A NOTE ON KOSMANN-LIE
DERIVATIVES OF WEYL SPINORS.

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ABSTRACT. Kosmann-Lie derivatives in the bundle of Weyl spinors are considered. It is shown that the basic spin-tensorial fields of this bundle are constants with respect to these derivatives.

1. INTRODUCTION.

Lie derivatives arise in studying continuous symmetries of various geometric structures on manifolds. They are also used in symmetry analysis of ordinary and partial differential equations (see [1]). In general relativity the bundle of Weyl spinors $SM$ is a special geometric structure built over the space-time manifold $M$. The main goal of this paper is to clarify the procedure of applying Lie derivatives to the basic attributes of this geometric structure, i.e. to the basic spin-tensorial fields associated with the bundle of Weyl spinors.

2. LIE DERIVATIVES OF SPATIAL STRUCTURES.

Let $M$ be a space-time manifold of general relativity. This means that it is a four-dimensional orientable manifold equipped with a Minkowski type metric $g$ and with a polarization. A polarization, which is typically not mentioned, is a geometric structure that marks the future half light cone in the tangent space $T_p(M)$ for each point $p \in M$ (see more details in [2]). A Lie derivative $L_X$ is usually given by some vector field $X$ in $M$. Once such a vector field $X$ is fixed, it produces a one-parametric local group of local diffeomorphisms:

$$\varphi_\varepsilon : M \rightarrow M.$$  \hspace{1cm} (2.1)

The letter $t$ is typically used for the parameter of this local group (see [3]), but here we use the Greek letter $\varepsilon$ since $t$ in physics is reserved for the time variable. The local diffeomorphisms (2.1) induce the local diffeomorphisms

$$\varphi_\varepsilon : TM \rightarrow TM,$$

$$\varphi_\varepsilon^* : T^*M \rightarrow T^*M.$$  \hspace{1cm} (2.2)

in tangent and cotangent bundles respectively. These induced diffeomorphisms (2.2) act as linear mappings in fibers of $TM$ and $T^*M$. For this reason they can be extended to local diffeomorphisms of tensor bundles:

$$\varphi_\varepsilon : T'_x M \rightarrow T'_x M.$$  \hspace{1cm} (2.3)

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Here in (2.3) through $T^r M$ we denote the following tensor product of $r$ copies of the tangent bundle $TM$ and $s$ copies of the cotangent bundle $T^*M$:

\[ T^r M = TM \otimes \ldots \otimes TM \otimes T^*M \otimes \ldots \otimes T^*M. \quad (2.4) \]

Let’s study the diffeomorphisms (2.1) and (2.2) in more details. Assume that $p$ and $q$ are two points of the space-time manifold $M$ such that $q = \varphi_\varepsilon(p)$. Then $p = \varphi_{-\varepsilon}(q)$ and we have the following commutative diagram:

\[
\begin{array}{ccc}
T_p(M) & \xrightarrow{\varphi_{\varepsilon*}} & T_q(M) \\
\pi & & \pi \\
p & \xrightarrow{\varphi_{\varepsilon}} & q \\
\pi & & \pi \\
T_p^s(M) & \xrightarrow{\varphi_{-\varepsilon}^*} & T_q^s(M).
\end{array}
\]

Assume that we have some local chart with the coordinates $x^0, x^1, x^2, x^3$ in some neighborhood of the point $q$. Assume also that $\varepsilon$ is small enough so that the point $p = \varphi_{-\varepsilon}(q)$ in (2.5) is covered by the same local chart. Then the coordinates of the points $p$ and $q$ in this chart are related to each other as follows:

\[\begin{align*}
x^0 &= u^0(\varepsilon, y^0, y^1, y^2, y^3), & y^0 &= u^0(-\varepsilon, x^0, x^1, x^2, x^3), \\
x^1 &= u^1(\varepsilon, y^0, y^1, y^2, y^3), & y^1 &= u^1(-\varepsilon, x^0, x^1, x^2, x^3), \\
x^2 &= u^2(\varepsilon, y^0, y^1, y^2, y^3), & y^2 &= u^2(-\varepsilon, x^0, x^1, x^2, x^3), \\
x^3 &= u^3(\varepsilon, y^0, y^1, y^2, y^3), & y^3 &= u^3(-\varepsilon, x^0, x^1, x^2, x^3).
\end{align*}\]

Using (2.6), we define the matrices $\Phi(\varepsilon)$ and $\Phi(-\varepsilon)$ with the components

\[\Phi^i_j(\varepsilon) = \frac{\partial u^i(\varepsilon, y^0, y^1, y^2, y^3)}{\partial y^j}, \quad \Phi^i_j(-\varepsilon) = \frac{\partial u^i(-\varepsilon, x^0, x^1, x^2, x^3)}{\partial x^j}.\]

Let $Y_{i_1 \ldots i_r}(x^0, x^1, x^2, x^3)$ be the components of some tensorial field $Y$ of the type $(r, s)$ in the local coordinates $x^0, x^1, x^2, x^3$ and let $\varphi_\varepsilon(Y)_{i_1 \ldots i_r}(x^0, x^1, x^2, x^3)$ be the components of its image $\varphi_\varepsilon(Y)$ under the mapping (2.3). Then we have

\[\varphi_\varepsilon(Y)_{i_1 \ldots i_r}(x^0, x^1, x^2, x^3) = \sum_{k_1, \ldots, k_s}^{3} \Phi^i_{h_1}(\varepsilon) \ldots \Phi^i_{h_r}(\varepsilon) \times \Phi^{k_1}_{j_1}(-\varepsilon) \ldots \Phi^{k_s}_{j_s}(-\varepsilon) Y_{j_1 \ldots j_s}(y^0, y^1, y^2, y^3).\]

According to [3], the Lie derivative $L_X$ applied to $Y$ is defined as follows:

\[L_X(Y) = \lim_{\varepsilon \to 0} \frac{Y - \varphi_\varepsilon(Y)}{\varepsilon} = - \frac{d \varphi_\varepsilon(Y)}{d \varepsilon} \bigg|_{\varepsilon=0}.\]
Let $X^0, X^1, X^2, X^3$ be the components of the vector field $X$ in the local coordinates $x^0, x^1, x^2, x^3$. Then for $\varepsilon \to 0$ we have the following Taylor expansions of the functions (2.6) representing the mappings $\varphi_\varepsilon$ and $\varphi_{-\varepsilon}$:

$$
\begin{align*}
&u^0(\varepsilon, y^0, y^1, y^2, y^3) = y^0 + X^0(y^0, y^1, y^2, y^3) \varepsilon + \ldots, \\
u^1(\varepsilon, y^0, y^1, y^2, y^3) = y^1 + X^1(y^0, y^1, y^2, y^3) \varepsilon + \ldots, \\
u^2(\varepsilon, y^0, y^1, y^2, y^3) = y^2 + X^2(y^0, y^1, y^2, y^3) \varepsilon + \ldots, \\
u^3(\varepsilon, y^0, y^1, y^2, y^3) = y^3 + X^3(y^0, y^1, y^2, y^3) \varepsilon + \ldots.
\end{align*}
$$

(2.10)

Applying (2.10) and (2.11) to (2.7), we derive

$$
\Phi_j^i(\varepsilon) = \delta_j^i + \frac{\partial X^i(x^0, x^1, x^2, x^3)}{\partial x^j} \varepsilon + \ldots, \\
\Phi_j^i(-\varepsilon) = \delta_j^i - \frac{\partial X^i(x^0, x^1, x^2, x^3)}{\partial x^j} \varepsilon + \ldots.
$$

(2.12) and (2.13)

Here $X^i(x^0, x^1, x^2, x^3)$ are the components of the vector field $X$ in the local coordinates $x^0, x^1, x^2, x^3$. Now if we denote by $L_X(Y)_{j_1 \ldots j_s}$ the components of $L_X(Y)$, then, applying (2.12) and (2.12) to (2.8) and taking into account (2.9), we obtain

$$
L_X(Y)_{j_1 \ldots j_s} = \sum_{m=1}^s \sum_{k_m=0}^3 \frac{\partial X^k_m}{\partial x^{j_m}} Y_{i_1 \ldots i_r}^{j_1 \ldots j_s} - \sum_{m=1}^r \sum_{k_m=0}^3 \frac{\partial X^1_m}{\partial x^{j_m}} Y_{i_1 \ldots i_r}^{j_1 \ldots j_s} + \sum_{k=0}^3 X^k \frac{\partial Y_{i_1 \ldots i_r}}{\partial x^k}.
$$

(2.14)

The Lie derivative $L_X$ given by the formula (2.14) possesses the following properties:

1. the Lie derivative $L_X$ preserves the type of a tensor field, i.e. $Y$ and $L_X(Y)$ are tensor fields of the same type;
2. $L_X(Y_1 \otimes Y_2) = L_X(Y_1) \otimes Y_2 + Y_1 \otimes L_X(Y_2)$ for arbitrary two tensorial fields $Y_1$ and $Y_2$;
3. $L_X(C(Y)) = C(L_X(Y))$, i.e. $L_X$ commute with contractions.

The properties (1)–(3) are easily derived with the use of the formula (2.14) itself.

3. Lie derivatives in frames formalism.

Let $x^0, x^1, x^2, x^3$ be the local coordinates of some local chart of the space-time manifold $M$. The coordinates $x^0, x^1, x^2, x^3$ induce the frame $X_0, X_1, X_2, X_3$ of the coordinate vector fields in the domain of these coordinates:

$$
X_0 = \frac{\partial}{\partial x^0}, \quad X_1 = \frac{\partial}{\partial x^1}, \quad X_2 = \frac{\partial}{\partial x^2}, \quad X_3 = \frac{\partial}{\partial x^3}.
$$

(3.1)
The frame composed by the vector fields (3.1) is a holonomic frame since these vector fields commute with each other:

\[ [X_i, X_j] = 0. \]

However, one can consider some non-holonomic frame \( \mathbf{\Upsilon}_0, \mathbf{\Upsilon}_1, \mathbf{\Upsilon}_2, \mathbf{\Upsilon}_3 \), i.e. a frame with non-commuting vector fields:

\[ [\mathbf{\Upsilon}_i, \mathbf{\Upsilon}_j] = \sum_{k=0}^{3} c^k_{ij} \mathbf{\Upsilon}_k. \]  (3.2)

The commutation coefficients \( c^k_{ij} \) in (3.2) are uniquely determined by the frame vector fields \( \mathbf{\Upsilon}_0, \mathbf{\Upsilon}_1, \mathbf{\Upsilon}_2, \mathbf{\Upsilon}_3 \) since they are linearly independent at each point of their domain. Our nearest goal is to derive the formula analogous to (2.14) for the case where all tensorial fields are represented by their components in some non-holonomic frame \( \mathbf{\Upsilon}_0, \mathbf{\Upsilon}_1, \mathbf{\Upsilon}_2, \mathbf{\Upsilon}_3 \).

Let \( \phi \) be a scalar field. Then the Lie derivative \( L_X \) of \( \phi \) is reduced to the differentiation of the function \( \phi \) along the vector \( X \):

\[ L_X(\phi) = \sum_{k=0}^{3} X^k \frac{\partial \phi}{\partial x^k}. \]  (3.3)

The formula (3.3) is easily derived by substituting \( Y = \phi \) with \( r = 0 \) and \( s = 0 \) into (2.14). Similarly, if \( Y \) is a vector field, from (2.14) we derive

\[ L_Y(Y) = [X, Y]. \]  (3.4)

Now assume that both of the vector fields \( X \) and \( Y \) are represented by their expansions in a non-holonomic frame \( \mathbf{\Upsilon}_0, \mathbf{\Upsilon}_1, \mathbf{\Upsilon}_2, \mathbf{\Upsilon}_3 \):

\[ X = \sum_{k=0}^{3} X^k \mathbf{\Upsilon}_k, \quad Y = \sum_{i=0}^{3} Y^i \mathbf{\Upsilon}_i. \]  (3.5)

Then, substituting (3.5) into (3.4), we derive

\[ L_X(Y)^k = \sum_{i=0}^{3} X^i L_{\mathbf{\Upsilon}_i}(Y^k) - \sum_{i=0}^{3} Y^i L_{\mathbf{\Upsilon}_i}(X^k) + \sum_{i=0}^{3} \sum_{j=0}^{3} c^k_{ij} X^i Y^j. \]  (3.6)

The Lie derivatives \( L_{\mathbf{\Upsilon}_i}(Y^k) \) and \( L_{\mathbf{\Upsilon}_i}(X^k) \) in (3.6) are calculated according to the formula (3.3), i.e. we substitute \( \phi = Y^k \) and \( \phi = X^k \) into (3.3). Note that \( X^k \) in (3.3) differ from that of (3.5). The components of \( X \) in the formula (3.3) are taken from the expansion of the vector field \( X \) in the holonomic frame (3.1). The non-holonomic version of the formula (3.3) looks like

\[ L_X(\phi) = \sum_{k=0}^{3} X^k L_{\mathbf{\Upsilon}_k}(\phi). \]
Let's denote by \( \eta^0, \eta^1, \eta^2, \eta^3 \) the dual frame for \( \Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3 \). This means that \( \eta^0, \eta^1, \eta^2, \eta^3 \) are four covectorial fields such that
\[
\eta^i(\Upsilon_j) = \langle \eta^i, \Upsilon_j \rangle = C(\eta^i \otimes \Upsilon_j) = \delta^i_j. \tag{3.7}
\]

Using the properties (1)–(3) from Section 2 and using (3.4), from (3.7) we derive
\[
L_{\Upsilon_i}(\eta^k) = -\sum_{j=0}^{3} c^k_{ij} \eta^i. \tag{3.8}
\]

The commutation coefficients \( c^k_{ij} \) in (3.8) are the same as in (3.2). Assume that \( \Upsilon \) is a covectorial field expanded in the frame \( \eta^0, \eta^1, \eta^2, \eta^3 \):
\[
\Upsilon = \sum_{j=1}^{3} \Upsilon_j \eta^j. \tag{3.9}
\]

From (3.7), (3.8), and (3.9) we derive
\[
L_{\Upsilon_j}(\Upsilon_k) = \sum_{i=0}^{3} \Upsilon_i L_{\Upsilon_j}(\Upsilon_k) = \sum_{i=0}^{3} \Upsilon_i L_{\Upsilon_j}(\Upsilon_i) - \sum_{j=0}^{3} \sum_{j=0}^{3} \Upsilon_j \Upsilon^i c^j_{ik}. \tag{3.10}
\]

Now, using the properties (1)–(3) again and combining (3.6) with (3.10), we can extend the formula (3.6) to the case of an arbitrary tensor field \( \Upsilon \) of the type \( r, s \):

\[
L_{\Upsilon_j}(\Upsilon_{ij}^{i_1 \ldots i_s}) = \sum_{i=0}^{3} \Upsilon_i L_{\Upsilon_j}(\Upsilon_{ij}^{i_1 \ldots i_s}) +
\sum_{m=1}^{r} \sum_{k_m=0}^{3} \left( L_{\Upsilon_{jm}}(\Upsilon_{km}) - \sum_{i=0}^{3} c_{ijm} \Upsilon^i \right) \Upsilon^{i_1 \ldots i_r}_{ij_1 \ldots j_m \ldots j_s}
- \sum_{m=1}^{r} \sum_{k_m=0}^{3} \left( L_{\Upsilon_{km}}(\Upsilon^{km}) - \sum_{i=0}^{3} c_{ikm} \Upsilon^i \right) \Upsilon^{i_1 \ldots i_r}_{ij_1 \ldots j_m \ldots j_s}. \tag{3.11}
\]

The formulas (3.6) and (3.10) are special cases of the formula (3.11). If the frame \( \Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3 \) coincides with the holonomic frame (3.1), then \( c^k_{ij} = 0 \) and the formula (3.11) reduces to (2.14).

Now let's return back to the formula (2.8). This formula is valid in frame presentation of tensor fields too. However, the matrices \( \Phi^i_j(\varepsilon) \) and \( \Phi^i_j(-\varepsilon) \) in this case are not given by the formulas (2.7). Here we use the formulas
\[
\Phi^i_j(\varepsilon) = \delta^i_j + \left( L_{\Upsilon_j}(\Upsilon^i) - \sum_{m=0}^{3} \Upsilon^m c_{mj} \right) \varepsilon + \ldots, \tag{3.12}
\]
\[
\Phi^i_j(-\varepsilon) = \delta^i_j - \left( L_{\Upsilon_j}(\Upsilon^i) - \sum_{m=0}^{3} \Upsilon^m c_{mj} \right) \varepsilon + \ldots. \tag{3.13}
\]

The formulas (3.12) and (3.13) are analogous to (2.12) and (2.13).
4. Tangent vector fields on vector bundles.

Let $\mathcal{V}M$ be an $n$-dimensional vector bundle over the space-time manifold $M$. Any trivialization of $\mathcal{V}M$ is given by $n$ sections $\Upsilon_1, \ldots, \Upsilon_n$ linearly independent at each point of their domain $U$. Let $v$ be a vector of the fiber $V_q(M)$:

$$v = v^1 \Upsilon_1 + \ldots + v^n \Upsilon_n. \quad (4.1)$$

Then $\tilde{q} = (q, v)$ is a point of $\mathcal{V}M$. If $x^0, x^1, x^2, x^3$ are some local coordinates within the domain $U \subset M$ and $v^0, v^1, v^2, v^3$ are taken from (4.1), then

$$x^0, \ldots, x^3, v^1, \ldots, v^n \quad (4.2)$$

are the coordinates of the point $\tilde{q} = (q, v)$. The coordinates (4.2) are naturally subdivided into two groups — the base coordinates $x^0, \ldots, x^3$ and the fiber coordinates $v^1, \ldots, v^n$. Let $\mathbf{X}$ be a tangent vector field on $\mathcal{V}M$. In the local coordinates (4.2) it is represented as the following differential operator:

$$\mathbf{X} = \sum_{i=0}^{3} X^i \frac{\partial}{\partial x^i} + \sum_{i=1}^{n} V^i \frac{\partial}{\partial v^i}. \quad (4.3)$$

Under the canonical projection $\pi: \mathcal{V}M \to M$ the vector field (4.3) is mapped to

$$\mathbf{X} = \sum_{i=0}^{3} X^i \frac{\partial}{\partial x^i}. \quad (4.4)$$

The vector field (4.3) produces the one-parametric local group of diffeomorphisms

$$\varphi_\varepsilon: \mathcal{V}M \to \mathcal{V}M \quad (4.5)$$

that extends the local group (2.1) produced by the vector field (4.4). Due to (4.5) the functions (2.6) are complemented with the functions

$$\begin{cases}
  u^1 = U^1(\varepsilon, y^0, \ldots, y^3, w^1, \ldots, w^n), \\
  \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
  u^n = U^n(\varepsilon, y^0, \ldots, y^3, w^1, \ldots, w^n),
\end{cases} \quad (4.6)$$

$$\begin{cases}
  w^1 = U^1(-\varepsilon, x^0, \ldots, x^3, v^1, \ldots, v^n), \\
  \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
  w^n = U^n(-\varepsilon, x^0, \ldots, x^3, v^1, \ldots, v^n). 
\end{cases} \quad (4.7)$$

**Definition 4.1.** The tangent vector field (4.3) on $\mathcal{V}M$ is called *concordant with the bundle structure* if the functions (4.6) and (4.7) are linear with respect to their arguments $v^1, \ldots, v^n$ and $w^1, \ldots, w^n$.

In the case of a concordant vector field $\mathbf{X}$ the local diffeomorphisms (4.5) break into the series of linear mappings

$$\varphi_\varepsilon: \mathcal{V}_p(M) \to \mathcal{V}_q(M), \quad (4.8)$$
where \( q = \varphi_\varepsilon(p) \). The functions (4.6) present the diffeomorphisms (4.8) in local coordinates. In the case of a concordant vector field they are given by the formulas

\[
U^i(\varepsilon, y^0, \ldots, y^3, w^1, \ldots, w^n) = \sum_{j=1}^n U^i_j(\varepsilon, y^0, \ldots, y^3) w^j. \tag{4.9}
\]

The vertical components of the tangent vector field (4.3) are given by the derivatives

\[
V^i = \left. \frac{dU^i(\varepsilon, x^0, \ldots, x^3, v^1, \ldots, v^n)}{d\varepsilon} \right|_{\varepsilon=0}. \tag{4.10}
\]

Substituting (4.9) into (4.10), we derive

\[
V^i = \sum_{j=1}^n V^i_j(x^0, \ldots, x^3) v^j, \quad \text{where} \quad V^i_j = \left. U^i_j \right|_{\varepsilon=0}. \tag{4.11}
\]

**Theorem 4.1.** The tangent vector field (4.3) on a vector bundle \( VM \) is concordant with the bundle structure if and only if its vertical components are linear functions with respect to \( v^1, \ldots, v^n \) given by the formula (4.11).

For the matrices \( U^i_j \) in (4.9), which represent the linear mappings (4.8) in the frame \( \Upsilon_1, \ldots, \Upsilon_n \), the formulas (4.10) and (4.11) yield

\[
U^i_j(\varepsilon) = \delta^i_j + V^i_j(x^0, \ldots, x^3) \varepsilon + \ldots, \tag{4.12}
\]

\[
U^i_j(-\varepsilon) = \delta^i_j - V^i_j(x^0, \ldots, x^3) \varepsilon + \ldots.
\]

The expansions (4.12) are similar to (2.12), (2.13), (3.12), and (3.13).

Let \( Y \) be a tensor field of the type \((r, s)\) associated with the vector bundle \( VM \), i.e. let \( Y \) be a section of \( V_r^s M \), where \( V_r^s M \) is the following tensor bundle:

\[
V_r^s M = VM \otimes \ldots \otimes VM \otimes V^* M \otimes \ldots \otimes V^* M. \tag{4.13}
\]

Assume that \( Y_{i_1 \ldots i_r}^{j_1 \ldots j_s} (x^0, x^1, x^2, x^3) \) are the components of the tensor field \( Y \) in the frame \( \Upsilon_1, \ldots, \Upsilon_n \). Then, using the quantities \( V^i_j \) from the expansions (4.12), we define the Lie derivative \( \mathcal{L}_X(Y) \) of the field \( Y \):

\[
\mathcal{L}_X(Y)^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \sum_{i=0}^3 X^i \left[ \mathcal{L}_{X^i} (Y_{j_1 \ldots j_s}^{i_1 \ldots i_r}) + \right. \\
\left. + \sum_{m=1}^n \sum_{k_m=1}^n V^{k_m}_{j_1 \ldots k_m \ldots j_s} \right] Y_{j_1 \ldots j_s}^{i_1 \ldots i_r}
\]

\[
+ \sum_{m=1}^n \sum_{k_m=1}^n V^{k_m}_{j_1 \ldots j_s} \ Y_{j_1 \ldots j_s}^{i_1 \ldots i_r} + \sum_{m=1}^n \sum_{k_m=1}^n V^{k_m}_{j_1 \ldots j_s} \ Y_{j_1 \ldots j_s}^{i_1 \ldots i_r}. \tag{4.14}
\]

The Lie derivative (4.14) is called natural if the quantities \( V^i_j \) are expressed in some natural way through the components of the vector field \( X \) in (4.4).
5. NATURAL LIFtings and Kosmann liftings.

In this section we apply the results of the previous section 4 to the tangent bundle $TM$, i.e. we set $VM = TM$. Comparing the formula (4.13) with (2.4) and the formula (4.14) with (3.11), we find that

$$V_j = L_{\psi_j}(X^i) - \sum_{m=0}^{3} X^m c^i_{mj}. \quad (5.1)$$

Now we substitute (5.1) into (4.11) and then we substitute (4.11) into (4.3):

$$X_N = \sum_{i=0}^{3} \sum_{m=0}^{3} X^m \psi_m \frac{\partial}{\partial x^i} + \sum_{i=0}^{3} \sum_{j=0}^{3} \left( L_{\psi_j}(X^i) - \sum_{m=0}^{3} X^m c^i_{mj} \right) v^j \frac{\partial}{\partial v^i}. \quad (5.2)$$

The tangent vector field (5.2) on $TM$ is a natural lifting of the vector field $X = \sum_{i=0}^{3} \sum_{m=0}^{3} X^m \psi_m \frac{\partial}{\partial x^i}$ from $M$ to $TM$. The Lie derivative (4.14) determined by the vector field (5.3) and by its lifting (5.2) is a natural Lie derivative coinciding with (3.11) and (2.14).

Now let's recall that the space-time manifold $M$ is equipped with the metric $g$. Its signature is $(+, -, -, -)$. For this reason each fiber $T_p(M)$ of the tangent bundle $TM$ is a pseudo-Euclidean linear vector space.

**Definition 5.1.** A lifting $\mathbf{X}_N$ of a vector field $\mathbf{X}$ from $M$ to $TM$ is called a Kosmann lifting if the linear mappings (4.8) associated with this lifting are isometries.

Kossmann liftings were first introduced by Yvette Kosmann in [4–7].

**Theorem 5.1.** The natural lifting (5.2) of a vector field $\mathbf{X}$ is a Kosmann lifting if and only if $\mathbf{X}$ is a Killing vector field.

The proof is trivial. By definition, Killing vector fields are those whose local diffeomorphisms preserve the metric tensor $g$. Hence, the mapping $\varphi_{\varepsilon^*}$ from (2.2) restricted to any fiber $T_p(M)$ is a linear isometry.

Let's study the isometry condition from the definition 5.1 in more details. Applying the linear mappings (4.8) to the metric tensor, we get the equality

$$g_{ij}(x^0, \ldots, x^3) = \sum_{r=0}^{3} \sum_{s=0}^{3} U^r_i(-\varepsilon) U^s_j(-\varepsilon) g_{rs}(y^0, \ldots, y^3). \quad (5.4)$$

Differentiating the formula (5.4) with respect to $\varepsilon$, we take into account the formulas (4.12), (2.6), (2.10), (2.11), (4.10), and (4.11). As a result we get

$$0 = \sum_{r=0}^{3} V^r_i g_{rj} + \sum_{r=0}^{3} V^r_j g_{ir} + \sum_{m=0}^{3} X^m L_{\psi_m}(g_{ij}). \quad (5.5)$$

The equality (5.5) can be simplified to

$$\mathcal{L}_\mathbf{X}(g) = 0, \quad (5.6)$$
where the Lie derivative $\mathcal{L}_X$ is calculated according to the formula (4.14). We shall treat the equality (5.6) neither as a condition for $X$ nor as a condition for $g$, but as a condition for $V^j_i$. For this purpose we denote

$$V_{ij} = \sum_{r=0}^{3} V^r_i g_{rj}. \quad (5.7)$$

Then the equality (5.5), which is equivalent to (5.6) is written as follows:

$$V_{ij} + V_{ji} = -\sum_{m=0}^{3} X^m \mathcal{L}_m (g_{ij}). \quad (5.8)$$

The equality (5.8) fixes the symmetric part of $V_{ij}$ for a Kosmann lifting of a vector field. The skew-symmetric part of $V_{ij}$ can be obtained by alternating (5.1). As a result we get the following two formulas:

$$V^\text{sym}_{ij} = -\frac{1}{2} \sum_{m=0}^{3} X^m \mathcal{L}_m (g_{ij}), \quad (5.9)$$

$$V^\text{skew}_{ij} = \frac{1}{2} \sum_{r=0}^{3} L_{X_r} (X^r) g_{rj} - \frac{1}{2} \sum_{r=0}^{3} L_{X_r} (X^r) g_{ri} - \frac{1}{2} \sum_{r=0}^{3} \sum_{m=0}^{3} X^m c^r_m g_{rj} + \frac{1}{2} \sum_{r=0}^{3} \sum_{m=0}^{3} X^m c^r_m g_{ri}. \quad (5.10)$$

Adding the formulas (5.9) and (5.10), we derive

$$V_{ij} = -\frac{1}{2} \sum_{m=0}^{3} X^m \mathcal{L}_m (g_{ij}) + \frac{1}{2} \sum_{r=0}^{3} L_{X_r} (X^r) g_{rj} - \frac{1}{2} \sum_{r=0}^{3} \sum_{m=0}^{3} L_{X_r} (X^r) g_{ri} - \frac{1}{2} \sum_{r=0}^{3} \sum_{m=0}^{3} X^m c^r_m g_{rj} + \frac{1}{2} \sum_{r=0}^{3} \sum_{m=0}^{3} X^m c^r_m g_{ri}. \quad (5.11)$$

In order to get back to $V^j_i$ we need to raise the index $j$ in (5.11):

$$V^j_i = -\frac{1}{2} \sum_{m=0}^{3} \sum_{r=0}^{3} X^m \mathcal{L}_m (g_{ir}) g^{rj} - \frac{1}{2} \sum_{r=0}^{3} \sum_{s=0}^{3} L_{X_r} (X^r) g^{rj} + \frac{1}{2} \sum_{r=0}^{3} \sum_{s=0}^{3} X^m c^r_m g_{rj} + \frac{1}{2} \sum_{r=0}^{3} \sum_{m=0}^{3} X^m c^r_m g_{ri}. \quad (5.12)$$

Note that in order to fit (4.11) we should exchange the indices $i$ and $j$ in (5.12):

$$V^i_j = -\frac{1}{2} \sum_{m=0}^{3} \sum_{r=0}^{3} g^{ir} X^m \mathcal{L}_m (g_{rj}) - \frac{1}{2} \sum_{r=0}^{3} \sum_{s=0}^{3} g^{rs} L_{X_r} (X^r) g_{rj} + \frac{1}{2} \sum_{r=0}^{3} \sum_{s=0}^{3} X^m c^r_m g_{rj} + \frac{1}{2} \sum_{r=0}^{3} \sum_{m=0}^{3} X^m c^r_m g_{ri}. \quad (5.13)$$
As a result, substituting (5.13) into (4.11), we derive the formula

\[ V^i = -\frac{1}{2} \sum_{m=0}^{3} \sum_{r=0}^{3} g^{ir} X^m L_{\mathcal{Y}_m}(g_{rj}) v^j + \frac{1}{2} \sum_{j=0}^{3} L_{\mathcal{Y}_j}(X^i) v^j - \frac{1}{2} \sum_{r=0}^{3} \sum_{s=0}^{3} \sum_{j=0}^{3} g^{is} L_{\mathcal{Y}_s}(X^r) g_{rj} v^j - \frac{1}{2} \sum_{m=0}^{3} \sum_{j=0}^{3} X^m e^i_{mj} v^j + \frac{1}{2} \sum_{r=0}^{3} \sum_{s=0}^{3} \sum_{m=0}^{3} \sum_{j=0}^{3} g^{is} X^m e^r_{ms} g_{rj} v^j. \]  

(5.14)

Applying (5.14) to (4.3) and taking into account that we choose \( VM = TM \), we get the following tangent vector field on the tangent bundle:

\[ \mathfrak{X}_K = \sum_{i=0}^{3} \sum_{m=0}^{3} X^m \mathcal{Y}^i_m \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_{i=0}^{3} \sum_{j=0}^{3} L_{\mathcal{Y}_j}(X^i) v^j - \frac{3}{2} \sum_{m=0}^{3} \sum_{r=0}^{3} \sum_{j=0}^{3} e^i_{mj} v^j + \frac{3}{2} \sum_{r=0}^{3} \sum_{s=0}^{3} \sum_{m=0}^{3} \sum_{j=0}^{3} g^{is} X^m e^r_{ms} g_{rj} v^j - \frac{3}{2} \sum_{j=0}^{3} \sum_{m=0}^{3} \sum_{r=0}^{3} \sum_{s=0}^{3} g^{ir} X^m L_{\mathcal{Y}_m}(g_{rj}) v^j \]  

(5.15)

**Definition 5.2.** The tangent vector field (5.15) on the tangent bundle \( TM \) is called the standard Kosmann lifting of the vector field (4.4) from \( M \) to \( TM \).

We can calculate the standard Kosmann lifting (5.15) using the holonomic frame (3.1) instead of the non-holonomic frame \( \mathcal{Y}_0, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3 \). Then (5.15) reduces to

\[ \mathfrak{X}_K = \sum_{i=0}^{3} X^i \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_{i=0}^{3} \sum_{j=0}^{3} \sum_{r=0}^{3} \sum_{s=0}^{3} g^{ir} \frac{\partial X^r}{\partial x^s} g_{rj} v^j + \frac{3}{2} \sum_{j=0}^{3} \sum_{m=0}^{3} \sum_{r=0}^{3} \sum_{s=0}^{3} g^{ir} X^m \frac{\partial g_{rj}}{\partial x^m} v^j \]  

(5.16)

Now let’s remember that the metric \( g \) is associated with the metric connection \( \Gamma \). It is known as the Levi-Civita connections. The components of the Levi-Civita connection in a holonomic frame (3.1) are given by the formula

\[ \Gamma^k_{ij} = \sum_{r=0}^{3} \frac{g^{kr}}{2} \left( \frac{\partial g_{rj}}{\partial x^i} + \frac{\partial g_{jr}}{\partial x^i} - \frac{\partial g_{ji}}{\partial x^r} \right). \]  

(5.17)

The formula (5.17) is easily derived from the following two conditions (see [8]):

\[ \Gamma^k_{ij} = \Gamma^k_{ji}, \quad \nabla_k g_{ij} = 0. \]  

(5.18)
Using the connection components $\Gamma_{ij}^k$, now we express the partial derivatives in the formula (5.16) through the corresponding covariant derivatives:

$$\frac{\partial X^r}{\partial x^s} = \nabla_s X^r - \sum_{m=0}^3 \Gamma_{sm}^r X^m,$$

(5.19)

Moreover, from the second equality (5.18) we derive

$$\frac{\partial g_{rj}}{\partial x^m} = 3 \sum_{s=0}^3 \Gamma_{mj}^s g_{rs} + \sum_{s=0}^3 \Gamma_{mr}^s g_{sj}.$$  

(5.20)

Substituting (5.19) and (5.20) back into (5.16) and taking into account the first equality in (5.17), we get the following formula:

$$X^K = \sum_{i=0}^3 X^i \frac{\partial}{\partial x^i} + \sum_{i=0}^3 \sum_{j=0}^3 \left( -\frac{1}{2} \sum_{r=0}^3 \sum_{s=0}^3 g^{is} \nabla_s X^r g_{rj} + \frac{1}{2} \nabla_j X^i - \sum_{m=0}^3 X^m \Gamma_{mj}^i \right) v^j \frac{\partial}{\partial v^i}.$$  

(5.21)

The components of the natural lifting (5.2) also can be expressed through the connection components $\Gamma_{ij}^k$ and covariant derivatives in the holonomic frame (3.1):

$$X_N = \sum_{i=0}^3 X^i \frac{\partial}{\partial x^i} + \sum_{i=0}^3 \sum_{j=0}^3 \left( \nabla_j X^i - \sum_{m=0}^3 X^m \Gamma_{mj}^i \right) v^j \frac{\partial}{\partial v^i}.$$  

(5.22)

Two different liftings (5.21) and (5.22) are associated with two different Lie derivatives $L_X$ and $L_X$ respectively. Here $L_X$ is the regular Lie derivative, while $L_X$ is called the standard Kosmann-Lie derivative. Both of them are differentiations of the algebra of tensor fields. Comparing (5.21) and (5.22), we get the relationship

$$L_X = L_X + S_X.$$  

(5.23)

Here $S_X$ is a degenerate differentiation in the sense of the proposition 3.3 in Chapter I of [3]. From (5.21) and (5.22) we derive that the degenerate differentiation $S_X$ in (5.23) is given by the tensor field $S_X$ with the following components:

$$S^i_j(X) = \frac{\nabla_j X^i + \nabla^i X_j}{2} = \frac{1}{2} \nabla_j X^i + \frac{1}{2} \sum_{r=0}^3 \sum_{s=0}^3 g^{is} \nabla_s X^r g_{rj}.$$  

(5.24)

The tensor field $S_X$ with the components (5.24) is equal to zero if and only if $X$ is a Killing vector field. This fact is easily derived from the theorem 5.1 or from the equality (5.6), which is fulfilled identically by definition.
Another special case for the formula (5.15) is the case of a non-holonomic, but orthonormal frame $\mathbf{\gamma}_0$, $\mathbf{\gamma}_1$, $\mathbf{\gamma}_2$, $\mathbf{\gamma}_3$. In such a frame the components of the metric tensor are constants. They are given by the Minkowski matrix:

$$g_{ij} = g^{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$  

(5.25)

For the components of the Levi-Civita connection in such a frame we have

$$\Gamma^k_{ij} = \frac{c^k_{ij}}{2} \sum_{r=0}^{3} \sum_{s=0}^{3} \frac{c^r_{is}}{2} g^{kr} g_{sj} - \sum_{r=0}^{3} \sum_{s=0}^{3} \frac{c^r_{js}}{2} g^{ks} g_{ri}.$$  

(5.26)

The formula (5.26) is derived from the general formula (6.3) in [9]. From the formulas (5.25) and (5.26) we derive the following equalities:

$$L_{\mathbf{\gamma}_m} (g_{ij}) = 0,$$

$$\Gamma^k_{ij} - \Gamma^k_{ji} = c^k_{ij}.$$  

(5.27)

Moreover, from $\nabla_m g_{ij} = 0$ in this case we derive

$$\sum_{r=0}^{3} \sum_{s=0}^{3} g^{is} \Gamma^r_{ms} g_{rj} = -\Gamma^i_{mj}.$$  

(5.28)

In the non-holonomic frame $\mathbf{\gamma}_0$, $\mathbf{\gamma}_1$, $\mathbf{\gamma}_2$, $\mathbf{\gamma}_3$ the formulas (5.19) are replaced by

$$L_{\mathbf{\gamma}_s} (X^r) = \nabla_s X^r - \sum_{m=0}^{3} \Gamma^r_{sm} X^m,$$

$$L_{\mathbf{\gamma}_j} (X^i) = \nabla_j X^i - \sum_{m=0}^{3} \Gamma^i_{jm} X^m.$$  

(5.29)

Applying (5.27), (5.28), and (5.29) to (5.15), we get the formula

$$\mathbf{\nabla}_K \mathbf{X} = \sum_{i=0}^{3} \sum_{m=0}^{3} X^m \Gamma^i_{m} \frac{\partial}{\partial x^i} + \sum_{i=0}^{3} \sum_{j=0}^{3} \left( \frac{1}{2} \nabla_j X^i - \frac{1}{2} \sum_{r=0}^{3} \sum_{s=0}^{3} g^{is} \nabla_s X^r g_{rj} - \sum_{m=0}^{3} X^m \Gamma^i_{mj} \right) v^j \frac{\partial}{\partial v^i}.$$  

(5.30)

Similarly, applying (5.29) to (5.2) and taking into account (5.27), we get

$$\mathbf{\nabla}_N \mathbf{X} = \sum_{i=0}^{3} \sum_{m=0}^{3} X^m \Gamma^i_{m} \frac{\partial}{\partial x^i} + \sum_{i=0}^{3} \sum_{j=0}^{3} \left( \nabla_j X^i - \sum_{m=0}^{3} X^m \Gamma^i_{mj} \right) v^j \frac{\partial}{\partial v^i}.$$  

(5.31)

The formulas (5.30) to (5.31) coincide with (5.21) to (5.22) respectively, though $\Gamma^i_{mj}$ are not symmetric with respect to $m$ and $j$ in this case. The formulas (5.30)
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and (5.31) again lead to the formula (5.23), which is valid irrespective to the choice of a holonomic or non-holonomic frame in $M$.

Note that the formulas (5.29) are the same as the formula (3.10) in [10]. The formula (5.29) resembles the formulas (3.11) and (3.12) in [10]. However, it doesn’t coincide with them. Unlike [10] and [11], for the sake of simplicity in this paper I do not use principal fiber bundles at all.

6. Commutation relationships for Kosmann-Lie derivatives.

Regular Lie derivatives acting upon tensorial fields form a representation of the Lie algebra of vector fields in $M$. This fact is expressed by the formula

$$[L_X, L_Y] = L_{[X,Y]}.$$  \hfill (6.1)

Commutation relationships for Kosmann-Lie derivatives are different from (6.1):

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]} + S_{X,Y}.$$ \hfill (6.2)

Here $S_{X,Y}$ is a degenerate differentiation given by the tensor field $S_{X,Y}$, where

$$S_{X,Y} = L_X(S_Y) - L_Y(S_X) - S_{[X,Y]} + [S_X, S_Y].$$ \hfill (6.3)

By means of direct calculations the formula (6.3) can be reduced to

$$S_{X,Y} = -[S_X, S_Y].$$ \hfill (6.4)

Applying (6.4) to (6.2), we get

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]} - [S_X, S_Y].$$ \hfill (6.5)

The commutator $[S_{X,Y}]$ in the formulas (6.3), (6.4), and (6.5) is understood as a commutator of two operator-valued tensorial fields:

$$[S_X, S_Y] = S_X \ast S_Y - S_X \ast S_Y.$$ \hfill (6.6)

In the coordinate representation the commutator (6.6) turns to the commutator of two matrices whose components are calculated according to the formula (5.24).

Note that in general case the commutator (6.6) is not zero. For this reason $[\mathcal{L}_X, \mathcal{L}_Y] \neq \mathcal{L}_{[X,Y]}$. This fact is pointed out in [10].

7. Kosmann-Lie derivatives for Weyl spinors.

The bundle of Weyl spinors is a two-dimensional complex vector bundle over the space-time manifold $M$. We denote it $SM$. The spinor bundle $SM$ is related to the tangent bundle $TM$ in some special way. The relation of $SM$ and $TM$ is formulated in terms of frames. It is based on the well-known group homomorphism

$$\phi : SL(2, \mathbb{C}) \to SO^+(1, 3, \mathbb{R}).$$ \hfill (7.1)
Let $\Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3$ be a positively polarized right orthonormal frame of the tangent bundle $TM$. By definition (see Section 5 in [12] or Section 1 in [13]) it is canonically associated with some frame $\Psi_1, \Psi_2$ of $SM$ in such a way that if two positively polarized right orthonormal frames $\Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3$ and $\tilde{\Upsilon}_0, \tilde{\Upsilon}_1, \tilde{\Upsilon}_2, \tilde{\Upsilon}_3$ are bound with the transition matrices $S$ and $T = S^{-1}$ in the formulas

$$
\tilde{\Upsilon}_i = \sum_{j=0}^{3} S^j_i \Upsilon_j, \quad \Upsilon_i = \sum_{j=0}^{3} T^j_i \tilde{\Upsilon}_j,
$$

then their associated spinor frames $\Psi_1, \Psi_2$ and $\tilde{\Psi}_1, \tilde{\Psi}_2$ are bound with the transition matrices $\mathcal{S}$ and $\mathcal{I} = \mathcal{S}^{-1}$ in the formulas

$$
\tilde{\Psi}_i = \sum_{j=1}^{2} \mathcal{S}^j_i \Psi_j, \quad \Psi_i = \sum_{j=1}^{2} \mathcal{I}^j_i \tilde{\Psi}_j,
$$

and the spacial transition matrices $S$ and $T$ are produced from the spinor transition matrices $\mathcal{S}$ and $\mathcal{I}$ through the group homomorphism (7.1):

$$
S = \phi(\mathcal{S}), \quad T = \phi(\mathcal{I}).
$$

A spinor frame $\Psi_1, \Psi_2$ canonically associated with some positively polarized right orthonormal tangent frame $\Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3$ is called an orthonormal frame of the spinor bundle $SM$. In order to visualize this canonical frame association in $SM$ and $TM$ we use the following diagram:

$$
\text{Orthonormal frames} \quad \rightarrow \quad \text{Positively polarized right orthonormal frames}.
$$

(7.2)

There are two basic spin-tensorial fields in the bundle of Weyl spinors:

| Symbol | Name                        | Spin-tensorial type |
|--------|-----------------------------|---------------------|
| $d$    | Skew-symmetric metric tensor | (0, 2)(0, 0)(0, 0)   |
| $G$    | Infeld-van der Waerden field | (1, 0)(1, 0)(0, 1)   |

(7.3)

Their role for $SM$ is similar to that of the metric tensor $g$ for $TM$. The spin-tensorial type in the table (3.1) specifies the number of indices in coordinate representation of fields. The first two numbers are the numbers of upper and lower spinor indices, the second two numbers are the numbers of upper and lower conjugate spinor indices, and the last two numbers are the numbers of upper and lower tensorial indices (they are also called spacial indices).

The spin-tensorial fields $d$ and $G$ are introduced by means of the explicit formulas for their components in canonically associated frame pairs (7.2). Let $\Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3$ be some positively polarized right orthonormal frame in $TM$ and let $\Psi_1, \Psi_2$ be its associated orthonormal spinor frame in $SM$. The components of the Infeld-van der Waerden field $G$ in such a frame pair composed by two canonically associated
frames are given by the following Pauli matrices:

\[
G_{0i}^{i\bar{j}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma_0, \\
G_{2i}^{i\bar{j}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2,
\]

\[
G_{1i}^{i\bar{j}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \\
G_{3i}^{i\bar{j}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3.
\] (7.4)

The skew-symmetric metric tensor \( d \) is given by the matrix

\[
d_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\] (7.5)

in any orthonormal spinor frame \( \Psi_1, \Psi_2 \). Unlike (7.4), the choice of the associated frame \( \Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3 \) is inessential for the components of the matrix (7.5) since \( d \) has no spatial indices at all. The dual metric tensor for \( d \) is given by the matrix

\[
d^{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\] (7.6)

The matrix (7.6) is inverse to the matrix (7.5).

Let \( \mathcal{S} \) be some matrix from the group \( \text{SL}(2, \mathbb{C}) \) and let \( S = \phi(\mathcal{S}) \) be its image under the homomorphism (7.1). Then we have the relationships

\[
2 \sum_{i=1}^{2} \sum_{j=1}^{2} \mathcal{S}_i^a \sigma_{m}^{i\bar{j}} \mathcal{S}_i^c \sigma_{k}^{a\bar{c}} = \sum_{k=1}^{3} S_k^m \sigma_k^{a\bar{a}},
\] (7.7)

\[
2 \sum_{i=1}^{2} \sum_{j=1}^{2} \mathcal{S}_i^a d_{ij} \mathcal{S}_i^b = d_{ab},
\] (7.8)

where \( \sigma_{m}^{i\bar{j}} \) and \( \sigma_k^{a\bar{a}} \) are the components of the Pauli matrices (7.4). In essential, the relationships (7.7) form a definition of the group homomorphism (7.1) (see [13] for more details). The relationships (7.8) are fulfilled due to \( \mathcal{S} \in \text{SL}(2, \mathbb{C}) \).

Now let's take some arbitrary vector field \( X \) on \( M \). Then \( TX_K \) is its Kosmann lifting to the tangent bundle \( TM \) given by the formula (5.30). It induces local one-parametric group of local diffeomorphisms

\[
\varphi_\varepsilon : TM \to TM
\] (7.9)

that extends (2.1), but does not coincide with (2.2). The Kosmann lifting \( TX_K \) of the vector field \( X \) is concordant with the bundle structure of the tangent bundle \( TM \) in the sense of the definition 4.1. For this reason the local diffeomorphisms (7.9) break into the series of linear mappings

\[
\varphi_\varepsilon : T_p(M) \to T_q(M),
\] (7.10)

where \( q = \varepsilon(p) \). According to the definition 5.1 the mappings (7.10) are isometries. Therefore, taking some positively polarized right orthonormal frame \( \Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3 \) in \( TM \) and applying \( \varphi_\varepsilon \) to it, we get another orthonormal frame \( \varphi_\varepsilon(\Upsilon_0) \),
\(\varphi_\varepsilon(\Upsilon_1), \varphi_\varepsilon(\Upsilon_2), \varphi_\varepsilon(\Upsilon_3)\) in \(T\mathcal{M}\). Note that \(\varphi_\varepsilon\) is homotopic to the identical mapping. For this reason it preserves the discrete properties like polarization and orientation, i.e. \(\varphi_\varepsilon(\Upsilon_0), \varphi_\varepsilon(\Upsilon_1), \varphi_\varepsilon(\Upsilon_2), \varphi_\varepsilon(\Upsilon_3)\) a positively polarized right orthonormal frame. Assume that \(\varepsilon\) is small enough so that both points \(p\) and \(q = \varphi_\varepsilon(p)\) belong to the domain of the frame \(\Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3\). Then at the point \(q\) we have

\[
\varphi_\varepsilon(\Upsilon_j) = \sum_{i=0}^{3} V_j^i(\varepsilon) \Upsilon_i.
\tag{7.11}
\]

The Lorentzian matrix \(V\) with the components \(V_j^i(\varepsilon) \in \text{SO}^+(1,3,\mathbb{R})\) in (7.11) is a coordinate presentation of the linear mapping (7.10) in the frame \(\Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3\).

Each of the two positively polarized right orthonormal frames \(\Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3\) and \(\varphi_\varepsilon(\Upsilon_0), \varphi_\varepsilon(\Upsilon_1), \varphi_\varepsilon(\Upsilon_2), \varphi_\varepsilon(\Upsilon_3)\) is associated with some orthonormal frame in \(S\mathcal{M}\). Using this fact, we define a linear mapping

\[
\varphi_\varepsilon: S_p(M) \rightarrow S_q(M)
\tag{7.12}
\]

closing the following commutative diagram of frame associations:

\[
\Psi_1, \Psi_2 \quad \longrightarrow \quad \Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3
\]
\[
\downarrow \varphi_\varepsilon \quad \quad \downarrow \varphi_\varepsilon
\]
\[
\varphi_\varepsilon(\Psi_1), \varphi_\varepsilon(\Psi_2) \quad \longrightarrow \quad \varphi_\varepsilon(\Upsilon_0), \varphi_\varepsilon(\Upsilon_1), \varphi_\varepsilon(\Upsilon_2), \varphi_\varepsilon(\Upsilon_3)
\tag{7.13}
\]

In the coordinate form the linear mapping (7.12), which closes the diagram (7.13), is presented by some matrix \(W \in \text{SL}(2,\mathbb{C})\):

\[
\varphi_\varepsilon(\Psi_j) = \sum_{i=1}^{2} W_j^i(\varepsilon) \Psi_i.
\tag{7.14}
\]

The horizontal arrows in the diagram (7.13) are canonical frame associations. For this reason the components of the matrices \(V\) and \(W\) in (7.11) and (7.14) should satisfy the relationships (7.7) and (7.8):

\[
\sum_{i=1}^{2} \sum_{a=1}^{2} W_i^a(\varepsilon) \sigma_m^i \sigma_k^a = \sum_{i=1}^{3} V_k^i(\varepsilon) \sigma_m^i,
\tag{7.15}
\]
\[
\sum_{i=1}^{2} \sum_{j=1}^{2} W_i^j(\varepsilon) d_{ij} W_j^k(\varepsilon) = d_{ab}.
\tag{7.16}
\]

According to the general recipe (4.12), now we pass from the matrices \(W(\varepsilon)\) and \(V(\varepsilon)\) to their expansions as \(\varepsilon \rightarrow 0\), i.e. we write

\[
W_j^i(\varepsilon) = \delta_j^i + W_j^i(x^0, x^1, x^2, x^3) \varepsilon + \ldots,
\]
\[
V_j^i(\varepsilon) = \delta_j^i + V_j^i(x^0, x^1, x^2, x^3) \varepsilon + \ldots.
\tag{7.17}
\]
Note that the quantities $V^i_j$ in (7.17) are already known. They are taken from (5.13). In our special case, where $\mathbf{Y}_0, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3$ is an orthonormal frame, we can use a more simple formula for $V^i_j$. It is extracted from (5.30):

$$V^i_j = -\frac{1}{2} \sum_{r=0}^{3} \sum_{s=0}^{3} g^{i_s} \nabla_s X^r g_{rj} + \frac{1}{2} \nabla_j X^i - \sum_{m=0}^{3} X^m \Gamma^i_m.$$  (7.18)

As for the quantities $W^i_j$, they should be calculated by substituting (7.17) back into (7.15) and (7.16). Like in the case of (5.7), we denote

$$W^i_{ij} = 2 \sum_{s=1}^{2} W^s_i d_{sj}.$$  (7.19)

Then, applying (7.17) to (7.16), we derive the following formula:

$$W^i_{ij} - W^j_{ji} = 0.$$  (7.20)

Due to the formula (7.20) the quantities (7.19) are symmetric with respect to the indices $i$ and $j$. Our next goal is to resolve the relationships (7.15) with respect to these quantities $W^i_{ij}$.

Let’s substitute the expansions (7.17) into (7.15) and collect the first order terms with respect to the parameter $\varepsilon$. As a result we get

$$\sum_{i=1}^{2} W^a_i \sigma^i_m + \sum_{i=1}^{2} \sigma^i_m W^a_i = \sum_{k=0}^{3} V^k_m \sigma^a_k.$$  (7.21)

Keeping in mind that we use the frames $\mathbf{Y}_0, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3$ and $\Psi_1, \Psi_2$ canonically associated to each other in the sense of the diagram (7.2), we replace the components of Pauli matrices by the components of Infeld-van der Waerden field in (7.21):

$$\sum_{i=1}^{2} W^a_i G^i_m + \sum_{i=1}^{2} G^a_i W^a_i = \sum_{k=0}^{3} V^k_m G^a_k.$$  (7.22)

In order to transform the formula (7.22) we multiply it by $G^m_{u\bar{u}}$ and sum over the index $m$. Doing it, we use one of the identities

$$\sum_{m=0}^{3} G^{a\bar{a}}_m G^m_{u\bar{u}} = 2 \delta^a_u \delta^{\bar{a}}_{\bar{u}}, \quad \sum_{u=1}^{2} \sum_{\bar{u}=1}^{2} G^a_{m\bar{u}} G^m_{u\bar{a}} = 2 \delta^n_m,$$  (7.23)

where $G^a_{m\bar{u}}$ are the components of the inverse Infeld-van der Waerden field. They are produced from $G^{a\bar{a}}_n$ by lowering upper spinor indices $a$ and $\bar{a}$ and by raising lower spacial index $n$ according to the following formula:

$$G^m_{u\bar{a}} = 2 \sum_{a=1}^{2} \sum_{\bar{a}=1}^{2} \sum_{n=0}^{3} G^{a\bar{a}}_n d_{au} \bar{d}_{\bar{a}u} g^{nm}.$$  (7.24)
The components of the conjugate spinor metric $\bar{d}$ in (7.24) are produced from the components of $d$ by means of the complex conjugation:

$$\bar{d}^i_j = \bar{d}^j_i, \quad \bar{d}^{ij} = d^{ij}.$$ 

The identities (7.23) are taken from [14]. Applying them to (7.22), we get

$$2 W^a_u \bar{d}^\bar{a} u + 2 \delta^a_u \bar{W}^a_u = \sum_{k=0}^{3} \sum_{m=0}^{3} V^i_k G^\bar{a}^{km} G^m_{u\bar{u}}. \quad (7.25)$$

In order to use the symmetry (7.20) we need to lower the indices $a$ and $\bar{a}$ in (7.25):

$$W^a_u \bar{d}^\bar{a} u + d^a_u \bar{W}^a_u = \sum_{k=0}^{3} \sum_{m=0}^{3} \sum_{s=1}^{2} \sum_{\bar{s}=1}^{2} V^i_k G^{s\bar{a}} V^k_m G^m_{u\bar{s}} + d^a_u \bar{d}^{s\bar{a}} G^m_{u\bar{u}}. \quad (7.26)$$

Note that $\bar{W}^a_u$ in (7.26) is symmetric with respect to the indices $u$ and $\bar{a}$, while $\bar{d}^{s\bar{a}}$ is skew-symmetric with respect to these indices. Therefore, we have

$$\sum_{u=1}^{2} \sum_{\bar{a}=1}^{2} \bar{W}^a_u \bar{d}^{s\bar{a}} = 0. \quad (7.27)$$

Applying (7.27), we multiply (7.26) by $\bar{d}^{s\bar{a}}$ and sum up over the indices $u$ and $\bar{a}$. As a result we get the following formula for $W^a_u$:

$$W^a_u = \frac{1}{4} \sum_{k=0}^{3} \sum_{m=0}^{3} \sum_{s=1}^{2} \sum_{\bar{s}=1}^{2} G^{s\bar{a}} V^k_m G^m_{u\bar{s}}. \quad (7.28)$$

Now let’s return back to the quantities $W^i_j$ by raising the index $a$ in (7.28). As a result of this standard procedure we obtain

$$W^i_j = \frac{1}{4} \sum_{k=0}^{3} \sum_{m=0}^{3} \sum_{s=1}^{2} \sum_{\bar{s}=1}^{2} G^{i\bar{a}} V^k_m G^m_{j\bar{s}}. \quad (7.29)$$

The next step is to substitute (7.18) into (7.29). Doing it, let’s recall that the metric connection $\Gamma$ has the unique extension $(\Gamma, \Lambda, \bar{\Lambda})$ to the spinor bundle $SM$. Its spinor components are given by the formula

$$A^i_{rj} = \sum_{k=0}^{3} \sum_{m=0}^{3} \sum_{s=1}^{2} \sum_{\bar{s}=1}^{2} G^{i\bar{a}} \Gamma^k_{rm} G^m_{j\bar{s}} - \sum_{s=1}^{2} \sum_{\bar{s}=1}^{2} \sum_{q=0}^{3} \frac{L_{\tau}(G^q_{\bar{a}}) G^q_{j\bar{s}}}{4} + \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{L_{\tau}(d^{ij}_{\bar{a}}) \bar{d}^{ij}_{\bar{a}}}{4}. \quad (7.30)$$

The formula is derived in [14] and is verified in [15]. In our special case the frames $\Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3$ and $\Psi_1, \Psi_2$ form a canonically associated pair in the sense of the diagram (7.2). In this case the formula (7.30) reduces to

$$A^i_{rj} = \frac{1}{4} \sum_{k=0}^{3} \sum_{m=0}^{3} \sum_{s=1}^{2} \sum_{\bar{s}=1}^{2} G^{i\bar{a}} \Gamma^k_{rm} G^m_{j\bar{s}}. \quad (7.31)$$
The formulas (7.31) and (7.29) are very similar. Therefore, substituting (7.18) into (7.29), due to the presence of $\Gamma_{m j}^k$ in (7.18) we can write

$$W^i_j = -\frac{1}{8} \sum_{q=0}^{3} \sum_{p=0}^{3} \sum_{k=0}^{3} \sum_{\bar{s}=1}^{2} G^i_k \, g^{k p} \, \nabla_p X^s \, g_{q m} \, G^m_{j s} +$$

$$+ \frac{1}{8} \sum_{k=0}^{3} \sum_{\bar{s}=1}^{2} G^i_k \, \nabla_m X^k \, G^m_{j s} - \frac{3}{3} \, X^m \, \Lambda_{m j}^i.$$  

Knowing the quantities (7.18) and (7.32) is sufficient to construct a spinor extension of the Kosmann-Lie derivative $\mathcal{L}_X$. Let $Y$ be a spin-tensorial field of the type $(\varepsilon, \eta, \sigma, \zeta, c, f)$. Then for the components of the field $\mathcal{L}_X(Y)$ we have the formula

$$\mathcal{L}_X(Y)_{b_1 \ldots b_6 | b_7 \ldots b_{10} d_1 \ldots d_f} = \frac{3}{3} \, X^m \, L_Y(X)_{b_1 \ldots b_6 | b_7 \ldots b_{10} d_1 \ldots d_f} -$$

$$- \sum_{\mu=1}^{\varepsilon} \sum_{\bar{v}_\mu=1}^{2} W_{\bar{v}_\mu}^{a_1 \ldots a_6 | a_7 \ldots a_{10} c_1 \ldots c_6} +$$

$$+ \sum_{\mu=1}^{\eta} \sum_{\bar{w}_\mu=1}^{2} W_{\bar{w}_\mu}^{a_1 \ldots a_6 | a_7 \ldots a_{10} c_1 \ldots c_6} -$$

$$+ \sum_{\mu=1}^{\sigma} \sum_{\bar{w}_\mu=1}^{2} W_{\bar{w}_\mu}^{a_1 \ldots a_6 | a_7 \ldots a_{10} c_1 \ldots c_6} +$$

$$- \sum_{\mu=1}^{\zeta} \sum_{\bar{w}_\mu=1}^{2} W_{\bar{w}_\mu}^{a_1 \ldots a_6 | a_7 \ldots a_{10} c_1 \ldots c_6} -$$

It is easy to see that (7.33) is a version of (4.14). In the case of a purely tensorial field $Y$, i.e. if $\varepsilon = 0$, $\eta = 0$, $\sigma = 0$, and $\zeta = 0$, the Kosmann-Lie derivative (7.33) reduces to (5.23). However, in general case we cannot use this formula (5.23) since the regular Lie derivative $L_X$ has no spinor extension yet. For this reason, instead of the formula (5.23), in this case we write

$$\mathcal{L}_X = \nabla_X + S_X.$$  

Like in (5.23), by $S_X$ in (7.34) we denote a degenerate differentiation. According to the results of [12], each degenerate differentiation extended to spinors is defined by three spin-tensorial field of the types $(1,1,0,0,0,0)$, $(0,0,1,1,0,0)$, and $(0,0,0,0,1,1)$. We denote them $\mathcal{S}_X$, $\mathcal{S}_X$, and $S_X$ respectively. Here are the components of $S_X$:

$$S^i_j(X) = \frac{\nabla^i_X \, X_j - \nabla_j \, X^i}{2}.$$  

(7.35)
Comparing (7.35) with (5.24), we see that \( S_X \) in (7.34) is different from that of (5.23). The formula (7.35) is extracted from (7.18). Similarly, looking at (7.32), we find the components of the spin-tensorial field \( S_X \):

\[
S^i_j(X) = \sum_{k=0}^{3} \sum_{m=0}^{3} \sum_{s=1}^{2} G^{i}_{ks} \frac{\nabla^k X_m - \nabla_m X^k}{8} G^m_{js}.
\]  

(7.36)

The components of \( \bar{S}_X \) are produced from (7.36) by means of the complex conjugation: \( \bar{S}^i_j(X) = \bar{S}^i_j(X) \). For this components we derive the formula

\[
\bar{S}^i_j(X) = \sum_{k=0}^{3} \sum_{m=0}^{3} \sum_{s=1}^{2} G^i_s \frac{\nabla^k X_m - \nabla_m X^k}{8} G^m_{sj}.
\]  

(7.37)

The formula (7.34) complemented with (7.35), (7.36), and (7.37) is equivalent to the formula (7.33).

Let’s consider a particular example of applying the formula (7.33). Assume that \( \psi \) is a spinor field, i.e. a field with the spin-tensorial type \((1,0)(0,0)\). Then for the components of the spinor field \( L_X(\psi) \) we have

\[
L_X(\psi)^i = \sum_{m=0}^{3} X^m \nabla_m \psi^i + \sum_{k=0}^{3} \sum_{m=0}^{3} \sum_{s=1}^{2} G^{i}_{ks} \frac{\nabla^k X_m - \nabla_m X^k}{8} G^m_{sj} \psi^j.
\]

This formula resembles the formula (3.19) in [10] and the formula (5.5’) in [11].

Note that the formulas (7.18) and (7.32) were derived under the assumption that \( \Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3 \) is a positively polarized right orthonormal frame in \( TM \) and \( \Psi_1, \Psi_2 \) its canonically associated orthonormal frame in \( SM \). However, these formulas remain valid for an arbitrary frame pair provided we use the general formula

\[
\Gamma^r_i = \sum_{r=0}^{3} \frac{g^{kr}}{2} (L_{\Upsilon}(g_{rij}) + L_{\Upsilon}(g_{ir}) - L_{\Upsilon}(g_{ij})) +
+ \frac{c_{ij}}{2} - \sum_{r=0}^{3} \sum_{s=0}^{2} \frac{c_{rs}^{i}}{2} g^{kr} g_{sj} - \sum_{r=0}^{3} \sum_{s=0}^{2} \frac{c_{rs}^{j}}{2} g^{kr} g_{si}.
\]

for \( \Gamma^i_{mj} \) in (7.18) instead of (5.26) and the general formula (7.30) for \( \Lambda^i_{mj} \) in (7.32) instead of (7.31). The formula (7.33) is also valid for an arbitrary frame pair under the same provisions.

8. Kosmann-Lie derivatives of the basic fields.

There are three basic field in the theory of Weyl spinors. Two of them \( \mathbf{d} \) and \( \mathbf{G} \) are listed in the table (7.3). The third is the metric tensor \( g \). Now we shall apply the Kosmann-Lie derivative (7.33) to these basic fields. For this purpose it is convenient to choose some canonically associated pair of frames \( \Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3 \) and \( \Psi_1, \Psi_2 \). In such a frame pair the components of all basic fields are constants.
Indeed, they are given by the formulas (5.25), (7.4), and (7.5). Therefore we have

\[ \sum_{m=0}^{3} X^m L \mathcal{Y}_m (d_{ij}) = 0, \]  
\[ \sum_{m=0}^{3} X^m L \mathcal{Y}_m (G^a \bar{a}) = 0, \]  
\[ \sum_{m=0}^{3} X^m L \mathcal{Y}_m (g_{ij}) = 0. \]  

Applying (7.33) to \( d \) and taking into account (8.1), (7.19), and (7.20), we obtain

\[ \mathcal{L}_X(d)_{ij} = \sum_{s=1}^{2} W_i^s d_{sj} + \sum_{s=1}^{2} W_j^s d_{is} = W_{ij} - W_{ji} = 0. \]  

Similarly, applying (7.33) to \( G \) and taking into account (8.2) and (7.22), we get

\[ \mathcal{L}_X(G)_{a \bar{a} m} = - \sum_{i=1}^{2} W_i^a G^{a \bar{a}} - \sum_{i=1}^{2} C_i^{a \bar{a}} + \sum_{k=0}^{3} V_k^i c_{k}^{a \bar{a}} = 0. \]  

And finally we apply the formula (7.33) to the metric tensor \( g \). As a result, taking into account (8.3), (5.7), and (5.8), we derive

\[ \mathcal{L}_X(g)_{ij} = \sum_{r=0}^{3} V_r^i g_{rj} + \sum_{r=0}^{3} V_r^j g_{ir} = V_{ij} + V_{ji} = 0. \]  

The formulas (8.4), (8.5), and (8.6) are summarized in the following theorem.

**Theorem 8.1.** For any vector field \( X \) in \( M \) the basic tensorial and spin-tensorial fields \( g, d, \) and \( G \) associated with the bundle of Weyl spinors \( SM \) are constant with respect to the Kosmann-Lie derivative \( \mathcal{L}_X \).

9. Some concluding remarks.

Note that the quantities \( V_j^i \) for (7.33) are taken from (7.18). However, they could be taken from (5.1) either. In the latter case the equality \( \mathcal{L}_X(d) = 0 \) would be preserved, but the equality \( \mathcal{L}_X(g) = 0 \) would be replaced by

\[ \mathcal{L}_X(g) = L_X(g). \]

As for the formula (8.5), it would be replaced by the following one:

\[ \mathcal{L}_X(G)_{a \bar{a} m} = \sum_{k=0}^{3} \frac{\nabla_m X^k + \nabla^k X_m}{2} G_k^{a \bar{a}}. \]

This choice of \( V_j^i \) is preferred in [11]. As for our choice of \( V_j^i \) in this paper, in [11] it is referred to as the “metric Lie derivative” introduced by Bourguignon and
Gauduchon in [16]. Since there are various approaches, I should regretfully conclude that there is no canonical definition of the Lie derivative for spinors thus far.

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