WEIGHTED NORM INEQUALITIES OF \((1, q)\)-TYPE FOR INTEGRAL AND FRACTIONAL MAXIMAL OPERATORS

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Dedicated to Richard L. Wheeden

ABSTRACT. We study weighted norm inequalities of \((1, q)\)-type for \(0 < q < 1\),
\[
\|Gv\|_{L^q(\Omega, d\sigma)} \leq C\|v\|
\]
for all positive measures \(v\) in \(\Omega\),
along with their weak-type counterparts, where \(\|v\| = v(\Omega)\), and \(G\) is an integral operator with nonnegative kernel,
\[
Gv(x) = \int_{\Omega} G(x, y) d\nu(y).
\]

These problems are motivated by sublinear elliptic equations in a domain \(\Omega \subset \mathbb{R}^n\) with non-trivial Green’s function \(G(x, y)\) associated with the Laplacian, fractional Laplacian, or more general elliptic operator.

We also treat fractional maximal operators \(M_\alpha (0 \leq \alpha < n)\) on \(\mathbb{R}^n\), and characterize strong- and weak-type \((1, q)\)-inequalities for \(M_\alpha\) and more general maximal operators, as well as \((1, q)\)-Carleson measure inequalities for Poisson integrals.

1. INTRODUCTION

In this paper, we discuss recent results on weighted norm inequalities of \((1, q)\)-type in the case \(0 < q < 1\),
\[
\|Gv\|_{L^q(\Omega, d\sigma)} \leq C\|v\|
\]
for all positive measures \(v\) in \(\Omega\), where \(\|v\| = v(\Omega)\), and \(G\) is an integral operator with nonnegative kernel,
\[
Gv(x) = \int_{\Omega} G(x, y) d\nu(y).
\]

Such problems are motivated by sublinear elliptic equations of the type
\[
\begin{cases}
-\Delta u = \sigma u^q & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]
in the case \(0 < q < 1\), where \(\Omega\) is an open set in \(\mathbb{R}^n\) with non-trivial Green’s function \(G(x, y)\), and \(\sigma \geq 0\) is an arbitrary locally integrable function, or locally finite measure in \(\Omega\).

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The only restrictions imposed on the kernel $G$ are that it is quasi-symmetric and satisfies a weak maximum principle. In particular, $G$ can be a Green operator associated with the Laplacian, a more general elliptic operator (including the fractional Laplacian), or a convolution operator on $\mathbb{R}^n$ with radially symmetric decreasing kernel $G(x,y) = k(|x-y|)$ (see [1], [12]).

In particular, we consider in detail the one-dimensional case where $\Omega = \mathbb{R}^+$ and $G(x,y) = \min(x,y)$. We deduce explicit characterizations of the corresponding $(1,q)$-weighted norm inequalities, give explicit necessary and sufficient conditions for the existence of weak solutions, and obtain sharp two-sided pointwise estimates of solutions.

We also characterize weak-type counterparts of (1.1), namely,

$$
\|G\nu\|_{L^q(\Omega, d\sigma)} \leq C \|\nu\|.
$$

Along with integral operators, we treat fractional maximal operators $M_\alpha$ with $0 \leq \alpha < n$ on $\mathbb{R}^n$, and characterize both strong- and weak-type $(1,q)$-inequalities for $M_\alpha$, and more general maximal operators. Similar problems for Riesz potentials were studied earlier in [6]–[8]. Finally, we apply our results for the integral operators to the Poisson kernel to characterize a $(1,q)$-Carleson measure inequality.

2. Integral Operators

2.1. Strong-Type $(1,q)$-Inequality for Integral Operators. Let $\Omega \subseteq \mathbb{R}^n$ be a connected open set. By $\mathcal{M}^+(\Omega)$ we denote the class of all nonnegative locally finite Borel measures in $\Omega$. Let $G: \Omega \times \Omega \to [0, +\infty]$ be a nonnegative lower-semicontinuous kernel. We will assume throughout this paper that $G$ is quasi-symmetric, i.e., there exists a constant $a > 0$ such that

$$
a^{-1} G(x,y) \leq G(y,x) \leq a G(x,y), \quad x, y \in \Omega.
$$

If $\nu \in \mathcal{M}^+(\Omega)$, then by $G\nu$ and $G^*\nu$ we denote the integral operators (potentials) defined respectively by

$$
G\nu(x) = \int_\Omega G(x,y) d\nu(y), \quad G^*\nu(x) = \int_\Omega G(y,x) d\nu(y), \quad x \in \Omega.
$$

We say that the kernel $G$ satisfies the weak maximum principle if, for any constant $M > 0$, the inequality

$$
G\nu(x) \leq M \quad \text{for all } x \in S(\nu)
$$

implies

$$
G\nu(x) \leq hM \quad \text{for all } x \in \Omega,
$$

where $h \geq 1$ is a constant, and $S(\nu) := \text{supp} \ \nu$. When $h = 1$, we say that $G\nu$ satisfies the strong maximum principle.

It is well-known that Green’s kernels associated with many partial differential operators are quasi-symmetric, and satisfy the weak maximum principle (see, e.g., [2], [3], [12]).
The kernel $G$ is said to be degenerate with respect to $\sigma \in \mathcal{M}^+(\Omega)$ provided there exists a set $A \subset \Omega$ with $\sigma(A) > 0$ and

$$G(\cdot, y) = 0 \quad d\sigma\text{-a.e. for } y \in A.$$ 

Otherwise, we will say that $G$ is non-degenerate with respect to $\sigma$. (This notion was introduced in [19] in the context of $(p, q)$-inequalities for positive operators $T : L^p \to L^q$ in the case $1 < q < p$.)

Let $0 < q < 1$, and let $G$ be a kernel on $\Omega \times \Omega$. For $\sigma \in \mathcal{M}^+(\Omega)$, we consider the problem of the existence of a positive solution $u$ to the integral equation

$$u = G(u^q d\sigma) \quad \text{in } \Omega, \quad 0 < u < +\infty \quad d\sigma\text{-a.e.,} \quad u \in L^q_{\text{loc}}(\Omega).$$

We call $u$ a positive supersolution if

$$u \geq G(u^q d\sigma) \quad \text{in } \Omega, \quad 0 < u < +\infty \quad d\sigma\text{-a.e.,} \quad u \in L^q_{\text{loc}}(\Omega).$$

This is a generalization of the sublinear elliptic problem (see, e.g., [4], [5], and the literature cited there):

$$\begin{cases}
-\Delta u = \sigma u^q \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega,
\end{cases}$$

where $\sigma$ is a nonnegative locally integrable function, or measure, in $\Omega$.

If $\Omega$ is a bounded $C^2$-domain then solutions to (2.5) can be understood in the “very weak” sense (see, e.g., [13]). For general domains $\Omega$ with a nontrivial Green function $G$ associated with the Dirichlet Laplacian $\Delta$ in $\Omega$, solutions $u$ are understood as in (2.3).

**Remark 2.1.** In this paper, for the sake of simplicity, we sometimes consider positive solutions and supersolutions $u \in L^q(\Omega, d\sigma)$. In other words, we replace the natural local condition $u \in L^q_{\text{loc}}(\Omega, d\sigma)$ with its global counterpart. Notice that the local condition is necessary for solutions (or supersolutions) to be properly defined.

To pass from solutions $u$ which are globally in $L^q(\Omega, d\sigma)$ to all solutions $u \in L^q_{\text{loc}}(\Omega, d\sigma)$ (for instance, very weak solutions to (2.5)), one can use either a localization method developed in [7] (in the case of Riesz kernels on $\mathbb{R}^n$), or modified kernels $\tilde{G}(x, y) = \frac{G(x, y)}{m(x)m(y)}$, where the modifier $m(x) = \min \{1, G(x, x_0)\}$ (with a fixed pole $x_0 \in \Omega$) plays the role of a regularized distance to the boundary $\partial \Omega$. One also needs to consider the corresponding $(1, q)$-inequalities with a weight $m$ (see [16]). See the next section in the one-dimensional case where $\Omega = (0, +\infty)$.

**Remark 2.2.** Finite energy solutions, for instance, solutions $u \in W^{1, 2}_0(\Omega)$ to (2.5), require the global condition $u \in L^{1+q}(\Omega, d\sigma)$, and are easier to characterize (see [6], [16]).

The following theorem is proved in [16]. (The case where $\Omega = \mathbb{R}^n$ and $G = (-\Delta)^{-\frac{q}{2}}$ is the Riesz potential of order $\alpha \in (0, n)$ was considered earlier in [7].)
Theorem 2.3. Let \( \sigma \in \mathcal{M}^+(\Omega) \), and \( 0 < q < 1 \). Suppose \( G \) is a quasi-symmetric kernel which satisfies the weak maximum principle. Then the following statements are equivalent:

1. There exists a positive constant \( \kappa = \kappa(\sigma) \) such that
   \[
   \| Gv \|_{L^\infty(\sigma)} \leq \kappa \| v \| \quad \text{for all } v \in \mathcal{M}^+(\Omega).
   \]

2. There exists a positive supersolution \( u \in L^q(\Omega, d\sigma) \) to (2.3).

3. There exists a positive solution \( u \in L^q(\Omega, d\sigma) \) to (2.3), provided additionally that \( G \) is non-degenerate with respect to \( \sigma \).

Remark 2.4. The implication (1) \( \Rightarrow \) (2) in Theorem 2.3 holds for any nonnegative kernel \( G \), without assuming that it is either quasi-symmetric, or satisfies the weak maximum principle. This is a consequence of Gagliardo’s lemma [10]; see details in [16].

Remark 2.5. The implication (3) \( \Rightarrow \) (1) generally fails for kernels \( G \) which do not satisfy the weak maximum principle (see examples in [16]).

The following corollary of Theorem 2.3 is obtained in [16].

Corollary 2.6. Under the assumptions of Theorem 2.3 if there exists a positive supersolution \( u \in L^q(\Omega, \sigma) \) to (2.4), then \( G\sigma \in L^{\frac{q}{q-1}}(\Omega, d\sigma) \).

Conversely, if \( G\sigma \in L^{\frac{q}{q-1}}(\Omega, d\sigma) \), then there exists a non-trivial supersolution \( u \in L^q(\Omega, \sigma) \) to (2.4) (respectively, a solution \( u \), provided \( G \) is non-degenerate with respect to \( \sigma \)).

2.2. The One-Dimensional Case. In this section, we consider positive weak solutions to sublinear ODEs of the type (2.5) on the semi-axis \( \mathbb{R}_+ = (0, +\infty) \). It is instructive to consider the one-dimensional case where elementary characterizations of \((1, q)\)-weighted norm inequalities, along with the corresponding existence theorems and explicit global pointwise estimates of solutions are available. Similar results hold for sublinear equations on any interval \((a, b) \subset \mathbb{R}\).

Let \( 0 < q < 1 \), and let \( \sigma \in \mathcal{M}^+(\mathbb{R}_+) \). Suppose \( u \) is a positive weak solution to the equation

\[
- u'' = \sigma u^q \quad \text{on } \mathbb{R}_+, \quad u(0) = 0,
\]

such that \( \lim_{x \to +\infty} \frac{u(x)}{x} = 0 \). This condition at infinity ensures that \( u \) does not contain a linear component. Notice that we assume that \( u \) is concave and increasing on \([0, +\infty)\), and \( \lim_{x \to 0^+} u(x) = 0 \).

In terms of integral equations, we have \( \Omega = \mathbb{R}_+ \), and \( G(x, y) = \min(x, y) \) is the Green function associated with the Sturm-Liouville operator \( \Delta u = u'' \) with zero boundary condition at \( x = 0 \). Thus, (2.6) is equivalent to the equation

\[
u(x) = G(u^q d\sigma)(x) := \int_0^{+\infty} \min(x, y) u(y)^q d\sigma(y), \quad x > 0,
\]

where \( \sigma \) is a locally finite measure on \( \mathbb{R}_+ \), and

\[
\int_0^a y u(y)^q d\sigma(y) < +\infty, \quad \int_{+\infty}^a u(y)^q d\sigma(y) < +\infty, \quad \text{for every } a > 0.
\]
This “local integrability” condition ensures that the right-hand side of (2.7) is well defined. Here intervals \((a, +\infty)\) are used in place of balls \(B(x, r)\) in \(\mathbb{R}^n\).

Notice that
\[
(2.9) \quad u'(x) = \int_x^{+\infty} u(y)^q d\sigma(y), \quad x > 0.
\]

Hence, \(u\) satisfies the global integrability condition
\[
(2.10) \quad \int_0^{+\infty} u(y)^q d\sigma(y) < +\infty
\]
if and only if \(u'(0) < +\infty\).

The corresponding \((1, q)\)-weighted norm inequality is given by
\[
(2.11) \quad \|Gv\|_{L^q(\sigma)} \leq \kappa \|v\|,
\]
where \(\kappa = \kappa(\sigma)\) is a positive constant which does not depend on \(v \in \mathcal{M}^+(\mathbb{R}_+)\).

Obviously, (2.11) is equivalent to
\[
(2.12) \quad \|H_+v + H_-v\|_{L^q(\sigma)} \leq \kappa \|v\| \quad \text{for all } v \in \mathcal{M}^+(\mathbb{R}_+),
\]
where \(H_\pm\) is a pair of Hardy operators,
\[
H_+(y) = \int_0^{+\infty} y^q d\sigma(y), \quad H_-(x) = x \int_x^{+\infty} d\nu(y).
\]

The following proposition can be deduced from the known results on two-weight Hardy inequalities in the case \(p = 1\) and \(0 < q < 1\) (see, e.g., [20]). We give here a simple independent proof.

**Proposition 2.7.** Let \(\sigma \in \mathcal{M}^+(\mathbb{R}_+)\), and let \(0 < q < 1\). Then (2.11) holds if and only if
\[
(2.13) \quad \kappa(\sigma)^q = \int_0^{+\infty} x^q d\sigma(x) < +\infty,
\]
where \(\kappa(\sigma)\) is the best constant in (2.11).

**Proof.** Clearly,
\[
H_+v(x) + H_-v(x) \leq x \|v\|, \quad x > 0.
\]

Hence,
\[
\|H_+v + H_-v\|_{L^q(\sigma)} \leq \left( \int_0^{+\infty} x^q d\sigma(x) \right)^{\frac{1}{q}} \|v\|,
\]
which proves (2.12), and hence (2.11), with \(\kappa = \left( \int_0^{+\infty} x^q d\sigma(x) \right)^{\frac{1}{q}}\).

Conversely, suppose that (2.12) holds. Then, for every \(a > 0\), and \(v \in \mathcal{M}^+(\mathbb{R}_+),
\[
\int_a^a x^q d\sigma(x) \left( \int_x^{+\infty} d\nu(y) \right)^q q \leq \int_0^{+\infty} (H_+v)^q d\sigma \leq \kappa^q \|v\|^q.
\]
For $v = \delta_{x_0}$ with $x_0 > a$, we get
\[
\int_0^a x^q \, d\sigma(x) \leq \kappa.
\]
Letting $a \to +\infty$, we deduce (2.13). \qed

Clearly, the Green kernel $G(x, y) = \min(x, y)$ is symmetric, and satisfies the strong maximum principle. Hence, by Theorem 2.3 equations (2.6) and (2.7) have a non-trivial (super)solution $u \in L^q(\mathbb{R}_+, \sigma)$ if and only if (2.13) holds.

From Proposition 2.7, we deduce that, for “localized” measures $d\sigma_a = \chi(0, a) \, d\sigma (a > 0)$, we have
\[
(2.14) \quad \kappa(\sigma_a) = \left( \int_a^{+\infty} x^q \, d\sigma(x) \right)^{\frac{1}{q}}.
\]

Using this observation and the localization method developed in [7], we obtain the following existence theorem for general weak solutions to (2.5), along with sharp pointwise estimates of solutions.

We introduce a new potential
\[
(2.15) \quad K\sigma(x) := x \left( \int_x^{+\infty} y^q \, d\sigma(y) \right)^{\frac{1}{1-q}}, \quad x > 0.
\]

We observe that $K\sigma$ is a one-dimensional analogue of the potential introduced recently in [7] in the framework of intrinsic Wolff potentials in $\mathbb{R}^n$ (see also [8] in the radial case). Matching upper and lower pointwise bounds of solutions are obtained below by combining $G\sigma$ with $K\sigma$.

**Theorem 2.8.** Let $\sigma \in \mathcal{M}^+ (\mathbb{R}_+)$, and let $0 < q < 1$. Then equation (2.5), or equivalently (2.6) has a nontrivial solution if and only if, for every $a > 0$,
\[
(2.16) \quad \int_0^a x \, d\sigma(x) + \int_a^{+\infty} x^q \, d\sigma(x) < +\infty.
\]
Moreover, if (2.16) holds, then there exists a positive solution $u$ to (2.5) such that
\[
(2.17) \quad C^{-1} \left[ \left( \int_0^x y \, d\sigma(y) \right)^{\frac{1}{1-q}} + K\sigma(x) \right] \leq u(x) \leq C \left[ \left( \int_0^x y \, d\sigma(y) \right)^{\frac{1}{1-q}} + K\sigma(x) \right].
\]
The lower bound in (2.17) holds for any non-trivial supersolution $u$.

**Remark 2.9.** The lower bound
\[
(2.18) \quad u(x) \geq (1 - q)^{\frac{1}{1-q}} \left[ G\sigma(x) \right]^{\frac{1}{1-q}}, \quad x > 0,
\]
is known for a general kernel $G$ which satisfies the strong maximum principle (see [11], Theorem 3.3; [16]), and the constant $(1 - q)^{\frac{1}{1-q}}$ here is sharp. However, the second term on the left-hand side of (2.17) makes the lower estimate stronger, so that it matches the upper estimate.
**Proof.** The lower bound

\[(2.20) \quad u(x) \geq (1-q) \frac{1}{r_q} \left[ \int_0^x y d\sigma(y) \right] ^{\frac{1}{r_q}}, \quad x > 0,\]

is immediate from (2.19).

Applying Lemma 4.2 in [7], with the interval \((a, +\infty)\) in place of a ball \(B\), and combining it with (2.14), for any \(a > 0\) we have

\[\int_a^{+\infty} u(y)^q d\sigma(y) \geq c(q) \int_0^{+\infty} y^q d\sigma(y) \geq c(q) \left[ \int_0^{+\infty} y^q d\sigma(y) \right] ^{\frac{1}{r_q}}.\]

Hence,

\[u(x) \geq G(u^q d\sigma) \geq x \int_0^{+\infty} y^q d\sigma(y) \left[ \int_0^{+\infty} y^q d\sigma(y) \right] ^{\frac{1}{r_q}}.\]

Combining the preceding estimate with (2.20), we obtain the lower bound in (2.17) for any non-trivial supersolution \(u\). This also proves that (2.16) is necessary for the existence of a non-trivial positive supersolution.

Conversely, suppose that (2.16) holds. Let

\[(2.21) \quad v(x) := c \left[ \int_0^x y d\sigma(y) \right] ^{\frac{1}{r_q}} + K\sigma(x), \quad x > 0,\]

where \(c\) is a positive constant. It is not difficult to see that \(v\) is a supersolution, so that \(v \geq G(v^q d\sigma)\), if \(c = c(q)\) is picked large enough. (See a similar argument in the proof of Theorem 5.1 in [3].)

Also, it is easy to see that \(v_0 = c_0(v^q d\sigma) ^{\frac{1}{r_q}}\) is a subsolution, i.e., \(v_0 \leq G(v^q d\sigma)\), provided \(c_0 > 0\) is a small enough constant. Moreover, we can ensure that \(v_0 \leq v\) if \(c_0 = c_0(q)\) is picked sufficiently small. (See details in [8] in the case of radially symmetric solutions in \(\mathbb{R}^n\).) Hence, there exists a solution which can be constructed by iterations, starting from \(u_0 = v_0\), and letting

\[u_{j+1} = G(u_j^q d\sigma), \quad j = 0, 1, \ldots.\]

Then by induction \(u_j \leq u_{j+1} \leq v\), and consequently \(u = \lim_{j \to +\infty} u_j\) is a solution to (2.22) by the Monotone Convergence Theorem. Clearly, \(u \leq v\), which proves the upper bound in (2.17). \qed

### 2.3 Weak-Type \((1,q)\)-Inequality for Integral Operators

In this section, we characterize weak-type analogues of \((1,q)\)-weighted norm inequalities considered above. We will use some elements of potential theory for general positive kernels \(G\), including the notion of inner capacity, \(\text{cap}(\cdot)\), and the associated equilibrium (extremal) measure (see [9]).

**Theorem 2.10.** Let \(\sigma \in \mathcal{M}^+(\mathbb{R}^n), 0 < q < 1, \) and \(0 \leq \alpha < n\). Suppose \(G\) satisfies the weak maximum principle. Then the following statements are equivalent:

1. There exists a positive constant \(\kappa_w\) such that

\[\|Gv\|_{L^{q,\omega}(\sigma)} \leq \kappa_w \|v\| \quad \text{for all } v \in \mathcal{M}^+(\mathbb{R}^n).\]
There exists a positive constant $c$ such that
\[ \sigma(K) \leq c \left( \text{cap}(K) \right)^q \]
for all compact sets $K \subset \mathbb{R}^n$.

\[ G\sigma \in L^{\frac{1}{1-q},\infty}(\sigma). \]

Proof: (1) $\Rightarrow$ (2) Without loss of generality we may assume that the kernel $G$ is strictly positive, that is, $G(x,x) > 0$ for all $x \in \Omega$. Otherwise, we can consider the kernel $G$ on the set $\Omega \setminus A$, where $A := \{ x \in \Omega: G(x,x) \neq 0 \}$, since $A$ is negligible for the corresponding $(1,q)$-inequality in statement (1). (See details in [16] in the case of the corresponding strong-type inequalities.)

We remark that the kernel $G$ is known to be strictly positive if and only if, for any compact set $K \subset \Omega$, the inner capacity $\text{cap}(K)$ is finite ([9]). In this case there exists an equilibrium measure $\lambda$ on $K$ such that
\[ G\lambda \geq 1 \text{ n.e. on } K, \quad G\lambda \leq 1 \text{ on } S(\lambda), \quad \|\lambda\| = \text{cap}(K). \]

Here n.e. stands for nearly everywhere, which means that the inequality holds on a given set except for a subset of zero capacity [9].

Next, we remark that condition (1) yields that $\sigma$ is absolutely continuous with respect to capacity, i.e., $\sigma(K) = 0$ if $\text{cap}(K) = 0$. (See a similar argument in [16] in the case of strong-type inequalities.) Consequently, $G\lambda \geq 1$ $d\sigma$-a.e. on $K$. Hence, by applying condition (1) with $\nu = \lambda$, we obtain (2).

(2) $\Rightarrow$ (3) We denote by $\sigma_E$ denotes the restriction of $\sigma$ to a Borel set $E \subset \Omega$. Without loss of generality we may assume that $\sigma$ is a finite measure on $\Omega$. Otherwise we can replace $\sigma$ with $\sigma_F$ where $F$ is a compact subset of $\Omega$. We then deduce the estimate
\[ \|G\sigma_F\|_{L^{\frac{1}{1-q},\infty}(\sigma_F)} \leq C < \infty, \]
where $C$ does not depend on $F$, and use the exhaustion of $\Omega$ by an increasing sequence of compact subsets $F_n \uparrow \Omega$ to conclude that $G\sigma \in L^{\frac{1}{1-q},\infty}(\sigma)$ by the Monotone Convergence Theorem.

Set $E_t := \{ x \in \Omega: G\sigma(x) > t \}$, where $t > 0$. Notice that, for all $x \in (E_t)^c$,
\[ G\sigma_{(E_t)^c}(x) \leq G\sigma(x) \leq t. \]

The set $(E_t)^c$ is closed, and hence the preceding inequality holds on $S(\sigma_{(E_t)^c})$. It follows by the weak maximum principle that, for all $x \in \Omega$,
\[ G\sigma_{(E_t)^c}(x) \leq G\sigma(x) \leq ht. \]

Hence,
\[ \{ x \in \Omega: G\sigma(x) > (h+1)t \} \subset \{ x \in \Omega: G\sigma_{E_t}(x) > t \}. \]

Denote by $K \subset \Omega$ a compact subset of $\{ x \in \Omega: G\sigma_{E_t}(x) > t \}$. By (2), we have
\[ \sigma(K) \leq c \left( \text{cap}(K) \right)^q \]
If $\lambda$ is the equilibrium measure on $K$, then $G\lambda \leq 1$ on $S(\lambda)$, and $\lambda(K) = \text{cap}(K)$ by (2.22). Hence by the weak maximum principle $G\lambda \leq h$ on $\Omega$. Using quasi-symmetry of the kernel $G$ and Fubini’s theorem, we have

$$\text{cap}(K) = \int_K d\lambda = \frac{1}{t} \int_K G\sigma d\lambda \leq \frac{a}{t} \int_{E_t} Gd\sigma \leq \frac{ah}{t} \sigma(E_t).$$

This shows that

$$\sigma(K) \leq \frac{c(ah)^q}{t^q} \left( \sigma(E_t) \right)^q.$$

Taking the supremum over all $K \subset E_t$, we deduce

$$\left( \sigma(E_t) \right)^{1-q} \leq \frac{c(ah)^q}{t^q}.$$

It follows from (2.23) that, for all $t > 0$,

$$t^{\frac{q}{1-q}} \sigma \left( \Omega: G\sigma(x) > (h+1)t \right) \leq t^{\frac{q}{1-q}} \sigma(E_t) \leq c^{\frac{1}{1-q}} (ah)^{\frac{q}{1-q}}.$$

Thus, (3) holds.

(3) $\Rightarrow$ (2) By Hölder’s inequality for weak $L^q$ spaces, we have

$$\|G\nu\|_{L^q(\sigma)} = \left\| \frac{G\nu}{G\sigma} G\sigma \right\|_{L^q(\sigma)} \leq \left\| \frac{G\nu}{G\sigma} \right\|_{L^{\frac{q}{1-q}}(\sigma)} \left\| G\sigma \right\|_{L^{\frac{q}{1-q}}(\sigma)} \leq C \left\| G\sigma \right\|_{L^{\frac{q}{1-q}}(\sigma)} \|\nu\|,$$

where the final inequality,

$$\left\| \frac{G\nu}{G\sigma} \right\|_{L^{\frac{q}{1-q}}(\sigma)} \leq C \|\nu\|,$$

with a constant $C = C(h, a)$, was obtained in [16], for quasi-symmetric kernels $G$ satisfying the weak maximum principle. □

3. FRACTIONAL MAXIMAL OPERATORS

We denote by $\mathcal{M}^+(\mathbb{R}^n)$ the class of positive locally finite Borel measures on $\mathbb{R}^n$. For $\nu \in \mathcal{M}^+(\mathbb{R}^n)$, we set $\|\nu\| = \nu(\mathbb{R}^n)$.

Let $\nu \in \mathcal{M}^+(\mathbb{R}^n)$, and let $0 \leq \alpha < n$. We define the fractional maximal operator $M_\alpha$ by

$$M_\alpha \nu(x) := \sup_{Q \ni x} \frac{|Q| \nu}{|Q|^{1-\frac{\alpha}{n}}}, \quad x \in \mathbb{R}^n,$$

where $Q$ is a ball in $\mathbb{R}^n$. This definition is consistent with the standard fractional maximal operator for $\alpha = n$. For $\alpha < n$, the operator $M_\alpha$ is defined in terms of the measure $\nu$ and the volume of balls in $\mathbb{R}^n$. The operator $M_\alpha$ is a non-trivial extension of the classical maximal function, and it plays a crucial role in the study of singular integrals and partial differential equations.
where $Q$ is a cube, $|Q|_v := v(Q)$, and $|Q|$ is the Lebesgue measure of $Q$. If $f \in L^1_{\text{loc}}(\mathbb{R}^n, d\mu)$ where $\mu \in \mathscr{M}^+(\mathbb{R}^n)$, we set $M_\sigma(f d\mu) = M_\sigma v$ where $d\nu = |f| d\mu$, i.e.,

$$M_\sigma(f d\mu)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f| d\mu, \quad x \in \mathbb{R}^n. \tag{3.2}$$

For $\sigma \in \mathscr{M}^+(\mathbb{R}^n)$, it was shown in [22] that in the case $0 < q < p$,

$$M_\sigma : L^p(dx) \to L^q(d\sigma) \iff M_\sigma \nu \in L^{q,n}(d\sigma), \tag{3.3}$$

$$M_\sigma : L^p(dx) \to L^{q,\infty}(d\sigma) \iff M_\sigma \nu \in L^{q,\infty,n}(d\sigma), \tag{3.4}$$

provided $p > 1$.

More general two-weight maximal inequalities

$$\|M_\sigma(f d\mu)\|_{L^q(\sigma)} \leq \kappa \|f\|_{L^p(\mu)}, \quad \text{for all } f \in L^p(\mu), \tag{3.5}$$

where characterized by E. T. Sawyer [18] in the case $p = q > 1$, R. L. Wheeden [24] in the case $q > p > 1$, and the second author [22] in the case $0 < q < p$ and $p > 1$, along with their weak-type counterparts,

$$\|M_\sigma(f d\mu)\|_{L^{q,\infty}(\sigma)} \leq \kappa_w \|f\|_{L^p(\mu)}, \quad \text{for all } f \in L^p(\mu), \tag{3.6}$$

where $\sigma, \mu \in \mathscr{M}^+(\mathbb{R}^n)$, and $\kappa, \kappa_w$ are positive constants which do not depend on $f$.

However, some of the methods used in [22] for $0 < q < p$ and $p > 1$ are not directly applicable in the case $p = 1$, although there are analogues of these results for real Hardy spaces, i.e., when the norm $\|f\|_{L^p(\mu)}$ on the right-hand side of (3.5) or (3.6) is replaced with $\|M_\nu f\|_{L^p(\mu)}$, where

$$M_\nu f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f| d\mu. \tag{3.7}$$

We would like to understand similar problems in the case $0 < q < 1$ and $p = 1$, in particular, when $M_\alpha : \mathscr{M}^+(\mathbb{R}^n) \to L^q(d\sigma)$, or equivalently, there exists a constant $\kappa > 0$ such that the inequality

$$\|M_\alpha v\|_{L^q(\sigma)} \leq \kappa \|v\| \tag{3.8}$$

holds for all $v \in \mathscr{M}^+(\mathbb{R}^n)$.

In the case $\alpha = 0$, Rozin [17] showed that the condition

$$\sigma \in L^{\frac{n}{n-\alpha}-1}(\mathbb{R}^n, dx)$$

is sufficient for the Hardy-Littlewood operator $M = M_0 : L^1(dx) \to L^q(\sigma)$ to be bounded; moreover, when $\sigma$ is radially symmetric and decreasing, this is also a necessary condition. We will generalize this result and provide necessary and sufficient conditions for the range $0 \leq \alpha < n$. We also obtain analogous results for the weak-type inequality

$$\|M_\alpha v\|_{L^{q,\infty}(\sigma)} \leq \kappa_w \|v\|, \quad \text{for all } v \in \mathscr{M}^+(\mathbb{R}^n), \tag{3.9}$$

where $\kappa_w$ is a positive constant which does not depend on $v$. 


We treat more general maximal operators as well, in particular, dyadic maximal operators

\[ M_\rho v(x) := \sup_{Q \in \mathcal{D}} \rho_Q |Q| v, \]

where \( \mathcal{D} \) is the family of all dyadic cubes in \( \mathbb{R}^n \), and \( \{\rho_Q\}_{Q \in \mathcal{D}} \) is a fixed sequence of nonnegative reals associated with \( Q \in \mathcal{D} \). The corresponding weak-type maximal inequality is given by

\[ \|M_\rho v\|_{L^{q^*}(\sigma)} \leq \nu_w \|v\|, \quad \text{for all } v \in \mathcal{M}(\mathbb{R}^n). \]

\[ u \in L^q(\sigma), \quad \text{and } u \geq M_\alpha(u^d \nu), \]

Moreover, \( u \) can be constructed as follows: \( u = \lim_{j \to \infty} u_j \), where \( u_0 := (M_\alpha \sigma)^{1/q} \), \( u_{j+1} \geq u_j \), and

\[ u_{j+1} := M_\alpha(u_j^d \sigma), \quad j = 0, 1, \ldots. \]

In particular, \( u \geq (M_\alpha \sigma)^{1/q} \).

Proof: \((\Rightarrow)\) We let \( u_0 := (M_\alpha \sigma)^{1/q} \). Notice that, for all \( x \in Q \), we have \( u_0(x) \geq \left( \frac{|Q|}{|Q|^1 - \frac{\alpha}{q}} \right)^{1/q} \). Hence,

\[ u_1(x) := M_\alpha(u_0^d \sigma)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1 - \frac{\alpha}{q}}} \int_Q u_0^d \sigma \geq \sup_{Q \ni x} \left( \frac{|Q|}{|Q|^1 - \frac{\alpha}{q}} \right)^{1/q} = u_0(x). \]

By induction, we see that

\[ u_{j+1} := M_\alpha(u_j^d \sigma) \geq M_\alpha(u_{j-1}^d \sigma) = u_j, \quad j = 1, 2, \ldots. \]

Let \( u = \lim u_j \). By \((3.8)\), we have

\[ \|u_{j+1}\|_{L^q(\sigma)} = \|M_\alpha(u_j^d \sigma)\|_{L^q(\sigma)} \leq \nu \|u_j\|_{L^q(\sigma)} \leq \nu \|u_j+1\|_{L^q(\sigma)}. \]

From this we deduce that \( \|u_{j+1}\|_{L^q(\sigma)} \leq \nu \|u_j\|_{L^q(\sigma)} \) for \( j = 0, 1, \ldots \). Since the norms \( \|u_j\|_{L^q(\sigma)} \) are uniformly bounded, by the Monotone Convergence Theorem, we have for \( u := \lim_{j \to \infty} u_j \) that \( u \in L^q(\sigma) \). Note that by construction \( u = M_\alpha(u^d \nu) \).

\((\Leftarrow)\) We can assume here that \( M_\alpha v \) is defined, for \( v \in \mathcal{M}(\mathbb{R}^n) \), as the centered fractional maximal function,

\[ M_\alpha v(x) := \sup_{r > 0} \frac{v(B(x, r))}{|B(x, r)|^{1 - \frac{\alpha}{q}}}, \]

since it is equivalent to its uncentered analogue used above. Suppose that there exists \( u \in L^q(\sigma) \) (\( u \neq 0 \)) such that \( u \geq M_\alpha(u^d \nu) \). Set \( \omega := u^d \nu \). Let \( v \in \mathcal{M}(\mathbb{R}^n) \).
We note that we have
\[
M_\alpha \nu(x) = M_\omega(x) = \sup_{r > 0} \frac{\|B(x, r)\|_\nu}{\|B(x, r)\|_\omega}.
\]

Thus,
\[
\|M_\alpha \nu\|_{L^q(\sigma)} = \frac{\|M_\alpha \nu\|}{\|M_\omega \nu\|_{L^q(\sigma)}} \leq \frac{\|M_\omega \nu\|}{\|M_\omega \nu\|_{L^q(\sigma)}} \leq C \|\nu\|_{L^1(\sigma)} \leq C \|\nu\|.
\]

by Jensen’s inequality and the \((1,1)\)-weak-type maximal function inequality for \(M_\sigma \nu\). This establishes (3.8). □

3.2. Weak-Type Inequality. For \(0 \leq \alpha < n\), we define the Hausdorff content on a set \(E \subset \mathbb{R}^n\) to be
\[
H^{n-\alpha}(E) := \inf \left\{ \sum_{i=1}^{\infty} r_i^{n-\alpha} : E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i) \right\}
\]
where the collection of balls \(\{B(x_i, r_i)\}\) forms a countable covering of \(E\).

Theorem 3.2. Let \(\sigma \in M^+(\mathbb{R}^n)\), \(0 < q < 1\), and \(0 \leq \alpha < n\). Then the following conditions are equivalent:

1. There exists a positive constant \(\kappa_w\) such that
\[
\|M_\alpha \nu\|_{L^q(\sigma)} \leq \kappa_w \|\nu\| \quad \text{for all } \nu \in \mathcal{M}(\mathbb{R}^n).
\]
2. There exists a positive constant \(C > 0\) such that
\[
\sigma(E) \leq C (H^{n-\alpha}(E))^q \quad \text{for all Borel sets } E \subset \mathbb{R}^n.
\]
3. \(M_\alpha \sigma \in L^{\frac{q}{q-1}}(\sigma)\).

Remark 3.3. In the case \(\alpha = 0\) each of the conditions (1)–(3) is equivalent to \(\sigma \in L^{\frac{q}{q-1}}(dx)\).

Proof. (1) \(\Rightarrow\) (2) Let \(K \subset E\) be a compact set in \(\mathbb{R}^n\) such that \(H^{n-\alpha}(K) > 0\). It follows from Frostman’s theorem (see the proof of Theorem 5.1.12 in [1]) that there exists a measure \(\nu\) supported on \(K\) such that \(\nu(K) \leq H^{n-\alpha}(K)\), and, for every
$x \in K$ there exists a cube $Q$ such that $x \in Q$ and $|Q|_v \geq c|Q|^{1-\frac{\alpha}{n}}$, where $c$ depends only on $n$ and $\alpha$. Hence,

$$M_\alpha v(x) \geq \sup_{Q \ni x} \frac{|Q|_v}{|Q|^{1-\frac{\alpha}{n}}} \geq c \quad \text{for all } x \in K,$$

where $c$ depends only on $n$ and $\alpha$. Consequently,

$$\frac{c^q \sigma(K)}{\|M_\alpha v\|_{L^q_\alpha}^q} \leq \varepsilon_0^q \left( H^{n-\alpha}(K) \right)^q.$$

If $H^{n-\alpha}(E) = 0$, then $H^{n-\alpha}(K) = 0$ for every compact set $K \subset E$, and consequently $\sigma(E) = 0$. Otherwise,

$$\sigma(K) \leq \varepsilon_0^q \left( H^{n-\alpha}(K) \right)^q \leq \varepsilon_0^q \left( H^{n-\alpha}(K) \right)^q,$$

for every compact set $K \subset E$, which proves (2) with $C = c^{-q} \varepsilon_0^q$.

(2) $\Rightarrow$ (3) Let $E_t := \{x : M_\alpha \sigma(x) > t\}$, where $t > 0$. Let $K \subset E_t$ be a compact set. Then for each $x \in K$ there exists $Q_x \ni x$ such that

$$\frac{\sigma(Q_x)}{|Q_x|^{1-(\frac{\alpha}{n})}} > t.$$

Now consider the collection $\{Q_x\}_{x \in K}$, which forms a cover of $K$. By the Besicovitch covering lemma, we can find a subcover $\{Q_i\}_{i \in I}$, where $I$ is a countable index set, such that $K \subset \bigcup_{i \in I} Q_i$ and $x \in K$ is contained in at most $b_n$ sets in $\{Q_i\}$. By (2), we have

$$\sigma(K) \leq \left[ H^{n-\alpha}(K) \right]^q,$$

and by the definition of the Hausdorff content we have

$$H^{n-\alpha}(K) \leq \sum_{i \in I} |Q_i|^{-(\alpha/n)}.$$

Since $\{Q_i\}$ have bounded overlap, we have

$$\sum_{i \in I} \sigma(Q_i) \leq b_n \sigma(K).$$

Thus,

$$\sigma(K) \leq \left( b_n \frac{\sigma(K)}{t} \right)^q,$$

which shows that

$$t^\frac{n}{\alpha} \sigma(K) \leq \left( b_n \right)^{\frac{1}{\alpha}} < +\infty.$$

Taking the supremum over all $K \subset E_t$ in the preceding inequality, we deduce $M_\alpha \sigma \in L^{\frac{n}{\alpha}}(\sigma)$.

(3) $\Rightarrow$ (1). We can assume again that $M_\alpha$ is the centered fractional maximal function, since it is equivalent to the uncentered version. Suppose that $M_\alpha \sigma \in L^{\frac{n}{\alpha}}(\sigma)$. Let $v \in M(\mathbb{R}^n)$. Then, as in the case of the strong-type inequality,

$$\frac{M_\alpha v(x)}{M_\alpha \sigma(x)} = \sup_{r > 0} \frac{|B(x,r)|_v}{|B(x,r)|^{1-\frac{\alpha}{n}}} \leq \sup_{r > 0} \frac{|B(x,r)|_\sigma}{|B(x,r)|^{1-\frac{\alpha}{n}}}.$$
Theorem 3.4. Of the proofs must be modified as indicated below. \[ \text{[22]} \] for weak-type inequalities remain valid in the case we estimate a sequence of non-negative reals. Then obviously

By Hölder’s inequality for weak $L^p$-spaces,

\[
\|M_\sigma v\|_{L^{q,-}(\sigma)} \leq \|M_\sigma \alpha\|_{L^{q,-}(\sigma)} \|M_\sigma v\|_{L^{1,-}(\sigma)} \\
\leq c \|M_\sigma \alpha\|_{L^{q,-}(\sigma)} \|v\|,
\]

where in the last line we have used the $1,1$-weak-type maximal function inequality for the centered maximal function $M_\sigma v$. \hfill $\square$

We now characterize weak-type $(1, q)$-inequalities (3.11) for the generalized dyadic maximal operator $M_\rho$ defined by (3.10). The corresponding $(p, q)$-inequalities in the case $0 < q < p$ and $p > 1$ were characterized in [22]. The results obtained in [22] for weak-type inequalities remain valid in the case $p = 1$, but some elements of the proofs must be modified as indicated below.

**Theorem 3.4.** Let $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$, $0 < q < 1$, and $0 \leq \alpha < n$. Then the following conditions are equivalent:

1. There exists a positive constant $\alpha_w$ such that (3.11) holds.
2. $M_\rho \sigma \in L^{q,-,w}(\sigma)$.

**Proof.** (2) $\Rightarrow$ (1) The proof of this implication is similar to the case of fractional maximal operators. Let $v \in \mathcal{M}(\mathbb{R}^n)$. Denoting by $Q, P \in \mathcal{D}$ dyadic cubes in $\mathbb{R}^n$, we estimate

\[
M_\rho v(x) = \sup_{Q \ni x} \langle \rho_Q |Q|_v \rangle = \sup_{P \ni x} \langle \rho_P |P|_\sigma \rangle \\
\leq \sup_{Q \ni x} \frac{|Q|_v}{|Q|_\sigma} =: M_\sigma v(x).
\]

By Hölder’s inequality for weak $L^p$-spaces,

\[
\|M_\rho v\|_{L^{q,-}(\sigma)} \leq \|M_\rho \alpha\|_{L^{q,-}(\sigma)} \|M_\rho v\|_{L^{1,-}(\sigma)} \\
\leq \|M_\rho \sigma\|_{L^{q,-}(\sigma)} \|M_\rho v\|_{L^{1,-}(\sigma)} \\
\leq c \|M_\rho \sigma\|_{L^{q,-}(\sigma)} \|v\|,
\]

by the $(1,1)$-weak-type maximal function inequality for the dyadic maximal function $M_\sigma$.

1. $\Rightarrow$ (2) We set $f = \sup_Q (\lambda_Q \chi_Q)$ and $dv = f \, d\sigma$, where $\{\lambda_Q\}_{Q \in \mathcal{D}}$ is a finite sequence of non-negative reals. Then obviously

\[
M_\rho v(x) \geq \sup_Q (\lambda_Q \rho_Q \chi_Q), \quad \|v\| \leq \sum_Q \lambda_Q |Q|_\sigma.
\]

By (1), for all $\{\lambda_Q\}_{Q \in \mathcal{D}},$

\[
\|\sup_Q (\lambda_Q \rho_Q \chi_Q)\|_{L^{q,-}(\sigma)} \leq \alpha_w \sum_Q \lambda_Q |Q|_\sigma.
\]
Hence, by Theorem 1.1 and Remark 1.2 in [22], it follows that (2) holds.

\[\square\]

4. CARLESON MEASURES FOR POISSON INTEGRALS

In this section we treat \((1, q)\)-Carleson measure inequalities for Poisson integrals with respect to Carleson measures \(\sigma \in \mathcal{M}^+(\mathbb{R}^{n+1}_+)\) in the upper half-space \(\mathbb{R}^{n+1}_+ = (x, y): x \in \mathbb{R}^n, y > 0\). The corresponding weak-type \((p, q)\)-inequalities for all \(0 < q < p\) as well as strong-type \((p, q)\)-inequalities for \(0 < q < p\) and \(p > 1\), were characterized in [23]. Here we consider strong-type inequalities of the type

\[
\|Pv\|_{L^q(\mathbb{R}^{n+1}_+)} \leq \kappa \|v\|_{\mathcal{M}^+(\mathbb{R}^n)}, \quad \text{for all } v \in \mathcal{M}^+(\mathbb{R}^n),
\]

for some constant \(\kappa > 0\), where \(Pv\) is the Poisson integral of \(v \in \mathcal{M}^+(\mathbb{R}^n)\) defined by

\[Pv(x, y) := \int_{\mathbb{R}^n} P(x-t, y)d\nu(t), \quad (x, y) \in \mathbb{R}^{n+1}_+\]

Here \(P(x, y)\) denotes the Poisson kernel associated with \(\mathbb{R}^{n+1}_+\).

By \(P^*\mu\) we denote the formal adjoint (balayage) operator defined, for \(\mu \in \mathcal{M}^+(\mathbb{R}^{n+1}_+)\), by

\[P^*\mu(t) := \int_{\mathbb{R}^{n+1}_+} P(x-t, y)d\mu(x, y), \quad t \in \mathbb{R}^n.
\]

We will also need the symmetrized potential defined, for \(\mu \in \mathcal{M}^+(\mathbb{R}^{n+1}_+)\), by

\[PP^*\mu(x, y) := P(\mu^*d\nu) := \int_{\mathbb{R}^{n+1}_+} P(x-\tilde{x}, y+\tilde{y})d\mu(\tilde{x}, \tilde{y}), \quad (x, y) \in \mathbb{R}^{n+1}_+.
\]

As we will demonstrate below, the kernel of \(PP^*\mu\) satisfies the weak maximum principle with constant \(h = 2^{n+1}\).

**Theorem 4.1.** Let \(\sigma \in \mathcal{M}^+(\mathbb{R}^{n+1}_+)\), and let \(0 < q < 1\). Then inequality (4.7) holds if and only if there exists a function \(u > 0\) such that

\[u \in L^q(\mathbb{R}^{n+1}_+, \sigma), \quad \text{and} \quad u \geq PP^*(u^q \sigma) \quad \text{in } \mathbb{R}^{n+1}_+.
\]

Moreover, if (4.7) holds, then a positive solution \(u = PP^*(u^q \sigma)\) such that \(u \in L^q(\mathbb{R}^{n+1}_+, \sigma)\) can be constructed as follows:

\[u = \lim_{j \to \infty} u_j, \quad \text{where}\]

\[u_{j+1} := PP^*(u_j^q \sigma), \quad j = 0, 1, \ldots, \quad u_0 := c_0(PP^*\sigma)^{\frac{1}{1-q}},
\]

for a small enough constant \(c_0 > 0\) (depending only on \(q\) and \(n\)), which ensures that \(u_{j+1} \geq u_j\). In particular, \(u \geq c_0(PP^*\sigma)^{\frac{1}{1-q}}\).

**Proof.** We first prove that (4.1) holds if and only if

\[
\|PP^*\mu\|_{L^q(\mathbb{R}^{n+1}_+, \sigma)} \leq \kappa \|\mu\|_{\mathcal{M}^+(\mathbb{R}^{n+1}_+)}, \quad \text{for all } \mu \in \mathcal{M}^+(\mathbb{R}^{n+1}_+).
\]

Indeed, letting \(v = P^*\mu\) in (4.1) yields (4.3) with the same embedding constant \(\kappa\).
Conversely, suppose that (4.3) holds. Then by Maurey’s factorization theorem (see [14]), there exists \( F \in L^1(\mathbb{R}_+^{n+1}, \sigma) \) such that \( F > 0 \, d\sigma \)-a.e., and

\[
\|F\|_{L^1(\mathbb{R}_+^{n+1}, \sigma)} \leq 1, \quad \sup_{(x,y) \in \mathbb{R}_+^{n+1}} \mathbb{P}^* (F^{1 - \frac{1}{q}}) (x,y) \leq \infty.
\]

By letting \( y \downarrow 0 \) in (4.4) and using the Monotone Convergence Theorem, we deduce

\[
\sup_{x \in \mathbb{R}^n} \mathbb{P}^* (F^{1 - \frac{1}{q}}) (x) \leq \infty.
\]

Hence, by Jensen’s inequality and (4.5), for any \( v \in \mathcal{M}^+(\mathbb{R}^n) \), we have

\[
\|Pv\|_{L^1(\mathbb{R}_+^{n+1}, \sigma)} \leq \|Pv\|_{L^1(\mathbb{R}_+^{n+1}, F^{1 - \frac{1}{q}} d\sigma)} = \|P^* (F^{1 - \frac{1}{q}})\|_{L^1(\mathbb{R}_+^n, d\nu)} \leq \kappa \|v\|_{\mathcal{M}^+(\mathbb{R}^n)}.
\]

We next show that the kernel of \( P^* \) satisfies the weak maximum principle with constant \( h = 2^{n+1} \). Indeed, suppose \( \mu \in \mathcal{M}^+(\mathbb{R}_+^{n+1}) \), and

\[
P^* \mu (\tilde{x}, \tilde{y}) \leq M, \quad \text{for all } (\tilde{x}, \tilde{y}) \in S(\mu).
\]

Without loss of generality we may assume that \( S(\mu) \subseteq \mathbb{R}_+^{n+1} \) is a compact set. For \( t \in \mathbb{R}^n \), let \((x_0, y_0) \in S(\mu) \) be a point such that

\[
| (t,0) - (x_0, y_0) | = \text{dist} \left( (t,0), S(\mu) \right).
\]

Then by the triangle inequality, for any \((\tilde{x}, \tilde{y}) \in S(\mu) \),

\[
| (x_0, y_0) - (\tilde{x}, -\tilde{y}) | \leq | (x_0, y_0) - (t,0) | + | (t,0) - (\tilde{x}, -\tilde{y}) | \leq 2 | (t,0) - (\tilde{x}, \tilde{y}) |.
\]

Hence,

\[
\sqrt{t - \tilde{x}}^2 + \tilde{y}^2 \geq \frac{1}{2} \sqrt{[x_0 - \tilde{x}]^2 + (y_0 + \tilde{y})^2}.
\]

It follows that, for all \( t \in \mathbb{R}^n \) and \((\tilde{x}, \tilde{y}) \in S(\mu) \), we have

\[
P(t - \tilde{x}, \tilde{y}) \leq 2^{n+1} P(x_0 - \tilde{x}, y_0 + \tilde{y}).
\]

Consequently, for all \( t \in \mathbb{R}^n \),

\[
\mathbb{P}^* \mu (t) \leq 2^{n+1} \mathbb{P}^* \mu (x_0, y_0) \leq 2^{n+1} M.
\]

Applying the Poisson integral \( P[dt] \) to both sides of the preceding inequality, we obtain

\[
\mathbb{P}^* \mu (x,y) \leq 2^{n+1} M \quad \text{for all } (x,y) \in \mathbb{R}_+^{n+1}.
\]

This proves that the weak maximum principle holds for \( P^* \) with \( h = 2^{n+1} \). It follows from Theorem 2.3 that (4.1) holds if and only if there exists a nontrivial \( u \in L^q(\mathbb{R}_+^{n+1}, \sigma) \) such that \( u \geq \mathbb{P}^* (u^q d\sigma) \). Moreover, a positive solution \( u = \mathbb{P}^* (u^q d\sigma) \) can be constructed as in the statement of Theorem 4.1 (see details in [16]).

\[\square\]

**Corollary 4.2.** Under the assumptions of Theorem 4.1, inequality (4.1) holds if and only if there exists a function \( \phi \in L^1(\mathbb{R}^n) \), \( \phi > 0 \) a.e., such that

\[
\phi \geq P^* \left[ (P \phi)^q d\sigma \right] \quad \text{a.e. in } \mathbb{R}^n.
\]
Moreover, if (4.1) holds, then there exists a positive solution \( \phi \in L^1(\mathbb{R}^n) \) to the equation \( \phi = P^*[(P\phi)^q]d\sigma \).

**Proof.** If (4.1) holds then by Theorem 4.1 there exists \( u = PP^*(u^q d\sigma) \) such that \( u > 0 \) and \( u \in L^q(\mathbb{R}^{n+1}_+, \sigma) \). Setting \( \phi = P^*(u^q d\sigma) \), we see that

\[
\mathbf{P}\phi = PP^*(u^q d\sigma) = u,
\]

so that \( \phi = P^*[(P\phi)^q]d\sigma \), and consequently

\[
\|\phi\|_{L^q(\mathbb{R}^n)} = \|u\|_{L^q(\mathbb{R}^{n+1}_+, \sigma)}^q = \int_{\mathbb{R}^n} u(x, y) dx < \infty.
\]

Conversely, if there exists \( \phi > 0, \phi \in L^1(\mathbb{R}^n) \) such that \( \phi \geq P^*[(P\phi)^q]d\sigma \), then letting \( u = \mathbf{P}\phi \), we see that \( u \) is a positive harmonic function in \( \mathbb{R}^{n+1}_+ \) so that

\[
u = \mathbf{P}\phi \geq PP^*(u^q d\sigma),
\]

and for all \( y > 0 \),

\[
\|u\|_{L^q(\mathbb{R}^{n+1}_+, \sigma)}^q = \int_{\mathbb{R}^n} \left[PP^*(u^q d\sigma)\right]^q (x, y) dx \leq \int_{\mathbb{R}^n} u(x, y) dx = \|\phi\|_{L^1(\mathbb{R}^n)} < \infty.
\]

Hence, inequality (4.1) holds by Theorem 4.1 \( \square \)

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