Geometric phases, gauge symmetries and ray representation

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Abstract

The conventional formulation of the non-adiabatic (Aharonov-Anandan) phase is based on the equivalence class \( \{ e^{i\alpha(t)}\psi(t,\vec{x}) \} \) which is not a symmetry of the Schrödinger equation. This equivalence class when understood as defining generalized rays in the Hilbert space is not generally consistent with the superposition principle in interference and polarization phenomena. The hidden local gauge symmetry, which arises from the arbitrariness of the choice of coordinates in the functional space, is then proposed as a basic gauge symmetry in the non-adiabatic phase. This re-formulation reproduces all the successful aspects of the non-adiabatic phase in a manner manifestly consistent with the conventional notion of rays and the superposition principle. The hidden local symmetry is thus identified as the natural origin of the gauge symmetry in both of the adiabatic and non-adiabatic phases in the absence of gauge fields, and it allows a unified treatment of all the geometric phases. The non-adiabatic phase may well be regarded as a special case of the adiabatic phase in this re-formulation, contrary to the customary understanding of the adiabatic phase as a special case of the non-adiabatic phase. Some explicit examples of geometric phases are discussed to illustrate this re-formulation.

1 Introduction

The study of geometric phases is an attempt to understand quantum mechanics better. The geometric phases have been mainly analyzed by using the adiabatic approximation \[1\]-\[7\], though the processes slightly away from adiabaticity have been considered in \[8\]. A definition of the non-adiabatic phase, which is closely related to the adiabatic phase but without assuming adiabaticity, has been proposed in \[9\]\[10\]. A generalization of geometric phases for noncyclic evolutions has also been proposed \[11\], where the old idea of Pancharatnam \[12\] played an important role.
These earlier works have been further elaborated by various authors, for example, in Refs. [13, 14, 15, 16, 17] and references therein.

It has been recently shown [18] that gauge symmetries involved in the adiabatic (Berry) phase and the non-adiabatic (Aharonov-Anandan) phase are quite different by using a second quantized formulation [19]. In this formulation the hidden local gauge symmetry, which appears as a result of the arbitrariness of the phase choice of the complete orthonormal basis set in field theory, provides a basis for the parallel transport and holonomy in the analysis of adiabatic phases [2]; this local symmetry itself is exact regardless of adiabatic or non-adiabatic processes.

In the present paper, we analyze the physical implications of these two different gauge symmetries appearing in the definitions of geometric phases. The gauge symmetry in the non-adiabatic phase is based on the equivalence class [9, 10, 11]

\[ \{ e^{i\alpha(t)} \psi(t, \vec{x}) \} \] (1)

instead of constant phases in the conventional definition of rays in the Hilbert space [20, 21]. Since the Schrödinger equation is not invariant under the equivalence class (1), one may consider an equivalence class of Hamiltonians \( \{ \hat{H} - \hbar \partial_t \alpha(t) \} \). The gauge symmetry means an assignment of the physical significance to those quantities invariant under gauge transformations. A convenient way to identify a gauge invariant quantity is to impose the parallel transport condition

\[
\int d^3x \bar{\psi}(t, \vec{x})^\dagger i\partial_t \psi(t, \vec{x}) = 0
\] (2)

by choosing a suitable parameter \( \alpha(t) \) in \( \bar{\psi}(t, \vec{x}) = e^{i\alpha(t)}\psi(t, \vec{x}) \). This \( \bar{\psi}(t, \vec{x}) \) is written as

\[
\bar{\psi}(t, \vec{x}) = \exp[i \int_0^t dt \int d^3x \psi(t, \vec{x})^\dagger i\partial_t \psi(t, \vec{x})] \psi(t, \vec{x})
\] (3)

and it is invariant up to a constant phase factor for any choice of \( \psi(t, \vec{x}) \) in the above equivalence class; the factor on the exponential plays a role of gauge field. This \( \bar{\psi}(t, \vec{x}) \) thus has the same property as the conventional Schrödinger amplitude \( \psi(t, \vec{x}) \) under the hidden local symmetry [18]. However, \( \bar{\psi}(t, \vec{x}) \) is non-local and non-linear in \( \psi(t, \vec{x}) \) and a linear superposition of \( \psi(t, \vec{x}) \) does not lead to a linear superposition of \( \bar{\psi}(t, \vec{x}) \) in general. The variable \( \bar{\psi}(t, \vec{x}) \) also satisfies

\[
i\hbar \frac{\partial}{\partial t} \bar{\psi}(t, \vec{x}) = [\hat{H}(t) - \int d^3x \bar{\psi}(t, \vec{x})^\dagger \hat{H}(t) \bar{\psi}(t, \vec{x}) / \int d^3x |\bar{\psi}(t, \vec{x})|^2] \bar{\psi}(t, \vec{x})
\] (4)
if \( \psi(t, \vec{x}) \) satisfies the ordinary linear Schrödinger equation. Even in the adiabatic limit, a linear superposition of two independent solutions of (4) does not generally satisfy (4).

We examine to what extent the equivalence class (1) is regarded as defining a generalization of conventional rays, and it is shown that the generalized rays thus defined are not generally consistent with the superposition principle both in the interference and polarization phenomena. It is also explained that the equivalence class (1) in the non-adiabatic phase is not reduced to the gauge symmetry in the adiabatic phase even in the adiabatic limit. As a result, these two gauge symmetries give rise to different constraints in the measurements of the adiabatic phase by interference.

To reconcile these complications with the attractive idea of the non-adiabatic phase, we suggest a re-formulation of the non-adiabatic phase on the basis of hidden local gauge symmetry arising from the arbitrariness of the choice of coordinates in the functional space [13]. The hidden local gauge symmetry keeps \( \psi(t, \vec{x}) \) invariant up to a constant phase, namely, \( \psi(t, \vec{x}) \rightarrow e^{i\alpha(0)} \psi(t, \vec{x}) \) in contrast to (1). We show that this re-formulation reproduces all the successful aspects of the non-adiabatic phase in a way manifestly consistent with the conventional notion of rays and the superposition principle. The hidden local gauge symmetry controls both of the adiabatic and non-adiabatic phases. We thus understand the natural origin of the gauge symmetry in geometric phases, which appears even in the absence of gauge fields. Conceptually, our re-formulation identifies both of the adiabatic and non-adiabatic phases as associated with the parallel transport and holonomy of an orthonormal basis set, rather than the Schrödinger amplitude itself, which specifies the coordinates of the functional space.

In the present paper, we first recapitulate the basic aspects of the hidden local gauge symmetry and the non-adiabatic phase in Sections 2 and 3. The consistency of the equivalence class (1), when understood as a generalized notion of rays, with the superposition principle is examined in Section 4. We then present the re-formulation of the non-adiabatic phase on the basis of hidden local symmetry in Section 5 and discuss some explicit examples of geometric phases to illustrate the re-formulation in Section 6.

2 Hidden local gauge symmetry

We start with the generic hermitian Hamiltonian \( \hat{H} = \hat{H}(\hat{p}, \hat{x}, X(t)) \) for a single particle theory in the background variable \( X(t) = (X_1(t), X_2(t), ...) \). The path integral for this theory for the time interval \( 0 \leq t \leq T \) in the second quantized
formulation is given by

\[ Z = \int \mathcal{D}\psi^{\dagger}\mathcal{D}\psi \exp \left\{ \frac{i}{\hbar} \int_{0}^{T} dt d^{3}x [\psi^{\dagger}(t, \vec{x}) i \hbar \frac{\partial}{\partial t} \psi(t, \vec{x}) \right. \\
- \left. \psi^{\dagger}(t, \vec{x}) \hat{H} \left( \frac{\hbar}{i} \frac{\partial}{\partial \vec{x}}, \vec{x}, X(t) \right) \psi(t, \vec{x})] \right\}. \] (5)

We then define a complete set of eigenfunctions

\[ \hat{H} \left( \frac{\hbar}{i} \frac{\partial}{\partial \vec{x}}, \vec{x}, X(t) \right) v_{n}(\vec{x}, X(t)) = \mathcal{E}_{n}(X(t)) v_{n}(\vec{x}, X(t)), \]
\[ \int d^{3}x v_{n}^{\dagger}(\vec{x}, X(t)) v_{m}(\vec{x}, X(t)) = \delta_{n,m}, \] (6)

and expand the classical field \( \psi(t, \vec{x}) \) in the path integral which is a Grassmann number for a fermion, for example, as

\[ \psi(t, \vec{x}) = \sum_{n} b_{n}(t) v_{n}(\vec{x}, X(t)). \] (7)

We then have \( \mathcal{D}\psi^{\dagger}\mathcal{D}\psi = \prod_{n} \mathcal{D}b_{n}^{\dagger}\mathcal{D}b_{n} \) and the path integral in the second quantized formulation is written as

\[ Z = \int \prod_{n} \mathcal{D}b_{n}^{\dagger}\mathcal{D}b_{n} \exp \left\{ \frac{i}{\hbar} \int_{0}^{T} dt \sum_{n} b_{n}^{\dagger}(t) i \hbar \frac{\partial}{\partial t} b_{n}(t) \right. \\
+ \left. \sum_{n,m} b_{n}^{\dagger}(t) \langle n | i \hbar \frac{\partial}{\partial t} | m \rangle b_{m}(t) - \sum_{n} b_{n}^{\dagger}(t) \mathcal{E}_{n}(X(t)) b_{n}(t) \right\} \] (8)

where the second term in the action, which is defined by

\[ \int d^{3}x v_{n}^{\dagger}(\vec{x}, X(t)) i \hbar \frac{\partial}{\partial t} v_{m}(\vec{x}, X(t)) \equiv \langle n | i \hbar \frac{\partial}{\partial t} | m \rangle, \]
stands for the term commonly referred to as Berry’s phase and its off-diagonal generalization. We take the time \( T \) as a period of the variable \( X(t) \) in the analysis of geometric phases, unless stated otherwise. The adiabatic process means that \( T \) is much larger than the typical time scale \( \hbar/\Delta \mathcal{E}_{n}(X(t)) \).

Translated into the operator formulation, we thus obtain the effective Hamiltonian (depending on Bose or Fermi statistics)

\[ \hat{H}_{\text{eff}}(t) = \sum_{n} \hat{b}_{n}^{\dagger}(t) \mathcal{E}_{n}(X(t)) \hat{b}_{n}(t) \]
\[ - \sum_{n,m} \hat{b}_{n}^{\dagger}(t) \langle n | i \hbar \frac{\partial}{\partial t} | m \rangle \hat{b}_{m}(t) \] (9)
with \([\hat{b}_n(t), \hat{b}_m^\dagger(t)]_z = \delta_{n,m}\). All the information about geometric phases is included in the effective Hamiltonian and thus purely dynamical. See also Berry [8] for a related observation. When one defines the Schrödinger picture \(\hat{H}_{\text{eff}}(t)\) by replacing all \(\hat{b}_n(t)\) by \(\hat{b}_n(0)\) in the above \(\hat{H}_{\text{eff}}(t)\), the second quantization formula for the evolution operator gives rise to [19]

\[
\langle m|T^* \exp\{-\frac{i}{\hbar} \int_0^T \hat{H}_{\text{eff}}(t) dt\}|n\rangle
\]

\[
= \langle m(T)|T^* \exp\{-\frac{i}{\hbar} \int_0^T \hat{H}(\hat{p}, \hat{x}, X(t)) dt\}|n(0)\rangle
\]

(10)

where \(T^*\) stands for the time ordering operation. The state vectors in the second quantization on the left-hand side are defined by \(|n\rangle = \hat{b}_n^\dagger(0)|0\rangle\), and the state vectors on the right-hand side stand for the first quantized states defined by \(\langle \vec{x}|n(t)\rangle = v_n(\vec{x}, (X(t)))\). Both-hand sides of the above equality (10) are exact, but the difference is that the geometric terms, both of diagonal and off-diagonal, are explicit in the second quantized formulation on the left-hand side.

The probability amplitude which satisfies Schrödinger equation with \(\psi_n(\vec{x}, 0; X(0)) = v_n(\vec{x}; X(0))\) is given by

\[
\psi_n(\vec{x}, t; X(t)) = \langle 0| T^* \exp\{-\frac{i}{\hbar} \int_0^t \hat{H}_{\text{eff}}(\vec{p}, \vec{x}, X(t)) dt\}|0\rangle
\]

(11)

since \(i\hbar \partial_t \hat{\psi} = \hat{H} \hat{\psi}\) in the present problem. To be explicit, we have

\[
\psi_n(\vec{x}, t; X(t))
\]

(12)

\[
= \sum_m v_m(\vec{x}; X(t)) \langle m|T^* \exp\{-\frac{i}{\hbar} \int_0^t \hat{H}_{\text{eff}}(t) dt\}|n\rangle
\]

by noting that (10) is given by \(\langle 0| \hat{b}_m(t) \hat{b}_n^\dagger(0)|0\rangle\). This formula is also derived by noting

\[
\psi_n(\vec{x}, t; X(t))
\]

\[
= \langle \vec{x}|T^* \exp\{-\frac{i}{\hbar} \int_0^t \hat{H}(\vec{p}, \vec{x}, X(t)) dt\}|n(0)\rangle
\]

\[
= \sum_m v_m(\vec{x}; X(t))
\]

\[
\times \langle m(t)|T^* \exp\{-\frac{i}{\hbar} \int_0^t \hat{H}(\vec{p}, \vec{x}, X(t)) dt\}|n(0)\rangle
\]

(13)
and the relation (10). In the adiabatic approximation, where we assume the dominance of diagonal elements, we have

\[ \psi_n(\vec{x}, t; X(t)) \]

\[ \simeq v_n(\vec{x}, X(t)) \exp\left\{ -\frac{i}{\hbar} \int_0^t \left[ \mathcal{E}_n(X(t)) - \langle n | i\hbar \frac{\partial}{\partial t} | n \rangle \right] dt \right\}. \]

The path integral formula (8) is based on the expansion (7) and the starting second-quantized path integral (5) depends only on the field variable \( \psi(t, \vec{x}) \), not on \( \{ b_n(t) \} \) and \( \{ v_n(\vec{x}, X(t)) \} \) separately. This fact shows that our formulation contains an exact hidden local gauge symmetry which keeps the field variable \( \psi(t, \vec{x}) \) invariant

\[ v_n(\vec{x}, X(t)) \rightarrow v'_n(t; \vec{x}, X(t)) = e^{i\alpha_n(t)} v_n(\vec{x}, X(t)), \]

\[ b_n(t) \rightarrow b'_n(t) = e^{-i\alpha_n(t)} b_n(t), \quad n = 1, 2, 3, \ldots, \] (15)

where the gauge parameter \( \alpha_n(t) \) is a general function of \( t \). This gauge symmetry (or substitution rule) states the fact that the choice of coordinates in the functional space is arbitrary and this symmetry by itself does not give any conservation law. This symmetry is exact under a rather mild condition that the basis set (6) is not singular, namely, it is exact not only for the adiabatic case but also for the non-adiabatic case. Consequently, physical observables should always respect this symmetry. Also, by using this local gauge freedom, one can choose the phase convention of the basis set \( \{ v_n(t, \vec{x}, X(t)) \} \) at one’s will such that the analysis of geometric phases becomes simplest.

Our next observation is that \( \psi_n(\vec{x}, t; X(t)) \) transforms under the hidden local gauge symmetry (15) as

\[ \psi'_n(\vec{x}, t; X(t)) = e^{i\alpha_n(0)} \psi_n(\vec{x}, t; X(t)) \] (16)

independently of the value of \( t \). This transformation is derived by using the exact representation (11), and it implies that \( \psi_n(\vec{x}, t; X(t)) \) is a physical object since \( \psi_n(\vec{x}, t; X(t)) \) stays in the same ray \[20, 21\] under an arbitrary hidden local gauge transformation. This transformation is explicitly checked for the adiabatic approximation (14) also.

The product \( \psi_n(\vec{x}, 0; X(0))^\dagger \psi_n(\vec{x}, T; X(T)) \) is thus manifestly independent of the choice of the phase convention of the basis set \( \{ v_n(t, \vec{x}, X(t)) \} \). For the adiabatic formula (14), the gauge invariant quantity is given by

\[ \psi_n(\vec{x}, 0; X(0))^\dagger \psi_n(\vec{x}, T; X(T)) \]

\[ = v_n(0, \vec{x}; X(0))^\dagger v_n(T, \vec{x}; X(T)) \]

\[ \times \exp\left\{ -\frac{i}{\hbar} \int_0^T \left[ \mathcal{E}_n(X(t)) - \langle n | i\hbar \frac{\partial}{\partial t} | n \rangle \right] dt \right\}. \] (17)
We then observe that by choosing the hidden gauge such that \( v_n(T, \vec{x}; X(T)) = v_n(0, \vec{x}; X(0)) \), the prefactor \( v_n(0, \vec{x}; X(0))\dagger v_n(T, \vec{x}; X(T)) \) becomes real and positive. Note that we are assuming the cyclic evolution of the external parameter, \( X(T) = X(0) \). Then the phase factor in (17) defines a physical quantity uniquely. See also Refs. [16, 17]. After this gauge fixing, the phase in (17) is still invariant under residual gauge transformations satisfying the periodic boundary condition \( \alpha_n(0) = \alpha_n(T) \), in particular, for \( \alpha_n(X(t)) \).

A change of the coordinates in the functional space more general than (15) is possible [18], and we utilize it to describe the non-adiabatic phase later.

3 Non-adiabatic phase

We recapitulate the basic aspects of non-adiabatic phases defined by Aharonov and Anandan [9, 10] and analyzed further by Samuel and Bhandari [11]. See also review [13].

The analysis in Ref. [9] starts with the wave function satisfying

\[
\int d^3x \psi(t, \vec{x})\dagger \psi(t, \vec{x}) = 1, \quad \psi(T, \vec{x}) = e^{i\phi} \psi(0, \vec{x}) \tag{18}
\]

with a real constant \( \phi \). For simplicity we restrict our attention to the unitary time-development as in (18). The condition (18) then implies the existence of a hermitian Hamiltonian

\[
i\hbar \frac{\partial}{\partial t} \psi(t, \vec{x}) = \hat{H}(\frac{\hbar}{i}, \frac{\partial}{\partial \vec{x}}, \vec{x}, X(t))\psi(t, \vec{x}) \tag{19}
\]

but now the variable \( X(t) \) need not be slowly varying. The mathematical basis of the non-adiabatic phase is the equivalence class, namely, the identification of all the state vectors of the form ("projective Hilbert space")

\[
\{ e^{i\alpha(t)} \psi(t, \vec{x}) \}. \tag{20}
\]

Note that they project \( \psi(t, \vec{x}) \) for each \( t \), which means local in time unlike the conventional notion of rays which is based on constant \( \alpha \) [20, 21]. Since the conventional Schrödinger equation is not invariant under this equivalence class, we may consider an equivalence class of Hamiltonians

\[
\{ \hat{H} - \hbar \frac{\partial}{\partial t} \alpha(t) \}. \tag{21}
\]
The equivalence class (20) means that we assign physical significance to those quantities invariant under the equivalence class. One can choose a suitable representative state vector $\tilde{\psi}(t, \vec{x}) = e^{-i\alpha(t)}\psi(t, \vec{x})$ such that

$$\tilde{\psi}(T, \vec{x}) = \tilde{\psi}(0, \vec{x})$$

by choosing $\alpha(T) - \alpha(0) = \phi$. This $\tilde{\psi}(t, \vec{x})$ is not invariant under (20), but it plays an important role in defining physical quantities.

One can also choose a representative state vector $\bar{\psi}(t, \vec{x}) = e^{i\alpha(t)}\psi(t, \vec{x})$ such that

$$\int d^3x \bar{\psi}^\dagger(t, \vec{x}) i\partial_t \bar{\psi}(t, \vec{x}) = \int d^3x \psi^\dagger(t, \vec{x}) i\partial_t \psi(t, \vec{x}) - \partial_t \alpha(t) = 0$$

namely [14, 15]

$$\bar{\psi}(t, \vec{x}) = \exp[i \int_0^T dt \int d^3x \psi(t, \vec{x}) \bar{\psi}^\dagger(t, \vec{x}) \psi(t, \vec{x})]$$

up to a constant phase factor $e^{i\alpha(0)}$. The exponential factor in (24) plays a role of gauge field, and under the equivalence class (or gauge transformation) $\psi(t, \vec{x}) \to e^{i\alpha(t)}\psi(t, \vec{x})$ one has

$$\tilde{\psi}(t, \vec{x}) \to e^{i\alpha(0)}\tilde{\psi}(t, \vec{x}).$$

This property (25), which is valid independently of the precise form of the Hamiltonian in (19) since we use only the property (18), implies that $\tilde{\psi}$ is a physical gauge invariant object up to a constant phase.

The manifestly gauge invariant quantity is then defined by

$$\bar{\psi}(0, \vec{x})^\dagger \bar{\psi}(T, \vec{x})$$

by following the prescription (17). By a suitable gauge transformation $\psi(t, \vec{x}) \to \tilde{\psi}(t, \vec{x}) = e^{-i\alpha(t)}\psi(t, \vec{x})$, we have $\tilde{\psi}(0, \vec{x}) = \tilde{\psi}(T, \vec{x})$ as in (22). The above gauge invariant quantity is then written as

$$\bar{\psi}(0, \vec{x})^\dagger \bar{\psi}(T, \vec{x}) = |\psi(0, \vec{x})|^2 \exp[i\beta]$$

$$= |\psi(0, \vec{x})|^2 \exp[i\beta]$$

(27)
with
\[
\beta = \oint dt \int d^3x \tilde{\psi}(t, \vec{x})^\dagger i \frac{\partial}{\partial t} \tilde{\psi}(t, \vec{x})
\] (28)
which extracts all the information about the phase from the gauge invariant quantity.
This quantity \(\beta\), which is still invariant under the residual gauge symmetry \(\alpha(t)\) with \(\alpha(0) = \alpha(T)\), is called “non-adiabatic phase” [9].

The Schrödinger equation for \(\psi(t, \vec{x}) = e^{i\gamma(t)} \tilde{\psi}(t, \vec{x})\)
\[
i\hbar \partial_t \psi(t, \vec{x}) = \hat{H} \psi(t, \vec{x})
\] (29)
with \(\gamma(T) - \gamma(0) = \phi\) implies
\[
\hbar \phi = \hbar \oint dt \int d^3x \tilde{\psi}(t, \vec{x})^\dagger i \frac{\partial}{\partial t} \tilde{\psi}(t, \vec{x})
- \int_0^T dt \int d^3x \psi^\dagger(t, \vec{x}) \hat{H} \psi(t, \vec{x})
= \hbar \beta - \int_0^T dt \int d^3x \psi^\dagger(t, \vec{x}) \hat{H} \psi(t, \vec{x}).
\] (30)
The last term \(\int_0^T dt \int d^3x \psi^\dagger(t, \vec{x}) \hat{H} \psi(t, \vec{x})\) on the right-hand side is called in [9] as a “dynamical phase”, though the total phase \(\hbar \phi\) is in fact generated by the Hamiltonian \(\hat{H}\) and thus dynamical. Eq.(30) defines the non-adiabatic phase and the “dynamical phase” simultaneously.

4 Ray representation and superposition principle
We examine the physical implications of the two different gauge symmetries, the hidden local gauge symmetry (15) and the equivalence class (20). The basic correspondence is
\[
v_n(\vec{x}; X(t)) \leftrightarrow \psi(t, \vec{x})
\] (31)
with the equivalence classes
\[
\{e^{i\alpha_n(t)}v_n(\vec{x}; X(t))\} \leftrightarrow \{e^{i\alpha(t)}\psi(t, \vec{x})\}.
\] (32)
The physical gauge invariant phases in the cyclic evolution are then given by, respectively, (17) and (27). The two formulations are thus very similar to each other, but there is a crucial difference: The true correspondence should be
\[
\psi_n(\vec{x}, t; X(t)) \leftrightarrow \psi(t, \vec{x}),
\] (33)
since both of $\psi_n(\vec{x}, t; X(t))$ in (11) and $\psi(t, \vec{x})$ stand for the Schrödinger probability amplitudes. Note that the probability amplitude need not be a linear superposition of basis vectors as is seen in the exact expression before approximation in (12). The hidden local symmetry (15) gives rise to the conventional notion of rays with constant phases, as is seen in (16).

We would like to understand the physical and conceptual basis for postulating the equivalence class (20). One may understand that the equivalence class is based on a generalization of the notion of rays in the Hilbert space. We examine this possibility. An important property of the Schrödinger amplitude is that one can consider a superposition of two probability amplitudes such as

$$\psi(t, \vec{x}) = c_1 e^{i\alpha_1} \psi_1(t, \vec{x}) + c_2 e^{i\alpha_2} \psi_2(t, \vec{x})$$

(34)

with two real constants $\alpha_1$ and $\alpha_2$ for the solutions of the Schrödinger equation

$$i\hbar \partial_t \psi_k(t, \vec{x}) = \hat{H} \psi_k(t, \vec{x}), \quad k = 1, 2.$$  

(35)

The superposition satisfies the same Schrödinger equation $i\hbar \partial_t \psi(t, \vec{x}) = \hat{H} \psi(t, \vec{x})$ and thus gives a probability amplitude. This superposition principle is based on the conventional notion of rays with constant phases.

On the other hand, for the generalized equivalence class we have

$$\psi'(t, \vec{x}) = c_1 e^{i\alpha_{1}(t)} \psi_1(t, \vec{x}) + c_2 e^{i\alpha_{2}(t)} \psi_2(t, \vec{x})$$

(36)

for the solutions of the Schrödinger equation

$$i\hbar \partial_t (e^{i\alpha_{k}(t)} \psi_k(t, \vec{x})) = (\hat{H} - \hbar \partial_t \alpha_k(t))(e^{i\alpha_{k}(t)} \psi_k(t, \vec{x})), \quad k = 1, 2$$

(37)

The superposition (of linearly independent $\psi_1$ and $\psi_2$) does not satisfy the Schrödinger equation of the general form

$$i\hbar \partial_t \psi'(t, \vec{x}) = (\hat{H} - \hbar \partial_t \alpha(t))\psi'(t, \vec{x})$$

(38)

except for the case

$$\partial_t \alpha_1(t) = \partial_t \alpha_2(t).$$

(39)

If one imposes this condition on the parameters $\alpha(t)$ for any combination of state vectors, the generalized ray is reduced to the conventional ray with a new Hamiltonian

$$\hat{H}' = \hat{H} - \hbar \partial_t \alpha_1(t).$$

(40)
Another important consequence of the equivalence class of states (20) is that one can always choose a representative \( \bar{\psi}(t, \vec{x}) = e^{i\alpha(t)}\psi(t, \vec{x}) \) which satisfies the parallel transport condition (23). Namely,

\[
\bar{\psi}(t, \vec{x}) = e^{i \int_0^t dt d^3x \psi(t, \vec{x})^\dagger i \partial_t \psi(t, \vec{x})} \psi(t, \vec{x})
\]  

(41)

up to a constant phase \( e^{i\alpha(0)} \). Given the Schrödinger equation

\[
i\hbar \partial_t \psi(t, \vec{x}) = H \psi(t, \vec{x}),
\]  

(42)

one has

\[
i\hbar \partial_t \bar{\psi}(t, \vec{x}) = (H - \hbar \partial_t \alpha(t)) \bar{\psi}(t, \vec{x})
\]

\[
= [H - \int d^3x \psi(t, \vec{x})^\dagger i\hbar \partial_t \psi(t, \vec{x})] \bar{\psi}(t, \vec{x})
\]

\[
= [H - \int d^3x \psi(t, \vec{x})^\dagger H \psi(t, \vec{x})] \bar{\psi}(t, \vec{x})
\]

\[
= [H - \int d^3x \bar{\psi}(t, \vec{x})^\dagger H \bar{\psi}(t, \vec{x})] \bar{\psi}(t, \vec{x}).
\]  

(43)

Namely, the representative which satisfies the parallel transport and gauge invariant conditions satisfies the non-linear Schrödinger equation\cite{22, 23}. One may also write this equation in the form of (4), which exhibits the symmetry under \( \bar{\psi}(t, \vec{x}) \rightarrow Z \bar{\psi}(t, \vec{x}) \) with a complex constant \( Z \)\cite{23}.

A linear superposition of two representatives

\[
c_1 \bar{\psi}_1(t, \vec{x}) + c_2 \bar{\psi}_2(t, \vec{x})
\]

(44)

of the two equivalence sets of states \( \{e^{i\alpha_1(t)}\psi_1(t, \vec{x})\} \) and \( \{e^{i\alpha_2(t)}\psi_2(t, \vec{x})\} \), where \( \psi_1(t, \vec{x}) \) and \( \psi_2(t, \vec{x}) \) are linearly independent, satisfies the same (non-linear) Schrödinger equation only for

\[
\int d^3x \psi_1(t, \vec{x})^\dagger i\hbar \partial_t \psi_1(t, \vec{x})
\]

\[
= \int d^3x \psi_2(t, \vec{x})^\dagger i\hbar \partial_t \psi_2(t, \vec{x})
\]  

(45)

which is consistent with \( \partial_t \alpha_1(t) = \partial_t \alpha_2(t) \) in (39). The superposition of two probability amplitudes which satisfy the parallel transport and gauge invariance conditions is regarded as the Schrödinger probability amplitude only under this condition.
The polarization measurement cannot distinguish $\psi'(t, \vec{x}) = e^{i\alpha(t)}\psi(t, \vec{x})$ and $\psi(t, \vec{x})$ in the sense that
\[
\psi'(t, \vec{x})^\dagger \sigma \psi'(t, \vec{x}) = \psi(t, \vec{x}) \sigma \psi(t, \vec{x})
\] (46)
and thus one may regard the generalized rays and the conventional rays are equivalent in the analysis of polarization phenomena. The situation is however more involved: An analysis of the movement of the polarization vector in the constant magnetic field $\vec{B}$ described by, for example,
\[
\hat{H} = -\mu \hbar \sigma \vec{B}
\] (47)
is based on the superposition of two states
\[
\psi(t) = \cos \frac{\theta}{2} \psi_+(t) + \sin \frac{\theta}{2} \psi_-(t),
\]
\[
i\hbar \partial_t \psi(t) = \hat{H} \psi(t)
\] (48)
with
\[
i\hbar \partial_t \psi_\pm(t) = H \psi_\pm(t),
\]
\[
H \psi_\pm(t) = \mp \mu \hbar \sigma B \psi_\pm(t)
\] (49)
If one uses different representatives in the conventional definition of rays with constant phases, $\{e^{i\alpha_1(t)}\psi_+(t)\}$ and $\{e^{i\alpha_2(t)}\psi_-(t)\}$, in (48) the phase factors are simply absorbed in the different choice of the superposition coefficients $\cos \frac{\theta}{2}$ and $\sin \frac{\theta}{2}$.

If one considers the equivalence classes in the notion of generalized rays
\[
\{e^{i\alpha_1(t)}\psi_+(t)\}, \quad \{e^{i\alpha_2(t)}\psi_-(t)\},
\] (50)
a linear superposition of two representatives
\[
\psi'(t) = \cos \frac{\theta}{2} e^{i\alpha_1(t)} \psi_+(t) + \sin \frac{\theta}{2} e^{i\alpha_2(t)} \psi_-(t)
\] (51)
does not satisfy the Schrödinger equation in general except for $\partial_t \alpha_1(t) = \partial_t \alpha_2(t)$, for which the generalized ray is reduced to the conventional ray for a modified Hamiltonian $\hat{H}' = \hat{H} - \hbar \partial_t \alpha_2(t)$. Incidentally, in the present case one cannot maintain the parallel transport condition (23) by choosing suitable $\alpha_1(t)$ and $\alpha_2(t)$ in $\{e^{i\alpha_1(t)}\psi_+(t)\}$ and $\{e^{i\alpha_2(t)}\psi_-(t)\}$, since for such a case one has to satisfy $\partial_t \alpha_1(t) = \partial_t \alpha_2(t)$ and
\[
\hbar \partial_t \alpha_1(t) = \int d^3x \psi_+(t, \vec{x})^* i\hbar \partial_t \psi_+(t, \vec{x}) = -\mu \hbar B,
\]
\[
\hbar \partial_t \alpha_2(t) = \int d^3x \psi_-(t, \vec{x})^* i\hbar \partial_t \psi_+(t, \vec{x}) = \mu \hbar B.
\] (52)
A solution of the Schrödinger equation is often written as a superposition of two or more other solutions of the Schrödinger equation. The notion of rays should be consistent with such a general situation. Only in the conventional definition of rays \[20, 21\], one can maintain consistency and describe the movement of the polarization vector consistently.

In the case of the explicit construction of the Schrödinger amplitude \(\psi_n(t, \vec{x})\) in (12), one can write
\[
\psi_n(t, \vec{x}) = \sum_m v_m(t, \vec{x}) G_{mn}(t),
\]
\[
i\hbar \partial_t \psi_n(t, \vec{x}) = H \psi_n(t, \vec{x}),
\]
(53)

where \(G_{mn}(t)\) stands for the unitary evolution operator (10) and \(\psi_n(t, \vec{x})\) is invariant under the hidden local symmetry up to a constant phase. On the other hand, the quantity \(\tilde{\psi}_n(t, \vec{x})\), which is invariant under the equivalence class (20) up to a constant phase, satisfies
\[
\tilde{\psi}_n(t, \vec{x}) = \exp[i \int_0^t \int d^3 x \psi_n^\dagger(t, \vec{x}) i \partial_t \psi_n(t, \vec{x})] \psi_n(t, \vec{x}),
\]
\[
i\hbar \partial_t \tilde{\psi}_n(t, \vec{x}) = [\hat{H}(t) - \int d^3 x \tilde{\psi}_n^\dagger \hat{H}(t) \tilde{\psi}_n(t, \vec{x}),
\]
\[
\int d^3 x \tilde{\psi}_n(t, \vec{x})^\dagger i \partial_t \psi_n(t, \vec{x}) = 0.
\]
(54)

From these expressions, one can clearly see the difference between the two gauge symmetries. One can also see that the gauge symmetry in the non-adiabatic phase is not reduced to the hidden local symmetry even in the adiabatic limit. The adiabatic formula
\[
\psi_n(\vec{x}, t; X(t)) \sim v_n(\vec{x}, X(t)) \exp\left\{-\frac{i}{\hbar} \int_0^t [\mathcal{E}_n(X(t)) - \langle n| i\hbar \frac{\partial}{\partial t}|n\rangle] dt\right\},
\]
(55)
is invariant under the hidden local symmetry (15) up to a constant phase, but this symmetry has nothing to do with the equivalence class \(\{e^{i\alpha(t)} \psi_n(\vec{x}, t; X(t))\}\).

Physically, the basic difference between the two gauge symmetries is that the quantity \(\psi_n^\dagger(0, \vec{x}) \psi_n(t, \vec{x})\) in (17) invariant under the hidden gauge symmetry is directly measurable as the interference term in
\[
|\psi_n^\dagger(0, \vec{x}) + \psi_n(t, \vec{x})|^2
\]
\[
= |\psi_n(0, \vec{x})|^2 + |\psi_n(t, \vec{x})|^2 + 2 \text{Re} \psi_n^\dagger(0, \vec{x}) \psi_n(t, \vec{x})
\]
(56)
by superposing two beams; for one of the beams one may choose $\hat{H} = 0$ and for the other one may choose $\hat{H} \neq 0$ with the identical kinematical phases which depend on the length of the two arms. On the other hand, the quantity $\bar{\psi}(0, \vec{x})^\dagger \psi(t, \vec{x})$ in (26) invariant under the equivalence class is not directly measured as the interference term in

$$|\bar{\psi}(0, \vec{x})^\dagger + \psi(t, \vec{x})|^2 = |\bar{\psi}(0, \vec{x})^\dagger|^2 + |\bar{\psi}(t, \vec{x})|^2 + 2\text{Re}\{\psi(0, \vec{x})^\dagger \bar{\psi}(t, \vec{x})\}$$

except for the case

$$\int d^3x \bar{\psi}(0, \vec{x})^\dagger \psi(t, \vec{x}) = 0$$

for all $t$, which ensures

$$ih\partial_t\{\exp[i \int_0^t dt \int d^3x \bar{\psi}(t, \vec{x})^\dagger \partial_t \psi(t, \vec{x})] \psi(t, \vec{x})\} = \dot{\hat{H}}\{\exp[i \int_0^t dt \int d^3x \bar{\psi}(t, \vec{x})^\dagger \partial_t \psi(t, \vec{x})] \psi(t)\}.$$  

Under the condition (58), the interference pattern in (57) agrees with the pattern in (56) dictated by quantum mechanics. This property (57) of the equivalence class differs from the conventional notion of gauge symmetry where only the gauge invariant quantity is directly measurable.

The last property (57) is also important in the analysis of non-adiabatic phases for non-cyclic processes [11] in the manner of Pancharatnam, where the measurement of interference provides a basic means to define the relative phase. To be precise, one can define a unique relative phase in the interference term only for the integrated quantity [18] in the case of the non-cyclic process

$$\int d^3x \bar{\psi}(0, \vec{x})^\dagger \psi(t, \vec{x})$$

but still such a phase is not directly measured by interference.
To summarize the analysis in this section, the basis of the equivalence class in the non-adiabatic phase may be understood as follows: Given any $\psi(t)$, one can consider the equivalence class

$$\{e^{i\alpha(t)}\psi(t)\} \quad (61)$$

for the specific $\psi(t)$, then the notion of the equivalence class provides a convenient means to extract the geometric property of the very specific $\psi(t)$. However, the equivalence class thus defined has no direct connection to a generalization of rays in the Hilbert space, and the physical origin of the equivalence class is not clear. Also, the gauge invariance is not a criterion of observables, as is exemplified by the gauge non-invariance of the Hamiltonian in (21). In this connection, we mention the “gauge independent formulation” on the basis of the density matrix [24]. A density matrix for a pure state $\psi(t)$

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| \quad (62)$$

is trivially invariant under the equivalence class (20). But the density matrix for the pure state does not tell how the pure state is formed, and the notion of rays and the superposition principle are crucial in the construction of the pure state. Also, the trivial invariance of the density matrix under the equivalence class means that the equivalence class by itself does not provide any useful information for the density matrix.

5 Non-adiabatic phase and hidden local symmetry

To reconcile the attractive idea of the non-adiabatic phase with the conventional notion of rays, we suggest to utilize a general unitary transformation of coordinates in the functional space [18]. Our observation is very simple: We start with the basic assumptions in (18) and (19)

$$i\hbar\partial_t\psi(t, \vec{x}) = \hat{H}(t)\psi(t, \vec{x}),$$

$$\int d^3 x \psi^\dagger(t, \vec{x})\psi(t, \vec{x}) = 1,$$

$$\psi(t, \vec{x}) = e^{i\phi(t)}\tilde{\psi}(t, \vec{x}),$$

$$\tilde{\psi}(T, \vec{x}) = \tilde{\psi}(0, \vec{x}),$$

$$\phi(T) = \phi, \quad \phi(0) = 0. \quad (63)$$
These assumptions combined with a constraint analogous to (30) gives

$$\psi(t, \vec{x}) = \tilde{\psi}(t, \vec{x}) \exp\left\{ -\frac{i}{\hbar} \int_0^t dt \int d^3 x \tilde{\psi}^\dagger(t, \vec{x}) \hat{H} \tilde{\psi}(t, \vec{x}) \right\}$$

which is transformed as $\psi(t, \vec{x}) \rightarrow e^{i\alpha(t)}\psi(t, \vec{x})$ under the equivalence class of Hamiltonians (21) with fixed $\tilde{\psi}(t, \vec{x})$. Our suggestion is rather to regard $\tilde{\psi}(t, \vec{x})$ as one of the basis vectors and incorporate the hidden local gauge symmetry (15) with fixed $\hat{H}$. Then $\psi(t, \vec{x})$ is invariant up to a constant phase under the hidden local symmetry, and the hidden local symmetry uniquely fixes the non-adiabatic phase as in the case of the adiabatic phase in (17).

We now explain the above construction. We first define a unitary transformation $U(t)$

$$w_n(t, \vec{x}) = \sum_m U(t)_{nm} v_m(t, \vec{x}),$$

$$w_n(0, \vec{x}) = w_n(0, \vec{x}),$$

$$\hat{H}(t) v_m(t, \vec{x}) = \mathcal{E}_m(t) v_m(t, \vec{x}),$$

$$\int d^3 x v_m^\dagger(t, \vec{x}) v_n(t, \vec{x}) = \delta_{m,n}$$

by taking the basis set $\{v_m(t, \vec{x})\}$ as a basic building block, for the sake of definiteness. We choose the unitary transformation such that the first element of the new complete orthonormal set $\{w_n(t, \vec{x})\}$ satisfies

$$w_1(t, \vec{x}) = \tilde{\psi}(t, \vec{x}),$$

which is possible since $\{v_m(t, \vec{x})\}$ forms a complete orthonormal set. The expansion of the field variable in second quantization is then given by

$$\hat{\psi}(t, \vec{x}) = \sum_m \hat{c}_m(t) w_m(t, \vec{x})$$

$$= \sum_m \hat{b}_m(t) v_m(t, \vec{x})$$

with

$$\hat{c}_m(t) = \sum_n \hat{b}_n(t) U(t)_{nm}^\dagger.$$
The variable $\psi(t, \vec{x})$ in (67) contains an exact hidden local symmetry

\[ w_m(t, \vec{x}) \to e^{i\alpha_m(t)} w_m(t, \vec{x}), \]
\[ \hat{c}_m(t) \to e^{-i\alpha_m(t)} \hat{c}_m(t) \]  

(69)

with general functions $\{\alpha_m(t)\}$. Following (13), we define

\[ \psi_n(t, \vec{x}) = \langle \vec{x} | T^n \exp \left\{ -\frac{i}{\hbar} \int_0^t \hat{H}(\hat{\rho}, \hat{\vec{x}}, X(t)) dt \right\} | n(0) \rangle \]

\[ = \sum_m w_m(t, \vec{x}) \times \langle m(t) | T^n \exp \left\{ -\frac{i}{\hbar} \int_0^t \hat{H}(\hat{\rho}, \hat{\vec{x}}, X(t)) dt \right\} | n(0) \rangle \]

\[ = \sum_m w_m(t, \vec{x}) \times \langle m | T^n \exp \left\{ -\frac{i}{\hbar} \int_0^t \hat{H}_{eff}(t) dt \right\} | n \rangle \]  

(70)

where $\hat{H}_{eff}(t)$ in the Schrödinger picture is obtained from

\[ \hat{H}_{eff}(t) = \sum_{n,m} \hat{c}_n(t) \left[ \int d^3 x w_n^\dagger(t, \vec{x}) \hat{H}(t) w_m(t, \vec{x}) \right. \]

\[ \left. - \int d^3 x w_n^\dagger(t, \vec{x}) i \hbar \partial_t w_m(t, \vec{x}) \right] \hat{c}_m(t) \]  

(71)

by replacing all $\hat{c}_n(t)$ by $\hat{c}_n(0)$. The state in the first quantization is defined by $\langle \vec{x} | n(t) \rangle = w_n(t, \vec{x})$ and the state in the second quantization is defined by $| n \rangle = c_n^\dagger(0) | 0 \rangle$ in (70).

The amplitudes thus defined satisfy

\[ i \hbar \partial_t \psi_n(t, \vec{x}) = \hat{H}(t) \psi_n(t, \vec{x}), \]
\[ \psi_n(0, \vec{x}) = w_n(0, \vec{x}). \]  

(72)

In particular, the amplitude $\psi_1(t, \vec{x})$ satisfies

\[ i \hbar \partial_t \psi_1(t, \vec{x}) = \hat{H}(t) \psi_1(t, \vec{x}), \]
\[ \psi_1(0, \vec{x}) = w_1(0, \vec{x}) = \psi(0, \vec{x}). \]  

(73)

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We thus have

\[
\psi(t, \vec{x}) = \psi_1(t, \vec{x}) = w_1(t, \vec{x}) \exp\left\{-\frac{i}{\hbar} \left[ \int_0^t dt \int d^3x w_1^\dagger(t, \vec{x}) \hat{H} w_1(t, \vec{x}) \right. \right.
\]

\[
\left. \left. - \int_0^t dt \int d^3x w_1^\dagger(t, \vec{x}) i\hbar \partial_t w_1(t, \vec{x}) \right]\right\} \tag{74}
\]

where the last structure is fixed by noting \( \psi(t, \vec{x}) = w_1(t, \vec{x}) e^{i\phi(t)} \) by assumption, namely, by the assumption that only the diagonal component survives for \( \psi_1(t, \vec{x}) \) in (70).

The above formulation makes it clear that \( \psi(t, \vec{x}) \) is fixed without referring to the equivalence class (20) or the notion of the equivalence class of Hamiltonians in (21). The geometric term in (74) is determined by the hidden local symmetry \(^1\) in (69) with a fixed Hamiltonian but without referring to any explicit form of the Hamiltonian. The explicit form of the Hamiltonian is however essential to ensure the periodicity of \( \psi(t, \vec{x}) = \psi_1(t, \vec{x}) \) up to a phase for arbitrary \( \vec{x} \). The amplitude \( \psi(t, \vec{x}) \) is invariant under the hidden local symmetry \( w_1(t, \vec{x}) \rightarrow e^{i\alpha_1(t)} w_1(t, \vec{x}) \) up to a constant phase, \( \psi(t, \vec{x}) \rightarrow e^{i\alpha_1(0)} \psi(t, \vec{x}) \), and satisfies the linear Schrödinger equation. The quantity

\[
\psi_\dagger(0, \vec{x}) \psi(T, \vec{x}) = w_1^\dagger(0, \vec{x}) w_1(T, \vec{x}) \exp\left\{-\frac{i}{\hbar} \left[ \int_0^T dt \int d^3x w_1^\dagger(t, \vec{x}) \hat{H} w_1(t, \vec{x}) \right. \right.
\]

\[
\left. \left. - \int_0^T dt \int d^3x w_1^\dagger(t, \vec{x}) i\hbar \partial_t w_1(t, \vec{x}) \right]\right\} \tag{75}
\]

is thus manifestly invariant under the hidden local symmetry with a fixed Hamiltonian. Note that the left-hand side of (75) is not invariant under the equivalence class (20). If one chooses the gauge such that \( w_1(0, \vec{x}) = w_1(T, \vec{x}) \) as in our starting

\(^1\)The hidden local symmetry (69) allows us to choose a representative \( \bar{w}_1(t, \vec{x}) = e^{i\alpha(t)} w_1(t, \vec{x}) \) which satisfies the parallel transport condition \( \int d^3\bar{w}_1(t, \vec{x}) i\partial_t \bar{w}_1(t, \vec{x}) = 0 \). Namely, \( \bar{w}_1(t, \vec{x}) = \exp\left[ i \int_0^t dt \int d^3x w_1^\dagger(t, \vec{x}) i\partial_t w_1(t, \vec{x}) \right] w_1(t, \vec{x}) \). This combination appears in (74), and the quantity manifestly invariant under the hidden local symmetry

\[
\bar{w}_1^\dagger(0, \vec{x}) \bar{w}_1(T, \vec{x}) = w_1^\dagger(0, \vec{x}) \exp\left[ i \int_0^T dt \int d^3x w_1^\dagger(t, \vec{x}) i\partial_t w_1(t, \vec{x}) \right] w_1(T, \vec{x}) \]

defines the non-adiabatic phase as holonomy for a cyclic evolution of the specific basis vector. Exactly the same consideration applies to the adiabatic phase in (17).
construction (65), the exponential factor in (75) extracts the entire phase from the
gauge invariant quantity and, in particular, the non-adiabatic phase is given by

$$\beta = \oint dt \int d^3x w_1^\dagger(t, \vec{x}) i \hbar \partial_t w_1(t, \vec{x}). \quad (76)$$

The hidden local symmetry, which is consistent with the linear Schrödinger
equation, is thus identified as the natural origin of the gauge symmetry in the
non-adiabatic phase without gauge fields. The basis set \{w_n(t, \vec{x})\} specify the coordinates in the functional space, and they do not satisfy the Schrödinger equation nor are the eigenvectors of \(\hat{H}\) in general. The non-adiabatic phase is regarded as a generalization of the adiabatic phase since it is defined without assuming adiabaticity in the sense of the slowness of the movement. At the same time, the non-adiabatic phase is also regarded as a special case of the adiabatic phase in that the exact periodicity of the specific state \(\psi_1(t, \vec{x})\) up to a phase is assumed and thus the exact adiabaticity in the sense of the absence of quantum mixing with other states is assumed. The adiabatic phase is rather universal in the sense that one can always define the adiabatic phase for any process as long as the (general) adiabaticity condition is satisfied.

We illustrate this re-formulation of geometric phases in the next section.

6 Explicit examples

6.1 Adiabatic phase

We have already explained that the gauge symmetry in the non-adiabatic phase
is not reduced to that in the adiabatic phase even in the adiabatic limit. We here analyze the implications of this difference. If one takes the equivalence class (20) as a gauge symmetry, one is allowed to choose representatives \(\tilde{\psi}_1(t, \vec{x})\) and \(\tilde{\psi}_2(t, \vec{x})\) in (24), which are gauge invariant up to a constant phase. When \(\int d^3x \psi^\dagger(t, \vec{x}) i \partial_t \psi(t, \vec{x})\)
is identical for \(\tilde{\psi}_1(t, \vec{x})\) and \(\tilde{\psi}_2(t, \vec{x})\), one has

$$c_1 \tilde{\psi}_1(t, \vec{x}) + c_2 \tilde{\psi}_2(t, \vec{x}) \quad \text{and}$$

$$= e^{i \int_0^t dt \int d^3x \psi^\dagger(t, \vec{x}) i \partial_t \psi(t, \vec{x}) \{c_1 \psi_1(t, \vec{x}) + c_2 \psi_2(t, \vec{x})\}$$

and one can assign the physical meaning to the absolute square of the superposition. Note that the observable interference pattern is unique and given by

$$|c_1 \psi_1(t, \vec{x}) + c_2 \psi_2(t, \vec{x})|^2 \quad (78)$$

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at any moment. The condition (77) needs to be satisfied precisely not only at \( t = 0 \) and \( t = T \) but also for all \( t \), since the system is not allowed to go away from quantum mechanics in the intermediate stage. See also (45). If one takes the equivalence class (20) literally and chooses representatives which satisfy the gauge invariance condition, the interference measurement of the non-adiabatic phase thus becomes equivalent to the *conventional* measurement of interference for a very limited set of amplitudes with the constraint (45) for any \( t \), which may also be expressed in terms of an on-shell value \( \int d^3x \psi^\dagger(t, \vec{x}) \hat{H}(t) \psi(t, \vec{x}) \) with an equivalence class of Hamiltonians.

In the adiabatic limit, a superposition of two independent adiabatic solutions

\[
c_1 \psi_1(\vec{x}, t; X(t)) + c_2 \psi_2(\vec{x}, t; X(t))
\]

\[
\simeq c_1 v_1(\vec{x}; X(t)) \exp\left\{-\frac{i}{\hbar} \int_0^t [\mathcal{E}_1(X(t)) - \langle 1| i\hbar \frac{\partial}{\partial t} |1\rangle] dt\right\}
+ c_2 v_2(\vec{x}; X(t)) \exp\left\{-\frac{i}{\hbar} \int_0^t [\mathcal{E}_2(X(t)) - \langle 2| i\hbar \frac{\partial}{\partial t} |2\rangle] dt\right\}
\] (79)

satisfies the condition (45) only when the “dynamical phase” \( \mathcal{E}_n(X(t)) \) is identical

\[
\mathcal{E}_1(X(t)) = \hbar \int d^3x \psi_1^\dagger(t, \vec{x}) i\partial_t \psi_1(t, \vec{x})
\]

\[
= \hbar \int d^3x \psi_2^\dagger(t, \vec{x}) i\partial_t \psi_2(t, \vec{x})
\]

\[
= \mathcal{E}_2(X(t))
\] (80)

for the two solutions. This gives a sufficient condition to measure the adiabatic phase described by \( \psi_n \), but actually only the identical integrated

\[
\int_0^T dt \mathcal{E}_1(X(t)) = \int_0^T dt \mathcal{E}_2(X(t))
\] (81)

is necessary for the direct measurement of the adiabatic phase, as is seen in (79). The stronger condition (80) arises from the non-locality of \( \tilde{\psi} \) in \( \psi \).

An explicit procedure \[1\] to measure the adiabatic phase is to separate the path of a particle into two in some region of space \( \vec{x} \); the external parameter \( X(t) \) in one of the beams, for example, may be chosen to be constant, and the superposition of two beams is measured later to extract the geometric phase. One then controls the ”dynamical phase” to be identical as in (81) for these two paths. Although the stronger condition (80) happens to be satisfied by an explicit example discussed in \[1\], only the weaker condition (81) is necessary for the direct measurement of the adiabatic phase in interference experiments. In our re-formulation of the non-adiabatic phase, we encounter only the weaker condition.
6.2 Spin polarization

Most of the experimental analyses \[25, 26, 27, 28, 29\] of geometric phases are based on the polarization measurements. We thus study the model described by

\[ \hat{H} = -\mu \hbar \vec{B}(t) \vec{\sigma} \]  \hspace{1cm} (82)

where \( \vec{\sigma} \) stand for Pauli matrices and \( \vec{B}(t) \) is generally a time dependent magnetic field.

We first briefly comment on the general aspects of the movement of polarization vectors and the holonomy for spinor basis vectors. The most general form of the normalized basis vectors are parameterized as

\[ v_+(t) = \left( \cos \frac{1}{2} \theta(t) e^{-i\varphi(t)} \right), \quad v_-(t) = \left( \sin \frac{1}{2} \theta(t) e^{-i\varphi(t)} \right) \]  \hspace{1cm} (83)

if one takes the hidden local symmetry (15) into account. It is also shown that

\[ v_+^\dagger(t) \vec{\sigma} v_+(t) = (\sin \theta \cos \varphi(t), \sin \theta \sin \varphi(t), \cos \theta), \]
\[ = -v_-^\dagger(t) \vec{\sigma} v_-(t), \]  \hspace{1cm} (84)

and

\[ \int v_\pm^\dagger(t) i \partial_t v_\pm(t) dt = -\frac{1}{2} \int (1 \mp \cos \theta) d\varphi + 2\pi \]
\[ = -\frac{1}{2} \Omega_\pm \]  \hspace{1cm} (85)

up to \( 2n\pi \); \( \Omega_\pm \) stand for the solid angles drawn by the closed movements of unit polarization vectors in (84).

By using the hidden local symmetry, one may choose a representative \( \bar{v}_\pm(t) = e^{i\alpha(t)} v_\pm(t) \) such that

\[ \bar{v}_+^\dagger(t) i \partial_t \bar{v}_+(t) = v_+^\dagger(t) i \partial_t v_+(t) - \partial_t \alpha(t) = 0. \]  \hspace{1cm} (86)

We thus have

\[ \bar{v}_\pm(t) = \exp[i \int_0^t dt v_\pm^\dagger(t) i \partial_t v_\pm(t)] v_\pm(t) \]  \hspace{1cm} (87)

and

\[ \bar{v}(0)_\pm \bar{v}_\pm(T) = \exp[-i \frac{1}{2} \Omega_\pm]. \]  \hspace{1cm} (88)

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where $T$ stands for the period of the movement of polarization vectors. This shows that the notion of parallel transport (86) and holonomy (88), which is analyzed without referring to any explicit Hamiltonian, is a notion for the basis vectors $[2, 7]$ rather than for the Schrödinger amplitudes. In the case of Schrödinger amplitudes, one needs to analyze the Schrödinger equation and the quantum transition between $v_\pm(t)$.

We now illustrate our re-formulation of non-adiabatic phases in the spin polarization phenomena.

(i) For the special case \[\vec{B}(t) = (0, 0, B)\] (89) in (82) with a constant $B$, one may consider

\[
\psi_+(t) = \cos \frac{1}{2} \theta v_+ e^{i \mu B t / \hbar} + \sin \frac{1}{2} \theta v_- e^{-i \mu B t / \hbar},
\]

\[
i \hbar \partial_t \psi_+ (t) = \hat{H} \psi_+ (t)
\]
with

\[
\hat{H} v_\pm = \mp \hbar \mu B v_\pm, \quad v_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

This model, though quite simple, is conceptually important [9], and we explain it in some detail. The amplitude $\psi_+(t)$ is written as

\[
\psi_+(t) = w_+(t) \exp [i \hbar \mu B t / \hbar]
\]

\[
= w_+(t) \exp \{- \frac{i}{\hbar} \int_0^t \left[ -\hbar \mu B \cos \theta - \hbar \mu B (1 - \cos \theta) \right] dt \}
\]

\[
= w_+(t) \exp \{- \frac{i}{\hbar} \int_0^t dt \left[ w_+(t)^\dagger \hat{H} w_+(t) - w_+(t)^\dagger i \hbar \partial_t w_+(t) \right] \}
\]

(92)

with

\[
w_+(t) = \cos \frac{1}{2} \theta v_+ + \sin \frac{1}{2} \theta v_- e^{-2i \mu B t / \hbar},
\]

\[
w_+(T) = w_+(0),
\]

\[
w_+(t)^\dagger i \hbar \partial_t w_+(t) = \mu B (1 - \cos \theta),
\]

\[
w_+(t)^\dagger \hat{H} w_+(t) = -\hbar \mu B \cos \theta
\]

(93)

where $T = \pi / \mu B$. Similarly, one may define

\[
\psi_-(t) = - \sin \frac{1}{2} \theta v_+ e^{i \mu B t / \hbar} + \cos \frac{1}{2} \theta v_- e^{-i \mu B t / \hbar},
\]

\[
i \hbar \partial_t \psi_- (t) = \hat{H} \psi_- (t)
\]

(94)
and
\[\psi_-(t) = w_-(t) \exp[-i\hbar \mu B t / \hbar] = w_-(t) \exp\left\{-\frac{i}{\hbar}[\hbar \mu B \cos \theta + \hbar \mu B (1 - \cos \theta)]t\right\} = w_-(t) \exp\left\{-\frac{i}{\hbar} \int_0^t dt [w_-(t) \hat{H} w_-(t) - w_-(t)^\dagger i \hbar \partial_t w_-(t)]\right\}\]

with
\[w_-(t) = -\sin \frac{1}{2} \theta v_+ e^{2i \hbar \mu B t / \hbar} + \cos \frac{1}{2} \theta v_-,
\]
\[w_-(T) = w_-(0),
\]
\[w_-(t)^\dagger i \partial_t w_-(t) = -\mu B (1 - \cos \theta),
\]
\[w_-(t)^\dagger \hat{H} w_-(t) = \hbar \mu B \cos \theta.\]  

We here performed the unitary transformation
\[
\begin{pmatrix}
  w_+(t) \\
  w_-(t)
\end{pmatrix} =
\begin{pmatrix}
  \cos \frac{1}{2} \theta & \sin \frac{1}{2} \theta e^{-2i \hbar \mu B t / \hbar} \\
  -\sin \frac{1}{2} \theta e^{2i \hbar \mu B t / \hbar} & \cos \frac{1}{2} \theta
\end{pmatrix}
\begin{pmatrix}
  v_+ \\
  v_-
\end{pmatrix} = U(t)
\begin{pmatrix}
  v_+ \\
  v_-
\end{pmatrix}
\]

This means that the expansion of the field variable in second quantization is given by
\[
\hat{\psi}(t) = \sum_n \hat{b}_n(t) v_n = \sum_n \hat{c}_n(t) w_n(t)
\]
with
\[
\begin{pmatrix}
  \hat{b}_+(t) \\
  \hat{b}_-(t)
\end{pmatrix} \equiv U^T(t)
\begin{pmatrix}
  \hat{c}_+(t) \\
  \hat{c}_-(t)
\end{pmatrix}
\]

where $U^T(t)$ stands for the transpose of $U(t)$. The effective Hamiltonian $\hat{H}_{eff}(t)$ in (73) in the present case is given by
\[
\hat{H}_{eff}(t) = \sum_{n=\pm} \hat{c}_n(t)^\dagger [w_n(t)^\dagger \hat{H} w_n(t) - w_n(t)^\dagger i \hbar \partial_t w_n(t)] \hat{c}_n(t)
\]
and the off-diagonal terms completely cancel. This formula is exact and thus non-adiabatic.

The variable $\hat{\psi}(t)$ in (98) is invariant under the hidden local gauge symmetry

$$w_n(t) \rightarrow e^{i\alpha_n(t)}w_n(t), \quad \hat{c}_n(t) \rightarrow e^{-i\alpha_n(t)}\hat{c}_n(t). \quad (101)$$

The expressions of $\psi_\pm(t)$ in (92) and (95) are invariant under the hidden gauge symmetry up to a constant phase. But no symmetry with respect to the equivalence class $\{e^{i\alpha(t)}\psi_\pm(t)\}$ in (20) nor the equivalence class of Hamiltonians (21) appear. We operate on a fixed Hamiltonian.

If one recalls that

$$w_\pm^\dagger(t)\vec{\sigma}w_\pm(t) = (\sin \theta \cos \varphi(t), \sin \theta \sin \varphi(t), \cos \theta),$$

$$= -w_\pm^\dagger(t)\vec{\sigma}w_\mp(t), \quad (102)$$

with $\varphi = -2\mu_B t$, the solid angles $\Omega_\pm$ subtended by the polarization vectors $w_\pm^\dagger(t)\vec{\sigma}w_\pm(t)$ around the z-axis during a cyclic motion are respectively given by

$$\int_0^T dt w_+(t)^\dagger i\partial_t w_+(t) = \pi(1 - \cos \theta)$$

$$= -\frac{1}{2} \int_0^{2\pi} d\varphi(1 - \cos \theta) = -\frac{1}{2}\Omega_+, \quad (103)$$

$$\int_0^T dt w_-(t)^\dagger i\partial_t w_-(t) = -\pi(1 - \cos \theta)$$

$$= \pi(1 + \cos \theta) - 2\pi = -\frac{1}{2}\Omega_-$$

up to $2n\pi$, where $T = \pi/\mu B$. The non-adiabatic phase contained in the manifestly gauge invariant $\psi_\dagger_+(0)\psi_+(T)$, for example, is measured by the interference in

$$|\psi_+(T) + \psi_+(0)|^2 = |\psi_+(T)|^2 + |\psi_+(0)|^2 + 2\text{Re}\psi_\dagger_+(0)\psi_+(T)$$

$$= 2 + 2\cos[\pi \cos \theta - \frac{1}{2}\Omega_+] \quad (104)$$

which reproduces the result of Aharonov and Anandan [9]. A separation of the non-adiabatic phase $-\frac{1}{2}\Omega_+$ from the “dynamical phase” $\int_0^T dt w_+(t)^\dagger \hat{H}w_+(t)/\hbar = -\pi \cos \theta$ is possible if one separates the beam into two and later superposes them with a suitable Hamiltonian in the second path which cancels the “dynamical phase” $-\pi \cos \theta$ in the first path. We can thus describe the non-adiabatic phase consistently in terms of the hidden local symmetry without referring to the equivalence class (20)
or the notion of the equivalence class of Hamiltonians in (21).

(ii) We next analyze (82) in the case

\[ \vec{B}(t) = B(\sin \theta \cos \varphi(t), \sin \varphi(t), \cos \theta) \]  

where \( \varphi(t) = \omega t \) with constant \( \omega, B \) and \( \theta \). We then have the effective Hamiltonian in (9)

\[ \hat{H}_{eff}(t) = [-\mu hB - \frac{(1 + \cos \theta)}{2} \hbar \omega] \hat{b}_+ \hat{b}_+ + [\mu hB - \frac{1 - \cos \theta}{2} \hbar \omega] \hat{b}_- \hat{b}_- \]

\[ - \frac{\sin \theta}{2} \hbar \omega [\hat{b}_+ \hat{b}_- + \hat{b}_- \hat{b}_+] \]  

with

\[ v_+(t) = \begin{pmatrix} \cos \frac{1}{2} \theta e^{-i \varphi(t)} \\ \sin \frac{1}{2} \theta \end{pmatrix}, \quad v_-(t) = \begin{pmatrix} \sin \frac{1}{2} \theta e^{-i \varphi(t)} \\ -\cos \frac{1}{2} \theta \end{pmatrix} \]

which satisfy \( \hat{H}(t)v_\pm(t) = \pm \mu hB v_\pm(t) \) and the relations

\[ v_+^\dagger(t) i \frac{\partial}{\partial t} v_+(t) = \frac{(1 + \cos \theta)}{2} \omega \]

\[ v_+^\dagger(t) i \frac{\partial}{\partial t} v_-(t) = \frac{\sin \theta}{2} \omega = v_-^\dagger(t) i \frac{\partial}{\partial t} v_+(t), \]

\[ v_-^\dagger(t) i \frac{\partial}{\partial t} v_-(t) = \frac{1 - \cos \theta}{2} \omega. \]

We next perform a unitary transformation

\[ \begin{pmatrix} \hat{b}_+(t) \\ \hat{b}_-(t) \end{pmatrix} = \begin{pmatrix} \cos \frac{1}{2} \alpha & -\sin \frac{1}{2} \alpha \\ \sin \frac{1}{2} \alpha & \cos \frac{1}{2} \alpha \end{pmatrix} \begin{pmatrix} \hat{c}_+(t) \\ \hat{c}_-(t) \end{pmatrix} \]

\[ \equiv U^T \begin{pmatrix} \hat{c}_+(t) \\ \hat{c}_-(t) \end{pmatrix} \]

where \( U^T \) stands for the transpose of \( U \). The eigenfunctions are transformed to

\[ \begin{pmatrix} w_+(t) \\ w_-(t) \end{pmatrix} = U \begin{pmatrix} v_+(t) \\ v_-(t) \end{pmatrix} \]

\[ = \begin{pmatrix} \cos \frac{1}{2} \alpha & \sin \frac{1}{2} \alpha \\ -\sin \frac{1}{2} \alpha & \cos \frac{1}{2} \alpha \end{pmatrix} \begin{pmatrix} v_+(t) \\ v_-(t) \end{pmatrix} \]

\[ = \begin{pmatrix} \cos \frac{1}{2} \alpha & \sin \frac{1}{2} \alpha \\ -\sin \frac{1}{2} \alpha & \cos \frac{1}{2} \alpha \end{pmatrix} \begin{pmatrix} v_+(t) \\ v_-(t) \end{pmatrix} \]

\[ = \begin{pmatrix} \cos \frac{1}{2} \alpha & \sin \frac{1}{2} \alpha \\ -\sin \frac{1}{2} \alpha & \cos \frac{1}{2} \alpha \end{pmatrix} \begin{pmatrix} v_+(t) \\ v_-(t) \end{pmatrix} \]
or explicitly
\[
\begin{align*}
w_+(t) &= \left( \cos \frac{1}{2} (\theta - \alpha) e^{-i \varphi(t)} \right) , \\
w_-(t) &= \left( \sin \frac{1}{2} (\theta - \alpha) e^{-i \varphi(t)} \right).
\end{align*}
\] (111)

The field variable \( \hat{\psi}(t, \vec{x}) \) in second quantization is given by
\[
\hat{\psi}(t, \vec{x}) = \sum_{n=\pm} \hat{b}_n(t) v_n(t)
\]
\[
= \sum_{n=\pm} \hat{c}_n(t) w_n(t)
\] (112)

which is invariant under the hidden local symmetry
\[
w_n(t) \to e^{i \alpha_n(t)} w_n(t), \quad \hat{c}_n(t) \to e^{-i \alpha_n(t)} \hat{c}_n(t).
\] (113)

We also have
\[
w^\dagger_\pm(t) \hat{H} w_\pm(t) = \mp \mu \hbar H \cos \alpha
\]
\[
w^\dagger_\pm(t) i \hbar \partial_t w_\pm(t) = \frac{\hbar \omega}{2} (1 \pm \cos(\theta - \alpha))
\] (114)

If one chooses the parameter \( \alpha \) in (109) as
\[
\tan \alpha = \frac{\hbar \omega \sin \theta}{2 \mu \hbar B + \hbar \omega \cos \theta}
\] (115)

one obtains a diagonal effective Hamiltonian
\[
\hat{H}_{\text{eff}}(t) = \sum_{n=\pm} \hat{c}_n^\dagger (\mp \mu \hbar B \cos \alpha - \frac{\hbar \omega}{2} (1 \pm \cos(\theta - \alpha))] \hat{c}_n
\]
\[
= \sum_{n=\pm} \hat{c}_n^\dagger (w^\dagger_\pm(t) \hat{H} w_\pm(t) - w^\dagger_\pm(t) i \hbar \partial_t w_\pm(t)) \hat{c}_n.
\] (116)

The above unitary transformation is time-independent and thus the effective Hamiltonian is not changed \( \hat{H}_{\text{eff}}(b^\dagger_\pm(t), b_\pm(t)) = \hat{H}_{\text{eff}}(c^\dagger_\pm(t), c_\pm(t)) \). We have the Schrödinger amplitudes in (72)
\[
\psi_\pm(t) = w_\pm(t) \exp\{-\frac{i}{\hbar} (\mp \mu \hbar B \cos \alpha - \frac{\hbar \omega}{2} (1 \pm \cos(\theta - \alpha))]t\}
\]
\[
= w_\pm(t) \exp\{-\frac{i}{\hbar} \int_0^t dt[w^\dagger_\pm(t) \hat{H} w_\pm(t) - w^\dagger_\pm(t) i \hbar \partial_t w_\pm(t)]\}.
\] (117)
These expressions are periodic with period $T = \frac{2\pi}{\omega}$ up to a phase, and they are exact and thus non-adiabatic. From the viewpoint of the diagonalization of the Hamiltonian, we have not completely diagonalized the exact Hamiltonian since $w_{\pm}(t)$ carry certain time-dependence. These formulas are invariant under the hidden local symmetry (113) up to a constant phase factor, but no invariance under the equivalence class (20) nor equivalence class of Hamiltonians (21) appear. We operate on a fixed Hamiltonian.

In the generic case with period $T = \frac{2\pi}{\omega}$, one can measure $\psi^+(0)\psi_+(T)$, for example, which is manifestly invariant under the hidden local symmetry by the interference in

$$|\psi_+(T) + \psi_+(0)|^2 = 2|\psi_+(0)|^2 + 2\text{Re}\psi^+_+(0)\psi_+(T) = 2 + 2\cos[(\mu B \cos \alpha)T - \frac{1}{2}\Omega_+]$$

where

$$\Omega_+ = 2\pi [1 - \cos(\theta - \alpha)]$$

stands for the solid angle drawn by $w^+_+(t)\sigma w_+(t)$ by noting

$$w^+_+(t)\sigma w_+(t) = (\sin(\theta - \alpha) \cos \varphi, \sin(\theta - \alpha) \sin \varphi, \cos(\theta - \alpha))$$

$$= -w^-_+(t)\sigma w^-_+(t).$$

The separation of the non-adiabatic phase and the “dynamical phase” in (118) is achieved by varying the parameters in the Hamiltonian, namely, $B$ and $\omega$ in the present case. The formula (118) however shows that both of the non-adiabatic phase and the “dynamical phase” depend on these parameters in a non-trivial way. In the limit $\hbar \omega \ll \mu \hbar B$, $\alpha \to 0$ in (115) and the above formula (118) is reduced to the familiar adiabatic phase. For $B \to$ small with fixed $T = 2\pi/\omega$, $\alpha \to \theta$ in (115) and the geometric phase becomes trivial [19]. More generally, in the extreme non-adiabatic limit $\hbar \omega \gg \mu \hbar B$, $\alpha \to \theta$ in (115) and the non-adiabatic phase becomes trivial. This fact holds independently of an explicit model: The phase in (76) becomes trivial $\beta \simeq 0$ in the extreme non-adiabatic limit defined by $\Delta E \ll 2\pi \hbar /T$ where $\Delta E$ stands for the level splitting of a two-level truncation of (9). The diagonalization of the dominant geometric term in (9) approximately diagonalizes the effective Hamiltonian which gives a trivial geometric phase [19]. The subtraction of $\int_0^T dt\mathcal{E}_n(X(t))$ then removes the almost degenerate “dynamical phase” in (9) and thus resulting in the trivial $\beta \simeq 0$.

The eigenfunctions $w_{\pm}(t)$ defined in (111) are periodic with period $T = \frac{2\pi}{\omega}$. By considering the difference of the energy factors on the shoulders of the exponential
in (117), a linear superposition of Schrödinger amplitudes $\psi_\pm(t)$ is periodic with period $T$ up to a phase only when

$$T\omega = 2\pi n, \quad T[2\mu B \cos \alpha + \omega \cos(\theta - \alpha)] = 2\pi m$$

(121)

with two integers $n$ and $m$. For the generic case, however, a linear superposition of $\psi_\pm(t)$ is not periodic and either one of $\psi_\pm(t)$ is an allowed periodic function with period $T = \frac{2\pi}{\omega}$. For the special case in (121), one may consider a linear combination

$$\Psi_+(t) = \cos \frac{\Theta}{2} \psi_+(t) + \sin \frac{\Theta}{2} \psi_-(t)$$

$$\Psi_-(t) = -\sin \frac{\Theta}{2} \psi_+(t) + \cos \frac{\Theta}{2} \psi_-(t)$$

(122)

both of which satisfy the Schrödinger equation, and one can repeat the analysis analogous to (92) and (95) but we forgo the details. The movement of the polarization vector in this case is induced by a superposition of Schrödinger amplitudes rather than by an attempt to diagonalize the evolution operator, and thus it is close to the non-adiabatic phase in the original sense of Aharonov and Anandan [9].

7 Conclusion

The notion of rays in the Hilbert space is based on the equivalence class

$$\{e^{i\alpha} \psi(t, \vec{x})\}$$

(123)

with constant phases $\alpha$ [20, 21]. One of the possible generalizations of the above equivalence class may be

$$\{e^{i(\alpha(t))} \psi(t, \vec{x})\}$$

(124)

which played a basic role in the definition of the non-adiabatic phase [9, 10, 11]. But the origin of this gauge symmetry and the consistency of imposing the gauge symmetry in the absence of gauge fields were not clear. In particular, a representative, which satisfies the parallel transport and gauge invariance conditions,

$$\tilde{\psi}(t, \vec{x}) = e^{i \int_0^t dt \int d^3x \psi^*(t, \vec{x}) i\partial_\psi(t, \vec{x}) \psi(t, \vec{x})} \psi(t, \vec{x})$$

(125)

is non-local and non-linear in the Schrödinger amplitude $\psi(t, \vec{x})$, and thus the consistency with the superposition principle was not obvious.
We proposed a re-formulation of the non-adiabatic phase on the basis of the hidden local gauge symmetry \[18\] arising from the arbitrariness of the choice of coordinates in the functional space. The equivalence class in this case is
\[
\{e^{i\alpha_n(t)}w_n(t, \vec{x})\}
\] (126)
where \(\{w_n(t, \vec{x})\}\) is a complete orthonormal basis set, and this gauge symmetry gives rise to the conventional equivalence class (123) for the Schrödinger amplitudes. The hidden local gauge symmetry maintains the consistency of the non-adiabatic phase with the conventional notion of rays and the superposition principle. This re-formulation clarifies the natural origin of the gauge symmetry in both of the adiabatic and non-adiabatic phases without gauge fields, and it allows a unified treatment of all the geometric phases.

As for other applications of the second quantized formulation, it has been shown elsewhere that the geometric phase and the quantum anomaly, which have been long considered to be closely related, in fact have nothing to do with each other \[30\].

References

[1] M.V. Berry, Proc. Roy. Soc. A\textbf{392}, 45 (1984).
[2] B. Simon, Phys. Rev. Lett. \textbf{51}, 2167 (1983).
[3] A.J. Stone, Proc. Roy. Soc. A\textbf{351}, 141 (1976).
[4] H. Longuet-Higgins, Proc. Roy. Soc. A\textbf{344}, 147 (1975).
[5] F. Wilczek and A. Zee, Phys. Rev. Lett. \textbf{52}, 2111 (1984).
[6] H. Kuratsuji and S. Iida, Prog. Theor. Phys. \textbf{74}, 439 (1985).
[7] J. Anandan and L. Stodolsky, Phys. Rev. D\textbf{35}, 2597 (1987).
[8] M.V. Berry, Proc. Roy. Soc. A\textbf{414}, 31 (1987).
[9] Y. Aharonov and J. Anandan, Phys. Rev. Lett. \textbf{58}, 1593 (1987).
[10] J. Anandan and Y. Aharonov, Phys. Rev. D\textbf{38}, 1863 (1988).
[11] J. Samuel and R. Bhandari, Phys. Rev. Lett. \textbf{60}, 2339 (1988).
[12] S. Pancharatnam, Proc. Indian Acad. Sci. A\textbf{44}, 247 (1956).
[13] J. Anandan, Nature 360, 307 (1992).

[14] I.J.R. Aitchison and K. Wanelik, Proc. Roy. Soc. A439, 25 (1992).

[15] N. Mukunda and R. Simon, Ann. Phys. (N.Y.) 228, 205 (1993).

[16] G. Garcia de Polavieja and E. Sjöqvist, Am. J. Phys. 66, 431 (1998).

[17] A. Mostafazadeh, J. Phys. A32, 8157 (1999).

[18] K. Fujikawa, Phys. Rev. D72, 025009 (2005).

[19] K. Fujikawa, Mod. Phys. Lett. A20, 335 (2005).

[20] P.A.M. Dirac, Principles of Quantum Mechanics (Oxford University Press, Oxford, 1958).

[21] R.F. Streater and A.S. Wightman, PCT, Spin and Statistics and All That (W.A. Benjamin, Inc., New York, 1964).

[22] I. Bialynicki-Birula and J. Mycielski, Ann. Phys. (N.Y.) 100, 62 (1976).

[23] S. Weinberg, Ann. Phys. (N.Y.) 194, 336 (1989).

[24] A.G. Wagh and V.C. Rakhecha, J. Phys. A: Math. Gen. 32, 5167 (1999).

[25] T. Bitter and D. Dubbers, Phys. Rev. Lett. 59, 251 (1987)

[26] R. Bhandari and J. Samuel, Phys. Rev. Lett. 60, 1211 (1988).

[27] R. Simon, H.J. Kimble, and E.C.G. Sudarshan, Phys. Rev. Lett. 61, 19 (1988); Phys. Rev. Lett. 63, 1021 (1989).

[28] A.G. Wagh, V.C. Rakhecha, P. Fischer, and A. Ioffe, Phys. Rev. Lett. 81, 1992 (1998); Phy. Rev. Lett. 83, 2090 (1999).

[29] A.G. Wagh, G. Badurek, V.C. Rakhecha, R.J. Buchelt, and A. Schricker, Phys. Lett. A268, 209 (2000).

[30] K. Fujikawa, Phys. Rev. D 73, 025017 (2006).