On the points without universal expansions

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Abstract

Let $1 < \beta < 2$. Given any $x \in [0, (\beta - 1)^{-1}]$, a sequence $(a_n) \in \{0, 1\}^\mathbb{N}$ is called a $\beta$-expansion of $x$ if $x = \sum_{n=1}^{\infty} a_n \beta^{-n}$. For any $k \geq 1$ and any $(b_1 b_2 \cdots b_k) \in \{0, 1\}^k$, if there exists some $k_0$ such that $a_{k_0+1}a_{k_0+2} \cdots a_{k_0+k} = b_1 b_2 \cdots b_k$, then we call $(a_n)$ a universal $\beta$-expansion of $x$. Sidorov [21], Dajani and de Vries [4] proved that given any $1 < \beta < 2$, then Lebesgue almost every point has uncountably many universal expansions. In this paper we consider the set $V_\beta$ of points without universal expansions. For any $n \geq 2$, let $\beta_n$ be the $n$-bonacci number satisfying the following equation: $\beta_n = \beta_{n-1} + \beta_{n-2} + \cdots + \beta + 1$. Then we have $\dim_H(V_{\beta_n}) = 1$, where $\dim_H$ denotes the Hausdorff dimension. Similar results are still available for some other algebraic numbers. As a corollary, we give some results of the Hausdorff dimension of the survivor set generated by some open dynamical systems. This note is another application of our paper [5].

1 Introduction

Let $1 < \beta < 2$. Given any $x \in [0, (\beta - 1)^{-1}]$, a sequence $(a_n) \in \{0, 1\}^\mathbb{N}$ is called a $\beta$-expansion of $x$ if

$$x = \sum_{n=1}^{\infty} a_n \beta^{-n}.$$ 

Sidorov [20] proved that given any $1 < \beta < 2$, then almost every point in $[0, (\beta - 1)^{-1}]$ has uncountably many expansions. If $(a_n)$ is the only $\beta$-expansion of $x$, then we call $x$ a univoque point with unique expansion $(a_n)$. Denote by $U_\beta$ all the univoque points in base $\beta$. For the unique expansions, there are many results, see [9] [12] and references therein.

Let $(a_n)$ be a $\beta$-expansion of $x$. If for any $k \geq 1$, and any $(b_1 b_2 \cdots b_k) \in \{0, 1\}^k$ there exists some $k_0$ such that

$$a_{k_0+1}a_{k_0+2} \cdots a_{k_0+k} = b_1 b_2 \cdots b_k,$$

then we call $(a_n)$ a universal $\beta$-expansion of $x$.

The dynamical approach is a good way which can generate $\beta$-expansions effectively. Define $T_0(x) = \beta x, T_1(x) = \beta x - 1$, see Figure 1.

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Let $x \in [0, (\beta - 1)^{-1}]$ with an expansion $(a_n)_{n=1}^\infty$, and set $T_{a_1a_2\ldots a_n} = T_{a_n} \circ T_{a_{n-1}} \circ \cdots \circ T_{a_1}$. We call $\{T_{a_1a_2\ldots a_n}(x)\}_{n=0}^\infty$ an orbit of $x$ in base $\beta$. For simplicity, we denote by $T_{a_n}(x) = x$.

Clearly, for different expansions, $x$ has distinct orbits. Evidently, for any $n \geq 1$, we have

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_n}{\beta^n} + \frac{T_{a_1a_2\ldots a_n}(x)}{\beta^n}.$$ 

The digits $(a_n)$ are chosen in the following way: if $T_{a_1a_2\ldots a_{j-1}}(x) \in [0, \beta^{-1})$, then $a_j = 0$, if $T_{a_1a_2\ldots a_{j-1}}(x) \in ((\beta - 1)^{-1}\beta^{-1}, (\beta - 1)^{-1}]$, then $a_j = 1$. However, if $T_{a_1a_2\ldots a_{j-1}}(x) \in [\beta^{-1}, (\beta - 1)^{-1}\beta^{-1}]$, then we may choose $a_j$ to be 0 or 1. Due to this observation, we call $[\beta^{-1}, (\beta - 1)^{-1}\beta^{-1}]$ the switch region. All the possible $\beta$-expansions can be generated in terms of this idea, see [8, 3]. If $x$ has exactly $k$ different expansions, then we say $x$ has multiple expansions [22, 6, 7].

For the set of unique expansions, one has criteria that characterizes this type of expansions [9, 12]. However, for the universal expansions and multiple expansions, few papers considered this aspect. In this paper, we shall use the dynamical approach to study the universal expansions.

Universal expansions have a close connection with the following discrete spectra

$$D = \left\{ \sum_{i=0}^n a_i\beta^i : a_i \in \{0, 1\}, n \geq 0 \right\}.$$ 

Denote by $D = \{y_0 = 0 < y_1 < y_2 < \cdots < y_k < \cdots\}$. Define

$$L^1(\beta) = \limsup_{k \to \infty} (y_{k+1} - y_k).$$

Erdős and Komornik [10] proved that if $L^1(\beta) = 0$, then all the points of $(0, (\beta - 1)^{-1})$ have universal expansions. Moreover, Erdős and Komornik [10] showed that if $1 < \beta \leq \sqrt{2} \approx 1.19$, then $L^1(\beta) = 0$. In particular, Erdős and Komornik [10] proved that $L^1(\sqrt{2}) = 0$. Sidorov and Solomyak [23] also considered some algebraic numbers for which $L^1(\beta) = 0$, and their result is improved by Komornik and Akiyama [1]. In [1], Akiyama and Komornik proved that if $1 < \beta \leq \sqrt{2} \approx 1.26$, then $L^1(\beta) = 0$. In [15], Feng utilized Akiyama and Komornik’s result [1], and implemented some ideas in fractal geometry showing that for any non-Pisot $\beta \in (1, \sqrt{2})$ if $\beta^2$ is not a Pisot number, then $L^1(\beta) = 0$. For the generic results, Sidorov [21] showed that given any $1 < \beta < 2$, almost every point in $[0, (\beta - 1)^{-1}]$ has at least one universal expansion. Dajani and de Vries [4], used a dynamical approach to show that for any $\beta > 1$, almost every point of $[0, (\beta - 1)^{-1}]$ has uncountably many universal expansions.
The results of Sidorov [20] and those of Dajani and de Vries [4] imply that the set of points without universal expansions has zero Lebesgue measure. In other words, the Lebesgue measure of $V_\beta$ is zero, where

$$V_\beta = \{ x \in [0, (\beta - 1)^{-1}] : x \text{ does not have a universal expansion} \}.$$ 

A natural question is to study the Hausdorff dimension of the set $V_\beta$. This is the main motivation of this paper. Using one property of Pisot numbers, we have the following result.

**Theorem 1.1.** For any $n \geq 2$, let $\beta_n$ be the $n$-bonacci number satisfying the following equation:

$$\beta_n = \beta^{n-1} + \beta^{n-2} + \cdots + \beta + 1,$$

then $\dim_H(V_{\beta_n}) = 1$.

For some Pisot numbers, we have similar results. For $1 < \beta < \frac{1 + \sqrt{5}}{2}$, interestingly, the Hausdorff dimension of $V_\beta$ has a close connection with an old conjecture posed by Erdős and Komornik [11].

**Conjecture 1.2.** For any non-Pisot $\beta \in \left(1, \frac{1 + \sqrt{5}}{2}\right)$, $L^1(\beta) = 0$.

This conjecture is true if $\beta \in (1, \sqrt{2}]$ and $\beta^2$ is not a Pisot number, see [13]. We make a brief discussion of the connection between the dimension of $V_\beta$ and this conjecture. If we were able to find some non-Pisot number $1 < \beta < \frac{1 + \sqrt{5}}{2}$ such that the Hausdorff dimension of $V_\beta$ is positive, then $L^1(\beta) > 0$. The reason is due to the fact that $L^1(\beta) = 0$ implies all the points of $(0, (\beta - 1)^{-1})$ have universal expansions. In other words, we disprove the Erdős-Komornik conjecture. Therefore, considering the Hausdorff dimension of $V_\beta$ is meaningful to this conjecture. The dimensional problem of $V_\beta$ has a strong relation with open dynamical systems. Roughly speaking, $V_\beta$ is a union of countable survivor sets generated by some open dynamical systems. These open dynamical systems are smaller than the usual open systems as we consider all the possible orbits, i.e. all the possible orbits should avoid some holes. In this paper, we shall make use of this tool to study the dimension of $V_\beta$.

The paper is arranged as follows. In section 2, we start with some necessary definitions and notation, then we state the main results of the paper. In section 3, we give the proofs, and in section 4 we give some final remarks.

## 2 Preliminaries and Main results

In this section, we give some notation and definitions. Let $\Omega = \{0, 1\}^\mathbb{N}$, $E = [0, (\beta - 1)^{-1}]$, and $\sigma$ be the left shift. The random $\beta$-transformation $K$ is defined in the following way, see [8].
Definition 2.1. \( K : \Omega \times E \to \Omega \times E \) is defined by

\[
K(\omega, x) = \begin{cases} 
(\omega, \beta x) & x \in [0, \beta^{-1}) \\
(\sigma(\omega), \beta x - \omega_1) & x \in [\beta^{-1}, \beta^{-1}(\beta - 1)^{-1}] \\
(\omega, \beta x - 1) & x \in (\beta^{-1}(\beta - 1)^{-1}, (\beta - 1)^{-1}] 
\end{cases}
\]

We call \([\beta^{-1}, \beta^{-1}(\beta - 1)^{-1}]\) the switch region since in this region we can choose the digit to be used and change from 0 to 1 or vica versa.

When the orbits of points hit or enter the switch region and we always choose the digit 1, then we call this algorithm the greedy algorithm. More precisely, the greedy map is defined in the following way: \( G : E \to E \) is defined by

\[
G(x) = \begin{cases} 
\beta x & x \in [0, \beta^{-1}) \\
\beta x - 1 & x \in [\beta^{-1}, (\beta - 1)^{-1}] 
\end{cases}
\]

Let \((\omega, x) \in \Omega \times E\). For any \( n \geq 1 \), we denote by \( K^n(\omega, x) = K(K^{n-1}(\omega, x)) \) the \( n \) iteration of \( K \), and let \( \pi(\omega, x) = x \) be the projection in the second coordinate. We can study \( \beta \)-expansions via the following iterated function system,

\[
f_j(x) = \frac{x + j}{\beta}, \quad j \in \{0, 1\}.
\]

The self-similar set [14] for this IFS is the interval \([0, (\beta - 1)^{-1}]\). This tool is useful in the proof of Lemma 3.12. Before we state our main results, we define some sets. Given \( 1 < \beta < 2 \) and any \( N \geq 3 \). Define

\[
E_{\beta,N} = \{ x \in [0, (\beta - 1)^{-1}] : \text{no orbit of } x \text{ hits } [0, \beta^{-N}(\beta - 1)^{-1}] \},
\]

\[
F_{\beta,N} = \{ x \in [0, (\beta - 1)^{-1}] : \text{the greedy orbit of } x \text{ does not hit } [0, \beta^{-N}(\beta - 1)^{-1}] \}.
\]

We can give a simple symbolic explanation of \( E_{\beta,N} \), namely, any \( \beta \)-expansion \((a_n)\) of any point in \( E_{\beta,N} \) does not contain the block \((00 \cdots 0)_N\). Let

\[
O = \{ \pi(K^n(\omega, 1)) \cup \pi(K^n(\omega, (\beta - 1)^{-1} - 1)) : n \geq 0, \omega \in \Omega \}
\]

be the set of all possible orbits of 1 and \((\beta - 1)^{-1} - 1\). An algebraic number \( \beta > 1 \) is called a Pisot number if all of its conjugates lie inside the unit circle. Now we state our main results of this paper.

**Theorem 2.2.** For any \( n \geq 2 \), let \( \beta_n \) be the \( n \)-bonacci number satisfying the following equation:

\[
\beta^n = \beta^{n-1} + \beta^{n-2} + \cdots + \beta + 1,
\]

then \( \dim_H(V_{\beta_n}) = 1 \).

The following result gives a sufficient condition under which the Hausdorff dimension of \( F_{\beta,N} \) can be calculated.

**Corollary 2.3.** Let \( \frac{1 + \sqrt{5}}{2} < \beta < 2 \). If all the possible orbits of 1 hit finite points, then given any \( N \geq 3 \), \( \dim_H(F_{\beta,N}) \) can be calculated explicitly. In particularly, for any Pisot number in \((1, 2)\), \( \dim_H(F_{\beta,N}) \) can be calculated.
This result is indeed a corollary of [5, Theorem 4.2]. Generally, calculating the Hausdorff dimension of $E_{\beta,N}$ is not an easy problem. By definition, $E_{\beta,N} \subset F_{\beta,N}$ for any $N \geq 3$. Hence, $E_{\beta,N}$ is a smaller survivor set, and it is difficult to calculate the dimension of this set. However, for the sequence $(\beta_n)$, we have the following asymptotic result.

**Theorem 2.4.** For any $n \geq 2$ and $N \geq 3$, let $\beta_n$ be the n-bonacci number, then $\dim_H(F_{\beta_n,N-1}) < \dim_H(E_{\beta_n,N}) \leq \dim_H(F_{\beta_n,N})$. Subsequently,

$$\lim_{N \to \infty} \dim_H(E_{\beta_n,N}) = \lim_{N \to \infty} \dim_H(F_{\beta_n,N}) = 1.$$ 

Moreover, for any $N > 2n+4$, $\dim_H(F_{\beta_n,N} \setminus E_{\beta_n,N}) > 0$. Furthermore, we can find some set with positive Hausdorff dimension such that every point in this set has uncountably many expansions, but none of them is a universal expansion.

The last statement strengthens one result of [21, Counterexample]. The following result is about the topological structure of $E_{\beta,N}$.

**Theorem 2.5.** Given any $N \geq 3$, for almost every $\beta \in (1,2)$, $E_{\beta,N}$ is a graph-directed self-similar set.

### 3 Proof of main theorems

In this section, we give a proof of Theorem 2.4. To begin with, we recall some classical results and notation. An expansion $(a_n)$ is called the quasi-greedy expansion if it is the largest infinite expansions, in the sense of lexicographical ordering. Denote by $\sigma((a_n)_{n=1}^\infty) = (a_n)_{n=2}^\infty$, and $\sigma^k((a_n)_{n=1}^\infty) = (a_n)_{n=k+1}^\infty$. Let $(\alpha_n)$ be the quasi-greedy expansion of 1. The following classical result was proved by Parry [18].

**Theorem 3.1.** Let $(a_n)_{n=1}^\infty$ be an expansion of $x \in [0,(\beta-1)^{-1}]$. Then $(a_n)_{n=1}^\infty$ is a greedy expansion if and only if $\sigma^k((a_n)_{n=1}^\infty) < (\alpha_n)_{n=1}^\infty$ if $a_k = 0$.

**Lemma 3.2.** For any $n \geq 2$, let $\beta_n$ be the n-bonacci number. Then for any $N \geq 3$,

$$F_{\beta_n,N-1} \subset E_{\beta_n,N}.$$ 

**Proof.** Since $\beta_n$ is the Pisot number satisfying the equation $\beta^n = \beta^{n-1} + \beta^{n-2} + \cdots + \beta + 1$, it follows that the quasi-greedy expansion of 1 is $(1^{n-1}0)\infty$. Hence the block $1^n \infty$ can appear in the greedy expansions. In other words, any expansion in base $\beta_n$ can be changed into the greedy expansions using the rule $10^n \sim 01^n$, i.e. the block $1^n \infty$ can be replaced by $01^n$ without changing the value of the corresponding number. Given any point $x \notin E_{\beta_n,N}$, there exists an expansion of $x$ such that its coding, say $(a_n)$, consists of a block $(0 \cdots 0)$ with length $N$, i.e. there exists some $k_0$ such that $a_{k_0+1} \cdots a_{k_0+N} = 0 \cdots 0$. If $(a_n)$ is the greedy expansion of $x$, then clearly $x \notin F_{\beta_n,N-1}$. Assume $(a_n)$ is not the greedy expansion. We can transform $(a_n)$ into the greedy expansion of $x$ by using the rule $10^n \sim 01^n$. Denote the acquired greedy expansion of $x$ by $(b_n)$. Notice that the used
transformation shrinks a block of zeros in the sequence \((a_n)\) by at most one term. To
be more precise, if \(a_{k_0+1} \cdots a_{k_0+N}a_{k_0+N+1}a_{k_0+N+n} = 0 \cdots 01^n\), then the corresponding
block is

\[
b_{k_0+1} \cdots b_{k_0+N}b_{k_0+N+1}b_{k_0+N+n} = 0 \cdots 01^n.
\]

Thus, \(x \notin F_{\beta,n,N-1}\). □

Next, we want to prove that

\[
\lim_{N \to \infty} \dim_H(F_{\beta,n,N}) = 1.
\]

This result can be obtained by the perturbation theory, it is essentially proved by Fer-
guson and Pollicott [11, Theorem 1.2].

**Lemma 3.3.** For any \(1 < \beta < 2\), \(\lim_{N \to \infty} \dim_H(F_{\beta,N}) = 1\).

Here we give a detailed proof of our desired limit,

**Lemma 3.4.**

\[
\lim_{N \to \infty} \dim_H(F_{\beta,n,N}) = 1.
\]

For simplicity, we assume \(n = 2\), for \(n \geq 3\) the proof is similar but the calculation is more
complicated. We give an outline of the proof of this lemma. First, we give a Markov
partition for \([0, (\beta - 1)^{-1}]\) using the orbit of 1. Hence, we can define an adjacency matrix
\(S\) and construct an associated subshift of finite type \(\Sigma\). Equivalently, we transform the
original space \([0, 1]^N\) into a subshift of finite type. Next, we define a submatrix \(S'\) of \(S\),
and construct a graph-directed self-similar set with the open set condition [17]. Finally,
we identify \(F_{\beta,n,N}\) with a graph-directed self-similar set, and prove the desired result.
Now we transform the symbolic space as follows.

**Lemma 3.5.** Let \(\beta = \frac{1 + \sqrt{5}}{2}\), and \(x \in [0, (\beta - 1)^{-1}]\). Then the greedy expansion of \(x\)
has a coding which is from some subshift of finite type.

**Proof.** Firstly, we give a Markov partition for the interval \([0, (\beta - 1)^{-1}]\) as follows: let

\[
a_1 = 0, a_i = \beta^{-N-2+i}(\beta - 1)^{-1}, 2 \leq i \leq n - 1, a_n = \beta^{-1} = \beta^{-2}(\beta - 1)^{-1},
\]

\(a_{n+1} = 1, a_{n+2} = (\beta - 1)^{-1}\). Define

\[
A_1 = [0, \beta^{-N}(\beta - 1)^{-1}], A_i = [\beta^{-N+i-2}(\beta - 1)^{-1}, \beta^{-N+i-1}(\beta - 1)^{-1}], 2 \leq i \leq N,
A_{N+1} = [1, (\beta - 1)^{-1}].
\]

It is easy to check that

\[
T_0(A_i) = A_1 \cup A_2, T_0(A_i) = A_{i+1}, 2 \leq i \leq N - 1,
\]

and that \(T_1(A_N) = \bigcup_{i=1}^{N-1} A_i, T_1(A_{N+1}) = A_N \cup A_{N+1}\). Hence, we have the following
adjacency matrix \(S = (s_{ij})_{(N+1) \times (N+1)}\)

\[
s_{ij} = \begin{cases} 
1 & i = 1, j = 1, 2 \\
1 & 2 \leq i \leq N - 1, j = i + 1 \\
1 & i = N, j = 1, 2, \ldots, N - 1 \\
1 & i = N + 1, j = N, N + 1 \\
0 & \text{else}
\end{cases}
\]
In terms of $S$, we can construct a subshift of finite type $\Sigma$. For any 
\[(\alpha_i) \in \{1, 2, \cdots, N + 1\}^N,\]
we call $\{A_{\alpha_i}\}_{i=1}^\infty$ an admissible path if there is some $T_k$, $k = 0$ or 1, such that 
\[T_k(A_{\alpha_i}) \supset A_{\alpha_{i+1}}\]
for any $i \geq 1$. In terms of this definition, we have that 
\[\Sigma = \{(\alpha_i)_{i=1}^\infty : \alpha_i \in \{1, 2, \cdots, N + 1\}, \{A_{\alpha_i}\}_{i=1}^\infty \text{ is an admissible path}\}.\]

\[\square\]

**Remark 3.6.** Usually, we take the elements of Markov partition $A_i$ closed on the left and open on the right, i.e. $A_i = [a_i, a_{i+1})$. Under our algorithm, one has a choice at the endpoints. For example the point $\beta^{-1}$ it is the right endpoint of $A_{N-3}$ and the left endpoint of $A_{N-2}$. For this point, we can implement $T_0$ on $A_k = [a_k, \beta^{-1}]$ or $T_1$ on $[\beta^{-1}, a_{k+2}]$. This adjustment is due to the proof of Lemma 3.8. When we construct a graph-directed self-similar set, we need the closed interval, see the graph-directed construction in [17]. This is the reason why we need some compromise here. Although our Markov partition is a little different from the usual definition, this adjustment does not affect our result.

By definition of $E_{\beta,N}$, for any point $x \in E_{\beta,N}$, all possible orbits of $x$ do not hit the hole $A_1 = [0, \beta^{-N}(\beta - 1)^{-1}]$, which is the first element of the Markov partition. By Lemma 3.5, $x$ also has a coding in the new symbolic space $\Sigma$. For simplicity, we denote this coding of $x$ in $\Sigma$ by $\{\alpha_n\}_{n=1}^\infty$. Since $x \in E_{\beta,N}$, the symbol 1 cannot appear in the coding $\{\alpha_n\}_{n=1}^\infty$. Motivated by this observation, we construct a new matrix as follows. We delete the first row and first column of $S$, and keep the rest of the matrix. Denote the new resulting matrix by $S'$, and the associated subshift generated by $S'$ is denoted by $\Sigma'$. $S'$ can be represented by a directed graph $(V, E)$. The vertex set consists of the underlying partition $\{A_i\}_{i=2}^k$. For two vertices, if one vertex is one of components of the image of another vertex, then we can find a similitude, which is the inverse of the expanding map, between these two vertices. For instance, for the vertices $A_2$ and $A_3$, if $T_0(A_2) = A_3$, then we can label a directed edge, from the vertex $A_2$ to $A_3$, by a similitude $f(x) = T_0^{-1}(x) = \frac{x}{\beta}$. We denote all admissible labels between two vertices by $E$. Then by Mauldin and Williams’ result [17], we can construct a graph-directed self-similar set $K'_N$ satisfying the open set condition, for the detailed construction, see [17]. Now we have the following lemma.

**Lemma 3.7.** Let $\beta = \frac{1 + \sqrt{5}}{2}$. $F_{\beta,N} = K'_N$ except for a countable set, i.e. there exists a countable set $C_1$ such that $F_{\beta,N} \subset K'_N \subset C_1 \cup F_{\beta,N}$.

**Proof.** Evidently, $F_{\beta,N} \subset K'_N$. Take $x \in K'_N$. Then by the definition of $K'_N$, the greedy orbit of $x$ does not hit $[0, \beta^{-N}(\beta - 1)^{-1})$. If the greedy orbit of $x$ does not hit the closed interval $[0, \beta^{-N}(\beta - 1)^{-1}]$, then $x \in F_{\beta,N}$. If there exists some $(i_1i_2\cdots i_n)$ such that 
\[T_{i_1i_2\cdots i_n}(x) = \beta^{-N}(\beta - 1)^{-1},\]
then
\[ x \in \bigcup_{n=1}^{\infty} \bigcup_{(i_1, \ldots, i_n) \in \{0,1\}^n} f_{i_1 \ldots i_n}(\beta^{-N}(\beta - 1)^{-1}), \]
where \( f_0(x) = \beta^{-1}x, f_1(x) = \beta^{-1}x + \beta^{-1} \). Therefore,
\[ K_N' \subset E_{\beta,N} \cup \bigcup_{n=1}^{\infty} \bigcup_{(i_1, \ldots, i_n) \in \{0,1\}^n} f_{i_1 \ldots i_n}(\beta^{-N}(\beta - 1)^{-1}). \]

Lemma 3.8. Let \( \beta = \frac{1 + \sqrt{5}}{2} \). Then
\[
\dim_H(F_{\beta,N}) = \frac{\log \lambda_N}{\log \beta},
\]
where \( \lambda_N \) is the largest positive root of the following equation
\[
x^{N-1} = \sum_{i=0}^{N-3} x^i.
\]
Moreover, \( \lim_{N \to \infty} \lambda_N = \frac{1 + \sqrt{5}}{2} = \beta \).

Proof. By Lemma 3.7, \( \dim_H(F_{\beta,N}) = \dim_H(K_N') \). \( K_N' \) is a graph-directed self-similar set with the open set condition, as such we can explicitly calculate its Hausdorff dimension, namely, \( \dim_H(F_{\beta,N}) = \frac{\log \lambda_N}{\log \beta} \), where \( \lambda_N \) is indeed the spectral radius of \( S' \), for the detailed method, see [17]. The second statement is a simple exercise. We finish the proof of Lemma 3.4 for the case \( n = 2 \). For \( n \geq 3 \), the proof is similar. \( \square \)

Similar result is available for the doubling map with hole [13]. Let \( D(x) = 2x \mod 1 \) be the doubling map defined on \([0,1)\). Given any \( \epsilon > 0 \), set
\[
D_\epsilon = \{ x \in [0,1) : D^n(x) \notin [0,\epsilon] \text{ for any } n \geq 0 \}. \]
Clearly \( \lim_{\epsilon \to 0} \dim_H(D_\epsilon) \) exists. Hence, it suffices to consider the following set
\[
D_{2^{-N}} = \{ x \in [0,1) : D^n(x) \notin [0,2^{-N}] \text{ for any } n \geq 0 \}
\]
if we want to find \( \lim_{\epsilon \to 0} \dim_H(D_\epsilon) \). We have the following result.

Example 3.9.
\[
\dim_H(D_{2^{-N}}) = \frac{\log \gamma_N}{\log 2},
\]
where \( \gamma_N \) is the \( N \)-bonacci number satisfying the equation
\[
x^n = x^{n-1} + x^{n-2} + \cdots + x + 1.
\]
It is easy to see that \( \lim_{N \to \infty} \gamma_N = 2 \). Therefore,
\[
\lim_{\epsilon \to 0} \dim_H(D_\epsilon) = \lim_{N \to \infty} \dim_H(D_{2^{-N}}) = 1.
\]
Proof of Theorem 1.1. Let $\beta_n$ be a $n$-bonacci number. By Lemmas 3.4 and 3.7 we have

$$E_{\beta_n,N} \subset F_{\beta_n,N} \subset K'_N \subset F_{\beta_n,N} \cup C_1.$$ 

Finally, by Lemma 3.4,

$$\lim_{N \to \infty} \dim_H(F_{\beta_n,N}) = 1.$$ 

Therefore,

$$\dim_H(F_{\beta_n,N}) = \dim_H(E_{\beta_n,N}) \leq \dim_H(V_{\beta_n}) \leq 1,$$

which implies that $\dim_H(V_{\beta_n}) = 1.$ \hfill $\square$

It is easy to show that when $\beta$ is a Pisot number, then all the possible orbits of $x \in \mathbb{Q}([\beta]) \cap [0, (\beta - 1)^{-1}]$ hit finitely many points only. The following lemma is standard. However for the sake of convenience, we give the detailed proof.

Lemma 3.10. Suppose $\beta$ is a Pisot number and $x \in \mathbb{Q}([\beta]) \cap [0, (\beta - 1)^{-1}]$, then the set

$$\{ \pi(K^n(\omega, x)) : n \geq 0, \omega \in \Omega \}$$

is a finite set.

Proof. Let $M(X) = X^d - q_1 X^{d-1} - \cdots - q_d$ be the minimal polynomial of $\beta$ with $q_i \in \mathbb{Z}$. Since $\mathbb{Q}([\beta])$ is generated by $\{\beta^{-1}, \cdots, \beta^{-d}\}$, there exist $a_1, a_2 \cdots, a_d \in \mathbb{Z}$ and $b \in \mathbb{N}$ such that

$$x = b^{-1} \sum_{i=1}^{d} a_i \beta^{-i}.$$ 

We assume that $b$ is as small as possible to ensure uniqueness. Let $\beta_1 = \beta$, and $\beta_2, \cdots, \beta_d$ the Galois conjugates of $\beta$, and set $B = (\beta_j^i)_{1 \leq i, j \leq d}$. Define for $n \geq 0$ and $\omega \in \Omega$,

$$r^{(1)}_n(\omega) = \beta^n \left( x - \sum_{k=1}^{n} b_k(\omega, x) \beta^{-k} \right)$$

and

$$r^{(j)}_n(\omega) = \beta^n_j \left( b^{-1} \sum_{i=1}^{d} a_i \beta^{-i}_j - \sum_{k=1}^{n} b_k(\omega, x) \beta^{-k}_j \right)$$

for $j = 2, 3, \cdots, d$. Consider the vector $R_n(\omega) = (r^{(1)}_n(\omega), \cdots, r^{(d)}_n(\omega))$. We first show that the set $\{R_n(\omega) : n \geq 0, \omega \in \Omega\}$ is uniformly bounded (in $n$ and $\omega$). First note that $r^{(1)}_n(\omega) = \pi(K^n(\omega, x))$, hence $|r^{(1)}_n(\omega)| \leq (\beta - 1)^{-1}$ for any $n$ and any $\omega$. Let $\eta = \max_{2 \leq j \leq d} |\beta_j|$, then $\eta < 1$. For $j = 2, \cdots, d$

$$|r^{(j)}_n(\omega)| = \left| \left( b^{-1} \sum_{i=1}^{d} a_i \beta^{-i}_j - \sum_{k=1}^{n} b_k(\omega, x) \beta^{-k}_j \right) \right| \leq \left| b^{-1} \sum_{i=1}^{d} a_i |\eta^{-i}| \right| + \left| \sum_{k=1}^{n} b_k(\omega, x) \eta^{-k} \right| \leq \frac{b^{-1} \max_{2 \leq j \leq d} |a_i| + 1}{1 - \eta}.$$ 

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Let $C = \max \left\{ (\beta - 1)^{-1}, \frac{b^{-1}\max_{1 \leq i \leq d} |a_i| + 1}{1 - \eta} \right\}$, then $r_n^{(j)} < c$ for any $1 \leq j \leq d, n \geq 0$ and $\omega \in \Omega$. Thus the set $\{R_n(\omega) : n \geq 0, \omega \in \Omega\}$ is uniformly bounded. Next we show that for each $\omega \in \Omega$ and $n \geq 0$, there exists $P(X) \in \mathbb{Z}_n(\omega) \in \mathbb{Z}^d$, then $\beta_2, \cdots, \beta_d$ are also roots of $P(X)$, it suffices to show that

$$r_n^{(1)} = b^{-1} \left( \sum_{k=1}^{d} z_n^{(k)}(\omega)\beta^{-k} \right)$$

for $z_n^{(k)} \in \mathbb{Z}$. The proof is done by contradiction. Let $n = 1$ and note that $1 = q_1\beta^{-1} + \cdots + q_d\beta^{-d}$. Now

$$r_1^{(1)}(\omega) = \beta x - b_1(\omega, x)$$

$$= \beta b^{-1} \sum_{k=1}^{d} a_k\beta^{-k} - b_1(\omega, x) \sum_{k=1}^{d} q_k\beta^{-k}$$

$$= b^{-1} \left( \sum_{k=1}^{d} (a_1 q_k - b_1(\omega, x) b q_k + a_{k+1})\beta^{-k} + (a_1 - b_1(\omega, x) b) q_d \beta^{-d} \right)$$

$$= b^{-1} \sum_{k=1}^{d} z_1^{(k)}(\omega)\beta^{-k}$$

with

$$z_1^{(k)}(\omega) = \begin{cases} (a_1 - b_1(\omega, x) b) q_k + a_{k+1} & \text{if } k \neq d \\ (a_1 - b_1(\omega, x) b) q_d & \text{if } k = d \end{cases}$$

Suppose now that $r_i^{(1)} = b^{-1} \sum_{k=1}^{d} z_i^{(k)}(\omega)\beta^{-k}$ for $z_i^{(k)} \in \mathbb{Z}$. Since $r_i^{(1)} = \pi(K^n(\omega, x))$ for all $n \geq 0$, we have

$$r_{i+1}^{(1)} = \beta r_i^{(1)} - b_{i+1}(\omega, x)$$

$$= \beta b^{-1} \sum_{k=1}^{d} z_i^{(k)}(\omega)\beta^{-k} - b_{i+1}(\omega, x) \sum_{k=1}^{d} q_k\beta^{-k}$$

$$= b^{-1} \left( \sum_{k=1}^{d-1} (z_i^{(k)} q_k - b_{i+1}(\omega, x) b q_k + z_i^{(k+1)})\beta^{-k} + (z_i^{(1)}(\omega) - b_{i+1}(\omega, x) b) q_d \beta^{-d} \right)$$

$$= b^{-1} \sum_{k=1}^{d} z_{i+1}^{(k+1)}(\omega)\beta^{-k}$$

with

$$z_{i+1}^{(k)}(\omega) = \begin{cases} (z_i^{(1)}(\omega) - b_{i+1}(\omega, x) b) q_k + z_i^{(k+1)}(\omega) & \text{if } k \neq d \\ (z_i^{(1)}(\omega) - b_{i+1}(\omega, x) b) q_d & \text{if } k = d \end{cases}$$

Thus, $z_{i+1}^{(k)}(\omega) \in \mathbb{Z}$. Setting $\mathbb{Z}(\omega) = (z_1^{(1)}, \cdots, z_d^{(d)})$, we have $\mathbb{Z}(\omega) \in \mathbb{Z}^d$ and $R_n(\omega) = b^{-1} \mathbb{Z}(\omega) B$. Since $B$ is invertible, and $R_n(\omega)$ is uniformly bounded in $n$ and $\omega$, we have that $\mathbb{Z}(\omega)$ is uniformly bounded, and hence takes only finitely many values. It follows that $(R_n(\omega))$ takes only finitely many values. Therefore, the set

$$\{\pi(K^n(\omega, x)) : n \geq 0, \omega \in \Omega\}$$

is finite. \hfill \Box
Corollary 3.11. Let $\beta \in (1,2)$ be a Pisot number. For any $a_{i_1}a_{i_2}\cdots a_{i_n} \in \{0,1\}^n$, the orbits of the endpoints of the interval $f_{a_{i_1}a_{i_2}\cdots a_{i_n}}([0, (\beta - 1)^{-1}])$ hit finite points.

Proof. By symmetry, we only need to prove that for the left endpoint $\sum_{j=1}^{n} a_{i_j} \beta^{-j}$, all of its orbits hit finite points. This is a directly consequence of Lemma 3.10. \hfill \square

Proof of Corollary 2.3 and Theorem 2.4. By Lemma 3.10 and Corollary 3.11 and the main result of Mauldin and Williams [17], we can calculate the Hausdorff dimension $H$ of $\dim$. Moreover, the following inclusion holds, i.e. we assume that $\beta_n$. Clearly, $D$ has uncountably many elements. Moreover, the following inclusion holds,

$$D = \{10^{i_1}10^{i_2}10^{i_3} \cdots : n+1 \leq i_k \leq N-1 \text{ and there are infinitely many } i_k = N-1\}.$$

By Theorem 3.1 all the codings in $D$ are greedy in base $\beta_n$. Using the rule $10^n \sim 01^n$, we have

$$x = (10^{N-1}10^{i_2}10^{i_3} \cdots)_\beta = (10^N1^{n-1}01^{i_2}10^{i_3} \cdots)_\beta \notin E_{\beta \cdot N} \cup E_{\beta},$$

where $(b_k)_\beta = \sum_{k=1}^{n} b_k \beta^{-k}$. Hence, $p(D) \cap E_{\beta \cdot N} \cup E_{\beta} = \emptyset$. Now we want to show that $p(D) \cap E_{\beta \cdot N} = \emptyset$ and $\dim H(F_{\beta \cdot N}) > 0$. By the definition of $D$, for any $10^{i_1}10^{i_2}10^{i_3} \cdots$ there are infinitely many $i_k = N-1$. Without loss of generality, we assume that $i_1 = N-1$, i.e. let

$$(a_k) = 10^{N-1}10^{i_2}10^{i_3} \cdots.$$

Using the rule $10^n \sim 01^n$, we have

$$x = (10^{N-1}10^{i_2}10^{i_3} \cdots)_\beta = (10^N1^{n-1}01^{i_2}10^{i_3} \cdots)_\beta \notin E_{\beta \cdot N} \cup E_{\beta},$$

where $(b_k)_\beta = \sum_{k=1}^{n} b_k \beta^{-k}$. Hence, $p(D) \cap E_{\beta \cdot N} \cup E_{\beta} = \emptyset$. In order to prove $\dim H(F_{\beta \cdot N}) > 0$, it suffices to show that $\dim H(p(D)) > 0$. Here, the set $(D, \sigma)$ is indeed a subset of some $S$-gap shift [10], i.e. $D \subset D'$, where

$$D' = \{10^{i_1}10^{i_2}10^{i_3} \cdots : n+1 \leq i_k \leq N-1\}.$$

The entropy of $D'$ can be calculated, i.e. $h(D') = \log \lambda$, where $\lambda$ is the largest positive root of the equation

$$1 = \sum_{k \in \{n+1, \ldots, N-1\}} x^{-k-1}.$$

Now we construct a subset of $p(D)$ as follows: let $J$ be the self-similar set with the IFS

$$\left\{ g_1(x) = \frac{x}{\beta^{n+2}} + \frac{1}{\beta} , g_2(x) = \frac{x}{\beta^{N}} + \frac{1}{\beta} \right\},$$

i.e.

$$J = g_1(J) \cup g_2(J).$$

By the definitions of $p(D)$ and $J$, $J \subset p(D)$. Let

$$E := (\beta^{N-1}(\beta^{N-1})^{-1}, \beta^{n+1}(\beta^{n+2} - 1)^{-1}).$$

Proof. By symmetry, we only need to prove that for the left endpoint $\sum_{j=1}^{n} a_{i_j} \beta^{-j}$, all of its orbits hit finite points. This is a directly consequence of Lemma 3.10. \hfill \square
It is easy to check that \( g_1(E) \cap g_2(E) = \emptyset \), and \( g_i(E) \subset E \). In other words, the IFS satisfies the open set condition \([14]\). Hence, \( \dim_H(J) = s > 0 \), where \( s \) is the unique solution of the equation \( \beta^{(-n-2)s} + \beta^{-Ns} = 1 \). Subsequently,

\[
0 < \dim_H(J) = s \leq \dim_H(p(D)).
\]

For the last statement of Theorem 2.4 it suffices to consider the set \( p(D) \). \( \square \)

Now we prove Theorem 2.5. We partition the proof into several lemmas. The following result is essentially proved in [2]. For convenience, we give the detailed proof.

**Lemma 3.12.** Given \( 1 < \beta < 2 \) and let \( N \geq 3 \). If there exists some \((\eta_1, \eta_2, \ldots, \eta_p) \in \{0, 1\}^p\) such that \( T_{\eta_1, \eta_2, \ldots, \eta_p}(\beta^{-N}(\beta - 1)^{-1}) \in (0, \beta^{-N}(\beta - 1)^{-1}) \), then \( E_{\beta,N} \) is a graph-directed self-similar set.

**Proof.** By assumption and the continuity of the \( T_j \)'s, there exists \( \delta > 0 \) such that

\[
T_{\eta_1, \eta_2, \ldots, \eta_p}(\beta^{-N}(\beta - 1)^{-1}, \beta^{-N}(\beta - 1)^{-1} + \delta) \subset (0, \beta^{-N}(\beta - 1)^{-1}).
\]

Set \( H = [0, \beta^{-N}(\beta - 1)^{-1} + \delta] \). We partition \( [0, (\beta - 1)^{-1}] \) in terms of the iterated function system

\[
f_j(x) = \frac{x + j}{\beta}, j \in \{0, 1\}.
\]

For any \( L \) we have

\[
[0, (\beta - 1)^{-1}] = \bigcup_{(i_1, \ldots, i_L) \in \{1, \ldots, m\}^L} f_{i_1} \circ \cdots \circ f_{i_L}([0, (\beta - 1)^{-1}]).
\]

We assume without loss of generality that \( L \) is sufficiently large such that

\[
|f_{i_1} \circ \cdots \circ f_{i_L}([0, (\beta - 1)^{-1}])| < \delta
\]

for all \((i_1, \ldots, i_L) \in \{0, 1\}^L\). Correspondingly, we partition the symbolic space \( \{0, 1\}^N \) provided by the cylinders of length \( L \). For every \((i_1, \ldots, i_L) \in \{0, 1\}^L \) let

\[
C_{i_1 \ldots i_L} = \left\{ (x_n) \in \{0, 1\}^N : x_n = i_n \text{ for } 1 \leq n \leq L \right\}.
\]

The set \( \{C_{i_1 \ldots i_L} \}_{i_1, \ldots, i_L} \in \{0, 1\}^L \) is a partition of \( \{0, 1\}^N \), and \( f_{i_1} \circ \cdots \circ f_{i_L}([0, (\beta - 1)^{-1}]) = \pi(C_{i_1 \ldots i_L}) \). Let

\[
\mathcal{F} = \left\{ (i_1, \ldots, i_L) \in \{1, \ldots, m\}^L : f_{i_1} \circ \cdots \circ f_{i_L}([0, (\beta - 1)^{-1}]) \cap [0, (\beta - 1)^{-1}] \neq \emptyset \right\}
\]

and

\[
\mathcal{F}' = \bigcup_{(i_1, \ldots, i_L) \in \mathcal{F}} \pi(C_{i_1 \ldots i_L}).
\]

By our assumptions on the size of our cylinders the following inclusions hold

\[
[0, \beta^{-N}(\beta - 1)^{-1}] \subset \mathcal{F}' \subset H.
\]

Using these inclusions we can show that \( x \notin E_{\beta,N} \) if and only if there exists \((\theta_1, \ldots, \theta_n) \in \{1, \ldots, m\}^n\) such that \( T_{\theta_1 \ldots \theta_n}(x) \in \mathcal{F}' \). If \( x \notin E_{\beta,N} \) then by the above observation, there
exists \((\theta_1, \ldots, \theta_{n_1}) \in \{1, \ldots, m\}^{n_1}\) such that \(T_{\theta_1 \ldots \theta_{n_1}}(x) \in \mathbb{F}'\). Therefore, \(x\) has a coding containing a block from \(\mathbb{F}\). Conversely, if there exists \((\theta_1, \ldots, \theta_{n_1}) \in \{1, \ldots, m\}^{n_1}\) such that \(T_{\theta_1 \ldots \theta_{n_1}}(x) \in \mathbb{F}'\), then the condition

\[
T_{\eta_1 \ldots \eta_p}(\beta^{-N}(\beta - 1)^{-1}, \beta^{-N}(\beta - 1)^{-1} + \delta) \subset (0, \beta^{-N}(\beta - 1)^{-1})
\]

yields \(x \notin E_{\beta,N}\). Taking \(\mathbb{F}\) to be the set of forbidden words defining a subshift of finite type, we see that \(E_{\beta,N}\) is a graph-directed self-similar set, see [5, 17].

Schmeling [19] proved the following result.

**Lemma 3.13.** For almost every \(\beta \in (1, 2)\), the greedy orbits of 1 and the lazy orbit of \(\bar{1} = (\beta - 1)^{-1} - 1\) are dense.

**Proof of Theorem 2.5.** Theorem 2.5 follows immediately from Lemmas 3.12 and 3.13.

\[\square\]

## 4 Final remarks

Similar results are available if we consider \(\beta\)-expansions with more than two digits. For some Pisot numbers, we may implement similar ideas which are utilized in Lemmas 3.4 and 3.2. Finally we pose a problem.

**Problem 4.1.** Does there exist \(\delta > 0\) such that for any \(\beta \in (2 - \delta, 2)\), \(\dim_H(V_{\beta}) = 1\).

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