INVERSE OPTIMAL CONTROL OF REGIME-SWITCHING
JUMP DIFFUSIONS

WENSHENG YIN\(^1\), JINDE CAO\(^{1,2,*}\) AND YONG REN\(^3\)

1. School of Mathematics, Southeast University, Nanjing 211189, China
2. Yonsei Frontier Lab, Yonsei University, Seoul 03722, South Korea
3. School of Mathematics & Statistics, Anhui Normal University, Wuhu 241000, China

Abstract. This paper studies the inverse optimal control using Legendre-Fenchel (in short, LF) translation method for regime-switching jump diffusions. Our approach is to first design inverse pre-optimal stabilization controllers and then obtain inverse optimal stabilizers, which avoids solving a Hamilton-Jacobi-Bellman equation. Finally, an application to stochastic Hamiltonian systems with Markov regime-switching is studied in detail for illustration.

1. Introduction. Jump processes provide mathematical model in many research fields such as mathematical biology, financial market, population processes, control problems, and so on, see for example, [1, 2, 3, 14, 16]. Due to the random change of the environment concerning system lives, regime-switching processes have attracted much interest lately, progress towards these problems have so far involved a wide range of ideas surround as biological and ecological systems, risk theory, stabilization theory, we refer to readers [4, 12, 15, 17, 18, 24] and the references therein. Inspired by problems of model complexity in random environments, in which both structural changes and sudden changes of parameters as well as large fluctuations coexist and are intertwined, this paper study regime-switching jump diffusion processes. The quantitative properties of these systems have greatly expanded over the previous few years, with most results making a connection between the control theory and the qualitative dynamics or long-time behavior of the associated dynamical system, see, for example, [5, 17, 18, 19, 20, 21, 22, 23, 25] and references therein.

As a well-defined dynamic system, there are few results on the optimal stabilization theory of regime-switching jump diffusion systems. It is the most intriguing topics to investigate the optimality of the regime-switching jump diffusion systems. It was shown that the dynamic programming method and the viscosity approach are two powerful tools to study stochastic optimal control problems by means of Hamilton-Jacobi-Bellman equation. However, those method have many limitations.
Because the optimal control problem is ultimately reduced to that of solving a partial differential equation (in short, PDE). Thus, a main drawback in the fields of stochastic optimal control problem is the fact that the solution of the PDE does not exist, or exists but is not unique. In other words, the premise of this method is to guarantee the existence of the regular solution to the Hamilton-Jacobi-Bellman equation. However, the existence and uniqueness of the solutions to the Hamilton-Jacobi-Bellman equation are available only for some special cases so far. This is not the case in general. This question is relaxed in the theory of inverse optimal control, which provides a powerful approach to solve optimal control problems, see [10]. Afterwards, Deng and Krstic [6, 7] extended the previous results to stochastic case and the design scheme of inverse optimal control for nonlinear systems are presented in detail. The main approach to inverse optimal stabilization is to design pre-inverse optimal stabilizer then design inverse optimal controller with the aid of LF transform. For more details on this topic, the readers are referred to Do [8, 9] and the references therein. The problem that we study in this paper is a continuation of [9, 13]. Compared with the work of [9], this framework addresses the influence of Markov regime-switching. It is also worth pointing out that the current results are similar to those in pure jump diffusion work, some additional care is needed due to the presence of regime switching. The gain function is closely related to the $LU(\cdot,\cdot,\cdot)$. The potential difficulty is to compute $LU(\cdot,\cdot,\cdot)$ associated with the process $(X,\Lambda)$. Consequently, the optimal stabilization problem is transferred to some existing results and techniques. There are several available work, see [5, 25]. In [13], the state feedback control is used to stabilize a given unstable hybrid SDE. In this article, our main purpose is to provide an optimization scheme for the design of the controller for hybrid SDEs with the state feedback control. More precisely, these problem can be split in two parts: the sufficient condition of the stabilization of system is obtained by Lyapunov function and then the corresponding controller is optimized by means of LF transform.

This paper is organized as following: In Section 2 we detail the additional notations used in this article. Section 3 contains the proof of our main results. Section 4 is devoted to example, some specific model to which our results apply. A summary of the key point is given in Section 5.

2. Preliminaries. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with filtration $\mathcal{F}^t$ satisfying the usual conditions. Let $B(\cdot)$ be a $R^m$-valued standard Wiener process, $\Xi$ be a subset of $(R^n-\{0\})$, and $N(\cdot,\cdot)$ defined on $R^+ \times (R^n-\{0\})$ is an $\mathcal{F}_t$-adapted Poisson random measure with the compensator $\tilde{N}(dt,dy) = N(dt,dy) - \nu(dy)dt$, where $\nu(dy)$ is a Lévy measure on $\Xi$. Let $\Lambda(\cdot)$ be a Markov chain taking values in $S = \{1,2,\cdots,n\}$.

We consider a stochastic control model is governed by the following equation:

$$
\begin{aligned}
dX(t) &= f(t,X(t),\Lambda(t))dt + g(t,X(t),\Lambda(t))dB(t) \\
&\quad + \int_{\Xi} h(t,X(t),\Lambda(t),y)N(dt,dy) + H(t,X(t),\Lambda(t))u(t)dt,
\end{aligned}
$$

where initial date $X(t_0) = X_0$, $\Lambda(t_0) = i_0$, $f(\cdot,\cdot,\cdot)$ : $[t_0,\infty) \times R^d \times S \rightarrow R^d$, $g(\cdot,\cdot,\cdot)$ : $[t_0,\infty) \times R^d \times S \rightarrow R^{d \times m}$, $h(\cdot,\cdot,\cdot,\cdot)$ : $[t_0,\infty) \times R^d \times S \times \Xi \rightarrow R^d$, $H(\cdot,\cdot,\cdot,\cdot)$ : $[t_0,\infty) \times R^d \times S \rightarrow R^{d \times m}$, $u$, valued in $R^m$, is the control process. The switching
process $\Lambda(\cdot)$ satisfies:
\[
\mathbb{P}\{A(t + \Delta) = j | A(t) = i\} = \begin{cases} q_{ij} \Delta + o(\Delta), & i \neq j \\ 1 + q_{ii} \Delta + o(\Delta), & i = j, \end{cases}
\]

as $\Delta \downarrow 0$, where $q_{ij}$ denotes the transition rate from $i$ to $j$ with $q_{ij} > 0$ if $i \neq j$.

Let $Q = (q_{ij})_{n \times n}$ be irreducible and conservative, i.e., $q_i = -q_{ii} = \sum_{j \neq i} q_{ij} < \infty$. In addition, we assume that the Brownian motion $B(\cdot)$, the Poisson random measure $N(\cdot, \cdot)$, and the Markov chain $\Lambda(\cdot)$ are mutually independent. To examine the well-posedness of global solutions to the SDE (1), we introduce some notations. For any $u \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{S}; \mathbb{R}^+)$, let us define the operator by
\[
LU(t, X, i) = U_t(t, X, i) + \nabla U f(t, X, i) + \nabla U H(t, X, i) u
+ \frac{1}{2} \text{Trac}(g^T(t, X, i) \nabla^2 U g(t, X, i))
+ \int_{\mathbb{E}} [U(t, X + h(t, X, i, y), i) - U(t, X, i)] \nu(dy) + \sum_{j=1}^n q_{ij} U(t, X, j),
\]

where $\nabla U$ and $\nabla^2 U$ denote the gradient and Hessian matrix of $U(t, X, i)$, respectively. To examine the properties of system (1), we need the following assumptions:

(A1) we suppose that
\[
|f(t, X, i) - f(t, Y, i)| + \sqrt{g(t, X, i) - g(t, Y, i)} + \sqrt{\int_{\mathbb{E}} |h(t, X, i, y) - h(t, Y, i, y)| \nu(dy)}/\sqrt{M} |X - Y|
\]

for $X, Y \in \mathbb{R}^d$ with $|X| \leq M$ and $i \in \mathbb{S}$, where $M \geq 1, C_M > 0$.

(A2) $H(t, X, i)u$ satisfies local Lipschitz condition.

Remark 1. Let $\tilde{f}(t, X, i) = H(t, X, i)u$. We can see that $\tilde{f}$ has the same conditions as $f$.

Lemma 2.1. ([25]) Assume that assumptions (A1), (A2) hold. There exists a function $U(\cdot, \cdot, \cdot) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{S}; \mathbb{R}^+)$ such that there is a constant $\beta > 0$ satisfying
\[
LU(t, X, i) \leq \beta U(t, X, i), \quad U_M := \inf_{|X| > M, i \in \mathbb{S}} U(t, X, i) \to \infty, \quad \text{as } M \to \infty.
\]

Then process $(X(t), \Lambda(t))$ is regular.

It follows from (A1), (A2) and Lemma 2.1, system (1) has a unique global solution, see [25] for details. For the sake of the concept of trivial solutions, we further assume that $f(t, 0, i) = 0$, $g(t, 0, i) = 0$, $h(t, 0, i, y) = 0$ and control function $u = \psi(t, X, i)$ continuous away from the origin, with $\psi(t, 0, i) = 0$ for each $i \in \mathbb{S}$.

Theorem 2.2. Let $p, k_1, k_2, k_3$ be positive constants. There exists a function $U(\cdot, \cdot, \cdot) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{S}; \mathbb{R}^+)$ such that

(i) $k_1 |X|^p \leq U(t, X, i) \leq k_2 |X|^p$,

(ii) $LU(t, X, i) \leq -k_3 |X|^p$

for $(t, X, i) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{S}$. Then it holds that $\mathbb{E}|X(t)|^p \leq \frac{k_2}{k_1} |X_0|^p e^{-k_3 t}$.

The results have been well studied, see Zhen et al. ([5], Theorem 3.1) and Zong et al. [25].

Lemma 2.3. ([9], [11]) Let us present Legendre-Fenchel (in short, LF) transform:

(1) $I_\eta(s) = s(\eta')^{-1}(s) - \eta((\eta')^{-1}(s)) = \int_0^s (\eta^{-1})(u) du$,

(2) $I_\eta(s) = \eta(s), \quad I_\eta(s)$ is a class $\mathcal{K}_\infty$ function,
(3) $J_\eta(\eta'(s)) = s\eta'(s) - \eta(s)$,
where $\eta(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is a class $\mathcal{K}_\infty$ function, $\eta'(s)$ denotes the derivative of $\eta(s)$, and $(\eta')^{-1}(s)$ is the inverse function of $\eta'(s)$.

Lemma 2.4. For any $X, Y \in \mathbb{R}^d$, then $X^T Y \leq \eta(\|X\|) + I_\eta(\|Y\|)$, “$\Rightarrow$” is obtained if and only if $Y = \eta'(\|X\|) \frac{X}{\|X\|}$.

Definition 2.5. There exist a positive definite function $\varphi : [0, T] \times \mathbb{R}^d \times \mathbb{S} \rightarrow [0, \infty)$ and $\mathcal{K}_\infty$ function $\eta(\cdot)$, a symmetric positive definite matrix-valued function $G(t, X, i)$ and feedback control $u$ such that $p$-th moment inverse optimal exponential stabilization for (1) is solvable and the gain function is minimal, where the gain function is given as

$$J(t, X, u, i) = E \left[ \int_0^\infty \left( \varphi(t, X, i) + \eta(\|G^\frac{1}{2}(t, X, i)\|u) \right) dt \right].$$

Denote by $\prod_{s, X, i}$ the collection of all admissible control. The corresponding value function is described as

$$V(s, X, i) = \inf_{u \in \prod_{s, X, i}} J(s, X, u, i).$$

An admissible control $u^* \in \prod_{s, X, i}$ is called optimal, if it holds

$$V(s, X, i) = J(s, X, u^*, i).$$

Remark 2. Definition 2.5 indicates that value function includes $\varphi(t, X, i)$, $\eta(r)$, $G(t, X, i)$ and feedback control $u$. When we search for different $\mathcal{K}_\infty$ function, this leads to different inverse optimal control law. Hence, the flexibility of the cost function is left to the designer.

3. Main results.

Theorem 3.1. There exists a Lyapunov function $U(t, X, i)$ satisfies condition (i) in Theorem 2.2. Let $G(t, X, i)$ be a symmetric positive definite matrix-valued function. Let

$$u := u(t, X, i)$$

be pre-optimal controller. Suppose that $u(t, X, i)$ ensures the $p$-th moment exponential stabilization of system (1). Let

$$u^* := u^*(t, X, i)$$

be optimal controller for all $\alpha \geq 2$. Then $u^*(t, X, i)$ ensures the $p$-th moment inverse optimal exponential stabilization of system (1) by minimizing the cost functional

$$J(t, X, u, i) = E \left[ \int_0^\infty \left( \varphi(t, X, i) + \alpha^2 \eta \left( \frac{2}{\alpha} \|G^\frac{1}{2}(t, X, i)u\| \right) \right) dt \right].$$
where

$$\varphi(t, X, i) = 2\alpha \left\{ I_q(\|\nabla UH(t, X, i)G^{-\frac{1}{2}}(t, X, i)\|) - U_i(t, X, i) - \nabla Uf(t, X, i) - \frac{1}{2} \text{Tr} (g^T(t, X, i) \nabla^2 Ug(t, X, i)) \right.$$ 

$$- \int_{\Xi} \left[ U(t, X + h(t, X, i), y) - U(t, X, i) \right] \nu(dy) 
- \sum_{j=1}^{n} q_{ij} U(t, X, j) \right\} + a(a - 2) I_q(\|\nabla UH(t, X, i)G^{-\frac{1}{2}}(t, X, i)\|).$$

(7)

Moreover, $U(t, X, i)$ satisfies the family Hamilton-Jacobi-Bellman

$$\frac{\varphi(t, X, i)}{2\alpha} - \frac{\alpha}{2} I_q(\|\nabla UH(t, X, i)G^{-\frac{1}{2}}(t, X, i)\|) 
+ U_i(t, X, i) + \nabla Uf(t, X, i) + \frac{1}{2} \text{Tr} (g^T(t, X, i) \nabla^2 Ug(t, X, i)) 
+ \int_{\Xi} \left[ U(t, X + h(t, X, i, y), i) - U(t, X, i) \right] \nu(dy) + \sum_{j=1}^{n} q_{ij} U(t, X, j) = 0$$

(8)

for $\alpha \geq 2, i \in \mathbb{S}$.

Proof. The proof is separated into three steps.

**Step 1**: In this step, we show the $J(t, X, u, i)$ is always meaningful. Since $u$ ensures that system (1) $p$-th moment exponential stabilization, then there exist a constant $\lambda$ and positive definite $U(t, X, i)$ such that

$$LU(t, X, i) |_{u} = U_i(t, X, i) + \nabla Uf(t, X, i) + \frac{1}{2} \text{Tr} (g^T(t, X, i) \nabla^2 Ug(t, X, i)) 
- \nabla UH(t, X, i)G^{-1}(t, X, i)(\nabla UH(t, X, i))^T I_q(\|\nabla UH(t, X, i)G^{-\frac{1}{2}}(t, X, i)\|) 
\|\nabla UH(t, X, i)G^{-\frac{1}{2}}(t, X, i)\|^2 + \int_{\Xi} \left[ U(t, X + h(t, X, i, y), i) - U(t, X, i) \right] \nu(dy) + \sum_{j=1}^{n} q_{ij} U(t, X, j)

= U_i(t, X, i) + \nabla Uf(t, X, i) + \frac{1}{2} \text{Tr} (g^T(t, X, i) \nabla^2 Ug(t, X, i)) 
- I_q(\|\nabla UH(t, X, i)G^{-\frac{1}{2}}(t, X, i)\|) 
+ \int_{\Xi} \left[ U(t, X + h(t, X, i, y), i) - U(t, X, i) \right] \nu(dy) + \sum_{j=1}^{n} q_{ij} U(t, X, j)
\leq -\lambda U(t, X, i).$$

(9)

In view of (7) and (9), for $\alpha \geq 2$, it holds that

$$\varphi(t, X, i) \geq 2\alpha \lambda U(t, X, i) + a(a - 2) I_q(\|G^{\frac{1}{2}}(t, X, i)u\|) + \frac{\alpha}{2} I_q(\|\nabla UH(t, X, i)G^{-\frac{1}{2}}(t, X, i)\|)$$

Therefore, $\varphi(t, X, i)$ is positive definite.
Step 2: This step is to show that the $u^*$ is also an admissible feedback control, which will be the desired optimal feedback control.

$$LU(t, X, i)_{|u^*} = U_i(t, X, i) + \nabla U f(t, X, i) + \frac{1}{2} \text{Trac}(g^T(t, X, i)\nabla^2 U g(t, X, i))$$

$$- \frac{\alpha}{2} \nabla U H(t, X, i)G^{-\frac{1}{2}}(t, X, i)(\nabla U H(t, X, i))^T$$

$$\times (\eta')^{-1}(\|\nabla U H(t, X, i)G^{-\frac{1}{2}}(t, X, i)\|)$$

$$\leq \frac{\alpha}{2} \left( I_\eta(\|\nabla U H(t, X, i)G^{-\frac{1}{2}}(t, X, i)\|) \right)$$

$$- \frac{\alpha}{2} \left( \eta \left( (\eta')^{-1}(\|\nabla U H(t, X, i)G^{-\frac{1}{2}}(t, X, i)\|) \right) \right)$$

$$+ \int_\xi [U(t, X + h(t, X, i, y), i) - U(t, X, i)] \nu(dy) + \sum_{j=1}^n q_{ij} U(t, X, j)$$

$$\leq LU(t, X, i)_{|u} - \frac{\alpha}{2} \left( \eta \left( (\eta')^{-1}(\|\nabla U H(t, X, i)G^{-\frac{1}{2}}(t, X, i)\|) \right) \right)$$

This means that $u^*$ is also an admissible feedback control and ensures system (1) $p$-th moment exponential stabilization. Since $U(t, X, i) \in C^{1,2}((0, T) \times \mathbb{R}^n)$, we have for all $(t, X) \in [0, T) \times \mathbb{R}^n$, $u^* \in \prod_{s, X, i}$, by Itô’s formula

$$U(t, X, i) - U(t_0, X_0, i_0) - LU(t, X, i)_{|u} = \int_{t_0}^t g(s, X(s), X(s))dB(s)$$

$$+ \int_\xi [U(s, X + h(s, X, i, y), i) - U(s, X, i)] \tilde{N}(ds, dy).$$

Taking the expectation yields that

$$\mathbb{E} \left( U(t, X, i) - U(t_0, X_0, i_0) - \int_{t_0}^t LU(s, X, i)_{|u} ds \right) = 0.$$ (12)

Furthermore, invoking the equalities (7) and (12) and substituting them into (6) lead to

$$J(t, X, u, i) = \mathbb{E} \left[ \int_{t_0}^\infty \left( \varphi(s, X, i) + \alpha^2 \eta \left( \frac{2}{\alpha} \|G^\frac{1}{2}(s, X, i)u\| \right) \right) ds \right]$$

$$= -2\alpha \lim_{t \to \infty} \mathbb{E} \left( U(t, X, i) - U(t_0, X_0, i_0) \right)$$

$$+ \mathbb{E} \left\{ \int_{t_0}^\infty \left[ \alpha^2 \eta \left( \frac{2}{\alpha} \|G^\frac{1}{2}(s, X, i)u\| \right) 

+ \alpha(\alpha - 2) \left( I_\eta(\|\nabla U H(s, X, i)G^{-\frac{1}{2}}(s, X, i)\|) \right) \right] ds \right\}. $$ (13)
Now using (4) yields that
\[
\nabla U H(t, X, i) u(t, X, i)
\]
\[= -\nabla U H(t, X, i) G^{-\frac{1}{2}}(t, X, i)(\nabla U H(t, X, i))^T I_\eta \left( \| \nabla U H(t, X, i) G^{-\frac{1}{2}}(t, X, i) \| \right).
\]
Directly computing, we obtain that
\[I_\eta \left( \| \nabla U H(t, X, i) G^{-\frac{1}{2}}(t, X, i) \| \right) = -\nabla U H(t, X, i) u(t, X, i).
\] (14)
Next, using (5) yields that
\[
\eta' \left( \frac{2}{\alpha} \| G^\frac{1}{2}(t, x, i) u^* \| \right) = \| \nabla U H(t, X, i) G^{-\frac{1}{2}}(t, X, i) \|.
\]
Further, we have
\[\frac{2}{\alpha} \| G^\frac{1}{2}(t, x, i) u^* \| = (\eta')^{-1} \left( \| \nabla U H(t, X, i) G^{-\frac{1}{2}}(t, X, i) \| \right).
\] (15)
Similarly, using (5) and (15), we have
\[\frac{2}{\alpha} G^\frac{1}{2}(t, x, i) u^* = -\nabla U H(t, X, i) G^{-\frac{1}{2}}(t, X, i))^T \frac{2}{\alpha} \| G^\frac{1}{2}(t, x, i) u^* \| \frac{\eta'}{\frac{2}{\alpha} \| G^\frac{1}{2}(t, x, i) u^* \|}.
\]
Then
\[
\nabla U H(t, X, i) G^{-\frac{1}{2}}(t, X, i) = -\eta' \left( \frac{2}{\alpha} \| G^\frac{1}{2}(t, x, i) u^* \| \right) \left( \frac{\frac{2}{\alpha} G^\frac{1}{2}(t, x, i) u^*)^T}{\frac{2}{\alpha} \| G^\frac{1}{2}(t, x, i) u^* \|} \right)
\]
Substituting (14), (15) and (16) into (13) that
\[
J(t, X, u, i)
\]
\[= -2\alpha \lim_{t \to \infty} E(U(t, X, i) - U(t_0, X_0, i_0))
\]
\[+ E \left\{ \int_{t_0}^\infty \alpha^2 \eta \left( \frac{2}{\alpha} \| G^\frac{1}{2}(s, X, i) u \| \right) \alpha^2 \left( I_\eta \left( \eta' \left( \| G^\frac{1}{2}(s, X, i) u^* \| \right) \right) \right.
\]
\[\left. -2\alpha \eta' \left( \frac{2}{\alpha} \| G^\frac{1}{2}(s, X, i) u^* \| \right) \frac{\left( \frac{2}{\alpha} G^\frac{1}{2}(s, X, i) u^*)^T}{\frac{2}{\alpha} \| G^\frac{1}{2}(s, X, i) u^* \|} \right) \right\}
\]
Invoking Lemma 2.3, we have
\[
J(t, X, u, i)
\]
\[\geq -2\alpha \lim_{t \to \infty} E(U(t, X, i) - U(t_0, X_0, i_0))
\]
\[+ E \left\{ \int_{t_0}^\infty \alpha^2 \eta \left( \frac{2}{\alpha} \| G^\frac{1}{2}(s, X, i) u \| \right) \alpha^2 \left( I_\eta \left( \eta' \left( \| G^\frac{1}{2}(s, X, i) u^* \| \right) \right) \right.
\]
\[\left. -2\alpha^2 \eta \left( \frac{2}{\alpha} \| G^\frac{1}{2}(s, X, i) u \| \right) \eta \left( \frac{2}{\alpha} \| G^\frac{1}{2}(s, X, i) u \| \right) \right\}
\]
\[= -2\alpha \lim_{t \to \infty} E(U(t, X, i) - U(t_0, X_0, i_0))
\] (18)
where “\(=\)” holds if and only if
\[
\| \eta' \left( G^\frac{1}{2}(s, X, i) u^* \right) \frac{\left( \frac{2}{\alpha} G^\frac{1}{2}(s, X, i) u^*)^T}{\frac{2}{\alpha} \| G^\frac{1}{2}(s, X, i) u^* \|} \right) = \| \eta' \left( G^\frac{1}{2}(s, X, i) u \right) \frac{\left( \frac{2}{\alpha} G^\frac{1}{2}(s, X, i) u \right)^T}{\frac{2}{\alpha} \| G^\frac{1}{2}(s, X, i) u \|}.
\] (19)
Since (19) holds for \( x \in \mathbb{R}^n \), we have \( u = u^* \). In addition, because \( u = u^* \) stabilizes the system (1), we obtain the desired limit with \( \lim_{t \to \infty} EU(t, X, i) = 0 \). In view of (18), the optimal cost functional are

\[
\arg\min_u J(t, X, i, u) = u^* \quad \text{and} \quad \min_u J(t, X, i, u) = 2\alpha EU(t_0, X_0, i_0).
\]

Let \( \bar{u} \) guarantee system (1) \( p \)-th moment inverse optimal exponential stabilization. Replacing \( u \) of (13) with \( \bar{u} \) yields that

\[
\begin{align*}
J(t, X, \bar{u}, i) &= \mathbb{E} \left[ \int_{t_0}^\infty 2\alpha \left( -U_s(s, X, i) - \nabla Uf(s, X, i) - \frac{1}{2} \text{Tr}(g^T(s, X, i)\nabla^2 Ug(s, X, i)) \right) \\
&\quad - \nabla UH(s, X, i)\bar{u} - \int_\mathbb{E} [U(s, X + h(s, X, i, z), i) - U(s, X, i)]\nu(dy) \\
&\quad - \sum_{j=1}^n q_{ij} U(s, X, j) \right] ds + \mathbb{E} \int_{t_0}^\infty \left[ \alpha^2 I_\eta(\|\nabla UH(s, X, i)G^{-\frac{1}{2}}(s, X, i)\|) \\
&\quad + 2\alpha \nabla UH(s, X, i)\bar{u} + \alpha^2 \eta \left( \frac{2}{\alpha} \|G^{\frac{1}{2}}(s, X, i)\bar{u}\| \right) \right] ds \\
&= -2\alpha \mathbb{E} \int_{t_0}^\infty LU(s, X, i)\|\bar{u}\| ds + \Upsilon \\
&= -2\alpha (\lim_{t \to \infty} \mathbb{E}(U(t, X, i) - U(t_0, X_0, i_0))) + \Upsilon \\
&= J(t, X, u^*, i) + \Upsilon,
\end{align*}
\]

where

\[
\Upsilon = \mathbb{E} \int_{t_0}^\infty \left[ \alpha^2 I_\eta(\|\nabla UH(t, X, i)G^{-\frac{1}{2}}(t, X, i)\|) + 2\alpha \nabla UH(t, X, i)\bar{u} \\
&\quad + \alpha^2 \eta \left( \frac{2}{\alpha} \|G^{\frac{1}{2}}(t, X, i)\bar{u}\| \right) \right] dt.
\]

Invoking Young’s inequality shows that

\[
2\alpha \nabla UH(t, X, i)\bar{u} \geq -\alpha^2 I_\eta(\|\nabla UH(t, X, i)G^{-\frac{1}{2}}(t, X, i)\|) - \alpha^2 \eta \left( \frac{2}{\alpha} \|G^{\frac{1}{2}}(t, X, i)\bar{u}\| \right).
\]

\("=\) holds only if

\[
\frac{2}{\alpha} \bar{u} = - (\eta')^{-1}(\|\nabla UH(t, X, i)G^{-\frac{1}{2}}(t, X, i)\|)(\nabla UH(t, X, i)G^{-\frac{1}{2}}(t, X, i))^T \frac{\nabla UH(t, X, i)G^{-\frac{1}{2}}(t, X, i)}{\|\nabla UH(t, X, i)G^{-\frac{1}{2}}(t, X, i)\|}.
\]

From (23), we conclude that \( \bar{u} = u^* \). Since \( J(t, X, i, u^*) = 2\alpha EU(t_0, X_0, i_0) \). It follows from (21),(22), (23) that \( \Upsilon \geq 0 \) and \( \Upsilon = 0 \) only if \( \bar{u} = u^* \).
**Step 3:** Proof of continuity

\[
\lim_{\|\nabla U_H(t,X,i)\|G^{-\frac{1}{2}}\to 0} \|u^*\|
\]

\[
= \lim_{\|\nabla U_H(t,X,i)\|G^{-\frac{1}{2}}\to 0} \left( \left\| \frac{\alpha}{2} G^{-1}(t,X,i)(\nabla U_H(t,X,i))^T \right\| \times (\eta')^{-1}(\|\nabla U_H(t,X,i)\|^T G^{-\frac{1}{2}}(t,X,i)) \right)
\]

\[
= \lim_{\|\nabla U_H(t,X,i)\|G^{-\frac{1}{2}}\to 0} \frac{\alpha}{2} G^{-\frac{1}{2}}(t,X,i)((\eta')^{-1}(\|\nabla U_H(t,X,i)\|G^{-\frac{1}{2}}(t,X,i))}
\]

\[
= 0.
\]

We complete the proof. \(\square\)

**Remark 3.** The theorem above indicates that the gain function \(J(t,X,u,i)\) and Hamilton-Jacobi-Bellman equation (8) are generated after the feedback control \(u\).

**Remark 4.** In summary, the main procedure for design of an inverse optimal control for regime-switching jump diffusions include the following steps:

1. Set \(u(t,X,i) = -A(\nabla U_H(t,X,i))^T, u^*(t,X,i) = -A^*(\nabla U_H(t,X,i))^T\), where

\[
A := L_g(\|\nabla U_H(t,X,i)\|^T G^{-\frac{1}{2}}(t,X,i)) G^{-1}(t,X,i)
\]

and

\[
A^* := \frac{\alpha}{2} (\|\nabla U_H(t,X,i)\|^T G^{-\frac{1}{2}}(t,X,i)) G^{-1}(t,X,i).
\]

It is obvious that \(A\) and \(A^*\) are positive define matrices. Choosing an appropriate Lyapunov function is employed to guarantee for \(LU \leq -\mu U\);

2. Searching for \(\mathcal{K}_\infty\) function \(\eta(r)\), then the \(G(t,X,i)\) is determined by the matrix \(A\).

3. Finally, inverse optimal controller is obtained by (4).

Now, we would like to present sufficient conditions of inverse optimal stabilizers.

**Theorem 3.2.** There exists a \(U(t,X,i)\) such that

\[
\inf_{u \in \mathbb{R}^m} LU(t,X,i) \leq -\beta_1 U(t,X,i)
\]

and

\[
\|\nabla U_H(t,X,i)\| \geq \beta_2 U(t,X,i), \tag{25}
\]

where \(\beta_1\) and \(\beta_2\) are two positive constants. Then, inverse optimal stabilization is solvable.

**Proof.** We choose \(\eta(s) = \frac{1}{2} s^2\), \(\alpha = 2\) and

\[
G = I \left\{ \left( 1 + \frac{1}{\|\nabla U_H(t,X,i)\|^T} \right)^{-0.5} \right\}, \nabla U_H(t,X,i) \neq 0, \tag{26}
\]

\[
\text{c, } \nabla U_H(t,X,i) = 0,
\]

\[
\text{where } \beta_1 \text{ and } \beta_2 \text{ are two positive constants. Then, inverse optimal stabilization is solvable.}
\]

\[
\text{Proof. We choose } \eta(s) = \frac{1}{2} s^2, \alpha = 2 \text{ and }
\]

\[
G = I \left\{ \left( 1 + \frac{1}{\|\nabla U_H(t,X,i)\|^T} \right)^{-0.5} \right\}, \nabla U_H(t,X,i) \neq 0, \tag{26}
\]

\[
\text{where } \beta_1 \text{ and } \beta_2 \text{ are two positive constants. Then, inverse optimal stabilization is solvable.}
\]
where
\[
\bar{L} = U_i(t, X, i) + \nabla U f(t, X, i) + \frac{1}{2} \text{Trac}(g^T( t, X, i) \nabla^2 U g(t, X, i)) \\
+ \int_{\Xi} [U(t, X + h(t, X, i, y), i) - U(t, X, i)] N(dy) + \sum_{j=1}^{n} q_j U(t, X, j)
\]
and \( c \) is a positive constant. According to (5), straightforward calculations yield that
\[
u^* = \begin{cases} 
-2 \left( 1 + \frac{1}{\|
abla U H(t, X, i)\|^2} \right) (\nabla U H(t, X, i))^T \nabla U H(t, X, i) \neq 0, \\
0, \quad \nabla U H(t, X, i) = 0.
\end{cases} \tag{27}
\]
Then we know immediately that the inverse pre-optimal control
\[
u = \begin{cases} 
- \left( 1 + \frac{1}{\|
abla U H(t, X, i)\|^2} \right) (\nabla U H(t, X, i))^T \nabla U H(t, X, i) \neq 0, \\
0, \quad \nabla U H(t, X, i) = 0.
\end{cases} \tag{28}
\]
Substituting (28) into (3), we have
\[
LU = \begin{cases} 
\bar{L} - \left( 1 + \frac{1}{\|
abla U H(t, X, i)\|^2} \right) \|
abla U H(t, X, i)\|^2, \|
abla U H(t, X, i)\| \neq 0, \\
L, \quad \nabla U H(t, X, i) = 0.
\end{cases} \tag{29}
\]
where we used conditions (24) and (25). Consequently, by Theorem 3.1, we conclude that a feedback control \( u \) guarantees system (1) \( p \)-th moment exponentially stable.

4. Application and example. We consider a Hamiltonian system subject to random perturbations:
\[
\ddot{z}(t) + a_{\Lambda(t)} \dot{z}(t) + \dot{z}(t) + \left( b_{\Lambda(t)} B(t) + c_{\Lambda(t)} \int_{\Xi} N(1, dy) \right) \dot{z}(t) + d_{\Lambda(t)} \sin z(t) = 0, \tag{30}
\]
where \( N(1, \Xi) \) is a Poisson random measure, and \( \Lambda(\cdot) \) is a Markov chain taking values in the state space \( \mathbb{S} = \{1, 2\} \) with the generator
\[
\mathbb{S} = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}.
\]
More precisely, let \( x_1(t) \) and \( x_2(t) \) denote respectively the position and velocity of a particle moving in \( \mathbb{R} \) at time \( t \geq 0 \). Introducing \( x(t) = (x_1(t), x_2(t))^T = (z(t), \dot{z}(t))^T \), we can rewrite (30) as the two-dimensional SDE
\[
\begin{align*}
dx(t) &= f(t, x(t), \Lambda(t)) dt + g(t, x(t), \Lambda(t)) dB(t) \\
&\quad + \int_{\Xi} h(t, x(t), \Lambda(t), y) N(dt, dy), \tag{31}
\end{align*}
\]
where \( f(t, x(t), i) = \begin{pmatrix} x_2(t) \\ -x_1(t) - d_i \sin x_1(t) - a_i x_2(t) \end{pmatrix} \),
g(t, x(t), i) = \begin{pmatrix} 0 \\ -b_i x_1(t) \end{pmatrix} \), \( h(t, x(t), i, y) = \begin{pmatrix} 0 \\ -c_i x_1(t) \end{pmatrix} \).
To design inverse optimal control stabilizers \( u \) for the system (31), the corresponding controlled system becomes

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= x_2(t), \\
\frac{dx_2(t)}{dt} &= (-x_1(t) - d_i \sin x_1(t) - a_i x_2(t) + u) dt - b_i x_1(t)dB(t) - \int_{\Xi} c_i x_1(t)N(dt, dy).
\end{align*}
\] (32)

The method of the designing is very technical so we break it into three steps. 

**Step 1:** Set \( \bar{x}_2 = x_2 - \hat{x}_2 \), where \( \hat{x}_2 \) is the virtual control of \( x_2 \). Choose 

\[
U_1(t, x_1, i) = \frac{1}{2} \theta_i |x_1|^2
\]

and \( \bar{x}_2 = -lx_1 \), where \( l > 0 \). We calculate 

\[
LU_1(t, x_1, i) = \theta_i x_1(\bar{x}_2 - lx_1) + \frac{1}{2} \sum_{j=1}^{2} q_{ij} \theta_j |x_1|^2,
\]

where \( LU_1(t, x_1, i) \) along with the first equation of (32). From the second equation of (32), we have 

\[
\begin{align*}
d\bar{x}_2(t) &= (-x_1(t) - d_i \sin x_1(t) - a_i x_2(t) + u + lx_2(t)) dt
\end{align*}
\]

\[
- b_i x_1(t)dB(t) - \int_{\Xi} c_i x_1(t)N(dt, dy).
\] (33)

**Step 2:** Set \( U_2(t, x, i) = U_1 + \frac{1}{2} \bar{\theta}_i |\bar{x}_2|^2 \). We calculate 

\[
LU_2(t, x, i) = \theta_i x_1(\bar{x}_2 - lx_1) + \frac{1}{2} \sum_{j=1}^{2} q_{ij} \theta_j |x_1|^2
\]

\[
+ \bar{\theta}_i \bar{x}_2(-x_1 - d_i \sin x_1 - a_i x_2 + u + lx_2) + \frac{1}{2} b_i^2 \bar{\theta}_i x_1^2
\]

\[
+ \frac{1}{2} \bar{\theta}_i \int_{\Xi} (|c_i x_1|^2 - 2\bar{x}_2 c_i x_1) \nu(dy) + \frac{1}{2} \sum_{j=1}^{2} q_{ij} \bar{\theta}_j |\bar{x}_2|^2. \] (34)

We design \( u \) as \( u = -\alpha_i \bar{\theta}_i \bar{x}_2 \), where \( \alpha_i \) is a positive number to be chosen later. Substituting \( u \) into (34) yields 

\[
LU_2(t, x, i)
\]

\[
= \theta_i x_1(\bar{x}_2 - lx_1) + \frac{1}{2} \sum_{j=1}^{2} q_{ij} \theta_j |x_1|^2
\]

\[
+ \bar{\theta}_i \bar{x}_2(-x_1 - d_i \sin x_1 - a_i x_2 - \alpha_i \bar{\theta}_i \bar{x}_2 + lx_2)
\]

\[
+ \frac{1}{2} b_i^2 \bar{\theta}_i x_1^2 + \frac{1}{2} \bar{\theta}_i \int_{\Xi} (|c_i x_1|^2 - 2\bar{x}_2 c_i x_1) \nu(dy) + \frac{1}{2} \sum_{j=1}^{2} q_{ij} \bar{\theta}_j |\bar{x}_2|^2
\]

\[
= \theta_i x_1(\bar{x}_2 - lx_1) + \frac{1}{2} \sum_{j=1}^{2} q_{ij} \theta_j |x_1|^2
\]

\[
+ \bar{\theta}_i \bar{x}_2(-x_1 - d_i \sin x_1 - (a_i - l)(\bar{x}_2 + lx_1) - \alpha_i \bar{\theta}_i \bar{x}_2)
\]

\[
+ \frac{1}{2} b_i^2 \bar{\theta}_i x_1^2 + \frac{1}{2} \bar{\theta}_i \int_{\Xi} (|c_i x_1|^2 - 2\bar{x}_2 c_i x_1) \nu(dy) + \frac{1}{2} \sum_{j=1}^{2} q_{ij} \bar{\theta}_j |\bar{x}_2|^2
\]

\[
\leq -m_1^2 |x_1|^2 - m_2^2 |\bar{x}_2|^2,
\] (35)
where

\[
m^i_1 = l\theta_i - \varepsilon_1 |\theta_i - \bar{\theta}_i + \bar{\theta}_i (|d_i|(a_i - l))| - \varepsilon_2 \bar{\theta}_i \int_\Xi |c_i|\nu(dy)
\]

\[-\frac{1}{2} b^2 \bar{\theta}_i - \frac{1}{2} \bar{\theta}_i \int_\Xi |c_i|^2\nu(dy) - \frac{1}{2} \sum_{j=1}^2 q_{ij}\bar{\theta}_j > 0,
\]

\[
m^i_2 = (a_i - l + \alpha_i\bar{\theta}_i)\bar{\theta}_i - \frac{1}{4\varepsilon_1} |\theta_i - \bar{\theta}_i + \bar{\theta}_i (|d_i|(a_i + l))| - \frac{1}{4\varepsilon_2} \bar{\theta}_i \int_\Xi |c_i|\nu(dy) - \frac{1}{2} \sum_{j=1}^2 q_{ij}\bar{\theta}_j > 0
\]

with \(\varepsilon, i = 1, 2\) being positive constants. Let \(X = (x_1, x_2)\). Thus (35) can write

\[
LU_2(t, X, i) \leq -m^iU_2(t, X, i),
\]

where \(m^i = 2(m^i_1 \wedge m^i_2)\).

**Step 3:** Comparing \(u = -\alpha_i\bar{\theta}_i\bar{x}_2\) with (4), we have

\[
G^{-1}(t, X, i)(\nabla U H(t, X, i))^T I_g(\|\nabla U H(t, X, i)^T G^{-\frac{1}{2}}(t, X, i)\|) = \alpha_i\bar{\theta}_i\bar{x}_2,
\]

where \(U = U_2\) and \(H(t, X, i) = (0, 1)^T\). Choosing \(\eta(r) = \frac{1}{2}r^2\), so \((\eta')^{-1}(r) = r\), \(I_g(r) = \frac{1}{2}r^2\). By (37), we obtain that

\[
G(t, X, i) = \frac{1}{2\alpha_i}.
\]

Then the inverse optimal controller is

\[
u^*(t, X, i) = -\alpha_i\bar{\theta}_i\bar{x}_2.
\]

5. **Conclusions.** The topic of this article is to extend jump diffusion systems to the case of regime-switching jump diffusion systems. In current research, an inverse optimal control strategy for SDEs with regime-switching jump diffusion problem is stated and we present optimality conditions for ensuring \(p\)-th moment inverse optimal stabilization and minimizing a cost functional. Specifically, we avoid solving Hamilton-Jacobi-Bellman equation. Finally, we employ the main theorem to attack the inverse optimal problem of stochastic Hamiltonian systems with Markov regime-switching jump diffusion.

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**REFERENCES**

[1] D. Applebaum, *Lévy Processes and Stochastic Calculus*, Cambridge Studies in Advanced Mathematics, 116. Cambridge University Press, Cambridge, 2009.

[2] D. Applebaum and M. Siakalli, Asymptotic stability of stochastic differential equations driven by Lévy noise, *J. Appl. Probab.*, 46 (2009), 1116–1129.

[3] J. Bao and C. Yuan, Stochastic population dynamics driven by Lévy noise, *J. Math. Anal. Appl.*, 391 (2012), 363–375.

[4] J. Bao and J. Shao, Permanence and extinction of regime-switching predator-prey models, *SIAM J. Math. Anal.*, 48 (2016), 725–739.
[5] Z. Chao, K. Wang, C. Zhu and Y. Zhu, Almost sure and moment exponential stability of regime-switching jump diffusions, SIAM J. Control Optim., 55 (2017), 3458–3488.

[6] H. Deng and M. Kratica, Stochastic nonlinear stabilization I: A backstepping design, Systems Control Lett., 32 (1997), 143–150.

[7] H. Deng and M. Krstić, Stochastic nonlinear stabilization-II: Inverse optimality, Systems Control Lett., 32 (1997), 151–159.

[8] K. D. Do, Global inverse optimal stabilization of stochastic nonholonomic systems, Systems Control Lett., 75 (2015), 41–55.

[9] K. D. Do, Inverse optimal control of stochastic systems driven by Lévy processes, Automatica, 107 (2019), 539–550.

[10] R. A. Freeman and P. V. Kokotovic, Inverse optimality in robust stabilization, SIAM J. Control Optim., 34 (1996), 1365–1391.

[11] G. Hardy, J. E. Littlewood and G. Polya, Inequalities, Cambridge University Press, 1989.

[12] L. Hu and X. Mao, Almost sure exponential stabilisation of stochastic systems by state-feedback control, Automatica J. IFAC, 44 (2008), 465–471.

[13] J. Hu, W. Liu, F. Deng and X. Mao, Advances in stabilization of hybrid stochastic differential equations by delay feedback control, SIAM J. Control Optim., 58 (2020), 735–754.

[14] R. C. Merton, Continuous-time finance, The Journal of Finance, 46 (1991), 1567–1570.

[15] X. Mao, Stabilization of continuous-time hybrid stochastic differential equations by discrete-time feedback control, Automatica J. IFAC, 49 (2013), 3677–3681.

[16] B. Øksendal and A. Sulem, Applied Stochastic Control of Jump Diffusions, Second edition. Universitext, Springer-Verlag, Berlin, 2005.

[17] J. Shao, Stabilization of regime-switching processes by feedback control based on discrete time observations, SIAM J. Control Optim., 55 (2017), 724–740.

[18] J. Shao and F. Xi, Stabilization of regime-switching processes by feedback control based on discrete time observations II: State-dependent case, SIAM J. Control Optim., 57 (2019), 1413–1439.

[19] H. Ji, J. Shao and F. Xi, Stability of regime-switching jump diffusion processes, J. Math. Anal. Appl., 484 (2020), 21pp.

[20] F. Wu, G. Yin and Z. Jin, Kolmogorov-type systems with regime-switching jump diffusion perturbations, Discrete Contin. Dyn. Syst. Ser. B, 21 (2016), 2293–2319.

[21] F. Xi, On the stability of jump-diffusions with Markovian switching, J. Math. Anal. Appl., 341 (2008), 588–600.

[22] F. Xi and C. Zhu, On Feller and strong feller properties and exponential ergodicity of regime switching jump diffusion processes with countable regimes, SIAM J. Control Optim., 55 (2017), 1789–1818.

[23] G. Yin and F. Xi, Stability of regime-switching jump diffusions, SIAM J. Control Optim., 48 (2010), 4525–4549.

[24] G. Yin and C. Zhu, Hybrid Switching Diffusions: Properties and Applications, Springer, New York, 2010.

[25] X. Zong, F. Wu, G. Yin and Z. Jin, Almost sure and pth-moment stability and stabilization of regime-switching jump diffusion systems, SIAM J. Control Optim., 52 (2014), 2595–2622.

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E-mail address: wenShengYin@126.com
E-mail address: jdcao@seu.edu.cn
E-mail address: renyong@126.com