In this paper, we address the problem of simultaneous stabilization with desired decoupling, robustness, and performance for a finite collection of unstable multi-input-multi-output adversely coupled uncertain plants with resource constraints. For this purpose, we present a robust simultaneously stabilizing decoupling (RSSD) static output feedback controller whose synthesis is based on finding a central plant (closest plant) from a set of plants to be simultaneously stabilized. This paper presents a tractable method to obtain a central plant from a finite set of stable/unstable plants. As the closeness between plants and their frequency characteristics can be simultaneously modified by cascading these plants with pre and post compensators, a simultaneous closeness-performance enhanced (SCP) central plant problem and its non-convex-non-smooth optimization problem are formulated to obtain suitable pre and post compensators and resulting SCP central plant. Further, the existence conditions of an RSSD static output feedback controller are developed by utilizing the maximum $\nu$-gap metric of the SCP central plant, the sufficiency condition of SCP central plant for the simultaneous stabilization, and the eigenstructure assignment algorithm for output feedback. These existence conditions are utilized to formulate a new non-smooth optimization problem for the synthesis of RSSD static output feedback controller. The performance index and the constraints of the non-smooth optimization problem are obtained from the existing conditions of robust simultaneously stabilizing static output feedback controller. A new genetic algorithm based iterative algorithm is developed to generate an RSSD static output feedback controller by sequentially solving the optimization problem of SCP central plant problem and RSSD static output feedback problem. The effectiveness of this iterative algorithm is then demonstrated through numerical simulations.

**Keywords** Decoupling · central plant · output feedback · robust simultaneous stabilization · $\nu$-gap metric
1 Introduction

Simultaneously stabilizing controller is a single controller that simultaneously stabilize a finite collection of plants. This type of controller is of interest when we design a flight control system for micro-air vehicles (MAVs), where the vehicle operating in different flight conditions is modeled as a collection of linear time-invariant (LTI) plants. The recent trends in micro air vehicles (MAVs) is the development of a new class of extremely small air vehicles called nano air vehicles (NAVs) that execute missions undetected with a high degree of agility. They are widely used for intelligence, battlefield surveillance and reconnaissance, and disaster assessment missions. MAVs and Navs have dimensional and weight constraints as their overall dimension and weight need to be less than 75 mm and 20 g [1]. The LTI plants of a NAV are generally unstable because of significant adverse coupling, dimensional constraints on the stabilizers, aerodynamic effects, and due to the limitation in keeping center-of-gravity (CG) at the desired location. The dynamics of a NAV has adverse coupling due to the significant gyroscopic coupling and counter torque of the propulsion unit. The adverse coupling degrades the stability and performance characteristics of different modes [2]-[3]. Further, the dynamics of these vehicles cannot be decoupled into lateral and longitudinal dynamics because of the adverse coupling.

The LTI plants of a NAV may have a different number of unstable poles as these plants are modeled for different operating conditions defined in the flight envelope. A NAV has resource constraints due to lack of lightweight sensors to measure every state variable (like an angle-of-attack, airspeed, and sideslip angle) of the plant. Besides, a NAV is unable to have an adequate number of actuators necessary for the successful operation due to the weight constraints. Additionally, an autopilot of a NAV may not possess sufficient computational power. The plants of a NAV have significant model uncertainty mainly because of unsteady flow, wind effect, and inaccurate measurements of forces and moments. This causes notable errors in the state variable estimations. In general, the LTI plants of NAVs are unstable multi-input-multi-output (MIMO) adversely coupled with a different number of unstable poles [4]-[6]. These small air vehicles require a feedback controller to accomplish a mission. However, conventional methods such as gain scheduling and full state feedback controller are not suitable because of the resource constraints and model uncertainty. Besides, the successive loop closure is also ill-suited due to the presence of significant adverse coupling in the dynamics. Furthermore, the limited computation power necessitates a computationally simple control law that requires less computer memory. Hence, a NAV needs a multivariable simultaneously stabilizing static output feedback controller. Besides, this controller needs to provide the desired decoupling, robustness, and performance to all the plants of a NAV.

The simultaneous stabilization problem was first studied in [7] and [8]. The simultaneous stabilization problem of more than two plants does not have any tractable solution due to its NP-hard nature [9]-[10]. Hence, a simultaneous stabilization problem is solved through numerical means. In [11], an iterative algorithm based on linear matrix inequality is developed to solve the simultaneous stabilization problem of a finite set of strictly proper MIMO plants with static full state feedback and output feedback. A decomposition strategy to solve a simultaneous stabilization problem is proposed in [12]. Here, a bi-level design optimization structure is selected in which the design of the single controllers for each individual plant is carried out at the bottom level whereas at the top-level optimization generates a single controller which is approximate of those individual controllers. Further solutions to the simultaneous stabilization problem employing optimization techniques can be found in [13]-[16]. The simultaneous stabilization problem can also be solved by first deriving a sufficiency condition based on central plant and then synthesize a controller that satisfies the sufficiency condition using robust stabilization theory.

Following this method, a single controller that satisfies the sufficiency condition of the central plant is synthesized in [17]. Here, the central plant is obtained by solving a 2-block optimization problem. In [18], H∞ controller design method is utilized to synthesize a simultaneous stabilization controller for a finite collection of plants correspond to the longitudinal dynamics of an unmanned aircraft based on a central plant. Note that the aforementioned methods are not suitable for the simultaneous stabilization of unstable plants with adverse coupling. Further, one can observe that finding an analytic or numerical method to generate a single controller that concurrently achieves simultaneous stabilization, robustness, performance, and decoupling is difficult.

In this paper, we formulate a robust simultaneous stabilization decoupling (RSSD) problem for a finite set of unstable MIMO adversely coupled resource constraints plants (with different number of unstable poles) of a NAV. The approach followed in this paper to solve this problem is about finding an RSSD static output feedback controller that satisfies the sufficiency condition of central plant for simultaneous stabilization. This paper has the following main contributions.

1. A tractable method to obtain a central plant from a finite set of plants, $P_i(s) \in R L_{\infty}^{n_i \times m_i}$, for simultaneous stabilization is developed using $\nu$-gap metric and robust stabilization theory. Following this tractable method, the non-convex-non-smooth optimization problem given in [19] is employed to obtain an SCP central plant.
2. The existence conditions of the RSSD output feedback controller are developed by exploiting the properties of $\nu$-gap metric of the SCP central plant, the sufficiency condition of SCP central plant for the simultaneous
stabilization, and the eigenstructure assignment algorithm for output feedback. These existence conditions are utilized to formulate a new non-smooth optimization problem for an RSSD output feedback controller.

3. A new tractable genetic algorithm based iterative algorithm (NN-RSSD) is developed to solve an RSSD problem. NN-RSSD algorithm finds the solution by sequentially solving two optimization problems of the SCP central plant problem and the RSSD output feedback controller problem.

4. The effectiveness of NN-RSSD algorithm is demonstrated by generating an RSSD output feedback controller for eight unstable MIMO adversely coupled resource constrained plants with a different number of unstable poles of a 75 mm wingspan fixed-wing NAV.

This paper is organized as follows. In Section 2 we describe an adversely coupled plant. The synthesizing of an RSSD static output feedback controller is explained in Section 3. In Section 4 numerical simulation results are presented. Finally, in Section 5 the key results of this paper are summarized.

2 Adversely Coupled Plant

Conventionally, the linear equations of motion of aerospace vehicles can be decoupled into longitudinal and lateral equations of motions when rotor gyroscopic effects, aerodynamic cross-coupling, and inertial coupling are negligibly small. These equations of motion are given by

\[ \dot{x}_{Lo} = A_{Lo}x_{Lo} + B_{Lo}\delta_{Lo} \]  
\[ \dot{x}_{La} = A_{La}x_{La} + B_{La}\delta_{La} \]  

where \( x_{Lo} \in \mathbb{R}^{\hat{a}}, x_{La} \in \mathbb{R}^{\hat{b}}, \delta_{Lo} \in \mathbb{R}^{\hat{c}}, \) and \( \delta_{La} \in \mathbb{R}^{\hat{d}} \) are longitudinal state vector, lateral state vector, longitudinal control vector, and lateral control vector, respectively. Besides, \( A_{Lo} \in \mathbb{R}^{\hat{a} \times \hat{a}}, A_{La} \in \mathbb{R}^{\hat{b} \times \hat{b}}, B_{Lo} \in \mathbb{R}^{\hat{a} \times \hat{c}}, \) and \( B_{La} \in \mathbb{R}^{\hat{b} \times \hat{d}} \) represent system and control matrices of longitudinal and lateral state-space models, respectively. However, significant cross-coupling results in a coupled equations of motion and is given as

\[ \dot{x} = A_{c}x + B_{c}\delta_{c} \]  

where \( x \in \mathbb{R}^{(\hat{a}+\hat{b})} = [x_{Lo}|x_{La}]^T, A_{c} \in \mathbb{R}^{(\hat{a}+\hat{b}) \times (\hat{a}+\hat{b})} = \left[ \begin{array}{cc} A_{Lo} & A_{La} \\ A_{Lo}^T & A_{La} \end{array} \right], B_{c} \in \mathbb{R}^{(\hat{a}+\hat{b}) \times (\hat{c}+\hat{d})} = \left[ \begin{array}{cc} B_{Lo} & B_{La} \\ B_{Lo}^T & B_{La} \end{array} \right], \) and \( \delta_{c} \in \mathbb{R}^{(\hat{c}+\hat{d})} = [\delta_{Lo}|\delta_{La}]^T. \) Here, \( A_{Lo}^{(1)} \in \mathbb{R}^{\hat{a} \times \hat{a}}, A_{La}^{(1)} \in \mathbb{R}^{\hat{b} \times \hat{a}}, B_{La}^{(1)} \in \mathbb{R}^{\hat{a} \times \hat{d}}, \) and \( B_{La}^{(1)} \in \mathbb{R}^{\hat{b} \times \hat{c}} \) symbolize longitudinal coupling block of \( A_{c}, \) lateral coupling block of \( A_{c}, \) longitudinal coupling block of \( B_{c}, \) and lateral coupling block of \( B_{c}, \) respectively. The longitudinal and lateral dynamics of an aerospace vehicle are coupled when the longitudinal state variable has significant influence from the lateral modes. Similarly, lateral state variable has significant influence from the longitudinal modes. If the coupling is not significant, then \( A_{La}^{(1)} \) and \( A_{La}^{(1)} \) can be approximated with a zero matrix. From flight dynamics point of view, a plant of an aerospace vehicle is coupled when

\[ \bar{x}_{i}(\lambda_{i}(\left[ \begin{array}{cc} A_{Lo} & 0 \\ 0 & A_{La} \end{array} \right]) \neq \bar{x}_{i}(\lambda_{i}(\left[ \begin{array}{cc} A_{Lo} & 0 \\ 0 & A_{La} \end{array} \right]))) \]  

for all \( i = 1, 2, \ldots, (\hat{a} + \hat{b}) \). Here, \( \bar{x}_{i} \) denotes the eigenvector of \( i \) th eigenvalue, \( \lambda_{i}. \) Further, the coupled plant has adverse coupling when any \( \lambda_{i}(\left[ \begin{array}{cc} A_{Lo} & 0 \\ 0 & A_{La} \end{array} \right]) \in \mathbb{C}_{-} \) migrates towards \( \mathbb{C}_{+} \) due to the presence of \( A_{Lo}^{(1)} \) and \( A_{La}^{(1)} \) in \( A_{c}. \) Generally, the plants of a NAV have adverse coupling. Note that one needs to refer \( \text{[2]} \) for further details about the adverse coupling in a NAV.

3 RSSD Static Output Feedback Controller

In this section, we formulate robust simultaneous stabilization decoupling problem for an unstable MIMO uncertain adversely coupled resource constrained plants of a NAV.

3.1 Problem Statement

We now consider a finite set, \( \mathcal{P} \), that contains LTI MIMO plants of a NAV. The plants belonging to \( \mathcal{P} \) are stabilizable, adversely coupled, and uncertain. Furthermore, unstable plants of \( \mathcal{P} \) may have a different number of unstable poles. We define \( \mathcal{P} \) as \( \mathcal{P} = \{P_{f}(s) \in \mathcal{R}L_{\infty}^\hat{r} \times \hat{m} | f = \{1, 2, \ldots, N\}, N \in \mathbb{Z}_{\geq 3}, N < \infty \} \). Here, \( \mathcal{R}L_{\infty}^{\hat{r} \times \hat{m}} \) symbolizes the space of proper, real-rational, \( \hat{r} \times \hat{m} \) matrix-valued functions of \( s \in \mathbb{C} \) which are analytic in \( \mathbb{C}_{+} \cup \mathbb{C}_{-}. \) The objective here is to find a static output feedback controller, \( K \in \mathbb{R}^{\hat{m} \times \hat{r}} \), that achieves the following for all the plants belonging to \( \mathcal{P} \).
1. Robust simultaneous stabilization.
2. Mode decoupling.
3. Specified performance characteristics.

3.2 Simultaneous Stabilization with Central Plant

Let the generalized stability margin, $b_{P_i,K}$, of $P_i(s) \in \mathcal{P}$ for a stabilizing controller, $K$ of $P_i(s)$ is defined as

$$b_{P_i,K} = \frac{1}{\left\| \left[ \begin{array}{cc} P_i(s) & I \\ \end{array} \right] (I - KP_i(s))^{-1} \left[ \begin{array}{cc} -I & K \end{array} \right] \right\|_{\infty}}$$

(4)

Now, we state the following definitions to explicate the simultaneous stabilization of the finite set of plants with a central plant.

**Definition 3.1.** Minimum simultaneous stabilization stability margin: The minimum simultaneous stabilization stability margin of a plant that belongs to $\mathcal{P}$ is the smallest generalized stability margin that needs to be achieved by a single controller synthesized about the same plant for simultaneously stabilizing all the plants in the set.

**Definition 3.2.** Central plant: A plant that has the smallest minimum simultaneous stabilization stability margin among the minimum simultaneous stabilization stability margins of all the plants in $\mathcal{P}$.

In this paper, the approach to solve a simultaneous stabilization problem is to derive the sufficiency condition for simultaneous stabilization in the first place and then synthesize a stabilizing controller that satisfies this condition.

3.2.1 Sufficiency Condition for the Simultaneous Stabilization

Consider $P_i(s) \in \mathcal{P}$ be the nominal plant and the other plants in $\mathcal{P} \setminus \{P_i(s)\}$ be its perturbed plants. Let $P_f(s) \in \mathcal{P} \setminus \{P_i(s)\}$. A perturbed plant model for $P_i(s)$ is required to develop the sufficiency condition. For this purpose, we have right coprime factor uncertainty model for $P_i(s)$. Note that, right coprime factor uncertainty model [21] does not require the same number of unstable poles. Let $\epsilon_{P_i,P_f}$ is the least upper bound on perturbations between $P_i(s)$ and $P_f(s)$. The robust stability condition of $P_i(s)$ is given as

$$b_{P_i,K} > \epsilon_{P_i,P_f}$$

(5)

We now proceed to state the following theorem to attain the sufficiency condition for simultaneous stabilization.

**Theorem 3.1.** A stabilizing controller $K$ of $P_i(s) \in \mathcal{P}$ that achieves the condition given in (5) simultaneously stabilizes all the plants in $\mathcal{P}$ if $\epsilon_{P_i,P_f} = \epsilon_{P_i}$. Here, $\epsilon_{P_i}$ is defined as

$$\epsilon_{P_i} = \max \{ \epsilon_{P_i,P_f} \mid P_f(s) = (N(s) + \Delta_{N_{P_i,P_f}}(s))(M(s) + \Delta_{M_{P_i,P_f}}(s))^{-1}, P_i(s) = N(s)M^{-1}(s),$$

$$\left\| \left[ \begin{array}{cc} \Delta_{N_{P_i,P_f}}(s) & \Delta_{M_{P_i,P_f}}(s) \end{array} \right]^{\top} \right\|_{\infty} \leq \epsilon_{P_i,P_f},$$

$$P_i(s), P_f(s) \in \mathcal{P} \forall f = \{1, 2, \ldots, N\} \}$$

In [2], $N(s) \in \mathcal{RH}_{\infty}^{m \times \bar{m}}$ and $M(s) \in \mathcal{RH}_{\infty}^{n \times \bar{n}}$ with det$(M(s) \neq 0)$ are the normalized right coprime factors of $P_i(s)$. Also, $\Delta_{N_{P_i,P_f}}(s) \in \mathcal{RH}_{\infty}^{m \times \bar{m}}$ and $\Delta_{M_{P_i,P_f}}(s) \in \mathcal{RH}_{\infty}^{n \times \bar{n}}$ are the right coprime factor perturbations of $N(s)$ and $M(s)$, respectively.

**Proof.** Let us assume $K$ satisfies (5), then $K$ stabilizes $P_f(s)$. Further, if $\epsilon_{P_i,P_f} = \epsilon_{P_i}$, then the least upper bound on the perturbation between $P_i(s)$ and any other plants belonging to $\mathcal{P} \setminus \{P_f(s)\}$ will be smaller than $\epsilon_{P_i,P_f}$. Then, by virtue of small gain theorem, $K$ simultaneously stabilizes all the plants in $\mathcal{P}$. This establishes the proof.

Therefore, (5) with $\epsilon_{P_i,P_f} = \epsilon_{P_i}$ forms the sufficiency condition of $P_i(s)$ for the simultaneous stabilization of all the plants in $\mathcal{P}$. Let $\inf_{K} b_{P_i,K}$ is the smallest generalized stability margin of $P_i(s)$. The strict sufficiency condition for simultaneous stabilization considering all the stabilizing controllers of $P_i(s)$ is defined as

$$\inf_{K} b_{P_i,K} > \epsilon_{P_i}$$

(7)
Thus, \( \inf_{\mathbf{K}} b_{\mathbf{P}, \mathbf{K}} \) becomes the minimum simultaneous stabilization stability margin of \( \mathbf{P}_i(s) \). Each plant in \( \mathcal{P} \) has a strict sufficiency condition. The main question is which strict sufficiency condition of a plant in \( \mathcal{P} \) is suitable for the synthesis of a simultaneously stabilizing controller. Equations (5) and (7) indicate a sufficiency condition with the smallest minimum simultaneous stabilization stability margin is suitable. The sufficiency condition of the central plant is therefore suitable for simultaneous stabilization. Hence, a tractable method for identifying a central plant in \( \mathcal{P} \) is required to solve the simultaneous stabilization problem.

### 3.2.2 Identification of a Central Plant

We now state the following theorem which establishes the relationship between a minimum simultaneous stability margin and a \( v \)-gap metric of the plants belonging to \( \mathcal{P} \).

**Theorem 3.2.** \( \mathbf{P}_i(s) \in \mathcal{P} \) has the smallest minimum simultaneous stabilization stability margin if \( \mathbf{P}_i(s) \) satisfies the following condition

\[
\max \{ \delta_v(\mathbf{P}_i(j\omega), \mathbf{P}_f(j\omega)) \} \forall f = \{1, 2, \ldots, N\}, \mathbf{P}_i(s), \mathbf{P}_f(s) \in \mathcal{P} \}
\]

\[= \min \tau \]

where \( \tau = \{ \max \{ \delta_v(\mathbf{P}_1(j\omega), \mathbf{P}_f(j\omega)) \} \forall f = \{1, 2, \ldots, N\}, \mathbf{P}_1(s), \mathbf{P}_f(s) \in \mathcal{P} \}, \max \{ \delta_v(\mathbf{P}_2(j\omega), \mathbf{P}_f(j\omega)) \} \forall f = \{1, 2, \ldots, N\}, \mathbf{P}_2(s), \mathbf{P}_f(s) \in \mathcal{P} \}, \ldots, \max \{ \delta_v(\mathbf{P}_N(j\omega), \mathbf{P}_f(j\omega)) \} \forall f = \{1, 2, \ldots, N\}, \mathbf{P}_N(s), \mathbf{P}_f(s) \in \mathcal{P} \} \}.\]

Here, \( \delta_v(\mathbf{P}_1(j\omega), \mathbf{P}_f(j\omega)) \) denotes \( v \)-gap metric [20] between \( \mathbf{P}_1(s) \) and \( \mathbf{P}_f(s) \).

**Proof.** The relation between \( \delta_v(\mathbf{P}_i(j\omega), \mathbf{P}_f(j\omega)) \) and the right coprime factor perturbations of \( \mathbf{P}_i(s) \) is given as

\[
\delta_v(\mathbf{P}_i(j\omega), \mathbf{P}_f(j\omega)) = \left\| \begin{bmatrix} \Delta_{N_iP_f} & \Delta_{M_iP_f} \end{bmatrix} \right\|_\infty \leq \epsilon_{P_iP_f}
\]

For the simultaneous stabilization problem, \( \mathbf{P}_f(s) \) is known although it is considered to be the perturbed plant of \( \mathbf{P}_i(s) \). Thus, one can rewrite (9) as

\[
\delta_v(\mathbf{P}_i(j\omega), \mathbf{P}_f(j\omega)) = \epsilon_{P_iP_f}
\]

Using (6) and (10), we rewrite (7) as

\[
\inf_{\mathbf{K}} b_{\mathbf{P}, \mathbf{K}} > \max \{ \delta_v(\mathbf{P}_i(j\omega), \mathbf{P}_f(j\omega)) \} \mid \mathbf{P}_i(s), \mathbf{P}_f(s) \in \mathcal{P} \quad \forall f = \{1, 2, \ldots, N\}\}
\]

Equation (11) indicates the minimum simultaneous stabilization stability margin of \( \mathbf{P}_i(s) \) needs to be greater than \( \max \{ \delta_v(\mathbf{P}_i(j\omega), \mathbf{P}_f(j\omega)) \} \mid \mathbf{P}_i(s), \mathbf{P}_f(s) \in \mathcal{P} \quad \forall f = \{1, 2, \ldots, N\}\} \) for the simultaneous stabilization. Similar to (11), the strict sufficiency condition of each plant in \( \mathcal{P} \) can be obtained. Let the set, \( \{ \inf_{\mathbf{K}} b_{\mathbf{P}_i \setminus \{ \mathbf{P}_i \}, \mathbf{K}} \} \) consists of the minimum simultaneous stabilization stability margin of all the plants in \( \mathcal{P} \setminus \{ \mathbf{P}_i \} \).

Then, for all the single stabilizing controller that solves the strict sufficiency condition associated with each plant in \( \mathcal{P} \), the following condition holds if \( \mathbf{P}_i(s) \) satisfies (6).

\[
\inf_{\mathbf{K}} b_{\mathbf{P}, \mathbf{K}} < \min \{ \inf_{\mathbf{K}} b_{\mathbf{P}_i \setminus \{ \mathbf{P}_i \}, \mathbf{K}} \}
\]

The condition specified in (12) establishes that if \( \mathbf{P}_i(s) \) satisfies (6), then the smallest minimum simultaneous stabilization stability margin of \( \mathcal{P} \) belongs to \( \mathbf{P}_i(s) \). The theorem is therefore proven.

**Corollary 3.1.** The plant in \( \mathcal{P} \) with the smallest minimum simultaneous stabilization stability margin has the maximum \( v \)-gap metric which is the minimum among the maximum \( v \)-gap metrics of all the plants in \( \mathcal{P} \).

**Proof.** Let us assume \( \mathbf{P}_i(s) \in \mathcal{P} \) has the smallest minimum simultaneous stabilization stability margin. Equation (11) needs to be satisfied by a simultaneously stabilizing controller for \( \mathbf{P}_i(s) \). Subsequently, \( \mathbf{P}_i(s) \) and its simultaneously stabilizing controller satisfy (12) as \( \mathbf{P}_i(s) \) has got the smallest minimum simultaneous stabilization stability margin.
The basic structure of \( W \) is the result of Corollary 3.1, the method to acquire the central plant is to perform
\[
\hat{v} \in \text{argmin} \quad \| W \|_{\infty} \quad \| W \|_{\infty} \leq \| W \|_{\infty} \leq \| W \|_{\infty}
\]
where \( v \) minimizes the \(-\) gap metric which is the minimum among the maximum \( v \)-gap metrics of the plants in \( \mathcal{P} \).

Equation (13) indicates the plant with the smallest minimum simultaneous stabilization stability margin has the maximum \( v \)-gap metric which is the minimum among the maximum \( v \)-gap metrics of the plants in \( \mathcal{P} \).

As the result of Corollary 3.1, the method to acquire the central plant is to perform \( v \)-gap metric analysis that gives a plant in \( \mathcal{P} \) with a maximum \( v \)-gap metric which is the minimum among maximum \( v \)-gap metrics of all the plants in \( \mathcal{P} \). Equation (13) suggests that smaller \( \min \bar{\tau} \) is better for the simultaneous stabilization problem. \( \min \bar{\tau} \) can be reduced by altering the \( v \)-gap metric between the plants in \( \mathcal{P} \). The pre and post compensators cascaded with the plants modify the \( v \)-gap metric and the frequency characteristics.

Let \( W_{in}(s) \in \mathcal{R} \mathcal{H} \infty \hat{\delta} \times \hat{\delta} \) and \( W_{ot}(s) \in \mathcal{R} \mathcal{H} \infty \hat{\delta} \times \hat{\delta} \) are the pre and post compensators of all the plants belong to \( \mathcal{P} \). The basic structure of \( W_{in}(s) \) and \( W_{ot}(s) \) is defined as
\[
W_{in} = \{ \text{diag}[w_{in1}, \ldots, w_{inm}] | w_{in1}, \ldots, w_{inm} \in \mathcal{R} \mathcal{H} \infty \} \\
W_{ot} = \{ \text{diag}[w_{ot1}, \ldots, w_{otm}] | w_{ot1}, \ldots, w_{otm} \in \mathcal{R} \mathcal{H} \infty \}
\]
where the elements of \( W_{in} \) and \( W_{ot} \) are expressed as
\[
w_{in1} = \frac{a_{in1}s + b_{in1}}{c_{in1}s + d_{in1}} \quad \forall h = \{1, \ldots, \hat{m}\}, \\
w_{ot1} = \frac{a_{ot1}s + b_{ot1}}{c_{ot1}s + d_{ot1}} \quad \forall l = \{1, \ldots, \hat{f}\},
\]

Let \( \kappa = \{ H_{e}(s) | H_{f}(s) \in \mathcal{R} \mathcal{L} \infty \hat{\delta} \times \hat{\delta} , H_{f}(s) = W_{ot}(s)P_{f}(s)W_{in}(s), P_{f}(s) \in \mathcal{P} \} \). Now, the simultaneous closeness-performance enhanced (SCP) central plant problem is stated as follows. Find \( W_{in}(s) \) and \( W_{ot}(s) \) such that the maximum \( v \)-gap metric of the central plant of \( \kappa \) (closeness-performance enhanced central plant) needs to be less than \( \bar{\epsilon} \). At the same time, \( W_{in}(s) \) and \( W_{ot}(s) \) need to induce desired frequency characteristics on all the plants (performance enhanced plants) that belongs to \( \kappa \).

In [19], an optimization problem to find these feasible compensators is given. However, this note does not present a profound theoretical explanation for the \( v \)-gap metric-based central plant identification method which forms the basis of the optimization problem. The optimization problem is given as

\[
\text{minimize} \quad J_1 = \min \bar{\epsilon} \\
\begin{align*}
a_{in1UB} &> a_{in1} &> a_{in1LB} &\quad \forall \ h = 1, \ldots, \hat{m} \\
b_{in1UB} &> b_{in1} &> b_{in1LB} &\quad \forall \ h = 1, \ldots, \hat{m} \\
c_{in1UB} &> c_{in1} &> c_{in1LB} &\quad \forall \ h = 1, \ldots, \hat{m} \\
d_{in1UB} &> d_{in1} &> d_{in1LB} &\quad \forall \ h = 1, \ldots, \hat{m} \\
a_{ot1UB} &> a_{ot1} &> a_{ot1LB} &\quad \forall \ l = 1, \ldots, \hat{f} \\
b_{ot1UB} &> b_{ot1} &> b_{ot1LB} &\quad \forall \ l = 1, \ldots, \hat{f} \\
c_{ot1UB} &> c_{ot1} &> c_{ot1LB} &\quad \forall \ l = 1, \ldots, \hat{f} \\
d_{ot1UB} &> d_{ot1} &> d_{ot1LB} &\quad \forall \ l = 1, \ldots, \hat{f} \\
Z_{W_{in}} \cap P_{W_{ot}} &= \emptyset \\
P_{W_{in}} \cap Z_{W_{ot}} &= \emptyset \\
(Z_{W_{in}} \cup Z_{W_{ot}}) \cap Z_P &= \emptyset \\
(P_{W_{in}} \cup P_{W_{ot}}) \cap P_P &= \emptyset
\end{align*}
\]

where \( \bar{\epsilon} \) is defined as
\[
\bar{\epsilon} = \{ \max \{ \delta_{e}(W_{ot}(j\omega)P_{1}(j\omega)W_{in}(j\omega), W_{ot}(j\omega)P_{f}(j\omega) \} \mid \forall \ h = 1, \ldots, N \} , \ldots, \max \{ \delta_{e}(W_{ot}(j\omega)P_{N}(j\omega)W_{in}(j\omega), W_{ot}(j\omega)P_{f}(j\omega)W_{in}(j\omega) \} \mid \forall \ h = 1, \ldots, N \} \}
\]

In (18), \( Q \) represents the set that contains the coefficients of \( w_{in1} \) and \( w_{ot1} \). \( Z_{W_{in}} \) and \( Z_{W_{ot}} \) signify the sets that consist of the zeros (corresponding to each input-output pair) of \( W_{in} \) and \( W_{ot} \), respectively. Likewise, \( P_{W_{in}} \) and
$P_{\text{W}_{\text{ot}}}$ comprise all the poles of $W_{\text{in}}$ and $W_{\text{ot}}$, respectively. The sets, $Z_P$ and $P_P$ contain all the poles and zeros (corresponding to each input-output pair) of all the plants in $\mathcal{P}$, respectively. In (18), $a_{\text{in}_{\text{h},L,B}}$ and $a_{\text{in}_{\text{h},L,L,B}}$ denote the upper and lower bounds of $a_{\text{in}_{h}}$ for all $h = \{1, \ldots, \hat{m}\}$. Similarly, $b_{\text{in}_{\text{h},U,B}}, b_{\text{in}_{\text{h},L,B}}, c_{\text{in}_{\text{h},U,B}}, c_{\text{in}_{\text{h},L,B}}$, and $d_{\text{in}_{\text{h},U,B}}, d_{\text{in}_{\text{h},L,B}}$ are the bounds of other coefficients of $w_{\text{in}_h}$ for all $h = \{1, \ldots, \hat{m}\}$. Likewise, $a_{\text{ot}_{U,B}}, a_{\text{ot}_{L,B}}, b_{\text{ot}_{U,B}}, b_{\text{ot}_{L,B}}, c_{\text{ot}_{U,B}}, c_{\text{ot}_{L,B}}, d_{\text{ot}_{U,B}}$, and $d_{\text{ot}_{L,B}}$ are the upper and lower bounds of the coefficients of $w_{\text{ot}_l}$ for all $l = \{1, \ldots, \hat{r}\}$. The constraints, $Z_{\text{W}_{\text{in}}} \cap P_{\text{W}_{\text{ot}}} = \emptyset$ and $P_{\text{W}_{\text{in}}} \cap Z_{\text{W}_{\text{ot}}} = \emptyset$ prevent any pole-zero cancellation between the pre-compensator and post-compensator. Furthermore, $(Z_{\text{W}_{\text{in}}} \cup Z_{\text{W}_{\text{ot}}}) \cap Z_P = \emptyset$ and $(P_{\text{W}_{\text{in}}} \cup P_{\text{W}_{\text{ot}}}) \cap P_P = \emptyset$ avert the pole-zero cancellation between the unshaped plants and the compensators. Other constraints are the bounds on the coefficients of $w_{\text{in}_h}$ and $w_{\text{ot}_l}$. These bounds are needed to provide desired frequency characteristics to the plants. The bounds on the coefficients prevent the minimization of $J_1$ with any $W_{\text{in}}(s)$ and $W_{\text{ot}}(s)$ that degrade the frequency characteristics of all the augmented plants. Furthermore, the performance index of (18) is non-convex and non-smooth and hence (18) requires an iterative algorithm to obtain the solution.

### 3.3 Decoupling and Simultaneous Stabilization with Output Feedback

Let us assume $W_{\text{in}}(s)$ and $W_{\text{ot}}(s)$ are the solution of (18). $H_{\text{cp}}(s) = W_{\text{ot}}(s)P_1(s)W_{\text{in}}(s)$ is the SCP central plant with $\hat{J}_1$ and $b_{H_{\text{cp}},K}$ as its maximum $\nu$-gap metric and generalized stability margin, respectively. The sufficiency condition for the simultaneous stabilization of $H_{\text{cp}}(s)$ is given as

$$b_{H_{\text{cp}},K} > \hat{J}_1$$

(20)

where $b_{H_{\text{cp}},K}$ is defined as

$$b_{H_{\text{cp}},K} = \frac{1}{\left\| \begin{bmatrix} H_{\text{cp}} & I \end{bmatrix} (I-KH_{\text{cp}})^{-1} \left[ \begin{bmatrix} -I & K \end{bmatrix} \right] \right\|_{\infty}}$$

(21)

Generally, a stabilizing controller of $H_{\text{cp}}(s)$ achieves the condition in (20) by minimizing the infinity norm shown in (21). Further, we require an output feedback controller that provides some degree of mode decoupling to the closed-loop plants apart from achieving the condition given in (20). Using eigenstructure assignment technique described in [22], $\hat{r}$ desired closed-loop eigenvalues and $\hat{m}$ entries of the corresponding eigenvectors are assigned with the output feedback controller, $K \in \mathbb{R}^{\hat{m} \times \hat{r}}$ given by

$$K = W(CR)^{-1}$$

(22)

where $C$ denotes the output matrix of $H_{\text{cp}}(s)$. $E = [E_1, \ldots, E_l, \ldots, E_r]$ and $R = [R_1, \ldots, R_l, \ldots, R_r]$ are defined as

if $\{ \lambda_l \forall l = \{1, 2, \ldots, r\} \}$ are real then

$$\begin{bmatrix} R_l & E_l \end{bmatrix} \in \{ S_l : \begin{bmatrix} A - \lambda_l I & B \end{bmatrix} \begin{bmatrix} R_l & E_l \end{bmatrix} = 0 \}$$

(23)

if $\{ \lambda_l \forall l = \{1, 2, \ldots, r\} \}$ are complex then

$$\begin{bmatrix} R_l & E_l \end{bmatrix} \in \{ S_l : \begin{bmatrix} \begin{bmatrix} A - \lambda_l^{re} I & \lambda_l^{im} I \\ -\lambda_l^{im} I & A - \lambda_l^{re} I \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} R_l^{re} & R_l^{im} \\ E_l^{re} & E_l^{im} \end{bmatrix} \end{bmatrix} = 0$$

(24)

where $R_l = [R_l^{re}, R_l^{im}]^T$ and $E_l = [E_l^{re}, E_l^{im}]^T$. $A$ and $B$ represent system and input matrices of $H_{\text{cp}}(s)$, respectively. $\lambda_l^{re}$ and $\lambda_l^{im}$ denote $l$th desired eigenvalue, real part of the $l$th desired eigenvalue, and imaginary part of the $l$th desired eigenvalue. $R_l$ and $E_l$ are computed using specific values assigned to $\hat{m}$ entries of eigenvectors of a desired closed-loop eigenvalue. For decoupling, these $\hat{m}$ entries need to be assigned with zero value.

The controller $K$ given by (22) depends on the allowable eigenvector space [22]. The allowable eigenvector space depends on the desired closed-loop eigenvalues, system, input, and output matrices of the plant. Hence, for a desired closed-loop eigenvalues and eigenvectors, $K$ is constant. However, a constant $K$ cannot be used to minimize the infinity norm shown in (21). In eigenstructure assignment technique, desired closed-loop eigenvalues require specific characteristics. For example, if these desired closed-loop eigenvalues need a damping ratio greater than or equal to $\zeta_{cl}$, then these eigenvalues lie in a region defined by the constant damping ratio line.
There are an infinite set of eigenvalues which have a damping ratio greater than or equal to $\zeta_{de}$. Hence, it is possible to construct a single controller for each set of these desired closed-loop eigenvalues. A search and minimization algorithm can minimize the infinity norm in (21) utilizing a set of controllers computed for different desired closed-loop eigenvalues located in the predefined region of the complex plane. This search and minimization algorithm requires a constraint to place $\hat{r}$ desired closed-loop eigenvalues with damping ratios greater than or equal to $\zeta_{de}$ in $C_-$. This constraint is given as

$$\lambda_l \in S_1 \forall l=\{1,2,\ldots,\hat{r}\}$$

(25)

In (25), $\lambda_l$ is the $l$th desired closed-loop eigenvalue that is placed by $K$. Additionally, $S_1$ is defined as

$$S_1 = \{ \lambda \in C_- \mid \zeta(\lambda) \geq \zeta_{de}, \Re(\lambda) < 0 \}$$

(26)

The controller given by (22) that achieves (20) needs to be a stabilizing controller of $H_{cp}(s)$. For that, the minimization algorithm must compute a $K$ that places remaining $n-\hat{r}$ ($n$ is the number of states of $H_{cp}(s)$) closed-loop eigenvalues in the predefined region of $C_-$. Nevertheless, there is no analytical solution to the output feedback problem [23]. Thus, to place $n-\hat{r}$ closed-loop eigenvalues in the predefined region of $C_-$, the search and minimization algorithm requires the constraint given below.

$$\lambda_i \in S_1 \forall i=\{\hat{r}+1,\hat{r}+2,\ldots,n\}$$

(27)

In (27), $\lambda_i$ is the $i$th closed-loop eigenvalue of the matrix, $A+BKC$. The $\lambda_i$ is not the desired closed-loop eigenvalue. In brief, if the search and minimizing algorithm obtains a $K$ given by (22) that achieves (20) with all $\lambda(A+BKC) \in S_1$, then $K$ provides desired decoupling to the closed-loop plant of $H_{cp}$ and simultaneous stabilization to all the plants in $\kappa$. When $K$ fails to provide the decoupling characteristics of the closed-loop plant of $H_{cp}(s)$ to all the closed-loop plants associated with the other plants belonging to $\kappa$, then a new solution of (13) is required. This solution needs to provide a new $H_{cp}(s)$ with a maximum $v-$gap metric that should be less than the previous $\hat{J}_1$. This is because the closed-loop characteristics of $H_{cp}(s)$ and the remaining plants of $\kappa$ will be similar when $\hat{J}_1$ of $H_{cp}(s)$ gets closer to zero.

### 3.4 Robust Simultaneously Stabilizing Decoupling Output Feedback Controller

In this section, we describe the existence conditions of RSSD output feedback controller and NN-RSSD output feedback algorithm.

#### 3.4.1 Existence Conditions

We state the following lemma for deriving the existence conditions of an RSSD output feedback controller.

**Lemma 3.1.** Let $W$ and $R$ are chosen to achieve desired decoupling. Then, the stabilizing output feedback controller of $H_{cp}(s)$ given by (22) accomplishes robust simultaneous stabilization of $N$ plants belonging to $\xi = \{H(s) \mid H(s) \in \kappa, \delta_v(H_{cp}(j\omega), H(j\omega)) \leq \hat{J}_1, H_{cp}(s) \in \kappa \}$ and provides exact desired decoupling to the closed-loop plant of $H_{cp}(s)$, if the following conditions are satisfied.

1. $\lambda_l \in S_1 \forall l=\{1,2,\ldots,\hat{r}\}$
2. $\lambda_i \in S_1 \forall i=\{\hat{r}+1,\hat{r}+2,\ldots,n\}$
3. $b_{H_{cp},K} > \hat{J}_1$

**Proof.** $K$ stabilizes $H_{cp}(s)$ when the conditions [1] and [2] are met. Concurrently, $K$ provides exact desired decoupling to $H_{cp}$ and simultaneously stabilizes $N$ plants belonging to $\xi$ when $K$ achieves all three conditions. Now to prove $K$ achieves robust simultaneous stabilization of $N$ plants, let us define the ball of plants, $B(H_{cp}(s), \hat{J}_1)$, as

$$B(H_{cp}(s), \hat{J}_1) = \{ H(s) \mid \delta_v(H_{cp}(j\omega), \hat{H}(j\omega)) \leq \hat{J}_1, H_{cp}(s) \in \xi, H(s) \in \mathcal{R}C_{\infty}^{[\hat{n} \times \hat{n}]} \}$$

(28)

When $K$ satisfies all the three conditions, any plant belonging to $\partial B$ (boundary of $B(H_{cp}(s), \hat{J}_1)$) is also stabilized by $K$ as $v-$gap metrics between the plants belonging to $\partial B$ and $H_{cp}(s)$ are $\hat{J}_1$. $v-$gap metrics between $H_{cp}(s)$ and the plants in $Int(B)$ (interior of $B(H_{cp}(s), \hat{J}_1)$) are less than $\hat{J}_1$, as $\hat{J}_1$ is the maximum $v-$gap metric of $H_{cp}(s)$. Therefore, $K$ simultaneously stabilizes all the plants belong to $B(H_{cp}(s), \hat{J}_1)$ if the conditions [1], [2], and [3] are met. As $\xi \subset B(H_{cp}(s), \hat{J}_1)$, $K$ simultaneously stabilizes all the $N$ plants. Even if any plant, $H(s) \in \xi$ is perturbed to form the plant, $\hat{H}(s) \in B(H_{cp}(s), \hat{J}_1)$ is also stabilized by $K$. Therefore, $K$ achieves robust simultaneous stabilization of $N$ plants belonging to $\xi$. This establishes the proof.
Lemma [3,1] indicates the existence conditions of RSSD output feedback controller are the conditions [1], [2], and [3]. Moreover, the structure of $\xi$ suggests any $K$ that satisfies the three conditions of Lemma [3,1] also achieves robust simultaneous stabilization of N plants of $P$.

### 3.4.2 NN-RSSD Output Feedback Algorithm

As there is no closed-form solution for the RSSD static output feedback problem, a new non-smooth optimization problem using the existence conditions of RSSD output feedback controller is formulated and is given as

$$
\begin{align*}
\text{minimize} & \quad J_2 = \left\| \begin{bmatrix} H_{cp} & I \end{bmatrix} (I - KH_{cp})^{-1} \begin{bmatrix} -I & K \end{bmatrix} \right\|_{\infty} \\
\text{subject to} & \quad \lambda_i \in S_1 \forall i = \{1, 2, \ldots, \hat{r}\} \\
& \quad K = W (CR)^{-1} \\
& \quad \lambda_i \in S_1 \forall i = \{\hat{r} + 1, \hat{r} + 2, \ldots, n\}
\end{align*}
$$

The solution of the robust simultaneous stabilization decoupling problem is a static output feedback controller, $K \in \mathbb{R}^{n \times \hat{r}}$ that reduces $J_2$ below $\bar{J}_1$ with all the constraints of (29) satisfied. The optimization problem in (29) requires the solutions of (18). Besides, the objective function of (29) is non-smooth. Hence, a genetic algorithm based iterative algorithm, NN-RSSD output feedback algorithm is developed for sequentially solving the SCP central plant problem given in (18) and the optimization problem of RSSD output feedback controller problem given in (29). At first, NN-RSSD output feedback algorithm solves (18). Thereafter, (29) is solved using the solution of (18). For this purpose, NN-RSSD algorithm has two population-based genetic algorithm (GA) solvers, namely, GA-SCP and GA-RSSD. GA-SCP solves the SCP central plant problem given in (18). GA-RSSD solves the optimization problem for the RSSD output feedback controller given in (29). In GA-SCP and GA-RSSD, GA employs the following steps.

1. Randomly generate initial values of search variables.
2. Compute the value of fitness function using the feasible values among the values of search variables.
3. Generate new values of search variables by examining the fitness value of search variables and applying genetic operators on the values of search variables.
4. Repeat 2 and 3 until the stopping criterion is satisfied.

During the fitness evaluation, if the obtained solution satisfies an adaptive constraint, then GA-SCP invokes GA-RSSD and passes $H_{cp}(s)$ and its maximum $v$-gap metric to it. Here, the adaptive constraint is fitness value of GA-SCP $< \bar{J}_1$. When GA-RSSD could not find a feasible controller for $H_{cp}(s)$ and its maximum $v$-gap metric, then $\bar{J}_1$ is assigned with the fitness value that previously satisfied the adaptive constraint.

Note that, new $\bar{J}_1 < \text{old } \bar{J}_1$ enables GA-SCP to search for newer $H_{cp}(s)$ with a maximum $v$-gap metric less than new $\bar{J}_1$. This enhances the possibility of obtaining a feasible RSSD output feedback controller with a new $H_{cp}(s)$ and its maximum $v$-gap metric. NN-RSSD algorithm terminates when the number of generation of GA-SCP exceeds its maximum value or when GA-RSSD generates a feasible RSSD output controller.

**Search Variables:** The search variables of GA-SCP are the coefficients of $w_{in1}$ and $w_{out1}$. The feasible values of these search variables are those satisfy all the constraints of the SCP central plant problem given in (18). The search variables for GA-RSSD are $\hat{r}$ desired closed-loop eigenvalues. Besides, in the NN-RSSD output feedback algorithm, the chosen desired eigenvector element for mode decoupling is not assigned a zero value but is confined in a bounded set and used as a search variable by GA-RSSD. This reduces the failure of the eigenstructure assignment algorithm in computing $K$ given by (22) when all the chosen desired eigenvector elements are assigned with zero. Further, the availability of the chosen desired eigenvector element as search variables provides additional freedom in the computation of the controller defined by (22). The bound on a chosen desired eigenvector element is determined by the magnitude of unfavorable dominance of a mode on the state variable. The feasible values of $\hat{r}$ eigenvalues and chosen desired eigenvectors are those belonging to $S_1$ and the user-defined bounded set, respectively.

**Fitness function:** The fitness function of GA-SCP is the performance index, $J_1$, of SCP central plant problem given in [18]. The fitness function computation of GA-SCP is given in Algorithm-1. The fitness function of GA-RSSD is the performance index, $J_2$, of the optimization problem given in (29). The fitness function computation of GA-RSSD is given in Algorithm-2.
The eigenvalues of the coupled spiral, coupled Dutch roll, and coupled short period modes of the NAV are assigned fitness function computation of GA-RSSD

Algorithm 1: Fitness function computation of GA-SCP

1: Input: \( P, \, \hat{J}_1 \), and feasible \( W_{in} \) and \( W_{ot} \) for \( i \)th generation
2: Compute: \( J_1 \) for fitness evaluation
3: if Fitness value < \( \hat{J}_1 \) then
4: Obtain: \( H_{cp} \) by performing \( v \)–gap metric analysis on \( \xi \)
5: Set \( \hat{J}_1 \) = Fitness value
6: Call: GA-RSSD
7: continue search
8: else
9: continue search
10: end if

Algorithm 2: Fitness function computation of GA-RSSD

1: Input: \( H_{cp}(s), \, \hat{J}_1, \, n, \, \hat{r}, \) and the feasible values of search variables of GA-II for \( \hat{h} \)th generation
2: if \( i \) ≤ maximum number of generation of GA-II then
3: Compute: \( W \) and \( R \) (using \( A \) and \( B \) of \( H_{cp}(s) \) and feasible values of search variables of GA-II)
4: Compute: \( K = W(CR)^{-1} \)
5: if \( \lambda_i(A + BK) \) \( \forall i \in \{\hat{r} + 1, \hat{r} + 2, \ldots, n\} \) \( \in S_1 \) then
6: Compute: \( J_2 \) for fitness evaluation
7: if Fitness value < \( \bar{J} \) then
8: Controller found \((K)\) and Exit
9: else
10: Continue search
11: end if
12: else
13: Continue search
14: end if
15: else
16: return
17: end if

4 Simulation Results

This section discusses the synthesis of an RSSD output feedback controller using NN-RSSD output feedback algorithm for eight unstable MIMO adversely coupled plants of the NAV mentioned in [2]. Among the output variables of these plants (pitch angle \((\theta)\), pitch rate \((q)\), yaw rate \((r)\), roll rate \((p)\), and roll angle \((\phi)\)), the controlled variables are \( \theta \) and \( \phi \). Besides, the frequency characteristics of the eight plants require modification as they do not have desired characteristics. \( P_2(s) \) among the eight plants is identified as the central plant as it has the smallest maximum \( v \)–gap metric of 0.4489 (\( \min \, \hat{\epsilon} \)).

The eigenvalues of the coupled spiral, coupled Dutch roll, and coupled short period modes of the NAV are assigned as these eigenvalues do not possess the desired stability and performance characteristics. For more details about these modes and their coupling characteristics, one should refer [2]. For these modes, the bounds on the desired eigenvector elements for mode decoupling are given in Table [1]. In Table [1] \( \hat{\theta}, \, \hat{w}, \, \hat{r}, \, \hat{p}, \) and \( \hat{\phi} \) denote the eigenvector elements associated with the state variables, \( u \) (translational velocity in x body-axis), \( w \) (translational velocity in w body-axis), \( r, \, p, \) and \( \phi \), respectively. We require the value of three chosen desired eigenvector elements associated with each mode for the computation of \( K \). In this direction, the remaining chosen desired eigenvector element of coupled Dutch roll is associated with \( r \). Likewise, remaining elements of the coupled short period mode are associated with \( u \) and \( \theta \) (pitch angle). Besides, these elements are also used as search variables. NN-RSSD output feedback algorithm is executed with inputs, \( \hat{r} = 5, \, m = 3, \, n = 19, \, \hat{J}_1 = \min \, \hat{\epsilon}, \, P \), the bounds on the coefficients of \( w_{in8} \) and \( w_{ot1} \), the bounds on the chosen desired eigenvector elements, the maximum number of generations of GA-SCP (20) and GA-RSSD (1000), and \( S_1 \) defined for the damping ratio, \( \zeta_{ot1} = 0.3 \). NN-RSSD output feedback algorithm terminates at

[4]To abide with journal page limit, the state space model of all the eight plants of the NAV, the flight control system design specifications, frequency characteristics of eight plants, \( v \)–gap metric analysis of eight plants, detailed descriptions of desired eigenvector elements and its bounds, and the bounds of the coefficients of pre and post compensators are given in the supporting material.
Table 1: Bounds on the chosen desired eigenvector elements for mode decoupling

| Mode                  | Eigenvector Element | Bound               |
|-----------------------|---------------------|---------------------|
|                       | $\bar{u}$           | -0.5279 ± 0.01      |
|                       | $\bar{w}$           | 0.3438 ± 0.1        |
|                       | $\bar{r}$           | 0.0126 ± 0.1        |
| Coupled Spiral        | $\bar{p}$           | -0.9821 ± 0.0000j  |
|                       | roll $\phi$         | 0.0688 ± 0.7723j   |
|                       | period $\bar{r}$    | 0.7723 ± 0.0000j   |

The 388th generation of GA-RSSD. Subsequently, the feasible compensators are given as

$$W_{in} = diag \begin{bmatrix} 1.0s + 7.36 & 14.71s + 59.78 & 0.992s + 5.091 \\ 0.007s + 10.1 & 0.002s + 0.999 & 0.0053s + 11.89 \end{bmatrix}$$

(30)

$$W_{ot} = diag \begin{bmatrix} 0.61s + 11 & 0.23s + 29.58 & 0.813s + 12.1 \\ 1.6s + 1.6 & 0.55s + 0.36 & 7.996s + 1.46 \\ 0.91s + 9.78 & 0.78s + 16.81 & 0.331s + 0.342 \end{bmatrix}$$

(31)

Also, the feasible RSSD static output feedback controller gain is given as

$$K = \begin{bmatrix} 1.13 & 0.78 & 0.26 & 0.14 & 0.13 \\ -1.07 & -0.78 & 0.74 & 0.009 & 0.54 \\ 0.19 & -0.06 & -0.002 & 1.29 & 0.04 \end{bmatrix}$$

(32)

Besides, the SCP central plant is $H_{cp}(s) = W_{ot}(s)P_3(s)W_{in}(s)$. The maximum $\nu$-gap metric of $H_{cp}(s)$ is 0.3184. As this value is less than 0.4489, one of the objectives of the pre and post compensators is achieved.

The frequency characteristics of $H_f(s) = W_{ot}(s)P_f(s)W_{in}(s)$ for all $f = \{1, 2, \ldots, 8\}$ are shown in Fig. 1. This figure indicates $\sigma(H_f(0)) > 6$ dB for all $f = \{1, 2, \ldots, 8\}$. As illustrated in Fig. 1, the 0 dB crossing frequencies of all the performance enhanced plants are within the desired limit as the range of 0 dB crossing frequencies of $\sigma(H_f(s))$ for all $f = \{1, 2, \ldots, 8\}$ are within 26-41.33 rad/s. Hence, $W_{in}$ and $W_{ot}$ given in (30) and (31) satisfy all the design requirements with respect to the performance enhanced plants.

![Figure 1: Singular value plots of all performance enhanced plants](image-url)
4.1 Stability analysis

The controller gain given in (32) simultaneously stabilizes eight plants of the NAV as all the eigenvalues of the closed-loop plants, \( \hat{H}_f = \frac{H_f K}{1 - H_f(j\omega)K} \) for all \( f = \{1, 2, \ldots, 8\} \) lie in \( \mathcal{C}_- \) as shown in Fig. 2. Additionally, this figure indicates the damping ratios of all the closed-loop eigenvalues are greater than or equal to 0.3.

![Figure 2: Eigenvalues of the eight closed-loop plants of the NAV](image)

The parametric and unmodeled dynamic uncertainties are represented by inverse input multiplicative and output multiplicative uncertainties, respectively. The tolerance level of the closed-loop plants against the output multiplicative uncertainty is analyzed using the plot of \( \frac{1}{\sigma(H_f(j\omega)K(1 - H_f(j\omega)K)^{-1})} \) for all \( f = \{1, 2, \ldots, 8\} \), depicted in Fig. 3.

![Figure 3: Uncertainty bounds of the eight closed-loop plants against output multiplicative uncertainty](image)

This figure depicts that the output multiplicative uncertainty tolerance levels increase with the frequency. This characteristic is required to tackle the output multiplicative uncertainty that occurs at high-frequency. Further, \( \hat{H}_3(s) \) has the worst-case tolerance level of about 51.8% at 2.22 rad/s. This tolerance level is in the low-frequency region (with respect to the bandwidth of the augmented plants) where the unmodeled dynamic uncertainty does not occur.

The tolerance level of the closed-loop plant against the inverse input multiplicative uncertainty is analyzed by studying the plots of \( \frac{1}{\sigma(1-KH_f(s))^{-1}} \) for all \( f = \{1, 2, \ldots, 8\} \) shown in Fig. 4.

![Figure 4: Tolerance levels of all the closed-loop plants against inverse input multiplicative uncertainty](image)

This figure shows \( \hat{H}_4(s) \) has the worst-case inverse input multiplicative uncertainty tolerance level of 62% at 2987 rad/s. This worst-case tolerance level occurs in the high-frequency region where parametric uncertainty does not exist. The tolerance levels of all the closed-loop plants are greater than 79% when the frequency is below 41.33 rad/s (maximum bandwidth of augmented plant). This satisfies the tolerance level requirement against the parametric uncertainties.
Hence, it is categorically established that the controller given in (32) robust simultaneously stabilizes all the eight plants of the NAV.

4.2 Performance and decoupling analysis

The performance of gain, $K$, is evaluated by analyzing the singular value plots of the output sensitivity function, $S_{of}(s) = (I - H_f(s)K)^{-1}$ shown in Fig. 5. It is noticeable from Fig. 5 that the worst 0 dB crossing frequency (from below) of $\overline{\sigma}(S_{of}(j\omega))$ corresponds to the controllable output channel is 197 rad/s. Hence, the controller rejects any disturbance with a frequency less than 197 rad/s acting on any controllable output channels of the NAV. The worst magnitudes of $\overline{\sigma}(S_{of}(0))$ associated with $\bar{r}$, $\bar{\phi}$, and $\bar{\theta}$ output channels are -7.8 dB, -44.6 dB, and -59.8 dB, respectively. Therefore, the worst maximum steady-state error for a step input at the controllable output channels are 40.7 %, 0.6 %, and 0.01 %, respectively.

The detailed performance analysis including the input sensitivity, $S_{if}(s)$ and $K$ are given in the supporting material.

The value of chosen desired eigenvector elements associated with the coupled spiral mode of the closed-loop plant of $H_{cp}$ are $\bar{u}=-0.0065$, $\bar{w}=0.0905$, and $\bar{r}=-0.0002$. For the coupled Dutch roll mode, the value of these elements are $\bar{p}=-0.0159 \pm 0.0423j$ and $\bar{\phi}=-0.0010 \pm 0.0039j$. Likewise, the chosen desired eigenvector element of coupled short period mode is $\bar{r}=-0.0003 \pm 0.0001j$. The values of these elements suggest that they are within the bounds given in Table 1. Hence, the RSSD static output feedback controller along with the pre and post compensators accomplish the desired performance and decoupling objectives.
4.3 Six-Degree-of-Freedom Simulations

In order to assess the performance of the controller, two cases of six-degree-of-freedom simulations are accomplished. In these simulations, the closed-loop nonlinear and linear plants of the NAV are forced to track a doublet $\theta$ and $\phi$ reference signals separately. The magnitude and pulse width of the doublet $\theta$ reference signal are $\pm 0.0873$ rad ($\pm 5$ deg) and 2 s, respectively. Similarly, the doublet $\phi$ reference signal has a magnitude of $\pm 0.0349$ rad ($\pm 2$ deg) and a pulse width of 2 s. The output time responses of the linear closed-loop plant of $H_{cp}$ are shown in Figs. 6(a)-7(e). These responses are offsetted with the trim values. Also, the output time responses of the corresponding nonlinear closed-loop plant are shown in Figs. 6(a)-7(e). The output time responses of all the closed-loop plants are given in the supporting material. After analyzing all these plots, the following conclusions are made.

1. The closed-loop nonlinear plants are stable as their time responses of output variables are not diverging in finite time. The time responses of the closed-loop nonlinear and linear (with offset) plants are almost identical which establishes that the linear plant accurately captures the behavior of the nonlinear plant around the trim point. Furthermore, the closed-loop plants are tracking their reference signals as the tracking variables are within the specified error band.

2. While tracking the doublet $\theta$ reference signal, the change in yaw rate responses is minimal when compared to the roll rate and pitch rate responses. This indicates that the coupled short period mode of all the closed-loop plants are decoupled from yaw rate response which is one of the design requirement.

3. Compared to the change in yaw rate responses, a substantial change in roll rate responses are visible when all the closed-loop plants track the doublet $\theta$ command. This large change is mainly due to the variation of counter torque due to the change in $\delta_T$ and thereby thrust. The coupling induced by counter torque affects the coupled roll modes of all the closed-loop plants and thereby the roll rates. The controller is not designed to decouple the coupled roll modes from roll rate responses which result in the excitation of the roll rates when the NAV is forced to track a doublet $\theta$ command. However, the converging (converging to the trim value) of roll rate responses establishes that the coupled roll modes of all the closed-loop plants are stable. The roll angle responses indicate the variations in the roll angle responses are minimal. This is because the effect of counter torque on the coupled spiral mode is minimal as $u$ and $w$ are decoupled from the coupled spiral mode. Furthermore, the time responses indicate that all the closed-loop plants have similar desired decoupling characteristics and thus the controller accomplishes one of the design objectives.

The simulations that assess the wind disturbance attenuation capability of the controller is given in supporting material.
Figure 6: Output responses of closed-loop nonlinear and linear plants of $H_{cp}$ while tracking doublet $\theta$ reference signal.

(a) Pitch angle response

(b) Roll angle response

(c) Roll rate response

(d) Pitch rate response

(e) Yaw rate response
5 Conclusions

In this paper, a new tractable method is proposed to synthesize an RSSD static output feedback controller that achieves simultaneous stabilization, desired decoupling, robustness, and performance for a finite set of unstable MIMO adversely coupled plants with resource constraints. For that, a method to identify a central plant from a finite set of plants is developed using the robust stabilization theory of coprime factorized plant and the properties of $v$-gap metric. This method is tractable and also suitable for stable/unstable plants with a varying/same number of unstable poles. The existence conditions of the RSSD static output feedback controller are developed by exploiting the properties of $v$-gap metric of SCP central plant, the sufficiency condition of SCP central plant for simultaneous stabilization, and the eigenstructure assignment algorithm for output feedback. The conditions thus developed are easily testable for a given set of plants. Further, these existence conditions are utilized to develop a new non-smooth optimization problem for the RSSD static output feedback controller. A new tractable genetic algorithm based iterative algorithm, NN-RSSD algorithm is developed for sequentially solving the SCP central plant and the RSSD output feedback optimization problems. The effectiveness of NN-RSSD output feedback algorithm is demonstrated by generating the RSSD static output feedback controller for the eight unstable plants of NAV. The pre and post compensator thus generated resulted in the SCP central plant whose maximum $v$-gap metric is less than the maximum $v$-gap metric of the central plant. Also, the frequency characteristics of the performance enhanced plants satisfy design requirements. The stability, performance, and decoupling analyzes of all the closed-loop plants demonstrate that they have desired design specifications. The six-degree-of-freedom simulations indicate a single controller provides simultaneous stabilization, desired decoupling, robustness, and performance to all the closed-loop non-linear and linear plants. The proposed...
method to generate RSSD static output feedback controller is generic and can be applied to any linear stable/unstable and minimum/non-minimum phase plants.

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