A criterion of quasi-infinite divisibility for discrete laws

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Abstract

We consider arbitrary discrete probability laws on the real line. We obtain a criterion of their belonging to a new class of quasi-ininitely divisible laws, which is a wide natural extension of the class of well known infinitely divisible laws through the Lévy type representations.

Keywords and phrases: discrete probability laws, characteristic functions, spectral Lévy type representations, quasi-infinitely divisible laws.

1 Introduction and problem setting

This note is devoted to the question posed in [1] about the description of quasi-infinitely divisible laws within the class of arbitrary univariate discrete probability laws.

The class of quasi-infinitely divisible laws is a rather wide extention of the class of well known infinitely divisible laws. Let $F$ be a distribution function on the real line with the characteristic function $f$. Following [12], $F$ and $f$ (and the corresponding probability law) are called quasi-infinitely divisible if $f$ admits the following Lévy type representation:

$$f(t) = \exp \left\{ it\gamma - \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R}\setminus\{0\}} (e^{itu} - 1 - itu \sin u) d\Lambda(u) \right\}, \ t \in \mathbb{R}, \quad (1)$$

for some $\gamma \in \mathbb{R}$ (the set of real numbers), $\sigma^2 \geq 0$, and function $\Lambda$ (the Lévy spectral function), which has a finite total variation on every interval $(-\infty, -r]$ and $[r, \infty)$, $r > 0$, and it satisfies

$$\lim_{u \to -\infty} \Lambda(u) = \lim_{u \to +\infty} \Lambda(u) = 0, \quad \text{and} \quad \int_{0<|u|<\delta} x^2 d|\Lambda|(u) < +\infty, \quad \delta > 0.$$ 

Here $f$ uniquely determines the triplet $(\gamma, \sigma^2, \Lambda)$. Quasi-infinitely divisible $F$ and $f$ are rationally infinitely divisible, i.e. there exist infinitely divisible distribution functions $F_1$ and $F_2$ with characteristic functions $f_1$ and $f_2$, respectively, such that $F_1 = F \ast F_2$, where “$\ast$” is the convolution, or in the equivalent form: $f(t) = f_1(t)/f_2(t), \ t \in \mathbb{R}$. It is clear that every infinitely divisible distribution

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function is quasi-infinitely divisible with non-decreasing \( \Lambda \) on every interval \((-\infty, 0)\) and \((0, \infty)\). The opposite is not true, the examples can be found in the classical monographs [7], [14] and [15].

The first detailed analysis of quasi-infinitely divisible laws based on their Lévy type representations was performed in [12], where the authors studied questions concerning supports, moments, continuity, and weak convergence for these laws. These results were generalized and complemented in the papers [1], [2], [3], [4], [5], [8], [9] and [10]. The quasi-infinitely divisible laws have already found a lot of interesting applications (see [5], [6], [13], [16], [17] and the references given there).

It is rather natural and interesting to investigate criteria of quasi-infinite divisibility for probability laws. The most deep results here were obtained for univariate discrete laws in [1], [9], and [12]. In particular, following [12], a discrete lattice probability law is quasi-infinitely divisible if its characteristic function \( f \) has no zeroes on the real line, i.e. \( f(t) \neq 0, t \in \mathbb{R} \). For arbitrary discrete laws only the sufficient condition is known due to [1]. Namely, if \( f \) is separated from zero, i.e. \( |f(t)| \geq \mu > 0 \) for some \( \mu \) and all \( t \in \mathbb{R} \), then the law is quasi-infinitely divisible (see the next section for more details). The latter result generalizes the previous, because in the lattice case the function \( t \mapsto |f(t)|, t \in \mathbb{R} \), is continuous and periodic, and the absence of zeroes is equivalent to the separatness from zero over the period segment (see [1] for more details). Anyway, however, the following interesting question remains here (it was posed in [1]). Are all characteristic functions of quasi-infinitely divisible laws separated from zero? In this note we will show that it is true. So we obtain a criterion of quasi-infinite divisibility for arbitrary discrete laws.

We will use the following notation. We denote by \( \mathbb{R} \), \( \mathbb{Z} \) and \( \mathbb{N} \) the sets of real numbers, integers, and positive integers, respectively. We write \( a_n \sim b_n \) for number sequences \( a_n \) and \( b_n \), \( n \in \mathbb{N} \), if \( a_n/b_n \to 1, n \to \infty \).

### 2 The result

Let us consider an arbitrary discrete distribution function

\[
F(x) := \sum_{k \in \mathbb{N} : x_k \leq x} p_{x_k}, \quad x \in \mathbb{R},
\]

where \( x_k, k \in \mathbb{N}, \) are distinct real numbers, \( p_{x_k} \geq 0, k \in \mathbb{N}, \) and \( \sum_{k=1}^{\infty} p_{x_k} = 1 \). Let \( f \) be the characteristic function of \( F \):

\[
f(t) = \sum_{k \in \mathbb{N}} p_{x_k} e^{itx_k}, \quad t \in \mathbb{R}.
\] (2)

Let \( X \) be the set of all points of growth of \( F \), i.e. \( X := \{ x_k : p_{x_k} > 0, k \in \mathbb{N} \} \neq \emptyset \). Let us introduce the set of all finite \( \mathbb{Z} \)-linear combinations of elements from the set \( X \):

\[
\langle X \rangle := \left\{ \sum_{k=1}^{m} c_k x_k : c_k \in \mathbb{Z}, x_k \in X, m \in \mathbb{N} \right\}.
\]

In other words, \( \langle X \rangle \) is a module over the ring \( \mathbb{Z} \) with the generating set \( X \). Since \( X \neq \emptyset \), \( \langle X \rangle \) is an infinite countable set. The following result was obtained in [1].
Theorem 1 Suppose that $\inf_{t \in \mathbb{R}} |f(t)| > 0$, i.e. there exists $\mu > 0$ such that $|f(t)| \geq \mu > 0$ for all $t \in \mathbb{R}$. Then $f$ admits the following representation

$$f(t) = \exp \left\{ it\gamma_0 + \sum_{u \in \langle X \rangle \setminus \{0\}} \lambda_u (e^{i tu} - 1) \right\}, \quad t \in \mathbb{R},$$

where $\gamma_0 \in \langle X \rangle$, $\lambda_u \in \mathbb{R}$ for all $u \in \langle X \rangle \setminus \{0\}$, and $\sum_{u \in \langle X \rangle \setminus \{0\}} |\lambda_u| < \infty$. Here $F$ is quasi-infinitely divisible and $f$ has representation (1) with

$$\gamma = \gamma_0 + \sum_{u \in \langle X \rangle \setminus \{0\}} \lambda_u \sin(u), \quad \sigma^2 = 0, \quad \Lambda(x) = \begin{cases} \sum_{u \in \langle X \rangle: u \leq x} \lambda_u, & x < 0, \\ -\sum_{u \in \langle X \rangle: u > x} \lambda_u, & x > 0. \end{cases}$$

Thus the condition $\inf_{t \in \mathbb{R}} |f(t)| > 0$ implies the quasi-infinite divisibility of $F$. We complement this fact by the following proposition.

Theorem 2 Suppose that $\inf_{t \in \mathbb{R}} |f(t)| = 0$. Then $F$ is not quasi-infinitely divisible.

Proof. It is known that characteristic functions of quasi-infinitely divisible laws have no zeroes on the real line (see [12]). Therefore if there exists $t_0 \in \mathbb{R}$ such that $f(t_0) = 0$, then $F$ is not quasi-infinitely divisible. Hence we focus only on the case $f(t) \neq 0$ for all $t \in \mathbb{R}$ with $\inf_{t \in \mathbb{R}} |f(t)| = 0$.

Suppose, contrary to our claim, that $F$ is a quasi-infinitely divisible distribution function. Then $f$ admits representation (1) with some triplet $(\gamma, \sigma^2, \Lambda)$. Let us consider for fixed $\tau \in \mathbb{R}$ the following function

$$\psi_\tau(t) := \frac{f(t + \tau)f(t - \tau)}{f(t)^2} = \exp \left\{ -\sigma^2 \tau^2 + 2 \int_{\mathbb{R} \setminus \{0\}} e^{i \tau x} (\cos(\tau x) - 1) d\Lambda(x) \right\}, \quad t \in \mathbb{R}.$$ 

Due to the conditions for $\Lambda$, for any fixed $\tau \in \mathbb{R}$ we have

$$\sup_{t \in \mathbb{R}} |\psi_\tau(t)| \leq \exp \left\{ -\sigma^2 \tau^2 + 2 \int_{\mathbb{R} \setminus \{0\}} (1 - \cos(\tau x)) d|\Lambda|(x) \right\} < \infty. \quad (4)$$

Let $(t_m)_{m \in \mathbb{N}}$ be an increasing sequence such that $t_m \to +\infty$ and $f(t_m) \to 0$, $m \to \infty$. Let us define $\varphi_m(\tau) := f(t_m + \tau)$, $\tau \in \mathbb{R}$, $m \in \mathbb{N}$. Since $f$ is an almost periodic function by (2), the sequence $(\varphi_m)_{m \in \mathbb{N}}$ is relatively compact in the topology of uniform convergence on $\mathbb{R}$ (see [11] pp. 23–24). So there exists a subsequence $(\varphi_{m_l})_{l \in \mathbb{N}}$ that uniformly converges to an almost periodic function $\varphi$, i.e.

$$\sup_{\tau \in \mathbb{R}} |f(t_{m_l} + \tau) - \varphi(\tau)| \to 0, \quad l \to \infty. \quad (5)$$

From this we have

$$\sup_{\tau \in \mathbb{R}} |f(t_{m_l} + \tau)f(t_{m_l} - \tau) - \varphi(\tau)\varphi(-\tau)| \to 0, \quad l \to \infty. \quad (6)$$

Indeed, for any $\tau \in \mathbb{R}$ and $l \in \mathbb{N}$ it holds that

$$f(t_{m_l} + \tau)f(t_{m_l} - \tau) - \varphi(\tau)\varphi(-\tau) = (f(t_{m_l} + \tau) - \varphi(\tau))f(t_{m_l} - \tau) + (f(t_{m_l} - \tau) - \varphi(-\tau))\varphi(\tau).$$
Due to $|f(t_{m_i} - \tau)| \leq 1$ and $|\varphi(\tau)| \leq 1$ for any $\tau \in \mathbb{R}$ (see (2) and (3)), we get the inequality

$$|f(t_{m_i} + \tau)f(t_{m_i} - \tau) - \varphi(\tau)\varphi(-\tau)| \leq |f(t_{m_i} + \tau) - \varphi(\tau)| + |f(t_{m_i} - \tau) - \varphi(-\tau)|$$

$$\leq 2\sup_{\tau \in \mathbb{R}} |f(t_{m_i} + \tau) - \varphi(\tau)|,$$

which (together with (3)) implies (4).

Suppose that $\varphi(\tau)\varphi(-\tau) = 0$ for any $\tau \in \mathbb{R}$, i.e.

$$\sup_{\tau \in \mathbb{R}} |f(t_{m_i} + \tau)f(t_{m_i} - \tau)| \to 0, \quad l \to \infty.$$ 

In particular, for any fixed $s \in \mathbb{R}$ we have

$$\left|f(t_{m_i} + \tau)f(t_{m_i} - \tau)\right|_{\tau = t_{m_i} + s} = |f(2t_{m_i} + s)f(-s)| \to 0, \quad l \to \infty.$$ 

Since $f(t) \neq 0$, $t \in \mathbb{R}$, we conclude that $|f(2t_{m_i} + s)| \to 0, l \to \infty$ for any fixed $s \in \mathbb{R}$. Next, from the sequence $(m_i)_{i \in \mathbb{N}}$ we can choose a subsequence $(m_i')_{i \in \mathbb{N}}$ such that the functions $s \mapsto f(2t_{m_i} + s)$ uniformly converge on $\mathbb{R}$ as $l \to \infty$ (see [11] pp. 23–24). It is clear that their uniform limit is zero, i.e.

$$\sup_{s \in \mathbb{R}} |f(2t_{m_i} + s)| \to 0, \quad l \to \infty.$$ 

So, in particular, $|f(0)| = |f(2t_{m_i} - 2t_{m_i})| \to 0, l \to \infty$, but $f(0) = 1$, a contradiction.

Thus there exists $\tau \in \mathbb{R}$ such that $\varphi(\tau)\varphi(-\tau) \neq 0$. Then, due to (4) and the convergence $f(t_{m_i}) \to 0, l \to \infty$, for this $\tau$ we have

$$|\psi_\tau(t_{m_i})| = \frac{|f(t_{m_i} + \tau)f(t_{m_i} - \tau)|}{|f(t_{m_i})|^2} \sim \frac{|\varphi(\tau)\varphi(-\tau)|}{|f(t_{m_i})|^2} \to \infty, \quad l \to \infty.$$ 

Thus we found $\tau$ such that $\sup_{t \in \mathbb{R}} |\psi_\tau(t)| = \infty$. This contradicts (4). Therefore $f$ can not be the characteristic function of a quasi-infinitely divisible law. \(\square\)

The examples of $f$ satisfying $\inf_{t \in \mathbb{R}} |f(t)| = 0$ were considered in [11] and [9].

Thus Theorem 1 and Theorem 2 yield the following criterion: \textit{discrete distribution function $F$ is quasi-infinitely divisible if and only if} $\inf_{t \in \mathbb{R}} |f(t)| > 0$, \textit{i.e.} there exists $\mu > 0$ such that $|f(t)| \geq \mu > 0$ for all $t \in \mathbb{R}$. Moreover, now we know that the characteristic functions of all discrete quasi-infinitely divisible laws admit representation (3) with some $\gamma_0$ and $\lambda_n$.

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