Next-to SV resummed Drell–Yan cross section beyond leading-logarithm

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Abstract We present the resummed predictions for inclusive cross section for Drell–Yan (DY) production up to next-to-next-to leading logarithmic (NNLL) accuracy taking into account both soft virtual (SV) and next-to SV (NSV) threshold logarithms. We restrict ourselves to resummed contributions only from quark anti-quark ($q\bar{q}$) initiated channels. The resummation is performed in Mellin-$N$-space. We derive the $N$-dependent coefficients and the $N$-independent constants to desired accuracy for our study. The resummed results are matched through the minimal prescription procedure with the fixed-order results. We find that the resummation, taking into account the NSV terms, appreciably increases the cross section while decreasing the sensitivity to renormalisation scale. We observe that, at 13 TeV LHC energies, the SV + NSV resummation at NLL (NNLL) gives about 8% (2%) corrections respectively to the NLO (NNLO) results for the considered $Q$ range: 150–3500 GeV. In addition, the absence of quark gluon initiated contributions to NSV part in the resummed terms leaves large factorisation scale dependence indicating their importance at NSV level. We also study the numerical impact of $N$-independent constants and explore the ambiguity involved in exponentiating them. Finally we present our predictions for the neutral Drell–Yan process at various center of mass of energies.

1 Introduction

Standard Model (SM) has been extremely successful in describing the physics of elementary particles. Thanks to precise predictions of various observables from SM and their measurements at the collider experiments with unprecedented accuracy, we could validate the SM and at the same time set stringent constraints on the parameters present in various beyond SM (BSM) scenarios. While there have been strenuous efforts in search of new physics signatures at the large hadron collider (LHC), it is important to improve the level of precision in SM and BSM predictions to arrive at sensible conclusions. Precise predictions of observables require the use of complex mathematical techniques and a deeper understanding of the underlying theory. The spin-offs include new developments in various branches of mathematics and other fields, and in addition, the perturbative predictions dealing with Feynman loop and phase space integrals demonstrate rich mathematical structure in gauge theories. In particular, these results have shed light on the underlying structure of the ultraviolet (UV) and infrared (IR) sectors of the SM.

Among innumerable final states produced in hadron collisions, leptons are relatively easy to observe due to the clean environment and the corresponding measurements are less plagued by experimental uncertainties. The production of a pair of leptons, called Drell–Yan (DY) production is customarily used for luminosity monitoring at the hadron colliders. Theoretically, for very long, the observables in DY production belong to the category of “well studied” quantities in the SM and as well as in various BSMs. Note that the next-to-next-to leading order (NNLO) quantum chromodynamics (QCD) correction \cite{1–3} to this process was computed more than three decades ago, see also \cite{1–17}. Similar results are also available in certain BSMs, see \cite{18–20}. More recently, a series of results on inclusive cross sections for the production of a pair of leptons, single Z/W\textsuperscript{$\pm$} at N$^3$LO in perturbative QCD has become available \cite{17}. These corrections \cite{17} are already found to be tiny, and at the invariant mass $Q = 150$ GeV of a pair of leptons, they reduce the cross section by 1%. The renormalisation and factorisation scale...
uncertainties as well as the uncertainties from the choice of PDFs give about 2.5%.

Dedicated efforts like in [17] to obtain perturbative QCD results provide a theoretical laboratory to understand the structure of the perturbation series. Due to the complexity involved in performing many body phase space integrals in higher order computations, one resorts to the method of threshold expansion. For example, at every perturbative order in the strong coupling constant, the Feynman diagrams are computed as a series expansion around the threshold region denoted by \( Q^2 \approx \bar{s} \), where \( Q \) is the invariant mass of the pair of leptons produced in the partonic reaction whose center of mass energy is \( \sqrt{\bar{s}} \). Such an expansion not only provides reliable estimates of the higher order effects but also shed light on the logarithmic structure in higher order perturbative results. The leading terms in the threshold region contain contributions from virtual subprocesses as well as from soft gluons from real emissions. These are often called soft plus virtual contributions (SV). The SV terms at third order were known for some time, see [10–13,15,21–24]. In addition, using the resummation framework developed in [25,26] for threshold logarithms in SV contributions, several numerical studies were carried out to NNLL accuracy to improve the predictions, see [15,27–29]. In [30], we reported the numerical impact of threshold corrections within the resummation framework developed in [25,26] for SV logarithms in the invariant mass distribution of a pair of leptons produced in hadron colliders can be expressed as a convolution of perturbatively calculable coefficient functions (CFs), \( \Delta_{ab} \), and non-perturbative flux \( \Phi_{ab} \). That is,

\[
\frac{d\sigma}{dQ}(q^2, \tau) = \sigma^{(0)}_{DY}(q^2) + \int_1^{\frac{1}{2}} \frac{dz}{z} \Phi_{ab} \left( \frac{\tau}{z}, \mu_F^2, \mu_F^2 \right) \Delta_{ab}(q^2, \mu_F^2, z) .
\]

(1)

Here \( a, b = q, \bar{q}, g \) refer to incoming partonic states and \( \sigma^{(0)}_{DY} \) is the Born cross section:

\[
\sigma^{(0)}_{DY} = \frac{2\pi}{n_c} \left[ \frac{Q}{\sqrt{s}} \mathcal{F}^{(0)} \right],
\]

(2)

with \( Q = \sqrt{q^2} \) being the invariant mass of the lepton pairs and \( n_c = 3 \) in QCD. The factor \( \mathcal{F}^{(0)} \) is found to be

\[
\mathcal{F}^{(0)} = \frac{4\alpha^2}{3q^2} \left[ Q_q^2 + \frac{2q^2(q^2 - M_Z^2)}{((q^2 - M_Z^2)^2 + M_Z^2 \Gamma_Z^2)} c_w^2 s_w^2 Q_q g_e^V g_q^V + \frac{q^4}{((q^2 - M_Z^2)^2 + M_Z^2 \Gamma_Z^2)} c_w^2 s_w^2 (g_e^V)^2 + (g_A^V)^2 \right] .
\]

(3)

with \( \alpha \) being the fine structure constant and \( c_w, s_w \) are respectively the sine and cosine of Weinberg angle. \( M_Z \) and \( \Gamma_Z \) are the mass and the decay width of the Z-boson. Also,
where \( Q_a \) being the electric charge and \( T^3_a \) is the weak isospin of the electron or quarks. The flux \( \phi_{ab} \) is defined in terms of parton distribution functions (PDF) \( f_a, f_b \) of incoming partons \( a \) and \( b \) respectively at the factorisation scale \( \mu_F \):

\[
\phi_{ab}(\tau, \mu_F^2) = \int \frac{dy}{y} f_a(y, \mu_F^2) f_b(\frac{\tau}{y}, \mu_F^2)
\]

where \( \tau = q^2/S \) is the hadronic scaling variable with \( S \) being the square of hadronic center of mass energy. The fits of non-perturbative PDFs are available from NNLO level while the CFs are perturbatively calculable in powers of renormalized strong coupling constant, \( a_s = g_s^2/16\pi^2 \):

\[
\Delta_{ab}(q^2, \mu_R^2, \tau) = \sum_{i=0}^{\infty} a_s^i(\mu_R^2)\Delta_{ab}^{(i)}(q^2, \mu_R^2, \mu_F^2, \tau),
\]

where \( g_s \) is the QCD coupling constant and \( \mu_R \) is the renormalisation scale. Higher order corrections from QCD are inevitable at hadron colliders as they are often large and they reduce uncertainties resulting from the scales \( \mu_R, \mu_F \), the choice of PDFs and \( a_s \). Note that DY predictions are stable with respect to factorization (\( \mu_F \)) and renormalisation (\( \mu_R \)) scales already at NNLO in QCD. One finds 2% uncertainty in the predictions at NNLO for a canonical variation of factorization and renormalisation scales compared to NLO where it is about 9.2%. Similarly, the K-factor increases marginally from 1.25 at NLO to 1.28 at NNLO. At third order, the scale uncertainties can be determined from previous order DY results. However, such estimates make sense only if we include the scale independent parts which originate genuinely at third order. The recent third order results \cite{17} predict decrease in the cross section by less than 1% for \( Q = 150 \text{ GeV} \) and the uncertainties from scales and from the choice of PDFs give about 2.5%. While this is a great improvement in the predictions, it is important to estimate other missing higher order effects.

2.1 Threshold expansion

Computing CFs beyond N^3LO is highly challenging, hence one can look for alternative approaches to identify the dominant contributions to the CFs. In the case of leptons with high invariant mass, the threshold expansion of CFs around \( z \approx 1 \) was observed to be a good alternative to exact computation. Here, \( z \) is the partonic scaling variable defined by \( z = q^2/\hat{s} \), with \( \hat{s} \) the square of partonic center of mass energy. At the threshold, we decompose the CFs as

\[
\Delta_{ab}(q^2, \mu_F^2, z) = \delta_{ab}\Delta_{ab}^{SV}(q^2, \mu_F^2, z) + \Delta_{ab}^{reg}(q^2, \mu_F^2, z).
\]

Here \( \Delta_{ab}^{SV} \) denotes the soft-virtual (SV) corrections which comprises pure virtual contributions from \( q + \bar{q} \to l^+l^- \) and leading threshold contributions from quark anti-quark initiated partonic channels with at least one emission of on-shell parton. The former depends on the scale \( z \) through \( \delta(1-z) \) while the latter through both \( \delta(1-z) \) and plus distributions \( \delta_k(z) \) defined by

\[
\delta_k(z) = \left( \frac{\ln^k(1-z)}{1-z} \right)_+,
\]

and are integrable with any regular function \( f(z) \):

\[
\int_0^1 dz f(z) \left( \frac{\ln^k(1-z)}{1-z} \right)_+ = \int_0^1 dz \left( f(z) - f(1) \right) \times \left( \frac{\ln^k(1-z)}{1-z} \right),
\]

The second term in (7), \( \Delta_{ab}^{reg} \), refers to regular CFs, which contain terms of the form \( (1-z)^m \ln^k(1-z), m, k = 0, 1, \ldots, \infty \) with rational and irrational constants. Expanding both SV and regular CFs perturbatively in powers of \( a_s \),

\[
\Delta_{ab}^J(q^2, \mu_F^2, z) = \sum_{i=0}^{\infty} a_s^i(\mu_F^2)\Delta_{ab}^{(i)}(q^2, \mu_R^2, \mu_F^2, z),
\]

we find

\[
\Delta_{ab}^{SV(\delta)}(z) = \delta_{ab}\left( \Delta_{ab, SV}(1-z) + \sum_{k=0}^{2i-1} \Delta_{ab, \delta}^{(i)}(z) \right)
\]

and

\[
\Delta_{ab}^{reg(\delta)}(z) = \sum_{k=0}^{2i-1} \sum_{l=0}^{\infty} \Delta_{ab, l, k}^{reg}(1-z)^l \ln^k(1-z).
\]

The above expansion is called threshold expansion. The systematic threshold expansion of CFs in partonic scaling variable \( z \) will be useful provided the partonic flux \( \Phi_{ab}(\tau/\hat{s}) \) that multiplies them to give hadronic cross sections also dominates in the same region for a given hadronic scaling variable \( \tau \).

In general, the CFs in inclusive cross sections such as DY and Higgs productions, the energy scales \( q^2, \mu_R^2 \) and \( \mu_F^2 \) appear as logarithms, in addition to the partonic scaling variable \( z \) appearing through \( \delta(1-z) \), plus distributions \( \delta_k(z) \) and regular functions of \( z \). The coefficients of these terms are perturbatively computable and are controlled by set of differential equations that depend on the UV and IR anomalous dimensions. The solutions to these equations demonstrate rich universal structure which can be exploited to understand the structure of the coefficients to all orders in
perturbation theory. The IR structure of multi-loop amplitudes beyond two loops [50–53] (see [54,55] for a QFT with mixed gauge groups), of inclusive cross sections to third order [1,3,17,56–58] provide better understanding of CFs. For complete list of Higgs production in gluon fusion see [3,10–12,16,33,46,56,59–71] and [1–17] for Drell–Yan production.

Among the aforementioned terms that contribute to CFs, the SV terms are known for several observables. In particular for DY and Higgs productions beyond second order, see [10–13,15,21–24]. We obtain these results from process dependent pure virtual subprocesses and soft gluons from real emissions in the threshold region. The latter is a universal quantity in such a sense that they do not depend on the hard process under study, but only on the nature of incoming states. The soft and collinear modes in a scattering process can be captured at the Lagrangian level using effective theory approach. For example, soft-collinear effective theory can be captured at the Lagrangian level using effective theories. The soft and collinear modes in a scattering process is an enhancement of single-logarithms at large \( \Delta \) logarithms from threshold regions to all orders.

These contributions are often called next-to SV (NSV) or next-to-leading power (NLP) contributions. There have been several studies to understand the NSV terms in inclusive processes [35,38–45]. The physical evolution equation was improved earlier in the seminal works by Sterman [25] and Catani and Trentedue [26] through reorganisation of the large logarithms to all order in the perturbative series, called the threshold resummation. There is a vast literature on this which is applied to variety of processes, see [27,32,75–78] for Higgs production in gluon fusion, [79,80] for bottom quark annihilation, for DY [15,27–30] and for DIS and SIA of \( e^+e^- \) [81]. Threshold resummation is conveniently performed in Mellin space where the conjugate variable to \( z \) is \( N \). In Mellin space, all the \( z \)-space convolutions become normal products. The threshold limit in Mellin space is when \( N \) goes large, which corresponds to \( z \rightarrow 1 \) in \( z \)-space. Due to smallness of \( \alpha_s(\mu_R^2) \), one finds that the exponent in the \( N \)-space at each order in \( a_\epsilon(\mu_R^2) \) contains \( \delta(1) \) terms defined by \( \omega = 2\alpha_s(\mu_R^2)\beta_0 \ln N \). Such terms spoil the truncation of the perturbative series. Using renormalisation group improved solution to RG of \( \alpha_s \), one can reorganise the perturbative series in the exponent wherein \( \omega \) terms are summed up at every order in \( a_\epsilon \). Following [25,26] one finds that for \( \Delta_{\gamma, N} = \int_0^1 \Delta z z^{-1} \Delta_{\gamma}(z) \),

\[
\lim_{N \to \infty} N g_{\gamma}^0 (a_s(\mu_R^2)) = \lim_{N \to \infty} \Delta g_{\gamma}^0 (a_s(\mu_R^2)) = N g_{\gamma}^0 (\omega)
\]

\[+ \sum_{i=0}^{\infty} a_i^2 (\mu_R^2) g_{\gamma,i+2}^0 (\omega), \quad (13)\]

where \( g_{\gamma}^0 (a_s(\mu_R^2)) \) is \( N \) independent. Inclusion of successive terms in (13) predicts the leading-logarithms (LL), next-to-leading (NLL) etc. logarithms to all orders in \( a_\epsilon \). The exponents \( g_{\gamma}^0 (\omega) \) depend on process independent/universal IR anomalous dimensions while the constant \( g_{\gamma}^0 \) depends on the specific hard process. Results for the resummation of threshold logarithms in \( N \)-space up to third order are available for variety of inclusive processes such as DY and Higgs productions to perform threshold resummation to next-to-next-to-next-to leading logarithmic (N3LL) accuracy [15,27,30,80]. Threshold resummation is also found to play important role for differential observables like rapidity [19,26,82–84]. Inclusion of these effects are shown to improve the fixed-order results.

2.2 Next-to SV

Perturbative predictions of beyond SV terms are available for partonic sub processes up to third order for a variety of hadronic cross sections, namely Drell–Yan production and bottom quark as well as gluon initiated Higgs boson productions at the hadron colliders. Like SV terms, these results not only play an important role to precisely predict the respective observables, but also shed light on the structure of beyond SV terms in the threshold expansion at higher orders. Among these, let us consider a class of leading terms:

\[
\Delta_{\gamma, N}^N (z) = \sum_{i=0}^{\infty} a_i^2 (\mu_R^2) \Delta_{\gamma, N}^N (i)(z), \quad (14)
\]

where \( \Delta_{\gamma, N}^N (i)(z) \) is defined by setting \( l = 0 \) in (12), i.e.,

\[
\Delta_{\gamma, N}^N (i)(z) = \sum_{k=0}^{2l-1} \Delta_{\gamma, N}^{reg, i}(k) \ln^k (1 - z). \quad (15)
\]

These contributions are often called next-to SV (NSV) or next to leading power (NLP) contributions. There have been several studies to understand the NSV terms in inclusive processes [35,38–45]. The physical evolution equation was exploited earlier in the work by [36] to understand the effect of these terms. A remarkable development was made by Moch and Vogt in [37] (and [33,46]) using the physical evolution kernels (PEK) and the fixed-order results that are available for DIS, semi-inclusive \( e^+e^- \) annihilation and Drell–Yan production of a pair of leptons in hadron collisions. They found that in kernels that govern the physical equations, there is an enhancement of single-logarithms at large \( z \) up to third order. Conjecturing that it will hold true to all orders around \( z = 1 \), the logarithms were systematically resummed to all orders exactly like the way of SV resummation. In addition, absence of certain powers of \( \ln (1 - z) \) terms in the kernel at a given order in \( a_\epsilon \) can be used [37] to predict certain next-to SV logarithms at higher orders.
Recently, in [49], we investigated the structure of NSV terms present in the quark anti-quark initiated channels in the inclusive production of pair of leptons in Drell–Yan process and gluon/bottom anti-bottom initiated ones for Higgs boson production. We also analyzed the all-order perturbative structure of the NSV logarithms in the coefficient functions of deep inelastic scattering (DIS) and semi-inclusive $e^+e^−$ annihilation (SIA) processes in [85]. The formalism is even extended in the context of rapidity distributions to study the all-order behaviour of the NSV terms in addition to the SV distributions in the aforementioned threshold processes, namely Drell–Yan and Higgs production through gluon fusion and bottom quark annihilation in [86]. We used the well known factorisation properties and renormalisation group invariance along with certain universal structure of real and virtual contributions obtained through Sudakov K + G equation. Like, SV terms, NSV terms do demonstrate rich perturbative structure with certain universal anomalous dimensions. We found that the NSV logarithms in Mellin space can also be resummed in a systematic fashion to all orders in perturbation theory. fixed-order results known up to third order for DY productions can be used to determine the threshold exponents from the NSV logarithms with third order logarithmic accuracy. The present article explores the numerical impact of these resummed results taking into account both SV and NSV logarithms in the quark anti-quark initiated channels for the DY process at the LHC.

3 Resummation of SV + NSV

In [49], some of us have developed a theoretical formalism to systematically resum the NSV contributions in the diagonal channels of inclusive cross sections of Drell–Yan and Higgs boson productions at the hadron colliders. For completeness, we briefly describe the formalism [49], which shows how the building blocks of perturbative results in the threshold region can be organised using their factorisation properties and the logarithmic structure. Thanks to a set of differential equations that govern these building blocks, it is possible to sum up certain class of threshold logarithms to all orders in perturbation theory. In particular, the solutions to such equations lead to a compact integral representation in $z$-space that captures SV and NSV terms of inclusive rates in these processes. The integral representation can be conveniently used in Mellin N-space to resum $O(1)$ terms that show up in large SV and NSV contributions at every order to obtain reliable theoretical predictions at colliders.

We begin by defining $\Delta_q$ as a sum of SV and NSV contributions to diagonal channels:

$$\Delta_q(z) = \Delta_q^{SV}(z) + \Delta_q^{NSV}(z)$$

(16)

where the SV part is defined in (11) and NSV in (14). Using the mass factorisation that separates collinear singular part from the bare partonic cross sections and the UV and IR renormalisation group equations that various building blocks satisfy we can cast the CFs of inclusive cross sections in dimensional regularisation ($n = 4 + \varepsilon$) as

$$\Delta_q(q^2, \mu_R^2, \mu_F^2, z) = \mathcal{G} \exp \left( \Psi^q(q^2, \mu_R^2, \mu_F^2, z, \varepsilon) \right) \bigg|_{\varepsilon=0} ,$$

(17)

where $\Psi^q$ is finite in the limit $\varepsilon \to 0$ and is given by

$$\Psi^q(q^2, \mu_R^2, \mu_F^2, z, \varepsilon) = \left( \ln \left( Z_{UV, q} (\hat{a}_s, \mu, \mu_R^2, \varepsilon) \right)^2 \right. + \ln \left( \hat{F}_q (\hat{a}_s, \mu^2, -q^2, \varepsilon) \right)^2 \delta(1-z)$$

$$+ 2 \Phi_q (\hat{a}_s, \mu^2, q^2, z, \varepsilon) - 2 \mathcal{G} \ln \Gamma_{qq} (\hat{a}_s, \mu^2, \mu_F^2, z, \varepsilon),$$

(18)

where $Z_{UV, q}$ is the overall renormalisation constant which is unity for vector/axial vector interactions in quark anti-quark initiated channels. The bare strong coupling constant, $\hat{a}_s = \frac{\hat{g}_s^2}{16\pi^2}$, with $\hat{g}_s$, the bare QCD coupling constant and the scale $\mu$ results from dimensional regularisation. The square of the form factor (FF), $\hat{F}_q$, encodes pure virtual contributions to $q + \bar{q} \to l^+ l^−$ while the soft-collinear function, $\Phi_q$, contains contributions from remaining partonic subprocesses normalised by square of the form factor. Thanks to the fact that mass factorisation terms required for the SV and NSV contributions to diagonal channels depend only on diagonal kernels $\Gamma_{qq}$ where we need to keep only diagonal splitting functions $P_{qq}$, the logarithm of these kernels completely decouples from the rest.

The symbol “$\mathcal{G}$” refers to convolution, which acting on any exponential of a function $f(z)$ takes the following expansion:

$$\mathcal{G} e^{f(z)} = \delta(1-z) + \frac{1}{1!} f(z) + \frac{1}{2!} \left( f \otimes f \right) (z) + \cdots$$

(19)

Since we have restricted ourselves to SV + NSV contributions to $\Delta_q$, we keep only those terms that are proportional to SV distributions namely $\delta(1-z)$, $\mathcal{G} f(z)$ and NSV terms $\ln^i(1-z)$ with $i = 0, 1, \ldots$ and drop rest of the terms resulting from the convolutions.

The form factor, soft-collinear function and Altarelli–Parisi (AP) kernels that contribute to $\Delta_q$ are computable order by order in $a_s$ in perturbation theory. One finds that each of them demonstrates rich infrared structure through certain differential equations. For example, the form factor satisfies Sudakov’s $K + G$ differential equation while the mass factorisation kernels satisfy AP evolution equations.
In addition, they are independently renormalisation group invariants. These differential equations are controlled by universal UV and IR anomalous dimensions that are perturbatively calculable. Thanks to these differential equations and the fact that $\Delta_q$ is finite, one finds that soft-collinear function $\Phi_q$ also satisfies $K + G$ like differential equation. The solution to the form factor is expressed in terms of cusp ($A^q$), soft ($f^q$), collinear ($B^q$) anomalous dimensions and process dependent constants, while for the mass factorisation kernels, one finds the solution in terms of diagonal AP splitting functions which contain only $\delta(1 - z)$, $\mathcal{D}_0(z)$ and $\ln^j(1 - z)$, $j = 0, 1$ terms. Unlike the form factor and AP kernels, the solution to $\Phi_q$ is hard to obtain without the knowledge of their kernels $\vec{K}$ and $\vec{G}$ (see [49]). The singular kernel $\vec{K}$ can be determined from singular terms of FF and $G$ (see [49]).

The singular $Cq$ terms of $\Phi_q$ where $\Psi$ is the solution to third order results of AP kernels while the finite part is obtained from the fixed-order IR divergences as poles in $\varepsilon$ dimensional regularisation. The resulting $z$ dependent solution of $\Phi_q$ depends on process independent anomalous dimensions $A^q$, $f^q$, $C^q$ and $D^q$ and certain process dependent quantities. Combining the solutions from all the differential equations one obtains an all order exponentiation of SV and NSV contributions to $\Delta_q$ as given in (17). While each piece contains both UV and IR divergences as poles in $\varepsilon$, the divergences cancel among themselves when $\varepsilon \to 0$, leaving finite $\Psi^q$. In [49], an integral representation for the function $\Psi^q$ in terms of $z$ was obtained:

$$\Delta_q(q^2, \mu_R^2, \mu_F^2, z) = C^q_0(q^2, \mu_R^2, \mu_F^2) \times \mathcal{C} \exp \left( 2\Psi^q_0(q^2, \mu_F^2, z) \right),$$

where

$$\Psi^q_0(q^2, \mu_F^2, z) = \frac{1}{2} \int_{\mu_F^2}^{\infty} \frac{d\lambda^2}{\lambda^2} \sum_{\omega} P^q_\omega(a_s(\lambda^2), z)$$

$$+ \mathcal{D}^q(a_s(q^2(1 - z)^2), z),$$

with

$$\mathcal{D}^q(a_s(q^2(1 - z)^2), z) = \left( \frac{1}{1 - z} \mathcal{C}^q_{SV}(a_s(q^2(1 - z)^2)) \right) + \psi_f, q(a_s(q^2(1 - z)^2), z).$$

The coefficient $C^q_0$ is $z$ independent and is expanded in powers of $a_s(\mu_R^2)$ as

$$C^q_0(q^2, \mu_R^2, \mu_F^2) = \sum_{i=0}^{\infty} a_s^i(\mu_R^2) C^q_{0i}(q^2, \mu_R^2, \mu_F^2).$$

The results for $C^q_0$ can be found in [15] and the coefficients $C^q_{0i}$ are given in Appendix C.1. The splitting function $P'_{qq}$ is related to the AP splitting functions $P_{qq}(z, \mu_F^2)$. Expanding the latter around $z = 1$ and dropping those terms that do not contribute to SV + NSV, we find

$$P_{qq}(z, a_s(\mu_F^2)) = 2B^q(a_s(\mu_F^2)) \delta(1 - z) + P'_{qq}(z, a_s(\mu_F^2)).$$

where,

$$P'_{qq}(z, a_s(\mu_F^2)) = 2 \left[ A^q(a_s(\mu_F^2)) \mathcal{D}_0(z) + C^q(a_s(\mu_F^2)) \ln(1 - z) + D^q(a_s(\mu_F^2)) \right].$$

(25)

The constants $C^q$ and $D^q$ can be obtained from the splitting functions $P'_{qq}$ which are known to three loops in QCD [87,88] (see [87–96] for the lower order ones). The cusp, soft and the collinear anomalous dimensions and the constants $C^q$ and $D^q$ are expanded in powers of $a_s(\mu_F^2)$:

$$X^q(a_s(\mu_F^2)) = \sum_{i=1}^{\infty} a_s^i(\mu_F^2) X^q_i, \quad X = A, f, B, C, D$$

(26)

where $X^q_i$ to third order are available in [87,88] and are listed in Appendix A. The function $\mathcal{C}^q_{SV}(a_s(q^2(1 - z)^2))$ is related to the threshold exponent $\mathcal{D}^q(a_s(q^2(1 - z)^2))$ via Eq. (46) of [11] (see Appendix A). The function $\psi_{f, q}$ in powers of $a_s$ is given by

$$\psi_{f, q}(a_s(q^2(1 - z)^2), z) = \sum_{i=1}^{\infty} a_s^i(q^2(1 - z)^2)$$

$$\times \sum_{k=0}^{i} \psi_{k, i}^{(k)} \ln^k(1 - z).$$

(27)

The coefficients $\psi_{k, i}^{(k)}$ are known to third order and are listed in Appendix B (see also [49]).

The resumation of threshold logarithms can be conveniently done in Mellin space, where $z \to 1$ translates to large $N$ limit. In the latter, $\mathcal{O}(1)$ terms from $\omega = 2\beta_0 a_s(\mu_R^2) \ln N$ show up at every order in $a_s$ spoiling the truncation of perturbative series in the exponent. This can be resolved by reorganising the series using the integral representation (21) and the resummed strong coupling constant. To include SV and NSV terms in the resummation in Mellin space, we need to keep $\ln N$ as well as $\mathcal{O}(1/N)$ terms in the large $N$ limit. We find that (21) can correctly predict only SV and NSV terms while the predictions beyond the NSV terms namely $\mathcal{O}(1) \ln^j(1 - z)$; $n, j \geq 0$ in z-space and terms of $\mathcal{O}(1/N^2)$ in $N$-space will not be correct. The Mellin moment of $\Delta_q$ was obtained in [49] and is given by
\[ \Delta_{q,N}(q^2, \mu_R^2, \mu_F^2) = C_0^q(q^2, \mu_R^2, \mu_F^2) \exp \left( \Psi_{q,N}^q(q^2, \mu_F^2) \right), \tag{28} \]

where

\[ \Psi_{q,N}^q(q^2, \mu_F^2) = 2 \int_0^1 dz z^{N-1} \Psi_{\beta^q}(q^2, \mu_F^2, z). \tag{29} \]

Also, \( C_0^q \) are \( N \)-independent constants coming from FF and the \( \delta(1-z) \) part of soft-collinear function and AP kernels. Note that the \( \delta(1-z) \) pieces in \( z \)-space translates to \( N \)-independent pieces in Mellin space. And the Mellin transformation of the plus distributions, given in (29), give rise to \( \ln N \) and \( N \)-independent constants. Hence expressing \( \Psi_{q,N}^q \) as

\[ \Psi_{q,N}^q = \Psi_{SV,N}^q + \Psi_{NSV,N}^q \tag{30} \]

where \( \Psi_{SV,N}^q \) contains \( \ln^j N, j = 0, 1, \ldots \) terms and while \( \Psi_{NSV,N}^q \) contains terms of the form \( \ln(N \ln N, j = 0, 1, \ldots) \). We find that \( \Psi_{SV,N}^q \) takes the form:

\[ \Psi_{SV,N}^q = \ln(g_0^q(a_s(\mu_R^2))) + g_1^q(\omega) \ln N \]

\[ + \sum_{i=0}^{\infty} a_i^q(\mu_R^2) g_{i+2}^q(\omega). \tag{31} \]

Here the \( g_i^q \) coefficients are universal and they depend only on the initial partons. The constants in \( g_0^q \) are the \( N \)-independent pieces obtained after Mellin transformation of \( \Psi_{SV}^q \) and they satisfy the condition \( \Psi_{SV,N}^q - \ln(g_0^q) = 0 \) when \( N = 1 \). Expanding them in powers of \( a_s \) we get,

\[ \ln(g_0^q(a_s(\mu_R^2))) = \sum_{i=1}^{\infty} a_i^q(\mu_R^2) g_{0,i}^q. \tag{32} \]

These exponents agree with those given in [26,27,80], and they are listed in the Appendices C.2 and C.4. In standard \( N \)-approach we absorb the \( N \)-independent pieces \( g^q_{0,i} \) into \( C_0^q \) and collectively define it as

\[ \tilde{g}_0^q(q^2, \mu_R^2, \mu_F^2) = C_0^q(q^2, \mu_R^2, \mu_F^2) g_0^q(a_s(\mu_R^2)). \tag{33} \]

Thus the resulting quantity comprises of \( \delta(1-z) \) contributions from the form factor, soft-collinear function, AP kernels and \( N \)-independent part of the Mellin moment of the distributions in \( \Psi_{\beta^q}(q^2, \mu_F^2, z) \).

The coefficients \( g_{0,i}^q \) are listed in the Appendix C.3.

The function \( \Psi_{NSV,N}^q \) in (30) is given by

\[ \Psi_{NSV,N}^q = \frac{1}{N} \sum_{i=0}^{\infty} a_i^q(\mu_R^2) \left( \tilde{g}_{i+1}^q(\omega) + \bar{h}_i^q(\omega, N) \right). \tag{34} \]

with

\[ h_i^q(\omega, N) = \sum_{k=0}^{i} h_{ik}^q(\omega) \ln^k N. \tag{35} \]

where \( \tilde{g}_{0}^q(\omega) \) and \( h_{0}^q(\omega) \) are presented in the Appendices C.5 and C.6 respectively. In each exponents, \( g_{0}^q(\omega) \), \( \bar{g}_{i}^q(\omega) \) and \( h_{ik}^q(\omega) \), we resum \( \mathcal{O}(1) \) term \( \omega \) in Mellin space to all orders in perturbation theory. This is possible because of the argument in the coupling constant \( a_i(q^2(1-z)^2) \) resulting the integrals and from the function \( \mathcal{Z}^q \).

In the SV part of the resummed result, the intrinsic ambiguity that exists while dealing with what needs to be exponentiated gives scope to explore their impact. Among different prescriptions, the standard approach is to exponentiate only large-\( N \) pieces coming from the threshold region. Also, for large \( N \), the expansion of Euler Gamma functions gives Euler–Mascheroni constant, \( \gamma_E \) and considering these large effects, one can exponentiate \( N \), which is defined as \( N = \exp(\gamma_E) \), instead of \( N \) without disturbing the fixed-order predictions. Numerically, however this can make a difference at the leading logarithmic accuracy as was already seen in [97] where the perturbative convergence was shown to improve with \( N \) terms. In a different scheme, called Soft exponentiation one exponentiates the Mellin moment of soft-collinear function [11,21] which contains all the plus distributions and \( \delta(1-z) \) terms. Alternatively, one can also exponentiate the complete form factor along with the soft-collinear function in the Mellin space. This approach was explored in [32,78] to study the inclusive Higgs boson production in gluon fusion. It was found to predict results that are less sensitive to the unphysical scales compared to the standard threshold approach. This approach is theoretically justified because the form factor satisfies the Sudakov K + G type equation [11,21,98–101] whose solution is an exponential of \( N \) independent constant. For the numerical study of DY production this approach was used in [102]. We give the expressions for different resummation schemes below.

- Standard \( N \) exponentiation: In this scheme, we exponentiate only the large-\( N \) pieces that contribute to the CFs and the exponent is devoid of \( N \)-independent pieces. Note that this will only change the \( \Psi_{SV,N}^q \). The \( \Psi_{NSV,N}^q \) will remain same as it contains only \( N \)-dependent terms. Hence we write:

\[ \Delta_{q,N}(q^2, \omega) = \tilde{g}_{0}^q(q^2) \exp \left( G_{SV,N}^q(q^2, \omega) + \Psi_{NSV,N}^q(q^2, \omega) \right), \tag{36} \]

where NSV part is defined in (34). And the SV part is given as,
which can be obtained from the Mellin transformation of $\Psi_{q}$ and keeping only those terms that vanish when $N = 1$. This is the standard approach adopted all along this paper and we compare other prescriptions with the $N$-exponentiation scheme in later sections.

Standard $\tilde{N}$ exponentiation: Here, the large logarithms are expressed in functions of $\tilde{N}$ instead of $N$, where $\tilde{N} = N \exp(\gamma_{E})$. Hence along with large $N$-pieces, the $\gamma_{E}$ terms are also taken into exponentiation as they also contribute to $O(1)$ terms. The resulting exponent takes the form:

$$\Delta_{q,N}(q^{2}, \omega) = \tilde{g}_{0}^{\tilde{N}}(q^{2}, \mu_{R}^{2}, \mu_{F}^{2}) \exp \left( G_{SV,N}^{q}(q^{2}, \omega) \right. \right.$$ 

$$\left. + \Psi_{NSV,N}^{q}(q^{2}, \omega) \right). \quad (38)$$

Here the $\tilde{g}_{0}^{\tilde{N}}(q^{2}, \mu_{R}^{2}, \mu_{F}^{2})$ are given by

$$\tilde{g}_{0}^{\tilde{N}}(q^{2}, \mu_{R}^{2}, \mu_{F}^{2}) = C_{0}^{q}(q^{2}, \mu_{R}^{2}, \mu_{F}^{2}) \tilde{g}_{0}(a_{i}(\mu_{R}^{2})), \quad (39)$$

where $\tilde{g}_{0}$ results from the $N$-independent part of the Mellin transformation of the distributions and the $\gamma_{E}$'s are all absorbed into the $\tilde{N}$-dependent functions. In doing so, the $\tilde{N}$-dependent part of SV, i.e., $G_{SV,N}^{q}$, is obtained by setting $\Psi_{SV,N}^{q} - \ln \tilde{g}_{0}^{\tilde{N}} = 0$ when $\tilde{N} = 1$. Hence, $G_{SV,N}^{q}$ and $\Psi_{NSV,N}^{q}$ are given by

$$G_{SV,N}^{q} = g_{0}^{q}(\tilde{\omega}) \ln \tilde{N} + \sum_{i=0}^{\infty} a_{i}^{q}(\mu_{R}^{2}) \tilde{g}_{i+2}^{q}(\tilde{\omega}), \quad (40)$$

$$\Psi_{NSV,N}^{q} = \frac{1}{N} \sum_{i=0}^{\infty} \tilde{h}_{i}^{q}(\mu_{R}^{2}) \left( \tilde{g}_{i+1}^{q}(\tilde{\omega}) + h_{i}^{q}(\tilde{\omega}, \tilde{N}) \right), \quad (41)$$

with

$$h_{i}^{q}(\tilde{\omega}, \tilde{N}) = \sum_{k=0}^{i} h_{ik}^{q}(\tilde{\omega}) \ln \tilde{N}. \quad (42)$$

The expansion parameter $\tilde{\omega} = 2\beta_{0}a_{i}(\mu_{R}^{2}) \ln \tilde{N}$. Numerically, this can make difference at every logarithmic accuracy. For the SV, the perturbative convergence was shown to improve with $\tilde{N}$ terms in $[30]$. We will address the impact of the same in NSV case in next section.

Soft exponentiation: Another scheme one can adopt is the Soft exponentiation, where we exponentiate the complete finite part of soft-collinear function $\Phi_{q}$. This includes the $N$-independent pieces arising after the Mellin transformation of $\Psi_{q}^{g}$, along with the $N$-dependent ones which we used in Standard $N$-exponentiation. For this scheme the resummed result takes the form:

$$\Delta_{q,N}(q^{2}, \omega) = \tilde{g}_{0}^{\tilde{N}}(q^{2}) \exp \left( \Psi_{SV,N}^{q}(q^{2}, \omega) \right.$$ 

$$\left. + \Psi_{NSV,N}^{q}(q^{2}, \omega) \right). \quad (43)$$

with

$$\Psi_{SV,N}^{q}(q^{2}, \omega) = \ln N g_{1}^{q}(\omega) + \sum_{i=0}^{\infty} a_{i}^{q} \tilde{g}_{i+2}(q^{2}, \omega). \quad (44)$$

Here the complete finite part coming from the soft-collinear function is absorbed into the resummed exponent, which thereby gives rise to the exponents, $g_{i}^{q}$. The remaining $N$-independent terms coming from finite part of form factor and AP kernels contribute to $\tilde{g}_{0}^{\tilde{N}}$, whose expansion in powers of $a_{i}$ is given as:

$$\tilde{g}_{0}(q^{2}) = 1 + \sum_{i=1}^{\infty} a_{i} \tilde{g}_{0}^{q}(q^{2}). \quad (45)$$

The $N$-independent constants $\tilde{g}_{0}^{q}(q^{2})$ and the resummed exponents $g_{i}^{q}(\omega)$ are listed in Appendix E.

All exponentiation: In light of (17), which came out as a consequence to the first-order differential equations satisfied by each of the building blocks results in an all-order exponential structure for $\Delta_{q,N}$. Hence it is natural to study the numerical impact of entire contribution taken in to Mellin space, which is done in All exponentiation scheme. The resummed result then takes the form:

$$\Delta_{q,N}(q^{2}, \omega) = \exp \left( \Psi_{SV,N}^{q}(q^{2}, \omega) + \Psi_{NSV,N}^{q}(q^{2}, \omega) \right). \quad (46)$$

where

$$\Psi_{SV,N}^{q}(q^{2}, \omega) = \ln N g_{1}^{q}(q^{2}) + \sum_{i=0}^{\infty} a_{i}^{q} \tilde{g}_{i+2}(q^{2}, \omega). \quad (47)$$

The resummed exponents $g_{i}^{q}(\omega)$ contain both $N$-dependent and independent terms and are listed in Appendix F. This scheme was explored in $[32,78]$ to study the inclusive cross section for the production of Higgs boson in gluon fusion at the LHC. For similar study for the DY in DIS and $\overline{\text{MS}}$ schemes, see $[102]$.

In $[30]$, we had studied how various schemes discussed so far can affect the predictions of invariant mass distribution of lepton pairs, inclusive $Z$ and $W^{\pm}$ production rates. In the present paper we extend this analysis in the presence of resummed NSV exponent and study the numerical impact.
on the production of a pair of leptons in DY process at the LHC. At NNLO level, we have the contributions from all the channels. Our numerical predictions are based on fixed-order NNLO results for the CFs and on parton distribution functions known to NNLO accuracy. The resummed results are matched to the fixed-order result in order to avoid any double counting of threshold logarithms. The resummed result at a given accuracy, say $N_{\text{LL}}$, is computed by taking the difference between the resummed result and the same truncated up to order $a_s^n$. Hence, it contains contributions from the SV and NSV terms to all orders in perturbation theory starting from $a_s^{n+1}$:

$$
\sigma_{N_{\text{LL}}}^{\text{NLO}+N_{\text{LL}}} = \sigma_{N_{\text{LL}}}^{\text{NLO}} + \sigma^{(0)}
\times \sum_{ab \in \{q, \bar{q}\}} \int_{c-i\epsilon}^{c+i\epsilon} \frac{dN}{2\pi i} (t)^{-N} \delta_{ab} f_a N(\mu_T^2) f_b N(\mu_T^2)
\times \left( \Delta_{q,N}^{N_{\text{LL}}} - \Delta_{q,N} \left|_{N_{\text{NLO}}} \right. \right),
$$

(48)

where $\sigma_N$ is the Mellin moment of $d\sigma/dQ$. The Mellin space PDF ($f_i, N$) can be evolved using QCD-PEGASUS [103]. Alternatively, we use the technique described in [26,66] to directly deal with PDFs in the $z$-space. The contour $c$ in the Mellin inversion can be chosen according to Minimal prescription [75] procedure. In the above (48) the second term represents the resummed result truncated to $N_{\text{LL}}$ order. To distinguish between SV and SV + NSV resummation, all along the paper, we denote the former by $N_{\text{NLO}}$ and the latter by $N_{\text{NLL}}$ for the $n^\text{th}$ level logarithmic accuracy.

In Tables 1 and 2, we list the resummed exponents which are required to predict the tower of SV and NSV logarithms respectively in $\Delta_{q,N}$ at a given logarithmic accuracy. Let us first review the predictions for the SV logarithms which are already known in the literature [15,25–30]. As can be seen from Table 1, using the first set of resummed exponents $\{\hat{g}_0, \hat{g}_1\}$ which constitute to the SV-LL resummation, we can predict the highest SV logarithm of $\Delta_{q,N}^{(i)}$ at every order in $a_s$ in perturbation theory, i.e., $L_{N}^{(i)}$ at the order $a_s^i$ for all $i > 1$. Similarly using the second set of resummed exponents $\{\hat{g}_0, \hat{g}_1\}$ along with the first set, one can predict the next-to-highest SV logarithms to all orders, i.e., two towers of logarithms namely $\{L_{N}^{(i−1)}, L_{N}^{(i−2)}\}$ in $\Delta_{q,N}^{(i)}$ for all $i > 2$. These towers of logarithms belong to the SV-NLL resummation. In general, using the $n$-th set $\{\hat{g}_0, \hat{g}_1, \hat{g}_2, h_0\}$ along with the previous sets, we can predict the highest $(2n + 1)$ towers of SV-logarithms, at every order in $a_s^i$ for all $i > n + 1$ with $n = 0,1,2\ldots$ and these towers constitute to the SV-$N_{\text{NLL}}$ resummation.

Now let us turn to the predictions for NSV logarithms present in $\Delta_{q,N}$. From Table 2, we can see that the first set of resummed exponents $\{\hat{g}_0, \hat{g}_1, \hat{g}_2, h_0\}$ which contributes to the $\Pi$ resummation, facilitates the prediction of highest NSV logarithms of $\Delta_{q,N}^{(i)}$ to all orders in perturbation theory, i.e., $L_{N}^{(i−1)}$ at $a^i_s$ for all $i > 1$. Furthermore, the second set of resummed exponents $\{\hat{g}_0, \hat{g}_1, \hat{g}_2, h_0\}$ along with the first set, predicts the next-to-highest NSV logarithms besides the highest NSV logarithms to all orders, namely the tower of $L_{N}^{(i−2)}$ for $\Delta_{q,N}^{(i)}$ with $i > 2$. This tower of logarithms belongs to the $N_{\text{LL}}$ resummation. In general, using the $n$-th set $\{\hat{g}_0, \hat{g}_1, \hat{g}_2, h_0\}$ along with the previous sets, we can predict the highest $(n + 1)$ towers of NSV logarithms at every order in $a_s^i$ for all $i > n + 1$ with $n = 0,1,2\ldots$. These $n + 1$ towers of NSV logarithms constitute to the $N_{\text{NLL}}$ resummation. In summary, the predictive nature of SV and NSV logarithms start to differ from next-to-leading logarithmic accuracy (which is $N_{\text{LL}}$ for SV and $N_{\text{LL}}$ for NSV) onwards. For the SV case, beyond LL accuracy, we can predict additional two towers of logarithms at each logarithmic accuracy to all orders in $a_s$. On the other hand, for the NSV case, only one additional tower of logarithms can be predicted at each logarithmic accuracy. This implies that the NSV resummation is less predictive compared to the SV counterpart, however this is the inherent property of the subleading logarithms present in the NSV part.

In the following, we expand the resummed results in power series in $a_s$ at various logarithmic accuracy and compare them against the known fixed-order results in the literature. Prior to this, we compare our result at $\Pi$ accuracy against that of [104]. In [104], within the framework of soft-collinear effective theory (SCET), the authors have obtained leading logarithmic terms at NSV for the quark–antiquark production channel of the DY process to all orders in $a_s$. This was achieved by extending the factorisation properties of the cross section to NSV level and using renormalisation group equations of NSV operators and soft functions. Using our $N$-space result, in the $\Pi$ approximation, we have

$$
\Delta_{q,N}^{\Pi} = \hat{g}_0 \exp \left[ \ln N \hat{g}_1 (\omega) + \frac{1}{N} \left( \hat{g}_1 (\omega) + h_0 (\omega, N) \right) \right].
$$

(49)

After expanding the exponents in powers of $a_s$ and keeping only terms of $O(1/N)$, we obtain

$$
\Delta_{q,N}^{\Pi} = \exp \left[ 8C_F a_s \left( \ln^2 N + \frac{\ln N}{N} \right) \right].
$$

(50)

The above $N$-space result can be Mellin inverted to $z$-space and it reads as

$$
\Delta_{q,N}^{\Pi, z} = \Delta_{q,N}^{\Pi, z, \text{SV}} - 16C_F a_s \exp \left[ 8C_F a_s \ln^2 (1-z) \right] \ln (1-z).
$$

(51)
The above result agrees exactly with Eq. (4.2) of [104] for \( Q = q \).

Next, we expand the resummed expression at \( \text{LL} \) accuracy given in (50) up to \( a_{q}^{4} \) (N4LO) and compare the predictions for leading NSV logarithms against those from fixed-order predictions. Note that, as can be seen from Table 2, the \( \text{LL} \) resummation which comprises of only one loop anomalous dimensions and SV + NSV coefficients from fixed-order NLO results, predicts the leading logarithms \( \ln^{3} N / N \), \( \ln^{2} N / N \), \( \ln^{7} N / N \) etc at \( a_{q}^{2} \) (NNLO), \( a_{q}^{3} \) (N3LO), \( a_{q}^{4} \) (N4LO) and so on respectively.

For example, at \( a_{q}^{2} \) (NNLO), in \( N \)-space, we obtain

\[
\Delta^{(2)}_{q,N|\text{LL}} = \Delta^{\text{SV},(2)}_{q,N|\text{LL}} + \left( \ln^{3} N / N \right) \left( 64 C_{F}^{2} \right),
\]

and the Mellin inverted result in \( z \)-space is

\[
\Delta^{(2)}_{q,z|\text{LL}} = \Delta^{\text{SV},(2)}_{q,z|\text{LL}} - \ln^{3}(1-z) \left( 128 C_{F}^{2} \right).
\]

The above predictions at NNLO level agrees with the known results obtained in [1–3]. Our prediction at \( a_{q}^{3} \) (N3LO) in \( N \) and \( z \)-spaces are

\[
\Delta^{(3)}_{q,N|\text{LL}} = \Delta^{\text{SV},(3)}_{q,N|\text{LL}} + \left( \ln^{5} N / N \right) \left( 256 C_{F}^{3} \right),
\]

\[
\Delta^{(3)}_{q,z|\text{LL}} = \Delta^{\text{SV},(3)}_{q,z|\text{LL}} - \ln^{5}(1-z) \left( 512 C_{F}^{3} \right),
\]

and are agreement with the results given in [58]. Now at \( a_{q}^{4} \) (N4LO), our predictions in \( N \) and \( z \)-spaces read

\[
\Delta^{(4)}_{q,N|\text{LL}} = \Delta^{\text{SV},(4)}_{q,N|\text{LL}} + \frac{\ln^{7} N}{N} \left[ \frac{2048}{3} C_{F}^{4} \right],
\]

\[
\Delta^{(4)}_{q,z|\text{LL}} = \Delta^{\text{SV},(4)}_{q,z|\text{LL}} + \ln^{7}(1-z) \left[ -\frac{4096}{3} C_{F}^{4} \right],
\]

and are found to be in agreement with the results obtained using the physical evolution kernels (PEK) approach by Moch and Vogt in [37]. In all of the above expressions, we have set \( \mu_{e}^{2} = \mu_{F}^{2} = q^{2} \).

We now consider the predictions of resummed result at \( \text{NLL} \) and \( \text{NNLL} \) up to order \( a_{q}^{4} \). At \( \text{NLL} \), the resummed expression takes the following form

\[
\Delta^{\text{NLL}}_{q,N} = \left( g_{0,0}^{q} a_{q}^{q} g_{0,1}^{q} \right) \exp \left[ \ln N \ g_{q}^{q}(\omega) + q_{q}^{q}(\omega) \right]
\]

\[
+ \frac{1}{N} \left( g_{1}^{q}(\omega) a_{q}^{q} g_{1}^{q}(\omega) + h_{2}^{q}(\omega, N) + a_{q}^{4} h_{4}^{q}(\omega, N) \right),
\]

where the resummed exponents can be found in the Appendices C.5 and C.6. Note that at \( \text{NLL} \) accuracy, we require anomalous dimensions up to two loops and second order SV + NSV coefficients obtained from NNLO results. Expanding the resummed result in powers of \( a_{q}^{4} \), we obtain the next-to leading logarithms \( \ln^{3} N / N \), \( \ln^{5} N / N \) etc at \( a_{q}^{3} \) (N3LO), \( a_{q}^{4} \) (N4LO) and so on respectively. At \( a_{q}^{4} \) (N4LO), in \( N \)

### Table 1

| Resummed exponents | Predictions – SV Logarithms | Logarithmic accuracy |
|--------------------|-----------------------------|----------------------|
| \( \tilde{g}_{0,0}^{q} \), \( \tilde{g}_{1}^{q} \) | \( L_{q}^{6} \), \( L_{q}^{8} \), \( L_{q}^{10} \), \( L_{q}^{12} \), \( L_{q}^{14} \), \( L_{q}^{16} \), \( L_{q}^{18} \), \( L_{q}^{20} \), \( L_{q}^{22} \), \( L_{q}^{24} \) | LL |

### Table 2

| Resummed exponents | Predictions – NSV Logarithms | Logarithmic accuracy |
|--------------------|-----------------------------|----------------------|
| \( \tilde{g}_{0,0}^{q} \), \( \tilde{g}_{1}^{q} \), \( h_{0}^{q} \) | \( L_{q}^{7} \), \( L_{q}^{8} \), \( L_{q}^{9} \), \( L_{q}^{10} \), \( L_{q}^{11} \), \( L_{q}^{12} \), \( L_{q}^{13} \), \( L_{q}^{14} \), \( L_{q}^{15} \), \( L_{q}^{16} \), \( L_{q}^{17} \), \( L_{q}^{18} \), \( L_{q}^{19} \), \( L_{q}^{20} \), \( L_{q}^{21} \), \( L_{q}^{22} \), \( L_{q}^{23} \), \( L_{q}^{24} \) | NNLL |
and $z$-spaces, the predictions are given below by setting $\mu_R^2 = \mu_F^2 = q^2$.

$$
\Delta_{q,N}^{(3)}|_{\text{NNLL}} = \Delta_{q,N}^{(3)}|_{\text{NLO}} + \Delta_{q,N}^{(3)}|_{\text{NLL}} + \frac{\ln^4 N}{N} \left\{ -\frac{3520}{9} C_A C_F^2 + \frac{27}{9} \frac{640}{9} C_F^2 n_f + \left( 248 + 1280 \gamma_E \right) C_F^2 \right\}, \\
\Delta_{q,z}^{(3)}|_{\text{NNLL}} = \Delta_{q,z}^{(3)}|_{\text{NLO}} + \Delta_{q,z}^{(3)}|_{\text{NLL}} + \ln^4 (1-z) \left\{ -\frac{1280}{9} C_F^2 n_f + 1728 C^2_f \right\},
$$

where $\gamma_E$ is the Euler–Mascheroni constant which arises due to the Mellin transformation in $N$-space. The above prediction agrees with the fixed-order result at order $\alpha_s^4$ in $[58]$. The predictions at $\alpha_s^4$ (N$^4$LO) in $N$ and $z$-spaces are as follows

$$
\Delta_{q,N}^{(4)}|_{\text{NNLL}} = \Delta_{q,N}^{(4)}|_{\text{NLO}} + \Delta_{q,N}^{(4)}|_{\text{NLL}} + \frac{\ln^6 N}{N} \left\{ \frac{19712}{9} C_A C_F^2 + \frac{19712}{9} C_A C_F^2 + \left( 1280 + \frac{14336}{3} \gamma_E \right) C_F^2 \right\}, \\
\Delta_{q,z}^{(4)}|_{\text{NNLL}} = \Delta_{q,z}^{(4)}|_{\text{NLO}} + \Delta_{q,z}^{(4)}|_{\text{NLL}} + \ln^4 (1-z) \left\{ \frac{39424}{9} C_A C_F^2 + \frac{7168}{9} C_F^2 n_f + 1728 C^2_f \right\},
$$

where we have set $\mu_R^2 = \mu_F^2 = q^2$. The above predictions at $\alpha_s^4$ are found to be in agreement with the results computed using the physical evolution kernels (PEK) approach in $[37]$.

Finally, using the NNLL resummation, which further embeds the three loop anomalous dimensions and third order SV + NSV coefficients obtained from N$^3$LO results, we predict the next-to-next-to-leading logarithms $\ln^5 N$ at $\alpha_s^4$ (N$^4$LO), $\ln^4 N$ at $\alpha_s^5$ (N$^5$LO) and so on. The resummed expression at NNLL accuracy is given by

$$
\Delta_{q,N}^{\text{NNLL}} = (\tilde{g}_{s,0,0}^q + a_s \tilde{g}_{s,0,1}^q + a_s^2 \tilde{g}_{s,0,2}^q) \exp \left[ \ln N \ g_1^q (\omega) + g_2^q (\omega) + a_s \ g_3^q (\omega) + \frac{1}{N} \ g_4^q (\omega) + a_s \ g_5^q (\omega) + a_s^2 \ g_6^q (\omega, N) + a_s h_1^q (\omega, N) + a_s^2 h_2^q (\omega, N) \right].
$$

The prediction for the next-to-next-to-leading logarithm $\ln^5 N$ at $\alpha_s^4$ (N$^4$LO) in $N$ and $z$-spaces is provided below

$$
\Delta_{q,N}^{\text{NNLL}} = \Delta_{q,N}^{(4)}|_{\text{NNLL}} + \Delta_{q,N}^{(4)}|_{\text{NLL}} + \frac{\ln^5 N}{N} \left\{ \frac{61952}{27} C_F^2 C_A + \frac{296192}{27} \gamma_E - 1536 \gamma_2 \right\} C_F^2 C_A + \left( -4992 + 10752 \gamma_2 + 14336 \gamma_2^2 + 4096 \gamma_2 \right) C_F^2 + \left( 22528 + \frac{7168}{3} \gamma_E \right) n_f C_F^2 + 2048 \gamma_2 n_f C_F^2 \right\}, \\
\Delta_{q,z}^{\text{NNLL}} = \Delta_{q,z}^{(4)}|_{\text{NNLL}} + \Delta_{q,z}^{(4)}|_{\text{NLL}} + \ln^5 (1-z) \left\{ -\frac{123904}{27} C_F^2 C_A - \frac{805376}{27} \right\} C_F^2 + \left( -3072 \gamma_2 \right) C_F^2 C_A + \left( -9088 + 20480 \gamma_2 \right) C_F^2 + \left( 45056 + \frac{139520}{27} n_f C_F^2 - \frac{4096}{27} n_f^2 C_F^2 \right) \right\},
$$

where we have set $\mu_R^2 = \mu_F^2 = q^2$. The above predictions at $\alpha_s^4$ agree with those of $[37]$ predicted using physical evolution equations. In the next section we present the numerical results of our predictions along with the scale uncertainties for the neutral DY process at the LHC.

### 4 Phenomenology

In this section, we perform a detailed numerical study on the impact of resummed soft virtual plus next-to-sofa soft virtual (SV + NSV) results for the production of di-leptons in neutral DY process at the LHC. We include all the partonic channels at the Fixed-order (FO) up to NNLO with off-shell photon and $Z$ boson intermediate states. We restrict ourselves to center of mass energy of 13 TeV at the LHC, however our analysis can be extended to other energies as well as to other colliders. We use the following electro-weak parameters for the vector boson masses and widths, Weinberg angle ($\theta_w$) and the fine structure constant ($\alpha$):

$$
m_Z = 91.1876 \text{ GeV}, \quad \Gamma_Z = 2.4952 \text{ GeV}, \quad \sin^2 \theta_w = 0.22343, \quad \alpha = 1/128.
$$

The parton distribution functions are directly taken from the lhapdf $[105]$ routine. All results are obtained using the MMHT2014 $[106]$ parton densities throughout. The strong coupling constant is evolved to the renormalisation scale $\mu_R$ using the three-loop QCD beta function in the MS-scheme with $n_f = 5$ active massless quark flavours.

We begin with a discussion on the relative contributions of SV and NSV terms in the fixed-order results at the hadronic level. We carry out this analysis in $N$-space as the resum-
mation is performed in the $N$-space. The partonic coefficient function, $\Delta_q$ given in (16), comprises of functions namely, SV distributions, $\{\delta(1 - z), \Phi_{k}(z)\}$ and NSV logarithms, $\ln^k(1 - z)$ in $z$-space. After performing the Mellin transformation on $\Delta_q$, the SV distributions and NSV logarithms transform to $\{1, \ln^k N, \ln^k N/N\}$ in the Mellin $N$-space. Here, ‘1’ stands for the Mellin moment of the SV distribution $\delta(1 - z)$. Before we present the corresponding contributions of SV and NSV terms to hadronic cross sections, we plot the following integral as a function of $\tau$:

$$\mathcal{F}(\tau) = \int_{\tau}^{\infty} \frac{dz}{z^2} \Phi_{q\bar{q}} \left(\frac{\tau}{z}\right) \mathcal{M}^{-1}[\mathcal{G}(N)],$$

where $\mathcal{G}(N) = \{1, \ln N, \ln N/N\}$.

(62)

In the above expression, $\mathcal{M}^{-1}$ stands for the Mellin inversion. The plot given in Fig. 1 demonstrates the hierarchical behaviour of the terms in the threshold expansion at the numerical level, in particular, it reflects to the fact that at the threshold, $\mathcal{F}$ gets larger contribution from the SV terms $\{1, \ln N\}$ as compared to the subleading $\ln N/N$ terms. This hierarchy remains the same when the corresponding coefficients from the perturbative results of CFs are taken into account along with these functions. This is shown in Table 3 for $\Delta_q^{(2)}$, i.e., at NNLO, we find that the SV logarithms contribute to 34.5% of the Born contribution, whereas the NSV distributions give 7.87% in $N$-space. We notice that this trend is same at other known orders too. For example at NLO, the total SV contributions amount to 22% whereas NSV gives rise to a total of 6% of the Born contribution in $N$-space for $Q = 200$ GeV. Similarly Table 4 shows the contributions from SV and NSV logarithms at the $N^3$LO level. Although the contribution of NSV logarithms is subdominant to the SV counterpart in $N$-space up to $N^3$LO, it is still numerically sizeable and hence cannot be ignored.

Having studied the numerical relevance of subleading NSV logarithms in the fixed-order results, we now turn to assess the impact of their resummation on the cross sections. Besides the theoretical motivation of resumming the large enhancements arising from these logarithms, it would be interesting to see how phenomenologically important the resummed NSV logarithms are, in addition to the well established SV resummation for the Drell–Yan cross section. We begin the analysis by addressing the following questions:

- In comparison to the fixed-order corrections, how large are the SV + NSV resummed effects on the cross sections?
- How do the resummed NSV terms change the predictions of SV resummed result?
- What are the impacts of different schemes on the SV + NSV resummation?

We will discuss each of these questions in great detail in subsequent sections.
4.1 Fixed-order vs resummed results

In the following we study how the inclusion of SV+NSV resummation modifies the predictions from fixed-order results for inclusive DY di-lepton pair production. For this purpose, we get the matched predictions by appropriately including the leading, the next-to-leading and the next-to-next-to-leading resummed results with the corresponding fixed-order results. By investigating how sensitive is the fixed-order results for inclusive DY di-lepton pair production. For this purpose, we get the matched predictions by appropriately including the leading, the next-to-leading and the next-to-next-to-leading resummed results with the corresponding fixed-order results. By investigating how sensitive is the fixed-order results for inclusive DY di-lepton pair production. For this purpose, we get the matched predictions by appropriately including the leading, the next-to-leading and the next-to-next-to-leading resummed results with the corresponding fixed-order results.

In Table 5 we present the K factors resulting from both fixed as well as resummed contributions at three different values of $Q$, namely $Q = 500, 1000, 2000$ GeV. We find that there is an increment of $[39.5\%, 36.5\%]$ in the cross section when we go from LO to NNLO for the $Q$ values $[500, 2000]$ GeV respectively. The inclusion of the resummed results, for the same values of $Q$, increases, the LO by $[6.2\%, 10.62\%]$ when we include $\mu_R, \mu_F$ the NLO by $[3.7\%, 5.2\%]$ due to NLL and the NNLO by $[0.94\%, 1.2\%]$ due to NNLL. This is reflected in Fig. 2, where the resummed curves can be found to lie above their corresponding fixed-order ones signifying the enhancement due to the resummed corrections.

Interestingly, we can also see from Table 5, that the K factors at NLO + NLL are closer to those of NNLO hinting that NLL from quark part mimics the contributions from entire second order. The resummed curves at NLO + NLL and NNLO + NNLL are closer compared to the fixed-order ones, namely NLO and NNLO. This accounts for the fact that the addition of resummed effect improves the reliability of perturbative predictions. The K factors at NNLO and NNLO + NNLL are more closer than those at NLO and NLO + NLL. This is attributed to the fact that the resummed correction decreases as we go for higher order resummed contributions.

7-point scale uncertainties of the resummed results

Both fixed-order as well as resummed results contain renormalisation and factorisation scales which are unphysical. We now turn to assess the impact of these scales on our predictions. The dependence on these scales quantifies the corresponding errors due to their presence. The standard approach to estimate this error is to use the canonical 7-point variation, where $\mu = \{\mu_F, \mu_R\}$ is varied in the range $\frac{1}{2} \leq \frac{\mu}{Q} \leq 2$, keeping the ratio $\mu_R/\mu_F$ not larger than 2 and smaller than 1/2.

The left panel of Fig. 3 contains the invariant mass distributions obtained using fixed-order CFs as a function of $\tau$ and the bands are due to 7-point scale variation, while the right panel is obtained using resummed results at various logarithmic accuracy. We find that the width of the resummed band at NLO + NLL is lesser than that of the corresponding fixed-order ones from $Q = 1000$ GeV onwards but the width of the NNLO+NNLL doesn’t show much improvement against the fixed-order ones. The reason for this large scale uncertainty can be attributed to the fact that the resummed predictions lack the off-diagonal counter part. We will discuss more on this point in detail in subsequent analysis while considering the effect of both the scales independently.

In Table 6 we quote both fixed-order and resummed predictions at various logarithmic accuracies along with asymmetric errors resulting from 7-point scale variation for two values of $Q$, namely $Q = 1000$ GeV and $Q = 2000$ GeV. We find that there is a systematic enhancement of the cross sections as we increase the order of perturbation. For example, there is an increment of $24.2\%$ when going from LO + LL to NLO + NLL accuracy, which further improves by $0.58\%$ at NNLO + NNLL for $Q = 2000$ GeV. In addition, the scale uncertainty gets reduced significantly while going from LO + LL to NNLO + NNLL. This is also reflected in Fig. 3 (right panel), where one finds the uncertainty band of NNLO + NNLL is contained within the NLO + NLL band throughout the considered $Q$-range. This was not the case for the fixed-order predictions, where NNLO band was found to differ from the NLO one at high energies. This hints
Table 5  The K-factor values for resummed result in comparison to the fixed-order ones

| \( \mu_R = \mu_F = Q\) (GeV) | LO + LL | NLO | NLO + NLL | NNLO | NNLO + NNLL |
|-----------------------------|---------|-----|-----------|------|-------------|
| 500                         | 1.0624  | 1.3425 | 1.3925 | 1.3950 | 1.4082      |
| 1000                        | 1.0728  | 1.3464 | 1.3995 | 1.4004 | 1.4138      |
| 2000                        | 1.1062  | 1.3064 | 1.3739 | 1.3652 | 1.3818      |

Fig. 3  7-point scale variation of the resummed result against fixed-order around the central scale choice \( \mu_R = \mu_F = (1, 1)Q \) for 13 TeV LHC. The dashed lines refer to the corresponding central scale \( Q = \mu_R = \mu_F \) at each order.

Table 6  Values of resummed cross section in \( 10^{-5}\) pb/GeV at various orders in comparison to the fixed-order results at different central scales \( Q = \mu_R = \mu_F = 1000 \) and 2000 GeV for 13 TeV LHC

| \( Q \) | LO | LO + LL | NLO | NLO + NLL | NNLO | NNLO + NNLL |
|-------|----|---------|-----|-----------|------|-------------|
| 1000  | 2.3476^{+4.10\%}_{-3.92\%} | 2.5184^{+4.25\%}_{-4.25\%} | 3.1609^{+1.79\%}_{-1.69\%} | 3.2857^{+2.08\%}_{-1.38\%} | 3.2876^{+0.20\%}_{-0.31\%} | 3.3191^{+1.13\%}_{-0.36\%} |
| 2000  | 0.0501^{+8.50\%}_{-7.46\%} | 0.0554^{+9.10\%}_{-7.91\%} | 0.0654^{+2.83\%}_{-2.98\%} | 0.0688^{+1.43\%}_{-1.23\%} | 0.0684^{+0.37\%}_{-0.62\%} | 0.0692^{+0.89\%}_{-0.78\%} |

to the notable NSV contributions coming from the resummation effects in diagonal channels. These conclusions might change if we include resummed effects from off-diagonal channels which are currently not available.

In the above analysis, fixed-order results used for the numerical predictions contained all the partonic channels while the resummed contributions are only from quark anti-quark initiated channels. In the absence of resummed contributions, under the 7-point scale variations, the scale dependence is expected to go down as we increase the order of perturbation. However, this may not be the case if we include resummed effects only in quark anti-quark initiated channels. The quark gluon and gluon gluon initiated channels are also important as they contribute significantly to the cross section and more importantly they improve the stability of perturbative predictions under the scale variations. In order to understand the role of these partonic contributions, we drop them in the previous analysis restricting to quark anti-quark initiated contributions and then compare the outcomes given in Fig. 4 and in Table 7 against Fig. 3 and the Table 6. We find that there is a systematic enhancement of 28.19% when going from LO + LL to NLO_q\bar{q} + NLL and 2.08% from NLO_q\bar{q} + NLL to NNLO_q\bar{q} + NNLL for \( Q = 2000 \) GeV. These increments are more compared to those in the case where all the channels are included in the fixed-order part. This stems from the fact that the cancellation which results due to the inclusion of other channels, in particular the qg-channel, is not considered here. As a result of which the error bands in Fig. 4 of NNLO_q\bar{q} + NNLL is wider than that of NLO_q\bar{q} + NLL in comparison to the resummed curves shown in Fig. 3 (left panel).

Hence in the 7-point variation, we find that the resummed result shows a systematic enhancement of the cross section as well as reduction of the uncertainties with the inclusion of each logarithmic corrections. But the scale uncertainties of the resummed result shows much improvement at the NLO+
Fig. 4 7-point scale variation of the resummed result against fixed-order around the central scale choice \((\mu_R, \mu_F) = (1, 1)Q\) for \(q\bar{q}\) channel.

Table 7 Values of resummed cross section at various orders in comparison to the fixed-order results in \(10^{-5}\) pb/GeV for \(q\bar{q}\) channel at different central scales \(Q = \mu_R = \mu_F = 1000\) and 2000 GeV for 13 TeV LHC

| \(Q = \mu_R = \mu_F\) (GeV) | NLO\(_{q\bar{q}}\) | NLO\(_{q\bar{q}} + \) NLL | NNLO\(_{q\bar{q}}\) | NNLO\(_{q\bar{q}} + \) NNLL |
|-------------------------------|-----------------|-----------------|-----------------|-----------------|
| 1000                          | 3.3204 \(\pm\) 1.91\% | 3.4452 \(\pm\) 4.18\% | 3.5260 \(\pm\) 3.46\% | 3.5576 \(\pm\) 4.46\% |
| 2000                          | 0.0676 \(\pm\) 2.17\% | 0.07102 \(\pm\) 2.33\% | 0.0717 \(\pm\) 1.63\% | 0.0725 \(\pm\) 2.37\% |

Fig. 5 \(\mu_F\) scale variation of the resummed results against the fixed-order with the scale \(\mu_R\) held fixed at \(Q\) for 13 TeV LHC.
NNLL than at the NNLO + NNLL level. To understand this and also the cause of the uncertainties better we now turn to analyze the effect of each scale individually on the resummed result.

Uncertainties of the resummed results with respect to \( \mu_R \) and \( \mu_F \)

So far, we have studied the importance of both fixed-order as well as resummed contributions using the K factor and the uncertainties arising from unphysical scales \( \mu_R \) and \( \mu_F \). The numbers in the Table 5 demonstrate the importance of not only the fixed-order contributions but also the effects coming from the resummed ones. The resummed results comprises of both SV and NSV logarithms and the importance of large coefficients of the NSV terms was illustrated for the case of fixed-order corrections in Table 3. From the analysis of 7-point variation, we have shown that inclusion of leading, next-to-leading and next-to-next-to-leading collinear logarithms, in addition to the SV distributions, from the \( q\bar{q} \) channel, to all orders in perturbation theory through resummation significantly enhances the cross section. But this enhancement comes with a price, namely the uncertainties resulting from unphysical scales. As we have seen in Fig. 3 that while the width of the resummed band at NLO + NNLL is less than that of the corresponding fixed-order ones from \( Q = 1000 \) GeV onwards, the width of the NNLO + NNLL doesn’t show much improvement even at higher values of \( Q \) as against the fixed-order result. In order to understand the reason behind this we aim to study the impact of the two scales separately. This is our next task.

The dependence of the cross section on \( \mu_F \) is plotted in Fig. 5, as a function of \( \tau \) with \( \mu_R \) held fixed at \( Q \). The bands are obtained by varying the scale \( \mu_F \) by a factor of two up and down around the central scale \( \mu_R = \mu_F = Q \). Here, the resummed bands look similar to that of Fig. 3 (right panel), however the width of NLO + NNLL and NNLO + NNLL bands become slightly thinner as compared to the 7-point scale variation. This suggest that the contribution to the width of the bands in Fig. 3 mainly comes from the uncertainties arising from the \( \mu_F \) variations. For instance, the uncertainty at NLO + NNLL with respect to \( \mu_F \) variation is \(+1.44\%\) whereas the uncertainties arising from the 7-point variation is \,+1.43\%\) for \( Q = 2000 \) GeV. Similarly at NNLO + NNLL, the uncertainty arising from \( \mu_F \) variation is \,+0.79\%\) and from 7-point variation is \,+0.89\%\) for the same value of \( Q \). Now let us compare the \( \mu_F \) uncertainty of the resummed result with respect to their fixed-order. We find that the width of the NLO band decreases with the inclusion of NNLL from \( Q = 1600 \) GeV onwards. But the \( \mu_F \) uncertainty for NNLO increases at NNLO + NNLL. The reason behind this stems from the contribution coming from \( qg \)-channel. The one-loop correction from the \( q\bar{q} \)-channel is 22.09\% of the NLO cross section, whereas the correction at the same order from the \( qg \)-channel is about \(-5.04\%\) of the NLO cross section. Now at NLO + NNLL, we sum up the collinear logarithms from the diagonal channel which is also the dominating channel at NLO and hence the improvement through resummation. But the scenario is different at NNLO level. The \( a_s^2 \) corrections from \( q\bar{q} \) and \( qg \)-channel contribute to 4.86\% and \(-2.47\%\) respectively to the NNLO cross section along with the other sub-dominating channels. However the sources of collinear logarithms at the threshold limit is only from \( q\bar{q} \) and \( qg \)-channels. Hence at NNLO there is a bigger cancellation between \( q\bar{q} \) and \( qg \)-channels than that at NLO. The cancellation at NNLO + NNLL is not matched due to the unavailability of the \( qg \) resummed collinear logarithms. Thus the \( \mu_F \) variation in Fig. 5 reflects the role of the other channels and the need for \( qg \) resummation for betterment of the result.

Let us try to understand why the inclusion of resummed contributions in quark gluon initiated channels is important. Note that in the above analysis, we had taken all the partonic channels for the fixed-order part and only quark anti-quark initiated channels for the resummed part. We found that in the fixed-order, the \( \mu_F \) dependence from the PDFs and from quark anti-quark initiated as well as from the quark-gluon initiated processes are expected to compensate each other according to renormalisation group equation with respect to factorisation scale. However, in the resummed part, this will not happen due the absence of resummed quark gluon counter part and this is the reason why one gets larger \( \mu_F \) dependence in the predictions at NNLO + NNLL level. A symmetric analysis where we keep only quark anti-quark initiated channels both in fixed and resummed contributions can demonstrate this better and hence the Fig. 6. As one can see easily, the bands in both fixed-order and resummed predictions are wider compared to those in the Fig. 5 indicating the importance of quark gluon initiated channel both in fixed as well as resummed parts.

Figure 7 shows the dependence of the cross section on \( \mu_R \) keeping \( \mu_F \) fixed at \( Q \). The bands are obtained by varying the scale \( \mu_R \) by a factor of two up and down around the central scale \( \mu_R = \mu_F = Q \). We observe that at NNLO + NNLL the error band becomes substantially thinner as compared to Fig. 5. This is because each partonic channel is invariant under \( \mu_R \) variation when taken to all orders and hence inclusion of more corrections within a channel is expected to reduce the uncertainty. We find that the \( \mu_R \) uncertainty at NLO + NNLL ranges between \(+1.35\%\) and \(-1.23\%\) whereas for NLO it is between \,+1.46\%\) and \,-1.28\%\) for \( Q = 2000 \) GeV. And at NNLO + NNLL the uncertainty is found to be \,+0.02\%\) and for NNLO it is \,+0.37\%\) for the same value of \( Q \). Hence from Fig. 7 one can see that \( \mu_R \) dependence goes down for \( Q = 800 \) GeV onwards.
Since different partonic channels do not mix under the variation of $\mu_R$, the symmetric analysis of keeping only quark anti-quark initiated channels will have similar behaviour as that of the case where other partonic channels are included in the fixed-order, which is given in Fig. 8. The $\mu_R$ uncertainty is significantly decreased as we go from NLO to NLO + NLL to NNLO to NNLO + NNLL. We find the uncertainty at NNLO $+0.48\% -0.57\%$ gets improved to $+0.0006\% -0.20\%$ for $Q = 1000$ GeV. And this improvement continues to grow even for higher values of $Q$. Therefore the inclusion of resummed result reduces the $\mu_R$ uncertainty remarkably as compared to the fixed-order ones.

Hence in conclusion to this section, we find that the uncertainty, which earlier manifested in the 7-point variation, shown in Fig. 3, was largely due to the $\mu_F$ uncertainty. Now as we know that different partonic channels mix under the variation of factorisation scale $\mu_F$ and so any uncertainty arising due to its variation only hints towards the “uncompensated” contributions from other channels. The fact that the resummed result at NLO + NLL shows improvement as compared to the fixed-order, emphasizes the importance of the resummed NSV logarithms. Similarly at the NNLO + NNLL level the large cancellation between different channels, mainly $q\bar{q}$ and $qg$, equally hints towards the importance of the collinear logarithms of the $qg$ channel. Similarly for the $\mu_R$ variation, where the dependency is supposed to get better with the inclusion of more corrections within a
partonic channel, we see a substantial improvement due to SV + NSV resummation in comparison to the fixed-order result. But now that we have seen the effect of the combined resummed result on the fixed-order let us analyze which part of the SV + NSV resummation, i.e., whether its the resummation of the distribution or of the NSV logarithms, plays the dominant role in any kind of improvement discussed so far.

4.2 SV resummation vs SV + NSV resummation

In the earlier section we have made a quantitative comparison of SV + NSV resummed results against the fixed-order ones. We found that there is a significant enhancement of the cross section and the $\mu_R$ scale uncertainty gets substantially improved with the inclusion of the resummed corrections. We also found that the uncertainties related to the $\mu_F$ variation shows betterment at NLO + NLL for higher $Q$ values but not at the NNLO+NNLL level. Now, in this section we turn to a detailed analysis on the inclusion of NSV resummation over SV resummation so as to estimate the effect of resummed collinear logarithms from the $q\bar{q}$-channel.

We begin with the K-factor, which is presented in Table 8, to examine the impact of resummed NSV logarithms. Keeping all the partonic channels in the fixed-order, we find that the inclusion of the resummed NSV logarithms enhances the SV resummed corrections significantly throughout the considered $Q$ range. In particular, for $Q = 2000$ GeV, there is a considerable amount of increment of 2.08% when we go from NLL to NLL and 0.64% from NNLL to NNLL.

Figure 9 demonstrates this for a wider range of $Q$ values. We can also observe that the curves corresponding to SV + NSV resummed results at NLO + NLL and NNLO + NNLL are closer compared to the SV counter parts, accounting for the perturbative convergence when NSV effects are taken into consideration. It also shows that the resummed correction decreases when we go to higher logarithmic accuracy as the resummed curves at the second order are closer than that of the first order ones.

We now analyze the scale uncertainties of the SV + NSV resummed results in comparison to the SV resummation. In Fig. 10 we plot the SV and SV + NSV resummed results at various logarithmic accuracy as a function of $\tau$ taking into account the respective 7-point scale variations. Note that the SV + NSV resummed predictions are more sensitive to the scales compared to the SV ones. For instance, for $Q = 2000$ GeV, we find the scale uncertainty of SV resummed results, which was in between $+0.32\%$ $-0.27\%$, is enhanced to $+0.89\%$ $-0.78\%$ when the resummed NSV corrections are added. This further hints towards our earlier findings in Sect. 4.1, that the absence of complete resummation of collinear logarithms, which includes both diagonal as well as off-diagonal contributions, causes the SV + NSV resummed predictions to be more sensitive to the unphysical scales.

However the SV resummation, which gets contributions only from the diagonal channel, shows improvement in 7-point scale uncertainties for higher values of $Q$. Similar findings are observed even if we restrict ourselves to only diagonal ($q\bar{q}$) channel as shown in right panel of Fig. 10. For comparison, we present the SV and SV + NSV resummed results along with the fixed-order predictions for $Q = 1000$, 2000 GeV with their respective percentage scale uncertainties in Table 9 at the second order.

The width of the band is expected to reduce when we fix the factorisation scale and vary only the renormalisation...
scale, since the effect of later gets cancelled within a given partonic channel. Hence we now turn to analyze the effect of each these scales separately on the resummed result.

As seen in Sect. 4.1, the large uncertainties in the 7-point variations were mostly from the $\mu_F$ variation and hence we compare in Fig. 11 the $\mu_F$ sensitivity of SV and SV + NSV resummed cross sections as a function of $\tau$, keeping $\mu_R$ fixed at $Q$. Note here that the bands are obtained by varying only $\mu_F$ by a factor of 2 up and down around the central scale $\mu_R = \mu_F = Q$. We find, that where the $\mu_F$ band of fixed-order, i.e., at NLO, starts widening for higher values of $Q$, the resummed bands i.e., at NLO + NLL and NLO + NNLL, shows a systematic reduction in the width for the same range of $Q$. But there is larger reduction in the SV resummed case in comparison to the SV + NSV. On the other hand the $\mu_F$ band of NNLO is better than both SV and SV + NSV for the considered $Q$ range. This clearly hints towards the level of cancellation between different partonic channels. At NNLO there is a significant cancellation between $q\bar{q}$ and $g\bar{g}$-channel which leads to the reduction of the $\mu_F$ width from NLO to NNLO and the SV resummation, which resums the distributions present only in the diagonal channel, doesn’t require any compensation from other channels. But when it comes to SV + NSV resummation there is a compensation required from the other partonic channels to enhance the $\mu_F$ stability. Note that in all these analysis, we studied the impact of fixed-order and resummed CFs using same PDF sets to desired logarithmic accuracy for both of them. For studies related to $\mu_F$ variations, it is worthwhile to consider resummed PDFs if they are available.

However, this is not the case if we keep the $\mu_F$ intact and study the sensitivity due to $\mu_R$ variation as the latter effects are supposed to cancel within a partonic channel. This is depicted in Fig. 12, where we vary the $\mu_R$ dependence of cross section as a function of $\tau$ with $\mu_F$ held fixed at $Q$. As can be seen from the plot, the width of the $\mu_R$ band is significantly reduced in case of resummed bands in comparison to the fixed-order band, leading to a reliable predictions from the resummation. And among the resummed bands, the uncertainty of SV + NSV resummed results are comparable to those of SV resummation. For instance, for $Q = 1000$ GeV, the uncertainty is $12\%$ whereas at NNLO + NLL whereas it is $15\%$ at NLO + NLL. Similarly, the uncertainty at NNLO + NNLL is $10\%$ whereas at NNLO+NNLL it is found to be $30\%$ for the same value of $Q$.

In order to see the NSV effects more clearly, let us focus on the sensitivity of the predictions to the choice of the scale $\mu_R$ within the $q\bar{q}$ channel as a function of $\tau$, which is depicted in Fig. 13. Interestingly, the behaviour of NNLO$_{q\bar{q}}$ + NNLL is significantly improved from the corresponding SV results, NNLO$_{q\bar{q}}$ + NNLL, for a wide range of $Q$. There is a large increment in the central value, in addition to the significant reduction of the $\mu_R$ uncertainty. To see this quantitatively, we quote in Table 10 both the fixed-order and SV and SV + NSV resummed predictions along with asymmetric errors resulting from $\mu_R$ variation with $\mu_F$ held fixed at $Q$, say $Q = 1000$ and 2000 GeV. The uncertainties, in Table 10, are obtained with respect to the $\mu_R$ scale variation with $\mu_F$ held fixed at $Q$. This evidently shows that there is a considerable improvement when adding NSV resummation over the existing SV results and hence leading to more reliable predictions.

To summarise, our analysis on the sensitivity of hadronic results to $\mu_F$ and $\mu_R$ scales where the resummed parts from $q\bar{q}$ channel is taken into account, helps us to understand the role of various channels as well as on the PDFs that contribute. As far as the $\mu_R$ scale is concerned, inclusion of SV as well as NSV resummed contributions in CFs alone helps to reduce the sensitive to this scale. However, this is not the case for $\mu_F$ as our numerical study shows the need for resummed NSV contributions to CFs of $q\bar{g}$-channel as well as the resummed PDFs.
Fig. 10 7-point scale variations of NNLL and NNLL matched to NNLO for all-channels (left panel) and $q\bar{q}$-channel (right panel)

Table 9 Values of SV and SV + NSV resummed cross section in $10^{-5}$ pb/GeV at second logarithmic accuracy in comparison to the fixed-order results at different central scales

| $Q = \mu_R = \mu_F$ | NNLO | NNLO + NNLL | NNLO + NNLL |
|---------------------|------|-------------|-------------|
| 1000                | 3.2876 $^{+0.20\%}_{-0.31\%}$ | 3.2993 $^{+0.36\%}_{-0.29\%}$ | 3.3191 $^{+1.13\%}_{-0.86\%}$ |
| 2000                | 0.0684 $^{+0.62\%}$ | 0.0687 $^{+0.32\%}_{-0.27\%}$ | 0.0692 $^{+0.89\%}_{-0.78\%}$ |

Fig. 11 $\mu_F$ variation between SV and SV + NSV resummed results matched to NLO (left panel) and NNLO (right panel) with the scale $\mu_R$ held fixed at $Q$
4.3 SV + NSV resummation in different schemes

This section is devoted to the study of SV + NSV resummation effects in different schemes, namely $N$ and -$\bar{N}$ exponentiations. The details of both the schemes are discussed in Sect. 3 and here we illustrate their numerical impact. The main point of difference between these two schemes arises from the “additional” resummation of $\gamma_E$, i.e., in -$\bar{N}$. We resum $\gamma_E$ terms along with $\ln N$ whereas for $N$ exponentiation we only resum the $\ln N$ terms. The effect of this “additional” resummation of $\gamma_E$ has shown better convergence for SV resummation, now here it would be interesting to see what changes does SV + NSV resummation bring to the observations made earlier in [30].

At first we begin with the analysis of K factor which is defined in (63). To differentiate between these two scheme we denote $K$ for $N$ and $K^1$ for -$\bar{N}$ exponentiation as given in Fig. 14. Earlier in Sect. 4.1, we found that the K factor for $N$ exponentiation shows certain hierarchy which grows from LO + LL to NLO + NLL to NNLO + NNLL. Now for -$\bar{N}$ we find, that the $K_{NLO+NLL}$ is greater than $K_{NNLO+NNLL}$. However at NNLO + NNLL, we observe a striking feature,

---

1 It is to be noted that both K and $\overline{K}$ follows the same definition as given in (63).

---

Fig. 12. $\mu_R$ variation between SV and SV + NSV resummed results matched to NLO (left panel) and NNLO (right panel) with the scale $\mu_F$ held fixed at $Q$.

Fig. 13. $\mu_R$ variation between SV and SV + NSV matched to NNLO$_{q\bar{q}}$ with the scale $\mu_F$ held fixed at $Q$ in $q\bar{q}$-channel.
namely the K factors for both N and \( \bar{N} \) exactly overlap on each other for the considered \( Q \) range. To quantify the overlap at NNLO + NNLL, we quote that for \( \bar{N} \) at \( Q = 2000 \text{ GeV} \), the K factor is 1.3823 while for \( N \) it is 1.3818 (see Table 5). This implies that the correction at NNLO + NNLL is independent of any schemes.

Moreover, like \( N \) exponentiation, we see an overall increment in the cross section for \( \bar{N} \) with the inclusion of higher order logarithmic corrections. For instance at \( Q = 2000 \text{ GeV} \), the LO prediction is enhanced by 27.34\%, the NLO by 6.88\% and similarly the NNLO by 1.31\%. As a whole the resummed result at NNLO + NNLL increases the LO prediction by 38.32\% for the same value of \( Q \). And the perturbative convergence between resummed curves in \( \bar{N} \) is better than \( N \) exponentiation by a very small margin as can be seen from Fig. 14.

Now we proceed to study the uncertainties resulting from \( \mu_R \) and \( \mu_F \) for both \( N \) and \( \bar{N} \). This is presented in Fig. 15. We find that there is a systematic enhancement in the cross section and in scale reduction for both the schemes. However, in contrast to \( N \) exponentiation, studied in sec. 4.1, we find that the bands of NNLO + NNLL is not contained within the band of NLO + NNLL in \( N \)-scheme, as the enhancement from NLO + NNLL to NNLO + NNLL is negative. For instance, there is an increment of 9.6\% from LO + LL to NLO + NNLL accuracy, which decreases by \(-0.85\% \) at NNLO + NNLL for \( Q = 2000 \text{ GeV} \). This can also be seen from Fig. 14.

Although the width of NNLO + NNLL band is smaller than that of NLO + NNLL yet the uncertainties associated with these two bands are more than \( N \) exponentiation (see Table 11). Hence in summary, we find that when we go from LO + LL to NNLO + NNLL, the cross section increases more in \( N \) scheme compared to \( \bar{N} \) and the uncertainty in \( N \) scheme is smaller compared to \( \bar{N} \). Interestingly at NNLO + NNLL level, their central values are very close to each other compared to previous order hinting better scheme independence as we increase the order of perturbation.

Besides \( N \) and \( \bar{N} \) there are two other schemes as we discussed in Sect. 3, which are Soft and All exponentiation. While in Soft -scheme, we exponentiate only the Mellin moment of complete soft-collinear function, in All -scheme, the form factor contributions are also taken into exponentiation, additionally. We have explored the NSV resummation under these two schemes along with the \( N \)- and \( \bar{N} \)-schemes for two different \( Q \)-values, namely \( Q = 1000 \text{ GeV} \) and 2000 GeV and the results are enlisted in Table 11.

### Table 10 Comparison of SV and SV + NSV resummed cross section in \( 10^{-5}\) pb/GeV for \( q\bar{q} \)-channel at different central scales

| \( Q = \mu_R = \mu_F \) | NNLO\( q\bar{q} \) | NNLO\( q\bar{q} \) + NNLL | NNLO\( q\bar{q} \) + NNLL |
|--------------------------|------------------|------------------|------------------|
| 1000                     | 3.5266 ± 0.49\% | 3.5376 ± 0.25\% | 3.5570 ± 0.006\% |
| 2000                     | 0.0717 ± 0.54\% | 0.0721 ± 0.19\% | 0.0725 ± 0.00\%  |

### Fig. 14 Comparison between the K-factors for the \( N \) and \( \bar{N} \) exponentiation at the central scale \( Q = \mu_R = \mu_F \)

for different logarithmic accuracies. Noting various values, we find that at LO+LL and NLO + NNLL, the \( \bar{N} \) scheme gives the large corrections, say \( 0.0638 \times 10^{-7} \) pb/GeV and \( 0.0699 \times 10^{-5} \) pb/GeV respectively at \( Q = 2000 \text{ GeV} \), whereas, at NNLO+NNLL i.e., after adding more logarithmic corrections, the large contributions are arising from All -scheme. However, the uncertainties coming from \( \mu_R \) and \( \mu_F \) variation are found to be least in \( N \) and All exponentiation at any logarithmic accuracy for the considered \( Q \)-range.

#### 4.4 Numerical results for different collider energies

In Tables 12, 13 and 14, we summarise the results for different collider energies and different values of the invariant mass of di-leptons. We estimate the theoretical uncertainty by independently varying the scales \( \mu_F = \langle \mu F, \mu_R \rangle \) up and down, by a factor of two i.e. \( \frac{1}{2} \leq \frac{\mu_F}{\mu_R} \leq 2 \). We find that the inclusion of SV NNLL increases the NNLO distribution by 0.24–0.55\% whereas the inclusion of NSV at NNLL increases it by 0.72–1.65\%, in the considered collider energies and the \( Q \) range.

#### 5 Discussion and conclusions

Production of pair of leptons in Drell–Yan process is one of the cleanest processes at the LHC, also well studied both in SM and beyond SM. The perturbative predictions taking into account radiative corrections from strong and elec-
Fig. 15 A comparison between 7-point scale variation in both $N$ and $\bar{N}$ exponentiation around the central scale choice ($\mu_R, \mu_F) = (1, 1)Q$ for 13 TeV LHC

Table 11 A comparison of resummed cross-sections in $10^{-5}$ pb/GeV between different resummation schemes up to NNLO + NNLL

| Order     | $Q = \mu_R = \mu_F$ (GeV) | N-Exp       | $\bar{N}$-Exp | Soft-Exp | All-Exp |
|-----------|---------------------------|-------------|---------------|----------|---------|
| LO + LL   | 1000                      | 2.5184 +4.49% | 2.8636 +5.53% | 2.5184 +4.49% | 2.5184 +4.49% |
|           | 2000                      | 0.0692 +0.89% | 0.0693 +1.13% | 2.5184 +4.49% | 0.0692 +0.89% |
| NLO + NLL | 1000                      | 3.2858 +2.08% | 3.3418 +2.13% | 3.3027 +2.14% | 3.0151 +1.24% |
|           | 2000                      | 0.0688 +1.44% | 0.0699 +1.51% | 3.3027 +2.14% | 0.0688 +1.44% |
| NNLO + NNLL | 1000                     | 3.3191 +1.13% | 3.3204 +1.16% | 3.3204 +1.13% | 3.3204 +1.13% |
|           | 2000                      | 0.0692 +0.89% | 0.0693 +1.13% | 3.3204 +1.13% | 0.0692 +0.89% |

Table 12 Resummed results for invariant mass distribution of di-lepton pair against fixed-order (in pb/GeV) at different centre of mass energies at the LHC for $Q = 200$ GeV

| $\sqrt{s}$ | NNLO       | NNLO + NLL | NNLO + NNLL |
|------------|------------|------------|-------------|
| 7 TeV      | $2.56 \times 10^{-2} +0.29%$ | $2.57 \times 10^{-2} +0.82%$ | $2.59 \times 10^{-2} +2.19%$ |
| 8 TeV      | $3.06 \times 10^{-2} +0.32%$ | $3.07 \times 10^{-2} +0.86%$ | $3.09 \times 10^{-2} +2.25%$ |
| 13 TeV     | $5.63 \times 10^{-2} +0.45%$ | $5.65 \times 10^{-2} +0.99%$ | $5.69 \times 10^{-2} +2.43%$ |
| 14 TeV     | $6.16 \times 10^{-2} +0.43%$ | $6.18 \times 10^{-2} +1.00%$ | $6.22 \times 10^{-2} +2.45%$ |
| 100 TeV    | $5.13 \times 10^{-1} +1.18%$ | $5.15 \times 10^{-1} +1.16%$ | $5.18 \times 10^{-1} +2.09%$ |

Table 13 Resummed results for invariant mass distribution of di-lepton pair against fixed-order (in pb/GeV) at different centre of mass energies at the LHC for $Q = 500$ GeV

| $\sqrt{s}$ | NNLO       | NNLO + NLL | NNLO + NNLL |
|------------|------------|------------|-------------|
| 7 TeV      | $3.25 \times 10^{-4} +0.22%$ | $3.26 \times 10^{-4} +0.40%$ | $3.28 \times 10^{-4} +1.36%$ |
| 8 TeV      | $4.13 \times 10^{-4} +0.19%$ | $4.14 \times 10^{-4} +0.42%$ | $4.17 \times 10^{-4} +1.49%$ |
| 13 TeV     | $9.04 \times 10^{-4} +0.21%$ | $9.07 \times 10^{-4} +0.54%$ | $9.13 \times 10^{-4} +1.58%$ |
| 14 TeV     | $10.09 \times 10^{-4} +0.23%$ | $10.13 \times 10^{-4} +0.56%$ | $10.19 \times 10^{-4} +1.59%$ |
| 100 TeV    | $1.214 \times 10^{-2} +0.35%$ | $1.217 \times 10^{-2} +0.78%$ | $1.223 \times 10^{-2} +1.23%$ |
The latter contains both resummed threshold terms to fixed-order LO, NLO and NNLO contributions increases the sensitivity of the predictions to these scales. A detailed investigation reveals that the resumed results are more sensitive to factorisation scales, we have computed both SV and NSV resummed contributions with and without $qg$-channels and performed 7-point and single scale variations. We find that at $a_s^2$, the corrections from $q\bar{q}$ and $qg$-channel are 4.86% and $-2.47\%$ respectively to the NNLO cross section along with the other sub-dominating channels. As these corrections are closer to each other at the fixed-order level, we expect that resumed NSV from $qg$-channel is as important as the one from $q\bar{q}$ channel.

Unlike in the case of $\mu_F$, different channels that contribute do not mix under the variation of $\mu_R$ as each of them is renormalisation group invariant with respect to $\mu_R$. Hence, we expect that the predictions based on single scale variation should be less sensitive to $\mu_R$ as we increase the order of perturbation. Our numerical study confirms this both for fixed-order as well as for resummed predictions.

We have also investigated the role of resummed NSV terms in the resummed predictions in comparison to the known SV resummation. We find that there is about 2% increment when we go from NLL to NNLL and about 0.64% from NNLL to NNLL demonstrating the importance of resummed NSV contributions from quark anti-quark initiated channels. The 7-point scale variation gives larger scale uncertainty if we add resummed contributions from NSV terms. This stems from the scale $\mu_F$ as resummed NSV part lack contributions from $qg$ channel. The single scale variations where we keep one scale fixed and vary the other scale confirms our interpretation.

One finds that the SV part of the resummed predictions depends on how one treats the $N$-independent part of the resummed results. Such terms come from both form factors.

| $\sqrt{s}$  | NNLO        | NNLO + NNLL | NNLO + NNLL |
|------------|-------------|-------------|-------------|
| 7 TeV      | $7.55 \times 10^{-6} \pm 0.42\%$ | $7.59 \times 10^{-6} \pm 0.32\%$ | $7.65 \times 10^{-6} \pm 0.99\%$ |
| 8 TeV      | $1.092 \times 10^{-5} \pm 0.36\%$ | $1.098 \times 10^{-5} \pm 0.33\%$ | $1.11 \times 10^{-5} \pm 1.02\%$ |
| 13 TeV     | $3.29 \times 10^{-5} \pm 0.28\%$ | $3.30 \times 10^{-5} \pm 0.36\%$ | $3.32 \times 10^{-5} \pm 1.13\%$ |
| 14 TeV     | $3.79 \times 10^{-5} \pm 0.19\%$ | $3.81 \times 10^{-5} \pm 0.36\%$ | $3.83 \times 10^{-5} \pm 1.15\%$ |
| 100 TeV    | $6.96 \times 10^{-4} \pm 0.21\%$ | $6.98 \times 10^{-4} \pm 0.59\%$ | $7.01 \times 10^{-4} \pm 1.50\%$ |
and SV part of the soft-collinear function. Different schemes are adopted to deal with such terms and they give different numerical predictions. We consider four schemes and study their numerical impact on the predictions taking into account NSV terms. Interestingly we find that the difference in predictions from $N$ and $\overline{N}$ schemes gets reduced as we increase the logarithmic accuracy signaling the scheme independence. Finally, we conclude our detailed numerical analysis by presenting predictions for invariant mass distributions at scales $Q = 1000, 2000\,\text{GeV}$ for different hadronic center of mass energies along with uncertainties resulting from unphysical scales.

In summary, our analysis provides the precise numerical predictions from resummed NSV terms upto NNLO+NNLL level for the first time to the invariant mass distribution of a pair of leptons produced at the LHC.

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**Appendix A: Anomalous dimensions**

Here we present all the anomalous dimensions used in performing the resummation.

**A.1 Cusp anomalous dimensions $A_i^q$**

In the following we list the cusp anomalous dimensions $A_i^q$ till four-loop level:

$$A_1^q = 4\, C_F,$$

$$A_2^q = C_F\, n_f \left( -\frac{40}{9} + C_F\, C_A \left( \frac{268}{9} - 8\, \xi_2 \right) \right).$$

**A.2 Collinear anomalous dimensions $B_i^q$**

The collinear anomalous dimensions $B_i^q$ are given till three-loop as,

$$B_1^q = 3\, C_F,$$

$$B_2^q = C_F\, n_f \left( -\frac{1}{3} + \frac{8}{3}\, \xi_2 \right) + C_F\, C_A \left( \frac{17}{6} - 12\, \xi_3 \right).$$

The quadratic Casimirs $C_F$ and $C_A$ are given by $\frac{n_c^2 - 1}{2\, n_c}$ and $n_c$ respectively. The quartic Casimirs are given by

$$A_{\mu}^{abc} = \frac{\delta_{\mu,0}}{8!} = \frac{n_c (n_c^2 + 6)}{48},$$

$$A_{\mu}^{abcd} = \frac{\delta_{\mu,0}}{2n_c^2 + 18} = \frac{(n_c^4 - 6n_c^2 + 18)}{96n_c^2},$$

with $N_A = n_c^2 - 1$ and $N_F = n_c$ where $n_c = 3$ for QCD.
A.3 Soft anomalous dimensions $f_i^q$

The soft anomalous dimensions $f_i^q$ till three-loop are given as

\[
\begin{align*}
f_1^q &= 0, \\
f_2^q &= C_F n_f \left( -\frac{112}{27} + \frac{4}{3} \zeta_3 \right) + C_F C_A \left( \frac{808}{27} - 28 \zeta_3 - \frac{22}{3} \zeta_2 \right), \\
f_3^q &= C_F n_f^2 \left( -\frac{2080}{729} + \frac{112}{27} \zeta_1 - \frac{40}{27} \zeta_2 \right) + C_F C_A n_f \left( -\frac{11842}{729} + \frac{728}{27} \zeta_3 + \frac{2828}{81} \zeta_1 - \frac{96}{5} \zeta_2 \right) + C_F C_A^2 \left( \frac{136781}{729} + 192 \zeta_3 - \frac{1316}{3} \zeta_2 - \frac{12650}{81} \zeta_1 + \frac{176}{3} \zeta_3 + \frac{352}{5} \zeta_2 \right) + C_F^2 n_f \left( -\frac{1711}{27} + \frac{304}{9} \zeta_1 + 4 \zeta_2 + \frac{32}{5} \zeta_2^3 \right). \tag{A.10}
\end{align*}
\]

A.4 NSV anomalous dimensions $C_i^q$ & $D_i^q$

The NSV anomalous dimensions $C_i^q$ and $D_i^q$ till three-loop are given as

\[
\begin{align*}
C_1^q &= 0, \\
C_2^q &= 16 C_F, \\
C_3^q &= C_F C_A \left( \frac{2144}{9} - 64 \zeta_2 \right) + n_f C_F^2 \left( -\frac{320}{9} \right), \tag{A.13}
D_2^q &= -4 C_F, \tag{A.15}
\end{align*}
\]

A.5 SV threshold exponents

The function $\bar{g}_{SV}^{q} (\alpha_s(q^2(1-z)^2), \epsilon)$ given in (22) is related to the threshold exponent $D_i^q (\alpha_s(q^2(1-z)^2), \epsilon)$ via Eq. (46) of [11] where the universal $D_i^q$ coefficients, till three-loop, are given as

\[
\begin{align*}
D_1^q &= 0, \\
D_2^q &= C_F \left(-\frac{3712}{729} + \frac{320}{27} \zeta_3 + \frac{640}{27} \zeta_2 \right) + C_F n_f \left( \frac{125252}{729} - \frac{29392}{81} \zeta_2 + \frac{736}{15} \zeta_2^2 \right) + C_F^2 \left( -\frac{594058}{729} + \frac{5384}{5} \zeta_3 - \frac{2992}{3} \zeta_2^3 \right), \tag{A.18}
\end{align*}
\]

\[
\begin{align*}
D_3^q &= C_F \left(-\frac{3712}{729} + \frac{320}{27} \zeta_3 + \frac{640}{27} \zeta_2 \right) + C_F n_f \left( \frac{125252}{729} - \frac{29392}{81} \zeta_2 + \frac{736}{15} \zeta_2^2 \right) + C_F^2 \left( -\frac{594058}{729} + \frac{5384}{5} \zeta_3 - \frac{2992}{3} \zeta_2^3 \right), \tag{A.19}
\end{align*}
\]

Appendix B: NSV coefficients

Here we present all the NSV coefficients $\varphi_{q,i}^{(k)}$ given in (27).

\[
\begin{align*}
\varphi_{q,1}^{(0)} &= 4 C_F, \\
\varphi_{q,1}^{(1)} &= 0, \\
\varphi_{q,1}^{(2)} &= C_F C_A \left( \frac{1402}{27} - 28 \zeta_3 - \frac{112}{3} \zeta_2 \right) + C_F^2 (-32 \zeta_2), \tag{A.14}
\end{align*}
\]

\[
\begin{align*}
\varphi_{q,2}^{(0)} &= 10 C_F C_A - 10 C_F^2, \tag{A.16}
\varphi_{q,2}^{(2)} &= -4 C_F^2. \tag{A.17}
\end{align*}
\]
\[ \varphi_{q,3}^{(0)} = C_F C_A \left( \frac{727211}{729} + 192 \zeta_3 - \frac{29876}{27} \zeta_3 - \frac{82868}{81} \zeta_2 \right) + \frac{176}{3} \zeta_2 \zeta_3 + 120 \zeta_2^2 + C_F C_A \left( - \frac{5143}{27} - \frac{2180}{9} \zeta_3 \right) - \frac{11584}{27} \zeta_2 + \frac{2272}{15} \zeta_2^2 + C_F^3 \left( 23 + 48 \zeta_3 - \frac{32}{3} \zeta_2 \right) - \frac{448}{15} \zeta_2^2 \]  
\[ + n_f C_F C_A \left( - \frac{155902}{729} + 129 \zeta_3 + \frac{26312}{81} \zeta_2 \right) - \frac{368}{15} \zeta_2^2 \]  
\[ + n_f C_F^2 \left( - \frac{1309}{9} + \frac{496}{3} \zeta_3 + \frac{2536}{27} \zeta_2 + \frac{32}{5} \zeta_2^2 \right) + n_f^2 C_F \left( \frac{12656}{729} - \frac{160}{27} \zeta_3 + \frac{704}{27} \zeta_2 \right). \]

\[ \varphi_{q,3}^{(1)} = C_F C_A \left( \frac{244}{9} + 24 \zeta_3 - \frac{8}{9} \zeta_2 \right) + C_F^2 C_A \left( - \frac{18436}{81} + \frac{544}{3} \zeta_3 \right) + 964 \zeta_3 + C_F^3 \left( - \frac{64}{3} - 64 \zeta_3 + \frac{80}{3} \zeta_2 \right) + n_f C_F C_A \left( - \frac{256}{9} - \frac{28}{9} \zeta_2 \right) + n_f C_F^2 \left( \frac{3952}{81} - \frac{160}{9} \zeta_2 \right). \]

\[ \varphi_{q,3}^{(2)} = C_F C_A \left( 34 - \frac{10}{3} \zeta_2 \right) + C_F^2 C_A \left( - 96 + \frac{52}{3} \zeta_2 \right) + C_F^3 \left( 16 \right) \]  
\[ + n_f C_F C_A \left( - \frac{10}{3} \right) + n_f C_F^2 \left( \frac{40}{3} \right). \]

\[ \varphi_{q,3}^{(3)} = C_F^2 C_A \left( - \frac{176}{27} \right) + n_f C_F^2 \left( \frac{32}{27} \right). \]  

**Appendix C: Resummation coefficients for the N exponentiation**

**C.1 The N-independent coefficients \( C_0^q \)**

The \( N \)-independent coefficients \( C_0^q \) in Eq. (23), till three-loop, are given by

\[ C_{00}^q = 1. \]  

\[ C_{01}^q = C_F \left\{ - 16 + 8 \zeta_2 + 6 L_q r - 6 L_f r \right\}. \]  

\[ C_{02}^q = C_F n_f \left\{ - \frac{127}{6} + 8 \zeta_3 - \frac{112}{9} \zeta_2 - \frac{34}{3} L_q r + 2 L_q r^2 + \frac{2}{3} L_f r \right\}. \]  

\[ C_{03}^q = C_F N_{n_f} \left\{ - \frac{16}{3} L_f r \zeta_2 - 2 L_q r^2 \right\} + C_F C_A \left\{ - \frac{1535}{12} + 28 \zeta_5 \right\} + \frac{592}{9} \zeta_2 - \frac{12}{5} \zeta_2^2 + \frac{193}{3} L_q r - 24 L_q r \zeta_3 - 11 L_q r \zeta_2 \]  
\[ - \frac{17}{3} L_f r + 24 L_f r \zeta_3 - \frac{88}{3} L_f r \zeta_2 + 11 L_f r \zeta_3 \]  
\[ + 511 \zeta_4 - 60 \zeta_5 - 70 \zeta_2 + \frac{72}{5} \zeta_2^2 - 93 L_q r + 48 L_q r \zeta_3 \]  
\[ + 24 L_q r \zeta_2 + 18 L_q r^2 + 93 L_f r - 48 L_f r \zeta_3 - 24 L_f r \zeta_2 \]  
\[ - 36 L_f r \zeta_4 + 18 L_f r^2 \right\}. \]  

\[ C_{03}^q = C_F N_{n_f} \left\{ - \frac{16}{3} \zeta_5 + \frac{28}{3} \zeta_3 + 20 \zeta_2 - \frac{4}{3} \zeta_2^2 \right\} + C_F n_f^2 \left\{ - 7081 \zeta_4 - 1264 \zeta_5 + 2416 \zeta_2 + 128 \zeta_2^2 \right\} + \frac{220}{9} L_q r + \frac{64}{9} L_q r \zeta_3 - \frac{32}{9} L_q r \zeta_2 - \frac{68}{9} L_q r + \frac{8}{9} L_q r \zeta_3 \]  
\[ + \frac{34}{9} L_f r + \frac{32}{9} L_f r \zeta_3 - \frac{160}{27} L_f r \zeta_2 - \frac{4}{9} L_f r + \frac{32}{9} L_f r \zeta_2 \]  
\[ - \frac{8}{9} L_f r \zeta_3 \right\} + C_F C_A n_f \left\{ - \frac{110651}{243} - \frac{8 \zeta_5}{3} - \frac{3052}{9} L_q r \zeta_3 \right\} + \frac{204 \zeta_3}{81} - \frac{82385}{3} \zeta_3 - \frac{400 \zeta_5}{3} + 843 \zeta_3 - \frac{884}{3} \zeta_3 \]  
\[ + \frac{14611}{135} \zeta_2^2 + \frac{3082}{3} L_q r - 80 L_q r \zeta_3 \]  
\[ + \frac{4952}{9} L_q r \zeta_3 - \frac{240}{3} L_q r \zeta_2 + \frac{68}{9} L_q r \zeta_2^2 - \frac{2429}{9} L_q r \zeta_2 \]  
\[ + \frac{88}{9} L_q r \zeta_3 + \frac{242}{9} L_q r + \frac{1657}{18} \zeta_2 - 80 L_f r \zeta_3 \]  
\[ + \frac{3104}{9} L_f r \zeta_3 - \frac{8992}{27} L_f r \zeta_2 + 4 L_f r \zeta_2^2 + \frac{493}{9} L_f r \]  
\[ + \frac{88}{9} L_f r \zeta_3 + \frac{968}{9} L_f r \zeta_2 - \frac{242}{9} L_f r \zeta_2^2 + C_F n_f^2 \right\} - \frac{421}{3} \]  
\[ - \frac{608}{9} \zeta_5 + \frac{6952}{27} L_q r \zeta_3 - \frac{2632}{27} \zeta_5 - \frac{896}{9} L_q r \zeta_3 - 3568 \zeta_5 \]  
\[ + \frac{230}{9} L_q r \zeta_3 + \frac{368}{3} L_q r \zeta_2 + \frac{176}{15} L_q r \zeta_2 + \frac{112}{5} L_q r \zeta_2 \]  
\[ - \frac{92}{3} L_q r \zeta_3 + 32 L_q r \zeta_3 + 12 L_q r - \frac{275}{3} L_f r + \frac{28}{3} L_f r \zeta_3 \]  
\[ - \frac{56}{3} L_f r \zeta_3 + \frac{176}{15} L_f r \zeta_2 + 72 L_f r L_q r + 32 L_f r L_q r \zeta_2 \]  
\[ - 12 L_f r L_q r^2 + 20 L_f r L_q r \zeta_3 - 32 L_f r L_q r \zeta_2 \]  
\[ - 12 L_f r L_q r^2 + 12 L_f r^2 \right\} + C_F C_A \left\{ \frac{74321}{36} - 5512 \zeta_5 \right\} \]  
\[ - \frac{34612}{27} \zeta_5 + \frac{592}{3} \zeta_3 - \frac{13186}{27} \zeta_2 + \frac{3392}{9} \zeta_2 \zeta_3 \]  
\[ - \frac{11}{3} \zeta_2^2 \zeta_3. \]
\[ g_{0,0}^{q} = 0, \quad g_{0,1}^{q} = C_F \left\{ 7 \zeta_2 - 8 \gamma_E L_{\text{qr}} + 8 \gamma_E L_f + 8 \gamma_E^2 \right\}, \quad g_{0,2}^{q} = C_F n_f \left\{ - \frac{64}{9} \zeta_2 + \frac{80}{9} \zeta_3 + \frac{16}{27} L_{\text{qr}} \zeta_2 - \frac{224}{27} \gamma_E \right\} \]

\[ + \frac{80}{9} \gamma_E L_{\text{qr}} - \frac{8}{3} \gamma_E^2 L_{\text{qr}} - \frac{80}{9} \gamma_E L_f + \frac{8}{9} \gamma_E^2 L_f - \frac{80}{9} \gamma_E^2 \]

\[ + \frac{16}{3} \gamma_E^2 L_{\text{qr}} - \frac{32}{9} \gamma_E^3 \} + C_F A \left\{ \frac{352}{9} \zeta_3 + \frac{536}{9} \zeta_2 \right\} \]

\[ - \frac{16}{3} \gamma_E^2 L_{\text{qr}} \zeta_2 - \frac{88}{3} \gamma_E L_{\text{qr}} - \frac{1616}{27} \gamma_E - 56 \gamma_E \zeta_3 \]

\[ - \frac{536}{9} \gamma_E L_{\text{qr}} + 16 \gamma_E L_{\text{qr}} \zeta_2 \]

\[ + \frac{44}{3} \gamma_E^2 L_{\text{qr}} + \frac{536}{9} \gamma_E L_f - 16 \gamma_E L_f \zeta_2 - \frac{44}{3} \gamma_E L_f \zeta_3 \]

\[ + \frac{536}{9} \gamma_E^2 - 16 \gamma_E \zeta_2 - \frac{88}{3} \gamma_E^2 L_{\text{qr}} + \frac{176}{9} \gamma_E^3 \} \]
Here, $\gamma_E$ is the Euler–Mascheroni constant.

C.3 The $N$-independent coefficients $\tilde{g}_q^q$

The $N$-independent coefficients $\tilde{g}_0^{q,i}$ in Eq. (33), till three-loop, are given by

$$\tilde{g}_{0,0}^q = 1,$$  \hspace{1cm} (C.30)

$$\tilde{g}_{0,1}^q = C_F \left\{ -16 + 16\xi_2 + 6L_{qe} - 6L_{fr} - 8\gamma_EL_{qe} + 8\gamma_EL_{fr} \right. \\
+ 8\gamma_E^2 \left\}.$$  \hspace{1cm} (C.31)

$$\tilde{g}_{0,2}^q = C_F n_f \left\{ \frac{127}{6} + \frac{8}{5} \xi_2 - \frac{64}{9} \zeta_2 + \frac{34}{3} \gamma_EL_{fr} + \frac{16}{3} \gamma_EL_{fr}^2 \right. \\
+ 2\frac{L_{fr}^2}{2} + \frac{2}{3} L_{fr} + \frac{16}{3} \gamma_EL_{fr} - 2L_{fr}^2 + \frac{224}{27} \gamma_E \\
\left. + \frac{80}{5} \gamma_EL_{fr}^2 - \frac{80}{9} \gamma_EL_{fr} + \frac{8}{3} \gamma_EL_{fr}^2 - \frac{80}{9} \gamma_E^2 \\
+ \frac{16}{5} \gamma_E^2 L_{fr}^2 - \frac{32}{5} \gamma_E^2 \right\} \frac{C_F A}{1535} + \frac{604}{9} \xi_2 \\
+ \frac{376}{3} \zeta_2 - \frac{92}{9} \xi_3 + \frac{193}{9} \gamma_EL_{fr}^3 - \frac{88}{9} \gamma_EL_{fr}^3 \\
- \frac{11L_{fr}^2}{2} + \gamma_EL_{fr} + \frac{24L_{fr}^3}{3} - \frac{88}{3} \xi_2 L_{fr} + \frac{11L_{fr}}{2} \\
\left. + \frac{1616}{27} \gamma_E - \frac{56L_{fr}}{9} + \frac{536}{9} \gamma_EL_{fr} - 16L_{fr} \gamma_EL_{fr} - \frac{44}{3} \gamma_EL_{fr}^2 \\
+ \gamma_EL_{fr}^2 - \frac{536}{9} \xi_2 - \frac{16\gamma_EL_{fr}}{3} + \gamma_EL_{fr} - \frac{176}{3} \gamma_E^2 + \frac{C_F A}{27} \right\} \\
\left. + \frac{511}{4} - \frac{1}{60} \xi_3 + 198\xi_2 + \frac{552}{5} \xi_2 - 93L_{fr} + 48L_{fr} \gamma_EL_{fr} \\
+ 72L_{fr} \gamma_EL_{fr} + 18L_{fr}^2 + \frac{93}{9} L_{fr} - 48L_{fr} \gamma_EL_{fr} - 72L_{fr} \gamma_EL_{fr} \\
- \frac{36L_{fr}L_{fr}^2 + 18L_{fr}^2 + 128L_{fr} \gamma_EL_{fr} - 128L_{fr} \gamma_EL_{fr}^2 \right\} \\
\left. - \frac{48\gamma_EL_{fr}^2 - 128\gamma_EL_{fr} - 128L_{fr} \gamma_EL_{fr}^2 - \gamma_EL_{fr}L_{fr}^2 \right\} \\
- \frac{48\gamma_EL_{fr}^2 - 128\gamma_EL_{fr} + 128L_{fr} \gamma_EL_{fr}^2 + 96L_{fr} \gamma_EL_{fr} - \gamma_EL_{fr}L_{fr}^2 \right\} \\
\left. - 64\gamma_E L_{fr} \gamma_EL_{fr} + 32\gamma_E^2 \gamma_EL_{fr} - 64\gamma_E^2 \gamma_EL_{fr} + 64\gamma_E \gamma_EL_{fr} + 32\gamma_E^2 \right\}.$$  \hspace{1cm} (C.32)

$$\tilde{g}_{0,3}^q = C_F n_f n_{f-1} \left\{ \frac{8 - \frac{10}{3} \xi_3 + \frac{28}{3} \xi_3 + 20 \xi_2 - \frac{4}{3} \xi_2}{3} \right\} \\
+ C_F n_f (\frac{7081}{243} + \frac{16}{8} \xi_3 + \frac{1072}{9} \xi_2 + \frac{448}{9} \xi_2) \\
+ \frac{220}{9} \gamma_EL_{fr} - \frac{64}{9} \gamma_EL_{fr} + 68L_{fr} \gamma_EL_{fr} + \frac{68}{9} \gamma_EL_{fr}^2 \\
+ \frac{32}{9} L_{fr} + \frac{34}{9} \gamma_EL_{fr} + \frac{34}{9} \gamma_EL_{fr}^2 + \frac{32}{9} \gamma_EL_{fr}^3 - 160L_{fr} \gamma_EL_{fr}^2 \\
+ \frac{4}{9} \gamma_EL_{fr}^3 + \frac{32}{9} L_{fr} \gamma_EL_{fr}^2 + \frac{8}{9} \gamma_EL_{fr}^4 + \frac{3712}{27} \gamma_E + \frac{64}{9} \xi_2 L_{fr} \\
- \frac{800}{81} \gamma_EL_{fr} \gamma_EL_{fr}^2 + 16 \gamma_EL_{fr}^2 \gamma_EL_{fr} + \frac{32}{27} \gamma_EL_{fr}^3 - 160 \gamma_EL_{fr}^3 \right\}.$$

\[ + \frac{32}{27} \gamma_EL_{fr} \gamma_EL_{fr}^3 + \frac{800}{81} \gamma_EL_{fr}^2 L_{fr} + \frac{32}{27} \gamma_EL_{fr}^2 L_{fr} + 16 \gamma_EL_{fr}^2 L_{fr} + \frac{32}{27} \gamma_EL_{fr}^3 L_{fr} + \frac{32}{27} \gamma_EL_{fr}^3 L_{fr}^2 \\
+ \frac{640}{81} \gamma_EL_{fr} + \frac{128}{27} \gamma_EL_{fr}^2 + \frac{64}{27} \gamma_EL_{fr}^3 + \frac{C_FC_A}{110651} \left\{ \frac{1}{243} \right\}. \]
The resummation exponents \( g_i^q \) in Eq. (31), till three-loop, are given by

\[
g_1^q = \frac{1}{\beta_0} C_F \left[ 8 + \frac{8}{\omega} - 8L_\omega \right]. \tag{C.34}
\]

\[
g_2^q = \frac{1}{\beta_0^2} C_F \left[ 4\omega + 4L_\omega + 2L^2 \right] + \frac{1}{\beta_0^3} C_F \left[ \frac{40}{9}\omega + 40L_\omega \right] + \frac{1}{\beta_0^4} C_F C_{\ln f} \left[ \frac{40}{9}\omega + 40L_\omega \right] + \frac{1}{\beta_0^5} C_F C_A \left[ \frac{1}{9}\omega - 8\omega_2 - 268L_\omega - 268L_\omega \omega_2 \right] + \frac{1}{\beta_0^6} C_F \left[ 4\omega L_{\ell r} + 4L_\omega - 8L_\omega \right]. \tag{C.35}
\]

\[
g_3^q = \frac{1}{(1 - \omega)} \left[ \frac{1}{\beta_0^2} C_F \left[ 2\omega^2 + 4L_\omega \omega + 2L^2 \right] + \frac{1}{\beta_0^3} C_F \left[ 4\omega \right] - 2\omega^2 \right] + \frac{1}{\beta_0^4} C_F C_{\ln f} \left[ \frac{40}{9}\omega + 20\omega_2 + 40L_\omega \right] + \frac{1}{\beta_0^5} C_F C_A \left[ \frac{1}{9}\omega - 8\omega_2 - 268L_\omega + 8L_\omega \omega_2 \right] + \frac{1}{\beta_0^6} C_F \left[ 4\omega L_{\ell r} + 4L_\omega - 8L_\omega \right]. \tag{C.36}
\]

Here, \( L_\omega = \ln(1 - \omega) \) and \( \omega = 2\beta_0\omega_1\left(\mu^2_R\right) \) in \( N \). And \( \beta_i \)'s are the QCD \( \beta \) functions which are given by

\[
\beta_0 = \frac{11}{3} C_A - \frac{2}{3} n_f,
\]

\[
\beta_1 = \frac{34}{3} C_A - 2n_f C_F - \frac{10}{3} n_f C_A,
\]

\[
\beta_2 = \frac{2857}{54} C_A^3 - \frac{1415}{54} C_A^2 n_f + \frac{79}{54} C_F n_f^2 + \frac{11}{9} C_F n_f^2 - \frac{205}{18} C_F C_A n_f + C_F^2 n_f. \tag{C.37}
\]

C.5 The NSV resummed exponents \( \tilde{g}_i^q \)

The resummation exponents \( \tilde{g}_i^q \) in (34), till three-loop, are given by

\[
\tilde{g}_1^q = \frac{1}{\beta_0} C_F \left[ 4L_\omega \right]. \tag{C.38}
\]

\[
\tilde{g}_2^q = \frac{1}{(1 - \omega)^2} \left[ \frac{1}{\beta_0^2} C_F C_{\ln f} \left[ \frac{40}{9}\omega - 40L_\omega \right] + \frac{1}{\beta_0^3} C_F C_A \left[ \frac{136}{3}\omega + \frac{136}{3} L_\omega \right] + \frac{1}{\beta_0^4} C_F^2 n_f \left[ -8\omega \right] - \frac{1}{\beta_0^5} C_F \left[ 8L_\omega - 8L_\omega \right] - \frac{1}{\beta_0^6} C_F \left[ 8L_\omega - 8L_\omega \right] \right]. \tag{C.39}
\]

\[
\tilde{g}_3^q = \frac{1}{(1 - \omega)^2} \left[ \frac{1}{\beta_0^2} C_F C_{\ln f} \left[ \frac{200}{9}\omega^2 - 200L_\omega \right] + \frac{1}{\beta_0^3} C_F C_A \left[ \frac{2312}{9}\omega^2 - \frac{2312}{9} L_\omega \right] + \frac{1}{\beta_0^4} C_F^2 n_f \left[ \frac{80}{3}\omega^2 - 80L_\omega \right] + \frac{1}{\beta_0^5} C_F^2 C_{\ln f} \left[ -\frac{272}{3}\omega^2 + \frac{272}{3} L_\omega \right] + \frac{1}{\beta_0^6} C_F^3 n_f \left[ \frac{400}{27}\omega - \frac{31}{3}\omega^2 + \frac{400}{27} L_\omega \right] \right].
\]
\( C.6 \) The NSV resummed exponents \( h^q_{ij} \)

The resummation constants \( h^q_{ij} \) in Eq. (34), till three-loop, are given by

\[
\begin{align*}
  h^q_{00} &= \frac{1}{\beta_0} C_F \left\{ -\frac{80}{9} \omega \right\} , \\
  h^q_{11} &= 0 .
\end{align*}
\]
\[-\frac{1072}{9} L_{fr, \omega} + 32 L_{fr, \omega} \tau_2 + \frac{536}{9} L_{fr, \omega}^2 - 16 L_{fr, \omega}^2 \tau_2 \\
+ \frac{892}{9} \gamma E - 32 \gamma E \tau_2 \right] + C_F^2 \left\{ -8 \tau_2 + 24 L_{fr} - 24 L_{fr} \\
+ 48 L_{fr, \omega} - 24 L_{fr, \omega}^2 - 28 \gamma E - 32 \gamma E L_{fr} + 32 \gamma E L_{fr} \\
- 64 \gamma E L_{fr, \omega} + 32 \gamma E L_{fr, \omega}^2 + 56 \gamma E^2 \right\} + \beta_0 \bar{C}_F \left\{ 16 \tau_2 \\
- 16 \gamma E L_{fr} + 4 \gamma E L_{fr, \omega} + 8 L_{fr, \omega} - 4 L_{fr, \omega}^2 - 4 L_{fr}^2, \\
+ 8 L_{fr, \omega}^2 - 4 L_{fr, \omega}^3 + 32 \gamma E L_{fr} + 16 \gamma E L_{fr} \right\} \right],
\tag{C.45}
\]

\[h_{21}^q = \frac{1}{(1 - \omega)^2} \left\{ \frac{1}{\beta_0} \beta_1 C_F^2 \left[ -32 \omega + 16 \omega^2 - 32 L_{\omega} \right] \\
+ 1 \frac{C_F^2 n_f}{\beta_0} \left\{ -640 \omega + 320 \omega^2 \right\} + \frac{1}{\beta_0} C_F^2 C_A \\
\times \left\{ 4288 \omega - 128 \omega \tau_2 - 2144 \omega^2 + 64 \omega^2 \tau_2 \right\} + C_F C_A \right. \\
\times \left\{ -20 \right. \}
+ \frac{C_F^2}{\beta_0} \left\{ 20 - 32 L_{fr} \\
+ 32 L_{fr} - 64 L_{fr, \omega} + 32 L_{fr, \omega}^2 + 48 \gamma E \right\} \right],
\tag{C.46}
\]

\[h_{22}^q = \frac{1}{(1 - \omega)^3} \left\{ \frac{1}{\beta_0} \beta_1 C_F^2 \left[ -32 \frac{27}{\omega} \right] \\
+ 1 \frac{C_F^2 n_f}{\beta_0} \left\{ 176 \frac{27}{\omega} \right\} \right].
\tag{C.47}
\]

**Appendix D: Resummation coefficients for the $\overline{N}$ exponentiation**

For the case of $\overline{N}$ exponentiation, all the $N$-dependent resummed exponents namely $g_0^q(\overline{\omega})$, $\tilde{g}_0^q(\overline{\omega})$, and $h_0^q(\overline{\omega})$ given in Eqs. (40) and (42), respectively can be obtained from the corresponding exponents in standard $N$-approach through setting all the $\gamma E$ terms to zero and replacing all the $\ln N$ terms by $\ln \overline{N}$ as well as all the $O(\ln N)$ terms by $\overline{\omega}$ as mentioned in Sect. 3. The $N$-independent constants $g_0^q(\overline{\omega})$ given in Eq. (39) can be obtained from their counterparts in standard $N$-approach by simply putting the $\gamma E$ terms equal to zero.

**Appendix E: Resummation coefficients for the soft exponentiation**

For the case of soft exponentiation, all the terms coming from the soft-collinear function $\Phi_s$ are exponentiated and hence this means all the contribution to the finite ($N$-independent) piece from the soft-collinear function is also being exponentiated. The resummation coefficients for the soft exponentiation denoted by $\tilde{g}_0^q, \tilde{h}_0^q$, and $\tilde{h}_0^q$ in Eqs. (45) and (44), respectively can be obtained from their counterparts in standard $N$ scheme as described below.
\[
\begin{align*}
&+ \frac{505087}{9} \zeta_3 - \frac{2272}{9} \zeta_3^2 + \frac{1193026}{9} \zeta_2^2 \\
&- \frac{2336}{3} \zeta_2 \zeta_3 - \frac{4303}{15} \zeta_2^3 + \frac{38272}{3} \zeta_2^2 \\
&+ \frac{1045955}{729} \Lfr \zeta_2 + \frac{727}{27} Lqr \zeta_2 - \frac{34928}{27} \Lqr \zeta_3 \\
&- \frac{68552}{81} \Lqr \zeta_2 + \frac{176}{3} \Lfr \zeta_3 + \frac{340}{3} \Lfr \zeta_2^2 \\
&- \frac{37364}{81} \Lfr^2 + \frac{176 \Lfr \zeta_3}{9} - \frac{1504}{9} \Lfr^2 \zeta_2 - \frac{88}{5} \Lfr \zeta_2^2 \\
&+ \frac{493}{9} \Lfr^2 \zeta_2 - \frac{88 \Lfr \zeta_3}{9} - \frac{968}{9} \Lfr^2 \zeta_2 - \frac{242}{9} \Lfr^2 \zeta_2 \\
&+ C_F^2 \left[ - \frac{56963}{486} + \frac{832}{9} \zeta_3 + \frac{26080}{81} \zeta_3 \\
&+ \frac{27410}{81} \zeta_2 - \frac{296}{3} \zeta_2 \zeta_3 - \frac{5852}{27} \zeta_2^2 + \frac{6947}{27} \Lqr \zeta_3 \\
&- \frac{1208}{9} \Lfr \zeta_3 + \frac{8120}{27} \Lfr \zeta_2 + \frac{1328}{3} \Lfr \zeta_2^2 \\
&- \frac{149448}{81} \Lfr^2 + \frac{136}{9} \Lfr \zeta_3 - \frac{1657}{18} \Lfr \zeta_2 + \frac{100}{9} \Lfr^2 \zeta_2 \\
&+ \frac{1676}{27} \Lfr^3 - \frac{152}{27} \Lfr^2 \zeta_2 + \frac{100}{9} \Lfr^2 + \frac{8}{9} \Lfr^3 \\
&- \frac{3131}{27} \Lfr \zeta_2 + \frac{88 \Lfr \zeta_3}{9} + \frac{32 \Lfr \zeta_2^2}{3} + \frac{656}{15} \Lfr \zeta_2 \\
&+ \frac{872}{9} \Lfr \zeta_4 + \frac{80}{3} \Lfr \zeta_2 \zeta_3 + \frac{32}{3} \Lfr \zeta_2 \zeta_2^2 \\
&+ \frac{8}{3} \Lfr Lfr \zeta_3 + \frac{20}{3} \Lfr \zeta_2 - \frac{32}{3} \Lfr \zeta_2 \zeta_3 - \frac{44}{3} \Lfr \zeta_2 \\
&- \frac{12}{3} \Lfr \zeta_4 - \frac{4 \Lfr \zeta_2 \zeta_2 - \frac{12}{3} \Lfr \zeta_2 \zeta_2}{3} + C_F^2 C_A \\
&\right] \times \left[ \frac{824281}{324} - \frac{5512}{27} \zeta_3 - \frac{52564}{27} \zeta_3 - \frac{592}{3} \zeta_3 \\
&- \frac{406507}{162} \zeta_2 + \frac{3380}{3} \zeta_2 \zeta_3 + \frac{184474}{135} \zeta_2^2 \\
&- \frac{11824}{63} \zeta_3 - \frac{14269}{6} \Lfr \zeta_2 + \frac{240 \Lfr \zeta_2 \zeta_3}{95227} \Lfr \zeta_3 \\
&+ \frac{48536}{27} \Lfr \zeta_2 - \frac{696 \Lfr \zeta_2 \zeta_3}{95227} \Lfr \zeta_2^2 \\
&+ \frac{208099}{162} \zeta_2^2 - \frac{5644}{9} \zeta_2 \zeta_3 - \frac{6752}{9} \zeta_2^2 \zeta_3 \\
&+ \frac{344}{5} \zeta_2 \zeta_2^2 + \frac{10340}{27} \zeta_2 + \frac{104 \zeta_2 \zeta_3}{4} \\
&+ \frac{1052}{9} \Lfr \zeta_3 + \frac{598}{9} \Lfr \zeta_3 - \frac{84 \zeta_2 \zeta_2 - \frac{44}{9} \Lfr \zeta_2 \zeta_2}{9} \\
&+ \frac{25988}{27} \Lfr - \frac{240 \Lfr \zeta_2}{9} - \frac{136 \Lfr \zeta_3}{9} - \frac{44 \Lfr \zeta_2 \zeta_3}{9} \\
&+ \frac{304 \Lfr \zeta_3 \zeta_3}{15} - \frac{3608}{15} \Lfr \zeta_3 - \frac{5396}{9} \Lfr Lfr \zeta_4 \\
&+ \frac{456 \Lfr \zeta_3 \zeta_3}{9} + \frac{300}{3} \Lfr \zeta_3 - \frac{48 \Lfr \zeta_2 \zeta_2}{9} \zeta_3
\right].
\end{align*}
\]
Appendix F: Resummation coefficients for the all exponentiation

For the case of All exponentiation, the complete $\tilde{g}_0^q$ is being exponentiated along with the large-$N$ pieces. This brings into modification only for the resummed exponent compared to the ‘Standard $N$ exponentiation’. We write the modified resummed exponents denoted by $g_{i}^{q,\text{All}}$ in Eq. (47) in terms of exponents in standard $N$ as,

$$g_{1}^{q,\text{All}} = g_{1}^{q},$$  \hspace{1cm} (F.59)

$$g_{2}^{q,\text{All}} = g_{2}^{q} + \Delta_{g_{2}^{q,\text{All}}},$$  \hspace{1cm} (F.60)

$$g_{3}^{q,\text{All}} = g_{3}^{q} + \Delta_{g_{3}^{q,\text{All}}},$$  \hspace{1cm} (F.61)

where $\Delta_{g_{i}^{q,\text{All}}}$ terms are found from exponentiating the complete $\tilde{g}_0^q$ prefactor and they are given as,

$$\Delta_{g_{2}^{q,\text{All}}} = \tilde{g}_0^q,$$  \hspace{1cm} (F.62)

$$\Delta_{g_{3}^{q,\text{All}}} = \left(-\frac{(\tilde{g}_0^q)^2}{2} + \tilde{g}_0^{q2}\right),$$  \hspace{1cm} (F.63)

where the coefficients $\tilde{g}_0^q$ are given in Appendix C.3.

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