Backward Filtering of Images by Iterating Forward Filters

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(Received October 1, 2015, revised December 1, 2015)

Abstract: Given a filtering operation for images, we consider a problem for inverting it. If the algorithm of the original, i.e. forward filter is explicitly expressed, its inversion is tractable. However, if its algorithm is unknown exactly, it is hard to be reversed. We present, in this paper, a simple solution for this problem only by iterating the forward filter. This method can be used for a wide class of image filters including ones built in photo-retouch softwares. Examples are shown for smoothing, sharpening, half-toning and resizing filters.

Keywords: Inverse filter, Iterative solution method, Contractive mapping theorem, Built-in filter

1. Introduction

Major classes of image filtering operation are smoothing and sharpening. The smoothing filter includes Gaussian, bilateral[1], non-local means[2], median filters and so on. Typical example of sharpening filter is the unsharp masking. In this paper, we consider the inverse of these filters. Except for linear filters such as the Gaussian filter, their inversion is not so simple. Furthermore, many filters implemented in general-purpose photo-retouch softwares such as the Photshop and the Paintshop are executed by built-in programs of which algorithm is unknown for users. This closedness inhibits their inversion.

A method[3] has been presented for inverting some filters such as the bilateral filter. We, in this paper, present a similar simple method for general image filters and analyzed the behavior of the method. In this paper, a scalar is denoted by a lower-case letter, vectors by thick lower-case letters and matrices by upper-case letters.

2. Fundamental Theory

Let \( x \) be a vector of pixel values in an image and \( y \) be a vector outputted by a filter as is illustrated in Fig. 1 where we call the ordinary filter “forward filter” in contrast to its reverse called “backward filter”. The vectors \( x \) and \( y \) satisfy an equation \( a(x, y) = 0 \) where \( 0 \) is the zero vector. In this paper, we classify \( a(x, y) \) to the following three classes:

- **type 1:** \( a(x, y) \) is nonlinear with respect to both of \( x \) and \( y \).
- **type 2:** \( a(x, y) = y - b(x) \), i.e. linear with respect to \( y \). The function \( b(x) \) takes nonnegative values because \( y \) is pixel values.
- **type 3:** \( a(x, y) = x - c(y) \) where \( c(y) \geq 0 \).

2.1 Iterative Methods

Iterative methods are commonly used for solving nonlinear equations. Iterations are generally expressed by

\[
\begin{align*}
\xi^{(k+1)} &= f(\xi^{(k)}) \\
\end{align*}
\]

where \( \xi \) is the iteration counter \( \xi = 0, 1, 2, \ldots \) and \( x^{(k)} \) is called \( \xi \)th iterant. General theorems for their convergence are the Liapunov’s theorem and the contractive mapping theorem[4], in which the latter one is mainly concerned in this paper except for the mode filter[5] for which the Liapunov’s theorem is resorted.

We use simple iterative methods for solving the equation \( a(x, y) = 0 \) as follows.

**type 1:**

\[
\text{forward filter} : \quad \text{we convert } a(x, y) = 0 \text{ to } y = y - ka(x, y) \text{ where } k > 0. \text{ The iteration is written by}
\]

\[
y^{(k+1)} = y^{(k)} - ka(x, y^{(k)})
\]

**backward filter**:

We convert \( a(x, y) = 0 \) to \( x = x + ka(x, y) \) for which iteration is

\[
x^{(k+1)} = x^{(k)} + ka(x^{(k)}, y)
\]

**type 2:**

**forward filter**:

The equation \( a(x, y) = y - b(x) = 0 \) leads to \( y = b(x) \) which needs no iteration.

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backward filter:

We convert \( b(x) = y \) to \( x = x - k[b(x) - y] \) and solve it with the iteration

\[
x^{(k+1)} = x^{(k)} - k[b(x^{(k)}) - y]
\]

\( k \) is a constant.

\( x^{(k)} \) decreases monotonically\[4\], i.e. \( d(x^{(k)}) \) is the Liapunov’s function for this iteration.

3. Smoothing Filters

Let a value of pixel \((i, j)\) in an input image be \( x_{ij} \), and an output of a filter be \( y_{ij} \). Arrays of these pixel values are \( x = [x_{ij}] \) and \( y = [y_{ij}] \). Some filters are derived from a statistical estimation as

\[
y_{ij} = \arg \min_{y_{ij}} d(x, y_{ij})
\]

3.1 Mode Filter

In the mode filter\[5\], the objective function in eq.(4) is

\[
d(x, y_{ij}) = -\sum_{i, j} \sum_{p, m = -p} s_{lm} e^{-|x_{ij} - x_{ij, j+m}|^2}
\]

3.2 Bilateral Filter

In the bilateral filter (BF)[1], the objective function in eq.(4) is

\[
d(x, y_{ij}) = \sum_{i, j} \sum_{p, m = -p} s_{lm} e^{-|x_{ij} - x_{ij, j+m}|^2} (y_{ij} - x_{ij, j+m})^2
\]

from which

\[
\alpha(x, y_{ij}) = \frac{d(x, y_{ij})}{s_{ij}} = \sum_{i, j} \sum_{p, m = -p} s_{lm} e^{-|x_{ij} - x_{ij, j+m}|^2} (y_{ij} - x_{ij, j+m})^2
\]

\[
\beta(x, y_{ij}) = \sum_{i, j} \sum_{p, m = -p} s_{lm} e^{-|x_{ij} - x_{ij, j+m}|^2}
\]

\[
y_{ij}^* = \frac{\sum_{i, j} \sum_{p, m = -p} s_{lm} e^{-|x_{ij} - x_{ij, j+m}|^2} x_{ij, j+m}}{\sum_{i, j} \sum_{p, m = -p} s_{lm} e^{-|x_{ij} - x_{ij, j+m}|^2}}
\]

This \( \alpha(x, y_{ij}) \) is nonlinear with respect to both of \( x \) and \( y_{ij} \), hence this mode filter is of type 1 mentioned above.

forward filter:

If we set \( k = 1 \), eq.(1) leads to

\[
y_{ij}^{(k+1)} = \frac{\sum_{i, j} \sum_{p, m = -p} s_{lm} e^{-|y_{ij}^{(k)} - x_{ij, j+m}|^2} x_{ij, j+m}}{\sum_{i, j} \sum_{p, m = -p} s_{lm} e^{-|y_{ij}^{(k)} - x_{ij, j+m}|^2}}
\]

which converges for arbitral initial values \( y_{ij}^{(0)} \) because \( d(x, y_{ij}^{(k)}) \) decreases monotonically\[4\], i.e. \( d(x, y_{ij}^{(k)}) \) is the Liapunov’s function for this iteration.

backward filter:

Similarly, by setting \( k = 1 \), eq.(2) becomes

\[
x_{ij}^{(k+1)} = x_{ij}^{(k)} + \frac{\sum_{i, j} \sum_{p, m = -p} s_{lm} e^{-|b(y_{ij}^{(k)}) - x_{ij, j+m}|^2} x_{ij, j+m}}{\sum_{i, j} \sum_{p, m = -p} s_{lm} e^{-|b(y_{ij}^{(k)}) - x_{ij, j+m}|^2}}
\]

which is written in a vector form as

\[
x^{(k+1)} = x^{(k)} + \frac{\sum_{i, j} \sum_{p, m = -p} s_{lm} e^{-|y_{ij}^{(k)} - x_{ij, j+m}|^2} x_{ij, j+m}}{\sum_{i, j} \sum_{p, m = -p} s_{lm} e^{-|y_{ij}^{(k)} - x_{ij, j+m}|^2}} + y_{ij}
\]

If we set \( k = 1 \), eq.(3) leads to

\[
x_{ij}^{(k+1)} = x_{ij}^{(k)} + \frac{\sum_{i, j} \sum_{p, m = -p} s_{lm} e^{-|y_{ij}^{(k)} - x_{ij, j+m}|^2} x_{ij, j+m}}{\sum_{i, j} \sum_{p, m = -p} s_{lm} e^{-|y_{ij}^{(k)} - x_{ij, j+m}|^2}} + y_{ij}
\]

which converges because its right hand side is a contractive function of \( x^{(k)} \). This convergence is ensured for an arbitral initial value, while we set \( x_{ij}^{(0)} = y_{ij} \) for avoiding wasteful computation.
Regularized Nonlinear Diffusion Filter

The regularized nonlinear diffusion filter (RNDF)[8] is defined by

\[ d(x_{ij}, y) = (x_{ij} - y_{ij})^2 - \phi \sum_{l = -p}^{p} \sum_{m = -p}^{p} s_{lm} e^{-\beta(y_{ij} - y_{ij+l+m})^2} \] (16)

from which we get

\[ a(x_{ij}, y) = \partial d(x_{ij}, y)/\partial y_{ij} \]
\[ = y_{ij} - x_{ij} + \beta \phi \sum_{l = -p}^{p} \sum_{m = -p}^{p} s_{lm}(y_{ij} - y_{ij+l+m}) e^{-\beta(y_{ij} - y_{ij+l+m})^2} \] (17)

which is linear with respect to \( x_{ij} \), hence this RNDF is of type 3.

**Forward filter**

From eq.(17), \( a(x_{ij}, y) = 0 \) leads to the iteration

\[ x_{ij}^{(e+1)} = x_{ij}^{(e)} - \beta \phi \sum_{l = -p}^{p} \sum_{m = -p}^{p} s_{lm} e^{-\beta(y_{ij} - y_{ij+l+m})^2} \]
\[ y_{ij}^{(e+1)} = \frac{y_{ij}^{(e)} + \beta \phi \sum_{l = -p}^{p} \sum_{m = -p}^{p} s_{lm} e^{-\beta(y_{ij} - y_{ij+l+m})^2}}{1 + \beta \phi \sum_{l = -p}^{p} \sum_{m = -p}^{p} s_{lm} e^{-\beta(y_{ij} - y_{ij+l+m})^2}} \] (18)

where \( \sum_{l, m} \) means the sum for \( l \) and \( m \) skipping the center pixel where \( l = 0 \) and \( m = 0 \). An output of this RNDF for image in Fig. 3(a) is shown in Fig. 3(b) where we set \( p = 5, \alpha = 0.01, \beta = 0.01, \phi = 0.02 \).

**Backward filter**

From \( a(x, y_{ij}) = 0 \) in eq.(17), we obtain

\[ x_{ij} = y_{ij} + \beta \phi \sum_{l = -p}^{p} \sum_{m = -p}^{p} s_{lm}(y_{ij} - y_{ij+l+m}) e^{-\beta(y_{ij} - y_{ij+l+m})^2} \] (19)

by which we can get the image in Fig. 3(a) from Fig. 3(b). Equation (19) is a sharpening filter called

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Figure 5: Inversion of anisotropic smoothing filter.

Figure 6: Inversion of image enhancer.

3.5 Finally Converged Value

The iteration converges for all of these filters. The iterant $x^{(t)}$ almost converges to the original $x$, thus the filters can be inverted. In some highly nonlinear filters, however, the final iterant $x^{(oo)}$ deviates from the original $x$. This is because such filters remove some components in $x$ completely. If we decompose $x$ into a robust signal component $x_r$ and a fragile component $x_f$ as $x = x_r + x_f$, then $b(x_f) = 0$ hence $b(x) = b(x_r + x_f) = b(x_r)$. The iteration for inverting the forward filter $y = b(x)$ converges to $x^{(oo)} = x_r$. An image with addition to 30% random noise to Fig. 3(a) is shown in Fig. 6(a) for which output of the median filter of 5 x 5 window is shown in Fig. 6(b). Noise in Fig. 6(a) is almost removed while some textures also disappear. Iterants are shown in Fig. 6(c),(d),(e) where the lost textures are gradually restored while noise still remains removed. Thus this iteration method is useful for removal of noise from images while maintaining textures.

4. Sharpening Filters

As is seen above, the coefficient $k$ in the iterations in section 2.1 can be safely set to 1 for ensuring their convergence for smoothing filters. This is intrinsically owing to the contractiveness of these filters. Sharpening filters, however, does not satisfy this condition as is explained in this section. Typical sharpening filter is the unsharp masking (UM), e.g. if we write eq.(11) in a vector form as $y = H(x)x$, the unsharp masking derived from it is expressed by

$$y = x + \mu[x - H(x)x] = [(1 + \mu)I - \mu H(x)]x$$

with $\mu > 0$. The eigenvalue of $(1 + \mu)I - \mu H(x)$ is $1 + \mu - \mu \lambda$ where $\lambda$ is the eigenvalue of $H(x)$. This eigenvalue $1 + \mu - \mu \lambda$ exceeds 1 because $0 < \lambda \leq 1$. This eigenvalue larger than 1 means that this filter magnifies high frequency components in $x$, i.e. UM is a sharpening filter. This UM is of type 2...
Inverting Filters by Iteration

Figure 8: Inverse halftoning.

Figure 9: Inverse resizing.

Figure 10: Iterative regularization.

Various sharpening filters are also included in photo-retouch software. The magnification factor is unknown for those built-in filters, for such filters $k$ should be safely set to a small value, typically $k = 0.5$.

An example of a foggy image as $x$ is shown in Fig. 7(a) and its enhanced output $y$ by an online software “One-Shot-Enhancer” in the webpage http://www.picturetopeople.org is shown in Fig. 7(b). Iterants $x^{(i)}$ are shown in Fig. 7(c) and (d) where we set $k = 0.5$.

5. Relaxed Iterative Methods

The above smoothing and sharpening are fundamental processing of images, hence its inversion is not so difficult. However, if filters are more advanced, the iteration formula should be extended.

5.1 Quantizing Filters

By quantizing $x$ to discrete levels, $y$ takes discrete values, extremely only two levels, black and white. For such quantization, halftoning techniques such as the error diffusion are used. Inversion of such quantizing filters becomes an integer programming (IP) problem which is hard to solve. We approximate it by a nonlinear programming (NP) problem by smoothing the discrete image $y$.

Let a quantizing operation be denoted by $y = q(x)$ which is rewritten as $q(x) - y = 0$. We relax this equation to $s(q(x) - y) = 0$ where $s(\cdot)$ is a smoothing filter. Next we convert $s(q(x) - y) = 0$ to $x = x - k[s(q(x) - y)]$. The iteration

$$x^{i+1} = x^i - k[s(q(x^i) - y)]$$

needs no complex computation demanded for many inverse halftoning methods[10][11] even without explicit programming by utilizing a photo retouch software. We experimented this iteration for the images in Fig. 8 where (a) is an input grayscale image $x$ and (b) is $y$ with the use of the

and eq.(3) reads

$$x^{(i+1)} = x^{(i)} - k[(1 + \mu)I - \mu H(x^{(i)})]x^{(i)} + ky$$

which converges if $-1 \leq 1 - k(1 + \mu - \mu I) \leq 1$ which holds if $0 < k \leq 2/(1 + \mu)$. The iterant $x^{(i)}$ converges with oscillation if $-1 \leq 1 - k(1 + \mu - \mu I) \leq 0$. We cannot stop such oscillating iteration at its midway due to errors. Hence $0 \leq 1 - k(1 + \mu - \mu I) \leq 1$ is required for monotonic convergence which can be safely stopped at any midpoint. Therefore, $0 < k \leq 1/(1 + \mu)$ is desired for smooth convergence.

Various sharpening filters are also included in photo-retouch software. The magnification factor is unknown for those built-in filters, for such filters $k$ should be safely set to a small value, typically $k = 0.5$.

An example of a foggy image as $x$ is shown in Fig. 7(a) and its enhanced output $y$ by an online software “One-Shot-Enhancer” in the webpage http://www.picturetopeople.org is shown in Fig. 7(b). Iterants $x^{(i)}$ are shown in Fig. 7(c) and (d) where we set $k = 0.5$.
Floyd-Steinberg error diffusion as \( q(\cdot) \). Some iterants are shown in Fig. 8 (c),(d) where we used the Gaussian filter as \( s(\cdot) \).

5.2 Resizing Filters When \( r(\cdot) \) is a resizing operation, \( y = r(x) \) cannot be directly inverted because image size is different for \( x \) and \( y \). Similarly to the above quantization, we relax \( r(x) - y = 0 \) to \( b(r(x) - y) = 0 \) where \( b(\cdot) \) is the inverse resizing of \( r(\cdot) \), i.e. if \( r(\cdot) \) changes the size \( M \times N \) to \( M' \times N' \), \( b(\cdot) \) alters \( M' \times N' \) to \( M \times N \). We convert \( b(r(x) - y) = 0 \) to \( x = x - b(r(x) - y) \) and solve it by the iteration \( x^{(i+1)} = x^i - b(r(x^i) - y) \). This iteration resembles the back-projection method for image super-resolution[12][13].

An example image shown in Fig.9(a) is shrunk to Fig.9(b) of which width and height are 1/4 of those of Fig.9(a). Jaggies obstruct the perceptual quality of Fig.9(b). Enlargement of Fig.9(b) to Fig.9(a) is an example called super-resolution. Iterants \( x^{(i)} \) are shown in Fig.9(c) and (d) where we set \( k = 0.5 \). We used the flow filter in Fig.5 for interpolating enlarged images. Jaggies are weakened in Fig.9(d).

5.3 Iterative Regularization Equation (22) is effective even when \( q(x) = x \). Image denoising is an operation for recovering \( x \) from \( y = x + n \) where \( n \) is noise. A conventional way of denoising is smoothing of \( y \) and outputting \( s(y) \) where \( s(\cdot) \) is a smoothing filter. This smoothing removes noise but also smoothes out signal components slightly.

A method called the iterative regularization[14] restores lost signal components by iterating

\[
x^{(i+1)} = x^{(i)} + s(y - x^{(i)})
\]

where its initial value is set to \( x^{(0)} = s(y) \). Note that this iteration should be stopped at its midway because \( x \) approaches gradually to \( y \) and eventually \( x^{(\infty)} = y \).

Addition of Gaussian noise of standard deviation 10 to the image in Fig. 3(a) yields Fig.10(a). Blurring this Fig.10(a) by a Gaussian filter, we get Fig. 10(b) where noise is reduced while textures disappear slightly. The iterant \( x^{(i)} \) and \( x^{(0)} \) of eq.(23) are shown in Fig.10(c),(d) where lost textures are partially recovered. We set \( s(\cdot) \) as the Gaussian filter of the standard deviation 3.

6. Conclusion

We have presented a method for inverting image filters by simply iterating them. An outstanding benefit of the proposed method lies in its applicability to wide class of filters including built-in softwares of which algorithm is unknown exactly. We have experimented this method for smoothing and enhancing filters, image halftoning and resizing operations. Automatic setting rule of the coefficient in the iteration is under study for image sharpening and enhancing filters.

Acknowledgment

This work was supported by JSPS KAKENHI Grant Number JP16K00241.

References

[1] C. Tomasi and R. Manduchi: “Bilateral filtering for gray and color images”, Proc. ICCV, pp.839-846, 1998.
[2] A. Buades, B. Coll and J. Morel: “Image denoising by non-local averaging”, Proc. ICASSP, pp.25-28, 2005.
[3] L. Hang, K. Inoue and K. Urahama: “Inverse cross-bilateral filter for cross-sharpening of images”, Proc. GCCE, pp.924-929, 2014.
[4] J. Ortega and W. Rheinboldt: “Iterative solutions of nonlinear equations in several variables”, Society for Industrial Mathematics, 1987.
[5] L. D. Griffin: “Mean, median and mode filtering of images”, Proc. R. Soc. Lond. A, vol.456, pp.2995-3004, 2000.
[6] A. Gadde, S. K. Narang and A. Ortega: “Bilateral filter: graph spectral interpretation and extensions”, Proc.ICIP, pp.1222-1226, 2013.
[7] P. Milanfar and H. T. Esfandarian: “A new class of image filters without normalization”, Proc. ICIP, pp.3294-3298, 2016.
[8] P. Perona and J. Malik: “Scale-space and edge detection using anisotropic diffusion”, IEEE Trans. Patt. Anal. Mach. Intell., vol.12, no.7, pp.629-639, 1990.
[9] H. Kang, S. Lee and C. Chui: “Flow-based image abstraction”, IEEE Trans. Vis. Comput. Graph., vol.15, no.1, pp.62-76, 2009.
[10] Z. Xiong, M. T. Orchard and K. Ramchandran: “Inverse halftoning using wavelets”, IEEE Trans. Image Process., vol.8, no.10, pp.1479-1483, 1999.
[11] J. Puzicha, M. Held, J. Ketterer, J. M. Buhmann and D. W. Fellner: “On spatial quantization of color images”, IEEE Trans. Image Process. vol.9, no.4, pp.666-682, 2000.
[12] M. Irani and S. Peleg: “Motion analysis for image enhancement: resolution, occlusion, and transparency”, J. Visual Comm., Image Repres., vol.4, no.4, pp.324-335, 1993.
[13] S. Dai M. Han, Y. Wu and Y. Gong: “Bilateral back-projection for single image super resolution”, Proc. ICME, pp.1039-1042, 2007.
[14] M. R. Charest and P. Milanfar: “On iterative regularization and its applications”, IEEE Trans. Circ. Syst. Video Tech., vol.18, no.3, pp.406-411, 2008.

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