Superconducting correlations out of repulsive interactions on a fractional quantum Hall edge

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We consider a fractional quantum Hall bilayer system with an interface between quantum Hall states of filling fractions (ν\text{top}, ν\text{bottom}) = (1,1) and (1/3,2), motivated by a recent approach to engineering artificial edges. We show that random tunneling and strong repulsive interactions within one of the layers will drive the system to a stable fixed point with two counterpropagating charge modes which have attractive interactions. As a result, slowly decaying correlations on the edge become predominantly superconducting. We discuss the resulting observable effects, and derive general requirements for electron attraction in Abelian quantum Hall states. The broader interest in fractional quantum Hall edge with quasi-long range superconducting order lies in the prospects of hosting exotic anyonic boundary excitations, that may serve as a platform for topological quantum computation.

Introduction. Combining superconductivity and fractional quantum Hall edge states opens the possibility to engineer exotic topological phases of matter with anyonic boundary excitations [2][3]. A possible route to this is by using the proximity effect with a bulk superconductor and a quantum well in a hybrid structure [9][10][11]. Another, less studied possibility is that of intrinsic superconductivity on the edge. Evidently, on a one-dimensional edge there is no true long-range order and correlation functions decay algebraically. One can nevertheless refer to a superconducting phase as the one where the slowest decaying correlation function is of superconducting nature, i.e., a pairing correlator [12]. Such power-law (or quasi-long-range) superconducting order may still be relevant for topological quantum computing applications [13], c.f. [14] in the context of Majorana bound states.

In a recent experimental work [1] it has been demonstrated that in an engineered bilayer system it is possible to structure and control co- and counterpropagating edge modes in both the integer and fractional quantum Hall regimes. The present work takes advantage of this new paradigm and shows that one can design chiral modes with bare repulsive interaction in the presence of disorder to induce attractive interaction between the resulting effective modes. This gives rise to a phase with algebraically-decaying superconducting order.

To describe our results qualitatively, let us recall the pioneering work [15] of Kane, Fisher, and Polchinski (KFP) for ν = 2/3 edge hosting counterpropagating ν = 1/3 and ν = 1 modes. Random tunneling and sufficiently strong interaction between the two modes can drive the system to a fixed point with decoupled neutral and charge modes. The charge of the latter, 2e/3, is determined by the constituent bare modes and charge conservation. The fixed point is approached upon, for example, lowering the temperature, and can be understood as a renormalization of the interaction between the neutral and charge mode (the interaction is an irrelevant perturbation and renormalizes to zero). The novel aspect in our proposal is to consider an additional ν = 1 (1e) charge mode interacting with the KFP modes, see Fig. 1. As in the conventional KFP theory, both charge modes decouple from the neutral mode upon decreasing temperature. However, now there is a set of KFP fixed points, parametrized by the interaction between the 2e/3 and 1e charge modes. Our main finding is that this fixed-point interaction can be attractive, even when the bare interactions of the high-temperature limit are repulsive. We further substantiate this claim by studying the renormalization group flow in a fine-tuned strongly-interacting model where the 2e/3 and the neutral mode are already decoupled on the level of the bare Hamiltonian. We then move on to study the new fixed point. We find that the fixed point has superconducting correlations of the charge modes: their pairing correlation function decays slower than any charge density correlation function. Finally, we outline how our model can be realized in an engineered quantum Hall bilayer system and how one can detect the attractive interactions at the fixed point by using 3 experimental probes: multiterminal shot noise, tunneling spectroscopy, and ground state charge in a quantum dot geometry.

Model. We consider a system with 3 relevant edge modes. We assume a right-moving ν = 1/3 mode and a pair of counterpropagating ν = 1 modes. In terms of a three-component chiral boson field \( \phi = (\phi_{1/3} \phi_{-1} \phi_1)^T \), our model is described by the imaginary-time action

\[
S = \int d\tau dx \frac{1}{4T} \left[ \partial_\tau \phi \mathbf{K} \partial_x \phi + \partial_x \phi \mathbf{V} \partial_\tau \phi \right] + \int d\tau dx \left[ \xi(x)e^{i\phi} + \xi^*(x)e^{-i\phi} \right] \tag{1}
\]

where in the first line \( \mathbf{K} = \text{diag}(3,-1,1) \) and the \( V- \)
ing Ref. [15], it is convenient to work in the basis where the elements of the \( W \) are diagonal. This is the basis of a right-moving neutral mode and a left-moving charge mode, 

\[
\phi_n = \frac{1}{\sqrt{2}} (3\phi_{1/3} + \phi_{-1}) , \quad \phi_{-2/3} = \frac{\sqrt{3}}{2} (\phi_{1/3} + \phi_{-1}) .
\]

The random tunneling, which conserves charge, only couples to \( \phi_n \): in this “neutral mode basis” \( \phi = (\phi_n \ \phi_{-2/3} \ \phi_{1/3})^T(n) \) and the tunneling vector becomes 

\[
c = (\sqrt{2}, 0, 0)\, (n). \quad \text{Also, } K = \text{diag}(1, -1, 1)\, (n) \quad \text{and}
\]

\[
V = \left( \begin{array}{ccc}
 v_n & U_{n,-2/3} & U_{n,1/3} \\
 U_{n,-2/3} & v_{2/3} & U_{n,-2/3,1} \\
 U_{n,1/3} & U_{n,-2/3,1} & v_{1}
\end{array} \right)\, (n),
\]

where the matrix elements are simple linear combinations of the elements from Eq. (2). In particular, the interaction between the charge modes \( \phi_{2/3}, \phi_{1/3} \) is \( U_{2/3,1} = \frac{1}{2\sqrt{3}} (3U_{1/3,1} - U_{1/3,1}) \). We see that the interaction is attractive, \( U_{-2/3,1} < 0 \), when \( U_{1/3,1} > 3U_{1/3,1} \). This can happen when the mode \( \phi_{1/3} \) is the nearest one to \( \phi_1 \), as in Fig. 1. As we show below, the attractive interaction between two charge modes makes the superconducting pair correlations between them the slowest decaying correlation function in the system, which we call superconducting state in 1D.

Evidently, in the bare non-renormalized \( V \)-matrix the seemingly attractive interaction is just a result of a basis change from a system with purely repulsive interactions. The off-diagonal elements \( U_{n,-2/3}, U_{n,1/3} \) that couple the neutral mode to the two charge modes ensure that there are no superconducting correlations. However, we will show next that under renormalization, the elements \( U_{n,-2/3}, U_{n,1/3} \) will flow to zero due to disorder in the neutral mode, while \( U_{-2/3,1} \) remains approximately constant. In the original basis this corresponds to \( U_{1/3,1}, U_{-1,1} \) flowing to negative values, i.e., attraction, see Fig. 2.

The weak-disorder RG flow of \( V^{(n)} \) was studied by Moore & Wen [17], who found that a relevant disorder operator \( e^{i\sqrt{2} \phi_n} \) drives the \( V \)-matrix towards a fixed point which is diagonal in the neutral sector. Therefore, \( U_{n,-2/3} \) and \( U_{n,1/3} \) are both irrelevant and flow to weak coupling [18]. Furthermore, the disorder operator \( e^{i\sqrt{2} \phi_n} \) commutes with \( \partial_x \phi_{-2/3} \partial_x \phi_{1/3} \), so we expect \( U_{-2/3,1} \) to be marginal. In the general case the RG equations are somewhat complicated and not very illustrative. However, we can obtain tractable results in the special fine-tuned (yet generic in terms of the resulting physics) point \( U_{1/3,1} = 3(v_{1/3} + v_{-1})/4 \), which corresponds to \( U_{n,-2/3} = 0 \) [or \( \Delta_{3,1,0} = 1 \)]. For this value of \( U_{1/3,1} \), we obtain the RG equations of the couplings in the limit of weak disorder and weak couplings [19]. Our findings agree with the above qualitative description:

**Neutral mode basis and perturbative RG.** Following Ref. [15], it is convenient to work in the basis where
Numerical solution of the RG equations produces the flow diagram shown in Fig. 2, presented in terms of the bare interactions in Eq. (7). We find the scaling dimension of a generic vertex operator $O = \exp i(c_\nu \phi_n + c_1 \phi_1 + c_{2/3} \phi_{-2/3})$ is

$$
\Delta_c = \frac{1}{4}(c_1 + c_{2/3})^2 \epsilon^{2x} + \frac{1}{4}(c_1 - c_{2/3})^2 \epsilon^{-2x} + \frac{1}{2} \epsilon^2.
$$

Using Eq. (7), one finds that the scaling dimension $\Delta$ of the attractive interaction $U_{-2/3,1}$ in Eq. (7) makes the pairing correlation function the slowest decaying one. The superconducting pairing correlation function in the original basis is $O_{SC} \sim e^{i(\phi_1 - \phi_{-1})}$ [this operator creates two counter-propagating electrons in the $\nu = 1$ modes]. Its dimension is calculated by first expressing $\phi_{-1}$ in terms of $\phi_n$ and $\phi_{-2/3}$: $e^{i(\phi_1 - \phi_{-1})} = e^{i(\phi_1 - \phi_{-1})/\sqrt{3}(\phi_{-2/3} - \phi_n)}$, and then using Eq. (7). We find the scaling dimension $\Delta_{SC} = \frac{1}{4}(1 + \sqrt{\frac{3}{2}})^2 \epsilon^{2x} + \frac{1}{4}(1 - \sqrt{\frac{3}{2}})^2 \epsilon^{-2x} + \frac{1}{4}$. For $\chi > 0.26$ we have $\Delta_{SC} < 1$, so the pairing correlator decays slower than the neutral mode correlator $e^{i(\phi_{1/3} + \phi_{-1})} = e^{i\chi \phi_1}$. Likewise, the diagonal density operator $O_{c,+1}(x) \sim \delta_\nu \phi_{+1}$ has $\Delta = 1$ irrespective of $U_{-2/3,1}$ and so the density perturbation decays faster than pairing. Finally, we consider the off-diagonal density operator [21] $O_{CDW} \sim e^{i(\phi_1 + \phi_{-1})}$. We find $\Delta_{CDW} = \frac{1}{4}(1 + \sqrt{\frac{3}{2}})^2 \epsilon^{2x} + \frac{1}{4}(1 - \sqrt{\frac{3}{2}})^2 \epsilon^{-2x} + \frac{1}{4}$. Since $\chi > 0$ for $U_{-2/3,1} < 0$ [Eq. (7)], we always have $\Delta_{SC} < \Delta_{CDW}$. Thus, superconducting pair correlations are the slowest decaying ones in the strong coupling fixed point. Next, we discuss the measurable effects of this attraction.

**Consequences of attraction.** The relatively long-ranged pairing correlations are a direct consequence of the attractive interaction $U_{-2/3,1} < 0$. One way to probe our proposed fixed point is to measure $U_{-2/3,1}$ or its sign. Since the fixed point action is that of a non-chiral spinless Luttinger liquid, one is faced with the known task of measurement of the Luttinger liquid parameter. Next, we outline three possible ways to do this. We focus on the experimentally relevant bilayer quantum Hall system, see Fig. 3.

**Signature of attraction in shot noise.** It is well-known that the interaction parameter in a non-chiral Luttinger liquid can be measured with a.c. shot noise [21,23]. The attractive interactions in our setup can be measured in a similar experiment, see Fig. 3. Employing the theory of inhomogeneous Luttinger liquid [23], we solve [19] the problem of a bare incoming mode $\phi_1$ scattered off an interacting region at the superconducting fixed point. In particular, the charge reflected into the neutral mode decouples. Such a fine-tuning yields $U_{-2/3,1} = \sqrt{2/3}U_{1/3,1} > 0$, assuming repulsive bare interactions. Thus, renormalization by random tunneling is essential for obtaining an attractive interaction.

The charge sector action can be diagonalized by a hyperbolic rotation

$$
\begin{pmatrix}
\phi_1 \\
\phi_{-2}
\end{pmatrix} = \begin{pmatrix}
\cosh \chi & \sinh \chi \\
\sinh \chi & \cosh \chi
\end{pmatrix}
\begin{pmatrix}
\phi_n \\
\phi_{-1}
\end{pmatrix}, \quad \text{tanh} 2\chi = -\frac{U_{1/3,1}}{v_2/3 + v_1}.
$$

At the fixed point we have a decoupled right-moving neutral mode $\phi_n$, a right-moving $\nu = 1$ charge mode $\phi_1$, and a left-moving charge mode $\phi_{-2/3}$. The latter two are coupled via an interaction that is attractive, $U_{-2/3,1} < 0$, as long as the bare interactions satisfy $3U_{1/3,1} < U_{1/3,1}$. The set of fixed point $V$-matrices [5] can also be obtained even without random tunneling by fine-tuning the bare interactions in Eq. (2) in such a way that the neutral mode decouples. Such a fine-tuning yields $U_{-2/3,1} = \sqrt{2/3}U_{1/3,1} > 0$, assuming repulsive bare interactions. Thus, renormalization by random tunneling is essential for obtaining an attractive interaction.
Signature of attraction in tunneling conductance. One can also measure $U_{-2/3,1}$ from tunneling conductance in the interacting region, for example by using a point-contact to an auxiliary $\nu = 1$ edge. For describing the tunneling Hamiltonian, consider the vertex operator $e^{i(\phi_1 - \phi_{-1}) + 3\phi_{1/3}/3}$ that creates an excitation of total charge $n_1 + n_{-1} + n_{1/3}$ on the edge; here $n_1, n_{-1}, n_{1/3} \in \mathbb{Z}$. The contribution to the tunneling current from the above operator exhibits a power-law bias voltage dependence \[ I \propto V^{2\alpha - 1} \] where $\alpha = \Delta_{n_1,n_{-1},n_{1/3}}$ is the scaling dimension \[ \Delta_{n_1,n_{-1},n_{1/3}} \] obtained from Eq. (6) after transforming the vertex operator into the neutral mode basis by using Eq. (3). The total tunneling current is a sum of elementary tunneling processes, but will be dominated at small voltages by those with a low value of $\alpha$. For moderate interaction strengths $\chi$ [Eq. (6)] the dominant contributions are the 1-electron tunneling operators $e^{i\phi_1}$ and $e^{-i\phi_{-1}}$, as well as the 2-electron tunneling operator $e^{i(\phi_1 - \phi_{-1})}$. Their respective tunneling amplitudes $t_1, t_{-1}$, and $t_1t_{-1}$, are in principle controllable by gating, so that different 1-electron contributions can be turned on and off. The signature of attractive interactions ($\chi > 0$) is that $\Delta_{1,1,0} < \Delta_{1,0,0} + \Delta_{0,-1,0}$, meaning that when tunneling to both $\nu = 1$ edges is present, the current is less suppressed by a small bias than one would expect from uncorrelated tunnelings to each edge separately.

Signature of attraction in a mesoscopic droplet. Finally, one can perform a fully thermodynamic measurement in a Coulomb blockaded quantum Hall droplet, see Fig. 3. This is akin to ideas of “attraction from repulsion” that have been implemented in other systems \[ 27 \], compare also proposals to probe neutral modes in the context of quantum Hall edges \[ 28 \]. The signature of attraction in the Coulomb blockaded droplet is $\nu e/\nu_1$-periodic charge transitions as a gate charge is varied \[ 19 \]. This signature can be measured in a thermodynamic capacitive measurement of the charge or in a transport measurement of the Coulomb peak spacings.

Discussion. Our proposal relies on the tunneling between the modes $\phi_{1/3}$ and $\phi_{-1}$ being the most RG relevant perturbation. Typically, the tunneling $e^{i(\phi_1 + \phi_{-1})}$ between the $\nu = 1$ modes is also relevant and leads to a trivial localization of the $\nu = 1$ modes. This effect should however be present only at very low temperatures since we expect the bare amplitude of the $\nu = 1$ tunneling to be very weak due to the large separation of the $\nu = 1$ modes. The $\nu = 1$ tunneling can also be entirely avoided by considering a setup with spin-polarized Landau levels where $\phi_1$ and $\phi_{-1}$ have opposite spins and the tunneling between them is forbidden by spin conservation. This can be achieved with an interface between $(\nu_{\text{top}}, \nu_{\text{bottom}}) = (1/3, 2)$ and $(\nu_{\text{top}}, \nu_{\text{bottom}}) = (1, 1)$, assuming the $\nu = 2$ state consists of opposite-polarized $\nu = 1$ states. This state also satisfies the requirement that $\nu_{\text{bottom}} \geq \nu_{\text{top}}$ holds on both sides \[ 11 \].

In our model we assumed that $U_{1/3,1}$ is the largest interaction while the other two were treated perturbatively, which ensures that $e^{i(3\phi_{1/3} + \phi_{-1})}$ is relevant and KFP fixed point is reached. Thus, we rely on the double-inequality $U_{1/3,1} > U_{1,3,1} > 3U_{1,-1,1}$ to approach the fixed point with attractive interactions. We find that Coulomb interaction screened by a nearby gate electrode \[ 29, 19 \] allows both inequalities to be satisfied.

One may ask how essential the bilayer construction is to manifest our theory. For example, edge reconstruction in a $\nu = 1/n$ Laughlin state can give rise to a $\nu = 1/m$ stripe in the bulk-vacuum interface. Disordered tunneling between the two inner modes gives rise to counterpropagating neutral and a charge modes. The propagation directions of these modes are determined by comparing the two filling fractions. If $n < m$, the charge mode is co-propagating with the outermost $\nu = 1/m$ mode. Therefore, there are no emerging superconducting correlations even if there is attraction between the two charge modes. (Interactions between co-propagating modes do not affect the scaling dimensions of the operators involved, since the $V$-matrix can be diagonalized with an orthogonal transformation \[ 28 \].) In the more interesting scenario $n > m$, the charge modes are counterpropagating and superconducting correlations may in principle emerge. In this case the the interaction is attractive when $U_{1,3,1} > \frac{3}{m}U_{1,-1,1}$. However, for an interaction falling monotonically with distance, we expect $U_{1,3,1} > U_{1,-1,1}$ because the outermost mode $1/m$ is closer to $-1/m$ rather than the bulk mode $1/n$. This is why we do not expect to find superconducting correlations in such a sim-
ple model of edge reconstruction. This problem is circumvented in the bilayer setup, see Fig. 1. Finally, we note that our proposal also works for an interface between $(\nu_{\text{top}}, \nu_{\text{bottom}}) = (2/3, 1)$ and $(\nu_{\text{top}}, \nu_{\text{bottom}}) = (0, 2)$, assuming that the $\nu = 2/3$ edge consists of counterpropagating $\nu = 1$ and $\nu = 1/3$ modes \[33, 34\].

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[20] See Supplementary Material, where we present the full RG equations at $U_{n-2/3} = 0$, discuss in more detail the signatures of attraction, and outline the geometrical requirements to find attraction from repulsion in a bilayer system.
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Compare with the emergence of negative currents in Ref. \[32\].
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SUPPLEMENTARY MATERIAL TO “SUPERCONDUCTING CORRELATIONS OUT OF REPULSIVE INTERACTIONS ON A FRACTIONAL QUANTUM HALL EDGE”

In this Supplementary Material, we present the full RG equations at the fine-tuned point $U_{n,-2/3} = 0$, discuss in more detail the signatures of attraction in the shot noise and Coulomb blockaded droplet, and finally outline the geometrical requirements to find attraction from repulsion in a bilayer system.

A. Renormalization group flow of the $V$-matrix

In this section we study the fine-tuned KFP fixed point which corresponds to a bare interaction $U_{1/3,-1} = 3(v_{1/3} + v_{-1})/4$ which means $\Delta_{3,1,0} = 1$. This fine-tuned point is easy to study because the neutral mode is automatically decoupled from $\phi_{-2/3}$, since in Eq. (4) of the main text we have $U_{n,-2/3} = 0$:

$$V = \begin{pmatrix} v_n & 0 & U_{n1} \\ 0 & v_{2/3} & U_{-2/3,1} \\ U_{n1} & U_{-2/3,1} & v_1 \end{pmatrix}_{(n)}.$$ (8)

For completeness, the non-zero interactions in the original basis are $U_{n1} = \frac{1}{\sqrt{2}}(U_{3/4,1} - U_{-1,1})$ and $U_{-2/3,1} = \frac{1}{\sqrt{3}}(3U_{-1,1} - U_{1,1})$. Treating the interactions $U_{n1}, U_{-2/3,1}$ perturbatively, we can diagonalize $V$ with a transformation $M$ that preserves $K = \text{diag}(1,-1,1)_{(n)}$:

$$V^{(D)} = MVM^T, \quad M = \begin{pmatrix} 1 & 0 & -a_1 \\ 0 & 1 & a_2 \\ a_1 & a_2 & 1 \end{pmatrix}_{(n)}.$$ (9)

where $a_1 = \frac{U_{n1}}{v_n - v_1}$, $a_2 = \frac{U_{-2/3,1}}{v_{2/3} + v_1}$. The tunneling vector $c = (\sqrt{2},0,0)_{(n)}$ transforms to $c^{(D)T} = M^T(\sqrt{2},0,0)^T_{(n)} = \sqrt{2}(1,0,-a_1)^T_{(n)}$.

Next, we follow Ref. [7] to find how $V$ flows upon renormalization. To first order in $U_{n1}, U_{-2/3,1}$, the flow is entirely due to $U_{n1}$ which couples to disordered neutral mode. The RG equation is

$$\frac{dV^{(D)}}{dl} = \begin{pmatrix} 0 & 0 & (v_n - v_1) \frac{d\theta}{dl} \\ 0 & 0 & 0 \\ (v_n - v_1) \frac{d\theta}{dl} & 0 & 0 \end{pmatrix}_{(n)}.$$ (10)

where

$$(v_n - v_1) \frac{d\theta}{dl} = 4\pi \frac{2a_1 v_n v_1}{v_n^2 v_{1/2}^2} W \approx 8\pi \frac{U_{n1}}{v_n - v_1} \frac{v_1}{v_n} W.$$ (11)

This corresponds to

$$\frac{dU_{n1}}{dl} = -8\pi \frac{1}{v_1 - v_n} \frac{v_1}{v_n} U_{n1} W.$$ (12)

Ignoring the term $U_{-2/3,1}$ in Eq. (8), the action corresponds to a neutral mode coupled to a co-moving charge mode. This system has a stable fixed point $[7] U_{n1} \to 0$ when $v_1 > v_n$.

To find beyond tree-level accuracy, we can diagonalize $V^{(n)}$ working to 2nd order accuracy. The tunneling vector is

$$c^{(D)} = \sqrt{2} \begin{pmatrix} 1 - \frac{U_{n1}^2}{2(v_n - v_1)^2}, \frac{U_{-2/3,1} U_{n1}}{(v_n - v_1)(v_{2/3} + v_1)}, \frac{U_{n1}}{v_n - v_1} \end{pmatrix}^T_{(n)},$$ (13)

which shows that $\frac{dU_{-2/3,1}}{dl}$ starts at order $\sim U_{-2/3,1} U_{n1}^2$. One therefore has to go to 3rd order to capture this accurately. Such a calculation gives

$$\frac{dU_{-2/3,1}}{dl} = -8\pi \frac{U_{-2/3,1} U_{n1}^2}{(v_n - v_1)^2 (v_{2/3} + v_n) \frac{v_1 v_{2/3}}{v_n} W.$$ (14)
Thus $U_{-2/3,1}$ is irrelevant but very marginally so: only to second order in $U_{-2}^2$. The 3rd order corrections to Eq. (12) modify its RHS by an overall factor $1 + \frac{U_{-2/3,1}^2}{(\nu_1 - \nu_n)(\nu_2 + \nu_n)} - \frac{2U_{-2}^2}{(\nu_1 - \nu_n)^2}$. This correction is inconsequential at small interaction strengths. Solving Eqs. (12) [including the aforementioned correction] and (14) numerically and expressing them in the original basis leads to Fig. 2 of the main text.

### B. Signature of attraction in shot noise

We focus on a geometry where the modes $\phi_{-1}$, $\phi_{1/3}$ are uncoupled at $x \to \pm \infty$. In the scattering region $0 < x < L$ the modes couple as described by the bare action [1], see Fig. 3a. We assume that the fixed point of a decoupled neutral mode is reached throughout the entire scattering region. Let us consider an incoming mode $\phi_{-1}$ from $x = + \infty$, which gets reflected at $x = L$ into modes $\phi_{1/3}$, $\phi_{1}$ upon encountering the scattering region. For $x > L$, the eigenmodes of the system are $\phi_{-1}$, $\phi_{1}$, $\phi_{1/3}$. They are described by an action $\mathcal{S} = \sqrt{3} \phi_{1/3}$

$$S = \frac{1}{4\pi} \int dx \int_0^L \left[ \partial_x \phi_{-1} \partial_x \phi_{1} + \partial_x \phi_{1/3} \partial_x \phi_{1/3} - \partial_x \phi_{-1} i \partial_x \phi_{-1} + v_1 (\partial_x \phi_{1})^2 + v_{1/3} (\partial_x \phi_{1/3})^2 + v_{-1} (\partial_x \phi_{-1})^2 \right].$$

For $x < L$, we have first the action of decoupled neutral mode and coupled charge modes,

$$S = \frac{1}{4\pi} \int dx \int_0^L \left[ \partial_x \phi_{-1} \partial_x \phi_{1} + \partial_x \phi_{1/3} \partial_x \phi_{1/3} - \partial_x \phi_{-2/3} i \partial_x \phi_{-2/3} + v_n (\partial_x \phi_{-1})^2 + v_{1/3} (\partial_x \phi_{1/3})^2 + v_{-2/3} (\partial_x \phi_{-2/3})^2 \right],$$

where $\phi_n = \frac{1}{\sqrt{2}} \left( \sqrt{3} \phi_{1/3} + \phi_{-1} \right)$ and $\phi_{-2/3} = \sqrt{\frac{3}{2}} \left( \frac{1}{\sqrt{3}} \phi_{1/3} + \phi_{-1} \right)$. Here, we can diagonalize the charge sector by

$$\begin{pmatrix} \phi_1 \\ \phi_{-2/3} \end{pmatrix} = \begin{pmatrix} \cos \chi \\ \sin \chi \end{pmatrix} \begin{pmatrix} \cosh \chi \\ \sinh \chi \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{\sqrt{2} - \sqrt{3}} \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix},$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{\sqrt{2}}{\sqrt{2} - \sqrt{3}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ 1 \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}.$$
From the last equation we find \( f_-(z) = \Theta(\frac{z-\nu}{\nu})/\frac{3}{\sqrt{2}} (\sqrt{2} + 3\sqrt{6}) \cosh \chi \) for \( z > 0 \). Note that \( \frac{3}{\sqrt{2}} (\sqrt{2} + 3\sqrt{6}) = \frac{3\sqrt{2}}{3\sqrt{3}-1} \). Thus, the function \( f_- \) is determined fully to be \( f_-(z) = \Theta(\frac{z-\nu}{\nu}) (3\sqrt{3} - 1) / 3\sqrt{2} \cosh \chi \). This is the transmitted wave. The remaining two equations yield \( B_1 = B_{1/3} = 0 \) and reflection coefficients

\[
R_1 = \frac{(3\sqrt{3} - 1) \tanh \chi}{3\sqrt{2}} \approx -1.011 \frac{1-g}{1+g}, \quad R_{1/3} = -\frac{1}{3},
\]

where \( \frac{(3\sqrt{3} - 1) \tanh \chi}{3\sqrt{2}} \approx 1.011 \) and we introduced the “Luttinger liquid parameter” \( g = \sqrt{\frac{2(u_2/3+v_1)-U_{-2/3,1}}{2(u_2/3+v_1)+U_{-2/3,1}}} \). The charges reflected into the drains 1 and 1/3 are respectively \( R_1 \) and \( R_{1/3} \). When \( U_{-2/3,1} > 0 \) [repulsive interaction], we have \( g < 1 \) and correspondingly \(-1.011 \lesssim R_1 < 0 \). On the other hand, when \( U_{-2/3,1} < 0 \) [attraction], we have \( g > 1 \) and \( 0 < R_1 \lesssim 1.011 \). Therefore, the crucial signature of attraction is the sign of \( R_1 \). It can be measured by measuring the current in the \( \nu = 1 \) edge in time-domain.

### C. Signature of attraction in a mesoscopic droplet

In a periodic finite system of length \( L \), we have the mode expansion [\( \nu = \pm 1, 1/3 \)]

\[
\phi_\nu(x) = \frac{2\nu}{L} N_\nu + \varphi_\nu + \delta \phi_\nu(x),
\]

where \([\varphi_\nu, N_\nu] = i \) and \( \varphi_\nu, N_\nu \) commute with \( \delta \phi_\nu \). Using the mode expansion, we find for the charge sector of the fixed point action [obtained from Eq. (5) of the main text, see also Eq. (16)]

\[
S_{\text{charge}} = \int d\tau dx \frac{1}{4\pi} \left[ \partial_x \delta \phi_1 i \partial_x \delta \phi_1 - \partial_x \delta \phi_{-2/3} i \partial_x \delta \phi_{-2/3} \right]
\]

\[
+ \int d\tau dx \frac{1}{4\pi} (v_{2/3}(\partial_x \delta \phi_{-2/3})^2 + v_1(\partial_x \delta \phi_1)^2 + 2U_{-2/3,1}(\partial_x \delta \phi_{-2/3})\partial_x \delta \phi_1)
\]

\[
+ \frac{1}{2L} \int d\tau dx \left[ N_1 i \partial_x \varphi_1 - v_{2/3} N_{-2/3} i \partial_x \varphi_1 \right]
\]

\[
+ \frac{\pi}{L} \int d\tau \left[ v_{2/3} N_{-2/3}^2 + v_1 N_1^2 + 2U_{-2/3,1} N_{-2/3} N_1 \right],
\]

where

\[
N_{-2/3} = \sqrt{\frac{3}{2}} \left( \frac{1}{3} N_{1/3} + N_{-1} \right), \quad \varphi_c = \sqrt{\frac{3}{2}} \left( \varphi_{1/3} + \varphi_{-1} \right).
\]

For the neutral sector we have [Eq. (5) of the main text, Eq. (16)]

\[
S_{\text{neutral}} = \int d\tau dx \frac{1}{4\pi} \left[ \partial_x \delta \varphi_n i \partial_x \delta \varphi_n + v_n(\partial_x \delta \varphi_n)^2 + \frac{2\pi}{L} N_n i \partial_x \varphi_n + v_n \left( \frac{2\pi}{L} N_n \right)^2 \right],
\]

where \( N_n = \sqrt{\frac{1}{2}} \left( N_{1/3} + N_{-1} \right) \) and \( \varphi_n = \sqrt{\frac{1}{2}} \left( 3\varphi_{1/3} + \varphi_{-1} \right) \).

The “charging” Hamiltonian obtained from above is

\[
H_c = \frac{\pi}{L} \left[ v_n N_n^2 + v_{2/3} N_{-2/3}^2 + v_1 N_1^2 + 2U_{-2/3,1} N_{-2/3} N_1 \right].
\]

The stability of \( V_{f,p} \) requires that \( \sqrt{v_{2/3} v_{-1}} > |U_{-2/3,1}| \). This ensures that \( H_c \) is positive,

\[
v_{2/3} N_{-2/3}^2 + v_1 N_1^2 + 2U_{-2/3,1} N_{-2/3} N_1 > \left( \sqrt{v_{2/3} N_{-2/3}^2} - \sqrt{v_1 N_1^2} \right)^2.
\]

In reality, the charging Hamiltonian is dominated by the total charging energy, which arises from the long-range Coulomb interaction. The total charge is given by \( N_{\text{tot}} = N_1 + N_{-1} + \frac{1}{3} N_{1/3} = N_1 + \sqrt{\frac{2}{3}} N_{-2/3} \), and the charging energy is then

\[
E_c(N_1 + \sqrt{\frac{2}{3}} N_{-2/3} - n^2),
\]
where \( g \) is the controllable induced gate charge. Let us see how the attractive interaction \( \frac{U}{2} N_{-2/3} N_{-2/3} N_1 \) affects the \( n_g \)-dependence of the ground state charge. Note that \( N_1 \) is integer while \( N_{-2/3} = \sqrt{3/2} \times \text{integer} \). The total energy is

\[
E_c(N_1 + \sqrt{\frac{2}{3}} N_{-2/3} - n_g)^2 + \frac{\pi}{L} \left[ v_{2/3} N_{2/3}^2 + v_1 N_1^2 + 2U_{-2/3,1} N_{-2/3} N_1 \right].
\] (33)

We include the terms \( v_{2/3}, v_1 \) to ensure bounded spectrum. In our model \( L \) is the system length. However, our bosonization description does not treat accurately the long-range Coulomb interaction so we cannot obtain quantitative estimates. The qualitative findings outlined below should however remain true.

For simplicity, let us focus on the four states \( (N_1, \sqrt{\frac{2}{3}} N_{-2/3}) = \{(0, 0), (1, 0), (0, 1), (1, 1)\} \). Relative to the \((0, 0)\) state, the other have energies

\[
E(1, 0) = E_c(1 - 2n_g) + \frac{\pi}{L} v_1,
\] (34)

\[
E(0, 1) = E_c(1 - 2n_g) + \frac{\pi}{L} v_{2/3},
\] (35)

\[
E(1, 1) = 4E_c(1 - n_g) + \frac{\pi}{L} \left[ v_{2/3} + v_1 + 2U_{-2/3,1} \right].
\] (36)

We have \( E(1, 1) = 0 \) when

\[
E_c(1 - 2n_g) = - \left( E_c + \frac{\pi}{2L} \left[ v_{2/3} + v_1 + 2U_{-2/3,1} \right] \right).
\] (37)

For this value of \( n_g \), the other two energies are

\[
E(1, 0) = -\frac{\pi}{L} U_{-2/3,1} - \left( E_c + \frac{\pi}{2L} \left[ v_{2/3} - v_1 \right] \right),
\] (38)

\[
E(0, 1) = -\frac{\pi}{L} U_{-2/3,1} - \left( E_c + \frac{\pi}{2L} \left[ -v_{2/3} + v_1 \right] \right),
\] (39)

which are positive when

\[
-U_{-2/3,1} > \frac{L}{\pi} E_c \pm \frac{1}{2} \left[ v_{2/3} - v_1 \right].
\] (40)

Under this condition we have a direct transition from \((0, 0)\) ground state to \((1, 1)\) ground state as \( n_g \) is tuned. We have an earlier condition \( \sqrt{v_{2/3} v_1} > |U_{-2/3,1}| \) from stability. This imposes the constraint

\[
2\sqrt{v_{2/3} v_1} > 2\frac{L}{\pi} E_c \pm \left[ v_{2/3} - v_1 \right].
\] (41)

This is a condition on \( E_c \). For example, when \( v_{2/3} > v_1 \), the signature \((0, 0) \rightarrow (1, 1)\) transition exists when

\[
2\frac{L}{\pi} E_c < 2\sqrt{v_{2/3} v_1} - \left[ v_{2/3} - v_1 \right] = 2v_1 - (\sqrt{v_{2/3}} - \sqrt{v_1})^2.
\]

D. Geometric requirements for the bilayer in the case of long-range Coulomb repulsion

In the main text we found the requirement \( 3U_{-1,1} < U_{1/3,1} \) in order to get attraction, \( U_{-2/3,1} < 0 \), between the charge modes at the disordered fixed point. In order to ensure that we flow to the disordered fixed point, we further require \( U_{1,-1} > U_{3,1} \). In this Section, we consider Coulomb interaction \( U(|r|) \) and find the requirements for the bilayer geometry. (Note however that our bosonization description assumes short-range interactions; our estimates in this Section are therefore mostly qualitatively.) We assume that the bilayers are separated by a distance \( z \), while in-plane the modes \( \phi_{-1}, \phi_{1/3}, \phi_{1} \) are at positions \( 0, y_0, y_0 + a \). Thus, we have the double inequality [recall that \( \phi_1 \) is separated by additional distance \( z \) in the perpendicular direction]

\[
U(y_0) > U(\sqrt{a^2 + z^2}) > 3U(\sqrt{(y_0 + a)^2 + z^2}).
\] (42)

This inequality cannot be satisfied for simple Coulomb interaction \( U(|r|) \sim 1/|r| \). However, for a faster decaying interaction, \( U(|r|) \sim 1/|r|^3 \), it can be satisfied \[29\]. (For example, when \( z \to 0 \), we find \( a > y_0 > (\sqrt{3} - 1)a \approx 0.44a \).)
The Coulomb interaction decays cubically when it is screened by an external gate. Let us consider a gate planar with the bilayer at a distance $d$ from the bottom quantum well. If we have a bottom gate $[d < z < 0]$ and the mode 1 lives in the bottom layer, we have for example $U_{-1,1} \propto \frac{1}{\sqrt{(y_0 + a)^2 + z^2}} - \frac{1}{\sqrt{(y_0 + a)^2 + (2d - z)^2}}$. Supposing that $y_0, a \gg |d|, |z|$ we have then for example $U_{-1,1} \propto \frac{1}{2} \frac{(2d - z)^2 - z^2}{(y_0 + a)^2}$ which decays cubically, as promised. The inequalities in this case become

$$U_{-1,1} \approx \frac{2d^2}{y_0^3} > U_{-1,1} \approx \frac{1}{2} \frac{(2d - z)^2 - z^2}{a^3} > 3U_{-1,1} \approx 3 \frac{1}{2} \frac{(2d - z)^2 - z^2}{(y_0 + a)^3},$$

(43)

or

$$\sqrt{\frac{1}{1 - \frac{z}{a}}} > \frac{y_0}{a} > \sqrt{3} - 1.$$  (44)

The right inequality corresponds to the condition $3U_{-1,1} < U_{1/3,1}$ for attraction. The left one ensures that the disordered fixed point should be reachable. The RHS is less than one, $\sqrt{3} - 1 \approx 0.44$. On the other hand, the LHS is always larger than one for a bottom gate, $d < z < 0$. For a top gate, $z < 0 < d$, there is an upper bound $d < \frac{|z|}{\sqrt{3} - 1} \approx 0.09 |z|$.