mathbb{P}^N(K) \), let \( p \) be a prime of \( K \), and write \( x = [x_0 : \cdots : x_N] \), where the \( x_i \) are normalized so that they are \( p \)-integral for all \( i \), and at least one \( x_i \) is a \( p \)-adic unit. Then the reduction of \( x \) modulo \( p \) is the point \( \tilde{x} = [\tilde{x}_0 : \cdots : \tilde{x}_N] \in \mathbb{P}^N(k_p) \), where \( k_p \) is the residue field at \( p \), and the \( \tilde{x}_i \) are the reductions of \( x_i \) modulo \( p \). If \( X \subseteq \mathbb{P}^N(K) \), then \( \tilde{X} := \{ \tilde{x} : x \in X \} \) is the reduction of \( X \) modulo \( p \).

Similarly, we define the reduction of a hyperplane as follows.

**Definition 1.3.** If \( H_a : a_Nx_N + \cdots + a_0x_0 = 0 \) is a hyperplane in \( \mathbb{P}^N_K \) corresponding to \( a = [a_0 : \cdots : a_N] \in \mathbb{P}^N(K) \), then we write \( \tilde{H}_a := H_a \) for the reduction of \( H_a \) in \( \mathbb{P}^N_{k_p} \).

Good reduction of \( X \) mod \( p \) describes points in a certain kind of general position retaining that property upon reduction modulo the primes above \( p \) in a field of definition for the points in \( X \).

**Definition 1.4.** Let \( F \) be a field. We say that a finite set of at least \( N+1 \) points \( X \subseteq \mathbb{P}^N(F) \) is in general linear position if no hyperplane contains \( N+1 \) points of \( X \).

If a set \( Y \) of \( N \) points determines a unique hyperplane in \( \mathbb{P}^N \), we denote this hyperplane by \( H_Y \). Note that if \( X \) is in general linear position and contains at least \( N+1 \) points, then any subset \( Y \) of size \( N \) determines a hyperplane \( H_Y \).

**Definition 1.5.** Let \( S \) be a finite set of primes in \( \mathcal{O}_K \). Let \( X \subseteq \mathbb{P}^N(K) \) be a finite Gal(\( K/K \))-invariant subset of size at least \( N+1 \) with field of definition \( K(X) \). We say \( X \) has good reduction outside \( S \) if for all primes \( p \notin S \) and all primes \( \mathfrak{p} \) in \( K(X) \) lying over \( p \), the reduction modulo \( \mathfrak{p} \) of \( X \) is in general linear position; that is, for all such primes \( \mathfrak{p} \) over \( p \), the map

\[
Y \subseteq X, \#Y = N \mapsto \tilde{H}_Y \mod \mathfrak{p},
\]

is injective.

Let \( m \geq \frac{N+2}{2} \). We denote by \( \mathcal{R}_{d,N}[m](K,S) \) the set of triples \( (f,X,Y) \) so that:

- \( f \) is a degree \( d \geq 2 \) morphism \( \mathbb{P}^N \to \mathbb{P}^N \) defined over \( K \),
- \( X \) is a finite set of at least \( N+1 \) points in \( \mathbb{P}^N(K) \),
- \( f(X) \) is a finite set of at least \( N+1 \) points in \( \mathbb{P}^N(K) \),
- \( Y \) is a finite set of at least \( N+1 \) points in \( \mathbb{P}^N(K) \) with \( \#Y = N \),
- \( X \cap f(X) \) is empty,
- \( Y \cup f(Y) = X \cup f(X) \),
- \( X \) and \( f(X) \) are in general linear position modulo reduction by all primes \( p \notin S \).

For each \( (f,X,Y) \) in \( \mathcal{R}_{d,N}[m](K,S) \), we define

\[
\varphi_{d,N}[m](K,S)(f,X,Y) := \tilde{H}_Y \mod \mathfrak{p},
\]

where \( \mathfrak{p} \) is any prime of \( K \) lying above \( p \).

The number \( \varrho_{d,N}[m](K,S) \) is defined as the number of distinct \( (f,X,Y) \) in \( \mathcal{R}_{d,N}[m](K,S) \) that satisfy the above conditions.
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- $|Y| = m$,
- $X = Y \cup f(Y)$,
- $X$ is a $\text{Gal}(\overline{K}/K)$-invariant subset of $\mathbb{P}^N(\overline{K})$,
- $f$ and $X$ have good reduction outside $S$,
- $Y$ is not contained in a hypersurface of degree at most $2d$.

It is not difficult to show that there exist configurations of $k$ points in $\mathbb{P}^N(K)$ satisfying the final condition provided $k \geq \binom{N+2d}{2d}$. (We prove this in the Appendix.) In fact, it is the generic expectation that a subset of at least $\binom{N+2d}{2d}$ distinct points is not contained in any degree $2d$ hypersurface, so this can be thought of as an additional general position condition.

Throughout this paper we write, for a ring $R$,

$$\text{PL}_{N+1}(R) := \text{GL}_{N+1}(R)/R^\times \hookrightarrow \text{PGL}_{N+1}(R).$$

When $\text{Pic}(R)$ is trivial, $\text{PL}_{N+1}(R) = \text{PGL}_{N+1}(R)$ [2, Corollary 2.7]. Given $\phi \in \text{PL}_{n+1}(\mathcal{O}_S)$ and $(f, X, Y) \in \mathcal{R}_{d,N}[m](K, S)$, define

$$(f, X, Y)^\phi := (\phi^{-1} \circ f \circ \phi, \phi^{-1}(X), \phi^{-1}(Y)).$$

Clearly $f$ has good reduction outside $S$ if and only if $\phi^{-1} \circ f \circ \phi$ has good reduction outside $S$. That $X$ has good reduction outside $S$ if and only if $\phi^{-1}(X)$ has good reduction outside $S$ follows from the fact that $\phi \in \text{PL}_{N+1}(\mathcal{O}_S)$ induces an automorphism of $\mathbb{P}^N(k_p)$ for the residue field $k_p$ of each $p \notin S$. We thus obtain an action of $\text{PGL}_{N+1}(\mathcal{O}_S)$ on $\mathcal{R}_{d,N}[m](K, S)$.

In this paper, we prove the following Shafarevich-style finiteness theorem.

**Theorem 1.6.** $\mathcal{R}_{d,N}[m](K, S)$ admits only finitely many $\text{PL}_{N+1}(\mathcal{O}_S)$-orbits.

Theorem 1.6 recovers [8, Theorem 2] in the case $N = 1$, restricting the latter to triples $(f, X, Y)$ for which $f$ has multiplicity 1 on $Y$. As in [8] we could allow multiplicities to be taken into account when $N = 1$, but there does not seem to be an appropriate higher dimensional analogue of this aspect that follows from our proof.

We note the necessity of the final condition defining $\mathcal{R}_{d,N}[m](K, S)$ in Theorem 1.6. Let $K = \mathbb{Q}$, let $f_c = [x_0^3 : c(x_0^2 + x_1^2 - x_2^2) + x_0^2 + x_1^2 : x_0^2 + x_1^2 - x_2^2]$ for $c \in \mathbb{Z}$, and let $Y \subseteq \mathbb{P}^2(\mathbb{Q})$ be any set of order at least 4 in general linear position which is contained in the hypersurface $Z : x_0^3 + x_1^3 - x_2^3 = 0$. Let $S$ be a finite set of places of $\mathbb{Q}$ containing the archimedean place such that $Y \cup f_c(Y)$ has good reduction outside of $S$. (Note that $f_c(Y)$ is independent of $c$, and that we can form such a set $Y$ by taking suitable Pythagorean triples.) Since $|Y| \geq 4$ and $Y$ is in general linear position, there are finitely many $\phi \in \text{PGL}_3(\overline{K})$ leaving $X := Y \cup f_c(Y)$
invariant. It follows that there are infinitely many $\text{PGL}_3(K)$-conjugacy classes of triples in $\{(f_c, X, Y) : c \in \mathbb{Z}\}$, and a fortiori infinitely many $\text{PL}_3(O_S)$-conjugacy classes of such triples. Since $f_c$ has good reduction outside $S_\infty$ for all $c \in \mathbb{Z}$, each of these triples would be an element of $\mathcal{R}_{2,2}[m](\mathbb{Q}, S)$ where $m = |Y|$—if the final condition defining $\mathcal{R}_{d,N}[m](K, S)$ were dropped, that is. Similar examples may be constructed for any $d, N \geq 2$. We thus see that in the absence of the final condition, there is no $M$ such that Shafarevich finiteness holds for all $K, S$ and $m \geq M$ as posited by Theorem 1.6.

The general strategy of our proof is similar to that of [8]. First, we use the Hermite–Minkowski theorem to reduce the problem to sets $X$ with a fixed splitting field $L$. This is the content of §2. Next, we show that large sets $X \subseteq \mathbb{P}^N(L)$ having good reduction outside $S$ yield many solutions to the classical $S$-unit equation over $L$. The choice of geometry behind our definition of good reduction is crucial to this connection. The finiteness of the number of solutions to the $S$-unit equation then allows us to show that, given $K, S$ and $n \gg 1$, and up to $\text{PL}_N(O_S)$-conjugacy, there are only finitely many Galois-invariant $X \subseteq \mathbb{P}^N(K)$ of order $n$ having good reduction outside of $S$. Finally, we show that given such an $X$ and $d \geq 2$, there are only finitely many morphisms $f : \mathbb{P}^N \to \mathbb{P}^N$ of degree $d$ defined over $K$ that realize $X$ as $Y \cup f(Y)$ for some $Y$ not contained in a hypersurface of degree at most $2d$.

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2. Reduction to finiteness for points defined over a single field extension

Inspired by [8, Sublemma 8], in this section we first prove that the Galois-invariant sets $X \subseteq \mathbb{P}^N$ of a given size having good reduction outside $S$ have all of their points defined over a fixed finite extension of $L/K$. This is achieved in Lemma 2.1. In Lemma 2.2 we reduce Theorem 1.6 to $L$ and the set of primes $T$ lying over $S$.

Lemma 2.1. There exists a fixed finite extension $L$ of $K$ such that for any $(f, X, Y) \in \mathcal{R}_{d,N}[m](K, S)$, each point of $X$ is defined over $L$.

Proof. Let $K(X)$ be the field of definition of the point set $X$ over $K$; that is, if $X = \{P_1, \ldots, P_m\}$ and $G_K := \text{Gal}(\overline{K}/K)$ then $K(X) := K(P_1, \ldots, P_m)$ is the subfield of $\overline{K}$
which is fixed by \( \{ \sigma \in G_K : \sigma(P_i) = P_i \quad \forall 1 \leq i \leq m \} \). As \( X \) is \( G_K \)-invariant, the extension degree \([K(X) : K]\) is bounded above by a constant depending only on \( m \) and \( N \).

If \( K(X)/K \) ramifies at a prime \( p \), then the inertia group is non-trivial, so there is a non-trivial element \( \sigma \) of \( G_K \) which preserves the valuation associated to a prime \( p \) lying over \( p \). Since the element \( \sigma \) is non-trivial and \( K(X) \) is the fixed field of all elements acting trivially on the points of \( X \), there is some \( P_i \neq P_j \) so that \( \sigma(P_i) = P_j \). By definition of the inertia group, \( P_i \equiv P_j \mod p \). By the assumption that \( X \) has good reduction outside \( S \), we conclude that \( K(X)/K \) is unramified outside \( S \).

Thus by Hermite–Minkowski [4, Theorem III.2.13], there are only finitely many such \( K(X) \), and we may take their compositum to assume all points of \( X \) are contained in a single field as claimed. \( \square \)

Let \( \mathcal{X}[n](K, S) \) be the collection of sets \( X \subseteq \mathbb{P}^N(K) \) such that

- \( |X| = n \)
- \( X \) is \( G_K \)-invariant
- \( X \) has good reduction outside \( S \).

**Lemma 2.2.** For any \( n, K, \) and \( S \), the natural map

\[
\mathcal{X}[n](K, S)/\text{PL}_{N+1}(\mathcal{O}_S) \to \{ X \subseteq \mathbb{P}^N(K) : \#X = n \}/\text{PGL}_{N+1}(K)
\]

is finite-to-one.

**Proof.** We generalize Silverman’s proof of [8, Sublemma 8]. Let \( L \) be as in Lemma 2.1. Assume without loss that \( S \) is large enough so that if \( T \) is the set of primes of \( L \) above \( S \), then \( \mathcal{O}_T \) and \( \mathcal{O}_S \) are PIDs. (We note that the proof of the Claim below shows that if \( S' \) is the appropriate enlargement of \( S \), then the natural map

\[
\mathcal{X}[n](K, S)/\text{PL}_{N+1}(\mathcal{O}_S) \to \mathcal{X}[n](K, S')/\text{PL}_{N+1}(\mathcal{O}_{S'})
\]

is injective.) Fix \( X_0 \in \mathcal{X}[n](K, S) \) and let

\[
\text{PGL}_{N+1}(K, S, X_0) = \{ \phi \in \text{PGL}_{N+1}(K) : \phi(X_0) \in \mathcal{X}[n](K, S) \}.
\]

It suffices to define a map from \( \text{PL}_{N+1}(K, S, X_0) \) to a finite set and to show that if two elements \( \phi, \phi_0 \in \text{PGL}_{N+1}(K, S, X_0) \) have the same image under this map, then they differ by an element of \( \text{PL}_{N+1}(\mathcal{O}_S) \). This then proves that \( \text{PGL}_{N+1}(K, S, X_0)/\text{PL}_{N+1}(\mathcal{O}_S) \) is finite.

We first claim that if \( \phi \in \text{PGL}_{N+1}(K, S, X_0) \), then \( \phi \in \text{PL}_{N+1}(\mathcal{O}_T) \). From Lemma 2.1, we have \( X_0, \phi(X_0) \subseteq \mathbb{P}^N(L) \). Any element of \( \text{PGL}_{N+1}(K) \) is determined by its values on \( N + 2 \)
points such that no $N + 1$ of them are contained in a hyperplane, so since $n \geq N + 2$, we have $\phi \in \text{PGL}_{N+1}(L)$.

Moreover, the good reduction assumptions on $X_0$ and $\phi(X_0)$ imply that for any prime $\mathfrak{P}$ of $L$ with $\mathfrak{P} \notin T$, any $N + 1$ points in the reduction of $X_0 \bmod \mathfrak{P}$ are not contained in a hyperplane, and similarly for $\phi(X_0)$. We claim that $\phi \in \text{PL}_{N+1}(\mathcal{O}_T)$. To see this, let $P_0, \ldots, P_{N+1}$ be $N + 2$ points in $X_0$ and let $Q_0, \ldots, Q_{N+1}$ satisfy $\phi(P_i) = Q_i$ for all $i$. Write the $P_i = [x_{i,0}, \ldots, x_{i,N}]$ in ‘normalized form’ in the sense that each $x_i$ is $\mathfrak{P}$-integral and at least one $x_i$ is a $\mathfrak{P}$-adic unit. This is possible because $\mathcal{O}_T$ is a PID. We claim that the following holds.

**Claim:** There is a linear transformation $\psi \in \text{PL}_{N+1}(\mathcal{O}_T)$ such that

$$
\psi(P_0) = [1 : 0 : \cdots : 0], \ldots, \psi(P_N) = [0 : \cdots : 0 : 1];
$$

and $\psi(P_{N+1}) = [1 : 1 : \cdots : 1]$.

Proof of claim: Let $\underline{P_i} = (x_{i,0}, \ldots, x_{i,N})$ be the corresponding points in $\mathbb{A}^{N+1}(L)$, and similarly for $\underline{\psi(P_i)}$. Since the $P_i$ are in general linear position, any $N + 1$ of them span $\mathbb{A}^{N+1}(L)$. Thus we have

$$
P_{N+1} = \sum_{i=0}^{N} \lambda_i P_i,
$$

where $\lambda_i \in L^*$ for all $i$. (If some $\lambda_i$ were equal to 0, then the $P_i$ would not be in general linear position, because $P_{N+1}$ would be in the linear subspace spanned by $P_0, \ldots, \hat{P_i}, \ldots, P_N$.) Rescaling the $P_i$ gives

$$
P_{N+1} = \sum_{i=0}^{N} P_i.
$$

Taking $M \in \text{GL}_{N+1}(L)$ to have columns $\underline{P_i}$ gives $MP_i = \psi(P_i)$ for all $i$. By the good reduction assumption on $X_0$, we deduce from this argument that $M$ has $\mathcal{O}_T$-integral columns (note that the good reduction assumption implies that the rescaling is by $T$-units, and so preserves the $\mathcal{O}_T$-integrality of the $\underline{P_i}$), and that for any $\mathfrak{P} \notin T$ with residue field $k_\mathfrak{p}$, the reduction of $M$ modulo $\mathfrak{P}$ yields an element of $\text{GL}_{N+1}(k_\mathfrak{p})$. Thus $M \in \text{GL}_{N+1}(\mathcal{O}_T)$. The corresponding transformation $\psi \in \text{PL}_{N+1}(\mathcal{O}_T)$ we are after is the image of $M$ in $\text{GL}_{N+1}(\mathcal{O}_T)/\mathcal{O}_T^\times$. Finally, we note that this argument has also shown that any non-invertible linear transformation over a field $F$ maps $N + 1$ points in general linear position in $\mathbb{P}^N(F)$ to a set of points that are not in general linear position. This is the key fact proving that (2.1) is injective. □
Resuming the proof of Lemma 2.2, we have established the above Claim. Similarly, if \( \lambda \in \text{PGL}_{N+1}(L) \) is a linear transformation with \( \lambda(Q_0) = [1 : 0 : \cdots : 0], \lambda(Q_1) = [0 : 1 : 0 : \cdots : 0], \ldots , \lambda(Q_N) = [0 : 0 : \cdots : 0 : 1] \), \( \lambda(Q_{N+1}) = [1 : 1 : \cdots : 1] \), then \( \lambda \in \text{PL}_{N+1}(O_T) \). Hence, \( \phi = \lambda^{-1} \circ \psi \in \text{PL}_{N+1}(O_T) \).

Next we define a map \( \text{PGL}_{N+1}(K, S, X_0) \to \text{Map}_{\text{Set}}(\text{Gal}(L/K), S_{X_0}) \), by \( \phi \mapsto (\sigma \mapsto \phi^{-1} \circ \phi^\sigma) \), where \( S_{X_0} \) denotes the set of permutations of \( X_0 \). Note that \( \text{Gal}(L/K) \) and \( S_{X_0} \) are both finite, so \( \text{Map}_{\text{Set}}(\text{Gal}(L/K), S_{X_0}) \) is finite. To see that this map is well-defined, we must check that for any \( \phi \in \text{PGL}_{N+1}(K, S, X_0) \) and any \( \sigma \in \text{Gal}(L/K) \), we have \( \phi^{-1} \circ \phi^\sigma : X_0 \to X_0 \).

Let \( \phi \in \text{PGL}_{N+1}(K, S, X_0) \) and \( \sigma \in \text{Gal}(L/K) \). The sets \( X_0, \phi(X_0) \subseteq \mathbb{P}^N(L) \) are both \( \text{Gal}(L/K) \)-invariant. Hence, for all \( \sigma \in \text{Gal}(L/K) \), we have \( \phi(X_0) = (\phi(X_0))^\sigma = \phi^\sigma(X_0) = \phi^\sigma(X_0) \). This yields \( \phi^{-1} \circ \phi^\sigma : X_0 \to X_0 \).

Now fix \( \phi_0 \in \text{PGL}_{N+1}(K, S, X_0) \), and suppose \( \phi \in \text{PGL}_{N+1}(K, S, X_0) \) has the same image as \( \phi_0 \) in \( \text{Map}_{\text{Set}}(\text{Gal}(L/K), S_{X_0}) \). Then for all \( \sigma \in \text{Gal}(L/K) \), \( \phi^{-1} \circ \phi^\sigma \) and \( \phi_0^{-1} \circ \phi_0^\sigma \) have the same action on \( X_0 \). Since \( X_0 \) has \( n \geq N + 2 \) points in general position, we must have \( \phi^{-1} \circ \phi^\sigma = \phi_0^{-1} \circ \phi_0^\sigma \) as elements of \( \text{PGL}_{N+1}(L) \). So \( \phi \circ \phi_0^{-1} = \phi^\sigma \circ (\phi_0^\sigma)^{-1} = (\phi \circ \phi_0^{-1})^\sigma \). Since this holds for all \( \sigma \in \text{Gal}(L/K) \), we have \( \phi \circ \phi_0^{-1} \in \text{PGL}_{N+1}(K) \). We have already seen that \( \phi, \phi_0 \in \text{PL}_{N+1}(O_T) \), and thus we have \( \phi, \phi_0 \in \text{PL}_{N+1}(O_S) \) as desired.

\[\square\]

3. Reduction to bounds on the number of solutions to S-unit equations

This section is dedicated to proving the finiteness (up to \( \text{PL}_{N+1}(O_S) \)-conjugacy) of the number of sets \( X \subseteq \mathbb{P}^N(K) \) having good reduction outside \( S \), with the number of such sets depending only on \( |X| \) and \( |S| \); see Corollary 3.2. We will show that this comes down to the finiteness of the number of solutions to the classical \( S \)-unit equation.

Recall that we have fixed \( N + 2 \) points \( P_0 = [1 : 0 : \cdots : 0], P_1 = [0 : 1 : 0 : \cdots : 0], \ldots , P_N = [0 : \cdots : 0 : 1], P_{N+1} = [1 : \cdots : 1] \). We denote the homogeneous coordinates of \( \mathbb{P}^N \) by \( z_0, \ldots , z_N \) and fix a union of hyperplanes

\[\mathcal{H} := \left( \bigcup_{0 \leq i \neq j \leq N} \{ z_i - z_j = 0 \} \right) \cup \left( \bigcup_{i=0}^N \{ z_i = 0 \} \right).\]

Note that \( P_0, \ldots , P_N \in \mathcal{H} \). In what follows we assume that \( S \) is large enough so that \( O_S \) is a PID, so that for \( x \in \mathbb{P}^N \) we write \( x = [x_0 : x_1 : \cdots : x_N] \) in \( S \)-normalized form; meaning:

- \( x_0, \ldots , x_N \in O_S \).
- \( \max \{|x_0|_v, \ldots , |x_N|_v\} = 1 \) for all non-archimedean places \( v \in M_K \setminus S \).
The following lemma will be crucial in the proof of Theorem 1.6.

**Lemma 3.1.** Let $N \in \mathbb{N}$, $K$ be a number field and $S$ a finite set of primes of $K$, with group of $S$-units denoted $U_S$. Let $X \subseteq \mathbb{P}^N(K)$ be such that:

1. $X$ has good reduction outside $S$.
2. $P_0, \ldots, P_N, P_{N+1} \in X$.

If $x \in X \setminus \mathcal{H}$, then for all $i, j \in \{0, \ldots, N\}$ with $i \neq j$ we have $x_i - x_j \in U_S$. Moreover, for all $i \in \{0, \ldots, N\}$, $x_i \in U_S$.

**Proof.** Let $v$ a non-archimedean place of $K$ outside $S$ and let $x \in X \setminus \mathcal{H}$. Let $i \neq j$. Note that the hyperplane $H : z_i - z_j = 0$ passes through the $N$ points $P_k \in X$ for $k \neq i, j$ and doesn’t pass through $x$. Since $X$ has good reduction outside $S$, we see that $x \mod v$ cannot lie in the hyperplane $H \mod v$. Therefore $x_i \neq x_j \mod v$. Recalling that $x_0, x_1, \ldots, x_N$ are $S$-integers and $v \notin S$ is arbitrary we get that $x_i - x_j \in U_S$ as claimed. Similarly, for each $i$, the hyperplane $H : z_i = 0$ passes through the $N$ points $P_k$ such that $z_i = 0$, and so the assumption that $X$ has good reduction outside $S$ means that any $x \in X \setminus \mathcal{H}$ satisfies $x_i \in U_S$. □

Although the proof of the following proposition assumes $O_S$ is a PID, in order to be able to invoke Lemma 3.1, it is clear that the result therefore holds a fortiori without this assumption.

**Corollary 3.2.** Let $N \geq 1$, $K$ be a number field and $S$ a finite set of primes of $K$. There is a finite set $\Pi = \Pi(K, S, N) \subseteq \mathbb{P}^N(K)$ such if $X \subseteq \mathbb{P}^N(K)$ satisfies:

1. $X$ has good reduction outside $S$.
2. $P_0, \ldots, P_N, P_{N+1} \in X$,

then $X \subseteq \Pi$.

**Proof.** First note that if $x \in X \cap \mathcal{H}$, then $x \in \{P_0, P_1, \ldots, P_{N+1}\}$, since otherwise, one of the hyperplanes in the union (3.1) defining $\mathcal{H}$ contains $N + 1$ points of $X$. Thus we can suppose without loss that $x = [x_0 : \cdots : x_N] \in X \setminus \mathcal{H}$, with coordinates $x_0, x_1, \ldots, x_N$ in $S$-normalized form as explained above. Consider the equation

$$(x_0) + (-x_1) = x_0 - x_1,$$
which we know to be an $S$-unit equation by Lemma 3.1. Dividing through by $x_0$ on both sides and rearranging, we obtain

$$\frac{x_0 - x_1}{x_0} + \frac{-x_1}{x_0} = 1.$$ 

By [1, Theorem 5.2.1], we conclude that there is a finite set $\Pi_0 = \Pi_0(S, K)$ such that $\frac{x_0}{x_0} \in \Pi_0$. The same argument applied to

$$(x_0) + (-x_i) = x_0 - x_i$$

implies that for each $0 \leq i \leq N$, $\frac{x_i}{x_0} \in \Pi_0$. □

4. Proof of Shafarevich finiteness

Proof of Theorem 1.6. Let $(f, X, Y) \in \mathcal{R}_{d,N}[m](K, S)$. Since $X$ is in general linear position, conjugating $(f, X, Y)$ by an element of $\text{PL}_{N+1}(O_S)$ if necessary and arguing as in Lemma 2.2 we may assume that $P_0, \ldots, P_{N+1} \in X$. Moreover, by Lemma 2.1, there is a fixed extension $L/K$ depending only on $S$ such that $X \subseteq \mathbb{P}^N(L)$. Let $T$ be the set of primes of $L$ lying above $S$. By Corollary 3.2, it follows that $X$ is contained in a finite set $\Pi' = \Pi'(L, T, N)$ of points in $\mathbb{P}^N(L)$. So we may assume $f$ induces a map $\sigma : Y \to \Pi'$, where $Y \subseteq \Pi'$. On the other hand, the subvariety given by $f = g$ for degree $d$ morphisms $f, g : \mathbb{P}^N \to \mathbb{P}^N$ is contained in a hypersurface of degree $2d$. Since by assumption $Y$ is not contained in any degree $2d$ hypersurface, we see that $f$ is uniquely determined. As $Y \subseteq \Pi'$, we have finitely many choices for $Y$, so only finitely many $\text{PL}_{N+1}(O_T)$-orbits of $(f, X, Y) \in \mathcal{R}_{d,N}[m](L, T)$. This implies, by Lemma 2.2, that there are only finitely many $\text{PL}_{N+1}(O_S)$-orbits of $(f, X, Y) \in \mathcal{R}_{d,N}[m](K, S)$. □

Appendix A.

The purpose of the following lemma is to show that in our definition of good reduction for $X := Y \cup f(Y) \subseteq \mathbb{P}^N(K)$, the assumption on $Y$ not being contained in any degree $2d$ hypersurface is generically satisfied, provided $|Y| \geq \binom{N+d}{d}$. 

Lemma A.1. Let $m, N, d \geq 1$ be integers, and let $K$ be an arbitrary field. If $m \geq \binom{N+d}{d}$, then there are infinitely many subsets of $\mathbb{P}^N(K)$ of order $m$ that are not contained in any hypersurface of degree at most $d$. If $m \leq \binom{N+d}{d} - 2$, then no such subsets exist.

Proof. Let $P_1, \ldots, P_m$ be distinct points in $\mathbb{P}^N(K)$, and let $F \in K[x_1, \ldots, x_{N+1}]$ be a homogeneous polynomial defining a (hypothetical) degree $d$ hypersurface in $\mathbb{P}^N(K)$ that contains...
Let $\sigma_1, \ldots, \sigma_r$ be the standard monic degree $d$ homogeneous monomials in the $N + 1$ variables $x_1, \ldots, x_{N+1}$, and write

$$F = \sum_{i=1}^{r} a_i \sigma_i$$

for $a_i \in K$. At least one coefficient $a_i$ is nonzero; without loss of generality, suppose $a_0 \neq 0$, and rescale the $a_i$ so that $a_0 = 1$. Then we have

$$\sigma_1(P_1) + a_2 \sigma_2(P_1) + \cdots + a_r \sigma_r(P_1) = 0$$

$$\vdots$$

$$\sigma_1(P_m) + a_2 \sigma_2(P_m) + \cdots + a_r \sigma_r(P_m) = 0.$$

Switching our perspective, we can view the $a_i$ for $2 \leq i \leq r$ as the coordinates of a vector $(a_2, \ldots, a_r) \in K^{r-1}$ that is a solution to the system of equations

$$\sigma_1(P_1) + A_2 \sigma_2(P_1) + \cdots + A_r \sigma_r(P_1) = 0$$

$$\vdots$$

$$\sigma_1(P_m) + A_2 \sigma_2(P_m) + \cdots + A_r \sigma_r(P_m) = 0.$$

where the $A_i$ are seen as indeterminates. But this is a system of $m$ equations in $r - 1$ variables, which we know will have a solution regardless of what $P_1, \ldots, P_m$ are if $m < r - 1$. As $r$ is the number of monic homogeneous monomials of degree $d$ in $N + 1$ variables, we have that $r = \binom{N+d}{d}$.

Now suppose $m = r - 1$. If $(\sigma_j(P_i))_{j \geq 2, i \geq 1}$ has nonzero determinant, then the system has a unique solution. Viewing the coordinates of the $P_i$ as indeterminates, such a determinant can be viewed as a polynomial in $(m-1)(N+1)$ variables, and so defines a hypersurface in $\mathbb{A}^{(m-1)(N+1)}(K)$. Clearly there are infinitely many choices of coordinates for the $P_i$ that yield a point not on this hypersurface. Supposing there is a unique solution to the system, we are then free to choose an $r$th point not on the hypersurface corresponding to this solution. We thus see that the collection of order $r$ subsets of $\mathbb{P}^N_K$, parametrized by $(\mathbb{P}^N_K)^r/S_r$ (allowing the points to occur with multiplicity), has a Zariski dense set of elements whose corresponding subsets of $\mathbb{P}^N_K$ are not contained in any single hypersurface of degree $d$. The statement for configurations of more than $r = \binom{N+d}{d}$ points follows from a similar argument. □
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