Chiral Rings and Physical States in $c < 1$ String Theory

Suresh Govindarajan, T. Jayaraman and Varghese John

The Institute of Mathematical Sciences
C.I.T. Campus, Taramani
Madras 600 113, INDIA

We show how the double cohomology of the String and Felder BRST charges naturally leads to the ring structure of $c < 1$ strings. The chiral ring is a ring of polynomials in two variables modulo an equivalence relation of the form $x^p \simeq y^{p+1}$ for the $(p+1, p)$ model. We also study the states corresponding to the edges of the conformal grid whose inclusion is crucial for the closure of the ring. We introduce candidate operators that correspond to the observables of the matrix models. Their existence is motivated by the relation of one of the screening operators of the minimal model to the zero momentum dilaton.
1. Introduction

The continuum formulation of $c < 1$ models coupled to gravity continues to be much less developed in comparison to the progress that was made using the matrix model approach. Physical states, physical operators and their correlation functions are not yet fully understood, especially if we demand that they are in one-to-one correspondence with the results of the matrix model. Clearly, the source of the problem lies in the fact that in the continuum formulation, the Liouville field is considered to be a scalar field and the full theory is then treated by the machinery of standard string theory. What in essence is necessary therefore, is some kind of consistent truncation of this theory to reproduce the results of the matrix models.

These problems were already visible from the work of Lian and Zuckerman [1] in their construction of the BRST invariant states of the minimal models coupled to gravity. The whole set of positive ghost number states had scaling dimensions that did not match those that came from the matrix model. On the other hand, even the negative ghost number states needed to be supplemented by those that came from the boundary of the Kac table. It seemed that a correlation function could not contain operators that matched those of the matrix model while at the same time conserving ghost number so that it did not vanish [2]. A further problem with these states was the fact that their construction was not immediately obvious. Some progress was also made in correlation functions when the work of Goulian and Li [3], Dotsenko [4] and Kitazawa [5] showed how three-point functions could be calculated using the Coulomb gas formalism and that they were the same as those of the matrix model. It was also found possible to do the calculations for states both inside and outside the Kac table [4] [5]. The calculation of higher point functions was however blocked by the necessity of analytic continuation in the number of screening operators to negative or fractional values.

Some of these problems began to be addressed by the use of the full power of the Coulomb gas method to represent the matter part of the operators. Apart from being useful to study the BRST cohomology itself [6], we showed that this method was helpful in indicating how these states could be explicitly written down [6]. This was particularly useful for states of ghost number different from $\pm \frac{1}{2}$ (these were constructed explicitly

\[\text{In our convention the state } |0\rangle_M \otimes |0\rangle_L \otimes c_1 |0\rangle_{gh} \text{ is defined to have zero ghost number with } c(b) \text{ having ghost number } +1(-1). \text{ Further, in going from states to operators, one increments the ghost number by } 1.\]
earlier by different methods [7][8]. Another advantage of the method was that it provided an alternate representation of these BRST invariant operators in terms of pure vertex operators that were easier to work with. Indeed it was shown that ghost number conserving correlation functions of the Lian-Zuckerman operators could be mapped into those involving only pure vertex operators (DK states).

In this paper we begin by reviewing the Coulomb gas construction of the BRST invariant states and operators using the double cohomology of the Felder BRST and the string BRST charges. We extend our considerations to the states that come from the boundary of the Kac table and show how they naturally appear in the Coulomb gas construction. The inclusion of both primary fields and their duals leads to the appearance of ghost number zero operators that provide the chiral ring structure for these models analogous to those first studied by Witten [9][10] in the context of $c = 1$ theories coupled to gravity. We show some explicit construction of these operators and provide a general cohomology argument for their existence. The appearance of this structure is extremely useful as it provides a means to compute quite general correlation functions of vertex operators for states both inside and outside the Kac table. In fact, such computations are crucial in showing that we can compute correlation functions with generalized screening operators which enables one to avoid the problem of analytic continuation in the number of screening operators[11]. This opens the route to computation of n-point functions in the continuum formulation.

We finally turn to a consideration of the true physical operators of the theory, namely the operators that can be both ghost number conserving and have the same scaling dimensions as the operators in the matrix model. Of course, we are not referring here to operators that arise purely as vertex operators as one may object that they are purely artefacts of the Coulomb gas method. What we are interested in are operators that can equally well be represented directly in terms of the primary and secondaries in the matter part of the theory. Thus we are interested in physical operators that are built out of the higher ghost number states that appear in the Lian and Zuckerman analysis. It appears that we have to go beyond the considerations of the original BRST analysis. We propose, in analogy with the dilaton of usual string theory, a set of operators that are BRST trivial in the holomorphic sector of the theory and have positive ghost number and are representative of the BRST cohomology of negative ghost number in the anti-holomorphic sector. We motivate the construction using in particular the relation of the screening operator to the zero momentum dilaton. We also discuss how these operators may be directly related to the matrix model observables. We relegate a mathematical argument to the appendix.
2. BRST Invariant States in $c < 1$

In the continuum formulation of $c < 1$ matter coupled to two dimensional gravity, the cohomology analysis of Lian and Zuckerman\cite{1} and subsequently by Bouwknegt, McCarthy and Pilch\cite{2} has shown that the physical states occur at non-trivial ghost number – one for each null state over the primaries. The Liouville dressing of these states of non-zero ghost number is the same as the dressing of the corresponding null states. We refer to these states (of non-trivial ghost number) as LZ states.

States outside the minimal conformal grid (Kac table), when provided with a suitable Liouville dressing will be called DK states.\footnote{DK states are sometimes referred to as tachyon states.} DK states have 0 ghost number in our convention. In this section, we will explain how DK states and ring elements (ghost number -1 states) provide an equally good representation of physical states. However, unlike LZ states, these states are easy to write down and one can actually attempt calculations involving these operators. In an earlier paper\cite{6}, we had clarified how DK states are actually related to LZ states by means of descent equations which follow from a double cohomology analysis. We shall now briefly summarise those results and then proceed to explain how ring elements are also related to LZ states via descent.

2.1. Double Cohomology

We consider two scalars $X$ (for matter) and $\phi$ (for the Liouville mode) with background charges $\alpha_0$ and $\beta_0$ respectively at infinity. The corresponding energy-momentum tensors are given by

\[
T^M = -\frac{1}{4} \partial X \partial X + i\alpha_0 \partial^2 X , \\
T^L = -\frac{1}{4} \partial \phi \partial \phi + i\beta_0 \partial^2 \phi ,
\]

with central charges $c_M = 1 - 24\alpha_0^2$ and $c_L = 1 - 24\beta_0^2$. For the $(p, p + 1)$ unitary minimal models

\[
\alpha_0^2 = \frac{1}{4p(p+1)} \text{ and } \beta_0^2 = -\frac{(2p+1)^2}{4p(p+1)} .
\]

The vertex operators $e^{i\alpha X}$ and $e^{i\beta \phi}$ have conformal weights $\alpha(\alpha - 2\alpha_0)$ and $\beta(\beta - 2\beta_0)$ respectively. The usual screening charges for matter are

\[
\alpha_+ = \sqrt{\frac{p+1}{p}} \text{ and } \alpha_- = -\sqrt{\frac{p}{p+1}} .
\]
The screening charges for the Liouville sector are given by

\[ \beta_+ = i\alpha_+ \quad \text{and} \quad \beta_- = -i\alpha_- \]

Following Felder\cite{13}, we consider the complex of Fock space (hereafter referred to as the Felder complex) in the matter sector given by \( \bigoplus F_{m', \pm m + 2np} \), where \( F_{m', m} \) is the Fock space built over the primary associated with the vertex operator \( e^{i\alpha_{m', m}X} \). Here

\[\alpha_{m', m} = \frac{1-m'}{2}\alpha_- + \frac{1-m}{2}\alpha_+ .\]

We will also need the dual Fock space obtained by \( F_{-m', -m} \). There exists an identity under the change of label given by \((m', m) \to (m' + p + 1, m + p)\) for the matter sector and \((m', m) \to (m' + p + 1, m - p)\) in the Liouville sector. These two seemingly distinct labels refer to the same vertex operator. This identity will prove to be useful later. There is one such Felder complex for every \( m', m \) restricted to the range \( 1 \leq m' \leq p \) and \( 1 \leq m \leq (p-1) \).

The screening operators \( Q^{+}_m = \int \prod_{i=1}^{m} dz_i e^{i\alpha_+X(z_i)} \) and similarly \( Q^{+}_{p-m} \) act on these Fock spaces. The irreducible Virasoro module \( L(m', m) \) (for a given \( c_m \) labelled by \( p \)) is given by \( \text{Ker} Q^{+}_m / \text{Im} Q^{+}_{p-m} \) on this complex. We shall refer to the screening operators loosely as \( Q_F \) except when necessary. We also have the Fock spaces of the Liouville and ghost sectors denoted by \( F(\beta) \) and \( F(gh) \) respectively. The string BRST operator \( Q_B \) given by

\[ Q_B = \oint : c(z)(T^M(z) + T^L(z) + \frac{1}{2}T_{gh}(z)) : \quad (2.2) \]

acts on the tensor product \( F(\alpha) \otimes F(\beta) \otimes F(gh) \).

Hence, in the the Coulomb gas realisation of minimal models coupled to gravity one has to work with the double complex associated with the two BRST operators.

2.2. Descent Equations

The analysis of the double cohomology of the string and Felder BRST\cite{1, 12, 13} provides a simple relation between LZ states of non-trivial ghost number and DK states. This is obtained by means of descent equations as was explicitly shown in \cite{13}. The existence of descent is seen from the isomorphism of cohomology classes\cite{12},

\[ H^{(n)}(H^{(0)}(F(\alpha)_M \otimes F(\beta)_L \otimes F_{gh}, Q_F), Q_B) \]

\[ \simeq H^{(n)}(H^{(0)}(F(\alpha)_M \otimes F(\beta)_L \otimes F_{gh}, Q_B), Q_F) \quad . \quad (2.3) \]
Since Felder\cite{13}, has shown that $H^0(F(\alpha_{m',m})) \simeq L(m',m)$, the LHS of (2.3) are the LZ states and the RHS are the DK states. This implies that the LZ and DK states are isomorphic to each other and hence the existence of descent.

The descent equations are

\begin{align}
Q_B|LZ\rangle^{-n} &= Q_F|I_1\rangle^{-n+1} \\
&\vdots \\
Q_B|I_{n-1}\rangle^{-1} &= (-)^{(n-1)}Q_F|DK\rangle^0
\end{align}

(2.4)

where we have denoted the ghost number of the state in the superscript. One can always use (2.4) to construct LZ states by “solving” the equations as has been demonstrated in \cite{3}. For the generic ghost number $-1$ LZ state, we have

\begin{align}
Q_B|LZ\rangle^{-1} &= Q_F|DK\rangle^0 = |u\rangle_M \otimes |v\rangle_L \otimes c_1|0\rangle_{gh}
\end{align}

(2.5)

where $|u\rangle_M$ is the non-vanishing matter null. One important point is that the LZ states thus obtained come with a precise choice of matter momenta. For the $-1$ case the matter momenta is such that there exists a non-vanishing null at the required level.

Consider the following example, already dealt with in \cite{3} for the case of $c_M = 0$. The descent equation for a ghost number $-2$ LZ state is

\begin{align}
Q_B|LZ\rangle &= Q_F|I\rangle \\
Q_B|I\rangle &= -Q_F|DK\rangle
\end{align}

(2.6)

Figure 1 shows the location of the matter part of these states in the Felder complex. Consider the DK state obtained from dressing $|v_{2,5}\rangle_M$. “Solving” (2.6) and after tedious but straightforward algebra, we obtain

\begin{align}
|DK\rangle &= |v_{2,5}\rangle_M \otimes |v_{-2,5}\rangle_L \otimes c_1|0\rangle_{gh} \\
|I\rangle &= \mathcal{L}_4^b|v_{2,3}\rangle_M \otimes |v_{-2,5}\rangle_L \otimes c_1|0\rangle_{gh} \\
|LZ\rangle &= \mathcal{L}_5^{2b}|v_{2,1}\rangle_M \otimes |v_{-2,5}\rangle_L \otimes c_1|0\rangle_{gh}
\end{align}

(2.7)

where

\begin{align}
\mathcal{L}_4^b &= \frac{94}{3}b_{-4} + b_{-3}(\frac{61}{3}L_{-1}^L + 3L_{-1}^M) \\
&\quad + b_{-2}(4L_{-2}^L - 4L_{-2}^M + \frac{20}{3}L_{-2}^L) \\
&\quad + b_{-1}(-3L_{-3}^L - \frac{41}{3}L_{-3}^M - \frac{20}{3}L_{-1}^L L_{-2}^M \\
&\quad + \frac{20}{3}L_{-2}^M L_{-1}^M - L_{-1}^L L_{-1}^M + L_{-1}^L L_{-1}^M L_{-2}^M)
\end{align}

(2.8)
\[ \mathcal{L}_5^{2b} = -\frac{5}{3} b_4 b_{-1} + 4 b_{-3} b_{-2} + b_{-3} b_{-1} \left( \frac{2}{3} L_{-1}^L - 15 L_{-1}^M \right) \\
+ b_{-2} b_{-1} \left( -4 L_{-2}^L - \frac{2}{3} L_{-1}^L L^2_{-1} - 6 L_{-1}^L L_{-1}^M + 6 L_{-1}^M L^2_{-1} \right). \quad (2.9) \]

These expressions are unique upto \( Q_B \) exact pieces that are also \( Q_F \) closed. One thus obtains the LZ state with matter momentum labelled by (2,1) as seen in (2.7).

The construction outlined above is tied down to a particular choice of matter momentum in the LZ state. Since there exists an equivalent LZ state with dual matter momentum, one would like to know what happens to descent if we had taken the dual \((2\alpha_0 - \alpha)\) in the matter momentum of the LZ state. The simple case of a ghost number \(-1\) state illustrates the point. Let \( |\tilde{LZ}\rangle \) represent the state with matter momenta dual to that obtained from the LZ state in (2.5). Now the operation of \( Q_B \) on \( |\tilde{LZ}\rangle \) again gives the matter null. But, this is a vanishing null and hence descent ends at this point. Thus the ghost number \(-1\) state itself is the other representative at the end point of the descent instead of a DK state.

This situation generalises to the ghost number \(-n\) case, with the descent ending at a ghost number \(-1\) state instead of a ghost number zero state (viz. a DK state) on flipping the matter momenta of the LZ state to its dual. The flip of matter momenta to its dual interchanges non-vanishing nulls with the vanishing nulls. Thus a descent from the LZ state would now give an intermediate state with different matter momenta and level different from the original one. For instance, in the contruction of a state of ghost number \(-2\) given earlier in (2.6) figure 2 depicts the descent on the dual Felder complex. Now we have

\[ |\tilde{LZ}\rangle^{-2} = \mathcal{L}_5^{2b} |v_{1,1}\rangle_M \otimes |v_{-2,5}\rangle_L \otimes c_1 |0\rangle_{gh} \]
\[ Q_B |\tilde{LZ}\rangle^{-2} = \mathcal{L}_3^{b} \mathcal{L}_2 |v_{1,1}\rangle_M \otimes |v_{-2,5}\rangle_L \otimes c_1 |0\rangle_{gh} \]
\[ = -4 \mathcal{L}_3^{b} |u_{1,1}\rangle_M \otimes |v_{-2,5}\rangle_L \otimes c_1 |0\rangle_{gh} \]
\[ = Q_F |R\rangle^{-1} \quad (2.10) \]

where
\[ \mathcal{L}_3^{b} = b_{-3} - b_{-2} L_{-1}^M + b_{-1} \left( L_{-2}^L + \frac{1}{6} \left( L_{-1}^L L_{-1}^L + L_{-1}^M L_{-1}^M \right) \right), \]
\[ \mathcal{L}_2 = (L_{-2}^M - \frac{3}{2} L_{-1}^M L_{-1}^2) . \]

In the second step of the descent in (2.6), we had the non-vanishing null over \( |v_{2,3}\rangle_M \) appearing in the construction. Here we obtain the vanishing third level null over \( |v_{1,3}\rangle_M \) and thus \( Q_B |R\rangle = 0 \). Thus the descent terminates one step earlier.

\[ ^4 \text{ We correct a typographical error in the expression for } \mathcal{L}_5^{2b} \text{ given in [6] } \]
From a constructionist viewpoint, one can now see how such a process can occur for states of arbitrary ghost number \(-n\). An examination of the levels of the appropriate null vectors in the Felder diagram is in agreement with a descent terminating earlier. A more general and rigorous argument using dimensionalities of cohomology classes in the double cohomology of \(Q_B\) and \(Q_F\) shows that indeed such ghost number \(-1\) states exist. We present the details of the argument in the Appendix. Hence there are two possible endpoints for descent: states at zero ghost number – DK states and states at ghost number \(-1\). The latter are precisely the ring elements for \(c < 1\). We shall discuss them in more detail in the following section.

So far the discussion has been restricted to the negative ghost number sector. The positive ghost number states are partners to the negative ghost number states in the sense that the norm on the sphere is obtained as

\[ ^+n \langle LZ | c_0 | LZ \rangle^{-n} \quad (2.11) \]

Given an LZ state of ghost number \(-n\), it is now trivial to construct a \(Q_B\) closed state of ghost number \(+n\) which has a non-zero norm with the given LZ state. This new state of ghost number \(+n\) is

\[ |LZ\rangle^{+n} = M^n |\hat{LZ}\rangle^{-n}, \quad (2.12) \]

where \(M = \{Q_B, c_0\}\) and \(|\hat{LZ}\rangle^{-n}\) is the LZ state with the matter and Liouville momenta flipped to their duals. The state given in (2.12) is obviously not exact and hence a good element of the cohomology. The Liouville dressing is \(\beta > \beta_0\) as given by the analysis of Lian and Zuckerman. It can also be shown that the LZ states of positive ghost number thus obtained are equivalent to those obtained by the construction described in \([6]\) up to exact pieces.

2.3. Edge States

The edge states are matter states corresponding to the boundary of the Kac table i.e, \((m',0)\) and \((0,m)\). In the minimal models before coupling to gravity one can use the fusion rules to show that the edge states decouple (along with the states outside the Kac table), and the states inside the Kac table form a closed algebra in the usual OPE sense.

However, on coupling to gravity the minimal model fusion rules seem to be erased \([3, 4]\). Hence we find that the edges (and the states outside the Kac Table) do not decouple by the standard arguments. In fact at the level of the three point functions it was shown
by Kitazawa [3] that even though the matter contribution to the three point function is
vanishing the Liouville contribution is infinite so that the full three point function is finite,
upto singular gamma functions corresponding to the leg factors of the external states. We
emphasise that this non-decoupling of the edge states is not dependent on the Coulomb
gas formalism. The zeros of the OPE coefficients are cancelled by the infinities of the
Liouville and these OPE coefficients carry intrinsic meaning independent of the Coulomb
gas method. Indeed the order of the zero, for instance can easily obtained from the
monodromy coefficients of the minimal model correlation functions.

We now look at the BRST analysis of these states on coupling to gravity. This has
been previously studied in [1];[12]. We recollect these results briefly in this section and then
indicate how descent equations can be used to construct the corresponding LZ states.

The Fock space resolutions of the edge states have been discussed in [14] and they are
of the form (see figure 3)

$$0 \xrightarrow{Q_F} \mathcal{F}_{0,m-(k+2)p} \xrightarrow{Q_F} \mathcal{F}_{0,m+kp} \xrightarrow{Q_F} 0$$

and in the dual space (see figure 4)

$$0 \xrightarrow{Q_F} \mathcal{F}_{0,-m-kp} \xrightarrow{Q_F} \mathcal{F}_{0,-m+(k+2)p} \xrightarrow{Q_F} 0$$

where $Q_F = \left( Q_- \right)^{(k+1)(p+1)}$.

Notice that since the Felder complex associated with the edge states has only finite
number of Fock spaces, there are only LZ states of ghost number $(\pm 1)$. The matter part
of the LZ states belong to the Fock space labelled $(0, m + kp)$ and its dual respectively.
Now we use the descent equations which arise from the double cohomology analysis of the
Felder and string BRST analysis (as was done for the states in the interior of the Kac table
in [3]). For the LZ state of positive ghost number we have (see figure 4)

$$|DK\rangle = Q_F |v_{0,-m-kp}\rangle_M \otimes |v_{0,-m+(k+2)p}\rangle_L \otimes c_1 |0\rangle_{gh} ,$$

$$|LZ\rangle^{+1} = Q_B |v_{0,-m-kp}\rangle_M \otimes |v_{0,-m+(k+2)p}\rangle_L \otimes c_1 |0\rangle_{gh} ,$$

(2.13)

The descent equation for the ghost number $-1$ is (see figure 3)

$$Q_B |LZ\rangle^{-1}_{(0,m+kp)} = Q_F |u'_1\rangle_{(0,m-(k+2)}}$$

(2.14)

from which we obtain the LZ state with ghost number $(1)$. As in the case of non-edge
states, on taking the matter dual in the LZ state of ghost number $-1$, we obtain the
ring elements corresponding to the edge states. These have matter momenta labelled by \((0, -m - kp)\).

A similar construction can be done for the \((m', 0)\) edges using \(Q_F = (Q_+)^{(k+1)p}\). Unlike the non-edge states, the descent seems tied down to a particular choice of resolution, i.e., descents involving \((0, m)\) edges use only \(Q_-\) and \((m', 0)\) use only \(Q_+\). Of course, this does not matter for the case of the ring elements as in this case there is no descent. Notice that in the construction of the LZ states of the edges we used Felder BRST operators of the form \(Q_F = (Q_-)^{(k+1)(p+1)}\) unlike in the construction for nonedge states where we always used Felder BRST operators of the form \(Q_F = (Q_+)^m\) with \(m < p\).

On comparing the scaling exponents of matrix models and topological minimal models coupled to gravity, only the \((0, m)\) edge states are required to complete the series of exponents. For the \((p+1, p)\) models, the Liouville momenta are parametrised by the series

\[
\frac{p(n - 2) + \alpha}{2\sqrt{p(p + 1)}}
\]

where \(n \geq 0\) and \(\alpha = 0, 1, \ldots, (p - 2)\). The edge states of type \((m', 0)\) are outside this set and correspond to \(\alpha = (p - 1)\). Thus, the two sets of edge states, although similar at first sight actually behave differently. However, it is not easy to see how the “wrong” edge states \(\text{-(m’, 0)}\) actually decouple in OPEs and correlation functions.

3. Chiral rings in \(c < 1\)

The general construction which argues for the presence of ghost number \(-1\) states in the Coulomb gas method allows us to develop a ring structure in analogy with the work of Witten for \(c = 1\). Following the suggestion of Kutasov, Martinec, and Seiberg, we define the two operators that generate the ring structure,

\[
x = R_{1,2} = (b_- c_1 + t(L_{-1}^L - L_{-1}^M)e^{i\alpha_{1,2}}Xe^{i\beta_{1,2}}
\]

\[
y = R_{2,1}(b_- c_1 + \frac{1}{t}(L_{-1}^L - L_{-1}^M)e^{i\alpha_{2,1}}Xe^{i\beta_{2,1}})
\]

i.e., \(x\) and \(y\) are the LZ\(^{-1}\) states with matter momenta labelled by \(\alpha_{1,2}\) and \(\alpha_{2,1}\) respectively. Before we proceed further we would like to point out that a target space boost of the \((X, \phi)\) system which transforms the \(c = 1\) theory to the appropriate \(c < 1\) theory, would in fact transform the generators \(x\) and \(y\) of Witten to precisely the ones we have written above. This fact is quite useful and we return to it later on.
For simplicity, let us consider the case of $c = 0$. In this case, $y$ is a physical operator in the theory while $x$ is not. The full set of Liouville momenta for this case (including the edge states of type $(0, m)$) fit the series $\frac{(n-2)}{\sqrt{6}}$. If we examine all the Liouville momenta that arise when we consider the powers $y^n$, we find that they precisely fit the set that appear for all LZ states of negative non-zero ghost number. Hence it appears that $y^n$ gives all the ghost number zero operators whose construction we described in the last section provided we work in the $Q_-$ resolution. One can also check the level of the oscillator and ghost part of the state $y^n$. It is easy to show by examining the vertex operator momenta for both the matter and Liouville part that the level is $(n + 1)$. A straightforward computation of the OPE of $y$ and $y$ in fact shows that $y^2$ is a ghost number -1 state of the type described in the previous section. One can expect therefore all $y^n$ to behave consistently in this fashion.

For the case of $c = 0$, $x$ as noted earlier is not a physical state. In fact, $x$ is a state from the edge of the Kac table, however this being the edge that does not appear in the set of operators that match the Liouville momenta that come from the matrix model scaling dimensions. However, if we compute $x^2$ by the usual OPE we find that it is a physical state. We can now check the Liouville momenta of the elements of the ring given by $x^{2n}$, $yx^{2n}$, and $y^2x^{2n}$ span the entire set of allowed Liouville momenta of the negative ghost number states of $c = 0$ coupled to matter. It thus appears that we have two sets of ghost number zero operators that describe the same set of states, even though the general cohomology argument assures us that there is only one ring element for a given Liouville momenta.

How are the two sets of ring elements related? The point is clarified by examining the two ring elements $x^2$ and $y^3$ in detail. Both these operators have the same Liouville momenta. The matter vertex operator momentum for the two operators are $(1, 3)$ and $(1, -1)$ respectively. The level of the two operators are 3 and 4 respectively. It is easy to see that if we operate with $Q_B$ on both these states we would get zero as there are null states at these levels. However using a $Q_+$ descent with $x^2$ and a $Q_-$ descent with $y^3$ we can go back and construct the same LZ state of ghost number $-2$. Thus we are led to an equivalence relation $x^2 \simeq y^3$. In general, all states which under $Q_+$ or $Q_-$ descent lead to the same LZ state would be identified with the same ring element. Thus the chiral ring in $c = 0$ matter coupled to Liouville is described by the elements $x^m y^n$ modulo the relation $x^2 \simeq y^3$ and we can have the elements $y^n$ and $xy^n$ or $x^{2n}$, $yx^{2n}$, $y^2x^{2n}$, $x^{2n+1}$, $x^{2n+1}y$ and $x^{2n+1}y^2$. The choice is dictated by the choice of resolution. Either set of elements forms a
ring. To reproduce only the allowed Liouville momenta the “wrong” edge states must be removed, namely the elements $xy^n$ or $x^{2n+1}$, $x^{2n+1}y$ and $x^{2n+1}y^2$.

These features clearly generalise to the arbitrary $(p + 1, p)$ model coupled to gravity. Thus in the general case we have the ring generated by the elements $x$ and $y$ and modded out by the equivalence relation

$$x^p \simeq y^{(p+1)}.$$  

This gives us the elements $y^n$, $xy^n$, $x^2y^n$, $x^{p-2}y^n$, $x^{p-1}y^n$. The Liouville scaling dimensions of these elements run over all the allowed Liouville scaling dimensions of the model given in (2.15) plus the momenta of the “wrong” edge states. Alternatively, we can have the elements of the dual resolution which can be simply obtained by using the equivalence relation (3.2) on the first set of elements. Both these sets of elements form a ring in themselves. Note however that unlike the case of $c = 0$, here the “wrong” edge states are needed for the ring to be closed under multiplication. Hence they cannot be excluded in any obvious way.

Let us return to the case of the ‘unphysical’ element $x$ in the case of $c = 0$. However this state is certainly in the $Q_B$ cohomology, except that the particular matter representation is simply not included in the theory. Hence it would seem that another way to represent the situation is to take the ring with elements $x^m y^n$ and mod out by the ideal generated by $x$.

These features can clearly be generalised for the arbitrary $(p + 1, p)$ model coupled to gravity. In the general case, we have the ring with elements $x^m y^n$ modded out by the ideal generated by the element $x^{p-1}$. It is easy to see that this construction gives us the elements $y^n$, $xy^n$, $x^2y^n$, $x^{p-2}y^n$. Here the “wrong” edge states clearly get excluded. It is useful to see the striking similarity with the structure of the states in topological minimal models coupled to topological gravity if we think of the $y^n$ as coming from the gravity part and the $x$, $x_{p-1}$ from the matter part.

Before we leave this discussion of the ring structure, we would like to return to the relation between the $c = 1$ coupled to gravity theory and the $c < 1$ coupled to gravity theory that was briefly referred to at the beginning of this section. The operators of the two theories are related by a Lorentz transformation of the $X$ and $\phi$ co-ordinates of the $c = 1$ theory that relates operators of that theory to those in $c < 1$ coupled to gravity. This
transformation has been noted by several authors earlier (See for instance [16] [17] [18]). The transformation of $X$ and $\phi$ can be written down

$$X \rightarrow X \cosh \theta + i\phi \sinh \theta$$
$$\phi \rightarrow iX(-\sinh \theta) + \phi \cosh \theta$$

(3.3)

where $\cosh \theta = \frac{2p+1}{2\sqrt{p(p+1)}}$ and $\sinh \theta = \frac{1}{2\sqrt{p(p+1)}}$. It is easy to see that this transformation rotates the energy-momentum tensor of the $c = 1$ model to that of the $c < 1$ model with the appropriate charge at infinity term. In the case of vertex operators, operators with specific matter and Liouville momenta are rotated from the $c = 1$ theory to the allowed matter and Liouville momenta of the $c < 1$ theory. A vertex operator of the form

$$V_k^- = e^{-ikX}e^{-(-1-k)\phi}$$

(3.4)

is transformed to

$$V_k^- = exp\left\{\frac{1}{2\sqrt{p(p+1)}}\left([2(p+1)k + 1]iX - [2(p+1)k + (2p+1)]\phi\right)\right\}$$

(3.5)

and

$$V_k^+ = e^{-ikX}e^{-(-1+k)\phi}$$

(3.6)

is transformed to

$$V_k^- = exp\left\{\frac{1}{2\sqrt{p(p+1)}}\left([2pk + 1]iX + [2pk - (2p+1)]\phi\right)\right\}$$

(3.7)

If we put $k = \frac{pn + \alpha + 1}{2p}$ for $V_k^+(k > 0)$ and $k = \frac{-pn - \alpha - 1}{2(p+1)}$ for $V_k^-(k < 0)$. where $n > 0$ and $\alpha = 0, \ldots, p - 2$, we obtain all the DK states of the $c < 1$ model. Though this parametrization (2.15) of the matter and Liouville momenta is somewhat unusual it is perfectly in accordance with requirements of comparing scaling dimensions that arise from matrix models. This transformation also rotates the ring generators $x$ and $y$ to those of the $c < 1$ model. Thus we can use some of the results of the $c = 1$ case directly in our cases. The ring multiplication trivially follows. So does the action of the ring element on the DK states. This invariance can easily be seen if one rewrites the ring elements in light-cone coordinates ($\phi \pm iX$). Now the Lorentz transformation corresponds to

$$(\phi + iX) \longrightarrow \gamma(\phi + iX)$$
$$(\phi - iX) \longrightarrow \frac{1}{\gamma}(\phi - iX)$$
where $\gamma = \sqrt{\frac{\nu}{\nu + 1}}$. The correlation function of N-DK states in $c < 1$ is identical to that of N tachyon operators in the $c = 1$ case if the values of momenta $k$ are chosen to be as given above. Of course, for these special values $(N, M)$ correlators no longer vanish and one has to extend the calculations of the $c = 1$ case. We can also use the ring elements exactly as in the $c = 1$ case to show that all the momentum dependence of the correlation functions is only in the form of leg factors\[19\]. These issues will be dealt with elsewhere [1].

4. Matrix model observables in the continuum

Notwithstanding the interesting structure of the space of states in this theory, the key question of the identification of the matrix model observables remains to be answered. We present in this section, a possible answer to the question, though a complete proof would be seen to be still lacking. As we shall soon see, these candidates are exact states but do not appear to decouple in correlation functions.

The LZ states have, as noted earlier $\beta \leq \beta_0$, with negative ghost number or $\beta \geq \beta_0$ with positive ghost number. However the matching of scaling dimensions of the operators with those of the matrix model requires that $\beta \leq \beta_0$. Since the Liouville scalar is non-compact, this must be true in both the holomorphic and non-holomorphic sectors. Clearly if we construct negative ghost number operators with $\beta \leq \beta_0$ in both holomorphic and anti-holomorphic sectors, then in any correlation function the ghost number cannot be balanced without introducing the positive ghost number operators with $\beta \geq \beta_0$ which of course will lead to the wrong scaling behaviour with respect to the matrix model. The question is whether we can introduce operators of positive ghost number that have $\beta \leq \beta_0$ in some systematic fashion.

In the construction of LZ states from the DK states described in [6], it was noticed that in the negative ghost number case, the LZ condition $\beta < \beta_0$ followed easily. But the same could not be said for the positive ghost number sector. The whole construction works for both the possible dressings. But this does not seem to agree with the cohomology analysis of LZ and BMP. This is resolved when one notices that for the case of dressing given by $\beta < \beta_0$, the same states can be constructed by other means which makes them exact. We shall show this by a simple example. Consider the oscillator state which replaces the null over the matter vertex operator labelled by $(1, 1)$. This is a level 1 oscillator state. This is given by

$$|w_{1,1}M = \partial X|v_{1,1}M \ .$$

(4.1)
The possible Liouville dressings are given by $\beta^\geq = \beta_{1,1}$ and $\beta^\leq = \beta_{-1,-1}$, the former corresponding to $\beta < \beta_0$. The LZ state at ghost number 1 is given by,

$$|W^\geq\rangle = \partial X |w_{1,1}\rangle_M \otimes |\beta^\geq\rangle \otimes c_1 |0\rangle_{gh} ,$$

$$|LZ^\geq\rangle = Q_B |W^\geq\rangle ,$$

$$= c_{-1} |w_{1,1}\rangle_M \otimes |\beta^\geq\rangle \otimes c_1 |0\rangle_{gh} \quad \text{and}$$

$$|W^\leq\rangle = \partial X |w_{1,1}\rangle_M \otimes |\beta^\leq\rangle \otimes c_1 |0\rangle_{gh} ,$$

$$|LZ^\leq\rangle = Q_B |W^\leq\rangle ,$$

$$= c_{-1} |w_{1,1}\rangle_M \otimes |\beta^\leq\rangle \otimes c_1 |0\rangle_{gh} . \quad (4.2)$$

The state $|LZ^\geq\rangle$ is not truly exact since $|W^\geq\rangle$ is not closed in the Felder cohomology. The state $|LZ^\geq\rangle$ would also seem to be not exact following similar arguments. But, it can be constructed by another means where it is clearly BRST exact.

$$|W_L\rangle = \partial \phi |w_{1,1}\rangle_M \otimes |\beta^\leq\rangle \otimes c_1 |0\rangle_{gh} ,$$

$$|LZ^\leq\rangle = Q_B |W_L\rangle . \quad (4.3)$$

Unlike, $|W^\leq\rangle$, $|W_L\rangle$ is closed under the action of $Q_F$ and hence the wrong dressing state is BRST exact. However a $Q_F$ map on both $|W^\geq\rangle$ and $|W^\leq\rangle$ gives a DK state that has the matter charge $\alpha_+$, namely that of one of the matter screening operators. The choice of wrong Liouville dressing therefore would give us a DK state that is one of the matter screening operators. In the other case, we would get an operator with the matter screening charge and the dual to the identity in the Liouville sector. This Liouville charge however will create problems in fixing the scaling behaviour of the correlation function.

Thus we see that at least one of the matter screening operators is associated with a LZ state that is not in the double cohomology. From the point of view of the DK type of states, the screening operator plays a crucial role in computing correlation functions. Hence a correlation function involving such an operator should presumably be non-vanishing. Thus a correlation function involving a operator of the form $LZ^{-n} \otimes LZ^{-n}$ and a string of operators of the form $c \partial c$ and $\bar{c} \partial \bar{c}$ would preserve ghost number and would have the same scaling dimensions as a one-point function of the corresponding DK operator. In general, one would expect such a correlator to be non-zero. We can see this by generalising the procedure of the repeated use of the $Q_B$ and $Q_F$ descent that was outlined in \[6\] for the

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\[5\] The bar refers to the anti-holomorphic sector.
case of the three-point function. In the case of four-point functions and higher we expect various subtleties of the contact terms of \( Q_B \) to come into play; however it is clear that we could expect it to be non-zero.

This procedure suggests a more general construction of a set of operators with ghost number zero where the string of \( c \partial c \) operators act effectively at the same point as the operator associated with the state \(|LZ\rangle^{-n} \otimes |\bar{L}Z\rangle^{-n}\). The physical operator in a symmetric form therefore could be written as

\[
(M - \bar{M})^n (LZ)^{-n} (\bar{L}Z)^{-n}.
\]

These operators are obviously closed under \( Q_B \) and are very similar to the zero-momentum dilaton in critical bosonic strings and the operators of topological gravity. They are also exact from the argument presented earlier for the exactness of the positive ghost number \( LZ \) operators with \( \beta < \beta_0 \).

5. Conclusions

There is still work to be done before the nature of the equivalence between the continuum formulation and the matrix model and topological formulations of \( c < 1 \) theories coupled to gravity is fully clear. However it is clear that the topological theory has a close relation to the underlying Coulomb gas formalism of the continuum formulation. The extra complications of screening etc. make the final picture less clear. The ring structure is particularly useful enabling more detailed calculations with DK states. Computations with the physical operators introduced in the last section are perhaps the most complicated, as is already evident from the work done on the zero-momentum dilaton of critical string theory\[20\] [21] the analog of the recursion relations of topological models would be reproduced here with these operators. We have also not isolated the algebras of \( W_n \) type that should appear in these models.

One of the most important problems is the implementation of the truncation of the "wrong" edge states. We believe that this is the source of the essentially non-linear algebra of \( W_n \) type that appears in these models. Without this truncation we would see only the essentially linear part of the algebra.

The formulation of the ring structure here is somewhat different from the approach of Sarmadi and Kanno [22] who do not use the Coulomb gas construction and have a ring of operators of different ghost numbers in contrast to our approach. It would be interesting
to study how these two different approaches are related. While readying this paper for publication, we also received their preprint [23].

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Appendix A. Cohomology argument for the existence of ring elements

We do not set out here to explain the full mathematical machinery that is needed, especially that of spectral sequences, for the analysis of the double cohomology. We merely take the results as given in Bouwknegt, McCarthy and Pilch [12]. Further mathematical details are given in the book of Bott and Tu [24]. The double cohomology can be analysed by either imposing the $Q_B$ cohomology first or the $Q_F$ cohomology on the double complex $(\mathcal{F}_{m',m} \otimes \mathcal{F}_L \otimes \mathcal{F}_{gh})$, henceforth denoted by $K$. We shall start with the latter. Note that we wish to include both the resolution $\mathcal{F}_{m',m}$ and the dual resolution $\mathcal{F}_{p'-m',p-m}$. The $Q_F$ cohomology is non-trivial only for $H_{Q_F}^{(0)}(K)$. However the dimensionality of this is 2 since we have included both the resolutions, viz. we get both the matter momenta $\alpha_{m',m}$ and its dual. We can now impose the $Q_B$ cohomology, obtaining $H_{Q_B}^{(n)}(H_{Q_F}^{(0)}(K))$. The total cohomology for negative ghost number is given by

$$H_D^{(-n)} = \sum_{p+q=-n} H^p_{rel}(H^q(\mathcal{F}_M \otimes \mathcal{F}_L \otimes \mathcal{F}_{gh}, Q_F), Q_B)$$

(A.1)

$$= H_{rel}^{(-n)}(H^{(0)}(\mathcal{F}_M \otimes \mathcal{F}_L \otimes \mathcal{F}_{gh}, Q_F), Q_B)$$

It turns out as we explain below that, in fact,

$$H_D^{(-n)} = \sum_{p+q=-n} H^q(H^p_{rel}(\mathcal{F}_M \otimes \mathcal{F}_L \otimes \mathcal{F}_{gh}, Q_B), Q_F)$$

(A.2)

$$= H^{-n+1}(H^1_{rel}(\mathcal{F}_M \otimes \mathcal{F}_L \otimes \mathcal{F}_{gh}, Q_B), Q_F)$$

$$+ H^{-n}(H^0_{rel}(\mathcal{F}_M \otimes \mathcal{F}_L \otimes \mathcal{F}_{gh}, Q_B), Q_F)$$

Since $H_D^{(-n)}$ computed both ways should be the same, and the DK states exhaust the elements of $H_{rel}^{(-n)}(H^{(0)}(\mathcal{F}_M \otimes \mathcal{F}_L \otimes \mathcal{F}_{gh}, Q_B), Q_F)$, the ring elements are clearly identified with the rest.
Before deriving (A.2), we first illustrate the technique of spectral sequences for filtered complexes by deriving (A.1). For the double cohomology in question, we have one spectral sequence \( \{ E_r, d_r \} \) where \( E_r \) is bigraded and

\[
\begin{align*}
E_1^{p,q} &= H_{Q_F}^{p,q}(K) \\
E_2^{p,q} &= H_{Q_B}^{p,q}H_{Q_F}(K) \\
& \vdots \\
E_r^{p,q} &= H_{Q_B}^{p,q}H_{Q_F}(K)
\end{align*}
\]

and \( d_r : E_r^{p,q} \rightarrow E_{r+1}^{p,q-r+1} \). Note that in this notation, \( H_{Q_F}^{p,q} \) is the \( p \)-th \( Q_F \) cohomology in the \( q \)-th complex and \( H_{Q_B}^{p,q} \) is the \( q \)-th \( Q_B \) cohomology in the \( p \)-th complex. Obviously, \( p \) is the tower number and \( q \) is the ghost number in the complex \( K \). Since \( H_{Q_F} \) is non-zero only for \( p = 0 \), we have

\[
E_2^{0,n} = H_{Q_B}^{0,n}H_{Q_F}(K)
\]

It is clear that \( d_2 \) is zero as there is no element in \( E_2^{2,n-1} \). Hence \( E_2 = E_3 = \ldots = E_\infty \) and \( H_{D}^{(n)} = H_{Q_B}^{0,n}H_{Q_F}(K) \). This is indeed (A.1). We can also construct another spectral sequence, \( \{ E_r', d_r' \} \), where \( E_r' \) is bigraded and

\[
\begin{align*}
E_1'^{p,q} &= H_{Q_B}^{p,q}(K) \\
E_2'^{p,q} &= H_{Q_F}^{p,q}H_{Q_B}(K) \\
& \vdots
\end{align*}
\]

and \( d_r' : E_r'^{p,q} \rightarrow E_{r+1}'^{p,q-r+q+r} \). If we are in the sector of Liouville momenta with \( \beta < \beta_0 \), we get non-zero elements only in \( E_2'^{(n,0)} = H_{Q_F}^{0,0}H_{Q_B}(K) \) and in \( E_2'^{(n-1,1)} = H_{Q_F}^{n-1,1}H_{Q_B}(K) \). Hence again \( d_2' \) is zero and \( E_2' = E_3' = \ldots = E_\infty' \). Thus

\[
H_{D}^{(n)}(K) = \oplus_{p+q=n} E_\infty'^{(p,q)}(K) = \oplus_{p+q=n} E_2'^{(p,q)}(K) = H_{Q_F}^{(n-1,1)}H_{Q_B}(K) \oplus H_{Q_B}(K).
\]

One can always rotate from one quadrant to another in the two gradings and hence we have (A.1) and (A.2).
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