A proof of the \((\alpha, \beta)\)–inversion formula conjectured by Hsu and Ma

Jin Wang\(^\dagger\) and Xinrong Ma\(^\ddagger\)

\(^\dagger\) Department of Mathematics, Soochow University, Suzhou 215006, P. R. China
E-mail: jinwang2016@yahoo.com

\(^\ddagger\) Department of Mathematics, Soochow University, Suzhou 215006, P. R. China
E-mail: xrma@suda.edu.cn

Abstract. In light of the well–known fact that the \(n\)th divided difference of any polynomial of degree \(m\) must be zero while \(m < n\), the present paper proves a conjecture by Hsu and Ma [14] analogous to the \((f, g)\)–inversion, which is called the \((\alpha, \beta)\)–inversion formula. It turns out to be, as applications, more useful in dealing with reciprocal relations of sequences such as elliptic divisible sequences, elliptic hypergeometric series, and partial theta functions.

Key words: matrix inversion, hypergeometric series, \((\alpha, \beta)\)–inversion formula, triple sum identity, quintuple sum identity, elliptic divisible sequences, elliptic hypergeometric series, partial theta function

2010 Mathematics Subject Classification: 33D15;05A10

1 Introduction

Throughout this paper, all operations are carried out on the complex field \(\mathbb{C}\). Recall that \(F = (F(n, k))_{n,k \in \mathbb{Z}}\) is an infinite–dimensional lower-triangular matrix over \(\mathbb{C}\), often denoted by \((F(n, k))_{n \geq k \in \mathbb{Z}}\), provided that each entry \(F(n, k) = 0\) unless \(n \geq k\). The matrix \(G = (G(n, k))_{n,k \in \mathbb{Z}}\) is the inverse matrix of \(F\) if

\[
\sum_{k \leq i \leq n} F(n,i)G(i,k) = \delta_{n,k} \text{ for all } n, k \in \mathbb{Z},
\]

where \(\delta_{n,k}\) denotes the usual Kronecker delta, \(\mathbb{Z}\) denotes the set of integers. Two such matrices, as pointed out by Henrici [11] and Gessel and Stanton [9, p.175, §2] independently, is equivalent to the Lagrange inversion formula and is often called an inversion formula or a reciprocal relation in the context of Combinatorics. In what follows, we call such a pair of matrices \(F\) and \(G\) with the reciprocal relation a matrix inversion. As many facts have shown that matrix inversions, called the inverse technique by Chu and Hsu [5], play very important roles in deriving summation and transformation formulas of various hypergeometric series. The reader may consult [2, 3, 4, 5, 7, 9, 13, 16, 17, 24] for more details.

It is worth noting that, after a long–term observation, Ma found

**Theorem 1.1** (The \((f, g)\)–inversion formula: Ma [13]). Preserve the above notation and assumptions. Suppose further \(g(x, y)\) is antisymmetric, i.e., \(g(x, y) = -g(y, x)\). Let \(F = (F(n, k))_{n \geq k \in \mathbb{Z}}\) and \(G = (G(n, k))_{n \geq k \in \mathbb{Z}}\) be two matrices with entries given by

\[
F(n, k) = \frac{\prod_{i=k}^{n-1} f(x_i, b_k)}{\prod_{i=k+1}^{n} g(b_i, b_k)} \quad \text{and} \quad G(n, k) = \frac{f(x_k, b_k) \prod_{i=k+1}^{n} f(x_i, b_n)}{f(x_n, b_n) \prod_{i=k}^{n-1} g(b_i, b_n)},
\]

respectively.
Then $F = (F(n,k))_{n \geq k \in \mathbb{Z}}$ and $G = (G(n,k))_{n \geq k \in \mathbb{Z}}$ is a matrix inversion if and only if for all $a, b, c, x \in \mathbb{C}$, there holds

$$f(x,a)g(b,c) + f(x,b)g(c,a) + f(x,c)g(a,b) = 0. \quad (1.3)$$

With a motivation to extend the range of validity of the $(f, g)$–inversion formula to arbitrary discrete sequences, Hsu and Ma \cite{14} conjectured a discrete analogue of Theorem 1.1. In this context, we call it the $(\alpha, \beta)$–inversion formula. However, it remains unproved so far.

**Conjecture 1.2** (The $(\alpha, \beta)$–inversion formula: Hsu and Ma \cite{14}). Let $\{\alpha_{n,k}\}_{n,k \in \mathbb{Z}}$ and $\{\beta_{n,k}\}_{n,k \in \mathbb{Z}}$ be two arbitrary double index sequences over $\mathbb{C}$ such that none of the terms $\alpha_{n,n}$ or $\beta_{n,k}$ is zero, and $\beta_{n,k}$ is antisymmetric, i.e., $\beta_{n,k} = -\beta_{k,n}$. Let $F = (F(n,k))_{n \geq k \in \mathbb{Z}}$ and $G = (G(n,k))_{n \geq k \in \mathbb{Z}}$ be two infinite–dimensional lower–triangular matrices with entries given by

$$F(n,k) = \frac{\prod_{i=k+1}^{n} \alpha_{i,k}}{\prod_{i=k}^{n-1} \beta_{i,k}} \quad \text{and} \quad G(n,k) = \frac{\alpha_{n,n} \prod_{i=k}^{n-1} \alpha_{i,n}}{\prod_{i=k}^{n} \beta_{i,n}}. \quad (1.4a)$$

Then $F = (F(n,k))_{n \geq k \in \mathbb{Z}}$ and $G = (G(n,k))_{n \geq k \in \mathbb{Z}}$ is a matrix inversion if and only if for arbitrary integers $n, k, p, q$,

$$\alpha_{n,p} \beta_{q,k} + \alpha_{n,q} \beta_{k,p} + \alpha_{n,k} \beta_{p,q} = 0. \quad (1.5)$$

The theme of this paper is to show that (1.5) is sufficient but not necessary to Conjecture 1.2, i.e., the $(\alpha, \beta)$–inversion formula. Our argument relies on the following general matrix inversion.

**Theorem 1.3.** Let $\{a_n\}_{n \in \mathbb{Z}}, \{b_n\}_{n \in \mathbb{Z}}$ and $\{s_n\}_{n \in \mathbb{Z}}, \{m_n\}_{n \in \mathbb{Z}}$ be four arbitrary sequences over $\mathbb{C}$ such that none of the terms both $a_n$ and $b_n$ is zero, $s_n$ are distinct from each other. Let $F = (F(n,k))_{n \geq k \in \mathbb{Z}}$ and $G = (G(n,k))_{n \geq k \in \mathbb{Z}}$ be two infinite–dimensional lower–triangular matrices with entries given by respectively

$$F(n,k) = \frac{b_n}{b_k} \prod_{i=k+1}^{n} \frac{m_i (s_k - s_{i-1} + a_{i-1} b_{i-1} m_{i-1})}{s_k - s_i} \quad (1.6a)$$

and

$$G(n,k) = \frac{a_k}{a_n} \prod_{i=k}^{n-1} \frac{m_i (s_n - s_{i+1} + a_{i+1} b_{i+1} m_{i+1})}{s_n - s_i}. \quad (1.6b)$$

Then $F$ and $G$ is a matrix inversion.

Several notation on convention are needed. In what follows, any product of the form, as for bilateral summation over $\mathbb{Z}$, is defined by (cf. \[8\])

$$\prod_{i=k}^{n} a_i = \begin{cases} a_k a_{k+1} \cdots a_n, & n \geq k; \\ 1, & n = k - 1; \\ 1/(a_{n+1} a_{n+2} \cdots a_{k-1}), & n \leq k - 2. \end{cases} \quad (1.7)$$

As for $q$-series, we employ the following standard notations for the $q$-shifted factorials: for $n \in \mathbb{Z}$,

$$(a; q)_\infty = \prod_{i=0}^{\infty} (1 - a q^i), (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty},$$

$$(a_1, a_2, \cdots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n.$$
As for theta and elliptic hypergeometric series, we adopt the standard concepts from [8, p.304, (11.2.5)/(11.2.6)] for the modified Jacobi, partial theta functions, and the elliptic analogue of the $q$–shifted factorial:

$$\theta(x; q) = (x, q/x; q)_{\infty},$$
$$\Theta(q; x) = \sum_{n \geq 0} (-1)^n q^{n(n-1)/2} x^n,$$
$$(x; q, p)_n = \prod_{k=0}^{n-1} \theta(xq^k; p),$$
as well as their multivariate analogues

$$\theta(a_1, a_2, \cdots, a_m; q) = \theta(a_1; q)\theta(a_2; q)\cdots\theta(a_m; q),$$
$$(a_1, a_2, \cdots, a_m; q, p)_n = (a_1; q, p)_n(a_2; q, p)_n\cdots(a_m; q, p)_n.$$

Our paper is organized as follows. In Sections 2 and 3, we will first show Theorem 1.3 and then show that (1.5) is sufficient but not necessary for Conjecture 1.2 to be true. Some specific matrix inversions, particularly these for elliptic hypergeometric series, elliptic divisible sequences and partial theta functions, derived by Conjecture 1.2 will be presented in Section 4.

## 2 Proof of Theorem 1.3

Our proof of Theorem 1.3 mainly involves the following well–known fact about the divided difference of polynomials. See [1, p.123] for further details.

**Lemma 2.1.** Let $H(x)$ be a polynomial in $x$ of degree no more than $n − 1$ and $x_0, x_1, \cdots, x_n$ be $n + 1$ distinct nodes. Then

$$[x_0, x_1, \cdots, x_n]H = \sum_{0 \leq i \leq n} \frac{H(x_i)}{\prod_{j=0, j \neq i}^{n} (x_i - x_j)} = 0, \quad (2.1)$$

where the classical divided difference of $H(x)$ with respect to $\{x_i|0 \leq i \leq n\}$ is recursively defined by

$$[x_0]H = H(x_0),$$
$$[x_0, x_1]H = \frac{H(x_0) - H(x_1)}{x_0 - x_1},$$
$$[x_0, x_1, x_2]H = \frac{[x_0, x_1]H - [x_1, x_2]H}{x_0 - x_2},$$
$$\cdots$$
$$[x_0, x_1, \cdots, x_n]H = \frac{[x_0, x_1, \cdots, x_{n-1}]H - [x_1, x_2, \cdots, x_n]H}{x_0 - x_n}.$$

Now write the polynomial $H(x)$ of degree $n − 1$ as

$$H(x) = \prod_{i=1}^{n-1} (x + a_i).$$

Then Lemma 2.1 is therefore rephrased explicitly

$$[x_0, x_1, \cdots, x_n]H = \sum_{0 \leq i \leq n} \frac{\prod_{j=1}^{n-1} (x_i + a_j)}{\prod_{j=0, j \neq i}^{n} (x_i - x_j)} = 0. \quad (2.2)$$
Now we are in a good position to show Theorem 1.3 by use of Lemma 2.1.

Proof. It only needs to check that (1.1) is true for all \( n \geq k \), which is self-evident for \( n = k \).

For \( n > k \), we compute in a straightforward way that

\[
\sum_{k \leq i \leq n} F(n, i)G(i, k) = \sum_{k \leq i \leq n} \frac{b_n}{b_i} \prod_{j=i+1}^{n} m_j \prod_{j=i+1}^{n} \frac{(s_i - s_{j-1} + a_{j-1}b_{j-1}m_{j-1})}{s_i - s_j} \times \frac{a_k}{a_i} \prod_{j=k}^{i-1} m_j \prod_{j=k}^{i-1} \frac{(s_i - s_{j+1} + a_{j+1}b_{j+1}m_{j+1})}{s_i - s_j}
\]

\[
= b_n a_k \prod_{j=k}^{n} m_j \sum_{k \leq i \leq n} \frac{1}{a_i b_i m_i} \prod_{j=k}^{n-1} (s_i - s_j + a_j b_j m_j) \prod_{j=k+1}^{n} (s_i - s_j + a_j b_j m_j) \prod_{j=k \neq i}^{n} (s_i - s_j)
\]

\[
= b_n a_k \prod_{j=k}^{n} m_j \sum_{k \leq i \leq n} \prod_{j=k}^{n-1} (s_i - s_j + a_j b_j m_j) \prod_{j=k \neq i}^{n} (s_i - s_j)
\]

\[
= b_n a_k \prod_{j=k}^{n} m_j \sum_{0 \leq i \leq n-k} \prod_{j=1}^{n-k-1} (s_{i+k} - s_{j+k} + a_{j+k} b_{j+k} m_{j+k}) \prod_{j=0 \neq i}^{n-k} (s_{i+k} - s_{j+k})
\]

Observe that

\[
\sum_{0 \leq i \leq n-k} \prod_{j=1}^{n-k-1} (s_{i+k} - s_{j+k} + a_{j+k} b_{j+k} m_{j+k}) \prod_{j=0 \neq i}^{n-k} (s_{i+k} - s_{j+k})
\]

is just a special case of Eq. (2.2) under the specifications that \( n \to n-k \) and

\[
(x_i, a_i) \to (s_{i+k}, -s_{i+k} + a_{i+k} b_{i+k} m_{i+k}).
\]

Hence we obtain

\[
\sum_{k \leq i \leq n} F(n, i)G(i, k) = 0.
\]

This gives the complete proof of the theorem. \( \blacksquare \)

Remark 2.2. As Krattenthaler pointed out, Theorem 1.3 can be derived directly from his well-known inversion formula. We refer the reader to [12] for further detail.

## 3 Results on Conjecture 1.2

Unlike the \((f, g)\)-inversion formula, we will show via the use of Theorem 1.3 that (1.5) is sufficient but not necessary to Conjecture 1.2 i.e., the \((\alpha, \beta)\)-inversion formula.

### 3.1 Proof of Conjecture 1.2 under (1.5)

For this purpose, it is convenient to introduce

**Definition 3.1.** Let \( \{\alpha_{n,k}\}_{n,k \in \mathbb{Z}} \) and \( \{\beta_{n,k}\}_{n,k \in \mathbb{Z}} \) be two arbitrary double index sequences over \( \mathbb{C} \). Then

1. **Triple sum identity (TSI)** If for any integers \( n, k, p, q \), it holds

\[
\alpha_{n,p} \beta_{q,k} + \alpha_{n,q} \beta_{p,k} + \alpha_{n,k} \beta_{p,q} = 0,
\]

then we say that \( \{\alpha_{n,k}\}_{n,k \in \mathbb{Z}} \) and \( \{\beta_{n,k}\}_{n,k \in \mathbb{Z}} \) satisfy the *triple sum identity*. 


(ii) Quintuple sum identity (QSI) If for any integers \( x, y, p, q, \) it holds

\[
\begin{align*}
\alpha_{x,p} \beta_{x,y} \beta_{x,p} \beta_{y,q} &+ \alpha_{x,p} \beta_{x,y} \beta_{p,y} \beta_{p,q} \\
- \alpha_{x,y} \beta_{x,p} \beta_{y,q} &- \alpha_{y,x} \beta_{x,p} \beta_{p,q} - \alpha_{x,x} \beta_{p,y} = 0,
\end{align*}
\]

then we say that \( \{ \alpha_{n,k} \}_{n,k \in \mathbb{Z}} \) and \( \{ \beta_{n,k} \}_{n,k \in \mathbb{Z}} \) satisfy the quintuple sum identity.

No matter how different these two identities would like, we are now able to show that both are equivalent to each other in certain condition. This equivalence is based on the following two facts. The first one is

**Lemma 3.2.** Two sequences \( \{ \alpha_{n,k} \}_{n,k \in \mathbb{Z}} \) and \( \{ \beta_{n,k} \}_{n,k \in \mathbb{Z}} \) with \( \beta_{n,k} = -\beta_{k,n} \) satisfy QSI (3.2) if and only if for any integers \( x, p, y, \)

\[
\alpha_{p,x} \beta_{y,p} + \alpha_{p,y} \beta_{p,x} + \alpha_{p,p} \beta_{x,y} = 0.
\]

**Proof.** Now that \( \{ \alpha_{n,k} \}_{n,k \in \mathbb{Z}} \) and \( \{ \beta_{n,k} \}_{n,k \in \mathbb{Z}} \) satisfy QSI (3.2), in which we may take \( y = x \) to get

\[
\alpha_{x,p} \alpha_{x,x} \beta_{q,x} \left( \beta_{x,p} + \beta_{p,x} \right) - \alpha_{x,x} \beta_{x,p} \{ \beta_{x,q} + \alpha_{p,q} \beta_{p,x} - \alpha_{p,p} \beta_{x,q} \} = 0,
\]

which can be simplified to (3.3) only noting the fact that \( \beta_{x,p} = -\beta_{p,x} \) and replacing \( q \) with \( y \). Conversely, suppose (3.3) holds. Then making the parametric replacement \( (x, y, p) \rightarrow (p, y, x) \) in (3.3), we have

\[
\alpha_{x,y} \beta_{p,y} = \alpha_{x,p} \beta_{x,y} - \alpha_{x,y} \beta_{x,p}.
\]

Alternatively, setting \( x \rightarrow q \) in (3.3), we have

\[
\alpha_{p,p} \beta_{q,y} = \alpha_{p,q} \beta_{p,y} - \alpha_{p,y} \beta_{p,q}.
\]

Multiplying (3.4) with (3.5), we find

\[
\alpha_{x,x} \alpha_{p,p} \beta_{p,y} \beta_{q,y} = (\alpha_{x,p} \beta_{x,y} - \alpha_{x,y} \beta_{x,p})(\alpha_{p,q} \beta_{p,y} - \alpha_{p,y} \beta_{p,q})
\]

\[
= \alpha_{x,p} \alpha_{p,q} \beta_{x,y} \beta_{p,y} - \alpha_{x,p} \alpha_{p,q} \beta_{x,y} \beta_{p,q} - \alpha_{x,y} \alpha_{p,q} \beta_{x,p} \beta_{y} + \alpha_{x,y} \alpha_{p,q} \beta_{x,p} \beta_{y}.
\]

Upon substituting these relations into the left-hand side of (3.2), we arrive at

\[
\text{LHS of (3.2)} = \alpha_{x,p} \beta_{q,y}(\alpha_{p,y} \beta_{x,p} + \alpha_{p,q} \beta_{x,p}) - \alpha_{x,p} \beta_{q,y}(\alpha_{p,y} \beta_{x,p} + \alpha_{p,q} \beta_{x,p})
\]

\[
= \alpha_{x,p}(\beta_{q,y} \alpha_{p,p} \beta_{x,y} - \beta_{x,y} \alpha_{p,p} \beta_{q,y}) = \alpha_{x,p} \alpha_{p,p}(\beta_{q,y} \beta_{x,y} - \beta_{x,y} \beta_{q,y}) = 0.
\]

In the antepenultimate equality, we have utilized (3.3). This completes the proof of the lemma.

Next, we will show that TSI (3.1) is also equivalent to (3.3).

**Lemma 3.3.** Sequences \( \{ \alpha_{n,k} \}_{n,k \in \mathbb{Z}} \) and \( \{ \beta_{n,k} \}_{n,k \in \mathbb{Z}} \) with \( \beta_{n,k} = -\beta_{k,n} \) satisfy TSI (3.1) if and only if they satisfy (3.3).

**Proof.** Actually, it only needs to derive TSI

\[
\alpha_{n,k} \beta_{x,y} + \alpha_{n,x} \beta_{y,k} + \alpha_{n,y} \beta_{k,x} = 0
\]

from (3.3). To do this, we first take into account the special case of (3.3)

\[
\beta_{y,x} = \frac{\alpha_{p,y} \beta_{p,x} - \alpha_{p,x} \beta_{p,y}}{\alpha_{p,p}}.
\]
And then by substituting (3.7) into (3.4), we see that

\[
\text{LHS of (3.6)} = \frac{\alpha_{n,k}}{\alpha_{p,p}} \{\alpha_{p,x} \beta_{p,y} - \alpha_{p,y} \beta_{p,x}\} + \frac{\alpha_{n,x}}{\alpha_{p,p}} \{\alpha_{p,y} \beta_{p,k} - \alpha_{p,k} \beta_{p,y}\} + \frac{\alpha_{n,y}}{\alpha_{p,p}} \{\alpha_{p,k} \beta_{p,x} - \alpha_{p,x} \beta_{p,k}\}.
\]

Rearranging the terms yields

\[
\text{LHS of (3.6)} = \frac{\beta_{p,k}}{\alpha_{p,p}} \{\alpha_{n,x} \alpha_{p,y} - \alpha_{n,y} \alpha_{p,x}\} + \frac{\beta_{p,x}}{\alpha_{p,p}} \{\alpha_{n,y} \alpha_{p,k} - \alpha_{n,k} \alpha_{p,y}\} + \frac{\beta_{p,y}}{\alpha_{p,p}} \{\alpha_{n,k} \alpha_{p,x} - \alpha_{n,x} \alpha_{p,k}\}.
\]

Observe that the left-hand side of (3.9) is independent of \(p\). On letting \(p = n\) in the last identity gives rise to the desired identity.

Summing up, we have

**Proposition 3.4.** Suppose \(\{\beta_{n,k}\}_{n,k \in \mathbb{Z}}\) is antisymmetric, i.e., \(\beta_{n,k} = -\beta_{k,n}\). If \(\{\alpha_{n,k}\}_{n,k \in \mathbb{Z}}\) and \(\{\beta_{n,k}\}_{n,k \in \mathbb{Z}}\) satisfy TSI (3.1), then they satisfy QSI (3.2). Vice versa.

**Proof.** Evidently, from Lemmas 3.2 and 3.3 it is easily found that TSI (3.1) and QSI (3.2) are equivalent to each other.

Now we are ready to show Conjecture 1.2 under (3.1), which we often refer to as TSI (3.1). That is, if (3.1) holds true, then both (1.4a) and (1.4b) just forms a matrix inversion.

**Proof of Conjecture 1.2 under (3.1).** To show the result in question, we define, for a fixed integer \(p\), that

\[
s_n = \frac{\alpha_{p,n}}{\beta_{p,n}}, \quad m_n = \frac{\alpha_{n,p}}{\beta_{n,p}}, \quad a_n = -\frac{\alpha_{n,n}}{\alpha_{n,p}}, \quad b_n = \frac{\alpha_{p,p}}{\alpha_{n,p}},
\]

such that \(F(n,k)\) and \(G(n,k)\) in Theorem 1.3 can be restated in the form given by Conjecture 1.2 namely,

\[
\frac{b_n}{b_k} \prod_{i=k+1}^{n} m_i (s_k - s_{i-1} + a_{i-1} b_{i-1} m_{i-1}) = \prod_{i=k+1}^{n-1} \frac{\alpha_{i,k}}{\beta_{i,k}} \quad \text{and}
\]

\[
\frac{a_k}{a_n} \prod_{i=k}^{n-1} m_i (s_n - s_{i+1} + a_{i+1} b_{i+1} m_{i+1}) = \frac{\alpha_{k,k}}{\alpha_{n,n}} \prod_{i=k+1}^{n} \frac{\alpha_{i,n}}{\beta_{i,n}}.
\]

Or equivalently,

\[
\prod_{i=k+1}^{n} \frac{b_i}{b_{i-1}} \frac{m_i (s_k - s_{i-1} + a_{i-1} b_{i-1} m_{i-1})}{s_k - s_i} = \prod_{i=k+1}^{n-1} \frac{\alpha_{i-1,k}}{\beta_{i,k}}, \quad (3.9)
\]

\[
\prod_{i=k}^{n-1} \frac{a_i}{a_{i+1}} \frac{m_i (s_n - s_{i+1} + a_{i+1} b_{i+1} m_{i+1})}{s_n - s_i} = \prod_{i=k}^{n-1} \frac{\alpha_{i,i}}{\alpha_{i+1,i+1}} \frac{\alpha_{i+1,n}}{\beta_{i,n}}. \quad (3.10)
\]

Next, we may easily deduce from (3.9) by induction on \(n\) that for integers \(i\),

\[
\frac{b_i}{b_{i-1}} \frac{m_i (s_k - s_{i-1} + a_{i-1} b_{i-1} m_{i-1})}{s_k - s_i} = \frac{\alpha_{i-1,k}}{\beta_{i,k}}.
\]
Using (3.10) and by induction on \( k \), we also obtain
\[
\frac{a_i}{a_{i+1}} \frac{m_i (s_n - s_{i+1} + a_{i+1} b_{i+1} m_{i+1})}{s_n - s_i} = \frac{\alpha_{i,i}}{\alpha_{i+1,i+1}} \frac{\alpha_{i+1,n}}{\beta_{i,n}}.
\]
On substituting (3.8) into and simplifying the resulted, we obtain
\[
-\alpha_{p,k} \beta_{p,i-1} \alpha_{i-1,p} + \alpha_{p-1,i-1} \beta_{p,k} \alpha_{i-1,1} \alpha_{i-1,i-1} \alpha_{p,p} \beta_{p,k}
= \frac{\alpha_{i-1,k}}{\beta_{i,k}}
\]
and
\[
-\alpha_{p,n} \beta_{p,i+1} \alpha_{i+1,p} + \alpha_{p+1,i+1} \beta_{p,n} \alpha_{i+1,i+1} \alpha_{p,p} \beta_{p,n}
= \frac{\alpha_{i+1,n}}{\beta_{i,n}},
\]
both of which turn out to be, after further simplification,
\[
\mathcal{L}(i - 1, k; p, i) = \mathcal{L}(i + 1, n; p, i) = 0,
\]
where \( \mathcal{L}(x; y; p, q) \) denotes the sum on the left-hand side of (3.2). It is asserted by the known condition of QSI (3.2). As Proposition 3.4 shows, the latter is equivalent to TSI (3.1). The conjecture is thus confirmed.

### 3.2 Why (1.5) is not necessary to Conjecture 1.2

In order to clarify this point, assuming that (1.5) is true while both (1.4a) and (1.4b) compose a matrix inversion, we now proceed to calculate \( \beta_{k,n} \) in two different ways provided that \( \{\alpha_{k,n}\}_{n,k \in \mathbb{Z}} \) and \( \{\beta_{k,n}\}_{k \in \mathbb{Z}} \) are given.

For this purpose, we start with a special case of Lemma 3.3 and the definition (1.1). At first, by (3.1), we may set up a recurrence relation for \( \{\beta_{k,n}\}_{k \leq n} \in \mathbb{Z} \) as below.

**Proposition 3.5.** Suppose that \( \{\alpha_{k,n}\}_{n,k \in \mathbb{Z}} \) and \( \{\beta_{k,n}\}_{n,k \in \mathbb{Z}} \) satisfy TSI (3.1), \( \beta_{n,k} = -\beta_{k,n} \).

Then for \( k \leq n \)
\[
\beta_{k,n} = \sum_{i=k+1}^{n} \left\{ \frac{\alpha_{i-1,k}}{\alpha_{i-1,i-1}} \prod_{j=i+1}^{n} \frac{\alpha_{j-1,j}}{\alpha_{j-1,j-1}} \right\} \beta_{i-1,i}.
\]

**Proof.** It suffices to make the parametric replacement
\[
(n, p, q, k) \rightarrow (n - 1, k, n, n - 1)
\]
in (3.11). We obtain at once
\[
\beta_{k,n} = \frac{\alpha_{n-1,n}}{\alpha_{n-1,n-1}} \beta_{k,n-1} + \frac{\alpha_{n-1,k}}{\alpha_{n-1,n-1}} \beta_{n-1,n}.
\]
Observe that (3.13) is recursive with respect to \( \{\beta_{k,n}\}_{n \geq 1} \). By iterating this recurrence repeatedly \( n - k \) times and then we obtain (3.12).

As we will see later, the recursive relation (3.12) may serve as a practical way to get \( (\alpha, \beta) \)-inversions. For that end, using the recursive relation (3.12), we now give two rather general kinds of solutions to TSI (3.1).

**Corollary 3.6.** Let \( \{a_n, b_n, x_n, y_n, t_n\}_{n \in \mathbb{Z}} \) are arbitrary sequences and
\[
\alpha_{k,n} = \frac{\prod_{i=1}^{n} x_i}{\prod_{i=1}^{n} y_i}, \quad \beta_{n-1,n} = t_n, \beta_{k,n} = -\beta_{n,k}.
\]
Then \( \{\alpha_{k,n}\}_{n,k \in \mathbb{Z}} \) and \( \{\beta_{k,n}\}_{n,k \in \mathbb{Z}} \) satisfy TSI (3.1) if and only if
\[
\beta_{k,n} = \sum_{i=k+1}^{n} t_i \prod_{j=i+1}^{n} x_j \prod_{j=i-1}^{n} x_j.
\]
Proof. Suppose that $\{\alpha_{k,n}\}_{n,k}\in\mathbb{Z}$ and $\{\beta_{k,n}\}_{n,k}\in\mathbb{Z}$ satisfy TSI (3.1). Then, from Proposition 3.5, the relation (3.15) follows directly. Conversely, let (3.15) be given. We need to check

$$\alpha_{n,p}\beta_{q,k} + \alpha_{n,q}\beta_{k,p} + \alpha_{n,k}\beta_{p,q} = 0.$$  \hspace{1cm} (3.16)

Without loss of generality, suppose that $p \geq k \geq q$. In view of the arbitrariness of $\{t_n\}_{n\in\mathbb{Z}}$, it is only need to show the coefficients of $t_m$ with $m: p \geq m \geq k$ in the sum on the left–hand side of (3.16), namely

$$\frac{\prod_{i=1}^{p} x_i}{\prod_{i=1}^{n} y_i} \sum_{i=q+1}^{k} t_i \frac{\prod_{j=i+1}^{k} x_j}{\prod_{j=q+1}^{n} x_j} + \frac{\prod_{i=1}^{q} x_i}{\prod_{i=1}^{n} y_i} \sum_{i=k+1}^{p} t_i \frac{\prod_{j=i+1}^{q} x_j}{\prod_{j=k+1}^{n} x_j} + \frac{\prod_{i=1}^{k} x_i}{\prod_{i=1}^{n} y_i} \sum_{i=p+1}^{q} t_i \frac{\prod_{j=i+1}^{k} x_j}{\prod_{j=p+1}^{n} x_j},$$

equals zero. Actually, this coefficient is

$$= \frac{\prod_{i=1}^{q} x_i \prod_{j=m+1}^{n} x_j}{\prod_{i=1}^{n} y_i \prod_{j=k+1}^{n} x_j} \frac{\prod_{i=1}^{k} x_i \prod_{j=m+1}^{n} x_j}{\prod_{i=1}^{n} y_i \prod_{j=p+1}^{n} x_j} \frac{\prod_{i=1}^{p} x_i \prod_{j=m+1}^{n} x_j}{\prod_{i=1}^{n} y_i \prod_{j=q+1}^{n} x_j}.$$

Thus, (3.16) is proved.

Corollary 3.7. Assume that $\beta_{n,k} = -\beta_{k,n}$ and

$$\alpha_{k,n} = x_k a_n + y_k b_n, \; \beta_{n-1,n} = a_n b_{n-1} - a_{n-1} b_n,$$  \hspace{1cm} (3.17)

where $\{a_n, b_n, x_n, y_n\}$ are arbitrary sequences. Then $\{\alpha_{k,n}\}_{n,k}\in\mathbb{Z}$ and $\{\beta_{k,n}\}_{n,k}\in\mathbb{Z}$ satisfy TSI (3.1) if and only if

$$\beta_{k,n} = a_n b_k - a_k b_n.$$  \hspace{1cm} (3.18)

Proof. It can be verified in a straightforward manner.

On the other hand, only using the definition (1.1), it is easily found that

Proposition 3.8. Suppose that $\{\alpha_{k,n}\}_{n,k}\in\mathbb{Z}$ and $\{\beta_{k,n}\}_{n,k}\in\mathbb{Z}$ forms an $(\alpha,\beta)$–inversion. Then for $k \leq n$,

$$\beta_{k,n} = -\frac{f(k,n;k) + f(k,n;n)}{\sum_{i=k+1}^{n-1} g(k,n;i)},$$  \hspace{1cm} (3.19)

where

$$f(k,n;i) = (-1)^{n-i} \prod_{k \leq j_1 < j_2 \leq n \atop j_1, j_2 \neq i} \beta_{j_1,j_2} \prod_{j=k+1}^{n} \alpha_{j,i},$$

$$g(k,n;i) = (-1)^{n-i} \prod_{k \leq j_1 < j_2 \leq n \atop j_2 - j_1 \leq n-k-1} \beta_{j_1,j_2} \prod_{j=k+1}^{n} \alpha_{j,i}.$$  \hspace{1cm} (3.20)
A proof of the \((\alpha, \beta)\)-inversion formula conjectured by Hsu and Ma

**Proof.** By the definition (1.1), we see that for \(n > k\),
\[
\sum_{i=k}^{n} \prod_{j=i+1}^{n} \alpha_{j,i} \times \frac{\alpha_{k,k} \prod_{j=k+1}^{i} \alpha_{j,i}}{\prod_{j=k}^{n} j} = 0,
\]
which, after simplified by the relation \(\beta_{k,n} = -\beta_{n,k}\), amounts to
\[
\sum_{i=k}^{n} f(k, n; i) = 0, \tag{3.22}
\]
where \(f(k, n; i)\) is given by (3.20). Further, we rearrange the sum on the left-hand side of (3.22) in the form
\[
f(k, n; k) + \sum_{i=k+1}^{n-1} f(k, n; i) + f(k, n; n) = 0.
\]
Observe that for \(k+1 \leq i \leq n-1\), the summand \(f(k, n; i)\) contains the factor \(\beta_{k,n}\), namely
\[
f(k, n; i) = g(k, n; i) \beta_{k,n}
\]
with \(g(k, n; i)\) defined by (3.21). This leads us to (3.19).

Now we are in a good position to explain why TSI (3.1), i.e., (1.5) is not necessary to Conjecture 1.2. It is because both (3.12) and (3.19) are two recursive relations for \(\{\beta_{k,k+n}\}n \geq 1\). Once \(\{\alpha_{k,n}\}n,k \in \mathbb{Z}\) and \(\{\beta_{k,k+1}\}k \in \mathbb{Z}\) are given as the initial conditions, these two recursive relations may lead us to two different results. This will contradict the uniqueness of \(\beta_{k,n}\) restricted by the known conditions.

The following is a short Mathematica program to find \(\{\beta_{k,n}\}k \leq n \in \mathbb{Z}\) recursively via (3.12) and (3.19).

```mathematica
(* Mathematica program *)
c[i_] := ti[i - 1]
d[k_, n_] := Sum[a[i - 1, k]/a[i - 1, i - 1] Product[a[j - 1, j]/a[j - 1, j - 1], {j, i + 1, n}] c[i], {i, k + 1, n}]
t2[k_] := (a[1 + k, 2 + k] t1[k] + a[1 + k, k] t1[1 + k])/a[1 + k, 1 + k]
t3[k_] := (a[1 + k, 3 + k] a[2 + k, 3 + k] t1[k] t2[k] t1[1 + k] - a[1 + k, k] a[2 + k, k] t1[1 + k] t2[1 + k])/(a[1 + k, 2 + k] a[2 + k, 2 + k] t1[k] t2[k] - a[1 + k, 1 + k] a[2 + k, 1 + k] t1[1 + k] t2[k])
t4[k_] := (a[1 + k, 4 + k] a[2 + k, 4 + k] a[3 + k, 4 + k] t1[k] t2[k] t3[k] t1[1 + k] t2[1 + k] t1[2 + k] + a[1 + k, k] a[2 + k, k] a[3 + k, k] t1[1 + k] t2[1 + k] t3[1 + k] t1[2 + k] t2[2 + k] t1[3 + k])/(a[1 + k, 3 + k] a[2 + k, 3 + k] a[3 + k, 3 + k] t1[k] t2[k] t3[k] t1[1 + k] t2[1 + k] t3[1 + k] t2[2 + k] - a[1 + k, 2 + k] a[2 + k, 2 + k] a[3 + k, 2 + k] t1[k] t3[k] t2[k] a[2 + k, 1 + k] a[3 + k, 1 + k] t1[2 + k] t3[1 + k] t2[2 + k] t1[3 + k])
```

As an example, we list some computational results to justify our argument.

**Example 3.9.** Set \(\alpha_{k,n} = k + n\) and \(\beta_{k,k+1} = k\). Then the output by the above program are
\[
\begin{align*}
t2[k] - d[k, k + 2] &= 0, \\
t3[k] - d[k, k + 3] &= \frac{8k^3 + 32k^2 + 32k + 5}{8k^3 + 36k^2 + 52k + 24}, \\
t4[k] - d[k, k + 4] &= \frac{2k + 7}{8(k + 1)(k + 2)(k + 3)(2k + 3)(2k + 5)} f(k).
\end{align*}
\]
where
\[
f(k) = 3072k^{11} + 56320k^{10} + 451904k^9 + 2085376k^8 + 6115168k^7 + 11884320k^6 \\
+ 15498308k^5 + 13457624k^4 + 7592100k^3 + 2669648k^2 + 540883k + 47328, \\
g(k) = 48k^7 + 544k^6 + 2452k^5 + 5656k^4 + 7216k^3 + 5232k^2 + 2175k + 464.
\]

4 Some explicit matrix inversions

To justify possibly applications of the $((\alpha, \beta) -$ inversion given by Conjecture 1.2, we now list some important concrete examples via the use of Corollaries 3.6 and 3.7.

There comes at first is Gasper’s matrix inversion which appeared in the bibasic hypergeometric series [6, Eqs. (3.1) and (3.2)]. Gasper obtained such a pair of matrix inversion in his extension of Euler’s transformation formula. Displayed as below, it is indeed a special case of the $((\alpha, \beta) -$ inversion formula.

**Example 4.1.** Let $F = (F(n,k))_{n \geq k \in \mathbb{Z}}$ and $G = (G(n,k))_{n \geq k \in \mathbb{Z}}$ be two matrices with entries given by
\[
F(n,k) = (-1)^{n-k} p^{-(n-k)k}(ap^k q^k, bp^{-k} q^k; q)_{n-k} / (p, bp^{-n-k}/a; p)^{n-k} \tag{4.1a}
\]
and
\[
G(n,k) = p^{-(n-k)+k}(1 - ap^k q^k)(1 - bp^{-k} q^k)(ap^n q^k, bq^k p^{-n}; q)_{n-k} / (1 - ap^k q^k)(1 - bp^{-n} q^k)(p, bp^{1-2n}/a; p)^{n-k} \tag{4.1b}
\]
respectively. Then $F = (F(n,k))_{n \geq k \in \mathbb{Z}}$ and $G = (G(n,k))_{n \geq k \in \mathbb{Z}}$ is a matrix inversion.

**Proof.** It only needs to take in the $((\alpha, \beta) -$ inversion formula
\[
\alpha_{i,k} = (1 - x_k q^i)(1 - y_k q^i), \\
\beta_{i,k} = (p^i - p^{-k})(1 - b p^{-k-i} / a),
\]
where $x_k = ap^k, y_k = bp^{-k}.$ Clearly, $\beta_{i,k} = -\beta_{k,i}.$ As such, it remains to check (1.5). The detail is left to the reader.

Another important $((\alpha, \beta) -$ inversion formula is the following result due to Schlosser, who has used it (cf. [20, Eqs. (7.18)/(7.19)]) successfully to set up transformation formulas of bilateral hypergeometric series.

**Example 4.2.** Let $F = (F(n,k))_{n \geq k \in \mathbb{Z}}$ and $G = (G(n,k))_{n \geq k \in \mathbb{Z}}$ be two matrices with entries given, respectively, by
\[
F(n,k) = \left( \frac{1}{b, c-a(a+bq^k); q} \right)_{n-k} \tag{4.2a}
\]
and
\[
G(n,k) = \lambda(n,k) \left( \frac{q^{k-n+1}, (a+bq^n)q^{k+1} - a(a+bq^k); q} {q, (a+bq^n)q^k; q} \right)_{n-k} \tag{4.2b}
\]
with
\[ \lambda(n, k) = (-1)^{n-k} q^{\binom{n-k}{2}} \frac{c - (a + bq^k)(a + q^k)}{c - (a + bq^n)(a + q^n)}. \]

Then \( F = (F(n, k))_{n \geq k \in \mathbb{Z}} \) and \( G = (G(n, k))_{n \geq k \in \mathbb{Z}} \) is a matrix inversion.

**Proof.** It follows from the \((\alpha, \beta)\)-inversion formula by specifying
\[
\alpha_{i,k} = (q^k - q^i/b)(c - (a + bq^k)(a + q^k)), \\
\beta_{i,k} = (q^k - q^i)(c - (a + bq^k)(a + bq^i)).
\]

The verification of (1.5) is left to the interested reader.

Of all matrix inversions useful to elliptic hypergeometric series is one due to Warnaar [24].

**Example 4.3.** (Warnaar’s elliptic matrix inversion) Let \( F = (F(n, k))_{n \geq k \in \mathbb{Z}} \) and \( G = (G(n, k))_{n \geq k \in \mathbb{Z}} \) be two infinite lower-triangular with the entries given by
\[
F(n, k) = \prod_{i=k}^{n-1} \frac{\theta(x_ibk; q)\theta(x_i; q)}{\theta(b_i; q)\theta(b_i; q)} \\ G(n, k) = \prod_{i=k}^{n-1} \frac{\theta(x_ibk; q)\theta(x_i; q)}{\theta(b_i; q)\theta(b_i; q)}.
\]

Then \( F = (F(n, k))_{n \geq k \in \mathbb{Z}} \) and \( G = (G(n, k))_{n \geq k \in \mathbb{Z}} \) is a matrix inversion.

**Proof.** Specify in Conjecture 1.2
\[
\alpha_{i,k} = b_k \theta(x_ibk; q)\theta\left(\frac{x_i}{b_k}; q\right) \quad \text{and} \quad \beta_{i,k} = b_k \theta(b_ibk; q)\theta\left(\frac{b_i}{b_k}; q\right).
\]

It is easy to check that \( \beta_{i,k} = -\beta_{k,i} \) and TSI (1.5) is asserted by the well–known theta function identity [8, Ex. 2.16(i)]:
\[
\theta(xy, x/y, uv, u/v; q) - \theta(xv, x/v, yu, u/y; q) = \frac{u}{y} \theta(xu, x/u, yv, y/v; q).
\]

As displayed in [21], Warnaar’s elliptic matrix inversion has been used successfully in the theory of elliptic hypergeometric series. Now, with the help of Corollary 3.6 we can obtain a new elliptic matrix inversion.

**Example 4.4.** Let \( F = (F(n, k))_{n \geq k \in \mathbb{Z}} \) and \( G = (G(n, k))_{n \geq k \in \mathbb{Z}} \) be two infinite lower-triangular with the entries given by
\[
F(n, k) = \prod_{i=k+1}^{n} \frac{1}{(x; q, p)_i(y; q, p)_i-1S_{i,k}} \\ G(n, k) = \prod_{i=k}^{n-1} \frac{1}{(x; q, p)_i(y; q, p)_iS_{i,n}}.
\]

Then \( F = (F(n, k))_{n \geq k \in \mathbb{Z}} \) and \( G = (G(n, k))_{n \geq k \in \mathbb{Z}} \) is a matrix inversion. Here, we define for any complex numbers \( x, p, q, \)
\[
S_{k,n} = \sum_{i=k+1}^{n} \frac{t_i}{(x; q, p)_i(x; q, p)_i}.
\]
Proof. It suffices to take in Corollary 3.6 that
\[ x_i = \theta(xq^{i-1}; p), \quad y_i = \theta(yq^{i-1}; p). \]
Therefore, we have that
\[ \alpha_{i,k} = \frac{(x; q, p)_k}{(y; q, p)_i} \quad \text{and} \quad \beta_{i,k} = (x; q, p)_i(x; q, p)_k S_{i,k}. \]

Going along with this line, we yet find another new theta matrix inversion arising from Schilling and Warnaar’s partial theta function identity.

Example 4.5. Let \( F = (F(n, k))_{n \geq k \in \mathbb{Z}} \) and \( G = (G(n, k))_{n \geq k \in \mathbb{Z}} \) be two infinite lower-triangular with the entries given by
\[
F(n, k) = \frac{\prod_{i=k}^{n-1} (a_i + \Theta(q; b_k))}{\prod_{i=k+1}^n (b_i - b_k)L(b_i, b_k)} \quad \text{and} \quad (4.6a)
\]
\[
G(n, k) = \frac{a_k + \Theta(q; b_k)}{a_n + \Theta(q; b_n)} \frac{\prod_{i=k+1}^n (a_i + \Theta(q; b_n))}{\prod_{i=k}^{n-1} (b_i - b_n)L(b_i, b_n)}. \quad (4.6b)
\]
Then \( F = (F(n, k))_{n \geq k \in \mathbb{Z}} \) and \( G = (G(n, k))_{n \geq k \in \mathbb{Z}} \) is a matrix inversion. Here, for any two complex numbers \( x, y \), we define
\[
L(x, y) = -(q, xq, yq; q)\sum_{n=0}^{\infty} \frac{(xy; q)_{2n}}{(q, xq, yq, xy; q)_n} q^n. \quad (4.7)
\]

Proof. It follows from the \( (\alpha, \beta) \)-inversion formula by specifying
\[
\alpha_{i,k} = a_i + \Theta(q; b_k) \quad \text{and} \quad \beta_{i,k} = (b_i - b_k)L(b_i, b_k).
\]
The validity of (\ref{eq:4.5}) results from the following partial theta function identity
\[
L(x, y) = \frac{\Theta(q, x) - \Theta(q, y)}{x - y}, \quad (4.8)
\]
which is given by Lemma 4.3 in \cite{18} due to Schilling and Warnaar.

We end this paper by an elliptic divisible sequence \( \{W_n\}_{n \in \mathbb{Z}} \) first introduced by M. Ward \cite{24}. This important sequence is defined recursively by
\[
\begin{cases}
W_{n+2}W_{n-2} = W_{n+1}W_{n-1}W_2^2 - W_1W_3W_n^2 \\
W_{-n} = -W_n; W_0 = 0, W_1 = 1.
\end{cases} \quad (4.9)
\]
See \cite{22} for details. Our only goal here is to give a general reciprocal relation for such kind of sequences. It convictively shows that the \( (\alpha, \beta) \)-inversion formula of Conjecture 1.2 has an advantage over the \( (f, g) \)-inversion of Theorem 1.1 as far as discrete sequences are concerned.

Example 4.6. Let \( F = (F(n, k))_{n \geq k \in \mathbb{Z}} \) and \( G = (G(n, k))_{n \geq k \in \mathbb{Z}} \) be two infinite lower-triangular with the entries given by
\[
F(n, k) = \frac{W_k^{2(n-k)}}{\prod_{i=2k+1}^{n+k} W_i \prod_{i=1}^{n-k} W_i} \quad \text{and} \quad (4.10)
\]
\[
G(n, k) = (-1)^{n-k} \frac{W_k^2 W_n^{2(n-k)} \prod_{i=1}^{n+k-1} W_i}{W_n^{2m-1} \prod_{i=1}^{m-1} W_i \prod_{i=1}^{n-k} W_i}. \quad (4.11)
\]
Then \( F = (F(n, k))_{n \geq k \in \mathbb{Z}} \) and \( G = (G(n, k))_{n \geq k \in \mathbb{Z}} \) is a matrix inversion.
A proof of the \((\alpha, \beta)\)-inversion formula conjectured by Hsu and Ma

**Proof.** It suffices to take in (1.4) of Conjecture 1.2, preforming as above,

\[
\alpha_{i,k} = W_k^2, \quad \beta_{i,k} = W_{i+k}W_{i-k}.
\]

Observe that \(\beta_{i,k} = -\beta_{k,i}\), because \(W_{-k} = -W_k\) while TSI (1.5) is just agreement with the property (cf. [22]) that for all integers \(k, p, q \in \mathbb{Z}\),

\[
W_k^2W_{p+q}W_{p-q} + W_p^2W_{q+k}W_{q-k} + W_q^2W_{k+p}W_{k-p} = 0. \tag{4.12}
\]

As claimed. \(\blacksquare\)

**Acknowledgements**

This work was supported by NSFC grant No. 11471237. The authors would like to thank Professor Ch. Krattenthaler for his detailed deduction of Theorem 1.3 from his famous inversion formula.

**References**

[1] Burden R. L., Faires J. D., Numerical Analysis, 7th Edition, Pacific Grove, CA: Brooks/Cole, 2001.

[2] Chu W. C., Inversion techniques and combinatorial identities, *Boll. Unione Mat. Italiana* 7-B(1993), 737-760.

[3] Chu W. C., Inversion techniques and combinatorial identities: strange evaluations of hypergeometric series, *Pure Math. Appl.* 4(1993), 409-428.

[4] Chu W. C., Inversion techniques and combinatorial identities: a unified treatment for the \(\tau F_q\)-series identities, *Collect. Math.* 45(1994), 13-43.

[5] Chu W. C., Hsu L.C., Some new applications of Gould-Hsu inversion, *J. Combin. Inform. System Sci.* 14(1989), 1-4.

[6] Gasper G., Summation, transformation, and expansion formulas for bibasic series, *Trans. Amer. Math. Soc.* 312(1989), 257-278.

[7] Gasper G., Schlosser M., Summation, transformation, and expansion formulas for multibasic theta hypergeometric series, *Adv. Stud. Contemp. Math.* 11(2005), 67-84.

[8] Gasper G., Rahman M., Basic Hypergeometric Series (second edition), Encyclopedia Math. Appl., Vol.96, Cambridge Univ. Press, Cambridge, 2004.

[9] Gessel Ir., Stanton D., Applications of \(q\)-Lagrange inversion to basic hypergeometric series, *Trans. Amer. Math. Soc.* 2111(1983), 173-203.

[10] Gould H. W., Hsu L. C., Some new inverse series relations, *Duke Math. J.* 40(1973), 885-891.

[11] Henrici P., Applied and Computational Complex Analysis, Vol.1, John Wiley& Sons Inc. New York, 1974.

[12] Krattenthaler Ch., A new matrix inverse, *Proc. Amer. Math. Soc.* 124 (1996), 47-59.

[13] Ma X. R., An extension of Warnaar’s matrix inversion, *Proc. Amer.Math.Soc.* 133 (2005), 3179-3189.

[14] Ma X. R., Hsu L.C., The \((\alpha, \beta)\)-inversion formula, *J. Math. Res. & Exposition* 25(4) (2005), 624-624.

[15] Ma X. R., The \((f, g)\)-inversion formula and its applications: the \((f, g)\)-summation formula, *Advances in Appl. Math.* 38 (2007), 227-257.

[16] Milne S. C., Inverse properties of triangular arrays of numbers, *International J. Anal. Appl.* 1 (1981), 1-7.

[17] Milne S. C., Bhatnagar G., A characterization of inverse relations, *Discrete Math.* 193 (1998), 235-245.

[18] Schilling A., Warnaar S. O., Conjugate Bailey pairs. From configuration sums and fractional–level string functions to Bailey’s lemma, *Contemp. Math.* 297 (2002) 227–255.

[19] Schlosser M., Multidimensional matrix inversions and \(A_r\) and \(D_r\) basic hypergeometric series, *Ramanujan J.* 1 (1997), 243-274.

[20] Schlosser M., Some new applications of matrix inversions in \(A_r\), *Ramanujan J.* 3 (1999), 405-461.

[21] Schlosser M., Inversion of bilateral basic hypergeometric series, *Electron. J. Comb.* 10 (2003), #R10, 27pp.
[22] Poorten Alfred J. Van Der, Swart Christine S., Recurrence Relations for Elliptic Sequences: Every Somos 4 is a Somos $k$, \textit{Bull. London Math. Soc.} \textbf{38} (2006), 546-554.

[23] Ward M., Memoir on elliptic divisibility sequences, \textit{Amer. J. Math.} \textbf{70} (1948), 31-74.

[24] Warnaar S. O., Summation and transformation formulas for elliptic hypergeometric series, \textit{Constr. Approx.} \textbf{18}(2002), 479-502.