Two Dimensional Quantum Chromodynamics on a Cylinder

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ABSTRACT

We study two dimensional Quantum Chromodynamics with massive quarks on a cylinder in a light–cone formalism. We eliminate the non–dynamical degrees of freedom and express the theory in terms of the quark and Wilson loop variables. It is possible to perform this reduction without gauge fixing. The fermionic Fock space can be defined independent of the gauge field in this light–cone formalism.
The problem of deriving the low energy properties of strong interactions from the lagrangian of Quantum Chromodynamics (QCD) remains as an important challenge to particle theorists. ’t Hooft \[1\] derived the meson spectrum in the large \(N_c\) limit from QCD in two dimensional Minkowski space. Recently, his diagrammatic method has been reformulated in the language of operators \[2\] and the baryon \[3\] spectrum in the large \(N_c\) limit has been derived as well. In this model there are no gluonic degrees of freedom; all the degrees of freedom in the Yang–Mills field can be gauged away.

If the space–time is not simply connected, there are some Yang–Mills degrees of freedom that cannot be gauged away: those associated to parallel transport around non–contractible loops. A simple special case is Yang–Mills theory on a cylinder, which was solved by canonical methods in Ref. \[4\]. In this case the only physical degree of freedom is the Wilson loop around the cylinder; Yang–Mills theory reduces to quantum mechanics on a group manifold. It is of interest to study Yang–Mills theory coupled to fermions on a cylinder, as it will provide a generalization of ’t Hooft’s model that contains some gluonic degrees of freedom. In this paper, we will reduce the action of Dirac–Yang–Mills theory on a cylinder to a form in which all but a finite number of degrees of freedom of the gauge have been eliminated. We believe that this form of the theory can be solved in the large \(N_c\) limit.

We will perform our analysis in a co–ordinate system different from the Cartesian \((x, t)\) co–ordinates used to solve pure Yang–Mills theory in Ref. \[4\].

We define \(u = t + |x|\), \(dt = du - \text{sgn}(x) dx\).

The metric, \(ds^2 = dt^2 - dx^2 = du^2 - 2\text{sgn}(x) du dx\). Thus the metric tensor,

\[
\eta_{\mu\nu} = \begin{pmatrix}
1 & \text{sgn}(x) \\
-\text{sgn}(x) & 0
\end{pmatrix}, \quad \eta^{\mu\nu} = \begin{pmatrix}
0 & -\text{sgn}(x) \\
-\text{sgn}(x) & -1
\end{pmatrix}
\]

\(\sqrt{-\eta} = 1\)

The points \((u, x)\) and \((u, x + 2L)\) are the same on the cylinder.

This system \((u, x)\) is a variant of the usual light–cone formalism. The disadvantage of the Cartesian co–ordinates in our case is that the energy appears quadratically in the mass shell condition:

\[
p_0^2 - p_1^2 = m^2 \quad (1)
\]
so that there are two values of energies, differing by a sign, for each momentum. Then, the fermionic Fock space has to be defined by filling the negative energy sea. Unfortunately, this can lead to trouble since the definition of the Dirac sea can depend on the gauge field [5]. It is an old observation of Dirac [6] (revived in the context of QCD in Ref. [7]) that in light–cone co–ordinates, the analogue of energy appears linearly so that this problem does not occur. If we were really to use the conventional light–cone co–ordinates, our method of eliminating the gauge field would not work. We will use $u$ as our evolution variable, so that the equal ‘time’ surfaces are still light–cones.

The mass–shell condition in our co–ordinate system is

$$-2\text{sgn}(x)p_x p_u - p_x^2 = m^2$$

so that $p_u$ has a unique solution:

$$p_u = -\frac{1}{2} \left[ \frac{m^2}{p_x} + p_x \right] \text{sgn}(x).$$

If we impose $\text{sgn}(x)p_x < 0$ on the one–particle Hilbert space of fermions, we guarantee that the Fermionic energy is positive. The equal $u$ surface is a cone and our condition says that the fermions move towards the future. For the moment we set aside this issue.

The Gamma matrices satisfying the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ are:

$$\gamma^u = \gamma^0 + \text{sgn}(x)\gamma^1 = \begin{pmatrix} 0 & 1 + \text{sgn}(x) \\ 1 - \text{sgn}(x) & 0 \end{pmatrix} \quad \text{and} \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Also,

$$\gamma^1\gamma^u = \begin{pmatrix} 1 - \text{sgn}(x) & 0 \\ 0 & -1 - \text{sgn}(x) \end{pmatrix}, \quad \gamma^u\gamma^1 = \begin{pmatrix} -1 - \text{sgn}(x) & 0 \\ 0 & 1 - \text{sgn}(x) \end{pmatrix}$$

With hindsight, in order to have only one component of the quark spinor to evolve in time, we choose the quark spinor as:

$$q = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi \\ \psi \end{pmatrix}, \quad x > 0$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \psi \\ -\chi \end{pmatrix}, \quad x < 0$$
On the cylinder, the points \((t, x) \sim (t, x+2L)\) i.e., \((u, x) \sim (u, x+2L)\). All fields are periodic with period \(2L\):

\[
q(u, x) = q(u, x + 2L)
\]

\[
A_\mu(u, x) = A_\mu(u, x + 2L)
\]

We choose \(A_\mu\) to be anti-hermitian.

The action of two dimensional QCD is,

\[
S = \int dudx \sqrt{-\eta} \left[ -\bar{q}i\gamma^\mu(\partial_\mu + A_\mu)q - m\bar{q}q + \frac{1}{4\alpha}\text{tr} F^{\mu\nu} F_{\mu\nu} \right]
\] (4)

In terms of the components of the quark spinor (and introducing \(E\) as an independent variable to get a first order action),

\[
S = \int dudx \left[ -\psi^\dagger i(\partial_u + A_u)\psi - \frac{\text{sgn}(x)}{2}\{\psi^\dagger i(\partial_1 + A_1)\psi - \chi^\dagger i(\partial_1 + A_1)\chi\} \\
- \frac{\text{sgn}(x)}{2}(\psi^\dagger \chi + \chi^\dagger \psi) \\
- \frac{1}{\alpha}\text{tr}E\{\partial_u A_1 - \partial_1 A_u + [A_u, A_1]\} + \frac{1}{2\alpha}\text{tr}E^2 \right]
\]

where

\[
E = \partial_u A_1 - \partial_1 A_u + [A_u, A_1]
\] (5)

is the electric field.

Our strategy will be similar to that in Ref. [4]: to eliminate the Yang–Mills field without imposing any gauge condition.

It will be useful to define the variable \(h\) such that,

\[
\frac{\partial h}{\partial x} + A_1 h = 0 \quad \text{with} \quad h(u, -L) = 1.
\]

Then,

\[
h(u, x) = P[e^{-\int_{-L}^{x} dy \quad A_1(u, y)}]
\]

The Wilson loop is then given by \(q = h^{-1}(u, L)\). In general, this \(q\) is not equal to one; in fact it contains all the gauge invariant degrees of freedom of the gauge field co–ordinates. (We will see that the boundary value of the electric field plays the role of a canonical conjugate to \(q\)).
Now, define new variables $\tilde{\psi}$ etc. by,

\[
\psi = h\tilde{\psi} \quad \chi = h\tilde{\chi} \\
A_u = h\tilde{A}_u h^{-1} \quad E = h\tilde{E} h^{-1}.
\]

The old variables satisfied the boundary conditions:

\[
\chi(L) = \psi(-L) \quad \psi(L) = -\chi(-L) \\
\chi(0^+) = \psi(0^-) \quad \psi(0^+) = -\chi(0^-)
\]

and

\[
E(L) = E(-L).
\]

In terms of the new field variables, we see that:

\[
\tilde{\chi}(L) = q\tilde{\psi}(-L) \quad \tilde{\psi}(L) = -q\tilde{\chi}(-L) \\
\tilde{E}(L) = q\tilde{E}(-L) q^{-1}
\]

We will define $E(-L) = e$.

After some calculations, the action can be written in terms of the new field variables as:

\[
S = \int dudx \left[ -\tilde{\psi}^\dagger i\partial_u \tilde{\psi} - \frac{\text{sgn}(x)}{2}(\tilde{\psi}^\dagger i\partial_1 \tilde{\psi} - \tilde{\chi}^\dagger i\partial_1 \tilde{\chi}) - \frac{m\text{sgn}(x)}{2}(\tilde{\psi}^\dagger \tilde{\chi} + \tilde{\chi}^\dagger \tilde{\psi}) \right] \\
+ \int dudx \left[ -\tilde{\psi}^\dagger iA\tilde{\psi} - \frac{1}{\alpha} \text{tr}\{(\partial_1 \tilde{E})A\} \right] + \frac{1}{2\alpha} \text{tr}\tilde{E}^2 \\
- \frac{1}{\alpha} \int du \text{ tr}(q^{-1}\partial_u q e) \tag{6}
\]

The action is written in the above form by using the identities

\[
\partial_1 A_u + [A_1, A_u] = h(\partial_1 \tilde{A}_u) h^{-1}
\]

and

\[
h^{-1}(\partial_u A_1) h = -\partial_1(h^{-1}\partial_u h).
\]

(Both of these can be seen using $\partial_1 h = -A_1 h$.)
Also we have redefined,
\[ \tilde{A}_u + h^{-1} \partial_u h = A \]

In this action, the variables $A$ and $\chi$ do not have derivatives with respect to the evolution variable $u$. Hence they are just Lagrange multipliers imposing some constraints and can be eliminated. Varying the action written above w.r.t $A$, we obtain the constraint equation (analogue of Gauss’ law):
\[ \frac{1}{\alpha} \partial_1 \tilde{E}^- : i \tilde{\psi} \tilde{\psi}^\dagger : = 0 \]

We can solve for $\tilde{E}(x)$:
\[ \tilde{E}(x) = \alpha \int_{x-L}^{x} dy : i \tilde{\psi} \tilde{\psi}^\dagger(y) : + e \]

We still have a constraint, coming from the periodicity condition $E(-L) = E(L)$:
\[ e + \alpha \int_{-L}^{L} dy : i \tilde{\psi} \tilde{\psi}^\dagger(y) : = q e q^{-1} \]

This generates a finite dimensional part of the gauge invariance that cannot be eliminated without running into Gribov ambiguities. At end we can recover this part of the gauge group by imposing an equivariance condition on the wave–functions.

Also, we can eliminate $\tilde{\chi}$ from our action using:
\[ -i \partial_1 \tilde{\chi} + m \tilde{\psi} = 0 \]

This eliminates half the fermion degrees of freedom as in the usual light cone formalism [2].

We have thus reduced the action to:
\[
S = \frac{1}{2\alpha} \int dudx \, \text{tr} \left[ e + \alpha \int_{-L}^{x} dy : i \tilde{\psi} \tilde{\psi}^\dagger(y) : \right]^2 - \frac{1}{\alpha} \int du \, \text{tr}(q^{-1} \partial_u q e)
+ \int dudx \, \left[ -\tilde{\psi}^\dagger i \partial_u \tilde{\psi} + \frac{\text{sgn}(x)}{2} \tilde{\psi}^\dagger (\hat{p} + \frac{m^2}{\hat{p}}) \tilde{\psi} \right]
\]

where $\hat{p} = -i \partial_1$. 
Now we can read off the canonical commutation relations on an equal u surface:

\[
[\tilde{\psi}^\dagger(x), \tilde{\psi}(y)]_+ = \delta(x - y) \quad [\text{tr} \lambda e, q] = \lambda q - q \lambda
\]

(10)

all others being zero. The hamiltonian is then

\[
H =: - \frac{1}{2\alpha} \int dx \quad \text{tr} \left[ e + \int_{-L}^{x} dy \quad : i\tilde{\psi}\tilde{\psi}^\dagger(y) : \right]^2 \\
- \int dx \quad \left[ \frac{\text{sgn}(x)}{2} \tilde{\psi}^\dagger(\hat{p} + \frac{m^2}{\hat{p}})\tilde{\psi} : \right]
\]

(11)

It should be of interest to diagonalize this hamiltonian numerically using the techniques of Ref. [8]. Alternately, one could ‘bosonize’ this using the techniques of Ref. [2]. This would yield a theory of mesons, baryons and ‘glueballs’ that is semi–classical in the large \(N_c\) limit.

We notice here that the current \(j(y) =: i\tilde{\psi}\tilde{\psi}^\dagger(y) :\) is not periodic in \(2L\) but,

\[
\tilde{E}(x) = e + \alpha \int_{-L}^{x} dy \quad : i\tilde{\psi}\tilde{\psi}^\dagger(y) : \text{ obeys the periodicity condition:}
\]

\[
\tilde{E}(L) = q\tilde{E}(-L)q^{-1}
\]

This is because the ‘electric field’ operator has a geometric meaning as a covariant derivative on the associated bundle over the space of gauge fields modulo gauge transformations.

While this paper was in preparation, we received a recent paper [9] also dealing with two dimensional gauge theories. However, the approach is different, as they do not use a light cone formalism.
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