Pseudo minimum phi-divergence estimator for multinomial logistic regression with complex sample design

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Abstract

This article develops the theoretical framework needed to study the multinomial logistic regression model for complex sample design with pseudo minimum phi-divergence estimators. Through a numerical example and simulation study new estimators are proposed for the parameter of the logistic regression model with overdispersed multinomial distributions for the response variables, the pseudo minimum Cressie-Read divergence estimators, as well as new estimators for the intra-cluster correlation coefficient. The results show that the Binder’s method for the intra-cluster correlation coefficient exhibits an excellent performance when the pseudo minimum Cressie-Read divergence estimator, with \( \lambda = \frac{2}{3} \), is plugged.

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1 Introduction

Multinomial logistic regression is frequently the method of choice when the response is a qualitative variable, with two or more mutually exclusive unordered response categories, and interest is in the relationship between the response variables with respect to their corresponding explanatory variables or covariates. The \( k \) explanatory variables of interest, \( x = (x_1, ..., x_k)^T \), may be binary, categorical, ordinal or continuos. The multinomial logistic regression procedure is based on assuming that the \((d+1)\)-dimensional response random variable \( Y = (Y_1, ..., Y_{d+1})^T \) is a multinomial random variable of a unique observation with parameters \( \pi_1(\beta), ..., \pi_{d+1}(\beta) \) being

\[
\pi_r(\beta) = \Pr(Y_r = 1|x) = \begin{cases} 
\frac{\exp\{x^T\beta_r\}}{1 + \sum_{s=1}^{d} \exp\{x^T\beta_s\}}, & r = 1, ..., d \\
\frac{1}{1 + \sum_{s=1}^{d} \exp\{x^T\beta_s\}}, & r = d+1
\end{cases},
\]

(1)
with $\beta = (\beta_1^T, ..., \beta_d^T)^T$, where $\beta_r = (\beta_{1r}, ..., \beta_{kr})^T$ is a $k$-dimensional real value vector of unknown parameters for $r = 1, ..., d$. An observation of $Y$, $y_i$ is any $(d + 1)$-dimensional vector with $d$ zeros and a unique one (classification vector), which is observed together with explanatory variables $x$. In order to make inferences about $\beta_r, r = 1, ..., d$, a random sample $(Y_i, x_i), i = 1, ..., n$ is considered, where $Y_i = (Y_{i1}, ..., Y_{i,d+1})^T$ and $x_i = (x_{i1}, ..., x_{ik})^T$. For more details about multinomial logistic regression models see for instance Agresti (2002), Amemiya (1981), Anderson (1972, 1982, 1984), Engel (1988), Lesaffre (1986), Lesaffre and Albert (1986, 1989), Liu and Agresti (2005), Mantel (1966), Theil (1969), McCullagh (1980). In that papers the inferences about the parameters are carried out on the basis of the maximum likelihood estimator in the case of the estimation and on the likelihood ratio test and Wald tests in the case of testing. In Gupta et al. (2006a, 2006b, 2007, 2008) new procedures for making statistical inference in the multinomial logistic regression were presented based on phi-divergences measures.

When the data have been collected not under the assumptions of simple random sampling but in a complex survey, with stratification, clustering, or unequal selection probabilities, for example, the estimation of the multinomial logistic regression coefficients and their estimated variances that ignore these features may be misleading. Discussions of multinomial logistic regression in sample surveys can be seen in Binder (1983), Roberts, Rao and Kumar (1987), Skinner, Holt and Smith (1989), Morel (1989), Lehtonen and Pahkinen (1995) and Morel and Neerchal (2012).

In this paper, we consider the multinomial logistic regression model with complex survey and we shall introduce for this model the pseudo minimum phi-divergence estimator for the regressions coefficients, deriving its asymptotic distribution. As a particular case, we shall obtain the asymptotic distribution of the pseudo maximum likelihood estimator. In Section 2 we present some notation as well as some results in relation to the maximum likelihood estimator. Section 5 is devoted to introduce the pseudo minimum phi-divergence estimator as an extension of the maximum likelihood estimator as well as its asymptotic distribution. In Section 4 and 5 the numerical example and simulation study are shown. Finally, in Section 6 some concluding remarks are given.

2 Multinomial logistic regression model for complex sample design

We shall assume that the population under consideration is divided into $H$ distinct strata. In each stratum $h$, the sample is consisted of $n_h$ clusters, $h = 1, ..., H$, and each cluster is comprised of $m_{hi}$ units, $h = 1, ..., H, i = 1, ..., n_h$. Let

$$y_{hij} = (y_{hij1}, ..., y_{hij,d+1})^T, h = 1, ..., H, i = 1, ..., n_h, j = 1, ..., m_{hi}$$

be the $(d+1)$-dimensional classification vectors, with $y_{hijr} = 1$ and $y_{hijr} = 0$ for $s \in \{1, ..., d+1\} - \{r\}$ if the $j$-th unit selected from the $i$-th cluster of the $h$-th stratum fall in the $r$-th category. Let $x_{hi} = (x_{hij1}, ..., x_{hijk})^T$ be a $k$-dimensional vector of explanatory variables associated with the $i$-th cluster in the $h$-th stratum for the $j$-th individual. We shall also denote by $w_{hi}$ the sampling weight from the $i$-th cluster of the $h$-th stratum. For
each $i$, $h$ and $j$, the expectation of the $r$-th element of $Y_{hij} = (Y_{hij1}, ..., Y_{hij,d+1})^T$, with a realization $y_{hij}$, is determined by the multinomial logistic regression relationship

$$
\pi_{hijr}(\beta) = \begin{cases} 
\frac{\exp(x_{hij}^T \beta_r)}{1 + \sum_{s=1}^d \exp(x_{hij}^T \beta_s)}, & r = 1, ..., d \\
1 + \sum_{s=1}^d \exp(x_{hij}^T \beta_s), & r = d + 1 
\end{cases},
$$

(3)

with $\beta_r = (\beta_{1r}, ..., \beta_{kr})^T \in \mathbb{R}^k$, $r = 1, ..., d$. We shall denote by $\pi_{hij}(\beta)$ the $(d+1)$-dimensional probability vector

$$
\pi_{hij}(\beta) = (\pi_{hij1}(\beta), ..., \pi_{hij,d+1}(\beta))^T.
$$

(4)

The parameter space associated to the multinomial logistic regression model considered in (3) is given by

$$
\Theta = \{\beta = (\beta_1^T, ..., \beta_d^T)^T, \beta_j = (\beta_{j1}, ..., \beta_{jk})^T \in \mathbb{R}^k, j = 1, ..., d\} = \mathbb{R}^{dk}.
$$

In this context and taking into account the weights $w_{hi}$, the pseudo log-likelihood, $L(\beta)$, for the multinomial logistic regression model given in (3) has the expression

$$
L(\beta) = \sum_{h=1}^H \sum_{i=1}^{n_h} \sum_{j=1}^{m_{hi}} w_{hi} \log \pi_{hij}^T(\beta) y_{hij},
$$

(5)

where $\log \pi_{hij}(\beta) = (\log \pi_{hij1}(\beta), ..., \log \pi_{hij,d+1}(\beta))^T$. For more details about $L(\beta)$ see for instance Morel (1989) and Morel and Neerchal (2012).

In practice, it is not a strong assumption to consider that the expectation of the $r$-th component of $Y_{hij}$ does not depend on $j$, i.e.,

$$
\pi_{hijr}(\beta) = \pi_{hir}(\beta), \quad j = 1, ..., m_{hi},
$$

where $\pi_{hir}(\beta) = \text{E}[Y_{hir}] = \text{Pr}(Y_{hir} = 1)$. This is related to a common vector of explanatory variables $x_{hi} = (x_{h1}, ..., x_{hk})^T$ for all the individuals in the $i$-th cluster of the $h$-th stratum and we shall denote $\pi_{hi}(\beta)$ instead of $\pi_{hij}(\beta)$ the vector mean associated to $Y_{hij}$. Let

$$
\hat{Y}_{hi} = \sum_{j=1}^{m_{hi}} Y_{hij} = \left(\sum_{j=1}^{m_{hi}} Y_{hij1}, ..., \sum_{j=1}^{m_{hi}} Y_{hij,d+1}\right)^T = (\hat{Y}_{h11}, ..., \hat{Y}_{h1,d+1})^T
$$

(6)

be the random vector of counts in the $i$-th cluster of the $h$-th stratum. Under homogeneity assumption within the clusters, the pseudo log-likelihood is

$$
L(\beta) = \sum_{h=1}^H \sum_{i=1}^{n_h} \sum_{j=1}^{m_{hi}} w_{hi} \log \pi_{hi}^T(\beta) y_{hij}
$$

$$
= \sum_{h=1}^H \sum_{i=1}^{n_h} w_{hi} \log \pi_{hi}^T(\beta) \hat{y}_{hi},
$$

(7)

The pseudo maximum likelihood estimator $\hat{\beta}_p$ of $\beta$ is obtained maximizing in $\beta$ the pseudo log-likelihood given in (7). This estimator can be obtained as the solution of the system of equations

$$
\sum_{h=1}^H \sum_{i=1}^{n_h} w_{hi} \frac{\partial \pi_{hi}^T(\beta)}{\partial \beta} \Delta^{-1}(\pi_{hi}^T(\beta)) r_{hi}(\beta) = 0_{dk},
$$

(8)
being
\[ \frac{\partial \pi_{hi}^T (\beta)}{\partial \beta} = \Delta (\pi_{hi}^* (\beta)) \otimes x_{hi}, \]
\[ \Delta (\pi_{hi}^* (\beta)) = \text{diag}(\pi_{hi}^* (\beta)) - \pi_{hi}^* (\beta) \pi_{hi}^T (\beta), \]
\[ r_{hi}^* (\beta) = \hat{y}_{hi}^* - m_{hi} \pi_{hi}^* (\beta). \]

With superscript * on a vector we denote the vector obtained deleting the last component from the initial vector, and thus \( \pi_{hi}^* (\beta) = (\pi_{h1} (\beta), \ldots, \pi_{hid} (\beta))^T \) and \( \hat{y}_{hi}^* = (\hat{y}_{h1}^*, \ldots, \hat{y}_{hid}^*)^T \). The system of equations (8) can be written as \( u (\beta) = 0_{dk} \), being
\[ u (\beta) = \sum_{h=1}^{H} \sum_{i=1}^{n_h} u_{hi} (\beta), \quad (9) \]
\[ u_{hi} (\beta) = w_{hi} r_{hi}^* (\beta) \otimes x_{hi}. \quad (10) \]

3. **Pseudo minimum phi-divergence estimator: asymptotic distribution**

In this Section we shall introduce, for the first time, the pseudo minimum phi-divergence estimator, \( \hat{\beta}_{\phi,P} \), of the parameter \( \beta \) as a natural extension of the pseudo maximum likelihood estimator \( \hat{\beta}_P \). We define the following theoretical probability vector
\[ \pi (\beta) = \frac{1}{\tau} (w_{11} m_{11} \pi_{11}^T (\beta), \ldots, w_{1n1} m_{1n1} \pi_{1n1}^T (\beta), \ldots, w_{H1mH1} \pi_{H1mH1}^T (\beta), \ldots, w_{HnH} m_{HnH} \pi_{HnH}^T (\beta))^T, \]
with
\[ \tau = \sum_{h=1}^{H} \sum_{i=1}^{n_h} w_{hi} m_{hi} \]
being a known value. Based on \( \hat{y}_{hi} \), observation of \( \hat{Y}_{hi} \) defined in (8), we consider the vector \( \hat{y}_h \) for each stratum \( h \),
\[ \hat{y}_h = (w_{h1} \hat{y}_{h1}^T, \ldots, w_{hn} \hat{y}_{hn}^T)^T. \]
We shall also consider the non-parametric probability vector
\[ \hat{p} = \frac{1}{\tau} (\hat{y}_1^T, \ldots, \hat{y}_H^T)^T = \frac{1}{\tau} (w_{11} \hat{y}_{11}^T, \ldots, w_{1n1} \hat{y}_{1n1}^T, \ldots, w_{H1mH1} \hat{y}_{H1mH1}^T, \ldots, w_{HnH} \hat{y}_{HnH}^T)^T. \]

The Kullback-Leibler divergence between the probability vectors \( \hat{p} \) and \( \pi (\beta) \) is given by
\[ d_{K-L} (\hat{p}, \pi (\beta)) = \frac{1}{\tau} \sum_{h=1}^{H} \sum_{i=1}^{n_h} \sum_{s=1}^{d+1} \hat{y}_{his} \log \frac{\hat{y}_{his}}{m_{hi} \pi_{his} (\beta)} \]
\[ = K - \frac{1}{\tau} \sum_{h=1}^{H} \sum_{i=1}^{n_h} \sum_{s=1}^{d+1} \hat{y}_{his} \log \pi_{his} (\beta) \]
\[ = K - \frac{1}{\tau} \sum_{h=1}^{H} \sum_{i=1}^{n_h} w_{hi} \log \pi_{hi}^* (\beta) \hat{y}_{hi}, \]

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with $K$ being a constant not dependent of $\beta$. Based on (7) and (12), we can define the pseudo maximum likelihood estimator for the multinomial logistic regression model given in (3) by

$$\hat{\beta}_P = \arg \min_{\beta \in \Theta} d_{K-L}(\hat{p}, \pi(\beta)).$$

(13)

But Kullback-Leibler divergence is a particular divergence measure in the family of phi-divergences, $d_{\phi}(\hat{p}, \pi(\beta)) = 1/\tau \sum_{h=1}^{H} \sum_{i=1}^{n_h} w_{hi} m_{hi} \sum_{s=1}^{d+1} \pi_{his}(\beta) \phi \left( \frac{\hat{y}_{his}}{m_{hi} \pi_{his}(\beta)} \right)$, (14)

where $\phi \in \Phi^*$ is the class of all convex functions $\phi(x)$, defined for $x > 0$, such that at $x = 1$, $\phi(1) = 0$, $\phi''(1) > 0$, and at $x = 0$, $0 \phi(0)/0 = 0$ and $0 \phi(p)/0 = \lim_{u \to \infty} \phi(u)/u$. For every $\phi \in \Phi^*$ differentiable at $x = 1$, the function

$$\varphi(x) \equiv \phi(x) - \phi'(1)(x-1)$$

also belongs to $\Phi^*$. Then we have $d_{\varphi}(\hat{p}, \pi(\beta)) = d_{\phi}(\hat{p}, \pi(\beta))$, and $\varphi$ has the additional property that $\varphi'(1) = 0$. Because the two divergence measures are equivalent, we can consider the set $\Phi^*$ to be equivalent to the set

$$\Phi \equiv \Phi^* \cap \{\phi : \phi'(1) = 0\}.$$

In what follows, we give our theoretical results for $\phi \in \Phi$, but we often apply them to choices of functions in $\Phi^*$.

An equivalent definition of (14) is a weighted version of phi-divergences between the cluster non-parametric probabilities and theoretical probabilities

$$d_{\phi}(\hat{p}, \pi(\beta)) = \sum_{h=1}^{H} \sum_{i=1}^{n_h} w_{hi} m_{hi} \sum_{s=1}^{d+1} \pi_{his}(\beta) \phi \left( \frac{\hat{y}_{his}}{m_{hi} \pi_{his}(\beta)} \right),$$

where

$$d_{\phi} \left( \frac{\hat{y}_{his}}{m_{hi}}, \pi_{hi}(\beta) \right) = \sum_{s=1}^{d+1} \pi_{his}(\beta) \phi \left( \frac{\hat{y}_{his}}{m_{hi} \pi_{his}(\beta)} \right).$$

For more details about phi-divergences measures see Pardo (2005).

Based on (13) and (14) we shall introduce, in this paper, the pseudo minimum phi-divergence estimator for the parameter $\beta$ in the multinomial logistic regression model under complex survey defined in (3) as follows,

**Definition 1** We consider the multinomial logistic regression model with complex survey defined in (3). The pseudo minimum phi-divergence estimator of $\beta$ is defined as

$$\hat{\beta}_{\phi,P} = \arg \min_{\beta \in \Theta} d_{\phi}(\hat{p}, \pi(\beta)), $$

where $d_{\phi}(\hat{p}, \pi(\beta))$, the phi-divergence measure between the probability vectors $\hat{p}$ and $\pi(\beta)$, is given in (14).

For $\phi(x) = x \log x - x + 1$ the associated phi-divergence (14) coincides with the Kullback-Leibler divergence (12), therefore the pseudo minimum phi-divergence estimator of $\beta$ based on $\phi(x)$ contains as special case the pseudo maximum likelihood estimator. With the same philosophy, the following result generalizes $u_{hi}(\beta)$ given in (10) and later this result plays an important role for the asymptotic distribution of the pseudo minimum phi-divergence estimator, $\hat{\beta}_{\phi,P}$. 

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**Theorem 2** The pseudo minimum phi-divergence estimator of $\beta$, $\hat{\beta}_{\phi,P}$, is obtained by solving the system of equations $u_\phi(\beta) = 0_{dk}$, where

$$u_\phi(\beta) = \sum_{h=1}^{H} \sum_{i=1}^{n_h} u_{\phi,h_i}(\beta),$$  \hspace{1cm} (15)

$$u_{\phi,h_i}(\beta) = \frac{w_{hi} m_{hi}}{\phi''(1)} \Delta(\pi_{hi}^*(\beta)) f_{\phi,h_i}(\hat{\pi}_{hi}(\beta)) \otimes x_{hi},$$  \hspace{1cm} (16)

where

$$f_{\phi,h_i}(\hat{\pi}_{hi}(\beta)) = (f_{\phi,h_1}(\hat{\pi}_{hi}(\beta)), \ldots, f_{\phi,h_d}(\hat{\pi}_{hi}(\beta)))^T,$$

$$f_{\phi,h_i}(x,\beta) = \frac{x}{\pi_{hi}(\beta)} \phi' \left( \frac{x}{\pi_{hi}(\beta)} \right) - \phi \left( \frac{x}{\pi_{hi}(\beta)} \right)$$  \hspace{1cm} (17)

**Proof.** The pseudo minimum phi-divergence estimator of $\beta$, $\hat{\beta}_{\phi,P}$, is obtained by solving the system of equations

$$\frac{\partial}{\partial \beta} d_\phi(\hat{\beta}, \pi(\beta)) = 0_{dk},$$

and then it is also obtained from $u_\phi(\beta) = 0_{dk}$, where

$$u_\phi(\beta) = -\frac{\tau}{\phi''(1)} \frac{\partial}{\partial \beta} d_\phi(\hat{p}, \pi(\beta)) = \sum_{h=1}^{H} \sum_{i=1}^{n_h} u_{\phi,h_i}(\beta),$$

with

$$u_{\phi,h_i}(\beta) = -\frac{w_{hi} m_{hi}}{\phi''(1)} \frac{\partial}{\partial \beta} d_\phi(\hat{\pi}_{hi}(\beta)) = \frac{w_{hi} m_{hi}}{\phi''(1)} \sum_{s=1}^{d+1} \frac{\partial \pi_{hs}(\beta)}{\partial \beta} f_{\phi,h_s}(\hat{\pi}_{hi}(\beta))$$

$$= \frac{w_{hi} m_{hi}}{\phi''(1)} \frac{\partial \pi_{hi}(\beta)}{\partial \beta} f_{\phi,h_i}(\hat{\pi}_{hi}(\beta)), \hspace{1cm} (18)$$

and

$$f_{\phi,h_i}(\hat{\pi}_{hi}(\beta)) = (f_{\phi,h_1}(\hat{\pi}_{hi}(\beta)), \ldots, f_{\phi,h_d}(\hat{\pi}_{hi}(\beta)))^T.$$  \hspace{1cm} (19)

Since

$$\frac{\partial \pi_{hi}(\beta)}{\partial \beta} = (I_{d \times d}, 0_{d \times 1}) \Delta(\pi_{hi}(\beta)) \otimes x_{hi},$$

the expression of $u_{\phi,h_i}(\beta)$ is rewritten as (16). \hspace{1cm} $\blacksquare$

**Remark 3** An important family of divergence measures is obtained by restricting $\phi$ from the family of convex functions to the Cressie-Read subfamily

$$\phi_{\lambda}(x) = \begin{cases} x^{\lambda+1} - x - \lambda(x-1), & \lambda \in \mathbb{R} - \{-1, 0\} \\ \frac{1}{\lambda(1+v)} \left[ x^{\lambda+1} - x - v(x-1) \right], & \lambda \in \{-1, 0\} \end{cases}$$  \hspace{1cm} (20)

We can observe that for $\lambda = 0$, we have

$$\phi_{\lambda=0}(x) = \lim_{\nu \to 0} \frac{1}{\nu(1+v)} \left[ x^{\nu+1} - x - v(x-1) \right] = x \log x - x + 1,$$

and the associated phi-divergence $[14]$, coincides with the Kullback-Leibler divergence $[12]$, therefore the pseudo minimum phi-divergence estimator of $\beta$ based on $\phi_{\lambda}(x)$ contains as special case the pseudo maximum likelihood estimator and $u_{hi}(\beta)$ given in (16) matches $u_{\phi,h_i}(\beta)$ given in (16). For the Cressie-Read subfamily it is established that for $\lambda \neq -1$, $u_{\phi,h_i}(\beta) = \sum_{h=1}^{H} \sum_{i=1}^{n_h} u_{\phi,h_i}(\beta)$, where

$$u_{\phi,h_i}(\beta) = \frac{m_{hi}}{(\lambda+1)m_{hi}} \frac{\partial \pi_{hi}(\beta)}{\partial \beta} \text{diag}^{-(\lambda+1)}(\pi_{hi}(\beta)) \hat{y}_{hi}^{\lambda+1},$$
since we have (18) with

\[
f_{\phi, hi}(\frac{\hat{y}_{hi}}{m_{hi}}, \beta) = \frac{1}{\lambda + 1} \left( \frac{1}{m_{hi}^{\lambda+1}} \text{diag}^{-\lambda - 1}(\pi_{hi}(\beta)) \hat{y}_{hi}^{\lambda+1} - 1_{d+1} \right),
\]

From (19) and

\[
\Delta(\pi_{hi}(\beta)) \text{diag}^{-\lambda - 1}(\pi_{hi}(\beta)) = \Delta(\pi_{hi}(\beta)) \text{diag}^{-\lambda}(\pi_{hi}(\beta)) \text{diag}^{-\lambda}(\pi_{hi}(\beta))
\]

it is concluded that

\[
u_{\phi, hi}(\beta) = \frac{w_{hi}}{(\lambda + 1)m_{hi}} \left( \text{diag}^{-\lambda}(\pi_{hi}(\beta)) \hat{y}_{hi}^{\lambda+1} - [1_{d+1}^T \text{diag}^{-\lambda}(\pi_{hi}(\beta)) \hat{y}_{hi}^{\lambda+1}] \pi_{hi}(\beta) \right) \otimes x_{hi},
\]

where

\[
\epsilon_{hi} = \text{diag}^{-\lambda}(\pi_{hi}(\beta)) \hat{y}_{hi}, \quad \hat{\epsilon}_{hi} = \text{diag}^{-\lambda}(\pi_{hi}(\beta)) \hat{y}_{hi}^T.
\]

Notice that replacing \( \lambda = 0 \) in \( \nu_{\phi, hi}(\beta) \) given in (22), \( u_{hi}(\beta) \) given in (17) is obtained. For \( \lambda = -1 \) in (22), we have

\[
\lim_{\lambda \to -1} f_{\phi, hi}(\frac{\hat{y}_{hi}}{m_{hi}}, \beta) = \log \left( \text{diag}^{-\lambda}(\pi_{hi}(\beta)) \hat{y}_{hi} \right),
\]

and therefore

\[
\lim_{\lambda \to -1} \nu_{\phi, hi}(\beta) = w_{hi}m_{hi} \Delta(\pi_{hi}(\beta)) \log \left( \text{diag}^{-\lambda}(\pi_{hi}(\beta)) \hat{y}_{hi} \right) \otimes x_{hi}.
\]

The family of pseudo minimum divergence estimators, obtained from \( \phi_{\lambda}(x) \) given in (20), will be called the pseudo minimum Cressie-Read divergence estimators and for \( \beta \) they will be denoted by \( \hat{\beta}_{\phi, \lambda} \). This family of estimators will be used in Sections 4 and 5.

In the following theorem we shall present the asymptotic distribution of the pseudo minimum phi-divergence estimator, \( \hat{\beta}_{\phi, \lambda} \).

**Theorem 4** Let \( \hat{\beta}_{\phi, \lambda} \) the pseudo minimum phi-divergence estimator of parameter \( \beta \) for a multinomial logistic regression model with complex survey, \( n = \sum_{h=1}^{H} n_h \) the total of clusters in all the strata of the sample and \( \eta_{h}^{*} \) an unknown proportion obtained as \( \lim_{n \to \infty} \frac{n_h}{n} = \eta_{h}^{*} \), \( h = 1, \ldots, H \). Then we have

\[
\sqrt{n}(\hat{\beta}_{\phi, \lambda} - \beta_{0}) \xrightarrow{n \to \infty} \mathcal{N}(0_{dh}, H^{-1}(\beta_{0}) G(\beta_{0}) H^{-1}(\beta_{0}))
\]

where

\[
H(\beta) = \lim_{n \to \infty} H_{n}(\beta) = \sum_{h=1}^{H} \eta_{h}^{*} \lim_{n_h \to \infty} H^{(h)}_{n_h}(\beta), \quad G(\beta) = \lim_{n \to \infty} G_{n}(\beta) = \sum_{h=1}^{H} \eta_{h}^{*} \lim_{n_h \to \infty} G^{(h)}_{n_h}(\beta),
\]

with

\[
H_{n}(\beta) = \frac{1}{n} \sum_{h=1}^{H} w_{hi} m_{hi} \Delta(\pi_{hi}^{(h)}(\beta)) \otimes x_{hi}^T, \quad H^{(h)}_{n_h}(\beta) = \frac{1}{n_h} \sum_{i=1}^{n_h} w_{hi} m_{hi} \Delta(\pi_{hi}^{(h)}(\beta)) \otimes x_{hi}^T,
\]

\[
\Delta(\pi_{hi}(\beta)) \text{diag}^{-\lambda - 1}(\pi_{hi}(\beta)) = \Delta(\pi_{hi}(\beta)) \text{diag}^{-\lambda}(\pi_{hi}(\beta)) \text{diag}^{-\lambda}(\pi_{hi}(\beta))
\]

and for \( \lambda = -1 \) in (22), we have

\[
\lim_{\lambda \to -1} f_{\phi, hi}(\frac{\hat{y}_{hi}}{m_{hi}}, \beta) = \log \left( \text{diag}^{-\lambda}(\pi_{hi}(\beta)) \hat{y}_{hi} \right),
\]

and therefore

\[
\lim_{\lambda \to -1} \nu_{\phi, hi}(\beta) = w_{hi}m_{hi} \Delta(\pi_{hi}(\beta)) \log \left( \text{diag}^{-\lambda}(\pi_{hi}(\beta)) \hat{y}_{hi} \right) \otimes x_{hi}.
\]
\[ G_n(\beta) = \frac{1}{n} \sum_{h=1}^{n} \sum_{i=1}^{n_h} V[U_{hi}(\beta)], \quad G_n^{(i)}(\beta) = \frac{1}{n} \sum_{h=1}^{n} \sum_{i=1}^{n_h} V[U_{hi}(\beta)], \quad V[U_{hi}(\beta)] = w_h^2 V[\hat{Y}_{hi}] \otimes x_{hi}x_{hi}^T. \]

\( H(\beta) \) is the Fisher information matrix, \( V[\cdot] \) denotes the variance-covariance matrix of a random vector and \( U_{hi}(\beta) \) is the random variable generator of \( u_{hi}(\beta) \), given by (10).

**Proof.** From Theorem 2 and by following the same steps of the linearization method of Binder (1983),

\[ G(\beta) = \lim_{n \to \infty} V[\frac{1}{\sqrt{n}} U_{\phi}(\beta)] \quad \text{and} \quad H(\beta) = -\lim_{n \to \infty} \frac{1}{n} \frac{\partial U_{\phi}^T(\beta)}{\partial \beta}, \]

where \( U_{\phi}(\beta) \) is the random vector generator of \( u_{\phi}(\beta) \), given by (15). Taking into account that \( f_{\phi_{hi}}(\pi_{hi}(\beta), \beta) = 0 \) and \( f_{\phi_{hi}}(\pi_{hi}(\beta), \beta) = \frac{1}{\pi_{hi}(\beta)} \frac{\partial f_{\phi_{hi}}}{\partial \beta}(1) \), a first Taylor expansion of \( f_{\phi_{hi}}(\hat{\pi}_{hi}, \beta) \) given in (17) is

\[ f_{\phi_{hi}}(\hat{\pi}_{hi}, \beta) = f_{\phi_{hi}}(\pi_{hi}(\beta), \beta) + \frac{1}{\pi_{hi}(\beta)} f_{\phi_{hi}}'(\pi_{hi}(\beta))(\hat{\pi}_{hi} - \pi_{hi}(\beta)) + o\left(\frac{1}{\pi_{hi}(\beta)}(\hat{\pi}_{hi} - \pi_{hi}(\beta))\right), \]

i.e.

\[ f_{\phi_{hi}}(\hat{\pi}_{hi}, \beta) = \frac{\partial f_{\phi_{hi}}}{\partial \beta}(1) \text{diag}^{-1}(\pi_{hi}(\beta))(\hat{\pi}_{hi} - \pi_{hi}(\beta)) + o\left(\frac{1}{\pi_{hi}(\beta)}(\hat{\pi}_{hi} - \pi_{hi}(\beta))\right), \]

and hence from (18)

\[ \frac{1}{\sqrt{n}} U_{\phi}(\beta) = \frac{1}{\sqrt{n}} \sum_{h=1}^{H} \sum_{i=1}^{n_h} w_{hi} m_{hi} \frac{\partial \pi_{hi}(\beta)}{\partial \beta} \text{diag}^{-1}(\pi_{hi}(\beta))(\hat{\pi}_{hi} - \pi_{hi}(\beta)) + \frac{1}{\sqrt{n}} \sum_{h=1}^{H} \sqrt{n_h} o \left(\frac{1}{\sqrt{n_h}} \left(\sum_{i=1}^{n_h} \hat{Y}_{hi} - \sum_{i=1}^{n_h} m_{hi} \pi_{hi}(\beta)\right)\right). \]

From the Central Limit Theorem given in Rao (1973, page 147)

\[ \frac{1}{\sqrt{n}} \left(\sum_{i=1}^{n_h} \hat{Y}_{hi} - \sum_{i=1}^{n_h} m_{hi} \pi_{hi}(\beta)\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0_{d+1}, \lim_{n_h \to \infty} \frac{1}{n_h} \sum_{i=1}^{n_h} V[\hat{Y}_{hi}]), \]

then

\[ o \left(\frac{1}{\sqrt{n}} \left(\sum_{i=1}^{n_h} \hat{Y}_{hi} - \sum_{i=1}^{n_h} m_{hi} \pi_{hi}(\beta)\right)\right) = o_p(1_{d+1}) = o_p(1_{d+1}). \]

and thus

\[ \frac{1}{\sqrt{n}} U_{\phi}(\beta) = \frac{1}{\sqrt{n}} \sum_{h=1}^{H} \sum_{i=1}^{n_h} w_{hi} \frac{\partial \log \pi_{hi}(\beta)}{\partial \beta} ((\hat{y}_{hi} - m_{hi} \pi_{hi}(\beta)) + o_p(1_{d+1}). \]

Since

\[ \frac{\partial \log \pi_{hi}(\beta)}{\partial \beta} \pi_{hi}(\beta) = \frac{\partial \pi_{hi}(\beta)}{\partial \beta} \text{diag}^{-1}(\pi_{hi}(\beta))\pi_{hi}(\beta) \]

\[ = \frac{\partial \pi_{hi}(\beta)}{\partial \beta} \frac{\partial (\pi_{hi}(\beta)1_{d+1})}{\partial \beta} = 0_{d+1}, \]

\[ \frac{\partial \log \pi_{hi}(\beta)}{\partial \beta} y_{hi} = \frac{\partial \pi_{hi}(\beta)}{\partial \beta} \text{diag}^{-1}(\pi_{hi}(\beta))\hat{Y}_{hi} \]

\[ = ((I_{d \times d}, 0_{d \times 1}) \Delta(\pi_{hi}(\beta)) \otimes x_{hi}) \text{diag}^{-1}(\pi_{hi}(\beta))\hat{Y}_{hi} \]

\[ = ((I_{d \times d}, 0_{d \times 1}) \Delta(\pi_{hi}(\beta)) \text{diag}^{-1}(\pi_{hi}(\beta))\hat{Y}_{hi} \otimes x_{hi} = ((I_{d \times d}, 0_{d \times 1}) \hat{Y}_{hi} - m_{hi} \pi_{hi}(\beta) \otimes x_{hi} \]

\[ = (\hat{y}_{hi} - m_{hi} \pi_{hi}(\beta)) \otimes x_{hi} = (\hat{y}_{hi} - m_{hi} \pi_{hi}(\beta)) \otimes x_{hi}, \]

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it follows that
\[
\frac{1}{\sqrt{n}}U_\phi (\beta) = \frac{1}{\sqrt{n}} \sum_{h=1}^{H} \sum_{i=1}^{n_h} w_{hi} (\bar{y}_{hi} - m_{hi} \pi_{hi}^*(\beta)) \otimes x_{hi} + o_p(1_{dk}),
\] 
(24)
Then \( H(\beta_0) \) is the limit of
\[
-\frac{1}{n} \frac{\partial}{\partial \beta} U_\phi^T (\beta) = -\frac{1}{n} \sum_{h=1}^{H} \sum_{i=1}^{n_h} w_{hi} m_{hi} \frac{\partial}{\partial \beta} \pi_{hi}^*(\beta) \otimes x_{hi} + o_p(1_{dk} x_{dk})
\]
\[
= -\frac{1}{n} \sum_{h=1}^{H} \sum_{i=1}^{n_h} w_{hi} m_{hi} \Delta(\pi_{hi}^*(\beta)) \otimes x_{hi} + o_p(1_{dk} x_{dk}),
\]
as \( n \) increases, and hence \( H(\beta) = \lim_{n \to \infty} H_n(\beta) \). On the other hand, from (24) it follows that
\[
\frac{1}{\sqrt{n}}U_\phi (\beta) = \frac{1}{\sqrt{n}} U (\beta) + o_p(1_{dk}),
\]
and this justifies that \( G(\beta) = \lim_{n \to \infty} G_n(\beta) \). \( \blacksquare \)

The following result justifies how to estimate \( G_n(\beta) \), in particular \( \hat{G}_n(\hat{\beta}_p) \) given in (20), which is provided by the SURVEYLOGISTIC procedure of SAS.

**Remark 5** Matrix \( G(\beta_0) \) of Theorem [4] can be consistently estimated as
\[
\hat{G}_n(\hat{\beta}_p) = \frac{1}{n} \sum_{h=1}^{H} \sum_{i=1}^{n_h} (u_{hi}(\hat{\beta}_p) - \frac{1}{n} u(\hat{\beta}_p)) \left( u_{hi}(\hat{\beta}_p) - \frac{1}{n} u(\hat{\beta}_p) \right)^T,
\] 
(25)
with \( \hat{\beta}_p \) being any pseudo minimum phi-divergence estimator of parameter \( \beta \). In particular, if \( \phi(x) = x \log x - x + 1 \),
\[
\hat{G}_n(\hat{\beta}_p) = \frac{1}{n} \sum_{h=1}^{H} \sum_{i=1}^{n_h} u_{hi}(\hat{\beta}_p) u_{hi}^T(\hat{\beta}_p),
\] 
(26)
since \( u(\hat{\beta}_p) = 0_{dk} \). On the other hand, matrix \( H(\beta_0) \) of Theorem [4] can be consistently estimated as
\[
H_n(\hat{\beta}_p) = \frac{1}{n} \sum_{h=1}^{H} \sum_{i=1}^{n_h} w_{hi} m_{hi} \Delta(\pi_{hi}^*(\hat{\beta}_p)) \otimes x_{hi} x_{hi}^T.
\]

Let \( \hat{\beta}_p \) denote the minimum phi-divergence estimator of \( \beta \) for simple random sampling within each cluster, i.e. multinomial sampling. By following Gupta and Pardo (2007), it can be seen that
\[
\lim_{n \to \infty} V[\sqrt{n} \hat{\beta}_p] = H^{-1}(\beta_0).
\]
The “design effect matrix” for the multinomial logistic regression model with sample survey design is defined as
\[
\lim_{n \to \infty} V[\sqrt{n} \hat{\beta}_p] V^{-1}[\sqrt{n} \hat{\beta}_p] = H^{-1}(\beta_0) G(\beta_0)
\] and the “design effect”, denoted by \( \nu \), for the multinomial logistic regression model with sample survey design is defined as \( \nu(\beta_0) = \frac{1}{dk} \text{trace} (H^{-1}(\beta_0) G(\beta_0)) \). In practice, \( H(\beta_0) \) and \( G(\beta_0) \) can be consistently estimated through the pseudo minimum phi-divergence estimator of parameter \( \beta \) as
\[
H_n(\hat{\beta}_p) = \frac{1}{n} \sum_{h=1}^{H} \sum_{i=1}^{n_h} w_{hi} m_{hi} \Delta(\pi_{hi}^*(\hat{\beta}_p)) \otimes x_{hi} x_{hi}^T,
\]
and \( \hat{G}_n(\hat{\beta}_p) \) given in (25). For more details about the design matrix in other models see for instance Rao and Scott (1984) or formula 7.6 in Rao and Thomas (1989).
Definition 6 A consistent estimator of the design effect matrix, $H^{-1}(\beta)G(\beta)$, based on the linearization method of Binder (1983) and the pseudo minimum phi-divergence estimator of parameter $\beta$, is

$$H_{\nu}^{-1}(\hat{\beta}_{\phi,p})\overline{G}_{n}(\hat{\beta}_{\phi,p}) = \left(\sum_{h=1}^{H}n_{h}m_{h}\Delta(\pi_{h,i}(\hat{\beta}_{\phi,p})) \otimes x_{h,i}x_{h,i}^{T}\right)^{-1}$$

$$\times \sum_{h=1}^{H}n_{h} \left(\bar{u}_{h}(\hat{\beta}_{\phi,p}) - \frac{1}{n}u(\hat{\beta}_{\phi,p})\right) \left(\bar{u}_{h}(\hat{\beta}_{\phi,p}) - \frac{1}{n}u(\hat{\beta}_{\phi,p})\right)^{T}.$$  

Similarly, a consistent estimator of the design effect, $\nu(\beta_{0}) = \frac{1}{dk}\text{trace}(H^{-1}(\beta_{0})G(\beta_{0}))$, based on the linearization method of Binder (1983) and the pseudo minimum phi-divergence estimator of parameter $\beta$, is

$$\nu(\hat{\beta}_{\phi,p}) = \frac{1}{dk}\text{trace}\left(H_{\nu}^{-1}(\hat{\beta}_{\phi,p})\overline{G}_{n}(\hat{\beta}_{\phi,p})\right).$$  

The estimator of the design effect is specially interesting for clusters such that

$$E[\hat{Y}_{hi}] = m_{h}\pi_{h,i}(\beta_{0})$$  and  $$V[\hat{Y}_{hi}] = \nu_{mh}m_{h}\Delta(\pi_{h,i}(\beta_{0})),$$

with $\nu_{mh}$ being the overdispersion parameter, $\rho_{h}^{2}$ being the intra-cluster correlation coefficient and equal cluster sizes in the strata, $m_{hi} = m_{h}$, $h = 1, ..., H$, $i = 1, ..., n_{h}$. Examples of distributions of $\hat{y}_{hi}$ verifying (28) are the so-called “overdispersed multinomial distributions” (see Alonso et al. (2016)). For these distributions, once the pseudo minimum phi-divergence estimator of parameter $\beta$, $\hat{\beta}_{\phi,p}$, is obtained, the interest lies in estimating $\rho_{h}^{2}$.

In Theorems 7 and 9 two proposals of families of estimates for $\nu_{mh}$ and $\rho_{h}^{2}$ are established. Both proposals are independent of the weights except for $\hat{\beta}_{\phi,p}$, and this fact has a logical explanation taking into account that the weights are constructed only for estimation of $\beta$.

Theorem 7 Let $\hat{\beta}_{\phi,p}$ the pseudo minimum phi-divergence estimate of parameter $\beta$ for a multinomial logistic regression model with “overdispersed multinomial distribution”. Assume that $w_{hi} = w_{h}$, $i = 1, ..., n_{h}$. Then

$$\hat{\nu}(\hat{\beta}_{\phi,p}) = \frac{1}{dk}\text{trace}\left(\sum_{i=1}^{n_{h}}m_{h}\Delta(\pi_{h,i}(\hat{\beta}_{\phi,p})) \otimes x_{h,i}x_{h,i}^{T}\right)^{-1}$$

$$\times \sum_{i=1}^{n_{h}} \left(\bar{v}_{h}(\hat{\beta}_{\phi,p}) - \bar{v}_{h}(\hat{\beta}_{\phi,p})\right) \left(\bar{v}_{h}(\hat{\beta}_{\phi,p}) - \bar{v}_{h}(\hat{\beta}_{\phi,p})\right)^{T}.$$  

with

$$v_{h,i}(\hat{\beta}_{\phi,p}) = r_{hi}^{*}(\beta) \otimes x_{h,i},$$

$$\bar{v}_{h}(\hat{\beta}_{\phi,p}) = \frac{1}{n} \sum_{k=1}^{n_{h}}v_{h,k}(\hat{\beta}_{\phi,p}),$$

is an estimator of $\nu_{mh}$ based on the “linearization method of Binder” and the pseudo minimum phi-divergence estimator of $\hat{\beta}_{\phi,p}$, and

$$\hat{\rho}_{h}^{2}(\hat{\beta}_{\phi,p}) = \frac{\hat{\nu}(\hat{\beta}_{\phi,p}) - 1}{n_{h} - 1}$$

is an estimator of $\rho_{h}^{2}$ based on the “linearization method of Binder” and the pseudo minimum phi-divergence estimator of $\hat{\beta}_{\phi,p}$.
Proof. If $V[\hat{Y}_{hi}] = \nu_m_h \Delta(\pi_{hi}(\beta_0))$, then from the expression of $G_{nh}^{(h)}(\beta_0)$ given in Theorem 5,

$$G_{nh}^{(h)}(\beta_0) = \frac{1}{n_h} \sum_{i=1}^{n_h} w_h^2 V[\hat{Y}_{hi}] \otimes x_{hi} x_{hi}^T = \nu_m h \frac{1}{n_h} \sum_{i=1}^{n_h} w_h m_h \Delta(\pi_{hi}^{(h)}(\beta_0)) \otimes x_{hi} x_{hi}^T = \nu_m h w_h H_{nh}^{(h)}(\beta_0).$$

Hence, from

$$\text{trace}(H_{nh}^{(h)}(\beta_0)^{-1} G_{nh}^{(h)}(\beta_0)) = \nu_m h w_h dk,$$

and consistency of $H_{nh}^{(h)}(\hat{\beta}_\phi,P)$ and $G_{nh}^{(h)}(\hat{\beta}_\phi,P)$,

$$\hat{\nu}_{nh}(\hat{\beta}_\phi,P) = \frac{1}{dk} \text{trace}\left(\frac{1}{w_h} H_{nh}^{(h)}(\hat{\beta}_\phi,P)^{-1} G_{nh}^{(h)}(\hat{\beta}_\phi,P)\right),$$

is proven with

$$\frac{1}{w_h} H_{nh}^{(h)}(\hat{\beta}_\phi,P)^{-1} G_{nh}^{(h)}(\hat{\beta}_\phi,P) = \left(\sum_{i=1}^{n_h} m_h \Delta(\pi_{hi}^{(h)}(\hat{\beta}_\phi,P)) \otimes x_{hi} x_{hi}^T\right)^{-1} \times \sum_{i=1}^{n_h} \left(\nu_{hi}(\hat{\beta}_\phi,P) - \bar{v}_h(\hat{\beta}_\phi,P)\right) \left(\nu_{hi}(\hat{\beta}_\phi,P) - \bar{v}_h(\hat{\beta}_\phi,P)\right)^T,$$

$$\nu_{hi}(\hat{\beta}_\phi,P) = \frac{1}{w_h} u_{hi}(\beta),$$

which is equivalent to (29). □

Remark 8 Since

$$\nu_{nh}(\hat{\beta}_\phi,P) = \frac{1}{w_h} \frac{1}{dk} \text{trace}\left(H_{nh}^{(h)}(\hat{\beta}_\phi,P)^{-1} G_{nh}^{(h)}(\hat{\beta}_\phi,P)\right) = \frac{1}{w_h} \hat{\nu}^{(h)}(\hat{\beta}_\phi,P),$$

(30)

unless $w_h = 1$, the overdispersion parameter $\nu_{nh}(\hat{\beta}_\phi,P)$ and the design effect $\hat{\nu}^{(h)}(\hat{\beta}_\phi,P)$ of the $h$-th stratum are not in general equivalent. Based on the expression of (29) $\nu_{nh}()$, does not depend on the weights except for that $\hat{\beta}_\phi,P$ is plugged in $\nu_{nh}()$, additionally based on (30) it is concluded that $\hat{\nu}^{(h)}(\hat{\beta}_\phi,P)$ is directly proportional to the weights.

Theorem 9 Let $\hat{\beta}_\phi,P$ the pseudo minimum phi-divergence estimate of parameter $\beta$ for a multinomial logistic regression model with “overdispersed multinomial distribution”. Then

$$\nu_{nh}(\hat{\beta}_\phi,P) = \frac{1}{n_h d} \sum_{i=1}^{n_h} \sum_{s=1}^{d+1} \left(\hat{y}_{his} - m_h \pi_{his}(\hat{\beta}_\phi,P)\right)^2$$

is an estimation of $\nu_{nh}$ based on the “method of moments” and the pseudo minimum phi-divergence estimator of $\hat{\beta}_\phi,P$, and

$$\hat{\rho}_h^{(h)}(\hat{\beta}_\phi,P) = \frac{\nu_{nh}(\hat{\beta}_\phi,P) - 1}{m_h - 1}$$

is an estimation of $\rho_h^2$ based on the “method of moments” and the pseudo minimum phi-divergence estimator of $\hat{\beta}_\phi,P$. 

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Proof. The mean vector and variance-covariance matrix of
\[ Z_{hi}^*(\beta_0) = \sqrt{m_h} \Delta^{-\frac{1}{2}}(\pi_{hi}^*(\beta_0))(\frac{\hat{\pi}_{hi}^*}{m_h} - \pi_{hi}^*(\beta_0)), \]
are respectively
\[ E[Z_{hi}^*(\beta_0)] = 0_d, \]
\[ V[Z_{hi}^*(\beta_0)] = \nu_{mh} I_d, \]
for \( h = 1, \ldots, H \). An unbiased estimator of \( V[Z_{hi}^*(\beta_0)] \) is
\[ \hat{V}[Z_{hi}^*(\beta_0)] = \frac{1}{n_h} \sum_{i=1}^{n_h} Z_{hi}^*(\beta_0)Z_{hi}^{*T}(\beta_0), \]
from which is derived
\[ E \left[ \text{trace} \hat{V}[Z_{hi}^*(\beta_0)] \right] = \text{trace} V[Z_{hi}^*(\beta_0)], \]
\[ E \left[ \frac{1}{n_h} \sum_{i=1}^{n_h} \text{trace} \left( Z_{hi}^*(\beta_0)Z_{hi}^{*T}(\beta_0) \right) \right] = \text{trace} (\nu_{mh} I_d), \]
\[ E \left[ \frac{1}{n_h} \sum_{i=1}^{n_h} Z_{hi}^{*T}(\beta_0)Z_{hi}^*(\beta_0) \right] = \nu_{mh} d, \]
\[ E \left[ \frac{1}{n_h d} \sum_{i=1}^{n_h} Z_{hi}^{*T}(\beta_0)Z_{hi}^*(\beta_0) \right] = \nu_{mh}. \]
This expression suggest using
\[ \tilde{\nu}_{mh}(\hat{\beta}_{\phi,P}) = \frac{1}{n_h d} \sum_{i=1}^{n_h} Z_{hi,\phi,P}^{*T}(\hat{\beta}_{\phi,P})Z_{hi,\phi,P}(\hat{\beta}_{\phi,P}) \]
\[ = \frac{1}{n_h d} m_h \left( \frac{\hat{\pi}_{hi}^* - \pi_{hi}^*(\beta_{\phi,P})}{m_h} \right)^T \Delta^{-\frac{1}{2}}(\pi_{hi}^*(\beta_{\phi,P})) \left( \frac{\hat{\pi}_{hi}^* - \pi_{hi}^*(\beta_{\phi,P})}{m_h} \right) \]
\[ = \frac{1}{n_h d} m_h \left( \frac{\hat{\pi}_{hi}^* - \pi_{hi}^*(\beta_{\phi,P})}{m_h} \right)^T \Delta^{-\frac{1}{2}}(\pi_{hi}^*(\beta_{\phi,P})) \left( \frac{\hat{\pi}_{hi}^* - \pi_{hi}^*(\beta_{\phi,P})}{m_h} \right), \]
\[ Z_{hi,\phi,P}^* = \sqrt{m_h} \Delta^{-\frac{1}{2}}(\pi_{hi}^*(\beta_{\phi,P})) \left( \frac{\hat{\pi}_{hi}^*}{m_h} - \pi_{hi}^*(\beta_{\phi,P}) \right). \]
Finally, since \( \Delta^{-\frac{1}{2}}(\pi_{hi}^*(\beta_{\phi,P})) = \text{diag}^{-\frac{1}{2}}(\pi_{hi}^*(\beta_{\phi,P})) \), is a possible expression for the generalized inverse, the desired result for \( \tilde{\nu}_{mh}(\hat{\beta}_{\phi,P}) \) is obtained. \( \blacksquare \)

4 Numerical Example

In this Section we shall consider an example, which appears in SAS Institute Inc. (2013, Chapter 95) as well as in An (2002), in order to illustrate how does the pseudo minimum phi-divergence estimator work for the multinomial logistic regression with complex sample survey.

A market research firm conducts a survey among undergraduate students at the University of North Carolina (UNC), at Chapel Hill, to evaluate three new web designs at a commercial web-site targeting undergraduate students. The total number of student in each class in the Fall semester of 2001 is shown in Table 1. The
sample design is a stratified sample with clusters nested on them, with the strata being the four students’ classes and the clusters the three web designs. Initially 100 students were planned to be randomly selected in each of the $n = 12$ web designs using sample random sampling (without replacement). For this reason, the weights for estimation are considered to be $w_1 = \frac{3734}{300}$, $w_2 = \frac{3565}{300}$, $w_3 = \frac{3903}{300}$, $w_4 = \frac{4196}{300}$. Since $m_{hi} = 100$ for $h = 1, 2, 3, 4 = H$ (strata), $i = 1, 2, 3 = n_h$ (clusters) except for $m_{12} = 90$ and $m_{43} = 97$, in practice some observations are missing values. Each student selected in the sample is asked to evaluate the three Web designs and to rate them ranging from dislike very much to like very much: (1) dislike very much, (2) dislike, (3) neutral, (4) like, (5 = $d + 1$) like very much. The survey results are collected and shown in Table 2 with the three different Web designs coded A, B and C. This table matches the one given in An (2002) and the version appeared in SAS Institute Inc. (2013, Chapter 95) is slightly different.

| Class   | Enrollment |
|---------|------------|
| Freshman| 3734       |
| Sophomore| 3565     |
| Junior  | 3903       |
| Senior  | 4196       |

Table 1: Number of student in each class of the target population for the survey.

| Strata   | Design | Rating Counts |
|----------|--------|---------------|
|          |        | 1 2 3 4 5     |
| Freshman | A      | 10 34 25 16 15|
|          | B      | 5 10 24 30 21 |
|          | C      | 11 14 20 34 21|
| Sophomore| A      | 19 12 26 18 25|
|          | B      | 10 18 32 23 17|
|          | C      | 15 22 34 9  20|
| Junior   | A      | 8 21 23 26 22 |
|          | B      | 1 14 25 23 37 |
|          | C      | 16 19 30 23 12|
| Senior   | A      | 11 14 24 33 18|
|          | B      | 8 15 35 30 12 |
|          | C      | 2  34 27 18 16|

Table 2: Evaluation of New Web Designs.

The explanatory variables are qualitative, and valid to distinguish the clusters within the strata. With respect to design A, it is given by $x_{h1}^T = x_1^T = (1, 0, 0)$, $h = 1, 2, 3, 4$; with respect to design B, by $x_{h2}^T = x_2^T = (0, 1, 0)$, $h = 1, 2, 3, 4$; with respect to design C, by $x_{h3}^T = x_3^T = (0, 0, 1)$, $h = 1, 2, 3, 4$. In Table 2 every
row represents the pseudo minimum Cressie-Read divergence estimates of the 5-dimensional probability vector \( \pi_h(\hat{\beta}_{\phi, P}) = \pi_i(\hat{\beta}_{\phi, P}) \), for the \( i \)-th cluster \( i = 1, 2, 3 \), for any stratum \( h = 1, 2, 3, 4 \), and a specific value in \( \lambda \in \{0, \frac{2}{3}, 1, 1.5, 2, 2.5\} \). Each column of Table 4 summarizes, first the pseudo minimum Cressie-Read divergence estimates of \( \beta = (\beta^T_1, \beta^T_2, \beta^T_3, \beta^T_4)^T \), with \( \beta^T_i = (\beta_{i1}, \beta_{i2}, \beta_{i3}) \) \( i = 1, 2, 3, 4 \) and \( \lambda \in \{0, \frac{2}{3}, 1, 1.5, 2, 2.5\} \), as well as the two versions of the intra-cluster correlation estimates according to Theorems 7 and 9 for the strata with the same cluster sizes, i.e. Sophomore (2) and Junior (3). Section 5 is devoted to study through simulation the best choice for the value of \( \lambda \) according to the root of the minimum square error of \( \tilde{\rho}^2(\hat{\beta}_{\phi, P}) \) and \( \tilde{\rho}^2(\hat{\beta}_{\phi, P}) \).

| \lambda | Design | 1     | 2     | 3     | 4     | 5     |
|---------|--------|-------|-------|-------|-------|-------|
| 0       | A      | 0.1185| 0.2016| 0.2445| 0.2363| 0.1991|
|         | B      | 0.0611| 0.1458| 0.2983| 0.2727| 0.2222|
|         | C      | 0.1083| 0.2276| 0.2791| 0.2124| 0.1727|
| \frac{2}{3} | A    | 0.1200| 0.2079| 0.2387| 0.2369| 0.1965|
|         | B      | 0.0660| 0.1439| 0.2931| 0.2672| 0.2297|
|         | C      | 0.1145| 0.2275| 0.2723| 0.2167| 0.1690|
| 1       | A      | 0.1208| 0.2109| 0.2359| 0.2371| 0.1952|
|         | B      | 0.0676| 0.1431| 0.2909| 0.2648| 0.2336|
|         | C      | 0.1163| 0.2279| 0.2695| 0.2188| 0.1675|
| 1.5     | A      | 0.1221| 0.2152| 0.2319| 0.2374| 0.1934|
|         | B      | 0.0693| 0.1420| 0.2879| 0.2616| 0.2392|
|         | C      | 0.1179| 0.2289| 0.2659| 0.2215| 0.1657|
| 2       | A      | 0.1234| 0.2191| 0.2282| 0.2376| 0.1917|
|         | B      | 0.0705| 0.1410| 0.2854| 0.2587| 0.2444|
|         | C      | 0.1188| 0.2301| 0.2630| 0.2240| 0.1641|
| 2.5     | A      | 0.1246| 0.2226| 0.2248| 0.2377| 0.1902|
|         | B      | 0.0714| 0.1402| 0.2831| 0.2562| 0.2491|
|         | C      | 0.1192| 0.2314| 0.2604| 0.2262| 0.1628|

Table 3: Pseudo minimum Cressie-Read divergence estimates of probabilities for any of the four strata.
| $\lambda$ | 0  | $\frac{2}{3}$ | 1 | 1.5 | 2 | 2.5 |
|------------|----|---------------|---|-----|---|-----|
| $\hat{\beta}_{11,P}$ | 0.5188 | -0.4933 | -0.4802 | -0.4604 | -0.4411 | -0.4228 |
| $\hat{\beta}_{12,P}$ | -1.2910 | -1.2475 | -1.2400 | -1.2381 | -1.2424 | -1.2494 |
| $\hat{\beta}_{13,P}$ | -0.4665 | -0.3889 | -0.3649 | -0.3397 | -0.3230 | -0.3116 |
| $\hat{\beta}_{21,P}$ | 0.0127 | 0.0564 | 0.0773 | 0.1069 | 0.1336 | 0.1573 |
| $\hat{\beta}_{22,P}$ | -0.4210 | -0.4676 | -0.4899 | -0.5213 | -0.5498 | -0.5750 |
| $\hat{\beta}_{23,P}$ | 0.2761 | 0.2974 | 0.3079 | 0.3233 | 0.3380 | 0.3517 |
| $\hat{\beta}_{31,P}$ | 0.2056 | 0.1947 | 0.1894 | 0.1816 | 0.1741 | 0.1670 |
| $\hat{\beta}_{32,P}$ | 0.2946 | 0.2438 | 0.2196 | 0.1857 | 0.1551 | 0.1280 |
| $\hat{\beta}_{33,P}$ | 0.4803 | 0.4770 | 0.4754 | 0.4733 | 0.4714 | 0.4697 |
| $\hat{\rho}_2^2(\hat{\beta}_{11,P})$ | 0.0119 | 0.0123 | 0.0127 | 0.0135 | 0.0142 | 0.0150 |
| $\hat{\rho}_2^2(\hat{\beta}_{12,P})$ | 0.0119 | 0.0048 | 0.0051 | 0.0056 | 0.0061 | 0.0067 |
| $\hat{\rho}_2^2(\hat{\beta}_{13,P})$ | 0.0088 | 0.0072 | 0.0066 | 0.0059 | 0.0054 | 0.0051 |
| $\hat{\rho}_2^2(\hat{\beta}_{21,P})$ | 0.0088 | 0.0014 | 0.0010 | 0.0006 | 0.0003 | 0.0000 |

Table 4: Pseudo minimum Cressie-Read divergence estimates of $\beta$ and $\rho^2$. 

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5 Simulation Study

In order to analyze the performance of the proposed estimators through root of the mean square errors (RMSE), an adapted design focused in the simulation experiment proposed in Morel (1989) is conducted. Based on a unique stratum with \( n \) clusters of the same size \( m \), three overdispersed multinomial distributions for \( \mathbf{Y}_i \) described as

\[
E[\mathbf{Y}_i] = m \mathbf{\pi}_i (\mathbf{\beta}_0) \quad \text{and} \quad \mathbf{V}[\mathbf{Y}_i] = \nu_m m \Delta (\mathbf{\pi}_i (\mathbf{\beta}_0)),
\]

\[
\nu_m = 1 + \rho^2 (m - 1),
\]

are considered for \( i = 1, \ldots, n \), the Dirichlet-multinomial (DM), the random-clumped (RC) and the \( m \)-inflated distribution (m-I), all of them with the same parameters \( \mathbf{\pi}_i (\mathbf{\beta}_0) \) and \( \rho \) (see Appendix of Alonso et al. (2016) for details of their generators). The value of the true probability associated with the distribution \( \mathbf{\pi}_i (\mathbf{\beta}_0) \)

\[
\pi_{ir} (\mathbf{\beta}_0) = \frac{\exp \{ x_i^T \mathbf{\beta}_{r,0} \}}{\sum_{r=1}^{3} \exp \{ x_i^T \mathbf{\beta}_{r,0} \}}, \quad r = 1, 2, 3, 4,
\]

\[
\mathbf{\beta} = (\mathbf{\beta}_1^T, \mathbf{\beta}_2^T, \mathbf{\beta}_3^T, \mathbf{\beta}_4^T)^T,
\]

\[
\mathbf{\beta}_1 = (-0.3, -0.1, 0.1, 0.2), \quad \mathbf{\beta}_2 = (0.2, -0.2, -0.2, 0.1), \quad \mathbf{\beta}_3 = (-0.1, 0.3, -0.3, 0.1), \quad \mathbf{\beta}_4 = (0, 0, 0, 0),
\]

\[
x_i \overset{ind}{\sim} \mathcal{N}(\mathbf{\mu}, \Sigma), \quad \mathbf{\mu} = (1, -2, 1, 5)^T, \quad \Sigma = \text{diag}\{0, 25, 25, 25\}, \quad i = 1, \ldots, n,
\]

while the value true intra-cluster correlation parameter, \( \rho^2 \), is different depending on the scenario. Notice that \( d = 3 \) and \( k = 4 \), and the values of \( n \) and \( m \) are different depending on the scenario.

- Scenario 1: \( n = 60, m = 21, \rho^2 \in \{0.05i\}_{i=0}^{19} \), DM and m-I distributions (Figures 1-3);
- Scenario 2: \( n \in \{10i\}_{i=1}^{10}, m = 21, \rho^2 = 0.25 \), RC distribution (Figure 4);
- Scenario 3: \( n = 60, m \in \{10i\}_{i=1}^{10}, \rho^2 = 0.25 \), RC distribution (Figures 5-6 above);
- Scenario 4: \( n = 60, m \in \{10i\}_{i=1}^{10}, \rho^2 = 0.75 \), RC distribution (Figures 5-6 middle);
- Scenario 5: \( n = 20, m \in \{10i\}_{i=1}^{10}, \rho^2 = 0.25 \), RC distribution (Figures 5-6 below).

In the previous scenarios the RMSE for the pseudo minimum Cressie-Read divergence estimators of \( \mathbf{\beta} \) with \( \lambda \in \{0, \frac{2}{5}, 1, 1.5, 2, 2.5\} \) are studied, as well as for the estimators of \( \rho^2 \), depending on the method (of moments or Binder) and the value of \( \lambda \) to estimate \( \mathbf{\beta} \) (ordinal axis of the plots). As expected from a theoretical point of view, the simulations show that the RMSE increases as \( \rho^2 \) increases, \( n \) decreases or \( m \) decreases.

For \( \mathbf{\beta} \), the interest of the pseudo minimum Cressie-Read divergence estimators is clearly justified for small-moderate sizes of \( n \) and strong-moderate intra-cluster correlation. The cluster size, \( m \), affects but not so much as the number of clusters, \( n \). More thoroughly, in these cases, the value of \( \lambda \in \{\frac{2}{5}, 1, 1.5, 2, 2.5\} \) exhibits better performance than the pseudo maximum likelihood estimator (\( \lambda = 0 \)).

For the estimators of the intra-cluster correlation coefficient two clear and important findings, valid for any value of \( n \), \( m \), or true value of \( \rho^2 \), are:
* The estimator of $\rho^2$ with the method of of moments is not recommended, since the estimator with the Binder’s method is much better.

* The best estimator of $\rho^2$ with the Binder’s method is obtained with $\lambda = \frac{2}{3}$.

6 Concluding remarks

Even though the multinomial logistic regression is an extensively applied model, in our knowledge there is no study which compares the method of moments and the Binder’s method for estimating the intrachuster correlation coefficient. The simulation study designed in this paper shows that the Binder’s method is by far the best choice.

As future research, we would like to extend the proposed method to be valid for estimating the $\beta$ and $\rho^2$ for different cluster sizes.
Figure 1: RMSEs of pseudo minimum Cressie-Read divergence estimators of $\beta$ for three distributions.
Figure 2: RMSEs of estimators of $\rho^2$ based on the method of moments for three distributions.
Figure 3: RMSEs of estimators of $\rho^2$ based on the method of Binder.
Figure 4: RMSEs of estimators of $\beta$ and $\rho^2$ when the total number of clusters, $n$, increases, for the random clumped distribution. Case $m = 21$, $\rho = 0.25$. 

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Figure 5: RMSEs of estimators of $\beta$ when the number of individuals within clusters, $m$, increases, for the random clumped distribution. Cases: $n = 60$, $\rho = 0.25$ (above), $n = 60$, $\rho = 0.75$ (middle), $n = 20$, $\rho = 0.25$ (below).
Figure 6: RMSEs of estimators of $\rho^2$ (Binder’s method) when the number of individuals within clusters, $m$, increases, for the random clumped distribution. Cases: $n = 60$, $\rho = 0.25$ (above), $n = 60$, $\rho = 0.75$ (middle), $n = 20$, $\rho = 0.25$ (below).
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