GEOMETRIC PROPERTIES OF DOMAINS RELATED TO $\mu$-SYNTHESIS

PAWEL ZAPALOWSKI

ABSTRACT. In the paper we study the geometric properties of a large family of domains, called the generalized tetrablocks, related to the $\mu$-synthesis, containing both the family of the symmetrized polydiscs and the family of the $\mu_{1,n}$-quotients $E_n$, $n \geq 2$, introduced recently by G. Bharali. It is proved that the generalized tetrablock cannot be exhausted by domains biholomorphic to convex ones. Moreover, it is shown that the Carathéodory distance and the Lempert function are not equal on a large subfamily of the generalized tetrablocks, containing i.a. $E_n$, $n \geq 4$. We also derive a number of geometric properties of the generalized tetrablocks as well as the $\mu_{1,n}$-quotients. As a by-product, we get that the pentablock, another domain related to the $\mu$-synthesis problem introduced recently by J. Agler, Z. A. Lykova, and N. J. Young, cannot be exhausted by domains biholomorphic to convex ones.

1. INTRODUCTION

A consequence of the celebrated Lempert theorem (cf. [21]) is the fact that if a domain $D$ can be exhausted by domains biholomorphic to convex ones, then the Carathéodory distance and the Lempert function coincide on $D$.

For more than 20 years it was an open conjecture that any bounded pseudoconvex domain $D$ with equality of the Carathéodory distance and the Lempert function can be exhausted by domains biholomorphic to convex ones.

Ten years ago A. Agler and N. J. Young introduced domain $G_2$, arising from the $\mu$-synthesis, called symmetrized bidisc (cf. [3]). In 2007 A. A. Abouhajar, M. C. White, N. J. Young introduced another domain related to $\mu$-synthesis problem, called tetrablock and denoted by $E$ (cf. [1]). Both domains are bounded, hyperconvex (cf. Section 3 for the definition of the hyperconvexity), and they cannot be exhausted by domains biholomorphic to convex ones. Nevertheless, the Lempert function and the Carathéodory distance coincide on them (see [3], [7], [10], [13], [25]). Further properties of these domains may be found in [19], [26] and [29].

$G_2$ and $E$ are—so far—the only counterexamples to the conjecture stated above.

A natural generalization of the symmetrized bidisc to higher dimensions is the symmetrized polydisc (cf. [8]). It turned out that in the family of the symmetrized polydiscs the symmetrized bidisc is the only counterexample for the converse to the Lempert theorem (see [22], [23], [24]). Further properties of the symmetrized polydisc may be found in [11].

Recently G. Bharali introduced another domain closely associated with an aspect of $\mu$-synthesis, denoted by $E_n$ and called $\mu_{1,n}$-quotient, $n \geq 2$ (cf. [6]). It is a natural generalization of the tetrablock, since $E_2 = E$.

This article is devoted to studying the complex geometry of bounded domains related to the $\mu$-synthesis, which form a large family, containing both the family of the symmetrized polydiscs and the family of the $\mu_{1,n}$-quotients. The domains considered in the paper are generated by the space $E$ of the scalar block diagonal matrices (see the formula (3) below). We shall call them the generalized tetrablocks and denote by $E_E$. In the engineering literature (e.g. [9]) the space $E$ of matrices is usually taken to be given by a block diagonal structure, which partially justifies our choice. Let us mention here that such a choice of the space $E$ implies the logarithmic plurisubharmonicity of the structured singular value $\mu_E$ (cf. Proposition 3.2). The relation of the generalized tetrablocks to the $\mu$-synthesis problem will be explained in Section 3.

Our first aim is to show that most of the generalized tetrablocks are not the counterexamples for the converse to the Lempert theorem. To be more precise, we show that the Carathéodory distance and the Lempert function are not equal on a large subfamily—denote it for a moment by $E$—of the generalized tetrablocks (cf. Proposition 3.10). We also show that none of the generalized tetrablock can be exhausted by domains biholomorphic to convex ones (cf. Theorem 3.12).

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We also prove that any generalized tetrablock from the family $E$ is neither $\mathbb{C}$-convex nor starlike about the origin, and that there is another subfamily of the generalized tetrablocks, containing i.a. the $\mu_{1,n}$-quotients, such that each member of this subfamily is linearly convex, and hence pseudoconvex (cf. Proposition 5.18), hyperconvex and polynomially convex (cf. Proposition 5.20).

As an application, we get that in the family of the $\mu_{1,n}$-quotients, bounded hyperconvex domains, there are at most two counterexamples to the converse of the Lempert theorem. More precisely, the Carathéodory distance and the Lempert function are not equal on $\mathbb{E}_n$, $n \geq 4$. Moreover, none of $\mathbb{E}_n$ can be exhausted by domains biholomorphic to convex ones (cf. Theorem 4.1 which collects also further properties of the $\mu_{1,n}$-quotients). All this properties make the family of the $\mu_{1,n}$-quotients very similar the family of the symmetrized polydiscs.

As a by-product of our considerations we get that the pentablock, another domain related to $\mu$-synthesis introduced recently by J. Agler, Z. A. Lykova, and N. J. Young in [4]—although it is not generated by the space of the scalar block diagonal matrices—is hyperconvex and yet cannot be exhausted by domains biholomorphic to convex ones (cf. Theorem 5.1 and Proposition 5.2).

Almost all results mentioned above are—more or less—easy consequence of the following, simple but powerful, fact saying that the generalized tetrablock $\mathbb{E}_E'$ generated by any subspace $E'$ of the vector space $E$ is an analytic retract of $\mathbb{E}_E$ (cf. Theorem 3.7). Another important tool we exploit in the paper are Propositions 2.1 and 2.2 which originate in A. Edigarian’s paper [12]. Since both propositions may be formulated in terms of arbitrary retracts, we put them into separate section.

The paper is organized as follows. In Section 2 we formulate two properties of general analytic retracts, we shall use in the sequel. In Section 3 we define the family of the generalized tetrablocks, show their relation to the $\mu$-synthesis problem, and give its geometric properties. In Section 4 we gather all results concerning the $\mu_{1,n}$-quotients, whereas the last section is devoted to the pentablock.

Here is some notation we shall use throughout the paper. By $\mathbb{D}$ we denote the open unit disc in the complex plane. Let $c_D$, $k_D$, and $l_D$ denote, respectively, the Carathéodory pseudodistance, the Kobayashi pseudodistance, and the Lempert function of a domain $D \subset \mathbb{C}^n$ (for the definition and main properties of $c_D$, $k_D$, and $l_D$ the Reader may consult [13]). For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ we use the standard notation

$$\lambda z := (\lambda z_1, \ldots, \lambda z_n), \quad z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$ 

Moreover, for $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$ and $\lambda \in \mathbb{C}$ denote the action on $\mathbb{C}^n$

$$m_\lambda z := (\lambda^{m_1} z_1, \ldots, \lambda^{m_n} z_n), \quad z = (z_1, \ldots, z_n) \in \mathbb{C}^n.$$ 

In the paper we will use the notion of quasibalanced domains. Recall that a domain $D \subset \mathbb{C}^n$ is called $m$-balanced, where $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$, if $m_\lambda z \in D$ whenever $z \in D$ and $\lambda \in \mathbb{D}$. A $(1, \ldots, 1)$-balanced domain is called balanced. A domain is called quasi-balanced, if it is $m$-balanced for some $m$.

## 2. Analytic retracts

A domain $G$ is said to be an analytic retract of a domain $D$ if there exist analytic maps $\theta : G \rightarrow D$, $\iota : D \rightarrow G$ such that $\iota \circ \theta = \text{id}_G$.

For a domain $G$ by $S(G)$ we denote the set of all holomorphic mappings $F : G \times G \rightarrow G$ such that $F(z, z) = z$, $F(z, w) = F(w, z)$ for any $z, w \in G$.

Moreover, $G$ is called taut if for any sequence $(f_j)_{j \in \mathbb{N}}$ of holomorphic mappings $f_j : \mathbb{D} \rightarrow G$ there exists a subsequence $(f_{j_k})_{k \in \mathbb{N}}$ convergent uniformly on compact sets to a holomorphic mapping $f : \mathbb{D} \rightarrow G$ or there exists a subsequence $(f_{j_k})_{k \in \mathbb{N}}$ that diverges uniformly on compact sets.

We shall make use of the following simple observation, which originates in A. Edigarian’s paper [12] and is interesting in its own right.

**Proposition 2.1.** Let $G$ be an analytic retract of $D$ such that $S(G) = \emptyset$. Then $D$ is not biholomorphic to a convex domain. If, additionally, $G$ is taut, then $D$ cannot be exhausted by domains biholomorphic to convex ones.

**Proof.** Suppose $\Omega$ is a convex domain and $f : D \rightarrow \Omega$ is biholomorphic. By assumption, there are holomorphic mappings $\theta : G \rightarrow D$, $\iota : D \rightarrow G$ with $\iota \circ \theta = \text{id}_G$. Define

$$F(z, w) := \iota \circ f^{-1} \left( \frac{f \circ \theta(z) + f \circ \theta(w)}{2} \right), \quad z, w \in G.$$ 

Observe that $F \in S(G)$—a contradiction.
Now assume $G$ is taut. Suppose $D_1 \subset D_2 \subset \ldots, \bigcup_{j \geq 1} D_j = D$, $\Omega_j$ is a convex domain and $f_j : D_j \to \Omega_j$ is biholomorphic, $j \geq 1$. Define $G_j := \theta^{-1}(D_j)$ and
\[
F_j(z, w) := \iota \circ f_j^{-1} \left( \frac{f_j \circ \theta(z) + f_j \circ \theta(w)}{2} \right), \quad z, w \in G_j, \quad j \geq 1.
\]
Observe that $F_j : G_j \times G_j \to G$ with $F_j(z, w) = F_j(w, z)$, $F_j(z, z) = z, z, w \in G_j, \quad j \geq 1$. It follows easily from Montel’s argument that there exists a holomorphic mapping $F : G \times G \to G$ such that $F(z, w) = F(w, z)$, $F(z, z) = z, z, w \in G$. Tautness of $G$ implies that either $F(G \times G) \subset G$ or $F(G \times G) \not\subset \partial G$. Since $F(z, z) = z \in G$, we conclude that the first case holds, i.e. $F \in \mathcal{S}(G)$—a contradiction.

Using holomorphic contractibility of the families of the Kobayashi pseudodistances and the Lempert functions we are able to prove

**Proposition 2.2.** Let $G$ be an analytic retract of $D$ such that $l_G$ is not a distance. Then $l_D$ is not a distance. In particular, $c_D \not\equiv l_D$ and $D$ cannot be exhausted by domains biholomorphic to convex ones.

**Proof.** Suppose $l_D$ is a distance, i.e. $k_D \equiv l_D$. By assumption, there are holomorphic mappings $\theta : G \to D$, $\iota : D \to G$ such that the holomorphic contractibility of the relevant families implies
\[
k_G(x_1, x_2) \geq k_D(\theta(x_1), \theta(x_2)) = l_D(\theta(x_1), \theta(x_2)) \geq l_G(\iota \circ \theta(x_1), \iota \circ \theta(x_2)) = l_G(x_1, x_2), \quad x_1, x_2 \in G,
\]
i.e. $k_G \equiv l_G$—a contradiction. $\square$

3. **The Generalized Tetrablock**

Consider positive integers $n \geq 2$, $s \leq n$, and $r_1, \ldots, r_s$ with $\sum_{j=1}^s r_j = n$. In the set $A(r_1, \ldots, r_s) := \{0, \ldots, r_1\} \times \cdots \times \{0, \ldots, r_s\} \setminus \{(0, \ldots, 0)\}$ we introduce the following order. Given two different $\alpha = (\alpha_1, \ldots, \alpha_s), \beta = (\beta_1, \ldots, \beta_s) \in A(r_1, \ldots, r_s)$ we write
\[
\alpha < \beta \iff \alpha_j < \beta_j, \quad \text{where} \quad j_0 := \max\{j : \alpha_j \neq \beta_j\}.
\]
Therefore we may write $A(r_1, \ldots, r_s) = \{\alpha^1, \ldots, \alpha^N\}$, where $\alpha^1 < \cdots < \alpha^N$ and $N := \prod_{j=1}^s (r_j + 1) - 1$.

Finally, for $x = (x_1, \ldots, x_N) \in \mathbb{C}^N$ and $z = (z_1, \ldots, z_s) \in \mathbb{C}^s$ put
\[
R_x(z) := 1 + \sum_{j=1}^N (-1)^{|\alpha^j|} x_j z^{\alpha^j},
\]
and define
\[
E_{n,r_1,\ldots,r_s} := \left\{ x \in \mathbb{C}^N : \forall z \in \mathbb{C}^s \quad R_x(z) \neq 0 \right\}.
\]
The set $E_{n,r_1,\ldots,r_s}$ we shall call the **generalized tetrablock**.

**Remark 3.1.** Note that $E_{2,1,2} = \mathbb{G}_2, E_{n,1,n} = \mathbb{G}_n, E_{2,2,1,1} = \mathbb{E}$, and $E_{n,2,n-1,1} = E_n$.

3.1. **Relation to the $\mu$-synthesis problem.** One of the central notions in the theory of robust control is the structured singular value, a matrix function denoted by $\mu$ and defined on $\mathbb{C}^{m \times n}$. In the definition of $\mu$ there is an underlying structure identified with linear subspace $E$ of $\mathbb{C}^{n \times m}$.

Let $E$ be a linear subspace of $\mathbb{C}^{n \times m}$. The **structured singular value** $\mu_E$ relative to $E$ is a function $\mu_E : \mathbb{C}^{m \times n} \to \mathbb{R}_+$ given by
\[
\mu_E(A) := \inf \{\|X\| : X \in E, \det(I_n - AX) = 0\}, \quad A \in \mathbb{C}^{m \times n},
\]
with the understanding that $\mu_E(A) = 0$ if $I_n - AX$ is always nonsingular. Here $\|\cdot\|$ denotes the operator norm. Recall that
- $\mu_E$ is upper semicontinuous,
- $\mu_E(\lambda A) = |\lambda| \mu_E(A)$ for any $\lambda \in \mathbb{C}, A \in \mathbb{C}^{m \times n}$.

In particular,
\[
\Omega_{\mu_E} := \{ A \in \mathbb{C}^{n \times m} : \mu_E(A) < 1 \}
\]
is a balanced domain and $\mu_E$ is its Minkowski functional (cf. [15], Remark 2.2.1). The space $E$ is usually taken to be given by a block diagonal structure (cf. [9] for basic properties of $\mu_E$ is this case). In this paper we consider only repeated scalar blocks. To be more precise, for a given
positive integers \( n \geq 2, s \leq n \), and \( r_1, \ldots, r_s \) with \( \sum_{j=1}^s r_j = n \), consider the vector subspace \( E \subset \mathbb{C}^{n \times n} \) consisting of the following scalar block diagonal matrices

\[
E = E(n; s; r_1, \ldots, r_s) := \{ \text{diag}[z_1 I_{r_1}, \ldots, z_s I_{r_s}] \} \subset \mathbb{C}^{n \times n} : z_1, \ldots, z_s \in \mathbb{C} \}.
\]

Throughout the paper \( E \) shall always denote the above subspace unless stated otherwise. For such a space \( E \),

- \( \rho = \rho_{E(n; i; a)} \leq \mu_E \leq \mu_{A^{n \times n}} = \| \cdot \| \), where \( \rho \) is the spectral radius,
- \( \mathbb{B}_{n \times n} \subset \Omega_{E} \subset \mathbb{E}_{n \times n} \), where \( \mathbb{B}_{n \times n} : = \{ X \in \mathbb{C}^{n \times n} : \| X \| < 1 \} \) is the unit ball and \( \Omega_{n} := \{ X \in \mathbb{C}^{n \times n} : \rho(X) < 1 \} \) is the spectral ball,
- \( \mu_E \) is continuous,
- \( \mu_E(A) = \max_{X \in \mathbb{B}_{n \times n}} \rho(XA) \) for any \( A \in E \).

**Proposition 3.2.** \( \log \mu_E \) is continuous plurisubharmonic and \( \Omega_{E} \) is pseudoconvex.

**Proof.** Recall that the spectral radius \( \rho \) is plurisubharmonic function (cf. [27]). Then the properties above imply that \( \mu_E \) is plurisubharmonic. Now, as \( \mu_E \) is plurisubharmonic Minkowski functional of the balanced domain \( \Omega_{E} \), we conclude that \( \Omega_{E} \) is pseudoconvex and \( \log \mu_E \) is plurisubharmonic (cf. [10], Proposition 2.2.22).

In the theory of robust control, the \( \mu \)-synthesis problem—an interpolation problem for analytic matrix functions, a generalization of the classical problems of Nevanlinna-Pick and Carathéodory-Fejér—is to construct an analytic matrix function \( F : \mathbb{D} \rightarrow \Omega_{E} \) satisfying a finite number of interpolation conditions.

There is a natural relation between \( \mathbb{E}_{n; s; r_1, \ldots, r_s} \) and the domain \( \Omega_{E} \).

For \( j \leq n \) let \( \mathcal{J}^j := \{ (i_1, \ldots, i_j) \in \mathbb{N}^j : 1 \leq i_1 < \cdots < i_j \leq n \} \). Moreover, for \( \alpha \in (r_1, \ldots, r_s) \) define

\[
\mathcal{J}^{|\alpha|}_\alpha := \{ (i_1, \ldots, i_{|\alpha|}) \in \mathcal{J}^{|\alpha|} : r_1 + \cdots + r_j - 1 + 1 \leq i_{|\alpha|} + \cdots + i_{|\alpha|+1} < \cdots < i_{|\alpha|+\ldots+i_{|\alpha|+1}} \leq r_1 + \cdots + r_j, j = 1, \ldots, s \}.
\]

(recall here that \( |\alpha| \leq n \)). It is elementary to see that

\[
\bigcup_{j=1}^n \mathcal{J}^j = \bigcup_{j=1}^{|\alpha|} \mathcal{J}^{|\alpha|}_{\alpha} \quad \text{and} \quad \mathcal{J}^{|\alpha|}_{\alpha} \cap \mathcal{J}^{|\beta|}_{\beta} = \emptyset \quad \text{whenever } \alpha \neq \beta.
\]

Finally, for \( I \in \mathcal{J}^j \) and \( A \in \mathbb{C}^{n \times n} \) let \( A_I \) denotes the \( j \times j \) submatrix of \( A \) whose rows and columns are indexed by \( I \).

Define a polynomial mapping \( \pi_E : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^N \) given by

\[
\pi_E(A) := \left( \sum_{I \in \mathcal{J}^{|\alpha|}_{\alpha}} \det(A_I), \ldots, \sum_{I \in \mathcal{J}^{|\alpha|}_{N}} \det(A_I) \right).
\]

**Proposition 3.3.** \( \pi_E(\Omega_{E}) \subset \mathbb{E}_{n; s; r_1, \ldots, r_s} \).

In view of the above proposition, to shorten the notation, we shall write \( \mathbb{E}_E := \mathbb{E}_{n; s; r_1, \ldots, r_s} \). In the sequel we shall use the following

**Lemma 3.4** (cf. [6]). If \( A \in \mathbb{C}^{n \times n} \) then

\[
\det(I_n - A \text{diag}[z_1, \ldots, z_n]) = 1 + \sum_{j=1}^n (-1)^j \sum_{I \in \mathcal{J}^j} \det(A_I) z_I,
\]

where \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \) and \( I := (i_1, \ldots, i_j) \) for \( I = (i_1, \ldots, i_j) \).

**Proof of Proposition 3.3** Let \( r > 0 \) and \( A \in \mathbb{C}^{n \times n} \). Observe that \( \mu_E(A) \leq 1/r \) iff \( \| X \| \geq r \) for any \( X \in E \) with \( \det(I_n - AX) = 0 \).

For \( z = (z_1, \ldots, z_s) \in \mathbb{C}^s \) define \( \mathbf{z} = (z_{r_1 \times 1}, \ldots, z_{s \times r_s}) \in \mathbb{C}^n \). Note that

\[
\mathbf{z}_I = z^{|\alpha|}, \quad I \in \mathcal{J}^{|\alpha|}_\alpha, \quad z \in \mathbb{C}^s, \quad \alpha \in (r_1, \ldots, r_s).
\]

For any \( X \in E \) there is \( z = (z_1, \ldots, z_s) \in \mathbb{C}^s \) such that

\[
X = \text{diag}[z_1, \ldots, z_1, z_2, \ldots, z_s] \in \mathbb{C}^{n \times n}.
\]
Lemma 3.4 together with (1) implies that for $X$ given by (3) we have

$$(6) \quad \det(I_n - AX) = 1 + \sum_{j=1}^{n} (-1)^{|\alpha|} \left( \sum_{I \in \mathcal{J}_n^{|\alpha|}} \det A_I \right) z^{\alpha_j},$$

Hence, by (1), $\mu_E(A) \leq 1/r$ iff the zero variety of the polynomial $\mu$ in $z_1, \ldots, z_s$ does not meet the open polydisc $(r\mathbb{D})^s$.

Suppose that $\mu_E(A) < 1$ and $x = \pi_E(A)$. For some $r > 1$ we have $\mu_E(A) \leq 1/r$, and so the zero variety of $\mu$ is disjoint from $(r\mathbb{D})^s$ and, consequently, from $\overline{\mathbb{D}}^s$. Thus $x \in \mathbb{E}_E$.

□

**Remark 3.5.** (a) If $n = 2$, $s = 1$, i.e. $E := \{zI_2 \in \mathbb{C}^{2 \times 2} : z \in \mathbb{C}\}$, then $N = 2$ and

$$\pi_E(A) = (\text{tr} A, \det A), \quad A \in \mathbb{C}^{2 \times 2}.$$

(b) If $n = s = 2$, i.e. $E := \{\text{diag}[z_1, z_2] \in \mathbb{C}^{2 \times 2} : z_1, z_2 \in \mathbb{C}\}$, then $N = 3$ and

$$\pi_E(A) = (a_{1,1}, a_{2,2}, \det A), \quad A = [a_{j,k}]_{j,k=1}^2 \in \mathbb{C}^{2 \times 2}.$$

(c) More general, if $s = 2$, $r_1 = n - 1$, $r_2 = 1$, i.e. $E := \{\text{diag}[z_1I_{n-1}, z_2] \in \mathbb{C}^{n \times n} : z_1, z_2 \in \mathbb{C}\}$, then $N = 2n - 1$ and

$$\pi_E(A) = \left( \sum_{I \in \mathcal{J}_1^{|\alpha|}, i_1 \geq 2} \det A_I, \ldots, \sum_{I \in \mathcal{J}^{s-1-1}, i_1 \geq 2} \det A_I, \sum_{I \in \mathcal{J}_1^{|\alpha|}, i_1 = 1} \det A_I, \ldots, \sum_{I \in \mathcal{J}^{s-1-1}, i_1 = 1} \det A_I \right), \quad A \in \mathbb{C}^{n \times n}.$$

(d) Recall that $\pi_E(\Omega_{\mu_E}) = \mathbb{E}_E$ for $\mathbb{E}_E \in \{\mathbb{G}_n, \mathbb{E}_n\}$, $n \geq 2$. It is an open question whether this equality holds for general $\mathbb{E}_E$.

(e) About 15 years ago J. Agler and N. J. Young in [2] devised a new approach to the Nevanlinna–Pick interpolation problem for $\Omega_{\mu_E}$. They reduced the given analytic interpolation problem for $\Omega_{\mu_E}$-valued functions with to one for $\mathbb{G}_n$-valued functions (if $n = 2$ and $s = 1$) or $\mathbb{E}_n$-valued functions (if $n = s = 2$). Recently, G. Bharali applied this reduction strategy in the case of $\mathbb{E}_n$-valued functions (if $s = 2$, $r_1 = n - 1$, $r_2 = 1$). Previous attempts to find analysable instances of $\mu$-synthesis have led to the study of the symmetrized bidisc, the tetrablock and the $\mu_{1,m}$-quotients. First two of these domains have turned out to have interesting function-theoretic properties. The genesis of this paper was to examine to what extend properties of $\mathbb{G}_n$ and $\mathbb{E}_n$ are inherited by their natural generalizations such as $\mu_{1,n}$-quotients $\mathbb{E}_n$ or the so-called generalized tetrablocks $\mathbb{E}_E$.

(f) Observe that $n \leq N \leq 2^n - 1$. Moreover, if $s = 1$ then $N = n$, whereas for $s = n$ we have $N = 2^n - 1$.

(g) Recall that one of two major effects of the idea introduced by J. Agler and N. J. Young is the reduction in the dimension complexity of the Nevanlinna–Pick interpolation problem for $\Omega_{\mu_E}$. (f) shows that this advantage disappears completely as the number of scalar blocks in $E$ increases. Moreover, the dimension may significantly increase when passing form $\Omega_{\mu_E}$ to $\mathbb{E}_E$ as $n^2 \ll 2^n - 1$ for big $n$.

3.2. **Geometry of the generalized tetrablock.**

**Proposition 3.6.** $\mathbb{E}_E$ is bounded $(|\alpha^1|, \ldots, |\alpha^N|)$-balanced domain.

**Proof.** First we show that $\mathbb{E}_E$ is $|\alpha|$-balanced, where $|\alpha| := (|\alpha^1|, \ldots, |\alpha^N|)$. Take $x = (x_1, \ldots, x_N) \in \mathbb{E}_E$ and $\lambda \in \overline{\mathbb{D}}$. Our aim is to show $|\alpha|_\lambda x \in \mathbb{E}_E$, i.e.

$$R_{|\alpha|_\lambda x}(z) \neq 0, \quad z = (z_1, \ldots, z_s) \in \overline{\mathbb{D}}^s.$$ 

But it is an immediate consequence of the following equality

$$R_{|\alpha|_x}(z) = 1 + \sum_{j=1}^{N} (-1)^{|\alpha|} |\alpha^j| x_j z^{|\alpha^j|} = 1 + \sum_{j=1}^{N} (-1)^{|\alpha|} x_j(\lambda z)^{|\alpha^j|} = R_x(\lambda z).$$

It remains to observe that $R_x(\lambda z) \neq 0$, since $\lambda z \in \overline{\mathbb{D}}^s$ for any $z \in \overline{\mathbb{D}}^s$. Thus $\mathbb{E}_E$ is $(|\alpha^1|, \ldots, |\alpha^N|)$-balanced set.

Since $\mathbb{E}_E$ is open by definition, we conclude that $\mathbb{E}_E$ is $(|\alpha^1|, \ldots, |\alpha^N|)$-balanced domain.
To see $E_E$ is bounded we proceed as follows. Take $m = (m_1, \ldots, m_s) \in \mathbb{N}^s$ such that
\[ z_0^{(m, \alpha') \neq z_0^{(m, \alpha')}, \quad j, k = 1, \ldots, N, \ j \neq k, \ z_0 \in \overline{D} \setminus \{0\}, \]
and
\[ (-1)^{[\alpha]} = (-1)^{(m, \alpha')}, \quad j = 1, \ldots, N. \]
Put $z(z_0, m) := (z_0^{m_1}, \ldots, z_0^{m_s}), \ z_0 \in \overline{D}$. Take $(x_1, \ldots, x_N) \in E_E$. Since
\[ R_s(z(z_0, m)) = 1 + \sum_{j=1}^N (-1)^{(m, \alpha')} x_j z_0^{(m, \alpha')} \neq 0, \ z_0 \in \overline{D}, \]
we conclude that $\tilde{x} := (\tilde{x}_1, \ldots, \tilde{x}_M) \in G_M$ with $M := \max\{\langle m, \alpha' \rangle : j = 1, \ldots, N\}$ and
\[ \tilde{x}_k := \begin{cases} x_k, & \text{if there is } j \text{ such that } \langle m, \alpha' \rangle = k, \ k = 1, \ldots, M. \end{cases} \]
The boundedness of the symmetrized polydisc $G_M$ finishes the proof. \hfill \square

Let
\[ E' := \{ \text{diag}[z_1^{1}, \ldots, z_s^{1}] \in \mathbb{C}^{n' \times n'} : z_1, \ldots, z_s \in \mathbb{C} \}, \]
for some $s' \leq s$ and $n' := \sum_{j=1}^{s'} r_j$. Let $N'$ be such that
\[ \alpha_j' = 0, \quad 1 \leq j \leq N', \ s' < \nu \leq s, \quad \text{and} \quad \alpha_{s'+1}' \neq 0. \]
Observe that $N' = \prod_{j=1}^{s'} (r_j + 1) - 1$. We define
\[ (\alpha')' := (\alpha_1', \ldots, \alpha_s'), \quad j = 1, \ldots, N'. \]
For $x \in \mathbb{C}^N$ write $x = (x', x'') \in \mathbb{C}^{N'} \times \mathbb{C}^{N''}$, where $N'' := N - N'$. For $z \in \mathbb{C}^s$ write $z = (z', z'') \in \mathbb{C}^{s'} \times \mathbb{C}^{s-s'}$. Finally, for $x' = (x_1, \ldots, x_N')$ define
\[ R_{x'}(z') := 1 + \sum_{j=1}^{N'} (-1)^{[\alpha']'} x_j (z')^{(\alpha')'}, \quad z' \in \mathbb{C}^{s'}. \]
Then
\[ E_{E'} = \{ z' \in \mathbb{C}^{N'} : \forall z \in \overline{D}, \ R_{x'}(z') \neq 0 \}. \]
Throughout the paper $E'$ will always denote the "subspace" \( \square \) of the space $E$ given by \( \square \) unless stated otherwise. We start with elementary but crucial

**Theorem 3.7.** The mappings
\[ E_{E'} \ni x' \overset{\theta}{\mapsto} (x', 0) \in E_E, \quad E_E \ni (x', x'') \mapsto x' \in E_{E'}, \]
are well defined. In particular, $E_{E'}$ is an analytic retract of $E_E$. Moreover, $E_E$ is a Hartogs domain over $E_{E'}$ with $N'$-dimensional $m$-balanced fibers, where
\[ m = \left[ \begin{array}{c} \alpha_1^{N'+1}, \ldots, \alpha_s^{N'+1} \end{array} \right]_{(N'+1) \times (N'+1)} \in \mathbb{Z}^{N''}, \]
\[ M = \prod_{j=s'+1}^s (r_j + 1) - 1. \]

**Proof.** Let $x' \in E_{E'}$. Consider the point $(x', 0) \in \mathbb{C}^N$. Then
\[ R_{(x', 0)}(z) = 1 + \sum_{j=1}^{N'} (-1)^{[\alpha']'} x_j z^{\alpha'} = 1 + \sum_{j=1}^{N'} (-1)^{[\alpha']'} x_j (z')^{(\alpha')'} = R_{x'}(z'), \]
for any $z = (z', z'') \in \mathbb{C}^s$. Consequently, since $R_{x'}(z') \neq 0$ for any $z' \in \overline{D}^{s'}$ then also $R_{(x', 0)}(z) \neq 0$ for any $z = (z', z'') \in \overline{D}^N$, i.e. $(x', 0) \in E_E$. Hence $\theta$ is well defined.
Now take $x \in \EuScript E_E$. Directly from the definition of $\EuScript E_E$ it follows that $R_x(z',0) \neq 0$ for all $z' \in \overline{\Omega'}$.

Note that
\[
R_x(z',0) = 1 + \sum_{j=1}^{N'} (-1)^{|\alpha^j|}x_jz^{\alpha^j} = 1 + \sum_{j=1}^{N'} (-1)^{(|\alpha^j|)}x_j(z')^{(\alpha^j)} = R_x(z'),
\]
whence $x' \in \EuScript E_{E'}$, i.e. $\nu$ is well defined, too.

So far we know that
\[
\EuScript E_E \cap (\mathbb{C}^{N'} \times \{0\}^{N''}) = \EuScript E_{E'} \times \{0\}^{N''}.
\]
To see that $\EuScript E_{E'}$ is a Hartogs domain over $\EuScript E_{E'}$ with $N''$-dimensional $m$-balanced fibers we proceed as follows. For $x' \in \EuScript E_{E'}$ define the fiber $D_{x'} := \{x'' \in \mathbb{C}^{N'} : (x', x'') \in \EuScript E_E\}$. It remains to see that $D_{x'}$ is $m$-balanced, $x' \in \EuScript E_{E'}$. Recall that
\[
N'' = N - N' = \prod_{j=1}^{s'} (r_1 + 1) \prod_{j=s'+1}^{s} (r_1 + 1) - 1 = (N' + 1)M.
\]
Let
\[
\beta^j := (\alpha^j(N'+1), \ldots, \alpha^j(N'+1)), \quad j = 1, \ldots, M,
\]
and observe that $|\beta^j| = |\alpha^j(N'+1)|$, $j = 1, \ldots, M$.

Fix $x' \in \EuScript E_{E'}$ and $x'' = (x_1, \ldots, x_{N''}) \in D_{x'}$. We aim at showing that
\[
R_{(x',|\beta|,x'')} (z) \neq 0, \quad z \in \overline{\Omega'}, \quad \lambda \in \overline{\Omega},
\]
where $|\beta| := (|\beta^1|, \ldots, |\beta^1|, \ldots, |\beta^M|, \ldots, |\beta^M|)$. In other words, we want to show that
\[
R_{(x',|\beta|,x'')} (z) \neq 0, \quad z \in \overline{\Omega'}, \quad \lambda \in \overline{\Omega}.
\]
But it is an immediate consequence of the following equality
\[
R_{(x',|\beta|,x'')} (z) = R_{(x',|\beta|,x'')} (z) + \sum_{k=1}^{M} (-1)^{|\beta^k|}x_k^{(N'+1)+j}(z')^{(\alpha^j)}.
\]
Indeed, using (11) we get
\[
R_{(x',|\beta|,x'')} (z) = R_{(x',|\beta|,x'')} (z') + \sum_{k=1}^{M} (-1)^{|\beta^k|}x_k^{(N'+1)+j}(z')^{(\alpha^j)}.
\]
Hence and from the fact that $(z', \lambda z'') \in \overline{\Omega'}$ for any $\lambda \in \overline{\Omega}$ and $(z', z'') \in \overline{\Omega''}$ we get (10). \hfill \Box

Remark 3.8. Note that in the above theorem instead of first $s'$ blocks $r_1, \ldots, r_{s'}$ that define the subspace $E'$ one may take arbitrary subset $\{r_{j_1}, \ldots, r_{j_s'}\}$ of $\{r_1, \ldots, r_s\}$.

Corollary 3.9. $G_{\text{max}(2, r_1, \ldots, r_s)}$ is an analytic retract of $\EuScript E_E$.

Proof. If $s = 1$ then $\EuScript E_E = \mathbb{G}_0$ and we are done. So assume that $s > 1$. If there is $j$ with $r_j = \text{max}(2, r_1, \ldots, r_s)$, without loss of generality we may assume that $r_1 = \text{max}(r_1, \ldots, r_s)$ and define $E' := \{z_{r_1} : z \in \mathbb{C}\}$. Then Theorem 3.7 implies that $\EuScript E_{E'} = \mathbb{G}_{r_1}$ is an analytic retract of $\EuScript E_E$.

Otherwise $s = n$ and $r_1 = \cdots = r_s = 1$. Then we define $E' := \{\text{diag}(z_1, z_2) : z_1, z_2 \in \mathbb{C}\}$ and either $\EuScript E_E = \EuScript E_{E'}$, or Theorem 3.7 implies that $\EuScript E_{E'} = \EuScript E$ is an analytic retract of $\EuScript E_E$. Moreover, $\mathbb{G}_2$ is an analytic retract of $\EuScript E$. Indeed, to see this consider the analytic mappings
\[
\mathbb{G}_2 \ni (s, p) \mapsto \theta \mapsto \left(\frac{s}{2}, \frac{s}{2}, p\right) \in \EuScript E, \quad \EuScript E \ni (x_1, x_2, x_3) \mapsto \iota \mapsto (x_1 + x_2, x_3) \in \mathbb{G}_2,
\]
whence $\iota \circ \theta = \text{id}_{\mathbb{G}_2}$. Consequently, $\mathbb{G}_2$ is an analytic retract of $\EuScript E_E$. \hfill \Box

Proposition 3.10. Assume there is $j$ such that $r_j \geq 3$. Then $l_{\EuScript E_E}$ is not a distance. In particular, $c_E \neq l_{\EuScript E_E}$. \hfill \Box
Proof. If \( s = 1 \) then \( E_E = G_n, n \geq 3 \), and we are done. So assume that \( s > 1 \). Without loss of generality we may assume that \( r_1 \geq 3 \). We apply Theorem 3.7 to \( E' \) with \( s' = 1 \). Then we use Proposition 2.2 with \( G = E_{E'} = G_{r_1}, D = E_E \) and the fact that \( G_{r_1} \) is not a distance (cf. [22]).

Moreover, in some cases we get more precise information.

**Proposition 3.11.** If there is \( j \) such that \( r_j = 3 \) then \( c_{E_E}(0, \cdot) \neq k_{E_E}(0, \cdot) \). In particular, \( c_{E_E}(0, \cdot) \neq k_{E_E}(0, \cdot) \).

Proof. If \( s = 1 \) then \( E_E = G_3 \) and we are done. So assume that \( s > 1 \). Without loss of generality we may assume that \( r_1 = 3 \). We apply Theorem 3.7 to \( E' \) with \( s' = 1 \). Then we use Proposition 2.2 with \( G = E_{E'} = G_3, D = E_E \) and the fact that \( c_{G_3}(0, \cdot) \neq k_{G_3}(0, \cdot) \) (cf. [23]).

Now we are in position to prove the following

**Theorem 3.12.** \( E_E \) cannot be exhausted by domains biholomorphic to convex ones.

Proof. In view of Proposition 3.10 it suffices to consider \( \max\{r_1, \ldots, r_s\} \leq 2 \). From Corollary 3.9 it follows that \( G_2 \) is an analytic retract of \( E_E \). Moreover, \( G_2 \) is taut and \( S(G_2) = \emptyset \) (cf. [12], Corollary 3). It remains to apply Proposition 2.1.

We conclude this subsection with some further basic geometric properties of the generalized tetrablocks \( E_E \).

**Corollary 3.13.** \( E_E \) is not circled.

Proof. Corollary 3.9 implies that, after the permutation of the variables if necessary, there is \( n \geq 2 \) such that \( (x, 0) \in E_E \) if \( x \in G_n \). It remains to observe that \( G_n \) is not circled.

Recall that a domain \( D \subset \mathbb{C}^n \) is called (cf. [15], [3])
- \( \mathbb{C} \)-convex if for any affine complex line \( L \) such that \( L \cap D \neq \emptyset \), the set \( L \cap D \) is connected and simply connected;
- linearly convex if its complement is a union of affine complex hyperplanes.

Note that any \( \mathbb{C} \)-convex domain is linearly convex.

**Proposition 3.14.** If there is \( j \) such that \( r_j \geq 3 \) then \( E_E \) is neither \( \mathbb{C} \)-convex nor starlike about the origin.

Proof. Without loss of generality we may assume that \( r_1 \geq 3 \). Let \( E' \) be given by (7) with \( s' = 1 \). It follows from Theorem 3.7 that
\[
E_E \cap (\mathbb{C}^{r_1} \times \{0\}^{N-r_1}) = G_{r_1} \times \{0\}^{N-r_1}.
\]
Since \( G_{r_1} \) is not \( \mathbb{C} \)-convex (cf. [25]), there is an affine complex line \( L' \subset \mathbb{C}^{r_1} \) such that \( L' \cap G_{r_1} \neq \emptyset \) and the set \( L' \cap G_{r_1} \neq \emptyset \) is not connected or is not simply connected. Consequently, \( L := L' \times \{0\}^{N-r_1} \subset \mathbb{C}^N \) is an affine complex line such that \( L \cap E_E \neq \emptyset \) and the set \( L \cap E_E \) either is not connected or is not simply connected.

To see \( E_E \) is not starlike about the origin, use (12) and the fact that \( G_{r_1} \) is not starlike about the origin (cf. [25]).

In [28] N. J. Young showed that \( E \) is not an analytic retract of the open unit ball of a \( J^* \)-algebra of finite rank (see [14] for a definition of a \( J^* \)-algebra). By careful analysis of Young’s proof L. Kosinski showed in [20] that the same property holds for \( G_2 \) (and hence for \( P \), since \( G_2 \) is an analytic retract of \( P \)). Consequently, we get

**Proposition 3.15.** Assume that there is \( j \) such that \( r_j = 2 \) or there are \( j, k, j \neq k \), such that \( r_j = r_k = 1 \). Then \( E_E \) is not an analytic retract of the open unit ball of a \( J^* \)-algebra of finite rank.

3.3. The case \( r_1 = \cdots = r_s = 1 \). Assume \( s \geq 2 \). Let \( E' \) be defined as in (7) with \( s' = s - 1 \). We write the polynomial (2) defining \( E_E \) in the form
\[
R_{s}(z) = R_{s}'(z') - z_{s}P_{s'}(z')
\]
and define the rational function
\[
\Psi_{s'}(x) := \frac{P_{s'}(z')}{R_{s}'(z')}, \quad x = (x', x'') \in \mathbb{C}^{N} \times \mathbb{C}^{N''}, \quad z' \in \mathbb{C}^{s-1}, \quad R_{s}'(z') \neq 0.
\]

There is the following immediate characterization of such \( E_E \), analogous to the one for the \( \mu_{1,n} \)-quotient \( E_n \) (cf. [11] and [8]).
Proposition 3.16. Let $s \geq 2$, $r_s = 1$, $E'$ be as in [7] with $s' = s - 1$, and let $(x', x'') \in \mathbb{C}^{N'} \times \mathbb{C}^{N''}$. Then the following are equivalent

(i) $(x', x'') \in \mathbb{E}_E$;

(ii) $x' \in \mathbb{E}_{E'}$, the function $z' \mapsto \Psi_z(x', x'')$ is holomorphic on $\mathbb{D}^{s-1}$, continuous on $\overline{\mathbb{D}}^{s-1}$, and

$$\max_{z' \in \overline{\mathbb{D}}^{s-1}} |\Psi_{z'}(x', x'')| = \max_{z' \in \overline{\mathbb{D}}^{s-1}} |\Psi_{z'}(x', x'')| < 1.$$

Proof. Condition (i) is equivalent to

$$R_{s'}(z') \neq z_{s'}P_{s''}(z'), \quad z' \in \mathbb{D}^{s-1}, \quad z_s \in \overline{\mathbb{D}},$$

i.e. $x' \in \mathbb{E}_{E'}$ and $1 \neq z_s\Psi_{z'}(x', x'')$ for all $z' \in \overline{\mathbb{D}}^{s-1}$, which is equivalent to (ii).

Let $\overline{\mathbb{E}}_E$ denote the closure of $\mathbb{E}_E$. Similarly we obtain

Proposition 3.17. Let $s \geq 2$, $r_2 = \cdots = r_s = 1$, $E'$ be as in [7] with $s' = s - 1$, and let $x = (x', x'') \in \mathbb{C}^{N'} \times \mathbb{C}^{N''}$. Then the following are equivalent

(i) $R_s(z) \neq 0$ for any $z \in \mathbb{D}^s$;

(ii) $x \in \overline{\mathbb{E}}_E$;

(iii) $x' \in \overline{\mathbb{E}}_{E'}$, the function $z' \mapsto \Psi_{z'}(x', x'')$ is holomorphic on $\mathbb{D}^{s-1}$, and

$$\sup_{z' \in \mathbb{D}^{s-1}} |\Psi_{z'}(x', x'')| \leq 1.$$

Proof. (i)$\Rightarrow$(ii) Take $z \in \overline{\mathbb{D}}$. Since $rz \in \mathbb{D}^s$ for any $r \in (0, 1)$, (i) implies $R_s(rz) \neq 0$. Recall that $R_s(rz) = R_{s'}(z)$, whence $|\Psi_{z}(x) \in \mathbb{E}_E$ for $r \in (0, 1)$, i.e. $x \in \overline{\mathbb{E}}_E$.

(ii)$\Rightarrow$(i) Using induction on $k$ we prove that for any $k = 2, \ldots, n$

$$x \in \overline{\mathbb{E}}_{E_k} \Rightarrow R_s(z) \neq 0 \text{ for any } z \in \mathbb{D}^k,$$

where

$$E_k = \{\text{diag}[z_1, z_2, \ldots, z_k] : z_1, \ldots, z_k \in \mathbb{C}\}, \quad k = 2, \ldots, s.$$

Now fix $2 \leq k < n$ and assume the implication (13) holds for $k$. Suppose $x \in \overline{\mathbb{E}}_{E_{k+1}}$ but $0 = R_s(z) = R'_{s'}(z') - z_{k+1}P_{s''}(z')$ for some $z' \in \mathbb{D}^k$, $z_{k+1} \in \mathbb{D}$. Then $x' \in \overline{\mathbb{E}}_{E_k}$, i.e. $R_{s'}(z') \neq 0$ by inductive assumption. Consequently, we may proceed as in the case $k = 2$.

The proof of (ii)$\Rightarrow$(iii) is much as for Proposition 3.16.

Argument used in [20] to prove linear convexity of the pentablock allows us to get

Proposition 3.18. If $r_2 = \cdots = r_s = 1$ then $\mathbb{E}_E$ is linearly convex and, consequently, pseudoconvex.

Proof. Without loss of generality we may assume that $s \geq 2$. Using induction on $k$ we prove that $\mathbb{E}_E \subset \mathbb{C}^N$ given by (13) is linearly convex, $k = 2, \ldots, s$.

First we show that $\mathbb{E}_E$ is linearly convex. Take $x_0 \notin \mathbb{E}_E$. We are looking for a complex hyperplane $H \subset \mathbb{C}^{N_2}$ such that $x_0 \in H$ and $H \cap \mathbb{E}_E = \emptyset$. It follows from Theorem 3.7 that if we write $x = (x', x'') \in \mathbb{E}_E \subset \mathbb{C}^{r_1} \times \mathbb{C}^{N_2-r_1}$ then $x' \in \mathbb{G}_{r_1}$.

Assume first that $x_0' \notin \mathbb{G}_{r_1}$. Since $\mathbb{G}_{r_1}$ is linearly convex (cf. [23]), there is a complex hyperplane $H' \subset \mathbb{C}^{r_1}$ such that $x_0' \in H'$ and $H' \cap \mathbb{G}_{r_1} = \emptyset$. Then $H := H' \times \mathbb{C}^{N_2-r_1}$ is a complex hyperplane we are looking for (use Theorem 3.7).

Now consider the case $x_0' \in \mathbb{G}_{r_1}$. It follows from Proposition 3.10 that $\Psi_{z_1}(x_0', x''_0)$ is well defined for any $z_1 \in \overline{\mathbb{D}}$ and $\omega \in \mathbb{C} \setminus \mathbb{D}$ such that $\Psi_{z_1}(x_0', x''_0) = \omega$ and $\Psi_{z_1}(x', x'') \neq \omega$ whenever $(x', x'') \in \mathbb{E}_E$. Thus

$$H := \{(x', x'') \in \mathbb{C}^{r_1} \times \mathbb{C}^{N_2-r_1} : \Psi_{z_1}(x', x'') = \omega\}$$

is a complex hyperplane satisfying desired properties.

Now fix $2 \leq k < s$ and assume that $\mathbb{E}_E$ is linearly convex. In order to show that $\mathbb{E}_{E_{k+1}}$ is linearly convex, we take $x_0 \notin \mathbb{E}_{E_{k+1}}$ and look for a complex hyperplane $H \subset \mathbb{C}^{N_{k+1}}$ such that $x_0 \in H$ and $H \cap \mathbb{E}_{E_{k+1}} = \emptyset$. Since $\mathbb{E}_E$ is assumed to be linearly convex we may proceed as in first inductive step replacing $\mathbb{G}_{r_1}$ with $\mathbb{E}_{E_k}$.

Pseudoconvexity of $\mathbb{E}_E$ is a consequence of Proposition 2.1.8 from [5].
For a given \( m = (m_1, \ldots, m_n) \in \mathbb{N}^n \) and an \( m \)-balanced domain \( D \subset \mathbb{C}^n \) define its \( m \)-Minkowski functional
\[
\mathfrak{h}_D(x) := \inf\{ \lambda > 0 : m_\lambda x \in D \}, \quad x \in \mathbb{C}^n.
\]
This function has similar properties as the standard Minkowski functional for balanced domains. Some of them may be found in [13]. In particular,
- \( \mathfrak{h}_D(m_\lambda x) = |\lambda| \mathfrak{h}_D(x), \ x \in \mathbb{C}^n, \ \lambda \in \mathbb{C}, \)
- \( D = \{ x \in \mathbb{C}^n : \mathfrak{h}_D(x) < 1 \}, \)
- if \( D \) is bounded and \( \partial D = \{ x \in \mathbb{C}^n : \mathfrak{h}_D(x) = 1 \} \) then \( \mathfrak{h}_D \) is continuous.

A bounded domain is called \textit{hyperconvex} if there exists a continuous negative plurisubharmonic exhaustion function. In particular, any hyperconvex domain is taut.

**Proposition 3.19.** If \( r_2 = \cdots = r_s = 1 \) then \( \mathbb{E}_E \) is continuous. In particular, \( \mathbb{E}_E \) is hyperconvex.

**Proof.** To prove the continuity of \( \mathbb{E}_E \) it suffices to show that
\[
\partial \mathbb{E}_E \subset \{ x \in \mathbb{C}^N : \mathbb{E}_E(x) = 1 \}.
\]
Suppose there is \( x \in \partial \mathbb{E}_E \) such that \( \mathbb{E}_E(x) > 1 \). Take \( 0 < \lambda < 1 \) such that \( \mathbb{E}_E(\lambda |x|) = \lambda \mathbb{E}_E(x) > 1 \). In particular, \( |\lambda|_\alpha x \notin \mathbb{E}_E \). On the other hand, \( x \in \mathbb{E}_E \), i.e. \( R_\lambda(z) \neq 0 \) for all \( z \in \mathbb{D}^s \), whence
\[
R_{\alpha |\lambda|_\alpha} x(z) = R_\alpha(\lambda z) \neq 0, \quad z \in \mathbb{D}^s,
\]
i.e. \( |\alpha|_\lambda x \notin \mathbb{E}_E \)—a contradiction.

To see \( \mathbb{E}_E \) is hyperconvex, observe that \( \log \mathbb{E}_E \) is continuous negative plurisubharmonic exhaustion function on \( \mathbb{E}_E \) (cf. [22], Proposition 1).

Recall that a set \( A \subset \mathbb{C}^n \) is \textit{polynomially convex} or a \textit{Runge domain} if for every compact \( K \subset A \) the polynomial hull \( \bar{K} \) of \( K \) is contained in \( A \). Using the same argument as in the Proposition 3.18 we are able to show

**Proposition 3.20.** If \( r_2 = \cdots = r_s = 1 \) then \( \mathbb{E}_E \) is polynomially convex.

**Proof.** Without loss of generality we may assume that \( s \geq 2 \). First, using induction on \( k \) as before, we prove that \( \mathbb{E}_{E_k} \subset \mathbb{C}^{N_k}, \ k = 2, \ldots, s \), where \( E_k \) is given by (14), are polynomially convex.

First we show that \( \mathbb{E}_{E_2} \) is polynomially convex. Take \( x_0 \notin \mathbb{E}_{E_2} \). We are looking for a polynomial \( f \) such that \( |f| \leq 1 \) on \( \mathbb{E}_{E_2} \) and \( |f(x_0)| > 1 \).

It follows from Theorem 3.7 that if we write \( x = (x', x'') \in \mathbb{E}_{E_2} \subset \mathbb{C}^{r_1} \times \mathbb{C}^{N_2-r_1} \) then \( x' \in \mathbb{E}_{r_2} \).

Assume first that \( x_0 \notin \mathbb{E}_{r_2} \). Since \( \mathbb{E}_{r_2} \) is polynomially convex (cf. [3] for the proof that \( \mathbb{E}_{r_2} \) is polynomially convex; polynomial convexity of closure of the symmetrized polydisc may be proved in the same way), there is a polynomial \( f' \) such that \( |f'| \leq 1 \) on \( \mathbb{E}_{r_2} \) and \( |f'(x_0)| > 1 \). Then \( f(x', x'') := f'(x') \) is the polynomial we are looking for (use Theorem 3.7).

Now consider the case \( x_0 \in \mathbb{E}_{r_2} \). It follows from Proposition 3.17 that there is a \( \zeta_1 \in \mathbb{D} \) such that \( |\Psi_{\zeta_1}(x_0', x_0'')| > 1 \) and \( |\Psi_{\zeta_1}(x', x'')| \leq 1 \) on \( \mathbb{E}_{E_2} \). Thus it suffices to approximate the rational function \( \Psi_{\zeta_1} \) by polynomials. But it is an immediate consequence of the following

**Lemma 3.21.** Let \( m = (m_1, \ldots, m_n) \in \mathbb{N}^n \), let \( D \subset \mathbb{C}^n \) be \( m \)-balanced domain, and let \( f : \mathbb{D} \to \mathbb{C} \) be holomorphic. Then
\[
f(z) = \sum_{k=0}^{\infty} Q_k(z), \quad z \in D,
\]
where
\[
Q_k(z) := \sum_{\alpha \in \mathbb{Z}_+^n : (\alpha, m) = k} \frac{1}{\alpha!} D^\alpha f(0) z^\alpha, \quad z \in \mathbb{D}^n,
\]
(it is understood \( Q_k = 0 \) if there is no \( \alpha \in \mathbb{Z}_+^n \) with \( (\alpha, m) = k \)); observe that \( Q_k : \mathbb{C}^n \to \mathbb{C} \) is an \( m \)-homogeneous polynomial of degree \( k \), i.e. \( Q_k(m_\lambda z) = \lambda^k Q_k(z), \ z \in \mathbb{C}^n, \lambda \in \mathbb{C} \). Moreover, for any compact \( K \subset D \) there exist \( C > 0 \) and \( \vartheta \in (0, 1) \) such that
\[
\|Q_k\|_K \leq C \vartheta^k, \quad k \in \mathbb{Z}_+.
\]
In particular, the series converges locally normally in \( D \).
The above lemma is well known in the case of balanced domains (cf. Proposition 1.8.4 in [17]). Since the quasibalanced case may be proved in the same way as the balanced case, we omit here its proof.

Now fix \( 2 \leq k < s \) and assume that \( \mathbb{E}_{E_k} \) is polynomially convex. In order to show that \( \mathbb{E}_{E_{k+1}} \) is polynomially convex, we take \( x_0 \notin \mathbb{E}_{E_{k+1}} \) and look for a polynomial \( f \) such that \( |f| \leq 1 \) on \( \mathbb{E}_{E_{k+1}} \) and \( |f(x_0)| > 1 \). Since \( \mathbb{E}_{E_k} \) is assumed to be linearly convex we may proceed as in first inductive step replacing \( \mathbb{E}_{E_k} \) with \( \mathbb{E}_{E_k} \).

Define

\[
\mathbb{E}^{(r)}_E := \{|a|_r : x \in \mathbb{E}_E\}, \quad r \in (0, 1).
\]

Observe that \( \mathbb{E}^{(r)}_E \) is polynomially convex and

\[
\bigcup_{r \in (0, 1)} \mathbb{E}^{(r)}_E = \mathbb{E}_E.
\]

Now consider any compact \( K \subset \mathbb{E}_E \). Then, for \( r \) sufficiently close to 1, \( K \subset \mathbb{E}^{(r)}_E \). Since \( \mathbb{E}^{(r)}_E \) is polynomially convex, we have \( \bar{K} \subset \mathbb{E}^{(r)}_E \subset \mathbb{E}_E \), i.e. \( \mathbb{E}_E \) is polynomially convex. \( \square \)

4. The \( \mu_{1,n} \)-quintents

In 2014 G. Bharali introduced the following domain

\[
\mathbb{E}_n := \{(x, y) \in \mathbb{C}^{n-1} \times \mathbb{C}^n : \forall z, w \in \mathbb{C} \; Q_x(z) - wP_y(z) \neq 0\}, \quad n \geq 2,
\]

where

\[
P_y(z) := \sum_{j=1}^{n-1} (-1)^{j-1} y_j z^{j-1}, \quad Q_x(z) := 1 + \sum_{j=1}^{n-1} (-1)^j x_j z^j.
\]

\( \mathbb{E}_n \) is called \( \mu_{1,n} \)-quintent. It is a natural generalization of the tetrablock, since \( \mathbb{E}_2 = \mathbb{E} \). On the other hand, \( \mu_{1,n} \)-quintent is a particular case of the generalized tetrablock. Indeed, \( \mathbb{E}_n = \mathbb{E}_E \) for \( E = \{\mathrm{diag}[z_1 \mathbb{1}_{n-1}, z_2] \in \mathbb{C}^{n \times n} : z_1, z_2 \in \mathbb{C}\} \).

Below we collect the geometric properties of \( \mu_{1,n} \)-quintents \( \mathbb{E}_n \), \( n \geq 2 \), which are immediate consequence of the results from the previous section.

**Theorem 4.1.**

(a) \( \mathbb{E}_n \) is bounded \((1, 2, \ldots, n-1, k+1, k+2, \ldots, k+n)\)-balanced domain, \( k \geq 0 \), but not circled.

(b) \( \mathbb{E}_n \) is a Hartogs domain in \( \mathbb{C}^{2n-1} \) over \( \mathbb{G}_{n-1} \) with \( n \)-dimensional balanced fibers.

(c) \( \mathbb{E}_n \) cannot be exhausted by domains biholomorphic to convex ones.

(d) Let \( n \geq 4 \). Then \( l_{\mathbb{E}_n} \) is not a distance. In particular, \( c_{\mathbb{E}_n} \neq l_{\mathbb{E}_n} \).

(e) \( c_{\mathbb{E}_n}(0, \cdot) \neq k_{\mathbb{E}_n}(0, \cdot) \). In particular, \( c_{\mathbb{E}_n}(0, \cdot) \neq l_{\mathbb{E}_n}(0, \cdot) \).

(f) If \( n \geq 4 \) then \( \mathbb{E}_n \) is neither \( \mathbb{C} \)-convex nor starlike about the origin.

(g) \( \mathbb{E}_n \) is linearly convex and hyperconvex.

(h) \( \mathbb{E}_n \) is polynomially convex.

(i) \( \mathbb{E}_3 \) is not an analytic retract of the open unit ball of a \( \mathcal{J}^* \)-algebra of finite rank.

**Remark 4.2.** In view of the above results, among the domains \( \mathbb{E}_n \), \( n \geq 2 \), the only interesting examples—from the point of view of the Lempert theorem—are \( \mathbb{E}_2 \) and, possibly, \( \mathbb{E}_3 \). Recall that \( c_{\mathbb{E}_2} = l_{\mathbb{E}_2} \) (cf. [13]) and \( \mathbb{E}_2 \) is \( \mathbb{C} \)-convex (cf. [29]). It is an open question whether \( \mathbb{E}_3 \) has also these properties.

5. The pentablock

Recently J. Agler, Z. A. Lykova, and N. J. Young introduced a new domain related to \( \mu \)-synthesis, called pentablock. Recall that the pentablock may be defined as follows (cf. [4])

\[
\mathcal{P} := \{(a, s, p) \in \mathbb{C}^3 : (s, p) \in \mathbb{G}_2 : |a| < \left| 1 - \frac{\beta s}{1 + \sqrt{1 - |\beta|^2}} \right|\},
\]

where

\[
\beta = \beta(s, p) := \frac{s - \overline{sp}}{1 - |p|^2}.
\]

Note that \( \mathcal{P} \) is a Hartogs domain over \( \mathbb{G}_2 \) with balanced fibers. Moreover, \( \mathcal{P} \) is bounded, nonconvex, \((k, 1, 2)\)-balanced, \( k \geq 0 \), starlike about the origin and polynomially convex (cf. [1]). Recently L. Kosiński in [20] showed that \( \mathcal{P} \) is linearly convex. In particular, \( \mathcal{P} \) is pseudoconvex.

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Proposition 2.1 implies immediately

**Theorem 5.1.** $\mathcal{P}$ cannot be exhausted by domains biholomorphic to convex ones.

**Proof.** Indeed, since $\mathcal{P}$ is a Hartogs domain over $G_2$, it suffices to take

$$G_2 \ni (s, p) \overrightarrow{\theta} (0, s, p) \in \mathcal{P}, \quad \mathcal{P} \ni (a, s, p) \overrightarrow{\varphi} (s, p) \in G_2,$$

and observe that $\mathcal{S}(G_2) = \emptyset$ (cf. [12], Corollary 3).

Moreover, we have the following simple

**Proposition 5.2.** Let $m = (k, 1, 2)$, $k \geq 1$. Then the $m$-Minkowski functional $\mathfrak{h}_\mathcal{P}$ is continuous. In particular, $\mathcal{P}$ is hyperconvex.

**Proof.** To prove the continuity of $\mathfrak{h}_\mathcal{P}$ it suffices to show that

$$\partial \mathcal{P} \subset \{ z \in \mathbb{C}^3 : \mathfrak{h}_\mathcal{P}(z) = 1 \}.$$

Suppose $\mathfrak{h}_\mathcal{P}(z) > 1$ for some $z \in \partial \mathcal{P}$ and take $0 < r < 1$ with $\mathfrak{h}_\mathcal{P}(m_r z) = r \mathfrak{h}_\mathcal{P}(z) > 1$. In particular, $m_r z \notin \mathcal{P}$. On the other hand, $z = (a, s, p) \in \overline{\mathcal{P}}$, i.e. $(s, p) \in \overline{G_2}$ and

$$|a| \leq 1 - \frac{\frac{1}{2} \beta |s|^2}{1 + \sqrt{1 - |\beta|^2}}$$

(cf. [4], Theorem 5.3). Write $m_r z = (r^k a, ra, r^2 p)$. Then $(rs, r^2 p) \in G_2$ and

$$|r^k a| < |a| \leq 1 - \frac{\frac{1}{2} \beta (rs, r^2 p)}{1 + \sqrt{1 - |\beta|^2}},$$

(last inequality follows from the fact that $\mathcal{P}$ is $(0, 1, 2)$-balanced, i.e. $(a, rs, r^2 p) \in \overline{\mathcal{P}}$, i.e. $m_r z \in \mathcal{P}$—a contradiction.

To see $\mathcal{P}$ is hyperconvex, observe that $\log \mathfrak{h}_\mathcal{P}$ is continuous negative plurisubharmonic exhaustion function on $\mathcal{P}$ (cf. [22], Proposition 1).

**Remark 5.3.** We end this section with some natural questions. Do the Carathéodory distance and the Lempert function coincide on the pentablock? Is pentablock a $\mathbb{C}$-convex domain? Can $\mathcal{P}$ be exhausted by strongly linearly convex domains?

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Faculty of Mathematics and Computer Science, Jagiellonian University, Lojasiewicza 6, 30-348 Kraków, Poland
E-mail address: Pawel.Zapalowski@im.uj.edu.pl