BINARY CYCLOTOMIC POLYNOMIALS: REPRESENTATION VIA WORDS AND ALGORITHMS

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ABSTRACT. Cyclotomic polynomials are basic objects in Number Theory. Their properties depend on the number of distinct primes that intervene in the factorization of their order, and the binary case is thus the first nontrivial case. This paper sees the vector of coefficients of the polynomial as a word on a ternary alphabet \{-1, 0, +1\}. It designs an efficient algorithm that computes a compact representation of this word. This algorithm is of linear time with respect to the size of the output, and, thus, optimal. This approach allows to recover known properties of coefficients of binary cyclotomic polynomials, and extends to the case of polynomials associated with numerical semi-groups of dimension 2.

1. INTRODUCTION

General description. Cyclotomic polynomials are defined as the irreducible factors in $\mathbb{Q}[x]$ of the polynomial $x^n - 1$. The identity

$$x^n - 1 = \prod_{d|n} \Phi_d(x)$$

holds between the $d$-th cyclotomic polynomials $\Phi_d$. The polynomial $\Phi_d(x)$ equals the product $\prod (x - \alpha)$ where $\alpha$ ranges over the primitive $d$-th roots of unity. It has integer coefficients, is self-reciprocal, and irreducible over $\mathbb{Q}[x]$. Its degree is $\varphi(d)$, where $\varphi$ stands for Euler’s totient function. The following holds, for any prime $p$,

$$\Phi_{pn}(x) = \Phi_{n}(x^p) \quad \text{if } p \text{ divides } n,$$

$$\Phi_{n}(x)\Phi_{pn}(x) = \Phi_{n}(x^p) \quad \text{if } p \text{ does not divide } n .$$

It is thus enough to compute $\Phi_d$ when $d$ is squarefree. Due to the equality

$$\Phi_p(x) = x^{p-1} + \cdots + x + 1,$$
that holds when \( d = p \) is prime, the first nontrivial case occurs when \( d \) is the product of two distinct primes \( p \) and \( q \), and defines what is called a \textit{binary} cyclotomic polynomial.

A cyclotomic polynomial \( \Phi_d \) is usually given by its dense representation, described by the vector of its coefficients in \( \mathbb{Q}[\phi(d)^{-1}] \). Many properties of interest hold on the dense representation of a binary cyclotomic polynomial \( \Phi_{pq} \) (see [3, 4, 6]), for instance:

(a) the polynomial \( \Phi_{pq} \) has all its coefficients in the ternary alphabet \( \{-1, 0, +1\} \);

(b) its nonzero coefficients alternate their signs; i.e. after a 1 follows a \(-1\).

(c) for \( p < q \), the maximum number of consecutive zeros in the vector of coefficient equals \( p - 2 \).

Many algorithms are designed for computing these polynomials, most of them being based on identities of type (1.1) and (1.2). Around 2010, Arnold and Monagan wished to study the coefficients of cyclotomic polynomials in an experimental way. They thus designed efficient algorithms that compute cyclotomic polynomials of very large order. Their work [1] is the main reference on this subject.

Our approach. We are concerned here with the representation and the computation of binary cyclotomic polynomials \( \Phi_{pq} \). We let \( m = \phi(pq) = (p - 1)(q - 1) \), and we consider the \textit{cyclotomic vector} \( a_{pq} \), (that is moreover palindromic) formed with coefficients of \( \Phi_{pq} \), from two points of view:

(i) first, classically, as a vector of \( \mathbb{Q}^{m+1} \),

(ii) but also as a word of length \( m + 1 \) on the ternary alphabet \( A = \{-1, 0, +1\} \).

With the first point of view, we use classical arithmetical operations on vectors. With the second point of view, we use operations on words as cyclic permutations, concatenations and fractional powers. In fact, we use both points of view and then all the operations of these two types. We have just to check that the sum of two words of \( A^{m+1} \) always belongs to \( A^{m+1} \). We will see in Lemma 1 that this is indeed the case in our context: the “bad” cases \((+1) + (+1)\) or \((-1) + (-1)\) never occur in the addition of two symbols of \( A \), and the sum remains “internal”.

We design an algorithm, called BCW, that \textit{only} uses simple operations on words (concatenation, shift, fractional power, internal addition). It takes as an input a pair \( (p, q) \) of two prime numbers \( p, q \) with \( p < q \), together with the quotient and the remainder of the division of \( q \) by \( p \); this input is thus of size \( \Theta(\log q) \). The algorithm outputs the cyclotomic word \( a_{pq} \) of size \( \Theta(pq) \); it performs a number \( \Theta(pq) \) of operations on symbols of \( A \). The complexity of the algorithm is thus linear with respect to the size of the output, and thus optimal in this sense. The \textit{proof} of the algorithm is based on arithmetical operations on polynomials, of type (1.1) or (1.2), that are further \textit{transfered} into operations on words; however, the algorithm itself \textit{does not perform} any polynomial multiplication or division.

To the best of our knowledge, our approach, based on combinatorics of words, is quite novel inside the domain of cyclotomic polynomials. The compact representation of binary cyclotomic polynomials described in Theorem 1 appears to be new and the algorithm BCW, whose complexity is \( \Theta(pq) \), is more efficient that the already existing algorithms, whose complexity is \( O(pq)E(p, q) \), where \( E(p, q) \) is polynomial in \( \log q \) (see Section 4).
2. Statement of the main results

We first define the main operations on words that will be used, then we state our two main results: the first one (Theorem 1) describes a compact representation of the cyclotomic word, whereas the second result (Theorem 2) describes the algorithm – so called the binary cyclotomic word algorithm (BCW algorithm for short) – that is used to obtain it. Theorem 1 will be proven in Section 3 and Theorem 2 in Section 4.

Operations on words. We first define the operations on words that we will use along the paper: concatenation and fractional power, cyclic permutation, addition.

Concatenation and fractional power. Given two words $u = u_0u_1\ldots u_{k-1}$ and $v = v_0v_1\ldots v_{\ell-1}$ from the alphabet $\mathcal{A}$, its concatenation is denoted by $u \cdot v$. For any $s \in \mathbb{N}$, the $s$-power of a word is denoted by $v^s$, and $v^0 = \varepsilon$ is the empty word.

Let $k$ and $p$ be positive integers. For any $v \in \mathcal{A}^p$, the fractional power $v^{k/p}$ of $v$ is the element of $\mathcal{A}^k$ defined as follows:

(a) For $k < p$, then $v^{k/p}$ is the truncation of the word $v$ to its prefix of length $k$.

(b) For $k \geq p$, then $v^{k/p} = (v^{k/p})_0 \cdot v^{(k \mod p)}/p$.

Circular permutation. The left circular permutation $\sigma$ is the mapping $\sigma : \mathcal{A}^p \to \mathcal{A}^p$ defined as

\[ \sigma^s = \sigma^{s \mod p}, \quad \sigma^p = \text{Id}. \]

Addition. The paper is based on a transfer between algebra and combinatorics on words and leads to define addition between words; we begin with the usual sum between two vectors of $\mathbb{Z}^k$, component by component: the sum $u + v$ between two words $u = u_0u_1\ldots u_{k-1}$ and $v = v_0v_1\ldots v_{k-1}$ is

\[ u + v = (u_0 + v_0)(u_1 + v_1)\cdots(u_{k-1} + v_{k-1}). \]

Then, the sum of two words from $\mathcal{A}$ may have symbols not in $\mathcal{A}$, due to the two bad cases $[+1 + (+1)]$ and $[(-1) + (-1)]$. However, we will prove in Lemma 3.2 that this will never occur in our framework. Then, an addition between two symbols of $\mathcal{A}$ will be always an internal operation, with a sum that stays inside $\mathcal{A}$ and coincides with the sum in $\mathbb{Z}$:

\[ 0 + 0 = 0; \quad 1 + 0 = 1; \quad -1 + 0 = -1; \quad -1 + 1 = 0; \quad 1 + (-1) = 0. \]

A compact representation of binary cyclotomic polynomials.

Theorem 2.1. Let $p < q$ be prime numbers. Consider the alphabet $\mathcal{A} = \{-1, 0, 1\}$ and the left circular permutation $\sigma$ defined in (2.1). With the $(p - 1)$ words $d_0, d_1, \ldots, d_{p-2} \in \mathcal{A}^p,$

\[ d_0 = 1(-1)0\cdots0, \quad d_i = \sigma^q(d_{i-1}) = \sigma^{i\cdot q}(d_0), \quad \text{for } i \geq 1, \]

define the $(p - 1)$ words $\omega_0, \omega_1, \ldots, \omega_{p-2},$

\[ \omega_0 = d_0, \quad \omega_i = \omega_{i-1} + d_i, \quad \text{for } i \geq 1. \]

Then, the following holds:
The words $\omega_0, \omega_1, \ldots, \omega_{p-2}$ belong to $A^p$: each addition $\omega_i + d_i$ is internal in $A$. These $(p-1)$ words only depend on the pair $(p, r = q \pmod{p})$.

Let $s = \lfloor q/p \rfloor$. The cyclotomic word $a_{pq}$ coincides with the prefix of length $\varphi(pq)+1$ of the word

$$b_{pq} = \omega_0^s \cdot \omega_1^s \cdot \omega_2^s \cdot \omega_{p-2}^s \cdot \omega_{p-2}^s \in A^{(p-1)q},$$

and is written in fractional power notation as

$$a_{pq} = \omega_0^{q/p} \cdot \omega_1^{q/p} \cdot \omega_{p-3}^{q/p} \cdot \omega_{p-2}^{(q-p+2)/p}.$$

The algorithm BCW. Given two positive primes $p$ and $q$ with $p < q$, together with the quotient $s = \lfloor q/p \rfloor$ and the remainder $r = q \pmod{p}$ of $q$ by $p$, the BCW algorithm proceeds in three main steps. The first two steps define the precomputation phase and only depend on the pair $(p, r = q \pmod{p})$, whereas the third step performs the computation itself and depends on the pair $(q, p)$.

Precomputation phase. It has two steps and computes

(i) the words $d_i$ for $i \in [0, p-2]$ defined in (2.2),
(ii) the words $\omega_i$ defined in (2.2) and their prefixes $\omega_i^r$ for $i \in [0, p-2]$.

Computation phase. It computes the word $b_{pq}$ defined in (2.3) with concatenating powers of $\omega_0, \ldots, \omega_{p-2}$. It depends on $s$. The cyclotomic word $a_{pq}$ is obtained from $b_{pq}$ by deleting its suffix of length $p-2$.

The BCW algorithm only performs the operations described at the beginning of this section, on words of length at most $p$, each one having a cost $\Theta(p)$. The precomputation phase performs

- cyclic permutations of any order of a word of length $p$;
- truncations of a suffix of length $\ell < p$;
- internal additions between two words of length $p$.

The computation phase performs concatenations of a word of length at most $p$ (at the end of an already existing word of any length);

Theorem 2 summarizes the analysis of the complexity of the BCW algorithm. It will be proven in Section 4.

**Theorem 2.2.** Suppose that the two prime numbers $p < q$ are given together with the quotient $s = \lfloor q/p \rfloor$ and the remainder $r = q \pmod{p}$. Then, the BCW algorithm computes the cyclotomic word $a_{pq}$ in $\Theta(pq)$ operations on words:

(a) The precomputation phase only depends on the pair $(p, r = q \pmod{p})$ and its cost is $\Theta(p^2)$.
(b) The computation phase performs $s(p-2)$ concatenations of words of length $p$, and its cost is $\Theta(pq)$. 

3. A compact representation for binary cyclotomic polynomials

This section is devoted to the proof of Theorem 1 and is organized into three main parts. We first consider the particular case $p = 2$, then we describe the general case $p > 2$, and we finally prove two auxiliary results (Proposition 1 and Lemma 1) that have been used in the proof of the general case.

**Particular case $p = 2$.** The polynomial $\Phi_{2q}$ satisfies the relation $\Phi_{2q}(x) = \Phi_q(-x)$. Since $\Phi_q(x) = 1 + x + \cdots + x^{q-1}$, it turns out that $\Phi_{2q}$ is a polynomial of degree $q - 1$ whose nonzero coefficients take the values 1 and $-1$ alternatively. The equality $m + 1 = \varphi(2q) + 1 = q$ holds and entails $\lfloor (m + 1)/q \rfloor = 1$. There is only one word $\omega_0 = 1(-1)$ which has to be concatenated with itself $\lceil q/2 \rceil$ times. The resulting word is

$$\omega_0^{\lfloor q/2 \rfloor} = 1(-1)1(-1) \cdots 1(-1),$$

which coincides with the vector $a_{2q}$ of coefficients of $\Phi_{2q}$. This ends the proof for the case $p = 2$.

**General case $p > 2$.** We let $m = \varphi(pq)$. We consider the set of words $A_{m+1}$ as embedded in $Q^{m+1}$ because we only deal with internal additions between words. This fact that is stated as the assertion (1) of Theorem 2.1 is proven in Lemma 3.2. We then do not distinguish between words or vectors, and the notation $v = v_0 \cdots v_m$ denotes both a word $v \in A_{m+1}$ and/or a vector of $Q^{m+1}$. We write $v_{i,j}$ to denote the $j$-th symbol (or coordinate) of a word $v_i$.

The proof is described along three main steps. In Step 0, we deal with linear algebra over the vector space $Q^{m+1}$, introduce the shift-matrix $S$ and recall its main properties. Then Step 1 transfers polynomial identities into linear algebra equations on $Q^{m+1}$. Finally, Step 2 adopts the point of view of words.

**Step 0. Starting with linear algebra.** We consider the vector space $Q^{m+1}$, and the linear map

$$S : Q^{m+1} \to Q^{m+1} \quad Sv = 0v_0 \cdots v_{m-1} \quad \text{for any } v = v_0 \cdots v_m \in Q^{m+1}.$$ 

We denote by $e_0, e_1, \ldots, e_m$ the vectors of the canonical basis of $Q^{m+1}$: the word $e_1$ is the word whose symbols satisfy $e_{i,i} = 1$ and $e_{i,j} = 0$ for $j \neq i$. We deal with the matrix associated with the linear map $S$ in the canonical basis that is also denoted by $S$ and called the shift-matrix. This is a nilpotent matrix of size $(m + 1) \times (m + 1)$ having 1-s along the main lower subdiagonal and 0-s everywhere else, that satisfies

$$S^{m+1} = 0 \in Q^{(m+1) \times (m+1)} \quad \text{and} \quad S^{\ell} e_0 = e_{\ell}, \quad \text{for any } \ell \in [0, m].$$

With a polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$, we associate the matrix $f(S) = a_0 \text{Id} + a_1 S + a_2 S^2 + \cdots + a_m S^m$ (here $\text{Id}$ is the identity matrix) together with the vector of coefficients $a = a_0 a_1 \ldots a_m \in Q^{m+1}$. The following relation holds

$$(3.1) \quad a = a_0 S^n e_0 + a_1 S^1 e_0 + \cdots + a_m S^m e_0 = f(S) e_0.$$ 

Moreover, $f(S)$ is invertible if and only if $a_0 \neq 0$. 
**Step 1. From polynomial identities to linear algebra equations.** Our starting point is the polynomial identity \((1.2)\). When applied to the case when \(p\) and \(q\) are distinct primes, it writes as
\[(1 - x^q)\Phi_{pq}(x) = (1 - x)\Phi_q(x^p) .\]
We now specialize the previous identity in the shift-matrix \(S\) of size \((m + 1) \times (m + 1)\), apply it to the vector \(e_0\), and obtain
\[(\text{Id} - S^q)\Phi_{pq}(S)e_0 = (\text{Id} - S)\Phi_q(S^p)e_0 .\]
Relation \((3.1)\) entails the equality
\[a_{pq} = \Phi_{pq}(S)e_0 = (\text{Id} - S^q)^{-1}(\text{Id} - S)\Phi_q(S^p)e_0 .\]
Since \(S\) is a nilpotent matrix, the inverse matrix \((\text{Id} - S^q)^{-1}\) equals
\[\left(\text{Id} - S^q\right)^{-1} = \sum_{i=0}^{[(m+1)/q]} S^{iq} .\]
Next, for \(p > 2\), the following holds
\[m + 1 = q(p - 2) + q - (p - 1) + 1, \quad 0 \leq q - (p - 1) + 1 < q ,\]
and entails the equality \([(m + 1)/q] = p - 2\). Following \((3.3)\) and \((3.4)\), the vector \(a_{pq}\) satisfies
\[a_{pq} = \Phi_{pq}(S)e_0 = \sum_{i=0}^{p-2} S^{iq}c_{pq} \quad \text{with} \quad c_{pq} = (\text{Id} - S)\Phi_q(S^p)e_0 .\]
The relation \(S^\ell e_0 = e_\ell\) holds for integer \(\ell \in [0, m]\) and yields
\[\Phi_q(S^p)e_0 = \sum_{j=0}^{[m+1]/p} e_{jp} .\]
Then the vector \(c_{pq} \in \mathbb{Q}^{m+1}\) defined in \((3.5)\) is
\[c_{pq} = \sum_{j=0}^{[m+1]/p} (e_{jp} - e_{jp+1}) = 1 - 10 \cdots 0 \underbrace{1 - 10 \cdots 0}_{p} \underbrace{1 - 10 \cdots 0}_{m+1 \text{ (mod } p)} .\]

**Step 2. From vectors to words.** With \((3.6)\), we now view the vector \(c_{pq}\) as a word of \(A^{m+1}\) that is written as a fractional power of the word \(d_0 \in A^p\) defined in the statement of Theorem 2.1:
\[c_{pq} = d_0^{(m+1)/p} \in A^{m+1} .\]
We now use the next proposition (Proposition 1) that states that such a fractional power can be written in terms of the transforms \(d_i^*\) of \(d_0\) via cyclic permutations. The proof of Proposition 1 will be found at the end of this section.
Proposition 3.1. The fractional power $a_0^{(m+1)/p}$ of $d_0$ is written as a concatenation of fractional powers of the cyclic transforms $d_i = \sigma^i (d_0)$:

$$c_{pq} = a_0^{(m+1)/p} = d_0^{q/p} \cdot d_1^{q/p} \cdots d_{p-2}^{q/p} \cdot d_{p-3}^{(q-p+2)/p}.$$

We then return to the proof. In (3.5), we now view, for each $i \in [0, p-2]$, $S^iqc_{pq}$ as a word in $A^{m+1}$, written as

$$S^iqc_{pq} = 0^iq \cdot d_0^{q/p} \cdot d_1^{q/p} \cdots d_{p-2-i}^{(q-p+2)/p},$$

where $0^iq$ is the word obtained with the concatenation of $iq$ consecutive zeros. From (3.5), the cyclotomic word $a_{pq}$ is expressed in terms of $c_{pq}$ as

$$a_{pq} = \sum_{i=0}^{p-2} S^iqc_{pq} = \sum_{i=0}^{p-2} 0^iq \cdot d_0^{q/p} \cdot d_1^{q/p} \cdots d_{p-2-i}^{(q-p+2)/p}.$$

We now explain how the sums $\omega_i$ that intervene in Theorem 2.1 appear. We consider the $(p-2)$ following vectors of $Q^q$

$$f_0 = d_0^{q/p}, \quad f_1 = d_1^{q/p}, \quad f_2 = d_2^{q/p}, \ldots, \quad f_{p-3} = d_{p-3}^{q/p},$$

together with the $(p-2)$ following vectors of $Q^{q-p+2}$:

$$g_0 = d_0^{(q-p+2)/p}, \quad g_1 = d_1^{(q-p+2)/p}, \ldots, \quad g_{p-2} = d_{p-2}^{(q-p+2)/p}.$$

We use the expression of the vector $a_{pq}$ in (3.8) as a sum that uses $(p-1)$ lines, indexed from 0 to $p-2$, each line for each term of the sum,

$$a_{pq} = f_0 \cdot f_1 \cdot f_2 \cdot f_3 \cdots f_{p-3} \cdot g_{p-2} + 0^q \cdot f_0 \cdot f_1 \cdot f_2 \cdots f_{p-4} \cdot g_{p-3} + 0^q \cdot 0^q \cdot f_0 \cdot f_1 \cdots f_{p-5} \cdot g_{p-4} \cdots + 0^q \cdot 0^q \cdot 0^q \cdots 0^q \cdot g_0.$$

Each line exactly follows the same pattern: it is formed with $(p-2)$ blocks of length $q$, followed with a block of length $(q-p+2)$. The line of index $i$ begins with $i$ blocks of $q$ zeroes, that perform a shift between blocks. Then, when we read the result of the previous sum by columns, indexed from 0 to $p-1$, the $j$-th column contains (for $0 \leq j < p-2$) the element $f_0 + \cdots + f_j$, whereas the last column equals the sum of elements defined in (3.9).

Provided that all the sums be internal, this implies the equality

$$a_{pq} = d_0^{q/p} \cdot (d_0 + d_1)^{q/p} \cdots (d_0 + \cdots + d_{p-3})^{q/p} \cdot (d_0 + \cdots + d_{p-2})^{(q-p+2)/p},$$

and introduces the words $\omega_i$ involved in Theorem 2.1.

The next lemma shows that the computation of the words $\omega_i = d_0 + \cdots + d_i$ defined in Theorem 1 indeed involves internal operations in $A$. It will be proven at the end of the section. It provides the proof of Statement (1) of Theorem 2.1.
Lemma 3.2. Under the assumptions and notations of Theorem 2.1, the words $\omega_i$ belong to $A^p$ for each $i \in [0, p-2]$.

Now, with Lemma 1, the cyclotomic word $a_{pq}$ is thus expressed as a concatenation of fractional powers,

$$a_{pq} = \omega_0^{q/p} \cdot \omega_1^{q/p} \cdots \omega_{p-3}^{q/p} \cdot \omega_{p-2}^{(q-p+2)/p},$$

it has length $m+1$ and coincides with the prefix of length $m+1$ of the word

$$b_{pq} = \omega_0^{q/p} \cdot \omega_1^{q/p} \cdots \omega_{p-3}^{q/p} \cdot \omega_{p-2}^{q/p}.$$  

This ends the proof of Theorem 1.

We now provide the proof of the two results that we have used.

Proof of Proposition 3.1. The first step of the proof is based on a simple fact which holds for any positive integers $m$, $p$ and $q$, with $m > p$ and $q \in [p+1, m]$, and any word $d_0 \in A^p$. The fact is the following: if $r = q \pmod{p}$, then

$$d_0^{(m+1)/p} = d_0^{q/p} \cdot d_1^{(m+1-q)/p} \quad \text{with} \quad d_1 = \sigma^q(d_0) = \sigma^r(d_0) \in A^p.$$  

Here we prove it. First, it is clear that

$$d_0^{(m+1)/p} = d_0^{q/p} \cdot v \quad \text{with} \quad v \in A^{m+1-q}.$$

Let $d_j$ and $v_j$ denote the $j$th coordinates of $d_0$ and $v$, respectively. It is clear that $v_j$ equals the $(j+r) \pmod{p}$ coordinate of $d_0$. This implies that $v = d_1^{(m+1-q)/p}$, with $d_1 = \sigma^q(d_0) = \sigma^r(d_0) = d_{r+1} \cdots d_p d_0 \cdots d_r$. This is what we wanted to prove.

For the particular choice $m = \varphi(pq)$, it turns out that $\lfloor (m+1)/q \rfloor = p-2$ and $m+1 = q - p + 2 \pmod{q}$ (recall that $p > 2$).

Now, we apply an inductive argument. For $i = 0, \ldots, p-2$, set $d_i = \sigma^r(d_{i-1})$. Then

$$d_0^{(m+1)/p} = d_0^{q/p} \cdot d_1^{(m+1-q)/p} = d_0^{q/p} \cdot d_1^{q/p} \cdots d_{i-3}^{q/p} \cdot d_{i-2}^{(q-p+2)/p}.$$  

Proof of Lemma 3.1. Fix $i \in [1, p-2]$ and let $j$ be in the interval $[0, p-1]$. With $r = q \pmod{p}$, the relation

$$d_k = \sigma^r(d_{k-1}) = \sigma^{kr}(d_0)$$

shows that

$$d_{k,j} = \begin{cases} 1 & \text{if } j + kr = 0 \pmod{p}, \\ -1 & \text{if } j + kr = 1 \pmod{p}, \\ 0 & \text{otherwise}. \end{cases}$$

Now, we proceed with an inductive argument. The statement of the lemma is clear for $\omega_0 = d_0$. Suppose that $\omega_{i-1} \in A^p$. From the definition of words $\omega_i$, given in (2.2), it is clear that

$$\omega_{i-1} = d_0 + d_1 + \cdots + d_{i-1}.$$  

Hence, if $\omega_{i-1,j} = 1$, then $j + kr = 0 \pmod{p}$ for some $k \in [0, i-1]$; and similarly, if $\omega_{i-1,j} = -1$, then $j + kr = 1 \pmod{p}$ for some $k \in [0, i-1]$. Remark that the converse might not be true because there might be cancellations between 1 and −1.
Since \( \gcd(p, r) = 1 \), we have \( kr \neq ir \pmod{p} \) for any \( k \in [0, i-1] \). It follows that there is no \( j \in [0, p-1] \) such that \( \omega_{i-1,j} = d_{i,j} = 1 \) or \( \omega_{i-1,j} = d_{i,j} = -1 \). Then, the sum \( \omega_i = \omega_{i-1} + d_i \in \mathcal{A}^p \), that is, the sums of two 1 or two -1 never occur.

4. Algorithms

This section considers algorithms, and analyses their complexities. It is organised into three main parts. It begins with the BCW algorithm, then explains how it may be used to compute tables of binary cyclotomic polynomials, and finally compares the BCW algorithms with other algorithms that have been previously proposed.

**Analysis of the BCW algorithm.** We first analyse the BCW algorithm designed in Section 2. This will provide the proof of Theorem 2.2. The analysis of the BCW algorithm follows from the next two lemmas.

**Lemma 4.1.** [Precomputation step]. The output \( O_{p,r} \) of the precomputation step on the input \( (p, r = q \pmod{p}) \) contains the words \( \omega_i \) together with their prefixes \( \omega_{r/p} \), for \( i = 0, \ldots, p-2 \). It is computed with \( \Theta(p^2) \) operations on words of \( \mathcal{A}^p \). The total cost for computing \( O_{p,r} \) is \( \Theta(p^3) \).

**Proof.** In Lemma 3.2, we already proved that all the additions between words \( \omega_i \) and words \( d_i \) are internal to the alphabet \( \mathcal{A} \). Given \( r = q \pmod{p} \), the \( p-2 \) words \( d_i \) are obtained by applying \( p-2 \) cyclic permutations of order \( r \) to a word of length \( p \). The words \( \omega_i \) are obtained with \( \Theta(p^2) \) internal additions of words in \( \mathcal{A}^p \). The computation of the prefixes \( \omega_{r/p} \) only involves \( p-2 \) truncations. All the words involved has length \( p \). Thus, the total cost is \( \Theta(p^2) \). \( \square \)

**Lemma 4.2.** [Concatenation step]. The concatenation step computes \( a_{pq} \) from the output \( O_{p,r} \) of the precomputation step in \( \Theta(q) \) words operations (mainly concatenations). The total cost of the concatenation step is \( \Theta(pq) \).

**Proof.** We consider the quotient \( s \) and the remainder \( r \) of the division of \( q \) by \( p \). The word \( b_{pq} \) is obtained by concatenating the words \( \omega_i \in \mathcal{A}^p \) or \( \omega_{r/q} \in \mathcal{A}^r \) according to the order prescribed in Theorem 2.1. There are \( \Theta(sp) = \Theta(q) \) such concatenations of words of length at most \( p \). Finally, to obtain \( a_{pq} \) from \( b_{pq} \), it suffices to delete the last \( p-2 \) symbols. The total cost of this step is \( \Theta(pq) \). \( \square \)

**Computing tables of binary cyclotomic polynomials with the BCW algorithm.**

We denote by \( O_p \) the total precomputation output relative to a fixed prime \( p \), that is the union of the precomputation outputs relative this fixed prime \( p \) and all possible values of \( r \in [1, p-1] \). The cost of building the total output \( O_p \) is \( \Theta(p^3) \).

We discuss the cost of computing a list of polynomials \( \Phi_{pq} \) for a fixed prime \( p_0 \) and a number \( t = p_0^b \) of primes \( q \) that all satisfy \( q \leq p_0^\alpha \) (with \( \alpha > 1 \)).
We now briefly describe algorithms that have been

Comparison with other algorithms.

Case (a). Consider first the case when we wish to compute the polynomials \( \Phi_{pq} \) for a number \( t = p_0^\beta \) of primes \( q \) that satisfy \( q \leq p_0^\alpha \) (with \( \alpha \geq 1 \)) and \( q \pmod{p} = r \). We then precompute only the output \( \mathcal{O}_{pq,r} \), with a cost \( \Theta(p_0^2) \), then we perform \( t \) computations of cost \( p_0^{1+\alpha} \). The total cost is

\[
\Theta(p_0^2) + \Theta(p_0^{1+\alpha + \beta}) = \Theta(p_0^{1+\beta+\alpha}).
\]

In this case, the cost of the precomputation is always smaller than the cost of the computation.

Case (b). Consider now the case when we wish to compute the polynomials \( \Phi_{pq} \) for a number \( t = p_0^\beta \) of primes \( q \) that all satisfy \( q \leq p_0^\alpha \), with \( \alpha \geq 1 \) and various values of \( q \pmod{p} \). We have to compute the total precomputation output \( \mathcal{O}_{pq} \) in \( \Theta(p_0^3) \) steps, then we perform \( t \) computations of cost \( p_0^{1+\alpha} \). The total cost is

\[
\Theta(p_0^3) + \Theta(p_0^{1+\alpha + \beta}) = \Theta(p_0^{1+\max(\alpha + \beta, 2)}).
\]

The cost of the precomputation is larger than the cost of the computation, when \( \alpha + \beta \) is smaller than 2.

Case (c). Consider finally the case when we wish to compute the polynomials \( \Phi_{pq} \) for any prime \( q \leq p_0^\alpha \), with \( \alpha \geq 1 \) with various values of \( q \pmod{p} \). We assume that the list of primes \( q \leq p_0^\alpha \) is already built. There are \( t = \Theta(1/\log p_0)p_0^\alpha \) such primes \( q \). We have to compute the total output \( \mathcal{O}_{pq} \) in \( \Theta(p_0^3) \) steps, then we perform \( t = \Theta(p_0^\delta) \) computations of cost \( p_0^{1+\alpha} \). The total cost is

\[
\Theta(p_0^3) + \Theta(p_0^{1+2\alpha}) = \Theta(p_0^{1+2\alpha}).
\]

The cost of the precomputation is always smaller than the cost of the computation.

Comparison with other algorithms. We now briefly describe algorithms that have been previously designed to compute cyclotomic polynomials. Even when they are designed in the case of a general cyclotomic polynomial, we only analyse their complexity in the binary case, and compare them to the complexity of the BCW algorithm. In the following, \( M(\ell) \) denotes the cost of multiplying two integers of bit-length at most \( \ell \). With fast-multiplication algorithms, \( M(\ell) \) is \( O(\ell \log \ell \log \log \ell) \).

(i) The first algorithm is based on a formula due to Lenstra-Lam-Leung [4, 5], and is only valid in the binary case. With this formula, the \( j \)-th coefficient of \( a_{pq} \) is computed via a solution \((x, y)\) of an equation of the type \( j = xp + yq \) with \(|x| < q \) and \(|y| < p \). The cost of computing one single coefficient is thus \( O(M(\log q)) \). The total cost for the whole vector \( a_{pq} \) is \( \Theta(pqM(\log q)) \).

(ii) The other two algorithms are due to Arnold and Monagan and are described in [1]. They compute the cyclotomic vector in the case of a general order \( n \), but we describe them for a binary order \( n = pq \). They do not perform any polynomial divisions.

(ii)(a) The first algorithm is a recursive algorithm, called the sparse power series algorithm (SPS algorithm, for short) because it expresses cyclotomic polynomials as products
of sparse power series. In the binary case, the recursion is made with \( \Theta(pq) \) additions between integers less than (but close to) \( \phi(pq) \). Then, its complexity is \( \Theta(pq) \log q \).

(ii)(b) The second algorithm is called the Big Prime algorithm (BP, for short). It is adapted to the case when the order \( n \) has a big prime as a factor. Here, in the binary case, the big prime is \( q \). The algorithm is based on Identity (1.1) and performs a precomputation step which involves \( \Theta(p) \) computations of residues modulo \( p \) of integers of size \( O(\log q) \). The cost of this step is then \( \Theta(p)M(\log q) \). The whole cyclotomic word \( a_{pq} \) is obtained via a recursive step which computes \( \phi(pq) \) residues modulo \( p \) of integers of size \( O(\log q) \). Thus, the cost of computing the whole vector \( a_{pq} \) is \( \Theta(pq)M(\log q) \).

The previous discussion shows that the existing algorithms are all of complexity \( O(pq) \cdot E(p, q) \) where the factor \( E(p, q) \) is polynomial in \( \log q \). The complexity of the BCW algorithm does not contain such a factor \( E(p, q) \).

5. Conclusions and further work

In this conclusion, we first explain how our results may be extended to another family of polynomials, related to semi-groups. We then discuss how the representation of Theorem 2.1 may entail results on the length of blocks of consecutive zeros in the cyclotomic words. The paper ends with a short discussion on the non-binary case.

Another polynomial \( F_{p, q} \). We consider two numbers \( (p, q) \) with \( 2 < p < q \), that are coprime but now not necessarily primes, and define the polynomial \( F_{p, q} \in \mathbb{Z}[x] \) as follows:

\[
F_{p, q}(x) = \frac{(x^{pq} - 1)(x - 1)}{(x^p - 1)(x^q - 1)}.
\]

A folklore result relates the polynomial \( F_{p, q} \) to numerical semi-groups (see, [6] for definitions and proofs). A numerical semi-group is the set

\[
S(p, q) = \{ap + bq \mid a, b \in \mathbb{Z}_{\geq 0}\}.
\]

The polynomial \( F_{p, q} \) is called the semi-group polynomial associated with \( S(p, q) \) due to the following identity

\[
F_{p, q}(x) = 1 + (x - 1) \sum_{s \in S(p, q)} x^s.
\]

Clearly, \( F_{p, q} \) coincides with the cyclotomic polynomial \( \Phi_{pq} \) when \( p \) and \( q \) are primes. Also, if we define \( F_q = 1 + x + \cdots + x^{q-1} \), the polynomial \( F_{p, q} \) satisfies the identity (3.2) that is our starting point in the proof of Theorem 2.1, namely

\[
(1 - x^q)F_{p, q}(x) = (1 - x)F_q(x^p).
\]

This entails that all our results also hold for semi-groups polynomials \( F_{p, q} \).

Proving other properties for \( \Phi_{pq} \) and \( F_{p, q} \). The representation provided in Theorem 2.1 implies that the coefficients of \( \Phi_{pq} \) belong to \( A \). A parameter of interest is the maximum
gap of these polynomials, that we now define: for a given polynomial \( f(x) = b_1x^{n_1} + b_2x^{n_2} + \cdots + b_kx^{n_k} \), with \( b_i \) all nonzero and \( n_1 < n_2 < \cdots < n_k \), the maximum gap is defined as
\[
g(f) = \max_{1 \leq i < k} (n_{i+1} - n_i), \quad g(f) = 0 \text{ if } k = 1.
\]
The fact that \( g(F_{p,q}) = p - 1 \) was proven in [3] for binary cyclotomic polynomials \( \Phi_{pq} \) and in [6] for semi-group polynomials. In [2, 7], it is proven that the number of maximum gaps in binary cyclotomic polynomials is \( 2\left\lfloor q/p \right\rfloor \).

Our description in terms of words easily entails the inequality \( g(F_{p,q}) \geq p - 1 \) and proves that there are, at least, \( 2\left\lfloor q/p \right\rfloor \) maximum gaps. The upper bounds will be recovered from our Theorem 2.1 provided that the possible cancellations \( 1 + (-1) \) be controlled in the sums \( \omega_{k-1} + d_1 \). This is the aim of a further work.

**Possible extensions to the non-binary case?** Our results seem to be specific to the binary case where the order is \( n = pq \). In the case when \( n \) is squarefree with at least three prime factors, the coefficients of the cyclotomic polynomial do not any longer belong to a finite alphabet \( A \). It seems thus difficult to easily deal with words on this alphabet \( A \).

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