Verifying Array Manipulating Programs by Tiling

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Abstract. Formally verifying properties of programs that manipulate arrays in loops is computationally challenging. In this paper, we focus on a useful class of such programs, and present a novel property-driven verification method that first infers array access patterns in loops using simple heuristics, and then uses this information to compositionally prove universally quantified assertions about arrays. Specifically, we identify \textit{tiles} of array accesses patterns in a loop, and use the tiling information to reduce the problem of checking a quantified assertion at the end of a loop to an inductive argument that checks only a slice of the assertion for a single iteration of the loop body. We show that this method can be extended to programs with sequentially composed loops and nested loops as well. We have implemented our method in a tool called \texttt{Tiler}. Initial experiments show that \texttt{Tiler} outperforms several state-of-the-art tools on a suite of interesting benchmarks.

1 Introduction

Arrays are widely used in programs written in imperative languages. They are typically used to store large amounts of data in a region of memory that the programmer views as contiguous, and which she can access randomly by specifying an index (or offset). Sequential programs that process data stored in arrays commonly use looping constructs to iterate over the range of array indices of interest and access the corresponding array elements. The ease with which data can be accessed by specifying an index is often exploited by programmers to access or modify array elements at indices that change in complex ways within a loop. While this renders programming easier, it also makes automatic reasoning about such array manipulating programs significantly harder. Specifically, the pattern of array accesses within loops can vary widely from program to program, and may not be easy to predict. Furthermore, since the access patterns often span large regions of the array that depend on program parameters, the array indices of interest cannot be bounded by statically estimated small constants. Hence, reasoning about arrays by treating each array element as a scalar
is not a practical option for analyzing such programs. This motivates us to ask if we can automatically infer program-dependent patterns of array accesses within loops, and use these patterns to simplify automatic verification of programs that manipulate arrays in loops.

A commonly used approach for proving properties of sequential programs with loops is to construct an inductive argument with an appropriate loop invariant. This involves three key steps: (i) showing that the invariant holds before entering the loop for the first time, (ii) establishing that if the invariant holds before entering the loop at any time, then it continues to hold after one more iteration of the loop, and (iii) proving that the invariant implies the desired property when the loop terminates. Steps (i) and (ii) allow us to inductively infer that the invariant holds before every iteration of the loop; the addition of step (iii) suffices to show that the desired property holds after the loop terminates. A significant body of research in automated program verification is concerned with finding invariants that allow the above inductive argument to be applied efficiently for various classes of programs.

For programs with loops manipulating arrays, the property of interest at the end of a loop is often a universally quantified statement over array elements. Examples of such properties include $\forall i \ ((0 \leq i < N) \rightarrow (A[i] \geq minVal) \land (A[i] \leq A[i + 1]))$, $\forall i \ ((0 \leq i < N) \land (i \mod 2 = 0) \rightarrow (A[i] = i))$ and the like. In such cases, a single iteration of the loop typically only ensures that the desired property holds over a small part of the array. Effectively, each loop iteration incrementally contributes to the overall property, and the contributions of successive loop iterations compose to establish the universally quantified property. This suggests the following approach for proving universally quantified assertions about arrays.

- We first **identify the region of the array where the contribution of a generic loop iteration is localized**. Informally, we call such a region a *tile* of the array. Note that depending on the program, the set of array indices representing a tile may not include all indices updated in the corresponding loop iteration. Identifying the right tile for a given loop can be challenging in general; we discuss more about this later.

- Next, we **carve out a “slice” of the quantified property that is relevant to the tile identified above**. Informally, we want this slice to represent the contribution of a generic loop iteration to the overall property. The inductive step of our approach checks if a generic iteration of the loop indeed ensures this slice of the property.

- Finally, we check that **the tiles cover the entire range of array indices of interest, and successive loop iterations do not interfere with each other’s contributions**. In other words, once a loop iteration ensures that the slice of the property corresponding to its tile holds, subsequent loop iterations must not nullify this slice of the property. Formalizing these “range covering” and “non-interference” properties allows us to show that the contributions of
different loop iterations compose to yield the overall quantified property at
the end of the loop.

The remainder of the paper describes a technique and a tool that uses the above
ideas to prove quantified assertions in a useful class of array manipulating pro-
grams. We focus on assertions expressed as universally quantified formulas on
arrays, where the quantification is over array indices. Specifically, suppose $I$ de-
notes a sequence of integer-valued array index variables, $A$ denotes an array and
$V$ denotes a sequence of scalar variables used in the program. We consider as-
sertions of the form $\forall I (\Phi(I) \implies \Psi(A,V,I))$, where $\Phi(I)$ is a quantifier-free
formula in the theory of arithmetic over integers, and $\Psi(A,V,I)$ is a quantifier-
free formula in the combined theory of arrays and arithmetic over integers. In-
formally, such an assertion states that for array indices satisfying condition $\Phi(I)$
(viz. even indices or indices greater than a parameter $N$), the corresponding ar-
ray elements satisfy the property $\Psi(A,V,I)$. The formal syntax of our assertions
is explained in Section 3. In our experience, assertions of this form suffice to
express a large class of interesting properties of array manipulating programs.

Although the general problem of identifying tiles in programs with array
manipulating loops is hard, we have developed some heuristics to automate tile
identification in a useful class of programs. To understand the generic idea behind
our tiling heuristic, suppose the program under consideration has a single loop,
and suppose the quantified property is asserted at the end of the loop. We
introduce a fresh counter variable that is incremented in each loop iteration.
We then use existing arithmetic invariant generation techniques, viz. [1,2], to
identify a relation between the indices of array elements that are accessed and/or
updated in a loop iteration, and the corresponding value of the loop counter.
This information is eventually used to define a tile of the array for the loop under
consideration.

In a more general scenario, the program under verification may have a se-
quence of loops, and the quantified property may be asserted at the end of the
last loop. In such cases, we introduce a fresh counter variable for each loop,
and repeat the above process to identify a tile corresponding to each loop. For
our tiling-based technique to work, we also need invariants, or mid-conditions,
between successive loops in the program. Since identifying precise invariants is
uncomputable in general, we work with candidate invariants reported by exist-
ing off-the-shelf annotation/candidate-invariant generators. Specifically, in our
implementation, we use the dynamic analysis tool Daikon [2] that informs us
of candidate invariants that are likely (but not proven) to hold between loops.
Our algorithm then checks to see if the candidate invariants reported after every
loop can indeed be proved using the tiling-based technique. Only those can-
didates that can be proved in this way are subsequently used to compose the
tiling-based reasoning across consecutive loops. Finally, tiling can be applied to
programs with nested loops as well. While the basic heuristic for identifying
tiles remains the same in this case, the inductive argument needs to be carefully
constructed when reasoning about nested loops. We discuss this in detail later
in the paper.
We have implemented the above technique in a tool called Tiler. Our tool takes as input a C function with one or more loops manipulating arrays. It also accepts a universally quantified assertion about arrays at the end of the function. Tiler automatically generates a tiling of the arrays for each loop in the C function and tries to prove the assertion, as described above. We have applied Tiler to a suite of 60 benchmarks comprised of programs that manipulate arrays in different ways. For most benchmarks where the specified assertion holds, Tiler was able to prove the assertion reasonably quickly. In contrast, two state-of-the-art tools for reasoning about arrays faced difficulties and timed out on most of these benchmarks. For benchmarks where the specified assertion does not hold, Tiler relies on bounded model checking to determine if an assertion violation can be detected within a few unwindings of the loops. There are of course corner cases where Tiler remains inconclusive about the satisfaction of the assertion. Overall, our initial experiments suggest tiling-based compositional reasoning can be very effective for proving assertions in a useful class of array manipulating programs.

The primary contributions of the paper can be summarized as follows.

– We introduce the concept of tiling for reasoning about quantified assertions in programs manipulating arrays in loops.
– We present a tiling-based practical algorithm for verifying a class of array manipulating programs.
– We describe a tool that outperforms several state-of-the-art tools for reasoning about arrays on a suite of benchmarks. Our tool performs particularly well on benchmarks where the quantified assertion holds.

2 Motivating Example

Fig. 1(a) shows a C function snippet adapted from an industrial battery controller. This example came to our attention after a proprietary industry-strength static analysis tool failed to prove the quantified assertion at the end of the function. Note that the function updates an array volArray whose size is given by COUNT. In general, COUNT can be large, viz. 100000. The universally quantified assertion at the end of the “for” loop requires that every element of volArray be either zero or at least as large as MIN. It is not hard to convince oneself through informal reasoning that the assertion indeed holds. The difficulty lies in proving it automatically. Indeed, neither Booster [3] nor Vaphor [4], which can reason about arrays with parameterized bounds, are able to prove this assertion within 15 minutes on a desktop machine. Bounded model checking tools like CBMC [5] and SMACK+Corral [6] are able to prove this assertion for arrays with small values of COUNT. For large arrays, viz. COUNT = 100000, these tools cannot prove the assertion within 15 minutes on a desktop machine. This is not surprising since a bounded model checker must unwind the loop in the function a large number of times if COUNT is large.

Let us now illustrate how tiling-based reasoning works in this example. We introduce a fresh auxiliary variable (say j) to denote the index used to up-
```c
void BatteryController () {
    int COUNT, MIN, i;
    int volArray[COUNT];

    if( COUNT %4 != 0) return;

    for (i =1;i <= COUNT /4; i ++) {
        if (5 >= MIN )
            volArray[i *4 -4] = 5;
        else
            volArray[i *4 -4] = 0;

        if (7 >= MIN )
            volArray[i *4 -3] = 7;
        else
            volArray[i *4 -3] = 0;

        if (3 >= MIN )
            volArray[i *4 -2] = 3;
        else
            volArray[i *4 -2] = 0;

        if (1 >= MIN )
            volArray[i *4 -1] = 1;
        else
            volArray[i *4 -1] = 0;
    }
}
```

```c
void BatteryControllerInst () {
    int COUNT, MIN, i,j;
    int volArray[COUNT];

    if( COUNT %4 != 0) return;
    assume (i >=1 && i <= COUNT /4);
    assume (4*i -4 <=j && j <4* i);

    if (5 >= MIN )
        volArray[i *4 -4] = 5;
    else
        volArray[i *4 -4] = 0;

    if (7 >= MIN )
        volArray[i *4 -3] = 7;
    else
        volArray[i *4 -3] = 0;

    if (3 >= MIN )
        volArray[i *4 -2] = 3;
    else
        volArray[i *4 -2] = 0;

    if (1 >= MIN )
        volArray[i *4 -1] = 1;
    else
        volArray[i *4 -1] = 0;

    assert ( volArray[j] >= MIN || volArray[j] ==0);
}
```

Fig. 1: Motivating example period-4

date an element of \texttt{volArray}. Using arithmetic invariant generation techniques, viz. INVGEN \cite{1}, we can now learn that for all array accesses in the \(i\)th loop iteration, the value of the index lies between \(4 \times i - 4\) and \(4 \times i\). Therefore, we choose \([4 \times i - 4, 4 \times i]\) as the tile corresponding to the \(i\)th iteration of the loop.

In order to successfully apply the tiling-based reasoning, we must ensure that our tiles satisfy certain properties.

- **Covers range**: This ensures that every tile contains only valid array indices, and that no array index of interest in the quantified assertion is left unaccounted for in the tiles. In our example, array indices range from 0 to \(COUNT-1\), while the loop (and hence, tile) counter \(i\) ranges from 1 to \(COUNT/4\). Since the \(i\)th tile comprises of the array indices \(4i - 4, 4i - 3, 4i - 2\) and \(4i - 1\), both the above requirements are met.

- **Sliced property holds for tile**: The sliced property in this case says that the elements of \texttt{volArray} corresponding to indices within a tile have values that are either 0 or at least \(MIN\). To prove that this holds after an iteration of the loop, we first obtain a loop-free program containing a single generic iteration of the loop, and check that the elements of \texttt{volArray} corresponding to the \(i\)th tile satisfy the sliced property after the execution of the \(i\)th loop iteration. The transformed program is shown in Fig.\text{(b)}. Note that this program has a fresh variable \(j\). The assume statements at lines 7-8 say that \(i\) is within the expected range and that \(j\) is an index in the \(i\)th tile. Since this program
is loop-free, we can use a bounded model checker like CBMC to prove
the assertion in the transformed program.

- **Non-interference across tiles**: To show this, we assume that the sliced property holds for the \(i\)-th tile, where \(0 \leq i' < i\), before the \(i\)-th loop iteration starts. This can be done by adding the following three extra assumptions after lines 7 and 8 in Fig. 1(b): (i) assume \((1 \leq i' < i)\), (ii) assume \((4\times i' - 4 \leq j' < 4\times i')\), and (iii) assume \((\text{volArray}[j'] \geq \text{MIN} || \text{volArray}[j'] == 0)\). We then assert at the end of the loop body that the sliced property for the \(i\)-th tile continues to hold even after the \(i\)-th iteration.

This can be done by replacing the assertion in line 26 of Fig. 1(b) by \textbf{assert} \((\text{volArray}[j'] \geq \text{MIN} || \text{volArray}[j'] == 0)\). As before, since the program in Fig. 1(b) is loop-free, this assertion can be easily checked using a bounded model checker like CBMC.

Once all the above checks have succeeded, we can conclude that the quantified assertion holds in the original program after the loop terminates. Note the careful orchestration of inductive reasoning to prove the sliced property, and compositional reasoning to aggregate the slices of the property to give the original quantified assertion. Our tiling-based tool proves the assertion in this example in less than a second.

### 3 Preliminaries

For purposes of this paper, an array-manipulating program \(P\) is a tuple \((\mathcal{V}, \mathcal{L}, \mathcal{A}, \text{PB})\), where \(\mathcal{V}\) is a set of scalar variables, \(\mathcal{L} \subseteq \mathcal{V}\) is a set of scalar loop counter variables, \(\mathcal{A}\) is a set of array variables, and \(\text{PB}\) is the program body generated by the following grammar:

\[
\text{PB} ::= \text{St} \\
\text{St} ::= v := E | A[E] := E | \text{assume}(\text{BoolE}) | \text{if}(\text{BoolE}) \text{ then } \text{St} \text{ else } \text{St} | \text{for } (\ell := 0; \ell < E; \ell := \ell + 1) \{\text{St}\} | \text{St} ; \text{St} \\
E ::= E \text{ op } E | A[E] | v | \ell | c \\
\text{BoolE} ::= E \text{ relop } E | \text{BoolE AND BoolE} | \text{NOT} \text{ BoolE} | \text{BoolE OR BoolE}
\]

Here, we assume that \(A \in \mathcal{A}, v \in \mathcal{V}\setminus\mathcal{L}, \ell \in \mathcal{L}\) and \(c \in \mathbb{Z}\). We also assume that “op” (resp. “relop”) is one of a set of arithmetic (resp. relational) operators. We wish to highlight the following features of programs generated by this grammar:

- There are no unstructured jumps, like those effected by \textbf{goto} or \textbf{break} statements in C-like languages. The effect of a \textbf{break} statement inside a loop in a C-like language can always be modeled by setting a flag, and by conditioning the execution of subsequent statements in the loop body on this flag being not set, and by using this flag to determine whether to exit the loop. The effect of a \textbf{break} statement in a conditional branch can also be similarly modeled. Therefore, we can mimic the behaviour of \textbf{break} statements in our programs.


– We can have sequences of possibly nested loops, with non-looping program fragments between loops. Furthermore, the body of a loop and the corresponding loop head, i.e. control location where the loop is entered, are easily identifiable.

– Every loop is associated with a scalar loop counter variable that is set to 0 when the loop is entered, and incremented after every iteration of the loop. We assume that each loop has a unique counter variable.

– The only assignments to loop counter variables happen when a loop is entered for the first time and at the end of an iteration of the corresponding loop body. Other assignment statements in the program cannot assign to loop counter variables. Loop counter variables can however be freely used in expressions throughout the program.

– The restriction on the usage of loop counter variables simplifies the analysis and presentation, while still allowing a large class of programs to be effectively analyzed. Specifically, whenever the count of iterations of a loop can be expressed in a closed form in terms of constants and variables not updated in the loop, we can mimic its behaviour using our restricted loops. As a generic example, suppose we are told that the loop for (i:=exp1; Cond; i:=exp2) { LoopBody } iterates exp3 times, where exp3 is an arithmetic expression in terms of constants and variables not updated in the loop. The behaviour of this loop can be mimicked using the following restricted loop, where l and flag are fresh variables not present in the original program:

\[
\text{for (l:=0; l<exp3 ; l:=l+1) } \\
\text{ { if (l=0) } \\
\text{ { i:=exp1; if (Cond) { LoopBody; i:=i-2} }} \\
\text{ else { if (l == 0) { i:=exp1; if (Cond) { LoopBody; i:=i-2} }} }
\]

To see a specific example of this transformation, suppose the program under verification has the loop:

\[
\text{for (i:=2*M; i>=0; i:=i-2) } \\
\text{ { LoopBody } }
\]

where M and i are variables not updated in LoopBody. Clearly, this loop iterates \((M + 1)\) times. Therefore, it can be modeled in our restricted language as:

\[
\text{for (l:=0; l<M+1; l:=l+1) } \\
\text{ { if (l==0) { i:=2*M; if (i >= 0) { LoopBody; i:=i-2} }} }
\]

For clarity of exposition, we abuse notation and use \(V\) and \(A\) to also denote a sequence of scalar and array variables, when there is no confusion. A verification problem for an array manipulating program is a Hoare triple \(\{\text{PreCond}\} P \{\text{PostCond}\}\), where each of PreCond and PostCond are quantified formulae of the form \(\forall I (\Phi(I) \implies \Psi(A, V, I))\). Here, \(I\) is assumed to be a sequence of array index variables, \(\Phi\) is a quantifier-free formula in the theory of arithmetic over integers, and \(\Psi\) is a quantifier-free formula in the combined theory of arrays and arithmetic over integers. The formula \(\Phi(I)\) identifies the relevant indices of the array where the property \(\Psi(A, V, I)\) must hold. This allows us to express a large class of useful pre- and post-conditions, including sortedness, which can be expressed as \(\forall j (0 \leq j < N) \rightarrow (A[j] \leq A[j+1])\).

Let \(\text{AtomSt}\) denote the set of atomic statements in a program generated by the above grammar. These are statements of the form \(v := E\), \(A[E] := E\) or \(\text{assume}(E)\). It is common to represent such a program by a control flow graph \(G = (N, E, \mu)\), where \(N\) denotes the set of control loca-
lations of the program, \( E \subseteq N \times N \times \{tt, ff, U\} \) represents the flow of control, and \( \mu : N \rightarrow \text{AtomSt} \cup \text{BoolE} \) annotates every node in \( N \) with either an assignment statement, an assume statement or a Boolean condition.

We assume there are two distinguished vertices called Start and End in \( N \), that represent the entry and exit points of control flow for the program. An edge \( (n_1, n_2, L) \) represents flow of control from \( n_1 \) to \( n_2 \) without any other intervening node. The edge is labeled \( tt \) or \( ff \) if \( \mu(n_1) \) is a Boolean condition, and it is labeled \( U \) otherwise. If \( \mu(n_1) \) is a Boolean condition, there are two outgoing edges labeled \( tt \) and \( ff \) respectively, from \( n_1 \). Control flows from \( n_1 \) to \( n_2 \) along \( (n_1, n_2, L) \) only if \( \mu(n_1) \) evaluates to \( L \). If \( \mu(n_1) \) is an assume or assignment statement, there is a single outgoing edge from \( n_1 \), and it is labeled \( U \). Henceforth, we use CFG to refer to a control flow graph.

A CFG may have cycles in general. A back-edge in a CFG is an edge from a node (control location) within the body of a loop to the node representing the corresponding loop head. Clearly, removing all back-edges from a CFG renders it acyclic. The target nodes of back-edges, i.e. nodes corresponding to loop heads, are also called cut-points of the CFG. Every acyclic sub-graph of a CFG that starts from a cut-point or Start and ends at another cut-point or End, and that does not pass through any other cut-points in between and also does not include any back-edge, is called a segment. For example, consider the CFG shown in Fig. 2 for clarity, edges labeled \( U \) are shown unlabeled in the figure. The cut-points in this CFG are nodes 1, 2 and 3, the back-edges are \( e_1, e_2 \) and \( e_3 \), and the segments are \( S \rightarrow 1, 1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow \{4, 6\} \rightarrow 5, 2 \rightarrow 7 \) and \( 1 \rightarrow E \). Note that every segment is an acyclic sub-graph of the CFG with a unique source node and a unique sink node.

4 A Theory of Tiles

In this section, we present a theory of tiles for proving universally quantified properties of arrays in programs that manipulate arrays within loops.

4.1 Tiling in a simple setting

Consider a program \( P \) as defined in the previous section that accesses elements of an array \( A \) in a loop \( L \). Suppose \( P \) has a single non-nested loop \( L \) with loop counter \( \ell \) and loop exit condition \( (\ell < \mathcal{E}_L) \), where \( \mathcal{E}_L \) is an arithmetic expression involving only constants and variables not updated in \( L \). Thus, the loop iterates \( \mathcal{E}_L \) times, with the value of \( \ell \) initialized to 0 at the beginning of the first iteration,
and incremented at the end of each iteration. Each access of an element of A in the loop is either a read access or a write access. For example, in the program shown in Fig. 3, the loop L (lines 2-11) has three read accesses of A (at lines 5, 6, 7), and three write accesses of A (at lines 3, 6, 7). In order to check an assertion about the array at the end of the loop (see, for example, line 12 of Fig. 3), we wish to tile the array based on how its elements are updated in different iterations of the loop, reason about the effect of each loop iteration on the corresponding tile, and then compose the tile-wise reasoning to prove/disprove the overall assertion.

Note that the idea of tiling an array based on access patterns in a loop is not new, and has been used earlier in the context of parallelizing and optimizing compilers [7,8]. However, its use in the context of verification has been limited [9]. To explore the idea better, we need to formalize the notion of tiles.

Let Indices\(_A\) denote the range of indices of the array A. We assume that this is available to us; in practice, this can be obtained from the declaration of A if it is statically declared, or from the statement that dynamically allocates the array A. Let Pre and Post denote the pre- and post-conditions, respectively, for the loop L under consideration. Recall from Section 3 that both Pre and Post have the form \(\Psi(j) \implies \Psi(A, V, j)\), where V denotes the set of scalar variables in the program. To keep the discussion simple, we consider Post to be of this specific form for the time being, while ignoring the form of Pre. We show later how the specific form of Pre can be used to simplify the analysis further. For purposes of simplicity, we also assume that the array A is one-dimensional; our ideas generalize easily to multi-dimensional arrays, as shown later. Let Inv be a (possibly weak) loop invariant for loop L. Clearly, if Pre \(\implies\) Inv and Inv \(\land\) \(\neg(\ell < E_L)\) \(\implies\) Post, then we are already done, and no tiling is necessary. The situation becomes interesting when Inv is not strong enough to ensure that Inv \(\land\) \(\neg(\ell < E_L)\) \(\implies\) Post. We encounter several such cases in our benchmark suite, and it is here that our method adds value to existing verification flows.

A tiling of A with respect to L, Inv and Post is a binary predicate \(\text{Tile}_{L, \text{Inv}, \text{Post}} : \mathbb{N} \times \text{Indices}_A \rightarrow \{\text{tt}, \text{ff}\}\) such that conditions T1 through T3 listed below hold. Note that these conditions were discussed informally in Section 2 in the context of our motivating example. For ease of notation, we use Tile instead of \(\text{Tile}_{L, \text{Inv}, \text{Post}}\) below, when L, Inv and Post are clear from the context. We also use \("\ell\)th tile" to refer to all array indices in the set \(\{j \mid (j \in \text{Indices}_A) \land \text{Tile}(\ell, j)\}\).

(T1) Covers range: Every array index of interest must be present in some tile, and every tile contains array indices in Indices\(_A\). Thus, the formula \(\eta_1 \land \eta_2\) must be valid, where 
\[
\eta_1 \equiv \forall j (j \in \text{Indices}_A) \land \Psi(j) \implies \exists \ell ((0 \leq \ell < E_L) \land \text{Tile}(\ell, j)),
\]
and
\[
\eta_2 \equiv \forall \ell ((0 \leq \ell < E_L) \land \text{Tile}(\ell, j) \implies (j \in \text{Indices}_A)).
\]
(T2) Sliced post-condition holds inductively: We define the sliced post-condition for the \( \ell \)-th tile as \( \text{Post}_{\text{Tile}(\ell, \cdot)} \triangleq \forall j (\text{Tile}(\ell, j) \land \Phi(j) \implies \Psi(A, V, j)) \). Thus, \( \text{Post}_{\text{Tile}(\ell, \cdot)} \) asserts that \( \Psi(A, V, j) \) holds for all relevant \( j \) in the \( \ell \)-th tile. We now require that if the (possibly weak) loop invariant \( \text{Inv} \) and the sliced post-condition for the \( \ell' \)-th tile for all \( \ell' \in \{0, \ldots, \ell - 1\} \) hold prior to executing the \( \ell \)-th loop iteration, then the sliced post condition for the \( \ell \)-th tile and \( \text{Inv} \) must also hold after executing the \( \ell \)-th loop iteration.

Formally, if \( L_{\text{body}} \) denotes the body of the loop \( L \), the Hoare triple given by \( \{\text{Inv} \land \bigwedge_{\ell' \in [0, \ell-1]} \text{Post}_{\text{Tile}(\ell', \cdot)}\} L_{\text{body}} \{\text{Inv} \land \text{Post}_{\text{Tile}(\ell, \cdot)}\} \) must be valid for all \( \ell \in \{0, \ldots, E_\ell - 1\} \).

(T3) Non-interference across tiles: For every pair of iterations \( \ell, \ell' \) of the loop \( L \) such that \( \ell' < \ell \), the later iteration \( (\ell) \) must not falsify the sliced post condition \( \text{Post}_{\text{Tile}(\ell', \cdot)} \) rendered true by the earlier iteration \( (\ell') \).

Formally, the Hoare triple \( \{\text{Inv} \land (0 \leq \ell' < \ell) \land \text{Post}_{\text{Tile}(\ell', \cdot)}\} L_{\text{body}} \{\text{Post}_{\text{Tile}(\ell, \cdot)}\} \) must be valid for all \( \ell \in \{0, \ldots, E_\ell - 1\} \).

Note that while tiling depends on \( L \), \( \text{Inv} \) and \( \text{Post} \) in general, the pattern of array accesses in a loop often suggests a natural tiling of array indices that suffices to prove multiple assertions \( \text{Post} \) using reasonably weak loop invariants \( \text{Inv} \). The motivating example in Section 2 illustrated this simplification.

Theorem 1. Suppose \( \text{Tile}_{L, \text{Inv}, \text{Post}} : \mathbb{N} \times \text{Indices}_A \to \{\tt, \ff\} \) satisfies conditions T1 through T3. If \( \text{Pre} \implies \text{Inv} \) also holds and the loop \( L \) iterates at least once, then the Hoare triple \( \{\text{Pre}\} L \{\text{Post}\} \) holds.

Proof sketch: The proof proceeds by induction on the values of the loop counter \( \ell \). The inductive claim is that at the end of the \( \ell \)-th iteration of the loop, the post-condition \( \bigwedge_{\ell' \in [0, \ell]} \text{Post}_{\text{Tile}(\ell, \cdot)} \) holds. The base case is easily seen to be true from condition T2 and from the fact that \( \text{Pre} \implies \text{Inv} \). Condition T3 and the fact that \( \ell \) is incremented at the end of each loop iteration ensure that once we have proved \( \text{Post}_{\text{Tile}(\ell, \cdot)} \) at the end of the \( \ell \)-th iteration, it cannot be falsified in any subsequent iteration of the loop. Condition T2 now ensures that the sliced post-condition can be inductively proven for the \( \ell \)-th tile. By condition T1, we also have \( \bigwedge_{0 \leq \ell' < \ell} \text{Post}_{\text{Tile}(\ell, \cdot)} \equiv \text{Post} \). Since the loop \( L \) iterates with \( \ell \) increasing from 0 to \( E_\ell - 1 \), it follows that \( \text{Post} \) indeed holds if \( \text{Inv} \) holds before the start of the first iteration. This is the compositional step in our approach. Putting all the parts together, we obtain a proof of \( \{\text{Pre}\} L \{\text{Post}\} \).

A few observations about the conditions are worth noting. First, note that there is an alternation of quantifiers in the check for T1. Fortunately, state-of-the-art SMT solvers like Z3 [10] are powerful enough to check this condition efficiently for tiles expressed as Boolean combinations of linear inequalities on \( \ell \) and \( V \), as is the case for the examples in our benchmark suite. We anticipate that
with further advances in reasoning about quantifiers, the check for condition T1 will not be a performance-limiting step.

The checks for T2 and T3 require proving Hoare triples with post-conditions that have a conjunct of the form \( \text{Post}_{\text{Tile}(e, \cdot)} \). From the definition of a sliced post-condition, we know that \( \text{Post}_{\text{Tile}(e, \cdot)} \) is a universally quantified formula. Additionally, the pre-condition for T2 has a conjunct of the form \( \bigwedge_{\ell' \preceq \ell} \text{Post}_{\text{Tile}(e', 
abla)} \), which is akin to a universally quantified formula. Therefore T2 and T3 can be checked using Hoare logic-based reasoning tools that permit quantified pre- and post-conditions, viz. \[13\] \[14\]. Unfortunately, the degree of automation and scalability available with such tools is limited today. To circumvent this problem, we propose to use stronger Hoare triple checks that logically imply T2 and T3, but do not have quantified formulas in their pre- and post-conditions. Since the program, and hence \( L_{\text{body}} \), is assumed not to have nested loops, state-of-the-art bounded model checking tools that work with quantifier-free pre- and post-conditions, viz. CBMC, can be used to check these stronger conditions. Specifically, we propose the following pragmatic replacements of T2 and T3.

\((T2^*)\) Let \( \text{RdAcc}_{\ell}(e) \) denote the set of array index expressions corresponding to read accesses of \( A \) in the \( \ell \)th iteration of the loop \( L \). For example, in Fig. 3 \( \text{RdAcc}_{\ell}(e) = \{\ell, \ell - 1\} \). Clearly, if \( L_{\text{body}} \) is loop-free, \( \text{RdAcc}_{\ell}(e) \) is a finite set of expressions. Suppose \( |\text{RdAcc}_{\ell}(e)| = k \) and let \( e_1, \ldots, e_k \) denote the expressions in \( \text{RdAcc}_{\ell}(e) \). Define \( \zeta(e) \) to be the formula \( \bigwedge_{\ell \in \text{RdAcc}_{\ell}(e)} \left( ((0 \leq \ell < \ell) \land \text{Tile}(e_k, \psi_{\ell}, e_k) \land \Phi_{\ell}(e_k) \Rightarrow \Psi(A, V, e_k)) \right) \), where \( e_k \) are fresh variables not used in the program. Informally, \( \zeta(e) \) states that if \( A[e_k] \) is read in the \( \ell \)th iteration of \( L \) and if \( e_k \) belongs to the \( \ell \)-th \( (\ell < \ell) \) tile, then \( \Phi(e_k) \Rightarrow \Psi(A, V, e_k) \) holds.

We now require the following Hoare triple to be valid, where \( j \) is a fresh free variable not used in the program.

\[ \{ \text{Inv} \land (0 \leq \ell < \ell) \land \zeta(e) \land \text{Tile}(e, j) \land \Phi(j) \} \quad L_{\text{body}} \quad \{ \text{Inv} \land \Psi(A, V, j) \} \]

\((T3^*)\) Let \( j' \) and \( \ell' \) be fresh free variables that are not used in the program. We require the following Hoare triple to be valid:

\[ \{ \text{Inv} \land (0 \leq \ell' < \ell < \ell) \land \text{Tile}(e', j') \land \Phi(j') \land \Psi(A, V, j') \} \quad L_{\text{body}} \quad \{ \Psi(A, V, j') \} \]

**Lemma 1.** The Hoare triple in \( T2^* \) implies that in \( T2 \). Similarly, the Hoare triple in \( T3^* \) implies that in \( T3 \).

The proof follows from the observation that a counterexample for validity of the Hoare triple in \( T2 \) or \( T3 \) can be used to construct a counterexample for validity of the triple in \( T2^* \) or \( T3^* \) respectively.

Observe that \( T2^* \) and \( T3^* \) require checking Hoare triples with quantifier-free formulas in the pre- and post-conditions. This makes it possible to use assertion checking tools that work with quantifier-free formulas in pre- and post-conditions. Furthermore, since \( L_{\text{body}} \) is assumed to be loop-free, these checks can also be discharged using state-of-the-art bounded model checkers, viz. CBMC. The scalability and high degree of automation provided by tools like CBMC make conditions T1, T2* and T3* more attractive to use.
4.2 Tiling in more general settings

The above discussion was restricted to a single uni-dimensional array accessed within a single non-nested loop in a program P. We now relax these restrictions and show that the same technique continues to work with some adaptations.

We consider the case where P is a sequential composition of possibly nested loops. To analyze such programs, we identify all segments in the CFG of P. Let \textbf{CutPts} be the set of cut-points of the CFG. Recall from Section 3 that a segment s is a sub-DAG of the CFG between a source node in \textbf{CutPts} \cup \{\text{Start}\} and a sink node in \textbf{CutPts} \cup \{\text{End}\}. Thus, a segment s corresponds to a loop-free fragment of P. Let \(\ell_s\) denote the loop counter variable corresponding to the innermost loop in which s appears. We assign \(\perp\) to \(\ell_s\) if s lies outside all loops in P. Let \textbf{OuterLoopCtrs}_s denote the set of loop counter variables of all outer loops (excluding the innermost one) that enclose (or nest) s. The syntactic restrictions of programs described in Section 3 ensure that \(\ell_s\) and \textbf{OuterLoopCtrs}_s are uniquely defined for every segment s.

Suppose we are given (possibly weak) invariants at every cut-point in P, where \(\text{Inv}_c\) denotes the invariant at cut-point c. We assume the invariants are of the usual form \(\forall I (\Phi(I) \implies \Psi(A, V, I))\), where I is a sequence of quantified array index variables, and A and V are sequences of array and scalar variables respectively. Let \(A_s\) be a sequence of arrays that are updated in the segment s between cut-points \(c_1\) and \(c_2\), and for which \(\ell_s \neq \perp\). We define a tiling predicate \(\text{Tile}_{s,\text{Inv}_{c_1},\text{Inv}_{c_2}} : \mathbb{N} \times \text{Indices}_{A_s} \rightarrow \{tt, ff\}\), where \(\text{Indices}_{A_s} = \prod_{A \in A_s} \text{Indices}_{A}\) plays a role similar to that of \(\text{Indices}_A\) in Section 4.1 (where a single array A was considered). The predicate \(\text{Tile}_{s,\text{Inv}_{c_1},\text{Inv}_{c_2}}\) relates values of the loop counter \(\ell_s\) of the innermost loop containing s to the index expressions that define the updates of arrays in \(A_s\) in the program segment s. The entire analysis done in Section 4.1 for a simple loop L can now be re-played for segment s, with \(\text{Inv}_{c_1}\) playing the role of \(\text{Inv}\), \(\text{Inv}_{c_2}\) playing the role of \(\text{Post}\), \(V \cup \text{OuterLoopCtrs}_s\) playing the role of V, and \(\ell_s\) playing the role of \(\ell\). If the segment s is not enclosed in any loop, i.e. \(\ell_s = \perp\), we need not define any tiling predicate for this segment. This obviates the need for conditions T1 and T3, and checking T2 simplifies to checking the validity of the Hoare triple \{\text{Inv}_{c_1}\} s \{\text{Inv}_{c_2}\}. In general, \text{Inv}_{c_1} and \text{Inv}_{c_2} might be universally quantified formulas. In such cases, the technique used to simplify condition T2 to T2* in Section 4.1 can be applied to obtain a stronger condition, say T2**, that does not involve any tile, and requires checking a Hoare triple with quantifier-free pre- and post-conditions. If the condition checks for all segments as described above succeed, it follows from Theorem 1 and Lemma 1 that we have a proof of \{\text{Pre}\} P \{\text{Post}\}.

Recall that in Section 4.1, we ignored the specific form of the pre-condition \text{Pre}. As defined in Section 3, \text{Pre} has the same form as that of the post-condition and invariants at cut-points considered above. Therefore, the above technique works if we treat \text{Pre} as \text{Inv}_{\text{Start}} and \text{Post} as \text{Inv}_{\text{End}}.

The extension to multi-dimensional arrays is straightforward. Instead of using one index variable \(j\) for accessing arrays, we now allow a tuple of index variables \((j_1, j_2, \ldots, j_r)\) for accessing arrays. Each such variable \(j_i\) takes values from its...
own domain, say $\text{Indices}_A$. The entire discussion about tiles above continues to hold, including the validity of Theorem 1 if we replace every occurrence of an array index variable $j$ by a sequence of variables $j_1, \ldots, j_r$ and every occurrence of $\text{Indices}_A$ by $\text{Indices}_A \times \text{Indices}_A \times \ldots \times \text{Indices}_A$.

5 Verification by Tiling

The discussion in the previous section suggests a three-phase algorithm, presented as Algorithm 1, for verifying quantified properties of arrays in programs with sequences of possibly nested loops manipulating arrays. In the first phase of the algorithm, we use bounded model checking with small pre-determined loop unrollings to check for assertion violations. If this fails, we construct the CFG of the input program $P$, topologically sort its cut-points and initialize the sets of candidate invariants at each cut-point to $\emptyset$.

Algorithm 1 TILEDVERIFY($P$ : program, $\text{Pre}$: pre-condn, $\text{Post}$: post-condn)

1: Let $G$ be the CFG for program $P = (A, V, L, \text{PB})$, as defined in Section 3.

2: Do bounded model checking with pre-determined small loop unrollings;
3: if counterexample found then return “Post condition violated!”;
4: CutPts := set of cut-points in $G$;  
5: Remove all back-edges from $G$ and topologically sort CutPts;  
6: for each $c$ in CutPts do
7:   $\text{CandInv}[c] := \emptyset$;  
8: $\text{CandInv}[\text{Start}] := \text{Pre}; \text{CandInv}[\text{End}] := \text{Post};$  
9: for each segment $s$ from $c_1$ to $c_2$, where $c_1, c_2 \in \text{CutPts} \cup \{\text{Start}, \text{End}\}$ and $c_1 \leq c_2$ do
10:   if $s$ lies within a loop then
11:     $\ell[S] :=$ loop counter of innermost nested loop containing $s$;
12:     OuterLoopCtrs[$s] :=$ loop counters of all other outer loops containing $s$;
13:   else
14:     $\ell[S] := 1; \text{OuterLoopCtrs}[s] := \emptyset;$
15:   $\text{ScalarVars}[s] := V \cup \text{OuterLoopCtrs}[s]$;
16:   $\text{CandInv}[c_2] := \text{CandInv}[c_2] \cup \text{findHeuristicCandidateInvariants}(s, c_2, \ell[S], \text{ScalarVars}[s], A);$

17: for each segment $s$ from $c_1$ to $c_2$ do
18:   if $s$ lies within a loop then
19:     $\text{CandTile}[s] := \text{findHeuristicTile}(s, \ell[S], \text{ScalarVars}[s], A);$  
20:     Check conditions $T1, T2^*$ and $T3^*$ for $\text{CandTile}[s]$, as described in Section 4.1
21:     if (not timed out) AND ($T1$ or $T3^*$ fail) then
22:       Re-calculate $\text{CandTile}[s]$ using different heuristics; goto 20
23:     if (not timed out) AND ($T2^*$ fail) AND ($c_2 \neq \text{End}$) then
24:       Re-calculate $\text{CandInv}[c_2]$ using different heuristics; goto 20
25:   else  

26:     Check condition $T2^*$, as described in Section 4.2
27:     if (not timed out) AND ($T2^*$ fails) AND ($c_2 \neq \text{End}$) then
28:       Re-calculate $\text{CandInv}[c_2]$ using different heuristics; goto 20
29:     if timed out then return “Time out! Inconclusive answer!”
30: return “Post-condition verified!”
In the second phase, we generate candidate invariants at each cut-point \( c \) by considering every segment \( s \) that ends at \( c \). For each such segment \( s \), we identify the loop counter \( \ell[s] \) corresponding to the innermost loop in which \( s \) appears, and the set of loop counters \( \text{OuterLoopCtrs}[s] \) corresponding to other loops that contain (or nest) \( s \). Note that when the program fragment in the segment \( s \) executes, the active loop counter that increments from one execution of \( s \) to the next is \( \ell[s] \). The loop counters in \( \text{OuterLoopCtrs}[s] \) can be treated similar to other scalar variables in \( \mathcal{V} \) when analyzing segment \( s \). We would like the candidate invariants identified at different cut-points to be of the form \( \forall I \ (\Phi(I) \implies \Psi(A, \mathcal{V}, I)) \), whenever possible. We assume access to a routine \text{findHeuristicCandidateInvariants} for this purpose. Note that the candidate invariants obtained from this routine may not actually hold at \( c_2 \). In the next phase, we check using tiling whether a candidate invariant indeed holds at a cut-point, and use only those candidates that we are able to prove.

**Algorithm 2** \text{findHeuristicTile}(s : segment, \ell : loop counter, \text{ScalarVars}: set of scalars, \( A \): set of arrays)

1: Let \( c_1 \) be the starting cut-point (or Start node) of \( s \);
2: for each array \( A \) updated in \( s \) do
3: \quad \text{UpdIndexExprs}^A[s] := \emptyset;
4: for each update of the form \( A[e] := e' \) at location \( e \) in \( s \) do \quad \triangleright e and \( e' \) are arith expns
5: \quad \quad \hat{e} := e \text{ in terms of \( \ell \), ScalarVars}, \( A \) at \( c_1 \) \quad \triangleright \text{Obtained by backward traversal from } e \text{ to } c_1
6: \quad \text{UpdIndexExprs}^A[s] := \text{UpdIndexExprs}^A[s] \cup (\hat{e})
7: \quad \text{InitTile}^A(\ell, j) := \text{Simplify}(\bigvee_{e \in \text{UpdIndexExprs}^A[s]} (j = e)); \quad \triangleright \text{Initial estimate of tile}
8: \quad for each \( e \in \text{UpdIndexExprs}^A[s] \) do
9: \quad \quad if \ (\text{InitTile}^A(\ell, e) \land \text{InitTile}^A(\ell + k, e) \land (0 \leq \ell < \ell + k < \mathcal{E}_c)) \text{ is satisfiable} \text{ then}
10: \quad \quad \quad \text{Remove } e \text{ from } \text{UpdIndexExprs}^A[s];
11: \quad \quad \text{Tile}^A(\ell, j) := \text{Simplify}(\bigvee_{e \in \text{UpdIndexExprs}^A[s]} (j = e)); \quad \triangleright \text{Refined tile}
12: \quad return \bigwedge_{A, A} \text{Tile}^A(\ell, \cdot);

In the third phase, we iterate over every segment \( s \) between cut-point \( c_1 \) and \( c_2 \) again, and use heuristics to identify tiles. This is done by a routine \text{findHeuristicTile}. The working of our current tiling heuristic is shown in Algorithm 2. For every array update \( A[e] := e' \) in segment \( s \), the heuristic traverses the control flow graph of \( s \) backward until it reaches the entry point of \( s \), i.e. \( c_1 \), to determine the expression \( e \) in terms of values of \( \ell[s] \), \( \mathcal{V} \), \( \text{OuterLoopCtrs}[s] \) and \( A \) at \( c_1 \). Let \( \text{UpdIndexExprs}^A[s] \) denote the set of such expressions for updates to \( A \) within \( s \). We identify an initial tile for \( A \) in \( s \) as \( \text{InitTile}^A(\ell[s], j) \equiv \bigvee_{e \in \text{UpdIndexExprs}^A[s]} (j = e) \). It may turn out that the same array index expression appears in two or more initial tiles after this step. For example, in Fig. 3 we obtain \( \text{InitTile}^A(\ell, j) \equiv (\ell \leq j \leq \ell + 1) \), and hence \( \text{InitTile}^A(\ell, \ell + 1) \land \text{InitTile}^A(\ell + 1, \ell + 1) \) is satisfiable. While the conditions T1, T2 and T3 do not forbid overlapping tiles in general (non-interference is different from non-overlapping tiles), our current tiling heuristic avoids them by refining the initial tile estimates. For each expression \( e \) in \( \text{UpdIndexExprs}^A[s] \), we check if \( \text{InitTile}^A(\ell[s], e) \land \text{InitTile}^A(\ell[s] + k, e) \) is satisfiable.
isfiable. If so, we drop $e$ from the refined tiling predicate, denoted $\text{Tile}^A(\ell[s], \cdot)$ in Algorithm 2. This ensures that an array index expression $e$ belongs to the tile corresponding to the largest value of the loop counter $\ell[s]$ when it is updated. The procedure Simplify invoked in lines 7 and 11 of Algorithm 2 tries to obtain a closed form linear expression (or Boolean combination of a few linear expressions) for $\vee_{e \in \text{UpdIndexExprs}^A[s]}(j = e)$, if possible. In the case of Fig. 3, this gives the tile $(j = \ell)$, which suffices for proving the quantified assertion in this example.

Sometimes, the heuristic choice of tiling or the choice of candidate invariants may not be good enough for the requisite checks ($T1, T2^*, T2^{**}, T3$) to go through. In such cases, Algorithm 1 allows different heuristics to be used to update the tiles and invariants. In our current implementation, we do not update the tiles, but update the set of candidate invariants by discarding candidates that cannot be proven using our tiling-based checks. It is possible that the tiles and candidate invariants obtained in this manner do not suffice to prove the assertion within a pre-defined time limit. In such cases, we time out and report an inconclusive answer.

6 Implementation and Experiments

Implementation: We have implemented the above technique in a tool called TILER. The tool is built on top of the LLVM/CLANG [13] compiler infrastructure. We ensure that input C programs are adapted, if needed, to satisfy the syntactic restrictions in Section 3. The current implementation is fully automated for programs with non-nested loops, and can handle programs with nested loops semi-automatically.

Generating candidate invariants: We use a template-based dynamic analysis tool, DAIKON[2], for generating candidate invariants. DAIKON supports linear invariant discovery among program variables and arrays, and reports invariants at the entry and exit points of functions. In order to learn candidate quantified invariants, we transform the input program as follows. The sizes of all arrays in the program are changed to a fixed small constant, and all arrays and program variables that are live are initialized with random values. We then insert a dummy function call at each cut-point. Our transformation collects all array indices that are accessed in various segments of the program and expresses them in terms of the corresponding loop counter(s). Finally, it passes the values of accessed array elements, the corresponding array index expressions and the loop counter(s) as arguments to the dummy call, to enable DAIKON to infer candidate invariants among them. The transformed program is executed multiple times to generate traces. DAIKON learns candidate linear invariants over the parameters passed to the dummy calls from these traces. Finally, we lift the candidate invariants thus identified to quantified invariants in the natural way.

As an example, consider the input program shown in figure 4(a). The transformed program is shown in figure 4(b). In the transformed program, arrays $a$ and $b$ are initialized to random values. The dummy function call in loop $L1$ has
void copynswap() {
    int s, i, tmp;
    int a[s], b[s], acopy[s];
    for (i = 0; i < s; i++) { //L1
        acopy[i] = a[i];
    }
    for (i = 0; i < s; i++) { //L2
        tmp = a[i], a[i] = b[i]; b[i] = tmp;
    }
    for (i = 0; i < s; i++) {
        assert(b[i] == acopy[i]);
    }
}

void dummy(int a_i, int b_i, int acopy_i, int i) { }

void copynswap() {
    int s = 10, i, tmp;
    int a[s], b[s], acopy[s];
    for (i = 0; i < s; i++) {
        a[i] = rand(); b[i] = rand();
    }
    for (i = 0; i < s; i++) { //L1
        acopy[i] = a[i];
        dummy(a[i], b[i], acopy[i], i);
    }
    for (i = 0; i < s; i++) { //L2
        tmp = a[i], a[i] = b[i]; b[i] = tmp;
    }
}

Fig. 4: (a)Input program (b) Transformed Program

four arguments $a[i], b[i], acopy[i]$ and $i$. Based on concrete traces, DAikon initially detects the candidate invariants $(a_i = acopy_i)$ and $(a_i \neq b_i)$ on the parameters of the dummy function. We lift these to obtain the candidate quantified invariants $\forall i. (a[i] = acopy[i])$ and $\forall i. (a[i] \neq b[i])$. In the subsequent analysis, we detect that $\forall i. (a[i] \neq b[i])$ cannot be proven. This is therefore dropped from the candidate invariants (line 24 of Algorithm 1), and we proceed with $\forall i. (a[i] = acopy[i])$, which suffices to prove the post-condition.

Tile generation and checking: Tiles are generated as in Algorithm 2. Condition T1 is checked using Z3 \cite{Z3}, which has good support for quantifiers. We employ CBMC\cite{CBMC} for implementing the checks T2*, T2**, and T3*.

**Benchmarks** We evaluated our tool on 60 benchmarks from the test-suites of BOOSTER\cite{Booster} and VAPHOR\cite{Vaphor}, as well as on programs from an industrial code base. The benchmarks from BOOSTER and VAPHOR test-suites (Table 1(a)) perform common array operations such as array initialization, reverse order initialization, incrementing array contents, finding largest and smallest elements, odd and even elements, array comparison, array copying, swapping arrays, swapping a reversed array, multiple swaps, and the like. Of the 135 benchmarks in this test suite, 66 benchmarks are minor variants of the benchmarks we report. For example, there are multiple versions of programs such as copy, init, copyinit, with different counts of sequentially composed loops. In such cases, the benchmark variant with the largest count is reported in the table. Besides these, there are 22 cases containing nested loops which can currently be handled only semi-automatically by our implementation, and 25 cases with post-conditions in a form that is different from what our tool accepts. Hence, these results are not reported here.

Benchmarks were also taken from the industrial code of a battery controller in a car (Table 1(b)). These benchmarks set a repetitive contiguous bunch of cells in a battery with different values based on the guard condition that gets satisfied. The size of such a contiguous bunch of cells varies in different models. The assertion checks if the cell values are consistent with the given specification.

All our benchmarks are within 100 lines of uncommented code. The programs have a variety of tiles such as $4i - 4 \leq j < 4i, 2i - 2 \leq j < 2i, j = size - i - 1,$
| BENCHMARK | #L | T | S+C | B | V |
|-----------|----|---|-----|---|---|
| init2pic.c | 1 | ✓0.5 | ✓0.01 | ✓1.0 |
| initmin.c | 2 | ✓0.8 | ✓0.01 | ✓0.7 |
| evenod.c | 1 | ✓0.4 | ✓0.01 | ✓0.84 |
| revdtrl.c | 1 | ✓0.6 | ✓0.01 | ✓0.79 |
| largest.c | 1 | ✓0.4 | ✓0.01 | ✓0.02 |
| smallest.c | 1 | ✓0.8 | ✓0.01 | ✓0.02 |
| cpy.c | 1 | ✓0.0 | ✓0.01 | ✓2.0 |
| cpynrev.c | 2 | ✓3.8 | ✓3.1 | ✓5.4 |
| cpynwsp.c | 2 | ✓4.2 | ✓12.3 | ✓3.8 |
| cpynwsp2.c | 3 | ✓10.2 | ✓198 | ✓7.2 |
| 01.c | 1 | ✓0.44 | ✓0.05 | ✓0.38 |
| 02.c | 1 | ✓0.65 | ✓0.02 | ✓2.3 |
| 06.c | 2 | ✓8.15 | ✓0.04 | ✓0.38 |
| 27.c | 1 | ✓0.41 | ✓0.01 | ✓0.12 |
| 43.c | 1 | ✓0.43 | ✓0.03 | ✓0.09 |
| maxinarr.c | 1 | ✓0.51 | ✓0.01 | ✓1.1 |
| mininarr.c | 1 | ✓0.53 | ✓0.02 | ✓1.3 |
| compare.c | 1 | ✓0.44 | ✓0.04 | ✓0.66 |
| palindrome.c | 1 | ✓0.52 | ✓0.02 | ✓0.39 |
| copy9.c | 9 | ✓44 | ✓0.46 | TO |
| init0.c | 9 | ✓29.2 | ✓0.34 | ✓0.16 |
| sequinit.c | 1 | ✓0.45 | ✓0.03 | ✓0.43 |
| nec40t.c | 1 | ✓0.50 | ✓0.06 | ✓0.48 |
| sumarr.c | 1 | ✓0.55 | ✓0.06 | ✓4.2 |
| vararr.c | 1 | ✓0.92 | ✓0.03 | ✓1.12 |
| find.c | 1 | ✓0.52 | ✓0.02 | ✓0.14 |
| running.c | 1 | ✓0.62 | ✓0.04 | ✓0.12 |
| revcp.c | 1 | ✓0.7 | ✓0.01 | ✓0.73 |
| revcpwsp.c | 2 | ✓6.0 | ✓0.02 | TO |
| revcpwsp2.c | 3 | ✓8.0 | ✓0.03 | TO |

| BENCHMARK | #L | T | S+C | B | V |
|-----------|----|---|-----|---|---|
| copy9u.c | 9 | ✓0.16 | ✓4.24 | ✓10.44 | ✓36.8 |
| revcpwspu.c | 2 | ✓0.18 | ✓3.11 | ✓8.01 | TO |
| skippedu.c | 1 | ✓0.81 | ✓2.94 | ✓0.02 | TO |
| melceu.c | 1 | ✓0.37 | ✓2.5 | ✓* | ✓* |
| poly1.c | 1 | TO | ✓15 | TO |
| poly2.c | 2 | ✓6.34 | ✓15.3 | TO |
| tcpy.c | 1 | ✓0.65 | ✓TO | ✓725.1 |
| skipped.c | 1 | ✓1.24 | ✓TO | TO |
| rewrev.c | 1 | ✓0.48 | ✓0.01 | TO |
| rewrev2.c | 1 | ✓0.39 | ✓TO | TO |
| rewnlr.c | 1 | ✓0.49 | ✓0.01 | TO |
| rewnlrrev.c | 1 | ✓0.28 | ✓0.01 | TO |
| rewnlrrev2.c | 1 | ✓0.47 | ✓0.01 | TO |
| pr2.c | 1 | ✓0.7 | ✓TO | TO |
| pr3.c | 1 | ✓0.7 | ✓TO | TO |
| pe4.c | 1 | ✓0.68 | ✓TO | TO |
| pe5.c | 1 | ✓1.32 | ✓TO | TO |
| nr2.c | 1 | ✓1.48 | ✓TO | TO |
| nr3.c | 1 | ✓1.48 | ✓TO | TO |
| nr5.c | 1 | ✓1.32 | ✓TO | TO |

Table 1: Results on selected benchmarks from (a) Booster & Vaphor test-suite and (b) industrial code. #L is the number loops (and sub-loops, if any) in the benchmark, T is Tiler, S+C is SMACK+Corral, B is Booster, and V is Vaphor. ✓ indicates assertion safety, ? indicates assertion violation, ? indicates unknown result, and * indicates unsupported construct. All the times are in seconds. TO is time-out. * indicates semi-automated experiments and the corresponding execution times are of the automated part. See text for explanation.

\[ j = i \text{ etc., with the last one being the most common tile, where } i \text{ denotes the loop counter and } j \text{ denotes the array index accessed.} \]

**Experiments** The experiments reported here were conducted on an Intel Core i5-3320M processor with 4 cores running at 2.6 GHz, with 4GB of memory running Ubuntu 14.04 LTS. A time-out of 900 seconds was set for Tiler, SMACK+Corral, Booster, and Vaphor. The memory limit was set to 1GB for all the tools. SPACER was used as the SMT solver for Horn formulas generated by Vaphor since this has been reported to perform well with Vaphor. In addition, C programs were manually converted to mini-Java, as required by Vaphor. Since SMACK+Corral is a bounded model checker, a meaningful comparison with Tiler can be made only in cases where the benchmark violates a quantified assertion. In such cases, the verifier option `svcomp` was used for Corral. In all other cases, we have shown a † in the column for SMACK+Corral in Table 1 to indicate that comparison is not meaningful.
Tiler takes about two seconds for verifying all single loop programs that satisfy their assertions. For programs containing multiple loops, 10 random runs of the program were used to generate candidate invariants using Daikon. The weak loop invariant \( \text{Inv} \), mentioned in Section 4, was assumed to be true. Tiler took a maximum of 35 seconds to output the correct result for each such benchmark. The execution time of Tiler includes instrumentation for Daikon, trace generation, execution of Daikon on the traces for extracting candidate invariants, translating these to assume statements for use in CBMC, proving the reported candidate invariants and proving the final assertion. The execution of Daikon and proving candidate invariants took about 95% of the total execution time.

To demonstrate the application of our technique on programs with nested loops, we applied it to the last four benchmarks in Table 1(b), each of which has a loop nested inside another. We used Tiler to automatically generate tiles for these programs. We manually encoded the sliced post-condition queries and ran CBMC. We did not have time to automate trace generation for Daikon and for making the above CBMC calls automatically for this class of programs. We are currently implementing this automation.

**Analysis** Booster and Vaphor performed well on benchmarks from their respective repositories. Although Vaphor could analyze the benchmark for reversing an array, as well as one for copying and swapping arrays, it could not analyze the benchmark for reverse copying and swapping. Since the arrays are reversed and then swapped, all array indices need to be tracked in this case, causing Vaphor to fail. Vaphor also could not verify most of the industrial benchmarks due to two key reasons that are not handled well by Vaphor: (i) at least two distinguished array cells need to be tracked in these benchmarks, and (ii) updates to the arrays are made using non-sequential index values.

Booster could analyze all the examples in which the assertion gets violated, except for a benchmark containing an unsupported construct (shift operator) indicated by *. This is not surprising since finding a violating run is sometimes easier than proving an assertion. Booster however could not prove several other industrial benchmarks because it could not accelerate the expressions for indices at which the array was being accessed. Tiler, on the other hand, was able to generate interesting tiles for almost all these benchmarks.

In our experiments, SMACK+Corral successfully generated counterexamples for all benchmarks in which the assertion was violated. As expected, it was unable to produce any conclusive results for benchmarks with parametric array sizes where the quantified assertions were satisfied.

**Limitations** There are several scenarios under which Tiler may fail to produce a conclusive result. Tiler uses CBMC with small loop unwinding bounds to find violating runs in programs with shallow counter-examples. Consequently, when there are no short counter-examples (e.g. in mclceu.c), Tiler reports an inconclusive answer. Tiler is also unable to report conclusively in cases where the tile generation heuristic is unable to generate the right tile (e.g. in tcpy.c),
when Daikon generates weak mid-conditions (e.g. in poly2.c) or when CBMC takes too long to prove conditions T2* or T3* (e.g. in poly1.c).

Our work is motivated by the need to prove quantified assertions in programs from industrial code bases, where we observed interesting patterns of array accesses. Our tile generation heuristic is strongly motivated by these patterns. There is clearly a need to develop more generic tile generation heuristics for larger classes of programs.

7 Related Work

The Vaphor tool [4] uses an abstraction to transform array manipulating programs to array-free Horn formulas, parameterized by the number of array cells that are to be tracked. The technique relies on Horn clause solvers such as Z3 [10], Spacer [15] and Eldarica [16] to check the satisfiability of the generated array-free Horn formulas. Vaphor does not automatically infer the number of array cells to be tracked to prove the assertion. It also fails if the updates to the array happen at non-sequential indices, as is the case in array reverse and swap, for example. In comparison, Tiler requires no input on the number of cells to be tracked and is not limited by sequential accesses. The experiments in [4] show that Horn clause solvers are not always efficient on problems arising from program verification. To be efficient on a wide range of verification problems, the solvers need to have a mix of heuristics. Our work brings a novel heuristic in the mix, which may be adopted in these solvers.

Booster [3] combines acceleration [17,18] and lazy abstraction with interpolants for arrays [19] for proving quantified assertions on arrays for a class of programs. Interpolation for universally quantified array properties is known to be hard [20,21]. Hence, Booster fails for programs where simple interpolants are not easily computable. Fluid updates [22] uses bracketing constraints, which are over- and under-approximations of indices, to specify the concrete elements being updated in an array without explicit partitioning. This approach is not property-directed and their generalization assumes that a single index expression updates the array.

The analysis proposed in [23,24] partitions the array into symbolic slices and abstracts each slice with a numeric scalar variable. These techniques cannot easily analyze arrays with overlapping slices, and they do not handle updates to multiple indices in the array or to non-contiguous array partitions. In comparison, Tiler uses state-of-the-art SMT solver Z3 [10] with quantifier support [25] for checking interference among tiles and can handle updates to multiple non-contiguous indices.

Abstract interpretation based techniques [26,29] propose an abstract domain which utilizes cell contents to split array cells into groups. In particular, the technique in [26] is useful when array cells with similar properties are non-contiguously present in the array. All the industrial benchmarks in our test-suite are such that this property holds. Template-based techniques [27] have been used to generate expressive invariants. However, this requires the user to supply the right templates, which may not be easy in general. In [28], a technique
to scale bounded model-checking by transforming a program with arrays and possibly unbounded loops to an array-free and loop-free program is presented. This technique is not compositional, and is precise only for a restricted class of programs.

There are some close connections between the notion of tiles as used in this paper and similar ideas used in compilers. For example, tiling/patterns have been widely used in compilers for translating loops into SIMD instructions \[8,29\]. Similarly, the induction variable pass in LLVM can generate all accessed index expressions for an array in terms of the loop counters. Note, however, that not all such expressions may be part of a tile (recall the tiles in Fig. 3). Hence, automatically generating the right tile remains a challenging problem in general.

8 Conclusion

Programs manipulating arrays are known to be hard to reason about. The problem is further exacerbated when the programmer uses different patterns of array accesses in different loops. In this paper, we provided a theory of tiling that helps us decompose the reasoning about an array into reasoning about automatically identified tiles in the array, and then compose the results for each tile back to obtain the overall result. While generation of tiles is difficult in general, we have shown that simple heuristics are often quite effective in automatically generating tiles that work well in practice. Surprisingly, these simple heuristics allow us to analyze programs that several state-of-the-art tools choke on. Further work is needed to identify better and varied tiles for programs automatically.

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