Supplementary to “Inference of Random Effects for Linear Mixed-Effects Models with a Fixed Number of Clusters”

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Supplementary Material

The supplementary materials consist of three appendices that prove all the theoretical results except for Theorem 2, whose proof is straightforward and is hence omitted. Appendix A contains auxiliary lemmas that are required in the proofs. Appendix B provides proofs for Example 1 and Theorems 1 and 3–5. Appendix C gives proofs for all the lemmas.

A Auxiliary Lemmas

We start with the following matrix identities, which will be repeated applied:

\begin{align*}
\det(A + cd') &= \det(A)(1 + d'A^{-1}c), \quad (A.1) \\
(A + cd')^{-1} &= A^{-1} - \frac{A^{-1}cd'A^{-1}}{1 + d'A^{-1}c}, \quad (A.2)
\end{align*}

where $A$ is an $n \times n$ nonsingular matrix, and $c$ and $d$ are $n \times 1$ column vectors. Note that (A.2) is applied iteratively to establish the decomposition of the precision matrix $H_{i}^{-1}(\gamma, \theta)$, where

\begin{equation}
H_{i}(\gamma, \theta) \equiv \sum_{k \in \gamma} \theta_{k}z_{i,k}z_{i,k}' + I_{n_i}. \quad (A.3)
\end{equation}

Heuristically speaking, let $z_{i,(s)}$; $s = 1, \ldots, q(\gamma)$ be the $s$-th column of $Z_{i}(\gamma)$ and

\begin{equation}
H_{i,t}(\gamma, \theta) = \sum_{s=1}^{t} \theta_{(s)}z_{i,(s)}z_{i,(s)}' + I_{n_i}; \quad t = 1, \ldots, q(\gamma), \quad (A.4)
\end{equation}

where $\theta_{(s)}$ denotes the $s$-th element of $\theta$; $s = 1, \ldots, q(\gamma)$. Suppose that $q(\gamma) = q$. Then by (A.2),

\begin{equation}
H_{i,q}^{-1}(\gamma, \theta) = H_{i,q-1}^{-1}(\gamma, \theta) - \frac{\theta_{q}H_{i,q-1}^{-1}(\gamma, \theta)z_{i,q}z_{i,q}'H_{i,q-1}^{-1}(\gamma, \theta)z_{i,q}}{1 + \theta_{q}z_{i,q}'H_{i,q-1}^{-1}(\gamma, \theta)z_{i,q}}. \quad (A.5)
\end{equation}

Applying (A.2) iteratively, we obtain the decomposition

\begin{equation}
H_{i,q}^{-1}(\gamma, \theta) = I_{n_i} - \sum_{k=1}^{q} \frac{\theta_{k}H_{i,k-1}^{-1}(\gamma, \theta)z_{i,k}z_{i,k}'H_{i,k-1}^{-1}(\gamma, \theta)z_{i,k}}{1 + \theta_{k}z_{i,k}'H_{i,k-1}^{-1}(\gamma, \theta)z_{i,k}}. \quad (A.6)
\end{equation}
note that $H_{1,0}(\gamma, \theta) = I_n$. The proofs of Lemmas 2, 3, and 4 are then based on the induction and the decomposition of (A.6).

The proofs of theorems in Section 3 heavily rely on the asymptotic properties of the quadratic forms, $x_{i,j}^t H_{i}^{-1}(\gamma, \theta) x_{i,j} - z_{i,k} H_{i}^{-1}(\gamma, \theta) z_{i,k}$, $\varepsilon_{i}^t H_{i}^{-1}(\gamma, \theta) \varepsilon_{i}$, and $x_{i,j}^t H_{i}^{-1}(\gamma, \theta) \varepsilon_{i}$, with $H_i(\gamma, \theta)$ defined in (A.3), for $i = 1, \ldots, m$; $j, j' = 1, \ldots, p$ and $k, k' = 1, \ldots, q$. The following lemmas give their convergence rates.

**Lemma 2** Consider the linear mixed-effects model $(\alpha, \gamma)$ of (2.4). Suppose that (A0)–(A3) hold. Then for $H_i(\gamma, \theta)$ defined in (A.3), we have

(i) For $i = 1, \ldots, m$ and $j, j' = 1, \ldots, p$,

$$\sup_{\theta \in (0, \infty)^q} |x_{i,j}^t H_i^{-1}(\gamma, \theta) x_{i,j'}| = \begin{cases} d_{i,j} n_{i}^{\xi} + o(n_{i}^{\xi}); & \text{if } j = j', \\ o(n_{i}^{\xi - \tau}); & \text{if } j \neq j'. \end{cases}$$

(ii) For $i = 1, \ldots, m$, $j = 1, \ldots, p$ and $k \notin \gamma$,

$$\sup_{\theta \in (0, \infty)^q} |x_{i,j}^t H_i^{-1}(\gamma, \theta) z_{i,k}| = o(n_{i}^{(\xi + \ell)/2 - \tau}).$$

(iii) For $i = 1, \ldots, m$, $j = 1, \ldots, p$ and $k \in \gamma$,

$$\sup_{\theta \in (0, \infty)^q} \theta_k |x_{i,j}^t H_i^{-1}(\gamma, \theta) z_{i,k}| = o_p(n_{i}^{(\xi - \ell)/2 - \tau}),$$

$$\sup_{\theta \in (0, \infty)^q} |x_{i,j}^t H_i^{-1}(\gamma, \theta) z_{i,k}| = o(n_{i}^{(\xi - \ell)/2 - \tau}).$$

**Lemma 3** Consider the linear mixed-effects model $(\alpha, \gamma)$ of (2.4). Suppose that (A0) and (A2) hold. Then for $H_i(\gamma, \theta)$ defined in (A.3), we have

(i) For $i = 1, \ldots, m$ and $k, k^* \notin \gamma$,

$$\sup_{\theta \in (0, \infty)^q} |z_{i,k}^t H_i^{-1}(\gamma, \theta) z_{i,k^*}| = \begin{cases} c_{i,k} n_{i}^{\xi} + o(n_{i}^{\xi}); & \text{if } k = k^*, \\ o(n_{i}^{\xi - \tau}); & \text{if } k \neq k^*. \end{cases}$$

(ii) For $i = 1, \ldots, m$ and $k \in \gamma$,

$$\sup_{\theta \in (0, \infty)^q} |\theta_k^2 z_{i,k}^t H_i^{-1}(\gamma, \theta) z_{i,k} - \theta_k| = O(n_{i}^{-\ell}),$$

$$\sup_{\theta \in (0, \infty)^q} |z_{i,k}^t H_i^{-1}(\gamma, \theta) z_{i,k}| = O(n_{i}^{\ell}).$$

(iii) For $i = 1, \ldots, m$ and $k, k^* \in \gamma$ with $k \neq k^*$,

$$\sup_{\theta \in (0, \infty)^q} \theta_k \theta_{k^*} |z_{i,k}^t H_i^{-1}(\gamma, \theta) z_{i,k^*}| = o(n_{i}^{-\ell - \tau}),$$

$$\sup_{\theta \in (0, \infty)^q} \theta_k |z_{i,k}^t H_i^{-1}(\gamma, \theta) z_{i,k^*}| = o(n_{i}^{-\tau}),$$

$$\sup_{\theta \in (0, \infty)^q} |z_{i,k}^t H_i^{-1}(\gamma, \theta) z_{i,k^*}| = o(n_{i}^{\ell - \tau}).$$

(iv) For $i = 1, \ldots, m$, $k \in \gamma$ and $k^* \notin \gamma$,

$$\sup_{\theta \in (0, \infty)^q} \theta_k |z_{i,k}^t H_i^{-1}(\gamma, \theta) z_{i,k^*}| = o(n_{i}^{-\tau}),$$

$$\sup_{\theta \in (0, \infty)^q} |z_{i,k}^t H_i^{-1}(\gamma, \theta) z_{i,k^*}| = o(n_{i}^{\ell - \tau}).$$
Lemma 4 Consider the linear mixed-effects model $(\alpha, \gamma)$ of (2.4). Suppose that (A0)–(A3) hold. Then for $H_i(\gamma, \theta)$ defined in (A.3), we have

(i) For $i = 1, \ldots, m$ and $k \in \gamma,$
\[ \sup_{\theta \in [0, \infty)^{q(\gamma)}} \theta_k \cdot |z_{i,k}^T H^{-1}_i(\gamma, \theta) \epsilon_i| = O_p(n_i^{-\ell/2}), \]
\[ \sup_{\theta \in [0, \infty)^{q(\gamma)}} |z_{i,k}^T H^{-1}_i(\gamma, \theta) \epsilon_i| = O_p(n_i^{\ell/2}). \]

(ii) For $i = 1, \ldots, m$ and $k \notin \gamma,$
\[ \sup_{\theta \in [0, \infty)^{q(\gamma)}} |z_{i,k}^T H^{-1}_i(\gamma, \theta) \epsilon_i| = O_p(n_i^{\ell/2}). \]

(iii) For $i = 1, \ldots, m$ and $j = 1, \ldots, p,$
\[ \sup_{\theta \in [0, \infty)^{q(\gamma)}} |x_{i,j}^T H^{-1}_i(\gamma, \theta) \epsilon_i| = O_p(n_i^{\ell/2}). \]

In addition,
\[ \sup_{\theta \in [0, \infty)^{q(\gamma)}} \left| \sum_{i=1}^m x_{i,j}^T H^{-1}_i(\gamma, \theta) \epsilon_i \right| = O_p \left( \left( \sum_{i=1}^m n_i^\ell \right)^{1/2} \right). \]

(iv) For $i = 1, \ldots, m,$
\[ \sup_{\theta \in [0, \infty)^{q(\gamma)}} \epsilon_i^T H^{-1}_i(\gamma, \theta) \epsilon_i = \epsilon_i^T \epsilon_i + O_p(q). \]

Note that Lemma 2 (i) implies that, for $(\alpha, \gamma) \in A \times \mathcal{G},$
\[ \sum_{i=1}^m X_i(\alpha)^T H^{-1}_i(\gamma, \theta) X_i(\alpha) = \left( \sum_{i=1}^m n_i^\ell \right) T(\alpha) + \left\{ a \left( \sum_{i=1}^m n_i^{-\tau} \right) \right\} p(\alpha) \times p(\alpha) \]
\[ = \left( \sum_{i=1}^m n_i^\ell \right) T(\alpha) + \left\{ a \left( n_{\min}^{-\tau} \sum_{i=1}^m n_i^\ell \right) \right\} p(\alpha) \times p(\alpha) \]
uniformly over $\theta \in [0, \infty)^{q(\gamma)},$ where $\{a\}_{k \times j}$ denotes a $k \times j$ matrix with elements equal to $a$ and $T(\alpha)$ is a diagonal matrix with diagonal elements bounded away from 0 and $\infty.$ Hence by (A.2) with $p(\alpha)$-vectors $c = \{o(n_{\min}^{-\tau/2})\} p(\alpha) \times 1$ and $d = \{o(n_{\min}^{-\tau})\} p(\alpha) \times 1,$ and a $p(\alpha) \times p(\alpha)$ diagonal matrix $A = T(\alpha),$ we have, for $(\alpha, \gamma) \in A \times \mathcal{G},$
\[ \left( \sum_{i=1}^m \frac{X_i(\alpha)^T H^{-1}_i(\gamma, \theta) X_i(\alpha)}{n_i^\ell} \right)^{-1} = \left( T(\alpha) + \left\{ o(n_{\min}^{-\tau}) \right\} p(\alpha) \times p(\alpha) \right)^{-1} \]
\[ = T^{-1}(\alpha) + \left\{ o(n_{\min}^{-\tau}) \right\} p(\alpha) \times p(\alpha) \]
uniformly over $\theta \in [0, \infty)^{q(\gamma)},$ which plays a key role in proving lemmas for theorems.

The following lemma shows that $\theta_k$ does not converge to 0 in probability for $k \in \gamma \cap \gamma_0,$ which allows us to restrict the parameter space of $\theta$ from $[0, \infty)^{q(\gamma)}$ to
\[ \Theta_\gamma = \{ \theta \in [0, \infty)^{q(\gamma)} : \theta(\gamma \cap \gamma_0) \in (0, \infty)^{q(\gamma \cap \gamma_0)} \}. \]
Lemma 5  Under the assumptions of Theorem 1, let $\theta_{0,i}^j$ be $\theta$ except that $\{\theta_{k,0} : k \in \gamma \cap \gamma_0\}$ are replaced by $\{\theta_{k,0} : k \in \gamma \cap \gamma_0\}$. Then for any $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}$, $\nu^2 > 0$, and $\theta \in [0, \infty) \cap \gamma$, with $\theta_k \rightarrow 0$ for some $k \in \gamma \cap \gamma_0$, we have

$$-2 \log L(\theta, \nu^2; \alpha, \gamma) - \{ -2 \log L(\theta_{0,i}^j, \nu^2; \alpha, \gamma) \} \to \infty$$

as $N \to \infty$, where $-2 \log L(\theta, \nu^2; \alpha, \gamma)$ is given in (2.7).

Based on Lemma 5, the following lemma is needed to develop the convergence rates of components of the likelihood equations given in (B.1) and (B.2), uniformly over $\Theta_\gamma$ defined in (A.8).

Lemma 6  Consider a mixed-effects model $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}$ with $H(\gamma, \theta)$ defined in (2.5) and $\Theta_\gamma$ defined in (A.8). Suppose that (A0)–(A3) hold. Then

(i) For $i, i^* = 1, \ldots, m$, $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}$ and $k, k^* \in \gamma$,

$$\sup_{\theta \in \Theta_\gamma} \theta_k \theta_{k^*} \left| h_{i,k}^{i,j} H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) h_{i^*,k^*} \right| = o \left( \frac{n_i^{(\xi - \ell)/2} (\xi - \ell)/2}{\sum_{i=1}^{m} n_i^\gamma} \right),$$

(ii) For $i, i^* = 1, \ldots, m$, $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}$, $k \in \gamma$ and $k^* \notin \gamma$,

$$\sup_{\theta \in \Theta_\gamma} \theta_k \left| h_{i,k}^{i,j} H^{-1}(\gamma, \theta) M(\alpha, \gamma, \gamma; \theta) h_{i^*,k^*} \right| = o \left( \frac{n_i^{(\xi - \ell)/2} (\xi - \ell)/2}{\sum_{i=1}^{m} n_i^\gamma} \right),$$

(iii) For $i = 1, \ldots, m$, $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}$ and $k \in \gamma$,

$$\sup_{\theta \in \Theta_\gamma} \theta_k \left| h_{i,k}^{i,j} H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) \epsilon \right| = o_p(n_i^{-\ell/2}),$$

(iv) For $i = 1, \ldots, m$, $(\alpha, \gamma) \in (\mathcal{A} \setminus \mathcal{A}_0) \times \mathcal{G}$ and $k \in \gamma$,

$$\sup_{\theta \in \Theta_\gamma} \theta_k \left| h_{i,k}^{i,j} H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) X(\alpha \setminus \alpha) \beta(\alpha \setminus \alpha) \right| = o(n_i^{(\xi - \ell)/2 - \gamma}),$$

(v) For $i = 1, \ldots, m$ and $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}$,

$$\sup_{\theta \in \Theta_\gamma} \epsilon \left| h_{i,k}^{i,j} H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) \epsilon \right| = O_p(p(\alpha)).$$

(vi) For $i = 1, \ldots, m$, $(\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}$ and $k \notin \gamma$,

$$\sup_{\theta \in \Theta_\gamma} \left| h_{i,k}^{i,j} H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) \epsilon \right| = o_p(n_i^{\ell/2}).$$
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(vi) For \((\alpha, \gamma) \in (A \setminus A_0) \times \mathcal{G}\),

\[
\sup_{\theta \in \Theta} \left| e^H(\gamma, \theta) M(\alpha, \gamma; \theta) X(\alpha_0 \setminus \alpha) \beta(\alpha_0 \setminus \alpha) \right| = o_p \left( \left( \sum_{i=1}^{m} n_i \right)^{1/2} \right).
\]

(viii) For \(i, i' = 1, \ldots, m, (\alpha, \gamma) \in A \times \mathcal{G}\) and \(k, k' \notin \gamma\),

\[
\sup_{\theta \in \Theta} \left| h_{i,k}^H(\gamma, \theta) M(\alpha, \gamma; \theta) h_{i',k'} \right| = o_p \left( \frac{n_{i+k} \cdot (\xi + \ell)^2 / n_{i+k}^{\xi + \ell} / 2 - \tau}{\sum_{i=1}^{m} n_i^{\xi - \tau}} \right).
\]

(ix) For \(i = 1, \ldots, m, (\alpha, \gamma) \in (A \setminus A_0) \times \mathcal{G}\) and \(k \notin \gamma\),

\[
\sup_{\theta \in \Theta} \left| h_{i,k}^H(\gamma, \theta) M(\alpha, \gamma; \theta) X(\alpha_0 \setminus \alpha) \beta(\alpha_0 \setminus \alpha) \right| = o_p \left( \frac{n_{i+k} \cdot (\xi + \ell)^2 / n_{i+k}^{\xi + \ell} / 2 - \tau}{\sum_{i=1}^{m} n_i^{\xi - \tau}} \right).
\]

(x) For \((\alpha, \gamma) \in (A \setminus A_0) \times \mathcal{G}\),

\[
\sup_{\theta \in \Theta} \left| \beta(\alpha_0 \setminus \alpha)' X(\alpha_0 \setminus \alpha)' H^{-1}(\gamma, \theta) \times M(\alpha, \gamma; \theta) X(\alpha_0 \setminus \alpha) \beta(\alpha_0 \setminus \alpha) \right| = o_p \left( \sum_{i=1}^{m} n_i^{\xi - \tau} \right).
\]

B Theoretical Proofs

B.1 Proof of Theorem 1

We shall focus on the asymptotic properties of \(\hat{\epsilon}^2(\alpha, \gamma)\) and \(\{\hat{\theta}_k(\alpha, \gamma) : k \in \gamma\}\), and derive the asymptotic properties of \(\{\hat{\sigma}_k^2(\alpha, \gamma) : k \in \gamma\}\) via \(\hat{\epsilon}^2(\alpha, \gamma) = \hat{\epsilon}^2(\alpha, \gamma) \hat{\theta}_k(\alpha, \gamma); k \in \gamma\).

If \(\hat{\epsilon}^2(\alpha, \gamma) > 0\) and \(\hat{\theta}_k(\alpha, \gamma) > 0; k \in \gamma\), then we can derive them using the likelihood equations. Differentiating the profile log-likelihood function of (2.7) with respect to \(\epsilon^2\) and \(\{\hat{\theta}_k : k \in \gamma\}\), we obtain

\[
\frac{\partial}{\partial \epsilon^2} \left( -2 \log L(\theta, \epsilon^2; \alpha, \gamma) \right) = \frac{N}{\epsilon^2} - \frac{y' H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma; \theta)) y}{\epsilon^4} \tag{B.1}
\]

and

\[
\frac{\partial}{\partial \theta_k} \left( -2 \log L(\theta, \epsilon^2; \alpha, \gamma) \right) = \sum_{i=1}^{m} \left[ z_{i,k}^H M(\alpha, \gamma; \theta) z_{i,k} - \frac{\{h_{i,k}^H M(\alpha, \gamma; \theta) y\}^2}{\epsilon^2} \right]. \tag{B.2}
\]

To derive \(\hat{\epsilon}^2(\alpha, \gamma)\) and \(\{\hat{\theta}_k(\alpha, \gamma) : k \in \gamma\}\), we must study the convergence rate of each term on the right-hand sides of both (B.1) and (B.2) by Lemmas 2–4 and Lemma 6.

We first prove (3.1) using (B.1). Consider the following decomposition of \(y' H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma; \theta)) y\) in (B.1):

\[
y' H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma; \theta)) y \\
= \mu_0' H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma; \theta)) \mu_0 \\
+ 2 \mu_0' H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma; \theta)) Z(\gamma_0) b(\gamma_0) + \epsilon \tag{B.3} \\
+ (Z(\gamma_0) b(\gamma_0) + \epsilon)' H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma; \theta)) Z(\gamma_0) b(\gamma_0) + \epsilon \\
- (Z(\gamma_0) b(\gamma_0) + \epsilon)' H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) Z(\gamma_0) b(\gamma_0) + \epsilon.
\]
The first two terms of (B.3) are zeros because
\[(I_N - M(\alpha, \gamma; \theta)) \mathbf{u}_0 = 0; \quad \alpha \in A_0, \quad (B.4)\]
which is obtained by treating \(\mathbf{u}_0 = X(\alpha) \beta(\alpha)\) for some \(\beta(\alpha) \in \mathbb{R}^b(\alpha)\) under \(\alpha \in A_0\), where note that by (2.9), \(M(\alpha, \gamma; \theta) X(\alpha) = X(\alpha)\). By Lemma 3 (ii)–(iii), Lemma 4 (i), and Lemma 4 (iv), the third term of (B.3) can be written as
\[\sum_{i=1}^{m} \mathbf{Z}_i(\gamma_0) \mathbf{b}_i(\gamma_0) + \mathbf{e}_i' \mathbf{H}^{-1}(\gamma, \theta) \mathbf{Z}_i(\gamma_0) \mathbf{b}_i(\gamma_0) + \mathbf{e}_i\]
uniformly over \(\theta \in \Theta_\gamma\). Note that by the Cauchy–Schwarz inequality,
\[\left( \sum_{i=1}^{m} \mathbf{n}_i^{(\gamma-\ell)/2} \right)^2 = O\left( \sum_{i=1}^{m} \mathbf{n}_i^{\ell} \sum_{i=1}^{m} \mathbf{n}_i^{\ell} \right). \quad (B.5)\]
Hence, by Lemma 6 (i), Lemma 6 (iii), and Lemma 6 (v), the last term of (B.3) can be written as
\[\mathbf{y}' \mathbf{H}^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma; \theta))\mathbf{y} = \mathbf{e}'\mathbf{e} + o_p\left( \sum_{k,k' \in \gamma_0} \frac{m}{\theta_k \theta_{k'}} \right) + O_p\left( \sum_{k \in \gamma_0} \frac{m}{\theta_k} \right) + O_p\left( p + mq \right).
\]
It follows from (B.1) that for \(\nu^2 \in (0, \infty)\),
\[\nu^2 \{ \frac{\partial}{\partial \nu} [-2 \log \psi(\theta, \nu^2; \alpha, \gamma)] \} = N\left( \nu^2 - \frac{\mathbf{e}'\mathbf{e}}{N} \right) + o_p\left( \sum_{k,k' \in \gamma_0} \frac{m}{\theta_k \theta_{k'}} \right) + O_p\left( \sum_{k \in \gamma_0} \frac{m}{\theta_k} \right) + O_p\left( p + mq \right).
\]
uniformly over \(\theta \in \Theta_\gamma\). This and Lemma 5 imply that
\[\nu^2(\alpha, \gamma) = \frac{\mathbf{e}'\mathbf{e}}{N} + O_p\left( \frac{p + mq}{N} \right). \quad (B.6)\]
Thus (3.1) follows by applying the law of large numbers to \(\mathbf{e}'\mathbf{e}/N\). In addition, the asymptotic normality of \(\nu^2(\alpha, \gamma)\) follows by \(p + mq = o(N^{1/2})\) and an application of the central limit theorem to \(\mathbf{e}'\mathbf{e}/N\) in (B.6).

Next, we prove (3.2), for \(k \in \gamma \cap \gamma_0\), using (B.2). By Lemma 6 (i) and Lemma 6 (iii), we have, for \(k \in \gamma \cap \gamma_0\),
\[\theta_k \mathbf{h}_k' \mathbf{H}^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) (\mathbf{Z}_i(\gamma_0) \mathbf{b}_i(\gamma_0) + \mathbf{e}) = \theta_k \mathbf{h}_k' \mathbf{H}^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) \mathbf{e} + o_p\left( \sum_{k \in \gamma_0} \frac{m}{\theta_k} \right)
\]
uniformly over \( \theta \in \Theta_{\gamma} \). This and (B.4) imply that for \( k \in \gamma \cap \gamma_0 \),
\[
\theta_k h_{i,k}^* H^{-1}(\gamma, \theta )|_{\Theta_{\gamma} - M(\alpha, \gamma; \theta)} y
\]
\[
= \theta_k h_{i,k}^* H^{-1}(\gamma, \theta )|_{\Theta_{\gamma} - M(\alpha, \gamma; \theta)} (Z(\gamma_0) b(\gamma_0) + \epsilon)
\]
\[
= \theta_k h_{i,k}^* H^{-1}(\gamma, \theta )|_{\Theta_{\gamma} - M(\alpha, \gamma; \theta)} (Z(\gamma_0) b(\gamma_0) + \epsilon)
\]
\[
+ o_p \left( \sum_{k^* \in \gamma_0} \frac{n_i}{\theta_{k^*} \sum_{i=1}^m n_i^*} \right) + o_p(n_i^{-\ell/2})
\]
\[
= \theta_k z_{i,k}^* H^{-1}(\gamma, \theta )|_{\Theta_{\gamma} - M(\alpha, \gamma; \theta)} b(\gamma_0) + \epsilon_i
\]
\[
+ o_p \left( \sum_{k^* \in \gamma_0} \frac{n_i}{\theta_{k^*} \sum_{i=1}^m n_i^*} \right) + o_p(n_i^{-\ell/2})
\]
\[
= b_{i,k} + O_p(n_i^{-\ell/2}) + o_p \left( \sum_{k^* \in \gamma_0} \frac{n_i}{\theta_{k^*} \sum_{i=1}^m n_i^*} \right)
\]
uniformly over \( \theta \in \Theta_{\gamma} \), where the last equality follows from Lemma 3 (ii)–(iii) and Lemma 4 (i). Hence, for \( k \in \gamma \cap \gamma_0 \),
\[
\theta_k^2 (h_{i,k}^* H^{-1}(\gamma, \theta )|_{\Theta_{\gamma} - M(\alpha, \gamma; \theta)} y)^2
\]
\[
= b_{i,k}^2 + O(n_i^{-\ell/2}) + o_p \left( \sum_{k^* \in \gamma_0} \frac{n_i \ell}{\theta_{k^*} \sum_{i=1}^m n_i^*} \right)
\]
uniformly over \( \theta \in \Theta_{\gamma} \). This together with Lemma 3 (ii) and (B.2) imply that for \( k \in \gamma \cap \gamma_0 \),
\[
\theta_k^2 \left\{ \frac{\partial}{\partial \theta_k} \left( -2 \log L(\theta, \nu; \alpha, \gamma) \right) \right\}
\]
\[
= m \left( \theta_k - \frac{1}{m} \sum_{i=1}^m \frac{\beta_{i,k}^2}{\nu_i^2} \right) + o_p \left( \sum_{i=1}^m n_i^{-\ell/2} \right)
\]
\[
+ o_p \left( \sum_{k^* \in \gamma_0} \frac{m}{\theta_{k^*} \sum_{i=1}^m n_i^*} \right)
\]
uniformly over \( \theta \in \Theta_{\gamma} \). By (B.5), Lemma 5 and setting (B.8) to 0, we obtain
\[
\hat{\theta}_k(\alpha, \gamma) = \frac{1}{m} \sum_{i=1}^m \frac{\beta_{i,k}^2}{\nu_i^2(\alpha, \gamma)} + o_p \left( \frac{1}{m} \sum_{i=1}^m n_i^{-\ell/2} \right), \quad k \in \gamma \cap \gamma_0.
\]
This proves (3.2), for \( k \in \gamma \cap \gamma_0 \). It remains to prove (3.2), for \( k \in \gamma \setminus \gamma_0 \). We prove by showing that (B.2) is asymptotically nonnegative, for \( \theta_k \in \{\mu_{\max}, \infty\} \); \( k \in \gamma \setminus \gamma_0 \) using a recursive argument. Let \( \theta^1 \) be \( \theta \) except that \( \{\theta_k : k \in \gamma \cap \gamma_0\} \) are replaced by \( \{\theta_k(\alpha, \gamma) : k \in \gamma \cap \gamma_0\} \). By Lemma 6 (i) and Lemma 6 (iii), we have, for \( k \in \gamma \setminus \gamma_0 \),
\[
\theta_k h_{i,k}^* H^{-1}(\gamma, \theta^1 )|_{\Theta_{\gamma} - M(\alpha, \gamma; \theta^1)} (Z(\gamma_0) b(\gamma_0) + \epsilon)
\]
\[
= \theta_k h_{i,k}^* H^{-1}(\gamma, \theta^1 )|_{\Theta_{\gamma} - M(\alpha, \gamma; \theta^1)} (Z(\gamma_0) b(\gamma_0) + \epsilon)
\]
\[
= \theta_k h_{i,k}^* H^{-1}(\gamma, \theta^1 )|_{\Theta_{\gamma} - M(\alpha, \gamma; \theta^1)} \left( \sum_{i=1}^m \sum_{i=1}^m b_{i,k}^+ h_{i,k}^+ \right)
\]
\[
= o_p \left( \frac{n_i \ell}{\theta_{k^*} \sum_{i=1}^m n_i^*} \right) + o_p(n_i^{-\ell/2})
\]
uniformly over \( \theta(\gamma \setminus \gamma_0) \in [0, \infty)^{(\gamma \setminus \gamma_0)} \). This and (B.4) imply that for \( k \in \gamma \setminus \gamma_0 \),

\[
\begin{aligned}
\theta h_k' H^{-1}(\gamma, \theta^1)(I_N - M(\alpha, \gamma; \theta^1))y \\
= \theta h_k' H^{-1}(\gamma, \theta^1)(I_N - M(\alpha, \gamma; \theta^1))(Z(\gamma_0)b(\gamma_0) + \varepsilon) \\
= \theta h_k' H^{-1}(\gamma, \theta^1)(Z(\gamma_0)b(\gamma_0) + \varepsilon) \\
+ o_p\left(\frac{n_1^{(\xi-\ell)/2} \sum_{i=1}^m n_i^{(\xi-\ell)/2}}{\sum_{i=1}^m n_i^{\xi}}\right) + o_p(n_i^{\ell/2}) \\
= \theta h_k' H^{-1}(\gamma, \theta^1)\left(\sum_{k' \in \gamma_0} z_{i,k'} b_{i,k'} + \varepsilon_i\right) \\
+ o_p\left(\frac{n_1^{(\xi-\ell)/2} \sum_{i=1}^m n_i^{(\xi-\ell)/2}}{\sum_{i=1}^m n_i^{\xi}}\right) + o_p(n_i^{\ell/2}) \\
= O_p(n_i^{\ell/2}) + o_p\left(\frac{n_1^{(\xi-\ell)/2} \sum_{i=1}^m n_i^{(\xi-\ell)/2}}{\sum_{i=1}^m n_i^{\xi}}\right)
\end{aligned}
\]  

(B.9)

uniformly over \( \theta(\gamma \setminus \gamma_0) \in [0, \infty)^{(\gamma \setminus \gamma_0)} \), where the last equality follows from Lemma 3 (iii) and Lemma 4 (i). Hence by (B.5), Lemma 3 (ii), and (B.2), we have, for \( \theta(\gamma \setminus \gamma_0) \in [0, \infty)^{(\gamma \setminus \gamma_0)} \) and \( k \in \gamma \setminus \gamma_0 \),

\[
\begin{aligned}
\left\{\frac{\partial}{\partial \theta_k} \{-2 \log L(\theta^1, \nu^2; \alpha, \gamma)\}\right\} \\
= m \theta_k + o_p\left(\sum_{i=1}^m n_i^{\ell}\right) + o_p\left(\frac{\sum_{i=1}^m n_i^{(\xi-\ell)/2} \sum_{i=1}^m n_i^{(\xi-\ell)/2}}{\sum_{i=1}^m n_i^{\xi}}\right) \\
= m \theta_k + o_p\left(\sum_{i=1}^m n_i^{\ell}\right) \\
= m \theta_k + o_p(m \log(n_{\text{min}})n_i^{-\ell})
\end{aligned}
\]

This implies that \(-2 \log L(\theta^1, \nu^2; \alpha, \gamma)\) is an asymptotically nondecreasing function on \( \theta_k \in (\log(n_{\text{min}})n_i^{-\ell}, \infty) \), for \( k \in \gamma \setminus \gamma_0 \) given other \( \theta(\gamma \setminus \gamma_0) \in [0, \infty)^{(\gamma \setminus \gamma_0)} \). It follows that \( \theta_k(\alpha, \gamma) \in [0, \log(n_{\text{min}})n_i^{-\ell}) \), \( k \in \gamma \setminus \gamma_0 \). The above convergence rate can be recursively improved. Without loss of generality, assume that \( n_{\text{min}} = n_1 \leq n_2 \leq \cdots \leq n_m = n_{\text{max}} \). We can restrict the parameter space of \( \theta_k \) in the next step to

\[
\theta_{\gamma,k,i} = \left\{\theta(\gamma \setminus \gamma_0) \in [0, \infty)^{(\gamma \setminus \gamma_0)} : \theta_k \leq \log(n_{\text{min}})n_i^{-\ell}\right\}
\]

(B.10)

with \( i = 1 \). Then, by Lemma 6 (i) and Lemma 6 (iii), we have, for \( k \in \gamma \setminus \gamma_0 \),

\[
\begin{aligned}
\theta h_k' H^{-1}(\gamma, \theta^1)M(\alpha, \gamma; \theta^1)(Z(\gamma_0)b(\gamma_0) + \varepsilon) \\
= o_p\left(\frac{n_1^{(\xi-\ell)/2} \sum_{i=1}^m n_i^{(\xi-\ell)/2}}{\sum_{i=1}^m n_i^{\xi}}\right) + o_p(\theta_k n_i^{\ell/2})
\end{aligned}
\]
uniformly over $\theta(\gamma \setminus \gamma_0) \in \Theta_{\gamma, k, 1}$. This and (B.4) imply that for $k \in \gamma \setminus \gamma_0$,

$$\theta_k h'_{1,k} H^{-1}(\gamma, \theta^1)(I_N - M(\alpha, \gamma; \theta^1))y$$

$$= \theta_k h'_{1,k} H^{-1}(\gamma, \theta^1)(I_N - M(\alpha, \gamma; \theta^1))(Z(\gamma_0)b(\gamma_0) + \epsilon)$$

$$= \theta_k z'_{1,k} H^{-1}(\gamma, \theta^1)(Z(\gamma_0)b(\gamma_0) + \epsilon)$$

$$+ o_p \left( \frac{n_1(\xi^{-\ell}/2 \sum_{i=1}^m n_i^{\xi^{-\ell}/2}} {\sum_{i=1}^m n_i^{\xi^{-\ell}/2}} \right) + o_p(\theta_k n_1^{\xi^{-\ell}/2})$$

$$= O_p(\theta_k n_1^{\xi^{-\ell}/2}) + o_p \left( \frac{n_1(\xi^{-\ell}/2 \sum_{i=1}^m n_i^{\xi^{-\ell}/2}} {\sum_{i=1}^m n_i^{\xi^{-\ell}/2}} \right)$$

uniformly over $\theta(\gamma \setminus \gamma_0) \in \Theta_{\gamma, k, 1}$, where the last equality follows from Lemma 3 (iii) and Lemma 4 (i). Hence by (B.5), Lemma 3 (ii), (B.2), and (B.9), we have

$$\theta_k^2 \left\{ \frac{\partial}{\partial \theta_k} \{-2 \log L(\theta^1, \nu^2, \alpha, \gamma) \} \right\}$$

$$= (m - 1) \theta_k + O_p(\theta_k^2 n_1^{i^{-\ell}}) + O_p \left( \sum_{i=2}^m n_i^{i^{-\ell}} \right) + o_p \left( \sum_{i=1}^m n_i^{i^{-\ell}} \right)$$

uniformly over $\theta(\gamma \setminus \gamma_0) \in \Theta_{\gamma, k, 1}$. Hence, setting the above equation equal to 0, we have

$$\theta_k^2 = \frac{1}{m - 1 + O_p(\log(n_{\text{min}}))} O_p \left( \sum_{i=2}^m n_i^{i^{-\ell}} \right) = O_p(n_2^{-\ell}).$$

Now we can further restrict the parameter space of $\theta_k$ to $\Theta_{\gamma, k, 2}$ in (B.10). Continuing this procedure, we can recursively obtain $\hat{\theta}_k(\alpha, \gamma) = O_p(n_i^{-\ell})$; $k \in \gamma \setminus \gamma_0$, for $i = 3, \ldots, m$. This completes the proof of (3.2), for $k \in \gamma \setminus \gamma_0$. Hence the proof of Theorem 1 is complete.

B.2 Proof of Example 1

Note that for $q = 1$, $Z_0 = z_{1,1}$ and $b_1 = b_{1,1}$. Note that by Lemma 5, we consider the sample space $(\sigma^2_1, \nu^2) \in (0, \infty)^2$. We first derive the explicit forms of the ML estimators $\hat{\theta}_1$ and $\hat{\nu}^2$. 
By (B.2), we have
\[
\frac{\partial}{\partial \theta_1} \{-2 \log L(\theta_1, \bar{v}^2)\} = \sum_{i=1}^{m} \frac{z_{i,1}' z_{i,1}}{1 + \hat{\theta}_1 z_{i,1}' z_{i,1}} - \frac{1}{v \cdot b} \sum_{i=1}^{m} \left\{ z_{i,1}' \left( I_n - \frac{\theta_1 z_{i,1} z_{i,1}'}{1 + \theta_1 z_{i,1} z_{i,1}'} \right) y_i \right\}^2
= \sum_{i=1}^{m} \frac{z_{i,1}' z_{i,1}}{1 + \hat{\theta}_1 z_{i,1}' z_{i,1}} - \frac{1}{v \cdot b} \sum_{i=1}^{m} \left\{ z_{i,1}' z_{i,1} b_{i,1} + \frac{z_{i,1}' e_i}{1 + \theta_1 z_{i,1} z_{i,1}} \right\}^2
= \sum_{i=1}^{m} \left\{ \frac{1}{\hat{\theta}_1} - \frac{1}{\theta_1 (1 + \theta_1 z_{i,1}' z_{i,1})} \right\}
- \frac{1}{v \cdot b} \sum_{i=1}^{m} \frac{b_{i,1} - b_{i,1}}{\theta_1 (1 + \theta_1 z_{i,1}' z_{i,1})} + \frac{z_{i,1}' e_i}{1 + \theta_1 z_{i,1} z_{i,1}} \right\}^2
= \sum_{i=1}^{m} \frac{m}{\theta_1} \sum_{i=1}^{m} b_{i,1}^2 \frac{1}{v \cdot b^2 \theta_1^2} + 2 \sum_{i=1}^{m} \frac{b_{i,1} z_{i,1}' e_i}{v^2 \theta_1 (1 + \theta_1 z_{i,1}' z_{i,1})} + R(\sigma_1^2, \bar{v}^2),
\]
where \(\sigma_1^2 = \theta_1 v^2\) and
\[
R(\sigma_1^2, \bar{v}^2) = - \sum_{i=1}^{m} \frac{1}{\theta_1 (1 + \theta_1 z_{i,1}' z_{i,1})} - \sum_{i=1}^{m} \frac{(z_{i,1}' e_i)^2}{v^2 (1 + \theta_1 z_{i,1}' z_{i,1})^2} + \sum_{i=1}^{m} \frac{2 b_{i,1} z_{i,1}' e_i}{v^2 \theta_1 (1 + \theta_1 z_{i,1}' z_{i,1})^2}
+ \sum_{i=1}^{m} \frac{2 b_{i,1}^2}{v^2 \theta_1^2 (1 + \theta_1 z_{i,1}' z_{i,1})} - \sum_{i=1}^{m} \frac{b_{i,1} z_{i,1}' e_i}{v^2 \theta_1 (1 + \theta_1 z_{i,1}' z_{i,1})^2}.
\]
Note that ML estimators \(\hat{\sigma}_1^2 = \hat{\theta}_1 \bar{v}^2\) and \(\hat{\bar{v}}^2\) satisfy
\[
0 = \frac{m}{\hat{\theta}_1} - \sum_{i=1}^{m} \frac{b_{i,1}^2}{v^2 \theta_1^2} + \sum_{i=1}^{m} \frac{2 b_{i,1} z_{i,1}' e_i}{v^2 \theta_1 (1 + \theta_1 z_{i,1}' z_{i,1})} + R(\hat{\sigma}_1^2, \hat{\bar{v}}^2),
\]
which implies that
\[
\hat{\sigma}_1^2 = \hat{\theta}_1 \hat{\bar{v}}^2 = \frac{1}{m} \sum_{i=1}^{m} b_{i,1}^2 + \frac{1}{m} \sum_{i=1}^{m} \frac{2 b_{i,1} z_{i,1}' e_i}{1 + \theta_1 z_{i,1}' z_{i,1}} + \frac{\hat{\bar{v}}^2}{m} R(\hat{\sigma}_1^2, \hat{\bar{v}}^2)
= \frac{1}{m} \sum_{i=1}^{m} b_{i,1}^2 + \frac{1}{m} \sum_{i=1}^{m} \frac{2 b_{i,1} z_{i,1}' e_i}{z_{i,1}' z_{i,1}} - \frac{1}{m} \sum_{i=1}^{m} \frac{2 b_{i,1} z_{i,1}' e_i}{(1 + \theta_1 z_{i,1}' z_{i,1}) z_{i,1}' z_{i,1}} + \frac{\hat{\bar{v}}^2}{m} R(\hat{\sigma}_1^2, \hat{\bar{v}}^2)
= \frac{1}{m} \sum_{i=1}^{m} b_{i,1}^2 + \frac{1}{m} \sum_{i=1}^{m} \frac{2 b_{i,1} z_{i,1}' e_i}{z_{i,1}' z_{i,1}} + R^*(\hat{\sigma}_1^2, \hat{\bar{v}}^2),
\]
where
\[
R^*(\hat{\sigma}_1^2, \hat{\bar{v}}^2) = - \frac{1}{m} \sum_{i=1}^{m} \frac{2 b_{i,1} z_{i,1}' e_i}{(1 + \theta_1 z_{i,1}' z_{i,1}) z_{i,1}' z_{i,1}} + \frac{\hat{\bar{v}}^2}{m} R(\hat{\theta}_1, \hat{\bar{v}}^2)
\]
with \(R(\sigma_1^2, \bar{v}^2)\) defined in (B.11). By (B.12), we have
\[
\frac{\sum_{i=1}^{m} b_{i,1}^2}{\hat{\sigma}_1^2} = O_p(1),
\]
\[
\frac{b_{i,1}^2}{\hat{\sigma}_1^2} = O_p(1),
\]
\[
\frac{(b_{i,1} z_{i,1}' e_i)^2}{1 + \theta_1 z_{i,1}' z_{i,1}} = O_p(1).
\]
By (B.13) and (B.14), we have

\[ R^*(\hat{\sigma}^2, \hat{\nu}^2) = o_p(n^{-1}). \]  

(B.15)

Similarly, by (B.1), we have

\[
\frac{\partial}{\partial \nu^2} (\hat{\theta}_1, \hat{\nu}^2) = \frac{N}{\nu^2} \sum_{i=1}^{m} \left( I_n - \theta_1 z_{i,1} z_{i,1} \right) y_i 
\]

\[
= \frac{N}{\nu^2} \sum_{i=1}^{m} z_{i,1} b_{i,1} + \epsilon_i \epsilon_i \left( I_n - \theta_1 z_{i,1} z_{i,1} \right) (z_{i,1} b_{i,1} + \epsilon_i) 
\]

\[
= \frac{N}{\nu^2} \sum_{i=1}^{m} \left( b_{i,1} z_{i,1} + \epsilon_i \epsilon_i + \frac{2 b_{i,1} z_{i,1} \epsilon_i}{1 + \theta_1 z_{i,1} z_{i,1}} + \epsilon_i \epsilon_i - \frac{\theta_1 (z_{i,1} \epsilon_i)^2}{1 + \theta_1 z_{i,1} z_{i,1}} \right) 
\]

The ML estimators \( \hat{\theta}_1 \) and \( \hat{\nu}^2 \) satisfy

\[
0 = \frac{N}{\nu^2} \sum_{i=1}^{m} \left( b_{i,1} z_{i,1} + \epsilon_i \epsilon_i + \frac{2 b_{i,1} z_{i,1} \epsilon_i}{1 + \theta_1 z_{i,1} z_{i,1}} + \epsilon_i \epsilon_i - \frac{\theta_1 (z_{i,1} \epsilon_i)^2}{1 + \theta_1 z_{i,1} z_{i,1}} \right) 
\]

which implies that

\[
\hat{\nu}^2 = \frac{1}{N} \sum_{i=1}^{m} \epsilon_i \epsilon_i + R^!(\hat{\sigma}^2, \hat{\nu}^2), \]  

(B.16)

with

\[
R^!(\hat{\sigma}^2, \hat{\nu}^2) = \frac{1}{N} \sum_{i=1}^{m} \left( b_{i,1} z_{i,1} + \epsilon_i \epsilon_i + \frac{2 b_{i,1} z_{i,1} \epsilon_i}{1 + \theta_1 z_{i,1} z_{i,1}} + \epsilon_i \epsilon_i - \frac{\theta_1 (z_{i,1} \epsilon_i)^2}{1 + \theta_1 z_{i,1} z_{i,1}} \right). 
\]

This together with (B.14) yields

\[
R^!(\hat{\sigma}^2, \hat{\nu}^2) = O_p(n^{-1}). \]  

(B.17)

We are now ready to compare the asymptotic behaviors between the LS predictors and the empirical BLUPs. Note that for \( i = 1, \ldots, m \), we have

\[
\hat{b}_{i,1} = (\hat{\sigma}_i^2 z_{i,1} z_{i,1})^{-1} z_{i,1} y_i, 
\]

\[
\hat{b}_{i,1}(\hat{\sigma}_i^2, \hat{\nu}^2) = \frac{\hat{\nu}^2}{\hat{\sigma}_i^2} z_{i,1} (\hat{\sigma}_i^2 z_{i,1} z_{i,1} + \hat{\nu}^2 I_n)^{-1} y_i 
\]

Hence

\[
\hat{z}_{i,1}(\hat{b}_{i,1} - b_{i,1}) = \frac{\hat{\sigma}_i^2 z_{i,1} \epsilon_i}{z_{i,1} \epsilon_i}, \]  

(B.18)

and

\[
\hat{b}_{i,1}(\hat{\sigma}_i^2, \hat{\nu}^2) - b_{i,1} = \frac{\hat{\sigma}_i^2 z_{i,1} (\hat{\sigma}_i^2 z_{i,1} z_{i,1} + \hat{\nu}^2 I_n)^{-1} (z_{i,1} b_{i,1} + \epsilon_i) - b_{i,1}}{1 + (\hat{\sigma}_i^2 / \hat{\nu}^2) z_{i,1} \epsilon_i}, \]

\[
= \left( \frac{\hat{\sigma}_i^2 z_{i,1} \epsilon_i}{1 + (\hat{\sigma}_i^2 / \hat{\nu}^2) z_{i,1} \epsilon_i} \right) \epsilon_i + \left( \frac{\hat{\sigma}_i^2 / \hat{\nu}^2}{z_{i,1} \epsilon_i} \right) \epsilon_i 
\]

\[
= \frac{(\hat{\sigma}_i^2 / \hat{\nu}^2) z_{i,1} \epsilon_i}{1 + (\hat{\sigma}_i^2 / \hat{\nu}^2) z_{i,1} \epsilon_i} 
\]
which implies that
\[
\mathbf{e}_{i,1} (b_{i,1} (\sigma_i^2, \sigma^2) - b_{i,1}) = \frac{\mathbf{e}_{i,1} \left( \left( \sigma_i^2 / \sigma^2 \right) \mathbf{e}_{i,1} - b_{i,1} \right)}{1 + (\sigma_i^2 / \sigma^2) \mathbf{e}_{i,1}^2}.
\] (B.19)

Note that by (B.18),
\[
\sum_{i=1}^{m} \left\| \mathbf{e}_{i,1} (b_{i,1} - b_{i,1}) \right\|^2 = \sum_{i=1}^{m} (b_{i,1} - b_{i,1})^2 \mathbf{e}_{i,1}^2 = \sum_{i=1}^{m} \left( \mathbf{e}_{i,1}^2 \mathbf{e}_{i,1}^2 \right)^2
\]
and by (B.19),
\[
\sum_{i=1}^{m} \left\| \mathbf{e}_{i,1} (b_{i,2} (\sigma_i^2, \sigma^2) - b_{i,1}) \right\|^2 = \sum_{i=1}^{m} \left( \left( \sigma_i^2 / \sigma^2 \right) \mathbf{e}_{i,1}^2 - b_{i,1} \right)^2 \mathbf{e}_{i,1}^2 \mathbf{e}_{i,1}^2 = \sum_{i=1}^{m} \left( (\sigma_i^2 / \sigma^2) \mathbf{e}_{i,1}^2 - b_{i,1} \right)^2 \mathbf{e}_{i,1}^2.
\]
which implies that
\[
D_1 (\sigma_1^2, \sigma^2) = \sum_{i=1}^{m} \left( \mathbf{e}_{i,1}^2 \mathbf{e}_{i,1}^2 \right)^2 \left( (\sigma_i^2 / \sigma^2) \mathbf{e}_{i,1}^2 - b_{i,1} \right)^2 \mathbf{e}_{i,1}^2 = \sum_{i=1}^{m} \left( (\sigma_i^2 / \sigma^2) \mathbf{e}_{i,1}^2 - b_{i,1} \right)^2 \mathbf{e}_{i,1}^2.
\]
we have
\[
D_1 (\sigma_1^2, \sigma^2) = \sum_{i=1}^{m} \left( \mathbf{e}_{i,1}^2 \mathbf{e}_{i,1}^2 \right)^2 \left( (\sigma_i^2 / \sigma^2) \mathbf{e}_{i,1}^2 - b_{i,1} \right)^2 \mathbf{e}_{i,1}^2 = \sum_{i=1}^{m} \left( (\sigma_i^2 / \sigma^2) \mathbf{e}_{i,1}^2 - b_{i,1} \right)^2 \mathbf{e}_{i,1}^2.
\]
with
\[
R_1 (\sigma_1^2, \sigma^2) = \sum_{i=1}^{m} \left( \mathbf{e}_{i,1}^2 \mathbf{e}_{i,1}^2 \right)^2 \left( (\sigma_i^2 / \sigma^2) \mathbf{e}_{i,1}^2 - b_{i,1} \right)^2 \mathbf{e}_{i,1}^2 = \sum_{i=1}^{m} \left( (\sigma_i^2 / \sigma^2) \mathbf{e}_{i,1}^2 - b_{i,1} \right)^2 \mathbf{e}_{i,1}^2.
\]
Note that by (B.14),
\[
R_1 (\sigma_1^2, \sigma^2) = O_p (n^{-3/2}).
\]

Further, by (B.12) and (B.16), we have

\[
\frac{2(\varepsilon_{i,1}', \varepsilon_i)^2}{(\varepsilon_{i,1}', \varepsilon_i) \sigma_i} \equiv \frac{2(\varepsilon_{i,1}', \varepsilon_i)^2}{(\zeta_{i,1}', \zeta_i) \sigma_i} \left(\frac{\sum_{k=1}^{m} \epsilon_k^2/e_k}{N}\right) \quad \text{and} \quad \frac{2(\varepsilon_{i,1}', \varepsilon_i)^2}{(\varepsilon_{i,1}', \varepsilon_i) \sigma_i} \equiv \frac{2(\varepsilon_{i,1}', \varepsilon_i)^2}{(\zeta_{i,1}', \zeta_i) \sigma_i} \left(\frac{\sum_{k=1}^{m} \epsilon_k^2/e_k}{N}\right)
\]

Similarly,

\[
\frac{2b_{k,1} \varepsilon_{i,1}' \epsilon_i}{(\sigma_1^2/\hat{\nu}^2)(\varepsilon_{i,1}', \varepsilon_i)\sigma_i} = \frac{2b_{k,1} \varepsilon_{i,1}' \epsilon_i \sum_{k=1}^{m} \epsilon_k^2/e_k}{n(\sum_{k=1}^{m} b_{k,1}^2 + 2 \sum_{k=1}^{m} b_{k,1} \varepsilon_{k,1}' \epsilon_k/(m \varepsilon_{k,1}' \varepsilon_k))} + R_{1,2}(\hat{\sigma}_1^2, \hat{\nu}^2), \quad \text{(B.24)}
\]

with

\[
R_{1,2}(\hat{\sigma}_1^2, \hat{\nu}^2) = \frac{2b_{k,1} \varepsilon_{i,1}' \epsilon_i \sum_{k=1}^{m} \epsilon_k^2/e_k}{n(\sum_{k=1}^{m} b_{k,1}^2 + 2 \sum_{k=1}^{m} b_{k,1} \varepsilon_{k,1}' \epsilon_k/(m \varepsilon_{k,1}' \varepsilon_k))} \times \left\{ R^1(\hat{\sigma}_1^2, \hat{\nu}^2) \left(\frac{\sum_{k=1}^{m} \epsilon_k^2}{m} + \sum_{k=1}^{m} \frac{b_{k,1}^2 \varepsilon_{k,1}' \epsilon_k}{m \varepsilon_{k,1}' \varepsilon_k} \right) - \hat{\nu}^2 \sum_{k=1}^{m} \frac{\epsilon_k^2}{N} \right\}.
\]

Hence by (B.14), (B.15), and (B.17), we have

\[
R_{1,1}(\hat{\sigma}_1^2, \hat{\nu}^2) = O_p(n^{-3/2}), \quad i = 1, 2, 3. \quad \text{(B.26)}
\]
Furthermore, we have

\[ \frac{1}{n} \left\{ \sum_{k=1}^{m} b_{k,1}^2 + 2 \sum_{k=1}^{m} b_{k,1} z_{k,1} \epsilon_k / (m z_{k,1} z_{k,1}) \right\} \left( z_{i,1} - \hat{z}_{i,1} \right) \]

\[ \frac{1}{n(z_{i,1} - \hat{z}_{i,1})} \left\{ \sum_{k=1}^{m} b_{k,1}^2 + 2 \sum_{k=1}^{m} b_{k,1} z_{k,1} \epsilon_k / (m z_{k,1} z_{k,1}) \right\} \left( z_{i,1} - \hat{z}_{i,1} \right) \]

\[ = \frac{1}{n(z_{i,1} - \hat{z}_{i,1})} \left\{ \sum_{k=1}^{m} b_{k,1}^2 + 2 \sum_{k=1}^{m} b_{k,1} z_{k,1} \epsilon_k / (m z_{k,1} z_{k,1}) \right\} \left( z_{i,1} - \hat{z}_{i,1} \right) \]

\[ + R_{i,4}, \quad (B.27) \]

with

\[ R_{i,4} = \frac{1}{n(z_{i,1} - \hat{z}_{i,1})} \left\{ \sum_{k=1}^{m} b_{k,1}^2 + 2 \sum_{k=1}^{m} b_{k,1} z_{k,1} \epsilon_k / (m z_{k,1} z_{k,1}) \right\} \left( z_{i,1} - \hat{z}_{i,1} \right) \]

Note that

\[ R_{i,4} = O_p(n^{-3/2}). \quad (B.28) \]

By (B.21), (B.23), (B.24), (B.25), and (B.27), we have

\[ nD(\hat{\theta}_i^2, \hat{v}_i^2) = A_{n,m} + nR_1(\hat{\theta}_i^2, \hat{v}_i^2) + n \sum_{i=1}^{m} \left\{ R_{i,1}(\hat{\theta}_i^2, \hat{v}_i^2) + R_{i,2}(\hat{\theta}_i^2, \hat{v}_i^2) - R_{i,3}(\hat{\theta}_i^2, \hat{v}_i^2) + R_{i,4} \right\} \]

\[ \equiv A_{n,m} + O_p(n^{-1/2}) \]

with

\[ A_{n,m} = \sum_{i=1}^{m} \left\{ \frac{2(z_{i,1} - \hat{z}_{i,1})^2 \sum_{k=1}^{m} b_{k,1}^2 + 2b_{k,1} z_{k,1} \epsilon_k / (m z_{k,1} z_{k,1}) \sum_{k=1}^{m} b_{k,1}^2}{n(z_{i,1} - \hat{z}_{i,1})} \right\} \]

where the last equality follows from (B.22), (B.26), and (B.28). Note that \( (\sum_{k=1}^{m} b_{k,1} \sigma^2_{i,1})^{-1} \) follows the inverse-chi-squared distribution with \( m \) degrees of freedom. We have

\[ E\left( \frac{1}{\sum_{k=1}^{m} b_{k,1}^2} \right) = \frac{1}{(m-2)\sigma^2_{i,1}}, \quad \text{provided } m > 2, \]

\[ E\left( b_{k,1}^2 / (\sum_{k=1}^{m} b_{k,1}^2)^2 \right) = \frac{1}{m(m-2)\sigma^2_{i,1}}, \quad \text{provided } m > 4. \quad (B.29) \]

By (B.29) and

\[ E\left( \left\{ \sum_{i=1}^{m} \epsilon_i^2 \right\}^2 \right) = (2mn + m^2 n^2)\sigma^2_{i,1}, \]

\[ E(\epsilon_i^2(z_{i,1} - \hat{z}_{i,1})) = 0, \]

\[ E(\epsilon_i^2(z_{i,1} - \hat{z}_{i,1})^2) = n^2 \sigma^2_{i,1} + o(n^2). \]
we have that for \( m > 0 \),

\[
E(A_{n,m}) = E \sum_{i=1}^{m} \left( \frac{2(z_{i,1}' \epsilon_i)^2 \sum_{k=1}^{m} \epsilon_i' \epsilon_k - b_{1,1}' \sum_{k=1}^{m} \epsilon_i' \epsilon_k}{n \sum_{k=1}^{m} b_{k,1}' b_{k,1}'} x_{i,1}' \frac{x_{i,1}'}{x_{i,1}' x_{i,1}} \right) + 2b_{1,1}' \sum_{k=1}^{m} \epsilon_i' \epsilon_k \sum_{k=1}^{m} b_{k,1}' b_{k,1}'
- 4b_{1,1}' \sum_{k=1}^{m} \epsilon_i' \epsilon_k \left( \sum_{k=1}^{m} b_{k,1}' b_{k,1}' \frac{x_{i,1}'}{x_{i,1}'} \right)
\]

\[
= 2m^2v_0^4 - \frac{m^2v_0^4}{(m-2)\sigma_{1,0}^2} + o(1)
\]

\[
= 2m^2v_0^4 - \frac{m^2v_0^4}{(m-2)\sigma_{1,0}^2} - E \left( \sum_{i=1}^{m} 4b_{i,1}' \sum_{k=1}^{m} \epsilon_i' \epsilon_k \left( \sum_{k=1}^{m} b_{k,1}' b_{k,1}' \frac{x_{i,1}'}{x_{i,1}'} \right) \right) + o(1)
\]

\[
= 2m^2v_0^4 - \frac{m^2v_0^4}{(m-2)\sigma_{1,0}^2} - \frac{4m^2v_0^4}{(m-2)\sigma_{1,0}^2} + o(1)
\]

\[
= \frac{m(m-4)v_0^4}{(m-2)\sigma_{1,0}^2} + o(1).
\]

This completes the proofs.

\section*{B.3 Proof of Theorem 5}

In this section, we first prove Theorem 5 to simplify the proofs of Theorems 3 and 4. As with the proof of Theorem 1, we shall focus on the asymptotic properties of \( \hat{v}^2(\alpha, \gamma) \) and \( \{\hat{\theta}_k(\alpha, \gamma) : k \in \gamma\} \), and derive them by solving the likelihood equations directly.

We first prove (3.11) using (B.1). For \((\alpha, \gamma) \in (A \setminus A_0) \times G\), we have

\[
(I_N - M(\alpha, \gamma; \theta))\mu_0 = (I_N - M(\alpha, \gamma; \theta))X(a_0 \setminus \alpha)\beta_0(a_0 \setminus \alpha),
\]

where \( \beta_0(a_0 \setminus \alpha) \) denotes the sub-vector of \( \beta_0 \) corresponding to \( a_0 \setminus \alpha \). Note that by the Cauchy–Schwarz inequality, we have

\[
\left( \sum_{i=1}^{m} \left( z_i \xi_i^0 + \xi_i^0 / \sqrt{2} \right)^2 \right)^2 = O \left( \sum_{i=1}^{m} \frac{m}{n_i^0} \sum_{i=1}^{m} n_i^0 \right).
\]
Hence by (B.31) and Lemma 6, we have
\[
\begin{align*}
(X(a_0 \backslash a)\beta_0(a_0 \backslash a) + Z(\gamma_0)b(\gamma_0) + \epsilon)'H^{-1}(\gamma, \theta)M(\alpha, \gamma; \theta) \\
\times (X(a_0 \backslash a)\beta_0(a_0 \backslash a) + Z(\gamma_0)b(\gamma_0) + \epsilon)
\end{align*}
\]
\[
= \beta_0(a_0 \backslash a)'(X(a_0 \backslash a)'H^{-1}(\gamma, \theta)M(\alpha, \gamma; \theta)X(a_0 \backslash a)\beta_0(a_0 \backslash a)
\]"
by (B.1), we have, for $v^2$ uniformly over $\theta$, the last equality follows from (B.5) and Lemmas 2–4. Hence by (B.1), we have, for $v^2 \in (0, \infty),\vphantom{\sum_{i=1}^{m} x_i}$

$$
\begin{align*}
\psi^4 \left\{ \frac{\partial}{\partial \nu^2} \{-2 \log L(\theta, \nu^2; \alpha, \gamma)\} \right\} &= N \left( v^2 - \frac{\epsilon^T \epsilon}{N} + \frac{1}{N} \sum_{i=1}^{m} \sum_{j \in \alpha \setminus \alpha} \beta_{i,0}^T d_{i,j} n_i^2 \epsilon + \frac{1}{N} \sum_{i=1}^{m} \sum_{k \in \theta_0 \setminus \gamma} b_{i,k}^T c_{i,k} n_i^2 \right) \\
&+ o_p \left( \sum_{i=1}^{m} n_i^2 \right) + \frac{m-1}{N} \sum_{i=1}^{m} n_i^2 + o_p \left( \frac{m}{N} \sum_{i=1}^{m} n_i^2 \right) + O_p(p + mq)
\end{align*}
$$

uniformly over $\theta \in \Theta$. This and Lemma 5 imply that

$$\hat{\psi}^2(\alpha, \gamma) = \frac{\epsilon^T \epsilon}{N} + \frac{1}{N} \sum_{i=1}^{m} \sum_{j \in \alpha \setminus \alpha} \beta_{i,0}^T d_{i,j} n_i^2 \epsilon + \frac{1}{N} \sum_{i=1}^{m} \sum_{k \in \theta_0 \setminus \gamma} b_{i,k}^T c_{i,k} n_i^2$$

$$+ o_p \left( \sum_{i=1}^{m} n_i^2 \right) + \frac{m-1}{N} \sum_{i=1}^{m} n_i^2 + o_p \left( \frac{m}{N} \sum_{i=1}^{m} n_i^2 \right) + O_p(p + mq). \tag{B.32}$$

Thus (3.11) follows by applying the law of large numbers to $\epsilon^T \epsilon/N$. In addition, if $(\xi, \ell) \in (0, 1/2) \times (0, 1/2)$, the asymptotic normality of $\hat{\psi}^2(\alpha, \gamma)$ follows by $p + mq = o(N^{1/2})$ and an application of the central limit theorem to $\epsilon^T \epsilon/N$ in (B.32).

Next, we prove (3.12), for $k \in \gamma \cap \gamma_0$, using (B.2). By (B.31) and Lemma 6 (i)–(iv), we have, for $k \in \gamma \cap \gamma_0$.

$$\begin{align*}
\theta_{h_{i,k}^T H^{-1}(\gamma, \theta)} M(\alpha, \gamma; \theta) (X(\alpha \setminus \alpha) b(\alpha \setminus \alpha) + Z(\gamma_0) b(\gamma_0) + \epsilon) \\
= \theta_{h_{i,k}^T H^{-1}(\gamma, \theta)} M(\alpha, \gamma; \theta) \\
\times (X(\alpha \setminus \alpha) b(\alpha \setminus \alpha) + \sum_{i^* = 1}^{m} \sum_{k^* \in \gamma_0} b_{i^*, k^*} h_{i^*, k^*} + \epsilon) \\
= o_p \left( \frac{n_i^{(\ell-\ell)/2}}{\sum_{i=1}^{m} n_i^{(\xi+\ell)/2}} \right) + o_p (n_i^{(\xi+\ell)/2}) + o_p (n_i^{\ell/2}) \\
= o_p \left( \frac{n_i^{(\ell-\ell)/2}}{\sum_{i=1}^{m} n_i^{(\xi-\ell)/2}} \right) + o_p (n_i^{(\ell-\ell)/2}) + o_p (n_i^{\ell/2})
\end{align*}$$
uniformly over $\theta \in \Theta_\gamma$. This and (B.30) imply that for $k \in \gamma \cap \gamma_0$,

$$
\theta_k h_{i,k}^t H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma, \theta)) y
\leq \theta_k h_{i,k}^t H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma, \theta))
\times (X(\alpha_0 \setminus \alpha) b_0(\alpha_0 \setminus \alpha) + Z(\gamma_0) b(\gamma_0) + \epsilon)
= \theta_k h_{i,k}^t H^{-1}(\gamma, \theta)(X(\alpha_0 \setminus \alpha) b_0(\alpha_0 \setminus \alpha) + Z(\gamma_0) b(\gamma_0) + \epsilon)
\leq \theta_k h_{i,k}^t H^{-1}(\gamma, \theta)(X(\alpha_0 \setminus \alpha) b_0(\alpha_0 \setminus \alpha) + \sum_{k' \in \gamma_0} z_{i,k'} b_{i,k'} + \epsilon)
\leq b_{i,k} + o_p(n_i^{(\xi - \ell)/2}) + o_p(n_i^{(\xi - \ell)/2}) + o_p(1)
$$

uniformly over $\theta \in \Theta_\gamma$, where the last equality follows from Lemma 2 (iii), Lemma 3 (ii)–(iv), and Lemma 4 (i). It follows that for $k \in \gamma \cap \gamma_0$,

$$
\theta_k^2 [h_{i,k}^t H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma, \theta)) y]^2
\leq b_{i,k}^t + o_p(n_i^{(\xi - \ell)/2}) + o_p(n_i^{(\xi - \ell)/2}) + o_p(1)
$$

uniformly over $\theta \in \Theta_\gamma$. Hence by Lemma 3 (ii) and (B.2), we have, for $k \in \gamma \cap \gamma_0$,

$$
\frac{\partial}{\partial \theta_k} \left( -2 \log L(\theta, \gamma, \alpha, \gamma) \right)
= m \left( \theta_k - \frac{1}{m} \sum_{i=1}^m b_{i,k}^t \frac{1}{\gamma^2(\alpha, \gamma)} \right) + o_p \left( \frac{1}{m} \sum_{i=1}^m n_i^{(\xi - \ell)/2} \left( 1 + \frac{\sum_{i=1}^m n_i^{(\xi - \ell)/2}}{\gamma^2(\alpha, \gamma)} \right) + o_p(m) \right)
$$

uniformly over $\theta \in \Theta_\gamma$. This implies that for $k \in \gamma \cap \gamma_0$,

$$
\hat{\theta}_k(\alpha, \gamma) = \frac{1}{m} \sum_{i=1}^m b_{i,k}^t \frac{1}{\gamma^2(\alpha, \gamma)} + o_p \left( \frac{1}{m} \sum_{i=1}^m n_i^{(\xi - \ell)/2} \left( 1 + \frac{\sum_{i=1}^m n_i^{(\xi - \ell)/2}}{\gamma^2(\alpha, \gamma)} \right) + o_p(1) \right)
$$

This proves (3.12), for $k \in \gamma \cap \gamma_0$.

It remains to prove (3.12), for $k \in \gamma \setminus \gamma_0$. Let $\theta^k$ be $\theta$ except that $\{\theta_k : k \in \gamma \cap \gamma_0\}$ are replaced by $\{\theta_k(\alpha, \gamma) : k \in \gamma \cap \gamma_0\}$. By (B.31) and Lemma 6 (i)–(iv), we have, for $k \in \gamma \setminus \gamma_0$,

$$
\theta_k h_{i,k}^t H^{-1}(\gamma, \theta^k) M(\alpha, \gamma, \theta^k)(X(\alpha_0 \setminus \alpha) b_0(\alpha_0 \setminus \alpha) + Z(\gamma_0) b(\gamma_0) + \epsilon)
= \theta_k h_{i,k}^t H^{-1}(\gamma, \theta^k) M(\alpha, \gamma, \theta^k)
\times (X(\alpha_0 \setminus \alpha) b_0(\alpha_0 \setminus \alpha) + \sum_{i', k' \in \gamma_0} b_{i',k'} h_{i',k'} + \epsilon)
\leq \theta_k h_{i,k}^t H^{-1}(\gamma, \theta^k)(X(\alpha_0 \setminus \alpha) b_0(\alpha_0 \setminus \alpha) + Z(\gamma_0) b(\gamma_0) + \epsilon)
\leq \theta_k h_{i,k}^t H^{-1}(\gamma, \theta^k)(X(\alpha_0 \setminus \alpha) b_0(\alpha_0 \setminus \alpha) + \sum_{i', k' \in \gamma_0} z_{i',k'} b_{i',k'} + \epsilon)
\leq b_{i,k} + o_p(n_i^{(\xi - \ell)/2}) + o_p(n_i^{(\xi - \ell)/2}) + o_p(1)
$$

This proves (3.12), for $k \in \gamma \setminus \gamma_0$. It follows that (3.12) holds for all $k$. Thus (3.12) holds.
uniformly over \( \theta \in [0, \infty)^{\theta(\gamma \setminus \gamma_0)} \). This and (B.30) imply that for \( k \in \gamma \setminus \gamma_0 \),

\[
\theta_k \left| H_k^{-1}(\gamma, \theta^\dagger)(I_N - M(\alpha, \gamma; \theta^\dagger))y \right. \\
= \theta_k \left| H_k^{-1}(\gamma, \theta^\dagger)(I_N - M(\alpha, \gamma; \theta^\dagger))(X(a_0 \setminus \alpha)\beta_0(a_0 \setminus \alpha) \\
+ Z(\gamma_0)b(\gamma_0) + \epsilon) \\
= \theta_k \left| H_k^{-1}(\gamma, \theta^\dagger)(X(a_0 \setminus \alpha)\beta_0(a_0 \setminus \alpha) + Z(\gamma_0)b(\gamma_0) + \epsilon) \\
+ o_p(1) \\
= \theta_k \left| Z_k^{-1}(\gamma, \theta^\dagger)(X_i(a_0 \setminus \alpha)\beta_0(a_0 \setminus \alpha) + \sum_{i^* = 1}^{m} \sum_{k^* \in \gamma_0} b_{i^*, k^*}h_{i^*, k^*} + \epsilon_i) \\
+ o_p \left( \left( \sum_{i=1}^{m} n_i \right)^{1/2} \left( \sum_{i=1}^{m} n_i \right)^{1/2} \right) \\
= o_p \left( n_i^{\xi-\ell} \left( \sum_{i=1}^{m} n_i \right) \right) + o_p(1)
\]

uniformly over \( \theta \in [0, \infty)^{\theta(\gamma \setminus \gamma_0)} \), where the last equality follows from Lemma 2 (iii), Lemma 3 (iii)-(iv), and Lemma 4 (i). Therefore,

\[
\theta_k^2 \left| h_k \left| H_k^{-1}(\gamma, \theta^\dagger)(I_N - M(\alpha, \gamma; \theta^\dagger))y \right. \right. \\
= o_p \left( n_i^{\xi-\ell} \left( \sum_{i=1}^{m} n_i \right) \right) + o_p(1)
\]

uniformly over \( \theta \in [0, \infty)^{\theta(\gamma \setminus \gamma_0)} \). Hence by Lemma 3 (ii) and (B.2), we have for \( k \in \gamma \setminus \gamma_0 \),

\[
\theta_k^2 \left( \frac{\partial}{\partial \theta_k} \left\{ -2 \log L(\theta^\dagger, \nu^2; \alpha, \gamma) \right\} \right. \\
= m \theta_k + o_p \left( \sum_{i=1}^{m} n_i^{\xi-\ell} \left( 1 + \sum_{i=1}^{m} n_i \right) \right) + o_p(m)
\]

uniformly over \( \theta \in [0, \infty)^{\theta(\gamma \setminus \gamma_0)} \). This implies that for \( k \in \gamma \setminus \gamma_0 \),

\[
\hat{\theta}_k(\alpha, \gamma) = o_p \left( \frac{1}{m} \sum_{i=1}^{m} n_i^{\xi-\ell} \left( 1 + \sum_{i=1}^{m} n_i \right) \right) + o_p(1).
\]

This completes the proof of (3.12). Thus the proof of Theorem 5 is complete.

**B.4 Proof of Theorem 3**

As with the proof of Theorem 1, we shall focus on the asymptotic properties of \( \hat{\theta}^2(\alpha, \gamma) \) and \( \{\hat{\theta}_k(\alpha, \gamma) : k \in \gamma \} \), and derive them by solving the likelihood equations directly.
We first prove (3.7) using (B.1). Hence by (B.31), Lemma 6 (i)–(iii), Lemma 6 (v)–(vi), and Lemma 6 (viii), we have
\[
(Z(\gamma_0)b(\gamma_0) + e)'H^{-1}(\gamma, \theta)M(\alpha, \gamma; \theta)(Z(\gamma_0)b(\gamma_0) + e)
\]
\[
= \left( \sum_{i=1}^{m} \sum_{k \in \gamma_0} b_{i,k} + e \right)'H^{-1}(\gamma, \theta)M(\alpha, \gamma; \theta) \left( \sum_{i=1}^{m} \sum_{k \in \gamma_0} b_{i,k} + e \right)
\]
\[
= o_p \left( \sum_{i=1}^{m} n_i^{\ell} \right) + o_p \left( \sum_{i=1}^{m} n_i^{\ell/2} \right) + O_p(p)
\]
\[
= o_p \left( \sum_{i=1}^{m} n_i^{\ell} \right) + O_p(p)
\]
uniformly over \( \theta \in \Theta_\gamma \).

This and Lemma 5 imply that for \((z, \ell) \in (0, 1] \times (0, 1],\)
\[
v^\ell \left\{ \frac{\partial}{\partial v} \{-2 \log L(\theta, v^2; \alpha, \gamma)\} \right\}
\]
\[
= N \left( v^2 - \frac{e^\ell}{N} + \frac{1}{N} \sum_{k \in \gamma_0} b_{i,k}^2 c_{i,k} n_i^{\ell} \right) + o_p \left( \sum_{i=1}^{m} n_i^{\ell} \right)
\]
\[
+ O_p \left( \sum_{k \in \gamma_0} \frac{m}{\theta_k} \right) + o_p \left( \sum_{k \in \gamma_0} \frac{m}{\theta_k} \right) + O_p(p + mq)
\]
uniformly over \( \theta \in \Theta_\gamma \).

This and Lemma 5 imply that for \((z, \ell) \in (0, 1] \times (0, 1],\)
\[
\hat{v}^\ell(\alpha, \gamma) = \frac{e^\ell}{N} + \frac{1}{N} \sum_{i=1}^{m} \sum_{k \in \gamma_0} b_{i,k}^2 c_{i,k} n_i^{\ell}
\]
\[
+ o_p \left( \frac{1}{N} \sum_{i=1}^{m} n_i^{\ell} \right) + O_p \left( \frac{p + mq}{N} \right).
\]
Thus (3.7) follows by applying the law of large numbers to \(e^\ell / N\).

In addition, if \( \ell \in (0, 1/2),\)
the asymptotic normality of \(\hat{v}^\ell(\alpha, \gamma)\) follows by \(p + mq = o(N^{1/2})\) and an application of the central limit theorem to \(e^\ell / N\) in (B.33).
Next, we prove (3.8), for $k \in \gamma \cap \gamma_0$, using (B.2). By (B.31) and Lemma 6 (i)–(iii), we have, for $k \in \gamma \cap \gamma_0$,

$$\theta_k h_{i,k}^i H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) (Z(\gamma_0) b(\gamma_0) + \epsilon)$$

$$= \theta_k h_{i,k}^i H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) \left( \sum_{i=1}^m \sum_{k' \in \gamma_0} b_{i,k,k'} h_{i,k,k'} + \epsilon \right)$$

$$= o_p \left( n_i^{(\xi-\ell)/2} \left( \frac{\sum_{i=1}^m n_i^k}{\sum_{i=1}^m n_i^k} \right)^{1/2} \right) + o_p(n_i^{\ell/2})$$

$$= o_p \left( n_i^{(\xi-\ell)/2} \left( \frac{\sum_{i=1}^m n_i^k}{\sum_{i=1}^m n_i^k} \right)^{1/2} \right) + o_p(1)$$

uniformly over $\theta \in \Theta_\gamma$. This and (B.4) imply that for $k \in \gamma \cap \gamma_0$,

$$\theta_k h_{i,k}^i H^{-1}(\gamma, \theta) (I_N - M(\alpha, \gamma; \theta)) y$$

$$= \theta_k h_{i,k}^i H^{-1}(\gamma, \theta) (I_N - M(\alpha, \gamma; \theta)) (Z(\gamma_0) b(\gamma_0) + \epsilon)$$

$$= \theta_k h_{i,k}^i H^{-1}(\gamma, \theta) (Z(\gamma_0) b(\gamma_0) + \epsilon) + o_p \left( n_i^{(\xi-\ell)/2} \left( \frac{\sum_{i=1}^m n_i^k}{\sum_{i=1}^m n_i^k} \right)^{1/2} \right) + o_p(1)$$

$$= \theta_k z_{i,k}^i H^{-1}(\gamma, \theta) \left( \sum_{k' \in \gamma_0} z_{i,k,k'} + \epsilon_i \right)$$

$$+ o_p \left( n_i^{(\xi-\ell)/2} \left( \frac{\sum_{i=1}^m n_i^k}{\sum_{i=1}^m n_i^k} \right)^{1/2} \right) + o_p(1)$$

$$= b_{i,k} + o_p \left( n_i^{(\xi-\ell)/2} \left( \frac{\sum_{i=1}^m n_i^k}{\sum_{i=1}^m n_i^k} \right)^{1/2} \right) + o_p(1)$$

uniformly over $\theta \in \Theta_\gamma$, where the last equality follows from Lemma 3 (ii)–(iv) and Lemma 4 (i). Hence, for $k \in \gamma \cap \gamma_0$,

$$\theta_k^2 \left( h_{i,k}^i H^{-1}(\gamma, \theta) (I_N - M(\alpha, \gamma; \theta)) y \right)^2$$

$$= b_{i,k}^2 + o_p \left( n_i^{\xi-\ell} \left( \frac{\sum_{i=1}^m n_i^k}{\sum_{i=1}^m n_i^k} \right) \right) + o_p(1)$$

uniformly over $\theta \in \Theta_\gamma$. Hence by Lemma 3 (ii) and (B.2), we have, for $k \in \gamma \cap \gamma_0$,

$$\theta_k^2 \left( \frac{\partial}{\partial \theta_k} \left( -2 \log L(\theta, v^2; \alpha, \gamma) \right) \right)$$

$$= m \left( \theta_k - \frac{1}{m} \sum_{i=1}^m \frac{b_{i,k}^2}{\ell^2} \right) + o_p \left( \frac{m}{n_i^k} \left( \frac{\sum_{i=1}^m n_i^k}{\sum_{i=1}^m n_i^k} \right) \right) + o_p(m)$$

uniformly over $\theta \in \Theta_\gamma$. Hence we have, for $k \in \gamma \cap \gamma_0$,

$$\theta_k(\alpha, \gamma) = \frac{1}{m} \sum_{i=1}^m \frac{b_{i,k}^2}{\ell^2(\alpha, \gamma)} + o_p \left( \frac{1}{m} \sum_{i=1}^m n_i^{\xi-\ell} \left( \frac{\sum_{i=1}^m n_i^k}{\sum_{i=1}^m n_i^k} \right) \right) + o_p(1).$$

This completes the proof of (3.8), for $k \in \gamma \cap \gamma_0$. 
It remains to prove (3.8), for \( k \in \gamma \setminus \gamma_0 \). Let \( \theta^1 \) be \( \theta \) except that \( \{ \tilde{\theta}_k : k \in \gamma \cap \gamma_0 \} \) are replaced by \( \{ \tilde{\theta}_k(\alpha, \gamma) : k \in \gamma \cap \gamma_0 \} \). By (B.31) and Lemma 6 (i)–(iii), we have, for \( k \in \gamma \setminus \gamma_0 \),

\[
\theta_k h_{i,k} H^{-1}(\gamma, \theta^1)M(\alpha, \gamma; \theta^1)(I_N - M(\alpha, \gamma; \theta^1))\{Z(\gamma_0)b(\gamma_0) + \epsilon\}
\]

\[
= \theta_k h_{i,k} H^{-1}(\gamma, \theta^1)M(\alpha, \gamma; \theta^1) \left( \sum_{i=1}^{m} \frac{n_i^*}{n_i} \right) + o_p(1)
\]

uniformly over \( \theta(\gamma \setminus \gamma_0) \in [0, \infty)^{q(\gamma \setminus \gamma_0)} \), where this equality follows from Lemma 3 (iii)–(iv) and Lemma 4 (i). Therefore,

\[
\theta_k^2 \{ h_{i,k} H^{-1}(\gamma, \theta^1)(I_N - M(\alpha, \gamma; \theta^1))y \}^2 = o_p \left( \frac{n_{\epsilon}^2 \sum_{i=1}^{m} n_i^*}{\sum_{i=1}^{m} n_i^*} \right) + o_p(1)
\]

uniformly over \( \theta(\gamma \setminus \gamma_0) \in [0, \infty)^{q(\gamma \setminus \gamma_0)} \). Hence by Lemma 3 (ii) and (B.2), we have, for \( k \in \gamma \setminus \gamma_0 \),

\[
\frac{\partial}{\partial \theta_k} \left( -2 \log L(\theta^1; \nu^2; \alpha, \gamma) \right) = m \theta_k + o_p \left( \frac{m n_{\epsilon}^2 \sum_{i=1}^{m} n_i^*}{\sum_{i=1}^{m} n_i^*} \right) + o_p(m)
\]

uniformly over \( \theta(\gamma \setminus \gamma_0) \in [0, \infty)^{q(\gamma \setminus \gamma_0)} \). This implies that, for \( k \in \gamma \setminus \gamma_0 \),

\[
\tilde{\theta}_k(\alpha, \gamma) = o_p \left( \frac{1}{n_0} \sum_{i=1}^{m} n_i^2 \sum_{i=1}^{m} n_i^* \right) + o_p(1).
\]

This completes the proof of (3.8). Hence the proof of Theorem 3 is complete.

### B.5 Proof of Theorem 4

As with the proof of Theorem 1, we shall focus on the asymptotic properties of \( \tilde{\nu}^2(\alpha, \gamma) \) and \( \{ \tilde{\theta}_k(\alpha, \gamma) : k \in \gamma \} \), and derive them by solving the likelihood equations directly.
We first prove (3.9) using (B.1). By Lemma 6 (i), Lemma 6 (iii)–(v), Lemma 6 (vii), and Lemma 6 (x), we have

\[
\begin{aligned}
&\left( X(\alpha_0 \setminus \alpha) \beta_0(\alpha_0 \setminus \alpha) + Z(\gamma_0) b(\gamma_0) + \epsilon \right)' H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) \\
&\times \left( X(\alpha_0 \setminus \alpha) \beta_0(\alpha_0 \setminus \alpha) + Z(\gamma_0) b(\gamma_0) + \epsilon \right) \\
&= \left( X(\alpha_0 \setminus \alpha) \beta_0(\alpha_0 \setminus \alpha) + \sum_{i=1}^{m} \sum_{k \in \gamma_0} b_{i,k} h_{i,k} + \epsilon \right)' H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) \\
&\times \left( X(\alpha_0 \setminus \alpha) \beta_0(\alpha_0 \setminus \alpha) + \sum_{i=1}^{m} \sum_{k \in \gamma_0} b_{i,k} h_{i,k} + \epsilon \right) \\
&= o\left( \sum_{i=1}^{n} n_i^\xi \right) + o_p \left( \sum_{k,k' \in \gamma_0} m \frac{\theta_{k,k'}}{\theta_k^{\gamma}} \right) + o_p \left( \sum_{k \in \gamma_0} m \frac{\theta_k^{\gamma}}{\theta_k} \right) + O_p(p)
\end{aligned}
\]

uniformly over \( \theta \in \Theta_\gamma \). This and (B.30) imply

\[
\begin{aligned}
y' H^{-1}(\gamma, \theta) (I_N - M(\alpha, \gamma; \theta)) y \\
&= \left( X(\alpha_0 \setminus \alpha) \beta_0(\alpha_0 \setminus \alpha) + Z(\gamma_0) b(\gamma_0) + \epsilon \right)' H^{-1}(\gamma, \theta) \\
&\times \left( X(\alpha_0 \setminus \alpha) \beta_0(\alpha_0 \setminus \alpha) + Z(\gamma_0) b(\gamma_0) + \epsilon \right) \\
&+ o\left( \sum_{i=1}^{n} n_i^\xi \right) + o_p \left( \sum_{k,k' \in \gamma_0} m \frac{\theta_{k,k'}}{\theta_k^{\gamma}} \right) + o_p \left( \sum_{k \in \gamma_0} m \frac{\theta_k^{\gamma}}{\theta_k} \right) + O_p(p)
\end{aligned}
\]

uniformly over \( \theta \in \Theta_\gamma \), where the last equality follows from Lemma 3 (ii)–(iv) and Lemma 4. Hence by (B.1), we have, for \( \nu^2 \in (0, \infty) \),

\[
\begin{aligned}
\nu^4 \left\{ \frac{\partial}{\partial \nu^2} \left[ -2 \log L(\theta, \nu^2; \alpha, \gamma) \right] \right\} \\
= \nu^2 \left[ - \epsilon' \epsilon + \frac{1}{N} \sum_{i=1}^{m} \sum_{j \in \alpha_0 \setminus \alpha} \beta_{j0}^2 d_{i,j} n_i^\xi \right] + o_p \left( \sum_{i=1}^{n} n_i^\xi \right) \\
+ o_p \left( \sum_{k,k' \in \gamma_0} m \frac{\theta_{k,k'}}{\theta_k^{\gamma}} \right) + O_p(p + mq)
\end{aligned}
\]
uniformly over $\theta \in \Theta_\gamma$. This and Lemma 5 imply that for $(\xi, \ell) \in (0, 1] \times (0, 1],$
\[
\hat{v}^2(\alpha, \gamma) = \frac{e^\ell}{N^2} + \frac{1}{N} \sum_{i=1}^m \sum_{j \in \alpha_i} \beta_{ij}^2 d_{ij} n_i^\xi
\]
\[+ o_p \left( \frac{1}{N} \sum_{i=1}^m n_i^\xi \right) + O_p \left( \frac{p + mq}{N} \right). \tag{B.34}\]

Thus (3.9) follows by applying the law of large numbers to $e^\ell/N$. In addition, if $\xi \in (0, 1/2)$, the asymptotic normality of $\hat{v}^2(\alpha, \gamma)$ follows by $p + mq = o(N^{1/2})$ and an application of the central limit theorem to $e^\ell/N$ in (B.34).

Next, we prove (3.10), for $k \in \gamma \cap \gamma_0$, using (B.2). By Lemma 6 (i) and Lemma 6 (iii)–(iv), we have, for $k \in \gamma \cap \gamma_0$,
\[
\theta_k h_{i,k}^\ell H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta)(X_0(\alpha \setminus \alpha) - b_0(\alpha_0 \setminus \alpha) + Z(\gamma_0) b(\gamma_0) + \epsilon)
\]
\[= \theta_k h_{i,k}^\ell H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta)
\times \left( X_0(\alpha \setminus \alpha) - b_0(\alpha_0 \setminus \alpha) + \sum_{k^* \in \gamma_0} b_{i,k^*} - b_{i,k^*} + \epsilon \right)
\]
\[= o_p \left( \sum_{k^* \in \gamma_0} n_i^\xi_{k^*} \right) + o_p(1)
\]
uniformly over $\theta \in \Theta_\gamma$. This and (B.30) imply that for $k \in \gamma \cap \gamma_0$,
\[
\theta_k h_{i,k}^\ell H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma; \theta)) y
\]
\[= \theta_k h_{i,k}^\ell H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma; \theta))
\times \left( X_0(\alpha \setminus \alpha) - b_0(\alpha_0 \setminus \alpha) + Z(\gamma_0) b(\gamma_0) + \epsilon \right)
\]
\[= \theta_k h_{i,k}^\ell H^{-1}(\gamma, \theta) X_0(\alpha \setminus \alpha) - b_0(\alpha_0 \setminus \alpha) + Z(\gamma_0) b(\gamma_0) + \epsilon
\]
\[+ o_p \left( \sum_{k^* \in \gamma_0} n_i^\xi_{k^*} \right) + o_p(1)
\]
\[= \theta_k h_{i,k}^\ell H^{-1}(\gamma, \theta) X_0(\alpha \setminus \alpha) - b_0(\alpha_0 \setminus \alpha) + \sum_{k^* \in \gamma_0} z_{i,k^*} b_{i,k^*} + \epsilon_i
\]
\[+ o_p \left( \sum_{k^* \in \gamma_0} n_i^\xi_{k^*} \right) + o_p(1)
\]
\[= \theta_k h_{i,k}^\ell H^{-1}(\gamma, \theta) X_0(\alpha \setminus \alpha) - b_0(\alpha_0 \setminus \alpha) + \sum_{k^* \in \gamma_0} z_{i,k^*} b_{i,k^*} + \epsilon_i
\]
\[+ o_p \left( \sum_{k^* \in \gamma_0} n_i^\xi_{k^*} \right) + o_p(1)
\]
uniformly over $\theta \in \Theta_\gamma$, where the last equality follows from Lemma 2 (iii), Lemma 3 (ii)–(iii), and Lemma 4 (i). Hence, for $k \in \gamma \cap \gamma_0$,
uniformly over $\theta \in \Theta_\gamma$. Hence by Lemma 3 (ii) and (B.2), we have, for $k \in \gamma \cap \gamma_0$,

$$
\theta_k' \left\{ \frac{\partial}{\partial \theta} \left\{ -2 \log L(\theta, \nu^2 ; \alpha, \gamma) \right\} \right\} \\
= m \theta_k - \frac{1}{m} \sum_{i=1}^m \bar{b}_{i,k}^2 \frac{1}{v^2} + o_p \left( \sum_{i=1}^m \sum_{k,k^* \in \gamma_0} \frac{\alpha_i - \epsilon}{\theta_k \theta_{k^*}} \right) + o_p(1)
$$

uniformly over $\theta \in \Theta_\gamma$. This and Lemma 5 imply that for $k \in \gamma \cap \gamma_0$,

$$
\theta_k(\alpha, \gamma) = m \frac{1}{m} \sum_{i=1}^m \bar{b}_{i,k}^2 \frac{1}{v^2} + o_p \left( \sum_{i=1}^m \sum_{k,k^* \in \gamma_0} \frac{\alpha_i - \epsilon}{\theta_k \theta_{k^*}} \right) + o_p(1).
$$

This completes the proof of (3.10) when $k \in \gamma \cap \gamma_0$.

It remains to prove (3.10), for $k \in \gamma \setminus \gamma_0$. Let $\theta^1$ be $\theta$ except that $\{\theta_k : k \in \gamma \cap \gamma_0\}$ are replaced by $\{\hat{\theta}_k(\alpha, \gamma) : k \in \gamma \cap \gamma_0\}$. By Lemma 6 (i) and Lemma 6 (iii)–(iv), we have, for $k \in \gamma \setminus \gamma_0$,

$$
\theta_k h_k H^{-1}(\gamma, \theta^1) M(\alpha, \gamma ; \theta^1) \left( X(\alpha_0 \setminus \alpha) \beta_0(\alpha_0 \setminus \alpha) + Z(\gamma_0) b(\gamma_0) + \epsilon \right)
$$

$$
= \theta_k h_k H^{-1}(\gamma, \theta^1) M(\alpha, \gamma ; \theta^1)
$$

$$
\times \left( X(\alpha_0 \setminus \alpha) \beta_0(\alpha_0 \setminus \alpha) + \sum_{i=1}^m \sum_{k,k^* \in \gamma_0} b_{i,k^*} h_k h_{k^*} + \epsilon \right)
$$

$$
= o_p \left( n_i(\xi - \ell) / 2 - \tau \right) + o_p \left( n_i - \ell / 2 \right)
$$

$$
= o_p \left( n_i(\xi - \ell) / 2 + o_p(1) \right)
$$

uniformly over $\theta(\gamma \setminus \gamma_0) \in [0, \infty)^{\gamma \setminus \gamma_0}$. This and (B.30) imply that for $k \in \gamma \setminus \gamma_0$,

$$
\theta_k h_k h_k^* H^{-1}(\gamma, \theta^1) (I_N - M(\alpha, \gamma ; \theta^1)) y
$$

$$
= \theta_k h_k h_k^* H^{-1}(\gamma, \theta^1) (I_N - M(\alpha, \gamma ; \theta^1))
$$

$$
\times \left( X(\alpha_0 \setminus \alpha) \beta_0(\alpha_0 \setminus \alpha) + Z(\gamma_0) b(\gamma_0) + \epsilon \right)
$$

$$
= \theta_k h_k h_k^* H^{-1}(\gamma, \theta^1) \left( X(\alpha_0 \setminus \alpha) \beta_0(\alpha_0 \setminus \alpha) + Z(\gamma_0) b(\gamma_0) + \epsilon \right)
$$

$$
+ o_p \left( n_i(\xi - \ell) / 2 + o_p(1) \right)
$$

$$
= \theta_k z_k^* h_k^* H^{-1}(\gamma, \theta^1) \left( X_i(\alpha_0 \setminus \alpha) \beta_0(\alpha_0 \setminus \alpha) + \sum_{k \in \gamma_0} z_i,k^* h_{i,k^*} + \epsilon_i \right)
$$

$$
+ o_p \left( n_i(\xi - \ell) / 2 + o_p(1) \right)
$$

$$
= o_p \left( n_i(\xi - \ell) / 2 + o_p(1) \right)
$$

uniformly over $\theta(\gamma \setminus \gamma_0) \in [0, \infty)^{\gamma \setminus \gamma_0}$, where the last equality follows from Lemma 2 (iii), Lemma 3 (iii), and Lemma 4 (i). Therefore,

$$
\theta_k^2 \left\{ \frac{\partial}{\partial \theta} \left\{ -2 \log L(\theta^1, \nu^2 ; \alpha, \gamma) \right\} \right\} = m \theta_k + o_p \left( \sum_{i=1}^m n_i \right) + o_p(m)
$$

uniformly over $\theta(\gamma \setminus \gamma_0) \in (0, \infty)^{\gamma(\gamma \setminus \gamma)}$. Hence by Lemma 3 (ii) and (B.2), we have, for $k \in \gamma \setminus \gamma_0$,

$$
\theta_k^2 \left\{ \frac{\partial}{\partial \theta} \left\{ -2 \log L(\theta^1, \nu^2 ; \alpha, \gamma) \right\} \right\} = m \theta_k + o_p \left( \sum_{i=1}^m n_i \right) + o_p(m)
$$
uniformly over $\theta(\gamma \setminus \gamma_0) \in [0, \infty)^q(\gamma \setminus \gamma_0)$. This and Lemma 5 imply that for $k \in \gamma \setminus \gamma_0$,

$$
\hat{\alpha}_k(\alpha, \gamma) = o_p\left(\frac{1}{m} \sum_{i=1}^{m} n_i^{\xi-\ell} \right) + o_p(1).
$$

This completes the proof of (3.10), for $k \in \gamma \setminus \gamma_0$. Hence the proof of Theorem 4 is complete.

C Proofs of Auxiliary Lemmas

C.1 Proof of Lemma 2

Let $z_{i,s} := s = 1, \ldots, q(\gamma)$ be the $s$-th column of $Z_i(\gamma)$ and $H_i,\ell(\gamma, \theta)$ defined in (A.4). For Lemma 2 (i)–(ii) to hold, it suffices to prove that for $k \notin \gamma$ and $j, j^* = 1, \ldots, p$,

$$
x'_{i,j} H_i^{-1}(\gamma, \theta) z_{i,j} = d_{i,j} n_i^\xi + o(n_i^\xi) + o(t n_i^{\xi-2\tau}), \quad (C.1)
$$

$$
x'_{i,j} H_i^{-1}(\gamma, \theta) z_{i,j^*} = o(n_i^{\xi-\tau}) + o(t n_i^{\xi-2\tau}), \quad (C.2)
$$

$$
x'_{i,j} H_i^{t}(\gamma, \theta) z_{i,k} = o(n_i^{(\xi + \ell)/2-\tau}) + o(t n_i^{(\xi + \ell)/2-2\tau}) \quad (C.3)
$$

uniformly over $\theta \in [0, \infty)^q(\gamma)$. We prove (C.1)–(C.3) by induction. For $j = 1, \ldots, p$ and $t = 1$, by (A.2) and (A1)–(A3), we have

$$
x'_{i,j} H_i^{-1}(\gamma, \theta) x_{i,j} = x'_{i,j} x_{i,j} - \frac{\theta(1)x'_{i,j} z_{i,1}(1)x'_{i,1}(1)x_{i,j}}{1 + \theta(1)x'_{i,1}(1)x_{i,1}(1)} = d_{i,j} n_i^\xi + o(n_i^\xi) + o(t n_i^{\xi-2\tau})
$$

uniformly over $\theta \in [0, \infty)^q(\gamma)$. For $j, j^* = 1, \ldots, p$, $j \neq j^*$ and $t = 1$, by (A.2) and (A1)–(A3), we have

$$
x'_{i,j} H_i^{-1}(\gamma, \theta) x_{i,j^*} = x'_{i,j} x_{i,j^*} - \frac{\theta(1)x'_{i,j} z_{i,1}(1)x'_{i,1}(1)x_{i,j^*}}{1 + \theta(1)x'_{i,1}(1)x_{i,1}(1)} = o(n_i^{\xi-\tau}) + o(n_i^{\xi-2\tau})
$$

uniformly over $\theta \in [0, \infty)^q(\gamma)$. For $j = 1, \ldots, p$, $k \notin \gamma$ and $t = 1$, by (A.2) and (A1)–(A3), we have

$$
x'_{i,j} H_i^{-1}(\gamma, \theta) z_{i,k} = x'_{i,j} z_{i,k} - \frac{\theta(1)x'_{i,j} z_{i,1}(1)x'_{i,1}(1)z_{i,k}}{1 + \theta(1)x'_{i,1}(1)x_{i,1}(1)} = o(n_i^{\xi-\tau}) + o(n_i^{(\xi + \ell)/2-\tau})
$$

uniformly over $\theta \in [0, \infty)^q(\gamma)$. Now suppose that (C.1)–(C.3) hold for $t = r$. Then for

$$
x'_{i,j} H_i^{t}(\gamma, \theta) z_{i,j} = x'_{i,j} H_i^{-1}(\gamma, \theta) z_{i,j} - \frac{\theta(r+1)x'_{i,j} H_i^{-1}(\gamma, \theta) z_{i,(r+1)} x'_{i,(r+1)} H_i^{t}(\gamma, \theta) z_{i,(r+1)}}{1 + \theta(r+1)x'_{i,(r+1)} H_i^{t}(\gamma, \theta) z_{i,(r+1)}} = d_{i,j} n_i^\xi + o(n_i^\xi) + o(t n_i^{\xi-2\tau})
$$

uniformly over $\theta \in [0, \infty)^q(\gamma)$. Hence the proof of Lemma 5 is complete.
uniformly over $\theta \in [0, \infty)^q(\gamma)$. For $j, j^* = 1, \ldots, p$, $j \neq j^*$, and $t = r + 1$, by (A.2) and (C.1)–(C.3) with $t = r$, and Lemma 3 (i), we have

$$x'_{i,j} H^{-1}_{r,t+1}(\gamma, \theta) x_{i,j^*} = \frac{\theta_r x'_{i,j} H^{-1}_{r,t}(\gamma, \theta) z_{i,(r+1)} x'_{i,j^*} H^{-1}_{r,t+1}(\gamma, \theta) z_{i,(r+1)}}{1 + \theta_r x'_{i,j} H^{-1}_{r,t}(\gamma, \theta) z_{i,(r+1)} x'_{i,j^*} H^{-1}_{r,t+1}(\gamma, \theta) z_{i,(r+1)}},$$

$$= o(n^{\xi-\tau}) + o((r+1)n^{\xi-2\tau})$$

uniformly over $\theta \in [0, \infty)^q(\gamma)$. For $j, j^* = 1, \ldots, p$, $k \not\in \gamma$, and $t = r + 1$, by (A.2) and (C.1)–(C.3) with $t = r$, and Lemma 3 (i), we have

$$x'_{i,j} H^{-1}_{r,t+1}(\gamma, \theta) z_{i,k} = \frac{\theta_r x'_{i,j} H^{-1}_{r,t}(\gamma, \theta) z_{i,(r+1)} x'_{i,j^*} H^{-1}_{r,t+1}(\gamma, \theta) z_{i,(r+1)}}{1 + \theta_r x'_{i,j} H^{-1}_{r,t}(\gamma, \theta) z_{i,(r+1)} x'_{i,j^*} H^{-1}_{r,t+1}(\gamma, \theta) z_{i,(r+1)}},$$

$$= o(n^{(\xi+\ell)/2-\tau}) + o((r+1)n^{(\xi+\ell)/2-2\tau})$$

uniformly over $\theta \in [0, \infty)^q(\gamma)$. This completes the proofs of (C.1)–(C.3). Hence the proofs of Lemma 2 (i)–(ii) are complete.

We finally prove Lemma 2 (iii). Without loss of generality, we assume that $q(\gamma) = q$, $t = q$, and $k = q$. Then by (A.2),

$$\theta(q) x'_{i,j} H^{-1}_{i,q}(\gamma, \theta) z_{i,(q)} = \theta(q) x'_{i,j} H^{-1}_{r,q-1}(\gamma, \theta) z_{i,(q)}$$

$$- \frac{\theta(q) x'_{i,j} H^{-1}_{r,q}(\gamma, \theta) z_{i,(q)} x'_{i,j^*} H^{-1}_{r,q-1}(\gamma, \theta) z_{i,(q)}}{1 + \theta(q) x'_{i,j} H^{-1}_{r,q}(\gamma, \theta) z_{i,(q)} x'_{i,j^*} H^{-1}_{r,q-1}(\gamma, \theta) z_{i,(q)}},$$

$$= \theta(q) x'_{i,j} H^{-1}_{i,q}(\gamma, \theta) z_{i,(q)}$$

where we note that $\theta(q)$ can be arbitrarily small and the dominant term of the denominator of the last equation can be equal to (i) $\theta(q) x'_{i,j} H^{-1}_{i,q-1}(\gamma, \theta) z_{i,(q)}$ or (ii) 1. For the case of (i), $\theta(q)n^{\xi}_q \to \infty$ by Lemma 3 (i); hence, using Lemma 2 (ii) and Lemma 3 (i), we have

$$\theta(q) x'_{i,j} H^{-1}_{i,q}(\gamma, \theta) z_{i,(q)} = o(n^{(\xi-\ell)/2-\tau}),$$

and thus

$$x'_{i,j} H^{-1}_{i,q}(\gamma, \theta) z_{i,(q)} = o(n^{(\xi+\ell)/2-\tau}).$$

For the case of (ii), $\theta(q) = O(n^{\xi}_q)$ by Lemma 3 (ii); hence, using Lemma 3 (i), we have

$$\theta(q) x'_{i,j} H^{-1}_{i,q}(\gamma, \theta) z_{i,(q)} = o(n^{(\xi+\ell)/2-\tau}),$$

which also gives the following two results:

$$x'_{i,j} H^{-1}_{i,q}(\gamma, \theta) z_{i,(q)} = o(n^{(\xi+\ell)/2-\tau}),$$

$$\theta(q) x'_{i,j} H^{-1}_{i,q}(\gamma, \theta) z_{i,(q)} = o(n^{(\xi-\ell)/2-\tau}).$$

In conclusion, we have

$$\theta(q) x'_{i,j} H^{-1}_{i,q}(\gamma, \theta) z_{i,(q)} = o(n^{(\xi-\ell)/2-\tau}),$$

$$x'_{i,j} H^{-1}_{i,q}(\gamma, \theta) z_{i,(q)} = o(n^{(\xi+\ell)/2-\tau})$$

uniformly over $\theta \in [0, \infty)^q(\gamma)$. This completes the proof.
C.2 Proof of Lemma 3

Let \( z_{i,(s)} : s = 1, \ldots, q(\gamma) \) be the \( s \)-th column of \( Z_i(\gamma) \) and \( H_{i,t}(\gamma, \theta) \) defined in (A.4). We first prove Lemma 3 (i). By (A.4), it suffices to prove that for \( k \notin \gamma \),

\[
 z_{i,k}^{-1}H_{i,t}^{-1}(\gamma, \theta)z_{i,k} = c_{i,k}n_k^t + o(n_k^t) + o(tn_k^{t-2\gamma}) \tag{C.5}
\]

and for \( k, k^* \notin \gamma \) and \( k \neq k^* \),

\[
 z_{i,k}^{-1}H_{i,t}^{-1}(\gamma, \theta)z_{i,k^*} = o(n_k^{t-\gamma}) + o(tn_k^{t-2\gamma}) \tag{C.6}
\]

uniformly over \( \theta \in [0, \infty)^{q(\gamma)} \) by induction. For \( t = 1 \) and \( k \notin \gamma \), by (A.2) and (A2), we have

\[
 z_{i,k}^{-1}H_{i,1}^{-1}(\gamma, \theta)z_{i,k} = z_{i,k}^{-1} \left( I_{n_k} - \frac{\theta(1)z_{i,(1)}^T z_{i,(1)}}{1 + \theta(1)z_{i,(1)}^T z_{i,(1)}} \right) z_{i,k} = z_{i,k}^{-1} - \frac{\theta(1)z_{i,k}^T z_{i,k} z_{i,(1)}^T z_{i,(1)}}{1 + \theta(1)z_{i,(1)}^T z_{i,(1)}} c_{i,k}n_k^1 + o(n_k^1) + o(n_k^{1-2\gamma})
\]

uniformly over \( \theta \in [0, \infty)^{q(\gamma)} \). For \( k, k^* \notin \gamma \) and \( k \neq k^* \), by (A.2) and (A2), we have

\[
 z_{i,k}^{-1}H_{i,1}^{-1}(\gamma, \theta)z_{i,k^*} = z_{i,k}^{-1} \left( I_{n_k} - \frac{\theta(1)z_{i,k^*}^T z_{i,k^*} z_{i,(1)}^T z_{i,(1)}}{1 + \theta(1)z_{i,(1)}^T z_{i,(1)}} \right) z_{i,k^*} = c_{i,k}n_k^1 + o(n_k^1) + o(n_k^{1-2\gamma})
\]

uniformly over \( \theta \in [0, \infty)^{q(\gamma)} \). Now suppose that (C.5) and (C.6) hold for \( t = r + 1 \), by (A.2), and (C.5) and (C.6) with \( t = r \), we have

\[
 z_{i,k}^{-1}H_{i,r+1}^{-1}(\gamma, \theta)z_{i,k} = z_{i,k}^{-1} \left( I_{n_k} - \frac{\theta(r+1)z_{i,(r+1)}^T z_{i,(r+1)}}{1 + \theta(r+1)z_{i,(r+1)}^T z_{i,(r+1)}} \right) z_{i,k} - \frac{\theta(r+1)z_{i,k}^T z_{i,k} z_{i,(r+1)}^T z_{i,(r+1)}}{1 + \theta(r+1)z_{i,(r+1)}^T z_{i,(r+1)}} c_{i,k}n_k^t + o(n_k^t) + o(tn_k^{t-2\gamma})
\]

uniformly over \( \theta \in [0, \infty)^{q(\gamma)} \). For \( k, k^* \notin \gamma \) and \( t = r + 1 \), by (A.2), and (C.5) and (C.6) with \( t = r \), we have

\[
 z_{i,k}^{-1}H_{i,r+1}^{-1}(\gamma, \theta)z_{i,k^*} = z_{i,k}^{-1} \left( I_{n_k} - \frac{\theta(r+1)z_{i,k^*}^T z_{i,k^*} z_{i,(r+1)}^T z_{i,(r+1)}}{1 + \theta(r+1)z_{i,(r+1)}^T z_{i,(r+1)}} \right) z_{i,k^*} = o(n_k^{t-\gamma}) + o(tn_k^{t-2\gamma})
\]

uniformly over \( \theta \in [0, \infty)^{q(\gamma)} \). This completes the proof of (C.5) and (C.6). Hence Lemma 3 (i) follows from (C.5), (C.6) with \( t = q(\gamma) \) and \( q = o(n_k^{\min}) \). This completes the proof of Lemma 3 (i).
We now prove Lemma 3 (ii). Without loss of generality, we assume that \( q(\gamma) = q \) and \( k = q \). Then by Lemma 3 (i) and (A.2),

\[
\theta^2(q)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q)} = \theta^2(q) \left\{ z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q)} - \frac{\theta(q)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q)}}{1 + \theta(q)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q)}} \right\}
\]

\[
= \frac{\theta^2(q)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q)}}{1 + \theta(q)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q)}} - \frac{\theta(q)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q)}}{1 + \theta(q)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q)}} = O(\theta^2(q)n_i^\ell)
\]

uniformly over \( \theta \in [0, \infty)^q \). Again, by Lemma 3 (i), we have

\[
\theta^2(q)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q)} = \frac{\theta^2(q)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q)}}{1 + \theta(q)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q)}}
\]

\[
= \frac{\theta(q)}{1 + \theta(q)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q)}} = \theta(q) + O(n_i^\ell)
\]

uniformly over \( \theta \in [0, \infty)^q \). This completes the proof of Lemma 3 (ii).

We now prove Lemma 3 (iii). Without loss of generality, we assume that \( q(\gamma) = q \), \( k = q \), and \( k^\tau = (q - 1) \). Then by (A.2),

\[
\begin{align*}
\theta(q)\theta(q-1)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q-1)} &= \theta(q)\theta(q-1)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q-1)} \\
&= \theta(q)\theta(q-1)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q-1)} \\
&= \frac{\theta(q)\theta(q-1)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q-1)}}{1 + \theta(q)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q-1)}}
\end{align*}
\]

where we note that \( \theta(q) \) and \( \theta(q-1) \) can be arbitrarily small and the dominant term of the denominator of the last equation can be equal to

(i) \( \frac{\theta(q)\theta(q-1)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q-1)}}{1 + \theta(q-1)z'_{i,(q-1)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q-1)}} \)

(ii) \( \frac{\theta(q)\theta(q-1)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q-1)}}{1 + \theta(q-1)z'_{i,(q-1)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q-1)}} \)

(iii) \( \frac{\theta(q)\theta(q-1)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q-1)}}{1 + \theta(q-1)z'_{i,(q-1)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q-1)}} \)

For the case of (i), \( \theta(q)n_i^\ell \rightarrow \infty \) and \( \theta(q-1)n_i^\ell \rightarrow \infty \) by Lemma 3 (i); hence, using Lemma 3 (i), we have

\[
\theta(q)\theta(q-1)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q-1)} = \text{op}(n_i^{\ell - \tau}),
\]

which also gives the following two results:

\[
\theta(q)z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q-1)} = \text{op}(n_i^{\ell - \tau}),
\]

\[
z'_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)z_{i,(q-1)} = \text{op}(n_i^{\ell - \tau}).
\]
For the case of (ii), \( \theta(q) n_i \rightarrow \infty \) and \( \theta(q) = O(n_i^{-\xi}) \) (or vice versa) by Lemma 3 (i); hence, using Lemma 3 (i), we have

\[
\theta(q) \theta(q-1) z_{i,q}^{-1}(\gamma, \theta) z_{i,q-1}(q) = o_p(\theta(q-1) n_i^{-\xi}),
\]

which gives the following three results:

\[
\theta(q) \theta(q-1) z_{i,q}^{-1}(\gamma, \theta) z_{i,q-1}(q) = o_p(n_i^{-\xi}),
\]

\[
\theta(q) \theta(q-1) z_{i,q}^{-1}(\gamma, \theta) z_{i,q-1}(q) = o_p(n_i^{-\xi}),
\]

\[
\theta(q) \theta(q-1) z_{i,q}^{-1}(\gamma, \theta) z_{i,q-1}(q) = o_p(n_i^{-\xi}).
\]

For the case of (iii), \( \theta(q) = O(n_i^{-\xi}) \) and \( \theta(q) = O(n_i^{-\xi}) \) by Lemma 3 (i); hence, using Lemma 3 (i), we have

\[
\theta(q) \theta(q-1) z_{i,q}^{-1}(\gamma, \theta) z_{i,q-1}(q) = o_p(\theta(q) \theta(q-1) n_i^{-\xi}),
\]

which also gives the following three results:

\[
\theta(q) \theta(q-1) z_{i,q}^{-1}(\gamma, \theta) z_{i,q-1}(q) = o_p(n_i^{-\xi}),
\]

\[
\theta(q) \theta(q-1) z_{i,q}^{-1}(\gamma, \theta) z_{i,q-1}(q) = o_p(n_i^{-\xi}),
\]

\[
\theta(q) \theta(q-1) z_{i,q}^{-1}(\gamma, \theta) z_{i,q-1}(q) = o_p(n_i^{-\xi}).
\]

In conclusion, we have

\[
\theta(q) \theta(q-1) z_{i,q}^{-1}(\gamma, \theta) z_{i,q-1}(q) = o_p(n_i^{-\xi}),
\]

\[
\theta(q) \theta(q-1) z_{i,q}^{-1}(\gamma, \theta) z_{i,q-1}(q) = o_p(n_i^{-\xi}),
\]

\[
\theta(q) \theta(q-1) z_{i,q}^{-1}(\gamma, \theta) z_{i,q-1}(q) = o_p(n_i^{-\xi}).
\]

uniformly over \( \theta \in [0, \infty]^q \). This completes the proof of Lemma 3 (iii).

We finally prove Lemma 3 (iv). Without loss of generality, it suffices to prove Lemma 3 (iv) by replacing \( H_i(\gamma, \theta) \) with \( H_i, q-1(\gamma, \theta) \) with \( q(\gamma) = q, k = (q - 1) \), and \( k^* = (q) \). Then by (A.2),

\[
\theta(q-1) z_{i,q}^{-1}(\gamma, \theta) z_{i,q-1}(q) = \theta(q-1) \left\{ \left( z_{i,q}^{-1}(\gamma, \theta) z_{i,q-1}(q) - \theta(q-1) z_{i,q}^{-1}(\gamma, \theta) z_{i,q-1}(q) \right) \cdot \left( H_i, q-1(\gamma, \theta) z_{i,q-1}(q) + \theta(q-1) z_{i,q}^{-1}(\gamma, \theta) z_{i,q-1}(q) \right) \right\}
\]

\[
= \theta(q-1) z_{i,q}^{-1}(\gamma, \theta) z_{i,q-1}(q) + \theta(q-1) z_{i,q}^{-1}(\gamma, \theta) z_{i,q-1}(q) \cdot H_i, q-1(\gamma, \theta) z_{i,q-1}(q).
\]

Hence, Lemma 3 (iv) follows from Lemma 3 (i) and arguments similar to the proof of (C.4). This completes the proof.

C.3 Proof of Lemma 4

Note that for \( k = 1, \ldots, q \) and \( j = 1, \ldots, p \),

\[
\epsilon_j^* z_{i,k} = O_p(n_i^{1/2}),
\]

\[
\epsilon_j^* z_{i,j} = O_p(n_i^{1/2}).
\]
Lemma 4 (ii)-(iii) then follow arguments similarly from the induction and the proofs of Lemma 2 (i) are hence omitted.

We next prove Lemma 4 (iv). Let $\mathbf{z}_{i,(s)}$ be the s-th column of $\mathbf{Z}_i(\gamma)$ and $\mathbf{H}_{i,t}(\gamma, \theta)$ be defined in (A.4). Without loss of generality, we assume $q(\gamma) = q$. Hence by (A.6), Lemma 3 (i), and Lemma 4 (ii), we have

$$
epsilon_i'\mathbf{H}^{-1}_{i,q}(\gamma, \theta)\mathbf{e}_i = \epsilon_i'\mathbf{e}_i - \sum_{k=1}^{q} \frac{\theta(k)\epsilon_i'\mathbf{H}^{-1}_{i,k-1}(\gamma, \theta)\mathbf{z}_{i,(k)}'\mathbf{e}_i}{1 + \theta(k)\epsilon_i'\mathbf{z}_{i,(k)}\mathbf{H}^{-1}_{i,k-1}(\gamma, \theta)\mathbf{e}_i} = \epsilon_i'\mathbf{e}_i + O_p(q)$$

uniformly over $\theta \in [0, \infty)^q$. This completes the proof of Lemma 4 (iv).

It remains to prove Lemma 4 (i). Again, without loss of generality, it suffices to prove Lemma 4 (i) for $q(\gamma) = q$ and $k = (q)$. Then by (A.2),

$$\theta(q)\epsilon_i'\mathbf{H}^{-1}_{i,q}(\gamma, \theta)\mathbf{z}_{i,(q)} = \frac{\theta(q)\epsilon_i'\mathbf{H}^{-1}_{i,q-1}(\gamma, \theta)\mathbf{z}_{i,(q)}}{1 + \theta(q)\epsilon_i'\mathbf{z}_{i,(q)}\mathbf{H}^{-1}_{i,q-1}(\gamma, \theta)\mathbf{e}_i}.$$

Hence, Lemma 4 (i) follows from Lemma 3 (i), Lemma 4 (ii), and arguments similar to the proof of (C.4). This completes the proof.

C.4 Proof of Lemma 5

We show the lemma for $(\alpha, \gamma) \in \mathcal{A}_0 \times \mathcal{G}_0$, where the proofs with respect to the remaining models are similar and hence omitted.

Let $\mathbf{z}_{i,(s)}$ be the s-th column of $\mathbf{Z}_i(\gamma)$ and $\mathbf{H}_{i,t}(\gamma, \theta)$ be defined in (A.4). Without loss of generality, we assume that $q(\gamma) = q$ and $\mathbf{Z}_i(\gamma_0)b_i(\gamma_0) = \sum_{s=q-q_0+1}^{q} \mathbf{z}_{i,(s)}b_{i,(s)}$. It then suffices to prove that for $(\alpha, \gamma) \in \mathcal{A}_0 \times \mathcal{G}_0$ and $v^2 > 0$

$$-2 \log L(\theta, v^2; \alpha, \gamma) - \{ -2 \log L(\theta^0, v^2; \alpha, \gamma) \} \xrightarrow{p} \infty, \quad (C.8)$$

as both $N \to \infty$ and $\theta(k) \to 0$ for some $k \in \{ q-q_0+1, \ldots, q \}$, where $\theta^0 \equiv (0, \ldots, 0, \theta_{(q-q_0+1)}, 0, \ldots, \theta_{(q)}, 0)'$, and $\theta_{(s),0}$ being the true value of $\theta_{(s)}$; $s = q-q_0+1, \ldots, q$. Note that by (A.3) and (A.1), we have

$$\det(\mathbf{H}_i(\gamma, \theta)) = \det \left( \mathbf{I}_{q_i} + \sum_{s=1}^{q} \theta_{(s)}\mathbf{z}_{i,(s)}\mathbf{z}_{i,(s)}' \right)$$

$$= \det(\mathbf{H}_{i,q-1}(\gamma, \theta) + \theta(q)\mathbf{z}_{i,(q)}\mathbf{z}_{i,(q)}')$$

$$= \det(\mathbf{H}_{i,q-1}(\gamma, \theta))(1 + \theta(q)\mathbf{z}_{i,(q)}(\gamma, \theta)\mathbf{z}_{i,(q)}).$$

Continuously expanding the above equation by (A.1) yields

$$\log \det(\mathbf{H}_i(\gamma, \theta)) = \log \left\{ \prod_{s=1}^{q} (1 + \theta_{(s)}\mathbf{z}_{i,(s)}'\mathbf{H}^{-1}_{i,s-1}(\gamma, \theta)\mathbf{e}_i) \right\}$$

$$= \sum_{s=1}^{q} \log(1 + \theta_{(s)}\mathbf{z}_{i,(s)}'\mathbf{H}^{-1}_{i,s-1}(\gamma, \theta)\mathbf{e}_i).$$
\[ -2 \log L(\theta, v^2; \alpha, \gamma) \]
\[ = N \log(2\pi) + N \log(v^2) + \log \det(H(\gamma, \theta)) + \frac{y' H^{-1}(\gamma, \theta) A(\alpha, \gamma; \theta)y}{v^2} \]
\[ = N \log(2\pi) + N \log(v^2) + \sum_{i=1}^{m} \sum_{s=q-90+1}^{q} \log(1 + \theta(s) z_{i,s}(H_{i,s-1}^{-1}(\gamma, \theta)z_{i,s}(s))) \]
\[ + \left( Z(\tau_0) b(\tau_0) + \epsilon \right)' \left( H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma; \theta)) \right) Z(\tau_0) b(\tau_0) + \epsilon. \]

Hence, we have, for \((\alpha, \gamma) \in A_0 \times G_0\),
\[ -2 \log L(\theta, v^2; \alpha, \gamma) = \sum_{i=1}^{m} \sum_{s=q-90+1}^{q} \log \left( 1 + \theta(s) z_{i,s}(H_{i,s-1}^{-1}(\gamma, \theta)z_{i,s}(s)) \right) \]
\[ + \frac{1}{v^2} \left( Z(\tau_0) b(\tau_0) + \epsilon \right)' \left( H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma; \theta)) \right) Z(\tau_0) b(\tau_0) + \epsilon, \]

where
\[ \left( Z(\tau_0) b(\tau_0) + \epsilon \right)' \left( H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma; \theta)) \right) Z(\tau_0) b(\tau_0) + \epsilon \]
\[ = b(\tau_0)' Z(\tau_0)' \left( H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma; \theta)) \right) Z(\tau_0) b(\tau_0) \]
\[ = b(\tau_0)' Z(\tau_0)' \left( H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma; \theta)) \right) Z(\tau_0) b(\tau_0) \]
\[ + 2b(\tau_0)' Z(\tau_0)' \left( H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma; \theta)) \right) \epsilon \]
\[ + \epsilon' \left( H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma; \theta)) \right) \epsilon. \]

Hence, for (C.8) to hold, it suffices to prove
\[ \epsilon' \left( H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma; \theta)) \right) \]
\[ - H^{-1}(\gamma, \theta)' \left( H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma; \theta)) \right) \epsilon = O_p(m) \]  
(C.9)

uniformly over \(\theta \in [0, \infty)^q\) and
\[ \sum_{i=1}^{m} \sum_{s=q-90+1}^{q} \log \left( 1 + \theta(s) z_{i,s}(H_{i,s-1}^{-1}(\gamma, \theta)z_{i,s}(s)) \right) \]
\[ + \frac{1}{v^2} \left( b(\tau_0)' Z(\tau_0)' \left( H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma; \theta)) \right) \right) \]
\[ - H^{-1}(\gamma, \theta)' \left( H^{-1}(\gamma, \theta)(I_N - M(\alpha, \gamma; \theta)) \right) Z(\tau_0) b(\tau_0) \]
\[ + O_p(m) \rightarrow_{p} \infty, \]  
(C.10)

as both \(N \to \infty\) and \(\theta(k) \to 0\) for some \(k \in \{q-90+1, \ldots, q\}\). Before proving (C.9) and (C.10), we prove the following equations, for \(H_{i,k}^{\text{th}}\) being defined in (2.5) and \(k = \ldots, q\).
uniformly over $\theta \in [0, \infty)^q$. It suffices to prove (C.11)–(C.14) for $k = q$. For (C.11) with $k = q$, we have

$$e' H^{-1}(\gamma, \theta)_{1:k} h_{1:i}(k) h'_{1:i}(k) H^{-1}(\gamma, \theta)_{1:k} M(\alpha, \gamma; \theta) = O_p(1) \quad (C.11)$$

and

$$e' H^{-1}(\gamma, \theta_{1:k}) X(\alpha) (X(\alpha)' H^{-1}(\gamma, \theta) X(\alpha))^{-1} \times X(\alpha)' H^{-1}(\gamma, \theta_{1:k}) h_{1:i}(k) H^{-1}(\gamma, \theta_{1:k}) X(\alpha) = O_p(1) \quad (C.13)$$

uniformly over $\theta \in [0, \infty)^q$, where the second last equality follows from Lemma 3 (i) and Lemma 4 (i)–(ii). For (C.12) with $k = q$, we have

$$e' H^{-1}(\gamma, \theta)_{1:k} h_{1:i}(q) h'_{1:i}(q) H^{-1}(\gamma, \theta)_{1:k} M(\alpha, \gamma; \theta) = O_p(1) \quad (C.12)$$

and

$$e' H^{-1}(\gamma, \theta_{1:k}) X(\alpha) (X(\alpha)' H^{-1}(\gamma, \theta) X(\alpha))^{-1} \times X(\alpha)' H^{-1}(\gamma, \theta_{1:k}) h_{1:i}(k) H^{-1}(\gamma, \theta_{1:k}) X(\alpha) = O_p(1) \quad (C.14)$$

uniformly over $\theta \in [0, \infty)^q$, where the second last equality follows from (2.9) and (A.5) and the third equality follows from (A.7), Lemma 2 (iii), and Lemma 4 (ii)–(iii). For (C.15) with
$k = q$, we have

$$
\epsilon' H^{-1}(\gamma, \theta_0^i) X(\alpha) (\epsilon' H^{-1}(\gamma, \theta) X(\alpha))^{-1}
\times X(\alpha)' H^{-1}(\gamma, \theta_0^i) h_{i,q}(\theta) H_{i,q}^{-1}(\gamma, \theta) \epsilon
= \left(\sum_{i=1}^{m} \epsilon' H_{i,q}^{-1}(\gamma, \theta_0^i) X_i(\alpha) \right) \left(\sum_{i=1}^{m} X_i(\alpha)' H_{i,q}^{-1}(\gamma, \theta) X_i(\alpha) \right)^{-1}
\times \left(\sum_{i=1}^{m} \epsilon' H_{i,q}^{-1}(\gamma, \theta_0^i) z_{i,q} \right) \left(\sum_{i=1}^{m} X_i(\alpha)' H_{i,q}^{-1}(\gamma, \theta_0^i) z_{i,q} \right)^{-1}
\times \left(\sum_{i=1}^{m} X_i(\alpha)' H_{i,q}^{-1}(\gamma, \theta_0^i) z_{i,q} \right)
= \{O_p(1)\}_{1 \times p(\alpha)} \{T^{-1}(\alpha) + \{o(n_{\min}^{-r})\}_{p(\alpha) \times p(\alpha)} \{o(n_{\min}^{\ell/2-r})\}_{p(\alpha) \times 1}\}
\times \{O_p(n_{i,q})\}_{1 \times 1}
= o_p(1)
$$

uniformly over $\theta \in [0, \infty)^q$, where the second equality follows from (A.7), Lemma 2 (iii), and Lemma 4 (ii)–(iii). For (C.14) with $k = q$,

$$
\epsilon' H^{-1}(\gamma, \theta_0^i) X(\alpha) (\epsilon' H^{-1}(\gamma, \theta_0^i) X(\alpha))^{-1} X(\alpha)' H^{-1}(\gamma, \theta_0^i) h_{i,q}(\theta) H_{i,q}^{-1}(\gamma, \theta) \epsilon
= \left(\sum_{i=1}^{m} \epsilon' H_{i,q}^{-1}(\gamma, \theta_0^i) X_i(\alpha) \right) \left(\sum_{i=1}^{m} X_i(\alpha)' H_{i,q}^{-1}(\gamma, \theta_0^i) X_i(\alpha) \right)^{-1}
\times \left(\sum_{i=1}^{m} \epsilon' H_{i,q}^{-1}(\gamma, \theta_0^i) z_{i,q} \right) \left(\sum_{i=1}^{m} X_i(\alpha)' H_{i,q}^{-1}(\gamma, \theta_0^i) z_{i,q} \right)^{-1}
\times \left(\sum_{i=1}^{m} X_i(\alpha)' H_{i,q}^{-1}(\gamma, \theta_0^i) z_{i,q} \right)
= \{O_p(1)\}_{1 \times p(\alpha)} \{T^{-1}(\alpha) + \{o(n_{\min}^{-r})\}_{p(\alpha) \times p(\alpha)} \{o(n_{\min}^{\ell/2-r})\}_{p(\alpha) \times 1}\}
\times \{O_p(n_{i,q})\}_{1 \times 1}
= o_p(1)
$$

uniformly over $\theta \in [0, \infty)^q$, where the second equality follows from (A.7), Lemma 2 (ii)–(iii), and Lemma 4 (iii). This completes the proofs of (C.11)–(C.14). We now prove (C.9). Note
that

\[
\epsilon' \{ \mathbf{H}^{-1}(\gamma, \theta_0^*) \mathbf{M}(\alpha, \gamma; \theta_0^*) - \mathbf{H}^{-1}(\gamma, \theta) \mathbf{M}(\alpha, \gamma; \theta) \} \epsilon \\
= \epsilon' \{ \mathbf{H}^{-1}(\gamma, \theta_0^*) \mathbf{M}(\alpha, \gamma; \theta_0^*) - \mathbf{H}^{-1}(\gamma, \theta_0^*) \mathbf{M}(\alpha, \gamma; \theta_0^*) \} \epsilon \\
+ \epsilon' \{ \mathbf{H}^{-1}(\gamma, \theta_0^*) \mathbf{M}(\alpha, \gamma; \theta_0^*) - \mathbf{H}^{-1}(\gamma, \theta_0^*) \mathbf{M}(\alpha, \gamma; \theta_0^*) \} \epsilon \\
= \epsilon' \mathbf{H}^{-1}(\gamma, \theta_0^*) \{ \mathbf{M}(\alpha, \gamma; \theta_0^*) - \mathbf{M}(\alpha, \gamma; \theta_0^*) \} \epsilon + o_p(m) \\
= \epsilon' \mathbf{H}^{-1}(\gamma, \theta_0^*) \{ \mathbf{M}(\alpha, \gamma; \theta_0^*) - \mathbf{M}(\alpha, \gamma; \theta_0^*) \} \epsilon + o_p(m) \\
= o_p(m)
\]

uniformly over \( \theta \in [0, \infty)^9 \), where the second equality follows from (C.12) that

\[
\epsilon' \{ \mathbf{H}^{-1}(\gamma, \theta_0^*) - \mathbf{H}^{-1}(\gamma, \theta) \} \mathbf{M}(\alpha, \gamma; \theta) \epsilon \\
= \epsilon' \mathbf{H}^{-1}(\gamma, \theta_0^*) \{ \mathbf{H}(\gamma, \theta) - \mathbf{H}(\gamma, \theta_0^*) \} \mathbf{H}^{-1}(\gamma, \theta_0^*) \mathbf{M}(\alpha, \gamma; \theta) \epsilon \\
= \sum_{i=1}^{m} \sum_{k=q}^{q_0, q+1} (\theta(k) - \theta(k), 0) \epsilon' \mathbf{H}^{-1}(\gamma, \theta) h_{i,(k)} h_{i,(k)}' \mathbf{H}^{-1}(\gamma, \theta_0^*) \mathbf{M}(\alpha, \gamma; \theta) \epsilon \\
= o_p(m)
\]

uniformly over \( \theta \in [0, \infty)^9 \), the second last equality follows from (C.13) that

\[
\epsilon' \mathbf{H}^{-1}(\gamma, \theta_0^*) \{ \mathbf{M}(\alpha, \gamma; \theta_0^*) - \mathbf{M}(\alpha, \gamma; \theta_0^*) \} \epsilon \\
= \epsilon' \mathbf{H}^{-1}(\gamma, \theta_0^*) \{ \mathbf{M}(\alpha, \gamma; \theta_0^*) - \mathbf{M}(\alpha, \gamma; \theta_0^*) \} \epsilon \\
= \sum_{i=1}^{m} \sum_{k=q}^{q_0, q+1} (\theta(k) - \theta(k), 0) \epsilon' \mathbf{H}^{-1}(\gamma, \theta_0^*) \mathbf{M}(\alpha, \gamma; \theta_0^*) \epsilon \\
= o_p(m)
\]
uniformly over $\theta \in [0, \infty)^q$, and the last equality follows from (C.14) that

$$
epsilon' H^{-1}(\gamma, \theta_0) \{M(\alpha, \gamma; \theta_0) - X(\alpha)X(\alpha)' H^{-1}(\gamma, \theta_0) \} \epsilon
= e'H^{-1}(\gamma, \theta_0) X(\alpha) X(\alpha)' H^{-1}(\gamma, \theta_0) X(\alpha) - 1
= e'H^{-1}(\gamma, \theta_0) X(\alpha) X(\alpha)' H^{-1}(\gamma, \theta_0) X(\alpha) - 1
= H^{-1}(\gamma, \theta_0) X(\alpha) X(\alpha)' H^{-1}(\gamma, \theta_0) X(\alpha) - 1
= \sum_{i=1}^m \sum_{k=q-00+1}^q (\theta(k), 0 - \theta(k)) e'H^{-1}(\gamma, \theta_0) h_{1, (k)} h_{1, (k)} H^{-1}(\gamma, \theta_0) X(\alpha) X(\alpha)' H^{-1}(\gamma, \theta_0) X(\alpha) - 1
= \epsilon \{H^{-1}(\gamma, \theta) - H^{-1}(\gamma, \theta_0)\} \epsilon
= e'H^{-1}(\gamma, \theta_0) H(\gamma, \theta_0) - H(\gamma, \theta) H^{-1}(\gamma, \theta_0) \epsilon
= e'H^{-1}(\gamma, \theta_0) \epsilon
= ε' \{H^{-1}(\gamma, \theta) - H^{-1}(\gamma, \theta_0)\} \epsilon
= e'H^{-1}(\gamma, \theta_0) H(\gamma, \theta_0) - H(\gamma, \theta) H^{-1}(\gamma, \theta_0) \epsilon
= e'H^{-1}(\gamma, \theta_0) \epsilon
= O_p(m)
uniformly over $\theta \in [0, \infty)^q$. Also, by (C.11),

$$
epsilon' H^{-1}(\gamma, \theta) \epsilon
= e'H^{-1}(\gamma, \theta_0) H(\gamma, \theta_0) - H(\gamma, \theta) H^{-1}(\gamma, \theta_0) \epsilon
= e'H^{-1}(\gamma, \theta_0) \epsilon
= ε' \{H^{-1}(\gamma, \theta) - H^{-1}(\gamma, \theta_0)\} \epsilon
= e'H^{-1}(\gamma, \theta_0) H(\gamma, \theta_0) - H(\gamma, \theta) H^{-1}(\gamma, \theta_0) \epsilon
= e'H^{-1}(\gamma, \theta_0) \epsilon
= O_p(m)
uniformly over $\theta \in [0, \infty)^q$. This together with (C.15) gives (C.9). We now prove (C.10). As with the proof of (C.15), we have

$$
b(\gamma_0)' Z(\gamma_0)' \{H^{-1}(\gamma, \theta_0) M(\alpha, \gamma; \theta_0) - H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta)\} Z(\gamma_0) b(\gamma_0) = O_p(m)
uniformly over $\theta \in [0, \infty)^q$. Hence

$$
b(\gamma_0)' Z(\gamma_0)' \{H^{-1}(\gamma, \theta) (I_N - M(\alpha, \gamma; \theta))
- H^{-1}(\gamma, \theta_0) (I_N - M(\alpha, \gamma; \theta_0))\} Z(\gamma_0) b(\gamma_0)
= b(\gamma_0)' Z(\gamma_0)' \{H^{-1}(\gamma, \theta) - H^{-1}(\gamma, \theta_0)\} Z(\gamma_0) b(\gamma_0) + O_p(m)
= \sum_{i=1}^m \sum_{s=q-00+1}^q (\theta(s), 0 - \theta(s)) b(\gamma_0)' Z(\gamma_0)' H^{-1}(\gamma, \theta) h_{1, (s)}
\times h_{1, (s)} H^{-1}(\gamma, \theta_0) Z(\gamma_0) b(\gamma_0) + O_p(m)
uniformly over $\theta \in [0, \infty)^q$. Hence, for (C.10) to hold, it suffices to prove that for $k = q - q_0 + 1, \ldots, q$ and $i = 1, \ldots, m$,

$$
\log \left( \frac{1 + \theta(k) z_{i, (k)} H^{-1}(\gamma, \theta) z_{i, (k)}}{1 + \theta(k) z_{i, (k)} H^{-1}(\gamma, \theta_0) z_{i, (k)}} \right)
= O_p \left( b(\gamma_0)' Z(\gamma_0)' H^{-1}(\gamma, \theta) h_{1, (k)} h_{1, (k)} H^{-1}(\gamma, \theta_0) Z(\gamma_0) b(\gamma_0) \right)
\quad (C.16)
as both $N \to \infty$ and $\theta(k) \to 0$ for some $k \in \{q - q_0 + 1, \ldots, q\}$. It suffices to prove (C.16) for $k = q$. By Lemma 3 (ii)–(iii), we have

\[
\begin{align*}
\frac{b(\gamma_0)'}{Z(\gamma_0)'H^{-1}(\gamma, \theta)h_{i,(q)}H^{-1}(\gamma, \theta)Z(\gamma_0)b(\gamma_0)} &= \left(\frac{b_{i,(q)}'z_{i,(q)}'H_{i,q}^{-1}(\gamma, \theta)z_{i,(q)}H_{i,q}^{-1}(\gamma, \theta)}{1 + \theta(q)z_{i,(q)}'H_{i,q}^{-1}(\gamma, \theta)z_{i,(q)}}\right)
\end{align*}
\]

Hence, for (C.16) with $k = q$ to hold, it suffices to prove that

\[
\log \left(1 + \theta(k)z_{i,(q)}'H_{i,q}^{-1}(\gamma, \theta)z_{i,(q)}\right) \frac{z_{i,(q)}'}{z_{i,(q)}'} \left(1 + \theta(q)z_{i,(q)}'H_{i,q}^{-1}(\gamma, \theta)z_{i,(q)}\right)^{-1} \to 0,
\]

as both $N \to \infty$ and $\theta(q) \to 0$, which follows from Lemma 3 (i) and L’Hospital’s rule. This completes the proof of (C.16). This completes the proof.

C.5 Proof of Lemma 6

We first prove Lemma 6 (i). For $i, i^* = 1, \ldots, m$, $(\alpha, \gamma) \in A \times G$ and $k, k^* \in \gamma$, we have

\[
\begin{align*}
\frac{\partial \theta_{k,k^*}H_{i,k}^{-1}(\gamma, \theta)X_i(\alpha)}{\sum_{i=1}^m n_i} &= \left(\frac{\sum_{i=1}^m X_i(\alpha)'H_{i,q}^{-1}(\gamma, \theta)X_i(\alpha)}{\sum_{i=1}^m n_i} \right)^{-1} \\
&\times \left(\frac{\sum_{i=1}^m n_i}{\sum_{i=1}^m n_i} \right)^{-1} \\
\end{align*}
\]

\[
\begin{align*}
&= \left\{o\left(n_i^\xi \tau/2 - \tau\right)\right\}_1 \times \left\{T^{-1}(\alpha) + \left\{o\left(n_{\min}^\tau\right)\right\}_p(\alpha) \times \left\{o\left(n_i^\tau\right)\right\}_p(\alpha)\right\} \\
&\times \left\{o\left(n_i^\xi \tau/2 - \tau\right)\right\}_1 \\
&= \left\{o\left(n_i^\xi \tau/2 - \tau\right)\right\}_1 \\
&= \left\{o\left(n_i^\xi \tau/2 - \tau\right)\right\}_1
\end{align*}
\]
uniformly over \(\theta \in \Theta_\gamma\), where the second equality follows from (A.7) and Lemma 2 (iii). Similarly, by (A.7) and Lemma 2 (iii), we have

\[
\begin{align*}
\theta_k h_{i,k}^* & = \left( \theta_k z_{i,k}^* h_{i,k}^{-1} (\gamma, \theta) X_i(\alpha) \right) \left( \sum_{i=1}^m X_i(\alpha)' H_{i,k}^{-1} (\gamma, \theta) X_i(\alpha) \right)^{-1} \\
& \quad \times \left( \frac{X_i(\alpha)' h_{i,k}^{-1} (\gamma, \theta) z_{i,k}^*}{\sum_{i=1}^m n_i^\xi} \right) \\
& = \{ o(n_i^{(\xi+\ell)/2-\tau}) \}_{1 \times p(\alpha)} \left( T^{-1}(\alpha) + \{ o(n_i^{1}) \}_{p(\alpha) \times p(\alpha)} \right) \\
& \quad \times \left\{ o\left( \frac{n_i^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^m n_i^\xi} \right) \right\}_{p(\alpha) \times 1} \\
& = o\left( \frac{n_i^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^m n_i^\xi} \right)
\end{align*}
\]

uniformly over \(\theta \in \Theta_\gamma\). Further, by (A.7) and Lemma 2 (iii), we have

\[
\begin{align*}
h_{i,k}^* & = \left( \theta_k z_{i,k}^* h_{i,k}^{-1} (\gamma, \theta) X_i(\alpha) \right) \left( \sum_{i=1}^m X_i(\alpha)' H_{i,k}^{-1} (\gamma, \theta) X_i(\alpha) \right)^{-1} \\
& \quad \times \left( \frac{X_i(\alpha)' h_{i,k}^{-1} (\gamma, \theta) z_{i,k}^*}{\sum_{i=1}^m n_i^\xi} \right) \\
& = \{ o(n_i^{(\xi+\ell)/2-\tau}) \}_{1 \times p(\alpha)} \left( T^{-1}(\alpha) + \{ o(n_i^{1}) \}_{p(\alpha) \times p(\alpha)} \right) \\
& \quad \times \left\{ o\left( \frac{n_i^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^m n_i^\xi} \right) \right\}_{p(\alpha) \times 1} \\
& = o\left( \frac{n_i^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^m n_i^\xi} \right)
\end{align*}
\]

uniformly over \(\theta \in \Theta_\gamma\). This completes the proof of Lemma 6 (i).

We now prove Lemma 6 (ii). For \(i, i^* = 1, \ldots, m, (\alpha, \gamma) \in A \times G\), \(k \in \gamma\) and \(k^* \notin \gamma\),

\[
\begin{align*}
\theta_k h_{i,k}^* & = \left( \theta_k z_{i,k}^* h_{i,k}^{-1} (\gamma, \theta) X_i(\alpha) \right) \left( \sum_{i=1}^m X_i(\alpha)' H_{i,k}^{-1} (\gamma, \theta) X_i(\alpha) \right)^{-1} \\
& \quad \times \left( \frac{X_i(\alpha)' h_{i,k}^{-1} (\gamma, \theta) z_{i,k}^*}{\sum_{i=1}^m n_i^\xi} \right) \\
& = \{ o(n_i^{(\xi+\ell)/2-\tau}) \}_{1 \times p(\alpha)} \left( T^{-1}(\alpha) + \{ o(n_i^{1}) \}_{p(\alpha) \times p(\alpha)} \right) \\
& \quad \times \left\{ o\left( \frac{n_i^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^m n_i^\xi} \right) \right\}_{1 \times p(\alpha)} \\
& = o\left( \frac{n_i^{(\xi+\ell)/2-\tau}}{\sum_{i=1}^m n_i^\xi} \right)
\end{align*}
\]
uniformly over \( \theta \in \Theta_\gamma \), where the second equality follows from Lemma 2 (ii–(iii)) and (A.7). Similarly, by (A.7) and Lemma 2 (ii–(iii)), we have

\[
\begin{align*}
\hat{h}_{i,k} H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) \hat{h}_{i,k}^* \\
= \left( \theta_k z_{i,k} H_{i,k}^{-1}(\gamma, \theta) X_i(\alpha) \right) \left( \sum_{i=1}^m X_i(\alpha)' H_{i,k}^{-1}(\gamma, \theta) X_i(\alpha) \right)^{-1} \\
\times \left( \frac{X_i(\alpha)' H_{i,k}^{-1}(\gamma, \theta) z_{i,k}^*}{\sum_{i=1}^m n_i^2} \right) \\
= \left\{ o(n_i^{(\ell+\ell)/2-\tau}) \right\}_{1 \times p(\alpha)} \left\{ T^{-1}(\alpha) + \{ o(n_{\gamma \min}^{-\tau}) \}_{p(\alpha) \times p(\alpha)} \right\} \{ O_p(1) \}_{p(\alpha) \times 1} \\
= o_p(n_i^{-\ell/2})
\end{align*}
\]

uniformly over \( \theta \in \Theta_\gamma \). This completes the proof of Lemma 6 (ii).

We now prove Lemma 6 (iii). For \((\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}\) and \(k \in \gamma\),

\[
\begin{align*}
\theta_k \hat{h}_{i,k}^* H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) \epsilon \\
= \left( \theta_k z_{i,k}^* H_{i,k}^{-1}(\gamma, \theta) X_i(\alpha) \right) \left( \sum_{i=1}^m X_i(\alpha)' H_{i,k}^{-1}(\gamma, \theta) X_i(\alpha) \right)^{-1} \\
\times \left( \frac{X_i(\alpha)' H_{i,k}^{-1}(\gamma, \theta) \epsilon_i}{\sum_{i=1}^m n_i^2} \right) \\
= \left\{ o(n_i^{(\ell+\ell)/2-\tau}) \right\}_{1 \times p(\alpha)} \left\{ T^{-1}(\alpha) + \{ o(n_{\gamma \min}^{-\tau}) \}_{p(\alpha) \times p(\alpha)} \right\} \{ O_p(1) \}_{p(\alpha) \times 1} \\
= o_p(n_i^{-\ell/2})
\end{align*}
\]

uniformly over \( \theta \in \Theta_\gamma \), where the second equality follows from (A.7), Lemma 2 (iii), and Lemma 4 (iii). Similarly, by (A.7), Lemma 2 (iii), and Lemma 4 (iii), we have

\[
\begin{align*}
\hat{h}_{i,k} H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) \epsilon \\
= \left( \frac{z_{i,k}^* H_{i,k}^{-1}(\gamma, \theta) X_i(\alpha)}{\sum_{i=1}^m n_i^2} \right)^{1/2} \left( \sum_{i=1}^m X_i(\alpha)' H_{i,k}^{-1}(\gamma, \theta) X_i(\alpha) \right)^{-1} \\
\times \left( \frac{X_i(\alpha)' H_{i,k}^{-1}(\gamma, \theta) \epsilon_i}{\sum_{i=1}^m n_i^2} \right)^{1/2} \\
= \left\{ o(n_i^{(\ell+\ell)/2-\tau}) \right\}_{1 \times p(\alpha)} \left\{ T^{-1}(\alpha) + \{ o(n_{\gamma \min}^{-\tau}) \}_{p(\alpha) \times p(\alpha)} \right\} \{ O_p(1) \}_{p(\alpha) \times 1} \\
= o_p(n_i^{-\ell/2})
\end{align*}
\]

uniformly over \( \theta \in \Theta_\gamma \). This completes the proof of Lemma 6 (iii).
We now prove Lemma 6 (iv). For \((\alpha, \gamma) \in (A \setminus A_0) \times G, k \in \gamma,\)

\[
\theta_k h^i_{i,k} H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) X(\omega_k \setminus \alpha) \beta(\omega_0 \setminus \alpha) \\
= \left( \theta_k z^i_{i,k} H^{-1}(\gamma, \theta) X_i(\alpha) \right) \left( \sum_{i=1}^{m} X_i(\alpha)' H^{-1}_i(\gamma, \theta) X_i(\alpha) \right)^{-1} \\
\times \left( \sum_{i=1}^{m} \sum_{j \in \omega_k \setminus \alpha} X_i(\alpha)' H^{-1}_i(\gamma, \theta) x_{i,j} \beta, 0 \right) \\
= \{o(n_1(\xi + \ell)/2 - \tau)\}_1 \times (\alpha) \left\{ T^{-1}(\alpha) + \{o(n_{\text{min}}^-)\}_1 \times (\alpha) \} \{o(n_{\text{min}}^-)\}_1 \times (\alpha) \right\} \\
= o_p(n_1(\xi + \ell)/2 - \tau)
\]

uniformly over \(\theta \in \Theta_\gamma,\) where the second equality follows from (A.7), Lemma 2 (i), and Lemma 2 (iii). Similarly, by (A.7) and Lemma 2 (i) and (iii), we have

\[
h^i_{i,k} H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) X(\omega_k \setminus \alpha) \beta(\omega_0 \setminus \alpha) \\
= \left( \theta_k z^i_{i,k} H^{-1}(\gamma, \theta) X_i(\alpha) \right) \left( \sum_{i=1}^{m} X_i(\alpha)' H^{-1}_i(\gamma, \theta) X_i(\alpha) \right)^{-1} \\
\times \left( \sum_{i=1}^{m} \sum_{j \in \omega_0 \setminus \alpha} X_i(\alpha)' H^{-1}_i(\gamma, \theta) x_{i,j} \beta, 0 \right) \\
= \{o(n_1(\xi + \ell)/2 - \tau)\}_1 \times (\alpha) \left\{ T^{-1}(\alpha) + \{o(n_{\text{min}}^-)\}_1 \times (\alpha) \} \{o(n_{\text{min}}^-)\}_1 \times (\alpha) \right\} \\
= o_p(n_1(\xi + \ell)/2 - \tau)
\]

uniformly over \(\theta \in \Theta_\gamma.\) This completes the proof of Lemma 6 (iv).

We now prove Lemma 6 (v). For \((\alpha, \gamma) \in A \times G,\) we have

\[
e^i H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) e^i \\
= \left( \sum_{i=1}^{m} e^i_{i,k} H^{-1}_i(\gamma, \theta) X_i(\alpha) \right) \left( \sum_{i=1}^{m} X_i(\alpha)' H^{-1}_i(\gamma, \theta) X_i(\alpha) \right)^{-1} \\
\times \left( \sum_{i=1}^{m} X_i(\alpha)' H^{-1}_i(\gamma, \theta) e^i \right) \\
= \{O_p(1)\}_1 \times (\alpha) \left\{ T^{-1}(\alpha) + \{o(n_{\text{min}}^-)\}_1 \times (\alpha) \} \{O_p(1)\}_1 \times (\alpha) \right\} \\
= O_p(\alpha)
\]

uniformly over \(\theta \in \Theta_\gamma,\) where the second equality follows from (A.7) and Lemma 4 (iii). This completes the proof of Lemma 6 (v).
We now prove Lemma 6 (vii). For \((\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}\) and \(k \neq \gamma\), we have

\[
\begin{align*}
& h_{i,k} H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) \epsilon \\
& = \left( z_{i,k}^T H^{-1}(\gamma, \theta) X_i(\alpha) \right) \left( \sum_{i=1}^m X_i(\alpha)^T H^{-1}(\gamma, \theta) X_i(\alpha) \right)^{-1} \\
& \times \left( \sum_{i=1}^m \frac{X_i(\gamma)^T H_{i,-1}^{-1}(\gamma, \theta) x_i}{n_k} \right) \\
& = \left\{ o\left( \frac{n_i^{\xi / 2}}{\xi} \right) \right\} \left\{ o\left( \frac{n_i^{-\tau}}{\tau} \right) \right\} \left\{ o\left( \frac{n_i^{\xi}}{\xi} \right) \right\} \left\{ o\left( \frac{n_i}{\tau} \right) \right\} \\
& = o_p\left( \frac{n_i^{\xi / 2}}{\xi} \right)
\end{align*}
\]
uniformly over \(\theta \in \Theta_n\), where the second equality follows from (A.7), Lemma 2 (ii), and Lemma 4 (iii). This completes the proof of Lemma 6 (vii).

We now prove Lemma 6 (viii). For \((\alpha, \gamma) \in (\mathcal{A} \setminus \mathcal{A}_0) \times \mathcal{G}\), we have

\[
\begin{align*}
e^{-T}(\gamma, \theta) M(\alpha, \gamma; \theta) X(\alpha) & = \left( \sum_{i=1}^m \frac{X_i(\alpha)'^T H^{-1}(\gamma, \theta) X_i(\alpha)}{n_k} \right)^{-1} \\
& \times \left( \sum_{i=1}^m \sum_{j \in \mathcal{A}_0 \setminus \mathcal{A}_0} X_i(\alpha)'^T H_{i,-1}^{-1}(\gamma, \theta) x_i, j \beta_j \theta \right) \\
& = \left\{ o_p\left( \frac{1}{\xi} \right) \right\} \left\{ o\left( \frac{n_i^{\xi / 2}}{\xi} \right) \right\} \left\{ o\left( \frac{n_i^{-\tau}}{\tau} \right) \right\} \left\{ o\left( \frac{n_i^{\xi}}{\xi} \right) \right\} \left\{ o\left( \frac{n_i}{\tau} \right) \right\} \\
& = o_p\left( \frac{1}{\xi} \right)
\end{align*}
\]
uniformly over \(\theta \in \Theta_n\), where the second equality follows from (A.7), Lemma 2 (i), and Lemma 4 (iii). This completes the proof of Lemma 6 (vii).

We now prove Lemma 6 (viii). For \(i, i^* = 1, \ldots, m\), \((\alpha, \gamma) \in \mathcal{A} \times \mathcal{G}\) and \(k, k^* \neq \gamma\), we have

\[
\begin{align*}
h_{i,k} H^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) h_{i^*, k^*} & = \left( z_{i,k}^T H_{i,-1}^{-1}(\gamma, \theta) X_i(\alpha) \right) \left( \sum_{i=1}^m \frac{X_i(\alpha)'^T H_{i,-1}^{-1}(\gamma, \theta) X_i(\alpha)}{n_k} \right)^{-1} \\
& \times \left( \frac{X_{i^*}(\gamma)'^T H_{i^*, -1}^{-1}(\gamma, \theta) x_{i^*, k^*}}{n_{i^*}} \right) \\
& = \left\{ o\left( \frac{n_i^{\xi / 2}}{\xi} \right) \right\} \left\{ o\left( \frac{n_i^{-\tau}}{\tau} \right) \right\} \left\{ o\left( \frac{n_i^{\xi / 2}}{\xi} \right) \right\} \left\{ o\left( \frac{n_i^{\xi / 2}}{\xi} \right) \right\} \left\{ o\left( \frac{n_i^{-\tau}}{\tau} \right) \right\} \\
& = o_p\left( \frac{n_i^{\xi / 2}}{\xi} \right)
\end{align*}
\]
uniformly over \(\theta \in \Theta_n\), where the second equality follows from (A.7) and Lemma 2 (ii). This completes the proof of Lemma 6 (viii).
We now prove Lemma 6 (ix). For \((\alpha, \gamma) \in (A \setminus A_0) \times G\), \(k \notin \gamma\), we have

\[
\begin{align*}
    h_{i,k}' H_i^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) X(\alpha \setminus \alpha) \beta(\alpha \setminus \alpha) \\
    &= (z_{i,k}' H_i^{-1}(\gamma, \theta) X_i(\alpha)) \left( \sum_{i=1}^m X_i(\alpha)' H_i^{-1}(\gamma, \theta) X_i(\alpha) \right)^{-1} \\
    &\quad \times \left( \sum_{i=1}^m \sum_{j \in \alpha \setminus \alpha} X_i(\alpha)' H_i^{-1}(\gamma, \theta) x_{i,j} \beta_{i,j} \right) \\
    &= \left\{ o(n_1^\xi/2^{-\tau}) \right\}_1 \times (T^{-1}(\alpha) + \{ o(n_{min}^-) \} p(\alpha) \times p(\alpha)) \\
    &\quad \times \{ o(n_{min}^-) \} p(\alpha) \times 1 \\
    &= o(n_1^\xi/2^{-\tau}) \}
\end{align*}
\]

uniformly over \(\theta \in \Theta_\gamma\), where the second equality follows from (A.7) and Lemma 2 (i)–(ii). This completes the proof of Lemma 6 (ix).

We finally prove Lemma 6 (x). For \((\alpha, \gamma) \in (A \setminus A_0) \times G\), we have

\[
\begin{align*}
    \beta(\alpha \setminus \alpha)' X(\alpha \setminus \alpha)' H_i^{-1}(\gamma, \theta) M(\alpha, \gamma; \theta) X(\alpha \setminus \alpha) \beta(\alpha \setminus \alpha) \\
    &= \left( \sum_{i=1}^m \sum_{j \in \alpha \setminus \alpha} \beta_{i,j} x_{i,j}' H_i^{-1}(\gamma, \theta) X_i(\alpha) \right) \left( \sum_{i=1}^m X_i(\alpha)' H_i^{-1}(\gamma, \theta) X_i(\alpha) \right)^{-1} \\
    &\quad \times \left( \sum_{i=1}^m \sum_{j \in \alpha \setminus \alpha} X_i(\alpha)' H_i^{-1}(\gamma, \theta) x_{i,j} \beta_{i,j} \right) \\
    &= \left\{ o\left( \sum_{i=1}^m n_1^{-\xi-\tau} \right) \right\}_1 \times (T^{-1}(\alpha) + \{ o(n_{min}^-) \} p(\alpha) \times p(\alpha)) \times \{ o(n_{min}^-) \} p(\alpha) \times 1 \\
    &= o\left( \sum_{i=1}^m n_1^{-\xi-\tau} \right)
\end{align*}
\]

uniformly over \(\theta \in \Theta_\gamma\), where the second equality follows from (A.7) and Lemma 2 (i). This completes the proof.