Relative dynamical degrees of correspondences over a field of arbitrary characteristic

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Abstract. Let $\mathbb{K}$ be an algebraically closed field of arbitrary characteristic, $X$ and $Y$ irreducible possibly singular algebraic varieties over $\mathbb{K}$. Let $f : X \dashrightarrow X$ and $g : Y \dashrightarrow Y$ be dominant correspondences, and $\pi : X \dashrightarrow Y$ a dominant rational map which semi-conjugate $f$ and $g$, i.e. so that $\pi \circ f = g \circ \pi$. We define relative dynamical degrees $\lambda_p(f|\pi) \geq 1$ for any $p = 0, \ldots, \dim(X) - \dim(Y)$. These degrees measure the relative growth of positive algebraic cycles, satisfy a product formula when $Y$ is smooth and $g$ is a multiple of a rational map, and are birational invariants. More generally, a weaker product formula is proven for more general semi-conjugacies, and for any generically finite semi-conjugacy $(\varphi, \psi)$ from $\pi_2 : (X_2, f_2) \rightarrow (Y_2, g_2)$ to $\pi_1 : (X_1, f_1) \rightarrow (Y_1, g_1)$ we have $\lambda_p(f_1|\pi_1) \geq \lambda_p(f_2|\pi_2)$ for all $p$. Many of our results are new even when $\mathbb{K} = \mathbb{C}$. Self-correspondences are abundant, even on varieties having not many self rational maps, hence these results can be applied in many situations. In the last section of the paper, we will discuss recent new applications of this to algebraic dynamics, in particular to pullback on $l$-adic cohomology groups in positive characteristics.

1. Introduction

One important tool in Complex Dynamics is dynamical degrees of dominant meromorphic selfmaps. They are bimeromorphic invariants of a meromorphic selfmap $f : X \dashrightarrow X$ of a compact Kähler manifold $X$. For a surjective holomorphic map $f$, the dynamical degree $\lambda_p(f)$ is simply the spectral radius of $f^* : H^{p,p}(X) \rightarrow H^{p,p}(X)$. Fundamental results of Gromov [24] and Yomdin [37] expressed the topological entropy of a surjective holomorphic map in terms of its dynamical degrees: $h_{\text{top}}(f) = \log \max_{0 \leq p \leq \dim(X)} \lambda_p(f)$. Since then, dynamical degrees have played a more and more important role in dynamics of meromorphic maps. In many results and conjectures in Complex Dynamics in higher dimensions, dynamical degrees play a central role. For a comprehensive survey, see [18].

The definition of dynamical degrees for meromorphic maps is more complicated. Let $X$ be a compact Kähler manifold of dimension $k$ with a Kähler form $\omega_X$, and let $f : X \dashrightarrow X$...
be a dominant meromorphic map. For $0 \leq p \leq k$, the $p$-th dynamical degree $\lambda_p(f)$ of $f$ is defined as follows:

\[ \lambda_p(f) = \lim_{n \to \infty} \left( \int_X (f^n)^* (\omega_X^p) \wedge \omega_X^{k-p} \right)^{1/n}. \]

The existence of the limit in the above expression is non-trivial and has been proven by Russakovskii and Shiffman [32] when $X = \mathbb{P}^k$, and by Dinh and Sibony [14, 15] when $X$ is compact Kähler. Both of these results use regularisation of positive closed currents. The limit in (1.1) is important in showing that dynamical degrees are birational invariants (see [15]), i.e. dynamical degrees of two birationally equivalent maps are the same.

For meromorphic maps of compact Kähler manifolds with invariant fibrations, a more general notion called relative dynamical degrees has been defined by Dinh and Nguyen in [12]. (Here, by a fibration we simply mean a dominant rational map, without any additional requirements.) Via “product formulas” proven in [12] when $X$ is complex projective and [13] when $X$ is compact Kähler, these relative dynamical degrees provide a very useful tool to check whether a meromorphic map is primitive (i.e. has no invariant fibrations over a base which is of smaller dimension and not a point). Roughly speaking, primitive maps (first defined in [40]) are those which do not come from smaller-dimensional manifolds, hence are “building blocks” from which all meromorphic maps can be constructed. In another direction, when $\mathbb{K} = \mathbb{C}$, Dinh and Sibony [17] defined dynamical degrees and topological entropy for meromorphic correspondences over irreducible varieties. Computations of dynamical degrees of so-called Hurwitz correspondences of the moduli spaces $\mathcal{M}_{0, N}$ were given in [30], wherein a proof that dynamical degrees of correspondences (over $\mathbb{K} = \mathbb{C}$, and for irreducible varieties) are birational invariants was also given.

Recently, work on birational maps of surfaces over an algebraically closed field of arbitrary characteristic has become more and more popular. As some examples, we refer the readers to [3, 19, 20, 28, 36]. In these results, (relative) dynamical degrees also play an important role. As in the case when $\mathbb{K} = \mathbb{C}$, for any algebraically closed field $\mathbb{K}$ of positive characteristic, relative dynamical degrees, via the product formula (see Theorem 1.3), can be used to easily show that birational maps $f$ of surfaces with $\lambda_1(f) > 1$ are primitive. With a view toward further applications, in this paper we define relative dynamical degrees for correspondences, in the most general setting of non-irreducible varieties and fields (not necessarily algebraically closed) of arbitrary characteristic. Note that even if rational maps may be rare for certain varieties, every variety has a lot of correspondences. The results in this paper provide recent applications to algebraic dynamics (see Section 6.5), in particular to pullback on $l$-adic cohomology groups in positive characteristics.

**Notation.** Before continuing, let us recall some notation. Let $\mathbb{K}$ be a field and $X, Y$ irreducible (not necessarily smooth or projective) varieties. A correspondence $f : X \rightrightarrows Y$ is given by an effective algebraic cycle $\Gamma_f = \sum_{i=1}^m \Gamma_i$ on $X \times Y$, where $m$ is a positive integer and $\Gamma_i \subset X \times Y$ are irreducible subvarieties of dimension exactly $\dim(X)$. (Note that this definition, which is modelled on rational maps and is good for doing dynamics, corresponds to those of degree 0 as defined in [23, Example 16.1.1].) We do not assume that $\Gamma_i$ are distinct, and hence may write the above sum as $\sum_{j} a_j \Gamma_j$, where $\Gamma_j$ are distinct and $a_j$ are positive integers. We will call $\Gamma_f$ the graph of $f$, by abusing the usual notation when $f$ is a rational map. If $f$ is a correspondence and $a \in \mathbb{N}$, we denote by $af$ the correspondence whose graph
is $a \Gamma_f$. In other words, if $\Gamma_f = \sum_i \Gamma_i$, then $\Gamma_{af} = \sum_i a \Gamma_i$. If $\Gamma_f = a \Gamma$, where $\Gamma$ is irreducible and $a \in \mathbb{N}$, we say that the correspondence $f$ is irreducible. A rational map $f$ is an irreducible correspondence, since its graph (defined as the closure in $X \times Y$ of the graph of the regular map $f|_U : U \to Y$, where $U \subset X$ is any non-empty Zariski-open set on which $f$ is a genuinely regular morphism) is irreducible. A correspondence is dominant if for each $i$ in the sum, the two natural projections from $\Gamma_i$ to $X, Y$ are dominant. The sum of two dominant correspondences, which is the correspondence whose graph is the sum of the graphs of the given two correspondences, is again dominant. Dominant correspondences can be composed similarly to the compositions of rational maps and the resulting correspondence is also dominant (see Section 3 for more detail). Two dominant correspondences $f : X \dasharrow X$ and $g : Y \dasharrow Y$ are semi-conjugate if there is a dominant rational map $\pi : X \to Y$ such that $\pi \circ f = g \circ \pi$. We will simply write $\pi : (X, f) \to (Y, g)$ to mean that $\pi$ is a dominant rational map semi-conjugating $(X, f)$ and $(Y, g)$.

Now we are ready to state the main results of the paper. For simplicity and to efficiently convey the main ideas, we will state here only the case when $\mathbb{K}$ is algebraically closed and $X, Y$ are irreducible projective varieties. The case when $X$ and $Y$ are irreducible but not projective readily follows, since we can restrict the attention to projective closures of affine Zariski-open sets. Extensions to the case when $X, Y$ are not irreducible or $\mathbb{K}$ is not algebraically closed will be discussed later in Section 6. If $\tau : X' \to X$ is a dominant rational map with generically finite fibres and $f : X \dasharrow X$ is a dominant correspondence, then by Section 3.2 the pullback $\tau^*(f) : X' \dasharrow X'$ is a well-defined dominant correspondence.

**Theorem 1.1.** Let $X$ and $Y$ be irreducible projective varieties, of corresponding dimensions $k$ and $l$, respectively, over an algebraically closed field $\mathbb{K}$. Let $\pi : (X, f) \to (Y, g)$ be a semi-conjugacy of dominant correspondences. Then, for any $0 \leq p \leq k - l$, there is a well-defined number $\lambda_p(f|\pi)$, called the $p$-th relative dynamical degree. If $Y$ is a point (so $l = 0$), we denote these simply by $\lambda_p(f)$ and call them dynamical degrees. They always satisfy the following:

1. The regular case. Assume that $X$ is smooth and $\pi$ is regular. Let $H_X$ be an ample divisor on $X$ and $H_Y$ be an ample divisor on $Y$. Then for each $p = 0, \ldots, k - l$ the limit

$$\lim_{n \to \infty} \deg((f^n)^*(H_X^p) \cdot \pi^*(H_Y^l))^{\frac{1}{n}}$$

exists and is equal to $\lambda_p(f|\pi)$. Here the degree of an algebraic cycle $\alpha$ of pure codimension $q$ on $X$ is computed as $\alpha \cdot H_X^q$.

2. Birational invariant. The relative dynamical degrees are birational invariants. More precisely, let two semi-conjugacies $\pi_1 : (X_1, f_1) \to (Y_1, g_1)$ and $\pi_2 : (X_2, f_2) \to (Y_2, g_2)$ together with two semi-conjugacies $\varphi : (X_2, f_2) \to (X_1, f_1)$ and $\psi : (Y_2, g_2) \to (Y_1, g_1)$ be given, where the two composition maps $\psi \circ \pi_2$ and $\pi_1 \circ \varphi$ are the same. Assume that $\psi : Y_2 \dasharrow Y_1$ is a generically finite rational map, and moreover that $\varphi$ is birational. Then $\lambda_p(f_1|\pi_1) = \lambda_p(f_2|\pi_2)$ for all $p = 0, \ldots, \dim(X_1) - \dim(Y_1)$.

3. Pullback invariant. Let $\tau : X' \dasharrow X$ be a dominant rational map with generically finite fibres. Then $\lambda_p(\pi^*(f)|\pi \circ \tau) = \deg(\tau) \lambda_p(f|\pi)$ for all $p$.

4. Uniqueness. Relative dynamical degrees are uniquely determined by (1) and (3) above.

5. $\lambda_p(f|\pi) \geq 1$ for all $p \in \{0, \ldots, k - l\}$. Moreover, $\lambda_p(f^m|\pi) = \lambda_p(f|\pi)^m$ for all $m \in \mathbb{N}$.
For a smooth projective variety $X$, let $N^p(X)$ be the group of algebraic cycles of codimension $p$ on $X$ modulo numerical equivalence (see the next section for more detail), and let $N^p_R(X) = N^p(X) \otimes \mathbb{R}$. From the first part of Theorem 1.1, it is standard to obtain that 

$$\lambda_p(f) = \lim_{n \to \infty} \|(f^n)_p\|^{\frac{1}{n}},$$

where $\| \cdot \|$ is a given norm on $N^p_R(X)$ and $\|(f^n)_p\|$ the corresponding norm of the linear map $(f^n)_p : N^p_R(X) \to N^p_R(X)$. Theorem 1.1, and its extensions in Section 6, thus extends the definition of (relative) dynamical degrees from dynamics on $\mathbb{K} = \mathbb{C}$ to an arbitrary field. In the case of correspondences, the definition of relative dynamical degrees is new even when $\mathbb{K} = \mathbb{C}$. When the field $\mathbb{K}$ is such that resolutions of singularities exist for varieties over $\mathbb{K}$ up to dimension $d$, we established previously in [34] the existence of relative dynamical degrees of semi-conjugate dominant rational maps over spaces of dimension $\leq d$ and some basic properties. The main ideas in [34] are to make various rational maps become regular morphisms and various varieties become smooth by using resolution of singularities, and to replace regularisation of positive closed currents by a version of Chow’s moving lemma in Roberts [31], which is summarised in Lemma 2.2 below. (By Hironaka’s theorem, resolutions of singularities exist in all dimensions when $\mathbb{K}$ has characteristic 0. On a field of positive characteristic, resolutions of singularities are established for dimensions up to 3 (see [1, 2, 5–7, 38, 39]) through the effort of many mathematicians.) The current paper uses, instead of resolution of singularities, de Jong’s alteration [9]. This allows us to again work with smooth varieties, but changes rational maps to correspondences. On the other hand, dealing with correspondences furnishes more applications. For example, in [35], we proposed an approach toward a generalisation of Weil’s Riemann hypothesis using correspondences (see Section 6.5).

In case $f$ is a rational map, its dynamical degrees are log-concave, that is, we have 

$$\lambda_{j+1}(f) \lambda_{j-1}(f) \leq \lambda_j(f)^2.$$ 

A generalisation of this is given in Lemma 5.7. Later, see Theorem 5.8, we give a stronger version of parts (2) and (3) in Theorem 1.1.

Relative dynamical degrees also satisfy a type of triangle inequality. This can be used to produce examples of primitive correspondences, discussed in more details in Section 6. We remark that the inequality in the theorem is strict in general (see Remark 1.4).

**Theorem 1.2.** Let $f_1, f_2 : X \dasharrow X$ be commuting correspondences $(f_1 \circ f_2 = f_2 \circ f_1)$, and $\pi : X \dasharrow Y$ a dominant rational map. Then $\lambda_p(f_1 + f_2|\pi) \leq \lambda_p(f_1|\pi) + \lambda_p(f_2|\pi)$ for all $p$.

The main use of relative dynamical degrees has been in showing that some rational maps do not accept non-trivial invariant fibrations [12, 13, 34]. This was achieved via a so-called product formula, which relates dynamical degrees between semi-conjugate rational maps via the relative dynamical degrees. We next extend this to the setting of correspondences.

**Theorem 1.3.** Let $\pi : (X, f) \to (Y, g)$ be a semi-conjugacy between dominant correspondences. Assume that $Y$ is smooth and that $g^n$ are irreducible for all $n$.

1. For all $p = 0, \ldots, \dim(X)$, we have

$$\lambda_0(g) \lambda_p(f) \geq \max_{0 \leq j \leq l, 0 \leq p-j \leq k-l} \lambda_j(g) \lambda_{p-j}(f|\pi).$$

2. Product formula. We assume moreover that $g = ag'$, where $a$ is a positive integer and $g' : Y \dasharrow Y$ a dominant rational map. Then, for every $0 \leq p \leq k$, we have

$$\lambda_0(g) \lambda_p(f) = \max_{0 \leq j \leq l, 0 \leq p-j \leq k-l} \lambda_j(g) \lambda_{p-j}(f|\pi).$$
Remarks. Note that if \( c > 0 \) is a constant such that
\[
c \lambda_p(f) \geq \max_{0 \leq j \leq l, 0 \leq p-j \leq k-l} \lambda_j(g) \lambda_{p-j}(f | \pi)
\]
for all \( p = 0, \ldots, \dim(X) \), then \( c \geq \lambda_0(g) \). In fact, by choosing \( p = 0 \) in the above we must have \( c \lambda_0(f) \geq \lambda_0(g) \lambda_0(f | \pi) \). From the proof of Theorem 6.1, we have \( \lambda_0(f) = \lambda_0(f | \pi) \), and hence \( c \geq \lambda_0(g) \) as wanted.

An equivalent form for the product formula (part (2) in the above theorem) is
\[
\lambda_p(f) = \max_{0 \leq j \leq l, 0 \leq p-j \leq k-l} \lambda_j(g') \lambda_{p-j}(f | \pi),
\]
because it can easily be computed that \( \lambda_j(g) = a \lambda_j(g') \) for all \( j \). In particular, if \( f \) is a dominant rational map (hence, so is \( g \)), then the product formula is satisfied with \( a = 1 \).

There are many examples for which we can apply the product formula in this general form, see part (4) of Remark 1.4. In this construction, generally we have \( a > 1 \).

Note also that the product formula does not hold in general when \( g \) is not irreducible, even for as simple cases as \( g = g_1 + g_2 \) where \( g_1, g_2 \) are two rational maps, see part (6) in Remark 1.4. The proof of part (1) of Theorem 1.3 suggests that it does not hold for general semi-conjugacies. However, at the moment we cannot yet find any counter-example. Under the assumptions on \( g \) as in the statement of part (1) of Theorem 1.3, it is also not clear to us whether the inequality may actually be always an equality.

Main idea for the proofs of Theorems 1.1 and 1.3, and the log-concavity. The main idea is to first establish the results in the case where \( X \) is smooth projective, where good properties can be utilised. Then we reduce the general case to the smooth case by making use of de Jong’s theorem.

By using some appropriate alterations \( \tau_1 : X_1 \to X \), we can find a semi-conjugacy \( \pi_1 : (X_1, \tau_1^* f) \to (Y, \deg(\tau_1)g) \) (where \( \tau_1^* f \) is the pullback correspondence, see Section 3 for more details), and moreover the map \( \pi_1 : X_1 \to Y \) is regular. Using part (1) of Theorem 1.1, we then would like to define
\[
\lambda_p(f | \pi) = \frac{1}{\deg(\tau_1)} \lambda_p(\tau_1^* f | \pi_1).
\]
For this to be well defined, we need to show that this quantity does not depend on the choice of the alterations. To this end, given any two pullback semi-conjugacies
\[
\tau_1 : (X_1, \tau_1^* f) \to (X, \deg(\tau_1) f) \quad \text{and} \quad \tau_2 : (X_2, \tau_2^* f) \to (X, \deg(\tau_2) f),
\]
we first construct an irreducible smooth projective variety \( X_3 \) together with three generically finite regular morphisms \( \varphi_1 : X_3 \to X_1 \), \( \varphi_2 : X_3 \to X_2 \) and \( \tau_3 : X_3 \to X \) such that one has \( \varphi_1 \circ \tau_1 = \tau_3 = \varphi_2 \circ \tau_2 \). (For example this can be done by choosing \( X_3 \) to be an alteration of a component of the pullback of the diagonal \( \Delta_X \subset X \times X \) by the generically finite regular morphism \( \tau_1 \times \tau_2 : X_1 \times X_2 \to X \times X \).) It follows from this that
\[
\deg(\varphi_1) \deg(\tau_1) = \deg(\tau_3) = \deg(\varphi_2) \deg(\tau_2).
\]
Then by results in Section 3 of this paper, we have two semi-conjugacies
\[
\varphi_1 : (X_3, \tau_3^* f) \to (X_1, \deg(\varphi_1) \tau_1^* f) \quad \text{and} \quad \varphi_2 : (X_3, \tau_3^* f) \to (X_2, \deg(\varphi_2) \tau_2^* f).
\]
These semi-conjugacies allow us to relate both \( \lambda_p(\tau_1^* f | \pi_1) \) and \( \lambda_p(\tau_2^* f | \pi_2) \) to \( \lambda_p(\tau_3^* f | \pi_3) \), via the help of part (1) and Lemma 5.4.
This proves the existence of relative dynamical degrees. Part (1) is proven in the course of this process. Part (2) is proven by observing that if \( \varphi : (X_2, f_2) \to (X_1, f_1) \) is a semi-conjugacy, where \( \varphi \) is birational, then \( f_2 = \varphi^* f_1 \). Part (3) is standard.

For the proof of the log-concavity, Lemma 5.7, let \( n_i \) be a sequence of increasing integers so that \( f^{n_i} \) are irreducible. If \( X \) is smooth, then we can apply the Grothendieck–Hodge index theorem to an alteration of \( \Gamma f^{n_i} \), together with part (1) of Theorem 1.1, to complete the proof. When \( X \) is not smooth, we observe that if \( \tau_1 : X_1 \to X \) is an alteration, then while \( (\tau_1^* (f))^{n_i} \) may not be irreducible, the pullbacks by different irreducible components of its are related in such a way that the Grothendieck–Hodge index theorem again applies.

The proof of the product formula, part (2) of Theorem 1.3, as in [12, 13], essentially follows from estimates on strict intersection and strict pullback by correspondences of algebraic cycles which will be given in Section 4. Note that there are complications when we consider general correspondences \( f \) and \( g \), since the equality \( f^*(\alpha \wedge \beta) = f^*(\alpha) \wedge f^*(\beta) \) (the right-hand side is the strict intersection of varieties, that is, the closure of the intersection of two cycles on the open set where they intersect properly, please see Section 4 for more detail) for appropriate positive cycles, which were used in [12, 13] for rational maps \( f \), simply do not hold for a general correspondence. We are able to prove a weaker version of this needed intermediate result under the assumption that \( g \) is a multiple of a rational map.

**Remark 1.4.** We now give some further remarks on Theorem 1.1.

1. In the case of complex projective manifolds, the dynamical degrees can be also computed using Dolbeault cohomology groups \( H^{p,p}(X) \). The proof of Theorem 1.1 in that setting is given using properties of positive closed currents. For a smooth projective variety defined on an arbitrary algebraically closed field of characteristic 0, \( H^{p,p}(X) \) can also be computed by the algebraic de Rham groups \( H^p(X, \Omega^p_X) \). In positive characteristic, since we lack the notions of positivity for the classes in \( H^p(X, \Omega^p_X) \), it is not clear that dynamical degrees can also be computed using algebraic de Rham groups. The same question can be asked for other cohomology groups, for example \( l \)-adic cohomology groups. We note that for automorphisms of surfaces over an algebraically closed field of arbitrary characteristic, dynamical degrees can be computed by using the \( l \)-adic cohomology groups as proven in Esnault and Srinivas [20].

The proof of this result, however, relies very largely on special properties of surfaces such as classification of surfaces and the possibility of lifting certain maps to characteristic 0.

Including the results in this paper, we are able to give very simple proofs of generalisations of their results, see Section 6.5.

2. In the literature, when \( \mathbb{K} = \mathbb{C} \), the relative dynamical degrees for rational maps were given in [12, 13] and the dynamical degrees for correspondences were given in [17]. With some efforts (in particular to obtain a universal constant \( C \) as in Lemma 4.5), all results on fields of characteristic 0 may be reduced to the case where \( \mathbb{K} = \mathbb{C} \) via the Lefschetz principle. However, the proofs so obtained are very indirect, with not very explicit bounds, and provide no further insight (in particular, since these results are stated in terms of cohomological groups) to how to extend to other fields, even when we assume that resolutions of singularities exist. A direct treatment of relative dynamical degrees for rational maps over fields when resolutions of singularities exist was given in [34].

The pullback of an effective divisor is effective, hence the first dynamical degree \( \lambda_1(f) \) of a dominant rational map \( f : X \to X \) (see e.g. [4] for surfaces) and by duality also the dynamical degree \( \lambda_{k-1}(f) \) (here \( k = \dim(X) \)) are well defined, over arbitrary characteristic.
(3) The following features of our results are new:

- Dynamical degrees are defined for correspondences over reducible varieties even when \( \mathbb{K} = \mathbb{C} \). Relative dynamical degrees are defined for correspondences, even when \( \mathbb{K} = \mathbb{C} \) and the involved varieties are irreducible.

- Dynamical degrees are defined for rational maps over a field of arbitrary characteristic.

- A different proof (using Chow’s moving lemma instead of regularisation of positive closed currents) is given for the known results over \( \mathbb{K} = \mathbb{C} \), with explicit estimates in terms of the dimensions of the varieties \( X, Y \) and the degrees of them in given embeddings \( X, Y \subseteq \mathbb{P}^N \).

- Theorem 5.8 is stated for the first time, even for the case where \( \mathbb{K} = \mathbb{C} \) (in this case [12] observed it for the special case of dynamical degrees of meromorphic maps).

- Theorems 1.3 and 1.2 are also stated for the first time, even when \( \mathbb{K} = \mathbb{C} \).

(4) There are many examples for which the above results can be applied. For instance, let \( \pi : X \to Y \) be a dominant rational map and \( g : Y \to Y \) a dominant correspondence. Let \( k = \dim(X), l = \dim(Y) \) and \( h : \mathbb{P}^{k-l} \to \mathbb{P}^{k-l} \) a dominant correspondence. Given a dominant rational map \( \tau : X \to \mathbb{P}^{k-l} \), we can pullback – via the generically finite dominant rational map \( (\pi, \tau) : X \to Y \times \mathbb{P}^{k-l} \) (see Section 3 for more details) – the correspondence \( g \times h : Y \times \mathbb{P}^{k-l} \to Y \times \mathbb{P}^{k-l} \) to a correspondence \( f : X \to X \), together with a semi-conjugacy \( \pi : (X, f) \to (Y, a, \deg(\tau, g)) \), where \( a = \lambda_0(h) \).

(5) The log-concavity is proven as soon as a version of Grothendieck–Hodge index theorem is available for the graphs of an infinite subset of \( \{ f^n : n \in \mathbb{N} \} \), thanks to the existence of the limit in Theorem 1.1 (1). Provided that \( f^n \) is irreducible, we can use Grothendieck–Hodge index theorem for an alteration of the graph of \( f^n \). The reason for the log-concavity to fail in the case \( f \) is reducible lies in the fact that the Grothendieck–Hodge index theorem fails for reducible smooth varieties.

It is certainly true that the assumption of Lemma 5.7 does not hold for a general correspondence. For example [17], if \( f \) is an irreducible correspondence which is symmetric with respect to the diagonal, then \( f^2 \) contains a multiple of the diagonal, hence is not irreducible. However, it is expected that the assumption holds in many cases of interest, other than the case where \( f \) is already a rational map. In fact, we expect that for a generic irreducible correspondence \( f : X \to X \), all iterates \( f^n \) are irreducible.

The log-concavity actually fails for a general (not irreducible) correspondence even when \( X \) is smooth. Here is one counter-example. Let \( g : X \to X \) be a dominant rational map such that there is a \( p = 1, \ldots, \dim(X) - 1 \) for which \( \lambda_{p-1}(g) + \lambda_{p+1}(g) > 2\lambda_p(g) \). (For example, \( g \) is an endomorphism of degree \( \geq 2 \) on \( \mathbb{P}^k \).) Let \( a \geq 1 \) be an integer, and \( f = g + a \Delta_X \), where \( \Delta_X \) is the diagonal. By Theorem 1.1 (1), it can be computed that \( \lambda_q(f) = \lambda_q(g) + a \) for all \( q = 0, \ldots, \dim(X) \). Then, from the assumptions on the dynamical degrees of \( g \), it follows that \( \lambda_p(f)^2 < \lambda_{p-1}(f)\lambda_{p+1}(f) \) provided \( a \) is large enough.

(6) Here we give a counter-example to the product formula in part (2) of Theorem 1.3 in case the correspondences \( g^n \) are always reducible. Let \( X = \mathbb{P}^2, Y = \mathbb{P}^1 \), and \( \pi : X \to Y \) the map \([x_0 : x_1 : x_2] \mapsto [x_0 : x_1] \). Given positive integers \( d_1, d_2 \), we let \( f_1, f_2 : X \to X \) be the maps

\[
\begin{align*}
    f_1[x_0 : x_1 : x_2] &\mapsto [x_0^{d_1} : x_1^{d_1} : x_2^{d_1}], \\
    f_2[x_0 : x_1 : x_2] &\mapsto [x_0^{d_2} : x_1^{d_2} : x_2^{d_2}]
\end{align*}
\]
and \( g_1, g_2 : Y \to Y \) the maps
\[
g_1[x_0 : x_1] \mapsto [x_0^{d_1} : x_1^{d_1}], \quad g_2[x_0 : x_1] \mapsto [x_0^{d_2} : x_1^{d_2}].
\]
It can be checked that \( \pi \circ f_1 = g_1 \circ \pi, \pi \circ f_2 = g_2 \circ \pi. \) Moreover, \( f_1, f_2 \) commute, and \( g_1, g_2 \) commute.

Given positive integers \( a_1, a_2, \) we consider the correspondences
\[
f = a_1 f_1 + a_2 f_2 : X \to X, \quad g = a_1 g_1 + a_2 g_2 : Y \to Y.
\]
It can easily be checked that \( \pi \circ f = g \circ \pi. \) Moreover, since the cohomology groups of \( X \) and \( Y \) are very simple, it readily follows that
\[
\lambda_0(f) = a_1 + a_2, \quad \lambda_1(f) = a_1 d_1 + a_2 d_2, \quad \lambda_2(f) = a_1 d_1^2 + a_2 d_2^2,
\]
\[
\lambda_0(g) = a_1 + a_2, \quad \lambda_1(g) = a_1 d_1 + a_2 d_2, \quad \lambda_1(\pi) = a_1 + a_2 d_2.
\]
If part (2) of Theorem 1.3 was to hold for \( \pi : (X, f) \to (Y, g), \) we would have in particular
\[
(a_1 + a_2)(a_1 d_1^2 + a_2 d_2^2) = \lambda_0(g)\lambda_2(f) = \lambda_1(g)\lambda_1(\pi) = (a_1 d_1 + a_2 d_2)^2.
\]
This is impossible for a generic choice of \( a_1, a_2, d_1, d_2. \)

(7) If \( f : X \to X \) is any correspondence and \( \Delta_X \) is the diagonal, then it can be checked that \( \lambda_p(f + \Delta_X) = \lambda_p(f) + 1 \) for all \( p = 0, \ldots, \dim(X). \) Hence the inequality in Theorem 1.2 may be the equality in certain examples.

However, the inequality is strict in general. For example, let \( f : X \to X \) be a birational map and let \( \pi : X \to Y \) be a dominant rational map so that there exists one number \( p \) for which \( \lambda_p(\pi), \lambda_p(f^{-1}) > 1. \) (There are many such maps, e.g. among pseudo-automorphisms in dimension 3. The latter are birational maps \( f : X \to X, \) where \( \dim(X) = 3, \) so that both \( f \) and \( f^{-1} \) have no exceptional divisors.) Then \( f \) and \( f^{-1} \) commute, and
\[
(f + f^{-1})^n = \sum_{i=0}^{n} C(n, i) f^i f^{-n+i} \leq C(n, \lfloor \frac{n}{2} \rfloor) \Delta_X + \sum_{i<\frac{n}{2}} C(n, i)(f^{-1})^n + \sum_{i>\frac{n}{2}} C(n, i) f^{2i-n},
\]
where \( C(n, i) \) are binomial coefficients and \( \Delta_X \) the diagonal. By using part (1) of Theorem 1.1, it readily follows that
\[
\lambda_p(f + f^{-1}) \leq \max \left\{ 2, \lambda_p(f), \lambda_p(f^{-1}), \lambda_p(f^{-1}) \right\} < \lambda_p(f) + \lambda_p(f^{-1}).
\]

(8) The assumption that \( \mathbb{K} \) is algebraically closed is essential in several parts in the proofs of our results. First, to be able to define strict transform of correspondences, we need to use that generic fibres of surjective regular morphisms are of correct dimensions (this replaces Sard’s theorem on singularities of regular maps, which holds only in characteristic 0), see Section 3 for more details. This property is known only for algebraically closed fields and for quasi-
projective varieties. Second, we need to use algebraic cycles and the groups $NP(X)$ have nice properties under the mentioned assumptions on $K$.

However, we can reduce the general case to the case where $K$ is algebraically closed by lifting to its algebraic closure. In doing this, we will need to deal with the case where the varieties considered are no longer irreducible. We will elaborate more on this in Section 6.

Organisation of the paper. The remaining of this paper is organised as follows. In Section 2, we present some preliminaries on algebraic cycles, Roberts’ version of Chow’s moving lemma and Grothendieck–Hodge index theorem. In Section 3, we present some facts about correspondences, semi-conjugacies between them and pullbacks of certain diagrams of dominant correspondences. In Section 4, we give some estimates on strict pullbacks of algebraic cycles by correspondences and strict intersection of algebraic cycles. Section 5 is devoted to the proofs of main results and the log-concavity. In Section 6, we discuss some extensions and applications. In particular, we will show that all correspondences of surfaces whose dynamical degrees violate the log-concavity (as constructed in Remark 1.4) are weakly primitive, and discuss a possible application of our results to the Hurwitz correspondences considered in [30]. In the last subsection, we discuss some further remarks concerning extending results from complex dynamics to algebraic dynamics.

Remark 1.5. (1) More recently, Nguyen-Bac Dang posted a preprint [8] presenting an alternative approach for relative dynamical degrees of rational maps on normal projective varieties. In [8], de Jong’s alteration is also used but for a different purpose.

(2) Some of the results in this paper (such as Theorems 1.3 and 5.8) require certain varieties involved to be smooth. Hence (as explained e.g. in Theorem 5.8), these results will be valid up to a dimension where resolutions of singularities are available. At the moment, the existence of resolutions of singularities in any characteristic up to dimension 3 is enough for applications of our results to singular varieties in small dimensions, for example to proving the primitivity of birational maps of surfaces with $\lambda_1 > 1$. On the other hand, correspondences allow flexibility in applications. For example, we proposed in [35] an approach using correspondences essentially (see Condition (A) there, and also Section 6.5 below) to solve a generalisation of the famous Weil’s Riemann hypothesis.

Besides resolutions of singularities, there are two other possible approaches toward getting rid of the smoothness assumptions in the results mentioned above. The first approach is to work directly on union of smooth varieties, extending what is done in Section 6.2. We can use Case 2 in Section 3.2 to always reduce to this case. The second approach is to combine the ideas in [8] and the current results in this paper to work directly on normal varieties. Since any variety has a normalisation, which is birational to it and is normal, we can always reduce to this case using Theorem 1.1. Since the actual installation of either approach may take a considerable amount of time and take us too far away from the original ideas and approaches in this paper, we defer it to a future work.

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2. Algebraic cycles

Throughout the section, we fix an algebraically closed field $\mathbb{K}$ of arbitrary characteristic. Recall that a smooth projective variety over $\mathbb{K}$ is a non-singular subvariety of a projective space $\mathbb{P}^N_{\mathbb{K}}$. We will recall the definition and some results on algebraic cycles, Roberts’ version of Chow’s moving lemma and the Grothendieck–Hodge index theorem. We then arrive at a useful result on the intersection of two cycles and define a norm $\| \cdot \|_1$ which will be used in the proof of Theorem 1.1.

2.1. Algebraic cycles. The reference for materials in this subsection is the book of Fulton [23]. Let $X \subseteq \mathbb{P}^N_{\mathbb{K}}$ be a smooth projective variety of dimension $k$ over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic. A $q$-cycle on $X$ is a finite sum $\sum n_i [V_i]$, where $V_i$ are $q$-dimensional irreducible subvarieties of $X$ and $n_i$ are integers. The group of $q$-cycles on $X$, denoted $Z^q(X)$, is the free abelian group on the $p$-dimensional subvarieties of $X$. A $q$-cycle $\alpha$ is effective if it has the form $\alpha = \sum n_i [V_i]$, where $V_i$ are irreducible subvarieties of $X$ and $n_i \geq 0$.

Let $X$ and $Y$ be smooth projective varieties, and let $f : X \to Y$ be a morphism. For any irreducible subvariety $V$ of $X$, we define the pushforward $f_* [V]$ as follows. Let $W = f(V)$. If $\dim(W) < \dim(V)$, then $f_* [V] = 0$. Otherwise, $f_* [V] = \deg(V/W) [W]$. This gives a pushforward map $f_* : Z_q(X) \to Z_q(Y)$ (see [23, Section 1.4]).

We refer the readers to [23] for the definitions of rational and algebraic equivalences of algebraic cycles. Roughly speaking, two algebraic cycles are rationally equivalent if they are elements of a family of algebraic cycles parameterised by $\mathbb{P}^1$. Similarly, two algebraic cycles are algebraically equivalent if they are elements of a family of algebraic cycles parameterised by a smooth algebraic variety. The groups of $q$-cycles modulo rational and algebraic equivalences are denoted by $A_q(X)$ and $B_q(X)$.

We write $Z^p(X)$, $A^p(X)$ and $B^p(X)$ for the corresponding groups of cycles of codimension $p$. Since $X$ is smooth, we have an intersection product $A^p(X) \times A^q(X) \to A^{p+q}(X)$, making $A^*(X)$ a ring, called Chow’s ring of $X$ (see [23, Sections 8.1 and 8.3]).

For a dimension 0 cycle $y = \sum m_i [p_i]$ on $X$, define its degree to be $\deg(y) = \sum m_i$. We say that a cycle $\alpha \in A^p(X)$ is numerically equivalent to zero if and only if $\deg(\alpha, \beta) = 0$ for all $\beta \in A^{k-p}(X)$ (see [23, Section 19.1]). The group of codimension $p$ algebraic cycles modulo numerical equivalence is denoted by $N^p(X)$. These are finitely generated free abelian groups (see [23, Example 19.1.4]). We will use the vector spaces $N^p_\mathbb{R}(X) = N^p(X) \otimes \mathbb{R}$ and $N^p_\mathbb{C}(X) = N^p(X) \otimes \mathbb{C}$ in defining dynamical degrees.

Remark 2.1. We have the following inclusions: rational equivalence $\subset$ algebraic equivalence $\subset$ numerical equivalence.
2.2. Roberts’ version of Chow’s moving lemma. Let $X$ be a smooth projective variety of dimension $k$ over $\mathbb{K}$. If $V$ and $W$ are two irreducible subvarieties of $X$, either $V \cap W = \emptyset$ or any irreducible component of $V \cap W$ has dimension at least $\dim(V) + \dim(W) - k$. We say that $V$ and $W$ are intersect properly if any component of $V \cap W$ has dimension exactly $\dim(V) + \dim(W) - k$. When $V$ and $W$ intersect properly, the intersection $V \cdot W$ is well defined as an effective $\dim(V) + \dim(W) - k$ cycle.

Given $\alpha = \sum_i m_i[V_i] \in Z_q(X)$ and $\beta = \sum_j n_j[W_j] \in Z_q'(X)$, we say that $\alpha \cdot \beta$ is well defined if $V_i$ and $W_j$ intersect properly for every $i, j$. The usual statement of Chow’s moving lemma is that we can always find $\alpha'$ which is rationally equivalent to $\alpha$ so that $\alpha' \cdot \beta$ is well defined. Since in the sequel we will need to use some quantitative properties of such cycles $\alpha'$, we will follow the construction of such cycles $\alpha'$ by Roberts [31]. See also the paper [22] for a generalisation to moving families of cycles of bounded degrees.

Fix an embedding $X \subset \mathbb{P}^N_\mathbb{K}$. We choose a linear subspace $L \subset \mathbb{P}^N_\mathbb{K}$ of dimension $N-k-1$ such that $L \cap X = \emptyset$. For any irreducible subvariety $Z$ of $X$, it follows that $L \cap Z = \emptyset$ as well and we denote by $C_L(Z)$ the cone over $Z$ with vertex $L$ (see [26, Example 6.17]). For any such $Z$, $C_L(Z) \cdot X$ is well defined and has the same dimension as $Z$, and moreover $C_L(Z) \cdot X - Z$ is effective (see [31, Lemma 2]).

Let $Y_1, Y_2, \ldots, Y_m$ and $Z$ be irreducible subvarieties of $X$; here $Y_1, \ldots, Y_m$ do not need to be closed. We define the excess $e(Z; Y_1, \ldots, Y_m)$, briefly denoted by $e(Z)$ if there is no confusion, of $Z$ relative to $Y_1, \ldots, Y_m$ to be the maximum of the integers

$$\dim(W) + k - \dim(Z) - \dim(Y_i),$$

where $i$ runs from 1 to $m$, and $W$ runs through all components of $Z \cap Y_i$, provided that one of these integers is non-negative. Otherwise, the excess is defined to be 0.

More generally, if $Z = \sum_i m_i[Z_i]$ is a cycle, where $Z_i$ are irreducible subvarieties of $X$, we define $e(Z) = \max_i e(Z_i)$. We then also define the cone $C_L(Z) = \sum_i m_iC_L(Z_i)$. We have the following Chow’s moving lemma.

**Lemma 2.2.** Let $\mathbb{K}$ be an algebraically closed field. Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension $k$. Let $Z$ be an effective algebraic cycle on $X$, and $Y_1, \ldots, Y_m$ (not necessarily closed) irreducible varieties of $X$, with the excess $e(Z)$ (relative to $Y_1, \ldots, Y_m$). For generic linear subspaces $L_i \subset \mathbb{P}^N$, $i = 1, \ldots, e = e(Z)$, and a generic automorphism $g$ of $\mathbb{P}^N$, one has that if the effective algebraic cycles $Z_i$ are recursively defined by $Z_0 = Z$ and $Z_i = C_{L_i}(Z_{i-1}) \cdot X - Z_{i-1}$, then as algebraic cycles

$$Z = Z_0 = (-1)^e Z_e + \sum_{i=1}^e (-1)^{i-1} C_{L_i}(Z_{i-1}) \cdot X.$$

Moreover, the algebraic cycle

$$Z' = (-1)^e Z_e + \sum_{i=1}^e (-1)^{i-1} (gC_{L_i}(Z_{i-1})) \cdot X$$

has $e(Z') = 0$, and is rationally equivalent to $Z$ via a rational family of algebraic cycles

$$Z(t) = (-1)^e Z_e + \sum_{i=1}^e (-1)^{i-1} (g(t)C_{L_i}(Z_{i-1})) \cdot X.$$

Here $g(t)$ is any rational curve of automorphisms of $\mathbb{P}^N$ joining the identity map 1 and $g$. 

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For the proof, we need the following result of Roberts.

**Lemma 2.3** ([31, p. 93]). *For any cycle* $Z$ *and any irreducible subvarieties* $Y_1, \ldots, Y_m$, *we have* $e(C_L(Z) \cdot X - Z) \leq \max(e(Z) - 1, 0)$ *for generic linear subspaces* $L \subset \mathbb{P}^N$ *of dimension* $N - k - 1$ *such that* $L \cap X = \emptyset$.

Now we can finish the proof of Lemma 2.2 as follows (see [31, Theorem, p.94]). Let $Y_1, \ldots, Y_m$ and $Z$ be irreducible varieties on $X$. If $e = e(Z) = 0$, then $Z$ intersect properly $Y_1, \ldots, Y_m$, hence we are done. Otherwise, $e \geq 1$. Applying the main lemma, we can find linear subspaces $L_1, \ldots, L_e \subset \mathbb{P}^N_K$ of dimension $N - k - 1$ such that if $Z_0 = Z$ and $Z_i = C_{L_i}(Z_{i-1}) \cdot X - Z_{i-1}$ for $i = 1, \ldots, e = e(Z)$, then $e(Z_i) \leq e - i$. In particular, we have $e(Z_e) = 0$. It is easy to see that

$$Z = Z_0 = (-1)^e Z_e + \sum_{i=1}^{e} (-1)^{i-1} C_{L_i}(Z_{i-1}) \cdot X.$$  

It is known that there are points $g \in \text{Aut}(\mathbb{P}^N_K)$ such that $(gC_{L_i}(Z_{i-1})) \cdot X$ and $(gC_{L_i}(Z_{i-1})) \cdot Y_j$ are well defined for $i = 1, \ldots, e$ and $j = 1, \ldots, m$. We can choose a rational curve in $\text{Aut}(\mathbb{P}^N_K)$ joining the identity map $1$ and $g$, thus see that $Z$ is rationally equivalent to

$$Z' = (-1)^e Z_e + \sum_{i=1}^{e} (-1)^{i-1} (gC_{L_i}(Z_{i-1})) \cdot X.$$  

By construction, $e(Z') = 0$, as desired.

**2.3. Grothendieck–Hodge index theorem.** Let $X \subset \mathbb{P}^N_K$ be a smooth projective variety of dimension $k$. Let $H \subset \mathbb{P}^N_K$ be a hyperplane, and let $\omega_X = H|_X$. We recall that $N^p(X)$, the group of codimension $p$ cycles modulo the numerical equivalence, is a finitely generated free abelian group. We define $N^p_{\mathbb{R}}(X) = N^p(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $N^p_{\mathbb{C}}(X) = N^p(X) \otimes_{\mathbb{Z}} \mathbb{C}$. These are real (and complex) vector spaces of real (and complex) dimension equal rank($N^p(X)$). For $p = 1$, it is known that dim$_{\mathbb{R}}(N^1_{\mathbb{R}}(X)) = \text{rank}(NS(X)) =: \rho$, the rank of the Néron–Severi group of $X$ (see [23, Example 19.3.1]).

We define for $u, v \in N^1_{\mathbb{C}}(X)$ the Hermitian form $\mathcal{H}(u, v) = \deg(u \cdot \overline{v} \cdot \omega_X^{k-2})$. Here the degree of a complex 0-cycle $\alpha + i\beta$ is defined to be the complex number $\deg(\alpha) + i \deg(\beta)$. The analogue of the Hodge index theorem for complex projective manifolds is the well-known Grothendieck–Hodge index theorem (see [25]), which says that $\mathcal{H}$ has signature $(1, \rho - 1)$.

**2.4. Some norms on the vector spaces** $N^p_{\mathbb{R}}(X)$ **and** $N^p_{\mathbb{C}}(X)$. Given an embedding $i : X \subset \mathbb{P}^N_K$ of a smooth projective variety $X$ of dimension $k$, let $H \in A^1(\mathbb{P}^N)$ be a hyperplane and $\omega_X = H|_X = i^*(H) \in A^1(X)$. For an irreducible subvariety $V \subset X$ of codimension $p$, we define the degree of $V$ to be $\deg(V)$ (the degree of the dimension 0 cycle $V \cdot \omega_X^{k-p}$), or equivalently $\deg(V)$ (the degree of the variety $i_*(V) \subset \mathbb{P}^N$). Similarly, we define for an effective codimension $p$ cycle $V = \sum_i m_i [V_i]$ (here $m_i \geq 0$ and $V_i$ are irreducible), the degree $\deg(V) = \sum_i m_i \deg(V_i)$. This degree is extended to vectors in $N^p_{\mathbb{R}}(X)$. Note that the degree map is a numerical equivalent invariant.

As a consequence of Lemma 2.2, we have the following result on intersection of cycles. This result will be used very frequently later, in particular in obtaining upper bounds for the
strict intersections of algebraic cycles in Lemma 4.1. The latter in turn is crucial in showing the limit in part (1) of Theorem 1.1 exists and in proving part (2) of Theorem 1.3.

**Lemma 2.4.** Define $C = k \deg(X)^k$, where $k = \dim(X)$ and $\deg(X)$ is the degree of $X$ in the given embedding $i$. For all irreducible subvarieties $V, W$ in $X$, the intersection class $V \cdot W$ is represented by $\alpha_1 - \alpha_2$, where $\alpha_1$ and $\alpha_2$ are effective algebraic cycles satisfying $\deg(\alpha_1), \deg(\alpha_2) \leq C \deg(V) \deg(W)$.

**Proof.** By using Lemma 2.2, $W$ is rationally equivalent to

$$W' = \sum_{i=1}^{e} (-1)^{i-1} gC_{L_i}(W_{i-1}) \cdot X + (-1)^e W_e,$$

where $W_0 = W$, $W_i = C_{L_i}(W_{i-1}) \cdot X - W_{i-1}$, $C_{L_i}(W_{i-1}) \subset \mathbb{P}^N$ is a cone over $W_{i-1}$, and $g \in \text{Aut}(\mathbb{P}^N)$ is an automorphism. Moreover, $gC_{L_i}(W_{i-1}) \cdot X, gC_{L_i}(W_{i-1}) \cdot V$ and $W_e \cdot V$ are all well defined. We note that $e \leq k = \dim(X)$, and for any $i = 1, \ldots, e$

$$\deg(W_i) \leq \deg(gC_{L_i}(W_{i-1}) \cdot X) \leq \deg(gC_{L_i}(W_{i-1})) \deg(X) = \deg(C_{L_i}(W_{i-1})).$$

Here we used the facts that $\deg(C_{L_i}(W_{i-1})) = \deg(W_{i-1})$ (see [26, Example 18.17]), and $\deg(gC_{L_i}(W_{i-1})) = \deg(C_{L_i}(W_{i-1}))$ because $g$ is an automorphism of $\mathbb{P}^N$ (hence a linear map).

Therefore, the degrees of $W_i$ are all $\leq (\deg(X))^k \deg(W)$. By definition, the intersection product $V \cdot W \in A^*(X)$ is given by $V \cdot W'$, which is well defined. We now estimate the degrees of effective cycles $gC_{L_i}(W_{i-1})X \cdot V$ and $W_e \cdot V$. Firstly, by the projection formula

$$\deg(gC_{L_i}(W_{i-1})|_X \cdot V) = \deg(\iota_*(gC_{L_i}(W_{i-1})|_X \cdot V)) = \deg(gC_{L_i}(W_{i-1}) \cdot \iota_*(V))$$

$$= \deg(gC_{L_i}(W_{i-1}) \cdot \deg(V) \leq \deg(X)^k \deg(W) \deg(V).$$

Finally, we estimate the degree of $W_e \cdot V$. Since $W_e \cdot V$ is well defined, we can choose a linear subspace $L \subset \mathbb{P}^N$ so that $C_{L}(W_e) \cdot X$ and $C_{L}(W_e) \cdot V$ are well defined. Recall that $C_{L}(W_e) - W_e$ is effective, we have

$$\deg(V \cdot W_e) \leq \deg(V \cdot C_{L}(W_e)|_X) = \deg(V). \deg(C_{L}(W_e))$$

$$\leq \deg(X)^k \deg(V) \deg(W).$$

From these estimates, we see that we can write

$$V \cdot W' = \alpha_1 - \alpha_2,$$

where $\alpha_1, \alpha_2$ are effective cycles and $\deg(\alpha_1), \deg(\alpha_2) \leq k \deg(X)^k \deg(V) \deg(W).$ $\square$

Using this degree map, we define for an arbitrary vector $v \in N_{\mathbb{R}}^p(X)$ the norm

$$(2.1) \quad \|v\|_1 = \inf(\deg(v_1) + \deg(v_2) : v = v_1 - v_2, v_1, v_2 \in N_{\mathbb{R}}^p(X) \text{ are effective}).$$

That this is actually a norm can be seen as follows. The non-trivial condition to check is that if $\|v_0\|_1 = 0$, then $v_0 = 0$. In fact, if $\|v_0\|_1 = 0$, then by definition there are effective cycles (with
3. Correspondences

In this section, we present results on correspondences which are needed to prove Theorem 1.1. We will be mostly concerned with correspondences between quasi-projective varieties over an algebraically closed field \( \mathbb{K} \). These assumptions allow us to use the following result (see [33, Theorem 1.25, Section I.6.3]) on dimensions of fibres of surjective regular morphisms.

**Theorem 3.1.** Let \( \mathbb{K} \) be an algebraically closed field, \( X, Y \) irreducible quasi-projective varieties over \( \mathbb{K} \), and \( f : X \to Y \) a surjective regular morphism. Then there exists a non-empty Zariski-open subset \( U \subset Y \) such that \( \dim(f^{-1}(y)) = \dim(X) - \dim(Y) \) for all \( y \in U \).

In this paper we will need to consider various intersection of varieties, and the condition for these to be well defined is that the set theoretical intersection has the correct dimension. Theorem 3.1 helps us to check that condition is satisfied over a non-empty Zariski-open set. The stronger version (Sard’s theorem – valid only in characteristic 0), is not needed.

### 3.1. Correspondences

Let \( X, Y \) be quasi-projective varieties of pure dimensions \( k, l \).

First, we assume that \( X, Y \) are irreducible. A correspondence \( f : X \rightrightarrows Y \) is represented by its graph \( \Gamma_f = \sum_{i=1}^{m} \Gamma_i \), viewed as an algebraic cycle, where each \( \Gamma_i \) is an irreducible subvariety of dimension \( k \) of \( X \times Y \). We do not assume that the varieties \( \Gamma_i \) are distinct. We say that \( f \) is irreducible if \( \Gamma_f = a_{\Gamma} \), where \( a \in \mathbb{N} \) and \( \Gamma \) is irreducible. A correspondence is dominant if for each \( i \), the two natural projections from \( \Gamma_i \) to \( X, Y \) are dominant. For example, if \( f : X \longrightarrow Y \) is a dominant rational map, then it is an irreducible dominant correspondence. If \( f \) is a correspondence whose graph is \( \Gamma_f = \sum_{i} \Gamma_i \), we denote by \( a_{f} \) the correspondence whose graph is \( \Gamma_{af} = \sum_{i} a_{\Gamma_i} \).

With respect to [23, Example 16.1.1], our correspondences are those in there with degree 0. In the sequel, we will mostly need to work with the case of dominant correspondences \( f : X \rightrightarrows Y \), where \( \dim(X) = \dim(Y) \) and \( X, Y \) are projective. In this case, correspondences are a kind of multi-valued maps. For generic \( x \in X \), the fibre \( \Gamma_{f,x} = \Gamma_f \cap (\{x\} \times Y) \) is a well-defined algebraic cycle of dimension 0, and \( f(x) \) is the push-forward of \( \Gamma_{f,x} \) to \( Y \).

Dominant correspondences can be composed, much as the compositions between dominant rational maps, and the resulting correspondence is also dominant. In fact, this can be done as follows. Let \( f : X \rightrightarrows Y \) and \( g : Y \rightrightarrows Z \) be two dominant correspondences. By Theorem 3.1, there are non-empty Zariski-open subsets \( U \subset X, U' \subset Y \) over which the fibres of the projections \( \Gamma_f \to X, \Gamma_g \to Y \) have the correct dimension, and \( f(U) \subset U' \). Then we define, for \( x \in U \), \( g \circ f(x) = g(f(x)) \). This is a correspondence from \( U \) to \( Z \); taking the closure of the graph of this in \( X \times Z \), we obtain the graph of the correspondence \( g \circ f : X \rightrightarrows Z \). On this
open set $U$, it can be checked that this definition of composition of correspondences is the same as that (pulling back the graphs of the two correspondences to $X \times Y \times Z$, intersecting them and then projecting the resulting cycle to $X \times Z$) given in [23, Definition 16.1.1]. However, in the whole $X$, our definition may be different from that given in [23]. The difference is of the same nature as that between total and strict pullbacks of the cohomology class of a variety by a surjective morphism.

Given two dominant correspondences $f : X \to X$ and $g : Y \to Y$, a dominant rational map $\pi : X \dashrightarrow Y$ is a semi-conjugacy of $f$ and $g$ if it satisfies $\pi \circ f = g \circ \pi$, and it is denoted by $\pi : (X, f) \to (Y, g)$. By definition, we see that $\pi \circ f = g \circ \pi$ if and only if there is a non-empty Zariski-open set $U \subset X$ such that $\pi \circ f(x) = g \circ \pi(x)$ for all $x \in U$.

The above definitions can be extended to the case where $X$ and $Y$ are reducible. In this case, let $X_1, \ldots, X_m$ be components of $X$ and $Y_1, \ldots, Y_p$ components of $Y$. Then a correspondence between $X$ and $Y$ is a collection of correspondences $f_{i,j} : X_i \to Y_j$, $(i, j) \in I$, where $I \subset \{1, \ldots, m\} \times \{1, \ldots, p\}$. The correspondence is dominant if and only if each $f_{i,j}$ is dominant, the union of all the domains of the collection is dense in $X$, and the union of all the ranges of the collection is dense in $Y$. If $f : X \to X$ is a dominant correspondence, then we can iterate $f$ in an obvious way. More precisely, if $f_{i,j}$ and $f_{k,l}$ are two irreducible components of $f$, we compose $f_{i,j} f_{k,l}$ whenever the range of $f_{k,l}$ is the domain of $f_{i,j}$; otherwise we do not compose them. A semi-conjugacy between two dominant correspondences $(X, f)$ and $(Y, g)$ is then a dominant rational map $\pi : X \dashrightarrow Y$, which in this case means a collection of dominant rational maps $\pi_i : X_i \dashrightarrow Y_{a(i)}$ for $i = 1, \ldots, m$ such that $Y = \bigcup_i Y_{a(i)}$ (hence each $X_i$ is in the domain of exactly one rational map $\pi_i$, while each $Y_i$ may be in the range of different such maps) satisfying $\pi \circ f = g \circ \pi$ for all $n \in \mathbb{N}$.

Since compositions of dominant correspondences are associative, that is, we have the identity $(f \circ g) \circ h = f \circ (g \circ h)$, no matter if the varieties involved are irreducible or not, we have the following.

**Lemma 3.2.** If the maps $(X, f) \to (Y, g)$ and $(Y, g) \to (Z, h)$ are semi-conjugate, then the induced $(X, f) \to (Z, h)$ is also semi-conjugate. If the map $\pi : (X, f) \to (Y, g)$ is a semi-conjugacy, then $\pi \circ f^n = g^n \circ \pi$ for all $n \in \mathbb{N}$.

To work with semi-conjugacies of correspondences over reducible varieties, it is convenient to consider semi-conjugacies of pairs. To this end, let $f : X_1 \dashrightarrow X_2$ and $g : Y_1 \dashrightarrow Y_2$ be dominant correspondences between irreducible varieties, where $\dim(X_1) = \dim(X_2)$ and $\dim(Y_1) = \dim(Y_2)$. A semi-conjugacy between $(X_1, X_2, f)$ and $(Y_1, Y_2, g)$ is then a pair of dominant rational maps $\pi_1 : X_1 \dashrightarrow Y_1$ and $\pi_2 : X_2 \dashrightarrow Y_2$ such that $\pi_2 \circ f = g \circ \pi_1$. We describe this fact by writing $(\pi_1, \pi_2) : (X_1, X_2, f) \to (Y_1, Y_2, g)$.

The following result characterises semi-conjugate correspondences of pairs.

**Lemma 3.3.** Let the map $(\pi_1, \pi_2) : (X_1, X_2, f) \to (Y_1, Y_2, g)$ be a semi-conjugacy, where $\dim(X_1) = \dim(X_2)$ and $\dim(Y_1) = \dim(Y_2)$. Write $\Gamma_g = \sum_i a_i G_i$, where $G_i$ are distinct irreducible varieties and $a_i \in \mathbb{N}$ is the multiplicity of $G_i$. For a generic $x \in X_1$, the cardinality of $f(x) \cap \pi_2^{-1}(z)$ is $a_i$, for every $z \in g \pi_1(x)$ such that $(\pi_1(x), z) \in G_i$.

**Proof.** Let $x \in X_1$ be generic. Define $y = \pi_1(x)$. Then, by assumption, we have

$$\pi_2 f(x) = g \pi_1(x) = g(y).$$
If \((y, z) \in G_i\), then \(z\) appears in \(g(y)\) with multiplicity \(a_i\). From this, it follows that there are exactly \(a_i\) points (counted with multiplicities) in \(f(x)\) which is mapped to \(z\) by the map \(\pi_2\). In other words, \(\# f(x) \cap \pi_2^{-1}(z) = a_i\).

### 3.2. Pullback of correspondences by rational maps.

In this subsection we define certain pullback operators of correspondences by dominant rational maps.

**Case 1: Equi-dimensional pullback.** Let \(\pi = (\pi_1, \pi_2) : (X_1, X_2) \rightarrow (Y_1, Y_2)\) be a dominant rational map of ordered tuples of varieties, and \(g : Y_1 \rightarrow Y_2\) a dominant correspondence. Assume that \(\dim(X_1) = \dim(X_2) = \dim(Y_1) = \dim(Y_2)\). The pullback of \(g\) by \(\pi\), denoted by \(f = \pi^*g : X_1 \rightarrow X_2\) is defined as follows: for \(U \subset X_1\) a suitable non-empty Zariski-open set, we define \(f(x_1) = \pi_2^{-1}g\pi_1(x_1)\). Since \(X_1, X_2, Y_1, Y_2\) are of same dimension, the cardinality of \(f(x_1)\) is finite. More formally, the graph of \(f\) is then the strict pullback of the graph of \(g\) by the generically finite rational map \(\pi_1 \times \pi_2\). That is, \(\Gamma_f\) is the closure of \((\pi_1 \times \pi_2)^*(\Gamma_g)\), where \(U \subset Y_1 \times Y_2\) is any non-empty Zariski-open set over which \(\pi_1 \times \pi_2\) has finite fibres.

**Lemma 3.4.** Let \(a\ be the degree of \(\pi_2\), i.e. the cardinality (counted with multiplicities) of a generic fibre of \(\pi_2\). Then \(\pi_2 \circ f = ag \circ \pi_1\). In other words, we have a semi-conjugacy \(\pi : (X_1, X_2, f = \pi^*g) \rightarrow (Y_1, Y_2, ag)\).

**Proof.** For a generic point \(x_1 \in X_1\), put \(y_1 = \pi_1(x_1)\) we have
\[
\pi_2 \circ f(x_1) = \pi_2 \circ \pi_2^{-1}g(y_1) = \pi_2 \circ a \pi_1(x_1),
\]
as desired.

**Case 2: Pullback of a triple of correspondences.** Let be given two dominant semi-conjugacies \(\pi : (X_1, X_2, f) \rightarrow (Y_1, Y_2, g)\) and \(\psi : (W_1, W_2, h) \rightarrow (Y_1, Y_2, g)\), where \(X_1, X_2, Y_1, Y_2, W_1, W_2\) are of pure dimensions and satisfy \(\dim(X_1) = \dim(X_2)\), \(\dim(Y_1) = \dim(Y_2)\) and \(\dim(W_1) = \dim(W_2)\). The following proposition allows us to pullback this triple.

**Proposition 3.5.** There is a dominant correspondence \(u : Z_1 \rightarrow Z_2\) (in general \(Z_1\) and \(Z_2\) may be reducible) for which there are semi-conjugacies \(\widetilde{\psi} : (Z_1, Z_2, u) \rightarrow (X_1, X_2, af)\) and \(\widetilde{\psi} : (Z_1, Z_2, u) \rightarrow (W_1, W_2, ah)\), where \(a\) is a positive integer depending only on the correspondence \(g : Y_1 \rightarrow Y_2\).

The following construction, which we will do under the additional assumptions that \(X_1 = X_2 = X\), \(Y_1 = Y_2 = Y\), \(W_1 = W_2 = W\), \(\pi_1 = \pi_2 = \pi\) and \(\psi_1 = \psi_2 = \psi\), extends easily to the general case. (For example, in the general case, we define \(Z_1\) as the pullback of the triple \(p_1 : X_1 \rightarrow Y_1\) and \(\psi_1 : W_1 \rightarrow Y_1\), and similarly for \(Z_2\).)

**Defining \(Z\).** On two appropriate non-empty Zariski-open sets \(U \subset X\) and \(U' \subset W\), we have that the (reduced) variety \(\{(x, w) \in U \times U' : \pi(x) = \psi(w)\}\) is of pure dimension \(\dim(X) + \dim(W) - \dim(Y)\). Let \(Z\) be the closure of this variety in \(X \times W\). In other words, \(Z\) is the usual pullback of the triple \(\pi : X \rightarrow Y\) and \(\psi : W \rightarrow Y\). In general \(Z\) is not irreducible.

We denote the natural projections from \(Z\) to \(X, W\) by \(\widetilde{\psi} : Z \rightarrow X\) and \(\widetilde{\psi} : Z \rightarrow W\).
Defining the correspondence $u : Z \to Z$. Let $\Gamma$ be the closure in $X \times W \times X \times W$ of the set $\{(x, w, x', w') : (x, x') \in V, (w, w') \in V', \pi(x) = \psi(w), \pi(x') = \psi(w')\}$, where $V$ and $V'$ are appropriate Zariski-open sets of $\Gamma_f$ and $\Gamma_h$, respectively. We can choose, e.g., $V = \{(x, x') \in \Gamma_f : x \in U\}$ for an appropriate non-empty Zariski-open set $U \subset X$, and similarly $V' = \{(w, w') \in \Gamma_h : w \in U'\}$ for some appropriate non-empty Zariski-open set $U' \subset W$.

We have the following properties of $\Gamma$.

(a) $\Gamma \subset Z \times Z$: This is clear from the definition.

(b) $\Gamma$ is non-empty and of pure dimension equal to $\dim(Z)$: Let $(x, w) \in U \times U'$, and $y = \pi(x) = \psi(w)$. Then, from $\pi f(x) = g(y) = \psi h(w)$, thanks to the two semi-conjugacies $(X, f) \to (Y, g)$ and $(W, h) \to (Y, g)$, it follows that for any $x' \in f(x)$ there is $w' \in h(w)$ such that $\pi(x') = \psi(w')$, and vice versa. Thus $\Gamma$ is non-empty. Moreover, since both $f(x)$ and $h(w)$ have bounded cardinalities, it follows that $\Gamma$ has dimension equal to $\dim(Z)$.

Let $\Gamma_g = \sum_{i \in I} a_i G_i$, where $I$ is a finite set, $\Gamma_i$ are distinct irreducible varieties and $a_i \in \mathbb{N}$. We define $a(g) := \prod_{i \in I} a_i$. Let $V$ be an irreducible component of the (reduced) variety $\Gamma$. Let $\tau : Z \to Y$ be $\tau = \psi \circ \pi = \pi \circ \tilde{\psi}$. Then the (strict) image of $V$ (see Section 4 for the definition) under the rational map $\tau \times \tau : Z \times Z \to Y \times Y$ must lie in exactly one of the components $G_i$ of $\Gamma_g$. Then we define $b(V) := a_i$, and

$$\Gamma_u := \sum_V a(g) V / b(V),$$

where $V$ runs all over the irreducible components of $\Gamma$. Since $a(g) = \prod_{i \in I} a_i$ and $b(V)$ is one of the coefficients $a_i$, it follows that $a(g) / b(V)$ are integers for all $V$. From the properties of $\Gamma$, it follows immediately that $\Gamma_u$ is the graph of a dominant correspondence $u : Z \to Z$.

**Lemma 3.6.** Let $G_i$, $a_i$ and $a(g)$ be as in the previous paragraph. Then we have the equalities $\tilde{\psi} \circ u = a(g)(f \circ \tilde{\psi})$ and $\pi \circ u = a(g)(h \circ \pi)$. In other words, we have two semi-conjugacies $\tilde{\pi} : (Z, u) \to (X, a(g) f)$ and $\tilde{\psi} : (Z, u) \to (W, a(g) h)$.

**Proof.** Let $z = (x, w) \in Z \cap (U \times U')$, where $U \subset X$ and $U' \subset W$ are appropriate non-empty Zariski-open sets. Define $y = \pi(x) = \psi(w)$. We have by definition

$$u(z) = \bigcup_{t \in g(y)} a(g) / a(t) (f(x) \cap \pi^{-1}(t)) \times (h(w) \cap \psi^{-1}(t)),$$

where $a(t) = a_i$ provided $(y, t) \in G_i$. Then $\#(f(x) \cap \pi^{-1}(t)) = a(t) = \#(h(w) \cap \psi^{-1}(t))$ for all $t \in g(y)$, by Lemma 3.3. Hence

$$\tilde{\pi} \circ u(z) = \bigcup_{t \in g(y)} a(g) / a(t) \times \#(f(x) \cap \pi^{-1}(t)) \times (h(w) \cap \psi^{-1}(t))$$

$$= a(g) \bigcup_{t \in g(y)} (h(w) \cap \psi^{-1}(t)) = a(g) h(w) = a(g) h \tilde{\pi}(z).$$

This shows that we have a semi-conjugacy $\tilde{\pi} : (Z, u) \to (W, a(g) h)$. Similarly, we have a semi-conjugacy $\tilde{\psi} : (Z, u) \to (X, a(g) f)$. \qed
4. Pullback by correspondences and strict intersections of varieties

In this section we consider pullback, strict transforms by correspondences and strict intersection of algebraic varieties. We assume throughout this section that the field \( \mathbb{K} \) is algebraically closed of arbitrary characteristic. We will be mostly concerned only with algebraic cycles over smooth projective varieties.

4.1. Strict intersection of varieties. In this subsection, we prove an estimate for strict intersection of algebraic cycles on a product space \( X \times Y \). We first define the notion of strict intersection of two varieties. Let \( Z \) be an irreducible smooth projective variety and let \( V, W, L \subset Z \) be irreducible varieties. Let \( U \subset Z \) be a non-empty Zariski-open set on which the set theoretical intersection \( V \cap W \) is of the correct dimension. Then \( V|_U \cap W|_U \) is well defined as a variety, and we define \( V \cap^U W \) as the closure in \( Z \) of \( V|_U \cap W|_U \). When \( U \) is the largest such open set, we denote by \( V \cap^o W \) the algebraic cycle \( V \cap W \).

The main result of this subsection is the following.

**Lemma 4.1.** Let \( X, Y \subset \mathbb{P}^N \) be irreducible smooth projective varieties over \( \mathbb{K} \) of dimensions \( k \) and \( l \), respectively. Let \( o_X \) and \( o_Y \) be generic hyperplane sections on \( X \) and \( Y \). There is a constant \( A > 0 \) such that for any pair of irreducible subvarieties \( V, W, L \subset X \times Y \) and an open subset \( \tilde{U} \subset X \times Y \) over which the intersection \( V|_\tilde{U} \cap W|_\tilde{U} \) is well defined, then in \( N^*(X \times Y) \),

\[
W \cap^U V \leq \left( A \sum_{0 \leq j \leq k, 0 \leq p(W) - j \leq l} \alpha_j(W) \omega_X^{k-j} \cdot \omega_Y^{p(W)+j} \right) \cdot [V].
\]

Here \( p(W) \) is the codimension of \( W \) in \( X \times Y \) and \( \alpha_j(W) = W \cdot \omega_X^j \cdot \omega_Y^{p(W)-j} \).

**Proof.** We will follow the ideas in [13, Section 2] (of regularising the diagonals) plus that in Lemma 2.2 above (of using cones and automorphisms). Let us define \( Z = X \times Y \). Let \( \Delta_Z \subset Z \times Z, \Delta_X \subset X \times X \) and \( \Delta_Y \subset Y \times Y \) be the diagonals, and let \( \pi_1, \pi_2 : Z \times Z \rightarrow Z \) be the projections. For simplicity, let us use \( o_X \) and \( o_Y \) to denote also the pullbacks, via the projections \( Z = X \times Y \rightarrow X, Y \), of \( o_X \) and \( o_Y \) to \( Z \). We have an isomorphism

\[
\phi : Z \times Z \rightarrow (X \times X) \times (Y \times Y)
\]

given by the formula \( \pi(x_1, y_1, x_2, y_2) = ((x_1, x_2), (y_1, y_2)) \), under which \( \Delta_Z \) is mapped to \( \Delta_X \times \Delta_Y \). We use the Segre embedding \( \mathbb{P}^N \times \mathbb{P}^N \rightarrow \mathbb{P}^{N'} \) to embed both \( X \times X \) and \( Y \times Y \) into \( \mathbb{P}^{N'} \).

As in Section 2.2, we can find a linear subspace \( L_1 \) of \( \mathbb{P}^{N'} \) of codimension \( 2k + 1 \) and a linear subspace \( L_2 \) of \( \mathbb{P}^{N'} \) of codimension \( 2l + 1 \) so that in \( \mathbb{P}^{N'} \) the cone \( C_{L_1}(\Delta_X) \) intersects properly with \( X \times X \), and the cone \( C_{L_2}(\Delta_Y) \) intersects properly with \( Y \times Y \). Then \( \Delta_X \) is a component of \( C_{L_1}(\Delta_X) \cap (X \times X) \), and \( \Delta_Y \) is a component of \( C_{L_2}(\Delta_Y) \cap (Y \times Y) \). Let \( U_{L_1} \subset \Delta_X \) be the (largest) non-empty Zariski-open set of \( \Delta_X \) which does not intersect with other components of \( C_{L_1}(\Delta_X) \cap (X \times X) \) (in other words, \( \Delta_X \) is the unique component in \( U_{L_1} \) – maybe with some multiplicity \( > 1 \) – of \( C_{L_1}(\Delta_X) \cap (X \times X) \)), and we define similarly the non-empty Zariski-open set \( U_{L_2} \subset \Delta_Y \). Note that there is a finite set \( S_1 \) of such linear subspaces \( L_1 \) so that \( \bigcup_{L_1 \in S_1} U_{L_1} = \Delta_X \). In fact, it suffices to show that for any \( x_0 \in \Delta_X \), there
is a linear subspace $L_1 \subset \mathbb{P}^{N'}$ as above for which $\Delta_X$ is the only component containing $x_0$ of $\mathcal{C}_{L_1}(\Delta_X) \cap (X \times X)$. To this end, it suffices to choose $L_1$ so that it does not intersect with the tangent space $T_{x_0} X \times X \subset \mathbb{P}^{N'}$. At such a point $x_0$, the corresponding linear projection $\pi_{L_1}: X \times X \rightarrow \mathbb{P}^{2k}$ is smooth, and hence the conclusion follows. Similarly, there is a finite set $\mathcal{S}_2$ of such linear subspaces $L_2$ so that $\bigcup_{L_2 \in \mathcal{S}_2} U_{L_2} = \Delta_Y$. Therefore,

$$\bigcup_{L_1 \in \mathcal{S}_1, L_2 \in \mathcal{S}_2} U_{L_1} \times U_{L_2} = \Delta_X \times \Delta_Y.$$

Now let $U \subset Z = \Delta_Z$ be any Zariski-open subset on which $V|_U \cap W|_U$ is well defined. This is the same as that the two varieties $\Delta_Z$ and $\pi_1^{-1}(V) \cap \pi_2^{-1}(W)$ have well-defined intersection on the Zariski-open set $\pi_1^{-1}(U) \cap \pi_2^{-1}(U) \subset Z \times Z$. For each $L_1 \in \mathcal{S}_1$ and $L_2 \in \mathcal{S}_2$ we define $U_{L_1, L_2} = \phi^{-1}(U_{L_1} \times U_{L_2})$. It follows from the choice of $\mathcal{S}_1$ and $\mathcal{S}_2$ in the above paragraph that

$$\bigcup_{L_1 \in \mathcal{S}_1, L_2 \in \mathcal{S}_2} U_{L_1, L_2} = \Delta_Z.$$

Via the obvious isomorphism $\Delta_Z = Z$, we regard $U_{L_1, L_2}$ also as a subset of $Z$. Hence, we obtain

$$V|_U \cap W|_U \leq \sum_{L_1 \in \mathcal{S}_1, L_2 \in \mathcal{S}_2} V|_{U \cap U_{L_1, L_2}} \cap W|_{U \cap U_{L_1, L_2}}.$$

To prove the lemma, it thus suffices to bound the individual strict intersections $V \cap U \cap U_{L_1, L_2} W$. From now on, we fix $L_1 \in \mathcal{S}_1$ and $L_2 \in \mathcal{S}_2$. Define

$$B(V, W) = \phi(\pi_1^{-1}(V) \cap \pi_2^{-1}(W))$$

and

$$B(U, L_1, L_2) = \phi(U \cap U_{L_1, L_2}) \subset \phi(U_{L_1, L_2}) = U_{L_1} \times U_{L_2} \subset \Delta_X \times \Delta_Y.$$

Since $\Delta_X \times \Delta_Y$ is the only component in $U_{L_1, L_2}$ of $C_{L_1} (\Delta_X) \times C_{L_2} (\Delta_Y) \mid_{Y \times Y}$, it follows that

$$\phi(\Delta_Z \cap V \cap U_{L_1, L_2} \cap W \cap U_{L_1, L_2}) \leq [C_{L_1} (\Delta_X) \mid_{X \times X} \times C_{L_2} (\Delta_Y) \mid_{Y \times Y}] \cap B(V, W) \cap B(U, L_1, L_2).$$

The algebraic group $Aut(\mathbb{P}^{N'}) \times Aut(\mathbb{P}^{N'})$ acts transitively on $\mathbb{P}^{N'} \times \mathbb{P}^{N'}$. It follows from [27, Corollary 4] that there are rational curves $g_1(t), g_2(t) (t \in D \subset \mathbb{P}^1)$ of automorphisms of $\mathbb{P}^{N'}$ with $g_1(0) = g_2(0) = Id$ such that $g_1(t) C_{L_1} (\Delta_X)$ intersects $X \times X$ properly, $g_2(t) C_{L_2} (\Delta_Y)$ intersects $Y \times Y$ properly, and $g_1(t) C_{L_1} (\Delta_X) \mid_{X \times X} \times g_2(t) C_{L_2} (\Delta_Y) \mid_{Y \times Y}$ intersects $B(V, W)$ properly, for all $t$ in a non-empty Zariski-open set $\Omega \subset D \subset \mathbb{P}^1$. Let $\Gamma_0$ be the algebraic cycle $\left\langle (X \times X) \times (Y \times Y) \times \Omega \right\rangle$ whose fibre over $t \in \Omega$ is

$$[g_1(t) C_{L_1} (\Delta_X) \mid_{X \times X} \times g_2(t) C_{L_2} (\Delta_Y) \mid_{Y \times Y}] \cap B(V, W),$$

and let $\Gamma$ be the closure of $\Gamma_0$ in $(X \times X) \times (Y \times Y) \times \mathbb{P}^1$. Note that $\Gamma_0$ can also be defined as the intersection between $B(V, W)$ and the algebraic cycle in $(X \times X) \times (Y \times Y) \times \Omega$ whose fibre over $t \in \Omega$ is $g_1(t) C_{L_1} (\Delta_X) \mid_{X \times X} \times g_2(t) C_{L_2} (\Delta_Y) \mid_{Y \times Y}$. Hence, by the choice of $\Omega$,
$\Gamma_0$ is equi-dimensional and so is $\Gamma$. For $t \in \mathbb{P}^1$ define by $\gamma(t)$ the fibre over $t$ of $\Gamma$. Then (note that 0 may not be in $\Omega$

$$
\gamma(0) \geq [C_{L_1}(\Delta_X)|_{X \times X} \times C_{L_2}(\Delta_Y)|_{Y \times Y}] \wedge B(V, W)|_{B(U, L_1, L_2)} \\
\geq \phi(\Delta_Z \wedge \pi_1^{-1}(V)|_{U \cap U_{L_1, L_2}} \wedge \pi_2^{-1}(W)|_{U \cap U_{L_1, L_2}}),
$$

and hence, applying the inverse map $\phi^{-1}$, since $\gamma(0)$ is an effective equi-dimensional algebraic cycle,

$$(\pi_1)_*(\phi^{-1}[\gamma(0)]) \geq (\pi_1)_*(\pi_1^{-1}(V)|_{U \cap U_{L_1, L_2}} \wedge \pi_2^{-1}(W)|_{U \cap U_{L_1, L_2}} \wedge \Delta Z)$$

$$= V|_{U \cap U_{L_1, L_2}} \wedge W|_{U \cap U_{L_1, L_2}}.$$ 

It then follows that $(\pi_1)_*(\phi^{-1}[\gamma(0)]) \geq V \wedge U_{L_1, L_2} W$.

Hence, to finish the proof it is enough to bound the class of $(\pi_1)_*(\phi^{-1}[\gamma(0)])$ in $N^*(Z)$. The class of $\gamma(0)$ is the same as the class of $\gamma(t)$ for every $t \in \Omega$. Let $H \subset \mathbb{P}^{N'}$ be a hyperplane; then in $\mathbb{P}^{N'}$, the class of $C_{L_1}(\Delta_X)$ is $A_1 H^k$ and the class of $C_{L_2}(\Delta_Y)$ is $A_2 H^l$ for some positive integers $A_1, A_2 > 0$. For $t \in \Omega$, since $g_1(t)$ and $g_2(t)$ are linear maps thus preserve degrees, we have in $N^*((X \times X) \times (Y \times Y))$ that

$$[\gamma(t)] = [g_1(t)C_{L_1}(\Delta_X)|_{X \times X}] \cdot [g_2(t)C_{L_2}(\Delta_Y)|_{Y \times Y}] \cdot [B(V, W)]$$

$$= A_1 A_2[H^k|_{X \times X}] \cdot [H^l_{Y \times Y}] \cdot [B(V, W)].$$

Hence, applying the inverse map $\phi^{-1}$, noting that

$$\phi^{-1}[B(V, W)] = \pi_1^{-1}(V) \times \pi_2^{-1}(W),$$

$$\phi^{-1}[H^k|_{X \times X}] = (\pi_1^*(\omega_X) + \pi_2^*(\omega_Y))^k,$$

$$\phi^{-1}[H^l|_{Y \times Y}] = (\pi_1^*(\omega_Y) + \pi_2^*(\omega_Y))^l,$$

we obtain from the above calculations that in $N^*(Z)$:

$$V \wedge U_{L_1, L_2} W \leq (\pi_1)_*(\phi^{-1}[\gamma(0)])$$

$$= A_1 A_2(\pi_1)_*[(\pi_1^*(\omega_X) + \pi_2^*(\omega_Y))^k \wedge (\pi_1^*(\omega_Y) + \pi_2^*(\omega_Y))^l]$$

$$\wedge \pi_1^{-1}(V) \times \pi_2^{-1}(W)].$$

The last expression is a sum of terms, modulo positive multiplicity constants involving $A_1 A_2$ and binomial numbers, of the form

$$(\pi_1)_*[\pi_1^*(\omega_X^{k-j} \wedge \omega_Y^l \wedge [V]) \wedge \pi_2^*(\omega_X^j \wedge \omega_Y^{l-i} \wedge [W])],$$

which by the projection formula is

$$[\omega_X^{k-j} \wedge \omega_Y^l \wedge [V] \wedge (\pi_1)_*\pi_2^*]\omega_X^j \wedge \omega_Y^{l-i} \wedge [W].$$

The key observation is that $(\pi_1)_*\pi_2^*[\omega_X^{j} \wedge \omega_Y^{l-i} \wedge [W]$ is non-zero only if $j + l - i = p(W)$, and in which case the term becomes $\alpha_j(W)$. Therefore, an upper bound for $V \wedge U_{L_1, L_2} W$ is

$$\left(A' \sum_{0 \leq j \leq k, 0 \leq p(W) - j \leq l} \alpha_j(W)\omega_X^{k-j} \cdot \omega_Y^{l-p(W)+j} \right) [V],$$

where $A'$ depends only on $A_1 A_2$ and the binomial numbers. Hence the lemma is proven. □
4.2. Pullback and strict transforms of algebraic cycles by correspondences. Let $X$ and $Y$ be two irreducible smooth projective varieties and $f : X \to Y$ a dominant correspondence. Let $\Gamma_f$ be the graph of $f$, and write $\Gamma_f = \sum_{i \in I} a_i G_i$, where $I$ is a finite set, $G_i$ are distinct irreducible varieties and $a_i \in \mathbb{N}$. Then we can define the pushforward operators $f_* : A_q(X) \to A_q(Y)$ and pullback operators $f^* : A^p(Y) \to A^p(X)$ (see [23, Chapter 16]). For example, there are two methods to define the pullback operators:

Method 1. Let $\pi_X, \pi_Y : X \times Y \to X, Y$ be the two projections, and let $\Gamma_f$ be the graph of $f$. For $\alpha \in A^p(Y)$, we define $f^*(\alpha) \in A^p(X)$ by the formula

$$f^*(\alpha) = (\pi_X)_*(\pi_Y^*(\alpha)) = \sum_{i \in I} a_i G_i^*(\alpha).$$

Method 2. Let $\tau_i : \Gamma_i \to G_i$ be an alteration of $G_i$, that is, $\Gamma_i$ is smooth and $\tau_i$ is generically finite of degree $d_i$. The results of de Jong assure the existence of alterations, and let $p_i, g_i : \Gamma_i \to X, Y$ be the induced morphisms. Then we define

$$f^*(\alpha) = \sum_{i \in I} \frac{a_i (p_i)_*(g_i^*(\alpha))}{d_i}.$$ 

For the convenience of the readers, we recall here an argument to show why these two methods give the same answer. It suffices to show for each $i$ that

$$G_i^*(\alpha) = \frac{(p_i)_*(g_i^*(\alpha))}{d_i}.$$ 

In fact, since $p_i = \pi_X \circ \tau_i$ and $g_i = \pi_Y \circ \tau_i$, we have

$$(p_i)_* g_i^*(\alpha) = (\pi_X)_*(\tau_i)_* \pi_Y^*(\alpha) = d_i (\pi_X)_*(\pi_Y^*(\alpha) \cdot G_i).$$

Here we use the projection formula $(\tau_i)_* \pi_Y^*(\beta) = d_i G_i \cdot \beta$ for algebraic cycles $\beta$ on $X \times Y$.

In defining dynamical degrees and proving some of their basic properties, we need to estimate the degrees of the pullback and of strict transforms by a correspondence of a cycle. We present these estimates in the remaining of this subsection. We fix an alteration $\Gamma_i$ of the component $G_i$ of $\Gamma_f$, and let $p_i, g_i : \Gamma_i \to X, Y$ be the induced morphisms. By Theorem 3.1, the sets $V_l = \{ y \in Y : \dim (g_i^{-1}(y)) \geq l \}$ are algebraic varieties of $Y$ (see [33, Corollary, p. 76]). We denote the critical image of $g_i$ by $C_{g_i} = \bigcup_{l > \dim(X) - \dim(Y)} V_l$. Before continuing, we mention a convention frequently used later.

Convention. Let $f : X \to Y$ be a surjective morphism, and $W \subset Y$ a subvariety so that the set $f^{-1}(W)$ has the correct dimension. Then the intersection $\pi_Y^{-1}(W) \cap \Gamma_f$ is well defined. By abusing the notation, we denote also by $f^{-1}(W)$ the effective algebraic cycle of pure dimension $(\pi_X)_*(\pi_Y^{-1}(W) \cap \Gamma_f)$, that is, the scheme theoretic inverse image in intersection theory (see [23]). For example, if $f : \mathbb{C} \to \mathbb{C}$ is the map $z \mapsto z^2$, then as a set $f^{-1}(0) = 0$ but as an algebraic cycle $f^{-1}(0) = 2(0)$. In the remaining of this paper, it will be clear from the context that by $f^{-1}(W)$ whether we mean a set or an algebraic cycle.

Lemma 4.2. Let $W$ be an irreducible subvariety of $Y$. If $W$ intersects properly every irreducible component of $V_l$ (for every $l > \dim(X) - \dim(Y)$), then $g_i^{-1}(W)$ is well defined as an effective algebraic cycle of pure dimension of $\Gamma_i$, and moreover the class $[g_i^{-1}(W)]$ represents the pullback $g_i^*(W)$ in $A^*(\Gamma_i)$.
Proof. (See also [23, Example 11.4.8].) By the intersection theory (see [23, Section 8.2] and [22, Theorem 3.4]), it suffices to show that the set $g_i^{-1}(W)$ has the correct dimension $\dim(X) - \dim(Y) + \dim(W)$. First, if $y \in W - \mathcal{C}_{g_i}$, then $\dim(g_i^{-1}(y)) = \dim(X) - \dim(Y)$ by the definition of $\mathcal{C}_{g_i}$. Hence we have $\dim(g_i^{-1}(W - \mathcal{C}_{g_i})) = \dim(W) + \dim(X) - \dim(Y)$. It remains to show that $g_i^{-1}(W \cap \mathcal{C}_{g_i})$ has dimension $\leq \dim(X) + \dim(W) - \dim(Y) - 1$. Let $Z$ be an irreducible component of $W \cap \mathcal{C}_{g_i}$. We define $l = \inf(\dim(g_i^{-1}(y)) : y \in Z)$. Then $l > \dim(X) - \dim(Y)$ and for generic $y \in Z$ we have $\dim(g_i^{-1}(y)) = l$ by Theorem 3.1.

Let $V$ be an irreducible component containing $Z$. By assumption, $V \cap W$ has dimension $\dim(V) + \dim(W) - \dim(Y)$, hence $\dim(Z) \leq \dim(V) + \dim(W) - \dim(Y)$. We obtain

$$\dim(g_i^{-1}(Z - V_{l+1})) = l + \dim(Z) \leq l + \dim(V) + \dim(W) - \dim(Y).$$

Since $g_i$ is surjective (because $f$ is dominant) and $V \neq Y$, it follows that

$$\dim(X) - 1 \geq \dim(g_i^{-1}(V)) \geq \dim(V) + l.$$ From these last two estimates we obtain

$$\dim(g_i^{-1}(Z - V_{l+1})) \leq l + \dim(V) + \dim(W) - \dim(Y) \leq \dim(X) - 1 + \dim(W) - \dim(Y).$$

Since there are only a finite number of such components, it follows that

$$\dim(g_i^{-1}(W \cap \mathcal{C}_{g_i})) \leq \dim(W) + \dim(X) - \dim(Y) - 1,$$

as claimed. \qed

We next estimate the degree of the pullback of a cycle. Fix an embedding $Y \subset \mathbb{P}_K^N$, and let $\iota : Y \subset \mathbb{P}_K^N$ be the inclusion. Let $H \subset \mathbb{P}_K^N$ be a generic hyperplane and let $\omega_Y = H|_Y$.

Lemma 4.3. Let $f : X \dashrightarrow Y$ be a correspondence.

(a) Let $p = 0, \ldots, \dim(Y)$, and let $Z \subset X$ be a proper subvariety. Then there is a linear subspace $H^p \subset \mathbb{P}_K^N$ of codimension $p$ such that $H^p$ intersects $Y$ properly, $f^*(\iota^*(H^p))$ is well defined as an algebraic cycle of $X$, and $f^*(\iota^*(H^p))$ has no component on $Z$. In particular, for any non-negative integer $p$, the pullback $f^*(\omega_Y^p) \in A^p(X)$ is effective.

(b) Let $W$ be an irreducible subvariety of codimension $p$ in $Y$. Then in $A^p(X)$, we can represent $f^*(W)$ by $\beta_1 \beta_2$, where $\beta_1, \beta_2$ are effective and $\beta_1, \beta_2 \leq C \deg(W) f^*(\omega_Y^p)$ for some constant $C > 0$ independent of the varieties $W$ and $X$, and the correspondence $f$.

Proof. We may assume that $\Gamma_f$ is irreducible. Let $\Gamma \to \Gamma_f$ be an alteration, with the induced morphisms $p : \Gamma \to X$ and $g : \Gamma \to Y$. Since by definition $f^*(W) = p_* g^*(W)$ and since $p_*$ preserves effective classes, it suffices to prove the lemma for the morphism $g$. We let the varieties $V_l$ be as those defined before Lemma 4.2.

(a) Let $H^p \subset \mathbb{P}_K^N$ be a generic codimension $p$ linear subspace. Then in $A^p(Y)$, $\omega_Y^p$ is represented by $\iota^*(H^p)$. We can choose such an $H^p$ so that $H^p$ intersects properly $Y$, $g(Z)$ and all irreducible components of $V_l$ and $g(Z) \cap V_l$ for all $l > \dim(X) - \dim(Y)$. By Lemma 4.2, the pullback $g^*(\iota^*(H^p)) = g^{-1}(\iota^*(H^p))$ is well defined as an algebraic cycle of $\Gamma$. Moreover, the dimension of $g^{-1}(\iota^*(H^p)) \cap Z$ is less than the dimension of $g^{-1}(\iota^*(H^p))$. In particular, $g^*(\iota^*(H^p))$ is effective and has no component on $Z$.
(b) It follows from Lemma 2.2 that \( W \) is rationally equivalent to \( i^*(\alpha_1) - i^*(\alpha_2) \pm W_e \), where \( \alpha_1, \alpha_2 \subseteq \mathbb{P}_K^N \) and \( W_e \subset Y \) are subvarieties of codimension \( p \), and they intersect properly \( Y \) and all irreducible components of \( V_l \) for all \( l > \dim(X) - \dim(Y) \). Moreover, we have \( \deg(\alpha_1), \deg(\alpha_2), \deg(W_e) \leq C \deg(W) \) for some \( C > 0 \) independent of \( W \), and we can find a codimension \( p \) variety \( \alpha \subset \mathbb{P}_K^N \) so that \( \alpha \) intersects properly with \( Y \) and all \( V_l \), \( i^*(\alpha) - W_e \) is effective, and \( \deg(\alpha) \leq C \deg(W_e) \). Note that in \( A^P(Y) \) we have \( i^*(\alpha_1) \sim \deg(\alpha_1)\omega_Y^p \), \( i^*(\alpha_2) \sim \deg(\alpha_2)\omega_Y^p \) and \( i^*(\alpha) \sim \deg(\alpha)\omega_Y^p \). Note also that \( 0 \leq g^*(W_e) \leq g^*(i^*(\alpha)) \). Therefore, in \( A^P(\Gamma) \),
\[
g^*(W) \sim \deg(\alpha_1)g^*(\alpha_Y^p) - \deg(\alpha_2)g^*(\omega_Y^p) \pm g^*(W_e),
\]
where each of the three terms on the right-hand side is effective and \( \leq C \deg(W)g^*(\omega_Y^p) \) for some \( C > 0 \) independent of \( W, X \) and \( f \).

Finally, we estimate the degree of a strict transform of a cycle by a correspondence. Again, it is sufficient to consider the case \( \Gamma_f \) is irreducible. Let \( \Gamma \to \Gamma_f \) be an alteration with the induced morphisms \( p : \Gamma \to X \) and \( g : \Gamma \to Y \). Define
\[
g_0 = g|_{\Gamma - g^{-1}(\mathcal{C}_g)} : \Gamma - g^{-1}(\mathcal{C}_g) \to Y - \mathcal{C}_g.
\]
It follows that \( g_0 \) is a proper morphism, and for any \( y \in Y - \mathcal{C}_g \), \( g_0^{-1}(y) \) has the correct dimension \( \dim(X) - \dim(Y) \). Let \( W \subset Y \) be a codimension \( p \) subvariety. The inverse image \( g_0^{-1}(W) = g^{-1}(W) \cap (\Gamma - g^{-1}(\mathcal{C}_g)) \subset \Gamma - g^{-1}(\mathcal{C}_g) \) is an effective algebraic cycle of pure codimension \( p \) of \( \Gamma - g^{-1}(\mathcal{C}_g) \), hence its closure \( \text{cl}(g_0^{-1}(W)) \subset \Gamma \) is an effective algebraic cycle of pure codimension \( p \), and we define \( f^o(W) = p_* \text{cl}(g_0^{-1}(W)) \). Note that a strict transform depends on the choice of the alteration \( \Gamma \) of the graph \( \Gamma_f \). (We can also define a strict transform more intrinsically using the graph \( \Gamma_f \) directly, as in [16].)

**Lemma 4.4.** Let \( W \subset Y \) be a codimension \( p \) subvariety. Then \( f^o(W) \) is an effective cycle, and in \( A^P(X) \) we have
\[
f^o(W) \leq C \deg(W) f^*(\omega_Y^p),
\]
where \( C > 0 \) is a constant independent of \( W, X \) and the correspondence \( f \).

**Proof.** Recall again that here we can assume that \( \Gamma_f \) is irreducible. That \( f^o(W) \) is an effective cycle follows from the definition. It suffices to prove the lemma for the morphism \( g : \Gamma \to Y \). By Lemma 2.2, we can decompose \( W \) as follows:
\[
W = \sum_{i=1}^{\epsilon} (-1)^{i-1} i^*(C_{L_i}(W_{i-1})) + (-1)^\epsilon W_e,
\]
where the variety \( W_e \) intersects properly all irreducible components of \( V_l \) for all \( l \) greater than \( \dim(X) - \dim(Y) \), and \( C_{L_i}(W_{i-1}) \subset \mathbb{P}_K^N \) are subvarieties of codimension \( p \) intersecting \( Y \) properly (but may not intersect properly the irreducible components of \( V_l \)). Moreover, we have the following bound on the degrees:
\[
\text{deg}(W_e), \deg(C_{L_i}(W_{i-1})) \leq C \deg(W)
\]
for all \( i \), where \( C > 0 \) is independent of \( W, X \) and \( f \).
By the definition of $g^0$ we have

$$g^0(W) = \sum_{i=1}^{e} (-1)^{i-1} g^0(t^*(C_{L_i}(W_{i-1}))) + (-1)^e g^0(W_e). \tag{4.2}$$

Note that $e \leq \dim(Y)$. We now estimate each term on the right-hand side of (4.2). Let $S \subset \mathbb{P}^N_K$ be a subvariety of codimension $p$ intersecting $Y$ properly (but may not intersect properly the components of $V_l$). We first show that, in $A^P(\Gamma)$, for any such $S$,

$$g^0(t^*(S)) \leq \deg(S)g^*(\omega_Y^P). \tag{4.3}$$

We can find a rational curve of automorphisms $\tau(t) \in \text{Aut}(\mathbb{P}^N_K)$ for $t \in D \subset \mathbb{P}^1_K$ such that for a dense Zariski-open dense subset $U \subset D \subset \mathbb{P}^1$, $\tau(t)S$ intersects properly $Y$ and all the irreducible components of $V_l$ (for $l > \dim(X) - \dim(Y)$) for all $t \in U$. Let $\mathcal{S} \subset Y \times \mathbb{P}^1$ be the closure of $\{g(t)z : t \in D, z \in S\}$, hence for $t \in U \subset \mathbb{P}^1$, $\mathcal{S}_t = t^*(\tau(t)S) \subset Y$. Since $\mathcal{S}$ intersects $Y$ properly, we have $\mathcal{S}_0 = t^*(S)$. By the choice of $\mathcal{S}$, for any $t \in U$ the pullback $g^*(\mathcal{S}_t)$ is well defined as an effective algebraic cycle of $\Gamma$.

We consider the induced map $G : \Gamma \times \mathbb{P}^1 \to Y \times \mathbb{P}^1$ given by the formula

$$G(z, t) = (g(z), t).$$

We define by $G_0$ the restriction map $G_0 : \Gamma \times U \to Y \times U$. By the choice of the variety $\mathcal{S}$, the pullback

$$G_0^{-1}(\mathcal{S}) = G^{-1}(\mathcal{S}) \cap (\Gamma \times U) \subset \Gamma \times U$$

is an effective algebraic cycle of pure codimension $p$, hence its closure $G^0(\mathcal{S}) \subset \Gamma \times \mathbb{P}^1$ is an effective algebraic cycle of pure codimension $p$. Moreover, for all $t \in U$ we have

$$G_0^0(\mathcal{S}_t) = g^*(\mathcal{S}_t).$$

Since $g_0 : \Gamma - g^{-1}(\mathcal{E}_g) \to Y - \mathcal{E}_g$ has all fibres of the correct dimension $\dim(X) - \dim(Y)$, it follows that

$$G_0^0(\mathcal{S})_0 \cap (\Gamma - g^{-1}(\mathcal{E}_g)) = g_0^{-1}(t^*(S)).$$

In fact, let $G_1$ be the restriction of $G$ to $(\Gamma - \mathcal{E}_g) \times \mathbb{P}^1$. Then

$$G_1^{-1}(\mathcal{S}) = G^{-1}(\mathcal{S}) \cap [(\Gamma - \mathcal{E}_g) \times \mathbb{P}^1]$$

is an effective algebraic cycle in $(\Gamma - \mathcal{E}_g) \times \mathbb{P}^1$ of pure codimension $p$. Hence its closure in $\Gamma \times \mathbb{P}^1$, denoted by $\overline{G}(\mathcal{S})$ is an effective algebraic cycle of pure codimension $p$. For $t \in U$, we have

$$\overline{G}(\mathcal{S})_t = g^*(\mathcal{S}_t) = G^0(\mathcal{S}_t),$$

because on the one hand $\overline{G}(\mathcal{S})_t \subset G^{-1}(\mathcal{S})_t = g^*(\mathcal{S}_t)$, and on the other hand $g^*(\mathcal{S}_t)$ has no component on $g^{-1}(\mathcal{E}_g)$ and $\overline{G}(\mathcal{S})_t \cap (\Gamma - g^{-1}(\mathcal{E}_g)) = g_0^{-1}(\mathcal{S}_t)$. Therefore $\overline{G}(\mathcal{S}) = G^0(\mathcal{S})$ as algebraic cycles on $\Gamma \times \mathbb{P}^1$. In particular,

$$G^0(\mathcal{S})_0 \cap (\Gamma - g^{-1}(\mathcal{E}_g)) = \overline{G}(\mathcal{S})_0 \cap (\Gamma - g^{-1}(\mathcal{E}_g)) = g_0^{-1}(t^*(S)),$$

as claimed. Hence

$$g^0(t^*(S)) \leq G^0(\mathcal{S})_0$$
as algebraic cycles on $\Gamma$. Since $G_0^0(\mathcal{S})$ is rationally equivalent to $G_0^0(\mathcal{S})_t$ for any $t$ in $U$, it follows that for all such $t$ we have

$$g^0(t^*(\mathcal{S})) \leq G_0^0(\mathcal{S})_t = g^*(\mathcal{S}_t) = \deg(S)g^*(\omega^p_Y)$$

in $A^p(\Gamma)$. Hence (4.3) is proven.

Now we continue the proof of the lemma. By (4.3) and the bound on degrees (4.1), for all $i = 1, \ldots, e$,

$$g^0(t^*(C_{L_i}(W_{i-1}))) \leq C \deg(W)g^*(\omega^p_Y)$$

in $A^p(\Gamma)$, where $C > 0$ is independent of $W, X$ and $f$. It remains to estimate $g^0(W_e)$. By the choice of $W_e$, the pullback $g^*(W_e)$ is well defined as a subvariety of $\Gamma$, hence by (b) of Lemma 4.3 and the bound on degrees (4.1) we have

$$g^0(W_e) \leq g^*(W_e) \leq C \deg(W)g^*(\omega^p_Y)$$

in $A^p(\Gamma)$, where $C > 0$ is independent of $W, X$ and $f$. Thus the proof is completed. \hfill $\Box$

**Lemma 4.5.** Let $X, Y, Z$ be irreducible smooth projective varieties, and let $f : Y \rightarrow X$ and $g : Z \rightarrow Y$ be dominant correspondences. We fix an embedding $Y \subset \mathbb{P}^M_K$ and let $\omega_Y$ be the pullback to $Y$ of a generic hyperplane in $\mathbb{P}^M_K$. Then in $A^P(Z)$,

$$(f \circ g)^*(\omega^p_X) \leq C \deg(f^*(\omega^p_Y))g^*(\omega^p_Y),$$

where $C > 0$ is independent of $f$ and $g$.

**Proof.** We can find proper subvarieties $V_X \subset X, V_Y \subset Y$ and $V_Z \subset Z$ so that the correspondences $f_0 : Y - V_Y \rightarrow X - V_X$ and $g_0 : Z - V_Z \rightarrow Y - V_Y$ have all fibres of the correct dimensions, by using Theorem 3.1. Then it follows that the composition $Z - V_Z \rightarrow X - V_X$ has all fibres of the correct dimension.

Define by $(f \circ g)_0$ the restriction of $f \circ g$ to $Z - V_Z$. Then it satisfies

$$(f \circ g)_0 = f_0 \circ g_0 : Z - V_Z \rightarrow X - V_X$$

and has all fibres of the correct dimension. We define the strict transforms $f^0, g^0$ and $(f \circ g)^0$ using these restriction correspondences $f_0, g_0$ and $(f \circ g)_0$.

By Lemma 4.3(a), we can find a linear subspace $H^P \subset \mathbb{P}^N_K$ so that $H^P$ intersects $X$ properly. $(f \circ g)^0(t^*(H^P))$ is well defined as a variety and has no component on $V_Z$, and $f^*(t^*(H^P))$ is well defined as a variety. Then

$$(f \circ g)^0(t^*(H^P)) = (f \circ g)^0(t^*(H^P))$$

is the closure of

$$\left(f \circ g\right)^0_0(t^*(H^P)) = (f_0 \circ g_0)^0_0(t^*(H^P)) = (g_0)^0_0(f_0)^0(t^*(H^P))$$

Therefore

$$(f \circ g)^*(t^*(H^P)) = g^0f^0(t^*(H^P)) \leq g^0f^*(t^*(H^P)).$$

as effective algebraic cycles of $Z$. By Lemma 4.4, we have the desired result. \hfill $\Box$
5. Basic properties of relative dynamical degrees

In this section we prove Theorems 1.1, 1.2 and 1.3. We also prove in Theorem 5.8 below a generalisation of parts (2) and (3) in Theorem 1.1. The first two sections contain the proofs of Theorem 1.1 and the log-concavity Lemma 5.7, and we proceed as follows.

- First, we will prove part (1) of Theorem 1.1, that is, to define relative dynamical degrees in the regular case.
- Then we will use de Jong’s alterations and part (1) to define relative dynamical degrees for the general case, and prove other parts of Theorem 1.1.

From now on, we assume that both $X$ and $Y$ are embedded in the same projective space $\mathbb{P}^N$. Let $H_N \subset \mathbb{P}^N$ be a generic hyperplane. We denote by $\omega_X$ the restriction of $H_N$ to $X$ and $\omega_Y$ the restriction of $H_N$ to $Y$.

### 5.1. The case $X$ is smooth and $\pi$ is a regular morphism.

Because $Y$ has dimension $l$, a generic complete intersection of hyperplane sections $\omega^l_Y$ is a union of $\deg(Y)$ points. The preimages $\pi^{-1}(\omega^l_Y) \subset X$ will play an important role in the definition of relative dynamical degrees. The advantage of the choice $\pi^{-1}(\omega^l_Y)$ is that these are rationally equivalent and that $\pi^{-1}(\omega_Y) = (\pi \circ \iota_Y)^*(H_N)$ is a nef class, since $\iota_Y \circ \pi : X \to \mathbb{P}^N$ is a regular morphism.

**Lemma 5.1.** Let $\omega^l_{X,1}$ and $\omega^l_{X,2}$ be two generic complete intersections of hyperplane sections of $X$ (that is, they are varieties of codimension $p$ which are complete intersections of $p$ hyperplane sections), and $\omega^l_{Y,1}$ and $\omega^l_{Y,2}$ two generic complete intersections of hyperplane sections of $Y$. Then the intersections $f^*(\omega^l_{X,1}) \cap \pi^{-1}(\omega^l_{Y,1})$ and $f^*(\omega^l_{X,2}) \cap \pi^{-1}(\omega^l_{Y,2})$ are well defined and have the same class in $N^*(X)$.

**Proof.** Let $\omega^l_{X,1}$ be a fixed complete intersection of hyperplanes sections of $X$ so that $f^{-1}(\omega^l_{X,1})$ has the correct dimension. This is possible by Lemma 4.3.

Let $Z \subset X$ be any proper subvariety of dimension $< \dim(X)$. We show that for a generic choice of $\omega^l_Y$, $\pi^{-1}(\omega^l_Y)$ intersects $Z$ properly. Otherwise, since $\pi^{-1}(\omega^l_Y)$ is a divisor on $X$ and since $Z$ has only a finite number of irreducible components, there will be an irreducible component of $Z$ belonging to all $\pi^{-1}(\omega^l_Y)$—when we vary $\omega^l_Y$—which is impossible.\(^1\) By using this, starting from $Z_0 = f^{-1}(\omega^l_{X,1})$, there is a generic $\omega^l_{Y,1}$ such that the intersection $Z_1 = Z_0 \cap \pi^{-1}(\omega^l_{Y,1})$ is well defined. Keeping doing this $l$ times, we obtain that for a generic choice of hyperplane sections $\omega^l_{Y,1}$, the intersection $f^{-1}(\omega^l_{X,1}) \cap \pi^{-1}(\omega^l_{Y,1})$ is well defined. The class in $N^*(X)$ is then the product of the two classes $[f^{-1}(\omega^l_{X,1})]$ and $[\pi^{-1}(\omega^l_{Y,1})]$, and hence is independent of $\omega^l_{Y,1}$.

By Lemma 5.1, the number

$$\deg_P(f|\pi) := \deg(f^*(\omega^l_X) \cap \pi^{-1}(\omega^l_Y))$$

is well defined and is independent of the choices of generic complete intersections of hyperplane sections $\omega^l_X$ and $\omega^l_Y$.

\(^1\) Here the assumption that $\pi$ is regular is used, in that if $A$ and $B$ are two subsets of $Y$, then one has $\pi^{-1}(A) \cap \pi^{-1}(B) = \pi^{-1}(A \cap B)$.
Our next objective is to show that the following limit exists:

$$\lambda_p(f \mid \pi) = \lim_{n \to \infty} \deg_p(f^n \mid \pi)^{\frac{1}{n}}.$$  

These limits are the relative dynamical degrees we seek to define. To prove the existence of these limits, we will make use of another quantity associated to $f$.

**Notation.** Let $\mathcal{B}_f \subset X$ be a proper subvariety, called a “bad” set for $f$, outside it the map $f$ has good properties needed for our purpose. For example, $\mathcal{B}_f$ may contain the critical and indeterminate set of $f$, but it may also contain the critical and indeterminate sets of some iterates of $f$ and the preimages of some “bad” set $\mathcal{B}_g \subset Y$. These “bad” sets are chosen specifically for each of the results proven below, so that the behaviour of certain iterates of $f$ is nice outside $\mathcal{B}_f$. For example, in the proof of Theorem 5.3 below, we choose $\mathcal{B}_f$ in such a way that the strict transforms $f_0^m \cdot f_0^n$ and $f_0^{m+n}$ are well defined, and $f_0^m \circ f_0^n = f_0^{m+n}$ outside $\mathcal{B}_f$. Since it will be quite cumbersome to list $\mathcal{B}_f$ in each case, we will not do so in the sequel.

Let $Y$ be a subvariety of $X$, each of whose irreducible components does not belong to the “bad” set $\mathcal{B}_f$. Then from the proof of Lemma 5.1, the strict pushforward $f_0(\omega_X^j \cap (V - \mathcal{B}_f))$ is well defined for a generic complete intersection of hyperplane sections $\omega_X^j$. Moreover, the class in $N^*(X)$ of the closure of $f_0(\omega_X^j \cap (V - \mathcal{B}_f))$ is independent of the choice of $\mathcal{B}_f$ and such a generic complete intersection of hyperplane sections $\omega_X^j$. We choose, in particular, $V = \pi^{-1}(\omega_Y^l)$, where $\omega_Y^l$ is a generic complete intersection of hyperplane sections of $Y$. Then the class in $N^*(X)$ of the closure of $f_0(\omega_X^j \cap (\pi^{-1}(\omega_Y^l) - \mathcal{B}_f))$ is independent of the choice of such $\mathcal{B}_f$, $\omega_X^j$ and $\omega_Y^l$. For $j = k - l - p$, we denote the degree of the closure of $f_0(\omega_X^j \cap (\pi^{-1}(\omega_Y^l) - \mathcal{B}_f))$ by $\deg_p(f \mid \pi)$.

**Lemma 5.2.** For all $0 \leq p \leq k - l$, we have $\deg_p(f \mid \pi) = \deg_p'(f \mid \pi)$.

**Proof.** We choose a generic complete intersection of hyperplane sections $\omega_X^{k-l-p}$ of $X$.

By definition,

$$\deg_p(f \mid \pi) = \omega_X^{k-l-p} \cdot Z,$$

where $Z$ is the closure of $f^*(\omega_Y^p) \cdot (\pi^{-1}(\omega_Y^l) - \mathcal{B}_f)$. Since $\omega_X^{k-l-p}$ is a generic complete intersection of hyperplane sections, we have

$$\omega_X^{k-l-p} \cap Z' = \emptyset,$$

where

$$Z' = Z - f^*(\omega_Y^p) \cdot (\pi^{-1}(\omega_Y^l) - \mathcal{B}_f).$$

Moreover, we can choose so that $\omega_X^{k-l-p} \cdot f^*(\omega_Y^p) \cdot (\pi^{-1}(\omega_Y^l) - \mathcal{B}_f)$, which is a 0-cycle, does not intersect any priori given subvariety of $X$.

Let $\Gamma_f$ be the graph of $f$, and let $\pi_1, \pi_2$ be the two projections $X \times X \to X$. We consider the following 0-cycle on $X \times X$:

$$\pi_1^*(\omega_X^{k-l-p} \cdot (\pi^{-1}(\omega_Y^l) - \mathcal{B}_f)) \cdot \pi_2^*(\omega_X^p) \cdot \Gamma_f.$$

Denote by $\alpha$ this 0-cycle. The degree of $(\pi_1)_*(\alpha)$ is $\deg_p(f \mid \pi)$ and the degree of $(\pi_2)_*(\alpha)$ is $\deg_p'(f \mid \pi)$. Then, the lemma follows from the fact that the degrees of the push forwards by $\pi_1$ and $\pi_2$ of a 0-cycle on $X \times X$ are the same. □
Now we define relative dynamical degrees. By the use of Theorem 3.1, we note that Sard–Bertini’s type theorem for characteristic 0 is not needed in the algebraic case (compare with the proof of Proposition 3.3. in [12]).

**Theorem 5.3.** For any $0 \leq p \leq k - l$, the following limit exists:

$$
\lambda_p(f|\pi) := \lim_{n \to \infty} \deg_p(f^n|\pi)^{\frac{1}{n}} = \lim_{n \to \infty} \deg_p(f^n|\pi)^{\frac{1}{n}}.
$$

We call $\lambda_p(f|\pi)$ the $p$-th relative dynamical degree of $f$ with respect to $\pi$.

**Proof.** By Lemma 5.2, we only need to show that the first limit exists. To this end, it suffices to show that there is a constant $C > 0$ independent of $m, n$ and $f$ such that

$$
(5.1) \quad \deg_p(f^{n+m}|\pi) \leq C \deg_p(f^n|\pi), \deg_p(f^m|\pi).
$$

We define $R$ to be the closure of $f_0^n(\omega_X^{k-l-p} \land (\pi^{-1}(\omega_Y^l) - \mathcal{B}))$, where $\mathcal{B}$ is a proper subvariety of $X$, depending on $f^n$, $f^m$ and $f^{n+m}$, so that the strict transform $f_0^m(R)$ is well defined and equal to the closure of the set $f_0^n(\omega_X^{k-l-p} \land (\pi^{-1}(\omega_Y^l) - \mathcal{B}))$. By choosing $\omega_X^{k-l-p}$ and $\omega_Y^l$ generically, we can assume that $R$ is an effective algebraic cycle of pure codimension $k - l - p$ of $\pi^{-1}(g_X^m(\omega_Y^l))$ and has proper intersections with the “bad” set $\mathcal{B}$, using the property $\pi \circ f = g \circ \pi$. We note that by definition the degree of $R$ is $\deg_p(f^n|\pi).

We observe that in $N^*(X)$,

$$
(5.2) \quad R \leq C \deg(R)\omega_X^{k-l-p} \land \pi^{-1}(g_X^m(\omega_Y^l))
$$

where $C > 0$ is a constant independent of $R, m, n$ and the maps $\pi, f$ and $g$. In fact, consider the embedding $\pi^{-1}(g_X^m(\omega_Y^l)) \subset \mathbb{P}^N$ induced from the embedding $X \subset \mathbb{P}^N$. Regardless whether $\pi^{-1}(g_X^m(\omega_Y^l))$ is smooth or not, it has the correct dimension $k - l$. We then can use Lemma 2.2 to have that if $L \subset \mathbb{P}^N$ is any linear subspace of dimension $n - (k - l) - 1$ and $C_L(R)$ is the cone over $R$ with vertex $L$, then $C_L(R) \cap \pi^{-1}(g_X^m(\omega_Y^l))$ has the correct dimension $\dim(R)$. In particular, since $R$ is one component of $C_L(R) \cap \pi^{-1}(g_X^m(\omega_Y^l))$ and the degree of $C_L(R)$ is bounded from above by $C \deg(R)$, we obtain equation (5.2).

Using the same arguments as in the proof of Lemma 4.4, we can show moreover that

$$
f_0^m(R) \leq C \deg(R)f_0^m(\omega_X^{k-l-p} \land \pi^{-1}(g_X^m(\omega_Y^l))
$$
in $N^*(X)$. This completes the proof of Theorem 5.3, by noting that $\pi^{-1}(g_X^m(\omega_Y^l))$ is contained in at most $\deg(Y)$ fibres of the form $\pi^{-1}(\omega_Y^l)$.

Hence, part (1) of Theorem 1.1 is proven, except we did not show that the limit is independent of the choice of the embeddings $\iota_X$ and $\iota_Y$. But this is clear since $\omega_X$ is ample and $\omega_Y^l$ has the same class in $N^*(Y)$ as a positive multiple of the class of a point, and relative dynamical degrees can be computed on numerical classes. That these numbers are at least 1 follows easily from the fact that they are limits of the $n$-th roots of degrees of non-trivial subvarieties of $X$, and the latter are positive integers.

**5.2. The general case.** In this subsection we prove Theorem 1.1 in the general case. Hence, we do not assume that $X$ or $Y$ is smooth or $\pi$ is regular. We only require that $X$ and $Y$ are irreducible. We need to make use of the following special case of Theorem 5.8.
Lemma 5.4. Let $X, Z$ be irreducible smooth projective varieties and $Y$ an (not necessarily smooth) irreducible projective variety. Assume that $\dim(X) = \dim(Z)$, and there are two semi-conjugacies $\varphi : (Z, h) \to (X, f)$ and $\pi : (X, f) \to (Y, g)$, where both $\varphi$ and $\pi$ are dominant regular morphisms. Then the relative dynamical degrees (as defined in the previous subsection) of $\pi : (X, f) \to (Y, g)$ and $\varphi \circ \varphi : (Z, h) \to (Y, g)$ are the same.

Proof. Let $U \subset X$ be a non-empty Zariski-open set such that all fibres $\varphi^{-1}(x)$ are finite. Define $U' = \varphi^{-1}(U)$. Let $(f^n)_0$ and $(h^n)_0$ be the strict pushforward with respect to $U$ and $\varphi^{-1}(U)$. Choose $\omega^*_Z, \omega^*_Y$, and $\omega^*_Y$ to be generic complete intersections of hyperplane sections. Then the relative dynamical degrees (as defined in the previous subsection) of $\pi : (X, f) \to (Y, g)$ and $\varphi \circ \varphi : (Z, h) \to (Y, g)$ are the same.

Fix $n \in \mathbb{N}$. We now estimate the number of points in $\omega^*_Z, \omega^*_Y$, and $\omega^*_Y$.

Define $\varphi' = \varphi|_{U'}$. Let $a$ be the degree of $\varphi$. Since $\omega^*_Z|_{U'} \cap (h^n)|_{U'}(\varphi^* \varphi^*(\omega^*_Y) \cap \omega^*_Z)$ is a finite number of points, and $\varphi : U' \to U$ is a finite map, it follows that the wedge intersection $\varphi'_*(\omega^*_Z)|_{U'} \cap \varphi'_*(h^n)|_{U'}(\varphi^* \varphi^*(\omega^*_Y) \cap \omega^*_Z)$ is well defined. Moreover, since the degree of $\varphi$ is $a$, for any $x \in U$ there are at most $a$ points in $\varphi^{-1}(x)$. Thus, we obtain

$$(\omega^*_Z, (h^n)_* \varphi^* \varphi^*(\omega^*_Y) \cap \omega^*_Z) = (f^n)|_{U'} \varphi'_*(\omega^*_Y) \cap \omega^*_Z.$$

Since

$$\varphi \circ h = f \circ \varphi,$$

we can assume also that

$$\varphi'_*(h^n)|_{U'}(\varphi^* \varphi^*(\omega^*_Y) \cap \omega^*_Z) = (f^n)|_{U'} \varphi'_*(\omega^*_Y) \cap \omega^*_Z.$$

We then obtain

$$(\omega^*_Z, (h^n)_* \varphi^* \varphi^*(\omega^*_Y) \cap \omega^*_Z) \leq a \varphi'_*(\omega^*_Z) \cap \varphi'_*(h^n)|_{U'}(\varphi^* \varphi^*(\omega^*_Y) \cap \omega^*_Z).$$

We can choose $U$ such that

$$(f^n)|_{U'}(\varphi^* \varphi^*(\omega^*_Y) \cap \omega^*_Z) = (f^n)|_{U'}(\varphi^* \varphi^*(\omega^*_Y) \cap \omega^*_Z).$$

Using Lemma 2.2, as in the proof of Lemma 4.1, we can find a cone $\mathcal{C}_L(\varphi'_*(\omega^*_Z))$ which intersects $(f^n)|_{U'}(\varphi^* \varphi^*(\omega^*_Y) \cap \omega^*_Z)$ properly on $U$, and by choosing a 1-family of automorphisms $g(t)$ of $\mathbb{P}_N$, parameterised by a rational curve $D$, we have that $g(t)\mathcal{C}_L(\varphi'_*(\omega^*_Z))|_X$ intersects $(f^n)|_{U'}(\varphi^* \varphi^*(\omega^*_Y) \cap \omega^*_Z)$ properly, for generic $t \in D \subset \mathbb{P}^1$. From this we obtain, where from the second line computations are in $N^*(X)$ and we use that the class...
of $g(t)C_L(\varphi'_* (\omega^P_Y)|X$ is a constant multiple (depending only on $\varphi$ and $\omega_Z$) of $\omega^P_X$:

$$\varphi'_* (\omega^P_Y) \cap (f^n)[U[\pi^*(\omega^L_Y) \cap \varphi'_* (\omega^k_{Z-l-p})]$$

$$\leq g(t)C_L(\varphi'_* (\omega^P_Y)|X \cdot (f^n)*[\pi^*(\omega^L_Y) \cap \varphi'_* (\omega^k_{Z-l-p})]$$

$$= ([g(t)C_L(\varphi'_* (\omega^P_Y)|X], (f^n)*[\pi^*(\omega^L_Y) \cap \varphi'_* (\omega^k_{Z-l-p})])$$

$$= \langle \omega^P_Y, (f^n)*[\pi^*(\omega^L_Y) \cap \varphi'_* (\omega^k_{Z-l-p})]$$

$$= \langle (f^n)*[\pi^*(\omega^L_Y) \cap \varphi'_* (\omega^k_{Z-l-p})]$$

$$\leq A([[(f^n)*[\pi^*(\omega^L_Y)], [\omega^k_{X-l-p}] \times \{\varphi_0 \omega^k_{X-l-p}, [\omega^k_{X}]\}]$$

$$\leq A'([[(f^n)*[\pi^*(\omega^L_Y) \cap \pi^*(\omega^k_{X})], [\omega^k_{X-l-p}] = A' \deg(f^n)[\pi]).$$

In the fourth line, we used the proof of Lemma 5.1 to represent the class $[(f^n)*[\omega_Y^P] \cap \pi^*(\omega^L_Y)]$ by the class of an effective algebraic cycle of pure dimension $(f^n)*[\omega_Y^P] \cap \pi^*(\omega^L_Y)$. The fifth line follows from the fourth line by applying Lemma 2.4. From the above, we obtain $\lambda_p(h|\pi \circ \varphi) \leq \lambda_p(f|\pi)$. The reverse inequality can be proven similarly (and easier).

We proceed to defining relative dynamical degrees in this general case. Let $\tau : X' \to X$ be an alteration, where $\tau$ is generically finite and so that $\pi \circ \tau$ is a regular morphism. Let $\tau^* f$ be the pullback of the correspondence $f$ by $\tau$, as defined in Section 3. We then define, using the regular case considered in the previous subsection,

$$\lambda_p(f|\pi) := \frac{1}{\deg(\tau)} \lambda_p(\tau f|\tau \circ \pi).$$

To show that this is well defined, we need to show the following. Let $\tau_1 : X_1 \to X$ and $\tau_2 : X_2 \to X$ be two alterations so that $\pi \circ \tau_1$ and $\pi \circ \tau_2$ are regular. Then

$$\frac{1}{\deg(\tau_1)} \lambda_p(\tau^*_1 f|\tau_1 \circ \pi) = \frac{1}{\deg(\tau_2)} \lambda_p(\tau^*_2 f|\tau_2 \circ \pi).$$

To this end, we first observe that there is another alteration $\tau_3 : X_3 \to X$ together with two generically finite morphisms $\varphi_1 : X_3 \to X_1, \varphi_2 : X_3 \to X_2$ such that $\tau_1 \circ \varphi_1 = \tau_3 = \tau_2 \circ \varphi_2$. In fact, let $\Gamma \subset X_1 \times X_2$ be an irreducible component of the correct dimension of the set $\{(x_1, x_2) \in X_1 \times X_2 : \tau_1(x_1) = \tau_2(x_2)\}$ and which is dominant over $X_1$ and $X_2$. Then any alteration $X_3$ of $\Gamma$ will satisfy the requirement.

Since $\tau_3 = \tau_1 \circ \varphi_1 = \tau_2 \circ \varphi_2$, we have $\varphi^*_1 (\tau^*_1 f) = \varphi^*_2 (\tau^*_1 f)$. By Lemma 3.4, we have semi-conjugacies

$$(X_3, \tau^*_3 f) \to (X_1, \deg(\varphi_1)\tau^*_1 f) \quad \text{and} \quad (X_3, \tau^*_3 f) \to (X_2, \deg(\varphi_2)\tau^*_2 f).$$

Writing $\pi_i = \pi \circ \tau_i$ for $i = 1, 2, 3$, by Lemma 3.4, we have the equality

$$\deg(\varphi_1) \lambda_p(\tau^*_1 f|\pi_1) = \lambda_p(\tau^*_1 f|\pi_3) = \deg(\varphi_2) \lambda_p(\tau^*_2 f|\pi_2).$$

which, together with $\deg(\varphi_1) \deg(\tau_1) = \deg(\tau_3) = \deg(\varphi_2) \deg(\tau_2)$, gives the desired one

$$\frac{1}{\deg(\tau_1)} \lambda_p(\tau^*_1 f|\pi_1) = \frac{1}{\deg(\tau_3)} \lambda_p(\tau^*_3 f|\pi_3) = \frac{1}{\deg(\tau_2)} \lambda_p(\tau^*_2 f|\pi_2).$$

Thus the relative dynamical degrees in this case are well defined. Parts (2), (3) and (4) of Theorem 1.1 follow from the following lemma.
Lemma 5.5. The following statements hold.

(a) Let two semi-conjugacies \( \pi_1 : (X_1, f_1) \to (Y_1, g_1) \), \( \pi_2 : (X_2, f_2) \to (Y_2, g_2) \) together with two semi-conjugacies \( \varphi : (X_2, f_2) \to (X_1, f_1) \), \( \psi : (Y_2, g_2) \to (Y_1, g_1) \) be given such that the two composition maps \( \psi \circ \pi_2 \) and \( \pi_1 \circ \varphi \) are the same, and assume that \( \psi : Y_2 \to Y_1 \) is a generically finite rational map, and moreover that \( \varphi \) is birational. Then \( \lambda_p(f_1|\pi_1) = \lambda_p(f_2|\pi_2) \) for all \( p = 0, \ldots, \dim(X_1) - \dim(Y_1) \).

(b) Let \( \tau : X' \to X \) be a dominant rational map with generically finite fibres. Then
\[
\lambda_p(\pi^*(f)|\tau \circ \pi) = \deg(\tau)\lambda_p(f|\pi)
\]
for all \( p \).

(c) Relative dynamical degrees are uniquely determined by (1) and (3) of Theorem 1.1.

Proof. (a) Since \( \varphi : X_2 \to X_1 \) is a birational map, it can easily be checked that
\[
f_2 = \varphi^*(f_1).
\]
Let \( \tau_2 : Z \to X_2 \) be an alteration (so \( Z \) is smooth and \( \tau_2 \) is a generically finite regular morphism) so that the three composition maps \( \tau_1 = \varphi \circ \tau_2 : Z \to X_1 \), \( \pi'_2 = \pi_2 \circ \tau_2 : Z \to Y_2 \) and \( \pi'_1 = \pi_1 \circ \tau_1 : Z \to Y_1 \) are all regular. Let \( h : Z \to Z \) be the pullback of \( f_2 \) by \( \tau_2 \), that is, \( h = \tau_2^*f_2 \). Then from \( \tau_1 = \varphi \circ \tau_2 \) and \( f_2 = \varphi^*(f_1) \), we have \( \tau_1^*(f_1) = \tau_1^*\varphi^*(f_1) = \tau_1^*(f_2) \).

Therefore, from \( \deg(\tau_1) = \deg(\tau_2) \) since \( \pi \) is birational, by definition we obtain
\[
\lambda_p(f_1|\pi_1) = \frac{1}{\deg(\tau_1)}\lambda_p(\tau_1^*f_1|\pi'_1) = \frac{1}{\deg(\tau_2)}\lambda_p(\tau_2^*f_2|\pi'_2) = \lambda_p(f_2|\pi_2).
\]

Here the second equality between \( \lambda_p(\tau_1^*f_1|\pi'_1) \) and \( \lambda_p(\tau_2^*f_2|\pi'_2) \) remains to be explained, since the spaces involved are different. Define \( f := \tau_1^*(f_1) : Z \to Z \), which is also the same as \( \tau^*(f_2) \). From the assumption of the lemma, we have \( \pi'_1 = \varphi \circ \pi'_2 \). Therefore, we obtain \( (\pi'_1)^*(\omega_{Y_1}^1) = (\pi'_2)^*(\varphi^*(\omega_{Y_2}^2)) \). By using Theorem 1.1 (1), which says that relative dynamical degrees can be computed using numerical classes, and the fact that in \( N^*(Y_2) \) the two classes of \( \varphi^*(\omega_{Y_1}^1) \) and \( \omega_{Y_2}^2 \) are positive multiples of each other, the desired equality follows.

(b) Let \( \tau_1 : X_1 \to X' \) be an alteration, where \( X_1 \) is a smooth projective variety and so that \( \tau_1' = \tau \circ \tau_1 \) and \( \pi'_1 = \pi \circ \tau'_1 \) are regular morphisms. By the definition of relative dynamical degrees, and the facts that \( (\tau_1')^*(f) = \tau_1^*(\tau^*(f)) \) and \( \deg(\tau_1') = \deg(\tau_1) \), we have
\[
\deg(\tau)\lambda_p(f|\pi) = \frac{1}{\deg(\tau_1')}\lambda_p((\tau_1')^*(f)|\pi_1') = \frac{1}{\deg(\tau_1')}\lambda_p((\tau_1')^*(\tau^*(f))((\pi \circ \tau) \circ \tau_1)) = \lambda_p(\tau^*(f)|\pi \circ \tau).
\]

(c) We need to show that if \( \lambda_p(f|\pi) \) are real numbers satisfying parts (1) and (3) of Theorem 1.1, then \( \lambda_p(f|\pi) = \lambda_p(f|\pi) \). Let \( \tau : X' \to X \) be an alteration, where \( X' \) is a smooth projective variety and \( \pi \circ \tau \) is a regular morphism. By part (3) of Theorem 1.1 we have \( \deg(\tau)\lambda_p(f|\pi) = \lambda_p(\tau^*(f)(\pi \circ \tau)) \), and the right-hand side is – by part (1) of Theorem 1.1 – the same as \( \lambda_p(\tau^*(f)|\pi \circ \tau) \), which in turn is the same as \( \deg(\tau)\lambda_p(f|\pi) \) by (5.3). Thus \( \lambda_p(f|\pi) = \lambda_p(f|\pi) \) is uniquely defined by (5.3).
In the next lemma we finish the proof of Theorem 1.1 by proving its part (5).

**Lemma 5.6.** For every $p \in \{0, \ldots, \dim(X) - \dim(Y)\}$ we have $\lambda_p(f|\pi) \geq 1$. Moreover, for every $m \in \mathbb{N}$, we have $\lambda_p(f^m|\pi) = \lambda_p(f|\pi)^m$.

**Proof.** Let $\tau : X' \rightarrow X$ be an alteration, where $\tau$ is generically finite and so that the composition $\pi' = \pi \circ \tau$ is a regular morphism as before. Let $d$ be the degree of $\tau$. Let $m \in \mathbb{N}$ and $p \in \{0, \ldots, \dim(X) - \dim(Y)\}$. Let $g = \tau^*(f)$ and $g_m = \tau^*(f^m)$. From the definition of the pullback of correspondences, it can easily be checked that $g^m = d^{m-1}g_m$ for all $m$. In fact, this follows by induction on $m$ if we can show that for any two dominant correspondences $f_1, f_2 : X \rightarrow Y$, then $(\tau^* f_1) \circ (\tau^* f_2) = d \tau^*(f_1 \circ f_2)$. To this end, from Case 1 in Section 3.2 and the fact that $\tau \circ \tau^{-1} = \text{deg}(\tau) \text{Id}$, for a generic point $x' \in X'$ we have as wanted:

$$
\tau^*(f_1) \circ \tau^*(f_2)(x') = [\tau^{-1}(f_1) \circ \tau^{-1}f_2 \tau(x')] = \tau^{-1}f_1[\tau \circ \tau^{-1}]f_2 \tau(x') = d[\tau^{-1}(f_1 \circ f_2) \tau(x')] = d \tau^*(f_1 \circ f_2)(x').
$$

Therefore, since $X'$ is smooth and $\pi'$ is a regular morphism as chosen, we have by part (1) of Theorem 1.1 that $\lambda_p(g^m|\pi') = d^{m-1}\lambda_p(g_m|\pi')$ for all $m \in \mathbb{N}$. From this and by the definition of relative dynamical degrees, we have

$$
\lambda_p(f|\pi)^m = \left[\frac{1}{d} \lambda_p(g|\pi') \right]^m = \frac{1}{d^m} \lambda_p(g_m|\pi') = \lambda_p(f^m|\pi)
$$

for all $m$. It remains to show that $\lambda_p(f|\pi) \geq 1$ for all $p \in \{0, \ldots, \dim(X) - \dim(Y)\}$. From part (1) of Theorem 1.1, it is obvious to see that $\lambda_p(g_m|\pi') \geq 1$ for all $m \in \mathbb{N}$. It follows that $\lambda_p(f^m|\pi) \geq \frac{1}{d}$ for all $m$. From what we have just proven, we have

$$
\lambda_p(f|\pi) = \lim_{m \to \infty} \left[\lambda_p(f^m|\pi)\right]^\frac{1}{m} \geq \lim_{m \to \infty} \left[\frac{1}{d}\right]^\frac{1}{m} = 1,
$$

as desired. \hfill \Box

Now we proceed to proving the log-concavity of relative dynamical degrees. We note that in the case the underlying field is $\mathbb{C}$, our proof here is different from and simpler than that used in [12, Proposition 3.6].

**Lemma 5.7.** Assume that there is an infinite sequence $n_i \in \mathbb{N}$ for which the graph of $f^{n_i}$ is irreducible for all $i$. Then the relative dynamical degrees are log-concave, that is,

$$
\lambda_{p+1}(f|\pi)\lambda_{p-1}(f|\pi) \leq \lambda_p(f|\pi)^2
$$

for all $1 \leq p \leq k - l - 1$.

**Proof.** If $X$ is smooth and the semi-conjugacy $\pi$ is regular, then the result follows by applying the Grothendieck–Hodge index theorem to an alternation of $\Gamma_{f^{n_i}}$, using part (1) of Theorem 1.1. In the general case we proceed as follows.

Let $\tau_1 : X_1 \rightarrow X$ be an alteration of degree $d_1$ so that $X_1$ is smooth and the composition $\pi_1 = \pi \circ \tau_1 : X_1 \rightarrow Y$ is a regular morphism. Define $f_1 = \tau_1^* f : X_1 \rightarrow X_1$. By definition, we have

$$
\lambda_p(f|\pi) = \frac{1}{d_1} \lambda_p(f_1|\pi_1)
$$
From the proof of Lemma 5.1, it can easily be shown that

\[ \lambda_p(f_1|\pi_1) = \lim_{n \to \infty} \langle (f_1^n)^* \tau_1^* (\omega_X^p) \wedge \pi_1^* (\omega_Y^p), \tau_1^* (\omega_X^{k-l-p}) \rangle^{\frac{1}{p}}. \]

Choose \( n_i \) to be a positive integer for which the graph \( \Gamma_{f^{n_i}} \) is irreducible. Let \( \Gamma \to \Gamma_{f^{n_i}} \) be an alteration of degree \( d \) of the graph of \( f^{n_i} \), and let \( p_1, p_2 : \Gamma \to X \) be the two projections. We write

\[ \Gamma_{f^{n_i}} = \sum_{q \in I_{n_i}} G_q \]

for some finite set \( I_{n_i} \), where \( G_q \) are irreducible components. For each \( q \in I_{n_i} \), there is an alteration \( \widetilde{G}_q \to G_q \) of degree \( a_q \) which is equipped with a regular morphism \( \alpha_q : \widetilde{G}_q \to \Gamma \). Denote by \( p_{1,q}, p_{2,q} : \widetilde{G}_q \to X_1 \) the natural projections. By definition,

\[ \langle (f_1^n)^* \tau_1^* (\omega_X^p) \wedge \pi_1^* (\omega_Y^p), \tau_1^* (\omega_X^{k-l-p}) \rangle = \sum_{q \in I_{n_i}} \frac{1}{a_q} p_{1,q}^* \pi_1^* (\omega_Y^p) \cdot p_{1,q}^* \tau_1^* (\omega_X^{k-l-p}) \cdot p_{2,q}^* \tau_1^* (\omega_X^p). \]

We now show that each summand on the right-hand side of the above equality is a positive multiple of \( p_{1,q}^* \pi_1^* (\omega_Y^p) \cdot p_{1,q}^* (\omega_X^{k-l-p}) \cdot p_{2,q}^* (\omega_X^p) \). Moreover, the multiples depend only on the degrees of the morphisms \( G_q \to \Gamma \), and are independent of \( p = \{0, \ldots, k-l\} \). This can be seen as follows. First, since all the maps \( p_{1,q}, \tau_1, p_1, \alpha_q \) are regular, \( \tau_1 \circ p_{1,q} = p_1 \circ \alpha_q \), and \( \omega_X \) is the pullback of a hyperplane divisor of some embedding \( X \subset \mathbb{P}^N \), we get

\[ p_{1,q}^* \tau_1^* (\omega_X^{k-l-p}) = (\tau_1 \circ p_{1,q})^* (\omega_X^{k-l-p}) = (p_1 \circ \alpha_q)^* (\omega_X^{k-l-p}) = \alpha_q^* p_{1,q}^* (\omega_X^{k-l-p}). \]

Similarly, we have

\[ p_{2,q}^* \tau_1^* (\omega_X^p) = \alpha_q^* p_{2,q}^* (\omega_X^p). \]

Finally, while \( \pi_1 \circ p_{1,q} = \pi \circ p_1 \circ \alpha_q \) only as rational maps (since we do not assume that \( \pi \) is regular), since the class of \( \omega_Y^p \) is a union of generic points in \( Y \), we still have as above

\[ p_{1,q}^* \pi_1^* (\omega_Y^p) = \alpha_q^* \pi_1^* (\omega_Y^p). \]

Putting all of these together, and using that \( \alpha_q \) is regular, we obtain

\[ p_{1,q}^* \pi_1^* (\omega_Y^p) \cdot p_{1,q}^* \tau_1^* (\omega_X^{k-l-p}) \cdot p_{2,q}^* \tau_1^* (\omega_X^p) = \deg(\alpha_q) p_{1,q}^* \pi_1^* (\omega_Y^p) \cdot p_{1,q}^* \tau_1^* (\omega_X^{k-l-p}) \cdot p_{2,q}^* (\omega_X^p). \]

Since the last number is an intersection of nef classes on the irreducible smooth variety \( \Gamma \), we can approximate them by \( \mathbb{Q} \)-ample classes, and use the usual Grothendieck–Hodge index theorem to prove the log-concavity. Here we recall briefly this well-known argument. By the Grothendieck–Hodge index theorem we have, for \( k = \dim(X) = \dim(\Gamma) \),

\[ [\alpha \cdot \alpha \cdot c_1 \cdot c_2 \cdots c_{k-2}] \times [\beta \cdot \beta \cdot c_1 \cdot c_2 \cdots c_{k-2}] \leq [\alpha \cdot \beta \cdot c_1 \cdot c_2 \cdots c_{k-2}]^2, \]

for all nef divisors \( \alpha, \beta, c_1, \ldots, c_{k-2} \) on the irreducible smooth variety \( \Gamma \). Applying this with \( \alpha = p_{2,q}^* (\omega_X^p) \), \( \beta = p_{1,q}^* (\omega_Y^p) \), \( c_1 = \cdots = c_l = p_{1,q}^* \pi_1^* (\omega_Y^p) \), \( c_{l+1} = \cdots = c_{k-p-1} = p_{1,q}^* (\omega_X^p) \), and \( c_{k-p} = \cdots = c_{k-2} = p_{2,q}^* (\omega_X^p) \), taking the \( n_i \)-th root on both sides and taking limit when \( n_i \to \infty \), we easily obtain the log-concavity. \( \square \)
Theorem 5.8. Let be given two semi-conjugacies
\[ \pi_1 : (X_1, f_1) \rightarrow (Y_1, g_1) \quad \text{and} \quad \pi_2 : (X_2, f_2) \rightarrow (Y_2, g_2) \]

Together with two semi-conjugacies \( \varphi : (X_2, f_2) \rightarrow (X_1, f_1) \) and \( \psi : (Y_2, g_2) \rightarrow (Y_1, g_1) \) such that the two composition maps \( \psi \circ \pi_2 \) and \( \pi_1 \circ \varphi \) are the same. Assume that \( \varphi : X_2 \rightarrow X_1 \) and \( \psi : Y_2 \rightarrow Y_1 \) are generically finite rational maps.

1. We have \( \lambda_p(f_1|\pi_1) \geq \lambda_p(f_2|\pi_2) \) for all \( p = 0, \ldots, \dim(X_1) - \dim(Y_1) \).
2. Assume moreover that at least one of the following conditions is satisfied:
   a) \( X_1 \) is smooth and \( \pi_1 \) is regular,
   b) resolution of singularities is available for varieties of dimension up to \( \dim(X_1) \).

Then \( \lambda_p(f_2|\pi_2) = \lambda_p(f_1|\pi_1) \) for all \( p = 0, \ldots, \dim(X_1) - \dim(Y_1) \).

Proof. Let \( \tau_2 : Z \rightarrow X_2 \) be an alteration (so \( Z \) is smooth and \( \tau_2 \) is a generically finite regular morphism) so that the composition maps \( \tau_1 = \varphi \circ \tau_2 : Z \rightarrow X_1, \pi_1' = \pi_2 \circ \tau_2 : Z \rightarrow Y_2 \) and \( \pi_1' = \pi_1 \circ \tau_1 : Z \rightarrow Y_1 \) are all regular.

1. Let us define two correspondences over \( Z \): \( h_1 = \tau_1^*(f_1) \) and \( h_2 = \tau_2^*(f_2) \). We have two semi-conjugacies \( (Z, h_1) \rightarrow (X_1, \deg(\tau_1) f_1) \) and \( (Z, h_2) \rightarrow (X_2, \deg(\tau_2) f_2) \). By the last part of the proof of Lemma 5.5, we have that \( \lambda_p(h_1|\pi_1') = \lambda_p(h_1|\pi_2') \) for all \( p \). Hence, by definition and the fact that \( \deg(\tau_1) = \deg(\tau_2) \deg(\varphi) \) since \( \tau_1 = \varphi \tau_2 \), we obtain

\[
\lambda_p(f_1|\pi_1) = \frac{1}{\deg(\tau_1)} \lambda_p(h_1|\pi_1') = \frac{1}{\deg(\tau_1) \deg(\varphi)} \lambda_p(h_1|\pi_2'),
\]

\[
\lambda_p(f_2|\pi_2) = \frac{1}{\deg(\tau_2)} \lambda_p(h_2|\pi_2').
\]

First we show that \( \Gamma_{h_1} - \Gamma_{h_2} \) is an effective algebraic cycle, that is, all irreducible components of \( \Gamma_{h_2} \) are contained in \( \Gamma_{h_1} \) and the multiplicity of that component in \( \Gamma_{h_2} \) is bounded from above by the corresponding in \( \Gamma_{h_1} \). To this end, by defining \( f_2' = \varphi_2^*(f_1) : X_2 \rightarrow X_2 \), we observe that \( h_1 = \tau_1^* \varphi_2^*(f_1) = \tau_2^*(f_2') \), therefore to show that \( \Gamma_{h_2} - \Gamma_{h_2} \) is an effective algebraic cycle it suffices to show that \( \Gamma_{f_2'} - \Gamma_{f_2} \) is effective. This follows easily from the fact that for a generic \( x_2 \in X_2 \),

\[
\varphi f_2(x_2) = f_1 \varphi(x_2),
\]

\[
f_2'(x_2) = \varphi^{-1} f_1 \varphi(x_2).
\]

From this, it readily follows that \( \Gamma_{h_2} - \Gamma_{h_2} \) is an effective algebraic cycle for all \( n \in \mathbb{N} \). Therefore, we obtain by the definition of relative dynamical degrees that \( \lambda_p(h_1|\pi_2') \geq \lambda_p(h_2|\pi_2') \) for all \( p \). The above consideration then implies that \( \deg(\varphi) \lambda_p(f_1|\pi_1) \geq \lambda_p(f_2|\pi_2) \) for all \( p \).

Now we finish the proof of part (1). Applying the above inequality to \( f_1^n \) and \( f_2^n \) instead of \( f_1 \) and \( f_2 \), we obtain the estimate \( \deg(\varphi) \lambda_p(f_1^n|\pi_1) \geq \lambda_p(f_2^n|\pi_2) \) for all \( n \in \mathbb{N} \). From this and the fact (following from part (1) of Theorem 1.1) that \( \lambda_p(f_1^n|\pi_1) = \lambda_p(f_1|\pi_1)^n \) and \( \lambda_p(f_2^n|\pi_2) = \lambda_p(f_2|\pi_2)^n \) for all \( n \in \mathbb{N} \), we have \( \lambda_p(h_1|\pi_1^n) \geq \lambda_p(h_2|\pi_2^n) \) for all \( p \).

2. Assume that condition (a) is satisfied, that is, \( X_1 \) is smooth and \( \pi_1 \) is regular. We consider the correspondence \( h = \tau_2^*(f_2) \). By Lemma 3.4, we have a semi-conjugacy

\[
(Z, h = \tau_2^*(f_2)) \rightarrow (X_2, \deg(\tau_2) f_2),
\]

which together with the given semi-conjugacy \( (X_2, f_2) \rightarrow (X_1, f_1) \) gives also a semi-conjugacy \( (Z, h) \rightarrow (X_1, \deg(\tau_2) f_1) \). By definition, we have \( \lambda_p(h|\pi_2') = \deg(\tau_2) \lambda_p(f_2|\pi_2) \). Under
condition (a), Lemma 5.4 can be applied, and we obtain \( \lambda_p(h|\pi'_1) = \deg(\tau_2)\lambda_p(f_1|\pi_1) \). The argument at the end of the proof of Lemma 5.5 helps to show that \( \lambda_p(h|\pi'_1) = \lambda_p(h|\pi'_2) \), and the latter gives that
\[
\lambda_p(f_1|\pi_1) = \lambda_p(f_2|\pi_2),
\]
as wanted.

Assume that condition (b) is satisfied. Then we can find a resolution \( \tau : X'_1 \to X_1 \) such that \( X'_1 \) is smooth and the induced map \( \pi_1 \circ \tau : X'_1 \to Y_1 \) is regular. Since \( \tau \) is birational, the semi-conjugacy \( (X_2, f_2) \to (X_1, f_1) \) lifts to a semi-conjugacy
\[
(X_2, f_2) \to (X'_1, f'_1 = \tau^*(f_1)).
\]
By Lemma 5.5, \( \lambda_p(f'_1|\pi_1 \circ \tau) = \lambda_p(f_1|\pi) \). Hence this case is reduced to the previous case, and the proof is completed. \( \square \)

5.3. Proof of Theorem 1.2. By using pullback to an alteration \( X_1 \) of \( X \) if necessary, we may assume that \( X \) is smooth. Since \( f_1, f_2 : X \to Y \) are commutative, we have
\[
(f_1 + f_2)^n = \sum_{i=0}^{n} C(n, i) f_1^i f_2^{n-i},
\]
where \( C(n, i) \) are the binomial coefficients. Similar to the proof of (5.1) in the proof of Theorem 5.3, by using \( f_1^i, f_2^{n-i} \) and their composition \( f_1^i \circ f_2^{n-i} \) in the place of \( f^n, f^m \) and their composition \( f^{n+m} \), we find that there is a constant \( A > 0 \) depending only on \( X \) and \( Y \) and the map \( \pi \) such that in the notation of Section 5.1 we have
\[
\deg_p((f_1 + f_2)^n) \leq A \sum_{i=0}^{n} C(n, i) \deg_p(f_1^i) \deg_p(f_2^{n-i}).
\]
This, together with part (1) of Theorem 1.1, concludes the proof of Theorem 1.2.

5.4. Proof of Theorem 1.3. (1) We observe that it is enough to prove [12, Proposition 4.5]. Checking the proof of that proposition carefully, we see that it suffices to show the following: for every \( a > \lambda_0(g) \),
\[
\alpha^n ((f^n)^*(\omega_{\mathcal{Y}}^j \wedge \omega_X^{p-j}), \pi^*(\omega_{\mathcal{X}}^{l-j} \wedge \omega_Y^{k-p+q+j}) \geq \pi^*((g^n)^*(\omega_{\mathcal{Y}}^j \wedge \omega_Y^{l-j} \wedge (f^n)^*(\omega_X^{p-j} \wedge \omega_X^{k-l-p+j}),
\]
provided that \( n \) is large enough and \( g^n \) is irreducible.

To this end, we follow the proof of Lemma 5.9. Let \( n \) be an integer for which \( g^n \) is irreducible, and write \( g^n = a_n G_n \), where \( G_n \subset Y \times Y \) is irreducible. Let \( \Gamma f_n = \Gamma_i \), where \( \Gamma_i \) are irreducible varieties. Let \( \Gamma \) be one among the varieties \( \Gamma_i \). Since \( \pi : (X, f) \to (Y, g) \) is a semi-conjugacy, the image of \( \Gamma \) under the regular morphism \( \pi \times \pi : X \times X \to Y \times Y \) must be \( G_n \). We let \( \tilde{G}_n \to G_n \) be an alteration of degree say \( d' \), and let \( p_1, p_2 : \tilde{G}_n \to Y \) be the two natural projections. We can choose an alteration \( \tilde{\Gamma} \to \Gamma \) of degree say \( d' \) which is equipped with a regular surjective morphism \( \gamma : \tilde{\Gamma} \to \tilde{G}_n \) so that the two compositions \( \tilde{\Gamma} \to \Gamma \) \( \to G_n \) and \( \tilde{\Gamma} \to \tilde{G}_n \to G_n \) coincide. Let \( p'_1, p'_2 : \tilde{\Gamma} \to X \) be the two natural projections. Let \( \beta \) be
the degree of the morphism $G_n \to Y$. Then
\[
\langle \Gamma^* (\pi^* (\omega_Y^l) \wedge \omega_X^{k-l+p+j}), \pi^* (\omega_Y^l) \wedge \omega_X^{k-l-p-j} \rangle \\
= \frac{1}{d'} (p_1')^* \pi^* (\omega_Y^l) \wedge (p_1')^* (\omega_X^{k-l+p+j}) \wedge (p_2')^* \pi^* (\omega_Y^l) \wedge (p_2')^* (\omega_X^{k-l-p-j}) \\
= \frac{1}{d'} \gamma^* (p_1^* (\omega_Y^l) \wedge p_2^* (\omega_Y^l)) \wedge \gamma^* (p_1')^* (\omega_X^{k-l-p-j}) \wedge (p_2')^* (\omega_X^{k-l-p-j}) \\
= \frac{1}{d'} \frac{1}{d n} \gamma^* (p_1^* (\omega_Y^l) \wedge p_2^* (\omega_Y^l)) \wedge \gamma^* (p_1')^* (\omega_X^{k-l-p-j}) \wedge \gamma^* (p_2')^* (\omega_X^{k-l-p-j}) \\
= \frac{1}{b_n} (p_1')^* \pi^* \left[ \frac{1}{d} (p_1 + p_2) (\omega_Y^l) \wedge \omega_Y^l \right] \wedge (p_1')^* (\omega_X^{k-l-p-j}) \wedge \left[ \frac{1}{d} (p_2')^* (\omega_X^{k-l-p-j}) \right] \\
= \frac{1}{b_n} \pi^* [G_n^* (\omega_Y^l) \wedge \omega_Y^l] \wedge \Gamma^* (\omega_X^{k-l-p-j}) \wedge \omega_X^{k-l-p-j}.
\]

In the above, we used that if $\tau : Y' \to Y$ is a generically finite surjective morphism and $\alpha$ is a finite union of points in $Y'$, then $\tau^* \tau_\alpha (\alpha) = \deg (\tau) \alpha$ have the same class in $N^* (Y')$. This is then applied to $Y' = G_n$, $\tau = p_1$, and $\alpha = p_1^* (\omega_Y^l) \wedge p_2^* (\omega_Y^l)$ (recall here that $\dim (Y) = l$).

Taking the sum all over $\Gamma$, and by observing that
\[
(g^n)^* = a_n G_n^* \quad \text{and} \quad \lim_{n \to \infty} (a_n b_n)^{1/n} = \lambda_0 (g),
\]
we obtain the inequality
\[
\lambda_0 (g) \lambda_p (f) \geq \max_{0 \leq j \leq l, 0 \leq p-j \leq k-l} \lambda_j (g) \lambda_{p-j} (f | \pi)
\]
for all $p = 0, \ldots, \dim (X)$.

(2) Provided that we can prove results from [13, Section 3] (cf. also [12, Section 4]), part (2) will follow easily. We can assume that $X$ is a smooth projective variety and $\pi : X \to Y$ is a surjective morphism. We observe that [12, Lemmas 4.3 and 4.5] follow if we can show the following: If $f$ is a dominant rational map (hence so is $g$), then the two numbers
\[
a^n \langle (f^n)^* (\pi^* (\omega_Y^{p-q}) \wedge \omega_X^q) \wedge \omega_X^{k-l-p} \rangle \\
\quad \text{and} \\
\langle (f^n)^* (\pi^* (\omega_Y^{p-q}) \wedge \omega_X^{k-l-p}) \wedge (f^n)^* (\omega_X^q) \wedge \omega_X^{k-l-p} \rangle
\]
are compatible. Using the arguments in the proof of Lemma 5.7 to each irreducible component of $\Gamma^* g^n$, we are able to achieve this estimate under the assumption that $g$ is a multiple of a rational map.

**Lemma 5.9.** Let $X$ and $Y$ be irreducible smooth projective varieties of corresponding dimensions $k$ and $l$ with $k \geq l$, and $\pi : (X, f) \to (Y, g)$ a semi-conjugacy, where $\pi : X \to Y$ is a surjective regular morphism. Assume that $g = a g'$, where $g' : Y \to Y$ is a dominant rational map. Let $\omega_X$ be an ample divisor on $X$ and $\omega_Y$ an ample divisor on $Y$. Then there are two positive constants $C_1, C_2$ such that for all $n \in \mathbb{N}$ and $p, q, r \in \mathbb{N}$ we have
\[
C_1 (f^n)^* \pi^* (\omega_Y^p) \wedge \pi^* (\omega_Y^q) \wedge \omega_X^{k-p-q-r} \\
\quad \leq a^n \langle (f^n)^* (\pi^* (\omega_Y^p) \wedge \omega_X^q) \wedge \omega_X^{k-p-q-r} \rangle \\
\quad \leq C_2 (f^n)^* \pi^* (\omega_Y^p) \wedge \pi^* (\omega_Y^q) \wedge \omega_X^{k-p-q-r}.
\]
Proof. If $f$ is a dominant rational map, the arguments in the proofs of [12, Lemmas 4.3 and 4.5] apply almost verbatim, pending the simplifications made in [13]. In the general case, we argue as follows.

Fix a positive integer $n$, and let $G_n \subset Y \times Y$ be the graph of the dominant rational map $(g^n) : Y \to Y$. Let $\Gamma_f = \sum_i \Gamma_i$, where $\Gamma_i$ are irreducible varieties. Let $\Gamma$ be one among the varieties $\Gamma_i$. Since $\pi : (X, f) \to (Y, g)$ is a semi-conjugacy, it follows that the image of $\Gamma$ under the regular morphism $\pi \times \pi : X \times X \to Y \times Y$ must be $G_n$. We let $G_n \to G_n$ be an alteration of degree say $d$, and let $p_1, p_2 : \widehat{G_n} \to Y$ be the two natural projections. We can choose an alteration $\widehat{\Gamma} \to \Gamma$ of degree say $d'$ which is equipped with a regular surjective morphism $\gamma : \widehat{\Gamma} \to G_n$ so that the two compositions $\widehat{\Gamma} \to \Gamma \to G_n$ and $\widehat{\Gamma} \to \widehat{G_n} \to G_n$ coincide. Let $p_1', p_2' : \widehat{\Gamma} \to X$ be the two natural projections. Then

$$
\begin{align*}
\langle \Gamma^* (\pi^* (\omega_Y^P) \wedge \omega_X^k), \pi^* (\omega_Y^P) \wedge \omega_X^k \rangle \\
= \frac{1}{d^2} (p_1')^* \pi^* (\omega_Y^P) \wedge (p_1')^* (\omega_X^k) \wedge (p_2')^* \pi^* (\omega_X^k) \wedge (p_2')^* (\omega_X^k) \\
= \frac{1}{d^2} \gamma^* (p_1^* (\omega_Y^P) \wedge p_2^* (\omega_Y^P)) \wedge (p_1')^* (\omega_X^k) \wedge (p_2')^* (\omega_X^k) \\
\sim \frac{1}{d^2} \frac{1}{d^2} \gamma^* (p_1^* (\omega_Y^P) \wedge p_2^* (\omega_Y^P)) \wedge (p_1')^* (\omega_X^k) \wedge (p_2')^* (\omega_X^k) \\
\sim (p_1')^* \pi^* \left[ \frac{1}{d} (p_1^* (\omega_Y^P) \wedge \omega_Y^P) \right] \wedge (p_1')^* (\omega_X^k) \wedge (p_2')^* (\omega_X^k) \\
\sim \pi^* [G_n^* (\omega_Y^P) \wedge \omega_Y^P] \wedge \Gamma^* (\omega_X^k) \wedge \omega_X^k.
\end{align*}
$$

Here $\sim$ means that the two numbers involved are positive multiples of each other, where the multiples are constants bounded in terms of $X$ and $Y$ only. We used in the fourth line of the above expression that

$$
\frac{1}{d} p_1^* (p_1) \wedge p_2^* (p_1) = p_1^* \wedge p_2^*
$$

(here $p_1^*$ on the far left means the strict pullback by $p_1$), which follows from the fact that the projection to the first factor $G_n \to Y$ is a birational morphism, since $G_n$ is the graph of a (dominant) rational map on $Y$.

If we take the sum all over $\Gamma$, the proof of the lemma is completed, by using Lemma 4.1 and the fact that $(g^n)^* = a^n G_n^*$.

\[ \square \]

6. Extensions and applications

In this section we collect some extensions and applications.

6.1. Correspondences over non-projective varieties. Let $\mathbb{K}$ be an algebraically closed field, $X$ an (not necessarily projective) irreducible variety and $f : X \to X$ a correspondence. By taking the closure of an affine Zariski-open subset of $X$, we can find a projective variety $X'$ which is birational to $X$, and hence define the dynamical degrees of $f$ via its pullback to $X'$ using (2) of Theorem 1.1. Similarly, if $X$ or $Y$ are not projective, we can define relative dynamical degrees by the same reduction.
6.2. Correspondences over reducible varieties. Let $K$ be an algebraically closed field, and $X, Y$ (reducible) projective varieties. Let $f : X \dashrightarrow X$ and $g : Y \dashrightarrow Y$ be dominant correspondences. Assume that $\pi : X \dashrightarrow Y$ is a dominant rational map such that $\pi \circ f = g \circ \pi$. (Recall from Section 3 that while the definition of iterates $f^n$ for correspondences over reducible varieties is quite delicate, we still have by Lemma 3.2 that $\pi \circ f^n = g^n \circ \pi$ for all $n \in \mathbb{N}$ as in the case of semi-conjugacy on irreducible varieties.) Then relative dynamical degrees can be defined and satisfy all the properties in Theorem 1.1. The reason for this is that in proving these results, the property we need is only an equality between strict transforms $f_n$ of $f$ and $g_n$ of $g$ outside certain bad sets, and this is certainly true for the general case considered here even though the iterates $f^n$ are more involved to define.

Let us elaborate on how to proceed. We discuss the case of dynamical degrees only, the general case of relative dynamical degrees is similar. Let $X = X_1 \sqcup \cdots \sqcup X_m$ be a reducible projective variety, where $X_i$ are distinct irreducible components of it. Let $f : X \dashrightarrow X$ be a dominant rational morphism. We choose alterations $i_i : Z_i \rightarrow X_i$ for $i = 1, \ldots, m$, where if $i \neq j$ we regard $Z_i$ and $Z_j$ as distinct (irreducible smooth) varieties even if they may be isomorphic to each other.

For each component $f_{i,j} : X_i \dashrightarrow X_j$ of $f$, we consider the pullback $g_{i,j} : Z_i \dashrightarrow Z_j$, where the graph of $g_{i,j}$ is the pullback of the variety $\Gamma_{f_i,j}$ by the generically finite morphism $\tau_i \times \tau_j : Z_i \times Z_j \rightarrow X_i \times X_j$. Define $d_{i,j}$ to be the degree of the morphism $\tau_i : Z_i \rightarrow X_i$. By Lemma 3.4, we have a semi-conjugacy of pairs $\tau_{i,j} = (\tau_i, \tau_j) : (Z_i, Z_j, g_{i,j}) \rightarrow (X_i, X_j, d_{j,i} f_{i,j}).$

We define

$$d = \prod_{(i,j) \in I} d_{i,j}$$

and

$$g = \sum_{(i,j) \in I} \frac{d}{d_{i,j}} g_{i,j}.$$ 

Let $Z = Z_1 \sqcup \cdots \sqcup Z_m$. Then $g$ is a correspondence on $Z$ and it is dominant since $f$ is dominant. Since $Z_i$ are distinct and $X_i$ are distinct, we see by definition that $f_{i,j}$ and $f_{k,l}$ can be composed if and only if $g_{i,j}$ and $g_{k,l}$ can be composed. From this, it can be checked that we have a semi-conjugacy

$$\tau : (Z, g) \rightarrow (X, df),$$

where $\tau = (\tau_1, \ldots, \tau_m) : (Z_1, \ldots, Z_m) \rightarrow (X_1, \ldots, X_m)$. Then, provided that we can define $\lambda_p(g)$, we can assign

$$\lambda_p(f) := \frac{1}{d} \lambda_p(g).$$

Now we indicate how to define $\lambda_p(g)$ and to show that the above definition is independent of the choice of $Z$. Let $\omega_i$ be an ample divisor on $Z_i$. We then define

$$\deg_p(g) = \sum_{(i,j) \in I} f_{i,j}^* (\omega_j^P) \wedge \omega_i^{k-p}.$$ 

Then it can be proven as before that the following limit exists:

$$\lim_{n \rightarrow \infty} (\deg_p(g^n))^{\frac{1}{n}},$$

and we assign $\lambda_p(g)$ to be this limit.
Now we show that the definition of $\lambda_p(f)$, as defined above, is independent of the choice of $Z$. Let be given another $Z' = Z'_1 \cup \cdots \cup Z'_m$ together with alterations $\tau'_i : Z'_i \to X_i$ and pullbacks $g'_{i,j} = (\tau'_i, \tau'_j)^* (f_{i,j})$. Let $d'_j$ be the degree of the morphism $\tau'_j : Z'_j \to X_j$, and let

$$d' = \prod_{(i,j) \in I} d'_j.$$ 

Then the correspondence $g' : Z' \vdash Z'$ is

$$g' = \sum_{(i,j) \in I} \frac{d'}{d_j} g'_{i,j}.$$ 

We can find another $\overline{Z} = (\overline{Z}_1, \ldots, \overline{Z}_m)$ together with alterations $\varphi_i : \overline{Z}_i \to Z_i$ and $\varphi'_i : \overline{Z}_i \to Z'_i$ such that $\tau_i \circ \varphi_i = \tau'_i \circ \varphi'_i$, and we denote by $\overline{\tau}_i : \overline{Z}_i \to X_i$ this common morphism. We then define

$$\overline{g}_{i,j} : \overline{Z}_i \vdash \overline{Z}_j$$

to be the pullback of $f_{i,j}$ by $(\overline{\tau}_i, \overline{\tau}_j) : (\overline{Z}_i, \overline{Z}_j) \to (X_i, X_j)$, $\overline{d}_i$ to be the degree of $\overline{\tau}_i : \overline{Z}_i \to X_i$, and

$$\overline{d} = \prod_{(i,j) \in I} \overline{d}_j.$$ 

The correspondence $\overline{g} : \overline{Z} \vdash \overline{Z}$ is finally

$$\overline{g} = \sum_{(i,j) \in I} \frac{\overline{d}}{d_j} \overline{g}_{i,j}.$$ 

Now we consider each $(i, j) \in I$ separately. We have a semi-conjugacy

$$\overline{g}_{i,j} \to \deg(\varphi_j) g_{i,j},$$ 

therefore we have a semi-conjugacy

$$\frac{\overline{d}}{d_j} g_{i,j} \to \frac{\overline{d}}{d_j} \deg(\varphi_j) g_{i,j} = \frac{\overline{d}}{d_j} g_{i,j} = d'_j \frac{d'}{d_j} g_{i,j}.$$ 

Taking the sum, we have a semi-conjugacy $\overline{g} \to d' g$, and moreover we have as argued before semi-conjugacies

$$\overline{g}^n \to (d')^n g^n$$

for all $n \in \mathbb{N}$. Similarly, we have semi-conjugacies

$$\overline{g}^n \to d^n (g')^n.$$ 

From this, we can define relative dynamical degrees and prove basic properties as in the case for irreducible varieties.

### 6.3. Fields that are not algebraically closed

Let $K$ be a field, not necessarily closed, and $\overline{K}$ its algebraic closure. Let $X$ be a variety over $K$, together with a dominant correspondence $f : X \vdash X$. Let $\overline{X}$ be the lift of $X$ and $\overline{f}$ the lift of $f$ to $\overline{K}$. We note that even if $X$ is irreducible, $\overline{X}$ may not be so. Hence in general we need to work with correspondences over reducible varieties. We then can define dynamical degrees for $f$ in terms of that for $\overline{f}$. 
6.4. Primitive correspondences. We say that a dominant correspondence \( f : X \looparrowright X \) is primitive if there is no semi-conjugacy \( \pi : (X, f) \to (Y, g) \), where \( Y \) is irreducible smooth and \( 0 < \dim(Y) < \dim(X) \).\(^2\) Note that any dominant correspondence \( f : X \looparrowright X \) is always semi-conjugate to a multiple of the identity map of a point, and hence in the above definition we may consider the special case where \( g \) is a multiple of a dominant rational map on \( Y \), or more generally a correspondence \( g \) so that \( g^n \) is irreducible for infinitely many \( n \in \mathbb{N} \). By the examples in Remark 1.4, there are many semi-conjugacies \( \pi : (X, f) \to (Y, g) \), where \( g \) is a multiple of a rational map but \( f \) is not a multiple of a rational map. In fact, such examples can always be constructed provided we are given two dominant rational maps \( \pi : X \to Y \) and \( g' : Y \to Y \), where \( \pi \) is not a birational map.

We say that a dominant correspondence \( g : Y \looparrowright Y \) is morphism-like if \( \lambda_0(g) \leq \lambda_1(g) \) and \( g^n \) is irreducible for infinitely many \( n \in \mathbb{N} \). For example, if \( g \) is a multiple of a dominant rational map, then it is morphism-like. We say that a dominant correspondence \( f : X \looparrowright X \) is weakly primitive if it cannot be semi-conjugate to a correspondence \( g : Y \to Y \), where \( 0 < \dim(Y) < \dim(X) \) and \( g \) is a morphism-like.

We have the following criterion for weakly primitivity. It says that if \( f \) is not morphism-like, then it must be weakly primitive.

**Theorem 6.1.** If \( \lambda_0(f) > \lambda_1(f) \), then \( f \) is weakly primitive.

**Proof.** This follows from Theorem 1.3, by using that for a morphism-like \( g \) we have \( \lambda_1(g) \geq \lambda_0(g) \) and that \( \lambda_0(f) = \lambda_0(f|_{\pi}) \). The latter can be seen as follows. We have by definition

\[
\lambda_0(f|_{\pi}) = \lim_{n \to \infty} \left( \frac{(f^n)^*g^\pi(X) \cdot \pi^*(\omega_{\pi}^X) \cdot \omega_{\pi}^{k-1}}{\pi^*} \right)^{1/n} = \lim_{n \to \infty} \frac{\|g^n(X)\|^{1/n}}{\pi} = \lambda_0(f),
\]

as desired. \( \Box \)

We note that under the assumptions of this theorem, it may still happen that \( f \) is imprimitive, being semi-conjugate to a correspondence which is not morphism-like. See Example 2 below for more detail.

Similarly, when \( f \) is a dominant rational map such that \( \lambda_1(f) > \lambda_2(f) \), it can be shown as in [29] that \( f \) is (strongly) primitive. We also have the following consequence of resolution of singularities for surfaces and threefolds. It can be applied to the case \( \dim(X) \leq 4 \) to show that certain maps are primitive.

**Corollary 6.2.** Let \( X \) and \( Y \) be irreducible projective varieties (not necessarily smooth) and let \( \pi : (X, f) \looparrowright (Y, g) \) be a semi-conjugacy of dominant correspondences, where \( g \) is a multiple of a rational map. Assume that \( \dim(Y) \leq 3 \). Then the product formula (part (2) in Theorem 1.3) is satisfied.

**Example 1.** For the Hurwitz correspondences \( f \) (for the definition and properties see [30]), it was shown in [30] that \( \lambda_0(f) \) is the largest dynamical degree, hence Theorem 6.1 may be applicable.

\(^2\) If resolution of singularities is available or if the discussion in Remark 1.5 can be carried out, then the assumption that \( Y \) is smooth is not needed.
**Example 2.** Let \( f' : X \to X \) be a dominant rational map such that

\[
\lambda_k(f') > \lambda_{k-1}(f'),
\]

where \( k = \dim(X) \). (For example, \( f' \) is a regular morphism of \( \mathbb{P}^k \) of degree \( d \geq 2 \)) Let \( f = (f')^{-1} \) be the reverse correspondence of \( f' \), i.e. the graph of \( f \) is the image of the graph of \( f' \) by the automorphism \((x_1, x_2) \to (x_2, x_1)\) of \( X \times X \). Then

\[
\lambda_0(f) = \lambda_k(f') > \lambda_{k-1}(f') = \lambda_1(f).
\]

Therefore, Theorem 6.1 is applicable.

From a given \( f : X \to X \) which satisfies the assumptions of Theorem 6.1, we can construct more in the following manner. Let \( f_1, \ldots, f_m : X \to X \) be dominant correspondences for which any pair among \( f, f_1, \ldots, f_m \) commute. (For example, choose the correspondences \( f_i \) among the iterates of \( f \)) Then the new correspondence \( af + a_1f_1 + \cdots + a_mf_m \) with \( a \gg \max\{a_1, \ldots, a_m\} \) will do the job, thanks to Theorem 1.2.

In these examples, it is possible that \( f \) is still imprimitive. In fact, let \( k \geq 2 \), \( X = \mathbb{P}^k \), \( Y = \mathbb{P}^{k-1} \), and let \( \pi : X \to Y \) be the dominant rational map

\[
[x_0 : x_1 : \cdots : x_{k-1} : x_k] \mapsto [x_0 : \cdots : x_{k-1}].
\]

Let \( d \geq 2 \), \( f' : X \to X \) be the surjective morphism

\[
[x_0 : \cdots : x_k] \mapsto [x_0^d : \cdots : x_k^d],
\]

and let \( g' : Y \to Y \) be the surjective morphism

\[
[x_0 : \cdots : x_{k-1}] \mapsto [x_0^d : \cdots : x_{k-1}^d].
\]

Let \( f \) be the reverse of \( f' \) and \( g \) the reverse of \( g' \). Then it can be checked that we have a semi-conjugacy \( \pi : (X, f) \to (Y, dg) \). Note that the correspondence \( g \) is not morphism-like: while \( g^n \) (being the reverse of \( (g')^n \)) is irreducible for all \( n \), the condition \( \lambda_0(g) \leq \lambda_1(g) \) is not satisfied.

**Example 3.** Even in the case where the dynamical degrees of a correspondence do not satisfy the assumptions of Theorem 6.1, a full use of the product formula for dynamical degrees may still help to show that a certain correspondence is primitive. For example, let \( f : X \to X \) be a correspondence on a surface. Assume that there is an irreducible curve \( Y \) together with a dominant rational map \( g' : Y \to Y \) and a dominant rational map \( \pi : X \to Y \) so that \( \pi \circ f = g \circ \pi \), where \( g = ag' \). By Corollary 6.2, the product formula for relative dynamical degrees is valid in this situation, even when \( Y \) is not smooth. Therefore,

\[
\begin{align*}
a \lambda_0(f) &= \lambda_0(g) \lambda_0(f | \pi), \\
a \lambda_1(f) &= \max \{ \lambda_0(g) \lambda_1(f | \pi), \lambda_1(g) \lambda_0(f | \pi) \}, \\
a \lambda_2(f) &= \lambda_1(g) \lambda_1(f | \pi).
\end{align*}
\]

In particular,

\[
\begin{align*}
a^2 \lambda_1(f)^2 &\geq (\lambda_0(g) \lambda_1(f | \pi)) (\lambda_1(g) \lambda_0(f | \pi)) \\
&= (\lambda_0(g) \lambda_0(f | \pi)) (\lambda_1(g) \lambda_1(f | \pi)) \\
&= a^2 \lambda_0(f) \lambda_2(f).
\end{align*}
\]
Hence, provided $\lambda_1(f)^2 < \lambda_0(f)\lambda_2(f)$ (that is, when the log-concavity fails), $f$ cannot be semi-conjugate to a multiple of a dominant rational map over an irreducible curve $Y$.

In Remark 1.4, we constructed many examples of correspondences whose dynamical degrees violate the log-concavity. Here is one specific example. We let $h : \mathbb{P}^2 \to \mathbb{P}^2$ be a regular morphism of algebraic degree $d \geq 2$. Let $a$ be a positive integer. Consider $f = h + a\Delta$, where $\Delta$ is the diagonal. Then $\lambda_0(f) = 1 + a$, $\lambda_1(f) = d + a$ and $\lambda_2(f) = d^2 + a$. It can be checked that $\lambda_1(f)^2 < \lambda_0(f)\lambda_2(f)$ for all choices of $a$.

**Example 4.** Similar to Example 3, if $\pi : (X, f) \to (Y, ag')$ is a semi-conjugacy, where $\dim(X) = 3$, $0 < \dim(Y) < 3$, and $g' : Y \to Y$ is a dominant rational map, then we must have $\lambda_0(f)\lambda_3(f) \leq \lambda_1(f)\lambda_2(f)$. A construction in the same line as that in Remark 1.4 and Example 2 provides many examples where the previous inequality is violated.

**6.5. An application toward algebraic dynamics.** We present in this subsection an application, recently proven in [35], toward pullback on étale cohomology groups. Let $X$ be an irreducible smooth projective variety over an algebraic closed field $\mathbb{K}$ of arbitrary characteristic. Let $f : X \to X$ be a regular endomorphism. We can define pullback on étale cohomology groups $f^* : H^i(X, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l)$, where $l$ is relatively prime to the characteristic of $\mathbb{K}$. Fix a field embedding $\mathbb{Q}_l \subset \mathbb{C}$. Define $\chi_i(f)$ to be the maximum of the absolute values of eigenvalues of $f^* : H^i(X, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l)$, where the absolute value is induced from the embedding of $\mathbb{Q}_l$ in $\mathbb{C}$. In [35], using the results in this paper and Deligne’s proof of Weil’s Riemann hypothesis [10, 11], we showed that

$$\max_{i=0,\ldots,\dim(X)} \chi_i(f) = \max_{i=0,\ldots,\dim(X)} \lambda_i(f). \tag{6.1}$$

This extends a previous result of Esnault and Srinivas [20] on automorphisms of surfaces, and answers one question posed in that paper. Moreover, assuming Standard Conjecture D (numerical vs. homological equivalence) of algebraic cycles, we proved the same result for the more general case when $f$ is a correspondence.

By comparing to the situation when $\mathbb{K} = \mathbb{C}$, equation (6.1) points to the possibility of a mysterious topology in positive characteristic (or more generally Grothendieck’s topos) so that the corresponding topological entropy is the same as $\ln \max_{i=0,\ldots,\dim(X)} \lambda_i(f)$. When $\mathbb{K}$ is non-Archimedean, we may consider the non-Archimedean topology. However, the corresponding topology does not always equal $\ln \max_{i=0,\ldots,\dim(X)} \lambda_i(f)$, see [21].

Since we always have $\chi_{2i}(f) \geq \lambda_i(f)$ for all $i$ and all $f$, the most reasonable explanation to always having equation (6.1) is that in fact we should have $\chi_{2i}(f) = \lambda_i(f)$ for all $i$ and all $f$. Note that, in turn, the famous Weil’s Riemann hypothesis (solved by Deligne [10, 11]) is a special case of this speculation for $f$ (the Frobenius map of $X$). This speculation is true for $\mathbb{K} = \mathbb{C}$. For more detail on this, please see [35].

It was shown in [35] that in order to have $\chi_{2i}(f) = \lambda_i(f)$ for all regular morphisms $f$ and for all $i$ (more generally for dominant correspondences, in which case the definition of $\chi_j(f)$ must be modified appropriately) on a given smooth projective variety $X$, it suffices to prove the following: Fix a norm on $H^{2i}(X, \mathbb{Q}_l)$. There exists a constant $C > 0$ so that for any two dominant correspondences $\Gamma_1$ and $\Gamma_2$, we always have

$$\| (\Gamma_1 + \Gamma_2)^*|_{H^{2i}(X,\mathbb{Q}_l)} \| \geq C \| \Gamma_1^*|_{H^{2i}(X,\mathbb{Q}_l)} \|. $$
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