On the coefficients of the cyclotomic polynomials of order three

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Abstract

We say that a cyclotomic polynomial $\Phi_n(x)$ has order three if $n$ is the product of three distinct primes, $p < q < r$. Let $A(n)$ be the largest absolute value of a coefficient of $\Phi_n(x)$ and $M(p)$ be the maximum of $A(pqr)$. In 1968, Sister Marion Beiter [3, 4] conjectured that $A(pqr) \leq \frac{p+1}{2}$. In 2008, Yves Gallot and Pieter Moree [8] showed that the conjecture is false for every $p \geq 11$, and they proposed the Corrected Beiter conjecture: $M(p) \leq \frac{2}{3}p$. Here we will give a sufficient condition for the Corrected Beiter conjecture and prove it when $p = 7$.

Key words: Cyclotomic polynomial; Ternary cyclotomic polynomial; Beiter’s conjecture; Corrected Beiter conjecture

1 Introduction

The $n$th cyclotomic polynomial is the monic polynomial whose roots are the primitive $n$th roots of unity and are all simple. It is defined by

$$\Phi_n(x) = \prod_{\substack{1 \leq a \leq n \\ (a,n) = 1}} (x - e^{\frac{2\pi ia}{n}}) = \sum_{i=0}^{\phi(n)} c_i x^i. \quad (1.1)$$

The degree of $\Phi_n$ is $\phi(n)$, where $\phi$ is the Euler totient function. It is known that the coefficients $c_i$, where $0 \leq i \leq \phi(n)$, are all integers.

Definition 1.1

$$A(n) = \max\{|c_i|, 0 \leq i \leq \phi(n)\}. \quad (1.2)$$

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For $n < 105$, $A(n) = 1$. It was once conjectured that this would hold for all $n$, however $A(105) = 2$. Note that 105 is the smallest positive integer that is the product of three distinct odd primes. In fact, it is easy to prove that $A(p) = 1$ and $A(pq) = 1$ for distinct primes $p, q$. Besides, we have the following useful propositions.

**Proposition 1.2** The nonzero coefficients of $\Phi_{pq}(x)$ alternate between $+1$ and $-1$.

**Proposition 1.3** Let $p$ be a prime.
If $p \mid n$, then $\Phi_{pn}(x) = \Phi_n(x^p)$, so $A(pn) = A(n)$.
If $p \nmid n$, then $\Phi_{pn}(x) = \Phi_n(x^p)/\Phi_n(x)$.
If $n$ is odd, then $\Phi_{2n}(x) = \Phi_n(-x)$, so $A(2n) = A(n)$.

**Proof.** See [11] for details.

By the proposition above, it suffices to consider squarefree values of $n$ to determine $A(n)$. For squarefree $n$, the number of distinct odd prime factors of $n$ is the order of the cyclotomic polynomial $\Phi_n$. Therefore the cyclotomic polynomials of order three are the first non-trivial case with respect to $A(n)$. We also call them ternary cyclotomic polynomials.

Assume $p < q < r$ are odd primes, Bang [2] proved the bound $A(pqr) \leq p - 1$. This was improved by Beiter [3, 4], who proved that $A(pqr) \leq p - \lfloor \frac{p}{4} \rfloor$, and made the following conjecture.

**Conjecture 1.4 (Beiter)** $A(pqr) \leq \frac{p+1}{2}$.

Beiter proved her conjecture for $p \leq 5$ and also in case either $q$ or $r \equiv \pm 1$ (mod $p$) [3]. If this conjecture holds, it is the strongest possible result of this form. This is because Möller [12] indicated that for any prime $p$ there are infinitely many pairs of primes $q < r$ such that $A(pqr) > \frac{p+1}{2}$. Define

$$M(p) = \max\{A(pqr) \mid p < q < r\},$$

where the prime $p$ is fixed, and $q$ and $r$ are arbitrary primes. Now with Möller’s result, we can reformulate Beiter’s conjecture.

**Conjecture 1.5** For $p > 2$, we have $M(p) = \frac{p+1}{2}$.

However, Gallot and Moree [8] showed that Beiter’s conjecture is false for every $p \geq 11$. For $p = 7$, it is still an open problem. In this paper, we will give an answer. Based on extensive numerical computations, they gave many counter-examples and proposed the Corrected Beiter conjecture.

**Conjecture 1.6 (Corrected Beiter conjecture)** We have $M(p) \leq \frac{2}{3}p$. 

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This is the strongest corrected version of Beiter’s conjecture because they also proved that for any \( \varepsilon > 0 \), \( \frac{2}{3}p(1 - \varepsilon) \leq M(p) \leq \frac{2}{3}p \) for every sufficiently large prime \( p \).

## 2 Preliminaries

Let \( p < q < r \) be odd primes. We will first give a lemma for computing the coefficients of \( \Phi_{pqr} \) explicitly. By Proposition 1.2, we can get

\[
\Phi_{pq}(x) = \frac{\Phi_{pq}(x^r)(x^p - 1)(x^q - 1)}{(x^pq - 1)(x - 1)} = \sum_i c_ix^i. \tag{2.1}
\]

Let

\[
f(x) = \Phi_{pqr}(x)(x - 1) = \sum_i (c_{i-1} - c_i)x^i = \sum_j b_jx^j \tag{2.2}
\]

and

\[
g(x) = f(x)(x^{pq} - 1) = \sum_j (b_j - pq - b_j)x^j = \sum_k a_kx^k. \tag{2.3}
\]

For \( i < 0 \) or \( i > \phi(pqr) = (p-1)(q-1)(r-1) \), \( j < 0 \) or \( j > (p-1)(q-1)(r-1)+1 \) and \( k < 0 \) or \( k > (p-1)(q-1)(r-1)+1+pq \), we set \( c_i = b_j = a_k = 0 \). Obviously, we have

\[
c_i = \sum_{j \geq i+1} b_j = \sum_{j \geq i+1} \sum_{k \equiv j \mod pq} a_k = \sum_{k \geq i+1+pq} a_k + \sum_{k \geq i+2pq} a_k + \cdots. \tag{2.4}
\]

Let

\[
\Phi_{pq}(x) = \sum_m d_m x^m. \tag{2.5}
\]

then

\[
g(x) = \Phi_{pq}(x^r)(x^p - 1)(x^q - 1) = \sum_m d_m x^{mr}(x^{p+q} - x^q - x^p + 1). \tag{2.6}
\]

For \( m < 0 \) or \( m > \phi(pq) = (p - 1)(q - 1) \), we set \( d_m = 0 \).

**Notation** \( \forall n \in \mathbb{Z}, \) let \( \pi \) be the unique integer such that \( 0 \leq \pi \leq pq - 1 \) and \( \pi \equiv n \mod pq \).

For any \( n \in \mathbb{Z} \), define a map

\[
\chi_n : \mathbb{Z} \to \{0, \pm 1\}
\]
by

\[ \chi_n(i) = \begin{cases} 
1 & \text{if there exists an integer } s_1 \text{ with } n + p + q \geq i + 1 + s_1pq > n + q \\
-1 & \text{if there exists an integer } s_2 \text{ with } n + p \geq i + 1 + s_2pq > n \\
0 & \text{otherwise.}
\end{cases} \]

Note that this map is well-defined. An elementary somewhat tedious argument then shows that alternatively one can define \( \chi_n \) by

\[ \chi_n(i) = \begin{cases} 
1 & \text{if } n + p + q \geq i + 1 > n + q \text{ or } \overline{i + 1} \leq n + p + q < \overline{n + q} \\
-1 & \text{if } n + p \geq i + 1 > \overline{n} \text{ or } \overline{i + 1} \leq n + p < \overline{n} \\
0 & \text{otherwise.}
\end{cases} \]

Now it is not difficult to verify the lemma below.

**Lemma 2.1** With notation as above, we have

\[ c_i = \sum_{mr+p+q \geq i+1+pq} d_m \chi_{mr}(i). \quad (2.7) \]

**Proof.** Combining (2.3) and (2.6) yields

\[ g(x) = \sum_k a_k x^k = \sum_m d_m x^{mr}(x^{p+q} - x^q - x^p + 1). \]

By (2.4), we know that to compute \( c_i \) it suffices to consider only the coefficients \( a_k \) of the terms of \( g(x) \) with exponents \( k \geq i + 1 + pq \). On the other hand, for \( d_m x^{mr}(x^{p+q} - x^q - x^p + 1) \), \( mr + p + q \geq i + 1 + pq \), the contribution to \( c_i \) is

\[ d_m (W_i(mr + p + q) - W_i(mr + q) - W_i(mp + p) + W_i(mr)), \quad (2.8) \]

where \( W_i(m_1) \) counts the number of integers \( s \geq 1 \) such that \( m_1 \geq i + 1 + spq \). Now note that

\[ W_i(mr + p + q) - W_i(mp + q) = \begin{cases} 
1 & \text{if there exists an integer } s_1 \text{ with } mr + p + q \geq i + 1 + s_1pq > mr + q; \\
0 & \text{otherwise,}
\end{cases} \]
and

\[-W_i(mr + p) + W_i(mr) = \begin{cases} 
-1 & \text{if there exists an integer } s_2 \text{ with } mr + p \geq i + 1 + s_2pq > mr; \\
0 & \text{otherwise.} \end{cases}\]

By the definition of \(\chi_n\), it then follows that the expression in (2.8) equals \(d_m\chi_{mr}(i)\), so we complete the proof of the lemma.

Especially, note that \(c_i = 0\) for \(i < 0\), so we can immediately obtain the following consequence which will be very important in the next section.

**Lemma 2.2** For any integer \(i\),

\[
\sum_m d_m\chi_{mr}(i) = 0. \tag{2.9}
\]

**Proof.** From either definition of \(\chi_n\), it is easy to find that the value of \(\chi_n(i)\) only depends on \(\bar{n}\) and \(\bar{r}\). That means that for any \(n', i' \in \mathbb{Z}, n' \equiv n \pmod{pq}, i' \equiv i \pmod{pq}\), we have

\[
\chi_{n'}(i') = \chi_n(i). \tag{2.10}
\]

For any integer \(i\), there exists an integer \(s\) such that \(mr + p + q \geq (i - spq) + 1 + pq\) for any non-negative integer \(m\). Observe that \(c_i = 0\) for \(i < 0\) and \(d_m = 0\) for \(m < 0\), hence we have

\[
\sum_m d_m\chi_{mr}(i) = \sum_m d_m\chi_{mr}(i - spq)
= \sum_{mr + p + q \geq (i - spq) + 1 + pq} d_m\chi_{mr}(i - spq)
= c_{i - spq}
= 0.
\]

\[\square\]

**Lemma 2.3** With the notation as above, we have

\[
A(pqr) = \max_{i,j \in \mathbb{Z}} \left| \sum_{m \geq j} d_m\chi_{mr}(i) \right|. \tag{2.11}
\]
Now it suffices to show that for any \( i, j \in \mathbb{Z} \),

\[
\left| \sum_{m \geq j} d_m \chi_{mr}(i) \right| \leq A(pqr).
\]

Let \( s \) be the largest integer such that \( jr + p + q \geq (i + spq) + 1 + pq \). If \((j - 1)r + p + q < (i + spq) + 1 + pq\), then

\[
\left| \sum_{m \geq j} d_m \chi_{mr}(i) \right| = \left| \sum_{m \geq j} d_m \chi_{mr}(i + spq) \right| = |c_{i+spq}| \leq A(pqr).
\]

If \((j - 1)r + p + q \geq (i + spq) + 1 + pq\) and \( \chi_{jr}(i) = 0 \), then \( jr + p > (j - 1)r + p + q \geq (i + spq) + 1 + pq \) (because \( r > q \)) and hence \((i + spq) + 1 + pq \leq jr < jr + q < (j + 1)r\). Let \( j_1 \) be the smallest integer such that \( j_1r + p + q \geq (i + (s + 1)pq) + 1 + pq \), then for \( j \leq m < j_1 \), \( \chi_{mr}(i) = 0 \). Therefore we have

\[
\left| \sum_{m \geq j} d_m \chi_{mr}(i) \right| = \left| \sum_{m \geq j_1} d_m \chi_{mr}(i) \right| = |c_{i+(s+1)pq}| \leq A(pqr).
\]

If \((j - 1)r + p + q \geq (i + spq) + 1 + pq\) and \( \chi_{jr}(i) \neq 0 \), then \( \chi_{jr}(i) = -1 \), that is, \( jr + p \geq (i + spq) + 1 + pq \geq jr \). Since \( p < q < r \), we get \((j - 2)r + p + q < jr < (i + spq) + 1 + pq\) and \((j + 1)r > jr + p \geq (i + spq) + 1 + pq\) which implies that for \( j + 1 \leq m < j_1 \), \( \chi_{mr}(i) = 0 \). It follows that

\[
\left| \sum_{m \geq j+1} d_m \chi_{mr}(i) \right| = \left| \sum_{m \geq j_1} d_m \chi_{mr}(i) \right| = |c_{i+(s+1)pq}| \leq A(pqr),
\]

and

\[
\left| \sum_{m \geq j} d_m \chi_{mr}(i) \right| = |c_{i+spq}| \leq A(pqr).
\]

If \( \chi_{(j-1)r}(i) = 0 \), then

\[
\left| \sum_{m \geq j} d_m \chi_{mr}(i) \right| = \left| \sum_{m \geq j-1} d_m \chi_{mr}(i) \right| = |c_{i+spq}| \leq A(pqr).
\]

If \( \chi_{(j-1)r}(i) \neq 0 \), then \( \chi_{(j-1)r}(i) = 1 \). By Proposition 1.2, we have \( d_{j-1}d_j \leq 0 \),
hence $d_{j-1} \chi_{j-1} r(i) d_j \chi_j r(i) \geq 0$, therefore

$$\left| \sum_{m \geq j} d_m \chi_{mr}(i) \right| \leq \max \left\{ \left| \sum_{m \geq j-1} d_m \chi_{mr}(i) \right|, \left| \sum_{m \geq j+1} d_m \chi_{mr}(i) \right| \right\} \leq A(pqr).$$

This completes the proof of the corollary. \qed

**Remark 2.4** If $q$ and $r$ interchange, we will have similar arguments as above. Lemma 2.1 and Lemma 2.2 still hold, but Lemma 2.3 should be modified. We can only get the trivial conclusion (2.12), but it is sufficient for estimating the upper bound of $A(pqr)$ to consider $\max_{i,j \in \mathbb{Z}} \left| \sum_{m \geq j} d_m \chi_{mr}(i) \right|$.

Based on the results above, we can establish explicitly the following Theorem 2.5 and Theorem 2.6 which have been proven by Kaplan [9].

**Theorem 2.5 (Nathan Kaplan, 2007)** Let $p < q < r$ be odd primes. Then for any prime $s > q$ such that $s \equiv \pm r \pmod{pq}$, $A(pqr) = A(pqs)$.

**Proof.** If $s \equiv r \pmod{pq}$, then $\chi_{mr}(i) = \chi_{ms}(i)$, by (2.11), we obtain

$$A(pqr) = \max_{i,j \in \mathbb{Z}} \left| \sum_{m \geq j} d_m \chi_{mr}(i) \right| = \max_{i,j \in \mathbb{Z}} \left| \sum_{m \geq j} d_m \chi_{ms}(i) \right| = A(pqs).$$

Next we consider the case $s \equiv -r \pmod{pq}$. From the definition of $\chi_n$, we can simply verify that

$$\chi_{mr}(i) = -\chi_{-mr}(-i + p + q - 1). \quad (2.13)$$

Therefore, by (2.10) and (2.13), we have

$$A(pqr) = \max_{i,j \in \mathbb{Z}} \left| \sum_{m \geq j} d_m \chi_{mr}(i) \right| = \max_{i,j \in \mathbb{Z}} \left| \sum_{m \geq j} d_m \chi_{-mr}(-i + p + q - 1) \right| = \max_{i,j \in \mathbb{Z}} \left| \sum_{m \geq j} d_m \chi_{ms}(-i + p + q - 1) \right| = A(pqs). \quad \square$$

**Theorem 2.6 (Nathan Kaplan, 2007)** Let $p < q$ and $r \equiv \pm 1 \pmod{pq}$ be odd primes. Then $A(pqr) = 1$. 

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Theorem 2.7 Let \( d_m = 0 \) for \( m < 0 \) or \( m > \phi(pq) \), by (2.9), we can get for any pair of integers \( i \) and \( j \leq 0 \) or \( j > \phi(pq) \),

\[
\sum_{m \geq j} d_m \chi_{mr}(i) = 0.
\]

Given \( i \) and \( 0 < j \leq \phi(pq) \), let

\[
M^+ = \{0 \leq m \leq \phi(pq)|\chi_{mr}(i) = 1\}, \quad M^- = \{0 \leq m \leq \phi(pq)|\chi_{mr}(i) = -1\}.
\]

Since \( r \equiv \pm 1 \mod pq \), the definition of \( \chi_n \) implies that both of \( M^+ \) and \( M^- \) are sets of consecutive integers, of cardinality at most \( p \). Let us first assume that \( j \in M^- \), then \( j \notin M^+ \). It follows that \( \chi_{mr}(i) \neq 1 \) either for all \( \phi(pq) \geq m \geq j \) or for all \( 0 \leq m < j \), hence \( \sum_{m \geq j, m \in M^+} d_m \chi_{mr}(i) \) or \( \sum_{m < j, m \in M^+} d_m \chi_{mr}(i) \) should be 0. On the other hand, by Proposition 1.2, we have

\[
\left| \sum_{m \geq j, m \in M^+} d_m \chi_{mr}(i) \right| \leq 1, \quad \left| \sum_{m < j, m \in M^+} d_m \chi_{mr}(i) \right| \leq 1, \quad (2.14)
\]

\[
\left| \sum_{m \geq j, m \in M^-} d_m \chi_{mr}(i) \right| \leq 1, \quad \left| \sum_{m < j, m \in M^-} d_m \chi_{mr}(i) \right| \leq 1. \quad (2.15)
\]

Combining the above observations and (2.9), we have

\[
\left| \sum_{m \geq j} d_m \chi_{mr}(i) \right| = \left| \sum_{m \geq j, m \in M^+} d_m \chi_{mr}(i) + \sum_{m \geq j, m \in M^-} d_m \chi_{mr}(i) \right|
\]

\[
= \left| \sum_{m < j, m \in M^+} d_m \chi_{mr}(i) + \sum_{m < j, m \in M^-} d_m \chi_{mr}(i) \right|
\]

\[
\leq 1.
\]

For the case \( j \notin M^- \), the argument is similar. Therefore by (2.12), we have \( A(pqr) \leq 1 \), thus \( A(pqr) = 1 \). This completes the proof. \( \square \)

Theorem 2.7 Let \( p < q < r \) be odd primes. Then \( A(pqr) \leq \min\{r, pq - r\} \).

Proof. Given \( i \) and \( 0 < j \leq \phi(pq) \), according to the proof of Theorem 2.6, there must exist a partition of \( [0, \phi(pq)] \), \( 0 = t_0 < t_1 < t_2 < \cdots < t_{\tau-1} \leq t_{\tau} = \phi(pq), t_k \in \mathbb{Z} \) for \( 0 \leq k \leq \tau \), such that

\[
M_k^+ = \{t_{k-1} < m \leq t_k|\chi_{mr}(i) = 1\}, 2 \leq k \leq \tau,
\]

\[
M_1^+ = \{t_0 \leq m \leq t_1|\chi_{mr}(i) = 1\}
\]

are all sets of consecutive integers, of cardinality at most \( p \). In fact, we can obtain this partition by induction. First, let \( m_1 \) be the smallest integer such
that \(0 \leq m_1 \leq \phi(pq)\) and \(\chi_{m_1}(i) = 1\). Then we can take \(t_1 + 1\) to equal the smallest integer such that \(m_1 \leq t_1 \leq \phi(pq)\) and \(\chi_{(t_1+1)r}(i) \neq 1\). Next let \(m_2\) be the smallest integer such that \(t_1 < m_2 \leq \phi(pq)\) and \(\chi_{m_2r}(i) = 1\). Then we can take \(t_2 + 1\) to equal the smallest integer such that \(m_2 \leq t_2 \leq \phi(pq)\) and \(\chi_{(t_2+1)r}(i) \neq 1\). Moreover, by the definition of \(\chi_n\), we have

\[
(m_2 - m_1)r \geq p(q - 1).
\]

(2.16)

Inductively, we can get \(m_3, t_3, \ldots, m_\tau, t_\tau\). Notice that if \(m_k\) does not exist or \(m_k = \phi(pq)\), then we can take \(t_k = t_{k+1} = \cdots = t_\tau = \phi(pq)\). Specially, if \(t_\tau < \phi(pq)\), we claim \(m_{\tau+1}\) does not exist. Otherwise, by (2.16) we have

\[
(m_{\tau+1} - m_1)r = (m_{\tau+1} - m_\tau + \cdots + m_2 - m_1)r \geq p(q - 1)r.
\]

(2.17)

On the other hand,

\[
(m_{\tau+1} - m_1)r \leq \phi(pq)r = (p - 1)(q - 1)r.
\]

This contradicts (2.17), so we can always take \(t_\tau = \phi(pq)\). Similarly, there also exists a partition of \([0, \phi(pq)]\), \(0 = s_0 < s_1 \leq s_2 \leq \cdots \leq s_{\tau-1} \leq s_\tau = \phi(pq), s_l \in \mathbb{Z}\) for \(0 \leq l \leq \tau\), such that

\[
M^-_l = \{s_{l-1} < m \leq s_l | \chi_{mr}(i) = -1\}, 2 \leq l \leq \tau,
\]

\[
M^-_1 = \{s_0 \leq m \leq s_1 | \chi_{mr}(i) = -1\}
\]

are all sets of consecutive integers, of cardinality at most \(p\).

Assume \(t_{k-1} < j \leq t_k\) for some \(1 \leq k \leq \tau\) and \(s_{l-1} < j \leq s_l\) for some \(1 \leq l \leq \tau\). Let us first assume \(j \in M^-_l\), then \(j \notin M^+_k\). By (2.14) and (2.15), we have

\[
\sum_{m \geq j} d_m \chi_{mr}(i) = \sum_{m \geq j, m \in M^+_k} d_m \chi_{mr}(i) + \cdots + \sum_{m \geq j, m \in M^+_\tau} d_m \chi_{mr}(i) + \sum_{m \geq j, m \in M^-_l} d_m \chi_{mr}(i) + \cdots + \sum_{m \geq j, m \in M^-_\tau} d_m \chi_{mr}(i)
\]

\[
\left| \sum_{m \geq j} d_m \chi_{mr}(i) \right| \leq \left| \sum_{m \geq j, m \in M^+_k} d_m \chi_{mr}(i) \right| + \cdots + \left| \sum_{m \geq j, m \in M^+_\tau} d_m \chi_{mr}(i) \right| + \left| \sum_{m \geq j, m \in M^-_l} d_m \chi_{mr}(i) \right| + \cdots + \left| \sum_{m \geq j, m \in M^-_\tau} d_m \chi_{mr}(i) \right|
\]

\[
\leq 2\tau - k - l + 2.
\]

Similarly we also have
\[ \sum_{m<j} d_m \chi_{mr}(i) \leq \sum_{m<j, m \in M_i^+} d_m \chi_{mr}(i) + \sum_{m<j, m \in M_i^-} d_m \chi_{mr}(i) + \sum_{m<j, m \in M_k^-} d_m \chi_{mr}(i) \]

\[ \leq k + l - 1. \]

Thus
\[ \left| \sum_{m \geq j} d_m \chi_{mr}(i) \right| + \left| \sum_{m < j} d_m \chi_{mr}(i) \right| \leq 2 \varpi + 1. \]

By (2.9), we certainly get
\[ \left| \sum_{m \geq j} d_m \chi_{mr}(i) \right| = \left| \sum_{m < j} d_m \chi_{mr}(i) \right|. \]

Therefore
\[ \left| \sum_{m \geq j} d_m \chi_{mr}(i) \right| \leq \varpi. \]

For the case \( j \notin M_i^- \), the argument is similar. Therefore (2.11) yields \( A(pqr) \leq \varpi \). On the other hand, by Dirichlet’s Prime Number Theorem, we know there exists a prime \( s > q \) satisfying \( \varpi = pq - \varpi \). That means \( s \equiv -r \pmod{pq} \), by Theorem 2.5 and the arguments above, we get
\[ A(pqr) = A(pqs) \leq \varpi = pq - \varpi. \]

We have thus proved the theorem. \( \square \)

3 Main result

Now to estimate the upper bound of \( A(pqr) \), we need to investigate the properties of the coefficients of \( \Phi_{pq} \). First we introduce some notation for the rest of the paper.

**Notation** For any distinct primes \( p \) and \( q \), let \( q_p^* \) be the unique integer such that \( 0 < q_p^* < p \) and \( qq_p^* \equiv 1 \pmod{p} \). Let \( \overline{q_p} \) be the unique integer such that \( 0 < \overline{q_p} < p \) and \( q \equiv \overline{q_p} \pmod{p} \).

About the coefficients of \( \Phi_{pq} \), Lam and Leung [10] showed

**Theorem 3.1 (T.Y. Lam and K.H. Leung, 1996)** Let \( \Phi_{pq}(x) = \sum_m d_m x^m \). For \( 0 \leq m \leq \phi(pq) \), we have
(A) $d_m = 1$ if and only if $m = up + vq$ for some $u \in [0, p_q^* - 1]$ and $v \in [0, q_p^* - 1]$;

(B) $d_m = -1$ if and only if $m + pq = u'p + v'q$ for some $u' \in [p_q^*, q - 1]$ and $v' \in [q_p^*, p - 1]$;

(C) $d_m = 0$ otherwise.

The numbers of terms of the former two kinds are, respectively, $p_q^* q_p^*$ and $(q - p_q^*)(p - q_p^*)$, with difference 1 since $(p - 1)(q - 1) = (p_q^* - 1)p + (q_p^* - 1)q$.

About $A(pqr)$, the best known general upper bound to date is due to Bartłomiej Bzdęga [5]. He gave the following important result:

**Theorem 3.2 (Bartłomiej Bzdęga, 2008)** Set

$$\alpha = \min\{q_p^*, r_p^*, p - q_p^*, p - r_p^*\}$$

and $0 < \beta < p$ satisfying $\alpha \beta qr \equiv 1 \pmod{p}$. Put $\beta^* = \min\{\beta, p - \beta\}$. Then we have

$$A(pqr) \leq \min\{2\alpha + \beta^*, p - \beta^*\}.$$

We can now prove our main result.

**Theorem 3.3** Let $p < q < r$ be odd primes. Suppose $\min\{q_p^*, r_p^*, p - q_p^*, p - r_p^*\} > \frac{p - 1}{3}$, then $A(pqr) \leq \frac{p + \beta^*}{2}$.

**Proof.** Let us first assume

$$1 \leq p - q_p^* \leq r_p^* < p - r_p^* \leq q_p^* \leq p - 1. \quad (3.1)$$

According to Theorem 3.2, it follows $\alpha = p - q_p^*$, $\beta = p - r_p^*$ and $\beta^* = r_p^*$. Suppose $A(pqr) > \frac{p + \beta^*}{2}$, so we easily get

$$\frac{p + \beta^*}{2} < p - \beta^*.$$  

This implies that

$$\beta^* < \frac{1}{3}p. \quad (3.2)$$

By (2.12), we know there exist a pair of integers $i, j$ such that

$$\left| \sum_{m \geq j} d_m \chi_{mr}(i) \right| > \frac{p + \beta^*}{2}. \quad (3.3)$$

By Theorem 3.1, we can divide the nonzero terms of $\Phi_{pq}(x)$ into $p$ classes depending on the value of $v$ or $v'$. From the definition of $\chi_n$, we can simply verify that for any given class, there is at most one term such that $\chi_{mr}(i) = 1$.  

\[11\]
For the case $\chi_{mr}(i) = -1$, we have the similar result. By (2.9) and (3.3), we immediately obtain
\[
\left| \sum_{m<j} d_m \chi_{mr}(i) \right| > \frac{p + \beta^*}{2}.
\] (3.4)

This implies that the number of the nonzero terms of $\Phi_{pq}(x)$ such that $\chi_{mr}(i) = \pm 1$ is more than $p + \beta^*$. Therefore there are more than $\beta^*$ classes such that each of them has two terms $d_m x^m$ and $d_m' x^{m'}$ such that $\chi_{mr}(i) = 1$ and $\chi_{m'r}(i) = -1$ respectively. Moreover, $m$ and $m'$ should satisfy $m \geq j, m' < j$ or $m < j, m' \geq j$, otherwise $d_m \chi_{mr}(i) + d_m' \chi_{m'r}(i) = 0$, thus their contributions to the left sides of (3.3) and (3.4) are both zero. For convenience of description, We call them the special classes.

Now we claim that the special classes contain not only the classes of $d_m = 1$, but also the classes of $d_m = -1$. In fact, by Theorem 3.1, the number of the classes of $d_m = -1$ is just $p - q_p^* \leq \beta^*$, so the special classes must contain the classes of $d_m = 1$. If there are more than $\beta^*$ classes of $d_m = 1$ in the special classes, then it implies that there exist $u_1 \in [0, p_q^* - 1], v_1 \in [0, q_p^* - 1]$ such that
\[
u_1 p + v_1 q \geq j,
\]
u_2 \in [0, p_q^* - 1], v_2 \in [0, q_p^* - 1] such that
\[
u_2 p + v_2 q < j
\]
and
\[
u_2 - v_1 \geq \beta^*.
\]
This yields
\[(u_1 - u_2)p + (v_1 - v_2)q > 0,
\]

hence
\[(u_1 - u_2)p > (v_2 - v_1)q \geq \beta^* q.
\]

On the other hand,
\[(u_1 - u_2)p \leq (p_q^* - 1)p = (p - q_p^*)q - p + 1 \leq \beta^* q - p + 1.
\]
The equality holds because $(p - 1)(q - 1) = (p_q^* - 1)p + (q_p^* - 1)q$. Therefore we derive a contradiction and prove our claim.

Since $(up + vq)r + p \equiv (up + (v - r_p^*)q)r + p + q \pmod{pq}$, we have
\[\chi_{mr}(i) = -1 \iff \chi_{(m-r_q^*)r}(i) = 1.
\] (3.5)

We claim that
\[
\sum_{m \geq j} d_m \chi_{mr}(i) < -\frac{p + \beta^*}{2}.
\] (3.6)
By (3.3), we know \( \sum_{m \geq j} d_m \chi_{mr}(i) > \frac{p+\beta^*}{2} \) or \( \sum_{m \geq j} d_m \chi_{mr}(i) < -\frac{p+\beta^*}{2} \). If the former holds, then

\[
\sum_{m \geq j, \beta^*} d_m \chi_{mr}(i) > \frac{p-\beta^*}{2} > \beta^*
\]

because

\[
\sum_{m \geq j, \beta^*} d_m \chi_{mr}(i) \leq \beta^*.
\]

Thus there must exist \( u \in [0, p_q^* - 1] \) and \( v \in [0, q_p^* - 1 - \beta^*] \) such that \( \chi_{(up+vq)}r(i) = 1 \). By (3.5), we have \( \chi_{(up+(v+r_p^*)q)}r(i) = -1 \) and \( v+r_p^* \in [0, q_p^* - 1] \) since \( \beta^* = r_p^* \). Hence \( d_{up+vq} \chi_{(up+vq)r}(i) + d_{up+(v+r_p^*)q} \chi_{(up+(v+r_p^*)q)r}(i) = 0 \), their contributions to the left side of (3.7) are zero. This is a contradiction, so we establish the second claim.

Combining the above arguments yields there exist \( u_1, u_2 \in [0, p_q^* - 1] \) and \( v \in [0, q_p^* - 1] \) such that \( u_1p + vq \geq j > u_2p + vq \), \( \chi_{(u_1p+vq)r}(i) = -1 \) and \( \chi_{(u_2p+vq)r}(i) = 1 \). This implies that

\[
(u_1p + vq)r + p + \bar{q}_p = (u_2p + vq)r + p + q
\]

or

\[
(u_1p + vq)r + p - (p - \bar{q}_p) = (u_2p + vq)r + p + q.
\]

Similarly, we also have there exist \( u'_1, u'_2 \in [p_q^*, q - 1] \) and \( v' \in [p_q^*, p - 1] \) such that \( u'_1p + v'q - pq \geq j > u'_2p + v'q - pq \), \( \chi_{(u'_1p+v'q-pq)r}(i) = 1 \) and \( \chi_{(u'_2p+v'q-pq)r}(i) = -1 \). This implies that

\[
(u'_1p + v'q - pq)r + p + q - \bar{q}_p = (u'_2p + v'q - pq)r + p
\]

or

\[
(u'_1p + v'q - pq)r + p + q + (p - \bar{q}_p) = (u'_2p + v'q - pq)r + p.
\]

If (3.8) and (3.10) hold simultaneously, then we get

\[
(u_1 + u'_1)p = (u_2 + u'_2)p.
\]

Hence

\[
q \mid (u_1 + u'_1 - u_2 - u'_2).
\]

This contradicts \( 0 < u_1 + u'_1 - u_2 - u'_2 \leq q - 2 \). Similarly (3.9) and (3.11) can not hold simultaneously, so without loss of generality, we assume (3.9) and (3.10) are correct. By (3.5), we have \( \chi_{(u'_1p+v'q-pq-r_p^*)r}(i) = 1 \) since \( \chi_{(u'_2p+v'q-pq)r}(i) = -1 \). It follows that the class of \( v' - r_p^* \in [0, q_p^* - 1] \) does not contain a term such that \( \chi_{mr}(i) = 1 \) since \( \chi_{((u'_2p+v'q-pq-r_p^*)r}(i) = 1 \). If it does not contain a term such that \( \chi_{mr}(i) = -1 \) either, then the contributions of this class to the left sides of (3.3) and (3.4) are both zero. It is easy to verify that there must exist
a special class of \( v' \in [q_p^*, p - 1] \) such that the class of \( v' - r_p^* \) contains a term such that \( m \geq j \) and \( \chi_{mr}(i) = -1 \). This implies that there exist \( u_3 \in [0, p_q^*-1] \) such that \( u_3 p + (v' - r_p^*)q \geq j \) and \( \chi_{(u_3p+(v'-r_p^*)q)r}(i) = -1 \), so

\[
(u_3p + (v' - r_p^*)q)r + p + 4q = ((u_2' - q)p + (v' - r_p^*)q)r + p + q
\]  

(3.12)

or

\[
(u_3p + (v' - r_p^*)q)r + p - (p - d_p) = ((u_2' - q)p + (v' - r_p^*)q)r + p + q.
\]

If the latter holds, by (3.9) we get

\[
(u_3 - u_1)p = (u_2' - q - u_2)p.
\]

Hence

\[
q \mid (u_3 + u_2 - u_1 - u_2').
\]

(3.13)

On the other hand, by

\[
(u_3p + (v' - r_p^*)q) \geq j > u_2'p + v'q - pq
\]

we have

\[
0 > (u_3 - u_2)p > (r_p^* - p)q.
\]

Note that

\[
0 > (u_2 - u_1)p \geq -(p_q^* - 1)p = -(p - q_p^*)q + p - 1 \geq -r_p^*q + p - 1,
\]

so we can get

\[
0 > (u_3 + u_2 - u_1 - u_2')p > -pq + p - 1.
\]

This contradicts (3.13) and establishes the validity of (3.12).

Similarly we have \( \chi_{(u_4'p + v'q - pq + r_p^*)q}(i) = -1 \) because \( \chi_{(u_4'p + v'q - pq)(i) = 1} \). The class of \( v' - p + r_p^* \in [0, q_p^*-1] \) does not contain a term such that \( \chi_{mr}(i) = -1 \), but it contains a term such that \( \chi_{mr}(i) = 1 \). This implies that there exist \( u_4 \in [0, p_q^*-1] \) such that \( u_4 p + (v' - p + r_p^*)q < j \) and \( \chi_{(u_4p+(v'-p+r_p^*)q)r}(i) = 1 \), so we can get

\[
(u_4'p + v'q - pq + r_p^*q)r + p + 4q = (u_4p + (v' - p + r_p^*)q)r + p + q.
\]

(3.14)

Combining (3.10), (3.12) and (3.14), we obtain

\[
(u_3p + (v' - r_p^*)q)r + p + 3q = (u_4p + (v' - p + r_p^*)q)r + p + q.
\]

Since \( \chi_{(u_3p+(v'-r_p^*)q)r}(i) = -1 \) and \( \chi_{(u_4p+(v'-p+r_p^*)q)r}(i) = 1 \), we know

\[
((u_4p + (v' - p + r_p^*)q)r + p + q) - ((u_3p + (v' - r_p^*)q)r + p) \leq p - 1
\]
or
\[
\frac{(u_3 p + (v - r_p^*) q) r + p) - (u_4 p + (v' - p + r_p^*) q) r + p + q}{r} \leq p - 1.
\]
That is,
\[3q_p \leq p - 1\]
or
\[pq - 3q_p \leq p - 1.\]
Note that \(0 < q_p < p\), hence it is obvious that the former inequality holds. Therefore \(q_p \leq \frac{p - 1}{3}\).

If (3.8) and (3.11) hold simultaneously, we similarly get \(p - q_p \leq \frac{p - 1}{3}\).

For the cases
\[1 \leq q_p^* \leq r_p^* \leq p - q_p^* \leq p - 1,\]
\[1 \leq p - q_p^* \leq r_p^* \leq q_p^* \leq p - 1\]
and
\[1 \leq q_p^* \leq p - r_p^* \leq r_p^* \leq p - q_p^* \leq p - 1,\]
we can get the above results similarly. Observe that, we can immediately obtain the remaining four cases provided that \(q_p^* \) and \(r_p^* \) interchange. In these cases, by Remark 2.4, it is not difficult to establish \(\tau_p \leq \frac{p - 1}{3}\) or \(p - \tau_p \leq \frac{p - 1}{3}\) similarly. Combining the above arguments yields
\[
\min\{q_p, p - q_p, \tau_p, p - \tau_p\} \leq \frac{p - 1}{3}.
\]
This is a contradiction and completes the proof of the theorem.

\[\square\]

**Corollary 3.4** Let \(p < q < r\) be odd primes. Suppose \(\min\{q_p, p - q_p, \tau_p, p - \tau_p\} > \frac{p - 1}{3}\), then \(A(pqr) \leq \frac{2}{3} p\).

**Proof.** By Theorem 3.2 and 3.3, we have
\[
A(pqr) \leq \min\{p - \beta^*, \frac{p + \beta^*}{2}\} \leq \frac{2}{3} p.
\]
\[\square\]

Now we can show in the special case \(p = 7\) that both Beiter’s conjecture and the Corrected Beiter conjecture are correct.

**Theorem 3.5** We have \(M(7) = 4\).

**Proof.** Suppose there exists a pair of primes \(7 < q < r\) such that \(A(7qr) \geq 5\). Then by Theorem 3.2, we must have \(\alpha = \beta^* = 2\). It implies that \(q, r \equiv \pm 3\)
(mod 7), hence \( \min\{q_7, 7 - q_7, r_7, 7 - r_7\} > \frac{7-1}{3} = 2 \). By Theorem 3.3, we have \( A(7qr) \leq \frac{7+2}{2} < 5 \). This is a contradiction, so \( M(7) \leq 4 \). Recall that Möller [12] indicated that for any prime \( p \) there are infinitely many pairs of primes \( q < r \) such that \( A(pqr) \geq \frac{p+1}{2} \). Therefore we have \( M(7) = 4 \). \( \square \)

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