1 Introduction

Electrodynamics in vacuum or in media with moving boundaries was the subject of numerous studies in the XXth century. It is sufficient to remember that the famous Einstein’s paper which gave birth to the relativity theory was entitled “Electrodynamics of moving bodies.” The total number of publications in this field is enormous, since it includes, in particular, such important problem as radio-location. In this review we confine ourselves to the problem of cavities with (ideal) reflecting moving boundaries. This means that we consider the fields confined in some limited volume, thus leaving aside the problem of the field propagation in the (semi)infinite space and reflection from single boundaries (having in mind that numerous references to the publications related to this “full space” problem can be found, e.g., in [2–5]).

The plan of the chapter is as follows. In the next section we give a brief historical review of the relevant studies, both in the fields of classical and quantum electrodynamics, supplying it with an extensive list of the known publications. Although we tried to give a more or less complete list, it is clear that some publications have been (undeliberately) missed. To excuse we could mention that for decades different groups of physicists and mathematicians performed the studies in their own fields of interest, not suspecting the existence of analogous results found in other areas. We hope that our review will serve to diminish this gap essentially.

It is clear that all results accumulated for several decades cannot be collected in one chapter. Therefore in the following sections we have decided to make an emphasis mainly on the detailed exposition of our own results concerning the recently obtained analytical solutions for the cavities with resonantly oscillating boundaries, since we hope that these solutions could be important for the further studies on the problem known under the names “Nonstationary Casimir Effect” or “Dynamical Casimir Effect.”

2 Brief history of studies on electrodynamics with moving boundaries

2.1 Classical fields in cavities with moving boundaries

The first exact solution of the wave equation \((c = 1)\)

\[
\frac{\partial^2 A}{\partial t^2} - \frac{\partial^2 A}{\partial x^2} = 0,
\]

in a time-dependent domain \(0 < x < L(t)\) (where \(L(t)\) is the given law of motion of the right boundary), satisfying the boundary conditions \(A(0, t) = A(L(t), t) = 0\)
was obtained by Nicolai [1] for

\[ L(t) = L_0(1 + \alpha t). \]  

The solution was interpreted in terms of the transverse vibrations of a string with a variable length. A few years later, these results were published in [3], where the extension to the case of electromagnetic field was also made. A similar treatment was given by Havelock [8] in connection with the problem of radiation pressure. Quarter a century later, the one-dimensional wave equation in the time-dependent interval interval \( 0 < x < a + bt \) was considered in [3] under the name “Spaghetti problem.”

A new wave of interest to the problem of electromagnetic field in a cavity with moving boundaries has arisen only in the beginning of 60s, being motivated, partially, by the experiments [10] on the “field compression,” accompanied by the frequency multiplication (for 2.3 times) due to multiple reflections of the initial \( H_{011} \) wave (\( \lambda_m = 10 \) cm) from the opposite sides of a resonator, one of which was a “plasma piston” moving uniformly with the velocity \( v \sim 2 \cdot 10^7 \) cm/s. Kurilko [11] studied linearly polarized electromagnetic field between two ideal infinite plates moving with equal constant velocities towards each other. He considered consecutive reflections of the waves from each boundary, writing explicit expressions for the finite time-space intervals, corresponding to zero, one, two, etc., reflections. Baláz [12] gave a detailed study of the string problem with the aid of the method similar to that used by Nicolai and Havelock. Besides considering the uniformly moving boundary, he has found an exact solution for \( L(t) = (t^2 + 1)^{1/2} \), and gave some graphical method of finding the solution for an arbitrary law of motion \( L(t) \). Greenspan [13] studied the one-dimensional string with the uniformly moving right boundary, assuming the boundary condition at the left point in the form \( A(0, t) = \sin(\omega t) \). The problem of the “field compression” between two ideal infinite moving walls (the same geometry as in [12]) was studied by Stetsenko [14, 15]. An approximate solution for the electromagnetic field in a rectangular waveguide cavity with a uniformly moving boundary was obtained in [4]. In this case one has to solve the equation \( c = 1 \)

\[ A_{xx} - A_{tt} = \kappa^2 A \]  

with the same boundary condition as in (2). The detailed paper by Baranov and Shirokov [17] can be considered, in a sense, as a concluding study of the one-dimensional problem with a uniformly moving boundary (although many publications on this subject continued to appear up to last years). The experiments on laser cavities with uniformly moving mirrors were described, e.g., in [18–22]. In these experiments, the constant velocity of the mirror varied from 7 cm/s [18] to 400 m/s [22].

In short note [23], Askar’yan has pointed out two possible effects of oscillating surfaces on the electromagnetic field inside the (laser) resonator cavities. The first one is the influence of oscillations on the generation and intensity of the laser radiation. It was extensively studied in many experiments, devoted, in particular, to such problems as the generation of optical pulses [24, 25], phase locking of laser modes (where the frequencies of the mirror oscillations varied from 50 Hz [26] to 500 KHz [27, 28] and 1 MHz [24, 29], see the review in [31]), or modulation of the laser radiation [32, 33] (in [33] the frequencies varied from 17 to 70 KHz). The theory of these phenomena was considered, e.g., in [34–36].

The second effect predicted by Askar’yan was the field amplification inside the cavity under the parametric resonance condition, when the mirror oscillates at twice the field eigenfrequency. It was not observed yet, as far as we know, and the main part of the present review is devoted just to the progress in the theoretical treatment of this phenomenon achieved in recent years.

In 1967, Grinberg [37] has proposed a general method of solving the wave equation in the case of an arbitrary law of motion of the boundary, based on expanding the solution over the complete set of “instantaneous modes.” Krasič’nikov [38] seems to be the first who gave rather detailed study of the electromagnetic vibrations in a spherical cavity with an oscillating boundary. The one-dimensional cavity with a resonantly oscillating boundary was considered with the aid of the method of characteristics in study [38], where it was shown that the energy is ‘pumped’ to the high-frequency modes at the expense of the lower-frequency ones.

A significant contribution was made in a series of papers by Vesnitskii and co-authors. In [40] he gave an exact solution for the problem of a rectangular waveguide with a uniformly moving lateral wall, i.e., for the equation and the boundary conditions

\[ A_{xx} + A_{zz} - A_{tt} = 0, \quad A\big|_{x=0} = A\big|_{x=L(t)} = 0 \]  

(5)
with \( L(t) = L_0(1 + \alpha t) \) (in this case, \( A \) is the \( E_y \) component of the field of the type \( H_{\text{in}} \)). A spherical resonator, whose radius linearly changed with time, was considered in [11]. A general (although implicit, in some sense) solution of the one-dimensional problem with an arbitrary law of motion of the boundary was given in [13], where it was shown that a complete set of solutions to the problem (1)-(2) can be expressed through the solution of some simple functional equation (see equation (7) in the next subsection). The solutions of the \textit{inhomogeneous} one-dimensional wave equation for the law of motion \( L(t) = L_0(1 + \alpha t)^{\pm 1} \), with an arbitrary inhomogeneity and arbitrary initial conditions, were given in [14]. A family of concrete laws of motion admitting simple explicit expressions for the mode functions was found in the framework of the \textit{inverse problem} in [14]. The two-dimensional rectangular membrane with a single uniformly moving boundary was considered in [15]. A possibility of the frequency modulation in the waveguide with a slowly oscillating boundary was studied in [16]. A review of the results obtained by Vesnitskii and co-authors was made in [17].

Exact solutions for the waveguide with \textit{nonuniformly} moving boundary (problem (3)) have been found by Barsukov and Grigoryan [18, 19]. They considered also the electromagnetic resonator with moving boundary [20]. Similar problems were studied in [21, 22]. Periodic solutions of a one-dimensional wave equation with homogeneous conditions on moving boundaries were considered in [23]. Transition processes in one-dimensional systems with moving boundaries were studied in [24]. Oscillations of a round membrane with a uniformly varying radius were considered in [25]. A nonlinear transformation was applied to solve the inhomogeneous problem of the forced resonance oscillations in a one-dimensional cavity with moving boundaries in [26]. The scaling transformation method, which reduces the problem with moving boundaries to that with fixed ones by means of the transformations such as \( x \to x/L(t) \), was considered in [27, 28].

In the last decade of the century, the problem of vibrating string with moving boundaries was studied in [29] (two supports moving toward each other with constant velocities) and [30–34] (oscillating supports). The electromagnetic (better to say, massless scalar) field in one dimensional ideal cavities with periodically moving boundaries were considered in [35, 36]. Analytical solutions for the field in circular waveguides with (linearly) moving boundaries were obtained recently in [37, 38]. The “dynamical” field modes in the one-dimensional and spherical cavities with uniformly moving boundaries were found once more time in [39]. The same problem for the expanding/contracting ideal spherical cavity, whose radius varies as \( R(t) = R_0\sqrt{1 + \alpha t} \), was solved recently in [40]. (A family of the laws of motion of the boundary, which includes, in particular, the dependences such as \( R(t) = R_0\sqrt{1 + \alpha t + \beta t^2} \), \( R(t) = D t + E + F(At + B)^{-1} \), and their combinations, was considered in the case of the \textit{diffusion-type} equations in [41, 42], where it was shown that this family admits \textit{exact} solutions of the problem.)

### 2.2 Quantum fields in the presence of moving boundaries

Moore’s paper [70] seems to be the first one devoted to the problem of \textit{quantum fields} in cavities with moving boundaries. It was motivated by the studies of a more general problem of the particle (in particular, photon) creation in the nonstationary universe [71] and in external intense fields (see, e.g., the books [72, 73] and reviews by Ritus and Nikishov in [31]), which is closely related to the problem of field quantization in the spaces with nontrivial (e.g., time-dependent) geometry. Considering a model of the “scalar electrodynamics” (when the field depends on a single space coordinate), Moore has found a complete set of solutions to the problem (3)-(4) in the form

\[
A_n(x,t) = C_n \left\{ \exp \left[ -i \pi n R(t-x) \right] - \exp \left[ -i \pi n R(t+x) \right] \right\},
\]

where function \( R(\xi) \) must satisfy the functional equation

\[
R(t + L(t)) - R(t - L(t)) = 2.
\]
of the boundary (the solution to Eq. (5), found by many authors cited above, reads (remember that we assume \( c = 1 \))

\[
R_\alpha(\xi) = \frac{2 \ln |1 + \alpha \xi|}{\ln |(1 + v)/(1 - v)|}, \quad v = \alpha L_0.
\]  

(8)

Evidently, if \( \alpha \to 0 \), this function goes to \( R_0(\xi) = \xi/L_0 \). For an arbitrary nonrelativistic law of motion one can find the solution in the form of the expansion over subsequent time derivatives of the wall displacement. We give it in the form obtained in [106]:

\[
R(\xi) = \xi \lambda(\xi) - \frac{1}{2} \xi^2 \dot{\lambda}(\xi) + \frac{1}{6} \xi \ddot{\lambda}(\xi) \left( \xi^2 - L^2(\xi) \right) + \cdots, \quad \lambda(\xi) \equiv L^{-1}(\xi).
\]  

(9)

In the special case \( L(t) = L_0/(1 + \alpha t) \), when \( \dot{\lambda}(\xi) = 0 \), Eq. (5) yields another exact solution, \( R(\xi) = L_0^{-1} (\xi + \frac{1}{2} \alpha \xi^2) \). Unfortunately, the expansions such as (3) cannot be used in the long-time limit \( \xi \to \infty \), since the terms proportional to the derivatives of \( \lambda(\xi) \) (which are supposed to be small corrections) become bigger than the unperturbed term \( \xi \lambda(\xi) \).

Castagnino and Ferraro [93] have found several solutions of the Moore equation (7) with the aid of the inverse method, which was used earlier by Vesnitskii [44]. In this method, one chooses some reasonable function \( R(\xi) \) and determines the corresponding law of motion of the boundary \( L(t) \) using the consequence of Eq. (5),

\[
\dot{L}(t) = \frac{R'(t - L(t)) - R'(t + L(t))}{R'(t - L(t)) + R'(t + L(t))}.
\]  

(10)

To solve differential equation (11) with some simple functions \( R(\xi) \) is more easy (at least this can be done numerically) than to solve the functional equation (7) for the given function \( L(t) \). However, not for any simple function \( R(\xi) \) the dependence \( L(t) \) appears admissible from the point of view of physics (the velocity may occur greater than the speed of light, or some discontinuities may arise). Actually, the cases considered in [113] correspond to some monotonous displacements of the mirror from the initial to final positions. Typical functions \( R(\xi) \) used in [113] were some combinations of \( \xi/L_0 \) and some trigonometric functions such as \( \sin(m \pi \xi/L_0) \). A large list of simple functions \( R(\xi) \) (rational, exponential, logarithmic, hyperbolic, trigonometrical and inverse trigonometrical) and corresponding to them functions \( L(t) \) was given in [113].

However, no one of these functions can be used in the parametric resonance case. The asymptotical solution of the Moore equation in the parametric resonance case \( L(t) = L_0 \left[ 1 + \epsilon \sin(\pi q t/L_0) \right] \), \( q = 1, 2, \ldots \), was found in [107] [110]. For \( \epsilon t \gg 1 \) it has the form (here \( L_0 = 1 \))

\[
R(t) = t - \frac{2}{\pi q} \mathrm{Im} \left\{ \ln \left[ 1 + \zeta + \exp(i \pi q t)(1 - \zeta) \right] \right\},
\]  

(11)

\[ \zeta = \exp \left[ (-1)^{q+1} \pi q t \right], \]

which clearly demonstrates that the asymptotical mode structure in the resonance case is quite different from the mode structure inside the cavity with unmoving walls. The solution (11) was improved and generalized to the case of two vibrating boundaries in [111] [112]. However, the form of this solution is not very convenient for the calculation of various sums giving the mean numbers of photons, energy, etc., so a lot of rather sophisticated calculations must be done before the final physical results could be obtained.

One of the first estimations of the number of photons which could be created from vacuum in a cavity whose boundary moves with nonrelativistic velocity have been performed by Rivlin [113], who considered the parametric amplification of the initial vacuum field oscillations in the framework of the classical approach. He gave an estimation of the number of created photons \( N \sim (\epsilon \omega_1 t)^2 \), where \( \epsilon \sim \delta L/L \) is the relative amplitude of the variation of the distance between the walls and \( \omega_1 \) is the fundamental unperturbed field eigenfrequency (provided the frequency of the wall vibrations \( \omega_w \) is close to \( 2 \omega_1 \)). A similar estimation was given by Sarkar [114], who used an approximate solution to the Moore equation (7) in the form of the asymptotic series with respect to a small parameter \( \epsilon \), found in [115] (such solutions were constructed earlier in [17] [23]). However, the tremendous numerical values obtained by Sarkar were quite unrealistic, since he used the value of \( \epsilon \) which was many orders of magnitude higher than those which can be achieved in the laboratory under real conditions. Moreover, the oversimplified approach of Rivlin and simple perturbative solutions of Sarkar are not valid, as a matter of fact, under the resonance conditions, due to the presence of
the secular terms (as it always happens for parametric systems). Actually, the structure of the ‘dynamical’ modes in the resonance case is completely different (in the most interesting long-time limit) from the simple standing waves existing in the cavity with unmoving walls. The same is true for the estimation made by Askar’yan [23] in the classical case. He evaluated the average work done by the moving wall on the field as $A \sim \int p v \, dt$, where $v = v_0 \sin(\omega_0 t + \phi)$ is the wall velocity, and $p(t)$ is the radiation pressure. Taking the monochromatic dependence $p(t) = p_0 \sin^2(2\omega_1 t)$, he obtained the linear dependence on time in the resonance case $\omega_w = 2\omega_1$: $A \sim p_0 v_0 g t \sin \phi + \text{const}$. However, these evaluations can serve only as some indication that in the resonance case the energy of the field can grow at the expense of the mechanical work done by the vibrating wall. The real time dependence can be quite different, because the relation $p(t) = p_0 \sin^2(2\omega_1 t)$ holds, as a matter of fact, only until $\omega_1 t \ll 1$, whereas for larger times the effect of the mode reconstruction must be taken into account. One of the goals of this review is to demonstrate what happens in reality in the resonance case.

Approximate solutions to the Moore equation, such as (1), were used to evaluate corrections to the famous Casimir attractive force between infinite ideal walls (17) (for the reviews on the Casimir effect see, e.g., [18, 12]) due to the nonrelativistic motion of the walls. The leading term of these corrections turned out proportional to the square of velocity of the boundary (i.e., of the order of $(v/c)^2$). The Casimir force in the relativistic case was calculated in (74, 75, 81, 90, 91, 93, 101, 103) and depends in the generic case not only on the instantaneous velocity, but on the whole time dependence $L(t)$ (through the function $R(\xi)$, i.e., on the acceleration and other time derivatives). The first calculations of the forces acting on the single mirrors moving with nonrelativistic velocities due to the vacuum or thermal fluctuations of the field were performed in the framework of the spectral approach (using the fluctuation-dissipation theorem) in (122) (three-dimensional case, the force proportional to the fifth-order derivative of the coordinate) and in (123) (one-dimensional model, the force proportional to the third-order derivative of the coordinate). It was mentioned that the force could be significantly amplified under the resonance conditions, either in the LC-contour (122) or in the Fabry–Perot cavity (123) (earlier, such a possibility was discussed in [124]). These studies were continued in (125, 133), where it was assumed that the velocity of the boundary is perpendicular to the surface. The Casimir force between two parallel plates, when their relative velocity is also parallel to the surfaces, was considered by Levitov (134). Later, the theory of “Casimir friction” was developed in (135, 139). The reviews of these approaches can be found in (140, 141).

Various quantum effects arising due to the motion of dielectric boundaries, including the modification of the Casimir force and creation of photons, both in one and three dimensions, and for different orientations of the velocity vector with respect to the surface, have been studied in detail in the series of papers by Barton and his collaborators (142, 148). In the paper (142) the term Mirror-Induced Radiation (MIR) has been introduced.

Another name, Nonstationary Casimir Effect (NSCE), was introduced earlier in (110) for the class of phenomena caused by the reconstruction of the quantum state of field due to a time dependence of the geometrical configuration (149, 152). Its synonym is the term Dynamical Casimir Effect, which became popular after the series of articles by Schwinger (153, 157), in which he tried to explain the phenomenon of sonoluminescence by the creation of photons in bubbles with time-dependent radii, oscillating under the action of acoustic pressure in the liquids (see a brief discussion of this subject in the last section).

A possibility of generating the “nonclassical” (in particular, squeezed) states of the electromagnetic field in the cavity with moving walls was pointed out in (106, 114, 122, 158, 161). The dynamical Casimir force was interpreted as a mechanical signature of the squeezing effect associated with the mirror’s motion in (122, 123) (see also 102).

It was suggested also in (106, 124, 159, 160) that a significant amount of photons could be created from vacuum even for quite small nonrelativistic velocities of the walls, provided the boundaries of a high-Q cavity perform small oscillations at a frequency proportional to some cavity unperturbed eigenfrequency, due to an accumulation of small changes in the state of the field for a long time. Indeed, using the asymptotical solutions of the Moore equation (11), it was shown in (107, 110) that the rate of photon generation in each mode becomes constant in the long-time limit, being linearly proportional to the product $\omega_1$, and that the photons are generated in a wide frequency band whose width grows exponentially fast in time (163). Other approximate or exact solutions of the Moore equation for the specific periodical time dependences of the cavity dimensions were found by different methods in studies (111, 112, 164, 168), which confirmed the effect of resonance generation of photons.
Moore’s approach is based on the decomposition of the field over the mode functions satisfying automatically the (one-dimensional) wave equation. There exists another approach (proposed in the framework of the classical problem as far back as in [17]), when the mode functions are chosen in such a way that they satisfy automatically the time-dependent boundary conditions: see Eq. (17) below. In this case it is possible to describe the behavior of the field with the aid of some effective Hamiltonian, which is an infinite-dimensional quadratic form of the boson creation/annihilation operators with time-dependent coefficients responsible for the coupling between different modes. Such an approach was considered for the first time (in the quantum case) by Razavy and Terning [169, 172] (the case of massive field was considered in [172], see also [173]). However, the resulting infinite set of coupled evolution equations for the annihilation and creation operators turned out rather complicated in the generic case. and for this reason they were treated only perturbatively [170]. Later, a similar method was used by Calucci [174], who confined, however, only with the case of adiabatically slow motion of the wall, when no photons could be created (recently, this adiabatic case was studied in detail in [173]). An essential progress in the development of the Hamiltonian approach (we shall call it also the instantaneous basis method or IBM-method) was achieved after the papers by Law [176, 177], who has demonstrated that the effective Hamiltonian can be significantly simplified under the resonance conditions. Nonetheless, even the reduced coupled equations of motion resulting from the simplified Hamiltonians have been treated for some time either perturbatively (i.e., in the form of the Taylor expansion with respect to the time variable) or numerically (actually, also in the short-time limit corresponding to the initial stage of the process): see the studies [176, 181] (for the 1D cavity model) and [182] (for a three-dimensional cavity and a waveguide). The general structure of the effective infinite-dimensional quadratic Lagrangians and Hamiltonians arising in the canonical approach to the dynamical Casimir effect was analyzed and classified in [183–185]. The methods of diagonalization of such Hamiltonians were considered in [186–188].

The first analytical solutions describing the field inside the 1D cavity with resonantly oscillating boundaries have been found in the simplest cases in [188–190]. More general solutions have been obtained in [191–193]. They hold for any moment of time (provided the amplitude of the wall vibrations is small enough; this limitation, however, is quite unessential under realistic conditions). Moreover, these solutions enable us not only to calculate the number of photons created from an arbitrary initial state (thus giving, for instance, the temperature corrections), but also to account for the effects of detuning from a strict resonance. Besides, they enable to calculate the degree of squeezing in the field quadrature components, to find the photon distribution function and the energy density distribution inside the cavity, etc. Therefore, one of the main purposes of this chapter is to give the detailed description of the analytical solutions found in [188–193] and to discuss their physical consequences.

Another goal is to consider the simplest models of the three-dimensional cavity, following the scheme given in [188–189] (for ideal boundaries) and in [194–195] (for lossy cavities). A more detailed study of the quantum properties of the electromagnetic field in rectangular 3D cavities, which takes into account the polarization of the field, was performed in [196–197], but only for the uniform motion of the walls. The periodic motion was considered in [198], also with account of the polarization and the influence of all three dimensions, but in the framework of some approximations equivalent to the short-time limit. The case of a three-dimensional rectangular cavity divided in two parts by an ideal mirror, which suddenly disappears, was considered in [199].

The name Motion Induced Radiation was given to the effect of radiation emission outside the cavity with vibrating walls [200]. Using the spectral approach, the authors of [200] showed that the radiation can be essentially enhanced under the resonance conditions (by the orders of magnitude, comparing with the case of a single mirror). The problem of the photon generation by a single perfectly reflecting mirror performing a bounded nonrelativistic motion was studied in [201–202], where the effects of polarization have been taken into account, and the spectral and angular distributions of the emitted photons have been found. Arbitrary space-time deformations of a single moving mirror have been treated in [203–205] with the aid of the path-integral approach. The same approach was used in the case of a cavity with deformable perfectly reflecting boundaries in [206].

The creation of photons or specific (e.g., squeezed) states of the electromagnetic field due to the motion of some effective mirrors made of the free electrons moving with (ultra)relativistic velocities was considered in [207–208]. Another kind of “effective moving mirrors” consisting of the electron-hole plasma generated in semiconductors under the action of powerful laser pulses was suggested in [208–209].
The influence of temperature on the dynamical Casimir effect was evaluated in [110, 210]; a more detailed analysis was given in [211] (in the framework of the Hamiltonian approach), [212] (using the thermofield dynamics), and especially in [192, 193]. The energy density distribution inside the cavity under the resonance conditions is no more uniform, on the contrary, the main part of the energy is concentrated in several sharp peaks which move from one boundary to another, becoming more and more narrow with the course of time. This effect was studied in the framework of numerical calculations in [112, 164–166, 168] (earlier, it was discussed in [44] for the classical field). The analytical form of the pulses, including their “fine structure” in the case of initial states different from the vacuum or thermal ones, was found in [193]. A similar pulse structure of radiation emitted from the high finesse vibrating cavity with partially transparent mirrors was studied in [210, 213].

The evolution of classical fields in the cavities filled in with media whose dielectric properties vary in time was considered, e.g., as far back as in [214]. Yablonovitch [215] proposed to use a medium with a rapidly decreasing in time refractive index (“plasma window”) to simulate the so-called Unruh effect [216], i.e., creation of quanta in an accelerated frame of reference. More rigorous and detailed studies of quantum phenomena in nonstationary (deformed) media have been performed in [159, 160, 217–229]. The case when the dielectric constant changes simultaneously with the distance between mirrors (in one dimension) was considered in [230, 231]. A comparison of the spectra of photons created due to the motion of mirrors and due to the time variations of the dielectric permeability was made in [232]. An analog of the nonstationary Casimir effect in the superfluid $^3$He, namely, the friction force on the moving interface between two different phases, was discussed in [233].

The problem of the interaction between the electromagnetic field created due to the NSCE and various detectors (harmonic oscillators, two-level systems, Rydberg atoms, etc.) placed inside the cavity with moving walls was studied by means of different methods in [188, 189, 234–238]. It is also discussed in this chapter.

The first experiments on the interaction between the powerful laser radiation and freely suspended light mirrors have been reported in [239, 240], where the effect of the optical bistability, similar to that observed usually in the so-called Kerr media, was observed. The first theoretical study of this phenomenon in the cavities whose walls can move under the action of the radiation pressure force was given in [241]. This subject received much attention for the last decade in connection with different problems, such as, e.g., the attenuation or elimination of noise in interferometers [242–253]. It is important, in particular, for the gravitational wave detectors and for the general problem of measuring weak forces acting on a quantum system: see, e.g., [254, 255] for more details. Another possible application could be the generation of the so-called “nonclassical states” of the field and the mirror itself (when it is considered as a quantum object, too) [253, 256–258]. The stability of such states, in turn, is closely related to the general problem of decoherence in quantum mechanics, and one of the mechanisms which can destroy the coherence of pure quantum superpositions is just the dynamical Casimir effect [259, 260]. The back reaction of the dynamical Casimir effect was recently studied in [261]. The influence of fluctuations of the positions of the walls on the field inside the cavity was considered in [113, 203, 202], whereas the Brownian motion of the walls due to the field fluctuations was studied in [140, 203]. The role of the dynamical Casimir effect in the cosmological problems was studied recently in [264, 267].

### 3 1D cavity with oscillating boundaries

Let us start with the case of a single space dimension. Consider a cavity formed by two infinite ideal plates moving in accordance with the prescribed laws

\[
\begin{align*}
x_{\text{left}}(t) &= u(t), & x_{\text{right}}(t) &= u(t) + L(t)
\end{align*}
\]

where $L(t) > 0$ is the time dependent length of the cavity. Taking into account only the electromagnetic modes whose vector potential is directed along $z$-axis (“scalar electrodynamics” [74]), one can write down the field operator in the Heisenberg representation $\hat{A}(x, t)$ at $t \leq 0$ (when both the plates were at rest at the positions $x_{\text{left}} = 0$ and $x_{\text{right}} = L_0$) as (we assume $c = \hbar = 1$)

\[
\hat{A}_n = 2 \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin \frac{n\pi x}{L_0} \hat{b}_n \exp (-i\omega_n t) + \text{h.c.}
\] (12)
where \( \hat{b}_n \) means the usual annihilation photon operator and \( \omega_n = \pi n / L_0 \). The choice of coefficients in equation (12) corresponds to the standard form of the field Hamiltonian

\[
\hat{H} = \frac{1}{8\pi} \int_0^{L_0} dx \left[ \left( \frac{\partial A}{\partial t} \right)^2 + \left( \frac{\partial A}{\partial x} \right)^2 \right] = \sum_{n=1}^{\infty} \omega_n \left( \hat{b}_n^\dagger \hat{b}_n + \frac{1}{2} \right).
\]  

(13)

For \( t > 0 \) the field operator can be written as

\[
\hat{A}(x, t) = 2 \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[ \hat{b}_n \psi_n(x, t) + \text{h.c.} \right].
\]  

(14)

To find the explicit form of functions \( \psi_n(x, t) \), \( n = 1, 2, \ldots \), one should take into account that the field operator must satisfy

i) the wave equation (1),

ii) the boundary conditions (2) or their generalization

\[
A(u(t), t) = A(u(t) + L(t), t) = 0,
\]  

(15)

iii) the initial condition (12), which is equivalent to

\[
\psi_n(x, t < 0) = \sin \frac{n \pi x}{L_0} \exp (-i \omega_n t).
\]  

(16)

Following the approach of Refs. [37, 170, 174, 176, 191] we expand the function \( \psi_n(x, t) \) in a series with respect to the \textit{instantaneous basis}:

\[
\psi_n(x, t > 0) = \sum_{k=1}^{\infty} Q_k^{(n)}(t) \sqrt{\frac{L_0}{L(t)}} \sin \left( \frac{\pi k [x - u(t)]}{L(t)} \right), \quad n = 1, 2, \ldots
\]  

(17)

with the initial conditions

\[
Q_k^{(n)}(0) = \delta_{kn}, \quad \dot{Q}_k^{(n)}(0) = -i \omega_n \delta_{kn}, \quad k, n = 1, 2, \ldots
\]  

This way we satisfy automatically both the boundary conditions (17) and the initial condition (16). Putting expression (17) into the wave equation (1), one can arrive after some algebra at an infinite set of coupled differential equations [180, 184, 191]

\[
\ddot{Q}_k^{(n)} + \omega_k^2(t) Q_k^{(n)} = 2 \sum_{j=1}^{\infty} g_{kj}(t) \dot{Q}_j^{(n)} + \sum_{j=1}^{\infty} \dot{g}_{kj}(t) Q_j^{(n)} + \mathcal{O}(g_{kj}^2),
\]  

(18)

where

\[
\omega_k(t) = k \pi / L(t)
\]

and the time dependent antisymmetric coefficients \( g_{kj}(t) \) read (for \( j \neq k \))

\[
g_{kj} = -g_{jk} = (1)^{k-j} \frac{2k j \left( \hat{L} + \dot{u} \epsilon_{kj} \right)}{(j^2 - k^2) L(t)}, \quad \epsilon_{kj} = 1 - (-1)^{k-j}.
\]  

(19)

For \( u = 0 \) (the left wall at rest) the equations like (18)-(19) were derived in [174, 177].

If the wall comes back to its initial position \( L_0 \) after some interval of time \( T \), then the right-hand side of equation (18) disappears, so at \( t > T \) one can write

\[
Q_k^{(n)}(t) = \xi_k^{(n)} e^{-i \omega_k(t+\delta T)} + \eta_k^{(n)} e^{i \omega_k(t+\delta T)}, \quad k, n = 1, 2, \ldots
\]  

(20)
where $c_k^{(n)}$ and $i_k^{(n)}$ are some constant complex coefficients. Consequently, at $t > T$ the initial annihilation operators $\hat{b}_n$ cease to be “physical”, due to the contribution of the terms with “incorrect signs” in the exponentials $\exp(i\omega_k t)$. Introducing a new set of “physical” operators $\hat{a}_m$ and $\hat{a}_m^\dagger$, which give the decomposition of the vector potential operator at $t > T$ in the form analogous to (22),

$$A(x,t) = \sum_{n=1}^{\infty} \frac{2}{n} \sin \left( \pi nx/L_0 \right) \left[ \hat{a}_n e^{-i\omega_n(t+\delta T)} + \text{h.c.} \right]$$

(21)

one can easily check that the two sets of operators are related by means of the Bogoliubov transformation

$$\hat{a}_m = \sum_{n=1}^{\infty} \left( \hat{b}_n \alpha_{nm} + \hat{b}_n^\dagger \beta_{nm}^* \right), \quad m = 1, 2, \ldots$$

(22)

with the coefficients

$$\alpha_{nm} = \sqrt{\frac{m}{n}} c_m^{(n)}, \quad \beta_{nm} = \sqrt{\frac{m}{n}} i_k^{(n)}.$$  

(23)

The unitarity of the transformation (22) implies the following constraints:

$$\sum_{m=1}^{\infty} (\alpha_{nm} \alpha_{km} - \beta_{nm}^* \beta_{km}) = \sum_{m=1}^{\infty} \frac{m}{n} \left( c_m^{(n)*} c_k^{(n)} - i_k^{(n)*} i_k^{(n)} \right) = \delta_{nk}$$

(24)

$$\sum_{n=1}^{\infty} (\alpha_{nm}^* \alpha_{nj} - \beta_{nm} \beta_{nj}) = \sum_{n=1}^{\infty} \frac{m}{n} \left( c_m^{(n)*} c_j^{(n)} - i_k^{(n)*} i_j^{(n)} \right) = \delta_{mj}$$

(25)

$$\sum_{n=1}^{\infty} (\beta_{nm}^* \alpha_{nk} - \beta_{nk} \alpha_{nm}) = \sum_{n=1}^{\infty} \frac{1}{n} \left( i_k^{(n)*} c_k^{(n)} - i_k^{(n)} c_k^{(n)*} \right) = 0$$

(26)

The mean number of photons in the $m$th mode equals the average value of the operator $\hat{a}_m^\dagger \hat{a}_m$ in the initial state $|\psi_n\rangle$ (remember that we use the Heisenberg picture), since just this operator has a physical meaning at $t > T$:

$$\mathcal{N}_m = \langle \psi_n | \hat{a}_m^\dagger \hat{a}_m | \psi_n \rangle$$

$$= \sum_n |\beta_{nm}|^2 + \sum_{n,k} \left[ (\alpha_{nm}^* \alpha_{km} + \beta_{nm}^* \beta_{km}) \langle \hat{b}_n \hat{b}_k \rangle + 2 \mathrm{Re} \left( \beta_{nm} \alpha_{km} \langle \hat{b}_n \hat{b}_k \rangle \right) \right]$$

$$= \sum_{n=1}^{\infty} \frac{m}{n} \eta_m^{(n)*} \eta_m^{(n)} + \sum_{n,k=1}^{\infty} \frac{m}{nk} \left( \xi_m^{(n)*} \xi_m^{(k)} + \eta_m^{(n)*} \eta_m^{(k)} \right) \langle \hat{b}_n \hat{b}_k \rangle$$

$$+ 2 \mathrm{Re} \sum_{n,k=1}^{\infty} \frac{m}{nk} \eta_m^{(n)*} \xi_m^{(k)} \langle \hat{b}_n \hat{b}_k \rangle.$$  

(27)

The first sum in the right-hand sides of the relations above describes the effect of the photon creation from vacuum due to the NSCE, while the other sums are different from zero only in the case of a nonvacuum initial state of the field.

To find the coefficients $c_k^{(n)}$ and $i_k^{(n)}$ one has to solve an infinite set of coupled equations (28) ($k = 1, 2, \ldots$) with time-dependent coefficients, moreover, each equation also contains an infinite number of terms. However, the problem can be essentially simplified, if the walls perform small oscillations at the frequency $\omega_\omega$ close to some unperturbed field eigenfrequency:

$$L(t) = L_0 \left( 1 - \varepsilon_L \sin [\omega_1 (1 + \delta)t] \right), \quad u(t) = \varepsilon_u L_0 \sin \left[ \omega_1 (1 + \delta)t + \varphi \right].$$

Assuming $|\varepsilon_L|, |\varepsilon_u| \sim \varepsilon \ll 1$, it is natural to look for the solutions of equation (28) in the form similar to (29),

$$Q_k^{(n)}(t) = \xi_k^{(n)} e^{-i\omega_k (1+\delta)t} + \eta_k^{(n)} e^{i\omega_k (1+\delta)t}.$$  

(28)
but now we allow the coefficients $\xi_k^{(n)}$ and $\eta_k^{(n)}$ to be slowly varying functions of time. The further procedure is well known in the theory of parametrically excited systems [266, 268]. First we put expression (29) into equation (23) and neglect the terms $\xi, \eta$ (having in mind that $\xi, \eta \sim \varepsilon$, while $\xi, \eta \sim \varepsilon^2$), as well as the terms proportional to $L^2 \sim \dot{u}^2 \sim \varepsilon^2$. Multiplying the resulting equation for $Q_k$ by the factors $\exp \{i\omega_k (1 + \delta) t\}$ and $\exp \{-i\omega_k (1 + \delta) t\}$ and performing averaging over fast oscillations with the frequencies proportional to $\omega_k$ (since the functions $\xi, \eta$ practically do not change their values at the time scale of $2\pi/\omega_k$) one can verify that only the terms with the difference $j - k = \pm p$ survive in the right-hand side. Consequently, for even values of $p$ the term $\dot{u}$ in $g_{kj}(t)$ does not make any contribution to the simplified equations of motion, thus only the rate of change of the cavity length $\dot{L}/L_0$ is important in this case. On the contrary, if $p$ is an odd number, then the field evolution depends on the velocity of the centre of the cavity $v_c = \dot{u} + \dot{L}/2$ and does not depend on $\dot{L}$ alone. These interference effects were discussed in the short time limit $\varepsilon \omega_1 t \ll 1$ (see also [200]). We assume hereafter that $u = 0$ (i.e. that the left wall is at rest), since this assumption does not change anything if $p$ is an even number, whereas one should simply replace $\dot{L}/L_0$ by $2v_c/L_0$ if $p$ is an odd number.

The final equations for the coefficients $\xi_k^{(n)}$ and $\eta_k^{(n)}$ contain only three terms with simple time independent coefficients in the right-hand sides:

$$\frac{d}{d\tau} \xi_k^{(n)} = (-1)^p \left[ (k + p)\xi_{k+p}^{(n)} - (k - p)\xi_{k-p}^{(n)} \right] + 2i\gamma k\xi_k^{(n)},$$

$$\frac{d}{d\tau} \eta_k^{(n)} = (-1)^p \left[ (k + p)\eta_{k+p}^{(n)} - (k - p)\eta_{k-p}^{(n)} \right] - 2i\gamma k\eta_k^{(n)}.$$

The dimensionless parameters $\tau$ (a “slow” time) and $\gamma$ read ($\varepsilon \equiv \varepsilon L$)

$$\tau = \frac{1}{2}\varepsilon \omega_1 t, \quad \gamma = \delta/\varepsilon.$$

The initial conditions are

$$\xi_k^{(n)}(0) = \delta_{kn}, \quad \eta_k^{(n)}(0) = 0.$$

Note, however, that uncoupled equations (29, 30) hold only for $k \geq p$. This means that they describe the evolution of all the Bogoliubov coefficients only if $p = 1$. Then all the functions $\eta_k^{(n)}(t)$ are identically equal to zero due to the initial conditions (32), consequently, no photon can be created from vacuum. Moreover, in the next section we show that the total number of photons (but not the total energy) is an integral of motion in this specific case.

4 “Semi-resonance” case ($p = 1$)

If $p = 1$, one has to solve the set of equations ($k, n = 1, 2, \ldots$)

$$\frac{d}{d\tau} \xi_k^{(n)} = (k - 1)\xi_{k-1}^{(n)} - (k + 1)\xi_{k+1}^{(n)} + 2i\gamma k\xi_k^{(n)}.$$

An immediate consequence of these equations and the condition $\xi_k^{(n)}(0) = \delta_{kn}$ is the identity

$$\sum_m m\xi_m^{(n)}(\tau)\xi_m^{(k)*}(\tau) \equiv n\delta_{nk},$$

which is nothing but the unitarity condition of the Bogoliubov transformation in this special case. Taking into account this identity, one can easily verify that the total average number of photons in all modes is conserved in time:

$$N = \sum_{nk} \sum_m n\xi_m^{(n)}(\tau)\xi_m^{(k)*}(\tau) = \sum_n \langle \sum_m \langle \hat{b}_n^\dagger \hat{b}_n \rangle | n \rangle.$$

A similar phenomenon in the classical case was discussed in [44], whereas the quantum case was considered in [176] and especially in [190]. Also, using Eqs. (33) and (34) one can verify that the total energy (normalized by $\omega_1$)

$$E = \sum_{nk} \sum_m n^2 \xi_m^{(n)}\xi_m^{(k)*} \langle \hat{b}_n^\dagger \hat{b}_n \rangle | n \rangle.$$
satisfies the simple equation (hereafter overdots mean the differentiation with respect to the dimensionless time $\tau$)

$$\dot{E} = 4a^2 E + 4\gamma^2 E(0) - 2\gamma \text{Im}(G_1), \quad (36)$$

where

$$a = \sqrt{1 - \gamma^2}, \quad (37)$$
$$G_1 = 2 \sum_{n=1}^{\infty} \sqrt{n(n+1)}(\hat{b}_n^\dagger \hat{b}_{n+1}). \quad (38)$$

The quantum averaging is performed over the initial state of the field (no matter pure or mixed). The initial values of the total energy and its first derivative (with respect to $\tau$) are given by

$$\mathcal{E}(0) = \sum_{n=1}^{\infty} n(\hat{b}_n^\dagger \hat{b}_n), \quad \dot{\mathcal{E}}(0) = \text{Re}(G_1). \quad (39)$$

Consequently, the solution to equation (36) can be expressed as

$$\mathcal{E}(\tau) = \mathcal{E}(0) + \frac{2 \sinh^2(2\tau)}{a^2} \left[ \frac{\mathcal{E}(0) - \gamma/2 \text{Im}(G_1)}{2a} \right] + \text{Re}(G_1) \frac{\sinh(2\tau)}{2a}. \quad (40)$$

One can easily prove that $|\dot{\mathcal{E}}(0)| \leq \mathcal{E}(0)$. Thus the total energy grows exponentially when $\tau \gg 1$ (provided $\gamma < 1$), although it can decrease at $\tau \ll 1$, if $\mathcal{E}(0) < 0$. Since the total number of photons is constant, such a behaviour is explained by the effect of pumping the highest modes at the expense of the lowest ones (in the classical case this effect was noticed in \[39\]).

We see that the total energy can be found without any knowledge of the Bogoliubov coefficients. However, these coefficients are necessary, if one wants to know the distribution of the energy or the mean photon numbers over the modes. To solve the infinite set of equations (33) we introduce the generating function

$$X^{(n)}(z, \tau) = \sum_{k=1}^{\infty} \xi_k^{(n)}(\tau) z^k \quad (41)$$

where $z$ is an auxiliary variable. Using the relation $kz^k = z(dz^k/dz)$ one obtains the first-order partial differential equation

$$\frac{\partial X^{(n)}}{\partial \tau} = (z^2 - 1 + 2i\gamma z) \frac{\partial X^{(n)}}{\partial z} + \xi_1^{(n)}(\tau) \quad (42)$$

whose solution satisfying the initial condition $X^{(n)}(0, z) = z^n$ reads

$$X^{(n)}(z, \tau) = \left[ \frac{z g(\tau) - S(\tau)}{g'(\tau) - zS(\tau)} \right]^n + \int_{0}^{\tau} \xi_1^{(n)}(x) \, dx \quad (43)$$

where

$$S(\tau) = \sinh(\alpha \tau)/a, \quad g(\tau) = \cosh(\alpha \tau) + i\gamma S(\tau). \quad (44)$$

Differentiating (43) over $z$ we find

$$\xi_1^{(n)}(\tau) = \frac{n[-S(\tau)]^{n-1}}{|g'(\tau)|^{n+1}} \quad (45)$$

Putting this expression into the integral in the right-hand side of equation (43) we arrive at the final form of the generating function

$$X^{(n)}(z, \tau) = \left[ \frac{z g(\tau) - S(\tau)}{g'(\tau) - zS(\tau)} \right]^n - \left[ \frac{-S(\tau)}{g'(\tau)} \right]^n \quad (46)$$

which satisfies automatically the necessary boundary condition $X^{(n)}(\tau, 0) = 0$. The right-hand side of (46) can be expanded into the power series of $z$ with the aid of the formula (269, vol. 3, section 19.6, equation (16))

$$(1 - t)^{b-c}(1 - t + xt)^{-b} = \sum_{m=0}^{\infty} \frac{t^m}{m!} c_m F(-m, b; c; x),$$
where \( F(a, b; c; x) \) is the Gauss hypergeometric function, and \((c)_k = \Gamma(c + k)/\Gamma(c)\). In turn, the function \((c)_m F(-m, b; c; x)\) with an integer \(m\) is reduced to the Jacobi polynomial in accordance with the formula (268, vol. 2, section 10.8, equation (16))

\[
(c)_m F(-m, b; c; x) = m!(-1)^m P^{(b-m, c-1)}_m(2x - 1).
\]

Consequently,

\[
(1 - t)^{b-c}(1 - t + xt)^{-b} = \sum_{m=0}^{\infty} (-t)^m P^{(b-m, c-1)}_m(2x - 1)
\]

and the coefficient \( \xi^{(n)}_m(\tau) \) reads

\[
\xi^{(n)}_m(\tau) = (-\kappa)^{n-m} \lambda^{n+m} P^{(n-m,-1)}_m (1 - 2\kappa^2)
\]

where

\[
\kappa(\tau) = \frac{S}{\sqrt{g^*}} \equiv \frac{S(\tau)}{\sqrt{1 + S^2(\tau)}}
\]

\[
\lambda(\tau) = \sqrt{g(\tau)/g^*(\tau)} \equiv \sqrt{1 - \gamma^2 \kappa^2 + i\gamma \kappa}, \quad |\lambda| = 1.
\]

The form (48) is useful for \( n \geq m \). To find a convenient formula in the case of \( n \leq m \) we introduce the two-dimensional generating function

\[
X(\tau, z, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} z^m y^n \xi^{(n)}_m(\tau) = \sum_{n=1}^{\infty} X^{(n)}(z, \tau) y^n
\]

\[
= \frac{g^*(\tau) + yS(\tau)[g^*(\tau) - g(\tau)y + S(\tau)(y - z)]}{g^*(\tau) + S(\tau)(y - z)}.
\]

The coefficient at \( z^m \) in (51) yields another one-dimensional generating function

\[
X_m(\tau, y) = \sum_{n=1}^{\infty} y^n \xi^{(n)}_m(\tau) = y\frac{[g(\tau)y + S(\tau)]^{m-1}}{[g^*(\tau) + S(\tau)]^{m+1}}.
\]

Then equation (47) results in the expression

\[
\xi^{(n)}_m = (1 - \kappa^2)\kappa^{m-n}\lambda^{n+m} P^{(m-n,1)}_{m-1} (1 - 2\kappa^2).
\]

Note that the functions \( S(\tau), \cosh(\alpha \tau) \) and \( \kappa(\tau) \) are real for any value of \( \gamma \). For \( \gamma > 1 \) it is convenient to use instead of (44) the equivalent expressions in terms of the trigonometrical functions:

\[
\tilde{S}(\tau) = \sin(\hat{a} \tau) / \hat{a}, \quad \tilde{g}(\tau) = \cos(\hat{a} \tau) + i\gamma \tilde{S}(\tau), \quad \hat{a} = \sqrt{\gamma^2 - 1}.
\]

In the special case \( \gamma = 1 \) one has \( S(\tau) = \tau \) and \( g(\tau) = 1 + i\tau \). In particular,

\[
\xi^{(n)}_m(\tau; \gamma = 1) = \frac{\tau^{m-n} (1 + i\tau)^{m-1}}{(1 - i\tau)^{m+1}} P^{(m-n,1)}_{m-1} \left( \frac{1 - \tau^2}{1 + \tau^2} \right).
\]

The knowledge of the two-dimensional generating function enables to verify the unitarity condition (24). Consider the product \( X^*(\tau, z_1, y_1) X(\tau, z_2, y_2) \), which is a four–variable generating function for the products \( \xi^{(n)*}_m \xi^{(k)}_l \). Taking \( y_1 = \sqrt{u} \exp(i\varphi) \), \( y_2 = \sqrt{u} \exp(-i\varphi) \) and integrating over \( \varphi \) from 0 to \( 2\pi \) one obtains a three–variable generating function \( \sum \zeta^{(n)*}_m \zeta^{(k)}_l \eta^{(n)}_m \). Dividing it by \( u \) and integrating the ratio over \( u \) from 0 to 1 one arrives finally at the relation

\[
\sum_{n,m,l=1}^{\infty} \zeta^{(n)*}_m \zeta^{(k)}_l \eta^{(n)}_m = - \ln(1 - z_1^* z_2) = \sum_{k=1}^{\infty} \frac{1}{k} (z_1^* z_2)^k,
\]

which is equivalent to the special case of (24) for \( \eta^{(k)}_m \equiv 0 \):

\[
\sum_{n} \frac{1}{n} \xi^{(n)*}_m(\tau) \xi^{(n)}_j(\tau) \equiv \frac{1}{m} \delta_{mj}.
\]
4.1 Examples

Suppose that initially there was a single excited mode labeled with an index $n$. Due to the linearity of the process one may assume that the mean number of photons in this mode was $\nu_n = 1$. Then the mean occupation number of the $m$-th mode at $\tau > 0$ equals

$$\mathcal{N}^{(n)}_m = \frac{m}{n} \left[ \xi^{(n)}_m \right]^2 = \frac{m}{n} \left[ (1 - \kappa^2) \kappa^{m-n} P_{n-1}^{(m-n-1)} \left( 1 - 2\kappa^2 \right) \right]^2,$$

where $\kappa$ is given by (53). For example, in the special case $\gamma = 0$ we have

$$\mathcal{N}^{(1)}_n = \frac{m(\tanh \tau)^{2m-2}}{(\cosh \tau)^4},$$
$$\mathcal{N}^{(2)}_n = \frac{m(\tanh \tau)^{2m-4}}{2(\cosh \tau)^4} \left( (m-1) - (m+1) \tanh^2 \tau \right)^2.$$

The maximum of function $\mathcal{N}^{(1)}_m(\tau)$ is achieved at $\sinh \tau_{\text{max}} = \sqrt{(m-1)/2}$. For $m \geq 1$ it equals $\mathcal{N}^{(1)}_m(\tau_{\text{max}}) \approx 4/(me^4)$. For a fixed value of $\tau \gg 1$, the occupation number distribution $\mathcal{N}^{(1)}_m$ reaches its maximum at $n_{max} = \cosh^2 \tau$, and $\mathcal{N}^{(1)}_{n_{max}} = (e\cosh^2 \tau)^{-1} \ll 1$. The maximum of the energy distribution is shifted to the right, $m_{max} = 2\cosh^2 \tau$, and its value is not decreased with time: $\xi^{(1)}_{\text{max}} = 4/e^2$. This explains the exponential growth of the total energy.

Although formula (58) seems asymmetrical with respect to the indices $m$ and $n$, actually the relation

$$\mathcal{N}^{(n)}_m = \mathcal{N}^{(m)}_n$$

holds. To prove it we calculate the generating function

$$Q(u, v) = \sum_{m,n=1}^{\infty} u^m v^n \mathcal{N}^{(n)}_m.$$

It is related to the function $X(z, y)$ (51) as follows

$$Q(u, v) = v \frac{d}{dv} \int_0^u d\tau \int_0^{2\pi} \frac{d\varphi d\psi}{(2\pi)^2} X \left( \sqrt{v}e^{i\varphi}, \sqrt{v}e^{i\psi} \right) X^* \left( \sqrt{u}e^{i\varphi}, \sqrt{u}e^{i\psi} \right).$$

Having performed all the calculations we arrive at the expression

$$2Q(u, v) = \frac{1 + uv - \kappa^2(u + v)}{\left[ 1 + uv - \kappa^2(u + v) \right]^2 - 4uv(1 - \kappa^2)^2} \left[ 1 + uv - \kappa^2(u + v) \right]^{1/2} - 1.$$

Then (59) is a consequence of the relation $Q(u, v) = Q(v, u)$.

The initial stage of the evolution of $\mathcal{N}^{(n)}_m(\tau)$ does not depend on the detuning parameter $\gamma$, since the principal term of the expansion of (58) with respect to $\tau$ yields

$$\mathcal{N}^{(n)}_{m \pm q}(\tau \to 0) = \frac{n + q}{n} \left[ n(n+1) \ldots (n+q+1) \right]^{\gamma^2} \tau^{2q}.$$

However, the further evolution is sensitive to the value of $\gamma$. If $\gamma \leq 1$, then the function $\mathcal{N}^{(n)}_m(\tau)$ has many maxima and minima (especially for large values of $m$ and $n$), but finally it decreases asymptotically as $mna^4 / \cosh^4(a\tau)$. On the contrary, if $\gamma > 1$, then the function $\mathcal{N}^{(n)}_m(\tau)$ is periodic with the period $\pi/a$, and it turns into zero for $\tau = k\pi/a$, $k = 1, 2, \ldots$ (excepting the case $m = n$). The magnitude of the coefficient $\mathcal{N}^{(n)}_m(\tau)$ decreases approximately as $\gamma^{-2|m-n|}$ for $\gamma \gg 1$.

Now let us assume for simplicity that $\gamma = 0$. Then Eq. (53) can be represented in the equivalent form

$$\xi^{(n)}_m = n \frac{(\tanh \tau)^{m-n}}{\cosh^2 \tau} \left( -1 \right)^{n-1} F \left( 1 - n, m + 1; 2; \frac{1}{\cosh^2 \tau} \right).$$
If \( m \gg n \to O(1) \), then \( (m)_k \approx m^k \) \((k \leq n)\), so the Gauss hypergeometric function in Eq. (62) can be replaced by the confluent hypergeometric function with a negative integral first index, which is reduced to the associated Laguerre polynomial \([269] L_n^{(1)}(\mu)\) of the scaled variable \( \mu = m/\cosh^2 \tau \). Using the approximation \((\tanh \tau)^{2m} \approx \exp \left(-m/\cosh^2 \tau\right)\) valid for \( \tau \gg 1 \), we arrive at a simplified expression

\[
\mathcal{E}^{(n)}_m = nN^{(n)}_m = \frac{1}{n} \mu^2 e^{-\mu} \left[ L_n^{(1)}(\mu) \right]^2,
\]

describing the energy distribution over the modes with large number \( m \).

The fluctuations of the occupation numbers can be calculated with the aid of the formula (again for \( \gamma = 0 \))

\[
\langle \hat{N}^2 \rangle_m(\tau) - \langle \hat{N} \rangle_m(\tau) = \frac{m^2}{n} \left[ \xi_m^{(n)}(\tau) \right]^4 \left[ \langle \hat{N}^2 \rangle_n(0) - \langle \hat{N} \rangle_n(0) \right],
\]

which is an immediate consequence of Eq. (57). Consequently, the type of the photon statistics (sub– or super–Poissonian) is conserved. In particular, if the initial mode was in a coherent state, then all other modes will be excited in the coherent states, too.

The assumption on the equidistant eigenmode spectrum can be justified to certain extent for the longitudinal modes of a Fabry-Perot resonator with perfect mirrors, if the order of interference (the mode number) is high enough. So let us suppose that initially some single mode with the period \( \pi/\tau \) was excited. Although the lowest modes are not equidistant in this case, for a limited period of time in the beginning of the process (until \( \tau \ll \log(2n) \)), the evolution of the chosen mode and its neighbours can be described by the solutions obtained above, since the influence of the “remote” modes becomes essential at sufficiently large times. Then the populations of the modes exhibit strong oscillations, since they are proportional to the squares of the Jacobi polynomials of large degrees. The asymptotics of these polynomials (\([269]\), Eq. (10.14.10)) yields

\[
N^{(n)}_m \approx \frac{2}{n \pi \sin(2\varphi)} \cos^2 \left( (m+n)\varphi - 2|m-n| + \frac{\pi}{4} \right), \quad \sin \varphi = \tanh \tau.
\]

This formula holds provided that \( n \gg |m-n| \sim O(1) \) and \( n \sin(2\varphi) \gg 1 \). For \( \tau \ll 1 \) we observe fast oscillations, which amplitude is modulated with the period \( \pi/\tau \), whereas for \( \tau \gg 1 \) we have more slow periodic variations of \( N^{(n)}_m \) as the function of \( m \) with the period \( \pi e^\tau/2 \).

An interesting problem is the field evolution in a cavity which was initially in the equilibrium state at a finite temperature, when the initial occupation numbers were given by the Planck distribution \( \nu_n = \{ \exp(\beta n) - 1 \}^{-1} \). Let us consider two limit cases. The first one corresponds to the low–temperature approximation \( \nu_n = \exp(-\beta n) \). Then the occupation number of the \( m \)-th mode is nothing but the coefficient at \( v^m \) in the expansion \( \{ \exp(\beta n) \}^{-1} \) with \( u = \exp(-\beta) \). Using the well known generating function of the Legendre polynomials \( P_m(z) \) (\([269]\), Eq. (10.10.39)), one can obtain the following expression (for \( \gamma = 0 \)):

\[
N^{(\beta)}_m = \frac{1}{2} \left( \frac{e^{-\beta} - \rho^2}{1 - e^{-\beta} \rho^2} \right)^m \left[ P_m(x) + P_{m-1}(x) \right], \quad \rho \equiv \tanh(\tau),
\]

\[
x = \frac{e^{-\beta} (1 - \rho^2)^2 + \rho^2 (1 - e^{-\beta})^2}{(e^{-\beta} - \rho^2)(1 - e^{-\beta} \rho^2)}.
\]

In particular, \( N^{(\beta)}_1 = e^{-\beta}(\cosh \tau)^{-4} (1 - e^{-\beta} \tanh^2 \tau)^{-2} \).

In the special case of a cavity filled in with a high-temperature thermal radiation, the initial distribution over modes reads \( \nu_n(\Theta) = \Theta/n \), constant \( \Theta \) being proportional to the temperature. Then \( N^{(\Theta)}_m = \sum_n \nu_n(\Theta) N^{(n)}_m \). This sum is nothing but \( \Theta \) multiplied by the coefficient at \( v^m \) in the Taylor expansion of the function

\[
\hat{Q}(v) = \int_0^1 \frac{du}{u} Q(u, v) = \ln \frac{1 - v \kappa^2(\tau)}{1 - v}.
\]

Thus we have

\[
\mathcal{E}^{(\Theta)}_m = nN^{(\Theta)}_m = \Theta (1 - [\kappa(\tau)]^{2m})
\]

We see that the resonance vibrations of the wall cause an effective cooling of the lowest electromagnetic modes (provided \( |\gamma| < 1 \)). The total number of quanta and the total energy in this example are formally infinite, due to the equipartition law of the classical statistical mechanics. In reality both these quantities are finite, since \( \nu_n(\Theta) < \Theta/n \) at \( n \to \infty \) due to the quantum corrections.
5  Generic resonance case $p \geq 2$

If $p \geq 2$, we have $p - 1$ pair of coupled equations for the coefficients with lower indices $1 \leq k \leq p - 1$

$$\frac{d}{dt} \xi^{(n)}_{k} = (-1)^{p} \left[ (k + p) \xi^{(n)}_{k+p} - (p - k) \eta^{(n)}_{p-k} \right] + 2i\gamma k \xi^{(n)}_{k}, \quad (64)$$

$$\frac{d}{dt} \eta^{(n)}_{k} = (-1)^{p} \left[ (k + p) \eta^{(n)}_{k+p} - (p - k) \xi^{(n)}_{p-k} \right] - 2i\gamma k \eta^{(n)}_{k}. \quad (65)$$

In this case some functions $\eta^{(n)}_{k}(t)$ are not equal to zero at $t > 0$, thus we have the effect of photon creation from the vacuum.

It is convenient to introduce a new set of coefficients $\rho^{(n)}_{k}$, whose lower indices run over all integers from $-\infty$ to $\infty$:

$$\rho^{(n)}_{k} = \begin{cases} 
\xi^{(n)}_{k}, & k > 0 \\
0, & k = 0 \\
-\eta^{(n)}_{k}, & k < 0 
\end{cases} \quad (66)$$

Then one can verify that equations (64), (65) and (67) can be combined in a single set of equation ($k = \pm 1, \pm 2, \ldots$) [91]

$$\frac{d}{dt} \rho^{(n)}_{k} = (-1)^{p} \left[ (k + p) \rho^{(n)}_{k+p} - (p - k) \rho^{(n)}_{p-k} \right] + 2i\gamma k \rho^{(n)}_{k} \quad (67)$$

with the initial conditions ($n = 1, 2, \ldots$)

$$\rho^{(n)}_{k}(0) = \delta_{kn}. \quad (68)$$

A remarkable feature of the set of equations (67) is that its solutions satisfy exactly the unitarity conditions (24)-(26) (although the coefficients $\xi^{(n)}_{k}$ and $\eta^{(n)}_{k}$ introduced via equation (28) have additional phase factors in comparison with the coefficients defined in equation (24), these phases do not affect the identities concerned), which can be rewritten as

$$\sum_{m=-\infty}^{\infty} m \rho^{(n)*}_{m} \rho^{(k)}_{m} = n \delta_{nk}, \quad n, k = 1, 2, \ldots \quad (69)$$

$$\sum_{n=1}^{\infty} \frac{m}{n} \left[ \rho^{(n)*}_{n} \rho^{(n)}_{j} - \rho^{(n)*}_{-n} \rho^{(n)}_{-j} \right] = \delta_{mj}, \quad m, j = 1, 2, \ldots \quad (70)$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \left[ \rho^{(n)*}_{j} \rho^{(n)}_{m} - \rho^{(n)*}_{m} \rho^{(n)}_{j} \right] = 0, \quad m, j = 1, 2, \ldots \quad (71)$$

For example, calculating the derivative $I = (d/d\tau) \sum_{m=-\infty}^{\infty} m \rho^{(n)*}_{m} \rho^{(k)}_{m}$ with the aid of equation (67) and its complex conjugated counterpart one can easily verify that $I = 0$. Then the value of the right-hand side of (69) is a consequence of the initial conditions (58). The identities (70) and (71) can be verified in a similar way, if one uses instead of (67) the recurrence relations between the coefficients $\rho^{(n)}_{m}$ with the same lower index $m$ but with different upper indices: see equations (71) and (88).

Due to the initial conditions (68) the solutions to (67) satisfy the relation

$$\rho^{(k+np)}_{j+kp} \equiv 0 \quad \text{if} \quad j \neq k \quad (72)$$

$$j, k = 0, 1, \ldots, p - 1, \quad m = 0, \pm 1, \pm 2, \ldots, \quad n = 0, 1, 2, \ldots$$

Consequently, the nonzero coefficients $\rho^{(n)}_{m}$ form $p$ independent subsets

$$\rho^{(n)}_{j+qp} \equiv \rho^{(n)}_{j+kp} \quad (73)$$

$$j = 0, 1, \ldots, p - 1, \quad q = 0, 1, 2, \ldots, \quad k = 0, \pm 1, \pm 2, \ldots$$
The subset \( y_k^{(q,0)} \) is distinguished, because \( y_k^{(q,0)} \equiv 0 \) for \( k \leq 0 \) and the upper index \( q \) begins at \( q = 1 \). One can verify that the functions \( y_m^{(n,0)}(\tau) \) with \( m \geq 1 \) are given by the formulae for \( \zeta_m^{(n)}(\tau) \) found in the preceding section, provided one replaces \( \tau \) by \((-1)^p \tau^r\) and \( \gamma \) by \((-1)^p \gamma \), whereas \( y_m^{(n,0)}(\tau) \equiv 0 \) for \( m \leq 0 \).

In the generic case \( j \neq 0 \) it is reasonable to introduce a generating function in the form of the Laurent series of an auxiliary variable \( z \)

\[
R^{(n,j)}(z, \tau) = \sum_{m=-\infty}^{\infty} y_m^{(n,j)}(\tau) z^m
\]

since the lower index of the coefficient \( y_m^{(n,j)} \) runs over all integers from \(-\infty\) to \( \infty \). One can verify that the function (74) satisfies the homogeneous equation

\[
\frac{\partial R^{(n,j)}}{\partial \tau} = \left[ \sigma \left( \frac{1}{z} - z \right) + 2i\gamma \right] \left( j + pz \frac{\partial}{\partial z} \right) R^{(n,j)}, \quad \sigma = (-1)^p.
\]

The solution to (73) satisfying the initial condition \( R^{(n,j)}(z, 0) = z^n \) reads

\[
R^{(n,j)}(z, \tau) = z^{-j/p} \left[ \frac{zg(p\tau) + \sigma S(p\tau)}{g'(p\tau) + z\sigma S(p\tau)} \right]^{n+j/p}
\]

where the functions \( S(\tau) \) and \( g(\tau) \) were defined in (44). The coefficients of the Laurent series (74) can be calculated with the aid of the Cauchy formula

\[
y_m^{(n,j)}(\tau) = \frac{1}{2\pi i} \oint_C \frac{dz}{z^{m+1}} R^{(n,j)}(z, \tau)
\]

where the closed curve \( C \) rounds the point \( z = 0 \) in the complex plane in the counterclockwise direction. Making a scale transformation one can reduce the integral (77) with the integrand (76) to the integral representation of the Gauss hypergeometric function (269, vol 1, section 2.1.3)

\[
F(a, b; c; x) = \frac{-i\Gamma(c) \exp(-i\pi b)}{2\sin(\pi b) \Gamma(c - b) \Gamma(b)} \int_1^{0+} \frac{t^{b-1}(1 - t)^{c-b-1}}{(1 - tx)^a} dt,
\]

where \( \text{Re}(c - b) > 0, b \neq 1, 2, 3, \ldots, \) and the integration contour begins at the point \( t = 1 \) and passes around the point \( t = 0 \) in the positive direction. After some algebra one can obtain the expression

\[
y_m^{(n,j)} = -\frac{\Gamma(-m - j/p)}{\pi \Gamma(1 + n - m)} \frac{\Gamma(1 + n + j/p) \sin[\pi (m + j/p)]}{\Gamma(1 + n - m)} \times (\sigma \kappa)^{n-m} \lambda^{m+n+2j/p} F(n + j/p, -m - j/p; 1 + n - m; \kappa^2).
\]

We assume hereafter \( \kappa \equiv \kappa(p\tau) \) and \( \lambda \equiv \lambda(p\tau) \), the functions \( \kappa(x) \) and \( \lambda(x) \) being defined as in (49) and (60). Using the known formula

\[
\Gamma(-z) \sin(\pi z) = -\pi / \Gamma(z + 1)
\]

one can eliminate the gamma-function of a negative argument:

\[
y_m^{(n,j)} = \frac{\Gamma(1 + n + j/p)}{\Gamma(1 + m + j/p) \Gamma(1 + n - m)} \times (\sigma \kappa)^{n-m} \lambda^{m+n+2j/p} F(n + j/p, -m - j/p; 1 + n - m; \kappa^2).
\]

The form (81) gives an explicit expression for the coefficient \( \zeta_j^{(j+pm)} \) with \( 0 \leq m \leq n \). Moreover, it clearly shows the fulfilment of the initial condition \( y_m^{(n,j)}(\tau = 0) = \delta_{mn} \). Transforming the hypergeometric function with the aid of the formula (269, 270)

\[
\lim_{c \to -n} \frac{F(a, b; c; x)}{\Gamma(c)} = \frac{(a)_{n+1}(b)_{n+1}x^{n+1}}{(n+1)!} \frac{F(a + n + 1, b + n + 1; n + 2; x)}{\Gamma(c)}
\]

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(n = 0, 1, 2, . . . ) and the identity (81) one obtains an equivalent expression

\[ y_{m}^{(n,j)} = \frac{\Gamma (m+j/p)(-\sigma \kappa)^{m-n}\lambda^{m+n+2j/p}}{\Gamma (n+j/p)\Gamma (1+m-n)} \times F (m+j/p, -n-j/p; 1+m-n; \kappa^2) \]  

(82)

which gives a convenient form of the coefficient \( \xi_{j+p/m}^{(n)} \) for \( m \geq n \). Formula (79) with negative values of the lower index gives an explicit expression for the nonzero coefficients \( \eta_{nk-j}^{(pn+j)} \) \( (k \geq 1, n \geq 0) \):

\[ \eta_{pk-j}^{(pn+j)} = - \frac{\Gamma (k-j/p)\Gamma (1+n+j/p)\sin [\pi (k-j/p)]}{\pi \Gamma (1+n+k)} \times (\sigma \kappa)^{n+k}\lambda^{n+k+2j/p} F (n+j/p, k-j/p; 1+n+k; \kappa^2). \]  

(83)

Note that the expressions (81)-(83) are valid for \( j = 0 \), too. In this case they coincide with the formulæ obtained in the preceding section. Formulas (81)-(83) immediately give the short-time behaviour of the Bogoliubov coefficients at \( \tau \to 0 \) it is sufficient to put \( \kappa \approx \rho \tau, \lambda \approx 1 \) and to replace the hypergeometric functions by 1. In this limit the detuning parameter \( \gamma \) drops out of the expressions (in the leading terms of the Taylor expansions).

At \( \tau \to \infty \) we have the following asymptotics of the functions \( \kappa (\rho \tau) \) and \( \lambda (\rho \tau) \) (if \( \gamma \leq 1 \))

\[ \kappa \approx 1 - \frac{1}{2} S^{-2}(\rho \tau) \to 1, \quad \lambda \to a + i \gamma, \quad \tau \to \infty. \]

Then equation (79) together with the known asymptotics of the hypergeometric function \( F (a, b; a+b+1; 1-x) \) at \( x \ll 1 \)\( ^{269, 270} \)

\[ F (a, b; a+b+1; 1-x) = \frac{\Gamma (a+b+1)}{\Gamma (a+1)\Gamma (b+1)} [1 + ab \ln(x) + O(x)] \]  

(84)

lead to the asymptotic expression for the Bogoliubov coefficients

\[ y_{m}^{(n,j)} (\tau \gg 1) = \frac{\sin [\pi (m+j/p)]}{\pi (m+j/p)} (a + i \gamma)^{m+n+2j/p} \sigma^{n-m} \times \left[ 1 + O \left( \frac{mn}{S^2} \ln S \right) \right] \]  

(85)

For \( \gamma < 1 \) the correction has an order \( mn\gamma \exp(-2ap\tau) \), while for \( \gamma = 1 \) it has an order \( mn \ln(\tau)/\tau^2 \).

One can verify that the generating function (74) satisfies the recurrence relation

\[ \frac{\partial R_{q,j}^{(n)}}{\partial \tau} = (j + q) \left\{ \sigma \left[ R_{q-1,j}^{(n-1)} - R_{q+1,j}^{(n+1)} \right] + 2i \gamma R_{q,j}^{(n)} \right\} \]  

(86)

Its immediate consequence is an analogous relation for the Bogoliubov coefficients with the same lower indices:

\[ \frac{d}{d\tau} \rho_{m}^{(n)} = n \left\{ \sigma \left[ \rho_{m}^{(n-p)} - \rho_{m}^{(n+p)} \right] + 2i \gamma \rho_{m}^{(n)} \right\}. \]  

(87)

Equation (85) is valid for \( n > p \) (when \( q \geq 1 \) and \( j \geq 1 \) in (86)), since the coefficients \( \rho_{m}^{(n)} \) are not defined when \( n < 0 \). However, using the chain of identities

\[ R_{-1,j}^{(0)} (z) = z^{-j/p} \left[ \frac{S + g^* z}{g^* + S z} \right]^{j/p-1} = \frac{1}{z} \left( \frac{1}{z} \right)^{j/p-1} \left[ \frac{S + g^* z}{g + S z} \right]^{1-j/p} \]

one can obtain the first \( p - 1 \) recurrence relations

\[ \frac{d}{d\tau} \rho_{m}^{(n)} = n \left\{ \sigma \left[ \rho_{m}^{(n-p)} - \rho_{m}^{(n+p)} \right] + 2i \gamma \rho_{m}^{(n)} \right\}, \quad n = 1, 2, \ldots, p - 1. \]  

(88)
To treat the special case \( n = p \) (it corresponds to the distinguished subset with \( j = 0 \)) one should take into account that \( R^{(0,0)}(z) = 1 \), which means formally that \( \rho_m^{(0)} = \delta_{m0} \). So the last recurrence relation reads
\[
\frac{d}{d\tau} \rho_m^{(p)} = p \left\{ -\sigma \rho_m^{(2p)} + 2i\epsilon \rho_m^{(p)} \right\}, \quad m \geq 1
\]
(remember that \( \rho_m^{(p)} = 0 \) for \( m < 0 \)). Now one can verify that the unitarity conditions \( (70)-(71) \) are the consequences of the equations \( 87 \) and \( 88 \).

### 6 Photon statistics

To evaluate the mean number of photons and the statistical properties of the quantum field created in the cavity at \( t > T \) we introduce the Hermitian quadrature component operators
\[
\hat{q}_m = (\hat{a}_m + \hat{a}_m^\dagger) / \sqrt{2}, \quad \hat{p}_m = (\hat{a}_m - \hat{a}_m^\dagger) / (i\sqrt{2}).
\]
Their variances are defined as
\[
U_m = \langle \hat{q}_m^2 \rangle - \langle \hat{q}_m \rangle^2, \quad V_m = \langle \hat{p}_m^2 \rangle - \langle \hat{p}_m \rangle^2,
\]
whereas the covariance is given by
\[
Y_m = \frac{1}{2} (\langle \hat{p}_m \hat{q}_m + \hat{q}_m \hat{p}_m \rangle - \langle \hat{p}_m \hat{q}_m \rangle).
\]
The average values must be calculated in the state defined with respect to the initial operators \( \hat{b}_n \) (remember that we use here the Heisenberg picture).

#### 6.1 Initial vacuum state

The vacuum state is defined by means of the relations \( \hat{b}_n |0\rangle = 0 \). In this case, \( U_m + V_m = 2\mathcal{N}_m^{(\text{vac})} + 1 \), where \( \mathcal{N}_m^{(\text{vac})} \) is the mean number of photons created from vacuum in the \( m \)th mode. It is given by the single sums over index \( n \) in Eq. \((27)\). Initially, \( U_m(0) = V_m(0) = 1/2, \ Y_m(0) = 0 \). Using \((22)\) and assuming for simplicity \( \omega_1 = 1 \), we obtain the following expressions for \( t > 0 \),
\[
U_m = \frac{m}{2} \sum_{n=1}^{\infty} \frac{1}{n} \left| \rho_m^{(n)} - \rho_m^{-n} \right|^2, \quad V_m = \frac{m}{2} \sum_{n=1}^{\infty} \frac{1}{n} \left| \rho_m^{(n)} + \rho_m^{-n} \right|^2;
\]
\[
Y_m = \sum_{n=1}^{\infty} \frac{m}{n} \text{Im} \left[ \rho_m^{(n)} \rho_m^{-n} \right]
\]
where the coefficients \( \rho_m^{(n)} \) should be taken at the moment \( T \), thus their argument is \( \tau_T \equiv \frac{1}{2}\omega_1 T \). Strictly speaking, the expressions \((89)-(90)\) have physical meanings at those moments of time \( T \) when the wall returns to its initial position, i.e., for \( T = N\pi/p(1 + \delta) \) with an integer \( N \). Consequently, the argument \( \tau_T \) of the coefficients \( \rho_m^{(n)} \) in \((29)-(30)\) assumes discrete values \( \tau^{(N)} = N\pi/[2p(1 + \delta)] \). One should remember, however, that something interesting in our problem happens for the values \( \tau \sim 1 \) (or larger). Then \( N \sim \varepsilon^{-1} \gg 1 \), and the minimal increment \( \Delta \tau \sim \varepsilon \) is so small that \( \tau_T \) can be considered as a continuous variable (under the realistic conditions, \( \varepsilon \leq 10^{-8} \)). For this reason, we omit hereafter the subscript \( T \), writing simply \( \tau \) instead of \( \tau_T \) or \( \tau^{(N)} \).

Differentiating the right-hand sides of equations \((83)\) and \((84)\) with respect to the ‘slow time’ \( \tau \), one can remove the fraction \( 1/n \) with the aid of the recurrence relations \((87)\) and \((88)\). After that, changing if necessary the summation index \( n \) to \( n \pm p \), one can verify that almost all terms in the right-hand sides are cancelled, and the infinite series are reduced to the finite sums:
\[
\frac{dU_m}{d\tau} = \sigma_m \sum_{n=1}^{p-1} \text{Re} \left( \rho_m^{(p-n)} \mp \rho_m^{(p-n)} \left[ \rho_m^{(n)} \mp \rho_m^{(-n)} \right] \right).
\]
\[ \frac{dY_m}{d\tau} = \sigma m \sum_{n=1}^{p-1} \text{Im} \left( \rho^{(n)*}_m \rho^{(p-n)*}_m + \rho^{(n)}_{-m} \rho^{(p-n)}_{-m} \right) \]

Now one should take into account the structure of the coefficients \( \rho^{(n)}_m \): they are different from zero provided the difference between the upper index \( n \) and the lower one \( m \) is some multiple of the number \( p \). If \( m = j + pk \) with \( j = 1, \ldots, p - 1 \) and \( k = 0, 1, 2, \ldots \), then only the terms with \( n = j \) or \( n = p - j \) survive in the sums above. Depending on whether \( j = p/2 \) or \( j \neq p/2 \), we obtain two different sets of explicit expressions for the derivatives of the (co)variances.

1) If \( m = j + pk \) but \( j \neq p/2 \) (in particular, for all odd values of \( p \)), then

\[ \frac{dU_m}{d\tau} = \frac{dV_m}{d\tau} = 2\sigma m \text{Re} \left( \rho^{(j)*}_m \rho^{(p-j)}_{-m} \right), \quad \frac{dY_m}{d\tau} = 0. \] (91)

In this case \( Y_m \equiv 0 \) and \( U_m = V_m = \mathcal{N}_m^{(\text{vac})} + 1/2 \).

2) A different situation happens in the distinguished modes with the numbers \( \mu = p(k + 1/2), k = 0, 1, 2, \ldots \):

\[ \frac{dU_\mu}{d\tau} = -\mu \text{Re} \left( \left( \rho^{(p/2)}_\mu - \rho^{(p/2)}_{-\mu} \right)^2 \right), \quad \frac{dV_\mu}{d\tau} = \mu \text{Re} \left( \left( \rho^{(p/2)}_\mu + \rho^{(p/2)}_{-\mu} \right)^2 \right), \] (92)

\[ \frac{dY_\mu}{d\tau} = \mu \text{Im} \left( \left( \rho^{(p/2)}_\mu \right)^2 + \left( \rho^{(p/2)}_{-\mu} \right)^2 \right). \] (93)

We shall call such modes as “principal” ones; they exist only if \( p \) is an even number. In the strict resonance case (\( \gamma = 0 \)) all the coefficients \( \rho^{(p/2)}_\mu \) are real, so \( Y_\mu = 0 \) and \( dU_\mu/d\tau \leq 0 \) in the whole interval \( 0 \leq \tau < \infty \), resulting in the inequality \( U_\mu(\tau) < 1/2 \), which tells us that the field occurs in the squeezed quantum state.

### 6.1.1 Squeezing in the “principal” modes

Note that the coefficients \( \rho^{(p/2)}_\mu \) depend on the parameter \( p \) only through the dependence of the the variable \( \kappa \) on the product \( p\tau \): see equations (74) or (81). Thus to study the squeezing properties of the field created due to the NCE, it is sufficient to consider the most important special case of the parametric resonance at the double fundamental frequency \( 2\omega_1 \) (i.e. \( p = 2 \)), since the formulae for \( p > 2 \) can be obtained by a simple rescaling of the ‘slow time’ (for the ‘principal’ modes). In this case, only the odd modes can be excited from the vacuum, and they do exhibit some squeezing.

Using equations (82) and (83) one can immediately find the Taylor expansions of the (co)variances at \( \tau \to 0 \) (assuming \( (-1)^{m+1} = 1 \)):

\[ \frac{U_{2m+1}}{V_{2m+1}} = \frac{1}{2} + \tau^{2m+1} \left[ \frac{(2m - 1)!!}{m!} \right]^2 \left[ 1 \pm \frac{2m + 1}{(m + 1)^2} \tau + \mathcal{O}(\tau^2) \right] \] (94)

\[ Y_{2m+1} = -2\gamma(2m + 1)\tau^{2(m+1)} \left[ \frac{(2m - 1)!!}{m!} \right]^2 + \cdots \] (95)

We see that the \( U \)-variances are always less than \( 1/2 \) at the initial stage, but the degree of their squeezing rapidly decreases with increase of the number \( m \). Note that the dependence on the detuning parameter \( \gamma \) in the short-time limit appears only in terms of the order of \( \tau^{2m+3} \) (and higher).

In the opposite limit \( \tau \to \infty \) (or \( \kappa \to 1 \)), using equations (82), (83) and the asymptotics of the Bogoliubov coefficients \( \tilde{\mathcal{X}}_j \) we obtain constant time derivatives

\[ \frac{dU_{2m+1}}{d\tau} \big|_{\tau \to \infty} = -\frac{16a}{\pi^2(2m + 1)} \sin^2 \left( \frac{m + 1}{2} \right) \phi \] (96)

\[ \frac{dV_{2m+1}}{d\tau} \big|_{\tau \to \infty} = -\frac{16a}{\pi^2(2m + 1)} \cos^2 \left( \frac{m + 1}{2} \right) \phi \] (97)

\[ \frac{dY_{2m+1}}{d\tau} \big|_{\tau \to \infty} = -\frac{8a}{\pi^2(2m + 1)} \sin [(2m + 1) \phi] \] (98)
where \( \phi \equiv \arcsin \gamma \). Consequently, all the (co)variances increase with time linearly, giving the constant photon generation rate in the ‘principal’ (odd) modes

\[
d\mathcal{N}_{2m+1}/d\tau|_{\tau \to \infty} = \frac{8a}{\pi^2(2m+1)}.
\] (99)

Equation (99) results in a simple estimation of the mean photon number in the \( \mu \)th mode at \( \tau > 1 \): \( \mathcal{N}_\mu(\tau) \approx a\tau/\mu \).

Since the covariance \( Y_\mu \) is different from zero if \( \gamma \neq 0 \), the initial vacuum state of the field is transformed to the correlated quantum state \( ^{273, 274} \). One should remember, however, that the values of \( U_\mu, V_\mu \) and \( Y_\mu \) yield the (co)variances of the field quadratures only at the moment \( t = T \) (when the wall stopped to oscillate). At the subsequent moments of time the quadrature variances exhibit fast oscillations with twice the frequency of the mode. For example (omitting the mode index),

\[
\sigma_q(t') = U \cos^2(\omega t') + V \sin^2(\omega t') + \gamma \sin(2\omega t'), \quad t' = t - T.
\]

Therefore the physical meanings have not the values \( U_\mu, V_\mu \) and \( Y_\mu \) themselves, but rather the minimal \( \sigma_{\min} \equiv u_\mu \) and maximal \( \sigma_{\max} \equiv v_\mu \) values of the quadrature variances during the period of fast oscillations \( ^{273, 274} \)

\[
\left(\begin{array}{c} u_\mu \\ v_\mu \end{array}\right) = \frac{1}{2} \left(U_\mu + V_\mu + \sqrt{(U_\mu - V_\mu)^2 + 4Y_\mu^2}\right).
\] (100)

Only in the special case of the strict resonance (\( \gamma = 0 \)) we have \( u_\mu = U_\mu \) and \( v_\mu = V_\mu \). In the generic case \( \gamma \neq 0 \), all three (co)variances, \( U_\mu, V_\mu \), and \( Y_\mu \), linearly increase with the interaction time \( T \) if \( \tau_T = \tau \gg 1 \), due to Eqs. (93)-(98). Nonetheless, the minimal variance \( u_\mu \) tends to a constant value at \( \tau \to \infty \). This is shown in the subsection \( ^{6.3} \). The examples of explicit time dependences of the coefficients \( u_\mu \) and \( v_\mu \) are given in subsection \( ^{6.3} \).

### 6.1.2 Mean photon number

Differentiating the “vacuum” part of sum (27) with respect to \( \tau \) and performing the summation over the upper index \( n \) with the aid of (97)-(98) (remembering that the coefficients \( \rho^{(n)}_m \) are different from zero provided the difference \( n - m \) is a multiple of \( p \)) one can obtain the formula for the photon generation rate from vacuum in each mode \( (0 \leq j \leq p - 1, \ q = 0, 1, 2, \ldots) \)

\[
\frac{d}{d\tau} \mathcal{N}_{j+pq}^{(\text{vac})} = -2\sigma(j + pq)Re \ \left[ \xi^{(j)}_{j+pq} \eta^{(p-j)}_{j+pq} \right] \\
= 2p\sqrt{1 - \gamma^2 \kappa^2} \sin(\pi j/p)\Gamma(q + j/p)\Gamma(1 + q + j/p)\Gamma(2 - j/p)\kappa^{2q+1} \\
\times F(q + j/p, -j/p; 1 + q; \kappa) F(q + j/p, 1 - j/p; 2 + q; \kappa^2)
\] (101)

We see that there is no photon creation in the modes with numbers \( p, 2p, \ldots \). In the short-time limit,

\[
\mathcal{N}_{j+pq}^{(\text{vac})} \sim \tau^{2q+1}, \quad \tau \ll 1.
\]

In the long-time limit the photon generation rate tends to the constant value (if \( \gamma < 1 \))

\[
\frac{d}{d\tau} \mathcal{N}_{j+pq}^{(\text{vac})} = \frac{2ap^2 \sin^2(\pi j/p)}{\pi^2(j + pq)} \left[ 1 + O\left( \frac{pq}{\sqrt{S}} \ln S \right) \right], \quad ap\tau \gg 1
\] (102)

For \( q \gg 1 \) and for a fixed value of \( \kappa \) one can simplify the right-hand side of (101) using Stirling’s formula for the Gamma-functions and the easily verified asymptotical formula

\[
F(a, b; c; z) \approx (1 - az/c)^{-b}, \quad a, c \gg 1.
\]

In this case

\[
\frac{d}{d\tau} \mathcal{N}_{j+pq}^{(\text{vac})} \approx 2p\sqrt{1 - \gamma^2 \kappa^2} \frac{\sin(\pi j/p)\Gamma(2 - j/p)\kappa^{2q+1}}{\pi \Gamma(j/p)q^2(1-j/p)(1 - \kappa^2)^{1-2j/p}}, \quad q \gg 1.
\] (103)
In particular, if \( q \gg S^2(p\tau) \gg 1 \), then

\[
\frac{d}{d\tau} N^{(\text{vac})}_{j+pq} \approx 2p^n \frac{\sin(\pi j/p)\Gamma(2 - j/p) (S^2/q)^{(2(1-j/p)}}}{\pi \Gamma(j/p)S^2} \exp\left(-\frac{q}{S^2}\right),
\]

(104)

Comparing (102) and (104) one can conclude that the number of the effectively excited modes (i.e. the modes with a time independent photon generation rate) increases in time exponentially, approximately as \( S^2(\tau)/\ln S(\tau) \).

The total number of photons generated from vacuum in all the modes equals

\[
N^{(\text{vac})} = \sum_{m,n=1}^{\infty} \frac{\kappa(n)}{m} |\eta_{m}|^2.
\]

(105)

Differentiating (105) with respect to \( \tau \) and performing the summation over \( m \) with the help of equations (29)- (65) or (67) one can obtain the formula

\[
\frac{dN^{(\text{vac})}}{d\tau} = 2\sigma \Re \sum_{n=1}^{\infty} \frac{1}{\pi} \sum_{m=1}^{p} m(p - m) \rho_{p-m}(\tau) \rho_{p-m}(\tau).
\]

(106)

Evidently, the right-hand side of this equation equals zero in the “semi-resonance” case \( p = 1 \).

Differentiating equation (106) once again over \( \tau \) one can perform the summation over the upper index \( n \) with the aid of equations (22), (57) to obtain a closed expression for the second derivative of the total number of “vacuum” photons

\[
\frac{d^2N^{(\text{vac})}}{d\tau^2} = 2\Re \sum_{m=1}^{p-1} m(p - m) \left[ \epsilon_{m} \xi_{p-m} + \eta_{m}^{*} \xi_{p-m} \right]
\]

\[
= 2 \sum_{m=1}^{p-1} m(p - m) \left\{ m(p - m) \left[ \frac{\kappa}{p} F \left( \frac{m}{p}, 1 - \frac{m}{p}; 2; \kappa^2 \right) \right]^2 \right.
\]

\[+ \left( 1 - 2\gamma^2 \kappa^2 \right) F \left( \frac{m}{p}, -\frac{m}{p}; 1; \kappa^2 \right) F \left( \frac{m}{p} - 1, 1 - \frac{m}{p}; 1; \kappa^2 \right) \right\}
\]

(107)

In the short-time limit one obtains

\[
\dot{N}^{(\text{vac})} = \frac{1}{3} p(p^2 - 1), \quad |ap\tau| \ll 1
\]

(108)

In the long-time limit the formulae (81), (84) and \( \sum_{m=1}^{p-1} \sin^2(\pi m/p) = p/2 \) lead to another simple expression (provided \( p \geq 2 \))

\[
\dot{N}^{(\text{vac})} = 2a^2 p^3/\pi^2, \quad ap\tau \gg 1, \quad a > 0
\]

(109)

Consequently, the total number of photons created from vacuum due to NSCE increases in time quadratically both in the short-time and in the long-time limits (although with different coefficients).

6.2 Arbitrary initial conditions

For an arbitrary initial state of the field one can write \( U_m = U_m^{(\text{vac})} + \Delta U_m \), where \( U_m^{(\text{vac})} \) is given by equation (89); similar expressions can be written for \( V_m \) and \( Y_m \). The corrections due to the nonvacuum initial states are given by

\[
\frac{\Delta U_m}{\Delta V_m} = \Re \sum_{n,j} \frac{m}{\sqrt{n} j} \left[ \rho_{m}^{(n)} \mp \rho_{-m}^{(n)} \right] \left[ \rho_{m}^{(j)} \mp \rho_{-m}^{(j)} \right] \left[ \langle \hat{b}_n \hat{b}_j \rangle - \langle \hat{b}_n \rangle \langle \hat{b}_j \rangle \right]
\]

\[+ \pm \left[ \rho_{m}^{(n)} \mp \rho_{-m}^{(n)} \right] \left[ \rho_{m}^{(j)} \mp \rho_{-m}^{(j)} \right] \left[ \langle \hat{b}_n \hat{b}_j \rangle - \langle \hat{b}_n \rangle \langle \hat{b}_j \rangle \right]
\]

(110)
\[
\Delta Y_m = \text{Im} \sum_{n,j} \frac{m}{\sqrt{n}j} \left[ \left( \rho_m^{(n)*} - \rho_m^{(j)*} \right) \left( \hat{b}_m^\dagger \hat{b}_j - \langle \hat{b}_m \rangle \langle \hat{b}_j \rangle \right) + \left( \rho_m^{(n)} - \rho_m^{(j)} \right) \left( \langle \hat{b}_n \rangle \langle \hat{b}_j \rangle - \langle \hat{b}_n \rangle \langle \hat{b}_j \rangle \right) \right] \]
\]

where the average values like \( \langle \hat{b}_m \rangle \rangle \) are calculated in the initial state. All the corrections disappear in the case of the initial coherent state, \( \hat{b}_n | \alpha \rangle = \alpha_n | \alpha \rangle \). If the initial density matrix is diagonal in the Fock basis (as happens e.g. for the Fock or thermal states) then \( \langle \hat{b}_m^\dagger \hat{b}_j \rangle = \delta_n \delta_m \) \( (v_n \geq 0) \), all other average values in (110) and (111) being equal to zero. In this case the double sums are reduced to the single ones:

\[
\Delta U_m = m \sum_n \nu_n | \rho_m^{(n)} - \rho_m^{-1} |^2, \quad \Delta V_m = m \sum_n \nu_n | \rho_m^{(n)} + \rho_m^{-1} |^2
\]

(112)

\[
\Delta Y_m = 2m \sum_n \nu_n \text{Im} \left[ \rho_m^{(n)*} \rho_m^{-1} \right] .
\]

(113)

We see that the initial fluctuations always increase both the variances \( U_m \) and \( V_m \) (for the diagonal density matrix). However, asymptotically at \( \tau \rightarrow \infty \) the corrections are bounded for the physical initial states having finite total numbers of photons, because the coefficients \( | \rho_m^{(n)} \pm \rho_m^{-1} |^2 \) and \( \text{Im} \left[ \rho_m^{(n)*} \rho_m^{-1} \right] \) do not depend on the summation index \( n \) in this limit. For example, if \( p = 2 \), then

\[
\rho_{2m+1}^{(n+1)} (\tau \gg 1) \approx \frac{2(-1)^m}{\pi(2m+1)} (a + i\gamma)^{m+n+1},
\]

(114)

\[
\rho_{2m-1}^{(n+1)} (\tau \gg 1) \approx \frac{2(-1)^m}{\pi(2m+1)} (a + i\gamma)^{n-m},
\]

(115)

and the exponent \( n \) disappears in the sums, because \( |a + i\gamma| = 1 \). Thus we have in the ‘principal’ \( \mu \)-modes (taking \( p = 2 \) for the sake of simplicity)

\[
\begin{align*}
\Delta U_\mu^{(\infty)} &= \frac{8Z}{\pi^2 \mu} \times \left\{ \begin{array}{c} 2 \sin^2 \left( \frac{\mu\phi}{2} \right) \\ 2 \cos^2 \left( \frac{\mu\phi}{2} \right) \\ -2 \sin \left( \frac{\mu\phi}{2} \right) \end{array} \right\}, \\
\Delta V_\mu^{(\infty)} &= \frac{2\sin \left( \frac{\mu\phi}{2} \right) + f}{\pi^2 \mu}, \quad \chi = \mu\phi.
\end{align*}
\]

(116)

where \( \phi \equiv \arcsin \gamma \), \( Z = \sum_{k=0}^{\infty} \frac{\nu_{2k+1}}{2k+1} \).

The expressions in (116) are very similar to those in equations (113)-(114). The consequence of Eq. (116) is the important result that in the limit \( \tau \rightarrow \infty \) the minimal variance \( u_\mu \) does not depend on the initial state of the field inside the cavity, provided the initial density matrix was diagonal in the Fock basis. Indeed, in this case, combining the equations (110)-(111) and (112), we can write the variances at \( \tau \gg 1 \) as (we omit the subscript \( \mu \))

\[
\begin{pmatrix}
U(\tau) \\
V(\tau) \\
Y(\tau)
\end{pmatrix} = \begin{pmatrix}
2F \sin^2 (\chi/2) + f \\
2F \cos^2 (\chi/2) + g \\
-F \sin \chi + h
\end{pmatrix}, \quad F = \frac{8(a\tau + Z)}{\pi^2 \mu}, \quad \chi = \mu\phi.
\]

(117)

The corrections \( f \), \( g \) and \( h \) can be found by integrating equations (12)-(13), therefore they do not depend on the initial state. At \( \tau \rightarrow \infty \) these corrections tend to finite limits, so they are much smaller than \( F \). Evidently, \( U + V = 2F + f + g \), whereas

\[
(U - V)^2 + 4Y^2 = 4F^2 + 4F(g - f) \cos \chi - 2h \sin \chi + (f - g)^2 + 4h^2.
\]

For \( F \gg f, g, h \) we have

\[
\sqrt{(U - V)^2 + 4Y^2} = 2F + (g - f) \cos \chi - 2h \sin \chi + O(1/F)
\]
so the minimal variance \( u(\tau) \) tends to the finite limit
\[
u(\infty) = f \cos^2(\chi/2) + g \sin^2(\chi/2) + h \sin \chi, \quad (118)
\]
which does not depend on \( Z \), i.e. on the initial state.

The correction to the mean number of photons in the \( m \)-th mode for the “diagonal” initial distributions is given by the sum
\[
\Delta N_m = m \sum_n \frac{\nu_n}{n} \left( |\rho_m^{(n)}|^2 + |\rho_{-m}^{(n)}|^2 \right).
\]

It tends to the limit \( \Delta N_m^{(\infty)} = 8Z/(\pi^2 \mu) \).

The total number of photons in all the modes equals \( N = N^{(\text{vac})} + N^{(\text{cav})} \), where
\[
N^{(\text{cav})} = N(0) + 2 \sum_{m,n,k=1}^{\infty} \frac{m}{\sqrt{nk}} \left[ \eta_m^{(n)*} \eta_m^{(k)} \langle \hat{\rho}_n \hat{\rho}_k \rangle + \text{Re} \left( \eta_m^{(n)} \eta_m^{(k)} \langle \hat{b}_n \hat{b}_k \rangle \right) \right] \quad (119)
\]
to obtain this formula one should use the identity \((21)\). Differentiating \((119)\) with respect to \( \tau \) and performing the summation over \( m \) with the help of equations \((23), (30), (64), (65), (67)\), or \((17)\), one can obtain the formula
\[
\frac{dN^{(\text{cav})}}{d\tau} = 2\sigma \sum_{n,k=1}^{\infty} \frac{\langle \hat{b}_n \hat{b}_k \rangle}{\sqrt{nk}} \sum_{m=1}^{\infty} n(p-m) \left[ \rho_{m-p}^{(n)*} \rho_{m-p}^{(k)} + \rho_{m-p}^{(n)} \rho_{m-p}^{(k)*} \right]
- 2\sigma \text{Re} \sum_{n,k=1}^{\infty} \frac{\langle \hat{b}_n \hat{b}_k \rangle}{\sqrt{nk}} \sum_{m=1}^{\infty} n(p-m) \left[ \rho_{m-p}^{(n)*} \rho_{m-p}^{(k)} + \rho_{m-p}^{(n)} \rho_{m-p}^{(k)*} \right]. \quad (120)
\]
Using equation \((120)\) and replacing the coefficients \( \rho_m^{(n)} \) by their asymptotical values \((87)\) one can obtain the expression
\[
\frac{dN^{(\text{cav})}}{d\tau} = \frac{4\sigma \nu^2}{\pi^2} \sum_{m=1}^{p-1} \sin^2(\pi m/p) \sum_{n,k=0}^{\infty} \frac{\sigma^{n+k}}{\sqrt{(m+pn)(m+pk)}} \times \left\{ \langle \hat{b}_m^{\dagger} \hat{b}_{m+pn} \rangle (a+i\gamma)^{k-n} - \sigma \text{Re} \left[ \langle \hat{b}_m \hat{b}_{m+pn} \rangle (a+i\gamma)^{k+n+1} \right] \right\} \quad (121)
\]
which holds provided \( \nu \gg 1 \) and \( a > 0 \). For the physical initial states the sum in the right-hand side of \((121)\) is finite. This is obvious if a finite number of modes was excited initially. But even if the cavity was initially in a high-temperature thermal state, so that \( \langle \hat{b}_n \hat{b}_k \rangle = \Theta n/k \), \( \langle \hat{b}_n \hat{b}_k \rangle = 0 \), the sum over \( n, k \) yields a finite value \( \Theta \sum_{n=0}^{\infty} (m+pn)^{-2} \). Consequently, the total number of “nonvacuum” photons increases in time \textit{linearly} at \( \nu \gg 1 \), whereas the total number of quanta generated from vacuum increases \textit{quadratically} in the long time limit.

6.3 The “principal resonance” \( (p=2) \)

Many formulas obtained above can be simplified in the special case \( p=2 \). In this case there are two subsets of nonzero Bogoliubov coefficients. The first one consists of the coefficients with even upper and lower indices \( \xi_k^{(2q)} \) which are reduced to the coefficients \( \xi_k^{(q)} \) of the “semi-resonance” case. However, since \( n_k^{(2q)} \equiv 0 \), this subset does not contribute to the generation of new photons. The second subset is formed by the “odd” coefficients which can be written as \( [\kappa \equiv \kappa(2\tau)] \)
\[
\xi_k^{(2n+1)} = \frac{\Gamma(n+3/2)\kappa^{n-m}\lambda^{m+n+1}}{\Gamma(m+3/2)\Gamma(1+n-m)} \times \frac{F(n+1/2, -m-1/2; 1+n-m; \kappa^2)}{m+n} \quad (122)
\]
For example, in the case of the quadrature variance $U_1$ relations (67).

The average number of photons in the first mode can be obtained by integrating this equation. It is convenient where

\[ c \rho \eta \] were defined in (50).

It is known \[275\] that the hypergeometric function $F(a, b; c; z)$ with ‘half-integral’ parameters $a, b$ and an integral parameter $c$ can be expressed in terms of the complete elliptic integrals

\[
\begin{align*}
K(\kappa) &= \int_{0}^{\pi/2} \frac{d\alpha}{\sqrt{1-\kappa^2\sin^2\alpha}} = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; \kappa^2\right) \quad (125) \\
E(\kappa) &= \int_{0}^{\pi/2} d\alpha \sqrt{1-\kappa^2\sin^2\alpha} = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; \kappa^2\right). \quad (126)
\end{align*}
\]

In particular,

\[
\begin{align*}
\xi_1^{(1)} &= \frac{2}{\pi} \lambda(\kappa) E(\kappa), \quad \eta_1^{(1)} = \frac{2}{\pi \kappa} [\tilde{\kappa}^2 K(\kappa) - E(\kappa)], \\
\rho_3^{(1)} &= \frac{2\lambda^2(\kappa)}{3\pi \kappa} \left[(1 - 2\kappa^2) E(\kappa) - \tilde{\kappa}^2 K(\kappa)\right] \\
\rho_{-3}^{(1)} &= -\frac{2}{3\pi \kappa^2 \lambda(\kappa)} \left[(2 - \kappa^2) E(\kappa) - 2\tilde{\kappa}^2 K(\kappa)\right] \quad (127) - (129)
\end{align*}
\]

where

\[
\tilde{\kappa} \equiv \sqrt{1-\kappa^2} = [1 + S^2(2\tau)]^{-1/2}, \quad (130)
\]

and $\lambda(\kappa)$ was defined in \[30\].

The general structure of the coefficients $\rho_{\mu}^{(1)}$ (we confine ourselves to the case $p = 2$) is as follows

\[
\begin{align*}
\rho_{2m+1}^{(1)} &= \frac{2\lambda^{m+1}(\kappa)}{\pi \kappa^m} \left[f_m(\kappa^2) E(\kappa) + \kappa^2 g_m(\kappa^2) K(\kappa)\right] \quad (131) \\
\rho_{-2m-1}^{(1)} &= \frac{2}{\pi \kappa^m \lambda(\kappa)} \left[r_m(\kappa^2) E(\kappa) + \kappa^2 s_m(\kappa^2) K(\kappa)\right] \quad (132)
\end{align*}
\]

where $f_m(x), g_m(x), r_m(x), s_m(x)$ are the polynomials of the degree $m$ which can be found from the recurrence relations \[13\].

The photon generation rate from vacuum in the principal cavity mode ($m = 1$) reads

\[
\frac{dN_1^{(vac)}}{d\tau} = -2\text{Re} \left[\eta_1^{(1)} \xi_1^{(1)}\right] = \frac{8\sqrt{1-\gamma^2\kappa^2}}{\pi^2 \kappa} E(\kappa) \left[E(\kappa) - \tilde{\kappa}^2 K(\kappa)\right]. \quad (133)
\]

The average number of photons in the first mode can be obtained by integrating this equation. It is convenient to integrate with respect to the variable $\kappa$, taking into account the relation (for $p = 2$, for instance)

\[
d\kappa = 2\beta \tilde{\kappa}^2 d\tau, \quad \beta = \text{Re} \lambda = \sqrt{1-\gamma^2\kappa^2}.
\]

For example, in the case of the quadrature variance $U_1$ we arrive at the equation

\[
\frac{dU_1}{d\kappa} = \frac{2}{\pi^2 \kappa^2 \kappa^2 \beta} \left\{ \kappa^2 \left[1 - 2\gamma^2 \kappa^2\right] + 1 - 2\beta \kappa \right\} E^2(\kappa) \]

\[
-2\tilde{\kappa}^2 \left[1 - \beta \kappa\right] E(\kappa) K(\kappa) + \kappa^2 \tilde{\kappa}^2 K(\kappa)\right\}. \quad (134)
\]

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Let us consider first the case $\gamma = 0$, when $\beta = 1$. Taking into account the differentiation rules we may suppose that the factor $\kappa^2$ in the denominator of the right-hand side of equation (134) comes from the derivative $dK/d\kappa$. Thus it is natural to look for the solution in the form

$$U_1 = \frac{2}{\pi^2\kappa} \left[ A(\kappa)K^2(\kappa) + B(\kappa)K(\kappa)E(\kappa) + C(\kappa)E^2(\kappa) \right],$$

where $A(\kappa)$, $B(\kappa)$ and $C(\kappa)$ are some polynomials of $\kappa$. Putting the expression (136) into equation (134) we obtain a set of coupled equations for the unknown functions $A, B, C$. Writing $A(\kappa) = a_0 + A_1(\kappa)$, $B(\kappa) = b_0 + B_1(\kappa)$, $C(\kappa) = c_0 + C_1(\kappa)$ we determine the constant coefficients $a_0$, $b_0$ and $c_0$ by putting $\kappa = 0$ in that equations. Then we obtain new equations for the functions $A_1(\kappa)$, $B_1(\kappa)$ and $C_1(\kappa)$ and repeat the procedure. After a few steps we arrive at the equations which have obvious trivial solutions $A_n = B_n = C_n = 0$. This confirms our hypothesis on the polynomial structure of the functions $A(\kappa)$, $B(\kappa)$ and $C(\kappa)$ and gives the final answer. The equations for the variances $U_\mu$, $V_\mu$, etc. with $\mu \geq 3$ can be integrated in the same manner, the only difference is that one should write $\kappa^n$ instead of $\kappa$ in the denominator of the expression like (138). In the generic case $\gamma \neq 0$ we notice that the factor $\beta$ can appear in the denominator of the expression (134) as a result of differentiating the function $\beta(\kappa)$, since $d\beta/d\kappa = -\gamma^2\kappa/\beta$. Therefore we split each function, $A, B, C$ in the ‘$\beta$-even’ and ‘$\beta$-odd’ parts like $A = A_e(\kappa) + \beta(\kappa)A_o(\kappa)$. The equations for the ‘even’ and ‘odd’ coefficients turn out independent, and we solve them using the procedure described above.

The results of the integrations are as follows, 

$$N_1^{(\text{vac})}(\kappa) = \frac{2}{\pi^2}K(\kappa) \left[ 2E(\kappa) - \kappa^2K(\kappa) \right] - \frac{1}{2},$$

$$U_1 = \frac{2}{\pi^2\kappa} \left[ \kappa^2(\beta - \kappa)K^2(\kappa) - 2(\beta - \kappa)K(\kappa)E(\kappa) + \beta E^2(\kappa) \right],$$

$$V_1 = \frac{2}{\pi^2\kappa} \left[ 2(\beta + \kappa)K(\kappa)E(\kappa) - \kappa^2(\beta + \kappa)K^2(\kappa) - \beta E^2(\kappa) \right],$$

$$Y_1 = \frac{2\gamma}{\pi^2} \left[ \kappa^2K^2(\kappa) - 2K(\kappa)E(\kappa) + E^2(\kappa) \right].$$

Making the transformation (269, 270)

$$K \left( \frac{1 - \kappa}{1 + \kappa} \right) = \frac{1 + \kappa}{2}K(\kappa), \quad E \left( \frac{1 - \kappa}{1 + \kappa} \right) = \frac{E(\kappa) + \kappa K(\kappa)}{1 + \kappa}$$

one can rewrite formulae (127) and (137) in the form found for the first time in (189) (in the special case of $\gamma = 0$). Using the asymptotical expansions of the elliptic integrals at $\kappa \to 1$ (270)

$$K(\kappa) \approx \ln \frac{4}{\kappa} + \frac{1}{4} \left( \ln \frac{4}{\kappa} - 1 \right) \kappa^2 + \cdots$$

$$E(\kappa) \approx 1 + \frac{1}{2} \left( \ln \frac{4}{\kappa} - 1 \right) \kappa^2 + \cdots$$

one can obtain the formula

$$N_1^{(\text{vac})}(\tau \gg 1) = \frac{8a}{\pi^2 \tau} + \frac{4}{\pi^2} \ln \left( \frac{2}{a} \right) - \frac{1}{2} + O \left( \tau^{-4a} \right), \quad a > 0.$$ 

In the special case of $\gamma = 1$ one can obtain the expansion

$$N_1^{(\text{vac})}(\tau \gg 1) = \frac{4}{\pi^2} \ln \tau + \frac{12}{\pi^2} \ln 2 - \frac{1}{2} + O \left( \tau^{-2} \right).$$
If \( \gamma > 1 \), the number of photons in the principal mode oscillates with the period \( \pi/(2\hat{a}) \). For \( \gamma \gg 1 \) one can write \( \kappa \approx \sin(2\hat{a}\tau)/\hat{a} \), i.e. \( |\kappa| \ll 1 \). In this case

\[
\mathcal{N}^1_{\text{vac}} \approx \frac{\kappa^2}{4} \approx \frac{\sin^2(2\hat{a}\tau)}{4\hat{a}^2} \ll 1.
\]

Equation (100) yields the minimal and maximal invariant variances

\[
u_1 = \frac{2}{\pi^2\kappa^4} \left[ \tilde{\kappa}^2(1 - \kappa)K^2(\kappa) - 2(1 - \kappa)K(\kappa)E(\kappa) + E^2(\kappa) \right],
\]
\[
\nu_1 = \frac{2}{\pi^2\kappa^4} \left[ 2(1 + \kappa)K(\kappa)E(\kappa) - \tilde{\kappa}^2(1 + \kappa)K^2(\kappa) - E^2(\kappa) \right],
\]

which depend on the detuning parameter \( \gamma \) only implicitly, through the dependence on \( \gamma \) of the function \( \kappa(\tau) \). In the short time limit \( \tau \ll 1 \) (then \( \kappa \approx 2\tau \)) we obtain, using the Taylor expansions of the complete elliptic integrals, \( u_1 = \frac{1}{2} - \tau + \tau^2 + \cdots \) and \( v_1 = \frac{1}{2} + \tau + \tau^2 + \cdots \) in accordance with (124). More precisely,

\[
\frac{u_1}{v_1} = \frac{1}{2} \left( 1 + \kappa + \frac{1}{2}\kappa^2 + \frac{1}{4}\kappa^3 + \frac{7}{32}\kappa^4 + \cdots \right)
\]

The minimal variance \( u_1 \) monotonously decreases from the value \( 1/2 \) at \( t = 0 \) to the constant asymptotic value \( 2/\pi^2 \) at \( \tau \gg 1 \), confirming qualitatively the evaluations of (107, 110) and giving almost 50% squeezing in the initial vacuum state. The variance of the conjugate quadrature monotonously increases, and for \( \tau \gg 1 \) it becomes practically linear function of time: \( v_1(\tau) = 1 \approx 0 \) (167/\pi^2). The asymptotical minimal value \( u_1(\tau = \infty) \) does not depend on \( \gamma \) provided \( \gamma \leq 1 \) (only the rate of reaching this asymptotical value decreases with \( \gamma \) as \( \sqrt{1 - \gamma^2} \)). In the strongly detuned case, \( \gamma > 1 \), the minimal variance oscillates as a function of \( \tau \) (being always greater than \( 2/\pi^2 \)), since in this case the function \( \kappa(\tau) \) oscillates between \( -\gamma^{-1} \) and \( \gamma^{-1} \).

The minimal variance does not go to zero when \( \tau \to \infty \) due to the strong intermode interaction, which results in a high degree of quantum mixing for each mode. Since the state originating from the initial vacuum state belongs to the class of quantum states (see the next subsection), the quantum ‘purity’ \( \chi_m \equiv \text{Tr} \hat{\rho}_m^2 \) of the \( m \)th field mode (described by means of the density matrix \( \hat{\rho}_m \)) can be expressed in terms of the (co)variances as (177)

\[
\chi_m = \left[ 4 \left( U_m V_m - Y_m^2 \right) \right]^{1/2}.
\]

Using equations (117)-(118) one can check that for \( \tau \gg 1 \), \( U/V = 2Fu(\infty) + \mathcal{O}(1) \approx \tau \). Consequently, the purity factor \( \chi \) asymptotically goes to zero as \( \tau^{-1/2} \). For instance, for \( m = 1 \) we have (writing simply \( K \) and \( E \) instead of \( K(\kappa) \) and \( E(\kappa) \))

\[
\chi_m = \frac{\pi^2}{4\kappa} \left[ 4K^3E + 4\tilde{\kappa}^4K^3E - 6\tilde{\kappa}^2K^2E^2 - E^4 - \tilde{\kappa}^2K^4 \right]^{-1/2}
\]

The initial dependence on \( \kappa \) is rather weak: \( \chi(\kappa \ll 1) = 1 - \frac{3}{\pi^2}\kappa^4 + \cdots \). But when \( \kappa \to 1 \), \( \chi \) rapidly goes to zero: \( \chi(\kappa \ll 1) \approx (8/\pi^2)\ln(4/\kappa) \approx 1/\gamma \), with \( d\chi/d\kappa \to -\infty \).

The expressions for the variances in the modes with numbers \( m \geq 3 \) are rather involved. Here we give only one explicit example – the variance \( U_3 \) for \( \gamma = 0 \):

\[
U_3 = \frac{2}{3\pi^2\kappa^4} \left[ 3\kappa^2(1 - \kappa) (4 + 10\kappa + 9\kappa^2) K^2(\kappa) \right. \\
\left. + (1 - \kappa) (4\kappa^4 - 14\kappa^2 - 20\kappa - 8) K(\kappa)E(\kappa) \right. \\
\left. + (4\kappa^4 + 6\kappa^3 - \kappa^2 + 6\kappa + 4) E^2(\kappa) \right]
\]

The Taylor expansion of the right-hand side of (142) coincides with the expansion (14). The asymptotical value at \( \tau \to \infty \) equals \( U_3(\kappa = 1) = 38/(9\pi^2) \approx 0.43 \). We see that the squeezing rapidly disappears with increase of the mode number \( \mu \). The variance \( V_3 \) can be obtained from (142) by means of a simple substitution \( \kappa \to -\kappa \). Therefore the mean number of photons in the third mode is given by

\[
N_3 = \frac{2}{3\pi^2\kappa^4} \left[ (3\kappa^2 - 2) K(2E - \tilde{\kappa}^2K) + 2(1 + \kappa^2) E^2 \right] - \frac{1}{2}
\]
The second derivative of the total mean number of photons created from vacuum in all the modes takes the form
\[
\frac{d^2N^{(\text{vac})}}{d\tau^2} = \frac{8}{\pi^2\kappa^2} \left[ \dot{\kappa}^2 K^2 - 2\dot{\kappa}^2 KE + (1 + \kappa^2 - 2\gamma^2\kappa^4) E^2 \right].
\] (144)
Integrating this equation with the account of the condition \(dN^{(\text{vac})}/d\tau = 0\) at \(\tau = 0\) [which is a trivial consequence of Eq. (106)], we obtain very simple expression
\[
N^{(\text{vac})} = \frac{2}{\pi^2} K(\kappa) [K(\kappa) - E(\kappa)].
\] (145)
In the limiting cases this formula yields
\[
N^{(\text{vac})}(\tau \ll 1) \approx \tau^2
\]
\[
N^{(\text{vac})}(\tau \gg 1) = 8a^2\tau^2/\pi^2 + O(\tau), \quad a > 0.
\]
If \(\gamma \gg 1\), then \(|\kappa| \ll 1\), but \(\gamma^2\kappa^2 \approx \sin^2(2\tilde{a}\tau) \sim O(1)\). In this case the Taylor expansion of the second derivative yields \(N^{(\text{vac})} = 2\cos(4\tilde{a}\tau) + O(\gamma^{-2})\). Integrating this equation with account of the initial conditions \(N^{(\text{vac})}(0) = N^{(\text{vac})}(0) = 0\) one obtains \(N^{(\text{vac})} \approx N_1^{(\text{vac})} \approx \sin^2(2\tilde{a}\tau)/(4\tilde{a}^2)\).

6.4 Photon distribution

Now let us turn to the photon distribution function (PDF) \(f(n) \equiv \langle n|\rho_m(t)|n\rangle\), where \(|n\rangle\) is the multimode Fock state, \(n = (n_1, n_2, \ldots)\), and \(\rho_m(t)\) is the time-dependent density matrix of the \(m\)th field mode in the Schrödinger picture. Note that all the calculations in the preceding sections were performed in the framework of the Heisenberg picture. Nonetheless, the available information is sufficient to calculate the PDF for the special (but very important) class of Gaussian initial states (defined as the states whose density matrices, or wave functions, or Wigner functions, are described by some Gaussian exponentials). This class includes coherent, squeezed and thermal states; in particular, it includes the vacuum state.

The solution is based on two key points. The first one is the statement that the field evolution in a cavity with moving boundaries can be described not only in the Heisenberg picture, but, equivalently, in the framework of the Schrödinger picture, with a quadratic multidimensional time-dependent Hamiltonian. The second key point is the fact that the evolution governed by quadratic Hamiltonians transforms any Gaussian state to another Gaussian state. Knowing these facts, it remains to take into account that the photon distribution function of any Gaussian state is determined completely by the average values of quadratures and by their variances, which obviously do not depend on the quantum mechanical representation.

In the most compact form the information on the photon distribution \(f(n)\) in some mode (we suppress here the mode index) is contained in the generating function
\[
G(z) = \sum_{n=0}^{\infty} f(n)z^n
\]
For a generic one-mode Gaussian state it can be expressed as
\[
G(z) = [G(z)]^{-1/2} \exp \left( \frac{1}{D} \left[ \frac{zg_1 - z^2g_2}{G(z)} - g_0 \right] \right)
\] (146)
where
\[
G(z) = \frac{1}{4} \left[ (1 + z)^2 + 4(UV - Y^2)(1 - z)^2 + 2(U + V)(1 - z^2) \right]
\] (147)
\[
D = 1 + 2(U + V) + 4(UV - Y^2) = 4G(0)
\]
\[
g_0 = \langle \hat{p} \rangle^2(2U + 1) + \langle \hat{q} \rangle^2(2V + 1) - 4\langle \hat{p} \rangle \langle \hat{q} \rangle Y
\]
\[
g_1 = 2\langle \hat{p} \rangle^2 \left( U^2 + Y^2 + U + \frac{1}{4} \right) + 2\langle \hat{q} \rangle^2 \left( V^2 + Y^2 + V + \frac{1}{4} \right)
\]
\[-4\langle \hat{p} \rangle \langle \hat{q} \rangle Y(U + V + 1)\]
$$g_2 = 2 \langle \hat{p} \rangle^2 \left( U^2 + Y^2 - \frac{1}{4} \right) + 2 \langle \hat{q} \rangle^2 \left( V^2 + Y^2 - \frac{1}{4} \right) - 4 \langle \hat{p} \rangle \langle \hat{q} \rangle Y (U + V).$$

In the generic case $f(n)$ is related to the two-dimensional ‘diagonal’ Hermite polynomials [280]:

$$f(n) = \frac{F_0}{n!} H_n^{(R)} (x, x^*)$$

where

$$F_0 = f(0) = 2D^{-1/2} \exp (-g_0/D)$$

$$x = \sqrt{2} \left\{ (2V - 1) \langle \hat{q} \rangle - 2Y \langle \hat{p} \rangle + i [(1 - 2U) \langle \hat{p} \rangle + 2Y \langle \hat{q} \rangle] \right\}$$

$$2(U + V) - 4(UV - Y^2) - 1$$

and $2 \times 2$ symmetric matrix $R$ has the elements

$$R_{11} = R_{22} = \frac{2}{D}(V - U - 2iY), \quad R_{12} = R_{21} = \frac{1}{D} \left[ 1 - 4(1 + UV - Y^2) \right]$$

The two-dimensional Hermite polynomials are defined via the expansion [280]

$$\exp \left( -\frac{1}{2} a R a + a R x \right) = \sum_{m,n=0}^{\infty} \frac{a_m a_n}{m! n!} H_n^{(R)} (x_1, x_2)$$

where $x = (x_1, x_2), a = (a_1, a_2)$. The properties of these polynomials were studied in [281,282]. In particular, they can be expressed as finite sums of the products of the usual (one-dimensional) Hermite polynomials.

The corresponding formula for the probabilities reads [280]

$$f(n) = F_0 \left( \frac{\Delta}{D} \right) \sum_{k=0}^{n} \left( \frac{S}{\Delta} \right)^k \frac{n!}{(n-k)!2^k} |H_{n-k}(\xi)|^2$$

where

$$\Delta = \sqrt{(U - V)^2 + 4Y^2}, \quad S = 4 (UV - Y^2) - 1$$

$$\xi = \frac{(2V + 1) \langle \hat{q} \rangle - 2Y \langle \hat{p} \rangle + i [(1 + 2U) \langle \hat{p} \rangle - 2Y \langle \hat{q} \rangle]}{[2D(V - U - 2iY)]^{1/2}}$$

If $\langle \hat{p} \rangle = \langle \hat{q} \rangle = 0$, then the generating function [140] is reduced to $[G(z)]^{-1/2}$, i.e. it has the same structure as the known generating function of the Legendre polynomials $P_n(x)$. In this case, we have the following expression for the photon distribution in the nth field mode:

$$f_m(n) = \frac{2 [(2u_m - 1)(2v_m - 1)]^{n/2}}{[(2u_m + 1)(2v_m + 1)]^{(n+1)/2}} P_n \left( \frac{4u_m v_m - 1}{\sqrt{(4u_m^2 - 1)(4v_m^2 - 1)}} \right)$$

It depends only on the invariant minimal and maximal variances $u_m$ and $v_m$. Note that the argument of the polynomial in [151] is always outside the ‘traditional’ interval $(-1, 1)$ (in particular, this argument is pure imaginary if $2v_m < 1$), being exactly equal to 1 for the ‘non-principal’ modes with $u_m = v_m = N_m + \frac{1}{2}$, when formula (151) transforms to the time-dependent Planck’s distribution

$$f_m(n; \tau) = \frac{N_m^n}{[N_m(\tau) + 1]^{n+\tau}}.$$

Only for the ‘principal’ $\mu$-modes the spectrum of photons is different from Planck’s one due to the squeezing effect.

The function (151) can be simplified in the long-time limit $\tau \gg 1$, when the average number of created photons $N \equiv \bar{n} \approx (V + U)/2$ exceeds 1. Then the mean-square fluctuation of the photon number has the same order of magnitude as the mean photon number itself, $\sqrt{\sigma_m} \approx \sqrt{2N}$, and the most significant part
of the spectrum corresponds to the values \( n \gg 1 \). Using the Laplace–Heine asymptotical formula for the Legendre polynomial \[ P_n(z) \approx \frac{(z + \sqrt{z^2 - 1})^{n+1/2}}{\sqrt{2\pi n (z^2 - 1)^{1/4}}}, \quad n \gg 1 \]

one can simplify \([151]\) for the fixed values of the invariant variances \( u \) and \( v \) as

\[
f(n) \approx \frac{1}{\sqrt{\pi n(u - v)}} \left( \frac{2v - 1}{2v + 1} \right)^{n+1/2} \approx \frac{1}{\sqrt{\pi n(u - v)}} \left( \frac{2v - 1}{2v + 1} \right)^{n+1/2}, \quad n \gg 1 \]

provided the positive difference \( v - u \) is not too small. Another approximate formula can be used if \( v \gg 1 \) but \( u \sim 1 \):

\[
f(n) \approx \frac{\sqrt{\pi}(2u - 1)^{n/2}}{\sqrt{v(2u + 1)^{(n+1)/2}}} e^{-n/(2v)} P_n \left( \frac{2u}{\sqrt{4u^2 - 1}} \right), \quad n \ll 8v^2. \]

The first and second derivatives of the generating function \((146)\) at \( z = 1 \) yield the first two moments of the photon distribution (hereafter we suppress subscript \( m \))

\[
\bar{n} = \frac{1}{2}(u + v - 1), \quad \sigma_n = \overline{n^2} - (\bar{n})^2 = \frac{1}{4} (2u^2 + 2v^2 - 1) \]

which result in the Mandel parameter \([284]\)

\[
Q \equiv \sigma_n / \bar{n} - 1 = \frac{u^2 + v^2 - u - v + 1/2}{u + v - 1}. \]

This parameter appears positive for all values of \( \tau \), so the photon statistics is super-Poissonian, with strong bunching of photons (the pair creation of photons in the NSCE was discussed in \([142, 198, 200, 213]\)). In particular,

\[
Q_{2m+1}(\tau \to 0) \approx [(m + 1)(2m - 1)!!/m!]^2 \tau^{2m}/(2m + 1), \quad Q_1(0) = 1,
\]

whereas \( Q_m \approx V_m(\tau) \gg 1 \) for \( \tau \gg 1 \) (if \( \gamma \ll 1 \)).

7 Energy and formation of packets

7.1 Energy density

The mean value of the energy density operator in one space dimension

\[
\hat{W}(x,t) = \frac{1}{8\pi} \left[ (\partial A/\partial t)^2 + (\partial^2 A/\partial x)^2 \right] \]

at \( t \geq T \) equals (hereafter we assume \( L_0 = 1 \), i.e. \( \omega_1 = \pi \))

\[
\hat{W}(x,t) = \pi \sum_{m,j=1}^{\infty} \sqrt{|m|} \left\{ \cos[\pi(m + j)x] \text{Re} \left[ \langle \hat{a}_m \hat{a}_j \rangle e^{-i\pi(m+j)t'} \right] \right. \\
+ \left. \frac{1}{2} \cos[\pi(m - j)x] \left[ \langle \hat{a}_m \hat{a}_j \rangle e^{i\pi(m-j)t'} + \langle \hat{a}_m \hat{a}_j \rangle e^{-i\pi(m-j)t'} \right] \right\} \]

where the quantum mechanical averaging \( \langle \cdot \cdot \rangle \) is performed over the initial state of the field (the Heisenberg picture) and \( t' \equiv t + \delta T \).
7.1.1 Regularization and Casimir’s energy

Due to the commutation relations $[\hat{a}_m, \hat{a}_j^\dagger] = \delta_{mj}$ the series (157) contains the vacuum divergent ‘diagonal’ part with $m = j$:

$$\hat{W}^{(\text{vac})} = (\pi/2) \sum_{m=1}^{\infty} m \exp(-i\pi mt' + i\pi mt).$$

(158)

The recipe how to regularize this divergence was given in [81]. One should write the first term in the argument of the exponential in (158) as it stands, but to replace $t'$ in the second term by $t' + i\eta$, $\eta > 0$ (this is the so-called ‘point-splitting method’). Then the sum becomes convergent, giving

$$\hat{W}^{(\text{vac})}(\eta) = (\pi/2) \sum_{m=1}^{\infty} m e^{-m\pi\eta} = (\pi/8) [\sinh(\pi\eta/2)]^{-2}.$$  

The Taylor expansion of this function reads $\hat{W}^{(\text{vac})}(\eta) = (2\pi\eta^2)^{-1} - \pi/24 + \mathcal{O}(\eta^2)$. According to [81], one should remove the divergent term $(2\pi\eta^2)^{-1}$ and after that proceed to the limit $\eta \to 0$. This limit value gives us the known expression for the one-dimensional negative vacuum Casimir energy [21, 117–120] (which does not depend on the coordinate $x$ in the case involved)

$$\hat{W}^{(\text{Cas})} = -\pi/24 \quad (\text{or} \quad -\frac{\pi\hbar c}{24L_0^2} \text{ in the dimensional units})$$  

(159)

Extracting this vacuum energy from $\hat{W}$ we arrive at the expression

$$W = \hat{W} - \hat{W}^{(\text{Cas})} = \pi \sum_{m,j=1}^{\infty} \sqrt{mj} \Re \left[ \langle \hat{a}_m \hat{a}_j \rangle \cos[\pi(m+j)x] e^{-i\pi(m+j)t'} + \langle \hat{a}_m^\dagger \hat{a}_j \rangle \cos[\pi(m-j)x] e^{i\pi(m-j)t'} \right].$$

(160)

The same expression (160) can be obtained if one calculates the mean value of the normally ordered (with respect to the operators $\hat{a}_n^\dagger$ and $\hat{a}_n$) counterpart of the operator (156) (cf. [164]). Then the total energy (without the vacuum part) assumes the usual form

$$\mathcal{E} = \int_0^{L_0} W(x,t) dx = \sum_{n=1}^{\infty} \omega_n \langle \hat{a}_n^\dagger \hat{a}_n \rangle$$

(161)

which justifies the choice of the normalization in (12) and (21).

Since the initial quantum state was defined with respect to the ‘in’-operators $\hat{b}_m^\dagger$ and $\hat{b}_n$, we must express the ‘out’ operators $\hat{a}_m^\dagger$ and $\hat{a}_n$ in terms of $\hat{b}_m^\dagger$ and $\hat{b}_n$ by means of formula (24). Thus we arrive at the expression containing a combination of the mean values $\langle \hat{b}_n \hat{b}_k \rangle$, $\langle \hat{b}_n^\dagger \hat{b}_k^\dagger \rangle$, $\langle \hat{b}_n \hat{b}_k \rangle$ and $\langle \hat{b}_n^\dagger \hat{b}_k^\dagger \rangle$ calculated in the initial quantum state. For the initial vacuum state defined according to the relations $\hat{b}_n(0) = 0$, $n = 1, 2, \ldots$, only the nonzero mean values are $\langle \hat{b}_n \hat{b}_k^\dagger \rangle = \delta_{nk}$. Then (160) is transformed into the triple sum

$$W_0(x,t) = \pi \sum_{m,n,j=0}^{\infty} \frac{mj}{n} \Re \left\{ \frac{\cos[\pi(m-j)x] e^{i\pi(m-j)t'} \rho_{-m}^{(n)} \rho_{-j}^{(n)*}}{\rho_{-m}^{(n)} \rho_{-j}^{(n)*}} - \cos[\pi(m+j)x] e^{i\pi(m+j)t'} \rho_{-m}^{(n)} \rho_{-j}^{(n)*} \right\}.$$  

(162)

Evidently, $W_0(x,t) = 0$ for $t \leq 0$. For an arbitrary initial state the energy density can be written as a sum of the ‘vacuum’ and ‘nonvacuum’ contributions

$$W = W_0 + W_1, \quad W_1 = \pi \sum_{n,k=1}^{\infty} \frac{1}{\sqrt{nk}} \Re \left[ \langle \hat{b}_n \hat{b}_k \rangle B^{(nk)} + \langle \hat{b}_n^\dagger \hat{b}_k^\dagger \rangle \bar{B}^{(nk)} \right]$$

(163)
Making the change of the summation index \( j \rightarrow -j \) in the first term of (162) we can write
\[
W_0(x,t) = -\pi \text{Re} \sum_{n = 1}^{\infty} \sum_{m = 1}^{\infty} \sum_{j = -\infty}^{\infty} \frac{mj}{n} \cos[\pi(m + j)x] e^{i\pi(m+j)x^2} \rho_m \rho_j.
\]

Similarly, changing the indices \( m \rightarrow -m \) or \( j \rightarrow -j \) in (164) and (165) we can reduce 4 sums with apparently different summands and the indices running from 1 to \( \infty \) to the unified sums whose two indices run from \( -\infty \) to \( \infty \):
\[
B^{(nk)} = \sum_{m,j=1}^{\infty} m j \cos[\pi(m + j)x] e^{-i\pi(m+j)x^2} \rho_m \rho_j
\]
\[
\tilde{B}^{(nk)} = \sum_{m,j=1}^{\infty} m j \cos[\pi(m - j)x] e^{i\pi(m-j)x^2} \rho_m \rho_j
\]

Now, replacing the cosine function by the sum of two imaginary exponentials we see that \( W(x,t) \) is actually the sum of two identical functions of the light-cone variables:
\[
W(x,t) = \frac{\pi}{2} [F(u;\tau) + F(v;\tau)], \quad u = t' + x, \quad v = t' - x,
\]
where
\[
F = F_0 + \sum_{n,k=1}^{\infty} \frac{1}{\sqrt{n}k} \text{Re} \left[ \langle \tilde{b}_n \tilde{b}_k \rangle F^{(nk)} + \langle \tilde{b}_n \tilde{b}_k \rangle \tilde{F}^{(nk)} \right]
\]
\[
F_0(u;\tau) = -\text{Re} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{j=-\infty}^{\infty} \frac{mj}{n} e^{i\pi(m+j)u} \rho_m^*(\tau) \rho_j(\tau)
\]
\[
F^{(nk)}(u;\tau) = \sum_{m=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} m j e^{-i\pi(m+j)u} \rho_m^*(\tau) \rho_j(\tau)
\]
\[
\tilde{F}^{(nk)}(u;\tau) = \sum_{m=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} m j e^{i\pi(m-j)u} \rho_m^*(\tau) \rho_j(\tau)
\]

The extra argument \( \tau \) in the above expressions is introduced in order to emphasize that the energy density depends not only on the value of the current time variable \( t \) (which must satisfy the condition \( t > (1+\delta)T \),
but also on the moment of time \( T \) when the wall stopped to move. It is worth mentioning that the variables \( t \) and \( \tau \) are independent, as well as \( u \) and \( \tau \) or \( v \) and \( \tau \).

Evidently, the double sums \((169)\) and \((170)\) are factorized to the products of independent sums over \( m \) and \( j \):

\[
F^{(nk)}(u; \tau) = G^{(n)}(u; \tau) G^{(k)}(u; \tau), \quad \hat{F}^{(nk)}(u; \tau) = G^{(n)*}(u; \tau) G^{(k)}(u; \tau),
\]

\[
G^{(n)}(u; \tau) = \sum_{m=-\infty}^{\infty} me^{-i\mu u} \rho_m^{(n)}(\tau) = \frac{i}{\pi} \frac{\partial}{\partial u} \sum_{m=-\infty}^{\infty} e^{-i\mu u} \rho_m^{(n)}(\tau)
\]

(171)

The last sum in \((171)\) can be easily expressed in terms of the generating function \((74)\) if one writes \( n = j + kp, \)

\( m = j + lp \) and \( z = \exp(-i\pi pu). \) Thus we obtain

\[
G^{(n)}(u; \tau) = \left. \frac{nz [zg^{(pr)} + \sigma S^{(pr)}]^{n/p-1}}{[g^{(pr)} + z\sigma S^{(pr)}]^{n/p+1}} \right|_{z = \exp(-i\pi pu)} = nf^{1/2} \Lambda^{n/p},
\]

(172)

where

\[
f(u; \kappa) = |g^{(pr)} + z\sigma S^{(pr)}|^{-4} = \frac{(1 - \kappa^2)^2}{[1 + \kappa^2 + 2\kappa \cos(p\pi u - \varphi)]^2},
\]

(173)

\[
\Lambda = \frac{zg^{(pr)} + \sigma S^{(pr)}}{g^{(pr)} + z\sigma S^{(pr)}} = e^{i(2\varphi - p\pi u)} \frac{1 + \sigma \exp[i(p\pi u - \varphi)]}{1 + \sigma \exp[i(\varphi - p\pi u)]},
\]

(174)

\[
\kappa = \frac{\sqrt{1 + S^2(pr)}}{\gamma}, \quad \exp(i\varphi) = \sqrt{1 - \gamma^2} \kappa^2 + i\gamma \kappa.
\]

In the ‘vacuum’ contribution \((168)\) we have some asymmetry between the indices \( m \) and \( j \), since \( m \) runs from 1 to \( \infty \), whereas \( j \) runs from \( -\infty \) to \( \infty \). This asymmetry can be eliminated if one differentiates both sides of equation \((168)\) with respect to the independent variable \( \tau \) at a fixed value of \( u \) and performs the summation over the superscript \( n \) with the aid of the recurrence relations \((72)\) and \((88)\). It is easy to verify that all the summands with \( n \geq p \) are canceled, so the infinite series over \( n \) can be reduced to the finite sum from 1 to \( (p - 1)\):

\[
\frac{\partial F_0(u; \tau)}{\partial \tau} = -\sigma \text{Re} \sum_{n=1}^{p-1} \sum_{m=1}^{\infty} \sum_{j=-\infty}^{\infty} mj e^{i\pi(m+j)u} \quad \times \left[ \rho_m^{(n)}(\tau) \rho_j^{(p-n)*}(\tau) + \rho_j^{(p-n)}(\tau) \rho_m^{(n)*}(\tau) \right]
\]

(175)

Making the change of summation indices \( m \rightarrow -m, j \rightarrow -j, n \rightarrow p - n \) in the first product inside the square brackets one can reduce two sums in the right-hand side of \((175)\) to the single series where both the indices \( m \) and \( j \) run from \(-\infty\) to \( \infty \). Moreover, the sums over \( m \) and \( j \) become completely independent, giving rise to the equation

\[
\frac{\partial F_0(u; \tau)}{\partial \tau} = -\sigma \text{Re} \sum_{n=1}^{p-1} G^{(n)}(u; \tau) G^{(p-n)}(u; \tau),
\]

(176)

where \( G^{(n)}(u; \tau) \) is given by \((171)\). Due to \((172)\) the sum in the right-hand side of \((176)\) is reduced to the sum \( \sum_{n=1}^{p-1} n(p-n) = \frac{1}{2} p(p^2 - 1) \). Introducing the variable \( \eta = \exp(2a\rho\tau) \) we obtain the explicit expression

\[
\frac{\partial F_0(u; \eta)}{\partial \eta} = -\frac{(p^2 - 1)a^4 \eta [\eta^2(1 + \alpha + \beta) + \alpha - \beta - 1]}{12 [p^2(1 + \alpha + \beta) - 2\eta(\gamma^2 + \beta) + 1 + \beta - \alpha]^3},
\]

(177)

where \( \alpha = \sigma a \cos(p\pi u) \) and \( \beta = \sigma \gamma \sin(p\pi u) \). Integrating \((177)\) with the initial condition \( F_0 = 0 \) at \( \tau = 0 \)

(178)
where function $f(u;\kappa)$ is given by (173). Finally we obtain the following expression for the function $F(u;\tau)$ defined by equation (166):

$$F = -\mathcal{B} + f(u;\tau) \left\{ \mathcal{B} + \sum_{n,k=1}^{\infty} \sqrt{n}k \text{Re} \left[ (\hat{\phi}_n^* \hat{b}_k) \Lambda^{n+k} + (\hat{b}_n^\dagger \hat{b}_k) \Lambda^{n-k} \right] \right\}. \quad (179)$$

In the special case of the initial states whose density matrix is diagonal in the Fock basis, so that $F_{n\kappa} = \nu_n \delta_{nk}$ and $F_{n\kappa}^* = 0$ (for example, the Fock or thermal states; $\nu_n$ is the mean number of quanta in the $n$th mode), the sum in (179) is proportional to the initial total energy $\mathcal{E}_0$ in all the modes (above the Casimir level):

$$F^{(diag)}(u;\tau) = -\mathcal{B} + f(u;\tau) [\mathcal{B} + N_0], \quad (180)$$

$$N_0 = \sum_{n=1}^{\infty} \nu_n = \mathcal{E}_0 / \pi \equiv \mathcal{E}_0 / (\hbar \omega_1).$$

### 7.2 Packet formation

Now let us analyse the expressions for the energy density obtained above. For the initial vacuum state we see immediately from equations (173) and (178) that in the generic case the function $W_0(x,t)$ with the fixed value of the ‘fast time’ $t$ has $p$ peaks in the interval $0 \leq x \leq 1$, whose positions are determined by the equations $\sigma \cos(p\pi u - \varphi) = -1$ and $\sigma \cos(p\pi v - \varphi) = -1$. Obviously, for $t > T$ the energy density is a periodic function of the time variable $t$, with the period $\Delta t = 1$ if $p$ is an even number and $\Delta t = 2$ if $p$ is odd.

For the even values of the resonance multiplicity $p$ we have $p/2$ peaks moving (with the light speed) in the positive direction and $p/2$ peaks moving in the negative direction. If $p$ is odd, then the numbers of peaks of each kind differ by 1. All the peaks have the same height

$$W_{\text{max}}^{(\text{vac})} = 2\pi \mathcal{B} \kappa / (\kappa - 1)^2 = \frac{\pi}{2} \mathcal{B} \left( e^{4\mathcal{B}\tau} - 1 \right) \quad (181)$$

(in this section the expressions containing $\tau$ are related to the special case of the strict resonance $\gamma = 0$), excepting some distinguished instants of time when two peaks moving in the opposite directions merge, forming a peak with double the height.

If $\kappa \rightarrow 1$ (i.e. $\tau > 1$ and $\gamma < 1$), then the energy density can be approximated in the vicinity of each peak by the Lorentz-like distribution

$$W_{\text{max}}^{(\text{vac})}(\delta x) = \frac{W_{\text{max}}^{(\text{vac})}}{\left[ 1 + (2\delta x / \Delta_{1/4})^2 \right]^2}, \quad (182)$$

where the width

$$\Delta_{1/4} = \frac{2}{p\pi} \frac{1 - \kappa}{\sqrt{\kappa}} \approx \frac{4}{p\pi} e^{-2p\tau}$$

of each peak is defined as the double distance between the position of the maximum and the point where the energy density decreases 4 times. One can introduce also the ‘energy width’ of each peak by means of the relation $W_{\text{max}} \Delta_E = \mathcal{E}(\tau)/p$. For $\kappa \rightarrow 1$ we obtain

$$\Delta_E \approx (1 - \kappa) / (2p\pi) \approx (\pi p)^{-1} e^{-2p\tau}.$$  

Except for narrow regions of the length $\Delta_+ \approx (\pi p)^{-1} \sqrt{1 - \kappa}$ of $\mathcal{E}(\tau) \approx \sqrt{2(\pi p)^{-1} e^{-2p\tau}}$ nearby the peaks the ‘dynamical’ energy density is less than its initial vacuum value, in agreement with the results of [112, 164, 165, 168] obtained in the framework of different approaches. The minimum values of $W_0$ far off the peaks are given by (taking into account the contributions of both the functions $F_0(u)$ and $F_0(v)$)

$$W_{\text{min}} = -4\pi \mathcal{B} \frac{\kappa}{(\kappa + 1)^2} = \pi \mathcal{B} \left( e^{-4\mathcal{B}\tau} - 1 \right). \quad (183)$$
If $\kappa \to 1$, $W_{\text{min}} \to -\pi (p^2 - 1)/24$. Adding to this expression the initial Casimir energy \cite{153} we obtain the total asymptotical minimum value (cf. \cite{164})

$$W^{(\alpha s)}_{\text{min}} = -\pi p^2/24. \quad (184)$$

For an arbitrary initial state the energy density has, besides the ‘vacuum’ part, the additional terms given in equation \cite{174}. Since these terms are proportional to the same functions $f(u; \kappa)$ or $f(v; \kappa)$ which determine the structure of the ‘vacuum’ part, the positions of the peaks are not changed (remember that $|\alpha| = 1$).

For the initial states with diagonal density matrices in the Fock basis (in particular, for the thermal states) all the peaks still have equal heights, increased by the quantity $\Delta W_{\text{max}} = 4 \mathcal{C}_0 (1 + \kappa) / (1 - \kappa)^2$, compared with the vacuum case. However, the asymptotical minimal value of the energy density at $\kappa \to 1$ does not depend on the initial state, being given by formula \cite{184} in all the cases.

If the initial density matrix in the Fock basis has nonzero off-diagonal elements (as happens, in particular, for any pure state different from the Fock one, e.g., for the coherent states), different terms in the sum \cite{174} can interfere. As a consequence, the peaks acquire some kind of ‘fine structure’. For example, if only the first mode was excited initially in the coherent state $|\alpha\rangle$, $\alpha = |\alpha| \exp(i\psi)$, then for $p = 2$ and $\gamma = 0$ (the strict resonance) we have

$$\Delta W \equiv W - W^{(\text{vac})} = \pi |\alpha|^2 \frac{(1 - \kappa^2)^2 \left[ \kappa \sin(z + \psi) + \sin(z - \psi) \right]^2}{\left[ (1 - \kappa^2 + 4 \kappa \sin^2 z) \right]^3}$$

where $z \equiv \pi (u - u_*)$ and $u_*$ is the position of the ‘vacuum’ peak determined above. If $\psi = \pi/2$, then we have the high maximum $\Delta W_{\text{max}}^{(\text{vac})} = \pi |\alpha|^2 (1 + \kappa)^2 / (1 - \kappa)^2$ at $z = 0$. However, if $\psi = 0$, then instead of a maximum we have the minimum $\Delta W = 0$ at the same point $z = 0$, and the peak is split in two symmetric humps with equal maximal heights $\Delta W_{\text{max}}^{0} = (1 + \kappa)^2 / 27 \kappa \Delta W_{\pi/2}^{\text{max}}$ located at the points $z = \pm (1 - \kappa) / \sqrt{8} \kappa$. In the intermediate case $0 < \psi < \pi/2$ asymmetric forms of the peaks are observed. If $p > 2$ or several modes were excited initially, the interference between different terms in \cite{174} can result in different heights of the peaks and more complicated ‘fine structures’ (provided $(b_n, b_k) \neq 0$ for some $n$ and $k$).

If the detuning $\gamma$ is different from zero, then some deformations of the form of peaks are observed, although the maximal heights are still of the same order of magnitude as in the case $\gamma = 0$, as far as $\gamma \leq 1$. But if the detuning exceeds the critical value $\gamma = 1$, the energy becomes an oscillating function of the ‘slow time’ $\tau$, the amplitude of oscillations being proportional approximately to $(\gamma^2 - 1)^{-1}$ \cite{191, 192}. The peaks become rather wide and low, since the parameter $\kappa$ is limited by the inequality $\kappa \leq \gamma^{-1}$ if $\gamma > 1$. The illustrations can be found in \cite{193}.

Since the components of the energy-momentum tensor $T_{00}$ and $T_{11}$ are given by similar expressions in the case of single space dimension, the force acting on each wall has the same time dependence as the energy density at the points $x = 0$ and $x = 1$. For the most part of time during the period of field oscillations $2L_0/c$ (where $L_0$ is the distance between the walls at rest) this force is negative, being less than the static Casimir force, with the maximal amplification coefficient $\kappa p^2$. However, the average value of the force over the period is positive due to the creation of real photons inside the cavity.

If the walls possess some small transmission coefficient, then a small part of radiation accumulated inside the cavity can leave it. In this case one could observe sharp pulses of radiation outside the cavity \cite{210}, whose amplitudes must be proportional to the heights of the peaks inside the cavity multiplied by the small transmission coefficient. The intensity of these pulses can be significantly increased, if the initial state is different from vacuum and possesses sufficient energy, like thermal states \cite{210, 211} or coherent states. However, to describe the form of the pulses exactly it is necessary to develop a more general theory which would take into account the boundary conditions corresponding to the partially transmitting walls (because the nonzero transmission coefficient can change significantly the pulse shape, just as the nonzero detuning deformed the form of packets in the examples considered above).

### 7.3 Total energy

The total energy \cite{164} of the field inside the cavity (above the initial Casimir level) can be obtained by integrating the density $W(x)$ \cite{164} over $x$. The contribution of the vacuum (function $F_0$ in \cite{167}) and
‘diagonal’ terms (given by the partial sum in (167) over \( n = k \)) can be calculated with the aid of the formula

\[
\int_0^\pi \frac{dx}{(a + b \cos x)^2} = \frac{\pi a}{(a^2 - b^2)^{1/2}}.
\]

To find the contribution of ‘nondiagonal’ terms \((n \neq k)\) it is convenient to replace the integration over \( x \) by the integration in the complex \( z \)-plane \((z = \exp[-i\pi p\mu] \text{ or } z = \exp[-i\pi pv])\) over the circle \(|z| = 1\). One can check that this circle is passed \( p \) times when \( x \) goes from 0 to 1 (if one takes into account both the ‘\( u \)'- and ‘\( v \)'-contributions). It turns out that the integrals of the ‘non-diagonal’ terms are different from zero if and only if the corresponding integrands in the \( z \)-plane have simple poles inside the circle \(|z| = 1\). This happens only when \( k + n = p \) in the first term inside the square brackets in (179) and \( k - n = p \) in the second term inside the same brackets.

Finally we obtain a simple expression

\[
E(\tau) = E_0 + 2S^2(p\tau) \left[ E_0 + \pi B + \frac{\gamma\sigma}{2} \text{Im}(\hat{G}) \right] - \frac{\sigma}{2} S(2p\tau)\text{Re}(\hat{G}),
\]

where

\[
\hat{G} = 2\pi \sum_{n=1}^{\infty} \sqrt{n(n+p)}(\hat{b}_n^\dagger \hat{b}_{n+p}) + \pi \sum_{n=1}^{p-1} \sqrt{n(p-n)}(\hat{b}_n \hat{b}_{p-n})
\]

(186)

(if \( p = 1 \), the last sum in (186) should be replaced by zero). Formula (185) was found for the first time in a different way in [191]. In particular, for the initial vacuum state of field we have

\[
E^{(\text{vac})}(\tau) = \frac{p^2 - 1}{12a^2} \sinh^2(p\tau).
\]

The total energy increases exponentially at \( \tau \to \infty \), provided \( \gamma < 1 \). In the special case \( \gamma = 0 \) such asymptotical behaviour of the total energy was obtained also in the frameworks of other approaches in [164, 166, 177]. Here we have found the exact dependence of the total energy on time in the whole interval \( 0 \leq \tau < \infty \), as well as a nontrivial dependence on the initial state of field, which is contained in the constant parameter \( \hat{G} \). This parameter is equal to zero for initial Fock or thermal states of the field. However, in a generic case \( \hat{G} \neq 0 \), and it can affect significantly the total energy, if \( E(0) \gg 1 \). Consider, for example, the case \( p = 2 \). If initially the first mode \((n = 1)\) was in the coherent state \(|\alpha\rangle\) with \( \alpha = |\alpha|e^{i\phi} \), \(|\alpha| \gg 1\), and all other modes were not excited, then \( E(0) = |\alpha|^2 \), \( \hat{G} = \alpha^2 \), so for \( \tau \gg 1 \) and \( \gamma = 0 \) (exact resonance) we have \( E(\tau \gg 1) \approx \frac{1}{4}|\alpha|^2e^{4\pi} [2 - \cos(2\phi)] \). The maximal value of the energy in this case is three times bigger than the minimal one, depending on the phase \( \phi \).

According to (185), the initial stage of the evolution does not depend on the detuning parameter \( \gamma \) for all states which yield \( \text{Im}(\hat{G}) = 0 \), since at \( \tau \to 0 \) one has

\[
E(\tau) \approx E(0) - \sigma \text{Re}(\hat{G})\tau r + 2 \left[ E(0) + \frac{p^2 - 1}{24} + \frac{\gamma\sigma}{2} \text{Im}(\hat{G}) \right] (\tau r)^2
\]

(188)

Formula (188) is exact in the case of \( \gamma = 1 \).

If \( \gamma > 1 \), then one should replace each function \( \sinh(ax)/a \) in (185) by its trigonometrical counterpart \( \sin(\tilde{a}x)/\tilde{a} \); see Eq. [24]. In this case the total energy oscillates in time with the period \( \pi/|p\tilde{a}| \), returning to the initial value at the end of each period. For a large detuning \( \gamma \gg 1 \) the amplitude of oscillations decreases as \( \gamma^{-1} \) if \( \text{Re}\hat{G} \neq 0 \) and as \( \gamma^{-2} \) otherwise.

Note that the total “vacuum” and “nonvacuum” energies increase exponentially with time, if \( \gamma < 1 \) and \( \tau > 1 \), whereas the total number of photons increases only as \( \tau^2 \) and \( \tau \), respectively, under the same conditions. The origin of such a great difference in the behaviours of the total energy and the total number of photons becomes clear, if one looks at the asymptotical formulae (103)-(104). They show that the rate of photon generation in the \( m \)th completely excited mode decreases approximately as \( \tilde{N}_m \sim 1/m \) (excepting the modes whose numbers are multiples of \( p \)), so the stationary rate of the energy generation \( \dot{E}_m = m\tilde{N}_m \) asymptotically almost does not depend on \( m \). In turn, the number of the effectively excited modes increases in time exponentially. These two factors lead to the exponential growth of the total energy (see also [163] in the special case \( \gamma = 0 \)).
8 Three-dimensional nondegenerate cavity

8.1 Empty cavity

Now let us proceed to the three-dimensional case. For definiteness we choose a rectangular cavity with dimensions $L_x$, $L_y$, $L_z$ (briefly designated by symbol $\{L\}$). If these dimensions do not depend on time, each field mode is determined by three integers $m, n, l$, responsible for the eigenfrequency

$$\omega_{mnl} = \pi \left[ \frac{(m/L_x)^2 + (n/L_y)^2 + (l/L_z)^2}{(m/L_x)^2 + (n/L_y)^2 + (l/L_z)^2} \right]^{1/2},$$

(189)

and by two orthogonal directions of polarization. In order to simplify the exposition and to get rid of extra unessential indices, let us consider the case when $L_z \ll L_x \sim L_y$. Then the frequencies with $l \neq 0$ are much greater than those with $l = 0$. It is clear that the interaction between low- and high-frequency modes in the nonstationary case is weak. Consequently, studying the excitation of the lowest modes we may confine ourselves to the case of $l = 0$. Then the only possible polarization of the vector potential is along $z$-axis, so the low-frequency part of the Heisenberg field operator at $t < 0$ reads

$$\hat{A}_z(x, y, t < 0) = \sum_n (2\pi/\omega_n)^{1/2} \psi_n(x, y|\{L\}) \left[ \hat{b}_n e^{-i\omega_n t} + \hat{b}_n^\dagger e^{i\omega_n t} \right].$$

(190)

The difference from the similar expression (12) is that now the suffix $n$ is replaced by its "vector" counterpart $\mathbf{n} = (m, n)$, and the function $\psi_n(x, y|\{L\})$ depends on two space coordinates:

$$\psi_n(x, y|\{L\}) = 2 \left( L_x L_y L_z \right)^{-1/2} \sin \frac{m\pi x}{L_x} \sin \frac{n\pi y}{L_y}.$$

The coefficients in Eq. (190) are chosen again in correspondence with the standard form of the field Hamiltonian (13).

Now let the dimension $L_x$ to depend on time according to the given law $L(t)$. To satisfy the boundary conditions

$$A_z|_{x=0} = A_z|_{x=L(t)} = A_z|_{y=0} = A_z|_{y=L_y} = 0$$

we write the field operator at $t > 0$ in the same functional form (190), but with the time-dependent parameter $L(t)$:

$$\hat{A}_z(x, y, t) = 2\sqrt{\pi} \sum_n \psi_n(x, y|L(t), L_y) \hat{Q}_n(t).$$

(191)

In the stationary case the operators $\hat{Q}_n(t)$ coincide with the (coordinate) quadrature components of the field mode operators. Putting (191) into the wave equation

$$\partial^2 A_z/\partial t^2 - \Delta A_z = 0$$

we arrive at the equation looking just as Eq. (18):

$$\hat{Q}_k^{(n)} + \omega_k^2(t) \hat{Q}_k^{(n)} = 2\lambda(t) \sum_j g_{kj} \hat{Q}_j^{(n)} + \hat{\lambda}(t) \sum_j g_{kj} \hat{Q}_j^{(n)}$$

$$+ \lambda^2(t) \sum_j g_{jk} g_{ji} \hat{Q}_i^{(n)}.$$  

(192)

Now $\lambda(t) = \dot{L}(t)/L(t)$, all the indices are "two-vectors", the frequencies are given by Eq. (189) with $l = 0$ and $L(t)$ instead of $L_x$, and the constant numerical coefficients $g_{kj}$ are given by the integrals

$$g_{kj} = L \int_0^{L_x} dx \int_0^{L_y} dy \int_0^{L_z} dz \psi_j(r|L) \frac{\partial \psi_k(r|L)}{\partial L}.$$  

The explicit form of "two-dimensional" coefficients $g_{kj}$ is more complicated than a simple formula (13). However, these coefficients remain antisymmetrical: $g_{kj} = -g_{jk}$, due to the normalization of functions $\psi_k$, 

$$\int_0^{\{L\}} dr \psi_m(r) \psi_n = \delta_{mn},$$  

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and due to zero boundary conditions at $x = L$. (Moreover, they do not depend on the cavity dimensions.)

Although we use the same notation as in the 1D case, the operators $\hat{Q}_n(t)$ in Eq. (191) differ from their analogs in a similar decomposition (17). Now $\hat{Q}_n(t)$ means the Hermitian operator coinciding with the ("coordinate") quadrature component of the field mode operator.

Suppose for simplicity that the wall oscillates at twice the eigenfrequency of some unperturbed mode,

$$L(t) = L_0 \left[ 1 - \epsilon_L \cos(2\omega_m t) \right], \quad |\epsilon_L| \ll 1$$

and let us look for the solution to Eq. (192) in the form

$$Q_k(t) = \xi_k(\epsilon_L t) \exp (-i\omega_k t) + \eta_k(\epsilon_L t) \exp (i\omega_k t)$$

(we have omitted "hats" over operators). Contrary to the one-dimensional case, now all the terms on the right-hand side of Eq. (192) disappear after averaging over fast oscillations, since the spectrum $\omega_j$ is not equidistant. Indeed, the first and the second sums on the right-hand side do not contain functions $Q_k$ due to the antisymmetry of coefficients $g_{kj}$, whereas the last sum is proportional to $\lambda^2 \sim \epsilon_L^2$. Consequently, after the multiplication by the proper exponential functions, the right-hand side will consist of the terms containing the factors such as $\exp \left( i \pm \omega_j \pm \omega_k \pm 2\omega_m t \right)$ with $j \neq k$. After averaging all these terms turn into zero. (Strictly speaking, the frequency spectrum (189) contains the "equidistant subset", corresponding to the indices $m, n, l$ multiplied by the same integral factors. However, this fact does not change the conclusion, because the "coupling constants" $g_{kj}$ between such modes are equal to zero.)

Consequently, in the resonance case the field problem is reduced to that of a one-dimensional parametric oscillator with the time dependence of the eigenfrequency in the form

$$\omega(t) = \omega_0 \left[ 1 + 2\tilde{\epsilon} \cos(2\omega_0 t) \right],$$

$\omega_0 \equiv \omega_{mn}$ being the unperturbed eigenfrequency of the resonance mode. Here the frequency modulation depth $\tilde{\epsilon}$ is related to the cavity length modulation depth $\epsilon_L$ as follows:

$$\tilde{\epsilon} = \frac{1}{2} \epsilon_L \left[ 1 + (nL_0/mL)^2 \right]^{-1/2}.$$ We use the notation $\tilde{\epsilon}$ in order not to confuse the dimensionless modulation parameters in the one-dimensional and three-dimensional cases.

At this point we may abandon the Heisenberg picture and proceed to the Schrödinger representation. Of course, both representations are equivalent, as soon as the field problem has been reduced to studying a finite-dimensional quantum system. However, the most of numerous investigations of the time-dependent quantum oscillator, since Husimi’s paper [285], were performed in the Schrödinger picture. So it is natural to use the known results. According to [279, 285, 286], all the characteristics of the quantum oscillator are determined completely by the complex solution of the classical oscillator equation of motion

$$\ddot{u} + \omega^2(t)u = 0,$$

satisfying the normalization condition

$$\dddot{u} - \ddot{u} = 2i.$$ Let us assume that function $\omega(t)$ takes the constant value $\omega_0$ at $t \leq 0$ and at $t > t_f > 0$. Moreover, it is convenient to choose the initial conditions for $u$-function as follows:

$$u(0) = 1/\sqrt{\omega_0}, \quad \dot{u}(0) = i\sqrt{\omega_0}.$$ Then the quantum mechanical average number of photons created from the ground state due to the time dependence of the frequency in the interval of time $0 < t < t_f$ is given by the formula

$$\langle n \rangle = \frac{1}{4\omega_0} \left( |\dot{u}|^2 + \omega_0^2 |u|^2 \right) - \frac{1}{2}.$$ Looking for the solution of Eq. (195) in the parametric resonance case (194) in the form

$$u(t) = \frac{1}{\sqrt{\omega_0}} \left[ \xi(t) e^{i\omega_0 t} + \eta(t) e^{-i\omega_0 t} \right]$$

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(the opposite signs in the arguments of the exponential functions in Eqs. (193) and (199) are due to the different representations: the former equation is written in the Heisenberg picture, while the latter – in the Schrödinger one) and using the method of averaging over fast oscillations, one can easily obtain the first order differential equations for the amplitudes (provided that $|\tilde{\varepsilon}| \ll 1$),

$$\dot{\tilde{\xi}} = i\omega_0 \tilde{\varepsilon} \eta, \quad \dot{\eta} = -i\omega_0 \tilde{\varepsilon} \xi.$$  \hspace{1cm} (200)

Their solutions satisfying initial conditions (197) (up to the terms of the order of $\tilde{\varepsilon}$) read (214, 287, 290)

$$\xi(t) = \cosh(\omega_0 \tilde{\varepsilon} t), \quad \eta(t) = -i \sinh(\omega_0 \tilde{\varepsilon} t).$$  \hspace{1cm} (201)

Due to Eqs. (198), (199) and (201), the average number of photons (and the total energy in the cavity) grows in time exponentially:

$$\langle n \rangle = |\eta|^2 = \sinh^2(\omega_0 \tilde{\varepsilon} t).$$  \hspace{1cm} (202)

It is well known that the initial vacuum state of the oscillator is transformed into the squeezed vacuum state, if the frequency depends on time (see, e.g., reviews [223, 279] and numerous references therein). Moreover, looking at Eq. (202) one can immediately recognize the combination $\omega_0 \tilde{\varepsilon}$ as the so called squeezing parameter. Therefore the probability to register $n$ photons exhibits typical oscillations:

$$P_{2m} = \frac{(\tanh(\omega_0 \tilde{\varepsilon} t))^{2m}}{\cosh(\omega_0 \tilde{\varepsilon} t)} \frac{(2m)!}{(2^m m!)^2}, \quad P_{2m+1} = 0.$$  \hspace{1cm} (203)

This distribution possesses the photon number variance $\sigma_n = \frac{1}{2} \sinh^2(2\omega_0 \tilde{\varepsilon} t)$. Similar formulas for the amount of photons created in a cavity filled with a medium with a time-dependent dielectric permeability (and stationary boundaries) were found in [222]. The quadrature variances change in time as (now $\tau = \tilde{\varepsilon}_0 \tilde{\varepsilon} t$)

$$U = \frac{1}{2} e^{-2\tau}, \quad V = \frac{1}{2} e^{2\tau}.$$  \hspace{1cm} (204)

An unlimited squeezing can be achieved in this case due to the absence of interaction with other modes.

### 8.2 Interaction with a probe oscillator inside the cavity

The situation changes drastically, if the field mode can interact with some detector placed inside the cavity. Following [188, 189] we demonstrate the effect in the framework of a simplified model, when a harmonic oscillator tuned to the frequency of the resonant mode is placed at the point of maximum of the amplitude mode function $\psi_{mn}(x,y|\{L\})$ in the 3D rectangular cavity.

Assuming the interaction between the oscillator and the field to be described by means of the standard minimal coupling term $-(e/mc)pA$, we arrive at the following two-dimensional Hamiltonian governing the evolution of the coupled system “field oscillator + detector":

$$H = \frac{1}{2} \left[ P^2 + \omega_0^2 Q^2 + p^2 + \omega_0^2 q^2 - 4\omega_0 \kappa p Q \right].$$  \hspace{1cm} (205)

Here $P, Q$ are the quadrature components of the field oscillator, and $p, q$ are those of the probe oscillator. We neglect the interaction with nonresonant modes, since it is reasonable to suppose that under the resonance conditions their contribution is not essential at $\tilde{\varepsilon} \ll 1$.

In general, the dimensionless coupling coefficient $\kappa$ must depend on time, due to the decomposition [191]. However, since this coefficient is small, its variations of the order of $\tilde{\varepsilon} \kappa$ can be neglected in comparison with the relative variation of the eigenfrequency $\delta \omega/\omega \sim \tilde{\varepsilon}$. So $\kappa$ is assumed to be constant.

Suppose that the lowest cavity mode is resonant. Then one can evaluate the dimensionless coupling constant as $\kappa \sim (e^2/2\pi mc^2L)^{1/2}$ (here we return to the dimensional variables). The maximum value of parameter $\tilde{\varepsilon}$ is (see the discussion in section 4) $\tilde{\varepsilon}_{max} \sim \delta_{max} v_s/2\pi c$, where $\delta_{max} \sim 0.01$ is the maximal possible relative deformation in the material of the wall, and $v_s \sim 5 \cdot 10^2$ m/s is the sound velocity inside the wall. Then the ratio $\tilde{\varepsilon}/\kappa$ cannot exceed the value $\delta_{max} \left(mv_s^2L/8\pi e^2\right)^{1/2} \sim 0.05$ for $L \sim 1$ cm and $m \sim$
the mass of electron (for these parameters $\kappa \sim 2 \cdot 10^{-7}$). Consequently, one may believe that in the real conditions $\tilde{\varepsilon}/\kappa \ll 1$.

In the time-independent case, $\omega(t) = \text{const} = \omega_0$, we have two eigenfrequencies

$$
\omega_{\pm} = \omega_0(1 \pm \kappa)
$$

(provided that $|\kappa| \ll 1$). Let us assume that the wall vibrates exactly at twice the lower frequency $\omega_-:

$$
\omega(t) = \omega_0 [1 + 2\tilde{\varepsilon}\cos(2\omega_- t)].
$$

Then the lower and upper modes practically do not interact in the limit of $\tilde{\varepsilon} \ll \kappa$.

The Schrödinger equation with Hamiltonian (205) can be solved in the framework of the general theory of multidimensional quantum systems with arbitrary quadratic Hamiltonians, first proposed in [278] and exposed in detail, e.g., in [279]. In particular, if both for the field and the probe oscillators were initially in their ground states,

$$
\psi(Q, q, 0) = \sqrt{\frac{1}{\pi}} \exp \left[ -\frac{1}{2} (Q^2 + q^2) \right],
$$

then the wave function of the coupled “field + probe oscillator” system at $t > 0$ can be written as [189]

$$
\psi(Q, q, t) = \sqrt{\frac{1}{\pi \cosh \mu}} \exp \left( -it - \frac{1}{2} [a(t)Q^2 + b(t)q^2 - 2c(t)qQ] \right),
$$

with the following coefficients:

$$
a(t) = 1 + \tanh \mu e^{-2i\varphi} - i\kappa e^{i\Phi} [\tanh \mu e^{-i\Phi} (1 + \tanh \mu \sin \Phi e^{i\varphi}) - \sin \varphi],
$$

$$
b(t) = 1 - \tanh \mu e^{-2i\varphi} - i\kappa e^{i\Phi} [\tanh \mu e^{-i\Phi} (1 - \tanh \mu \sin \Phi e^{i\varphi}) + \sin \varphi],
$$

$$
c(t) = \tanh \mu e^{-2i\varphi} + i\kappa \left[ 1 - \cos \varphi e^{-i\Phi} + i\tanh^2 \mu \sin \Phi e^{i(\varphi - 2\Phi)} \right].
$$

Here

$$
\Phi = (\omega_+ + \omega_-) t = 2\omega_0 t, \quad \varphi = (\omega_+ - \omega_-) t = 2\omega_0 \kappa t = \kappa \Phi,
$$

$$
\mu = \tilde{\varepsilon} \omega_0 t, \quad \Delta = 1 - \cosh \mu \cos \Phi.
$$

In all the formulas above, the terms of the order of $\kappa^2$ were neglected, as well as the terms proportional to $\tilde{\varepsilon}$ (excepting, of course, the arguments of the hyperbolic functions). Evidently, $\Phi \gg \varphi \gg \mu$. Hereafter we confine ourselves to the most interesting long-time limit case, when $\mu \gg 1$. Then all the terms proportional to $\kappa$ can be neglected, so one can write

$$
a(t) = 1 + i\chi, \quad b(t) = 1 - i\chi, \quad c(t) = \chi, \quad \chi = \tanh \mu e^{-2i\varphi}.
$$

Eq. (209) shows that the coupled system turns out in a two-mode squeezed state at $t > 0$. The properties of this state, as well as of any Gaussian state are determined completely by its covariance matrix

$$
M = \|M_{\alpha\beta}\| = \begin{pmatrix} M_{\pi\pi} & M_{\pi x} \\ M_{x\pi} & M_{xx} \end{pmatrix}, \quad M_{\alpha\beta} = \frac{1}{2} \langle \hat{z}_\alpha \hat{z}_\beta + \hat{\bar{z}}_\alpha \hat{\bar{z}}_\beta \rangle,
$$

where the 4-dimensional (in the present case) vector $z$ is defined as follows: $z = (\pi, \chi) = (P, p, Q, q)$ (evidently, $\langle z \rangle = 0$ in the case under study). Using the general formulas for multidimensional Gaussian states given in [279], we have obtained (15) the following explicit expressions for the two-dimensional blocks of matrix $M$ in the long-time limit $\mu \gg 1$:

$$
M_{\pi\pi} = \frac{1}{2} \cosh^2 \mu \begin{pmatrix} 1 + \tanh \mu \sin \phi & -\tanh \mu \cos \phi \\ -\tanh \mu \cos \phi & 1 - \tanh \mu \sin \phi \end{pmatrix},
$$

$$
M_{xx} = \frac{1}{2} \cosh^2 \mu \begin{pmatrix} 1 - \tanh \mu \sin \phi & \tanh \mu \cos \phi \\ \tanh \mu \cos \phi & 1 + \tanh \mu \sin \phi \end{pmatrix},
$$
\[ M_{xx} = \tilde{M}_{xx} = \frac{1}{4} \sinh(2\mu) \begin{pmatrix} -\cos \phi & -\tanh \mu - \sin \phi \\ \tanh \mu - \sin \phi & \cosh \phi \end{pmatrix}, \]

where \( \phi = 2\varphi_\mu \), and \( \tilde{M} \) means the transposed matrix. Consequently, there exists a strong correlation between the field and probe oscillators in the long-time limit. For instance, the correlation coefficient between the quadrature components reads

\[ r_{qQ} \equiv \frac{\langle qQ \rangle}{\sqrt{\langle q^2 \rangle \langle Q^2 \rangle}} = \frac{\sinh \mu \cos \phi}{\sqrt{1 + (\sinh \mu \cos \phi)^2}}. \]

(If \( \phi \approx \pi/2 \), this coefficient, as well as other analogous elements of the covariance matrix, does not turn exactly into zero; in such a special case \( r_{qQ} \approx \kappa \), due to neglected terms of the order of \( \kappa \).)

It is clear that the density matrix of the probe oscillator (which is obtained from the density matrix of the total system \( \rho(Q, q; Q', q') = \psi(Q, q) \psi^\ast(Q', q') \) by putting \( Q = Q' \) and integrating over \( Q \)) has also the Gaussian form. Its properties are determined completely by the reduced covariance matrix (accidentally, it coincides with \( M_{xx} \) when \( \omega_0 = 1 \))

\[ M_{pr} = \frac{1}{2} \cos^2 \mu \begin{pmatrix} 1 - \tanh \mu \sin \phi & \tanh \mu \cos \phi \\ \tanh \mu \cos \phi & 1 + \tanh \mu \sin \phi \end{pmatrix}. \] (210)

A similar matrix for the field oscillator can be obtained from Eq. (210) by means of changing the sign of parameter \( \mu \). As was shown in [28], the photon statistics in Gaussian one-mode states is determined completely by two invariants of the covariance matrix, according to the relation

\[ s \equiv 2\langle q^2 \rangle = T - \sqrt{T^2 - 4d}. \] (213)

Both subsystems have identical invariants:

\[ T = 4d = \cosh^2 \mu, \]

so for \( \mu \gg 1 \) they appear in highly mixed quantum states, with rather moderate degree of squeezing, which tends asymptotically to 50% (cf. the one-dimensional case described in subsection 6.3):

\[ s = e^{-\mu} \cosh \mu = \frac{1}{2} \left( 1 + e^{-2\mu} \right). \]

The average number of quanta in each subsystem equals

\[ \langle n \rangle = \frac{1}{2}(T - 1) = \frac{1}{2} \sinh^2 \mu, \]

i.e., twice less than in the case of an empty cavity. The variance of the number of quanta (photons) equals

\[ \sigma_n \equiv \langle n^2 \rangle - \langle n \rangle^2 = \frac{1}{4} \left( 2T^2 - 4d - 1 \right) = \frac{1}{4} \sinh^2 \mu \cosh(2\mu). \]
Mandel’s parameter (defined by Eq. (157)) turns out much greater than unity for \( \mu \gg 1 \), indicating that the photon statistics is highly super-Poissonian:
\[
Q \approx \sinh^2(\mu).
\]

The photon distribution function can be expressed in terms of the Legendre polynomials, according to the general formula (151):
\[
\mathcal{P}_n = \frac{2(iz)^n}{\sqrt{1 + 3 \cosh^2 \mu}} P_n(-iz),
\]
where
\[
z = \frac{\sinh \mu}{\sqrt{1 + 3 \cosh^2 \mu}}
\]
Actually the right-hand side of Eq. (214) is a polynomial of degree \( n \) with respect to the variable \( z^2 \), due to the recurrence relation
\[
n \mathcal{P}_n = z^2 [(2n - 1) \mathcal{P}_{n-1} + (n - 1) \mathcal{P}_{n-2}].
\]
If \( \mu \gg 1 \), then \( z^2 \approx 1/3 \). The behavior of the distribution function (214) was shown in [189]. Since the argument of the Legendre polynomial is pure imaginary, \( \mathcal{P}_n \) has no oscillations, in contradistinction to the vacuum squeezed state.

8.3 Interaction with a two-level detector

Another model of the detector, which has only two energy levels, was considered in [188]. The most significant features can be described, in the rotating wave approximation, in the framework of the following generalization of the Jaynes-Cummings Hamiltonian:
\[
H = a^\dagger a + \frac{1}{2} \Omega \sigma_z + \kappa (a \sigma_+ + a^\dagger \sigma_-) + \frac{\bar{\epsilon}}{2} \sin(\omega_w t) \left[ a^2 + (a^\dagger)^2 \right].
\]
The eigenfrequency of the unperturbed mode is assumed \( \omega_0 = 1 \), \( \Omega \) is the energy level difference of the detector, \( \kappa \) and \( \bar{\epsilon} \ll \kappa \) have the same meaning as above; \( a, a^\dagger \) and \( \sigma_+, \sigma_- \), \( \sigma_z \) are the standard photon and spin operators. The wave function of the system “field + detector” can be written as
\[
\psi(t) = \sum_{n=0}^{\infty} \left( c_n^-(t)|n, -\rangle + c_n^+(t)|n, +\rangle \right),
\]
with a clear meaning of the symbols. If \( \bar{\epsilon} = 0 \), then the known solution of the JC-model reads [29]
\[
c_0^-(t) = c, \quad c_n^{-(+)}(t) = a_n \cos \vartheta_n \exp \left[-itE_n^+\right] - b_n \sin \vartheta_n \exp \left[-itE_n^-\right],
\]
\[
E_n^\pm = n + \frac{1}{2} \pm \lambda_n, \quad \lambda_n = \left[ \frac{1}{4} (1 - \Omega)^2 + \kappa^2 (n + 1) \right]^{1/2},
\]
\[
\tan \vartheta_n = \left( \frac{2\lambda_n - 1 + \Omega}{2\lambda_n + 1 - \Omega} \right)^{1/2}.
\]
We suppose that initially the system was in the ground state with the only nonzero coefficient \( c_0^-(0) = 1 \), and that the frequency of wall’s vibrations is close to twice the frequency of the unperturbed mode: \( \omega_w = 2 - \nu \). Looking for the solution at \( \bar{\epsilon} \neq 0 \) in the same form (214)-(218), but with time-dependent coefficients, and neglecting the rapidly oscillating terms containing \( \exp(2it) \), we get the following equation for the coefficient \( c(t) \):
\[
\dot{c} = \frac{\sqrt{3}}{4} \bar{\epsilon} \left[ b_1 \sin \vartheta_1 \exp \left[ it \left( \lambda_1 - \nu \right) \right] - a_1 \cos \vartheta_1 \exp \left[ -it \left( \lambda_1 + \nu \right) \right] \right].
\]
Assuming \( \nu = \lambda_1 = \left[ \frac{1}{4}(1 - \Omega)^2 + 2\kappa^2 \right]^{1/2} \) and neglecting the terms oscillating with the frequencies of the order of \( \kappa \), one can check that the infinite system of equations for \( a_n \) and \( b_n \) is reduced to the following two equations:

\[
\dot{c} = \frac{\sqrt{2}}{4} \varepsilon \sin \vartheta_1 b_1, \quad \dot{b}_1 = -\frac{\sqrt{2}}{4} \varepsilon \sin \vartheta_1 c.
\]

Consequently, in the resonance case we have only three nonzero amplitudes:

\[
c_{0}^{(-)} = \cos(\alpha t), \quad c_{2}^{(-)} = \sin \vartheta_1 \sin(\alpha t) \exp \left[ -itE_{1}^{(-)} \right], \quad c_{1}^{(+)} = -\cos \vartheta_1 \sin(\alpha t) \exp \left[ -itE_{1}^{(-)} \right],
\]

where \( \alpha = \sin \vartheta_1 \varepsilon \sqrt{2}/4 \). No more than two photons can be created, and the probability of finding the detector in an excited state \( P^{(+)} \) is always less than \( 1/2 \). All the probabilities, in contradistinction to the first example, are periodically oscillating functions of time:

\[
P_1 = P^{(+)} = \cos^2 \vartheta_1 \sin^2(\alpha t), \quad P_2 = \sin^2 \vartheta_1 \sin^2(\alpha t).
\]

It is interesting that the upper level of the detector never can be populated with 100% probability, since \( \vartheta_n > 0 \) for all values of parameters. For \( \Omega = 1, \vartheta_n = \pi/4 \), and \( P_1 = P_2 = P^{(+)} = \frac{1}{4} \sin^2(\varepsilon t/4) \). Large detuning, \( 1 - \Omega \gg \kappa \), results in increasing \( P^{(+)} \), since \( \vartheta_1 \to 0 \). However, in such a case \( \alpha \to 0 \), as well, and the applicability of JC-model to the description of the interaction between the detector and field becomes questionable. Recently, a more general model was considered in [238].

### 9 Influence of damping

The complete theory of the field quantization in media with moving nonideal boundaries is not available at present. The field quantization in spatially inhomogeneous but nonabsorbing dielectrics was studied in [292, 293], and the same problem for nonabsorbing media with time dependent parameters was considered in [217, 219, 222]. The case of absorbing media was analysed in [294, 297]. The theory of the field quantization in leaky cavities was developed in [298, 300]. However, in all those studies the boundaries were fixed. Due to the complexity of the problem, only a few simplest models have been considered up to now in the case of moving walls.

For example, one can try, as the first step, to neglect coupling between different field modes inside the cavity. Such an approximation can be justified, e.g. for an adiabatic motion of the cavity walls, when the characteristic mechanical frequency, \( \omega_m \), is many orders of magnitude less than the electromagnetic field eigenfrequency, \( \omega_c \). However, no new photons can be created under the condition \( \omega_m \ll \omega_c \) (and the photon number distribution cannot be changed, as well), since the photon number operator is the adiabatic invariant in this case.

Fortunately, as was shown in the preceding section, the interaction between different field modes can be neglected also in the case of a three-dimensional cavity with a nonequidistant spectrum of the field eigenfrequencies, under the parametric resonance condition \( \omega_m \approx 2\omega_c \). In such a case, one can infer some quantitative information on the behavior of the field in the cavity, studying the problem of the parametrically excited oscillator with damping.

The influence of an environment on the parametric amplification was considered in detail, e.g., in [301], where an explicit coupling with a heat bath consisting of harmonic oscillators was introduced. More general models of the environment were studied, e.g., in the framework of the influence functional approach [302], mainly in connection with cosmological problems. It was shown in [299, 300, 303, 304] that the influence of the “modes of the universe” outside the cavity with fixed mirrors can be described effectively in the framework of the Heisenberg-Langevin equation of motion for the photon annihilation and creation operators \( \hat{a}, \hat{a}^\dagger \).

An equivalent description in the Schrödinger picture is achieved in the framework of the “standard master equation” [305, 310], whose simplest form reads

\[
\dot{\rho} = \frac{i}{\hbar} [\rho, \hat{H}] + \frac{\gamma}{2} \left[ 2\hat{a}\rho\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\rho - \rho\hat{a}^\dagger\hat{a} \right],
\]

(219)
where $\dot{\rho}$ is the statistical operator of the distinguished field mode, $\hat{H}$ is the Hamiltonian, and the damping coefficient $\gamma$ absorbs all the details of the loss mechanism: the transmissivity of the mirror, the coupling between the field and the atoms inside the wall, etc., so that it is proportional to the reciprocal of the dissipation time scale. It was assumed in [256, 257] that Eq. (214) can be used as well in the case of the moving mirrors, with the same value of the damping coefficient as in the case of the fixed boundaries. Following the same line as in [256, 257], we also assume that the time evolution of the mixed quantum state of the resonance field mode is governed by a linear master equation (although one cannot exclude a possibility that such an approach is oversimplified: see, e.g., [311, 312]). However, instead of using the operator equation like (214), we consider the most general linear equation of the Fokker–Planck type for the Wigner function $W(q, p, t)$ [277, 308, 313] ($q, p$ are the quadrature components of the field mode),

$$
\frac{\partial W}{\partial t} = \frac{\partial}{\partial q} (\gamma q \partial q W) + \frac{\partial}{\partial p} (\gamma p + \omega^2(t) q) W 
+ D_{qq} \frac{\partial^2 W}{\partial q^2} + D_{pp} \frac{\partial^2 W}{\partial p^2} + 2D_{qp} \frac{\partial^2 W}{\partial q \partial p} \tag{220}
$$

The coefficients $\gamma_i$ and $D_{ij} = D_{ji}$ depend on the concrete form of the microscopic interaction between the system involved and an environment [305–308, 314]. For example, the simplest models of the damped optical oscillator with a constant frequency $\omega_0$ yield the following set of coefficients [308, 313]:

$$
\begin{align*}
\gamma_p &= \gamma_q = \pi s(\omega_0)|g(\omega_0)|^2, \\
D_{pp} &= \omega_0^2 D_{qq} = \gamma_q \mathcal{E}_{eq}(\omega_0), \quad D_{pq} = 0, \\
\end{align*}
$$

where $s(\omega)$ is the density of states of the reservoir, $g(\omega_0)$ is the function describing the intensity of coupling between the distinguished oscillator and the reservoir degrees of freedom, and $\mathcal{E}_{eq}(\omega_0)$ is the equilibrium energy of the oscillator with frequency $\omega_0$ at temperature $T$,

$$
\mathcal{E}_{eq}(\omega_0) = \frac{1}{2} \hbar \omega_0 \coth \left( \frac{\hbar \omega_0}{2k_B T} \right). \tag{223}
$$

Choosing different couplings between the oscillator under study and the reservoir, one can obtain various other sets of the drift and diffusion coefficients [308, 314], but all of them must obey the constraint [277, 318–319]

$$
D_{pp} D_{qq} - D_{pq}^2 \geq \hbar^2 (\gamma_p + \gamma_q)^2 / 16, \tag{224}
$$

which guarantees an absence of nonphysical solutions violating the uncertainty relations and corresponding to nonpositively definite density matrices. Besides, the coefficients $D_{pp}$ and $D_{qq}$ must be positive. However, it will be shown in this section, that in the case of a weak damping, the evolution depends on two combinations of the damping and diffusion coefficients only,

$$
\gamma = \frac{1}{2} (\gamma_p + \gamma_q), \quad \mathcal{E}_* = \frac{1}{2\gamma} \left( D_{pp} + \omega_0^2 D_{qq} \right). \tag{225}
$$

Due to the fluctuation-dissipation theorem, the diffusion coefficients are proportional to the damping coefficients, therefore $\mathcal{E}_*$ does not depend on $\gamma$, at least up to small corrections of an order of $\gamma^2$. The physical meaning of the parameters $\gamma$ and $\mathcal{E}_*$ is elucidated below: $2\gamma$ is the reciprocal energy relaxation time of the cavity due to all possible mechanisms (a real dissipation in the walls and the leakage through the boundaries), i.e., $2\gamma = \omega_0/Q$, $Q$ being the cavity quality factor, whereas $\mathcal{E}_* = \mathcal{E}_{eq}(\omega_0)$.

Solutions to Eq. (220) with a constant frequency and different sets of constant diffusion coefficients were obtained by many authors; they were analysed in detail in [277, 308], where other references can be found. Since Eq. (220) looks like a two-dimensional Schrödinger equation with a quadratic (although nonhermitian) Hamiltonian, its propagator can be calculated in the framework of the method of quantum integrals of motion [278]. The explicit form of this propagator in the generic case of time dependent coefficients can be found in [277, 324]. Here we confine ourselves to calculating the second order statistical moments and the energy (the number of photons) of the field oscillator.
9.1 Evolution of the energy and the second order moments

The time dependence of the energy and the second order statistical moments (variances) of the field mode quadrature components, \( \sigma_{ab} \equiv \frac{1}{2} \langle ab + ba \rangle - \langle a \rangle \langle b \rangle \), is governed by the equations following from Eq. (220),

\[
\begin{align*}
\dot{\sigma}_{qq} &= 2\sigma_{pq} - 2\gamma_0 \sigma_{qq} + 2D_{qq}, \\
\dot{\sigma}_{pq} &= \sigma_{pp} - \omega^2(t) \sigma_{pq} - 2\gamma_0 \sigma_{pq} + 2D_{pq}, \\
\dot{\sigma}_{pp} &= -2\omega^2(t) \sigma_{pq} - 2\gamma_0 \sigma_{pp} + 2D_{pp}.
\end{align*}
\]

We assume that the oscillator eigenfrequency depends on time as

\[
\omega(t) = \omega_0 \left[ 1 + 2\tilde{\varepsilon} \sin(\Omega t) \right], \quad \Omega = 2(\omega_0 + \delta), \quad |\delta| \ll \omega_0, \quad |\tilde{\varepsilon}| \ll 1,
\]

where \( \omega_0 \) is the unperturbed field eigenfrequency, and \( \Omega \) is the frequency of the wall vibrations. Then one could suppose that the damping and diffusion coefficients must depend on time, as well. We argue, however, that in the case under study the coefficients \( \gamma_a \) and \( D_{ab} \) can be considered as time independent. For example, let us look at the expressions \( (221) \) and \( (222) \). In the case of a vibrating cavity, the time variable could enter the coefficients \( \gamma_a \) and \( D_{ab} \) through the coupling function \( g(\omega_0) \), which can depend on the variable length of the cavity \( L(t) = L_0 \left[ 1 + \xi_0 \tilde{\varepsilon} \sin(\Omega t) \right] \), \( \xi_0 \) being a numerical coefficient. Then one could expect a similar time dependence of the coefficients of the Fokker–Planck equations, \( \gamma(t) = \gamma_0 \left[ 1 + \xi_1 \tilde{\varepsilon} \sin(\Omega t) \right] \), \( D(t) = D_0 \left[ 1 + \xi_2 \tilde{\varepsilon} \sin(\Omega t) \right] \), \( \xi_1 \) and \( \xi_2 \) being some other numerical coefficients. One should remember, however, that the modulating parameter \( \tilde{\varepsilon} \) is very small under the realistic conditions: its absolute value cannot exceed \( 10^{-8} \). The set of equations \( (226)-(228) \) contains three small dimensionless parameters, \( \tilde{\varepsilon}, \delta/\omega_0 \), and \( \gamma_0/\omega_0 \), and we are interested in the weak damping case, when these parameters are of the same order of magnitude. Under this condition, the time dependent parts of the coefficients \( \gamma_a \) and \( D_{ab} \) are proportional to the products \( \tilde{\varepsilon} \gamma_a/\omega_0 \sim O(\tilde{\varepsilon}^2) \sim 10^{-16} \), so it seems reasonable to neglect these extremely small terms.

A significant time dependence of the damping and diffusion coefficients could arise in the specific case of an unstable reservoir, provided that the reservoir oscillators having the frequencies close to \( \omega_0 \) could be also excited due to some resonance processes between the vibrating surface of the wall and the reservoir. In such a case, the energy of the resonant oscillators would increase in time, resulting in increasing values of the damping and diffusion coefficients (see Eq. \( (222) \)). As a consequence, we would obtain an additional amplification of the energy of the field mode due to the interaction with the reservoir. However, such a model seems unrealistic, since it implies that some distinguished degrees of freedom of the reservoir are practically isolated from the rest of the reservoir. This conjecture contradicts the usual concept of the reservoir consisting of a great number of strongly interacting particles, so that the state of the reservoir is not sensitive to small external perturbations. Therefore, we assume that the only time dependent coefficient in Eqs. \( (226)-(228) \) is \( \omega(t) \).

The set of equations \( (226)-(228) \) is equivalent to the third order differential equation for the variance

\[
\sigma_{qq}^{\ast} \equiv \sigma
\]

\[
\frac{d^3 \sigma}{dt^3} + 6\gamma \frac{d^2 \sigma}{dt^2} + 4\omega^2(t) \frac{d\sigma}{dt} + 4\omega_0(\dot{\omega} + 2\gamma_0 \omega) \sigma = 8\gamma \mathcal{E}_\ast,
\]

where the coefficients \( \gamma \) and \( \mathcal{E}_\ast \) are given by Eq. \( (223) \) (we neglect the terms of the second order with respect to \( \gamma, D_{ab}, \) and \( \dot{\varepsilon} \)). To find an approximate explicit solution to Eq. \( (230) \) with function \( \omega(t) \) given by Eq. \( (229) \), we use the well known method of slowly varying amplitudes, which was exposed, e.g. in textbooks \[260–268\] and applied to the quantum parametric oscillator in \[289–301\]. Following this method, we write

\[
\sigma_{qq}(t) = A(t) + B(t) \cos(\Omega t) + C(t) \sin(\Omega t).
\]

so that the function \( (231) \) with \( A, B, C = \text{const} \) is an exact solution to Eq. \( (230) \) for \( \dot{\varepsilon} = \gamma = \delta = 0 \). Supposing that the dimensionless small parameters \( \gamma/\omega_0 \) and \( \delta/\omega_0 \) have the same orders of magnitude as the small parameter \( \tilde{\varepsilon} \), i.e. \( \gamma = \tilde{\gamma} \tilde{\varepsilon} \omega_0, \delta = \tilde{\delta} \tilde{\varepsilon} \omega_0, \tilde{\gamma}, \tilde{\delta} \sim O(1) \), we assume that the amplitude coefficients \( A, B, C \) are functions of the “slow time” \( \tau = \tilde{\varepsilon} t \), so that the time derivatives \( \frac{d^k A}{dt^k}, \frac{d^k B}{dt^k}, \frac{d^k C}{dt^k} \) are proportional to \( \tilde{\varepsilon}^k \) \( (k = 1, 2, 3) \). Then we put the function \( (231) \) into Eq. \( (230) \) and neglect the terms proportional to \( \tilde{\varepsilon}^2 \) and \( \tilde{\varepsilon}^3 \). Besides, we perform averaging over fast oscillations with the frequency \( \Omega \), in order to eliminate higher harmonics with frequencies \( m\Omega, m = 2, 3, \ldots, \) whose amplitudes are proportional
This system can be easily solved for an arbitrary time dependent function \( \gamma(t) \), since the substitution \( \tilde{A}(t) = \hat{A}(t) \exp \left[ -2 \int \gamma(\tau) d\tau \right] \) removes the function \( \gamma(t) \) from the homogeneous parts of Eqs. (232)-(234). However, we consider the case of constant coefficients only, due to the physical reasons discussed above.

To simplify the formulas, we assume hereafter \( \hbar = \omega_0 = 1 \); thence \( \Omega = 2 \) in the amplitude coefficients. Neglecting small terms of the order of \( \gamma/\Omega, \kappa/\Omega, \) and \( \delta/\Omega \) in the amplitude coefficients, one can express the variances \( \sigma_{pp} \) and \( \sigma_{pq} \) as follows

\[
\begin{align*}
\sigma_{pp}(t) &= A(t) - B(t) \cos(\Omega t) - C(t) \sin(\Omega t), \\
\sigma_{pq}(t) &= C(t) \cos(\Omega t) - B(t) \sin(\Omega t).
\end{align*}
\]

Then the initial conditions for the set (232)-(234) read

\[
A(0) = [\sigma_{qq}(0) + \sigma_{pp}(0)]/2, \quad B(0) = [\sigma_{qq}(0) - \sigma_{pp}(0)]/2, \quad C(0) = \sigma_{pq}(0).
\]

Evidently, \( A(t) \) coincides with the mean energy to within small corrections of the order of \( \delta, \gamma, \) and \( \delta \):

\[
E(t) \equiv (\sigma_{pp} + \sigma_{qq})/2 = A(t).
\]

In order to elucidate the meaning of the coefficients \( B \) and \( C \), consider the determinant of the invariance matrix (cf. Eq. (211))

\[
d = \sigma_{pp} \sigma_{qq} - \sigma_{pq}^2 \geq \hbar^2/4.
\]

Here the last inequality holds due to the Schrödinger–Robertson uncertainty relation [271,278,321,322]. The meaning of the parameter \( d \) as the universal quantum invariant is discussed in [224]. The minimal invariant variance (100) can be expressed in terms of \( E \) and \( d \) as

\[
u = E - \sqrt{E^2 - d} = \frac{d}{E + \sqrt{E^2 - d}},
\]

and one can easily verify the relations

\[
d = A^2 - B^2 - C^2, \quad u = A - \sqrt{B^2 + C^2}.
\]

The solutions to Eqs. (232)-(234) read

\[
\begin{align*}
A(t) &= A_\ast + e^{-2\gamma t} \left[ a_0 \delta + a_+ \kappa e^{2\nu t} + a_- \kappa e^{-2\nu t} \right], \\
B(t) &= B_\ast + e^{-2\gamma t} \left[ a_+ \nu e^{2\nu t} - a_- \nu e^{-2\nu t} \right], \\
C(t) &= C_\ast + e^{-2\gamma t} \left[ a_0 \kappa + a_+ \delta e^{2\nu t} + a_- \delta e^{-2\nu t} \right],
\end{align*}
\]

where \( \nu = \sqrt{\kappa^2 - \delta^2} \), and

\[
\begin{align*}
a_0 &= \frac{1}{\nu^2} \left[ \kappa C(0) - \delta A(0) + E_\ast \delta \right], \\
a_\pm &= \frac{1}{2\nu^2} \left[ \kappa A(0) - \delta C(0) \pm \nu B(0) - \frac{\kappa \gamma E_\ast}{\gamma^2 + \nu} \right], \\
A_\ast &= \frac{\gamma^2 + \delta^2}{\gamma^2 - \nu^2} E_\ast, \quad B_\ast = \frac{\kappa \gamma E_\ast}{\gamma^2 - \nu^2}, \quad C_\ast = \frac{\kappa \delta E_\ast}{\gamma^2 - \nu^2}.
\end{align*}
\]
The meanings of the parameters $\gamma$ and $\mathcal{E}_*$ become clear, if one considers the special case of the oscillator with time independent coefficients, $\kappa = \delta = \nu = 0$. Then we see that $2\gamma$ is the energy relaxation coefficient, so that it can be expressed in terms of the cavity $Q$-factor by means of the relation $2\gamma = \omega_0/Q$. The energy of the oscillator in this special case tends to $\mathcal{E}_*$ as $t \to \infty$, and this value can be identified with the thermodynamic equilibrium oscillator energy $\mathcal{E}_{eq}$ given by Eq. (228) (up to corrections of the order of $(\gamma/\omega_0)^2$ [277, 308].

The sign of the difference $\nu - \gamma$ determines the regions of stable and unstable solutions of Eq. (230) in the space of parameters $\kappa, \delta, \gamma$ (for small values of these parameters). The stable (limited in time) solutions exist for large values of the damping or detuning coefficients, $\nu < \gamma$, i.e.

$$
\gamma^2 + \delta^2 > \kappa^2. \tag{245}
$$

In this case, the final state of the oscillator does not depend on the initial conditions. The asymptotical values of the energy, the $d$-factor, and the minimal invariant variance read

$$
\mathcal{E}(\infty) = \frac{\gamma^2 + \delta^2}{\gamma^2 - \nu^2} \mathcal{E}_{eq}, \quad d(\infty) = \frac{\gamma^2 + \delta^2}{\gamma^2 - \nu^2} \mathcal{E}_{eq}^2, \quad u(\infty) = \frac{\sqrt{\gamma^2 + \delta^2}}{\kappa + \sqrt{\gamma^2 + \delta^2}} \mathcal{E}_{eq}. \tag{246}
$$

At zero temperature, when $\kappa$ tends to the threshold, the minimal variance $u$ goes to the value $1/4$, which is twice less than in the coherent state.

Above the threshold, i.e. in the instability region $\nu > \gamma$, the energy (or the number of photons) increases exponentially for $(\nu - \gamma)t \gg 1$,

$$
\mathcal{E}(t) = a_+ \kappa e^{2(\nu - \gamma)t}, \tag{247}
$$

and it depends on the initial conditions through the coefficient $a_+$. Using Eq. (244) and taking into account the uncertainty relation $d \geq 1/4$ (for $\hbar = 1$), one can verify that the coefficient $a_+$ is bounded from below by a positive value,

$$
a_+ > \frac{\kappa \mathcal{E}_{eq}}{2 \nu^2 (\nu - \gamma)}. \tag{248}
$$

Consider a special case of initial thermal equilibrium state. Then

$$
a_+^{(eq)} = \frac{\kappa \mathcal{E}_{eq}}{2 \nu (\nu + \gamma)}, \quad a_0^{(eq)} = 0, \quad d(0) = \mathcal{E}_{eq}^2, \tag{249}
$$

thus $d$-factor [238] depends on time as

$$
d(t) = \frac{\mathcal{E}_{eq}^2}{\nu (\nu^2 - \gamma^2)} \left[ 2 \kappa^2 \gamma e^{-2\gamma t} \sinh(2\nu t) + \kappa^2 \nu e^{-4\gamma t} - \nu (\gamma^2 + \delta^2) \right]. \tag{250}
$$

If $\gamma > 0$ and $(\nu - \gamma)t \gg 1$, then due to Eq. (238), the minimal variance tends asymptotically to a constant value

$$
u_\infty = \frac{\gamma}{\gamma + \nu} \mathcal{E}_{eq}. \tag{251}
$$

Consequently, a large squeezing can be achieved even for a high temperature initial state, if $\nu \gg \gamma$. If $\gamma = 0$, then $d$ does not depend on time, $d \equiv \mathcal{E}_{eq}^2$, and for $\nu t \gg 1$ the minimal variance goes asymptotically to zero as $u \approx \mathcal{E}_{eq} (\nu/\kappa)^2 \exp(-2\nu t)$. One should remember, nonetheless, that the solutions indicating the exponential growth of the energy are justified until $t \ll t_2 \sim (\omega_0 \delta^2)^{-1}$, since for larger times the neglected second order terms in Eqs. (230), (232)-(234) could become important. However, the time $t_2$ is very large under the realistic conditions.

10 Discussion

We have demonstrated a significant progress in our understanding and quantitative description of quantum processes in cavities with moving boundaries, achieved 30 years after the pioneer paper by Moore [76] and almost 80 years after the first papers on classical electrodynamics in such cavities by Nicolai and Havelock [6, 8]. We have shown that quanta of electromagnetic field can be created from vacuum in a cavity with
vibrating walls under the resonance condition (and cannot be generated for nonresonance nonrelativistic laws of boundary motion, in particular, in the case of a large detuning from the resonance), and the quantum state of field exhibits the “nonclassical” properties.

The possibility of observing the effect depends crucially on the achievable values of the wall displacement amplitude. For the cavity dimensions of the order of 1 ÷ 100 centimeters, the resonance frequency \(\omega_0/\pi\) varies from 30 GHz to 300 MHz. It is difficult to imagine that the wall could be forced to oscillate as a whole at such a high frequency. Rather, one could think on the oscillations of the surface of the cavity wall. In such a case one has to find a way of exciting a sufficiently strong standing acoustic wave at frequency \(\omega_w = 2\omega_0\) inside the wall. The amplitude \(a\) of this wave (coinciding with the amplitude of oscillations of the free surface) is connected with the relative deformation amplitude \(\delta\) inside the wall as \(\delta = \omega_w a/v_s\), where \(v_s\) is the sound velocity. Since the usual materials cannot bear the deformations exceeding the value \(\varepsilon\) (in the asymptotical regime), we get 600 photons/sec.

The rate of photon generation in the principal mode of a 1D cavity can be estimated as

\[
\langle \frac{dP_1}{dt} \rangle_{\text{max}} = \frac{4}{\pi^2} \frac{v_s \delta_{\text{max}} \omega_1}{c} \approx 6 \times 10^{-8} \omega_1/2\pi.
\]  

(251)

It is proportional to the frequency. For \(\omega_1/2\pi = 10\,\text{GHz}\) (corresponding to a distance between the plates of the order of several centimeters) we get 600 photons/sec.

This number can be significantly increased in a 3D cavity, due to the exponential law \(202\). For the same frequency \(\omega_0/2\pi = 10\,\text{GHz}\), the maximal value of parameter \(\mu = \gamma\omega_0 t\) equals \(\mu_{\text{max}} \approx 600t\), time \(t\) being expressed in seconds. Even if the amplitude of the vibrations were 100 times less than the maximal possible value, in \(t = 1\,\text{s}\) one could get about \(\sinh^2(6) \approx 4 \times 10^4\) photons in an empty cavity. Obviously, the concrete shape of a 3D cavity is not important. The significant requirements are: i) the nondegenerate character of the eigenfrequency spectrum, and ii) the condition of the parametric resonance between the oscillating wall and some electromagnetic mode. The total energy of photons created in the 1D cavity is approximately the same as in the three-dimensional case. The difference is that in the 1D case this energy is spread over many interacting modes, resulting in moderate numbers of quanta in each mode. The rate of photon generation in the \(m\)-th (odd) mode of the 1D cavity is approximately \(m\) times less than in the principal mode with \(m = 1\) (in the asymptotical regime).

To create the above-mentioned 600 or \(4 \times 10^4\) photons, one should vibrate the wall for not less than 1 second. The necessary Q-factor of the cavity must be \(Q \sim 3 \times 10^{10}\). This value was achieved in experiments already several years ago \(323\). An unsolved problem is how to excite the high-frequency surface vibrations with a sufficiently large amplitude. One could think, for instance, on using some kind of piezoeffect. This method was successfully applied in early experiments devoted to solving the mode-locking and pulse production problems in lasers with the aid of vibrating mirrors. The displacements of the mirror from 0.1 \(\mu\text{m}\) to 0.7 \(\mu\text{m}\) at the frequency 500 KHz were achieved in \(27\). In \(28\) the resonance vibrations of the mirror in a laser with the length 250 cm (i.e., at a frequency about 100 MHz) were excited with the aid of a quartz transducer. However, for our purposes the frequency 100 MHz is too small, since the parameter \(\mu\) becomes 100 times less, comparing with the estimations given above (remember that \(\varepsilon_{\text{max}}\) does not depend on the frequency).

Fortunately, the results of recent studies \(19\,21\,21\) show that the influence of temperature is not so significant as it could appear at first glance. Moreover, in certain situations the initial temperature fluctuations could be used to amplify the effect \(21\). However, the resonance requirements are rather hard: for the values of the frequency \(\omega \sim 10^{10}\,\text{Hz}\) and the maximal possible modulation parameter \(\varepsilon \sim 10^{-8}\) the admissible detuning should not exceed 100 Hz during the whole time interval \(\sim 1\,\text{s}\), necessary to accumulate the resonance effect.

One of the reasons of the studies on the dynamical Casimir effect for the last few years was Schwinger’s hypothesis \(15\,17\) that this effect could explain the sonoluminescence phenomenon, i.e., the emission of bright short pulses of the visible light from the gas bubbles in the water, when the bubbles pulsate due to the pressure oscillations in a strong standing acoustic wave. (For the reviews and numerous references related to this effect see, e.g., \(12\,22\,23\)) There are several publications, e.g., \(23\,33\), whose authors considered the models giving tremendous numbers of photons which could be produced even in the visible range due to
the fast motion of the boundaries. However, the analysis of these models shows that they are based on such
laws of motion of the boundaries which imply the superluminal velocities, so they are not realistic.

Although the results of this chapter, being obtained in the framework of simplified one-dimensional and
three-dimensional models, cannot be applied directly to the analysis of the sonoluminescence problem, they
are not in favor of Schwinger’s hypothesis. The main difficulty is connected with quite different time scales
of the phenomena. The accumulation of the ‘dynamical Casimir energy’ is a very slow process, which needs
a great number of wall oscillations, whereas the sonoluminescence pulses (containing up to \(10^7\) photons)
have the duration of the order of picoseconds. Moreover, the wall oscillations must be in extremely fine
tuned resonance with the field eigenfrequencies, since the detuning \(\delta > \epsilon\) completely destroys the energy
growth \[191\]. In particular, if the frequency of the wall oscillations \(\omega_{\text{wall}}\) is much less than the minimal
field eigenfrequency \(\omega_1\), then the field variation is adiabatic, and the mean number of created photons is
proportional to \[39\] \(\epsilon^2 (\omega_{\text{wall}}/\omega_1)^4 \ll 1\). These features survive in the three-dimensional model considered
in this chapter, too. Therefore, it is difficult to believe that very specific conditions of the parametric resonance
described above could arise naturally in the sonoluminescence case. For other discussions of the problem
see, e.g., \[32\] and the contributions in Ref. \[33\].

Actually, the main obstacle to produce the ‘Casimir light’ is the very low ratio of the wall velocity to
the speed of light in possible laboratory experiments. If the velocity of the boundary were of the order of
c, then a sufficient number of photons could be created from vacuum practically for any law of motion. For
the nonrelativistic velocities the only possibility is to accumulate the effect gradually under the resonance
conditions. Nonetheless, perhaps, the experimental situation could be improved in the case of using some
kinds of ‘effective mirrors’, such as, e.g., the layers made of the electron-hole plasma \[20\], or some others.
Therefore, we cannot exclude a possibility that in a not very remote future one could assist a show of
“quantum magics,” when some “quantum magician” takes an empty box, then shakes it well, opens — and
an astonished audience would see a great number of photons which have appeared “from nothing” due to
the Nonstationary Casimir Effect.

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