EPW SEXTICS AND HILBERT SQUARES OF K3 SURFACES

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ABSTRACT. We prove that the Hilbert square $S^{[2]}$ of a very general primitively polarized K3 surface $S$ of degree $d(n) = 2(4n^2 + 8n + 5)$, $n \geq 1$ is birational to a double Eisenbud-Popescu-Walter sextic. Our result implies a positive answers, in the case when $r$ is even, to a conjecture of O’Grady: On the Hilbert square of a very general K3 surface of degree $r^2 + 2$, $r \geq 1$ there is an antisymplectic involution. We explicitly give this involution on $S^{[2]}$ in term of the corresponding EPW polarization on it.

1. Introduction and motivations

O’Grady conjectured in [OG1] that on the Hilbert square of a K3 surface of genus $g = r^2 + 2$, $r \geq 0$ there exists an antisymplectic involution (see (4.3.3) in [OG1]).

We show here the following theorem, that in particular implies that O’Grady conjecture is true in the case when $r$ is even.

Theorem 1.1. The Hilbert square of a very general K3 surface of degree $d(n) = 2(4n^2 + 8n + 5)$, $n \geq 1$ is birational to a double EPW sextic.

Indeed, for $d(n) = 8n^2 + 16n + 10 = 2(4(n + 1)^2 + 1)$, the genus of $S$ is $g(n) = d(n)/2 + 1 = 4n^2 + 8n + 6 = (2n + 2)^2 + 2$, and for $n \geq 1$, $r = 2n + 2$ covers all even numbers $r \geq 4$, and the antisymplectic involution is determined in term of the EPW polarization (see Section 3.1).

Notice that while the case $r = 0$ is well known, and the case $r = 2$ is studied in detail by O’Grady (see e.g. §4.3 in [OG1]), in the cases of odd $r$ very little is known: only the case $r = 1$ is studied in [De] and [Fe].

To show our main result we will follow O’Grady study of the case $r = 2$, considering a double EPW sextic associated to a special K3 surface of degree 10, together with the methods used by Hassett in [Ha1].

The proof of our main theorem is given in Section 3 while notations and basic facts and properties of double EPW sextic are recalled in Section 2.

2. Fano fourfolds $X_{10}$, EPW sextics and K3 surfaces

2.1. Fano fourfolds $X_{10}$. By $X_{10}$ we denote a prime Fano fourfold of index two and degree 10. By [Mu] and [Gus], any smooth $X_{10}$ is either a complete intersection of the Grassmannian $G = G(2,5) \subset \mathbb{P}^9$ with a hyperplane and a quadric (the 1-st type) or a double covering of the smooth Fano fourfold $W_5 = G(2,5) \cap \mathbb{P}^7$ branched along
a quadratic section of $W_5$ (the 2-nd, or the Gushel’ type). The moduli stack $\mathcal{S}_{10}$ of smooth $X_{10}$ is of dimension 24, and in $\mathcal{S}_{10}$ the general $X_{10}$ is from the first type. The condition for $X_{10}$ to be of the second type is of codimension 2, and the general $X_{10}$ of the second type is a smooth deformation from $X_{10}$ of the first type.

Let $X$ be a fourfold of type $\mathcal{S}_{10}$. By the Hodge-Riemann bilinear relations, the Hodge structure on $H^4(X, \mathbb{Z})$ has weight 2, and the intersection form on $X$ endows the 4-th integral cohomology of $X$ with a structure of the lattice $\Lambda = H^4(X, \mathbb{Z}) = I_{22,2}$, where $I_{22,2}$ denotes the lattice $22\langle 1 \rangle \oplus 2\langle -1 \rangle$.

By [DIM1], the lattice $H^4(X, \mathbb{Z})$ contains the fixed rank two polarization sublattice $\Lambda_2 := H^4(G, \mathbb{Z})|_X$ spanned on the restrictions to $X$ of the two Schubert cycles $\sigma_1, \sigma_2$ on $G = G(2, 5)$. In the basis $(u, v) = (\sigma_1|_X, \sigma_2|_X - \sigma_1|_X)$, the intersection form of the lattice $\Lambda_2 = H^4(G, \mathbb{Z})|_X = Zu + Zv$ is given by

$$u^2 = v^2 = 2, \quad uv = 0.$$  

For $X = X_{10}$, the primitive cohomology lattice with respect to the lattice polarization $\Lambda_2$, or the vanishing cohomology lattice is

$$\Lambda_0 = H^4(X, \mathbb{Z})_{\text{van}} = \Lambda_2^\perp = 2E_8 \oplus 2U \oplus 2\langle 2 \rangle,$$

ibid. $\Lambda_0$ is even of signature $(20, 2)$.

2.2. **EPW sextics.** Eisenbud-Popescu-Walter sextics, or in short EPW sextics, are special hypersurfaces of degree six in $\mathbb{P}^5$, first introduced in [EPW] as examples of Lagrangian degeneracy loci. These hypersurfaces are singular in codimension two, but O’Grady realized in [OG1] [OG2] that they admit smooth double covers which are irreducible holomorphic symplectic fourfolds. We will refer to this double covering as double EPW sextic. In fact, the first examples of such double covers were discovered by Mukai in [Mu1], who constructed them as moduli spaces of stable rank two vector bundles on a polarized K3 surface of degree 10.

Moreover O’Grady showed in [OG2] that the generic such double cover is a deformation of the Hilbert square of a K3 and that the family of double EPW sextics is a locally versal family of projective deformations of such an Hilbert square of a K3 surface.

Let $V$ be a 6-dimensional complex vector space and let us choose a volume-form on $V$

$$\text{vol} : \wedge^6 V \to \mathbb{C}$$

and let us equip $\wedge^3 V$ with the symplectic form $(\alpha, \beta)_V := \text{vol}(\alpha \land \beta)$.

Let $LG(\wedge^3 V)$ be the symplectic Grassmannian parametrizing Lagrangian subspaces of $\wedge^3 V$. Given a non-zero $v \in V$ let

$$F_v := \{ \alpha \in \wedge^3 V | v \land \alpha = 0 \}$$

be the sub-space of $\wedge^3 V$ consisting of multiples of $v$. $(\ , \ )_V$ is zero on $F_v$ and $\dim(F_v) = 10$, thus $F_v \in LG(\wedge^3 V)$. Let

$$F \subset \wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$$

be the sub-vector-bundle with fiber $F_v$ over $[v] \in \mathbb{P}(V)$. Then

$$\det F \cong \mathcal{O}_{\mathbb{P}(V)}(-6)$$
Given \( A \in \text{LG}(\wedge^3 V) \) we let \( Y_A = \{ [v] \in \mathbb{P}(V) | F_v \cap A \neq \{0\} \} \). Thus \( Y_A \) is the degeneracy locus of the map \( \lambda_A : F \to (\wedge^3 V/A) \otimes \mathcal{O}_{\mathbb{P}(V)} \) where \( \lambda_A \) is given by inclusion (1) followed by the quotient map

\[
\wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)} \to (\wedge^3 V/A) \otimes \mathcal{O}_{\mathbb{P}(V)}
\]

Since the vector-bundles appearing in (3) have equal rank the determinant of \( \lambda_A \) makes sense and \( Y_A = V(\det \lambda_A) \); this formula shows that \( Y_A \) has a natural structure of closed subscheme of \( \mathbb{P}(V) \). By \( \text{[OG2]} \) we have \( \det \lambda_A \in H^0(\mathcal{O}_{\mathbb{P}(V)}(6)) \) and hence \( Y_A \) is either a sextic hypersurface or \( \mathbb{P}(V) \). An \textit{EPW sextic} is a sextic hypersurface in \( \mathbb{P}^5 \) which is projectively equivalent to \( Y_A \) for some \( A \in \text{LG}(\wedge^3 V) \), and a \textit{double EPW sextic} is its associated double covering studied by O’Grady.

### 2.3. Necessary conditions and negative Pell’s equations.

Next we will look for necessary conditions to have a birational map between an Hilbert square of a K3 surface of degree \( d \) and a double EPW sextic. We will follow Mukai (\text{[Mu1]}).

**Proposition 2.1.** Let \( \tilde{Y} \to Y \) be a double EPW sextic which is smooth and birational to \( S^{[2]} \) for a primitively polarized K3 surface \( S \) of degree \( d = 2g - 2 \geq 10 \) and Picard number 1. Then the negative Pell’s equation

\[
y^2 - (g-1)x^2 = -1
\]

has an integer solution.

**Proof.** By a result of Mukai (see \text{[Mu1]} Corollary 5.9), if \( Y \) is birational to \( S^{[2]} \), then there exists an isometry between the Neron-Severi lattices \( \text{NS}(Y) \cong \text{NS}(S^{[2]}) \). Recall that \( \text{NS}(S^{[2]}) = \mathbb{Z} h + \mathbb{Z} \delta \), where \( (h, h) = d = 2g - 2 \), \((h, \delta) = 0 \), and \((\delta, \delta) = -2 \).

Let \( \pi : \tilde{Y} \to Y \) be the double covering defined by the antisymplectic involution, as in \text{[OG1]}, \text{[OG2]}. The EPW polarization \( \gamma \) on \( \tilde{Y} \) is the preimage of the hyperplane class on the EPW sextic \( Y \subset \mathbb{P}^5 \). Therefore the intersection index

\[
\gamma^4 = \text{deg} \pi \cdot \text{deg}(Y) = 2 \cdot 6 = 12.
\]

Since the double EPW sextic \( \tilde{Y} \) is a deformation of a Hilbert square of a K3 surface (see \text{[OG2]}) then the Fujiki constant \( c(\tilde{Y}) = c(S^{[2]}) = 3 \), see (1.0.1) and (4.1.4) in \text{[OG1]}. Therefore for the Beauville form \((\cdot, \cdot)\) on \( \text{NS}(\tilde{Y}) \) one will have:

\[
12 = \gamma^4 = c(\tilde{Y})(\gamma, \gamma)^2 = 3(\gamma, \gamma)^2,
\]

which yields

\[
(\gamma, \gamma) = 2.
\]

By the isometry \( \text{NS}(\tilde{Y}) \cong \text{NS}(S^{[2]}) \) we can identify \( \gamma \) with an element of \( \text{NS}(S^{[2]}) \), i.e. the birationality of \( \tilde{Y} \) with the Hilbert square of a K3 surface as above implies that there exist integers \( x, y \) such that \( \gamma = xh - y\delta \). Then

\[
2 = (\gamma, \gamma) = (xh - y\delta, xh - y\delta) = dx^2 - 2y^2 = (2g - 2)x^2 - 2y^2,
\]

from where

\[
y^2 - (g-1)x^2 = -1.
\]
Remark 2.2. It is well known that if $p$ is prime then the negative Pell’s equation $y^2 - px^2 = -1$ has a solution if and only if $p = 2$ or $p \equiv 1 \pmod{4}$, see e.g. Theorem 3.4.2 in [AAC].

Below we use the case when $p = 5$ which corresponds to a double EPW sextic birational to the Hilbert square of a K3 surface of degree 10, see §4.3 in [OG1]. For $p = 5$, the minimal solution of $y^2 - 5x^2 = -1$ is $(y, x) = (2, 1)$, and all solutions $(y_n, x_n), n \geq 0$ to $y^2 - 5y^2 = -1$ are given by

$$
2y_n = (1 + 2\sqrt{5})(2 + \sqrt{5})^{2n} + (1 - 2\sqrt{5})(2 - \sqrt{5})^{2n},
$$

$$
2x_n = (2 + 1/\sqrt{5})(2 + \sqrt{5})^{2n} + (2 - 1/\sqrt{5})(2 - \sqrt{5})^{2n},
$$

see e.g. Theorem 3.4.1 on p.141 and the formulas on p.305 in [AAC].

3. Double EPW sextics and Hilbert squares of K3 surfaces

We can now state main result of the paper, which is the following:

**Theorem 3.1.** The Hilbert square of a very general K3 surface of degree $d = d(n) = 2(4n^2 + 8n + 5), n \geq 1$ is birational to a double EPW sextic $\tilde{Y}$.

The proof of Theorem 3.1 uses similar methods used by Hassett in [Ha1] to show a (stronger, in some sense) similar result for the variety of lines on a cubic fourfold. Our main observation is that the same approach can be used also in the case of double EPW sextics. We divide the proof into several parts:

3.1. The birational involution on $S^{[2]}$ for a K3 surface $S$ of degree 10. For a K3 surface $S$ with a polarization $f$ of degree $f^2 = d = 2g - 2$ and Picard number 1, any curve $C \subset |f|$ defines a divisor $F_C = \{\xi \in S^{[2]} : \text{Supp}(\xi) \cap C \neq \emptyset\}$ on $S^{[2]}$. All divisors $F_C$ belong to the same class $f \in \text{NS}(S^{[2]})$. We use the same notation for the class $f$ and for the polarization $f$ on $S$. The class of the diagonal $\Delta = \{\xi \in S^{[2]} : \text{Supp}(\xi) = \text{point}\}$ is divisible by two in $\text{NS}(S^{[2]})$, and if $\Delta = 2\delta$ then

$$
\text{NS}(S^{[2]}) = Zf + Z\delta.
$$

If $(\ldots)$ is the Beauville form on $\text{NS}(S^{[2]})$, then

$$(f, f) = d, (f, \delta) = 0, (\delta, \delta) = -2.$$

If on $S$ there is a polarization $f$ of degree $d = 10$, then there exists a birational involution $j : S^{[2]} \to S^{[2]}$. For the general pair $(x, y)$ of points on the general $S$ the involution $j$ can be described geometrically as follows (for more detail see [OG2]):

Let $G = G(2, 5) = G(1 : \mathbb{P}^4) \subset \mathbb{P}^9$ be the grassmannian of lines in $\mathbb{P}^4$. By [Mu1], the general smooth K3 surface $S$ of degree 10 is a quadratic section $S = V_5 \cap Q$ of the unique smooth del Pezzo threefold $V_5 = G \cap \mathbb{P}^6$, which is a prime Fano threefold of index 2 and degree 5. By the general choice of $S \subset V_5$, the general non-ordered pair of points $(x, y)$ on $S \subset V_5$ is a general pair of points on $V_5$. The del Pezzo threefold $V_5$ has the property that through the general pair of points on $V_5$ passes a unique conic $q = q_{x,y}$. Indeed, let $l_x, l_y$ be the two lines in $\mathbb{P}^4$ representing the points $x, y \in V_5 \subset G = G(1 : \mathbb{P}^4)$. By the general choice of $x, y$, the lines $l_x$ and $l_y$ do not intersect each other and span a 3-space $\mathbb{P}^{3}_{x,y} \subset \mathbb{P}^4$. Any conic $q \subset G$ which passes
through $x$ and $y$ lies in the Plücker quadric $G(2, 4)_{x,y} = G(1 : \mathbb{P}^3_{x,y}) \subset G$. In addition, since $V_5 = G \cap \mathbb{P}^6$ then any conic on $V_5$ which passes through $x$ and $y$ lies on the codimension 3 subspace $\mathbb{P}^6 \cap \mathbb{P}^9 = \text{Span}(G)$. Therefore the set of conics on $V_5$ which pass through $x$ and $y$ sweep out the intersection $q_{x,y} = G(2, 4)_{x,y} \cap \mathbb{P}^6$, which by the general choice of $x, y$ is a codimension 3 linear section of the 4-dimensional quadric $G(2, 4)_{x,y}$, i.e. a conic. Since $S = V_5 \cap Q$ is a quadratic section of $V_5$, the conic $q_{x,y}$ intersects $S$ at $x, y$ and a pair of other 2 points $x', y'$. This defines a birational involution

$$j : S^{[2]} \to S^{[2]} \quad j(x, y) = (x', y').$$

Let $r = f - 2\delta \in \text{NS}(S^{[2]}) = \mathbb{Z}f + \mathbb{Z}\delta$. Then

$$(r, r) = (f - 2\delta, f - 2\delta) = (f, f) + 4(\delta, \delta) = 2.$$ 

By Propositions 4.1 and 4.21 in [OG1], on $\text{NS}(S^{[2]}) = \mathbb{Z}f + \mathbb{Z}\delta$ the involution $j$ is given by the reflection with respect to $r$

$$j : z \mapsto j(z) = -z + (z, r)r = -z + (z, f - 2\delta)(f - 2\delta).$$

We keep the same notation for the involution $j$ on $S^{[2]}$ and the involution $j$ on $\text{NS}(S^{[2]})$. In particular,

$$j(f) = -f + (f, f - 2\delta)(f - 2\delta) = -f + 10(f - 2\delta) = 9f - 20\delta,$$

$$j(\delta) = -\delta + (\delta, f - 2\delta)(f - 2\delta) = -\delta + 4(f - 2\delta) = 4f - 9\delta.$$ 

### 3.2. The Hilbert square of a K3 surface of degree 10 as a double EPW sextic.

Let $S \subset V_5 \subset G = G(1 : \mathbb{P}^4)$ be a very general K3 surface with a polarization $h$ of degree 10, where $V_5 = G \cap \mathbb{P}^6$ is as above. By [OG1], [OG2], the Hilbert square $S^{[2]}$ is a special case (as a birational equivalence class) of a double EPW sextic. The double covering is defined by the involution $j$ on $S^{[2]}$, and can be described as follows.

Let $\mathbb{P}^5 = |I_S(2)|$ be the projective space of quadrics in $\mathbb{P}^6$ which contain $S \in |\mathcal{O}_{V_5}(2)|$. In $\mathbb{P}^5$, the quadrics which contain $V_5$ form a hyperplane identified with the space of Pfaffian quadrics. Let $\xi \in S^{[2]}$, and let $\mathbb{P}^1_\xi = \text{Span}(\xi)$. Then $\xi$ defines a hyperplane

$$\mathbb{P}^4_\xi = |I_{S \cup \mathbb{P}^1_\xi}(2)| \subset |I_S(2)| = \mathbb{P}^5.$$ 

If $S$ does not contain lines, which is the general case, then the map

$$\pi : S^{[2]} \to \mathbb{P}^5, \quad \xi \mapsto \mathbb{P}^4_\xi$$

is well defined for any $\xi \in S^{[2]}$. The map $\pi$ is (generically) the double covering defined by the involution $j$. We shall show only that the images of two involutive elements by $\pi$ coincide; for more detail see [OG1] and [Mu1]. Indeed, if $j(\xi)$ is the involutive of $\xi$, then the lines $\mathbb{P}^1_\xi = \text{Span}(\xi)$ and $\mathbb{P}^1_{j(\xi)} = \text{Span}(j(\xi))$ intersect each other, since by construction of $j(\xi)$, $\xi + j(\xi)$ lie on a conic – see above. Since the lines $\mathbb{P}^1_\xi$ and $\mathbb{P}^1_{j(\xi)}$ are bisecant or tangent to $S$ and intersect each other, any quadric which contains $S$ together with one of these two lines contains also the other line. By the definition of $\pi$, the last yields that the images $\pi(\xi)$ and $\pi(j(\xi))$ coincide.

By §4.3 in [OG1], the image $Y_0 \subset \mathbb{P}^5$ of the double covering $\pi$ is an EPW sextic, defining the double EPW sextic

$$\tilde{Y}_0 \to Y_0,$$

which is birational to the Hilbert square $S^{[2]}$, see also Theorem 4.15 in [OG3].
In the sequel we will also need the following result of O’Grady:

**Lemma 3.2.** (see Proposition 4.21 and Corollary 5.21 in [OG1]). The class \( \gamma \) of the EPW polarization \( \pi^*(\mathcal{O}_{\gamma_0}(1)) \in \text{NS}(S^{[2]}) = Zf + Z\delta \) is \( \gamma = h - 2\delta \).

**Remark 3.3.** Here we assume that \( \text{NS}(S) \cong Z \) and denote by \( h \) the ample generator. The EPW-polarization \( \gamma = xh - y\delta \) is \( j \)-invariant, i.e. \( j(\gamma) = \gamma \), where
\[
j : z \mapsto -z + (z, r)r
\]
is the involution defined by \( r = h - 2\delta \), which interchanges the two preimages of the general point \( p \in Y_0 \), see Subsection 3.2. The equality \( \gamma = j(\gamma) = -\gamma + (r, \gamma)r \) yields \( 2\gamma = (r, \gamma)r \), i.e. \( \gamma \) is proportional to \( r = h - 2\delta \). Since \( \gamma \) is primitive, i.e. not divisible by an integer, \( \gamma = r \).

3.3. **K3 surfaces with two polarizations of degree 10.**

**Lemma 3.4.** Let \( R \) be the rank two lattice \( R = Zf + Zh \) with intersection form
\[
\begin{array}{ccc}
  f & h \\
  10 & n + 10 & 10 \\
  h & n + 10 & 10 \\
\end{array}
\]
where \( n \geq 1 \). Then there exists a K3 surface \( S \) with \( \text{NS}(S) = Zf + Zh \), such that \( f \) and \( h \) are two very ample polarizations on \( S \).

**Proof.** Let \( \Lambda = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \) be the K3 cohomology lattice. By Theorem 2.4 in [LP], there exists an embedding \( R \subset \Lambda \). By the surjectivity of the period map for K3 surfaces one can assume that e.g. \( f \) is a very ample polarization on a K3 surface \( S \). Since \( (f, h) > 0 \) then the divisor class \( h \) is effective, and one needs to see that \( h \) is very ample. If \( h \) is not ample then on \( S \) will exist a \((-2)\)-curve \( E \) such that \( h \cdot E \leq 0 \). If then \( k = h \cdot E = 0 \) then \( R_0 = Zf + ZE \) will be a sublattice of \( R \) of discriminant \( d(R_0) = -20 \). Since \( R_0 \subset R \) then \( d(R) = -n(n + 20) \) divides \( d(R_0) = -20 \), which is not possible. It remains the possibility when \( hE = -k < 0 \). Since \( E^2 = -2 \), then \( E \) defines a reflection
\[
r_E : x \mapsto \tilde{x} = x + (x, E)E,
\]
x \( \in \text{NS}(S) \supset R \). In particular, \( \tilde{h} = h - kE \), \( (\tilde{h}, \tilde{h}) = (h, h) = 10 \), and \( (f, \tilde{h}) = (f, h - kE) = (f, h) - k(f, E) < (f, h) \) since \( f \) is (very) ample and \( E \) is effective. Since \( \tilde{h} \in R \) then \( R' = Zf + Z\tilde{h} \) is a sublattice of \( R = Zf + Zh \). Therefore \( d(R) \) divides \( d(R') \), and since both \( d(R) \) and \( d(R') \) are negative, then \( d(R') \leq d(R) \). But
\[
d(R') = (f, f)(\tilde{h}, \tilde{h}) - (f, \tilde{h})^2 = (f, f)(h, h) - (f, h)^2 > (f, f)(h, h) - (f, h)^2 = d(R),
\]
contradiction. This proves the Lemma. For more detail see Lemma 4.3.3 and §6 in [Hä1].

3.4. **Proof of Theorem 3.1.** Let \( S \) be a very general K3 surface with a primitive polarization \( h \) of degree 10 as in 3.2. Denote by \( \tilde{Y}_0 \) the corresponding double EPW sextic to \( S^{[2]} \). Let \( \tilde{Y}_t \) be a local deformation of \( \tilde{Y}_0 \) in the polarization \( \gamma = h - 2\delta \) as a double EPW sextic \( \pi_t : \tilde{Y}_t \to Y_t \). Since \( \tilde{Y}_t \) is a deformation of a Hilbert square of a K3 surface, the Fujiki constant \( c(\tilde{Y}_t) = c(S^{[2]}) = 3 \), and as in the proof of Proposition 2.1 we get \( (\gamma, \gamma) = 2 \).
Let $S$ be a very general K3 surface with two polarizations $f$ and $h$ (generating the Neron-Severi lattice) as in Lemma 3.4. By above, e.g. in the polarization $h$, the Hilbert square $S^{[2]}$ is birational to a double EPW sextic $\tilde{Y}_0$. By Proposition 2.2 and Theorem 4.15 in [OG3], $Y_0$ belongs to the locus $\Delta - \Sigma$ (ibid. (0.0.7)-(0.0.8)), and by Proposition 6.3 of [OG2] has a unique singular point $p_0$ of multiplicity three.

The Hilbert square $S^{[2]} \to \tilde{Y}_0$ is a small resolution of $p_0$ which is a contraction of a Lagrangian plane on $S^{[2]}$ to the point $p_0$.

Next, we proceed as in the proof of Theorem 6.1.4 in [Ha1] for families of lines on cubic fourfolds, adapted to the case of double EPW sextics.

By [OG3] the period map for double EPW sextics extends regularly around the period point of $S^{[2]}$. Let $\mathcal{F}_{g(n)}$ be the moduli space of primitively polarized K3 surfaces $S'$ of genus $g(n) = d(n)/2 + 1$. By the surjectivity of the period map for K3 surfaces (see [LP]), one can consider $h_2 = \gamma + (2n + 2)\delta_2 \in \Pi$ as the (quasi) polarization of a K3 surface of genus $g(n)$, with $\delta_2 = 4f - 9\delta \in \Pi$ the class of the half-diagonal on its Hilbert square, see also the proof of Proposition 7 in [BHT]. By Proposition 10, Theorem 6 and Remark 2 on p. 779-780 of [Be] (see also Theorem 6.1.2 in [Ha1]) in the 20-dimensional local moduli space $\mathcal{M}$ of double EPW sextics $\tilde{Y}_t$ around $Y_0$ the condition that $\delta_2 = 4f - 9\delta$ remains algebraic, i.e. an element of $NS(\tilde{Y}_t)$, describes locally a smooth component of the divisor in $\mathcal{M}$ on which $\tilde{Y}_t$ remains birational to a Hilbert square of a K3 surface $S_t$ of genus $g(n)$.

For the general double EPW sextic $\tilde{Y}_t$ as above, the lattice $NS(\tilde{Y}_t)$ has rank two, and is the saturation of the rank two sublattice

$$\Pi = \mathbb{Z}\gamma + \mathbb{Z}\delta_2 = \mathbb{Z}(h - 2\delta) + \mathbb{Z}(4f - 9\delta).$$

Since $\Pi$ is saturated, then $NS(\tilde{Y}_t)$ coincides with $\Pi$, in particular the discriminant $d(NS(\tilde{Y}_t))$ is the discriminant of $\Pi$. By using $(\gamma, \gamma) = 2$ and $(\delta_2, \delta_2) = -2$, and the intersection table from Lemma 3.4 we compute

$$(\gamma, \delta_2) = (h - 2\delta, 4f - 9\delta) = 4(h, f) + 18(\delta, \delta) = 4(n + 10) - 36 = 4n + 4.$$ 

Therefore

$$d(NS(\tilde{Y}_t)) = d(\Pi) = det \begin{pmatrix} (\gamma, \gamma) & (\gamma, \delta_2) \\ (\delta_2, \gamma) & (\delta_2, \delta_2) \end{pmatrix} = (\gamma, \gamma)(\delta_2, \delta_2) - (\gamma, \delta_2)^2 = 2(-2) - (4n + 4)^2 = (-2)(8n^2 + 16n + 10).$$

Therefore $\tilde{Y}_t$ is birational to the Hilbert square of a K3 surface $S_t$ of degree

$$d(n) = 2g(n) - 2 = 8n^2 + 16n + 10 = 2(4(n + 1)^2 + 1).$$

This proves Theorem 3.1.

Remark 3.5. Let $NS(S^{[2]}_t) = \mathbb{Z}h_2 + \mathbb{Z}\delta_2$, where $h_2$ is the primitive polarization class on $S^{[2]}_t$. By using that

$$NS(S^{[2]}_t) \cong NS(\tilde{Y}_t) \cong \Pi,$$
we can compute directly the degree \(d(n) = (h_2, h_2)\) of the K3 surface \(S_t\). Since \(h_2\) is primitive and orthogonal to the half-diagonal class \(\delta_2\), and since
\[
\Pi \cap \delta_2^2 = Z(\gamma + (2n + 2)\delta_2),
\]
then \(h_2 = \gamma + (2n + 2)\delta_2\). From here, and by the intersection table from Lemma 3.4 we get again
\[
d(n) = (h_2, h_2) = (\gamma, \gamma) + 2(2n + 2)(\gamma, \delta_2) + (2n + 2)^2(-2) =
2 + 8(n + 1)^2 = 8n^2 + 16n + 10.
\]
The intersection matrix of \(\Pi\) in the base \(h_2, \delta_2\), is
\[
\begin{pmatrix}
h_2 & \delta_2 \\
\delta_2 & d(n)
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
0 & -2
\end{pmatrix}.
\]

**Remark 3.6.** Theorem 3.1 implies that on the Hilbert square \(S^{[2]}\) of a general K3 surface \(S\) of degree \(d(n) = 8n^2 + 16n + 10\), \(n \geq 1\) the EPW polarization \(\gamma = h_2 - (2n + 2)\delta_2\) defines an antisymplectic birational involution.

This proves the O’Grady conjecture that on the Hilbert square of a K3 surface of genus \(g = r^2 + 2\), \(r \geq 0\) there exists an antisymplectic involution (see (4.3.3) in [OG1]), in the case when \(r\) is even. Indeed, for \(d(n) = 8n^2 + 16n + 10 = 2(4(n + 1)^2 + 1)\), the genus of \(S\) is \(g(n) = d(n)/2 + 1 = 4n^2 + 8n + 6 = (2n + 2)^2 + 2\), and for \(n \geq 1\), \(r = 2n + 2\) covers all even numbers \(r \geq 4\). The case \(r = 0\) is well known, and the case \(r = 2\) is studied in detail by O’Grady, see e.g. §4.3 in [OG1]. The odd case \(r = 1\) is studied in [De] and [Fe].

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