Study of corrections to the dust model via perturbation theory

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Abstract

This work reports on the application of the Eulerian perturbation theory to a recently proposed model of cosmological structure formation by gravitational instability (Domínguez 2000). Its physical meaning is discussed in detail and put in perspective of previous works. The model incorporates in a systematic fashion corrections to the popular dust model due to multistreaming and, more generally, the small-scale, virialized degrees of freedom. It features a time-dependent length scale \( L(t) \) estimated to be \( L/r_0 \sim 10^{-1} \) \( (r_0(t) \) is the nonlinear scale, at which \( \langle \delta^2 \rangle = 1 \)). The model provides a new angle on the dust model and allows to overcome some of its limitations. Thus, the scale \( L(t) \) works as a physically meaningful short-distance cutoff for the divergences appearing in the perturbation expansion of the dust model when there is too much initial power on small scales. The model also incorporates the generation of vorticity by tidal forces; according to the perturbational result, the filtered vorticity for standard CDM initial conditions should be significant today only at scales below \( \sim 1 \, h^{-1}\text{Mpc} \).

Key words: cosmology: theory – large-scale structure of Universe – gravitation – instabilities

1 INTRODUCTION

The evolution of cosmological structures by gravitational instability is usually modelled with the popular dust model (Peebles 1980; Sahni & Coles 1995): it is simply the hydrodynamic Eqs. for a fluid under the influence of no other force but its own gravity (no pressure, no viscosity, no heat flows). The usual justification of this model is that (long-range) gravity is the overwhelmingly dominant force on the cosmological large scales and that, while the gravitational instability is not too advanced, the matter distribution looks like a continuum. It is found, however, that the success of the dust model extends beyond the expectations raised by this argumentation, as evidenced by the comparison with N-body simulations of the perturbation theory predictions, both Eulerian (Juszkiewicz et al. 1993; Bernardeau 1994; Gaztañaga & Baugh 1995; Bernardeau 1996) and Lagrangian (Buchert et al. 1994; Melott et al. 1995; Bouchet et al. 1995; Weiss et al. 1996): some perturbation predictions hold even close to the nonlinear scale \( r_0 \), when the matter distribution does not look homogeneous any more. But the dust model has of course its own shortcomings, notably that the solutions typically develop singularities: this already happens in the first order Lagrangian solution (Zel’dovich 1970; Buchert 1989; Moutarde et al. 1991) (the singularities are generically sheet-like – ‘pancakes’ in the cosmological literature); the absence of any dissipative term in the dust model Eqs. suggests that this is a property of the model and not an artifact of the perturbation expansion. Nevertheless, this problem could be satisfyingly (when compared to simulations) tackled by a phenomenological correction, the adhesion model (Gurba-tov et al. 1989), which adds a viscosity to the dust model.

Recently, I proposed a novel approach (Domínguez 2000; Paper I hereafter). The new model is characterized by a correction term to the dust model featuring a length scale \( L \). It relies on a formal expansion in powers of \( L \) which I call the small-size expansion (SSE)\(^\dagger\). I showed in Paper I that the correction behaves like an effective viscosity for the generic sheet-like collapse configuration; in this sense, it represents a derivation of a generalized adhesion-like model which continues the line of research of (Buchert & Domínguez 1998; Buchert et al. 1999). In that work I also raised two questions, which will be the goal of this paper: a better physical

\( \dagger \) This anticipates the discussion in Sec. 5; in Paper I the denomination was the ‘large-scale expansion’, because it can be viewed also as a formal expansion in powers of \( k \) in Fourier space. But this designation is unfortunate, since it can be mistaken for the large-scale cosmological expansion.

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understanding of the length $L$ and the possibility that the corrections may act as a source of vorticity.

The analytical approach is hindered by the nonlinear nature of the dust model and the corrections: in Sec. 2 I apply the Eulerian perturbation expansion in the nonlinearities, which will allow a direct comparison to the results for the dust model obtained with the same technique. The calculations are mathematically a bit more involved, but there are no essential differences. The skewness of the density contrast is computed to lowest order in Sec. 3, since this example provides a test case for comparison of the SSE predictions with the dust-model ones. Sec. 4 studies the SSE prediction to lowest perturbational order for the vorticity of the peculiar velocity. This is most interesting because the vorticity vanishes in the dust model and is thus entirely due to the correction. Sec. 5 provides a detailed discussion of the physical meaning and interpretation of the SSE and the scale $L$. Finally, Sec. 6 summarizes the conclusions and points out several possible lines of future research. App. A briefly repeats the derivation of the SSE in Paper I. App. B collects mathematical manipulations.

## 2 Correction to Dust and Perturbation Theory

The model with corrections to dust which I proposed in Paper I introduces a comoving length scale $L$ through smoothing of the microscopic mass and peculiar-momentum density fields. Then it is assumed that the coupling of the small-scale ($< L$) degrees of freedom with the large-scale ($> L$) ones is weak. This allows a certain expansion in powers of $L$ (this is the SSE) to be truncated. App. A contains a brief derivation of the model, slightly generalized over the one in Paper I by allowing for a time-dependent smoothing length $L(t)$. It yields the following Eqs. in standard comoving coordinates for the mass density field $\rho(x,t)$ and the peculiar velocity field $\mathbf{u}(x,t)$ to first order in the SSE ($\delta_1 \equiv \partial \rho / \partial x_1$, and a summation is implied by the repeated index $i$):

$$
\frac{\partial \rho}{\partial t} = -3H\rho - \frac{1}{a} \nabla \cdot (a\mathbf{u}) + \frac{L}{a} B L^2 \nabla^2 \rho,
$$

$$
\frac{\partial (a\mathbf{u})}{\partial t} = -4H a \mathbf{u} + \partial \mathbf{u} - \frac{1}{a} \nabla \cdot (a\mathbf{u}\mathbf{u}) +
\frac{L}{a} B L^2 \left\{ (\nabla\rho \cdot \nabla) \mathbf{w} - \frac{1}{a} \nabla \cdot [\rho (\partial_i \mathbf{u})(\partial_i \mathbf{u})] + \frac{L}{a} \nabla^2 (a\mathbf{u}) \right\},
$$

$$
\nabla \cdot \mathbf{w} = -4\pi G a (g - 2),
$$

$$
\nabla \times \mathbf{w} = 0,
$$

where $B$ is a constant of order unity fixed by the shape of the smoothing window. When $L$ is formally set to zero, Eqs. (1) reproduce the dust model and this means physically that the structure below the scale $L$ is dynamically unimportant. The terms proportional to $\partial_i \mathbf{u}$ to first order in the SSE have a simple geometrical meaning, as discussed in App. A. The other two correction terms represent the influence of the small-scale structure on the dynamical evolution of the large scales: $(\nabla\rho \cdot \nabla) \mathbf{w}$ is a tidal correction to the ‘macroscopic’ gravitational field $\mathbf{w}$; $\nabla \cdot [\rho (\partial_i \mathbf{u})(\partial_i \mathbf{u})]$ is a correction to the advective term $\nabla \cdot (a\mathbf{u} \mathbf{u})$ due to the velocity dispersion induced by shears in the ‘macroscopic’ velocity field $\mathbf{u}$. Conceptually, these corrections have the same origin as pressure, viscosity, heat conduction in usual hydrodynamics, namely, the dynamical effect of the microscopic degrees of freedom neglected in the coarse-grained description. But of course, their form is very different because of the qualitatively disparate underlying physics (long vs. short range dominant interaction).

The analytical approach is hindered by the nonlinear nature of the dust model and this means physically that the structure below the scale $L$ is dynamically unimportant. This allows a certain expansion in powers of $L$ (this is the SSE) to be truncated. App. A contains a brief derivation of the SSE in Paper I. App. B collects mathematical manipulations.

The perturbation expansions (2) are inserted in Eqs. (1). The initial conditions at time $t_{in}$ read

$$
\delta_1(x,t_{in}) = \delta(x,t_{in}), \quad \mathbf{w}_1(x,t_{in}) = 0, \quad \lambda \geq 2,
$$

and similarly for $\mathbf{u}$. To lowest order the linearized version of Eqs. (1) is found:

$$
\frac{\partial \delta_1}{\partial t} = -\frac{1}{a} \nabla \cdot \mathbf{w}_1 + B L^2 \nabla^2 \delta_1,
$$

$$
\frac{\partial \mathbf{w}_1}{\partial t} = -H \mathbf{u}_1 + \mathbf{w}_1 + B L^2 \nabla^2 \mathbf{u}_1,
$$

$$
\nabla \cdot \mathbf{w}_1 = -4\pi G a \rho_0 \delta_1,
$$

$$
\nabla \times \mathbf{w}_1 = 0.
$$

These Eqs. differ from the well-known linear dust model only in the terms $\propto L$. The solutions are more easily written down for the Fourier transform of the fields (to avoid burdening the notation, the same symbol will be used for a quantity and its Fourier transform. Which one is meant should be clear from the argument of the function and the context).

$$
\phi(k) := \int d^3x \ e^{ik \cdot x} \phi(x),
$$

In the long-time limit ($t \gg t_{in}$), but still within the validity regime of the perturbation expansion, the solution reads ($L_t \equiv L(t), k \equiv |k|$):

$$
\begin{align*}
&\delta = \sum_{n=1}^{\infty} \varepsilon^n \delta_n, & u = \sum_{n=1}^{\infty} \varepsilon^n u_n.
\end{align*}
$$

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\[\delta_2(k, t) = b_t e^{-\frac{1}{2}BL_1^2 k^2} \delta_{\text{mic}}(k, t_{\text{in}}),\]
\[u_1(k, t) = -a_t b_t \frac{i k}{k^2} e^{\frac{1}{2} BL_1^2 k^2} \delta_{\text{mic}}(k, t_{\text{in}}),\]

where \(b_t\) is the growing mode of \(b + 2Hb - 4\pi G_0 b = 0\), with the normalization \(a(t_{\text{in}}) = b(t_{\text{in}}) = 1\), and

\[\delta_{\text{mic}}(k, t_{\text{in}}) = e^{\frac{1}{2} BL_1^2 k^2} \delta(k, t_{\text{in}})\]

is the unsmoothed initial density contrast, Eqs. (A3). (This identification is rigorously true only for a Gaussian window; otherwise, it is part of the approximation behind the SSE, see end paragraph in App. A). Compared to the usual linear solutions, Eqs. (4-5) exhibit the explicit time-dependent smoothing. This spoils the separability of the time and wavevector dependences and renders the calculations a bit more messy.

Knowing the solutions (4-5), the first-order nonlinear corrections obey a set of linear, inhomogeneous, partial differential Eqs.:

\[\frac{\partial \delta_2}{\partial t} = -\frac{1}{a} \nabla \cdot u_2 + BL_1 \nabla^2 \delta_2 - \frac{1}{a} \nabla \cdot (\delta_1 u_1),\]
\[\frac{\partial u_2}{\partial t} = -H u_2 + u_1 + BL_1 \nabla u_2 - \frac{1}{a} (\nabla u_1) \nabla u_1 + BL_1^2 \left\{ (\nabla \delta_1) \nabla u_1 - \frac{1}{a} \nabla \cdot [(\partial_t u_1)(\partial_t u_1)] + \frac{1}{L^2} \nabla \delta_1 \cdot \nabla u_1 \right\} ,\]
\[\nabla \cdot u_2 = -4\pi G a_0 \delta_2,\]
\[\nabla \times u_2 = 0.
\]

These Eqs. will be studied in the following Secs. As remarked, the time-dependence of \(L\) complicates a little the calculations. However, the qualitative features of the results to follow are the same as if \(L = 0\).

### 3 SKEWNESS OF THE DENSITY CONTRAST

As an example of how to proceed, Eqs. (6) are solved for \(\delta_3\) in this Sect. and the skewness of the density contrast is computed to leading-order. The terms \(\propto L\) are eliminated by introducing auxiliary variables:

\[\Delta_2(k, t) := \delta_2(k, t) e^{\frac{1}{4} BL_1^2 k^2}, \quad U_2(k, t) := u_2(k, t) e^{\frac{1}{4} BL_1^2 k^2}\]

Eqs. (6) then read:

\[\frac{\partial \Delta_2}{\partial t} = \frac{1}{a} \delta T U_2 + Q_\Delta,\]
\[\frac{\partial U_2}{\partial t} = -H U_2 - 4\pi G a_0 \frac{i k}{k^2} \Delta_2 + Q_U,\]

with the sources:

\[Q_\Delta(k, t) := e^{\frac{1}{4} BL_1^2 k^2} \int \frac{dq}{(2\pi)^3} [ik \cdot u_1(q, t)] \delta_1(k - q, t),\]
\[Q_U(k, t) := e^{\frac{1}{4} BL_1^2 k^2} \int \frac{dq}{(2\pi)^3} \left\{ \frac{1}{a_t} [iq \cdot u_1(k - q, t)] u_1(q, t) + BL_1^2 [q \cdot (k - q)] \left[ 4\pi G a_0 \delta_1(q, t) \right] \delta_1(k - q, t) \frac{iq}{q^2} - \frac{2L_1}{L^2} \delta_1(k - q, t) u_1(q, t) \int \right\}.\]

Eqs. (7) now look the same as those of the dust model except for the specific form of the sources. The solution for \(\Delta_2\) with the initial condition \(\Delta_2(k, t_{\text{in}}) = 0\) is

\[\Delta_2(k, t) = b_t \int_{t_{\text{in}}}^t d\tau \int_{t_{\text{in}}}^\tau d\tau' \left\{ \frac{a_\tau b_\tau}{a_{\tau'} b_{\tau'}} \right\}^2 \left( \frac{Q_\Delta(k, \tau')}{b_{\tau'}} \right),\]
\[Q(k, t) := 2H \Delta_2(k, t) + \dot{\Delta}_2(k, t) + \frac{1}{a_t} (i k \cdot Q_U(k, t)) = \]
\[= b_t^2 \int \frac{d\tau}{(2\pi)^3} \delta_{\text{mic}}(q, t_{\text{in}}) \delta_{\text{mic}}(q - k, t_{\text{in}}) e^{\frac{1}{2} BL_1^2 (k^2 - q^2 - \delta(k - q)^2)} \times k \cdot \frac{q}{q^2} \left[ 1 - BL_1^2 k \cdot (k - q) \right] \left[ 4\pi G a_0 \left( \frac{b_t}{b_{\tau'}} \right)^2 + \frac{k \cdot (k - q)}{|k - q|^2} \right],\]

where the linear solutions (4-5) have been inserted. To simplify the solution further, we assume an Einstein–de Sitter background, \(b_t = a_t = (t/t_{\text{in}})^{2/3}\). Then, one can write finally:

\[\Delta_2(k, t) = \int \frac{d\tau}{(2\pi)^3} \delta_{\text{mic}}(q, t_{\text{in}}) \delta_{\text{mic}}(k - q, t_{\text{in}}) \times \delta_3(k - q, q) (F_3(k - q, q, t),\]

with the kernel

\[\delta_3(k, q) := \frac{q \cdot (k + q)}{q^2} \left[ \frac{3}{2} + \frac{k \cdot (k + q)}{k^2} \right],\]

and the function

\[F_3(k, q, t) := a_t \frac{e^{-\frac{1}{4} BL_1^2 k^2 + q^2}}{a_t} \int \frac{d\tau}{(2\pi)^3} \left( \frac{a_\tau}{a_{\tau'}} \right)^4 \frac{a_{\tau'}^2}{a_{\tau}} \times [1 - BL_1^2 \delta_3(k, q)] e^{\frac{1}{4} BL_1^2 (k^2 - q^2 - k^2 - q^2)}.
\]

The corrections to dust are collected in the function \(F_3\). In the dust limit, \(L \to 0\), it becomes a function of time only: \(F_3 \to (2/7)a_t^2\) (in the long-time limit \(t \gg t_{\text{in}}\)).

The second-order solution can be employed to estimate the skewness of the density contrast, defined as \(S_3 := \langle k^3 \rangle / \langle k^2 \rangle^2\), where \(\langle \cdot \cdot \cdot \rangle\) denotes an ensemble average over initial conditions. When the initial density contrast is Gaussian distributed, \(\langle \delta^2 \rangle = 0\) by Eq. (4). Hence, the skewness of the evolved density contrast is generated solely by the nonlinear evolution of the gravitational instability and this motivates the interest in its computation.

The variance and the skewness of the density contrast smoothed on a scale \(R\) read:

\[\sigma^2(R, t) = \int \frac{dk}{(2\pi)^3} (\delta(k, t) \delta(q, t)) \tilde{W}(Rk) \tilde{W}(Rq),\]
\[S_3(R, t) = \frac{1}{\sigma^3(R, t)} \int \frac{dk}{(2\pi)^3} (\delta(k, t) \delta(q, t) \delta(p, t)) \times \tilde{W}(Rk) \tilde{W}(Rq) \tilde{W}(Rp),\]

where \(\tilde{W}(\cdot)\) is the Fourier transform of the smoothing window. As mentioned previously, the use of the perturbation theory solutions (2) requires \(R\) to be large enough. Furthermore, initial Gaussian inhomogeneities are assumed, characterized by the power spectrum

\[\langle \delta_{\text{mic}}(k, t_{\text{in}}) \delta_{\text{mic}}(q, t_{\text{in}}) \rangle = (2\pi)^3 P(k) \delta^{(3)}(k + q).
\]
With the perturbation solutions (4) and (8), the variance to leading order reduces to the linear solution,
\[ \sigma_1^2(R, t) = a^2 \int \frac{dk}{(2\pi)^3} P(k)|\hat{W}(Rk)|^2 e^{-BLk^2}, \]
while the skewness reads
\[ S_3(R, t) = \frac{6a^2}{\sigma_1^3(R, t)} \int \frac{dkdq}{(2\pi)^3} P(k)P(q)\cal G_3(k, q) \times \]
\[ \times F_\delta(k, q)\hat{W}(Rk)\hat{W}(Rq)\hat{W}(R|k - q|). \]
Comparison with N-body simulations has shown that the dust prediction \((L = 0)\) is very good (Juszkiewicz et al. 1993), and not only for very large \(R\) but also even close to the nonlinear scale \(R_0\), defined by the condition \(\sigma(R_0, t) = 1\), where perturbation results would be expected to break down. In fact, the limit \(L \to 0\) can be taken inside the integrals in Eqs. (11) and (12), since the smoothing on scale \(R\) already assures UV convergence. Taylor-expansion provides a correction to the dust prediction of order \((L/R)^2\), and this suggest that the scale \(L\) in the SSE is smaller than \(R_0\).

The correction to dust is barely noticeable in the leading-order skewness because the scale \(R\) already works as a short-distance cutoff. But this is of course not the case at higher perturbational orders or with other measurable quantities. The simplest example is \(\sigma^2(R)\): it can be argued (Valageas 2002) that the perturbational correction in the dust model always exhibits an UV divergence at a sufficiently large order, in spite of the smoothing over a scale \(R\) well in the linear regime. Indeed, the next-to-leading perturbational contribution to \(\sigma^2(R)\) (of order \(e^4\), which requires going to third order in the perturbation expansion) exhibits the following scaling when \(P(k) \propto k^n\) (self-similar power spectrum) within the dust model (Lokas et al. 1996; Scoccimarro & Frieman 1996b):
\[ \sigma^2(R, t) - \sigma^2_t \sim \begin{cases} \left[ \frac{r_0(t)}{R} \right]^{2(n+3)}, & -3 < n < -1, \\ \left[ \frac{r_0(t)}{R} \right]^{2(n+4)} \left[ \frac{L}{R} \right]^{-(n+1)}, & -1 < n, \end{cases} \]
with an UV cutoff \(L \ll R\). If \(n < -1\), the smoothening over the large scale \(R\) is enough to avoid short-distance divergences and the scale-dependence can be guessed from self-similarity arguments. However, if \(n > -1\), there is too much small-scale power and the non-linear couplings require an additional UV cutoff \(L\).

In the next Sec., an example is studied where the deviations from the dust model are particularly evident: the vorticity of the velocity field.

## 4 VORTICITY OF THE PECULIAR VELOCITY

The dust model Eqs. lack a source of vorticity, \(\omega := \nabla \times \mathbf{u}\); it can arise only from an initial vorticity, which is nevertheless strongly damped in the linear regime. This tiny initial vorticity may be amplified in high-density, collapsing regions (Buchert 1992; Barrow & Saich 1993), but then caustics develop (multistream flow, the field \(\mathbf{u}\) becomes multivalued) and the dust model itself breaks down. Thus, the corrections to the dust model are most relevant in this case. As I will show, Eqs. (1) do indeed generate vorticity, and the leading-order perturbational contribution will be computed.

Let us take the curl of Eq. (6) for \(\mathbf{u}_2\) and use that \(\omega_1 = 0\) by virtue of the solution (5), to get
\[ \frac{\partial \omega_2}{\partial t} + H\omega_2 - BL\nabla^2\omega_2 = Q_\omega, \]
\[ Q_\omega := BL^2\nabla \times \left\{ (\nabla \delta_1 \cdot \nabla)\mathbf{u}_1 - \frac{1}{a} \nabla \cdot [(\partial_i u_1)(\partial_j u_1)] + \right\} \]
\[ + 2\frac{L}{a}(\nabla \delta_1 \cdot \nabla)\mathbf{u}_1. \]
Upon introducing the traceless shear tensor of the peculiar velocity field,
\[ \sigma_{ij} := \frac{1}{2} (\partial_i u_j + \partial_j u_i) - \frac{1}{3} (\nabla \cdot \mathbf{u}) \delta_{ij}, \]
and the tidal tensor of the peculiar gravitational acceleration,
\[ E_{ij} := \frac{1}{2} (\partial_i w_j + \partial_j w_i) - \frac{1}{3} (\nabla \cdot \mathbf{w}) \delta_{ij}, \]
one can write the source term alternatively as
\[ Q_\omega = BL^2\nabla \times \left\{ E_{11} \nabla \delta_1 - \frac{1}{a} \nabla \cdot \left[ \frac{2}{3} (\nabla \cdot \mathbf{u}_1) \sigma_1 + \sigma_1 \cdot \sigma_1 \right] + \right\} \]
\[ + 2\frac{L}{a}(\nabla \delta_1 \cdot \nabla)\mathbf{u}_1. \]
Since \(\sigma_1\) is in fact proportional to \(E_{11}\) by Eq. (5), the ultimate dynamical source of vorticity is the gravitational tidal force, whose effect is manifested directly (first term) and indirectly through the induced velocity dispersion (second and third terms). As already remarked, the fourth term has a purely geometrical origin.

Inserting the linear solutions (4-5) in Eq. (15) and Fourier transforming, this source reads
\[ Q_\omega(k, t) = BL^2 \frac{F_\omega(t)}{a^4} \int \frac{dq}{(2\pi)^3} \delta_{\text{nic}}(q, t) \delta_{\text{nic}}(k - q, t) \times \]
\[ \times \left[ \frac{q}{a} \cdot \mathbf{q} \right] \left( k \times q \right) e^{-\frac{1}{2} BLq^2 (q^2 + |k - q|^2)} \]
\[ \times \left[ \frac{4\pi G}{b^2} \right] \left[ \frac{b}{b} \right]^2 + \frac{\frac{2}{b} \frac{L}{bL}}{bL}. \]
Eq. (14) can be integrated immediately for the Fourier transform of the vorticity with the initial condition \(\omega_2(k, t_0) = 0\):
\[ \omega_2(k, t) = \int_{t_0}^t d\tau \frac{\partial \omega_2}{\partial \tau} e^{-\frac{1}{2} BL \tau^2 (\tau^2 - L^2)} k^2 Q_\omega(k, \tau). \]
An estimate of the strength of the vorticity on a scale \(R\) is provided by its variance,
\[ \sigma_2^2(R, t) := \int \frac{dkdq}{(2\pi)^3} (\omega(k, t) \cdot \omega(q, t)) \hat{W}(Rk)\hat{W}(Rq). \]
Inserting the perturbation solution (16-17) and simplifying:
\[ \sigma_2^2(R, t) = \int \frac{dkdq}{(2\pi)^3} P(k)P(q)\cal G_2(k, q) \hat{W}(R|k - q|) \times \]
\[ \times |\mathbf{E}(k - q, t)|^2 e^{-BLk^2 |k - q|^2}, \]
with the positive, symmetrized kernel.
\[ G(\mathbf{k}, \mathbf{q}) := \frac{1}{2} |\mathbf{k} \times \mathbf{q}|^2 (\mathbf{k} \cdot \mathbf{q})^2 \left[ \frac{1}{k^2} - \frac{1}{q^2} \right]^2, \]

and the function
\[
\Xi(z, t) := \frac{1}{\alpha_t} \int_{t_m}^t d\tau B L_r^2 F_\omega(\tau) e^{-B L_r^2 \tau}.
\]  

To gain insight into this result, it will be worked out for the particular case of a self-similar model: an Einstein–de Sitter background and an initial power spectrum \( P(k) \propto k^n \) are assumed. It remains to specify the smoothing length \( L(t) \) in a self-similar model, the only relevant scale is \( r_0(t) \) and the natural choice is \( L(t) = \lambda r_0(t) \), where \( \lambda \) is a time-independent proportionality constant. This choice will be supported by the discussion of the physical meaning of the SSE in the next Sec.

The details of the mathematical analysis are collected in App. B. The physically interesting results can be summarized in the asymptotic behavior of \( \sigma_\omega \) in the limit \( R \gg L_t \) and \( t \gg t_m \):

\[
\frac{\sigma_\omega(R, t)}{\sigma(t)} \sim \begin{cases} 
\left( \frac{L_t}{R} \right)^2 \left( \frac{r_0(t)}{R} \right)^{n+3} \frac{L_t}{R}^{-n/2}, & -3 < n < -\frac{3}{2}, \\
\left( \frac{L_t}{R} \right)^{n+3} \frac{L_t}{R}^{-n/2}, & -3 < n,
\end{cases}
\]  

(20)

up to a dimensionless factor which depends only on the spectral index \( n \). The lower bound \(-3 < n \) prevents an \textit{infrared} (IR) divergence. The crossover at \( n = -3/2 \) is related to the UV behavior of the integrals, and we are in a similar situation as with Eq. (13). If \( n < -3/2, R \) is enough to regularize possible UV divergences, the scale \( L \) only appears in the prefactor \( BL_r^2 \) of the source (15) and the scale-dependence of \( \sigma_\omega \) can be guessed by simple power-counting. If \( n > -3/2, \) however, \( L \) is relevant also as UV cutoff.

Fig. 1 shows the numerical computation of \( \sigma_\omega \) with Eq. (18), confirming the asymptotic scaling (20); in particular, that the \( R \)-dependence is insensitive to the spectral index when \( n > -3/2 \). It also shows how well the asymptotic scaling is already obeyed when \( R/L_t \approx 10 \). In the opposite limit \( R \rightarrow 0 \) (known in the literature as the ‘unsmoothed’ limit), \( \sigma_\omega \) approaches a nonvanishing constant.

To address the relevance of the generated vorticity, it will be compared to the dust prediction for the divergence of the velocity field, \( \theta := \nabla \cdot \mathbf{u} \). The variance \( \sigma_\theta^2 := \langle \theta^2 \rangle \) on a scale \( R \) is defined in analogy with Eq. (9). The variance \( \sigma_\omega^2 \), of perturbational order \( \varepsilon^3 \), must be compared with the next-leading order in \( \sigma_\theta^2 \), denoted by \( \sigma_\theta^2 \). This term scales like \( \tilde{a}^2 \times \text{Eq. (13)} \); more generally, each term in the perturbational expansion of \( \sigma_\theta^2 \) is proportional to the corresponding one in the perturbational expansion of \( (\tilde{a} \sigma)^2 \) (Scoccimarro & Frieman 1996a; Fosalba & Gaztañaga 1998). Thus

\[
\frac{\sigma_\omega}{\sigma_\theta} \sim \begin{cases} 
\left( \frac{L_t}{R} \right)^2, & -3 < n < -\frac{3}{2}, \\
\left( \frac{L_t}{R} \right)^{-n+3/2}, & -\frac{3}{2} < n < -1, \\
\left( \frac{L_t}{R} \right)^{-3/2+1}, & -1 < n,
\end{cases}
\]  

(21)

up to a factor of order unity. The exponent of the ratio \( L_t/R \) is in all cases positive. At fixed resolution \( R \), the vorticity becomes more and more relevant in time; at fixed time, it becomes subdominant on large scales.

A more realistic CDM initial power spectrum at the scales of physical interest can be approximated well by a power-law \( P(k) \propto k^n \) with \( n \approx -1, -2 \). This initial condition is set at a redshift \( z \approx 10^3 \), and so the present epoch corresponds to \( a_t \sim 10^2 \); well in the long-time regime (see App. B). Realistic deviations from an exact Einstein–de Sitter background are expected to affect only weakly the results, as happens with predictions by the dust model. As will be argued in Sec. 5, the ratio \( \lambda = L_t/r_0(t) \) is \( \approx 0.1 \). With a value \( r_0(t) \) (today) \( \approx 8 \text{ h}^{-1} \text{Mpc} \), the estimate (21) suggests that the vorticity becomes significant today on scales \( R \approx L_t \approx 1 \text{ h}^{-1} \text{Mpc} \); the physical vorticity \( \omega/a \) has then a value of the order of 1-10 times the Hubble constant today, as can be immediately read from Fig. 1. This conclusion does not disprove the approximation of potential flow at weakly nonlinear, large scales \( (R \gg r_0(t)) \), which is invariably done both in theoretical (Sahni & Coles 1995) and observational (Dekel 1994) studies. Both the length scale of relevance and the magnitude of the vorticity agree with the result by Pichon & Bernardeau (1999), who studied the growth of vorticity with the Zel’Dovich approximation at times just after the formation of the first caustic. But the SSE offers a more straightforward and systematic method than the one employed by these authors.

5 DISCUSSION

As follows from the derivation in App. A, the SSE relies on a dynamically negligible effect of the substructure of the coarsening cells of size \( \approx L \); the only dynamically relevant properties of the cells are the mass, \( \propto \xi \), and the center-of-mass velocity, \( \mathbf{u} \). On the other hand, one may notice that Eqs. (A1-A2) for the evolution of \textit{point particles} under its own gravity can be formally written as the dust model Eqs. for the fields \( \varrho_{mic} \) and \( \mathbf{u}_{mic} \), Eqs. (A3). Therefore, an in-
terpretation of the assumption behind the SSE is that the evolution of the scales \( L \) corresponds, in the lowest approximation, to that of a set of ‘effective point particles’, plus corrections due to the nonvanishing particle size \( \approx L \) and their internal structure. This interpretation motivates the appellation ‘small-size expansion’. In a bottom-up scenario, it is natural to identify the effective particles with the most recently (and thus largest) formed clusters. In fact, Peebles (1974; 1980, Sec. 28) had already shown the mutual cancellation of the nonlinear couplings to the small scales in the Eq. for \( \delta \) when these small scales consist of virialized clusters.

It must be noticed that this provides a different physical interpretation of the dust model and extends its validity. Indeed, the dust model is usually justified starting from the ideal fluid model; then the pressure term is dropped because self-gravity is dominant on scales much larger than the Jeans’ length. This argumentation applies in top-down (‘pancake’) models because, at a given time, matter is smoothly distributed on small scales and the free-flying of particles gives rise to a kinetic pressure: the ideal fluid model is a good approximation. This does not hold, however, in the opposite limit of a bottom-up (‘hierarchical’) evolution, for the small scales are highly structured. Nevertheless, the SSE argues how the dust model can still be a good description when the evolution is observed through the right spatial resolution \( \sim L^{-1} \): then particles appear trapped in the virialized clusters and to a first approximation, there is no free-fly, no kinetic pressure. In both extreme scenarios is the dust model the lowest-order approximation, the differences show up only in the corrections.

The Zel’dovich approximation (Zel’dovich 1970) was originally devised to address structure formation in top-down scenarios (HDM models). It is the lowest-order term in a Lagrangian perturbation expansion of the dust model (Buchert 1989; Moutarde et al. 1991). Thus, it came out as unexpected and worth remarking that it provided also a good description of the large-scale structure in bottom-up simulations, particularly if the initial power spectrum was first smoothed on a scale \( \sim r_0 \); this is the truncated Zel’dovich approximation (P. Coles et al. 1993; Melott et al. 1994; Pauls & Melott 1995). The SSE offers a physical explanation for this convergence and, as a novel aspect, implements in a systematic way the role of the ‘truncation’ (smoothing) and the ‘residual’ coupling to the neglected small scales.

We see that smoothing is an essential component of the dust model itself in a hierarchical scenario. The use of an UV-cutoff in the perturbation expansion is not new, but it was mainly introduced as a matter of practical purposes; e.g., to regularize the divergences in the perturbation expansion of the dust model as the initial small-scale power is increased (Sec. 3). These divergences are regarded unphysical, since N-body simulations show nothing special in the observables. This is usually interpreted as a failure of the perturbation expansion, assuming the results from simulations to be qualitatively right, and so it is conjectured that divergences could be cured by resumming the expansion (Bernardeau 1996). The SSE can be viewed as such a resummation: it takes account of a highly nonlinear feature — the virialized clusters, and its dynamical effect on the evolution of the large scales.

According to the physical interpretation of the SSE, the smoothing length \( L(t) \) should scale with \( r_0(t) \), as the natural measure of the size of the largest virialized structures. Consequently, the self-similar behavior expected when \( P(k) \propto k^n \) on an Einstein–de Sitter background is not spoiled. Such a cutoff was in fact employed by Jain & Bertschinger (1996), and criticized in turn by Scoccimarro & Frieman (1996b), who rejected its ad hoc time dependence. The latter authors suggest that self-similarity breaks down in general in perturbation theory, because the UV cutoff is just a property of the initial conditions. The SSE approach reconciles self-similarity with perturbation theory by realizing that the dynamically generated UV cutoff \( L(t) \propto r_0(t) \) is itself a defining ingredient of the dust model in a hierarchical scenario.

Structures of size \( L(t) \) should behave approximately as ‘particles’ (clusters); hence, \( L(t) \) must be defined through a condition which explicitly tests somehow the degree of ‘dynamical isolation’ of the structure below a given scale. In Sec. 3 it was concluded that \( L \) is different from the nonlinear scale \( r_0 \); then, \( \sigma^2 = 1 \) is not such a tester. Nevertheless, a possible suggestion of how the function \( \sigma(R, t) \) could be useful is the following: Let us assume that, in the case of a scale-invariant initial power spectrum and an Einstein–de Sitter background, \( \sigma(R, t) \) is given by the stable clustering ansatz (Davis & Peebles 1977) in the nonlinear regime \( L < r_0 \). Then, the recently formed clusters, and thus the length \( L \), may be identified by the scale \( R \) at which \( \sigma(R, t) \) starts following the stable clustering prediction. Fig. 3 in (Colombi et al. 1996) and Fig. 1 in (Jain 1997) provide the estimate \( L/r_0 \approx 10^{-1} \). N-body simulations seem to support the stable clustering hypothesis (Elstathiou et al. 1988; Colombi et al. 1996; Jain 1997; Valageas et al. 2000). But this point is not settled yet and there are evidences that it only holds for the recently formed clusters (Munshi et al. 1998; Ma & Fry 2000); even so, however, the previous reasoning would still be functional. The numerical results for the vorticity, Sec. 4, also favor a ratio \( L/r_0 \approx 0.1 \).

This estimate is consistent with the fact that the dust model predictions agree with N-body simulations when the former are not UV-divergent, in some cases also up to the limit \( R \approx r_0 \) (beyond the a priori expected range of validity, \( R \gg r_0 \)) as the example of the skewness in Sec. 3 shows, corrections are then of order \( (L/r_0)^2 \) when pushing \( R \gg r_0 \), and thus possibly unobservable (e.g., the estimated absolute error of the measured skewness is about 0.1 (Juszkiewicz et al. 1993)). On the one hand, this good agreement of the convergent dust-model predictions supports the hypothesis of the mode-decoupling at the basis of the SSE. On the other hand, it suggests that even the dust-model UV-divergent perturbation results may provide good approximations when regularized in the framework of the SSE.

6 CONCLUSIONS AND OUTLOOK

I have explored the application of Eulerian perturbation theory to a recently proposed model of cosmological structure formation, the small-size expansion (SSE), which incorporates corrections to the dust model. The SSE features a length scale \( L(t) \) which is identified with the typical size of the most recently formed virialized structures. The per-
turbation results suggest $L/r_0 \sim 0.1$ with $r_0$ the nonlinear scale.

When the dust-model perturbation results are finite, the corrections are irrelevant. However, the length $L(t)$ works as a natural UV-cutoff which regularizes perturbation expansions in the dust model which would be otherwise divergent. The SSE may also address multistreaming effects perturbationally, which are absent in the dust model. As an example, I have computed the vorticity generated by tidal torques: its relative importance is predicted to diminish when the cosmic flow is observed on increasingly larger scales, and to be significant today on scales $\sim 1\ h^{-1}\text{Mpc}$ with a value $\sim$ Hubble constant.

The SSE provides a new perspective and a possibly useful tool in the analytical understanding of the evolution by gravitational instability. The model based on the SSE must still be certainly subject to further investigation to check its internal consistency and eventual success: the results of this work can be contrasted to N-body simulations. As well as the predictions for the vorticity, it will be particularly interesting to compare regularized perturbation corrections, e.g., for the density variance $\sigma^2$. Finally, another interesting line of future research on the SSE is the application of the Lagrangian perturbation theory: this should help clarify the relationship to the truncated Zel’dovich approximation.

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REFERENCES

Barrow J. D., Saich P., 1993, Class. Quantum Grav. 10, 79
Bernardeau F., 1994, ApJ 433, 1
Bernardeau F., 1996, Proc. XXXIth Moriond meeting, astro-ph/9607004
Bernardeau F., Colombi S., Gaztañaga E., Scocciarino R., 2002, Phys. Rep. (to be published), astro-ph/0112551
Bouchet F. R., Colombi S., Hivon E., Juszkiewicz R., 1993, A&A 293, 641
Bouchet F. R., Colombi S., Hivon E., Juszkiewicz R., 1996, ApJ 419, L9
Buchert T., Domínguez A., 1997, MNRAS 287, 687
Buchert T., 1992, MNRAS 254, 729
Buchert T., Domínguez A., 1998, A&A 335, 395
Buchert T., Domínguez A., Pérez-Mercader, J., 1999, A&A 349, 343
Buchert T., Melott A. L., Weiss A. C., 1994, A&A 288, 349
Coles P., Melott A. L., Shandarin S. F., 1993, MNRAS 260, 675
Colombi S., Bouchet F. R., Hernquist L., 1996, ApJ 465, 14
Davis M., Peebles P. J., 1977, ApJS 34, 425
Dekel A., 1994, ARA&A 32, 371
Domínguez A., 2000, Phys. Rev. D 62, 103501 (Paper I)
Efstathiou G., Frenk C. S., White S. D. M., Davis M., 1988, MNRAS 235, 749
Ehlers J., Buchert T., 1997, Gen. Relativ. Grav. 29, 733
Fosalba P., Gaztañaga E., 1998, MNRAS 301, 535
Gaztañaga E., Baugh C. M., 1995, MNRAS 273, L1
Gurbatov S. N., Saichev A. I., Shandarin S. F., 1989, MNRAS 236, 385
Jain B., 1997, MNRAS 287, 687

APPENDIX A: DERIVATION OF THE SMALL-SIZE EXPANSION (SSE)

To make this work as self-contained as possible, I provide in this App. the derivation in Paper I leading to Eqs. (1). It is however more general in that a time-dependent smoothing length is allowed.

The basic model is a system of nonrelativistic, identical point particles which (i) are assumed to interact with each other via gravity only, (ii) look homogeneously distributed on sufficiently large scales, which thus are assumed to evolve according to an expanding Friedmann-Lemaître cosmological background, and (iii) deviations to homogeneity are relevant only on scales small enough that a Newtonian approximation is valid to follow their evolution. Let $a(t)$ denote the expansion factor of the Friedmann-Lemaître cosmological background, $H(t) = \dot{a}/a$ the associated Hubble function, and $\phi_0(t)$ the homogeneous (background) density on large scales, $x_\alpha$ is the comoving spatial coordinate of the $\alpha$-th particle, $u_\alpha$ is its peculiar velocity, and $m$ its mass. In terms of these variables the evolution is described by the following set of equations ($\nabla_\alpha$ denotes a partial derivative with respect to $x_\alpha$):}

\begin{equation}
\dot{x}_\alpha - \frac{1}{a} u_\alpha = \frac{H}{a} u_\alpha - H u_\alpha, \quad (A1)
\end{equation}

\begin{equation}
\nabla_\alpha \cdot w_\alpha = -4\pi G a \left[ \frac{m}{a^3} \sum_{\beta \neq \alpha} \delta(x_\alpha - x_\beta) - \phi_0 \right], \quad (A2)
\end{equation}

\begin{equation}
\nabla_\alpha \times w_\alpha = 0,
\end{equation}

where $w_\alpha$ is the peculiar gravitational acceleration acting on the $\alpha$-th particle. Finally, Eqs. (A2) must be subjected to periodic boundary conditions in order to yield a Newtonian description consistent with the Friedmann-Lemaître solution at large scales (Ehlers & Buchert 1997).

One can formally define a microscopic mass density field and a microscopic peculiar velocity field,
with the constant \( u \). Coarsening cells of comoving size \( \sim L(t) \) are defined with the help of a smoothing window \( W(z) \) (see Paper I for details on the properties required on the window). The coarse-grained fields associated to the microscopic fields are respectively

\[
\varrho_{\text{mic}}(x, t) = \frac{m}{a(t)^3} \sum_{\alpha} \delta^{(3)}(x - x_\alpha(t)),
\]

\[
\varrho_{\text{mic}}u_{\text{mic}}(x, t) = \frac{m}{a(t)^3} \sum_{\alpha} u_\alpha(t) \delta^{(3)}(x - x_\alpha(t)).
\]  

(A3)

Coarsening cells of comoving size \( \sim L(t) \) are defined with the help of a smoothing window \( W(z) \) (see Paper I for details on the properties required on the window). The coarse-grained fields associated to the microscopic fields are respectively

\[
\varrho(x, t) = \int \frac{dy}{L^3} W \left( \frac{|x - y|}{L} \right) \varrho_{\text{mic}}(y, t),
\]

\[
\varrho u(x, t) = \int \frac{dy}{L^3} W \left( \frac{|x - y|}{L} \right) \varrho_{\text{mic}} u_{\text{mic}}(y, t).
\]  

(A4)

Notice in particular that \( \varrho \) is defined by a volume-average of the peculiar momentum density (i.e., by a mass-average of the velocity) and thus its physical meaning is transparent: \( \varrho \) is the center-of-mass velocity of the coarsening cell. These fields obey the following exact evolution Eqs. (conservation of mass and momentum, following from Eqs. (A1)):

\[
\frac{\partial \varrho}{\partial t} = -3H \varrho - \frac{1}{a} \nabla \cdot (\varrho u) + \frac{i}{L} \nabla \cdot s,
\]

\[
\frac{\partial (\varrho u)}{\partial t} = -4H \varrho u + f - \frac{1}{a} \nabla \cdot P + \frac{i}{L} \nabla \cdot D.
\]  

(A5)

New fields have arisen, whose physical meaning is also clear from its definitions: two vector fields,

\[
s(x, t) = \int \frac{dy}{L^3} W \left( \frac{|x - y|}{L} \right) (y - x) \varrho_{\text{mic}}(y, t),
\]

\[
f(x, t) = \int \frac{dy}{L^3} W \left( \frac{|x - y|}{L} \right) \varrho_{\text{mic}} u_{\text{mic}}(y, t),
\]

and two second-rank tensor fields,

\[
P(x, t) = \int \frac{dy}{L^3} W \left( \frac{|x - y|}{L} \right) \varrho_{\text{mic}} u_{\text{mic}} u_{\text{mic}}(y, t),
\]

\[
D(x, t) = \int \frac{dy}{L^3} W \left( \frac{|x - y|}{L} \right) (y - x) \varrho_{\text{mic}} u_{\text{mic}}(y, t).
\]  

(A6)

(The definition of \( u_{\text{mic}} \) is Eq. (A3) with the replacement \( u_\alpha \to u_\alpha \)). If the evolution Eqs. for these fields are computed, new fields appear, and so on ad infinitum. A physical assumption is required to cut the hierarchy: the non-linear coupling in Eqs. (A5) to the degrees of freedom below the smoothing scale \( L \) is assumed to be weak. To implement this idea mathematically, notice that Eqs. (A4) can be formally inverted to yield the following expansion in \( L \) (this is most easily derived in Fourier space):

\[
\varrho_{\text{mic}} = \left[ 1 - \frac{1}{2} B \left( L \nabla \right)^2 + o(L \nabla)^4 \right] \varrho,
\]

\[
u_{\text{mic}} = \left[ 1 - \frac{1}{2} B \left( L \nabla \right)^2 - BL^3 \nabla \cdot \nabla + o(L \nabla)^4 \right] u,
\]  

(A7)

with the constant

\[
B = \frac{1}{3} \int dz \, z^2 W(z) = \frac{4\pi}{3} \int_0^{+\infty} dz \, z^4 W(z).
\]

These expansions just show how the microscopic fields can be recovered by taking into account higher and higher derivatives of the smoothed fields, i.e., finer and finer details in the spatial distribution. Now consider, e.g., the tensor field \( P \), encompassing nonlinear mode-mode couplings: the weak-coupling assumption above then means that the dominant dynamical contribution of \( P \) to the evolution Eq. (A5) arises from the coupling between modes on scales \( \geq L \). Hence, \( P \) can be replaced in this Eq. by the result of inserting the expansions (A7) in the definition (A6):

\[
P \to \varrho uu + BL^2 g(\partial u)(\partial u) + o(L \nabla)^4.
\]

Applying the same reasoning to the other fields:

\[
f \to \varrho w + BL^2 (\nabla \cdot \nabla)w + o(L \nabla)^4,
\]

\[
\nabla \cdot w = -4\pi G a (\varrho - \varrho_0),
\]

\[
\nabla \times w = 0,
\]

and

\[
s \to BL^2 \nabla \varrho + o(L \nabla)^4,
\]

\[
D \to BL^2 \nabla(\varrho u) + o(L \nabla)^4.
\]  

(A8)

Taking these results in Eqs. (A5), one obtains a formal expansion in \( L \) (called the small-size expansion in the main text). Truncating after the second term, Eqs. (1) are recovered.

The fields \( s \) and \( D \) have a purely geometrical origin and describe how the smoothed fields change simply because the smoothing length is varied. Its diffusion-like expression is thus barely surprising. In fact, truncating the expansions (A8) is exact for a Gaussian window, \( W(z) = \exp(-\pi z^2) \) and \( B = 1/2\pi \). In general, the approximation (A8) is equivalent to replacing the arbitrary window \( W(z) \) by a Gaussian one with a width fixed by the constant \( B \).

**APPENDIX B: VORTICITY VARIANCE**

In this App. I collect the mathematical details of the study of Eq. (18).

We consider first the IR behavior. When \( P(k) \sim k^\alpha \) as \( k \to 0 \), the integrals behave asymptotically in the limit \( q \to 0 \), \( (k/q) \to 0 \) as \( s \equiv \cos \phi \), \( \phi \equiv (k, q) \)

\[
\sim |\Xi(0, t)|^2 \int_0^{k_{\alpha+2}} dq \, q^{n+6} \int_{-1}^{+1} ds \, s^2 (1 - s^2),
\]

which is finite and nonvanishing if \( n > -3 \), independently of the values of \( R \) and \( L \).

To address the UV behavior \( (k, q) \to +\infty \), one makes the change of variable \( q \to p = k - q \). Assuming \( P(k) \sim k^\alpha \) as \( k \to +\infty \), we have asymptotically

\[
\sim \int_0^{+\infty} dp \, p^4 e^{-BL^2 p^2} |\bar{W}(Rp)|^2 \times \int_{-1}^{+1} ds \, s^2 (1 - s^2).
\]

The smoothing window over scale \( R \) renders the \( p \)-integral UV-convergent, no matter the value of \( L \). From the definition of the function \( \Xi \), it follows that \( \Xi(k^2, t) \leq \Xi(0, t) e^{-BL^2 k^2} \) (since \( \bar{L} > 0 \) by assumption), so that the \( k \)-integral is also convergent as long as \( L_m \neq 0 \), as expected. However, in the dust limit \( L \equiv 0 \), the \( k \)-integral converges only if \( n < -3/2 \).
In what follows, the particular case of an Einstein–de Sitter background is considered with the self-similar initial power spectrum

\[ P(k) = A_n r_{in}^{n+3} k^n. \]

where \( r_{in} \equiv r_0(t_{in}) \). A Gaussian window is assumed, \( W(z) = \exp(-\pi z^2) \), and then

\[ A_n = \frac{(2\pi)^{\frac{1-n}{2}}}{\Gamma(\frac{2-n}{2})}. \]

In Eq. (18), three angular integrations can be performed immediately. Then, one exploits the symmetry of the integrand under exchange \( k \leftrightarrow q \) and changes to new variables \( k \rightarrow k/L_t, q \rightarrow kp \):

\[
\sigma_\omega^2 = \frac{2A_n^2 r_{in}^{2(n+3)} L_t^{2(n+5)}}{(2\pi)^3} \int_0^{+\infty} dk k^{2n+9} \int_0^1 dp \, p^{n+2} (1 - p^2)^2 \times \\
\times \int_{-1}^{+1} ds \, s^2 (1 - s^2) \left| \frac{k^2 ps}{L_t^2} \right|^2 e^{-B \left( 1 + \frac{B}{2} s^2 \right)} (1 + p^2 - 2ps)k^2.
\]

Inserting Eq. (B1) in Eq. (11), one finds the linear dust prediction (i.e., up to corrections of order \( (L/R)^2 \) : \( r_0(t) = r_{in} t^{2(n+3)} \)). As remarked in Sec. 4, a constant ratio \( \lambda = L_t/r_0(t) \) is taken. Then, the function \( \Xi \) can be written in terms of the incomplete gamma function. However, it proves a better idea (particularly in order to compute \( \sigma_\omega \) numerically) to perform first the \( k \)-integral, and next, in the integral defining \( \Xi \), Eq. (19), to change variables \( \tau = [r_0(t)/r_{in}] \tau = a_t^{2(n+3)} \tau \). One finally gets:

\[
\sigma_\omega^2 = \frac{\Gamma(n+5)}{2n+3} \left[ \frac{(5n+23)\lambda \tau}{8F(n+\frac{3}{2})} \right]^2 \left[ \frac{r_0(t)}{L_t} \right]^{2(n+3)} \left[ 1 + \left( \frac{R}{L_t} \right)^2 \right]^{-n+5} \times \\
\times \int_{r_{in}/r_0}^{r_{in}/r_{0}^2} d\tau_1 d\tau_2 \left( \tau_1 \tau_2 \right)^{\frac{5(n+3)}{8}} H_n \left( \frac{\tau_1 + \tau_2}{1 + (R/L_t)^2} \right),
\]

with the function

\[ H_n(z) := \int_0^1 dz \, z^{n+2} \sqrt{1 - z^2} \int_{-1}^{+1} ds \, s^2 (1 - s^2) \left( 1 - \frac{2 - y}{2} zs \right)^{-(n+5)}, \]

obtained with another change of variable: \( z := 2p/(1 + p^2) \).

Only the domain \( 0 \leq y \leq 2 \) is interesting. The function may exhibit a singularity as \( y \to 0^+ \). A straightforward study of the convergence of the integrals near the limits \( s = 1, z = 1 \) provides the asymptotic behavior

\[ H_n(y \to 0^+) \sim \begin{cases} 1, & -3 < n < -\frac{3}{2} \\ y^{-n-\frac{3}{2}}, & -\frac{3}{2} < n. \end{cases} \]

With this information, one can easily derive the scaling behavior (20) from Eq. (B2). The long-time limit, \( t \gg t_{in} \), is obtained by setting the lower limit of the \( \tau \)-integrals to zero (no divergence arises with the constraint \( -3 < n \)). In the exponent range \(-3 < n < 4\), the long-time approximation is accurate to the 1% level for times such that \( a_t > 5 \).

The integral (B2) is also well suited for numerical evaluation. In fact, the \( s \)-integral in the function \( H_n \) can be computed analytically. And the introduction of the variable \( u := \tau_1 + \tau_2 \) allows to compute one of the \( \tau \)-integrals analytically too. In the end, the numerical computation reduces to

\[ a_t^{2(n+3)} \tau \]