New error bounds for the extended vertical LCP

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Abstract

In this paper, by making use of this fact that for \(a_j, b_j \in \mathbb{R}, j = 1, 2, \ldots, n\), there are \(\lambda_j \in [0, 1]\) with \(\sum_{j=1}^{n} \lambda_j = 1\) such that

\[
\min_{1 \leq j \leq n} \{a_j\} - \min_{1 \leq j \leq n} \{b_j\} = \sum_{j=1}^{n} \lambda_j (a_j - b_j),
\]

some new error bounds of the extended vertical LCP under the row \(W\)-property are obtained, which cover the error bounds in [Math. Program., 106 (2006) 513-525] and [Comput. Optim. Appl., 42 (2009) 335-352]. Not only that, these new error bounds skillfully avoid the inconvenience caused by the row rearrangement technique for error bounds to achieve the goal of reducing the computation workload, which was introduced in the latter paper mentioned above. Besides, with respect to the row \(W\)-property, two new sufficient and necessary conditions are obtained.

Keywords: The extended vertical LCP; row \(W\)-property; error bound

AMS classification: 90C33, 65F10, 65F50, 65G40

1 Introduction

For \(A_j \in \mathbb{R}^{n \times n}\) and \(q_j \in \mathbb{R}^n\) \((j = 0, 1, 2, \ldots, k)\) being given known matrices and the source terms, the extended vertical linear complementarity problem is to find \(x \in \mathbb{R}^n\) such that

\[
r(x) := \min\{A_0x + q_0, A_1x + q_1, \ldots, A_kx + q_k\} = 0,
\]

This research was supported by National Natural Science Foundation of China (No.11961082).

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where min is the component minimum operator. Here, Eq. (1.1) is denoted by EVLCP(A, q) for short, where

\[ A = (A_0, A_1, \ldots, A_k) \quad \text{and} \quad q = (q_0, q_1, \ldots, q_k). \]

When \( A_0 = I \) and \( q_0 = 0 \) in (1.1), where \( I \) denotes the identity matrix, the EVLCP \((A, q)\) reduces to the vertical LCP, which was introduced by Cottle and Dantzig [1], also see [2]. Further, when \( A_0 = I \) and \( q_0 = 0 \) and \( k = 1 \) in (1.1), the EVLCP \((A, q)\) comes back to the standard LCP \((A_1, q_1)\), see [3,4].

So far, it has been found that the EVLCP \((A, q)\) has more and more applications in the many fields, like, such as nonlinear networks [5], control theory [6], the mixed lubrication problem [7], stochastic impulse control games [8], the boundary value problem [9], the generalized bimatrix games [10], the generalized Leontief input-output model [11], the discrete HJB equations [12], volterra ecosystem [13] and so on. There exist many literatures to pay attention to the existence of solutions and algorithms for the EVLCP \((A, q)\), see [2][6][8][14][16].

For the EVLCP \((A, q)\), another important and interested topic in theory is error bound, which has drawn widespread attention because of an important tool in theoretical analysis, including convergence analysis, sensitive analysis and verification of the solutions. At present, there have some results about error bounds in the literature. For instance, assume that \( x^* \) is the unique solution of the LCP \((A_1, q_1)\), for \( A_1 \) being \( P \)-matrix, a well-know global error bound was given in [17] by Mathias and Pang, and described below

\[ \| x - x^* \|_{\infty} \leq \frac{1 + \| A_1 \|_{\infty}}{\alpha(A_1)} \| r(x) \|_{\infty}, \quad \text{for any} \quad x \in \mathbb{R}^n, \tag{1.2} \]

where \( \alpha(A_1) := \min_{\| x \|_{\infty} = 1} \{ \max_{1 \leq i \leq n} x_i(A_1 x)_i \} \).

By the equivalent form of the minimum function, Chen and Xiang in [18] obtained the following error bound in \( \| \cdot \|_p \) (\( p \geq 1 \), or \( p = \infty \)) norms,

\[ \| x - x^* \|_p \leq \max_{d \in [0,1]^n} \| (I - D + DA_1)^{-1} \|_p \| r(x) \|_p, \quad \text{for any} \quad x \in \mathbb{R}^n, \tag{1.3} \]

where \( D = \text{diag}(d) \) with \( d \in [0,1]^n \), which is sharper than (1.2) in \( \| \cdot \|_{\infty} \); see [18]. Moreover, for \( A_1 \) being an \( H_\ast \)-matrix, Chen and Xiang confirmed

\[ \max_{d \in [0,1]^n} \| (I - D + DA_1)^{-1} \|_p \leq \| A_1 \|_1^{-1} \max(\wedge_1, I) \|_p, \tag{1.4} \]

where \( \wedge_1 \) is the diagonal part of \( A_1 \) and \( \langle A_1 \rangle \) is its comparison matrix (i.e., \( \langle A_1 \rangle_{ij} = |(A_1)_{ii}|, \langle A_1 \rangle_{ij} = -(A_1)_{ij} \) for \( i \neq j \)).

Recently, by the row rearrangement technique introduced by Zhang et al. in [19], i.e., \( A' = (A'_0, A'_1, \ldots, A'_k) \) is called a row rearrangement of \( A = (A_0, A_1, \ldots, A_k) \) if for any \( i \in N := \{1, 2, \ldots, n\} \),

\[ (A'_{ij})_i = (A_{ji})_i \in \{(A_0)_i, (A_1)_i, \ldots, (A_k)_i\} = \{(A'_0)_i, (A'_1)_i, \ldots, (A'_k)_i\}, \]
where \((\cdot)_i\) means the \(i\)-th row of a given matrix, for the EVLCP \((A, q)\), assume that \(A = (A_0, A_1, \ldots, A_k)\) has the row \(W\)-property (see the following definition), Zhang et al. in [19] presented the following result,

\[\|x - x^*\| \leq \alpha(A)\|r(x)\|, \text{ for any } x \in \mathbb{R}^n, \tag{1.5}\]

where

\[
\alpha(A) = \max_{A' \in \mathcal{R}(A)} \max_{j \in \{0, 1, \ldots, k\}} \max_{d \in [0, 1]^n} \|(I - D)A'_j + A'_i\|^{-1}, \tag{1.6}
\]

\(A'_j, A'_i \in \mathbb{R}^{n \times n}\) are any two blocks in \(A' \in \mathcal{R}(A)\) with \(\mathcal{R}(A)\) being the set of all row rearrangements of \(A\). When the norm in (1.5) is taken as \(\|\cdot\|_{\infty}\), we denote

\[
\alpha_\infty(A) = \max_{A' \in \mathcal{R}(A)} \max_{j \in \{0, 1, \ldots, k\}} \max_{d \in [0, 1]^n} \|(I - D)A'_j + A'_i\|_{\infty}^{-1}.
\]

There is no doubt that Eq. (1.5) presents a general result for the upper global error bound of the EVLCP \((A, q)\) under the row \(W\)-property. By investigating Eq. (1.5), clearly, it is easy to know that before obtaining Eq. (1.5), we have to calculate and obtain the exact value of (1.6). Whereas, in the implementations, calculating the exact value of (1.6) is a very difficult task because there is a rearrangement for the row of matrix \(A\), in particular, when the order of matrix \(A\) is large. It is a main motivation of this present paper. In this paper, to overcome this disadvantage caused by the row rearrangement technique in essence, we have to carve out a new approach to obtain the error bound for the EVLCP \((A, q)\).

Our approach inspired by the work in [18], we first develop a general equivalent form of the minimum function. Then, based on this, some new error bounds for the EVLCP \((A, q)\) are obtained. Not only that, these new error bounds cover some existing results in [18] and [19] as well. Meanwhile, it avoids the row rearrangement of the system matrix \(A\). Incidentally, for the row \(W\)-property, two new sufficient and necessary conditions are given.

The rest of the article expands as follows. First, from the view of the calculation time, section 2 further discusses the result in (1.5). Secondly, section 3 provides some error bounds of the EVLCP \((A, q)\) under the row \(W\)-property by a general equivalent form of the minimum function, which is entirely different from Eq. (1.5). Thirdly, section 4 presents some numerical examples to show the feasibility of the error bound. Finally, section 5 is a brief conclusion.

By the way, the following notations, definitions and results will be used throughout the paper, which can be founded in [20]. Let \(A = (a_{ij})\) and \(B = (b_{ij}) \in \mathbb{R}^{n \times n}\). Then \(A \geq (>) B\) means \(a_{ij} \geq (>) b_{ij}\) for \(i, j = 1, 2, \ldots, n\). We indicate \(|A| = (|a_{ij}|)\). Matrix \(A = (a_{ij})\) is called a strictly diagonal dominant matrix if \(|a_{ii}| > \sum_{j \neq i}|a_{ij}|, i \in \mathbb{N} := \{1, 2, \ldots, n\}\). \(\rho(\cdot)\) indicates the spectral radius of the matrix. A block matrix \(A = (A_0, A_1, \ldots, A_k)\) has the row \(W\)-property if

\[
\min(A_0x, A_1x, \ldots, A_kx) \leq 0 \leq \max(A_0x, A_1x, \ldots, A_kx) \Rightarrow x = 0.
\]

The EVLCP \((A, q)\) has a unique solution for any \(q\) if and only if \(A\) has the row \(W\)-property.
2 Old error bound

In this section, we will discuss the result in (1.5) from the angle of the calculation time.

To calculate $\alpha(A)$, a natural question is how many times you need to calculate $\|(I - D)A_j' + A_i'\|^{-1}$. Further, when confronting Eq. (1.6), the first thought is to calculate the number of elements in $\mathcal{R}(A)$, that is, we need to know that how many the row rearrangements of $A = (A_0, A_1, \ldots, A_k)$ there are in $\mathcal{R}(A)$. To answer this question, we get Proposition 2.1.

**Proposition 2.1** Let $A = (A_0, A_1, \ldots, A_k)$ with $A_j \in \mathbb{R}^{n \times n} (j = 0, 1, \ldots, k)$. Then the cardinality of $\mathcal{R}(A)$ is $[(k + 1)!]^n$.

**Proof.** If only one row is rearranged, noting that a total number of matrices are $k + 1$, there are a total of $(k + 1)!$ different sorting methods. Because there are a total of $n$ rows and the arrangement of different rows can be arbitrarily combined, there are a total of $[(k + 1)!]^n$ different sorting methods. Therefore, matrix $A$ has $[(k + 1)!]^n$ different row rearrangements, that is, the cardinality of $\mathcal{R}(A)$ is $[(k + 1)!]^n$. □

In fact, once a certain $A'$ is selected, there exist the total $C_{k+1}^2 = \frac{k(k+1)}{2}$ kinds of options for choosing $A_j', A_i'$. From Proposition 2.1, a total of elements in $\mathcal{R}(A)$ are $[(k + 1)!]^n$, so the calculation times of $\|(I - D)A_j' + A_i'\|^{-1}$ are at most $\frac{k(k+1)}{2}[(k + 1)!]^n$ times to obtain $\alpha(A)$. Therefore, the calculation times of Theorem 4.1 in [19] for $k = 1$ achieve at most $2^n$ times. Whereas, the calculation time of Theorem 3.1 in [19] is 1. The next question is whether Theorem 4.1 in [19] for $k = 1$ is in contradiction with Theorem 3.1 [19] because the form of Theorem 4.1 in [19] for $k = 1$ is different from in Theorem 3.1 in [19] as well. To answer this question, we require Proposition 2.2.

**Proposition 2.2** Let $A = (A_0, A_1)$ with $A_0, A_1 \in \mathbb{R}^{n \times n}$. Then

$$\max_{A' \in \mathcal{R}(A)} \max_{d \in [0, 1]^n} \|(I - D)A_0' + A_1\|^{-1} = \max_{d \in [0, 1]^n} \|(I - D)A_0 + DA_1\|^{-1}.$$  

**Proof.** For any $A' = (A_0', A_1')$, $D = \text{diag}(d)$, $d \in [0, 1]^n$, $i = 1, 2, \cdots, n$, we have

$$((I - D)A_0' + DA_1)_i = (1 - d_i)(A_0)_i + d_i (A_1)_i.$$  

According to the definition of row rearrangement, we get

$$\{(A_0)_i, (A_1)_i\} = \{(A_0)_i, (A_1)_i\}.$$  

When $(A_0)_i = (A_0)_i, (A_1)_i = (A_1)_i$, we take $\hat{d}_i = d_i$; when $(A_0)_i = (A_1)_i, (A_1)_i = (A_0)_i$, we take $d_i = 1 - d_i, \hat{d}_i \in [0, 1]$. Further, we have

$$1 - d_i)(A_0)_i + d_i(A_1)_i = (1 - \hat{d}_i)(A_0)_i + \hat{d}_i(A_1)_i.$$  

Let $\hat{D} = \text{diag}(\hat{d})$. Then

$$(I - D)A_0' + DA_1' = (I - \hat{D})A_0 + \hat{D}A_1.$$
Therefore, from the arbitrariness of $A'$, we obtain
\[
\max_{A' \in \mathcal{R}(A)} \max_{d \in [0,1]^n} \|((I - D)A_0 + A_1')^{-1}\| = \max_{d \in [0,1]^n} \|((I - D)A_0 + DA_1')^{-1}\|.
\]
This completes the proof. \hfill \Box

Based on Proposition 2.2, in essence, Theorem 4.1 in [19] for $k = 1$ is in line with Theorem 3.1 [19]. The calculation times of both are the same, 1 time.

According to the process of the proof of Proposition 2.2, we can see that when calculating $\|((I - D)A_j' + DA_l')^{-1}\|$, as long as two of the $k + 1$ elements in each row are chosen, no matter how many $A_j, A_l'$ they can be combined into, ultimately, their calculation results are the same, so there is no need to reconsider. Therefore, calculating $\alpha(A)$ only needs to calculate $(C^2_{k+1})^n$ times $\|((I - D)A_j' + A_l')^{-1}\|$. From this, we present Proposition 2.3.

**Proposition 2.3** Let $A = (A_0, A_1, \ldots, A_k)$. Then there needs to calculate $\alpha(A)$ at most $(C^2_{k+1})^n$ times $\|((I - D)A_j' + DA_l')^{-1}\|$, i.e.,
\[
\alpha(A) = \max_{A' \in \mathcal{R}(A)} \max_{B_1, B_2 \in [0,1]^n} \|((I - D)B_1 + DB_2)^{-1}\|,
\]
where $(B_1)_i. = (A_j)_i., (B_2)_i. = (A_l)_i., j < l \in \{0, 1, \ldots, k\}$.

To further explain Proposition 2.3, Example 2.1 is provided.

**Example 2.1** Let $A = (A_0, A_1, A_2)$, where
\[
A_0 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.
\]

When using Eq. (2.1), from Proposition 2.3, its calculation times are 9. Specifically as follows:

Let
\[
B_1 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.
\]
Then
\[
\mu_1 = \max_{d \in [0,1]^n} \|((I - D)B_1 + DB_2)^{-1}\|_\infty = 3;
\]
let
\[
B_1 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.
\]
Then
\[
\mu_2 = \max_{d \in [0,1]^n} \|((I - D)B_1 + DB_2)^{-1}\|_\infty = 2;
\]
let 
\[ B_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}. \]

Then 
\[ \mu_3 = \max_{d \in [0,1]^n} \|(I - D)B_1 + DB_2\|_\infty = 3; \]

let 
\[ B_1 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \]

Then 
\[ \mu_4 = \max_{d \in [0,1]^n} \|(I - D)B_1 + DB_2\|_\infty = 2; \]

let 
\[ B_1 = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \]

Then 
\[ \mu_5 = \max_{d \in [0,1]^n} \|(I - D)B_1 + DB_2\|_\infty = 3; \]

let 
\[ B_1 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}. \]

Then 
\[ \mu_6 = \max_{d \in [0,1]^n} \|(I - D)B_1 + DB_2\|_\infty = \frac{3}{2}; \]

let 
\[ B_1 = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}. \]

Then 
\[ \mu_7 = \max_{d \in [0,1]^n} \|(I - D)B_1 + DB_2\|_\infty = 2; \]

let 
\[ B_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}. \]

Then 
\[ \mu_8 = \max_{d \in [0,1]^n} \|(I - D)B_1 + DB_2\|_\infty = 3; \]
let
\[ B_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}. \]
Then
\[ \mu_9 = \max_{d \in [0,1]} \| (I - D)B_1 + DB_2 \|_\infty = 2. \]

Based on the above computational results, we obtain
\[ \alpha_\infty(A) = \max\{\mu_1, \mu_2, \ldots, \mu_9\} = 3. \]

However, directly using Eq. (1.5), its calculation times are \( \frac{k(k+1)}{2}[(k+1)!]^n = 108 \), this is because a lot of repetitive work are done. For example, let,
\[ A'_0 = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}, \quad A'_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad A'_2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \]
then \( A' = (A'_0, A'_1, A'_2) \in \mathcal{R}(A) \). Similarly, let,
\[ A''_0 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad A''_1 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad A''_2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \]
then \( A'' = (A''_0, A''_1, A''_2) \in \mathcal{R}(A) \). Of course, \( A = (A_0, A_1, A_2) \in \mathcal{R}(A) \). When using Eq. (1.5), we have to calculate
\[ \max_{d \in [0,1]} \| (I - D)A_0 + DA_1 \|_\infty, \quad \max_{d \in [0,1]} \| (I - D)A'_0 + DA'_1 \|_\infty \]
and
\[ \max_{d \in [0,1]} \| (I - D)A'' + DA''_1 \|_\infty. \]
Whereas, from the proof of Proposition 2.2, the above three formulas are equivalent, so they belong to the repeated calculation. Proposition 2.3 just avoids this.

Example 2.1 is simple, but it tells us that for any block matrix \( A \) containing three two-by-two matrices, i.e., \( k = 2 \) and \( n = 2 \), once the row rearrangement of \( A \) is required, its calculation times are at most 9 to obtain the corresponding error bound. It’s easy to imagine that for the block matrix \( A \) containing \( k + 1 \) \( n \)-by-\( n \) matrices, making use of Proposition 2.3 to compute the error bound of the EVLCP \((A, q)\), once the row rearrangement occurs, the corresponding computational cost should be highly expensive and unacceptable for the sufficiently large \( k \) or \( n \)! Therefore, to face this headwind, we have to exploit a new and effective tool to obtain the error bound of the EVLCP \((A, q)\).
3 New error bound

In this section, to avoid Proposition 2.3, we will give some new error bounds for the EVLCP \((A, q)\). For this goal, we first present some requisite lemmas.

**Lemma 3.1** Let all \(a_j, b_j \in \mathbb{R}, j = 1, 2, \ldots, n\). Then there are \(\lambda_j\) with \(\lambda_j \in [0, 1]\) such that

\[
\min_{1 \leq j \leq n} \{a_j\} - \min_{1 \leq j \leq n} \{b_j\} = \sum_{j=1}^{n} \lambda_j (a_j - b_j).
\]  

(3.1)

**Proof.** The result in Lemma 3.1 is given directly from the mean value theorem of Lipschitz functions with the generalized gradient. \(\square\)

**Lemma 3.2** Let \(a_j, b_j, t_j \in \mathbb{R}\) with \(a_j > 0\), \(t_j \in [0, 1]\), \((j = 1, 2, \ldots, n)\) and \(\sum_{j=1}^{n} t_j = 1\). Then

\[
\frac{\sum_{j=1}^{n} t_j b_j}{\sum_{j=1}^{n} t_j a_j} \leq \max_{1 \leq j \leq n} \left\{ \frac{|b_j|}{a_j} \right\}.
\]

(3.2)

In addition, if \(b_j = 1\), then the inequality (3.2) simplifies as

\[
\frac{1}{\sum_{j=1}^{n} t_j a_j} \leq \max_{1 \leq j \leq n} \left\{ \frac{1}{a_j} \right\}.
\]

**Proof.** The proof is straightforward. \(\square\)

In addition, it is noted that the following inequality is still true, i.e.,

\[
\sum_{j=2}^{n} t_j b_j \leq \max_{2 \leq j \leq n} \{|b_j|\},
\]

where \(t_j \in [0, 1]\) and \(\sum_{j=2}^{n} t_j \leq 1\). In fact, by the simple calculation, we have

\[
\sum_{j=2}^{n} t_j b_j = \frac{\sum_{j=2}^{n} t_j b_j}{1 - \sum_{j=2}^{n} t_j + \sum_{j=2}^{n} t_j} = \frac{\sum_{j=2}^{n} t_j b_j}{1 - \sum_{j=2}^{n} t_j + \sum_{j=2}^{n} t_j} = \sum_{j=2}^{n} \frac{t_j b_j}{t_2 b_2 + t_3 b_3 + \ldots + t_n b_n}
\]

\[
\leq \sum_{j=2}^{n} \frac{t_j b_j}{(1 - \sum_{j=2}^{n} t_j + t_2) + (1 - \sum_{j=2}^{n} t_j + t_3) + \ldots + (1 - \sum_{j=2}^{n} t_j + t_n)}
\]

\[
\leq \sum_{j=2}^{n} \frac{t_j b_j}{(1 - \sum_{j=2}^{n} t_j + t_2) + (1 - \sum_{j=2}^{n} t_j + t_3) + \ldots + (1 - \sum_{j=2}^{n} t_j + t_n)}
\]

\[
\leq \max_{2 \leq j \leq n} \{|b_j|\}.
\]
Lemma 3.3 [21] Matrix $A = (A_0, A_1, \ldots, A_k)$ has the row $W$-property if and only if for arbitrary nonnegative diagonal matrix $X_0, X_1, \ldots, X_k$ with $\text{diag}(X_0 + X_1 + \ldots + X_k) > 0$, \[
\det(X_0A_0 + X_1A_1 + \ldots + X_kA_k) \neq 0.
\]

To obtain the error bound for the EVLCP $(A, q)$, we require a new sufficient and necessary for the row $W$-property, see Lemma 3.4.

Lemma 3.4 Matrix $A = (A_0, A_1, \ldots, A_k)$ has the row $W$-property if and only if $D_0A_0 + D_1A_1 + \ldots + D_kA_k$ is nonsingular for arbitrary nonnegative diagonal matrices $D_j = \text{diag}(d_j)$ with $d_j \in [0, 1]^n$, $(j = 0, 1, \ldots, k)$ and $\sum_{j=0}^k D_j = I$.

**Proof.** The proof is straightforward by making use of Lemma 3.3. □

**Remark 3.1** When $k = 1$ in Lemma 3.4, the result in Lemma 3.4 goes back to Lemma 2.2 in [19]. Clearly, Lemma 3.4 is a generalization of their result. Next, we discuss the error bound for the EVLCP $(A, q)$.

Assume that $x^*$ is the unique solution of the EVLCP $(A, q)$. Based on Lemma 3.1, we set \[
a_j = A_jx + q_j, \quad b_j = A_jx^* + q_j, \quad j = 0, 1, \ldots, k.
\] (3.3)

Substituting (3.3) into (3.1) yields \[
r(x) = \min_{0 \leq j \leq k} \{A_jx + q_j\} - \min_{0 \leq j \leq k} \{A_jx^* + q_j\} = (D_0A_0 + D_1A_1 + \ldots + D_kA_k)(x - x^*),
\] (3.4)

where $D_j = \text{diag}(d_j)$ with $d_j \in [0, 1]^n$ $(j = 0, 1, \ldots, k)$ are nonnegative diagonal matrices and $\sum_{j=0}^k D_j = I$.

Combining (3.4) with Lemma 3.4, we immediately gain the upper global error bound of the EVLCP $(A, q)$ under the row $W$-property.

Theorem 3.1 Let $D_0, D_1, \ldots, D_k$ satisfy the conditions of Lemma 3.4. If $A = (A_0, A_1, \ldots, A_k)$ has the row $W$-property, then for any $x \in \mathbb{R}^n$, \[
\|x - x^*\| \leq \max \|D_0A_0 + D_1A_1 + \ldots + D_kA_k\|^{-1} \|r(x)\|.
\] (3.5)

**Remark 3.2** When $A_0 = I$ and $k = 1$ in (3.5), Theorem 3.1 reduces to (1.3), see Eq. (2.3) on page 516 in [18] as well. Further, when $k = 1$ in (3.5), Theorem 3.1 reduces to Theorem 3.1 in [19]. In addition, comparing (3.5) with (1.5), the former advantage over the latter is that the former no longer requires the row rearrangement of the matrix $A$. That is to say, (3.5) successfully avoids the row rearrangement of the matrix $A$. From the view of the calculation time, in general, Eq. (1.5) requires $(k+1)^n$ times, Eq. (3.5) requires only 1 time. Compared with the error bound (1.5) by the row rearrangement technique, our new error bound (3.5) greatly reduces the computation workload in a way.

In addition, we also obtain the lower global error bound for the EVLCP $(A, q)$ under the row $W$-property. By making use of (3.4), we easily find that \[
\|x - x^*\| \geq \max \|D_0A_0 + D_1A_1 + \ldots + D_kA_k\|^{-1} \|r(x)\|.
\]
Theorem 3.1 provides a new result for the upper global error bound of the EVLCP \((A, q)\) under the row \(\mathcal{W}\)-property, whereas, it contains arbitrary nonnegative diagonal matrices \(D_0, D_1, \ldots, D_k\) such that it seems be an impossible task to find \(\max \| (D_0 A_0 + D_1 A_1 + \ldots + D_k A_k)^{-1} \|\) under certain conditions. To turn around this negative situation, an effective approach is to limit the range of \(A\). That is to say, once we are able to choose the suitable \(A\), Theorem 3.1 is able to get rid of this unfavorable situation, on condition that the error can be permitted.

In the following, we consider the upper global error bound of the EVLCP \((A, q)\) from two aspects: (I) \(A = (A_0, A_1, \ldots, A_k)\) with the diagonal part of every matrix \(A_j\) being positive; (II) \(A = (A_0, A_1, \ldots, A_k)\) with every matrix \(A_j\) being strictly diagonally dominant. Under these two cases, interestingly, we obtain the upper global error bound the same as these in [19], see Theorem 3.2 and Theorem 3.3. Not only that, The proof of Theorem 3.2 is simpler than the proof of Theorem 4.3 in [19].

**Theorem 3.2** Let \(A_j = \land_j - C_j\) in \(A\) with \(\land_j > 0\), where \(\land_j\) is the diagonal part of \(A_j\), \(j=0,1,\ldots,k\). If \(A_j = \land_j - C_j\) satisfy

\[
\rho(\max_{0 \leq j \leq k} \{\land_j^{-1} | C_j|\}) < 1, \quad (3.6)
\]

then \(A = (A_0, A_1, \ldots, A_k)\) has the row \(\mathcal{W}\)-property and

\[
\max \| (D_0 A_0 + D_1 A_1 + \ldots + D_k A_k)^{-1} \| \leq \| (I - \max_{0 \leq j \leq k} \{\land_j^{-1} | C_j|\})^{-1} \max_{0 \leq j \leq k} \{\land_j^{-1}\} \|. \quad (3.7)
\]

**Proof.** Let \(V = \sum_{j=0}^{k} D_j \land_j\) and \(U = \sum_{j=0}^{k} D_j C_j\). Then

\[
(D_0 A_0 + D_1 A_1 + \ldots + D_k A_k)^{-1} = (V - U)^{-1} = (I - V^{-1} U)^{-1} V^{-1}.
\]

Under the condition (3.6), together with Lemma 3.2, we have

\[
V^{-1} \leq \max_{0 \leq j \leq k} \{\land_j^{-1}\}, \quad V^{-1} |U| \leq \max_{0 \leq j \leq k} \{\land_j^{-1} | C_j|\},
\]

and

\[
| (I - V^{-1} U)^{-1} | = | I + (V^{-1} U) + (V^{-1} U)^2 + ... |
\leq | I + (V^{-1} |U|) + (V^{-1} |U|)^2 + ... |
\leq | I + (\max_{0 \leq j \leq k} \{\land_j^{-1} | C_j|\}) + (\max_{0 \leq j \leq k} \{\land_j^{-1} | C_j|\})^2 + ... |
\leq | I - \max_{0 \leq j \leq k} \{\land_j^{-1} | C_j|\} |^{-1}.
\]

So,

\[
| (I - V^{-1} U)^{-1} V^{-1} | \leq | I - \max_{0 \leq j \leq k} \{\land_j^{-1} | C_j|\} |^{-1} \max_{0 \leq j \leq k} \{\land_j^{-1}\}.
\]
Hence,
\[
\| (I - V^{-1}U)^{-1}V^{-1} \| \leq \| (I - V^{-1}U)^{-1} \| \leq \| I - \max_{0 \leq j \leq k} \{ \land_j^{-1} | C_j | \} \|^{-1} \max_{0 \leq j \leq k} \{ \land_j^{-1} \},
\]
which implies that (3.7) is true. By Lemma 3.4, it is easy to see that \( A = (A_0, A_1, \ldots, A_k) \) has the row \( \mathcal{W} \)-property.

In fact, the process of the proof of Theorem 3.2 is also suitable for the proof of Theorem 2.1 in [18] and Theorem 3.2 in [19].

**Theorem 3.3** Let \( A = (A_0, A_1, \ldots, A_k) \) with every matrix \( A_j \) being strictly row diagonally dominant and \( \text{sign}(\land_0) = \text{sign}(\land_j) \), where \( \land_j \) is a diagonal part of \( A_j \), \( j = 0, 1, \ldots, k \). Then \( A = (A_0, A_1, \ldots, A_k) \) has the row \( \mathcal{W} \)-property and
\[
\max \| (D_0A_0 + D_1A_1 + \ldots + D_kA_k)^{-1} \|_\infty \leq \frac{1}{\min_{i \in N} \min((\langle A_0 \rangle e)_i, (\langle A_1 \rangle e)_i, \ldots, (\langle A_k \rangle e)_i)}.
\]  

(3.8)

Here, the proof of Theorem 3.3 is omitted, one can see the proof of Theorem 3.3 in [19] for more details.

Comparing Theorem 4.4 in [19] with Theorem 3.3, the latter no longer requires “each \( i \in N, (A_j)_{ii}(A_l)_{ii} > 0 \), for any \( j < l \in \{0, 1, \ldots, k\} \)”, just needs to keep \( \land_j \) the same sign.

In [22], Xiu and Zhang extended Eq. (1.2) to the EVLCP \((A, q)\) for \( k = 1 \) under the row \( \mathcal{W} \)-property:
\[
\| x - x^* \|_\infty \leq \frac{\| A_0 + A_1 \|_\infty}{\alpha \{A_0, A_1\}} \| r(x) \|_\infty, \text{ for any } x \in \mathbb{R}^n,
\]  

(3.9)

where
\[
\alpha \{A_0, A_1\} := \min_{\| x \|_\infty = 1} \{ \max_{1 \leq i \leq n} \langle A_0 x \rangle (A_1 x)_i \}.
\]

In the sequel, what we’re interested in is whether we will extend (3.9) to the EVLCP \((A, q)\) under the row \( \mathcal{W} \)-property, meanwhile, and avoid the row rearrangement technique as well. To this end, we require the following lemma, see Lemma 3.5.

**Lemma 3.5** If matrix \( \tilde{A} = (A_0, D_1A_1 + D_2A_2 + \ldots + D_kA_k) \) has the row \( \mathcal{W} \)-property, where matrices
\( D_j = \text{diag}(d_j) \) with \( d_j \in [0, 1]^n \) \((j = 1, \ldots, k) \) are arbitrary nonnegative diagonal matrices and \( \sum_{j=1}^k D_j = I \), if and only if matrix \( A = (A_0, A_1, \ldots, A_k) \) has the row \( \mathcal{W} \)-property.

**Proof.** \((\Rightarrow)\) If \( \tilde{A} = (A_0, D_1A_1 + D_2A_2 + \ldots + D_kA_k) \) has the row \( \mathcal{W} \)-property, matrices \( D_j = \text{diag}(d_j) \) with \( d_j \in [0, 1]^n \) \((j = 1, \ldots, k) \) are arbitrary nonnegative diagonal matrices and \( \sum_{j=1}^k D_j = I \), then, by using Lemma 3.4, for arbitrary nonnegative diagonal matrix \( \hat{D} = \text{diag}(\hat{d}) \) with \( \hat{d} \in [0, 1]^n \), matrix
\[
(I - \hat{D})A_0 + \hat{D}(D_1A_1 + D_2A_2 + \ldots + D_kA_k)
\]
is nonsingular. For arbitrary nonnegative diagonal matrix \( \bar{D}_j = \text{diag}(\bar{d}_j) \) with \( \bar{d}_j \in [0, 1]^n \) \((j = 0, 1, \ldots, k)\) and \( \sum_{j=0}^{k} D_j = I \), we set \( \bar{D} = \text{diag}(\bar{d}) = I - \bar{D}_0 \), i.e., \( \bar{d}_i = 1 - (\bar{d}_0)_i \), \( \bar{d} \in [0, 1]^n \). Let \( D_l = \text{diag}(d_l), l = 1, 2, \ldots, k \), where \( (d_l)_i = \frac{(d_i)}{d_l} \) if \( d_l \neq 0 \); \( (d_l)_i = \frac{1}{k} \) if \( d_l = 0 \). Further,

\[
D_0A_0 + D_1A_1 + \ldots + D_kA_k = (I - \bar{D})A_0 + \bar{D}(D_1A_1 + D_2A_2 + \ldots + D_kA_k),
\]

and \( \sum_{j=1}^{k} D_j = I \). So matrix

\[
\bar{D}_0A_0 + \bar{D}_1A_1 + \ldots + \bar{D}_kA_k
\]
is nonsingular. By using Lemma 3.4, \( A \) has the row \( W \)-property.

\((\Leftarrow) \) If \( A \) has the row \( W \)-property, then for arbitrary nonnegative diagonal matrix \( \bar{D}_j = \text{diag}(\bar{d}_j) \) with \( \bar{d}_j \in [0, 1]^n \) \((j = 0, 1, \ldots, k)\) and \( \sum_{j=0}^{k} D_j = I \), matrix

\[
D_0A_0 + D_1A_1 + \ldots + D_kA_k
\]
is nonsingular. In the following, we prove that \( \bar{A} = (A_0, D_1A_1 + D_2A_2 + \ldots + D_kA_k) \) has the row \( W \)-property, where matrices \( D_j = \text{diag}(d_j) \) with \( d_j \in [0, 1]^n \) \((j = 1, \ldots, k)\) are arbitrary nonnegative diagonal matrices and \( \sum_{j=0}^{k} D_j = I \). In fact, for arbitrary nonnegative diagonal matrix \( \hat{D} = \text{diag}(\hat{d}) \) with \( \hat{d} \in [0, 1]^n \),

\[
(I - \hat{D})A_0 + \hat{D}(D_1A_1 + D_2A_2 + \ldots + D_kA_k) = (I - \hat{D})A_0 + \hat{D}D_1A_1 + \hat{D}D_2A_2 + \ldots + \hat{D}D_kA_k.
\]

Let \( D_0 = I - \hat{D}, D_1 = \hat{D}D_1, \ldots, D_k = \hat{D}D_k \). Then \( \hat{d}_j \in [0, 1]^n \) \((j = 0, 1, \ldots, k)\), \( \sum_{j=0}^{k} D_j = I \), and matrix

\[
(I - \hat{D})A_0 + \hat{D}(D_1A_1 + D_2A_2 + \ldots + D_kA_k) = \hat{D}_0A_0 + \hat{D}_1A_1 + \ldots + \hat{D}_kA_k
\]
is nonsingular. By using Lemma 3.4 again, \( \bar{A} \) has the row \( W \)-property. \( \square \)

From Lemma 3.5, together with Lemma 3.1, for the \((3.9)\) type error bound, we give a general result for the EVLCP \((A, q)\) under the row \( W \)-property.

**Theorem 3.4** Let matrix \( A = (A_0, A_1, \ldots, A_k) \) have the row \( W \)-property. Then for any \( x \in \mathbb{R}^n \),

\[
\|x - x^*\|_\infty \leq \min_{D_1, \ldots, D_k} \left\{ \frac{\|A_0 + \sum_{j=1}^{k} D_jA_j\|_\infty}{\alpha\{A_0, A_1, \ldots, A_k\}} \right\} \|r(x)\|_\infty,
\]

where matrices \( D_j = \text{diag}(d_j) \) with \( d_j \in [0, 1]^n \) \((j = 1, \ldots, k)\) are arbitrary nonnegative diagonal matrices and \( \sum_{j=1}^{k} D_j = I \), and

\[
\alpha\{A_0, A_1, \ldots, A_k\} = \min_{\|x\|_\infty = 1} \left\{ \max_{1 \leq i \leq n} \left( \frac{\max (A_0x)_i \left( \sum_{j=1}^{k} D_jA_j \right) x_i}{\|x\|_\infty} \right) \right\}.
\]
Proof. Since matrix $A = (A_0, A_1, \ldots, A_k)$ has the row $W$-property, from Lemma 3.5, matrix $\bar{A} = (A_0, D_1 A_1 + D_2 A_2 + \ldots + D_k A_k)$ has the row $W$-property, where matrices $D_j = \text{diag}(d_j)$ with $d_j \in [0,1]^n$ ($j = 1, \ldots, k$) are arbitrary nonnegative diagonal matrices and $\sum_{j=1}^k D_j = I$. As done in [22], we introduce a quantity $\alpha\{A_0, A_1, \ldots, A_k\}$ below

$$\alpha\{A_0, A_1, \ldots, A_k\} = \min_{\|x\|_{\infty} = 1} \left\{ \max_{1 \leq i \leq n} (A_0 x)_i \left( \left( \sum_{j=1}^k D_j A_j \right) x \right)_i \right\}.$$ 

Further, we take

$$s(x) = (A_0 x + q_0) - r(x) \text{ and } t(x) = \min_{1 \leq j \leq k} \{ A_j x + q_j \} - r(x).$$

Then

$$s(x) \geq 0, t(x) \geq 0, s(x)^T t(x) = 0.$$ 

Thus, for any $x \in \mathbb{R}^n$ and each $i = 1, 2, \ldots, n$, together with Lemma 3.1, we obtain

$$0 \geq (s_i(x) - s_i(x^*)) (t_i(x) - t_i(x^*))$$

$$= ((A(x - x^*))_i - r_i(x) + r_i(x^*)) \left( \left( \sum_{j=1}^k D_j A_j \right) (x - x^*) \right)_i - r_i(x) + r_i(x^*))$$

$$\geq ((A(x - x^*))_i \left( \sum_{j=1}^k D_j A_j \right) (x - x^*))_i$$

$$- (r_i(x) - r_i(x^*)) \left( A + \sum_{j=1}^k D_j A_j \right) (x - x^*))_i$$

$$\geq ((A(x - x^*))_i \left( \sum_{j=1}^k D_j A_j \right) (x - x^*))_i$$

$$- \|r(x)\|_{\infty} \cdot \|A_0 + \sum_{j=1}^k D_j A_j\|_{\infty} \cdot \|x - x^*\|_{\infty},$$

from which it follows that

$$\|A_0 + \sum_{j=1}^k D_j A_j\|_{\infty} \cdot \|r(x)\|_{\infty} \cdot \|x - x^*\|_{\infty} \geq \max_{1 \leq i \leq n} ((A(x - x^*))_i \left( \sum_{j=1}^k D_j A_j \right) (x - x^*)),$$

$$\geq \alpha\{A_0, A_1, \ldots, A_k\} \|x - x^*\|_{\infty}.$$ 

This yields the desired inequality (3.10). $\square$
4 Numerical examples

Since the advantages and disadvantages of Theorem 3.2 and Theorem 3.3 for numerical examples have been presented in [19], we here compare Theorem 3.1 with Eq. (1.5), also see Theorem 4.1 in [19]. Numerical examples used by us are from two aspects: (1) on the one hand, we still adopt a example in [19]; on the other hand, we list some new examples.

Example 4.1 [19] Let $A = (A_0, A_1, A_2)$, where

$$A_0 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}.$$ 

Since $(A_0)_2 = (A_1)_2 = (A_2)_2$, this particularity avoids the row rearrangement. Even so, by Eq. (1.5), their calculations take 3 times:

$$\mu_1 = \max_{d \in [0, 1]^n} \|(I - D)A_0 + DA_1)^{-1}\|_\infty = 1,$$

$$\mu_2 = \max_{d \in [0, 1]^n} \|(I - D)A_0 + DA_2)^{-1}\|_\infty = 1,$$

$$\mu_3 = \max_{d \in [0, 1]^n} \|(I - D)A_1 + DA_2)^{-1}\|_\infty = 1.$$

Based on this,

$$\alpha_\infty(A) = \max\{\mu_1, \mu_2, \mu_3\} = 1.$$

Directly using Theorem 3.1, for $d_1 + d_2 \leq 1$ with $d_1, d_2 \in [0, 1]$, we calculate

$$\max_{d \in [0, 1]^n} \|(D_0A_0 + D_1A_1 + D_2A_2)^{-1}\|_\infty = \max\left\{\frac{2 + 2d_2}{2 + 2d_2 + d_1}, \frac{2 + d_1}{2 + d_1 + 2d_2}\right\} = 1 = \alpha_\infty(A).$$

Whereas, once the system matrix occurs the row rearrangement, the calculation times by Eq. (1.5) increases dramatically, see Example 2.1 and Example 4.3.

Example 4.2 Here, Example 2.1 is used as Example 4.2 to investigate Theorem 3.1. Because the main diagonal elements of $A_0, A_1, A_2$ are positive, the second element in the first row is non-negative, and the first element in the second row is non-positive, so, for matrix $D_0A_0 + D_1A_1 + D_2A_2$, its main diagonal elements are positive, the second element in the first row is non-negative, and the first element in the second row is non-positive for any $D_j = \text{diag}(d_j)$ with $d_j \in [0, 1]^n$ $(j = 0, 1, 2)$ being nonnegative diagonal matrices and $\sum_{j=0}^2 D_j = I$. According to the definition of the determinant, we can get $\det(D_0A_0 + D_1A_1 + D_2A_2) > 0$, i.e., $D_0A_0 + D_1A_1 + D_2A_2$ is nonsingular. From Lemma 3.4, the block matrix $A = (A_0, A_1, A_2)$ has row $\mathcal{W}$-property.

Directly using Theorem 3.1, for $D_j = \text{diag}(d_j)$ with $d_j \in [0, 1]^n$ $(j = 0, 1, 2)$ are nonnegative diagonal matrices and $\sum_{j=0}^2 D_j = I$, we calculate

$$\max_{d \in [0, 1]^n} \|(D_0A_0 + D_1A_1 + D_2A_2)^{-1}\|_\infty = \max\{w_1, w_2\} = 3,$$

where

$$w_1 = \frac{2 - (d_1)_{11}}{1 + (d_2)_{11} + (1 - (d_1)_{11})(1 + (d_1)_{22} - (d_2)_{22})}.$$
and

\[ w_2 = \frac{2 + (d_1)_{22}}{1 + (d_2)_{11} + (1 - (d_1)_{11})(1 + (d_1)_{22} - (d_2)_{22})}. \]

When using Eq. (1.5), although \( \alpha_\infty(A) = 3 \), see Example 2.1, which is equal to our this result, its calculation times are 9.

It is easy to check that \( A_0, A_1 \) in Example 4.2 are not strictly diagonal dominant, and that

\[ \rho(\max\{\Lambda^{-1}_0|C_0|, \Lambda^{-1}_1|C_1|, \Lambda^{-1}_2|C_2|\}) = 1.4142 > 1. \]

Hence the error bounds in Theorem 3.2, Theorem 3.3, Theorem 4.3 and Theorem 4.4 of [19] cannot work for this case.

**Example 4.3** Let \( A = (A_0, A_1, A_2, A_3) \), where

\[
A_0 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, 
A_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, 
A_2 = \begin{bmatrix} 4 & 1 \\ 0 & 1 \end{bmatrix}, 
A_3 = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}
\]

With the same analysis method as in Example 4.3, it is easy to get that \( A = (A_0, A_1, A_2, A_3) \) has row \( \mathcal{W} \)-property. Directly using Theorem 3.1, for \( D_0, D_1, D_2, D_3 \) are nonnegative diagonal matrices with \( d_j \in [0, 1]^n \) (\( j = 0, 1, 2, 3 \)) and \( \sum_{j=0}^{3} D_j = I \), we calculate

\[
\max_{d \in [0,1]^n} \|(D_0 A_0 + D_1 A_1 + D_2 A_2 + D_3 A_3)^{-1}\|_\infty = \max \{\bar{w}_1, \bar{w}_2\} = 4 = \alpha_\infty(A).
\]

where

\[
\bar{w}_1 = \frac{2 - (d_1)_{22} - (d_2)_{22} + 2(d_3)_{22}}{1 + 3(d_2)_{11} + (d_3)_{11} + (1 + (d_1)_{22} - (d_2)_{22} + 2(d_3)_{22})(1 - (d_1)_{11} - (d_3)_{11})}
\]

and

\[
\bar{w}_2 = \frac{2 - (d_1)_{11} + 3(d_2)_{11}}{1 + 3(d_2)_{11} + (d_3)_{11} + (1 + (d_1)_{22} - (d_2)_{22} + 2(d_3)_{22})(1 - (d_1)_{11} - (d_3)_{11})}.
\]

Of course, we can adopt Eq. (1.5) to obtain \( \alpha_\infty(A) = 4 \), whereas, its calculation times are 36. Here, the computational process of Eq. (1.5) is omitted.

It is easy to check that \( A_0, A_1, A_3 \) are not strictly diagonal dominant, and that

\[ \rho(\max\{\Lambda^{-1}_0|C_0|, \Lambda^{-1}_1|C_1|, \Lambda^{-1}_2|C_2|, \Lambda^{-1}_3|C_3|\}) = 2.3028 > 1. \]

For Example 4.3, the error bounds in Theorem 3.2, Theorem 3.3, Theorem 4.3 and Theorem 4.4 of [19] cannot work for this case as well.

From Example 2.1 and Example 4.3, it is easy to find that although the latter has one more matrix than the former, the calculation times of the latter are more 27 times than the former.
5 Conclusion

In this paper, by introducing a general equivalent form of the minimum function, some new error bounds for the EVLCP($A, q$) under the row $W$-property have been presented. These new error bounds not only cover some existing results, but also keep away from the row rearrangement of the system matrix. In addition, with respect to the row $W$-property, we also obtain two new sufficient and necessary conditions. Finally, by numerical examples, we show that the new error bound is feasible. Compared with the error bound by the row rearrangement technique, our new error bound greatly reduces the computation workload in a way.

In addition, by a lot of numerical experiments, we find that the error bound (3.5) is equal to the error bound (1.5). There exists an interested problem how to prove the equality of both, which is necessary for us to study it in the future.

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