Some Remarks on Quantum Tomography in Laser Cooling

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Abstract

In this article we take into consideration the evolution model of 3-level quantum systems known as laser cooling. The evolution in this model is given by an equation which is a special case of the general Kossakowski-Lindblad master equation. The explicit knowledge about the evolution makes it possible to apply the stroboscopic tomography to laser cooling. In this article we present some remarks concerning the minimal number of observables and moments of measurement for quantum tomography in laser cooling.

1 Introduction

By the term quantum tomography one understands a wide variety of methods which aim to reconstruct an accurate representation of a quantum system by taking a series of measurements. One of the most fundamental models of quantum tomography, the so-called static tomography model, enables to reconstruct the density matrix of a quantum system provided one can measure $N^2 - 1$ distinct observables (where $N = \dim \mathcal{H}$). This approach can be found in many papers and books, such as [1, 2, 3]. A completely new approach to quantum tomography originated in 2011 in the paper [4], where it was shown that a wave function can be obtained directly if one employs the idea of weak measurement. Later, it was proved that this approach can be generalized so that it can also be applied to mixed states identification [5].

The most important property that all tomography models should possess is practicability, which means that a theoretical model can be in the future implemented in an experiment. Therefore, in this article we employ the stroboscopic approach to quantum tomography, which for the first time appeared in [6] and then it was developed in other research papers such as [7] and [8]. One can also look up a very well-written review paper [9]. Recently some new ideas concerning the stroboscopic approach has been presented in [10].

The stroboscopic tomography concentrates on determining the optimal criteria for quantum tomography of open quantum systems. The data for reconstruction is provided by mean values of some hermitian operators $\{Q_1, \ldots, Q_p\}$, where obviously $Q_i = Q_i^*$. The underlying assumption behind this approach claims that if one has the knowledge about the evolution of the system, each observable can be measured a certain number of times at different instants. Although there are many possible aspects concerning this problem, in this article we are mainly interested in the minimal number of distinct observables and moments of measurement required for quantum tomography. One can recall the theorem concerning the minimal number of observables [7].
Theorem 1.1. For a quantum system with dynamics given by a master equation of the form
\[ \dot{\rho} = L\rho \]
(1)
one can calculate the minimal number of distinct observables for quantum tomography from the formula
\[ \eta := \max_{\lambda \in \sigma(L)} \{ \dim \ker(L - \lambda I) \}, \]
(2)
which means that for every generator $L$ there exists a set of observables $\{Q_1, \ldots, Q_\eta\}$ such that their expectation values determine the initial density matrix. Consequetly, they also determine the complete trajectory of the state.

The operator $L$ that appears in the equation (1) shall be called the generator of evolution. The number $\eta$ is usually refered to as the index of cyclicity of a quantum system.

One can also recall the theorem on the upper limit of moments of measurement [8].

Theorem 1.2. In order to reconstruct the density matrix of an open quantum system the number of times that each observable from the set $\{Q_1, \ldots, Q_\eta\}$ should be measured (denoted by $M_i$ for $i = 1, \ldots, \eta$) fulfills the inequality
\[ M_i \leq \mu(\lambda, L), \]
(3)
where by $\mu(\lambda, L)$ one denotes the degree of the minimal polynomial of $L$.

The above theorem gives the upper boundary concerning the number of measurements of each distinct observable. Naturally, another problem relates to the choice of the time instants. Some considerations about this issue can be found in [8].

The knowledge about the stroboscopic tomography shall be applied in the next chapter to the evolution model known as laser cooling.

2 Quantum tomography in laser cooling - initial results

An example often studied in the area of laser spectroscopy is a quantum system subject to laser cooling with three energy levels ($\dim \mathcal{H} = 3$). The evolution of the density matrix of such a three level system is given by
\[ \frac{d\rho}{dt} = -i[H(t), \rho] + \gamma_1 \left( E_1 \rho E_1^* - \frac{1}{2} \{E_1^T E_1, \rho\} \right) + \gamma_2 \left( E_2 \rho E_2^* - \frac{1}{2} \{E_2^T E_2, \rho\} \right), \]
(4)
where $E_1 = |1\rangle \langle 2|$ and $E_2 = |3\rangle \langle 2|$.

For simplicity it will be assumed that
\[ |1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad |2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad |3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \]
(5)

Moreover in this analysis we take $H(t) = [0]$, where $[0]$ denotes a 3—dimensional matrix with all entries equal 0. This assumption means that we shall analyze only the Lindbladian part of the equation of evolution.

Based on vectorization theory [11] the quantum Liouville operator of the system with dynamics given by (4) can be explicitly expressed as
\[ L = \gamma_1 \left( E_1 \otimes E_1 - \frac{1}{2} (I \otimes E_1^T E_1 + E_1^T E_1 \otimes I) \right) + \gamma_2 \left( E_2 \otimes E_2 - \frac{1}{2} (I \otimes E_2^T E_2 + E_2^T E_2 \otimes I) \right). \]
(6)
Taking into account the assumptions (5) the matrix form of the quantum generator $L$ can be obtained

$$
L = \begin{bmatrix}
0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2}(\gamma_1 + \gamma_2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2}(\gamma_1 + \gamma_2) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\gamma_1 - \gamma_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2}(\gamma_1 + \gamma_2) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\gamma_1 - \gamma_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}(\gamma_1 + \gamma_2) & 0 \\
\end{bmatrix}.
$$

(7)

Having the explicit form of quantum generator $L$ one can easily calculate its eigenvalues

$$
\sigma(L) = \{0, 0, 0, 0, -(\gamma_1 + \gamma_2), -\frac{1}{2}(\gamma_1 + \gamma_2), -\frac{1}{2}(\gamma_1 + \gamma_2), -\frac{1}{2}(\gamma_1 + \gamma_2)\}.
$$

(8)

Since in this case the operator $L$ is not self-adjoint, the algebraic multiplicity of an eigenvalue does not have to be equal to its geometric multiplicity. But one can quickly determine that there are four linearly independent eigenvectors that correspond to the eigenvalue $0$. Therefore we can find the index of cyclicity for the operator in question

$$
\max_{\lambda \in \sigma(L)} \{\dim \text{Ker}(L - \lambda I)\} = 4,
$$

(9)

which means that we need at least four distinct observables to perform quantum tomography on the analyzed system. One can instantly notice that if the static approach was applied to laser cooling, one would have to measure 8 distinct observables. If one thinks of potential applications in experiments, then this result means that one would have to prepare 4 different experimental set-ups instead of 8. This observation suggests that the stroboscopic approach has an obvious advantage over the static approach.

The next issue that we are interested in is the minimal polynomial for operator $L$. Assuming that this polynomial has the form

$$
d_3L^3 + d_2L^2 + d_1L + d_0 = 0,
$$

(10)

one can get

$$
d_3 = 1, \quad d_2 = \frac{3}{2}(\gamma_1 + \gamma_2), \quad d_1 = \frac{1}{2}(\gamma_1 + \gamma_2)^2, \quad d_0 = 0.
$$

(11)

Thus we see that $\mu = \deg \mu(\lambda, L) = 3$. This means that each observable should be measured at most at three different time instants. Thus one can conclude that the total number of measurements for quantum tomography in laser cooling cannot exceed 12.

Having found these results we can conclude that in order to reconstruct the density matrix of the system in question we need 4 observables $Q_1, Q_2, Q_3, Q_4$ that fulfill the condition [9]

$$
\bigoplus_{i=0}^{4} K_3(L, Q_i) = B_+(\mathcal{H}),
$$

(12)

where by $K_3(L, Q_i)$ one should understand a Krylov subspace which is expressed as

$$
K_3(L, Q_i) = \{Q_i, L^*Q_i, (L^*)^2Q_i\}.
$$

(13)

The equation (12) gives the necessary condition that the set of observables $\{Q_1, \ldots, Q_4\}$ needs to fulfill in order for one to be able to perform quantum tomography in laser cooling. This condition can be used to determine the explicit forms of these observables.
3 Summary

In this paper we presented some remarks concerning quantum tomography in laser cooling. The stroboscopic approach was applied to determine to optimal criteria for quantum observability. The results shall be developed in next paper towards the complete quantum tomography model for laser cooling.

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