Heterotic Matrix String Theory and Riemann Surfaces

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**Abstract:** We extend the results found for Matrix String Theory to Heterotic Matrix String Theory, i.e. to a 2d $O(N)$ SYM theory with chiral (anomaly free) matter and $\mathcal{N} = (8,0)$ supersymmetry. We write down the instanton equations for this theory and solve them explicitly. The solutions are characterized by branched coverings of the basis cylinder, i.e. by compact Riemann surfaces with punctures. We show that in the strong coupling limit the action becomes the heterotic string action plus a free Maxwell action. Moreover the amplitude based on a Riemann surface with $p$ punctures and $h$ handles is proportional to $g_s^{2-2h-p}$, as expected for the heterotic string interaction theory with string coupling $g_s = 1/g$. 
1. Introduction

The purpose of this paper is to extend to Heterotic Matrix String Theory (HMST), the results obtained recently for Matrix String Theory (MST), (see also the related papers [16, 17, 18, 19, 20, 21]). In the latter it was shown what had been previously conjectured in [22, 23, 24, 25], namely that MST, i.e. $U(N)$ SYM theory in two dimensions with adjoint matter and $\mathcal{N} = (8,8)$ supersymmetry, describes in the strong coupling limit IIA superstring theory and all its string interactions. The relevant Riemann surfaces are dynamically generated via BPS instantons, which are referred to throughout as stringy instantons. Each Riemann surface (with $p$ punctures and $h$ handles) is associated to a corresponding string process in an obvious way. To verify the correctness of this association the action was expanded about an instanton; it was shown in [14] that the resulting amplitude is proportional to $g_s^{-\chi}$, where $g_s$ is the string coupling constant (i.e. the inverse of the Yang-Mills coupling) and $\chi = 2 - 2h - p$ is the Euler characteristic of the Riemann surface. This is what one expects from perturbative string interaction theory. Relying on this basic result, in [15], we set out to analyze the moduli space of stringy instantons and found that it does not exactly coincide with the moduli space of compact Riemann surfaces with punctures, which is relevant to closed string theory, but it mimics it very closely: in fact it is a discretized version of the latter in the sense that some of the moduli are discrete. However these moduli become continuous in the $N \to \infty$ limit. We argued in [15] that in this limit the moduli space of stringy instantons does reproduce the moduli space of Riemann surfaces relevant to the type IIA theory.
In the present paper we extend the above result to the HMST. The latter is an $O(N)$ SYM theory with chiral (anomaly free) matter and $\mathcal{N} = (8,0)$ supersymmetry. We first write down the instanton equations for this theory. Then we show how one can solve them explicitly. The solutions are of course different from the MST case, but they span the same kind of branched coverings of the basis cylinder, i.e. the same kind of compact Riemann surfaces with punctures, as in MST. We then expand the action about any such instanton in the strong coupling limit. The details are different from the MST case and somewhat subtler (see Appendix), however the final result is similar to MST. The strong coupling limit action is the heterotic string action plus a free Maxwell action. Moreover the amplitude based on a Riemann surface with $p$ punctures and $h$ handles is proportional to $g^2h + p - 2s$, as expected for the heterotic string interaction theory. At this point the analysis of the moduli space of stringy instantons goes through as in [15]. This moduli space is a discretized approximation of the moduli space of Riemann surfaces in heterotic string theory. The approximation gets better and better as $N$ becomes larger.

The paper is organized as follows. Section 2 is devoted to heterotic stringy instantons. In section 3 we expand the action around any instanton solution and compute the strong coupling limit. The Appendix contains a few details of the relevant calculations.

2. The 2d SYM model and classical supersymmetric configurations

The Heterotic Matrix String Theory (HMST) is specified by an $O(N)$ SYM model in a 2d Euclidean space-time. We will choose $N$ even, $N = 2n$. The action is

$$S = \frac{1}{\pi} \int d^2w \left\{ \text{Tr} \left( D_w X^i D_{\bar{w}} X^i + \frac{1}{4g^2} F_{w\bar{w}}^2 - \frac{g^2}{2} [X^i, X^j]^2 + i(\theta_s D_w \theta_s - \theta_c D_w \theta_c) - 2g\theta_s \Gamma_i [X^i, \theta_s] + i\chi D_w \chi \right) \right\},$$

(2.1)

where we use the complex notation

$$w = \frac{1}{2}(\tau + i\sigma), \quad \bar{w} = \frac{1}{2}(\tau - i\sigma), \quad A_w = A_0 - iA_1, \quad A_{\bar{w}} = A_0 + iA_1,$$

where $\sigma$ and $\tau$ are the world-sheet coordinate on the cylinder. In (2.1) $g$ is the gauge coupling, $X^i$ with $i = 1, \ldots, 8$ are real symmetric $N \times N$ matrices. $A_0, A_1$ are real antisymmetric $N \times N$ matrices. The covariant derivative is defined as $D_w X^i = \partial_w X^i - [A_w, X^i]$. $F_{w\bar{w}}$ is the gauge curvature. Summation over the $i,j$ indices is understood. $\theta_s(\theta_c)$ represents $8$ $N \times N$ symmetric (antisymmetric) matrices whose entries are 2D real spinors and simultaneously transform in the $\mathbf{8}_s$ and $\mathbf{8}_c$ representations of $SO(8)$. $\chi$ are 2D fermions transforming according to the fundamental representation of $O(N)$ and the fundamental representation of $SO(32)$. The matrices $\Gamma_i$ are the $16 \times 16$ $SO(8)$ gamma matrices. For definiteness we will write the matrices $\Gamma_i$ in the form

$$\Gamma_i = \begin{pmatrix} 0 & \gamma_i \\ \tilde{\gamma}_i & 0 \end{pmatrix},$$

(2.2)

and $\gamma_i, \tilde{\gamma}_i$ are the same as in Appendix 5B of [27]. All the fields in HMST are periodic on the cylinder.
The action (2.1) is chiral, however the fermion representations are such that the gauge anomaly vanishes (see also Appendix). It is invariant under the supersymmetry transformations ($\mathcal{N} = (8,0)$ supersymmetry)

$$
\delta X^i = \frac{i}{g} \epsilon^j \gamma^j \theta_s \\
\delta \theta_s = -\frac{1}{g} D_w X^i \gamma_i \epsilon_c \\
\delta \theta_c = (\frac{i}{2g^2} F_{w\bar{w}} - \frac{i}{2} [X^i, X^j] \bar{\gamma}_{ij}) \epsilon_c \\
\delta A_{\bar{w}} = -2 \epsilon_c \theta_c \\
\delta A_w = 0, \quad \delta \chi = 0,
$$

(2.3)

where

$$
\gamma_{ij} = \frac{1}{2}(\gamma_i \bar{\gamma}_j - \gamma_j \bar{\gamma}_i), \quad \gamma_{ij} = \frac{1}{2}(\gamma_i \gamma_j - \gamma_j \gamma_i).
$$

The HMST, in the strong coupling limit, is expected to represent heterotic string theories. We can enrich the content of (2.1) by introducing Wilson lines. The corresponding term in the integrand of (2.1) is

$$
\chi B \chi = \sum_{a=1}^{32} \chi^a B_{ab} \chi^b, \text{ where } B \text{ is a real antisymmetric matrix.}
$$

A specific choice of $B$ leads to a theory in which, for example, $SO(32)$ is broken to $SO(16) \times SO(16)$. Via a T-duality transformation this theory can be related to the ten-dimensional $E_8 \times E_8$ heterotic theory broken down to $SO(16) \times SO(16)$ by another suitable Wilson line, see [7, 8, 9, 10]. In the absence of Wilson lines it is expected to represent $SO(32)$ heterotic string theory compactified to nine dimension on a very small circle.

All this, as well as the s-duality connection with type IA and type IB theories [28, 29], is well-known. However the introduction of Wilson lines does not affect our subsequent derivation. Therefore we will drop them throughout and resume them at the end of our derivation.

### 2.1 Interpretation of the strong coupling states

The naive strong coupling limit ($g \to \infty$) in the action, after rescaling $A \to g A$, tells us that all the $X^i, A_w, A_{\bar{w}}$ and $\theta$ commute, and $\chi A_w \chi = 0$.

Denoting with a bar the fields in the strong coupling limit, one sees that there are two types of solutions. Let us denote by $\mathbf{1}$ the $2 \times 2$ identity matrix and by $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The first type of solution is: $\bar{A}_w = \bar{a} \otimes \epsilon, \bar{\theta}_c = \bar{\theta}_c \otimes \epsilon, \bar{X}^i = \bar{x}^i \otimes \mathbf{1}$ and $\bar{\theta}_s = \bar{\theta}_s \otimes \mathbf{1}$, where $\bar{a}, \bar{x}^i, \bar{\theta}_c$ and $\bar{\theta}_s$ are diagonal matrices. Moreover $\bar{\chi}$ is such that $\bar{\chi} A_w \bar{\chi} = 0$, which implies that half of the degrees of freedom of $\bar{\chi}$ must vanish. For a reason that will be explained below, we actually scale out $\bar{\theta}_c$ by multiplying it by a suitable power of $1/g$, so that in the above formulas it is understood $\bar{\theta}_c = 0$.

A second group of solution is characterized by $\bar{A}_w = 0$. Consequently $\bar{X}^i$ and $\bar{\theta}_s$ are generical diagonal matrices (without the two by two identification of eigenvalues as before). Supersymmetry then requires that $\bar{\theta}_c = 0$. 

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In the following we refer to these two group of solutions as first and second branch, respectively. Although one can hardly attach too much importance to these naive elaborations, they actually turn out to be very suggestive and not at odds with the results of the more appropriate treatment presented later.

It is well-known that in the strong coupling theory there is a residual gauge freedom, which contains the Weyl group of $O(N)$ and allows a variety of boundary conditions. Each one of them, in the first branch, defines a string configuration. Let us consider, for example, a solution from the first branch. We have in particular $\tilde{X}^i = \text{Diag}(x^i_1, \ldots, x^i_n) \otimes 1$.

The distinct eigenvalues of the latter can be interpreted as free strings of various lengths. For instance, let us consider the effect on $X^i$ of the element $\hat{P} \equiv \mathcal{P} \otimes 1$ of the Weyl group of $O(N)$, where

$$
\mathcal{P} = \begin{pmatrix}
0 & 0 & \cdots & \cdots & 1 \\
1 & 0 & \cdots & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}.
$$

The boundary condition $\tilde{X}^i(2\pi) = \hat{P}\tilde{X}^i(0)\hat{P}^{-1}$ implies that $x^i_k(2\pi) = x^i_{k-1}(0)$, and so the $x^i_k$ form a unique long string of length $2\pi n$. A parallel interpretation holds for the fermionic degrees of freedom. We see that, apart from the gauge field, we get the spectrum of the free heterotic superstring. The gauge degrees of freedom need further specification. Here we only anticipate that the gauge field will turn out to be essential for the interpretation of the strong coupling limit of HMST as heterotic string interaction theory.

One may wonder at this point why above we scaled out $\tilde{\theta}_c$ and not in its stead the full set of $\tilde{\chi}$ degrees of freedom. We would have obtained in this way, apart from the gauge degrees of freedom, the spectrum of the free type IIA superstring theory. The answer is that there are various indications that the strong coupling limit in this case would be discontinuous. For example, the HMST is a chiral fermionic theory, while the strong coupling limit would not be chiral. Moreover the supersymmetry transformations inherited by those of HMST (see (2.3) are not those of the type IIA. Therefore we exclude the possibility of such a strong coupling limit.

We can proceed similarly also for the second branch solutions. They have apparently a spectrum which coincides with the heterotic superstring spectrum. However, one sees immediately that the residual discrete gauge transformations, which form the Weyl group of $O(N)$, are not the expected ones for a heterotic string interpretation of this branch. This is a spy of of the fact that, as will become clear later on, the strong coupling limit of the second branch, if it exists, cannot be interpreted as a string theory.

### 2.2 2D instantons and Hitchin systems

We look now for classical Euclidean supersymmetric configurations that preserve half supersymmetry. To this end we set $\theta = \chi = 0$ and look for solutions of the equations $\delta \theta = 0$, $\delta \chi = 0$.

\footnote{Here we use the same explicit example as in [13] in order to point out the analogies and differences between the two cases}
i.e., from (2.3),
\[
\left( \frac{i}{2g^2} F_{w\bar{w}} - \frac{i}{2} [X^i, X^j] \bar{\gamma}_{ij} \right) \epsilon_c = 0, \quad D_w X^i \gamma_i \epsilon_c = 0.
\] (2.4)

Solutions of these equations that preserve one half supersymmetry are the following ones. Set \( X^i = 0 \) for all \( i \) except two, for definiteness \( X^i \neq 0 \) for \( i = 1, 2 \); remark that \( \gamma_{12} \) is an antisymmetric \( 8 \times 8 \) matrix, and \( \gamma_2^2 = -1 \) and therefore its eigenvalues are \( \pm i \) (moreover \( \bar{\gamma}_{12} = \gamma_{12} \)). It is easy to show that there exists \( \epsilon \), with four independent components, such that
\[
\gamma_{12} \epsilon = i \epsilon, \quad \gamma_1 \epsilon = -i \gamma_2 \epsilon.
\]

We replace this \( \epsilon \) in eq. (2.4). It is convenient to introduce the complex notation \( X = X^1 + i X^2, \quad \bar{X} = X^1 - i X^2 = X^\dagger \). \( X, \bar{X} \) are complex symmetric matrices. Then the conditions to be satisfied in order to preserve one half supersymmetry are
\[
F_{w\bar{w}} + g^2 [X, \bar{X}] = 0, \quad D_w X = 0,
\] (2.5)

It is easy to verify that, if non-trivial solutions to such equations exist, they satisfy the equations of motion of the action (2.1). The action with \( \theta = 0, \chi = 0, X^i = 0 \) for \( i = 3, \ldots 8 \) can be normalized as follows
\[
S_{\text{inst}} = \frac{1}{2\pi} \int d^2 w \text{Tr} \left( -X D_w D_{\bar{w}} \bar{X} - \bar{X} D_w D_{\bar{w}} X + \frac{1}{2g^2} F_{w\bar{w}}^2 + \frac{g^2}{2} [X, \bar{X}]^2 \right).
\] (2.6)

It is elementary to prove that \( S_{\text{inst}} \) vanishes in correspondence to solutions of (2.5).

3. Heterotic stringy instantons and Riemann surfaces

3.1 Heterotic stringy instantons

We recall that \( N \) is even, \( N = 2n \). We want to construct solutions of eqs. (2.5) that interpolate between any two asymptotic string states \( (w = \pm \infty) \) as the ones considered in sec.2.1. It turns out that the right (heterotic stringy) strong coupling solutions of (2.5) are of the type:
\[
X \to X^{(b)}, \quad A \to A^{(b)} = 0,
\] (3.1)

where \( X^{(b)} = \hat{X} \otimes 1 \) and \( \hat{X} = \text{Diag}(x_1, \ldots, x_n) \). We construct the instanton that reduces to it in the strong coupling limit in close analogy with [13, 14].

Our purpose is to construct a symmetric matrix \( X \) and an antisymmetric connection \( A \) that satisfy (2.5). Let us start from the canonical \( n \times n \) matrix \( M \) whose eigenvalues coincide with those of \( \hat{x} \),
\[
M = \begin{pmatrix}
-a_{n-1} & -a_{n-2} & \ldots & \ldots & -a_0 \\
1 & 0 & \ldots & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix}.
\] (3.2)
We map the problem from the cylinder to the disk: \( w \to \bar{z} = e^w \), and require \( \partial \bar{z} M = 0 \), and therefore \( \partial \bar{z} a_i = 0 \) for each \( i = 0, \ldots, n - 1 \), after mapping . The matrix \( M \) defines a branched covering of the cylinder \( \mathcal{C} \), which is a Riemann surface with punctures, as discussed at length in [13, 14, 15].

Now we identify the eigenvalues of \( \hat{X} \) with those of \( M \). We diagonalize \( M \)

\[
M = S \hat{M} S^{-1}, \quad \hat{M} = \text{Diag}(x_1, \ldots, x_n)
\]

by means, [11], of the following matrix \( S \in SL(n, \mathbb{C}) \):

\[
S = \Delta^{-\frac{1}{n}} \begin{pmatrix}
\frac{x_1^{n-1}}{x_1} & \frac{x_2^{n-1}}{x_2} & \cdots & \frac{x_n^{n-1}}{x_n} \\
\frac{x_1^{n-2}}{x_1} & \frac{x_2^{n-2}}{x_2} & \cdots & \frac{x_n^{n-2}}{x_n} \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1
\end{pmatrix}, \tag{3.3}
\]

where

\[
\Delta = \prod_{1 \leq i < j \leq n} (x_i - x_j). \tag{3.4}
\]

Next we set

\[
S = B Q,
\]

where \( B = \sqrt{SS} = \hat{B} \). Here tilde stands for transposition. One has \( QQ = B^{-1}SSB^{-1} = 1 \).

In other words we decompose \( S \) into the product of a complex symmetric matrix \( B \) and a complex orthogonal matrix \( Q \).

As a consequence

\[
M_s = B^{-1}MB = Q\hat{M}Q^{-1} \tag{3.5}
\]

is a symmetric complex matrix. We choose \( M_s \) as our starting point in the construction of the instanton, in the same way as we used \( M \) in [13, 14]. At this point it would seem that we have to accompany \( M_s \) with a corresponding connection \( A_a = -Q\partial Q^{-1} \). However one can prove that the latter identically vanishes. This follows from the fact that \( Q \) and \( Q^{-1} \) contain strictly fractional singularities in \( z \), say \( (z - a)^{k/j} \) with \( k, j \) relatively prime integers, but never \( (z - a)^{-l} \), with \( l \) positive integer. Therefore, in the sense of complex distributions, \( \partial \bar{z} Q^{-1} = 0 \).

Now the complex orthogonal matrix \( Q \) can be decomposed as follows

\[
Q = HO, \quad H = \sqrt{QQ^\dagger}. \tag{3.6}
\]

As a consequence, \( H = H^\dagger \), \( OO^\dagger = 1 \) and \( O\bar{O} = 1 \). Therefore \( O \in O(n, \mathbb{R}) \).

Our complete ansatz for the instanton is defined by the couple \( X = Y^{-1}M_s Y \otimes 1 \) and \( A = -Y^{-1}\partial_w Y \otimes 1 \), where \( Y = HL \), and \( L \) is a complex orthogonal matrix. In particular

\[
X = Y^{-1}M_s Y \otimes 1 = L^{-1}O\hat{M}O^{-1}L \otimes 1. \tag{3.7}
\]

Since \( L \) is a complex orthogonal matrix, \( X \) is symmetric and \( A \) is antisymmetric, as desired.
The points where $\Delta$ vanishes, i.e. where two eigenvalues coincide, are branch points of the covering defined by the matrix $M$. They are therefore characterized by a monodromy. Let us check that the solutions we propose are not affected by such monodromies, i.e. are single-valued. Notice that under a monodromy transformation around any point of the basis cylinder, we have

$$S \rightarrow S \Lambda, \quad B \rightarrow B, \quad Q \rightarrow Q \Lambda, \quad H \rightarrow H, \quad O \rightarrow O \Lambda, \quad \hat{M} \rightarrow \Lambda^{-1} \hat{M} \Lambda,$$

where $\Lambda$ is the monodromy matrix about the given point (one such monodromy matrix is, for example, the matrix $P$ introduced above).

Therefore $M$ and $X$, as well as $A$, are single-valued provided $L$ is. Let us see how one can construct such an $L$.

In order for our ansatz to satisfy (2.5), the entries of the complex orthogonal matrix $L$ must be single-valued fields that satisfy field equations of the WZNW type, with delta-function-type sources at the branch points. We have already discussed the existence of solutions of such equations in [13, 14, 15]. We have seen that in the strong coupling limit $L$ tends to $1$ outside an arbitrary small neighborhood of the branch points. Therefore, excluding a small neighborhood of the branch points, we can set

$$X = O \tilde{X} O^{-1} \otimes 1, \quad A = -H^{-1} \partial_w H \otimes 1 = -O \partial_w O^{-1} \otimes 1$$

since, as we have seen, $\partial_w Q = 0$. Therefore in the strong coupling limit, outside the branch points, we recover $A^{(b)}$ and $X^{(b)}$ up to a gauge transformation.\textsuperscript{2}

We also remark that eqs. (2.3) are invariant under orthogonal $O(N)$ transformations. Therefore the instantons solutions we have found are understood to be defined up to arbitrary orthogonal $O(N)$ transformations.

### 3.2 Other branches

With the term strong coupling solutions we mean the solutions of (2.5) in which $F_{w\bar{w}} = 0$ and $[X, \tilde{X}] = 0$ separately. There are many other possible strong coupling solutions of (2.5) beside (3.1). Therefore it is important to understand why in this paper we limit ourselves to (3.1). The strong coupling configurations split into two branches, just as the solutions studied in naive strong coupling limit in sec.2.1. The first branch is given, in the diagonal representation, by

$$A_w = A_w^{(b)} = \hat{A} \otimes \epsilon, \quad X \equiv X^{(b)} = \tilde{X} \otimes 1,$$

where $\hat{A}$ and $\tilde{X}$ are diagonal matrices. The second branch corresponds to configurations $(X^{(b)}, A^{(b)})$ in which $A^{(b)} = 0$ and $X^{(b)}$ is a generic diagonal matrix (i.e. without two by two identification of the eigenvalues as in (3.1)). This classification closely parallels the one found in section 2.1 in the naive strong coupling limit.

The strong coupling solutions (3.1) correspond to the intersection of the two branches. In the same way as above we constructed the instantons corresponding to any strong

\textsuperscript{2}Of course, if $O \in O(n)$, $O \otimes 1 \in O(N)$.  

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coupling solution (3.1), we can construct instantons corresponding to any strong coupling solution in the two branches. This has been done and will be reported elsewhere. However a generic instanton in the two branches needs not lead in the strong coupling limit to the heterotic string theory. The instantons in the intersection (i.e. the stringy instantons) do.

Let us clarify what we mean by this. As is clear from [13, 14] and from the next section, our aim is to expand the action about a given instanton solution and to extract the strong coupling limit. The instanton is really stringy if the Riemann surface it contains (as a branched covering of the cylinder) is in accord with the string interpretation as a scattering process, in the strong coupling limit. This requires that the corresponding amplitude be proportional to $g^{-\chi}$, where $\chi$ is the Euler characteristic of the Riemann surface, as explained in the introduction. This factor can only come from a Maxwell field theory on the covering of the cylinder, which, in turn, can only be a consequence of a surviving part of the original gauge symmetry of the theory. We recall that in [13, 14] the $U(N)$ gauge symmetry of the theory breaks down, in the strong coupling configurations on the basis cylinder, to $(U(1))^N$, and that this is the basis for the persistence of a $U(1)$ gauge symmetry on the covering in the strong coupling limit. In the instantons constructed in the previous subsection, the strong coupling configurations (3.1) break down the gauge symmetry $O(N)$ in such a way that a $(O(2))^n$ symmetry survives. This, as will be seen, guarantees that a Maxwell theory will survive on the corresponding covering in the strong coupling limit. This is the reason why we limit ourselves in this paper to strong coupling configurations (3.1) represented by $X^{(b)}$’s in which the eigenvalues are identified two by two, i.e. are in the intersection of the two branches.

Should we consider instead diagonal $X^{(b)}$’s in which all eigenvalues are different (generic solutions of the second branch), any non-discrete gauge symmetry would be destroyed in the strong coupling limit. This would not leave any room for a Maxwell field on the covering, therefore such configurations cannot trigger a string interpretation. As for generic solutions of the first branch, they contain a non-vanishing background gauge potential. The corresponding coverings may possibly be interpreted in terms of scattering of heterotic strings with non-perturbative objects.

In conclusion, only in the intersection of the two branches a genuine heterotic string interpretation of the strong coupling limit seems to be possible. That is the reason why we called heterotic stringy the instantons corresponding to the intersection between the two branches.

### 3.3 Lifting to the spectral covering

Following [14], we show now that the HMST in the strong YM coupling limit, can be lifted to the covering $\Sigma$ of the basis cylinder. To this end we first rewrite the action in the following form

$$
S = \frac{1}{\pi} \int d^2 w \left\{ \text{Tr} \left( D_w X^I D_w^\dagger X^J - \frac{g^2}{2} [X^I, X^J]^2 - g^2 [X^I, X] [X^J, \bar{X}] + D_w X D_w^\dagger \bar{X} \right) + \frac{1}{4g^2} \left( F_{w\bar{w}} + g^2 [X, \bar{X}] \right)^2 + i(\theta_+ D_w \theta_- - \theta_-^\dagger D_w^\dagger \theta_+^\dagger) - g\theta^T \Gamma_i [X^I, \theta] \right\} + i \chi D_w \chi,
$$

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where $I = 3, 4, \ldots, 8$. We now expand the action around a generic instanton configuration writing any field $\Phi$ as

$$\Phi = \Phi^{(b)} + \phi^i + \phi^n \equiv \Phi^{(b)} + \phi \equiv \Phi^c + \phi^a,$$

(3.9)

where $\Phi^{(b)}$ is the background value of the field at infinite coupling, $\phi^i$ are the fluctuations along the Cartan directions (for $A$ and $\theta_c$) or the directions which commute with the Cartan generators (for $X^i$ and $\theta_s$), while $\phi^n$ are the fluctuations along the complementary directions in Lie algebra $\mathfrak{o}(N)$ and in the relevant representations. So, in particular,

$$a^i_w = a^0_w \otimes \epsilon, \quad x^i = x^0 \otimes 1, \quad \theta^{\mathfrak{t}}_c = \theta^0_c \otimes \epsilon, \quad \theta^{\mathfrak{t}}_s = \theta^0_s \otimes 1,$$

(3.10)

where $a^0_w, x^0, \theta^0_c, \theta^0_s$ are diagonal $n \times n$ matrices. Therefore for instance $a^0_w$ has $2n(n - 1)$ components, while $x^0$ have $2n^2$ components each.

As for $\chi$ the above splitting has a specific meaning. As a vector in the fundamental representation of $O(N)$, we split it as follows

$$\chi^i = \frac{\chi_0}{\sqrt{2}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \chi^n = \frac{\chi_1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

(3.11)

where $\chi_0, \chi_1$ are $n$-component vectors. This splitting is actually rather arbitrary: we could replace $\chi_0$ and $\chi_1$ with any two linearly independent combinations of them. However, the final result below would not change.

In the strong coupling we consider the above action on the base space $C_0$, i.e. the initial cylinder with a small disk cut out around any branch point (in other words we insert a regulator defined by the radius of such disks). Then the only field with a non-vanishing background part is $X^{(b)}(b) = \hat{X} \otimes 1$ and its complex conjugate, since $A^{(b)} = 0$ dressed with the orthogonal gauge transformation $O$, see (3.4). We can get rid of $O$ by means of a gauge transformation in the action. But we have to exercise some care because in so doing we would $O$-rotate all the fluctuations (both Cartan and non-Cartan). In order to simplify things we will understand that the fluctuations in (3.9) are defined up to the gauge transformation $O^{-1}$.

Now, since the value of the action on the background is zero and since the background is a solution of the equation of motion, the expansion of the action starts with quadratic terms in the fluctuations. Next we proceed, exactly, as in [14]. We use the same gauge fixing

$$\mathcal{G} = D^0_{\bar{w}} a_{\bar{w}} + D^0_w a_w + g^2([X^0, \bar{X}] + [\bar{X}^0, X]) + 2g^2[X^0, x^I] = 0,$$

(3.12)

where $D^0$ stands for the covariant derivative with respect to $A^0$. Next we introduce the Faddeev-Popov ghost and antighost fields $c$ and $\bar{c}$. They will be in the antisymmetric representation of $O(N)$, and will therefore be expanded as $a_w$ and $a_{\bar{w}}$. Then we add to the action the gauge fixing term

$$S_{gf} = -\frac{1}{4\pi g^2} \int d^2w \: \mathcal{G}^2$$

(3.13)
and the corresponding Faddeev-Popov ghost term

\[ S_{\text{ghost}} = \frac{1}{2\pi g^2} \int d^2 w \ c \delta \bar{G} \delta c, \]  

(3.14)

where \( \delta \) represents the gauge transformation with parameter \( c \).

At this point, to single out the strong coupling limit of the action, we rescale the fields in appropriate manner.

\[ A_w = g a_w^t + a_w^n, \quad A_{\bar{w}} = g a_{\bar{w}}^t + a_{\bar{w}}^n, \]

\[ c = g c^t + \sqrt{g} c^n, \quad \bar{c} = g \bar{c}^t + \frac{1}{\sqrt{g}} \bar{c}^n. \]  

(3.15)

and

\[ X^i = X^{i(b)} + \frac{1}{g} x^n, \quad \theta_s = \theta_s^t + \frac{1}{\sqrt{g}} \theta_s^n \]

\[ \theta_c = \frac{1}{\sqrt{g}} \theta_c^t + \frac{1}{g} \theta_c^n, \quad \chi = \chi^t + \frac{1}{g} \chi^n. \]  

(3.16)

The reason why we scale out \( \theta_c \), and not \( \chi_1 \), has been explained in section 2.1. We will make a further comment on this point in the Appendix.

After these rescalings the action becomes

\[ S = S_{\text{sc}} + Q_n + \cdots, \]  

(3.17)

where ellipsis denote non-leading terms in the strong coupling limit,

\[ S_{\text{sc}} = \frac{1}{\pi} \int_{C_0} d^2 w \ \left\{ \frac{1}{2} \text{Tr} \left[ \partial_w x^t \partial_{\bar{w}} x^t + \partial_w \bar{x}^t \partial_{\bar{w}} \bar{x}^t + i \theta_s^t \partial_{\bar{w}} \theta_s^t \right. \right. \]

\[ \left. - \partial_w a_w^t \partial_{\bar{w}} a_w^t - \partial_w \bar{c}^t \partial_{\bar{w}} \bar{c}^t \right] + i \chi^t \partial_w \chi^t \} \]  

(3.18)

and \( Q_n \) is the purely quadratic term in the \( \phi^n \) fluctuations. They can be integrated over and give exactly 1. Although at first sight everything looks like [14], there are several subtle differences. In fact the derivation of (3.18) and the integration of \( Q_n \) are not as straightforward as in [14]. In order not to interrupt the exposition we have collected the relevant details in the Appendix.

In conclusion, in the strong coupling limit \( (g \to \infty) \), the action becomes

\[ S_{\text{sc}} = \frac{1}{\pi} \int_{C_0} d^2 w \ \sum_{p=1}^{n} \left\{ \partial_w \phi^t(p) \partial_{\bar{w}} \phi^t(p) + \partial_w \phi^3 \partial_{\bar{w}} \phi^3 + i \partial_w \phi^0 \partial_{\bar{w}} \phi^0 \right. \]

\[ + \partial_w a_w^2 \partial_{\bar{w}} a_w^2 + \partial_w \bar{c}^2 \partial_{\bar{w}} \bar{c}^2 \right] + i \chi^t \partial_w \chi^t \} \].  

(3.19)

The matrices which appear in this action are diagonal \( n \times n \) matrices, while \( \chi^n \) is an \( n \)-component vector. We can therefore rewrite the action in terms of the component modes \( \phi^0 = \phi(1), \ldots, \phi(n) \):

\[ S_{\text{sc}} = \frac{1}{\pi} \int_{C_0} d^2 w \ \sum_{p=1}^{n} \left\{ \partial_w \phi^t(p) \partial_{\bar{w}} \phi^t(p) + \partial_w \phi^3 \partial_{\bar{w}} \phi^3 + i \partial_w \phi^0 \partial_{\bar{w}} \phi^0 \right. \]

\[ + \partial_w a_w^2 \partial_{\bar{w}} a_w^2 + \partial_w \bar{c}^2 \partial_{\bar{w}} \bar{c}^2 \right\} \].  

(3.20)
At first sight this looks like a theory of free fields on \(C_0\). However, as pointed out in \[14\], this is not correct. The fields \(x^i_{(p)}...\) are not single-valued on the cylinder: by going around a branch point some of the \(x^i_{(p)}\) is mapped to some adjacent one, and this is precisely the way a branch point describes a string interaction. The right interpretation was given in \[14\]: every \(n\)-tuple of fields \(x^i_{(p)} (\theta_{s(p)}, \chi_{(p)}, a_{w(p)}, c_{(p)})\) form a unique well defined field \(\tilde{x}^i (\tilde{\theta}_s, \tilde{\chi}, \tilde{a}_w, \tilde{c})\) on the Riemann surface \(\Sigma\), which is the covering of the cylinder defined by the background we have expanded about. At this point we have a well defined action on all of \(\Sigma\), which is smooth also in correspondence with the branch points. Therefore we can remove without harm the regulator introduced before.

Finally we can write the strong coupling action (3.20) as follows

\[
S_{\Sigma}^{sc} = S_{\Sigma}^{het} + S_{\Sigma}^{Maxwell},
\]

(3.21)

\[
S_{\Sigma}^{het} = \frac{1}{\pi} \int_{\Sigma} d^2 z \left( \partial_z \tilde{x}^i \partial_{\bar{z}} \tilde{x}^i + i(\tilde{\theta}_s \partial_z \tilde{\theta}_s + \tilde{\chi} \partial_z \tilde{\chi}) \right),
\]

(3.22)

\[
S_{\Sigma}^{Maxwell} = \frac{1}{\pi} \int_{\Sigma} d^2 z \left( g_z \bar{g}_{\bar{z}} \partial_z \tilde{a} \partial_{\bar{z}} \tilde{a} + \partial_z \tilde{c} \partial_{\bar{z}} \tilde{c} \right),
\]

(3.23)

In (3.22) a \(\sqrt{\omega_z}\) (resp. \(\sqrt{\omega_{\bar{z}}}\)) factor has been absorbed in \(\tilde{\theta}_s\) (resp. \(\tilde{\chi}\)) which is a \((\frac{1}{2}, 0)\) (resp. \((0, \frac{1}{2})\)) differential on \(\Sigma\) and the metric in the Maxwell term is \(g_z \bar{g}_{\bar{z}} = \omega_z \omega_{\bar{z}}\).

Summarizing, what we have obtained in this subsection is that the strong coupling effective theory is given by the Green-Schwarz heterotic string action on the branched covering worldsheet plus a decoupled Maxwell theory on the same surface.

3.4 String amplitudes

The structure of the strong coupling theory is now parallel to the one obtained for the Matrix String Theory in \[14\] and we can quickly draw our conclusions by simply taking the results of \[14, 13\] and applying them to HMST. The Riemann surface \(\Sigma\), generated by the instanton, is taken to represent a heterotic string process. The surface has punctures (i.e. the points of the covering that correspond to \(z = 0\) and \(z = \infty\)) which represent the incoming and outgoing strings: the length of each asymptotic string, i.e. the multiplicity of the corresponding branch point plus one \[15\], is physically interpreted as its light-cone momentum \(p_+\). Naturally, in order to fully describe the asymptotic string states, we have to introduce in the path integral the corresponding vertex operators. Each vertex is constructed out of the \(\tilde{x}, \tilde{\theta}, \tilde{\chi}\) fields and, moreover, specifies the string transverse momentum. Since the vertices do not depend on the Maxwell field and ghosts, we can integrate them out (the non-Cartan modes have been integrated out above). Since the action is free, the integration produces a ratio of determinants, which turns out to be a constant. However we have to take account of the zero modes for the fields that have been rescaled. The rescaled fields on \(\Sigma\) (3.15) are

\[
\tilde{a}_z \rightarrow g \tilde{a}_z, \quad \tilde{a}_{\bar{z}} \rightarrow g \tilde{a}_{\bar{z}}, \quad \tilde{c} \rightarrow g \tilde{c}, \quad \tilde{\bar{c}} \rightarrow g \tilde{\bar{c}}.
\]

(3.24)
As a consequence the Maxwell partition function rescales too with a factor depending on the zero modes. By counting the zero modes of the Maxwell fields and ghosts we conclude, as in [14], that (thanks to the Maxwell partition function) the string amplitude corresponding to $\Sigma$ is proportional to $g^{2-2h-p}$, where $h$ and $p$ are respectively the number of handles and punctures of $\Sigma$. Being the string coupling $g_s = g^{-1}$, this multiplicative factor is exactly what we need in order to correctly reproduce string interaction theory.

Of course this is not the end of the story. In order to obtain the complete amplitude we have to sum over all the instantons of the same topological type $(h, p)$. Let us call $V_1, \ldots, V_k$ the vertex operators corresponding to $k$ incoming and outgoing strings, and insert them into the path integral. The genus $h$ amplitude (in the strong coupling limit) will schematically be:

$$\langle V_1, \ldots, V_k \rangle_h = g_s^{-\chi} \int_{\mathcal{M}_N^{(h, p)}} dm \int D[\bar{x}, \bar{\theta}, \bar{a}, \bar{c}] V_1 \ldots V_k e^{-S_{het}}, \quad (3.25)$$

Integrating over $\mathcal{M}_N^{(h, p)}$ means integrating over all distinct instantons which underlie the given string process for fixed $N$, that is to say with assigned incoming and outgoing strings and string interactions. In ordinary string interaction theory $\mathcal{M}_N^{(h, p)}$ is nothing but the moduli space of Riemann surfaces of genus $h$ with $p$ punctures, a complex space of dimension $3h - 3 + p$. In HMST $\mathcal{M}_N^{(h, p)}$ is the same as in MST and we can simply summarize the result obtained in [15]: $\mathcal{M}_N^{(h, p)}$ is a discrete slicing of the moduli space of Riemann surfaces of genus $h$ with $p$ punctures, every slice having complex dimension $2h - 3 + p$; in other words $h$ moduli are discrete in HMST; we have argued in [15] that when $N \to \infty$ these discrete parameters become continuous and the full moduli space of Riemann surfaces is recovered.

Therefore we can say that for large $N$ the strong coupling limit of HMST reproduces the heterotic string interaction theory.

As pointed out in section 2, in the absence of Wilson lines, this is the $SO(32)$ theory compactified to nine dimensions on a circle of very small radius.

To obtain other heterotic string theories, one must introduce Wilson lines $\chi B \chi$, which at strong coupling become $\chi B \chi \to \chi_0 B \chi_0$. The latter term is lifted to the covering as $\tilde{\chi} B \tilde{\chi}$ and accounts for the breaking of $SO(32)$ to some suitable subgroup. As remarked at the beginning, with a standard choice of Wilson lines, one can break $SO(32)$ to $SO(16) \times SO(16)$, and relate the model to the ten-dimensional $E_8 \times E_8$ string also broken to $SO(16) \times SO(16)$.

A. Appendix

In this Appendix we discuss the derivation of the strong coupling action (3.17) and the integration of the $Q_a$ therein. The problems we have to face are related to the presence in the action of chiral fermions and to the absence in the HMST of half the supersymmetry, compared to MST. In (2.1) all fermions appear quadratically, therefore they can be

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3Fermionic zero modes are absent if $2h + p - 2 > 0$
(at least formally) integrated and give an overall well defined fermion determinant, since chiral anomalies from different multiplets cancel. If we were able to explicitly expand it in powers of $1/g$, we would start from this well defined expansion and then integrate also over the bosons. Unfortunately we do not know how to do that. Therefore we can only carry out the expansion directly in the action. However, if we do so, we break the abovementioned well-defined determinant to pieces and, in particular, we have to treat each chiral species separately. This, in turn, leads to the longstanding problem of representing chiral fermions in field theory. Various solutions have been proposed over the years: dynamical mechanisms based on an infinite number of flavors \[30\] or on a generalized Pauli-Villars regularization \[31\] (see the reviews \[32, 33\], and the references therein); in two dimensions, in particular, the twistor formalism has been extensively used \[34\]. Any of these choices would require a long technical detour from the mainstream of our paper. Therefore we take a simpler course. We operate formally on the action, by making sure, however, that our procedure leads at every step to a well-defined result.

We warn the reader that a complete confirmation of the correctedness of this way of proceeding would come from the full expansion of the path integral in powers of $1/g$. This is however beyond our present ability.

We deal first with the action term $\chi D w \chi$, which turns out to be the most delicate. Let us use (3.11), (3.10) and rewrite it as

$$S_\chi = \frac{i}{\pi} \int \chi D w \chi = \frac{i}{\pi} \int (\chi^t + \chi^n) \left( \partial_w - a_w^t - a_w^n \right) (\chi^t + \chi^n)$$

$$= \frac{i}{\pi} \int (\chi^t \partial_w \chi^t - \chi^t a_w^t \chi^t - \chi^t a_w^n \chi^n - \chi^n a_w^t \chi^t + \cdots)$$

$$= \frac{i}{\pi} \int (\chi_0 \partial_w \chi_0 - \chi_0 a_w \chi_0 + 2 \chi_0 a_w^0 \chi_1 + \cdots) . \quad (A.1)$$

Dots represent non-leading terms in the strong coupling limit once the rescalings (3.15), (3.16) are applied, so for simplicity we drop them right away. However it should be kept in mind that the rescalings will become effective only later on. The matrix $a_w$ is $n \times n$ antisymmetric. Its elements are given by

$$(a_w)_{ij} = (a_w^n)_{2i,2j} + (a_w^n)_{2i,2j+1} + \cdots + (a_w^n)_{2i+1,2j} + (a_w^n)_{2i,2j+1} + \cdots .$$

Looking at the last line of (A.1) a difficulty is immediately evident. The second term should simply not be there in the strong coupling limit, the first term should end up in $S_{sc}$ and the last term should account for the overall result of 1 in the integration of $Q_n$. But first and last term are intertwined. We have to disentangle them. In so doing we will solve also the problem of the second term.

To this end let us, for simplicity, single out the Grassmann path integral involving the terms (A.1).

$$\int D\chi_0 D\chi_1 \ e^{-S_\chi} .$$

\[4\]Our wording is probably too poor in regard to fermion determinants: by well-defined we mean that a determinant is a function rather than a section of some nontrivial line bundle.
In this path integral we introduce a delta function \( \delta(\chi_0 - \psi) \) where \( \psi \) is a fermionic field of the same type as \( \chi_0 \) (i.e. a vector with \( n \) components, each of which is a real spinor in the fundamental of \( \text{SO}(32) \)), and integrate over \( \psi \). Then we lift the delta function to the exponent by means of a Lagrange multiplier \( \lambda \), which is itself a spinor of the same type as \( \chi_0 \) and \( \psi \). The path integral is equivalent to the initial one, provided we integrate over the modes of \( \chi_0, \chi_1, \psi \) and \( \lambda \) and provided the action (A.1) is replaced by

\[
S'_{\chi} = \frac{i}{\pi} \int (\chi_0 \partial_w \chi_0 - \psi a_w \chi_0 + 2 \psi a_w^0 \chi_1 - \lambda \chi_0 + \lambda \psi + \cdots).
\]

Now we redefine \( \chi_1 \) as

\[
\chi_1' = \chi_1 \frac{1}{2} a_w^0 \lambda - \frac{1}{2} a_w^0 a_w \chi_0
\]

and obtain

\[
S'_{\chi} = \frac{i}{\pi} \int (\chi_0 \partial_w \chi_0 + 2 \psi a_w^0 \chi_1' - \lambda \chi_0 + \cdots). \tag{A.2}
\]

Another way to get rid of the term \( \chi_0 a_w \chi_0 \) is to integrate first over the \( a^n_w \) degrees of freedom and their conjugate: a well-known procedure to englobe linear terms in a gaussian integration leads to a quartic term in \( \chi_0 \), which is however vanishing.

Summarizing, the path integral involving the \( \chi \) modes has become

\[
\int \mathcal{D}\chi_0 \mathcal{D}\chi_1 \mathcal{D}\lambda \mathcal{D}\psi e^{\frac{i}{\pi} \int (\chi_0 \partial_w \chi_0 + 2 \psi a_w^0 \chi_1 - \lambda \chi_0 + \cdots)} . \tag{A.3}
\]

It remains for us to rescale the fields according according to (3.15), (3.16) and, in addition, \( \lambda \) according to \( \lambda \to \frac{1}{g} \lambda \). As a consequence, all terms represented by dots, as well as the \( \lambda \chi_0 \) term, will drop out in the strong coupling limit, leaving a zero volume integration constant.

We will comment about this later on. Disregarding it for a moment, the \( \chi \) path integral in the \( g \to \infty \) limit becomes

\[
\int \mathcal{D}\chi_0 \mathcal{D}\chi_1 \mathcal{D}\psi e^{\frac{i}{\pi} \int (\chi_0 \partial_w \chi_0 - 2 \psi a_w^0 \chi_1)} . \tag{A.4}
\]

Now the first term is shuffled to \( S_{sc} \) in (3.18), while the second term is exactly what we need in order to get \( 1 \) from the path integration of \( Q_n \). Let us concentrate now on the latter.

\( Q_n \) has the form

\[
Q_n = Q_n^{(\text{matter})} + Q_n^{(\text{gauge})}
\]

\[
Q_n^{(\text{matter})} = \frac{1}{\pi} \int d^2 w \text{Tr} \left[ x^n Q x^n + x^n Q x^n + i (\theta_s^n, \theta_c^n) A \left( \theta_s^n, \theta_c^n \right) - 2 \psi a_w^0 \chi_1 \right] \tag{A.5}
\]

\[
Q_n^{(\text{gauge})} = -\frac{1}{\pi} \int d^2 w \text{Tr} \left[ a_w^n Q a_w^n + \bar{e}^n Q e^n \right],
\]

where

\[
Q = \text{ad}_x \circ \text{ad}_x - \text{ad}_{a_w} \circ \text{ad}_{a_w} + \text{ad}_{x^n} \circ \text{ad}_{x^n}
\]
\[ A = \begin{pmatrix} -\text{ad}_{\psi}^\dagger & i\gamma_i\text{ad}_{\chi^{\dagger}} \\ i\gamma_i\text{ad}_{\chi^{\dagger}} & \text{ad}_{\psi}^\dagger \end{pmatrix}. \]

It is understood that one must integrate \( Q_n \) over all the \( n \) modes, including \( \chi_1 \) (remember (3.11)), and \textit{in addition} over \( \psi \). In a very obvious way \( \psi \) can be written as 
\[
\psi^t = \sqrt{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]

so that 
\[
2\psi^t a_{\psi}^d \chi_1 = -\psi^t a_{\psi}^d \chi_1 - \chi_1 a_{\psi}^d \psi^t.
\]

Our aim is to show that the integration over the bosonic degrees of freedom in the path integral of \( Q_n \) is exactly compensated for by the integration over the fermionic ones. As for \( Q_n^{(\text{gauge})} \) the argument is the same as in [14] and will not be repeated here: the integral over the gauge and ghost modes gives 1.

As for the matter part, integrating formally we obtain a ratio of determinants of \( A \) and \( Q \). However there are here some subtleties that we have to explain. In \( Q_n^{(\text{matter})} \) the operator \( A \) is a chiral operator: there is an asymmetry between \( \theta^\alpha \) and \( \theta^\beta \). The asymmetry is measured in the same way we do for chiral anomalies: we compute the trace of the square generators in the symmetric and adjoint representations of the Lie algebra \( \mathfrak{g}(N) \).

Setting \( \theta = 1 \) the corresponding quantity in the fundamental representation, we get \( N + 2 \) in the symmetric representation and \( N - 2 \) in the adjoint. Therefore the determinant that comes from the formal integration over these modes is not well defined.

To see this point more clearly we need a further splitting of the non-Cartan degrees of freedom.

\[
\phi^n = \phi^0_{\tau} \otimes \tau + \phi^0_{\sigma} \otimes \sigma + \phi^0 + \phi^n',
\]

where \( \phi^0_{\tau} \) and \( \phi^0_{\sigma} \) are diagonal \( n \times n \) matrices,
\[
\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

and \( \phi^n' \) denotes the remaining non-Cartan directions. It is clear that the components \( \phi^0_{\tau} \) and \( \phi^0_{\sigma} \) are present only in the fields \( X^i \) and \( \theta^s \), not in the others.

Now, computing the trace of the square generators in the symmetric and adjoint representations, it is easy to see that the chiral asymmetry of \( (N + 2) - (N - 2) = 4 \) between the two representations is accounted for exactly by the modes \( \vartheta^\alpha_\sigma \) and \( \vartheta^\alpha_{\psi} \). But, calculating the trace in the fundamental representation, we see that this asymmetry is exactly compensated for by the \( \chi_1 \) and \( \psi \) modes. (Naturally these compensations are the same that cooperate to cancel the chiral anomaly in the initial action (2.1)).

Therefore the operator \( A \), as it appears in (A.3) is non-chiral and the corresponding determinant well defined. We can therefore proceed from now on as in [14]. What we have to evaluate is the ratio \( ((\det A)^{16}/(\det Q)^8)^{2N^2+N} \). The involved operators do not have zero modes. The \( \det A \) in the numerator should be understood as \( \sqrt{\det(-AA^\dagger)} \). But \( AA^\dagger = A^\dagger A = -Q \). Therefore the net result of integrating over the non-Cartan modes is 1. This is the result expected from supersymmetry in the absence of zero modes.
As for the overall Jacobian arising from the rescalings (3.15), (3.16) and the one of $\lambda$, it gives rise to a factor $g^{60n}$. This factor in the $g \to \infty$ limit is an infinite factor that must be related to the abovementioned zero volume factor due to integration over $\lambda$. Such infinite and zero factors do not exist for finite $g$. They are an artifact of the $g \to \infty$ limit. Therefore it is sensible to assume that they just compensate for each other and give a constant which we can choose to be 1.

Resuming now the considerations at the beginning of this section, we remark that formal manipulations of the path integral have led us, nevertheless, to a blameless definition of the integration over the non-Cartan modes. As for the surviving Cartan modes, lifted to the covering they give rise to a well-defined theory, the heterotic superstring.

Finally we would like to make a comment on the possibility, mentioned in section 2.1, of an alternative strong coupling limit which would be obtained by scaling away all the $\chi$ degrees of freedom while keeping $\theta^c$. This would surprisingly lead to a type IIA theory. We have already ruled out above this possibility on the basis of some simple arguments. Here we would like to strengthen the argument that the strong coupling limit should be a chiral theory. This can be done by a request of anomaly matching condition similar to 't Hooft’s one [3]. The ‘flavor’ symmetry of our theory at $g = 0$ can be thought of as a global $SO(8)_R \times SO(8)_L \times SO(32)_L$. Suppose we gauge this symmetry (for instance, by coupling $\theta^c$ to a gauge potential $C^{ij}_w$ via $\theta^c C^{ij}_w \gamma^{ij} \theta^c$, and so on); it is elementary to see that the corresponding anomalies are reproduced in the heterotic strong coupling limit with the same relative coefficient, while they would not reproduced in the would-be IIA strong coupling limit.

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After completing this work, we saw a new paper by T. Wynter [36] dealing with the strong coupling of MST.

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