A quark coalescence model for polarized vector mesons and baryons

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A non-relativistic quark coalescence model is formulated for polarized vector mesons and baryons of spin-1/2 octet and spin-3/2 decuplet. With the spin density matrix, one can compute in a uniform way the polarizations of vector mesons and baryons from those of quarks and antiquarks with explicit momentum dependence. The results are compared to that obtained from kinetic and statistical models for hadrons.

I. INTRODUCTION

The system in non-central heavy-ion collisions at high energies have large orbital angular momenta [1–6] (see Ref. [7] for a recent review). Some of the angular momenta are transferred to the hot and dense matter and lead to polarization of quarks along the direction of the orbital angular momentum or the event plane. This type of polarization is called the global polarization. Recently the STAR collaboration has measured the global polarization of Λ and Λ hyperons in the beam energy scan program [8, 9]. At all energies below 62.4 GeV, non-vanishing values of the global polarization for Λ and Λ have been measured [8].

Strong magnetic fields are also produced in non-central heavy ion collisions [10–19]. The magnetic fields are on the average in the same direction as the orbital angular momenta. But such strong magnetic fields decay very quickly after two nuclei pass through each other at high energies. So they are expected to have effects only in the partonic phase during the early stage of the evolution [10, 20–26]. The strong magnetic fields can also polarize quarks through their magnetic moments [27, 28].

How these polarized quarks form and relate to polarized hadrons is an important question to understand the polarization phenomenon in the final state of hadrons. Most previous calculations of the global polarizations of the Λ and Λ hyperons [29–34] are based on statistical-hydro models [35–41]. In these calculations the polarizations of hadrons are assumed to arise from the thermal (Fermi-Dirac) distribution with spin degrees of freedom due to the vorticity fields which are obtained by hydrodynamic [42–44] or transport models [32, 45, 46]. In an early paper [1] by Liang and one of us, an estimate of the hyperon polarization from polarized quarks was made in a quark recombination model. The same quark recombination model was used to compute the spin density matrix elements of vector mesons [2]. But momentum dependence was not included in such a quark recombination model.

In this paper we will formulate a non-relativistic quark coalescence model with explicit momentum dependence based on the spin density matrix, with which we can compute the polarizations of vector mesons and baryons of spin-1/2 octet and spin-3/2 decuplet through coalescence of polarized quarks and anti-quarks in a systematic way. The conventional quark coalescence models [47, 48] or recombination models [49, 50] do not include the spin degrees of freedom. We focus on vector mesons and baryons of the spin-1/2 octet and spin-3/2 decuplet. The generalization to other meson and baryon multiplets are straightforward. With this coalescence model we can compute the polarizations of vector mesons and baryons as functions of their momenta from the quark polarizations obtained from other approaches such as Wigner function approach [26, 27, 51–55]. The polarizations include those induced by the vorticity and by the magnetic field.

The paper is organized as follows. In section II we introduce the density matrix which we will use to compute a particle’s polarization. In Section III we formulate a quark coalescence model for the spin density matrix of vector mesons with explicit momentum dependence. In Section IV we formulate the same model for the polarizations of baryons in the spin-1/2 octet and spin-3/2 decuplet. In Section V we show how to compute the quark polarization from Wigner functions. In Section VI we discuss in some approximations the quark and hadron polarizations from the vorticity and from the magnetic field separately. We also give some features of the spin density matrix elements ρ00 for vector mesons. In the last section we conclude by giving a summary of the main results.

II. BASICS OF SPIN DENSITY MATRIX

Since we make use of the spin density matrix throughout the paper to compute the particle polarization, in this section we give some basics about the spin density matrix.
We consider an ensemble of particles with spin quantum number \( S \). The normalized spin states are labeled by \( |\psi_i\rangle \) from which the spin density operator can be defined

\[
\rho = \sum_i P_i |\psi_i\rangle\langle \psi_i|,
\]

where \( P_i \) is the probability of the spin state \( |\psi_i\rangle \). We list here the properties of the spin density operator \( \rho \) \([56, 59]\): (a) \( \rho \) is Hermitian, \( \rho = \rho^\dagger \); (b) The trace of \( \rho \) is 1, \( \text{Tr} \rho = \sum_i |\psi_i\rangle \rho |\psi_i\rangle = \sum_i P_i = 1 \); (c) \( \rho \) is a positive-semidefinite operator, i.e., for any state \( |\phi\rangle \), we have \( \langle \phi | \rho | \phi \rangle = \sum_i P_i |\phi_i\rangle \rho |\phi_i\rangle \geq 0 \).

The dimension of the spin space for a spin-\( S \) particle is \( 2S + 1 \), the spin density operator is a \( (2S + 1) \times (2S + 1) \) matrix with \( 2(2S + 1)^2 \) real parameters. The conditions that \( \rho \) is Hermitian and has unit trace reduce the number of real parameters to \( 4S(S+1) \). The positive-semidefinite condition can also impose restrictions on these real parameters.

If the particle is unpolarized, the spin density matrix \( \rho \) has a simple form

\[
\rho = \frac{1}{2S + 1} \text{diag}(1, 1, \cdots, 1).
\]

Any deviation from (2) indicates some degree of spin alignment or polarization.

Let us look at a few examples. The first example is the spin-1/2 particle. The spin density matrix has three real parameters and can be written as

\[
\rho = \frac{1}{2}(1 + \vec{\mathcal{P}} \cdot \sigma),
\]

where \( \vec{\mathcal{P}} \) is a unit three-vector serving as the polarization vector and \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) are Pauli matrices. The polarization can be read out by

\[
\vec{\mathcal{P}} = \frac{\text{Tr}(\rho \sigma)}{\text{Tr}(\rho)}.
\]

The octet hyperon \( \Lambda^0 \) is a spin-1/2 particle whose polarization can be measured by its weak decay, \( \Lambda^0 \to p + \pi^- \) \([\,][\,]\). In Section [IV] we will use the quark coalescence model to compute the polarization of \( \Lambda^0 \) through its spin density matrix.

The second example is the spin-1 particle whose spin density matrix has eight real parameters. We look at the vector meson \( K^{*0} \) and \( \phi \) for illustration with following strong decay modes

\[
\begin{align*}
K^{*0} & \to K^+ + \pi^-, \quad (\sim 100\%), \\
\phi & \to K^+ + K^-, \quad (\sim 49%).
\end{align*}
\]

where the fractions are the branching ratios. The polarization of \( K^{*0} \) and \( \phi \) can be measured through these decays \([58, 59]\). We note that parity is conserved in strong decays so the final states must be in the p-wave with \( L = 1 \). We take \( K^{*0} \) for an example, all discussions apply to \( \phi \) equally. The decay amplitude can be described by the \( \mathcal{S} \) matrix

\[
\langle K^+, \pi^- | \mathcal{S} | K^{*0}, S_z \rangle = Y_{1, S_z}(\theta, \phi)
\]

where \( S_z = -1, 0, 1, (\theta, \phi) \) denote the polar and azimuthal angle of one decay daughter, and \( Y_{1, S_z}(\theta, \phi) \) denote the spherical harmonic functions of \( L = 1 \). The angular distribution of \( K^+ \) or \( \pi^- \) for a specific initial spin state \( |1, S_z\rangle \) of \( K^{*0} \) is

\[
\frac{dN}{d\Omega} = |\langle K^+, \pi^- | \mathcal{S} | K^{*0}, S_z \rangle|^2 = |Y_{1, S_z}(\theta, \phi)|^2.
\]

We now consider an ensemble of \( K^{*0} \) at rest having the probability \( P_i \) in the spin state \( |\psi_i\rangle \). Then the spin density operator \( \rho \) is given by Eq. (1). The angular distribution for the decay daughter can be written as

\[
\frac{dN}{d\Omega} = \sum_i P_i |\langle K^+, \pi^- | \mathcal{S} | \psi_i \rangle|^2
\]

\[
= \sum_i P_i \langle K^+, \pi^- | \mathcal{S} | \psi_i \rangle \langle \psi_i | \mathcal{S}^\dagger | K^+, \pi^- \rangle
\]

\[
= \langle K^+, \pi^- | \mathcal{S} \rho \mathcal{S}^\dagger | K^+, \pi^- \rangle.
\]


Inserting a completeness relation \( \sum_{S_z} |K^*; S_z\rangle \langle K^*; S_z| = 1 \) into Eq. (3), we obtain Eq. (10) of Ref. [60],

\[
\frac{dN}{d\Omega} = \frac{1}{8\pi} \left[ \frac{3}{2} \left( (1 - \rho_{00}) + (3\rho_{00} - 1) \cos^2 \theta \right. \right. \\
- 2 \text{Re} \rho_{-1,1} \sin^2 \theta \cos(2\phi) - 2 \text{Im} \rho_{-1,1} \sin^2 \theta \sin(2\phi) \\
+ \sqrt{2} \text{Re} (\rho_{-1,0} - \rho_{01}) \sin(2\theta) \cos \phi \\
+ \sqrt{2} \text{Im} (\rho_{-1,0} - \rho_{01}) \sin(2\theta) \sin \phi \left. \right] ,
\]

where \( \rho_{S_{1z},S_{2z}} \equiv \langle K^*; S_{2z}|\rho|K^*; S_{2z}\rangle \) denote the spin density matrix elements for \( K^* \). Note that \( dN/d\Omega \) is automatically normalized to 1, \( \int d\Omega(dN/d\Omega) = 1 \). If the parent particle is unpolarized, \( \rho_{S_{1z},S_{2z}} = \delta_{S_{1z},S_{2z}}/3 \), then we have \( dN/d\Omega = 1/(4\pi) \).

In principle, if one measures \( dN/d\Omega \) in experiments, one can determine five real parameters out of eight ones in the spin density matrix for vector mesons according to Eq. (10).

\[
\rho_{00}, \text{ Re}\rho_{-1,1}, \text{ Im}\rho_{-1,1}, \text{ Re}(\rho_{-1,0} - \rho_{01}), \text{ Im}(\rho_{-1,0} - \rho_{01}).
\]

The polarization vector \( \vec{\mathcal{P}} \) can be determined from the spin density matrix [30]

\[
\vec{\mathcal{P}} = \frac{1}{S} \frac{\text{Tr}(\rho \hat{S})}{\text{Tr}(\rho)},
\]

where \( S = 1 \) and the spin operators for vector mesons, \( \hat{S} = (\hat{S}_1, \hat{S}_2, \hat{S}_3) \), are defined by

\[
\hat{S}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{S}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{S}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

Then we obtain the polarization \( \vec{\mathcal{P}} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3) \),

\[
\mathcal{P}_1 = \sqrt{2} \frac{1}{\text{Tr}(\rho)} \text{Re}(\rho_{-1,0} + \rho_{01}), \\
\mathcal{P}_2 = \sqrt{2} \frac{1}{\text{Tr}(\rho)} \text{Im}(\rho_{-1,0} + \rho_{01}), \\
\mathcal{P}_3 = \frac{1}{\text{Tr}(\rho)} (\rho_{11} - \rho_{-1,-1}).
\]

Note that the above do not match the five quantities in [10] that can be measured from \( dN/d\Omega \). Therefore the polarization of vector mesons cannot be measured from \( dN/d\Omega \) in strong decays in which the parity is conserved. This is very different from \( \Lambda^0 \) in the weak decay where the broken parity can be used to determine the polarization.

When the azimuthal angle \( \phi \) in Eq. (9) is integrated out, we obtain the polar angle distribution

\[
\frac{dN}{d\cos \theta} = \int_0^{2\pi} d\phi \frac{dN}{d\Omega} = \frac{3}{4} \left[ (1 - \rho_{00}) + (3\rho_{00} - 1) \cos^2 \theta \right].
\]

So one can measure \( \rho_{00} \) from the polar angle distribution of the daughter particle. Any deviation from 1/3 for \( \rho_{00} \) may indicate some degree of polarization of vector mesons. This type of measurements have been performed in the STAR experiment [58].

The third example is the spin-3/2 decuplet baryon \( \Delta^{++} \) which mostly decays into a proton and a \( \pi^+ \) by strong interaction. The spin-parity of \( \Delta^{++} \) is \((3/2)^+\), while those of the proton and pion are \((1/2)^+\) and \(0^-\) respectively. By the parity conservation in the strong decay the angular distribution of the proton must be in the P-wave state with
The spin density matrix of the spin-3/2 particle like $\Delta^{++}$ is a $4 \times 4$ complex matrix with fifteen independent parameters. The angular distribution for the decay daughter can be written as

$$\frac{dN}{d\Omega} = \sum_{S} \langle \hat{p}, S^p, \pi^+ | \hat{\mathcal{G}} \rho \mathcal{G} \hat{p}, S^p, \pi^+ \rangle$$

$$= \sum_{S,z_1,z_2} \rho_{S,z_1} \rho_{S,z_2} \sum_{S^p} f(S^p, S_z) f^*(S^p, S_{z^2}),$$

where the final state of the proton and pion is denoted as $|\hat{p}, S^p, \pi^+ \rangle$ with $\hat{p}$ being the proton’s momentum direction and $S^p$ denoting the proton’s spin state along the z-direction. In deriving the second line of Eq. (15) we have inserted a complete set of spin states of $\Delta^+$ into the first line of Eq. (15), $\sum_{S} |\Delta^+, S_z \rangle$. The spin density matrix elements for all spin states of $\Delta^+$ cannot be determined by the daughter particle’s momentum distribution alone.

We see that the angular distribution involves only five parameters out of fifteen independent ones, the rest for all spin states of $\Delta^+$. This is the feature of the parity conserved decay of $\Delta^{++}$, same as in the strong decay of $K^{*0}$ in (4).

III. SPIN DENSITY MATRIX FOR VECTOR MESONS

We consider an ensemble of free quarks and antiquarks in a deconfined system. These quarks and antiquarks will combine to form hadrons in the hadronization process. For mesons we define a quark-antiquark state with momenta and spins along a fixed direction (the z-direction)

$$|q_1, \bar{q}_2; s_1, s_2; \mathbf{p}_1, \mathbf{p}_2 \rangle \equiv |q_1, \bar{q}_2; s_1, s_2 \rangle \langle \mathbf{p}_1, \mathbf{p}_2 \rangle,$$

where $q_1 = u, d, s$ and $\bar{q}_2 = \bar{u}, \bar{d}, \bar{s}$ denote the flavors of the quark and anti-quark respectively, $s_1, s_2 = \pm 1/2$ denote the spin of the quark and antiquark in the z-direction respectively, and ‘$q$’ labels the quark-antiquark momentum state. The momentum states of the quark and antiquark in coordinate representation are plane waves,

$$\langle x_1, x_2 | q_1, \mathbf{p}_1, \mathbf{p}_2 \rangle = \frac{1}{V} \exp \left( i \mathbf{p}_1 \cdot x_1 + i \mathbf{p}_2 \cdot x_2 \right),$$

where $V$ is the volume. Now we can write down the density operator for quarks and antiquarks,

$$\rho = V^2 \sum_{s_1, s_2} \sum_{q_1, \bar{q}_2} \int \frac{d^4 p_1}{(2\pi)^3} \frac{d^4 p_2}{(2\pi)^3} w_{q_1, s_1}(p_1) w_{\bar{q}_2, s_2}(p_2)$$

$$\times |q_1, \bar{q}_2; s_1, s_2; \mathbf{p}_1, \mathbf{p}_2 \rangle \langle q_1, \bar{q}_2; s_1, s_2; \mathbf{p}_1, \mathbf{p}_2 |,$$

where $w_{q_1, s_1}(p_1)/w_{\bar{q}_2, s_2}(p_2)$ are the weight functions for the quark/antiquark which satisfy the normalization conditions

$$\sum_s w_{q,s}(p) = 1, \quad \sum_s w_{\bar{q},s}(p) = 1.$$

The weight functions are related to the polarization of the quark and antiquark by

$$w_{q, \pm 1/2}(p) = \frac{1}{2} \left[ 1 \pm \mathcal{P}_q(p) \right],$$

$$w_{\bar{q}, \pm 1/2}(p) = \frac{1}{2} \left[ 1 \pm \mathcal{P}_{\bar{q}}(p) \right],$$

where $\mathcal{P}_q(p)$ and $\mathcal{P}_{\bar{q}}(p)$ are the polarization of the quark and antiquark in the z-direction respectively. We will use shorthand notations for the weight functions, $w_{q(\bar{q}), \pm} \equiv w_{q(\bar{q}), \pm 1/2}$, in the rest of the paper.

Now we look at the meson state defined by

$$|M; S, S_z, \mathbf{p} \rangle \equiv |M; S, S_z \rangle |M; \mathbf{p} \rangle,$$
where $|M; S, S_z⟩$ denotes the meson’s spin-flavor wave function and ’M’ labels the meson state. For a vector meson we have $S = 1$ and $S_z = -1, 0, 1$ for the spin state in the z-direction. In the quark model, the momentum state of the meson in coordinate representation is given by

$$\langle x_1, x_2 | M; p \rangle = \frac{1}{\sqrt{V/2}} \exp(i p \cdot x) \varphi_M(y).$$  \hfill (22)

Here $x = (x_1 + x_2)/2$ is the center position of the quark and antiquark, $y = x_1 - x_2$ is their distance, and $\varphi_M(y)$ is the meson wave function satisfying the normalization condition $\int d^2y |\varphi_M(y)|^2 = 1$. We choose the Gaussian form for $\varphi_M(y)$ whose Fourier transform is given by $\int_0^\infty d^3q e^{-i q \cdot y} \varphi_M(y) = \left(\frac{2\sqrt{\pi}}{a_M}\right)^{3/2} \exp\left(-\frac{q^2}{2a^2_M}\right)$.

We can compute the spin density matrix element $\rho_{00}^{S=1}$ on the vector meson state with, for example, the quark flavor $q\bar{q}$, where $S_{z1}, S_{z2} = -1, 0, 1$. The element $\rho_{00}$ for $S_{z1} = S_{z2} = 0$ is of special interest since it can be measured in experiments (as discussed in Section II).

\begin{align*}
\rho_{00}^{S=1}(p) &= \langle M; 1, 0, p | \rho | M; 1, 0, p \rangle \\
&= \frac{1}{2} \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} | w_{q^+} (p_1) w_{\bar{q}^-} (p_2) + w_{q^-} (p_1) w_{\bar{q}^+} (p_2) | \langle q; p_1, p_2 | M; p \rangle^2 \\
&= \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \left[ w_{q^+} (p_2 + q) w_{\bar{q}^-} (p_2 - q) + w_{q^-} (p_2 + q) w_{\bar{q}^+} (p_2 - q) \right] |\varphi_M(q)|^2, \hfill (24)
\end{align*}

where we have used the Clebsch-Gordan coefficients in the quark spin wave function of the vector meson state $|S = 1, S_z = 0⟩$ and the amplitude of a vector meson state projected onto the quark-antiquark state

\begin{align*}
\langle q; p_1, p_2 | M; p \rangle &= \int d^3x_1 d^3x_2 \langle q; p_1, p_2 | x_1, x_2 \rangle \langle x_1, x_2 | M; p \rangle \\
&= \frac{(2\pi)^3}{\sqrt{V/2}} \delta^{(3)}(p - p_1 - p_2) \varphi_M(q), \hfill (25)
\end{align*}

where we have used Eqs. \ref{eq:17} with $q = (p_1 - p_2)/2$. From Eq. \ref{eq:20} we obtain

\begin{align*}
|\langle q; p_1, p_2 | M; p \rangle|^2 &= \frac{(2\pi)^3}{\sqrt{V/2}} \delta^{(3)}(p - p_1 - p_2) |\varphi_M(q)|^2, \hfill (26)
\end{align*}

where we have used $\delta^{(3)}(0) = V/(2\pi)^3$. Eq. \ref{eq:26} has been used in Eq. \ref{eq:24}.

Similarly we can obtain other elements of the spin density matrix for vector mesons as well as for scalar (spin-0) mesons. We list all diagonal elements as follows,

\begin{align*}
\rho_{00}^{S=0}(p) &= \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \left[ w_{q^+} (p_2 + q) w_{\bar{q}^-} (p_2 - q) + w_{q^-} (p_2 + q) w_{\bar{q}^+} (p_2 - q) \right] |\varphi_M(q)|^2, \\
\rho_{00}^{S=1}(p) &= \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \left[ w_{q^+} (p_2 + q) w_{\bar{q}^-} (p_2 - q) + w_{q^-} (p_2 + q) w_{\bar{q}^+} (p_2 - q) \right] |\varphi_M(q)|^2, \\
\rho_{11}^{S=1}(p) &= \int \frac{d^3q}{(2\pi)^3} \left[ w_{q^+} (p_2 + q) w_{\bar{q}^-} (p_2 - q) + w_{q^-} (p_2 + q) w_{\bar{q}^+} (p_2 - q) \right] |\varphi_M(q)|^2, \\
\rho_{-1,-1}^{S=1}(p) &= \int \frac{d^3q}{(2\pi)^3} \left[ w_{q^+} (p_2 + q) w_{\bar{q}^-} (p_2 - q) + w_{q^-} (p_2 + q) w_{\bar{q}^+} (p_2 - q) \right] |\varphi_M(q)|^2. \hfill (27)
\end{align*}
The first line implies that the spin and flavor part of the three-quark state is independent of its momentum part. The determinate $P$ determine $P$, see Eq. (14). However, by measuring $\rho_{b}$ baryon in the deconfined quark system, we see that if the quark and antiquark are not polarized ($P$, Eq. (27), we have from Eq. (27),

\[ \langle \bar{s} s | P_0 P_0 | s \bar{s} \rangle = \frac{1}{3} - \frac{4}{9} \int \frac{d^3q}{(2\pi)^3} \mathcal{P}_q \left( \frac{P}{2} + q \right) \mathcal{P}_q \left( \frac{P}{2} - q \right) |\varphi_M(q)|^2. \]

If we only consider vector mesons, we can obtain the normalized spin density matrix element $\tilde{\rho}_{00}^{S=1}(p)$ (for the vector meson which is related to the quark and antiquark polarization functions. The normalized spin density matrix element $\tilde{\rho}_{00}^{S=1}(p)$ is given by

\[ \tilde{\rho}_{00}^{S=1}(p) = \frac{\int d^3q [1 - \mathcal{P}_q(p/2 + q) \mathcal{P}_q(p/2 - q)] |\varphi_M(q)|^2}{\int d^3q [1 + \mathcal{P}_q(p/2 + q) \mathcal{P}_q(p/2 - q)] |\varphi_M(q)|^2}. \]

We see that if the quark and antiquark are not polarized ($P_q = \mathcal{P}_q = 0$), we have $\tilde{\rho}_{00}^{S=1} = 1/3$. If the quark polarization is small, $\mathcal{P}_q \sim \mathcal{P}_q \ll 1$, Eq. (29) can be approximated as

\[ \tilde{\rho}_{00}^{S=1}(p) \approx \frac{1}{3} - \frac{4}{9} \int \frac{d^3q}{(2\pi)^3} \mathcal{P}_q \left( \frac{P}{2} + q \right) \mathcal{P}_q \left( \frac{P}{2} - q \right) |\varphi_M(q)|^2. \]

If the quark polarization is independent of momentum, Eq. (30) becomes

\[ \tilde{\rho}_{00}^{S=1} \approx \frac{1}{3} - \frac{4}{9} \mathcal{P}_q \mathcal{P}_q. \]

From Eqs. (30,31) we see that the deviation from non-polarized case $\tilde{\rho}_{00}^{S=1} = 1/3$ is of second order in $\mathcal{P}_q$ or $\mathcal{P}_q$. For $\phi(1020)$ with the quark content $s\bar{s}$, we obtain its polarization along z-direction from Eq. (13),

\[ \mathcal{P}_\phi = \rho_{11}^{\phi} - \rho_{-1,-1}^{\phi}, \]

where we have from Eq. (27),

\[ \rho_{11}^{\phi} - \rho_{-1,-1}^{\phi} = \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} |\varphi_M(q)|^2 \left[ \mathcal{P}_q \left( \frac{P}{2} + q \right) + \mathcal{P}_q \left( \frac{P}{2} - q \right) \right], \]

\[ \rho_{11}^{\phi} + \rho_{00}^{\phi} + \rho_{-1,-1}^{\phi} = \frac{3}{4} + \frac{1}{4} \int \frac{d^3q}{(2\pi)^3} |\varphi_M(q)|^2 \left( \mathcal{P}_q \left( \frac{P}{2} + q \right) + \mathcal{P}_q \left( \frac{P}{2} - q \right) \right). \]

For the $\phi(1020)$ meson, $\rho_{00}^{\phi}$ is given by Eq. (30) or Eq. (31) with replacement $\mathcal{P}_q \rightarrow \mathcal{P}_s$ and $\mathcal{P}_q \rightarrow \mathcal{P}_s$. If the quark polarization is independent of momentum, Eq. (32) becomes

\[ \mathcal{P}_\phi = \frac{2(\mathcal{P}_s + \mathcal{P}_s)}{3 + \mathcal{P}_s \mathcal{P}_s}. \]

From Section IV, we know that one cannot measure the polarization $\mathcal{P}_\phi$ in the strong decay [5] except the element $\rho_{00}$, see Eq. (17). However, by measuring $\rho_{00}^{\phi}$ one can determine the values of $\mathcal{P}_s$ and $\mathcal{P}_s$ from which one can indirectly determine $\mathcal{P}_\phi$.

\section{IV. Spin Density Matrix for Octet and Decuplet Baryons}

Similar to the quark-antiquark state defined in Eqs. (16,17) for a meson, we can define a three-quark state for a baryon in the deconfined quark system,

\[ |q_1, q_2, q_3; s_1, s_2, s_3; p_1, p_2, p_3 \rangle = |q_1, s_1, p_1 \rangle |q_2, s_2, p_2 \rangle |q_3, s_3, p_3 \rangle = |q_1, q_2, q_3; s_1, s_2, s_3 \rangle |q_1, p_1, p_2, p_3 \rangle, \]

\[ |x_1, x_2, x_3; p_1, p_2, p_3 \rangle = \frac{1}{\sqrt{3!}} \exp \left( ip_1 \cdot x_1 + ip_2 \cdot x_2 + ip_3 \cdot x_3 \right), \]

where $s_{1,2,3} = \pm 1/2$ and $q_{1,2,3} = u, d, s$ denote the spin states in the z-direction and flavors of three quarks respectively. The first line implies that the spin and flavor part of the three-quark state is independent of its momentum part. The
angular momentum is respect to any interchange of two quark labels since their color wave function is anti-symmetric and their orbital belong to the 56-plet of SU(6) \cite{61}. The spin-flavor quark wave functions of these baryons must be symmetric with form in coordinate representation

$$S$$

For a spin-1/2 baryon, we have amplitudes of the spin-flavor parts and those of the momentum parts separately. Due to the factorization forms of the quark and baryon wave function in Eqs. (34,36), we can calculate the overlapping

$$|\langle q_1, q_2, q_3; s_1, s_2, s_3; p_1, p_2, p_3 | B; S_2, S_1, p \rangle|^2$$

where $$w_{q,s}$$ are the weight functions for quarks related to the quark polarization $$\mathcal{P}_q$$ by Eq. (20).

Now let us look at a baryon state defined by

$$|B; S, S_z, p\rangle = |B; S, S_z\rangle |B; p\rangle,$$

where $$|B; S, S_z\rangle$$ is the spin-flavor wave function of the baryon and $$|B; p\rangle$$ is the baryon wave function in momentum space. For a spin-1/2 baryon, we have $$S = 1/2$$ and $$S_z = \pm 1/2$$. The momentum state of the baryon has the following form in coordinate representation

$$|B; S, S_z, p\rangle = \frac{1}{V^{1/2}} \exp(i p \cdot x) \varphi_B(x, y, z),$$

where $$x = (x_1 + x_2 + x_3)/3, y = (x_1 + x_2)/2 - x_3, and z = x_1 - x_2$$ whose conjugate momenta are $$p = p_1 + p_2 + p_3, r = (p_1 + p_2 - 2p_3)/3, and q = (p_1 - p_2)/2$$, respectively. Note that the Jacobian between $$(x_1, x_2, x_3)$$ and $$(x, y, z)$$ is 1, so is the Jacobian between $$(p_1, p_2, p_3)$$ and $$(p, r, q)$$. The baryon wave function $$\varphi_B(y, z)$$ is normalized as

$$\int d^3y d^3z |\varphi_B(y, z)|^2 = 1,$$

where $$\chi$$ and $$\rho$$ are the SU(6) quark wave functions and the SU(6) quark wave functions are in the form \cite{47,48}.

The spin density matrix elements for the baryon are given by projecting the baryon states on the spin density operator,

$$\rho_{s_1, s_2} (p) = \langle B; S, S_z, p | \rho | B; S, S_z, p \rangle = V^3 \sum_{s_1, s_2, s_3} \sum_{q_1, q_2, q_3} d^3p_1 d^3p_2 d^3p_3 \langle q_1, q_2, q_3; s_1, s_2, s_3; p_1, p_2, p_3 | B; S, S_z, p \rangle \times \langle B; S, S_z, p | B; S, S_z, p \rangle,$$

where $$\rho$$ is the SU(6) or spin-flavor quark wave function and $$\chi$$ denotes the spin wave function with specific symmetry for interchange of two quark labels, the superscript 8 and 10 mean octet and decuplet respectively, and the subscript S, MS and MA mean symmetric, mixed symmetric and mixed anti-symmetric respectively.

Due to the factorization forms of the quark and baryon wave function in Eqs. (34,36), we can calculate the overlapping amplitudes of the spin-flavor parts and those of the momentum parts separately.

We now look at the spin-flavor quark wave functions of spin-1/2 octet baryons and spin-3/2 decuplet baryons which belong to the 56-plet of SU(6) \cite{61}. The spin-flavor quark wave functions of these baryons must be symmetric with respect to any interchange of two quark labels since their color wave function is anti-symmetric and their orbital angular momentum is $$L = 0$$. Therefore the SU(6) or spin-flavor quark wave functions are in the form \cite{61}.

$$\begin{align*}
|B; S = \frac{1}{2}, S_z \rangle & = \frac{1}{\sqrt{2}} (F_{MS} \chi_{MS} + F_{MA} \chi_{MA}) \\
|B; S = \frac{3}{2}, S_z \rangle & = F_S \chi_S
\end{align*}$$

where $$F$$ denotes the flavor wave function and $$\chi$$ denotes the spin wave function with specific symmetry for interchange of two quark labels, the superscript 8 and 10 mean octet and decuplet respectively, and the subscript S, MS and MA mean symmetric, mixed symmetric and mixed anti-symmetric respectively.

Then we compute the spin density matrix elements for baryons in Eq. (39). We take the $$\Lambda$$ hyperon ($$B = \Lambda$$) as an example, whose flavor-spin parts of the spin density matrix elements can be evaluated by the SU(6) quark wave
function of $\Lambda$. The projection amplitude of a baryon momentum state on the three-quark one is given by

$$
\langle q; p_1, p_2, p_3| B; p \rangle = \int d^3x_1 d^3x_2 d^3x_3 \langle q; p_1, p_2, p_3|x_1, x_2, x_3 \rangle \langle x_1, x_2, x_3| B; p \rangle
$$

$$
= \frac{1}{\sqrt{2}} (2\pi)^3 \delta^{(3)}(p - p_1 - p_2 - p_3) \varphi_B(r, q),
$$

where the modulus square is

$$
| \langle q; p_1, p_2, p_3| B; p \rangle |^2 = \frac{(2\pi)^3}{\sqrt{3}} \delta^{(3)}(p - p_1 - p_2 - p_3) |\varphi_B(r, q)|^2.
$$

Then we obtain the diagonal element $\rho_{++}^\Lambda (\equiv \rho_{\frac{3}{2}+}^\Lambda )$ in Eq. (39),

$$
\rho_{++}^\Lambda (p) = \left\langle \Lambda; \frac{1}{2} - \frac{1}{2}, \frac{1}{2} \right| \rho \left| \Lambda; \frac{1}{2} - \frac{1}{2}, \frac{1}{2} \right\rangle
$$

$$
= \frac{1}{24} \int \frac{d^3r}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} |\varphi_1 (r, q)|^2
$$

$$
\times \{ w_+ (p_1) [2 - \mathcal{P}_u (p_2) \mathcal{P}_d (p_3) - \mathcal{P}_u (p_3) \mathcal{P}_d (p_2)] 
$$

$$
+ w_+ (p_2) [2 - \mathcal{P}_u (p_3) \mathcal{P}_d (p_1) - \mathcal{P}_u (p_1) \mathcal{P}_d (p_3)] 
$$

$$
+ w_+ (p_3) [2 - \mathcal{P}_u (p_1) \mathcal{P}_d (p_2) - \mathcal{P}_u (p_2) \mathcal{P}_d (p_1)] \},
$$

where we have applied $p_1 = p/3 + r/2 + q$, $p_2 = p/3 + r/2 - q$ and $p_3 = p/3 - r$. Note that the subscript 's, +' of $w_{s,+}$ means the s quark with spin up (please do not confuse s with the quark spin). Similarly we obtain another diagonal element $\rho_{--}^\Lambda (\equiv \rho_{\frac{3}{2}-}^\Lambda )$

$$
\rho_{--}^\Lambda (p) = \left\langle \Lambda; \frac{1}{2} - \frac{1}{2}, \frac{1}{2} \right| \rho \left| \Lambda; \frac{1}{2} - \frac{1}{2}, \frac{1}{2} \right\rangle
$$

$$
= \frac{1}{24} \int \frac{d^3r}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} |\varphi_1 (r, q)|^2
$$

$$
\times \{ w_- (p_1) [2 - \mathcal{P}_u (p_2) \mathcal{P}_d (p_3) - \mathcal{P}_u (p_3) \mathcal{P}_d (p_2)] 
$$

$$
+ w_- (p_2) [2 - \mathcal{P}_u (p_3) \mathcal{P}_d (p_1) - \mathcal{P}_u (p_1) \mathcal{P}_d (p_3)] 
$$

$$
+ w_- (p_3) [2 - \mathcal{P}_u (p_1) \mathcal{P}_d (p_2) - \mathcal{P}_u (p_2) \mathcal{P}_d (p_1)] \},
$$

All off-diagonal elements are vanishing, i.e. $\rho_{+-} = \rho_{-+} = 0$

The polarization formula for spin-1/2 baryons can be expressed as

$$
\mathcal{P}_{B,1/2}(p) = \frac{\rho_{++}^B (p) - \rho_{--}^B (p)}{\rho_{++}^B (p) + \rho_{--}^B (p)},
$$

where the superscript 'B' denotes a type of spin-1/2 baryons. For the spin-3/2 baryons, the polarization is given by Eq. (A3), namely,

$$
\mathcal{P}_{B,3/2}(p) = \frac{1}{2} \left[ \rho_{\frac{3}{2}+}^B (p) - \rho_{\frac{3}{2}-}^B (p) + \rho_{\frac{3}{2}+}^B (p) - \rho_{\frac{3}{2}-}^B (p) \right] + \rho_{\frac{3}{2}+}^B (p) + \rho_{\frac{3}{2}-}^B (p) + \rho_{\frac{3}{2}+}^B (p) + \rho_{\frac{3}{2}-}^B (p),
$$

where the superscript 'B' denotes a type of spin-3/2 baryons.

For the $\Lambda$ hyperon we have

$$
\rho_{++}^\Lambda (p) + \rho_{--}^\Lambda (p) = \frac{1}{24} \int \frac{d^3r}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} |\varphi_{\Lambda} (r, q)|^2
$$

$$
\times \{ 6 - \mathcal{P}_u (p_2) \mathcal{P}_d (p_3) - \mathcal{P}_u (p_3) \mathcal{P}_d (p_2) 
$$

$$
- \mathcal{P}_u (p_3) \mathcal{P}_d (p_1) - \mathcal{P}_u (p_1) \mathcal{P}_d (p_3) 
$$

$$
- \mathcal{P}_u (p_1) \mathcal{P}_d (p_2) - \mathcal{P}_u (p_2) \mathcal{P}_d (p_1) \},
$$

$$
\rho_{++}^\Lambda (p) - \rho_{--}^\Lambda (p) = \frac{1}{24} \int \frac{d^3r}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} |\varphi_{\Lambda} (r, q)|^2
$$

$$
\times \{ \mathcal{P}_u (p_1) [2 - \mathcal{P}_u (p_2) \mathcal{P}_d (p_3) - \mathcal{P}_u (p_3) \mathcal{P}_d (p_2)] 
$$

$$
+ \mathcal{P}_u (p_2) [2 - \mathcal{P}_u (p_3) \mathcal{P}_d (p_1) - \mathcal{P}_u (p_1) \mathcal{P}_d (p_3)] 
$$

$$
+ \mathcal{P}_u (p_3) [2 - \mathcal{P}_u (p_1) \mathcal{P}_d (p_2) - \mathcal{P}_u (p_2) \mathcal{P}_d (p_1)] \},
$$

(47)
Now we look at the limit that the quark polarization is much smaller than 1. In this case we can neglect the quadratic terms in the quark polarization, from Eq. (45) the polarization of $\Lambda$ can be approximated as

$$\mathcal{P}_\Lambda(x,p) \approx \frac{1}{3} \int \frac{d^3r}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} |\varphi_\Lambda(r,q)|^2 \left[ \mathcal{P}_u(p_1) + \mathcal{P}_d(p_2) + \mathcal{P}_s(p_3) \right],$$

(48)

which shows that the polarization of $\Lambda$ is mainly determined by that of the s-quark. For antihyperons, we can make the replacement $\Lambda \rightarrow \bar{\Lambda}$ and $u,d,s \rightarrow \bar{u},\bar{d},\bar{s}$ in Eqs. (46-48).

Similarly with the SU(6) quark wave functions of the octet baryons $\Sigma^0$ and $\Xi^-$ and the decuplet baryons $\Delta^{++}$ and $\Omega$, we can obtain their polarizations from Eq. (45) and (46) respectively.

If the quark polarization is independent of momentum, we obtain

$$\mathcal{P}_u = \mathcal{P}_d = \mathcal{P}_s,$$

$$\mathcal{P}_{\Sigma^0} = \mathcal{P}_{\Xi^-} = \mathcal{P}_{\Omega} = \mathcal{P}_{\Delta^{++}} = \mathcal{P}_{\Lambda} = \mathcal{P}_s,$$

(49)

If we further have $\mathcal{P}_u = \mathcal{P}_d = \mathcal{P}_s$, the above polarization results are consistent to Ref. [1].

V. FERMION POLARIZATIONS FROM WIGNER FUNCTIONS

The covariant Wigner function method [62, 63] for spin-1/2 fermions is a useful tool to study the chiral magnetic effect, the chiral vortical effect and other related effects [26, 27, 51, 53–55]. The Wigner function is equivalent to the spin tensor component of a fermionic system.

The Wigner function for spin-1/2 fermions in an external electromagnetic field is given by [62–65]

$$W_{\alpha\beta}(x,p) = \frac{1}{(2\pi)^3} \int d^3y e^{-i p \cdot y} \left\langle \tilde{\psi}_\beta \left( x + \frac{y}{2} \right) U(A;x + \frac{1}{2} y, x - \frac{1}{2} y) \psi_\alpha \left( x - \frac{y}{2} \right) \right\rangle,$$

(50)

where $\psi$ denotes the fermionic field and $\alpha, \beta$ the spinor indices, $U(A;x_2, x_1) = \exp \left[ i Q \int_{x_1}^{x_2} dx^\mu A_\mu(x) \right]$ is the gauge link which renders the Wigner function gauge invariant with $A_\mu$ being the gauge potential of the electromagnetic field, and $\left\langle \hat{O} \right\rangle$ denotes the ensemble average of the operator $\hat{O}$ over thermal states. The spin tensor component of the Wigner function can be extracted as

$$\mathcal{M}^{\mu\nu}(x,p) = \frac{1}{2} \text{Tr} \left[ \{\gamma^\mu, S^{\alpha\beta}\} W(x,p) \right]$$

$$= -\frac{i}{2} e^{\mu\nu\rho\sigma} \mathcal{A}_\rho(x,p),$$

(51)

where $\mathcal{A}_\rho(x,p)$ is the axial vector component of the Wigner function and can be obtained by $\mathcal{A}_\mu(x,p) = \text{Tr}[\gamma^\mu \gamma_5 W(x,p)]$, here we have used $\{\gamma^\mu, S^{\alpha\beta}\} = i e^{\mu\nu\rho\sigma} \lambda^\rho \lambda^\sigma$ and $S^{\alpha\beta} = \frac{1}{4} \{\gamma^\alpha, \gamma^\beta\}$. Note that $\mathcal{M}^{\mu\nu}(x,p)$ are real quantities. The spin angular momentum tensor of a fermionic system is given by

$$M^{\rho\sigma} = \int d^3x \int d^3p \int dp_0 u_\lambda M^{\lambda\rho\sigma}(x,p)$$

$$= \frac{1}{2} u_\lambda \epsilon^{\rho\sigma\alpha} \int d^3x \int d^3p \int dp_0 \mathcal{A}_\alpha(x,p).$$

(52)

We see that $\int dp_0 u_\lambda M^{\lambda\rho\sigma}(x,p)$ plays the role of the spin tensor density in phase space. Here $u^\lambda$ is the fluid velocity and $p_0 = u \cdot p$. A four-momentum can be decomposed as $p^\mu = \bar{p}^\mu + (p \cdot u) u^\lambda$. So in a general frame the spatial
momentum integral in Eq. \((52)\) means \(d^3p = d^3\tilde{p}\). Then an on-shell momentum satisfies \(p^2 = m^2\) which can be rewritten as \(p_0^2 + \tilde{p}_0^2 = m^2\). The on-shell energy is then given by \(E_p = \sqrt{m^2 - \tilde{p}^2}\). In the local rest frame of the fluid we have \(u^\mu = (1, \mathbf{0}), \tilde{p}^\lambda = (0, \mathbf{p})\), \(\tilde{p}^2 = -|\mathbf{p}|^2\), \(E_p = \sqrt{|\mathbf{p}|^2 + m^2}\). In this section we sometime use the covariant form and sometime boldface version of a four-vector. By boldface of the four-vector we refer to its the spatial component in the local rest frame of the fluid.

The axial vector component at the zeroth or leading order in powers of the field strength tensor and space-time derivative is given by \([27]\)

\[
\omega^\mu_{(0)} = \text{Tr}[(\gamma^\mu \gamma^5)W_{(0)}] = m \theta(p_0) n^\mu(p, \mathbf{n}) - \theta(-p_0) n^\mu(-p, -\mathbf{n}) \delta(p^2 - m^2)(f_+ - f_-),
\]

where \(p_0 \equiv u \cdot p\) and \(f_\pm\) is given by

\[
f_\pm \equiv \frac{2}{(2\pi)^4} \left[ \theta(p_0)f_{\text{FD}}(p_0 - \mu_\pm) + \theta(-p_0)f_{\text{FD}}(-p_0 + \mu_\pm) \right].
\]

In Eqs. \((53,55)\) upper/lower sign denotes the spin state along \(\pm\mathbf{n}\) (\(\mathbf{n}\) is the spin quantization direction) in the particle’s rest frame, \(\mu_\pm\) denotes the corresponding chemical potentials, and \(f_{\text{FD}}(E) = 1/(e^{\beta E} + 1)\) is the Fermi-Dirac distribution. In Eq. \((55)\) \(n^\mu(p, \mathbf{n})\) is the spin quantization direction in the local frame and given by

\[
n^\mu(p, \mathbf{n}) = \Lambda^\mu_\nu(-\mathbf{v}_p)n^\nu(0, \mathbf{n}) = \left(\frac{\mathbf{n} \cdot \mathbf{p}}{m}, \frac{\mathbf{n} + (\mathbf{n} \cdot \mathbf{p})\mathbf{p}}{m(m + E_p)}\right).
\]

Here \(\Lambda^\mu_\nu(-\mathbf{v}_p)\) is the Lorentz transformation for \(\mathbf{v}_p = \mathbf{p}/E_p\) and \(n^\nu(0, \mathbf{n}) = (0, \mathbf{n})\) is the four-vector of the spin quantization direction in the particle’s rest frame. One can check that \(n^\mu(p, \mathbf{n})\) satisfies \(n^2 = -1\) and \(n \cdot p = 0\), so it behaves like a spin four-vector up to a factor of \(1/2\).

The first or next-to-leading order contribution for the axial vector component of the Wigner function for massive fermions can be obtained by generalizing the solution for massless fermions \([24,27,51,53]\),

\[
\omega^\mu_{(1)}(x, p) = \frac{1}{2} \hbar \beta \Omega^\alpha_\mu p_\sigma \delta(p^2 - m^2) \frac{df_+ + f_-}{d(\beta p_0)} - \frac{Q\hbar\tilde{F}_\alpha^\lambda p_\lambda}{p^2} \delta(p^2 - m^2)(f_+ + f_-),
\]

where \(\tilde{F}_\alpha^\lambda = \frac{1}{2} \epsilon^{\alpha\lambda\mu\nu} F_{\mu\nu}\) and \(\tilde{\Omega}^\alpha_\mu = \frac{1}{2} \epsilon^{\mu\nu\sigma\rho} \Omega^\nu_\sigma\) with \(\Omega^\nu_\sigma = \frac{1}{2}(\partial^\nu u_\sigma - \partial^\sigma u_\nu)\). Here \(\epsilon^{\mu\nu\sigma\beta}\) and \(\epsilon_{\mu\nu\sigma\beta}\) are antisymmetric tensors with \(\epsilon^{0123} = 1(-1)\) and \(\epsilon_{0123} = -1(1)\) for even (odd) permutations of indices 0123, so we have \(\epsilon^{0123} = -\epsilon_{0123} = 1\). In Eq. \((56)\) we have also assumed that \(\beta = 1/T\) is a constant.

We now compute \(\int dp_0 A^\alpha_{(0)}(x, p)\) from Eqs. \((53,55)\). At the leading order, we obtain

\[
\int dp_0 A^\alpha_{(0)} = \frac{1}{(2\pi)^3} \frac{m}{E_p} \left[ \frac{n^\alpha(p, \mathbf{n})}{e^{\beta(E_p - \mu_+)} + 1} - n^\alpha(-\mathbf{p}, -\mathbf{n}) \frac{1}{e^{\beta(E_p + \mu_+)} + 1} \right].
\]

If \(\mu_\pm = \mu\) does not depend on the spin state along \(\pm\mathbf{n}\), we see from Eq. \((57)\) that \(A^\alpha_{(0)} = 0\). In this case the non-vanishing polarization can only come from the first-order contribution \(A^\alpha_{(1)}(x, p)\) in Eq. \((56)\) whose vorticity part reads

\[
\int dp_0 A^\alpha_{(1)} = -\frac{1}{2} \hbar \beta \int dp_0 \Omega^\alpha_\mu p_\sigma \frac{df_+ + f_-}{d(\beta p_0)} \delta(p^2 - m^2)
\]

\[
= \frac{1}{(2\pi)^3} \frac{1}{E_p} \Omega^\alpha_\mu \left\{ p_\sigma \left[ e^{\beta(E_p - \mu)} \frac{e^{\beta(E_p - \mu)} + 1}{e^{\beta(E_p - \mu)} + 1} - p_\sigma \left[ e^{\beta(E_p + \mu)} \frac{e^{\beta(E_p + \mu)} + 1}{e^{\beta(E_p + \mu)} + 1} \right] \right\}
\]

\[
= \frac{1}{(2\pi)^3} \frac{\hbar \beta}{E_p} \Omega^\alpha_\mu p_\sigma \left\{ e^{\beta(E_p - \mu)} \frac{e^{\beta(E_p - \mu)} + 1}{e^{\beta(E_p - \mu)} + 1} + e^{\beta(E_p + \mu)} \frac{e^{\beta(E_p + \mu)} + 1}{e^{\beta(E_p + \mu)} + 1} \right\},
\]

where in the last line we have made the replacement \(\mathbf{p} \rightarrow -\mathbf{p}\) in the antifermion term and implied the same on-shell momentum \(p^\sigma = (E_p, \mathbf{p})\) for both fermions and antifermions. This makes no difference when carrying out the integral over momentum. We have assumed the chemical potentials are the same for both spin orientations, i.e. \(\mu_\pm = \mu\), so the sum over spin states gives a factor 2. Note that we can further simplify Eq. \((58)\) by keeping the time-like
component of the momentum, \( p^\alpha \to u^\alpha E_p \), in the last line if there is no dependence of \( f_{FD}(E_p \mp \mu) \) on the direction of \( p \). This will result in that the polarization density is proportional to \( \omega^\alpha = \hat{\Omega}^\alpha_\sigma u^\sigma \). The electromagnetic field part \( \int dp_0 A^\alpha_{(EM)} \) from Eq. \( (56) \) gives

\[
\int dp_0 A^\alpha_{(EM)} = -Q\hbar \int dp_0 F^{\alpha\lambda}_\gamma p_\gamma \frac{\delta(p^2 - m^2)}{p^2 - m^2}(f_+ + f_-) - \frac{1}{(2\pi)^3} \beta Q h \frac{1}{E_p} \tilde{F}^{\alpha\lambda}_\gamma p_\gamma \left\{ \frac{e^{\beta(E_p-\mu)}}{[e^{\beta(E_p-\mu)} + 1]^2} - \frac{e^{\beta(E_p+\mu)}}{[e^{\beta(E_p+\mu)} + 1]^2} \right\},
\]

where we have used the same on-shell momentum \( p^\lambda = (E_p, p) \) for both fermions and antifermions. The derivation of \( (59) \) is given in Appendix B.

We now obtain the spin tensor density in phase space, \( \int dp_0 \sigma_\alpha(x, p) \), in Eq. \( (52) \) as

\[
\int dp_0 \sigma_\alpha(x, p) = \frac{1}{2} u_\lambda \epsilon^{\lambda\sigma\alpha} \int dp_0 \sigma_\alpha(x, p),
\]

where \( \int dp_0 \sigma_\alpha(x, p) \) are given by Eqs. \( (68, 59) \). It is convenient to distinguish fermions from antifermions, for which we obtain their spin tensor densities in phase space

\[
M^{\sigma\sigma}_{\pm}(x, p) = -\frac{1}{2(2\pi)^3} \beta \frac{h}{E_p} \frac{1}{2} u_\lambda \epsilon^{\lambda\sigma\sigma} \left( \hat{\Omega}_{\alpha\sigma} \pm Q \frac{1}{E_p} \tilde{F}_{\alpha\sigma} \right) p^\sigma \left[ 1 - f_{FD}(E_p \mp \mu) \right],
\]

where the upper/lower sign is for the fermion/antifermion. Using the number density for fermions and antifermions in phase space,

\[
n_{\pm}(x, p) = \frac{1}{(2\pi)^3} \frac{2}{e^{\beta(E_p \mp \mu)} + 1},
\]

we obtain the spin tensor per particle in phase space

\[
\tilde{M}^{\sigma\sigma}_{\pm}(x, p) = \frac{M^{\sigma\sigma}_{\pm}(x, p)}{n_{\pm}(x, p)} = -\frac{h\beta}{4E_p} \frac{1}{2} u_\lambda \epsilon^{\lambda\sigma\sigma} \left( \hat{\Omega}_{\alpha\sigma} \pm Q \frac{1}{E_p} \tilde{F}_{\alpha\sigma} \right) p^\sigma \left[ 1 - f_{FD}(E_p \mp \mu) \right].
\]

We then obtain the spin vector per particle in phase space for the massive fermion or antifermion according to the Pauli-Lubanski pseudovector,

\[
S^\mu_{\pm}(x, p) = -\frac{1}{2m} \epsilon^{\mu\sigma\nu} M^{\pm}_{\sigma\nu} = \frac{h\beta}{4m} \left( \hat{\Omega}^{\alpha\lambda} \pm Q \frac{1}{E_p} \tilde{F}^{\alpha\lambda} \right) p_\lambda \left[ 1 - f_{FD}(E_p \mp \mu) \right],
\]

where \( p \) is an on-shell four-momentum for both fermion and antifermion and \( E_p \equiv \sqrt{m^2 - p^2} = p \cdot u \) in a general frame.

In the non-relativistic limit and the local rest frame of the fluid, the spatial component of the spin per particle in momentum space takes the form

\[
S_{\pm}(p) \approx \frac{h\beta}{4m} (E_p \omega \pm QB) \left[ 1 - f_{FD}(E_p \mp \mu) \right],
\]

where we have dropped the coordinate dependence of the spin vector for simplicity. The polarization per particle in momentum space can be obtained by

\[
\vec{\mathcal{P}}_{\pm}(p) = 2S_{\pm}(p).
\]

We can apply Eqs. \( (65, 66) \) to quarks and antiquarks and use Eq. \( (40) \) to compute \( \rho_{00} \) for vector mesons and use Eq. \( (45) \) to compute the polarization of \( \Lambda \) (\( \bar{\Lambda} \)).
VI. POLARIZATIONS: FROM QUARKS TO VECTOR MESONS AND BARYONS

In this section we look closely at the quark and hadron polarizations induced by the vorticity and magnetic field by applying Eqs. (63,66). We look at the contributions from the vorticity and from the magnetic field separately. For the simplicity of illustration we take a limit with three conditions: (a) small constant polarizations; (b) Boltzmann limit with $1 - f_{FD}(E_p + \mu) \approx 1$; (c) non-relativistic limit with $E_p \approx m_q$ ($m_q$ is the quark mass). At these limits the magnitudes of quark polarizations have the simple form by Eqs. (63,66) as

$$\mathcal{P}_q(\omega) \approx \frac{1}{2} \beta \omega,$$

$$\mathcal{P}_q(B) \approx \beta \mu_{mq} B,$$

where $\mu_{mq} = Q_q/(2m_q)$ is the magnetic moment of the quark with the electric charge $Q_q$, and we have used $\omega = |\omega|$ and $B = |B|$. Substituting Eq. (67) into Eq. (19), we obtain the polarizations of baryons to the leading order in the quark polarization. We can also obtain the spin density matrix elements and polarizations for the $\phi$ meson through Eqs. (31,33). All these results are listed in Table I.

We make some remarks about these results. The first remark is about the spin density matrix element of the $\phi$ meson $\rho_{00}$. The deviations from the non-polarized value, $\rho_{00} - 1/3$, have the opposite sign from the vorticity and magnetic field: $\rho_{00}(\omega) - 1/3 \approx -(\beta \omega)^2/9 < 0$ while $\rho_{00}(B) - 1/3 \approx 4\beta^2 \mu_{ms}^2 B^2/9 > 0$. Our result for $\rho_{00}(\omega)$ is consistent with the hadron statistical model [30] in which the mesons are treated as elementary particles, but our result for $\rho_{00}(B)$ is different from Eqs. (35,36) in the hadron statistical model [30], see Appendix C for a brief introduction to the particle polarization in the hadron statistical model from the magnetic field and vorticity.

In general cases, $\rho_{00}$ for other vector mesons is given by Eq. (31). Since the vorticity contribution $\rho_{00}(\omega)$ is independent of quark flavors, so we always have

$$\rho_{00}(\omega) = \frac{1}{3} - \frac{1}{9} (\beta \omega)^2 < \frac{1}{3},$$

for all vector mesons. In contrast the magnetic field contribution $\rho_{00}(B)$ does depend on the electric charges of the quark and antiquark inside the vector meson,

$$\rho_{00}(B) = \frac{1}{3} - \frac{4}{9} \beta^2 \mu_{ms}^2 m_{mq}^2 B^2$$

$$= \frac{1}{3} - \frac{1}{9} (\beta \omega)^2 Q_1 Q_2 B^2,$$

where $Q_1$ and $Q_2$ are the electric charge of the quark and antiquark in the vector meson respectively, and $m_1$ and $m_2$ are the mass of the quark and antiquark respectively. So for electrically neutral vector mesons such as $\rho^0$, $K^{*0}$, $K^+\pi^-$ and $\phi$, we have $\rho_{00}(B) > 1/3$. But for electrically charged vector mesons such as $\rho^+, \rho^-, K^{+\pi^0}$ and $K^{*-\pi^0}$, we have $\rho_{00}(B) < 1/3$. So we may conclude that for electrically charged vector mesons we have $\rho_{00} < 1/3$, while the situation is inconclusive for electrically neutral vector mesons, i.e. the magnitude of $\rho_{00}$ can be either $\rho_{00} < 1/3$ or $\rho_{00} > 1/3$ depending on the competition between the vorticity and magnetic field contribution.

The second remark is about the polarizations of baryons in magnetic fields which we can express in terms of the baryon magnetic moments with following formula in the constituent quark model [61]:

$$\mu_{m\Lambda} = \mu_{ms},$$

$$\mu_{m\Sigma} = \frac{1}{3}(2\mu_{ms} + 2\mu_{md} - \mu_{ms}),$$

$$\mu_{m\Xi} = \frac{1}{3}(4\mu_{ms} - \mu_{md}),$$

$$\mu_{m\Delta^+} = 3\mu_{ms},$$

$$\mu_{m\Omega} = 3\mu_{ms}.$$  

(70)

By these relations in the constituent quark model about baryon magnetic moments, our results for these baryons’ polarizations are consistent with the statistical model for hadrons [30] in which hadrons are treated as elementary particles, see Appendix C for details. Our results for baryon polarizations from the vorticity are also consistent with the hadron statistical model [30]. It is also interesting to look at the ratios among the polarizations of different baryons:

$$\mathcal{P}(\omega) \rightarrow \Lambda : \Sigma^0 : \Xi^- : \Delta^+: \Omega = \frac{1}{2} : \frac{1}{2} : \frac{1}{2} : \frac{5}{6} : \frac{5}{6},$$

$$\mathcal{P}(B) \rightarrow \Lambda : \Sigma^0 : \Xi^- : \omega\Delta^+: \Omega = \mu_{m\Lambda} : \mu_{m\Sigma} : \mu_{m\Xi} : \frac{5}{9} \mu_{m\Delta^+} : \frac{5}{9} \mu_{m\Omega},$$  

(71)
Table I: The polarizations of the φ meson and baryons in the quark coalescence model. The spin density matrix elements $\rho_0^\phi$ for the φ meson are also listed. In the first/second line are listed the results for the φ meson and baryons from the vorticity/magnetic field. In the third line are listed the polarizations of baryons in terms of baryon magnetic moments given by the constituent quark model. These results are consistent with those from the hadron statistical model in which hadrons are treated as elementary particles. In the third line are also listed the results for the φ meson in the hadron statistical model in which φ is treated as an elementary particle. The results are all expanded to the leading order in the small polarization.

| $P(\omega)$ or $\rho_0^\phi(\omega)$ | $P_\phi$ | $P_0^\phi$ | $P_{20}$ | $P_{30}$ | $P_{40}$ | $P_{31}$ |
|--------------------------------------|----------|-------------|----------|----------|----------|----------|
| $P(B)$ or $\rho_0^B(B)$              | 0        | $\frac{1}{3}(\beta_{\mu\alpha}B)^2$ | $\frac{1}{8}\beta_\mu B\frac{1}{2}B(2\mu_\mu\mu + 2\mu_\mu - \mu_\mu)$ | $\frac{1}{2}B(4\mu_\mu - \mu_\mu)$ | $\frac{1}{2}\beta_\mu B$ | $\frac{1}{4}\beta_\mu B$ |

where the first line is the ratios for the vorticity contributions while the second line is for the magnetic field contributions.

The final remark is that the polarization of any hadron from the vorticity in the hadron statistical model is consistent with that in the quark coalescence model under the three conditions as listed in the beginning of this section.

VII. SUMMARY AND CONCLUSIONS

In this paper we formulate a non-relativistic quark coalescence model with explicit momentum dependence based on the spin density matrix, with which one can describe the spin alignments of vector mesons and polarizations of baryons in a uniform way. The building blocks of the quark coalescence model is the quark and antiquark polarizations as the functions of momenta. The quark and antiquark polarizations can be calculated from the spin-tensor component of the Wigner function. Then we obtain the quark and antiquark polarizations induced by the vorticity and the magnetic field, with which the polarizations of vector mesons and baryons can be built up.

For vector mesons we start from the density matrix of quark-antiquark states with probability functions related to the quark and antiquark polarizations. Then we project vector meson states onto the density matrix to obtain the spin density matrix element with momentum dependence. The overlapping amplitude between the vector meson state and the quark-antiquark state has to be evaluated. The final result for the spin density matrix elements of vector mesons are obtained as a functional of the quark and antiquark polarization functions. In the same way, we can also describe the baryon polarizations in terms of the quark and antiquark polarization functions. By projection of the baryon state onto the density matrix of three-quark states we obtain the spin density matrix of baryons as a functional of the quark and antiquark polarization functions. The overlapping amplitude between the baryon state and the three-quark state has to be evaluated. From the spin density matrix of baryons we can compute the baryon polarizations.

Note that the current quark coalescence model is non-relativistic and valid for mesons and baryons with small momenta compared to their masses. For hadrons with large momenta, one has to formulate a relativistic version of the model. The current model can be the basis for further numerical simulations with event generators to give realistic predictions about global polarizations of vector mesons and baryons, which we will investigate in the future.

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Appendix A: Spin amplitudes and polarization vector of $\Delta^{++}$

In this appendix we give the spin amplitudes of the decay $\Delta^{++} \rightarrow p + \pi^+$ and the polarization vector of $\Delta^{+++}$.

The initial spin states of $\Delta^{++}$ are written as $|\phi, S_z\rangle$, where $S_z = \pm \frac{3}{2}, \pm \frac{1}{2}$. The angular momentum states of the proton and pion are in the multiplet of total angular momentum $J = 3/2$ which we denote as $|J, J_z\rangle$. The state $|J, J_z\rangle$ is a coupled state of the proton spin state $|\frac{1}{2}, S_z\rangle_p$ and its orbital angular momentum state $|L, L_z\rangle$, with
$L = 1$ and $L_z = 0, \pm 1$,

$$\left| \frac{3}{2} \frac{3}{2} \right\rangle_f = |1,1\rangle_L \left| \frac{1}{2} \frac{1}{2} \right\rangle_p,$$

$$\left| \frac{3}{2} \frac{1}{2} \right\rangle_f = |1,-1\rangle_L \left| \frac{1}{2} \frac{1}{2} \right\rangle_p,$$

$$\left| \frac{3}{2} \frac{1}{2} \right\rangle_f = \sqrt{\frac{1}{3}} |1,1\rangle_L \left| \frac{1}{2} \frac{1}{2} \right\rangle_p + \sqrt{\frac{2}{3}} |1,0\rangle_L \left| \frac{1}{2} \frac{1}{2} \right\rangle_p,$$

$$\left| \frac{3}{2} \frac{1}{2} \right\rangle_f = \sqrt{\frac{2}{3}} |1,0\rangle_L \left| \frac{1}{2} \frac{1}{2} \right\rangle_p + \sqrt{\frac{1}{3}} |1,-1\rangle_L \left| \frac{1}{2} \frac{1}{2} \right\rangle_p.$$  \hspace{1cm} (A1)

We can define the decay transition matrix $\mathcal{T}$ as

$$\mathcal{T} |\Delta^{++}, S_z\rangle = \left| \frac{3}{2}, S_z \right\rangle_f,$$  \hspace{1cm} (A2)

from which we can compute the decay transition amplitude between the initial and final state,

$$f(S^p_z, S_z) = \langle \hat{p}, S^p_z; \pi^+ | \mathcal{T} |\Delta^{++}, S_z\rangle$$

$$= \langle \hat{p}, S^p_z; \pi^+ | \left| \frac{3}{2}, S_z \right\rangle_f = \langle \hat{p}, S^p_z | \left| \frac{3}{2}, S_z \right\rangle_f,$$  \hspace{1cm} (A3)

where the final state can be denoted as $|\hat{p}, S^p_z; \pi^+ \rangle \equiv \{\theta, \phi, S^p\}$ with $\hat{p} = (\theta, \phi)$ being the proton’s momentum direction in the polar and azimuthal angle. With $\langle \hat{p} | L = 1, L_z = Y_{1-L_z}(\theta, \phi)$, we obtain $f(S^p_z, S_z)$ as

$$f \left( \frac{1}{2}, \frac{3}{2} \right) = Y_{1,1}(\theta, \phi), \quad f \left( -\frac{1}{2}, -\frac{3}{2} \right) = 0,$$

$$f \left( -\frac{1}{2}, -\frac{3}{2} \right) = Y_{1,-1}(\theta, \phi), \quad f \left( \frac{1}{2}, -\frac{3}{2} \right) = 0,$$

$$f \left( \frac{1}{2}, 1 \right) = \sqrt{\frac{2}{3}} Y_{1,0}(\theta, \phi), \quad f \left( -\frac{1}{2}, 1 \right) = \sqrt{\frac{1}{3}} Y_{1,1}(\theta, \phi),$$

$$f \left( \frac{1}{2}, -1 \right) = \sqrt{\frac{1}{3}} Y_{1,-1}(\theta, \phi), \quad f \left( -\frac{1}{2}, -1 \right) = \sqrt{\frac{2}{3}} Y_{1,0}(\theta, \phi).$$  \hspace{1cm} (A4)

With the decay amplitudes \textbf{(A4)}, we can evaluate Eq. \textbf{(15)} and obtain the angular distribution as

$$\frac{dN}{d\Omega} = \frac{3}{8\pi} \left\{ \left[ 1 - \frac{2}{3} \left( \rho_{-\frac{1}{2}, -\frac{1}{2}} + \rho_{\frac{1}{2}, \frac{1}{2}} \right) \right] - \left[ 1 - 2 \left( \rho_{-\frac{1}{2}, -\frac{1}{2}} + \rho_{\frac{1}{2}, \frac{1}{2}} \right) \right] \right\} \cos^2 \theta$$

$$+ \frac{2}{\sqrt{3}} \left( \text{Re} \rho_{-\frac{1}{2}, -\frac{1}{2}} - \text{Re} \rho_{\frac{1}{2}, \frac{1}{2}} \right) \sin(2\theta) \cos \phi$$

$$+ \frac{2}{\sqrt{3}} \left( \text{Im} \rho_{-\frac{1}{2}, -\frac{1}{2}} - \text{Im} \rho_{\frac{1}{2}, \frac{1}{2}} \right) \sin(2\theta) \sin \phi$$

$$- \frac{2}{\sqrt{3}} \left( \text{Re} \rho_{-\frac{1}{2}, \frac{1}{2}} + \text{Re} \rho_{\frac{1}{2}, -\frac{1}{2}} \right) \sin^2 \theta \cos(2\phi)$$

$$- \frac{2}{\sqrt{3}} \left( \text{Im} \rho_{-\frac{1}{2}, \frac{1}{2}} + \text{Im} \rho_{\frac{1}{2}, -\frac{1}{2}} \right) \sin^2 \theta \sin(2\phi) \right\},$$  \hspace{1cm} (A5)

where the spin density matrix elements for $\Delta^{++}$ are defined by

$$\rho_{S_{1z}, S_{2z}} = \langle \Delta^{++}, S_{1z} | \rho | \Delta^{++}, S_{2z} \rangle.$$  \hspace{1cm} (A6)

The polarization vector $\vec{p} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ for $\Delta^{++}$ can be determined from Eq. \textbf{(11)} with $S = 3/2$ and the spin operators for spin-3/2 particles being defined by

$$\hat{S}_1 = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & \sqrt{3} \end{pmatrix}, \quad \hat{S}_2 = \frac{i}{2} \begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad \hat{S}_3 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix},$$  \hspace{1cm} (A7)
whose result is

\[ P_1 = \frac{1}{\text{Tr}(\rho)} \left[ \frac{2}{\sqrt{3}} \left( \text{Re} \rho_{-\frac{1}{2},-\frac{1}{2}} + \text{Re} \rho_{\frac{1}{2},\frac{1}{2}} \right) + \frac{4}{3} \text{Re} \rho_{-\frac{1}{2},\frac{1}{2}} \right], \]

\[ P_2 = \frac{1}{\text{Tr}(\rho)} \left[ \frac{2}{\sqrt{3}} \left( \text{Im} \rho_{-\frac{1}{2},-\frac{1}{2}} + \text{Im} \rho_{\frac{1}{2},\frac{1}{2}} \right) + \frac{4}{3} \text{Im} \rho_{-\frac{1}{2},\frac{1}{2}} \right], \]

\[ P_3 = \frac{1}{\text{Tr}(\rho)} \left[ \frac{1}{3} \left( \rho_{\frac{1}{2},\frac{1}{2}} - \rho_{-\frac{1}{2},-\frac{1}{2}} \right) + \rho_{\frac{1}{2},\frac{1}{2}} - \rho_{-\frac{1}{2},-\frac{1}{2}} \right]. \]  

(\text{A8})

Appendix B: Derivation of Eq. (59)

We give a detailed derivation of Eq. (59). To this end we look at the four-momentum integral,

\[
\int d^4 p \sigma^\mu_{(EM)} = \frac{Q}{2\pi^3} \int d^4 \rho \delta^\prime (p^2 - m^2) (f_+ + f_-)
\]

\[
= \frac{4}{(2\pi)^3} Q \int d^4 p \delta^\prime (p^2 - m^2) \left[ \left[ \theta(u \cdot p) f_{FD}(u \cdot p - \mu) + \theta(-u \cdot p) f_{FD}(-u \cdot p + \mu) \right] \right]
\]

\[
= \frac{4}{(2\pi)^3} Q \int d^4 p \delta^\prime (p^2 - m^2) \theta(u \cdot p) [f_{FD}(u \cdot p - \mu) - f_{FD}(u \cdot p + \mu)]
\]

\[
= \frac{2}{(2\pi)^3} Q \int d^4 p \delta (p^2 - m^2) \delta (u \cdot p) [f_{FD}(u \cdot p - \mu) - f_{FD}(u \cdot p + \mu)]
\]

\[
= \frac{2}{(2\pi)^3} Q \beta \int \frac{d^4 p}{E_p} \frac{p}{E_p} \left[ \sum f_{FD}(E_p - \mu) \right] \left[ (1 - f_{FD}(E_p - \mu)) - f_{FD}(E_p + \mu) \right] (1 - f_{FD}(E_p + \mu))].
\]  

(\text{B1})

In the second line we have used \( \mu_{1,1} = \mu \), then the sum over spin states gives a factor 2. In the fifth line we have used the integral by part and dropped the complete derivative term. In the last equality we have replaced \( u_{\nu} \rightarrow p_{\nu}/E_p \) and carried out the integral over \( p_0 \equiv p \cdot u \), where \( E_p = \sqrt{m^2 - p^2} \). Then we arrive at Eq. (59).

Appendix C: Particle polarization in statistical model for hadrons

In the statistical model for hadrons, we treat a hadron as an elementary particle with inner structure. We take the nonrelativistic limit and consider a hadron with the spin \( S \) and the magnetic moment \( \mu_m = \mu_m S/S \) in a magnetic field \( B = Be_z \) along the \( z \)-direction. We know that the spin along the \( z \)-direction takes the value \( S_z = -S, -S + 1, \ldots, S - 1, S \). Suppose that the ensemble of the particles is in an equilibrium state with the temperature \( T \), so the probability for the state with a specific value of \( S_z \) is

\[
w(S_z) = \frac{e^{\beta \mu_m B S_z/S}}{\sum_{S_z=-S}^{S} e^{\beta \mu_m B S_z/S}}. \]  

(\text{C1})

Then the average polarization along the \( z \)-direction is

\[
P_S = \frac{1}{S} \sum_{S_z=-S}^{S} S_z w(S_z)
\]

\[
= \left( 1 + \frac{1}{2S} \right) \coth \left[ \left( 1 + \frac{1}{2S} \right) \beta \mu_m B \right] - \frac{1}{2S} \coth \left( \frac{\beta \mu_m B}{2S} \right).
\]  

(\text{C2})
If the magnetic field is weak, in the leading order we have
\[ \mathcal{P}_S \approx \frac{(S + 1)}{3S} \beta \mu_m B. \tag{C3} \]

For spin-1/2 and 3/2 particles, the above becomes
\[ \mathcal{P}_{S=1/2} = \beta \mu_{m,1/2} B, \quad \mathcal{P}_{S=3/2} = \frac{5}{9} \beta \mu_{m,3/2} B, \tag{C4} \]

which are consistent with the quark coalescence model.

For spin-1 particles, we have
\[ \mathcal{P}_S = \frac{2}{3} \beta \mu_{m,1} B. \tag{C5} \]

We can also obtain the density matrix element in the weak magnetic field
\[ \rho_{00}^{S=1} = w(S = 1, S_z = 0) = \frac{1}{1 + 2 \cosh(\beta \mu_m B)} \approx \frac{1}{3} - \frac{1}{9} (\beta \mu_m B)^2 \tag{C6} \]

where the equality holds for particles without the magnetic moment. For the \( \phi \) meson, we have \( \mathcal{P}_\phi = \frac{2}{3} \beta \mu_{m, \phi} B \) and \( \rho_\phi = \frac{1}{4} - \frac{1}{8} (\beta \mu_{m, \phi} B)^2 \).

The polarization from the vorticity can be obtained by replacing \( \mu_m B/S \) by \( \omega \).

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