DISJOINT $n$-AMALGAMATION AND PSEUDOFINITE COUNTABLY CATEGORICAL THEORIES

ALEX KRUCKMAN

ABSTRACT. Disjoint $n$-amalgamation is a condition on a complete first-order theory specifying that certain locally consistent families of types are also globally consistent. In this paper, we show that if a countably categorical theory $T$ admits an expansion with disjoint $n$-amalgamation for all $n$, then $T$ is pseudofinite. All theories which admit an expansion with disjoint $n$-amalgamation for all $n$ are simple, but the method can be extended, using filtrations of Fra"ıssé classes, to show that certain non-simple theories are pseudofinite. As case studies, we examine two generic theories of equivalence relations, $T^{\text{feq}}$ and $T^{\text{CPZ}}$, and show that both are pseudofinite. The theories $T^{\text{feq}}$ and $T^{\text{CPZ}}$ are not simple, but they are NSOP$_1$. This is established here for $T^{\text{CPZ}}$ for the first time.

1. INTRODUCTION

The theory $T_{RG}$ of the random graph (also called the Rado graph) arises naturally in two distinct ways. First, the random graph is the Fra"ıssé limit of the class of all finite graphs $G$: the unique countable ultrahomogeneous graph which embeds a copy of each finite graph. Second, $T_{RG}$ is the almost-sure theory of finite graphs, in the sense of zero-one laws: letting $G(n)$ be the set of (labeled) graphs of size $n$ and $\mu_n$ the uniform measure on $G(n)$, we have

$$\lim_{n \to \infty} \mu_n(\{G \in G(n) \mid G \models \varphi\}) = 1 \iff \varphi \in T_{RG}.$$ 

The latter observation shows that $T_{RG}$ is pseudofinite; that is, every sentence in the theory has a finite model. In fact, the probabilistic argument shows that each sentence $\varphi \in T_{RG}$ has many finite models. For large $n$, most finite graphs of size $n$ satisfy $\varphi$.

The situation is very different for the class $G_\Delta$ of finite triangle-free graphs. $G_\Delta$ has a Fra"ıssé limit, the generic triangle-free graph (also called the Henson graph). $G_\Delta$ also has a zero-one law for the uniform measures $\mu_n$ on $G_\Delta(n)$, but its almost-sure theory diverges from its generic theory. Indeed, Erdös, Kleitman, and Rothschild [14] showed that almost all large finite triangle-free graphs are bipartite, and hence do not contain any cycles of odd length, in contrast to the generic triangle-free graph.

So the probabilistic argument that showed that the theory of the random graph is pseudofinite fails for the generic triangle-free graph. In fact, it is still unknown whether the theory of the generic triangle-free graph is pseudofinite (see [9][11]). This state of affairs suggests the following very general question:

**Question 1.1.** When does a Fra"ıssé limit have a pseudofinite theory?

There are, essentially, two ways to show that a theory $T$ is pseudofinite. First, one can construct a sequence of finite structures which satisfy more and more of $T$. An example in the case of the random graphs is the sequence of Paley graphs: For each prime power $q \equiv 1(\text{mod } 4)$ define a graph with domain the finite field $\mathbb{F}_q$, putting an edge between distinct elements $a$ and $b$ just in case $a - b$ is a square in $\mathbb{F}_q$. Then the theories of the Paley graphs converge to $T_{RG}$ (see [3] for details, and [4] for other explicit constructions).

The second way is via a probabilistic argument. Usually, this amounts to specifying a probability measure $\mu_n$ on the class of $L$-structures of size $n$ for each $n$ (or some other sequence of classes $K(n)$) such that

$$\lim_{n \to \infty} \mu_n(\{A \in K(n) \mid A \models \varphi\}) = 1 \iff \varphi \in T.$$ 

The first method has the advantage of being more explicit, and the constructions may be of combinatorial interest. But the second method tells us something more, assuming that the measures $\mu_n$ are natural enough: not only do the sentences of $T$ have finite models, but most large structures in some class satisfy the sentences in $T$. Of course, the meaning of “natural” is left intentionally vague. For example, the measure $\mu_n$ should not concentrate on the $n^{th}$ element of some explicit sequence! Refining our question,
Question 1.2. When does a Fraïssé limit have a pseudofinite theory for a good probabilistic reason? For example, when is the almost-sure theory for a natural sequence \((K(n), \mu_n)_{n \in \omega}\) of classes of finite structures equipped with probability measures?

An example of a Fraïssé limit which is pseudofinite, but not for a good probabilistic reason, is the vector space \(V\) of countably infinite dimension over a finite field. The finite models of sentences in \(\text{Th}(V)\) are few and far between, existing only in certain finite sizes and unique up to isomorphism in those sizes. This is one of a whole family of examples of a similar character, the smoothly approximable structures studied by Kantor, Liebeck, and Macpherson in [19] and classified by Cherlin and Hrushovski in [10]. Smoothly approximable structures are essentially algebraic: they are coordinatized by certain geometries coming from vector spaces equipped with bilinear forms.

The main purpose of this paper is advance a claim that “combinatorial” Fraïssé limits (in contrast to the algebraic smoothly approximable structures) which are pseudofinite tend to be pseudofinite for a good probabilistic reason, and moreover that this good probabilistic reason tends to rely on a combinatorial condition, disjoint \(n\)-amalgamation, which generalizes the disjoint (or “strong”) amalgamation property for Fraïssé classes with trivial algebraic closure.

The starting point is Theorem 3.10 which shows that for a countably categorical theory, disjoint \(n\)-amalgamation for all \(n\) is a sufficient condition for pseudofiniteness. The hypothesis of disjoint \(n\)-amalgamation for all \(n\) is very strong; however, almost all examples of “combinatorial” countably categorical theories which are known to be pseudofinite either have disjoint \(n\)-amalgamation for all \(n\), or are reducts of theories with disjoint \(n\)-amalgamation for all \(n\). Such theories are simple (in the sense of the model-theoretic dividing line), see Theorem 3.14. The only exceptions (that I am aware of at the time of this writing) are built from equivalence relations.

In [22], Kim and Pillay made the “rather outrageous conjecture” that every pseudofinite countably categorical theory is simple. The generic theory of a parameterized family of equivalence relations, \(T_{\text{eq}}^*\), was suggested by Shelah as a counterexample to this conjecture. However, to my knowledge, no proof that \(T_{\text{eq}}^*\) is pseudofinite has appeared in the literature.

In this paper, I show pseudofiniteness of \(T_{\text{eq}}^*\) (Section 4.2), as well as another generic theory of equivalence relations, \(T_{\text{CPZ}}\) (Section 4.3), which was introduced (and shown to not be simple) by Casanovas, Peláez, and Ziegler [17]. In both cases, the argument relies on a method of filtering the relevant Fraïssé class as a union of simpler Fraïssé classes, each of which admits an expansion to a countably categorical theory with disjoint \(n\)-amalgamation for all \(n\). This shows that the pseudofiniteness of these examples, too, can be viewed as a consequence of a probabilistic argument involving disjoint \(n\)-amalgamation. An interesting feature of this method is that each sentence of the theory is shown to be in the almost-sure theory for a sequence \((K(n), \mu_n)_{n \in \omega}\) of classes of finite structures equipped with probability measures, and hence is pseudofinite for a good probabilistic reason, but different sentences require different sequences.

Countably categorical pseudofinite theories do not have the strict order property. Since I am not aware of a reference for this folklore result, I will give a quick proof here:

Proposition 1.3. No countably categorical pseudofinite theory has the strict order property.

Proof. If \(T\) has the strict order property, then it interprets a partial order with infinite chains. So it suffices to show that no countably categorical partial order \((P, <)\) with infinite chains is pseudofinite.

By compactness, we can find an infinite increasing chain \(\{a_i \mid i \in \mathbb{N}\}\) with \(P \models a_i < a_j\) if and only if \(i < j\). In a countably categorical theory, automorphism-invariant properties are definable, so there is a formula \(\varphi(x)\), with \(\varphi(x) \in \text{tp}(a_i)\) for all \(i\), such that \(P \models \varphi(b)\) if and only if there is an infinite increasing chain above \(b\).

Now \(P \models (\exists x \varphi(x)) \land (\forall x \varphi(x) \rightarrow \exists y (x < y \land \varphi(y)))\), but this sentence cannot hold in any finite structure, since it implies the existence of an infinite increasing chain. \(\square\)

In [13], Džamonja and Shelah introduced the property \(\text{SOP}_1\) (a theory is \(\text{NSOP}_1\) if it does Not have the 1-Strong Order Property). It is the first in a linearly ordered hierarchy of dividing lines \(\text{SOP}_n\) in the complexity of first-order theories, lying between simplicity and the strict order property (simple \(\Rightarrow\) \(\text{NSOP}_1\) \(\Rightarrow\) \(\ldots\) \(\Rightarrow\) \(\text{NSOP}_n\) \(\Rightarrow\) \(\ldots\) \(\Rightarrow\) strict order property). The generic triangle-free graph is \(\text{SOP}_3\) but \(\text{NSOP}_4\) [26]. Chernikov and Ramsey [12] gave a independence relation criterion for \(\text{NSOP}_1\) and used it to show that \(T_{\text{eq}}^*\) is \(\text{NSOP}_1\). The theory \(T_{\text{CPZ}}\) was not considered in [12], but the methods there
also suffice to show that $T_{\text{CPZ}}$ is NSOP$_1$ (see Corollary 4.10). On the other hand, almost nothing is known about pseudofiniteness of countably categorical theories in the region between SOP$_1$ and the strict order property. While acknowledging that we have a paucity of other examples, I think it is reasonable to update the outrageous conjecture of Kim and Pillay in the following way:

**Conjecture 1.4.** Every pseudofinite countably categorical theory is NSOP$_1$.

In Section 2, I review the relevant background on Fraïssé theory. I introduce disjoint $n$-amalgamation in Section 3.1 and prove Theorem 3.10 in Section 3.2. Section 3.3 contains some context about the role of $n$-amalgamation properties in model theory, as well as an explanation of how Theorem 3.10 generalizes and unifies previous work. In Section 4, I introduce the notion of a filtered Fraïssé class and give the applications to generic theories of equivalence relations ($T^*_\text{eq}$ and $T_{\text{CPZ}}$), along with a negative result, Proposition 4.4 showing that this method cannot be used to show that the generic triangle-free graph is pseudofinite.

**Acknowledgements:** I would like to thank Tom Scanlon for his support and for many helpful discussions. Nick Ramsey also had a great influence on this paper: he told me about the theory Conjecture 1.4, and, after some effort, convinced me that SOP$_1$ is a natural dividing line.

## 2. Preliminaries

In this section, I give a brief review of Fraïssé theory. The “canonical language” described in Definition 2.7 provides the bridge to general countably categorical theories. For proofs, see [6] Sections 2.6-8 or [17] Section 7.1.

Let $L$ be a relational language (not necessarily finite), and let $K$ be a class of finite $L$-structures which is closed under isomorphism.

- $K$ has the **hereditary property** if it is closed under substructure.
- $K$ has the **joint embedding property** if for all $A, B \in K$, there exists $C \in K$ and embeddings $A \hookrightarrow C$ and $B \hookrightarrow C$.
- $K$ has the **amalgamation property** (or 2-amalgamation) if for all $A, B, C \in K$ and embeddings $f: A \hookrightarrow B$ and $g: A \hookrightarrow C$, there exists $D \in K$ and embeddings $f': B \hookrightarrow D$ and $g': C \hookrightarrow D$ such that $f' \circ f = g' \circ g$.
- $K$ has the **disjoint amalgamation property** (or disjoint 2-amalgamation) if it has the amalgamation property and, additionally, the images of $B$ and $C$ in $D$ can always be taken to be disjoint over the image of $A$ in $D$: $(f' \circ f)[A] = f'[B] \cap g'[C] = (g' \circ g)[A]$.
- $K$ is a **weak Fraïssé class** if it is countable up to isomorphism and has the hereditary property, the joint embedding property, and the amalgamation property.
- $K$ is a **Fraïssé class** if it is a weak Fraïssé class and additionally $K$ contains only finitely many structures of size $n$ up to isomorphism for all $n \in \omega$.

**Remark 2.1.** My use of terminology here is slightly nonstandard: what I call a weak Fraïssé class is often simply called a Fraïssé class. However, as we are only interested in Fraïssé classes with countably categorical generic theory, it is convenient to include the finiteness condition in the definition. In a finite relational language, the notions coincide. Also, in many sources the disjoint amalgamation property is called the strong amalgamation property.

Let $M$ be a countable $L$-structure.

- The **age** of $M$ is the class of all finite structures which embed in $M$.
- $M$ is **ultrahomogeneous** if every isomorphism between finite substructures of $M$ extends to an automorphism of $M$.
- $M$ has **trivial acl** if $acl(A) = A$ for all $A \subseteq M$.

**Theorem/Definition 2.2.** $K$ is a weak Fraïssé class if and only if there is a countable ultrahomogeneous structure $M_K$ with age $K$. In this case, $M_K$ is unique up to isomorphism and is called the *Fraïssé limit* of $K$. We call $T_K = \text{Th}(M_K)$ the *generic theory* of $K$. $K$ is a (strong) Fraïssé class if and only if $T_K$ is countably categorical. In this case, $T_K$ has quantifier elimination. A Fraïssé class $K$ has the disjoint amalgamation property if and only if $M_K$ has trivial acl.
Given a Fraïssé class $K$ and $n \in \omega$, I will write $K(n)$ for the (finite) set of structures in $K$ with domain $[n] = \{1, \ldots, n\}$. Note that I include the empty structure in the case $n = 0$. It will be convenient to identify these structures with their quantifier-free $n$-types: for $A \in K(n)$,

$$\text{qf-tp}(A) = \{ \varphi(x_1, \ldots, x_n) \mid \varphi \text{ is quantifier-free, and } A \models \varphi(1, \ldots, n) \}.$$ 

Since $T_K$ has quantifier elimination, we can further identify the structures in $K(n)$ with the set of first-order $n$-types over the empty set relative to $T_K$ which are non-redundant, in the sense that they contain the formulas $\{x_i \neq x_j \mid i \neq j\}$.

Now each $n$-type relative to $T_K$ is isolated by a quantifier-free formula. In other words, each structure $A \in K(n)$ is distinguished from the others by a single quantifier-free formula $\theta_A(x_1, \ldots, x_n)$. In the case that $L$ is finite, we may take $\theta_A$ to be the conjunction of the quantifier-free diagram of $A$. If $L$ is infinite, a large enough part of the quantifier-free diagram suffices.

**Theorem 2.3.** In this notation, the generic theory $T_K$ can be explicitly axiomatized as follows:

- The universal theory of $K$: This amounts to the sentences, for $n \in \omega$,

$$\forall x_1, \ldots, x_n \left( \bigwedge_{i \neq j} x_i \neq x_j \right) \rightarrow \left( \bigvee_{A \in K(n)} \theta_A(\overline{x}) \right),$$

with $\theta_A$ determined by how $\theta_A$ determines the other quantifier-free formulas. That is, for each $n$, $A \in K(n)$, and quantifier-free formula $\varphi(\overline{x}) \in \text{qf-tp}(A)$,

$$\forall \overline{x} \theta_A(\overline{x}) \rightarrow \varphi(\overline{x}).$$

- One-point extension axioms: For all $A \in K(n)$ and $B \in K(n+1)$, we say that $(A, B)$ is a one-point extension if $A$ is the induced substructure of $B$ with domain $[n]$. Given a one-point extension $(A, B)$, we have the axiom

$$\forall \overline{x} \exists y \theta_A(\overline{x}) \rightarrow \theta_B(\overline{x}, y).$$

**Definition 2.4.** $T_{K,n}$ is the (incomplete) theory axiomatized by

1. The sentences in the universal theory of $K$ in at most $n$ universal quantifiers.
2. All one-point extension axioms for $K$ (with no restriction on the sizes of $A$ and $B$).

A model of $T_{K,n}$ satisfies all the one-point extension axioms over substructures satisfying the formulas $\theta_A$ for $A \in K$, but its age need only agree with $K$ up to substructures of size at most $n$. We will see in Theorem 3.10 below that basic disjoint amalgamation up to level $n$ implies pseudofiniteness of $T_{K,n}$.

It will be useful to consider expansions of $T_K$ at the level of the Fraïssé class $K$.

**Definition 2.5.** Let $K$ and $K'$ be Fraïssé classes in languages $L$ and $L'$, respectively, such that $L \subseteq L'$. We say that $K'$ is a Fraïssé expansion of $K$ if

1. $K = \{ A \mid L \mid A \in K' \}$
2. For all one-point extensions $(A, B)$ in $K$, and every expansion of $A$ to a structure $A'$ in $K'$, there is an expansion of $B$ to a structure $B'$ in $K'$ such that $(A', B')$ is a one-point extension.

**Theorem 2.6.** $K'$ is an expansion of $K$ if and only if the Fraïssé limit $M_{K'}$ of $K'$ is an expansion of the Fraïssé limit $M_K$ of $K$.

**Proof.** Suppose that $M_{K'} \models L = M_K$. Then $K = \text{Age}(M_K) = \{ A \mid L \mid A \in \text{Age}(M_{K'}) \}$, and $\text{Age}(M_{K'}) = K'$. Given a one-point extension $(A, B)$ and an expansion $A'$ of $A$, we can find a substructure of $M_{K'}$ isomorphic to $A'$. In the reduct, this substructure is isomorphic to $A$, and, since the appropriate extension axiom is true of $M_K$, it extends to a copy of $B$. We can take $B'$ to be the $L'$-structure on this subset of $M_{K'}$.

Conversely, to show that $M_{K'}$ is an expansion of $M_K$, by countable categoricity it suffices to show that $M_{K'} \models L$ satisfies the theory $T_K$. It clearly satisfies the universal part, since $\text{Age}(M_{K'} \models L) = \{ A \mid L \mid A \in K' \} = K$. For the extension axioms, suppose $(A, B)$ is a one-point extension, and we have a copy of $A$ in $M_{K'} \models L$. Let $A'$ be the $L'$-structure on this subset of $M_{K'}$. Since $K'$ is a Fraïssé expansion of $K$, we can find a $B'$ such that $(A', B')$ is a one-point extension, and, since the appropriate extension axioms is true of $M_{K'}$, our copy of $A'$ extends to a copy of $B'$. Hence, in the reduct, our copy of $A$ extends to a copy of $B$. \( \square \)
**Definition 2.7.** Let $T$ be any countably categorical $L$-theory, and let $M$ be its unique countable model. The canonical language for $T$ is the language $L'$ with one $n$-ary relation symbol $R_p$ for each $n$-type $p(\bar{x})$ realized in $M$.

We make $M$ into an $L'$-structure $M'$ in the natural way by setting $M' \models R_p(\bar{x})$ if and only if $\bar{x}$ realizes $p(\bar{x})$ in $M$. Let $T' = \text{Th}_{L'}(M')$. Then $T$ and $T'$ are interdefinable, $M'$ is ultrahomogeneous, and hence is the Frasé limit of its age $K$, and $K$ has the disjoint amalgamation property if and only if $M$ has trivial acl. Note that for each $A \in K(n)$, we may take the isolating formula $\theta_A$ to be one of the basic $n$-ary relation symbols $R_p$.

3. Disjoint $n$-amalgamation

3.1. Definitions. To fix notation, $[n] = \{1, \ldots, n\}$, $\mathcal{P}([n])$ is the powerset of $[n]$, and $\mathcal{P}^-(n)$ is the set of all proper subsets of $[n]$. A family $\mathcal{F} \subseteq \mathcal{P}([n])$ of subsets of $[n]$ is downwards closed if $S' \subseteq S$ whenever $S' \subseteq S$ and $S \in \mathcal{F}$.

Let $T$ be a theory and $A$ a set of parameters in a model of $T$. We say that a type $p(\bar{x})$ over $A$ in the variables $\{x_i \mid i \in I\}$ is non-redundant if it contains the formulas $\{x_i \neq x_j \mid i \neq j \in I\}$ and $\{x_i \neq a \mid i \in I, a \in A\}$. Given a downwards closed family of subsets $\mathcal{F} \subseteq \mathcal{P}([n])$, and variables $\bar{x}_1, \ldots, \bar{x}_n$, a coherent $\mathcal{F}$-family of types over $A$ is a set $\{p_S \mid S \in \mathcal{F}\}$ such that each $p_S$ is a non-redundant type over $A$ in the variables $\bar{x}_i = (\bar{x}_i \mid i \in S)$, and $p_S \subseteq p_{S'}$ when $S' \subseteq S$. Here each $\bar{x}_i$ is a tuple of variables, possible empty or infinite, but such that $\bar{x}_i$ is disjoint from $\bar{x}_j$ when $i \neq j$.

For $n \geq 2$, a disjoint $n$-amalgamation problem is a coherent $\mathcal{P}^-(n)$-family of types over a set $A$. A basic disjoint $n$-amalgamation problem is a disjoint $n$-amalgamation problem over the empty set in the singleton variables $x_1, \ldots, x_n$.

A solution to a (basic) disjoint $n$-amalgamation problem is an extension of the coherent $\mathcal{P}^-(n)$-family of types to a coherent $\mathcal{P}([n])$-family of types; that is, a non-redundant type $p\in \mathcal{P}([n])$ such that $p_S \subseteq p_{[n]}$ for all $S$. We say $T$ has (basic) disjoint $n$-amalgamation if every (basic) $n$-amalgamation problem has a solution.

If we replace $\mathcal{P}^-(n)$ by another downwards closed family of subsets $\mathcal{F}$ in the definitions above, we call the amalgamation problem partial.

First, some remarks on the definitions:

**Remark 3.1.** In any coherent $\mathcal{F}$-family of types over $A$, the type $p_0$ is a $0$-type in the empty tuple of variables - i.e., it simply specifies the elementary diagram of the parameters $A$.

**Remark 3.2.** To specify a disjoint $n$-amalgamation problem, it would be sufficient to give the types $p_S$ for all $S$ with $|S| = n - 1$ and check that they agree on intersections, in the sense that $p_S \mid \bar{x}_{S \cap S'} = p_{S'} \mid \bar{x}_{S \cap S'}$ for all $S$ and $S'$. However, it is sometimes notationally convenient to keep the intermediate stages around.

**Remark 3.3.** A Frasé class $K$ has the disjoint amalgamation property if and only if $T_K$ has disjoint 2-amalgamation. Indeed, given $A, B, C \in K$ and embeddings $f : A \rightarrow B$ and $g : A \rightarrow C$, we take $A$ to be the base set of parameters, so $p_0 = \text{qf-tp}(A)$, and we set $p_{[1]}(\bar{x}_1) = \text{qf-tp}(B \setminus A)/A$ and $p_{[2]}(\bar{x}_2) = \text{qf-tp}(C \setminus A)/A$, identifying $A$ with its images in $B$ and $C$ under $f$ and $g$. By quantifier elimination, these quantifier-free types determine complete types relative to $T_K$. A solution to this disjoint 2-amalgamation problem is the same as a structure $D$ in $K$ into which $B$ and $C$ embed disjointly over the image of $A$.

**Remark 3.4.** In a Frasé class $K$, we have identified $K(n)$, the structures in $K$ with domain $[n]$, with the set of non-redundant quantifier-free $n$-types relative to $T_K$. For $n \geq 2$, a basic disjoint $n$-amalgamation problem relative to $T_K$ is a $\mathcal{P}^-(n)$-family of quantifier-free types $P = \{p_S \mid S \in \mathcal{P}^-(S)\}$ in the variables $x_1, \ldots, x_n$, where each type $p_S$ is the quantifier-free type of a structure in $K$ of size $|S|$. We write $K(n, P) = \{p_{[n]}(x_1, \ldots, x_n) \in K(n) \mid p_S \subseteq p_{[n]}\}$ for the set of solutions to the amalgamation problem $P$, each of which is the quantifier-free type of a structure in $K$ of size $n$. Then to say that $T_K$ has basic disjoint $n$-amalgamation is to say that $K(n, P)$ is nonempty for all $P$.

It will be useful to observe that disjoint amalgamation gives solutions to partial amalgamation problems as well.

**Lemma 3.5.** Suppose that $T$ has (basic) disjoint $k$-amalgamation for all $2 \leq k \leq n$. Then every partial (basic) disjoint $n$-amalgamation problem has a solution.
Proposition 3.6. We are given a partial disjoint $n$-amalgamation problem over $A$ in variables $\overline{x}_1, \ldots, \overline{x}_n$; that is, a coherent $\mathcal{F}$-family of types $\{p_S \mid S \subseteq \mathcal{F}\}$, with $\mathcal{F} \subseteq \mathcal{P}^-(|[n]|)$ downwards closed.

We build a solution to the partial $n$-amalgamation problem from the bottom up. By induction on $1 \leq k \leq n$, I claim that we can extend this family to a coherent $\mathcal{F}_k$-family of types, where $\mathcal{F}_k = \mathcal{F} \cup \{S \subseteq [n] \mid |S| \leq k\}$. When $k = n$, we have a coherent $\mathcal{P}([-n])$-family of types, as desired.

When $k = 1$, if there is any $i$ such that $i \notin S$ for all $S \in \mathcal{F}$, then the original $\mathcal{F}$-family of types says nothing about the variables $\overline{x}_i$. We add $\{i\}$ into $\mathcal{F}_1$ and choose any non-redundant type $p_{\{i\}}$ over $A$ in the variables $\overline{x}_i$. If $\emptyset \notin \mathcal{F}$ (which only happens if $\mathcal{F}$ is empty) we also add it into $\mathcal{F}_1$, along with the unique 0-type $p_\emptyset$ containing the elementary diagram of $A$.

Given a coherent $\mathcal{F}_k$-family of types by induction, with $2 \leq k \leq n$, we wish to extend to a coherent $\mathcal{F}_k$-family of types. If there is any set $S \subseteq [n]$ with $|S| = k$ such that $S \notin \mathcal{F}_{k-1}$, then all proper subsets of $S$ are in $\mathcal{F}_{k-1}$. Hence we have types $\{p_R \mid R \in \mathcal{P}^-(S)\}$ which form a coherent $\mathcal{P}^-(S)$-family. Using $k$-amalgamation, we can find a non-redundant type $p_S$ in the variables $\overline{x}_S$ extending the types $p_R$. Doing this for all such $S$ gives a coherent $\mathcal{F}_k$-family of types, as desired.

Disjoint $n$-amalgamation is apparently more general and natural, but it is basic disjoint $n$-amalgamation which is relevant in the proof of Theorem 3.10 We are largely interested in theories with disjoint $n$-amalgamation for all $n$, and in this case the two notions agree.

Proposition 3.6. $T$ has disjoint $n$-amalgamation for all $n$ if and only if $T$ has basic disjoint $n$-amalgamation for all $n$.

Proof. One direction is clear, since basic disjoint $n$-amalgamation is a special case of disjoint $n$-amalgamation.

In the other direction, note first that there is a solution to the $n$-amalgamation problem $\{p_S \mid S \subseteq \mathcal{P}^-(|[n]|)\}$ if and only if the partial type

$$\bigcup_{S \subseteq \mathcal{P}^-(|[n]|)} p_S(\overline{x}_S) \cup \bigcup_{x \neq x'} x \neq x'$$

is consistent (actually, we could omit the formulas asserting non-redundancy when $n > 2$). Hence, by compactness, we can reduce to the case that $A$ is finite, and each tuple of variables $\overline{x}_i$ is finite.

Let $N = |A| + \sum_{i=1}^n |\overline{x}_i|$, where $|\overline{x}_i|$ is the length of the tuple $\overline{x}_i$. Introduce variables $y_1, \ldots, y_N$, where $y_1, \ldots, y_{|A|}$ enumerate $A$ and the remaining variables relabel the $x$ variables. Now each type $p_S$ over $A$ determines a type in some subset of the $y$ variables, by replacing the parameters from $A$ and the $x$ variables by the appropriate $y$ variables. Closing downward under restriction to smaller sets of variables, we obtain a partial disjoint $N$-amalgamation problem over the empty set in the singleton variables $y_1, \ldots, y_N$. By Lemma 3.3 and basic disjoint $N$-amalgamation, this partial amalgamation problem has a solution, a type $p_{\emptyset_N}(y_1, \ldots, y_N)$ over the empty set. Once again replacing the $y$ variables with the original parameters from $A$ and $x$ variables, we obtain a type $p_{\emptyset}|_S$ over $A$ which is a solution to the original $n$-amalgamation problem.

Example 3.7. The class $\mathcal{G}_\Delta$ of triangle-free graphs does not have disjoint 3-amalgamation: the non-redundant 2-types determined by $x_1 E x_2$, $x_2 E x_3$, and $x_1 E x_3$ cannot be amalgamated. Generalizing, let $K_n^k$ be the class of $K_n$-free $k$-hypergraphs: the language consists of a single $k$-ary relation $R(x_1, \ldots, x_k)$, and the structures in $K_n^k$ are hypergraphs (so $R$ is symmetric and antireflexive) containing no complete hypergraph on $n$ vertices. Note that $\mathcal{G}_\Delta$ is $K_3^2$.

For $n > k$, $K_n^k$ satisfies basic disjoint $m$-amalgamation for $m < n$, but fails basic disjoint $n$-amalgamation, since the first forbidden configuration has size $n$. However, $K_n^k$ already fails disjoint $(k+1)$-amalgamation. Over a base set $A$ consisting of a complete hypergraph on $(n-k-1)$ vertices, the $k$-type which describes, together with $A$, a complete hypergraph on $(n-1)$ vertices is consistent, but $(k+1)$ copies of it cannot be amalgamated.

Example 3.8. There are countably categorical theories which do not have disjoint $n$-amalgamation for all $n$, but which admit countably categorical expansions with disjoint $n$-amalgamation for all $n$.

As a simple example, consider the theory of a single equivalence relation with $k$ infinite classes. Transitivity is a failure of disjoint 3-amalgamation: the non-redundant 2-types determined by $x_1 E x_2$, $x_2 E x_3$, and $\neg x_1 E x_3$ cannot be amalgamated. But if we expand the language by adding $k$ new unary relations $C_1, \ldots, C_k$ in such a way that each class is named by one of the $C_i$, the resulting theory has disjoint $n$-amalgamation for all $n$. 
For a more interesting example, the random graph (which is easily seen to have disjoint \( n \)-amalgamation for all \( n \)) in its canonical language has a reduct to a ultrahomogeneous 3-hypergraph, where the relation \( R(a, b, c) \) holds if and only if there are an odd number of the three possible edges between \( a, b, \) and \( c \). This structure turns out to be ultrahomogeneous in the language \( \{ R \} \), and its age is the class of all finite 3-hypergraphs with the property that on any four vertices \( a, b, c, \) and \( d \), there are an even number of the four possible 3-edges. Hence this class fails to have disjoint 4-amalgamation. For more information on this example, see [24], where it is called the homogeneous two-graph. More examples of this kind can be found in the literature on reducts of homogeneous structures, e.g. [27].

3.2. Pseudofiniteness.

**Definition 3.9.** A theory \( T \) is **pseudofinite** if for every sentence \( \varphi \) such that \( T \models \varphi \), \( \varphi \) has a finite model.

**Theorem 3.10.** Below is stated in a fine-grained way: amalgamation just up to level \( n \) gives pseudofiniteness of the theory \( T_{K,n} \) (see Definition 2.4). The proof involves a probabilistic construction of a structure of size \( N \) for each \( N \) “from the bottom up”. This is the same idea as in the proof of Lemma 3.9, but there we could fix an arbitrary \( k \)-type extending a given coherent family of \( l \)-types for \( l < k \). Here we introduce randomness by choosing an extension uniformly at random.

The probabilistic calculation is essentially the same as the one used in the classical proofs of the zero-one laws for graphs and general \( L \)-structures (see [17, Lemma 7.4.6]). The key point is that the amalgamation properties allow us to make all choices as independently as possible: the quantifier-free types assigned to subsets \( A \) and \( B \) of \( [N] \) are independent when conditioned on the quantifier-free type assigned to \( A \cap B \). It is this independence which makes the calculation go through.

Formally, we construct a probability measure on the space \( L[N] \) of \( L \)-structures with domain \( [N] \). Given a formula \( \varphi(\bar{x}) \) and a tuple \( \bar{a} \) from \( [N] \), we write \( \{ M \in L[N] \mid M \models \varphi(\bar{a}) \} \) for each \( l \)-tuple \( \bar{a} \). We choose the quantifier-free 1-type of \( \{ i \} \) uniformly at random from \( K(1) \). Now proceed inductively: Having assigned quantifier-free \( l \)-types to all subsets of size \( l \) with \( l < k \), we wish to assign quantifier-free \( k \)-types. For each \( k \)-tuple \( i_1, \ldots, i_k \) of distinct elements from \( [N] \), let \( P = \{ p_S \mid S \in P^-(\{\bar{i}\}) \} \) be the collection of quantifier-free types assigned to all proper substructures \( ps(\bar{a}s) = qf-tp(\{i_j \mid j \in S\}) \). If \( T_K \) has basic disjoint \( k \)-amalgamation, \( M(k, P) \) is nonempty and finite, and we may choose the quantifier-free \( k \)-type of \( i_1, \ldots, i_k \) uniformly at random from \( M(k, P) \).

Now if \( T_K \) has basic disjoint \( k \)-amalgamation for all \( k \), we can continue this construction all the way up to \( k = N \), so that the resulting structure \( M_N \) is in \( K(N) \). Call this the unbounded case. On the other hand, if \( T_K \) has basic disjoint \( k \)-amalgamation only for \( k \leq n \), then we stop at \( k = n \). To complete the construction, we assign any remaining relations completely freely at random. That is, for each relation \( R \) (of arity \( r > n \)) and \( r \)-tuple \( i_1, \ldots, i_r \) containing at least \( n + 1 \) distinct elements, we set \( R(i_1, \ldots, i_r) \) with probability \( \frac{1}{2} \). The result is an \( L \)-structure \( M_N \) which may not be in \( K \), but the induced structures of size at most \( n \) are guaranteed to be in \( K \). Call this the bounded case.

I claim that if \( \varphi \) is one of the axioms of \( T_{K,n} \) (in the bounded case) or \( T_K \) (in the unbounded case), then

\[
\lim_{N \to \infty} \mu_N([\varphi]) = 1.
\]

Each universal axiom \( \varphi \) has the form \( \forall x_1, \ldots, x_k \psi(\bar{x}) \) (with \( k \leq n \) in the bounded case), where \( \psi \) is quantifier-free and true on all tuples from structures in \( K \). Since all substructures of our random structure of size \( k \) are in \( K \), \( \varphi \) is always satisfied by \( M_N \), and so \( \mu_N([\varphi]) = 1 \) for all \( N \).

Now suppose \( \varphi \) is the one-point extension axiom \( \forall \bar{x} \exists y \theta_A(\bar{x}) \to \theta_B(\bar{x}, y) \). Let \( \bar{a} \) be a tuple of \( |A| \)-many distinct elements from \( [N] \) and \( b \) any other element. Conditioning on the event that \( M_n \models \theta_A(\bar{a}) \), there is a positive probability \( \varepsilon \) that \( M_N \models \theta_B(\bar{a}, b) \).

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Indeed, in the unbounded case, or when $|A| < n$ in the bounded case, $\theta_B$ specifies the quantifier-free $|B|$-type of the tuple $\vec{b}$ among those allowed by $K$. There is a positive probability ($\frac{1}{|K(k,P)|}$) that the correct 1-type is assigned to $b$, and, given that the correct $l$-type has been assigned to all subtuples of $\vec{b}$ of length $l < k$, there is a positive probability ($\frac{1}{|K(k,P)|}$) for the appropriate family $P$ that the correct $k$-type is assigned to a given subtuple of length $k$. Then $\varepsilon$ is the product of all these probabilities for $1 \leq k \leq |B|$. When $|A| \geq n$ in the unbounded case, the above reasoning applies for the subtuples of $\vec{b}$ of length at most $n$. On longer tuples, since $\theta_B$ only mentions finitely many relations, and the truth values of these relations are assigned freely at random, there is some additional positive probability that these will be decided in a way satisfying $\theta_B$ (at least $\frac{m}{m}$, where $m$ is the minimum number of additional instances of relations which need to be decided positively or negatively to ensure satisfaction of $\theta_B$).

Moreover, for distinct elements $b$ and $b'$, the events that $\vec{b}$ and $\vec{b}'$ satisfy $\theta_B$ are conditionally independent, since the quantifier-free types of tuples involving elements from $\vec{a}$ and $b$ but not $b'$ are decided independently from those of tuples involving elements from $\vec{a}$ and $b'$ but not $b$, conditioned on the quantifier-free type assigned to $\vec{a}$.

Now we compute the probability that $\varphi$ is not satisfied by $M_N$. Conditioned on the event that $M_N \models \theta_A(\vec{a})$, the probability that $M_N \not\models \exists y \theta_B(\vec{a}, y)$ is $(1 - \varepsilon)^{|A| - |\varphi|}$, since there are $N - |A|$ choices for the element $b$, each with independent probability $(1 - \varepsilon)$ of failing to satisfy $\theta_B$. Removing the conditioning, the probability that $M_N \not\models \exists y \theta_A(\vec{a}) \rightarrow \theta_B(\vec{a}, y)$ for any given $\vec{a}$ is at most $(1 - \varepsilon)^{|A| - |\varphi|}$, since the formula is vacuously satisfied when $\vec{a}$ does not satisfy $\theta_A$. Finally, there are $N^{||A|}$ possible tuples $\vec{a}$, so the probability that $M_N \not\models \forall \vec{a} \theta_A(\vec{a}) \rightarrow \theta_B(\vec{a}, y)$ is at most $N^{|A|}(1 - \varepsilon)^{|A| - |\varphi|}$. Since $|A|$ is constant, the exponential decay dominates the polynomial growth, and $\lim_{N \rightarrow \infty} \mu_N(|\neg \varphi|) = 0$, so $\lim_{N \rightarrow \infty} \mu_N(|\varphi|) = 1$.

To conclude, any sentence $\psi \in T_{K,n}$ is a logical consequence of finitely many of the axioms $\varphi_1, \ldots, \varphi_m$ considered above. We need only pick $N$ large enough so that $\mu_N([\varphi_i]) > 1 - \frac{1}{m}$ for all $i$. Then $\mu_N([\bigwedge_{i=1}^m \varphi_i]) > 0$, so the conjunction $\bigwedge_{i=1}^n \varphi_i$, and hence also $\psi$, has a model of size $N$. In the unbounded case, our construction ensures that this model is in $K$.

Corollary 3.11. Any countably categorical theory $T$ with disjoint $n$-amalgamation for all $n \geq 2$ is pseudofinite.

Proof. Let $T'$ be the equivalent of $T$ in the canonical language. Then it suffices to show that $T'$ is pseudofinite, since pseudofiniteness is preserved under interdefinability. But $T'$ is the generic theory for a Fraïssé class with basic disjoint $n$ amalgamation for all $n$, so by Theorem 3.10, it is pseudofinite.

Remark 3.12. Since pseudofiniteness is preserved under reduct, the examples described in Example 3.8 are pseudofinite.

3.3. Relationship to other notions. The notion of $n$-amalgamation has been studied in other model-theoretic contexts, usually in the form of independent $n$-amalgamation. Given some notion of independence, $\perp$, the main example being nonforking independence in a simple theory, an independent $n$-amalgamation problem is given by a coherent $P^-(\{n\})$-family of types over $A$, with the non-redundancy condition replaced by the condition that any realization $\{\vec{a}_i \mid i \in S\}$ of $p_S(\vec{a}_S)$ is an independent set over $A$ with respect to $\perp$.

In the case $n = 3$, independent 3-amalgamation over models is often called the independence theorem. It is a well-known theorem of Kim and Pillay that the independence theorem, along with a few other natural properties, characterizes forking in simple theories.

Theorem 3.13 (21, Theorem 4.2). Let $T$ be a complete theory and $\perp_A$ a ternary relation, written $a \perp_A B$, where $a$ is a finite tuple and $A$ and $B$ are sets. As usual, all tuples and sets come from some highly saturated model of $T$. Suppose that $\perp_A$ satisfies the following properties:

1. (Invariance) If $a \perp_A B$ and $tp(a' A'B') = tp(aAB)$, then $a' \perp_A B'$.
2. (Local character) For all $a, B$, there is $A \subseteq B$ such that $|A| \leq |T|$ and $a \perp_A B$.
3. (Finite character) $a \perp_A B$ if and only if for every finite tuple $b$ from $B$, $a \perp_A Ab$.
4. (Extension) For all $a$, $A$, and $B$, there is $a'$ such that $tp(a'/A) = tp(a/A)$ and $a' \perp_A B$.
5. (Symmetry) If $a \perp_A Ab$, then $b \perp_A Aa$.
6. (Transitivity) If $A \subseteq B \subseteq C$, then $a \perp_A B$ and $a \perp_B C$ if and only if $a \perp_A C$. 

1. (Independence theorem) Let $M \models T$ be a model, $a$ and $a'$ tuples such that $tp(a/M) = tp(a'/M)$ and $A$ and $B$ sets. If $A \downarrow_M B$, $a \downarrow_M A$, and $a' \downarrow_M B$, then there exists $a''$ such that $tp(Aa''/M) = tp(Aa/M)$, $tp(Ba''/M) = tp(Ba'/M)$, and $a'' \downarrow_M AB$.

Then $T$ is simple, and $\downarrow$ is nonforking ($\downarrow = \downarrow'$).

Disjoint $n$-amalgamation is a strong form of independent amalgamation, where the relevant independence relation is the disjointness relation $\dashv$, defined by $A \dashv C$ if and only if $A \cap B \subseteq C$. We say a theory has trivial forking if $\downarrow = \downarrow'$.

**Theorem 3.14.** A countably categorical theory $T$ with disjoint 2-amalgamation (i.e. trivial acl) and disjoint 3-amalgamation is simple with trivial forking.

**Proof.** We can use Theorem 3.13 to show that $\downarrow = \downarrow''$. Most of the conditions are straightforward to check, so I’ll only remark on a few of them. For local character, we can take $A = a \cap B$, so $A$ is finite and $a \dashv_A B$. For extension, we find $a'$ by realizing the type $tp(a/A) \cup \{a_i \neq b | a_i \text{ from } a \text{ and } b \in B\}$. This is consistent by trivial acl and compactness. Finally, for the independence theorem, we apply disjoint 3-amalgamation to amalgamate the three 2-types $p_{(12)} = tp(aA/M)$, $p_{(13)} = tp(a'B/M)$, $p_{(23)} = tp(AB/M)$ (first removing any redundant elements of $M$ from $a$, $a'$, $A$, and $B$). \hfill \Box

**Remark 3.15.** A consequence of Theorem 3.14 is that the class of triangle-free graphs $\mathcal{G}_\Delta$ does not admit an expansion to a class with disjoint $n$-amalgamation for all $n$, since its generic theory is not simple [20].

**Remark 3.16.** Motivated by the fact that many examples of simple theories (such as $T_{RG}$ and ACFA [8]) satisfy independent $n$-amalgamation for $n \geq 3$, Kolesnikov [23] and Kim, Kolesnikov, and Tsuboi [20] developed a hierarchy of notions of $n$-simplicity for $1 \leq n \leq \omega$, where 1-simplicity coincides with simplicity. If a countably categorical theory $T$ has disjoint $k$-amalgamation for all $2 \leq k \leq n$, then it is $(n-2)$-simple with trivial forking, and if it has disjoint $n$-amalgamation for all $n$, then it is $\omega$-simple.

In the context of abstract elementary classes, independent $n$-amalgamation of models goes by the name *excellence* (see [2], for example). In the context of stable theories, Goodrick, Kim, and Kolesnikov have uncovered a connection between existence and uniqueness of independent $n$-amalgamation and definable polygroups [16], generalizing earlier work of Hrushovski on 3-amalgamation [18].

The observation that disjoint $n$-amalgamation is sufficient for pseudofiniteness generalizes and unifies a number of earlier observations. I will note a few here:

- Oberschelp [25] identified an unusual syntactic condition which is sufficient for the almost-sure theory of a class of finite structures under the uniform measures to agree with its generic theory. A universal sentence is called *parametric* if it is of the form $\forall x_1, \ldots, x_n (\bigwedge_{i \neq x} x_i \neq x_j) \rightarrow \varphi(\bar{\tau})$ where $\varphi$ is a Boolean combination of atomic formulas $R(y_1, \ldots, y_m)$ such that each variable $x_i$ appears among the $y_j$. For example, reflexivity $\forall x R(x, x)$ and symmetry $\forall x, y R(x, y) \rightarrow R(y, x)$ are parametric conditions, while transitivity $\forall x, y, z (R(x, y) \land R(y, z)) \rightarrow R(x, z)$ is not a parametric condition, since each atomic formula appearing only involves two of the three quantified variables. A *parametric class* is the class of finite models of a set of parametric axioms.

- Any parametric class has $n$-amalgamation for all $n$. It is easiest to see this by checking basic disjoint $n$-amalgamation: The restrictions imposed by a parametric theory on the relations involving non-redundant $n$-tuples and $m$-tuples are totally independent when $n \neq m$.

- In their work on the random simplicial complex, Brooke-Taylor & Testa [5] introduced the notion of a local Fraïssé class and showed that the generic theory of a local class is pseudofinite, by methods similar to those in the proof of Theorem 5.10. A universal sentence is called *local* if it is of the form $\forall x_1, \ldots, x_n (R(x_1, \ldots, x_n) \rightarrow \psi(\bar{\tau}))$, where $R$ is a relation in the language and $\psi$ is quantifier-free. A *local class* is the class of finite models of a set of local axioms.

- Again, any local class has $n$-amalgamation for all $n$. A local theory only imposes restrictions on tuples which satisfy some relation. So disjoint $n$-amalgamation problems can be solved “freely” by simply not adding any further relations.

- In [1], Ahlman has investigated countably categorical theories in a binary relational language (one with no relation symbols of arity greater than 2) which are simple with SU-rank 1 and trivial pregeometry. In the case when acl$^a(\emptyset) = \emptyset$, this agrees with what I call a simple theory with trivial forking ($\downarrow = \downarrow'$) above.
Ahlman shows that in such a theory $T$ there is a $0$-definable equivalence relation $\xi$ with finitely many infinite classes such that $T$ can be axiomatized by certain “$(\xi, \Delta)$-extension properties” describing the possible relationships between elements in different classes. Further, he shows that these theories are pseudofinite. The definition of $(\xi, \Delta)$-extension property is somewhat technical, so I will not give it here. Instead I will note that the condition implies that $T$ has an expansion (obtained by naming the finitely many classes of $\xi$) with $n$-amalgamation for all $n$. The fact that the language is binary ensures that describing the possible relationships between pairs of elements suffices.

4. Generic theories of equivalence relations

4.1. Filtered Fraïssé classes. We will extend the disjoint $n$-amalgamation argument for pseudofiniteness to certain non-simple theories, using the notion of a filtered Fraïssé class.

Definition 4.1. A Fraïssé class $K$ is filtered by a chain $K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots$ if each $K_n$ is a Fraïssé class, and $\bigcup_{n \in \omega} K_n = K$.

Theorem 4.2. Let $K$ be a Fraïssé class filtered by $\{K_n \mid n \in \omega\}$. Then $\varphi \in T_K$ if and only if $\varphi \in T_{K_n}$ for all sufficiently large $n$.

Proof. It suffices to check for each of the axioms of $T_K$ given in Theorem 2.3. Since each $K_n$ is a subclass of $K$, every universal sentence in $T_K$ is also in $T_{K_n}$. Let $(A, B)$ be a one-point expansion with corresponding axiom $\varphi$. For large enough $n$, the structures $A$ and $B$ are in $K_n$, so $(A, B)$ is also a one-point expansion in $K_n$, and $\varphi \in T_{K_n}$. 

Pseudofiniteness is preserved in filtered Fraïssé classes.

Corollary 4.3. If a Fraïssé class $K$ is filtered by $\{K_n \mid n \in \omega\}$ and each generic theory $T_{K_n}$ is pseudofinite, then the generic theory $T_K$ is pseudofinite.

Proof. Each sentence $\varphi$ in $T_K$ is also in $T_{K_n}$ for sufficiently large $n$, and hence $\varphi$ has a finite model. □

As a consequence, if $K$ is filtered by $\{K_n \mid n \in \omega\}$, and each $K_n$ admits a Fraïssé expansion with disjoint $n$-amalgamation for all $n$, then $K$ is pseudofinite. This argument is used in the next two sections to establish pseudofiniteness of the theories $T^{\ast}_{\text{eq}}$ and $T_{\text{CPZ}}$.

It is worth noting that this method cannot be used to show that the theory of the generic triangle-free graph is pseudofinite. Let $G_1$, $G_2$, and $G_3$ be the graphs on three vertices with a single edge, two edges, and three edges respectively. For any filtration $\{K_n \mid n \in \omega\}$ of the Fraïssé class $G_\Delta$ of triangle-free graphs, some $K_n$ must include the graphs $G_1$ and $G_2$ but not $G_3$. But Proposition 4.4 shows that such a class does not admit a Fraïssé expansion with disjoint $n$-amalgamation for all $n$.

Proposition 4.4. Let $K$ be a Fraïssé class consisting of graphs (in the language with a single edge relation $E$), and suppose that $K$ contains the graphs $G_1$ and $G_2$ but not $G_3$. Then no Fraïssé expansion of $K$ has disjoint 2-amalgamation and disjoint 3-amalgamation.

Proof. Suppose for contradiction that $K$ has a Fraïssé expansion $K'$ in the language $L'$ with disjoint 2-amalgamation and disjoint 3-amalgamation. Let $p(x)$ be any quantifier-free 1-type in $K'$, then by disjoint 2-amalgamation we can find some quantifier-free 2-type $q(x, y)$ in $K'$ such that $q(x, y) \models p(x) \land p(y) \land x \neq y$.

Now, letting $p_0$ be the unique quantifier-free 0-type in $K'$, the family of types $\{p_0, p(x), p(y), p(z), q(x, y), q(y, z), q(x, z)\}$ is a basic disjoint 3-amalgamation problem for $K'$. Then we must have $q(x, y) \models \neg x Ey$, for otherwise the reduct to $L$ of any solution to the 3-amalgamation problem would be a copy of $G_3$ in $K$.

Let $H$ be the graph on two vertices, $v_1$ and $v_2$, with no edge. Note that $H$ is in $K$. Labeling the vertices of $G_1$ by $v_1$, $v_2$, $v_3$, so that the unique edge is $v_2 Ev_3$, $G_1$ is a one-point extension of $H$. Now $H$ admits an expansion to a structure in $K'$ (described by $q(v_1, v_2)$) in which both vertices $v_1$ and $v_2$ have quantifier-free type $p$, so since $K'$ is a Fraïssé expansion of $K$, $G_1$ admits a compatible expansion to a structure in $K'$, call it $G'_1$. Let $p'(y) = qf-tp_{G'_1}(v_1)$, and let $q_i(x, y) = qf-tp_{G'_1}(v_i, v_3)$ for $i = 1, 2$. Note that we have $q_i(x, y) \models p(x) \land p'(y)$ for $i = 1, 2$, but $q_1(x, y) \models \neg x Ey$, while $q_2(x, y) \models x Ey$. That is, the pair of 1-types $p(x)$ and $p'(y)$ are consistent with both $x Ey$ and $\neg x Ey$. We will use this situation to build a triangle.

Again, labeling the vertices of $G_2$ by $v_1$, $v_2$, $v_3$, so that $\neg v_1 Ev_2$, $G_2$ is a one-point extension of $H$. Since $H$ admits an expansion to a structure $H'$ in $K'$ so $H' \models q_1(v_1, v_2)$, $G_2$ admits a compatible expansion to
a structure \( G'_2 \) in \( K' \). Let \( p'' = \text{qf-tp}_{G'_2}(v_3) \), \( r_1(x,z) = \text{qf-tp}_{G'_2}(v_1,v_3) \), and \( r_2(y,z) = \text{qf-tp}_{G'_2}(v_2,v_3) \). Note that \( r_1(x,z) = p(x) \land p''(z) \land xEz \) and \( r_2(y,z) = p'(x) \land p''(z) \land yEz \).

Now the family of types \( \{p_0,p(x),p''(z),q_2(x,y),r_1(x,z),r_2(y,z)\} \) is a basic disjoint 3-amalgamation problem for \( K' \). But the reduct to \( L \) of any solution is a copy of \( G'_3 \) in \( K \).

\[ \square \]

4.2. The theory \( T_{\text{feq}} \). Let \( L \) be the language with two sorts, \( O \) and \( P \) (for “objects” and “parameters”), and a ternary relation \( E_x(y,z) \), where \( x \) is a variable of sort \( P \) and \( y \) and \( z \) are variables of sort \( O \). Then \( K_{\text{feq}} \) is the class of finite \( L \)-structures with the property that for all \( a \) of sort \( P \), \( E_a(y,z) \) is an equivalence relation on \( O \).

\( K_{\text{feq}} \) is a Fraïssé class. We define \( T_{\text{feq}} \) to be the generic theory of \( K_{\text{feq}} \). Our aim is to show that it is pseudofinite. Before giving the details of the proof, I will describe the simple idea: filter the class \( K_{\text{feq}} \) by the subclasses \( K_n \) in which each equivalence relation in the parameterized family has at most \( n \) classes. Expand these classes by parameterized predicates naming each class. The resulting class has \( n \)-amalgamation for all \( n \), and hence has pseudofinite generic theory.

**Theorem 4.5.** \( T_{\text{feq}} \) is pseudofinite.

**Proof.** For \( n \geq 1 \), let \( K_n \) be the subclass of \( K_{\text{feq}} \) consisting of those structures with the property that for all \( a \) of sort \( P \), the equivalence relation \( E_a \) has at most \( n \) classes. Let’s check that \( K_n \) is a Fraïssé class.

It clearly has the hereditary property. For the disjoint amalgamation property, suppose we have embeddings \( f: A \to B \) and \( g: A \to C \) of structures in \( K_n \). We specify a structure \( D \) with domain \( A \cup (B \setminus f[A]) \cup (C \setminus g[A]) \) into which \( B \) and \( C \) embed in the obvious way over \( A \). That is, for each parameter \( a \) in \( P(D) \), we must specify an equivalence relation on \( O(D) \). If \( a \) is in \( P(A) \), it already defines equivalence relations on \( B \) and \( C \). First, number the \( E_a \)-equivalence classes in \( A \) by 1,...,\( l \). Then, if there are further unnumbered \( E_a \)-classes in \( B \) and \( C \), number them by \( l+1,\ldots,m_B \) and \( l+1,\ldots,m_C \) respectively. Note that \( m_B,m_C \leq n \). Now define \( E_a \) in \( O(D) \) to have \( \text{max}(m_B,m_C) \) classes by merging the classes assigned the same number in the obvious way. The situation is even simpler if \( a \) is not in \( P(A) \). Say without loss of generality it is in \( P(B) \). Then we can extend \( E_a \) to \( O(C) \) by adding all elements of \( O(C \setminus g[A]) \) to a single existing \( E_a \)-class. The joint embedding property follows from the amalgamation property by taking \( A \) to be the empty structure.

Since for every structure in \( K_{\text{feq}} \), there is some finite bound on the number of equivalence classes for any equivalence relation in the parameterized family, \( K_{\text{feq}} = \bigcup_{n=1}^\infty K_n \). So \( K_{\text{feq}} \) is a filtered Fraïssé class, and by Corollary \ref{cor:filtered} it suffices to show that each \( T_{K_n} \) is pseudofinite.

Let \( L'_n \) be the expanded language which includes, in addition to the relation \( E \), \( n \) binary relation symbols \( C_1(x,y),\ldots,C_n(x,y) \), where \( x \) is a variable of sort \( P \) and \( y \) is a variable of sort \( O \). Let \( K'_n \) be the class of finite \( L'_n \)-structures which are expansions of structures in \( K_n \) such that for all \( a \) of sort \( P \), each of the \( E_a \)-classes is picked by one of the formulas \( C_i(a,y) \).

We need to check that \( K'_n \) is a Fraïssé expansion of \( K_n \). Certainly we have \( K_n = \{ A \mid L \mid A \in K'_n \} \), since every structure in \( K_n \) can be expanded to one in \( K'_n \) by labeling the classes for each equivalence relation. Suppose now that \( (A,B) \) is a one-point extension in \( K_n \), and \( A' \) is an expansion of \( A \) to a structure in \( K'_n \). If the new element \( b \in B \) is in \( P(B) \), then it defines a new equivalence relation \( E_b \) on \( O(A) = O(B) \), and we can expand \( B \) to \( B' \) in \( K'_n \) by labeling the \( E_b \) classes arbitrarily. On the other hand, suppose \( b \) is in \( O(B) \). Then for each parameter \( a \), either \( b \) is an existing \( E_a \)-class labeled by \( C_i(a,y) \), in which case we set \( C_i(a,b) \), or \( b \) is in a new \( E_a \)-class, in which case we set \( C_j(a,b) \) for some unused \( C_j \).

Finally, note that \( T_{K'_n} \) has disjoint 2-amalgamation, since it is a Fraïssé class with the disjoint amalgamation property. I claim that it also has disjoint \( n \)-amalgamation for all \( n \geq 3 \). Indeed, the behavior of the ternary relation \( E_x(y,z) \) is entirely determined by the behavior of the binary relations \( C_i(x,y) \), and an \( L'_n \)-structure \( (P(A),O(A)) \) is in \( K'_n \) if and only if for every \( a \) in \( P(A) \) and \( b \) in \( O(a) \), \( C_i(a,b) \) holds for exactly one \( i \). So any inconsistency is already ruled out at the level of the 2-types. Since every pair of variables is contained in one of the types in a coherent \( P^-([n]) \)-family of types for \( n \geq 3 \), every disjoint \( n \)-amalgamation problem has a solution.

So \( T_{K'_n} \) has disjoint \( n \)-amalgamation for all \( n \), and hence it and its reduct \( T_{K_n} \) are pseudofinite by Theorem \ref{thm:feq}. \( \square \)
A natural question is whether $T_{\text{feq}}^*$ is, in fact, the almost-sure theory for the class $K_{\text{feq}}$ for the uniform measures. It is not, as the following proposition shows. Of course, since we have described $K_{\text{feq}}$ as a two-sorted language, there is some ambiguity as to what we mean by the uniform measures. For maximum generality, let us fix two increasing functions $f, g : \omega \to \omega$. For $n \in \omega$, let $K_{\text{feq}}(f(n), g(n))$ be the structures in $K_{\text{feq}}$ with object sort of size $f(n)$ and parameter sort of size $g(n)$, and let $\mu_{f(n), g(n)}$ be the uniform measure on $K_{\text{feq}}(f(n), g(n))$.

**Proposition 4.6.** There is a sentence $\varphi$ in $T_{\text{feq}}^*$ such that

$$\lim_{n \to \infty} \mu_{f(n), g(n)}(\{A \in K_{\text{feq}}(f(n), g(n)) \mid A \vDash \varphi\}) = 0.$$  

**Proof.** An example of such a sentence $\varphi$ is

$$\forall (x : P) \forall (x' : P) \forall (y : O) \forall (y' : O) \exists (z : O) ((y \neq y') \rightarrow E_x(y, z) \land E_{x'}(y', z)),$$

which expresses that any two equivalence classes for distinct equivalence relations intersect. $\varphi$ is in $T_{\text{feq}}^*$, since for any structure in $K_{\text{feq}}$ with parameters $a \neq a'$ and objects $b, b'$ (possibly $b = b'$), we can add a new object element $c$ which is $E_a$-equivalent to $b$ and $E_{a'}$-equivalent to $b'$, so $\varphi$ is implied by the relevant one-point extension axioms.

I will sketch the asymptotics: the measure $\mu_{f(n), g(n)}$ on amongs picking $g(n)$ equivalence relations on a set of size $f(n)$ uniformly and independently. The expected number of equivalence classes in an equivalence relation on a set of size $n$, chosen uniformly, grows asymptotically as $\frac{n}{\log(n)} (1 + o(1))$ [13 Proposition VIII.8]. Thus, most of the $g(n)$ equivalence relations have equivalence classes which are much smaller (with average size approximately $\log(n)$) than the number of classes, and the probability that every $E_a$-class is large enough to intersect every $E_b$-class nontrivially is for all distinct $a$ and $b$ limits to 0.

Proposition [13] shows that $T_{\text{feq}}^*$ is not the almost-sure theory of $K_{\text{feq}}$ for the measures $\mu_{f(n), g(n)}$, but it would be interesting to know whether such an almost-sure theory exists.

**Question 4.7.** Does the class $K_{\text{feq}}$ have a first-order zero-one law for the measures $\mu_{f(n), g(n)}$? If so, does the almost-sure theory depend on the relative growth-rates of $f$ and $g$?

### 4.3. The theory $T_{\text{CPZ}}$

Let $L$ be the language with a symbol $E_n(\emptyset, \emptyset)$ of arity $2n$ for all $n \geq 1$. Then $K_{\text{CPZ}}$ is the class of finite $L$-structures with the property that $E_n$ is an equivalence relation on $n$-tuples for all $n$, and there is a single $E_n$-equivalence class consisting of all $n$-tuples which do not consist of $n$ distinct elements.

$K_{\text{CPZ}}$ is a Fraïssé class. We define $T_{\text{CPZ}}$ to be the generic theory of $K_{\text{CPZ}}$. In [7], Casanovas, Peláez, and Ziegler introduced the theory $T_{\text{CPZ}}$ and showed that it is NSOP$_2$ and not simple. For completeness, I will show how to combine the “independence lemma” from [7] with the 3-amalgamation criterion due to Chernikov and Ramsey [12] to show that $T_{\text{CPZ}}$ is, in fact, NSOP$_1$.

We write $\downarrow^c$ for coher independence: given a model $M$ and tuples $a$ and $b$, $a \downarrow^c_M b$ if and only if $tp(a/Mb)$ is finitely satisfiable in $M$; that is, for every formula $\varphi(x, m, b) \in tp(a/Mb)$, there exists $m' \in M$ such that $\models \varphi(m', m, b)$.

**Theorem 4.8** ([12], Theorem 5.7). $T$ is NSOP$_1$ if and only if for every $M \models T$ and $b_0c_0 \equiv_M b_1c_1$ such that $c_1 \downarrow^c_M c_0$, $c_0 \downarrow^c_M b_0$, and $c_1 \downarrow^c_M b_1$, there exists $b$ such that $b_0c_0 \equiv_M bc_0 \equiv_M bc_1 \equiv_M b_1c_1$.

For our purposes, the reader can take the independent 3-amalgamation condition in Theorem [12] as the definition of NSOP$_1$. For the original definition and further discussion of this property, see [12] or [13].

**Lemma 4.9** ([7], Lemma 4.2). Let $a, b, c, d', d''$ be tuples and $F$ a finite set from a model $M \models T_{\text{CPZ}}$. Assume that $a$ and $c$ have only elements of $F$ in common ($a \vdash^c_F c$). If $d' a \equiv_F d'b \equiv_F d'' b \equiv_F d'' c$, then there exists $d$ such that $d'a \equiv_F d'a \equiv_F d'' c$.

**Corollary 4.10.** $T_{\text{CPZ}}$ is NSOP$_1$.

**Proof.** Suppose we are given $M \models T_{\text{CPZ}}$ and $d'a \equiv_M d'' c$ such that $c \downarrow^c_M a$, $a \downarrow^c_M d'$, and $c \downarrow^c_M d''$. Let $p(x, y) = tp(d'a/M) = tp(d'' c/M)$. To verify the condition in Theorem [12] we need to show that $p(x, a) \cup p(x, c)$ is consistent.

Suppose it is inconsistent. Then there is some finite subset $F \subseteq M$ such that letting $q(x, y) = tp(d'a/F) = tp(d'' c/F)$, $q(x, a) \cup q(x, c)$ is inconsistent. Since $c \downarrow^c_M a$, we certainly have $c \downarrow^c_M a$. By increasing $F$,
we may assume that $c \models_F a$. By countable categoricity, $q$ is isolated by a single formula over $F$, and $q(d',y) \in \text{tp}(a/\text{Md}')$, so by finite satisfiability there exists $b$ in $M$ such that $\models q(d',b)$. Since $d' \equiv_M d''$, we also have $\models q(d'',b)$.

Now the assumptions of Lemma 4.9 are satisfied, and we can find $d$ such that $\models q(d,a) \land q(d,c)$, which contracts the assumption of inconsistency.

Now we return to pseudofiniteness of $T_{\text{CPZ}}$. The strategy is the same as in Section 12, filter the Fraïssé class $K_{\text{CPZ}}$ by bounding the number of equivalence classes, and expand to a class with disjoint $n$-amalgamation for all $n$ by naming the classes.

**Theorem 4.11.** $T_{\text{CPZ}}$ is pseudofinite.

*Proof.* For $n \geq 1$, let $K_n$ be the subclass of $K_{\text{CPZ}}$ consisting of those structures with the property that for all $k$, the equivalence relation $E_k$ has at most $n$ classes, in addition to the class of redundant tuples.

$K_n$ has the hereditary property, and the joint embedding property follows from the amalgamation property by taking $A$ to be the empty structure. For the disjoint amalgamation property, we wish to amalgamate embeddings $f: A \rightarrow B$ and $g: A \rightarrow C$ of structures in $K_n$. We specify a structure $D$ with domain $A \cup (B \setminus f[A]) \cup (C \setminus g[A])$ into which $B$ and $C$ embed in the obvious way over $A$. Since the relations $E_k$ are independent, we can do this separately for each. Make sure to put all redundant $k$-tuples into the $E_k$-class reserved for them, number the $E_k$-classes which intersect $A$ nontrivially, then go on to number the classes which just appear in $B$ and $C$, and merge those classes which are assigned the same number.

Since for every structure in $K_{\text{CPZ}}$, there is some finite bound on the number of equivalence classes for any equivalence relation, $K_{\text{eq}} = \bigcup_{n=1}^{\infty} K_n$. So $K_{\text{CPZ}}$ is a filtered Fraïssé class, and by Corollary 4.3 it suffices to show that each $T_{K_n}$ is pseudofinite.

Let $L'_n$ be the expanded language which includes, in addition to the relations $E_k$, $(n+1)$-ary relation symbols $C^1_k(\overline{x}), \ldots, C^n_k(\overline{x})$ for each $k$. Let $K'_n$ be the class of finite $L'_n$-structures which are expansions of structures in $K_n$ such that for all $k$, each $E_k$-class is picked out by one of the $C^i_k$, with the class of redundant tuples picked out by $C^n_k$.

We have $K_n = \{ A \mid L \mid A \in K'_n \}$, since every structure in $K_n$ can be expanded to one in $K'_n$ by labeling the classes for each equivalence relation. Suppose now that $(A,B)$ is a one-point extension in $K_n$, and $A'$ is an expansion of $A$ to a structure in $K'_n$. If any $k$-tuple involving the new element $b$ is part of a class which exists in $A$, we label it by the appropriate $C^i_k$. If adding the new element adds new $E_k$-classes, we simply label these classes by unused $C^1_k$ (by the bound $n$ on the number of classes, there will always be enough of the $C^1_k$). So $K'_n$ is a Fraïssé expansion of $K_n$.

It remains to show that $T_{K'_n}$ has disjoint $n$-amalgamation for all $n$. Suppose we have a coherent $\mathcal{P}^{-\{[n]\}}$-family of types. As noted before, the relations $E_k$ are independent, so we can handle them each separately. And the behavior of $E_k$ is entirely determined by the behavior of the relations $C^i_k$, so it suffices to set these. But the only restriction here is that every $k$-tuple should satisfy exactly one $C^i_k$, and it should be $C^n_k$ if and only if the tuple is redundant. So to solve our amalgamation problem, we simply assign relations from the $C^i_k$ arbitrarily to those non-redundant $k$-tuples which are not already determined by the types in the family.

Hence $T_{K'_n}$ has disjoint $n$-amalgamation for all $n$, so it and its reduct $T_{K_n}$ are pseudofinite.

**Proposition 4.12.** There is a sentence $\varphi$ in $T_{\text{CPZ}}$ such that

$$\lim_{n \rightarrow \infty} \mu_n(\{ A \in K_{\text{CPZ}}(n) \mid A \models \varphi \}) = 0.$$

*Proof.* An example of such a sentence $\varphi$ is $\forall x \forall y \forall y' \exists z E_1(x,z) \land E_2(y,y';x,z)$, which as usual is implied by the relevant one-point extension axioms. This sentence says that for all $x$, the function $\rho_x$ mapping an element $z$ in the $E_1$-class of $x$ to the $E_2$-class of $xz$ is surjective onto the $E_2$-classes.

The measure $\mu_n$ amounts to picking an equivalence relation on the $k$-tuples of distinct elements from a set of size $n$ for each $k$ uniformly and independently. Since our sentence only involves $E_1$ and $E_2$, we just need consider the equivalence relations on 1-tuples (there are $n$ of them) and the non-redundant 2-tuples (of which there are $n^2 - n$). Citing again the fact that the expected number of equivalence classes in a random equivalence relation grows asymptotically as $\frac{n}{\log(n)}(1 + o(1))$ [13 Proposition VIII.8], we see that with high probability there are more $E_2$-classes $(\frac{n^2}{\log(n^2 - n)}(1 + o(1)))$ than the size of the average $E_1$-class $(\log(n))$, in which case the function $\rho_x$ is not surjective for all $x$, and the probability that $\varphi$ is satisfied limits to 0.

□
In this case, too, it would be interesting to know whether there is a zero-one law for the uniform measures.

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