MAGNETIC FIELD OF RELATIVISTIC NONLINEAR PLASMA WAVE

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Longitudinal and transverse behavior of magnetic field of relativistic nonlinear three-dimensional plasma wave is investigated. It is shown that the magnetic field of the wave is different from zero and performs higher frequency oscillations compared to the plasma electron frequency. An increase in the nonlinearity leads to strengthening of magnetic field. The oscillations of magnetic field in transverse direction arise, that caused by the phase front curving of nonlinear plasma wave. The numerical results well conform with predictions of the analytical consideration of weakly-nonlinear case.

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The progress in the technology of ultrahigh intensity lasers and high current relativistic charged bunch sources permits the use of laser pulses\(^1\) or charged bunches\(^2\) for excitation of strong plasma waves. The excited plasma waves can be used, for example, for acceleration of charged particles and focusing of bunches.\(^2\) The amplitude of longitudinal electric field \(E_{\text{max}}\) of relativistic plasma waves excited in cold plasma is limited by the relativistic wave-breaking field\(^3\) \(E_{\text{rel}} = [2(\gamma - 1)]^{1/2}E_{\text{WB}}/\beta\), where \(\gamma = (1 - \beta^2)^{-1/2}\) is a relativistic factor, \(\beta = v_{\text{ph}}/c\) is a dimensionless phase velocity of the wave, \(E_{\text{WB}} = n_p\omega_{\text{pe}}v_{\text{ph}}/e\) \((E_{\text{WB}}[V/cm] \approx 0.96n_p^{1/2}[cm^{-3}])\) is the conventional non-relativistic wave-breaking field, \(\omega_{\text{pe}} = (4\pi n_p e^2/m_e)^{1/2}\) is the electron plasma frequency, \(n_p\) is the equilibrium density of plasma electrons, \(m_e\) and \(e\) are the mass and absolute value of the electron charge.

The linear plasma wave theory is valid when \(E_{\text{max}} \ll E_{\text{WB}}\). It is well known that a three-dimensional linear plasma wave in a cold plasma and in the absence of external fields is potential.\(^4\) In this case the magnetic field of plasma wave is zero. The magnetic field is absent also in one-dimensional nonlinear case due to the symmetry of the problem. However, magnetic field of nonlinear three-dimensional plasma wave with the relativistic phase velocity \((\beta \approx 1)\) was not studied up to now and our aim is to investigate this problem.
We shall study nonlinear plasma waves (wake waves) excited in cold plasma by relativistic electron bunches or intense laser pulses (drivers) and suppose the azimuthal symmetry of the problem. In this case for non-zero components of plasma electrons momentum and electromagnetic field of the wave we have the following set of equations:\(^5,6\):

\[
\beta \frac{\partial P_z}{\partial z} - \frac{\partial \gamma_e}{\partial z} - \beta^2 E_z = 0, \tag{1}
\]

\[
\beta \frac{\partial P_r}{\partial z} - \frac{\partial \gamma_e}{\partial r} - \beta^2 E_r = 0, \tag{2}
\]

\[
- \frac{\partial H_\theta}{\partial z} + \beta \frac{\partial E_r}{\partial z} + \beta r N_e = 0, \tag{3}
\]

\[
\mathbf{\nabla}_\perp H_\theta + \beta \frac{\partial E_z}{\partial z} + \beta z N_e + \beta \alpha = 0, \tag{4}
\]

\[
\beta \frac{\partial H_\theta}{\partial z} - \frac{\partial E_r}{\partial z} + \frac{\partial E_z}{\partial r} = 0, \tag{5}
\]

\[
N_e = 1 - \alpha - \mathbf{\nabla}_\perp E_r - \frac{\partial E_z}{\partial z}. \tag{6}
\]

As usual, the Eqs. (1) and (2) were derived taking into account that the curl of the generalized momentum is zero, \(\beta^2 \mathbf{H} - \text{rot} \mathbf{P} = 0\), or in our case

\[
\beta^2 H_\theta + \frac{\partial P_z}{\partial r} - \frac{\partial P_r}{\partial z} = 0. \tag{7}
\]

In Eqs. (1) - (6) \(\gamma_e = (1 + P_z^2 + P_r^2 + a^2/2)^{1/2}, \beta_{z,r} = P_{z,r} / \gamma_e\) and \(N_e = n_e/n_p\) are respectively a relativistic factor, dimensionless components of velocity and dimensionless density of plasma electrons, \(z = (\omega_{pe}/v_{ph})(Z - v_{ph}t)\), \(\alpha = n_b(z,r)/n_p, n_b\) is density of bunch electrons, \(a = e E_0(z,r)/m_e c \omega_0\), where \(E_0\) and \(\omega_0\) are the amplitude and frequency of laser pulse, \(\mathbf{\nabla}_\perp = \partial/\partial r + 1/r\). Also the following dimensionless variables have been used: the space variables are normalized on \(\lambda_p/2\pi = v_{ph}/\omega_{pe}\), where \(\lambda_p\) is the linear plasma wavelength, the momenta and velocities - respectively on \(m_e c\) and the velocity of light and the strengths of electric and magnetic fields - on the nonrelativistic wave-breaking field \(E_{WB}\). In the general case the analytical consideration of the problem seems impossible. First we consider weakly-nonlinear case and then present numerical results.
In the weakly-nonlinear case we suppose $\beta = 1$ and use the generalization of the well known expansion\textsuperscript{7} which was used to study one-dimensional nonlinear relativistic-plasma-waves\textsuperscript{8}:

\begin{align*}
    u(r, z) &= \varepsilon u_1(r, \Psi) + \varepsilon^2 u_2(r, \Psi) + \varepsilon^3 u_3(r, \Psi) + \ldots, \\
    \partial \Psi / \partial z &= 1 + \varepsilon k_1(r) + \varepsilon^2 k_2(r) + \ldots,
\end{align*}

where $u$ stands for normalized values $P_{z,r}, E_{z,r}$, or $H_{\theta}$, $\varepsilon = E_{z}^{\max} \ll 1$ is the small parameter, $P_{z,r}, E_{z,r}, H_{\theta} \ll 1$, $\Lambda_p$ is the nonlinear plasma wavelength. Substituting expansion (8) into equations (1)-(5) we have:

\begin{align*}
    \frac{\partial P_{z,i}}{\partial \Psi} - E_{zi} &= S_{1,i}, \\
    \frac{\partial P_{r,i}}{\partial \Psi} - E_{ri} &= S_{2,i} \\
    - \frac{\partial H_{\theta i}}{\partial \Psi} + \frac{\partial E_{ri}}{\partial \Psi} + P_{ri} &= S_{3,i} \\
    \nabla_{\perp} H_{\theta i} + \frac{\partial E_{zi}}{\partial \Psi} + P_{zi} &= S_{4,i} \\
    \frac{\partial H_{\theta i}}{\partial \Psi} - \frac{\partial E_{ri}}{\partial \Psi} + \frac{\partial E_{zi}}{\partial \Psi} &= S_{5,i},
\end{align*}

where subscript $i$ denotes the order of approximation and $S_{1-5,i}$ are the non-linear functions of $u_{i-1}, u_{i-2}, \ldots$, and $u_1$. In the linear case ($i = 1$) $S_{1-5,1} = 0$ and one can obtain well known solutions (see, e.g., Ref. 9): $P_{z1} = -R \sin \Psi$, $P_{r1} = (dR/dr) \cos \Psi$, $E_{z1} = -R \cos \Psi$, $E_{r1} = -(dR/dr) \sin \Psi$, and $H_{\theta 1} = 0$, where $R(r)$ depends on the radial distribution in the exciting source, i.e., on $\alpha(r)$ or $\alpha^2(r)$. In the second approximation, from the condition of absence of resonance terms (proportional to $\sin \Psi$ or $\cos \Psi$) one can find that $k_1 = 0$. Then

\begin{align*}
    S_{1,2} &= (1/2) \partial(P_{r1}^2 + P_{z1}^2) / \partial \Psi, \\
    S_{2,2} &= (1/2) \partial(P_{r1}^2 + P_{z1}^2) / \partial r, \\
    S_{3,2} &= P_{r1}(\nabla_{\perp} E_{r1} + \partial E_{z1}/\partial \Psi), \\
    S_{4,2} &= P_{z1}(\nabla_{\perp} E_{r1} + \partial E_{z1}/\partial \Psi), \\
    S_{5,2} &= 0.
\end{align*}
and for magnetic field strength we obtain the following equation:

\[ [(\partial/\partial r)\nabla_\perp - 1]H_{\theta 2} = \partial S_{4,2}/\partial r - \partial S_{3,2}/\partial \Psi. \quad (14) \]

Solution of this equation is

\[ H_{\theta 2} = f_1(r) + f_2(r) \cos(2\Psi), \quad (15) \]

where \( f_{1,2}(r) \) satisfies to equations

\[
\begin{align*}
(\Delta_\perp - r^{-2} - 1)f_1 &= (1/2)(d/dr)[R(\Delta_\perp - 1)R], \\
(\Delta_\perp - r^{-2} - 1)f_2 &= (1/2)[(dR/dr)\Delta_\perp R - R(d/dr)\Delta_\perp R],
\end{align*}
\]

here \( \Delta_\perp = \nabla_\perp (d/dr) \) is the transverse part of Laplacian. In the \( i-th \) approximation \( H_{\theta i} \sim \cos(i\Psi) \). The dependence \( k_2(r) \) can be obtained in the third approximation. So, the weakly-nonlinear theory predicts that the magnetic field of 3D nonlinear plasma wave is not zero, in opposite to the 3D linear case and to 1D nonlinear one, and oscillates at the harmonics of the linear plasma frequency; the nonlinear wavelength changes in the radial direction.

We have solved Eqs. (1) - (6) numerically choosing the Gaussian profile of the driver both in longitudinal and transverse directions:

\[ A(z, r) = A_0 \exp[-(z - z_0)^2/\sigma_z^2] \exp(-r^2/\sigma_r^2), \quad (16) \]

where \( A(z, r) \) stands for \( \alpha \) or \( a^2 \). Fig. 1 shows three-dimensional linear plasma wave excited by relativistic electron bunch. The linear numerical solution obtained well agreed with predictions of the linear theory. As is seen in Figure 1(b), in the linear case the magnetic field excited in plasma is localized in the range occupied by the bunch and in the wake \( E_z^{\text{max}} \gg H_{\theta}^{\text{max}} \approx 0 \), that correspond to the linear theory.

In the nonlinear regime the behavior of plasma waves is qualitatively changed. In Fig. 2 one can see a nonlinear plasma wave that is excited by an intense laser pulse. The main difference here from the linear case is the change of shape and length of the wave with the radial coordinate \( r \) [see Fig. 2(a)]. The magnetic field strength in the nonlinear plasma wave as shown in Figure 2(b) is different from zero and has the magnitude comparable with that of other components of the field. The magnitude of higher frequency
oscillations (as compared to the plasma frequency) performed by the magnetic field along $z$ grows in proportion to the nonlinearity. Such a behavior of magnetic field is a purely nonlinear effect. Indeed, in the linear case $H_\theta = 0$, i.e., the contribution of momentum components at the plasma frequency in the expression (7) is compensated. The nonlinearity of the wave implies a rise of higher harmonics in $P_z$ and $P_r$. According to (7), the rise of magnetic field is due to these harmonics and this accounts for frequent oscillations seen in Figure 2(b), that conforms with the weakly-nonlinear analytical consideration presented above. On the other hand, the non-zero magnetic field means, according to (7), that the motion of plasma electrons in the nonlinear wave is turbulent ($\text{rot} \mathbf{P} \neq 0$). The degree of turbulence (the measure of which is $H_\theta$) grows in proportion to the nonlinearity.

It is easy to see that due to the dependence of the wavelength on $r$, the field in the radial direction grows more chaotic as the distance from the driver increases. In fact, the oscillations of plasma for different $r$ are ”started” behind the driver with nearly equal phases but different wavelengths [see Fig. 2(a)]. As $|z|$ increases, the change of phase in transverse direction (for fixed $z$) becomes more and more marked. This leads to a curving of the phase front and to ”oscillations” in the transverse direction.\(^6\)\(^{10}\) The radial behavior of the longitudinal electric field and magnetic field of the nonlinear plasma wave is presented in Fig. 3. Qualitatively the radial dependence of the field differs from that of the linear case by the change of sign and ”steepening” of fields along $r$ and that is connected with the curvature of wave phase front.

So, we have found that the magnetic field of nonlinear plasma wave is different from zero and performs higher frequency oscillations as compared to the plasma frequency. The latter qualitatively differs from the linear case and one-dimensional nonlinear one. The obtained numerical results well agreed with the predictions of weakly-nonlinear theory.

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Figure 1. The linear plasma wave excited by an electron bunch with gaussian profile; \( \alpha_0 = 0.1, \sigma_z = 2, \sigma_r = 0.5, \gamma = 10 \). (a). The dimensionless strength of longitudinal electric field. 1—\( E_z(z) \) at the axis, \( r = 0 \); 2—\( r = 1 \); 3—\( r = 2 \). (b). The strength of azimuthal magnetic field. 1—\( r = 0.2 \); 2—\( r = 0.75 \); 3—\( r = 2 \).

Figure 2. The nonlinear plasma wave excited by laser pulse. The pulse parameters are: \( a_0^2 = 3.6, \sigma_z = 2, \sigma_r = 5, \gamma = 10 \). (a). The longitudinal electric field \( E_z \). \( r = 0, 2, 4 \) and 5 in the order of magnitude reduction. (b). The magnetic field strength. 1—\( r = 2 \); 2—\( r = 4 \).

Figure 3. The radial behavior of field for the case given in Figure 2 for \( z = -25 \). 1—longitudinal electric field \( E_z(z = -25, r) \); 2—magnetic field \( H_\theta(z = -25, r) \).
Figure 1.
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Figure 2.
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(a) 

(b)
Figure 3.
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