Electrodynamics of Thin Sheets of Twisted Material

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We construct a minimal theory describing the optical activity of a thin sheet of a twisted material, the simplest example of which is twisted bilayer graphene. We introduce the notion of “twisted electrical conductivity,” which parametrizes the parity-odd response of a thin film to a perpendicularly falling electromagnetic waves with wavelength much larger than the thickness of the sheet. We show that the low-frequency Faraday rotation angle has different behaviors in different phases. For an insulator, the Faraday angle behaves as $\propto \omega^2$ at low frequencies, with the coefficient being determined by the linear relationship between a component of the electric quadrupole moment and the external electric field. For superconductors, the Faraday rotation angle is constant when the frequency of the incoming EM waves is below the superconducting gap and is determined by the coefficient of the Lifshitz invariant in the Ginzburg-Landau functional describing the superconducting state. In the metallic state, we show that the twisted conductivity is proportional to the “magnetic helicity” (scalar product of the velocity and the magnetic moment) of the quasiparticle, averaged around the Fermi surface. The theory is general and is applicable to strongly correlated phases.

\textit{Introduction}.—Recently, twisted quasi-two-dimensional materials have attracted intense interest. The most well-known example of such materials is twisted bilayer graphene, where flat bands have been predicted and found and correlated insulating and superconducting states observed \cite{1-6}. Flat bands have also been discovered in the other twisted systems \cite{7, 8}, including twisted double bilayer graphene \cite{9, 11} and twisted trilayer graphene \cite{12}.

One probe of the states of the twisted quasi-2D materials is through their interaction with the external electromagnetic field. Due to the twisted structure, a circularly polarized electromagnetic wave passing through such a material can experience different responses depending on the sign of the polarization—the so-called Faraday rotation and circular dichroism \cite{13, 14}. In this paper we study the question: what does the chiral response of a twisted quasi-2D material to long-wavelength electromagnetic waves reveal about the structure of the low-energy excitations in such a materials?

We first construct here a minimal theory that describes the chiral response of a thin sheet of material with thickness $d$ much smaller than the wavelength $\lambda$ of the incoming light. We show that in addition to the usual 2D electrical conductivity, one just needs a new kinetic coefficient which we dub “twisted conductivity,” which determines the electric current generated by a gradient of the electric field, or the time dependence of the magnetic field. The same quantity also determines the “dipole current,” to be defined, in an external electric field.

In the metallic phase this twisted conductivity is shown to be proportional to the “magnetic helicity” of the quasiparticle excitations on the Fermi surface, defined as the product of the velocity and the in-plane magnetic moment of the Landau quasiparticle.

Previous theoretical work include Refs. \cite{15–17}, where one considers a model of two layers, the current on one layer is linearly dependent on the electric field on that layer and on the other layer. This approach is limited to two-layer systems within the approximation that the electrons are localized in a very thin shell around each layer, whose thickness is much less than the distance between the layers. In Ref. \cite{18} the Kubo formula for the chiral response is derived for commensurate twist angles.

\textit{General consideration}.—Consider a thin sheet of material, stretched along the $(x, y)$ directions. We will have in mind twisted bilayer graphene as a prototype. We are interested in the chiral response of this materials to electromagnetic waves which have wavelength much larger than the thickness of the sheet. For example, we want to understand Faraday’s rotation of the plane of polarization of an incoming electromagnetic wave.

We will be mostly interested in the case when the incoming electromagnetic waves fall perpendicularly to the plane of the sheet. In this case the is electric field parallel to the sheet.

We assume that the sheet is invariant under a discrete group of rotations around the $z$ axis, sufficiently large for the conductivity tensor to be isotropic (for example $C_{2z}$ in the case of twisted graphene sheets). The twisted sheet is assumed to be not invariant under 2D reflection but only under a 2D reflection (say, $x \rightarrow x, y \rightarrow -y$) combined with $z \rightarrow -z$, which is simply the two-fold rotation around the $x$ axis $C_{2x}$. This symmetry is present in twisted bilayer graphene \cite{19}. We will assume that this symmetry is not spontaneously broken. We will also assume time reversal invariance.

In the standard treatment, the response of a single thin layer (say, of graphene) to an incident electromagnetic wave is characterized by the 2D (complex) conductivity $\sigma(\omega)$ \cite{20–22}. To treat effects like Faraday rotation or circular dichroism, the surface conductivity is not suffi-
cient. We now develop a formalism, which deals exclusively with two-dimensional quantities, but still allows one to capture these effects.

Let us denote the current in a finite-width piece of material by $J(z,x)$. If the sheet is thin, we can define the 2D current as

$$j(x) = \int dz J(z,x),$$

and the linear electric response is given by the frequency-dependent electrical conductivity: $j = \sigma E$. The transmission and reflection amplitudes of an electromagnetic wave falling perpendicularly onto the sheet can then be expressed through the real and imaginary parts of $\sigma$ \cite{22}. To go beyond $\sigma$, we introduce a new current

$$\zeta(x) = \int dz z J(z,x).$$

For a system which consists of two layers separated by a distance $d$, if the thickness of the electron orbitals on each of the layers is much smaller than $d$, then $\zeta = \frac{\pi}{c^2} (j_1 - j_2)$, where $j_1$ and $j_2$ are the electric currents on the layers 1 and 2. We term this quantity the “dipole current,” because if there is no tunneling between the two layers, then the electric dipole moment along the $z$ direction is conserved, and $\zeta$ is the current of that conserved charge.

The source of the current $j$ are the electric field $E$, and the source of the dipole current $\zeta$ is the gradient of the electric field along the perpendicular direction $\partial_z E$. This can be seen by noticing that the Ohmic heat generated in the system per unit area is

$$\int dz J(z) \cdot E(z) \approx \int dz J(z) \cdot (E + z \partial_z E) = j \cdot E + \zeta \cdot \partial_z E.$$ \hspace{1cm} (3)

We now introduce the conductivities $\sigma, \sigma_1, \tilde{\sigma},$ and $\sigma_2$:

$$j = \sigma E - \sigma_1 \partial_z E \times \hat{z},$$

$$\zeta = \tilde{\sigma} (E \times \hat{z}) + \sigma_2 \partial_z E.$$ \hspace{1cm} (4)

In general the conductivities are functions of the frequency. Time reversal invariance implies the Onsager relation $\tilde{\sigma}(\omega) = \sigma_1(\omega)$. Positivity of entropy production implies that both $\sigma$ and $\sigma_2$ have positive real parts, and $\text{Re} \sigma \text{Re} \sigma_2 \geq (\text{Re} \sigma_1)^2$.

On dimensional ground, we expect that typically $\sigma_1 \sim d \sigma, \sigma_2 \sim d^2 \sigma$. For electromagnetic waves with wavelength $\lambda$, the effects of $\sigma_1$ and $\sigma_2$ will be suppressed by $d/\lambda$ and $(d/\lambda)^2$, respectively. However, since $\sigma_d$ is the first transport coefficient that breaks reflection symmetry, we need to keep it in order to compute, e.g., Faraday rotation. Although the effect $\sigma_d$ is always small, we will however keep it in our formalism for the sake of symmetry.

Considering an electromagnetic plane wave with frequency $\omega$ falling perpendicularly onto the plane. We can work in the gauge where $A_0 = A_z = 0$, where the Maxwell equation for the perpendicular components of the vector potential is (we use the Gauss units)

$$-\partial_z^2 A - \frac{\omega^2}{c^2} A = \frac{4\pi}{c} J.$$ \hspace{1cm} (6)

(We assume here that the thin sheet is immerse in vacuum. Our calculation can be trivially modified when the medium on one or both sides of the sheet has a nontrivial dielectric constant.) To find the boundary conditions at $z = 0$ we can integrate over $z$ from $z = -\epsilon$ to $z = +\epsilon$

$$-A_1^i|_{-\epsilon}^{+\epsilon} = \frac{4\pi \omega}{c^2} (\sigma A - \sigma_1 A' \times \hat{z}).$$ \hspace{1cm} (7)

We can also multiply Eq. (6) by $z$ and integrate over $z$ to find

$$\delta A = \frac{4\pi}{c^2} \zeta = \frac{4\pi \omega}{c^2} (\sigma_1 A \times \hat{z} + \sigma_2 A'),$$ \hspace{1cm} (8)

where we have denoted the jump of a quantity $A$ across the sheet by $\delta A \equiv A(+\epsilon) - A(-\epsilon)$. These boundary conditions can be written for the two circular polarizations $A_{x} = A_x \pm i A_y$ separately,

$$-A_{x}^i|_{-\epsilon}^{+\epsilon} = \frac{4\pi \omega}{c^2} (\sigma A_{x} \pm i \sigma_1 A'_{x}),$$ \hspace{1cm} (9)

$$A_{y}^i|_{-\epsilon}^{+\epsilon} = \frac{4\pi i\omega}{c^2} (\mp i \sigma_1 A_{x} + \sigma_2 A'_{x}).$$ \hspace{1cm} (10)

We note here some peculiarities of these boundary conditions. In the usual problem, only $\sigma$ is nonzero, and the boundary condition (9) specifies that $A_x$ is continuous across $z = 0$, while its first derivative $A'_x$ has a discontinuity proportional to the value of $A_y$ at $z = 0$. This leads to a consistent mathematical problem. But with nonzero $\sigma_1$ and $\sigma_2$, Eqs. (9) and (10) imply that both $A_x$ and $A'$ are discontinuous, and their discontinuity is proportional to their values at $z = 0$. The boundary conditions become ambiguous: at which value of $z$ should the right-hand sides of Eqs. (9) and (10) be calculated?

To fully resolve the ambiguity, one needs to solve the scattering problem for a slab of finite width and carefully take the limit of small width. This is done in the Supplementary Materials. We will use a shortcut that leads to the correct answer when the jumps of $A$ and $A'$ are small compared to their values. This requires

$$\frac{4\pi}{c^2} \sigma_1 \omega \ll 1.$$ \hspace{1cm} (11)

This condition is usually satisfied. For example for a sheet a few atomic layer thick, if $\sigma \sim e^2/h$ then $\sigma_1 \sim \frac{e^2}{\pi d}$. Then

$$\frac{4\pi \omega}{c^2} \sigma_1 \sim \frac{e^2 d}{\hbar c \lambda} \ll 1.$$ \hspace{1cm} (12)

When the jump of $A$ is much smaller than $A$, we can replace $A$ on the right-hand side of the boundary condition
by the average values on the two sides. This prescription
is similar to Griffiths’s treatment of the so-called
$\delta'(x)$ potential in quantum mechanics [23]. The boundary
conditions can now be written as

$$-\delta A'_\pm = \frac{4\pi i\omega}{c^2} (\sigma \tilde{A}_\pm \pm i\sigma_1 \tilde{A}'_\pm), \quad (13a)$$

$$\delta A_\pm = \frac{4\pi i\omega}{c^2} (\mp i\sigma_1 \tilde{A}_\pm + \sigma_2 \tilde{A}'_\pm). \quad (13b)$$

where we have denoted $\tilde{A} \equiv \frac{1}{2} [A(\mp \omega) + A(\mp \omega)]$.

Let us now consider a problem of scattering of a plane
wave onto the plane. A plane wave coming from $z = \infty$
gives rise to a transmitted and a scattered waves.

$$A_\pm (z) = \begin{cases} e^{ikz} + R_\pm e^{-ikz}, & z < 0, \\ T_\pm e^{ikz}, & z > 0. \end{cases} \quad (14)$$

With the boundary conditions, we find

$$T_\pm = \frac{1 \pm \tilde{\sigma}_1 k + \frac{1}{2}(\tilde{\sigma}_1^2 - \tilde{\sigma}_2)k^2}{1 + \frac{1}{2}\tilde{\sigma} + \frac{1}{4}(2\tilde{\sigma}_2 - \tilde{\sigma}_1^2 + \tilde{\sigma}_2)k^2}, \quad (15a)$$

$$R_\pm = -\frac{1}{2} \frac{1 + \tilde{\sigma}_1 k + \frac{1}{2}(\tilde{\sigma}_1^2 - \tilde{\sigma}_2)k^2}{1 + \frac{1}{2}\tilde{\sigma} + \frac{1}{4}(2\tilde{\sigma}_2 - \tilde{\sigma}_1^2 + \tilde{\sigma}_2)k^2}, \quad (15b)$$

where

$$\tilde{\sigma} = \frac{4\pi}{c} \sigma, \quad \tilde{\sigma}_1 = \frac{4\pi}{c} \sigma_1, \quad \tilde{\sigma}_2 = \frac{4\pi}{c} \sigma_2. \quad (16)$$

As a consistency check of the symmetrized prescription
(13), one can verify that if $\text{Re} \sigma = \text{Re} \sigma_1 = \text{Re} \sigma_2 = 0$, implying no dissipation, then $|T|^2 + |R|^2 = 1$.

Ignoring $\sigma_2$ and assuming $\tilde{\sigma}_1 k \ll 1$, we have

$$T_\pm = \frac{1 \pm \frac{4\pi}{c} \sigma_1 k}{1 + \frac{2\pi}{c} \sigma}, \quad R_\pm = -\frac{\frac{4\pi}{c} \sigma}{1 + \frac{2\pi}{c} \sigma}. \quad (17)$$

The Faraday rotation angle is then

$$\theta_F = \frac{1}{2} \arg \frac{T_+}{T_-} = \frac{4\pi \omega}{c^2} \text{Im} \sigma_1. \quad (18)$$

while the ratio of the absolute value of the transmission
amplitude of the two polarizations (which is different
from 1 if there is circular dichroism) is

$$\frac{|T_+|^2}{|T_-|^2} = 1 + \frac{16\pi \omega}{c^2} \text{Re} \sigma_1. \quad (19)$$

Both Eqs. (18) and (19) do not require $\sigma$ to be small to
be valid, but only (11).

In the Supplementary Materials we reproduce the
formulas by solving the Maxwell equations for a slab of finite
thickness in the limit of long wavelength. Equations (18)
and (19), as written, are correct also in the presence of a
dielectric constant on one or or both sides of the sheet.

**Metallic state.**—Consider a state with a Fermi surface.
In the regime where the incoming photon has energy less
than the Fermi energy (typically that means wavelength
in the infrared range) we can use the Fermi liquid theory
to treat the problem. In this description, we have quasiparticles states $|p, \alpha\rangle$ where $p$ is a momentum along
the sheet and $\alpha$ is an internal index (for twisted bilayer
graphene $\alpha$ would corresponds to the spin, valley, and
layer degeneracies). The state has a magnetic moment
$\mu_p$, parallel to the sheet. One can visualize this magnetic
moment as arising from the spiral-like motion of a wave
packet when it moves along the sheet, jumping back and
forth between the two layers. The energy in a magnetic
field $B$ has the form (from now on we suppress the $\alpha$
index)

$$E_p = \epsilon_p - \mu_p \cdot B. \quad (20)$$

Note that for fields that are constant on the $(x, y)$ plane,
$B = -\partial_z A \times \hat{z}$. Ignoring Fermi-liquid effects, the kinetic equation for the distribution function $f_p$ in the
relaxation-time approximation is

$$\frac{\partial f_p}{\partial t} + e\mathbf{E} \cdot \nabla_p f_p = -\frac{f_p - f_0(E_p)}{\tau}, \quad (21)$$

where $f_0(\epsilon) = [e^{\beta(\epsilon - \mu)} + 1]^{-1}$. Linearizing the equation:

$$\frac{\partial f_p}{\partial t} = -\frac{1}{1 - i\omega \tau} (-e\mathbf{E} \cdot \nabla_p + \mu_p \cdot \mathbf{B}) \left( -\frac{\partial f_0}{\partial \epsilon} \right). \quad (22)$$

To compute the current, we use

$$j = -e \int_p \frac{\partial E_p}{\partial \mathbf{p}} (f_0 + \delta f_p), \quad (23)$$

where $f_p \equiv \int d^2p/(2\pi)^2$, and the group velocity is computed using the full dispersion (20). Inserting Eq. (22) we find

$$j = e\tau \int_p \left[ \epsilon (\mathbf{E} \cdot \nabla_p) - i\omega (\mu_p \cdot \mathbf{B}) \right] v_{p, \alpha} \left( -\frac{\partial f_0}{\partial \epsilon} \right). \quad (24)$$

By using the Maxwell equation $\nabla \times \mathbf{E} = -c^{-1} \mathbf{B}$, one gets

$$\sigma = \frac{e^2 \tau \nu(\epsilon_p)}{2 (1 - i\omega \tau)} (v_{p, \alpha}^2), \quad \sigma_1 = -\frac{e^2 \tau \nu(\epsilon_p)}{2 (1 - i\omega \tau)} (v_{p, \alpha} \cdot \mu_p). \quad (25)$$

Here $\nu(\epsilon_p)$ is the density of state at the Fermi level and $\langle \cdot \rangle$ means averaging over the Fermi surface: $\langle A \rangle \equiv \nu^{-1}(\epsilon_F) \int_p A \delta(\epsilon_p - \epsilon_F)$. We will call the scalar product of the velocity $v_p$ and the in-plane orbital magnetic moment $\mu_p$ the “magnetic
c helicity” of the quasiparticle carrying momentum $p$. The average of the magnetic helicity around the Fermi surface $\langle v_{p, \alpha} \cdot \mu_p \rangle$ encodes the correlation between between the
direction of motion of the quasiparticles and their magnetic
moment. Such correlation leads to the appearance of a magnetic moment density when an electric current
flows through the system. Equation (25) implies that the twisted conductivity is related to this locking between velocity and magnetic moment. This locking is also the origin of the gyrotropic magnetic effect in 3D [24, 25] and 2D [18].

Note that the ratio of $\sigma_1$ and $\sigma$ does not depend on the mean free time $\tau$,

$$\frac{\sigma_1}{\sigma} = \frac{c}{\epsilon} \frac{\langle \mathbf{v}_p \cdot \mathbf{\mu}_p \rangle}{\langle \mathbf{v}_p^2 \rangle},$$  

(26)

and can be read out just from the wave functions of modes at the Fermi surface. This has the consequence that if $\sigma \ll \frac{c}{\epsilon} \approx 137^{\omega/\nu}$, circular dichroism, defined as $CD = (A_+ - A_-)/[2(A_+ + A_-)]$ where $A_+$ and $A_-$ are the absorption coefficient of the corresponding helicities, is linear in frequency

$$CD = \frac{\omega \langle \mathbf{v}_p \cdot \mathbf{\mu}_p \rangle}{\epsilon} \langle \mathbf{v}_p^2 \rangle.$$  

(27)

In the regime $\omega \tau \ll 1$, $\text{Im} \sigma_1 \sim \omega$, so the Faraday rotation angle is of order $\omega^2$. In the opposite regime $\omega \tau \gg 1$, the conductivities become purely imaginary and inversely proportional to the frequency $\omega$. In particular, the twisted conductivity $\text{Im} \sigma_1 \sim \omega^{-1}$. This means that the Faraday rotation angle is constant in this regime

$$\theta_F = -\frac{2\pi c}{\epsilon} \nu(\varepsilon_F) \langle \mathbf{v}_p \cdot \mathbf{\mu}_p \rangle.$$  

(28)

To illustrate the result, consider the case of twisted bilayer graphene. Here $\nu(\varepsilon_F)$ contains a factor of 8 from the valley, spin, and layer degeneracies. For large twisting angle $\theta$ (larger than the angle $1.1^\circ$ where flat bands appear) and small Fermi momentum $p_F \ll k_0 = 2k_D \sin \frac{\theta}{2}$ where $k_D$ is the Dirac momentum, the quasiparticles are localized on one layer, with small admixture from the other layer. The magnetic moment of the quasiparticle is mostly perpendicular to the momentum, but there is a small component along the quasiparticle’s momentum, leading to a nonzero average magnetic helicity. The angle of Faraday rotation in this case is (see the Supplementary Materials)

$$\theta_F = -\frac{240}{\alpha} \frac{v_0}{c} \frac{k_0 d}{h} \left( \frac{p_F}{k_0} \right)^2 \frac{w_{AA} w_{AB}}{(v_0 k_0)^2},$$  

(29)

where $\alpha \approx \frac{1}{137}$ is the fine structure constant, $v_0$ is the velocity of the Dirac fermion in single-layer graphene, $w_{AA}$ and $w_{AB}$ are the interlayer coupling parameters, and $d$ is the distance between the layers. For example, at twist angle $\theta = 2^\circ$, using the standard numerical values for the parameters (see, e.g., Ref. [26]) one gets $\theta_F \sim 10^{-5}$ when $p_F$ is of the same order of magnitude as $k_0$. The sign of $\theta_F$ is such that the plane of polarization is rotated in the same direction as the direction, with respect to which the graphene layer farther from the source is rotated with respect to the one closer to the source. Analogously, we obtain $CD \sim 10^{-4}$ for $\omega \sim v_0 k_0$.

**Superconducting case.**—We now consider the case of a superconducting thin layer. Even when we do not know the microscopic mechanism for superconductivity in twisted bilayer graphene and other twisted materials, we can treat the problem of scattering of long-wavelength electromagnetic waves (with frequency less than the superconducting gap, thus typically in the microwave range) phenomenologically using the Ginzburg-Landau theory. Keeping only the phase $\varphi$ of the order parameter, the Ginzburg-Landau energy functional has the form [27, 28]

$$H = \int d^3x \delta(z) \left[ \frac{\hbar^2 n_s}{2m} (D_a \varphi)^2 + \kappa n_s B_a D_a \varphi \right],$$  

(30)

where the covariant derivatives are $D_i \varphi = \partial_i \varphi + \frac{2e}{\hbar} A_i$, $D_a \varphi = \partial_a \varphi + \frac{2e}{\hbar} A_a$, and the index $a$ runs $x, y$. The term proportional to $\kappa$ in Eq. (30) is the so-called “Lifshitz invariant” term. One can imagine that this term arises from a nonminimal coupling of the order parameter $\psi$ with the electromagnetic field: $\frac{1}{2} \kappa B_a (\psi \partial_a \psi - D_a \psi \psi)$. This type of coupling has been considered previously in the treatment of non-centrosymmetric superconductors [27, 28]. In that context, the term is associated with spin-orbit coupling.

For twisted bilayer graphene and other twisted materials, from symmetry arguments one expects the Lifshitz invariant to appear independent of the mechanism of superconductivity. Physically, the Lifshitz invariant term implies that a moving condensate possesses a nonzero density of magnetic moment. This is expected if the superconductivity is formed by Cooper pairing of quasiparticles with nonzero average magnetic helicity, which, as we recall, parametrizes the locking between the velocity and the magnetic moment.

One can couple this action with the electromagnetic field in the bulk and solve the combined system of equations for the gauge field and the phase $\varphi$. Equivalently, one can simply compute the twist conductivity from Eq. (30). Differentiating the action with respect to $A_a$, and setting all fields to be spatially homogeneous along the directions of the sheet, in particular $\partial_a \varphi = 0$ (which can be done when the electromagnetic wave falls perpendicularly onto the sheet), one obtains a generalized London equation

$$J_a = \frac{\partial H}{\partial A_a} = \delta(z) \left[ \left( \frac{2e}{mc} n_s A_a - 2 \frac{\kappa n_s}{\hbar} B_a \right) \right]$$  

$$+ 2e \frac{\kappa n_s}{\hbar} \epsilon^{ab} \partial_b \delta(z) A_a, \quad (31)$$

from which one finds

$$\sigma = \frac{i (2e)^2 n_s}{m \omega}, \quad \sigma_1 = \frac{i c}{\omega} 2e \frac{\kappa n_s}{\hbar}.$$  

(32)
Since $\sigma_1 \sim \omega^{-1}$, the Faraday rotation angle is frequency-independent. This can be seen by applying simple field-theoretical power counting to the Lifshitz invariant term: as $B_b$ has dimension 2 and $D_a \varphi$ has dimension one, $kn_s$ must be dimensionless. Since the Faraday rotation angle is dimensionless and proportional to $kn_s$, there should be no frequency dependence. This is exactly the same behavior as for metals in the regime $\omega \tau \gg 1$.

The uniqueness of the Lifshitz invariant allows one to compute also the change of the plane of polarization when the incoming light falls onto the sheet at any angle. This calculation is done in the Supplementary Materials.

The Lifshitz invariant also leads to another effect—the chiral magnetic Josephson effect in which the Josephson junction is built up from two chiral superconductors linked by a uniaxial ferromagnet. Josephson current appears even with zero phase difference dues to a phase offset [29, 30]. This phase offset originates from the parity breaking term and is proportional to $\kappa$. This effect is similar to the chiral magnetic effect in hot QCD [31].

**Insulating states.**—In the insulating state, there is no low-energy degree of freedom living on the thin layer, so the effective action is just a local function of the electromagnetic field and its derivatives. Instead of the Lifshitz invariant, the leading term which breaks the 2D reflection symmetry, but preserve the combination of 2D reflection invariance, the leading term which breaks the 2D reflection symmetry, but preserve the combination of 2D reflection invariance, is

$$\delta S = \int dt \delta z \delta (z) \beta \epsilon^{ab} E_a \partial_z E_b,$$

where $\beta$ is some constant. The coefficient $\beta$ has dimension $-2$, thus one concludes that the Faraday rotation angle behaves like $\beta \omega^2$ for waves with frequency less than the gap. This prediction is independent of the nature of the insulating state, whether one is dealing with a band insulator or a strongly correlated insulator. Calculations similar to the one we have done for the superconducting case give $\theta_F = 4\pi \beta (\omega/c)^2$.

To give a physical interpretation to the term (33) in the effective action, let us imagine immersing a finite piece of thin twisted insulator into an uniform static parallel electric field $E_a$. For simplicity let us also assume that the boundary is also gapped. The effective action of this finite piece of material is described by the same Eq. (33) with $\beta$ replaced by the function $\beta(x)$, where $\beta(x) = \beta$ inside the piece and 0 outside. The charge density induced by a constant electric field field is

$$\rho = \frac{\delta S}{\delta A_0} = \delta' (z) \epsilon^{ab} E_a \partial_b \beta(x).$$

This charge distribution corresponds to zero total charge and dipole moment, but leads to nonzero value of the electric quadrupole moment

$$Q^{az} = \int dz d^2x \, z \epsilon^{az} \rho(z, x) = -\beta S \epsilon^{ab} E_b,$$

where $S$ is the total area of the piece of material. (This quadrupole moment is well defined as it is along directions perpendicular to the direction of the dipole moment.) Thus $\beta$ is the coefficient determining the quadrupole polarization induced by an electric field, which provides an independent way of measuring $\beta$.

**Conclusion.**—We have constructed the minimal theory describing the chiral response of a thin sheet of a twisted material, applicable to twisted bilayer graphene and related materials. In addition to the 2D electrical conductivity of the sheet, one needs to introduce another transport coefficient, which we term the “twisted conductivity.” For a metal, the twisted conductivity is proportional to the average magnetic helicity (i.e., scalar product of the velocity and the in-plane orbital magnetic moment) of the quasiparticles around the Fermi line. For a superconductor, the chiral response is shown to be related to the Lifshitz invariant, which parametrizes the magnetic moment density created by a supercurrent. For the insulating phase, the chiral response is related to the quadrupole moment induced by an in-plane electric field.

The discussion in this paper is general and insensitive to the details of microscopic physics. It would be interesting to see if the chiral response of the correlated phases is sensitive to the mechanism underlying these phases.

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In this Section, we consider the problem of the scattering of electromagnetic waves falling perpendicularly onto a slab of a finite width. Then taking the limit where the thickness of the slab is much smaller than the wavelength of the incoming radiation, we reproduce the formulas for the Faraday rotation angle and circular dichroism obtained in the main text.

Consider a layer of material, whose response to an external electric field is given by a nonlocal conductivity $\sigma(\omega; z, z')$

$$j_a(\omega; z) = \int dz' \sigma_{ab}(\omega; z, z')E_b(\omega, z').$$  \hfill (S1)

We assume that the layer has finite thickness $d$, so $\sigma_{ab} = 0$ when $|z| > d/2$ or $|z'| > d/2$, or both. We will omit the argument $\omega$ in further formulas. We want to derive the formula for the transmission and reflection coefficient in the limit of small $k$, $kd \ll 1$.

First, we can decompose $\sigma_{ab}$ into symmetric and antisymmetric parts

$$\sigma_{ab} = \sigma_S \delta_{ab} + \sigma_A \epsilon_{ab}. \hfill (S2)$$

In the helicity basis

$$j_{\pm} = j_x \pm i j_y, \quad E_{\pm} = E_x \pm i E_y, \hfill (S3)$$

Eq. (S1) can be written as

$$j_{\pm}(z) = \int dz' \sigma_{\pm}(z, z')E_{\pm}(z'), \hfill (S4)$$

where we defined

$$\sigma_{\pm} = \sigma_S \mp i\sigma_A. \hfill (S5)$$

The symmetry with respect to reflection around the $y$ axis combined with the exchange of the upper and lower layers $z \rightarrow -z$ (which is the two-fold rotation around the $x$ axis) implies

$$\sigma_{+}(z, z') = \sigma_{-}(-z, -z'). \hfill (S6)$$

Time reversal invariance implies $\sigma_{ab}(\omega; z, z') = \sigma_{ba}(\omega; z', z)$, which means

$$\sigma_{+}(z, z') = \sigma_{-}(z', z). \hfill (S7)$$

Combining two symmetries, we find

$$\sigma_{\pm}(z, z') = \sigma_{\pm}(-z', -z). \hfill (S8)$$

From now on we focus on the positive helicity, and drop the helicity index $+$ in formulas. Let us also define

$$\sigma(z) = \int dz' \sigma(z, z'). \hfill (S9)$$

From Eq. (S8) we find

$$\int dz \sigma(z, z') = \sigma(-z'). \hfill (S10)$$
The Maxwell equation can be written as
\[ \partial_z^2 A(z) + k^2 A(z) = -ik \int dz' \tilde{\sigma}(z, z') A(z'), \quad \tilde{\sigma}(z, z') = \frac{4\pi}{c} \sigma(z, z'). \tag{S11} \]

We will be looking for solution to this equation with the asymptotics
\[ A(z) \to e^{ikz}, \quad z \to +\infty, \tag{S12} \]
which means that in the opposite limit the asymptotics of \( A(z) \) is
\[ A(z) = \frac{1}{T} e^{ikz} + \frac{R}{T} e^{-ikz}, \tag{S13} \]
where \( T \) and \( R \) are the transmission and reflection amplitudes, respectively. Using Green’s function, Eq. (S11) can be written as
\[ A(z) = e^{ikz} - i \int dz_1 \, dz_2 \, \theta(z_1 - z) \sin k(z_1 - z) \tilde{\sigma}(z_1, z_2) A(z_2), \tag{S14} \]
from which we find
\[ \frac{1}{T} = 1 + \frac{1}{2} \int dz_1 \, dz_2 \, e^{-ikz_1} \tilde{\sigma}(z_1, z_2) A(z_2), \tag{S15a} \]
\[ \frac{R}{T} = -\frac{1}{2} \int dz_1 \, dz_2 \, e^{ikz_1} \tilde{\sigma}(z_1, z_2) A(z_2). \tag{S15b} \]

The Maxwell equation (S11) can be solved by iteration, where the solution is presented as an infinite series
\[ A(z) = \sum_{n=0}^{\infty} A^{(n)}(z), \tag{S16} \]
where we define
\[ A^{(0)}(z) = e^{ikz}, \tag{S17} \]
\[ A^{(n)}(z) = -i \int dz_1 \, dz_2 \, \theta(z_1 - z) \sin k(z_1 - z) \tilde{\sigma}(z_1, z_2) A^{(n-1)}(z_2). \tag{S18} \]

In the limit of small \( k \) the expansion (S16) is an expansion over \( k \). First consider \( z \) inside the slab, \( |z| < d/2 \), then \( \sin k(z - z_1) \sim kd \) since \( z_1 \) has to be inside the slab, we have
\[ A^{(n)}(z) \sim k A^{(n-1)}(z) \Rightarrow A^{(n)}(z) \sim (kd)^n, \quad |z| < \frac{d}{2}. \tag{S19} \]

Let us first compute \( T \) and \( R \) to order \( k^0 \). We replace in Eq. (S15) \( A(z) \to A^{(0)}(z) = e^{ikz} \). To order \( k^0 \) we can replace \( e^{\pm ikz_1} \) and \( e^{ikz_2} \) by 1 since \( z_1, z_2 \lesssim d \). We find
\[ \frac{1}{T} = 1 + \frac{1}{2} \int dz_1 \, dz_2 \, \tilde{\sigma}(z_1, z_2), \tag{S20} \]
\[ \frac{R}{T} = -\frac{1}{2} \int dz_1 \, dz_2 \, \tilde{\sigma}(z_1, z_2). \tag{S21} \]

Comparing with Eqs. (17) we can identify
\[ \sigma = \int dz_1 \, dz_2 \, \sigma(z_1, z_2). \tag{S22} \]

To compute \( T \) and \( R \) to order \( k^1 \) we need to replace, in Eqs. (S15) \( A(z) \) by \( A^{(0)}(z) + A^{(1)}(z) \), where \( A^{(0)} \) is computed to next-to-leading order in \( k \), and \( A^{(1)} \) to leading order,
\[ A^{(0)}(z) = 1 + ikz + O(k^2), \tag{S23} \]
\[ A^{(1)}(z) = -ik \int dz_1 \, \theta(z_1 - z)(z_1 - z) \tilde{\sigma}(z_1) + O(k^2). \tag{S24} \]
We find
\[ \frac{1}{T} = 1 + \frac{1}{2} \int dz \dot{\sigma}(z) - ik \int dz z \dot{\sigma}(z) - \frac{ik}{2} \int dz dz' \theta(z + z')(z + z') \dot{\sigma}(z) \dot{\sigma}(z'), \] (S25)
\[ \frac{R}{T} = -\frac{1}{2} \int dz \dot{\sigma}(z) + \frac{ik}{2} \int dz dz' \theta(z + z')(z + z') \dot{\sigma}(z) \dot{\sigma}(z'). \] (S26)

Let us now divide \( \dot{\sigma}(z) \) into symmetric and antisymmetric parts
\[ \sigma(z) = \sigma_s(z) + \sigma_a(z), \quad \sigma_s(z) = \sigma_s(-z), \quad \sigma_a(z) = -\sigma_a(-z) \] (S27)

Then we find
\[ \frac{1}{T} = \left[ 1 + \frac{1}{2} \int dz \ddot{\sigma}_s(z) \right] \left[ 1 - ik \int dz z \ddot{\sigma}_a(z) \right] - \frac{ik}{2} \int dz dz' \theta(z + z')(z + z') [\ddot{\sigma}_s(z) \ddot{\sigma}_s(z') + \ddot{\sigma}_a(z) \ddot{\sigma}_a(z')], \] (S28)
\[ \frac{R}{T} = -\frac{1}{2} \int dz \ddot{\sigma}(z) \left[ 1 - ik \int dz z \ddot{\sigma}_a(z) \right] + \frac{ik}{2} \int dz dz' \theta(z + z')(z + z') [\ddot{\sigma}_s(z) \ddot{\sigma}_s(z') + \ddot{\sigma}_a(z) \ddot{\sigma}_a(z')]. \] (S29)

If one now identifies
\[ \dot{\sigma} = \int dz \ddot{\sigma}(z) - ik \int dz dz' \theta(z + z')(z + z') [\ddot{\sigma}_s(z) \ddot{\sigma}_s(z') + \ddot{\sigma}_a(z) \ddot{\sigma}_a(z')], \] (S30)
\[ \ddot{\sigma}_1 = i \int dz z \ddot{\sigma}_a(z), \] (S31)
then equations (S28) and (S29) can be written as
\[ \frac{1}{T} = \left( 1 + \frac{\ddot{\sigma}}{2} \right) (1 - k \ddot{\sigma}_1), \] (S32)
\[ \frac{R}{T} = -\frac{\ddot{\sigma}}{2} (1 - k \ddot{\sigma}_1), \] (S33)

which coincide, to order \( O(k) \), with Eq. (17). For negative helicity \(-\), it is easy to find from Eq. (6) that \( \ddot{\sigma} \) remains the same, but \( \ddot{\sigma}_1 \) flips sign.

**TWISTED BILAYER GRAPHENE**

We compute the orbital magnetic moment of the quasiparticles twisted bilayer graphene. We use the continuum theory [1, 26]. We follow the notation of Ref. [26]. The system consists of two layers; the upper layer is rotated clockwise by an angle \( \theta > 0 \) with respect to the lower layer. The single-particle Hamiltonian is
\[ H = \begin{pmatrix} -iv_0 \sigma_{\theta/2} \cdot \nabla T(x) \\ T^\dagger(x) \end{pmatrix} \begin{pmatrix} \sigma_x T(x) \\ -iv_0 \sigma_{-\theta/2} \cdot \nabla \end{pmatrix}. \] (S34)

Here we denoted
\[ \sigma_{\theta/2} = e^{-i \theta / 4} \begin{pmatrix} \sigma_x \\ \sigma_y \end{pmatrix} e^{i \theta / 4}. \] (S35)

We also used
\[ T(x) = \sum_{a=1}^3 T_a e^{-i q_a \cdot x}, \] (S36)

where we defined the momentum \( q_a \) as
\[ q_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} k_y, \quad q_2 = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} k_y, \quad q_3 = \begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \end{pmatrix} k_y, \] (S37)
The expectation value of the magnetic moment is then

\[
T_1 = \begin{pmatrix}
w_{AA} & w_{AB} \\
w_{AB} & w_{AA}
\end{pmatrix}, \quad T_2 = \begin{pmatrix}
w_{AA} & w_{AB}e^{-2\pi i/3} \\
w_{AB}e^{2\pi i/3} & w_{AA}
\end{pmatrix}, \quad T_3 = T_2^\dagger.
\]

The matrix \( T_a \) can be written as

\[
T_a = w_{AA} + w_{AB}(\hat{z} \times \hat{q}_a) \cdot \sigma,
\]

where \( \hat{q}_a = q_a/k_\theta \) is the unit vector directed along the direction of \( q_a \). It is convenient to rotate the basis to transform the Hamiltonian into

\[
H \rightarrow \begin{pmatrix} e^{i\frac{\theta}{2}\sigma_z} & 0 \\ 0 & e^{-i\frac{\theta}{2}\sigma_z} \end{pmatrix} H \begin{pmatrix} e^{-i\frac{\theta}{2}\sigma_z} & 0 \\ 0 & e^{i\frac{\theta}{2}\sigma_z} \end{pmatrix} = \begin{pmatrix} -iv_\theta \sigma \cdot \nabla & T_\theta(x) \\ T_\theta^\dagger(x) & -iv_\theta \sigma \cdot \nabla \end{pmatrix},
\]

where

\[
T_\theta = \sum_a T^a_\theta e^{-i\hat{q}_a \cdot \hat{x}}, \quad T^a_\theta = w_{AA} e^{i\theta/2} + w_{AB}(\hat{z} \times \hat{q}_a) \cdot \sigma.
\]

To find the orbital magnetic moment of a state \( |\psi\rangle \), one can turn on a small in-plane magnetic field \( B \) by turning on opposite gauge potentials on the two layers,

\[
A_u = -\frac{d}{2}(\hat{z} \times B), \quad A_d = \frac{d}{2}(\hat{z} \times B),
\]

where \( u \) and \( d \) indicates upper and lower layers, and \( d \) is the distance between the two layers. The interaction of the system with the gauge field \( A_u \) and \( A_d \) is given by

\[
\delta H = -\frac{1}{c}(J_u \cdot A_u + J_d \cdot A_d) = \frac{1}{2c}(J_u - J_d) \cdot (\hat{z} \times B).
\]

By comparing this with \( \delta H = -\mu \cdot B \), we find

\[
\mu = \frac{d}{2c} \hat{z} \times (J_u - J_d) = -\frac{eo}{2c} \hat{z} \times (\sigma_u - \sigma_d).
\]

In the regime \( w_{AA}, w_{AB} \ll v_\theta k_\theta \), we can use perturbation theory over \( T \). To first order in perturbation the eigenstate of a Hamiltonian \( H = H_0 + H_1 \) is

\[
|0\rangle' = |0\rangle - \sum_{n \neq 0} \frac{|n\rangle\langle n|H_1|0\rangle}{E_n - E_0} + \mathcal{O}(T^2).
\]

If \( |0\rangle \) is a state on the upper layer with \( p \approx 0 \), then \( |n\rangle \) are states on the lower layer with momentum \( p + q_a \approx q_a \), \( a = 1, 2, 3 \). Thus we need to consider only 8 bands \( |0, \alpha\rangle \) and \( |a, \alpha\rangle \) with \( \alpha = 1, 2 \) being the spinor index. The unperturbed Hamiltonian is

\[
\langle 0, \alpha|H_0|0, \beta\rangle = v_\theta p \cdot \sigma_{\alpha\beta}, \quad \langle a, \alpha|H_0|b, \beta\rangle = v_\theta (q_a + p) \cdot \sigma_{\alpha\beta} \delta_{ab},
\]

while the perturbation part of the Hamiltonian is

\[
\langle 0, \alpha|H_1|a, \beta\rangle = (T^a_\theta)_{\alpha\beta}.
\]

The expectation value of the magnetic moment is then

\[
\mu = -\frac{d}{2c} \hat{z} \times \left( \langle 0|\sigma_u|0\rangle - \sum_{mn} \frac{\langle 0|H_1|n\rangle\langle m|\sigma_d|m\rangle\langle n|H_1|0\rangle}{(E_m - E_0)(E_n - E_0)} \right).
\]

Note that \( \langle 0|\sigma|0\rangle = \hat{p} \), and therefore does not contribute to the magnetic helicity. Thus, we only need to compute the second term inside the bracket. In each sector with fix \( a \), \( H_0 \) has two energy eigenvectors

\[
H_0|a, \pm\rangle = \pm v_\theta p_a |a, \pm\rangle, \quad p_a \equiv |q_a + p|.
\]
The sum over intermediate states in $\alpha$-sector gives
$$S_\alpha \equiv \sum_{\alpha=\pm} \frac{|a,\alpha\rangle\langle a,\alpha|}{E_\alpha - E_0} = \frac{1}{2v_0(p_a - p)}(1 + \sigma \cdot \hat{p}_a) - \frac{1}{2v_0(p_a + p)}(1 - \sigma \cdot \hat{p}_a).$$
\hfill (S50)

Putting everything into the expression for the orbital magnetic moment, we then find
$$\langle \mathbf{v}_p \cdot \mu_p \rangle = \frac{d}{2c} \mathbf{v}_p \cdot \sum_{\alpha=1}^3 \langle 0| T^\theta_a (\hat{z} \times \sigma) S_a (T^\theta_a)^\dagger |0 \rangle. \hfill (S51)$$

For $p \ll k_\theta$, this can be evaluated to give
$$\langle \mathbf{v}_p \cdot \mu_p \rangle = \frac{6w_{AA}w_{AB}}{(v_0k_\theta)^2} \frac{p edv_0^2}{k_\theta 2c} + O(p^2).$$
\hfill (S52)

The presence of a slip (vector $\mathbf{d}$ in Ref. [1]) changes the overall phase of the matrix $T(x)$ and hence according to Eq. (S51) does not change the result for the magnetic helicity.

**FARADAY ROTATION OF THE SUPERCONDUCTING PHASE WITH A FINITE INCIDENT ANGLE $\alpha$**

**Setup and equation of motions**

We consider a thin chiral superconductor with thickness $2\varepsilon$. The scattering light lives on the $(y,z)$ plane and makes an angle $\alpha$ with the $z$ axis as shown in Fig S1. We recall the action of a thin chiral superconductor interacting with the electromagnetic field
$$S = \int d^4x \left\{ -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + \delta(z) \left[ \frac{f^2}{2} (D_t \varphi)^2 - \frac{n_s}{2m} (D_a \varphi)^2 - n_s \kappa B_a D_\alpha \varphi \right] \right\}, \hfill (S53)$$
where we used $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\delta_\theta = \frac{1}{2}\delta$. We chose subscripts $a,b$ to denote inplane spacial directions. We chose the Coulomb gauge $A_0 = 0$, instead of $\partial_\alpha \varphi = 0$ for $\alpha = 0$ as in the main text, the equation of motion of $\varphi$ is non-trivial. The components of incident, reflection and transmission lights has the following ansatz
$$A_x^I e^{-i(\omega t - k \cdot r)}, \quad A_y^I e^{-i(\omega t - k \cdot r)}, \quad A_z^I e^{-i(\omega t - k \cdot r)},$$
$$A_x^R e^{-i(\omega t - k' \cdot r)}, \quad A_y^R e^{-i(\omega t - k' \cdot r)}, \quad A_z^R e^{-i(\omega t - k' \cdot r)},$$
$$A_x^T e^{-i(\omega t - k \cdot r)}, \quad A_y^T e^{-i(\omega t - k \cdot r)}, \quad A_z^T e^{-i(\omega t - k \cdot r)}. \hfill (S54)$$

\hfill (S55)
where we denote \( \mathbf{k} = (0, k_y, k_z) \) and \( \mathbf{k}' = (0, k_y, -k_z) \), with \( k_y = k \sin \alpha, k_z = k \cos \alpha \). Our task is finding \( A_i^R \) and \( A_i^T \) in term of \( A_i^I \). The equation of motion (e.o.m) of \( A_0 \) gives

\[
0 = \frac{1}{c} \partial_t \partial_z A_i + 2\delta(z)f^2 \partial_t \varphi. \tag{S57}
\]

Equation (S57) is nothing but the Gauss law which implies \( \nabla_i A_i = 0 \) outside the superconductor. The e.o.m of \( \varphi \) is

\[
0 = \delta(z) \left[ -f^2 \partial_t^2 \varphi + \frac{n_s}{m} \partial_z D_a \varphi + n_s \kappa \partial_a B_a \right]. \tag{S58}
\]

Finally the e.o.m of \( A_i \) are

\[
0 = - \frac{1}{4\pi c^2} \partial_t^2 A_z - \frac{e^{ab}}{4\pi} \partial_y B_b - n_s \kappa \delta(z) e^{ab} \partial_a \partial_b \varphi, \tag{S59}
\]

\[
0 = - \frac{1}{4\pi c^2} \partial_t^2 A_x - \frac{(\partial_y B_z - \partial_z B_y)}{4\pi} - \delta(z) \left( \frac{2n_s e}{m \hbar c} D_x \varphi + \frac{2n_s e}{\hbar c} \kappa B_x \right) + n_s \kappa \partial_z (\delta(z) D_y \varphi), \tag{S60}
\]

\[
0 = - \frac{1}{4\pi c^2} \partial_t^2 A_y - \frac{(\partial_z B_x - \partial_x B_z)}{4\pi} - \delta(z) \left( \frac{2n_s e}{m \hbar c} D_y \varphi + \frac{2n_s e}{\hbar c} \kappa B_y \right) - n_s \kappa \partial_z (\delta(z) D_x \varphi). \tag{S61}
\]

The combination of Gauss law (S57) and the e.o.m for \( A_i \) gives us

\[
\frac{1}{c^2} \partial_t^2 A_i - \Delta A_i = 0 \tag{S62}
\]

outside the superconductor. It is satisfied automatically with photon’s dispersion relation \( \omega = c |\mathbf{k}| \). In current setup, one can replace \( \partial_x \rightarrow 0 \). We also replace \( \partial_y \rightarrow i k_y \) and \( \partial_t \rightarrow -i \omega \) to convert above e.o.m to the momentum space. We employ the long-wave length limit approximations

\[
A_i(0) = \frac{1}{2} (A_i(\varepsilon) + A_i(-\varepsilon)), \quad A_i'(0) = \frac{1}{2} (A_i'(\varepsilon) + A_i'(-\varepsilon)), \tag{S63}
\]

with the notation \( A_i' = \partial_z A_i \). Integrating equations (S57)-(S61) from \(-\varepsilon\) to \(\varepsilon\), we obtain

\[
-i \omega \left( (A_z(\varepsilon) - A_z(-\varepsilon)) + 2e \omega f^2 \varphi \right) = 0, \tag{S64}
\]

\[
f^2 \omega^2 \varphi - \frac{n_s}{m} \left[ k_y^2 \varphi - \frac{i}{c} k_y (A_y(\varepsilon) - A_y(-\varepsilon)) \right] + i k_y n_s \kappa \frac{1}{2} (A_x'(\varepsilon) + A_x'(-\varepsilon)) = 0, \tag{S65}
\]

\[
-i k_y (A_y(\varepsilon) - A_y(-\varepsilon)) + i k_y A \frac{n_s e}{\hbar c} \kappa (A_x(\varepsilon) + A_x(-\varepsilon)) = 0, \tag{S66}
\]

\[
A_x'(\varepsilon) - A_x'(-\varepsilon) - 4\pi \left\{ \frac{2n_s e^2}{m \hbar c^2} \left[ A_x(\varepsilon) + A_x(-\varepsilon) \right] - \frac{n_s e}{\hbar c} \kappa \left[ -i k_y (A_x(\varepsilon) + A_x(-\varepsilon)) + (A_y(\varepsilon) + A_y'(-\varepsilon)) \right] \right\} = 0, \tag{S67}
\]

\[
A_y'(\varepsilon) - A_y'(-\varepsilon) - i k_y (A_x(\varepsilon) - A_x(-\varepsilon)) - 4\pi \left\{ \frac{2n_s e}{m \hbar c} \left[ i k_y \varphi + \frac{e}{\hbar c} (A_y(\varepsilon) + A_y(-\varepsilon)) \right] + \frac{n_s e}{\hbar c} \kappa (A_x'(\varepsilon) + A_x'(-\varepsilon)) \right\} = 0. \tag{S68}
\]

Multiplying equations (S60)-(S61) by \( z \) then integrating from \(-\varepsilon\) to \(\varepsilon\), we obtain

\[
-(A_x(\varepsilon) - A_x(-\varepsilon)) + 4\pi n_s \kappa \left[ -i k_y \varphi - \frac{e}{\hbar c} (A_y(\varepsilon) + A_y(-\varepsilon)) \right] = 0, \tag{S69}
\]

\[
-(A_y(\varepsilon) - A_y(-\varepsilon)) + 4\pi \frac{n_s e}{\hbar c} \kappa (A_x(\varepsilon) + A_x(-\varepsilon)) = 0. \tag{S70}
\]

We now have 9 equations including (S64)-(S70), and the Gauss law for \( z < -\varepsilon \) and \( z > \varepsilon \)

\[
A_i^R k_y - A_i^R k_z = 0, \quad A_i^T k_y + A_i^T k_z = 0, \tag{S71}
\]

to find 7 unknowns \( \varphi, A_i^R, A_i^T \) from \( A_i^I \) and \( k_y, k_z \) with the condition \( A_i^I k_y + A_i^I k_z = 0 \) implied by the Gauss law. One can check that 2 of above 9 equations are redundant. We can solve uniquely \( A_i^R, A_i^T \) from \( A_i^I \). We will summarize the results in the next subsection.
Transmission of linear polarized light

Due to the Gauss law, we define in-plane vector potential as

\[ A_{in}^I = \frac{A_y}{\cos \alpha}, \quad A_{in}^R = \frac{A_y}{\sin \alpha}, \quad A_{in}^T = \frac{A_y}{\cos \alpha} = -\frac{A_z}{\sin \alpha}. \]  

(S72)

We also define the out of plane vector potential as

\[ A_{out}^I = -A_x, \quad A_{out}^R = A_x, \quad A_{out}^T = -A_x. \]  

(S73)

The polarization function of the transmission light is

\[ P_T = A_{out}^T A_{in}^T. \]  

(S74)

From the definition (S74), we see that real (pure imaginary) \( P \) correspond to linear (elliptical) polarized light. Especially, \( P = i \) \((P = -i)\) corresponds to right-hand circle \((\text{left-hand circle})\) polarized light. We consider the linear polarized incident light with either in-plane polarization or out-of-plane. We calculate the polarization function using equations in the previous section and quote the result at the leading orders in the momentum \( k \):

**In-plane incident light** \((A_{in}^I \neq 0, A_{out}^I = 0)\)

We obtain the result of polarization functions as

\[
\mathcal{P}^T = \frac{\kappa mc}{e \cos^2(\alpha)} \left[ -64\pi^2 \frac{f^2}{c^2} \frac{nse^2}{mhc^2} \cos(\alpha) + ik \sin^2(\alpha) \left( 4\pi f^2 \frac{1}{c^2} \left( 16\pi^2 \frac{K^2 nse^2}{mhc^2} + 1 \right) - \frac{4\pi nse^2}{mhc^2} \right) \right] + O(k^2).
\]  

(S75)

**Out-of-plane incident light** \((A_{in}^I = 0, A_{out}^I \neq 0)\)

We obtain the result of polarization functions as

\[
1/\mathcal{P}^T = \frac{\kappa mh c}{e \cos^2(\alpha)} \left[ 64\pi^2 \frac{f^2}{c^2} \frac{nse^2}{mhc^2} \cos(\alpha) + ik \sin^2(\alpha) \left( 4\pi f^2 \frac{1}{c^2} \left( 16\pi^2 \frac{K^2 nse^2}{mhc^2} + 1 \right) - \frac{4\pi nse^2}{mhc^2} \right) \right] \frac{2f^2}{c^2} \left( 16\pi^2 \frac{K^2 nse^2}{mhc^2} - 1 \right).
\]  

(S76)

From equation (S75) and (S76), at the incident angle \( \alpha = 0 \), we obtain the Faraday rotation angle of the linear polarized light (both in-plane and out-of-plane)

\[
\theta_F = \arctan \left( \frac{8\pi \kappa nse}{h c} \right) \approx 8\pi \frac{\kappa nse}{h c},
\]  

(S77)

which is the same as the Faraday rotation angle in the main text for the superconducting phase by combining Eqs. (32) and (18). With a general value of \( \alpha \), the transmission light is neither linearly nor elliptically polarized.