A simplicial version of the 2–dimensional Fulton–MacPherson operad

NATHANIEL BOTTMAN
A simplicial version of the 2–dimensional Fulton–MacPherson operad

NATHANIEL BOTTMAN

We define an operad in Top, called $FM^W_2$. The spaces in $FM^W_2$ come with CW decompositions such that the operad compositions are cellular. In fact, each space in $FM^W_2$ is the realization of a simplicial set. We expect, but do not prove here, that $FM^W_2$ is isomorphic to the 2–dimensional Fulton–MacPherson operad $FM_2$. Our construction is connected to the author’s work on the symplectic $(A_{\infty}, 2)$–category, and suggests a strategy toward equipping the symplectic cochain complex with the structure of a homotopy Batalin–Vilkovisky algebra.

18M75, 55P48; 53D37

1 Introduction

Getzler and Jones [1994] introduced the Fulton–MacPherson operad

$$FM_2 = (FM_2(k))_{k \geq 1},$$

where $FM_2(k)$ is the compactification à la Fulton and MacPherson [1994] of the configuration space of $k$ distinct labeled points in $\mathbb{R}^2$, modulo translations and dilations. Getzler and Jones proposed in the same paper a collection of cellular decompositions of the spaces in $FM_2$, such that these decompositions are compatible with the operad maps $\circ_i : FM_2(k) \times FM_2(l) \to FM_2(k + l - 1)$. These decompositions formed the basis for a significant amount of work related to the Deligne conjecture, including a proof in [Getzler and Jones 1994] of that conjecture.

Unfortunately, Tamarkin found an error in Getzler and Jones’ decomposition. In particular, in the 9–dimensional space $FM_2(6)$, there are two disjoint open 6–cells $C_1$ and $C_2$ with the property that $\overline{C_1} \cap C_2$ is nonempty, as described in [Voronov 2000, Section 1.2.2]. Salvatore [2022] used meromorphic differentials to construct cellular decompositions of the spaces in FM. His approach is completely different from Getzler and Jones’.

We construct an operad of CW complexes, which we conjecture to be isomorphic in Top to $FM_2$. Under this expected isomorphism, our decompositions are refinements of Getzler and Jones’ attempted decompositions. The context for the current paper is the author’s program (as developed in [Bottman 2015; 2019a; 2019b; 2020; Bottman and Carmeli 2021; Bottman and Oblomkov 2019; Bottman and Wehrheim 2018]) to construct Symp, the symplectic $(A_{\infty}, 2)$–category. Specifically, the author plans to use the decompositions of FM that we construct here to understand the axioms for identity 1–morphisms
in an \((A_{\infty}, 2)\)–category. In the context of \(\text{Symp}\), this suggests a strategy toward endowing symplectic cohomology with a chain-level homotopy Gerstenhaber (and eventually, homotopy BV) algebra structure that is finite in each arity, thus answering Conjecture 2.6.1 of [Abouzaid 2015]. We note that our approach is compatible with the operations in \(\text{Symp}\), unlike Salvatore’s; in addition, we expect our approach to generalize to the Fulton–MacPherson operad of any dimension.

1.1 Getzler and Jones’ attempted decomposition

Getzler and Jones’ attempted decomposition is an adaptation to the case of \(\text{FM}_2\) of Fox and Neuwirth’s decomposition [1962] of the one-point compactification of the configuration space \((\mathbb{R}^2)^k \setminus \Delta\) of \(k\) points in \(\mathbb{R}^2\), where \(\Delta\) is the fat diagonal. A Fox–Neuwirth cell corresponds to a choice of which subsets of the points \(p_1, \ldots, p_k\) should be vertically aligned, the left-to-right order in which these subsets of points should appear, and the top-to-bottom order in which each subset of the points should appear. For instance, Figure 1 is a real-codimension-3 cell in \((((\mathbb{R}^2)^6 \setminus \Delta)^*)\). Getzler and Jones observed that the Fox–Neuwirth cells are invariant under translations and dilations, and moreover that one can define a similar type of cell for the boundary locus. The elements in the boundary of \(\text{FM}_2(k)\) are trees of “screens”, and these “boundary cells” are defined by partitioning and ordering the points on each of the screen in the same way as with Fox–Neuwirth cells.

1.2 Tamarkin’s counterexample

As described in [Voronov 2000], Tamarkin observed a way in which Getzler and Jones’ supposed decomposition fails. Consider \(\text{FM}_2(6)\), the open locus of which parametrizes configurations of six distinct points in \(\mathbb{R}^2\), up to translations and dilations. Next, we consider the two 6–cells \(C_1\) and \(C_2\) in Figure 2 (we omit the numberings). The \(j\)th bubble in \(C_2\) (for \(j = 1, 2\)) carries a modulus \(\lambda_j\) defined in the following way: by translating and dilating, we can move the left and right lines to \(x = 0\) and \(x = 1\), respectively; we then denote by \(\lambda_j\) the position of the middle line. The intersection \(\overline{C_1} \cap C_2\) is the codimension-1 locus in \(C_2\) in which \(\lambda_1 = \lambda_2\). What Getzler and Jones proposed is therefore not a cellular decomposition, because the intersection of the closures of two distinct \(n\) cells should be contained in the \((n-1)\)–skeleton.
In our construction, $C_1$, $C_2$, and $C_1 \cap C_2$ will each be a union of cells.

### 1.3 An overview of our construction

We construct a collection of CW complexes $FM^W_2(k)$ and maps

$$\circ_i : FM^W_2(k) \times FM^W_2(l) \to FM^W_2(k + l - 1) \quad \text{for } 1 \leq i \leq k.$$  

Here is our main result:

**Main Theorem** The spaces $(FM^W_2(k))_{k \geq 1}$ together with the composition operations $\circ_i$ form a non-$\Sigma$ operad, and the composition maps

$$\circ_i : FM^W_2(k) \times FM^W_2(l) \to FM^W_2(k + l - 1)$$

are cellular.

We will now give a brief overview of the definition of $FM^W_2(k)$.

(i) First, we define a “$W$–version” $W^W_n$ of the 2–associahedra by the analogy

$$K_r : W(\operatorname{Ass}) :: W^W_n : W^W_n.$$  

Here $K_r$ is the $(r-2)$–dimensional associahedron, and $W(\operatorname{Ass})$ is the Boardman–Vogt $W$–construction applied to the associative operad, which is defined in terms of metric stable trees and yields an operad of CW complexes that is isomorphic to the associahedral operad $K$ in $\text{Top}$. $W^W_n$ is an $(|n|+r-3)$–dimensional 2–associahedron, and $W^W_n$ is a CW complex that we define in Section 2 in terms of metric stable tree-pairs and which we expect to be homeomorphic to $W^W_n$. We then refine the CW structure on $W^W_n$ to a simplicial decomposition.

(ii) Toward our construction of $FM^W_2(k)$, we decompose $FM_2(k)$ into Getzler–Jones cells, then identify each open Getzler–Jones cell with a product of open 2–associahedra. We then replace each such product by the corresponding product of interiors of the spaces $W^W_n$ described in the previous step. This product
comes with a decomposition into products of simplices, and we refine this to a simplicial structure. Finally, we attach these decomposed Getzler–Jones cells together to produce $\text{FM}_2(W)(k)$. This part of the construction appears in Section 3.

The essential property of $\text{FM}_2(W)(k)$ that we must verify is that our CW decomposition is valid. It is clear that our putative open cells disjointly decompose our space, and that they are homeomorphic to open balls. The only nontrivial check we need to make is that the $n$–cells are attached to the $(n-1)$–skeleton. This is where Getzler and Jones’ attempted decomposition fails: the 6–cell $C_1$ that we described in Section 1.2 is not attached to the 5–skeleton. Our decomposition satisfies this property by construction: we attach a given $n$–cell by taking a closed $n$–simplex, then attaching it to the existing skeleton via quotient maps from the boundary $(n-1)$–simplices to the $(n-1)$–skeleton. In fact, the boundary of an $n$–cell is a union of cells of dimension at most $n - 1$.

### 1.4 The relationship between our construction and $\text{Symp}$

The genesis of the construction of $\text{FM}_2(W)$ was a connection between the symplectic $(A_\infty, 2)$–category $\text{Symp}$ and $E_2$ suggested by Jacob Lurie in 2016. (The construction of $\text{Symp}$ is a long-term project of the author, building on work of Ma’u, Wehrheim, and Woodward; see [Bottman 2015; 2019a; 2019b; 2020; Bottman and Carmeli 2021; Bottman and Wehrheim 2018; Ma’u et al. 2018].) We can express this connection concretely, via a collection of maps

$$f^W_\sigma: W_n^W \to \text{FM}_2(W(|n|)),$$

where $\sigma$ is a 2–permutation, as defined in Section 3.2. The idea of this map is very simple. The map $f_\sigma$ forgets the data of the lines, then labels the points according to the 2–permutation $\sigma$. Then $f_\sigma$ extends continuously to the boundary of $W_n$; it is an embedding on the interior of its domain, but contracts some boundary cells.

**Example 1.1** In Figure 3, we depict $W_{111}$ and its image under an appropriate map $f_\sigma$. More precisely, we depict their nets — to “assemble” both CW complexes, one would cut them out, then glue together like-numbered edges. As is evident, most of the 2–cells of $W_{111}$ are contracted by $f_\sigma$.

While it would take us too far afield to explain the relationship between $\text{FM}_2$ and $\text{Symp}$ (and their $W$–counterparts) in detail, let us indicate the basic idea. $\text{Symp}$, being an $(A_\infty, 2)$–category, assigns to a chain in a 2–associahedron $W_n$ an operation on 2–morphisms. (For instance, the objects of $\text{Symp}$ are symplectic manifolds, and given two objects $M_0$ and $M_1$, the 1–morphism category is $\text{Fuk}(M_0^- \times M_1)$; 2–associahedra $W_n$, where $n$ is a single positive integer, act on this Fukaya category by the usual $A_\infty$–operations.) The current definition of an $(A_\infty, 2)$–category, appearing in [Bottman and Carmeli 2021], does not equip identity 1–morphisms with all the possible structure. Indeed, when defining operations on 2–morphisms in the situation where some of the 1–morphisms are identities, those 1–morphisms should be allowed to be “moved past” the other 1–morphisms. To make this precise, one exactly needs to understand the maps $f_\sigma$, and to equip their targets with a CW structure so that $f_\sigma$ is cellular. One way to...
A simplicial version of the 2–dimensional Fulton–MacPherson operad

1187

Figure 3

proceed toward this goal is to first decompose \( \text{FM}_k^W \) so that \( f^W_\sigma \) is cellular, and next construct coherent homeomorphisms \( W_n \cong W_n^W \) and \( \text{FM}_2(k) \cong \text{FM}_2^W(k) \).

The following result therefore shows the way toward a connection between the symplectic \( (A_\infty, 2) \)–category and \( \text{FM}_2^W \). It is an immediate consequence of our construction of \( W_n^W \) and \( \text{FM}_2^W(k) \), and it forms the content of Remark 3.14.

**Proposition**  Fix \( r \geq 1, n \in \mathbb{Z}_{\geq 0} \setminus \{0\} \), and a 2–permutation \( \sigma \) of type \( n \). Then the associated map

\[
(6) \quad f^W_\sigma : W_n^W \to \text{FM}_2^W(|n|)
\]

is cellular.

1.5 Future directions

The author plans to develop several aspects of the current paper. In particular:

- With several collaborators, the author plans to extend this work to produce cellular decompositions of \( \text{FM}_k^W \) for all \( k \geq 1 \), and to show that \( \text{FM}_k^W \) is isomorphic to \( \text{FM}_k \) in \( \text{Top} \).

- This paper can be construed as a way of incorporating identity 1–morphisms into the symplectic \( (A_\infty, 2) \)–category. The author plans to formalize this in future work on the algebra of \( (A_\infty, 2) \)–categories.
We plan to upgrade this work to give a cellular model for the framed analogue of the Fulton–MacPherson operad. This suggests a way of endowing symplectic cohomology with a chain-level BV algebra structure, which is the subject of Conjecture 2.6.1 of [Abouzaid 2015].

Acknowledgments

This paper is a solution to homework problem #12 from Paul Seidel’s course on Categorical dynamics and symplectic topology at MIT in Spring 2013. The author thanks Prof. Seidel for his patience. Jacob Lurie drew an analogy that suggested to the author that there must be a link between $(A_\infty, 2)$–categories and $E_2$–algebras. Alexander Voronov explained to the author the colorful history surrounding this problem. A conversation with Naruki Masuda, Hugh Thomas, and Bruno Vallette led the author to think about replacing $\text{FM}_2$ with a “$W$–construction version” thereof. The author thanks Dean Barber, Michael Batanin, Sheel Ganatra, Ezra Getzler, Mikhail Kapranov, Ben Knudsen, Paolo Salvatore, and Dev Sinha for their interest and encouragement.

The author was supported by an NSF Mathematical Sciences Postdoctoral Research Fellowship and by an NSF Standard Grant (DMS-1906220). He thanks the Institute for Advanced Study, the Mathematical Sciences Research Institute, and the University of Southern California for providing excellent working conditions during the period when this work was carried out.

2 A “$W$–version” of the 2–associahedra

In this section, we construct a “$W$–version” of the 2–associahedra. (The 2–associahedra were originally defined in [Bottman 2019a].) This is an essential ingredient in our definition of $\text{FM}_2^W(k)$, which will appear in Section 3.

2.1 A warm-up: $K^W$, ie $W(\text{Ass})$, ie a $W$–version of the associahedra

In this subsection, we recall a certain operad, which we will denote by $K^W = (K^W_r)_{r \geq 1}$. This is simply the Boardman–Vogt $W$–construction applied to the associative operad Ass. We construct only $K^W$ rather than recalling the general definition of the $W$–construction, because this one-off construction will be a useful warm-up to our construction of $W^W$ later in this section. As noted in [Barber 2013], $K^W$ is isomorphic in Top to the associahedral operad $K$.

The following proposition summarizes what we will prove about $K^W$:

**Proposition 2.1** The spaces $(K^W_r)_{r \geq 1}$ form a non-$\Sigma$ operad of CW complexes, and the composition maps

\[ \circ_i : K^W_r \times K^W_s \to K^W_{r+s-1} \]

defined in Definition 2.11 are cellular.

We will prove Proposition 2.1 at the end of the current subsection.
We begin with a definition of rooted ribbon trees. Stable rooted ribbon trees with \( r \) leaves index the strata of the associahedron \( K_r \), and they will be an integral part of the definition of \( K^W_r \).

**Definition 2.2** [Bottman 2019a, Definition 2.2] A rooted ribbon tree (RRT) is a tree \( T \) with a choice of a root \( \alpha_{\text{root}} \in T \) and a cyclic ordering of the edges incident to each vertex; we orient such a tree toward the root. We say that a vertex \( \alpha \) of an RRT \( T \) is interior if the set \( \text{in}(\alpha) \) of its incoming neighbors is nonempty, and we denote the set of interior vertices of \( T \) by \( T_{\text{int}} \). An RRT \( T \) is stable if every interior vertex has at least two incoming edges. We define \( K^\text{tree}_r \) to be the set of all isomorphism classes of stable rooted ribbon trees with \( r \) leaves.

We denote the \( i \)th leaf of an RRT \( T \) by \( \lambda_i^T \). For any \( \alpha, \beta \in T \), \( T_{\alpha \beta} \) denotes those vertices \( \gamma \) such that the path \( [\alpha, \gamma] \) from \( \alpha \) to \( \gamma \) passes through \( \beta \). We define \( T_{\alpha} := T_{\alpha_{\text{root}} \alpha} \).

**Remark 2.3** Ribbon trees (resp. rooted ribbon trees) are often referred to as planar trees (resp. planted trees).

Next, we define a version of RRTs with internal edge lengths:

**Definition 2.4** A metric RRT \((T, (\ell_e))\) is the data of
- an RRT \( T \), and
- for every edge \( e \) of \( T \) not incident to a leaf (but possibly incident to the root), a length \( \ell_e \in [0, 1] \).

We call this a metric RRT of type \( T \).

Now we will define a “dimension” function \( d \) on stable RRTs:

**Definition 2.5** [Bottman 2019a, Definition 2.4] For \( T \) a stable RRT in \( K^\text{tree}_r \), we define its dimension \( d(T) \in [0, r - 2] \) like so:

\[
d(T) := r - \# T_{\text{int}} - 1.
\]

**Definition 2.6** Given a stable tree \( T \), the cell associated to \( T \) is denoted by \( C_T \) and is defined to consist of all metric RRTs of type \( T \).

Note that we can canonically identify \( C_T \) with the closed cube of dimension equal to the number of internal edges of \( T \). That is:

\[
C_T \cong [0, 1]^{\# T_{\text{int}} - 1} = [0, 1]^{r - 2 - d(T)}.
\]

As we will see, \( K^W_r \) is \((r - 2)\)-dimensional; it follows that \( d(T) \) is the codimension of \( C_T \) in \( K^W_r \). (The unfortunate clash of terminology between “dimension” and “codimension” is due to the fact that, in \( K_r \), the cell indexed by \( T \) has dimension \( d(T) \).)

We now define \( K^W_r \) by taking the union of the cells \( C_T \) for \( T \) any stable RRT with \( r \) leaves, then collapsing edges of length 0.

*Algebraic & Geometric Topology, Volume 24 (2024)*
Definition 2.7 Given \( r \geq 1 \), we define \( K^W_r \) to be the following quotient:

\[
K^W_r := \left( \bigsqcup_{T \in K^\text{tree}_r} C_T \right) / \sim.
\]

Here \( \sim \) identifies \((T, (\ell_e))\) and \((T', (\ell'_e))\) if, after collapsing all edges \( e \) of \( T \) with \( \ell_e = 0 \) and all edges \( e \) of \( T' \) with \( \ell'_e = 0 \), both metric RRTs reduce to the same metric RRT \((T'', (\ell''_e))\).

Example 2.8 In Figure 4, we depict the CW complex \( K^W_4 \). Note that this is a refinement of \( K_4 \), which (as a CW complex) is a pentagon. We have labeled the open top cells by the metric stable RRTs that they parametrize, where each \( a \) and \( b \) is allowed to vary in \([0, 1]\). The closed top cells are glued together along the cells where some of the edge lengths are 0—for instance, we have indicated how the top and top-right cubes are joined along the internal edge of the pentagon where the edge length \( b \) in both cells becomes 0. The boundary of \( K^W_r \) is the union of cells where at least one edge length is 1.

Finally, we define a simplicial refinement of the CW structure on \( K^W_r \). To approach this, we note that if \( P \) is the poset \( \{0, 1\}^k \), where \( \sigma_1 < \sigma_2 \) if \( \sigma_2 \) can be gotten by changing some of the 0s of \( \sigma_1 \) to 1s, then the nerve of \( P \) is a simplicial decomposition of the cube \([0, 1]^k\). More concretely, the top simplices are the sets of the form

\[
\{(x_1, \ldots, x_k) \in [0, 1]^k \mid 0 < x_{\sigma(1)} < \cdots < x_{\sigma(k)} < 1\},
\]

where \( \sigma \) is a permutation on \( k \) letters. The remaining simplices are the result of replacing some of these inequalities by equalities.

Definition 2.9 We refine the CW structure on \( K^W_r \) by decomposing each cell \( C_T \) in \( K^W_r \) like so: we make the identification \( C_T \cong [0, 1]^{r-2-d(T)} \), then perform the simplicial decomposition described in the previous paragraph. This refinement equips \( K^W_r \) with a simplicial decomposition.
Example 2.10  In Figure 5, we depict the simplicial complex $K_r^W$. This is the refinement of our initial cubical CW decomposition of $K_r^W$ gotten by subdividing each of the five squares into two triangles. We indicate the new edges by coloring them blue.

Now that we have constructed the spaces $K_r^W$, we can prove Proposition 2.1, which states that $(K_r^W)$ is a non-$\Sigma$ operad and that the operad maps are cellular.

Definition 2.11  Fix $r$, $s$, and $i \in [1, r]$. We wish to define the composition map

\[
\circ_i : K_r^W \times K_s^W \to K_{r+s-1}^W.
\]

We do so cell by cell. That is, fix cells $C_T \subset K_r^W$ and $C_{T'} \subset K_s^W$. Define $T''$ to be the result of grafting $T'$ to the $i^{th}$ leaf of $T$. Then we define $\circ_i$ on $C_T \times C_{T'}$ like so: given collections of edge lengths on $T$ and $T'$, combine them to produce a collection of edge lengths on $T''$, where we assign to the single newly formed interior edge the length 1.

Proof of Proposition 2.1  Fix $r$, $s$, and $i \in [1, r]$, and consider the composition map

\[
\circ_i : K_r^W \times K_s^W \to K_{r+s-1}^W.
\]

To show that $\circ_i$ is cellular, let’s consider the restriction of $\circ_i$ to a product $C_T \times C_{T'}$ of closed cubes, for $T \in K_r$ and $T' \in K_s$. Denote by $T''$ the tree obtained by grafting the root of $T'$ to the $i^{th}$ leaf of $T$. Then $\circ_i$ includes $C_T \times C_{T'}$ into $C_{T''}$ as the face gotten by requiring the outgoing edge of the root of $T'$ to have length 1. The CW structure of this face of $C_{T''}$ is finer than that of $C_T \times C_{T''}$, so $\circ_i$ is indeed cellular.
2.2 Metric tree-pairs and the definition of $W_n^W$

Just as we defined $K_r^W$ to be the parameter space of metric stable RRTs, we will define $W_n^W$ to parametrize metric stable tree-pairs. The definition of metric stable tree-pairs is somewhat involved, so we devote the current subsection to this definition.

Before defining metric stable tree-pairs, we recall the definition of stable tree-pairs:

**Definition 2.12** [Bottman 2019a, Definition 3.1] A stable tree-pair of type $n$ is a datum $2T = T_b \xleftarrow{f} T_s$, with $T_b$, $T_s$, and $f$ described below:

- The **bubble tree** $T_b$ is an RRT whose edges are either solid or dashed, which must satisfy these properties:
  - The vertices of $T_b$ are partitioned as $V(T_b) = V_{\text{comp}} \sqcup V_{\text{seam}} \sqcup V_{\text{mark}}$, where
    * every $\alpha \in V_{\text{comp}}$ has at least 1 solid incoming edge, no dashed incoming edges, and either a dashed or no outgoing edge;
    * every $\alpha \in V_{\text{seam}}$ has zero or more dashed incoming edges, no solid incoming edges, and a solid outgoing edge; and
    * every $\alpha \in V_{\text{mark}}$ has no incoming edges and either a dashed or no outgoing edge.
  - Stability If $\alpha$ is a vertex in $V_{\text{comp}}^1$ and $\beta$ is its incoming neighbor, then $\# \text{in}(\beta) \geq 2$; if $\alpha$ is a vertex in $V_{\text{comp}}^{\geq 2}$ and $\beta_1, \ldots, \beta_l$ are its incoming neighbors, then there exists $j$ with $\# \text{in}(\beta_j) \geq 1$.

- The **seam tree** $T_s$ is an element of $K_{\text{tree}}^r$.

- The **coherence map** is a map $f : T_b \rightarrow T_s$ of sets having these properties:
  - $f$ sends root to root, and if $\beta \in \text{in}(\alpha)$ in $T_b$, then either $f(\beta) \in \text{in}(f(\alpha))$ or $f(\alpha) = f(\beta)$.
  - $f$ contracts all dashed edges, and every solid edge whose terminal vertex is in $V_{\text{comp}}^1$.
  - For any $\alpha \in V_{\text{comp}}^{\geq 2}$, $f$ maps the incoming edges of $\alpha$ bijectively onto the incoming edges of $f(\alpha)$, compatibly with $<\alpha$ and $<f(\alpha)$.
  - $f$ sends every element of $V_{\text{mark}}$ to a leaf of $T_s$, and if $\lambda_i^{T_s}$ is the $i$th leaf of $T_s$, then $f^{-1}\{\lambda_i^{T_s}\}$ contains $n_i$ elements of $V_{\text{mark}}$, which we denote by $\mu_{i1}^{T_b}, \ldots, \mu_{in_i}^{T_b}$.

We denote by $W_n^{\text{tree}}$ the set of isomorphism classes of stable tree-pairs of type $n$. Here an isomorphism from $T_b \xleftarrow{f} T_s$ to $T_b' \xleftarrow{f'} T_s'$ is a pair of maps $\varphi_b : T_b \rightarrow T_b'$ and $\varphi_s : T_s \rightarrow T_s'$ that fit into a commutative square in the obvious way and that respect all the structure of the bubble trees and seam trees.

Next, we define metric stable tree-pairs. This notion is more subtle than that of metric stable RRTs, because we must impose conditions on the edge-lengths. (This should be compared to Bottman and Oblomkov’s similar constraints [2019, Section 3], imposed in order to define local charts on a complexified version of $W_n$.)
Definition 2.13  A metric stable tree-pair $(2T, (L_e), (\ell_e))$ is the following data:

- $2T$ is a stable tree-pair.
- We have, for every interior dashed edge $e$ of $T_b$, a length $L_e \in [0, 1]$, and, for every interior edge $e$ of $T_s$, a length $\ell_e \in [0, 1]$, subject to the following coherence conditions (where for convenience we set $L_\alpha := L_e$ for $\alpha \in V_{\text{comp}}(T_b) \setminus \{\alpha_{\text{root}}\}$ and $e$ the outgoing edge of $\alpha$, and similarly for the edge-lengths in $T_s$):
  - For every $\alpha_1, \alpha_2 \in V_{\text{comp}}^2(T_b)$ and $\beta \in V_{\text{comp}}^1(T_b)$ with $f(\alpha_1) = f(\alpha_2) = f(\beta)$, we require
    \begin{equation}
    \max_{\gamma \in [\alpha_1, \beta]} L_\gamma = \max_{\gamma \in [\alpha_2, \beta]} L_\gamma.
    \end{equation}
  - For every $\rho \in V_{\text{int}}(T_s) \setminus \{\rho_{\text{root}}\}$ and $\alpha \in V_{\text{comp}}^2(T_b)$ with $f(\alpha) = \rho$, we require
    \begin{equation}
    \ell_\rho = \max_{\gamma \in [\alpha, \beta_\alpha]} L_\gamma,
    \end{equation}
    where we define $\beta_\alpha$ to be the first element of $V_{\text{comp}}^2(T_b)$ that the path from $\alpha$ to $\alpha_{\text{root}}$ passes through.

Finally, we recall the dimension of a stable tree-pair. Similarly to the dimension of a stable RRT, this will be the codimension in $W_n^W$ of the cell corresponding to the stable tree-pair in question.

**Definition 2.14**  [Bottman 2019a, Definition 3.3]  For $2T$ a stable tree-pair, we define the dimension $d(2T) \in [0, |n| + r - 3]$ like so:

\begin{equation}
    d(2T) := |n| + r - #V_{\text{comp}}^1(T_b) - #(T_s)_{\text{int}} - 2.
\end{equation}

We are now prepared to define $W_n^W$, the “$W$–version” of the 2–associahedron. We will define $W_n^W$ by attaching together the cells $C_{2T}$, which consist of metric stable tree-pairs.

**Definition 2.15**  Given a stable tree-pair $2T$, the cell associated to $2T$ is the collection of all metric stable tree-pairs of type $2T$. We denote this cell by $C_{2T}$.

Note that we can identify $C_{2T}$ with the subset of the cube $[0, 1]^k$ defined by the equalities (14) and (15), where $k$ is the number of interior dashed edges of $T_b$ plus the number of interior edges of $T_s$.

**Definition 2.16**  Fix $r \geq 1$ and $n \in \mathbb{Z}_{\geq 0}^r \setminus \{0\}$. We define $W_n^W$ similarly to how we defined $K_r^W$ in **Definition 2.7**:

$$W_n^W := \left( \bigsqcup_{2T \in \mathcal{W}_n^{\text{tree}}} C_{2T} \right) / \sim.$$ 

The quotient here is somewhat subtler than the quotient that appeared in **Definition 2.7**, specifically when it comes to $T_b$. In $T_s$, we simply contract any edges of length 0. We indicate in **Figure 6** how to perform the necessary contractions in $T_b$ when some edge-lengths are 0. The reader should think of the left contraction as undoing a type-1 move (as in [Bottman 2019a, Section 3.1]), whereas the right contraction undoes either a type-2 or a type-3 move. Note that we are using the coherences enforced in

*Algebraic & Geometric Topology, Volume 24 (2024)*
**Definition 2.13** — for instance, these mean that we do not have to consider a situation as in the right-hand side of the above figure, but where only some of the edge-lengths in this portion of $T_b$ are 0.

**Example 2.17** In Figure 7, we depict the CW complex $W_{21}^W$. Each of the parameters $a$ and $b$ lie in $[0, 1]$; they do not have the same meaning across different cells. The eight interior edges (resp. sixteen boundary edges) correspond to the loci in the top cells where a parameter goes to 0 (resp. to 1).

Finally, we refine the CW structure on $W_n^W$ to a simplicial decomposition.

**Lemma 2.18** Fix a stable tree-pair $2T$. For every simplex $S$ in the standard simplicial decomposition of $[0, 1]^k \supset C_{2T}$, $S$ is either contained in $C_{2T}$ or disjoint from it. The collection of such simplices that are contained in $C_{2T}$ form a simplicial decomposition of $C_{2T}$.

**Proof** Fix a simplex $S$. $S$ is defined by a collection of equalities and inequalities of the form

$$0 \times x_{\sigma(1)} \times \cdots \times x_{\sigma(k)} \times 1,$$

where each “*” is either a “<” or an “=”, and where $\sigma$ is a permutation on $k$ letters. After imposing these (in)equality constraints, the left- and right-hand sides of the equalities (14) and (15) become single variables. This collection of equalities will either be always satisfied or never satisfied, depending on the constraints in (17). Depending on which of these is the case, $S$ is either contained in $C_{2T}$ or disjoint from it.

It follows immediately that the collection of simplices that are contained in $C_{2T}$ form a simplicial decomposition of $C_{2T}$.

---

*Algebraic & Geometric Topology, Volume 24 (2024)*
**Example 2.19** In Figure 8, we illustrate the closed cell in $W_{40}^W$ associated to the underlying tree-pair of the (top-dimensional) metric tree-pair shown on the right. The restriction on the lengths $a, b, c, d \in [0, 1]$ is that they must satisfy $\max(a, b) = \max(c, d)$; as a result, this cell has the CW type of a square pyramid.

We indicate the simplicial refinement of this cell: the square pyramid is subdivided into eight 3–simplices, which are defined by imposing inequalities and equalities as shown in this figure.
3 The construction of $\text{FM}_2^W$

In this final section, we will construct a collection of CW complexes $(\text{FM}_2^W (k))_{k \geq 1}$ and a collection of operations

\begin{equation}
\circ_i: \text{FM}_2^W (k) \times \text{FM}_2^W (l) \to \text{FM}_2^W (k + l - 1)
\end{equation}

such that these data form an operad.

We will now give an overview of our construction of $\text{FM}_2^W (k)$. This is an expansion of step (ii) in the overview we gave in Section 1.3, and we label the parts accordingly:

(iia) Each open Getzler–Jones cell in $\text{FM}_2^W (k)$ can be identified with a product of open 2–associahedra, ie a product of the form $\hat{W}_{m_1} \times \cdots \times \hat{W}_{m_a}$ (where “$\hat{X}$” is our notation for the interior of a space $X$). For each such open cell, we replace these 2–associahedra by their $W$–construction equivalents thusly: $\hat{W}_{m_1} \times \cdots \times \hat{W}_{m_a}$. This product comes with the product CW structure, and we refine this in a way that endows $\hat{W}_{m_1} \times \cdots \times \hat{W}_{m_a}$ with the structure of a simplicial complex.

(iib) While an open Getzler–Jones cell can be identified with a product $\hat{W}_{m_1} \times \cdots \times \hat{W}_{m_a}$ of 2–associahedra, their compactifications (in $\text{FM}_2^W (k)$ and $W_{m_1} \times \cdots \times W_{m_a}$, respectively) are different: the compactification of the former is smaller than the compactification of the latter. This is reflected in how we glue our products $\hat{W}_{m_1} \times \cdots \times \hat{W}_{m_a}$ together. Specifically, we perform this gluing by applying a quotient map to each simplex in the boundary of $W_{m_1} \times \cdots \times W_{m_a}$. This quotient map is closely related to the maps $f_\sigma: W_n \to \text{FM}_2^W (k)$ that we described in Section 1.4: they reflect the fact that the compactification used to define $W_n$ allows lines with no marked points, whereas the compactification of a Getzler–Jones cell does not allow this.

The following is the main result of this section, which we stated in the introduction and record again here:

**Main Theorem** The spaces $(\text{FM}_2^W (k))_{k \geq 1}$ together with the composition operations $\circ_i$ defined in Definition 3.11 form a non-$\Sigma$ operad, and the composition maps

\begin{equation}
\circ_i: \text{FM}_2^W (k) \times \text{FM}_2^W (l) \to \text{FM}_2^W (k + l - 1)
\end{equation}

are cellular.

**Proof** Combine Lemmata 3.12 and 3.13 below. \hfill \square

3.1 Quotient maps on 2–associahedra

Before we can define the quotient involved in (24), we will define for every cell $F$ in $\partial W_n^W$ a map $q_F$ from $F$ to a certain product of 2–associahedra, where this target will vary for difference choices of $F$.

We begin with two preliminary definitions:

**Definition 3.1** Fix $r \geq 1$ and $n \in \mathbb{Z}_{\geq 0} \setminus \{0\}$, and fix $i \in [1, r]$ such that $n_i = 0$. Define $\tilde{n} := (n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_r)$. We then define a map of posets $\pi^{\text{tree}}_i: W_n^{\text{tree}} \to W_{\tilde{n}}^{\text{tree}}$ by applying the following procedure to $2T = T_b \overset{f}{\to} T_s \in W_n^{\text{tree}}$:

*Algebraic & Geometric Topology, Volume 24 (2024)*
(i) Denote by $e_0$ the edge in $T_s$ incident to the $i^{th}$ leaf $x_i^{T_s}$. If $e$ is a solid edge in $T_b$ that is mapped identically under $f$ to $e_0$, then we delete $e$. Next, we delete $e_0$. We modify $f$ in the obvious way.

(ii) After performing these deletions, our tree-pair may no longer be stable. We rectify this in $T_b$ (resp. $T_s$) by performing the contractions indicated on the left (resp. right):

```
  |   →   |
```

Specifically, we perform these contractions as many times as necessary for the tree-pair to be stable. Denoting the end result of this procedure by $\tilde{2T}$, we define $\pi_i^{\text{tree}}(2T) := \tilde{2T}$.

Next, we define another map of posets. Fix $r \geq 1$ and $n \in \mathbb{Z}_{\geq 0} \setminus \{0\}$. Denote by $\tilde{n}$ the result of deleting all the zeroes from $n$, and set $\tilde{r}$ to be the length of $\tilde{n}$. We define $\pi^{\text{tree}}_n : W_n^{\text{tree}} \to W_{\tilde{n}}^{\text{tree}}$ by applying the map $\pi_i^{\text{tree}}$ once for each $i$ with $n_i = 0$.

It is not hard to check that the choices implicit in this definition do not matter, and that the resulting maps are indeed maps of posets.

**Definition 3.2** Fix $r \geq 1$ and $n \in \mathbb{Z}_{\geq 0} \setminus \{0\}$. We define a map $\pi^W : W_n^W \to W_{\tilde{n}}^W$ in the same fashion as $\pi^{\text{tree}}$, with the provision that when we contract adjacent edges of lengths $\ell_1$ and $\ell_2$ (whether in $T_b$ or $T_s$) we equip the resulting edge with length $\max(\ell_1, \ell_2)$.

Next, we recall a $W$–version analogue of two properties of the 2–associahedra:

**$W$–version analogue of the forgetful property of** [Bottman 2019a, Theorem 4.1] Fix $r \geq 1$ and $n \in \mathbb{Z}_{\geq 0} \setminus \{0\}$. There is a surjection $W_n^W \to K_r^W$ which sends a metric stable tree-pair $(T_b \xrightarrow{f} T_s, (L_e), (\ell_e))$ to the metric stable RRT $(T_s, (\ell_e))$.

**$W$–version analogue of the recursive property of** [Bottman 2019a, Theorem 4.1] Fix a stable tree-pair $2T = T_b \xrightarrow{f} T_s \in W_n^{\text{tree}}$. There is an inclusion of CW complexes

$$\Gamma_2^T : \prod_{\alpha \in V_{\text{comp}}^1(T_b) \setminus \text{in}(\alpha) = (\beta)} W_{\#_{\text{in}}(\beta)}^W \times \prod_{\rho \in V_{\text{in}}(T_s)} \prod_{\alpha \in V_{\text{comp}}^\geq 2(T_b) \setminus f^{-1}\{\rho\} \setminus \text{in}(\alpha) = (\beta_1, \ldots, \beta_{\#_{\text{in}}(\rho)})} W_{\#_{\text{in}}(\beta_1), \ldots, \#_{\text{in}}(\beta_{\#_{\text{in}}(\rho)})}^{W} \hookrightarrow W_n^W,$$

where the superscript on one of the product symbols indicates that it is a fiber product with respect to the maps in the description of the forgetful property above.

The map $\Gamma_2^T$ defined in [Bottman 2019a], which is defined for the posets $W_n^{\text{tree}}$, is defined by attaching stable tree-pairs together in a way specified by the stable tree-pair $2T$. This map is similar, but we are attaching together metric stable tree-pairs. We assign the length 1 to the edges along which we attach the trees. (The image of $\Gamma_2^T$ is a union of cells in $\partial W_n^W$.)
We can now define the quotient maps $q_F$ on $W_n^W$:

**Definition 3.3** Fix $r \geq 1$, $n \in \mathbb{Z}_{\geq 0} \setminus \{0\}$, a stable type-$n$ tree-pair $\tilde{2T}$, and a face $F$ of the associated cell $C_{\tilde{2T}}$ in $W_n^W$ with the property that $F$ lies in $\partial W_n^W$. (Equivalently, the metric tree-pairs in $F$ have at least one length that is identically equal to 1.) The *quotient map associated to $F$* is a map $q_F$ from $F$ to a product of $2$–associahedra. Given a metric stable tree-pair $(2T, (L_e), (\ell_e))$, we define its image under $\pi$ in the following fashion:

(i) Break up $T_b$ and $T_s$ along the edges that are identically 1 in $F$. Equivalently, choose $2T$ of minimal dimension with the property that $F$ lies in the image of $2T$, then identify $F$ as a top cell in a product of fiber products of the following form:

\[
\prod_{\alpha \in V_{\text{comp}}(T_b)} W^{W}_{\#_{\text{in}}(\beta)} \times \prod_{\rho \in V_{\text{in}}(T_s)} \prod_{\alpha \in V_{\text{comp}}(T_s) \cap f^{-1}(\rho)} W^{W}_{\#_{\text{in}}(\beta_1), \ldots, \#_{\text{in}}(\beta_{\#_{\text{in}}(\alpha)})}.
\]

As a result, we obtain a list of metric stable tree-pairs, which we can regard as lying inside a product $W_{m_1}^W \times \cdots \times W_{m_a}^W$.

(ii) We then apply the map $\pi^W$ to each of the factors in the product just recorded, hence producing an element of $W_{m_1}^W \times \cdots \times W_{m_a}^W$. (As in Definitions 3.1 and 3.2, $m^i$ denotes the result of removing the 0s from $m^i$.)

Note that for two cells $F_1$ and $F_2$ in the boundary of $W_n^W$, the targets of $q_{F_1}$ and $q_{F_2}$ are typically different.

**Example 3.4** In Figure 9, we illustrate several things about $W_{21}^W$. Initially, $W_{21}^W$ is an octagon, decomposed into eight squares; this is indicated by the black lines. The simplicial refinement divides each square into two 2–simplices. We have indicated the metric tree-pairs that correspond to each of the eight squares, as well as those corresponding to the sixteen 1–simplices that comprise $\partial W_{21}^W$. (Some dashed edges are not labeled; these should be interpreted as having length max($a, b$).)

Finally, we have indicated the behavior of the quotient maps on $W_{21}^W$. These maps are the identity on every edge except for those indicated in red. Each pair of red edges is contracted to a point. One reflection of this is that in Example 1.1, the octagons in $W_{111}$ are taken to the (cellular) hexagons in the Getzler–Jones cell indicted on the right.

### 3.2 The construction of $\text{FM}_2^W(k)$

In this subsection, we tackle the construction of $\text{FM}_2^W(k)$. First, we will describe our version of the Getzler–Jones cells. Next, we will explain how to glue these spaces together.

To define the Getzler–Jones cells, we must introduce 2–permutations, which will allow us to enforce the alignment and ordering of special points on screens as in Figure 1.

*Algebraic & Geometric Topology, Volume 24 (2024)*
Definition 3.5  Fix a finite set $A$. A 2–permutation $\sigma$ on $A$ is the data

- an ordered decomposition

\begin{equation}
A = A_1 \sqcup \cdots \sqcup A_r,
\end{equation}

where $A_r$ is allowed to be empty, and

- for each $i$, a linear order on $A_i$.

We define the type of $\sigma$ to be the vector $n := (|A_1|, \ldots, |A_r|)$. If $\sigma$ is a 2–permutation whose type $n$ has no zero entries, then we say that $\sigma$ has no empty part.
Remark 3.6  A type-$(1,\ldots, 1)$ 2–permutation is exactly the data of a permutation on $r$ letters. The same is true of a type-$(n)$ 2–permutation.

Next, we define a Getzler–Jones datum, the set of which indexes the Getzler–Jones cells in $\text{FM}^{W}_2(k)$.

Definition 3.7  Fix $k \geq 2$. A Getzler–Jones datum consists of

- a stable rooted tree $T$ with $k$ leaves, together with a numbering of its leaves from 1 through $k$, and
- for every interior vertex $v \in T_{\text{int}}$, a 2–permutation $\sigma$ on its incoming vertices $V_{\text{in}}(T)$ such that $\sigma$ has no empty part.

We denote the type of the 2–permutation associated to $v$ by $n(v)$. We will abuse notation and denote the entire Getzler–Jones datum by $T$.

Finally, we can define the Getzler–Jones cells of type $k$:

Definition 3.8  Fix $k \geq 2$ and a Getzler–Jones datum $T$. Then we define

\[
\text{GJ}_T := \prod_{v \in T_{\text{int}}} W_{n(v)}^W \quad \text{and} \quad \tilde{\text{GJ}}_T := \prod_{v \in T_{\text{int}}} W_{n(v)}^W.
\]

We call $\text{GJ}_T$ the Getzler–Jones cell $\text{GJ}_T$ associated to $T$, and refer to $\text{GJ}_T$ as a type-$k$ Getzler–Jones cell.

In Lemma 2.18 we equipped $W_{n}^W$ with the structure of a simplicial complex, which induces a CW structure on $\text{GJ}_T$ and $\tilde{\text{GJ}}_T$. We refine these to equip $\text{GJ}_T$ and $\tilde{\text{GJ}}_T$ with simplicial decompositions, in the fashion of Lemma 2.18.

Remark 3.9  The reason why we do not refer to $\tilde{\text{GJ}}_T$ as a “closed Getzler–Jones cell” is because it is not the closure in $\text{FM}^{W}_2(k)$ of $\text{GJ}_T$. In fact, it is larger than this closure. Our reason for making this second definition is that $\text{GJ}_T$ will be an integral part of our definition of $\text{FM}^{W}_2(k)$.

We will define $\text{FM}^{W}_2(k)$ as a quotient of the following form, where $T$ varies over type-$k$ Getzler–Jones data:

\[
\text{FM}^{W}_2(k) := \left( \bigsqcup_T \tilde{\text{GJ}}_T \right) / \sim.
\]

The remaining ingredient is the collection of maps that we will use to attach these spaces. As a consequence of the definition of these maps, $\text{FM}^{W}_2(k)$ will decompose as a set into the union of all type-$k$ Getzler–Jones cells.

Finally, we come to the definition of $\text{FM}^{W}_2(k)$:

Definition 3.10  Fix $k \geq 2$. We construct $\text{FM}^{W}_2(k)$ like so:

(i)  Begin with the following disjoint union, where $T$ varies over type-$k$ Getzler–Jones data:

\[
\bigsqcup_T \text{GJ}_T.
\]
Fix a type-$k$ Getzler–Jones datum $T$, and fix a cell $F$ in the boundary of $\tilde{GJ}_T = \coprod_{v \in T_{\text{int}}} W^W_{n(v)}$. $F$ lies inside a product of cells in the 2–associahedra that comprise $\tilde{GJ}_T$—that is, we may write $F \subset \coprod_{v \in T_{\text{int}}} F_v \subset \coprod_{v \in T_{\text{int}}} W^W_{n(v)}$, where $F_v$ is a cell in $W^W_{n(v)}$. For every $v$, we have a map $q_v$ from $W^W_{n(v)}$ to a product of 2–associahedra; by combining these, we obtain a map from $F$ to a product of 2–associahedra. In fact, we can regard the target of this map as a Getzler–Jones cell.

(iii) We take the quotient of the disjoint union in (25) by attaching the constituent spaces together via the maps we defined in the last step.

We define $\text{FM}_2^W(1)$ to be a point.

It is a consequence of the simplicial structure of the $\tilde{GJ}_T$ that each $\text{FM}_2^W(k)$ has the structure of a CW complex. As noted above, a result of our definition is that $\text{FM}_2^W(k)$ decomposes as a union of Getzler–Jones cells, over all Getzler–Jones data of type $k$.

### 3.3 The operad structure on $\text{FM}_2^W$

**Definition 3.11** Fix $k$, $l$, and $i \in [1, k]$. We wish to define the map

$$
\circ_i : \text{FM}_2^W(k) \times \text{FM}_2^W(l) \to \text{FM}_2^W(k + l - 1).
$$

To do so, fix Getzler–Jones data $T$ and $T'$ of types $k$ and $l$, respectively, and fix cells $F \subset \text{GJ}_T$ and $F' \subset \text{GJ}_{T'}$. We will define $\circ_i$ on

$$
\text{GJ}_T \times \text{GJ}_{T'} = \coprod_{v \in T_{\text{int}} \sqcup T'_{\text{int}}} W^W_{n(v)}.
$$

Define $T'''$ to be the result of grafting $T'$ to the $i$th leaf of $T$, and completing it to a Getzler–Jones datum in the obvious way. We define $\circ_i$ on $\text{GJ}_T \times \text{GJ}_{T'}$ to be the identification of $\text{GJ}_T \times \text{GJ}_{T'}$ with $\text{GJ}_{T''}$.

**Lemma 3.12** Taken together, the spaces $(\text{FM}_2^W(k))_{k \geq 1}$ together with the composition operations $\circ_i$ form a non-$\Sigma$ operad.

**Proof** This is immediate from the definition. □

**Lemma 3.13** The composition maps

$$
\circ_i : \text{FM}_2^W(k) \times \text{FM}_2^W(l) \to \text{FM}_2^W(k + l - 1)
$$

are cellular.

**Proof** This is similar to the proof of Proposition 2.1. □

**Remark 3.14** Fix $r \geq 1$, $n \in \mathbb{Z}^r_{\geq 0} \setminus \{0\}$, and a 2–permutation $\sigma$ of type $n$. Then the associated forgetful map

$$
f^W_{\sigma} : W^W_n \to \text{FM}_2^W(|n|)
$$

Algebraic & Geometric Topology, Volume 24 (2024)
is cellular. This map is defined in the obvious way: we first identify $W_n^W$ with the corresponding $\tilde{G}_T$, where $T$ is a Getzler–Jones datum whose associated tree $T$ is a corolla with $|n|$ leaves. Then, we include $\tilde{G}_T$ into the disjoint union $\bigsqcup_T \tilde{G}_T$, and finally take the quotient to land in $F^W_{m^W(|n|)}$.

References

[Abouzaid 2015] M Abouzaid, Sympolctic cohomology and Viterbo’s theorem, from “Free loop spaces in geometry and topology” (J Latschew, A Oancea, editors), IRMA Lect. Math. Theor. Phys. 24, Eur. Math. Soc., Zürich (2015) 271–485 MR Zbl

[Barber 2013] D A Barber, A comparison of models for the Fulton–MacPherson operads, PhD thesis, University of Sheffield (2013) Available at https://core.ac.uk/download/pdf/131322152.pdf

[Bottman 2015] N S Bottman, Pseudoholomorphic quilts with figure eight singularity, PhD thesis, Massachusetts Institute of Technology (2015) Available at http://hdl.handle.net/1721.1/101823

[Bottman 2019a] N Bottman, 2–Associahedra, Algebr. Geom. Topol. 19 (2019) 743–806 MR Zbl

[Bottman 2019b] N Bottman, Moduli spaces of witch curves topologically realize the 2–associahedra, J. Symplectic Geom. 17 (2019) 1649–1682 MR Zbl

[Bottman 2020] N Bottman, Pseudoholomorphic quilts with figure eight singularity, J. Symplectic Geom. 18 (2020) 1–55 MR Zbl

[Bottman and Carmeli 2021] N Bottman, S Carmeli, $(A_\infty, 2)$–categories and relative 2–operads, High. Struct. 5 (2021) 401–421 MR Zbl

[Bottman and Oblomkov 2019] N Bottman, A Oblomkov, A compactification of the moduli space of marked vertical lines in $C^2$, preprint (2019) arXiv 1910.02037

[Bottman and Wehrheim 2018] N Bottman, K Wehrheim, Gromov compactness for squiggly strip shrinking in pseudoholomorphic quilts, Selecta Math. 24 (2018) 3381–3443 MR Zbl

[Fox and Neuwirth 1962] R Fox, L Neuwirth, The braid groups, Math. Scand. 10 (1962) 119–126 MR Zbl

[Fulton and MacPherson 1994] W Fulton, R MacPherson, A compactification of configuration spaces, Ann. of Math. 139 (1994) 183–225 MR Zbl

[Getzler and Jones 1994] E Getzler, J D S Jones, Operads, homotopy algebra and iterated integrals for double loop spaces, preprint (1994) arXiv hep-th/9403055

[Ma'u et al. 2018] S Ma'u, K Wehrheim, C Woodward, $A_\infty$ functors for Lagrangian correspondences, Selecta Math. 24 (2018) 1913–2002 MR Zbl

[Salvatore 2022] P Salvatore, A cell decomposition of the Fulton MacPherson operad, J. Topol. 15 (2022) 443–504 MR Zbl

[Voronov 2000] A A Voronov, Homotopy Gerstenhaber algebras, from “Conférence Moshé Flato 1999, II: Quantization, deformations, and symmetries” (G Dito, D Sternheimer, editors), Math. Phys. Stud. 22, Kluwer, Dordrecht (2000) 307–331 MR Zbl

Max Planck Institute for Mathematics
Bonn, Germany
natebottman@gmail.com

Received: 2 April 2022 Revised: 27 August 2022

Geometry & Topology Publications, an imprint of mathematical sciences publishers
See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2024 is US $705/year for the electronic version, and $1040/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers
nonprofit scientific publishing

https://msp.org/

© 2024 Mathematical Sciences Publishers
Comparing combinatorial models of moduli space and their compactifications

DANIELA EGAS SANTANDER and ALEXANDER KUPERS

Towards a higher-dimensional construction of stable/unstable Lagrangian laminations

SANGJIN LEE

A strong Haken theorem

MARTIN SCHARLEMMANN

Right-angled Artin subgroups of right-angled Coxeter and Artin groups

PALLAVI DANI and IVAN LEVCOVITZ

Filling braided links with trisected surfaces

JEFFREY MEIER

Equivariantly slicing strongly negative amphichiral knots

KEEGAN BOYLE and AHMAD ISSA

Computing the Morava K–theory of real Grassmannians using chromatic fixed point theory

NICHOLAS J KUHN and CHRISTOPHER J R LLOYD

Slope gap distributions of Veech surfaces

LUIS KUMANDURI, ANTHONY SANCHEZ and JANE WANG

Embedding calculus for surfaces

MANUEL KRANNICH and ALEXANDER KUPERS

Vietoris–Rips persistent homology, injective metric spaces, and the filling radius

SUNHYUK LIM, FACUNDO MÉMOLI and OSMAN BERAT OKUTAN

Slopes and concordance of links

ALEX DEGTYAREV, VINCENT FLORENS and ANA G LECUONA

Cohomological and geometric invariants of simple complexes of groups

NANSEN PETROSYAN and TOMASZ PRTUŁA

On the decategorification of some higher actions in Heegaard Floer homology

ANDREW MANION

A simplicial version of the 2–dimensional Fulton–MacPherson operad

NATHANIEL BOTTMAN

Intrinsically knotted graphs with linklessly embeddable simple minors

THOMAS W MATTMAN, RAMIN NAIMI, ANDREI PAVELESCU and ELENA PAVELESCU