WIENER’S LEMMA FOR INFINITE MATRICES II

QIYU SUN

Abstract. In this paper, we introduce a class of infinite matrices related to the Beurling algebra of periodic functions, and we show that it is an inverse-closed subalgebra of \( B(\ell^q_w) \), the algebra of all bounded linear operators on the weight sequence space \( \ell^q_w \), for any \( 1 \leq q < \infty \) and any discrete Muckenhoupt \( A_q \)-weight \( w \).

1. Introduction

Let us begin the sequel to \[43\] by introducing a new class of infinite matrices,

(1.1) \( B(\mathbb{Z}^d, \mathbb{Z}^d) := \{ A := (a(i, j))_{i, j \in \mathbb{Z}^d} \mid \|A\|_{B(\mathbb{Z}^d, \mathbb{Z}^d)} < \infty \} \),

where \( d \geq 1, |x|_\infty := \max(|x_1|, \ldots, |x_d|) \) for \( x := (x_1, \ldots, x_d) \in \mathbb{R}^d \), and

(1.2) \[ \|A\|_{B(\mathbb{Z}^d, \mathbb{Z}^d)} := \sum_{k \in \mathbb{Z}^d} \left( \sup_{|i-j|_\infty \geq |k|_\infty} |a(i, j)| \right). \]

It is observed that a Toeplitz matrix \( A := (a(i - j))_{i, j \in \mathbb{Z}} \) associated with a sequence \( a := (a(n))_{n \in \mathbb{Z}} \) belongs to \( B(\mathbb{Z}, \mathbb{Z}) \) if and only if the Fourier series \( \hat{a}(\xi) := \sum_{n \in \mathbb{Z}} a(n) \exp(-\sqrt{-1} n \xi) \) belongs to the algebra

(1.3) \[ A^*(\mathbb{T}) := \left\{ \sum_{n=-\infty}^{\infty} a(n) e^{-\sqrt{-1} n \xi} \mid \sum_{k=0}^{\infty} \sup_{|n| \geq k} |a(n)| < \infty \right\}. \]

The above algebra \( A^*(\mathbb{T}) \) was introduced by A. Beurling for establishing contraction properties of periodic functions \[8\], and was used in considering pointwise summability of Fourier series \[10, 15, 16, 40, 48\]. So the class \( B(\mathbb{Z}^d, \mathbb{Z}^d) \) of infinite matrices can be interpreted as a non-commutative matrix extension of the Beurling algebra \( A^*(\mathbb{T}) \).

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Define the Gröchenig-Schur class $S(\mathbb{Z}^d, \mathbb{Z}^d)$ by
\begin{align}
S(\mathbb{Z}^d, \mathbb{Z}^d) := \left\{ (a(i, j))_{i,j \in \mathbb{Z}^d} \right\} \\
\left( \sup_{i \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d} |a(i, j)|, \sup_{j \in \mathbb{Z}^d} \sum_{i \in \mathbb{Z}^d} |a(i, j)| \right) < \infty \right\}
\end{align}
(1.4)
[25, 37, 41, 43], and the Gohberg-Baskakov-Sjöstrand class $C(\mathbb{Z}^d, \mathbb{Z}^d)$ by
\begin{align}
C(\mathbb{Z}^d, \mathbb{Z}^d) := \left\{ (a(i, j))_{i,j \in \mathbb{Z}^d} \right\} \\
\left( \sum_{k \in \mathbb{Z}^d} \left( \sup_{i, j = k} |a(i, j)| \right) \right) < \infty \right\}
\end{align}
(1.5)
[6, 20, 25, 32, 39, 43]. The above two classes of infinite matrices appeared in the study of Gabor time-frequency analysis, nonuniform sampling, and algebra of pseudo-differential operators etc (see [2, 4, 24, 28, 39, 42] for a sample of papers). From (1.1), (1.4) and (1.5) it follows that
\begin{align}
B(\mathbb{Z}^d, \mathbb{Z}^d) \subset C(\mathbb{Z}^d, \mathbb{Z}^d) \subset S(\mathbb{Z}^d, \mathbb{Z}^d).
\end{align}
(1.6)
This shows that any matrix in $B(\mathbb{Z}^d, \mathbb{Z}^d)$ belongs to the Gröchenig-Schur class $S(\mathbb{Z}^d, \mathbb{Z}^d)$ and also the Gohberg-Baskakov-Sjöstrand class $C(\mathbb{Z}^d, \mathbb{Z}^d)$.

An equivalent way of defining $B(\mathbb{Z}^d, \mathbb{Z}^d)$ is the existence of a radially decreasing sequence $\{b(i)\}_{i \in \mathbb{Z}^d}$ for any infinite matrix $A := (a(i, j))_{i,j \in \mathbb{Z}^d} \in B(\mathbb{Z}^d, \mathbb{Z}^d)$ such that
\begin{align}
|a(i, j)| \leq b(i - j) \quad \text{for all } i, j \in \mathbb{Z}^d,
\end{align}
(1.7)
\begin{align}
\sum_{i \in \mathbb{Z}^d} b(i) < \infty,
\end{align}
(1.8)
and
\begin{align}
b(i) = h(|i|_\infty) \text{ for some decreasing sequence } \{h(n)\}_{n=0}^\infty.
\end{align}
(1.9)
Therefore any infinite matrix in $B(\mathbb{Z}^d, \mathbb{Z}^d)$ is dominated by a convolution operator associated with a summable, radial and (radially) decreasing sequence. We remark that any infinite matrix in the Gohberg-Baskakov-Sjöstrand class $C(\mathbb{Z}^d, \mathbb{Z}^d)$ is dominated by a convolution operator associated with a summable sequence [6, 20, 25, 32, 39, 43].

A positive sequence $w := (w(i))_{i \in \mathbb{Z}^d}$ is said to be a discrete $A_q$-weight for $1 < q < \infty$ if there exists a positive constant $C$ such that
\begin{align}
\left( \sum_{i \in \mathbb{Z}^d} w(i) \right)^{N^{-d}} \left( \sum_{i \in \mathbb{Z}^d} (w(i))^{(q-1)} \right)^{1/(q-1)} \leq C
\end{align}
(1.10)
hold for all \( a \in \mathbb{Z}^d \) and \( 1 \leq N \in \mathbb{Z} \), and to be a discrete \( A_1 \)-weight if there exists a positive constant \( C \) such that
\[
(1.11) \quad N^{-d} \sum_{i \in a + [0,N-1]^d} w(i) \leq C \inf_{i \in a + [0,N-1]^d} w(i)
\]
hold for all \( a \in \mathbb{Z}^d \) and \( 1 \leq N \in \mathbb{Z} \) \([18, 40]\). The smallest constant \( C \) for which \((1.10)\) holds when \( 1 < q < \infty \), and respectively for which \((1.11)\) holds when \( q = 1 \), to be denoted by \( A_q(w) \), is the discrete \( A_q \)-bound. The positive sequences \( ((1 + |i|_\infty^\alpha)_{i \in \mathbb{Z}^d} \) with \(-d < \alpha < d(q-1)\) if \( 1 < q < \infty \), and respectively with \(-d < \alpha \leq 0 \) if \( q = 1 \), are discrete \( A_q \)-weights.

For \( 1 < q < \infty \), a positive locally integrable function \( w \) on \( \mathbb{R}^d \) is said to be an \( A_q \)-weight if there exists a positive constant \( C \) such that
\[
(1.12) \quad \left( \frac{1}{|Q|} \int_Q w(x) dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(q-1)} dx \right) \leq C
\]
for all cubic \( Q \subset \mathbb{R}^d \), where \(|Q|\) denotes the Lebesgue measure of the cubic \( Q \) \([39]\). Similarly for \( q = 1 \), a positive locally integrable function \( w \) is said to be an \( A_1 \)-weight if there exists a positive constant \( C \) such that
\[
(1.13) \quad \frac{1}{|Q|} \int_Q w(y) dy \leq C w(x), \quad x \in Q
\]
for all cubic \( Q \subset \mathbb{R}^d \) \([14]\). One may then verify that a positive sequence \( w := (w(i))_{i \in \mathbb{Z}^d} \) is a discrete \( A_q \)-weight if and only if \( w(x) := \sum_{i \in \mathbb{Z}^d} w(i) \chi_{[-1/2,1/2]^d}(x - i) \) is an \( A_q \)-weight, where \( 1 \leq q < \infty \) and \( \chi_E \) is the characteristic function on a set \( E \) \([33]\).

For \( 1 \leq q < \infty \) and a positive sequence \( w := (w(i))_{i \in \mathbb{Z}^d} \) on \( \mathbb{R}^d \), let \( \ell^q_w := \ell^q_w(\mathbb{Z}^d) \) be the space of all weighted \( q \)-summable sequences on \( \mathbb{Z}^d \), i.e.,
\[
(1.14) \quad \ell^q_w(\mathbb{Z}^d) := \left\{ (c(i))_{i \in \mathbb{Z}^d} \mid \|c\|_{q,w} := \left( \sum_{i \in \mathbb{Z}^d} |c(i)|^q w(i) \right)^{1/q} < \infty \right\}.
\]

For the trivial weight \( w_0 \) (i.e. \( w_0(i) = 1 \) for all \( i \in \mathbb{Z}^d \)), we will use \( \ell^q \) and \( \| \cdot \|_q \) instead of \( \ell^q_{w_0} \) and \( \| \cdot \|_{q,w_0} \) for brevity. Define the discrete maximal function by
\[
(1.15) \quad M c(i) := \sup_{0 \leq N \in \mathbb{Z}} \frac{1}{(2N+1)^d} \sum_{k \in i + [-N,N]^d} \left| c(k) \right| \quad \text{for } c := (c(i))_{i \in \mathbb{Z}^d}.
\]
Similar to the characterization of an \( A_q \)-weight on \( \mathbb{R}^d \) via the standard maximal operator \([34]\), the discrete \( A_q \)-weight can be characterized via
the discrete maximal function on the weighted $\ell^q$ space. More precisely, a positive sequence $w := (w(i))_{i \in \mathbb{Z}^d}$ is a discrete $A_q$-weight if and only if the discrete maximal operator $c \mapsto M c$ is of weak-type $(\ell^q_w, \ell^q_w)$, i.e., there exists a positive constant $C$ such that

$$\sum_{M(c(i)) \geq \alpha} w(i) \leq C^\alpha \|c\|_{\ell^q_w}^q$$

for all $\alpha > 0$ and $c \in \ell^q_w$.

Moreover for $1 < q < \infty$, the discrete maximal operator $M$ is of strong type $(\ell^q_w, \ell^q_w)$ for a discrete $A_q$-weight $w$, i.e., there exists a positive constant $C'$ such that

$$\|M c\|_{\ell^q_w} \leq C'|c|_{\ell^q_w}$$

for all $c \in \ell^p_w$.

The reader may refer to [18] for a complete account of the theory of weighted inequalities and its ramification.

Now let us present our results for infinite matrices in $B(\mathbb{Z}^d, \mathbb{Z}^d)$. In Section 3 we establish the following algebraic properties for the class $B(\mathbb{Z}^d, \mathbb{Z}^d)$ of infinite matrices.

**Theorem 1.1.** Let $1 \leq q < \infty$, and let $w$ be a discrete $A_q$-weight. Then $B(\mathbb{Z}^d, \mathbb{Z}^d)$ is a unital Banach algebra under matrix multiplication, and also a subalgebra of $B(\ell^q_w, \ell^q_w)$, the algebra of all bounded linear operators on the weight sequence space $\ell^q_w$.

By Theorem 1.1 every infinite matrix $A \in B(\mathbb{Z}^d, \mathbb{Z}^d)$ defines a bounded operator on $\ell^q_w$, for any $1 \leq q < \infty$ and for any discrete $A_q$-weight $w$, i.e., there exists a positive constant $C$ such that

$$\|A c\|_{\ell^q_w} \leq C \|c\|_{\ell^q_w}$$

for all $c \in \ell^q_w$.

Besides the above boundedness of an infinite matrix on the weighted sequence space $\ell^q_w$, it is natural to consider $\ell^q_w$-stability. Here for $1 < q < \infty$, and $w$ is a discrete sequence on $\mathbb{Z}^d$, we say that an infinite matrix $A$ has $\ell^q_w$ stability if there exists a positive constant $C$ such that

$$C^{-1} \|c\|_{\ell^q_w} \leq \|A c\|_{\ell^q_w} \leq C \|c\|_{\ell^q_w}$$

for all $c \in \ell^q_w$.

The $\ell^q_w$-stability is one of the basic assumptions for infinite matrices arising in the study of spline approximation, Gabor time-frequency analysis, nonuniform sampling, and algebra of pseudo-differential operators etc (see [2, 4, 24, 28, 39, 42, 43] and the references therein.) In Section 4, we establish the equivalence of $\ell^q_w$-stabilities of any infinite matrix in $B(\mathbb{Z}^d, \mathbb{Z}^d)$ for different exponents $1 \leq q < \infty$ and for different discrete $A_q$-weights $w$. 
**Theorem 1.2.** Let \( A \in B(\mathbb{Z}^d, \mathbb{Z}^d) \). If \( A \) has \( \ell^q_w \)-stability for some \( 1 \leq q < \infty \) and for some discrete \( A_q \)-weight \( w \), then it has \( \ell^{q'}_{w'} \)-stability for all \( 1 \leq q' < \infty \) and for all discrete \( A_{q'} \)-weights \( w' \).

The reader may refer to [1, 38, 49] for the equivalence of unweighted \( \ell^q_{w} \)-stability of infinite matrices in the Gohberg-Baskakov-Sjöstrand class \( C(\mathbb{Z}^d, \mathbb{Z}^d) \). If \( A \in B(\ell^q_w) \) has a left inverse \( B \in B(\ell^q_w) \), i.e., \( BA = I \), then \( A \) has \( \ell^q_w \)-stability. The converse is not true in general, unless \( q = 2 \).

As an application of Theorem 1.2, we show that the converse holds for any infinite matrix \( A \) in \( B(\mathbb{Z}^d, \mathbb{Z}^d) \).

**Corollary 1.3.** Let \( 1 \leq q < \infty \), and let \( w \) be a discrete \( A_q \)-weight. Then an infinite matrix in \( B(\mathbb{Z}^d, \mathbb{Z}^d) \) has \( \ell^q_w \)-stability if and only if it has a left inverse in \( B(\ell^q_w) \).

Given a Banach algebra \( B \), a subalgebra \( A \) of \( B \) is said to be inverse-closed if \( A \in A \) and the inverse \( A^{-1} \) of the element \( A \) exists in \( B \) implies that \( A^{-1} \in A \) [19, 35, 47]. The next question following the \( \ell^q_w \)-stability of an infinite matrix in \( B(\mathbb{Z}^d, \mathbb{Z}^d) \) is whether its inverse, if exists in \( B(\ell^q_w) \), belongs to \( B(\mathbb{Z}^d, \mathbb{Z}^d) \), or in the other word, whether \( B(\mathbb{Z}^d, \mathbb{Z}^d) \) is an inverse-closed subalgebra of \( B(\ell^q_w) \).

The inverse-closedness for the subalgebra of absolutely convergent Fourier series in the algebra of bounded periodic functions was first studied in [9, 19, 35, 50]. The inverse-closed property (=Wiener’s lemma) has been established for infinite matrices satisfying various off-diagonal decay conditions, see [3, 5, 6, 17, 23, 25, 29, 39, 41, 43] for a sample of papers. Inverse-closedness occurs under various names (such as spectral invariance, Wiener pair, local subalgebra) in many fields of mathematics, see the survey [21].

The inverse-closed property for non-commutative matrix subalgebra has been shown to be crucial for the well-localization of dual wavelet frames and dual Gabor frames [4, 24, 29, 30, 31], the algebra of pseudo-differential operators [22, 28, 39], the fast implementation in numerical analysis [12, 13, 27], and the local reconstruction in sampling theory [2, 42, 45].

It mixes art and hard mathematical work to consider the inverse-closed subalgebra of \( B(\ell^q_w) \). The art is to guess the off-diagonal decay of infinite matrices in an algebra \( A \), while the work is to prove the inverse-closedness of the algebra \( A \) in \( B(\ell^q_w) \). There are several approaches to prove the inverse-closedness of a subalgebra of \( B(\ell^q_w) \). Here are three of them: (i) the indirect approach, such as the Gelfand’s technique [17, 19, 23]; (ii) the semi-direct approach, such as the bootstrap
argument [29] and the derivation trick [23]; (iii) the direct approach, such as the commutator trick [6, 39] and the asymptotic estimate technique [38, 41, 43]. Each approach has its advantages and limitations. For instance, the Gelfand technique and the asymptotic estimate technique work well for inverse-closed subalgebras of $B(\ell^q)$ with $q = 2$, but they are not directly applicable for inverse-closed subalgebras of $B(\ell^q)$ with $q \neq 2$. The commutator trick is applicable to establish Wiener’s lemma for subalgebra of $B(\ell^q)$, $1 \leq q \leq \infty$ [6, 39]. In Section 5, we combine the commutator trick, the asymptotic estimate technique and the equivalence of $\ell^q_w$-stability for different exponents $q$ and for different discrete $A_q$-weights $w$, and then establish Wiener’s lemma for subalgebras of $B(\ell^q_w)$, where $1 \leq q < \infty$ and $w$ is a discrete $A_q$-weight.

**Theorem 1.4.** Let $1 \leq q < \infty$ and let $w$ be a discrete $A_q$-weight. Then $B(\mathbb{Z}^d, \mathbb{Z}^d)$ is an inverse-closed subalgebra of $B(\ell^q_w)$.

As an application of Theorem 1.4, we obtain Wiener’s lemma for the Beurling algebra $A^*(\mathbb{T})$ of periodic functions [7].

**Corollary 1.5.** If $f \in A^*(\mathbb{T})$ and $f(\xi) \neq 0$ for all $\xi \in \mathbb{R}$ then $1/f \in A^*(\mathbb{T})$.

As applications of Theorems 1.2 and 1.4, we establish the equivalence between the $\ell^q_w$-stability of an infinite matrix in $B(\mathbb{Z}^d, \mathbb{Z}^d)$ and the existence of its left inverse in $B(\mathbb{Z}^d, \mathbb{Z}^d)$.

**Corollary 1.6.** Let $1 \leq q \leq \infty$, and let $w$ be a discrete $A_q$-weight. Then an infinite matrix in $B(\mathbb{Z}^d, \mathbb{Z}^d)$ has $\ell^q_w$-stability if and only if it has a left inverse in $B(\mathbb{Z}^d, \mathbb{Z}^d)$.

2. A class of infinite matrices

In this section, we introduce a class of infinite matrices with off-diagonal decay, which includes the class $B(\mathbb{Z}^d, \mathbb{Z}^d)$ in the Introduction as a special case.

A weight matrix on $\mathbb{Z}^d \times \mathbb{Z}^d$, or a weight matrix for brevity, is a positive matrix $u := (u(i, j))_{i, j \in \mathbb{Z}^d}$ with each entry not less than one, i.e.,

$$u(i, j) \geq 1 \quad \text{for all } i, j \in \mathbb{Z}^d.$$  

(2.1)

For $1 \leq p \leq \infty$ and a weight matrix $u := (u(i, j))_{i, j \in \mathbb{Z}^d}$, define

$$B_{p, u}(\mathbb{Z}^d, \mathbb{Z}^d) := \left\{ A := (a(i, j))_{i, j \in \mathbb{Z}^d} \mid \| A \|_{B_{p, u}(\mathbb{Z}^d, \mathbb{Z}^d)} < \infty \right\},$$  

(2.2)
where
\begin{equation}
\|A\|_{\mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)} := \left\| \left( \sup_{|i-j|\leq |k|} |a(i, j)|u(i, j) \right)_{k \in \mathbb{Z}^d} \right\|_p.
\end{equation}

If \( p = 1 \) and \( u \equiv 1 \) (i.e., all entries of the weight matrix \( u \) are equal to 1), then
\begin{equation}
\mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) = \mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d) \quad \text{and} \quad \| \cdot \|_{\mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)} = \| \cdot \|_{\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)}.
\end{equation}

In this paper, we use \( \mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) \), \( \| \cdot \|_{\mathcal{B}_{p,u}} \) instead of \( \mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d) \), \( \mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d) \), \( \| \cdot \|_{\mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)} \), \( \| \cdot \|_{\mathcal{B}(\mathbb{Z}^d, \mathbb{Z}^d)} \) for brevity.

**Remark 2.1.** Let \( 1 \leq p \leq \infty \) and \( u \) be a weighted matrix. Define the Gröchenig-Schur class \( \mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) \) of infinite matrices by
\begin{equation}
\mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) := \left\{ A := (a(i, j))_{i, j \in \mathbb{Z}^d} \mid \|A\|_{\mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)} < \infty \right\},
\end{equation}
where
\begin{equation}
\|A\|_{\mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)} := \max \left( \sup_{i \in \mathbb{Z}^d} \left\| (a(i, j)u(i, j))_{j \in \mathbb{Z}^d} \right\|_p, \sup_{j \in \mathbb{Z}^d} \left\| (a(i, j)u(i, j))_{i \in \mathbb{Z}^d} \right\|_p \right).
\end{equation}

For \( p = 1 \), the class \( \mathcal{S}_{1,u}(\mathbb{Z}^d, \mathbb{Z}^d) \) were introduced by Schur \( \cite{37} \) for weight matrices \( u := (w(i)/w(j))_{i, j \in \mathbb{Z}^d} \) generated by positive sequences \( w := (w(i))_{i \in \mathbb{Z}^d} \), and by Gröchenig and Leinert \( \cite{25} \) for weight matrices \( u := (v(i - j))_{i, j \in \mathbb{Z}^d} \) associated with positive functions \( v \) on \( \mathbb{R}^d \). For \( 1 \leq p \leq \infty \), the class \( \mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) \) was introduced by Sun \( \cite{41} \) for polynomial weights \( u := ((1 + |i - j|)\alpha)_{i, j \in \mathbb{Z}^d} \) with \( \alpha > d(1 - 1/p) \) in \( \cite{41} \) and for any weighted matrix \( u \) in \( \cite{43} \). From the above definition of the Gröchenig-Schur class \( \mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) \), the following inclusion follows:
\begin{equation}
\mathcal{B}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) \subset \mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)
\end{equation}
for any \( 1 \leq p \leq \infty \) and for any weight matrix \( u \).

**Remark 2.2.** Let \( 1 \leq p \leq \infty \) and \( u \) be a weighted matrix. Define the Gohberg-Baskakov-Sjöstrand class \( \mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) \) of infinite matrices by
\begin{equation}
\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) := \left\{ A := (a(i, j))_{i, j \in \mathbb{Z}^d} \mid \|A\|_{\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)} < \infty \right\},
\end{equation}
where
\begin{equation}
\|A\|_{\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)} := \left\| \left( \sup_{i-j=k} (|a(i, j)|u(i, j)) \right)_{k \in \mathbb{Z}^d} \right\|_p.
\end{equation}

For \( p = 1 \) and the trivial weight matrix \( u_0 \) (i.e., \( u_0(i, j) = 1 \) for all \( i, j \in \mathbb{Z}^d \)), the class \( \mathcal{C}_{1,u_0}(\mathbb{Z}^d, \mathbb{Z}^d) \) was introduced by Gohberg, Kaashok, and Woerdeman \( \cite{20} \) as a generalization of
the class of Toeplitz matrices associated with summable sequences. It was reintroduced by Sjöstrand [39] in considering algebra of pseudodifferential operators. For $p = 1$ and nontrivial weight matrices $u := (v(i - j))_{i,j \in \mathbb{Z}^d}$ associated with positive functions $v$ on $\mathbb{R}^d$, the class $C_{1,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ was introduced and studied by Baskakov [6] and Kurbatov [32] independently, see also [25]. The above definition of the Gohberg-Baskakov-Sjöstrand class $C_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ is given by Sun [43] for any $1 \leq p \leq \infty$ and any weight matrix $u$. From the definition of the Gohberg-Baskakov-Sjöstrand class $C_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$, we have the following inclusion:

\[(2.10) \quad B_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) \subset C_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)\]

for any $1 \leq p \leq \infty$ and for any weight matrix $u$.

**Remark 2.3.** The inclusions (2.7) and (2.10) become equalities for $p = \infty$, i.e.,

\[(2.11) \quad B_{\infty,u}(\mathbb{Z}^d, \mathbb{Z}^d) = C_{\infty,u}(\mathbb{Z}^d, \mathbb{Z}^d) = S_{\infty,u}(\mathbb{Z}^d, \mathbb{Z}^d) =: J_u(\mathbb{Z}^d, \mathbb{Z}^d)\]

The class $J_u(\mathbb{Z}^d, \mathbb{Z}^d)$ of infinite matrices is usually known as the Jaffard class [6, 13, 23, 25, 29, 41, 43]. The Jaffard class $J_u(\mathbb{Z}^d, \mathbb{Z}^d)$ with polynomial weight $u := ((1 + |i - j|)^{\alpha})_{i,j \in \mathbb{Z}^d}$ was introduced by Jaffard [29] in considering wavelets on an open domain. The Jaffard class $J_u(\mathbb{Z}^d, \mathbb{Z}^d)$ with weight matrices $u := (v(i - j))_{i,j \in \mathbb{Z}^d}$ associated with positive functions $v$ on $\mathbb{R}^d$ was introduced by Baskakov [6] independently, and later applied nontrivially in the study of localization of frames [25], adaptive computation [13], and nonuniform sampling [42].

For the class $B_{p,u}$ of infinite matrices, we have the following proposition.

**Proposition 2.4.** Let $\alpha \in \mathbb{C}$, $1 \leq p \leq \infty$, $u$ be a weight matrix, and let $A := ((a(i, j)))_{i,j \in \mathbb{Z}^d}$ and $B := (b(i, j))_{i,j \in \mathbb{Z}^d}$ belong to $B_{p,u}$. Then

(i) $\|A + B\|_{B_{p,u}} \leq \|A\|_{B_{p,u}} + \|B\|_{B_{p,u}}$.

(ii) $\|\alpha A\|_{B_{p,u}} = |\alpha| \|A\|_{B_{p,u}}$.

(iii) $\|A^*\|_{B_{p,u}} = \|A\|_{B_{p,u}}$ where $A^* := (\overline{a(j, i)})_{i,j \in \mathbb{Z}^d}$ is the conjugate transpose of the matrix $A$.

(iv) $\|A\|_{B_{p,u}} \leq \|B\|_{B_{p,u}}$ if $|A| \leq |B|$, i.e., $|a(i, j)| \leq |b(i, j)|$ for all $i, j \in \mathbb{Z}^d$.

All conclusions in the above proposition follow directly from (2.2) and (2.5). From the conclusions (i) and (ii) of the above proposition, we see that $\| \cdot \|_{B_{p,u}}$ is a norm on the class $B_{p,u}$ of infinite matrices. The properties in the conclusion (iv) is usually known as the *solidness* of the matrix norm $\| \cdot \|_{B_{p,u}}$. 
3. Algebraic properties

In this section, we establish some algebraic properties for the class $\mathcal{B}_{p,u}$ of infinite matrices and give a proof of Theorem 1.1.

Let us first recall the concept of a $p$-submultiplicative weight matrix $u$ [25, 43, 44]. For $1 \leq p \leq \infty$, a weight matrix $u := (u(i,j))_{i,j \in \mathbb{Z}^d}$ is said to $p$-submultiplicative if there exists another weight matrix $v := (v(i,j))_{i,j \in \mathbb{Z}^d}$ such that

\begin{align}
(3.1) & \quad v(i,j) \geq 1 \text{ for all } i,j \in \mathbb{Z}^d, \\
(3.2) & \quad u(i,j) \leq u(i,k)v(k,j) + v(i,k)u(k,j) \text{ for all } i,j,k \in \mathbb{Z}^d, \\
(3.3) & \quad C_p(v,u) := \left\| \sup_{|i-j|, |k|} \left( v(i,j)(u(i,j))^{-1} \right) \right\|_{p/(p-1)} < \infty.
\end{align}

For $p = 1$, we simply say that a weight matrix is submultiplicative instead of 1-submultiplicative. We call the weight matrix $v$ satisfying (3.1), (3.2) and (3.3) a companion weight matrix of the $p$-submultiplicative weight matrix $u$. Denote by $C(u)$ the set of all companion weights of a $p$-submultiplicative weight matrix $u$, and define the $p$-submultiplicative bound $M_p(u)$ by

\begin{equation}
M_p(u) := \inf_{v \in C(u)} C_p(v,u).
\end{equation}

One may verify that $C(u)$ is a convex set and the infimum of $C_p(v,u)$ in the set $C(u)$ can be attained for some companion weight matrix $v$. So from now on, except stated explicitly, we always assume that the companion weight $v$ of a $p$-submultiplicative weight matrix $u$ is the one satisfying

\begin{equation}
M_p(u) = C_p(v,u).
\end{equation}

Remark 3.1. From the definitions of $p$-submultiplicative weight matrices on $\mathbb{Z}^d \times \mathbb{Z}^d$, we have the following:

(i) A $p$-submultiplicative weight matrix is $q$-submultiplicative for all $1 \leq q \leq p$.

(ii) A necessary condition for a weight matrix $u := (u(i,j))_{i,j \in \mathbb{Z}^d}$ to be $p$-submultiplicative is $u(i,j) \leq Cu(i,k)u(k,j)$ for all $i,j,k \in \mathbb{Z}^d$ and for some positive constant $C$. When $p = 1$, the above necessary condition is also a sufficient condition [25].

(iii) Let $1 \leq p \leq \infty, \delta \in (0,1)$, and let $\alpha$ be a number with the property that $\alpha > d - d/p$ if $1 < p \leq \infty$, and $\alpha \geq 0$ if $p = 1$. Then the Toeplitz matrices $p_\alpha := (1 + |i-j|\delta)^\alpha)_{i,j \in \mathbb{Z}^d}$
generated by the polynomial weight \((1 + |x|_\infty)^\alpha\), and \(e_\delta := \left(\exp(|i - j|_\infty^\delta)\right)_{i,j \in \mathbb{Z}^d}\) generated by the sub-exponential weight \(\exp(|x|_\infty^\delta)\), are \(p\)-submultiplicative \([43]\).

Now we state the main result of this section, an extension of Theorem 1.1.

**Theorem 3.2.** Let \(1 \leq p, q \leq \infty\), \(u\) be a \(p\)-submultiplicative weight matrix with the \(p\)-submultiplicative bound \(M_p(u)\), and let \(w\) be a discrete \(A_q\)-weight with the \(A_q\)-bound \(A_q(w)\). Then the following statements hold.

(i) If \(v\) is a companion weight matrix of the \(p\)-submultiplicative weight matrix \(u\), then

\[
\|AB\|_{B_{p,u}} \leq 2^{\frac{d}{p}}\left(\|A\|_{B_{p,u}}\|B\|_{B_{1,v}} + \|A\|_{B_{1,v}}\|B\|_{B_{p,u}}\right)
\]

for all \(A, B \in B_{p,u}\).

(ii) \(B_{p,u}\) is (and hence \(B\) is also) an algebra. Moreover

\[
\|AB\|_{B_{p,u}} \leq 2^{\frac{d}{p}}\left(\frac{1}{M_p(u)}\right)\|A\|_{B_{p,u}}\|B\|_{B_{p,u}}
\]

for all \(A, B \in B_{p,u}\).

(iii) \(B_{p,u}\) is a subalgebra of \(B\). Moreover

\[
\|A\|_{B_{p,u}} \leq M_p(u)\|A\|_{B_{p,u}}
\]

for all \(A \in B_{p,u}\).

(iv) \(B_{p,u}\) is (and hence \(B\) is also) a subalgebra of \(B(\ell^q_w)\). Moreover

\[
\|Ac\|_{q,w} \leq 2^{\frac{d}{p}+\frac{3}{q}}\left(\frac{1}{A_q(w)}\right)^{\frac{1}{q}}\|A\|_{B_{p,u}}\|c\|_{q,w}
\]

for all \(A \in B_{p,u}\) and \(c \in \ell^q_w\).

Before we give the proof of the above theorem, let us next have some remarks on the unital Banach algebra property of the algebra \(B_{p,u}\), on the equality of spectral radii in the algebras \(B_{p,u}\) and \(B_{1,v}\), and on the inclusion \(B_{p,u} \subset B(\ell^q_w)\).

**Remark 3.3.** For \(1 \leq p < \infty\) and a \(p\)-submultiplicative weight matrix \(u := (u(i, j))_{i, j \in \mathbb{Z}^d}\), following the standard procedure \([19, 35]\) we define \(\|A\|_{B_{p,u}} := \sup_{\|B\|_{B_{p,u}} = 1} \|AB\|_{B_{p,u}}\) for \(A \in B_{p,u}\). Then

\[
\|AB\|_{B_{p,u}} \leq \|A\|_{B_{p,u}}\|B\|_{B_{p,u}}\|E_{p,u}\|
\]

for all \(A, B \in B_{p,u}\).

If the weight matrix \(u\) further satisfies

\[
M := \sup_{i \in \mathbb{Z}^d} u(i, i) < \infty,
\]

then the identity matrix \(I\) belongs to \(B_{p,u}\), and the norms \(\|\cdot\|_{B_{p,u}}\) and \(\|\cdot\|_{B_{p,u}}\) on \(B_{p,u}\) are equivalent to each other, because

\[
M^{-1}\|A\|_{B_{p,u}} \leq \|A\|_{B_{p,u}} \leq 2^{\frac{d}{p}+\frac{3}{q}}\left(\frac{1}{M_p(u)}\right)\|A\|_{B_{p,u}}
\]

for all \(A \in B_{p,u}\).
by the conclusion (ii) of Theorem 3.2 and the fact that \( \|I\|_{B_{p,u}} = M \).
Therefore if \( 1 \leq p \leq \infty \) and \( u \) is a \( p \)-submultiplicative weight matrix satisfying (3.11), then the class \( B_{p,u} \) of infinite matrices endowed with the norm \( \| \cdot \|_{B_{p,u}} \) becomes a unital Banach algebra.

**Remark 3.4.** Let \( 1 \leq p \leq \infty \), \( u \) be a \( p \)-submultiplicative weight matrix, and \( v \) be its companion weight matrix. If the companion weight matrix \( v \) is submultiplicative, then both \( B_{p,u} \) and \( B_{1,v} \) are algebras by the conclusion (ii) of Theorem 3.2 and \( B_{p,u} \) is a subalgebra of \( B_{1,v} \) since

\[
\|A\|_{B_{1,v}} \leq C_p(v,u)\|A\|_{B_{p,u}} \quad \text{for all} \ A \in B_{p,u}.
\]

Applying (3.12) with \( A \) replaced by \( A^n \) and then taking \( n \)-th roots and the limit as \( n \to \infty \) yields

\[
\rho_{B_{1,v}}(A) := \lim_{n \to \infty} \sup(\|A^n\|_{B_{1,v}})^{1/n} \leq \lim_{n \to \infty} \sup(\|A^n\|_{B_{p,u}})^{1/n} =: \rho_{B_{p,u}}(A).
\]

Applying the conclusion (i) of Theorem 3.2 gives

\[
\|A^n\|_{B_{p,u}} \leq 2^{1+2/p} \|A^n\|_{B_{p,u}} \|A^n\|_{B_{1,v}},
\]

and then taking \( n \)-th roots and letting \( n \to \infty \) lead to the inequality

\[
\rho_{B_{p,u}}(A) \leq \rho_{B_{1,v}}(A).
\]

This concludes that if \( u \) is a \( p \)-submultiplicative weight matrix and its companion weight matrix \( v \) is submultiplicative, then the spectral radii \( \rho_{B_{p,u}}(A) \) and \( \rho_{B_{1,v}}(A) \) are the same for any \( A \in B_{p,u} \), i.e., \( \rho_{B_{1,v}}(A) = \rho_{B_{p,u}}(A) \) for all \( A \in B_{p,u} \). The above procedure to establish the equality of spectral radii in the algebras \( B_{p,u} \) and \( B_{1,v} \) from the inequality in the conclusion (i) of Theorem 3.2 is known as Brandenburg’s trick [11, 23]. Another technique to prove the equality of spectral radii in two algebras \( A_1 \) and \( A_2 \) with the same identity is by showing that

\[
\|A\|_{A_2} \leq C\|A\|_{A_1}
\]

and

\[
\|A^n\|_{A_1} \leq C\|A\|_{A_2}^{1+\theta}\|A^n\|_{A_2}^{1-\theta} \quad \text{for all} \ A \in A_1,
\]

where \( \| \cdot \|_{A_1} \) and \( \| \cdot \|_{A_2} \) are norms in the algebra \( A_1 \) and \( A_2 \) respectively, and where \( C \in (0, \infty) \) and \( \theta \in [0, 1) \) are constants independent of \( A \in A \). The estimates in (3.14) and (3.15) for \( A_2 = B(\ell^2) \) and \( A_1 = S_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) \) or \( C_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d) \) are established in [11, 43], while the ones for \( A_2 = B(\ell^2) \) and \( A_1 = B_{p,u}, 1 \leq p \leq \infty \), are given in Lemma 5.3.

**Remark 3.5.** The conclusion (iv) of Theorem 3.2 about the boundedness of an infinite matrix in \( B \) on the weight sequence space \( \ell^p_w \) is a simplified discrete version of the second conclusion in [10, Proposition 2 of Chapter 10]. The reader may refer to [26, Lemma 3.1] for a general result on the boundedness of an infinite matrix on sequence spaces.
We conclude this section by giving the proof of Theorem 3.2.

**Proof of Theorem 3.2.** (i): Let $1 \leq p \leq \infty$, $u$ be a $p$-submultiplicative weight matrix, and let $v$ be a companion weight matrix of the weight matrix $u$. Take $A := (a(i, j))_{i,j \in \mathbb{Z}^d}$ and $B := (b(i, j))_{i,j \in \mathbb{Z}^d}$ in $\mathcal{B}_{p,u}$, and write $AB := (c(i, j))_{i,j \in \mathbb{Z}^d}$. Then it follows from (3.2) that

$$|c(i, j)|u(i, j) = \left| \sum_{k \in \mathbb{Z}^d} a(i, k)b(k, j) \right|u(i, j)$$

$$\leq \sum_{k \in \mathbb{Z}^d} |a(i, k)||u(i, k)||b(k, j)||v(k, j)\right|$$

(3.16)

$$+ \sum_{k \in \mathbb{Z}^d} |a(i, k)||v(i, k)||b(k, j)||u(k, j)\right|$$

for all $i, j \in \mathbb{Z}^d$.

For $1 \leq p < \infty$, we obtain from (3.16) that

$$\sum_{k \in \mathbb{Z}^d} |a(i, k)||u(i, k)||b(k, j)||v(k, j)\right|$$

$$\leq \left( \sum_{k' \in \mathbb{Z}^d} (|a(i, k')||u(i, k')|^p||b(k', j)||v(k', j)) \right)^{1/p}$$

$$\times \left( \sum_{k'' \in \mathbb{Z}^d} |b(k'', j)||v(k'', j)) \right)^{(p-1)/p}$$

$$\leq (\|B\|_{\mathcal{B}_{1,v}})^{(p-1)/p} \left\{ \left( \sum_{k' \in \mathbb{Z}^d} |b(k', j)||v(k', j)\right) \right\}$$

$$\times \left( \sup_{|i' - j'| \geq |i - j|/2} (|a(i', j')||u(i', j')) \right)^p$$

$$+ \left( \sup_{|i' - j'| \geq |i - j|/2} (|b(i', j')||v(i', j')) \right)$$

$$\times \left( \sum_{k' \in \mathbb{Z}^d} (|a(i, k')||u(i, k'))^p \right)^{1/p}$$

$$\leq (\|B\|_{\mathcal{B}_{1,v}})^{(p-1)/p} \left\{ \|B\|_{\mathcal{B}_{1,v}} \left( \sup_{|i' - j'| \geq |i - j|/2} (|a(i', j')||u(i', j')) \right)^p \right\}$$

$$+ (\|A\|_{\mathcal{B}_{p,u}})^p \left\{ \sup_{|i' - j'| \geq |i - j|/2} (|b(i', j')||v(i', j'))) \right\}^{1/p}.$$
and
\[
\sum_{k \in \mathbb{Z}^d} |a(i, k)|v(i, k)|b(k, j)|u(k, j)
\leq \left( \|A\|_{\mathcal{B}_1,v} \right)^{(p-1)/p} \left\{ \|A\|_{\mathcal{B}_1,v} \left( \sup_{i'-j' \geq |i-j|/2} |b(i', j')|u(i', j')|^p \right) \\
+ \left( \|B\|_{\mathcal{B}_{p,u}} \right)^p \left( \sup_{i'-j' \geq |i-j|/2} |a(i', j')|v(i', j') \right) \right\}^{1/p}.
\]

Combining the above two estimates with (3.16) leads to
\[
\|AB\|_{\mathcal{B}_{p,u}} = \left( \sup_{i'-j' \geq |i-j|/2} (c(i, j)|u(i, j)) \right) \leq \left( \|B\|_{\mathcal{B}_1,v} \right)^{(p-1)/p} \left\{ \|B\|_{\mathcal{B}_1,v} \left( \sup_{i'-j' \geq |i-j|/2} |b(i', j')|u(i', j')|^p \right) \right\}^{1/p} \]
\[
+ \left( \|A\|_{\mathcal{B}_{p,u}} \right)^p \left( \sup_{i'-j' \geq |i-j|/2} |a(i', j')|v(i', j') \right) \right\}^{1/p}
\leq 2^{2/p} 5^{(d-1)/p} \left( \|A\|_{\mathcal{B}_{p,u}} \|B\|_{\mathcal{B}_1,v} + \|A\|_{\mathcal{B}_1,v} \|B\|_{\mathcal{B}_{p,u}} \right),
\]
where we have used the fact that
\[
\left( \sup_{i'-j' \geq |i-j|/N} |a(i, j)| \right) \leq N(2N + 1)^{d-1} \left( \sup_{i'-j' \geq |i-j|/N} |a(i, j)| \right)
\]
for any integer \(N \geq 1\) and \(A := (a(i, j)) \in \mathcal{B}\). This proves (3.6) for \(1 \leq p < \infty\).

For \(p = \infty\), it follows from (3.16) that
\[
\|AB\|_{\mathcal{B}_{\infty,u}} \leq \|A\|_{\mathcal{B}_{\infty,u}} \|B\|_{\mathcal{B}_1,v} + \|A\|_{\mathcal{B}_1,v} \|B\|_{\mathcal{B}_{\infty,u}}.
\]
Hence (3.6) for \(p = \infty\) is proved.

(ii) Let \(v\) be the companion weight matrix of the \(p\)-submultiplicative weight \(u\) that satisfies (3.1)–(3.3) and (3.3). Then
\[
\|A\|_{\mathcal{B}_{1,v}} \leq M_p(u) \|A\|_{\mathcal{B}_{p,u}} \quad \text{for all } A \in \mathcal{B}_{p,u},
\]
because
\[
\sup_{|i-j| \geq |k|} |a(i, j)| v(i, j) \leq \left( \sup_{|i'-j'| \geq |k|} |a(i', j')| u(i', j') \right) \\
\times \left( \sup_{|i'-j'| \geq |k|} v(i', j') (u(i', j'))^{-1} \right)
\]
(3.20)
hold for all \( k \in \mathbb{Z}^d \). Combining (3.6) and (3.19) proves (3.7).

(iii) Let \( v \) be the companion weight matrix of the \( p \)-submultiplicative weight \( u \) that satisfies (3.1)–(3.3) and (3.5). Then

\[
\|A\|_{B} \leq \|A\|_{B_1,v} \quad \text{for all } A \in B_1,v
\]
by (3.1) for the weight matrix \( v \). This together with (3.19) gives (3.8) and hence proves the conclusion (iii).

(iv) By (iii), it suffices to prove

\[
\|Ac\|_{q,w} \leq 2^{2d} 3^{d/q} (A_q(w))^{1/q} \|A\|_{B} \|c\|_{q,w}
\]
for all \( A := (a(i, j))_{i,j \in \mathbb{Z}^d} \in B \) and \( c \in \ell^q_w \). Set \( h(n) := \sup_{|i-j| \geq n} |a(i, j)| \).

Then \( \{h(n)\}_{n=0}^{\infty} \) is a decreasing sequence, i.e., \( h(n+1) \leq h(n) \) for all \( n \geq 0 \), and

\[
\sum_{l=1}^{\infty} h(2^{l-1}) 2^{(l+1)d} \leq 2^{2d} h(1) + 2^{d+2} \sum_{l=2}^{\infty} \left( \sum_{2^{l-2} < s \leq 2^{l-1}} h(s) \right) 2^{(d-1)}
\]
\[
\leq 2^{2d} h(1) + 2^{2d} \sum_{s=2}^{\infty} h(s) s^{d-1}
\]
\[
\leq 2^{2d} h(1) + 2^{d-1} \sum_{s=2}^{\infty} \sum_{k \in \mathbb{Z}^d \text{ with } |k|_{\infty} = s} h(|k|_{\infty})
\]
(3.23)
\[
\leq 2^{2d} (\|A\|_{B} - h(0)).
\]
For \( 1 < q < \infty \) and a discrete \( A_q \)-weight \( w \),

\[
\|Ac\|_{q,w} \leq \left\{ \sum_{i \in \mathbb{Z}^d} \left( \sum_{j \in \mathbb{Z}^d} h(|i-j|_{\infty}) |c(j)| \right)^q w(i) \right\}^{1/q}
\]
\[
\leq h(0) \left\{ \sum_{i \in \mathbb{Z}^d} |c(i)|^q w(i) \right\}^{1/q} + \left\{ \sum_{i \in \mathbb{Z}^d} w(i) \left( \sum_{l=1}^{\infty} h(2^{l-1}) 2^{(l+1)d} \right)^{q-1} \right\}^{1/q}
\times \left( \sum_{l=1}^{\infty} h(2^{l-1}) 2^{-(l+1)d(q-1)} \left( \sum_{2^{l-1} \leq |i-j|_{\infty} < 2^l} |c(j)|^q \right) \right)^{1/q}.
\]
Thus
\[
\|Ac\|_{q,w} \leq h(0)\|c\|_{q,w} + 2^{2d(q-1)/q}(\|A\|_{B} - h(0))^{(q-1)/q} \\
\times \left\{ \sum_{i \in \mathbb{Z}^d} \sum_{l=1}^{\infty} w(i)h(2^{l-1})2^{-(l+1)d(q-1)} \\
\times \left( \sum_{2^{l-1} \leq |i-j| \leq 2^l} |c(j)|^{q} w(j) \right) \right\}^{1/q},
\]

This together with the discrete $A_q$-weight assumption leads to
\[
\|Ac\|_{q,w} \leq h(0)\|c\|_{q,w} + 2^{2d(q-1)/q}(\|A\|_{B} - h(0))^{(q-1)/q} (A_q(w))^{1/q} \\
\times \left\{ \sum_{l=1}^{\infty} h(2^{l-1})2^{(l+1)d} \sum_{i \in \mathbb{Z}^d} \frac{w(i)}{\sum_{|i-j| \leq 2^l} w(j')} \right\}^{1/q} \\
\times \left( \sum_{2^{l-1} \leq |i-j| \leq 2^l} |c(j)|^{q} w(j) \right) \right\}^{1/q},
\]

\[
\leq h(0)\|c\|_{q,w} + 2^{2d(q-1)/q}(\|A\|_{B} - h(0))^{(q-1)/q} (A_q(w))^{1/q} \\
\times \left\{ \sum_{l=1}^{\infty} h(2^{l-1})2^{(l+1)d} \left( \sum_{j \in \mathbb{Z}^d} |c(j)|^{q} w(j) \right) \right\}^{1/q} \\
\times \left( \sum_{c \in \{-1,0,1\}^d} \sum_{|i-j| \leq 2^{l-1}} \frac{w(i)}{\sum_{|i-j| \leq 2^{l-1}} w(j')} \right) \right\}^{1/q},
\]

\[
\leq 2^{2d} 3^{d/p} (A_q(w))^{1/q} \|A\|_{B} \|c\|_{q,w},
\]

and hence (3.9) for $1 < q < \infty$ is established.

The conclusion (3.9) for $q = 1$ can be proved by similar arguments. We omit the details here. □

4. $\ell^q_w$-stability

In this section, we prove the following theorem, a slight generalization of Theorem 1.2 and Corollary 1.3. We also provide a characterization to the $\ell^q_w$-stability of a Toeplitz matrix in $B$.

**Theorem 4.1.** Let $1 \leq p \leq \infty$, $1 \leq q, q' < \infty$, let the weight matrix $u$ be $p$-submultiplicative, and $w, w'$ be discrete $A_q$-weight and $A_{q'}$-weight respectively. If $A \in B_{p,u}$ has $\ell^q_w$-stability, then $A$ has $\ell^{q'}_{w'}$-stability.
As the trivial weight \( w_0 \) (i.e. \( w_0(i) = 1 \) for all \( i \in \mathbb{Z}^d \)) is a discrete \( A_q \)-weight for any \( 1 \leq q < \infty \), we have the following corollary of Theorem 4.1. Similar result about \( \ell^q \)-stability for different exponents \( q \in [1, \infty) \) is established by Aldroubi, Baskakov and Krishtal [1] for infinite matrices in the Gohberg-Baskakov-Sjöstrand class \( C_{1,p_0}(\mathbb{Z}^d, \mathbb{Z}^d) \) with \( \alpha > (d + 1)^2 \), by Tessera [49] for \( \alpha > 0 \), and by Shin and Sun [38] for \( \alpha \geq 0 \), where \( p_\alpha = ((1 + |i - j|_\infty)^\alpha)_{i,j \in \mathbb{Z}^d} \).

Corollary 4.2. If \( A \in \mathcal{B} \) has \( \ell^q \)-stability for some \( 1 \leq q < \infty \), then \( A \) has \( \ell^{q'} \)-stability for all \( 1 \leq q' < \infty \).

The \( \ell^q_w \)-stability is one of the basic assumptions for infinite matrices arising in many fields of mathematics (see [2, 4, 24, 28, 29, 39, 42, 43, 44] for a sample of papers), but little is known about practical criteria for the \( \ell^q_w \)-stability of an infinite matrix, see [46] for the diagonal-blocks-dominated criterion for the \( \ell^2 \)-stability of infinite matrices in the Gohberg-Baskakov-Sjöstrand class \( C(\mathbb{Z}^d, \mathbb{Z}^d) \). As an application of Theorem 1.2, we have the following characterization to the \( \ell^q_w \)-stability of a Toeplitz matrix in \( \mathcal{B} \).

Corollary 4.3. Let \( 1 \leq q < \infty \), \( A := (a(i - j))_{i,j \in \mathbb{Z}^d} \) be a Toeplitz matrix in \( \mathcal{B} \), and let \( w \) be a discrete \( A_q \)-weight. Then \( A \) has \( \ell^q_w \)-stability if and only if \( \hat{a}(\xi) := \sum_{n \in \mathbb{Z}^d} a(n) e^{-\sqrt{-1} \pi n^t \xi} \neq 0 \) for all \( \xi \in \mathbb{R}^d \).

To prove Theorem 4.1 we recall a characterization for discrete \( A_q \)-weights.

Lemma 4.4. [18, 40] Let \( 1 \leq q < \infty \). Then \( w := (w(i))_{i \in \mathbb{Z}^d} \) is a discrete \( A_q \)-weight with the \( A_q \)-bound \( A_q(w) \) if and only if

\[
\left( N^{-d} \sum_{i \in a + [0,N-1]^d} |c(i)| \right)^q \left( N^{-d} \sum_{i \in a + [0,N-1]^d} w(i) \right) \leq A_q(w) N^{-d} \sum_{i \in a + [0,N-1]^d} |c(i)|^q w(i)
\]

(4.1) hold for all \( a \in \mathbb{Z}^d \), \( 1 \leq N \in \mathbb{Z} \) and sequences \( c := (c(i))_{i \in \mathbb{Z}^d} \).

To prove Theorem 4.1 we need a technical lemma about estimating a bounded sequence \( c \) via the sequence \( Ac \), which will also be used later in the proof of Theorem 1.4. Similar estimate is given in [39] when the infinite matrix \( A \) belongs to the Gohberg-Baskakov-Sjöstrand class \( C(\mathbb{Z}^d, \mathbb{Z}^d) \) and has \( \ell^p_w \)-stability for the trivial weight \( w \equiv 1 \).

Lemma 4.5. Let \( 1 \leq q < \infty \), and \( w \) be a discrete \( A_q \)-weight. If \( A \in \mathcal{B} \) has \( \ell^q_w \)-stability, then there exists a nonnegative sequence \( \{g(i)\}_{i \in \mathbb{Z}^d} \) on
such that

(4.2) \[ \sum_{k \in \mathbb{Z}^d} \left( \sup_{|i|_\infty \geq |k|_\infty} g(i) \right) < \infty \]

and

(4.3) \[ |c(i)| \leq \sum_{j \in \mathbb{Z}^d} g(i - j) |(Ac)(j)|, \ i \in \mathbb{Z}^d, \]

where \( c \in \ell^\infty. \)

**Proof.** Without loss of generality, we assume that

(4.4) \[ \|c\|_{q,w} \leq \|Ac\|_{q,w} \quad \text{for all } c \in \ell_q^q. \]

Let \( h(x) = \min(\max(2 - |x|_\infty, 0), 1) \) and \( N \) be a sufficiently large integer chosen later. Define linear operators \( \Psi^N_n, n \in N\mathbb{Z}^d, \) on \( \ell^q_w \) by

(4.5) \[ \Psi^N_n c := \left( h\left( \frac{j - n}{N} \right) c(j) \right)_{j \in \mathbb{Z}^d} \quad \text{for } c := (c(j))_{j \in \mathbb{Z}^d} \in \ell^q_w. \]

Then for \( c := (c(j))_{j \in \mathbb{Z}^d} \in \ell^q_w \) and \( |n - n'|_\infty \leq 8N, \)

\[
\| (\Psi^N_n A - A\Psi^N_n) \|_{q,w} c \|_{q,w} = \left\{ \sum_{i \in \mathbb{Z}^d} \left| \sum_{j \in \mathbb{Z}^d} \left( h\left( \frac{i - n}{N} \right) - h\left( \frac{j - n}{N} \right) \right) \right| \times \left| \frac{j - n'}{N} \right|^q w(i) \right\}^{1/q} \leq N^{-1/2} \left\{ \sum_{i \in \mathbb{Z}^d} \left( \sum_{|i - j|_\infty \leq \sqrt{N}} |a(i,j)||c(j)| \right)^q w(i) \right\}^{1/q} + \left\{ \sum_{i \in \mathbb{Z}^d} \left( \sum_{|i - j|_\infty > \sqrt{N}} |a(i,j)||c(j)| \right)^q w(i) \right\}^{1/q} \leq \left\{ 2^{2d+2d/q} N^{-1/2} (A_q(w))^{1/q} \|A\|_B + 2^{3d+2d/q+1} (A_q(w))^{1/q} \right\} \times \left( \sum_{|k|_\infty \geq \sqrt{N}/2} \sup_{|i - j|_\infty \geq |k|_\infty} |a(i,j)| \right) \|c\|_{q,w}, \tag{4.6} \right. \]
where the last inequality follows from Theorem 3.2 and the following estimate:

\[
\sum_{k \in \mathbb{Z}^d} \sup_{|i-j|_\infty \geq \max(|k|_\infty, \sqrt{N})} |a(i, j)| \\
\leq (2\sqrt{N} + 1)^d \sup_{|i-j|_\infty \geq \sqrt{N}} |a(i, j)| + \sum_{|k|_\infty > \sqrt{N}} \sup_{|i-j|_\infty \geq |k|_\infty} |a(i, j)| \\
\leq 2^{d+1} \sum_{|k|_\infty \geq \sqrt{N}/2} \sup_{|i-j|_\infty \geq |k|_\infty} |a(i, j)|.
\]

Similarly for \( c := (c(j))_{j \in \mathbb{Z}^d} \in \ell^q_w \) and \( |n - n'|_\infty > 8N \),

\[
\|(\Psi_n^N A - A\Psi_n^N)\Phi_n^N c\|_{q,w} \\
= \left( \sum_{i \in \mathbb{Z}^d} \left| \sum_{j \in \mathbb{Z}^d} h(i - n\frac{j}{N}) a(i, j) h(j - n\frac{n'}{N}) c(j) \right|^q w(i) \right)^{1/q} \\
\leq \left( \sup_{|i' - j'|_\infty \geq |n - n'|_\infty/2} |a(i', j')| \right) \left( \sum_{|i-n|_\infty < 2N} \left( \sum_{|j-n'|_\infty < 2N} |c(j)| \right)^q w(i) \right)^{1/q} \\
\leq 2^{2d} N^d (A_q(w))^{1/q} \left( \sup_{|i' - j'|_\infty \geq |n - n'|_\infty/2} |a(i', j')| \right) \left( \frac{\sum_{|i' - n|_\infty < 2N} w(i')}{\sum_{|i' - n'|_\infty < 2N} w(i')} \right)^{1/q} \|c\|_{q,w}.
\]

Define

\[
\alpha_n := \sum_{|i' - n|_\infty < 2N} w(i'), \quad n \in N\mathbb{Z}^d.
\]

and the linear operator \( \Phi_N \) on \( \ell^p_w \) by

\[
\Phi_N c := \left( \sum_{n \in N\mathbb{Z}^d} \left( h(j - n\frac{n}{N}) \right)^2 \right)^{-1} c(j) \quad \text{for} \quad c := (c(j))_{j \in \mathbb{Z}^d} \in \ell^p_w.
\]

Then for all \( n' \in N\mathbb{Z}^d \) with \( |n - n'| \leq 8N \),

\[
\alpha_n \leq \sum_{|i' - n'|_\infty < 10N} w(i') \leq 6^{da} A_q(w) \alpha_{n'}
\]

by (4.11), and

\[
\|\Phi_N c\|_{q,w} \leq \|c\|_{q,w} \quad \text{for all} \quad c \in \ell^q_w.
\]

Note that \( \Psi_n^N c \in \ell^p_w \) for any \( c \in \ell^\infty \) and \( n \in N\mathbb{Z}^d \), and

\[
\|\Psi_n^N c\|_{q,w} \leq \alpha_n^{1/q} \|c\|_\infty, \quad n \in N\mathbb{Z}^d.
\]
Then for $c \in \ell^\infty$, combining (4.4), (4.6), (4.7), (4.10), and (4.11) leads to

\begin{equation}
(4.13) \quad \alpha_n^{-1/q} \| \Psi_N^N c \|_{q,w} \leq \alpha_n^{-1/q} \| A \Psi_N^N c \|_{q,w} \\
\leq \alpha_n^{-1/q} \| \Psi_N^N A c \|_{q,w} + \alpha_n^{-1/q} \| (\Psi_N^N A - A \Psi_N^N) c \|_{q,w} \\
\leq \alpha_n^{-1/q} \| \Psi_N^N A c \|_{q,w} + \alpha_n^{-1/q} \sum_{n' \in \mathbb{N}Z^d} \| (\Psi_N^N A - A \Psi_N^N) \Psi_N^N \Phi_N \Psi_N^N c \|_{q,w} \\
\leq \alpha_n^{-1/q} \| \Psi_N^N A c \|_{q,w} + 2^{2d+2d/q} 6^d (A_q(w))^{2/q} \sum_{|n' - n|_\infty \leq 8N} \alpha_n^{-1/q} \| \Psi_N^N c \|_{q,w} \\
\times \bigg\{ N^{-1/2} \| A \|_B + 2^{d+1} \sum_{|k|_\infty \leq \sqrt{N}/2} \sup_{|i' - j'|_\infty \geq |k|_\infty} |a(i', j')| \bigg\} \\
+ \sum_{|n' - n|_\infty > 8N} \alpha_n^{-1/q} \| \Psi_N^N c \|_{q,w} \\
\times \bigg\{ 2^{2d} N^d (A_q(w))^{1/q} \sup_{|i' - j'|_\infty \geq |n - n'|_\infty / 2} |a(i', j')| \bigg\} \\
=: \alpha_n^{-1/q} \| \Psi_N^N A c \|_{q,w} + \sum_{n' \in \mathbb{N}Z^d} V_N(n - n') \alpha_n^{-1/q} \| \Psi_N^N c \|_{q,w}.
\end{equation}

Define sequences $V_N^l := (V_N^l(n))_{n \in \mathbb{N}Z^d}, l \geq 1$, as follows:

\begin{equation}
(4.14) \quad \begin{cases}
V_N^0(n) := V_N(n) & \text{if } l = 1 \text{ and } n \in \mathbb{N}Z^d, \\
V_N^l(n) := \sum_{n' \in \mathbb{N}Z^d} V_N(n - n') V_N^{l-1}(n') & \text{if } l \geq 2 \text{ and } n \in \mathbb{N}Z^d.
\end{cases}
\end{equation}

Then for $c \in \ell^\infty$, applying (4.13) repeatedly yields

\begin{equation}
(4.15) \quad \alpha_n^{-1/q} \| \Psi_N^N c \|_{q,w} \\
\leq \alpha_n^{-1/q} \| \Psi_N^N A c \|_{q,w} + \sum_{l = 0}^{l_0} \sum_{n' \in \mathbb{N}Z^d} V_N^l(n - n') \alpha_n^{-1/q} \| \Psi_N^N c \|_{q,w} \\
\sum_{n' \in \mathbb{N}Z^d} V_N^{l_0+1}(n - n') \alpha_n^{-1/q} \| \Psi_N^N c \|_{q,w}, \quad l_0 \geq 1.
\end{equation}

Set

\begin{equation}
(4.16) \quad \epsilon_N^l := \sum_{k \in \mathbb{N}Z^d} \sup_{|n|_\infty \geq |k|_\infty} |V_N^l(n)|.
\end{equation}

Inductively for $l \geq 2$,

\begin{equation}
\epsilon_N^l \leq \epsilon_N^{l-1} \sum_{k \in \mathbb{N}Z^d} \sup_{|n|_\infty \geq |k|_\infty / 2} |V_N(n)| \\
+ \epsilon_N \sum_{k \in \mathbb{N}Z^d} \sup_{|n|_\infty \geq |k|_\infty / 2} |V_N^{l-1}(n)| \leq 5^d \epsilon_N \epsilon_N^{l-1},
\end{equation}

as required.
where we have used (3.17) to obtain the last inequality. This shows that

\[ (4.17) \quad \epsilon_N^l \leq (5^d \epsilon_N^1)^l \quad \text{for all } l \geq 1. \]

Note that

\[ \epsilon_N^1 \leq 2^{4d+2d/q}3^{3d}(A_q(w))^{2/q} \left\{ N^d \left( \sup_{|i'-j'|_\infty > 4N} |a(i', j')| \right) \right. \]

\[ \left. + 2^{d+1} \left( \sum_{k' \in \mathbb{Z}^d, |k'|_\infty \geq \sqrt{N}/2} |a(i', j')| \right) \right. \]

\[ + N^{-1/2} \| A \|_{\mathcal{B}} \right\} + 2^{2d}(A_q(w))^{1/q} \]

\[ \times \left\{ \sum_{|k|_\infty > 8N} N^d \left( \sup_{|i'-j'|_\infty \geq |k|_\infty /2} |a(i', j')| \right) \right\} \]

\[ \leq 2^{4d+2d/q}3^{3d}(A_q(w))^{2/q} \left\{ N^{-1/2} \| A \|_{\mathcal{B}} \right. \]

\[ \left. + 2^{d+2} \left( \sum_{k' \in \mathbb{Z}^d, |k'|_\infty \geq \sqrt{N}/2} |a(i', j')| \right) \right. \]

\[ + 2^{2d+1}(A_q(w))^{1/q} \sum_{k' \in \mathbb{Z}^d, |k'|_\infty \geq 7N} \left( \sup_{|i'-j'|_\infty \geq 7|k'|_\infty /16} |a(i', j')| \right) \]

\[ \leq 2^{6d+3d}(A_q(w))^{2/q}N^{-1/2} \| A \|_{\mathcal{B}} + 2^{7d+3}3^{3d}(A_q(w))^{2/q} \]

\[ \times \left( \sum_{k' \in \mathbb{Z}^d, |k'|_\infty \geq \sqrt{N}/2} \left( \sup_{|i'-j'|_\infty \geq |k'|_\infty} |a(i', j')| \right) \right) \]

\[ \rightarrow 0 \quad \text{as } N \rightarrow +\infty \]

by the assumption \( A \in \mathcal{B} \). Let \( N \) be the integer chosen sufficiently large so that

\[ (4.18) \quad \epsilon_N^1 < 5^{-d}. \]

Taking the limit as \( l_0 \rightarrow \infty \) in (4.15), and using (4.12), (4.17) and (4.18) lead to

\[ \alpha_n^{-1/q} \| \Psi_n^N c \|_{q,w} \leq \alpha_n^{-1/q} \| \Psi_n^N Ac \|_{q,w} \]

\[ + \sum_{n' \in \mathbb{N}^d} \left( \sum_{i=1}^{\infty} V_N^i(n - n') \right) \alpha_{n'}^{-1/q} \| \Psi_{n'}^N Ac \|_{q,w} \]

\[ =: \sum_{n' \in \mathbb{N}^d} W_N(n - n')\alpha_{n'}^{-1/q} \| \Psi_{n'}^N Ac \|_{q,w}, \]
and
\[ (4.20) \quad \sum_{k \in \mathbb{N}^d} \left( \sup_{|n|_\infty \geq |k|_\infty} |W_N(n)| \right) < \infty. \]

Given any \( i \in \mathbb{Z}^d \), let \( n(i) \) be the unique integer in \( \mathbb{N}^d \) with \( i \in n(i) + \{0, \ldots, N - 1\}^d \). Then
\[ (4.21) \quad \alpha_n(i) \leq \sum_{|i'-i|_\infty < 3N} w(i') \leq (6N)^{dq} A_q(w) w(i) \]
by (4.1). This together with (4.19) implies that for any \( c \in \ell^\infty \),
\[ |c(i)| \leq (6N)^d (A_q(w))^{1/q} \alpha_n(i)^{-1/q} \Psi_n(i) c_{q,w} \]
\[ \leq (6N)^d (A_q(w))^{1/q} \sum_{n' \in \mathbb{N}^d} W_N(n(i) - n') \times \left( \sum_{j \in \mathbb{Z}^d} h((j - n')/N) |(Ac)(j)| \right) \]
\[ \leq (6N)^d (A_q(w))^{1/q} \left\{ \sum_{j \in \mathbb{Z}^d} \left( \sum_{\epsilon \in \{-4, \ldots, 4\}^d} W_N(n(i) - j + \epsilon N) |(Ac)(j)| \right) \right\} \]
\[ (4.22) \quad =: \sum_{j \in \mathbb{Z}^d} g(i - j) |(Ac)(j)|. \]

Then the sequence \( \{g(i)\}_{i \in \mathbb{Z}^d} \) just defined satisfies all requirements in Lemma 4.5 by (4.20) and (4.22).

Now we proceed to prove Theorem 4.1.

**Proof of Theorem 4.1.** By Theorem 3.2, it suffices to prove the conclusion for any infinite matrix \( A \in \mathcal{B} \).

By Theorem 3.2
\[ (4.23) \quad \|Ac\|_{q',w'} \leq 2^{2d} 3^{d/q'} (A_q(w'))^{1/q'} \|A\|_{\mathcal{B}} \|c\|_{q',w'} \quad \text{for all } c \in \ell_{w'}^{q'}. \]

Let \( \{g(i)\}_{i \in \mathbb{Z}^d} \) be the sequence in Lemma 4.5 and set
\[ (4.24) \quad A_0 := \sum_{k \in \mathbb{Z}^d} \left( \sup_{|i|_\infty \geq |k|_\infty} g(i) \right) < \infty. \]

Then
\[ \|c\|_{q',w'} \leq \left\| \left( \sum_{j \in \mathbb{Z}^d} g(i - j) |(Ac)(j)| \right)_{i \in \mathbb{Z}^d} \right\|_{q',w'} \]
\[ \leq 2^{2d} 3^{d/q'} A_0 (A_q(w'))^{1/q'} \|Ac\|_{q',w'} \quad \text{for all } c \in \ell^\infty \cap \ell_{w'}^{q'}. \]
where the first inequality follows from \((4.3)\) and the second inequality holds by Theorem \(3.2\).

Combining \((4.23)\) and \((4.25)\) proves the \(\ell_{w'}^q\)-stability for the infinite matrix \(A \in \mathcal{B}\).

Finally we prove Corollary \(1.3\).

**Proof of Corollary 1.3.** The necessity is well known, while the sufficiency follows from Theorem \(3.2\) and Corollary \(1.6\), whose proof will be given in the next section. \(\square\)

5. **Inverse-closedness**

In this section, we prove Theorem \(1.4\), Corollaries \(1.5\) and \(1.6\), and the following Wiener’s lemma for the subalgebra \(\mathcal{B}_{p,u}\) of \(\mathcal{B}(\ell_w^q)\).

**Theorem 5.1.** Let \(1 \leq p, q < \infty\), \(w\) be a discrete \(A_q\)-weight, \(u := (u(i,j))_{i,j \in \mathbb{Z}^d}\) be a \(p\)-submultiplicative weight matrix that satisfies \((3.1), (3.2), (3.3)\) and

\[
M := \sup_{i \in \mathbb{Z}^d} u(i,i) < \infty,
\]

and \(v := (v(i,j))_{i,j \in \mathbb{Z}^d}\) be a companion weight matrix of the \(p\)-submultiplicative weight matrix \(u\) that satisfies \((3.5)\). If there exist \(D \in (0, \infty)\) and \(\theta \in (0,1)\) such that

\[
\inf_{N \geq 1} (A_N + B_N(p)t) \leq Dt^\theta \text{ for all } t \geq 1
\]

where

\[
A_N := \sum_{|k| \leq N} \sup_{|k| \leq |i'| \leq |j'| \leq N} v(i',j')
\]

and

\[
B_N(p) := \left\| \left( \sup_{|i'| \leq |k| \leq |j'| \leq N} v(i',j')(u(i',j'))^{-1} \right)_{|k| \geq N/2} \right\|_{p/(p-1)},
\]

then \(\mathcal{B}_{p,u}\) is an inverse-closed subalgebra of \(\mathcal{B}(\ell_w^q)\).

One may verify that the weight matrices \(((1 + |i-j|)^\alpha)_{i,j \in \mathbb{Z}^d}\) with \(\alpha > d(1 - 1/p)\), and \((\exp(|i-j|^\delta))_{i,j \in \mathbb{Z}^d}\) with \(\delta \in (0,1)\), and their companion weight matrices satisfy the conditions on weight matrices required in Theorem \(5.1\) \([43]\). Hence we have the following corollary of Theorem \(5.1\).

**Corollary 5.2.** Let \(1 \leq p, q < \infty\), \(w\) be a discrete \(A_q\)-weight, and let \(u\) be either \(((1 + |i-j|)^\alpha)_{i,j \in \mathbb{Z}^d}\) with \(\alpha > d(1 - 1/p)\) or \((\exp(|i-j|^\delta))_{i,j \in \mathbb{Z}^d}\) with \(\delta \in (0,1)\). Then \(\mathcal{B}_{p,u}\) is an inverse-closed subalgebra of \(\mathcal{B}(\ell_w^q)\).
5.1. Proof of Theorem 1.4. Let $A \in B$ have an inverse $A^{-1} \in B(\ell^q_w)$. Then $\|c\|_{q,w} \leq \|A^{-1}\|_{B(\ell^q_w)} \|Ac\|_{q,w}$ for all $c \in \ell^q_w$, where $\|\cdot\|_{B(\ell^q_w)}$ is the operator norm on $B(\ell^q_w)$. Therefore $A$ has $\ell^q_w$-stability. By Lemma 4.5, there exists a sequence $\{g(i)\}_{i \in \mathbb{Z}^d}$ such that (4.2) and (4.3) hold.

Write $A^{-1} := (b(i,j))_{i,j \in \mathbb{Z}^d}$, set $c_j := (b(i,j))_{i \in \mathbb{Z}^d}$, and let $l_0 \geq 1$ define $c_j^0 := (b_0(i,j))_{i \in \mathbb{Z}^d}, j \in \mathbb{Z}^d$, where $b_0(i,j) := b(i,j)$ if $|i - j| \leq l_0$ and 0 otherwise. Then $c_j^0 \in \ell^q \cap \ell^q_w$ and

$$
\lim_{l_0 \to +\infty} \|c_j^0 - c_j\|_{q,w} = 0.
$$

Applying (4.3) to $c_j^0$ gives

$$
|b_0(i,j)| \leq \sum_{i' \in \mathbb{Z}^d} g(i - i')|A(c_j^0)(i')|, \quad i \in \mathbb{Z}^d.
$$

By (4.2), (5.5), and Theorem 3.2

$$
\begin{align*}
\sum_{i' \in \mathbb{Z}^d} g(i - i')|A(c_j^0 - c_j)(i')| &\leq w(i)^{-1/q} \left( \sum_{i' \in \mathbb{Z}^d} |g(i' - i'')(A(c_j^0 - c_j)(i''))| \right)_{i'' \in \mathbb{Z}^d} \|q,w
\leq 2^{2d/\omega} q w(i)^{-1/q} (A_q(w))^{2/q} \|A\|_B
\times \left( \sup_{|j'| \geq |k|} |g(j')| \right)_{k \in \mathbb{Z}^d} \|c_j^0 - c_j\|_{q,w}
\end{align*}
$$

(5.7) → 0 as $l_0 \to +\infty$.

Letting $l_0 \to +\infty$ in (5.6) and applying (5.7) gives

$$
|b(i,j)| \leq g(i - j) \quad \text{for all } i, j \in \mathbb{Z}^d.
$$

Hence the conclusion $A^{-1} \in B$ follows from (4.2) and (5.8). □

5.2. Proof of Corollary 1.5. Write $f(\xi) = \sum_{n \in \mathbb{Z}} a(n) e^{-\sqrt{-1}n\xi}$. Then $A := (a(i - j))_{i,j \in \mathbb{Z}}$ belongs to $B$ and has bounded inverse in $B(\ell^2)$. Moreover, $A^{-1} = (b(i - j))_{i,j \in \mathbb{Z}}$ for the sequence $b := (b(n))_{n \in \mathbb{Z}}$ determined by $1/f(\xi) = \sum_{n \in \mathbb{Z}} b(n) e^{-\sqrt{-1}n\xi}$. By Theorem 1.4, $A^{-1} \in B$ which in turn proves the desired conclusion that $1/f \in A^\ast(\mathbb{T})$. □

5.3. Proof of Corollary 1.6. The necessity follows from Theorem 3.2. Now the sufficiency: Let $1 \leq q < \infty$, $w$ be a discrete $A_q$-weight, and let $A \in B$ have $\ell^q_w$-stability. Then $A$ has $\ell^2$-stability by Theorem 1.2 i.e., there exists a positive constant $C$ such that

$$
C^{-1} \|c\|_2 \leq \|Ac\| \leq C \|c\|_2 \quad \text{for all } c \in \ell^2.
$$
This implies that $A^*A$ has bounded inverse in $B(\ell^2)$. On the other hand, $A^*A$ belong to $\mathcal{B}$ by Proposition 2.4 and Theorem 3.2. Therefore
\[
(A^*A)^{-1} \in \mathcal{B}
\]
by Theorem 1.4. Now we prove that $B := (A^*A)^{-1}A^*$ is the desired left inverse of the infinite matrix $A$ in $\mathcal{B}$. The conclusion that $B \in \mathcal{B}$ follows from (5.9), Proposition 2.4 and Theorem 3.2. From the definition of the infinite matrix $B$, it defines a left inverse in $\mathcal{B}(\ell^2)$, it belongs to $\mathcal{B}(\ell^2_w)$ by Theorem 3.2 and $B \in \mathcal{B}$, and the set $\ell^2 \cap \ell^2_w$ is dense in $\ell^2_w$. Therefore the infinite matrix $B$ is a left inverse in $\mathcal{B}(\ell^2_w)$.

5.4. Proof of Theorem 5.1. To prove Theorem 5.1, we need a technical lemma. Similar results are established in [41, 43] for infinite matrices in the Gröchenig-Schur class $\mathcal{S}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$ and the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_{p,u}(\mathbb{Z}^d, \mathbb{Z}^d)$, see also Remark 3.4.

Lemma 5.3. Let $1 \leq p \leq \infty$. If the weight matrix $u$ satisfies (3.1), (3.2), (3.3), (5.1) and (5.2) for some positive constants $D \in (0, \infty)$ and $\theta \in (0, 1)$, then
\[
\|A^2\|_{\mathcal{B}_{p,u}} \leq 2^{2+2/p}2^{(d-1)/p}D\|A\|^{1+\theta}_{\mathcal{B}_{p,u}}\|A\|^{1-\theta}_{B(\ell^2)} \quad \text{for all } A \in \mathcal{B}_{p,u}.
\]

Proof. Let $A := (a(i, j))_{i,j \in \mathbb{Z}^d} \in \mathcal{B}_{p,u}$, and let $A_N$ and $B_N(p)$ be as in (5.3) and (5.4) respectively. Recall that $|a(i, j)| \leq \|A\|_{\mathcal{B}(\ell^2)}$ for all $i, j \in \mathbb{Z}^d$. Then for $1 < p < \infty$,
\[
\sum_{k' \in \mathbb{Z}^d} |a(i, k')|u(i, k')|a(k', j)|v(k', j)
\]
\[
\leq \inf_{N \geq 1} \left\{ \|A\|_{\mathcal{B}(\ell^2)} \sum_{|k' - j|_\infty \leq N} |a(i, k')|u(i, k')v(k', j) \right. \\
+ \sum_{|k' - j|_\infty > N} |a(i, k')|u(i, k')|a(k', j)|v(k', j) \left. \right\}
\]
\[
\leq \inf_{N \geq 1} \left\{ \|A\|_{\mathcal{B}(\ell^2)} \left( \sum_{|k'' - j|_\infty \leq N} v(k'', j) \right)^{(p-1)/p} \right. \\
\times \left( \sum_{|k' - j|_\infty \leq N} (|a(i, k')|u(i, k')^p v(k', j))^{1/p} \right)^{1/p} \\
+ \left( \sum_{|k'' - j|_\infty > N} |a(k'', j)|v(k'', j) \right)^{(p-1)/p} \\
\times \left( \sum_{|k' - j|_\infty > N} (|a(i, k')|u(i, k')^p |a(k', j)|v(k', j))^{1/p} \right). \right\}.
\]
Therefore we obtain

\[
\left\{ \sum_{k \in \mathbb{Z}^d} \sup_{|i-j| \geq k} \left( \sum_{k' \in \mathbb{Z}^d} |a(i, k')|u(i, k')|a(k', j)|v(k', j)\right)^p \right\}^{1/p} \\
\leq \inf_{N \geq 1} \left\{ \|A\|_{\mathcal{B}(\ell^2)}(A_N)^{(p-1)/p} \right. \\
\times \left( A_N \sum_{k \in \mathbb{Z}^d} \sup_{|i'-j'| \geq k/2} (|a(i', j')|u(i', j'))^p \right) \right. \\
+ \|A\|_{\mathcal{B}_{p,u}}^p \sum_{k \in \mathbb{Z}^d} \sup_{|i'-j'| \geq k/2 \leq |i'-j'| \leq N} v(i', j') \right)^{1/p} \\
+ \|A\|_{\mathcal{B}_{p,u}}^{(p-1)/p} (B_N(p))^{(p-1)/p} \\
\times \left( \|A\|_{\mathcal{B}_{p,u}} B_N(p) \sum_{k \in \mathbb{Z}^d} \sup_{|i'-j'| \geq \max(|k|/2, N)} (|a(i', j')|u(i', j'))^p \right) \\
+ \|A\|_{\mathcal{B}_{p,u}}^{p+1} \left( \sum_{k \in \mathbb{Z}^d} \sup_{|i'-j'| \geq \max(|k|/2, N)} \right) \left( \frac{v(i', j')^{p/(p-1)}}{u(i', j')} \right)^{(p-1)/p} \right)^{1/p} \\
\leq 2^{1+2/p} 5^{(d-1)/p} \|A\|_{\mathcal{B}_{p,u}} \inf_{N \geq 1} \left( \|A\|_{\mathcal{B}(\ell^2)} A_N + B_N(p) \|A\|_{\mathcal{B}_{p,u}} \right) \\
\leq 2^{1+2/p} 5^{(d-1)/p} D \|A\|_{\mathcal{B}_{p,u}}^{1+\theta} \|A\|_{\mathcal{B}(\ell^2)}^{1-\theta}.
\]

Similarly, we have

\[
\left\{ \sum_{k \in \mathbb{Z}^d} \sup_{|i-j| \geq k} \left( \sum_{k' \in \mathbb{Z}^d} |a(i, k')|v(i, k')|a(k', j)|u(k', j)\right)^p \right\}^{1/p} \\
\leq 2^{1+2/p} 5^{(d-1)/p} D \|A\|_{\mathcal{B}_{p,u}}^{1+\theta} \|A\|_{\mathcal{B}(\ell^2)}^{1-\theta}.
\]

Combining the above two estimates and applying (3.16) with $B = A$, we then get the desire conclusion (5.10) for $1 < p < \infty$.

The conclusion (5.10) for $p = 1$ and for $p = \infty$ can be established similarly. We omit the details here. 

Having the above technical lemma, we can combine the arguments in [41, 43] and Wiener’s lemma for $\mathcal{B}$ to prove Theorem 5.1.

**Proof of Theorem 5.1.** Let $A \in \mathcal{B}_{p,u}$ and $A^{-1} \in \mathcal{B}(\ell^2_{w})$. Then $A^{-1} \in \mathcal{B} \subset \mathcal{B}(\ell^2)$ by Theorems 1.4 and 3.2. This implies that $C_1 I \leq A^* A \leq C_2 I$ for some positive constants $C_1$ and $C_2$, where $A^*$ is the conjugate
transpose of the matrix $A$ and $I$ is the identity matrix. Now set

$$B := I - \frac{2}{C_1 + C_2} A^* A.$$  

Then

$$\|B\|_{B(\ell^2)} \leq \frac{C_2 - C_1}{C_2 + C_1} := r_0 < 1.$$  

On the other hand, $A^* A \in B_{p,u}$ by Proposition 2.4 and Theorem 3.2 and $I \in B_{p,u}$ by (5.1). This shows that

$$\|B\|_{B_{p,u}} < \infty.$$  

Given any integer $n \geq 1$, write $n = \sum_{l=0}^{l_0} \epsilon_l 2^l$ with $\epsilon_l \in \{0, 1\}$. Applying Theorem 3.2 and Lemma 5.3 iteratively gives

$$\|B^n\|_{B_{p,u}} \leq (C \|B\|_{B_{p,u}})^{\epsilon_0} \|B^{n-\epsilon_0}\|_{B_{p,u}}$$
$$\leq C (C \|B\|_{B_{p,u}})^{\epsilon_0} (\|B\|_{B(\ell^2)})^{(1-\theta) \sum_{l=0}^{l_0-1} \epsilon_{l+1} 2^l}$$
$$\times (\|B\sum_{l=0}^{l_0-1} \epsilon_{l+1} 2^l\|_{B_{p,u}})^{(1+\theta)}$$
$$\leq \cdots \leq C^{l_0} (C \|B\|_{B_{p,u}})^{\sum_{l=0}^{l_0} \epsilon_l (1+\theta)^l} (\|B\|_{B(\ell^2)} \sum_{l=0}^{l_0} \epsilon_l (2^l - (1+\theta)^l)$$

$$\leq C^\log_2 n (Cr_0^{-1} \|B\|_{B_{p,u}})^{n \log_2 (1+\theta)} r_0^n,$$

where $C = \max(2^{2+2/p} 5^{(d-1)/p} D, 2^{1+2/p} 5^{(d-1)/p} M_p(u))$). This together with (5.12) and (5.13) shows that

$$\|A^{-1}\|_{B_{p,u}} = \|(A^* A)^{-1} A^*\|_{B_{p,u}}$$
$$= \frac{C_1 + C_2}{2} \left( \|A^* \|_{B_{p,u}} + \sum_{n=1}^{\infty} B^n \right) A^* \|_{B_{p,u}}$$
$$\leq \frac{C_1 + C_2}{2} \left( \|A^*\|_{B_{p,u}} + C \|A^*\|_{B_{p,u}} \right.$$  

$$\times \left( \sum_{n=1}^{\infty} C^{\log_2 n} (Cr_0^{-1} \|B\|_{B_{p,u}})^{n \log_2 (1+\theta)} r_0^n \right) < \infty.$$  

Hence the conclusion $A^{-1} \in B_{p,u}$ is proved. □

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CENTRAL FLORIDA, ORLANDO, FL 32816, USA

*E-mail address*: qsun@mail.ucf.edu