Deciding k-colourability of $P_5$-free graphs in polynomial time

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Abstract

The problem of computing the chromatic number of a $P_5$-free graph is known to be NP-hard. In contrast to this negative result, we show that determining whether or not a $P_5$-free graph admits a $k$-colouring, for each fixed number of colours $k$, can be done in polynomial time. If such a colouring exists, our algorithm produces it.

Keywords: graph colouring, dominating clique, polynomial-time algorithm, $P_5$-free graph

1 Introduction

A $k$-colouring of a graph $G$ is an assignment of $k$ colours to the vertices of $G$ so that no two adjacent vertices receive the same colour. The $k$-COLOURABILITY is the problem of determining whether or not a given graph $G$ admits a $k$-colouring. The optimization version of the problem asks to find a $k$-colouring of $G$ with minimum $k$, called the chromatic number of $G$ and denoted $\chi(G)$.

The $k$-COLOURABILITY is one of the central problems of algorithmic graph theory with numerous applications [4]. It is also one of the most difficult problems: it is NP-complete in general [12] and its optimization version is even hard to approximate [13]. Moreover, the problem remains difficult in many restricted graph families, for example triangle-free graphs [17] or line graphs [11] (in which case it coincides with the EDGE $k$-COLOURABILITY). On the other hand, when restricted to some other classes, such as graphs of vertex degree at most $k$ [2] or perfect graphs [8], the problem can be solved in polynomial time. Efficient polynomial-time algorithms for finding optimal colourings are available for many particular subclasses of perfect graphs, including chordal graphs [6], weakly chordal graphs [9], and comparability graphs [5].

All the aforementioned examples refer to graph classes possessing the property that with any graph $G$ they contain all induced subgraphs of $G$. Such classes are known in the literature under the name of hereditary classes. Any

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In this section we provide the necessary background and definitions used in the rest of the paper. For starters, we conclude with a summary of our results in Section 4 along with a list of open problems.

Sgall and Woeginger showed in [21] that 5-COLOURABILITY is NP-complete for \( P_5 \)-free graphs and 4-COLOURABILITY is NP-complete for \( P_{12} \)-free graphs. The last result was improved in [16], where the authors claim that by modifying the reduction from [21] 4-COLOURABILITY can be shown to be NP-complete for \( P_5 \)-free graphs.

On the other hand, the \( k \)-COLOURABILITY problem can be solved in polynomial time for \( P_t \)-free graphs as they constitute a subclass of perfect graphs. For \( t = 5, 6, 7 \), the complexity of the problem is generally unknown, except for the case of 3-COLOURABILITY of \( P_5 \)-free [20, 21] and \( P_6 \)-free graphs [19]. Known results on the \( k \)-COLOURABILITY problem in classes of \( P_t \)-free graphs are summarized in Table 1 (under columns 5 and 6, \( \alpha \) is matrix multiplication exponent known to satisfy \( 2 < \alpha < 2.376 \) [3]).

Table 1: Known complexities for \( k \)-colourability of \( P_t \)-free graphs

| \( k \) \( \setminus \) \( t \) | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | \ldots |
|----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|--------|
| 3              | \( O(m) \) | \( O(m) \) | \( O(n^\omega) \) | \( O(mn^\omega) \) | ? | ? | ? | ? | ? | ? | \ldots |
| 4              | \( O(m) \) | \( O(m) \) | ? | ? | ? | \( NP_c \) | \( NP_c \) | \( NP_c \) | \( NP_c \) | \( NP_c \) | \ldots |
| 5              | \( O(m) \) | \( O(m) \) | ? | ? | ? | \( NP_c \) | \( NP_c \) | \( NP_c \) | \( NP_c \) | \( NP_c \) | \ldots |
| 6              | \( O(m) \) | \( O(m) \) | ? | ? | ? | \( NP_c \) | \( NP_c \) | \( NP_c \) | \( NP_c \) | \( NP_c \) | \ldots |
| 7              | \( O(m) \) | \( O(m) \) | ? | ? | ? | \( NP_c \) | \( NP_c \) | \( NP_c \) | \( NP_c \) | \( NP_c \) | \ldots |
| \ldots         | \ldots | \ldots | \ldots | \ldots | \ldots | \ldots | \ldots | \ldots | \ldots | \ldots | \ldots |

In this paper, we focus on the minimal class from Table 1 where the \( k \)-COLOURABILITY problem is unsolved, i.e., the class of \( P_5 \)-free graphs. This class is “stubborn” with respect to various graph problems. For instance, \( P_5 \)-free graphs constitute a unique minimal class defined by a single forbidden induced subgraph with unknown complexity of the MAXIMUM INDEPENDENT SET and MINIMUM INDEPENDENT DOMINATING SET problems. Many algorithmic problems are known to be NP-hard in the class of \( P_5 \)-free graphs, which includes, among others, DOMINATING SET [14] and CHROMATIC NUMBER [15]. In contrast to the NP-hardness of finding the chromatic number of a \( P_5 \)-free graph, we show that \( k \)-COLOURABILITY can be solved in this class in polynomial time for each particular value of \( k \). In the case of a positive answer, our algorithm yields a valid \( k \)-colouring. Along with the mentioned result on 3-COLOURABILITY of \( P_5 \)-free graphs, our solution generalizes several other previously studied special cases of the problem, such as 4-COLOURABILITY of \((P_5, C_5)\)-free graphs [16] and 4-COLOURABILITY of \( P_5 \)-free graphs containing a dominating clique on four vertices [10]. We also note the algorithm in [7] that colours a \((P_5, C_5)\)-free graph \( G \) with \( \chi(G)^2 \) colours.

The remainder of the paper is organized as follows. In Section 2 we give relevant definitions, concepts, and notations. In Section 3 we present our recursive polynomial time algorithm that answers the \( k \)-colourability question for \( P_5 \)-free graphs. The difficult step in the algorithm is detailed using two different approaches. We conclude with a summary of our results in Section 4 along with a list of open problems.

## 2 Background and Definitions

In this section we provide the necessary background and definitions used in the rest of the paper. For starters, we assume that \( G = (V, E) \) is a simple undirected graph where \(|V| = n \) and \(|E| = m \). If \( A \) is a subset of \( V \), then
we let $G(A)$ denote the subgraph of $G$ induced by $A$. A stable set is a set of vertices such that there is no edge joining any two vertices in it.

**Definition 1** A set of vertices $A$ is said to dominate another set $B$, if every vertex in $B$ is adjacent to at least one vertex in $A$.

The following structural result about $P_5$-free graphs is from Bacsó and Tuza [1]:

**Theorem 1** Every connected $P_5$-free graph has either a dominating clique or a dominating $P_3$.

**Definition 2** Given a graph $G$, an integer $k$ and for each vertex $v$, a list $l(v)$ of $k$ colours, the $k$-list colouring problem asks whether or not there is a colouring of the vertices of $G$ such that each vertex receives a colour from its list.

**Definition 3** The restricted $k$-list colouring problem is the $k$-list colouring problem in which the lists $l(v)$ of colours are subsets of $\{1, 2, \ldots, k\}$.

Our general approach is to take an instance of a specific colouring problem $\Phi$ for a given graph and replace it with a polynomial number of instances $\phi_1, \phi_2, \phi_3, \ldots$ such that the answer to $\Phi$ is “yes” if and only if there is some instance $\phi_k$ that also answers “yes”.

For example, consider a graph with a dominating vertex $u$ where each vertex has colour list $\{1, 2, 3, 4, 5\}$. This listing corresponds to our initial instance $\Phi$. Now, by considering different ways to colour $u$, the following four instances will be equivalent to $\Phi$:

- $\phi_1$: $u = 1$ and the remaining vertices have colour lists $\{2, 3, 4, 5\}$,
- $\phi_2$: $u = 2$ and the remaining vertices have colour lists $\{1, 3, 4, 5\}$,
- $\phi_3$: $u = 3$ and the remaining vertices have colour lists $\{1, 2, 4, 5\}$,
- $\phi_4$: $u = \{4, 5\}$ and the remaining vertices have colour lists $\{1, 2, 3, 4, 5\}$.

In general, if we recursively apply such an approach we would end up with an equivalent set with an exponential number of colouring instances.

### 3 The Algorithm

Let $G$ be a connected $P_5$-free graph. This section describes a polynomial time algorithm that decides whether or not $G$ is $k$-colourable. Our strategy is as follows. First, we find a dominating set $D$ of $G$ which is a clique with at most $k$ vertices or a $P_3$. There are only a finite number of ways to colour the vertices of $D$ with $k$ colours. For each of these colourings of $D$, we recursively check if it can be extended to a colouring of $G$. Each of these subproblems can be expressed by a restricted list colouring problem. We now describe the algorithm in detail.
The algorithm is outlined in 3 steps. Step 2 requires some extra structural analysis and is presented using two different approaches in the following subsections.

**Algorithm**

1. First, we check if \( G \) contains a dominating set of size at most \( k \geq 3 \). If no such a set is found, then \( G \) is not \( k \)-colourable. Otherwise, let \( D \) be a dominating set in \( G \), which is either a clique with at most \( k \) vertices or a \( P_3 \). Let the vertices of the dominating set be \( d_1, d_2, \ldots, d_t \) with \( t \leq k \). Since \( D \) is a dominating set, we can partition the remaining vertices into fixed sets \( F_1, F_2, \ldots, F_r \), \( r \leq t \), such that vertices in \( F_1 \) are adjacent to \( d_1 \), and for \( i > 1 \), vertices in \( F_i \) are adjacent to \( d_i \) but not to \( \{d_1, \ldots, d_{i-1}\} \). The colour list of the vertices in the fixed sets have size at most \( k - 1 \) since each vertex in \( D \) is already assigned a colour. This gives rise to our original restricted list-colouring instance \( \Phi \).

2. Two vertices are dependent if there is an edge between them and the intersection of their colour lists is non-empty. In this step, we remove all dependencies between each pair of fixed sets. This process will create a set \( \{\phi_1, \phi_2, \phi_3, \ldots\} \), equivalent to \( \Phi \), of a polynomial number of colouring instances. Two different methods for performing this step are outlined in the following subsections.

3. For each instance \( \phi_i \) from Step 2 the dependencies between each pair of fixed sets have been removed which means that the vertices within each fixed set can be coloured independently. Thus, for each instance \( \phi_i \) we recursively see if each fixed set can be coloured with the corresponding restricted colour lists (the base case is when the colour lists are a single colour). If one such instance provides a valid \( k \)-colouring then return the colouring. Otherwise, the graph is not \( k \)-colourable.

As mentioned, the difficult part is reducing the dependencies between each pair of fixed sets (Step 2). We present two different approaches to handle Step 2. The first is conceptually simpler while the second includes additional structural results.

### 3.1 Removing the Dependencies Between Two Fixed Sets: Method I

Let \( \text{col}(C) \) be the set of colours that appear in the lists of vertices of a set \( C \). Let \( A \) and \( B \) be two fixed sets. Note that \(|\text{col}(A)| \leq k - 1 \) and \(|\text{col}(B)| \leq k - 1 \). We remove dependencies between \( A \) and \( B \) by applying the following procedure.

**Procedure One**

1. Find a \((k-1)\)-colouring of \( A \) (respectively, \( B \)) with stable sets \( A_1, A_2, \ldots, A_{k-1} \) (respectively, \( B_1, B_2, \ldots, B_{k-1} \)). If \( A \) or \( B \) cannot be \((k-1)\)-coloured, then \( G \) cannot be \( k \)-coloured.

2. For each \( i = 1, 2, \ldots, k - 1 \) and each \( j = 1, 2, \ldots, k - 1 \), remove dependencies between \( A_i \) and \( B_j \).

Now, we describe how to remove dependencies between two stable sets \( X = A_i \) and \( Y = B_j \). Let \( X' \) (respectively, \( Y' \)) be the set of vertices of \( X \) (respectively, \( Y \)) that are dependent on some vertices of \( Y \) (respectively, \( X \)). Note that \( X' \) is non-empty if and only if \( Y' \) is non-empty.

**Lemma 1** If \( X' \neq \emptyset \), there exists a vertex in \( X' \) that is adjacent to all vertices in \( Y' \).
Proof. Let \( x_1 \) be a vertex of \( X' \) with a maximal neighborhood in \( Y' \). Assume there exists a vertex \( y_2 \in Y' \) that is not adjacent to \( x_1 \). Then, there must exist a vertex \( x_2 \in X' \) (different than \( x_1 \)) adjacent to \( y_2 \). Also, by the choice of \( x_1 \), there must exist a vertex \( y_1 \in Y' \) that is adjacent to \( x_1 \) but not \( x_2 \). Since \( X \) and \( Y \) belong to different fixed sets, there exists a vertex \( v \) in the dominating set such that either \( v \) is adjacent to \( x_1, x_2 \) but not \( y_1, y_2 \), or \( v \) is adjacent to \( y_1, y_2 \) but not \( x_1, x_2 \). But then \( G(\{v, x_1, x_2, y_1, y_2\}) \) is an induced \( P_5 \); a contradiction. \[ \square \]

Lemma\[\text{[1]}\] states that as long as \( X' \) and \( Y' \) are non-empty, we can find a vertex \( x \in X' \) that dominates \( Y' \). Now given such a vertex \( x \), we can create new equivalent colouring instances by assigning to \( x \) (i) a colour from \( l(x) \cap \text{col}(Y') \) and (ii) the list \( l(x) - \text{col}(Y') \). In the former instances the vertices in \( Y' \) lose the colour assigned to \( x \) from their lists i.e., \( |\text{col}(Y')| \) decreases by one. In the latter instance, the vertex \( x \) is no longer dependent on any vertex in \( Y' \) and is thus removed from \( X' \). In this case, we recursively repeat this process until \( X' \) is empty by finding a new vertex in \( X' \) that dominates \( Y' \). This will result in at most \( kn \) new colouring instances where either \( X' \) is empty or \( |\text{col}(Y')| \) has decreased by one from its initial state. To reduce \( |\text{col}(Y')| \) to zero, we repeatedly apply this process at most \( k \) times. Thus, we can completely remove the dependencies between \( X' \) and \( Y' \) by producing at most \((kn)^k\) new equivalent colouring instances.

Analysis. To remove the dependencies between each \( A_i \) and \( B_j \) requires \((kn)^k\) new equivalent instances. Thus, to remove the dependencies between each pair of fixed sets (Step 2 of Procedure One) requires \((kn)^{k^3}\) new equivalent instances. Since there are \( k \) fixed sets, there are less than \( k^2 \) pairs of fixed sets. Thus, to remove dependencies between each pair of fixed sets (given the stable sets for each fixed set) requires \((kn)^{k^2}\) equivalent instances. To find the stable sets for each fixed set requires a single recursive \( k-1 \) colouring on the graph \( G \) with the initial dominating set combined with the edges between the fixed sets removed.

Now, let \( T(k) \) denote the number of subproblems produced by the Algorithm where \( k \) is the number of colours used on a graph with \( n \) vertices. From the previous analysis we arrive at the following recurrence where \( T(1) = 1 \):

\[
T(k) = (kn)^5 T(k - 1) + T(k - 1).
\]

A proof by induction shows \( T(k) = O((kn)^{k^5}) \), implying our algorithm runs in polynomial time.

3.2 Removing the Dependencies Between Two Fixed Sets: Method II

For our second method for removing the dependencies between a pair of fixed sets, it will be convenient to associate a fixed set \( F_i \) to the colours in its lists. For this purpose, let \( S_{list} \) denote a fixed set of vertices with colour list given by \( list \). We partition each such fixed set into \textbf{dynamic sets} \( P_i \) that each represents a unique subset of the colours in \( list \). For example: \( S_{123} = P_{123} \cup P_{12} \cup P_{13} \cup P_{23} \cup P_1 \cup P_2 \cup P_3 \). Initially, \( S_{123} = P_{123} \) and the remaining sets in the partition are empty. However, as we start removing dependencies, these sets will dynamically change. For example, if a vertex \( u \) is initially in \( P_{123} \) and one of its neighbors gets coloured 2, then \( u \) will be removed from \( P_{123} \) and added to \( P_{13} \).

Recall that our goal is to remove the dependencies between two fixed sets \( S_p \) and \( S_q \). To do this, we remove the dependencies between each pair \((P, Q)\) where \( P \) is a dynamic subset of \( S_p \) and \( Q \) is a dynamic subset of \( S_q \). By visiting these pairs in order from largest to smallest with respect to \( |\text{col}(P)| \) and then \( |\text{col}(Q)| \), we ensure that we only need to consider each pair once. Applying this approach, the crux of the reduction process is to remove the dependencies between a pair \((P, Q)\) by creating at most a polynomial number of equivalent colourings.

Now, observe that there exists a vertex \( v \) from the dominating set found in Step 1 of the algorithm that dominates every vertex in one set, but is not adjacent to any vertex in the other. This is because \( P \) and \( Q \) are subsets of different fixed sets. Without loss of generality assume that \( v \) dominates \( Q \). Now, consider the (connected)
Figure 1: Illustration of the graph $H$ from two dynamic sets components of $G(P)$ and $G(Q)$. If a component $Z$ in $G(P)$ is not adjacent to any vertex in $Q$ then the vertices in $Z$ have no dependencies with $Q$. The same applies for such components in $Q$. Since these components have no dependencies, we focus on the induced subgraph $H = G(P \cup Q \cup \{v\})$ with these components removed. This graph is illustrated in Figure 1 where the small rectangles represent the components in $G(P)$ and $G(Q)$ respectively. It is easy to observe that $H$ is connected (if not, then there are components $H_1, H_2$ of $H$, each of which contains a vertex in $P$ and a vertex in $Q$; it follows there are edges $(a, b)$ of $H_1$ and $(c, d)$ of $H_2$ such that $a, b, v, d, c$ induce a $P_5$).

THEOREM 2 Let $H$ be a connected $P_5$-free graph partitioned into three sets $P$, $Q$ and $\{v\}$ where $v$ is adjacent to every vertex in $Q$ but not adjacent to any vertex in $P$. Then there exists at most one component in $G(P)$ that contains two vertices $a$ and $b$ such that $a$ is adjacent to some component $Y_1 \in G(Q)$ but not adjacent to another component $Y_2 \in G(Q)$ while $b$ is adjacent to $Y_2$ but not $Y_1$.

PROOF: The proof is by contradiction. Suppose that there are two unique components $X_1, X_2 \in G(P)$ with $a, b \in X_1$ and $c, d \in X_2$ and components $Y_1 \neq Y_2$ and $Y_3 \neq Y_4$ from $G(Q)$ such that:

- $a$ is adjacent to $Y_1$ but not adjacent to $Y_2$,
- $b$ is adjacent to $Y_2$ but not adjacent to $Y_1$,
- $c$ is adjacent to $Y_3$ but not adjacent to $Y_4$,
- $d$ is adjacent to $Y_4$ but not adjacent to $Y_3$.

Let $y_i$ denote an arbitrary vertex from the component $Y_i$. Since $H$ is $P_5$-free, there must be edges $(a, b)$ and $(c, d)$, otherwise $a, y_1, v, y_2, b$ and $c, y_3, v, y_4, d$ would be $P_5$s. An illustration of these vertices and components is given in Figure 2, the solid lines.

Now, if $Y_2 = Y_3$, then there exists a $P_5 = a, y_2, c, d$. Thus, $Y_2$ and $Y_3$ must be unique components, and $Y_1, Y_4$ must be different as well for the same reason. Similarly $Y_2 \neq Y_4$. Now since $b, y_2, v, y_3, c$ cannot be a $P_5$, either $b$ is adjacent to $Y_3$ or $c$ must be adjacent to $Y_2$. Without loss of generality, suppose the latter. Now $a, b, y_2, v, y_4$ implies that either $a$ or $b$ is adjacent to $Y_4$. If the latter, then $a, b, y_4, d, c$ would be a $P_5$ which implies that $a$ must
be adjacent to $Y_1$ anyway. Thus, we end up with a $P_5 = a, y_4, v, y_2, c$ which is a contradiction to the graph being $P_5$-free. □

From Theorem 2, there is at most one component $X$ in $G(P)$ that contains two vertices $a$ and $b$ such that $a$ is adjacent to some component $Y_1 \in G(Q)$ but not adjacent to another component $Y_2 \in G(Q)$ while $b$ is adjacent to $Y_2$ but not $Y_1$. If such a component exists, then we can remove the vertices in $X$ from $P$ by applying the following general method for removing a component $C$ from a dynamic set $D$.

### Procedure RemoveComponent

Since $C$ is $P_5$-free, it has a dominating clique or $P_3$ (Theorem 1). If this dominating set $D$ can be coloured with the list $col(D)$, we consider all such colourings (otherwise we report there is no valid colouring for the given instance). For each case the colouring will remove all vertices in the component from $D$ to other dynamic sets represented by smaller subsets of available colours. Observe that since $k$ is fixed, the number of such colourings is constant.

If there are still dependencies between $P$ and $Q$, then we make the following claim (observing that the graph $H$ dynamically changes as $P$ and $Q$ change):

**Claim 1** There exists a vertex $x \in P$ that is adjacent to all components in $H(Q)$. Moreover, $x$ dominates all components of $H(Q)$ except at most one.

**Proof:** Let $x \in P$ be adjacent to a maximal number of components in $H(Q)$. If it is not adjacent to all components, then there must exist another vertex $x' \in P$ and components $Y_1, Y_2 \in Q$ such that $x$ is adjacent to $Y_1$ but not $Y_2$ and $x'$ is adjacent to $Y_2$ but not $Y_1$. This implies that there is a $P_5 = x, y_1, v, y_2, x'$ where $y_1 \in Y_1$ and $y_2 \in Y_2$ unless $x$ and $x'$ are adjacent. However by Theorem 2 they cannot belong to the same component in $H(P)$ since such a component would already have been removed - a contradiction.

Now, suppose that there are two components $Y_1$ and $Y_2$ in $H(Q)$ that $x$ does not dominate. Then there exists edges $(y_1, y_1') \in Y_1$ and $(y_2, y_2') \in Y_2$ such that $x$ is adjacent to $y_1$ and $y_2$, but not $y_1'$ nor $y_2'$. This however, implies the $P_5 = y_1', y_1, x, y_2, y_2'$ - a contradiction. □

Now we identify such an $x$ outlined in this claim and create equivalent new colouring instances by assigning $x$ with each colour from $col(P) \cap col(Q)$ and then with the list $col(P) - col(Q)$. If $x$ is assigned a colour from
col(P) ∩ col(Q), then all but at most one component will be removed from H(Q). If one component remains, then we can remove it from Q by applying Procedure RemoveComponent. In the latter case, where x is assigned the colour list col(P) − col(Q), x will be removed from P. If there are still dependencies between P and Q, we repeat this step by finding another vertex x. In the worst case we have to repeat this step at most |P| times. Therefore, the process for removing the dependencies between two dynamic sets creates at most O(n) new equivalent colouring instances.

Analysis. We have just shown that we require at most O(n) new equivalent colouring instances to remove the dependencies between two dynamic sets. Since each fixed set contains at most O(2^{k-1}) dynamic sets, there are O(2^{2(k-1)}) pairs of dynamic sets to consider between each pair of fixed sets. Thus, removing the dependencies between two fixed sets produces O(n^{22(k-1)}) subproblems. Since there at most k^2 pairs of fixed sets, this means that to remove the dependencies between all fixed sets creates O(n^{k^2.22(k-1)}) subproblems.

As with the previous method, let T(k) denote the number of subproblems produced by the Algorithm where k is the number of colours used on a graph with n vertices. From the previous analysis we arrive at the following recurrence where T(1) = 1:

T(k) ≤ cn^{k^2.22(k-1)} T(k-1).

A proof by induction proves that T(k) = O(n^{k^3.d(k-1)}), implying our algorithm runs in polynomial time.

Theorem 3 The restricted k-list colouring problem for P_5-free graphs, for a fixed integer k, can be solved in polynomial time.

Corollary 1 Determining whether or not a P_5-free graph can be coloured with k-colours, for a fixed integer k, can be decided in polynomial time.

4 Summary

The algorithm presented in this paper brings us one step closer to completely answering the question of when there exists a polynomial time algorithm for the k-COLOURABILITY problem for P_t-free graphs, given fixed k and t. In particular, we now know that there exists a polynomial time algorithm when t = 5 for any fixed value of k.

Continuing with this vein of research, the following open problems are perhaps the next interesting avenues for future research:

- Does there exist a polynomial time algorithm determine whether or not a P_7-free graph can 3-coloured.
- Does there exist a polynomial time algorithm determine whether or not a P_6-free graph can 4-coloured.
- Is the problem of k-colouring a P_7-free graph NP-complete.

Two other related open problems are to determine the complexities of the MAXIMUM INDEPENDENT SET and MINIMUM INDEPENDENT DOMINATING SET problems on P_5-free graphs.
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