A Phase Space Approach to the Conformal Construction of Non-Vacuum Initial Data Sets in General Relativity

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Abstract

We present a uniform (and unambiguous) procedure for scaling the matter fields in implementing the conformal method to parameterize and construct solutions of Einstein constraint equations with coupled matter sources. The approach is based on a phase space representation of the spacetime matter fields after a careful $n + 1$ decomposition into spatial fields $B$ and conjugate momenta $\Pi_B$, which are specified directly and are conformally invariant quantities. We show that if the Einstein-matter field theory is specified by a Lagrangian which is diffeomorphism invariant and involves no dependence on derivatives of the spacetime metric in the matter portion of the Lagrangian, then fixing $B$ and $\Pi_B$ results in conformal constraint equations that, for constant-mean curvature initial data, semi-decouple just as they do for the vacuum Einstein conformal constraint equations. We prove this result by establishing a structural property of the Einstein momentum constraint that is independent of the conformal method: For an Einstein-matter field theory which satisfies the conditions just stated, if $B$ and $\Pi_B$ satisfy the matter Euler-Lagrange equations, then (in suitable form) the right-hand side of the momentum constraint on each spatial slice depends only on $B$ and $\Pi_B$ and is independent of the spacetime metric. We discuss the details of our construction in the special cases of the following models: Einstein-Maxwell-charged scalar field, Einstein-Proca, Einstein-perfect fluid, and Einstein-Maxwell-charged dust. In these examples we find that our technique gives a theoretical basis for scaling rules, such as those for electromagnetism, that have worked pragmatically in the past, but also generates new equations with advantageous features for perfect fluids that allow direct specification of total rest mass and total charge in any spatial region.

1 Introduction

Initial data for the Cauchy problem in general relativity consist of a Riemannian metric $h$ on an $n$-dimensional initial time slice $\Sigma$ along with a symmetric $(0, 2)$-tensor $K$ that indirectly encodes the momentum conjugate to $h$ [ADM59] and that directly specifies the second fundamental form of the embedding of $\Sigma$ into the ambient spacetime $M$. For reasons analogous to those leading to the Gauss equation of electromagnetism, initial data

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(h, K) for the Cauchy problem cannot be freely specified, but are instead subject to the Einstein constraint equations

\[ R_h - |K|^2_h + (\text{tr}_h K)^2 = 16\pi E \]  \hspace{1cm} \text{[Hamiltonian constraint]} \hspace{1cm} (1.1)

\[ \text{div}_h K - d(\text{tr}_h K) = -8\pi J \]  \hspace{1cm} \text{[momentum constraint]} \hspace{1cm} (1.2)

where \( R_h \) and \( \text{tr}_h \) are the scalar curvature and trace operator of \( h \), \( d \) is the exterior derivative on \( \Sigma \) and, as recalled in Section 2.2, \( E \) and \( J \) are the energy and momentum densities of the matter fields.

There is a range of methods for constructing solutions \((h, K)\) of the constraint equations (1.1)–(1.2), including gluing [CD03][CIP05][CS06] and parabolic techniques[Ba93], but the conformal method [Li44][Yo73][Yo99][PY03][Ma14] stands out as a versatile workhorse and is widely used, for example, in numerical relativity. It is especially effective for generating constant-mean curvature (CMC) solutions, those satisfying \( \text{tr}_h K \) is constant. Subject to this restriction, the conformal method provides a satisfactory parameterization of the set of vacuum solutions of the constraint equations in vacuum in a number of asymptotic settings [Is95][Yo99][AC96].

The aim of this paper is to demonstrate how to effectively apply the conformal method to construct CMC non-vacuum initial data sets. Of course, the conformal method has long been used in the non-vacuum case ([IN77] is an early reference), but the community has not standardized on a single approach. Indeed alternative methods of scaling and unscaling sources appear in the literature [CB09]. Moreover, within the method of scaling sources, there are choices that need to be made for how various matter fields are scaled. Perfect fluids are a notable example, where the approaches used in the literature to pick a scaling (e.g., [DN02] or our own previous work [IMP05]) can seem ad-hoc, motivated by analytical practicality rather than a unifying principle. In this work we present a uniform and unambiguous approach to applying the method of scaling sources to the conformal method with coupled matter source fields. As we discuss below, this approach is closely tied to the Hamiltonian formulation of the matter source fields, with the phase space representation of the initial data for the source field playing a key role. Applying our technique, we recover well-known scaling formulas for certain common matter models (electromagnetism, most importantly), but also derive new equations for perfect fluids and, in particular, even for dust. That is, our work provides a theoretical foundation for what has worked practically in the past, and also derives previously unknown equations that extend the reach of the conformal method. Section 7.3 shows, for example, how to apply our techniques to charged dust coupled to electromagnetism and ensure that in the resulting solution of the constraint equations, the fluid’s charge to mass ratio is constant; previous applications of the conformal method to fluid matter models are unable to ensure this final condition on the charge density. We note additionally that our central results (Theorems 3.4 and 4.4) are not specific to the conformal method; they concern the structure of the momentum density \( J \) and are of independent interest.

To illustrate the situation, first consider the case of vacuum initial data on a compact time slice \( \Sigma \). One can take the gravitational seed data for the conformal method to consist of a Riemannian metric \( h \), a mean curvature function \( \tau \), a positive function \( N \), and a symmetric, trace-free (0,2)-tensor \( U \); we leave this data unmotivated for the moment and defer a more thorough discussion to Section 6. One then seeks a positive
conformal factor $\phi$ and a vector field $Z$ satisfying the Lichnerowicz-Choquet-Bruhat-York (LCBY) equations

$$-2\kappa q \Delta_h \phi + R_h \phi - \frac{1}{4N^2} |U + L_h Z|^2_h \phi^{-q-1} + \kappa \tau^2 \phi^{q-1} = 0$$  \hspace{1cm} \text{[LCBY Hamiltonian constraint]} \hspace{1cm} (1.3)$$

$$\text{div}_h \left( \frac{1}{2N} L_h Z \right) - \phi^q \kappa d\tau = - \text{div}_h \left( \frac{1}{2N} L_h U \right)$$  \hspace{1cm} \text{[LCBY momentum constraint]} \hspace{1cm} (1.4)$$

where $\Delta_h$ is the Laplacian of $h$, $L_h$ is the conformal Killing operator (equation (6.6) below), and where $\kappa$ and $q$ are constants depending on the dimension $n \geq 3$ of $\Sigma$:

$$q = \frac{2n}{n-2}, \quad \kappa = \frac{n-1}{n}. \hspace{1cm} (1.5)$$

For each solution $(\phi, Z)$ of equations (1.3)–(1.4) there is a corresponding physical solution $(h^*, K^*)$ of the constraint equations with $h^* = \phi^{q-2} h$ and with $K^*$ determined from $h, \phi, U, N, \tau$, and $Z$; Section 6.2 contains the details.

When $\tau$ is constant system (1.3)–(1.4) simplifies considerably because the LCBY momentum constraint no longer depends on $\phi$ and can be first solved for $Z$. On a compact manifold, there is a unique solution of the LCBY momentum constraint, up to the addition of a conformal Killing field (i.e., an element of the kernel of $L_h$). Moreover, because $Z$ only appears in the LCBY Hamiltonian constraint via $L_h Z$, this potential degeneracy is unimportant, and the analysis of the existence and multiplicity of solutions hinges only on the (scalar) LCBY Hamiltonian constraint. This decoupling of the momentum constraint is a key feature of the conformal method, and leads to a satisfactory parameterization of the CMC solutions of the constraint equations on compact manifolds where generic seed data leads to a unique solution and the exceptional cases are predictable and well-understood [Is95]. A similar analysis applies in the asymptotically Euclidean and asymptotically hyperbolic settings as well [Yo99][AC96]. When $\tau$ is not constant, however, the conformal method appears to suffer from a number of undesirable features [Ma11][Ng18][DHKM17], and for this reason we focus our attention on the CMC case. See also [Ma21] for an alternative generalization of the CMC conformal method to non-CMC initial data.

In the presence of matter, the LCBY equations become

$$-2\kappa q \Delta_h \phi + R_h \phi - \frac{1}{4N^2} |U + L_h Z|^2_h \phi^{-q-1} + \kappa \tau^2 \phi^{q-1} = 16\pi \phi^{q-1} E^*$$  \hspace{1cm} (1.6)$$

$$\text{div}_h \left( \frac{1}{2N} L_h Z \right) - \phi^q \kappa d\tau = - \text{div}_h \left( \frac{1}{2N} U \right) - 8\pi \phi^{q} J^* \hspace{1cm} (1.7)$$

where we have decorated the physical energy and momentum densities $E^*$ and $J^*$ with stars to indicate that they are generally expressions that depend on the solution metric $h^*$ (and hence on $h$ and $\phi$) along with certain seed data related to the matter model. For example, in the case of a scalar field we can take the matter seed data to consist of the field value $\psi$ along with a function $P$ that prescribes the normal derivative of the
scalar field in the ambient spacetime. In this case,

\[ \mathcal{E}^* = \frac{1}{2} P^2 + \frac{1}{2} |d\phi|^2_{h^*}, \]

where

\[ \mathcal{J}^* = -P \, d\phi. \] (1.9)

However, the LCBY momentum constraint now no longer decouples from the rest of the system because of the term \( \phi^q \mathcal{J}^* = -\phi^q P d\phi \) appearing in equation (1.7). Pragmatically, the obvious approach to recover the decoupling is to allow \( P \) to conformally transform so that the scaling \( h^* = \phi^{-2} h \) is associated with a scaling \( P^* = \phi^{-q} P \) for some initially specified function \( P \). Replacing \( P \) with \( P^* \) in equation (1.9), and assuming additionally that \( \tau \) is constant, equations (1.6)–(1.7) become

\[ -2\kappa q \Delta h \phi + R_{\alpha\beta} \phi - \frac{1}{4N^2} |U + L_h Z|_h^2 \phi^{-q-1} + \kappa T^2 \phi^{-1} = 8\pi \left[ \phi^{-q-1} P^2 + \phi |d\phi|^2_{h^*} \right] \] (1.10)

\[ \text{div}_h \left( \frac{1}{2N} L_h Z \right) = - \text{div}_h \left( \frac{1}{2N} U \right) + 8\pi P d\phi \] (1.11)

which achieves the desired decoupling of the momentum constraint. On a compact manifold, in the generic case where \( h \) has no conformal Killing fields, equation (1.11) can be solved independently for \( Z \), and there is a unique solution. Thus, as in the vacuum setting, the analysis reduces to the Lichnerowicz equation. It is worth noting that although the decoupling simplifies the analysis, the resulting Lichnerowicz equation can still harbor interesting features. Indeed, for the scalar field LCBY Hamiltonian constraint (1.10) the term \( \phi |d\phi|^2_{h^*} \) interacts nontrivially with the scalar curvature term and we refer to, e.g., [HPP08] for an analyses.

A similar scaling/decoupling procedure can be applied to electromagnetism. Now the matter fields can be specified by a 1-form \( E \) (the electric field) and a 2-form \( B \) (the magnetic field). In this case

\[ \mathcal{E}^* = \frac{1}{8\pi} \left[ |E|_{h^*}^2 + \frac{1}{2} |B|_{h^*}^2 \right], \]

\[ \mathcal{J}^* = \frac{1}{4\pi} \langle E, \cdot \cdot \cdot B \rangle_{h^*}, \]

The right-hand side of the LCBY momentum constraint then contains a term \( \phi^2 \langle E, \cdot \cdot \cdot B \rangle \) which again leads to coupling of the full LCBY system, even in in the CMC setting. To circumvent this difficulty we introduce a conformally scaling electric field so that the transformation \( h^* = \phi^{-2} h \) is associated with a transformation \( E^* = \phi^{-q} E \) for some initially specified 1-form \( E \). Moreover, \( \text{div}_{h^*} E^* = \text{div}_h \phi^{-2} E = \phi^{-q} \text{div}_h E \) and hence if \( E \) is \( h^* \)-divergence free (as is required by Gauss’ Law when there are no electric sources), then \( E^* \) is \( h^* \)-divergence free as well. It is seemingly remarkable that the scaling law that leads to decoupling the LCBY equations simultaneously leads to preservation of the electrovac constraint \( \text{div}_{h^*} E^* = 0 \).

The success of the scaling/decoupling procedure for these and other matter models naturally leads to the question of whether there is an underlying principle that unifies these examples. Moreover, one would
like to know if there is a rule that determines directly (without ad-hoc manipulation) how to scale the matter fields in general. Our main results give a positive resolution to these questions under the following physically natural hypotheses:

- the matter fields have an associated Lagrangian,
- the matter Lagrangian is diffeomorphism invariant,
- the matter Lagrangian is minimally coupled to gravity (so no derivatives of the spacetime metric appear in it).

A Lagrangian in this setting depends on the spacetime metric $g$ along with matter fields $\mathcal{F}$ and, as recalled in Section 2.2, the momentum density $\mathcal{J}$ is a well-defined quantity on a time slice $\Sigma$ depending on $g$ and $\mathcal{F}$ regardless of whether the Einstein equations or the equations of motion for the matter fields are satisfied. On this same slice, the spacetime metric determines a Riemannian metric $h$, and our main results stem from an invariance property of the combination $\mathcal{J} dV_h$ under metric perturbations. Indeed, in Sections 3 and 4 we prove two theorems that are concrete expressions of the following imprecise meta-theorem, which is a structural statement about the Einstein equations and is not simply a fact about the conformal method.

**Meta-Theorem 1.1.** Under the above hypotheses, the spacetime matter fields can be decomposed into certain 'spatial' variables $B$ with conjugate momenta $\Pi_B$ such that if $B$ and $\Pi_B$ satisfy the Euler-Lagrange equations of the matter Lagrangian, then the covector-valued $n$-form

$$\mathcal{J} dV_h (1.12)$$

on each time slice is determined by $B$ and $\Pi_B$ alone, and is independent of the metric.

That is, although the spatial metric volume form $dV_h$ and the momentum density $\mathcal{J}$ each depend on the metric, the combination $\mathcal{J} dV_h := \mathcal{J} \otimes dV_h$ depends only on the state and momenta of the matter fields after a suitable decomposition into 'spatial' variables. In fact, we give an explicit relation for how $\mathcal{J} dV_h$ is determined from the matter fields, equation (1.13) below, but these details are not needed to see how the meta-theorem resolves the question of decoupling. Indeed, consider a pair $h$ and $h^*$ of conformally related metrics: $h^* = \phi^2 h$. Fixing $B$ and $\Pi_B$, the meta-theorem implies

$$\mathcal{J}^* dV_{h^*} = \mathcal{J} dV_h,$$

where $\mathcal{J}^*$ and $\mathcal{J}$ are the momentum densities determined by the fixed matter fields along with $h^*$ and $h$ respectively. A computation shows $dV_{h^*} = \phi^2 dV_h$ and as a consequence $\mathcal{J}^* = \phi^{-2} \mathcal{J}$. Thus, so long as $B$ and $\Pi_B$ remain fixed, the term $\phi^2 \mathcal{J}^*$ appearing on the right-hand side of the LCBY momentum constraint (1.7) is the unique value $\mathcal{J} = \mathcal{J}(h, B, \Pi_B)$ regardless of the choice of conformal factor, and this leads to decoupling of the momentum constraint for CMC data.

The scaling law for the matter terms $B$ and $\Pi_B$ is the simplest one imaginable: scale nothing. Matter seed data consists of fixed field values and conjugate momenta, and in this sense the method of scaling sources is something of a misnomer: presented properly, the method would be better described as that of non-scaling.

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1The conjugate momentum of a field is the derivative of a Lagrangian with respect to the field's time derivative; formal definitions appear just prior to Theorem 3.4 and Corollary 4.3 below.
sources, except that this name is already used in the literature for something quite different and less practical. One may wish to represent the matter field in terms of different variables that mix in the metric, in which case there would be a non-trivial conformal scaling. In the case of a scalar field, it turns out that the momentum conjugate to $\psi$ is $\Pi_\psi = 2P dV_h$ and hence if $\Pi_\psi$ remains fixed but $h$ conformally transforms to $h^* = \phi^{\xi-2} h$, then $P$ must conformally transform to $P^* = \phi^q P$, exactly the scaling law used to derive equations (1.10)–(1.11). For electromagnetism, the important dynamical field is a $U(1)$ connection represented locally on a time slice by a spatial 1-form $A$, and its curvature is the 2-form $B$. The momentum conjugate to $A$ is the vector valued $n$-form $\Pi_A = -1/(4\pi) \langle E, \cdot \rangle dV_h$. One readily verifies that if $\Pi_A$ remains fixed and $h$ conformally transforms to $h^* = \phi^{\xi-2} h$, then we must conformally transform $E$ to $E^* = \phi^{-2} E$, which is again the scaling law used in the discussion prior to Meta-Theorem 1.1. Gauss’ Law $\text{div}_h E = 0$ is, in fact, a constraint on $\Pi_A$, and since $\Pi_A$ remains fixed, Gauss’ Law remains satisfied.

The vagueness of Meta-Theorem 1.1 comes from the breadth of what can be included under the umbrella of physical field theory. We prove variations of the meta-theorem in the following cases:

1. The matter fields take on values in a separate manifold (Theorem 3.4).
2. The matter fields are tensor-valued (Theorem 4.4).

Certain matter models, such as electromagnetism coupled with a charged fluid, involve both tensor-valued and manifold-valued fields and can be addressed by a straightforward generalization of our results. Nevertheless, the framework for field theory that we use is not enough to include all conceivable cases and the Vlasov matter model is a notable exception that we will address in future work.

The mechanism of how $J dV_h$ is determined from $B$ and $\Pi_B$ in Meta-Theorem 1.1 and its precise forms Theorems 3.4 and 4.4 is a little subtle. In Proposition 4.2 concerning tensor-valued fields, we show that so long as $B$ and $\Pi_B$ solve the Euler-Lagrange equations on the ambient manifold, then for any compactly supported vector field $\delta X$ on a time slice $\Sigma$

$$\int_{\Sigma} J(\delta X) dV_h = -\frac{1}{2} \int_{\Sigma} \Pi_B(\text{Lie}_{\delta X} B).$$  \hfill (1.13)

One would like to claim that

$$J dV_h = -\frac{1}{2} \Pi_B(\text{Lie} B)$$  \hfill (1.14)

but $\text{Lie}_{\delta X}$ is not tensorial in $\delta X$ and an exact expression for $J dV_h$ usually requires some integration by parts. The hypothesis that $B$ and $\Pi_B$ solve the Euler-Lagrange equations on the ambient manifold is seemingly out of place for a theorem about initial data, where $B$ and $\Pi_B$ are specified on a single time slice. However, the Euler-Lagrange equations may imply constraints on the initial data, such as Gauss’ law, and it turns out that these constraints must be satisfied in order for equation (1.13) to hold; see Section 5 where this is illustrated explicitly for a charged scalar field coupled to electromagnetism. On the other hand, the situation for manifold-valued fields is somewhat simpler; no $(n+1)$-decomposition of the spacetime matter field $\mathcal{F}$ is involved and in Proposition 3.3 we show

$$J dV_h = -\frac{1}{2} \Pi_B \circ \mathcal{F}.$$  

Some historical remarks concerning our results are in order. In fact, the general notion that matter fields and conjugate momenta should remain fixed in order to obtain decoupling was known in the 1970s, and an
imprecise statement along the lines of equation (1.14) appears in earlier work by one of us [IN77]. Although [IN77] cites the physics-style paper [Ku76] as justification, the literature does not appear to contain a mathematical proof. Indeed, there are important considerations needed to arrive at equation (1.14), including a specific form of the \((n + 1)\)-decomposition, that appear to have been previously overlooked. Moreover, the underlying principle seems to have been forgotten by the community generally, and is not mentioned in modern texts, e.g. [CB09]. Indeed, it was recently independently rediscovered by one of us (Maxwell). The current work therefore serves two purposes: first, to advertise the principle and second to give it rigorous justification.

The remainder of the paper has the following structure:

- Section 2 contains preliminaries to fix notation and recall important facts about the constraint equations. This can be skimmed, taking into account our notation \(j_\ast\) for a projection defined in Section 2.1 that is non-standard but used pervasively in the subsequent sections.
- Section 3 contains Theorem 3.4, the version of Meta-Theorem 1.1 that applies to matter sources taking on values in a separate manifold. While this is not a common setting, fluids are a notable application.
- Section 4 contains Theorem 4.4, the version of Meta-Theorem 1.1 that applies to tensor-valued matter sources.
- Section 5 illustrates our main results in the context of electromagnetism coupled to a charged scalar field (EMCSF). It serves as a concrete example to motivate the somewhat technical details of Section 4, especially the assumptions needed to obtain equation (1.13).
- Section 6 provides a more in-depth overview of the conformal method, and links it to a technique for solving a constraint appearing in the EMCSF model of Section 5. Section 6.3 contains remarks about important differences, even with the good scaling provided by Meta-Theorem 1.1, between the non-vacuum CMC conformal method versus its vacuum formulation.
- Finally, Section 7 contains applications of our results in three interesting cases in the context of the conformal method. In Section 7.1 we treat perfect fluids and, using the Lagrangian formulation for fluids, derive equations different from those found in [DN02] or [IMP05], including a new equation for dust. Our construction directly prescribes the total particle number in every spatial region, whereas past constructions do not preserve even the total particle number. Section 7.2 applies our results to the Proca model of a massive, spin-1 particle. The resulting equations contain an unexpected and potentially insightful appearance of the ‘densitized lapse’, perhaps the least-well understood component of the conformal method. Finally, Section 7.3 considers electromagnetism coupled to charged dust. Because we directly specify both the total rest mass and charge in any spatial region, we ensure that the mass to charge ratio of the particles is constant, a distinguishing feature of our approach.

# 2 Preliminaries

## 2.1 Slicing and Spatial Tensors

The focus of this work is essentially local and we therefore take, once and for all, \(M = \Sigma \times I\) where \(\Sigma\) is an open subset of \(\mathbb{R}^n\) with coordinates \(x^r\) and where \(I\) is an open interval in \(\mathbb{R}\) with coordinate \(t\). Each \(t \in I\) is
associated with the time slice $\Sigma_t = \Sigma \times \{t\}$. The split into space and time, while involving an arbitrary choice, is essential to the Hamiltonian approach that underlies our approach. With these fixed coordinates in hand, we use them whenever expedient to shorten a definition or argument, rather than relying on coordinate-free notations.

Although our main concern is initial data, our construction is tightly linked to the evolution problem and for the most part we work with tensors on the ambient spacetime $M$. Tensor fields on $M$ are assumed, for simplicity, to be smooth. A tensor $T$ on $M$ is \textbf{spatial} if it vanishes when one of its arguments is either of $dt$ or $\partial_t$. Equivalently, a spatial tensor consists of tensor products of $dx^i$ and $\partial_{x^i}$, and spatial tensors define, unambiguously, tensors intrinsic to each slice $\Sigma_t$.

We define a projection $j_*$ onto spatial tensors by declaring

$$j_* dt = 0,$$

$$j_* dx^i = dx^i,$$

$$j_* \partial_t = 0,$$

$$j_* \partial_{x^i} = \partial_{x^i},$$

and by extending $j_*$ to higher rank tensors by $j_* T(\cdot, \cdot \cdot, \cdot) = T(j_* \cdot, \cdot \cdot, j_* \cdot)$. In coordinates, $j_* T$ is obtained from $T$ by dropping any component containing either $dt$ or $\partial_t$.

The following elementary observations can all be established by working in coordinates.

\textbf{Lemma 2.1.}

1. Suppose $B$ is a spatial tensor field. Then $\partial_t B := \text{Lie}_{\partial_t} B$ is spatial, as is $\text{Lie}_S B$ for any spatial vector field $S$.

2. For any tensor field $T$, $\partial_t j_* T = j_* \partial_t T$.

3. If $T$ is a tensor field that vanishes on a slice $\Sigma_t$, then for any spatial vector field $S$, $\text{Lie}_S T = 0$ on $\Sigma_t$.

Consequently, if $T_1$ and $T_2$ are arbitrary tensor fields that are equal on a slice $\Sigma_t$, then $\text{Lie}_S T_1 = \text{Lie}_S T_2$ on $\Sigma_t$.

Note that the projection $j_*$ is a coordinate-dependent operation, but is independent of any metric.

The Hamiltonian approach to the evolution problem in general relativity requires splitting tensor fields into spatial components defined on the slices $\Sigma_t$, and this applies equally to the spacetime metric. A Lorentzian metric $g$ on $M$ is \textbf{slice compatible} if the restriction of $g$ to each slice $\Sigma_t$ is Riemannian. A slice-compatible Lorentzian metric has a timelike unique normal vector field $\nu$ defined by $\nu(dt) > 0$, $g(\nu, \nu) = -1$ and $j_* g(\nu, \cdot) = 0$. The vector field $\partial_t$ can then be written

$$\partial_t = N\nu + X$$

where the \textbf{shift} $X$ is spatial and where the \textbf{lapse} $N$ satisfies $\nu(dt) = 1/N$. The $n + 1$ \textbf{decomposition} of $g$ consists of the spatial tensors $h = j_* g$, $N$ and $X$. As is clear from the coordinate representation $g = -N^2 dt^2 + h_{ab}(dx^a + X^a dt)(dx^b + X^b dt)$, the metric is uniquely determined by its $n + 1$ decomposition, and we write $g(h, N, X)$ for the Lorentzian metric reconstructed from a spatial Riemannian metric $h$, a lapse $N > 0$. 

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and a spatial vector $X$. Our notation here conflicts mildly with that used in the introduction, where $h$ and $dV_h$ are tensor fields defined on a single time slice $\Sigma$, whereas here they are spatial tensors defined on all of $M$. Moreover, $h$ is not a Riemannian metric in the usual sense because $h(\partial_t, \cdot) = 0$, but it becomes one when restricted to any slice $\Sigma$.

The manifold $M$ is orientable and we fix an orientation on it. Let $dV_g$ be the positively oriented volume form of $g$ on $M$. We then define

$$dV_h = N^{-1} \partial_t \cdot dV_g$$

and observe that $dV_h$ is spatial and $N dt \wedge dV_h = dV_g$.

### 2.2 Einstein Constraint Equations

In this section we give a brief overview of the origin of the Einstein constraint equations, with a particular focus on deriving relation (2.4) below, which is a key tool in our work.

A Lagrangian for a field theory on $M$ is defined in terms of an $n + 1$-form on $M$. In particular, the Einstein-Hilbert Lagrangian for a Lorentzian metric $g$ is

$$L_{EH}(g, \partial g, \partial^2 g) = R_g dV_g$$

where $R_g$ is the scalar curvature of $g$. Recall that we have a fixed frame $\partial_k = \partial_k x$ of vector fields, and $\partial g$ denotes the collection of Lie derivatives $(\partial_1 g, \ldots, \partial_n g, \partial_t g)$, with $\partial^2 g$ defined similarly.

Let $L_{\text{Matter}}(\mathcal{F}, g)$ be a Lagrangian on $M$ depending on a section $\mathcal{F}$ of some fiber bundle over $M$ along with a Lorentzian metric $g$. In general, $L$ depends on ‘derivatives’ of $\mathcal{F}$, and we are more precise about this in applications. Regarding the metric, however, we assume the matter Lagrangian is minimally coupled and therefore depends pointwise on the values of $g$ but not its derivatives. The total Lagrangian for the metric and matter is then

$$L_{EH} + 8\pi L_{\text{Matter}}.$$

Recall that the action for the total Lagrangian is stationary at $g$ if it satisfies the Einstein equation

$$G = 8\pi T$$

where $G_g = \text{Ric}_g - R_g g$ is the Einstein tensor and where (as in [Wa84] Appendix E) the stress-energy tensor $T$ is defined by varying $L_{\text{Matter}}$ with respect to the inverse metric $g^{-1}$:

$$T dV_g = - \frac{\partial L_{\text{Matter}}}{\partial g^{-1}}$$

More explicitly, we are considering the map

$$g^{-1} \mapsto L_{\text{Matter}}(\mathcal{F}, g(g^{-1}))$$

pointwise on $M$ and linearizing with respect to $g^{-1}$. Hence $\partial L_{\text{Matter}}/\partial g^{-1}$ can be identified with a symmetric $(0,2)$ tensor-valued $n + 1$-form, and $T$ is obtained by dividing by $-dV_h$. 

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An inverse metric $g^{-1}$ that is Riemannian on the slices $\Sigma$ can be written as a function of a spatial Riemannian inverse metric $h^{-1}$ and a vector field $\nu$ transverse to the slices:

$$g^{-1} = -\nu \otimes \nu + h^{-1}.$$  

With respect to these variables,

$$\frac{\partial g^{-1}}{\partial \nu} [\delta \nu] = -(\nu \otimes \delta \nu + \delta \nu \otimes \nu)$$

and the Einstein constraint equations arise from variations of the total action with respect to $\nu$:

$$G(\nu, \cdot) = 8\pi T(\nu, \cdot).\quad (2.1)$$

The scalar **energy density** $E$ and covector **momentum density** $J$ of the matter fields seen by an observer with tangent $\nu$ are, by definition,

$$E = T(\nu, \nu)$$

$$J = -j^* T(\nu, \cdot).\quad (2.2)\quad (2.3)$$

On a slice $\Sigma_t$ we have the induced metric $h = j_* g$ and the second fundamental form $K = \frac{1}{2} \text{Lie}_\nu h$. The Gauss and Codazzi equations allow the the left-hand side of equation (2.1) to be written in terms of $g$ and $K$, whereas the right-hand side is written in terms $E$ and $J$ essentially by fiat. The Einstein constraint equations (1.1)–(1.2) then arise from identifying $\Sigma$ with $\Sigma_t = (\iota_t)_* \Sigma$ and spatial tensors on $\Sigma_t$ with their pullbacks onto $\Sigma$.

Writing $\nu = (\partial_t - X)/N$, the momentum density $J$ can be computed directly by varying $L_{\text{Matter}}$ with respect to the shift $X$ and leaving $h$ (or equivalently $h^{-1}$) and $N$ fixed. Indeed, unwinding definitions we compute the fundamental identity

$$\frac{\partial L_{\text{Matter}}}{\partial X}[\delta X] = -2T(\nu, \delta X/N) dV_g$$

$$= -2T(\nu, \delta X) dt \wedge dV_h$$

$$= 2J(\delta X) dt \wedge dV_h\quad (2.4)$$

for any spatial vector field $\delta X$. A related computation shows

$$\frac{\partial L_{\text{Matter}}}{\partial N} = -2E dt \wedge dV_h.\quad (2.5)$$

### 3 Lagrangians depending on manifold-valued fields

In this section we prove Theorem 3.4, the concrete version of Meta-Theorem 1.1 in the case where matter sources take on values in a separate manifold $F$ unrelated to spacetime itself.

Loosely, a Lagrangian in this context consumes a Lorentzian metric $g$ together with a map $F$ from $M$ to $F$ and yields an $n + 1$ form $L(F, \partial F, g)$ that depends on the values and first derivatives of $F$. To formalize this construction, let $E$ be the bundle over $F$ consisting of $n + 1$ Whitney sums

$$E = TF \oplus \cdot \cdot \cdot \oplus TF \quad \text{for } n + 1 \text{ times.}$$
A minimally coupled Lagrangian with field values from $F$ is a bundle map

$$L : (E \times M) \times_M G(M) \to \Lambda^{n+1}(M)$$

where $E \times M$ is the trivial bundle over $M$ with fiber $E$, $G(M)$ is the bundle of Lorentzian metrics, $\Lambda^{n+1}(M)$ is the bundle of $n+1$-forms and $\times_M$ denotes the fiber product. In practice, we take the domain of $L$ to be the subbundle of $(F \times M) \times_M (E \times M) \times_M G(M)$ where the first factor records the common base point in $F$ of the tangent vectors from the second factor.

A section $\mathcal{F}$ of $F \times M$ (i.e., a map from $M$ to $F$) and a Lorentzian metric $g$ determine a field of $n+1$ forms

$$L(\mathcal{F}, \partial \mathcal{F}, g)$$

where $\partial \mathcal{F}$ is shorthand for $(\mathcal{F}, \partial_1, \ldots, \mathcal{F}, \partial_n, \mathcal{F}, \partial_t)$. The Lagrangian is diffeomorphism invariant if whenever $\Psi$ is an orientation preserving diffeomorphism from some neighborhood $U_1$ of $M$ to a neighborhood $U_2$ of $M$, then for all $\mathcal{F}$ and $g$ defined on $U_2$,

$$\Psi^* L(\mathcal{F}, \partial \mathcal{F}, g) = L(\Psi^* \mathcal{F}, \partial \Psi^* \mathcal{F}, \Psi^* g) \text{ on } U_1.$$

The spacetime Lagrangian reduces to a slice Lagrangian on slices $\Sigma_t$ as follows. Let $p \in M$, and let

1. $h$ be a spatial Riemannian metric at $p$,
2. $N > 0$,
3. $X$ be a spatial vector at $p$
4. $\mathcal{F} \in F$,
5. $\mathcal{F}(i) \in T_F F, 1 \leq i \leq n$,
6. $\dot{\mathcal{F}} \in T_F F$.

The associated value of the slice Lagrangian at $p$ is then

$$L_{\Sigma_t}(\mathcal{F}, (\mathcal{F}(1), \ldots, \mathcal{F}(n)), \dot{\mathcal{F}}, h, N, X) = \partial_t \cdot L(\mathcal{F}, (\mathcal{F}(1), \ldots, \mathcal{F}(n), \dot{\mathcal{F}}), g(h, N, X));$$

the primary distinction between the bundle maps $L$ and $L_{\Sigma_t}$ is that all tensors associated with $L_{\Sigma_t}$, including its value as an $n$-form, are spatial. Additionally, the arguments corresponding to spatial derivatives of $\mathcal{F}$ are separated from the time derivative, and we use the notation $\partial_\mathcal{F} = (\mathcal{F}, \partial_1, \ldots, \mathcal{F}, \partial_n)$ whenever $\mathcal{F} : M \to F$.

Proposition 3.3 below concerns the structure of a diffeomorphism-invariant Lagrangian and its proof employs diffeomorphisms that preserve $t$ (i.e., diffeomorphisms $\Psi$ for which $\Psi^* t = t$). Before proving it, we require the following two technical lemmas concerning such diffeomorphisms, the first of which describes their interaction with the projection $j_\mathcal{F}$. Note that here and elsewhere we allow $\Psi^*$ to act on vectors by $\Psi^* V = \Psi^{-1}_* V$, and by extension $\Psi^*$ is well defined on general tensors.

**Lemma 3.1.** Suppose $\Psi$ is a $t$-preserving diffeomorphism from a neighborhood $U$ of $M$ to a neighborhood $V$ of $M$

1. If $\omega$ is a covariant tensor, then

$$j_\mathcal{F} \Psi^* \omega = j_\mathcal{F} \Psi^* j_\mathcal{F} \omega.$$
2. If \( \omega \) is an \( n + 1 \) form and \( S \) is a spatial vector, then
\[
j_\ast \Psi^\ast (S \wedge \omega) = 0.
\]
3. If additionally \( \Psi \) is the identity on some slice \( \Sigma_{t_0} \cap U \), and if \( B \) is a spatial tensor on \( V \), then
\[
j_\ast \Psi^\ast B = B \quad \text{on } \Sigma_{t_0} \cap U.
\]

**Proof.** We prove part 1 for a 1-form \( \omega \); the general case follows since \( j_\ast \) and \( \Psi^\ast \) distribute over tensor products. If \( S \) is a spatial vector, so is \( \Psi_\ast S \) and
\[
(j_\ast \Psi^\ast \omega)(S) = \Psi^\ast \omega(S) = \omega(\Psi_\ast S) = (j_\ast \omega)(\Psi_\ast S) = \Psi^\ast (j_\ast \omega)(S) = (j_\ast \Psi^\ast j_\ast \omega)(S).
\]
We conclude that \( j_\ast \Psi^\ast \omega = j_\ast \Psi^\ast j_\ast \omega \) since both vanish on \( \partial_t \) as well.

To establish part 2, observe that if \( S \) is spatial and \( \omega \) is an \( n + 1 \) form, then \( S \wedge \omega \) vanishes when all of its arguments are spatial. So \( S \wedge \omega = dt \wedge \eta \) for some spatial \( n - 1 \) form \( \eta \) and from part 1 we have
\[
j_\ast \Psi^\ast (S \wedge \omega) = j_\ast \Psi^\ast (j_\ast dt \wedge \eta) = 0.
\]

Finally, we turn to part 3 and assume that \( \Psi \) is the identity on \( \Sigma_{t_0} \cap U \). If \( S \) is a spatial vector on \( \Sigma_{t_0} \) then \( \Psi_\ast S = S \). Hence for any 1-form \( \omega \),
\[
j_\ast \Psi^\ast \omega(S) = \Psi^\ast \omega(S) = \omega(\Psi_\ast S) = \omega(S).
\]
If in addition \( \omega \) is spatial, then \( \omega(\partial_t) = 0 \), as is \( j_\ast \Psi^\ast \omega(\partial_t) \), and we conclude that \( j_\ast \Psi^\ast \omega = \omega \). On the other hand, for a spatial vector \( V \) we have the obvious identity \( \Psi^\ast V = V \) on \( \Sigma_{t_0} \). Hence \( j_\ast \Psi^\ast V = j_\ast V = V \). This establishes equation \( (3.1) \) if \( B \) is a spatial 1-form or vector, and the general result follows from our earlier observation that \( j_\ast \Psi^\ast \) distributes over tensor products.

The following lemma concerns the \( n + 1 \) decomposition of the pullback of a metric by a \( t \)-preserving diffeomorphism, and we note the appearance of \( j_\ast \) in it because the pullback of a spatial 1-form by a \( t \)-preserving diffeomorphism need not be spatial.

**Lemma 3.2.** Suppose \( \Psi \) is a \( t \)-preserving diffeomorphism from a neighborhood \( U \) of \( M \) to a neighborhood \( V \) of \( M \). If \( g \) is a slice-compatible Lorentzian metric on \( V \) with \( (n + 1) \) decomposition \( (h, N, X) \) then \( \hat{g} = \Psi^\ast g \) is a slice-compatible Lorentzian metric on \( U \) with \( (n + 1) \) decomposition \( (\hat{h}, \hat{N}, \hat{X}) \) satisfying
\[
\hat{h} = j_\ast \Psi^\ast h \quad \text{(3.2)}
\]
\[
\hat{N} = \Psi^\ast N \quad \text{(3.3)}
\]
\[
\hat{X} = \Psi^\ast (X + S) \quad \text{(3.4)}
\]
where \( S \) is the spatial vector field satisfying \( \Psi_\ast \partial_t = \partial_t + S \).

We have the additional identities
\[
\Psi^\ast (\partial_t - X) = \partial_t - \hat{X}, \quad \text{(3.5)}
\]
and
\[
dV_{\hat{h}} = j_\ast \Psi^\ast dV_h. \quad \text{(3.6)}
\]
Proof. First observe that if \( W \) is a nonzero spatial vector on \( U \), then \( \Psi_*W \) is nonzero and spatial on \( V \) and hence \( \Psi^*g \) is a Lorentzian metric satisfying \( \Psi^*g(W,W) = g(\Psi_*W,\Psi_*W) > 0 \). So it is also slice compatible. Moreover, if \( W \) and \( Z \) are spatial vectors at some point we find

\[
\hat{h}(W,Z) = \hat{g}(W,Z) = g(\Psi_*W,\Psi_*Z) = h(\Psi_*W,\Psi_*Z) = (\Psi^*h)(W,Z).
\]

The equality \( \hat{h} = j_*\Psi^*h \) now follows since both tensors vanish when some argument is \( \partial_t \).

Since \( \Psi \) preserves \( t \), \( \Psi_*\partial_t = \partial_t + S \) for some spatial vector \( S \). Consequently \( \Psi^*\partial_t = \partial_t + \Psi^*S \). If \( \nu \) and \( \hat{\nu} \) are the unit normals associated with \( g \) and \( \hat{g} \) it is easy to see that \( \hat{\nu} = \Psi^*\nu \). Hence

\[
\hat{\nu} = \Psi^* \left( \frac{\partial_t - X}{N} \right) = \frac{\partial_t - \Psi^*(S) - \Psi^*(X)}{\Psi^*N}
\]

and we identify \( \hat{N} = \Psi^*N \) and \( \hat{X} = \Psi^*(X + S) \).

Equality (3.5) follows from equality (3.4) and the identity \( \Psi_*\partial_t = \partial_t + S \), so it only remains to establish equation (3.6). Recall that \( dV_h \) is defined by \( N dV_h = \partial_t \wedge dV_g \) and satisfies \( N dV_h = dV_g \) Hence

\[
\hat{N} dV_h = \partial_t \wedge dV_g = \Psi^*((\partial_t + S) \wedge dV_g)
\]

\[
= \Psi^*((\partial_t + S) \wedge (N dt \wedge dV_h))
\]

\[
= \Psi^*N \Psi^*(dV_h - dt \wedge (S \wedge dV_h))
\]

\[
= \Psi^*N \left[ \Psi^*(dV_h) - dt \wedge \Psi^*(S \wedge dV_h) \right].
\]

Applying \( j_* \) to both sides and using the fact that \( \Psi^*N = \hat{N} \) we find \( j_*dV_h = j_*\Psi^*dV_h \). But \( dV_h \) is spatial and equation (3.6) therefore follows.

The following key proposition describes the very limited way in which the slice Lagrangian of a diffeomorphism invariant Lagrangian can depend on the shift.

**Proposition 3.3.** Let \( F \) be a manifold and let \( \mathcal{L} \) be a minimally coupled diffeomorphism invariant Lagrangian with field values in \( F \). Let \( g \) be a slice compatible Lorentzian metric on \( M \) with \((n+1)\) decomposition \((h,N,X)\) and let \( \mathcal{F} : M \to F \). Then

\[
L_2(\mathcal{F}, \partial_t \mathcal{F}, \partial_t \mathcal{F}, h, N, X) = L_2(\mathcal{F}, \partial_t \mathcal{F}, \partial_t \mathcal{F}, h, N, X) = L_2(\mathcal{F}, \partial_t \mathcal{F}, \partial_t \mathcal{F}, h, N, X).
\]

**Proof.** Fix some \( p \in M \) and let \( \Psi \) be a diffeomorphism from a neighborhood \( U \) of \( p \) to a neighborhood \( V \) of \( p \) that fixes the slice \( \Sigma_0 \cap U \) containing \( p \) and such that \( \Psi_*\partial_t = \partial_t - X \). Such a diffeomorphism can be constructed using the integral curves of \( \partial_t - X \).

Let \( \hat{\mathcal{F}} = \mathcal{F} \circ \Psi, \hat{g} = \Psi^*g \), and let \((\hat{h}, \hat{N}, \hat{X})\) be the \((n+1)\) decomposition of \( \hat{g} \). We compute on \( \Sigma_0 \cap U \)

\[
L_2(\hat{\mathcal{F}}, \partial_t \hat{\mathcal{F}}, \partial_t \hat{\mathcal{F}}, \hat{h}, \hat{N}, \hat{X}) = j_*(\partial_t \wedge L(\Psi^*\hat{\mathcal{F}}, \partial \Psi^*\hat{\mathcal{F}}, \Psi^*g))
\]

\[
= j_*(\partial_t \wedge L(\mathcal{F}, \partial \mathcal{F}, \hat{g}))
\]

\[
= j_*\Psi^* [((\partial_t - X) \wedge L(\mathcal{F}, \partial \mathcal{F}, \hat{g}))]
\]

\[
= j_*\Psi^* [L_2(\mathcal{F}, \partial_t \mathcal{F}, \partial_t \mathcal{F}, h, N, X) - X \wedge L(\mathcal{F}, \partial \mathcal{F}, \hat{g})]
\]

\[
= L_2(\mathcal{F}, \partial_t \mathcal{F}, \partial_t \mathcal{F}, h, N, X)
\]
where Lemma 3.1 Parts 2 and 3 have been applied at the final step.

Since \( \Psi \) is the identity on \( \Sigma_{t_0} \cap U \), we have the following relations on \( \Sigma_{t_0} \cap U \):

- \( \hat{F} = F \) and \( \partial_x \hat{F} = \partial_x F \)
- \( \hat{h} = h \) and \( \hat{N} = N \) (Lemma 3.2 and Lemma 3.1 Part 3).

Moreover, since \( \Psi_e \partial_t = \partial_t - X \), Lemma 3.2 equation (3.4) implies \( \hat{X} = 0 \) on all of \( U \). Hence

\[
L_{\Sigma}(\mathcal{F}, \partial_x \mathcal{F}, \partial_t \mathcal{F}, h, N, 0) = L_{\Sigma}(\mathcal{F}, \partial_x \mathcal{F}, \partial_t \mathcal{F}, h, N, X)
\]
on \( \Sigma_{t_0} \cap U \). Noting that

\[
\partial_t \hat{F} = (\Psi^* F) \partial_t \mathcal{F} = \mathcal{F}(\partial_t - X) = \partial_t \mathcal{F} - \mathcal{F}_e X
\]
we obtain the desired equality at the arbitrary point \( p \) and therefore generally.

In order to connect the previous result to the momentum constraint we recall the classical mechanics notion of momentum conjugate to \( F \). Fix a slice \( \Sigma_{t_0} \) along with the following data on \( \Sigma_{t_0} \):

- a map \( F : \Sigma_{t_0} \to F \)
- a spatial Riemannian metric \( h \)
- a positive function \( N \)
- a spatial vector field \( X \).

At a point \( p \in \Sigma_{t_0} \) we then obtain a map \( T_{F(p)}(N) \to \Lambda^n_{\Sigma}(M) \) given by

\[
\mathcal{F} \mapsto L_{\Sigma}(\mathcal{F}, \partial_x \mathcal{F}, \mathcal{F}_e, g, N, X).
\]
The momentum conjugate to \( \mathcal{F} \) at \( p \) is the linearization

\[
\Pi_{\mathcal{F}} = \frac{\partial L_{\Sigma}}{\partial \mathcal{F}}
\]
which can be identified with a \( T^*F_{(p)} \)-valued spatial n-form. That is, fixing an arbitrary choice \( \omega \) of a non-vanishing n-form on \( \Sigma_{t_0} \) (e.g. \( \omega = dV_h \), though this is not necessary) we have

\[
\Pi_{\mathcal{F}} = \eta \otimes \omega
\]
for some \( \eta \in T^*_{\mathcal{F}(p)} F \). If \( Z \in T_{\mathcal{F}(p)} F \) we define

\[
\Pi_{\mathcal{F}}(Z) = \eta(Z) \omega,
\]
and the value is evidently independent of the choice of the pair \( (\eta, \omega) \) representing \( \Pi_{\mathcal{F}} \).

**Theorem 3.4.** Using the notation of the preceding discussion,

\[
\mathcal{J} dV_h = -\frac{1}{2} \Pi_{\mathcal{F}} \circ \mathcal{F}_e
\]

(3.7)
Proof. Let \( X_s \) be a one-parameter family of shifts at some point \( p \) with \( X_0 = X \) and \( \frac{d}{dt} \bigg|_{s=0} X_s = \delta X \). Although Proposition 3.3 is a statement about fields defined on \( M \), local extension arguments imply that at \( p \)

\[
L_\mathcal{F}(\mathcal{F}, \partial_t \mathcal{F}, \mathcal{F}, h, N, X_s) = L_\mathcal{F}(\mathcal{F}, \partial_t \mathcal{F}, \mathcal{F} - X_s, g, N, 0).
\]

Taking a derivative with respect to \( s \) at \( s = 0 \) and using equation (2.4) we find

\[
\mathcal{J}(\delta X)dV_h = -\frac{1}{2}\Pi g(\mathcal{F}, \delta X).
\]

\( \square \)

4 Lagrangians depending on tensor fields

The aim of this section is to prove a generalization of Theorem 3.4 to matter fields represented by sections of tensor bundles. The work is more involved because the fields transform nontrivially under diffeomorphism, and the first step is a careful decomposition into spatial tensors.

4.1 \((n + 1)\) decomposition of tensors

There is more than one way to decompose a tensor on \( M \) into spatial tensors representing its components. One could use, e.g., the coordinate representation of the tensor in terms of \( \partial t \) and \( \partial_\nu \). Alternatively, with a Lorentzian metric \( g \) in hand one could use the slice normal \( \nu \) in place of \( \partial_t \). The normal \( \nu \) depends on both the lapse \( N \) and shift \( X \), however, and it turns out to be convenient for our purposes to not involve the lapse in the tensor decomposition. Thus we describe below a decomposition based on \( \partial_t - X \) rather than the full normal \( \nu \).

Let \( P_X \) be the \( g \)-orthogonal projection of tangent vectors to vectors tangential to slices \( \Sigma_t \). If \( Z = a\partial_t + W \) where \( W \) is spatial, then

\[
Z = aN\nu + aX + W
\]

and \( P_X(Z) = aX + W \). The adjoint \( P_X^* \) acts on 1-forms \( \omega \) by declaring \( (P_X^*\omega)(X) = \omega(P_XZ) \) for arbitrary vectors \( Z \). Note that the notation \( P_X \) is justified since, in terms of the \( n + 1 \) decomposition \((h, N, X)\), the projection depends only on the shift.

Given an arbitrary tensor in \( T^k(M) \) of rank \( k = k_1 + k_2 \), and given a slice-compatible Lorentzian metric \( g \) with \( n + 1 \) decomposition \((h, N, X)\) we obtain \( 2^k \) spatial tensors of varying rank less than or equal to \( k \) by inserting one of \( \partial_t - X \) or \( j_\pm \) for each contravariant argument and one of \( dt \) or \( P_X^* \) for each covariant argument. For example, a 1-form \( \omega = bdt + \eta \) where \( \eta \) is spatial decomposes to

\[
(b - \eta(X), \eta)
\]

and a vector \( Z = a\partial_t + W \) where \( W \) is spatial decomposes to

\[
(a, W + aX).
\]
Fixing some arbitrary ordering for the resulting tensors (e.g., lexicographically based on the interior product/projection choice made for each argument) we write \( S(T;X) \) for the collection of \( 2^k \) spatial tensors obtained this way and call the result the \( n + 1 \) \textbf{decomposition} of \( T \) determined by the time function \( t \) and metric \( g \), noting that the metric is only involved via the shift \( X \).

A tuple of tensors \( B \) representing a possible value of \( S \) is a \textbf{decomposed tensor} (of unified type \( T^k_k \)), and the set of all decomposed tensors is a fiber bundle over \( M \) consisting of a Whitney sum of spatial subbundles of tensor bundles.

From the explicit representations above it is clear that \( S(\cdot;X) \) is invertible for 1-forms and vector fields, and a proof by induction shows that it is invertible for arbitrary tensor fields. Writing \( S^{-1}(\cdot;X) \) for the recomposition operation, \( S^{-1}(B;X) \) can be written in terms of linear combinations of tensors obtained from \( B \) and \( X \) using the following operations:

1. interior products with \( X \),
2. tensor products with \( X, dt \) and \( \partial_t \),
3. argument reordering.

In particular, \( S^{-1}(B;X) \) depends smoothly on \( B \).

The \((n+1)\) decomposition map \( S \) is natural under \( t \)-preserving diffeomorphisms in the following sense.

\textbf{Lemma 4.1.} Let \( \Psi \) be a \( t \)-preserving diffeomorphism from a neighborhood \( U \) to a neighborhood \( V \) of \( M \) and let \( g \) be a slice-compatible Lorentzian metric on \( V \). Given a tensor field \( T \) on \( V \) define

\[
B = S(T;X) \quad \hat{B} = S(\Psi^*T;\hat{X})
\]

where \( \hat{X} \) is the shift vector of the slice-compatible Lorentzian metric \( \Psi^*g \). Then

\[
\hat{B} = j_* \Psi^* B. \tag{4.1}
\]

If additionally \( \Psi|_{\Sigma_{t_0} \cap U} = \text{Id} \) for some \( t_0 \), then the following equations hold on \( \Sigma_{t_0} \cap U \):

\[
\hat{B} = B \quad \text{Lie}_S \hat{B} = \text{Lie}_S B \tag{4.2}
\]

for all spatial vector fields \( S \).

\textbf{Proof.} To establish equation (4.1) we demonstrate below that for 1-forms \( \omega \),

\[
(\partial_t - \hat{X}) \circ \Psi^* \omega = \Psi^*((\partial_t - X) \circ \omega) \tag{4.4}
\]

and that for vector fields \( Z \)

\[
dt \circ \Psi^* Z = \Psi^*(dt \circ Z) \tag{4.6}
\]

\[
(\Psi^* Z) \circ \mathcal{P}_X^\circ = \Psi^*(Z \circ \mathcal{P}_X^\circ). \tag{4.7}
\]
Together these formulas nearly yield $\hat{B} = \Psi^* B$ for 1-forms and vectors, but equation (4.5) leads to the weaker statement $\hat{B} = j_\ast \Psi^* B$. The same arguments used to establish equations (4.4)–(4.7) apply to each component of a tensor and lead to the general result.

Equation (4.4) follows from the identity $\partial_t - \hat{X} = \Psi^* (\partial_t - X)$ from Lemma 3.2, equation (4.5) is immediate from Lemma 3.1 part 1, and equation (4.6) is a consequence of the identity $\Psi^* dt = dt$. Turning to equation (4.7), consider a vector $Z$ and let $W = \mathcal{P}_X(Z)$. Then $Z = a(\partial_t - X) + W$ for some number $a$ and $\Psi^*(Z) = a\Psi^*(\partial_t - X) + \Psi^* W = a(\partial_t - \hat{X}) + \Psi^* W$. Since $\Psi$ preserves $t$, $\Psi^* W$ is spatial as well and consequently

$$\mathcal{P}_X^\ast \Psi^* Z = \Psi^* W = \Psi^* (\mathcal{P}_X Z).$$

Equation (4.7) then follows from the definition of the adjoint.

Finally suppose $\Psi = \text{Id}$ on $U \cap \Sigma_{t_0}$ for some $t_0$. Equation (4.2) on $U \cap \Sigma_{t_0}$ follows from equation (4.1) and Lemma 3.1 part 3. But then Lemma 2.1 part 3 implies equation (4.3) on $\Sigma_{t_0} \cap U$ as well.

### 4.2 Structure of the momentum density

Let $E$ be a tensor bundle $T^{k_1}_{k_2}(M)$. A **minimally coupled Lagrangian depending on a field with values $E$** is a smooth bundle map

$$\mathcal{L} = E \oplus (E \oplus \cdots \oplus E) \oplus G(M) \to \Lambda^{n+1}(M)$$

where $\oplus$ denotes the Whitney sum and $G(M)$ is the bundle of Lorentzian metrics. A section $T$ of $E$ and Lorentzian metric $g$ then determine the $n+1$ form

$$\mathcal{L}(T, \partial T, g)$$

where $\partial T$ is shorthand for the tuple of Lie derivatives $(\partial x^1 T, \ldots, \partial x^n T, \partial T)$.

We say that $\mathcal{L}$ is **diffeomorphism invariant** if whenever $\Psi$ is an orientation preserving diffeomorphism from a neighborhood $U_1$ of $M$ to a neighborhood $U_2$ of $M$,

$$\Psi^* \mathcal{L}(T, \partial T, g) = \mathcal{L}(\Psi^* T, \partial(\Psi^* T), \Psi^* g)$$

on $U_1$ whenever $T$ and $g$ are defined on $U_2$.

To describe the slice Lagrangian in this context, consider a point $p \in M$ and the following data at $p$:

- $B$, a decomposed tensor of unified type $T^{k_1}_{k_2}(M)$,
- $B_{(i)}$, $i = 1, \ldots, n$, decomposed tensors of the same unified type as $B$,
- $\dot{B}$, a decomposed tensor of the same unified type as $B$,
- $h$, a spatial Riemannian metric,
- $N > 0$,
- vectors $X, X_{(1)}, \ldots, X_{(n)}$ and $\dot{X}$. 

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Let $\hat{B}$ be a smooth extension of $B$ near $p$ with $\partial_i B = B(0)$ for each $i$ and $\partial_t B = B$ at $p$; an easy coordinate argument shows that such an extension exists. Let $\hat{X}$ be a similar extension of $X$. The associated value of the slice Lagrangian $L_{\Sigma}$ at $p$ is

$$L_{\Sigma}(B, (B(1), \ldots, B(m)), h, N, X, (X(1), \ldots, X(m)), \hat{X}) = \partial_i \circ L(S^{-1}(\hat{B}, \hat{X}), \partial S^{-1}(\hat{B}, \hat{X}), g(h, N, \hat{X})).$$

This map is well defined because the derivatives of $S^{-1}(\hat{B}, \hat{X})$ at $p$ are computable solely from the values and derivatives of $\hat{B}$ and $\hat{X}$ at $p$ and hence are independent of the choice of extension. Because the derivatives of $S^{-1}(\hat{B}, \hat{X})$ at $p$ depend smoothly on the derivatives of $\hat{B}$ and $\hat{X}$ at $p$, $L_{\Sigma}$ is a smooth bundle map taking on values in the subbundle of spatial $n$-forms. If $B$ and $X$ are defined in a neighborhood of $p$ then they serve as their own extensions and we find

$$L_{\Sigma}(B, \partial_i B, \partial_i h, N, X, \partial_i X, \partial_i X) = \partial_i \circ L(S^{-1}(B, X), \partial S^{-1}(B, X), g(h, N, X))$$

at $p$ and indeed on the whole neighborhood.

We have the following analog of Proposition 3.3.

**Proposition 4.2.** Let $E$ be a tensor bundle $T^k_{\Sigma}(M)$ and suppose $L$ is a minimally coupled diffeomorphism invariant Lagrangian with field values from $E$. Let $g$ be a slice-compatible metric on $M$ with $(n + 1)$ decomposition $(h, N, X)$, let $T$ be a section of $E$, and let $B = S(T; X)$. Then

$$L_{\Sigma}(B, \partial_i B, \partial_i h, N, X, \partial_i X, \partial_i X) = L_{\Sigma}(B, \partial_i B, \partial_i h - \text{Lie}_X B, h, N, 0, 0, 0).$$

**Proof.** Let $T$ and $g$ be given and consider some point $p$ in $M$. As in the proof of Proposition 3.3, let $\Psi$ be a diffeomorphism from a neighborhood $U$ of $p$ to a neighborhood $V$ of $p$ that fixes the slice $\Sigma_0 \cap U$ containing $p$ and such that $\Psi_* \partial_t = \partial_t - X$.

Let $\hat{g} = \Psi^* g$ with $(n + 1)$ decomposition $(\hat{h}, \hat{N}, \hat{X})$ and let $\hat{B} = S(\Psi^* T; \hat{X})$. Using diffeomorphism invariance we compute on $\Sigma_0 \cap U$

$$L_{\Sigma}(B, \partial_i \hat{B}, \partial_i \hat{h}, \hat{N}, \hat{X}, \partial_i \hat{X}, \partial_i \hat{X}) = j_* (\partial_i \circ L(\Psi^* T, \partial \Psi^* T, \hat{g}))$$

$$= j_* (\partial_i \circ L(\Psi^* T, \partial T, g))$$

$$= j_* \Psi^* \left[ (\partial_i - X) \circ L(T, \partial T, g) \right]$$

$$= j_* \Psi^* L_{\Sigma}(B, \partial_i B, \partial_i h, N, X, \partial_i X, \partial_i X) - j_* \Psi^* X \circ L(T, \partial T, g)$$

$$= L_{\Sigma}(B, \partial_i B, \partial_i h, N, X, \partial_i X, \partial_i X).$$

The first equality in the chain uses the fact that $L_{\Sigma}$ is spatial (and hence $j_* L_{\Sigma} = L_{\Sigma}$); the final equality follows from Lemma 3.1 Parts 2 and 3.

The same argument as in Proposition 3.3 shows $\hat{h} = h$ and $\hat{N} = N$ on $\Sigma_0 \cap U$ and that $\hat{X} = 0$ on all of $U$. From the final conclusions of Lemma 4.1 we find $\hat{B} = B$ and $\partial_i \hat{B} = \partial_i B$ on $\Sigma_0 \cap U$, and combining these observations we conclude that on $U$,

$$L_{\Sigma}(B, \partial_i B, \partial_i h, N, 0, 0, 0, \partial_t) = L_{\Sigma}(B, \partial_i B, \partial_i h, N, X, \partial_i X, \partial_t X).$$

We wish to rewrite the remaining hatted term, $\partial_i \hat{B}$, in terms of unhatted quantities. Lemma 4.1, Lemma 2.1 part 2 and Lemma 3.2 equation (3.5) along with the fact that $\hat{X} = 0$ imply

$$\partial_i \hat{B} = \partial_i j_* \Psi^* B = j_* \partial_i \Psi^* B = j_* \Psi^* ((\partial_t - \text{Lie}_X)B).$$

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Since $B$ is spatial, $\partial_t B$ and $\text{Lie}_X B$ are both spatial (Lemma 2.1 Part 1). Hence Lemma 3.1 part 3 implies

$$j_* \Psi^* ((\partial_t - \text{Lie}_X) B) = (\partial_t - \text{Lie}_X) B$$

on $\Sigma_{s_0} \cap U$. That is, $\partial_t \dot{B} = (\partial_t - \text{Lie}_X) B$ at $p$. \hfill $\square$

As a consequence of Proposition 4.2, we drop the dependence of $L_\Sigma$ on $\dot{X}$. We would like to generalize Theorem 3.4, and to do this we first recall the notion of momentum in this context. Fix $B$, $h$, $N$, and $X$, on a slice $\Sigma_{s_0}$. Since $X$ is spatial, the Lie derivatives $\partial_t X$ are well-defined on $\Sigma_{s_0}$ and we have at each $p$ on $\Sigma_{\text{int}}$ a smooth map

$$\dot{B} \mapsto L_\Sigma(B, \partial_t B, \dot{B}, B, h, N, X, \partial_s X) N dV$$

with linearization

$$\Pi_B := \frac{\partial L_\Sigma}{\partial B}.$$  

This linearization is, by definition, the momentum conjugate to $B$. Fix a nonvanishing spatial $n$-form $\omega$ on $\Sigma_{s_0}$. Because $B$ is a tuple $(B_1, \ldots, B_2)$ of tensor fields, $\Pi_B$ can be represented as a tuple $(\Pi^i B_1 \omega, \ldots, \Pi^i B_2 \omega)$ of tensor field valued $n$-forms, with the starred tensor fields dual to those of $B$. If $\delta B = (\delta B_1, \ldots, \delta B_2)$ is a decomposed tensor of the same unified type as $B$ we define

$$\Pi_B(\delta B) = \sum_{j=1}^2 B^*_j(\delta B_j) \omega$$

with $B^*_j(\delta B_j)$ denoting tensor contraction. From the close coupling of $\partial_t B$ and $\text{Lie}_X B$ found in Proposition 4.2 we have the following immediate conclusion.

**Corollary 4.3.** On $\Sigma_{s_0}$ let $B$ and $B$ be decomposed tensor fields and let $(h, N, X)$ be a decomposed metric. Suppose $X_s$ is a one-parameter family of shifts with $X_0 = X$ and define $\delta X = \frac{d}{ds} \big|_{s=0} X_s$. Then

$$\frac{d}{ds} \bigg|_{s=0} L_\Sigma(B, \partial_s B, \dot{B}, h, N, X_s, \partial_s X_s) = -\Pi_B(\text{Lie}_\delta X B).$$

Corollary 4.3 is unfortunately not an immediate statement about the momentum constraint. The momentum constraint arises from varying the spacetime Lagrangian with respect to the shift, leaving $T$, $h$ and $N$ fixed. But the variation appearing in Corollary 4.3 leaves $B$ fixed, and $T = S^{-1}(B; X)$ changes as $X$ changes. In order to apply Corollary 4.3 to the momentum constraint we restrict our attention to fields $T$ that are stationary for $L$. That is, we assume that if $T_s$ is a path of tensor fields with $T_0 = T$ and $\delta T := \frac{d}{ds} \big|_{s=0} T_s$ compactly supported,

$$\int_M \left. \frac{d}{ds} \right|_{s=0} L(T_s, \partial T_s, g) = 0.$$  

**Theorem 4.4.** Using the notation and hypotheses of Proposition 4.2, suppose that $L$ is stationary at a tensor field $T$. Then for any compactly supported spatial vector field $\delta X$ on a slice $\Sigma_t$

$$\int_{\Sigma_t} \mathcal{J}(\delta X) dV_h = -\frac{1}{2} \int_{\Sigma_t} \Pi_B(\text{Lie}_\delta X).$$  

(4.8)
Proof. Let $X_t$ be a path of spatial vector fields on all of $M$ with $\frac{d}{ds}\big|_{s=0} X_s = \delta X$ and let $B = S(T;X)$. To simplify notation we write

$$\frac{\partial L_S}{\partial X} \left[ \delta X \right] = \frac{d}{ds}igg|_{s=0} L_S(B, \partial_s B, \partial_s B, h, N, X_s, \partial_s X_s)$$

$$\frac{\partial L}{\partial X} \left[ \delta X \right] = \frac{d}{ds}igg|_{s=0} L(T, \partial_s T, g(h, N, X_s))$$

$$\frac{\partial S^{-1}(B;X)}{\partial X} \left[ \delta X \right] = \frac{d}{ds}igg|_{s=0} S^{-1}(B;X_s).$$

Note that $S(B;X_s)$ determines a path $T_s$ of tensor fields. Given an arbitrary such path we set $\delta T = \frac{d}{ds}\big|_{s=0} T_s$ and write

$$\frac{\partial L}{\partial T} \left[ \delta T \right] = \frac{d}{ds}igg|_{s=0} L(T_s, \partial T_s, g).$$

Equation (4.8) is established by computing $\int_M dt \wedge \frac{\partial L_{\text{sr}}}{\partial X}$ two different ways. First, pointwise

$$dt \wedge \frac{\partial L_{\text{sr}}}{\partial X} = \frac{d}{ds}igg|_{s=0} L(S^{-1}(B,X_s), \partial S^{-1}(B,X_s), g(h, N, X_s))$$

$$= \int_M \left[ \frac{\partial L}{\partial T} \left[ \delta S^{-1}(B;X_s) \wedge \frac{\partial L}{\partial X} \left[ \delta X \right] \right] \right] dt \wedge dV_h.$$ 

Integrating over $M$ and using equation (2.4) and the fact that $L$ is stationary at $T$ we conclude

$$\int_M dt \wedge \frac{\partial L_{\text{sr}}}{\partial X} = \int_M \left[ \frac{\partial L}{\partial T} \left[ \delta X \right] \right] dt \wedge dV_h = 2 \int_M \mathcal{J}(\delta X) dt \wedge dV_h.$$ 

On the other hand, from Corollary 4.3 we have

$$\int_M dt \wedge \frac{\partial L_S}{\partial X} \left[ \delta X \right] = -\int_M dt \wedge \Pi_B(\text{Lie}_{\delta X} B)$$

and hence

$$-2 \int_M \mathcal{J}(\delta X) dt \wedge dV_h = \int_M dt \wedge \Pi_B(\text{Lie}_{\delta X} B). \quad (4.9)$$

This integration is on all of $M$, whereas equation (4.8) involves integration on a single slice. Nevertheless, since $\delta X$ is spatial, for an arbitrary function $f(t)$, $\text{Lie}_{f(t)\delta X} B = f(t) \text{Lie}_{\delta X} B$, and equation (4.8) therefore follows from equation (4.9) and a concentration argument.

Except in rare circumstances, the integrand on the right-hand side of equation (4.8) is not a tensorial expression in $\delta X$ and we cannot claim $\mathcal{J}(\delta X) dV_h = -\frac{1}{2} \Pi_B(\text{Lie}_{\delta X} B)$ pointwise. Nevertheless, $\mathcal{J} dV_h$ is completely determined by $\Pi_B$ and $B$. Indeed, in local coordinates one can write an expression for $\mathcal{J} dV_h$ in terms of $\Pi_B$ and $B$ using integration by parts to remove derivatives from $\delta X$ that appear in $\text{Lie}_{\delta X} B$. Although this
operation is done in coordinates, the result is a manifestly natural operation, though it is not one that is fa-
miliar to us in its full generality. Moreover, the exact expression appearing in the momentum constraint can
involve rewriting terms appearing after integration by par-
ts using constraints satisfied by the matter fields.
See Sections 7.2 and 7.3 where we carry out this process explicitly for concrete examples.

5 Electromagnetism-Charged-Scalar Field (EMCSF)

There are subtleties in the results of the previous section, and it is helpful to illustrate them in the context of
a concrete matter model: electromagnetism coupled to a charged scalar field (EMCSF).

Consider a trivial $\mathbb{C}$ bundle over $M$ with structure group $U(1)$ and let $D$ be a connection on the bundle. By
parallel transport we can choose a common length scale on the fibers of the bundle and subsequently select
a global section $f$ over $M$ with $|f| = 1$ on each fiber. The connection $D$ can be represented by means of a
1-form $A$ via

$$Df = i\varepsilon Af,$$

where the constant $\varepsilon$ is the charge of the scalar field.

Once the choice of frame $f$ is made an arbitrary section $S$ of the bundle can be represented by means of a
section $z$ of the trivial bundle $\mathbb{C} \times M$:

$$S = zf.$$

Under a global change of frame $\tilde{f} = e^{i\Xi f}$ we have the transformations

$$\tilde{z} = e^{-i\overline{\Xi} z},$$
$$\tilde{A} = A + d\Xi.$$  \hfill (5.1)

Let $g$ be a fixed (slice compatible) background Lorentzian metric on $M$ with $(n + 1)$ decomposition $(h, N, X)$. Leav-
ing the background frame $f$ for $E$ implicit, the matter fields for EMCSF consist of a 1-form $A$ and a
complex field $z$, and the EMCSF Lagrangian is

$$\mathcal{L}_{\text{EMCSF}} = -\frac{1}{8\pi} |dA|^2_g + |dz + i\varepsilon zA^2|_{g, g}^2 dV_g.$$  \hfill (5.2)

The Lagrangian is easily seen to be diffeomorphism invariant, and is also invariant under gauge transformations of the form (5.1). It depends on tensor-valued fields ($A$ and $f$) and hence the EMCSF model falls in
the category of matter fields, treated in Section 4.

5.1 Illustration of Proposition 4.2

Proposition 4.2 concerns the structure of the Lagrangian after the matter fields have carefully split into spatial
components. Following the procedure of Section 4.1, the scalar field $z$ is left untouched but we decompose
$A$ as follows:

$$A = j_+ A$$
$$A_+ = (\partial_t - X) \cup A.$$  \hfill (5.3)

Note that \((\partial_t - X) = N\nu\) where \(\nu\) is the slice unit normal, but the use of \(A_\perp\) rather than, e.g., \(A_\perp := \nu \perp A\) or \(A_0 := \partial_t \perp A\), was an essential tool in the proof of Proposition 4.2.

Proposition 4.2 asserts that when the slice Lagrangian \(L_\Sigma = \partial_t \perp L_{\text{EMCSF}}\) is written in terms of the spatial metric \(h\), the lapse \(X\), the shift \(N\), and the spatial variables \(z, A_\perp\), and \(A\), then the shift \(X\) and the Lie time derivatives \(\dot{A} = \partial_t A\), \(\dot{A}_\perp = \partial_t A_\perp\) and \(\dot{z} = \partial_t z\) appear in the slice Lagrangian only in the following combinations:

- \(\dot{A} - \text{Lie}_X A\),
- \(\dot{A}_\perp - \text{Lie}_X A_\perp\),
- \(\dot{z} - \text{Lie}_X z = \dot{z} - X(z)\).

Additionally, the lapse appears only algebraically in the slice Lagrangian; its spatial and time derivatives are absent. We now show by direct computation that these assertions indeed hold for the EMCSF model.

For convenience define the electric covector field \(E\) and its analog \(P\) for the scalar field via

\[
E = -j_*(\nu \perp dA) = \nu \perp (dz + i\epsilon zA)
\]

Using the notation \(d = j_*d\) and the identities \(\partial_t = N\nu + X\) and \(\text{Lie}_X A = d(X \perp A) + X \perp dA\), we find

\[
E = -\frac{1}{N} \left[ \dot{A} - X \perp dA - d(X \perp A) - dA_\perp \right] = -\frac{1}{N} \left[ \dot{A} - \text{Lie}_X A - dA_\perp \right]
\]

and

\[
P = \frac{1}{N} \left[ \dot{z} - X(z) + i\epsilon zA_\perp \right].
\]

The slice Lagrangian then becomes

\[
L_\Sigma = \frac{1}{4} \left[ 2|E|^2_h - |dA|^2_h + 2|P|^2_{C,h} - 2|dz + i\epsilon zA|^2_{C,h} \right] NdV_h.
\]

Noting that \(E\) and \(P\) in equation (5.7) are merely shorthand for the full expressions (5.5) and (5.6) we observe that the shift \(X\) and the time derivatives \(\dot{A}, \dot{A}_\perp\) and \(\dot{z}\) indeed appear in the Lagrangian only in the combinations \(\dot{A} - \text{Lie}_X A\) and \(\dot{z} - X(z) = \dot{z} - \text{Lie}_X z\). The combination \(\dot{A}_\perp - \text{Lie}_X A_\perp\) would also have been allowed by Proposition 4.2, but the Lagrangian does not depend on \(A_\perp\), a reflection of the previously mentioned gauge freedom. Finally we observe that although the slice Lagrangian depends on derivatives of \(X\) via the Lie derivatives, it depends only algebraically on \(N\). Thus we have illustrated Proposition 4.2 in this special case.

### 5.2 Illustration of Theorem 4.4

Conjugate momenta arise from varying the slice Lagrangian \(L_\Sigma\) with respect to the time derivatives of the field variables, whereas the momentum density arises from a variation of the slice Lagrangian with respect to
the shift via equation (2.4). The structural form of the slice Lagrangian implied by Proposition 4.2, wherein time derivatives of field variables appear only in certain combinations with the shift, is the key tool needed to relate conjugate momenta to the momentum density. This specific relationship is the content of Theorem 4.4, which asserts for the EMCSF Lagrangian that the integral relationship
\[ -2 \int_{\Sigma_t} \mathcal{J}(\delta X) dV_h = \int_{\Sigma_t} \left[ \Pi_A (\text{Lie}_{\delta X} A) + \Pi_{A_\perp} (\text{Lie}_{\delta X} A_{\perp}) + \Pi_z (\text{Lie}_{\delta X} z) \right] \]  
holds for any compactly supported vector field \( \delta X \) on a slice \( \Sigma_t \). In particular, if the conjugate momenta \( \Pi_A, \Pi_{A_\perp} \) and \( \Pi_z \) are known, then so is \( J dV_h \), a manifestation of Meta-Theorem 1.1.

There is an important caveat, however. Theorem 4.4 assumes that we are working with fields for which the Lagrangian is stationary. In particular, any constraints on the spatial variables implied by the Euler-Lagrange equations must hold. For the EMCSF fields, gauge freedom leads to the constraint
\[ \int_{\Sigma_t} \left[ -\Pi_A (d\theta) + \Pi_z (i\varepsilon z \theta) \right] = 0 \]  
which is essentially Gauss' Law and, as seen below, is necessary to establish equation (5.8).

We now illustrate Theorem 4.4 by deriving equation (5.8) directly. Recall that the momenta \( \Pi_A, \Pi_{A_\perp} \) and \( \Pi_z \) are obtained by varying the slice Lagrangian \( L_{\Sigma_t} \) with respect to \( \dot{A}, \dot{A_\perp} \) and \( \dot{z} \) respectively. We find
\[ \Pi_A = -\frac{1}{2\pi} \langle E, d\cdot \rangle dV_h \]  
\[ \Pi_{A_\perp} = 0 \]  
\[ \Pi_z = 2 \text{Re}(P \cdot) dV_h. \]  
Equation (2.4) implies that \( \mathcal{J} dV_h \) can be computed by varying \( L_{\text{EMCSF}} \) with respect to the shift, and from it we find
\[ \frac{\partial L_{\Sigma_t}}{\partial X} [\delta X] = 2\mathcal{J}(\delta X) dV_h. \]

Naively, one would like to simply substitute expressions (5.5) and (5.6) for \( E \) and \( P \) into equation (5.7) and take a derivative with respect to \( X \). Doing so we would find that for any compactly supported spatial vector field \( \delta X \)
\[ -2\mathcal{J}(\delta X) dV_h = \Pi_A (\text{Lie}_{\delta X} A) + \Pi_z (\text{Lie}_{\delta X} z) + \Pi_{A_\perp} (\text{Lie}_{\delta X} A_{\perp}), \]  
which is almost equation (5.8). This cannot be correct, however, because \( \text{Lie}_{\delta X} A \) is not tensorial in \( \delta X \).

To correct this error, and see why (5.8) only holds under an integral sign, and only under the additional constraint (5.9), recall that the variation in equation (2.4) leaves the matter fields \( A \) and \( z \) fixed. But we have written \( E \) and \( P \) in terms of the variable \( A_{\perp} = (\partial_t - X) \lhd A \). Although this choice is important for obtaining Proposition 4.2, and for deriving equation (5.7), it is not useful to continue using it here because fixing \( A_{\perp} \) while varying \( X \) implies that \( A \) changes.

For the purpose of computing the variation with respect to \( X \), we introduce \( A_0 = \partial_t \lhd A \), a variable independent of \( X \). The field \( A \) is determined by \( A, A_0 \) and \( \partial_t \) alone. Since
\[ A_{\perp} = A_0 - X \lhd A = A_0 - X \lhd A \]
and we can rewrite
\[
E = -\frac{1}{N} \left[ A - X \, dA - dA_0 \right]
\]
\[
P = \frac{1}{N} \left[ \frac{z - X(z)}{\epsilon} - i\epsilon z(X \, dA) + i\epsilon zA_0 \right].
\]
The essential point is that this form of \(E\) and \(P\) is written so that even if \(X\) is changed, the field \(A\) does not.

We can now vary the Lagrangian (5.7) with respect to \(X\) while holding \(A\) and \(A_0\) (and therefore also \(A\)) fixed. A computation using equation (2.4) and equations (5.10)–(5.12) then implies
\[
-2 \mathcal{J}(\delta X)dV_h = \Pi_A(\delta X \, dA) + \Pi_z(\delta X(z) + i\epsilon z(\delta X \, dA)),
\]
which is the true form of the momentum density. It is indeed tensorial in \(\delta X\), but the clean structure of equation (5.13) has been lost. To recover it in the integral sense we need to take into account a constraint arising from equation (5.11).

Because the momentum \(\Pi_{A_\cdot}\) vanishes identically, a solution of the EMCSF Euler-Lagrange equations necessarily satisfies on each slice
\[
\int_{\Sigma_t} \frac{\partial L_{\Sigma}}{\partial A_{t\mu}} \delta A_{t\mu} = 0
\]
for compactly supported functions \(\delta A_{t\mu}\). As a consequence, a computation leads to the constraint (5.9) for any compactly supported function \(\theta\) on a slice \(\Sigma_t\). Integrating equation (5.14) and applying the constraint (5.9) with \(\theta = \delta X \, dA\), and also using equation (5.11), we find
\[
-2 \int_{\Sigma_t} \mathcal{J}(\delta X)dV_h = \int_{\Sigma_t} \Pi_A(\delta X \, dA) + \Pi_z(\delta X(z) + i\epsilon z(\delta X \, dA))
\]
\[
= \int_{\Sigma_t} \Pi_A(\delta X \, dA) + \Pi_z(\delta X(z)) + \Pi_A(\delta X \, dA)
\]
\[
= \int_{\Sigma_t} \Pi_A(\text{Lie}_{dX} A) + \Pi_z(\text{Lie}_{dX} z)
\]
\[
= \int_{\Sigma_t} \Pi_A(\text{Lie}_{dX} A) + \Pi_z(\text{Lie}_{dX} z) + \Pi_{A_\cdot}(\text{Lie}_{dX} A_{t\cdot}).
\]
Thus we have derived equation (5.8). The important point is that equation (5.13) only holds under an integral sign, and only if the matter fields satisfy any constraints imposed by the matter Euler-Lagrange equations.

Concerning the constraint (5.9), Section 6.1 below outlines a systematic approach to constructing its solutions. We show additionally in Section 6.2 how core features of the conformal method of solving the gravitational field constraints are direct analogs of the EMCSF constraint solving procedure.

### 5.3 Field Values and Conformal Changes

As mentioned at the end of the introduction, in applications it may be more convenient to describe matter in terms of variables different from the ones used thus far in this section. For example,
\[
\Pi_A = \frac{1}{2\pi} \langle E, \cdot \rangle_h dV_h = \frac{1}{2\pi} E^4 \otimes dV_h
\]
and hence the electric field expressed either as a covector \((E)\) or a vector \((E^\sharp)\) encodes \(\Pi_A\), so long as the metric is also understood. If \(\Pi_A\) remains fixed but \(h \mapsto h^\ast = \phi^{q-2} h\), we obtain the conformal transformation laws

\[
E^\ast = \phi^{-2} E \\
(E^\ast)^\sharp = \phi^{-q} E^\sharp.
\]

Similarly, \(P\) encodes \(\Pi_z\) with the conformal transformation rule \(P^\ast = \phi^{-q} P\). On a slice \(\Sigma_t\), the constraint (5.9) can be expressed in metric terms as

\[
\text{div}_h E = -4\pi \varepsilon \text{Im}(\mathcal{F} z), \tag{5.16}
\]

where we are using the same notation for spatial fields on \(M\) and their their restrictions to \(\Sigma_t\). The fact that the constraint (5.9) does not actually involve the metric implies that this equation must be conformally invariant. Indeed, the invariance of equation (5.16) follows from the conformal transformation laws for \(E\) and \(P\) along with the rule \(\text{div}_h \phi^{-2} = \phi^{-q} \text{div}_h\) when acting on covector fields.

\section{The Conformal Method and Phase Space}

Theorems 3.4 and 4.4 concern the Einstein equations generally and are not statements about the conformal method. As far as the conformal method is concerned, we use the following consequence of these theorems: when momentum density is written as a function of the slice metric \(h\) along with certain suitably decomposed matter fields \(B\) and conjugate momenta \(\Pi_B\), then

\[
\mathcal{J}(\phi^{q-2} h, B, \Pi_B) = \phi^{-q} \mathcal{J}(h, B, \Pi_B). \tag{6.1}
\]

At this point, the hurried reader could continue to the applications of Section 7 after scanning the equations of the conformal method in its Lagrangian and Hamiltonian forms (equations (6.9)–(6.10) and (6.11)–(6.12) below) to verify that, as outlined in the introduction, decoupling of the momentum constraint occurs under the the transformation rule (6.1).

The larger picture of our work, however, concerns the very close association of the conformal method with the Hamiltonian formulation of the evolution problem: initial data for the matter variables are profitably specified by fields and their conjugate momenta, and (at least for CMC data) an analogous statement is true for the gravitational variables. Moreover, the conformal method itself is a rather direct analog of a natural technique for parameterizing solutions of the EMCSF constraint (5.9) from Section 5. We therefore outline the EMCSF construction in Section 6.1 as a means of motivating the conformal method, which we summarize in Section 6.2. Section 6.3 discusses the CMC specialization of the conformal method, and flags some important differences between the vacuum and non-vacuum settings.

\subsection{Solution of the EMCSF constraint}

Recall that the EMCSF field variables are a 1-form \(\mathcal{A}\) and a complex function \(z\). If \((\mathcal{A}, z)\) is a solution of the Euler-Lagrange equations for the Lagrangian (5.2), then so is \((\mathcal{A} + d\Xi, e^{-i\Xi} z)\) for any function \(\Xi\); it represents the same physical solution but with respect to a different frame. If we consider a gauge transformation
with $\Xi \equiv 0$ at some time $t_0$ then we find that the spatial variables $(A, A_\nu, z)$ corresponding to the spacetime fields $(\mathcal{A}, z)$ at $t = t_0$ are unaffected but their velocities transform as

$$
\begin{align*}
A &\to A + d\Xi \\
A_\nu &\to A_\nu + \Xi \\
\dot{z} &\to \dot{z} - i\epsilon\Xi z.
\end{align*}
$$

Thus, because of gauge freedom, the state of the physical system is invariant under velocity transformations

$$
(\dot{A}, \dot{A}_\nu, \dot{z}) \to (\dot{A}, \dot{A}_\nu, \dot{z}) + (d\theta, \theta, -i\epsilon\Xi \theta)
$$

with the arbitrary function $\theta$ playing the role of $\Xi$. Field velocities are meaningful only modulo such transformations. Because $\Pi_{A_\nu}$ always vanishes for this system, the constraint (5.9) at some $t$ can be written

$$
\int_{\Sigma_t} \Pi_A (d\theta) + \Pi_{A_\nu} (\theta) + \Pi_z (-i\epsilon\Xi \theta) = 0
$$

and consequently the total system momentum is insensitive to velocity perturbations of the form (6.2).

One approach to building solutions of the constraint is to specify field values $(A, A_\nu, z)$ along with the system velocity. Specifically, we provide $(\dot{A}, \dot{A}_\nu, \dot{z})$ and the system velocity is the equivalence class

$$(\dot{A}, \dot{A}_\nu, \dot{z}) + (d\theta, \theta, -i\epsilon\Xi \theta)$$

where $\theta$ is an arbitrary function. Recall that velocity and momentum are related to each other in classical mechanics by the Legendre transformation of the system [GPS02], and the constraint (6.3) is really a constraint on the momentum. Thus the strategy is to write down the momentum in terms of a velocity with the arbitrary function $\theta$ built into the expression, and then use the constraint to determine the choice of $\theta$. The end result is a system momentum that satisfies the constraint and that is related, via the Legendre transformation, to the specified velocity.

The Legendre transformation of the EMCSF system is effectively given by equations (5.5) and (5.6). Indeed, assuming the metric data $(h, N, X)$ is known, then $\dot{A}$ and $\dot{A}_\nu$ determine $E$ and $P$ according to these equations, and $E$ and $P$ determine $\Pi_A$, $\Pi_z$ via equations (5.10) and (5.12). Gauge freedom manifests itself in a degeneracy in the Legendre transformation, and $\Pi_{A_\nu}$ is always 0 and is unaffected by changes in $\dot{A}_\nu$.

Rewriting equations (5.5) and (5.6) and inserting the arbitrary function $\theta$ we find

$$
\dot{A} + d\theta = -NE + \text{Lie}_X A + dA_\nu \\
\dot{z} - i\epsilon\Xi \theta = NP + \text{Lie}_X z - i\epsilon\Xi A_\nu.
$$

Then, introducing $\Theta = A_\nu - \theta$, constraint (5.9) in the form of (5.16) can be written on $\Sigma_t$ as

$$
- \text{div}_h \left( \frac{1}{N} d\theta \right) + 4\pi \epsilon \frac{|z|^2}{N} \Theta = - \text{div}_h \left( \frac{1}{N} (A - \text{Lie}_X A) \right) + \frac{4\pi \epsilon}{N} \text{Im} \left( \overline{(\dot{z} - \text{Lie}_X z)} z \right).
$$

As we have done in arriving at (5.16), we are using the same notation for spatial tensor fields on $M$ and for their restrictions to $\Sigma_t$, and we note that the ‘spatial’ exterior derivative $d$ on $M$ has been replaced with the exterior derivative $d$ intrinsic to $\Sigma_t$. Equation (6.4) is an elliptic PDE for $\Theta$ with a favorable sign on the
potential term \(4\pi \varepsilon^2 |z|^2 / N\). When supplemented with suitable boundary conditions, one can find a unique solution \(\Theta\) of equation (6.4). The remaining momentum \(\Pi_A\) always vanishes, and we are therefore able to construct a full set of momenta from field values \((A, A_t, z)\) and velocities \((A, A_t, \dot{z})\) along with the metric data \((h, N, X)\). This strategy for solving the EMCSF constraint is analogous to the Lagrangian formulation of the conformal method found in the following section.

An alternative, closely related, parameterization involves starting with momenta \((\tilde{\Pi}_A, \tilde{\Pi}_z)\) that do not satisfy the constraint, but such that \(\tilde{\Pi}_A\) satisfies the vacuum constraint

\[
\int_{\Sigma} \tilde{\Pi}_A (d\theta) = 0
\]

for all functions \(\theta\) on \(\Sigma\). We then seek momenta \((\Pi_A, \Pi_z)\) that solve the constraint and which have a corresponding system velocity (up to gauge) that is the same as what would be determined by \((\tilde{\Pi}_A, \tilde{\Pi}_z)\) in the vacuum case via the inverse Legendre transformation. An argument similar to the one of the previous paragraph shows that this reduces to seeking \((E, P)\) of the form

\[
E = \tilde{E} - \frac{1}{N} d\theta \\
P = \tilde{P} - \frac{1}{N} i\varepsilon \theta
\]

where \((\tilde{E}, \tilde{P})\) are the metric representations of \((\tilde{\Pi}_A, \tilde{\Pi}_z)\), and so \(\text{div}_h \tilde{E} = 0\). The constraint (5.9) is satisfied when \(\theta\) solves the PDE

\[
- \text{div}_h \left( \frac{1}{N} d\theta \right) + 4\pi \varepsilon^2 |z|^2 \frac{1}{N} \theta = -4\pi \varepsilon \text{Im} \left( \tilde{P}z \right).
\]  

This variation of parameterizing solutions of the EMCSF constraint (5.9) corresponds to the Hamiltonian formulation of the conformal method.

### 6.2 Conformal Method

Although we have until this point decomposed the spacetime metric \(g\) into spatial variables \((h, N, X)\), the conformal method is based on an alternative, closely related, decomposition \((h, \alpha, X)\) where the spatial metric \(h\) and shift \(X\) have the same meaning as before but where \(\alpha\) and the lapse \(N\) are related by

\[
\alpha = dV_h / N.
\]

We call \(\alpha\) the **slice energy density** because a particle with unit mass and with velocity equal to the unit surface normal \(\nu = (\partial_t - X)/N\) has energy \(1/N\) in the background coordinate system \((x', t)\). If the metric \(h\) conformally transforms to \(\phi^{\ell - 2}h\), and if \(\alpha\) remains fixed, then the conformal transformation law for volume \(dV_h \to \phi^\ell dV_h\) implies the lapse transforms via \(N \to \phi^\ell N\); this is called a **densitized lapse**. We find that there are conceptual advantages to working with the conformally invariant slice energy density \(\alpha\) rather than the more commonly used densitized lapse, but these objects are equivalent to each other.

Although the reasons for the importance of slice energy density remain unclear, a hint can be found in [Ma21] where it is shown that the use of slice energy density as a variable instead of the usual lapse allows for a clean separation between conformal and volume components of the kinetic energy part of the
ADM gravitational Lagrangian. This separation affects the associated Legendre transformation in a way that facilitates specifying conformal class data (values and momenta) independently from volume information. In the conformal method, the metric $h$ is then decomposed into two variables, the conformal class $h = [h]$ and the volume form $dV_h$, with the conformal class becoming the primary gravitational variable, and the volume form determined by the constraint equations.

In the Lagrangian formulation of the conformal method, analogous to the EMCSF construction (6.4), we specify the following data:

- A conformal class $h$ and a conformal class velocity $\dot{h}$ analogous to EMCSF data $A$ and $\dot{A}$,
- gauge data $\alpha$ and $X$ analogous to $A_v$,
- a mean curvature function $\tau$ with no direct EMCSF analog,
- matter field information comparable to $z$ and $\Pi$, in the EMCSF construction.

The anomalous role of the mean curvature in this collection stands out, and the absence of a classical mechanical interpretation is perhaps related to the deficiencies of the conformal method in the non-CMC setting. Notably, however, when $\tau$ is constant it encodes a momentum conjugate to volume $\alpha dV_h$.

The gauge group for the Einstein gravitational field equations consists of spatial diffeomorphisms rather than EMCSF frame transformations, but the strategy remains the same. We specify the conformal class velocity $\dot{h}$ only up to gauge. The constraint equations impose a condition on the conjugate momentum of the conformal class, and the Legendre transformation of the system connects the conformal class velocity to the conformal class momentum. The gauge freedom in the conformal class velocity is then used to find a conformal class momentum that is related, via the Legendre transformation, to the conformal class velocity and that satisfies the constraint.

To make this procedure concrete, and to derive a PDE system that one can actually analyze, in practice one specifies the following seed data:

- A metric $h$ determining the conformal class $h = [h]$.
- A trace-free $(0, 2)$ tensor $U$ that, together with $h$, encodes the conformal class velocity $\dot{h}$ as explained below.
- Gauge data $\alpha$ and $X$, from which we define $N = dV_h/\alpha$.
- A mean curvature function $\tau$ and matter field information as specified for various example cases below.

Following [Ma14], the conformal class velocity is an equivalence class of pairs $(\phi^\theta h, \phi^\theta U)$ where $\phi$ is an arbitrary conformal factor; the trace-free condition on $U$ arises because a path of metrics with constant volume form has a trace-free derivative, and the conformal scaling on $U$ arises naturally as a consequence of scaling the metric. One can think of $(h, U, X, N, \tau)$ as the gravitational seed data, which specify the same conformally invariant information as $(\phi^\theta h, \phi^\theta U, \phi^\theta N, X, \tau)$, namely

$$(h = [h], \dot{h} = [(h, U)], \alpha = (1/N)dV_h, X, \tau).$$
Diffeomorphism gauge freedom manifests itself in that rather than specifying a conformal class velocity $[(h, U)]$, we specify $[(h, U + L_h W)]$ where $W$ is an arbitrary vector field and where $L_h W$ is the trace-free part of Lie$_W h$, namely

$$(L_h W)_{ab} = \nabla_a^b W_b + \nabla_b^a W_a - \frac{2}{N} (\text{div}_h W) h_{ab}. \quad (6.6)$$

We seek a solution $(h^*, K^*)$ of the constraint equations where $[h^*] = [h]$ and where the conformal class velocity of the initial data equals $[(h, U + L_h W)]$ for some vector field $W$ to be determined. Recall $K^*$ encodes the normal derivative of any extension of $h^*$ off of the initial slice via

$$2K^* = \frac{1}{N^*} (\partial_t h^* - \text{Lie}_X h^*) \quad (6.7)$$

where $N^* = dV_{h^*} / \alpha$ is the lapse determined by $h^*$ and $\alpha$. This is essentially the Legendre transformation at the level of the metric, and its trace-free part encodes the Legendre transformation at the level of conformal classes. Let $U^*$ be the trace-free part of $\partial_t h^*$ and let $\xi^*$ be the trace-free part of $K^*$. The conformal class velocity of the initial data is $[(h^*, U^*)]$ and equation (6.7) implies

$$2\xi^* = \frac{1}{N^*} (U^* - L_{\partial_t h^*} W).$$

By ansatz, $[h^*] = [h]$ and $[(h^*, U^*)] = [(h, U + L_h W)]$ and therefore $h^* = \phi^{q-2} h$ and $U^* = \phi^{q-2} (U + L_h W)$ for some conformal factor $\phi$. Using the relations $N^* = \phi^q N$ and $L_{\partial_t h^*} = \phi^{-2} L_h$ we find

$$\xi^* = \frac{1}{2N^*} (U^* - L_{\partial_t h^*} X) = \phi^{-2} \frac{1}{2N} (U - L_h X + L_h W). \quad (6.8)$$

Inserting $h^* = \phi^{q-2} h$ and $K^* = \xi^* + (\tau/n) h^*$ into the constraint equations (1.1)–(1.2), and using equation (6.8) to rewrite all quantities in terms of $(h, U, N, X, \tau)$ along with the unknowns $\phi$ and $W$, yields

$$-2\kappa q \Delta h + R_{h\phi} - \frac{1}{4N^2} |U + L_h Z|_h^2 \phi^{-q-1} + \kappa^2 \phi^{-q-1} = 16\pi \phi^q \xi^* \quad \text{(6.9)}$$

$$\text{div}_h \left( \frac{1}{2N} L_h Z \right) = -\text{div}_h \left( \frac{1}{2N} U \right) + \phi^q k d\tau - 8\pi \phi^q J^*, \quad \text{(6.10)}$$

where we have introduced $Z = W - X$ as a new unknown, and where the matter terms $\xi^*$ and $J^*$ are functions of the matter seed variables in addition to, conceivably, $(h, N, X, \phi, U, Z, \tau)$ and their derivatives. Equation (6.10) corresponds to equation (6.4) with $Z$ in equation (6.10) to $\Theta$ in equation (6.4). This presentation of the conformal method is often referred to as the conformal thin-sandwich method [Yo99], but it is identical, in a way made precise in [Ma14], to the historical conformal method developed in the 1970s.

As an alternative to specifying a conformal class velocity $[(h, U)]$, one can provide a vacuum conformal class momentum (modulo spatial diffeomorphisms) and request that the solution of the constraints has the same conformal class velocity as would be determined by the vacuum momentum. This is the Hamiltonian formulation of the conformal method from [PY03] and is an analog of the technique that leads to equation (6.5) for the EMCSF system. Vacuum conformal class momentum can be expressed, with respect to the conformal class representative $h$, by a symmetric, trace-free, divergence-free (0,2) tensor $\sigma^*$; see, e.g., [Ma14]. The variable $\sigma^*$ is analogous to $\bar{E}$ in the technique leading to equation (6.5) and we seek a second-fundamental
form

$$K^* = \phi^{-2} \sigma + \frac{1}{2N^*} L_{h^*} W + \frac{\tau h^*}{n}$$

$$= \phi^{-2}(\sigma + \frac{1}{2N} L_h W) + \phi^{q-2} \frac{\tau}{n} h$$

where the vector field $W$ is to be determined. The LCBY equations become

$$-2\kappa q \Delta_h \phi + R_{h \phi} - \left| \sigma + \frac{1}{2N} L_h W \right|^2 \phi^{-q-1} + \kappa \tau^2 \phi^{q-1} = 16\pi \phi^{q-1} \mathcal{E}^* \tag{6.11}$$

$$\text{div}_h \left( \frac{1}{2N} L_h W \right) = \phi^{q} \kappa \sigma - 8\pi \phi^{q} \mathcal{J}^*. \tag{6.12}$$

and in this (Hamiltonian) formulation, the momentum constraint (6.12) is an analog of equation (6.5).

In the same way that $A_\tau$ does not appear in either of equations (6.4) or (6.5), the shift $X$ is no longer present in either formulation (6.9)-(6.10) or (6.11)-(6.12). It can be freely traded with $W$ in the Lagrangian formulation and only the difference $Z = X - W$ matters. In the Hamiltonian formulation the shift plays no role in the first place. So although we have formally included it as seed data to help describe the conformal method, it is not truly a parameter and can be ignored. By contrast, the role of the slice energy density $\alpha$ is essential. It appears in the LCBY equations implicitly via $N = dV_h/\alpha$ and changing $\alpha$ affects the solution in the same way that changing $N$ in the EMCSF equations (6.4) and (6.5) affects the solution. Ultimately, both formulations of the conformal method specify a conformal class velocity (explicitly in the Lagrangian case and implicitly in the Hamiltonian case), but the Legendre transformation linking the conformal class momentum and then conformal class velocity requires additional metric information encoded in $\alpha$. Without this additional gauge information the problem posed by the conformal method is not well specified.

### 6.3 CMC Specialization

We are mostly interested in the constant mean curvature (CMC) case where $\tau$ is constant and where the conformal method has been maximally successful. In this case the LCBY momentum constraint (6.12) in the Hamiltonian formulation becomes

$$\text{div}_h \left( \frac{1}{2N} L_h W \right) = -8\pi \phi^{q} \mathcal{J}^*. \tag{6.13}$$

Under the scaling law $\mathcal{J}^* = \phi^{-q} \mathcal{J}$ discussed at the start of Section 6 (and implied by Meta-Theorem 1.1), this equation becomes

$$\text{div}_h \left( \frac{1}{2N} L_h W \right) = -8\pi \mathcal{J}, \tag{6.14}$$

which can potentially be solved for $W$ independent of $\phi$, in which case analysis of the system reduces to the Lichnerowicz equation (6.11) alone. In the CMC Lagragian formulation, again assuming the same scaling law for $\mathcal{J}$, the momentum constraint is

$$\text{div}_h \left( \frac{1}{2N} L_h Z \right) = -\text{div}_h \left( \frac{1}{2N} U \right) - 8\pi \mathcal{J} \tag{6.15}$$
which is similarly independent of \( \phi \).

There are two important differences between the vacuum and the non-vacuum initial data in the CMC conformal method that we wish to highlight here. In vacuum,

- the presence of conformal Killing fields (nontrivial solutions of \( L_h W = 0 \)) plays no role, and
- the choice of slice energy density \( \alpha \) plays no role in the Hamiltonian formulation.

Both of these statements are false in the non-vacuum setting, a fact that is easy to overlook for practitioners accustomed to working with vacuum data sets.

Concerning conformal Killing fields, consider the vacuum CMC Lagrangian LCBY momentum constraint

\[
\text{div}_h \left( \frac{1}{2N} L_h Z \right) = - \text{div}_h \left( \frac{1}{2N} U \right)
\]

and for simplicity, consider the case of a compact manifold. The left-hand side of equation (6.16) is formally self-adjoint with its kernel consisting of conformal Killing fields of the metric \( h \), should these exist. Integration by parts shows that the right-hand side is \( L^2 \) orthogonal to any conformal Killing fields and hence there exists solutions to equation (6.16) whether or not conformal Killing fields exist for \( h \). One readily verifies that multiple solutions to equation (6.16) exist if and only if \( h \) admits conformal Killing fields, since a pair of solutions \( Z' \) and \( Z \) differ by a conformal Killing field. Since a solution \( Z \) only appears in the LCBY Hamiltonian constraint in the form \( L_h Z = L_h Z' \), the potential multiplicity of solutions is irrelevant. On the other hand, for the full equation (6.15) we see that conformal Killing fields pose a genuine obstruction to solution. The equation is solvable if and only if \( J(h, B, \Pi_B) \) is \( L^2 \) orthogonal to any conformal Killing fields, and this poses an additional constraint on the matter fields that needs to be satisfied in advance.

Concerning the role of the slice energy density \( \alpha \) in the vacuum CMC Hamiltonian setting (again, on a compact manifold for simplicity), observe that the solutions \( W \) of equation (6.14) with \( \mathcal{J} = 0 \) are exactly conformal Killing fields. Since \( \alpha \) appears in the Lichnerowicz equation (6.11) only via the lapse \( N \) in the form \( 1/(2N) L_h W \), it follows that the solutions of this system, if any, are independent of the choice of \( \alpha \). But in the non-vacuum case the choice of \( \alpha \) has an impact on the resulting solution: changing \( \alpha \) changes \( N \) in equation (6.14) and, as can be seen directly in simple examples, thereby changes solutions of the non-homogeneous equation. This goes on to impact the term \( 1/(2N) L_h W \) in the Lichnerowicz equation and ultimately the resulting solution of the constraint equations. The significance of the dependence of the solution on the choice of \( \alpha \) is not yet well understood, and deserves further study.

Notably, these differences between the vacuum and non-vacuum CMC cases also appear in the non-CMC setting, even in vacuum. The slice energy density \( \alpha \) appears throughout the coupled system in the form of \( N \), and the presence of conformal Killing fields interferes with finding solutions of the LCBY vacuum momentum constraint

\[
\text{div}_h \left( \frac{1}{2N} L_h W \right) = \phi^{ \gamma k} d\tau
\]

because the right-hand side need not remain in the image of \( \text{div}_h \) as the unknown \( \phi \) varies. Because of these connections, it may be that a better understanding of the CMC non-vacuum conformal method leads to insights regarding an improved variation of the conformal method for non-CMC solutions.
7 Applications

We now apply our construction to derive the right-hand sides of the LCBY equations for a selection of concrete matter models. For definiteness, consider the Hamiltonian formulation in the CMC setting discussed in Sections 6.2 and 6.3. Thus the metric seed data can be taken to consist of a Riemannian metric $h$, a transverse-traceless tensor $\sigma$, a constant mean curvature $\tau$ and a positive lapse $N$; the LCBY equations then read

$$\begin{align*}
-2\kappa q\Delta h + R_h \phi - |\sigma + \frac{1}{2N} L_h W| h^{\phi^{-1}} + \kappa \tau^2 \phi^{\sigma^2} - 16\pi \phi^{\sigma-1} E^* & = 16\pi \phi^{\sigma-1} E^* \quad (7.1) \\
\text{div}_h \left( \frac{1}{2N} L_h W \right) & = -8\pi \phi^\sigma J^*. \quad (7.2)
\end{align*}$$

Our aim for each matter model is to write down the form of $E^*$ and $J^*$ as functions of $\phi$ along with other seed data to derive the final equations.

In each case the broad program begins with the following steps:

1. Identify the Lagrangian $L$ for the matter model along with the spacetime field variable(s).
2. For tensor-valued fields, decompose into spatial variables by the procedure of Section 4.1.
3. Determine the energy and momentum densities $E$ and $J$ for the model either by computing the stress-energy tensor and using equations (2.2)–(2.3) or by taking derivatives of the Lagrangian with respect to the lapse and shift via equations (2.4)–(2.5).
4. Letting $B$ denote the (potentially spatially decomposed) matter fields, identify the conjugate momenta $\Pi_B$ by writing the Lagrangian in terms of $B$ and its derivatives and computing the derivative of the Lagrangian with respect to $\dot{B}$.
5. Identify constraints for the matter fields arising from the Euler-Lagrange equations for the Lagrangian and determine if they are invariant under conformal changes; a failure of conformal invariance at this step would require some other procedure to construct initial data.

At this point, perhaps after a computation, we can write

$$\begin{align*}
E & = E(B, \Pi_B, h, N, X) \quad (7.3) \\
J & = J(B, \Pi_B, h, N, X). \quad (7.4)
\end{align*}$$

The quantities $E^*$ and $J^*$ in equations (7.1)–(7.2) are then obtained by replacing $h$ with $h^* = \phi^{\sigma-2} h$ and $N^* = \phi^\sigma N$ in equations (7.3)–(7.4):

$$\begin{align*}
E^* & = E(B, \Pi_B, \phi^{\sigma-2} h, \phi^\sigma N, X) \\
J^* & = J(B, \Pi_B, \phi^{\sigma-2} h, \phi^\sigma N, X),
\end{align*}$$

which completes the procedure. Because we have fixed $B$ and $\Pi_B$, Theorems 3.4 and 4.4 imply that the momentum constraint with this form of $J^*$ will have decoupled from the Hamiltonian constraint.
In practice, it is frequently cumbersome to write $E$ and $J$ in terms of $B$ and $\Pi_B$ directly, in part because familiar notation for matter models use quantities that encode $B$ and $\Pi_B$ in terms of variables that mix in the metric. For example, the electric field $E$ determines a momentum $-1/(2\pi) \langle E, \cdot \rangle_h \, dV_h$. This momentum can be held fixed when $h$ conformally changes so long as $E$ conformally changes as well; $h^* = \phi^{-2} h$ leads to $E^* = \phi^{-2} E$. In the case of fluids, there is a conformally invariant $n$-form $\omega$ determining particle number density, and this is determined by a conformally transforming scalar $n$ via $\omega = n^* \, dV_{h^*}$; maintaining $\omega = n^* \, dV_{h^*}$ requires $n^* = \phi^{-q} n$. In general we work with conformally transforming variables $C$ and transformation rules chosen so that $(C, h, N) \mapsto (C^*, h^*, N^*) = (C^*, h, \phi, N, \phi\theta^{-2} h, \phi\theta N)$ ensures $B$ and $\Pi_B$ remain fixed. The right-hand sides of the LCBY equations are then computed using

$$E^* = E(C^*(C, h, \phi, N), \phi\theta^{-2} h, \phi\theta N, X) \quad (7.5)$$

$$J^* = J(C^*(C, h, \phi, N), \phi\theta^{-2} h, \phi\theta N, X). \quad (7.6)$$

7.1 Perfect Fluids

Perfect fluids admit a Lagrangian description in which the field can be thought of as taking values in a separate manifold, the so-called material manifold. Hence Theorem 3.4 applies to these fields. Because the Lagrangian formulation for fluids is perhaps less familiar than the stress-energy tensor it determines, we begin with a brief summary of it. For more details the reader is referred to [Br93] and the approachable notes [Sc15].

7.1.1 Lagrangian Formulation

The $n$-dimensional material manifold $\Sigma$ is a set of labels for infinitesimal clusters of fluid particles. We make the important simplifying hypothesis ² that the fluid fills all of spacetime $M = \Sigma \times I$, in which case we can take $\Sigma = \Sigma$. The number density of particles on $\Sigma$ is determined by a fixed positive $n$-form $\omega$. Together, $(\Sigma, \omega)$ can be thought of as some instantaneous reference configuration of the fluid.

A fluid configuration on $M$ is determined by a submersion $\Psi : M \to \Sigma$ and the world line of a particle cluster $q \in \Sigma$ is $\Psi^{-1}(q)$. The map $\Psi$ here corresponds to the map $A$ of Section 3, with $\Sigma$ playing the role of $F$ from that section. The flux of particles through $n$-dimensional subspaces of $M$ is determined by the (non-vanishing) $n$-form

$$\omega = \Psi^* \overline{\omega},$$

and the fact that $d(\Psi^* \omega) = \Psi^*(d\omega) = 0$ reflects conservation of particle number for the fluid. Note that this description of the fluid configuration is completely independent of any spacetime metric.

If we have a spacetime metric $g$ on hand, it is convenient to express other tensorial quantities derived from $\omega$ in terms of $g$. First, the particle flux $Q$ is the unique vector satisfying

$$Q \cdot dV_g = \omega. \quad (7.7)$$

²For example, we may be modeling the interior of a fluid body. A Lagrangian description of a fluid only partly filling space requires a formulation of field theory more general than the one employed in Sections 3 and 4.
The particle flux is tangent to the worldlines of the fluid particles as can be seen by observing that \( \Psi_* Q = 0 \). Indeed:

\[
\Psi^* ((\Psi_* Q) \cdot \omega) = Q \cdot \omega = Q \cdot Q \cdot dV_g = 0,
\]

and the conclusion that \( \Psi_* Q = 0 \) then follows from the observations that \( \Psi \) is a submersion and that \( \omega \) is a non-vanishing top-level form. We assume that \( Q \) is timelike and future pointing, which is an open compatibility condition between \( \Psi \) and \( g \). Note that

\[
(div_g Q) dV_g = d(Q \cdot dV_g) = d\omega = 0
\]

and hence \( div_g Q = 0 \); this is a metric-biased reflection of the metric-free fact that particle number is conserved. The rest particle density is the scalar \( n_0 > 0 \) defined by

\[
n_0^2 = -g(Q, Q)
\]

and the unit vector \( U \) defined by

\[
Q = n_0 U
\]

is the fluid spacetime velocity.

The Lagrangian for an isentropic perfect fluid is determined by a function \( \rho(n_0) \) that expresses mass-energy density in terms of rest number density:

\[
\mathcal{L}_{\text{Fluid}}(\Psi, \partial \Psi, g) = -2\rho(n_0(\Psi, \partial \Psi, g)) dV_g.
\]

Note that \( n_0 \) is implicitly a function of both the metric (equations (7.7) and (7.8)) as well as the field configuration \( \Psi \) and its derivatives (\( n_0 \) depends on \( \omega = \Psi^* \omega \)).

The stress-energy tensor is obtained by varying the Lagrangian with respect to the (inverse) metric, and to do this we need to compute the variation of \( n_0 \) with respect to the metric. This computation is simplified by fixing an arbitrary non-vanishing \( n + 1 \)-form \( \Omega \) on spacetime so that \( \Psi^* \omega = Y \cdot \Omega \) for some vector field \( Y \) that does not depend on the metric. Observing \( Q = Y(\Omega/dV_g) \) we can write

\[
n_0^2 = -g(Y, Y) \left( \frac{\Omega}{dV_g} \right)^2.
\]

This expression for \( n_0^2 \) has the advantage that the dependence on \( g \) is now explicit; \( Y \) depends on \( \Psi \) and our arbitrary choice of \( \Omega \) but is independent of the metric. Thus

\[
\frac{\partial n_0^2}{\partial g^{-1}}_{ab} = Y_a Y_b \left( \frac{\Omega}{dV_g} \right)^2 - g(Y, Y) \left( \frac{\Omega}{dV_g} \right)^2 g_{ab} = Q_a Q_b - g(Q, Q) g_{ab} = n_0^2 (U_a U_b + g_{ab})
\]

and as a consequence

\[
T_{ab} dV_g = -\left( \frac{\partial \mathcal{L}_{\text{Fluid}}}{\partial g^{-1}} \right)_{ab} = (-\rho(n_0) g_{ab} dV_g + \rho'(n_0)n_0(U_a U_b + g_{ab})) dV_g
\]

\[
= \rho(n_0) U_a U_b + (\rho'(n_0)n_0 - \rho(n_0))[U_a U_b + g_{ab}] dV_g. \tag{7.10}
\]
We recognize this as the stress energy tensor of a perfect fluid with energy density \( \rho(n_0) \) and pressure \( p \) determined by the rest number density:

\[
p(n_0) = \rho'(n_0)n_0 - \rho(n_0). \tag{7.11}
\]

Noting that we are working with an isentropic fluid, the equation of state (7.11) is consistent with the first law of thermodynamics; compare with [MTWK17] equations (22.6) and (22.7a). So long as \( \rho \) is monotone in \( n_0 \), equation (7.11) implicitly gives a more familiar equation of state \( p = p(\rho) \). For example, linear equations of state \( p = (\gamma - 1)\rho \) used in cosmology arise from

\[
\rho(n_0) = Cn_0^\gamma
\]

for some constant \( C \). The special case of a pressureless fluid (dust) corresponds to \( \gamma = 1 \) and a stiff fluid with the speed of sound equal to the speed of light arises at the upper threshold value \( \gamma = 2 \); in Sections 7.1.4 and 7.1.5 we derive the right-hand sides of the LCBY equations for these two cases.

Before writing down the energy and momentum densities implied by equation (7.10) it is helpful to decompose the particle flux:

\[
Q = n(v + V) \tag{7.12}
\]

where \( v \) is the unit normal to slices of constant \( t \), \( V \) is the spatial velocity of the particle cluster measured by an observer traveling orthogonal to the slices of constant \( t \), and \( n \) is the particle density seen by the same observer. Then

\[
\langle U, v \rangle_g = \frac{1}{n_0} \langle Q, v \rangle_g = -\frac{n}{n_0} \tag{7.13}
\]

and consequently

\[
\mathcal{E} = T(v, v) = \rho(n_0) + \left(\frac{n}{n_0}\right)^2 n_0\rho'(n_0). \tag{7.14}
\]

Since \( j_\ast \langle v, \cdot \rangle_g = 0 \) and \( j_\ast \langle V, \cdot \rangle_g = \langle V, \cdot \rangle_h \) it follows that

\[
\mathcal{J} = -j_\ast T(v, \cdot) = \rho'(n_0)\left(\frac{n^2}{n_0}\right) \langle V, \cdot \rangle_h. \tag{7.15}
\]

Equations (7.8) and (7.12) imply the fundamental relation determining rest particle density from observed particle density and velocity:

\[
n_0^2 = n^2(1 - |V|^2_h). \tag{7.16}
\]

Hence we have an alternative form of the energy density:

\[
\mathcal{E} = \rho(n_0) + \frac{n^2}{n_0}\rho'(n_0)|V|^2_h. \tag{7.17}
\]

### 7.1.2 Conjugate Momentum for Perfect Fluids

Because our approach to constructing initial data involves fixing the field value \( \Psi \) and its conjugate momentum \( \Pi_\Psi \), we need to identify this latter quantity. Setting \( L_\Sigma = \partial_t + \mathcal{L}_{\text{Fluid}} \), by definition

\[
\Pi_\Psi = \frac{\partial L_\Sigma}{\partial \dot{\Psi}} = -2\rho'(n_0)\frac{\partial n_0}{\partial \Psi} NdV_h \tag{7.18}
\]
Derivatives of $\Psi$ appear in $n_0$ because it depends on $\omega = \Psi^\ast \overline{\omega}$ and we need to isolate the dependence on $\Psi$ while keeping $\partial_t \Psi$ fixed. Since $n_0$ is determined from $n$ and $V$ along with the metric (equation (7.16)), it suffices to determine how $n$ and $V$ depend on $\dot{\Psi}$.

**Lemma 7.1.** The following relations hold:

\begin{align*}
    n \, dV_h &= j_\ast 
    \Psi^\ast (n_0) \
    \Psi &= \Psi_\ast (-NV + X). 
\end{align*}

*Proof.* To establish (7.19) we compute

\[ \Psi^\ast \overline{\omega} = Q \, \partial V_g = n(n + V) \, \partial V_g = (n/N)(\partial_t + Z) \, \partial V_g \]

where $Z$ is an unimportant spatial vector. Lemma 3.1 part 2 implies $j_\ast Z \, \partial V_g = 0$ and hence

\[ j_\ast \Psi^\ast \overline{\omega} = \frac{n}{N} j_\ast \partial_t \, \partial V_g = n \, dV_h. \]

Turning to equation (7.20) we recall $\Psi_\ast Q = 0$. Since $Q = n(n + V)$ it follows that $\Psi_\ast V = -\Psi_\ast V$ and equation (7.20) results from an easy computation using the identities $V = (\partial_t - X)/N$ and $\dot{\Psi} = \Psi_\ast \partial_t$. 

Lemma 7.1 has two implications for the computation of $\Pi_{\Psi}$. First, $n$ is computable from $\partial_t \Psi$ and $h$ alone and is independent of $\dot{\Psi}$. So equation (7.16) implies

\[ \frac{\partial n_0^2}{\partial \Psi} = -2n_0 \left( V, \partial V/\partial \Psi \right)_h. \]

Second, equation (7.20) permits computation of $\partial V / \partial \Psi$. To do so, define

\[ \psi(t) = \Psi|_{\Sigma_t}. \]

Since $\Psi$ is, by hypothesis, a submersion and since the (assumed timelike) vector $Q$ lies in the kernel of $\Psi_\ast$, a dimension argument shows that for each $t$, $\psi(t)$ is a local diffeomorphism. Thus $\psi(t)$ is well-defined on tangent vectors and equation (7.20) implies

\[ \psi(t)^\ast \dot{\Psi} = (-NV + X)|_{\Sigma_t}. \]

Tangent vectors on $\Sigma_t$ can be identified with tangent vectors on $M$ and we conclude

\[ V = -\frac{\psi^\ast \dot{\Psi} - X}{N} \]

where $\psi^\ast \dot{\Psi}$ is defined to be the unique spatial vector $Y$ on $M$ with $\Psi_\ast Y = \dot{\Psi} \in T\Sigma$. Noting that $\psi^\ast$ depends on $\partial_t \Psi$ but not on $\Psi$, we conclude from equations (7.18), (7.21) and (7.22) that

\[ \Pi_{\Psi} = -2\rho'(n_0) \frac{n_0^2}{n_0} \left( V, \psi^\ast \right)_h \, dV_h. \]
This object is a little unwieldy; it is a $T^\ast \Sigma$-valued $n$-form on $M$. To mitigate this, we introduce the related covector field

$$P_\Psi = \rho'(n_0)\frac{n}{n_0} \langle V, \cdot \rangle_h$$

so that

$$\Pi_\Psi = -2(P_\Psi \circ \psi^\ast)n \, dV_h$$

and we interpret $P_\Psi$ as the fluid momentum density per unit particle. Equation (7.19) implies $n \, dV_h$ is metric-independent and hence $P_\Psi$ is computable from $\Psi$ and $\Pi_\Psi$ alone without using a spacetime metric.

We can now rewrite the energy and momentum densities from equations (7.15) and (7.17) as functions of $\Psi$, $\Pi_\Psi$ and $h$ as follows:

$$E = \rho(n_0) + \frac{n_0}{\rho'(n_0)} |P_\Psi|_h^2$$

$$J = n \ P_\Psi$$

where $n$ is determined from $\Psi$ and $h$ via $n \, dV_h = j \, \Psi^\ast \omega$, where $P_\Psi$ is computed from $\Psi$ and $\Pi_\Psi$ (but not $h$) as discussed above, and where $n_0$ is implicitly determined from $n$, $P_\Psi$ and $h$ by equations (7.16) and (7.24):

$$\left(\frac{n}{n_0}\right)^2 = 1 + \frac{1}{(\rho'(n_0))^2} |P_\Psi|_h^2.$$  

Note that taking equation (7.25) into account, the momentum constraint is exactly of the form of equation (3.7) from Theorem 3.4.

### 7.1.3 Application of the Conformal Method

We work on a single slice $\Sigma_{t_0}$ which we identify canonically with $\Sigma$. Spatial tensors defined along on $\Sigma_{t_0}$ in $M$ in the discussion above can then be unambiguously identified with tensors on $\Sigma$, and we do so without comment. Assuming that the initial fluid configuration $\psi(t_0) = \Psi|_{\Sigma_{t_0}}$ is a diffeomorphism, without loss of generality we can take $\Sigma = \Sigma$ and $\psi(t_0) = \text{Id}$. Hence the reference number density $\omega$ is simply the initial number density which, in a mild abuse of notation, we write as $\omega$. Having chosen $\omega$, the momentum $\Pi_\Psi$ is determined by choosing a covector field $P_\Psi$ according to equations (7.19) and (7.25) with $\psi = \text{Id}:

$$\Pi_\Psi = -2P_\Psi \otimes \omega.$$  

That is, the metric-independent matter field and the conjugate momentum are determined from $(\omega, P_\Psi)$, which are taken as the fixed non-metric seed data for the conformal method.

Since $E$ and $J$ in equations (7.26)–(7.27) are written in terms of $(n, n_0, P_\Psi, h)$ and do not explicitly involve our fixed variable $\omega$, we follow the procedure discussed at the start of Section 7 and introduce conformal transformation rules

$$(n, n_0, P_\Psi, h) \mapsto (n^\ast, n_0^\ast, P_\Psi, h^\ast = \phi^{g-2} h)$$

3Strictly speaking, in terms of our notation used up to this point the initial number density is the spatial tensor $j_\ast \omega$.
that are equivalent to the rule \((\omega, P_P, h) \mapsto (\omega, P_P, h^\ast)\). In place of \(\omega\) for the seed data we specify the function \(n\) and set \(\omega = n\,dV_h\). Keeping \(\omega\) fixed we conclude from \(\omega = n^\ast\,dV_h\) that \(n^\ast = \phi^{-q}n\). At this point, \(n^\ast_0(\phi)\) is determined by substituting starred variables into the implicit relation (7.28) and we find

\[
\left( \frac{n}{n^\ast_0(\phi)} \right)^2 \phi^{-2q} = 1 + \frac{1}{(\rho'(n^\ast_0(\phi)))^2} \phi^{2-q} |P_P|_h^2.
\]  

(7.29)

The right-hand side of the momentum constraint is

\[-8\pi\phi^q J^\ast = -8\pi\phi^q n^\ast P_P = -8\pi n П_P\]

(7.30)

and consequently the momentum constraint reads

\[\text{div}_h \left( \frac{1}{2N} L_h W \right) = -8\pi n П_P.\]

(7.31)

As expected, equation (7.31) does not involve \(\phi\) and has decoupled from the Hamiltonian constraint. Note that equation (7.31) holds regardless of the fluid’s equation of state.

By contrast, the form of the Hamiltonian constraint is sensitive to the choice of \(\rho(n_0)\), and we compute

\[E^\ast = \rho(n^\ast_0(\phi)) + \frac{n^\ast_0(\phi)}{\rho'(n^\ast_0(\phi))} \phi^{2-q} |P_P|_h^2\]

with \(n^\ast_0(\phi)\) determined implicitly by equation (7.29). This implicit relationship complicates the analysis of the conformal method, and a full treatment is beyond the scope of this paper. We content ourselves with two examples that illustrate some of the possibilities, dust and stiff fluids.

### 7.1.4 Dust

Dust arises from an equation of state \(\rho(n_0) = mn_0\) where \(m\) is the particle rest mass, and a computation shows

\[E = \frac{n^2}{n_0}.\]

(7.32)

Moreover, because \(\rho'(n_0) \equiv m\), we can solve equation (7.28) explicitly for \(n/n_0\) to obtain

\[E = n \sqrt{m^2 + |P_P|_h^2}.\]

Therefore

\[E^\ast = n^\ast \sqrt{m^2 + |P_P|_h^2} = n \phi^{-q} \sqrt{m^2 + \phi^{2-q} |P_P|_h^2}\]

and the Lichnerowicz equation (7.1) becomes

\[-2\kappa q \Delta_\phi + R_\phi - \left| \sigma + \frac{1}{2N} L_h W \right|_h^2 \phi^{-q-1} + \kappa \tau^2 \phi^{q-1} = 16\pi n \sqrt{m^2 \phi^{-2} + \phi^{-q} |P_P|_h^2}.\]

(7.33)

Because the right-hand side of this equation is monotone decreasing in \(\phi\), barrier techniques such as those employed in [Is95] can be used to establish existence and uniqueness of \(\phi\) in many contexts.
Equation (7.33) that we have derived here is manifestly different from the Lichnerowicz equation obtained for fluids in the earlier works, e.g. [DN02] or [IMP05]. These papers treat $\mathcal{E}$ and $\mathcal{J}$ as conformally transforming quantities with transformation rules independent of the fluid equation of state. For example, [IMP05] uses the rule
\begin{align*}
\mathcal{E}^* &= \phi^{1-3q/2} \mathcal{E} \\
\mathcal{J}^* &= \phi^{-q} \mathcal{J}.
\end{align*}

The transformation law for $\mathcal{J}$ is chosen simply for pragmatic reasons to ensure that the momentum constraint decouples, and the transformation law for $\mathcal{E}$ is chosen to ensure
\begin{equation}
\frac{|\mathcal{J}|^2_{\mathcal{E}}}{(\mathcal{E}^*)^2} = \frac{|\mathcal{J}|^2_{\mathcal{J}}}{\mathcal{E}^2}
\end{equation}
and hence the dominant energy condition (DEC) holds for $(\mathcal{E}^*, \mathcal{J}^*)$ if it does for $(\mathcal{E}, \mathcal{J})$. Alternatively, [DN02] uses the same transformation law for $\mathcal{J}$ but keeps $\mathcal{E}$ conformally invariant, motivated by important considerations arising at the boundary of a compact fluid body not filling spacetime. In both works, once a physical $\mathcal{E}^*$ and $\mathcal{J}^*$ have been determined by solving the LCBY equations, physical fluid variables (energy $\rho$ and velocity $V$) are extracted by inverting the relationship that defines $(\mathcal{E}^*, \mathcal{J}^*)$ from $(\rho, V)$ via the fluid equation of state; this operation is shown to be possible under natural physical assumptions about the equation of state, assuming $(\mathcal{E}^*, \mathcal{J}^*)$ satisfies the DEC. For example, the deconstruction procedure in [IMP05] implies the conformal transformation rules
\begin{align}
\rho^* &= \phi^{1-3q/2} \rho \\
V^* &= \phi^{q/2-1} V.
\end{align}

Although the approach we have described here is somewhat more involved than these earlier approaches, it enjoys a clearer connection between what is prescribed from matter in the seed data versus what appears in the finally constructed initial data. We are specifying a distribution of particles $\omega$ along with a conjugate momentum that are conformally invariant. In particular, the particle count $\int_\Omega \omega$ is preserved at the end of the construction in any region $\Omega \subseteq \Sigma$, and this better connects the Lichnerowicz equation to the Poisson equation of Newtonian gravity. By contrast, when $\rho$ and $V$ conformally transform with the rules (7.34)–(7.35) the final particle count depends on the determined conformal factor; e.g., if $V = 0$ a computation shows $\omega^* = \phi^{1-q/2} \omega$.

### 7.1.5 Stiff Fluids

In the earlier constructions of [DN02] and [IMP05] the fluid’s equation of state plays a compartmentalized role: it is used initially to determine $\mathcal{E}$ and $\mathcal{J}$, and it is used after solving the LCBY equations to deconstruct $\mathcal{E}^*$ and $\mathcal{J}^*$ into fluid variables. But it plays no role in the solvability of the LCBY system. So, from a certain perspective, with such a strategy all fluids are equivalent, assuming they satisfy the hypotheses needed to deconstruct $(\mathcal{E}^*, \mathcal{J}^*)$. In our procedure, the equation of state is an important component of the construction, and in this section we show how the LCBY equations for stiff fluids are closer in nature to those for scalar fields than they are to those for dust.
The equation of state of a stiff fluid is $\rho(n_0) = \mu n_0^2$ for some constant $\mu$ and equation (7.14) implies

$$\mathcal{E} = \mu(2n^2 - n_0^2).$$

Equation (7.28) for $n_0$ can be rewritten

$$n^2 = n_0^2 + \frac{1}{\mu^2} |P\psi|^2,$$

and as a consequence

$$\mathcal{E} = \mu n^2 + \frac{1}{\mu} |P\psi|^2.$$

Similarly

$$\mathcal{E}^* = \mu(n^*)^2 + \frac{1}{\mu} |P\psi|^2,$$

and the Hamiltonian constraint becomes

$$-2\kappa q \Delta_h \phi + R_h \phi - \left|\sigma + \frac{1}{2N} L_h W\right|^2 \phi^{-q-1} + \kappa \tau^2 \phi^{-1} = 16\pi \left[\phi^{-q-1} \mu m + \frac{1}{\mu} \phi |P\psi|^2\right].$$

(7.36)

The powers of $\phi$ appearing in the terms on the right-hand side of equation (7.36) strikingly correspond to those on the left-hand side. The $\phi^{-q-1}$ term is monotone decreasing in $\phi$ and therefore poses no difficulty for solvability. By contrast, the $\phi^1$ term is monotone increasing and needs to be accounted for by techniques such as those found in [HPP08] for working with scalar field matter sources. Indeed, equation (7.36) has a structure identical to that of the Lichnerowicz equation for a scalar field, equation (1.10). The correspondence found here between stiff fluids and scalar fields echoes well-known connections between the two models in the associated evolution problem ([Ch07], e.g.) and is absent in previous initial data constructions.

### 7.2 Proca

The Proca field describes a massive connection on a $U(1)$ bundle, generalizing the connection of Maxwell’s equations. The field is a 1-form $A$ on spacetime and the Lagrangian is the sum $\mathcal{L}_{Proca} = \mathcal{L}_{EM} + \mathcal{L}_m$ of the standard electromagnetism Lagrangian

$$\mathcal{L}_{EM} = -\frac{1}{8\pi} |dA|^2 dV_g$$

and a term involving the constant mass $m$ of the field

$$\mathcal{L}_m = -\frac{m}{4\pi} |A|^2 dV_g.$$
As in Section 5 we use the following the notation:

\[
E = -j_r (\nu \lrcorner dA)
\]

\[
A = j_r A
\]

\[
A_0 = \partial_t \lrcorner A
\]

\[
A_\perp = \nu \lrcorner A
\]

\[
A_r = (\partial_r - X) \lrcorner A = A_0 - X \lrcorner A
\]

\[
\dot{A} = \partial_t A.
\]

These are all spatial tensors, and we can identify them with tensors on an initial slice \( \Sigma_0 \), which we additionally identify with \( \Sigma \). The computation of Section 5 shows that on \( \Sigma \)

\[
E = -\frac{1}{N} \left( A - dA_r - \text{Lie}_X A \right)
\]

\[
= -\frac{1}{N} \left( A - dA_0 - X \lrcorner dA \right)
\]

where \( d \) is the exterior derivative intrinsic to \( \Sigma \). Using these variables, the slice Lagrangian \( L_\Sigma = \partial_t \lrcorner L_{\text{Proca}} \) can be written

\[
L_\Sigma = \left[ \frac{1}{4\pi} |E|_h^2 - \frac{1}{8\pi} |dA|_h^2 + \frac{m^2}{4\pi} \left( A_\perp^2 - |A|_h^2 \right) \right] N dV_h.
\]

(7.39)

In light of equation (7.37) and the identity \( A_\perp = A_r / N \) one can think of \( L_\Sigma \) as a function of metric variables \( (h, N, X) \) and field variables \( (A, A_r, \dot{A}, \dot{A}_r) \). Alternatively, using equation (7.38) and the formula \( A_\perp = (A_0 - X \lrcorner A) / N \) we can treat \( L_\Sigma \) as a function of \( (g, N, X) \) and \( (A, A_0, \dot{A}, \dot{A}_0) \).

The energy and momentum densities can be found via the equations

\[
\mathcal{E} dV_h = -\frac{1}{2} \frac{\partial L_\Sigma}{\partial N}
\]

\[
\mathcal{J} dV_h = \frac{1}{2} \frac{\partial L_\Sigma}{\partial X}
\]

taking care in the computation of \( \mathcal{J} \) to write \( L_\Sigma \) using \( A_0 \) instead of \( A_r \). We find

\[
\mathcal{E} = \frac{1}{8\pi} \left[ |E|_h^2 + \frac{1}{2} |dA|_h^2 + m^2 A_\perp^2 + m^2 |A|_h^2 \right]
\]

\[
\mathcal{J} = \frac{1}{4\pi} \left[ \langle E, \cdot \lrcorner dA \rangle_h - m^2 A_\perp A \right].
\]

(7.40)

(7.41)

Because the additional mass term \( L_m \) does not involve derivatives of \( A \), the momenta of the system are the same as for standard electromagnetism:

\[
\Pi_A = -\frac{1}{2\pi} \langle E, \cdot \rangle_h dV_h
\]

\[
\Pi_{A_r} = 0.
\]

The vanishing momentum \( \Pi_{A_r} \) implies a constraint: \( L_\Sigma \) must be stationary with respect to \( A_r \) and we find

\[
\int_{\Sigma} \left[ \langle E, dA_r \rangle + m^2 (A_r / N) \delta A_r \right] dV_h = 0
\]

(7.42)
for any compactly supported function \( \delta \). Hence
\[
\text{div}_h E = m^2 A_\perp.
\]
Substituting \( m^2 A_\perp = \text{div}_h E \) in the momentum constraint (7.41) we verify by an argument analogous to that of equation (5.15) that for any compactly supported vector field \( \delta X \),
\[
\int_\Sigma \mathcal{J}(\delta X) dV_h = \frac{1}{4\pi} \int_\Sigma \left[ \langle E, \delta X \cdot dA \rangle - (\text{div}_h E)A(\delta X) \right] dV_h
\]
\[
= \frac{1}{4\pi} \int_\Sigma \left[ \langle E, \delta X \cdot dA \rangle + \langle E, d(\delta X) \rangle \right] dV_h
\]
\[
= -\frac{1}{2} \int_\Sigma \left[ \Pi_A(\text{Lie}_{\delta X}A) + \Pi_{\delta A_\perp}(\text{Lie}_{\delta X}A_{\perp}) \right].
\]
Hence the conclusion of Theorem 4.4 indeed holds for this model.

To see how the constraint (7.42) is conformally invariant, equation (7.42) can be rewritten in terms of the momentum as
\[
\int_\Sigma \left[ -2\pi \Pi_A(\delta A_{\perp}) + m^2 \langle A_{\perp}/N, \delta A_{\perp} \rangle dV_h \right] = 0.
\]
Let \( P_A \) be the unique \( n - 1 \)-form such that \( P_A \land \eta = \Pi_A(\eta) \) for any 1-form \( \eta \); it is easy to see that \( P_A \) and \( A_{\perp} \) uniquely determine each other independent of the metric. Integration by parts then implies
\[
dP_A + \frac{m^2}{2\pi} \frac{dv_h}{N} = 0. \tag{7.43}
\]
At first glance, this constraint involves the spacetime metric via \( h \) and \( N \) and cannot be solved in advance in the context of the conformal method. Remarkably, however, the term \( dv_h/N \) already has a prominent role in the conformal method: it is the reference slice energy density \( \alpha \) from Section 6.2 and is the conformally fixed object that leads to the densitized lapse. Hence \( P_A \) satisfies \( dP_A + m^2 A_{\perp} = 0 \), and solutions of this constraint can be found \textit{a-priori} by the same techniques as Section 6.1. The densitized lapse of the conformal method is perhaps its least-well understood feature, and its natural appearance here for the Proca model may provide an avenue for clarifying its role.

We finish by deriving the right-hand sides of the LCBY equations. Rather than write equations (7.40)–(7.41) explicitly in terms of \( (A, A_{\perp}, \Pi_A, \Pi_{\perp}) \), we follow the procedure outlined at the start of Section 7: we keep the variables appearing in equations (7.40)–(7.41) but introduce conformal transformation rules
\[
(A, A_{\perp}, E, h, N) \mapsto (A, A_{\perp}^*, E^*, h^* = \phi^{\theta-2}h, N^* = \phi^\theta N)
\]
that are equivalent to the desired transformation
\[
(A, A_{\perp}, \Pi_A, \Pi_{\perp}, h, N) \mapsto (A, A_{\perp}, \Pi_A, \Pi_{\perp}, h^*, N^*).
\]
We take \( (A, A_{\perp}, E) \) as the matter seed data and define \( \Pi_A = -1/(2\pi) \langle E, \cdot \rangle_h \) and \( A_{\perp} = NA_{\perp} \). Fixing \( \Pi_A \) and \( A_{\perp} \) we require \( \Pi_A = -1/(2\pi) \langle E^*, \cdot \rangle_{h^*} \) and \( A_{\perp} = N^*A_{\perp}^* \) which leads to
\[
E^* = \phi^{-2}E
\]
\[
A_{\perp}^* = \phi^{-\theta}A_{\perp}.
\]
The right-hand sides of the LCBY equations (7.1)–(7.2) are then

\[16\pi \phi^{q-1} \mathcal{E}^* = 2 \phi^{q-1} \left[ |E^*|^2 + \frac{1}{2} |dA|^2 + m^2 (A_\perp)^2 + m^2 |A|^2 \right],\]

\[= 2 \left[ \phi^{-3} |E|^2 + \frac{1}{2} \phi^{3-q} |dA|^2 + m^2 \phi^{-q-1} (A_\perp)^2 + m^2 |A|^2 \right],\]

\[-8\pi \phi^{q} \mathcal{J}^* = -2 \phi^{q} \left[ \langle E^*, \cdot \rangle_{\cdot} dA - m^2 A_\perp A \right],\]

As implied by Theorem 4.4, \(-8\pi \phi^{q} \mathcal{J}^*\) does not involve \(\phi\) and the momentum constraint has decoupled from the Hamiltonian constraint. The mass term \(L_m\) in the Lagrangian leads to two terms on the right-hand side of the Hamiltonian constraint that are structurally identical to those of a scalar field.

The Proca field is treated in [IN77] in a more complex setting that also allows torsion, and it arrives at analogous LCBY equations. That work predates our current understanding of the densitized lapse as a part of the conformal method ([Yo99][PY03][Ma14]) and it is perhaps curious that the role of the densitized lapse is not observed there as it is here. In [IN77] the constraint (7.43) is written in the form \(dP_{A} + m^2 / (2\pi) A_\perp dV\) which is then used to eliminate \(A_\perp\) from the momentum constraint. Declaring this constraint to hold by fiat in fact implies the conformal transformation law \(A_\perp^* = \phi^{-q} A_\perp\) employed here but is arrived at by alternative means via an additional assumption (do whatever is necessary to \(A_\perp\) to maintain the constraint while leaving \(P_A\) fixed). A novel feature of our work is the decomposition of Section 4.1 that leads to \(A_\perp\) rather than \(A_\perp\) or \(A_0\) as a variable that should be kept fixed. Our approach is parsimonious, inasmuch as the conformal transformation rule for \(A_\perp\) then arises naturally; we simply fix \(A_\perp\) and conformally transform the lapse without any additional ansatizes.

### 7.3 Electromagnetism-Charged Dust (EMCD)

Our results can be generalized to matter fields having both tensor-valued and manifold-valued components, and we present an example of this here, a fluid consisting of dust with constant charge \(\varepsilon\) per unit particle coupled to electromagnetism. A notable feature of our construction is that the particle density of the solution is directly prescribed and the ratio of charge to rest mass is spatially constant.

The Lagrangian is a sum \(L_{\text{EMCD}} = L_{\text{EM}} + L_{\text{Dust}} + L_{I}\) of the standard Lagrangians for electromagnetism and dust together with a third interaction term:

\[L_{\text{EM}} = -\frac{1}{8\pi} |dA|^2 dV\]

\[L_{\text{Dust}} = -2m n_0 dV\]

\[L_{I} = 2 \varepsilon A(Q) dV.\]

The fluid variables \(n_0\) and \(Q\) (the rest particle density and the particle flux respectively) are defined in Section 7.1 and \(A\), as usual, is the electromagnetism connection 1-form. More fundamentally, the field variables consist of \(A\) and the fluid configuration map \(\Psi : M \to \Sigma\) defined at the start of Section 7.1.

Recall that the particle flux \(Q\) is defined by

\[Q \cdot dV = \Psi^* \omega\]
where \( \bar{\omega} \) is the reference particle density on the material manifold \( \Sigma \). A computation then shows we can alternatively write

\[
L_I = 2\varepsilon A \wedge \Psi^* \bar{\omega},
\]

an expression that does not involve the metric and which shows that the charge density of the fluid is proportional to the number density. Alternatively, a separate reference charge density \( \bar{\omega}_e \) could be specified on the material manifold and the interaction term would be \( 2A \wedge \Psi^* \bar{\omega}_e \).

Because the interaction term does not involve the metric, it has no impact on the stress-energy tensor, which is simply a sum of the stress-energy tensors for electromagnetism and dust. Consequently on a slice \( \Sigma_{t_0} \), now identified with \( \Sigma \),

\[
E = \frac{1}{8\pi} \left[ |E|^2 + \frac{1}{2} |dA|^2 \right] + n \sqrt{m^2 + |P_\Psi|^2}
\]

(7.44)

\[
\mathcal{J} = \frac{1}{4\pi} (E \cdot \partial dA) + n P_\Psi;
\]

(7.45)

the notation here follows that of Sections 7.1.4 and 7.2. In particular, \( A \) has been decomposed into \( (A, A_\perp) \), the electric field has the form of equation (7.40), the material manifold is \( \Sigma \) itself, and the fluid variables \( n \), which encodes the initial number density \( \omega \), and \( P_\Psi \), which represents the fluid momentum density, are discussed in detail in Section 7.1.3. As in Section 7.1.3 we have assumed, as we are free to do, that the initial fluid configuration satisfies \( \psi(t_0) = \text{Id} \).

The interaction term \( L_I \) does not involve derivatives of \( A \) and therefore does not impact the EM momenta. We have

\[
\Pi_A = -\frac{1}{2\pi} \langle E, \cdot \rangle_h dV_h
\]

\[
\Pi_{A\perp} = 0.
\]

On the other hand, \( L_I \) does contain derivatives of \( \Psi \), and this affects the fluid momentum. Using equation (7.22) we compute

\[
\mathcal{A}(Q) = \frac{n}{N} \left[ A_\perp - A(\psi^* \bar{\Psi} - X) \right]
\]

and therefore

\[
\partial_t \lrcorner L_I = 2n \varepsilon \mathcal{A}(Q) - A(\psi^* \bar{\Psi} - X) dV_h.
\]

(7.46)

Taking a derivative with respect to \( \bar{\psi} \) we conclude that \( \Pi_{\bar{\Psi}} \) picks up an additional term

\[-2n \varepsilon \mathcal{A} \circ \psi^* dV_h.\]

The contribution to \( \Pi_{\bar{\Psi}} \) from \( L_{Dust} \) is the same as in Section 7.1.3 and, recalling that on the initial slice we have assumed \( \psi \) = \text{Id}, we conclude

\[
\Pi_{\bar{\Psi}} = -2(P_\Psi + \varepsilon A) n dV_h.
\]

As argued in Section 7.1.3, \( n dV_h \) is metric independent; it is the initial number density \( \omega \) and therefore \( \Pi_{\bar{\Psi}} \) can be specified without using the metric by \( P_\Psi, \omega \) and \( A \).

Because the momentum conjugate to \( A_\perp \) vanishes there is a constraint; for any compactly supported function \( \delta A_\perp \),

\[
\int_\Sigma \frac{\partial L_\Sigma}{\partial A_\perp} [\delta A_\perp] = 0
\]

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where $L_\Sigma$ is the slice Lagrangian $\partial_t \cdot L_{EMCD}$. Using equation (7.46), a computation analogous to that leading to (7.42) implies

$$\int_\Sigma \left[ \frac{1}{2\pi} (E, d\delta A_v) + 2\epsilon n \delta A_v \right] dV_h \quad \text{(7.47)}$$

which implies Gauss’ Law

$$\text{div}_h E = 4\pi \epsilon n.$$

The constraint (7.47) can be alternatively written

$$\int_\Sigma \left[ -\Pi_\Sigma (d \delta A_v) + 2\epsilon \delta A_v \omega \right]$$

which is a metric-free statement and hence a conformally invariant condition relating $\Pi_A$ and $\omega$.

Although neither Theorem 3.4 nor Theorem 4.4 apply directly to this model, we can still verify that a natural generalization of equations (3.7) and (4.8) holds here. Using equation (7.45) and the definitions of $\Pi_A$ and $\Pi_\Psi$ we find that for any compactly supported vector field $\delta X$

$$-2 \int_\Sigma J(\delta X) dV_h = \int_\Sigma \left[ \Pi_A (\delta X \cdot dA) + \Pi_\Psi (\delta X) \right] + \int_\Sigma +2\epsilon \int_\Sigma A(\delta X)n dV_h.$$

Gauss’ Law and integration by parts implies

$$2\epsilon \int_\Sigma A(\delta X)n dV_h = \frac{1}{2\pi} \int_\Sigma (\text{div}_h E)A(\delta X) dV_h$$

$$= -\frac{1}{2\pi} \int_\Sigma (E, d(\delta X \cdot A)) dV_h$$

$$= \int_\Sigma \Pi_A (d(\delta X \cdot A)).$$

Finally, these last two equations along with the vanishing of $\Pi_{A_v}$, the formula $\text{Lie}_\delta X = d(\delta X \cdot A) + \delta X \cdot dA$ and the fact that the initial fluid configuration is the identity yields the desired relation

$$-2 \int_\Sigma J(\delta X) dV_h = \int_\Sigma \left[ \Pi_A (\text{Lie}_\delta X A) + \Pi_\Psi (\text{Lie}_\delta X \cdot A_v) + \Pi_\Psi (\Psi_\ast \delta X) \right].$$

Because the interaction term does not impact the energy and momentum densities, the right-hand sides of the LCBY equations are straightforward combinations of those for electrovac and dust. We can take as seed data for the matter fields $E, A, n$ and $P_\Psi$. Fixing $\Pi_A$ is equivalent to the transformation rule $E^* = \phi^{-2} E$ and fixing $\omega$ is implied by the transformation rule $n^* = \phi^{-\omega} n$. As remarked above, fixing $\Pi_\Psi$ is equivalent to fixing $\omega, P_\Psi$ and additionally $A$. Thus the variables and transformation rules are identical to those found in Sections 7.1.3, 7.1.4 and 7.2 and we find by the same computation as in those sections that the right-hand side of the LCBY equations (7.1)–(7.2) are

$$16\pi \phi \phi^{-1} E^* = 2\phi^{-3} |E|_h + \phi^{3-q} |dA|_h^2 + 16\pi n \sqrt{m^2 \phi^{-2} + \phi^{-q} |P_\Psi|_h^2}$$

$$-8\pi \phi^q J^* = -2 (E, \cdot dA) - 8\pi n P_\Psi.$$

The right-hand side of the momentum constraint has decoupled, and the right-hand side of the Lichnerowicz equation is monotone decreasing in $\phi$ and can be handled by barrier techniques.
The number density is fixed throughout this construction: it is \( n \, dV_h = n^* \, dV_h \). Hence so are the mass and charge densities, which are proportional to the number density by the constants \( m \) and \( \varepsilon \), and the fluid has a constant charge to mass ratio. The final electric field satisfies \( \text{div}_h \, E^* = 4\pi \varepsilon \, n^* \), and arranging for this along with the constant charge to mass ratio requires compatibility between two different scaling rules. The standard scaling \( E^* = \phi^{-2} \hat{E} \) for the electric field implies, since \( \text{div}_h (\phi^{-2} \hat{E}) = \phi^{-q} \text{div}_h \hat{E} \), that the (scalar) charge density must scale like \( \phi^{-q} \). But the constant charge to mass ratio then implies that the scalar mass density must also scale like \( \phi^{-q} \), which fortuitously agrees with the scaling we used for uncharged fluids in Section 7.1.3: \( n^* = \phi^{-q} n \). If the approaches of \([DN02]\) or \([IMP05]\) had been applied to the fluid variables, the scalar charge density would have scaled like \( \phi^{-q} \), but the scalar mass density would not have, and the resulting solution of the constraints would have had a mass density unrelated to the charge density. As remarked above, our construction can alternatively arrange for an arbitrary charge to mass ratio as a function of space by specifying a separate reference charge density on the material manifold.

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