RESULTS ABOUT PERSYMMETRIC MATRICES OVER $\mathbb{F}_2$ AND RELATED EXPONENTIALS SUMS

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1. EXPONENTIAL SUMS AND RANK OF PERSYMMETRIC MATRICES OVER $\mathbb{F}_2$

Abstract. Let $\mathbb{K}$ be the field of Laurent Series $\mathbb{F}_2((T^{-1}))$. We compute in particular exponential sums in $\mathbb{K}$ of the form

$$\sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ)$$

where $t$ is in the unit interval of $\mathbb{K}$, by showing that they only depend on the rank of some associated persymmetric matrices with entries in $\mathbb{F}_2$. A matrix $[\alpha_{i,j}]$ is persymmetric if $\alpha_{i,j} = \alpha_{i+j}$ for $i+j = r+s$. Besides we establish rank properties of a partition of persymmetric matrices. We use these results to compute the number $\Gamma_i$ of persymmetric matrices over $\mathbb{F}_2$ of rank $i$. We recover in this particular a general formula given by D.E. Daykin. Our proof is as indicated very different since it relies on rank properties of a partition of persymmetric matrices. We also prove that the number $R$ of representations in $\mathbb{F}_2[T]$ of 0 as a sum of quadratic forms associated to the exponential sums is given by an integral over the unit interval, and is a linear combination of the $\Gamma_i$s. We then compute explicitly the number $R$. Similar results are also obtained for $n+1$ dimensional $\mathbb{K}$-vector spaces. We finish the paper by computing explicitly the number of rank $i$ matrices of the form $\left[\begin{array}{cccc} A & \cdots & \cdots & \cdots \\ \vdots & \ddots & \vdots & \vdots \\ \cdots & \cdots & A \end{array}\right]$, where $A$ is persymmetric.

1.1. An outline of the main results.
Theorem 1.1. The number $\Gamma_i^{s \times k}$ of persymmetric $s \times k$ matrices over $\mathbb{F}_2$

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{k+s-3} & \alpha_{k+s-2} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{k+s-2} & \alpha_{k+s-1}
\end{pmatrix}
\]

of rank $i$ is given by

\[
\begin{cases}
1 & \text{if } i = 0, \\
3 \cdot 2^{(i-1)} & \text{if } 1 \leq i \leq s - 1, \\
2^{k+s-1} - 2^{2s-2} & \text{if } i = s (s \leq k).
\end{cases}
\]

Remark 1.2. David E. Daykin has already proved this result over any finite field $\mathbb{F}$ with the number 2 in the formula replaced by $|\mathbb{F}|$, and the number 3 replaced by $|\mathbb{F}|^2 - 1$. Our proof is different and proper to the finite field with two elements.

Theorem 1.3. Let $(j_1, j_2, j_3, j_4) \in \mathbb{N}^4$, then

\[
\# \left( \begin{array}{c}
j_1 \\
j_2 \\
j_3 \\
j_4 \\
\end{array} \right) \mathbb{F}/\mathbb{F}_{k+s-1} = \begin{cases}
1 & \text{if } j_1 = j_2 = j_3 = j_4 = 0, \\
2^{2j-1} & \text{if } j_1 = j_2 = j_3 = j_4 \in \{j, j+1\}, 1 \leq j \leq s - 1, \\
2^{2j-3} & \text{if } j_1 = j - 2, j_2 = j + 1, j_3 = j - 1, j_4 = j, 2 \leq j \leq s, \\
2^{k+s-1} - 2^{s-1} & \text{if } j_1 = j_2 = j_3 = j_4 = s, \\
0 & \text{otherwise},
\end{cases}
\]

Theorem 1.4. Let $h_{s,k}(t) = h(t)$ be the quadratic exponential sum in $\mathbb{P}$ defined by

\[
t \in \mathbb{P} \longmapsto \sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \in \mathbb{Z}.
\]

Then

\[
h(t) = 2^{k+2s-r(D_{s,k}(t))}
\]

and

\[
\int_{\mathbb{P}} h^q(t) dt = 2^{(q-1)(k+s)+1} \sum_{i=0}^{s} \Gamma_i^{s \times k} 2^{-qi}.
\]

Let $R$ denote the number of solutions $(Y_1, Z_1, \ldots, Y_q, Z_q)$ of the polynomial equation

\[
Y_1 Z_1 + Y_2 Z_2 + \ldots + Y_q Z_q = 0
\]

satisfying the degree conditions

\[
\deg Y_i \leq k - 1, \quad \deg Z_i \leq s - 1 \quad \text{for } 1 \leq i \leq q.
\]

Then

\[
R = \int_{\mathbb{P}} h^q(t) dt
\]
Theorem 1.5. Let $g_{s,k}(t) = g(t)$ be the quadratic exponential sum in $\mathbb{P}$ defined by
\[
te \in \mathbb{P} \mapsto \sum_{\deg Y = k-1} \sum_{\deg Z = s-1} E(tYZ) \in \mathbb{Z}.
\]
Then
\[
g(t) = \begin{cases} 
  2^{s+k-j-2} & \text{if } r(D(s-1) \times (k-1)(t)) = r(D(s) \times (k-1)(t)) = r(D(s) \times k(t)) = j, \\
  -2^{s+k-j-2} & \text{if } r(D(s-1) \times (k-1)(t)) = r(D(s) \times (k-1)(t)) = r(D(s) \times k(t)) = j \text{ and } r(D(s) \times k(t)) = j + 1, \\
  0 & \text{otherwise},
\end{cases}
\]
and
\[
\int_{\mathbb{P}} g^2(t) dt = 2^{s(k-2) + 2^{q(j+1) - 6j}} \sum_{j=0}^{s-1} \# \left( \frac{j}{j} \right)_{\mathbb{P} / \mathbb{P}_{k+s-1}} \cdot 2^{-2q}.
\]

Theorem 1.6. Let $g_{m,k}(t, \eta) = g(t, \eta)$ be the exponential sum in $\mathbb{P} \times \mathbb{P}$ defined by
\[
(t, \eta) \in \mathbb{P} \times \mathbb{P} \mapsto \sum_{\deg Y \leq k-1} \sum_{\deg Z \leq m} E(tYZ) \sum_{\deg U = 0} E(\eta YU) \in \mathbb{Z}.
\]
Then
\[
g(t, \eta) = \begin{cases} 
  2^{k+m+1 - r(D((1+m) \times k)(t)))} & \text{if } r(D((1+m) \times k)(t)) = r(D(1+m) \times k(t)), \\
  0 & \text{otherwise},
\end{cases}
\]
and
\[
\int_{\mathbb{P}} \int_{\mathbb{P}} g^2(t, \eta) dtd\eta = 2^{q(k+m+1) - 2^{k-m}} \inf_{i=0}^{\inf(k,1+m)} \sigma_{i,k}^{1+m} \times k \cdot 2^{-2q}.
\]

Theorem 1.7. Let $f_{m,k}(t, \eta) = f(t, \eta)$ be the exponential sum in $\mathbb{P} \times \mathbb{P}$ defined by
\[
(t, \eta) \in \mathbb{P} \times \mathbb{P} \mapsto \sum_{\deg Y \leq k-1} \sum_{\deg Z \leq m} E(tYZ) \sum_{\deg U \leq 0} E(\eta YU) \in \mathbb{Z}.
\]
Then
\[
f(t, \eta) = 2^{k+m+2 - r(D(1+m) \times k(t, \eta))}
\]
and
\[
\int_{\mathbb{P}} \int_{\mathbb{P}} f^2(t, \eta) dtd\eta = 2^{q(k+m+2) - 2^{k-m}} \inf_{i=0}^{\inf(k,2+m)} \Gamma_i^{1+m} \times k \cdot 2^{-2q}.
\]

Theorem 1.8. We have the following formula for all $0 \leq i \leq \inf(k, 2 + m)$
\[
\Gamma_i^{1+m} \times k = (2^i - 2) \cdot \Gamma_{i-1}^{(1+m) \times k} + 2 \Gamma_i^{(1+m) \times k}.
\]
The case $k = 2$

\[
\Gamma_i^{1+m} \times 2 = \begin{cases} 
1 & \text{if } i = 0, \\
9 & \text{if } i = 1, \\
2^{4+m} - 10 & \text{if } i = 2.
\end{cases}
\]

The case $m = 0$, $k \geq 2$

\[
\Gamma_i^{1} \times k = \begin{cases} 
1 & \text{if } i = 0, \\
3 \cdot (2^k - 1) & \text{if } i = 1, \\
2^{2k} - 3 \cdot 2^k + 2 & \text{if } i = 2.
\end{cases}
\]

The case $m = 1$, $k \geq 3$

\[
\Gamma_i^{1+1} \times k = \begin{cases} 
1 & \text{if } i = 0, \\
2^k + 5 & \text{if } i = 1, \\
11 \cdot (2^k - 1) & \text{if } i = 2, \\
2^{2k+1} - 3 \cdot 2^{k+2} + 2^4 & \text{if } i = 3.
\end{cases}
\]

The case $3 \leq k \leq 1+m$

\[
\Gamma_i^{1+m} \times k = \begin{cases} 
1 & \text{if } i = 0, \\
2^k + 5 & \text{if } i = 1, \\
3 \cdot 2^{k+2i-4} + 21 \cdot 2^{3i-5} & \text{if } 2 \leq i \leq k - 1, \\
2^{2k+m} - 5 \cdot 2^{3k-5} & \text{if } i = k.
\end{cases}
\]

The case $2 \leq m \leq k - 2$

\[
\Gamma_i^{1+m} \times k = \begin{cases} 
1 & \text{if } i = 0, \\
2^k + 5 & \text{if } i = 1, \\
3 \cdot 2^{k+2i-4} + 21 \cdot 2^{3i-5} & \text{if } 2 \leq i \leq m, \\
11 \cdot [2^{k+2m-2} - 2^{3m-2}] & \text{if } i = m + 1, \\
2^{2k+m} - 3 \cdot 2^{k+2m} + 2^{3m+1} & \text{if } i = m + 2.
\end{cases}
\]

Theorem 1.9. Let $\Gamma_i^{1+m} \times k$ denote the number of matrices of the form $[A \; B]$ of rank $i$ such that $A$ is a $(1+m) \times k$ persymmetric matrix and $B$ is a $n \times k$ matrix over $\mathbb{F}_2$, and where $\Gamma_i^{1+m} \times k$ denotes the number of $(1+m) \times k$ persymmetric matrices over $\mathbb{F}_2$ of rank $i$.

Then $\Gamma_i^{1+m} \times k$ expressed as a linear combination of the $\Gamma_i^{(1+m) \times k}$ is equal to

\[
\sum_{j=0}^{n} 2^{(n-j)-(i-j)} a_j^{(n)} \prod_{l=1}^{j} (2^{k_l} - 2^{i_l}) \cdot \Gamma_i^{(1+m) \times k} \quad \text{for} \quad 0 \leq i \leq \inf(k, n + m + 1)
\]

where

\[
a_j^{(n)} = \sum_{s=0}^{j-1} (-1)^s \prod_{l=0}^{j-s-1} 2^{(n-j)-(i-j)} - 2^{i_l} \cdot 2^{(n-j) + \frac{j-1}{2} + \frac{j}{2} + \frac{j-1}{2} + \frac{j}{2} + \frac{j-1}{2}} \quad \text{for} \quad 1 \leq j \leq n-1.
\]

We set

\[
a_0^{(n)} = a_n^{(n)} = 1
\]

and $\Gamma_i^{(1+m) \times k} = 0$ if $i - j \notin \{0, 1, 2, \ldots, \inf(k, 1+m)\}$. 

Corollary 1.10. We have the following formulas for \( n = 1, 2, 3, 4, 5 \):

\[
\Gamma_i^{[1+m]} \times k = 2^i \Gamma_i^{[1+m]} + (2^k - 2^{i-1}) \cdot \Gamma_{i-1}^{[1+m]} \quad \text{for} \quad 0 \leq i \leq \inf(k, 2 + m),
\]

\[
\Gamma_i^{[2+m]} \times k = 2^{2i} \Gamma_i^{[1+m]} + 3 \cdot 2^{i-1} (2^k - 2^{i-1}) \cdot \Gamma_{i-1}^{[1+m]} + (2^k - 2^{i-1})(2^k - 2^{i-2}) \cdot \Gamma_{i-2}^{[1+m]} \quad \text{for} \quad 0 \leq i \leq \inf(k, 3 + m),
\]

\[
\Gamma_i^{[3+m]} \times k = 2^{3i} \Gamma_i^{[1+m]} + 7 \cdot 2^{i-1} (2^k - 2^{i-1}) \cdot \Gamma_{i-1}^{[1+m]} + 7 \cdot 2^{i-2} (2^k - 2^{i-1})(2^k - 2^{i-2}) \cdot \Gamma_{i-2}^{[1+m]} + (2^k - 2^{i-1})(2^k - 2^{i-2})(2^k - 2^{i-3}) \Gamma_{i-3}^{[1+m]} \quad \text{for} \quad 0 \leq i \leq \inf(k, 4 + m),
\]

\[
\Gamma_i^{[4+m]} \times k = 2^{4i} \Gamma_i^{[1+m]} + 15 \cdot 2^{i-1} (2^k - 2^{i-1}) \cdot \Gamma_{i-1}^{[1+m]} + 15 \cdot 2^{i-3} (2^k - 2^{i-1})(2^k - 2^{i-2}) \cdot \Gamma_{i-2}^{[1+m]} + 15 \cdot 2^{i-4} (2^k - 2^{i-1})(2^k - 2^{i-2})(2^k - 2^{i-3}) \Gamma_{i-3}^{[1+m]} + (2^k - 2^{i-1})(2^k - 2^{i-2})(2^k - 2^{i-3})(2^k - 2^{i-4}) \Gamma_{i-4}^{[1+m]} \quad \text{for} \quad 0 \leq i \leq \inf(k, 5 + m),
\]

\[
\Gamma_i^{[5+m]} \times k = 2^{5i} \Gamma_i^{[1+m]} + 31 \cdot 2^{i-1} (2^k - 2^{i-1}) \cdot \Gamma_{i-1}^{[1+m]} + 155 \cdot 2^i (2^k - 2^{i-1})(2^k - 2^{i-2}) \cdot \Gamma_{i-2}^{[1+m]} + 155 \cdot 2^{i-6} (2^k - 2^{i-1})(2^k - 2^{i-2})(2^k - 2^{i-3}) \Gamma_{i-3}^{[1+m]} + 155 \cdot 2^{i-6} (2^k - 2^{i-1})(2^k - 2^{i-2})(2^k - 2^{i-3})(2^k - 2^{i-4}) \Gamma_{i-4}^{[1+m]} + (2^k - 2^{i-1})(2^k - 2^{i-2})(2^k - 2^{i-3})(2^k - 2^{i-4})(2^k - 2^{i-5}) \Gamma_{i-5}^{[1+m]} \quad \text{for} \quad 0 \leq i \leq \inf(k, 6 + m),
\]

Theorem 1.11. Let \( f_{m,k}(t, \eta_1, \eta_2, \ldots, \eta_n) \) be the exponential sum in \( \mathbb{P}^{n+1} \) defined by

\[
\sum_{\deg Y \leq k-1} \sum_{\deg Z \leq m} E(tYZ) \sum_{\deg U_1 \leq 0} E(\eta_1 YU_1) \sum_{\deg U_2 \leq 0} E(\eta_2 YU_2) \cdots \sum_{\deg U_n \leq 0} E(\eta_n YU_n).
\]

Set

\[
(t, \eta_1, \eta_2, \ldots, \eta_n) = (\sum_{i \geq 1} \alpha_i T^{-i}, \sum_{i \geq 1} \beta_{1i} T^{-i}, \ldots, \sum_{i \geq 1} \beta_{ni} T^{-i}) \in \mathbb{P}^{n+1}.
\]

Then

\[
f_{m,k}(t, \eta_1, \eta_2, \ldots, \eta_n) = 2^{k+m+n+1-r} D^{[1+m] \times k}_{(t, \eta_1, \eta_2, \ldots, \eta_n)}(t, \eta_1, \eta_2, \ldots, \eta_n)
\]

where

\[
D^{[1+m] \times k}_{(t, \eta_1, \eta_2, \ldots, \eta_n)}
\]
denotes the following \((1 + n + m) \times k\) matrix

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{1+m} & \alpha_{2+m} & \alpha_{3+m} & \ldots & \alpha_{k+m-1} & \alpha_{k+m} \\
\beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{k-1} & \beta_k \\
\beta_2 & \beta_3 & \beta_4 & \ldots & \beta_{k-1} & \beta_k \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{n1} & \beta_{n2} & \beta_{n3} & \ldots & \beta_{nk-1} & \beta_{nk}
\end{pmatrix}.
\]

Then the number denoted by \(R_q(n, k, m)\) of solutions

\((Y_1, Z_1, U_1^{(1)}, U_2^{(2)}, \ldots, U_n^{(2)}, Y_2, Z_2, U_1^{(2)}, U_2^{(2)}, \ldots, U_n^{(2)}, \ldots, Y_q, Z_q, U_1^{(q)}, U_2^{(q)}, \ldots, U_n^{(q)})\)

of the polynomial equations

\[
\begin{align*}
Y_1 Z_1 + Y_2 Z_2 + \ldots + Y_q Z_q &= 0 \\
Y_1 U_1^{(1)} + Y_2 U_1^{(2)} + \ldots + Y_q U_1^{(q)} &= 0 \\
Y_1 U_2^{(1)} + Y_2 U_2^{(2)} + \ldots + Y_q U_2^{(q)} &= 0 \\
&\vdots \\
Y_1 U_n^{(1)} + Y_2 U_n^{(2)} + \ldots + Y_q U_n^{(q)} &= 0
\end{align*}
\]

satisfying the degree conditions

\[
\deg Y_i \leq k - 1, \quad \deg Z_i \leq m, \quad \deg U_j^i \leq 0, \quad \text{for} \quad 1 \leq j \leq n \quad 1 \leq i \leq q
\]

is equal to the following integral over the unit interval in \(\mathbb{K}^{n+1}\)

\[
\int_{\mathbb{P}^{n+1}} f_{m,k}^q(t, \eta_1, \eta_2, \ldots, \eta_n) dt d\eta_1 d\eta_2 \ldots d\eta_n.
\]

Observing that \(f_{m,k}^q(t, \eta_1, \eta_2, \ldots, \eta_n)\) is constant on cosets of \(\mathbb{P}_{k+m} \times \mathbb{P}_k\), the above integral is equal to

\[
2^{q(k+m+n+1)-(n+1)} k^{-m} \inf_{i=0}^{n} \frac{i_{1+m}}{i_{1+m}} \gamma_k 2^{-iq} = R_q(n, k, m)
\]

Example. The number \(R_q(0, k, m)\) of solutions \((Y_1, Z_1, \ldots, Y_q, Z_q)\) of the polynomial equation

\[
Y_1 Z_1 + Y_2 Z_2 + \ldots + Y_q Z_q = 0
\]

satisfying the degree conditions

\[
\deg Y_i \leq k - 1, \quad \deg Z_i \leq m \leq k - 1 \quad \text{for} \ 1 \leq i \leq q.
\]
is equal to the following integral

\[
\int_{P} \left[ \sum_{\deg Y \leq k-1} \sum_{\deg Z \leq m} E(tYZ) \right]^q dt = 2^{(q-1)(k+m+1)+1} \sum_{i=0}^{1+m} \Gamma_i^{(1+m)\times k} 2^{-qi}
\]

\[
= \left\{ \begin{array}{ll}
2^k + 2^{1+m} - 1 & \text{if } q = 1, \\
2^{2k} + 3 \cdot (m + 1) \cdot 2^{k+m} & \text{if } q = 2, \\
2^{(q-1)(k+m+1)+1} \left[ 1 + \frac{3(2^{2-q})^m}{2^{2q-2}} + (2^{k+m} - 2^{2m})2^{-q(1+m)} \right] & \text{if } 3 \leq q.
\end{array} \right.
\]

**Example.** The number \( \Gamma_i^{\left[ \frac{1}{1+2} \right] \times 3} \) of rank i matrices of the form

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 \\
\alpha_2 & \alpha_3 & \alpha_4 \\
\alpha_3 & \alpha_4 & \alpha_5 \\
\beta_1 & \beta_2 & \beta_3 \\
\end{pmatrix}
\]

is equal to

\[
\begin{cases}
1 & \text{if } i = 0, \\
13 & \text{if } i = 1, \\
66 & \text{if } i = 2, \\
176 & \text{if } i = 3.
\end{cases}
\]

The number \( R_q(1, 3, 2) \) of solutions \( (Y_1, Z_1, U_1, \ldots, Y_q, Z_q, U_q) \) of the polynomial equations

\[
\begin{aligned}
&Y_1 Z_1 + Y_2 Z_2 + \ldots + Y_q Z_q = 0 \\
&Y_1 U_1 + Y_2 U_2 + \ldots + Y_q U_q = 0
\end{aligned}
\]

satisfying the degree conditions

\( \deg Y_i \leq 2, \quad \deg Z_i \leq 2, \quad \deg U_i \leq 0 \quad \text{for } 1 \leq i \leq q \)

is equal to the following integral

\[
\int_{P} \int_{P} \left[ \sum_{\deg Y \leq 2} \sum_{\deg Z \leq 2} E(tYZ) \sum_{\deg U \leq 0} E(\eta YU) \right]^q dtd\eta = 2^{7q-8} \sum_{i=0}^{3} \Gamma_i^{\left[ \frac{1}{1+2} \right] \times 3} 2^{-iq}
\]

\[
= 2^{4q-8} \cdot \left[ 2^{3q} + 13 \cdot 2^{2q} + 66 \cdot 2^q + 176 \right].
\]

**Example.** The number \( \Gamma_i^{\left[ \frac{5}{1+2} \right] \times 4} \) of rank i matrices of the form

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\
\alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\
\beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\
\beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \\
\beta_{31} & \beta_{32} & \beta_{33} & \beta_{34} \\
\beta_{41} & \beta_{42} & \beta_{43} & \beta_{44} \\
\beta_{51} & \beta_{52} & \beta_{53} & \beta_{54} \\
\end{pmatrix}
\]
is equal to

\[
\begin{cases}
1 & \text{if } i = 0, \\
561 & \text{if } i = 1, \\
65670 & \text{if } i = 2, \\
3731208 & \text{if } i = 3, \\
63311424 & \text{if } i = 4.
\end{cases}
\]

The number \(R_3(5, 4, 2)\) of solutions

\[(Y_1, Z_1, U_1^{(1)}, U_2^{(1)}, U_3^{(1)}, U_4^{(1)}, U_5^{(1)}, Y_2, Z_2, U_1^{(2)}, U_2^{(2)}, U_3^{(2)}, U_4^{(2)}, U_5^{(2)}, Y_3, Z_3, U_1^{(3)}, U_2^{(3)}, U_3^{(3)}, U_4^{(3)}, U_5^{(3)})\]

of the polynomial equations

\[
\begin{align*}
Y_1Z_1 + Y_2Z_2 + Y_3Z_3 &= 0, \\
Y_1U_1^{(1)} + Y_2U_1^{(2)} + Y_3U_1^{(3)} &= 0, \\
Y_1U_2^{(1)} + Y_2U_2^{(2)} + Y_3U_2^{(3)} &= 0, \\
Y_1U_3^{(1)} + Y_2U_3^{(2)} + Y_3U_3^{(3)} &= 0, \\
Y_1U_4^{(1)} + Y_2U_4^{(2)} + Y_3U_4^{(3)} &= 0, \\
Y_1U_5^{(1)} + Y_2U_5^{(2)} + Y_3U_5^{(3)} &= 0,
\end{align*}
\]
satisfying the degree conditions

\[
degY_i \leq 3, \quad degZ_i \leq 2, \quad degU_i \leq 0 \quad \text{for} \quad 1 \leq j \leq 5 \quad 1 \leq i \leq 3
\]
is equal to the following integral over the unit interval in \(\mathbb{K}^6\)

\[
\int_{\mathbb{K}^6} f_3(t, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5) dt d\eta_1 d\eta_2 d\eta_3 d\eta_4 d\eta_5 = 2^{10} \sum_{i=0}^{4} \Gamma_i^{[5/2]} \times 2^{-i^3} = 24413824.
\]
2. EXPONENTIAL SUMS AND RANK OF DOUBLE PERSYMMETRIC MATRICES OVER $\mathbb{F}_2$

Résultats. Soit $K^2$ l'espace vectoriel de dimension 2 où $K$ dénote le corps des séries de Laurent formelles $\mathbb{F}_2((T^{-1}))$. Nous calculons en particulier des sommes exponentielles (dans $K^2$) de la forme

$$\sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U \leq s+m-1} E(\eta YU)$$

où $(t, \eta)$ est dans la boule unité de $K^2$.

Nous démontrons qu'elles dépendent uniquement du rang de matrices doubles persymétriques avec des entrées dans $\mathbb{F}_2$, c'est-à-dire des matrices de la forme

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

où $A$ est une matrice $s \times k$ persymétrique et $B$ une matrice $(s+m) \times k$ persymétrique (une matrice $[a_{i,j}]$ est persymétrique si $a_{i,j} = a_{r,s}$ pour $i+j = r+s$). En outre, nous établissons plusieurs formules concernant des propriétés de rang de partitions de matrices doubles persymétriques, ce qui nous conduit à une formule récurrente du nombre $\Gamma_{i}^s$ des matrices de rang $i$ de la forme

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

où $A$ est persymétrique et $b$ une matrice ligne avec entrées dans $\mathbb{F}_2$. Nous déduisons de cette formule récurrente que si $0 \leq i \leq \min(s-1, k-1)$, le nombre $\Gamma_{i}^s$ dépend uniquement de $i$. D'autre part, si $i \geq s+1, k \geq i$, $\Gamma_{i}^s$ peut être calculé à partir du nombre $\Gamma_{i'}^{s'}$ de matrices de rang $(s'+1)$ de la forme

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$$

où $A'$ est une matrice $s' \times k'$ persymétrique et $B'$ une matrice $(s'+m') \times k'$ persymétrique, où $s', m'$ et $k'$ dépendent de $i$, $s$, $m$ et $k$. La preuve de ce résultat est basée sur une formule (donnée dans [4]) du nombre de matrices de rang $i$ de la forme

$$\begin{pmatrix} A \\ B \end{pmatrix}$$

où $A$ est persymétrique et $b$ une matrice ligne avec entrées dans $\mathbb{F}_2$. Nous montrons également que le nombre $R$ de représentations dans $\mathbb{F}_2[T]$ des équations polynomiales

$$\begin{cases} YZ + Y_1Z_1 + \ldots + Y_{q-1}Z_{q-1} = 0 \\ YU + Y_1U_1 + \ldots + Y_{q-1}U_{q-1} = 0 \end{cases}$$

associées aux sommes exponentielles

$$\sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U \leq s+m-1} E(\eta YU)$$

est donné par une intégrale sur la boule unité de $K^2$ et est une combinaison linéaire de $\Gamma_{i}^s$ pour $i \geq 0$. Nous pouvons alors calculer explicitement le nombre $R$. 


Theorem 2.1. Let $q$ be a rational integer $\geq 1$, then

$$g_{k,s,m}(t, \eta) = g(t, \eta) = \sum_{\text{deg}Y \leq k-1} \sum_{\text{deg}Z \leq s-1} E(tYZ) \sum_{\text{deg}U \leq s+m-1} E(\eta YU) = 2^{2s+m+k-r} \Gamma_i \left[ \frac{s}{s+m} \right] \left[ k \right],$$

$$\int_{\mathbb{F}_2^2} g_{k,s,m}^q(t, \eta) dtd\eta = 2^{2s+m+k(q-1)} \cdot 2^{-k+2} \cdot \sum_{i=0}^{\inf(2s+m,k)} \Gamma_i \left[ \frac{s}{s+m} \right] \left[ k \right] \cdot 2^{-qi}.$$
Theorem 2.3. Let \( (2.3) \)

such that \( A \) is a \( s \times k \) persymmetric matrix and \( B \) a \((s + m) \times k\) persymmetric matrix with entries in \( F_2 \):

\[
\Gamma_i^{[s + m] \times k} = 2 \cdot \Gamma_{i-1}^{[s-1+(m+1)] \times k} + 4 \cdot \Gamma_{i-1}^{[s+(m-1)] \times k} - 8 \cdot \Gamma_{i-2}^{[s-1+m] \times k} + \Delta_i^{[s + m] \times k}
\]

where the remainder \( \Delta_i \) is equal to

\[
\sigma_{i,i,i}^{[s + m] \times k} - 3 \cdot \sigma_{i-1,i-1,i-1}^{[s + m] \times k} + 2 \cdot \sigma_{i-2,i-2,i-2}^{[s + m] \times k}
\]

Recall that

\[
\sigma_{i,i,i}^{[s - 1 + m] \times k}
\]

is equal to the cardinality of the following set

\[
\left\{ (t, \eta) \in \mathbb{F}/\mathbb{F}_{k+1} \times \mathbb{F}/\mathbb{F}_{k+m+1} \mid r(D^{[s-1+m] \times k}(t, \eta)) = r(D^{[s+m] \times k}(t, \eta)) = r(D^{[s+m] \times k}(t, \eta)) = i \right\}.
\]

**Theorem 2.3.** Let \( s \geq 2 \) and \( m \geq 0 \), we have in the following two cases:

**The case** \( 1 \leq k \leq 2s + m - 2 \)
Theorem 2.4. The remainder $\Delta_i^k \left[ \frac{s}{s+m} \right]$ in the recurrent formula is equal to

\[
\begin{aligned}
&1 \quad \text{if } i = 0, \quad k \geq 1, \\
-12 \cdot \Gamma_2^i \left[ \frac{s-1}{s-1+m} \right] &- 12 \cdot \Gamma_2^i \left[ \frac{s-1}{s-1+m} \right] + 2 \quad \text{if } i = 2, \quad k \geq 3, \\
-14 \cdot \Gamma_2^i \left[ \frac{s-1}{s-1+m} \right] &- 14 \cdot \Gamma_2^i \left[ \frac{s-1}{s-1+m} \right] + 8 \cdot \Gamma_2^i \left[ \frac{s-1}{s-1+m} \right] \quad \text{if } i = 2, \quad k \geq 2, \\
-14 \cdot \Gamma_2^{i-2} \left[ \frac{s-1}{s-1+m} \right] &+ 8 \cdot \Gamma_2^{i-2} \left[ \frac{s-1}{s-1+m} \right] \quad \text{if } i \geq 3, \quad k \geq i + 1, \\
-14 \cdot \Gamma_2^{i-2} \left[ \frac{s-1}{s-1+m} \right] &+ 8 \cdot \Gamma_2^{i-2} \left[ \frac{s-1}{s-1+m} \right] \quad \text{if } i \geq 3, \quad k = i, \\
-14 \cdot \Gamma_2^{i-2} \left[ \frac{s-1}{s-1+m} \right] &+ 8 \cdot \Gamma_2^{i-2} \left[ \frac{s-1}{s-1+m} \right] \quad \text{if } i = 2s + m - 2, \quad k \geq i, \\
-14 \cdot \Gamma_2^{i-2} \left[ \frac{s-1}{s-1+m} \right] &+ 8 \cdot \Gamma_2^{i-2} \left[ \frac{s-1}{s-1+m} \right] \quad \text{if } i = 2s + m - 2, \quad k \geq i.
\end{aligned}
\]

Theorem 2.5. We have

\[
\begin{aligned}
\Delta_i^k \left[ \frac{s}{s+m} \right] &\times (i+1) = \Delta_i^k \left[ \frac{s}{s+m} \right] \quad \text{for } i \in [0, 2s + m - 3], \quad k \geq i + 1, \\
\Delta_i^k \left[ \frac{s}{s+m} \right] &\times i = \Delta_i^k \left[ \frac{s}{s+m} \right] \quad \text{for } i \in \{2s + m - 2, 2s + m - 1, 2s + m\}, \quad k \geq i.
\end{aligned}
\]
**Theorem 2.6.** We have for all \( m \geq 0 \)

\[
\Gamma_j^{[s+m] \times (k+1)} - \Gamma_j^{[s+m] \times k} = 0 \quad \text{if} \quad 0 \leq j \leq s-1, \; k > j.
\]

We have in the cases \( m \in \{0,1\} \)

\[
\Gamma_j^{[s] \times (k+1)} - \Gamma_j^{[s] \times k} = \begin{cases} 
3 \cdot 2^{k+s-1} & \text{if } j = 0, \; k > s, \\
21 \cdot 2^{k+s+3j-4} & \text{if } 1 \leq j \leq s-1, \; k > s+j, \\
3 \cdot 2^{2k+2s-2} - 3 \cdot 2^{k+4s-4} & \text{if } j = s, \; k > 2s,
\end{cases}
\]

In the case \( m \geq 2 \)

\[
\Gamma_j^{[s+m] \times (k+1)} - \Gamma_j^{[s+m] \times k} = \begin{cases} 
2^{k+s-1} & \text{if } j = 0, \; k > s, \\
3 \cdot 2^{k+s+2j-3} & \text{if } 1 \leq j \leq m-1, \; k > s+j, \\
11 \cdot 2^{k+s+2m-3} & \text{if } j = m, \; k > s+m,
\end{cases}
\]

**Theorem 2.7.** We have for \( m \geq 1 \)

\[
\Gamma_j^{[s+m] \times k} = 8^j \cdot \Gamma_j^{[s+(m-(j-1))] \times (k-(j-1))} \quad \text{if } 1 \leq j \leq m, \; k \geq s+j,
\]

\[
\Gamma_j^{[(s+(m-(j-1))] \times (s+1)} = 2^{4s+(m-(j-1))} - 3 \cdot 2^{3s-1} + 2^{2s-1} \quad \text{if } 1 \leq j \leq m, \; k = s+j,
\]

\[
\Gamma_j^{[s+(m-(j-1))] \times (k-(j-1))} = 3 \cdot 2^{k-j+s} + 21 \cdot [2^{3s-1} - 2^{2s-1}] \quad \text{if } 1 \leq j \leq m-1, \; k > s+j,
\]

\[
\Gamma_j^{[s+1] \times (k-(m-1))} = 11 \cdot 2^{k-m+s} + 21 \cdot 2^{3s-1} - 11 \cdot 2^{2s-1} \quad \text{if } j = m, \; k > s+m.
\]
Theorem 2.11. We have

\[ (2.19) \quad \Gamma\left[ \frac{s+m}{s+m+j} \right] \times k = 8^{2j+m} \cdot \Gamma\left[ \frac{s-j}{s-j+1} \right] \times (k-m-2j) \quad \text{if} \quad 0 \leq j \leq s-1, \ k \geq s+m+1+j, \]

\[ (2.20) \quad \Gamma\left[ \frac{s-j}{s-j+1} \right] \times (s-j+1) = 2^{4s-4j} - 3 \cdot 2^{3s-3j-1} + 2^{2s-2j-1} \quad \text{if} \quad 0 \leq j \leq s-1, \ k = s+m+1+j, \]

\[ (2.21) \quad \Gamma\left[ \frac{s-j}{s-j+1} \right] \times (k-m-2j) = 21 \cdot [2^{k-m-3j+s-1} + 2^{3s-3j-1} - 5 \cdot 2^{2s-2j-1}] \quad \text{if} \quad 0 \leq j \leq s-2, \ k > s+m+1+j, \]

\[ (2.22) \quad \Gamma\left[ \frac{1}{2} \right] \times (k-m-2s+2) = 2^{2(k-m)-4s+4} - 3 \cdot 2^{k-m-2s+2} + 2 \quad \text{if} \quad j = s-1, \ k > 2s+m. \]

Theorem 2.9. We have

\[ (2.23) \quad \Gamma\left[ \frac{s}{s} \right] \times k = \begin{cases} 
1 & \text{if} \ i = 0, \ k \geq 1, \\
21 \cdot 2^{3i-4} - 3 \cdot 2^{2i-3} & \text{if} \ 1 \leq i \leq s-1, \ k > i, \\
3 \cdot 2^{k+s-1} + 21 \cdot 2^{3s-4} - 27 \cdot 2^{2s-3} & \text{if} \ i = s, \ k > s, \\
21 \cdot [2^{k-2s+3i-4} + 2^{3i-4} - 5 \cdot 2^{4i-2s-5}] & \text{if} \ s+1 \leq i \leq 2s-1, \ k > i, \\
2^{2k+2s-2} - 3 \cdot 2^{k+4s-4} + 2^{6s-5} & \text{if} \ i = 2s, \ k > 2s. 
\end{cases} \]

Theorem 2.10. We have

\[ (2.24) \quad \Gamma\left[ \frac{s}{s} \right] \times i = \begin{cases} 
2^{2s+2i-2} - 3 \cdot 2^{3i-4} + 2^{2i-3} & \text{if} \ 1 \leq i \leq s, \\
2^{2s+2i-2} - 3 \cdot 2^{3i-4} + 2^{4i-2s-5} & \text{if} \ s+1 \leq i \leq 2s. 
\end{cases} \]

Theorem 2.11. We have

\[ (2.25) \quad \Gamma\left[ \frac{s}{s+1} \right] \times k = \begin{cases} 
1 & \text{if} \ i = 0, \ k \geq 1, \\
21 \cdot 2^{3i-4} - 3 \cdot 2^{2i-3} & \text{if} \ 1 \leq i \leq s-1, \ k > i, \\
2^{k+s-1} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3} & \text{if} \ i = s, \ k > s, \\
11 \cdot 2^{k+s-1} + 21 \cdot 2^{3s-1} - 53 \cdot 2^{2s-1} & \text{if} \ i = s+1, \ k > s+1, \\
21 \cdot [2^{k-2s+3i-5} + 2^{3i-4} - 5 \cdot 2^{4i-2s-6}] & \text{if} \ s+2 \leq i \leq 2s, \ k > i, \\
2^{2k+2s-1} - 3 \cdot 2^{k+4s-2} + 2^{6s-2} & \text{if} \ i = 2s+1, \ k > 2s+1. 
\end{cases} \]

Theorem 2.12. We have
Example. Let $q = 3$, $k = 4$, $s = 3$, $m = 2$. Then

$$\Gamma_i^\left[\frac{s}{s+1}\right]_i = \begin{cases} 2^{2s+2i-1} - 2^{3s-4} + 2^{2i-3} & \text{if } 1 \leq i \leq s + 1, \\ 2^{2s+2i-2} - 2^{3s-4} + 2^{4i-2s-6} & \text{if } s + 2 \leq i \leq 2s + 1. \end{cases}$$

Theorem 2.13. We have for $m \geq 2$

$$\Gamma_i^\left[\frac{s}{s+m}\right]_i = \begin{cases} 1 & \text{if } i = 0, k \geq 1, \\ 2^{2} \cdot 2^{3i} - 3 \cdot 2^{2i-3} + 2^{2s-1} & \text{if } 1 \leq i \leq s - 1, k > i, \\ 3 \cdot 2^{2k} + 2^{2s-1} - 3 \cdot 2^{2i} & \text{if } i = s, k > s, \\ 2^{k+s-1} & \text{if } s + 1 \leq i \leq s + m - 1, k > i, \\ 11 \cdot 2^{k+s} + 2^{2s-4} - 2^{3i-4} - 2^{3i-4} & \text{if } i = s + m, k > s + m, \\ 2^{2k} + 2^{s+m-2} & \text{if } s + m + 1 \leq i \leq 2s + m - 1, k > i, \\ 2^{2k} + 2^{s+m-2} - 3 \cdot 2^{2s+3m-4} + 2^{6s+3m-5} & \text{if } i = 2s + m, k > 2s + m. \end{cases}$$

Theorem 2.14. We have for $m \geq 2$

$$\Gamma_i^\left[\frac{s}{s+m}\right]_i = \begin{cases} 2^{2s+2i+m-2} - 3 \cdot 2^{3i} + 2^{2i-3} & \text{if } 1 \leq i \leq s + 1, \\ 2^{s+2i+m-2} - 3 \cdot 2^{3i} + 2^{3i-4} & \text{if } s + 2 \leq i \leq s + m + 1, \\ 2^{2s+2i+m-2} - 3 \cdot 2^{3i} + 2^{4i-2s-5} & \text{if } s + m + 2 \leq i \leq 2s + m. \end{cases}$$

Theorem 2.15. We denote by $R_q(k, s, m)$ the number of solutions $(Y_1, Z_1, U_1, \ldots, Y_q, Z_q, U_q)$ of the polynomial equations

$$\begin{cases} Y_1 Z_1 + Y_2 Z_2 + \ldots + Y_q Z_q = 0, \\ Y_1 U_1 + Y_2 U_2 + \ldots + Y_q U_q = 0, \end{cases}$$

satisfying the degree conditions

$$\deg Y_i \leq k - 1, \quad \deg Z_i \leq s - 1, \quad \deg U_i \leq s + m - 1 \quad \text{for } 1 \leq i \leq q.$$ 

Then

$$R_q(k, s, m) = \int_{\mathbb{P} \times \mathbb{P}} g_k^{(s, k, m)}(t, \eta) dt d\eta$$

$$= 2^{(2s+m+k)(q-1)} \cdot 2^{-k+2} \cdot \sum_{i=0}^{\inf(2s+m,k)} \Gamma_i^\left[\frac{s}{s+m}\right]_i \cdot 2^{-qi}.$$ 

Example. Let $q = 3$, $k = 4$, $s = 3$, $m = 2$. Then

$$\Gamma_i^\left[\frac{3}{3+2}\right]_i = \begin{cases} 1 & \text{if } i = 0, \\ 9 & \text{if } i = 1, \\ 78 & \text{if } i = 2, \\ 648 & \text{if } i = 3, \\ 15648 & \text{if } i = 4. \end{cases}$$
Hence the number $R_3(4, 3, 2)$ of solutions
\[ (Y_1, Z_1, U_1, Y_2, Z_2, U_2, Y_3, Z_3, U_3) \] of the polynomial equations
\[
\begin{align*}
Y_1 Z_1 + Y_2 Z_2 + Y_3 Z_3 &= 0, \\
Y_1 U_1 + Y_2 U_2 + Y_3 U_3 &= 0,
\end{align*}
\]
satisfying the degree conditions
\[ \text{deg}Y_i \leq 3, \quad \text{deg}Z_i \leq 2, \quad \text{deg}U_i \leq 4 \quad \text{for} \quad 1 \leq i \leq 3 \]
is equal to
\[
\int_{P \times P} g_{4, 3, 2}(t, \eta) dtd\eta = 2^{22} \cdot \sum_{i=0}^{4} \Gamma_i^{\left[\frac{3}{2}\right]} \times 2^{-3i}
\]
\[
= 2^{22} \cdot [1 + 9 \cdot 2^{-3} + 78 \cdot 2^{-6} + 648 \cdot 2^{-9} + 15648 \cdot 2^{-12}]
\]
\[
= 35356672.
\]

Example. Let $q = 4$, $k = 6$, $s = 5$, $m = 0$. Then
\[
\Gamma_i^{\left[\frac{5}{3}\right]} = 6 \begin{cases} 
1 & \text{if } i = 0, \\
9 & \text{if } i = 1, \\
78 & \text{if } i = 2, \\
648 & \text{if } i = 3, \\
5280 & \text{if } i = 4, \\
42624 & \text{if } i = 5, \\
999936 & \text{if } i = 6.
\end{cases}
\]

Hence the number $R_4(6, 5, 0)$ of solutions
\[ (Y_1, Z_1, U_1, Y_2, Z_2, U_2, Y_3, Z_3, U_3, Y_4, Z_4, U_4) \] of the polynomial equations
\[
\begin{align*}
Y_1 Z_1 + Y_2 Z_2 + Y_3 Z_3 + Y_4 Z_4 &= 0, \\
Y_1 U_1 + Y_2 U_2 + Y_3 U_3 + Y_4 U_4 &= 0,
\end{align*}
\]
satisfying the degree conditions
\[ \text{deg}Y_i \leq 5, \quad \text{deg}Z_i \leq 4, \quad \text{deg}U_i \leq 4 \quad \text{for} \quad 1 \leq i \leq 4 \]
is equal to
\[
\int_{P \times P} g_{4, 5, 0}(t, \eta) dtd\eta
\]
\[
= 2^{44} \cdot \sum_{i=0}^{6} \Gamma_i^{\left[\frac{5}{3}\right]} \times 2^{-4i}
\]
\[
= 2^{44} \cdot [1 + 9 \cdot 2^{-4} + 78 \cdot 2^{-8} + 648 \cdot 2^{-12} + 5280 \cdot 2^{-16} + 42624 \cdot 2^{-20} + 999936 \cdot 2^{-24}]
\]
\[
= 37014016 \cdot 2^{20}.
\]
Example. The fraction of square double persymmetric matrices $\begin{bmatrix} s \\ s + m \end{bmatrix} \times (2s + m)$ matrices which are invertible is equal to

\[
\frac{\Gamma \left[ \begin{bmatrix} s \\ 2s + m \end{bmatrix} \times (2s + m) \right]}{\sum_{i=0}^{2s+m} \Gamma \left[ \begin{bmatrix} i \\ s + m \end{bmatrix} \times (2s + m) \right]} = \frac{3}{8}.
\]
3. Exponential sums and rank of triple persymmetric matrices over $\mathbb{F}_2$

Abstract. Notre travail concerne une généralisation des résultats obtenus dans:
Exponential sums and rank of double persymmetric matrices over $\mathbb{F}_2$
arXiv: 0711.1937.

Soit $\mathbb{K}^3$ le $\mathbb{K}$-espace vectoriel de dimension 3 où $\mathbb{K}$ dénote le corps des séries de
Laurent formelles $\mathbb{F}_2(T^{-1})$. Nous calculons en particulier des sommes exponentielles (dans $\mathbb{K}^3$) de la forme
$\sum_{\text{deg } Y \leq k-1} \sum_{\text{deg } Z \leq s-1} E(tYZ) \sum_{\text{deg } U \leq s+m-1} E(\eta YU) \sum_{\text{deg } V \leq s+m+l-1} E(\xi YV)$
où $(t, \eta, \xi)$ est dans la boule unité de $\mathbb{K}^3$.

Nous démontrons qu'elles dépendent uniquement du rang de matrices triples persymétriques avec des entrées dans $\mathbb{F}_2$, c'est-à-dire des matrices de la forme $\begin{pmatrix} A & B & C \\ B & \alpha & \beta \\ C & \beta & \gamma \end{pmatrix}$
où $A$ est une matrice $s \times k$ persymétrique, $B$ une matrice $(s + m) \times k$ per-
symétrique et $C$ une matrice $(s + m + l) \times k$ persymétrique (une matrice $(A_{i,j})$
est persymétrique si $A_{i,j} = A_{r,s}$ pour $i+j = r+s$). En outre, nous établissons plusieurs formules concernant des propriétés de rang de partitions de matrices triples persymétriques, ce qui nous conduit à une formule récurrente du nombre
$\Gamma_i^{s+m+l} \times k$
de matrices de rang $i$ de la forme $\begin{pmatrix} \frac{A}{E} \end{pmatrix}$ Nous déduisons de cette
formule récurrente que si $0 \leq i \leq \inf(s-1, k-1)$, le nombre $\Gamma_i^{s+m+l} \times k$
dépend uniquement de $i$. D'autre part, si $i \geq 2s + m + 1, k \geq i$, $\Gamma_i^{s+m+l} \times k'$ peut être
calculé à partir du nombre $\Gamma_{i'}^{s'+m'+l'} \times k'$ de matrices de rang $(2s'+m'+1)$ de
la forme $\begin{pmatrix} A' \\ B' \\ C' \end{pmatrix}$ où $A'$ est une matrice $s' \times k'$ persymétrique, $B'$ une matrice $(s'+m') \times k'$ persymétrique et $C'$ une matrice $(s'+m'+l') \times k'$ où $s'$, $m'$, $l'$
et $k'$ dépendent de $i, s, m, l$ et $k$. La preuve de ce résultat est basée sur une formule
du nombre de matrices de rang $i$ de la forme $\begin{pmatrix} \frac{A}{B} \end{pmatrix}$ où $A$ est une matrice
double persymétrique et $b.$ une matrice ligne avec entrées dans $\mathbb{F}_2$. Nous mon-
trons également que le nombre $R$ de représentations dans $\mathbb{F}_2[T]$ des équations
polynomiales
$$\begin{cases}
YZ + Y_1Z_1 + \ldots + Y_{q-1}Z_{q-1} = 0 \\
YU + Y_1U_1 + \ldots + Y_{q-1}U_{q-1} = 0 \\
YV + Y_1V_1 + \ldots + Y_{q-1}V_{q-1} = 0
\end{cases}$$
associées aux sommes exponentielles
$\sum_{\text{deg } Y \leq k-1} \sum_{\text{deg } Z \leq s-1} E(tYZ) \sum_{\text{deg } U \leq s+m-1} E(\eta YU) \sum_{\text{deg } V \leq s+m+l-1} E(\xi YV)$
est donné par une intégrale sur la boule unité de $\mathbb{K}^3$ et est une combinaison linéaire
de $\Gamma_i^{s+m+l} \times k$ pour $i \geq 0$. Nous pouvons alors calculer explicitement le nombre
$R$. Notre article est, pour des raisons de longueur, limité au cas $m \geq 0$, $l = 0$. 
RESULTS ABOUT PERSYMMETRIC MATRICES OVER $F_2$ AND RELATED EXPONENTIAL SUMS

ABSTRACT. Our work concerns a generalization of the results obtained in : Exponential sums and rank of double persymmetric matrices over $F_2$ arXiv : 0711.1937.

Let $K^3$ be the 3-dimensional vectorspace over $K$ where $K$ denotes the field of Laurent Series $F_2((T^{-1}))$. We compute in particular exponential sums, (in $K^3$) of the form

$$
\sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U \leq s+m-1} E(\eta YU) \sum_{\deg V \leq s+m+1-l} E(\xi YV)
$$

where $(t, \eta, \xi)$ is in the unit interval of $K^3$. We show that they only depend on the rank of some associated triple persymmetric matrices with entries in $F_2$, that is matrices of the form $\begin{bmatrix} \frac{A}{B} & \frac{C}{D} \end{bmatrix}$ where $A$ is a $s \times k$ persymmetric matrix, $B$ a $(s+m) \times k$ persymmetric matrix and $C$ is a $(s+m+l) \times k$ persymmetric matrix ($A$ matrix $[a_{i,j}]$ is persymmetric if $a_{i,j} = a_{r,s}$ for $i + j = r + s$). Besides, we establish several formulas concerning rank properties of partitions of triple persymmetric matrices, which leads to a recurrent formula for the number $\Gamma_i(n+k)$ of rank $i$ matrices of the form $\begin{bmatrix} \frac{A}{B} \end{bmatrix}$. We deduce from the recurrent formula that if

$$0 \leq i \leq \inf(s-1, k-1) \text{ then } \Gamma_i \left( \begin{bmatrix} s+m & \quad k \end{bmatrix} \right)$$

depends only on $i$. On the other hand, if $i \geq 2s+m+1, k \geq i$, $\Gamma_i \left( \begin{bmatrix} s+m & \quad k \end{bmatrix} \right)$ can be computed from the number

$$\Gamma_{2s+m+1} \left( \begin{bmatrix} s'+m' & \quad k' \end{bmatrix} \right)$$

of rank $(2s'+m'+1)$ matrices of the form $\begin{bmatrix} \frac{A'}{B'} \end{bmatrix}$ where $A'$ is a $s' \times k'$ persymmetric matrix, $B'$ a $(s' + m') \times k'$ persymmetric matrix and $C'$ a $(s' + m' + l') \times k'$ persymmetric matrix , where $s, m, l$ and $k$ depend on $i, s, m, l$ and $k$. The proof of this result is based on a formula of the number of rank $i$ matrices of the form $\begin{bmatrix} \alpha \end{bmatrix}$ where $A$ is double persymmetric and $b_-$ a one-row matrix with entries in $F_2$. We also prove that the number $R$ of representations in $F_2[T]$ of the polynomial equations

$$
\begin{cases}
YZ + Y_1Z_1 + \ldots + Y_{q-1}Z_{q-1} = 0 \\
YU + Y_1U_1 + \ldots + Y_{q-1}U_{q-1} = 0 \\
YV + Y_1V_1 + \ldots + Y_{q-1}V_{q-1} = 0
\end{cases}
$$

associated to the exponential sums

$$
\sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U \leq s+m-1} E(\eta YU) \sum_{\deg V \leq s+m+1-l} E(\xi YV)
$$

is given by an integral over the unit interval of $K^3$, and is a linear combination of

$$\Gamma_n \left( \begin{bmatrix} s+m & \quad k \end{bmatrix} \right) \quad \text{for } i \geq 0.$$

We can then compute explicitly the number $R$. Our article is for reasons of length limited to the case $m \geq 0, l = 0$. 


3.1. A recurrent formula for the number of rank $i$ matrices of the form 
\[ \binom{m}{i} \], where $A$, $B$ and $C$ are persymmetric matrices over $\mathbb{F}_2$.

**Lemma 3.1.** Let $s \geq 2$, $m \geq 0$, $l \geq 0$, $k \geq 1$ and $0 \leq i \leq \inf(3s + 2m + l, k)$.

Then we have the following recurrent formula for the number $\Gamma \left[ \binom{s+m}{i} \right] \times k$ of rank $i$ matrices of the form $\binom{s+m}{i}$ such that $A$ is a $s \times k$ persymmetric matrix over $\mathbb{F}_2$, $B$ a $(s+m) \times k$ persymmetric matrix and $C$ a $(s+m+l) \times k$ persymmetric matrix

\[
\Gamma \left[ \binom{s+m}{i} \right] \times k
= \left[ 2 \cdot \Gamma \left[ \binom{s-1}{i-1} \right] \times k \right] + \left[ 4 \cdot \Gamma \left[ \binom{s-1}{i} \times k \right] \right] + \left[ 8 \cdot \Gamma \left[ \binom{s-1}{i} \times k \right] \right] - \left[ 8 \cdot \Gamma \left[ \binom{s-1}{i} \times k \right] \right] + 16 \cdot \Gamma \left[ \binom{s-1}{i} \times k \right] + 32 \cdot \Gamma \left[ \binom{s-1}{i} \times k \right] \\
+ 64 \cdot \Gamma \left[ \binom{s-1}{i} \times k \right] + \Delta \left[ \binom{s+m}{i} \times k \right]
\]

where

\[
\Delta \left[ \binom{s+m}{i} \times k \right] = \sigma \left[ \binom{s-1}{i} \times k \right] \cdot \sigma \left[ \binom{s-1}{i} \times k \right] - 7 \cdot \sigma \left[ \binom{s-1}{i} \times k \right] \cdot \sigma \left[ \binom{s-1}{i} \times k \right] + 14 \cdot \sigma \left[ \binom{s-1}{i} \times k \right] \cdot \sigma \left[ \binom{s-1}{i} \times k \right] - 8 \cdot \sigma \left[ \binom{s-1}{i} \times k \right] \cdot \sigma \left[ \binom{s-1}{i} \times k \right]
\]

Recall that

\[
\sigma \left[ \binom{s+m}{i} \times k \right] = \left\{ (t, \eta, \xi) \in \mathbb{F}/\mathbb{P}^{s+m-l-1} | r(D) \left[ \binom{s+m-1}{i} \times k \right] \right\} = i
\]

3.2. An outline of the main results in the case $m = l = 0$. 

Theorem 3.2. The number $\Gamma_i^{\left[\frac{s}{i}\right] \times k}$ of triple persymmetric $3s \times k$ matrices over $\mathbb{F}_2$

$$
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \ldots & \alpha_{s+k-3} & \alpha_{s+k-2} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \ldots & \alpha_{s+k-2} & \alpha_{s+k-1}
\end{pmatrix}
$$

of rank $i$ is given by

$$
(3.3) \begin{cases}
1 & \text{if } i = 0, \\
105 \cdot 2^{4i-6} - 21 \cdot 2^{3i-5} & \text{if } 1 \leq i \leq s - 1, \ k \geq i + 1, \\
7 \cdot 2^{k+s-1} - 7 \cdot 2^{2s} + 105 \cdot 2^{4s-6} - 21 \cdot 2^{3s-5} & \text{if } i = s, \ k \geq s + 1, \ s \geq 1, \\
147 \cdot (5 \cdot 2^{i-1} - 1) \cdot 2^{k+s+3j-6} & \text{if } i = s + j, \ 1 \leq j \leq s - 1, \ k \geq s + j + 1, \\
+21 \cdot \left[ 5 \cdot 2^{4s+4j-6} - 2^{3s+3j-5} - (155 \cdot 2^{2j-1} - 35 \cdot 2^{2s+4j-7} \right] & \text{if } i = 2s, \ k \geq 2s + 1, \\
7 \cdot 2^{2k+2s-2} + 21 \cdot \left[ 35 \cdot 2^{k+5s-7} - 39 \cdot 2^{k+4s-6} \right] & \text{if } i = 2s + 1 + j, \ k \geq 2s + 2 + j, \ 0 \leq j \leq s - 2, \\
+105 \cdot (2^{2k+2s+4j-2} - 31 \cdot 2^{7s+5j-3} + 93 \cdot 2^{6s+6j-3} \right) & \text{if } i = 3s, \ k \geq 3s \\
2^{3k+3s-3} - 7 \cdot 2^{k+6s-6} + 7 \cdot 2^{k+9s-8} - 2^{12s-9} & \text{if } i = 3s + j, \ 1 \leq j \leq s + 1, \\
2^{6s+3j-3} - 7 \cdot 2^{4i-6} + 3 \cdot 2^{3i-5} & \text{if } i = s + j, \ 1 \leq j \leq s + 1, \\
2^{6s+3j-3} + 7 \cdot 2^{2s+5j-8} - 7 \cdot 2^{2s+4j-7} - 7 \cdot 2^{4s+4j-6} + 3 \cdot 2^{3s+3j-5} & \text{if } i = 2s + 1 + j, \ 0 \leq j \leq s - 1,
\end{cases}
$$

We have for $0 \leq j \leq s - 2, \ k \geq 2s + 2 + j$

$$
(3.5) \Gamma_i^{\left[\frac{s}{i}\right] \times k} = 16^{3j} \cdot \Gamma_{2s+1+j}^{\left[\frac{s-j}{s-j}\right] \times (k-3j)}
$$

We have for $0 \leq j \leq s - 1$

$$
(3.6) \Gamma_i^{\left[\frac{s}{i}\right] \times (2s+1+j)} = 16^{3j} \cdot \Gamma_{2s+1+j}^{\left[\frac{s-j}{s-j}\right] \times (2s-j)}
$$
We have for $k \geq 3s$.

\[ \Gamma_{3s}^\frac{1}{2} = 16^{3(s-1)} \cdot \Gamma_{\frac{1}{3}}^\frac{1}{2} \times (k-3(s-1)) \]

**Theorem 3.3.** Let $q$ be a rational integer $\geq 1$, then

\[ g_{k,s}(t, \eta, \xi) = g(t, \eta, \xi) = \sum_{\text{deg}Y \leq k-1} \sum_{\text{deg}Z \leq s-1} E(YZ) \sum_{\text{deg}U \leq s-1} E(\eta U) \sum_{\text{deg}V \leq s-1} E(\eta V) = 2^{3s+k-r(D) \times (t, \eta, \xi)}, \]

\[ \int_{\mathbb{R} \times \mathbb{R} \times \mathbb{R}} g^q(t, \eta, \xi) dt dy d\xi = 2^{(3s+k)q} \cdot 2^{-3k-3s+3} \cdot \sum_{i=0}^{\inf(3s,k)} \Gamma_i^\frac{1}{2} \cdot 2^{-qi}. \]

**Theorem 3.4.** We denote by $R_q(k, s)$ the number of solutions 

\[ (Y_1, Z_1, U_1, \ldots, Y_q, Z_q, U_q, V_q) \]

of the polynomial equations

\[ \begin{cases} Y_1 Z_1 + Y_2 Z_2 + \ldots + Y_q Z_q = 0, \\ Y_1 U_1 + Y_2 U_2 + \ldots + Y_q U_q = 0, \\ Y_1 V_1 + Y_2 V_2 + \ldots + Y_q V_q = 0, \end{cases} \]

satisfying the degree conditions

\[ \text{deg}Y_i \leq k-1, \; \text{deg}Z_i \leq s-1, \; \text{deg}U_i \leq s-1, \; \text{deg}V_i \leq s-1 \quad \text{for} \quad 1 \leq i \leq q. \]

Then

\[ R_q(k, s) = \int_{\mathbb{R} \times \mathbb{R} \times \mathbb{R}} g^q_{k,s}(t, \eta, \xi) dt dy d\xi = 2^{(3s+k)q} \cdot 2^{-3k-3s+3} \cdot \sum_{i=0}^{\inf(3s,k)} \Gamma_i^\frac{1}{2} \cdot 2^{-qi}. \]

**Example.** $s = 1$, $k \geq i + 1$ for $0 \leq i \leq 2$

\[ \Gamma_i^\frac{1}{2} = \begin{cases} 1 & \text{if } i = 0 \\ 7 \cdot (2^k - 1) & \text{if } i = 1 \\ 7 \cdot (2^k - 1) \cdot (2^k - 2) & \text{if } i = 2 \\ 2^{3k} - 7 \cdot 2^{2k} + 7 \cdot 2^{k+1} - 2^3 & \text{if } i = 3, \; k \geq 3 \end{cases} \]

**Example.** $s = 2$, $k \geq i + 1$ for $0 \leq i \leq 5$

\[ \Gamma_i^\frac{2}{k} = \begin{cases} 1 & \text{if } i = 0 \\ 21 & \text{if } i = 1 \\ 147 \cdot 2^{k+1} + 1344 & \text{if } i = 3 \\ 7 \cdot 2^{2k+2} + 651 \cdot 2^{k+2} - 22624 & \text{if } i = 4 \\ 105 \cdot 2^{2k+2} - 315 \cdot 2^{k+5} + 53760 & \text{if } i = 5 \\ 23k^2 - 7 \cdot 2^{2k+6} + 7 \cdot 2^{k+10} - 32768 & \text{if } i = 6, \; k \geq 6 \end{cases} \]
Example. $s = 2$, $k = 6$.

The number $\Gamma_i \left[ \frac{3}{2} \right] \times 6$ of rank $i$ matrices of the form

$$
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\
\alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\
\beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 \\
\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\
\gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7
\end{pmatrix}
$$

is equal to

$$
\begin{cases}
1 & \text{if } i = 0 \\
21 & \text{if } i = 1 \\
1162 & \text{if } i = 2 \\
20160 & \text{if } i = 3 \\
258720 & \text{if } i = 4 \\
1128960 & \text{if } i = 5 \\
688128 & \text{if } i = 6
\end{cases}
$$

The number of solutions $(Y_1, Z_1, U_1, V_1, \ldots, Y_q, Z_q, U_q, V_q)$ of the polynomial equations

$$
\begin{align*}
Y_1 Z_1 + Y_2 Z_2 + \ldots + Y_q Z_q &= 0, \\
Y_1 U_1 + Y_2 U_2 + \ldots + Y_q U_q &= 0, \\
Y_1 V_1 + Y_2 V_2 + \ldots + Y_q V_q &= 0,
\end{align*}
$$

satisfying the degree conditions

$$
deg Y_i \leq 5, \quad deg Z_i \leq 1, \quad deg U_i \leq 1, \quad deg V_i \leq 1 \quad \text{for} \quad 1 \leq i \leq q.
$$

is equal to

$$
R_q(6, 2) = \int_{P \times P \times P} g_{6,2}^q(t, \eta, \xi) dt d\eta d\xi = 2^{12q-21} \cdot \sum_{i=0}^{6} \Gamma_i \left[ \frac{3}{2} \right] \times 6 \cdot 2^{-qi}
$$

$$
= 2^{12q-21} \cdot (1 + 21 \cdot 2^{-q} + 1162 \cdot 2^{-2q} + 20160 \cdot 2^{-3q} + 258720 \cdot 2^{-4q} + 1128960 \cdot 2^{-5q} + 688128 \cdot 2^{-6q})
$$

$$
= 2^{6q-21} \cdot (2^{6q} + 21 \cdot 2^{5q} + 1162 \cdot 2^{4q} + 20160 \cdot 2^{3q} + 258720 \cdot 2^{2q} + 1128960 \cdot 2^q + 688128)
$$

Example. $s = 3$, $k \geq i + 1$ for $0 \leq i \leq 8$

$$
\Gamma_i \left[ \frac{3}{1} \right] \times k = 
\begin{cases}
1 & \text{if } i = 0 \\
21 & \text{if } i = 1 \\
378 & \text{if } i = 2 \\
7 \cdot 2^{k+2} + 5936 & \text{if } i = 3 \\
147 \cdot 2^{k+2} + 84672 & \text{if } i = 4 \\
147 \cdot 9 \cdot 2^{k+3} + 959616 & \text{if } i = 5 \\
7 \cdot 2^{2k+4} + 2121 \cdot 2^{k+6} + 5863424 & \text{if } i = 6 \\
105 \cdot 2^{2k+4} + 2625 \cdot 2^{k+9} - 92897280 & \text{if } i = 7 \\
105 \cdot 2^{2k+8} - 315 \cdot 2^{k+14} + 220200960 & \text{if } i = 8 \\
2^{4k+6} - 7 \cdot 2^{2k+12} + 7 \cdot 2^{k+19} - 134217728 & \text{if } i = 9, \ k \geq 9
\end{cases}
$$
Example. $s = 3$, $k = 5$, $q = 3$.

The number $\Gamma_i^{\left[ \frac{3}{3} \right] \times 5}$ of rank $i$ matrices of the form

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\
\alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\
\alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\
\beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\
\beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 \\
\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 \\
\gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\
\gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 
\end{pmatrix}
\]

is equal to

\[
\begin{cases}
1 & \text{if } i = 0 \\
21 & \text{if } i = 1 \\
378 & \text{if } i = 2 \\
6832 & \text{if } i = 3 \\
103488 & \text{if } i = 4 \\
1986432 & \text{if } i = 5
\end{cases}
\]

The number of solutions $(Y_1, Z_1, U_1, V_1, Y_2, Z_2, U_2, V_2, Y_3, Z_3, U_3, V_3)$ of the polynomial equations

\[
\begin{align*}
Y_1Z_1 + Y_2Z_2 + Y_3Z_3 &= 0, \\
Y_1U_1 + Y_2U_2 + Y_3U_3 &= 0, \\
Y_1V_1 + Y_2V_2 + Y_3V_3 &= 0,
\end{align*}
\]

satisfying the degree conditions

$\deg Y_i \leq 4$, $\deg Z_i \leq 2$, $\deg U_i \leq 2$, $\deg V_i \leq 2$ for $1 \leq i \leq 3$.

is equal to

\[
R_3(5, 3) = \int_{\mathbb{P} \times \mathbb{P} \times \mathbb{P}} g_{Y, Z, U, V}^3(t, \eta, \xi) dtd\eta d\xi = 2^{33} \cdot \sum_{i=0}^{5} \Gamma_i^{\left[ \frac{3}{3} \right] \times 5} \cdot 2^{-3i}
\]

\[
= 2^{33} \cdot (1 + 21 \cdot 2^{-3} + 378 \cdot 2^{-6} + 6832 \cdot 2^{-9} + 103488 \cdot 2^{-12} + 1986432 \cdot 2^{-15}) = 3563904 \times 2^{18}
\]
3.3. An outline of the main results in the case $m = 1$, $l = 0$.

**Theorem 3.5.** We have

\[
\Gamma_i\left[\frac{s^j}{s^i + 1}\right] \times k = \begin{cases}
1 & \text{if } i = 0, \\
105 \cdot 2^{4i-6} - 21 \cdot 2^{k-5} & \text{if } 1 \leq i \leq s - 1, \ k \geq i + 1,
\end{cases}
\]

if $i = s$, $k \geq s + 1$, $s \geq 1$,

if $i = s + 1$, $k \geq s + 2$,

if $i = s + 2$, $k \geq s + 3$,

if $i = s + 3$, $k \geq s + 4$,

if $i = s + 4$, $k \geq s + 5$, $s \geq 2$,

\[
2 \geq j \leq s, \ k \geq s + j + 1,
\]

\[
3 \cdot 2^{2s-1} \cdot (2^{2k-2^{s+4}+4} + (735 \cdot 2^{5s-5} - 393 \cdot 2^{4s-4}) \cdot (2^k - 2^{s+2})
\]

if $i = 2s + 1$, $k \geq 2s + 2$,

\[
5 \cdot 2^{2s-1} \cdot (2^{2k-2^{s+4}+6} + (735 \cdot 2^{5s-1} - 1629 \cdot 2^{4s-1}) \cdot (2^k - 2^{s+3})
\]

if $i = 2s + 2$, $k \geq 2s + 3$,

\[
105 \cdot (2^{2k+2s+4}+2+7 \cdot 2^{k+5s+4}j+3-31 \cdot 2^{k+4s+5}j+3)
\]

if $i = 2s + 3 + j$, $k \geq 2s + 4 + j$, $0 \leq j \leq s - 2$,

if $i = 3s + 2$, $k \geq 3s + 2$.

\[
\Gamma_i\left[\frac{s^j}{s^i + 1}\right] \times i = \begin{cases}
2^{3s+3i-1} - 7 \cdot 2^{4i-6} + 3 \cdot 2^{3i-5} & \text{if } 1 \leq i \leq s + 1,
\end{cases}
\]

if $i = s + j$, $2 \leq j \leq s + 3$,

if $i = 2s + 1$,

if $i = 2s + 2$,

if $i = 2s + 3$,

if $i = 2s + 3 + j$, $0 \leq j \leq s - 1$.

We have the following reduction formulas

\[
\Gamma_{2s+2+j}\left[\frac{s^j}{s^i + 1}\right] = 16^{2j}\Gamma_{2s+2+j}\left[\frac{s^j}{s^i + 1}\right] \times (k-2j)
\]

if $0 \leq j \leq 1$.

\[
\Gamma_{2s+3+j}\left[\frac{s^j}{s^i + 1}\right] = 16^{2j+3}\Gamma_{2s+3+j}\left[\frac{s^j}{s^i + 1}\right] \times (k-2j+3)
\]

if $0 \leq j \leq s - 1$. 

Example. We have for \( s = 3 \):

\[
\Gamma_i^{\left[ \frac{3}{3+1} \right]^k} = \begin{cases} 
1 & \text{if } i = 0, \ k \geq 1, \\
21 & \text{if } i = 1, \ k \geq 2, \\
378 & \text{if } i = 2, \ k \geq 3, \\
2^{k+2} + 6320 & \text{if } i = 3, \ k \geq 4, \\
33 \cdot 2^{k+2} + 100416 & \text{if } i = 4, \ k \geq 5, \\
630 \cdot 2^{k+2} + 1524096 & \text{if } i = 5, \ k \geq 6, \\
1365 \cdot 2^{k+5} + 21224448 & \text{if } i = 6, \ k \geq 7, \\
96 \cdot 2^{2k} + 163008 \cdot 2^{k+2} + 1029 \cdot 2^{18} & \text{if } i = 7, \ k \geq 8, \\
1696 \cdot 2^{2k} + 2176512 \cdot 2^{k+2} + 5723 \cdot 2^{18} & \text{if } i = 8, \ k \geq 9, \\
105 \cdot 2^{2k+8} + 2625 \cdot 2^{k+15} - 90720 \cdot 2^{18} & \text{if } i = 9, \ k \geq 10, \\
105 \cdot 2^{2k+12} - 315 \cdot 2^{k+20} + 215040 \cdot 2^{18} & \text{if } i = 10, \ k \geq 11, \\
2^{3k+8} - 7 \cdot 2^{2k+16} + 7 \cdot 2^{k+25} - 2^{35} & \text{if } i = 11, \ k \geq 11.
\end{cases}
\]
3.4. An outline of the main results in the case \( m \geq 2, l = 0 \).

**Theorem 3.6.** The number \( \Gamma \left[ \frac{3s + 2m}{s + m} \right] \times k \) of triple persymmetric \((3s + 2m) \times k\) matrices over \( \mathbb{F}_2 \) of the form

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \ldots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \ldots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \ldots & \alpha_{s+k-3} & \alpha_{s+k-2} \\
\alpha_s & \alpha_{s+1} & \ldots & \alpha_{s+k-2} & \alpha_{s+k-1} \\
\beta_1 & \beta_2 & \ldots & \beta_{k-1} & \beta_k \\
\beta_2 & \beta_3 & \ldots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \ldots & \beta_{k+m-1} & \beta_{k+m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \ldots & \beta_{s+m+k-3} & \beta_{s+m+k-2} \\
\beta_{s+m} & \beta_{s+m+1} & \ldots & \beta_{s+m+k-2} & \beta_{s+m+k-1} \\
\gamma_1 & \gamma_2 & \ldots & \gamma_{k-1} & \gamma_k \\
\gamma_2 & \gamma_3 & \ldots & \gamma_k & \gamma_{k+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\gamma_{m+1} & \gamma_{m+2} & \ldots & \gamma_{k+m-1} & \gamma_{k+m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\gamma_{s+m-1} & \gamma_{s+m} & \ldots & \gamma_{s+m+k-3} & \gamma_{s+m+k-2} \\
\gamma_{s+m} & \gamma_{s+m+1} & \ldots & \gamma_{s+m+k-2} & \gamma_{s+m+k-1}
\end{pmatrix}
\]

is given by
\[(3.15)\]
\[
\begin{align*}
1 & \quad \text{if } i = 0, k \geq 1 \\
105 \cdot 2^{4i-6} - 21 \cdot 2^{3i-5} & \quad \text{if } 1 \leq i \leq s-1, k \geq i+1, \\
2^{k+s-1} - 2^s + 21 \cdot (5 \cdot 2^{4s-6} - 2^{3s-5}) & \quad \text{if } i = s, k \geq s + 1 \\
(21 \cdot 2^{j-1} - 3) \cdot 2^{k+s+2j-4} & \quad \text{if } i = s + j, 1 \leq j \leq m - 1, \\
+21 \cdot (5 \cdot 2^{4s+4j-6} - 2^{3s+3j-5} - 5 \cdot 2^{2s+4j-6} + 2^{2s+3j-5}) & \quad k \geq s + j + 1 \\
(21 \cdot 2^s + 3m - 5 + 45 \cdot 2^{s+2m-4}) \cdot (2^k - 2^{s+m+1}) & \quad \text{if } i = s + m, k \geq s + m + 1 \\
+105 \cdot 2^{4s+4m-6} - 21 \cdot 2^{3s+3m-5} - 21 \cdot 2^{2s+4m-6} + 9 \cdot 2^{2s+3m-5} & \quad \text{if } i = s + m + j, 1 \leq j \leq s - 1, \\
21 \cdot [2^{k+s+3m+3j-5} + 35 \cdot 2^{k+s+2m+4j-7} - 9 \cdot 2^{k+s+2m+3j-6} & \quad k \geq s + m + j + 1 \\
+5 \cdot 2^{4s+4m+4j-6} - 2^{3s+3m+3j-5} - 5 \cdot 2^{2s+4m+4j-6} & \\
-155 \cdot 2^{2s+3m+5j-8} + 45 \cdot 2^{2s+3m+4j-7}] & \quad \text{if } i = 2s + m, k \geq 2s + m + 1 \\
3 \cdot 2^{2k+2s+m-2} + 21 \cdot 2^{k+4s+3m-5} + 735 \cdot 2^{k+5s+2m-7} - 477 \cdot 2^{k+4s+2m-6} & \quad 0 \leq j \leq m - 2, k \geq 2s + m + 2 + j \\
+105 \cdot 2^{8s+4m-6} - 105 \cdot 2^{8s+4m-6} - 3255 \cdot 2^{7s+3m-8} + 1629 \cdot 2^{6s+3m-7} & \quad \text{if } i = 2s + m + 1 + j, \\
21 \cdot 2^{2k+2s+m+3j-2} + 21 \cdot 2^{k+4s+3m+3j-2} + 735 \cdot 2^{k+5s+2m+4j-3} & \\
-945 \cdot 2^{k+4s+2m+4j-3} + 105 \cdot 2^{8s+4m+4j-2} - 105 \cdot 2^{6s+4m+4j-2} & \\
-3255 \cdot 2^{7s+3m+5j-3} + 3255 \cdot 2^{6s+3m+5j-3} & \quad \text{if } i = 2s + 2m, k \geq 2s + 2m + 1 \\
53 \cdot 2^{2s-1} \cdot (2^{2k+4m-4} - 2^{4s+8m-2}) & \\
+ (735 \cdot 2^{5s-1} - 1629 \cdot 2^{4s-1}) \cdot (2^{k+6m-6} - 2^{2s+8m-5}) & \\
+21 \cdot [5 \cdot 2^{8s+8m-6} + 3 \cdot 2^{6s+8m-8} - 15 \cdot 2^{7s+8m-8}] & \quad \text{if } i = 2s + 2m + 1 + j, \\
105 \cdot (2^{2k+2s+4m+4j-2} + 7 \cdot 2^{k+5s+6m+4j-3} - 31 \cdot 2^{k+4s+6m+5j-3}) & \\
+105 \cdot (2^{8s+8m+4j-2} - 31 \cdot 2^{7s+8m+5j-3} + 93 \cdot 2^{6s+8m+6j-3}) & \quad 0 \leq j \leq s - 2, k \geq 2s + 2m + 2 + j \\
2^{3k+2m+3s-3} - 7 \cdot 2^{2k+4m+6s-6} + 7 \cdot 2^{k+6m+9s-8} - 2^{8s+12s-9} & \quad \text{if } i = 3s + 2m, k \geq 3s + 2m
\end{align*}
\]
We have for $0 \leq j \leq m - 2$, $k \geq 2s + m + 2 + j$

\[\Gamma_l^{\left[\frac{s \cdot m}{s + m}\right]} \times k = 16^2 \Gamma_l^{\left[\frac{s \cdot (m-j)}{s + m-j}\right]} \times (k-2j)\]

We have for $j = m - 1$, $k \geq 2s + 2m + 1$

\[\Gamma_l^{\left[\frac{s \cdot m}{s + m}\right]} = 16^{2\cdot m-2} \Gamma_l^{\left[\frac{s \cdot (m+1)}{s + 1}\right]} \times (k-2(m-1))\]

We have for $0 \leq j \leq s - 2$, $k \geq 2s + 2m + 2 + j$

\[\Gamma_l^{\left[\frac{s \cdot m}{s + m}\right]} \times k = 16^{2\cdot m+3} \Gamma_l^{\left[\frac{s \cdot (m-j)}{s + m-j}\right]} \times (k-2(m-3j))\]
We have for \( j = s - 1, \ k \geq 3s + 2m \)

\[
(3.20) \quad \Gamma_{3s+2m} \times k \left[ \begin{array}{c} \frac{s+m}{s+m} \\ \end{array} \right] = 16^{2m+3s-3} \Gamma_3 \times (k-2m-3s+3)
\]

\[
(3.21)
\]

Let \( q \) be a rational integer \( \geq 1 \), then

\[
g_{k,s,m}(t, \eta, \xi) = g(t, \eta, \xi) = \sum_{\deg Y \leq k-1} \sum_{\deg Z \leq s-1} E(tYZ) \sum_{\deg U \leq s+m-1} E(\eta YU) \sum_{\deg V \leq s+m-1} E(\eta YV)
\]

\[
= 2^{3s+2m+k-r(D)} \left[ \begin{array}{c} \frac{s+m}{s+m} \\ \end{array} \right] \times k (t, \eta, \xi)
\]

and

\[
\int_{\mathbb{P}^3} g^q(t, \eta, \xi) d\eta d\xi =
\]

\[
= \sum_{(t, \eta, \xi) \in \mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}} 2^{(k+3s+2m-r(D))} \left[ \begin{array}{c} \frac{s+m}{s+m} \\ \end{array} \right] \times k (t, \eta, \xi)
\]

\[
\inf(3s+2m) = \sum_{i=0}^{q \leq 3s+2m} \sum_{(t, \eta, \xi) \in \mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}} 2^{(k+3s+2m-i)} q \int_{\mathbb{P}_{k+s-1}} dt \int_{\mathbb{P}_{k+s+m-1}} d\eta \int_{\mathbb{P}_{k+s+m-1}} d\xi
\]

\[
= 2^{(k+3s+2m)} q^{-3(k+3s+2m-3)} \sum_{i=0}^{q \leq 3s+2m} \Gamma_{i} \left[ \begin{array}{c} \frac{s+m}{s+m} \\ \end{array} \right] \times k \cdot 2^{-iq}
\]

We denote by \( R_q(k, s, m) \) the number of solutions

\[
(Y_1, Z_1, U_1, V_1, \ldots, Y_q, Z_q, U_q, V_q) \text{ of the polynomial equations}
\]

\[
\left\{ \begin{array}{l}
Y_1 Z_1 + Y_2 Z_2 + \ldots + Y_q Z_q = 0,
Y_1 U_1 + Y_2 U_2 + \ldots + Y_q U_q = 0,
Y_1 V_1 + Y_2 V_2 + \ldots + Y_q V_q = 0.
\end{array} \right.
\]

satisfying the degree conditions

\[
\deg Y_i \leq k-1, \ \deg Z_i \leq s-1, \ \deg U_i \leq s+m-1, \ \deg V_i \leq s+m-1 \text{ for } 1 \leq i \leq q.
\]

Then

\[
(3.22) \quad R_q(k, s, m) = \int_{\mathbb{P} \times \mathbb{P}} g^q_{k,s,m}(t, \eta, \xi) d\eta d\xi = 2^{(k+3s+2m)} q^{-3(k+3s+2m-3)} \sum_{i=0}^{q \leq 3s+2m} \Gamma_{i} \left[ \begin{array}{c} \frac{s+m}{s+m} \\ \end{array} \right] \times k \cdot 2^{-iq}
\]
Example. We have for $s = 3$, $m = 4$, $k = 10$:

$$
\Gamma_i^{\left[ \frac{s+4}{3+4} \right]} \times 10 = \begin{cases} 
1 & \text{if } i = 0 \\
21 & \text{if } i = 1 \\
378 & \text{if } i = 2 \\
10416 & \text{if } i = 3 \\
140352 & \text{if } i = 4, \\
1994112 & \text{if } i = 5, \\
29598720 & \text{if } i = 6 \\
458661888 & \text{if } i = 7, \\
109389 \cdot 2^{16} & \text{if } i = 8, \\
213759 \cdot 2^{19} & \text{if } i = 9, \\
2^{44} - 14273 \cdot 2^{23} & \text{if } i = 10
\end{cases}
$$

Example. $s = 3$, $m = 4$, $k = 7$, $q = 3$

The number $\Gamma_i^{\left[ \frac{s+4}{3+4} \right]} \times 7$ of rank $i$ matrices of the form

$$
\left( \begin{array}{cccccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\
\alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \\
\alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 & \alpha_9 \\
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 \\
\beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 & \beta_8 \\
\beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 & \beta_8 & \beta_9 \\
\beta_4 & \beta_5 & \beta_6 & \beta_7 & \beta_8 & \beta_9 & \beta_{10} \\
\beta_5 & \beta_6 & \beta_7 & \beta_8 & \beta_9 & \beta_{10} & \beta_{11} \\
\beta_6 & \beta_7 & \beta_8 & \beta_9 & \beta_{10} & \beta_{11} & \beta_{12} \\
\beta_7 & \beta_8 & \beta_9 & \beta_{10} & \beta_{11} & \beta_{12} & \beta_{13} \\
\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 \\
\gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 \\
\gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 \\
\gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} \\
\gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} \\
\gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} & \gamma_{12} \\
\gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13}
\end{array} \right)
$$

is equal to

$$
\begin{cases} 
1 & \text{if } i = 0 \\
21 & \text{if } i = 1 \\
378 & \text{if } i = 2 \\
6832 & \text{if } i = 3 \\
108096 & \text{if } i = 4, \\
1714560 & \text{if } i = 5, \\
27276288 & \text{if } i = 6 \\
2^{35} - 3553 \cdot 2^{13} & \text{if } i = 7,
\end{cases}
$$
The number of solutions
\((Y_1, Z_1, U_1, V_1, Y_2, Z_2, U_2, V_2, Y_3, Z_3, U_3, V_3)\) of the polynomial equations
\[
\begin{align*}
Y_1 Z_1 + Y_2 Z_2 + Y_3 Z_3 &= 0, \\
Y_1 U_1 + Y_2 U_2 + Y_3 U_3 &= 0, \\
Y_1 V_1 + Y_2 V_2 + Y_3 V_3 &= 0,
\end{align*}
\]
satisfying the degree conditions
\[
\text{deg} Y_i \leq 6, \quad \text{deg} Z_i \leq 2, \quad \text{deg} U_i \leq 6, \quad \text{deg} V_i \leq 6 \quad \text{for} \quad 1 \leq i \leq 3.
\]
is equal to
\[
R_3(7,3,4) = \int_{P^3 \times P^3} g_{7,3,4}^3(t,\eta,\xi) dt d\eta d\xi = 2^{37} \cdot \sum_{i=0}^{7} \Gamma_{[\frac{3+i}{3+i}]} \cdot 2^{-3i}
\]
\[
= 2^{37} \cdot (1 + 21 \cdot 2^{-3} + 378 \cdot 2^{-6} + 6832 \cdot 2^{-9} + 108096 \cdot 2^{-12} + 1714560 \cdot 2^{-15}
\]
\[
+ 27276288 \cdot 2^{-18} + (2^{35} - 3553 \cdot 2^{13}) \cdot 2^{-21}) = 4243395 \cdot 2^{29}
\]

**Example.** The fraction of square triple persymmetric \(
\left[\begin{array}{cc}
\frac{s}{s+m} & \\
\frac{s}{s+m} & \\
\end{array}\right] \times (3s + 2m)\) matrices

which are invertible is equal to
\[
\frac{\Gamma_{[\frac{s}{s+m}]} \times (3s + 2m)}{\sum_{i=0}^{3s+2m} \Gamma_{i} [\frac{s}{s+m}] \times (3s + 2m)} = \frac{21}{64}
\]

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