CATEGORICAL NOTIONS OF
LAYERED TROPICAL ALGEBRA AND GEOMETRY

ZUR IZHAKIAN, MANFRED KNEBUSCH, AND LOUIS ROWEN

Abstract. This paper supplements [17], showing that categorically the layered theory is the same as
the theory of ordered monoids (e.g., the max-plus algebra) used in tropical mathematics. A layered
theory is developed in the context of categories, together with a “tropicalization functor” which permits
us to pass from usual algebraic geometry to the tropical world. We consider tropical varieties from this
categorical viewpoint, with emphasis on polynomial functions and their roots.

1. Introduction

Tropical geometry has led to considerable mathematical success in degenerating various combinatoric
questions. At the algebraic level, the degeneration often has led to the max-plus algebra, but in certain
ways this is too crude a process to preserve many important algebraic properties. Over the last few
years, the theory of supertropical algebras has been developed in a series of papers including [16], [22],
[23], [24], and [25], in which classical notions of commutative algebra pertaining to algebraic varieties,
matrices, and valuations, carry over intrinsically to the “tropical” world. This degeneration still is too
crude to handle several issues, such as multiple roots of polynomials. A more refined structure, called
$L$-layered domains, was introduced in [17] together with its basic traits, in order to be able to preserve
more algebraic properties.

Recent years have seen considerable progress in the algebraic perspective of tropical geometry. Notably,
building on work of Bieri-Groves [5], Berkovich [3] and Payne [36] have shown how to view the analyti-
fication of an affine variety algebraically, in terms of valuations and multiplicative seminorms extending
valuations.

This paper is part of a series including [17] and [20]. In [17] we showed by example how the layered
structure can cope with algebraic aspects of tropical mathematics that are inaccessible to less refined
structures; some of these examples are reviewed here for the reader’s convenience.

Our main purpose in this paper is to provide a more formal, unified foundation for further study,
for both the algebraic and geometric aspects. Since category theory pervades modern mathematics so
thoroughly, one feels obligated to describe the theory in categorical terms, and indeed this language
provides valuable information as to how the theory should progress, thereby throwing further light on
tropical geometry. We aim to understand those categories arising from algebraic considerations, focusing
on those algebraic aspects of the theory that reflect most directly on tropical geometry, largely via
a Zariski-type correspondence. To describe these categories in full detail would involve an inordinate
amount of technical detail, so we often make simplifying assumptions when they do not impact on the
tropical applications. Even so, each aspect of the theory involves its corresponding categories, and so

Date: May 2, 2014.

2010 Mathematics Subject Classification. Primary 06F20, 11C08, 12K10, 14T05, 14T99, 16Y60; Secondary 06F25,
16D25.

Key words and phrases. Tropical categories, tropical algebra, tropical geometry, valued monoids, valuation, tropicaliza-
tion, Zariski topology.

This research of the first and third authors is supported by the Israel Science Foundation (grant No. 448/09).
The research of the first author has been supported by the Oberwolfach Leibniz Fellows Programme (OWLF), Mathem-
atisches Forschungsinstitut Oberwolfach, Germany.
The second author was supported in part by the Gelbart Institute at Bar-Ilan University, the Minerva Foundation at
Tel-Aviv University, the Mathematics Dept. of Bar-Ilan University, and the Emmy Noether Institute. Research on this
paper was carried out by the three authors in the Research in Pairs program of the MFO in Oberwolfach.

We thank Steve Shnider and Erez Sheiner for explaining the proof of Payne’s theorem in our tropical seminar.
there are many categories to be described here. Although the language involves some technicalities, we try to keep it to a minimum, leaving subtler matters to [20]. Another related paper is [19], which delves into considerable detail in the supertropical setting, for which we generalize parts to the layered setting.

To obtain the appropriate functors, we need first to make categories of the classical “algebraic world” and the “tropical world.” Informally, the classical “algebraic world” is described by the categories associated to classical algebraic geometry, often over the complex numbers $\mathbb{C}$.

A deep connection between tropical geometry and valuation theory is already implicit in [5], and it is convenient to work over algebraically closed fields with valuation. Thus, as the algebraic aspect of the tropical theory has developed, $\mathbb{C}$ has been replaced by the field of Puiseux series, an algebraically closed field endowed with a (nonarchimedean) valuation, whose target is an ordered group, so it makes sense to work with ordered groups, or, slightly more generally, ordered monoids.

There has been considerable recent interest in developing algebraic geometry over arbitrary monoids [4, 8], and we shall draw on their work. One theme of this paper is how the assumption of an order on the monoid enriches the theory. The ordered monoid most commonly used in the tropical world is the “max-plus” algebra $\mathcal{M}$ (or its dual, the “min-plus” algebra), cf. [1], [2], [14], and [30]. Our first main result (Proposition 3.11), which sets the flavor for the paper, is that the category of ordered cancellative monoids is isomorphic to the category of bipotent semirings (without a zero element). Any ordered monoid can be viewed as a semiring, where multiplication is the given monoid operation and addition is defined by taking $a + b$ to be $\max\{a, b\}$ in Proposition 3.11. The universal of the appropriate forgetful functor is constructed in this context, in Proposition 3.13.

Since the underlying algebraic structures now are semirings, we switch to the language of semirings in order to be able to adapt concepts from ring theory and module theory, such as polynomials and matrices. We find it more convenient to work in the category of semirings†, defined as semirings not necessarily having a zero element, for the following reasons:

- The duality given in Proposition 3.12 holds for semirings† but not for semirings;
- Proofs are usually neater for semirings†, since the zero element 0 of a semiring needs special treatment;
- Many important examples (such as Laurent series and tori) are defined over semirings† but not over semirings (and in particular, Bieri-Groves‘ main theorem [5, Theorem A] is given for multiplicative groups);
- Once we get started with the layered theory, it is more natural to utilize a 0-layer (an ideal comprised of several elements) rather than a single element 0; anyway, one can recover the element 0 by inserting it into the 0-layer.

One might counter that various critical aspects of geometry such as intersection theory (which involve curves such as $xy = 0$) require a zero element. This turns out to be less important in the tropical theory since the zero element, $-\infty$, already is artificial, and can be dealt with at the appropriate time.

To describe tropicalization categorically, we utilize the category ValField describing fields with valuation, or, slightly more generally, the category ValDom describing integral domains with valuations. (In the sequel [20] to this paper, we proceed still further, with valued rings.) The theory is applicable to fields with valuation, in particular to the Puiseux series field. Intuitively, the corresponding tropical category just reformulates the valuation, where the operations are taken from the target of the valuation. Our category ValMon (cf. §4) is described in the language of monoids, in order to permit other tropicalization techniques.

At this point, let us stress that one principal role of the tropical algebra is to provide an intrinsic algebraic setting for studying valuations in ValField and their extensions, as described in [6,2] via Maslov dequantization [20] or the degeneration of “amoebas” [11, 32], and [40]. In a few words, one takes the power valuation to pass from the Puiseux series field to $\mathbb{Q}$, viewed as the max-plus algebra. This is formalized in Remark 4.9 as the functor $\mathcal{F}_{\text{val}}$ from the category of valued monoids to the category of ordered monoids (or, equivalently, bipotent semirings).

Unfortunately, the algebraic theory of bipotent semirings† is too weak to provide much information without additional structure. Accordingly, the algebra $\mathcal{M}$ was extended to extended tropical arithmetic [15] which evolved into the supertropical domain [22] and then to the layered domain† $\mathcal{R}(L, G)$.
of an ordered monoid \(G\) with respect to an indexing semiring\(^\dagger\) \(L\), called the the \textit{sorting set}, cf. [17] Definition 3.5. \(L\)-layered domains\(^{\dagger}\) become max-plus algebras when \(L\) is \(\{1\}\) and become supertropical domains when \(L\) is \(\{1, \infty\}\).

The general \(L\)-layered theory, set forth in \([15, 16]\) has many advantages over the other theories, as shown in \([17]\), because it enables us to distinguish among different ghost levels. This is really a linguistic distinction, as is explained in the next paragraph. Nevertheless, there is a definite advantage in making use of the tools available in the language of layered semirings.

Whereas the supertropical domain enables us to distinguish multiple tropical roots (say in the polynomial \(f(\lambda) = (\lambda + 3)^3\) from single roots, it does not say anything about the multiplicity of the corner root 3. Thus, it would not enable us to tell intrinsically whether 3 is a root or a pole of the function \(\frac{(\lambda+3)^3}{(\lambda+3)^2}\), whereas questions of this sort are answered at once in the layered structure. More sophisticated geometric examples are given in Example 7.24.

For the reader’s convenience let us point also to several applications of the layered structure from [17]:

- [17] Theorem 8.25 The \(\nu\)-multiplicativity of the resultant of tropical polynomials (in one indeterminate).
- [17] Theorem 8.33 The multiplicativity of the resultant of products of primary tropical polynomials (in one indeterminate).
- [17] Theorem 9.8 The computation of the layered discriminant of a tropical polynomial.
- [17] Example 10.6 Multiplicity of roots of tropical polynomials, by means of [17, Equation (9.1)].
- [17] Example 10.8 Unique factorization in many cases, as well as integration being defined.

Thus, we rely heavily on Construction 5.1 in order to pass back and forth between cancellative ordered monoids and \(L\)-layered domains\(^{\dagger}\), and this should be considered the main thrust of the layering procedure. Note that Construction 5.1 is formulated for semirings\(^{\dagger}\) without \(0\), in order to avoid complications. The more general theory is given in [20].

Intuitively, to obtain the appropriate layered category one might expect to take morphisms to be semiring\(^{\dagger}\) homomorphisms that preserve the layers, and these indeed play a key role to be described below. The category of main interest for the tropical algebraic theory is the category ULayBidom\(^{\dagger}\) of uniform \(L\)-layered bi-domains\(^{\dagger}\), which by Theorem 6.3 is isomorphic to the category OMon\(^{+}\) of cancellative ordered monoids, under the natural functor that restricts a uniform \(L\)-layered bi-domain\(^{\dagger}\) to its submonoid of tangible elements. In this way, we begin to see how identifications of categories help guide us in developing the theory.

This leads us to a delicate side issue. Although the ordered monoids of interest in the tropical theory are cancellative, such as the real max-plus algebra or, more generally, any ordered group, homomorphic images of cancellative monoids need not be cancellative. Thus, for a rich algebraic theory, we need a way of passing from arbitrary ordered monoids to layered semirings\(^{\dagger}\). Unfortunately the naive generalization of Construction 5.1 is not a semiring\(^{\dagger}\) since distributivity fails! In order not to go too far afielde in this paper, we stick with cancellative monoids, and consider noncancellative monoids in [20].

In \([6.2]\) we get to the functor \(\mathcal{F}_{LTrop}: \text{ValMon}^{+} \rightarrow \text{ULayBidom}^{\dagger}\), which describes the passage to the layered tropical world. \(\mathcal{F}_{LTrop}\) is applied to fields with valuation, in particular the field of Puiseux series, and enables one to translate equality to the “surpassing relation” described in [17, \S3.2]. In full generality the functor \(\mathcal{F}_{LTrop}\) involves subtleties discussed in [20].

The functor \(\mathcal{F}_{LTrop}\) is not faithful, since it only measures the action of the given valuation, and does not enable us to distinguish among elements having the same value. Thus in \([8]\) we also introduce briefly the \textit{exploded tropicalization functor} utilized in \([31]\) and another functor \(\mathcal{F}_{LTropunit}\) which retains extra information given in Proposition 6.10 such as is contained in “coamoebas.” Borrowing from classical valuation theory, we describe the exploded tropicalization functor in terms of the associated graded algebra, noting that in the case of the field of Puiseux series, the components of the associated graded algebra can be identified with the base field.

Although monoids recently have been seen to provide much of the underpinning for algebraic geometry, cf. [1, 8, 33, 34] for example, classical algebraic geometry relies for a large part on roots of polynomials, which can be understood more easily using semirings\(^{\dagger}\). Our approach to tropical geometry is to define affine varieties as sets of “ghost roots” of polynomials.
As is well known, and discussed in detail in [22], in contrast to the classical situation for polynomials over algebras over an infinite field, different tropical polynomials over a semiring $\mathbb{R}$ often take on the same values identically, viewed as functions. Furthermore, in max-plus situations one often wants to use variants such as Laurent polynomials (involving $\lambda^{-1}$ as well as $\lambda$) or polynomials with rational exponents, or even more generally one could talk in the language of the lattice of characters and its dual lattice, cf. [35, §2.2]. Also, as in classical algebraic geometry, we often want to limit the domain of definition to a given subset of $\mathbb{R}^n$ such as an algebraic variety. Thus, we work directly with functions from a set $\mathcal{S}$ to a layered domain, denoted $\text{Fun}(\mathcal{S}, \mathbb{R})$, or, more specifically, polynomially defined functions, denoted $\text{Pol}(\mathcal{S}, \mathbb{R})$ or Laurent polynomially defined functions, denoted $\text{Laur}(\mathcal{S}, \mathbb{R})$. In Proposition 7.8 we check that passing to the monoid of functions from $\mathcal{S}$ to an ordered monoid $\mathcal{M}$ and then translating to semirings yields the same categorical theory as moving first to a bipotent semiring and then passing to its function semiring.

Thus, taking $\mathcal{S} \subset \mathbb{R}(n)$, we redefine polynomials and monomials over $\mathbb{R}$ intrinsically as functions from $\mathcal{S}$ to $\mathbb{R}$, leading to an analog of the Zariski topology in Definition 9.2. This enables us to define a coordinate semiring via Definition 9.5.

Our view of tropical geometry relies largely on Kapranov’s Theorem, as extended by [36], which describes the 1:1 correspondence between roots of polynomials and corner roots of their tropicalizations. This process is understood categorically in terms of the supertropical structure, in §8. Ironically, although the Kapranov-Payne Theorem can be stated in the language of the tropicalization functor $F_{\text{LTrop}}$, the exploded tropicalization functor is needed (at least implicitly) in order to carry out the proof.

1.1. Overview of the major categories and functors in this paper.

In summary, let us review the main algebraic categories and their uses.

- The category $\text{ValField}$ (resp. $\text{ValMon}$) describing fields with valuation (resp. integral domains) with valuation. This is the ultimate arena of investigation, but its theory often is very difficult, thereby historically giving rise to tropical mathematics, which can be thought of as a degeneration of $\text{ValField}$.

- The category $\text{OMon}^+$ of cancellative ordered monoids, which is isomorphic to the category $\text{Bipot}^+$ of bipotent semirings. This is the traditional algebraic category underpinning tropical mathematics, but is too coarse a degeneration for many algebraic arguments, and often requires returning to $\text{ValField}$ in proofs.

- The category $\text{ULayBidom}^+$ of uniform $L$-layered bi-domains. For $L = \{1\}$ this is just the category $\text{OMon}^+$. For $L = \{1, \infty\}$ or $L = \{0, 1, \infty\}$ we get the supertropical theory, which suffices in linear algebra for the investigation of nonsingular matrices, bases, characteristic polynomials, and related notions. In order to discuss multiple roots of polynomials and singularity of curves, one needs to take $L$ containing $\mathbb{N}$.

- The category of exploded $L$-layered bi-domains. This is used by Sheiner and Shnider for the proof of the Kapranov-Payne theorem, as well as other deep results in the theory.

Here is a diagram of the categories under discussion in this paper, and the main functors connecting them.

\[
\begin{align*}
\text{ValField} & \xrightarrow{\text{Fun}_{\text{val}}} \text{ValMon}^+ \xrightarrow{F_{\text{LTrop}}} \text{OMon}^+ \xrightarrow{F_{\text{OMon}}} \text{Bipot}^+ \subset \text{Semir}^+ \\
\text{ValDom} & \xrightarrow{F_{\text{LTrop}}, \text{exp}} \text{ULayBidom}^+ \xrightarrow{\text{Fun}_{\text{Mon}}(\mathcal{S})} \text{OMon}^+ \xrightarrow{\text{Fun}_{\text{Semir}}(\mathcal{S})} \text{Semir}^+ \end{align*}
\]
\[ F_{\text{val}} \], which can be viewed as the customary tropicalization procedure, formalizes the order valuation on Puiseux series, which in most recent research has replaced the logarithm as the means of tropicalizing a variety.

\[ F_{\text{LTrop}} \], perhaps the most important functor in our theory, takes us from the classical world of algebraic geometry to the layered tropical world of this paper.

\[ F_{\text{OMon}} \] is the functor that enables us to pass from ordered monoids to bipotent semirings, thereby putting tools of semiring theory (such as polynomials) at our disposal.

\[ F_{\text{lay}} \] is the functor that enables us to “layer” an ordered monoid, and thus pass to the layered theory.

\[ F_{\text{LTrop;exp}} \] is the “exploded” functor, which preserves the leading coefficient of the original polynomial when tropicalizing, and thus permits Payne’s generalization of Kapranov’s theorem (and its application to tropical varieties).

The \( \text{Fun} \) functors take us to semirings\(^1\) of functions, thereby enabling us to treat polynomials (as functions).

At the conclusion of this paper, we consider how the layered category \( \text{LayBidom} \) enables us to define corner varieties, and we relate the categorical and geometric categories along the classical lines of algebraic geometry, obtaining a Zariski-type correspondence in Proposition 9.8.

There also is a functor \( F_{\text{LTrop;unit}} : \text{ValDom} \to \text{ULayBidom} \times \text{ValMon} \), which we did not put into the diagram, whose justification is given in the discussion after Proposition 6.10.

2. Background

We start with the category \( \text{Mon} \) of monoids and their monoid homomorphisms, viewed in the context of universal algebras, cf. Jacobson [27, §2].

**Definition 2.1.** A **semigroup** is a set with an associative operation, usually written multiplicatively as \( \cdot \). A **monoid** \( M := (M, \cdot) \) is a semigroup with a unit element \( 1_M \). A semigroup \( M \) is (left) **cancellative** with respect to a subset \( S \) if for every \( a_1, a_2, \in M, b \in S \),

\[
    b \cdot a_1 = b \cdot a_2 \quad \text{implies} \quad a_1 = a_2.
\]

\( M \) is **cancellative** if \( M \) is cancellative with respect to itself.

An element \( a \) of \( M \) is **absorbing** if \( ab = ba = a \) for all \( b \in M \). Usually the absorbing element (if it exists) is denoted as the zero element \( 0_M \), but it could also be identified with \(-\infty \). A semigroup \( M \) is **pointed** if it has an absorbing element \( 0_M \). A pointed semigroup \( M \) is **cancellative** if \( M \) is cancellative with respect to \( M \setminus \{0_M\} \). A subset \( a \subset M \) is a left (right) **semigroup ideal** if \( aM \subset a \) (\( aM \subset a \)).

The semigroups (as well as monoids) in this paper are presumed commutative, so left semigroup ideals are semigroup ideals.

A semigroup \( M := (M, \cdot) \) is **divisible** if for every \( a \in M \) and \( m \in \mathbb{N} \) there is \( b \in M \) such that \( b^m = a \).

A **semigroup homomorphism** is a map \( \phi : M \to M' \) satisfying

\[
    \phi(a_1 a_2) = \phi(a_1) \phi(a_2), \quad \forall a_1, a_2 \in M.
\]

(When dealing with pointed semigroups, we also require that \( \phi(0_M) = 0_{M'} \).) A **monoid homomorphism** is a semigroup homomorphism \( \phi : M \to M' \) also satisfying \( \phi(1_M) = 1_{M'} \).

2.1. **Semirings without zero.** We ultimately work in the environment of semirings (or, more precisely, semirings without a zero element, which we call a **semiring**\(^1\)). A standard general reference for the structure of semirings is [12]; also cf. [9]. Thus, a semiring\(^1\) \( (R, +, \cdot, 1_R) \) is a set \( R \) equipped with two binary operations \( + \) and \( \cdot \), called addition and multiplication, together with a unit element \( 1_R \) such that:

1. \( (R, +) \) is an Abelian semigroup;
2. \( (R, \cdot, 1_R) \) is a monoid with unit element \( 1_R \);
3. Multiplication distributes over addition on both sides.

When the multiplicative monoid \( (R, \cdot, 1_R) \) is cancellative, we say that \( (R, +, \cdot, 1_R) \) is a **domain**\(^1\); when \( (R, \cdot, 1_R) \) is also an Abelian group, we say that \( (R, +, \cdot, 1_R) \) is a **semifield**\(^1\). As customary, \( \mathbb{N} \) denotes the positive natural numbers, which is a cancellative domain\(^1\).
Definition 2.2. A homomorphism \( \varphi : R \to R' \) between two semirings\(^\dagger\) is defined as a homomorphism of multiplicative monoids that also preserves addition, to wit,
\[
\varphi(a + b) = \varphi(a) + \varphi(b) \quad \text{for all } a, b \in R.
\]
A semiring\(^\dagger\) isomorphism is a semiring\(^\dagger\) homomorphism that is 1:1 and onto.

Thus, we have the category \( \text{Semir}^\dagger \) of semirings\(^\dagger\) and their homomorphisms. This is closely related to the category \( \text{Semir} \) of semirings and semiring homomorphisms, especially since the semirings \( R \) of interest in the tropical theory, besides being multiplicatively cancellative, have the property that \( a + b \neq 0_R \) unless \( a = b = 0_R \); in other words, \( R \setminus \{0_R\} \) is closed under addition.

Remark 2.3. Any semiring\(^\dagger\) \( R \) can be embedded in a semiring \( R \cup \{0\} \) by formally adjoining a zero element \( 0 \) satisfying \( 0 + a = a = 0 + a \) and \( 0 \cdot a = a \cdot 0 = 0 \), \( \forall a \in R \cup \{0\} \). Conversely, if \( R \) is a semiring such that \( R \setminus \{0_R\} \) is closed under multiplication and addition, then \( R \setminus \{0_R\} \) is a semiring\(^\dagger\).

Proposition 2.4. The category \( \text{Semir}^\dagger \) is isomorphic to a subcategory of the category \( \text{Semir} \).

Proof. We just apply Remark 2.3 noting that any semiring\(^\dagger\) homomorphism \( \varphi : R \to R' \) can be extended to a semiring homomorphism \( \varphi : R \cup \{0_R\} \to R' \cup \{0_{R'}\} \) by putting \( \varphi(0_R) = 0_{R'} \). \( \square \)

An ideal \( a \) of a semiring\(^\dagger\) \( R \) is defined to be a sub-semigroup of \((R,+)\) which is also a (multiplicative) semigroup ideal of \((R,\cdot,1_R)\). (Clearly, when \( R \) has a zero element \( 0_R \), then \( 0_R \in a \).)

Example 2.5. If \( R \) is a semiring, then \( \{0_R\} \) is an ideal of the semiring \( R \cup \{0_R\} \) of Remark 2.3.

The tropical theory is closely involved with certain kinds of semirings\(^\dagger\).

Definition 2.6. A semiring\(^\dagger\) \( R \) is idempotent if
\[
a + a = a \quad \forall a \in R;
\]
\( R \) is bipotent if
\[
a + b \in \{a,b\} \quad \forall a,b \in R.
\]

The max-plus algebra is the prototype of a bipotent semiring\(^\dagger\).

2.2. Congruences. Unfortunately, kernels, such an important feature of category theory, play virtually no role in the general structure theory of semirings. In ring theory, the kernel \( \varphi^{-1}(0_{R'}) \) of any onto homomorphism \( \varphi : R \to R' \) is an ideal \( a \) of \( R \), and furthermore one can recover \( R' \) as isomorphic to \( R/a \).

This is not the case with semirings\(^\dagger\). Ideals do not play such a powerful role in the structure theory of semirings\(^\dagger\), since the construction \( R/a \) is problematic for an arbitrary ideal \( a \) (the difficulty arising from the fact that distinct cosets need not be disjoint).

Instead, one needs to consider more generally equivalence relations preserving the semiring\(^\dagger\) operations. From the general theory of universal algebra, one defines a congruence \( \Omega \) of an algebraic structure \( \mathcal{A} \) to be an equivalence relation \( \equiv \) which preserves all the relevant operations and relations; we call \( \equiv \) the underlying equivalence of \( \Omega \). Equivalently, a congruence \( \Omega \) is a sub-semiring\(^\dagger\) of \( \mathcal{A} \times \mathcal{A} \) that contains the diagonal
\[
\text{diag}(\mathcal{A}) := \{(a,a) : a \in \mathcal{A}\}
\]
as described in Jacobson \([27, \S 2]\). In other words, writing the underlying equivalence relation as \( a \equiv b \) whenever \( (a,b) \in \Omega \), we require that \( \equiv \) preserves all the relevant operations and relations.

Remark 2.7. We recall some key results of \([27, \S 2]\):

- Given a congruence \( \Omega \) of an algebraic structure \( \mathcal{A} \), one can endow the set
\[
\mathcal{A}/\Omega := \{[a] : a \in \mathcal{A}\}
\]
of equivalence classes with the same (well-defined) algebraic structure, and the map \( a \mapsto [a] \) defines an onto homomorphism \( \mathcal{A} \to \mathcal{A}/\Omega \). (For this reason, Berkovich \([1]\) calls them “ideals,” but this terminology conflicts with some of the literature, and we prefer to reserve the usage of “ideal” for the usual connotation.)
For any homomorphism \( \varphi : A \to A' \), one can define a congruence \( \Omega \) on \( A \) by saying that \( a \equiv b \iff \varphi(a) = \varphi(b) \). Then \( \varphi \) induces a 1:1 homomorphism

\[
\bar{\varphi} : A/\equiv \to A',
\]

via \( \bar{\varphi}(a) = \varphi(a) \).

We repeat the definition of congruence in each specific case that we need. Thus, a congruence \( \Omega \) on a semigroup \( M \) is an equivalence relation that preserves multiplication, in the sense that if \( a_1 \equiv b_1 \) and \( a_2 \equiv b_2 \), then \( a_1 a_2 \equiv b_1 b_2 \). In this case, the set of equivalence classes \( M/\Omega \) becomes a semigroup under the operation

\[
[a][b] = [ab],
\]

and there is a natural semigroup homomorphism given by \( a \mapsto [a] \). When \( M \) is a monoid, this becomes a monoid homomorphism, since \([1_M]\) is the multiplicative unit of \( M/\Omega \). When \( M \) is pointed, then \( M/\Omega \) is also pointed, with absorbing element \([0_M]\).

Here is another instance of a congruence that comes up in the passage from arbitrary monoids to cancellative monoids.

**Example 2.8.** Given an equivalence relation \( \equiv \) on a semigroup \( M \), and a sub-semigroup \( S \) of \( M \), we define the equivalence \( \equiv_S \) given by \( b_1 \equiv_S b_2 \) if \( b_1 s \equiv b_2 s \) for some \( s \in S \). When \( \equiv \) defines a congruence \( \Omega \), then \( \equiv_S \) also defines a congruence \( \Omega_S \). This congruence then identifies \( b_1 \) and \( b_2 \), thereby eliminating instances of non-cancellation, and is a useful tool.

### 2.2.1. Congruences over semirings

The congruence \( \Omega \) is a semiring\(^1\) congruence on a semiring\(^1\) \( R \) iff

\[
\begin{align*}
a_1 &\equiv a_2 \text{ and } b_1 \equiv b_2 \quad \text{ imply } \quad \begin{cases} a_1 + b_1 \equiv a_2 + b_2, \\ a_1 b_1 \equiv a_2 b_2. \end{cases} 
\end{align*}
\]

(2.1)

**Lemma 2.9.** To verify the conditions in (2.1) for commutative semirings\(^1\), it is enough to assume \( b_1 = b_2 \) and show for all \( a_1, a_2, \) and \( b \) in \( R \):

\[
\begin{align*}
a_1 &\equiv a_2 \quad \text{ implies } \quad a_1 + b \equiv a_2 + b; \quad (2.2) \\
a_1 &\equiv a_2 \quad \text{ implies } \quad a_1 b \equiv a_2 b. \quad (2.3)
\end{align*}
\]

Proof. \( a_1 + b_1 \equiv a_2 + b_1 \equiv a_2 + b_2 \). Likewise, \( a_1 b_1 \equiv a_2 b_1 \equiv a_2 b_2 \).

It often turns out that (2.2) enables us to obtain (2.3). On the other hand, in the case of semifields\(^1\), multiplicative cosets are more easily described than additive cosets, as is described in detail in [15]. To wit, let \( N := \{ a \in R : a \equiv 1_R \} \). For any \( a \in R \) we have \( ab_1 \equiv ab_2 \iff b_1 b_2^{-1} \in N \).

We write \( \text{diag}(M) \) for \( \{ (a, a) : a \in M \} \). As Berkovich [4] points out, any semigroup ideal \( a \) of a semigroup \( M \) gives rise to the congruence \( (a \times a) \cup \text{diag}(M) \), which corresponds to the Rees factor semigroup, and the analogous statement holds for monoids. A wrinkle emerges when we move to bipotent semirings\(^1\), since \( (a \times a) \cup \text{diag}(M) \) need not be closed under addition. Thus, the applications are limited, and are discussed in [20].

**Definition 2.10.** An identity \( f = g \) of a semiring\(^1\) \( R \) is an elementary sentence \( f(x_1, \ldots, x_m) = g(x_1, \ldots, x_m) \) that holds for all \( x_1, \ldots, x_m \) in \( R \).

**Remark 2.11.** Suppose we want to force a semiring\(^1\) \( R \) to satisfy a particular identity, in the sense that we want a semiring\(^1\) \( R \) in which \( f = g \) is an identity, together with a surjective homomorphism \( \varphi : R \to \bar{R} \) satisfying the universal property that any homomorphism of \( R \) to a semiring\(^1\) satisfying the identity \( f = g \) factors through \( \varphi \).

Intuitively, one must mod out the relation \( f = g \) by putting \( f(a_1, \ldots, a_m) \equiv g(a_1, \ldots, a_m) \) for all \( a_i \) in \( R \). For semirings\(^1\), in view of Lemma 2.9, since we are dealing with congruences, we must mod out the equivalence relation obtained by putting \( f(a_1, \ldots, a_m) + c \equiv g(a_1, \ldots, a_m) + c \) and \( f(a_1, \ldots, a_m)c \equiv g(a_1, \ldots, a_m)c \) for all \( c \) and \( a_i \) in \( R \).
Example 2.12. Consider the additive idempotence identity $x + x = x$. We attain this by imposing the equivalence relation given by $a + a \equiv a, \forall a \in R$. The congruence that it generates must also satisfy the relation $a + a + c \equiv a + c, \forall a \in R$.

But then we also get (2.3), since $ab + ab = (a + a)b$ and $(a + a + c)b = ab + ab + cb \equiv ab + cb = (a + c)b$.

Thus, (2.2) already defines the congruence. (This observation is to be elaborated shortly.)

Lemma 2.13. The bipotent semirings comprise a full subcategory Bipot of Semir.

Proof. If $\varphi : R \rightarrow R'$ is a semiring homomorphism and $a + b \in \{a, b\}$, then $\varphi(a) + \varphi(b) = \varphi(a + b) \in \{\varphi(a), \varphi(b)\}$.

\[\square\]

2.3. Hom and Adjoint functors. We need to use some well-known facts about categories.

Definition 2.14. For any category $\mathcal{C}$ and some given object $A$ in $\mathcal{C}$, we recall the well-known covariant functor

$$\text{Hom}(A, \_ : \mathcal{C} \rightarrow \text{Set},$$

which sends an object $B$ in $\mathcal{C}$ to Hom$(A, B)$, and which sends the morphism $\phi : B \rightarrow B'$ to the morphism Hom$(\__, \phi) : \text{Hom}(A, B) \rightarrow \text{Hom}(A, B')$ given by $f \mapsto \phi f$ for $f : A \rightarrow B$.

Likewise, given an object $B$ in $\mathcal{D}$, we define the contravariant functor

$$\text{Hom}(\__, B : \mathcal{D} \rightarrow \text{Set},$$

which sends an object $A$ to Hom$(A, B)$, and which sends the morphism $\varphi : A' \rightarrow A$ to the morphism Hom$(\varphi, \__ : \text{Hom}(A', B) \rightarrow \text{Hom}(A, B)$ given by $f \mapsto f \varphi$ for $f : A \rightarrow B$.

Recall that a functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is a left adjoint to $\mathcal{H} : \mathcal{D} \rightarrow \mathcal{C}$ (and $\mathcal{H}$ is a right adjoint to $\mathcal{F}$) if there is a canonical identification $\Psi : \text{Hom}(\mathcal{F}(A), B) \rightarrow \text{Hom}(A, \mathcal{H}(B))$ for all objects $A$ of $\mathcal{C}$ and $B$ of $\mathcal{D}$, for which the following diagrams are always commutative for all morphisms $\varphi : A \rightarrow A'$ and $\phi : B \rightarrow B'$:

- \[\text{Hom}(\mathcal{F}(A), B) \xrightarrow{\text{Hom}(\mathcal{F}(\varphi), \__)} \text{Hom}(\mathcal{F}(A'), B) \xrightarrow{\Psi} \text{Hom}(A, \mathcal{H}(B)) \]
- \[\text{Hom}(\mathcal{F}(A), B) \xrightarrow{\text{Hom}(\__, \phi)} \text{Hom}(\mathcal{F}(A), B') \xrightarrow{\Psi} \text{Hom}(A, \mathcal{H}(B')) \]

It is well-known that any left adjoint functor is unique up to isomorphism.

2.4. Universals. Recall from [27] §1.7 and §1.8 that the adjoint functor of a functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is obtained by identifying the appropriate universal $U$ of $\mathcal{F}$ together with the canonical morphisms $\iota : D \rightarrow \mathcal{F}(U(D))$, for objects $D$ in $\mathcal{D}$, satisfying the property that for any morphism $f : D \rightarrow \mathcal{F}(C)$ in $\mathcal{D}$ and object $C$ in $\mathcal{C}$, there is a morphism $g : U(D) \rightarrow C$ in $\mathcal{C}$ such that

$$\mathcal{F}(g \circ \iota) = f.$$
Example 2.15. We define the forgetful functor \( \text{Semir}^+ \to \text{Mon} \), by forgetting addition. The appropriate universal in this case is the monoid semiring \( \mathbb{N}[\mathcal{M}] \), defined analogously to the monoid algebra.

Definition 2.16. \([\mathcal{M}]\) denotes the semiring\(^1\) obtained by taking \( \mathbb{N}[\mathcal{M}] \) modulo the additive idempotence congruence of Example 2.12. Explicitly, \([\mathcal{M}]\) is comprised of formal sums of distinct elements of the monoid \( \mathcal{M} \), i.e.,

\[
[\mathcal{M}] = \left\{ \sum_{a \in S} a \mid S \subset \mathcal{M} \right\}
\]

equipped with addition

\[
\sum_{a \in S} a + \sum_{a \in S'} a = \sum_{a \in S \cup S'} a,
\]

and multiplication is obtained from the original multiplication in \( \mathcal{M} \), extended distributively.

Example 2.17. Since additive idempotence defines an identity, one has the category of additively idempotent semirings\(^1\); the forgetful functor to \( \text{Mon} \) now has the universal \([\mathcal{M}]\). The customary way to view tropical mathematics is by means of the max-plus semiring\(^1\), which is additively idempotent.

3. Pre-ordered semigroups, monoids, and semirings

Recall that a partial pre-order (a transitive relation \( (\leq) \)) is a pre-order if it is antisymmetric, i.e., \( a \leq b \) and \( b \leq a \) imply \( a = b \). We write \( a < b \) when \( a \leq b \) but \( a \neq b \).

A partial pre-order is called a preorder if any two elements are comparable. A (total) order is a preorder which is also a preorder.

3.1. Pre-ordered semigroups. We work with pre-ordered semigroups in this paper. The natural definition in terms of universal algebra is the following:

Definition 3.1. A semigroup \( \mathcal{M} := (\mathcal{M}, \cdot) \) (or a monoid \( \mathcal{M} := (\mathcal{M}, \cdot, 1_\mathcal{M}) \)) is partially pre-ordered (resp. partially ordered, pre-ordered, ordered) if it has a partial pre-order \( \leq \) (resp. partial order, pre-order, order) such that

\[
b \leq c \text{ implies } ab \leq ac \text{ and } ba \leq ca, \quad \forall a \in \mathcal{M}. \tag{3.1}
\]

We denote an ordered semigroup by \((\mathcal{M}, \cdot, \leq)\). Thus, totally ordered semigroups satisfy the following property:

\[
a \max\{b, c\} = \max\{ab, ac\}, \quad \forall a, b, c \in \mathcal{M}. \tag{3.2}
\]

We say that the relation \( (\leq) \) is strict if

\[
b < c \text{ implies } ab < ac \text{ and } ba < ca, \quad \forall a \in \mathcal{M}. \tag{3.3}
\]

Note 3.2. This definition requires that all elements of \((\mathcal{M}, \cdot, \leq)\) are positive or 0, an implicit assumption made throughout this paper, to be discussed after Definition 3.8.

Lemma 3.3. A total order \( (\leq) \) on a semigroup \( \mathcal{M} \) is strict iff the semigroup \( \mathcal{M} \) is cancellative.

Proof. \((\Rightarrow)\) Suppose \( ab = ac \). By symmetry, we may assume that \( b \leq c \). But if \( b < c \) then \( ab < ac \), a contradiction, so we conclude \( b = c \).

\((\Leftarrow)\) If \( b < c \) then \( ab \neq ac \), implying \( ab < ac \).

Let us construct the appropriate categories.

Definition 3.4. An order-preserving semigroup homomorphism is a homomorphism \( \phi : \mathcal{M} \to \mathcal{M}' \) satisfying the condition (where \( \leq \) denotes the partial order on the appropriate semigroup):

\[
a \leq b \text{ implies } \phi(a) \leq \phi(b), \quad \forall a, b \in \mathcal{M}. \tag{3.4}
\]

 \( \text{PPreOMon}, \text{PreOMon}, \text{POMon}, \text{POMon}^+, \text{OMon}, \) and \( \text{OMon}^+ \) denote the respective categories of partially pre-ordered, pre-ordered, partially ordered, cancellative partially ordered, ordered, and cancellative ordered monoids, whose morphisms are the order-preserving homomorphisms.

By definition, \( \text{OMon}^+ \) is a full subcategory both of \( \text{OMon} \) and of \( \text{POMon}^+ \), each of which is a full subcategory of \( \text{POMon} \), which is a full subcategory of \( \text{PPreOMon} \).
Remark 3.5. The forgetful functor from the category $\text{POMon}$ to the category $\text{PPreOMon}$ is obtained by viewing any partially ordered monoid $M$ naturally in $\text{PPreOMon}$.

We also can go in the other direction.

Remark 3.6. For the class of partially pre-ordered semigroups, our congruences $\Omega$ also satisfy the property that if $a_1 \leq a_2$ and $b_1 \equiv a_i$, then $b_1 \leq b_2$. In this case, $M/\Omega$ inherits the partial pre-order given by $[a] \leq [b]$ iff $a \leq b$.

Proposition 3.7. There is a retraction $F : \text{PPreOMon} \to \text{POMon}$ to the forgetful functor of Remark 3.5. Namely, we take the congruence $\Omega$ on a pre-ordered monoid $M$ given by $a \equiv b$ when $a \leq b$ and $b \leq a$, and define $F(M) := M/\Omega$.

Proof. It is easy to see that $\equiv$ is an equivalence relation that preserves the operation and the order, so is an ordered monoid congruence, and thus induces a partial order on $M/\Omega$ according to Remark 3.6.

We claim that any order-preserving homomorphism $\phi : M \to M'$ induces an order-preserving homomorphism $\tilde{\phi} : M/\Omega \to M'/\Omega$. Indeed, if $a \equiv b$, then $a \leq b$ and $b \leq a$, implying $\phi(a) \leq \phi(b)$ and $\phi(b) \leq \phi(a)$, yielding $\phi(a) \equiv \phi(b)$.

The functor $F$ is a retraction to the forgetful functor, since it acts trivially on any total ordered monoid. □

3.2. Pre-ordered semirings†.

Definition 3.8. We say that a semiring† $R$ is pre-ordered (resp. partially ordered, ordered) if it has a pre-order $\geq$ (resp. partial order, order) with respect to which both the monoid $(R, \cdot, 1_R)$ and the semigroup $(R, +)$ satisfy Condition 3.1 of Definition 3.1.

In other words, both multiplication and addition preserve the pre-order. There is a delicate issue in this definition. In the rational numbers, viewed as a multiplicative monoid, we have $1 < 2$ but $(-1)1 > (-1)2$. This difficulty is dealt with in [17], in which we define the order in terms of a cone of “positive” elements. Definition 3.8 is the special case in which all elements of $R$ are positive or 0, and is reasonable for tropical mathematics since the “zero” element (when it is included) is minimal. We use Definition 3.8 here because it is more appropriate to our categorical treatment.

A semiring† $R$ has the infinite element $\infty$ if
\[
\infty + a = \infty = \infty \cdot a = a \cdot \infty, \quad \text{for some } a \in R.
\]
(3.5)

Recall from [17] Corollary 2.15] that if $R$ has a unique infinite element, then
\[
\infty + a = \infty = \infty \cdot a = a \cdot \infty, \quad \forall a < \infty.
\]

Nonzero positive elements of an ordered semiring† need not be finite, and we could have several infinite elements (as can be seen easily by means of ordinals). We do not deal with such issues in this paper, and assume there is at most one infinite element $\infty$.

The following observation is implicit in [13] Theorem 4.2.

Proposition 3.9. There is a natural functor $\text{Semir}^\dagger \to \text{PPreOMon}$, where we define the preorder on a semiring† $R$ given by

\[
a \leq b \quad \text{iff} \quad a = b \quad \text{or} \quad b = a + c \quad \text{for some } c \in R.
\]

This functor always yields the trivial partial preorder on rings, since then $b = a + (b - a)$. The situation is quite different for the semirings† arising in tropical mathematics, because of bipotence.

Proposition 3.10. Suppose $R$ is an idempotent semiring†.

(i) $a \leq b$ iff $a + b = b$.

(ii) $\leq$ is a partial order, which is total when $R$ is bipotent.

Proof. (i): ($\Leftarrow$) Take $c = b$.

($\Rightarrow$) Suppose $a + c = b$. Then
\[
a + b = a + (a + c) = (a + a) + c = a + c = b.
\]
(ii): Transitivity follows because \( a + b = b \) and \( b + c = c \) imply
\[
a + c = a + (b + c) = (a + b) + c = b + c = c.
\]

It remains to prove antisymmetry. Suppose \( a \leq b \) and \( b \leq a \). Then, in view of (i), \( b = a + b = a \). \(\square\)

We are ready for a key identification of categories.

**Proposition 3.11.** There is a faithful functor \( \mathcal{F}_{\text{Mon}} : \text{Mon} \to \text{Semir}^1 \), whose image is \( \text{Bipot}^1 \).

**Proof.** Given any totally ordered monoid \((\mathcal{M}, \cdot, \geq, 1_{\mathcal{M}})\) we define \( a + b \) to be \( \max\{a, b\} \). Then \((\mathcal{M}, +)\) is a semigroup, since
\[
(a + b) + c = \max\{a, b, c\} = a + (b + c).
\]

Finally, this gives rise to a semiring\(^1\), since, by Equation (3.2),
\[
(a + b)c = \max\{a, b\}c = \max\{ac, bc\} = ab + ac.
\]

Any order-preserving monoid homomorphism \( \varphi : \mathcal{M} \to \mathcal{M}' \) is a semiring\(^1\) homomorphism, since for \( a \leq b \) we have
\[
\varphi(a + b) = \varphi(b) = \varphi(a) + \varphi(b).
\]

Conversely, given a bipotent semiring\(^1\) \( R \), the relation \( \leq \) of Proposition 3.9 is a total order by Proposition 3.10. Furthermore, \( \leq \) is preserved under multiplication, since \( b \leq c \) implies \( b + c = c \) and thus \( ab + ac = a(b + c) = ac \), yielding \( ab \leq ac \). Any semiring\(^1\) homomorphism \( \varphi : R \to R' \) is an order-preserving monoid homomorphism, for if \( a \leq b \), then
\[
\varphi(a) + \varphi(b) = \varphi(a + b) = \varphi(b),
\]
implying \( \varphi(a) \leq \varphi(b) \).

Note that we have just reconstructed the max-plus algebra. We will rephrase this result in the layered setting, as Theorem 6.3.

Proposition 3.11 enables one to pass back and forth between categories of totally ordered monoids and bipotent semirings. The first category enables us to exploit techniques from valuation theory, whereas the second enables us to introduce concepts from ring theory such as polynomials, modules, matrices, and homology theory.

**Proposition 3.12.** There is a functor \( \text{Bipot}^1 \to \text{Bipot}^1 \) sending a semiring\(^1\) to its dual bipotent semiring\(^1\) obtained as the same multiplicative monoid, but reversing the bipotence in addition; i.e., if originally \( a + b = a \), now we put \( a + b = b \).

**Proof.** This is seen readily by defining the pre-order given by \( a \geq b \) iff \( a + b = a \); then the dual bipotent semiring\(^1\) corresponds to the reverse pre-order, and any homomorphism preserves the (reverse) order. \(\square\)

For example, the dual semiring\(^1\) of \((\mathbb{R}, \max, +, 0)\) is \((\mathbb{R}, \min, +, 0)\). (The number 0 is really the unit element \(1_{\mathbb{R}}\).)

3.3. **The universal for the Frobenius property.** Usually one works with commutative, totally ordered monoids and semirings\(^1\). In this case, recall the well-known Frobenius property, cf. [22, Remark 1.1]:
\[
(a + b)^m = a^m + b^m
\]
(3.6)
for any \( m \in \mathbb{N} \).

These are identities of \( R \), so we could define the Frobenius monoid semiring\(^1\) \([\mathcal{M}]\) of an arbitrary monoid \( \mathcal{M} \), in which, in view of Remark 2.11, we impose on \([\mathcal{M}]\) (defined in Definition 2.10) the relations
\[
\left( \sum_{a \in S} a \right)^m + c = \left( \sum_{a \in S} a^m \right) + c, \quad \left( \sum_{a \in S} a \right)^m c = \sum_{a \in S} a^m c,
\]
for \( S \subset \mathcal{M} \) finite.

Note that when \( R \) is divisible, these relations are formal consequences of (3.6) since writing \( c = d^m \) we have
\[
d^m + \left( \sum_{a \in S} a \right)^m = \left( d + \sum_{a \in S} a \right)^m = d^m + \sum_{a \in S} a^m; \quad d^m \left( \sum_{a \in S} a \right)^m = \left( d \sum_{a \in S} a \right)^m = d^m \sum_{a \in S} a^m.
\]
The Frobenius monoid semiring $\mathcal{M}$ satisfies the following universal property:

**Proposition 3.13.** Suppose $\varphi : \mathcal{M} \to \mathcal{M}'$ is a monoid homomorphism, where the monoid $\mathcal{M}'$ is totally ordered. Viewing $\mathcal{M}'$ as a bipotent semiring via Proposition 3.11, we have a natural homomorphism

$$\hat{\varphi} : [\mathcal{M}] \to \mathcal{M}'$$

given by

$$\hat{\varphi} \left( \sum_{a \in S} a \right) = \sum_{a \in S} \varphi(a),$$

satisfying the universal property that $\varphi$ factors as

$$\varphi : \mathcal{M} \longrightarrow [\mathcal{M}] \longrightarrow \mathcal{M}'$$

*Proof.* The map given by $\sum_{a \in S} a \mapsto \sum_{a \in S} \varphi(a)$ is the desired semiring homomorphism, since $\mathcal{M}'$ satisfies the Frobenius property. □

On the other hand, the same argument shows that $[\mathcal{M}]$ itself is not ordered as a monoid, since one can provide any ordered monoid $\mathcal{M}$ with two orders, one order making $a + b = a$ and the reverse order making $a + b = b$, as shown in Remark 3.12. The point here is that the Frobenius property, being an algebra identity, permits the definition of a universal. Furthermore, one could define the category of semirings satisfying the Frobenius property, which comprises a full subcategory of Semir. Although Proposition 3.13 indicates that this is the “correct” category in which to conduct much of the investigation in tropical algebraic geometry, we forego further consideration of this category in this paper.

4. Integral domains and monoids with valuation

We turn to the main notion of “tropicalization.” As indicated in the introduction, we need to consider integral domains $W$ with valuation $v : W \setminus \{0_W\} \to \mathcal{G}$, having cancellative (ordered) value monoid $\mathcal{G}$. In valuation theory it is customary to write the operation of the value monoid $\mathcal{G}$ as addition, and to utilize the axiom

$$v(a + b) \geq \min\{v(a), v(b)\}.$$  

Note that we can replace the valuation $v$ by $v' := -v$ to get the dual equation

$$v'(a + b) \leq \max\{v'(a), v'(b)\}. \quad (4.1)$$

We adjust the notation of valuation theory to fit in with the algebraic language of semirings. Thus, from now on in this paper, we use multiplicative notation, written $\mathcal{G} := (\mathcal{G}, \cdot , \geq , 1_\mathcal{G})$, for the value monoid $\mathcal{G}$ with unit element $1_\mathcal{G}$, which can be viewed as a semiring via Proposition 2.13, and use $(4.1)$ for the valuation axiom, since it fits in better with the semiring approach. (But several authors, such as Sturmfels and his school, have used the min-plus algebra instead, in order to forego taking the negative.)

**Definition 4.1.** The algebra of Puiseux series $K$ over an algebraically closed field $K$ is the field of series of the form

$$p := \sum_{\tau \in T} c_{\tau} t^\tau, \quad c_{\tau} \in K,$$

with $T \subset \mathbb{R}$ well-ordered (from below).

Sometimes one takes $T \subset \mathbb{Q}$; any totally ordered field will do. We will take $\mathbb{R}$ in this paper. For any field $F$, we write $K^x$ for $K \setminus \{0\}$. The tropical connection is that the max-plus algebra appears as the target of the valuation

$$\text{Val} : K^x \longrightarrow \mathbb{R}$$

given by sending $p(t) \neq 0_K$ to the negative of the lowest exponent of its monomials having nonzero coefficient;

$$\text{Val}(p) := -\min\{\tau \in T : c_{\tau} \neq 0_K\}. \quad (4.3)$$
Remark 4.2. There is a natural multiplicative map \( \pi : k^\times \to k^\times \), sending a Puiseux series \( p = \sum_{\tau \in T} c_{\tau}t^\tau \) to \( c_{\val(p)} \).

This gives an extra important piece of information, since for any two Puiseux series \( p, q \) we must have \( \val(p + q) = \max\{\val(p), \val(q)\} \) unless \( \val(p) = \val(q) \) and \( \pi(p) = \pi(q) \), in which case \( \val(p + q) < \max\{\val(p), \val(q)\} \). In this way, \( \pi \) measures how much bipotence is lost with respect to \( \val \).

4.1. Valued monoids. We view the previous observations in a somewhat more general setting.

Definition 4.3. A monoid \( M = (\cdot, 1_M) \) is \emph{m-valued} with respect to a totally ordered monoid \( G := (\cdot, \geq, 1_G) \) if there is an onto monoid homomorphism \( v : M \to G \). This set-up is notated as the triple \((M, G, v)\).

Note 4.4. The hypothesis that \( v \) is onto can always be attained by replacing \( G \) by \( v(M) \) if necessary.

Given a field with valuation, or more generally, an integral domain \( W \) with valuation \( v : W \setminus \{0_W\} \to G \), we take \( M = \{0_M\} \), a cancellative submonoid of \( W \), to obtain the triple \((M, G, v)\) as in Definition 4.3. When \( W \) is an arbitrary commutative ring with valuation, we must assume that the monoid \( G \) is pointed, and take the triple \((W, G, v)\).

Example 4.5. Another major example of an m-valued monoid is \( M = (\mathbb{C}^\times, \cdot, 1) \), \( G = (\mathbb{R}_{\geq 0}, +, \geq, 0) \), and \( v : \mathbb{C}^\times \to \mathbb{R}_{\geq 0} \), given by \( v(z) = \log_t(|z|) \), where \( t \) is a given positive parameter. This leads us to the theory of complex amoebas, cf. Passare [32].

The category of m-valued monoids is quite intricate, since the morphisms should include all maps preserving monoid homomorphism \( \phi \) and their restriction to cancellative monoids, and whose morphisms are ring homomorphisms which restrict to morphisms in \( \ValMon^+ \) and each has its respectful forgetful functor to \( \ValMon^+ \).

Definition 4.6. \( \ValMon \) is the category of m-valued monoids whose objects are triples \((M, G, v)\) as in Definition 4.3 for which a morphism

\[
\phi : (M, G, v) \to (M', G', v')
\]

is comprised of a pair \((\phi_M, \phi_G)\) of a monoid homomorphism \( \phi_M : M \to M' \), as well as an order-preserving monoid homomorphism \( \phi_G : G \to G' \), satisfying the compatibility condition

\[
v'(\phi_M(a)) = \phi_G(v(a)), \quad \forall a \in M.
\]

\( \ValMon^+ \) is the full subcategory of \( \ValMon \) in which the target monoid \( G \) is cancellative.

Thus, we have the categories \( \ValField \) (resp. \( \ValDom \)) whose objects are fields (resp. integral domains) with valuations to cancellative monoids, and whose morphisms are ring homomorphisms which restrict to morphisms in \( \ValMon^+ \), and each has its respectful forgetful functor to \( \ValMon^+ \).

Remark 4.7. If \((M, G, v)\) is a triple, then \( v \) induces a pre-order \( \leq \) on \( M \), given by \( a \leq b \) iff \( v(a) \leq v(b) \) in \( G \).

Lemma 4.8. There is a fully faithful functor of categories \( \ValMon \to \PreOMon \).

Proof. The functor is given by Remark 4.7.

Conversely, given any monoid \( M \) with pre-order, we define \( G := M/\equiv \) as in Proposition 3.7. Then we define \( v : M \to G \) by \( a \mapsto [a] \); clearly \((M, G, v)\) is a triple. One sees easily that the morphisms match. \( \square \)

On the other hand, as we have observed, the main idea of tropicalization is the following observation:

Remark 4.9. The valuation itself provides a forgetful functor \( \mathcal{F}_\val : \ValMon^+ \to \OMon^+ \), where we remember only the target monoid \( G \) from the triple \((M, G, v)\).

Remark 4.10. Let us recall some valuation theory, which we can state in terms of an integral domain \( W \) with valuation \((W, G, v)\). The \emph{valuation ring} \( R \) (resp. \emph{valuation ideal} \( p \)) is the set of elements of \( W \) having value \( \geq 0 \) (resp. > 0); the \emph{residue domain} \( \bar{W} \) is \( R/p \).

The residue domain is an integral domain. When \( W \) is a field \( F \), the residue domain \( \bar{F} \) is also a field. For example, the valuation ideal \( p \) of \( k \) of Definition 4.1 is the set of Puiseux series having value > 0, and the residue field can be identified with \( K \).

Here is another example, to illustrate some subtler aspects of the definitions.
Example 4.11.

(i) The integral domain \( \mathbb{C}[\lambda_1, \lambda_2] \) has the natural valuation to \( \mathbb{Z} \times \mathbb{Z} \), ordered via the lexicographic order where \( v(\lambda_1) = (1, 0) \) and \( v(\lambda_2) = (0, 1) \). On the other hand, there is the valuation \( \tilde{v} : \mathbb{C}[\lambda_1, \lambda_2] \to \mathbb{Z} \) given by \( \tilde{v}(\sum_{i,j} \alpha_{i,j} \lambda_1^i \lambda_2^j) = k \) for that smallest \( k = i + j \) such that \( \alpha_{i,j} \neq 0 \). (In other words, \( \tilde{v}(\lambda_1) = \tilde{v}(\lambda_2) = 1 \).) The identity map \((1, 0, 1)_{\mathbb{Z} \times \mathbb{Z}}(\lambda) \) (where we replace the valuation \( v \) by \( \tilde{v} \)) is not a morphism in ValMon\(^+\) since it is not order-preserving. \( v(\lambda_1) = (1, 0) > (0, 2) = v(\lambda_2) \) whereas \( \tilde{v}(\lambda_1) = 1 < 2 = \tilde{v}(\lambda_2) \).

(ii) In (i), we take a different order on \( \mathbb{Z} \times \mathbb{Z} \), where two pairs are ordered first by the sum of their coordinates and then only secondarily via the lexicographic order. Now the identity map (where we replace the valuation \( v \) by \( \tilde{v} \)) is a morphism in ValMon\(^+\) since it is order-preserving. Note however that it is not strictly order-preserving, since \( v(\lambda_1) = (1, 0) > (0, 1) = v(\lambda_2) \) whereas \( \tilde{v}(\lambda_1) = 1 = \tilde{v}(\lambda_2) \).

5. The layered structure

We are ready to bring our leading player. In this section we describe the algebraic category in whose context we may formulate all the algebraic structure we need (including matrices and polynomials) for the layered theory. To simplify notation and avoid technical complications, we work with a semiring\(^1\) \( L \) without a zero element, even though information is lost; the full theory is given in \[20\]. Much of the layered theory stems from the following fundamental construction from \[17\], which is inspired by \[2\].

Construction 5.1. \( R := \mathcal{R}(L, \mathcal{G}) \) is defined set-theoretically as \( L \times \mathcal{G} \), where we denote the element \((\ell, a)\) as \([\ell]a\) and, for \( k, \ell \in L, a, b \in \mathcal{G}, \) we define multiplication componentwise, i.e.,

\[ [k]_a \cdot [\ell]_b = [k\ell]_{ab} \quad (5.1) \]

Addition is given by the usual rules:

\[ [k]_a + [\ell]_b = \begin{cases} [k]_a & \text{if } a > b, \\ [\ell]_b & \text{if } a < b, \\ [k+\ell]_a & \text{if } a = b. \end{cases} \quad (5.2) \]

We define \( R_\ell := \{\ell\} \times \mathcal{G}, \) for each \( \ell \in L. \) Namely \( R = \bigcup_{\ell \in L} R_\ell. \)

This is to be our prototype of a layered bi-domain\(^1\), and should be borne in mind throughout the sequel. Nevertheless, one should also consider the possibility that the monoid \( \mathcal{G} \) is non-cancellative, in which case, as noted in \[17\], Construction 5.1 fails to satisfy distributivity and thus is not a semiring\(^1\). This difficulty can be resolved, but the ensuing category becomes rather technical, so we defer it to \[20\].

5.1. Layered pre-domains\(^1\). We axiomatize Construction 5.1 in order to place it in its categorical framework.

Definition 5.2. Suppose \((L, \geq)\) is a partially pre-ordered semiring\(^1\). A \( L \)-layered pre-domain\(^1\)

\[ R := (R, L, (v_{m,\ell})), \]

is a semiring\(^1\) \( R, \) together with a partition \( \{R_\ell : \ell \in L\} \) into disjoint subsets \( R_\ell \subset R, \) called layers, such that

\[ R := \bigcup_{\ell \in L} R_\ell, \quad (5.3) \]

and a family of sort transition maps

\[ \nu_{m,\ell} : R_\ell \to R_m, \quad \forall m \geq \ell > 0, \]

such that

\[ \nu_{\ell, \ell} = \text{id}_{R_\ell} \]

for every \( \ell \in L, \) and

\[ \nu_{m,\ell} \circ \nu_{\ell, k} = \nu_{m, k}, \quad \forall m \geq \ell \geq k, \]

whenever both sides are defined, satisfying the following axioms A1–A4, and B.
We say that any element \(a\) of \(R_k\) has layer \(k\) \((k \in L)\). We write \(a \equiv_\nu b\) for \(b \in R_\ell\), whenever \(\nu_{m,k}(a) = \nu_{m,\ell}(b)\) in \(R_m\) for some \(m \geq k, \ell\). (This notation is used generically: we write \(a \equiv_\nu b\) even when the sort transition maps \(\nu_{m,\ell}\) are notated differently.)

Similarly, in line with Remark 5.5, we write \(a \leq_\nu b\) if \(\nu_{m,k}(a) + \nu_{m,\ell}(b) = \nu_{m,\ell}(b)\) in \(R_m\) for some \(m \geq k, \ell\).

The axioms are as follows:

\[
\begin{align*}
A1. & \ 1_R \in R_1. \\
A2. & \ \text{If } a \in R_k \text{ and } b \in R_\ell, \text{ then } ab \in R_{k\ell}. \\
A3. & \ \text{The product in } R \text{ is compatible with sort transition maps: Suppose } \ a \in R_k \text{ and } b \in R_\ell, \text{ with } m \geq k \text{ and } m' \geq \ell. \text{ Then } \\
& \quad \nu_{m,k}(a) \cdot \nu_{m',\ell}(b) = \nu_{mm',k\ell}(ab). \\
A4. & \ \nu_{\ell,k}(a) + \nu_{\ell',k}(a) = \nu_{\ell+\ell',k}(a) \text{ for all } a \in R_k \text{ and all } \ell, \ell' \geq k.
\end{align*}
\]

B. (Supertropicality) Suppose \(a \in R_k, b \in R_\ell, \text{ and } a \equiv_\nu b\). Then \(a + b \in R_{k+\ell}\) with \(a + b \equiv_\nu a\).

If moreover \(k = \infty\), then \(a + b = a\).

\(L\) is called the sorting semiring\(^{\dagger}\) of the \(L\)-layered pre-domain\(^{\dagger}\) \(R = \bigcup_{\ell \in L} R_\ell\).

For convenience, we assume in the sequel that \(L = L_{\geq 0}\), i.e., all nonzero elements of \(L\) are positive. Often \(L\) is \(\mathbb{N}\) or \(\mathbb{N}^+\).

**Remark 5.3.** The \(L\)-layered pre-domain\(^{\dagger}\) \(R\) has the special layer \(R_1\), which is a multiplicative monoid, called the monoid of tangible elements, and acts with the obvious monoid action (given by multiplication) on each layer \(R_k\) of \(R\).

Thus, in one sense, \(R\) extends its monoid of tangible elements. Although we have given up bipotence, and Axiom B provides us the slightly weaker notion of \(\nu\)-bipotence, which says that \(a + b \equiv_\nu a\) or \(a + b \equiv_\nu b\) for all \(a, b \in R\).

**Definition 5.4.** An \(L\)-layered bi-domain\(^{\dagger}\) is a \(\nu\)-bipotent \(L\)-layered pre-domain\(^{\dagger}\) which is (multiplicatively) cancellative. (We use the prefix “bi” in this paper to stress the \(\nu\)-bipotence.)

An \(L\)-layered bi-domain\(^{\dagger}\) \(R\) is called an \(L\)-layered bi-semifield\(^{\dagger}\) if \((R_1, \cdot)\) is an Abelian group.

Note that according to this definition, an \(L\)-layered bi-semifield\(^{\dagger}\) need not be a semifield\(^{\dagger}\) unless \(L\) itself also is a multiplicative group (and thus a semifield\(^{\dagger}\)). When \(R\) is an \(L\)-layered bi-semifield\(^{\dagger}\), the action of \(R_1\) is simply transitive, in the sense that for any \(a, b \in R_\ell\) there is a unique element \(r \in R_1\) for which \(ar = b\).

**Definition 5.5.** We write \(R_{> \ell}\) (resp. \(R_{\geq \ell}\)) for \(\bigcup_{k > \ell} R_k\) (resp. \(\bigcup_{k \geq \ell} R_k\)).

**Definition 5.6.** The layer \(R_1\) of an \(L\)-layered pre-domain\(^{\dagger}\) \(R\) is called the monoid of tangible elements.

We are interested in the case that \(R_1\) generates \(R\).

**Lemma 5.7.** If \(M\) is any submonoid of a \(L\)-layered pre-domain\(^{\dagger}\) \(R := (R, L, (\nu_{m,\ell}))\), then the additive sub-semigroup \(M\) of \(R\) generated by \(M\) is also a \(L\)-layered pre-domain\(^{\dagger}\).

**Proof.** \(M\) is a semiring\(^{\dagger}\), by distributivity. Axiom A1 is given, and the other axioms follow a fortiori. \(\square\)

**Definition 5.8.** The tangibly generated sub-semiring\(^{\dagger}\) is the sub-semiring\(^{\dagger}\) generated by \(R_1\); if this is \(R\), we say that \(R\) is tangibly generated.

**Remark 5.9.** Several initial observations are in order.

(i) The layered structure resembles that of a graded algebra, with two major differences: On the one hand, the condition that \(R\) is the disjoint union of its layers is considerably stronger than the usual condition that \(R\) is the direct sum of its components; on the other hand, Axioms \(A_4\) and \(B\) show that the layers are not quite closed under addition.
(ii) This paper is mostly about L-layered bi-domains\(^1\) (in particular, L-layered bi-semifields\(^1\)). However, since \(\nu\)-bipotence does not hold for polynomials, one considers the more general L-layered pre-domains\(^1\) when studying polynomial semifields\(^1\).

(iii) For each \(\ell \in L\) we introduce the sets

\[
R_{>\ell} := \bigcup_{m > \ell} R_m \quad \text{and} \quad R_{\geq \ell} := \bigcup_{m \geq \ell} R_m.
\]

Many of our current examples satisfy \(L = L_{\geq 1}\) and thus \(R = R_{\geq 1}\). When \(R\) is an L-layered bi-domain\(^1\), we claim that \(R_{\geq 1}\) is an \(L_{\geq 1}\)-layered sub-bi-domain\(^1\) of \(R\), and \(R_{>k}\) and \(R_{\geq k}\) are semifring\(^1\) ideals of \(R_{\geq 1}\), for each \(k \in L_{\geq 1}\). Indeed, this is an easy verification of the axioms, mostly from Axiom A2.

(iv) Given any L-layered bi-domain\(^1\) \(R\) and any multiplicative submonoid \(M\) of \(R_{\geq 1}\), we want to define the L-layered sub-bi-domain\(^1\) of \(R\) generated by \(M\). First we take

\[
M' := \{ \nu_{k,1}(a) : \ell \geq k \in L, \ a \in M \cap R_k \},
\]

which is a submonoid closed under the transition maps. Then we take

\[
M'' := M' \cup \{ a + b : a, b \in M' \text{ with } a \cong_{\nu} b \},
\]

This is closed under all the relevant operations, so is the desired L-layered bi-domain\(^1\). Note that the second stage is unnecessary for \(a = b\), in view of Axiom A4.

(v) Although ubiquitous in the definition, the sort transition maps get in the way of computations, and it is convenient to define the elements

\[
e_\ell := \nu_{\ell,1}(1_R) \quad (\ell \geq 1).
\]

If \(a \in R_k\), \(\ell \in L\), and \(\ell \geq 1\), we conclude by Axiom A3 that

\[
\nu_{k,1}(a) = \nu_{k,1}(1_R) \cdot \nu_{k,1}(a) = \nu_{k,1}(1_R) \cdot a = e_\ell a.
\]

Thus the sort transition map \(\nu_{k,k}\) means multiplication by \(e_\ell\).

Note that \(e_k + e_\ell = e_{k+\ell}\) by Axiom A4.

The element \(e_\ell\) is a (multiplicative) idempotent of \(R\) iff \(\ell^2 = \ell\) in \(L\). In particular, \(e_1\) and \(e_\infty\) (when \(\infty \in L\)) are idempotents of \(R\).

Let us introduce the sorting map \(s : R \to L\), which sends every element \(a \in R_\ell\) to its sort index \(\ell\), and we view the semifring\(^1\) \(R\) as an object fibered by \(s\) over the sorting semifring\(^1\) \(L\).

Remark 5.10. Axioms A1 and A2 yield the conditions

\[
s(1_R) = 1_L, \quad s(ab) = s(a)s(b), \quad \forall a, b \in R.
\]

Also, Axiom A4 yields \(s(a + a) = s(a) + s(a) = 2s(a)\), thereby motivating us to view addition of an element with itself as doubling the layer. Applying \(\nu\)-bipotence to Axiom B shows that

\[
s(a + b) \in \{ s(a), s(b), s(a) + s(b) \}.
\]

To emphasize the sorting map, as well as the order on \(L\), we sometimes write \((R, L, s, P, (\nu_{m,\ell}))\) for a given L-layered bi-domain\(^1\) \(R\) with sort transition maps \((\nu_{m,\ell} : m \geq \ell)\) and their accompanying sorting map \(s : R \to L\).

5.2. Uniform L-Layered bi-domains\(^1\).

There are two main examples coming from tropical mathematics.

(a) Let \(R = \mathcal{B}(L, G)\) (corresponding to the “naive” tropical geometry). By Construction 5.1, the sort transition maps \(\nu_{m,\ell}\) are all bijective.

(b) Suppose \(K\) is the field of Puiseux series \(\{ f := \sum_{u \in \mathbb{Q}} a_u \lambda^u : f\ \text{has well-ordered support} \}\) over a given field \(F\). Then we have the \(m\)-valuation \(v : K \to \mathbb{Q}\) taking any Puiseux series \(f\) to the lowest real number \(a\) in its support.

We incorporate \(K\) into the structure of \(R\), by putting \(R_\ell\) to be a copy of \(K\) for \(\ell \leq 1\) and \(R_\ell\) to be a copy of \(G\) for \(\ell > 1\). We take the \(\nu_{m,\ell}\) to be \(v\) whenever \(m > 1 \geq \ell\), and the identity otherwise. In this way, \(v\) is preserved within the structure of \(R\).
We focus on the first case, since one can reduce to it anyway via the equivalence given below in Definition 5.19 (which takes us from the usual algebraic world to the tropical world).

Definition 5.11. An L-layered pre-domain $R$ is uniform if $\nu_{\ell,k}$ is 1:1 and onto for each $\ell > k$.

Example 5.12. Example 5.1 is a uniform $L$-layered bi-domain $\nu$, when the monoid $G$ is cancellative.

Let us see how the layered theory simplifies for uniform $L$-layered bi-domains, enabling us to remove the sort transition maps $\nu_{\ell,k}$ from the picture.

Lemma 5.13. Any element $a \in R_{\ell}$ can be written uniquely as $e_{\ell}a_1 = \nu_{\ell,1}(a_1)$ for $a_1 \in R_1$.

Proof. Existence and uniqueness of $a_1$ are clear since $\nu_{\ell,1}$ is presumed to be 1:1. The last assertion follows from Axiom A3.

Proposition 5.14. In a uniform $L$-layered bi-domain $\nu$, if $a \equiv_{\nu} b$ for $a \in R_k$ and $b \in R_{\ell}$ with $\ell \geq k$ then $b = \nu_{\ell,k}(a)$. In particular, if $a \equiv_{\nu} b$ for $a, b \in R_{\ell}$, then $a = b$.

Proof. An immediate application of Lemma 5.13.

Now we can remove the sort transition maps from the definition, when we write $R = \bigcup_{\ell \in L} e_{\ell}R_1$.

Proposition 5.15. If $(L, \cdot)$ is a multiplicative group, then one has $\nu_{\ell,k}(ak) \equiv_{\nu} e_mak$ where $\ell = mk$.

In a uniform $L$-layered bi-domain (for $L$ arbitrary), the transition map $\nu_{\ell,\ell}$ is given by $e_{\ell}a_1 \to e_{m}a_1$.

Proof. If $\ell = k$ there is nothing to prove, so we assume that $\ell > k$ are non-negative, and write $\ell = k + p$ for $p \in L$. Then $\ell = k(1 + pk^{-1})$, and $m = 1 + pk^{-1}$. Now $e_mak \in R_{\ell}$, and

$e_mak \equiv_{\nu} ak \equiv_{\nu} \nu_{\ell,k}(ak)$.

The second assertion now is clear.

Thus the sort transition maps have been replaced by multiplication by the $e_m$. Note that $\nu$-bipotence and Axiom B′ could then be used as the definition for addition in $R$, and we summarize our reductions:

Proposition 5.16. A uniform $L$-layered bi-domain $\nu$ can be described as the semiring $R := \bigcup_{\ell \in L} R_{\ell}$, where each $R_{\ell} = e_{\ell}R_1$, $(R_1, \cdot)$ is a monoid, and there is a 1:1 correspondence $R_1 \to R_{\ell}$ given by $a \mapsto e_{\ell}a$ for each $a \in R_1$.

Proposition 5.17. In the uniform case, axioms A2 and A3 can be replaced by the respective axioms:

A2’. If $a = e_{\ell}a_1 \in R_k$ and $b = e_{\ell}b_1 \in R_{\ell}$, for $a_1, b_1 \in R_1$, then $ab = (a_1b_1)e_{k\ell}$.

A3’. $e_{\ell}e_k = e_{k\ell}$ for all $k, \ell \in L$.

Furthermore, Axiom A4 now is equivalent to Axiom B, which we can reformulate as:

B’. If $a = e_{\ell}a_1$ and $b = e_{\ell}a_1$ (so that $a \equiv_{\nu} b$), then $a + b = e_{k+\ell}a_1$.

The operations in $R$ are given by Axioms A2’, A3’, B’, and $\nu$-bipotence.

Proof. Axiom A3 follows from the observation that $e_{\ell}a_1e_{k}b_1 = e_{k}(a_1b_1)$; when $a_1, b_1 \in R_1$ then $a_1b_1 \in R_1$.

5.3. Reduction to the uniform case. In one sense, we can reduce the general case of an $L$-layered pre-domain $R$ to the uniform case. First we cut down on superfluous elements. Note that if $\nu_{k,1}$ are onto for all $k \geq 1$, then all the $\nu_{\ell,k}$ are onto for all $\ell \geq k$. Indeed, if $a \in R_{\ell}$ then writing $a = \nu_{\ell,1}(a_1)$ we have

$a = \nu_{\ell,k}(\nu_{k,1}(a_1))$.

Remark 5.18. Suppose $L = L_{\geq 1}$. For any $L$-layered pre-domain $R := (R, L, (\nu_{m,l}))$, if we replace $R_k$ by $\nu_{k,1}(R_1)$ for each $k \in L$, we get an $L$-layered bi-domain for which all the $\nu_{\ell,k}$ are onto.
Definition 5.19. Define the equivalence relation
\[ a \equiv b \quad \text{when} \quad s(a) = s(b) \quad \text{and} \quad a \cong \nu b. \]

In view of Proposition 5.14, this relation is trivial in case \( R \) is a uniform \( L \)-layered bi-domain\(^1\).

Proposition 5.20. The binary relation \( \prec \nu \) on an \( L \)-layered pre-domain\(^1\) \( R \) induces a pre-order on the semiring\(^1\) of equivalence classes \( R/\equiv \). Furthermore, if \( a \equiv b \), then \( ac \equiv bc \) and \( a + c \equiv b + c \) for all \( c \in R \).

Thus, \( \equiv \) is a congruence.

Proof. The first assertion is immediate. For the second assertion, \( s(ac) = s(a)s(c) = s(b)s(c) = s(bc) \) and \( ac \cong bc \), proving \( ac \equiv bc \).

Next, we consider addition. If \( a > \nu c \), then
\[ a + c = a \equiv b = b + c. \]

If \( a < \nu c \), then \( a + c = c + b + c \). If \( a \cong \nu c \), then
\[ s(a + c) = s(a) + s(c) = s(b) + s(c) = s(b + c), \]
and \( a + c \cong \nu b \cong \nu b + c. \)

\[ \square \]

Let us summarize.

Corollary 5.21. When the transition maps \( \nu_{\ell,k} \) are onto, one can reduce to uniform \( L \)-layered bi-domains\(^1\), by means of the equivalence relation \( \equiv \) of Definition 5.19.

Proof. Any \( \nu \)-equivalent elements having the same sort are identified. Then Proposition 5.20 shows that \( R/\equiv \) is an \( L \)-layered bi-domain\(^1\), under the natural induced layering, and the transition maps on \( R/\equiv \) clearly are bijective.

\[ \square \]

5.3.1. Ghosts and the surpassing relation. We want a layered version of ghosts.

Definition 5.22. An \( \ell \)-ghost sort is an element \( \ell + k \in L \) for some \( 0 \neq k \in L \). An element \( b \in R \) is an \( \ell \)-ghost (for given \( \ell \in L \)) if \( s(b) \in L \) is an \( \ell \)-ghost sort. A ghost element of \( R \) is a 1-ghost.

Thus, the relation \( (\geq) \) on \( L \) satisfies
\[ m \geq \ell \quad \text{when} \quad \begin{cases} m = \ell \\ \text{or} \\ m \text{ is an } \ell \text{-ghost sort.} \end{cases} \]

Here is a key relation in the theory, even though it does not play a major role in this discussion.

Definition 5.23. The \( L \)-surpassing relation \( \models_L \) is given by

\[ a \models_L b \quad \text{iff} \quad \begin{cases} a = b + c \quad \text{with} \ c \ an \ s(b)\text{-ghost}, \\ a = b, \\ a \cong \nu b \quad \text{with} \ a \ an \ s(b)\text{-ghost}. \end{cases} \quad (5.6) \]

It follows that if \( a \models_L b \), then \( a + b \) is \( s(b)\)-ghost. When \( a \neq b \), this means \( a \geq \nu b \) and \( a \) is an \( s(b)\)-ghost.

Remark 5.24. If \( a \models_L b \), then clearly \( a + c \models_L b + c \) and \( ac \models_L bc \). Thus \( \models_L \) respects the semiring\(^1\) operations.

Remark 5.25. If \( a > \nu b \), then \( (a + b)^m = a^m \). Hence, the Frobenius property \( (a + b)^m = a^m + b^m \) is satisfied in an \( L \)-layered pre-domain\(^1\) whenever \( a \neq \nu b \). We always have \( (a + b)^m \models_L a^m + b^m \).
5.4 Layered homomorphisms. In line with the philosophy of this paper, we would like to introduce the category of \( L \)-layered pre-domains\(^1\). This entails finding the correct definition of morphism. We start with the natural definition from the context of domains\(^1\). Although this definition is good enough for the purposes of this paper, a more sophisticated analysis would require us to consider the notion of “supervaluation” from [19], and how this relates to morphisms that preserve the layers. Here we take the morphisms in this category to be semiring\(^1\) homomorphisms which respect the order on the sorting semiring \( L \):

**Definition 5.26.** A layered homomorphism of \( L \)-layered pre-domains\(^1\) is a map

\[
\Phi := (\varphi, \rho) : (R, L, s, (\nu_{m, \ell})) \to (R', L', s', (\nu'_{m', \ell'}))
\]

such that \( \rho : L \to L' \) is a semiring\(^1\) homomorphism, together with a semiring\(^1\) homomorphism \( \varphi : R \to R' \) such that

1. If \( \varphi(a) \notin R'_0 \), then \( s'(\varphi(a)) \geq \rho(s(a)) \).
2. If \( a \cong \nu b \), then \( \varphi(a) \cong \nu \varphi(b) \). (This is taken in the context of the \( \nu'_{m', \ell'} \).

We always denote \( \Phi = (\varphi, \rho) \) as \( \Phi : R \to R' \) when unambiguous. In most of the following examples, the sorting semirings\(^1\) \( L \) and \( L' \) are the same. Accordingly, we call the layered homomorphism \( \Phi \) an \( L \)-homomorphism when \( L = L' \) and \( \rho = 1_L \).

**Proposition 5.27.** Any layered homomorphism \( \varphi \) preserves \( \nu \), in the following sense:

If \( a \cong \nu b \), then \( \varphi(a) \cong \nu \varphi(b) \).

**Proof.** \( \varphi(a) \cong \nu \varphi(a + b) = \varphi(a) + \varphi(b) \), implying \( \varphi(a) \cong \nu \varphi(b) \). \( \square \)

**Proposition 5.28.** Suppose \( \varphi : R \to R' \) is a layered homomorphism, and \( R \) is tangibly generated (cf. Definition 5.28). Then \( \varphi \) is determined by its restriction to \( R_1 \), via the formula

\[
\varphi(a + b) = \varphi(a) + \varphi(b), \quad \forall a, b \in R_1.
\]

**Proof.** It is enough to check sums, in view of Lemma 5.7. We get the action of \( \varphi \) on all of \( R \) since \( R_1 \) generates \( R \). \( \square \)

**Remark 5.29.** The definition given here of layered homomorphism is too strict for some applications. One can weaken the definition of layered homomorphism by utilizing the surpassing relation, requiring merely that \( \varphi(a) + \varphi(b) \vdash_L \varphi(a + b) \), but various technical difficulties arise, so we defer the study of this category to [20].

Before continuing, let us see how this definition encompasses various prior tropical situations.

**Example 5.30.** We assume throughout that \( R \) is an \( L \)-layered pre-domain\(^1\).

(i) In the max-plus situation, when \( L = \{1\} \), \( \rho \) must be the identity, and \( \Phi \) is just a semiring\(^1\) homomorphism.

(ii) In the “standard supertropical situation,” when \( L = \{1, \infty\} \), \( \Phi \) must send the ghost layer \( R_\infty \) to \( R'_\infty \). If \( a \in R \), one could take \( R'_1 := R \setminus a \) and \( R'_\infty := R'_\infty \cup a \). The identity map is clearly a layered homomorphism; its application “expands the ghost ideal” to \( a \), thereby taking the place of a semiring homomorphism to the factor semiring\(^1\).

(iii) Generalizing (ii), we obtain layered homomorphisms by modifying the layering. We say a re-sorting map of a uniform \( L \)-layered pre-domain\(^1\) \( R \) is a map \( s' : R \to L \) satisfying the following properties:

(a) \( s'(1_R) = 1 \),
(b) \( s'(R_1) \subseteq L_{\geq 1} \),
(c) \( s'(ab) = s'(a)s'(b) \), \( \forall a, b \in R_1 \).
(d) \( s'(e_\ell a) = \ell s'(a), \forall a \in R_1. \)

Then the following properties also are satisfied:

(a) \( s'(e_\ell) = \ell \) for all \( \ell \in L. \)
(b) \( s'(ab) \geq s'(a)s'(b) \) for all \( a, b \in R. \)
(c) \( s'(a) \geq s(a), \forall a \in R. \)

To see this, take \( a \in R_k \) and \( b \in R_\ell, \) and write \( a = e_k a_1 \) and \( b = e_\ell b_1 \) for \( a_1, b_1 \in R_1. \) Then

\[
s'(e_\ell) = s'(e_\ell 1_R) = \ell s'(1_R) = \ell \cdot 1 = \ell
\]

Taking \( c_1 = a_1 b_1 \in R_1, \) we have

\[
s'(ab) = s'(e_\ell e_\ell c_1) = k\ell s'(c_1) = k\ell s'(a_1)s'(b_1) = s'(a)s'(b),
\]
and \( s'(a) = s'(e_k a_1) = k s'(a_1) \geq k. \)

(iv) The natural injections \( R_{k \geq 1} \to R \) and \( \{ \bigcup_k R_\ell : \ell \in \mathbb{N} \} \to R \) are all examples of layered homomorphisms.

(v) The \( L \)-truncation map of [17, §3] is a layered homomorphism.

(vi) Suppose \( R \) is a layered pre-domain\(^1\). We adjoin \( \infty \) to \( L, \) and take \( R_\infty \) and \( \nu_{\infty,k} \) to be the direct limit of the \( R_k \) and \( \nu_{\ell,k}, \) and write \( \nu \) for the various \( \nu_{\infty,k}. \) An element in \( a \in R_1 \) is \( \nu \)-non-
cancellative if \( ab \equiv_\nu c \) for suitable \( b,c, \) where \( b \not\equiv_\nu c. \) We define the map \( \varphi : R \to R \) which is
the identity on \( \nu \)-cancellative elements but \( \varphi(a) = a^\nu \) for all \( \nu \)-non-cancellative elements \( a \in R. \)
In particular, \( \varphi(R) \) is comprised precisely of the \( \nu \)-cancellative tangible elements.

We claim that \( \varphi \) is a homomorphism. If \( ab \) is \( \nu \)-cancellative this is clear, so we may assume
that \( a \) is \( \nu \)-non-cancellative. Then

\[
\varphi(ab) = (ab)^\nu = a^\nu \varphi(b) = \varphi(a)\varphi(b).
\]

Certainly \( \varphi(a + b) = \varphi(a) + \varphi(b) \) by bipotence unless \( a \equiv_\nu b, \) in which case

\[
\varphi(a + b) = \varphi(a^\nu) = a^\nu + \varphi(a) + \varphi(b).
\]

Furthermore, \( \varphi(R) \) is a layered pre-domain\(^1\) which is \( \nu \)-cancellative with respect to \( \varphi(R) \).
Indeed, if \( \varphi(a)\varphi(b) \equiv_\nu \varphi(a)\varphi(c) \) with \( \varphi(a) \in \varphi(R)_1, \) then \( \varphi(b) \equiv_\nu \varphi(c). \)
Note that this is not the same example used in [19].

Our main example for future use is to be given in Remark [7,10]

6. THE LAYERED CATEGORIES AND THE CORRESPONDING TROPICALIZATION FUNCTORS

Having assembled the basic concepts, we are finally ready for the layered tropical categories. Our
objective in this section is to introduce the functor that passes from the “classical algebraic world” of
integral domains with valuation to the “layered world,” taking the cue from [22, Definition 2.1], which
we recall and restate more formally.

Here are our first main layered categories, starting with the more encompassing and proceeding to the
specific. In each case the morphisms are the relevant layered homomorphisms.

**Definition 6.1.**

(a) \( \text{LayPreD}^\dagger \) is the category whose objects are layered pre-domains\(^1\).

(b) \( \text{LayBidom}^\dagger \) is the full subcategory of \( \text{LayPreD}^\dagger \) whose objects are layered bi-domains\(^1\).

(c) \( \text{ULayBidom}^\dagger \) is the full subcategory of \( \text{LayPreD}^\dagger \) whose objects are uniform layered bi-domains\(^3\).
6.1. Identifications of categories of monoids and layered pre-domains.

Remark 6.2. We define the forgetful functor ULayBidom† → OMon† given by sending any uniform L-layered bi-domain† \( R := \LD(L, G) \) to \( R_1 \).

We want retracts for this forgetful functor. By Proposition 5.28, any layered homomorphism corresponds to a homomorphism of the underlying monoid of tangible elements, thereby indicating an identification between categories arising from the construction of layered bi-domains† from pre-ordered monoids.

Theorem 6.3. There is a faithful layering functor

\[ F_{\text{lay}} : \OMon^+ \rightarrow \ULayBidom^\dagger, \]

given by sending any monoid \( G \) to \( \LD(L, G) \), and sending the ordered homomorphism \( \varphi : G \rightarrow G' \) to the layered homomorphism \( \LD(L, G) \rightarrow \LD(L, G') \) induced by \( \varphi \). The functor \( F_{\text{lay}} \) is a left retract of the forgetful functor of Remark 6.2.

Proof. The image of a cancellative ordered monoid \( G \) is a layered bi-domain†, in view of [17, Proposition 2.3], and one sees easily that \( F_{\text{lay}} \varphi \) is a layered morphism since, for \( a \geq \nu \ b \),

\[ F_{\text{lay}} \varphi(\sum |k| a + |\ell| b) \cong \nu F_{\text{lay}} \varphi(\sum |k| a) \cong \nu \varphi(\sum |k| a) + \varphi(\sum |\ell| b), \]

and \( s'(F_{\text{lay}} \varphi(\sum |k| a + |\ell| b)) \geq k \).

Also, the morphisms match. The functor \( F_{\text{lay}} \) is faithful, since one recovers the original objects and morphisms by applying the forgetful functor of Remark 6.2. \( \square \)

More subtly, at times we want to forget the order on our monoids, to apply the theory of [8]. Even so, we have a universal construction with respect to “universal characteristic.”

Example 6.4. Given a cancellative monoid \( G \) and a partially ordered semiring \( L \), define the semiring† \( U_L(G) \) as follows:

Each element of \( U_L(G) \) is a formal sum of elements of \( G \), each supplied with its layer, i.e., has the form

\[ \left\{ \sum_{a \in S} [\ell a] : S \subset G \right\}. \]

Addition is given by the rule

\[ \sum_{a \in S} [\ell a] + \sum_{a \in S'} [\ell' a] = \sum_{a \in S \cup S'} [\ell a + \ell' a]. \tag{6.1} \]

Here we formally define \( \ell a + \ell' a \) to be \( \ell a \) (resp. \( \ell' a \)) if \( a \notin S' \) (resp. if \( a \notin S \)).

Multiplication is given by

\[ [k] a \cdot [\ell] b = [k \ell] (ab), \tag{6.2} \]

extended via distributivity.

We want \( U_L(G) \) to be the universal to the forgetful functor of Corollary 6.3. This is “almost” true, with a slight hitch arising from (6.1).

Proposition 6.5. Given any monoid bijection \( \varphi : \mathcal{M} \rightarrow \mathcal{G} \) where \( \mathcal{G} := (\mathcal{G}, \cdot, \leq, 1_\mathcal{G}) \) is a totally ordered monoid, viewed as a bipotent semiring† as in Proposition 5.17, there is a natural homomorphism

\[ \tilde{\varphi} : U_L(\mathcal{M}) \rightarrow \LD(L, \mathcal{G}) \]

given by

\[ \tilde{\varphi}\left(\sum_{a \in S} [\ell a]\right) = \sum_{a \in S} [\ell a] \varphi(a_1). \]

The composite

\[ \mathcal{M} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\tilde{\varphi}} \LD(L, \mathcal{G}). \]
also factors naturally as
\[ \mathcal{M} \xrightarrow{\varphi} (U_L(\mathcal{M}), \cdot) \xrightarrow{\hat{\varphi}} \mathcal{R}(L, \mathcal{G}). \]

In case \( L = \{1\} \) (the max-plus setting) or \( L = \{1, \infty\} \) (the standard supertropical setting), the previous assertion holds more generally for any monoid homomorphism \( \varphi : \mathcal{M} \rightarrow \mathcal{G} \).

**Proof.** The multiplication rules match, so the verifications follow formally, cf. Remark 6.5. The last assertion is true because in these particular situations the Frobenius property is an identity, holding even when \( a \cong_v b \). \( \square \)

**Remark 6.6.** Since the Frobenius property is an identity, one could just mod it out from our construction of \( U_L(\mathcal{M}) \) utilizing Remark 2.11 and thus get a universal with respect to satisfying the Frobenius property.

6.2. The layered tropicalization functor. Having our categories in place, we can get to the heart of tropicalization.

**Definition 6.7.** Given a semiring \( L \), the \emph{L-tropicalization functor}
\[ \mathcal{F}_{L\text{Trop}} : \text{ValMon}^+ \rightarrow \text{ULayBidom}^\dagger \]
from the category of valued monoids (with cancellation in the target) to the category of uniform layered bi-domains \(^\dagger\) is defined as follows: \( \mathcal{F}_{L\text{Trop}} : (\mathcal{M}, \mathcal{G}, v) \mapsto \mathcal{R}(L, \mathcal{G}) \) and \( \mathcal{F}_{L\text{Trop}} : \phi \mapsto \alpha_\phi \), where given a morphism \( \phi : (\mathcal{M}, \mathcal{G}, v) \rightarrow (\mathcal{M}', \mathcal{G}', v') \) we define \( \alpha_\phi : \mathcal{R}(L, \mathcal{G}) \rightarrow \mathcal{R}(L, \mathcal{G}') \), by
\[ \alpha_\phi(\{\ell\}a) := \{\ell\} \phi(a), \quad a \in \mathcal{G}, \quad (6.3) \]
cf. Formula (6.4).

We also consider \( \mathcal{F}_{L\text{Trop}} \) as acting on individual elements of \( \mathcal{M} \), whereby
\[ \mathcal{F}_{L\text{Trop}}(a) = [1]_v a. \quad (6.4) \]
This is indeed a functor, in view of [17, Theorem 4.9].

Note that the tropicalization functor \( \mathcal{F}_{L\text{Trop}} \) factors as \( \text{ValMon}^+ \rightarrow \text{OMon}^+ \rightarrow \text{ULayBidom}^\dagger \).

6.3. More comprehensive layered tropicalization functors. The basic layered tropicalization functor only recognizes the image in \( \mathcal{G} \), so loses much information about the original monoid \( \mathcal{M} \). In analogy to [31], in order to preserve information, we can encode extra information, motivated by the residue field in valuation theory.

6.3.1. The unit tropicalization functor.

**Definition 6.8.** Given a monoid \( \mathcal{M} := (\mathcal{M}, \cdot, 1_\mathcal{M}) \) with m-valuation \( v : \mathcal{M} \rightarrow \mathcal{G} \), we define its \emph{unit submonoid}
\[ \mathcal{M}_{(1)} := \{ a \in \mathcal{M} : v(a) = 1_\mathcal{G} \}, \]
the submonoid of \( \mathcal{M} \) on which the restriction of \( v \) is the trivial valuation.

When \( \mathcal{M} \) is a group, then \( \mathcal{M}_{(1)} \), also is a group. In particular, the category \( \text{ValMon}_{(1)} \) of unit monoids with m-valuation is a full subcategory of the category \( \text{ValMon} \).

**Example 6.9.** In Example 4.3, \( C_{(1)} \) is the complex unit circle.

The following observation is now clear.

**Proposition 6.10.** There is a functor
\[ \mathcal{F}_{L\text{Trop;unit}} : \text{ValMon}^+ \rightarrow \text{ULayBidom}^\dagger \times \text{ValMon}_{(1)}, \]
given as follows: \( \mathcal{F}_{L\text{Trop;unit}}((\mathcal{M}, \mathcal{G}, v)) = (\mathcal{R}(L, \mathcal{G}), \mathcal{M}_{(1)}) \) and \( \mathcal{F}_{L\text{Trop;unit}}(\phi) = (\alpha_\phi, \phi|_{\mathcal{M}_{(1)}}) \), where the morphism \( \alpha_\phi : \mathcal{R}(L, \mathcal{G}) \rightarrow \mathcal{R}(L', \mathcal{G}') \) is given by Equation (6.3).

**Proof.** We piece together the two functors. \( \square \)
This functor could be interpreted as separating the m-valuation \( v \) into two components, corresponding to the value monoid and the residue domain. Tropicalization in its original form involved taking the logarithm of the absolute value of \( re^{i\theta} \), which is just \( \log |r| \). Thus, the argument \( e^{i\theta} \) is lost, and researchers dealt with that separately. Since these all have absolute value 1, it seems appropriate in the valuation-theory analog to have \( \mathcal{F}_{\text{LTrop} \text{unit}} \) at our disposal.

A more direct approach in the terminology of Remark 6.3.2 Given two Puiseux series \( p, q \in K \) with \( \text{Val}(p) = \text{Val}(q) \), we see that \( \text{Val}(p) = \text{Val}(q) \) iff \( \text{Val}(pq^{-1}) = \text{Val}(1) = \mathbb{Z}_K \), i.e., \( pq^{-1} - 1 \) is in the valuation ideal of the valuation \( \text{Val} \). Thus, Proposition 6.10 gives us a way of understanding \( \mathcal{F}_{\text{LTrop} \text{unit}} \) in terms of \( \text{Val} \). Namely, we check whether two Puiseux series have the same lowest order exponent, and then can check whether their lowest coefficients are the same by means of the residue field.

Remark 6.11. Suppose \( W \) is an arbitrary integral domain with valuation \( v : W \setminus \{0_W\} \to \mathcal{G} \), with valuation ring \( R \) and residue domain \( W \). Take the unit submonoid \( W_1 \) of \( W \), cf. Definition 6.3. Clearly

\[
W_1 = \{ r \in R : r + p = 1 + p \}.
\]

When \( W \) is a field, \( W_1 \) is a multiplicative subgroup of \( W \) which could be thought of as the “first congruence subgroup” in valuation theory. Then, for \( b \neq 0, aW_1 = bW_1 \) iff \( v(a) = v(b) \) and \( 1 - ab^{-1} \in p \), which relates to the condition of the previous paragraph.

6.3.2. The exploded tropicalization functor. One could preserve more information, according to Parker [31], who introduced “exploded” tropical mathematics, and Payne [36]. This entails taking the leading coefficient of Puiseux series.

E. Sheiner introduced a related structure \( \mathcal{R}(K, R) \) on Puiseux series, in which he uses the residue field \( K \) as the sorting set. Define the map \( K \to \mathcal{R}(K, R) \) by \( p \mapsto \hat{a}v(p) \) where \( \alpha \) is the coefficient of the lowest monomial of the Puiseux series \( p \). This map, generalizing the Kapranov map, keeps track of the “leading coefficient” of the Puiseux series \( p \) in terms of when the image of \( p \) has layer 0.

From this perspective, the \( 0_K \) layer represents the “corner ghosts.” Thus, Sheiner has “exploded” the notion of valuation, and it is not difficult to define the “exploded functor” and transfer the statement and proof of Payne [36] to this context, to be indicated in [38]. Let us describe this procedure in algebraic terms, which means working in the associated graded algebra.

Definition 6.12. Given a valued monoid \( v : M \to \mathcal{G} \), and \( g \in \mathcal{G} \), we write \( M_{\geq g} \) for the \( M \)-module \( \{ a \in M : v(a) \geq g \} \), and \( M_{> g} \) for its submodule \( \{ a \in M : v(a) > g \} \).

When \( M \) is the multiplicative monoid of an integral domain \( W \), we can define the associated graded algebra

\[
\text{gr}(W) := \bigoplus_{a \in \mathcal{G}} W_{\geq g}/W_{> g},
\]

where operations are given by

\[
(a + W_{> g})(b + W_{> h}) = ab + W_{> gh} \quad (a + W_{> g}) + (b + W_{> h}) = a + b + W_{> gh}.
\]

Remark 6.13. It is well known that the associated graded algebra is an algebra, with the natural valuation \( \hat{v} \) induced by \( v \), i.e., \( \hat{v}(a + W_{> g}) = v(a) \). When the valuation \( v \) is discrete, each component \( W_{\geq g}/W_{> g} \) is (multiplicatively) isomorphic to \( W \).

Let us interpret “explosion” with respect to Puiseux series. For any real number \( \alpha \), the component \( K_{\geq \alpha}/K_{> \alpha} \) can be identified with \( K t^\alpha \), which as a module is isomorphic to \( K \), by means of taking the coefficient of the monomial of lowest order in a Puiseux series.

Definition 6.14. Notation as in Remark 6.11 define the exploded layered domain \( \mathcal{R}(W, \mathcal{G}) \). In other words, we sort the elements according to \( W \), with multiplication following the given multiplication in \( \mathcal{G} \) and addition given by the following rules:

\[
[a]_x + [b]_y = \begin{cases} 
[a]_x & \text{if } x > y, \\
[b]_y & \text{if } x < y, \\
[a+b]_x & \text{if } x = y.
\end{cases}
\]
Remark 6.15. Note that addition here is the classical addition induced from the integral domain $W$, so although this structure has a tropical aroma, it does preserve some of the original algebraic structure of the residue domain $W$.

Proposition 6.16. There is a functor
\[ F_{\text{LTrop};\exp}: \text{ValDom} \to \text{ULayBidom} \times \text{Ring}, \]
given as follows: $F_{\text{LTrop};\exp}(W, G, v) = (\mathcal{R}(L, G, \bar{W})$) and $F_{\text{LTrop};\exp}(\phi) = (\alpha_\phi, \overline{\phi})$, where the morphism $\alpha_\phi : \mathcal{R}(L, G) \to \mathcal{R}(L', G')$ is given by Equation (6.3) and $\overline{\phi}$ is the induced map on the residue domains.

Proof. As in Proposition [6.10], we piece together the two functors. To preserve even more information, one could sort instead with $\text{gr}(W)$.

7. The function category

We assume throughout that $R$ is an $L$-layered domain\(^1\). In the next section we describe layered varieties in terms of corner roots of ideals of polynomials over $R$. Thus, we need some preliminaries about the polynomial semiring\(^1\) over a layered bi-domain\(^1\); this is no longer bipotent.

7.1. The layered function monoid and domain\(^1\). As noted in the introduction, one significant difference between the tropical theory and “classical” algebra is that different tropical polynomials can agree as functions (whereas for algebras over an infinite field, any two distinct polynomials are also distinct as functions). The clearest way of coping with this phenomenon is to treat polynomials directly as functions from some subset of $R^{(n)}$ to an extension of $R$, and this enables us to unify various other constructions related to polynomials.

Definition 7.1. For any set $S$ and monoid $M$, $\text{Fun}(S, M)$ denotes the set of functions from $S$ to $M$.

Remark 7.2.

(i) $\text{Fun}(S, M)$ becomes a monoid, under pointwise multiplication, i.e.,
\[ (fg)(a) = f(a)g(a), \quad \forall f, g \in \text{Fun}(S, R), \quad \forall a \in S. \tag{7.1} \]

(ii) If the monoid $M$ is partially ordered, then $\text{Fun}(S, M)$ is also partially ordered, with respect to taking $f \geq g$ when $f(a) \geq g(a)$ for all $a \in S$.

When moreover $R$ is a semiring\(^1\), $\text{Fun}(S, R)$ also becomes a semiring\(^1\), under pointwise addition, i.e.,
\[ (f + g)(a) = f(a) + g(a), \quad \forall a \in S, \tag{7.2} \]
cf. [17] Definition 5.1.

Lemma 7.3.

(i) If a monoid $M$ is cancellative, then the function monoid $\text{Fun}(S, M)$ is cancellative.

(ii) If a layered pre-domain\(^1\) $R$ is cancellative, then the function semiring\(^1\) $\text{Fun}(S, R)$ is also a cancellative layered pre-domain\(^1\) (but not bipotent!)

(iii) If a semiring\(^1\) $R$ satisfies the Frobenius property \[17, \text{Definition 5.1}\], then $\text{Fun}(S, R)$ also satisfies the Frobenius property.

Proof. (i): By pointwise verification. For cancellation, note that if $fg = fh$, then $f(a)g(a) = f(a)h(a)$ for all $a \in S$, implying $g(a) = h(a)$ and thus $g = h$.

(ii): Same verification as in (i).

(iii): For the Frobenius property,
\[ (f + g)^n(a) = ((f + g)(a))^n = f(a)^n + g(a)^n, \quad \forall f, g \in \text{Fun}(S, R), \tag{7.3} \]
for all positive $n \in \mathbb{N}$. \(\square\)

There is a natural semiring\(^1\) injection $R \to \text{Fun}(S, R)$, given by viewing $r \in R$ as the constant function $f_r$ given by $f_r(a) = r, \forall a \in S$. In this way, we view $R$ as a sub-semiring\(^1\) of $\text{Fun}(S, R)$. At first, we take $S$ to be $R^{(n)}$. Later we will take $S$ to be a given “layered variety.” More generally, following Payne \[35, \S 2.2\], one could take the set $S$ to be the lattice of characters of an algebraic torus.
7.2. Functorial properties of the function monoid and semiring. We categorize the discussion of Section 3.11. First we define the function and polynomial categories.

Definition 7.4. $F := \text{Fun}_{\text{Mon}}(S, _) \text{ is the functor from Mon to Mon given by } M \mapsto \text{Fun}(S, M)$ for objects, and such that for any morphism $\varphi : M \to M'$, we define $F\varphi : \text{Fun}(S, M) \to \text{Fun}(S, M')$ to be given by

$$F\varphi(f)(a) = \varphi(f(a)).$$

The functor $F := \text{Fun}_{\text{Semir}}(S, _) : \text{Semir}^\dagger \to \text{Semir}^\dagger$ is given by $R \mapsto \text{Fun}(S, R)$ for objects, and again such that for any morphism $\varphi : R \to R'$, $F\varphi : \text{Fun}(S, R) \to \text{Fun}(S, R')$ is given by

$$F\varphi(f)(a) = \varphi(f(a)).$$

Lemma 7.5. $\text{Fun}_{\text{Mon}}(S, _)$ and $\text{Fun}_{\text{Semir}}(S, _)$ are functors. Furthermore, $\text{Fun}_{\text{Mon}}(S, _)$ restricts to a functor from $\text{OMon}^+$ to $\text{POMon}^+$.

Proof. The verifications are straightforward, in view of Remark 7.2(ii) and Lemma 7.3.

Definition 7.6. We denote the respective images of $\text{OMon}^+$ and $\text{Semir}^\dagger$ under the functors $\text{Fun}_{\text{Mon}}(S, _)$ and $\text{Fun}_{\text{Semir}}(S, _)$ as $\text{Fun}(S, \text{OMon}^+)$ and $\text{Fun}(S, \text{Semir}^\dagger)$, which are respective subcategories of $\text{POMon}^+$ and $\text{Semir}^\dagger$.

Now Proposition 3.11 says:

Proposition 7.7. There is a faithful functor

$$\mathcal{F}_{(S, \text{OMon}^+)} : \text{Fun}(S, \text{OMon}^+) \to \text{Fun}(S, \text{Semir}^\dagger),$$

induced by the functor $\mathcal{F}_{\text{OMon}}$ of Proposition 3.11, as described in the proof.

Proof. We define $\mathcal{F}_{(S, \text{OMon})}(\text{Fun}(S, M)) = \text{Fun}(S, M)$ (viewing $M$ as a semiring) and, for any monoid homomorphism $\varphi : M \to M'$, $\mathcal{F}_{(S, \text{OMon})}(\varphi) : f \mapsto \varphi \circ f$. This is clearly a functor, and is faithful since $M$ is embedded into $\text{Fun}(S, M)$.

Proposition 7.8. The functors $\mathcal{F}_{(S, \text{OMon})}(S, _)$ and $\text{Fun}_{\text{Semir}}(S, _)$ commute with $\mathcal{F}_{\text{OMon}}$ of Proposition 3.11, in the sense that

$$\mathcal{F}_{(S, \text{OMon})} \text{Fun}_{\text{Mon}}(S, _)(\mathcal{M}) = \text{Fun}_{\text{Semir}}(S, _)(\mathcal{M}).$$

Proof. Letting $R$ be the semiring $^\dagger$ of Proposition 3.11 we have $\mathcal{F}_{(S, \text{OMon})} \mathcal{F}_{\text{OMon}}(S, R)(\mathcal{M}) = \text{Fun}(S, R) = \text{Fun}_{\text{Semir}}(S, R)$.

Lemma 7.9. Construction 7.7 is functorial, in the sense that

$$\text{Fun}(S, (\mathcal{A}(L, G))) \approx (\text{Fun}(S, L), \text{Fun}(S, G)).$$

Proof. Any $f \in \text{Fun}(S, (\mathcal{A}(L, G)))$ is given by $f(a) = [k]b$ for suitable $k \in L$ and $b \in G$; we define $f_L$ and $f_G$ by $f_L(a) = k$ and $f_G(a) = b$. Now $f \mapsto [f_L]f_G$ defines a semiring $^\dagger$ homomorphism

$$\text{Fun}(S, (\mathcal{A}(L, G))) \to (\text{Fun}(S, L), \text{Fun}(S, G)).$$

Conversely, given $f_L \in \text{Fun}(S, L)$ and $f_G \in \text{Fun}(S, G)$ we define $f \in \text{Fun}(S, (\mathcal{A}(L, G)))$ by putting

$$f(a) = [f_L(a)]f_G(a).$$

One sees that the sorts are preserved.

7.3. Sorting the function semiring $^\dagger$.

Remark 7.10. If $S' \subseteq S$, there is a natural semiring $^\dagger$ homomorphism

$$\text{Fun}(S, R) \to \text{Fun}(S', R)$$
given by $f \mapsto f|_{S'}$. In particular, for $S' = \{a\}$, we have the evaluation homomorphism at $a$.

One main interest in the layered theory is the nature of these homomorphisms. To understand them, we need to introduce the appropriate sorting function.
Remark 7.11. When \( L \) is a partially ordered semiring\(^\dagger \), \( \text{Fun}(S,L) \) is also a semiring\(^\dagger \) (whose unit element is the constant function \( 1 \)), which by Remark 7.2(iii) is partially ordered by the relation:
\[
f \leq_S g \quad \text{if} \quad f(a) \leq g(a) \quad \text{for all} \quad a \in S.
\]
When \( L \) is directed from above, this partial order also is directed from above, since \( f(a), g(a) \in L \) are bounded by \( \max\{f(a), g(a)\} \).

If \( R \) is \( L \)-layered, then \( \text{Fun}(S,R) \) inherits the layered structure from \( R \) pointwise with respect to \( \text{Fun}(S,L) \), in the following sense taken from [17, Remark 5.3]:

**Definition 7.12.** The \( L \)-layering map of a function \( f \in \text{Fun}(S,R) \) is the map \( \vartheta_f : S \to L \) given by
\[
\vartheta_f(a) := s(f(a)), \quad a \in S.
\]

For a set \( \mathcal{I} \subset \text{Fun}(S,R) \) we define
\[
\vartheta_{\mathcal{I}}(a) := \min\{\vartheta_f(a) : f \in \mathcal{I}\}.
\]

In the layered theory, we only consider functions that are \( \nu \)-compatible, in the sense that if \( a \cong_{\nu} a' \), then \( f(a) \cong_{\nu} f(a') \).

**Example 7.13.** \( \vartheta_{\{1,2\}} \) is the given sorting map on \( R \).

**Example 7.14.** Take \( R = \mathbb{R}[\mathbb{N}, \mathbb{R}] \). Assume that \( S = R_{\{2\}}^{(2)} = \mathbb{R}^{(2)} \). The examples are written in logarithmic notation; e.g., \( 1 := 0 \) is the multiplicative unit, and \( 2 \cdot 3 = 5 \).

(i) Take \( f_k = \lambda_k^2 + \lambda_2 + 0 \) for \( k \in \mathbb{N} \), and \( a = (a_1, a_2) \in S \).
\[
\vartheta_{f_k}(a) = \begin{cases} 3 & \text{for } a_1 = a_2 = 0; \\ 2 & \text{for } a_1 = 0 > a_2 \quad \text{or} \quad a_1 = 0 > a_2 \quad \text{or} \quad a_1^k = a_2 > 0; \\ 1 & \text{otherwise.} \end{cases}
\]

(ii) Take \( \mathcal{I} = \{f_k : k \in \mathbb{N}\} \), \( a = (a_1, a_2) \in S \). In view of (i),
\[
\vartheta_{\mathcal{I}}(a) = \begin{cases} 3 & \text{for } a_1 = a_2 = 0; \\ 2 & \text{for } a_1 = 0 > a_2 \quad \text{or} \quad a_1 = 0 > a_2; \\ 1 & \text{otherwise.} \end{cases}
\]

Thus, the 2-layer is the union of two perpendicular rays.

(iii) Take \( \mathcal{I} = \{\lambda_1 + 2, \lambda_1 + 3\} \). The layering map \( \vartheta_{\mathcal{I}} \) restricted to the tangible elements is identically 1, the same as that of a tangible constant, although the ideal generated by \( \mathcal{I} \) does not contain any constants.

Nevertheless, we can distinguish between \( \mathcal{I} \) and tangible constants, by assuming that \( S \) contains elements of \( R \) having layer \( > 1 \). For example, \( \vartheta_{\mathcal{I}}(\{2\}) = 2 \) whereas \( \vartheta_f \) for a tangible constant function \( f \) is identically 1.

As noted in [17], we layer the semiring\(^\dagger \) \( \text{Fun}(S,R) \) with respect to the sorting semiring\(^\dagger \) \( \text{Fun}(S,L) \), by sending \( f \mapsto \vartheta_f \).

Given \( f, g \in \text{Fun}(S,L) \), write \( \tilde{k} = \vartheta_f \) and \( \tilde{\ell} = \vartheta_g \). When \( \tilde{\ell} > \tilde{k} \) we define the transition map
\[
\nu_{\tilde{\ell}, \tilde{k}} : \text{Fun}(S,R)_{\tilde{k}} \to \text{Fun}(S,R)_{\tilde{\ell}}
\]
by
\[
\nu_{\tilde{\ell}, \tilde{k}}(f) : a \mapsto \nu_{\tilde{k}, \tilde{\ell}}(f(a)), \quad \forall a \in S.
\]

**Lemma 7.15.** If \( R \) is a layered pre-domain\(^\dagger \) with partial pre-order \( \geq_{\nu} \), then we can extend \( \cong_{\nu} \) and \( \geq_{\nu} \) respectively to an equivalence and a partial pre-order on \( \text{Fun}(S,R) \) as follows:

(i) \( f \cong_{\nu} g \) iff \( f(a) \cong_{\nu} g(a) \), \( \forall a \in S \);

(ii) \( f \geq_{\nu} g \) iff \( f(a) \geq_{\nu} g(a) \), \( \forall a \in S \).

**Proof.** An easy point-by-point verification. \( \square \)

We usually start with a given layered domain\(^\dagger \) \( R \), and then apply Lemma 7.15. This rather general framework encompasses some very useful concepts, including polynomials, Laurent polynomials, etc.
7.4. Polynomials. We want to understand tropical algebraic geometry in terms of roots of polynomials. Specifically, we work in the sub-semiring $S$ of polynomials of semirings $L$ (language of semirings $L$). Let $R$ be a semifield. Specifically, we work in the sub-semiring $S$ of polynomials of semirings $L$. If we adjoin the symbol $-1$ (for multiplicative inverse), then $L = \mathbb{R}$ denotes the image of the Laurent polynomials $R[\lambda, \lambda^{-1}] := R[\lambda^1, \lambda^{-1}]$. If our language includes the symbol $\sqrt{-1}$, i.e., if we are working over a divisible monoid, then we would consider polynomials with rational powers, which are well-defined in view of Equation (3.6): although this case is important, we do not treat it here explicitly because of the extra notation involved.

Thus, we are working in the full subcategories $\text{Pol}(S, \text{OMon}^+)$ and $\text{Laur}(S, \text{OMon}^+)$ of polynomials of semirings $L$. If we adjoin the symbol $-1$ (for multiplicative inverse), then $\text{Laur}(S, \text{OMon}^+)$ of polynomials of semirings $L$. The functor of $\text{Proposition 7.7}$ restricts to faithful functors $\text{Pol}(S, \text{OMon}^+) \to \text{Pol}(S, \text{Semir}^+)$ and $\text{Laur}(S, \text{OMon}^+) \to \text{Laur}(S, \text{Semir}^+)$. The difficulty with treating polynomials (as well as Laurent polynomials) as functions could be that two polynomial functions may agree on $R$ but differ on some extension semiring $S$ of $R$. Fortunately, in [17, Theorem 5.33 and Corollary 5.34] saw that taking $R \to S$ to be the 1-divisible closure of the bi-semifield of fractions of $R$, if two polynomial functions differ on some extension of $R$, then they already differ on $R$. Thus, it suffices to look at $\text{Pol}(S, \ell)$ and $\text{Laur}(S, \ell)$. Strictly speaking, this was proved only for the specific construction used in [17], so to work with layered bi-domains $S$ we need to generalize the construction of 1-divisible closure to $L$-layered bi-domains $S$.

Example 7.16 (1-localization). If $R$ is an $L$-layered bi-domain $S$, then taking any multiplicative sub-monoid $S$ of $R$, we can form the localization $S^{-1}R$ as a monoid, and define addition via

$$a + b = a \frac{uv}{uw}$$

for $a, b \in R, u, v \in S$. $S^{-1}R$ becomes an $L$-layered bi-domain $S$ when we define $s(\frac{a}{v}) = s(a)$. There is a natural layered homomorphism $R \to S^{-1}R$ given by $a \mapsto \frac{a}{uv}$, which is injective since $R$ is cancellative.

Taking $S = R_1$, we call $S^{-1}R$ the $L$-layered bi-semifield of fractions of $R$; this construction shows that any uniform $L$-layered bi-domain $S$ can be embedded into a uniform $L$-layered bi-semifield $R$.

Example 7.17 ($\nu$-divisible closure). We say that an $L$-layered bi-domain $R$ is $\nu$-divisible if for each $a \in R$ and $n \in \mathbb{N}$ there is a $b \in R$ such that $b^n \equiv a$ under the equivalence of Definition 7.19. Note that if $s(a) = \ell$ then $s(b) = \sqrt{\ell}$. This implies that $L$ must be closed under taking $n$-th roots for each $n$. Assuming that $L$ is a group satisfying this condition, one can construct the $\nu$-divisible closure, sketched as follows:

Step 1: Given $a \in R_\ell$, adjoin a formal element $b \in R_{\sqrt{\ell}}$, and consider all formal sums

$$f(b) := \sum_i \alpha_i b^i : \alpha_i \in R.$$  \hfill (7.4)

($\sum_i \alpha_i b^i$ is to be considered as the $n$-th root of $\sum_i \alpha_i^n a^i$.) Define $R_b$ to be the set of all elements of the form (7.4), where any $\alpha \in R$ is identified with $\alpha b^n$. We can define the sorting map $s : R_b \to L$ via

$$s(f(b)) = \sum_i s(\alpha_i) \sqrt{\ell} \in L.$$  

We define $\equiv_\nu$ on $R_b$ (notation as in (7.4)) by saying $f_b \equiv_\nu \sum_{j=0}^n \alpha_j b^j$ if $\sum_i \alpha_i^n a^i \equiv_\nu \sum_j \alpha_j^n a^j$.

In particular, $f_b \equiv_\nu c$ for $c \in R$ if $\sum_i \alpha_i^n a^i \equiv_\nu c^n$. Likewise, we write $f_b >_\nu f'_b := \sum_{j=0}^n \alpha_j b^j$ if $\alpha_i^n a^i >_\nu \alpha_j^n a^j$.

Now we can define addition on $R_b$ so as to be $\nu$-bipotent, where for $\nu$-equivalent elements we define $f(b) + g(b)$ to be their formal sum (combining coefficients of the same powers of $b$); multiplication is then defined in the obvious way, via distributivity over addition. Now $R_b$ is an $L$-layered bi-domain $S$, in view of Proposition 5.13.

Step 2: Using Step 1 as an inductive step, one can construct the $\nu$-divisible closure by means of Zorn’s Lemma, analogously to the well-known construction of the algebraic closure, cf. [37, Theorem 4.88].
Example 7.18 (Completion). One can construct the completion of any L-layered domain\(^{+}\) \(R\) as follows:
We define \(\nu\)-Cauchy sequences in \(R\) to be those sequences \((a_i) := \{a_1, a_2, \ldots\}\) which become Cauchy sequences modulo \(\equiv_{\nu}\), but which satisfy the extra property that there exists an \(m\) (depending on the sequence) for which \(s(a_i) = s(a_{i+1}), \forall i \geq m\). This permits us to define the sort of the \(\nu\)-Cauchy sequence to be \(s(a_n)\). Then we define the null \(\nu\)-Cauchy sequences in \(R\) to be those sequences \((a_i) := \{a_1, a_2, \ldots\}\) which become null Cauchy sequences modulo \(\equiv_{\nu}\), and the completion \(\hat{R}\) to be the factor group.

We also extend our given pre-order \(\nu\) to \(\nu\)-Cauchy sequences by saying that \((a_i) \equiv_{\nu} (b_i)\) if \((a_i b_i^{-1})\) is a \(\nu\)-Cauchy sequence, and, for \((a_i) \not\equiv_{\nu} (b_i)\), we say \((a_i) >_{\nu} (b_i)\) when there is \(m\) such that \(a_i >_{\nu} b_i\) for all \(i > m\). The completion \(\hat{R}\) becomes an L-layered bi-domain\(^{1}\) under the natural operations, i.e., componentwise multiplication of \(\nu\)-Cauchy sequences, and addition given by the usual rule that

\[
(a_i) + (b_i) = \begin{cases} 
(a_i) & \text{if } (a_i) >_{\nu} (b_i), \\
(b_i) & \text{if } (a_i) <_{\nu} (b_i), \\
\nu_{s(a_i)+s(b_i), s(a_i+b_i)}(a_i + b_i) & \text{if } (a_i) \equiv_{\nu} (b_i).
\end{cases}
\]

(In the last line, we arranged for the layers to be added when the \(\nu\)-Cauchy sequences are \(\nu\)-equivalent.)

It is easy to verify \(\nu\)-bipotence for \(\hat{R}\).

These constructions are universal, in the following sense:

Proposition 7.19. Suppose there is an embedding \(\varphi : R \to F'\) of a uniform L-layered domain\(^{1}\) \(R\) into a 1-divisible, uniform L-layered bi-semifield\(^{1}\) \(F'\), and let \(F\) be the 1-divisible closure of the bi-semifield\(^{1}\) of fractions of \(R\). Then \(F'\) is an extension of \(F\). If \(F'\) is complete with respect to the \(\nu\)-pre-order, then we can take \(F'\) to be an extension of the completion of \(F\).

Proof. This is standard, so we just outline the argument. First we embed the L-layered bi-semifield\(^{1}\) of fractions of \(R\) into \(F'\), by sending \(\frac{b}{a_1} \to \frac{\varphi(b)}{\varphi(a_1)}\). This map is 1:1, since if \(\frac{b}{a_1} = \frac{d}{c_1}\), then \(c_1 b = a_1 d\), implying \(\varphi(c_1 b) = \varphi(a_1 d)\), and thus \(\frac{\varphi(b)}{\varphi(a_1)} = \frac{\varphi(d)}{\varphi(c_1)}\). Thus, we may assume that \(R\) is an L-layered bi-semifield\(^{1}\). Now we define the map \(F \to F'\) sending \(\sqrt[n]{a} \to \sqrt[n]{\varphi(a)}\), for each \(a \in F\). This is easily checked to be a well-defined, 1:1 layered homomorphism.

In case \(F'\) is complete, then we can embed the completion of \(F\) into \(F'\). (The completion of a 1-divisible bi-semifield\(^{1}\) is 1-divisible, since taking roots of a \(\nu\)-Cauchy sequence in \(F\) yields a \(\nu\)-Cauchy sequence.)

Corollary 7.20. Any L-layered homomorphism \(\Phi : R \to R'\) of uniform L-layered bi-domains\(^{1}\) extends uniquely to an L-layered homomorphism \(\Phi : \hat{R} \to \hat{R}'\).

Proof. Each of the constructions of bi-semifield\(^{1}\) of fractions, 1-divisible closure, and completion are universal, so applying them in turn yields us a unique ordered monoid homomorphism from the tangible component \(R_1\) to \(R'_1\), which readily extends (uniquely) to all of \(R\) since \(e_i \mapsto e'_i\).

7.5. Roots and layered varieties. In order to understand affine layered geometry, we need to know more about the affine layered algebraic sets. We fix the sorting semiring\(^{1}\) \(L\) for convenience, although one could also let \(L\) vary, and think of \(S\) as a subset of \(R_1^{(n)}\), where \(R\) is an L-layered bi-domain\(^{1}\). Standard tropical geometry can be recaptured by taking \(S \subseteq R_1^{(n)}\).

Definition 7.21. Suppose \(f, g \in \text{Pol}(S, R)\). We say that \(f\) dominates \(g\) at \(a \in S\) if \(f(a) \geq_{\nu} g(a)\). Write \(f = \sum f_i\) as a sum of monomials. The dominant part \(f_a\) of \(f\) at \(a\) is the sum of all those \(f_i\) dominating \(f\) at \(a\), i.e., for which \(f_i(a) \geq_{\nu} f(a)\). We write \(f|_X\) for the restriction of \(f\) to a nonempty subset \(X\) of \(S\).

Definition 7.22. An element \(a \in S\) is a corner root of \(f = \sum f_i\) if \(f(a) = s(f_i(a))\)-ghost for every monomial \(f_i\) of \(f\). (Thus, \(f_a\) contains at least two \(f_i\).) The (affine) corner locus of \(f \in \text{Pol}(S, R)\) with respect to the set \(S\) is

\[
Z_{\text{corn}}(f; S) := \{a \in S : a \text{ is a corner root of } f\}.
\]

We write \(Z_{\text{corn}}(f)\) for \(Z_{\text{corn}}(f; R_1^{(n)})\).
An element \( a \in S \) is a \textbf{cluster root} of \( f \) if \( f(a) = f_i(a) \) is 1-ghost for some monomial \( f_i \) of \( f_a \). (Thus, \( f_a \) is comprised of a single monomial \( f_i \).) The \textbf{combined ghost locus} with respect to the set \( S \) is

\[
\mathcal{Z}_{\text{comb}}(f; S) := \{ a \in S : a \text{ is a cluster or corner root} \}.
\]

The \textbf{(affine) corner algebraic set} and the \textbf{(affine) algebraic set} of a subset \( A \subseteq \text{Pol}(S, R) \) with respect to the set \( S \), are respectively,

\[
\mathcal{Z}_{\text{corn}}(A; S) := \bigcap_{f \in A} \mathcal{Z}_{\text{corn}}(f; S), \quad \mathcal{Z}_{\text{comb}}(A; S) := \bigcap_{f \in A} \mathcal{Z}_{\text{comb}}(f; S).
\]

When \( S \) is unambiguous (usually \( R^{(n)} \)), we write \( \mathcal{Z}_{\text{corn}}(A) \) and \( \mathcal{Z}_{\text{comb}}(A) \) for \( \mathcal{Z}_{\text{corn}}(A; S) \) and \( \mathcal{Z}_{\text{comb}}(A; S) \) respectively.

\textbf{Example 7.23.} Here are basic examples of affine layered algebraic sets.

(i) We view \( R^{(n)} \) as \( \mathcal{Z}_{\text{comb}}(\emptyset) \). Note that also \( R^{(n)} = \mathcal{Z}_{\text{comb}}(\{a\}) \) for any “ghost” constant \( a \in R_{>1} \).

(ii) The empty set is an algebraic set: \( \emptyset = \mathcal{Z}(\{a\}) \) for any \( a \in R_1 \).

(iii) A single point \( a = (a_1, \ldots, a_n) \in S \), where \( S \subseteq R^{(n)}_1 \), is a corner algebraic set:

\[
a = \mathcal{Z}(\{\lambda_1 + a_1, \ldots, \lambda_n + a_n\}).
\]

(iv) The familiar \textbf{tropical line} in the affine plane is \( \mathcal{Z}_{\text{corn}}(f; S) \) where \( f \) is linear of the form

\[
\alpha \lambda_1 + \beta \lambda_2 + \gamma,
\]

with \( \alpha, \beta, \gamma \in R_1 \), and \( S = R_1^{(2)} \). On the other hand, for \( I \) as in Example 7.14 \[ \mathcal{Z}_{\text{corn}}(I; S) \]

restricted to \( R_1^{(2)} \) is the union of two perpendicular rays, and does not satisfy the celebrated “balancing condition” of tropical geometry.

\textbf{Example 7.24.} A more sophisticated example: Whereas in the standard supertropical theory we have

\[
(x + y + z)(xy + xz + yz) = (x + y)(x + z)(y + z),
\]

they differ in the layered theory, since \( xyz \) has layer 3 in the left side but only layer 2 in the right side. Thus the layered theory permits greater refinement in reducing tropical varieties.

8. The \textbf{tropicalization functor on polynomials and their roots}

The tropicalization map \( \mathcal{F}_{LTrop} \) of \[ \text{[6,2] Equation [6,4]} \]

extends readily to polynomials, i.e., to the functor \( \tilde{\mathcal{F}}_{LTrop} : \text{Pol}(\mathcal{S}, \text{ValMon}^+) \to \text{Pol}(\mathcal{S}, \text{LayBidom}^+) \), where we define

\[
\tilde{\mathcal{F}}_{LTrop} \left( \sum_{i} a_i \lambda_i^{i_1} \cdots \lambda_n^{i_n} \right) = \sum_{i} \mathcal{F}_{LTrop}(a_i) \lambda_i^{i_1} \cdots \lambda_n^{i_n} = \sum_{i} \left[ [1] v(a_i) \lambda_i^{i_1} \cdots \lambda_n^{i_n} \right],
\]

for \( i = (i_1, \ldots, i_n) \), (and analogously for morphisms).

If \( a \in F^{(n)} \) is a root of \( f \in F[A] \), then clearly \( v(a) \) is a corner root of \( \tilde{\mathcal{F}}_{LTrop}(f) \). We are interested in the opposite direction. One of the key results of tropical mathematics is Kapranov’s theorem \[ \text{[11]} \], which says that for any polynomial \( f(\lambda_1, \ldots, \lambda_n) \), any corner root of the tropicalization of \( f \) has a pre-image which is a root of \( f \). This assertion also works for finite sets of polynomials, and thus for ideals, in view of \[ \text{[30]} \]. Our objective in this section is to understand this result in terms of the appropriate layered categories.

\textbf{Remark 8.1.} Let \( A := F[\lambda_1, \ldots, \lambda_n] \). Then the \textbf{Puiseux series valuation} \( \text{Val} \) extends naturally to a map \( \text{Val} : A \to R[\lambda_1, \ldots, \lambda_n] \), where each \( \lambda_i \) is fixed. If \( I \) is an ideal of \( A \), then \( \Phi(I) \) is an ideal of \( \Phi(A) \), so this “tropicalization” process sends ideals of algebras to semiring ideals, and transfers many properties from the “classical algebraic” world to the “tropical” world. One property which it does not preserve is generation of ideals. For example, two different polynomials \( f, g \) of \( I \) might have the same leading monomial and the same tropicalization, and then \( \Phi(f + (-g)) \) cannot be described in terms of \( \Phi(f) \) and \( \Phi(g) = \Phi(f) \). One can bypass this particular difficulty by using Gröbner bases (since they are comprised of polynomials of different lowest orders), but the necessity of choosing the “right” generators raises serious issues in tropical geometry. Fortunately, this concern is not critical in the current paper, since we do not require generators for studying the relevant categories.
Remark 8.2. We start with a triple \((F, \mathcal{G}, v)\), where \(F\) for example may be the algebra of Puiseux series over \(\mathbb{C}\), \(\mathcal{G} = (\mathbb{R}, +)\), and \(v : F \to \mathcal{G}\) the valuation \(\text{Val}\). Any point \((\alpha_1, \ldots, \alpha_n) \in F^{(n)}\) can be considered as a valuation \(\hat{v}\) extending \(v\), where \(\hat{v}(\lambda_i) = \alpha_i\). This can be extended to the group \(G\) generated by \(\lambda_i\) and \(\lambda_i^{-1}\) in the ring of Laurent series over \(F\). But if \(p\) is a prime ideal of \(F[\lambda] := F[\lambda_1, \ldots, \lambda_n]\), then the natural image of \(G\) in the field of fractions \(K = F[\lambda]/p\) is a group \(\hat{G}\), and Bieri-Groves \([5]\) describe the possible extensions of \(v\) to \(F[\hat{G}]\). Namely, \(\hat{G}\), being a finitely generated Abelian group, can be written as the direct sum of a free Abelian group of some rank \(m\) and a torsion group \(T\). We let \(F[\hat{G}]\) denote the \(F\)-subalgebra of \(K\) generated by \(\hat{G}\). After extending the valuation \(v\) to the free Abelian group, one sees by an exercise of Bourbaki \([6]\) that further extensions to \(F[\hat{G}]\) correspond to corner roots of the polynomials of \(p\).

This is explained in the proof of \([5]\) Theorem A], and can be explained tropically in terms of the proof of Bourbaki’s exercise:

If \(f(a) = 0\) then two of the monomials of \(\mathcal{F}_{\text{LTrop}}(f)\) must be equal and dominant when evaluated at \(a\), say \(\alpha a^i = \beta a^j\), so one can extend \(v\) to a valuation \(\hat{v}\) on \(F[a]\) given by \(\hat{v}(a) = \frac{\log |\alpha|}{\log |\beta|}\).

This discussion could be formulated in the language of \([16], [18], [19]\), as elaborated in \([20, \text{Remark 6.6}]\).

Definition 8.3. As in Remark \([6,17]\), suppose \(F\) is an arbitrary field with valuation \(v : F \to \mathcal{G}\), having valuation ring \(R\) and associated graded algebra \(\text{gr}(F)\). For any \(f \in F[\lambda]\), we define \(\hat{f}\) to be its natural image in \(\text{gr}(F)[\lambda]\). For an \(a \in F[\lambda]\), we define the exploded tropicalization \(\hat{a}\) of \(a\) to be

\[
\{ \hat{f} : f \in a \}.
\]

An element \(a := (a_1, \ldots, a_n) \in \text{gr}(F)\) is a graded root of a polynomial \(f \in \text{gr}(F)[\lambda]\) if

\[
\hat{f}(a_1 + F_{> s(a_1)}, \ldots, a_n + F_{> s(a_n)}) = 0
\]

in \(\text{gr}(F)\). (Intuitively, \(s(f(a_1, \ldots, a_n))\) is larger than expected.)

We take \(F\) to be a Henselian field with respect to a valuation \(v\) whose residue field is algebraically closed. For example, we could take \(F = K\), the field of Puiseux series over \(\mathbb{C}\). We have two areas of interest when studying Puiseux series – the semifield \(\mathbb{F}\) (which corresponds to the value group) and the residue field, which is a copy of \(\mathbb{C}\). We can combine these using the ‘exploded’ structure of Definition \([6,14]\).

Given a polynomial \(f \in \text{gr}(F)[\lambda]\), we define its corner exploded roots to be

\[
\{ \text{Graded roots } a = (\hat{a}_1, \ldots, \hat{a}_n) \in F_1^{(n)} : s(f(\hat{a}_1, \ldots, \hat{a}_n)) = 0 \},
\]

cf. Remark \([6,15]\). The corner exploded variety of an ideal \(\hat{a}\) of \(\text{gr}(F)\) is the set of common corner exploded roots of the polynomials of \(\hat{a}\).

The standard valuation-theoretic proofs of Kapranov’s theorem show that any corner root \(x\) of \(\mathcal{F}_{\text{LTrop}}(f)\) is the tropicalization of a point in the variety \(Z\) defined by \(f\). In other words, \(x\) lifts to an exploded root of \(f\). Payne’s theorem \([36]\) can be stated as follows:

Suppose \(X\) is an affine variety defined by a proper ideal \(a\) of \(F[\lambda]\), and \(\hat{a}\) is a graded root of the exploded tropicalization \(\hat{a}\) of \(\hat{a}\). Then the preimage of any point defined by \(\hat{a}\), if nonempty, is Zariski dense in \(X\). This is the algebraic essence of Parker’s ‘exploded’ approach.

9. The category of affine layered geometry

Our goal in this section is to connect affine layered geometry to a category which can be studied by means of standard algebraic techniques. This ties in with the algebraic categories of the previous sections, by means of the coordinate semifield \(\mathbb{F}\), which is to be studied more thoroughly in a subsequent paper.

Throughout, let \(F\) denote a layered bi-semifield \(\mathbb{F}\).

9.1. The Zariski topology. We want to mimic the classical Zariski theory as far as we can, starting with our layered bi-semifield \(\mathbb{F}\) and describing a topology on \(\mathcal{S}\), a given subset of \(F^{(n)}\).

Actually, there are several natural topologies on \(\mathcal{S}\).

Definition 9.1. Suppose \(f = \sum_i f_i\) is written as a sum of monomials in \(\text{Pol}(\mathcal{S}, R)\). The \(f_i\)-component of \(f\) is

\[
D_{f_i}(\mathcal{S}) := \{ a \in \mathcal{S} : f_i(a) = f(a) \}.
\]
Any root of $f$ in a component $D_f$, must be a cluster root.

**Remark 9.2.** In [17, Definition 6.5], we defined the L-layered component topology to have as its sub-base the components of polynomials of $\text{Pol}(S,R)$. Note that different components of a polynomial are disjoint, so open sets here are not necessarily dense in the component topology. Thus, although it provides useful information, the component topology is too fine to permit us to develop tropical algebraic geometry along classical lines. We rectify the situation by defining the principal corner open sets to be

$$\mathcal{D}(f;S) = S \setminus Z_{\text{corn}}(f) = \bigcup_{i \in I} D_f,$$

where $f = \sum_{i \in I} f_i$ is written as a sum of monomials in $\text{Pol}(S,R)$. Put another way,

$$\mathcal{D}(f;S) = \{ a \in S : s(f(a)) = s(f_i(a)) \text{ for some monomial } f_i \text{ of } f \},$$

The principal corner open sets form a base for a topology on $S$, which we call the (L-layered) corner Zariski topology, whose closed sets are affine corner algebraic sets.

Analogously, one could respectively take cluster roots and use $Z_{\text{comb}}(f;S)$ in place of corner roots and $Z_{\text{corn}}(f;S)$ to define the combined Zariski topology, whose closed sets are the algebraic sets. This is a somewhat finer topology, but the corner Zariski topology provides a closer analog to the usual notions of tropical geometry, so we will use that.

**Lemma 9.3.** The intersection of any two principal corner open sets contains a nonempty principal corner open set.

**Proof.** If $f(a) = f_i(a)$ on $D_{f,i}$ and $g(a) = g_j(a)$ on $D_{g,j}$, then clearly $fg(a) = f_i(a)g_j(a)$ on $D_{f,i} \cap D_{g,j}$ and nowhere else. □

**Proposition 9.4.** All open sets in the (L-layered) corner Zariski topology are dense.

**Proof.** Immediate from the lemma. □

**9.2. The coordinate semiring\(^1\).** We can return to algebra via the coordinate semiring\(^1\), just as in classical algebraic geometry.

**Definition 9.5.** The coordinate semiring\(^1\) of an affine layered algebraic set $X \subseteq S$, denoted $F[X]$, is the natural image of the semiring\(^1\) $\text{Pol}(X,F)$. The Laurent coordinate semiring\(^1\) $F(X)$ is the natural image of $\text{Laur}(X,F)$; its elements are called the regular functions of the algebraic set.

$\text{Pol}(X,F)$ can be identified with classes of polynomials over $X$ whose representatives are polynomials having no inessential monomials, cf. [17, Definition 5.5]. We say that a function in $F[X]$ (resp. $F(X)$) is tangible if it can be written as a tangible polynomial (resp. Laurent polynomial), i.e., having coefficients only in $F_1$.

**Remark 9.6.** When $X \subseteq Y$ we have a natural map $F[Y] \to F[X]$ obtained by restricting the domain of the function from $Y$ to $X$.

Coordinate semirings\(^1\) correspond naturally to congruences on $\text{Pol}(S,F)$ in the following manner.

**Definition 9.7.** A nonempty subset $X \subseteq S$ defines the congruence of $X$ on $\text{Fun}(S,F)$, denoted $\Omega_X$, whose underlying equivalence $\equiv_X$ is given by

$$f \equiv_X g \iff f(a) = g(a) \text{ for every } a \in X.$$

Conversely, given a congruence $\Omega$ on $\text{Fun}(S,F)$, define the variety of the congruence

$$V(\Omega) := \{ a \in S : f(a) = g(a), \forall (f,g) \in \Omega \} \subseteq S.$$

It is readily checked that

$$\Omega_{S_X} = \Omega_X \cap \Omega_Y,$$

for any $X, Y \subseteq S$.

**9.3. Zariski correspondences.** We have various correspondences between varieties and the algebraic structure.
9.3.1. The Zariski correspondence with ideals. Inspired by the layered Nullstellensatz given in [17, Theorem 6.14], the naive approach would be to define the corner ideal $I_{\text{corn}}(S)$ of a set $S$ to be
\[ \{ f \in \text{Pol}(S,F) : \text{a is a corner root of } f, \ \forall a \in S \}, \]
and $I_{\text{comb}}(S)$ to be
\[ \{ f \in \text{Pol}(S,F) : \text{a is in the combined ghost locus of } f, \ \forall a \in S \}. \]

This approach misses the mark, somewhat. On the one hand, different congruences can define the same ideal which is the pre-image of 0. On the other hand, there are “too many” ideals, in the sense that not every ideal defines a variety, and the correct algebraic approach is to utilize congruences rather than ideals. Furthermore, we need somehow to filter out those varieties obtained by degenerate intersections of hypersurfaces; this is treated in a later paper under preparation.

9.3.2. The Zariski correspondence with congruences. As just noted, it makes more sense to deal with congruences instead of ideals. We have the usual straightforward but important inverse Zariski correspondence:

**Proposition 9.8.** If $\Omega_1 \supseteq \Omega_2$, then $V(\Omega_1) \subseteq V(\Omega_2)$. Conversely, if $Y \supseteq X$, then $\Omega_Y \subseteq \Omega_X$. Consequently,
\[ V(\Omega_{V(\Omega)}) = V(\Omega) \quad \text{and} \quad \Omega_{V(\Omega_X)} = \Omega_X. \]

It follows that there is a 1:1 correspondence between congruences of varieties and varieties of congruences, given by $X \mapsto \Omega_X$ and $\Omega \mapsto V(\Omega)$. Furthermore, the coordinate semiring $F$ satisfies
\[ F[V(\Omega)] \cong \text{Pol}(S,F)/\Omega. \]

**Proof.** The inverse correspondence is immediate, and the next assertion is immediate. The 1:1 correspondence is then formal. To see the last assertion, note that two polynomials in $f, g$ are identified in $F[V(\Omega)]$ iff they agree on $V(\Omega)$, which by definition is the point set on which every pair $(f,g) \in \Omega$ agree; namely, $f$ and $g$ are identified in $\text{Pol}(S,F)/\Omega$. \qed

By the proposition, one sees that for any nonempty subset $X \subseteq S$ we have
\[ F[X] \cong \text{Pol}(S,F)/\Omega_X. \]  

**Definition 9.9.** A morphism of affine layered algebraic sets $\Phi : X \to Y$ is a continuous function that preserves (i.e., pulls back) regular functions, in the sense that if $U$ is an open subset of $Y$ and $\psi \in F(U)$, then $\psi \circ \Phi \in F(\Phi^{-1}(U))$.

$L$-TropAff is the category whose objects are the affine layered algebraic sets $X \subset F^{(n)}$ and whose morphisms $\Phi : X \to Y$ are morphisms of layered affine algebraic sets.

**Proposition 9.10.** Any morphism $\Phi : X \to Y$ of affine layered algebraic set gives rise to a natural algebra homomorphism $\Phi^* : F(\Phi(U)) \to F(U)$, by $\psi \mapsto \Phi^*(\psi)$, where $\Phi^*(\psi)(a) = \psi(\Phi(a))$, for every $a \in U$.

**Proof.** $\Phi^*(\psi + \varphi) = \Phi^*(\psi) + \Phi^*(\varphi)$ and $\Phi^*(\psi \varphi) = \Phi^*(\psi)\Phi^*(\varphi)$. \qed

We conclude by introducing the functor linking the algebraic and geometric (affine) categories.

**Definition 9.11.** $F_{L-\text{Coord}}$ is the contravariant functor from $L$-TropAff to ULayBidom$^\dagger$ given by sending an affine layered algebraic set $X$ to its coordinate semiring$^\dagger$ $F[X]$, and any morphism $\Phi : X \to Y$ of affine layered algebraic sets to the layered semiring$^\dagger$ homomorphism $\Phi^* : F[Y] \to F[X]$, i.e., $f \mapsto f_\Phi$ where $f_\Phi(a) = f(\Phi(a))$.

Many subtleties lie behind this definition; for example, which affine layered varieties correspond to the coordinate semirings$^\dagger$ of tropical varieties satisfying the balancing condition? This question is to be treated in a subsequent paper.
