New dissipation function for weakly turbulent wind-driven seas

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A generalization of the kinetic equation \cite{1} is proposed for explaining observed shapes of wind wave spectra. The approach allows to fix a critical uncertainty in modeling wind wave spectra using a trivial condition of equilibrium of nonlinear transfer and wave dissipation due to breaking. The first results of the generalized kinetic equation simulation are presented in order to show relevance of the proposed theory to the observed Phillips’ spectrum $E \sim \omega^{-5}$ for wind waves.

I. INTRODUCTION

Wind waves in seas and lakes represent an extremely complicated natural phenomenon governed by a great number of physical mechanisms. Complete physical theories for many of these mechanisms are not proposed yet that makes particular asymptotic cases extremely valuable. Phillips\textsuperscript{2} considered a limiting case of saturation of a random wave field due to wave breaking. The dimensional consideration gave the famous Phillips energy spectrum $E(\omega) = \alpha q^2 \omega^{-5}$. Phillips himself emphasized the idea of ‘hard, saturated limit of the spectrum’ where ‘any excursion above this limit’ is ‘relieved immediately by breaking’ [see p.506 of \textsuperscript{3}].

The conceptual passage from the idea of saturation to the concept of an equilibrium of random wave field brought Phillips himself\textsuperscript{2} to the model of spectral balance where all the mechanisms, wave input by wind, dissipation due to breaking and nonlinear transfer due to resonant wave-wave interactions compete on equal terms. This model lead to other fundamental exponent $-4$ of wind-wave spectral equilibrium range. The essential flaw of the Phillips theory of spectral equilibrium, in its own words [sect.5 of \textsuperscript{3}], is a number of empirical parameters responsible for wind-wave spectral balance. All these parameters should be matched each other. Phillips formulated the corresponding criteria of consistency of his model but pointed out the difficulties of this matching and deficiency of our knowledge of wind-wave physics.

In contrast to the Phillips model\textsuperscript{2} of wind-wave spectra $\omega^{-4}$ [as well as the semi-empirical model by Toba\textsuperscript{4}] that gives the same spectral slope] the weakly turbulent Kolmogorov-Zakharov cascade solutions\textsuperscript{5,6} appear to be ‘self-sufficient’ for explaining exponents $-4$ of direct and $-11/3$ of inverse cascades as inherent features of deep water wave spectra governed by constant spectral flux in an equilibrium range of wave scales. Thorough analysis of wind-wave balance shows that this equilibrium range where nonlinear resonant interactions are dominating does exist for typical conditions of wind seas and for wave scales we are really interested in\textsuperscript{7,9}. It must be admitted that the scheme of wind-wave balance by Phillips\textsuperscript{2} remains the most recognized one in the wind-wave community in spite of a number of unresolved questions on this scheme correctness. The most critical questions concern mechanisms of wave input and dissipation. Today, our knowledge of these mechanisms is mostly empirical and their description is mostly based on quasi-linear parameterizations. Thus, one has a conceptual gap between this quasi-linear vision of spectra close to $\omega^{-4}$ and the ‘utterly nonlinear’ authentic\textsuperscript{2} spectrum $\omega^{-5}$. The present paper is aimed at filling this gap following the theory of wave turbulence\textsuperscript{10} and its recent extension\textsuperscript{1}.\textsuperscript{2}

One of the point of critics of the weakly turbulent approach is restrictions on wave scales (or time of evolution) when the corresponding asymptotic expansions are formally valid. Generally, the basic assumptions of the theory fail to be valid at long time and/or very short or very long wave scales. At the same time, for particular solutions these basic assumptions keep their formal validity for infinitely long time in the whole range of wave scales. Newell and Zakharov\textsuperscript{1} proposed to consider the corresponding wave spectra as generalized Phillips spectra in wave turbulence.

In this paper we use the extension of the wave turbulence theory\textsuperscript{1} for explaining the authentic Phillips’ spectrum $\omega^{-5}$ as a result of competition of wave-wave resonant interactions and inherently nonlinear dissipative processes. The new dissipation function can be found in physically consistent way as one corresponding to the generalized Phillips’ spectrum. This new dissipation function has remarkable features. First, the balance of wave-wave resonances and nonlinear dissipation fixes the dissipation rate for the classic Phillips spectrum $\omega^{-5}$. This surprising result is in dramatic contrast with conventional Phillips model where the dissipation rate is assumed to be arbitrary. Secondly, in
the generalized Phillips model of this paper the new dissipation absorbs completely the spectral energy flux directed to infinitely high wave numbers. Thus, the extended wave turbulence approach appears to be able to describe the whole range of wave scales and explain different exponents of the observed wave spectra [11, 12].

In § 2 we give basic equations and necessary comments to the concept of the generalized Phillips’ spectra. A simple isotropic model of the generalized Phillips spectrum of deep water waves is discussed in § 3. Stationary and self-similar solutions for this model are analyzed as a basis for further comparison with results of simulations of the extended kinetic equation for deep water waves. Numerical simulation of wind-wave spectra within the kinetic equation with the new dissipation function are presented in § 4. Discussion and conclusions are given in § 5.

II. GENERALIZED PHILLIPS’ SPECTRUM OF DEEP WATER WAVES

In this section we reproduce results of Newell & Zakharov [1] for particular problem of statistical description of wind-wave field. We consider the Hasselmann kinetic equation for spatial spectrum of wave action $N_k$ of wind-driven waves in the following form

$$\frac{\partial N_k}{\partial t} + \nabla_k \omega_k \nabla_r N_k = S_{nl} + S_{in} + S_{diss}$$

(1)

Subscripts $k$, $r$ for $\nabla$ are used for gradients in wavevector $k$ and coordinate $r$ spaces correspondingly. For $N_k(t)$ and $\omega_k$ the subscript $k$ means dependence on wavevector. The term $S_{nl}$ in (1) is responsible for four-wave resonant interactions. Terms $S_{in}$ and $S_{diss}$ describe correspondingly input of wave action from wind and its dissipation. This description is based on phenomenological parameterizations mainly [see 12]. It gives very high dispersion of estimates of the terms [see fig. 1 in 7] and a number of physically different compositions of $S_{nl}$, $S_{in}$, $S_{diss}$. Validity and physical correctness of the empirically based terms are generally beyond critical consideration: quantitative aspects are dominating over obvious questions on their physical relevance. It is not the case of the theoretically based term $S_{nl}$ which explicit form provokes questions on limits of the asymptotic model of wave-wave interactions.

For deep water waves four-wave resonant interactions are described by the collision integral

$$S_{nl} = \frac{\pi g^2}{3} \int |T_{0123}|^2 (N_0 N_1 N_2 + N_1 N_2 N_3 - N_0 N_1 N_2 - N_0 N_1 N_3) \times \delta(k + k_1 - k_2 - k_3) \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3) dk_1 dk_2 dk_3$$

(2)

Here we use convention [14] for normalization of spectral densities $N(k)$ and kernels $T_{0123}$. We also use subscripts $i = 0 \ldots 3$ to denote dependence on argument $k_i$ [see 17]. Collection of cumbersome expressions for the interaction kernel $T_{0123}$ can be found in Appendix of [7].

A. Homogeneity property of $S_{nl}$ and validity of asymptotic statistical description

Homogeneity properties of $S_{nl}$ are of key importance for our consideration. First, wave frequencies and wavevectors in (2) obey the power-law dispersion relation for linear deep water waves

$$\omega(k) = \sqrt{g|k|}.$$  \hspace{1cm} (3)

Interaction kernel $|T_{0123}|^2$ is also a homogeneous function of power 6, i.e.

$$|T(\kappa k_0, \kappa k_1, \kappa k_2, \kappa k_3)|^2 = \kappa^6 |T(k_0, k_1, k_2, k_3)|^2$$

(4)

Correspondingly, for the collision integral (2) that is cubic in wave action spectral density one has the well-known re-scaling property in terms of wavenumber dependence

$$S_{nl}[\kappa k, \nu N_k] = \kappa^{19/2} \nu^3 S_{nl}[k, N_k]$$

or in terms of dependence on frequency $\omega$

$$S_{nl}[\nu \omega, \nu N_\omega] = \nu^{17} \nu^3 S_{nl}[\omega, N_\omega]$$

(6)

One can see that the collision integral depends heavily on wave scales and amplitudes that may break validity of the asymptotic approach during long-time evolution of wave field. The basic assumption of the asymptotic approach is smallness of wave period $T$ as compared with time scale of nonlinear interactions, i.e.

$$\frac{T}{T_{nl}} = \frac{1}{\omega_k N_k} \frac{dN_k}{dt} = \frac{1}{\omega_k N_k} S_{nl} \ll 1$$

(7)
It can be checked that for

\[ N_k \sim |k|^{-9/2} \quad \text{or} \quad N_\omega \sim \omega^{-5} \quad (8a, b) \]

conditions (6) are satisfied for any stretching parameter \( \kappa \) and \( \upsilon \). In other words, the asymptotic approach appears to be formally valid at any wave scale. Moreover, one can prove that this is true at any order of the asymptotic approach. For the collision integral asymptotic series

\[ S_{nl} = \sum_{n=4}^{\infty} S_{nl}^{(n)} \quad (9) \]

conditions similar to (7) remain valid for every term \( S_{nl}^{(n)} \) that represents resonant interaction of \( n \) waves [1]. Thus, the generalized Phillips spectrum (8) correspond to a special case when asymptotic description of wave-wave interaction can be formally extended over the whole range of wave scale. It gives a chance to propose a model where weak nonlinearity acts as a strong physical factor and is able to concur with other strong physical mechanisms, say, with dissipation due to wave breaking.

It should be stressed that the found Phillips spectrum for deep water waves differs from the classic Kolmogorov-Zakharov solutions for direct and inverse cascading [5, 6]

\[ N^{(1)}(k) = C_p P^{1/3} g^{-2/3} |k|^{-4} \quad (10) \]

\[ N^{(2)}(k) = C_q Q^{1/3} g^{-1/2} |k|^{-23/6} \quad (11) \]

\((P, Q – \text{energy and wave action flux}, C_p, C_q – \text{the corresponding Kolmogorov’s constants})\). Collision integral \( S_{nl} \) for solutions (10, 11) is plain zero and estimates of the kinetic equation validity (7) requires special care. Following the simplest (but not trivial) way proposed recently [9, 14] split nonlinear transfer term \( S_{nl} \) into two parts – nonlinear forcing \( F_k \) and nonlinear damping \( \Gamma_k N_k \) \((\Gamma_k – \text{nonlinear damping rate})\) as follows

\[ S_{nl} = F_k - \Gamma_k N_k \]

Evidently, the relaxation rate \( \Gamma_k \) gives a physically correct estimate of time scale of nonlinear wave-wave interactions in the kinetic equation (11). Thus, the KZ solution breaks when \[ \text{see eq.17 in 9} \]

\[ \Gamma_k \omega \simeq 4\pi g |k|^9 N_k^2 \simeq 1 \]

that gives

\[ 4\pi C_p^2 g^{-1/3} P^{2/3} |k_{br}| \simeq 1 \]

for direct cascade solution [5] and

\[ 4\pi C_q Q^{2/3} |k_{br}|^{4/3} \simeq 1 \]

for the inverse cascade [6]. The generalized Phillips’ spectrum (8) being steeper than its KZ counterparts cannot be balanced ‘on its own’. In contrast to KZ solutions (10, 11) it requires an additional term to provide a full balance in the right-hand side of the kinetic equation.

**B. Matching the dissipation function with nonlinear transfer term**

Matching a dissipation function can be done from different physical positions. First, one can rely completely on dimensional consideration and to construct a nonlinear dissipation function in the spirit of conventional models of wind-wave modeling as a local nonlinear function of spectral density \( N(k) (N(\omega)) \), frequency \( \omega \) (or wavenumber \( |k| \) and gravity acceleration \( g \) as the only physical scale of deep water waves. To match homogeneity conditions (7) of the nonlinear transfer term for the Phillips spectrum (8) one has

\[ S_{diss}[k, N_k] = \gamma_k N_k = \Phi_k (g^{1/2} k^{3/2} N_k) \omega N_k, \quad (12) \]

\[ S_{diss}[\omega, N_\omega] = \gamma_\omega N_\omega = C_\omega \Phi_\omega (\omega^6 N_\omega / g^7) \omega N_\omega \quad (13) \]
with non-dimensional constants $C_k$, $C_\omega$ and arbitrary functions $\Phi_k$ or $\Phi_\omega$ of non-dimensional arguments. Such ‘conventional’ approach implies that wave dissipation is determined by instantaneous state of wave field and, in a sense, represents the Phillips authentic idea of saturated spectra [2].

An alternative way to introduce the nonlinear dissipation for the Phillips spectrum follows the Phillips idea [3] of an equilibrium of different physical mechanisms, in our case, wave dissipation and nonlinear transfer. It can be regarded as ‘more physical’: we assume that flux of energy (or wave action) due to nonlinear transfer is damped by nonlinear dissipation. Thus, one has a dynamical equilibrium provided by these two physical mechanisms. Such vision of wave balance can be formulated quite naturally in terms of wave turbulence where spectral flux of energy $P$ plays a key role

$$S_{diss}[\omega, E_\omega] = C_{PH} \Psi(P\omega^3/g^2) \frac{P}{\omega}$$

where $\Psi$ is arbitrary function of non-dimensional argument.

III. THE PHILLIPS DISSIPATION FUNCTION AND PHILLIPS’ SPECTRUM

Consider the simplest model of balance of nonlinear transfer and wave dissipation [14] that satisfy condition (7) and provides the Phillips spectrum (8) formally in the whole range of wave scales

$$\frac{dE(\omega)}{dt} = -\frac{\partial P(\omega)}{\partial \omega} - \Psi(P\omega^3/g^2) \frac{P(\omega)}{\omega}$$

The right-hand side of (15) operates with formally non-local quantity of spectral flux of wave energy

$$P(\omega) = -\int_0^\omega \int_{-\pi}^\pi S_{nl}[\omega, \theta, E_\omega]d\omega d\theta$$

In fact, this non-local value can be interpreted quite naturally using homogeneity properties (5, 6). For gravity waves $P \sim E^3(\omega)\omega^{12}$ (see 10) one has

$$\frac{P\omega^3}{g^2} \sim \frac{E^3(\omega)\omega^{15}}{g^6} = \mu_w^6$$

i.e. the non-dimensional argument of $\Psi$ is proportional to wave steepness in power 6. Note, that we use differential frequency-dependent steepness $\mu_w$ in (17) in contrast to integral steepness

$$\mu_p^2 = \frac{E_{tot}\omega_p^4}{g^2}$$

($\omega_p$ spectral peak frequency) or cumulative steepness

$$\mu_c^2 = g^{-2} \int_0^{\omega_p} E(\omega)\omega^4d\omega.$$ 

The steepness definition (18) in terms of total energy $E_{tot}$ gives always finite values. Typically, for growing wind sea it is below 0.1 and is decaying with time or fetch. Two alternative definitions are useful for description high-frequency part of wave spectra. For weakly turbulent direct cascade solution [10] the differential and cumulative steepnesses are infinitely growing functions $\mu_w \sim \mu_c \sim \sqrt{\omega}$. The Phillips spectrum $E(\omega) \sim \omega^{-5}$ gives saturation of differential steepness $\mu_w$ and logarithmic growth of cumulative one $\mu_c$.

Our interpretation of function $\Psi(P\omega^3/g^2)$ equalizes two different models of wave dissipation: local in spectral density [12, 13] and non-local flux-dependent one [14]. We fix this parallel below in notation

$$\frac{P\omega^3}{g^2} = \mu.$$
A. Stationary solutions of the generalized kinetic equation

The stationary solution of (15) for an arbitrary dependence $\Psi(\mu)$ gives nothing but dependence of steepness $\mu = P \omega^3 / g^2$ on wave frequency

$$\int_0^\mu \frac{6d\mu}{\mu(3 - \Psi(\mu^0))} = \ln \left( \frac{\omega}{\omega_1} \right)$$

(20)

A finite $\omega_1$ in (20) is introduced from dimensional consideration only. The denominator of the integrand in (20) must not be zero at finite frequency $\omega$ for the integrand convergence and, hence, for physically correct values of $\mu$ (strictly positive). Thus, function $\Psi(\mu)$ has to tend to a constant value. Any cases but $\Psi(\mu) \to 3$ give power-law dependence of $\mu$ on frequency $\omega$. As far as $\mu$ has a sense of wave steepness the case $\Psi(\mu) \to 3$ is seen as the only physically relevant one. In this case the steepness for infinitely short waves is decaying with $\omega$.

Important results can be obtained for power-law dependence of function $\Psi$ on non-dimensional spectral flux $P \omega^3 / g^2$.

Let

$$\Psi = C \left( \frac{P \omega^3}{g^2} \right)^R$$

(21)

Stationary solution of (15) obeys

$$C \left( \frac{P \omega^3}{g^2} \right)^R = 3$$

(22)

Thus, the dissipation function $\Psi(P \omega^3 / g^2)$ should be constant for the stationary solution for any exponent $R$ in (21).

Note, that for $R = 0$ in (21) the formal stationary solution gives an arbitrary exponent that is determined by coefficient $C$, i.e.

$$P \sim \omega^{-C}$$

Mathematically, an arbitrary coefficient $C$ can give an arbitrary exponent for frequency spectrum. But from physical viewpoint we should take care because at $R = 0$ the key physical parameter – gravity acceleration $g$ falls out of the model. To avoid ‘the non-physical freedom’ one should take a formal limit at $R \to 0$ that fix exponent $C = 3$.

For $R \neq 0$ exponent $(-3)$ and dissipation rate $C$ are fixed by condition (22). Surprisingly, the condition of stationarity does not depend on choice of power-like function $\Psi$, i.e. on dependence of dissipation on wave steepness.

From homogeneity property (17) one gets immediately the Phillips spectrum

$$E(\omega) = \alpha_{ph} g^2 \omega^{-5}$$

(23)

where the Phillips $\alpha_{ph}$ can be determined from the relationship between energy spectrum and spectral flux.

B. Self-similar solutions for the generalized kinetic equation

Stationary solutions cannot be considered as a final argument for the Phillips spectrum: solutions that evolve in time and space should be studied as routes to possible equilibria states of wave spectra. Fortunately, for $R = 0$ the generalized kinetic equation (15) has self-similar solutions and re-scaling property (17) can be used in its full.

Consider spatially homogeneous (the so-called duration-limited) solutions of (15) in the form

$$E \sim t^{p+q} U_p(\omega t^q)$$

(24)

Total energy of this solution grows as

$$E_{tot} \sim t^p$$

(25)

Looking for similar homogeneity properties of terms in the right-hand side one gets from (17) the following conditions for exponents $p, q$

$$p = \frac{9q - 1}{2}$$

(26)
that is the same as for the family of self-similar solutions of the authentic conservative kinetic equation [see \[7\] for details]. The second condition single out special exponents for self-similar solutions of the generalized kinetic equation

\[ R(q - 1) = 0 \]  

(27)

The exponent \( q = 1 \) is unrealistic as soon as it corresponds to very fast growth of energy \( (E_{\text{tot}} \sim t^4) \) while \( R = 0 \) looks quite attractive for the model of wave dissipation.

The advantage of the proposed dissipation function in (15)

\[ S_{\text{diss}} = -C_{P h} \frac{P(\omega)}{\omega} \]  

(28)

is two-fold. First, self-similar solutions are seen usually as physically relevant asymptotics of solutions with arbitrary initial and boundary conditions. The second feature of the dissipation function is more important for further study. While the nonlinear transfer and dissipation terms have similar properties of homogeneity they can be balanced without additional fitting parameters. Thus, choosing the dissipation function (28) we follow the Phillips idea ‘like cures like’.

Note, that \[3\] realized this principle for all three terms in the right-hand side of the kinetic equation by fitting essentially nonlinear collision integral \( S_{\text{nll}} \) and quasi-linear functions \( S_{\text{in}}, S_{\text{diss}} \) to each other. The outcome of such composition was the well-known energy spectrum \( E(\omega) \sim \omega^{-4} \) associated with a number of experimental findings [e.g. \[16,18\]]. Extremely high price for this sophisticated composition was a number of uncertainties in empirical parameters of functions of wave input and dissipation [see sect.5 ‘Constraints on the empirical constants’ in \[3\]].

Following the Phillips principle ‘like cures like’ for the case of self-similar evolution with dissipation function (28) we avoid the Phillips problem:

\[ * \text{the spectral balance in our model does not require any fitting parameters.} * \]

C. Universality of dissipation function for the Phillips spectrum \( E \sim \omega^{-5} \)

Important analytical results can be acquired from analysis developed by \[19\]. Let us estimate collision integral \( S_{\text{nll}} \) for power-like isotropic functions

\[ E(\omega) = E_0 \left( \frac{\omega}{\omega_0} \right)^{-x} \]  

(29)

One can find an explicit expression

\[ S_{\text{nll}}^{(c)} = 8\pi^2 g^{-4} E_0^3 \omega_0^{11} \left( \frac{\omega}{\omega_0} \right)^{11-3x} F(x) \]  

(30)

where function \( F(x) \) is calculated by integration in parameter \( q \) – ‘base vector’ of a resonant quadruplet. Here we used superscript for collision integral in the kinetic equation \[11\] written for energy. Function \( F(x) \) is shown in fig. \[11\].

This function tends to infinity at \( x = 1 \) and \( x = 11/2 \) when the corresponding integral in parameter \( q \) diverges. The first case \( x = 1 \) gives divergence in high frequencies while \( x = 11/2 \) – in low frequencies. Two zeroes of function \( F(x) \) correspond to classic Kolmogorov-Zakharov solutions for direct \((x = 4)\) and inverse \((x = 11/3)\) cascades (see eqs. \[10\], \[11\]).

Similar analytical expression can be obtained for spectral flux of wave energy \( P \)

\[ P = 8\pi^2 g^{-4} E_0^3 \omega_0^{12} \left( \frac{\omega}{\omega_0} \right)^{12-3x} \frac{F(x)}{3x-12} \]  

(31)

For the inverse cascade case \( x = 11/3 \) the energy flux vanishes \( F(x) = 0 \) – there is no but wave action flux to low frequencies. For \( x = 4 \) – the direct cascade \( F(x) \) tends to zero together with the denominator in (31) that gives a finite constant of wave energy flux.

Comparing (30) and (31) one can get a ratio that can be related with dissipation coefficient \( C_{P h} \) in (28) for formally stationary solutions

\[ \frac{\omega \partial P/\partial \omega}{P} = 3x - 12 = C_{P h} \]  

(32)
FIG. 1: Function $F(x)$. In the right panel the plot is zoomed in order to show its behavior near points corresponding to the Kolmogorov-Zakharov solutions $x = 11/3$ and $x = 4$.

For the Phillips spectrum with $x = 5$ this ratio gives $C_{phillips} = 3$, i.e. exactly the coefficient for an arbitrary dependence of dissipation function on non-dimensional argument $P_\omega^3/g^2$ (see [22]).

Below we present results of simulations where dissipation will be expressed by ‘conventional’ dependencies on spectral densities [12,13] or differential wave steepness [17]. For

$$S_{diss}[\omega, E(\omega)] = C_{phillips} \mu_4^w \omega E(\omega)$$

(33)

this dissipation function balances the explicit collision integral (30) when

$$C_{phillips} = 8\pi^2 F(5) \approx 2.03$$

(34)

Here we used an approximate expression for $F(x)$ at $x \to 11/2$

$$F(x) = \frac{0.0129}{11/2 - x}, \quad x \to 11/2$$

Our estimate assumes the balance of nonlinear transfer and dissipation in the whole frequency range that is not the case of wind waves where strong inherently nonlinear dissipation occurs for relatively short waves as a result of wave breaking. More realistic dissipation function with a cutoff fixed frequency $\omega_{diss}$

$$S_{diss} = C_{diss} \mu_4^w \omega E(\omega) \Theta(\omega - \omega_{diss})$$

(35)

takes this phenomenon into account in the simplest way. Here $\Theta(x - x_0)$ is Heaviside function. This form of the dissipation function will be used below in our simulations. The theoretical estimate of the Phillips dissipation coefficient $C_{phillips}$ [34] gives a good reference in our numerical study.

IV. NUMERICAL RESULTS

Numerical simulation of the kinetic equation [11] has been carried out in order to check the simple theory presented above. The WRT algorithm [20, 21] was used for calculation collision integral. The dissipation function was calculated in accordance with definition [33], i.e. in terms of ‘observable’ parameters of wave steepness. In this paper we present the very first results of simulations in the simplest setup. We studied duration limited growth (spatially homogeneous) of initially isotropic spectrum. The initial spectrum is given by JONSWAP-like formula that correspond to frequency spectrum of energy

$$E(\omega) = \frac{\alpha g^2}{\omega_p^4} \omega^{-4} \exp \left( -\frac{5}{4} \left( \frac{\omega}{\omega_p} \right)^{-4} \right) \gamma^G$$

(36)

where

$$G = \exp \left( -\frac{(\omega - \omega_p)^2}{2\sigma_p^2\omega_p^2} \right)$$

(37)
with default peakedness parameters $\gamma = 3.3$, $\sigma_p$. The initial wavelength corresponds to rather long swell $\lambda \approx 240$ m. Minimal frequency in simulations $f_{\text{min}} = 0.02$Hz, maximal $f_{\text{max}} = 2$Hz, peak frequency $f_p = 0.079$Hz, the dissipation cutoff is set at four times higher frequency $f_{\text{diss}} = 0.319$Hz. With logarithmic grid each frequency domain (low bound $f_{\text{min}}$–peak, peak–dissipation cutoff) contains approximately the same number of grid points. In absence of wave input slow spectral downshift does not change essentially this division.

The reference case of ‘pure swell’ is presented in fig. 2. As it was demonstrated in [7] the solution evolves to a self-similar shape. In our case we see a transition to a self-similar behavior rather than the self-similarity itself because of relatively short time of evolution and realistic low wave steepness. Energy spectra (top fig. 2) show downshifting and a peakedness growth. The energy spectral flux tends to constant magnitudes in a wide range of non-dimensional frequencies ($3 < \omega/\omega_p < 20$). Integral wave steepness $\mu_p$ (circles in bottom fig. 2) is going down with time as well as curves of cumulative steepness $\mu_c$ which is growing function of frequency. The differential steepness shows similar behavior – steep growth with frequency.

Fig. 3 present results for dissipation function with coefficient $C_{\text{diss}} = 1.22 = 0.6 C_{\text{Phillips}}$ and dissipation cutoff frequency $\omega_{\text{diss}} = 2\text{rad}^{-1}$. This coefficient is almost two times less than our estimate of the Phillips coefficient $C_{\text{Phillips}} = 2.03$ defined for the Phillips exponent $(-5)$. One can see appearance of essentially steeper spectrum tail and dramatic reduction of all the steepness parameters defined above while for cumulative steepness $\mu_c$ one can see a sort of saturation. At the same time, the differential steepness $\mu_w$ shows a gradual growth with frequency. In bottom-right fig. 3 we give curves in accordance with two possible definition of dissipation, first, in terms of spectral flux [14], second, as a function of wave steepness $\mu_w$ [13] and conventional presentation of dissipation in wave models [12 13]. Higher dissipation rate $C_{\text{diss}} = 1.9 - \approx 5\%$ lower than the Phillips dissipation coefficient $C_{\text{Phillips}} = 2.03$ shows very strong tendency to Phillips spectrum $E(\omega) \sim \omega^{-5}$. The bottom-right panel fixes two key points: first, the differential steepness is tending to saturation, second, two alternative definitions of the parameter, in terms of spectral flux (bold lines) and as function of wave amplitude (thin line), appear to be close to each other.

One more example illustrate the effect of dissipation cutoff. In fig. 5 all the parameters of simulation are the same
FIG. 3: The kinetic equation solutions for dissipation function (33) with $C_{\text{Ph}} = 1.22$ and dissipation cutoff $\omega_{\text{diss}} = 2\text{rad}^{-1}$: top-left – frequency spectra of energy, hard straight line $- E(\omega) \sim \omega^{-4}$, dashed $- \omega^{-5}$; top-right – energy flux (curves), straight line corresponds to law $P \sim \omega^{-3}$; bottom-left – cumulative (lines) and integral wave steepness (symbols at $\omega/\omega_p = 1$); bottom-right – differential steepness defined in terms of spectral flux (bold line) and parameter $\mu_w$. Time in seconds is given in legend.

as in previous case but dissipation cutoff is set at $\omega_{\text{diss}} = 3\text{rad}^{-1}$. One can see the same effect: very strong tendency of solution to Phillips’ asymptotics and saturation of all the steepness parameters ($\mu_p$, $\mu_c$, $\mu_w$)

V. CONCLUSIONS AND DISCUSSION

In this work we proposed to reanimate the Phillips idea of wave spectra as a result of a balance of different physical mechanisms. We combine effects of nonlinear transfer due to four-wave resonant interactions of weakly nonlinear deep water waves and dissipation dealing with wave breaking. These two effects can be co-existing in quite good symbiosis for the famous Phillips spectrum $E(\omega) \sim \omega^{-5}$: just this spectral shape makes asymptotic kinetic equation to be formally valid in the whole range of wave scales. Thus, we have a good example of the generalized Phillips spectrum in this case.

There is one more point of interest in competition of dissipation and nonlinearity for the Phillips spectrum. Stationarity of this spectrum requires certain magnitude of dissipation coefficient – the Phillips coefficient. This is in contrast to widely accepted models of wave dissipation where dissipation function can be arbitrary: the spectral shaping does not depend on the dissipation magnitude. We gave analytical estimate of the Phillips dissipation coefficient $C_{\text{Phillips}} \approx 2.03$.

Our simulation within the simplest setup of slowly evolving isotropic wave field justified our theoretical results. Shape of wave spectra are shown to be sensitive to magnitude of wave dissipation. For values close to the found theoretical value $C_{\text{Phillips}} \approx 2.03$ the spectra are tending to the Phillips classic tail $\omega^{-5}$ and appeared to be long-lived. Deviation from this value give other spectral slopes. Variety of spectral slope is well-known from experimental studies. Collection of spectral slopes derived from buoy data have been presented, say, by Paul Liu long time ago [11, 12]. His fig. 8 in [12] shows a wide range of the exponents: 90% of them fall into range 3 – 7 with a maximum between 4 and 5. This high dispersion has been attributed to ‘an equilibrium range’ of wave spectra with no clear
idea what mechanisms are responsible for this equilibrium and why the Phillips spectrum $\omega^{-5}$ is emphasized in this experimental collection.

Our very first numerical experiments with the new dissipation function showed robustness of the effect of nonlinear dissipation. Its effect leads to saturation of wave steepness both in integral ($\mu_p$) or differential ($\mu_w$) quantities. This feature is of great importance for further simulation of wave spectra with both effects of wave input and dissipation taken into account. High nonlinearity of the new dissipation function suppresses effect of wave pumping in high frequency completely and, thus, the classic Phillips spectrum continues to play important role in general case.

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[1] A. C. Newell and V. E. Zakharov, Phys. Lett. A 372, 4230 (2008).
[2] O. Phillips, J. Fluid Mech. 4, 426 (1958).
[3] O. M. Phillips, J. Fluid Mech. 156, 505 (1985).
[4] Y. Toba, J. Oceanogr. Soc. Japan 28, 109 (1972).
[5] V. E. Zakharov and N. N. Filonenko, Soviet Phys. Dokl. 160, 1292 (1966).
[6] V. E. Zakharov and M. M. Zaslavsky, Izv. Atmos. Ocean. Phys. 19, 207 (1983).
[7] S. I. Badulin, A. N. Pshukarev, D. Resio, and V. E. Zakharov, Nonl. Proc. Geophys. 12, 891 (2005).
[8] I. R. Young and G. van Vledder, Phil. Trans. Roy. Soc. London 342, 505 (1993).
[9] V. E. Zakharov and S. I. Badulin, Doklady Earth Sciences 440, 1440 (2011).
[10] V. E. Zakharov, G. Falkovich, and V. Lvov, *Kolmogorov spectra of turbulence. Part I* (Springer, Berlin, 1992).
FIG. 5: The kinetic equation solutions for dissipation function \( E(\omega) \sim \omega^{-5} \), energy dissipation cutoff \( \omega_{diss} = 3 \) rad\(^{-1} \): top-left – frequency spectra of energy, hard straight line – \( E(\omega) \sim \omega^{-4} \), dashed – \( \omega^{-5} \); top-right – energy flux (curves), straight line corresponds to law \( P \sim \omega^{-3} \); bottom-left – cumulative (lines) and integral wave steepness (symbols at \( \omega/\omega_p = 1 \)); bottom-right – differential steepness defined in terms of spectral flux (bold line) and parameter \( \mu_w \). Time in seconds is given in legend.

[11] P. C. Liu, J. Geophys. Res. 94, 5017 (1989).
[12] P. C. Liu, in Proceedings of the International Twenty-first Coastal Engineering Conference, June 20-25, 1988, Costa del Sol-Malaga, Spain, edited by B. L. Edge (M.ASCE, (Edge and Co., Inc., 901 Plantation Lane, Mt. Pleasant, SC), New York, NY, 1989), vol. 3, pp. 1045–1057.
[13] L. Cavaleri, J.-H. G. M. Alves, F. Ardhuin, A. Babanin, M. Banner, K. Belibassakis, M. Benoit, M. Donelan, J. Groeneveld, T. H. C. Herbers, et al., Prog. Ocean. 75 (2007).
[14] V. E. Zakharov, Phys. Scr. T142, 014 (2010).
[15] V. P. Krasitskii, J. Fluid Mech. 272, 1 (1994).
[16] J. A. Battjes, T. J. Zitman, and L. H. Holthuijsen, J. Phys. Oceanogr. 17, 1288 (1987).
[17] M. A. Donelan, J. Hamilton, and W. H. Hui, Phil. Trans. Roy. Soc. Lond. A 315, 509 (1985).
[18] Y. Toba, J. Oceanogr. Soc. Japan 29, 70 (1973).
[19] V. V. Geogjaev and V. E. Zakharov (unpublished paper).
[20] B. Tracy and D. Resio, WES Rep. 11, US Army, Engineer Waterways Experiment Station, Vicksburg, MS (1982).
[21] D. J. Webb, Deep Sea Res. 25, 279 (1978).