Asymptotic orderings and approximations of the Master kinetic equation for large hard spheres systems

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In this paper the problem is posed of determining the physically-meaningful asymptotic orderings holding for the statistical description of a large $N$–body system of hard spheres, i.e., formed by $N \equiv \Theta \gg 1$ particles, which are allowed to undergo instantaneous and purely elastic unary, binary or multiple collisions. Starting point is the axiomatic treatment recently developed [Tessarotto et al., 2013-2016] and the related discovery of an exact kinetic equation realized by Master equation which advances in time the 1–body probability density function (PDF) for such a system. As shown in the paper the task involves introducing appropriate asymptotic orderings in terms of $\varepsilon$ for all the physically-relevant parameters. The goal is that of identifying the relevant physically-meaningful asymptotic approximations applicable for the Master kinetic equation, together with their possible relationships with the Boltzmann and Enskog kinetic equations, and holding in appropriate asymptotic regimes. These correspond either to dilute or dense systems and are formed either by small–size or finite-size identical hard spheres, the distinction between the various cases depending on suitable asymptotic orderings in terms of $\varepsilon$.

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1 - INTRODUCTION

In a series of papers [1–7] (see also Refs. [8, 9]) a new kinetic equation has been established for hard sphere systems subject to elastic instantaneous collisions, denoted as Master kinetic equation. Its basic features are that, unlike the Boltzmann and Enskog kinetic equations [11, 12], the same equation and its corresponding Master collision operator are exact, i.e., they hold for an arbitrary $N$–body hard-sphere system $S_N$ which is isolated, namely for which the number of particles ($N$) is constant, while furthermore that its solutions are entropy-preserving [6] and globally defined [7]. Concerning, in particular, the first feature this means that in such a context $S_N$ is allowed to have an arbitrary finite number $N$ of finite-size and finite-mass hard spheres, namely each one characterized by finite diameter $\sigma > 0$ and mass $m > 0$. In addition, by assumption $S_N$ is immersed in a bounded configuration domain $\Omega$, subset of the Euclidean space $\mathbb{R}^3$ which has a finite canonical measure $L^3_o \equiv \mu(\Omega) > 0$ ($L_o$ denoting a finite configuration-domain characteristic scale length) and is endowed with a stationary and rigid boundary $\partial \Omega$. However, the total volume occupied by hard-spheres cannot exceed the configuration-space volume. Hence, the parameters $N, L_o$ and $\sigma$ must necessarily satisfy the inequality

\[ \Delta \equiv \frac{N \mu(\Phi)}{\mu(\Omega)} = \frac{4\pi N\sigma^3}{3L_o^3} \leq 1, \]  

with $\mu(\Phi) = \frac{4\pi x^3}{3}$ denoting the volume of a single hard sphere and $\Delta$ the global diluteness parameter. For such an equation the particle correlations appearing through the $2$–body probability density function (PDF) are exactly taken into account by means of suitably-prescribed $1$– and $2$–body occupation coefficients which are position-dependent only. These peculiar features follow uniquely as a consequence of the new approach to classical statistical mechanics developed in Refs. [1–3] and referred to as "ab initio" axiomatic approach. As shown in the same references (for a review see also Ref. [9]), this is based on the adoption of appropriate extended functional setting and physics-based modified collision boundary conditions (MCBC) [1, 2] which are prescribed in order to advance in time across arbitrary (unary, binary or multiple) collision events the $N$–body probability density function (PDF). The physical origin of MCBC follows from the requirement that the deterministic $N$–body Dirac delta is included among the physically-admissible $N$–body PDF’s for $S_N$ [2]. Its physical interpretation is intuitive [4] being viewed as the jump condition for the $N$–body PDF along the phase-space Lagrangian trajectory $\{x(t)\}$ for an ensemble of $N$ tracer particles [10] undergoing an arbitrary collision event.
Based on the discovery of the Master kinetic equation, a host of new developments have opened up. These concern, first of all, the investigation of novel conceptual aspects of the same equation (for an extended discussion see Refs. [4–8]). However, it goes almost without saying that possible applications of the new equation are potentially ubiquitous. Many of these applications typically concern large systems, i.e., which are formed by a large number $N \equiv \frac{1}{\varepsilon} \gg 1$ of particles.

The goal of this paper is that of identifying the relevant physically-meaningful asymptotic approximations for the same kinetic equation which may correspond to physically-relevant practical applications of the theory. As shown below, this task involves the adoption of appropriate asymptotic orderings in terms of $\varepsilon$ for all the physically-relevant parameters. These include, besides the configuration-space scale length $L_o$ and the hard-sphere diameter $\sigma$ also the characteristic scale length $L_\rho$ which is associated with the spatial variations of the 1–body PDF which is prescribed so that

$$
\frac{1}{L_\rho} = \sup \left\{ \frac{\partial \ln \rho^{(N)}_1(x_1, t)}{\partial r_1} \right\}, \quad \forall (x_1, t) \in \Gamma_{1(1)} \times I \right\}.
$$

Notice that hereon $\frac{\partial \ln \rho^{(N)}_1(x_1, t)}{\partial r_1}$ is assumed to be bounded for all $(x_1, t)$ spanning the extended 1–body phase space $\Gamma_{1(1)} \times I$ (see notations below). In the present paper the issues are investigated which concern the classification of $N$–body systems which are characterized respectively by suitable asymptotic orderings. More precisely:

- A) $\sigma$–ordering regime: in which the diameter $\sigma$ of the hard spheres is either small-size or finite size, in the sense that either $\sigma \sim O(\varepsilon^\alpha)$ with suitable $\alpha > 0$ or $\sigma \sim O(\varepsilon^0)$;
- B) $\Delta$–ordering regime: in which the parameter $\Delta$ is either $\Delta \sim O(\varepsilon^\beta)$ with $\beta > 0$ or $\Delta \sim O(\varepsilon^0)$, namely the hard-sphere system is dilute or dense;

A type of alternate asymptotic ordering equivalent to B) is provided also by:

- C) $K_n$–ordering regime: in which a suitably-defined Knudsen number $K_n$ (see below) may be either of order $O(\varepsilon^\gamma)$, being $\gamma$ a suitable positive, either vanishing or negative real number. Accordingly the hard-sphere systems will be denoted as weakly-collisional, collisional or strongly collisional.

The topics indicated above include in particular:

**ISSUE #1:** the search of possible asymptotic (i.e., approximate) kinetic equations holding in cases A, B and C;

**ISSUE #2:** the possible asymptotic evaluation of the 1– and 2–body occupation coefficients in the same cases.

The goal is also to display the possible relationship of the Master kinetic equation with well-known kinetic equations, i.e., the Boltzmann and Enskog kinetic equations [11–13]. In particular, although in the case of finite-size hard spheres the strict validity of both the Boltzmann and Enskog equations "per se" is ruled out [2], this concerns:

**ISSUE #3:** the determination of the asymptotic modifications which enter the Boltzmann equation in the case the particle diameter $\sigma$ is suitably small.

**ISSUE #4:** the identification of the relevant asymptotic parameter sub-domains in which the Enskog equation still applies, albeit in a suitable approximate sense.

### 2 - DIMENSIONLESS REPRESENTATION OF THE MASTER KINETIC EQUATION

The prerequisite for carrying out the tasks outlined above is setting the Master kinetic equation in dimensionless form. To this end let us first notice that by construction the 1–body PDF $\rho^{(N)}_1(x_1, t)$ depends on the extended Newtonian state $(x_1 \equiv (r_1, v_1)_1, t)$. In particular this means that:

1) the Newtonian position vector and velocity vectors $r_1$ and $v_1$ which label the center of mass position and velocity of a representative particle, span the Euclidean configuration and velocity spaces $\Omega \subset \mathbb{R}^3$ and $U_{1(1)} \equiv \mathbb{R}^3$, while $t$ belongs to the Galilean time axis $I \equiv \mathbb{R}$. As a consequence the Galilean structure of $\Omega \times I$, i.e., the Euclidean distance in $\Omega$ and the time-interval in $I$, remains uniquely determined.

2) by construction $\rho^{(N)}_1(x_1, t)$ is a scalar with respect to the group of Galilei transformation which preserves the Galilean structure of the set $\Omega \times I$.

Next, let us introduce the characteristic scale length

$$
L \equiv \min \{L_o, L_\rho\},
$$

(3)
and a suitable constant characteristic time scale $\tau$ (whose definition remains in principle arbitrary). Then all the Newtonian variables $(x_1 \equiv \{r_1, v_1\}, t)$ can be conveniently replaced with the corresponding dimensionless quantities $\bar{r}_1 = \frac{L}{2} r_1, \bar{v}_1 = \frac{v}{v_1}$ and $\bar{t} = \frac{t}{\tau}$. This implies that the phase-space volume element must transform as $d\bar{r}_1 d\bar{v}_1 = \frac{L^2}{2} d r_1 d v_1$, while, in order to warrant the conservation of probability $d\bar{r}_1 d\bar{v}_1 p_1^{(N)}(r_1, v_1, t) = d\bar{r}_1 d\bar{v}_1 \bar{p}_1^{(N)}$, the dimensionless form, of the PDF $\bar{p}_1^{(N)}$ must be identified with $p_1^{(N)} = \frac{L^2}{2} \bar{r}_1^{(N)}(r_1, v_1, t)$. Notice here, however, that to preserve the scalar property of the transformed PDF $\bar{p}_1^{(N)}$ with respect to the Galilei group the latter should depend explicitly on the extended Newtonian state $(r_1, v_1, \bar{t})$ rather then the transformed state $(\bar{r}_1, \bar{v}_1, \bar{t})$. In fact due to their arbitrariness, the parameters $L$ and $\tau$ change the Galilei structure of space-time, i.e., the Euclidean distance and the time interval. Hence $\bar{p}_1^{(N)}$ still depends, as $p_1^{(N)}(r_1, v_1, t)$, on the same extended state $(x_1 \equiv \{r_1, v_1\}, t)$, and therefore is of the form $\bar{p}_1^{(N)} = \bar{p}_1^{(N)}(r_1, v_1, t)$. In terms of such a prescription the Master kinetic equation (see Ref.3) can therefore be formally represented in the dimensionless form

$$\mathcal{T}_1 \bar{p}_1^{(N)} = \mathcal{C}_1(\bar{p}_1^{(N)}) | p_1^{(N)}),$$

with $\mathcal{T}_1$ and $\mathcal{C}_1(\bar{p}_1^{(N)}) | p_1^{(N)})$ denoting respectively the free-streaming and Master collision operators. Both are cast in the dimensionless representation, i.e., so that $\mathcal{T}_1 = \frac{\partial}{\partial r} + \bar{v}_1 \cdot \frac{\partial}{\partial v}$, and

$$\mathcal{C}_1(\bar{p}_1^{(N)}) | p_1^{(N)} = K_n \int \bar{v}_2 \int d\Sigma_2 \left[ \bar{p}_2^{(N)}(\bar{x}; r_1, v_1, t) \right] \bar{v}_1 \cdot \sigma \cdot \mathcal{S}.$$

In addition, in the Eq. (4) $K_n$ identifies the Knudsen number

$$K_n = \frac{(N - 1) \sigma^2}{L^2},$$

while $\bar{p}_2^{(N)}(\bar{x}; r_1, v_1, t) = f(r_1, r_2, t) \times p_2^{(N)}(r_1, v_1, t)$, denoting the position-dependent dimensionless weight-factor

$$f(r_1, r_2, t) = \frac{k_2^{(N)}(r_1, r_2, t)}{k_1^{(N)}(r_1, t)} k_1^{(N)}(r_2, t).$$

The remaining notations are standard 3. Thus $U_1(k) = \mathbb{R}^3$ is the 1–body velocity space for the $k$–th particle, the symbol $\int d\Sigma_2$ denotes integration on the subset of the solid angle of incoming particles namely for which $v_{12} \cdot n_{12} < 0$, $\mathcal{S}$ denotes $\mathcal{S} = \mathcal{S} \equiv \mathcal{S}(\{r_2 - \frac{2}{3} n_2\})$, with $\mathcal{S}(x)$ being the strong theta function, while everywhere in the operator $\mathcal{C}_1(\bar{p}_1^{(N)}) | p_1^{(N)}$, $r_2$ is identity by construction with $r_2 = r_1 + \sigma n_2$. Furthermore $k_1^{(N)}(r_1, t)$, $k_1^{(N)}(r_2, t)$ and $k_1^{(N)}(r_2, t)$ identify the dimensionless 1– and 2–body occupation coefficients, whose definitions in terms of the dimensionless 1–body PDF are respectively:

$$k_1^{(N)}(r_1, t) = \int_{\Gamma_1(2)} d\Sigma_2 \frac{p_2^{(N)}(x; x_2, t)}{k_1^{(N)}(r_2, t)} k_2^{(N)}(r_1, r_2, t),$$

$$k_2^{(N)}(r_1, r_2, t) = \int_{\Gamma_1(3)} d\Sigma_3 \frac{p_3^{(N)}(x; x_3, t)}{k_1^{(N)}(r_3, t)} \ldots \int_{\Gamma_1(N)} d\Sigma_N \frac{p_N^{(N)}(x; x_N, t)}{k_1^{(N)}(r_N, t)}.$$

where once the position of particle 1 is assumed prescribed, $\Gamma_1(2), \Gamma_1(3), \ldots \Gamma_1(N)$ are the admissible subsets of the 1–body phase spaces of particles 2, 3, $\ldots$, $N$, $\Gamma_1(2), \Gamma_1(3), \ldots \Gamma_1(N)$ obtained by subtracting respectively the forbidden subsets $\Phi_{12}$ (from $\Gamma_1(2)$), $\Phi_{13} \cup \Phi_{23}$ (from $\Gamma_1(3)$), $\bigcup_{i=2, N-1} \Phi_{iN}$ (from $\Gamma_1(N)$).
A remark is in order concerning the comparison with the analogous dimensionless representation introduced originally by Grad [14] for the BBGKY hierarchy and the Boltzmann equation in particular (see also Ref. [15]). The basic departure of Eq. (3) with respect to the latter equation lies of course in the different realization of the collision operator. However, another major difference arises because of the explicit introduction of the characteristic scale length \( L \) in the definition of the Knudsen number given in Eq. (6). Such a choice is actually required in order to permit the distinction between different asymptotic ordering regimes (see next Section), while, in contrast, Grad’s approach dealt only with the so-called Boltzmann-Grad limit. Indeed, as shown below, it was actually appropriate for the treatment of the so-called dilute-gas asymptotic ordering only, namely for the case in which both the scale length \( L_o \) and \( L \) are considered of order \( O(\varepsilon) \).

3 - CLASSIFICATION OF THE ASYMPTOTIC ORDERING REGIMES

In this section the relevant asymptotic orderings are determined which are applicable in the case of large hard-sphere systems, i.e., for which \( N \equiv \frac{1}{\varepsilon} \gg 1 \) and subject to the validity of the volume constraint inequality (1). Based on the prescription of the Knudsen number (6) we are now able to show that the classification in terms of \( K_n \) pointed out above corresponds to suitably prescribe the magnitude of the ratio dimensionless \( \sigma/L \).

More precisely weakly-collisional, collisional or strongly collisional asymptotic regimes are obtained requiring that \( \sigma/L \) be of order \( O(\varepsilon^{\gamma - \frac{1}{2}}) \) with \( \gamma \) being respectively \( \gamma < 0, \gamma = 0 \) and \( \gamma > 0 \). For definiteness let us initially focus on the collisional \( K_n \)-ordering regime. In this case the following two possible dilute-gas ordering regimes (3A and 3B) can be distinguished.

3A - Dilute-gas small-size \( \sigma \)-ordering regime

Let us consider a first possible realization of the ”small-size” \( \sigma \)-ordering regime. This is obtained assuming that the diameter of the hard-spheres \( \sigma \) is considered \( \ll 1 \) in a suitable sense, while the scale-length \( L \) is ordered according to the prescriptions that \( \sigma/L \sim O(\varepsilon^{\frac{1}{2}}) \) and letting

\[
\begin{align*}
\sigma & \sim O(\varepsilon^\alpha) \\
L & \sim O(\varepsilon^{\alpha - \frac{1}{2}}) \\
L & \sim L_o \sim L_p
\end{align*}
\]

(10)

Introducing the dimensionless parameter

\[
\Delta L = \frac{4\pi N \sigma^3}{3L^3}
\]

(11)

this implies necessarily that \( \Delta L \sim O(\varepsilon^{\frac{1}{2}}) \) and hence, due to the inequalities \( L \leq L_o \), and \( \Delta \leq \Delta L \), also \( \Delta \lessgtr O(\varepsilon^{\frac{1}{2}}) \). Therefore the orderings (10) necessarily equivalently identify a dilute-gas ordering, which can therefore be characterized as a collisional, dilute-gas and small-size \( \sigma \)-ordering regime. In particular, when \( \alpha = \frac{1}{2} \) and \( L \sim L_o \) it follows that \( \Delta L \sim \Delta \sim O(\varepsilon^{\frac{1}{2}}) \) and \( K_n \sim O(\varepsilon^0) \), so that the customary dilute-gas ordering considered originally by Grad [14] (see also Refs. [3, 7]) is recovered. This is obtained requiring

\[
\begin{align*}
N\sigma^2 & \sim O(\varepsilon^0) \\
L & \sim L_o \sim O(\varepsilon^0)
\end{align*}
\]

(12)

3B - Dilute-gas finite-size \( \sigma \)-ordering regime

Another type of ordering regime is obtained requiring \( \sigma \) to be finite while prescribing again the scale-length \( L \) in such a way to satisfy the requirement \( K_n \sim O(\varepsilon^0) \). Let us require

\[
\begin{align*}
\sigma & \sim O(\varepsilon^0) \\
L & \sim O(\varepsilon^{-\frac{1}{2}}) \\
L & \sim L_p \lesssim L_o
\end{align*}
\]

(13)
Notice that again $\Delta_L \sim O(\varepsilon^{\frac{1}{2}})$ and hence $\Delta \lesssim O(\varepsilon^{\frac{1}{2}})$. Therefore the ordering [13] corresponds to a dilute-gas ordering which will be referred to as collisional, dilute-gas and finite-size $\sigma-$ordering regime.

Finally, for completeness we point out possible realizations of dense-gas ordering regimes.

3C - Dense-gas ordering regimes

Let us require for this purpose that the parameter $\Delta$ is of order $O(\varepsilon^0)$, i.e., that the hard-sphere system is dense. This happens in the case in which $\sigma/L_o \sim O(\varepsilon^\frac{1}{4})$. Since by construction due to the inequality $L \leq L_o$ also $\Delta \leq \Delta_L$ it follows necessarily that it must be $\Delta \sim \Delta_L$ and hence $L \sim L_o$ too. This means that $K_n \sim O(\varepsilon^{-\frac{1}{4}})$ which therefore corresponds necessarily to a strongly-collisional regime. Let us assume for this purpose that the following orderings apply:

$$
\begin{align*}
\sigma & \sim O(\varepsilon^0) \\
L & \sim O(\varepsilon^{-\frac{1}{4}}) \\
L & \sim L_o \\
\alpha & \in [0, \infty].
\end{align*}
$$

Therefore this implies that necessarily $\Delta \sim \Delta_L \sim O(0^0)$. In particular, if $\alpha > 0$ this corresponds to a small-size $\sigma-$ordering regime, to be referred to as strongly-collisional, dense-gas and small-size $\sigma-$ordering regime. A special case is provided by the choice $\alpha = 0$, which corresponds instead to a finite-size $\sigma-$ordering regime for which

$$
\begin{align*}
\sigma & \sim O(\varepsilon^0) \\
L & \sim O(\varepsilon^{-\frac{1}{2}}) \\
L & \sim L_o.
\end{align*}
$$

This can therefore be characterized as strongly-collisional, dense-gas and finite-size $\sigma-$ordering.

4 - ASYMPTOTIC APPROXIMATIONS OF THE MASTER KINETIC EQUATION

In this section we intend to pose the problem of the construction of the asymptotic approximations of the Master kinetic equation which are appropriate for the treatment of most of the asymptotic regimes discussed in subsections 3A, 3B and 3C. In detail we intend to show that:

- **First asymptotic approximation:** in the ordering regime [13] to lowest order in $O(\varepsilon)$ the Master equation reduces to the Boltzmann kinetic equation, with the Master collision operator being approximated in this case by the collision operator

$$
\overline{C}_{1MB}(p^{(N)}_1|p^{(N)}_1) = \frac{N\sigma^2}{L^2} \int_{U_{(2)}} d\nu_2 \int_0^{(+)} d\Sigma_{21} f(r_1, r_2 = r_1, t).
$$

$$
\left[ p^{(N)}_1(r_1, v_1^{(-)}, t) p^{(N)}_1(r_1, v_2^{(-)}, t) - p^{(N)}_1(r_1, v_1, t) p^{(N)}_1(r_1, v_2, t) \right] |\nu_{21} \cdot n_{21}|,
$$

where $f(r_1, r_2, t)$ is the strictly-positive weight-factor prescribed by Eq. [8]. In the expression of same equation the occupation coefficients are here approximated as follows:

$$
\begin{align*}
\kappa^{(N)}_1(r_1, t) & \equiv 1 - \frac{N}{2} \int_{\Phi_{12}} dr_2 n^{(N)}_1(r_2, t), \\
\kappa^{(N)}_2(r_1, r_2, t) & \equiv 1 - \frac{3N}{2} \int_{\Phi_{12}} dr_3 n^{(N)}_1(r_3, t) - \frac{3N}{2} \int_{\Phi_{23}} dr_2 n^{(N)}_1(r_2, t),
\end{align*}
$$

with $\Phi_{ij}$ denoting the hard-sphere interior domain $\Phi_{ij} = \{r_j : |r_j - r_i| < \sigma\}$. This yields therefore the asymptotic approximation

$$
f(r_1, r_2, t) \approx \frac{1 - \frac{3N}{2} \int_{\Phi_{12}} dr_3 n^{(N)}_1(r_3, t) - \frac{3N}{2} \int_{\Phi_{23}} dr_2 n^{(N)}_1(r_2, t)}{1 - \frac{N}{2} \int_{\Phi_{12}} dr_2 n^{(N)}_1(r_2, t) - \frac{N}{2} \int_{\Phi_{23}} dr_3 n^{(N)}_1(r_3, t)}.\]
The following remarks are in order regarding the collision operator \( C_{1MB}^{(N)}(\rho_1^{(N)}|\rho_1^{(N)}) \). First one notices that it provides a generalization of the Boltzmann collision operator (issue \#2). In particular, one can readily show (see the proof reported below) that to order \( O(\varepsilon^2) \) it coincides by construction with the customary Boltzmann collision operator, since then the weight-factor \( f(r_1, r_2, t) \) can be approximated with unity. The asymptotic approximate formula \( 18 \) for the weight-factor \( f(r_1, r_2, t) \) given by Eq. \( 8 \) retains, instead, also leading-order corrections which are produced by the 1− and 2−body occupation coefficients (issue \#3). Second, the structure of the collision operator \( 19 \) has also formal analogies with the one introduced by Enskog in his namesake equation. The key feature in this case lies in the prescription of the weight-factor \( f(r_1, r_2, t) \) which is here provided by Eq. \( 18 \) while remaining in principle undetermined in the context of the Enskog kinetic equation. Thus, provided, the same prescription indicated above is made for \( f(r_1, r_2, t) \), the collision operator \( 19 \) can be viewed as realizing also an approximate representation of the Enskog collision operator (issue \#4).

- **Second asymptotic approximation**: in validity of the ordering regimes \( 13 \) the Master equation reduces, instead, to an asymptotic Master kinetic equation determined by the collision operator

\[
C_{1MB}^{(N)}(\rho_1^{(N)}|\rho_1^{(N)}) = \frac{N \sigma^2}{L^2} \int_{U_{1(2)}} d\Sigma_2 \int^{(-)} d\Sigma_{21} f(r_1, r_2, t) \times 
\left[ \rho_1^{(N)}(r_1, v_1^{(-)}, t) \rho_1^{(N)}(r_2, v_2^{(-)}, t) - \rho_1^{(N)}(r_1, v_1^{+}, t) \rho_1^{(N)}(r_2, v_2^{+}, t) \right] \left| \nabla \cdot \mathbf{n}_{21} \right| \] \tag{19}

in which the weight-factor \( f(r_1, r_2, t) \) is expressed in terms of the asymptotic estimate \( 18 \) and due to the requirement that \( \sigma \) remains finite, so that necessarily \( r_2 = r_1 + \sigma \mathbf{n}_{21} \). This implies that although Eq. \( 19 \) has formal analogies with the customary form of the Enskog collision operator, two major differences arise. The first one lies in the prescription of the weight-factor itself, which in the present case is determined by Eq. \( 18 \) while its choice remains unspecified in the context of the Enskog statistical approach. The second follows because of the adoption of MCBC requiring that in Eqs. \( 19 \) the solid-angle integration must be carried out on the subset \( \int^{(-)} d\Sigma_{21} \) of incoming particles for which \( v_{12} \cdot \mathbf{n}_{12} < 0 \) instead on the complementary set \( \int^{(+)} d\Sigma_{21} \) as done in the Enskog collision operator (issue \#4).

- **Third asymptotic approximation**: in the ordering regime \( 14 \) subject to the requirement \( \alpha > 0 \) to leading order in \( O(\varepsilon) \) the Master equation reduces to the asymptotic Master kinetic equation, expressed in terms of the collision operator which to leading order in \( O(\varepsilon) \) reads

\[
C_{1MB}^{(N)}(\rho_1^{(N)}|\rho_1^{(N)}) = \frac{N \sigma^2}{L^2} \int_{U_{1(2)}} d\Sigma_2 \int^{(+)} d\Sigma_{21} f(r_1, r_2 = r_1, t) \times 
\left[ \rho_1^{(N)}(r_1, v_1^{(-)}, t) \rho_1^{(N)}(r_2, v_2^{(-)}, t) - \rho_1^{(N)}(r_1, v_1^{+}, t) \rho_1^{(N)}(r_2, v_2^{+}, t) \right] \left| \nabla \cdot \mathbf{n}_{21} \right| \] \tag{20}

where \( f(r_1, r_2, t) \) is prescribed again by Eq. \( 8 \). However, now in difference with the two cases indicated above the asymptotic approximations \( 17 \) and \( 18 \) do not hold, so that the occupation coefficients \( k_1^{(N)}(r_1, t) \) and \( k_2^{(N)}(r_1, r_2, t) \) need to be determined iteratively in terms of Eqs. \( 9 \).

Finally, we mention that the case represented by the ordering \( 15 \) must be treated separately, in the sense that no approximation is actually possible on the functional form of the Master collision operator \( 5 \).

4A - Proof of the first asymptotic approximation

Let us first prove the validity of Eqs. \( 16 \) and \( 18 \). For this purpose one first notices that thanks to the ordering regime \( 10 \) the 1−body PDFs \( \rho_1^{(N)}(r_2, v_2^{+}, t) \) and \( \rho_1^{(N)}(r_2, v_2^{-}, t) \) can be approximated in terms of \( \rho_1^{(N)}(r_1, v_1^{+}, t) \) and \( \rho_1^{(N)}(r_1, v_1^{-}, t) \) respectively. For the same reason the occupation coefficients \( k_1^{(N)}(r_2, t) \) and \( k_2^{(N)}(r_1, r_2, t) \) can be approximated in terms of \( k_1^{(N)}(r_1, t) \) and \( k_2^{(N)}(r_1, r_2 \equiv r_1, t) \). Second, again thanks to Eqs. \( 10 \), in the Master collision operator the solid-angle integration on the sub-domain \( v_{12} \cdot \mathbf{n}_{12} < 0 \) (namely \( \int^{(-)} d\Sigma_{21} \)) can be equivalently exchanged with the corresponding complementary subset \( v_{12} \cdot \mathbf{n}_{12} \geq 0 \), i.e., \( \int^{(+)} d\Sigma_{21} \), while the domain theta
function $\Theta^+(x_2)$ becomes $\Theta^-(x_2) \equiv \Theta(|x_1|)$ so that its contribution to the collision integral is ignorable. Third, to prove the asymptotic estimate (18), let us notice that in validity of the ordering (11) it follows that $\frac{d\Phi_2^{(N)}(r_2, t)}{d\Omega_2} \sim \pi_1^{(N)}(r_2, t) \frac{4\pi \sigma^3}{3L^3} \equiv \frac{\sigma^3}{L^3} \hat{x}$, with $\hat{x}$ a suitable mean value such that $\hat{x} \sim O(e^0)$. As a consequence from Eqs. (17) in order of magnitude it follows that

$$\begin{align*}
\{ k_1^{(N)}(r_1, t) &\sim 1 - \frac{N \sigma^3 \hat{x}}{2L^3} \\
\} \quad \Rightarrow \\
k_2^{(N)}(r_1, r_2, t) &\sim 1 - \frac{3N \sigma^3 \hat{x}}{2L^3}
\end{align*}$$

which implies

$$f(r_1, r_2, t) \sim 1 - \frac{N \sigma^3 \hat{x}}{2L^3}.$$  

The proof of the asymptotic estimates (21) is straightforward. In fact, Eq. (9) then requires, based on the mean-value theorem, implies that

$$\begin{align*}
\left(k_1^{(N)}(r_1^*, t)\right)^N &\sim \int d\xi_2 p_1^{(N)}(x_2, t) \int d\xi_3 p_1^{(N)}(x_3, t) \int d\xi_4 p_1^{(N)}(x_4, t) \\
&\equiv \int d\xi_2 n_1^{(N)}(r_2, t) \int d\xi_3 n_1^{(N)}(r_3, t) \int d\xi_4 n_1^{(N)}(r_4, t),
\end{align*}$$

with $k_1^{(N)}(r_1^*, t)$ a suitable mean-value. Therefore the same equation yields the asymptotic estimate

$$\left(k_1^{(N)}(r_1^*, t)\right)^N \sim \left(1 - \frac{\sigma^3 \hat{x}}{L^3}\right) \left(1 - \frac{2 \sigma^3 \hat{x}}{L^3}\right) \cdots \left(1 - \frac{(N - 1) \sigma^3 \hat{x}}{L^3}\right)$$

which is manifestly consistent with Eq. (21). The proof of the asymptotic estimates for $k_2^{(N)}(r_1, r_2, t)$ and (22) is analogous, thus yielding the consistency of the asymptotic approximations (17) and (18).

4B - Proof of the second asymptotic approximation

The proof of Eq. (19) is similar as far as the asymptotic estimate (22) is concerned. Now, however, due to the finite size of the hard spheres (see Eqs. (13)) the correct spatial dependences must be retained in the 1-body PDF’s $\rho_1^{(N)}(r_2, v_2^{(+)}), t)$ and $\rho_1^{(N)}(r_2, v_2, t)$ which must both be evaluated at the position $r_2 = r_1 + \sigma n_2$, As a consequence the corresponding asymptotic approximation (19) manifestly holds for the Master collision operator.

4C - Proof of the third asymptotic approximation

The proof of Eq. (20) is similarly straightforward. In fact, first one notices that in close analogy with case 4A, thanks to the small-size assumption introduced for $\sigma$, the 1-body PDFs $\rho_1^{(N)}(r_2, v_2^{(+)}), t), \rho_1^{(N)}(r_2, v_2, t)$ as well as the occupation coefficients $k_1^{(N)}(r_2, t)$ and $k_2^{(N)}(r_1, r_2, t)$ can all be approximated replacing $r_2 \rightarrow r_1$. As a consequence again the solid-angle integration $\int d\Sigma_2$ can be equivalently be evaluated in terms of the outgoing-particle subset $\int d\Sigma_2$ while the contribution of the theta function $\Theta^+$ is ignorable. Finally, due to the dense-gas asymptotic ordering included in (14) no obvious asymptotic approximation is available for the occupation coefficients $k_1^{(N)}(r_1, t)$ and $k_2^{(N)}(r_1, r_2 = r_1, t)$. Therefore their exact expression following from Eqs. (9) must be retained in Eq. (20).

5 - CONCLUSIONS

In this paper the problem has been addressed of identifying possible physically-meaningful asymptotic approximations of the Master kinetic equation which apply to, i.e., formed by $N \equiv \frac{1}{\varepsilon} \gg 1$ hard-spheres. The statistical
approach has been based on the "ab initio" axiomatic statistical theory recently developed\cite{1-9}. As a result, once the Master kinetic equation is cast in dimensionless form, the existence of multiple asymptotic ordering regimes for the same equation has been pointed out which hold for large $N$–body systems. These regimes correspond to appropriate prescriptions of the relevant physical parameters of the same equation and include, as a particular possible realization, the customary dilute-gas ordering originally introduced by Grad\cite{14} for his construction of the Boltzmann kinetic equation. The new ordering regimes encompass either small or finite-size hard-spheres as well as dilute or dense, collisional or strongly-collisional particle systems. In particular possible realizations include:

- **the dilute-gas small-size $\sigma$–ordering regime** (prescribed by the ordering Eqs. (10));
- **the dilute-gas finite-size $\sigma$–ordering regime** (in the sense of Eqs. (13));
- **the dense-gas ordering regime** (see Eqs. (14) in the case in which $\alpha > 0$).

Corresponding asymptotic approximations have been determined for the Master collision operator, displaying also their relationship/difference with respect to the Boltzmann and Enskog collision operators.

The present results are believed to be crucial both in kinetic theory and fluid dynamics. Indeed, regarding possible challenging future developments one should particularly mention possible applications both of the Master kinetic equation itself as well as of the asymptotic approximations here pointed out for the first time. The hard-sphere kinetic statistical treatment based on these equations is expected to successfully apply to a variety of complex fluid-dynamics systems as well as to neutral and/or ionized gases of interest for laboratory research and astrophysics.

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