Construction of a class of forward performance processes in stochastic factor models, and an extension of Widder’s theorem

Levon Avanesyan1 · Mykhaylo Shkolnikov1 · Ronnie Sircar1

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Abstract We consider the problem of optimal portfolio selection under forward investment performance criteria in an incomplete market. Given multiple traded assets, the prices of which depend on multiple observable stochastic factors, we construct a large class of forward performance processes, as well as the corresponding optimal portfolios, with power-utility initial data and for stock–factor correlation matrices with eigenvalue equality (EVE) structure, which we introduce here. This is done by solving the associated nonlinear parabolic partial differential equations (PDEs) posed in the “wrong” time direction. Along the way, we establish on domains an explicit form of the generalised Widder theorem of Nadtochiy and Tehranchi (Math. Finance 27:438–470, 2015, Theorem 3.12) and rely for that on the Laplace inversion in time of the solutions to suitable linear parabolic PDEs posed in the “right” time direction.

Keywords Factor models · Forward performance processes · Generalised Widder theorem · Hamilton–Jacobi–Bellman equations · Ill-posed partial differential equations · Incomplete markets · Merton problem · Optimal portfolio selection · Positive eigenfunctions · Time-consistency

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M. Shkolnikov
mykhaylo@princeton.edu

L. Avanesyan
levon.avanaesyan23@gmail.com

R. Sircar
sircar@princeton.edu

1 Department of Operations Research and Financial Engineering, Princeton University, Princeton, NJ 08544, USA
1 Introduction

In this paper, we study the optimal portfolio selection problem under forward investment criteria of power-utility form in incomplete markets, specifically stochastic factor models with a stock–factor correlation structure named EVE, which we introduce here. Our setup is that of a continuous-time market model with multiple stocks whose returns and volatilities are functions of multiple observable stochastic factors which follow jointly a diffusion process. The incompleteness arises from the imperfect correlation between the Brownian motions driving the stock prices and the factors. The factors themselves can model various market inputs, including stochastic interest rates, stochastic volatility and major macroeconomic indicators, such as inflation, GDP growth or the unemployment rate.

The optimal portfolio problem in continuous time was originally considered by Merton [19, 20] in his pioneering work, and is commonly referred to as the Merton problem. In this framework, an investor looks to maximise her expected terminal utility from wealth acquired in the investment process within a geometric Brownian motion market model. Good compilations of classical results can be found in the books by Duffie [5, Chap. 9], and Karatzas and Shreve [16, Chap. 3]. As fundamental as this setup is, it has two important drawbacks. First, the investor must decide on her terminal utility function before entering the market, and therefore cannot adapt it to changes in market conditions. Second, before settling on an investment strategy, the investor must firmly set her time horizon. That is, the portfolio derived in this framework is optimal only for one specific utility function over one time horizon.

External factors such as the economic cycle, natural disasters and the political climate can lead to dynamic changes in one’s preferences. This would change the terminal utility function, thereby affecting the optimal portfolio allocation. Moreover, the investor might want to alter the terminal time itself. In order to solve portfolio optimisation problems with an uncertain investment horizon, forward investment performance criteria were introduced and developed by Musiela and Zariphopoulou [22, 23] as well as Henderson and Hobson [12]. Instead of optimising the expectation of a deterministic utility function at a single terminal point in time, this approach maximises the expectation of a stochastic utility function at every single point in time. Forward performance processes (FPPs), as defined in Musiela and Zariphopoulou [24], capture the time evolutions of such stochastic utility functions.

A comprehensive description of all FPPs remains a challenging open problem. Much work towards this goal has been carried out throughout the last ten years; see Berrier et al. [2], El Karoui and Mrad [6, 7], Henderson and Hobson [12], Musiela and Zariphopoulou [26] and Žitković [34] for some important results. In [26], Musiela and Zariphopoulou proposed a construction of FPPs by means of solutions to a stochastic partial differential equation (SPDE). To find all the FPPs characterised by the SPDE, one would have to find all forward volatility processes, along with initial utility functions, for which the SPDE has a classical solution. The case of zero forward volatility yields time-monotone FPPs, and was extensively discussed in Musiela and Zariphopoulou [24, 25], as well as more recently in Källblad et al. [13] in the presence of model uncertainty.

We consider factor-driven market models and FPPs into which the randomness enters only through the underlying stochastic factors. Assuming such a form, with

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a compatible forward volatility process, the SPDE mentioned above reduces to an HJB equation set in the “wrong” time direction. In a complete market, one can use the Fenchel–Legendre transform to linearise the HJB equation and arrive at a linear second-order parabolic PDE set in the “wrong” time direction (see Nadtochiy and Tehranchi [27]). In an incomplete market, no such linearising transformation is available in general. To the best of our knowledge, the only exception is the special case of power utility in a one-factor market model where a linearisation is possible through a distortion transformation, as discovered in Zariphopoulou [33] for the Merton problem, and used for the construction of FPPs in Nadtochiy and Tehranchi [27], Nadtochiy and Zariphopoulou [28] and Shkolnikov et al. [31]. Construction and representation of FPPs in multi-factor incomplete market models have recently been addressed in Shkolnikov et al. [31] and Liang and Zariphopoulou [18]. The former deals with a two-factor case and provides asymptotic results for different time scales. The latter allows an arbitrary number of factors and trading constraints and gives backward stochastic differential equation (BSDE) representations of FPPs. All these papers assume a power-type (or homothetic) utility structure, as we also do in this paper.

We introduce a new class of multi-factor market models which we call **EVE correlation models** (see Definition 2.4). In this framework, we reduce the fully nonlinear HJB equation to a linear second-order parabolic PDE. Thereby, we obtain explicit characterisations of FPPs in such models. Our analysis also applies to the Merton problem, whose value function solves the same HJB equation posed in the “right” time direction.

In one-factor market models, Nadtochiy and Tehranchi [27, Theorem 3.12] exhibited a characterisation of all positive solutions to the linear parabolic equations posed in the “wrong” time direction that arise in the construction of FPPs of power-utility type. Their theorem constitutes a generalisation of the celebrated Widder theorem (see Widder [32]), which describes all positive solutions of the heat equation set in the “wrong” time direction. The generalised Widder theorem reveals that positive solutions of a linear second-order parabolic equation set in the “wrong” time direction must be integrals, with respect to a positive finite Borel measure, of exponentially scaled positive eigenfunctions for the corresponding elliptic operator. Moreover, for each solution, this representation is unique.

In our first main theorem (Theorem 2.14), we give a new version of [27, Theorem 3.12] on domains in the multi-stock multi-factor EVE setup with an initial utility function of power type to describe a new class of FPPs. Note that generalised Widder theorems do not provide a way to find the eigenfunctions and the measure that make up the representation of this class. Our second set of results addresses this issue: Theorem 2.17 gives the Laplace transform of the measure in terms of the solution to a linear parabolic equation set in the “right” time direction, and we provide a method (see Remark 2.18) of finding the only possible corresponding eigenfunctions as well. Thus we indeed obtain a large explicit class of FPPs.

The rest of the paper is structured as follows. In Sect. 2, we state our main results, postponing their proofs to later sections. In Sect. 3, we introduce relevant facts about FPPs and subsequently prove Theorem 2.14. In Sect. 4, we show Theorem 2.17, summarise some results from the theory of linear elliptic operators and use them...
to establish Propositions 2.20, 2.22, 2.24 and 2.25. In Sect. 5, we discuss the Merton problem within the framework of our market model. Lastly, in Sect. 6, we discuss EVE correlation models and construct explicit FPPs in affine multi-stock multi-factor market models of EVE type.

2 Main results

2.1 Model

Consider an investor with initial capital $X_0 = x > 0$ aiming to invest in a market with $n \geq 1$ stocks, the prices of which follow a process $S$, and a riskless bank account with zero interest rate. The stock prices depend on an observable $k$-dimensional stochastic factor process $Y$ taking values in $D \subseteq \mathbb{R}^k$, and are driven by a $d_W$-dimensional standard Brownian motion $W$. The factor process $Y$ itself is driven by a $dB$-dimensional standard Brownian motion $B$, whose correlation with $W$ is given by a fixed constant matrix

$$\text{corr}(W, B) = (\rho_{ji})_{j,i=1}^{d_W,d_B} =: \rho,$$

where $\rho_{ij} \in [-1, 1]$. Without loss of generality, we assume that $d_W \geq n$ (see Karatzas [14, Remark 0.2.6]). The investor’s filtration $(\mathcal{F}_t)_{t \geq 0}$ is generated by a pair $(S, Y)$ of processes satisfying

$$\frac{dS^i_t}{S^i_t} = \mu_i(Y_t) \, dt + \sum_{j=1}^{d_W} \sigma_{ji}(Y_t) \, dW^j_t, \quad i = 1, 2, \ldots, n, \quad (2.1)$$

$$dY_t = \alpha(Y_t) \, dt + \kappa(Y_t)^\top dB_t, \quad (2.2)$$

$$B_t = \rho^\top W_t + A^\top W^\perp_t, \quad (2.3)$$

where the superscript $\top$ denotes transposition and $W^\perp$ is a $d_{W^\perp}$-dimensional standard Brownian motion independent of $W$. We write $\mu$ for $(\mu_1, \mu_2, \ldots, \mu_n)^\top$ and $\sigma$ for $(\sigma_{ji})_{j,i=1}^{d_W,n}$ throughout.

Remark 2.1 It is straightforward to show that the positive semidefiniteness of the correlation matrix of the Brownian motion $(W, B)$ implies that the singular values of $\rho$ are in $[0, 1]$.

For the convenience of the reader, we summarise the dimensions of all the quantities we have introduced thus far:

$$\mu(\cdot) - n \times 1, \quad \sigma(\cdot) - d_W \times n, \quad W - d_W \times 1, \quad \alpha(\cdot) - k \times 1, \quad \kappa(\cdot) - dB \times k,$$

$$B - dB \times 1, \quad \rho - d_W \times dB, \quad A - d_{W^\perp} \times dB, \quad W^\perp - d_{W^\perp} \times 1.$$

Note that there is no loss of generality in using the representation (2.3) for the standard Brownian motion $B$, since we can let $A$ be the square root of the positive semidefinite matrix $I_{dB} - \rho^\top \rho$ (recall that the singular values of $\rho$ belong to $[0, 1]$), and $d_{W^\perp} = dB$.
The functions \( \mu : D \to \mathbb{R}^n \), \( \sigma : D \to \mathbb{R}^{d_W \times n} \) are continuous, and the stochastic differential equation (SDE) (2.2) has a unique weak solution. Moreover, the columns of \( \rho \) belong to the range of left-multiplication by \( \sigma(y) \) for all \( y \in D \).

Remark 2.3 The last condition in Assumption 2.2 holds only if the column rank of \( \rho \) is less than or equal to the column rank of \( \sigma \), and implies \( \sigma(y)\sigma(y)^{-1}\rho = \rho \) for all \( y \in D \), where \( \sigma(y)^{-1} \) is the Moore–Penrose pseudoinverse of \( \sigma(y) \). Indeed, \( \sigma(y)\sigma(y)^{-1}\sigma(y) = \sigma(y) \), so that the columns of \( \sigma(y) \) (and consequently the vectors in their span, that is, the range of the left-multiplication by \( \sigma(y) \)) are invariant under the left-multiplication by \( \sigma(y)\sigma(y)^{-1} \).

Our main result is for a particular class of multi-factor models, which we define next.

Definition 2.4 We call a market model an eigenvalue equality (EVE) correlation model if for some \( p \in [0, 1] \),

\[
\rho^\top \rho = p I_d_B. \tag{2.4}
\]

Note that in Definition 2.4, \( p \) must be between 0 and 1 since the singular values of \( \rho \) are in \([0, 1]\) (see Remark 2.1).

Remark 2.5 Note that for EVE correlation models, since \( \rho \) is a \( d_W \times d_B \)-matrix, at least one of the following two must hold:

(i) \( d_W \geq d_B \).
(ii) \( p = 0 \).

Finally, we remark that when \( d_B = 1 \), \( \rho \) is a vector and \( p := \rho^\top \rho \in [0, 1] \) so that (2.4) holds automatically.

The name EVE comes from the fact that the only restriction is on the eigenvalues of the matrix \( \rho^\top \rho \). For any orthonormal \( d_B \times d_B \) matrix \( O \), we may replace \( \kappa(\cdot) \) by \( O\kappa(\cdot) \) and \( B \) by \( \tilde{B} = OB \) in (2.2) without changing the dynamics of the pair \((S, Y)\). Since \( \tilde{B} \) is a \( d_B \)-dimensional standard Brownian motion and \( \text{corr}(W, \tilde{B}) = O^\top \rho^\top \rho O \) is diagonal for an appropriate choice of \( O \), we could have assumed without loss of generality from the very beginning that \( \rho^\top \rho \) is diagonal.

Section 6 is devoted to a further discussion of EVE correlation models.

2.2 Forward performance processes

The investor dynamically allocates her wealth in the market using a self-financing trading strategy that at any time \( t \geq 0 \) yields a portfolio allocation \( \pi_t = (\pi^1_t, \ldots, \pi^n_t) \) among the \( n \) stocks with the associated wealth process

\[
\frac{dX^\pi_t}{X^\pi_t} = (\sigma(Y_t)\pi_t)^\top \lambda(Y_t)dt + (\sigma(Y_t)\pi_t)^\top dW_t, \quad X^\pi_0 = x, \tag{2.5}
\]
where \( \lambda(Y_t) = (\sigma(Y_t)^\top)^{-1}\mu(Y_t) \) is the Sharpe ratio. Apart from the self-finance-ability, we impose additional conditions on the trading strategies to ensure that their wealth processes \( X^\pi \) are well defined by (2.5).

**Definition 2.6** An \((\mathcal{F}_t)_{t \geq 0}\)-progressively measurable self-financing trading strategy is called *admissible* if its portfolio allocation \( \pi \) among the \( n \) stocks fulfills

\[
\forall t \geq 0 : \int_0^t |\pi_s^\top \sigma(Y_s)^\top \lambda(Y_s)| \, ds < \infty \quad \text{and} \quad \int_0^t |\sigma(Y_s)\pi_s| \, ds < \infty
\]

with probability one. In this case, we write \( \pi \in \mathcal{A} \).

Next we recall the definition of FPPs given in Musiela and Zariphopoulou [24]. These capture how the utility function of an investor evolves over time as she continues to invest in the financial market.

**Definition 2.7** A (local) *forward performance process* (FPP) is an \((\mathcal{F}_t)_{t \geq 0}\)-progressively measurable process \( U(\cdot) : [0, \infty) \times (0, \infty) \rightarrow \mathbb{R} \) such that

(i) with probability one, all functions \( x \mapsto U_t(x) \), \( t \geq 0 \), are strictly concave and increasing;

(ii) for each \( \pi \in \mathcal{A} \), the process \( U_t(X^\pi) \), \( t \geq 0 \), is a (local) \((\mathcal{F}_t)_{t \geq 0}\)-supermartingale;

(iii) there exists an optimal \( \pi^* \in \mathcal{A} \) for which \( U_t(X^{\pi^*}) \), \( t \geq 0 \), is a (local) \((\mathcal{F}_t)_{t \geq 0}\)-martingale.

**Remark 2.8** We refer to \( \pi^* \) as an optimal portfolio since

\[
\mathbb{E}[U_t(X^{\pi^*})] = U_0(X_0) \geq \mathbb{E}[U_t(X^\pi)] \quad \text{for all} \; \pi \in \mathcal{A}.
\]

We refer to Musiela and Zariphopoulou [24, 26] and Nadtochiy and Zariphopoulou [28] for motivation and explanation of the above definition. We consider (local) FPPs of factor form into which the randomness enters only through the stochastic factor process, that is,

\[
U_t(x) = V(t, x, Y_t), \quad t \geq 0,
\]

for a deterministic function \( V : [0, \infty) \times (0, \infty) \times D \rightarrow \mathbb{R} \). Throughout the paper, we look for FPPs where the initial utility function is of product form and a power function in the wealth variable, i.e.,

\[
U_0(x) = V(0, x, Y_0) = \gamma^\frac{x^{1-\gamma}}{1-\gamma} h(Y_0) \quad \text{for some} \; \gamma \in (0, \infty) \setminus \{1\}.
\]

The crucial simplification arising from the structure in (2.7) lies in its propagation to positive times. In this paper, we construct (local) FPPs of the following form.

**Definition 2.9** We say that a (local) FPP \( U(\cdot) \) is of *separable power factor form* if

\[
U_t(x) = V(t, x, Y_t) = \gamma^\frac{x^{1-\gamma}}{1-\gamma} g(t, Y_t)
\]
for some $g$ that is continuously differentiable in $t$ (its first argument) and twice continuously differentiable in $y$ (its second argument).

In this paper, we characterise all separable power factor form local FPPs for EVE correlation models as introduced in Definition 2.4.

### 2.3 Characterising separable power factor form FPPs

In order to describe our construction of separable power factor form FPPs, we need to introduce some quantities related to linear elliptic operators of the second order. Consider on $C^2(D)$ such an operator

$$
  L = \frac{1}{2} \sum_{i,j=1}^{k} a_{ij}(y) \partial y_i y_j + \sum_{i=1}^{k} b_i(y) \partial y_i + P(y) \tag{2.8}
$$

under the following assumption.

**Assumption 2.10** There exists a $C^3$-diffeomorphism $\mathcal{X} : D \to \mathbb{R}^k$ such that the functions

$$
  \overline{a}_{ij}(z) := \left( (\nabla \mathcal{X}_i)^\top a \nabla \mathcal{X}_j \right)(\mathcal{X}^{-1}(z)),
$$

$$
  \overline{b}_i(z) := \left( (\nabla \mathcal{X}_i)^\top b \right)(\mathcal{X}^{-1}(z)) + \frac{1}{2} \text{trace}(\text{Hess}(\mathcal{X}_i) a)(\mathcal{X}^{-1}(z)),
$$

$$
  \overline{P}(z) := P(\mathcal{X}^{-1}(z)) \tag{2.9}
$$

are uniformly bounded and uniformly $\eta$-Hölder-continuous over $\mathbb{R}^k$ and the matrices $\overline{a}(z) := (\overline{a}_{ij}(z))_{i,j=1}^k$ are nondegenerate uniformly in $z \in \mathbb{R}^k$. That is, with the notation $\overline{b}(\cdot) = (\overline{b}_1(\cdot), \overline{b}_2(\cdot), \ldots, \overline{b}_k(\cdot))^\top$ and for some $\eta \in (0, 1)$, we have

(i) $\sup_{z \in \mathbb{R}^k} |\overline{a}(z)|, \sup_{z \in \mathbb{R}^k} |\overline{b}(z)|, \sup_{z \in \mathbb{R}^k} |\overline{P}(z)| < \infty$;

(ii) $\|\overline{a}\|_{\eta, \mathbb{R}^k}, \|\overline{b}\|_{\eta, \mathbb{R}^k}, \|\overline{P}\|_{\eta, \mathbb{R}^k} < \infty$, where $\|f\|_{\eta, \mathbb{R}^k} = \sup_{z, z' \in \mathbb{R}^k, z \neq z'} \frac{|f(z) - f(z')|}{|z - z'|^{\eta}}$;

(iii) $\inf_{z \in \mathbb{R}^k, |v| = 1} v^\top \overline{a}(z)v > 0$.

**Remark 2.11** Assumption 2.10 entails that the domain $D$ is $C^3$-diffeomorphic to $\mathbb{R}^k$. Furthermore, the operator $L$ on $D$ can be obtained as a pushforward under a $C^3$-diffeomorphism of a uniformly elliptic operator

$$
  \overline{L} = \frac{1}{2} \sum_{i,j=1}^{k} \overline{a}_{ij}(z) \partial z_i z_j + \sum_{i=1}^{k} \overline{b}_i(z) \partial z_i + \overline{P}(z) \tag{2.10}
$$

on $\mathbb{R}^k$ with uniformly bounded and uniformly $\eta$-Hölder-continuous coefficients. For a star-shaped domain $D$, it is well known (see e.g. [8, Sect. 10.1]) that one can find $C^\infty$-diffeomorphisms $\mathcal{X}^{-1}$ mapping $\mathbb{R}^k$ onto $D$. However, whether a locally uniformly elliptic operator $\overline{L}$ with locally bounded and locally $\eta$-Hölder-continuous coefficients is a pushforward under $\mathcal{X}^{-1}$ of an operator $\overline{L}$ with coefficients satisfying
the conditions (i)–(iii) of Assumption 2.10 needs to be checked on a case-by-case basis. As an example, consider the operator $\frac{1}{2} \sum_{i=1}^{k} y_i^{c_i} (1 - y_i)^{c'_i} \partial_{y_i} + P(y)$ on $(0, 1)^k$ with constants $c_i, c'_i \in (4, \infty)$ and a bounded $\eta$-Hölder-continuous potential $P$. Then for the $C^\infty$-diffeomorphism $\Xi : (0, 1)^k \to \mathbb{R}^k$, $y \mapsto (\tan(\pi y_i - \pi/2))_{i=1}^k$, it is elementary to verify that the coefficients of the resulting operator

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{k} \frac{\pi^2 y_i^{c_i} (1 - y_i)^{c'_i}}{\cos(\pi y_i - \pi/2)^4} \partial_{y_i} + \frac{\pi^2 y_i^{c_i} (1 - y_i)^{c'_i} \sin(\pi y_i - \pi/2)}{\cos(\pi y_i - \pi/2)^3} \partial_{y_i} + P((\arctan z_i)_{i=1}^k)$$

fulfil the conditions (i)–(iii) of Assumption 2.10.

**Remark 2.12** Whenever $D = \mathbb{R}^k$ (as for instance in [27, Sect. 3.1]), it is standard in the literature to assume that the conditions (i)–(iii) of Assumption 2.10 hold for $a(\cdot), b(\cdot), P(\cdot)$. This implies the set of conditions in Assumption 2.10 by taking $\Xi$ to be the identity map. Moreover, in this case, the SDE (2.2) admits a unique weak solution (see [15, Chap. 5, Remarks 4.17 and 4.30]).

We define the Hölder space $C^{2,\eta}(D) \subset C^2(D)$ as the subspace consisting of functions whose second-order partial derivatives are $\eta$-Hölder-continuous (in the same sense as in condition (ii) of Assumption 2.10) on compact subsets of $D$. Next, we introduce the sets of positive eigenfunctions for the operator $\mathcal{L}$ which correspond to eigenvalues $\lambda \in \mathbb{R}$ and are normalised at some fixed $y_0 \in D$, i.e.,

$$C_{\mathcal{L} - \lambda}(D) = \{ \psi \in C^{2,\eta}(D) : \psi(\cdot) > 0, \psi(y_0) = 1, (\mathcal{L} - \lambda)\psi = 0 \}.$$

Moreover, we define the spectrum of $\mathcal{L}$ associated with positive eigenfunctions as

$$\mathbb{S}_\mathcal{L}(D) = \{ \lambda \in \mathbb{R} : C_{\mathcal{L} - \lambda}(D) \neq \emptyset \}.$$

In Sect. 4.2, we provide some well-known results about the structure of the eigenfunction spaces $C_{\mathcal{L} - \lambda}(D)$ and the set of eigenvalues $\mathbb{S}_\mathcal{L}(D)$. In particular, Proposition 4.4 yields that in our setup, $\mathbb{S}_\mathcal{L}(D)$ is a half-line.

Finally, we call a functional $\Psi : \mathbb{S}_\mathcal{L}(D) \times D \to (0, \infty)$ with $\Psi(\xi, \cdot) \in C_{\mathcal{L} - \lambda}(D)$ for all $\xi \in \mathbb{S}_\mathcal{L}(D)$ a *selection of positive eigenfunctions*, and we recall the definition of Bochner-integrability in this setting.

**Definition 2.13** Given a positive finite Borel measure $\nu$ on $\mathbb{S}_\mathcal{L}(D)$, a selection of positive eigenfunctions $\Psi : \mathbb{S}_\mathcal{L}(D) \times D \to (0, \infty)$ is $\nu$-Bochner-integrable if for all compact $K \subseteq D$, $\int_{\mathbb{S}_\mathcal{L}(D)} \|\Psi(\xi, \cdot)\|_K \, \nu(d\xi) < \infty$, where $\|f\|_K = \sup_{y \in K} |f(y)|$.

In preparation for our main result, we define

$$a(\cdot) = \kappa(\cdot)^T \kappa(\cdot), \quad b(\cdot) = \alpha(\cdot) + \Gamma \kappa(\cdot)^T \rho^T \lambda(\cdot), \quad P(\cdot) = \frac{\Gamma}{2q} \lambda(\cdot)^T \lambda(\cdot), \quad (2.11)$$

where $\Gamma = \frac{1-q}{q}$ and $q = \frac{1}{1+\Gamma p}$.
**Theorem 2.14** Consider an EVE correlation model (2.1)–(2.3) with a correlation matrix $\rho$ satisfying Assumption 2.2. Suppose the second-order linear elliptic operator $L$ in (2.8) with coefficients provided in (2.11) satisfies Assumption 2.10. Then, given a function $h : D \rightarrow (0, \infty)$, there exists a local FPP of separable power factor form with the initial condition

$$U_0(x) = \gamma^\gamma \frac{x^{1-\gamma}}{1-\gamma} h(Y_0)^q$$

if and only if there exist a positive finite Borel measure $\nu$ on $S_L(D)$ and a $\nu$-Bochner-integrable selection of positive eigenfunctions $\Psi : S_L(D) \times D \rightarrow (0, \infty)$ such that

$$h(y) = \int_{S_L(D)} \Psi(\zeta, y) \nu(d\zeta).$$

(2.12)

Furthermore, each local FPP of separable power factor form is uniquely identified by such a pairing $(\Psi, \nu)$ and is given by

$$U_t(x) = \gamma^\gamma \frac{x^{1-\gamma}}{1-\gamma} \left( \int_{S_L(D)} e^{-\xi} \Psi(\zeta, Y_t) \nu(d\zeta) \right)^q.$$  

(2.13)

Any $\pi^*$ that satisfies

$$\sigma(Y_t)\pi^*_t = \frac{1}{\gamma} \left( \lambda(Y_t) + q\rho\kappa(Y_t) \frac{\int_{S_L(D)} e^{-\xi} (\nabla_y \Psi)(\zeta, Y_t) \nu(d\zeta)}{\int_{S_L(D)} e^{-\xi} \Psi(\zeta, Y_t) \nu(d\zeta)} \right)$$

(2.14)

is an associated optimal portfolio.

**Remark 2.15** We note that Equation (2.14) for optimal portfolios $\pi^*$ does not involve the initial wealth $x$. This is a consequence of the local FPP being of separable power factor form. In the setting of the Merton problem, the same statement is true (and well known) for terminal utility functions of power form.

**Remark 2.16** A solution to the optimal portfolio equation (2.14) can be obtained as follows. Since $\sigma(\cdot)^{-1} = (\sigma(\cdot)^\top \sigma(\cdot))^{-1} \sigma(\cdot)^\top$, one can write $\lambda(\cdot) = (\sigma(\cdot)^\top)^{-1} \mu(\cdot)$ as $\sigma(\cdot)(\sigma(\cdot)^\top \sigma(\cdot))^{-1} \mu(\cdot)$. In addition, by Assumption 2.2 and the Borel selection result of [4, Theorem 6.9.6], one can find a measurable $\zeta : D \rightarrow \mathbb{R}^{n \times dB}$ satisfying $\sigma(\cdot)\zeta(\cdot) = \rho$ which renders

$$\pi^*_t = \frac{1}{\gamma} \left( (\sigma(Y_t)^\top \sigma(Y_t))^{-1} \mu(Y_t) + q\zeta(Y_t)\kappa(Y_t) \frac{\int_{S_L(D)} e^{-\xi} (\nabla_y \Psi)(\zeta, Y_t) \nu(d\zeta)}{\int_{S_L(D)} e^{-\xi} \Psi(\zeta, Y_t) \nu(d\zeta)} \right)$$

(2.15)

The above theorem shows that for given admissible initial conditions, one can construct separable factor-form FPPs in EVE correlation models with general factor domains $D \subseteq \mathbb{R}^k$, while also providing necessary and sufficient conditions for such
admissibility. In particular, an investor with risk aversion \( \gamma \) and dependence \( h(Y_0) \) of her current utility function on the initial value of the factor process can extrapolate the future values of her utility function according to (2.13) and acquire a portfolio fulfilling (2.14) (e.g. the portfolio in (2.15)), provided \( h \) is of the form (2.12). It is therefore crucial to understand which functions \( h \) admit the representation (2.12) and to be able to determine the pairings \((\Psi, \nu)\) for them.

Note that condition (iii) in Assumption 2.10 and the invertibility of the Jacobian matrix of \( \Xi \) yield that \( \kappa(y) \) has full column rank \( k \) at each point \( y \), and thus \( k \leq d_B \). If \( p \neq 0 \), this combined with the observations in Remarks 2.3 and 2.5 implies the dimensional relationship \( k \leq d_B \leq n \leq d_W \) in our model (2.1)–(2.3).

To the best of our knowledge, the only other paper addressing explicitly FPPs in multi-factor models is Liang and Zariphopoulou [18]. In the models they consider, the factors are exponentially ergodic and live on the full space \( D = \mathbb{R}^k \), and they have the dimensional relationship \( n \leq d_W = d_B = k \). In addition, the form of the SPDE in [18] (compare [18, Equation (10)] to e.g. [27, Equation (3)]) implies that \( \sigma \sigma^{-1} = I_{d_W} \), and hence \( n = d_W \). Moreover, in [18], \( \rho = \text{corr}(W, B) = I_{d_W} \), and thus their model fits into the EVE framework with \( p = 1 \). The main difference of the setup in [18] from ours is the possibility of constraints on the set of admissible portfolios. Without constraints, it is possible to linearise the semi-linear PDE in [18, Equation (13)] through the exact same steps as in the proof of our Proposition 3.3 below. For general constraints, this is not possible. The authors circumvent this issue by representing FPPs as functions of the solutions to appropriate infinite-horizon BSDEs instead.

Another major difference from our results is in the set of allowable initial conditions from which FPPs can be constructed. In the absence of constraints, the results in [18] require the measure \( \nu \) (in the terminology of our Theorem 2.14) to be a multiple of a Dirac mass on an element of the set of eigenvalues \( \mathbb{S}_L(\mathbb{R}^k) \), thereby restricting the function \( h \) to be a positive eigenfunction of the elliptic operator \( L \). Our Theorem 2.14 characterises all admissible initial conditions through Equation (2.12). In addition, our factors live on general domains \( D \subseteq \mathbb{R}^k \) and are not required to be ergodic.

### 2.4 Finding selections of positive eigenfunctions \( \Psi \) and measures \( \nu \)

The next set of results addresses the problem of solving (2.12) for the pairing \((\Psi, \nu)\), when it exists. Equation (2.12) stems from a new variant of the generalised Widder theorem of Nadtochiy and Tehranchi [27, Theorem 3.12] (see Theorem 3.4 below), and thus our results can be viewed as yielding explicit versions of such theorems. The following theorem is also of independent interest, as it relates the pairing \((\Psi, \nu)\) arising in the positive solution of a linear second-order parabolic PDE posed in the “wrong” time direction to the solution of the same PDE posed in the “right” time direction.

**Theorem 2.17** Let \( \mathcal{L} \) satisfy Assumption 2.10 and let \( h \in C^{2,\eta}(\Omega) \) be a positive function such that

\[
(t, y) \mapsto \mathbb{E}[h(Z_t) \mathbb{1}_{\{\tau > t\}} | Z_0 = y] \text{ is locally bounded on } [0, \varepsilon] \times \Omega \quad (2.16)
\]

\( \Omega \) Springer
for the weak solution $Z$ of the SDE associated with $L_0 := L - P(y)$ and $\varepsilon > 0$, where $\tau$ is the first exit time of $Z$ from $D$. Then there is a positive classical solution of

$$\partial_t u + Lu = 0 \quad \text{pointwise on } [-\varepsilon, 0] \times D, \text{ with } u(0, \cdot) = h. \quad (2.17)$$

For a positive finite Borel measure $\nu$ on $S_L(D)$ and a $\nu$-Bochner-integrable selection of positive eigenfunctions $\Psi : S_L(D) \times D \to (0, \infty)$, the function $h$ can be expressed as

$$\int_{S_L(D)} e^{-\zeta t} \Psi(\zeta, \cdot) \nu(d\zeta), \quad t \in (-\varepsilon, 0]. \quad (2.18)$$

In this case, we have in particular that

$$u(t, y_0) = \int_{S_L(D)} e^{-\zeta t} \nu(d\zeta), \quad t \in (-\varepsilon, 0]. \quad (2.19)$$

Remark 2.18 Theorem 2.17 reveals that whenever a pairing $(\Psi, \nu)$ exists, it can be inferred by finding the measure $\nu$ through a one-dimensional Laplace inversion of $u(\cdot, y_0)$ (recall that the values of the Laplace transform on a non-trivial interval determine the underlying positive finite Borel measure, see e.g. [3, Sect. 30]) and then the functions $\Psi(\cdot, y), y \in D \setminus \{y_0\}$, from $u(\cdot, y), y \in D \setminus \{y_0\}$, through additional one-dimensional Laplace inversions.

As a by-product, we obtain the following uniqueness result for linear second-order parabolic PDEs posed in the “wrong” time direction by combining the generalised Widder theorem on domains (Theorem 3.4 below) with Theorem 2.17 and the uniqueness of the Laplace transform ([3, Sect. 30]).

Corollary 2.19 For any operator $L$ satisfying Assumption 2.10 and any strictly positive $h \in C^{2,\eta}(D)$ such that the function in (2.16) is locally bounded on a non-trivial cylinder $[0, \varepsilon] \times D$, there is at most one positive solution $\tilde{u}$ of the problem

$$\partial_t \tilde{u} + L\tilde{u} = 0 \quad \text{on } [0, \infty) \times D, \text{ with } \tilde{u}(0, \cdot) = h. \quad (2.20)$$

We stress that Corollary 2.19 is not an immediate consequence of the generalised Widder theorem on domains (Theorem 3.4) by itself. The latter does ensure that every pairing $(\Psi, \nu)$ corresponds to exactly one positive solution $\tilde{u}$ of (2.20). However, it is not clear a priori whether the representation $h = \int_{S_L(D)} \Psi(\zeta, \cdot) \nu(d\zeta)$ is unique for all functions $h$ with the property (2.16). Theorem 2.17 and the uniqueness of the Laplace transform ([3, Sect. 30]) show that this representation is indeed unique.

For arbitrary operators, relatively little is known about the sets of positive eigenfunctions $C_{L-\zeta}(D)$. Nevertheless, in certain situations, additional information on the sets $C_{L-\zeta}(D)$ is available and can be exploited to find the selection of positive eigenfunctions $\Psi$ for a given function $h$ by a finite number of Laplace inversions.
Proposition 2.20 Let $\mathcal{L}$ satisfy Assumption 2.10. Then
\[
\zeta_c(D) := \inf\{\xi \in \mathbb{R} : \xi \in \mathcal{S}_{\mathcal{L}}(D)\} \in \mathcal{S}_{\mathcal{L}}(D).
\] (2.21)

If in addition the potential $P$ is constant and $L_0 := \mathcal{L} - P$ is such that the corresponding solution of the generalised martingale problem on $D$ (see Pinsky [29, Sect. 1.13]) is recurrent, then $\zeta_c(D) = -P$ and $|C_{\mathcal{L} - \zeta_c(D)}(D)| = 1$.

Remark 2.21 The quantity $\zeta_c(D)$ of (2.21) is commonly referred to as the critical eigenvalue of the operator $\mathcal{L}$ on $D$.

The structure of the eigenspaces $C_{\mathcal{L} - \zeta}(D)$ can differ widely depending on the choice of the dimension $k$, the restrictions on the operator $\mathcal{L}$ and the domain $D$. The case $k = 1$ corresponds to having a single factor and leads to eigenspaces of dimension at most 2.

Proposition 2.22 Suppose $\mathcal{L}$ satisfies Assumption 2.10 on a domain $D \subseteq \mathbb{R}$. Then

the number of extreme points of the convex set $C_{\mathcal{L} - \zeta}(D)$ is 2 for all $\zeta > \zeta_c(D)$ and belongs to $\{1, 2\}$ for $\zeta = \zeta_c(D)$.

Proposition 2.22 reveals that in the setting of Theorem 2.17 with $k = 1$, one can determine the pairing $(\Psi, \nu)$ via a three-step procedure: First, one recovers $\nu$ by a one-dimensional Laplace inversion of $u(\cdot, y_0)$; second, one finds $\Psi(\zeta, y_1)\nu(d\zeta)$ by a one-dimensional Laplace inversion of $u(\cdot, y_1)$ for an arbitrary $y_1 \in D \setminus \{y_0\}$; third, for all $\zeta \geq \zeta_c(D)$ in the support of $\nu$, one solves the second-order linear ordinary differential equation for $\Psi(\zeta, \cdot)$ with the obtained boundary conditions at $y_0$ and $y_1$ to end up with the selection $\Psi$.

When $k \geq 2$, the variability in the dimensionality of the eigenspaces is illustrated by the following two scenarios, in which the eigenspaces have dimensions 1 and $\infty$, respectively.

Definition 2.23 A potential $P(\cdot)$ on $\mathbb{R}^k, k \geq 2$, is called principally radially symmetric if

$P = P_0 + P_1$,

where the functions $P_0$ and $P_1$ are locally integrable to power $d$ for some $d > k/2$, with $P_0$ being radially symmetric ($P_0(y) = \tilde{P}_0(|y|)$ for some $\tilde{P}_0$) and $P_1$ vanishing outside of a compact set.

Proposition 2.24 Let $k \geq 2$ and consider a positive $\phi \in C^2(\eta)(\mathbb{R}^k)$ with bounded $\nabla \phi$ and $\Delta \phi$, as well as an operator $\tilde{\mathcal{L}} := \Delta + P(y)$ on $\mathbb{R}^k$ with a locally $\eta$-Hölder-continuous bounded principally symmetric potential $P(\cdot)$. Then $\mathcal{L} := \frac{1}{\phi}\tilde{\mathcal{L}}\phi$ has the property $|C_{\mathcal{L} - \zeta}(\mathbb{R}^k)| = 1$ for any $\zeta \geq \zeta_c(\mathbb{R}^k)$ such that

\[
\int_1^\infty t^{k-3}g_0(t)^2\left(\int_t^\infty s^{1-k}g_0(s)^{-2}ds\right)dt = \infty,
\] (2.22)
where \( g_0 \) is the unique solution of

\[
g_0''(r) + \frac{k-1}{r}g_0'(r) - \left( \xi - \tilde{P}_0(r) \right)g_0(r) = 0 \quad \text{on } (0, \infty),
\]

with \( g_0(r) = 1 + o(r) \) as \( r \downarrow 0 \).

In the situation of Proposition 2.24, we must pick \( \Psi(\xi, \cdot) \) as the unique element of \( C_{L-\xi}(\mathbb{R}^d) \). On the other hand, in the case of a multidimensional factor process on a bounded domain \( D \) with a Lipschitz boundary, the eigenspaces are infinite-dimensional.

**Proposition 2.25** Let \( D \subseteq \mathbb{R}^k \), \( k \geq 2 \), be a bounded domain with a Lipschitz boundary and the coefficients \( a(\cdot), b(\cdot), P(\cdot) \) of \( L \) obey (i)–(iii) in Assumption 2.10 on \( D \). Then the convex set \( C_{L-\xi}(D) \) has infinitely many extreme points for all \( \xi > \xi_c(D) \).

Thus one cannot assert that the number of extreme points of \( C_{L-\xi}(D) \) is finite in general. Therefore, the procedure of Remark 2.18 cannot always be reduced to a finite number of one-dimensional Laplace inversions. In such cases, we propose to determine the FPP on a finite number of grid points \( y \in D \). First, one computes the Borel measure \( \nu \) by applying the inverse Laplace transform to the left-hand side of (2.19). Next, for each \( y \) on the grid, one calculates the selection of eigenfunctions \( \Psi(\cdot, y) \) by taking the inverse Laplace transform of the left-hand side in (2.18). From here, one can identify the value of the FPP on the grid by plugging the obtained values into (2.13).

### 3 Proof of Theorem 2.14 and a new Widder theorem

The goal of this section is to prove Theorem 2.14. Recall that we are interested in separable power factor form local FPPs as in Definition 2.9. We start by focusing on the function \( V \) in (2.6) and give a sufficient condition for \( (V(t, x, Y_t)) \) to be a local FPP.

**Proposition 3.1** Under Assumption 2.2, let \( V : [0, \infty) \times (0, \infty) \times D \to \mathbb{R} \) be continuously differentiable in \( t \) (its first argument) and twice continuously differentiable in \( x \) and \( y \) (the second and third arguments). Suppose further that \( V \) is strictly concave and increasing in \( x \) and a classical solution of the HJB equation

\[
\partial_t V + L_y V - \frac{1}{2} \frac{|\lambda \partial_x V + \rho \kappa \partial_x \nabla_y V|^2}{\partial_{xx} V} = 0 \quad \text{on } [0, \infty) \times (0, \infty) \times D, \quad (3.1)
\]

where \( L_y \) is the generator of the factor process \( Y \). Then \( (V(t, x, Y_t)) \) is a local FPP. Moreover, the corresponding optimal portfolio allocations \( \pi^* \) among the \( n \) stocks are of feedback form and characterised by

\[
\sigma(Y_t)\pi_t^* = -\frac{\lambda(Y_t) \partial_x V(t, X_t^\pi^*, Y_t) + \rho \kappa(Y_t) \partial_x \nabla_y V(t, X_t^\pi^*, Y_t)}{X_t^\pi^* \partial_{xx} V(t, X_t^\pi^*, Y_t)}. \quad (3.2)
\]
Proof For the first statement, one only needs to repeat the derivation of [31, Equation (1.6)] mutatis mutandis and to use \( \sigma(\cdot)\sigma(\cdot)^{-1}\rho = \rho \) (see Remark 2.3). For the second, we apply Itô’s formula to \( V(t, X^\pi_t, Y_t) \) and substitute \( \frac{1}{2} \left\{ \lambda \partial_{xx} V(t, X^\pi_t, Y_t) + \rho \kappa \partial_x \nabla_y V(t, X^\pi_t, Y_t) \right\}^2 \) for \( \partial_t V + \mathcal{L}_y V \) to conclude that the drift coefficient of \( V(t, X^\pi_t, Y_t) \) is \( \frac{1}{2} \partial_{xx} V(t, X^\pi_t, Y_t) \) multiplied by
\[
\left| \frac{\lambda(Y_t)\partial_x V(t, X^\pi_t, Y_t) + \rho \kappa(Y_t)\partial_x \nabla_y V(t, X^\pi_t, Y_t)}{\partial_{xx} V(t, X^\pi_t, Y_t)} + X^\pi_t \sigma(Y_t)\pi_t \right|^2.
\]
(3.3)
The process \( (V(t, X^\pi_t, Y_t)) \) is a local martingale if and only if the expression in (3.3) vanishes, which happens if and only if (3.2) holds.

Remark 3.2 The process \( (V(t, x, Y_t)) \) of Proposition 3.1 is a true FPP if the process \( (V(t, X^\pi_t, Y_t)) \) is a true supermartingale for every \( \pi \in \mathcal{A} \) and a true martingale for every optimal portfolio allocation \( \pi^* \) of (3.2). In view of Fatou’s lemma, the supermartingale property is fulfilled if \( \inf_{s \in [0, t]} V(s, X^\pi_s, Y_s) \) is integrable for all \( t \geq 0 \) and \( \pi \in \mathcal{A} \). The martingale property holds if all the diffusion coefficients \( \partial_x V(t, X^\pi_*^t, Y_t)X^\pi_*^t (\sigma(Y_t)\pi_*^t) \top, \nabla_y V(t, X^\pi_*^t, Y_t)\kappa(Y_t) \top \) of \( V(t, X^\pi_*^t, Y_t) \) are \((dt \times d\mathbb{P})\)-square-integrable on \([0, t] \times \Omega \) for each \( t \geq 0 \).

The HJB equation (3.1) is a fully nonlinear PDE and one does not expect to find explicit formulas for its solutions in general. However, in EVE correlation market models, for initial conditions of separable power type, (3.1) can be linearised.

Proposition 3.3 Let \( \rho \) be an EVE correlation matrix as in Definition 2.4, and let \( \Gamma = \frac{1-\gamma}{\gamma} \) and \( q = \frac{1}{1+1_p} \). Then the HJB equation (3.1) with an initial condition \( V(0, x, y) = \gamma \frac{x^{1-\gamma}}{1-\gamma} h(y) \), where \( h > 0 \), has a classical solution in separable power form, \( V(t, x, y) = \gamma \frac{x^{1-\gamma}}{1-\gamma} g(t, y) \), with \( g > 0 \) if and only if there exists a positive solution to the linear PDE problem
\[
\partial_t u + \mathcal{L} u = 0 \quad \text{on} \ [0, \infty) \times D, \ \text{with} \ u(0, \cdot) = h
\]
posed in the “wrong” time direction. Here \( \mathcal{L} \) is the linear elliptic operator of second order with the coefficients of (2.11). In that case, the two solutions are related through
\[
V(t, x, y) = \gamma \frac{x^{1-\gamma}}{1-\gamma} g(t, y).
\]
Proof Since we are looking for solutions of the HJB equation (3.1) in separable power form, we plug in the ansatz \( V(t, x, y) = \gamma \frac{x^{1-\gamma}}{1-\gamma} g(t, y) \) to arrive at
\[
\partial_t g + \mathcal{L}_y g + \frac{\Gamma}{2} \lambda \top \lambda g + \Gamma \lambda \top \rho \kappa \nabla_y g + \Gamma \frac{\nabla_y g \kappa \top \rho \kappa \nabla_y g}{2g} = 0,
\]
\[
g(0, \cdot) = h^q.
\]
(3.5)
Next, we employ the distortion transformation \( g(t, y) = u(t, y)^q \) and get the PDE

\[
q u^{q-1} \partial_t u + \frac{1}{2} \sum_{i,j=1}^{k} (\kappa^\top \kappa)_{ij} \left( q u^{q-1} \partial_{y_i y_j} u + q(q - 1) u^{q-2} (\partial_{y_i} u)(\partial_{y_j} u) \right) \\
+ \frac{1}{2} q^2 u^{q-2} (\nabla_y u)^\top \kappa^\top \rho \kappa \nabla_y u + q(\alpha + \Gamma \kappa^\top \rho \lambda)^\top u^{q-1} \nabla_y u + \frac{\Gamma}{2} \lambda^\top \lambda u^q 
\]

\( = 0 \) \hspace{1cm} (3.6)

with the initial condition \( u(0, \cdot) = h \). Moreover, the assumed positivity of \( g \) translates to \( u > 0 \) so that we can divide both sides of (3.6) by \( u^{q-1} \). In addition, we insert the identity \( \rho^\top \rho = p I_{d_B} \) of Definition 2.4 to end up with

\[
\partial_t u + \frac{1}{2} \sum_{i,j=1}^{k} (\kappa^\top \kappa)_{ij} \partial_{y_i y_j} u + (\alpha + \Gamma \kappa^\top \rho \lambda)^\top \nabla_y u + \frac{\Gamma}{2q} \lambda^\top \lambda u \\
+ \frac{1}{2u}(q + \Gamma pq - 1)(\nabla_y u)^\top \kappa^\top \kappa \nabla_y u = 0.
\] \hspace{1cm} (3.7)

The crucial observation is now that the nonlinear term in the PDE (3.7) drops out thanks to \( q = \frac{1}{1+1p} \). Hence, \( u \) is a positive solution of (3.4). The converse follows by carrying out the transformations we have used in the reverse order.

Proposition 3.3 reduces the task of finding solutions of the HJB equation (3.1) in separable power form to solving the linear PDE problem (3.4) set in the “wrong” time direction. The latter has been studied in Widder [32] with \( \mathcal{L} \) being the Laplace operator on \( \mathbb{R}^k \) and in Nadtochiy and Tehranchi [27] for more general linear second-order elliptic operators on \( \mathbb{R}^k \). We establish subsequently a variant of [27, Theorem 3.12] that allows linear second-order elliptic operators on domains \( D \subseteq \mathbb{R}^k \).

**Theorem 3.4** Under Assumption 2.10, a function

\[
u : \{(0, y_0)\} \cup \{(0, \infty) \times D\} \rightarrow (0, \infty)
\]

is a classical solution of \( \partial_t u + \mathcal{L} u = 0 \) with \( u(0, y_0) = 1 \) if and only if it admits the representation

\[
u(t, y) = \int_{\mathcal{S}_{\mathcal{L}}(D)} e^{-t \xi} \Psi(\xi, y) v(d\xi), \hspace{1cm} (3.8)
\]

where \( v \) is a Borel probability measure on \( \mathcal{S}_{\mathcal{L}}(D) \) and \( \Psi : \mathcal{S}_{\mathcal{L}}(D) \times D \rightarrow (0, \infty) \) is a \( v \)-Bochner-integrable selection of positive eigenfunctions. In this case, the pairing \((\Psi, v)\) is uniquely determined by the function \( \nu \).

**Proof** Let \( \nu \) as above be a classical solution of \( \partial_t u + \mathcal{L} u = 0 \) with \( u(0, y_0) = 1 \). Recalling the \( C^3 \)-diffeomorphism \( \Xi : D \rightarrow \mathbb{R}^k \) from Assumption 2.10 and taking without loss of generality \( \Xi(y_0) = 0 \) (otherwise we translate \( \Xi \) by \(-\Xi(y_0))\), we define

\[
\overline{u} : \{(0, 0)\} \cup \{(0, \infty) \times \mathbb{R}^k\} \rightarrow (0, \infty), \hspace{1cm} (t, z) \mapsto u(t, \Xi^{-1}(z)).
\]
Then \( \partial_t u(t, y) = \partial_t \overline{u}(t, \Xi(y)), \partial_{y_j} u(t, y) = \sum_{j=1}^k \partial_{z_{i_j}} \overline{u}(t, \Xi(y)) \partial_{y_j} \Xi_{i_j}(y) \) and

\[
\partial_{y_i y_j} u(t, y) = \sum_{i', j' = 1}^k \partial_{z_{i_j'}} \overline{u}(t, \Xi(y)) \partial_{y_i y_j} \Xi_{i_j'}(y).
\]

Plugging these into the PDE for \( u \), we conclude that \( \overline{u} \) is a classical solution of \( \partial_t \overline{u} + \mathcal{L} \overline{u} = 0 \) with \( \overline{u}(0, 0) = 1 \), where \( \mathcal{L} \) is the operator of (2.10), (2.9) on \( \mathbb{R}^k \). From [27, Theorem 3.12], we infer that

\[
\overline{u}(t, z) = \int_{\mathbb{S}_\mathcal{L}(\mathbb{R}^k)} e^{-t\zeta} \overline{\Psi}(\zeta, z) \nu(d\zeta)
\]

with a Borel probability measure \( \nu \) on \( \mathbb{S}_\mathcal{L}(\mathbb{R}^k) \) and a \( \nu \)-Bochner-integrable selection of positive eigenfunctions \( \overline{\Psi} : \mathbb{S}_\mathcal{L}(\mathbb{R}^k) \times \mathbb{R}^k \to (0, \infty) \) for the operator \( \overline{\mathcal{L}} \) (note that \( \overline{\Psi}(\zeta, \cdot) \in C^{2,\eta}(\mathbb{R}^k) \) by the Schauder interior estimate; see e.g. [11, inequality (6.23)]).

Next, we express \( \overline{\mathcal{L}} \overline{\Psi}(\zeta, \cdot) \) as

\[
\frac{1}{2} \sum_{i, j = 1}^k ((\nabla \Xi_i)^\top a \nabla \Xi_j)(\Xi^{-1}(\cdot)) \partial_{z_{i_j}} \overline{\Psi}(\zeta, \cdot)
\]

\[
+ \sum_{i = 1}^k \left( ((\nabla \Xi_i)^\top b)(\Xi^{-1}(\cdot)) + \frac{1}{2} \text{trace}(\text{Hess}(\Xi_i)a)(\Xi^{-1}(\cdot)) \right) \partial_{z_i} \overline{\Psi}(\zeta, \cdot)
\]

\[
+ P(\Xi^{-1}(\cdot)) \overline{\Psi}(\zeta, \cdot)
\]

\[
= \mathcal{L} \overline{\Psi}(\zeta, \Xi(y)) \big|_{y = \Xi^{-1}(\cdot)}
\]

and see that \( \overline{\mathcal{L}} \overline{\Psi}(\zeta, \cdot) = \zeta \overline{\Psi}(\zeta, \cdot) \) is equivalent to \( \mathcal{L} \overline{\Psi}(\zeta, \Xi(\cdot)) = \zeta \overline{\Psi}(\zeta, \Xi(\cdot)) \). Thus \( \mathbb{S}_{\overline{\mathcal{L}}}(\mathbb{R}^k) = \mathbb{S}_{\mathcal{L}}(D) \) and

\[
uu(t, y) = \overline{u}(t, \Xi(y)) = \int_{\mathbb{S}_{\mathcal{L}}(D)} e^{-t\zeta} \overline{\Psi}(\zeta, \Xi(y)) \nu(d\zeta),
\]

where \( \Psi(\cdot, \cdot) := \overline{\Psi}(\cdot, \Xi(\cdot)) : \mathbb{S}_{\mathcal{L}}(D) \times D \to (0, \infty) \) is a \( \nu \)-Bochner-integrable selection of positive eigenfunctions for the operator \( \mathcal{L} \) (observe that the images of compact sets under \( \Xi \) are compact).

Conversely, for a function \( u \) as in (3.8), we have \( u(0, y_0) = 1 \). Moreover, defining the function \( \overline{u} \) as before, we find that

\[
\overline{u}(t, z) = \int_{\mathbb{S}_{\mathcal{L}}(D)} e^{-t\zeta} \Psi(\zeta, \Xi^{-1}(z)) \nu(d\zeta).
\]
As above, we have $S_L(D) = S_L(\mathbb{R}^k)$ and that
\[ \overline{\Psi}(\cdot, \cdot) := \Psi(\cdot, \mathbb{E}^{-1}(\cdot)) : S_L(\mathbb{R}^k) \times \mathbb{R}^k \to (0, \infty) \]
provides a $\nu$-Bochner-integrable selection of positive eigenfunctions for the operator $\overline{L}$. By [27, Theorem 3.12], $\overline{u}$ is a classical solution of $\partial_t \overline{u} + \overline{L} \overline{u} = 0$ with $\overline{u}(0, 0) = 1$ (here, we again assume without loss of generality that $\mathbb{E}(\gamma_0) = 0$). Since $(\partial_t \overline{u} + \overline{L} \overline{u})(t, \mathbb{E}(y)) = (\partial_t \overline{u} + \overline{L} \overline{u})(t, y)$, the function $u$ is a classical solution of $\partial_t u + \overline{L} u = 0$. Lastly, due to [27, Theorem 3.12], the function $\overline{u}(\cdot, \cdot) = u(\cdot, \mathbb{E}^{-1}(\cdot))$ uniquely determines the pairing $(\overline{\Psi}(\cdot, \cdot), \nu) = (\Psi(\cdot, \mathbb{E}^{-1}(\cdot)), \nu)$, and thus the function $u$ uniquely determines the pairing $(\Psi, \nu)$. □

We now have all the ingredients needed to prove Theorem 2.14.

**Proof of Theorem 2.14** Take a function $h : D \to (0, \infty)$ and consider
\[ V(t, x, y) = \gamma^{x^1 - \gamma} \frac{1}{1 - \gamma} g(t, y), \quad \text{where } g > 0 \text{ and } g(0, y) = h(y)^q. \]

First, we show that $(V(t, x, Y_t))$ is a separable power factor form local FPP if and only if $V(t, x, y)$ is a classical solution to the HJB equation (3.1). Sufficiency follows trivially from Proposition 3.1. To prove necessity, consider a portfolio allocation $\pi \in \mathcal{A}$. We apply Itô’s formula to $\gamma^{x^1 - \gamma} \frac{1}{1 - \gamma} g(t, Y_t)$ and infer from the conditions (ii) and (iii) in Definition 2.7 that the resulting drift coefficient must be nonpositive for all $\pi \in \mathcal{A}$ and equal to 0 for any maximiser $\pi^* \in \mathcal{A}$. Equating the maximum of the drift coefficient over all $\pi \in \mathcal{A}$ to 0, we end up with the PDE in (3.5) for $g$; see Lemma A.1 in the Appendix. Thus $V$ is a classical solution to the HJB equation (3.1).

It follows by Proposition 3.3 that $(V(t, x, Y_t))$ is a separable power factor form local FPP if and only if $g(t, y) = u(t, y)^q$, where
\[ \partial_t u + \overline{L} u = 0 \quad \text{on } (0, \infty) \times D, \text{ with } u(0, \cdot) = h(\cdot). \quad (3.9) \]

By Theorem 3.4, each solution $u$ of (3.9) is uniquely identified with a pairing $(\Psi, \nu)$, and is given by the right-hand side of (3.8). This yields the necessity and sufficiency of the representation (2.12), as well as the identity (2.13). Finally, the characterisation (2.14) of the optimal portfolios is a direct consequence of (3.2) and (2.13). □

4 Proof of Theorem 2.17 and further ramifications

4.1 Proof of Theorem 2.17

We start our analysis of the pairing $(\Psi, \nu)$ by establishing Theorem 2.17.

**Proof of Theorem 2.17** Let $D' \subseteq D$ be a bounded subdomain with a $C^3$-boundary $\partial D' \subseteq D$ and $\psi : D' \to [0, 1]$ a three times continuously differentiable function with
compact support in $D'$. Then Assumption 2.10 and the formulas

$$a_{ij}(y) = \left( (\nabla \Xi_i^{-1})^\top \nabla \Xi_j^{-1} \right)(\Xi(y)),$$

$$b_i(y) = \left( (\nabla \Xi_i^{-1})^\top \nabla \Xi_i^{-1} \right)(\Xi(y)) + \frac{1}{2} \text{trace}(\text{Hess}(\Xi_i^{-1}) a)(\Xi(y)).$$

(4.1)

render Ladyžhenskaya et al. [17, Chap. IV, Theorem 5.2] applicable to the problem (posed in the “right” time direction)

$$\partial_t u_{D'} + \mathcal{L} u_{D'} = 0 \quad \text{pointwise on } [-\varepsilon, 0] \times D',$$

with $u_{D'}|_{[-\varepsilon, 0] \times \partial D'} = 0$, $u_{D'}(0, \cdot) = h\psi$, yielding a unique classical solution with $\eta$-Hölder-continuous $\partial_t u_{D'}$, $\partial_{y_j} u_{D'}$ in the $y$-variable, $\frac{\eta}{2}$-Hölder-continuous $\partial_{y_i} u_{D'}$, $\partial_{y_i y_j} u_{D'}$ in the $t$-variable and $\frac{1+\eta}{2}$-Hölder-continuous $\partial_{y_i} u_{D'}$ in the $t$-variable. In particular, $u_{D'}$ obeys the Feynman–Kac formula

$$u_{D'}(-t, y) = \mathbb{E}\left[ e^{\int_0^t P(Z_s) \text{d}s} h(\psi)(Z_t) 1_{\{\tau_{D'} > t\}} \big| Z_0 = y \right], \quad (t, y) \in [0, \varepsilon] \times D',$$

where $\tau_{D'}$ is the first exit time of $Z$ from $D'$.

Using the above construction for a sequence of subdomains $D'$ and functions $\psi$ increasing to $D$ and $1_D$, respectively, we arrive at the monotone limit

$$u(-t, y) = \mathbb{E}\left[ e^{\int_0^t P(Z_s) \text{d}s} h(\psi)(Z_t) 1_{\{\tau_D > t\}} \big| Z_0 = y \right], \quad (t, y) \in [0, \varepsilon] \times D,$$

of $u_{D'}$, which is locally bounded on $[0, \varepsilon] \times D$ by assumption. Thanks to this and the local regularity estimate from [17, Chap. IV, inequality (10.5)] on every fixed set $(-\varepsilon, 0) \times D'$ (and hence on its closure $[-\varepsilon, 0] \times \overline{D}'$), we can extract a subsequence of $u_{D'}$ converging uniformly together with $\partial_t u_{D'}$, $\partial_{y_i} u_{D'}$ and $\partial_{y_i y_j} u_{D'}$ on every fixed set $[-\varepsilon, 0] \times \overline{D}'$. Thus $u$ is a positive classical solution of the problem (2.17).

Now consider an arbitrary positive classical solution $u$ of the problem (2.17) and suppose there exist pairings $(\Psi^{(i)}, \nu^{(i)})$ and $(\Psi^{(2)}, \nu^{(2)})$ such that for all $y \in D$,

$$h(y) = \int_{\mathbb{S}_L(D)} \Psi^{(1)}(\xi, y) v^{(1)}(d\xi) = \int_{\mathbb{S}_L(D)} \Psi^{(2)}(\xi, y) v^{(2)}(d\xi).$$

In view of [29, Chap. 4, Theorem 3.2 and Exercise 4.16] (see also Sect. 4.2 below for more details), the elements of $\mathbb{S}_L(D)$ are bounded from below, so that the functions $\tilde{u}^{(i)}(t, y) := \int_{\mathbb{S}_L(D)} e^{-\zeta t} \Psi^{(i)}(\xi, y) v^{(i)}(d\xi)$, $i = 1, 2$, are finite on $[0, \infty) \times D$. By Theorem 3.4, each $\tilde{u}^{(i)}$ is a classical solution of

$$\partial_t \tilde{u}^{(i)} + \mathcal{L} \tilde{u}^{(i)} = 0 \quad \text{on } \{(0, y_0)\} \cup (0, \infty) \times D).$$

(4.2)

Moreover, each

$$v^{(i)}(t, y) := \begin{cases} u(t, y) & \text{for } (t, y) \in [-\varepsilon, 0] \times D, \\ \tilde{u}^{(i)}(t, y) & \text{for } (t, y) \in (0, \infty) \times D \end{cases}$$
is a positive classical solution of the PDE $\partial_t v^{(i)} + \mathcal{L} v^{(i)} = 0$ on $[-\epsilon, \infty) \times D$. Indeed, on the sets $[-\epsilon, 0] \times D$ and $(0, \infty) \times D$, this PDE holds by construction, whereas

$$
\partial_t \tilde{u}^{(i)}(0, y) = \lim_{t \downarrow 0} \partial_t \tilde{u}^{(i)}(t, y) = -\lim_{t \downarrow 0} \mathcal{L} \tilde{u}^{(i)}(t, y) = -\mathcal{L} \tilde{u}^{(i)}(0, y), \quad y \in D,
$$

by the interior Schauder estimate of [27, Theorem 6.2].

Shifting the time by $\epsilon$ and renormalising $v^{(i)}$, $i = 1, 2$, we get the functions $\tilde{v}^{(i)}(t, y)$, $i = 1, 2$, which solve the PDE (4.2) on $[0, \infty) \times D$. By Theorem 3.4, there exist pairings $(\tilde{\Psi}^{(i)}, \tilde{\nu}^{(i)})$, $i = 1, 2$, such that

$$
\tilde{v}^{(i)}(t, y) = \int_{S_L(D)} e^{-\xi t} \tilde{\Psi}^{(i)}(\xi, y) \tilde{\nu}^{(i)}(d\xi)
$$

for $i = 1, 2$. In particular, for $(t, y) \in (0, \infty) \times D$ and $i = 1, 2$,

$$
\int_{S_L(D)} e^{-\xi (t+\epsilon)} \tilde{\Psi}^{(i)}(\xi, y) \tilde{v}^{(i)}(d\xi) = \tilde{v}^{(i)}(t + \epsilon, y) = \frac{v^{(i)}(t, y)}{v^{(i)}(-\epsilon, y_0)} = \frac{\int_{S_L(D)} e^{-\xi t} \Psi^{(i)}(\xi, y)v^{(i)}(d\xi)}{v^{(i)}(-\epsilon, y_0)}. \tag{4.3}
$$

Plugging in first $y = y_0$, then $y \in D \setminus \{y_0\}$, and relying on the uniqueness of the Laplace transform (see [3, Sect. 30]), we read off $\tilde{\nu}^{(i)}(d\xi) = \frac{e^{\xi \epsilon}}{v^{(i)}(-\epsilon, y_0)} v^{(i)}(d\xi)$ and $\tilde{\Psi}^{(i)} = \Psi^{(i)}$, $i = 1, 2$, from (4.3). Hence for $(t, y) \in (-\epsilon, 0] \times D$ and $i = 1, 2$, we get

$$
u(t, y) = v^{(i)}(t, y) = v^{(i)}(-\epsilon, y_0) \tilde{v}^{(i)}(t + \epsilon, y)$$

$$= v^{(i)}(-\epsilon, y_0) \int_{S_L(D)} e^{-\xi (t+\epsilon)} \tilde{\Psi}^{(i)}(\xi, y) \tilde{v}^{(i)}(d\xi)$$

$$= \int_{S_L(D)} e^{-\xi t} \Psi^{(i)}(\xi, y)v^{(i)}(d\xi). \tag{4.4}$$

In particular, we get from the latter equation that

$$
\int_{S_L(D)} e^{-\xi t} \Psi^{(1)}(\xi, y)v^{(1)}(d\xi) = \int_{S_L(D)} e^{-\xi t} \Psi^{(2)}(\xi, y)v^{(2)}(d\xi).
$$

Just like above, plugging in $y = y_0$, then $y \in D \setminus \{y_0\}$, and using the uniqueness of the Laplace transform, we get $v^{(1)}(d\xi) = v^{(2)}(d\xi) = v(d\xi)$ and $\Psi^{(1)} = \Psi^{(2)} =: \Psi$. Combining this with (4.4), we obtain that any positive classical solution to the problem (2.17) must be as given in (2.18). This yields uniqueness as desired, and in the special case of $y = y_0$, we obtain (2.19).

□

4.2 Preliminaries on positive eigenfunctions

As a preparation for the proofs of Propositions 2.20, 2.22, 2.24 and 2.25, we recall some facts about the sets $S_L(D)$ and $C_{L-\xi}(D)$, $\xi \in S_L(D)$, from positive harmonic
Throughout this subsection, we let \( \mathcal{L} \) satisfy Assumption 2.10 and infer from (4.1) that \( \mathcal{L} \) is then locally uniformly elliptic with locally bounded and locally \( \eta \)-Hölder-continuous coefficients.

**Definition 4.1** Consider the solution \( Z \) of the generalised martingale problem on \( D \) associated with \( \mathcal{L}_0 = \mathcal{L} - P(y) \) (see [29, Sect. 1.13]). If

\[
D' \mapsto \mathbb{E} \left[ \int_0^\infty e^{\int_0^t P(Z_s) \, ds} \, 1_{D'}(Z_t) \, dt \mid Z_0 = y \right] < \infty
\]

(4.5)

for all bounded subdomains \( D' \subseteq D \) with \( \overline{D'} \subseteq D \) and \( y \in D \), then the positive Borel measure defined by (4.5) is called the Green measure for \( \mathcal{L} \) on \( D \). The density \( G(y,z) \) of the Green measure, if it exists, is referred to as the Green function.

By [29, Chap. 4, Theorem 3.1 and Exercise 4.16] for the operators \( \mathcal{L} - \zeta \), \( \zeta \in \mathbb{R} \), we have the next proposition.

**Proposition 4.2** If \( \zeta \in \mathbb{R} \) is such that the Green function exists for \( \mathcal{L} - \zeta \), then \( C_{L-\zeta}(D) \neq \emptyset \).

We proceed to the corresponding classification of the operators \( \mathcal{L} - \zeta \), \( \zeta \in \mathbb{R} \).

**Definition 4.3** An operator \( \mathcal{L} - \zeta \) on \( D \) is called

(i) **subcritical** if it possesses a Green function;
(ii) **critical** if it is not subcritical, but \( C_{L-\zeta}(D) \neq \emptyset \);
(iii) **supercritical** if it is neither critical nor subcritical.

Thus we are interested in the values of \( \zeta \) for which \( \mathcal{L} - \zeta \) is subcritical or critical, that is, \( \zeta \in \mathbb{S}_L(D) \). As it turns out, \( \mathbb{S}_L(D) \) is a half-line under Assumption 2.10.

**Proposition 4.4** (Pinksy [29], Chap. 4, Theorem 3.2 and Exercise 4.16) There exists a critical eigenvalue \( \zeta_c = \zeta_c(D) \in \mathbb{R} \) such that \( \mathcal{L} - \zeta \) is subcritical for \( \zeta > \zeta_c \), supercritical for \( \zeta < \zeta_c \), and either critical or subcritical for \( \zeta = \zeta_c \).

When the potential \( P \) is nonpositive, more information about the classification of the operator \( \mathcal{L} \) is available.

**Proposition 4.5** (Pinksy [29], Chap. 4, Theorem 3.3 and Exercise 4.16) For an operator \( \mathcal{L} \) with \( P \leq 0 \), one of the following holds:

(i) \( P \leq 0, \ P \not\equiv 0 \) and \( \mathcal{L} \) is subcritical.
(ii) \( P \equiv 0 \), the solution of the generalised martingale problem on \( D \) associated with \( \mathcal{L} \) is transient and \( \mathcal{L} \) is subcritical.
(iii) \( P \equiv 0 \), the solution of the generalised martingale problem on \( D \) associated with \( \mathcal{L} \) is recurrent and \( \mathcal{L} \) is critical.

**Remark 4.6** When \( \gamma > 1 \), the potential term in (2.11) is nonpositive. This, put together with Proposition 4.5, yields \( 0 \in \mathbb{S}_L \). Thus \( [0, \infty) \subseteq \mathbb{S}_L \) by Proposition 4.4.
4.3 Proofs of Propositions 2.20, 2.22, 2.24 and 2.25

At this point, we can read off Propositions 2.20, 2.22 and 2.24 from appropriate results in [21] and [29].

**Proof of Proposition 2.20** By Propositions 4.2 and 4.4,
\[
\inf \{ \zeta \in \mathbb{R} : \zeta \in \mathcal{S}(D) \} = \zeta_c(D) \in \mathcal{S}(D).
\]

If \( P \) is constant and the solution of the generalised martingale problem on \( D \) for \( \mathcal{L} - P \) is recurrent, then \( \mathcal{L} - P \) is critical by Proposition 4.5, and hence \( \zeta_c(D) = -P \). In this case, [29, Chap. 4, Theorem 3.4 and Exercise 4.16] yield
\[
|C_{\mathcal{L} - \zeta}(D)| = 1.
\]

□

**Proof of Proposition 2.22** It suffices to put together Proposition 4.4 with [29, Chap. 4, Remark 2 on p. 149, Theorem 3.4 and Exercise 4.16]. □

**Proof of Proposition 2.24** Note that for any \( \zeta \geq \zeta_c(\mathbb{R}^k) \) and \( f \in C_{\mathcal{L} - \zeta} \), one has \( \tilde{f} \in C_{\mathcal{L} - \zeta} \). Therefore, it is enough to prove \( |C_{\mathcal{L} - \zeta}| = 1, \zeta \geq \zeta_c(\mathbb{R}^k) \), which is readily obtained by combining Proposition 4.4 with Murata [21, Theorem 5.3]. □

**Remark 4.7** The condition (2.22) on \( \mathcal{L} \) and \( \zeta \) needs to be verified on a case-by-case basis. For example, consider a locally \( \eta \)-Hölder-continuous nonpositive bounded radially symmetric potential \( P_0 \) with \( \tilde{P}_0(r) = cr^{-2}, r \geq 1 \), for some \( c < 0 \). For some \( \phi \) as in Proposition 2.24 and \( \zeta = 0 \), take \( \mathcal{L} = \frac{1}{\phi}(\Delta + P_0)\phi \). Then we get
\[
g_0(r) = c_1r^{\frac{2-k+\sqrt{(k-2)^2-4c}}{2}} + c_2r^{\frac{2-k-\sqrt{(k-2)^2-4c}}{2}}, \quad r \geq 1,
\]
for some \( c_1, c_2 \in \mathbb{R} \). By [21, Theorem 4.6 (iii) and Theorem 2.4 (ii)], the operator \( \Delta + P_0 \) is subcritical so that \( c_1 \neq 0 \) by [21, Theorem 3.1 (ii)]. Thus (2.22) holds.

In the context of Proposition 2.25, the structure of the sets \( C_{\mathcal{L} - \zeta}(D), \zeta > \zeta_c(D) \), has been described in [1, Theorems 6.1 and 6.3], which we briefly recall for the convenience of the reader.

**Definition 4.8** A function \( f \in C_{\mathcal{L} - \zeta}(D) \) is called a minimal eigenfunction if \( \tilde{f} \leq f \) implies \( \tilde{f} = f \) for all \( \tilde{f} \in C_{\mathcal{L} - \zeta}(D) \).

**Proposition 4.9** (Ancona [1], Theorems 6.1 and 6.3) In the setting of Proposition 2.25, every minimal \( f \in C_{\mathcal{L} - \zeta}(D) \) has the property \( \lim_{z \to y} f(z) > 0 \) for exactly one point \( y \in \partial D \) and is uniquely determined by \( y \). We denote such an \( f \) by \( f_y \) and call it the minimal eigenfunction associated with \( y \). In addition, for every \( f \in C_{\mathcal{L} - \zeta}(D) \), there exists a unique Borel probability measure \( \xi \) on \( \partial D \) such that
\[
f(\cdot) = \int_{\partial D} f_y(\cdot) \xi(dy).
\]
Proposition 2.25 is a direct consequence of Proposition 4.9.

Proof of Proposition 2.25 The uniqueness of the Borel probability measure $\xi$ in the representation (4.6) shows that the extreme points of $C_{L-\xi}(D)$ are precisely the minimal eigenfunctions $f_y$, $y \in \partial D$. Clearly, $|\{f_y : y \in \partial D\}| = |\partial D| = \infty$. □

5 Merton problem in stochastic factor models

In this section, we consider the framework of the Merton problem, in which an investor aims to maximise her expected terminal utility from the wealth acquired through investment, i.e.,

$$\sup_{\pi \in A} \mathbb{E}[\nu_T(X^\pi_T, Y_T)].$$

Here, the time horizon $T$ and the utility function $\nu_T$ are chosen once and for all at time zero. It is well known (see e.g. [10, Sect. IV.3]) that the dynamic programming equation for the Merton problem within the Markovian diffusion model (2.1)–(2.3) takes the shape of the HJB equation

$$\partial_t V + \mathcal{L}_V - \frac{1}{2} \frac{\lambda \partial_x V + \rho \kappa \partial_x \nabla_y V}{\partial_{xx} V} = 0.$$  \hspace{1cm} (5.1)

In contrast to the preceding discussion, the HJB equation is here equipped with a terminal condition $V(T, \cdot, \cdot) = \nu_T(\cdot, \cdot)$ and hence posed in the backward (“right”) time direction. It turns out that under Definition 2.4, we can reduce the backward problem to a linear second-order parabolic PDE posed in the “right” time direction if the terminal utility function is of separable power form, i.e., $\nu_T(x, y) = \gamma \gamma x \gamma^{1-\gamma} h(y)^q$, and if appropriate technical assumptions hold.

Theorem 5.1 Let $\gamma \in (0, 1)$. Suppose the market model (2.1)–(2.3), the correlation matrix $\rho$ and the linear elliptic operator of the second order $\mathcal{L}$ with the coefficients

$$a(\cdot) = \kappa(\cdot) \top \kappa(\cdot), \quad b(\cdot) = \alpha(\cdot) + \Gamma \kappa(\cdot) \top \lambda(\cdot), \quad P(\cdot) = \frac{\Gamma}{2q} \lambda(\cdot) \top \lambda(\cdot)$$

satisfy Assumptions 2.2, 2.4 and 2.10, respectively, where $\Gamma = \frac{1-\gamma}{\gamma}$ and $q = \frac{1}{1+1/p}$. Suppose further that the volatility matrix $\kappa(\cdot)$ of the factor process is bounded, the weak solution $Z$ of the SDE associated with $\mathcal{L}_0 = \mathcal{L} - P(y)$ remains in $D$, and the terminal utility function is of separable power form, i.e., $\nu_T(x, y) = \gamma \gamma x \gamma^{1-\gamma} h(y)^q$, with an $h \in C^{2,\eta}(D)$ which is bounded above and below by positive constants and such that

$$(t, y) \mapsto \nabla_y \mathbb{E}[e^{\int_0^t P(Z_s) ds} h(Z_t) \bigg| Z_0 = y]$$

is bounded on $[0, T] \times D$. Then the value function for the corresponding Merton problem, $V(t, x, y) = \sup_{\pi \in A} \mathbb{E}[\nu_T(X^\pi_T, Y_T) \big| X^\pi_t = x, Y_t = y]$, can be written as

$$V(t, x, y) = \gamma \gamma x \gamma^{1-\gamma} \frac{u(t, y)^q}{1-\gamma}.$$  \hspace{1cm} (5.2)
Here, \( u \) is a classical solution of the linear PDE problem
\[
\partial_t u + Lu = 0 \quad \text{on } [0, T] \times D, \quad \text{with } u(T, \cdot) = h.
\]

Moreover, every portfolio allocation \( \pi^* \) fulfilling
\[
\sigma(Y_t)\pi_t^* = \frac{1}{\gamma} \left( \lambda(Y_t) + q\rho\kappa(Y_t) \frac{\nabla_y u(t, Y_t)}{u(t, Y_t)} \right)
\]
(5.3)
is optimal.

**Proof** By the classical verification paradigm (see e.g. [10, Chap. IV, proof of Theorem 3.1]), it is enough to show that for every portfolio allocation \( \pi \in \mathcal{A} \), the process \( V(t, X_{\pi}^t, Y_t) \), \( t \in [0, T] \), is a supermartingale and that for every solution \( \pi^* \) of (5.3), the process \( V(t, X_{\pi^*}^t, Y_t) \), \( t \in [0, T] \), is a martingale.

We follow the proof of Proposition 3.3 in the reverse direction and find that \( g(t, y) := u(t, y) q \) is a classical solution of (3.5), whereas the function \( V \) defined by (5.2) is a classical solution of the HJB equation (5.1) with \( V(T, \cdot, \cdot) = \nu_T (\cdot, \cdot) \).

For any \( \pi \in \mathcal{A} \), we may now apply Itô’s formula to \( V(t, X_{\pi}^t, Y_t) \) and replace \( \partial_t V + L_y V \) by \( \frac{1}{2} \sum_{x,y} \partial_{xx} V(t, X_{\pi}^t, Y_t) \) to see that the drift coefficient of \( (V(t, X_{\pi}^t, Y_t)) \) is the product of \( \frac{1}{2} \sum_{x,y} \partial_{xx} V(t, X_{\pi}^t, Y_t) \) with the expression in (3.3) and in particular nonpositive. Hence, the local martingale part of \( (V(t, X_{\pi}^t, Y_t)) \) is bounded below by \( -V(0, x, y) \) and consequently a supermartingale. Thus \( (V(t, X_{\pi}^t, Y_t)) \) is a supermartingale as well because it is a local supermartingale and bounded from below.

Next, we deduce from the proof of Theorem 2.17 that \( u(t, y) \) admits the stochastic representation
\[
u(t, y) = \mathbb{E} \left[ e^{\int_0^{T-t} P(Z_s) \, ds} h(Z_{T-t}) \mid Z_0 = y \right]
\]
(recall that \( Z \) remains in \( D \) by assumption). In addition, our further assumptions imply that \( \nabla_y u \) is bounded on \( [0, T] \times D \) and that \( u \) is bounded from above and below by positive constants on \( [0, T] \times D \). Together with the boundedness of the volatility matrix \( \kappa(\cdot) \) of the factor process and the Sharpe ratio \( \lambda(\cdot) \) (see Assumption 2.10 (i)), this yields the boundedness of \( (\sigma(Y_t)\pi_t^*) \) via (5.3). Finally, the drift coefficient of \( (V(t, X_{\pi^*}^t, Y_t)) \) vanishes and the quadratic variation of its local martingale part is computed to be
\[
\frac{1}{2} \sum_{x,y} \partial_{xx} V(t, X_{\pi^*}^t, Y_t) \left( \gamma^2 (X_{\pi^*}^s)^{2-2\gamma} |\sigma(Y_s)\pi_s^*|^2 \right.
\]
\[
+ \frac{\gamma^2 q^2}{(1 - \gamma)^2} (X_{\pi^*}^s)^{2-2\gamma} u(s, Y_s)^{2q-2} |\kappa(Y_s)\nabla_y u(s, Y_s)|^2
\]
\[
+ \frac{2\gamma^2 q}{1 - \gamma} (X_{\pi^*}^s)^{2-2\gamma} u(s, Y_s)^{q-1} \left( \sigma(Y_s)\pi_s^* \right) \top \rho\kappa(Y_s) \nabla_y u(s, Y_s) \left. \right) \, ds.
\]
(5.4)

The expectation of the latter integral is finite for all \( t \in [0, T] \) since \( (\sigma(Y_s)\pi_s^*) \) and \( (u(s, Y_s)^{q-1} |\kappa(Y_s)\nabla_y u(s, Y_s)|) \) are bounded, while \( \sup_{t \in [0, T]} \mathbb{E}[(X_{\pi^*}^t)^{2-2\gamma}] < \infty \).
thanks to the boundedness of \((\sigma(Y_s)\pi^*_s)\) and \((\lambda(Y_s))\) in
\[
X_t^{\pi^*} = \exp\left( \int_0^t (\sigma(Y_s)\pi^*_s)^\top \lambda(Y_s) \, ds + \int_0^t (\sigma(Y_s)\pi^*_s)^\top \, dW_s \right.
\]
\[
- \frac{1}{2} \int_0^t |\sigma(Y_s)\pi^*_s|^2 \, ds \right). 
\]
We conclude that \((V(t, X_t^{\pi^*}, Y_t))\) is a true martingale.

6 Discussion of EVE assumption

This last section is devoted to a thorough investigation of Definition 2.4 that plays a key role in the proof of Theorem 2.14. It is instructive to start with the two extreme cases corresponding to \(p = 1\) and \(p = 0\) there. Suppose first that \(A = 0\) in (2.3); in other words, the components of the Brownian motion \(B\) driving the factors are given by linear combinations of the components of the Brownian motion \(W\) driving the stock prices. We can then reparametrise the model such that \(B = W\), \(\rho = I_{dW}\) and \(\rho^\top \rho = I_{dW}\). Consequently, Definition 2.4 holds with \(p = 1\). The resulting market is complete, and we find ourselves in the framework of Nadtochiy and Tehranchi [27, Sect. 2.3]. It is therefore not surprising that the HJB equation (3.1) can be reduced to a linear PDE, even though the linearisation in Proposition 3.3 differs from the one in [27, Sect. 2.3]. On the other hand, when \(\rho = 0\) in (2.3), the Brownian motions \(B\) and \(W\) become independent, leading to an incomplete market. Nonetheless, Definition 2.4 is still satisfied with \(p = 0\). Thus the linearisation in Proposition 3.3 goes far beyond the complete market setup.

More generally, Definition 2.4 can be put to use as follows. In practice, the correlation matrix \(\rho\) can have hundreds or thousands of entries and hence might be difficult to estimate accurately in its entirety. However, one can attempt to obtain a less noisy estimate by projecting an estimate for \(\rho\) onto the submanifold of \(dW \times dB\) matrices fulfilling Definition 2.4. Restricting attention to the non-trivial case \(dW \geq dB\) (see Remark 2.5), with the exception of the zero matrix, the latter matrices can be written uniquely as \(rQ\), where \(r \in (0, 1]\) and \(Q\) has orthonormal columns, thereby forming a \((1 + dW dB - dB(dB + 1)/2)\)-dimensional submanifold of \(\mathbb{R}^{dW \times dB}\). As it turns out, the most tractable projection onto this submanifold is that with respect to the Frobenius norm (also known as the Hilbert–Schmidt norm) on \(\mathbb{R}^{dW \times dB}\).

6.1 Choice of \(r\) and \(Q\)

Let us equip the space \(\mathbb{R}^{dW \times dB}\) with the Frobenius norm
\[
|A|_F = \left( \sum_{i=1}^{dW} \sum_{j=1}^{dB} a_{ij}^2 \right)^{1/2} = \text{trace}(A^\top A)^{1/2}.
\]
For an estimate \(\hat{\rho}\) of \(\rho\), we can find a constant \(r \in [0, 1]\) and a matrix with orthonormal columns \(Q\) such that \(rQ\) minimises the distance induced by the Frobenius norm.
**Proposition 6.1** Consider the minimisation problem

\[
\min |\tilde{\rho} - rQ|_F \quad \text{subject to } r \in [0, 1] \text{ and } Q^\top Q = I_{dB}.
\]

Then \( r^* = \frac{\text{trace}(\hat{\rho}^\top \hat{\rho})^{1/2}}{dB} \) and \( Q^* = \hat{\rho}(\hat{\rho}^\top \hat{\rho})^{-1/2} \) are the minimisers.

**Proof** Equivalently, we can consider the problem

\[
\min |\tilde{\rho} - \tilde{Q}|_F^2 \quad \text{subject to } \tilde{Q}^\top \tilde{Q} = r^2 I_{dB}
\]

for fixed \( r \in [0, 1] \) and minimise over \( r \in [0, 1] \) subsequently. Applying the method of Lagrange multipliers with a \( dB \times dB \) Lagrange multiplier matrix \( \Lambda \), we get

\[
2(\tilde{Q} - \tilde{\rho}) = \tilde{Q}(\Lambda + \Lambda^\top) \iff \tilde{Q}(2I_{dB} - \Lambda - \Lambda^\top) = 2\hat{\rho}.
\]

(6.1)

Passing to the transpose on both sides of (6.1), taking the product of the resulting equation with the original equation and recalling the constraint, we see that

\[
r^2(2I_{dB} - \Lambda - \Lambda^\top)^2 = 4\hat{\rho}^\top \hat{\rho} \iff r(2I_{dB} - \Lambda - \Lambda^\top) = 2(\hat{\rho}^\top \hat{\rho})^{1/2},
\]

where \((\hat{\rho}^\top \hat{\rho})^{1/2}\) is the \( dB \times dB \) square root of the matrix \( \hat{\rho}^\top \hat{\rho} \). Together with (6.1) and the notation \((\hat{\rho}^\top \hat{\rho})^{-1/2}\) for the inverse of \((\hat{\rho}^\top \hat{\rho})^{1/2}\), this yields

\[
\tilde{Q} = r\hat{\rho}(\hat{\rho}^\top \hat{\rho})^{-1/2}.
\]

(6.2)

Plugging the formula for \( \tilde{Q} \) back into the objective function, we are left with the minimisation problem

\[
\min_{r \in [0, 1]} |\tilde{\rho} - r\hat{\rho}(\hat{\rho}^\top \hat{\rho})^{-1/2}|_F^2 \iff \min_{r \in [0, 1]} \left( \text{trace}(\hat{\rho}^\top \hat{\rho}) - 2r \text{trace}(\hat{\rho}^\top \hat{\rho})^{1/2} + r^2 dB \right).
\]

Consequently, the optimal \( r \) is \( \frac{\text{trace}(\hat{\rho}^\top \hat{\rho})^{1/2}}{dB} \), that is, the average of the singular values of \( \hat{\rho} \), whereas \( \tilde{Q} \) should be picked according to (6.2).

**\( \Box \)**

6.2 Choice of \( p \)

If one is only interested in the parameter \( p \) from Definition 2.4, then it is most natural to minimise \( |\hat{\rho}^\top \hat{\rho} - pI_{dB}| \) for a selection of a norm \( |\cdot| \) on \( \mathbb{R}^{dB \times dB} \). When \( |\cdot| \) is the operator norm (also known as the spectral radius or the Ky Fan 1-norm), the objective function becomes

\[
|\hat{\rho}^\top \hat{\rho} - pI_{dB}| = \max_{1 \leq i \leq dB} |\theta_i - p|,
\]

where \( \theta_1 \leq \theta_2 \leq \cdots \leq \theta_{dB} \) are the ordered eigenvalues of \( \hat{\rho}^\top \hat{\rho} \) (or, equivalently, the ordered squared singular values of \( \hat{\rho} \)). In this case, \( |\hat{\rho}^\top \hat{\rho} - pI_{dB}| \) is minimised by \( p = (\theta_1 + \theta_{dB})/2 \). When \( |\cdot| \) is the Frobenius norm, the objective function becomes

\[
|\hat{\rho}^\top \hat{\rho} - pI_{dB}| = \left( \sum_{i=1}^{dB} |\theta_i - p|^2 \right)^{1/2}.
\]
The minimiser for the latter is \( p = (\theta_1 + \theta_2 + \cdots + \theta_{dB})/dB \). When \( |\cdot| \) is the trace norm (also known as the nuclear norm or the Ky Fan dB-norm), the objective function becomes

\[
|\hat{\rho}^\top \hat{\rho} - p I_{dB}| = \sum_{i=1}^{dB} |\theta_i - p|,
\]

which is smallest for the median of \( \{\theta_1, \theta_2, \ldots, \theta_{dB}\} \).

### 6.3 Example: affine factor models

We conclude by illustrating the use of the EVE assumption in the framework of affine market models with nonnegative factors. In that situation, both the forward investment problem and the Merton problem can be reduced to the solution of a system of Riccati ordinary differential equations (ODEs). Consider the affine specialisation of the factor model (2.1)–(2.3) given by

\[
\frac{dS_i}{S_i} = \mu_i(Y_t) dt + \sum_{j=1}^{dW} \sigma_{ji}(Y_t) dW^j_t, \quad i = 1, 2, \ldots, n, \tag{6.3}
\]

\[
dY_t = (M^\top Y_t + w) dt + \kappa(Y_t)^\top dB_t, \tag{6.4}
\]

\[
B_t = \rho^\top W_t + A^\top W^\perp_t,
\]

where \( M \) has nonnegative off-diagonal entries, \( w \in [0, \infty)^k \), and \( \mu(\cdot), \sigma(\cdot), \kappa(\cdot), \rho \) are such that

\[
\lambda(y)^\top \lambda(y) = \mu(y)^\top \sigma(y)^{-1}(\sigma(y)^\top)^{-1}\mu(y) = \Lambda^\top y,
\]

\[
\kappa(y)^\top \kappa(y) = \text{diag}(L_1 y_1, L_2 y_2, \ldots, L_k y_k) \quad \text{with} \ L_1, L_2, \ldots, L_k \geq 0, \tag{6.5}
\]

\[
\Gamma \kappa(y)^\top \rho^\top \lambda(y) = N^\top y. \tag{6.6}
\]

**Remark 6.2** The condition (6.5) is necessary for the process \( Y \) of (6.4) to have values in \( [0, \infty)^k \) and be affine (see [9, Theorem 3.2]). Conversely, the SDE (6.4) with volatility coefficients satisfying (6.5) has a unique weak solution, which is affine and takes values in \( [0, \infty)^k \) (see [9, Theorem 8.1]).

Suppose now that the initial utility function for the forward investment problem or the terminal utility function for the Merton problem is of separable power form with \( h(y) = \exp(H^\top y + h_0) \). Under the EVE assumption, the HJB equation (3.1) arising in the two problems can be transformed into the linear second-order parabolic PDE of (3.4) (see the proof of Proposition 3.3), which in the setting of (6.3)–(6.6) amounts to

\[
\partial_t u + \frac{1}{2} \sum_{i=1}^k L_i y_i \partial_{y_i y_i} u + y^\top (M + N) \nabla_y u + w^\top \nabla_y u + \frac{\Gamma}{2q} y^\top \Lambda u = 0.
\]
Inserting the exponential-affine ansatz \( u(t, y) = \exp(\Phi^T_t y + \Theta_t) \), we obtain

\[
y^\top \dot{\Phi}_t + \dot{\Theta}_t + \frac{1}{2} \sum_{i=1}^k L_i y_i (\Phi^T_i)^2 + y^\top (M + N) \Phi_t + w^\top \Phi_t + \frac{\Gamma}{2q} y^\top \Lambda = 0.
\]

Equating the linear and the constant terms in \( y \) to 0 leads to the system of Riccati ODEs

\[
\dot{\Phi}_i^j + \frac{1}{2} L_i (\Phi^T_i)^2 + \sum_{j=1}^k (M + N)_{ij} \Phi^j_i + \frac{\Gamma}{2q} \Lambda_i = 0, \quad i = 1, 2, \ldots, k, \tag{6.7}
\]

\[
\dot{\Theta}_t + w^\top \Phi_t = 0.
\]

We note that \( \Theta \) is completely determined by the solution \( \Phi \) of the system (6.7). The latter can be solved numerically in general, and for special kinds of \( M \) and \( N \) even explicitly. For example, when \( M \) and \( N \) are diagonal, the system (6.7) splits into \( k \) one-dimensional Riccati ODEs

\[
\dot{\Phi}_i^j + \frac{1}{2} L_i (\Phi^T_i)^2 + (M_{ii} + N_{ii}) \Phi^j_i + \frac{\Gamma}{2q} \Lambda_i = 0, \quad i = 1, 2, \ldots, k. \tag{6.8}
\]

These ODEs can be solved by a separation of variables and subsequent integration. For instance, when \( \gamma \in (0, 1) \) and the discriminants \( D_i := (M_{ii} + N_{ii})^2 - L_i \frac{\Gamma}{q} \Lambda_i \) associated with the quadratic equations \( \frac{1}{2} L_i z^2 + (M_{ii} + N_{ii}) z + \frac{\Gamma}{2q} \Lambda_i = 0 \) are positive for all \( i \), we obtain the (real) roots

\[
z_{+,i} = -M_{ii} - N_{ii} + \frac{\sqrt{D_i}}{L_i}, \quad z_{-,i} = -M_{ii} - N_{ii} - \frac{\sqrt{D_i}}{L_i}.
\]

The general solution of (6.8) then becomes

\[
\Phi^j_i = \frac{z_{+,i} - \chi_i z_{-,i} e^{-\sqrt{D_i}t}}{1 - \chi_i e^{-\sqrt{D_i}t}}, \quad i = 1, 2, \ldots, k, \tag{6.9}
\]

and one can find the constants \( \chi_i \) by setting \( \Phi \) to \( H \) at the terminal time (for the Merton problem) or at time 0 (for the forward investment problem).

We conclude by discussing, in the latter setting and with \( M_{ii} + N_{ii} \geq 0 \) for all \( i \), the true FPP property of the process

\[
V(t, x, Y_t) = \gamma^\gamma x^{1-\gamma} u(t, Y_t)^q = \gamma^\gamma x^{1-\gamma} \exp(q \Phi^T_t Y_t + q \Theta_t). \tag{6.10}
\]

By arguing as in the second paragraph of the proof of Theorem 5.1, we conclude that \( (V(t, X^\pi_t, Y_t)) \) is a true supermartingale for each \( \pi \in \mathcal{A} \). It remains to see if \( (V(t, X^\pi_* t, Y_t)) \) is a true martingale for some \( \pi_* \in \mathcal{A} \) as in (3.2). To this end, we consider the expectation of the integral in (5.4). In view of the Cauchy–Schwarz inequality and Fubini’s theorem, it suffices to control the expectations of the two
summands in the first line of (5.4) uniformly over \( s \in [0, t] \). The random variables entering the two summands are in the case at hand given by

\[
X_s^{\pi_\gamma} = x \exp \left( \int_0^s \left( \frac{\Lambda^\top}{\gamma} + \frac{q \Phi_r^\top N^\top}{1 - \gamma} \right) Y_r \, dr + \int_0^s (\sigma(Y_r)\pi_r^\gamma)^\top dW_r \right) - \frac{1}{2} \int_0^s |\sigma(Y_r)\pi_r^\gamma|^2 \, dr,
\]

\[
|\sigma(Y_s)\pi_s^\gamma|^2 = \frac{1}{\gamma^2} \left( \Lambda^\top Y_s + \frac{2q}{\Gamma} \Phi_s^\top N^\top Y_s + pq^2 \sum_{i=1}^k L_i (\Phi_i^j)^2 Y_s^i \right),
\]

\[
|\kappa(Y_s)\nabla_s u(s, Y_s)|^2 = u(s, Y_s)^2 \sum_{i=1}^k L_i (\Phi_i^j)^2 Y_s^i.
\]

With \( p_1, p_2 > 1 \) satisfying \( p_1^{-1} + p_2^{-1} = 1, \tilde{\gamma} := 1 - \gamma \) and \( p_3 := 1 - 2\tilde{\gamma}p_2 < 1 \), we bound the expectation of the first summand in the first line of (5.4) using Hölder’s inequality and the supermartingale property of stochastic exponentials (see e.g. [15, Chap. 3, discussion before Proposition 5.12]) by \( \gamma^2 x^2 \tilde{\gamma} \) times

\[
\mathbb{E} \left[ |\sigma(Y_s)\pi_s^\gamma|^{2p_1} \exp \left( 2p_1 \tilde{\gamma} \int_0^s (\sigma(Y_r)\pi_r^\gamma)^\top \lambda(Y_r) \, dr \right. \right.
\]

\[
- p_1 \tilde{\gamma} p_3 \int_0^s |\sigma(Y_r)\pi_r^\gamma|^2 \, dr \right) \right]^{\frac{1}{p_1}}
\]

\[
\times \mathbb{E} \left[ \exp \left( 2p_2 \tilde{\gamma} \int_0^s (\sigma(Y_r)\pi_r^\gamma)^\top dW_r - p_2 \tilde{\gamma} (1 - p_3) \int_0^s |\sigma(Y_r)\pi_r^\gamma|^2 \, dr \right) \right]^{\frac{1}{p_2}}
\]

\[
\leq \mathbb{E} \left[ |\sigma(Y_s)\pi_s^\gamma|^{2p_1} \exp \left( 2p_1 \tilde{\gamma} \int_0^s \left( \frac{\Lambda^\top}{\gamma} + \frac{q \Phi_r^\top N^\top}{1 - \gamma} \right) Y_r \, dr \right. \right.
\]

\[
- p_1 \tilde{\gamma} p_3 \int_0^s |\sigma(Y_r)\pi_r^\gamma|^2 \, dr \right) \right]^{\frac{1}{p_1}}.
\]

For every \( i \) and \( r \), the coefficient of \( Y_s^i \) in the latter exponential admits the estimate

\[
p_1 \tilde{\gamma} \left( \frac{2\Lambda_i}{\gamma} + \frac{2q N_{ii} \Phi_i^r}{1 - \gamma} - \frac{p_3}{\gamma^2} \left( \Lambda_i + \frac{2q}{\Gamma} N_{ii} \Phi_i^r + pq^2 L_i (\Phi_i^j)^2 \right) \right)
\]

\[
\leq p_1 \tilde{\gamma} \left( \Lambda_i \frac{2\gamma - p_3}{\gamma^2} + N_{ii} c_{i,1} \Phi_i^r \frac{2q (\gamma - p_3)}{\gamma^2 (1 - \gamma)} - L_i (c_{i,2}^\Phi)^2 \frac{pq^2 p_3}{\gamma^2} \right) := \beta_i,
\]

where

\[
c_{i,1} = \begin{cases} z_{+,i} & \text{if } N_{ii} \geq 0, \\ z_{-,i} & \text{if } N_{ii} < 0, \end{cases}
\]

and

\[
c_{i,2}^\Phi = \begin{cases} z_{+,i} & \text{if } p_3 \geq 0, \\ z_{-,i} & \text{if } p_3 < 0 \end{cases}
\]

(note that (6.9) and \( M_{ii} + N_{ii} \geq 0 \) imply \( z_{-,i} \leq \Phi_i^j \leq z_{+,i} \leq 0 \)).
In view of the uniform boundedness of any given moment of $Y_s$ over $s \in [0, t]$ (see Filipović and Mayerhofer [9, Lemma A.1, Lemma 2.3 (iv) and Theorem 3.2]) and Hölder’s inequality, it suffices to control the exponential moment of $\int_0^s Y_r \, dr$ of an order slightly larger (componentwise) than $\beta := (\beta_1, \beta_2, \ldots, \beta_k)$ uniformly over $s \in [0, t]$. With the explicit solution

$$-M_{ii} + \sqrt{\Delta_i} \tan \left( \frac{\arctan \frac{M_{ii}}{\sqrt{\Delta_i}} + \frac{\sqrt{\Delta_i}}{2}}{t} \right), \quad i = 1, 2, \ldots, k,$$

to the system of Riccati ODEs in [9, Theorem 4.1 (ii), third line of display (4.5)], where $\Delta_i = 2L_{ii}\beta_i - M_{ii}^2$, we find that the exponential moment in consideration is bounded uniformly over $s \in [0, t]$ as long as

$$t < \min_{i=1,2,\ldots,k} \frac{\pi - 2 \arctan(M_{ii}/\sqrt{\Delta_i})}{\sqrt{\Delta_i}}. \quad (6.11)$$

Similarly, the expectation of the second summand in the first line of (5.4) is less than or equal to $\frac{\gamma^2 q}{(1-\gamma)^2} e^{2q\Theta_s}$ times

$$\mathbb{E} \left[ \left( \sum_{i=1}^k L_i(\Phi^2_i) Y_i^2 \right)^{p_1} \times \exp \left( 2p_1 \frac{\gamma}{1-\gamma} \int_0^s \frac{\Lambda^T}{\gamma} + \frac{q\Phi^T N^T}{1-\gamma} Y_r \, dr - p_1 \frac{\gamma}{1-\gamma} \int_0^s \sigma(Y_r) \pi^* \, dr \right) \right]^{\frac{1}{p_1}},$$

which is also bounded uniformly over $s \in [0, t]$ as long as (6.11) holds. All in all, the process in (6.10) is a true FPP at least until (but possibly not including) the time on the right-hand side of (6.11).

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**Appendix A**

**Lemma A.1** Let all the conditions of Theorem 2.14 hold and let $Y$ be a stochastic process on $D$ with dynamics as in (2.2) under a probability measure $\mathbb{P}$. Then for any $t > 0$ and any set $A \subseteq D$ of positive Lebesgue measure, we have $\mathbb{P}[Y_t \in A] > 0$.

**Proof** Let us argue by contradiction. Suppose there exist $t > 0$ and a set $A \subseteq D$ of positive Lebesgue measure such that $\mathbb{P}[Y_t \in A] = 0$. Consider the measure $\tilde{\mathbb{P}}$ defined by the Radon–Nikodým derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E} \left( \int \Gamma \rho^T \lambda(Y_s) dB_s \right)_t,$$
where $\mathcal{E}$ denotes the stochastic exponential. From Assumption 2.10, it follows that $\lambda$ is bounded; hence Novikov’s condition yields $\tilde{\mathbb{P}} \approx \mathbb{P}$. Thus $\mathbb{P}[Y_t \in A] = 0$ holds if and only if $\tilde{\mathbb{P}}[Y_t \in A] = 0$. Note that under $\tilde{\mathbb{P}}$, the process $Y$ has the dynamics

$$dY_t = (\alpha(Y_t) + \Gamma \kappa(Y_t) \top \rho \top \lambda(Y_t))dt + \kappa(Y_t) \top dB_t.$$ 

Let the set $C \subseteq \mathbb{R}^k$ be the image of the set $A \subseteq D$ under the diffeomorphism $\Xi : D \rightarrow \mathbb{R}^k$ and denote by $Z$ the image $\Xi(Y)$ of the process $Y$. Then $\mathbb{P}[Y_t \in A] = 0$ is equivalent to $\tilde{\mathbb{P}}[Z_t \in C] = 0$. Since $\Xi$ is a diffeomorphism, it follows that $C$ has positive Lebesgue measure. The process $Z$ is a diffusion on $\mathbb{R}^k$ with the generator

$$\mathcal{L}_Z = \frac{1}{2} \sum_{i,j=1}^k \tilde{a}_{ij}(z) \partial_{z_i z_j} + \sum_{i=1}^k \tilde{b}_i(z) \partial_{z_i},$$

where $\tilde{a}(\cdot)$, $\tilde{b}(\cdot)$ are as in (2.9) and $a(\cdot)$, $b(\cdot)$ are as in (2.11). Since $a(\cdot)$, $b(\cdot)$ satisfy Assumption 2.10 and $C$ has positive Lebesgue measure, it follows from [30, Theorem A] that $\tilde{\mathbb{P}}[Z_t \in C] > 0$, which is the desired contradiction. 

\[ \square \]

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