DEFORMATIONS OF SMOOTH FUNCTION ON 2-TORUS WHOSE KR-GRAPH IS A TREE

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Abstract. Let $f : T^2 \to \mathbb{R}$ be Morse function on 2-torus $T^2$, and $O(f)$ be the orbit of $f$ with respect to the right action of the group of diffeomorphisms $D(T^2)$ on $C^\infty(T^2)$. Let also $O_f(f, X)$ be a connected component of $O(f, X)$ which contains $f$. In the case when Kronrod-Reeb graph of $f$ is a tree we obtain the full description of $\pi_1 O_f(f)$.

This result also holds for more general class of smooth functions $f : T^2 \to \mathbb{R}$ which have the following property: for each critical point $z$ of $f$ the germ $f$ of $z$ is smoothly equivalent to some homogeneous polynomial $\mathbb{R}^2 \to \mathbb{R}$ without multiple points.

1. Introduction

Let $M$ be a smooth compact surface, $X$ be a closed (possible empty) subset of $M$, $D(M, X)$ be the group of diffeomorphisms of $M$ fixed on some neighborhood of $X$. Then the group $D(M, X)$ acts on the space of smooth functions $C^\infty(M)$ on $M$ by the following rule:

$$\gamma : C^\infty(M) \times D(M, X) \to C^\infty(M), \quad \gamma(f, h) = f \circ h. \quad (1.1)$$

For $f \in C^\infty(M)$, let

$$S(f, X) := \{ h \in D(M, X) \mid f \circ h = f \}, \quad \text{and} \quad O(f, X) := \{ f \circ h \mid h \in D(M, X) \}$$

be respectively the stabilizer and the orbit of $f$ with respect to the action $\gamma$.

If $X$ is the empty set, then we put

$$D(M) := D(M, \emptyset), \quad S(f) := S(f, \emptyset), \quad O(f) := O(f, \emptyset),$$

and so on. Endow spaces $D(M, X)$ and $C^\infty(M)$ with the corresponding Whitney topologies; these topologies induce certain topologies on $S(f, X)$ and $O(f, X)$.

Let $S_{id}(f, X)$ and $D_{id}(f, X)$ be respectively connected components of $S(f, X)$ and $D(f, X)$ which contain the identity map $id_M$, and $O_f(f, X)$ be the connected component of $O(f, X)$ containing $f$. We also set $S'(f, X) = S(f) \cap D_{id}(M, X)$.

Let $\mathcal{F}(M) \subset C^\infty(M)$ be a set of smooth functions satisfying the following conditions:

(B) the function $f$ takes the constant value at each connected component of the boundary $\partial M$;

(P) for each critical point $z$ of $f$ the germ of $f$ in $z$ is smoothly equivalent to some homogeneous polynomial $f_z : \mathbb{R}^2 \to \mathbb{R}$ without multiple factors.

Let $\text{Morse}_0(M)$ be the set of Morse functions satisfying condition (B). It is known that the set $\text{Morse}_0(M)$ is a dense subset of $C^\infty(M)$. By Morse Lemma, every non-degenerate singularity of Morse function is smoothly equivalent to the polynomial $\pm x^2 \pm y^2$ in some chart representation around this critical point. Thus, we have the inclusion $\text{Morse}_0(M) \subset \mathcal{F}(M) \subset C^\infty(M)$.

Theorem 1.1 ([7, 9, 10]). Let $f \in \mathcal{F}(M)$ and $X$ be a finite (possible empty) union of regular components of level-sets of $f$. Then the following statements hold:

1. the map $p : D(M, X) \to O(f, X)$, $p(h) = f \circ h$ is a Serre fibration with the fiber $S(f, X)$;

2. the restriction $p|_{D_{id}(M, X)} : D_{id}(M, X) \to O_f(f, X)$ of the map $p$ is also a Serre fibration;

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3. consider that \(X = \emptyset\) and either \(f\) has the critical point which is not non-degenerate local extremum, or \(M\) is non-orientable. Then \(S_{\alpha}(f)\) is contractible, \(\pi_nO_f(f) \cong \pi_nM\), for \(n \geq 3\), \(\pi_2O_f(f) = 0\), and for \(\pi_1O_f(f)\) we have the following sequence

\[1 \to \pi_1D_{\text{id}}(M) \overset{p}{\to} \pi_1O_f(f) \overset{\partial}{\to} \pi_0S'(f) \to 1;\]  
(1.2)

4. consider that \(\chi(M) < 0\) or \(X \neq \emptyset\). Then \(D_{\text{id}}(M, X)\) and \(S_{\alpha}(f, X)\) are contractible, \(\pi_nO_f(f, X) = 0\) for \(n \geq 2\), and the map \(\partial : \pi_1O_f(f, X) \to \pi_0S'(f, X)\) is the isomorphism.

In the serious of papers [7, 8, 9, 10, 11, 12, 13] Maksymenko described homotopy type of stabilizers of the action (1.1).

Maksymenko and the author in papers [14, 15] describes fundamental group of the orbit \(\pi_1O_f(f)\) of the function \(f\) in \(\mathcal{F}(\mathbb{T}^2)\) in the case when KR-graph of the function \(f\) has a cycle. In the case when KR-graph of \(f\) is a tree, the authors [6] found conditions under which the sequence (1.2) splits. The aim of the paper is to describe the group \(\pi_1O_f(f)\) of the function \(f\) from \(\mathcal{F}(\mathbb{T}^2)\) whose KR-graph is a tree.

**Remark 1.2.** Let also \(w : (Ik^{k}, \partial Ik^{k}, 0) \to (D_{\text{id}}(M, X), S_{\alpha}(f, X), \text{id}_M)\) be a continuous map of triples, \(k \geq 0\). From (2) of Theorem 1.1 follows that for every \(k \geq 0\) there is an isomorphism

\[\lambda_k : \pi_k(D_{\text{id}}(M, X), S_{\alpha}(f, X), \text{id}_M) \to \pi_kO_f(f, X), \quad \lambda_k([\omega]) = [f \circ \omega],\]

see for example [5, § 4.1, Theorem 4.1]. In the text we will identify \(\pi_1O_f(f)\) with \(\pi_1(D_{\text{id}}(M), S'(f))\).

**Structure of the paper.** In Section 2 we collects some preliminaries: wreath products of special type, needed for formulation of the result, Kronrod-Reeb graph of smooth functions and its automorphism group. The main result of the paper is Theorem 2.5 which is proved in Section 3.

## 2. Preliminaries

### 2.1. Wreath products \(G \wr_{\omega \times \omega} \mathbb{Z}^2\).

Let \(G\) be a group, and \(1\) be the unit of \(G\), Map(\(\mathbb{Z}_n \times \mathbb{Z}_m, G\)) be the group of all maps \(\mathbb{Z}_n \times \mathbb{Z}_m \to G\) with the point-wise multiplication, i.e., for \(\alpha, \beta : \mathbb{Z}_n \times \mathbb{Z}_m \to G\) from Map(\(\mathbb{Z}_n \times \mathbb{Z}_m, G\)) we have \((\alpha \beta)(i,j) = \alpha(i,j)\beta(i,j)\), where \((i,j) \in \mathbb{Z}_n \times \mathbb{Z}_m\), \(n, m \geq 1\).

The group \(\mathbb{Z}^2\) acts from the right on Map(\(\mathbb{Z}_n \times \mathbb{Z}_m, G\)) be the following rule: if \(\alpha \in \text{Map}(\mathbb{Z}_n \times \mathbb{Z}_m, G)\) and \((k, j) \in \mathbb{Z}^2\), then the result of this action \(\alpha^{k,j}\) is given by

\[\alpha^{k,j}(i, j) = \alpha(i + k \cdot \text{mod} \; n, j + l \cdot \text{mod} \; m), \quad (i, j) \in \mathbb{Z}^2.\]

The semi-direct product Map(\(\mathbb{Z}_n \times \mathbb{Z}_m, G) \rtimes \mathbb{Z}^2\), which corresponds to this \(\mathbb{Z}^2\)-action we denote by

\[G \wr_{\omega \times \omega} \mathbb{Z}^2 \triangleq \text{Map}(\mathbb{Z}_n \times \mathbb{Z}_m, G) \rtimes \mathbb{Z}^2,\]

and will call the wreath product of \(G\) and \(\mathbb{Z}^2\) under \(\mathbb{Z}_n \times \mathbb{Z}_m\).

Thus \(G \wr_{\omega \times \omega} \mathbb{Z}^2\) is a Cartesian product with the operation:

\[(\alpha, (k_1, k_2))(\beta, (l_1, l_2)) = (\alpha \beta^{k_1,k_2}, (k_1 + l_1, k_2 + l_2))\]

for all \((\alpha, (k_1, k_2)), (\beta, (l_1, l_2)) \in \text{Map}(\mathbb{Z}_n \times \mathbb{Z}_m, G) \times \mathbb{Z}^2\). Moreover we have the following exact sequence

\[1 \to \text{Map}(\mathbb{Z}_n \times \mathbb{Z}_m, G) \overset{\sigma}{\to} G \wr_{\omega \times \omega} \mathbb{Z}^2 \overset{p}{\to} \mathbb{Z}^2 \to 1,\]

where \(\sigma(\alpha) = (\alpha, (0, 0))\) is the inclusion, and \(p(\alpha, (a_1, a_2)) = (a_1, a_2)\) is the projection.

### 2.2. Kronrod-Reeb graph of smooth functions.

Let \(f \in \mathcal{F}(M)\) and \(c\) be a real number. Recall that a connected component \(C\) of level-set \(f^{-1}(c)\) is called *critical* if \(C\) contains at almost one critical point of \(f\), otherwise \(C\) is called *regular*.

Let \(\Delta\) be a partition of \(M\) onto connected components of level-sets of the function \(f\). It is well known that the quotient-space \(M/\Delta\) is 1-dimensional CW complex called Kronrod-Reeb graph of \(f\), or simply, KR-graph of \(f\); we will denote it by \(\Gamma_f\). From the definition of \(\Gamma_f\) follows that vertices of
\( \Gamma_f \) are critical components of level-sets of \( f \). Let also \( p_f : M \to \Gamma_f \) be the projection map. So, the function \( f \) can be represented as the following composition:

\[
\begin{align*}
  f &= \phi_f \circ p_f : M \xrightarrow{p_f} \Gamma_f \xrightarrow{\phi_f} \mathbb{R},
\end{align*}
\]

where \( \phi_f \) is the map induced by \( f \). Note that \( \phi_f \) is monotone on the set of edges of \( \Gamma_f \).

### 2.3. Actions of \( S(f) \) on \( \Gamma_f \)

Let \( f \) be a smooth function from \( \mathcal{F}(M) \), and \( h \in S'(f) \). Then \( f \circ h = f \) by definition of \( h \), and hence, \( h(f^{-1}(c)) = f^{-1}(c) \) for all \( c \in \mathbb{R} \). Then \( h \in S'(f) \) interchanges level-sets of \( f \). So, \( h \) induces the homomorphism \( \rho(h) \) of KR-graph \( \Gamma_f \) such that the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{p_f} & \Gamma_f & \xrightarrow{\phi_f} & \mathbb{R} \\
\downarrow{h} & & \downarrow{\rho(h)} & & \\
M & \xrightarrow{p_f} & \Gamma_f & \xrightarrow{\phi_f} & \mathbb{R}
\end{array}
\]

is commutative. In other words, we have a homomorphism: \( \rho : S'(f) \to \text{Aut}(\Gamma_f) \). Denote by \( G \) the image of \( S'(f) \) in \( \text{Aut}(\Gamma_f) \) under the map \( \rho \).

Let \( v \) be the vertex of \( \Gamma_f \) and \( G_v = \{ g \in G \mid g(v) = v \} \) be a stabilizer of \( v \) under the action \( G \). By star \( \text{st}(v) \) of the vertex \( v \) we will mean a closed connected \( G_v \)-invariant neighborhood of \( v \) in \( \Gamma_f \), which does not contains the other vertices of \( \Gamma_f \). The set \( G_v^{\text{loc}} = \{ g|_{\text{st}(v)} \mid g \in G_v \} \) is a subgroup of \( \text{Aut}(\text{st}(v)) \), which consists of the restrictions of elements of \( G_v \) onto \( \text{st}(v) \). We will call it the local stabilizer of \( v \) under the action of \( G \). Note that the group \( G_v^{\text{loc}} \) does not depends on the choice of the star \( \text{st}(v) \) of the vertex \( v \). In particular, the following diagram

\[
\begin{array}{ccc}
S'(f) & \xrightarrow{\rho} & G & \xrightarrow{\text{Aut}(\Gamma_f)} \\
\downarrow{p_f} & & \downarrow{\rho_0} & & \\
\pi_0S'(f) & \xrightarrow{\bar{\rho}} & G_v^{\text{loc}} & \xrightarrow{\text{Aut}(\text{st}(v))} \\
\end{array}
\]

is commutative, where \( p \) is a projection, \( r \) is the restriction map onto \( \text{st}(v) \), \( \rho = \rho_0 \circ r \), and \( \bar{\rho} = r \circ \rho \).

For the function \( f \in \mathcal{F}(T^2) \) on 2-torus the following result holds:

**Lemma 2.4** (Proposition 1, [6]). Let \( f \in \mathcal{F}(T^2) \) be such that its KR-graph \( \Gamma_f \) is a tree. Then there exists the unique vertex \( v \) of the graph \( \Gamma_f \) such that each component of \( T^2 - p_f^{-1}(v) \) is an open 2-disk.

The vertex \( v \) of \( \Gamma_f \) and the critical component \( V = p_f^{-1} \) of \( f^{-1}(\phi_f(v)) \), which corresponds to \( v \) we will call special.

The main result of our paper is the following result:

**Theorem 2.5.** Let \( f \in \mathcal{F}(T^2) \) be such that \( \Gamma_f \) is a tree, and \( v \) be the special vertex of \( \Gamma_f \). Then

1. \( G_v^{\text{loc}} \cong \mathbb{Z}_n \times \mathbb{Z}_m \) for some \( n, m \in \mathbb{N} \);
2. there exist closed 2-disks \( \{ D_1, D_2, \ldots, D_r \} \subset T^2 \) such that \( f|_{D_i} \in \mathcal{F}(D_i) \), \( i = 1, 2, \ldots, r \), and there is an isomorphism

\[
\xi : \pi_1 \mathcal{O}_f(f) \cong \prod_{i=1}^{r} \pi_0 S'(f|_{D_i}, \partial D_i) \cong \mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}^2.
\]

In particular, in the case \( G_v^{\text{loc}} = 1 \) we have the isomorphism \( \xi : \pi_1 \mathcal{O}_f(f) \cong \pi_0 S'(f) \times \mathbb{Z}^2 \), see Theorem 2 [6].
2.6. Combinatorial actions of finite groups on surfaces. Now we recall some results from [3]. Let \( f \in \mathcal{F}(M) \). Suppose that its Kronrod-Reeb graph \( \Gamma_f \) contains a special vertex \( v \), and \( V \) be the special component of level set of \( f \) which corresponds to \( v \).

Let \( S_V(f) = \{ h \in S(f) \mid h(V) = V \} \) be a subgroup of \( S(f) \) leaving \( V \) invariant. It is easy to see that \( \rho(S_V(f)) \subseteq G_v \). We denote by \( \phi \) the map

\[
\phi = r \circ \rho : S_V(f) \xrightarrow{\rho} G_v \xrightarrow{r} G_v^{loc}.
\]

Let \( H \) be a subgroup of \( G_v^{loc} \) and \( \mathcal{H} = \phi^{-1}(H) \) be a subgroup of \( S_V(f) \). We will say that the group \( \mathcal{H} \) has property (C) if the following conditions hold.

(C) Let \( h \in \mathcal{H} \), and \( E \) be a 2-dimensional element of \( \Xi \). Suppose that \( h(E) = E \). Then \( h(e) = e \) for all other \( e \in \Xi \), and the map \( h \) preserves orientation of each element of \( \Xi \).

Proposition 2.7 (Theorem 2.2 [3]). Suppose \( f \in \mathcal{F}(M) \) is such that its KR-graph \( \Gamma_f \) contains a special vertex \( v \), and \( G_v^{loc} \) be the local stabilizer of \( v \). Let also \( H \) be a subgroup of \( G_v^{loc} \), and \( \mathcal{H} = \phi^{-1}(H) \) be a subgroup of \( S_V(f) \) satisfying condition (C). Then there exists a section \( s : H \rightarrow \mathcal{H} \) of the map \( \phi \), i.e., the map \( s \) is a homomorphism satisfying the condition \( \phi \circ s = \text{id}_H \).

3. Proof of statement (1) of Theorem 2.5

Let \( f \in \mathcal{F}(T^2) \) be such that its KR-graph \( \Gamma_f \) is a tree, and \( v \) is the special vertex of the graph \( \Gamma_f \), and \( V \) be the connected component of \( f \), which corresponds to \( v \). We need to show that \( G_v^{loc} \cong \mathbb{Z}_n \times \mathbb{Z}_m \) for some \( n, m \in \mathbb{N} \).

Note that from Lemma 2.4 follows that \( V \) gives the partition of \( T^2 \): 0-dimensional and 1-dimensional cells are vertices and edges of \( V \) respectively, and 2-dimensional cells are connected component of \( T^2 \setminus V \).

From [7, Theorem 7.1] follows that for each \( h \in \ker(r \circ \rho) \) the following conditions hold:

1. \( h(e) = e \) for each cell \( e \),
2. the map \( h : e \mapsto h(e) \) preserves orientations of cells \( e \) of dimension \( \dim e \geq 1 \).

Let \( h \in S'(f) \) be a diffeomorphism. According to [9, Proposition 5.4] either all cells are \( h \)-invariant, or the number of invariant cells under \( h \) is equal to Lefschetz number \( L(h) \). Since \( h \) is isotopic to the identity map, it follows that \( L(h) = \chi(T^2) = 0 \). Thus, «combinatorial» action of \( h \) on the set of all cells defines by the action of \( h \) on any fixed 2-cell, i.e., by the action \( \rho(h) \) on the edge of \( st(v) \). Therefore, from Proposition 2.7 follows that there exists the section \( s : G_v^{loc} \rightarrow S'(f) \) of the map \( r \circ \rho \) such that \( s(G_v^{loc}) \) freely acts on \( T^2 \). In particular, the quotient-map \( q : T^2 \rightarrow T^2/G_v^{loc} \) is the covering map, and hence, \( T^2/G_v^{loc} \) is either 2-torus \( T^2 \), or Klein bottle. But since \( G_v^{loc} \)-action on \( T^2 \) is the action by diffeomorphisms which preserve orientation and are isotopic to \( \text{id}_{T^2} \) it follows that the quotient-map \( T^2/G_v^{loc} \) is a torus. In particular, we have the following short exact sequence:

\[
1 \longrightarrow \pi_1 T^2 \xrightarrow{q} \pi_1 T^2/G_v^{loc} \longrightarrow G_v^{loc} \longrightarrow 1.
\]

Since \( q \) is the monomorphism, it follows that the proposition (1) of Theorem 2.5 is the consequence of the following result:

Lemma 3.1 (Chapter E, [16]). Let \( A, B \) be free abelian group of the rank 2, and \( q : A \rightarrow B \) be an inclusion. Then there exists \( L, M \in A \) and \( X, Y \in B \) such that \( A = \langle L, M \rangle, \ B = \lambda X, Y, \rangle, \) and

\[
q(L) = nX, \quad q(M) = mnY
\]

for some \( n, m \in \mathbb{N} \), in particular \( B/A \cong \mathbb{Z}_n \times \mathbb{Z}_{mn} \).
4. Proof of statement (2) of Theorem 2.5

Step 1. Choice of generators of $\pi_1T^2$ and $\pi_1T^2/G_v^{loc}$. Fix a point $y \in T^2$. Let $z = q(y) \in T^2/G_v^{loc}$.

Then we have the following diagram:

$$
\begin{array}{c}
0 \xrightarrow{q} \pi_1(T^2, y) \xrightarrow{q} \pi_1(T^2/G_v^{loc}, z) \xrightarrow{\partial} G_v^{loc} \xrightarrow{q} 0 \\
0 \xrightarrow{q} \mathbb{Z}^2 \xrightarrow{q} \mathbb{Z}^2 \xrightarrow{\partial} \mathbb{Z}_n \times \mathbb{Z}_{nm} \xrightarrow{q} 0
\end{array}
$$

where $q : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ and $\partial : \mathbb{Z}^2 \rightarrow \mathbb{Z}_n \times \mathbb{Z}_{nm}$ are defined by the formulas:

$q(\lambda, \mu) = (n\lambda, mn\mu)$, \quad $\partial(x, y) = (x \bmod n, y \bmod nm)$.

Let $\hat{X}, \hat{Y} : T^2/G_v^{loc} \times [0, 1] \rightarrow T^2/G_v^{loc}$ be isotopies such that $\hat{X}_0 = \hat{X}_1 = \hat{Y}_0 = \hat{Y}_1 = \text{id}_{T^2/G_v^{loc}}$, and $\hat{X}_s \circ \hat{Y}_t = \hat{Y}_t \circ \hat{X}_s$ for all $s, t \in [0, 1]$. Moreover loops $\hat{X}_t, \hat{Y}_t : I \rightarrow T^2/G_v^{loc}$, defined by $\hat{X}_0(t) = \hat{X}(z, t)$ and $\hat{Y}_2(t) = \hat{Y}(z, t)$, represent elements $[\hat{X}_2] = (1, 0)$ and $[\hat{Y}_2] = (0, 1)$ in $\pi_1(T^2/G_v^{loc}, z)$.

Extend $\hat{X}$ and $\hat{Y}$ to maps $X, Y : T^2/G_v^{loc} \times \mathbb{R} \rightarrow T^2/G_v^{loc}$ by formulas:

$$
X(x, t) = \hat{X}(x, t \bmod 1), \quad Y(x, t) = \hat{Y}(x, t \bmod 1).
$$

Let $L, M : T^2 \times \mathbb{R} \rightarrow T^2$ be the unique liftings of $X$ and $Y$ respectively with respect to the map $q$ such that $L$ and $M$ commutes and $L_0 = M_0 = \text{id}_{T^2}$, i.e., $L_t \circ q = q \circ L_t$ and $Y_t \circ q = q \circ M_t$.

Let $s : G_v^{loc} \rightarrow S'(f)$ be a section of the map $r \circ \rho$, see Proposition 2.7. Since by (1) of Theorem 2.5, $G_v^{loc} \cong \mathbb{Z}_n \times \mathbb{Z}_{nm}$ for some $n, m \in \mathbb{N}$, it follows that $L_t \circ M_{tv} = M_{tv} \circ L_t$ for all $t, t' \in \mathbb{R}$ and

$$
L_k = s(k \bmod n, 0), \quad M_k = s(0, k \bmod mn)
$$

for all $k \in \mathbb{Z}$. In particular $L_{kn} = M_{knm} = \text{id}_{T^2}$, $k \in \mathbb{Z}^2$, and loops $L_z : [0, n] \rightarrow T^2$ and $M_z : [0, mn] \rightarrow T^2$ represent elements $[L_z] = (1, 0)$ and $[M_z] = (0, 1)$ in $\pi_1(T^2, y) \cong \mathbb{Z}^2$.

Since $G_v^{loc}$ freely acts on $T^2$, it follows that connected components of $T^2 - V$ we can enumerate by three indexes $D_{ijk}$ such that $i = 1, 2, \ldots, r$, $j = 0, 1, \ldots, n - 1$, and $k = 0, 1, \ldots, mn - 1$. Moreover if $\gamma = (a, b) \in \mathbb{Z}_n \times \mathbb{Z}_{nm} \cong G_v^{loc}$, then

$$
\gamma(D_{ijk}) = D_{i+1,j+a,k+b},
$$

where second index takes modulo $n$, and third by modulo $mn$.

We put $S_{ijk} = \pi_0S'(f|_{D_{ijk}}, \partial D_{ijk})$ and $S = \prod_{i=1}^{r} \prod_{j=0}^{n-1} \prod_{k=0}^{mn-1} S_{ijk}$. Next, define the homomorphism

$$
\tau : S \rightarrow \text{Map}(G_v^{loc}, \prod_{i=1}^{r} S_{000})
$$

by the formula: if $\alpha = (h_{ijk}) \in S$, then the map

$$
\tau(\alpha) : \mathbb{Z}_n \times \mathbb{Z}_{nm} \rightarrow \prod_{i=1}^{r} S_{000}
$$

is given by the formula

$$
\tau(\alpha)(a, b) = (M_k^{-1} \circ L_j^{-1} \circ h_{ijk} \circ L_j \circ M_k, i = 1, 2, \ldots, r),
$$

where $(a, b) \in \mathbb{Z}_n \times \mathbb{Z}_{nm} \cong G_v^{loc}$. After direct verifying we see that $\tau$ is the isomorphism.

Step 2. Epimorphism $\psi$. Let $h : I \rightarrow D_{id}(T^2)$ be a loop in $D_{id}(T^2)$ such that $h(0) = h(1) = \text{id}_{T^2}$, i.e., $h$ is an isotopy $h : T^2 \times I \rightarrow T^2$ of the torus. Let $x$ be a point in $T^2$. Then $h_x : \{x\} \times I \rightarrow T^2$ is a loop in $T^2$ with the starting point $x$. Define the map $\ell : \pi_1 D_{id}(T^2) \rightarrow \pi_1 T^2$ by the formula:

$$
\ell([h]) = [h_x] \in \pi_1 T^2.
$$

It is known that the map $\ell$ is the isomorphism, see [1, 2, 4].
Lemma 4.1. There exists the epimorphism $\psi : \pi_1(D_{id}(T^2), S'(f)) \to \pi_1 T^2 / G_v^{\text{loc}}$ such that the following diagram is commutative

$$
\begin{array}{ccc}
1 & \longrightarrow & \pi_1 D_{id}(T^2) \\ \\
\varepsilon & \downarrow & \pi_1(D_{id}(T^2), S'(f)) \downarrow \psi \\
1 & \longrightarrow & \pi_1 T^2 \\
\end{array}
\begin{array}{ccc}
& & 1 \\
\end{array}
$$

(4.2)

\begin{equation}
1 \longrightarrow \pi_1 D_{id}(T^2) \longrightarrow \pi_1(D_{id}(T^2), S'(f)) \longrightarrow \pi_0 S'(f) \longrightarrow 1
\end{equation}

and rows are exact sequences.

Proof. Fix any vertex $z$ of $V$ and define the map $\psi_0 : D_{id}(T^2) \to T^2 / G_v^{\text{loc}}$ by $\psi_0(h) = q(h(z))$, where $h \in D_{id}(T^2)$ and $q : T^2 \to T^2 / G_v^{\text{loc}}$ is a quotient-map induced by free action of $G_v^{\text{loc}}$ on $T^2$. Obviously, $\psi_0$ is continuous map. Since $G_v^{\text{loc}}$-action and $S'(f)$-action coincide on vertices of $V$, it follows that $\psi_0(h)$ belongs to some $G_v^{\text{loc}}$-orbit of the point $z$ for $h \in S'(f)$. Then the map $\psi_0$ induces the map of triple

$$
\psi_0 : (D_{id}(T^2), S'(f), \text{id}_{T^2}) \to (T^2 / G_v^{\text{loc}}, z, z), \quad \psi_0(h) = q(h(z)).
$$

In particular, $\phi_0$ induces the homomorphism

$$
\phi : \pi_1(D_{id}(T^2), S'(f), \text{id}_{T^2}) \to \pi_1(T^2 / G_v^{\text{loc}}, z, z)
$$

Since rows in diagram 4.2 are exact sequences, the map $\ell$ is the isomorphism, the map $\rho_0$ is the epimorphism, it follows that, by 5-lemma, the map $\phi$ is the epimorphism.

Step 3. Kernel of $\psi$. Let $f(V) = c$, $\epsilon > 0$ and $N$ be a connected component of $f^{-1}([c - \epsilon, c + \epsilon])$ contains $V$. We will call $N$ the $f$-regular neighborhood of $V$. Recall that $S'(f, N) : = \{ h \in S'(f) \mid h|_N = \text{id}_N \}$.

The following lemma describes the kernel of $\psi$.

Lemma 4.2. There exist isomorphisms between these five groups

$$
\ker \psi \xrightarrow{\zeta} \ker \rho_0 \leftrightarrow \pi_0 S'(f, N) \xrightarrow{\sigma} S \xrightarrow{\tau} \text{Map}(G_v^{\text{loc}}, \prod_{i=1}^r S_{00})
$$

Proof. (1) First we construct the isomorphism $\zeta : \ker \psi \to \ker \rho_0$. Consider the following diagram with exact rows and columns:

$$
\begin{array}{cccc}
& & 1 & 1 \\
& & \pi_1 D_{id}(T^2) & \longrightarrow \pi_1 T^2 \\
& & \ker \psi & \longrightarrow \pi_1(D_{id}(T^2), S'(f)) \\
& & \zeta & \cong \partial \circ \lambda_1^{-1} \\
1 & \ker \rho_0 & \longrightarrow \pi_0 S'(f) \\
& & \rho_0 & \longrightarrow G_v^{\text{loc}} \\
& & 1 & 1
\end{array}
$$

Since $\ell$ is the isomorphism, it follows that by $3 \times 3$-lemma, the homomorphism $\zeta = \partial \circ \lambda_1^{-1}|_{\ker \psi}$ is the isomorphism.

(2) Note that the following map is the isomorphism

$$
\sigma : S'(f, N) \cong \prod_{i,j,k} S'(f|_{D_{i,j,k}}, \partial D_{i,j,k}), \quad \sigma(h) = (h|_{D_{i,j,k}})_{i,j,k},
$$
which induces the an isomorphism

\[ \sigma : \pi_0 \mathcal{S}'(f, N) \cong \prod_{i=1}^{r} \prod_{j=0}^{n-1} \prod_{k=0}^{m-1} S_{ijk} = \mathcal{S}. \]

(3) It sufficient to show that the inclusion \( \iota : \mathcal{S}'(f, N) \to \ker(r \circ \rho) \) is a homotopy equivalence. Hence it induces the isomorphism \( \iota : \pi_0 \mathcal{S}'(f, N) \to \pi_0 \ker(r \circ \rho) = \ker \rho_0 \). Now we show that there exists an isotopy \( H : \ker(r \circ \rho) \times I \to \ker(r \circ \rho) \) such that the following conditions hold:

(i) \( H_0 = \text{id}_{T^2} \);
(ii) \( H_i(S'(f, N)) \subset S'(f, N) \) for all \( t \in I \);
(iii) \( H_i(\ker(r \circ \rho)) \subset \mathcal{S}'(f, N) \).

Let \( F \) be the Hamiltonian vector field of the function \( f \in \mathcal{F}(T^2) \), \( F : T^2 \times \mathbb{R} \to T^2 \) be the flow of \( F \), and \( N, N' \) be \( f \)-regular neighborhoods of \( V \) such that \( \overline{N} \subset \text{Int}(N') \). For each smooth function \( \gamma : T^2 \to \mathbb{R} \) define the map \( F \gamma : T^2 \to T^2 \) by the formula \( F \gamma(x) = F(x, \gamma(x)) \).

From [9, Claim 1] follows that for each \( h \in \ker(r \circ \rho) \) there exists a unique smooth function \( \beta_h \in C^\infty(N') \) such that \( h = F \beta_h \) on \( N' \), i.e., \( h(x) = F(x, \beta_h(x)) \), \( x \in N' \). Moreover the map \( s : \ker(r \circ \rho) \to C^\infty(N') \) defined by \( s(h) = \beta_h \) is continuous map with respect \( C^\infty \)-topologies. Furthermore, if \( h \) is fixed on \( N \), then \( \beta_h = 0 \) on \( N \).

Extend the function \( \beta_h \) to the smooth function \( \alpha_h \in C^\infty(M) \) such that \( \alpha_h|_N = \beta_h \) and \( \alpha_h = 0 \) on \( T^2 - N' \) in the following way. Let \( \varepsilon : T^2 \to [0, 1] \) be a smooth function on \( T^2 \) such that

1. \( \varepsilon \) is constant on orbits of flow \( F \);
2. \( \varepsilon = 1 \) on \( N \);
3. \( \varepsilon = 0 \) on \( T^2 - N' \).

Define \( \alpha_h = \varepsilon \beta_h \) on \( N' \) and \( \alpha_h = 0 \) on \( T^2 - N' \). Obviously, the correspondence \( h \mapsto \alpha_h \) is the continuous map \( \alpha : \ker(r \circ \rho) \to C^\infty(T^2) \). Furthermore from condition (1) on the function \( \varepsilon \) follows that the map \( F_{\alpha_h} : T^2 \to T^2 \) defined by the formula \( F_{\alpha_h}(x) = F(x, t \alpha_h(x)) \) is the diffeomorphism for all \( t \in I \), see [7, Claim 4.14.1]. From conditions (2) and (3) we have that

\[ F(x, t \alpha_h(x)) = \begin{cases} h(x), & x \in N, \\ x, & x \in T^2 - N'. \end{cases} \]

Define the isotopy \( H : \ker(r \circ \rho) \times I \to \ker(r \circ \rho) \) by the formula \( H(x, t) = h \circ F_{\alpha_h}^{-1} \). It remains to prove that \( H \) satisfies conditions (i)-(iii). Indeed,

(i) \( H_0(h) = h \circ F_0^{-1} = h \), i.e., \( H_0 = \text{id}|_{\ker(r \circ \rho)} \);
(ii) Consider that \( h \in S'(f, N) \). Then \( \beta_h = t \alpha_t = 0 \) in \( N \), and hence, \( F_{\alpha_t} \mid N = \text{id} \) for all \( t \in I \). In particular \( H_i(h)|_N = H_i|_N = \text{id} \).

(iii) \( H_1(h)|_N = h \circ F^{-1}|_N = h \circ h^{-1}|_N = \text{id} \).

So lemma is proved. \( \square \)

**Step 4. Defining the map \( \xi \).** Define the map

\[ \xi : \text{Map}(G_{\text{loc}}, \prod_{i=1}^{r} S_{i00}) \times \pi_1(T^2/G_{\text{loc}}) \to \pi_1(D_{\text{id}}(T^2), \mathcal{S}'(f), \text{id}_{T^2}) \]

by the following way. Let \( \alpha : \mathbb{Z}_n \times \mathbb{Z}_m \cong G_{\text{loc}} \to \prod_{i=1}^{r} S_{i00} \) be any map. For each triple \((i, j, k)\) we chose \( h_{ijk} \in S'(f|_{D_{i00}}, \partial D_{i00}) \) so that

\[ \alpha(i, j) = ([h_{1jk}], [h_{2jk}], \ldots, [h_{rjk}]), \]

and let \( h'_{ijk} : D_{i00} \to D_{i00} \) be any isotopy between \( h_{ijk} \) and \( h'_{ijk} = h_{ijk} \). Define the map

\[ h : (I, \partial I, 0) \to (D_{\text{id}}(T^2), \mathcal{S}'(f), \text{id}_{T^2}) \]

by the formula:

\[ h(t)(x) = \begin{cases} M_{k+at} \circ L_{j+bt} \circ h'_{ijk}(x) \circ L_j^{-1} \circ M_k^{-1}(x), & x \in D_{ijk}, \\ M_{at} \circ L_{bt}(x), & x \in N. \end{cases} \]
It is easy to see that $h$ is well defined. We set
\[ \xi(\alpha, (a, b)) = [h] \in \pi_1(D_{id}(T^2), S'(f), \text{id}_{T^2}). \]
Also it is no difficult to verify that the map \( \xi \) is the homomorphism. Furthermore from Lemma 4.1 and the formula for the map \( \tau \) follows that the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Map}(G_{v}^{\text{loc}}, \prod_{i=1}^{r} S_{i00}) & \xrightarrow{\tau \circ \sigma \circ \iota^{-1} \circ \zeta^{-1}} & \ker \psi \\
\downarrow & & \downarrow \\
\text{Map}(G_{v}^{\text{loc}}, \prod_{i=1}^{r} S_{i00} \rtimes \pi_1 T^2/G_{v}^{\text{loc}}) & \xrightarrow{\xi} & \pi_1(D_{id}(T^2), S'(f)) \\
\downarrow^{pr} & & \downarrow \\
\pi_1 T^2/G_{v}^{\text{loc}} & \xrightarrow{\pi_1 T^2/G_{v}^{\text{loc}}} & \pi_1 T^2/G_{v}^{\text{loc}}
\end{array}
\]

By 5-lemma we have that \( \xi \) is the isomorphism. Theorem 2.5 was proved.

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