On the Smallest Number of Functions Representing Isotropic Functions of Scalars, Vectors and Tensors

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Abstract
In this paper, we address the open problem (stated in Pennisi and Trovato, 1987. Int. J. Engng Sci., 25(8), 1059-1065) associated with the irreducibility of representations for isotropic functions. In particular, we prove that for isotropic functions that depend on $P$ vectors, $N$ symmetric tensors and $M$ non-symmetric tensors (a) the number of irreducible invariants for a scalar-valued isotropic function is $3P + 9M + 6N - 3$ (b) the number of irreducible vectors for a vector-valued isotropic function is 3 and (c) the number of irreducible tensors for a tensor-valued isotropic function is at most 9. The irreducible numbers in given (a), (b) and (c) are much lower than those obtained in the literature. This significant reduction in the number of irreducible scalar/vector/tensor-valued functions have the potential to substantially simplify modelling complexity.

1 Introduction
Mathematical modelling of physical conditions often requires representations for isotropic functions [5, 7]. In view of this much has been published on this subject (see, for example reference [4], and references therein). However, the derived number of isotropic functions in an irreducible basis (see definition of an irreducible basis in [25]) is still an open problem as stated by Pennisi and Trovato [4], where they state that: "Among all irreducible complete representations previously published in the literature (2.1)-(2.4) is that with fewer elements; but it is still an open problem to find, among all possible irreducible complete representations, that (if it exists) with fewer elements".

In this paper, we address this open problem and prove that only a few elements are required in irreducible bases. The proofs given here are simple (compared to the proofs given in the literature) and they are based on a spectral approach associated with the author’s work [9, 11, 13, 19]. This substantial reduction in numbers of elements in irreducible bases could radically reduce modelling complexity.

2 Preliminaries
Let $V$ be a 3-dimensional vector space. We define $\text{Lin}$ to be the space of all linear transformations (second-order tensors) on $V$ with the inner product $A : B = \text{tr}(AB^T)$, where $A, B \in \text{Lin}$ and $B^T$ is the transpose of $B$. We
define
\[ Sym = \{ A \in \text{Lin}|A = A^T \}, \quad Orth = \{ Q \in \text{Lin}|Q = Q^{-T} \}. \] (1)

The vectors considered here belong to the 3-dimensional Euclidean space \( \mathbb{E}^3 \), i.e., the vector space \( V \) furnished by the scalar product \( a \cdot b \), where \( a, b \in V \).

The summation convention is not used here and, all subscripts \( i, j \) and \( k \) take the values 1, 2, 3 unless stated otherwise.

3 Symmetric Tensors and Vectors

3.1 Scalar

The scalar function \( W(A_r, a_s), (r = 1, 2, \ldots, N; s = 1, 2, \ldots, P) \), where \( A_r \in Sym \) and \( a_s \in \mathbb{E}^3 \) are, respectively, symmetric tensors and vectors, is said to be scalar-valued isotropic function if
\[ W(A_r, a_s) = W(QA_rQ^{-T}, Qa_s) \] (2)

for all rotation tensor \( Q \in Orth \). Boehler [1] has shown that every scalar-valued isotropic function can be written as a function of invariants given in the following list:
\[ a_\alpha \cdot a_\alpha, \quad a_\alpha \cdot a_\beta, \quad \text{tr} A_i, \quad \text{tr} A_i^2, \quad \text{tr} A_i^3, \quad \text{tr} A_i A_j A_k, \quad a_\alpha \cdot A_i A_j a_\alpha, \quad a_\alpha \cdot A_i A_j a_\beta, \quad a_\alpha \cdot A_i^2 a_\beta, \quad a_\alpha \cdot (A_i A_j - A_j A_i) a_\beta, \] (3)

\( i, j, k = 1, 2, \ldots, N \) with \( i < j < k \) and \( \alpha, \beta = 1, 2, \ldots, P \) with \( \alpha < \beta \). However, Shariff [19] has shown that, for unit vectors \( v_\alpha \), only \( 2P + 6N - 3 \) of the invariants in (3) are independent and that the number of invariants in the irreducible functional basis is at most \( 2P + 6N - 3 \); far lower than the number of invariants given in (3). In the case when \( v_\alpha \) are not unit vectors it can be easily shown that only \( 3P + 6N - 3 \) of the invariants in (3) are independent. Below, for the sake of easy reading, we prove (similar to the work of Shariff [19]) that every scalar-valued isotropic function can be written as a function of at most \( 3P + 6N - 3 \) number of invariants. This significant reduction in number of scalar invariants (when compared to the list in (3)) could greatly assist in reducing modelling complexity (see for example references [8, 10, 12, 15, 17, 18, 20, 21, 22, 23])

**Proof**

For \( N \geq 1 \). Let express (say)
\[ A_1 = \sum_{i=1}^{3} \lambda_i v_i \otimes v_i, \] (4)

where \( \lambda_i \) and \( v_i \) are eigenvalues and (unit) eigenvectors of \( A_1 \), respectively and \( \otimes \) represents a dyadic product. Using \( \{v_1, v_2, v_3\} \) as a basis, we can express
\[ A_r = \sum_{i,j=1}^{3} A_{ij}^{(r)} v_i \otimes v_j, \quad a_s = \sum_{i=1}^{3} a_i^{(s)} v_i, \quad r = 2, 3, \ldots, N, \quad s = 1, 2, \ldots, P. \] (5)

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It is clear that the components $A_{ij}^{(r)}$ and $a_i^{(s)}$ are invariants, since

$$A_{ij}^{(r)} = v_i \cdot A_r v_j = Q v_i \cdot Q A_r Q^T v_j, \quad a_i^{(r)} = a_r \cdot v_i = Q a_r \cdot Q v_i. \quad (6)$$

Since,

$$\lambda_i, \quad A_{ij}^{(r)}, \quad a_i^{(s)}, \quad r \geq 2, \quad i, j = 1, 2, 3 \quad (7)$$

are "component" invariants, we can express

$$W(A_r, a_s) = W(Q A_r Q^T, Q a_s) = \hat{W}(\lambda_i, A_{ij}^{(r)}, a_i^{(s)}), \quad r \geq 2 \quad i, j = 1, 2, 3. \quad (8)$$

All invariant functions in (6) can be explicitly expressed in terms of the spectral invariants given below; for example, we can express the function

$$a_\alpha \cdot A_i^2 a_\beta = \sum_{p,q,m=1}^3 a_p^{(s)} A_p A_q A_q a_m^{(s)}, \quad i \neq 1 \quad (9)$$

Hence, the set of invariants in (7) is a complete representation for the scalar-valued isotropic function and since the terms in (7) are independent (invariant) components, the set is irreducible, i.e., incapable of being reduced. Hence, every scalar-valued isotropic function can be written as a function of at most $3P + 6N - 3$ number of invariants, far less than the number of invariants given in (3). The spectral invariants in (7) have been used in continuum modelling [8, 10, 12, 15, 17, 18, 20, 21, 22, 23] and spectral derivatives, associated with these spectral invariants, are given in [14, 16].

Since all of Boehler’s invariants (3) can be explicitly expressed in terms of the spectral invariants (7), this further validate our claim that the irreducible basis contains only $6N + 3P - 3$ invariants.

**Word of caution:** The function

$$\hat{W}(\lambda_i, A_{ij}^{(r)}, a_i^{(s)}) \quad (10)$$

must satisfy the $P$-property given in [12] and (for the benefit of the readers) in Appendix A. In this paper, we call a scalar-valued isotropic function that satisfies the $P$-property, a $P$-scalar-valued isotropic function. In general, the invariants appearing (as they are) in (7) are not $P$-scalar-valued isotropic functions.

In the case when $N = 0$, we have $W$ depends on $a_s$ only. In this case, we select the vector $a_1$ (say) and spectrally express

$$a_1 \otimes a_1 = \lambda v_1 \otimes v_1 + 0v_2 \otimes v_2 + 0v_3 \otimes v_3, \quad \lambda = a_1 \cdot a_1, \quad v_1 = \frac{a_1}{\sqrt{\lambda}} \quad (11)$$

and, $v_2$ and $v_3$ are any two (non-unique) orthonormal vectors that are perpendicular to $a$. Hence, for $N = 0$, we have $3P - 2$ irreducible invariants, i.e.,

$$\lambda, \quad a_i^{(s)}, \quad s = 2, 3, \ldots, P, \quad i = 1, 2, 3. \quad (12)$$

In the case where all of the vectors $a_s$ are unit vectors, we have only $2P - 2$ irreducible spectral invariants.

**Example 1:** Consider the strain energy function $W$ of a transversely isotropic elastic solid. We then have,

$$W(U, a \otimes a) = \tilde{W}(U, a) = \hat{W}(\lambda_i, a_i), \quad a_i = v_i \cdot a, \quad (13)$$
where $a_1 = a$ is the preferred direction unit vector, $A_1 = U$ is the right stretch tensor and
\[
\sum_{i=1}^{3} a_i^2 = 1. \tag{14}
\]

It is clear from (13) and (14), and if we consider the positive and negative values of $a_i$ as distinct single-valued functions then we can conclude that the number of invariants in the irreducible functional basis is 5.

**Example 2:** If we consider in Example 1, $A_1 = a \otimes a$ and $A_2 = U$, we have
\[
\lambda_1 = 1, \quad \lambda_2 = \lambda_3 = 0, \quad v_1 = a, \tag{15}
\]
$v_2$ and $v_3$ are any two (non-unique) orthonormal vectors that are perpendicular to $a$ and we then have
\[
W(a \otimes a, U) = \hat{W}(U_{ij}), \quad U_{ij} = v_i \cdot U v_j. \tag{16}
\]

We note that there are 6 (instead of 5) spectral invariants in (16). However, since $\hat{W}$ must satisfy the $P$-property, we can express $\hat{W}$ in terms of 5 independent invariants, that satisfy the $P$-property. For example, we can express $\hat{W}$ in terms of the 5 independent invariants
\[
I_1 = \sum_{i=1}^{3} U_{ii}, \quad I_2 = \sum_{i,j=1}^{3} U_{ij} U_{ji}, \quad I_3 = \sum_{i,j,k=1}^{3} U_{ij} U_{jk} U_{ki}, \quad I_4 = U_{11}, \quad I_5 = \sum_{i=1}^{3} U_{ii} U_{i1}. \tag{17}
\]

### 3.2 Vector

The vector function $g(A_r, a_s)$ is said to be vector-valued isotropic function if
\[
Qg(A_r, a_s) = g(QA_r Q^T, Qa_s) \tag{18}
\]
for all rotation tensor $Q$.

Smith [24] has shown that every vector-valued isotropic function can be written as a linear combination of the following vectors
\[
a_m, \quad A_i a_m, \quad A_i^2 a_m, \quad (A_i A_j - A_j A_i) a_m, \quad i, j = 1, 2, \ldots, N : i < j, \quad m = 1, 2, \ldots, P. \tag{19}
\]

It is understood that the coefficients in these linear combinations are $P$-scalar-valued isotropic functions.

Smith [24] and Pennisi and Trovato [4] claimed that the set of vectors in (19) is irreducible; we claim that the irreducible set contains only three linearly independent vectors. Below, we show via a theorem that every vector-valued isotropic function can be written as a linear combination of at most three linearly independent spectral vectors.

**Theorem 1** $g$ is an isotropic tensor function if and only if it has the representation
\[
g(A_r, a_s) = \sum_{i=1}^{3} g_i v_i, \tag{20}
\]
where $v_i$ is an eigenvector of $A_1$ and $g_i$ are isotropic invariants of the set
\[
S = \{A_1, A_2, \ldots A_N, a_1, a_2, \ldots, a_P\}. \tag{21}
\]
Proof:
(a) If (20) holds \( g \) is clearly a vector-valued isotropic function, since the coefficients \( g_i \) are isotropic invariants of the set \( S \) (21).

(b) For \( N \geq 1 \) and \( P \geq 0 \). Let \( v_i \) be unit eigenvectors of the symmetric tensor \( A_1 \) (see (4)). Hence we can write

\[
g(A_r, a_s) = \sum_{i=1}^{3} [g(A_r, a_s) \cdot v_i] v_i, \quad r = 1, 2, \ldots, N, \quad s = 1, 2, \ldots, P
\]  

(22)

and

\[
g(QA_rQ^T, Qa_s) = \sum_{i=1}^{3} [g(QA_rQ^T, Qa_s) \cdot Qv_i] Qv_i.
\]  

(23)

Let scalar function

\[
g_i(A_r, a_s) = g(A_r, a_s) \cdot v_i.
\]  

(24)

We then have

\[
g_i(QA_rQ^T, Qa_s) = g(QA_rQ^T, Qa_s) \cdot Qv_i.
\]  

(25)

In view of (18), (22) and (23), and since \( Q \) is arbitrary, we must have

\[
g_i(A_r, a_s) = g_i(QA_rQ^T, Qa_s),
\]  

(26)

which implies that \( g_i \) are functions of isotropic invariants of the vector and tensor set \( S \) given in (21). Note that, in view of the \( P \)-property, the functions \( g_i \) must also be \( P \)-scalar-valued isotropic functions.

In the case when \( N = 0 \), we consider the vectors \( v_i \) obtained similar to (11) and express

\[
g(a_r) = \sum_{i=1}^{3} g_i v_i, \quad g_i = g \cdot v_i.
\]  

(27)

All Smith’s vectors given in (19) can be expressed in terms of the unit vectors \( v_1, v_2 \) and \( v_3 \). For example the vector

\[
A_i a_m = \sum_{r=1}^{3} \sum_{s=1}^{3} A_r^{(i)} a_s^{(m)} v_r.
\]  

(28)

Hence, when a vector-valued function is expressed in terms of a linear combinations of Smith’s functions given in (19), it can then be expressed in terms of a linear combination of the symmetric spectral vectors \( v_1, v_2 \) and \( v_3 \); this further validates our claim that the irreducible basis contains only three vectors.

3.3 Symmetric Tensor

The symmetric tensor function \( G(A_r, a_s) \), is said to be tensor-valued isotropic function if

\[
QG(A_r, a_s)Q^T = G(QA_rQ^T, Qa_s)
\]  

(29)

5
for all rotation tensor $Q$.

Smith [24] has shown that every symmetric tensor-valued isotropic function can be written as a linear combination of the following symmetric tensors

\[
I, \quad A_i, \quad A_i^2, \quad A_iA_j + A_jA_i, \quad A_i^2A_j + A_jA_i^2, \quad A_iA_j + A_jA_i^2
\]

\[
a_m \otimes a_m, \quad a_m \otimes a_n + a_n \otimes a_m, \quad a_m \otimes A_i a_m + A_i a_m \otimes a_m, \quad a_m \otimes A_i^2 a_m + A_i^2 a_m \otimes a_m,
\]

where $(i, j = 1, 2, \ldots, N; i < j)$, $(p, q = 1, 2, \ldots, M; p < q)$, $(m, n = 1, 2, \ldots, P; m < n)$ and $I$ is the identity tensor. Smith [24] and Pennisi and Trovato [4] claimed that the set of symmetric tensors in (30) is irreducible; we, however, claim via Theorem 2 below, that the irreducible set contains only six linearly independent symmetric tensors.

**Theorem 2** $G$ is an isotropic tensor function if and only if it has the representation

\[
G(A_r, a_s) = \sum_{i,j=1} t_{ij} v_i \otimes v_j,
\]

where $v_i$ is an eigenvector of $A_1$ and $t_{ij}$ are functions of $P$-scalar-valued isotropic functions of the vector and tensor set given in (21).

**Proof**

(a) If (31) holds, since $t_{ij}$ are scalar invariants of the set $S$, then $G$ is clearly and isotropic tensor function.

(b) Using the basis $\{v_1, v_2, v_3\}$ obtained from (4), we can express

\[
G(A_r, a_s) = \sum_{i,j=1} t_{ij} v_i \otimes v_j,
\]

where

\[
t_{ij} = g_{ij}(A_r, a_s) = v_i \cdot G(A_r, a_s)v_j.
\]

Similarly, we can express

\[
G(QA_rQ^T, Qa_s) = \sum_{i,j=1} \bar{t}_{ij} Qv_i \otimes Qv_j,
\]

where

\[
\bar{t}_{ij} = Qv_i \cdot G(QA_rQ^T, Qa_s)Qv_j = g_{ij}(QA_rQ^T, Qa_s).
\]

If (29) holds then

\[
\sum_{i,j=1} \bar{t}_{ij} Qv_i \otimes Qv_j = \sum_{i,j=1} t_{ij} Qv_i \otimes Qv_j.
\]
Since $Q$ is arbitrary, we have

$$g_{ij}(A_r, a_s) = g_{ij}(QA_rQ^T, Qa_s) \quad (37)$$

which implies that the functions $t_{ij} = g_{ij}$ must depend on $P$-scalar-valued isotropic functions of $S$. Since $g_{ij} = g_{ji}$, all tensor-valued isotropic functions can be written as a linear combination of only six symmetric tensors

$$v_i \otimes v_i \quad (i = 1, 2, 3), \quad v_i \otimes v_j + v_j \otimes v_i \quad (i = 1, 2; j = 2, 3, i < j). \quad (38)$$

Hence, we can express

$$G(A_r, a_s) = \sum_{i=1}^{3} g_{ii} v_i \otimes v_i + \sum_{i<j} g_{ij} (v_i \otimes v_j + v_j \otimes v_i) \quad (i = 1, 2; j = 2, 3). \quad (39)$$

All symmetric tensors in (30) generated by Smith [24] can be expressed in terms of the six symmetric tensors given in (38), for example, the symmetric tensor

$$A_i A_j + A_j A_i = \sum_{p=1}^{3} g_{pp} v_p \otimes v_p + \sum_{p<q} g_{pq} (v_p \otimes v_q + v_q \otimes v_p) \quad (p = 1, 2; q = 2, 3), \quad (40)$$

where

$$g_{pp} = 2 \sum_{m=1}^{3} A_{pm}^{(i)} A_{mp}^{(j)}, \quad g_{pq} = \sum_{m=1}^{3} (A_{pm}^{(i)} A_{mq}^{(j)} + A_{qm}^{(i)} A_{mp}^{(j)}). \quad (41)$$

Hence, when a tensor-valued function is expressed in terms of a linear combinations of Smith’s functions given in (30), it can then be expressed in terms of a linear combination of the six symmetric spectral tensors given in (38); this further validates our claim that the irreducible basis contains only six symmetric tensors.

The above theorem proves that the irreducible set contains only six linearly independent symmetric tensors. This drastically reduce the complexity in physical modelling. For example, Merodio and Rajagopal [3] modelled viscoelastic solids, where the Cauchy stress $T$ depends on $A_1 = B$ (left Cauchy-Green stretch tensor), $A_2 = D$ (the symmetric part of the velocity gradient), $a_1 = m$ and $a_2 = n$ (preferred directions). Using Smith tensors (30), the Cauchy stress $T$ is described using 36 tensors obtained from (30) and, due to this large number of 36 tensors and 37 scalar invariants, the model is complicated; there is a dire need to simplify the model. Sometimes this is done by omission of invariants and tensors. However, the discrimination in selection of invariants and tensors is often debated, and neglecting the influence of some invariants and tensors may result in an incomplete representation of the full range of mechanical response subjected to a continuum. However, using the results obtained here, modelling viscoelastic solids is greatly simplified, we only require 15 scalar invariants and 6 symmetric tensors to fully describe the Cauchy stress $T$.

Remark:

Since both the scalars $g_i$ and $g_{ij}$ are, respectively, vector and tensor components, the vector $g$ and tensor $G$ are uniquely expressed in terms of the basis $\{v_1, v_2, v_3\}$ even though two or three of the vectors $v_1, v_2$ and $v_3$ may not be unique due to coalescence of eigenvalues.

The theorem below has been proven in the literature (see for example references Itskov [2] and Ogden [6]), however, for the benefit of the readers we prove it again here.
Theorem 3  If $G(V)$ is an isotropic tensor function then $G(V)$ is coaxial with $V$ and hence

$$VG = GV.$$

(42)

Proof

Let $v_1$ be an eigenvector of $V$ and choose

$$Q = 2v_1 \otimes v_1 - I = Q^T.$$  

(43)

In view of $V v_1 = \lambda_1 v_1$, we have

$$QV = VQ \rightarrow QVQ^T = V.$$  

(44)

From (29) we get

$$QG(V)Q^T = G(V).$$  

(45)

Hence

$$G(V)v_1 = \left( \sum_{i,j=1} t_{ij} Q v_i \otimes v_j Q^T \right) v_1.$$  

(46)

Note that $Q^Tv_1 = v_1$ and hence we have

$$G(V)v_1 = \sum_{i=1} t_{i1} Q v_i = \sum_{i=1} t_{i1} (2v_1 \otimes v_1 - I)v_i = 2t_{11} v_1 - G(V)v_1.$$  

(47)

Hence

$$G(V)v_1 = t_{11} v_1,$$  

(48)

which implies that $v_1$ is an eigenvector of $G(V)$ and $t_{11}$ is an eigenvalue of $G$. In a similar fashion, choosing $Q = 2v_r \otimes v_r - I = Q^T$, $r = 2, 3$, we can easily derive that

$$G(V) = \sum_{i=1} t_{ii} v_i \otimes v_i,$$  

(49)

and the theorem is proved.

Below is a theorem, which we believe is not found in the literature.

Theorem 4  Let $\lambda_i$ be the eigenvalues of $V$ and let

$$G(V) = \sum_{i=1}^3 t_i(\lambda_1, \lambda_2, \lambda_3) v_i \otimes v_i,$$  

(50)

be a symmetric isotropic tensor function, where $v_i$ is an eigenvector of $V$.

(a) If $\lambda_i = \lambda_j \neq \lambda_k$, $(i \neq j \neq k \neq i)$, then

$$t_i = t_j$$  

(51)
and we can uniquely express
\[ G(V) = t_i I + (t_k - t_i) v_3 \otimes v_3. \] (52)

(b) If \( \lambda_1 = \lambda_2 = \lambda_3 \) then
\[ t_1 = t_2 = t_3 \] (53)
and we can uniquely express
\[ G(V) = t_1 I. \] (54)

Proof
Consider the case \( \lambda_1 = \lambda_2 = \lambda \neq \lambda_3 \). In view of this, \( v_1 \) and \( v_2 \) are not unique and have infinitely many values. In view of the relation
\[ v_1 \otimes v_1 + v_2 \otimes v_2 + v_3 \otimes v_3 = I, \] (55)
we can write
\[ G(V) = t_1 I + (t_2 - t_1) v_2 \otimes v_2 + (t_3 - t_1) v_3 \otimes v_3. \] (56)

Since \( v_2 \) is not unique, we must have \( t_1 = t_2 \) to give \( G(V) \) a unique value. In a similar fashion, we can show for the cases \( \lambda_1 = \lambda_3 \) and \( \lambda_2 = \lambda_3 \). Hence, theorem (a) is proved.

In the case when \( \lambda_1 = \lambda_2 = \lambda_3 \), \( v_3 \) is also arbitrary, hence from (56) we must have \( t_1 = t_2 = t_3 \) and theorem (b) is proved.

We can see that in case when the classical invariants \( I_1 = \text{tr} V, I_2 = \text{tr} V^2 \) and \( I_3 = \text{tr} V^3 \) are used, we have [6]
\[ G(V) = \phi_0 I + \phi_1 V + \phi_2 V^2 = \sum_{i=1}^{3} (\phi_0 + \phi_1 \lambda_i + \phi_2 \lambda_i^2) v_i \otimes v_i = \sum_{i=1}^{3} t_i v_i \otimes v_i, \] (57)
\[ t_i = \phi_0 + \phi_1 \lambda_i + \phi_2 \lambda_i^2, \] (58)
where \( \phi_0, \phi_1 \) and \( \phi_2 \) depend on \( P \)-scalar-valued isotropic functions, \( I_1, I_2 \) and \( I_3 \). It is clear from (58) that \( t_i = t_j \) when \( \lambda_i = \lambda_j \).

4 Isotropic Functions of Non-symmetric Tensors

4.1 Scalar

The scalar function \( W(H_t, A_r, a_s), (r = 1, 2, \ldots, N; t = 1, 2, \ldots M; s = 1, 2, \ldots, P) \) is said to be a scalar-valued isotropic function if
\[ W(H_t, A_r, a_s) = W(QH_tQ^T, QA_rQ^T, Qa_s) \] (59)
for all rotation tensor $Q \in \text{Orth}$, where $H_t \in \text{Lin} \ (t = 1, \ldots, M)$ is a nonsymmetric second order tensor.

In the case when $M, N \geq 1$, we can easily proved, based on Section 3.1 that
\[ W(H_t, A_r, a_s) = W(QH_tQ^T, QA_rQ^T, QA_s) = W(H^{(t)}_{ij}, A^{(r)}_{ij}, a^{(s)}_i), \quad r = 2, 3, \ldots, N, \]
where the invariants
\[ \lambda_i, A^{(r)}_{ij}, a^{(s)}_i \]
are given in (7) and the invariants
\[ H^{(t)}_{ij} = v_i \cdot H_t v_j = Qv_i \cdot QH_tQ^TQv_j, \quad i, j = 1, 2, 3. \]

Since the above invariants are independent components, the irreducible basis consists of at most $3P + 9M + 6N - 3$ invariants. Note that Boehler [1] consider the isotropic function
\[ W(W_t, A_r, a_s), \]
where $W_t$ is a skew-symmetric tensor. He claimed that the irreducible set contains the "complicated" set of invariants
\[ a_{\alpha} \cdot a_{\alpha}, \quad a_{\alpha} \cdot a_{\beta}, \quad \text{tr} A_i, \quad \text{tr} A_i^2, \quad \text{tr} A_i A_j, \quad \text{tr} A_i^2 A_j, \quad \text{tr} A_i^2 A_j, \quad \text{tr} A_i A_j A_k, \quad \text{tr} A_i^2 W_p, \quad \text{tr} W_p W_q W_r, \quad a_{\alpha} \cdot A_i a_{\alpha}, \quad a_{\alpha} \cdot A_i^2 a_{\alpha}, \quad a_{\alpha} \cdot (A_i A_j - A_j A_i) a_{\beta}, \quad a_{\alpha} \cdot W_p^2 a_{\alpha}, \quad a_{\alpha} \cdot W_p W_q a_{\alpha}, \quad a_{\alpha} \cdot W_p^2 W_q a_{\alpha}, \quad a_{\alpha} \cdot W_p W_q^2 a_{\alpha}, \quad a_{\alpha} \cdot W_p a_{\beta}, \quad a_{\alpha} \cdot W_p^2 a_{\beta}, \quad a_{\alpha} \cdot (W_p W_q - W_q W_p) a_{\beta}, \quad \text{tr} A_i W_p^2, \quad \text{tr} A_i^2 W_p, \quad \text{tr} A_i^2 W_p A_i, \quad \text{tr} A_i W_p W_q, \quad \text{tr} A_i W_p W_q, \quad \text{tr} A_i W_p W_q, \quad \text{tr} A_i^2 W_p A_i, \quad \text{tr} A_i^2 W_p A_i, \quad \text{tr} A_i W_p W_q, \quad \text{tr} A_i W_p W_q, \quad \text{tr} A_i^2 W_p a_{\alpha}, \quad a_{\alpha} \cdot W_p A_i a_{\alpha}, \quad a_{\alpha} \cdot A_i^2 W_p a_{\alpha}, \quad a_{\alpha} \cdot (A_i W_p - W_p A_i) a_{\beta}, \]
where $i, j, k = 1, 2, \ldots, N$ with $i < j < k$; $p, q, r = 1, 2, \ldots, M$ with $p < q < r$ and $\alpha, \beta = 1, 2, \ldots, P$ with $\alpha < \beta$. However, prove that irreducible set contains only $3P + 3M + 6N - 3$ invariants and they are:
\[ \lambda_i, \quad A^{(r)}_{ij}, \quad a^{(s)}_i, \quad W^{(t)}_{kl} = v_k \cdot W_k v_i, \quad i, j, k, l = 1, 2, 3, \quad k < l, \quad r \geq 2. \]
The invariants in (65) are obtained from (61) and (62), by replacing $H_t$ with $W_t$ and taking note that
\[ v_i \cdot W_k v_i = 0, \quad v_i \cdot W_k v_j = -v_j \cdot W_k v_i, \quad i \neq j, \quad i, j = 1, 2, 3. \]

In the case when $N = 0$, we have
\[ W(H_t, a_s). \]
In this case, we let the orthonormal vectors $v_i$ to be the eigenvectors of the symmetric tensor $H_1 H_1^T$ (or alternatively $H_1^T H_1$), i.e.,
\[ H_1 H_1^T = \sum_{i=1}^{3} \lambda_i v_i \otimes v_i, \quad \lambda_i \geq 0. \]
The irreducible set contains at most $9M + 3P$ invariants
\[ H^{(t)}_{ij}, \quad a^{(s)}_i, \quad i, j = 1, 2, 3. \]
4.2 Vector

For a vector-valued isotropic function, it can be easily prove that, following Section 3.2,

\[ g(H_t, A_r, a_s) = \sum_{i=1}^{3} g_i v_i, \]  

(70)

where \( g_i \) are functions of the invariants in (61) and (62) or (69), as appropriate. Hence, the irreducible basis for \( g \) contain only the three vectors \( v_i \). Note that for \( H_t = W_t \), Smith [24] claimed that the irreducible basis for \( g \) contain the vectors

\[ a_m, A_i a_m, A_i^2 a_m, (A_i A_j - A_j A_i) a_m, W_p a_m, W_p^2 a_m, \] 

(71)

where \( i, j = 1, 2, \ldots, N; i < j \), \( p, q = 1, 2, \ldots, M; p < q \), \( m = 1, 2, \ldots, P \). This claim is incorrect since all the vectors in (71) can be written terms of the vectors \( v_1, v_2 \) and \( v_3 \).

4.3 Tensor

Following the method in Section 3.3, we can easily prove that the for any tensor in \( Lin \), with \( M, N \geq 1 \),

\[ H(H_t, A_r, a_s) = \sum_{i,j=1}^{3} h_{ij} v_i \otimes v_j, \]  

(72)

where, \( v_i \) is an eigenvector of \( A_1 \), and in general \( h_{ij} = v_i \cdot H v_j \neq h_{ji} \) are functions of the invariants in (61) and (62).

Hence, the irreducible basis for \( H \) contains, at most, 9 tensors, \( v_i \otimes v_j \). In the case when \( H \) is symmetric, the irreducible basis contains at most 6 symmetric tensors given in (38). In the case when \( H \) is a skew-symmetric tensor, the irreducible basis contains at most 3 skew-symmetric tensors, i.e.

\[ v_i \otimes v_j - v_j \otimes v_i \quad (i = 1, 2; j = 2, 3, i < j). \]  

(73)

Alternatively, for \( M \geq 1 \) and \( N \geq 0 \), using the singular value decomposition

\[ H_1 = \sum_{i=1}^{3} \lambda_i v_i \otimes u_i, \]  

(74)

we can easily prove that

\[ H(H_t, A_r, a_s) = \sum_{i,j=1}^{3} \hat{h}_{ij} v_i \otimes u_j, \]  

(75)

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where \( u_j \) are the unit eigenvectors of \( H_1^T H_1 \), \( v_i \) are the unit eigenvectors of \( H_1 H_1^T \) and the invariants
\[
\hat{h}_{ij} = v_i \cdot H u_j \neq \hat{h}_{ji}
\]
(76)
are functions of the \( 9M + 6N + 3P - 3 \) invariants
\[
\lambda_i, \quad u_i \cdot v_i, \quad v_i \cdot H u_j \quad (t \geq 2), \quad v_i \cdot A_r u_j, \quad a_s \cdot v_i.
\]
(77)

Smith [24] claimed for a symmetric tensor \( H \) and skew-symmetric tensors \( H_1 = W_1 \), the irreducible basis for symmetric \( H \) contains the set of symmetric tensors
\[
I, \quad A_i, \quad A_i^2, \quad A_i A_j + A_j A_i, \quad A_i^2 A_j + A_j A_i^2, \quad A_i A_j^2 + A_j A_i^2,
\]
\[
a_m \otimes a_m, \quad a_m \otimes a_n + a_n \otimes a_m, \quad a_m \otimes A_i a_m + A_i a_m \otimes a_m, \quad a_m \otimes A_i^2 a_m + A_i^2 a_m \otimes a_m,
\]
\[
A_i( a_m \otimes a_n - a_n \otimes a_m ) - (a_m \otimes a_n - a_n \otimes a_m) A_i,
\]
\[
W_p^2, \quad W_p W_q + W_q W_p, \quad W_p W_q - W_q W_p, \quad W_p^2 W_q - W_q^2 W_p,
\]
\[
A_i W_p - W_p A_i, \quad W_p A_i W_p, \quad A_i^2 W_p - W_p A_i^2, \quad W_p A_i^2 W_p - W_p^2 A_i W_p,
\]
\[
W_p a_m \otimes W_p a_m, \quad a_m \otimes W_p a_m + W_p a_m \otimes a_m, \quad W_p a_m \otimes W_p^2 a_m + W_p^2 a_m \otimes W_p a_m,
\]
\[
W_p( a_m \otimes a_n - a_n \otimes a_m ) + (a_m \otimes a_n - a_n \otimes a_m) W_p
\]
(78)
where \((i, j = 1, 2, \ldots, N; i < j)\), \((p, q = 1, 2, \ldots, M; p < q)\) and \((m, n = 1, 2, \ldots, P; m < n)\). In (78), it is clear that there is a large number of "complicated" symmetric tensors in the Smith [24] irreducible basis and this number is far greater than 6, the number of symmetric tensors in our irreducible basis. We note that all of Smith’s symmetric tensors in (78) can be expressed in terms of the six symmetric tensors given in (38).

5 Potential Vectors and Tensors

In this Section, we consider vectors and tensors that can be obtained from differentiating a scalar-valued isotropic function \( W \), i.e.,
\[
g = \frac{\partial W}{\partial a}, \quad G = \frac{\partial W}{\partial V}, \quad H = \frac{\partial W}{\partial F},
\]
(79)
where \( a \) is a vector, \( V \) is a symmetric tensor and \( F \) is a non-symmetric tensor. We called these vectors/tensors, potential vectors/tensors. For example, in non-linear hyper-elasticity, the potential nominal stress \( S = \frac{\partial W(c)}{\partial F} \), where \( F \) is the deformation gradient tensor and \( W(c) \) is the strain energy function.
5.1 Vector

Let \( W(H_t, A_r, a_s) \), be a scalar-valued isotropic function and let \( a = a_1 \). From Appendix B and following the work of Shariff [14], we obtain the relation

\[
g(H_t, A_r, a_s) = \frac{\partial W}{\partial \lambda} = \frac{\partial W}{\partial a} v_1 + \left( \frac{1}{\lambda} \frac{\partial W}{\partial v_1} \cdot v_2 \right) v_2 + \left( \frac{1}{\lambda} \frac{\partial W}{\partial v_1} \cdot v_3 \right) v_3
\]

\[
= \frac{\partial W}{\partial \lambda} v_1 + \frac{1}{\lambda} \left( (I - v_1 \otimes v_1)^T \frac{\partial W}{\partial v_1} \right) v_1^{(80)}
\]

where \( \lambda = \sqrt{a \cdot a} \). It is clear from (80), since the coefficients of \( v_i \) are scalar-valued isotropic functions, \( g \) is a vector-valued isotropic function.

5.2 Symmetric Tensor-Valued Isotropic Function \( G \)

In this case, we let \( V = A_1 = \sum_{i=1}^{3} \lambda_i v_i \otimes v_i \). Shariff [14] has shown that tensor-valued isotropic function

\[
G(H_t, A_r, a_s) = \frac{\partial W}{\partial V}
\]

\[
= \sum_{i=1}^{3} \frac{\partial W}{\partial \lambda_i} v_i \otimes v_i + \sum_{i,j=1, i < j}^{3} \frac{1}{2(\lambda_i - \lambda_j)} \left( \frac{\partial W}{\partial v_i} \cdot v_j - \frac{\partial W}{\partial v_j} \cdot v_i \right) (v_i \otimes v_j + v_j \otimes v_i) \tag{81}
\]

5.3 Non-symmetric Tensor-Valued Isotropic Function \( H \)

In this case, in view of singular value decomposition, we have \( H_1 = F = \sum_{i=1}^{3} \lambda_i v_i \otimes u_i \), where \( \lambda_i \) are the square root of the eigenvalues of \( FF^T \). \( v_i \) is a unit eigenvector of \( FF^T \) and \( u_i \) is a unit eigenvector of \( F^T F \). Shariff [14] (using a derivative convention used in Itskov [2]) has shown that tensor-valued isotropic function

\[
H(H_t, A_r, a_s) = \frac{\partial W}{\partial F}
\]

\[
= \sum_{i=1}^{3} \frac{\partial W}{\partial \lambda_i} v_i \otimes u_i + \sum_{i,j=1, i \neq j}^{3} \frac{\left( \lambda_i \left( \frac{\partial W}{\partial u_i} \cdot u_j - \frac{\partial W}{\partial u_j} \cdot u_i \right) + \lambda_j \left( \frac{\partial W}{\partial v_i} \cdot v_j - \frac{\partial W}{\partial v_j} \cdot v_i \right) \right)}{\lambda_i^2 - \lambda_j^2} v_i \otimes u_j \tag{82}
\]
6 Remark

In this communication we have shown that we need only 3 linearly independent vectors to represent both potential and non-potential vectors and a maximum of only 9 linearly independent tensors to represent both potential and non-potential tensors. However, the number of functions in a Smith [24] or Boehler [1] irreducible basis required to represent a potential vector/tensor is generally not the same as that required to represent a non-potential vector/tensor. For example, consider finite strain transversely isotropic elasticity with the preferred direction \( \mathbf{a} \) in the undeformed configuration. Let \( S(C, L) \) be the second Piola-Kirchhoff stress tensor, where \( C \) is the right Cauchy-Green tensor and \( \mathbf{L} = \mathbf{a} \otimes \mathbf{a} \). Using Smith [24] and Boehler [1] tensor functions, we have

\[
S = \alpha_0 I + \alpha_1 \mathbf{L} + \alpha_2 C + \alpha_3 C^2 + \alpha_4 (C \mathbf{L} + \mathbf{L} C) + \alpha_5 (C^2 \mathbf{L} + \mathbf{L} C^2),
\]

where \( \alpha_0 - \alpha_5 \) are isotropic invariants of the set \( \{C, \mathbf{L}\} \). For a hyperelastic material, there exist a strain energy function

\[
W(C, L) = \hat{W}(I_1, I_2, I_3, I_4, I_5),
\]

where the invariants

\[
I_1 = \text{tr} C, \quad I_2 = \text{tr} C^2, \quad I_3 = C^3, \quad I_4 = \text{tr} (C \mathbf{L}), \quad I_5 = \text{tr} (C^2 \mathbf{L}).
\]

The second (potential) Piola-Kirchhoff stress tensor then has the relation

\[
S = \frac{\partial W}{\partial E} = 2 \frac{\partial \hat{W}}{\partial I_1} I + 4 \frac{\partial \hat{W}}{\partial I_2} C + 6 \frac{\partial \hat{W}}{\partial I_3} C^2 + 2 \frac{\partial \hat{W}}{\partial I_4} L + \frac{\partial \hat{W}}{\partial I_5} (C \mathbf{L} + \mathbf{L} C), \quad E = \frac{1}{2} (C - I).
\]

Comparing (83) and (86), we observe that the representation for the hyperelastic material does not include the last term in (83), i.e., \( C^2 \mathbf{L} + \mathbf{L} C^2 \). It seems on the onset, if we use Smith [24] and Boehler [1] irreducible functions, the constitutive equation (83) cannot be described by a strain energy function (see comments made in Itskov [2] page 144). However, if we express the tensors

\[
I, \quad \mathbf{L}, \quad C, \quad C^2, \quad C \mathbf{L} + \mathbf{L} C, \quad C^2 \mathbf{L} + \mathbf{L} C^2
\]

in terms of the tensors \( v_i \otimes v_j \) (\( v_i \) is an eigenvector of \( C \)), their scalar coefficients are isotropic invariants of the set \( \{C, \mathbf{L}\} \), we could easily equate (83) with (86), which suggest that, when express in terms of the basis functions \( v_i \otimes v_j \), the constitutive equation (83) can be described by a strain energy function.

In general, following the above example, it can be easily shown that a non-potential vector/tensor can always be represented by a potential vector/tensor.

Appendix A: \( P \)-property

The description of the \( P \)-property uses the eigenvalues \( \lambda_i \) and eigenvectors \( v_i \) of the symmetric tensor \( A_1 \). A general anisotropic invariant, where its arguments are expressed in terms spectral invariants with respect to the basis \( \{v_1, v_2, v_3\} \) can be written in the form

\[
\Phi = \hat{W}(\lambda_1, v_i \cdot A_r v_j, v_i \cdot \mathbf{a}_s) = \hat{W}(\lambda_1, \lambda_2, \lambda_3, v_1, v_2, v_3),
\]

(A1)
where
\[ r = 2, \ldots, M, \quad s = 1, 2, \ldots, P, \] (A2)
and, in Eqn. (A1), the appearance of \( A_r \) and \( a_s \) is suppressed to facilitate the description of the \( P \)-property. \( \tilde{W} \) must satisfy the symmetrical property
\[ \tilde{W}(\lambda_1, \lambda_2, \lambda_3, v_1, v_2, v_3) = \tilde{W}(\lambda_2, \lambda_1, \lambda_3, v_2, v_1, v_3) = \tilde{W}(\lambda_3, \lambda_2, \lambda_1, v_3, v_2, v_1). \] (A3)

In view of the non-unique values of \( v_i \) and \( v_j \) when \( \lambda_i = \lambda_j \), a function \( \tilde{W} \) should be independent of \( v_i \) and \( v_j \) when \( \lambda_i = \lambda_j \), and \( \tilde{W} \) should be independent of \( v_1, v_2 \) and \( v_3 \) when \( \lambda_1 = \lambda_2 = \lambda_3 \). Hence, when two or three of the principal stretches have equal values the scalar function \( \Phi \) must have any of the following forms
\[
\Phi = \begin{cases} 
W(a)(\lambda, \lambda_k, v_k), & \lambda_i = \lambda_j = \lambda, i \neq j \neq k \\
W(b)(\lambda), & \lambda_1 = \lambda_2 = \lambda_3 = \lambda 
\end{cases}
\]

For example, consider
\[ \Phi = aA_1a = \sum_{i=1}^{3} \lambda_i(a \cdot v_i)^2, \] (A4)

where \( a \) is a fixed unit vector and
\[ \sum_{i=1}^{3} (a \cdot v_i)^2 = 1. \] (A5)

If
\[ \lambda_1 = \lambda_2 = \lambda, \] (A6)
we have
\[ \Phi = W(a)(\lambda, \lambda_3, v_3) = \lambda + (\lambda_3 - \lambda)(a \cdot v_3)^2 \] (A7)
and in the case of \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda, \)
\[ \Phi = W(b)(\lambda) = \lambda. \] (A8)

Hence, the invariant (A4) satisfies the \( P \)-property and we note that all the classical invariants described in Spencer [25] satisfy the \( P \)-property. In reference [16], the \( P \)-property described here is extended to non-symmetric tensors such as the two-point deformation tensor \( F \).

**Appendix B**

A dyadic product \( a \otimes a \) has the spectral representation
\[ a \otimes a = \lambda v_1, \quad \lambda = \sqrt{a \cdot a}, \quad v_1 = \frac{1}{\lambda}a. \] (B1)
The unit eigenvectors $v_2$ and $v_3$, associated with zero eigenvalues, are non-unique. In view of (B1), we have
\[ da = d\lambda v_1 + \lambda dv_1 = d\lambda v_1 + \lambda(da_2 v_2 + da_3 v_3). \] (B2)

Note that the above expression, have used the relation, for arbitrary,
\[ dv_1 = da_2 v_2 + da_3 v_3, \] (B3)

where $da_2$ and $da_3$ are arbitrary. We can write
\[ da = \sum_{i=1}^{3} (da)_i v_i, \quad (da)_1 = d\lambda, \quad (da)_2 = \lambda da_2, \quad (da)_3 = \lambda da_3. \] (B4)

For a scalar isotropic function $W = W(a) = W(s)(\lambda, v_1)$. Express
\[ \frac{\partial W(a)}{\partial a} = \sum_{i=1}^{3} \left( \frac{\partial W(a)}{\partial a} \right)_i v_i, \quad \left( \frac{\partial W(a)}{\partial a} \right)_i = \frac{\partial W(a)}{\partial a} v_i. \] (B5)

We then have
\[ dW = \sum_{i=1}^{3} \left( \frac{\partial W(a)}{\partial a} \right)_i (da)_i = \frac{\partial W(s)}{\partial \lambda} d\lambda + \frac{\partial W(s)}{\partial v_1} \cdot dv_1. \] (B6)

Using (B4) to (B5) and since $d\lambda, da_2$ and $da_3$ are arbitrary, we obtain the relations
\[ \left( \frac{\partial W(a)}{\partial a} \right)_1 = \frac{\partial W(a)}{\partial \lambda}, \quad \left( \frac{\partial W(a)}{\partial a} \right)_2 = \frac{1}{\lambda} \frac{\partial W(s)}{\partial v_1} \cdot v_2, \quad \left( \frac{\partial W(a)}{\partial a} \right)_3 = \frac{1}{\lambda} \frac{\partial W(s)}{\partial v_1} \cdot v_3. \] (B7)

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