A NOTE ON RESTRICTED ONLINE RAMSEY NUMBERS OF MATCHINGS

VOJTECH DVOŘÁK

Abstract. The restricted online Ramsey numbers were introduced by Conlon, Fox, Grinshpun and He [2] in 2019. In a recent paper [1], Briggs and Cox studied the restricted online Ramsey numbers of matchings and determined a general upper bound for them. They proved that for \( n = 3r - 1 = R_2(rK_2) \) we have \( \tilde{R}_2(rK_2; n) \leq n - 1 \) and asked whether this was tight. In this short note, we provide a general lower bound for these Ramsey numbers. As a corollary, we answer this question of Briggs and Cox, and confirm that for \( n = 3r - 1 \) we have \( \tilde{R}_2(rK_2; n) = n - 1 \). We also show that for \( n' = 4r - 2 = R_3(rK_2) \) we have \( \tilde{R}_3(rK_2; n') = 5r - 4 \).

1. Introduction

For families of graphs \( G_1, ..., G_t \), the Ramsey number \( R(G_1, ..., G_t) \) is the smallest integer \( n \) such that any colouring of \( K_n \) with colours \( 1, ..., t \) contains a graph \( G_i \) in colour \( i \) for some \( G_i \in G_t \) and some \( i \in \{1, ..., t\} \). The Ramsey numbers of graphs have been studied extensively, see for instance a survey of Conlon, Fox and Sudakov [3].

Many variants of the Ramsey numbers have been considered. In particular, in 2019, Conlon, Fox, Grinshpun and He [2] introduced the so-called restricted online Ramsey numbers. For families of graphs \( G_1, ..., G_t \) and integer \( n \) such that \( n \geq R(G_1, ..., G_t) \), the restricted online Ramsey number \( \tilde{R}(G_1, ..., G_t; n) \) is the smallest integer \( k \) for which Builder can always guarantee a win within the first \( k \) moves of the following game between Builder and Painter. In each turn, Builder picks an edge of initially uncoloured \( K_n \) and Painter chooses any colour out of \( 1, ..., t \) and colours the edge with this colour. Builder wins once there is a graph \( G_i \) in colour \( i \) for some \( G_i \in G_t \) and some \( i \in \{1, ..., t\} \). We note that the definitions of \( \tilde{R}(G_1, ..., G_t; n) \) differ slightly between the previous papers on this topic [1, 2, 4], but it is easy to see that all are equivalent.

Briggs and Cox [1] studied the restricted online Ramsey numbers of matchings and trees. They proved the following general theorem.

Theorem 1.1. Fix \( t \geq 2 \) and positive integers \( r_1, ..., r_t \). If \( n \geq R(r_1K_1, ..., r_tK_t) \), then

\[
\tilde{R}(r_1K_1, ..., r_tK_t; n) \leq \frac{2t - 1 + (t - 3) \log_2(t - 2)}{t + 1} n
\]

with the convention that \( \log_2 0 = 0 \).

Briggs and Cox [1] describe a further refinements of their proof in the cases \( t = 2, 3, 4 \), which imply the following result for \( r_1 = ... = r_t = r \) and \( n = R_t(rK_2) \).

Theorem 1.2. Fix \( r \geq 1 \) and let \( n_2 = R_2(rK_2) = 3r - 1, n_3 = R_3(rK_2) = 4r - 2, n_4 = R_4(rK_2) = 5r - 3 \). Then \( \tilde{R}_2(rK_2; n_2) \leq 3r - 2 = n_2 - 1, \tilde{R}_3(rK_2; n_3) \leq 5r - 4 \) and \( \tilde{R}_4(rK_2; n_4) \leq 7r - 5 \).

They ask whether we have \( \tilde{R}_2(rK_2; n_2) = n_2 - 1 \). The aim of this short note is to verify that this indeed holds. We also show that the bound \( \tilde{R}_3(rK_2; n_3) \leq 5r - 4 \)
is tight and that the bound $\tilde{R}_4(rK_2; n_4) \leq 7r - 5$ is tight except possibly for the exact value of the additive constant.

By describing a suitable strategy of Painter, we prove the following more general lower bound.

**Theorem 1.3.** Fix $t \geq 2$ and positive integers $r_1, \ldots, r_t$. If $n \geq R(r_1 K_1, \ldots, r_t K_t)$, then $\tilde{R}(r_1 K_2, \ldots, r_t K_2; n) \geq 3(\sum_{i=1}^{t} r_i - t + 1) - n$.

As a corollary, we answer the question of Briggs and Cox [1].

**Corollary 1.4.** Fix $r \geq 1$ and let $n_2 = R_2(rK_2) = 3r - 1$, $n_3 = R_3(rK_2) = 4r - 2$, $n_4 = R_4(rK_2) = 5r - 3$. Then $\tilde{R}_2(rK_2; n_2) = 3r - 2 = n_2 - 1$, $\tilde{R}_3(rK_2; n_3) = 5r - 4$ and $\tilde{R}_4(rK_2; n_4) \in \{7r - 6, 7r - 5\}$.

It remains unclear whether for $t$ and $r$ large and $n = R_t(rK_2)$, the magnitude of $\tilde{R}_t(rK_2; n)$ is closer to the upper bound from Theorem 1.1 or to the lower bound from Theorem 1.3.

### 2. Proof of Theorem 1.3

Consider the game played with $t$ colours on the edges of an initially uncoloured $K_n$. To prove Theorem 1.3, we will describe a strategy of Painter that ensures that after $T = 3(\sum_{i=1}^{t} r_i - t + 1) - n - 1$ moves (where by a move we mean Builder choosing some still uncoloured edge and Painter colouring it), there is no $r_i K_2$ of colour $i$ for $i = 1, \ldots, t$.

While taking her turns (and to help her with her colouring decisions), Painter will moreover assign the following states to the coloured edges of $K_n$ and to all the vertices of $K_n$. Coloured edges are either free, or rooted. Every rooted edge is characterized by its root, which is a vertex of $K_n$. Painter will assign (and update) the states of the coloured edges according to the strategy described below.

Vertices are of three types, characterized in the following way.

- If a vertex $v$ is a root of at least one coloured edge, it is of type I.
- If a vertex $v$ is not of type I, but there is at least one free edge with endpoint $v$, it is of type II.
- If a vertex $v$ is neither of type I nor of type II, it is of type III.

In particular, note that initially all the vertices are of type III, since no edges are coloured at the start of the game.

For $0 \leq j \leq \binom{n}{2}$ and $i = 1, \ldots, t$, let $A_j(i)$ be a number of type I vertices that are roots to at least one edge of colour $i$ after $j$ moves and let $B_j(i)$ be a number of free edges of colour $i$ after $j$ moves. Let $A_j = \sum_{i=1}^{t} A_j(i)$ and $B_j = \sum_{i=1}^{t} B_j(i)$.

Assume Builder chooses the edge $ab$ in $(k + 1)$st turn of hers (where $0 \leq k \leq \binom{n}{2} - 1$). Without loss of generality (as we could otherwise switch $a$ and $b$), we can assume that if $b$ is of type I, then $a$ is also of type I; and if $b$ is of type II, then $a$ is of type I or of type II. Painter chooses the colour of an edge and updates the states of the coloured edges as follows.

(i) If $a$ is a vertex of type I, we declare the edge $ab$ to be rooted at $a$. By definition, there exists at least one other edge rooted at $a$, of some colour $c_1$ (if there are more edges rooted at $a$, pick one arbitrarily). We colour $ab$ by colour $c_1$.

(ii) If $a$ is a vertex of type II, there exists by definition a free edge $ac$ for some $c$, of some colour $c_2$ (if there are more free edges with endpoint $a$, pick one arbitrarily). We declare both edges $ab, ac$ to be rooted at $a$ and colour $ab$ in $c_2$. 


(iii) If $a$ is a vertex of type III, then the edge $ab$ is declared to be free. It is coloured in any colour $c_3$ such that $A_k(c_3) + B_k(c_3) \leq r_{c_3} - 2$ if at least one such colour exists, and if not in an arbitrary colour.

The next two observations are straightforward.

**Observation 2.1.** The number of vertices of type III:
- stays the same during move (i)
- increases by 1 or stays the same during move (ii)
- decreases by 2 during move (iii)

**Observation 2.2.** If move $j$ was (i) or (ii), we have $A_j(i) + B_j(i) = A_{j-1}(i) + B_{j-1}(i)$ for $i = 1, \ldots, t$. If move $j$ was (iii) and Painter used colour $c$, we have $A_j(c) + B_j(c) = A_{j-1}(c) + B_{j-1}(c) + 1$ and for any $c' \neq c$ we have $A_j(c') + B_j(c') = A_{j-1}(c') + B_{j-1}(c')$.

Using Observation 2.1 and Observation 2.2, we prove the key lemma.

**Lemma 2.3.** We have $A_T + B_T \leq \sum_{i=1}^{t} r_i - t$.

**Proof.** Let $C_2$ be the number of moves (ii) up to time $T$, and let $C_3$ be the number of moves (iii) up to time $T$. At time $T$, by Observation 2.1 we have at most $n + C_2 - 2C_3$ vertices of type III. That implies $n + C_2 - 2C_3 \geq 0$. Since we further have $C_2 + C_3 \leq T$, we must have $C_3 \leq \frac{4 + T}{3}$.

Now by Observation 2.2, $A_T + B_T \leq C_3 \leq \frac{2 + T}{3} = \sum_{i=1}^{t} r_i - t + \frac{2}{3}$, and since $A_T + B_T$ is an integer, we have $A_T + B_T \leq \sum_{i=1}^{t} r_i - t$ as required. \hfill \Box

Continuing the proof of Theorem 1.3, we are now ready to show that after $T$ moves, there is no $r_iK_2$ of colour $i$ for $i = 1, \ldots, t$.

Note that the existence of $r_mK_2$ of colour $m$ would in particular imply that $A_T(m) + B_T(m) \geq r_m$. Because of the strategy of Painter and Observation 2.2, that would imply that $A_T(i) + B_T(i) \geq r_i - 1$ for $i = 1, \ldots, t$. Hence we would have $A_T + B_T \geq (r_1 - 1) + \ldots + r_m + \ldots + (r_t - 1) = \sum_{i=1}^{t} r_i - t + 1$, contradicting Lemma 2.3. Thus the proof of Theorem 1.3 is finished.

**Acknowledgements**

The author would like to thank his PhD supervisor professor Béla Bollobás for his support.

**References**

[1] J. Briggs, C. Cox Restricted online Ramsey numbers of matchings and trees, Electronic Journal of Combinatorics 27(3)(2020), P3.49.
[2] D. Conlon, J. Fox, A. Grinshpun, X. He Online Ramsey numbers and the subgraph query problem, In Building Bridges II (2019), 159–164.
[3] D. Conlon, J. Fox, B. Sudakov Recent developments in graph Ramsey theory, In A. Czumaj, A. Georgakopoulos, D. Kráľ, V. Lozin, O. Pikhurko (Eds.), Surveys in Combinatorics (2015) 49–118.
[4] D. Gonzalez, X. He, H. Zheng An upper bound for the restricted online Ramsey number, Discrete Mathematics 342(9) (2019), 2564—2569.

(Vojtěch Dvořák) Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, UK

Email address, Vojtěch Dvořák: vd273@cam.ac.uk