PI control of stable nonlinear plants using projected dynamical systems *

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Abstract

This paper presents a novel anti-windup proportional-integral controller for stable multi-input multi-output nonlinear plants. We use tools from projected dynamical systems theory to force the integrator state to remain in a desired (compact and convex) region, such that the plant input steady-state values satisfy the operational constraints of the problem. Under suitable monotonicity assumptions on the plant steady-state input-output map, we use singular perturbation theory results to prove the existence of a sufficiently small controller gain ensuring closed-loop (local) exponential stability and reference tracking for a feasible set of constant references. We suggest a particular controller design, which embeds (when possible) the right inverse of the plant steady-state input-output map. The relevance of the proposed controller scheme is validated through an application in the power systems domain, namely, the output (active and reactive) power regulation for a grid-connected synchronverter.

Key words: nonlinear systems, PI control, integral control, singular perturbations, windup, projected dynamical systems.

1 Introduction

One of the fundamental problems in control theory is the regulator problem, where the objective is to design a controller that forces the output of a plant to track a reference signal, while rejecting possible disturbances. It is often convenient to assume that the reference and the disturbance signals are generated by a fictitious system called the exosystem, which is an expression of our previous knowledge about these signals. When it is reasonable to assume that the signals originate from an exosystem, then the internal model principle (see Davison (1970); Francis (1975)), in its version for linear time-invariant (LTI) systems, states that the regulator problem is solved if all the unstable eigenvalues of the exosystem are poles of the controller. In the case of constant signals, this principle suggests that an integral controller is needed to solve the regulator problem.

For LTI systems, if the plant DC-gain sign is known and

the plant is stable, then, for sufficiently small controller gains of suitable sign, the closed-loop system formed by the plant with the integral controller is stable, and the regulator problem is solved (see Morari (1985)). Similar results have been established in Desoer and Lin (1985) for multi-input multi-output (MIMO) globally stable nonlinear systems, using singular perturbation (SP) theory. In their work, the plant DC-gain sign assumption is replaced with an assumption on the monotonicity of the plant steady-state input-output mapping. The same approach has been used in our paper Lorenzetti and Weiss (2022) for the stability analysis of a stable nonlinear plant connected in feedback with a single-input single-output (SISO) low-gain anti-windup (AW) proportional-integral (PI) controller. The recent paper Simpson-Porco (2020) on low-gain integral control for stable nonlinear systems, also employing SP tools, has generalized the main result from Desoer and Lin (1985). The assumption on the monotonicity of the plant steady-state input-output mapping has been replaced with the uniform infinitesimal contracting property of the reduced dynamics. It has then been shown how this relaxed assumption recovers the one in Morari (1985) for a linear plant. (For an extension of this result to nonlinear discrete-time systems with input constraints see Simpson-Porco (2021).) Other interesting results for linear systems with input-output nonlinearities and low-gain integral controllers are in Logemann et al (1999); Guiver et al. (2017).
An ubiquitous problem in control applications is windup. This happens when there is a mismatch between the controller output and the actual plant input, e.g., due to actuator limitations, causing long transients, oscillations and even instability (Kothare et al. (1994)). Several AW techniques have been proposed, resulting in a vast literature on the topic. It is not in the scope of this paper to provide a detailed review on AW control, instead we refer to Aaström and Rundqvist (1989); Edwards and Postlethwaite (1998); Kothare et al. (1994); Tarbouriech and Turner (2009); Zaccarian and Teel (2002) (and the references therein). Even though such AW control strategies prove to be effective, they mainly deal with linear plants and they often require the solution of an LMI-based optimization problem.

Our aim is to formulate a simple novel AW MIMO low-gain PI controller for stable nonlinear systems, which forces the integrator state (and, thus, the steady-state plant input) to be in a desired (compact and convex) region, using tools from projected dynamical systems (PDS) theory. We provide a rigorous closed-loop stability analysis, following the approach of Desoer and Lin (1985), Lorenzetti and Weiss (2022), Simpson-Porco (2021), by using SP theory results. We derive a sufficient condition on the controller gain that ensures closed-loop stability and reference tracking for a “naturally” feasible set of constant references. This result generalizes Lorenzetti and Weiss (2022), where a similar PI AW control strategy, using the saturating integrator introduced in Lorenzetti et al. (2020), was formulated for SISO stable nonlinear plants. The PI SISO saturating integrator performs remarkably well in several applications, see, e.g., Lorenzetti and Weiss (2022); Natarajan and Weiss (2017). We expect similar successful performance (and more) for its MIMO formulation shown in Fig. 3. A preliminary version of this paper, considering only integral control ($\tau_p = 0$), without the block $N$, and with no numerical example, has been presented in our recent conference paper Lorenzetti and Weiss (2021). Here, we introduce an additional degree of freedom in the controller design, namely, the block $N$ from Fig. 3, which we exploit to embed (when possible) the right inverse of the plant steady-state input-output map in the controller. We use a numerical example from the power electronics domain to illustrate the relevance of the proposed controller design.

Related to our work is the recent contribution Wang et al. (2020), where a bounded MIMO integral controller is presented (not based on PDS theory), called bounded integral controller (BIC) (this is an extension of the SISO BIC presented in Konstantopoulos et al. (2016)). The MIMO BIC enforces a sum-of-squares-type of constraint with time-varying input weights for the controller states, and input-to-state practical stability (ISpS) is guaranteed (using a small-gain argument) when the BIC is connected in feedback with an ISpS nonlinear plant. On the other hand, the (natural) link between PDS theory and AW design has been pointed out before. Investigations in a similar direction have been carried out in Teo and How (2011), where a gradient projection AW (GA-PW) scheme has been proposed. In particular, they provide sufficient conditions under which the following holds: Assume that the nominal unconstrained closed-loop system is stable and achieves tracking in correspondence of a certain equilibrium point, then the region of attraction of the same equilibrium point in the associated GA-PW closed-loop system is “larger” than the one of the unconstrained system. Although this result is of interest, the sufficient conditions provided are difficult to verify in practice (as pointed out by the authors), see Teo and How (2011) Theorem 2. Recently, the connection between PDS and AW schemes has also been investigated in Hauswirth et al. (2020), with applications to feedback optimization problems. In particular, they show that the closed-loop solutions of a high-gain integral AW control scheme uniformly converge to those of a PDS as the gain tends to infinity, see Remark 3.2. Finally, a low-gain projected integral control scheme for exponentially stable discrete-time nonlinear systems is presented in the recent contribution Simpson-Porco (2021).

The paper is organized as follows. In Sect. 2 we present some background on PDS theory. In Sect. 3 the PI SISO saturating integrator from Lorenzetti and Weiss (2022) is reformulated as a MIMO controller, using tools from PDS theory, and the control problem is described in precise terms. Sect. 4 contains our main result: the stability of the closed-loop system (and the consequent reference tracking), proved using SP theory. Finally, in Sect. 5 we illustrate the performance of the proposed controller through an application from power electronics.

2 Background on PDS theory

We present some background on PDS theory, taken mainly from Nagurney and Zhang (1995, Ch. 2).

**Notation.** Let $X \subset \mathbb{R}^q$ be closed and convex. Denote the boundary (interior) of $X$ by $\partial X$ (int $X$). Define the set of inward normals to $X$ at $x \in \partial X$ by

$$n(x) = \{ \gamma \mid \| \gamma \| = 1, \text{ and } \langle \gamma, x - y \rangle \leq 0, \forall y \in X \}. $$

**Definition 2.1** (Nagurney and Zhang (1995)) Let $X \subset \mathbb{R}^q$ be closed and convex, and let $z \in X$, $v \in \mathbb{R}^q$. Define the projection operator $P_X$ onto $X$ as

$$P_X(v) = \arg \min_{w \in X} \| v - w \|, \quad (2.1)$$

and the directional derivative of $P_X$ at $z$, along $v$, as

$$\Pi_X(z, v) = \lim_{\delta \to 0^+} \frac{P_X(z + \delta v) - z}{\delta}. \quad (2.2)$$

**Lemma 2.2** (Nagurney and Zhang, 1995 Lemma 2.1). Let $X, z, v, \Pi$ be as in Definition 2.1. Then:
1. We show how \( \Pi_X(z, \cdot) \) maps \( v \) (pointing outward) to \( w \), when \( z \in \partial X \). (Here the set \( n(z) \) is a singleton.)

(1) If \( z \in \text{int} \ X \), then \( \Pi_X(z, v) = v \).

(2) If \( z \in \partial X \), then \( \Pi_X(z, v) = v + \beta(z)n^*(z) \), where \( \beta(z) = \max\{0, (v, -n^*(z))\} \), \( n^*(z) = \arg \max_{n \in n(z)} (v, -n) \).

From Lemma 2.2, for any \( z \in X \) and \( v \in \mathbb{R}^q \), we have
\[
\Pi_X(z, kv) = k\Pi_X(z, v) \quad \forall k > 0. \tag{2.3}
\]

To help the reader in understanding the intuition behind Lemma 2.2, we show in Fig. 1 the resulting vector \( w = \Pi_X(z, v) \) when, e.g., \( z \in \partial X \) and \( v \) points outward.

**Definition 2.3** *(Nagurney and Zhang, 1995, Definition 2.5)*. Let \( X \subset \mathbb{R}^q \) be a closed and convex set, \( z \in X \), and \( F : X \rightarrow \mathbb{R}^q \) a vector field. The function \( z : [0, \infty) \rightarrow X \) is a Carathéodory solution to the equation
\[
\dot{z} = \Pi_X(z, -F(z)) \tag{2.4}
\]
if \( z(\cdot) \) is absolutely continuous and if
\[
\dot{z}(t) = \Pi_X(z(t), -F(z(t))),
\]
save on a set of Lebesgue measure zero (of points \( t > 0 \)).

For any \( z_0 \in X \) as initial value, we associate with (2.4) an initial value problem defined as:
\[
\dot{z} = \Pi_X(z, -F(z)), \quad z(0) = z_0. \tag{2.5}
\]

**Remark 2.4** If (2.5) has a solution, then such a solution is constrained in \( X \) for all \( t \geq 0 \).

**Definition 2.5** *(Nagurney and Zhang, 1995, Definition 2.6)*. Let \( X \) and \( F \) be as above. Define a projected dynamical system \( \text{PDS}(F, X) \) as a map \( \Phi : X \times \mathbb{R} \rightarrow X \), such that \( \Phi(z_0, t) = \Phi(z_0, t) \) is a Carathéodory solution of (2.5), so that for almost every \( t > 0 \)
\[
\dot{z}_0(t) = \Pi_X(\Phi_{z_0}(t), -F(\Phi_{z_0}(t))), \quad \Phi_{z_0}(0) = z_0.
\]

We show in Fig. 2 the portrait of a classical dynamical system and the portrait of the corresponding PDS.

**Definition 2.6** *(Nagurney and Zhang, 1995, Definition 2.7)*. The vector \( z^* \in X \) is an equilibrium point of the \( \text{PDS}(F, X) \) if \( \Pi_X(z^*, -F(z^*)) = 0 \).

![Fig. 1. We show how \( \Pi_X(z, \cdot) \) maps \( v \) (pointing outward) to \( w \), when \( z \in \partial X \). (Here the set \( n(z) \) is a singleton.)](image)

**Fig. 1.** We show how \( \Pi_X(z, \cdot) \) maps \( v \) (pointing outward) to \( w \), when \( z \in \partial X \). (Here the set \( n(z) \) is a singleton.)

**Remark 2.7** As pointed out in Nagurney and Zhang (1995), \( z^* \in X \) is an equilibrium point of the \( \text{PDS}(F, X) \) if the vector field \( F \) vanishes at \( z^* \). The converse, however, is only true when \( z^* \in \text{int} \ X \). Indeed, when \( z^* \in \partial X \), we may have \( F(z^*) \neq 0 \), but \( \Pi_X(z^*, -F(z^*)) = 0 \).

**Theorem 2.8** *(Nagurney and Zhang, 1995, Theorem 2.5)*. Assume that there exists a \( B > 0 \) such that the vector field \( F : X \rightarrow \mathbb{R}^q \) satisfies:
\[
\|F(z)\| \leq B(1 + \|z\|) \quad \forall z \in X,
\]
\[
\langle -F(x) + F(y), x - y \rangle \leq B\|x - y\|^2 \quad \forall x, y \in X.
\]
Then:

(1) For any \( z_0 \in X \), there exists a unique solution \( z : [0, \infty) \rightarrow X \) to the initial value problem (2.5).

(2) If \( z_n \rightarrow z_0 \) as \( n \rightarrow \infty \), then \( z(t; z_n) \) converges to \( z(t; z_0) \) uniformly on every compact set in \( [0, \infty) \).

**Remark 2.9** The definition of \( \Pi_X \) can be extended for \( z, v \in \mathbb{R}^q \) as follows:
\[
\Pi_X(z, v) = \frac{P_X(z) - z}{\|P_X(z) - z\|} \quad \forall z \in \mathbb{R}^q \setminus X. \tag{2.6}
\]

Suppose that (2.5) has a solution for any \( z_0 \in X \), given by \( \Phi \) from Definition 2.5. If \( z_0 \in \mathbb{R}^q \setminus X \), then the solution of (2.5) (with \( \Pi \) extended as in (2.6) and \( z(0) = z_0 \)) will move with unit velocity towards \( P_X(z_0) \), until it reaches it (in finite time). Then it will follow the flow \( \Phi \). With this extension, Theorem 2.8 remains valid for all \( z_0 \in \mathbb{R}^q \).

**Remark 2.10** The (uniform) Lipschitz continuity of \( F \) on \( X \subset \mathbb{R}^q \) implies the assumptions of Theorem 2.8.

**Remark 2.11** For the setting of this paper (i.e., \( X \) closed and convex, and \( F \in C^1 \)), the theory on PDS developed in Nagurney and Zhang (1995) is sufficient to derive our main result. However, for the interested readers, we refer to the contribution Hauswirth et al. (2021), where the work of Nagurney and Zhang (1995) is generalized in several directions. In particular, the conditions of Theorem 2.8 are relaxed and the existence and uniqueness of Krasovskii (and, when possible, Carathéodory) solutions to (2.5) is proved under milder assumptions on the set \( X \), on the vector field \( F \), and for a more general Riemannian metric, see (Hauswirth et al., 2021, Table 1). Using
the equivalent formulation of $\Pi_X$ from Hauswirth et al. (2021), it can be checked that $\Pi_X(z,\cdot)$ is a contraction. Indeed, defining the tangent cone $T_zX$ at $z \in X \subset \mathbb{R}^q$ as in Hauswirth et al. (2021) Definition 2.1, with $X$ closed and convex, and using as metric $\| \cdot \|$ the Euclidean norm, then $\Pi_X(z,v)$ from (2.2) can be formulated as in Hauswirth et al. (2021) Definition 3.1, i.e.,

$$\Pi_X(z,v) = \arg\min_{w \in T_zX} \|w - v\|. \quad (2.7)$$

It is a well-known result that the above operator and, equivalently, our (2.2), is a contraction, i.e.,

$$\|\Pi_X(z,v_1) - \Pi_X(z,v_2)\| \leq \|v_1 - v_2\|, \quad (2.8)$$

for all $v_1, v_2 \in \mathbb{R}^q$ and for all $z \in X$.

### 3 Problem formulation

Consider the nonlinear plant $P_0$ described by

$$\dot{x} = f_0(x,v), \quad y = g(x), \quad (3.1)$$

with $f_0 \in C^2(\mathbb{R}^n \times U; \mathbb{R}^n)$, $g \in C^1(\mathbb{R}^p; \mathbb{R}^p)$, where $U \subset \mathbb{R}^n$ is an open domain with $m \geq p$.

The control objective is to make the plant output signal $y$ track a constant reference signal $r \in Y \subset \mathbb{R}^p$, while making sure that the plant input signal $v$ converges to a steady-state value in a desired compact set $V \subset \mathbb{R}^m$ (e.g., determined by operational constraints). This tracking property should hold for all plant initial states $x_0$ in a reasonably large open set in $\mathbb{R}^n$.

To achieve this control objective, we form the closed-loop system shown in Fig. 3, described by the equations

$$\dot{x} = f_0(x,v), \quad y = g(x)$$

$$\dot{u}_t = \Pi_U(u_t + k(r - g(x)))$$

where $U \subset \mathbb{R}^p$ is an open domain, $U \subset \mathcal{U}$ is a compact and convex set ($U$ and $U$ to be defined), $\Pi_U$ is the operator from (2.2), with the extension (2.6), $\mathcal{N} \in C^2(\mathcal{U}, V)$ (to be defined), $V = \mathcal{N}(U)$, $k > 0$ and $\tau_0 \geq 0$. The state space of (3.2) is $\mathbb{R}^n \times \mathcal{U}$ and its state is $z(t) = \frac{x(t)}{u_t}$. As is often the case in nonlinear systems theory, the first equation in (3.2) only makes sense on a “region of interest” in the state space, namely on the open set

$$\mathcal{D}_r := \{ [\frac{x}{u_t}] \in \mathbb{R}^n \times U | u_t + \tau_p(k(r - g(x))) \in \mathcal{U} \}.$$

It will be convenient to introduce the “new plant” $P$ as the cascade of $\mathcal{N}$ and $P_0$, described by

$$\dot{x} = f(x,u), \quad y = g(x), \quad (3.3)$$

where $f(x,u) := f_0(x,\mathcal{N}(u)) \in C^2(\mathbb{R}^n \times \mathcal{U}; \mathbb{R}^n)$.

**Proposition 3.1** Consider the closed-loop system (3.2), with $k, \tau_p \in \mathbb{R}$, $r \in \mathbb{R}^p$. Then for every $[\frac{x_0}{u_0}] \in \mathcal{D}_r$ with $u_0 \in U$, there exists $\tau \in (0, \infty]$ such that (3.2), with initial conditions $z(0) = [\frac{x_0}{u_0}]$, has a unique Carathéodory solution (or state trajectory) $z = [\frac{x}{u}]$ defined on $[0, \tau)$. If $\tau$ is finite and maximal (i.e., the state trajectory cannot be continued beyond $\tau$), then $\limsup_{t \to \tau} ||z(t)|| = \infty$, or the signal $u(t) = u_t(t) + \tau_p k(r - g(x(t)))$ approaches $\partial \mathcal{U}$:

$$\liminf_{t \to \tau} d(u(t), \partial \mathcal{U}) = 0, \quad (3.4)$$

where $d$ denotes the distance in $\mathbb{R}^p$.

**Proof.** We introduce the closed and convex set $X \subset \mathbb{R}^n \times U$. An equivalent representation of (3.2) (for $u_t(t) \in U$) is

$$\dot{z} = \Pi_X(z, -F(z)), \quad (3.5)$$

where

$$-F(z) := \begin{bmatrix} f(x,u_t + \tau_p k(r - g(z))) \\ k(r - g(z)) \end{bmatrix},$$

so that (3.5) makes sense as long as $z(t) \in \mathcal{D}_r$. For any $\delta > 0$, $B_\delta$ denotes the closed ball of radius $\delta$ in $\mathbb{R}^n$, and also in $\mathbb{R}^p$ (the dimension will be clear from the context). We fix $[\frac{x_0}{u_0}] \in \mathcal{D}_r$ such that $u_0 \in U$. Define

$$X_\delta := (x_0 + B_\delta) \times ([u_0 + B_\delta] \cap \mathcal{U}).$$

We choose $\delta$ small enough so that $X_\delta \subset \mathcal{D}_r$. We have $F \in C^1(\mathcal{D}_r, \mathbb{R}^n \times \mathbb{R}^p)$ and $X_\delta$ is compact and convex, thus it follows from Theorem 2.8 and Remark 2.10 that (3.5), but with $X_\delta$ in place of $X$, has a unique solution $z : [0, \infty) \to X_\delta$ that satisfies $z(0) = [\frac{x_0}{u_0}].$ As long as $||z(t) - z(0)|| < \delta$, this solution $z$ is also a solution of the original (3.5). From here it follows that there exists $\tau > 0$ such that (3.5) has a unique state trajectory $z$ defined on $[0, \tau)$, starting from the initial state $z(0) = [\frac{x_0}{u_0}].$

Suppose that $\tau > 0$ as above is finite and maximal. If $\limsup_{t \to \tau} ||x(t)||$ is finite, then the trajectory $z$ is bounded on $[0, \tau)$ (because $u_t(t) \in U$ for all $t \in [0, \tau]$). If (3.4) were not true, then there exists $\epsilon > 0$ such that $d(u_t(t), \partial \mathcal{U}) \geq \epsilon$ for all $t \in [0, \tau].$ This implies that the closure of $\{z(t) | t \in [0, \tau]\}$ is a compact subset of $\mathcal{D}_r$. Since $F$ is continuous on $\mathcal{D}_r$, there exists $M > 0$ such that $||F(z(t))|| \leq M$ for all $t \in [0, \tau].$ Let $(t_j)$ be an increasing sequence such that $t_j \in [0, \tau)$, $t_j \to \tau$. Using (2.8) we obtain that for $j > k$

$$||z(t_j) - z(t_k)|| \leq \int_{t_k}^{t_j} ||\Pi_X(z(t), -F(z(t)))|| dt \leq M(t_j - t_k).$$
Thus, \((z(t_j))\) is a Cauchy sequence, so that it converges to a limit \(z(\tau) \in \mathcal{D}..\). It is easy to see that the limit is independent of the choice of \((t_j)\), and that the function \(z\), extended to \([0, \tau]\), is a Carathéodory solution of (3.5) on \([0, \tau]\). We could extend this solution even further, using the argument in the first part of this proof. This would contradict the maximality of \(\tau\), hence our assumption that (3.4) is false has led us to a contradiction. Thus, if \(\limsup_{t \to \tau} \|x(t)\|\) is finite, then (3.4) holds.

\[ \text{Remark 3.2} \hspace{1em} \text{The closed-loop system (3.2), with } \tau_p = 0 \text{ and } \mathcal{N} = I, \text{ can be approximated by the equations} \]
\[ \dot{x} = f(x, P_r(u_I)), \]
\[ \dot{u}_I = k(r - g(x)) - \frac{1}{K}(u_I - P(U(u_I))], \]
with \(K > 0\) small and \(P_r\) from (2.1). Indeed, for every initial state \((x_0, u_0) \in \mathbb{R}^n \times U\), the solution of (3.6) converges uniformly to that of (3.2) (with \(\tau_p = 0\), and \(\mathcal{N} = I\)) for \(K \to 0^+\), see (Hauswirth et al., 2020, Theorem 2).

\section{Closed-loop stability analysis}

In this section we present our main result, namely, we derive an upper bound for the gain \(k\) ensuring the existence of a (locally) exponentially stable equilibrium point for the closed-loop system (3.2), for each constant reference \(r \in \mathbb{R}^p\) \((Y \text{ to be defined}). We further characterize a subset of the region of attraction of this equilibrium point such that if the initial state is in this region, then the plant output \(y\) tracks \(r\). This result generalizes (Lorenzetti and Weiss, 2022, Theorem 4.3), which was formulated for the SISO saturating integrator. As in Lorenzetti and Weiss (2022), our stability analysis employs SM methods (see Appendix A for the details), which can be found, e.g., in (Kokotović et al., 1999, Ch. 7), (Khalil, 2002, Ch. 11).

\begin{assumption}
There exists a function \(\Xi \in C^1(\mathbb{R}^p; \mathbb{R}^n)\) such that
\[ f_0(\Xi(v), v) = 0 \quad \forall \ v \in \mathcal{V}. \] (4.1)
Moreover, the equilibrium points \(\{\Xi(v) \in \mathcal{V} \mid v \in \mathcal{V}\}\) are uniformly exponentially stable. This means that there exist \(\varepsilon_0 > 0, l > 0\) and \(\rho \geq 1\) such that for each constant input \(v_0 \in \mathcal{V}\), the following holds:
If \(\|x(0) - \Xi(v_0)\| < \varepsilon_0\), then for every \(t \geq 0\),
\[ \|x(t) - \Xi(v_0)\| \leq \rho e^{-lt}\|x(0) - \Xi(v_0)\|. \] (4.2)
\end{assumption}

\begin{remark}
Assumption 1 guarantees the stability of the boundary-layer system associated to the closed-loop system (3.2) (see (A.6) in Appendix A). This is a standard assumption in the framework of SM theory (see, for instance, Desoer and Lin (1983), Khalil (2002, Ch. 11), (Kokotović et al., 1999, Ch. 7)).
\end{remark}

\begin{remark}
The (uniform) exponential stability condition (4.2) can be checked by linearization: If the Jacobian matrices
\[ A(v_0) = \left. \frac{\partial f_0(x, v)}{\partial x} \right|_{x = \Xi(v_0)} \in \mathbb{R}^{n \times n} \]
have eigenvalues bounded away from the right half-plane,
\[ \max \text{Re}(A(v_0)) \leq l_0 < 0 \quad \forall \ v_0 \in \mathcal{V}, \]
then \(\Xi(v_0)\) is a uniformly exponentially stable equilibrium point of \(P_0\), for all \(v_0 \in \mathcal{V}\), see (Khalil, 2002, eq. (11.16)).
\end{remark}

\begin{assumption}
If \(\Xi\) satisfies (4.1) and (4.2), then \(\Xi \in C^2\) thanks to the implicit function theorem (since \(f_0 \in C^2\)).
\end{assumption}

\begin{notation}
Let \(G(v) = g(\Xi(v)) \in C^1(\mathcal{V}; \mathbb{R}^p)\) denote the steady-state input-output map corresponding to \(P_0\).
\end{notation}

We choose \(U \subset \mathcal{U}\) to be compact, convex, with \(\text{int} \ U \neq \emptyset\). We let \(Y = G(N(U))\), and, for any \(r \in Y\), we define
\[ u_r := (G \circ N)^{-1}(r) \quad x_r := \Xi(N(u_r)), \]
which are well-defined since \(G \circ N\) is strictly monotone on \(U\) (hence one-to-one). From Assumption 1, \((x_r, u_r)\) is an equilibrium point of the closed-loop system (3.2).

Some commentary on the sets \(\mathcal{V}, \mathcal{U}, V, U, \text{ and } Y\). The set \(\mathcal{V} \subset \mathbb{R}^m\) is a set of inputs for which we have steady-state stability of the plant \(P_0\) (see Assumption 1). The set \(\mathcal{U} \subset \mathbb{R}^p\) is where Assumption 2 holds and, thus, where we would like to constrain the state of the integrator \(u_I\) in order to obtain closed-loop stability. The set \(\mathcal{U}\) may be too large, and, to satisfy operational constraints, we impose \(u_I(t) \in U\), where \(\mathcal{V}\) is chosen as above. We denote \(V = N(U)\). Finally, \(Y = G(N(U))\) is the natural set of feasible references, since \(y = G(N(u_I))\) at steady-state.

\begin{remark}
Assumption 2 guarantees the stability of the reduced-order model associated to the closed-loop system (3.2) (see (A.5) in Appendix A for the details). This is a common assumption when SM tools are used to investigate the stability of a nonlinear plant connected in feedback with an integral controller, see Desoer and Lin (1983); Huang et al. (2019). The work of Simpson-Porco (2020) has extended the result from Desoer and Lin (1983), by replacing the monotonicity assumption on the input-output steady-state map with the infinitesimal contracting property of the reduced dynamics. However, as discussed in Simpson-Porco (2020, Sect. 3), if the infinitesimal contracting property is stated with respect to the standard Euclidean norm, then the conditions of Simpson-Porco (2020) reduce to those of Desoer and Lin (1983). In our framework, the two are equivalent.
\end{remark}
Remark 4.5 For a matrix $M \in \mathbb{R}^{p \times p}$, define $\text{Re} M = \frac{1}{2}(M + M^\top)$. The (strict) monotonicity of $G \circ \mathcal{N} \in C^1(\mathbb{U}; \mathbb{R}^p)$ is equivalent to the fact that $\text{Re} \frac{\partial(G \circ \mathcal{N})}{\partial u}$ is strongly positive, i.e., there exists a $\mu > 0$ such that
\[
\langle \frac{\partial(G \circ \mathcal{N})}{\partial u} w, w \rangle \geq \mu \|w\|^2 \quad \forall w \in \mathbb{R}^p, \forall u \in \mathbb{U},
\]
see [Nagurney and Zhang, 1993, Proposition 2.5].

There are several ways to choose $\mathcal{N}$ (the case $\mathcal{N} = I$ was considered in our recent conference paper [Lorenzetti and Weiss, 2021]). Assume that $G$ admits a right inverse $G^{-1}_{\text{right}} \in C^2(\mathbb{G}(\mathcal{V}); \mathcal{V})$, i.e., $G \circ G^{-1}_{\text{right}} = I$ (the identity on $\mathcal{V}$). Then, we suggest the choice $\mathcal{N} = G^{-1}_{\text{right}}$, for which Assumption 2 trivially holds, and $U = Y$. As shown in Sect. 5, the choice $\mathcal{N} = G^{-1}_{\text{right}}$ can be very convenient.

Remark 4.6 If $P \circ 0$ is linear, described by the matrices $A, B, C$ in the usual way ($x = Ax + Bv, y = Cx$), then Assumption 1 reduces to the fact that $A$ is Hurwitz. The functions $G, H$ from Assumptions 1 and 2 are given by
\[
\Xi(v) = (-A)^{-1}Bv, \quad G(v) = P(v),
\]
where $P(s) = C(sI - A)^{-1}B$ is the plant transfer function. In this case, if $P(0)$ is onto, then $\mathcal{N}$ can be chosen as $\mathcal{N} = P(0)^{-1}G(0)P(0)^{-1}$, so that, again, $G \circ \mathcal{N} = I$, and Assumption 2 is trivially satisfied.

Theorem 4.7 Consider the closed-loop system (3.2), where $P \circ 0$ satisfies Assumption 2. Then there exists a $\kappa > 0$ such that if the gain $k \in (0, \kappa]$, then for any $r \in \mathcal{Y} = G(\mathcal{N}(\mathcal{U}))$, $(\Xi(\mathcal{N}(u)), u_r)$ is a (locally) exponentially stable equilibrium point of the closed-loop system (3.2), with state space $\mathcal{X} = \mathbb{R}^n \times \mathcal{U}$. If the initial state $[u_0, v_0] \in \mathcal{D}_r$ ($\mathcal{D}_r$ from Sect. 3) of the closed-loop system satisfies $u_0 \in \mathcal{U}$ and $\|x_0 - \Xi(\mathcal{N}(u_0))\| \leq \varepsilon_0$, then
\[
x(t) \rightarrow \Xi(\mathcal{N}(u_r)), \quad u_t(t) \rightarrow u_r, \quad y(t) \rightarrow r, \quad (4.3)
\]
and this convergence is at an exponential rate.

For the proof see Appendix A. Note that, clearly, (4.3) implies that $u(t) \rightarrow u_r$ (since $e(t) = r - g(x(t)) \rightarrow 0$).

Remark 4.8 The results from Theorem 4.7 can be extended globally, following the procedure of [Lorenzetti and Weiss, 2022, Sect. V], if $P \circ 0$ satisfies the asymptotic gain property (introduced in Sontag and Wang, 1996) around each equilibrium point $\Xi(v_0)$, for all $v_0 \in \mathcal{V}$.

5 Power regulation for a grid-connected synchronverter

We present an application of the proposed control strategy for the (active and reactive) power regulation of a grid-connected synchronverter, when the grid is modelled as an infinite bus. In our simulations, we assume that the power set points for the synchronverter control algorithm are provided by an external control loop (e.g., using optimal power flow considerations), which we do not model. The synchronverter output active and reactive powers have to track these set points, whilst making sure to not leave the safe operating region. We compare the behaviour of the closed-loop system formed by the synchronverter model $P \circ 0$, our saturating integrator $\int \Pi_{\tau_i}$ (here $\tau_i = 0$), and the nonlinear gain $\mathcal{N} = G^{-1}_{\text{right}}$ (to be defined), with the one formed by $P \circ 0$, a classical integrator ($\Pi_{\tau_i} = I$, $\tau_i = 0$), and a static linear gain $\mathcal{N} = K \in \mathbb{R}^{p \times p}$ (to be defined).

5.1 Description of the synchronverter model

Synchronverters, see [Zhong and Weiss, 2011], are a particular type of virtual synchronous machines, i.e., inverters with a control algorithm that causes them to behave towards the power grid like synchronous generators. Among the different grid-connected synchronverter models in the literature, we refer to the fourth order grid-connected synchronverter model from Natarajan and Weiss, 2017, eq. (3.1), [Lorenzetti et al, 2012, eq. (13)], where the grid is modelled as an infinite bus. Due to lack of space, we omit the physical meaning of the equations, which can be found in the just cited references.

Let $P \circ 0$ be the fourth order grid-connected synchronverter model with state
\[
x = [i_d \ i_q \ \delta]^\top \in \mathbb{R}^4, \quad (5.1)
\]
where $i_d$ and $i_q$ are the d and q components of the stator currents, $\delta$ is the (virtual) rotor angular velocity, and $\delta$ is the power angle (regarded modulo $2\pi$, i.e., $\delta$ and $\delta + 2\pi$ are considered to be the same angle). The input is
\[
v = [T_m \ i_f]^\top \in \mathbb{R} \times (0, \infty), \quad (5.2)
\]
where $T_m$ is the (virtual) prime mover torque, and $i_f$ is the (virtual) field current. The output is
\[
y = [P \ Q]^\top \in \mathbb{R}^2, \quad (5.3)
\]
where $P$ is the active power, and $Q$ is the reactive power. The plant $P \circ 0$ is described by the equations
\[
H \dot{x} = A(x, v)x + h(x, v), \quad y = g(x), \quad (5.4)
\]
with
\[
H = \begin{bmatrix} L & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & J & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad h(x, v) = \begin{bmatrix} V \sin \delta \\ V \cos \delta \\ T_m + D_p \phi_n \\ -\phi_g \end{bmatrix},
\]
and
\[
A(x, v) = \begin{bmatrix} -R & \phi L & 0 & 0 \\ -\phi L & -R & -mi_f & 0 \\ 0 & mi_f & -D_p & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},
\]
and
\[
g(x) = -V \begin{bmatrix} \cos \delta & \sin \delta \\ -\sin \delta & \cos \delta \end{bmatrix} \begin{bmatrix} i_q \\ i_d \end{bmatrix}.
\]

Here \(L > 0\) is the total stator inductance, \(R > 0\) is the total stator resistance, \(J > 0\) is the rotor moment of inertia, \(V > 0\) is the rms value of the line voltage, \(D_p > 0\) is the frequency droop constant, \(\phi_n\) is the nominal grid frequency, \(\phi_g\) is the grid frequency, and \(m = \sqrt{3/2M_f}\), where \(M_f > 0\) is the peak mutual inductance between the virtual rotor winding and any one stator winding.

**Synchroverter parameters.** We use the synchroverter parameters from the numerical example [Lorenzetti et al. 2022 Subsect. VI-A], chosen for a synchroverter designed to supply a nominal active power of 9kW to a grid with nominal frequency \(\phi_n = 100\pi\text{rad/sec (50Hz)}\) and line voltage \(V = 230\sqrt{3}\)Volts. The parameters are: \(J = 0.2\text{Kg-m}\text{^2/\text{rad}}\), \(D_p = 3\text{N-m/(\text{rad/sec})}\), \(R = 1.875\Omega\), \(L = 56.75\text{mH}\), \(m = 3.5\), and \(\phi_g = \phi_n\).

### 5.2 Formulation of the control problem

The control problem that we address is the regulation of the synchroverter output \(y\) to the reference signal
\[
r = [P_{\text{set}}\ Q_{\text{set}}]^\top \in \mathbb{R}^2,
\]
while keeping the synchroverter input \(v\) in a safe (compact and convex) operating region \(V \subset \mathbb{R}^2 (m = p = 2)\). We form a closed-loop system as in Fig. 3 (here \(\tau_p = 0\)), and we are interested in studying its stability and tracking properties using Theorem 4.7. To this aim, we first verify whether \(P_0\) from (5.4) satisfies Assumptions 1, 2.

**Verification of Assumption 1.** The equilibrium points of the grid-connected synchroverter model (5.4) have been studied in Lorenzetti et al. [2022]. In particular, in [Lorenzetti et al. 2022 Prop. 3.1 and Prop. 3.3] it is shown that for each \(T_m > -\frac{\pi}{2\phi_m}\) there is a finite interval \(I_f \subset (0, \infty)\) such that for \(i_f \in \text{int} I_f\), the model \(P_0\) has two equilibrium points, of which at most one is stable. We denote by \(V\) the subset of \(\mathbb{R} \times (0, \infty)\) such that if \(v \in V\), then \(P_0\) has an exponentially stable equilibrium point corresponding to the constant input signal \(v\).

We mention that in [Natarajan and Weiss 2018 Theorem 6.3] sufficient conditions were given for an equilibrium point of \(P_0\) to be almost globally asymptotically stable. In this paper, (local) exponential stability is what we need, and that can be checked with relative ease, using the linearization of (5.4), according to Remark 4.2.

The function \(\Xi : V \rightarrow \mathbb{R}^4\) is given by

\[
\Xi(v) = \begin{bmatrix}
\frac{TM}{\sin} + \frac{V\sin(\text{arccos} L(v) - \phi)}{R} \\
-\frac{TM}{\sin} \\
\phi_g \\
\text{arccos} L(v) - \phi
\end{bmatrix},
\]

where \(\phi \in (0, \frac{\pi}{2})\) such that \(\tan \phi = \frac{\phi}{R}\).

\[
L(v) = -\frac{TM}{\xi_f} \frac{L\sqrt{p^2 + \phi_g^2}}{V} + \frac{m\xi_f\phi_g}{V\sqrt{p^2 + \phi_g^2}}, \quad p = \frac{R}{L}.
\]

Our numerical explorations indicate that within the rectangle \(R := [-60, 70] \times [0.01, 1.2]\) shown in Fig. 4,

\[
V \cap \mathcal{R} = \{[T_m\ i_f]^\top \in \mathcal{R} | \ |L(v)| < 1\}.
\]

For more details on the mapping \(\Xi\), see [Lorenzetti et al. 2022 Proposition 3.1 and eq. (48)] (\(T_m\) there is equivalent to \(T_m\) here, since we have assumed \(\phi_m = \phi_g\)).

**Verification of Assumption 2.** As suggested after Remark 4.5, we choose \(\mathcal{N} = G_{\text{right}}^{-1} \subset C^\infty(U, V)\), given by

\[
\mathcal{N}(u) = \left[\frac{4R^2\|u-C\|^2 - V^2}{4W\|u-M\|^2}, \frac{V^2 - R}{V\|u-M\|^2}\right],
\]

where

\[
C = \left[\frac{V^2}{\phi_g}\right], \quad Z = \left[\frac{R}{\phi_g}\right], \quad M = \frac{V^2}{\|Z\|^2} Z.
\]

![Fig. 4. A glance of the set \(V\) (in light green), when \([T_m\ i_f]^\top \in \mathcal{R} = [-60, 70] \times [0.01, 1.2]\).](image)

![Fig. 5. The set \(U\) (in light green) when \(\mathcal{N} = G_{\text{right}}^{-1}\), from (5.6), the set \(U \subset \mathcal{U}\) (in light blue), and the circle \(p^2 + q^2 = (15\text{kHz})^2\) (in dashed blue). The points \(C\) and \(M\) are as in (5.7). (Note that the lower edge of \(U\) is chosen slightly above the line passing through \(C\) and \(M\).)](image)
Fig. 6. A glance of the set $U$ (in light green) when $N = K$ from (5.8). The circle $P^2 + Q^2 = (15\text{kW})^2$ is in dashed blue. Comparing the above with the set $U$ from Fig. 5, it is clear that a large part (depicted here in red) of the half-plane above the line passing through $C$ and $M$ is missing.

Table 1

| $P_{set,j}$ [kW] | $Q_{set,j}$ [kVAR] |
|------------------|-------------------|
| -4              | 9                 |
| -5              | 17                |
| 3               | 12                |
| 5               | 16                |
| 6               | 12                |
| 10              | 17                |
| 11              | 12                |
| 17              | 2                 |
| 12              | -7                |
| 5               | -2                |

so that Assumption 2 is satisfied with $U = G(V)$. For more details on $G^{-1}_{\text{right}}$, see [Lorenzetti et al., 2022 Theorem 3.6, Remark 3.7]. The relevant portion of the set $U$ is shown in Fig. 5. (Outside the rectangular boundaries of Fig. 5, the powers are too large to have practical significance for the synchronverter considered here.) For more details on $U$ see [Lorenzetti et al., 2022 Subsect. VI-A].

Remark 5.1 We mention that an alternative choice for $N$ could be, e.g., $N = K \in \mathbb{R}^{2 \times 2}$ given by

$$K = \begin{bmatrix} \frac{1}{50} & 0 \\ 0 & \frac{1}{500} \end{bmatrix}.$$  

(5.8)

However, this choice leads to a smaller set $U$, see Fig. 6, where the points satisfying Assumption 2 (with $N = K$) are depicted in green. Thus, the advantage of using $N = G^{-1}_{\text{right}}$ is twofold: the resulting set $U$ is larger and there is no need to search (numerically) for the set $U$, by computing the region in which $\frac{\partial(G_{K})}{\partial u} > 0$, since $U = G(V)$.

The set $U$. Due to current limitations, the safe synchronverter operating region in the $(P, Q)$ plane is described by a disk of radius $15\text{kW}$. Thus, we choose $U \subset \mathbb{R}^{2 \times 2}$ closed and convex such that $P^2 + Q^2 \leq (15\text{kW})^2$ in $U$. The set $U$ is shown (in light blue) in Fig. 5. (For convenience, we chose the set $U$ to be a convex polyhedron.)

5.3 Simulation results

We choose as reference signal $r$ a sequence of ten different values for $(P_{set}, Q_{set})$ (shown in Table 1), which we assume to be generated by an external control loop (not modelled here), each kept constant for 10 seconds. In Fig. 7(a) we show the comparison (in the $(P, Q)$ plane) between the state trajectory of the saturating integrator from (3.2), in blue, with $N = G^{-1}_{\text{right}}$ (given in (5.6)) and $k = 2$, and the state trajectory of a classical integrator ($\Pi_{U} = I$ in (3.2)), in green, with $N = K$ (given in (5.8)) and $k = 1$ (in both cases $\tau_p = 0$). It is interesting to note that the reference point $(P_{set,9}, Q_{set,9}) = (12\text{kW}, -7\text{kVAR})$, which generates an unstable equilibrium point for the closed-loop system with a classical integrator, and a stable equilibrium point for the closed-loop system with the saturating integrator, is outside the set $U$ from Fig. 6, corresponding to $N = K$, but inside the set $U$ from Fig. 5, corresponding to $N = G^{-1}_{\text{right}}$. In Fig. 7(b) the same comparison is shown for the signal $v$ in the $(T_m, T_f)$ plane. Finally, we show in Fig. 8(a), 8(b) the output active power $P$ and the output reactive power $Q$ values (in time), for both scenarios.

Remark 5.2 The step reference $r$ described above is clearly not constant. However, it can be proved (see [Lorenzetti and Weiss, 2022, Prop. 4.5]) that the result from Theorem 4.7 can be extended for step references (with values in $Y$) whose discontinuity points are “sufficiently far” from each other.

6 Conclusions

A novel MIMO PI anti-windup controller for a stable nonlinear plant has been proposed, based on PDS theory, which extends our previous work [Lorenzetti and Weiss, 2022]. Under standard assumptions, we have used SP tools to derive a sufficient condition on the controller gain ensuring (local) closed-loop stability and constant reference tracking. We propose to embed the right inverse of the plant steady-state input-output map in the controller, and we have shown the advantages of this choice through a numerical example, namely, the output power regulation for a grid-connected synchronverter.

A Proof of Theorem 4.7

We rewrite the closed-loop system (3.2) as a standard SP model, as in [Lorenzetti and Weiss, 2022 Sect. III].

We introduce the variables

$$\tilde{x} := x - x_r, \quad \tilde{u}_I := u_I - u_r, \quad (A.1)$$

the functions (recall $f$ from (3.3))

$$\tilde{g}(\tilde{x}) := g(\tilde{x} + x_r), \quad \Pi_U(\tilde{u}_I, \cdot) := \Pi_U(\tilde{u}_I + u_r, \cdot),$$

$$\tilde{h}(\tilde{u}_I, \tilde{x}) := \Pi_U(\tilde{u}_I, \cdot - \tilde{g}(\tilde{x})), \quad \tilde{f}(\tilde{u}_I, \tilde{x}) := f(\tilde{x} + x_r, \tilde{u}_I + u_r),$$

$$\tilde{\beta}(\tilde{u}_I, \tilde{x}, k) := \tilde{f}(\tilde{u}_I + \tau_p k (r - \tilde{g}(\tilde{x})), \tilde{x}) - \tilde{f}(\tilde{u}_I, \tilde{x}),$$

where $x, u_I, u_r$ denote the state, the output of the integrator, and the reference input, respectively.
(a) The state trajectories of the saturating integrator $\Pi_U$ (in blue) and of the classical integrator (in green). We indicate (in red) the values of the reference $r$, to be tracked. As expected, the state of the saturating integrator is never leaving the set $U$ from Fig. 5 (shown here in light blue).

(b) The values of the signal $v$ in the two closed-loop systems described in the main caption. We indicate (in red) the values of $G_{\text{right}}(r)$, to be tracked by $v$. As expected, the signal $v$, in the presence of a saturating integrator, is never leaving the set $V = N(U)$ (shown here in light blue).

Fig. 7. The comparison between the signals $u_I$ (Subfig. a) and the signals $v$ (Subfig. b) for two different closed-loop systems (as in Fig. 3): one formed by $P_0$ in feedback with the saturating integrator $\int \Pi_U$ (from Fig. 5), $N = G_{\text{right}}^{-1}$ from (5.6), and $k = 2$, whose signals are indicated in blue, and the other one formed by $P_0$ in feedback with a classical integrator ($\Pi = I$), $N = K$ from (5.8), and $k = 1$, whose signals are indicated in green ($\tau_p = 0$ in both closed-loops).

and we change the time-scale of (3.2) introducing $s := k \cdot t$. Thus, using (2.3), we can rewrite (3.2) as

$$\frac{d\hat{u}_I}{ds} = \hat{h}(\hat{u}_I, \hat{x}), \quad k \frac{d\hat{x}}{ds} = \hat{f}(\hat{u}_I, \hat{x}) + \hat{\beta}(\hat{u}_I, \hat{x}, k). \quad (A.2)$$

For small $k > 0$, this is a standard singular perturbation model according to (Khalil, 2002, Sect. 11.5). We point out that in (Khalil, 2002, Sect. 11.5) the functions describing the singularly perturbed closed-loop systems are required to be locally Lipschitz, which is not the case here (because of $\Pi_U$). However, our system (A.2) fits the framework of (Kokotović et al., 1999, Ch. 7), where it is only required that a unique (local) closed-loop solution exists, which we have proved in Prop. 3.1.

Following (Khalil, 2002, Sect. 11.5), let

$$\tilde{\Xi}(\hat{u}_I) := \Xi(N(\hat{u}_I + u_r)) - x_r,$$

and define the fast variable

$$\tilde{x}_f := \hat{x} - \tilde{\Xi}(\hat{u}_I).$$

Using the notation introduced above, we reformulate our (A.2) like (Khalil, 2002, eqs. (11.35), (11.36)), i.e.,

$$\frac{d\hat{u}_I}{ds} = \hat{h}(\hat{u}_I, \tilde{x}_f + \tilde{\Xi}(\hat{u}_I)), \quad (A.3)$$

$$k \frac{d\tilde{x}_f}{ds} = \hat{f}(\hat{u}_I, \tilde{x}_f + \tilde{\Xi}(\hat{u}_I)) + \beta(\hat{u}_I, \tilde{x}_f + \tilde{\Xi}(\hat{u}_I), k)$$

$$- k \frac{d\tilde{\Xi}}{d\hat{u}_I} \hat{h}(\hat{u}_I, \tilde{x}_f + \tilde{\Xi}(\hat{u}_I)), \quad (A.4)$$
which has an equilibrium point at \((\bar{u}_f, \bar{x}_f) = (0, 0)\). In accordance with the change of variables (A.1), we define
\[
\tilde{U} := U - u_r \subset \mathbb{R}^p \quad \text{and} \quad \hat{U} := U - u_c \subset \tilde{U},
\]
which contain the origin. Thus, the state space of the closed-loop system (A.3)-(A.4) is \(X' := \hat{U} \times \mathbb{R}^n\).

Using standard arguments, see (Khalil 2002, Ch. 11), (Kokotović et al. 1999, Ch. 7) or (Lorenzetti and Weiss 2022, Sect. III), we identify the reduced model and the boundary-layer system associated to (A.3)-(A.4). Recall \(G\) from Assumption 2. Define the function
\[
\hat{G}(\tilde{u}_t) := \tilde{g}(\tilde{X}(\tilde{u}_t)) = G(N(\tilde{u}_t + u_r)).
\]
The reduced (slow) model associated to (A.3)-(A.4) is obtained by rewriting (A.4) in the original fast time scale \(t\) and then taking \(k = 0\), which yields
\[
\dot{\tilde{x}}_f = \tilde{f}(\tilde{u}_t, \tilde{x}_f + \tilde{X}(\tilde{u}_t)),
\]
where \(\tilde{u}_t \in \tilde{U}\) is treated as a fixed parameter.

We are now ready to prove the stability of the equilibrium point \((x_r, u_r)\) of the closed-loop system (3.2), using SP theory. We follow the arguments in (Lorenzetti and Weiss 2022, Sect. IV), which are based on the guidelines of (Khalil 2002, Sect. 11.5). (Note that the fast variable \(z\) in Lorenzetti and Weiss 2022 is denoted here by \(\tilde{x}_f\).)

Define the set \(\tilde{U}_\delta := \hat{U} + B_\delta\), where \(B_\delta\) denotes the closed ball of radius \(\delta > 0\) in \(\mathbb{R}^p\). We choose \(\delta\) such that \(\tilde{U}_\delta \subset \hat{U}\). We will use (Lorenzetti and Weiss 2022, Th. 4.2), but with \(\tilde{U}_\delta \subset \mathbb{R}^p\) (instead of \(\hat{U} \subset \mathbb{R}^p\)). To check this extension of (Lorenzetti and Weiss 2022, Th. 4.2), it is enough to replace the Lipschitz property of the saturating integrator \(\mathcal{J}\) in Lorenzetti and Weiss (2022) with the contraction property (2.8) of the operator \(\Pi_U\) in the proof of (Lorenzetti and Weiss 2022, Th. 4.2).

**Step 1: Stability of the reduced model (A.5).** Let \(r \in V\). The aforementioned extension of (Lorenzetti and Weiss 2022, Th. 4.2) demands the existence of a Lyapunov function \(V\) for (A.5) (defined on \(\tilde{U}_\delta\)) such that
\[
c_1\|\tilde{u}_t\|^2 \leq V(\tilde{u}_t) \leq c_2\|\tilde{u}_t\|^2,
\]
\[
d\frac{dV}{d\tilde{u}_t}(\tilde{u}_t, \tilde{X}(\tilde{u}_t)) \leq -c_3\|\tilde{u}_t\|^2, \quad \left\| \frac{dV}{d\tilde{u}_t} \right\| \leq c_4\|\tilde{u}_t\|,
\]
for all \(\tilde{u}_t \in \tilde{U}_\delta\), where \(c_1, \ldots, c_4\) are positive constants.

As in (Lorenzetti and Weiss 2022, Subsec. IV-A), we consider the candidate Lyapunov function
\[
V(\tilde{u}_t) = \frac{1}{2}\|\tilde{u}_t\|^2 \quad \forall \, \tilde{u}_t \in \tilde{U}_\delta.
\]
Its derivative along the trajectories of (A.5) is
\[
\frac{dV}{ds} = (\Pi_U(\tilde{u}_t, \hat{G}(0) - \hat{G}(\tilde{u}_t)), \tilde{u}_t).
\]
The (unique) equilibrium point of (A.5) is \(0 \in \text{int} \tilde{U}_\delta\) and \(\hat{G}\) is strictly monotone from Assumption 2. Therefore, the operator \(\Pi_U\) behaves like the identity and the block \(\hat{G}\) reduces to a classical integrator. Thus
\[
\frac{dV}{ds} = (\hat{G}(0) - \hat{G}(\tilde{u}_t), \tilde{u}_t) \leq -\mu\|\tilde{u}_t\|^2,
\]
and the conditions (A.7) are easily seen to hold.

**Step 2: Stability of the boundary-layer system (A.6).** Lorenzetti and Weiss (2022, Th. 4.2) requires the existence of a Lyapunov function \(W\) for (A.6) (defined on \(\tilde{U}_\delta \times B_{\varepsilon_0}\)) such that
\[
b_1\|\tilde{x}_f\|^2 \leq W(\tilde{u}_t, \tilde{x}_f) \leq b_2\|\tilde{x}_f\|^2,
\]
\[
\frac{\partial W}{\partial \tilde{x}_f}(\tilde{u}_t, \tilde{x}_f + \tilde{X}(\tilde{u}_t)) \leq -b_3\|\tilde{x}_f\|^2,
\]
\[
\left\| \frac{\partial W}{\partial \tilde{x}_f} \right\| \leq b_4\|\tilde{x}_f\|, \quad \left\| \frac{\partial W}{\partial \tilde{u}_t} \right\| \leq b_5\|\tilde{x}_f\|^2,
\]
for all \((\tilde{u}_t, \tilde{x}_f) \in \tilde{U}_\delta \times B_{\varepsilon_0}\) (recall \(\varepsilon_0\) from Assumption 1), where \(b_1, \ldots, b_5\) are positive constants. As in Lorenzetti and Weiss (2022), we want to use (Khalil 2002, Lemma 9.8) to guarantee the existence of a function \(W\) such that (A.8) holds. To check its assumptions, we use the arguments of (Lorenzetti and Weiss 2022, Subsec. IV-B), with the difference that here \(\tilde{U}_\delta \subset \mathbb{R}^p\) (instead of \(\tilde{U}_\delta \subset \mathbb{R}^p\)). Thus, we can simply replace \(F_j(z, \tilde{u}_t)\) there with \(F_{ij}(\tilde{x}_f, \tilde{u}_t) := \frac{\partial p}{\partial \tilde{u}_t}\) here, for all \(l \in \{1, 2, \ldots, p\}\) and for all \(j \in \{1, 2, \ldots, n\}\), to guarantee that the assumptions of (Khalil 2002, Lemma 9.8) are met and, thus, that a function \(W\) satisfying (A.8) exists.

**Step 3: Stability of the closed-loop system (3.2).** We complete the proof of Theorem 4.1 by following step-by-step that of (Lorenzetti and Weiss 2022, Th. 4.3), and using the extension of (Lorenzetti and Weiss 2022, Th. 4.2) discussed before Step 1.

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