GL(2) WEYL BOUND VIA A MULTIPLICATIVE CHARACTER DELTA METHOD

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ABSTRACT

We use a trivial delta method with multiplicative characters for congruence detection to prove the Weyl bound for GL(2) in $t$-aspect for a holomorphic or Hecke-Maass cusp form of arbitrary level and nebentypus. This parallels the work of Aggarwal [Agg18] in 2018, with the difference being multiplicative character has a more natural connection to the twisted $L$-function. This provides another viewpoint to understand and explore the trivial and other delta methods.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $f$ be a fixed level $M$ Hecke cusp form of weight $k$ with nebentypus $\psi$ and $L(s,f)$ be its associated $L$-function. By the Phragmen-Lindelof principle, we have the convexity bound

$$L\left(\frac{1}{2} + it, f\right) \ll_{f, \epsilon} t^{\frac{1}{2} + \epsilon}$$

for any $\epsilon > 0$. Any bound of the form

$$L\left(\frac{1}{2} + it, f\right) \ll_{f, \epsilon} t^{\frac{1}{2} - \delta}$$

for some $\delta > 0$ is called a subconvexity bound, and the current best bound is the Weyl-type bound, saying

$$L\left(\frac{1}{2} + it, f\right) \ll_{f, \epsilon} t^{\frac{3}{4} + \epsilon}.$$ 

This Weyl-type bound appears to be a natural barrier to many boundary problems.

This problem has been studied extensively and the Weyl bound has been established by a lot of people. Good [Goo82] was the first one to establish such bound for the full modular group with the tools from spectral theory. Later Jutila [JF87] used Farey fractions and Voronoi summations to get the same result, which was then extended by Meurman [Meu90] to cover Maass forms. Afterwards, Jutila [Jut97] extended Good’s proof to cover Maass forms. Recently, many people have established results on this problem, with various constraints on the levels, see for example [Ach+20; AS17; BMN19]. As better machinery get developed, these constraints, which are believed to be mere annoyance, are gradually relaxed. In 2018 Aggarwal [Agg18] established the Weyl bound for holomorphic or Hecke-Maass cusp form of arbitrary level and nebentypus.

In [Agg18], Aggarwal used the trivial delta method [Agg18, Lemma 1.1] stated as follows.

**Lemma 1.1.** Let $V$ be a smooth real valued function, compactly supported inside $\mathbb{R}^+$ such that $V$ has bounded derivatives and $\int_0^\infty V(x)dx = 1$. Let $X > 1$ and $q \in \mathbb{N}$ be such that $q > X^{1+\epsilon}$ for some $\epsilon > 0$. Then

$$\delta(n = 0) = \frac{1}{q} \sum_{\alpha \mod{q}} e\left(\frac{n\alpha}{q}\right) \int_0^\infty V(x)e\left(\frac{nx}{X}\right)dx + O_A(X^{-A})$$
for any $A > 0$ and here $e(x) = e^{2\pi i x}$.

Other than this problem, this delta method and its variant has also been successfully applied to achieve results on related problems, see for example [Agg+20b; Agg+20a].

The idea behind is as follows, for $q > X^{1+\epsilon}$,

$$\delta(n = 0) = \delta(q|n)\delta(|n| < X),$$

and one uses additive characters to detect the divisibility condition $\delta(q|n)$ and the $x$-integral to detect $\delta(|n| < X)$. However one can also detect the divisibility condition by multiplicative characters, giving us the following variant of trivial delta method.

**Lemma 1.2.** (Trivial Delta Method with Multiplicative Characters) Let $V \geq 0$ be a fixed smooth function compactly supported on $\mathbb{R}^+$. Let $n, r, p, N \in \mathbb{N}, K > 1$ such that $n, r \sim N, pK > N^{1+\epsilon}$, we have for $p \nmid r$,

$$\delta(n = r) = \frac{1}{\phi(p)} \sum_{\chi \mod p} \chi(n)r \int_{0}^{\infty} V(v) \left(\frac{n}{r}\right)^{-iv} dv + O(N^{-A})$$

for any $A > 0$.

**Proof.** Repeated integration by parts on the $v$-integral gives for any $j \in \mathbb{N}$,

$$\int_{0}^{\infty} V(v) \left(\frac{n}{r}\right)^{-iv} dv \ll j \left(K \left(\log n - \log r\right)^{-j}\right) \ll j \left(\frac{N}{K|n-r|}\right)^{j},$$

which gives us arbitrary saving unless $|n-r| \propto \frac{N^{1+\epsilon}}{X}$. The sum over $\chi \mod p$ detects $n \equiv r \pmod{p}$ as $p \nmid r$. The condition $pK > N^{1+\epsilon}$ makes sure such congruence condition is an equality. \qed

Using this delta method, we prove the Weyl bound for $GL(2)$ $L$-function in $t$ aspect.

**Theorem 1.3.** Let $f$ be a level $M$ Hecke cusp form of weight $k$ with nebentypus $\psi$ and $L(s, f)$ be its associated $L$-function. Then

$$L\left(\frac{1}{2} + it, f\right) \ll_{f, \epsilon} t^{\frac{1}{2}+\epsilon}$$

for any $\epsilon > 0$.

In the process, we proved the following Theorem as an intermediate step.

**Theorem 1.4.** Let $f$ be a level $M$ Hecke cusp form of weight $k$ with nebentypus $\psi$ and $\lambda_f(n)$ be its Hecke eigenvalues. Let $V$ be a smooth function compactly supported on $\mathbb{R}^+$. Then for $N < t^{1+\epsilon}$,

$$S(N) := \sum_{n} \lambda_f(n)n^{-it}V\left(\frac{n}{N}\right) \ll_{f, \epsilon} \min\left\{Nt^\epsilon, \sqrt{N}t^{\frac{1}{4}+\epsilon}\right\}.$$
used to establish great results especially in the recent years, few have tried to use multiplicative characters to detect congruences. Moreover, as the usage of delta methods gain increasing attention, many have questioned the underlying reasons why delta methods provide a means to solve boundary problems and how to understand the steps involved.

In this paper, we hope to provide another viewpoint as multiplicative characters have a more natural linkage with the $L$-functions. In particular, when one apply the trivial delta with multiplicative character on this $GL(2)$ aspect problem, one encounters the sums of the form

$$\sum_n \frac{\lambda_f(n)\chi(n)}{n^s} \text{ and } \sum_r \frac{\chi(r)}{r^s}$$

with a sum of $\chi \pmod{p}$. These sums are equal to $L(s, f \otimes \chi)$ and $L(s', \chi)$ respectively when $\text{Re}(s), \text{Re}(s') > 1$. One can then view these sums as $L$-functions and relates them as members of a family of $L$-functions as $\chi$ runs over the Dirichlet characters $\pmod{p}$. This is a small step towards understanding delta methods by family approach, which is vastly used in moment methods, and such methods is traditionally used to solve boundary problems before the delta methods seen more play.

Another difference in using multiplicative character instead of additive character in a congruence detection allows us to use functional equation of various $L$-functions instead of Voronoi summation for dual summation. This completely removes the needs of dealing with Bessel transforms that appear in Voronoi summation, but one has to deal with Gamma functions instead. While the Bessel function is related to the Gamma functions, and the analysis of either one is not particularly complicated in this fixed $GL(2)$ problem, not having Bessel functions can make the analysis simpler in certain cases.

Finally, we would like to point out that detecting congruences with multiplicative characters is not limited to the trivial delta method shown in this paper, but any delta methods that uses arithmetic congruences. While we don’t anticipate real improve in results established by using additive characters for congruence, this may simplify the analysis in certain cases and hopefully provide a better viewpoint in understanding how delta methods work as explained above.

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2. Notations

Let $c > 0$, $a, b \in \mathbb{C}$, $c \in \mathbb{Z}$ and $t \to \infty$.

| Symbol | Meaning |
|--------|---------|
| $a \ll b$ | There exists constant $c > 0$ such that $|a| \leq c|b|$ |
| $a \sim b$ | $b \ll a \ll b$ |
| $a \sim c$ | $bt^c \ll a \ll bt^c$ |
| $(a, b)$ | The greatest common divisor of $a$ and $b$ |
| $e(x)$ | $e^{2\pi ix}$ |
| $\sum_{\alpha \pmod{c}^*}$ | Sum over $\alpha \in (\mathbb{Z}/c)^*$ |
Sum over primitive characters Mod $p$

$$S(a, b; c) = \sum_{\chi \pmod{p}} \sum_{a \pmod{c}} e \left( \frac{a\chi + b\overline{\chi}}{c} \right)$$

We say $f$ is $X$-inert if it is smooth and satisfies for any $x \in \mathbb{R}$, $j \geq 0$,

$$x^j f^{(j)}(x) \ll X^j t^c.$$ 

Note that a similar definition is introduced by [KPY19], we include $t^c$ for convenience.

3. Proof Sketch

As $f$ is fixed and $t \to \infty$, we are going to assume $M = 1$ in the sketch. We first apply an approximate functional equation to reduce the proof of Theorem 1.3 to proving Theorem 1.4. Then we apply the trivial delta method in Lemma 1.2 with an average over the primes $p$ to separate the variables $n$ and $r$, giving us roughly

$$S(N) \sim \frac{1}{p} \sum_{p \nmid p} \sum_{n \leq N} \lambda_f(n) \chi(n) n^{iKv} \sum_{r \leq N} \Psi(r) r^{-i(t+Kv)} dv,$$

for $PK > N$. This is done in Section 5. The bound at this stage is $N^2$.

Next we perform, in Section 6, dual summations to the $n$-sum and $r$-sum by doing Mellin inversion and using functional equations of the $L$-functions $L(s, \chi)$ and $L(s, f \otimes \chi)$ respectively. The new length of $n$ becomes $\frac{PK^2}{N}$ and that of $r$ becomes $\frac{P}{N}$ as we take $K < t$. This implies we save $\frac{N}{PK}$ and $\frac{N}{\sqrt{K}}$ in the $n$ and $r$ sum respectively. Summing over $\chi \pmod{p}$ and cleaning up the $v$-integral by stationary phase analysis gives us $\sqrt{P}$ and $\sqrt{K}$ saving respectively, and we end up with roughly

$$S(N) \sim \frac{N^2}{p^3 K^2} \sum_{i=0}^{\infty} p^{-it} \sum_{n \leq N} \lambda_f(n) \sum_{r \leq N} r^{it} e \left( -\frac{n\pi}{p} - \sqrt{\frac{2nt}{\pi pr}} - \frac{n}{2pr} \right).$$

The bound at this stage is $P\sqrt{K}$, which can also be seen by the savings from $N^2$ from the process above. This is not good enough as we have the constraint $PK > N$. We have to save a bit more than $\sqrt{P}$ to beat convexity.

To break the structure, we perform Cauchy-Schwartz inequality in Section 7 to eliminate the GL(2) coefficients $\lambda_f(n)$ by taking out the $n$-sum. This gives us roughly

$$S(N) \leq \frac{N^{1/2}}{p^2 \sqrt{Kt}} \left( \sum_{p \nmid p} \sum_{n \leq N} \sum_{r \leq N} r^{it} e \left( -\frac{n\pi}{p} - \sqrt{\frac{2nt}{\pi pr}} - \frac{n}{2pr} \right) \right)^1.$$

Opening up the square, we perform Poisson summation on the $n$-sum, giving us roughly

$$S(N) \leq \frac{\sqrt{N}}{P \sqrt{T}} \left( \sum_{\alpha \in \mathbb{R}} \sum_{p_1-p_2 \neq p} \sum_{p_1-p_2 \neq p} \left( \frac{p_1}{r_1} \right)^{it} \sum_{r_1 \leq N} \sum_{r_2 \leq N} \frac{1}{r_2} \delta \left( n \equiv r_2 p_1 - r_1 p_2 \pmod{p_1 p_2} \right) \times \text{some integral} \right)^1.$$ 

We then split the treatment into whether $p_1 = p_2$ or not. For the case $p_1 = p_2$, we subdivide it into the diagonal case with $r_1 = r_2$ and the case $r_1 \neq r_2$. The diagonal case is bounded by $\sqrt{NK}$ as we get square root saving in $p$ and $r$-sums, and the $r_1 \neq r_2$ case is bounded by $K^{1/2} \sqrt{T}$ as we get $\sqrt{P}$ saving from $p$-sum, $\sqrt{P}$ by the congruence and $K^{1/2}$ from the integral. For the offdiagonal, we lose $\sqrt{N^2}$ on the $n$-sum length, but save $\sqrt{P}$ by the congruence on each
$r_1$ and $r_2$ and also saves $K^{\frac{1}{2}}$ in the integral, giving us the bound $\sqrt{Nt}K^{-\frac{1}{2}}$. The optimal choice of $K = t^{\frac{2}{3}}$ gives the desired bound $S(N) \ll \sqrt{Nt}^{\frac{1}{2}+\epsilon}$.

4. Preliminaries and Lemmas

Here we collect the facts we need about Hecke cusp forms, $L$-functions and the standard tools we need for this paper.

4.1. Hecke cusp forms. For a Hecke cusp form $f$, the $L$-function associated to $f$ is defined by analytic continuation of the series given by

$$L(s, f) = \prod_p \left(1 - \frac{\alpha_{f,1}(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha_{f,2}(p)}{p^s}\right)^{-1} = \sum_{n \geq 1} \lambda_f(n) n^{-s} \quad \text{for Re}(s) > 1.$$ 

If $f$ is holomorphic, $\lambda_f(n)$ satisfies the following bound by Deligne,

$$|\lambda_f(n)| \leq \tau(n) \ll n^\epsilon$$

for any positive integer $n$. When $f$ is primitive, $\lambda_f(n)$ is equal to the Hecke eigenvalues of $f$.

If $f$ is a Maass cusp form, a proof for a bound similar to (1) has not been established. But they satisfy "Ramanujan-Petersson on average" by Rankin-Selberg theory, i.e.

$$\sum_{1 \leq n \leq x} |\lambda_f(n)|^2 \ll f(x)^{1+\epsilon}.$$ (2)

4.2. Functional Equations. Let $\chi$ be a primitive Dirichlet character mod $p$ and $f$ be a level $M$ holomorphic cusp form of weight $k$ of nebentypus $\psi$ or a level $M$ Hecke-Maass cusp form with Laplace eigenvalue $\frac{1}{4} + r^2$ of nebentypus $\psi$. Let

$$a = \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ 1 & \text{if } \chi(-1) = -1, \end{cases}$$

$$\delta_f = \begin{cases} 0 & \text{if } f \text{ is even Maass form} \\ 1 & \text{if } f \text{ is odd Maass form} \end{cases}$$

and let

$$\gamma_f(s) = \begin{cases} \Gamma \left(s + \frac{k-1}{2}\right) & \text{if } f \text{ is holomorphic} \\ \Gamma \left(s + \frac{k}{2} + i\delta_f - i\Gamma\right) \Gamma \left(s + \frac{k}{2} - i\Gamma\right) & \text{if } f \text{ is Maass.} \end{cases}$$

Define

$$\Lambda(s, \chi) = \left(\frac{\pi}{p}\right)^{\frac{\alpha a}{2}} \Gamma \left(s + \frac{a}{2}\right) L(s, \chi),$$

$$\Lambda(s, f) = \pi^{-\frac{s}{2}} M^s \gamma_f(s) L(s, f)$$

$$\Lambda(s, f \otimes \chi) = \left(\frac{\sqrt{M} p}{2\pi}\right)^s \gamma_f(s) L(s, f \otimes \chi).$$

and let $\overline{f}$ be the dual cusp form defined by $\lambda_f(n) = \overline{\lambda_f(n)}$. Then we have for some root number $\epsilon(f) \in \mathbb{C}$ satisfying $|\epsilon(f)| = 1$, we have...
Lemma 4.1. *(Functional Equation)*

\[ \Lambda(s, \chi) = i^{-a} \epsilon \Lambda(1 - s, \overline{\chi}) \]

\[ \Lambda(s, f) = \epsilon(f) \Lambda(1 - s, \overline{f}) \]

\[ \Lambda(s, f \otimes \chi) = \epsilon(f) \psi(p) \chi(M) \epsilon_{\chi}^{2} \Lambda(1 - s, \overline{f} \otimes \overline{\chi}). \]

For proof, see for example [KMV+02; Blo+18; Li75].

4.3. **Approximate Functional Equation.** In order to express \( L(\frac{1}{2} + it, f) \) in a way we can deal with, we apply the following approximate functional equation. See for example [IK04, Thm 5.3, Prop. 5.4] for proof.

**Lemma 4.2.** Let \( G(u) \) be an even holomorphic function bounded in the strip \(-4 < \text{Re}(u) < 4\) and normalized by \( G(0) = 1 \). Then for \( s \) in the strip \( 0 \leq \text{Re}(s) \leq 1 \), we have

\[ L(s, f) = \sum_{n} \frac{\lambda_f(n)}{n^s} V_s \left( \frac{n}{\sqrt{M}} \right) + \epsilon(f) M^{1-s} \gamma_f(1-s) \sum_{n} \frac{\lambda_f(n)}{n^{1-s}} V_{1-s} \left( \frac{n}{\sqrt{M}} \right) + R, \]

where \( R \ll 1 \) and \( V_j(y) \) satisfies for any \( j \geq 0, A > 0, \)

\[ y^j V_j^{(j)}(y) \ll \left( 1 + \frac{y}{\sqrt{t}} \right)^{-A}. \]

4.4. **Gamma Function.** To analyse the Gamma function, we have the Stirling’s approximation, which is as follows.

**Lemma 4.3.** Let \( a_0 = 1, a_1 = -\frac{1}{12} \) and let \( z \to \infty \) with \( |\arg(z)| \leq \pi - \delta \) for any \( \delta > 0 \). There exists \( a_n \in \mathbb{C} \) for any \( n \geq 2 \) such that for any \( N \geq 1 \),

\[ \Gamma(z) = \sqrt{2\pi} \frac{z^{z-\frac{1}{2}} e^{-z}}{\left( \sum_{n=0}^{N-1} a_n \frac{1}{z^n} + O(|z|^{-N}) \right)}. \]

By Stirling’s approximation, we have the following two lemmas.

**Lemma 4.4.** Let \( \alpha, \beta, \tau \in \mathbb{R} \) such that \( \alpha, \beta \) is fixed. Then

\[ \left| \frac{\Gamma(\alpha + i \tau)}{\Gamma(\beta - i \tau)} \right| \ll (1 + |\tau|)^{\alpha - \beta}. \]

*Proof.* When \( |\tau| \ll 1 \), this is trivial. So we are left with the case \( |\tau| \to \infty \). By Stirling’s approximation, we have

\[ \left| \frac{\Gamma(\alpha + i \tau)}{\Gamma(\beta - i \tau)} \right| \ll (\alpha^2 + \tau^2)^{\frac{\beta - 1}{2}} e^{-\frac{\beta}{2} \arctan \frac{\alpha}{\beta} + \tau \arctan \frac{1}{\beta} \frac{\beta - \alpha + \beta}{2}} \ll |\tau|^{\alpha - \beta}. \]

Here we used the Taylor expansion \( \arctan(x) = \frac{\pi}{2} - \arctan(x^{-1}) = \frac{\pi}{2} - x^{-1} + O(x^{-3}) \) to get the second inequality.

**Lemma 4.5.** Let \( \alpha, \beta, \tau \in \mathbb{R} \) such that \( \alpha \) is fixed, \( \tau \ll t^\epsilon \ll \beta t^{-\epsilon} \). Then

\[ \Gamma(\alpha - i(\beta + \tau)) = \frac{|\beta|^{-2i\tau}}{e} |\beta|^{-2i\tau} e \left( -\frac{\tau^2}{2\pi \beta} \right) W_\alpha(\beta) + O(\beta^{-2}), \]

where

\[ W_\alpha(\beta) = e \left( \frac{1}{2\pi} \log \left( \frac{\alpha}{\beta} \right) \right) \left( 1 - i \frac{\alpha^2}{\beta} \right)^{\frac{1}{12 a - 1}} \left( 1 + i \frac{1}{12 a - 1} \right). \]
is $1$-inert in $\beta$.

**Proof.** By Stirling’s Approximation, we have

$$
\frac{\Gamma(\alpha - i(\beta + \tau))}{\Gamma(\alpha + i(\beta + \tau))} = \left(\frac{\alpha^2 + (\beta + \tau)^2}{e^2}\right)^{-i(\beta + \tau)} e^{(2\alpha - 1)i\arg(\alpha - i(\beta + \tau))} \left(\frac{1 + \frac{1}{12(\alpha - i(\beta + \tau))}}{1 + \frac{1}{12(\alpha + i(\beta + \tau))}} + O(\beta^{-2})\right).
$$

(3)

By Taylor expansions, we get

$$
\frac{1 + \frac{1}{12(\alpha - i(\beta + \tau))}}{1 + \frac{1}{12(\alpha + i(\beta + \tau))}} = 1 + \frac{1}{12(\alpha + i(\beta + \tau))} + O(\beta^{-2}),
$$

and

$$
e^{\left(\frac{2\alpha - 1}{2\pi} \arg(\alpha - i(\beta + \tau))\right)} = e^{\left(-\frac{2\alpha - 1}{2\pi} \arctan\left(\frac{\beta + \tau}{\alpha}\right)\right)}
= e^{\left(\frac{1}{2} - 2\alpha \left(\frac{\pi}{2} - \arctan\left(\frac{\alpha}{\beta + \tau}\right)\right)\right)} = e^{\left(1 - 2\alpha \left(\frac{\pi}{2} - \frac{\alpha}{\beta}\right)\right)} + O(\beta^{-2})
$$

and

$$
\left(\frac{\alpha^2 + (\beta + \tau)^2}{e^2}\right)^{-i(\beta + \tau)} = \left(\frac{|\beta|}{e}\right)^{-2i(\beta + \tau)} \left(1 + \frac{\tau}{\beta}\right)^{-2i(\beta + \tau)} \left(1 + \left(\frac{\alpha}{\beta}\right)^2\right)^{-i(\beta + \tau)}
= \left(\frac{|\beta|}{e}\right)^{-2i(\beta + \tau)} e^{-\frac{\beta + \tau}{\pi} \log\left(1 + \frac{\tau}{\beta}\right)} \left(1 - i(\beta + \tau)\left(\frac{\alpha}{\beta}\right)^2 + O(\beta^{-2})\right)
= \left(\frac{|\beta|}{e}\right)^{-2i(\beta + \tau)} e^{-\frac{\beta + \tau}{\pi} \left(\frac{\tau^2}{2\beta^2} + O(\beta^{-3})\right)} \left(1 - i|\alpha|^2\right) + O(\beta^{-2})
= \left(\frac{|\beta|}{e}\right)^{-2i(\beta + \tau)} e^{-\frac{\tau^2}{2\pi\beta}} \left(1 - i|\alpha|^2\right) + O(\beta^{-2}).
$$

Putting the above into (3), we get

$$
\frac{\Gamma(\alpha - i(\beta + \tau))}{\Gamma(\alpha + i(\beta + \tau))} = \left(\frac{|\beta|}{e}\right)^{-2i(\beta + \tau)} e^{-\frac{\tau^2}{2\pi\beta}} \left(1 + \frac{1}{12(\alpha + i(\beta + \tau))} + O(\beta^{-2})\right).
$$

\[\Box\]

**Remark.** We only wrote out the first two terms in the asymptotic expansion of $\Gamma$ to get the above lemma. If we need a better error term we can always write out more terms in the asymptotic expansion.

4.5. **Mellin transform.** Let $V$ be a $t^\varepsilon$-inert function compactly supported on $(t^\varepsilon, t^\varepsilon)$. 

$$
\tilde{V}(s) = \int_0^\infty V(x)x^{s-1}dx
$$

be the Mellin transform of $V$. Then repeated integration by parts on the $x$-integral gives

**Lemma 4.6.** Let $s = \sigma + i\tau$ with $\sigma$ fixed. For any $j \geq 0$, we have

$$
\tilde{V}(s) \ll j \left(\frac{t^\varepsilon}{1 + |	au|}\right)^j.
$$

4.6. **Oscillatory Integrals.** The first tool to study oscillatory integrals is stationary phase analysis as shown in the following lemma, which is Lemma 3.1 in [KPY19].
Lemma 4.7. Suppose that \( w \) is a \( X \)-inert function compact support on \([Z, 2Z]\), so that \( w^{(j)}(t) \ll (Z/X)^{-j} \). Also, suppose that \( \phi \) is smooth and satisfies \( \phi^{(j)}(t) \ll \frac{1}{Z^j} \) for some \( \frac{1}{Z^j} \geq R \geq 1 \) and all \( t \) in the support of \( w \). Let
\[
I = \int_{-\infty}^\infty w(t)e^{i\phi(t)}\,dt.
\]

(1) If \(|\phi'(t)| \gg \frac{1}{Z^j} \) for all \( t \) in the support of \( w \), then \( I \ll A(ZR^{-A}) \) for any \( A > 0 \).

(2) If \( \phi''(t) \gg \frac{1}{Z^{2j}} \) for all \( t \) in the support of \( w \), and there exists \( t_0 \in \mathbb{R} \) such that \( \phi'(t_0) = 0 \), then
\[
I = e^{i\phi(t_0)}\sqrt{\phi''(t_0)}F(t_0) + O(A(ZR^{-A}))
\]
where \( F \) is a \( X \)-inert function (depending on \( A \)) supported on \( t_0 \sim Z \).

Remark.

- By applying a smooth dyadic subdivision, the condition of \( w \) being compactly supported on \([Z, 2Z]\) for Lemma 4.7 can be relaxed to \( w \) being compactly supported on \((t^{-\varepsilon}, t^\varepsilon)\). And we will apply Lemma 4.7 with \( w \) being a \( t^\varepsilon \)-inert function supported on \((t^{-\varepsilon}, t^\varepsilon)\).

- Notice that there is a small difference of \( t^\varepsilon \) in the definition of Inert we are using and that of \([KPY19]\) as described in the notation section, but that does not affect the statement or proof of the lemma above.

We also have the following standard simple second derivative bound that requires much less conditions, see for example \([Sri65, Lemma 5]\).

Lemma 4.8. Let \( V \) be a \( t^\varepsilon \)-inert function compactly supported in \((t^{-\varepsilon}, t^\varepsilon)\) and let \( f \) be a real smooth function. Let \( r > 0 \) such that \(|f''(x)| > r\) for any \( x \in \text{supp}(V)\), then
\[
\int_0^\infty V(x)e(f(x))\,dx \ll \frac{t^\varepsilon}{\sqrt{r}}.
\]

5. Setup

In this paper, we present the calculations primarily focusing on holomorphic cusp form for the ease of demonstration and the calculations in \([Agg18]\) focused on Hecke Maass cusp form. We will point out the essential adjustment needed for Hecke Maass cusp form in the proof.

Let \( f \) be a fixed level \( M \) holomorphic cusp form of weight \( k \) with nebentypus \( \psi \). For \( N < t^{1+\varepsilon} \). By the approximate functional equation in Lemma 4.2, the bound for Gamma function in Lemma 4.4 and smooth dyadic subdivision, we have
\[
L\left(\frac{1}{2} + it, f\right) \ll_f \sup_{1 \leq N \ll t^{1+\varepsilon}} \frac{S(N)}{\sqrt{N}},
\]
where
\[
S(N) := \sum_n \lambda_f(n) n^{-it} V\left(\frac{n}{N}\right)
\]
for some smooth function \( V \) compactly supported on \((1, 2)\). This shows that Theorem 1.4 implies Theorem 1.3.

Note that by (1) or (2) for the Maass form case, we have
\[
S(N) \ll N t^\varepsilon.
\]
Let $U$ be a fixed smooth function compactly supported on $\left(\frac{1}{2}, \frac{5}{2}\right)$ and $U(x) = 1$ for $x \in [1, 2]$. Then

$$S(N) = \sum_n \lambda_f(n)V \left( \frac{n}{N} \right) \sum_r r^{-it}U \left( \frac{r}{N} \right) \delta(n = r).$$

Now we apply Lemma 1.2 to separate the oscillation in $S(N)$. Let $P, K \geq t^\varepsilon$ be parameters, $\mathcal{P}$ be the set of primes in $[P, 2P]$ and let $P^* = |\mathcal{P}|$, then $P^* > Pr^{-\varepsilon}$. Applying the above delta method together with an average of $p \in \mathcal{P}$, we get

$$S(N) = S^*(N) + S_0(N) + S_1(N),$$

where

$$S_0(N) = \frac{1}{P^*} \sum_{p \in \mathcal{P}} \sum_{\nu|n} \lambda_f(n)V \left( \frac{n}{N} \right) \ll \frac{Nt^\varepsilon}{P}$$

by (1) or (2) for the Maass form case,

$$S_1(N) = \frac{1}{P^*} \sum_{p \in \mathcal{P}} \frac{1}{\phi(p)} \sum_{\nu|n} \lambda_f(n)n^{iK\nu}V \left( \frac{n}{N} \right) \sum_r r^{-i(t+K\nu)}U \left( \frac{r}{N} \right) dv$$

and

$$S^*(N) = \frac{1}{P^*} \sum_{p \in \mathcal{P}} \frac{1}{\phi(p)} \sum_{\nu|n} \lambda_f(n)n^{iK\nu}V \left( \frac{n}{N} \right) \sum_r \chi(r)r^{-i(t+K\nu)}U \left( \frac{r}{N} \right) dv.$$

First we bound $S_1(N)$. By repeated integration by parts on the $v$-integral as shown in the proof of Lemma 1.2, we get arbitrary saving unless $|n - r| < \frac{K}{2}t^\varepsilon$. Bounding $\lambda_f(n)$ with (1) or (2) for the Maass form case, we get for $K < Nt^{-\varepsilon}$,

$$S_1(N) \ll \frac{N^2t^\varepsilon}{PK}.$$  

Hence for $K < Nt^{-\varepsilon}$, we have

$$S(N) = S^*(N) + O\left( \frac{Nt^\varepsilon}{P} + \frac{N^2t^\varepsilon}{PK} \right).$$

### 6. Dual Summations and Clean Up

Now we are left to bound

$$S^*(N) = \frac{1}{P^*} \sum_{p \in \mathcal{P}} \frac{1}{\phi(p)} \sum_{\nu|n} \lambda_f(n)n^{iK\nu}V \left( \frac{n}{N} \right) \sum_r \chi(r)r^{-i(t+K\nu)}U \left( \frac{r}{N} \right) dv.$$

We perform dual summations by using functional equations of various $L$-functions.

#### 6.1. $r$-Sum Functional Equation

Let

$$\tilde{U}(s) = \int_0^\infty U(x)x^{s-1}dx$$

be the Mellin transform of $U$. By Mellin inversion and the $L$-function representation, we have for any $\sigma > 0$,

$$\sum_r \chi(r)r^{-i(t+K\nu)}U \left( \frac{r}{N} \right) = \frac{1}{2\pi i} \int_{(\sigma)} \tilde{U}(s)N^s \sum_r \chi(r)r^{-i(t+K\nu)+s} ds = \frac{1}{2\pi i} \int_{(\sigma)} \tilde{U}(s)N^sL(s + i(t + K\nu), \chi) ds.$$  

(14)
Applying the functional equation of Dirichlet $L$-function in Lemma $4.1$, the $r$-sum becomes

$$
\frac{i^{-a} \Gamma}{2 \pi i} \int_{(\sigma)} \tilde{U}(s) N^s \left( \frac{\pi}{p} \right)^{s - \frac{1}{2} + i(t + K \nu)} \gamma(s + i(t + K \nu), a) L(1 - s - i(t + K \nu), \chi) ds,
$$

where

$$
\gamma(s, a) = \frac{\Gamma \left( \frac{1 - s + a}{2} \right)}{\Gamma \left( \frac{s + a}{2} \right)}.
$$

By Lemma $4.6$ on $\tilde{V}$ and Lemma $4.4$, giving $|\gamma(\sigma + i(t + K \nu + \tau), a)| \ll (1 + |t + K \nu + \tau|)^{-\sigma}$, we get arbitrary saving unless $|\text{Im}(s)| \ll t^\varepsilon$. Hence we can shift the contour to $-M$ for some $M > 0$, without hitting any pole as $\chi$ is primitive, to get

$$
\frac{i^{-a} \Gamma}{2 \pi i} \left( \frac{\pi}{p} \right)^{t + i(t + K \nu)} \sum_r \chi(r) r^{-1 + i(t + K \nu)} \int_{(-M)} \tilde{U}(s) \left( \frac{\pi N r}{p} \right)^{s} \gamma(s + i(t + K \nu), a) ds.
$$

Again bounding $|\gamma(\sigma + i(t + K \nu + \tau), a)| \ll (1 + |t + K \nu + \tau|)^{\sigma}$, we get arbitrary saving unless $r \ll \frac{N r^1 \varepsilon}{t^\varepsilon}$ as we have chosen $K \ll N t^{-10 \varepsilon} < t^{1-\varepsilon}$. Similarly, by shifting the contour to the right without hitting any pole as $|\text{Im}(s)| \ll t^\varepsilon$, we get arbitrary saving unless $r \gg \frac{N t^{1-\varepsilon}}{t^\varepsilon}$. Restricting the length of $r \sim \frac{p t}{N}$, we shift the contour back to $\frac{1}{2}$, giving us

$$
\frac{i^{-a} \Gamma}{2 \pi i} \left( \frac{\pi}{p} \right)^{t + i(t + K \nu)} \sum_r \chi(r) r^{-1 + i(t + K \nu)} V_c \left( \frac{r N}{p t} \right) \int_{\left| \text{Im}(s) \right| < t^\varepsilon} \tilde{U}(s) \left( \frac{\pi N r}{p} \right)^{s} \gamma(s + i(t + K \nu), a) ds + O(t^{-20})
$$

where $V_c$ is a $k^\varepsilon$-inert function compactly supported on $(t^{-2\varepsilon}, t^{2\varepsilon})$ and is equal to 1 on $[t^{-\varepsilon}, t^\varepsilon]$. 

Apply Lemma $4.5$ for $s = \frac{1}{2} + i \tau$ with $|\tau| \ll t^\varepsilon$, we have

$$
\gamma(s + i(t + K \nu), a) = \frac{\Gamma \left( \frac{1 - s + i(t + K \nu + \tau) + a}{2} \right)}{\Gamma \left( \frac{1 + i(t + K \nu + \tau) + a}{2} \right)}
$$

$$
= \left( \frac{t + K \nu}{2e} \right)^{i(t + K \nu)} \left( \frac{t + K \nu}{2} \right)^{-i\tau} e^{-\frac{\tau^2}{4\pi(t + K \nu)}} W_{\frac{1}{2} + \frac{i}{2}} \left( \frac{t + K \nu}{2} \right) + O(t^{-2})
$$

with $W_{\frac{1}{2} + \frac{i}{2}} \left( \frac{t + K \nu}{2} \right)$ being the one defined in Lemma $4.5$ is 1-inert. Hence we have

$$
\int_{\left| \text{Im}(s) \right| < t^\varepsilon} \tilde{U}(s) \left( \frac{\pi N r}{p} \right)^{s} \gamma(s + i(t + K \nu), a) ds
$$

$$
=W_{\frac{1}{2} + \frac{i}{2}} \left( \frac{t + K \nu}{2} \right) \left( \frac{t + K \nu}{2e} \right)^{i(t + K \nu)} \int_{\left| \text{Im}(s) \right| < t^\varepsilon} \tilde{U}(s) \left( \frac{\pi N r}{p} \right)^{s} \left( \frac{t + K \nu}{2} \right)^{-i\tau} e^{-\frac{\tau^2}{4\pi(t + K \nu)}} (1 + O(t^{-2})) ds
$$

$$
=i \sqrt{\frac{\pi N r}{p}} W_{\frac{1}{2} + \frac{i}{2}} \left( \frac{2 \pi r}{p} \right) \left( \frac{2 \pi r}{p} \right)^{i(t + K \nu)} \int_{|\tau| < t^\varepsilon} \tilde{U} \left( \frac{1}{2} + i \tau \right) e^{\frac{2 \pi r \nu}{p(t + K \nu)}} d\tau + O \left( \frac{t^\varepsilon}{\sqrt{N t}} \right)
$$

(17)
Putting this back into the $r$-sum after shifting the contour to $\frac{1}{2}$ in (16), and bounding the error term trivially with absolute value, we get
\[
\sqrt{N} \sum_{r} \frac{\chi(r)}{\sqrt{r}} \left( \frac{p(t + K v)}{2\pi \sigma r} \right)^{-i(t + K v)} V_{c} \left( \frac{r N}{P t} \right) W_{1, p, a}(r, v) + O \left( \frac{\sqrt{\tau}}{\sqrt{N} \tau^{1-c}} \right),
\] (18)

where
\[
W_{1, p, a}(r, v) = \frac{i^{-a}}{2\pi} W_{1} + \frac{i}{t^2} \left( t + \frac{K v}{2} \right) \int_{|\tau| < t^2} \tilde{U} \left( \frac{1}{2} + i \tau \right) \left( \frac{2\pi \tau t}{p(t + K v)} \right)^{i \tau} e \left( -\frac{\tau^2}{4\pi t (t + K v)} \right) d \tau
\]
is $t^\epsilon$-inert in both $r$ and $v$. 

Putting the $r$-sum back into $S^*(N)$ and bounding the error term with (1) or (2) for the Maass form case, we get
\[
S^*(N) = \frac{\sqrt{N}}{P} \sum_{p \in \mathcal{D}(\phi)(p)} \sum_{\chi(p)} e_{\tau} \int_{0}^{\infty} V(v) \sum_{n} \lambda_f(n) \chi(n) n^{i K v} V \left( \frac{n}{N} \right) \sum_{r} \frac{\chi(r)}{\sqrt{r}} \left( \frac{p(t + K v)}{2\pi \sigma r} \right)^{-i(t + K v)} V_{c} \left( \frac{r N}{P t} \right) W_{1, p, a}(r, v) dv
\]
\[+ O \left( \frac{\sqrt{P} \sqrt{N}}{t^{1-c}} \right). \] (19)

6.2. $n$-sum functional equation. Again by Mellin inversion, we have for any $\sigma > 1$,
\[
\sum_{n} \lambda_f(n) \chi(n) n^{i K v} V \left( \frac{n}{N} \right) \Gamma_{\chi}(s) N^{s} \sum_{n} \lambda_f(n) \chi(n) n^{-s+i K v} ds
= \frac{1}{2\pi i} \int_{(\sigma)} \tilde{V}(s) N^{s} L(s - i K v, f \otimes \chi) ds.
\]
Notice as $p \sim P > t^\epsilon$ and $M$ is fixed, we have $(M, p) = 1$. Let
\[
\gamma_{k}(s) = \frac{\Gamma(1-s+k-1)}{\Gamma(s+k-1)}.
\]
Apply the functional equation for twisted $L$-function in Lemma 4.1, and shift contour to $\sigma < 0$, the $n$-sum becomes
\[
\frac{e(f) \psi(p) \chi(M) c_{x}^{2}}{2\pi i} \left( \frac{p}{2\pi} \right)^{1-2s+2i K v} \gamma_{k}(s-i K v) L(1-s+i K v, \tilde{f} \otimes \tilde{\chi}) ds
= \frac{e(f) \psi(p) \chi(M) c_{x}^{2}}{2\pi i} \left( \frac{p}{2\pi} \right)^{1-2i K v} \sum_{n} \lambda_f(n) \chi(n) n^{-1-i K v} \frac{1}{\chi(n)} \chi(n) n^{i K v} \tilde{V}(s) N^{s} \left( \frac{p}{2\pi} \right)^{-2s} n^{s} \gamma_{k}(s-i K v) ds
\]
Apply Lemma 4.6 on $\tilde{V}$ and Lemma 4.1 gives $|\gamma_{k}(\sigma+i \tau-i K v)| \ll (1+|K v+i \tau|)^{-2\sigma}$, we get arbitrary saving unless $|\text{Im}(s)| \ll t^\epsilon$. So by contour shift to $-M$ for some $M > 0$, we get arbitrary saving unless $n \ll \frac{p^{2} K^{2} t^{\epsilon}}{N}$. Similarly, by shifting the contour to the right without hitting any pole as $|\text{Im}(s)| \ll t^\epsilon$, we get arbitrary saving unless $n \gg \frac{p^{2} K^{2}}{N}$. Hence we can insert the function $V_{c} \left( \frac{n N}{P K^{2}} \right)$ introduced in the previous section with the cost of arbitrary small error. Shifting the contour back to $\frac{1}{2}$ and truncating $|\text{Im}(s)| \ll t^\epsilon$ by $\tilde{V}$ with the cost of arbitrary small error, the $n$-sum is equal to
\[
\frac{e(f) \psi(p) \chi(M) c_{x}^{2}}{2\pi i} \left( \frac{p}{2\pi} \right)^{1+2i K v} \sum_{n} \lambda_f(n) \chi(n) n^{-1-i K v} V_{c} \left( \frac{n N}{P K^{2}} \right)
\times \int_{|\text{Im}(s)| < t^\epsilon} \tilde{V}(s) N^{s} \left( \frac{p}{2\pi} \right)^{-2s} n^{s} \gamma_{k}(s-i K v) ds + O(t^{-2002}).
\] (20)
Now by lemma \[4.5\] we get

\[
\gamma_k(s - iKv) = \frac{\Gamma\left(\frac{s}{2} - i(-Kv + \tau)\right)}{\Gamma\left(\frac{s}{2} + i(-Kv + \tau)\right)}
= \left(\frac{Kv}{e}\right)^{2iKv} (Kv)^{-2it} e\left(\frac{\tau^2}{2\pi K}v\right) W_{\frac{s}{2}}(-Kv) + O(K^{-2}),
\]

with \(W_{\frac{s}{2}}(-Kv)\) as defined in Lemma \[4.5\]. Hence the \(s\)-integral becomes

\[
\int_{|\text{Im}(s)| < \varepsilon} (s) N^s \left(\frac{p}{2\pi}\right)^{-2s} n^s \gamma_k(s - iKv) ds
= \left(\frac{Kv}{e}\right)^{2iKv} W_{\frac{s}{2}}(-Kv) \int_{|\text{Im}(s)| < \varepsilon} (s) N^s \left(\frac{p}{2\pi}\right)^{-2s} n^s (Kv)^{-2it} e\left(\frac{\tau^2}{2\pi K}v\right) (1 + O(K^{-2})) ds
\]

Bounding the error term with Deligne Bound or Rankin-Selberg Bound, the \(n\)-sum becomes

\[
e(f)\psi(p)\chi(M)e_{\tau} \frac{N}{\sqrt{n}} \sum_n \frac{\Lambda_f(n)\chi(n)}{n} \left(\frac{p^2 K^2 v^2}{4\pi^2 e^2 n}\right)^{iKv} V_\epsilon\left(\frac{nN}{p^2 K^2}\right) W_{2,\nu}(n, \nu) + O\left(\frac{p\sqrt{N}}{K^{1+\varepsilon}}\right),
\]

where

\[
W_{2,\nu}(n, \nu) = \frac{1}{2\pi} W_{\frac{s}{2}}(-Kv) \int_{|\tau| < \varepsilon} \tilde{V}(1 + i\tau) \left(\frac{4\pi^2 nN}{p^2 K^2}V\right)^{it} e\left(\frac{\tau^2}{2\pi K}\right) d\tau
\]

is \(t\)-inert in \(n, \nu\).

Putting this back into \(S^*(N)\) in (19) and bounding the error term with absolute value, we get

\[
S^*(N) = \frac{Ne(f)}{P^*} \sum_{p \leq \varepsilon} \psi(p) e^{\tau} \sum_{\nu(p)} \chi(M)e_{\tau} \int_0^{\infty} V(\nu) \sum_{\tau} \frac{\chi(\tau)}{\sqrt{\tau}} \left(\frac{p(t + K)}{2\pi e r}\right)^{-it+K} V_\epsilon\left(\frac{rN}{P^*}\right) W_{1,\nu}(r, \nu)
\]

\[
\times \sum_n \frac{\Lambda_f(n)\chi(n)}{n} \left(\frac{p^2 K^2 v^2}{4\pi^2 e^2 n}\right)^{iKv} V_\epsilon\left(\frac{nN}{p^2 K^2}\right) W_{2,\nu}(n, \nu) d\nu + O\left(\frac{p^2 \sqrt{N}}{K^{1+\varepsilon}} + \frac{\sqrt{PN}}{t^{1-\varepsilon}}\right)
\]

\[
= \frac{Ne(f)(2\pi e)^{it}}{P^*} \sum_{p \leq \varepsilon} \psi(p) e^{\tau} \sum_{\nu(p)} \epsilon_\nu \chi(-M) \sum_n \frac{\Lambda_f(n)\chi(n)}{n} \left(\frac{pK^2 v^2}{2\pi e n(t + K)}\right)^{iKv} V_\epsilon\left(\frac{rN}{P^*}\right) W_{1,\nu}(r, \nu)
\]

\[
\times \int_0^{\infty} V(\nu) W_{1,\nu}(r, \nu) W_{2,\nu}(n, \nu) \left(\frac{pK^2 v^2 r}{2\pi e n(t + K)}\right)^{iKv} (t + K)^{-it} d\nu + O\left(\frac{p^2 \sqrt{N}}{K^{1+\varepsilon}} + \frac{\sqrt{PN}}{t^{1-\varepsilon}}\right).
\]

**Remark.** For the Maass form case with Laplace eigenvalue \(\frac{s}{2} + t^2\), one have to replace \(\gamma_k(s)\) by

\[
\Gamma\left(\frac{1-s+i\tau+it}{2}\right) \Gamma\left(\frac{1-s+i\tau-it}{2}\right)
\]

where

\[
\delta_f = \begin{cases} 
0 & \text{if } f \text{ is an even Maass form} \\
1 & \text{if } f \text{ is an odd Maass form}
\end{cases}
\]
as defined before Lemma 4.7. While this slightly changes the sum we get, but the analysis and steps are the same for the rest of the paper to yield the same bound.

6.3. character sum. Now we evaluate the \( \chi \)-sum. Recall that the definition of \( a \) depends on \( \chi(-1) = \pm 1 \), and hence we have to split the sum into odd and even character sum. First consider the following sum,

\[
\mathcal{C}_\pm := \frac{1}{\phi(p)} \sum_{\chi(p)} \frac{\chi(-1) \pm 1}{2} \varepsilon_\chi \chi(n) \chi(Mr)
= \frac{1}{2\phi(p)} \sum_{\chi(p)} \varepsilon_\chi \left( T(n) \chi(-Mr) \pm \overline{T(n)} \chi(Mr) \right).
\]

(22)

Writing \( \varepsilon_\chi = \frac{1}{\sqrt{p}} \sum_{a[p]} e \left( \frac{a}{p} \right) \chi(a) \), we get

\[
\mathcal{C}_\pm = \frac{1}{2\sqrt{p}} \sum_{a[p]} e \left( \frac{a}{p} \right) \sum_{\chi(p)} \left[ T(n) \chi(-Mr) \pm \overline{T(n)} \chi(Mr) \right]
= \frac{1}{2\sqrt{p}} \delta(p \mid n) \sum_{a[p]} e \left( \frac{a}{p} \right) \delta(n \equiv -Mr \pmod{p}) \pm \delta(n \equiv Mr \pmod{p})
= \frac{1}{2\sqrt{p}} \delta(p \mid n, r) \left( e \left( \frac{nMr}{p} \right) \pm e \left( \frac{nMr}{p} \right) \right).
\]

Putting this back into \( S^* (N) \) in (21) by adding and subtracting the trivial character, we get

\[
S^* (N) = S_0^* (N) + S_1^* (N) - T(N) + O \left( \frac{p^2 \sqrt{N}}{K} + \frac{\sqrt{PN}}{t^{1-\epsilon}} \right),
\]

(23)

where for \( a = 0, 1 \),

\[
S_a^* (N) = \frac{Ne(f)(2\pi e)^{it}}{2P^{it}} \sum_{p \leq P} p^{-it} \sum_{p \mid n} \frac{\lambda_f(n)}{\sqrt{n}} V_e \left( \frac{nN}{p^2 K^2} \right) \sum_{p \mid r} r^{-\frac{i}{2}+it} V_e \left( \frac{rN}{p^t} \right) \left[ e \left( \frac{-nMr}{p} \right) \pm (-)^a e \left( \frac{nMr}{p} \right) \right]
\times \int_0^\infty V(v) W_1(p, a)(r, v) W_2(p, n, v) \left( \frac{pK^2 \sqrt{r}}{2\pi e n(t+Kv)} \right)^{ikv} (t+Kv)^{-it} dv,
\]

(24)

and

\[
T(N) = \frac{Ne(f)(2\pi e)^{it}}{P^{it}} \sum_{p \leq P} \frac{1}{p^{it} \phi(p)} \sum_n \frac{\lambda_f(n)}{\sqrt{n}} V_e \left( \frac{nN}{p^2 K^2} \right) \sum_{p \mid r} r^{-\frac{i}{2}+it} V_e \left( \frac{rN}{p^t} \right)
\times \int_0^\infty V(v) W_1(p, 0)(r, v) W_2(p, n, v) \left( \frac{pK^2 \sqrt{r}}{2\pi e n(t+Kv)} \right)^{ikv} (t+Kv)^{-it} dv.
\]

(25)

6.4. Treatment of \( T(N) \). Consider the \( n \)-sum in \( T(N) \),

\[
\sum_n \frac{\lambda_f(n)}{n^{\frac{1}{2}-iKv}} V_e \left( \frac{nN}{p^2 K^2} \right) W_2(p, n, v).
\]

By Mellin inversion and the \( L \)-function representation, we have for any \( \sigma > 1 \),

\[
\sum_n \frac{\lambda_f(n)}{n^{\frac{1}{2}-iKv}} \left( \frac{nN}{p^2 K^2} \right)^{\sigma} = \frac{1}{2\pi i} \int_{(\sigma)} \hat{V}(s) \left( \frac{p^2 K^2}{N} \right)^s \sum_n \frac{\lambda_f(n)}{n^{\frac{1}{2}-iKv-s}} ds
= \frac{1}{2\pi i} \int_{(\sigma)} \hat{V}(s) \left( \frac{p^2 K^2}{N} \right)^s L \left( \tau, s + \frac{1}{2} + iKv \right) ds.
\]

(26)
Applying Lemma 4.6 on $\tilde{V}$ gives arbitrary saving unless $|\text{Im}(s)| < t^\epsilon$, hence we can shift the contour to $-M$ for some $M > 0$ and apply functional equation to get (26) is equal to
\[
\frac{\epsilon(f)}{2\pi i} \int_{(-M)} \tilde{V}(s) e^{2 \pi i K_v s} \left( \frac{p^2 K_v^2}{N} \right)^s \frac{\Gamma \left( \frac{1}{2} - s - i K_v \frac{k-1}{2} \right)}{\Gamma \left( s + \frac{1}{2} + i K_v \frac{k-1}{2} \right)} L \left( f, \frac{1}{2} - s - i K_v \right) ds
\]
\[
= \frac{\epsilon(f)}{2\pi i} \int_{(-M)} \tilde{V}(s) e^{2 \pi i K_v s} \left( \frac{p^2 K_v^2}{N} \right)^s \frac{\Gamma \left( \frac{1}{2} - s - i K_v \frac{k-1}{2} \right)}{\Gamma \left( s + \frac{1}{2} + i K_v \frac{k-1}{2} \right)} \sum_n \lambda_f(n) n^{s-iK_v-1} ds. \tag{27}
\]
Bounding $\left| \frac{\Gamma \left( \frac{1}{2} - s + i K_v \frac{k-1}{2} \right)}{\Gamma \left( s + \frac{1}{2} + i K_v \frac{k-1}{2} \right)} \right| \ll 1 + |K_v + \tau^2 \text{Re}(s)|$, and repeated integration by parts on $\tilde{V}$ giving us arbitrary saving unless $|\tau| < t^\epsilon$, we get arbitrary saving by taking $M$ large enough unless
\[
n \ll \frac{N t^\epsilon}{p^2}.
\]
Hence by our choosing $p^2 > N t^\epsilon$, we get arbitrary saving, i.e.
\[
T(N) = O \left( k^{-2020} \right). \tag{28}
\]

**Remark.** One can directly apply Voronoi summation to get the same result here. But we intentionally avoid doing so for the purpose of this paper.

**Remark.** One has to adjust the Gamma function for the Maass form case in a similar fashion mentioned in the end of Section 6.2. Same analysis goes through.

### 6.5. Analyzing the $v$-integral

Now we analyze the $v$-integral given by
\[
I(p, n, r) := \int_0^\infty V(v) W_{1, p, a}(r, v) W_{2, p}(n, v) \left( \frac{p K^2 v^2 r}{2 \pi en(t + K_v)} \right)^{i K_v} (t + K_v)^{-it} dv = \int_0^\infty g(v) e(h(v)) dv \tag{29}
\]
where
\[
g(v) = V(v) W_{1, p, a}(r, v) W_{2, p}(n, v)
\]
\[
2\pi h(t) = 2Kv \log(v) - K \log(t + K_v) + K \log \left( \frac{p K^2 v^2 r}{2 \pi en} \right) - t \log(t + K_v)
\]
Now
\[
2\pi h'(v) = 2K \log(v) + 2K - K \log(t + K_v) - \frac{K^2 v}{t + K_v} + K \log \left( \frac{p K^2 v^2 r}{2 \pi en} \right) - \frac{K t}{t + K_v}
\]
\[
= 2K \log(v) - K \log(t + K_v) + K \log \left( \frac{p K^2 v^2 r}{2 \pi en} \right)
\]
\[
2\pi h''(v) = \frac{K(2t + K_v)}{v(t + K_v)}
\]
\[
h^{(j)}(v) \ll j K
\]
for $j \geq 2$. With out choice of $K < t$, $h''(v) \sim K$. Solving $h'(v_0) = 0$, we get the stationary phase is
\[
v_0 = \frac{\pi n}{p K r} \left( \sqrt{\frac{2 p r t}{\pi n}} + 1 + 1 \right)
\]
\[
h(v_0) = -Kv_0 - t \log(t + K_v).
\]
By Lemma 4.7 there exists some flat function $F$ such that,

$$I(p, n, r) = V(v_0)W_{1,p,a}(r, v_0)W_{2,p}(n, v_0)\frac{F(v_0)}{\sqrt{h''(v_0)}} e^{-it}$$

$$\times e\left(-\frac{n}{2pr} \left(\sqrt{\frac{2pr}{\pi n}} + 1 + 1\right) + \frac{t}{2\pi} \log\left(1 + \frac{\pi n}{2pr} \left(\sqrt{\frac{2pr}{\pi n}} + 1 + 1\right)\right)\right) + O(t^{-2020}).$$

For the ease of later computation, we rewrite

$$e\left(-\frac{n}{2pr} \left(\sqrt{\frac{2pr}{\pi n}} + 1 + 1\right)\right) = e\left(-\frac{nt}{2\pi pr} \left(1 + \frac{\pi n}{2pr} - \frac{n}{2pr}\right)\right)$$

$$= e\left(-\frac{nt}{2\pi pr}У_{1,p}(n, r)\right),$$

where

$$У_{1,p}(n, r) = e\left(-\frac{nt}{2\pi pr} \left(\sqrt{1 + \frac{\pi n}{2pr} - 1}\right)\right)$$

is $\frac{\kappa^3}{\ell^3}$-inert, and by Taylor expansions,

$$e\left(-\frac{t}{2\pi} \log\left(1 + \frac{\pi n}{2pr} \left(\sqrt{\frac{2pr}{\pi n}} + 1 + 1\right)\right)\right) = e\left(-\frac{t}{2\pi} \left(\frac{2\pi n}{pr} \sqrt{1 + \frac{\pi n}{2pr} + \frac{\pi n}{2pr} - 1}\right)\right)$$

$$+ \sum_{j=3}^{\infty} \frac{(-1)^{j-1}}{j} \left(\frac{\pi n}{pr} \left(\sqrt{1 + \frac{\pi n}{2pr}} + 1 + 1\right)^j\right)$$

$$= e\left(-\frac{nt}{2\pi pr}У_{2,p}(n, r)\right),$$

where

$$У_{2,p}(n, r) = e\left(-\frac{t}{2\pi} \sum_{j=2}^{\infty} \frac{(-1)^{j-1}}{j} \left(\frac{\pi n}{pr} \left(\sqrt{1 + \frac{\pi n}{2pr}} + 1 + 1\right)^j\right) + \frac{\pi n}{pr}\right)$$

is $\frac{\kappa^3}{\ell^3}$-inert. Putting this back into $I(p, n, r)$ in (29), we have

$$I(p, n, r) = V(v_0)W_{1,p,a}(r, v_0)W_{2,p}(n, v_0)\frac{F(v_0)}{\sqrt{h''(v_0)}} e^{-it}$$

$$\times e\left(-\frac{nt}{2\pi pr} - \frac{n}{2pr}У_{1,p}(n, r)У_{2,p}(n, r)\right) + O(t^{-2020}).$$

Putting this back into $S_a^*(N)$ in (24), we get

$$S_a^*(N) = \frac{\pi^2 e(f)(2\pi e)^i}{2} \cdot \left\{\prod_{p \in \mathbb{P}} \psi(p)\right\} \cdot \sum_{p \mid n} \lambda_f(n) V_e \left(\frac{nN}{p^2K^2}\right) \sum_{p \mid r} e^{it} V_e \left(\frac{rN}{p^2 K^2}\right) \left(e\left(-\frac{nM_r}{p}\right) + (-1)^a e\left(\frac{nM_r}{p}\right)\right)$$

$$\times e\left(-\frac{2nt}{\pi pr} - \frac{n}{2pr}\right) W_p(n, r) + O(t^{-2020}),$$

where

$$W_p(n, r) = \frac{p^2 K^2 \sqrt{r}}{N\sqrt{pr}} V(v_0)W_{1,p,a}(r, v_0)W_{2,p}(n, v_0)У_{1,p}(n, r)У_{2,p}(n, r)\frac{F(v_0)}{\sqrt{h''(v_0)}}.$$
Differentiating, we see that $W_p(n, r)$ is bounded by $t^\epsilon$ and $\left(1 + \frac{r^3}{n^2}\right)t^\epsilon$-inert in $n, r$.

At this point, bounding everything with absolute value yields

$$S^*_a(N) \ll \frac{N^2t^\epsilon}{p^3K^{\frac{2}{3}}t^{\frac{1}{2}}N} = C_{t^\epsilon}^{\frac{1}{2}}$$

which is not good enough yet as we have the constraint $PK > Nt^\epsilon$.

7. **CAUCHY SCHWARTZ**

Apply Cauchy Schwartz inequality and take out the $n$-sum, we get

$$S^*_a(N)^2 \leq \sum_{\pm} \frac{N^3}{p^4Kt^{1-\epsilon}} \sum_n V_e \left(\frac{nN}{p^2K^2}\right) \left| \sum_{p \in \mathbb{P}} \psi(p) \sum_{p_1 \in \mathbb{P}} v_1 \sum_{p_2 \in \mathbb{P}} v_2 \int \frac{rN}{pt} dW_p(n, r) \right|^2$$

$$= \sum_{\pm} \frac{N^3}{p^4Kt^{1-\epsilon}} \sum_n V_e \left(\frac{nN}{p^2K^2}\right) \left| \sum_{p \in \mathbb{P}} \psi(p) \sum_{p_1 \in \mathbb{P}} v_1 \sum_{p_2 \in \mathbb{P}} v_2 \int \frac{rN}{pt} dW_p(n, r) \right|^2$$

Perform Poisson summation on the $n$-sum, we get

$$\sum_n V_e \left(\frac{nN}{p^2K^2}\right) e \left(\pm \frac{nM_1}{p_1} \mp \frac{nM_2}{p_2} - \frac{2nt}{\pi p_1 r_1} - n + \frac{2nt}{\pi p_2 r_2} + n \right) W_{p_1}(n, r_1) W_{p_2}(n, r_2)$$

$$= \sum_{\gamma \mod p_1 p_2} V_e \left(\frac{\gamma}{p_1 p_2}\right) \left| \sum_{n = \gamma \mod p_1 p_2} \frac{\gamma(M_1 r_2 - M_2 r_1)}{p_1 p_2} \right|^2$$

Now we analyse the the integral $J_{p_1, p_2}(n, r_1, r_2)$. We will choose $K < t^{\frac{\delta}{\epsilon}}$, which implies $W_{p_1}, W_{p_2}$ are both $t^\epsilon$-inert function. By repeated integration by parts, we get arbitrary saving unless

$$|n| \ll \frac{Nt^\epsilon}{K}.$$
Hence we get

\[
S_d^*(N)^2 \leq \sum_{\epsilon = \pm 1} \frac{N^2 K}{p^2 t^{1-\epsilon}} \sum_{n \leq \frac{N}{K}} \sum_{\substack{p \leq \sqrt{p} \mid p_1 p_2 \mid p_1 p_2 | p \mid \epsilon}} \psi(p_1) \overline{\psi(p_2)} \left( \frac{p_2}{p_1} \right)^{it} \sum_{p_1 \mid r_1, p_2 \mid r_2} V_c \left( \frac{r_1 N}{p_1 t} \right) V_c \left( \frac{r_2 N}{p_2 t} \right) \left( \frac{r_1}{r_2} \right)^{it} \times \delta \left( \pm n \equiv M_{p_2} r_1 - M_{p_1} r_2 \pmod{p_1 p_2} \right) J_{p_1, p_2}(n, r_1, r_2) + O(t^{-2020}).
\]

Let

\[
H(w) = -\sqrt{\frac{2p^2 K^2 t w}{\pi N p_1 r_1}} - \frac{p^2 K^2 w}{2N p_1 r_1} + \sqrt{\frac{2p^2 K^2 t w}{\pi N p_2 r_2}} + \frac{p^2 K^2 w}{2N p_2 r_2} - \frac{n p^2 K^2 w}{p_1 p_2 N}
\]

be the phase function. Differentiating, we get

\[
H'(w) = \sqrt{\frac{2p^2 K^2 t}{8\pi N p_1 p_2 r_1 r_2 w^3}} \left( \frac{p_2 r_2 - p_1 r_1}{\sqrt{p_1 r_1} + \sqrt{p_2 r_2}} \right) \gg \frac{NK}{p^2 t^{1-\epsilon}} |p_2 r_2 - p_1 r_1|.
\]

Thus for \( K < t^{\frac{1}{2}+\epsilon} \), the second derivative bound in Lemma 4.8 gives

\[
J_{p_1, p_2}(n, r_1, r_2) \ll \min \left\{ 1, \sqrt{\frac{p^2 t}{NK|p_2 r_2 - p_1 r_1|}} \right\} t^\epsilon.
\]

Let \( \mathcal{D} \) be the contribution of the case \( p_1 = p_2 = p \) to the bound of \( S_d^*(N)^2 \) in (36), i.e.

\[
\mathcal{D} := \sum_\epsilon \frac{N^2 K}{p^2 t^{1-\epsilon}} \sum_{n \leq \frac{N}{K}} \sum_{\substack{p \leq \sqrt{p} \mid p_1 \mid p \mid r_1 \mid p_2 \mid r_2}} V_c \left( \frac{r_1 N}{p_1 t} \right) V_c \left( \frac{r_2 N}{p_2 t} \right) \left( \frac{r_1}{r_2} \right)^{it} \times \delta \left( \pm n \equiv M_{p_2} r_1 - M_{p_1} r_2 \pmod{p} \right) J_{p, p}(n, r_1, r_2).
\]

The congruence condition implies \( p \mid n \) and \( n \equiv M_{p_2} r_1 - M_{p_1} r_2 \pmod{p} \). Since \( PK > N t^\epsilon \), this implies \( n = 0 \). Hence the congruence condition reduces to \( r_1 \equiv r_2 \pmod{p} \). Further splitting \( \mathcal{D} \) into whether \( r_1 = r_2 \) or not, we get

\[
\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_1,
\]

where

\[
\mathcal{D}_0 = \frac{N^2 K}{p^2 t^{1-\epsilon}} \sum_{p \leq \sqrt{p} \mid r_1 \mid p \mid r_2} V_c \left( \frac{r N}{p t} \right)^2 J_{p, p}(0, r, r) \ll NK t^\epsilon
\]

by (38), and

\[
\mathcal{D}_1 = \frac{N^2 K}{p^2 t^{1-\epsilon}} \sum_{p \leq \sqrt{p} \mid r_1 \mid p \mid r_2} \sum_{r_1 \equiv r_2 \pmod{p}} \left( \frac{r_1}{r_2} \right)^{it} V_c \left( \frac{r_1 N}{p_1 t} \right) V_c \left( \frac{r_2 N}{p_2 t} \right) J_{p, p}(0, r_1, r_2).
\]
Now for \( r_1 \neq r_2 \), in order to use \((36)\) to bound the \( J_{p_1, p_2}(n, r_1, r_2) \), we split the difference \( \left| \frac{p_2 - r_1}{p_1} \right| \) into dyadic segments to get

\[
\mathcal{D}_2 = \frac{N^2 K}{p^2 t^{1-\epsilon}} \sum_{0 < \eta < t^\epsilon} \sum_{0 < \eta < t^\epsilon} \sum_{p \mid \eta} \sum_{p \mid \eta} \frac{r_1}{r_2} \frac{r_1 N}{Pt} \psi(p) \psi(p) \frac{r_2 N}{Pt} J_{p, p}(0, r_1, r_2) \delta \left( p 2^\eta < |r_2 - r_1| \leq p 2^{\eta+1} \right)
\]

Let \( \mathcal{O} \) be the contribution of the case \( p_1 \neq p_2 \) to the bound of \( S^*_\alpha(N)^2 \) in \((36)\), i.e.

\[
\mathcal{O} := \frac{N^2 K}{p^2 t^{1-\epsilon}} \sum_{|n| \leq 2t^\epsilon} \sum_{p_1 \neq p_2 \neq p_1 \neq p_2} \psi(p_1) \psi(p_2) \frac{p_2}{p_1} \sum_{r_1 \leq r_2} \frac{r_1}{r_2} \frac{r_1 N}{Pt} \frac{r_2 N}{Pt} 
\]

\[
\times \delta \left( \pm n = M_{p_1} r_1 - M_{p_2} r_2 \mod (p_1, p_2) \right) J_{p_1, p_2}(n, r_1, r_2)
\]

As \( p_1 \neq p_2 \), we have \( (p_1, p_2) = 1 \). The congruence condition implies that \( n \neq 0, p_1 r_1 \neq p_2 r_2, r_1 \equiv \pm M n p_1 \mod p_1 \) and \( r_2 \equiv \pm M n p_2 \mod p_2 \). In order to use \((36)\) to bound the \( J_{p_1, p_2}(n, r_1, r_2) \), we split the difference \( |p_2 r_2 - p_1 r_1| \) into dyadic segments,

\[
\mathcal{O} = \frac{N^2 K}{p^2 t^{1-\epsilon}} \sum_{|n| \leq 2t^\epsilon} \sum_{p_1 \neq p_2 \neq p_1 \neq p_2} \psi(p_1) \psi(p_2) \frac{p_2}{p_1} \sum_{r_1 \leq r_2} \frac{r_1}{r_2} \frac{r_1 N}{Pt} \frac{r_2 N}{Pt} 
\]

\[
\times \delta \left( r_1 \equiv \pm M n p_1 \mod p_1, r_2 \equiv \pm M n p_2 \mod p_2, 2^\eta \leq |p_2 r_2 - p_1 r_1| < 2^{\eta+1} \right) J_{p_1, p_2}(n, r_1, r_2).
\]

Applying \((36)\), we get

\[
\mathcal{O} \ll \frac{N^2 K}{p^2 t^{1-\epsilon}} \sum_{|n| \leq 2t^\epsilon} \sum_{p_1 \neq p_2 \neq p_1 \neq p_2} \sum_{r_1 \leq r_2} \sum_{r_1 \leq r_2} \frac{r_1}{r_2} \frac{r_1 N}{Pt} \frac{r_2 N}{Pt} 
\]

\[
\times \min \left\{ 1, \sqrt{\frac{p^2 t}{NK2^\eta}} \right\} \leq \frac{N t^{1+\epsilon}}{\sqrt{K}}.
\]

8. Final Bound

Combining the above cases \( \mathcal{D}_0, \mathcal{D}_1, \) and \( \mathcal{O} \), collecting all the constraints we have set for \( P \) and \( K \), we get for \( KP > N t^\epsilon, p^2 > N t^\epsilon, K < t^{1+\epsilon}, K < N t^{-\epsilon} \),

\[
S^*_\alpha(N)^2 \ll NK t^\epsilon + \sqrt{K} t^{1+\epsilon} + \frac{N t^{1+\epsilon}}{\sqrt{K}}.
\]

Putting this bound and \((28)\) into \((29)\), we get

\[
S^*_\alpha(N) \ll \left( \sqrt{\frac{NK}{K} t^{1+\epsilon} + \frac{\sqrt{N t^\epsilon} K}{t^2} + \frac{p^2 \sqrt{N}}{K} + \frac{\sqrt{P N}}{t} } \right) t^\epsilon.
\]
Putting this bound into (12), we get
\[ S(N) \ll \left( \frac{N}{P} + \frac{N^2}{PK} + \sqrt{N}K + K^{\frac{1}{2}} \sqrt{t} + \frac{\sqrt{N}t}{K^{\frac{1}{2}}} + \frac{P^2 \sqrt{N}}{K} + \frac{\sqrt{PN}}{t} \right) t^\epsilon. \] (49)

We will use this bound when \( N > t^{\frac{1}{2}+\epsilon} \), and in such case the optimal choices of the parameters under the constraints is

\[ P = t^{\frac{1}{2}+\epsilon} \text{ and } K = t^{\frac{1}{2}-\epsilon}, \]

giving us for \( N > t^{\frac{1}{2}+\epsilon} \),

\[ S(N) \ll \sqrt{N} t^{\frac{1}{2}+\epsilon}. \] (50)

Combining with the bound (6) for the case \( N < t^{\frac{1}{2}+\epsilon} \), we get

\[ \sup_{N \leq t^{\frac{1}{2}+\epsilon}} S(N) \ll \min \left\{ N t^\epsilon, \sqrt{N} t^{\frac{1}{2}+\epsilon} \right\}. \] (51)

This concludes Theorem 1.4 and hence conclude Theorem 1.3 by (4).

**Remark.** The constraints \( P^2 > N t^\epsilon \), \( K < t^{\frac{1}{2}+\epsilon} \) can be relaxed and the error term \( \frac{N^2 t^\epsilon}{PK}, \frac{P^2 \sqrt{N}}{K}, \frac{\sqrt{PN}}{t} \) can be improved by writing out more terms when doing asymptotic expansions. However, they are not affecting the final bound as the main term \( \sqrt{N}K + K^{\frac{1}{2}} \sqrt{t} + \frac{\sqrt{N}t}{K^{\frac{1}{2}}} \) dominates all these terms and constraints.

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