CROSS-SECTIONS, QUOTIENTS, AND REPRESENTATION RINGS OF SEMISIMPLE ALGEBRAIC GROUPS

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"Is Steinberg’s theorem [...] only true for simply connected groups [...]? What happens for $\text{GP}(1)$, for instance? Is there a rational section of $G$ over $I(G)$ ("invariants") in this case? [...] Is it true that $I(G)$ is a rational variety [...]?"
A. Grothendieck, Letter to J.-P. Serre, January 15, 1969, [GS, pp. 240–241].

Abstract. Let $G$ be a connected semisimple algebraic group over an algebraically closed field $k$. In 1965 Steinberg proved that if $G$ is simply connected, then in $G$ there exists a closed irreducible cross-section of the set of closures of regular conjugacy classes. We prove that in arbitrary $G$ such a cross-section exists if and only if the universal covering isogeny $\hat{\tau}: \hat{G} \to G$ is bijective; this answers Grothendieck’s question cited in the epigraph. In particular, for char $k = 0$, the converse to Steinberg’s theorem holds. The existence of a cross-section in $G$ implies, at least for char $k = 0$, that the algebra $k[G]^G$ of class functions on $G$ is generated by $\text{rk} G$ elements. We describe, for arbitrary $G$, a minimal generating set of $k[G]^G$ and that of the representation ring of $G$ and answer two Grothendieck’s questions on constructing generating sets of $k[G]^G$. We prove the existence of a rational (i.e., local) section of the quotient morphism for arbitrary $G$ and the existence of a rational cross-section in $G$ (for char $k = 0$, this has been proved earlier); this answers the other Grothendieck’s question cited in the epigraph. We also prove that the existence of a rational section is equivalent to the existence of a rational $W$-equivariant map $T \dashrightarrow G/T$ where $T$ is a maximal torus of $G$ and $W$ the Weyl group.

1. Introduction

Below all algebraic varieties are taken over an algebraically closed field $k$. We use the standard notation and conventions of [Bor] and [Sp].

Let $G$ be a connected semisimple algebraic group, $G \neq \{e\}$. Let $(G//G, \pi_G)$ be a categorical quotient for the conjugating action of $G$ on itself, i.e., $G//G$ is an affine variety and

$$\pi_G: G \longrightarrow G//G$$

a surjective morphism such that $\pi_G^*(k[G//G])$ is the algebra $k[G]^G$ of class functions on $G$. Every fiber of $\pi_G$ is then the closure of a regular conjugacy class (i.e., that of the maximal dimension) and such classes in general position are closed [Ste1, Theorem 6.11, Cor. 6.13, and Sect. 2.14].

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Definition 1.1. A closed irreducible subvariety $S$ of $G$ is called a cross-section (of the collection of fibers of $\pi_G$) in $G$ if $S$ intersects at a single point every fiber of $\pi_G$.

The elements of $S$ are the “canonical forms” of the elements of a dense constructible subset of $G$ with respect to conjugation. The image of any section of $\pi_G$ (i.e., a morphism $\sigma : G//G \to G$ such that $\pi_G \circ \sigma = \text{id}_{G//G}$) is an example of such $S$; moreover, this $S$ has the property that $\pi_G|_S : S \to G//G$ is an isomorphism. For char $k = 0$, every cross-section in $G$ is obtained in this manner (see Remark 1 in Section 6).

In 1965 Steinberg gave an explicit construction of a section of $\pi_G$ for every simply connected semisimple group $G$ (see his celebrated paper [Ste1]). Its image is a cross-section that intersects every regular conjugacy class and does not intersect other conjugacy classes.

In this paper we explore what happens in the general case, i.e., when $G$ is not necessarily simply connected. In this case the following two facts about cross-sections in $G$ for char $k = 0$ are known.

First, by [CTKPR, Theorem 0.3] in every connected semisimple algebraic group $G$ there is a rational section of $\pi_G$, i.e., a section over a dense open subset of $G//G$ (local section).

Second, by Kostant’s theorem [K, Theorem 0.10] there is an infinitesimal counterpart of Steinberg’s cross-section: for the adjoint action of $G$ on its Lie algebra Lie $G$, there is a closed irreducible subvariety in Lie $G$ that intersects every regular $G$-orbit at a single point.

In order to formulate our result consider the universal covering of $G$, i.e., an isogeny $\tau : \hat{G} \to G$

such that $\hat{G}$ is a simply connected semisimple algebraic group and the composition of $\tau$ with every projective rational representation of $G$ lifts to a linear one of $\hat{G}$.

We prove the following

Theorem 1.2. Let $G$ be a connected semisimple algebraic group.

(i) The following properties are equivalent:
   (a) there is a cross-section in $G$;
   (b) the isogeny $\tau$ is bijective.

(ii) If $\sigma : G//G \to G$ is a section of $\pi_G$, then the cross-section $\sigma(G//G)$ in $G$ intersects every regular conjugacy class and does not intersect other conjugacy classes.

Remark 1.3. The isogeny $\tau$ is bijective if and only if it is either an isomorphism or purely inseparable (radical). The latter holds if and only if char $k = p > 0$ and $p$ divides the order of the fundamental group of $G$.

The next corollary answers the first Grothendieck’s question cited in the epigraph and a question posed in [CTKPR, p. 4].

Corollary 1.4. Let $G$ be a connected semisimple algebraic group.

(i) If a section of $\pi_G$ exists, then $\tau$ is bijective.

(ii) For char $k = 0$, the following properties are equivalent:
   (a) there is a section of $\pi_G$;
   (b) there is a cross-section in $G$;
   (c) $G$ is simply connected.

Theorem 1.2 is proved in Section 2. One can show (see below Lemma 3.1) that if a cross-section in $G$ exists, then, at least for char $k = 0$, the variety $G//G$ is smooth (the converse is not true). The known
criterion of smoothness of $G//G$ (Theorem 3.2) may be interpreted as that of the existence of $\text{rk} G$ generators of $k[G]^G$. In Section 3 we consider the general case and describe a minimal generating set of $k[G]^G$ and singularities of $G//G$ for any $G$. This is based on the property that actually $G//G$ is a toric variety of a maximal torus $T$ of $G$. In particular, it also implies the affirmative answer to the last Grothendieck’s question cited in the epigraph:

**Corollary 3.7.** $G//G$ and $T/W$ are the rational varieties.

Here $W$ is the Weyl group of $G$, i.e., the quotient of $T$ in its normalizer $N_G(T)$, acting on $T$ via conjugation.

Parallel to this we describe a minimal generating set of the representation ring $R(G)$ of $G$. Note that finding generators of $R(G)$ attracted people’s attention during long time, in particular, because of the bearing on the $K$-theory (cf., e.g., [Hus, Chap. 13] where the generators of $R(G)$ are found for some classical $G$’s utilizing the ad hoc bulky arguments; see also [A]). Singularities of $G//G$ attracted people’s attention as well (see [Sl, Sects. 3.15, 4.5]).

The precise formulations of these results are given below in Theorems 3.5 and 3.9 and Lemma 3.10.

Constructing generating sets of $k[G]^G$ is the topic of two further questions of Grothendieck asked in [GS, p. 241]. In Section 4 we answer the first question in the negative and the second in the positive.

In Section 5 we consider rational (i.e., local) sections of $\pi_G$ and rational cross-sections in $G$, i.e., irreducible closed subsets $S$ of $G$ that intersect at a single point every fiber of $\pi_G$ over a point of a dense open subset of $G//G$ (depending on $S$). The closure of the image of a rational section of $\pi_G$ is an example of such $S$; moreover, this $S$ has the property that $\pi_G|_S: S \to G//G$ is a birational isomorphism. For char $k = 0$, every rational cross-section in $G$ is obtained in this way.

First, we show that the existence of a rational section of $\pi_G$ is equivalent to another property. Namely, note that $W$ also acts on $G/T$ as follows:

$$w \cdot gT := gw^{-1}T,$$

where $w \in N_G(T)$ is a representative of $w$. We prove

**Theorem 1.5.** Let $G$ be a connected semisimple algebraic group. The following properties are equivalent:

(i) there is a rational section of $\pi_G$;

(ii) there is a $W$-equivariant rational map $T \dashrightarrow G/T$.

Then we consider the existence problem and prove the following.

First, the next theorem answers the third Grothendieck’s question cited in the epigraph.

**Theorem 1.6.** For every connected semisimple algebraic group $G$, there is a rational section of $\pi_G$.

For char $k = 0$, this theorem has been proved earlier in [CTKPR, Theorem 0.3]. In our proof we use the relevant characteristic free results from [CTKPR], but bypass Theorem 2.12 from this paper (whose proof is based on the assumption char $k = 0$) by exploring properties of $\pi_G$ and proving that versality of $G$ holds in arbitrary characteristic (Lemma 5.8); this permits us to use Steinberg’s section of $\hat{\pi}_G$ in Lie $G$ used in [CTKPR].

**Corollary 1.7.** In every connected semisimple algebraic group $G$ there is a rational cross-section $S$ such that $\pi_G|_S: S \to G//G$ is a birational isomorphism.
Second, Theorems 1.5 and 1.6 yield the following

**Theorem 1.8.** For every connected semisimple algebraic group $G$, there is a $W$-equivariant rational map $T \rightarrow G/T$.

Section 6 contains some remarks, questions, and an example of a cross-section $S$ in $G$ such that $\pi_G|_S$ is not separable (hence $S$ is not the image of a section of $\pi_G$).

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2. Cross-sections in $G$

Fix a choice of Borel subgroup $\hat{B}$ of $\hat{G}$ and maximal torus $\hat{T} \subset \hat{B}$. Denote by $X(\hat{T})$ the character lattice of $\hat{T}$ in additive notation. For $\lambda \in X(\hat{T})$, denote by $t^\lambda$ the value of $\lambda: \hat{T} \rightarrow G_m$ at $t \in \hat{T}$. Let $\varpi_1, \ldots, \varpi_r \in X(\hat{T})$ be the system of fundamental weights of $\hat{T}$ with respect to $\hat{B}$.

Let $\varrho_i: \hat{G} \rightarrow \text{GL}(V_i)$ be an irreducible representation of $\hat{G}$ with $\varpi_i$ as the highest weight. Let $\chi_{\varpi_i} \in k[\hat{G}]^{G}$ be the character of $\varrho_i$.

Let $\hat{C}$ be the center of $\hat{G}$; it is a finite subgroup of $\hat{T}$. The conjugating action of $\hat{G}$ on itself commutes with the action of $\hat{C}$ on $\hat{G}$ by left translations. Therefore the latter action descends to $\hat{G}/\hat{G}$ and

$$\pi_{\hat{G}}: \hat{G} \rightarrow \hat{G}/\hat{G}$$

becomes a $\hat{C}$-equivariant morphism.

Endow the $r$-dimensional affine space $\mathbf{A}^r$ with the linear action of $\hat{T}$ by the formula

$$t \cdot (a_1, \ldots, a_r) := (t^{\varpi_1}a_1, \ldots, t^{\varpi_r}a_r), \quad t \in \hat{T}, \quad (a_1, \ldots, a_r) \in \mathbf{A}^r. \quad (2)$$

**Lemma 2.1.**

(i) The $\hat{T}$-stabilizer of the point $(1, \ldots, 1) \in \mathbf{A}^r$ is trivial. In particular, the considered action of $\hat{T}$ on $\mathbf{A}^r$ is faithful.

(ii) There is a $\hat{C}$-equivariant isomorphism

$$\lambda: \hat{G}/\hat{G} \rightarrow \mathbf{A}^r.$$  

**Proof.** Since $\varpi_1, \ldots, \varpi_r$ generate $X(\hat{T})$, we have

$$\bigcap_{i=1}^r \{t \in T | t^{\varpi_i} = 1\} = \{e\}. \quad (3)$$

But (2) entails that the $\hat{T}$-stabilizer of the point $(1, \ldots, 1)$ coincides with the right-hand side of equality (3). This proves (i).

By [Ste1, Theorems 6.1, 6.16] the $k$-algebra $k[\hat{G}]^{G}$ is freely generated by $\chi_{\varpi_1}, \ldots, \chi_{\varpi_r}$ and the morphism

$$\theta: \hat{G} \rightarrow \mathbf{A}^r, \quad \theta(g) = (\chi_{\varpi_1}(g), \ldots, \chi_{\varpi_r}(g)),$$

is surjective. Hence there is an isomorphism $\lambda: \hat{G}/\hat{G} \rightarrow \mathbf{A}^r$ such that the following diagram is commutative:

$$\begin{array}{ccc}
\hat{G}/\hat{G} & \xrightarrow{\pi_{\hat{G}}} & \hat{G} \\
\downarrow{\lambda} & & \downarrow{\theta} \\
\mathbf{A}^r & \xrightarrow{\theta} & \mathbf{A}^r
\end{array} \quad (4)$$

The morphism $\theta$ is $\hat{C}$-equivariant. Indeed, let $c \in \hat{C}$. Since $\varrho_i$ is irreducible, SCHUR’s lemma entails that $\varrho_i(c) = \mu_{i,c} \text{id}_{V_i}$ for some $\mu_{i,c} \in k$. On the other hand,
since \( c \in \hat{T} \), any highest vector in \( V_1 \) with respect to \( \hat{B} \) is an eigenvector of \( c \) with the eigenvalue \( c^{\omega_1} \). Hence \( \mu_{t,c} = c^{\omega_1} \). Therefore, for every \( g \in \hat{G} \), by (2) we have
\[
\theta(cg) = (\chi_{\omega_1}(cg), \ldots, \chi_{\omega_r}(cg)) \\
= (\text{trace} (\hat{g}_1(cg)), \ldots, \text{trace} (\hat{g}_r(cg))) \\
= (\chi_{\omega_1}(cg), \ldots, \chi_{\omega_r}(cg)) \\
= (\text{trace} (c^{\omega_1}g_1(g)), \ldots, \text{trace} (c^{\omega_r}g_r(g))) \\
= (c^{\omega_1}\text{trace} (g_1(g)), \ldots, c^{\omega_r}\text{trace} (g_r(g))) \\
= (c^{\omega_1}\chi_{\omega_1}(g), \ldots, c^{\omega_r}\chi_{\omega_r}(g)) \\
= c \cdot \theta(g),
\]
as claimed.

Since both \( \theta \) and \( \pi_{\hat{G}} \) are \( \hat{C} \)-equivariant and \( \pi_{\hat{G}} \) is surjective, commutativity of diagram (4) entails that \( \lambda \) is \( \hat{C} \)-equivariant as well. This proves (ii). \( \square \)

**Corollary 2.2.** Let \( g \) be a nonidentity element of \( \hat{C} \). Then there is no \( g \)-stable cross-section in \( \hat{G} \).

**Proof.** Assume the contrary and let \( \hat{S} \) be a \( g \)-stable cross-section in \( \hat{G} \). Since \( \pi_{\hat{G}} \) is \( \hat{C} \)-equivariant, \( \pi_{\hat{G}}|_S: \hat{S} \to \hat{G}/\hat{G} \) is a bijective \( g \)-equivariant morphism. As, by Lemma 2.1(ii), there is a point of \( \hat{G}/\hat{G} \) fixed by \( \hat{C} \), hence by \( g \), this implies that there is a point of \( \hat{S} \) fixed by \( g \). But for the action of \( \hat{C} \) on \( \hat{G} \) by left translations, the stabilizer of every point is trivial, a contradiction with \( g \neq e \). \( \square \)

Given an element \( h \) of an algebraic group \( H \), we shall denote its conjugacy class in \( H \) by \( H(h) \):
\[
H(h) := \{ \text{shs}^{-1} \mid s \in H \}.
\]

**Lemma 2.3.** Let \( H \) and \( \tilde{H} \) be connected algebraic groups and let \( \sigma: \tilde{H} \to H \) be an isogeny. Then the following properties hold:

(i) \( \sigma \) is a finite morphism;

(ii) \( \sigma(\tilde{H}(h)) = H(\sigma(h)) \) and \( \dim \tilde{H}(h) = \dim H(\sigma(h)) \) for every \( h \in \tilde{H} \);

(iii) if \( \tilde{H}(h) \) is a regular conjugacy class in \( \tilde{H} \) (i.e., that of the maximal dimension), then \( \sigma(\tilde{H}(h)) \) is a regular conjugacy class in \( H \);

(iv) if \( H \) and \( \tilde{H} \) are semisimple, then for every \( h \in \tilde{H} \),
\[
\sigma(\pi_{\tilde{H}}^{-1}(\pi_{\tilde{H}}(h))) = \pi_H^{-1}(\pi_H(\sigma(h))).
\]

**Proof.** The varieties \( H \) and \( \tilde{H} \) are normal (even smooth) and the fiber of \( \sigma \) over every point of \( H \) is a finite set whose cardinality does not depend on this point. Hence (cf. [G1, Sect. 2, Cor. 3]) \( \tilde{H} \) is the normalization of \( H \) in the field of rational functions on \( \tilde{H} \) and \( \sigma \) is the normalization map. This proves (i).

The first equality in (ii) holds as \( \sigma \) is an epimorphism of groups. The second follows from the first and theorem on dimension of fibers [Bor, AG 10.1]. This proves (ii).

As \( \sigma \) is surjective, (iii) follows from (ii).

Since the fibers of \( \pi_{\tilde{H}} \) and \( \pi_H \) are the closures of regular conjugacy classes and, by (i), the map \( \sigma \) is closed, (iv) follows from (iii). \( \square \)

**Corollary 2.4.** Let \( \tilde{G} \) be a connected semisimple algebraic group and let \( \sigma: \tilde{G} \to G \) be a bijective isogeny.

(i) If \( \tilde{S} \) is a cross-section in \( \tilde{G} \), then \( \sigma(\tilde{S}) \) is a cross-section in \( G \).
(ii) If $S$ is a cross-section in $G$, then $\sigma^{-1}(S)$ is a cross-section in $\hat{G}$.

The same holds if “cross-section” is replaced with “rational cross-section”.

Proof. By Lemma 2.3(i) the bijective map $\sigma$ is closed. Hence it is a homeomorphism. Both claims follow from this, the definitions of cross-section and rational cross-section, and Lemma 2.3(iv). \hfill \Box

Lemma 2.5. Assume that there is a subgroup $Z$ of $\hat{G}$ such that $G = \hat{G}/Z$ and $\tau$ is the quotient morphism $\hat{G} \to \hat{G}/Z$. Then there is a morphism

$$\varphi: \hat{G}/G \to G/G$$

such that

(i) $(G/G, \varphi)$ is a categorical quotient for the action of $Z$ on $\hat{G}/G$;

(ii) the following diagram is commutative:

$$\begin{array}{ccc}
\hat{G} & \xrightarrow{\tau} & G \\
\pi_{\hat{G}} \downarrow & & \downarrow \pi_G \\
\hat{G}/G & \xrightarrow{\varphi} & G/G
\end{array} \quad (7)
$$

(iii) for every point $x \in \hat{G}/G$, the following equality holds:

$$\tau(\pi_G^{-1}(x)) = \pi_G^{-1}(\varphi(x)). \quad (8)$$

Proof. As $\tau^*$, $\pi_{\hat{G}}^*$, and $\pi_G^*$ are injections, there is a unique morphism (6) such that $\tau^* \circ \pi_{\hat{G}}^* = \pi_G^* \circ \varphi^*$, i.e., diagram (7) is commutative.

Consider the action of $\hat{G}$ on $G$ via the isogeny $\tau$ and the conjugating action of $G$ on itself. The isogeny $\tau$ is then $\hat{G}$-equivariant and $G$-orbits in $\hat{G}$ are $G$-conjugacy classes, so we have $k[G]^\hat{G} = k[G]^G$. Since the conjugating action of $\hat{G}$ on itself commutes with the action of $Z$ by left translations, we have

$$\pi_{\hat{G}}^*(\varphi^*(k[G/\hat{G}])) = \tau^*(\pi_{\hat{G}}^*(k[G/\hat{G}])) = \tau^*(k[G]^\hat{G}) = \tau^*(k[G]^G) = (\tau^*(k[G]))^\hat{G}.$$

Thus, $\varphi^*(k[G/\hat{G}]) = k[\hat{G}/G]^Z$. This proves (i) and (ii). Lemma 2.3(iv) and commutativity of diagram (7) imply (iii). \hfill \Box

Below, given a variety $Z$, we denote by $T_{z,Z}$ the tangent space of $Z$ at a point $z$.

Proof of Theorem 1.2. First, we shall prove criterion (i).

1. By Steinberg’s theorem, $\hat{G}$ has a cross-section. Hence, by Corollary 2.4, if $\tau$ is bijective, then there exists a cross-section in $G$ as well.

So we may assume that $\tau$ is not bijective and we then have to prove that there is no cross-section in $G$. Solving this problem, we may assume that $\tau$ is separable. Indeed, if this is not the case, then by [Bor, Prop.17.9] there exist a connected semisimple algebraic group $\hat{G}$ and a commutative diagram of isogenies

$$\begin{array}{ccc}
\tilde{G} & \xrightarrow{\tau} & G \\
\mu \downarrow & & \downarrow \sigma \\
\hat{G} & \xleftarrow{\mu} & \hat{G}
\end{array} \quad (9)
$$

where $\mu$ is separable and $\sigma$ is purely inseparable. As $\sigma$ is bijective, Corollary 2.4 then reduces the problem to proving that there is no cross-section in $\hat{G}$, i.e., we may replace $G$ by $\hat{G}$ and $\tau$ by $\mu$. 

So from now on we may (and shall) assume that $\tau$ is a separable isogeny of degree $\geq 2$. This means that there is a nontrivial subgroup $Z$ of $\widehat{C}$ such that $G = \widehat{G}/Z$ and $\tau$ is the quotient morphism $G \to \widehat{G}/Z$.

2. Now, arguing on the contrary, assume that there is a cross-section $S$ in $G$.

**Claim 1.** (i) For every point $x \in \widehat{G}/\widehat{G}$, the intersection

$$\pi^{-1}_G(x) \cap \tau^{-1}(S)$$

is a nonempty subset of a single $Z$-orbit; in particular, it is finite.

(ii) There is a nonempty open subset $\widehat{U}$ of $\widehat{G}/\widehat{G}$ such that, for every $x \in \widehat{U}$, intersection (10) is a single point.

**Proof of Claim 1.** Consider diagram (7). Since $S \cap \pi^{-1}_G(\varphi(x))$ is a single point $g$, we deduce from (8) that intersection (10) is contained in $\tau^{-1}(g)$. This proves (i) as the fibers of $\tau$ are $Z$-orbits.

By Lemma 2.1(i) there is a nonempty open subset $U$ in $\widehat{G}/\widehat{G}$ such that the $\widehat{C}$-stabilizer of every point of $U$ is trivial. Take a point $x \in U$. Assume that intersection (10) contains two points $g_1$ and $g_2 \neq g_1$. By (i) there exists an element $z \in Z$ such that $g_2 = zg_1$. As $\pi_\widehat{G}$ is $\widehat{C}$-equivariant, $x = \pi_\widehat{G}(g_2) = \pi_\widehat{G}(zg_1) = z \cdot \pi_\widehat{G}(g_1) = z \cdot x$. Thus, $z$ belongs to the $\widehat{C}$-stabilizer of $x$. The definition of $U$ then implies that $z = e$. Hence $g_1 = g_2$, a contradiction. This proves (ii). \[\square\]

3. Since all the fibers of $\tau$ are finite, every irreducible component of $\tau^{-1}(S)$ has dimension $\leq \dim S = r$ and at least one of them has dimension $r$.

**Claim 2.** (i) There is a unique $r$-dimensional irreducible component $\widehat{S}$ of $\tau^{-1}(S)$.

(ii) $\tau(\widehat{S}) = S$.

**Proof of Claim 2.** Let $\widehat{S}$ be an $r$-dimensional irreducible component of $\tau^{-1}(S)$. Then $\tau(\widehat{S})$ contains an open subset of $S$. Since $\tau$ is closed, this proves (ii).

From (ii) we conclude that

$$\pi_G(\tau(\widehat{S})) = G/G.$$  

(11)

But by Lemma 2.5 the fibers of $\varphi$ in commutative diagram (7) are finite. This and (11) imply that $\pi_G(\widehat{S})$ contains a nonempty open subset of $\widehat{G}/\widehat{G}$.

Let now $\widehat{S}'$ be another $r$-dimensional irreducible components of $\tau^{-1}(S)$. Then, as above, $\pi_G(\widehat{S}')$ contains a nonempty open subset of $\widehat{G}/\widehat{G}$ as well. Therefore, $\pi_G(\widehat{S}) \cap \pi_G(\widehat{S}')$ contains a nonempty open subset $V$ of $\widehat{G}/\widehat{G}$. We may assume that $V \subseteq U$ for $U$ from Claim 1(ii). The latter then yields that $\pi^{-1}_G(V) \cap \widehat{S} = \pi^{-1}_G(V) \cap \widehat{S}'$. As both sides of this equality are the open subsets of respectively $\widehat{S}$ and $\widehat{S}'$, we infer that $\widehat{S} = \widehat{S}'$. This proves (i). \[\square\]

4. As $\widehat{S}$ is a unique $r$-dimensional irreducible component of the $Z$-stable variety $\tau^{-1}(S)$, we conclude that $\widehat{S}$ is $Z$-stable. We shall now show that $\widehat{S}$ is a cross-section in $\widehat{G}$. As this property contradicts Corollary 2.2, the proof of (i) will be then completed.

5. Let $x$ be a point of $\widehat{G}/\widehat{G}$. As $S$ is a section of $G$, the intersection $S \cap \pi^{-1}_G(\varphi(x))$ is a single point $g \in G$. By Claim 2(ii) there is a point $\widehat{g} \in \widehat{S}$ such that $\tau(\widehat{g}) = g$. Commutativity of diagram (7) then entails that $x$ and $\widehat{x} := \pi_\widehat{G}(\widehat{g})$ are in the same fiber of $\varphi$. Since the fibers of $\varphi$ are $Z$-orbits, there is an element $z \in Z$ such that $x = z \cdot \widehat{x}$. As $\pi_\widehat{G}$ is $Z$-equivariant, this yields $\pi_\widehat{G}(z\widehat{g}) = x$. But $z\widehat{g} \in \widehat{S}$ as $\widehat{S}$ is $Z$-stable.
and $\hat{g} \in \hat{S}$. Hence $\pi^{-1}_G(x) \cap \hat{S} \neq \emptyset$, i.e.,

$$\pi_G(\hat{S}) = \hat{G}/\hat{G}.$$  

6. It follows from Claim 1(i),(ii) and (12) that $\pi_G|\hat{S}$ is the surjective morphism with finite fibers, bijective over an open subset of $\hat{G}/\hat{G}$. As $\hat{G}$ is normal, $\hat{G}/\hat{G}$ is normal as well. Let $\nu: \hat{S} \to \hat{S}$ be the normalization. Then the surjective morphism $\pi_G|\hat{S} \circ \nu: \hat{S} \to \hat{G}/\hat{G}$ of normal varieties has finite fibers and is bijective over an open subset of $\hat{G}/\hat{G}$. Hence $\pi_G|\hat{S} \circ \nu$ is bijective (see [G1, Sect. 2, Cor. 2]). Whence $\pi_G|\hat{S}$ is bijective as well, i.e., $\hat{S}$ is a cross-section in $\hat{G}$. This completes the proof of (i).

We now turn to the proof of (ii).

Let $S := \sigma(G/G)$. Take a point $x \in S$ and put $y := \pi_G(x)$. As $\pi_G|S: S \to G/G$ is the isomorphism ($\sigma$ is its inverse), $(d\pi_G)_x$ is the isomorphism as well. Hence $(d\pi_G)_x$ is surjective. As $\dim T_{y,G/G} \geq \dim G/G = r$, this implies that there are functions $f_1, \ldots, f_r \in k[G]^G$ such that $(df_1)_x, \ldots, (df_r)_x$ are linearly independent. By [Ste1, Theorem 8.7] this yields that $x$ is regular. As $S$ intersects every fiber of $\pi_G$ at a single point and every such fiber contains a unique regular orbit, this proves (ii). Thus, the proof of Theorem 1.2 comes to a close. $\square$

3. SINGULARITIES OF $G/G$ AND GENERATORS OF $k[G]^G$ AND $R(G)$

The following lemma shows that there is a link between the existence of a cross-section in $G$ and smoothness of $G/G$.

**Lemma 3.1.** Let $\text{char } k = 0$. If a surjective morphism $\alpha: X \to Y$ of irreducible varieties admits a section $\sigma: Y \to X$, then smoothness of $X$ implies smoothness of $Y$.

**Proof.** Arguing on the contrary, assume that $y$ is a singular point of $Y$, i.e.,

$$\dim T_{y,Y} > \dim Y.$$  

(13)

Put $x = \sigma(y) \in X$. Since $\alpha \circ \sigma = \text{id}_Y$, the composition $d\alpha_x \circ d\sigma_y$ is the identity map of $T_{y,Y}$. Hence $d\alpha_x$ is surjective, i.e., $\text{rk } d\alpha_x = \dim T_{y,Y}$. By (13) this yields

$$\text{rk } d\alpha_x > \dim Y.$$  

(14)

As $\text{char } k = 0$, there is a dense open subset $U$ of $X$ such that $\text{rk } d\alpha_z = \dim Y$ for every point $z \in U$, see [H, 14.4]. As $z \mapsto \dim \ker d\alpha_z$ is the upper semi-continuous function [H, 14.6], we conclude that smoothness of $X$ implies that $\text{rk } d\alpha_z \leq \dim Y$ for every point $z \in X$. This contradicts (14). $\square$

This prompts to explore smoothness of $G/G$. The answer is known:

**Theorem 3.2** ([Ste3, §3], [R1, Prop. 4.1], [R2, Prop. 13.3]). Let $\text{char } k \neq 2$. The following properties are equivalent:

(i) $G/G$ is smooth;

(ii) $G/G$ is isomorphic to the affine space $A^n$;

(iii) $G = G_1 \times \cdots \times G_s$ where every $G_i$ is either a simply connected simple algebraic group or isomorphic to $\text{SO}_{n_i}$ for an odd $n_i$.

This criterion of smoothness of $G/G$ may be also interpreted as that of the existence of $r$ generators of the algebra of class functions on $G$. Below we describe a minimal system of generators of this algebra and singularities of $G/G$ in the general case. This also yields a minimal system of generators of the representation ring of $G$.

Let $B := \tau(\hat{B})$ and $T := \tau(\hat{T})$. This is respectively a Borel subgroup and a maximal torus of $G$. We consider the lattice $X(T)$ of characters of $T$ as the sublattice of $X(\hat{T})$.
identifying $\mu \in X(T)$ with $\tau^*(\mu) \in X(\hat{T})$. Then $X(\hat{T})$ is the weight lattice of $X(T)$. The monoid of highest weights of simple $\hat{G}$-modules (with respect to $\hat{B}$ and $\hat{T}$) is

$$\hat{D} := N \varpi_1 + \cdots + N \varpi_r, \quad N = \{0, 1, 2, \ldots\}. \quad (15)$$

and that of simple $G$-modules (with respect to $B$ and $T$) is

$$D := \hat{D} \cap X(T). \quad (16)$$

Let $W$ be the Weyl group of $\hat{G}$, i.e., the quotient of $\hat{T}$ in its normalizer, acting on $\hat{T}$ via conjugation. The Weyl group of $T$ is identified naturally with $W$ (see [Bor, Prop. 11.20 and Cor. 2(d) in 13.17]).

If $\varpi \in D$ and $E(\varpi)$ is a simple $G$-module with $\varpi$ as the highest weight, we denote by $\chi_{\varpi} \in k[G]^{\times}$ the character of $E(\varpi)$.

Given a nonzero commutative ring $A$ with identity element and a commutative monoid $M$, we denote by $A[M]$ the semigroup ring of $M$ over $A$. We identify $A[M]$ with $A \otimes_{\mathbb{Z}} \mathbb{Z}[M]$ in the natural way. If $S$ is a submonoid of the multiplicative monoid of $A[M]$ whose elements are linearly independent over $A$, then the subring of $A[M]$ generated by $S$ is identified naturally with $A[S]$. In particular, we consider $A[X(T)]$ and $A[D]$ as the subrings of $A[X(\hat{T})]$. The former is stable with respect to the natural action of $W$ on $\mathbb{Z}[X(T)]$. Using the notation and terminology of Bourbaki [Bou2, VI.3], we denote by $e^\mu$ the element of $\mathbb{Z}[X(\hat{T})]$ corresponding to $\mu \in X(\hat{T})$ and put

$$S(e^\mu) := \sum_{\nu \in W, \mu} e^\nu \in \mathbb{Z}[X(\hat{T})]^W. \quad (17)$$

Given an algebraic group $H$, we denote by $R(H)$ the representation ring of $H$: its additive group is the Grothendieck group of the category of finite dimensional algebraic $H$-modules with respect to exact sequences and the multiplication is induced by tensor product of modules. Using $\tau$, we identify $R(G)$ in the natural way with the subring of $R(\hat{G})$.

If $E$ is a finite dimensional algebraic $G$-module and $E_\mu$ is its weight space of a weight $\mu \in X(T)$, then the formal character of $E$,

$$\text{ch}_G[E] := \sum_{\mu \in X(T)} (\dim E_\mu) e^\mu, \quad (18)$$

is an element of $\mathbb{Z}[X(T)]^W$ depending only on the class $[E]$ of $E$ in $R(G)$. Clearly,

$$\text{ch}_G[E \otimes E'] = \text{ch}_G[E] \text{ch}_G[E']. \quad (19)$$

According to [Se, 3.6], the homomorphism of $\mathbb{Z}$-modules

$$\text{ch}_G: R(G) \rightarrow \mathbb{Z}[X(T)]^W, \quad [E] \mapsto \text{ch}_G[E], \quad (20)$$

is an isomorphism. By (19) it is an isomorphism of rings.

**Definition 3.3.** Let $\varpi \in \hat{D}$. We say that an element $x \in \mathbb{Z}[X(\hat{T})]^W$ is $\varpi$-sharp if the following property (M) holds:

(M) $e^\varpi$ is a unique maximal term of $x$.

**Example 3.4.** The elements $S(e^\varpi)$ and $\text{ch}_G[E(\varpi)]$ are $\varpi$-sharp (this follows, e.g., from [Bou2, VI.1.6, Prop. 18] and [Hum1, 31.3, Theorem]). \(\square\)

Property (M) implies that the support of a $\varpi$-sharp element $x$ lies in $\varpi + X(T)$. This and [Bou2, VI.3.4, formula (6)] yield

$$x = S(e^\varpi) + \text{sum of some } S(e^\varpi') \text{'s with } \varpi' \in \hat{D}, \varpi' < \varpi. \quad (21)$$
By [Bou2, VI.3.2, Lemma 2] if an element \( x' \) is \( \varpi' \)-sharp, then \( xx' \) is \( (\varpi + \varpi') \)-sharp.

Now fix a \( \varpi \)-sharp element \( x_\varpi \in \mathbb{Z}[X(T)]^W \), \( i = 1, \ldots, r \), and put
\[
x_\varpi := x_{\varpi_1} \cdots x_{\varpi_r} \quad \text{for} \quad \varpi = m_1 \varpi_1 + \cdots + m_r \varpi_r \in \hat{D}.
\]

By [Bou2, VI.3.4, Theorem 1] the set \( \{ x_\varpi \mid \varpi \in \hat{D} \} \) is then a basis of the \( \mathbb{Z} \)-module \( \mathbb{Z}[X(T)]^W \). As \( \{ e^\mu \mid \mu \in X(T) \} \) is a basis of the \( \mathbb{Z} \)-module \( \mathbb{Z}[X(T)] \) and the support of \( x_\varpi \) lies in \( \varpi + X(T) \), we deduce from this and (16) that \( \{ x_\varpi \mid \varpi \in D \} \) is a basis of the \( \mathbb{Z} \)-module \( \mathbb{Z}[X(T)]^W \). Hence the homomorphism of the \( \mathbb{Z} \)-modules
\[
\vartheta : \mathbb{Z}[X(T)]^W \to \mathbb{Z}[D], \quad \vartheta(x_\varpi) = e^\varpi \quad \text{for} \quad \varpi \in D,
\] (22)
is an isomorphism. Since \( x_{\varpi + \varpi'} = x_\varpi x_{\varpi'} \), it is, in fact, an isomorphism of rings.

As by Dedekind’s theorem \( \{ f_\mu : T \to k, t \mapsto t^\mu \mid \mu \in X(T) \} \) is a basis of the vector space \( k[T] \) over \( k \), the \( k \)-linear map \( k[T] \to k[X(T)] \), \( f_\mu \mapsto e^\mu \), is the isomorphism of \( k \)-algebras. We identify them by means of this isomorphism. So we have \( k[T] = k[X(T)] \) and
\[
k[T/W] = k[T]^W = k[X(T)]^W. \tag{23}
\]

Finally, take into account that by [Ste1, 6.4] the restriction map
\[
\text{res} : k[G/G] = k[G]^G \to k[T]^W, \quad \text{res}(f) = f|_T, \tag{24}
\]
is an isomorphism of \( k \)-algebras. Summing up, we obtain

**Theorem 3.5.**

(i) \( G/G \) and \( T/W \) are the affine toric varieties of \( T \) whose algebras of regular functions are isomorphic to \( k[D] \).

(ii) In the diagram
\[
\begin{array}{ccc}
\text{k}[G/G] & \xrightarrow{\text{res}} & k[T/W] \xrightarrow{\text{id} \otimes \vartheta} k[D]
\end{array}
\]
(see (24), (23), (22)) both maps are the isomorphisms of \( k \)-algebras.

(iii) Let \( F \) be the simple subring of \( k \). Then the image of \( F \otimes_{\mathbb{Z}} R(G) \) in \( k[G]^G \) under the composition of the ring isomorphisms
\[
k \otimes_{\mathbb{Z}} R(G) \xrightarrow{\text{id} \otimes \text{ch}_G} k[X(T)]^W = k[T]^W \xrightarrow{\text{res}^{-1}} k[G]^G
\] (25)
is an \( F \)-form of \( k[G/G] \) isomorphic to \( F \otimes_{\mathbb{Z}} R(G) \). In particular, if \( \text{char} k = 0 \), it is a \( \mathbb{Z} \)-form of \( k[G/G] \) isomorphic to \( R(G) \).

**Remark 3.6.** The fact that “multiplicative invariants” of finite reflection groups are semigroup algebras is already in the literature, first implicitly, then explicitly, see the historical account in [L1, Introduction]. Essentially, the main ingredients date back to [Ste1, §6] and [Bou2, VI, §3].

Since toric varieties are rational, Theorem 3.5(i) yields

**Corollary 3.7.** \( G/G \) and \( T/W \) are rational varieties.

In the next statement Theorem 3.5 is applied to finding a minimal system of generators of the algebra \( k[G]^G \) and that of the ring \( R(G) \).

Let \( \mathcal{H} \) be the Hilbert basis of the monoid \( D \), i.e., the set of all its indecomposable elements:
\[
\mathcal{H} = D_+ \setminus 2D_+ \quad \text{where} \quad D_+ := D \setminus \{0\}, \quad 2D_+ := D_+ + D_+.
\] (26)
The set \( \mathcal{H} \) is finite, generates \( D \), and every generating set of \( D \) contains \( \mathcal{H} \) (see, e.g., [L2, 3.4]).

**Remark 3.8.** There is an algorithm for computing \( \mathcal{H} \), see [Stu, 13.2] (cf. also Example 3.11 below).
2.1(i). Let \( y_i \) yield

From (2) we deduce that \( k \) with 

\[ \{ \chi_\varpi \mid \varpi \in \mathcal{H} \} \]

is a generating set of the ring \( R(G) \).

(ii) \[ \{ \chi_\varpi \mid \varpi \in \mathcal{H} \} \] is a generating set of the algebra \( k[G]^G \).

Proof. (i) Let \( Y \) be the affine toric variety of \( T \) with \( k[Y] = k[D] \). The linear span \( I \) 

of \( \{ e^\varpi \mid \varpi \in D_+ \} \) over \( k \) is a maximal \( T \)-invariant ideal in \( k[Y] \). Hence \( I/I^2 \) is the 

cotangent space of \( Y \) at the \( T \)-fixed point \( v \) where \( I \) vanishes. As \( I^2 \) is the linear span 

of \( \{ e^\varpi \mid \varpi \in 2D_+ \} \) over \( k \), this and (26) yield 

\[ \dim T_{v,Y} = \dim I/I^2 = |\mathcal{H}|. \] (27)

Now take into account that, given an affine algebraic variety \( X \), the algebra \( k[X] \) can be generated by \( d \) elements if and only if \( X \) admits a closed embedding in \( \mathbb{A}^d \). Hence 

\( d \geq \dim T_{x,X} \) for every point \( x \in X \). This, Theorem 3.5(i),(iii), and (27) prove (i).

(ii) Let \( \mu \in D \). As \( \mathcal{H} \) generates \( D \), we deduce from Example 3.4 that there is a 

\( \mu \)-sharp monomial \( M^\mu \) in the elements of the set \( \{ \chi_\varpi \mid \varpi \in D \} \). By (21) we have 

\[ M^\mu = S(e^{\mu'}) + \text{sum of some } S(e^{\mu'})'s \text{ with } \mu' \in D, \mu' < \mu. \] (28)

But \( \{ S(e^{\mu'}) \mid \mu \in D \} \) is a basis of the \( \mathbb{Z} \)-module \( \mathbb{Z}[X(T)]^W \) (see [Bou, VI.3.4, Lemma 3]). By [Bou, VI.3.4, Lemma 4] and (28) we then conclude that the set \( \{ M^\mu \mid \mu \in D \} \) 

generates the \( \mathbb{Z} \)-module \( \mathbb{Z}[X(T)]^W \). This means that the ring \( \mathbb{Z}[X(T)]^W \) is generated by 

the set \( \{ \chi_\varpi \mid \varpi \in \mathcal{H} \} \). As (20) is an isomorphism of rings, this proves (ii).

(iii) It follows from (ii) that the set \( \{ 1 \otimes [E(\varpi)] \mid \varpi \in \mathcal{H} \} \) generates the ring 

\( k \otimes_\mathbb{Z} R(G) \). But formula (18) shows that \( \chi_\varpi \) is the image of \( 1 \otimes [E(\varpi)] \) under 

the composition of the ring isomorphisms in diagram (25). This proves (iii). \( \square \)

Since the Weyl chambers are simplicial cones, Theorem 3.5(i) implies, at least for 

\( \text{char } k = 0 \), that \( G/\mathbb{G} \) and \( T/W \) are isomorphic to the quotient of \( \mathbb{A}^r \) by a linear 

action of a certain finite abelian group (see, e.g., [O, Prop. 1.25]). In particular, \( G/\mathbb{G} \) 

and \( T/W \) may have only finite quotient singularities. Below this finite group and its 

action on \( \mathbb{A}^r \) are explicitly described assuming that \( \tau \) is separable.

The latter assumption means that there is a subgroup \( Z \) of \( \hat{G} \) such that \( G = \hat{G}/Z \) and 

\( \tau \) is the quotient morphism \( \hat{G} \to \hat{G}/Z \). In this situation we have 

\[ X(T) = \{ \mu \in X(\hat{T}) \mid e^\mu = 1 \text{ for every } c \in Z \}. \] (29)

Consider the \( \hat{T} \)-orbit map of the point \( (1, \ldots, 1) \in \mathbb{A}^r \):

\[ \iota: \hat{T} \to \mathbb{A}^r, \quad \iota(t) = t \cdot (1, \ldots, 1). \] (30)

The map \( \iota^*: k[\mathbb{A}^r] \to k[\hat{T}] = k[X(\hat{T})] \) is an embedding as \( \iota \) is dominant by Lemma 

2.1(i). Let \( y_1, \ldots, y_r \) be the standard coordinate functions on \( \mathbb{A}^r \). Then (2) and (30) yield 

\[ \iota^*(y_i) = e^{\varpi_i}. \] (31)

From (2) we deduce that \( k[\mathbb{A}^r]^Z \) is the linear span over \( k \) of all monomials \( y^{m_1} \cdots y^{m_r} \) 

with \( m_1, \ldots, m_r \in \mathbb{N} \) such that \( e^{m_1 \varpi_1 + \cdots + m_r \varpi_r} = 1 \) for every \( c \in Z \). By (29) the 

latter condition is equivalent to the inclusion \( m_1 \varpi_1 + \cdots + m_r \varpi_r \in X(T) \). This, (31), 

(15), and (16) imply that \( \iota^*(k[\mathbb{A}^r]^Z) = k[D] \). Thus, taking into account Theorem 3.5, 

we obtain the isomorphisms of \( k \)-algebras

\[ k[\mathbb{A}^r]^Z \xrightarrow{\iota^*} k[D] \xrightarrow{(\text{id} \otimes \delta)^{-1}} k[T/W] \xrightarrow{(\text{res})^{-1}} k[G/\mathbb{G}]. \]
that, in turn, induce the isomorphisms of varieties $G//G \rightarrow T/W \rightarrow A^r/Z$.

By means of a special parametrization of $\hat{T}$ one obtains an explicit description of the elements of $\hat{C}$ well adapted for computing $k[A^r]^Z$. Since $\hat{G} = \hat{G}_1 \times \cdots \times \hat{G}_s$ and $\hat{C} = \hat{C}_1 \times \cdots \times \hat{C}_s$ where every $\hat{G}_i$ is a nontrivial normal simply connected simple subgroup of $\hat{G}$ and $\hat{C}_i$ is the center of $\hat{G}_i$, it suffices to describe this parametrization for simple groups $\hat{G}$. The answer is given below in Lemma 3.10.

Namely, let $\hat{\alpha}_1, \ldots, \hat{\alpha}_r \in X(\hat{T})$ be the system of simple roots of $\hat{T}$ with respect to $\hat{B}$ and let $\hat{\alpha}_i^\vee : G_m \rightarrow \hat{T}$ be the coroot corresponding to $\hat{\alpha}_i$. Then, for every $s \in G_m$,

$$((\hat{\alpha}_i^\vee(s)))^{\pi_i} = \begin{cases} s & \text{if } i = j, \\ 1 & \text{if } i \neq j. \end{cases} \quad (32)$$

If $(\cdot, \cdot)$ is the natural pairing between the lattices of characters and cocharacters of $\hat{T}$, we put $n_{ij} := (\hat{\alpha}_i, \hat{\alpha}_j^\vee)$. So $(n_{ij})_{i,j=1}^r$ is the Cartan matrix of $\hat{G}$.

By [Ste2, Lemma 28(b),(d) and its Cor. (a)] the map

$$\nu : G_m^r \rightarrow \hat{T}, \quad \nu(s_1, \ldots, s_r) = \hat{\alpha}_1^\vee(s_1) \cdots \hat{\alpha}_r^\vee(s_r), \quad (33)$$

is an isomorphism of groups and

$$\hat{C} = \{\hat{\alpha}_1^\vee(s_1) \cdots \hat{\alpha}_r^\vee(s_r) | s_1^{n_11} \cdots s_r^{n_{rr}} = 1 \text{ for every } i = 1, \ldots, r\}. \quad (34)$$

By (32) and (33) we have

$$((\nu(s_1, \ldots, s_r)))^{\pi_i} = (\hat{\alpha}_1^\vee(s_1))^{\pi_i} \cdots (\hat{\alpha}_r^\vee(s_r))^{\pi_i} = s_i.$$

This and (2) imply that, for every $s = (s_1, \ldots, s_r) \in G_m^r$ and $(a_1, \ldots, a_r) \in A^r$, we have

$$\nu(s) \cdot (a_1, \ldots, a_r) = (s_1a_1, \ldots, s_ra_r).$$

**Lemma 3.10.** For every simple simply connected group $\hat{G}$, the subgroup $\nu^{-1}(\hat{C})$ of the torus $G_m^r$ is described in the following Table 1 (simple roots in (33) are numerated as in [Bou2]):

| Table 1. |
|-----------------|-----------------|
| type of $\hat{G}$ | $\nu^{-1}(\hat{C})$ |
| $A_r$ | $\{(s, s^2, s^3, \ldots, s^r) | s^{r+1} = 1\}$ |
| $B_r$ | $\{(1, \ldots, 1, s^2) | s^2 = 1\}$ |
| $C_r$ | $\{(1, 1, s, 1, \ldots, s^{r \mod 2}) | s^2 = 1\}$ |
| $D_r, r \text{ odd}$ | $\{(s^2, 1, s^2, 1, \ldots, s^2, s, s^{-1}) | s^4 = 1\}$ |
| $D_r, r \text{ even}$ | $\{(s, 1, s, 1, \ldots, s, s, s^t, t) | s^2 = t^2 = 1\}$ |
| $E_6$ | $\{(s, 1, s^{-1}, 1, s, s^{-1}) | s^3 = 1\}$ |
| $E_7$ | $\{(1, s, 1, s, 1, 1, s) | s^2 = 1\}$ |
| $E_8$ | $\{(1, 1, 1, 1, 1, 1, 1, 1)\}$ |
| $F_4$ | $\{(1, 1, 1, 1)\}$ |
| $G_2$ | $\{(1, 1)\}$ |
Proof. By (34) an element \((s_1, \ldots, s_r) \in G_\nu^r\) lies in \(\nu^{-1}(\hat{C})\) if and only if \((s_1, \ldots, s_r)\) is a solution of the system of equations
\[
x_1^{n_1} \cdots x_r^{n_r} = 1, \\
\vdots \\
x_1^{n_1} \cdots x_r^{n_r} = 1,
\]
where \((n_{ij})_{i,j=1}^r\) is the Cartan matrix of \(G\).

Let, for instance, \(G\) be of type \(D_r\) for even \(r\). Using the explicit form of the Cartan matrix [Bou_2, Planche IV], one immediately verified that every element of \(C' := \{(s, 1, s, 1, \ldots, s, 1, st, t) \mid s^2 = t^2 = 1\}\) is a solution of (35). Hence, \(C' \subseteq \nu^{-1}(\hat{C})\). On the other hand, the fundamental group of the root system of type \(D_r\) is isomorphic to \(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\) (see [Sp. 8.1.11] and [Bou_2, Planche IV]). Hence, the Smith normal form of \((n_{ij})_{i,j=1}^r\) is diag \((1, 1, 1, 2, 2)\). Therefore, there is a basis \(\beta_1, \ldots, \beta_r\) of the coroot lattice of \(\hat{T}\) such that, for \((s_1, \ldots, s_r) \in G_\nu^r\), we have \(\beta_1(s_1) \cdots \beta_r(s_r) \in \hat{C}\) if and only if \((s_1, \ldots, s_r)\) is a solution of the system
\[
x_1 = 1, \ldots, x_{r-2} = 1, x_{r-1}^2 = 1, x_r^2 = 1.
\]
This yields \(|C'| = |\hat{C}|\); whence \(C' = \nu^{-1}(\hat{C})\).

For the groups of the other types the proofs are similar. \(\square\)

The following examples illustrate how this can be applied to exploring singularities of \(G\) and finding the minimal generating sets \(\{\chi_\omega \mid \omega \in \mathcal{H}\}\) and \(\{[E(\omega)] \mid \omega \in \mathcal{H}\}\) of, respectively, the algebra of class functions on \(G\) and the representation ring of \(G\).

Examples 3.11.

(1) Let \(G\) be of type \(B_r\) (where \(B_1 := A_1\)) and let char \(k \neq 2\). Table 1 implies that \(\nu^{-1}(\hat{C})\) is generated by \((1, \ldots, 1, -1)\). Whence \(k[A^C] = k[y_1, \ldots, y_{r-1}, y_r]\). Therefore, for the adjoint \(G\), i.e., for \(G = SO_{2r+1}\), the variety \(G\) is isomorphic to \(A^r\) (this agrees with Theorem 3.2) and
\[
\mathcal{H} = \{w_1, \ldots, w_{r-1}, 2w_r\}.
\]

(2) Let \(G\) be of type \(D_r\), \(r \geq 3\). Let char \(k \neq 2\) and let \(Z := \{t \in \hat{C} \mid t^{\nu^{-1}} = 1\}\). Then \(G := \hat{G}/Z = SO_{2r}\). Table 1 implies that \(\nu^{-1}(Z)\) is generated by \((1, \ldots, 1, -1, -1)\). Whence \(k[A^Z] = k[y_1, \ldots, y_{r-2}, y_{r-1}^2, y_r, y_{r-1}]\). Therefore, \(G\) is isomorphic to \(A^{r-2} \times X\) where \(X\) is a nondegenerate quadratic cone in \(A^3\) and
\[
\mathcal{H} = \{w_1, \ldots, w_{r-2}, 2w_{r-1}, 2w_r, w_{r-1} + w_r\}.
\]

(3) Let \(G\) be of type \(D_r\) with even \(r = 2d \geq 4\). Let char \(k \neq 2\) and let \(Z := \{t \in \hat{C} \mid t^{\nu^{-1}} = 1\}\). Then \(G := \hat{G}/Z\) is the half-spinor group \(Spin_{2d}^{1/2}\). Table 1 implies that \(\nu^{-1}(Z)\) is generated by \((-1, -1, -1, 1, \ldots, -1)\). Whence \(k[A^Z] = k[y_1, \ldots, y_{d-2}, y_{d-1}^2, y_d, y_{d-1}y_d]\). Therefore, \(G\) is isomorphic to \(A^d \times Y\) where \(Y\) is the affine cone over the Veronese variety \(\nu_2(P^{d-1})\) in \(P^{(d-1)(d+2)/2}\) and
\[
\mathcal{H} = \{w_i \mid i \text{ is even}\} \cup \{w_l + w_m \mid l, m \text{ are odd}\}.
\]

(4) Let \(G\) be of type \(E_7\) and let char \(k \neq 2\). Table 1 implies that \(k[A^C] = \text{minimally generated by } y_1, y_3, y_4, y_6\) and all monomials of degree 2 in \(y_2, y_5, y_7\). Therefore, if \(G\) is adjoint, then \(G\) is isomorphic to \(A^4 \times Y\) where \(Y\) is the affine cone over the Veronese variety \(\nu_2(P^2)\) in \(P^5\) (in particular, the tangent space of the 7-dimensional variety.
$G/G$ at the unique fixed point of $T$, see Theorem 3.5(i) and (27), is 10-dimensional) and
\[ H = \{ \varpi_1, \varpi_3, \varpi_4, \varpi_6, 2\varpi_2, 2\varpi_5, 2\varpi_7, \varpi_2 + \varpi_5, \varpi_2 + \varpi_7, \varpi_5 + \varpi_7 \}. \]

4. TWO FURTHER QUESTIONS OF GROTHENDIECK

Theorem 3.9 describes a minimal generating set of the algebra $k[G]^G$ of class functions on $G$. Constructing generating sets of $k[G]^G$ is the topic of two further questions of Grothendieck in [GS, p. 241]:

“[..] When $G$ is an adjoint group, is it possible to generate the affine ring of $I(G)$ with coefficients of the Killing polynomial? In the general case, is it enough to take the coefficients of analogous polynomials for certain linear representations (perhaps arbitrary faithful representations)? [...]”

Below we answer these questions.

Let $\varrho: G \to \text{GL}(V)$ be a finite dimensional linear representation of $G$. Define the set
\[ C_\varrho := \{ c_{\varrho,i} \in k[G] \mid i = 1, \ldots, \dim V \} \]
by the equality
\[ \det(xI - \varrho(g)) = \sum_{i=0}^{\dim V} c_{\varrho,i}(g)x^{\dim V - i} \quad \text{for every } g \in G, \quad (36) \]
where $x$ is a variable. If $V = E(\varpi)$ (here and below we use the notation of Section 3) and $\varrho$ determines the $G$-module structure of $E(\varpi)$, we put $C_\varpi := C_\varrho$.

Clearly, $c_{\varrho,i} \in k[G]^G$ and $c_{\varrho,1}$ is the character of $\varrho$. Hence by Theorem 3.9(iii)
\[ \bigcup_{\varpi \in H} C_\varpi \]
is a generating set of the algebra $k[G]^G$. This answers the second Grothendieck’s question in the affirmative.

In order to answer the first one in the negative it is sufficient to find an adjoint $G$ and two elements $z_1, z_2 \in T$ such that
(i) $z_1$ and $z_2$ are not in the same $W$-orbit;
(ii) the spectra of the linear operators $\text{Ad}_G z_1$ and $\text{Ad}_G z_2$ on the vector space $\text{Lie} G$ coincide.

Indeed, property (i) implies that there is a function $f \in k[T]^W$ such that $f(z_1) \neq f(z_2)$. Given isomorphism (24), this means that there is a function $\tilde{f} \in k[G]^G$ such that $\tilde{f}(z_1) \neq \tilde{f}(z_2)$. On the other hand, (36) and property (ii) imply that
\[ c_{\text{Ad}_G,i}(z_1) = c_{\text{Ad}_G,i}(z_2) \quad \text{for every } i. \]

Therefore, $\tilde{f}$ is not in the subalgebra of $k[G]^G$ generated by $C_{\text{Ad}_G}$, i.e., the latter is not a generating set of $k[G]^G$.

The following two examples show that one indeed can find $G$, $z_1$, and $z_2$ sharing properties (i) and (ii).

Examples 4.1.

(1) Let $G = H \times H$ where $H$ is a connected adjoint semisimple algebraic group. Let $T = S \times S$ where $S$ is a maximal torus of $H$. Let $W_S$ be the Weyl group of $H$ acting naturally on $S$. Take any two elements $a, b \in S$ that are not in the same $W_S$-orbit and put $z_1 := (a, b)$, $z_2 := (b, a) \in T$. As $W = W_S \times W_S$, property (i)
holds. On the other hand, clearly, for every \( i = 1, 2 \), the spectrum of \( \text{Ad}_G z_i \) is the union of the spectra of \( \text{Ad}_H a \) and \( \text{Ad}_H b \); whence property (ii) holds.

(2) In this example \( G \) is simple, namely, \( G = \text{PGL}_3 \). Let \( \alpha_1, \alpha_2 \in X(T) \) be the simple roots of \( T \) with respect to \( B \). As the map \( T \to G^2_m, t \mapsto (t^{\alpha_1}, t^{\alpha_2}) \), is surjective (in fact, an isomorphism), for every \( u, v \in k \), \( uv \neq 0 \), there are \( z_1, z_2 \in T \) such that \( z_1^{\alpha_1} = u, z_1^{\alpha_2} = v \) and \( z_2^{\alpha_1} = v, z_2^{\alpha_2} = u \). For these \( z_1, z_2 \), property (ii) holds as the set of roots of \( G \) with respect to \( T \) is \( \{ \pm \alpha_1, \pm \alpha_2, \pm (\alpha_1 + \alpha_2) \} \). Now take \( u \) and \( v \) such that all elements \( u, u^{-1}, v, v^{-1}, uv, u^{-1}v^{-1} \) are pairwise different. Then property (i) holds as there are no \( w \in W \) such that \( w(\alpha_1) = \alpha_2 \) and \( w(\alpha_2) = \alpha_1 \).

5. Rational cross-sections

Recall from [Ste1, 2.14, 2.15] that an element \( x \in G \) is called strongly regular if its centralizer \( G_x \) is a maximal torus. Such \( x \) is regular and semisimple. Strongly regular elements form a dense open subset \( G_0 \) of \( G \) stable with respect to the conjugating action of \( G \). Every \( G \)-orbit in \( G_0 \) is regular and closed in \( G \). We put

\[
(G//G)_0 := \pi_G(G_0) \quad \text{and} \quad T_0 := T \cap G_0.
\]
Abusing the notation, we denote \( \pi_G|_{G_0} \) still by \( \pi_G \):

\[
\pi_G : G_0 \to (G//G)_0. \tag{37}
\]

**Lemma 5.1.**

(i) \((G//G)_0\) is an open smooth subset of \( G//G \).

(ii) \( \pi_G|_{T_0} : T_0 \to (G//G)_0 \) is a surjective étale map.

(iii) \((G//G)_0, \pi_G)\) is the geometric quotient for the action of \( G \) on \( G_0 \).

**Proof.** Since \( G//G \) is normal and all fibers of \( \pi_G \) are of constant dimension and irreducible, \( \pi_G \) is an open map (see [Bor, AG.18.4]). Hence \((G//G)_0\) is open in \( G//G \).

As every element of \( G_0 \) is semisimple, it is conjugate to an element of \( T_0 \); whence \( \pi_G|_{T_0} \) is surjective.

The set \( T_0 \) is open in \( T \) and \( W \)-stable. For every point \( t \in T_0 \), we have \( G_t = T \), hence the \( W \)-stabilizer of \( t \) is trivial. Thus, the action of \( W \) on \( T_0 \) is set theoretically free. Since \( T \) is smooth, \( G//G \) is normal, and \((G//G, \pi_G|_T)\) is the quotient for the action of \( W \) on \( T \); hence \((G//G)_0, \pi_G)\) is étale and hence \((G//G)_0\) is smooth. This proves (i) and (ii).

By (ii) the map \( \pi_G : G_0 \to (G//G)_0 \) is separable and surjective. As its fibers are \( G \)-orbits and \((G//G)_0\) is normal, (iii) follows from [Bor, 6.6]. \( \square \)

The group \( W \) acts on \( G/T \times T_0 \) diagonally with the action on the first factor defined by formula (1). The group \( G \) acts on \( G/T \times T_0 \) via left translations of the first factor. These two actions commute with each other.

Consider the \( G \)-equivariant morphism

\[
\psi : G/T \times T_0 \to G_0, \quad (gT, t) \mapsto gt g^{-1}. \tag{38}
\]

The proofs of Lemma 5.2 and Corollary 5.4 reproduce that from my letter \([P_2]\).

**Lemma 5.2.** \( \psi \) is a surjective étale map.

**Proof.** As every \( G \)-orbit in \( G_0 \) intersects \( T_0 \), surjectivity of \( \psi \) follows from (38).

Take a point \( z \in G/T \times T_0 \). We shall prove that \( d\psi_z \) is an isomorphism. As \( G/T \times T_0 \) and \( G_0 \) are smooth, this is equivalent to proving that \( \psi \) is étale at \( z \). Using that \( \psi \) is \( G \)-equivariant, we may assume that \( z = (eT, s), s \in T_0 \).

\[CROSS-SECTIONS, QUOTIENTS, AND REPRESENTATION RINGS \ 15\]
Let $U_\alpha$ be the one-dimensional unipotent root subgroup of $G$ corresponding to a root $\alpha$ with respect to $T$ and let $\theta_\alpha : G_\alpha \to U_\alpha$ be the isomorphism of groups such that
\[ t\theta_\alpha(x)t^{-1} = \theta_\alpha(t^\alpha x) \quad \text{for all } t \in T, x \in G_\alpha, \]
see [Bor, IV.13.18]. Put
\[ C_\alpha := \{ (\theta_\alpha(x)T, s) \in G/T \times T_0 \mid x \in G_\alpha \}, \]
\[ D := \{ (eT, t) \in G/T \times T_0 \mid t \in T_0 \}. \]

The linear span of all $T_{z,G}$'s and $T_{z,D}$ is $T_{z,G/T \times T_0}$. We have
\[ \psi(\theta_\alpha(x)T, s) = \theta_\alpha(x)s\theta_\alpha(x)^{-1} = \theta_\alpha(x)s\theta_\alpha(-x) \]
\[ = \theta_\alpha(x)\theta_\alpha(-s^\alpha x)s = \theta_\alpha((1 - s^\alpha)x)s. \] (39)

Since $s$ is regular, $s^\alpha \neq 1$. Hence (39) shows that $\psi$ maps the curve $C_\alpha$ isomorphically onto the curve
\[ \psi(C_\alpha) = \{ \theta_\alpha((1 - s^\alpha)x)s \mid x \in G_\alpha \}. \]

Clearly, $\psi(D) = T_0$ and $\psi|_D : D \to T_0$ is the isomorphism. But $T_{e,G}$ is the linear span of $T_{e,T}$ and the tangent spaces of the curves $\{ \theta_\alpha(x) \mid x \in G_\alpha \}$ at $e$. Hence $T_{e,G}$ is the linear span of $T_{z,T}$ and the tangent spaces at $e$ of the right translations of these curves by $s$. This implies the claim of the lemma. \(\square\)

**Corollary 5.4.** $(G_0, \psi)$ is the quotient for the action of $W$ on $G/T \times T_0$.

**Proof.** By [Bor, Prop. II.6.6], as $G_0$ is normal and $\psi$ is surjective and separable, it suffices to prove that the fibers of $\psi$ are $W$-orbits.

Using (1) and (38) one immediately verifies that the fibers of $\psi$ are $W$-stable. On the other hand, let $\psi(g_1T, t_1) = \psi(g_2T, t_2)$. By (38) this equality is equivalent to $(g_1^{-1}g_2)t_2(g_1^{-1}g_2)^{-1} = t_1$. By [Ste1, 6.1] the latter, in turn, implies that there is an element $w \in W$ such that
\[ wt_2w^{-1} = (g_1^{-1}g_2)t_2(g_1^{-1}g_2)^{-1}. \]
Hence $g_1^{-1}g_2 = wz$ for $z \in G_{t_2}$. As $t_2 \in T$ is strongly regular, this yields that $z \in T$. Therefore,
\[ (g_2T, t_2) = (g_1wT, t_1w^{-1}t_1w) = w^{-1}(g_1T, t_1). \]
Thus, $(g_1T, t_1)$ and $(g_2T, t_2)$ are in the same $W$-orbit. This completes the proof. \(\square\)

Let $\pi_2 : G/T \times T_0 \to T_0$ be the second projection. Clearly, $(T_0, \pi_2)$ is the geometric quotient for the action of $G$ on $G/T \times T_0$. As $\psi$ is $G$-equivariant, this implies that there is a morphism $\phi : T_0 \to G//G$ such that the following diagram is commutative:
\[ \begin{array}{ccc}
G/T \times T_0 & \xrightarrow{\psi} & G_0 \\
\pi_2 \downarrow & & \downarrow \pi_G \\
T_0 & \xrightarrow{\phi} & (G//G)_0 
\end{array} \] (40)

**Lemma 5.5.**

(i) $\phi = \pi_G|_{T_0}$.

(ii) For every point $t \in T_0$, the restriction of $\psi$ to $\pi_2^{-1}(t)$ is a $G$-equivariant isomorphism $\pi_2^{-1}(t) \to \pi_G^{-1}(\phi(t))$. 


Proof. Take a point \( t \in T_0 \). Commutativity of diagram (40) and formula (38) yield that \( \pi_G(t) = \pi_G(\psi(eT, t)) = \phi(\pi_2(eT, t)) = \phi(t) \). This proves (i).

Commutativity of diagram (40) implies that the restriction of \( \psi \) to \( \pi_2^{-1}(t) \) is a \( G \)-equivariant morphism \( \pi_2^{-1}(t) \to \pi_G^{-1}(\phi(t)) \). As both \( \pi_2^{-1}(t) \) and \( \pi_G^{-1}(\phi(t)) \) are the \( G \)-orbits and the stabilizers of their points are conjugate to \( T \), this morphism is bijective. By Lemma 5.2 it is separable. Then, as \( \pi_G^{-1}(\phi(t)) \) is normal, it is an isomorphism. This proves (ii).

\[ \square \]

Proof of Theorem 1.5. Assume that (i) holds. Let \( \sigma : G \sslash G \to G \) be a rational section of \( \pi_G \), i.e., a section of \( \pi_G \) over a dense open subset \( U \) of \( (G \sslash G)_0 \). Let \( S \) be the closure of \( \sigma(U) \). Put \( \rho := \pi_G|_S : S \to (G \sslash G)_0 \). Since \( \pi_G \circ \sigma = \text{id} \), shrinking \( U \) if necessary, we may assume that, for every point \( x \in U \), the following properties hold:

(a) \( S \cap \pi_G^{-1}(x) \) is a single point \( s \);

(b) \( d\rho_s \) is an isomorphism.

Since \( \psi \) is an isomorphism on the fibers of \( \pi_2 \), property (a) implies that, for every point \( t \in \phi^{-1}(U) \), the \( W \)-stable closed set \( \psi^{-1}(S) \) intersects \( \pi_2^{-1}(t) \) at a single point. From this we infer that \( \psi^{-1}(S) \) has a unique irreducible component \( \tilde{S} \) whose image under \( \pi_2 \) is dense in \( T_0 \) — the argument is the same as that in the proof of Claim 2(i) in Section 2. Due to the uniqueness, this \( \tilde{S} \) is \( W \)-stable.

Let \( V \subseteq \pi_2(\tilde{S}) \cap \phi^{-1}(U) \) be an open subset of \( T_0 \). Replacing it, if necessary, by \( \bigcap_{w \in W} w(V) \), we may assume that \( V \) is \( W \)-stable. Let \( \tilde{\rho} : \pi_2^{-1}(V) \cap \tilde{S} \to V \) be the restriction of \( \pi_2 \) to \( \pi_2^{-1}(V) \cap \tilde{S} \). Then \( \tilde{\rho} \) is a bijective \( W \)-equivariant morphism. We claim that it is separable and hence, by Zariski’s Main Theorem, an isomorphism (as \( V \) is normal). Indeed, take a point \( \tilde{s} \in \pi_2^{-1}(V) \cap \tilde{S} \) and put \( \pi_2(\tilde{s}) = t \). Then property (b), Lemma 5.2, and commutativity of diagram (40) imply that \( d\tilde{\rho}_s : T_{\tilde{s}, \tilde{S}} \to T_{t, V} \) is an isomorphism; whence the claim by [Bor, AG.17.3].

Thus, \( \tilde{\rho}^{-1} : V \to \pi_2^{-1}(V) \cap \tilde{S} \) is a rational \( W \)-equivariant section of \( \pi_2 \). Its composition with the first projection \( G/T \times T_0 \to G/T \) is then a \( W \)-equivariant rational map \( T \to G/T \). This proves (i) \( \Rightarrow \) (ii).

Conversely, assume that (ii) holds. Let \( \gamma : T \to G/T \) be a \( W \)-equivariant rational map. Then \( \varsigma := (\gamma, \text{id}) : T_0 \to G/T \times T_0 \) is a \( W \)-equivariant rational section of \( \pi_2 \), i.e., a section of \( \pi_2 \) over a dense open subset \( V \) of \( T_0 \). We may assume that \( \varsigma(V) \) and \( S := \psi(\varsigma(V)) \) are open in their closures; \( \varsigma : V \to \varsigma(V) \) is an isomorphism, and the subsets \( \phi(V), \pi_G(S) \) of \( G \sslash G \) are open and coincide. As above, we may also assume that \( V \) is \( W \)-stable.

Taking into account that \( \varsigma \) is \( W \)-equivariant, \( \varsigma(V) \cap \pi_2^{-1}(t) \) is a single point for every \( t \in V \), and \( \psi \) is an isomorphism on the fibers of \( \pi_2 \), we conclude that property (a) holds for every \( x \in \varsigma(V) \). Thus, \( \rho := \pi_G|_S : S \to \phi(V) \) is a bijection.

We claim that \( \rho \) is separable, hence an isomorphism as \( \phi(V) \) is normal by Lemma 5.1(i). Indeed, \( d\rho_x \) is an isomorphism by Lemma 5.5(i) and Lemma 5.1(ii). Let \( s = \psi(\varsigma(t)) \in S \). Since the restriction of \( (d\pi_2)_x(t) \) to \( T_{\varsigma(t), \varsigma(V)} \) is an isomorphism with \( T_{t, V} \), commutativity of diagram (40) and Lemma 5.2 imply that property (b) holds; whence the claim.

Thus, the composition of \( \rho^{-1} : \phi(V) \to S \) and the identical embedding \( S \hookrightarrow G \) is a rational section of \( \pi_G \). This proves (ii) \( \Rightarrow \) (i) and completes the proof of the theorem.

\[ \square \]

Recall some definitions from [CTKPR, Sects. 2.2, 2.3, and 3].

Let \( P \) be a linear algebraic group acting on a variety \( X \) and let \( Q \) be its closed subgroup. \( X \) is called a \((P, Q)\)-variety if in \( X \) there is a dense open \( P \)-stable subset \( U \), called a friendly subset, such that a geometric quotient \( \pi_U : U \to U/P \) exists and...
\( \pi_U \) becomes the second projection \( P/Q \times \hat{U}/P \to \hat{U}/P \) after a surjective étale base change \( \hat{U}/P \to U/P \). If there is a rational section of \( \pi_U \), one says that \( X \) admits a rational section. \( X \) is called a versal \((P, Q)\)-variety if \( U/P \) is irreducible and, for every its dense open subset \((U/P)_0 \) \((P, Q)\)-variety \( Y \), there is a friendly subset \( V \) of \( Y \) such that \( \pi_V \) is induced from \( \pi_U \) by a base change \( V \to (U/H)_0 \).

Now we shall give the characteristic free proofs of the following two statements proved in [CTKPR] for \( \text{char} = 0 \).

**Lemma 5.6.** Let \( X \) be an irreducible variety endowed with a faithful action of a finite algebraic group \( H \). Then

(i) \( X \) is an \((H, \{e\})\)-variety;

(ii) \( X \) is a versal \((H, \{e\})\)-variety in each of the following cases:

(a) \( X \) is a free \( H \)-module;

(b) \( X \) is a linear algebraic torus and \( H \) acts by its automorphisms.

**Proof.** (i) Replacing \( X \) by its smooth locus, we may assume that \( X \) is smooth.

As \( H \) is finite, for any nonempty open affine subset \( U \) of \( X \), the set \( \bigcap_{h \in H} h(U) \) is \( H \)-stable, affine, and open in \( X \). So, replacing \( X \) by it, we may assume that \( X \) is affine. Then, as is well known, for the action of \( H \) on \( X \) there is a geometric quotient \( \pi: X \to X/H \) (see, e.g., [Bor, Prop. 6.15]). As \( X \) is normal, \( X/H \) is normal as well.

Since \( H \) is finite and the action is faithful, the points with trivial stabilizer form an open subset of \( X \). Replacing \( X \) by it, we may also assume that the action is set-theoretically free, i.e., the \( H \)-stabilizer of every point of \( X \) is trivial. As \( X \) and \( X/G \) are normal, by [G3, Exp. I, Théorème 9.5(ii)] and [Bou1, V.2.3, Cor. 4] this property implies that \( \pi \) is étale and hence \( X/H \) is smooth.

For every base change \( \beta: Y \to X/H \) of \( \pi \), the group \( H \) acts on \( X \times X/H Y \) via \( X \).

As the action of \( H \) on \( X \) is set-theoretically free, taking \( Y = X \) and \( \beta = \pi \), we obtain

\[
X \times X/H X = \bigsqcup_{h \in H} h(D) \quad \text{where} \quad D := \{(x, x) \mid x \in X\}.
\]

From this we deduce that in the commutative diagram

\[
\begin{array}{ccc}
H \times X & \xrightarrow{\alpha} & X \times X/H X \\
\downarrow & & \downarrow \\
X & & X
\end{array}
\]

where \( \alpha(h, x) := (h(x), x) \) and two other maps are the second projections, \( \alpha \) is an \( H \)-equivariant isomorphism. This proves (i).

The proofs of (ii)(a) and (ii)(b) are the same as that of (b) and (d) in [CTKPR, Lemma 3.3] if one replaces in them the references to [CTKPR, Theorem 2.12] (whose proof is based on the assumption \( \text{char} k = 0 \)) by the references to statement (i) of the present lemma. \( \square \)

**Remark 5.7.** The proof of (i) shows that, for finite group actions, set-theoretical freeness coincides with that in the sense of GIT, [MF, Def. 0.8].

**Lemma 5.8.** \( G \) is a versal \((G, T)\)-variety.

**Proof.** First, we shall give a characteristic free proof of the fact that \( G \) is a \((G, T)\)-variety (the proof given in [CTKPR] is based on the assumption \( \text{char} k = 0 \)). By Lemma 5.1(iii) this is equivalent to proving the existence of a dense open subset \( U \) of \( (G/G)_0 \) such that after a surjective étale base change \( U' \to U \) morphism (37) becomes the second projection \( G/T \times U' \to U' \).
Consider the base change of $\pi_G$ in (40) by means of $\phi$. Lemma 5.5(i) implies that

$$F := G_0 \times_{(G/\!/G)} T_0 = \{(g, t) \in G_0 \times T_0 \mid G(g) = G(t)\} \quad (41)$$

(see (5)). We have the canonical map corresponding to commutative diagram (40):

$$\gamma := \psi \times \text{id}: G/T \times T_0 \to F, \quad (gT, t) \mapsto (tg^{-1}, t). \quad (42)$$

It follows from (41) that $\gamma$ is surjective; whence $F$ is irreducible. But if for $t \in T_0$ and $g_1, g_2 \in G$ we have $g_1tg_1^{-1} = g_2tg_2^{-1}$, then $g_1T = g_2T$ since $G_t = T$. Therefore, $\gamma$ is bijective. Lemma 5.2 and (42) show that $d\gamma_x$ is injective for every $x \in G/T \times T_0$. Hence if $\gamma(x)$ lies in the smooth locus $F_{\text{sm}}$ of $F$, then $d\gamma_x$ is the isomorphism. This implies that $\gamma$ is separable and then, by ZARISKI’s Main Theorem, that the restriction of $\gamma$ to $\gamma^{-1}(F_{\text{sm}})$ is an isomorphism $\gamma^{-1}(F_{\text{sm}}) \to F_{\text{sm}}$.

As $F_{\text{sm}}$ is $G$-stable and $\gamma$ is $G$-equivariant, $\gamma^{-1}(F_{\text{sm}})$ is a $G$-stable open subset of $G/T \times T_0$. Hence it is of the form $G/T \times U'$ for an open subset $U'$ of $T_0$. But Lemmas 5(ii) and 5.5(i) imply that $U := \phi'(U')$ is open in $(G/\!/G)_0$ and $\phi|_{U'}: U' \to U$ is étale. This proves that after the étale base change $\phi|_{U'}: U' \to U$ morphism (37) becomes the second projection $G/T \times U' \to U'$. Hence $G$ is a $(G, T)$-variety.

By Lemma 5.6(b), $T$ is a versal $(W, \{e\})$-variety. The characteristic free arguments from [CTKPR, proof of Prop. 4.3(c)] then show that this fact implies versality of the $(G, T)$-variety $G$. This completes the proof of the lemma. □

Proof of Theorem 1.6. By [BT, Prop. (2.24)(ii)] the isogeny $\tau$ is central. Hence, the natural morphism $\hat{G}\hat{T} \to G\!/T$ is an isomorphism by [Bor, Props. 6.13, 22.5].

Using $\tau$, every action of $G$ lifts naturally to an action of $\hat{G}$ on the same variety. In particular, $G$ is endowed with an action of $\hat{G}$. But $G$ is a $(G, T)$-variety by Lemma 5.8(i). As $\hat{G}\hat{T}$ and $G/T$ are isomorphic, this means that $G$ is a $((G, T))$-variety. But $\hat{G}$ is a versal $(\hat{G}, \hat{T})$-variety (by Lemma 5.8) that admits a rational section (by Lemma 5.1(iii) and [Ste1, Theorem 1.4]). Hence by [CTKPR, Theorem 3.6(a)] (the proof of this result is characteristic free) every $(\hat{G}, \hat{T})$-variety admits a rational section. In particular, this is so for $G$. This proves (ii) and completes the proof of the theorem. □

Proof of Corollary 1.7. By Theorem 1.6 there is a rational section $\sigma: G\!/G \dasharrow G$ of $\pi_G$. The closure of the image of $\sigma$ is then the desired cross-section $S$ (see Section 6). □

6. COMPLEMENTS

1. Cross-sections versus sections. If there is a section $\sigma: G\!/G \to G$ of $\pi_G$, then $\sigma(G\!/G)$ is a cross-section in $G$. Indeed, as $\text{id}_{k[G\!/G]}$ is the composition of the homomorphisms

$$k[G\!/G] \xrightarrow{\pi_G} k[G] \xrightarrow{\sigma} k[G\!/G],$$

$\pi_G$ is surjective; by [G2, Cor. 4.2.3] this means that $\sigma$ is a closed embedding.

The cross-section $\sigma(G\!/G)$ has the property that the restriction of $\pi_G$ to $\sigma(G\!/G)$ is an isomorphism $\sigma(G\!/G) \to G\!/G$. Conversely, let $S$ be a cross-section in $G$. If $\pi_G|S: S \to G\!/G$ is separable, then, since $\pi_G|S$ is bijective and $G\!/G$ is normal, ZARISKI’s Main Theorem implies that $\pi_G|S$ is an isomorphism (cf. [Bor, AG 18.2]). So in this case the composition of $(\pi_G|S)^{-1}$ with the identity embedding $S \hookrightarrow G$ is a section of $\pi_G$ whose image is $S$. In particular, if char $k = 0$, then every cross-section in $G$ is the image of a section of $\pi_G$. If char $k > 0$, then in the general case this is not true.
Example 6.1. Let $G = \text{SL}_3$ and char $k = p > 0$. Then for every integer $d > 0$,

$$S := \{ s(a_1, a_2) \mid a_1, a_2 \in k \}, \quad \text{where} \quad s(a_1, a_2) := \begin{pmatrix} a_1 & a_2 & 1 \\ 1 & a_1^p - a_1 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

is a cross-section in $G$ such that $\pi_G|S$ is not separable. Indeed, as $\chi_\pi, (g)$ is the sum of principal $i$-minors of $g \in G$, we have (see Lemma 2.1(ii))

$$(\lambda \circ \rho)(s(a_1, a_2)) = (a_1^d, a_1(a_1^d - a_1) - a_2). \quad \Box$$

Similarly, if $\sigma : G//G \rightarrow G$ is a rational section of $\pi_G$ and $S$ is the closure of its image, then $S$ is a rational cross-section in $G$ such that the restriction of $\pi_G$ to it is a birational isomorphism with $G//G$.

6.1. 2. Group action on the set of cross-sections. Let $\text{Mor}(G//G, G)$ be the group of morphisms $G//G \rightarrow G$. If $S$ is a cross-section in $G$ and $\gamma \in \text{Mor}(G//G, G)$, then

$$\gamma(S) := \{ \gamma(s)s\gamma(s)^{-1} \mid s \in S \}$$

is a cross-section in $G$. This defines an action of $\text{Mor}(G//G, G)$ on the set of cross-sections in $G$. If char $k = 0$, then by [FM] this action is transitive. If char $k > 0$, then in the general case this is not true: in Example 6.1, STEINBERG’s section and $S$ are not in the same $\text{Mor}(G//G, G)$-orbit since, for the former, the restriction of $\pi_G$ is separable [Ste1, Theorem 1.5], but, for the latter, it is not.

3. Lifting $T$-action. By Theorem 3.5 there is an action of $T$ on $T//W$ determining a structure of a toric variety. This action cannot be lifted to $T$ making $\pi_T : T \rightarrow T//W$ equivariant. This follows from the fact that the automorphism group of the underlying variety of $T$ is $\text{GL}_r(\mathbb{Z}) \ltimes T$.

4. Image of a rational cross-section in $G$ under $\pi_G$. Assume that $\tau$ is not birational (for char $k = 0$, this means that $G$ is not simply connected). Let $S$ be a rational cross-section in $G$ such that $\varphi := \pi_G|S : S \rightarrow G//G$ is a birational isomorphism ($S$ exists by Corollary 1.7). Let $D$ be the closure of the complement of $\pi_G(S)$ in $G//G$.

The following shows that $D$ cannot be “too small”.

**Theorem 6.2.** $\text{codim}_{G//G} D = 1$.

**Proof.** Assume the contrary. Take a function $f \in k[S]$. Since $\varphi$ is a birational isomorphism, $f = \varphi^*(h)$ for some function $h \in k(G//G)$. As $G//G$ is normal, $h$ is regular at every point of $(G//G) \setminus D$, see [P1, Sect. 2, Lemma]. Using again that $G//G$ is normal, we then deduce from $\text{codim}_{G//G} D > 1$ that $h \in k[G//G]$. As $G$ and $G//G$ are affine, this shows that $\varphi$ is an isomorphism. Hence $S$ is a (global) cross-section in $G$. As $\tau$ is not birational, the latter contradicts Theorem 1.2(i). $\Box$

5. Questions. Given Theorem 1.8, it would be interesting to construct explicitly an example of a $W$-equivariant rational map $T \rightarrow G/T$.

- Is there such a map defined on $T_0$?
- Is there a rational section of $\pi_G$ defined on $(G//G)_0$?

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CROSS-SECTIONS, QUOTIENTS, AND REPRESENTATION RINGS

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