A NOTE ON COMPACT CR YAMABE SOLITON

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Abstract. In this paper, we show that the Webster scalar curvature of any compact CR Yamabe soliton must be constant.

1. Introduction

Given a compact Riemannian manifold \((M, g)\), the Yamabe problem is to find a metric conformal to \(g\) such that it has constant scalar curvature. This was solved by Aubin, Schoen, and Trudinger in [1, 23, 25]. The (unnormalized) Yamabe flow was introduced to study the Yamabe problem, which is defined as follows:

\[
\frac{\partial}{\partial t} g(t) = -R_{g(t)} g(t) \quad \text{for} \quad t \geq 0, \quad g(0) = g.
\]

Here \(R_{g(t)}\) is the scalar curvature of \(g(t)\). The existence and convergence of the Yamabe flow have been studied in [2, 3, 8, 24, 26]. Yamabe soliton is a self-similar solution to the Yamabe flow. More precisely, \(g(t)\) is called a Yamabe soliton if there exist a smooth function \(\sigma(t)\) and a 1-parameter family of diffeomorphisms \(\{\psi_t\}\) of \(M\) such that

\[
g(t) = \sigma(t) \psi_t^* g
\]

is the solution of the Yamabe flow \([11]\), with \(\sigma(0) = 1\) and \(\psi_0 = id_M\). The following is an alternative definition: \((M, g)\) is called Yamabe soliton if there exist a vector field \(X\) and a constant \(\rho \in \mathbb{R}\) such that

\[
(R_g - \rho) g = \mathcal{L}_X g.
\]

Here \(R_g\) is the scalar curvature of the metric \(g\), and \(\mathcal{L}_X\) is the Lie derivative in the direction of \(X\). Note that these two definitions are equivalent (see \([11]\) for the proof). Yamabe soliton has been studied by many authors. See \([1, 5, 9, 10, 11, 16, 17, 21, 22]\) and the references therein. In particular, we mention the following theorem related to the main result in this paper, which was obtained independently by di Cerbo and DiConzzi in \([11]\) and by Hsu in \([16]\):

**Theorem 1.1.** Any compact Yamabe soliton must have constant scalar curvature.

Suppose \((M, \theta)\) is a strictly pseudoconvex CR manifold of real dimension \(2n + 1\). The CR Yamabe problem is to find a contact form conformal to \(\theta\) such that it has constant Webster scalar curvature. This was solved by Jerison-Lee and Gamara-Yacoub in \([12, 13, 18, 19, 20]\). The (unnormalized) CR Yamabe flow is defined as...
the evolution equation of the contact form \( \theta(t) \):

\[
\frac{\partial}{\partial t} \theta(t) = -R_{\theta(t)} \theta(t) \quad \text{for } t \geq 0, \quad \theta(0) = \theta.
\]

Here \( R_{\theta(t)} \) is the Webster scalar curvature of the contact form \( \theta(t) \). The CR Yamabe flow was introduced to tackle the CR Yamabe problem. See [6, 7, 15] and the references therein. As in the Riemannian case, CR Yamabe soliton is a self-similar solution to the CR Yamabe flow: we call \( \theta(t) \) a CR Yamabe soliton if there exist a smooth function \( \sigma(t) \) and a 1-parameter family of CR diffeomorphisms \( \{ \psi_t \} \) of \( M \) such that

\[
\theta(t) = \sigma(t) \psi_t^* \theta
\]

is the solution of the CR Yamabe flow \( (1.3) \), with \( \sigma(0) = 1 \) and \( \psi_0 = id_M \).

The following is our main result, which is the CR version of the result of di Cerbo and Disconzi in [11] and Hsu in [16] that we mentioned above.

**Theorem 1.2.** If \((M, \theta(t))\) is a compact strictly pseudoconvex CR manifold satisfying \((1.4)\), then the Webster scalar curvature of \((M, \theta(t))\) is constant.

2. **Proof**

In this section, we are going to prove Theorem 1.2. We will consider the evolution of the quantity

\[
\frac{\int_M R_{\theta(t)} dV_{\theta(t)}}{\left( \int_M dV_{\theta(t)} \right)^{n+1}}
\]

along the CR Yamabe flow \((1.3)\). Note that if \( \theta(t) = u(t)^{\frac{n}{n+2}} \theta \) is the solution of the CR Yamabe flow \((1.3)\), then \( u(t) \) satisfies the following evolution equation:

\[
\frac{\partial}{\partial t} u(t) = -\frac{n}{2} R_{\theta(t)} u(t) \quad \text{for } t \geq 0.
\]

Therefore, by \((2.2)\), the volume form \( dV_{\theta(t)} \) of \( \theta(t) \) satisfies

\[
\frac{\partial}{\partial t} (dV_{\theta(t)}) = dV_{\theta(t)} = \frac{2n + 2}{n} u(t)^{\frac{n+2}{n}} \frac{\partial u(t)}{\partial t} = -(n+1) R_{\theta(t)} dV_{\theta(t)},
\]

which implies that

\[
\frac{d}{dt} \left( \int_M dV_{\theta(t)} \right) = -(n+1) \int_M R_{\theta(t)} dV_{\theta(t)}.
\]

Since \( \theta(t) = u(t)^{\frac{n}{n+2}} \theta \), \( u(t) \) satisfies the CR Yamabe equation:

\[-(2 + \frac{2}{n}) \Delta_{\theta} u(t) + R_{\theta} u(t) = R_{\theta(t)} u(t)^{1+\frac{n}{n+2}}\]

where \( \Delta_{\theta} \) is the sub-Laplacian of the contact form \( \theta \). Differentiate it with respect to \( t \), one can derive that the following evolution equation of the Webster scalar curvature \( R_{\theta(t)} \) of \( \theta(t) \): (see [14] or [15] for the case of normalized CR Yamabe flow)

\[
\frac{\partial}{\partial t} R_{\theta(t)} = (n+1) \Delta_{\theta(t)} R_{\theta(t)} + R_{\theta(t)}^2 R_{\theta(t)}.
\]
Here $\Delta_{\theta(t)}$ is the sub-Laplacian of the contact form $\theta(t)$. Therefore, we have

$$\frac{d}{dt} \left( \int_M R_{\theta(t)}dV_{\theta(t)} \right)$$

$$= \int_M \left( \frac{\partial}{\partial t} R_{\theta(t)} \right) dV_{\theta(t)} + \int_M R_{\theta(t)} \frac{\partial}{\partial t} (dV_{\theta(t)})$$

$$= \int_M \left( (n+1)\Delta_{\theta(t)} R_{\theta(t)} + R_{\theta(t)}^2 \right) dV_{\theta(t)} - (n+1) \int_M R_{\theta(t)}^2 dV_{\theta(t)}$$

$$= -n \int_M R_{\theta(t)}^2 dV_{\theta(t)}$$

(2.6)

where we have used (2.3) and (2.5). Combining (2.4) and (2.6), we obtain

$$\frac{d}{dt} \left( \int_M R_{\theta(t)}dV_{\theta(t)} \right) = -n \left( \int_M R_{\theta(t)}^2 dV_{\theta(t)} \right) \left( \int_M dV_{\theta(t)} \right)^{n+1} \leq 0$$

(2.7)

where the last inequality follows from Cauchy-Schwarz inequality. This shows that the quantity in (2.1) is decreasing along the unnormalized CR Yamabe flow (1.3).

On the other hand, the quantity in (2.1) is invariant under the CR Yamabe soliton (1.4). To see this, note that if $\theta(t) = \sigma(t)\psi^*_\tau(\theta)$ for some smooth function $\sigma(t)$ and a 1-parameter family of CR diffeomorphisms $\{\psi_\tau\}$ of $M$, then $R_{\sigma(t)\psi^*_\tau(\theta)} = \sigma(t)^{-1} R_{\psi^*_\tau(\theta)}$ and $dV_{\sigma(t)\psi^*_\tau(\theta)} = \sigma(t)^{n+1} dV_{\psi^*_\tau(\theta)}$, which implies that

$$\frac{\int_M R_{\theta(t)}dV_{\theta(t)}}{\left( \int_M dV_{\theta(t)} \right)^{n+1}} = \frac{\int_M R_{\psi^*_\tau(\theta)}dV_{\psi^*_\tau(\theta)}}{\left( \int_M dV_{\psi^*_\tau(\theta)} \right)^{n+1}} = \frac{\int_M R_\theta dV_\theta}{\left( \int_M dV_\theta \right)^{n+1}}$$

Therefore, we have

$$\frac{d}{dt} \left( \int_M R_{\theta(t)}dV_{\theta(t)} \right) = 0$$

under the CR Yamabe soliton (1.4). This implies that the inequality in (2.7) is equality. In particular, $R_{\theta(t)}$ must be constant by the equality case of the Cauchy-Schwarz inequality in (2.7). This completes the proof of Theorem 1.2.

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