Weak Orlicz-Hardy Martingale Spaces

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Abstract

In this paper, several weak Orlicz-Hardy martingale spaces associated with concave functions are introduced, and some weak atomic decomposition theorems for them are established. With the help of weak atomic decompositions, a sufficient condition for a sublinear operator defined on the weak Orlicz-Hardy martingale spaces to be bounded is given. Further, we investigate the duality of weak Orlicz-Hardy martingale spaces and obtain a new John-Nirenberg type inequality when the stochastic basis is regular. These results can be regarded as weak versions of the Orlicz-Hardy martingale spaces due to Miyamoto, Nakai and Sadasue.

1 Introduction

The Lebesgue’s theory of integration has taken a center role in modern analysis, which leads the more extensive classes of function spaces and martingale spaces to naturally arise. As is well known, as a generalization of $L_p$-space, the Orlicz space was introduced in [2]. Since then, the Orlicz spaces have been widely used in probability, partial differential equations and harmonic analysis; see [1, 3, 14, 16, 18], and so forth. In particular, Takashi, Eiichi and Gaku very recently studied the Orlicz-Hardy martingale spaces in [11], and using the atomic decomposition they obtained some very interesting martingale inequalities as well as the duality, and proved a generalized John-Nirenberg type inequality for martingale when the stochastic basis is regular. Let us briefly recall the main results of [11].

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Let $G$ be the set of all functions $\Phi : [0,\infty) \to [0,\infty)$ satisfying $\Phi(0) = 0$, $\lim_{r \to \infty} \Phi(r) = \infty$. The Orlicz space $L_\Phi$ is defined as the collection of all measurable functions $f$ with respect to $(\Omega, \mathcal{F}, P)$ such that $\mathbb{E}(\Phi(|f|)) < \infty$ for some $c > 0$ and
\[
\|f\|_{L_\Phi} = \inf \left\{ c > 0 : \mathbb{E}(\Phi(|f|/c)) \leq 1 \right\},
\]
where $\mathbb{E}$ denotes the expectation with respect to $\mathcal{F}$. For $q \in [1,\infty)$ and a function $\phi : (0,\infty) \to (0,\infty)$, the generalized Campanato martingale spaces $L_{q,\phi}$ is defined by
\[
L_{q,\phi} = \{ f \in L_q : \|f\|_{L_{q,\phi}} < \infty \},
\]
where
\[
\|f\|_{L_{q,\phi}} = \sup_{n \geq 1} \sup_{A \in \mathcal{F}_n} \frac{1}{\phi(P(A))} \left( \frac{1}{P(A)} \int_A |f - \mathbb{E}_n f|^q dP \right)^{1/q},
\]
with the convention that $\mathbb{E}_0 f = 0$. We refer to the recent paper [13] for the Morrey-Campanato spaces. Denote by $G_\ell$ the set
\[
G_\ell = \left\{ \Phi \in G : \exists c_\Phi \geq 1 \text{ and } \ell \in (0,1] \text{ s.t. } \Phi(tr) \leq c_\Phi \max\{t^\ell, t\} \Phi(r) \text{ for } t, r \in [0,\infty) \right\}.
\]
Then for $\Phi \in G_\ell$ and $\phi(r) = \frac{1}{\phi^{-1}(1/r)}$, where and in what follows $\Phi^{-1}$ denotes the inverse function of $\Phi$, the following duality holds,
\[
(H_2^q)^* = L_{2,\phi}.
\]
See Section 2 for the notation $H_2^q$. Moreover, the John-Nirenberg type inequality holds when the stochastic basis is regular, namely, $L_{q,\phi}$ are equivalent for all $1 \leq q < \infty$. It should be mentioned that Miyamoto, Nakai and Sadasue’s results above are exactly the generalization in [20] due to Weisz when $\Phi(t) = t^p$, $0 < p \leq 1$.

The main goal of this present paper is to deal with the weak Orlicz-Hardy martingale spaces, which are more inclusive classes than Orlicz-Hardy martingale spaces, and give the weak version of Miyamoto, Nakai and Sadasue’s results. In 2008 the weak Orlicz spaces and weak Orlicz-Hardy martingale spaces generated by nice young functions satisfying the $\mathcal{M}\triangle$-condition were first introduced in [5], and interpolation theorems and inequalities were proved for these spaces; Liu et al investigated the boundedness of some sublinear operators defined on weak Orlicz-Hardy martingale spaces in [9] and the first named author studied some embedding relationships between them in [7] in 2011; however, the existing results about weak Orlicz-Hardy martingale spaces are all associated with convex functions. In the present paper we are interested in the case $\Phi$ is not convex. We denote
\[
t_\phi^q(x) = \frac{1}{\phi(x)} x^{-1/q} \sup_{P(\nu < \infty) \leq x} \|f - f^\nu\|_q,
\]
where $\nu$ is a stopping time and $f^\nu$ is the stopped martingale. Very differently from (1.1), we define the weak generalized Campanato martingale space $wL_{q,\phi}$ as follows.
Definition 1.1. For $q \in [1, \infty)$ and a function $\phi : (0, \infty) \to (0, \infty)$, let

$$wL_{q,\phi} = \left\{ f \in L_q : \|f\|_{wL_{q,\phi}} = \int_0^\infty \frac{t^q_\phi(x)}{x} dx < \infty \right\}.$$ 

Then for $\Phi \in G\ell$ and $\phi(r) = \frac{1}{r\Phi^{-1}(1/r)}$, we have

$$(wH^s\Phi)^* = wL^2_{2,\phi}.$$ 

See Section 2 for the notation of $wH^s\Phi$. Furthermore, we by the duality obtain the weak type John-Nirenberg inequality when the stochastic basis is regular. That is, $wL_{q,\phi}$ are equivalent for all $1 \leq q < \infty$. We note that our theorems can deduce Weisz’s main results in [19].

It is well known that the method of atomic decompositions plays an important role in martingale theory; see for example, [4, 6, 21, 22]. In the present paper, the important step is to establish the weak atomic characterizations of weak Orlicz-Hardy martingale spaces. To this end, the main difficulty encountered is that there is no similar result to replace Lemma 3.1 or Remark 3.2 in [11]. Inspired by [19], we adopt a different method and apply some technical estimates. Particularly, we note that $\frac{\Phi^{-1}(t)}{t^p}$ are increasing and $\frac{\Phi^{-1}(t)}{t^q}$ are decreasing on $(0, \infty)$ for $\Phi \in G\ell$ with $q < \infty$, where $p = p\Phi^{-1}$ and $q = q\Phi^{-1}$ denote the lower index and the upper index of convex function $\Phi^{-1}$, respectively; see also Section 2 for the definitions of the lower and upper index.

This paper is organized as follows. Section 2 is on preliminaries and notations. Section 3 is devoted to the weak atomic decompositions of weak Hardy-Orlicz martingale spaces. By the atomic decompositions, a sufficient condition for a sublinear operator defined on weak Hardy-Orlicz martingale spaces to be bounded is given in Section 4. In Section 5, we deduce the new John-Nirenberg type inequality by duality.

We end this section by an open question. For $q \in [1, \infty)$ and a function $\phi : (0, \infty) \to (0, \infty)$, define $L_{q,\phi} = \{ f \in L_q : \|f\|_{L_{q,\phi}} < \infty \}$, where

$$\|f\|_{L_{q,\phi}} = \sup_{\nu} \frac{1}{\phi(P(\nu < \infty))} \left( \frac{1}{P(\nu < \infty)} \int_{\{\nu < \infty\}} |f - f_\nu|^q dP \right)^{1/q}. $$

The supremum is taken all the stopping times $\nu$. Then by the Proposition 2.12 in [11],

$$\|f\|_{L_{q,\phi}} \leq \|f\|_{L_{q,\phi}} \leq C_\phi \|f\|_{L_{q,\phi}},$$

where $\phi$ satisfies $\phi(r) \leq c_\phi \phi(s)$ for $0 < r \leq s < \infty$. Now we denote

$$u_{\phi}(x) = \frac{1}{\phi(x)} x^{-1/q} \sup_{n \geq 1} \sup_{A \in F_n} \sup_{P(A) \leq x} \left( \int_A |f - E_n f|^q dP \right)^{1/q},$$

where $\nu$ is a stopping time and $f_\nu$ is the stopped martingale. It is natural to define

$$wL_{q,\phi} = \left\{ f \in L_q : \|f\|_{wL_{q,\phi}} = \int_0^\infty \frac{u_{\phi}^q(x)}{x} dx < \infty \right\}.$$
In the time of writing this paper, we do not know if there is a result similar to (1.2).

Throughout this paper, \( \mathbb{Z} \) and \( \mathbb{N} \) denote the integer set and nonnegative integer set, respectively. We denote by \( C \) the positive constant, which can vary from line to line.

## 2 Preliminaries

Let \( (\Omega, \mathcal{F}, P) \) be a probability space, and \( \{\mathcal{F}_n\}_{n \geq 0} \) be a non-decreasing sequence of sub-\( \sigma \)-algebras of \( \mathcal{F} \) such that \( \mathcal{F} = \sigma(\bigcup_{n \geq 0} \mathcal{F}_n) \). The expectation operator and the conditioned expectation operator are denoted by \( E \) and \( E \mid \mathcal{E} \), respectively. For a martingale \( f = (f_n)_{n \geq 0} \) relative to \( (\Omega, \mathcal{F}, P; (\mathcal{F}_n)_{n \geq 0}) \), we denote its martingale difference by \( df_i = f_i - f_{i-1} \) \( (i \geq 0, \text{with convention } f_{-1} = 0) \). Then the maximal function, the quadratic variation and the conditional quadratic variation of martingale \( f \) are defined by

\[
M_n(f) = \sup_{0 \leq i \leq n} \left| f_i \right|, \quad M(f) = \sup_{i \geq 0} \left| f_i \right|, \\
S_n(f) = \left( \sum_{i=0}^{n} \left| df_i \right|^2 \right)^{1/2}, \quad S(f) = \left( \sum_{i=0}^{\infty} \left| df_i \right|^2 \right)^{1/2}, \\
s_n(f) = \left( \sum_{i=0}^{n} \mathbb{E}_{i-1} \left| df_i \right|^2 \right)^{1/2}, \quad s(f) = \left( \sum_{i=0}^{\infty} \mathbb{E}_{i-1} \left| df_i \right|^2 \right)^{1/2}.
\]

The stochastic basis \( (\mathcal{F}_n)_{n \geq 0} \) is said to be regular, if for \( n \geq 0 \) and \( A \in \mathcal{F}_n \), there exists \( B \in \mathcal{F}_{n-1} \) such that \( A \subset B \) and \( P(B) \leq R P(A) \), where \( R \) is a positive constant independent of \( n \). A martingale is said to be regular if it is adapted to a regular \( \sigma \)-algebra sequence. This amounts to saying that there exists a constant \( R > 0 \) such that

\[
f_n \leq R f_{n-1}
\]

for all nonnegative martingales \( (f_n)_{n \geq 0} \) adapted to the stochastic basis \( (\mathcal{F}_n)_{n \geq 0} \).

Recall that \( \mathcal{G} \) is the collection of all functions \( \Phi : [0, \infty) \rightarrow [0, \infty) \) satisfying \( \Phi(0) = 0 \), \( \lim_{r \rightarrow \infty} \Phi(r) = \infty \). A function \( \Phi \) is said to satisfy \( \Delta_2 \)-condition, denoted by \( \Phi \in \Delta_2 \), if there exists a constant \( C > 0 \) such that \( \Phi(2t) \leq C \Phi(t), \forall t > 0 \). A function \( \Phi : [0, \infty) \rightarrow [0, \infty) \) is said to be subadditive if \( \Phi(r+s) \leq \Phi(r) + \Phi(s) \) for all \( r, s \in [0, \infty) \).

We note that all concave functions are subadditive. Let \( \Phi_1 \) and \( \Phi_2 \) belong to \( \mathcal{G} \), which are said to be equivalent if there exists a constant \( C \geq 1 \) such that \( \Phi_1(t)/C \leq \Phi_2(t) \leq C \Phi_1(t) \) for all \( t \geq 0 \).

**Definition 2.1.** Let \( \Phi \in \mathcal{G} \), then the weak Orlicz space \( wL_\Phi \) is defined as the set of all measurable functions \( f \) with respect to \( (\Omega, \mathcal{F}, P) \) such that \( \|f\|_{wL_\Phi} < \infty \), where

\[
\|f\|_{wL_\Phi} := \inf \left\{ c > 0 : \sup_{t>0} \Phi \left( \frac{t}{c} \right) P(|f| > t) \leq 1 \right\}.
\]
If $\Phi(t) = t^p$, $0 < p < \infty$, then $wL_\Phi = wL_p$, where the weak $L_p$ space $wL_p$ consists of all measurable functions $f$ for which
\[ \|f\|_{wL_p} := \sup_{t>0} tp(|f| > t)^{1/p} < \infty. \]

It was proved in [9] that the functional $\| \cdot \|_{wL_\Phi}$ is a complete quasi-norm on $wL_\Phi$ when $\Phi \in \Delta_2$ and $\Phi$ is convex. In this paper we are interested in the case $\Phi$ is not convex. We assume that $\Phi$ is of lower type $\ell$ for some $\ell \in (0,1]$ and upper type 1, i.e., there exist a constant $c_\Phi \in [1,\infty)$ and some $\ell \in (0,1]$ such that
\[ \Phi(t) \leq c_\Phi \max\{t^{\ell}, t\} \Phi(r) \text{ for } t, r \in [0,\infty). \]

Let $G_\ell$ be the set of all $\Phi \in G$ satisfying the above inequality. For example, $\Phi(t) = t^p \left( \log(e + t) \right)^q$ is in $G_\ell$ if $0 < \ell \leq p < 1$ and $q \geq 0$. Let $\Phi \in G_\ell$, from [15] we know that $\Phi$ is equivalent to a concave function in $G_\ell$. Further, we can verify the functionals $\| \cdot \|_{wL_\Phi}$ and $\| \cdot \|_{wL_{\Phi}}$ are equivalent if $\Phi, \Psi \in G_\ell$ are equivalent. Therefore, we always assume that $\Phi \in G_\ell$ is concave in our theorems below. Thus $\Phi$ is subadditive, increasing, continuous and bijective from $[0,\infty)$ to itself when $\Phi \in G_\ell$.

Obviously, $G_\ell \subset \Delta_2$. It was shown in [11] that for any concave function $\Phi \in G_\ell$, $L_\Phi$ is a $\ell$-quasi norm. Here the functional $\| \cdot \|_{wL_\Phi}$ satisfies the following properties:

(i) $\|f\|_{wL_\Phi} \geq 0$, and $\|f\|_{wL_\Phi} = 0$ if and only if $f = 0$;

(ii) $\|cf\|_{wL_\Phi} = |c| \|f\|_{wL_\Phi}$;

(iii) $\|f + g\|_{wL_\Phi} \leq (2c_\Phi)^{1/\ell} \cdot 2(\|f\|_{wL_\Phi} + \|g\|_{wL_\Phi})$.

Indeed, (i) and (ii) can be proved easily. Here we only prove the generalized triangle inequality, namely (iii). Suppose that $\|f\|_{wL_\Phi} = a$, $\|g\|_{wL_\Phi} = b$, $a, b > 0$, and $K = (2c_\Phi)^{1/\ell}$. Then $\forall t > 0$,
\[
\Phi\left(\frac{t}{K \cdot 2(a + b)}\right) P(|f + g| > t) \leq \Phi\left(\frac{t}{K \cdot 2(a + b)}\right) \left( P\left(|f| > \frac{t}{2}\right) + P\left(|g| > \frac{t}{2}\right) \right)
\]
\[
\leq c_\Phi \cdot \frac{1}{2c_\Phi} \left( \Phi\left(\frac{t}{2a}\right) P\left(|f| > \frac{t}{2}\right) + \Phi\left(\frac{t}{2b}\right) P\left(|g| > \frac{t}{2}\right) \right) \leq 1.
\]

**Proposition 2.2.** Let $\Phi \in G_\ell$, then $L_1 \subset L_\Phi \subset wL_\Phi$.

**Proof.** Suppose that $f \in L_1$, then
\[
\int_\Omega \Phi\left(\frac{|f|}{\|f\|_1}\right) dP = \int_{\{|f| \leq \|f\|_1\}} \Phi\left(\frac{|f|}{\|f\|_1}\right) dp + \int_{\{|f| > \|f\|_1\}} \Phi\left(\frac{|f|}{\|f\|_1}\right) dp \leq \int_{\{|f| \leq \|f\|_1\}} \Phi(1) dp + \int_{\{|f| > \|f\|_1\}} c_\Phi \max\left\{ \frac{|f|}{\|f\|_1}, \left(\frac{|f|}{\|f\|_1}\right)^{\ell} \right\} \Phi(1) dp 
\]
\[
\leq \Phi(1) + c_\Phi \Phi(1) \int_{\{|f| > \|f\|_1\}} \frac{|f|}{\|f\|_1} dP \leq \Phi(1) + c_\Phi \Phi(1). 
\]
Taking $C_0 = \max\{\Phi(1) + c_\Phi \Phi(1), 1\}$, then
\[
\int_\Omega \Phi\left(\frac{|f|}{(C_0 \cdot c_\Phi)^{1/\ell}\|f\|_1}\right) dP \leq c_\Phi \cdot \frac{1}{C_0 c_\Phi} \int_\Omega \Phi\left(\frac{|f|}{\|f\|_1}\right) dP \leq 1,
\]
which means
\[
\|f\|_{L_\Phi} \leq (C_0 \cdot c_\Phi)^{1/\ell}\|f\|_1.
\]
Suppose that $f \in L_\Phi$ and $t > 0$. Then
\[
\Phi\left(\frac{t}{\|f\|_{L_\Phi}}\right) P(|f| > t) \leq \int_{\{|f| > t\}} \Phi\left(\frac{|f|}{\|f\|_{L_\Phi}}\right) dP \leq \int_\Omega \Phi\left(\frac{|f|}{\|f\|_{L_\Phi}}\right) dP \leq 1,
\]
which implies $L_\Phi \subset wL_\Phi$. The proof is complete.

Remark 2.3. It was proved in [16] that the Orlicz space $L_\Phi$ has absolute continuous norm when $\Phi \in \Delta_2$, that is,
\[
\lim_{P(A) \to 0} \|f\chi_A\|_{L_\Phi} = 0, \quad \forall f \in L_\Phi.
\]
But not every element in $wL_\Phi$ has absolute continuous quasi norm in spire of $\Phi \in \Delta_2$. For instance, let $\Omega = (0, 1]$ and $P$ be the Lebesgue measure on $\Omega$. Consider $wL_p$ ($0 < p < \infty$) and function $f(x) = x^{-1/q}$. A simple computation shows that $f \in wL_p$ when $q \geq p$, and $f$ has absolutely continuous norm in $wL_p$ when $q > p$, but it has not when $q = p$.

Definition 2.4. Let $wL_\Phi$ be the set of all $f \in wL_\Phi$ having the absolute continuous quasi norm defined by
\[
wL_\Phi := \left\{f \in wL_\Phi : \lim_{P(A) \to 0} \|f\chi_A\|_{wL_\Phi} = 0\right\}.
\]
It is easy to check $wL_\Phi$ is a linear space. Moreover $wL_\Phi$ is a closed subspace of $wL_\Phi$ when $\Phi \in G_\ell$. Indeed, suppose that $f_n \in wL_\Phi$, and $\|f_n - f\|_{wL_\Phi} \to 0$ as $n \to \infty$. Then $f \in wL_\Phi$ and
\[
\|f\chi_A\|_{wL_\Phi} \leq (2c_\Phi)^{1/\ell} \cdot 2(\|f_n\chi_A\|_{wL_\Phi} + \|(f_n - f)\chi_A\|_{wL_\Phi}) \leq (2c_\Phi)^{1/\ell} \cdot 2(\|f_n\chi_A\|_{wL_\Phi} + \|f_n - f\|_{wL_\Phi}).
\]
Since $f_n \in wL_\Phi$ has absolute continuous norm and $\|f_n - f\|_{wL_\Phi} \to 0$ as $n \to \infty$, we get
\[
\lim_{P(A) \to 0} \|f\chi_A\|_{wL_\Phi} = 0,
\]
namely, $f \in wL_\Phi$, which means $wL_\Phi$ is closed. Moreover, $L_1 \subset L_\Phi \subset wL_\Phi$ when $\Phi \in G_\ell$.

The following is an extension of Lebesgue controlled convergence theorem, which will be used to describe the quasi norm convergence (See Remark 3.2).
Proposition 2.5. (See [10], Theorem 3.2) Let \( \Phi \in \mathcal{G} \). \( f_n, g \in \mathcal{L}_\Phi \) and \( |f_n| \leq g \). If \( f_n \) converges to \( f \) almost everywhere, then
\[
\lim_{n \to \infty} \|f_n - f\|_{w, L_\Phi} = 0.
\]

In order to describe our results, we need the lower index and upper index of \( \Phi \). Let \( \Phi \in \mathcal{G} \), the lower index and the upper index of \( \Phi \) are respectively defined by
\[
p_\Phi = \inf_{t > 0} \frac{t \Phi'(t)}{\Phi(t)}, \quad q_\Phi = \sup_{t > 0} \frac{t \Phi'(t)}{\Phi(t)}.
\]

It is well known that \( 1 \leq p_\Phi \leq q_\Phi \leq \infty \) if \( \Phi \) is convex and \( 0 < p_\Phi \leq q_\Phi \leq 1 \) if \( \Phi \) is concave.

We note that the important observation below.

Lemma 2.6. Let \( \Phi \) be concave and \( q_\Phi - 1 < \infty \). Denote \( p = p_\Phi - 1 \), \( q = q_\Phi - 1 \). Then \( \Phi^{-1} \) are increasing on \( (0, \infty) \) and \( \Phi^{-1} \) are decreasing on \( (0, \infty) \).

Proof. It is easy to see that \( \Phi^{-1} \) is convex. Thus \( 1 \leq p \leq q < \infty \). From [10], we obtain that \( \Phi^{-1} \) is increasing on \( (0, \infty) \) and \( \Phi^{-1} \) is decreasing on \( (0, \infty) \). Replacing \( t \) with \( \Phi(t) \), we immediately get that \( \Phi^{-1} \) is increasing on \( (0, \infty) \) and \( \Phi^{-1} \) is decreasing on \( (0, \infty) \).

We now introduce the weak Orlicz-Hardy martingale spaces. Denote by \( \Lambda \) the collection of all sequences \( (\lambda_n)_{n \geq 0} \) of non-decreasing, non-negative and adapted functions with \( \lambda_\infty = \lim_{n \to \infty} \lambda_n \). As usual, the weak Orlicz-Hardy martingale spaces are defined as follows:
\[
\begin{align*}
\wH_\Phi &= \{ f = (f_n)_{n \geq 0} : \|f\|_{\wH_\Phi} = \|M(f)\|_{wL_\Phi} < \infty \}; \\
\wH_\Phi^S &= \{ f = (f_n)_{n \geq 0} : \|f\|_{\wH_\Phi^S} = \|S(f)\|_{wL_\Phi} < \infty \}; \\
\wH_\Phi_S &= \{ f = (f_n)_{n \geq 0} : \|f\|_{\wH_\Phi_S} = \|s(f)\|_{wL_\Phi} < \infty \}; \\
\wQ_\Phi &= \{ f = (f_n)_{n \geq 0} : \exists (\lambda_n)_{n \geq 0} \in \Lambda, \ s.t. \ S_n(f) \leq \lambda_n - 1, \lambda_\infty \in wL_\Phi \}, \\
\|f\|_{\wQ_\Phi} &= \inf_{(\lambda_n) \in \Lambda} \|\lambda_\infty\|_{wL_\Phi}; \\
\wD_\Phi &= \{ f = (f_n)_{n \geq 0} : \exists (\lambda_n)_{n \geq 0} \in \Lambda, \ s.t. \ |f_n| \leq \lambda_n - 1, \lambda_\infty \in wL_\Phi \}, \\
\|f\|_{\wD_\Phi} &= \inf_{(\lambda_n) \in \Lambda} \|\lambda_\infty\|_{wL_\Phi}.
\end{align*}
\]

Remark 2.7. We can get the Orlicz-Hardy martingale space \( H_\Phi^s \) when \( \|s(f)\|_{wL_\Phi} \) is replaced by \( \|s(f)\|_{L_\Phi} \) in the definition above. In order to describe the duality, we define
\[
\wH_\Phi^s = \{ f = (f_n)_{n \geq 0} : s(f) \in wL_\Phi \}.
\]

It is easy to see \( \wH_\Phi^s \) is a closed subspace of \( \wH_\Phi^s \). Similarly, we have \( \wH_\Phi \) and \( \wH_\Phi^S \), which are closed subspaces of \( \wH_\Phi \) and \( \wH_\Phi^S \), respectively.
Definition 2.8. A measurable function \( a \) is said to be a \( w-1 \)-atom (or \( w-2 \)-atom, \( w-3 \)-atom, resp.) if there exists a stopping time \( \nu \) such that

\[
\begin{align*}
(a1) \quad & a_n = E_n a = 0 \text{ if } \nu \geq n, \\
(a2) \quad & \|s(a)\|_{\infty} < \infty \text{ (or } \|S(a)\|_{\infty} < \infty, \|M(a)\|_{\infty} < \infty \text{ resp.).}
\end{align*}
\]

Now we define the weak Orlicz-Hardy spaces associated with weak atoms.

Definition 2.9. Let \( \Phi \in \mathcal{G}_{\ell} \) with \( \ell \in (0, 1] \). We define \( \mathcal{wH}_{\Phi,\text{at}}^{s},\mathcal{wH}_{\Phi,\text{at}}^{S},\mathcal{wH}_{\Phi,\text{at}}^{M} \) as the space of all \( f \in \mathcal{wL}_{\Phi} \) which admit a decomposition

\[
f = \sum_{k \in \mathbb{Z}} a^k \quad \text{a.e.}
\]

with for each \( k \in \mathbb{Z}, a^k \) is a \( w-1 \)-atom (\( w-2 \)-atom, \( w-3 \)-atom resp.) and satisfying \( s(a^k)(S(a^k), M(a^k) \text{ resp.}) \leq A \cdot 2^k \) for some \( A > 0 \), and

\[
\inf \left\{ c > 0 : \sup_{k \in \mathbb{Z}} \Phi \left( \frac{2^k}{c} \right) P(\nu_k < \infty) \leq 1 \right\} < \infty,
\]

where \( \nu_k \) is the stopping time corresponding to \( a^k \).

Moreover, define

\[
\|f\|_{\mathcal{wH}_{\Phi,\text{at}}^{s}} = \|f\|_{\mathcal{wH}_{\Phi,\text{at}}^{S}} = \inf \left\{ c > 0 : \sup_{k \in \mathbb{Z}} \Phi \left( \frac{2^k}{c} \right) P(\nu_k < \infty) \leq 1 \right\}
\]

where the infimum is taken over all decompositions of \( f \) described above.

Recall that for \( q \in [1, \infty) \) and a function \( \phi : (0, \infty) \to (0, \infty) \),

\[
\mathcal{wL}_{q,\phi} := \left\{ f \in L_q : \|f\|_{\mathcal{wL}_{q,\phi}} = \int_0^{\infty} t^q_{\phi}(x) dx < \infty \right\},
\]

where

\[
t^q_{\phi}(x) = \frac{1}{\phi(x)} x^{-1/q} \sup_{P(\nu < \infty) \leq x} \|f - f^\nu\|_{q},
\]

and \( \nu \) is a stopping time. Then we have

**Proposition 2.10.** If \( 1 \leq q_1 \leq q_2 < \infty \), then

\[
\|f\|_{\mathcal{wL}_{q_1,\phi}} \leq \|f\|_{\mathcal{wL}_{q_2,\phi}}.
\]

**Proof.** By Hölder’s inequality,

\[
t^q_{\phi}(x) = \frac{1}{\phi(x)} x^{-1/q_1} \sup_{P(\nu < \infty) \leq x} \left( \mathbb{E}(|f - f^\nu|^{q_1} \chi(\nu < \infty)) \right)^{1/q_1}
\]

\[
\leq \frac{1}{\phi(x)} x^{-1/q_1} \sup_{P(\nu < \infty) \leq x} \left( \mathbb{E}(|f - f^\nu|^{q_2})^{1/q_2} P(\nu < \infty)^{(1-q_1/q_2)/(1/q_1)} \right)
\]

\[
\leq \frac{1}{\phi(x)} x^{-1/q_2} \sup_{P(\nu < \infty) \leq x} \|f - f^\nu\|_{q_2} = t^{q_2}_{\phi}(x),
\]

which shows the proposition.
3 Weak Atomic Decompositions

We now are in a position to prove the weak atomic decomposition of the weak martingale Orlicz-Hardy spaces.

**Theorem 3.1.** Let \( \Phi \in G_\ell \) with \( \ell \in (0, 1] \) and \( q_{\Phi^{-1}} < \infty \). Then \( f \in wH^s_\Phi \) if and only if there exist a sequence of \( w-1 \)-atoms \( \{a^k\}_{k \in \mathbb{Z}} \) and corresponding stopping times \( \{\nu_k\}_{k \in \mathbb{Z}} \) such that

(i) \( f_n = \sum_{k \in \mathbb{Z}} E_n a^k \) a.e., \( \forall n \in \mathbb{N} \);

(ii) \( s(a^k) \leq A \cdot 2^k \) for some \( A > 0 \) and

\[
\inf \left\{ c > 0 : \sup_{k \in \mathbb{Z}} \Phi \left( \frac{2^k}{c} \right) P(\nu_k < \infty) \leq 1 \right\} < \infty.
\]

Moreover,

\[
\|f\|_{wH^s_\Phi} \approx \inf \left\{ c > 0 : \sup_{k \in \mathbb{Z}} \Phi \left( \frac{2^k}{c} \right) P(\nu_k < \infty) \leq 1 \right\},
\]

where the infimum is taken over all the preceding decompositions of \( f \). Consequently,

\( wH^s_\Phi = wH^s_{\Phi, at} \) with equivalent quasi norms.

**Proof.** Assume that \( f = (f_n)_{n \geq 0} \in wH^s_\Phi \). For \( k \in \mathbb{Z} \) and \( n \geq 0 \), let

\[
\nu_k = \inf \{ n : s_{n+1}(f) > 2^k \}, \quad a^k_n = f^\nu_{n+1} - f^\nu_n
\]

Then it’s clear that \( \{\nu_k\} \) is nondecreasing and that for any fixed \( k \in \mathbb{Z} \), \( a^k = (a^k_n)_{n \geq 0} \) is a martingale. Further we can see

\[
s(f^\nu_k) = s_{\nu_k}(f) \leq 2^k
\]

and

\[
\sum_{k \in \mathbb{Z}} a^k_n = \sum_{k \in \mathbb{Z}} (f^\nu_{n+1} - f^\nu_n) = f_n, \text{ a.e.}
\]

for all \( n \geq 0 \). Since \( s(f^\nu_k) \leq 2^k \), we have

\[
s(a^k) = \left( \sum_{n=1}^\infty \mathbb{E}_{n-1} |da^k_n|^2 \right)^{1/2} = \left( \sum_{n=1}^\infty \mathbb{E}_{n-1} |d(f^\nu_{n+1} - f^\nu_n)|^2 \right)^{1/2}
\]

\[
= \left( \sum_{n=1}^\infty \mathbb{E}_{n-1} |df_n \chi_{\{\nu_k < n \leq \nu_{k+1}\}}|^2 \right)^{1/2} = \left( \sum_{n=1}^\infty \mathbb{E}_{n-1} |df_n|^2 \right)^{1/2}
\]

\[
\leq s(f^\nu_{k+1}) \leq 2^{k+1} = 2 \cdot 2^k.
\]
Thus, \((a_n^k)_{n \geq 0}\) is \(L_2\)-bounded. Denote the limit still by \(a^k\). Then \(a_n^k = E_n a^k\) for all \(n \geq 0\). For \(\nu_k \geq n\), \(a_n^k = f_{\nu_{k+1}} - f_{\nu_k} = 0\). So \(a^k\) is a \(w\)-1-atom and (i) holds. Since \(\{\nu_k < \infty\} = \{s(f) > 2^k\}\), for any \(k \in \mathbb{Z}\) we have

\[
\Phi\left(\frac{2^k}{\|f\|_{wH^s}}\right) P(\nu_k < \infty) = \Phi\left(\frac{2^k}{\|f\|_{wH^s}}\right) P(s(f) > 2^k) \leq 1.
\]

Thus

\[
\inf \left\{ c > 0 : \sup_{k \in \mathbb{Z}} \Phi\left(\frac{2^k}{c}\right) P(\nu_k < \infty) \leq 1 \right\} \leq \|f\|_{wH^s} < \infty.
\]

The main part of the proof is the converse. Suppose that there exists a sequence of \(w\)-1-atoms such that (i) and (ii) hold. Let

\[
M = \inf \left\{ c > 0 : \sup_{k \in \mathbb{Z}} \Phi\left(\frac{2^k}{c}\right) P(\nu_k < \infty) \leq 1 \right\}.
\]

Without loss of generality, we may assume that \(A = 2^{N_A}\), where \(N_A \geq 0\) (Since for any \(A > 0\), there exist \(N_A \geq 0\) such that \(A \leq 2^{N_A}\)). For any \(\lambda > 0\), choose \(j \in \mathbb{Z}\) such that \(2^j \leq \lambda < 2^{j+1}\). Now let

\[
f_n = \sum_{k=-\infty}^{\infty} a_n^k = \sum_{k=-\infty}^{j-1} a_n^k + \sum_{k=j}^{\infty} a_n^k = g_n + h_n \quad (n \in \mathbb{N}). \tag{3.2}
\]

Then we have \(s(f) \leq s(g) + s(h)\) by the sublinearity of \(s(f)\). And thus

\[
P(s(f) > 2A\lambda) \leq P(s(g) > A\lambda) + P(s(h) > A\lambda).
\]

By (ii) we obtain

\[
s(g) \leq \sum_{k=-\infty}^{j-1} s(a^k) \leq \sum_{k=-\infty}^{j-1} A \cdot 2^k \leq A \cdot 2^j \leq A\lambda.
\]

So \(P(s(g) > A\lambda) = 0\). Since \(a_n^k = E_n a^k = 0\) on the set \(\{\nu_k \geq n\}\), thus \(s(a^k) = 0\) on \(\{\nu_k = \infty\}\). Denote \(p = p_{\Phi^{-1}}, q = q_{\Phi^{-1}}\), then by Lemma 2.6 we have

\[
\Phi\left(\frac{2A\lambda}{4M}\right) P(s(f) > 2A\lambda) \leq \Phi\left(\frac{A\lambda}{M}\right) P(s(h) > A\lambda)
\]

\[
\leq \Phi\left(\frac{A\lambda}{M}\right) \sum_{k=j}^{\infty} P(\nu_k < \infty)
\]

\[
\leq \Phi\left(\frac{A\lambda}{M}\right) \sum_{k=j}^{\infty} \frac{1}{\Phi\left(\frac{2^k}{M}\right)}.
\]

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Using the assumption $A = 2^{NA_0}$, we obtain $2^j \leq 2^{NA_0+j} \leq A\lambda < 2^{NA_0+j+1}$, where $N_A \geq 0$. Thus,

$$\Phi \left( \frac{2A\lambda}{4M} \right) P(s(f) > 2A\lambda) \leq \Phi \left( \frac{A\lambda}{M} \right) \sum_{k=j}^{N_A+j} \frac{1}{\Phi \left( \frac{2k}{M} \right)}$$

$$+ \Phi \left( \frac{A\lambda}{M} \right) \sum_{k=N_A+j+1}^{\infty} \frac{1}{\Phi \left( \frac{2k}{M} \right)}$$

$$= I + II.$$

Now let’s estimate $I$ and $II$ respectively. Using Lemma 2.6 again, we get

$$I = \sum_{k=j}^{N_A+j} \frac{1}{\Phi \left( \frac{2k}{M} \right)} \leq \sum_{k=j}^{N_A+j} \left( \frac{2\lambda}{2k} \right)^{\frac{1}{q}}$$

$$\leq \sum_{k=j}^{N_A+j} \left( \frac{2^{N_A+j+1-k}}{2k} \right)^{\frac{1}{q}} \leq \frac{2^{\frac{1}{q}(N_A+1)}}{1 - 2^{-\frac{1}{q}}}$$

$$= C_1$$

and

$$II = \sum_{k=N_A+j+1}^{\infty} \frac{1}{\Phi \left( \frac{2k}{M} \right)} \leq \sum_{k=N_A+j+1}^{\infty} \left( \frac{A\lambda}{2k} \right)^{\frac{1}{q}}$$

$$\leq \sum_{k=N_A+j+1}^{\infty} \left( \frac{2^{N_A+j+1-k}}{2k} \right)^{\frac{1}{q}} = \frac{1}{1 - 2^{-\frac{1}{q}}}$$

$$= C_2.$$

Let $C_0 = C_1 + C_2$. It’s easy to see that $C_0 > 1$. Thus

$$\Phi \left( \frac{2A\lambda}{(C_0 \cdot c_\Phi)^{1/4}M} \right) P(s(f) > 2A\lambda) \leq \frac{1}{C_0 \cdot c_\Phi} \Phi \left( \frac{2A\lambda}{4M} \right) P(s(f) > 2A\lambda)$$

$$\leq \frac{1}{C_0 \cdot c_\Phi} \cdot C_0 = 1.$$

And so we obtain

$$\|f\|_{wH^s_\Phi} \leq (C_0 \cdot c_\Phi)^{1/4} \cdot M. \quad (3.3)$$

Consequently, (3.1) holds. The proof of Theorem 3.1 is complete.

**Remark 3.2.** If $f \in wH^s_\Phi$ in Theorem 3.1, then not only (i) and (ii) hold, but also the sum $\sum_{k=m}^{n} a^k$ converges to $f$ in $wH^s_\Phi$ as $m \rightarrow -\infty$, $n \rightarrow \infty$. Indeed,

$$\sum_{k=m}^{n} a^k = \sum_{k=m}^{n} (f^{\nu_{k+1}} - f^{\nu_k}) = f^{\nu_{n+1}} - f^{\nu_m}.$$
By the sublinearity of $s(f)$ we have
\[ \| f - \sum_{k=m}^{n} a^k \|_{wH_\Phi^S} = \| s(f - f^{\nu_{n+1}} + f^{\nu_m}) \|_{wL_\Phi} \leq \| s(f - f^{\nu_{n+1}}) \|_{wL_\Phi} + \| s(f^{\nu_m}) \|_{wL_\Phi} \]
\[ \leq (2c_\Phi)^{1/\ell} \cdot 2 \left( \| s(f - f^{\nu_{n+1}}) \|_{wL_\Phi} + \| s(f^{\nu_m}) \|_{wL_\Phi} \right). \]

Since $s(f - f^{\nu_{n+1}})^2 = s(f)^2 - s(f^{\nu_{n+1}})^2$, then $s(f - f^{\nu_{n+1}}) \leq s(f)$, $s(f^{\nu_m}) \leq s(f)$ and $s(f - f^{\nu_{n+1}}), s(f^{\nu_m}) \to 0$ a.e. as $m \to -\infty, n \to \infty$. Thus by Proposition 2.5, we have
\[ \| s(f - f^{\nu_{n+1}}) \|_{wL_\Phi}, \| s(f^{\nu_m}) \|_{wL_\Phi} \to 0 \quad \text{as } m \to -\infty, n \to \infty, \]
which means $\| f - \sum_{k=m}^{n} a^k \|_{wH_\Phi^S} \to 0$ as $m \to -\infty, n \to \infty$. Further, for $k \in \mathbb{Z}$, $a^k = (a^k_n)_{n \geq 0}$ is $L_2$ bounded, hence $H_2^S = L_2$ is dense in $wH_\Phi^S$.

Recall that if $(\mathcal{F}_n)_{n \geq 0}$ is regular, then for any non-negative adapted sequence $\gamma = (\gamma_n)_{n \geq 0}$ and $\lambda \geq \| \gamma_0 \|_\infty$, there is a stopping time $\nu$ such that
\[ \{ \gamma^* > \lambda \} \subset \{ \nu < \infty \}, \quad \gamma^*_\nu \leq \lambda, \quad P(\nu < \infty) \leq RP(\gamma^* > \lambda) \]
(see [8]). Moreover, if $\lambda_1 \leq \lambda_2$, then we can take two stopping times $\nu_{\lambda_1}$ and $\nu_{\lambda_2}$ such that $\nu_{\lambda_1} \leq \nu_{\lambda_2}$. Therefore, if $(\mathcal{F}_n)_{n \geq 0}$ is regular, we get the atomic decompositions for $wH_\Phi^S$ and $wH_\Phi$.

**Theorem 3.3.** Let $\Phi \in \mathcal{G}_\ell$ with $\ell \in (0, 1]$ and $q_{\Phi^{-1}} < \infty$. Then, if $(\mathcal{F}_n)_{n \geq 0}$ is regular, we have
\[ wH_\Phi^S = wH_\Phi^S_{\text{at}} \quad \text{with equivalent quasi norms;} \]
\[ wH_\Phi = wH_\Phi^S_{\text{at}} \quad \text{with equivalent quasi norms.} \]

Moreover, if $f \in w\mathcal{H}_\Phi^S$ (or $w\mathcal{H}_\Phi$ resp.), the sum $\sum_{k=m}^{n} a^k$ converges to $f$ in $wH_\Phi^S$ (or $wH_\Phi$ resp.) as $m \to -\infty, n \to \infty$.

**Proof.** The proof shall be given for $wH_\Phi^S$, only, since it is just slightly different from the one for $wH_\Phi$. Let $f \in wH_\Phi^S$. Then for sequence $S_n(f)$ and $k \in \mathbb{Z}$, take stopping times $\nu_k$ such that
\[ \{ S(f) > 2^k \} \subset \{ \nu_k < \infty \}, \quad S_{\nu_k}(f) \leq 2^k, \quad P(\nu_k < \infty) \leq RP(S(f) > 2^k) \]
and $\nu_k \leq \nu_{k+1}$, $\nu_k \uparrow \infty$. Still define $a^k_n = f^{\nu_k+1}_n - f^{\nu_k}_n$, then $a^k = (a^k_n)_{n \geq 0}$ is a martingale and
\[ S(a^k) = \left( \sum_{n=1}^{\infty} |da^k_n|^2 \right)^{1/2} = \left( \sum_{n=1}^{\infty} |df_n|_{\nu_k < \nu_{k+1}}^2 \right)^{1/2} \leq S(f) \leq 2^{k+1} = 2 \cdot 2^k. \]
Thus, $(a^k_n)_{n \geq 0}$ is $L_2$-bounded. Denote the limit still by $a^k$. Then $a^k_n = E_na^k$ for all $n \geq 0$. For $\nu_k \geq n$, $a^k_n = f^{\nu_k+1}_n - f^{\nu_k}_n = f_n - f_n = 0$. So $a^k$ is a w-2-atom and
\[ f_n = \sum_{k \in \mathbb{Z}} a_k^n \text{ a.e.} \] Since \( P(\nu_k < \infty) \leq RP(S(f) > 2^k) \). Then let \( C_0 = \max\{R, 1\} \), we have

\[
\Phi \left( \frac{2^k}{(c\Phi C_0)^{1/\ell} \|f\|_{wH^S_{\phi}}} \right) P(\nu_k < \infty) \leq c\Phi \cdot \frac{1}{c\Phi C_0} \Phi \left( \frac{2^k}{\|f\|_{wH^S_{\phi}}} \right) \cdot RP(s(f) > 2^k)
\]

\[
\leq \Phi \left( \frac{2^k}{\|f\|_{wH^S_{\phi}}} \right) P(s(f) > 2^k) \leq 1,
\]

which means

\[
\inf \left\{ c > 0 : \sup_{k \in \mathbb{Z}} \Phi \left( \frac{2^k}{c} \right) P(\nu_k < \infty) \leq 1 \right\} \leq (c\Phi C_0)^{1/\ell} \|f\|_{wH^S_{\phi}} < \infty.
\]

Conversely, suppose that \( f \in wH^S_{\phi, at} \). Let

\[ M = \inf \left\{ c > 0 : \sup_{k \in \mathbb{Z}} \Phi \left( \frac{2^k}{c} \right) P(\nu_k < \infty) \leq 1 \right\}. \]

Without loss of generality, here we also assume that \( A = 2^{N_A} \), where \( N_A \geq 0 \). For any \( \lambda > 0 \), choose \( j \in \mathbb{Z} \) such that \( 2^j \leq \lambda < 2^{j+1} \). Define \( f^N \) in the same way as in (5.2), then similarly, we obtain

\[ S(g) \leq \sum_{k=-\infty}^{j-1} S(a^k) \leq A\lambda. \]

And since \( a_k^n = \mathbb{E}_n a^k = 0 \) on the set \( \{\nu_k \geq n\} \), \( S(a^k) = 0 \) on \( \{\nu_k = \infty\} \). Thus,

\[ \Phi \left( \frac{2A\lambda}{4M} \right) P(S(f) > 2A\lambda) \leq \Phi \left( \frac{A\lambda}{M} \right) P(S(h) > A\lambda) \]

\[ \leq \Phi \left( \frac{A\lambda}{M} \right) \sum_{k=j}^{\infty} P(\nu_k < \infty) \]

Dealing with the last inequality in the same way as in Theorem 3.1, we obtain

\[ \|f\|_{wH^S_{\phi}} \leq CM. \]

Further, if \( f \in w\mathcal{H}^S_{\phi} \), just as Remark 3.2, by Proposition 2.5 we get that the sum \( \sum_{k=m}^{n} a^k \) converges to \( f \) in \( wH^S_{\phi, at} \) as \( m \to -\infty, n \to \infty \). The proof of Theorem 3.2 is complete.

**Theorem 3.4.** Let \( \Phi \in \mathcal{G}_\ell \) with \( \ell \in (0, 1] \) and \( q_{\Phi^{-1}} < \infty \). Then

\[ w\mathcal{Q}_{\Phi} = wH^S_{\phi, at} \quad \text{with equivalent quasi norms;} \]

\[ w\mathcal{D}_{\Phi} = wH_{\phi, at} \quad \text{with equivalent quasi norms.} \]
Proof. The proof of Theorem 3.4 is similar to that of Theorem 3.1. So we just sketch the outline and omit the details. Suppose that \( f = (f_n)_{n \geq 0} \in wQ_\Phi \) (or \( f = (f_n)_{n \geq 0} \in wD_\Phi \) resp.). Let \( \nu_k = \inf\{n : \lambda_n > 2^k\} \), where \( \lambda_n \) is the sequence in the definition of \( wQ_\Phi \) (or \( wD_\Phi \), resp.). For \( k \in \mathbb{Z} \) and \( n \geq 0 \), we still define \( a_n^k = f_n^{k+1} - f_n^k \).

Then, in the same way as in Theorem 3.1, we can prove that there exists \( A > 0 \) such that \( S(a^k) \leq A \cdot 2^k \) (or \( M(a^k) \leq A \cdot 2^k \) resp.), and that \( \|f\|_{wH^S_{\Phi, at}} \leq C\|f\|_{wQ_\Phi} \) (or \( \|f\|_{wH^S_{\Phi, at}} \leq C\|f\|_{wD_\Phi} \), resp.).

For the converse part, assume that \( f = (f_n)_{n \geq 0} \in wH^S_{\Phi, at} \) (or \( f = (f_n)_{n \geq 0} \in wH_{\Phi, at} \) resp.), and let \( \lambda_n = \sum_{k \in \mathbb{Z}} \chi(\nu_k \leq n)\|S(a^k)\|_\infty \) (or \( \lambda_n = \sum_{k \in \mathbb{Z}} \chi(\nu_k \leq n)\|M(a^k)\|_\infty \) resp.). Then \( (\lambda_n)_{n \geq 0} \) is a non-negative, non-decreasing and adapted sequence with \( S_{n+1}(f) \leq \lambda_n \) (or \( M_{n+1}(f) \leq \lambda_n \) resp.). For \( y > 0 \), choose \( j \in \mathbb{Z} \) such that \( 2^j \leq y < 2^{j+1} \). Then \( \lambda_\infty = \lambda^{(1)} + \lambda^{(2)} \) with \( \lambda^{(1)} = \sum_{k = -\infty}^{j-1} \chi(\nu_k \leq n)\|S(a^k)\|_\infty \) \( \lambda^{(2)} = \sum_{k = j}^{\infty} \chi(\nu_k \leq n)\|M(a^k)\|_\infty \) (or \( \lambda^{(1)} = \sum_{k = -\infty}^{j-1} \chi(\nu_k \leq n)\|M(a^k)\|_\infty \) \( \lambda^{(2)} = \sum_{k = j}^{\infty} \chi(\nu_k \leq n)\|M(a^k)\|_\infty \) resp.). Similar to the argument of (3.3) (replacing \( s(g) \) and \( s(h) \) by \( \lambda^{(1)} \) and \( \lambda^{(2)} \), resp.), we obtain \( \|f\|_{wQ_\Phi} \leq C\|f\|_{wH^S_{\Phi, at}} \) (or \( \|f\|_{wD_\Phi} \leq C\|f\|_{wH_{\Phi, at}} \), resp.).

Theorem 3.3 and Theorem 3.4 together give the following corollary.

**Corollary 3.5.** Let \( \Phi \in G_\ell \) with \( \ell \in (0, 1] \) and \( q_{\Phi^{-1}} < \infty \). If \( (F_n)_{n \geq 0} \) is regular, then
\[
\|f\|_{wQ_\Phi} = \|f\|_{wH^S_{\Phi, at}}, \quad \|f\|_{wD_\Phi} = \|f\|_{wH_{\Phi, at}}.
\]

### 4 Bounded operators on weak Orlicz-Hardy spaces

As one of the applications of the atomic decompositions, we shall obtain a sufficient condition for a sublinear operator to be bounded from the weak martingale Orlicz-Hardy spaces to \( wL_\Phi \) spaces. Applying the condition to \( M(f), S(f) \) and \( s(f) \), we deduce a series of martingale inequalities.

An operator \( T : X \to Y \) is called a sublinear operator if it satisfies
\[
|T(f + g)| \leq |Tf| + |Tg|, \quad |T(\alpha f)| \leq |\alpha||Tf|,
\]
where \( X \) is a martingale spaces, \( Y \) is a measurable function space.

**Theorem 4.1.** Let \( 1 \leq r \leq 2 \) and \( T : L_r(\Omega) \to L_r(\Omega) \) be a bounded sublinear operator. If
\[
P(|Ta| > 0) \leq CP(\nu < \infty)
\]
for all \( w-1 \)-atoms, where \( \nu \) is the stopping time associated with \( a \) and \( C \) is a positive constant, then, for \( \Phi \in G_\ell \) with \( q_{\Phi^{-1}} < \infty \) and \( 1/p_{\Phi^{-1}} < r \), there exists a positive constant \( C \) such that
\[
\|Tf\|_{wL_\Phi} \leq C\|f\|_{wH^S_{\Phi}}, \quad f \in wH^S_{\Phi}.
\]

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Proof. Assume that $f \in wH^s_\Phi$. By Theorem 3.1, $f$ can be decomposed into the sum of a sequence of w-1-atoms. For any fixed $\lambda > 0$, choose $j \in \mathbb{Z}$ such that $2^{j-1} \leq \lambda < 2^j$ and let

$$f = \sum_{k=-\infty}^{\infty} a^k = \sum_{k=-\infty}^{j-2} a^k + \sum_{k=j}^{\infty} a^k := g + h.$$  

It follows from the sublinearity of $T$ that $|Tf| \leq |Tg| + |Th|$, so

$$P(|Tf| > 2\lambda) \leq P(|Tg| > \lambda) + P(|Th| > \lambda).$$

In Theorem 3.1, we have proved that $s(a^k) \leq A \cdot 2^k$ for some $A > 0$ and $s(a^k) = 0$ on the set $\{\nu_k = \infty\}$. Denote $p = p_{\Phi-1}$, $q = q_{\Phi-1}$. Remember that

$$\|a^k\|_r \leq C s(a^k)_{\infty}, \quad 1 \leq r \leq 2.$$  

(4.2)

It results from Lemma 2.6 that

$$\|g\|_r \leq \sum_{k=-\infty}^{j-1} \|a^k\|_r \leq C \sum_{k=-\infty}^{j-1} \|s(a^k)\|_r$$

$$\leq C \sum_{k=-\infty}^{j-1} 2^k P(\nu_k < \infty)^{1/r}.$$  

Since $T$ is bounded on $L_r(\Omega)$, then

$$\Phi\left(\frac{\lambda}{\|f\|_{wH^s_\Phi}}\right) P(|Tg| > \lambda) \leq \Phi\left(\frac{\lambda}{\|f\|_{wH^s_\Phi}}\right) \|Tg\|_r^r$$

$$\leq C \Phi\left(\frac{\lambda}{\|f\|_{wH^s_\Phi}}\right) \|g\|_r^r.$$  

By the estimate of $\|g\|_r$ above, we get

$$\Phi\left(\frac{\lambda}{\|f\|_{wH^s_\Phi}}\right) P(|Tg| > \lambda) \leq C \Phi\left(\frac{\lambda}{\|f\|_{wH^s_\Phi}}\right) \left(\sum_{k=-\infty}^{j-1} 2^k P(\nu_k < \infty)^{1/2}\lambda\right)^r$$

$$= C \Phi\left(\frac{\lambda}{\|f\|_{wH^s_\Phi}}\right) \left(\sum_{k=-\infty}^{j-1} 2^k \Phi\left(\frac{2^k}{\|f\|_{wH^s_\Phi}}\right)^{1/2} P(\nu_k < \infty)^{1/2}\frac{\lambda}{\lambda^r}\right)^r$$

$$\leq C \left(\sum_{k=-\infty}^{j-1} 2^k \Phi\left(\frac{2^k}{\|f\|_{wH^s_\Phi}}\right)^{1/2}\frac{\lambda}{\lambda^r}\right)^r.$$
Using Lemma 2.6 and the condition $\frac{1}{p} = \frac{1}{p_{\Phi^{-1}}} < r$, we obtain

$$
\Phi\left(\frac{\lambda \|f\|_{wH_\Phi}}{C_1\|f\|_{wH_\Phi}}\right) P(|Tg| > \lambda) \leq C \left( \sum_{k=-\infty}^{j-1} \left( \frac{2^k}{\lambda} \right)^{\frac{1}{p}} \right)^r
= C \lambda^{\frac{1}{p} - r} \cdot \left( \sum_{k=1-j}^{\infty} \left( \frac{1}{2} \right)^{1 - \frac{1}{p}} k \right)^r
\leq C \lambda^{\frac{1}{p} - r} \cdot (2^{j-1})^{r - \frac{1}{p}} \leq C_1.
$$

Taking $C_1 = (2c_{\Phi} \max\{C_1, 1\})^{1/\ell}$, then

$$
\Phi\left(\frac{\lambda \|f\|_{wH_\Phi}}{C_1\|f\|_{wH_\Phi}}\right) P(|Tg| > \lambda) \leq \frac{1}{2} C_{\Phi} \max\{C_1, 1\} \Phi\left(\frac{\lambda \|f\|_{wH_\Phi}}{\|f\|_{wH_\Phi}}\right) P(|Tg| > \lambda)
\leq \frac{1}{2} C_{\Phi} \max\{C_1, 1\} C_1 \leq \frac{1}{2}.
$$

On the other hand, since $|Th| \leq \sum_{k=j}^{\infty} |Ta^k|$, we get

$$
\Phi\left(\frac{\lambda \|f\|_{wH_\Phi}}{C_1\|f\|_{wH_\Phi}}\right) P(|Th| > \lambda) \leq \Phi\left(\frac{\lambda \|f\|_{wH_\Phi}}{\|f\|_{wH_\Phi}}\right) P(|Th| > 0)
\leq \Phi\left(\frac{\lambda \|f\|_{wH_\Phi}}{\|f\|_{wH_\Phi}}\right) \sum_{k=j}^{\infty} P(|Ta^k| > 0)
\leq C \Phi\left(\frac{\lambda \|f\|_{wH_\Phi}}{\|f\|_{wH_\Phi}}\right) \sum_{k=j}^{\infty} P(\nu_k < \infty)
\leq C \sum_{k=j}^{\infty} \frac{\Phi\left(\frac{\lambda \|f\|_{wH_\Phi}}{\|f\|_{wH_\Phi}}\right)}{\Phi\left(\frac{2^k \|f\|_{wH_\Phi}}{\|f\|_{wH_\Phi}}\right)} \Phi\left(\frac{2^k \|f\|_{wH_\Phi}}{\|f\|_{wH_\Phi}}\right) P(\nu_k < \infty)
\leq C \sum_{k=j}^{\infty} \lambda^{\frac{1}{q}} \leq C_2.
$$

Taking $C_2 = (2c_{\Phi} \max\{C_2, 1\})^{1/\ell}$, then

$$
\Phi\left(\frac{\lambda \|f\|_{wH_\Phi}}{C_2\|f\|_{wH_\Phi}}\right) P(\Phi(|Th|) > \lambda) \leq \frac{1}{2}.
$$

Since $T$ is sublinear,

$$
\Phi\left(\frac{2\lambda}{2(C_1 + C_{II})\|f\|_{wH_\Phi}}\right) P(|Tf| > 2\lambda) \leq \Phi\left(\frac{\lambda \|f\|_{wH_\Phi}}{C_1\|f\|_{wH_\Phi}}\right) P(|Tg| > \lambda)
+ \Phi\left(\frac{\lambda \|f\|_{wH_\Phi}}{C_{II}\|f\|_{wH_\Phi}}\right) P(|Th| > \lambda)
\leq 1.
$$
Hence,
\[ \|T f\|_{wL^\Phi} \leq C \|f\|_{wH^\Phi}, \quad f \in wH^\Phi. \]
The proof is complete.

**Remark 4.2.** Similarly, if \( T : L_r(\Omega) \to L_r(\Omega) \) is a bounded sublinear operator, \( 1 \leq r < \infty \), and (4.1) holds for all \( w \)-2-atoms (or \( w \)-3-atoms). Then for \( \Phi \in \mathcal{G}_\ell \) with \( q_{\Phi^{-1}} < \infty \) and \( \frac{1}{p_{\Phi^{-1}}} < r \), there exists a constant \( C > 0 \) such that
\[ \|T f\|_{wL^\Phi} \leq C \|f\|_{wQ}, \quad f \in wQ, \]
(\text{or} \quad \|T f\|_{wL^\Phi} \leq C \|f\|_{wD}, \quad f \in wD). \]
In this case, we do not need to restrict \( 1 \leq r \leq 2 \); in fact, (4.2) is replaced by
\[ \|a_k\|_r \leq C \|S(a_k)\|_r \quad (or \quad \|a_k\|_r \leq C \|M(a_k)\|_r), \]
which always holds for \( 1 \leq r < \infty \) by the Burkholder-Davis-Gundy inequalities.

**Theorem 4.3.** Let \( \Phi \in \mathcal{G}_\ell \) with \( \ell \in (0, 1] \) and \( q_{\Phi^{-1}} < \infty \). Then for all martingales \( f = (f_n)_{n \geq 0} \) the following inequalities hold:
\[ \|f\|_{wH^\Phi} \leq C \|f\|_{wH^\Phi}, \quad \|f\|_{wH^\Phi} \leq C \|f\|_{wH^\Phi}; \]  \( (4.3) \)
\[ \|f\|_{wH^\Phi} \leq C \|f\|_{wQ}, \quad \|f\|_{wH^\Phi} \leq C \|f\|_{wQ}; \]  \( (4.4) \)
\[ \|f\|_{wH^\Phi} \leq C \|f\|_{wD}, \quad \|f\|_{wH^\Phi} \leq C \|f\|_{wD}; \]  \( (4.5) \)
\[ C^{-1} \|f\|_{wD} \leq \|f\|_{wQ} \leq C \|f\|_{wD}. \]  \( (4.6) \)
Moreover, if \( \{F_n\}_{n \geq 0} \) is regular, then \( wH^S = wQ = wD = w\Phi = wH^\Phi \).

**Proof.** First we show (4.3). Let \( f \in wH^\Phi \). The maximal operator \( T f = M f \) is sublinear. It’s well known that \( T \) is \( L_2 \)-bounded. If \( a \) is a \( w \)-1-atom and \( \nu \) is the stopping time associated with \( a \), then \( \{|Ta| > 0\} = \{M(a) > 0\} \subset \{\nu < \infty\} \) and hence (4.1) holds. Since \( \Phi \in \mathcal{G}_\ell \), the condition \( \frac{1}{p_{\Phi^{-1}}} < 2 \) always holds for convex function \( \Phi^{-1} \). Thus it follows from Theorem 4.1 that
\[ \|f\|_{wH^\Phi} = \|T f\|_{wL^\Phi} \leq C \|f\|_{wH^\Phi}. \]
Similarly, considering the operator \( T f = S f \) we get the second inequality of (4.3).

Next we show (4.4) and (4.5). Choose \( r \) such that \( \frac{1}{p_{\Phi^{-1}}} < 2 \) and \( r < \infty \). Noticing that the operator \( M f, S f \) and \( sf \) are \( L_r \) bounded. Taking \( T f = M f, S f \) or \( sf \), resp. By Remark 4.2, we get (4.3) and (4.5).

To prove (4.6), we use (4.4) and (4.5). The method used below is the same as the proof of Theorem 3.5 in [17]. Assume that \( f = (f_n)_{n \geq 0} \in wQ \), then there exists an optimal control \( (\lambda^{(1)}_n)_{n \geq 0} \) such that \( S_n(f) \leq \lambda^{(1)}_n \) with \( \lambda^{(1)}_n \in wL^\Phi \). Since
\[ |f_n| \leq f^{*}_{n-1} + \lambda^{(1)}_n, \]
by \((4.4)\) we have
\[
\|f\|_{wD_\Phi} \leq C(\|f\|_{wH_\Phi} + \|\lambda_\infty^{(1)}\|_{wL_\Phi}) \leq C\|f\|_{wQ_\Phi}.
\]
On the other hand, if \(f = (f_n)_{n \geq 0} \in wD_\Phi\), then there exists an optimal control \((\lambda^{(2)}_n)_{n \geq 0}\)

such that \(|f_n| \leq \lambda^{(2)}_{n-1}\) with \(\lambda^{(2)}_n \in wL_\Phi\). Notice that
\[
S_n(f) \leq S_{n-1}(f) + 2\lambda^{(2)}_{n-1}.
\]
Using \((4.5)\) we can get the other side of \((4.6)\).

Further, suppose that \(\{\mathcal{F}_n\}_{n \geq 0}\) is regular. Then for any martingale \(f = (f_n)_{n \geq 0}\), we have \(|df_n|^2 \leq \frac{R-1}{2}E_{n-1}|df_n|^2\) (see [21], pp 31, Proposition 2.19). Thus
\[
S_n(f) \leq \sqrt{\frac{R-1}{2}}s_n(f).
\]
Since \(s_n(f) \in \mathcal{F}_{n-1}\), by the definition of \(wQ_\Phi\) we have
\[
\|f\|_{wQ_\Phi} \leq \|s(f)\|_{wL_\Phi} = \|f\|_{wH_\Phi^*}.
\]
Using \((4.4)\), \((4.6)\) and Corollary 3.5, we conclude that
\[
wH_\Phi^s = wQ_\Phi = wD_\Phi = wH_\Phi = wH_\Phi^s.
\]

5 The duality results

In this section, we investigate the dual of weak martingale Orlicz-Hardy spaces and give a new John-Nirenberg theorem.

**Theorem 5.1.** Let \(\Phi \in \mathcal{G}_\ell\) with \(\ell \in (0, 1]\), \(q_\Phi-1 < \infty\) and \(\phi(r) = 1/(r\Phi^{-1}(1/r))\). Then
\[
(w, \mathcal{H}^s_\Phi)^* = w\mathcal{L}_{2,\phi}.
\]

**Proof.** Let \(g \in w\mathcal{L}_{2,\phi}\), then \(g \in H^s_2\). Define
\[
l_g(f) = E\left(\sum_{n=1}^{\infty} df_n dg_n\right), \quad f \in H^s_2.
\]
From Theorem 3.1, there is a sequence of \(w\)-\(1\)-atoms \(a^k\) and corresponding stopping times \(\nu_k\), where \(k \in \mathbb{Z}\), such that
\[
s(a^k) \leq 2^{k+1}, \quad \Phi\left(\frac{2^k}{\|f\|_{wH_\Phi^*}}\right) P(\nu_k < \infty) \leq 1
\]
and

\[ df_n = \sum_{k=\infty}^{\infty} da_n^k \text{ a.e.} \]

for all \( n \in \mathbb{N} \). The last series also converges to \( df_n \) in \( H_2^s \)-norm. Hence

\[ l_g(f) = \sum_{n=1}^{\infty} \sum_{k=\infty}^{\infty} \mathbb{E}(da_n^k dg_n). \]

Applying the Hölder inequality and the definition of weak atoms, we get that

\[
|l_g(f)| \leq \sum_{k=\infty}^{\infty} \mathbb{E} \left( \sum_{n=1}^{\infty} |da_n^k| \chi_{\{\nu_k < n\}} |dg_n| \right)
\leq \sum_{k=\infty}^{\infty} \left( \mathbb{E} \sum_{n=1}^{\infty} |da_n^k|^2 \right)^{1/2} \left( \mathbb{E} \sum_{n=1}^{\infty} |dg_n|^2 \chi_{\{\nu_k < n\}} \right)^{1/2}
= \sum_{k=\infty}^{\infty} \left( \mathbb{E} \sum_{n=1}^{\infty} |da_n^k|^2 \right)^{1/2} \|S(g - g^{\nu_k})\|_2.
\]

Since \( \left( \mathbb{E} \sum_{n=1}^{\infty} |da_n^k|^2 \right)^{1/2} = \left( \mathbb{E}(S^2(a^k)) \right)^{1/2} = \|s(a^k)\|_2 \leq 2^{k+1} P(\nu_k < \infty)^{1/2} \) and \( P(\nu_k < \infty) \leq 1/\Phi \left( \frac{2^k}{\|f\|_{wH^s_\Phi}} \right) \), then

\[
|l_g(f)| \leq \sum_{k=\infty}^{\infty} 2^{k+1} P(\nu_k < \infty)^{1/2} \|g - g^{\nu_k}\|_2
\leq \sum_{k=\infty}^{\infty} \left( 2^k \Phi \left( \frac{\|f\|_{wH^s_\Phi}}{2^k} \right) \right)^{-1/2} \|g - g^{\nu_k}\|_2.
\]

Let \( A_k = 1/\Phi \left( \frac{2^k}{\|f\|_{wH^s_\Phi}} \right) \) and still denote \( p = p_{\Phi-1}, q = q_{\Phi-1} \). By Lemma 2.6, we obtain

\[
|l_g(f)| \leq 2 \|f\|_{wH^s_\Phi} \sum_{k=\infty}^{\infty} 2^k \frac{2^k}{\|f\|_{wH^s_\Phi}} \cdot A_k^{1/2} \|g - g^{\nu_k}\|_2
\leq 2 \|f\|_{wH^s_\Phi} \sum_{k=\infty}^{\infty} \left( \frac{1}{\phi(A_k)} A_k^{-1/2} \right) \sup_{P(\nu_k < \infty) \leq A_k} \|g - g^{\nu_k}\|_2
= 2^q \|f\|_{wH^s_\Phi} \sum_{k=\infty}^{\infty} t_\phi^2(A_k).
\]

Using Lemma 2.6 again, we get

\[
\frac{A_{k+1}}{A_k} = \frac{\Phi \left( \frac{2^k}{\|f\|_{wH^s_\Phi}} \right)}{\Phi \left( \frac{2^{k+1}}{\|f\|_{wH^s_\Phi}} \right)} \leq \left( \frac{2^k}{2^{k+1}} \right)^{1/q} = \left( \frac{1}{2} \right)^{1/q}.
\]
Thus,

\[
\sum_{k=-\infty}^{\infty} t_{\phi}^2(A_k) = \sum_{k=-\infty}^{\infty} \frac{t_{\phi}^2(A_k)(A_k - A_{k+1})}{A_k - A_{k+1}} \leq \frac{1}{1 - (\frac{1}{2})^{1/q}} \sum_{k=-\infty}^{\infty} \frac{t_{\phi}^2(A_k)(A_k - A_{k+1})}{A_k}
\]

\[
\leq C \int_{0}^{\infty} \frac{t_{\phi}^2(x)}{x} dx = C \|g\|_{wL_{2,\phi}}.
\]

And so

\[
|l_g(f)| \leq C \|f\|_{wH_{\phi}^s} \|g\|_{wL_{2,\phi}}.
\]

Since $H_{2,\phi}^s$ is dense in $wH_{\phi}^s$ (see Remark 3.2), $l$ can be uniquely extended to a continuous linear functional on $wH_{\phi}^s$.

Conversely, let $l \in (wH_{\phi}^s)^*$. Note that $H_{2,\phi}^s \subset wH_{\phi}^s$, hence $l \in (H_{2,\phi}^s)^*$. That means there exists $g \in H_{2,\phi}^s$ such that

\[
l(f) = E\left(\sum_{n=1}^{\infty} d f_n d g_n\right), \quad f \in H_{2,\phi}^s.
\]

Let $\nu_k$ be the stopping times satisfying $P(\nu_k < \infty) \leq 2^{-k}$ ($k \in \mathbb{Z}$). For $k \in \mathbb{Z}$, we define

\[
a_k = \frac{g - g_{\nu_k}}{(2^k)^{1/2} \frac{1}{\Phi^{-1}(2^k)} \|s(g - g_{\nu_k})\|_2}.
\]

The function $a_k$ is not necessarily a weak atom. However, it satisfies the condition (i) of Definition 2.8, namely, $a_k = 0$ on the set $\{\nu_k \geq n\}$. For $\lambda > 0$, choose $j \in \mathbb{Z}$ such that $2^j \leq \lambda < 2^{j+1}$ and define the martingales $f^N, g^N$ and $h^N$, respectively, by

\[
f^N_n = \sum_{k=-N}^{N} a^k_n, \quad g^N_n = \sum_{k=-N}^{j-1} a^k_n \quad \text{and} \quad h^N_n = \sum_{k=j}^{N} a^k_n. \quad (5.1)
\]

Then we have $\Phi(s(f)) \leq \Phi(s(g)) + \Phi(s(h))$ by the sublinearity of $s(f)$ and $\Phi(t)$. And thus

\[
P(\Phi(s(f^N)) > 2\lambda) \leq P(\Phi(s(g^N)) > \lambda) + P(\Phi(s(h^N)) > \lambda).
\]

Since

\[
\|s(g^N)\|_2 \leq \sum_{k=-N}^{j-1} \|s(a^k)\|_2 \leq \sum_{k=-N}^{j-1} (2^{-k})^{1/2} \Phi^{-1}(2^k),
\]

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then
\[
P(s(g^N) > \Phi^{-1}(\lambda)) \leq \frac{1}{(\Phi^{-1}(\lambda))^2} \|s(g^N)\|_2^2
\]
\[
\leq \frac{1}{(\Phi^{-1}(\lambda))^2} \left( \sum_{k=-N}^{j-1} (2^{-k})^{1/2} \Phi^{-1}(2^k) \right)^2
\]
\[
= \left( \sum_{k=-N}^{j-1} \frac{(2^{-k})^{1/2} \Phi^{-1}(2^k)}{\Phi^{-1}(\lambda)} \right)^2
\]
\[
\leq \left( \sum_{k=-N}^{j-1} (2^{-k})^{1/2} \left( \frac{2^k}{\lambda} \right)^p \right)^2
\]
\[
= \lambda^{-2p} \left( \sum_{k=-N}^{j-1} (2^{p-\frac{1}{2}})^k \right)^2 \leq C_1 \lambda^{-1}.
\]

In the last inequality above, we used $1/2 < 1 \leq p \leq q < \infty$, which results from that $\Phi^{-1}$ is a convex function. Denote $C_1 = (2c_\Phi \max\{C_1, 1\})^{1/\ell}$, then
\[
\Phi\left(\frac{\Phi^{-1}(\lambda)}{C_1}\right) P(s(g^N) > \Phi^{-1}(\lambda)) \leq c_\Phi \frac{1}{2c_\Phi \max\{C_1, 1\}} \lambda P(s(g^N) > \Phi^{-1}(\lambda)) \leq \frac{1}{2}.
\]

Noticing that $a_n^k = 0$ on $\{\nu_k \geq n\}$ and $P(\nu_k < \infty) \leq 2^{-k}$, we get
\[
P(s(h^N) > \Phi^{-1}(\lambda)) \leq \sum_{k=j}^{N} P(\nu_k < \infty) \leq \sum_{k=j}^{N} 2^{-k} = 2^{1-j} \leq 4\lambda^{-1}.
\]

Denote $C_\Pi = (8c_\Phi)^{1/\ell}$, then
\[
\Phi\left(\frac{\Phi^{-1}(\lambda)}{C_\Pi}\right) P(s(h^N) > \Phi^{-1}(\lambda)) \leq \frac{1}{2}.
\]

Let $C = 2^q \max\{C_1, C_\Pi\}$, then
\[
\Phi\left(\frac{\Phi^{-1}(2\lambda)}{C}\right) P(s(f^N) > \Phi^{-1}(2\lambda)) \leq \Phi\left(\frac{\Phi^{-1}(2\lambda)}{C}\right) \left( P(s(g^N) > \Phi^{-1}(\lambda)) + P(s(h^N) > \Phi^{-1}(\lambda)) \right)
\]
\[
\leq \Phi\left(\frac{2^q\Phi^{-1}(\lambda)}{2^qC_1}\right) P(s(g^N) > \Phi^{-1}(\lambda))
\]
\[
+ \Phi\left(\frac{2^q\Phi^{-1}(\lambda)}{2^qC_\Pi}\right) P(s(h^N) > \Phi^{-1}(\lambda))
\]
\[
\leq \frac{1}{2} + \frac{1}{2} = 1.
\]
which implies $\|f^N\|_{wH_\Phi} \leq C$. Since

\[
    l(f^N) = \mathbb{E} \sum_{n=1}^{\infty} d^N_{f_n} d_{g_n} = \mathbb{E} \sum_{n=1}^{N} \sum_{k=-N}^{k} d^N_{a_n} d_{g_n}
\]

\[
    = \sum_{k=-N}^{N} \mathbb{E} \sum_{n=1}^{n} \| |d_{g_n} - d_{g^\nu_k}|^2 \|_{(2k)^{1/2}\Phi^{-1}(2k)}\|S(g - g^\nu_k)\|_2
\]

\[
    = \sum_{k=-N}^{N} (2^{-k})^{1/2} \frac{1}{\Phi^{-1}(2k)} \|g - g^\nu_k\|_2;
\]

then

\[
    C\|l\| \geq l(f^N) = \sum_{k=-N}^{N} \frac{1}{\Phi^{-1}(2k)}(2^{-k})^{-1/2}\|g - g^\nu_k\|_2.
\]

Taking over all $N \in \mathbb{N}$ and the supremum over all of such stopping times such that $P(\nu_k < \infty) \leq 2^{-k}, k \in \mathbb{Z}$, we obtain

\[
    \|g\|_{wL_{1,\phi}} = \int_0^{\infty} \frac{t^2_\phi(x)}{x} dx \leq C \sum_{k=-\infty}^{\infty} t^2_\phi(2^{-k}) \leq C\|l\|.
\]

The proof of Theorem 5.1 is complete.

To obtain the new John-Nirenberg theorem, we first prove two lemmas.

**Lemma 5.2.** Let $\Phi \in \mathcal{G}_\ell$ with $\ell \in (0, 1], q_{\Phi^{-1}} < \infty$ and $\phi(r) = 1/(r\Phi^{-1}(1/r))$. If $\{\mathcal{F}_n\}_{n \geq 0}$ is regular, then

\[
    (w\mathcal{H}_\phi)^* = wL_{1,\phi}.
\]

**Proof.** Let $g \in wL_{1,\phi}$ and define

\[
    l_g(f) = \mathbb{E}(fg), \quad f \in L_\infty.
\]

Then

\[
    |l_g(f)| = |\mathbb{E}(fg)| = \sum_{k=-\infty}^{\infty} \mathbb{E}(a_k(g - g^\nu_k))| \leq \sum_{k=-\infty}^{\infty} \|a_k\|_\infty \|g - g^\nu_k\|_1 \leq \sum_{k=-\infty}^{\infty} 2^{k+1}\|g - g^\nu_k\|_1 \leq 2\|f\|_{wH_\phi} \sum_{k=-\infty}^{\infty} \frac{2^k}{\|f\|_{wH_\phi}} \|g - g^\nu_k\|_1.
\]

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Denote $A_k = 1/\Phi \left( \frac{2^k}{\|f\|_{wH_{\Phi}}} \right)$, we get

$$
|l_g(f)| \leq 2\|f\|_{wH_{\Phi}} \sum_{k=-\infty}^{\infty} \frac{1}{\phi(A_k)} A_k^{-1} \|g - g^{\nu_k}\|_1
$$

$$
\leq 2\|f\|_{wH_{\Phi}} \sum_{k=-\infty}^{\infty} t^1_{\phi}(A_k)
$$

$$
\leq C\|f\|_{wH_{\Phi}} \|g\|_{wL_{1,\phi}}
$$

Conversely, suppose that $l \in (w\mathcal{H}_{\Phi})^*$. Since $L_2$ is dense in $w\mathcal{H}_{\Phi}$, there exists $g \in L_2 \subset L_1$ such that

$$
l(f) = \mathbb{E}(fg), \quad f \in L_\infty.
$$

Let $\nu_k$ be the stopping times satisfying $P(\nu_k < \infty) \leq (\Phi(2^k))^{-1} (k \in \mathbb{Z})$. For $k \in \mathbb{Z}$, define

$$
h_k = \text{sign}(g - g^{\nu_k}), \quad a^k = 2^k(h_k - h_k^{\nu_k}).
$$

It is easy to see that each $a^k (k \in \mathbb{Z})$ is a $w$-3-atom. Thus, by Theorem 3.3, if $f^N$ is again defined by (5.1), then $\|f^N\|_{wH_{\Phi}} \leq C$. Therefore

$$
C\|l\| \geq \|l(f^N)\| = |\mathbb{E}(f^Ng)| = |\sum_{k=-N}^{N} \mathbb{E}(a^k g)| = |\sum_{k=-N}^{N} 2^k \mathbb{E}((h_k - h_k^{\nu_k})g)|
$$

$$
= |\sum_{k=-N}^{N} 2^k \mathbb{E}(h_k(g - g^{\nu_k}))| = \sum_{k=-N}^{N} 2^k \|g - g^{\nu_k}\|_1
$$

$$
= \sum_{k=-N}^{N} \phi \left( \frac{1}{\Phi(2^k)} \right) \left( \frac{1}{\Phi(2^k)} \right)^{-1} \|g - g^{\nu_k}\|_1
$$

Taking over all $N \in \mathbb{N}$ and the supremum over all of such stopping times such that $P(\nu_k < \infty) \leq (\Phi(2^k))^{-1}$, $k \in \mathbb{Z}$, we obtain

$$
\|g\|_{wL_{1,\phi}} = \int_{0}^{\infty} t^1_{\phi}(x) dx \leq C \sum_{k=-\infty}^{\infty} t^1_{\phi}(\frac{1}{\Phi(2^k)}) \leq C\|l\|.
$$

The proof of Lemma 5.2 is complete.

**Lemma 5.3.** Let $\Phi \in \mathcal{G}_\ell$ with $\ell \in (0, 1]$, $q_{\Phi-1} < \infty$ and $\phi(r) = 1/(r\Phi^{-1}(1/r))$. Then for $q \in [1, \infty)$, if $\{\mathcal{F}_n\}_{n \geq 0}$ is regular, we have

$$
(w\mathcal{H}_{\Phi})^* = w\mathcal{L}_{q,\phi}.
$$

**Proof.** If $g \in w\mathcal{L}_{q,\phi}$ and

$$
l_g(f) := \mathbb{E}(fg), \quad f \in L_{q'},
$$

then
where \( q' = q/(q-1) \), then by Proposition 2.10 and Theorem 5.2 we have

\[
|l(f)| = |\mathbb{E}(fg)| \leq C\|f\|_{wH_\Phi}\|g\|_{wL_1,\phi} \leq C\|f\|_{wH_\Phi}\|g\|_{wL_q,\phi}.
\]

Conversely, suppose that \( l \in (wH_\Phi)^* \). Since \( L_q^* \subset L_1 \subset L_\Phi \subset wL_\Phi \), \( L_q^* \) can be embedded continuously in \( wH_\Phi \). Thus there exists \( g \in L_q \) such that \( l \) equals \( l_g \) on \( L_q^* \). Let \( \nu_k \) be the stopping times satisfying \( P(\nu_k < \infty) \leq 2^{-k} \) (\( k \in \mathbb{Z} \)). Define

\[
h_k = \frac{|g - g^{\nu_k}|^{q-1}\text{sign}(g - g^{\nu_k})}{\|g - g^{\nu_k}\|_q^{q-1}}, \quad a^k = \Phi^{-1}(2^k)(2^k)^{-1/q'}(h_k - h_k^{\nu_k}).
\]

Then \( \|h_k\|_q = 1 \) and \( a^k = 0 \) on the set \( \{\nu_k = \infty\} \). For \( \lambda > 0 \), choose \( j \in \mathbb{Z} \) such that \( 2^j \leq \lambda < 2^{j+1} \). Define \( f^N, g^N \) and \( h^N \) (\( N \in \mathbb{N} \)) again by (5.1), then

\[
\|M(g^N)\|_{q'} \leq \|g^N\|_{q'} \leq \sum_{k=-N}^{j-1} \|a^k\|_{q'} \leq 2 \sum_{k=-N}^{j-1} \Phi^{-1}(2^k)(2^k)^{-1/q'}
\]

and

\[
P(M(g^N) > \Phi^{-1}(\lambda)) \leq \frac{1}{(\Phi^{-1}(\lambda))^{q'}}\|M(g^N)\|_{q'}^{q'} \leq \frac{2^{q'}}{(\Phi^{-1}(\lambda))^{q'}} \left( \sum_{k=-N}^{j-1} \Phi^{-1}(2^k)(2^k)^{-1/q'} \right)^{q'} \leq 2^{q'} \left( \sum_{k=-N}^{j-1} \frac{(2^{-k})^{1/q'} \Phi^{-1}(2^k)}{\Phi^{-1}(\lambda)} \right)^{q'} \leq C \left( \sum_{k=-N}^{j-1} (2^{-k})^{1/q'} \left( \frac{2^k}{\lambda} \right)^p \right)^{q'} = C \cdot \lambda^{-q'p} \left( \sum_{k=-N}^{j-1} (2^{p-q'})^k \right)^2 \leq C\lambda^{-1}.
\]

The last inequality holds since \( 1/q' < 1 \leq p \). Applying the method used in Theorem 5.1, we conclude that \( \|f^N\|_{wH_\Phi} \leq C \). Consequently,

\[
C\|l\| \geq |l(f^N)| = \left| \sum_{k=-N}^N \mathbb{E}(a^kg) \right| = \left| \sum_{k=-N}^N \Phi^{-1}(2^k)(2^k)^{-1/q'} \mathbb{E}((h_k - h_k^{\nu_k})g) \right| = \left| \sum_{k=-N}^N \Phi^{-1}(2^k)(2^k)^{-1/q'} \|g - g^{\nu_k}\|_q \right| = \sum_{k=-N}^N \frac{1}{\phi(2^{-k})^{1/q'}} \|g - g^{\nu_k}\|_q
\]

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Taking over all $N \in \mathbb{N}$ and the supremum over all of such stopping times such that $P(\nu_k < \infty) \leq 2^{-k} (k \in \mathbb{Z})$, we obtain

$$\|g\|_{w\mathcal{L}_{q, \phi}} = \int_0^\infty \frac{t_\phi^q(x)}{x} dx \leq C \sum_{k=-\infty}^\infty t_\phi^q(2^{-k}) \leq C\|l\|.$$  

The proof of the theorem is complete.

We finally formulate the weak version of the John-Nirenberg theorem, which directly results from Lemma 5.2 and Lemma 5.3.

**Theorem 5.4.** If there exists $\Phi \in \mathcal{G}_\ell$ with $q_{\Phi^{-1}} < \infty$ such that $\phi(r) = \frac{1}{r^{\Phi^{-1}(1/r)}}$ for all $r \in (0, \infty)$, and $\{\mathcal{F}_n\}_{n \geq 0}$ is regular. Then $w\mathcal{L}_{q, \phi}$ spaces are equivalent for all $1 \leq q < \infty$.

**Remark 5.5.** Considering $\Phi(t) \equiv 1$, we obtain the John-Nirenberg theorem, Corollary 8 in [13] due to Weisz.

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