MEAN CURVATURE FLOW IN FUCHSIAN MANIFOLDS

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Abstract. Motivated by questions in detecting minimal surfaces in hyperbolic manifolds, we study the behavior of geometric flows in complete hyperbolic three-manifolds. In most cases the flows develop singularities in finite time. In this paper, we investigate the mean curvature flow in a class of complete hyperbolic three-manifolds (Fuchsian manifolds) which are warped products of a closed surface of genus at least two and \( \mathbb{R} \). In particular, we prove that there exists a large class of closed initial surfaces, as geodesic graphs over the totally geodesic surface \( \Sigma \), such that the mean curvature flow exists for all time and converges to \( \Sigma \). This is among the first examples of converging mean curvature flows of compact hypersurfaces in Riemannian manifolds. We also provide some useful calculations for the general warped product setting.

1. Introduction

1.1. The setting. The mean curvature flow has been studied extensively in various ambient Riemannian manifolds and in most cases the flow of closed submanifolds develops singularities in finite time by the avoidance principle for the mean curvature flow. For instance, we know that the finite time singularity has to occur for any compact initial hypersurface in Euclidean space under the mean curvature flow ([Hui84]). The study of singularity formation has been a focal point of the field, see for instance [Hui86, HS09, CM12] and many others. Hyperbolic manifolds are known to possess extremely rich geometric structures. Our motivation is to better understand the mean curvature flow in hyperbolic manifolds, hoping that in further studies we can detect interesting geometric objects by running the mean curvature flow or similar flows to time infinity without developing any singularity or after handling possible singularities.

As a first step, we focus on the Fuchsian manifolds in this paper. Fuchsian manifolds are probably the most elementary complete, non-simply connected hyperbolic three-manifolds. A Fuchsian manifold is obtained as a quotient space of \( \mathbb{H}^3 \) by a Fuchsian group. Let \( M^3 \) be a Fuchsian manifold, and we always assume the genus of any incompressible surface of \( M^3 \) is at least

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two so that it carries its own hyperbolic metric. From differential geometry point of view, it is a warped product of a hyperbolic surface $\Sigma$ with $\mathbb{R}$, with the metric

$$ds^2 = dr^2 + \cosh^2(r)g_0,$$

where $g_0$ is the hyperbolic metric on $\Sigma$. Therefore the surface $\Sigma$ is totally geodesic in $M^3$. Clearly it is the only such surface in $M^3$.

Our main analytical tool is the mean curvature flow equation, which has the following form:

$$\frac{\partial}{\partial t} F(x,t) = -H(x,t)\nu(x,t),$$

$$F(\cdot,0) = F_0,$$

where $H(x,t)$ and $\nu(x,t)$ are the mean curvature and unit normal vector respectively at $F(x,t)$ of the evolving surface $S(t)$, and our convention of the mean curvature is the sum of the principal curvatures.

**Definition 1.1.** A smooth closed surface $S_0$ in $M^3$ is a (geodesic) graph over the totally geodesic surface $\Sigma$ if there is a constant $c_0 > 0$ such that the angle function $\Theta_0 = \langle n, \nu_0 \rangle \geq c_0$, where $\langle \cdot, \cdot \rangle$ is the metric over $M^3$, $n = \frac{\partial}{\partial r}$ is the unit normal vector field over $M$ which is perpendicular to $\Sigma$ and $\nu_0$ is the unit normal vector on $S_0$ of our choice.

Note that $\Theta_0 \in (0, 1]$ if $S_0$ is a graph, and $\Theta_0 \equiv 1$ if and only if $S_0$ is equidistant from $\Sigma$ (sometimes called parallel to $\Sigma$, or a level surface to $\Sigma$).

The mean curvature flow in warped product manifolds was also investigated by other authors, see e.g. [BM12]. Note that the warped product structures in [BM12] are completely different from ours. Geometrically, their warped structure can be thought as a real line bundle over a surface, while ours is a surface bundle over the real line. So the evolving hypersurfaces in their case are equidistant graphs over a reference hypersurface, while our evolving hypersurfaces are the more natural geodesic graphs. One does not in general expect a geodesic graph to stay geodesic graphs under the mean curvature flow. In fact, it was mentioned in [BM12] that in [Unt98] Unterberger gave an example of hypersurface which is a geodesic graph but loses this graphical property when it evolves under the mean curvature flow. On the other hand, we will see in this work that there indeed exists a large class of closed initial surfaces $S_0$’s in Fuchsian manifolds, as geodesic graphs over the totally geodesic surface $\Sigma$ with an explicit lower bound on the angle of $S_0$, such that the mean curvature flow starting from $S_0$ remains as geodesic graphs for all time and converges smoothly to $\Sigma$. 
1.2. Main Result. In this paper, we prove that if the angle function on the initial surface has a positive lower bound depending only on its maximal distance to the reference surface $\Sigma$, then the mean curvature flow with such an initial surface exists for all time and converges smoothly to the totally geodesic surface $\Sigma$ in a Fuchsian manifold $M^3 = (\mathbb{R} \times_{\cosh(r)} \Sigma, dr^2 + \cosh^2(r) g_0)$. More precisely, we have

**Theorem 1.2.** Let $M^3$ be a Fuchsian manifold and $\Sigma$ be the unique closed totally geodesic surface in $M^3$. Then for any $a_0 > 0$, if the initial smooth closed surface $S_0 \subset M^3$ has hyperbolic distance no larger than $a_0$ to $\Sigma$ and the minimum of the initial angle satisfies

\[
\min_{p \in S_0} \Theta_0(p) \geq \tanh(a_0),
\]

then the mean curvature flow with initial surface $S_0$ exists for all time, remains as geodesic graph over $\Sigma$ and converges continuously to $\Sigma$. Moreover, the convergence is smooth if the above inequality is strict.

**Remark 1.3.** Notably, in the special case where $\min_{p \in S_0} \Theta_0 = 1$, namely the initial surface $S_0$ is equidistant from the totally geodesic surface $\Sigma$, the evolving surfaces $S(t)$ remains equidistant from $\Sigma$ (i.e., $\Theta(t) = \langle n, \nu \rangle(t) \equiv 1$ for all $t \geq 0$), see the explicit solution (3.2). Moreover, interestingly we note that this result yields some geometric interactions between the flow and the ambient space, namely, the lower bound of the angle function $\Theta_0$ of $S_0$ on the right hand side of (1.2) is just the principle curvature of the equidistant surface $\Sigma(a_0)$, where $a_0$ is the maximal distance of $S_0$ to $\Sigma$.

Our techniques can be generalized to higher dimensional warped product manifolds of similar structure, possibly with appropriate variations of curvature conditions. In this paper, we stay with the current setting of Fuchsian manifolds.

1.3. Interaction between geometry and analysis. We want to highlight the interaction between analytical methods and geometric structures. Our setting of Fuchsian manifolds allows us to take advantage of its hyperbolic geometry in several stages of this work. It is a basic fact that the level sets $\{(\Sigma(r), \cosh^2(r)g_0)\}_{r \in \mathbb{R}}$ of the totally geodesic surface $\Sigma = \Sigma(0)$ form an equidistant foliation of the Fuchsian manifold $M^3$. Moreover, each fiber of the foliation is umbilic, which enables us to obtain the mean curvature flow with initial surface $\Sigma(r)$ an explicit solution for any fixed $r \in \mathbb{R}$ (see (3.2)). We use this special mean curvature flow as barriers and the avoidance principle for the mean curvature flows (see e.g. [Hui86]) to push flow
to the destination $\Sigma$ (see §3.1). Furthermore, we use the presence of a special vector field $V = \cosh(r) \frac{\partial}{\partial r}$ (see (2.3)) in a Fuchsian manifold to derive explicitly the evolution equation for the angle $\Theta$ (see (3.6)).

1.4. **Outline of the paper.** We provide some preliminary results in §2. Heart of the matter is to prove the preservation of graphical property of the flow, and we prove our main result Theorem 1.2 in §3. The scheme is the following. We first show the evolving surface must stay in a bounded region in $M^3$ for all time (the Squeeze Lemma 3.1) as long as the flow exists, then we derive the evolution equation for the angle function $\Theta(\cdot, t)$ (Theorem 3.6), and then most of the work is devoted to prove that the evolving surfaces stay graphical under the initial condition on distance and angle. Note that the uniform positive lower bound of the angle function $\Theta = \langle \mathbf{n}, \nu \rangle$ locally gives the uniform $C^1$-estimate of the graph function which represents the evolving surface. Therefore once we have established uniform bound for $\Theta$, standard parabolic theory ([LSU68]) gives bounds for all higher derivatives. In particular the second fundamental form for the evolving surface $S_t$ in $M^3$ is uniformly bounded. Huisken’s theorem ([Hui86]) then guarantees that the mean curvature flow exists for all time. The Squeeze Lemma 3.1 then gives the convergence of the flow. The smoothness will follow.

Finally, in §4 we remove the assumption of the angle function on the initial graph and illustrate the possible formation of singularities.

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2. Preliminaries

In this section, we fix our notations, and introduce some preliminary facts that will be used in this paper.
2.1. **Fuchsian manifolds.** A Fuchsian manifold $M^3$ is defined as a warped product space $\mathbb{R} \times \Sigma$. This is a complete hyperbolic three-manifold of fundamental importance in hyperbolic geometry and Kleinian group theory. The metric on a Fuchsian manifold $M^3$ is explicitly given as:

\[(M^3, g_M) = (\mathbb{R} \times \cosh(r) \Sigma, dr^2 + \cosh^2(r) g_0),\]

where $g_0$ is the induced metric on the surface $\Sigma$ which is hyperbolic. A fascinating feature of the geometry of the Fuchsian manifold $M^3$ is that the level set $\Sigma(r)$ of the totally geodesic surface $\Sigma(0) = \Sigma$ forms a global foliation of $M^3$, and each fiber surface $\Sigma(r)$ is umbilic, with constant principal curvature of $\tanh(r)$, cf. [O’N83].

Another important fact which we will make use of in the paper is the existence of the special vector field $V$ in $M^3$ (see for instance [Bre13]). Namely,

\[(2.2) \quad V = \cosh(r)n\]

satisfies that

\[(2.3) \quad \bar{\nabla} X V = \sinh(r)X,\]

for any smooth vector field $X$, and $\bar{\nabla}$ is the Levi-Civita covariant derivative in $M^3$. From this, one can furthermore deduce the following well-known formula. However, we include a direct proof in our setting for completeness:

**Lemma 2.1.** Let $X$ be any tangent vector field in a Fuchsian manifold $M^3$, then we have

\[(2.4) \quad \bar{\nabla}_X n = \tanh(r)(X - \langle X, n \rangle n).\]

**Proof.** Since both sides of the identity are linear in $X$, we only have to verify the identity by taking a local frame. Fix any point $\bar{q} \in M^3$ and let $\{\bar{e}_1, \bar{e}_2\}$ be a local normal frame at the corresponding point $q \in \Sigma$, then $\{\bar{e}_1, \bar{e}_2, \bar{e}_3 = n\}$ forms a local normal frame at $\bar{q}$. Then (2.4) is obviously true for $X = n$. When $X = \bar{e}_1$, we have

$$\bar{\nabla}_{\bar{e}_1} n = \bar{\Gamma}^3_{31} n + \bar{\Gamma}^j_{31} \bar{e}_j.$$ 

Since the metric on $M^3$ is given explicitly as (2.1), its Christoffel symbols $\bar{\Gamma}^k_{ij}$ can be computed explicitly. In our case, we have $\bar{\Gamma}^3_{31} = 0$ and $\bar{\Gamma}^j_{31} \bar{e}_j = \tanh(r)\bar{e}_j$. One sees that (2.4) holds for $X = \bar{e}_1$. Similarly we can verify the case of $X = \bar{e}_2$. \qed
2.2. Mean curvature flow. Let $F_0 : S \to M^3$ be the immersion of an incompressible surface $S$ in a Fuchsian manifold $M^3$. We assume that $S_0 = F_0(S)$ is a graph over $\Sigma$ with respect to $\mathbf{n}$, i.e., $\langle \mathbf{n}, \nu \rangle \geq c_0 > 0$, here $c_0$ is a constant to be determined later.

We consider a family of immersions of surfaces in $M^3$ moving under the mean curvature flow (1.1):

$$F : S \times [0, T) \to M^3, \quad 0 \leq T \leq \infty$$

with $F(\cdot, 0) = F_0$. For each $t \in [0, T)$, $S(t) = \{F(x, t) \in M^3 \mid x \in S\}$ is the evolving surface at time $t$, and $S(0)$ is the initial surface $S_0$.

The short-time existence of the solutions to (1.1) is standard for closed immersions, see e.g. [HP99]. Initial compact surface develops singularities in finite time along the mean curvature flow in Euclidean space, and in fact the norm of the second fundamental form blows up if the singularity occurs in finite time, see [Hui84, Hui86].

2.3. Evolution equations. In this subsection, for completeness we collect and derive a number of evolution equations of some quantities on $S(t)$, $t \in [0, T)$, which are involved in our calculations. We include here standard evolution equations for the mean curvature $H(\cdot, t)$, and the square norm of the second fundamental form $|A(\cdot, t)|^2$.

**Theorem 2.2.** ([Hui86])

\begin{equation}
\left( \frac{\partial}{\partial t} - \Delta \right) H = H(|A|^2 - 2),
\end{equation}

\begin{equation}
\left( \frac{\partial}{\partial t} - \Delta \right) |A|^2 = -2|\nabla A|^2 + 2|A|^4 + 4(|A|^2 - H^2).
\end{equation}

**Proof.** These equations are deduced for general Riemannian manifolds in [Hui86]. In our case of hyperbolic three-manifold, the ambient space $M^3$ has constant sectional curvature $-1$, and the Ricci curvature $\text{Ric}(\nu, \nu) = -2$ for any unit normal vector $\nu$.

The lemma then follows from combining these explicit curvature conditions and curvature equations $\bar{R}_{3i3j} = -g_{ij}$, as well as the well-known Simons’ identity (see e.g. [Sim68] or [SSY75]), satisfied by the second fundamental form $a_{ij}$.

Our estimates are mainly for the height function $u(x, t)$ and the angle function, which is the cosine of the genuine angle, $\Theta(x, t)$ on $S(t)$:

\begin{equation}
u(x, t) = \tau(F(x, t)),
\end{equation}

\begin{equation}
\Theta(x, t) = \langle \nu, \mathbf{n} \rangle(F(x, t)).
\end{equation}
Here \( r(p) = \pm \text{dist}(p, \Sigma) \) for all \( p \in M^3 \), the \textit{signed distance} to the fixed reference surface \( \Sigma \).

We have \( \Theta(x, t) \in [0, 1] \) by the choice. It is clear that \( S(t) \) is a geodesic graph over \( \Sigma \) if \( \Theta > 0 \) on \( S(t) \). \( \Theta(\cdot, 0) = \Theta_0 \) is for the initial surface.

**Theorem 2.3.** The evolution equations of \( u \) and \( \Theta \) have the following form:

\[
\frac{\partial}{\partial t} u = - H \Theta,
\]

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \Theta = (|A|^2 - 2) \Theta + \mathbf{n}(H_n) - H \langle \nabla_{\nu} \mathbf{n}, \nu \rangle
\]

\[
= (|A|^2 - 2) \Theta + \frac{1}{2} (\nabla_{\nu} L_n g(e_i, e_i) - (\nabla_{e_i} L_n g)(\nu, e_i))
\]

\[
- a_{ij} L_n g(e_i, e_j).
\]

Here \( \mathbf{n}(H_n) \) is the variation of mean curvature function of \( S(t) \) under the deformation vector field \( \mathbf{n} \).

**Proof.** The first equation is self-evident. The second equation can be found in (Bar84, EH91) in the Lorentzian setting. We write a proof in the appendix in our Riemannian setting for completeness.

We remark that the equation for \( \Theta \) in Theorem 2.3 is general, and quite difficult to work with, especially the term \( \mathbf{n}(H_n) \), as seen in the Appendix. We will take advantage of the explicit nature of the ambient warped product metric and derive more workable equations in our case (see (3.10)).

### 3. Proof of Main Theorem

We prove the main theorem in this section. In §3.1, we use the explicit solution to the mean curvature flow when the initial surface is parallel to the totally geodesic surface \( \Sigma \) to conclude (Theorem 3.1) that the mean curvature flow starting from any closed initial surface \( S_0 \) stays in a compact region in \( M^3 \) as long as it exists. This is standard \( C^0 \)-estimate using the avoidance principle for the mean curvature flow. In §3.2, we derive the key evolution equation (Theorem 3.6) for the angle function \( \Theta(\cdot, t) \). In §3.3, we prove the preservation of graphical property. We finally prove Theorem 1.2 in §3.4.

3.1. **The squeeze.** The following theorem is probably known previously, but we include the proof here for the sake of completeness.

**Theorem 3.1.** Let \( S_0 \) be as in Theorem 1.2, then as long as the flow exists we have

\[
\sinh^{-1}(e^{-2t} \sinh(a_0)) \leq u(\cdot, t) \leq \sinh^{-1}(e^{-2t} \sinh(a_0)).
\]
Proof. It follows from basic hyperbolic geometry that, if we denote \( \Sigma(r) \) (resp. \( \Sigma(-r) \)) the parallel surface equidistant \( r \) to \( \Sigma(0) = \Sigma \) which stays in the positive (resp. negative) side of \( \Sigma \), then \( \Sigma(r) \) (resp. \( \Sigma(-r) \)) is an umbilic surface of constant principal curvature \( \tanh(r) \) (resp. \( -\tanh(r) \)).

Now we consider the mean curvature flow with initial surface \( \Sigma(a_0) \) such that \( a_0 \geq \text{dist}(x, \Sigma) \) for any \( x \in S_0 \). Since the initial surface \( \Sigma(a_0) \) is umbilic, the mean curvature flow equation is reduced to the following ODE:

\[
\frac{dR}{dt} = -2 \tanh(R),
\]

with initial condition \( R(0) = a_0 \), which yields an explicit solution:

\[
R(t) = \sinh^{-1}(e^{-2t} \sinh(a_0)).
\]  

(3.2)

Similar calculations hold for \( \Sigma(-a_0) \). One sees that such mean curvature flow exists for all time, and all evolving surfaces are umbilic and converge to \( \Sigma \) as \( t \to \infty \).

Now by assumption, the initial surface \( S_0 \) lies between umbilic slices \( \Sigma(a_0) \) and \( \Sigma(-a_0) \), the conclusion then follows from the avoidance principle for the mean curvature flow. \( \square \)

Next we derive a general equation for \( \Delta u \).

**Lemma 3.2.** Let \( S \subset M^3 \) be a closed surface that is a geodesic graph over \( \Sigma \), and \( u(x) \) is the signed distance of \( x \in S \) to \( \Sigma \). Then we have:

\[
\Delta u = \tanh(u)(1 + \Theta^2) - H\Theta,
\]

where \( \Delta \) is the Laplace operator on \( S \) with respect to the induced metric.

**Proof.** For any point \( x \in S \), choose \( \{e_1, e_2\} \) (with \( e_3 = \nu \)) to be a local normal frame of \( S \) at \( x \). Then at \( x \) we can compute

\[
\Delta u = \sum_{i=1}^{2} \nabla_{e_i} \nabla_{e_i} u
\]

\[
= \sum_{i=1}^{2} \nabla_{e_i} \langle n, e_i \rangle
\]

\[
= \sum_{i=1}^{2} (\langle \bar{n} e_i, n \rangle + \langle n, \bar{n} e_i \rangle)
\]

\[
= \sum_{i=1}^{2} (\tanh(u)(e_i - \langle n, e_i \rangle n), e_i) + \sum_{i=1}^{2} \langle n, \bar{n} e_i e_i \rangle
\]

\[
= 2 \tanh(u) - \tanh(u)(1 - \Theta^2) - H\Theta
\]

(3.4)

\[
= \tanh(u)(1 + \Theta^2) - H\Theta,
\]
Remark 3.3. Combining with (2.9), we have the evolution equation for the hyperbolic distance function \( u \) of \( S(t) \) along the mean curvature flow:

\[
(3.5) \quad u_t - \Delta u = -\tanh(u)(1 + \Theta^2),
\]

which yields similar decay of \( u \) as in Theorem 3.1.

3.2. Evolution equation for the angle function. In this subsection, we take advantage of the presence of a special vector field \( V = \cosh(r)n \) (see (2.3)) to derive the evolution equation for the angle function \( \Theta(\cdot, t) \). Recall that, on the evolving surface \( S(t) \), it is given by \( \Theta(\cdot, t) = \langle n, \nu \rangle(\cdot, t) \). We find the following function more convenient to work with in our hyperbolic setting:

\[
(3.6) \quad \eta(\cdot, t) = \cosh(u)\Theta(\cdot, t) = \langle V, \nu \rangle.
\]

Lemma 3.4. The function \( \eta(\cdot, t) \) on the evolving surface \( S(t) \) satisfies the following equation:

\[
(3.7) \quad \Delta \eta = \sinh(u)H - |A|^2\eta + \langle V, \nabla H \rangle.
\]

Here \( \Delta \) is the Laplace operator on \( S(t) \) with respect to the induced metric.

Proof. For any point \( p \in S(t) \), we choose \( \{e_1, e_2\} \) (with \( e_3 = \nu \)) to be a local normal frame at \( p \). Then at \( p \) we have

\[
\Delta \eta = \sum_{i=1}^{2} \nabla_{e_i} \nabla_{e_i} \langle V, \nu \rangle \\
= \sum_{i=1}^{2} \langle \nabla_{e_i} \nabla_{e_i} V, \nu \rangle + 2 \langle \nabla_{e_i} V, \nabla_{e_i} \nu \rangle + \langle V, \nabla_{e_i} \nabla_{e_i} \nu \rangle \\
= \sum_{i=1}^{2} \langle \nabla_{e_i} (\sinh(u)e_i), \nu \rangle + 2 \sum_{i,k=1}^{2} \sinh(u)\langle e_i, a_{ik}e_k \rangle + \sum_{i,k=1}^{2} \langle V, \nabla_{e_i} (a_{ik}e_k) \rangle \\
= -\sinh(u)H + 2\sinh(u)H + \sum_{i,k=1}^{2} a_{ik} \langle V, \nabla_{e_i} e_k \rangle + a_{ik,i} \langle V, e_k \rangle \\
= \sinh(u)H - \sum_{i,k=1}^{2} a_{ik}^2 \eta + a_{ii,k} \langle V, e_k \rangle \\
= \sinh(u)H - |A|^2\eta + \langle V, \nabla H \rangle,
\]

where in the second to the last equality we have used the Codazzi equation and the fact that Fuchsian manifold \( M^3 \) is of constant curvature \(-1\).
We next compute $\Delta \Theta$ over the surface $S(t)$.

**Lemma 3.5. With the above notations, we have:**

$$
\Delta \Theta = \langle \nabla H, n \rangle - |A|^2 \Theta + \tanh(u)(1 + \Theta^2)H - \frac{\Theta(1 - \Theta^2)}{\cosh^2(u)}
- 2 \tanh(u) \langle \nabla \Theta, n \rangle - 2 \tanh^2(u) \Theta.
$$

(3.8)

**Proof.** A direct calculation yields

$$
\Delta \eta = \Delta (\cosh(u) \Theta)
= \cosh(u) \Delta \Theta + 2 \sinh(u) \langle \nabla \Theta, n \rangle + \Theta (\sinh(u) \Delta u + \cosh(u) |\nabla u|^2).
$$

(3.9)

Isolating $\Delta \Theta$, we have

$$
\Delta \Theta = \frac{\Delta \eta}{\cosh(u)} - 2 \tanh(u) \langle \nabla \Theta, n \rangle - \Theta \tanh(u) \Delta u - \Theta (1 - \Theta^2)
- \tanh(u) H - |A|^2 \Theta + \langle \nabla H, n \rangle - 2 \tanh(u) \langle \nabla \Theta, n \rangle
- \tanh^2(u) \Theta(1 + \Theta^2) + \tanh(u) H \Theta^2 - \Theta (1 - \Theta^2).
$$

Here we have used the fact that $|\nabla u|^2 = 1 - \Theta^2$. Now standard hyperbolic trigonometric identities give

$$
-\Theta (1 - \Theta^2) - \tanh^2(u) \Theta(1 + \Theta^2) = -\frac{\Theta(1 - \Theta^2)}{\cosh^2(u)} - 2 \tanh^2(u) \Theta.
$$

This completes the proof.

We can now derive the evolution equation for $\Theta$ along the flow.

**Theorem 3.6. With the above notations, we have:**

$$
\frac{\partial \Theta}{\partial t} - \Delta \Theta = |A|^2 \Theta - 2 \tanh(u) H + 2 \tanh(u) \langle \nabla \Theta, n \rangle
+ \frac{\Theta(1 - \Theta^2)}{\cosh^2(u)} + 2 \tanh^2(u) \Theta.
$$

(3.10)

**Proof.** Recall that for the mean curvature flow we have $\frac{\partial}{\partial t} \nu = \nabla H$, and geometrically, one can view $\frac{\partial}{\partial t}$ as the spatial covariant derivative $-H \nu$ here. Therefore

$$
\frac{\partial \Theta}{\partial t} = \frac{\partial}{\partial t} \langle \nu, n \rangle = \left( \frac{\partial}{\partial t} \nu, n \right) + \langle \nu, \nabla_{-H \nu} n \rangle = \langle \nabla H, n \rangle - H \langle \nu, \nabla \nu n \rangle.
$$

Using (2.4) we have

$$
\nabla \nu n = \tanh(u)(\nu - \Theta n),
$$

and thus

$$
\frac{\partial \Theta}{\partial t} = \langle \nabla H, n \rangle - H \tanh(u)(1 - \Theta^2).
$$

(3.11)

Now the conclusion follows from Lemma 3.5.
3.3. **Preserving graphical property.** In this subsection, we discuss the preservation of the graphical property. We formulate the a priori estimate on the angle $\Theta$ as follows.

**Theorem 3.7.** Let $M^3$ be a Fuchsian manifold and $S_0$ be a smooth closed surface which is a geodesic graph over the unique totally geodesic surface $\Sigma$ in $M^3$, and suppose there is a positive constant $a_0$ such that $S_0$ lies entirely between $\Sigma(\pm a_0)$. Then whenever the initial surface $S_0$ satisfies $\Theta_0 \geq \tanh(a_0)$, the mean curvature flow with initial surface $S_0$ remains as geodesic graph over $\Sigma$, namely $\Theta(\cdot, t) > 0$ as long as the flow exists.

**Proof.** We have $-a_0 \leq u(x, 0) \leq a_0$ for any $x \in S_0$. By Theorem 3.1, we have:

$$|u(x, t)| \leq \sinh^{-1}(e^{-2t} \sinh(a_0)). \quad (3.12)$$

We want to find a positive lower bound for $\Theta$ at the initial time $t_0$ to guarantee a positive lower bound for all time and hence the convergence.

It is equivalent to work with the function $\alpha = \Theta^2$ whose evolution equation is slightly more convenient to work with. With the evolution equation (3.10) for $\Theta$, we easily deduce the evolution equation for $\alpha$:

$$\frac{\partial \alpha}{\partial t} - \Delta \alpha = 2|A|^2 \alpha - 4 \tanh(u) H \Theta + 4 \tanh(u) \Theta \langle \nabla \Theta, n \rangle + \frac{2\alpha(1-\alpha)}{\cosh^2(u)} + 4 \tanh^2(u) \alpha - 2|\nabla \Theta|^2. \quad (3.13)$$

Let

$$\phi(t) = \min_{S(t)} \alpha$$

and we only need to consider the case of $\phi \in (0, 1)$ in search of a priori estimate. At the spatial minimum point of $\alpha$, we have $\nabla \Theta = 0$ and $\Delta \alpha \geq 0$, and so for $t > 0$ (using Hamilton’s trick):

$$\frac{d\phi}{dt} \geq \frac{\partial \alpha}{\partial t} - \Delta \alpha$$

$$= 2|A|^2 \alpha - 4 \tanh(u) H \Theta + \frac{2\alpha(1-\alpha)}{\cosh^2(u)} + 4 \tanh^2(u) \alpha$$

$$\geq 2|A|^2 \Theta^2 - 4 \sqrt{2} \tanh(|u|) |A| \Theta + \frac{2\alpha(1-\alpha)}{\cosh^2(u)} + 4 \tanh^2(u) \alpha$$

$$= 2(|A| \Theta - \sqrt{2} \tanh(|u|))^2 - 4 \tanh^2(u) + \frac{2\alpha(1-\alpha)}{\cosh^2(u)} + 4 \tanh^2(u) \alpha$$

$$\geq -4(1-\alpha) \tanh^2(u) = -4(1-\phi) \tanh^2(u).$$
Combining with $|u(x, t)| \leq \sinh^{-1}(e^{-2t}\sinh(a_0))$, we end up with the following ordinary differential inequality

\[
\begin{cases}
\left(\frac{1}{1-\phi}\right) \frac{d\phi}{dt} \geq -\frac{4e^{-4t}\sinh^2(a_0)}{1+e^{-4t}\sinh^2(a_0)} \\
\phi(0) = \phi_0,
\end{cases}
\]

where by assumption we have

\[
1 > \phi_0 = \min_{S_0} \alpha \geq \tanh^2(a_0) = \frac{\sinh^2(a_0)}{1 + \sinh^2(a_0)}.
\]

With a proper choice of $\epsilon \in [0, 1)$, we have

\[
\phi_0 = \frac{\epsilon + \sinh^2(a_0)}{1 + \sinh^2(a_0)},
\]

where $\epsilon > 0$ if (3.15) is a strict inequality.

Straightforward calculations yield:

\[
\phi(t) \geq \frac{\epsilon + \sinh^2(a_0)e^{-4t}}{1 + \sinh^2(a_0)e^{-4t}} > \epsilon
\]

for all $t \geq 0$ and so we have the following a priori estimate for $t \in [0, \infty)$:

\[
\Theta^2(\cdot, t) > \epsilon \geq 0,
\]

which provides a positive lower bound for the angle $\Theta$ in any finite time interval, and the evolving surface remains as geodesic graph over $\Sigma$ as long as it exists. This completes our proof.

### 3.4. Long-time existence and convergence.

Now we can re-assemble the ingredients and complete the proof of Theorem 1.2.

**Proof.** (of Theorem 1.2) We have shown in Theorem 3.7 that the mean curvature flow (1.1) stays graphical as long as it exists. This provides the gradient estimate for the mean curvature flow for any finite time interval. By the classical theory of parabolic equations in divergent form (for instance [LSU68]), the higher regularity and a priori estimates of the solution follow immediately. This yields the long time existence of the flow by Huisken ([Hui86]). Then by Theorem 3.1 and the avoidance principle, the continuous convergence of the flow also follows. When the inequality in the assumption is strict, the proof of Theorem 3.7 gives the uniform estimate of the angle for all time and so the higher order estimates are also uniform for all time, which provides the smooth convergence. The proof of Theorem 1.2 is completed.
4. General case and possible singularities

In this section, we discuss the general case after removing the assumption on the angle function $\Theta_0$ of the initial graph in Theorem 1.2 and illustrate the possible formation of singularities.

In light of the evolution equation (3.13), in order to rule out singularities, it will be enough to get a proper bound of $H$ at the point where $\Theta$ takes the spatial minimum in $(0, 1)$.

For the angle function $\Theta$ of any surface in $M^3$, we have the following calculation of its gradient over the surface in general:

$$\nabla \Theta = \nabla \langle n, \nu \rangle$$

$$= \langle \nabla_{e_i} \nu, n \rangle e_i + \langle \nabla_{e_i} n, \nu \rangle e_i$$

$$= \langle a_{ij} e_j, n \rangle e_i + \langle \tanh(u)(e_i - \langle e_i, n \rangle n), \nu \rangle e_i$$

$$= a_{ij} \langle e_j, n \rangle e_i - \tanh(u) \langle e_i, n \rangle \Theta e_i$$

(4.1)

where $\{e_1, e_2\}$ is any orthonormal frame at the point of interest on the surface, and we have used Lemma 2.1 in the second to the last step. By choosing $e_1$ and $e_2$ to be two principal directions, i.e., the second fundamental form $(a_{ij}) = \text{diag}\{a, b\}$, we have

$$\nabla \Theta = (a - \tanh(u) \Theta) \langle e_1, n \rangle e_1 + (b - \tanh(u) \Theta) \langle e_2, n \rangle e_2.$$  

(4.2)

Notice that the spatial maximum of $\Theta$ is clearly 1 and obtained clearly at the points with extremal height, where both $e_i$’s are perpendicular to $n$. Also recall $|\nabla u|^2 = \langle e_1, n \rangle^2 + \langle e_2, n \rangle^2 = 1 - \Theta^2$.

Meanwhile, at the spatial minimal point of $\Theta$, denoted by $\theta$ and assumed to be in $(0, 1)$, we see below that exactly one of $e_1$ and $e_2$ is perpendicular to $n$. Pick a tangent vector of the surface $S$, $\hat{e}_1$ perpendicular to $n$, which is unique up to sign as it is also perpendicular to $\nu$. Then take $\hat{e}_2$ accordingly so that they form an orthonormal frame of the tangent space of $S$. Use $\hat{a}_{ij}$ to denote the coefficients for second fundamental form with respect to this basis. Applying (4.1), we have

$$\hat{e}_1(\Theta) = a_{12} \langle \hat{e}_2, n \rangle,$$

which implies $a_{12} = 0$ since $\langle \hat{e}_2, n \rangle^2 = 1 - \theta^2 > 0$. So such chosen $\hat{e}_1$ and $\hat{e}_2$ are also principal directions and can be taken as $e_1$ and $e_2$ as above. So we can have $\langle e_1, n \rangle = 0$. By (4.2), we have $b = \tanh(L) \theta$ where $L$ is the height of the spatial minimal point under consideration.

As the flow starts with a graph, singularities can occur only when there is no positive lower bound for $\Theta$ by the discussion at the end of Section 3.
So by defining
\[ T = \sup \{ t \in (0, \infty) \mid \theta \geq C \text{ in } [0, t] \text{ for some } C > 0 \} \in (0, \infty), \]
we know the flow exists exactly in \([0, T)\). In the following, we focus on the case of \(T < \infty\), i.e., the flow fails to be graphical in finite time.

In this case, we can choose a time sequence \(\{t_i\}_{i=1}^{\infty}\) approaching \(T\) as \(i \to \infty\), such that \(\theta(t_i) \to 0\) as \(i \to \infty\), i.e., \(\langle n, \nu \rangle \to 0\) at the spatial minimal point \(p_i\) for \(\Theta\). Now we analyze \(p_i \in S(t_i)\) at time \(t_i\) more carefully.

For simplicity of notations, we frequently omit the index \(i\) below, and the limit is always taken as \(i \to \infty\).

We already know that there is one principal direction \(e_1\) such that \(\langle e_1, n \rangle = 0\) at the spatial minimal point. Geometrically, \(e_1\) is the direction of the curve as the intersection of the evolving graph \(S(t)\) and the equidistance graph \(\Sigma(L)\) where \(L\) is the height of the spatial minimal point.

As \(\langle e_1, n \rangle^2 + \langle e_2, n \rangle^2 = 1 - \theta^2\), we have \(\langle e_2, n \rangle^2 \to 1\), i.e., \(e_2 \to n\) by reversing \(e_2\) if necessary. Consider the geodesics on \(S(t_i)\) starting at \(p_i\) in the direction of \(e_2\). Since at \(p_i\) we have \(\langle \nabla_{e_2} e_2, \nu \rangle = -\langle \nabla_{e_2} \nu, e_2 \rangle = -b = -\tanh(L)\theta\), which approaches 0 by the decay of height and \(\theta(t_i) \to 0\), we have \(\nabla_{e_2} e_2 \to 0\) where by abuse of notation, \(e_2\) also stands for the unit tangent vector field along the geodesic. Together with \(e_2 \to n\), we know in the infinitesimal way at \(p_i\), this geodesic on \(S(t_i)\) approaches the \(r\)-curve which is geodesic of \(M^3\) in the direction \(n\). Intuitively, this is the direct consequence of the loss of graphical property.

Meanwhile, the “reason” for the loss of graphical property should be the “relative” blow-up of the principle curvature in the \(e_1\) direction, namely the quantity \(a\) in (4.2). We hope to illustrate in the following. Using (3.10) and \(b = \tanh(L)\theta\), we have

\[
\frac{d\theta}{dt} \geq (a^2 + \tanh^2(L)\theta^2)\theta - 2a \tanh(L) + \frac{\theta(1 - \theta^2)}{\cosh^2(L)}
\]

If \(|a| \leq -\theta \log \theta + C\theta\) for some \(C > 0\) at any spatial minimal point of \(\Theta\) as long as the flow exists, then we have

\[
\frac{d\theta}{dt} \geq C\theta \log \theta - C\theta.
\]

Direct calculation yields that \(\theta \geq Ce^{-Ce^Ct} > 0\), which rules out the loss of graphical property in finite time and we have the long time existence together with the continuous convergence. Motivated by this, for the case of \(T < \infty\), we set

\[
I = \{ t \in [0, T) \mid |a| > -\theta \log \theta + C\theta \text{ at all spatial minimal points of } \Theta(\cdot, t) \}.
\]
Note that \( \mathbb{I} \) has the closure in \( \mathbb{R} \) containing \( T \), since otherwise we can derive a positive lower bound for \( \theta \) for any finite time interval as above, contradicting \( T < \infty \). So we can choose the time sequence \( \{t_i\}_{i=1}^{\infty} \subset \mathbb{I} \) approaching \( T \) as \( i \to \infty \) and consider at the point \( q_i \) where \( \Theta \) achieves the spatial minimum on \( S(t_i) \) and \( |a| > -\theta \log \theta + C\theta \). We can further make sure that \( \theta \to 0 \) for this time sequence, since otherwise \( \theta \) will have a uniform positive lower bound for \( t \in \mathbb{I} \) close to \( T \), and \( \theta \) will then have a uniform positive lower bound for \( t \in [0, T) \setminus \mathbb{I} \) by applying the above argument for the interior of \( [0, T) \setminus \mathbb{I} \), contradicting the choice of \( T \). Here we make use of the continuity of \( \theta \) with respect to time.

In light of \( \theta \to 0 \), the scale of \( b = \tanh(L)\theta \) is way smaller than that of \( a \), i.e., “relative” blow-up. In other words, after a proper “blowing-down” (by the scale of \( \theta( - \log \theta)^{1/2} \), for example), we have the possible singularity modeled as a cylinder with the circle on the equidistant surface \( \Sigma(L) \) and pointing in the \( \mathbf{n} \) direction in the infinitesimal way.

In future works, we hope to provide more precise local and global pictures for the possible singularities. The goal is to either rule them out or come up with interesting examples.

5. Appendix

In this appendix, we give a detailed proof for the evolution equation for the angle function in Theorem 2.3, i.e., the equations (2.10) and (2.11). The calculation is carried out for the mean curvature flow of graphical hypersurface of general dimension, \( n \), in the ambient manifold \( M^{n+1} \) with a general warped product metric. We still use \( \mathbf{n} \) and \( \boldsymbol{\nu} \) to denote the unit normal vectors for the warped product foliation and the evolving hypersurface. Also we sum over all repeated indices in this section.

The following computation is done for \( F(p, t) \) for time \( t \). We choose the normal frame \( \{e_i\}_{i=1}^{n} \) for the evolving hypersurface. Then we require \( L_n e_i = [\mathbf{n}, e_i] = 0 \) to extend the frame to a neighborhood of \( F(p, t) \) in \( M^{n+1} \), i.e., using \( \mathbf{n} \) to generate a family of hypersurfaces with the initial one being the evolving hypersurface at time \( t \). Then the vector field \( \boldsymbol{\nu} \) below means the normal vector field for this family of hypersurfaces. This won’t affect the result for \( \Delta \Theta \) at \( F(p, t) \) for time \( t \).

\[
\Delta \Theta = e_i e_i (\boldsymbol{\nu}, \mathbf{n}) \\
= e_i (\boldsymbol{\nu}, \nabla e_i \mathbf{n}) + (\nabla e_i \boldsymbol{\nu}, \mathbf{n}) \\
= e_i (\nabla e_i \boldsymbol{\nu}, \mathbf{n}) + (\nabla e_i \boldsymbol{\nu}, e_j) \cdot (\mathbf{n}, e_j) \\
= e_i (\nabla e_i \boldsymbol{\nu}, \mathbf{n}) + (\nabla e_j \boldsymbol{\nu}, e_i) \cdot (\mathbf{n}, e_j) \\
= (\nabla e_i \boldsymbol{\nu}, \nabla n e_i) + (\boldsymbol{\nu}, \nabla e_i \nabla n e_i) + (\nabla e_i \nabla e_j \boldsymbol{\nu}, e_i) \cdot (\mathbf{n}, e_j)
\]
We consider each term separately below:

\[ \frac{\langle \nabla e_i \nu, \nabla e_i e_j \rangle \cdot \langle n, e_j \rangle + \langle \nabla e_j \nu, e_i \rangle \cdot \langle n, e_i e_j \rangle + \langle \nabla e_j \nu, e_i \rangle \cdot \langle n, \nabla e_i e_j \rangle}{2} = \langle \nabla e_i \nu, \nabla n e_i \rangle + \langle \nabla e_i \nu, \nabla e_i e_j \rangle + \langle \nabla e_i \nu, \nabla e_i e_i \rangle \cdot \langle n, e_j \rangle \]

where we have used \( \nabla e_i n = \nabla n e_i \) for the third equality, \( a_{ij} = \langle \nabla e_i \nu, e_j \rangle = \langle \nabla e_j \nu, e_i \rangle \) for the fourth equality and \( \nabla e_i e_i = -a_{ii} \nu = -H \nu \) for the last equality. For these terms, we have

\[ \langle \nu, \nabla e_i \nabla n e_i \rangle = \langle R(e_i, n) e_i, \nu \rangle + \langle \nu, \nabla n \nabla n e_i \rangle, \]

where \( R \) is the Riemannian curvature tensor, and we also have

\[ e_j \langle H \rangle = e_j \langle \nabla e_i \nu, \nu \rangle = \langle \nabla e_j \nabla e_i \nu, e_i \rangle + \langle \nabla e_i \nu, \nabla e_j e_i \rangle = -\langle \nabla (e_i, e_j) \nu, e_i \rangle + \langle \nabla e_i \nabla e_j \nu, e_i \rangle, \]

using \( [n, e_i] = 0 \), \( \nabla_{e_i} e_j = \nabla_{e_j} e_i = -a_{ij} \nu \) and \( [e_i, e_j] = 0 \). We also find:

\[ \langle \nabla e_i \nu, \nabla n e_i \rangle = \langle \nabla e_i \nu, e_j \rangle \cdot \langle \nabla n e_i, e_j \rangle = \langle \nabla e_i \nu, e_i \rangle \cdot \langle \nabla e_i n, e_j \rangle = \langle \nabla n e_i, e_j \rangle a_{ij} = \frac{a_{ij}}{2} n \langle (e_i, e_j) \rangle. \]

So the previous computation for \( \Delta \Theta \) can be continued as follows:

\[ \Delta \Theta = \langle \nabla e_i \nu, \nabla n e_i \rangle + \langle \nabla e_i \nabla e_i \nu, e_i \rangle + \langle \nabla e_i \nabla e_i \nu, \nabla e_i e_i \rangle \cdot \langle n, e_j \rangle \]

\[ + \langle \nabla e_j \nu, e_i \rangle \cdot \langle \nabla e_i n, e_j \rangle + \langle \nabla e_j \nu, e_i \rangle \cdot \langle n, \nabla e_i e_j \rangle = a_{ij} n \langle (e_i, e_j) \rangle + \langle R(e_i, n) e_i, \nu \rangle + \langle \nu, \nabla n \nabla e_i e_i \rangle \]

\[ + \langle n, e_j \rangle \cdot (e_j \langle H \rangle + \langle R(e_i, e_j) \nu, e_i \rangle) + \langle \nabla e_j \nu, e_i \rangle \cdot \langle n, \nabla e_i e_j \rangle. \]

We consider each term separately below:

\[ a_{ij} n \langle (e_i, e_j) \rangle = -a_{ij} n (g^{ij}), \]

\[ \langle R(e_i, n) e_i, \nu \rangle = -\text{Ric}(n, \nu), \]

\[ \langle \nu, \nabla n \nabla e_i e_i \rangle = n \left( \langle \nu, \nabla e_i e_i \rangle \right), \]

\[ \langle n, e_j \rangle \cdot e_j \langle H \rangle = \langle n, \nabla H \rangle, \]

\[ \langle n, e_j \rangle \cdot \langle R(e_i, e_j) \nu, e_i \rangle = \text{Ric}(n^t, \nu), \]

\[ \langle \nabla e_j \nu, e_i \rangle \cdot \langle n, \nabla e_i e_j \rangle = a_{ij} \cdot \langle n, -a_{ij} \nu \rangle = -\sum_{i,j} |a_{ij}|^2 \cdot \langle n, \nu \rangle, \]

where \( \nabla e_i e_i = -a_{ii} \nu = -H \nu \) is used for the third one, \( (g^{ij}) \) is the inverse matrix of \( (g_{ij} = \langle e_i, e_j \rangle) \) and \( n^t \) is the projection of \( n \) in the direction of the evolving hypersurface.
Remark 5.1. The equality \( \langle \nu, \nabla_{e_i} e_i \rangle = -H \) holds only at \( F(p,t) \) for time \( t \), and so \( n \left( \langle \nu, \nabla_{e_i} e_i \rangle \right) \) is NOT equal to \(-n(H)\). Nevertheless, we still have \( \langle \nu, \nabla_{e_i} e_i \rangle = -\langle \nabla_{e_i} \nu, e_i \rangle \) by the construction of \( \nu \) at the beginning of this appendix.

Now we can finish the computation for \( \Delta \Theta \):

\[
\Delta \Theta = -a_{ij} n(g^{ij}) + n \left( \langle \nu, \nabla_{e_i} e_i \rangle \right) - \langle n, \nu \rangle \cdot \text{Ric}(\nu, \nu)
+ \langle n, \nabla H \rangle - \sum_{i,j} |a_{ij}|^2 \cdot \langle n, \nu \rangle
= -n(H_n) + (\text{Ric}(\nu, \nu) - \sum_{i,j} |a_{ij}|^2) \Theta + \langle n, \nabla H \rangle,
\]

where \( n(H_n) = a_{ij} n(g^{ij}) - n \left( \langle \nu, \nabla_{e_i} e_i \rangle \right) \).

Next we further clarify the term \( n(H_n) \). It stands for the variation of mean curvature for the family of hypersurfaces starting with the evolving hypersurface under consideration at time \( t \) and flowing out by the vector field \( n \). This is the same family of hypersurfaces considered in the previous calculation, and we are only interested in the initial hypersurface which is the hypersurface evolving along the mean curvature flow at time \( t \). Clearly, we have

\[
n(H_n) = n(g^{ij} a_{ij}) = g^{ij} n(a_{ij}) + a_{ij} n(g^{ij}) = n(a_{ii}) + a_{ij} n(g^{ij})
\]
at the point since \( g^{ij} \) is the identity matrix at the point under consideration, and \( a_{ii} \) is not equal to \( H \) nearby. The computation for \( n(H_n) \) is as follows, still for just that point.

\[
n(H_n) = n(a_{ii}) + a_{ij} n(g^{ij})
= n \left( \langle \nabla_{e_i} \nu, e_i \rangle \right) - a_{ij} n \left( \langle e_i, e_j \rangle \right)
= \langle \nabla_n \nabla_{e_i} \nu, e_i \rangle + \langle \nabla_{e_i} \nu, \nabla_n e_i \rangle - a_{ij} L_n g(e_i, e_j)
= -a_{ij} (L_n g)(e_i, e_j) + e_i \left( \langle \nu, \nabla_n e_i \rangle + \langle e_i, \nabla_n \nu \rangle \right) - \langle \nabla_{e_i} e_i, \nabla_n \nu \rangle
- \langle \nu, \nabla_{e_i} \nabla_n e_i \rangle - \langle e_i, \nabla_{e_i} \nabla_n \nu \rangle + \langle e_i, \nabla_n \nabla_{e_i} \nu \rangle
= -a_{ij} (L_n g)(e_i, e_j) - \langle \nu, \nabla_{e_i} \nabla_n e_i \rangle + \langle R(n, e_i) \nu, e_i \rangle
= -a_{ij} (L_n g)(e_i, e_j) - e_i \left( \langle \nu, \nabla_n e_i \rangle + \langle \nabla_{e_i} \nu, \nabla_n e_i \rangle + \langle R(\nu, e_i) n, e_i \rangle \right)
- a_{ij} \langle L_n g \rangle(e_i, e_j) - e_i \left( \langle \nu, \nabla_n e_i \rangle + \langle \nabla_{e_i} \nu, \nabla_n e_i \rangle \right)
+ \langle \nabla_{e_i} \nabla_n e_i, e_i \rangle - \langle \nabla_{e_i} \nabla_{e_i} n, e_i \rangle - \langle \nabla_{e_i} \nabla_{e_i} n, e_i \rangle + \langle \nabla_{e_i} \nu \nabla n, e_i \rangle,
\]

where \( n(g^{ij}) = -g^{ik} n(g_{k\ell}) g^{\ell j} \) and \( (g^{ij}) \) being identity at the point are used for the first equality; \( n, e_i \) = 0 is used for the third equality; \( \langle e_i, \nu \rangle = 0 \), \( |\nu| = 1 \) and at \( F(p,t) \), \( \nabla_{e_i} e_i = -a_{ii} \nu = -H \nu \) are used for the fifth equality.
We have a few more terms to sort out.

\[-a_{ij} L_n g(e_i, e_j) = -\langle e_i, \bar{\nabla}_j \nu \rangle \left( \langle \bar{\nabla}_e n, e_j \rangle + \langle \bar{\nabla}_j n, e_i \rangle \right) \]

\[= -\langle \bar{\nabla}_{e_j} \nu n, e_j \rangle - \langle \bar{\nabla}_e n, \bar{\nabla}_j \nu \rangle, \]

\[
\frac{1}{2} \nu (L_n g(e_i, e_i)) = \nu \left( \langle \bar{\nabla}_e n, e_i \rangle \right) = \langle \bar{\nabla}_e \bar{\nabla}_e n, e_i \rangle + \langle \bar{\nabla}_e n, \bar{\nabla}_e e_i \rangle, \\
- L_n g(\bar{\nabla}_e e_i, e_i) = -\langle \bar{\nabla}_{\bar{\nabla}_e} n, e_i \rangle - \langle \bar{\nabla}_e n, \bar{\nabla}_e e_i \rangle, \\
- e_i (L_n g(\nu, e_i)) = -e_i \left( \langle \bar{\nabla}_e n, e_i \rangle + \langle \bar{\nabla}_e n, \nu \rangle \right) \\
= -\langle \bar{\nabla}_e e_i \bar{\nabla}_e n, e_i \rangle - \langle \bar{\nabla}_e n, \bar{\nabla}_e e_i \rangle - e_i \left( \langle \bar{\nabla}_e n, \nu \rangle \right). \]

Now it is easy to calculate:

\[\mathbf{n}(H_n) = \frac{1}{2} \nu (L_n g(e_i, e_i)) - L_n g(\bar{\nabla}_e e_i, e_i) - e_i (L_n g(\nu, e_i)) + \langle \bar{\nabla}_e n, \bar{\nabla}_e e_i \rangle \]

\[= \frac{1}{2} \langle \bar{\nabla}_e L_n g \rangle (e_i, e_i) - e_i (L_n g(\nu, e_i)) + \langle \bar{\nabla}_e n, \bar{\nabla}_e e_i \rangle \]

\[= \frac{1}{2} \langle \bar{\nabla}_e L_n g \rangle (e_i, e_i) + \langle \bar{\nabla}_e \nu n, \bar{\nabla}_e e_i \rangle \\
- \langle \bar{\nabla}_e L_n g \rangle (\nu, e_i) - L_n g(\bar{\nabla}_e \nu, e_i) - L_n g(\nu, \bar{\nabla}_e e_i) \]

\[= \frac{1}{2} \langle \bar{\nabla}_e L_n g \rangle (e_i, e_i) + \langle \bar{\nabla}_e \nu n, \bar{\nabla}_e e_i \rangle \\
- \langle \bar{\nabla}_e L_n g \rangle (\nu, e_i) - L_n g(a_{ij} e_j, e_i) - L_n g(\nu, -H \nu). \]

In light of \(\langle \bar{\nabla}_e n, \bar{\nabla}_e e_i \rangle = -H \langle \bar{\nabla}_e n, \nu \rangle = -\frac{1}{2} H L_n g(\nu, \nu),\) we conclude

\[\mathbf{n}(H_n) = \frac{1}{2} \langle \bar{\nabla}_e L_n g \rangle (e_i, e_i) - \langle \bar{\nabla}_e L_n g \rangle (\nu, e_i) - a_{ij} L_n g(e_i, e_j) + \frac{1}{2} H L_n g(\nu, \nu). \]

Noticing \(\langle \nu, \bar{\nabla}_e \nu n \rangle = \frac{1}{2} L_n g(\nu, \nu),\) we have

\[\mathbf{n}(H_n) = \frac{1}{2} \langle \bar{\nabla}_e L_n g \rangle (e_i, e_i) - \langle \bar{\nabla}_e L_n g \rangle (\nu, e_i) - a_{ij} L_n g(e_i, e_j) + H \langle \nu, \bar{\nabla}_e n \rangle. \]

We note that the advantage of computing this way is that the terms now depend mostly on the evolving hypersurface. The vector field \(\mathbf{n}\) only appears in \(L_n g.\)

In the following we compute \(\frac{\partial \bar{\nabla}}{\partial t}\) in detail. We use the coordinate system used in [Hui86], i.e., a normal coordinate system \(\{y_\alpha\}\) for \(F(p, t)\) in \(M\) with the frame vector for the first coordinate is \(-\nu\) at time \(t.\)

Let \(\nu = \nu^\alpha \frac{\partial}{\partial y^\alpha}\) and \(\mathbf{n} = n^\alpha \frac{\partial}{\partial y^\alpha}.\) We have \(\frac{\partial \bar{\nabla}}{\partial t} = \frac{\partial (g^{\alpha \beta} \nu_\alpha \nu_\beta)}{\partial t}.\) There are three terms from Leibniz rule.

\[
\frac{\partial g_{\alpha \beta}}{\partial t} = \frac{\partial}{\partial t} \left( \left\langle \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta} \right\rangle \right) = -H \nu \left( \left\langle \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta} \right\rangle \right) = 0,
\]

because the Christoffel symbols vanish at the point.
Define $\frac{\partial \nu}{\partial t}$ to be $\frac{\partial \nu}{\partial t} \frac{\partial}{\partial y^\alpha}$ and $\frac{\partial n}{\partial t}$ to be $\frac{\partial n}{\partial t} \frac{\partial}{\partial y^\alpha}$, and we see

$$\frac{\partial \Theta}{\partial t} = \langle \frac{\partial \nu}{\partial t}, n \rangle + \langle \nu, \frac{\partial n}{\partial t} \rangle.$$ 

We have $\frac{\partial \nu}{\partial t} = \nabla H$, and

$$\frac{\partial n}{\partial t} = \frac{\partial n}{\partial t} \frac{\partial}{\partial y^\alpha} = -H \nu(n^\alpha) \frac{\partial}{\partial y^\alpha} = -H \nabla_{\nu} n,$$

where the last equality is true again by the choice of \{y^\alpha\}.

In light of all these, we can now conclude (2.10) and (2.11).

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