ISOPERIMETRIC PROBLEMS IN SECTORS WITH DENSITY

Alexander Díaz, Nate Harman, Sean Howe, David Thompson

Abstract. We consider the isoperimetric problem in planar sectors with density $r^p$, and with density $a > 1$ inside the unit disk and 1 outside. We characterize solutions as a function of sector angle. We also solve the isoperimetric problem in $\mathbb{R}^n$ with density $r^p$, $p < 0$.

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1. Introduction

A density on the plane is a function weighting both perimeter and area, or, on a general manifold, a function weighting surface area and volume. Manifolds with density occur broadly in mathematics: from probability, where the classical example of Gauss space ($\mathbb{R}^n$ with density $ce^{-a^2r^2}$) plays an important role, to Riemannian geometry, where they appear as quotient spaces of Riemannian manifolds, and even to physics, where they describe spaces with different mediums. They have recently been in the spotlight for their role in Perelman’s proof of the Poincaré Conjecture. (For general references on these topics, see [M5] or better [M1, Ch. 18].)

We study the isoperimetric problem in planar sectors with certain densities. In this context, the isoperimetric problem seeks to enclose prescribed (weighted) area with least (weighted) perimeter (not counting the boundary of the sector). Solutions are known for very few surfaces with densities (see Sect. 2 below). Our major theorem after Dahlberg et al. [DDNT, Thm. 3.16] characterizes isoperimetric curves in a $\theta_0$-sector with density $r^p$, $p > 0$:

**Theorem.** (4.17) Given $p > 0$, there exist $0 < \theta_1 < \theta_2 < \infty$ such that in the $\theta_0$-sector with density $r^p$, isoperimetric curves are (see Fig. 1.1):

1. for $0 < \theta_0 < \theta_1$, circular arcs about the origin,
2. for $\theta_1 < \theta_0 < \theta_2$, undularies,
3. for $\theta_2 < \theta_0 < \infty$, semicircles through the origin.

We give bounds on $\theta_1$ and $\theta_2$ in terms of $p$, and a conjecture on their exact values. Section 5 gives further results on constant generalized curvature curves. Sectors with density $r^p$ are related to $L^p$ spaces (see e.g. Cor. 4.24), have vanishing
The three isoperimetric curves for sectors with density \( r^p \): a circular arc about the origin for small sectors, an undulatory for medium sectors, and a semicircle through the origin for large sectors.

In Theorems 6.3, 6.4, and 6.5, we provide a relatively minor result after Cañete et al. [CMV, Thm. 3.20] characterizing isoperimetric curves in a \( \theta_0 \)-sector with density \( a > 1 \) inside the unit disk and density 1 outside the unit disk. An interesting property of this problem is that it deals with a noncontinuous density. There are five different kinds of isoperimetric regions depending on \( \theta_0, a \), and the prescribed area, shown in Figure 6.1.

In Section 7, we discuss basic results in \( \mathbb{R}^n \) with radial density. We use a simple averaging technique of Carroll et al. [CJQW, Prop. 4.3] to give a short proof of a result of Betta et al. [BBMP, Thm. 4.3] and our Thm. 7.4 and to solve the isoperimetric problem in \( \mathbb{R}^n \) with density \( r^p, p < 0 \) (Thm. 7.5).
about the origin and semicircles through the origin in a \( \pi \)-sector. In this paper, we consider \( \theta_0 \)-sectors for general \( 0 < \theta_0 < \infty \).

For \( p \in (-\infty, -2) \cup (0, \infty) \), existence in the \( \theta_0 \)-sector follows from standard compactness arguments (Prop. 2.4). Lemma 4.7 limits the possibilities to circular arcs about the origin, semicircles through the origin, and undularies (nonconstant positive polar graphs with constant generalized curvature). Proposition 4.12 shows that if the circle is not uniquely isoperimetric for some angle \( \theta_0 \), it is not isoperimetric for all \( \theta > \theta_0 \). Therefore, transitional angles \( 0 \leq \theta_1 \leq \theta_2 \leq \infty \) exist. Isoperimetric regions that depend on sector angle have been seen before, as in the characterization by Lopez and Baker [LB, Thm. 6.1, Fig. 10] of perimeter-minimizing double bubbles in the Euclidean cone of varying angles, which is equivalent to the Euclidean sector. Theorem 4.17 provides estimates on the values of \( \theta_1 \) and \( \theta_2 \).

We conjecture (Conj. 4.18) that \( \theta_1 = \pi/\sqrt{p + 1} \) and \( \theta_2 = \pi(p + 2)/(2p + 2) \). Proposition 4.16 proves that the circle about the origin has positive second variation for all \( \theta_0 < \pi/\sqrt{p + 1} \), and Proposition 4.10 proves the semicircle through the origin is not isoperimetric for all \( \theta_0 < \pi(p + 2)/(2p + 2) \). Using the characterization of constant generalized curvature curves in Section 5, we wrote a computer program to predict isoperimetric curves in sectors with density \( r^p \), and the results of this program also support our conjecture as can be seen for \( p = 1 \) in Figure 1.2.

An easy symmetry argument (Prop. 3.2) shows that the isoperimetric problem in the \( \theta_0 \)-sector is equivalent to the isoperimetric problem in the \( 2\theta_0 \)-cone, a cone over \( S^1 \). We further note that the isoperimetric problem in the cone over \( S^1 \) with density \( r^p \) is equivalent to the isoperimetric problem in the cone over the product of \( S^1 \) with a \( p \)-dimensional manifold \( M \) among regions symmetric under a group of isometries acting transitively on \( M \). This provides a classical interpretation of the problem, which we use in Proposition 4.9 to obtain an improved bound for \( \theta_1 \) in the \( p = 1 \) case by taking \( M \) to be a rectangular two-torus.

Theorem 4.20 classifies isoperimetric regions of small area in planar polygons with density \( r^p \). Proposition 4.21 through Corollary 4.24 reinterpret Theorem 4.17 and Conjecture 4.18 in terms of the plane with differing area and perimeter densities and in terms of analytic inequalities.

1.2. Constant Generalized Curvature Curves. Section 5 provides further details on constant generalized curvature curves in the sector with density \( r^p \), which are of interest since isoperimetric curves must have constant generalized curvature (see Sect. 2). Proposition 5.2 proves Conjecture 4.18 under the hypothesis that undulary periods are bounded by the conjectured values of \( \theta_1 \) and \( \theta_2 \).

1.3. The Sector with Disk Density. Section 6 considers a sector of the plane with density \( a > 1 \) inside the unit disk and 1 outside. Cañete et al. [CMV] Sect. 3.3 consider this problem in the plane, which is equivalent to the \( \pi \)-sector. Proposition 6.2 gives the five possibilities of Figure 6.3. Our Theorems 6.3, 6.4, and 6.5 classify isoperimetric curves in a \( \theta_0 \)-sector, depending on \( \theta_0 \), density \( a \), and area.

1.4. \( \mathbb{R}^n \) with Radial Density. Section 7 considers the isoperimetric problem in \( \mathbb{R}^n \) with radial density. We use spherical symmetrization to reduce the problem to a two-dimensional isoperimetric problem in a plane with density. Using ideas from Carroll et al. [CJQW] Prop. 4.3, in Theorem 7.4 we give a new proof of a result
Theorem 4.17 says that there are angles $\theta_1$ and $\theta_2$ such that isoperimetric curves are circular arcs until $\theta_1$, undularies between $\theta_1$ and $\theta_2$, and semicircles after $\theta_2$. For $p = 1$, Conjecture 4.18 says that $\theta_1 = \pi / \sqrt{2}$ and $\theta_2 = 3 \pi / 4$. Numerical predictions of isoperimetric regions (above) agree with this conjecture.

of Betta et al. [BBMP, Thm 4.3] on perimeter densities in $\mathbb{R}^n$. We then use the same technique to prove that hyperspheres about the origin are isoperimetric in $\mathbb{R}^n$ with density $r^p$, $p < -n$. In Proposition 7.3 we provide a nonexistence result for $-n \leq p < 0$, and we conclude by conjecturing that hyperspheres through the origin are isoperimetric for $p > 0$ (Conj. 7.6).

1.5. Open Questions. We provide here a concise list of open questions related to this paper:

1. How can one prove Conjecture 4.18 on the values of the transitional angles $\theta_1$ and $\theta_2$?
2. Could the values of $\theta_1$ and $\theta_2$ be proven numerically for fixed $p$?
3. If a circular arc about the origin is isoperimetric in the $\theta_0$-sector with density $r^p$, is it isoperimetric in the same $\theta_0$-sector with density $r^q$, $q < p$?
4. Are circles about the origin isoperimetric in the Euclidean plane with perimeter density $r^p$, $p \in (0, 1)$? (See Rmk. after Conj. 4.22)
5. In the plane with density $r^p$, do curves with constant generalized curvature near that of the semicircle have half period $T \approx \pi (p + 2) / (2p + 2)$? (See Conj. 4.18 and Rmk. 5.3 for the corresponding result near the circular arc.)
6. Are spheres through the origin isoperimetric in $\mathbb{R}^n$ with density $r^p$, $p > 0$? (See Conj. 7.6)
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2. **Isoperimetric Problems in Manifolds with Density**

A *density* on a Riemannian manifold is a nonnegative, lower semicontinuous function $\Psi(x)$ weighting both volume and hypersurface area. In terms of the underlying Riemannian volume $dV_0$ and area $dA_0$, the weighted volume and area are given by $dV = \Psi dV_0$, $dA = \Psi dA_0$. Manifolds with densities arise naturally in geometry as quotients of Riemannian manifolds, in physics as spaces with different mediums, in probability as the famous Gauss space $\mathbb{R}^n$ with density $\Psi = ce^{-a^2r^2}$, and in a number of other places as well (see [M5] or better [M1, Ch. 18]).

For a curve in a two-dimensional manifold $M$ with density $e^\psi$, we define the generalized curvature $\lambda$ by

$$\lambda = \kappa - \frac{d\psi}{dn},$$

where $\kappa$ is the usual curvature and $n$ is the unit normal vector. This is the correct generalization of curvature to manifolds with density in that it provides a generalization of variational formulae [CHHSX, Prop 3.2].

The *isoperimetric problem* on a two-dimensional manifold with density seeks to enclose a given weighted area with the least weighted perimeter. As in the Riemannian case, for a smooth density isoperimetric curves have constant generalized curvature [CHHSX, Prop 4.2]. The solution to the isoperimetric problem is known only for a few manifolds with density including Gauss space (see [MI, Ch. 18]) and the plane with a handful of different densities (see Betta et al. [BBMP], Cañete et al. [CMV, Sect. 3], Dahlberg et al. [DDNT] Thm 3.16], Engelstein et al. [EMMP, Cor. 4.9], Rosales et al. [RCBM, Thm. 5.2], and Maurmann and Morgan [MM, Cor. 2.2]). Existence and regularity are discussed in our report [DHHT, Sect. 3], and the major results are given here.

**Proposition 2.1.** In the $\theta_0$-sector with density $r^p$ for $p \in (-2, 0)$, no isoperimetric regions exist.

**Proof.** Carroll et al. [CJQW, Prop. 4.2] prove this in the plane by constructing curves of arbitrarily low perimeter bounding any area. Their argument extends immediately to the sector. \hfill $\square$

The next theorem gives a general existence condition on isoperimetric surfaces in manifolds with volume density.
Theorem 2.2. Suppose $M^n$ is a smooth connected possibly non-complete $n$-dimensional Riemannian manifold with isoperimetric function $I$ such that $\lim_{A \to \infty} I(A) = \infty$. Suppose furthermore that every closed geodesic ball of finite radius in $M$ has finite volume and finite boundary area and there is some $C \in \mathbb{Z}^+$ and $x_0 \in M$ such that the complement of any closed geodesic ball of finite radius about $x_0$ contains finitely many connected components and at most $C$ unbounded connected components. Then for standard boundary area and any lower semi-continuous positive volume density $f$ such that

1. for some $x_0 \in M^n$ sup$\{f(x)\text{dist}(x, x_0) > R\}$ goes to 0 as $R$ goes to $\infty$;
2. for some $\epsilon > 0$, $\{x|B(x, \epsilon) \text{ is not complete}\}$ has finite weighted volume;

isoperimetric regions exist for any positive volume less than the volume of $M$.

Proof. Using standard compactness arguments from geometric measure theory we see that it suffices to show that no weighted volume can escape outside of an increasing sequence of compact sets whose union is $M^n$. That no weighted volume can escape to points of noncompleteness is immediate from the second condition on the density. To show that none can escape to infinity we first isolate a finite number of “ends” of the manifold and then show that in each of these no weighted volume can disappear either because there is only finite weighted volume there or because we can use the standard isoperimetric inequality combined with the density approaching 0.

The following proposition establishes an equivalence between various sectors with possibly differing perimeter and area densities.

Proposition 2.3. For any $n \in \mathbb{R}\backslash\{0\}$ the $\theta_0$-sector with perimeter density $r^p$ and area density $r^q$ is equivalent to the $|n|\theta_0$-sector with perimeter density $r^{(p+1)/n-1}$ and area density $r^{(q+2)/n-2}$.

Proof. Make the change of coordinates $w = z^n/n$.

The next two propositions give the existence and regularity of isoperimetric curves in the main cases that we examine in the rest of the paper.

Proposition 2.4. In the $\theta_0$-sector with density $r^p$ for $p \in (-\infty, -2) \cup (0, \infty)$ isoperimetric curves exist and are smooth curves with constant generalized curvature.

Proof. By Proposition 2.3 the $\theta_0$-sector with density $r^p$ is equivalent to a $(p+1)\theta_0$-sector with Euclidean perimeter and area density $r^q$ for $q \in (-2, 0)$. As we will see in Proposition 2.2 the isoperimetric problem in the sector with radial density is equivalent to the isoperimetric problem in a circular cone of twice the angle with the same radial density. Furthermore it suffices to consider the cone with the vertex deleted, which is a smooth manifold. Existence then follows from Theorem 2.2 and smoothness from Morgan [M6, Cor. 3.8, Sect. 3.10]. Constant generalized curvature follows from variational arguments.

Proposition 2.5. In the $\theta_0$-sector with density $a > 1$ inside the unit disk $D$ and 1 outside, isoperimetric curves exist for any given area. These isoperimetric curves are smooth except at the boundary of $D$, where they obey the following Snell refraction rule (see Fig. 2.7):

$$\frac{\cos \alpha_+}{\cos \alpha_-} = \frac{1}{a},$$
where $\alpha_+$ is the angle of intersection from inside of $D$ and $\alpha_-$ is the angle of intersection from outside of $D$.

**Proof.** Cañete et al. [CMV, Thm. 3.18] prove existence for the plane. The same result and proof hold for the $\theta_0$-sector. The regularity and Snell refraction rule follow from [CMV, Prop. 2.14].\qed

## 3. Isoperimetric Regions in Sectors with Density

We study properties of isoperimetric regions in planar sectors with radial densities. Proposition 3.2 shows there is a one-to-one correspondence between isoperimetric curves in the $\theta_0$-sector and in the $2\theta_0$-cone, modulo rotations. Propositions 3.4 and 3.5 provide some regularity results.

**Lemma 3.1.** Given an isoperimetric region in the $2\theta_0$-cone with density $f(r)$, there exist two rays from the origin separated by an angle of $\theta_0$ that divide both the area and perimeter of the region in half.

**Proof.** First we show that there are two rays that separate the area of the region in half. Take any two rays from the origin separated by an angle of $\theta_0$. Rotate them, keeping an angle of $\theta_0$ between them. The areas bounded by the rays and the boundary of the isoperimetric region vary continuously as we rotate, and so does their difference. Let the initial difference in areas be $\Delta A$. When we rotate the rays by an angle of $\theta_0$, the difference in areas is $-\Delta A$. So, there is a point where the difference is 0, implying that there are two rays separated by an angle $\theta_0$ that divide the area in half. If one side had less perimeter than the other, we could reflect it to obtain a region with the same area and less perimeter than our original region, violating the assumption that it is isoperimetric.\qed

**Proposition 3.2.** An isoperimetric region of area $2A$ in the cone of angle $2\theta_0$ with density $f(r)$ has perimeter equal to the twice the perimeter of an isoperimetric region of area $A$ in the sector of angle $\theta_0$ with density $f(r)$. Indeed, the operation
of doubling a sector to form a cone provides a one-to-one correspondence between isoperimetric regions in the sector and isoperimetric regions in the cone, modulo rotations.

Proof. Given an isoperimetric region in the sector, take its reflection into the cone to obtain a region in the cone with twice the area and twice the perimeter. This region must be isoperimetric for the cone, for if there were a region in the cone with less perimeter for the same area we could divide its area in half by two rays separated by an angle \( \theta_0 \) as in Lemma 3.1, take the side with at most half the perimeter to obtain a region in the \( \theta_0 \)-sector with the same area and less perimeter than our original isoperimetric region.

Conversely, given an isoperimetric region in the cone, divide its area and perimeter in half by the two rays described in Lemma 3.1. Both regions must be isoperimetric in the sector, for if there were a region with less perimeter for the same area, taking its double would yield a region in the cone with less perimeter than our original region for the same area. □

We shall find many occasions to use the following simple generalization of a proposition of Dahlberg et al.

**Proposition 3.3.** Consider a circular cone with smooth radial density \( e^{\psi(r)} \). A constant generalized curvature curve is symmetric under reflection across every line through the origin and a critical point of the radius function, in the sense that if \( \gamma \) is an arclength parameterization of the curve and \( s_0 \) is a critical point of the curve’s radius then for all \( s \) such that \( \gamma(s_0 + s) \) and \( \gamma(s_0 - s) \) are defined we have

\[
\gamma(s_0 + s) = R(\gamma(s_0 - s))
\]

where \( R \) is reflection about that line.

Proof. As in [DDNT, Lem. 2.1], this follows from the uniqueness of solutions to ordinary differential equations applied to the constant curvature equation. □

**Proposition 3.4.** In the \( \theta_0 \)-sector with smooth density \( f(r) \), isoperimetric curves meet the boundary perpendicularly.

Proof. By Proposition 3.2 there is a one-to-one correspondence between isoperimetric curves in the sector and the cone, meaning the double of this curve in the cone of angle \( 2\theta_0 \) is isoperimetric and hence smooth (Prop. 2.4). But the double can only be smooth at the points corresponding to the boundary points of the original curve if the intersection of the original curve with the boundary is perpendicular. □

**Proposition 3.5.** In the \( \theta_0 \)-sector with smooth density \( f(r) \), if isoperimetric curves are nonconstant polar graphs (in the sense that any radial ray intersects the curve exactly once), they do not contain a critical point of radius on the interior.

Proof. Assume there is an isoperimetric curve with a critical point of radius on the interior. Then by doubling as in Proposition 3.2 we get an isoperimetric curve on the \( 2\theta_0 \) cone, given in polar coordinates by a function \( r \). Since by \( 3.4 \) the initial curve met the boundary perpendicularly, \( r \) has critical points at 0 and \( \theta_0 \), as well as the given critical point between 0 and \( \theta_0 \). Since constant generalized curvature curves are symmetric under reflection across a line through the origin and a critical point of \( r \) (Prop. 3.3), we see that this curve has at least four critical points. By symmetry, critical points must be strict extrema. Let \( C \) be a circle about the axis intersecting the curve in at least four points. \( C \) divides the curve into at least two regions above \( C \) and two regions below \( C \). Interchanging one region above \( C \)
with a region below \( C \) results in a region with the same perimeter and area whose boundary is not smooth. Since isoperimetric curves must be smooth, \( r \) cannot be isoperimetric.

\[ \square \]

**Corollary 3.6.** If an isoperimetric curve \( r(\theta) \) in the \( \theta_0 \)-sector with smooth density is a nonconstant polar graph, \( r \) must be strictly monotonic.

**Remark.** Proposition 3.5 and Corollary 3.6 strengthen some arguments of Adams et al. \[ACDLV\], Lem. 3.6.

### 4. The Isoperimetric Problem in Sectors with Density \( r^p \)

Our main result, Theorem 4.17, characterizes isoperimetric regions in planar sectors with density \( r^p \), \( p > 0 \). The subsequent results consider reformulations of the problem in terms of differing perimeter and area densities in the plane and in terms of analytic inequalities.

**Proposition 4.1.** In the half plane with density \( r^p \), \( p > 0 \), semicircles through the origin are the unique isoperimetric curves.

**Proof.** By Proposition 3.2, there is a one-to-one correspondence between isoperimetric curves in the \( \theta_0 \)-sector and the \( 2\theta_0 \)-cone; in particular there is a correspondence between isoperimetric curves in the half plane and isoperimetric curves in the \( 2\pi \)-cone, i.e., the plane. Since circles through the origin are uniquely isoperimetric in the plane (see Sect. 1.1), semicircles through the origin are uniquely isoperimetric in the half plane.

\[ \square \]

**Proposition 4.2.** For density \( r^p \), if the circular arc about the origin is not uniquely isoperimetric in the \( \theta_0 \)-sector, for all \( \theta > \theta_0 \) it is not isoperimetric.

**Proof.** Let \( \gamma \) be a non-circular isoperimetric curve in the \( \theta_0 \)-sector, and let \( C \) be a circular arc bounding the same area as \( \gamma \). For any angle \( \theta > \theta_0 \), transition to the \( \theta \)-sector via the map \( \alpha \to \alpha \theta/\theta_0 \). This map multiplies area by \( \theta/\theta_0 \), and scales tangential perimeter. Therefore, if \( \gamma \) had the same or less perimeter than \( C \) in the \( \theta_0 \)-sector, its image under this map has less perimeter than a circular arc about the origin in the \( \theta \)-sector.

\[ \square \]

**Remark.** An equivalent statement of Proposition 4.2 is that if the circular arc about the origin is isoperimetric for some sector angle \( \theta_0 \), then it is uniquely isoperimetric for all \( \theta < \theta_0 \).

**Proposition 4.3.** In the \( \theta_0 \)-sector with density \( r^p \), \( p > 0 \), an isoperimetric region contains the origin, and its boundary is a polar graph.

**Proof.** By Proposition 2.3, the \( \theta_0 \)-sector with density \( r^p \) is equivalent to the \( (p+1)\theta_0 \) sector with Euclidean perimeter and area density \( r^{-q} \), \( q = p/(p+1) \). Since the properties in question are preserved by this change of coordinates, we can work in this space. Since the area density is strictly decreasing away from the origin, an isoperimetric region must contain the origin. Any isoperimetric region is bounded by a smooth curve of constant generalized curvature (Thm. 2.4), and since here generalized curvature is just classical curvature divided by the area density \( \text{CJQW} \), Def. 3.1, Prop. 3.2], the classical curvature does not change sign, and thus the curve is convex. As the isoperimetric region contains the origin and its boundary is convex, its boundary must be a polar graph.

\[ \square \]
Proposition 4.4. In the $\theta_0$-sector with density $r^p$, $p > 0$, the least-perimeter ‘isoperimetric’ function $I(A)$ satisfies

$$I(A) = cA^{p+1}/(p+2).$$

Proof. Dahlberg et al. prove this in the plane \[\text{DDNT, Lem. 3.7}\], and the same radial scaling argument works in the sector. □

Definition 4.5. The isoperimetric ratio of a $\theta_0$-sector with density $r^p$ is the constant $c$ from Proposition 4.4.

Definition 4.6. An undulary is a nonconstant positive polar graph with constant generalized curvature.

We note here a useful result from Dahlberg et al. \[\text{DDNT, Prop. 2.11}\]: In a planar domain with density $r^p$, $p > 0$, if a constant generalized curvature closed curve passes through the origin, it must be a circle.

Lemma 4.7. In the $\theta_0$-sector with density $r^p$, $p > 0$, isoperimetric curves are either circles about the origin, semicircles through the origin, or undularies.

Proof. By Proposition 2.4, minimizers exist, and by Proposition 4.3, they must be polar graphs with constant generalized curvature bounding regions that contain the origin. If the minimizer has constant radius, it is a circular arc. If the minimizer has nonconstant radius, it either remains positive, in which case it is an undulary, or goes through the origin. If the minimizer goes through the origin, it must be part of a circle through the origin (\[\text{DDNT, Prop. 2.11}\]). In order to meet the boundary perpendicularly, the curve must consist of an integer number of semicircles. Suppose we have $n$ semicircles each bounding a region of area $A_i$ with perimeter $P_i$. Since all semicircles through the origin have the same ratio $P_i/A_i^{(p+1)/(p+2)} = c$, we have

$$P = \sum P_i = c \sum A_i^{(p+1)/(p+2)} > c \left( \sum A_i \right)^{(p+1)/(p+2)} = P_1,$$

where $P_1$ is the perimeter of a single semicircle through the origin bounding area $A = \sum A_i$. Therefore, if a minimizer passes through the origin, it is a single semicircle. □

Theorem 4.8. In the $\pi/(p+1)$-sector with density $r^p$, $p > 0$, isoperimetric curves are circular arcs.

Proof. We change coordinates as in Proposition 2.4 to the half plane with Euclidean perimeter. By Proposition 4.3, an isoperimetric curve is a polar graph $r(\theta)$. Suppose that it is not a circular arc. By Proposition 3.3, $r(0) \neq r(\pi)$, and $r'(\theta) = 0$ at 0 and $\pi$, and nowhere else. Reflect $r$ over the x-axis, obtaining a closed curve. By the four-vertex theorem \[O\], this curve has at least four extrema of classical curvature. Generalized curvature in Euclidean coordinates is classical curvature divided by the area density \[CJQW, Def. 3.1\]. That is:

$$\kappa_\varphi = cr^p/p+1\kappa$$

for some $c > 0$. At an extremum of classical curvature we see

$$0 = \frac{d}{d\theta}\kappa_\varphi = \kappa'cr^p/p+1 + c\frac{p}{p+1}r^{-1/p+1}\kappa = c\frac{p}{p+1}r^{-1/p+1}r'\kappa,$$

which implies either $r' = 0$ or $\kappa = 0$. However, if $\kappa = 0$, the curve is the geodesic, which is a straight line in Euclidean coordinates, which cannot be isoperimetric.
Therefore \( r' = 0 \), meaning \( r \) must have a critical point other than 0 and \( \pi \), so it cannot be isoperimetric. \( \square \)

**Remark.** After finding this proof and examining the isoperimetric inequality in Proposition 4.23 we came across a more geometric proof. We show that in the Euclidean \( \pi \)-sector with area density \( cr^{-p/p+1} \), isoperimetric curves are circles about the origin. When \( p = 0 \), a semicircle about the origin is isoperimetric. Now, for any \( p > 0 \), suppose some region \( R \) is isoperimetric. Take a semicircle about the origin bounding the same Euclidean area; clearly it will have less perimeter. However, it also has more weighted area, because we have moved sections of \( R \) that were further away from the origin towards the origin. Since the area density is strictly decreasing in \( r \), we must have increased area. Therefore isoperimetric curves are circles about the origin for the Euclidean \( \pi \)-sector with area density \( cr^{-p/p+1} \), implying that isoperimetric curves are circular arcs in the \( \pi/(p + 1) \)-sector with density \( r^p \).

**Proposition 4.9.** In the sector of 2 radians with density \( r \), isoperimetric curves are circles about the origin.

**Proof.** Morgan [M3, Prop. 1] proves that in cones over the square torus \( T^2 = S^1(a) \times S^1(a) \) isoperimetric regions are balls about the origin as long as \( |\mathbb{T}^2| \leq |S^2(1)| \). We note that this proof still holds for rectangular tori \( T^2 = S^1(a) \times S^1(b) \), \( a \geq b \) so long as the ratio \( a : b \) is at most \( 4 : \pi \). Taking the cone over the rectangular torus with side lengths 4 and \( \pi \), and modding out by the shorter copy of \( S^1 \) we get the cone over an angle 4 with density \( \pi r \). Since isoperimetric regions are balls about the origin in the original space, their images, circles about the origin, are isoperimetric in this quotient space. \( \square \)

**Remark.** Morgan and Ritoré [MR, Rmk. 3.10] ask whether \( |M^n| \leq |\mathbb{S}^n(1)| \) is enough to imply that balls about the origin are isoperimetric in the cone over \( M \). Trying to take the converse to the above argument we found an easy counterexample to this question. Namely taking \( M \) to be a rectangular torus of area \( 4\pi \) with one very long direction and one very short direction, we see that balls about the origin are not isoperimetric, as you can do better with a circle through the origin across the short direction.

Along the same lines as Proposition 4.9 one might hope to obtain bounds for other values of \( p \) by examining when balls about the origin are isoperimetric in cones over \( S^1(\theta) \times M^p \) for \( M \) compact with a transitive isometry group and then modding out by the symmetry group of \( M \) to get the cone over \( S^1(\theta) \) with density proportional to \( r^p \). In order to see when balls about the origin are isoperimetric in the cone over \( T^2 \), Morgan uses the Ros product theorem with density [M2, Thm. 3.2], which requires the knowledge of the isoperimetric profile of the link (in his case \( T^2 \)). One such manifold of the form \( S^1 \times M \) for which the isoperimetric problem is solved is \( S^1 \times S^2 \) [PR, Thm. 4.3]. The three types of isoperimetric regions in \( S^1 \times S^2 \) are balls or complements of balls, tubular neighborhoods of \( S^1 \times \{ \text{point} \} \), or regions bounded by two totally geodesic copies of \( S^2 \). By far the most difficult of the three cases to deal with are the balls, where unlike the other two cases we cannot explicitly compute the volume and surface area. Fixing the volume of \( S^1 \times S^2 \) as \( 2\pi^2 = |\mathbb{S}^3| \) we can apply the same argument Morgan uses in [M3, Lemma 2] to show that if the sectional curvature of \( S^1 \times S^2 \) is bounded above by 1 (by taking \( S^2 \) large and \( S^1 \) small), then balls in \( S^1 \times S^2 \) do worse than balls in \( \mathbb{S}^3 \), as desired, but unfortunately tubular neighborhoods of \( S^1 \) then sometimes beat balls in \( \mathbb{S}^3 \).
for certain volumes, making it so we cannot apply the Ros product theorem. So without a better way to deal with balls in $S^1 \times S^2$ that does not depend on such a strong assumption on the sectional curvature, this method does not work even for the $p = 2$ case. Perhaps there is another way to prove when balls about the vertex are isoperimetric in the cone over $S^1 \times S^2$ without using the Ros product theorem.

**Proposition 4.10.** In the $\theta_0$-sector with density $r^p$, semicircles through the origin are not isoperimetric for $\theta_0 < \pi (p + 2) / (2p + 2)$.

**Proof.** In Euclidean coordinates, semicircles through the origin terminate at the angle $\theta = (\pi/2) (p + 1)$. Since the semicircle approaches this axis tangentially, for any $\theta_0 < (\pi/2) (p + 2)/(p + 1)$, there is a line normal to the boundary $\theta_0(p + 1)$ in Euclidean coordinates which intersects the semicircle at a single point $b$. Replacing the segment of the semicircle from $b$ to the origin with this line decreases area while decreasing perimeter. Therefore semicircles are not isoperimetric. $\square$

**Lemma 4.11.** If the $\theta_0$-sector with density $r^p$ has isoperimetric ratio $I_0$ and 

$$I_0 < \left( \frac{p + 2}{p + 1} \right) (p + 2)^{(p+1)/(p+2)} \theta_0^{(p+2)/(p+2)},$$

then there exists an $\epsilon > 0$ such that the isoperimetric ratio of any $(\theta_0 + t)$-sector with $0 < t < \epsilon$ is greater than or equal to the isoperimetric ratio of the $\theta_0$-sector, with equality if and only if the semicircle is isoperimetric in the $\theta_0$-sector, in which case the semicircle is uniquely isoperimetric in the $(\theta_0 + t)$-sector.

**Proof.** Consider an isoperimetric curve $\gamma$ in the $(\theta_0 + t)$-sector bounding area 1. By reflection we can assume it is nonincreasing (Cor. 4.9). We partition the $(\theta_0 + t)$-sector into a $\theta_0$-sector followed by a $t$-sector and we let $\alpha_t$ denote the area bounded by $\gamma$ in the $\theta_0$-sector and $\alpha_t$ the area bounded by $\gamma$ in the $t$-sector. Note $\alpha_0 + \alpha_t = 1$, and since $\gamma$ must be either an undulatory, a circular arc, or a semicircle, $\alpha_t = 0$ if and only if $\gamma$ is a semicircle. Then we can bound the isoperimetric ratio $I_t$ of the $(\theta_0 + t)$-sector by using the isoperimetric ratios $I_0$ for the $\theta_0$-sector and $R_t$ for the $t$-sector as follows:

$$I_t = P(\gamma) \geq I_0 \alpha_0^{(p+1)/(p+2)} + R_t \alpha_t^{(p+1)/(p+2)}.$$

Since the radius of $\gamma$ is nonincreasing we know that the $\theta_0$-sector contains at least its angular proportion of the area and thus $\alpha_0 \geq \theta_0/(\theta_0 + t)$. We now substitute $\alpha_t = 1 - \alpha_0$ and look at the right side of the inequality as a function of $\alpha_0$:

$$f_t(\alpha_0) = I_0 \alpha_0^{(p+1)/(p+2)} + R_t(1 - \alpha_0)^{(p+1)/(p+2)}.$$

This function is concave, so it attains its minimum at an endpoint. Since $\alpha_0$ is bounded between $\theta_0/(\theta_0 + t)$ and 1, we see that $I_t$ is greater than or equal to the minimum of $f_t(\theta_0/(\theta_0 + t))$ and $f_t(1)$. Since $f_t(1) = I_0$, we want to show that there is some $\epsilon$ such that $t < \epsilon$ implies $f_t(\theta_0/(\theta_0 + t)) > f(1) = I_0$. So we define a new function

$$g(t) = f_t(\theta_0/(\theta_0 + t)) - I_0.$$

We want to show that there is some positive neighborhood of 0 where $g(t) > 0$. For $t < \pi/(p + 1)$, isoperimetric regions are circular arcs, so for $t \in (0, \delta)$, $R_t = (p +
Corollary 4.13. The semicircle is isoperimetric in the “half-infinite parking garage” and $g$ is differentiable. Furthermore, $\lim_{t \to 0} g(t) = 0$ and so it suffices to prove that $\lim_{t \to 0} g'(t) > 0$. We calculate

$$
\lim_{t \to 0} g'(t) = -I_0 \left( \frac{p+1}{p+2} \right) \theta_0^{-1} + \left( p + 2 \right)^{(p+1)/(p+2)} \theta_0^{-1}\left( p + 1 \right) / \left( p + 2 \right)
$$

and deduce that this is greater than 0 if and only if

$$I_0 < \left( \frac{p+2}{p+1} \right) \left( p + 2 \right)^{(p+1)/(p+2)} \theta_0^{1/(p+2)}.
$$

Thus, we get a neighborhood where $g(t)$ is positive. Now, recall $I_t \geq f_t(\alpha_0)$ where $\alpha_0$ is the area bounded by the isoperimetric curve $\gamma$ in the $\theta_0$-sector, and $\alpha_0 \in [\theta_0/(\theta_0 + t), 1]$. Since $g(t)$ is positive, $f_t(\alpha_0) \geq f_t(1) = I_0$, with equality holding if and only if $\alpha_0 = 1$, that is, only if $\gamma$ is a semicircle. If $\gamma$ is a semicircle and the semicircle is not isoperimetric in the $\theta_0$-sector then we must have $I_t > I_0$, since otherwise the semicircle would be isoperimetric in the $\theta_0$-sector. (Note that by Proposition 4.11 if $\theta_0 < \pi/2$ then we can decrease the size of our neighborhood so that the semicircle is not isoperimetric anywhere in it and thus we do not have to worry about non-existence of the semicircle here). On the other hand if the semicircle is isoperimetric in the $\theta_0$-sector then we see that it is uniquely isoperimetric in the $(\theta_0 + t)$-sector since any other curve will have $\alpha_0 < 1$.

**Proposition 4.12.** For fixed $p > 0$, the isoperimetric ratio of the $\theta_0$-sectors with density $r^p$ is an increasing function of $\theta_0$ for $\theta_0 < \theta_2$, where $\theta_2$ is the angle at which the semicircle is first isoperimetric. On $[\theta_2, \infty)$ the isoperimetric ratio is constant and the semicircle is uniquely isoperimetric on $(\theta_2, \infty)$.

**Proof.** For a circle about the origin $P = \theta_0^1/(\theta_0^2 + 1) \left( p + 2 \right)^{(p+1)/(p+2)} A^{(p+1)/(p+2)}$, so we see that for all $p > 0$ the isoperimetric ratio $I_0$ satisfies the conditions of Lemma 4.11 for every $\theta_0$. By Proposition 4.11 the semicircle is isoperimetric in the $\pi$-sector, and since the isoperimetric ratio is continuous in $\theta_0$ there is some minimum $\theta_2$ where the semicircle is isoperimetric. By Lemma 4.11 the semicircle is isoperimetric on $[\theta_2, \infty)$. Thus, the isoperimetric ratio is the same for all of these sectors, and so the functions $f_t$ from the proof of the lemma are the same for all $\theta_0 \geq \theta_2$. Thus we see that we can pick the $\epsilon$ given by the lemma uniformly so that for all $\theta_0 \geq \theta_2$, the semicircle uniquely minimizes on $(\theta_0, \theta_0 + \epsilon)$, giving us that the semicircle uniquely minimizes on $(\theta_2, \infty)$. On the other hand, since the isoperimetric ratio is continuous in $\theta_0$, it is clear from Lemma 4.11 that it is strictly increasing on $\theta_0 < \theta_2$.

**Corollary 4.13.** The semicircle is isoperimetric in the “half-infinite parking garage” $\{(\theta, r)|\theta \geq 0, r > 0\}$ with density $r^p$.

**Proof.** Suppose $\gamma$ is isoperimetric in the half-infinite parking garage. For any $\theta_0 \geq \pi$ the restriction of $\gamma$ to the $\theta_0$ sector has isoperimetric ratio greater than or equal to that of the semicircle. Since $\lim_{\theta_0 \to \infty} P(\gamma|\theta_0) = P(\gamma)$ and $\lim_{\theta_0 \to \infty} A(\gamma|\theta_0) = A(\gamma)$ the limit of the isoperimetric ratios of $\gamma|\theta_0$ is the isoperimetric ratio of $\gamma$ and so we see it is also greater than or equal to that of the semicircle. Since the semicircle exists in the half-infinite parking garage we are done.

**Remark.** Studying the isoperimetric ratio turns out to be an extremely useful tool in determining the behavior of semicircles for $\theta_0 > \pi$. However, it is not the only
such tool. Here we give an entirely different proof that semicircles are isoperimetric for all \( n\pi \)-sectors.

**Proposition 4.14.** In the \( \theta_0 \)-sector with density \( r^p \), \( p > 0 \), even when allowing multiplicity greater than one, isoperimetric regions will not have multiplicity greater than one.

**Proof.** A region \( R \) with multiplicity may be decomposed as a sum of nested regions \( R_j \) with perimeter and area [M1] (Fig. 10.1.1):

\[
P(R) = \sum P(R_j), \\
A(R) = \sum A(R_j).
\]

Let \( R' \) be an isoperimetric region of multiplicity one and the same area as \( R \). Since isoperimetric regions remain isoperimetric under scaling, for each region \( P_j \geq cA_j^{p+1)/(p+2) \), where \( c = P_{R'}/A_{R'}^{(p+1)/(p+2)} \) (Prop. 4.14). By concavity

\[
P(R) = \sum P(R_j) \geq c \sum \left( A(R_j)^{p+2 \over p+1} \right) \geq c \left( \sum A(R_j) \right)^{p+2 \over p+1} = P(R'),
\]

with equality only if \( R \) has multiplicity one. Therefore no isoperimetric region can have multiplicity greater than one. \( \square \)

**Remark.** For \( p < -2 \) the isoperimetric function \( I(A) = cA^{(p+1)/(p+2)} \) is now convex. Therefore, regions with multiplicity greater than one can do arbitrarily better than regions with multiplicity one.

**Corollary 4.15.** In the \( n\pi \)-sector (\( n \in \mathbb{Z} \)), semicircles through the origin are uniquely isoperimetric.

**Proof.** Assume there is an isoperimetric curve \( r(\theta) \) bounding a region \( R \) which is not the semicircle. Consider \( R \) as a region with multiplicity in the half plane by taking \( r(\theta) \to r(\theta \mod \pi) \). Since semicircles are uniquely isoperimetric in the half plane, by Proposition 4.14 \( r \) cannot be isoperimetric. This implies that \( r \) could not have been isoperimetric in the \( n\pi \)-sector. \( \square \)

**Proposition 4.16.** In the \( \theta_0 \)-sector with density \( r^p \), \( p > -1 \), circular arcs about the origin have positive second variation if and only if \( \theta_0 < \pi/\sqrt{p+1} \). When \( p < -1 \), circular arcs about the origin always have positive second variation.

**Proof.** By Proposition 3.2 we think of the \( \theta_0 \)-sector as the cone of angle \( 2\theta_0 \). A circle of radius \( r \) in the \( \theta_0 \)-sector corresponds with a circle about the axis with radius \( r\theta_0/\pi \), giving the cone the metric \( ds^2 = dr^2 + (r\theta_0/\pi)^2 d\theta^2 \). For a smooth Riemannian disk of revolution with metric \( ds^2 = dr^2 + f(r)^2 d\theta^2 \) and density \( e^{\psi(r)} \), circles of revolution at distance \( r \) have positive second variation if and only if \( Q(r) = f'(r)^2 - f(r)f''(r) - f(r)^2 \psi''(r) \leq 1 \) [EMMP Thm. 6.3]. This corresponds to \( (\theta_0/\pi)^2 + p(\theta_0/\pi)^2 < 1 \), which, for \( p < -1 \), always holds. When \( p > -1 \), the condition becomes \( \theta_0 < \pi/\sqrt{p+1} \), as desired. \( \square \)

The following theorem is the main result of this paper.

**Theorem 4.17.** Given \( p > 0 \), there exist \( 0 < \theta_1 < \theta_2 < \infty \) such that in the \( \theta_0 \)-sector with density \( r^p \), isoperimetric curves are (see Fig. 1.1):

1. for \( 0 < \theta_0 < \theta_1 \), circular arcs about the origin,
2. for \( \theta_1 < \theta_0 < \theta_2 \), undularies,
3. for $\theta_2 < \theta_0 < \infty$, semicircles through the origin.

Moreover,
\[
\frac{\pi}{(p + 1)} \leq \theta_1 \leq \pi / \sqrt{p + 1},
\]
\[
\frac{\pi(p + 2)}{(2p + 2)} \leq \theta_2 \leq \pi.
\]

When $p = 1$, $\theta_1 \geq 2 > \pi/2 \approx 1.57$.

**Remark.** For $p < -2$, circular arcs about the origin bounding area away from the origin are isoperimetric for all sectors, and for $-2 \leq p < 0$ isoperimetric regions do not exist. The proofs given by Carroll et al. [CJQW, Prop. 4.3] generalize immediately from the plane to the sector. In Section 7 we give a generalization to $\mathbb{R}^n$, $n \geq 2$ of these statements about $p < 0$.

**Proof.** By Lemma 4.7, minimizers exist and must be circles, undularies, or semicircles. As $\theta$ increases, if the circle is not minimizing, it remains not minimizing (Prop. 4.2). If the semicircle is minimizing, it remains uniquely minimizing (Prop. 4.12). Therefore transitional angles $0 \leq \theta_1 \leq \theta_2 \leq \infty$ exist. Strict inequalities are trivial consequences of the following inequalities:

To prove $\theta_1 \geq \pi / (p + 1)$, note that circular arcs are the unique minimizers for $\theta_0 = \pi / (p + 1)$ (Thm. 4.8).

To prove $\theta_1 \leq \pi / \sqrt{p + 1}$, recall that circular arcs do not have nonnegative second variation for $\theta_0 > \pi / \sqrt{p + 1}$ (Prop. 4.10). To prove $\theta_2 \geq \pi(p + 2) / (2p + 2)$, recall that semicircles cannot minimize for $\theta_0 < \pi(p + 2) / (2p + 2)$ (Prop. 4.10). To prove $\theta_2 \leq \pi$, recall that semicircles minimize for $\theta_0 = \pi$ (Prop. 4.1). Since $\pi / \sqrt{p + 1} < \pi(p + 2) / (2p + 2)$ for all $p > 0$, we have $\theta_1 < \theta_2$. For $\theta_1 < \theta_0 < \theta_2$, neither the circle nor the semicircle minimizes, so minimizers are undularies. Finally, when $p = 1$, circles minimize for $\theta_0 = 2$ (Prop. 4.9). □

We conjecture that the circle is isoperimetric as long as it has nonnegative second variation, and that the semicircle is isoperimetric for all angles greater than $\pi(p + 2) / (2p + 2)$.

**Conjecture 4.18.** In Theorem 4.17, the transitional angles $\theta_1$, $\theta_2$ are given by $\theta_1 = \pi / \sqrt{p + 1}$ and $\theta_2 = \pi(p + 2) / (2p + 2)$.

**Remark.** This conjecture is supported by numeric evidence as in Figure 1.2.

**Remark.** Our isoperimetric undularies give explicit examples of the abstract existence result of Rosales et al. [RCBM, Cor. 3.13] of isoperimetric regions not bounded by lines or circular arcs.

One potential avenue for proving this conjecture is discussed in Proposition 5.2. We also believe the transition between the circle and the semicircle is parametrized smoothly by curvature, which is discussed in Section 5.

Theorem 4.20 allows us to apply Theorem 4.17 to the problem of classifying isoperimetric regions of small area in planar polygons with density $r^p$, as suggested to us by Antonio Cañete. It is similar to an isoperimetric theorem of Morgan on regions of small area in polytopes [M2, Thm. 3.8]. First we need a lemma:

**Lemma 4.19.** Consider a polygon in the plane with density $f$. Then, for any region $R$ inside the polygon where the density is bounded away from 0 and $\infty$, there exists a constant $c > 0$ such that any sub-region $S$ bounding Euclidean area less than...
half the Euclidean area of the polygon has weighted area $A$ and weighted perimeter $P$ satisfying

\[ P \geq cA^{1/2} \]

**Proof.** Let $P'$ and $A'$ denote the Euclidean perimeter and area of $S$ and let $m$ be the minimum of the density in $R$ and $M$ the maximum of the density in $R$. Then $P \geq m \cdot P'$ and $A \leq M \cdot A'$. By a standard result $P' \geq cA'^{1/2}$ for some constant $c$ (see, e.g., [M1, p. 112]), and combining the three inequalities we see that for a new constant the desired inequality holds. □

**Theorem 4.20.** Let $B$ be a planar polygon containing the origin and let $p > 0$.

Let $M$ be:

- The plane with density $r^p$ if the origin is contained in the interior of $B$
- The half plane with density $r^p$ if the origin is contained in the boundary of $B$ but not at a vertex.
- The $\theta_0$-sector with density $r^p$ if the origin is a vertex of $B$ of angle $\theta_0$.

Then for sufficiently small areas, isoperimetric regions in $B$ with density $r^p$ are the same as those in $M$ in the natural sense.

**Proof.** Let $C_1$ and $C_2$ be circles (or semi-circles or circular arcs, depending on $M$) about the origin of radii $r_1 < r_2$ and such that the following hold:

1. $C_2$ (and therefore $C_1$) intersects the polygon only at the side(s) containing the origin, or not at all if the origin is on the interior of $B$.
2. The weighted length of any curve between $C_1$ and $C_2$ is greater than $P_1$ where $P_1$ is the weighted perimeter of $C_1$.
3. $C_2$ contains less than half of the Euclidean area of $B$.

Now, suppose we have a single closed curve inside $B$ (by which we mean either closed in the traditional sense or intersecting the boundary at both endpoints) of weighted length less than $P_1$. Then because of 2 in the above list, the curve must lie completely inside of $C_2$ or completely outside of $C_1$. Because the density is bounded away from 0 and $\infty$ outside of $C_1$, if we take the length of the curve to be small enough that it cannot contain all of $C_1$ on its side of smaller Euclidean area, then by applying Lemma 4.19 we find that for any such curve completely outside of $C_1$, $P \geq c_1 A^{1/2}$ for some constant $c_1 > 0$ where $P$ and $A$ are the weighted perimeter and weighted area of the region bounded by the curve with smaller Euclidean area. On the other hand, we see by 3 in the above list that any curve contained completely in $C_2$ bounds its region of smaller Euclidean area completely inside $C_2$, and so by Prop. 4.13 satisfies $P \geq c_2 A^{(p+1)/(p+2)}$ for some constant $c_2 > 0$ where $P$ and $A$ are as before.

Now, $B$ has finite weighted area, and so by standard geometric measure theory isoperimetric regions exist for all possible weighted areas. So, let $R$ be an isoperimetric region of weighted area $A_0$ smaller than the weighted area of $C_1$, and let $P_0$ be the weighted perimeter of a circle about the origin of weighted area $A_0$. We will take $A_0$ to be small enough such that for any closed curve in $B$ (again closed in the sense that it separates two regions in $B$) with weighted length less than $P_0$ the region it bounds with smaller weighted area also has smaller Euclidean area. Now, we note that $R$ must have weighted perimeter less than or equal to $P_0$ and thus less than $P_1$. 


Consider a single connected component of $R$. Suppose there is no curve in its boundary such that the entire component is contained in the region of smaller weighted area bounded by the curve. Then the weighted area of the component is equal to the total weighted area of $B$ minus the sum of all the smaller of the weighted areas bounded by the curves forming the boundary. Because of the inequalities we have proven above for weighted perimeter in terms of weighted area and because of the inequalities we have had strictly larger weighted perimeter).

Conjecture 4.22. In the plane with perimeter density $r^k$, $k > -1$, and area density $r^m$ the following are isoperimetric curves:

1. for $m \in (-\infty, -2] \cup (2k, \infty)$ there are none.
2. for $k \in (-1, 0)$ and $m \in (-2, 2k)$ the circle about the origin.
3. for $k \in [0, \infty)$ and $m \in (-2, k - 1)$ the circle about the origin.
4. for $k \in [0, \infty)$ and $m \in [k, 2k]$ pinched circles through the origin.

We also obtain a conjecture on the area density range between $k - 1$ and $k$, which is missed by Proposition 4.21 by doing the same analysis with Proposition 2.3 and the values for $\theta_1$ and $\theta_2$ in Conjecture 4.18.

Conjecture 4.22. In the plane with perimeter density $r^k$, $k > -1$, and area density $r^m$ the following are isoperimetric curves:

1. for $k \in [0, \infty)$ and $m \in (k - 1, k - 1 + \frac{1}{k + 1}]$ the circle about the origin.
2. for $k \in [0, \infty)$ and $m \in (k - 1 + \frac{1}{k + 1}, k - 1 + \frac{2k + 1}{2k + 1})$ undularies.
3. for $k \in [0, \infty)$ and $m \in [k - 1 + \frac{k + 1}{2k + 1}, k]$ pinched circles through the origin.

Remark. The circular arc being isoperimetric up to the $\pi/(p/2 + 1)$ sector is equivalent to the circle being isoperimetric in the Euclidean plane with any perimeter density $r^p$, $p \in [0, 1]$ and Euclidean area. As $p/2 + 1$ is the tangent line to $\sqrt{p + 1}$ at 0, this is the best possible bound we could obtain which is linear in the denominator.
We now consider a more analytic formulation of the isoperimetric problem in the $\theta_0$-sector with density $r^p$, and give an integral inequality that is equivalent to proving the conjectured angle of $\theta_1$.

**Proposition 4.23.** In the $\theta_0$-sector with density $r^p$, $p > 0$, circles about the origin are isoperimetric if and only if the inequality

$$\left[ \int_0^1 r^{p+2} d\alpha \right]^{\frac{p+1}{p+2}} \leq \int_0^1 \sqrt{r^2 + \frac{r'^2}{[(p +1)\theta_0]^2}} d\alpha$$

holds for all $C^1$ functions $r(\alpha)$.

**Proof.** By Proposition 4.3, an isoperimetric curve is a polar graph. The rest follows from manipulating the integral formulas for weighted area and perimeter for polar graphs. □

**Remark.** This gives a nice analytic proof that circles about the origin are isoperimetric for $\theta_0 = \pi/(p + 1)$; letting $\theta_0 = \pi/(p + 1)$, we have

$$\left[ \int_0^1 r^{p+2} d\alpha \right]^{\frac{p+1}{p+2}} \leq \int_0^1 \sqrt{r^2 + \frac{r'^2}{\pi^2}} d\alpha.$$  

When $p = 0$, this corresponds to the isoperimetric inequality in the half-plane with density 1. As pointed out by Leonard Schulman of CalTech, the left hand side is nonincreasing as a function of $p$, meaning the inequality holds for all $p > 0$.

**Corollary 4.24.** In the $\theta_0$-sector with density $r^p$, $p > 0$, circles about the origin are isoperimetric for $\theta_0 = \pi/\sqrt{p + 1}$ if and only if the inequality

$$\left[ \int_0^1 r^q d\alpha \right]^{1/q} \leq \int_0^1 \sqrt{r^2 + (q - 1) \frac{r'^2}{\pi^2}} d\alpha$$

holds for all $C^1$ functions $r(\alpha)$ for $1 < q \leq 2$.

**Proof.** In the inequality from Corollary 4.23 let $q = (p + 2)/(p + 1)$, and let $\theta_0 = \pi/\sqrt{p + 1}$. □

**Remark.** We wonder if an interpolation argument might work here. When $q = 1$, the result holds trivially (equality for all functions $r$), and when $q = 2$, the inequality follows from the isoperimetric inequality in the half plane.

5. **Constant Generalized Curvature Curves**

We look at constant generalized curvature curves in greater depth. Theorem 5.1 classifies constant generalized curvature curves. Proposition 5.2 proves that if the half period of constant generalized curvature curves is bounded above by $\pi(p + 2)/(2p + 2)$ and below by $\pi/\sqrt{p + 1}$, then Conjecture 4.18 holds. Our major tools for studying constant generalized curvature curves are the second order constant generalized curvature equation and its first integral.

**Theorem 5.1.** In the plane with density $r^p$, a curve with constant generalized curvature $\lambda$ normal at $(1, 0)$ is (see Fig. 5.1):

1. for $\lambda \in (0, p + 2) - \{p + 1\}$, a periodic undulary,
2. for $\lambda \notin [0, p + 2]$, a periodic nodoid,
3. for $\lambda = p + 2$, a circle through the origin,
Types of constant generalized curvature curves: (a) for $0 < \lambda < p + 2$ nonconstant periodic polar graphs (undularies), (b) for $\lambda < 0$ or $\lambda > p + 2$ periodic nodoids, (c) for $\lambda = p + 2$, a circle through the origin, for $\lambda = p + 1$ a circle about the origin, for $\lambda = 0$ a curve asymptotically approaching the radial lines $\theta = \pm \pi/(2p + 2)$.

(4) for $\lambda = p + 1$, a circle about the origin,
(5) for $\lambda = 0$, the geodesic

$$r(\theta) = \left(\sec((p + 1)\theta)\right)^{1/(p+1)}$$

which asymptotically approaches the radial line $\theta = \pm \pi/(2p + 2)$.

Proof. These results follow from the differential equations for constant generalized curvature curves. Two points deserve special mention. First, reflection and scaling provides a correspondence between curves outside the circle and curves inside the circle. Second, to prove a curve $\gamma$ which is not the circle, semicircle or geodesic is periodic, it suffices to show $\gamma$ has a critical point after $(1,0)$. It follows directly from the differential equation that there is a radius at which $\gamma$ would attain a critical point. To prove $\gamma$ actually attains this radius, we note that otherwise $\gamma$ would have to asymptotically spiral to this radius. This implies $\gamma$ has the same generalized curvature as a circle of this radius, which the equations confirm cannot happen. 

Isoperimetric curves in the $\theta_0$-sector are polar graphs with constant generalized curvature that intersect the boundary perpendicularly. Given these restrictions and the classification of Theorem 5.1, we can give a sufficient condition for Conjecture 4.18 on the value of the transitional angles.

Proposition 5.2. In the plane with density $r^p$, assume the half period of constant generalized curvature curves normal at $(1,0)$ with generalized curvature $0 < \lambda < p + 1$ is bounded below by $\pi/\sqrt{p+1}$ and above by $\pi(p + 2)/(2p + 2)$. Then the conclusions of Conjecture 4.18 hold.

Proof. Given the correspondence mentioned in the proof of Theorem 5.1 it suffices to only check curves with $0 < \lambda < p + 1$. By Proposition 4.7 isoperimetric curves are either circles about the origin, semicircles through the origin, or undularies. An undulary is only in equilibrium when the sector angle is equal to its half period. If there are no undularies with half period less than $\pi/\sqrt{p+1}$ or greater than $\pi(p + 2)/(2p + 2)$, the only possible isoperimetric curves outside of that range
are the circle and the semicircle. The proposition follows by the bounds given in Theorem 4.17.

Remark 5.3. We now discuss some results on the periods of constant generalized curvature curves, and provide numeric evidence. Using the second order constant generalized curvature equation, one can prove that curves with generalized curvature near that of the circle have periods near $\pi/\sqrt{p+1}$. Similarly, using the first integral of the constant generalized curvature equation for $dr/d\theta$, we see that the half period of a curve with constant generalized curvature is given by

$$T = \int_{1}^{r_1} \frac{dr}{r \sqrt{\frac{1-r^{p+1}}{1-r^{p+2}} (1-r^{p+2}) - 1}}$$

where $r_1$ is the curve’s maximum radius. Using this formula, we generated numeric evidence for the bounds given in Proposition 5.2, as seen in Figure 5.2. This plot is given for $p = 2$, and is representative of all $p$. Moreover, the integral seems to be monotonic in $r_1$ which would imply that every undulary with $0 < \lambda < p + 2$ is isoperimetric for exactly one $\theta_0$-sector. Francisco López has suggested studying the integral above with the techniques of complex analysis.

6. THE ISOPERIMETRIC PROBLEM IN SECTORS WITH DISK DENSITY

In this section we classify the isoperimetric curves in the $\theta_0$-sector with density 1 outside the unit disk $D$ centered at the origin and $a > 1$ inside $D$. Cañete et al. [CMV, Sect. 3.3] consider this problem in the plane, which is equivalent to the $\pi$-sector. Proposition 6.2 gives the potential isoperimetric candidates. Theorems 6.3, 6.4, 6.5 classify the isoperimetric curves for every area and sector angle.
**Definition 6.1.** A *bite* is an arc of $\partial D$ and another internal arc (inside $D$), the angle between them equal to $\arccos(1/a)$ (see Fig. 6.1(c)).

![Figure 6.1](image_url)

**Figure 6.1.** Isoperimetric sets in sectors with disk density: (a) an arc about the origin inside or outside $D$, (b) an annulus inside $D$, (c) a bite, (d) a semicircle on the edge disjoint from the interior of $D$, (e) a semicircle centered on the x-axis enclosing $D$ for $\theta_0 = \pi$.

**Proposition 6.2.** In the $\theta_0$-sector with density $a > 1$ inside the unit disk $D$ and 1 outside, for area $A > 0$, an isoperimetric set is one of the following (see Fig. 6.1): (a) an arc about the origin inside or outside $D$; (b) an annulus inside $D$ with $\partial D$ as a boundary; (c) a bite; (d) a semicircle on the edge of the sector disjoint from the interior of $D$; (e) a semicircle centered on the x-axis enclosing $D$ for $\theta_0 = \pi$.

**Proof.** Any component of an isoperimetric curve has to meet the sector edge perpendicularly since any component that meets the boundary at a different angle contradicts Proposition 3.4, and any component that does not meet the sector edge can be rotated about the origin (which will still give an isoperimetric region) until it hits the sector edge at which point, since by Proposition 2.5 the curve is smooth, the intersection will be tangent so it will then contradict Proposition 3.4. Since each part of the boundary has to have constant generalized curvature it must be made up of circular arcs (here we mean any circular arcs, not necessarily arcs of circles about the origin as earlier in the paper). We can discard the possibility of combinations of circular arcs within the same density since one circular arc is better
than $n$ circular arcs (just as in the proof of Lemma 4.7). Therefore, there are five possible cases:

(1) A circular arc from one boundary edge to itself (including possibly the origin).
There are three possibilities according to whether the semicircle has 0, 1, or 2 endpoints inside the interior of $D$. A semicircle with two endpoints inside $D$ has an isoperimetric ratio of $2\pi a$. For a semicircle with one endpoint inside $D$ (Fig. 6.2), by Proposition 2.5 Snell’s Law holds. Then the only possible curve that intersects the boundary normally would also have to intersect $D$ normally. Its perimeter and area satisfy:

$$P = r(\pi - \beta + a\beta), \quad A = \frac{r^2}{2} (\pi - \beta + a\beta),$$

$$\frac{P^2}{A} = 2(\pi + \beta(a - 1)).$$

Therefore a semicircle (d) outside $D$ with isoperimetric ratio $2\pi$ is the only possibility.

(2) A circular arc from one boundary edge to another.
An arc (a) or annulus (b) about the origin are the only possibilities for any $\theta_0 < \pi$ or $A \leq a\theta_0/2$. At $\theta_0 = \pi$ and area $A > a\theta_0/2$ a semicircle (e) centered on the x-axis enclosing $D$ is equivalent to a semicircle centered at the origin. For $\theta_0 > \pi$ and $A > a\theta_0/2$, a semicircle tangent to $\partial D$ together with the rest of $\partial D$ (Fig. 6.3) is in equilibrium, but we will show that it is not isoperimetric. Its perimeter and area satisfy:

$$P = \pi R + (\theta_0 - \pi), \quad A = \frac{\pi}{2} (R^2 - 1) + \frac{\theta_0 a}{2}.$$

Comparing it with an arc (a) about the origin we see that it is not isoperimetric.

(3) Two circular arcs (c) meeting along $\partial D$ according to Snell’s Law (Prop. 2.5).
Theorem 6.3. For some $\theta_2 < \pi$, in the $\theta_0$-sector with density $a > 1$ inside the unit disk $D$ and 1 outside, for $\theta_0 \leq \pi$, there exists $0 < A_0 < A_1 < a\theta_0/2$, such that an isoperimetric curve for area $A$ is (see Fig. 6.1):

1. if $0 < A < A_0$, an arc about the origin if $\theta_0 < \pi/a$, semicircles on the edge disjoint from the interior of $D$ if $\theta_0 > \pi/a$, and both if $\theta_0 = \pi/a$;
2. if $A = A_0$, both type (1) and (3);
3. if $A_0 \leq A < A_1$, a bite, if $A_1 < A < a\theta_0/2$ an annulus inside $D$, and if $A = A_1$ both; if $\theta_0 > \theta_2$, $A_1 = a\theta_0/2$ and the annulus is never isoperimetric;
4. if $A \geq a\theta_0/2$, an arc about the origin; at $\theta_0 = \pi$ any semicircle centered on the $x$-axis enclosing $D$.

Theorem 6.4. In the $\theta_0$-sector with density $a > 1$ inside the unit disk $D$ and 1 outside, for $\pi < \theta_0 \leq a\pi$, there exists $A_0$, $A_1$, such that an isoperimetric curve for area $A$ is (see Fig. 6.7):

1. if $0 < A < A_0$, a semicircle on the edge disjoint from the interior of $D$;
2. if $A = A_0$, both type (1) and (3);
3. if $A_0 < A < a\theta_0/2$, a bite;
4. if $a\theta_0/2 \leq A < \theta_0^2(a-1)/2(\theta_0 \mp \pi)$, an arc about the origin;
5. if $A = \theta_0^2(a-1)/2(\theta_0 \mp \pi)$, both type (4) and (6);
6. if $A > \theta_0^2(a-1)/2(\theta_0 \mp \pi)$, a semicircle on the edge disjoint from the interior of $D$.

Theorem 6.5. In the $\theta_0$-sector with density $a > 1$ inside the unit disk $D$ and 1 outside, for $\theta_0 > a\pi$, the isoperimetric curves for area $A$ are semicircles on the edge disjoint from the interior of $D$ (Fig. 6.7k).
For $\theta_0 < f$ an arc about the origin is isoperimetric, while for $\theta_0 > f$ a semicircle in the edge is isoperimetric. For area infinitesimally less than $a\theta_0/2$, for $\theta_0 < g$ an annulus inside $D$ is isoperimetric, while for $\theta_0 > g$ a bite is isoperimetric.

Given Proposition 6.2, most of the proofs of Theorems 6.3, 6.4, 6.5 are by simple perimeter comparison, which we omit here. Figure 6.4 shows numerically the transition between different isoperimetric curves. The hardest comparison is handled by the following sample lemma.

**Lemma 6.6.** In the $\theta_0$-sector with density $a > 1$ inside the unit disk $D$ and $1$ outside, for a fixed angle and area $A < a\theta_0/2$, once a bite encloses more area with less perimeter than an annulus inside $D$, it always does.

**Proof.** Increasing the difference $a\theta_0/2 - A$, it will be sufficient to prove that a bite is better than a scaling of a smaller bite because an annulus changes by scaling. Therefore, there will be at most one transition.

To prove that a bite does better than a scaling of a smaller bite we show that it has less perimeter and more area. Take a smaller bite and scale it by $1 + \epsilon$ (see Fig. 6.5) and eliminate the perimeter outside $D$. Another bite is created with perimeter $P_a$ and area $A_a$:
$$P_a < (1 + \epsilon)P, \quad A_a = (1 + \epsilon)^2 (A_1 + A_2) - A_3 > (1 + \epsilon)^2 A_1,$$

Thus it is better than scaling a smaller bite. \hfill \Box

7. Isoperimetric Problems in \( \mathbb{R}^n \) with Radial Density

In this section we look at the isoperimetric problem in \( \mathbb{R}^n \) with a radial density. We use symmetrization to show that if an isoperimetric region exists, then an isoperimetric region of revolution exists (Lem. 7.1), thus reducing the problem to a planar problem (Lem. 7.2).

We also consider the specific case of \( \mathbb{R}^n \) with density \( r^p \). We first give a nonexistence result (Prop. 7.3) for \( -n \leq p < 0 \). The methods used by Carroll et al. to study planar densities \( r^{-p} \), \( p < -2 \) [CJQW, Prop. 4.3] can be used to recover an earlier result of Betta et al. on perimeter densities in \( \mathbb{R}^n \) [BBMP, Thms. 2.1, 4.2, 4.3], and we provide a proof along these lines in Theorem 7.4. We then adapt this proof in Proposition 7.5 to show that hyperspheres about the origin bounding volume away from the origin are uniquely isoperimetric in \( \mathbb{R}^n \) with density \( r^p \), \( p < -n \). We conclude by conjecturing that hyperspheres through the origin are isoperimetric in \( \mathbb{R}^n \) with density \( r^p \), \( p > 0 \). (Conj. 7.6). We'd like to thank Bruno Volzone and Alexander Kolesnikov for bringing the paper by Betta et al. to our attention, as well as Robin Walters and Frank Morgan for providing initial versions of our averaging proof of Theorem 7.4.

Lemma 7.1. In \( \mathbb{R}^n \) with a radial density, if there exists an isoperimetric region, then there exists an isoperimetric region of revolution.

Proof. Using spherical symmetrization [BZ Sect. 9.2] generalized with density [MBHH], we can replace the intersection of the isoperimetric region with each sphere about the origin by a polar cap of the same volume. \hfill \Box

Lemma 7.2. The isoperimetric problem in \( \mathbb{R}^n \) with density \( f(r) \) is equivalent to the isoperimetric problem in the half plane \( y > 0 \) with density \( y^{n-2}f(r) \).

Proof. Up to a constant, this is the quotient space of \( \mathbb{R}^n \) with density \( f(r) \) modulo rotations about an axis. By Lemma 7.1 there exists an isoperimetric region bounded by surfaces of revolution, so a symmetric isoperimetric region in \( \mathbb{R}^n \) corresponds to an isoperimetric region in the quotient space and vice-versa. \hfill \Box

Isoperimetric curves in the plane with density \( r^p \) are circles about the origin when \( p < -2 \), do not exist when \( -2 \leq p < 0 \), and are circles through the origin when \( p > 0 \) ([CJQW, Props. 4.2, 4.3], [DDNT Thm. 3.16]). We now look at the isoperimetric problem in \( \mathbb{R}^n \) with density \( r^p \).

Proposition 7.3. For \( -n \leq p < 0 \) in \( \mathbb{R}^n \) with density \( r^p \), isoperimetric regions do not exist: you can enclose any volume with arbitrarily small perimeter.

Proof. Consider a sphere \( S \) of radius \( R > 0 \) not containing the origin. Let \( r_{\min}, r_{\max} \) be the minimum and maximum values of \( r \) attained on \( S \), note that \( 0 < r_{\min} = r_{\max} - 2R \). Let \( V, P \) be the volume and perimeter respectively of \( S \). We have that:

$$V > c_0 R^n \min_S(r^p) = c_0 R^n r_{\max}^p.$$
Working with a fixed $V$, we get that for any sphere of radius $R$ with volume $V$,
\[ r_{\max} > c_1 R^q \]
where $q = -n/p \geq 1$ and $c_1 = (V/c_0)^{1/p}$. So, if we fix $V$ large enough so that $c_1 > 3$, we obtain
\[ r_{\min} > c_1 R^q - 2R > R^q \]
for any $R > 1$. Looking at $P$ we have
\[
P < c_2 R^{n-1} \max_s s(r^p) = c_2 R^{n-1} r_{\min}^p < c_2 R^{n-1} (R^q)^p = c_2 R^{-1},
\]
which can be made as small as we want by taking $R$ large, and so since there are spheres of arbitrarily large radius not containing the origin and having volume $V$, we see that there are regions of arbitrarily small perimeter enclosing this fixed volume $V$. Then, by scaling these regions, we obtain the result for all volumes. \(\square\)

**Remark.** A proof that applies for $-n < p < -n + 1$ was given by [CJQW, Prop. 4.2]. Their next proposition, [CJQW, Prop 4.3], which extends immediately to sectors, is by Proposition 2.3 equivalent to the $r^p$ case of a theorem of Betta et al. [BBMR, Thm. 4.3]. Here we use the averaging technique of Carroll et al. to provide a short proof of the whole theorem of Betta et al.

**Theorem 7.4** ([BBMR, Thm. 4.3]). Suppose $a$ is a non-negative non-decreasing smooth function on the positive real axis with $a(s) > 0$ for $s > 0$ and with
\[
\left[ a(s^{1/n}) - a(0) \right] s^{1-1/n}
\]
convex. Then hyperspheres about the origin are uniquely isoperimetric in $\mathbb{R}^n$ with Euclidean volume and surface area density $a$.

**Proof.** As in Betta et al., we can assume $a(0)=0$, since if we let
\[
a_1(s) = a(s) - a(0),
\]
then $a$-weighted surface area is just the sum of $a_1$-weighted surface area and a constant times Euclidean surface area, and so if hyperspheres are isoperimetric for both these densities and uniquely isoperimetric for one then they will also be uniquely isoperimetric for $a$.

Let $B$ be a region bounded by rectifiable hypersurfaces. We will show that a ball about the origin $B'$ of the same volume has less surface area than $B$ with equality only when $B$ is also a ball. In fact, we will prove a stronger statement: Note that the surface area of $B$ is greater than or equal to the tangential surface area of $B$, i.e. the component of surface area perpendicular to radial lines, with equality only when the boundary of $B$ consists only of hypersurfaces. We will show that $B'$ has less tangential surface area than $B$.

In $\mathbb{R}^n$ with radial surface area density $a$ and unit volume density, in polar coordinates the volume element is
\[
dV = r^{n-1} \cdot dr d\Theta,
\]
and the tangential surface area element is
\[
dQ = a(r)r^{n-1} d\Theta
\]
where $d\Theta$ is the element of surface area on the unit sphere. Under the change of coordinates $s = r^n$, the volume element becomes

$$dV = \frac{1}{n} ds d\Theta$$

and the tangential surface area element becomes

$$dQ = f d\Theta$$

where $f(s) = a(s^{1/n}) s^{1-1/n}$. Now, in the $s$-variable we obtain

$$\int_B dV = \int_{\partial B} \pm s \cdot \frac{1}{n} d\Theta$$

where $s$ is signed according to the radial orientation of the boundary (positive if pointing away from the origin, negative if pointing towards the origin). Let $t(\Theta)$ be the sign-weighted sum of all the $s$ values of points in the intersection of $\partial B$ and a radial ray in the direction of $\Theta$. Note that this is not defined in directions where $\partial B$ is radial, but this is a set of $(n-1)$-dimensional Lebesgue measure 0 in the $\Theta$-space so it will not effect our computations. We then have that

$$\int_B dV = \int_{\partial B} t(\Theta) \cdot \frac{1}{n} d\Theta,$$

where the latter integral is taken over all possible directions. Now, let $t_{avg}$ be the average value of $t$ over all directions. Then in these coordinates we see that $B'$, the ball about the origin with the same volume as $B$, is actually the ball about the origin of radius $t_{avg}$:

$$\int_B dV = \int_{\partial B} t(\Theta) \cdot \frac{1}{n} d\Theta = \int_{t_{avg}} t_{avg} \cdot \frac{1}{n} d\Theta.$$

Now, we get

$$\int_{\partial B} dQ \geq \int f(t(\Theta)) \cdot d\Theta \geq \int f(t_{avg}) \cdot d\Theta = \int_{\partial B'} dQ,$$

the first inequality coming from $f$ increasing, and the second inequality from $f$ convex (note that the middle integral should in general be taken only over the directions where the intersection with $\partial B$ is non-empty, but in this case since $f$ is zero at the origin we can in fact take it over all directions). So, we see that $B'$ has tangential surface area less than or equal to that of $B$, and since all of the surface area of $B'$ is counted in the tangential surface area ($B'$ is a ball about the origin), we conclude that $B'$ has surface area less than or equal to that of $B$ with equality possible only if all of the surface area of $B$ is also tangential, that is if $\partial B$ consists only of hyperspheres about the origin. But if $\partial B$ consists of multiple hyperspheres about the origin then the first inequality in (7.1) must be strict, and thus we conclude that equality holds if and only if $B = B'$.

**Proposition 7.5.** In $\mathbb{R}^n$ with density $r^p$ for $p < -n$, hyperspheres about the origin are uniquely isoperimetric (bounding volume away from the origin).

**Proof.** We use the substitution $s = r^{p+n}$. This will give us tangential surface area element

$$dQ = s^{1+\frac{1}{p+n}} d\theta$$
and the rest of the proof of Theorem 7.4 will carry through identically since
\[ -\frac{1}{\sqrt{p+n}} > 0. \]

\[ \square \]

Remark. The proof of Theorem 7.4 also gives this result in the \( \theta_0 \)-sector and even more generally in any cone over a subset of \( S^{n-1} \), but we may have to change the convexity condition: if hyperspheres about the origin are not isoperimetric in such a space with Euclidean surface area, then we cannot perform the trick of subtracting off \( a(0) \), and thus we will want to impose the stronger condition instead that \( a(s^{1/n})s^{1-1/n} \) is convex. We note that if we take the cone over a convex subset of \( S^{n-1} \) then the Euclidean case hyperspheres about the origin will be isoperimetric (\([LP] \) Thm 1.1], see also \([M7] \) Rmk. after Thm. 10.6]) and therefore the original convexity condition will suffice. The proof of Proposition 7.5 goes through in any of these spaces with no new conditions.

For \( \mathbb{R}^n \) with density \( r^p \), \( p > 0 \), we make a conjecture analogous to the planar case which was solved by Dahlberg et al. \([DDNT] \):

**Conjecture 7.6.** In \( \mathbb{R}^n \) with density \( r^p \), \( p > 0 \), hyperspheres through the origin are uniquely isoperimetric.

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Alexander Díaz, Department of Mathematical Sciences,
University of Puerto Rico, Mayagüez, PR 00681
E-mail address: alexander.diaz1@upr.edu

Nate Harman, Department of Mathematics and Statistics,
University of Massachusetts, Amherst, MA 01003
E-mail address: nateharman1234@yahoo.com

Sean Howe, Department of Mathematics,
Université Paris-Sud 11, Orsay, France 91405
E-mail address: seanpkh@gmail.com

David Thompson, Department of Mathematics and Statistics,
Williams College, Williamstown, MA 01267
E-mail address: dat1@williams.edu

Mailing Address: c/o Frank Morgan, Department of Mathematics and Statistics,
Williams College, Williamstown, MA 01267
E-mail address: Frank.Morgan@williams.edu