Recent scanning tunnelling microscopy experiments in NbN thin disordered superconducting films found an emergent inhomogeneity at the scale of tens of nanometers. This inhomogeneity is mirrored by an apparent dimensional crossover in the paraconductivity measured in transport above the superconducting critical temperature $T_c$. This behavior was interpreted in terms of an anomalous diffusion of fluctuating Cooper pairs, that display a quasi-confinement (i.e., a slowing down of their diffusive dynamics) on length scales shorter than the inhomogeneity identified by tunnelling experiments. Here we assume this anomalous diffusive behavior of fluctuating Cooper pairs and calculate the effect of these fluctuations on the electron density of states above $T_c$. We find that the density of states is substantially suppressed up to temperatures well above $T_c$. This behavior, which is closely reminiscent of a pseudogap, only arises from the anomalous diffusion of fluctuating Cooper pairs in the absence of stable preformed pairs, setting the stage for an intermediate behavior between the two common paradigms in the superconducting-insulator transition, namely the localization of Cooper pairs (the so-called bosonic scenario) and the breaking of Cooper pairs into unpaired electrons due to strong disorder (the so-called fermionic scenario).

I. INTRODUCTION

The physics of the Superconductor-Insulator Transition (SIT) in disordered superconducting thin films is attracting an ever increasing interest both for applicative purposes and for fundamental reasons. While in structurally granular thin films the intrinsic granularity plays an evident role, the situation is more involved in nominally homogeneous (i.e., non-granular) disordered thin films. On the one hand, what is often referred to as the “fermionic” scenario proposes that the SIT is driven by the reduced screening of the Coulomb repulsion with increasing disorder, that leads to a weakening of pairing and to a reduction of the critical temperature $T_c$. In this case, the insulating state hosts localized fermions and standard paraconductive fluctuations are expected above $T_c$, due to Gaussian-distributed short-lived Cooper pairs. On the other hand, tightly-bound Cooper pairs survive the SIT in a “bosonic” scenario, in which the gap persists above $T_c$, despite the loss of phase coherence. In this framework, the bosonic pairs either localize because the disorder-enhanced Coulomb interaction destroys their phase-coherent motion at large scales or disorder itself blurs the phase coherence without any relevant role of the Coulomb repulsion. In the latter case, it was also proposed that the superconducting state is characterized by an emergent disordered glassy phase with filamentary superconducting current. An anomalous distribution of the superconducting order parameter was proposed by theorists, and observed experimentally. A numerical approach to uniformly disordered superconductors has also shown that there is a continuous evolution from the weak-disorder limit, where the system has a rather homogeneous fermionic character, to the strong-disorder limit, where marked inhomogeneities appear in the superconducting order parameter, with an emergent bosonic nature characterized by a single-particle gap persisting on the insulating side of the SIT. A great deal of experimental activity has been devoted to this more disordered realization of the SIT. The intermediate situation, where Cooper pairs begin to evolve into bosonic pairs, but keep their fermionic character, has been recently investigated by scanning tunnelling microscopy (STM) and transport measurements in NbN thin films. STM revealed the occurrence of an emergent inhomogeneous state and pseudogap effects over scales of a few tens of nanometers. This intermediate-scale inhomogeneity affected the transport properties with the paraconductivity displaying a crossover from a seemingly zero-dimensional Aslamazov-Larkin behaviour to the expected two-dimensional behaviour when $T_c$ was approached from above. This crossover was interpreted in terms of an anomalous slowing down of the dynamics of fluctuating Cooper pairs at length scales smaller than those set by the emergent inhomogeneity. Fig. pictorially illustrates this effect.

In this paper we explore other consequences of this intriguing emergence of inhomogeneity. Specifically, we consider the effects of this anomalous dynamics of the fluctuating Cooper pairs on the electron density of states above $T_c$, showing that, in the presence of disorder, the slowing down of Cooper pair fluctuations may give rise to pseudogap effects on the same length scale over which the Cooper pairs are quasi-confined. This will explain the occurrence of the pseudogap observed in STM experiments despite the persistence of purely fluctuating Cooper pairs.
FIG. 1. Sketch of the anomalous diffusion of the fluctuating Cooper pairs induced by the emergent inhomogeneity in NbN thin films. (a) The fluctuating Cooper pairs above $T_c$ diffuse in the system, but are slowed inside the emergent inhomogeneities (represented by the darker regions). The multiple scattering inside these regions pictorially represents the diffusion slowdown leading to a quasi-confinement of the fluctuating Cooper pairs. This mechanism is expected to give rise to a pseudogap and to the zero-dimensional character of the paraconductive fluctuations. (b) Over longer distance and time scales a “coarse-grained” standard diffusive behaviour of the fluctuating Cooper pairs is recovered, leading to the usual two-dimensional AL behaviour of the paraconductivity.

The core idea of the phenomenological theoretical framework used in Ref.[19] to fit the paraconductivity of NbN thin film is that there exist some regions, inside the superconducting film, where the lifetime of fluctuating Cooper pairs above the superconducting critical temperature $T_c$ is longer than expected for a standard diffusion process, resulting in a quasi-confinement of the fluctuating Cooper pairs within nanoscopic regions, dubbed supergrains, to emphasize the fact that they do not correspond to the structural grains. Indeed, these regions, of a linear dimension $L_i \approx 50$ nm, are way larger than the typical disorder length scale of a few nanometers structurally present in nominally homogeneous NbN films. In this scenario, fluctuations with characteristic wavelength smaller than the typical inhomogeneity dimension $L_i$ are more long-lived than they would be in a standard diffusion process, thus giving an explanation to the experimentally observed paraconductivity anomalies, resembling a zero-dimensional behaviour.

It is well established from the theory of fluctuations in superconductors, that the propagator of the fluctuating Cooper pairs in the weak-coupling (BCS) limit has the following form

$$L(q, \Omega) \simeq \frac{8T}{N_0} \frac{1}{\tau_{GL}^{-1} + \epsilon(q) - i\Omega},$$

where $N_0$ is the electron DOS at the Fermi level, $\tau_{GL}^{-1} = \frac{8T}{\pi \log \frac{T}{T_c}}$ is the inverse Ginzburg-Landau lifetime, and $\epsilon(q)$ is the dispersion law, determining the increase of the inverse diffusion time with increasing magnitude of the wave vector $q$ of the fluctuating mode. In this work we adopt units such that the Planck constant $\hbar$ and the Boltzmann constant $k_B$ are unity.

The standard results from fluctuation theory are obtained with a quadratic diffusion law $\epsilon(q) = Dq^2$ (where $q \equiv |q|$). On the other hand, the anomalous diffusion should give higher lifetimes for wavelengths smaller than $L_i$ and recover the standard behaviour at larger scales. It was found that a good description of experimental paraconductivity measurements on NbN thin films was achieved through the following anomalous diffusive expression

$$\epsilon(q) = D \bar{q}^2 \log \left( 1 + \frac{q^2}{\bar{q}^2} \right),$$

where $D$ is the diffusion constant and $\bar{q} \approx L_i^{-1}$. Fig. 2 compares the standard quadratic diffusion law (black dashed line) with the anomalous diffusion of Eq. (2) (solid blue line). We point out that the value of the diffusion constant $D$ in the presence of disorder may be severely suppressed with respect to the standard BCS value. As long as $D$ stays finite, however, its value does not appear in the AL paraconductivity in two dimensions, as the consequence of a cancellation enforced by gauge

II. THEORETICAL BACKGROUND

The structure of this paper is as follows. In the Sec. II we present our phenomenological scheme involving the anomalous diffusion of fluctuating Cooper pairs. Sec. III is devoted to the theoretical many-body calculation that determines the effects of the fluctuating Cooper pairs on the electron density of states (DOS). This section will also present the systematic analysis of these effects and a comparison with the experimental results of STM on NbN. Sec. IV contains our final remarks and conclusions.
Disorder may also introduce corrections to the BCS value of $\tau_{GL}$, as it seems indeed to be the case in the more strongly disordered NbN films.\textsuperscript{20}

At small momenta the two lines coincide since the system is normally diffusive over large length scales. At $q \geq \bar{q}$ instead, the anomalous diffusion becomes much smaller than the quadratic one, indicating that for these modes the diffusion occurs with characteristic frequencies $\Omega \sim \tau_{GL}^{-1} + \varepsilon(q)$ much smaller (i.e., with much longer characteristic times) than in the standard case $\Omega \sim \tau_{GL}^{-1} + Dq^2$.

The present work aims at understanding the effects of such an anomalous diffusion of fluctuating Cooper pairs on the electron density of states of a two-dimensional superconductor.

III. DENSITY OF STATES FROM ANOMALOUS DIFFUSION OF FLUCTUATING COOPER PAIRS

Following a standard quantum many-body approach, the density of states of an electronic system is given by the imaginary part of its retarded Green function $G^R(k, \omega)$:

$$N(\omega) = -\frac{1}{2\pi} \int \frac{d^2k}{(2\pi)^2} \text{Im} [G^R(k, \omega)].$$  \hfill (3)

The Feynman diagrams of figure 3 give the fluctuative contribution to the Green function in the dirty limit.\textsuperscript{23–25} The wavy line corresponds to the propagator of fluctuating Cooper pairs $L(q, \Omega_n)$. While here we adopt the finite-temperature formalism with Matsubara frequencies $\Omega_n$, the form in Eq. (1) has been analytically continued to real frequency $\Omega$ in order to obtain a diffusive pole for the fluctuating Cooper pairs above $T_c$. The shaded semicircles are vertices coupling the Cooper pair fluctuations with the electron quasiparticles. In the presence of disorder due to quenched impurities and in the so-called ladder approximation\textsuperscript{20,21} they read

$$\Lambda(q, \omega_m, \Omega_n) = \frac{1}{\tau} \frac{\varepsilon(q)}{2\omega_m + \frac{\text{sign}(\omega_m)}{\tau} - \Omega_n}.$$  \hfill (4)

FIG. 3. Feynman diagram for the contribution of fluctuating Cooper pairs to the electron Green function. The thick/thin solid line represents the dressed/bare electron Green function, the wavy line represents the propagator of fluctuating Cooper pair, and the shaded semicircles represent the impurity ladders that dress the electron-Cooper pair fluctuation vertex and embody the effect of microscopic disorder.

Starting from the numerical evaluation of Eq. (6), we systematically investigate the role of disorder (namely of
the scattering time $\tau$) and of the inhomogeneity length scale ($L_i$).

One first generic finding is that the anomalous diffusion of fluctuating Cooper pairs substantially increases the size and the extension in temperature above $T_c$ of the pseudogap effects (i.e., of the partial DOS suppression that occurs above $T_c$ because of superconducting fluctuations) in comparison with the standard diffusion law $\varepsilon(q) = Dq^2$. These effects are markedly larger in the disordered case than in the clean case (i.e., in the $\tau \to \infty$ limit): Only in the disordered case the DOS suppression can become quantitatively comparable to the suppression observed by STM experiments in NbN. This is why in the following we will focus on the disordered case only.

In Fig. 4, different $\delta N(\omega)$ curves are compared, at $T/T_c = 1.84$, rather far above $T_c$. The black line represents the curve obtained with the standard diffusion law, while the coloured ones have been obtained with different values of $L_i = 20, 30, 50$ nm, [i.e. different $\bar{q}$ in Eq. (2)], all in the range of the inhomogeneity sizes observed in Ref. 19.

FIG. 4. Effects of (anomalously diffusing) Cooper pair fluctuations on the quasiparticle DOS at $T/T_c = 1.84$, at fixed level of disorder. The elastic scattering rate is $1/\tau = 20$ meV $\approx 0.067 E_F$, where $E_F = v_F k_F$ is the Fermi energy. The black line shows the result for standard diffusion, with barely any visible effect as compared to the much stronger suppression observed in the case of anomalous diffusion.

It is evident that anomalous diffusion greatly enhances the pseudogap effects. Quite far above $T_c$, at $T/T_c = 1.84$, the DOS suppression in the standard diffusion case (black solid line) is barely appreciable adopting a common scale, being much smaller than the suppression observed in the case of anomalous diffusion. This demonstrates that these pseudogap effects are quite robust in temperature. We also notice that increasing the parameter $L_i$ we obtain a more pronounced DOS suppression, as this reduces $\bar{q}$, thus extending the region in $q$ space where the anomalous diffusion of fluctuating Cooper pairs occurs.

In Fig. 5, we report the results of our analysis on the effects of disorder. It is evident that, for a given fixed value of $L_i = 30$ nm, at $T/T_c = 1.3$, the DOS suppression increases by increasing the elastic scattering rate $1/\tau$. This indicates that the homogeneously distributed microscopic disorder represented by the impurities is cooperative with the large-scale inhomogeneity producing a stronger effect of the anomalous diffusion of Cooper pair fluctuations.

FIG. 5. Effect of the disorder strength on the Cooper fluctuation contribution to the DOS at $T/T_c = 1.3$ and $L_i = 30$ nm.

The results reported above show that the pseudogap effects induced by the anomalous diffusion of fluctuating Cooper pairs are large enough to account for the sizable pseudogap effects observed by STM in NbN. On the other hand, the DOS corrections shown in the Figs. 4 and 5 systematically display a suppression (i.e., a negative correction) with respect to the reference DOS of the normal metallic state at high temperature. Therefore, result cannot account for the experimental observation of coherence effects that rather symmetrically produce an enhancement of DOS at finite energy (finite bias in tunnel experiments) above or below the Fermi level ($\omega = 0$). Of course the spectral weight is not lost in our calculations, but is simply distributed over a very broad range, larger than the range of the figures. To obtain more realistic tunnel spectra, it is crucial to recall that in the transport experiments of Ref. 19, the transition temperature in the two-dimensional regime that sets in when the true $T_c$ is approached is larger than the critical temperature of the anomalously diffusing fluctuating Cooper pairs that shape the paraconductivity data at higher temperatures further away from $T_c$, in the regime where fluctuations have the seemingly zero-dimensional character. This observation, substantiated in Fig. 4b of Ref. 19, is not surprising, because the 0D regime occurs at higher temperatures and is dominated by the anomalous diffusion of fluctuating Cooper pairs over length scales shorter...
that $L_i$ and at these scales the systems is seemingly unaware of the actual critical temperature $T_c$ at which the true two-dimensional global superconducting state is established.

Therefore, in order to mimic this ‘flowing’ of the critical temperature when shorter and shorter length scales (i.e., larger and larger inverse time scales) dominate the Cooper fluctuations, we assumed a smooth step-like energy dependent $T_c(\omega)$, as shown in Fig. 6.

![Fig. 6. The curve representing the proposed $T_c(\omega)$. A smooth change from the two-dimensional (2D) regime to the zero-dimensional (0D) one can be tuned using an arctangent interpolating function. The values of the asymptotic critical temperatures are those obtained in Ref. 19.](image)

With this phenomenological assumption, we then obtain the fitting of tunnel spectrum reported in Fig. 7 for $T = 1.1T_c$. The data were obtained taking the experimental spectra of the large pseudogap regions (the supergrains) and dividing them by the corresponding measurements in the small pseudogap regions, to extract the contribution of anomalously diffusing Cooper pair fluctuations and get rid of any background contribution. We used $L_i$ and $\tau$ as adjustable parameters. The fitting DOS suppression (normalised to the DOS in the metallic state) was obtained for $L_i = 16.5$ nm, quite comparable with the typical size of supergrains observed in NbN films, and for $1/\tau = 5$ meV, for which we have no independent determination. Fig. 7 reports the comparison between the theoretical calculations and the experimental data. The theoretical curve has also been convoluted with a gaussian with variance $\sigma = 0.4$ meV to account for the experimental resolution.

![Fig. 7. Comparison of the theoretical curve (red line) with experimental data (×) at a temperature $T = 1.1T_c$ and with $L_i = 16.5$ nm as typical size of the inhomogeneity. Here $1/\tau = 5$ meV. The theoretical curve has been convoluted with a gaussian with variance $\sigma = 0.4$ meV to account for the experimental resolution.](image)

**IV. CONCLUSIONS AND OUTLOOKS**

In this paper, we presented a theory for the DOS suppression due to anomalously diffusing fluctuating Cooper pairs above the critical temperature $T_c$ of a two-dimensional superconductor. Our theoretical results highlight the effectiveness of the anomalous diffusion of fluctuating Cooper pairs in enhancing the pseudogap at temperatures well above $T_c$. Since a similar effect was observed experimentally in some ultrathin NbN films, we compared our curves with STM experiments, in order to get an estimate of the parameters of the theory. Good agreement with experimental data is achieved, especially after introducing the natural idea that Cooper fluctuations at different energies may 'perceive' different critical temperatures. This led us to the phenomenological assumption that the critical temperature should, in our theory, be frequency dependent, as a consequence of the 0D-2D crossover, with the asymptotic high-frequency (0D) and low-frequency (2D) values fitted to match the experimental paraconductivity, and a smooth interpolating behaviour at intermediate-frequency. This work has been carried out on a phenomenological basis in the form of the anomalous diffusive law of Cooper pairs and of $T_c(\omega)$ and the understanding of the actual microscopical reasons giving rise to the inhomogeneities, their length scales and their effects on the above laws is lacking. Nevertheless, we point out that, our work opens the way to a new microscopic understanding of the SIT, in which Cooper pairs are not stable bosonic entities, but keep a fluctuating character. It is their slowing down that produces marked pseudogap effects mimicking that of preformed pairs and strengthening the idea of a gradual and continuous evolution between
the bosonic and fermionic scenarios.

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