Bearing-Only Network Localization: Localizability, Sensitivity, and Distributed Protocols

Shiyu Zhao, Daniel Zelazo,

Faculty of Aerospace Engineering, Technion - Israel Institute of Technology, Haifa, Israel.

Abstract

This paper addresses self-localization of stationary sensor networks based on inter-neighbor bearings and anchor nodes whose locations are known. In our work, we formulate the bearing-only network localization problem as a linear least-squares problem and consider measurement models with and without errors. We provide necessary and sufficient conditions for the localizability of a network with both algebraic and rigidity theoretic interpretations. The proposed conditions fully describe the relationship between the localizability and the bearing rigidity properties of a network. We also analyze the sensitivity of the localization problem to constant measurement errors. Upper bounds for the localization error and the bearing errors that a network can tolerate are presented. Finally, we propose distributed protocols to globally localize bearing-only networks. All the results presented in the paper are applicable to networks in arbitrary dimensions. This work is validated with numerical simulations.

Key words: Network localization, bearing rigidity, bearing measurement, localizability, distributed estimation

1 Introduction

In recent years, there has been a growing research interest in bearing-based distributed control and estimation in network systems. For example, distributed formation control using bearing-only measurements has been studied in [1–8]. Distributed localization of bearing-only sensor networks has been investigated by [9–15]. The research on bearing-based distributed control and estimation can be potentially applied to vision-based formation control of multi-vehicle systems and self-localization of camera or angle-of-arrival sensor networks.

The instrumental tool for analyzing bearing-based control and estimation problems is bearing rigidity theory (also known as parallel rigidity theory). The basic problem that bearing rigidity theory studies is whether the shape of a network can be uniquely determined based on inter-neighbor bearings. Most of the previous studies on bearing rigidity theory only focused on networks in two-dimensional ambient spaces [3, 10, 16–19]. In our recent work [20], we extended the previous studies to arbitrary dimensions and showed that a network can be uniquely determined up to a translation and a scaling factor if and only if the network is infinitesimally bearing rigid. Bearing rigidity theory has been successfully applied to solve the bearing-only formation control problem in [20]. The utility of bearing rigidity theory will be further demonstrated in this paper by its application to bearing-only network localization.

In this paper, we focus on self-localization of stationary bearing-only sensor networks in arbitrary dimensions. One basic problem to be addressed is whether all the positions of the nodes in a network can be localized merely based on the inter-neighbor bearing measurements and a number of anchor nodes whose positions are known (by, for example, GPS). This problem leads to a fundamental concept termed localizability. A network must be localizable in order to be localized by either centralized or decentralized protocols. The localizability of a bearing-only network is closely related to the bearing rigidity of the network. It has been shown in [12–14] that a network is localizable if the network is infinitesimally bearing rigid and has at least two anchor nodes. It has also been observed by [14, Corollary 10] that infinitesimal bearing rigidity is merely sufficient but not necessary to guarantee localizability. However, the relationship between bearing rigidity and network localizability has not been fully understood up to now. Moreover, the existing studies mainly focused on the localization of networks in two-dimensional spaces, whereas general results for arbitrary dimensions are still lacking. Finally, we would like to mention that the problem setup of the bearing-only network localization problem considered in...
this paper is similar to [12–15], where the networks are stationary, but different from bearing-based localization of mobile sensor networks [21], bearing-based target localization [22–24], or self-calibration of camera networks which requires a set of visual features that can be detected by multiple cameras simultaneously [25].

The main contributions of the paper are as follows. (i) We show that the problem of bearing-only network localization can be converted to a linear algebraic problem. This linear algebraic problem is further formulated as linear least-squares problems which enable us to analyze cases with and without measurement errors in a unified framework. (ii) We prove both algebraic and rigidity-based necessary and sufficient conditions for network localizability. The proposed conditions fully describe the relationship between the localizability and the bearing rigidity of a network. (iii) The sensitivity of the localization problem to measurement errors is also analyzed. One basic conclusion of the sensitivity analysis is that the optimal estimate of the network location will be sufficiently close to the true value when the measurement errors are sufficiently small. We also present specific upper bounds for the localization error and the bearing errors that a network can tolerate. (iv) We propose continuous-time protocols to distributedly localize bearing-only networks. In the absence of measurement errors, the proposed protocols can globally and exponentially localize a network if and only if the network is localizable. In the presence of measurement errors, the proposed protocols can still give maximum likelihood estimates of the network location. Finally, a novel contribution of this work is that all the results are applicable to networks in arbitrary dimensions.

The rest of the paper is organized as follows. Section 2 provides preliminaries to the bearing rigidity theory. Section 3 formally states the problem of bearing-only network localization. Section 4 and Section 5 present localizability and sensitivity analysis, respectively. Distributed localization protocols are presented in Section 6. Simulation examples are given in Section 7 and conclusions are drawn in Section 8.

Notations. Given $A_i \in \mathbb{R}^{p \times q}$ for $i = 1, \ldots, n$, denote diag$(A_i) \triangleq \text{blkdiag}\{A_1, \ldots, A_n\} \in \mathbb{R}^{np \times nq}$. Let Null$(\cdot)$ and Range$(\cdot)$ be the null space and range space of a matrix, respectively. Denote $I_d \in \mathbb{R}^{d \times d}$ as the identity matrix, and $1_d \triangleq [1, \ldots, 1]^T \in \mathbb{R}^d$. Let $\parallel \cdot \parallel$ be the Euclidian norm of a vector or the spectral norm of a matrix, and $\otimes$ be the Kronecker product.

An undirected graph, denoted as $G = (V, E)$, consists of a vertex set $V$ and an edge set $E \subseteq V \times V$. Let $n = |V|$ and $m = |E|$. The set of neighbors of vertex $i$ is denoted as $N_i \triangleq \{j \in V : (i, j) \in E\}$. An orientation of an undirected graph is the assignment of a direction to each edge. An oriented graph, denoted as $G^o = (E^o, V)$, is an undirected graph together with an orientation. The incidence matrix of an oriented graph is denoted as $H \in \mathbb{R}^{m \times n}$. For a connected graph, one always has $H1 = 0$ and $\text{Rank}(H) = n - 1$. Furthermore, define the inflated incidence matrix as $H \triangleq H \otimes I_d$.

2 Preliminaries to Bearing Rigidity Theory

Bearing rigidity theory plays a key role in bearing-only network localization problems. In this section we revisit some important concepts and results from bearing rigidity theory. For details, please see [20].

2.1 An Orthogonal Projection Operator

First of all, we introduce an important orthogonal projection operator which will be widely used in this paper. For any nonzero vector $x \in \mathbb{R}^d$ ($d \geq 2$), define the orthogonal projection operator $P : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ as

$$P(x) \triangleq I_d - \frac{x x^T}{\|x\|^2}.$$ 

For notational simplicity, we denote $P_x \triangleq P(x)$. Note that $P_x$ is an orthogonal projection matrix that geometrically projects any vector onto the orthogonal complement of $x$ (see Figure 1 for an illustration). The useful properties of $P_x$ are listed as below.

Lemma 1 For any nonzero vector $x \in \mathbb{R}^d$, the orthogonal projection matrix $P_x$ satisfies

(a) $P_x^T = P_x$ and $P_x^2 = P_x$;
(b) Null($P_x$) = span $\{x\}$;
(c) $P_x$ is positive semi-definite with eigenvalues $\{0, 1, \ldots, 1\}$.

Proof. See Appendix A

Lemma 2 Any nonzero vectors $x, y \in \mathbb{R}^d$ are parallel if and only if $P_x y = 0$ (or equivalently $P_y x = 0$).

Proof. It directly follows from Null($P_x$) = span $\{x\}$.
Lemma 3 Denote by $\theta \in [0, \pi]$ the angle between any two nonzero vectors $x, y \in \mathbb{R}^d$ (i.e., $x^T y = \|x\| \|y\| \cos \theta$). Then

$$\|P_x - P_y\| = \sin \theta.$$ 

Proof. See Appendix B. □

Lemma 3 is useful for analyzing the perturbation of the orthogonal projection operator.

### 2.2 Bearing Rigidity Theory

Consider a set of points $\{p_i\}_{i=1}^n$ in $\mathbb{R}^d$ ($n \geq 2$ and $d \geq 2$) and assume no two points are collocated. Denote $p = [p_1^T, \ldots, p_n^T]^T \in \mathbb{R}^{dn}$. Let $G = (V, E)$ be an undirected graph with $V = \{1, \ldots, n\}$.

Definition 1 (Network) A network, denoted as $G(p)$, is a combination of the graph $G$ and the position $p$, where node $i$ in the graph is mapped to the point $p_i$, $\forall i \in V$.

A network is also referred to as a framework or point formation in the literature. For a network $G(p)$, define the edge vector and the bearing, respectively, as

$$e_{ij} \triangleq p_j - p_i, \quad g_{ij} \triangleq e_{ij}/\|e_{ij}\|, \quad \forall (i, j) \in E.$$ 

The bearing $g_{ij}$ is a unit vector. Note $e_{ij} = -e_{ji}$ and $g_{ij} = -g_{ji}$. It is often helpful to consider an oriented graph, $G^* = (V, E^*)$, and express the edge vector and the bearing for the $k$th directed edge $(i, j) \in E^*$ as $e_k \triangleq p_j - p_i$, $g_k \triangleq e_k/\|e_k\|$, $\forall k \in \{1, \ldots, m\}$, respectively. For an arbitrary oriented graph, define the bearing function $F_B : \mathbb{R}^{dn} \rightarrow \mathbb{R}^{dm}$ as

$$F_B(p) \triangleq [g_1^T, \ldots, g_m^T]^T.$$ 

The bearing function describes all the bearings in the network. The bearing rigidity matrix is defined as the Jacobian of the bearing function,

$$R_B(p) \triangleq \frac{\partial F_B(p)}{\partial p} \in \mathbb{R}^{dm \times dn}. \quad (1)$$

Two important properties of the bearing rigidity matrix are given as below.

Lemma 4 ([20]) The bearing rigidity matrix in (1) can be expressed as $R_B(p) = \text{diag}(P_{g_i}/\|e_k\|) H$.

Lemma 5 ([20]) For any network $G(p)$, the bearing rigidity matrix satisfies $\text{Rank}(R_B) \leq dn - d - 1$ and span $\{1 \otimes I_d, p\} \subseteq \text{Null}(R_B)$.

Let $\delta p$ be a variation of $p$. If $R_B(p) \delta p = 0$, then $\delta p$ is called an infinitesimal bearing motion of $G(p)$. A motion is an infinitesimal bearing motion if and only if the motion does not change the bearings between any neighbors in $G(p)$. Any network always has two kinds of trivial infinitesimal bearing motions: translation and scaling of the entire network. We next define one of the most important concepts in bearing rigidity theory.

### Definition 2 (Infinitesimal Bearing Rigidity) A network is infinitesimally bearing rigid if the infinitesimal bearing motions of the network are trivial.

The following is a necessary and sufficient condition for infinitesimal bearing rigidity.

Theorem 1 ([20]) For any network $G(p)$, the following statements are equivalent:

(a) $G(p)$ is infinitesimally bearing rigid;
(b) $\text{Rank}(R_B) = dn - d - 1$;
(c) $\text{Null}(R_B) = \text{span} \{1 \otimes I_d, p\}$.

Theorem 2 ([20]) An infinitesimally bearing rigid network can be uniquely determined by the inter-neighbor bearings up to a translation and a scaling factor.

As will be shown later, infinitesimal bearing rigidity plays an important role in bearing-only network localization. For additional results and properties from bearing rigidity theory, please see [20].

### 3 Bearing-Only Network Localization

In this section, we formally state the problem of bearing-only network localization as a nonlinear algebraic problem, and then prove it is equivalent to a linear algebraic problem. It is finally reformulated as a linear least-squares problem such that the cases with and without measurement errors can be analyzed in a unified framework.

Consider a network of $n$ fixed nodes in $\mathbb{R}^d$ ($n \geq 2$, $d \geq 2$). Suppose the first $n_a$ nodes, called anchors, can directly measure their own positions with, for example, GPS sensors. The remaining $n_f$ nodes, called followers, cannot measure their positions. Note $0 \leq n_a \leq n$ and $n_a + n_f = n$. Denote $V_a = \{1, \ldots, n_a\}$ and $V_f = \{n_a + 1, \ldots, n\}$ as the index sets for the anchors and followers, respectively. The entire vertex set is thus $V = V_a \cup V_f$. Let $p_i \in \mathbb{R}^d$ be the position of node $i \in V$. Then $p_a = [p_1^T, \ldots, p_{n_a}^T]^T$, $p_f = [p_{n_a+1}^T, \ldots, p_n^T]^T$, and $p = [p_1^T, p_f^T]^T$.

Suppose the underlying graph $G = (V, E)$ is fixed and undirected. Assume node $i, \forall i \in V$, can measure the relative bearings of its neighbors, $\{g_{ij}\}_{j \in V_i}$. In practice, the bearings can be measured by an angle-of-arrival sensor
or a camera. We assume that there is a \textit{global orientation}
that can be measured by all the nodes. As a result, all
the bearings \( \{g_{ij}\}_{(i,j)\in E} \) can be expressed in a common
reference frame.

The problem of bearing-only network localization is for-
more stated as a nonlinear algebraic problem as below.

**Problem 1 (Bearing-Only Network Localization)**

Consider a network \( \mathcal{G}(p) \) where the positions of the follo-
wer nodes, \( \{p_i\}_{i\in V_f} \), are unknown. The bearing-only
network localization problem is to determine \( \{p_i\}_{i\in V_f} \)
given the bearings between all neighbor pairs, \( \{g_{ij}\}_{(i,j)\in E} \),
and the positions of the anchor nodes, \( \{p_i\}_{i\in V_a} \). Equiva-
antly, the network localization problem is to find \( \{\hat{p}_i\}_{i\in V} \)
that satisfy

\[
\begin{align*}
\hat{p}_j - \hat{p}_i &= g_{ij}, & \forall (i, j) \in E, \\
\hat{p}_i &= p_i, & \forall i \in V_a.
\end{align*}
\]  

(2)

Observe the true position \( p \) of the network is a feasible
solution to the above nonlinear algebraic problem. How-
ever, there may exist an infinite number of other feasible
solutions which makes the problem complicated. In par-
ticular, finding a solution to the constraints specified in
(2) does not guarantee it is the desired solution (i.e., the
true position \( p \)). This leads to the following important
concept we term \textit{localizability}.

**Definition 3 (Localizability)** A network \( \mathcal{G}(p) \) is called
localizable if the true position \( p \) is the unique feasible
solution to (2).

In order to analyze the nonlinear algebraic problem (2),
we show that the nonlinear constraints are related to the
following set of linear constraints,

\[
\begin{align*}
P_{g_{ij}} (\hat{p}_j - \hat{p}_i) &= 0, & \forall (i, j) \in E, \\
\hat{p}_i &= p_i, & \forall i \in V_a.
\end{align*}
\]  

(3)

Observe that the first equation in (3) is obtained by multi-
plying \( P_{g_{ij}} \) on both sides of the first equation in (2). In gen-
eral, the linear algebraic problem (3) and the non-
linear one (2) may not have the same feasible solutions.
However, the two problems are equivalent when the true
position \( p \) is the unique feasible solution as shown below.

**Lemma 6** The true position \( p \) is the unique solution to
(2) if and only if \( p \) is also the unique solution to (3).

**Proof.** (Sufficiency) Suppose \( p \) is the unique solution to
(3). The first equation in (3) is obtained by multiplying
\( P_{g_{ij}} \) on both sides of the first equation in (2). As a result,
any solution to (2) is also a solution to (3). Hence, the
set of the solutions to (2) is a subset of that of (3). There-
fore, if \( p \) is the unique solution to (3), it is also the
unique solution to (2).

(Necessity) Suppose \( p \) is the unique solution to (2). We
next prove by contradiction that \( p \) is also the unique
solution to (3). Assume that in addition to \( p \) there exists
another solution \( p' \) to (3). Define \( \delta p \triangleq p' - p \)
and

\[
\delta p' \triangleq p + k\delta p, \quad k \in \mathbb{R}.
\]

The rest of the proof shows that \( \delta p' \) with |\( k \)| sufficiently
small is another solution to (2), which is a contradiction.
Firstly, since both \( p \) and \( p' \) are solutions to (3), it can be
easily seen that \( \delta p' \) is also a solution to (3). As a result,
we have \( P_{g_{ij}} (\delta p') = 0, \forall (i, j) \in E \) which implies

\[
\frac{\delta p'_j - \delta p'_i}{\|\delta p'_j - \delta p'_i\|} = \pm g_{ij}, \quad \forall (i, j) \in E.
\]

If |\( k \)| is sufficiently small such that \( \delta p' \) is sufficiently
close to \( p \) and the sign of each entry in \( \frac{\delta p'_j - \delta p'_i}{\|\delta p'_j - \delta p'_i\|} \) is the same as
that of \( p_j - p_i \), we would have \( (\delta p'_j - \delta p'_i)/\|\delta p'_j - \delta p'_i\| = g_{ij}, \forall (i, j) \in E \).
In the special case where any entry of \( p_j - p_i \) is zero, the corresponding entry of \( \delta p_j - \delta p_i \) is also
zero because \( \delta p_j - \delta p_i \) is parallel to \( p_j - p_i \). Therefore,
\( \delta p' \) with |\( k \)| sufficiently small is another solution to (2),
which is a contradiction. \( \Box \)

Based on Lemma 6, we can study the bearing-only net-
work localization problem by analyzing the linear alge-
braic problem (3). In order to make our analysis more
general, we also consider the case where there may exist
measurement errors for the bearings \( \{g_{ij}\}_{(i,j)\in E} \) and the
anchor positions \( \{p_i\}_{i\in V_a} \). In the remainder of this sec-
tion, we reformulate the linear algebraic problem to lin-
ear least-squares problems to analyze the cases with and
without measurement errors in a unified way. In particu-
lar, we will consider two scenarios:

(A) The positions of the anchors are accurately known
and we only need to localize the followers.

(B) The measurements of the anchor positions may be
inaccurate and hence we need to localize both the
anchors and followers.

In both scenarios, the bearing measurements can be ei-
er either accurate or inaccurate. If a bearing or position
measurement is inaccurate, we always assume the measure-
ment error is \textit{constant}.

**Scenario A: Localizing the Followers**

Suppose the positions of the anchors are accurately
known and only the followers need to be localized. De-
note \{\hat{g}_{ij}\}_{(i,j)\in \mathcal{E}} as the measurements of \{g_{ij}\}_{(i,j)\in \mathcal{E}}. The measurement \hat{g}_{ij} is a unit vector which can be obtained by rotating the true bearing \(g_{ij}\) by a fixed, but unknown, angle. The measurement \(\hat{g}_{ij}\) can be either accurate or inaccurate. Although the true bearings satisfy \(g_{ij} = -g_{ji}\), if the bearing measurements are inaccurate, the relation \(\hat{g}_{ij} = -\hat{g}_{ji}\) does not hold in general.

The bearing-only network localization problem can be formulated as the following constrained optimization problem:

\[
\min_{\hat{p} \in \mathbb{R}^d} \quad J(\hat{p}) = \frac{1}{2} \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} \|P_{\hat{g}_{ij}}(\hat{p}_i - \hat{p}_j)\|^2 \quad (4)
\]

subject to \(\hat{p}_i = p_i, \quad i \in \mathcal{V}_a\).

It is clear that \(J(\hat{p}) \geq 0\) and \(J(\hat{p}) = 0\) when \(\|P_{\hat{g}_{ij}}(\hat{p}_i - \hat{p}_j)\| = 0, \forall (i,j) \in \mathcal{E}\). In the accurate case where \(\hat{g}_{ij} = g_{ij}, \forall (i,j) \in \mathcal{E}\), it then follows that \(J(p) = 0\) and hence the true position \(p\) is a global minimizer.

In the general case the minimizer may not occur at the true position \(p\). In this direction, express the objective function as the quadratic form

\[
J(\hat{p}) = \frac{1}{2} \hat{p}^T M \hat{p},
\]

where \(M \in \mathbb{R}^{dn \times dn}\) and the \(ij\)th block submatrix of \(M\) is

\[
[M]_{ij} = \begin{cases} 
M_{ij} = 0, & (i,j) \notin \mathcal{E}, \\
M_{ij} = -(P_{\hat{g}_{ij}} + P_{\hat{g}_{ji}}), & (i,j) \in \mathcal{E}, \\
M_{ii} = \sum_{j \in \mathcal{N}_i} (P_{\hat{g}_{ij}} + P_{\hat{g}_{ji}}), & i \in \mathcal{V}.
\end{cases} \quad (5)
\]

The matrix \(M\) resembles a weighted Laplacian matrix, where the weights are the projection matrices \(P_{\hat{g}_{ij}}\). The matrix is thus symmetric and positive semi-definite. Partition \(M\) as

\[
M = \begin{bmatrix} M_{aa} & M_{af} \\ M_{fa} & M_{ff} \end{bmatrix},
\]

where \(M_{aa} \in \mathbb{R}^{dn_a \times dn_a}, M_{af} = M_{fa}^T \in \mathbb{R}^{dn_a \times dn_f}\), and \(M_{ff} \in \mathbb{R}^{dn_f \times dn_f}\). By substituting \(\hat{p}_a = p_a\) into \(J(\hat{p})\), the constrained optimization problem (4) can be converted to the unconstrained problem

\[
\min_{\hat{p}_f \in \mathbb{R}^{dn_f}} \quad J(\hat{p}_f) = \frac{1}{2} (\hat{p}_f^T M_{ff} \hat{p}_f + 2 \hat{p}_a^T M_{af} \hat{p}_f + \hat{p}_a^T M_{aa} p_a).
\]

The solution to the least-squares problem is characterized as the follows.

**Theorem 3 (Condition for Unique Minimizer)** The least-squares problem (4) has a unique global minimizer if and only if the matrix \(M_{ff}\) is nonsingular. Moreover, the unique global minimizer is expressed by

\[
\hat{p}_f = -M_{ff}^{-1} M_{fa} p_a. \quad (6)
\]

When there are no measurement errors (i.e., when \(\hat{g}_{ij} = g_{ij}\)), \(\hat{p}_f = p_f\).

**Proof.** The gradient of the objective function can be calculated as \(\nabla_{\hat{p}_f} J(\hat{p}_f) = M_{ff} \hat{p}_f + M_{fa} p_a\). We can find the global minimizers by setting \(\nabla_{\hat{p}_f} J(\hat{p}_f) = 0\), which immediately leads to the following conclusions: (i) If \(M_{ff}\) is nonsingular, the global minimizer is unique and given by \(\hat{p}_f = -M_{ff}^{-1} M_{fa} p_a\). When there are no measurement errors, since the true position \(p_f\) is a feasible solution and the solution is unique, we know \(\hat{p}_f = p_f\). (ii) If \(M_{ff}\) is singular, the global minimizer is not unique and can be expressed as \(\hat{p}_f^* = -M_{ff}^{-1} M_{fa} p_a + x\) where \(M_{ff}^{-1}\) is the pseudoinverse of \(M_{ff}\) and \(x\) is an arbitrary vector in \(\text{Null}(M_{ff})\). □

Two important questions naturally follow from Theorem 3. The first is when \(M_{ff}\) is nonsingular, and the second is how large the localization error \(\|\hat{p}_f - p_f\|\) is. These two questions will be analyzed in detail in Sections 4 and 5.

**Scenario B: Localizing both the Anchors and Followers**

Suppose both of the bearing and the anchor position measurements may have constant errors. In this case, we need to estimate the positions of both the followers and the anchors. Denote \(\{\hat{p}_i\}_{i \in \mathcal{V}_a}\) as the measurements of \(\{p_i\}_{i \in \mathcal{V}_a}\).

The problem of bearing-only network localization in this scenario can be formulated as the unconstrained least-squares optimization problem,

\[
\min_{\hat{p} \in \mathbb{R}^d} \quad J(\hat{p}) = \frac{1}{2} \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} \|P_{\hat{g}_{ij}}(\hat{p}_i - \hat{p}_j)\|^2 \\
+ \frac{1}{2} \sum_{i \in \mathcal{V}_a} w_i \|\hat{p}_i - \hat{p}_a\|^2, \quad (7)
\]

where \(w_i > 0\) is a constant weight for the measurement \(\hat{p}_i\). If we are confident about the measurement \(\hat{p}_i\), the weight \(w_i\) should be large and vice versa. In the accurate case where \(\hat{g}_{ij} = g_{ij}, \forall (i,j) \in \mathcal{E}\) and \(\hat{p}_i = p_i, \forall i \in \mathcal{V}_a\), we have \(J(p) = 0\) and hence the true position \(p\) is a global minimizer. In the general case, in order to solve this
to be localized in either centralized or distributed ways. The localizability of a network is jointly determined by the topological structure and Euclidean location of the network as well as the selection of the anchors. In this section, we analyze and propose a variety of conditions for the localizability of bearing-only networks in arbitrary dimensions.

### 4.1 Bearing Laplacian

In the accurate case where \( \hat{g}_{ij} = g_{ij}, \forall (i, j) \in \mathcal{E} \), the matrix \([L]_{ij} = -(P_{gi} + P_{gj})\) defined in (5) becomes \([L]_{ij} = -2P_{gij}, \forall (i, j) \in \mathcal{E}\). As a result, the matrix \( M \) becomes \( M = 2L \), where \( L \in \mathbb{R}^{dn \times dn} \) and the \( ij \)-th block submatrix of \( L \) is

\[
\begin{align*}
[L]_{ij} & = 0, \quad (i, j) \notin \mathcal{E}, \\
[L]_{ij} & = -P_{gij}, \quad (i, j) \in \mathcal{E}, \\
[L]_{ii} & = \sum_{j \in \mathcal{N}_i} P_{gij}, \quad i \in \mathcal{V}.
\end{align*}
\]

The matrix \( L \) can be interpreted as a matrix-weighted graph Laplacian. Due to this special structure, we call \( L \) the bearing Laplacian since \( L \) carries the information of both the underlying graph and the bearings of the network. The matrix \( L \) is completely determined by the topological and Euclidean structure of the network. As will be seen later, the bearing Laplacian plays a fundamental role in the bearing-only network localization problem.

The bearing Laplacian \( L \) can be partitioned into

\[
L = \begin{bmatrix}
L_{aa} & L_{af} \\
L_{fa} & L_{ff}
\end{bmatrix},
\]

where \( L_{aa} \in \mathbb{R}^{dn_a \times dn_a}, L_{af} = \mathbf{1}_{dn_a}^T \in \mathbb{R}^{dn_a \times dn_f}, \) and \( L_{ff} \in \mathbb{R}^{dn_f \times dn_f} \). The useful properties of \( L \) are listed as below.

**Lemma 8** For any network \( \mathcal{G}(p) \), the bearing Laplacian \( L \) satisfies the following:

(a) \( L \) is symmetric positive semi-definite,  
(b) \( \text{Rank}(L) \leq dn - d - 1 \) and \( \text{Null}(L) \supseteq \text{span} \{ \mathbf{1} \otimes I_d, p \} \),  
(c) \( \text{Rank}(L) = dn - d - 1 \) and \( \text{Null}(L) = \text{span} \{ \mathbf{1} \otimes I_d, p \} \)  
if and only if \( \mathcal{G}(p) \) is infinitesimally bearing rigid.

**Proof.** Consider an arbitrary orientation of the graph \( \mathcal{G} \). The bearings can be indexed as \( \{ g_k \}_{k=1}^m \). Then, the bearing Laplacian \( L \) can be written as \( L = H^T \text{diag}(P_{gk}) H \). From property (a) of Lemma 1,
\[ P_{gk} = P_{gk}^T P_{gk}, \] and thus \( L \) can be expressed as
\[ L = \bar{H}^T \text{diag}(P_{gk}^T) \text{diag}(P_{gk}) \bar{H} = \mathcal{R}^T \mathcal{R}. \]

Note \( \mathcal{R} = \text{diag} \left( \| e_a \|_{I_d} R_B \right) \) where \( R_B \) is the bearing rigidity matrix. As a result, the matrix \( \mathcal{R} \) has the same rank and the same null space as \( R_B \). It follows from Lemma 5 that \( \text{Rank}(\mathcal{R}) \leq dn - d - 1 \) and \( \text{Null}(\mathcal{R}) \supseteq \text{span} \{ 1 \otimes I_d, p \} \). By Theorem 1, we know \( \text{Rank}(\mathcal{R}) = dn - d - 1 \) and \( \text{Null}(\mathcal{R}) = \text{span} \{ 1 \otimes I_d, p \} \) if and only if \( \mathcal{G}(p) \) is infinitesimally bearing rigid. Since \( \mathcal{L} \) has the same rank and null space as \( \mathcal{R} \), the proof is complete. \( \square \)

**Lemma 9** The matrix \( L_{ff} \) is positive semi-definite and satisfies
\[ L_{ff} p_f = -L_{fa} p_a, \]
where \( p_a \) and \( p_f \) are the positions of the anchors and followers, respectively.

**Proof.** For any nonzero \( x \in \mathbb{R}^{dn_f} \), denote \( \bar{x} = [0, x^T] \in \mathbb{R}^{dn} \). Since \( L \geq 0 \), we have \( x^T L_{ff} x = \bar{x}^T L \bar{x} \geq 0 \). As a result \( L_{ff} \) is positive semi-definite. It follows from Lemma 8 that \( L p = 0 \), which implies \( L_{fa} p_a + L_{ff} p_f = 0 \). \( \square \)

### 4.2 Conditions for Localizability

We first present an algebraic necessary and sufficient condition for localizability. This condition actually establishes the equivalency between the nonsingularity of \( L_{ff} \) and the localizability of the network.

**Theorem 5 (Algebraic Condition for Localizability)**
For any network \( \mathcal{G}(p) \), the following statements are equivalent:

(a) The network \( \mathcal{G}(p) \) is localizable;

(b) The matrix \( L_{ff} \) is nonsingular;

(c) The true position \( p \) is the unique global minimizer of the optimization problems (4) and (7) in the accurate case.

**Proof.** The equivalence between statements (a) and (c) follows immediately from Definition 3 and Lemma 6. The equivalence between statements (b) and (c) follows from Theorems 3 and 4. \( \square \)

A fundamental problem that naturally follows Theorem 5 is what kind of networks have nonsingular \( L_{ff} \) or what a localizable network intuitively looks like. This problem has been partially solved by some existing works. It has been shown in [13, 14] that a network is localizable if the network is infinitesimally bearing rigid and has more than two anchors. This condition, however, is sufficient but not necessary. As one main contribution of this paper, we next propose a necessary and sufficient rigidity condition. This rigidity condition is mathematically equivalent to the algebraic condition in Theorem 5, but it can give us more intuition on what localizable networks look like.

**Theorem 6 (Rigidity Condition for Localizability)**
For any network \( \mathcal{G}(p) \), the following statements are equivalent:

(a) The network \( \mathcal{G}(p) \) is localizable;

(b) Every infinitesimal bearing motion of the network involves at least one anchor. Equivalently, for any nonzero infinitesimal bearing motion \( \delta p \in \text{Null}(L) \), with
\[ \delta p = \begin{bmatrix} \delta p_a \\ \delta p_f \end{bmatrix}, \]
one has \( \delta p_a \neq 0 \), where \( \delta p_a \) corresponds to the anchors.

**Proof.** Since the network is localizable if and only if \( L_{ff} \) is nonsingular, we can prove the result by showing that \( L_{ff} \) is singular if and only if there exists an infinitesimal bearing motion \( \delta p \in \text{Null}(L) \) with \( \delta p_a = 0 \). Firstly, suppose \( L_{ff} \) is singular and there exists a nonzero vector \( x \in \mathbb{R}^{dn_f} \) such that \( L_{ff} x = 0 \). Let \( \delta p = [0, x^T] \in \mathbb{R}^{dn} \). Then it follows from \( \delta p^T L \delta p = x^T L_{ff} x \) that \( \delta p^T L \delta p = 0 \). Therefore, \( \delta p \) is an infinitesimal bearing motion with \( \delta p_a = 0 \) and \( \delta p_f = x \). Secondly, suppose there exists an infinitesimal motion \( \delta p \) with \( \delta p_a = 0 \) and \( \delta p_f \neq 0 \). Then it follows from \( \delta p_f^T L_{ff} \delta p_f = \delta p^T L \delta p \) and \( \delta p^T L \delta p = 0 \) that \( \delta p_f^T L_{ff} \delta p_f = 0 \), which implies that \( L_{ff} \) is singular. \( \square \)

Theorem 6 fully describes the relationship between localizability and bearing rigidity. As suggested by Theorem 6, localizability does not require the network to be infinitesimally bearing rigid because it allows non-trivial infinitesimal bearing motions as long as the bearing motions involve at least one anchor.

The intuition behind Theorem 6 is as follows. If there is an infinitesimal bearing motion for a network, then there exist different networks having exactly the same bearings as the true network. As a result, the infinitesimal bearing motion introduces some uncertainties to the localization of the true network. However, if the infinitesimal motion involves some anchors, these uncertainties can be resolved by the anchors whose positions are known.
We now illustrate the rigidity condition in Theorem 6 with some examples. Figure 2 shows examples of non-localizable networks. These networks are non-localizable because for each network there exist infinitesimal bearing motions that only correspond to the followers. For example, the follower in the network Figure 2(a) can move in the horizontal direction without changing the bearings. The two followers in the network Figure 2(b) can move together in the vertical direction without changing any bearings. For the network Figure 2(c), the inner and the outer triangles are concentric. The scale of the inner triangle of the three followers can be changed without affecting the bearings. Finally, in the network Figure 2(d), the scale of the right and left triangles can be changed without affecting the bearings. It is worth mentioning that all the four networks are not infinitesimally bearing rigid.

Figure 3 shows examples of localizable networks. The networks in Figure 3(a)-(d) are obtained by modifying the networks in Figure 2. As can be seen, by adding extra edges or assigning anchors to different nodes, a non-localizable network can become localizable. Finally, in the network Figure 2(e), the scale of the right and left triangles can be changed without affecting the bearings. It is worth mentioning that all the four networks are not infinitesimally bearing rigid.

Corollary 1 If the network $\mathcal{G}(p)$ is localizable, then
\[ n_a \geq \frac{\dim(\text{Null}(L))}{d} > 1. \]

Proof. Let $k = \dim(\text{Null}(L))$ and $B \in \mathbb{R}^{d_a \times k}$ be a basis matrix of $\text{Null}(L)$ (i.e., $\text{Range}(B) = \text{Null}(L)$). Then any infinitesimal bearing motion in $\text{Null}(L)$ is also in $\text{Range}(B)$ and hence can be expressed as $Bx$, where $x \in \mathbb{R}^k, x \neq 0$. Partition $B$ and express $Bx$ as
\[
Bx = \begin{bmatrix} B_a x \\ B_f x \end{bmatrix},
\]
where $B_a \in \mathbb{R}^{d_n \times k}$. According to Theorem 6, the network is localizable if and only if $B_a x \neq 0, \forall x \in \mathbb{R}^k, x \neq 0$. As a result, the matrix $B_a$ must have full column rank, which requires $B_a$ to be a tall matrix with $dn_a \geq k$. Finally, since $k = \dim(\text{Null}(L)) \geq d + 1$ by Lemma 8, we have $n_a \geq k/d > 1$. □

Corollary 1 implies two important facts. The first fact is that there must exist at least two anchors for a localizable network. The second fact is that more anchors are required for localizability when the dimension of $\text{Null}(L)$ increases. Intuitively speaking, the dimension of $\text{Null}(L)$ can be viewed as a measure of the “degree of bearing rigidity”; the dimension of $\text{Null}(L)$ is the smallest when the network is infinitesimally bearing rigid and it increases otherwise. As a result, the second fact suggests an intuition that the localizability of a network is jointly determined by two factors: the bearing rigidity and the anchors. As long as the two factors can be well balanced, the network localizability can be guaranteed. We can consider an extreme case where all the nodes in the network are anchors. In this case, the location of the entire network is already known and the network is always localizable even when the edge set $\mathcal{E}$ is empty. Another extreme case is when there are only two (i.e., the minimum number) anchors. In this case, the network must be infinitesimally bearing rigid in order to guarantee localizability, as we will show later.
The following result shows that infinitesimal bearing rigidity is sufficient to guarantee localizability.

**Corollary 2** When \( n_a \geq 2 \), if \( \mathcal{G}(p) \) is infinitesimally bearing rigid, then it is localizable.

**Proof.** Suppose \( \mathcal{G}(p) \) is infinitesimally bearing rigid. Then we have \( \text{Null}(L) = \text{span}\{1 \otimes I_d, p\} \). As a result, any infinitesimal bearing motion \( \delta p \in \text{Null}(L) = \text{span}\{1 \otimes I_d, p\} \) can be expressed as a linear combination of \( 1 \otimes I_d \) and \( p \). Since no two anchors have the same position, there does not exist a linear combination having \( \delta p_a = 0 \) if \( n_a \geq 2 \). Therefore, the network is localizable by Theorem 6. □

The intuition behind Corollary 2 is obvious. If a network is infinitesimally bearing rigid, it follows from the bearing rigidity theory that the network can be uniquely determined up to a translation and a scaling factor. Furthermore, if there are more than one anchors, the translation and the scale of the network can be determined by the anchors and thus the entire network can be uniquely determined.

It is already known that the condition in Corollary 2 is sufficient but not necessary for localizability (see, for example, Figure 3(c)-(e)). We next present a more relaxed condition than Corollary 2. To do that, we need to first introduce the following notion.

**Definition 4 (Augmented Network)** Given a network \( \mathcal{G}(p) \) with \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \), denote by \( \bar{\mathcal{G}}(p) \) an augmented network with \( \bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}) \) where \( \mathcal{E} \subseteq \bar{\mathcal{E}} \) and \( \bar{\mathcal{E}} \) is obtained by connecting every pair of anchors to each other (i.e., \( (v_i, v_j) \in \bar{\mathcal{E}}, \forall v_i, v_j \in \mathcal{V}_a \)).

The notion of an augmented network is demonstrated in Figure 4. It should be noted that connecting any pair of anchors in a network only changes \( L_{aa} \) but not \( L_{ff} \) in the bearing Laplacian. As a result, \( \mathcal{G}(p) \) and \( \bar{\mathcal{G}}(p) \) have exactly the same \( L_{ff} \) and hence they have the same localizability properties.

**Corollary 3** When \( n_a \geq 2 \), if \( \bar{\mathcal{G}}(p) \) is infinitesimally bearing rigid, then \( \mathcal{G}(p) \) is localizable.

**Proof.** If \( \bar{\mathcal{G}}(p) \) is infinitesimally bearing rigid, it follows from Corollary 2 that the network \( \bar{\mathcal{G}}(p) \) is localizable and hence \( L_{ff} \) is nonsingular. Since \( \mathcal{G}(p) \) and \( \bar{\mathcal{G}}(p) \) have the exactly the same \( L_{ff} \), the network \( \mathcal{G}(p) \) is also localizable. □

Corollary 3 is more relaxed than Corollary 2 because it does not require \( \mathcal{G}(p) \) to be infinitesimally bearing rigid. Corollary 3 can be viewed as a generalization of the result [14, Corollary 10] which is applicable to two-dimensional cases.

It is suggested by Corollary 3 that the infinitesimal bearing rigidity of \( \bar{\mathcal{G}}(p) \) is sufficient to ensure the localizability of \( \mathcal{G}(p) \). An important yet unexplored question is whether it is also necessary. The answer is negative when \( n_a \geq 3 \). For example, see the localizable network in Figure 3(e), where the three anchors are collinear and hence the augmented network \( \bar{\mathcal{G}}(p) \) is not bearing rigid. However, as shown by the following result, when there are exactly two anchors, the infinitesimal bearing rigidity of \( \bar{\mathcal{G}}(p) \) is both sufficient and necessary for the localizability of \( \mathcal{G}(p) \).

**Theorem 7** When \( n_a = 2 \), a network \( \mathcal{G}(p) \) is localizable if and only if the augmented network \( \bar{\mathcal{G}}(p) \) is infinitesimally bearing rigid.

**Proof.** The sufficiency has already been proved in Corollary 3. We next prove the necessity by contradiction. Suppose \( \mathcal{G}(p) \) is localizable and so is \( \bar{\mathcal{G}}(p) \). Assume \( \mathcal{G}(p) \) is not infinitesimally bearing rigid. Then \( \bar{\mathcal{G}}(p) \) has a non-trivial infinitesimal bearing motion \( \delta p \) which is not in span \( \{1 \otimes I_d, p\} \). Write \( \delta p \) as

\[
\delta p = \begin{bmatrix}
\delta p_1 \\
\delta p_2 \\
(*)
\end{bmatrix},
\]

where \( \delta p_1, \delta p_2 \in \mathbb{R}^d \) corresponds to the two anchors. Because the infinitesimal motion \( \delta p \) preserves all the bearings including the bearing between \( p_1 \) and \( p_2 \), we know that \( \delta p_1 - \delta p_2 \) is parallel to \( p_1 - p_2 \). As a result, there exists a nonzero scalar \( k \) such that \( \delta p_1 - \delta p_2 = k(p_1 - p_2) \). Construct \( \delta p' \) as

\[
\delta p' \triangleq \delta p + 1_n \otimes (k_{p_2} - k_{p_2}) - kp = \begin{bmatrix}
\delta p_1 \\
\delta p_2 \\
(*)
\end{bmatrix} + \begin{bmatrix}
k_{p_2} - k_{p_2} \\
k_{p_2} - k_{p_2} \\
(*)
\end{bmatrix} - \begin{bmatrix}
k_{p_1} \\
k_{p_2} \\
(*)
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
(*)
\end{bmatrix}.
\]

Note \( \delta p' \) is an infinitesimal bearing motion because it is a linear combination of \( \delta p, 1 \otimes I_d, \) and \( p \). Moreover,
and the localization error. Before we proceed, it is of the measurement errors on the nonsingularity of we consider the inaccurate case and analyze the impact lent to the localizability of the network. In this section, and showed that the nonsingularity of \( M \) in the previous section, we considered the accurate case works.

In fact, the triangulation-based method is related to a more complicated problem, construction of localizable networks, which will be studied specifically in future works.

2 Sensitivity Analysis

To conclude this section, we present an intuitive method for examining the localizability of a network without any mathematical calculations. This method is based on a basic localization technique known as triangulation [26]. The triangulation technique is demonstrated by the example shown in Figure 5(a). In this example, there are two anchors and one follower. Given the bearings of the two edges, the follower node can be triangulated as the intersection point of the two bearings. Since two nonparallel bearings only have one intersection point, the follower node can be uniquely localized. The triangulation method can be iteratively applied to check the localizability of the three-dimensional cubic network shown in Figure 5(b). The three red nodes can be firstly localized by triangulation based on the two anchors. Then, the two blue nodes can be further triangulated based on the red nodes. The green node can be finally determined by triangulation based on the two blue nodes. Therefore, the network is localizable.

It turns out that the triangulation-based method works well for many simple networks such as those in Figure 3. In fact, the triangulation-based method is related to a more complicated problem, construction of localizable networks, which will be studied specifically in future works.

5 Sensitivity Analysis

In the previous section, we considered the accurate case and showed that the nonsingularity of \( M_{ff} \) is equivalent to the localizability of the network. In this section, we consider the inaccurate case and analyze the impact of the measurement errors on the nonsingularity of \( M_{ff} \) and the localization error. Before we proceed, it is worth mentioning that \( M_{ff} \) can be nonsingular even for a non-localizable network given certain bearing measurement errors. In our work, we only study the impact of measurement errors on localizable networks.

Recall \( M_{ff} = 2L_{ff} \) and \( M_{fa} = 2L_{fa} \) in the accurate case. In the presence of measurement errors, the matrices \( M_{ff} \) and \( M_{fa} \) can be viewed as perturbations of \( L_{ff} \) and \( L_{fa} \), respectively. In particular, we can write \( M_{ff} = 2(L_{ff} + \Delta L_{ff}) \) and \( M_{fa} = 2(L_{fa} + \Delta L_{fa}) \). Therefore, the optimal estimate given in (6) is the solution to the perturbed linear system

\[
(L_{ff} + \Delta L_{ff})\hat{p}_f = -(L_{fa} + \Delta L_{fa})p_a.
\]

Perturbed linear systems have been well studied [27, Section III]. But in order to obtain useful conclusions for bearing-only network localization, we need to consider the specific features of the above perturbed system. First of all, an intuitive conclusion that can be immediately drawn from the perturbed linear system is that if the network is localizable (i.e., \( L_{ff} \) is nonsingular) and the bearing errors are sufficiently small (i.e., \( \|\Delta L_{ff}\| \) and \( \|\Delta L_{fa}\| \) are small), then the matrix \( M_{ff} \) would be nonsingular and the optimal estimate \( \hat{p}_f \) would be sufficiently close to the true value \( p_f \).

Let \( \theta_{ij} \in [0, \pi] \) be the angle between \( \tilde{g}_{ij} \) and \( g_{ij} \) (i.e., \( g_{ij}^T\tilde{g}_{ij} = \cos \theta_{ij} \)). In our work, we use the angle \( \theta_{ij} \) to represent the inconsistency between \( \tilde{g}_{ij} \) and \( g_{ij} \). This representation is valid for arbitrary dimensions. Note \( \theta_{ij} \neq \theta_{ji} \) in general. We define the total bearing measurement error for the followers as

\[
\epsilon_1 \triangleq \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} (\sin \theta_{ij} + \sin \theta_{ji}).
\]

We define \( \epsilon_1 \) in this way is because of the fact that \( \|P_{\tilde{g}_{ij}} - P_{g_{ij}}\| = \sin \theta_{ij} \) by Lemma 3. The next lemma shows how \( \epsilon_1 \) can be used to characterize \( \Delta L_{ff} \) and \( \Delta L_{fa} \).

**Lemma 10** For an arbitrary network \( \mathcal{G}(p) \) with arbitrary bearing measurements \( \{\tilde{g}_{ij}\}_{(i,j) \in \mathcal{E}} \),

\[
\|\Delta L_{ff}\| \leq \epsilon_1, \quad \|\Delta L_{fa}\| \leq \frac{1}{2} \epsilon_1.
\]

**Proof.** Denote \( \Delta P_{g_{ij}} \triangleq P_{\tilde{g}_{ij}} - P_{g_{ij}}, \forall (i,j) \in \mathcal{E} \). Then we have \( \|\Delta P_{g_{ij}}\| = \sin \theta_{ij} \) by Lemma 3. It follows from
Theorem 8 suggests that a large \( \lambda_{\min}(L_{ff}) \) would give the network a large tolerance to bearing measurement errors. Therefore, \( \lambda_{\min}(L_{ff}) \) is an important parameter of the network and it is meaningful to study how to maximize it. One possible way is to optimally select the anchors. This topic is, however, out of the scope of this paper.

\[
M_{ff} = 2(L_{ff} + \Delta L_{ff})
\]

That
\[
\|2\Delta L_{ff}\| = \|M_{ff} - 2L_{ff}\|
\]

\[
\leq \sum_{i \in V_f} \sum_{j \in V_f \cap N_i} \|\Delta P_{g_{ij}} + \Delta P_{g_{ji}}\|
\]

\[
+ \sum_{i \in V_f} \sum_{j \in N_i} (\|\Delta P_{g_{ij}}\| + \|\Delta P_{g_{ji}}\|)
\]

\[
\leq 2 \sum_{i \in V_f} \sum_{j \in N_i} (\|\Delta P_{g_{ij}}\| + \|\Delta P_{g_{ji}}\|)
\]

\[
= 2 \sum_{i \in V_f} \sum_{j \in N_i} (\sin \theta_{ij} + \sin \theta_{ji}) = 2 \epsilon_1.
\]

It follows from \( M_{fa} = 2(L_{fa} + \Delta L_{fa}) \) that

\[
\|2\Delta L_{fa}\| = \|M_{fa} - 2L_{fa}\|
\]

\[
\leq \sum_{i \in V_f} \sum_{j \in V_f \cap N_i} \|\Delta P_{g_{ij}} + \Delta P_{g_{ji}}\|
\]

\[
\leq \sum_{i \in V_f} \sum_{j \in N_i} (\|\Delta P_{g_{ij}}\| + \|\Delta P_{g_{ji}}\|)
\]

\[
= \sum_{i \in V_f} \sum_{j \in N_i} (\sin \theta_{ij} + \sin \theta_{ji}) = \epsilon_1.
\]

\[\square\]

We are now ready to present a sufficient condition to guarantee the nonsingularity of \( M_{ff} \) in the presence of bearing measurement errors.

**Theorem 8** Given a localizable network with \( L_{ff} \) nonsingular, suppose \( \lambda_{\min}(L_{ff}) > 0 \) is the smallest eigenvalue of \( L_{ff} \). The matrix \( M_{ff} \) is nonsingular if

\[
\epsilon_1 < \lambda_{\min}(L_{ff}).
\]

**Proof.** Since \( \|\Delta L_{ff}\| < \epsilon_1 \) by Lemma 10, it follows from (9) that \( \|\Delta L_{ff}\| < \lambda_{\min}(L_{ff}) = 1/\|L_{ff}^{-1}\| \), which further implies \( \|L_{ff}^{-1}\Delta L_{ff}\| < \|L_{ff}^{-1}\| \|\Delta L_{ff}\| < 1 \). As a result, the spectral radius \( \rho(L_{ff}^{-1}\Delta L_{ff}) < 1 \) and hence the matrix \((I + L_{ff}^{-1}\Delta L_{ff})\) is nonsingular. Therefore, \( M_{ff} = 2(L_{ff} + \Delta L_{ff}) = 2L_{ff}(I + L_{ff}^{-1}\Delta L_{ff}) \) is nonsingular. \( \square \)

Theorem 8 suggests that a large \( \lambda_{\min}(L_{ff}) \) would give the network a large tolerance to bearing measurement errors. Therefore, \( \lambda_{\min}(L_{ff}) \) is an important parameter of the network and it is meaningful to study how to maximize it. One possible way is to optimally select the anchors. This topic is, however, out of the scope of this paper.

The following result gives an upper bound for the localization error of the optimal estimate in (6).

**Theorem 9** Given a localizable network with \( L_{ff} \) nonsingular, if the total bearing measurement error satisfies (9), the optimal estimates \( \hat{p}_a = -M_{ff}^{-1}M_{fa}p_a \) given in (6) satisfies

\[
\|\hat{p}_a - p_a\| \leq \epsilon_1 \frac{\epsilon_1}{\lambda_{\min}(L_{ff}) - \epsilon_1} \left(\frac{1}{2}\|p_a\| + \|p_f\|\right)
\]

**Proof.** See Appendix D. \( \square \)

The next result gives a upper bound of the localization error of the optimal estimate \( \hat{p}_a = (M + S^TWS)^{-1}S^T\hat{p}_a \) in (8). The proof is similar to Theorems 8 and 9 and omitted here.

**Theorem 10** Given a localizable network with \( L_{ff} \) nonsingular, define \( \epsilon_2 = 2 \sum_{i \in \cal{V}} \sum_{j \in \cal{N}_i} (\sin \theta_{ij} + \sin \theta_{ji}) \). If

\[
\epsilon_2 < \lambda_{\min}(2L + S^TWS),
\]

then the matrix \((M + S^TWS)\) is nonsingular and the optimal solution \( \hat{p}_a \) satisfies

\[
\|\hat{p}_a - p_a\| \leq \frac{1}{\lambda_{\min}(2L + S^TWS) - \epsilon_2} \left(\|p_a\| + \|W\|\|\Delta p_a\|\right),
\]

where \( \Delta p_a = \hat{p}_a - p_a \) is the error of the anchor position measurement.

**Remark 1** Recall (8) is nonsingular if and only if \( M_{ff} \) is nonsingular. As a result, the condition in Theorem 8 can also ensure the nonsingularity of \((M + S^TWS)\). However, in order to derive a upper bound for \( \|\hat{p}_a - p_a\| \), we define a new error \( \epsilon_2 \) in Theorem 10.

At the end of this section, we would like to discuss the impact of the errors in the anchor position measurements on the optimal estimate (6). It is assumed in the optimization problem (4) that each anchor can measure its own position accurately. However, if this assumption cannot be satisfied in practice, it is meaningful to discuss how the errors would affect the estimate. For the sake of simplicity, we only consider the case where all the bearing measurements are accurate. If the anchor position measurements are accurate, then the optimal estimate (6) is \( p_f = -L_{ff}^{-1}L_{fa}p_a \); otherwise, the final estimate becomes

\[
\hat{p}_f = -L_{ff}^{-1}L_{fa}(p_a + \Delta p_a) = p_f - L_{ff}^{-1}L_{fa}\Delta p_a,
\]

where \( \Delta p_a \) denotes the anchor position error. We next
analyze the localization error \( -L_{ff}^{-1}L_{fa}\Delta p_a \) in three typical cases.

(a) Suppose \( \Delta p_a \) represents a common translational error for all the anchors such that \( \Delta p_a \) can be written as \( \Delta p_a = 1_{na} \otimes r \) where \( r \in \mathbb{R}^d \). It follows from \( L(1_n \otimes r) = 0 \) by Lemma 8 that \( -L_{ff}^{-1}L_{fa}(1_n \otimes r) = 1_{nf} \otimes r \) and consequently

\[
\hat{p}_f = p_f + 1_{nf} \otimes r.
\]

The above equation indicates that a common translational error in the anchor measurements would cause the same translational error for the estimates of followers. This conclusion is illustrated by the example shown in Figure 6(a).

(b) Suppose \( \Delta p_a \) represents a scaling error of \( p_a \) such that \( \Delta p_a \) can be written as \( \Delta p_a = kp_a \) where \( k \in \mathbb{R} \). It is easy to see

\[
\hat{p}_f = p_f + kp_f.
\]

The above equation indicates that a scaling error in the anchor measurements would cause the same scaling error in the estimates of the followers.

(c) When \( \Delta p_a \) corresponds to a more complicated error, the effect of \( \Delta p_a \) on \( \hat{p}_f \) would also be complicated. As demonstrated by Figure 6(b)-(c), the errors in the anchor position measurements may or may not cause localization errors for the followers. The mathematical interpretation of example (c) is that \( \Delta p_a \in \text{Null}(L_{fa}) \) and hence \( -L_{ff}^{-1}L_{fa}\Delta p_a = 0 \).

6 Distributed Network Localization Protocols

In this section, we propose distributed localization protocols to globally localize bearing-only networks in arbitrary dimensions. Similar to Section 3, we consider two scenarios and propose continuous-time protocols to distributedly solve the least-squares problems (4) and (7), respectively.

\[
-P_{g_{ij}}(\hat{p}_i(t) - \hat{p}_j(t))\]

\[
\hat{p}_i(t) = -\sum_{j \in \mathcal{N}_i} (P_{g_{ij}} + P_{g_{ji}})(\hat{p}_i(t) - \hat{p}_j(t)), \quad i \in \mathcal{V}_f.
\]

Two remarks regarding protocol (11) are given below. First, the protocol is distributed because it only requires the quantities of node \( i \) and node \( i \)'s neighbors. The quantities \( \{\tilde{g}_{ij}\}_{j \in \mathcal{N}_i} \) can be directly measured by node \( i \). The other quantities including \( \{\tilde{g}_{ji}\}_{j \in \mathcal{N}_i} \) need to be transmitted via communication to node \( i \) from its neighbors. Second, the protocol has a clear geometric interpretation. To see that, consider the accurate case where \( \hat{g}_{ij} = g_{ij} \) and \( \hat{g}_{ji} = g_{ji} \), and the protocol becomes

\[
\hat{p}_i(t) = -2 \sum_{j \in \mathcal{N}_i} P_{g_{ij}}(\hat{p}_i(t) - \hat{p}_j(t)).
\]

As illustrated in Figure 7, the term \(-P_{g_{ij}}(\hat{p}_i(t) - \hat{p}_j(t))\) is the orthogonal projection of \( \hat{p}_i(t) - \hat{p}_j(t) \) onto the orthogonal compliment of \( g_{ij} \), and hence it acts to reduce the bearing error. In addition, protocol (12) can be viewed as an extension of the protocol proposed in [14].

**Theorem 11** If \( M_{ff} \) is nonsingular, the estimate \( \hat{p}_f(t) \) given by protocol (11) globally and exponentially converges to the optimal estimate \( \hat{p}_f^* = -M_{ff}^{-1}M_{fa}p_a \). If \( M_{ff} \) is singular, the estimate \( \hat{p}_f(t) \) converges to \( \hat{p}_f = -M_{ff}^{-1}M_{fa}p_a + x(\hat{p}_f(0)) \), where \( M_{ff}^\dagger \) denotes the pseudoinverse of \( M_{ff} \), and \( x(\hat{p}_f(0)) \) is the orthogonal projection of the initial estimate \( \hat{p}_f(0) \) onto the null space of \( M_{ff} \).

**Proof.** If \( M_{ff} \) is nonsingular, the proof is obvious. If \( M_{ff} \) is singular, let \( M_{ff} = USU^T \) be the singular value decomposition and partition \( U \) and \( \Sigma \) to \( U = [U_0, U_1] \).
and $\Sigma = \text{diag}(\Sigma_0, \Sigma_1)$ where $\Sigma_0 = 0$ and $\Sigma_1 > 0$. If $M_{ff}$ is singular, the solution to (10) can be straightforwardly calculated as for all $t \geq 0$

$$\dot{\hat{p}}_f(t) = (U_0U_0^T + U_1e^{-\Sigma_1t}U_1^T) \dot{\hat{p}}_f(0) - (U_0U_0^T + U_1 \Sigma_1^{-1}(I - e^{-\Sigma_1t}U_1^T)) M_{fa}p_a.$$  

Firstly, the term $e^{-\Sigma_1t}$ converges to zero as $t \to \infty$. Secondly, the term $-(U_0U_0^T M_{fa} p_a)t = 0$ since any $x \in \text{Null}(M_{ff})$ always satisfies $x^TM_f = 0$. To see that, denote $\bar{x} = [0, x^T]^T \in \mathbb{R}^{dn}$ and then $M_{ff}x = 0 \Leftrightarrow x^TM_f \bar{x} = 0 \Leftrightarrow M_{af}x + M_{ff}x = 0 \Rightarrow M_{af}x = 0 \Rightarrow x^TM_{fa} = 0$. As a result, the final estimate given by protocol (11) is

$$\dot{\hat{p}}_f(\infty) = -(U_1 \Sigma_1^{-1}U_1^T) M_{fa} p_a + U_0U_0^T \dot{\hat{p}}_f(0),$$

where $U_1 \Sigma_1^{-1}U_1^T = M_{ff}^\dagger$ is the pseudoinverse of $M_{ff}$ and $U_0U_0^T \dot{\hat{p}}_f(0) = x(\dot{\hat{p}}_f(0))$ is the orthogonal projection of the initial guess $\tilde{p}_f(0)$ onto the null space of $M_{ff}$. $\square$

It is notable that protocol (11) requires $\{\tilde{g}_j\}_{j \in \mathcal{N}_i}$ which cannot be measured by node $i$. In the case where $\{\tilde{g}_j\}_{j \in \mathcal{N}_i}$ are not available to node $i$, we can use the following distributed protocol to localize the network:

$$\dot{\hat{p}}_i(t) = - \sum_{j \in \mathcal{N}_i} P_{g_j} (\hat{p}_i(t) - \hat{p}_j(t)), \quad i \in \mathcal{V}_f. \quad (13)$$

In the absence of measurement errors, protocol (13) is the same as the protocol (11) (with the additional gain of 2). In the presence of measurement errors, protocol (13) cannot achieve the optimal estimate that minimizes the least squares problem (4) in general. However, this protocol can be interpreted as a local gradient descent protocol for each node aiming at minimizing their own local objective functions. More specifically, define a local objective function for each follower as

$$J_i = \frac{1}{2} \sum_{j \in \mathcal{N}_i} \| P_{g_j} (\hat{p}_i - \hat{p}_j) \|^2, \quad i \in \mathcal{V}_f.$$  

It is easy to verify that protocol (13) satisfies $\dot{\hat{p}}_i = -\nabla_{\hat{p}_i} J_i$. The analysis of protocol (13) is analogous to protocol (11) and omitted here.

Scenario B: Distributed Localization of both Anchors and Followers

The least squares problem (7) can be distributedly solved by the following gradient descent protocol:

$$\dot{\hat{p}}(t) = -\nabla_{\hat{p}} J(\hat{p}) = -(M + S^T W S)\hat{p}(t) + S^T W \tilde{p}_a,$$  

(14)

whose elementwise expression is

$$\dot{\hat{p}}_i(t) = - \sum_{j \in \mathcal{N}_i} (P_{g_j} + P_{\tilde{g}_j})(\hat{p}_i(t) - \hat{p}_j(t)) - w_i(\hat{p}_i(t) - \tilde{p}_i), \quad i \in \mathcal{V}_a,$$

$$\dot{\hat{p}}_f(t) = - \sum_{j \in \mathcal{N}_i} (P_{g_j} + P_{\tilde{g}_j})(\hat{p}_i(t) - \hat{p}_j(t)), \quad i \in \mathcal{V}_f. \quad (15)$$

The above protocol for the followers is exactly the same as protocol (11). The convergence of the above protocol is given as below.

**Theorem 12** If $M_{ff}$ is nonsingular, the estimate $\hat{p}(t)$ given by protocol (15) globally and exponentially converges to the optimal estimate $\hat{p}^* = (M + S^T W S)^{-1} S^T W \tilde{p}_a$. If $M_{ff}$ is singular, the estimate $\hat{p}(t)$ converges to $\hat{p} = (M + S^T W S)^{\dagger} S^T W \tilde{p}_a + x(\hat{p}(0))$, where $\dagger$ denotes the pseudoinverse and $x(\hat{p}(0))$ is the orthogonal projection of the initial estimate $\hat{p}(0)$ onto the null space of $M + S^T W S$.

**Proof.** The proof is similar to that of Theorem 11 and omitted. $\square$

It is notable that protocol (15) requires $\{\tilde{g}_j\}_{j \in \mathcal{N}_i}$ which cannot be measured by node $i$. In the case where $\{\tilde{g}_j\}_{j \in \mathcal{N}_i}$ are not available to node $i$, we can use the following distributed protocol to localize the network:

$$\dot{\hat{p}}_i(t) = - \sum_{j \in \mathcal{N}_i} P_{g_j} (\hat{p}_i(t) - \hat{p}_j(t)) - w_i(\hat{p}_i(t) - \tilde{p}_i), \quad i \in \mathcal{V}_a,$$

$$\dot{\hat{p}}_f(t) = - \sum_{j \in \mathcal{N}_i} P_{g_j} (\hat{p}_i(t) - \hat{p}_j(t)), \quad i \in \mathcal{V}_f. \quad (16)$$

In the absence of measurement errors, protocol (16) is the same as the protocol (15) (the only difference is a constant control gain as two). But in the presence measurement errors, protocol (16) cannot achieve the optimal solution to the least squares problem (7) in general. However, the protocol (16) can be interpreted as a local gradient descent protocol for each node aiming at minimizing their own local objective functions. More specifically, define a local objective function for each node as

$$J_i = \frac{1}{2} \sum_{j \in \mathcal{N}_i} \| P_{g_j} (\hat{p}_i - \hat{p}_j) \|^2 + \frac{1}{2} w_i \| \hat{p}_i - \tilde{p}_i \|^2, \quad i \in \mathcal{V}_a,$$

$$J_i = \frac{1}{2} \sum_{j \in \mathcal{N}_i} \| P_{g_j} (\hat{p}_i - \hat{p}_j) \|^2, \quad i \in \mathcal{V}_f.$$  

It can be verified that protocol (16) satisfies $\dot{\hat{p}}_i = -\nabla_{\hat{p}_i} J_i$. The analysis of protocol (16) is analogous to protocol (15) and omitted here.
7 Simulation Examples

In this section, we present simulation results to demonstrate the localization protocols (11) and (15). The network to be localized in the simulation is a three-dimensional cubic network (see the blue network in Figure 8). This network consists of \(n = 8\) nodes and \(m = 13\) edges. Two nodes are anchors and the rest are followers. It can be calculated that \(\text{Rank}(R_B) = 20 = 3n - 4\) and hence the network is infinitesimally bearing rigid. As a result, the network is localizable by Corollary 2.

For the protocol (11), the anchor positions are known and only the followers need to be localized. The initial estimate of the network location is given in Figure 8(a). In the accurate case where there are no bearing measurement errors, the followers can be successfully localized as shown in Figure 9(a)-(b). In the inaccurate case where the bearing measurements are corrupted by constant errors, there is a final localization error as shown in Figure 9(c)-(d). In the simulation, the inaccurate bearing measurements are randomly generated in a way that the angle \(\theta_{ij}, \forall (i, j) \in E\) is drawn from a unified distribution on \([0, 10]\) degree.

For the protocol (15), both of the anchors and the followers need to be localized. The initial estimate of the network location is given in Figure 8(b), where the initial estimates of the anchor positions are not correct. In the accurate case, both of the anchors and the followers can be successfully localized as shown in Figure 10(a)-(b). In the inaccurate case, there is a final localization error as shown in Figure 10(c)-(d). In this simulation, the inaccurate bearing measurements are randomly generated in a way that the angle \(\theta_{ij}, \forall (i, j) \in E\) is drawn from a unified distribution on \([0, 10]\) degree. The inaccurate anchor position measurements are the two hollow squares as shown in Figure 10(d).

Some important observations can be obtained from the simulation. Firstly, the accuracy of the measurements of the anchor positions has a great impact on the final localization error. As demonstrated in Figure 10(d),
when the anchor position measurements are inaccurate, the localization protocol would translate or scale the entire network to suppress \( \|\hat{p}_i(t) - p_i\|, \forall i \in V_a \). Secondly, the measurement errors may cause localization error but will never cause instability. This is consistent with previous analysis of the localization protocols. Thirdly, in the accurate case, the localization error \( \|\hat{p}(t) - p\| \) will always decreases monotonically as shown in Figure 9(a) and Figure 10(a), but it may not be true for the inaccurate case as shown in Figure 9(c) and Figure 10(c). For example, protocol (11), in the accurate case, becomes \( \hat{p}_f = -2L_{ff} + 2L_{fa} p_a \). Define \( V = \|\hat{p}_f - p_f\|^2/4 \). Then \( V = -(\hat{p}_f - p_f)^T L_{ff}(\hat{p}_f - p_f) \leq 0 \) and hence \( \|\hat{p}_f - p_f\| \) decreases monotonically. But in the inaccurate case, we do not have the same result for protocol (11) though it can be guaranteed that \( J(\hat{p}) = \hat{p}^T M \hat{p} \) decreases monotonically. Fourthly, the sufficient conditions for the nonsingularity of \( M_{ff} \) given in Theorems 8 and 10 are conservative. For the example in Figure 9(c), the bearing measurement error \( \epsilon_1 = 3.43 \) is greater than \( \lambda_{\min}(L_{ff}) = 0.035 \). For the example in Figure 10(c), the bearing error \( \epsilon_2 = 11.64 \) is greater than \( \lambda_{\min}(M + S^T W S) = 0.045 \). But the protocols still work well given the large bearing measurement errors.

8 Conclusions

This paper studied the problem of the localization of stationary bearing-only sensor networks in arbitrary dimensions. We showed that this problem can be formulated as a linear least-squares problem. One main contribution of the paper is to propose a variety of conditions for localizability. The sensitivity of the network localization to constant measurement errors has also been analyzed. Distributed protocols have been proposed to globally localize bearing-only networks in arbitrary dimensions.

There are several important topics for future research. First, we assume in this paper that the underlying sensing graph is undirected. It is meaningful to study the case of directed sensing graphs. Moreover, other factors such as time delay and switching graphs should also be considered in the future. Finally, we assume in this paper that each sensor can measure the bearings of their neighbors in a global reference frame. The case without a global reference frame is also an interesting topic.

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A Proof of Lemma 1

Proof. It is easy to examine (a). We next prove (b)-(c). Firstly, it is straightforward to examine \( P_x x = 0 \). Secondly, suppose \( \{z_1, \ldots, z_{d-1}\} \) is an orthogonal basis of the orthogonal compliment of \( x \). Because \( z_i \perp x \) \( (i = 1, \ldots, d-1) \), we have \( P_x z_i = z_i \). Therefore, the zero eigenvalue is simple, and all the other \( d-1 \) eigenvalues are 1. Finally, for an arbitrary vector \( z \in \mathbb{R}^d \), we have \( z^T P_x z = \tilde{z}^T \tilde{P}_x \tilde{z} = z^T P_x z = \|P_x z\|^2 \geq 0 \) and hence \( P_x \) is positive semi-definite. □

B Proof of Lemma 3

Proof. Without loss of generality, assume \( x \) and \( y \) are two unit vectors satisfying \( \|x\| = \|y\| = 1 \). Then, we have \( P_x = I_d - xx^T \), \( P_y = I_d - yy^T \), and
\[
\|P_x - P_y\| = \|xx^T - yy^T\|.
\]
There always exists an orthogonal matrix \( U \in \mathbb{R}^{d \times d} \) such that the two vectors \( x \) and \( y \) can be orthogonally transformed to
\[
Ux = [1, 0, 0, \ldots, 0]^T,
\]
\[
Uy = [\cos \theta, \sin \theta, 0, \ldots, 0]^T.
\]
Since the spectral norm is invariant to orthogonal matrices, we have
\[
\|P_x - P_y\| = \|U(xx^T - yy^T)U^T\|
\]
\[
= \left\|\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \sin \theta \cos \theta & \sin^2 \theta \end{bmatrix}\right\|
\]
\[
= \sin \theta \|Q\|.
\]
where
\[
Q = \begin{bmatrix} \sin \theta & -\cos \theta \\ -\cos \theta & -\sin \theta \end{bmatrix}.
\]
It is easy to see \( Q^T Q = I_2 \) and hence \( Q \) is an orthogonal matrix. Then, \( \|P_x - P_y\| = \sin \theta \|Q\| = \sin \theta \|I\| = \sin \theta \). □

C Proof of Lemma 7

Proof. Since \( M \) is symmetric positive semi-definite, there exists \( A \in \mathbb{R}^{dn \times dn} \) such that \( M = A^T A \). Partition \( A \) into \( A = [A_0, A_I] \) where \( A_I \in \mathbb{R}^{dn \times dn} \). Then, we have \( M_{ff} = A_I^T A_I \). Therefore, \( M_{ff} \) is nonsingular if and only if \( A_I \) has full column rank.
On the other hand, denote
\[
\tilde{A} \triangleq \begin{bmatrix} A & A_f \\ W^\frac{1}{2}S & \end{bmatrix} = \begin{bmatrix} A_0 & A_f \\ W^\frac{1}{2} & 0 \end{bmatrix}.
\]
It is clear that \( M + S^T W S = \tilde{A}^T \tilde{A} \). As a result, the matrix \( M + S^T W S \) is nonsingular if and only if \( \tilde{A} \) has full column rank. Because \( W^\frac{1}{2} \) is diagonal, the first \( d_{u_a} \) columns of \( \tilde{A} \) are linear independent to the rest. Therefore, \( \tilde{A} \) has full column rank if and only if \( A_f \) has full column rank.

Combining the above two aspects completes the proof. \( \square \)

D Proof of Theorem 9

Proof. Recall \( p_f = -L_{ij}^{-1}L_{fa}p_a \). Rewrite \( \tilde{p}_f \) as 
\[
\tilde{p}_f = -L_{ij}^{-1}L_{fa}p_a + \tilde{p}_f
\]
substituting which into \( \tilde{p}_f \) gives
\[
\tilde{p}_f = -L_{ij}^{-1}L_{fa}p_a - L_{ij}^{-1}L_{ff}p_f + L_{ij}^{-1}L_{ff} \left( I + L_{ij}^{-1}L_{ff} \right)^{-1}L_{ij}^{-1}L_{fa}p_a
\]
\[
\tilde{p}_f = p_f - \left( I + L_{ij}^{-1}L_{ff} \right)^{-1}L_{ij}^{-1}L_{fa}p_a
\]
\[
\tilde{p}_f = L_{ij}^{-1}L_{ff} \left( I + L_{ij}^{-1}L_{ff} \right)^{-1}L_{ij}^{-1}L_{fa}p_a
\]
It follows that
\[
\| \tilde{p}_f - p_f \| \leq \left( I + L_{ij}^{-1}L_{ff} \right)^{-1}L_{ij}^{-1}L_{fa}p_a
\]
Substituting \( \| \Delta L_{ff} \| \leq \epsilon_1 \) and \( \| L_{fa} \| \leq \epsilon_1/2 \) as shown in Lemma 10, and
\[
\| (I + L_{ij}^{-1}L_{ff})^{-1} \| \leq \frac{1}{1 - \| L_{ij}^{-1}L_{ff} \| \| \Delta L_{ff} \|}
\]
by [29, Lemma 2.3.3] into the above inequality gives
\[
\| \tilde{p}_f - p_f \| \leq \frac{\| L_{ij}^{-1}L_{fa}p_a \| + \| p_f \| \epsilon_1}{1 - \| L_{ij}^{-1}L_{fa} \| \epsilon_1},
\]
Substituting \( \| L_{ij}^{-1}L_{fa} \| = 1/\lambda_{\text{min}}(L_{ff}) \) completes the proof. \( \square \)

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