ON EFFECTS OF GAUGING ON SYMPLECTIC STRUCTURE, THE HOPF TERM COUPLED TO $CP^1$ MODEL, AND FRACTIONAL SPIN

B.Chakraborty and A.S.Majumdar

S.N.Bose National Centre for Basic Sciences
Block JD, Sector III, Salt Lake, Calcutta 700091, India

Abstract
We couple the Hopf term to the relativistic $CP^1$ model and carry out the Hamiltonian analysis at the classical level. The symplectic structure of the model given by the set of Dirac Brackets among the phase space variables is found to be the same as that of the pure $CP^1$ model. This symplectic structure is shown to be inherited from the global $SU(2)$ invariant $S^3$ model, and undergoes no modification upon gauging the $U(1)$ subgroup, except the appearance of an additional first class constraint generating $U(1)$ gauge transformation. We then address the question of fractional spin as imparted by the Hopf term at the classical level. For that we construct the expression of angular momentum through both symmetric energy-momentum tensor as well as through Noether’s prescription. Both the expressions agree for the model indicating no fractional spin is imparted by this term at the classical level—a result which is at variance with what has been claimed in the literature. We provide an argument to explain the discrepancy and corroborate our argument by considering a radiation gauge fixed Hopf term coupled to $CP^1$ model, where the desired fractional spin is reproduced and is given in terms of the soliton number. Finally, by making the gauge field of the $CP^1$ model dynamical by adding the Chern-Simons term, the model ceases to be a $CP^1$ model, as is the case with its nonrelativistic counterpart. This model is also shown to reveals the existence of ‘anomalous’ spin. This is however given in terms of the total charge of the system, rather than any soliton number.

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1e-mail:biswajit@bose.ernet.in
2e-mail:archan@bose.ernet.in
1 Introduction

Physics of $2 + 1$ dimensional systems have attracted much attention in recent years. This is because they provide critical insights into a large variety of problems in diverse phenomenological areas such as condensed matter physics and quantum gravity [1,2,3]. In condensed matter physics one finds applications of these systems, for example, in quantum Hall effect and anyonic superconductivity [2]. The fact that $2 + 1$ dimensional systems have very distinctive properties compared to the corresponding higher dimensional cases is due to the strange nature of the Poincare group $ISO(2,1)$ in $2 + 1$ dimensions [1,3]. For instance, in $2 + 1$ dimensions there exists the possibility of having fractional spin and statistics [3] arising from the occurrence of a multiply connected configuration space which leads to the presence of nontrivial phase.

These possibilities are realized by coupling the Chern-Simons(CS) or the Hopf term to various matter fields. For the CS term, apart from studying the respective Galilean/Poincare covariance, the existence of fractional spin has been revealed by carrying out a canonical Hamiltonian analysis in the gauge fixed[4] as well as the gauge independent[5,6,7] scheme at the classical level, and extending the analysis to the quantum level. On the other hand, models involving the Hopf term were initially analysed in the path integral formalism [3]. For example, Wilczek and Zee(WZ) [8] had considered the $O(3)$ nonlinear sigma model (NLSM) coupled to the Hopf term. They showed that the system acquires a nontrivial phase upon an adiabatic rotation of $2\pi$, signalling existence of fractional spin. Later, the same system was considered by Bowick et al [9] where a Hamiltonian analysis revealed the same fractional spin as obtained by WZ. Although their analysis was carried out at the quantum level, the fractional spin they obtained was not a typical quantum effect unlike the WZ case. This analysis can indeed be carried out at the classical level itself to obtain the same result. In their gauge fixed analysis [9], the existence of fractional spin was demonstrated by computing the difference between the two expressions of angular momentum obtained by using the symmetric definition of the energy momentum(EM) tensor ($J^s$), and by the Noether prescription ($J^N$) respectively. In [9] ($J^N$) is just the orbital angular momentum, as NLSM consists of scalar fields only. One usually regards the former expression of angular momentum ($J^s$) as the physical one, as the corresponding EM tensor is obtained by functional differentiation of the action with respect to the metric, and is thus gauge invariant by construction. On the other hand, the latter expression of angular momentum $J^N$ which usually turns out to be gauge invariant only on the constraint surface and that too under those gauge transformations which reduce to identity asymptotically [7].

For the case of the CS term, the above approach has been adopted for several models [5-7] to reveal the existence of fractional spin in a gauge independent manner. The gauge independent scheme has certain advantages over the corresponding gauge fixed scheme. In the latter method, the transformation properties of the basic fields under symmetry transformations get affected by the gauge fixing condition used. One is thus forced to look at the transformation properties of gauge invariant objects to uncover the underlying symmetry. At the quantum level it therefore
becomes rather nontrivial to disentangle the terms arising due to anomalies from those terms which are artefacts of gauge fixing conditions. Furthermore the symplectic structure given by the set of Dirac Brackets (DB) among the independent phase space variables usually turn out, in the gauge fixed scheme, to be more complicated functions of the fields than their counterparts in the gauge independent scheme. Quantization by elevating the DB's to quantum commutators in the gauge fixed scheme (reduced phase space scheme) is therefore liable to possess comparatively more operator ordering ambiguities than their counterpart in the gauge independent scheme. The latter scheme corresponds to Dirac quantization, where the physical states are taken to be gauge invariant by definition and therefore annihilated by the first class constraints. These two schemes of quantization are not necessarily equivalent [5,10].

It is therefore desirable to have a gauge independent formulation of the $O(3)$ NLSM coupled to the Hopf term, just as has been done with models involving the CS term. However, a gauge independent formulation of models involving the Hopf term is not possible in general. To understand the difficulty, recall that the conserved current ($\partial_\mu j^\mu = 0$) given in terms of the matter fields of any field theoretical model can be expressed as the curl of a fictitious gauge field $a_\mu$ ($j^\mu \sim \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda$). Coupling the current $j^\mu$ to the field $a_\mu$ defines the Hopf term ($\sim j^\mu a_\mu$). Although this term has formal resemblance with the CS term, for the case of the Hopf term one should be careful not to regard $a_\mu$ as an additional variable in the configuration space. Rather, $a_\mu$ here is determined in terms of $j^\mu$, a procedure which necessitates a gauge fixing condition to be used a priori in order to invert the above relation. In this way the Hopf term is quite distinct to the CS term. Once this inversion is carried out, the Hopf term becomes a nonlocal expression (quadratic in current $j^\mu$) thereby representing a nonlocal current-current interaction. No wonder, in [9] the radiation gauge condition was used right at the beginning to define the model. It is clear that a gauge independent analysis of the $O(3)$ NLSM coupled to the Hopf term is not possible at this stage.

Nevertheless, one can use an alternative $CP^1$ description [11] of the same NLSM, which is a $U(1)$ gauge theory having an enlarged phase space. The advantage here is that unlike in the usual Hopf model described above, the gauge field, which is the Dirac monopole connection in the $U(1)$ principle bundle over $S^2$ [12], gets directly related to the matter fields of the $CP^1$ model in a gauge independent manner. The current is therefore constructed as a curl of this gauge field, and finally the Hopf term is obtained by contracting the current with $a_\mu$ as above. Consequently, one can do away with the inversion and the accompanying gauge fixing condition at any intermediate step. The Hopf term in the $CP^1$ description thus becomes a local expression where no gauge fixing condition is required a priori.

We thus feel motivated to carry out a gauge independent Hamiltonian analysis of the $CP^1$ model coupled to the Hopf term, so that Dirac quantization of the model could eventually be carried out. In this paper however, we confine our analysis at the classical level, and look for the presence of any fractional spin. As mentioned earlier, since the Hopf term has a formal similarity to the CS term, we would also like to compare the symplectic structure of the above
model with the one where the gauge filed is given dynamics through the CS term. While undertaking this job we are confronted with the following related questions. Since the $CP^1$ manifold is a coset space ($CP^1 \sim SU(2)/U(1)$), the $CP^1$ model can be thought of as obtained by gauging the $U(1)$ subgroup of another model enjoying a global $SU(2)$ symmetry. So how does the symplectic structure (Dirac Brackets) of the latter model compare with that of the $CP^1$ model? In other words we wish to investigate the effects of gauging on symplectic structure. Next, we would also need to know in what way the symplectic structure gets affected by the presence of the Hopf or CS term.

In order to address the above issues, we begin by presenting certain mathematical preliminaries towards model building in section 2. Here we discuss following Balachandran et al, how the line elements on $S^3$ and $S^2 \sim CP^1$ can be associated with the $SU(2)$ invariant $S^3$ model and the NLSM or its equivalent $CP^1$ model respectively. We also discuss how the Hopf term arises in the $2 + 1$ dimensional context. We then carry out the Hamiltonian analysis of the $S^3$ model in section 3. In section 4 we analyze the symplectic structure of the $CP^1$ model with or without the Hopf term. In the former situation, we examine the possibility of fractional spin by the explicit construction of angular momentum through both the Noether prescription and the symmetric energy momentum tensor, firstly, for the original $CP^1$ model with the Hopf term, and later for a gauge fixed version of the Hopf term as well. In section 5 make the gauge field of the $CP^1$ Lagrangian dynamical by adding the CS term and study its effects on the symplectic structure, together with investigating whether the model retains its $CP^1$ nature or not, in analogy with its nonrelativistic counterpart [12]. Section 6 is reserved for some concluding remarks.

2 Mathematical preliminaries in model building

In this section we shall review in detail some of the mathematical properties of groups and coset spaces to be used to construct a hierarchy of models whose Hamiltonian analysis shall be carried out in the subsequent sections. For this, we shall primarily follow [13] where a general framework for the construction of nonlinear models have been provided. We will be particularly interested in the group $SU(2)$ and its coset $CP^1$. In [12] a method of projection due to Atiyah [14] was used to derive the form of the $U(1)$ connection (monopole connection) on the $CP^1$ manifold. Here we shall provide an alternative derivation of the same treating $CP^1$ as a coset space ($CP^1 \sim S^2 \sim SU(2)/U(1)$) where we shall use the techniques of differential geometry on Lie group manifolds and coset spaces [13, 15]. Within this framework we shall also provide a geometrical interpretation for the equivalence between the relativistic $CP^1$ model and the $O(3)$ nonlinear sigma model [11]. We shall also review the relevant mathematics[15] required for the construction of the Hopf term.

Let us consider a Lie group $G$ and its subgroup $H$, such that $G/H$ is a homogeneous
coset space. We further assume that $G/H$ is a symmetric space implying that the generators $T_\alpha (\hat{\alpha} = 1, \ldots, \text{dim}[G])$ of $G$ satisfying

$$[T_\alpha, T_\beta] = i f_{\alpha \beta}^\gamma T_\gamma$$

(2.1)

can be split into the generators $T_\bar{\alpha}$ of $H(\bar{\alpha} = 1, \ldots, \text{dim}[H])$ and the complements $T_\alpha (\alpha = 1, \ldots, \text{dim}([G] - [H]))$ in such a way that

$$[T_\bar{\alpha}, T_\beta] = i f_{\bar{\alpha} \beta}^\gamma T_\gamma$$

$$[T_\alpha, T_\beta] = i f_{\alpha \beta}^\gamma T_\gamma$$

$$[T_\alpha, T_\beta] = i f_{\alpha \beta}^\gamma T_\gamma$$

(2.2)

The rest of the structure constants vanish, i.e.,

$$f_{\bar{\alpha} \bar{\beta}}^\gamma = f_{\bar{\alpha} \beta}^\gamma = f_{\alpha \beta}^\gamma = 0$$

(2.3)

If $g \in G$, one can construct the following Lie algebra valued left invariant Maurer-Cartan one form

$$g^{-1}dg = ie^{\hat{\alpha}}T_\alpha = i(e_\alpha T_\alpha + e^\alpha T_\alpha)$$

(2.4)

where $e_\alpha$ is an orthonormal basis on the cotangent space over a point in the coset space $G/H$ [15], provided the generators $T_\alpha$ are properly normalized [14]. $e_\alpha$ the $H$ gauge fields on $G/H$ [13,17], and $e^{\hat{\alpha}}$ represents the orthonormal basis on the group manifold $G$.

We now apply the formalism to the $CP^1$ manifold which is a symmetric space. The $CP^1$ manifold can also be considered as a coset space $SU(2)/U(1)$. The Pauli matrices $\sigma_\alpha$'s ($\alpha = 1, 2, 3$) which are the generators of $SU(2)$ satisfy (2.2,2.3). $\sigma^3$ is the generator of the $U(1)$ subgroup. The $CP^1$ manifold is given by the set of all non-zero complex doublets $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ satisfying the normalization condition

$$Z^\dagger Z = |z_1|^2 + |z_2|^2 = 1$$

(2.5)

and with the identification $Z \sim e^{i\theta}Z$, where $e^{i\theta} \in U(1)$ is any unimodular number in the complex plane. (2.5) represents $S^3$, i.e., the $SU(2)$ group manifold. Thus one identifies $CP^1$ with the coset space $SU(2)/U(1) \sim S^2$.

Note that given such a normalized doublet $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ satisfying (2.5), one can associate the element

$$g = \begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{pmatrix} \in SU(2)$$

(2.6)
The properly normalized elements for the $SU(2)$ Lie algebra are the $\sigma_a$'s themselves as they satisfy $tr(\sigma_a\sigma_b) = 2\delta_{ab}$. Therefore using (2.4) we see that the orthonormal basis $e_a$'s for the $SU(2)$ group manifold are given by

$$g^{-1}dg = ie^a\sigma_a$$ (2.7)

One can write the line element $ds^2$ on the $SU(2)$ group manifold as

$$ds^2 = e^ae^a$$ (2.8)

which can be simplified using (2.6) and (2.7) to get

$$ds^2 = -\frac{1}{2}tr(g^{-1}dg^{-1}dg) = \frac{1}{2}tr(dg^{-1}dg) = dZ^\dagger dZ$$ (2.9)

Following [13], one can associate a nonlinear $S^3$ model to this line element (2.9),

$$\mathcal{L} = e^a e^{a\mu} \partial_\mu Z^\dagger \partial^\mu Z$$ (2.10)

enjoying global $SU(2)$ invariance and subject to the constraint (2.5).

The $U(1)$ connection one form $A$ on $CP^1$ is given by $e^3$ which can also be obtained similarly using (2.7) and (2.6) to get

$$A = e^3 = -\frac{i}{2}tr(g^{-1}dg_3) = -iZ^\dagger dZ$$ (2.11)

This form agrees with the one obtained in [12] by using the method of projection [14].

The line element on $CP^1$ is given by

$$ds^2 = dM_a dM_a$$ (2.12)

where $M_a$ is a unit 3-vector, which can be obtained from the doublet $Z$ by using the Hopf map $M_a = Z^\dagger \sigma_a Z$ The $M_a$'s also satisfy

$$g\sigma_3 g^{-1} = M_a \sigma_a$$ (2.13)

where the use of (2.6) has been made. Using (2.12), (2.7), and (2.13) one can write

$$ds^2 = \frac{1}{2}tr[d(M_a\sigma_a)d(M_b\sigma_b)] = 4(e^1)^2 + (e^2)^2$$ (2.14)

Again following [13], one can associate to this line element the model

$$\mathcal{L} = e^a e^{a\mu} \partial_\mu M_a \partial^\mu M^a$$ (2.15)

which is precisely the NLSM.

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The presence of the factor of 4 in (2.14) may be understood in the following manner. Let us parametrize the doublet $Z$ satisfying (2.5) as

$$Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} e^{i\alpha}\cos\phi \\ -e^{-i\beta}\sin\phi \end{pmatrix}$$

(2.16a)

so that the associated $SU(2)$ group element (2.6) is

$$g = \begin{pmatrix} e^{i\alpha}\cos\phi & e^{i\beta}\sin\phi \\ -e^{-i\beta}\sin\phi & e^{-i\alpha}\cos\phi \end{pmatrix} \in SU(2)$$

(2.16b)

Substituting in (2.7) one can solve for $e^a$’s to get

$$e^1 = \sin\phi\cos\phi\cos(\alpha - \beta)d\alpha + \sin\phi\cos\phi\cos(\alpha - \beta)d\beta - \sin(\alpha - \beta)d\phi$$

$$e^2 = \sin\phi\cos\phi\sin(\alpha - \beta)d\alpha + \sin\phi\cos\phi\sin(\alpha - \beta)d\beta + \cos(\alpha - \beta)d\phi$$

$$e^3 = \cos^2\phi d\alpha + \sin^2\phi d\beta$$

(2.17)

The $e^a$’s and the $M_a$’s, parametrized by the polar coordinates $\Theta$ and $\Phi$ [12], and constructed through the Hopf map, are given in the gauge $z_2^* = z_2(\beta = 0)$ by

$$e^1 = \sin\phi\cos\phi\cos(\alpha - \beta)d\alpha - \sin\phi d\phi$$

$$e^2 = \sin\phi\cos\phi\sin(\alpha - \beta)d\alpha + \cos\phi d\alpha$$

$$e^3 = \cos^2\phi d\alpha + \sin^2\phi d\beta$$

(2.18a, b, c)

$$M_1 = -\sin2\phi\cos\alpha = \sin\Theta\cos\Phi$$

$$M_2 = \sin2\phi\sin\alpha = \sin\Theta\sin\Phi$$

$$M_3 = \cos2\phi = \cos\Theta$$

(2.19a, b, c)

From (2.19) the variables $\Theta$ and $\Phi$ can be easily identified as

$$\Theta = -2\phi; \Phi = -\alpha$$

(2.20)

Substituting these back in (2.18c), one gets the connection one-form, valid for the ‘southern’ hemisphere as

$$A^{(-)} = e^3 = \frac{1}{2}(-1 - \cos\Theta)d\Phi$$

(2.21a)

Proceeding similarly for the gauge $z_1 = z_1^*(\alpha = 0)$, one gets the corresponding expression for the connection one-form, valid for the ‘northern’ hemisphere, as

$$A^{(+)} = e^3 = \frac{1}{2}(1 - \cos\Theta)d\Phi$$

(2.21b)
\( A^{(+)} \) and \( A^{(-)} \), related by the gauge transformation \((A^{(+)} - A^{(-)} = d\Phi)\), define the Dirac magnetic monopole configuration on \( \mathbb{CP}^1 \sim S^2 \).

Now it is easy to see that in the gauge \( z^*_2 = z_2 \), \( e^1 \) and \( e^2 \) satisfy

\[
(e^1)^2 + (e^2)^2 = \frac{1}{4} [\sin^2 \Theta d\Phi^2 + d\Theta^2]
\]

(2.22)

where the use of (2.20) has been made. (2.22) has an overall factor of ‘1/4’, and when substituted in (2.14) indeed produces the line element of \( S^2 \). A similar result holds for the other gauge \( z^*_1 = z_1 \) also. The lesson is that although the \( e^a \)’s in (2.8) provide an orthonormal basis on \( S^3 \), the relevant \( e^a \)’s restricted to \( S^2 \), i.e., \( e^1|_{\beta=0} \) and \( e^2|_{\beta=0} \) are not orthonormal. Rather, \( (2e^1|_{\beta=0}, 2e^2|_{\beta=0}) \) provide an orthonormal basis on \( S^2 \). The factor of ‘2’ can be traced to the first relation involving \( \Theta \) and \( \phi \) in (2.20). The line element (2.12) on \( S^2 \), written in terms of the gauge invariant \( M_a \) variables, automatically gets \( e^1 \) and \( e^2 \) projected onto the cotangent space of \( S^2 \). Finally, we note that the line element on \( S^2 \) (2.14) can be expressed alternatively using (2.8) as

\[
d\bar{s}^2 = 4[ds^2 - (e^3)^2]
\]

(2.23)

Further, using (2.5) and (2.11), one can rewrite (2.23) as

\[
\frac{1}{4} dM_a dM_a = [dZ^\dagger dZ - (A)^2] = (dZ - iAZ)^\dagger (dZ - iAZ)
\]

(2.24)

Now, this implies that we have the following identity

\[
\frac{1}{4} \partial_\mu M_a \partial^\mu M_a = (D_\mu Z)^\dagger (D^\mu Z)
\]

(2.25a)

where \( D_\mu = (\partial_\mu - iA_\mu) \) stands for the covariant derivative operator, and

\[
A_\mu = -iZ^\dagger \partial_\mu Z
\]

(2.25b)

is the \( U(1) \) gauge field obtained by pulling back the connection (2.11) on \( \mathcal{M} \). The right hand side of (2.25a) corresponds to the \( \mathbb{CP}^1 \) model having a local \( U(1) \) invariance. From (2.25) it is clear that Lagrangian densities of the \( \mathbb{CP}^1 \) model and the NLSM (2.15) are the same. Note that both the NLSM and the \( \mathbb{CP}^1 \) Lagrangian (upto a factor) have been obtained from the same line element (2.12) or its equivalent (2.23) using the prescription of [13]. Thus these two models are classically equivalent [11]. Note that we do not have any dynamical term for the gauge field in the \( \mathbb{CP}^1 \) model.

We have thus constructed physical models associated with the line elements of \( S^3 \) and its coset \( S^2 \). Let us now address the issue of the existence of solitonic configurations [13, 16] for the models constructed above. For the case of the \( O(3) \) NLSM, it is necessary for the fields
$M_a$ to tend to a constant configuration asymptotically for finite energy static solutions to exist. With this requirement, the two-dimensional plane $D$ gets effectively compactified to $S^2$, so that the configuration space $C$ is the set of all maps $\{f\}$:

$$f : S^2 \rightarrow S^2 (field \ manifold)$$

(2.26)

Clearly, the splits into a disjoint union of path connected spaces [16] as

$$\Pi_0(C) = \Pi_2(S^2) = Z$$

(2.27)

Hence, there exist solitons or skyrmions in this model, characterized by the set of integers $N \in Z$ given by

$$N = \int d^2 x j^0(x)$$

(2.28)

where $j^0$ is the time component of the identically conserved current ($\partial_\mu j^\mu = 0$) given by

$$j^\mu = \frac{1}{8\pi} \epsilon^{\mu\nu\lambda} \epsilon_{abc} M_a \partial_\nu M_b \partial_\lambda M_c$$

(2.29)

Note that the conservation of $j^\mu$ holds irrespective of any equation of motion. $N$, referred to as the soliton number, labels disconnected pieces of the configuration space $C$. Parametrizing the unit vector $M_a$ by polar coordinates ($\Theta, \Phi$) as in (2.19), $j^\mu$ can be expressed as the curl of the gauge fields $A_\mu$ as

$$j^\mu = \frac{1}{2\pi} \epsilon^\mu_{\nu\lambda} \partial_\nu A_\lambda$$

(2.30)

where the use of (2.21) has been made.

In any soliton number sector, the fundamental group of the configuration space $C$ is non-trivial [16] since

$$\Pi_1(C) = \Pi_3(S^2) = Z$$

(2.31)

This implies that loops based at any point in the configuration space fall into separate homotopy classes labelled by another integer $H$. This integer can be given a representation by the so called Hopf term [3]

$$H = \int j^\mu A_\mu d^3x$$

(2.32)

This term can be added to the NLSM to impart fractional spin and statistics to the solitons [8,9] - a possibility arising out of (2.31). Note that in adding the Hopf term (2.32), we do not enlarge the configuration space. This is because the $A_\mu$ field is not treated as an independent variable in the configuration space; rather is determined in terms of $j_\mu$ by inverting (2.30). This is the reason why the Hopf term provides a nonlocal current-current interaction in general. However, in the $CP^1$ version the Hopf term is local since $A_\mu$ (2.36b), and hence $j_\mu$ (2.30) are given in
terms of local expressions of the $Z$ fields. Making use of (2.25b) and (2.30) in (2.32), one gets the following expression for the Hopf term in the hermitian form

$$H = -\frac{1}{4\pi} \int d^3x \epsilon^{\mu\nu\lambda} Z^\dagger \overset{\leftrightarrow}{\partial}_\mu Z \partial_{\nu} Z^\dagger \partial_{\lambda} Z$$

(2.33)

This Hopf term can be expressed alternatively in terms of the Cartan-Maurer form (2.4) \(\sim \int_M \text{tr}(g^{-1}dg)^3\) as has been shown in [13].

In the following two sections we shall consider the $S^3$ model and the $CP^1$ model extended by the Hopf term respectively, and carry out their Hamiltonian analysis. Apart from studying the similarities in their symplectic structure, our aim is to look for any fractional spin generated by the Hopf term.

3 Hamiltonian analysis of the $S^3$ model

The $S^3$ model is given by the Lagrange density

$$\mathcal{L} = \partial_\mu Z^\dagger \partial^\mu Z - \lambda (Z^\dagger Z - 1)$$

(3.1)

This is obtained from (2.10) by incorporating the constraint \((Z^\dagger Z = 1)\) (2.5) by the Lagrange multiplier $\lambda$. The model posses a global $SU(2)$ invariance but no local symmetry. The Legendre transformed Hamiltonian $\mathcal{H}_c$ is given by

$$\mathcal{H}_c = \pi_\alpha \pi_\alpha + \partial_i Z^\dagger \partial_i Z + \lambda (Z^\dagger Z - 1)$$

(3.2)

The canonically conjugate momenta of the configuration space variables $z_\alpha$, $z_\alpha^*$ and $\lambda$ are given by

$$\pi_\alpha = \frac{\delta \mathcal{L}}{\delta \dot{z}_\alpha} = \dot{z}_\alpha^*$$

(3.3a)

$$\pi_\alpha^* = \frac{\delta \mathcal{L}}{\delta \dot{z}_\alpha^*} = \dot{z}_\alpha$$

(3.3b)

$$\pi_\lambda = \frac{\delta \mathcal{L}}{\delta \dot{\lambda}} = 0$$

(3.3c)

respectively. Thus the only primary constraint of the model given by

$$\pi_\lambda \approx 0$$

(3.4)

Time preserving (3.4) with respect to the Hamiltonian (3.2) we get the secondary constraint

$$C_1(x) = Z^\dagger Z - 1 \approx 0$$

(3.5)
Repeating the above procedure, we obtain the following tertiary constraint

\[ C_2(x) = \pi^*_\alpha z^*_\alpha + \pi_\alpha z_\alpha \approx 0 \]  \hspace{1cm} (3.6)

It is apparent that the constraints (3.5) and (3.6) are second class as the Poisson Bracket (PB)

\[ C_{ij}(x, y) = \{C_i(x), C_j(y)\} = 2\epsilon_{ij}\delta(x - y) \]  \hspace{1cm} (3.7a)
does not vanish. The inverse of this matrix is given by

\[ (C^{-1})_{ij}(x, y) = -\frac{1}{2}\epsilon_{ij}\delta(x - y) \]  \hspace{1cm} (3.7b)

Clearly, there exist no more constraints. The second class constraints can now be ‘strongly’ implemented by using the appropriate Dirac Brackets (DB). The DB between two quantities \(A(x)\) and \(B(y)\) is given by [10]

\[ \{A(x), B(y)\} = \{A(x), B(y)\}_{PB} - \int dudv \{A(x), C_i(u)\}(C^{-1})_{ij}(u, v)\{C_j(v), B(y)\} \]  \hspace{1cm} (3.8)

Using (3.7b) and (3.8) one finds the following DB’s among the phase space variables

\[ \{z_\alpha(x), \pi_\beta(y)\} = (\delta_{\alpha\beta} - \frac{1}{2}z^*_\beta z_\alpha)\delta(x - y) \]

\[ \{\pi_\alpha(x), \pi_\beta(y)\} = \frac{1}{2}(\pi_\alpha z^*_\beta - z^*_\alpha \pi_\beta)\delta(x - y) \]

\[ \{z_\alpha(x), z_\beta(y)\} = \{z^*_\alpha(x), z_\beta(y)\} = 0 \]

\[ \{z_\alpha(x), \pi^*_\beta(y)\} = -\frac{1}{2}z_\alpha(x)z_\beta(x)\delta(x - y) \]

\[ \{\pi_\alpha(x), \pi^*_\beta(y)\} = \frac{1}{2}(\pi_\alpha z_\beta - \pi^*_\beta z^*_\alpha)\delta(x - y) \]  \hspace{1cm} (3.9)

This provides us with the symplectic structure of this model.

With these DB’s the total Hamiltonian can now be written as

\[ \mathcal{H} = \pi^i \pi + \partial_i Z^i \partial_i Z + u\pi_\lambda \]  \hspace{1cm} (3.10)

where \(u\) is an arbitrary Lagrange multiplier for the constraint (3.4). This constraint is the only first class constraint of the model (3.1). It gives vanishing brackets with all the phase space variables of the model except the Lagrange multiplier \(\lambda\). The time evolution of \(\lambda\) is therefore given by

\[ \dot{\lambda} = \int d^2x \{\lambda, \mathcal{H}\} = u \]  \hspace{1cm} (3.11)
which is again arbitrary. Without loss of generality, one can therefore put

\[ \lambda = 0 \]  \hspace{1cm} (3.12)

Clearly, (3.4) along with (3.12) form a new second class pair which is strongly implemented by an additional DB

\[ \{ \lambda, \pi_\lambda \} = 0 \]  \hspace{1cm} (3.13)

So finally the Hamiltonian reduces to

\[ \mathcal{H} = \pi^\dagger \pi + \partial_i Z^\dagger \partial_i Z \]  \hspace{1cm} (3.14)

Note that the model has no non-trivial first class constraint, and hence, no gauge symmetry. This is expected, since the Lagrangian (3.1) has only a global SU(2) symmetry.

\section{The \( CP^1 \) model with or without the Hopf term}

In this section we perform a Hamiltonian analysis of the \( CP^1 \) model coupled to the Hopf term, the essential features of which have been outlined in [18]. In addition to working out the complete constraint algebra, here we also wish to comment on some technical as well as physical subtleties in the calculation. We also present the corresponding analysis for the pure \( CP^1 \) case. Our aim is to highlight any difference in the symplectic structure made by the presence of the Hopf term, and also to investigate the existence of fractional spin in the model. Towards the end of this section we consider the example of a radiation gauge-fixed Hopf-\( CP^1 \) model where fractional spin is revealed by the construction of the angular momentum operator.

\subsection{Constraint analysis of the model}

The model is given by the Lagrangian

\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_H - \lambda (Z^\dagger Z - 1) \]  \hspace{1cm} (4.1a)

where

\[ \mathcal{L}_0 = (D_\mu Z)^\dagger (D^\mu Z) \]  \hspace{1cm} (4.1b)

corresponds to the right hand side of (2.25a) and

\[ \mathcal{L}_H = \theta \epsilon^{\mu \nu \lambda} \left[ Z^\dagger \partial_\mu Z \partial_\nu Z^\dagger \partial_\lambda Z + \partial_\mu Z^\dagger Z \partial_\nu Z^\dagger \partial_\lambda Z \right] \]  \hspace{1cm} (4.1c)
is the Hopf term corresponding to (2.33) with strength \( \theta \). This Hopf term was coupled to the nonrelativistic version of the \( CP^1 \) model in [19] where it was shown to alter the spin algebra. The configuration space variables are \( z_\alpha, z_\alpha^*, A_i, A_0, \) and \( \lambda \). The corresponding momenta are

\[
\pi_\alpha = \frac{\delta \mathcal{L}}{\delta \dot{z}_\alpha} = (D_0 z_\alpha)^* + \frac{\delta \mathcal{L}_H}{\delta \dot{z}_\alpha} = (D_0 z_\alpha)^* + \theta \epsilon^{ij} \left[ \partial_i Z^\dagger \partial_j Z z_\alpha^* + Z^\dagger \partial_j Z \partial_j z_\alpha - \partial_i Z^\dagger Z \partial_j z_\alpha^* \right]
\]

(4.2)

\[
\pi_\alpha^* = \frac{\delta \mathcal{L}}{\delta \dot{z}_\alpha^*} = (D_0 z_\alpha) + \frac{\delta \mathcal{L}_H}{\delta \dot{z}_\alpha^*} = (D_0 z_\alpha) + \theta \epsilon^{ij} \left[ -Z^\dagger \partial_j Z \partial_j z_\alpha + \partial_j Z^\dagger Z \partial_j z_\alpha + \partial_i Z^\dagger Z \partial_j z_\alpha \right]
\]

(4.3)

\[
\pi_i = \frac{\delta \mathcal{L}}{\delta A^i} = 0
\]

(4.4)

\[
\pi_0 = \frac{\delta \mathcal{L}}{\delta A^0} = \pi_0
\]

(4.5)

\[
\pi_\lambda = \frac{\delta \mathcal{L}}{\delta \lambda} = 0
\]

(4.6)

The Eqs.(4.4-4.6) represent the primary constraints of this model. Note that \( \mathcal{L}_H \) contains terms which are first order in the time derivative involving either \( \dot{z}_\alpha \) or \( \dot{z}_\alpha^* \). One can thus write

\[
\frac{\delta \mathcal{L}_H}{\delta \dot{z}_\alpha} \dot{z}_\alpha + \frac{\delta \mathcal{L}_H}{\delta \dot{z}_\alpha^*} \dot{z}_\alpha^* = \mathcal{L}_H
\]

(4.7)

Using this the Legendre transformed Hamiltonian can be obtained as

\[
\mathcal{H}_c = \pi_\alpha^* \dot{z}_\alpha + \pi_\alpha \dot{z}_\alpha - \mathcal{L} = (D_0 Z)^\dagger \dot{Z} + \dot{Z}^\dagger (D_0 Z) - (D_\mu Z)^\dagger (D^\mu Z) + \lambda (Z^\dagger Z - 1)
\]

(4.8)

Writing (4.8) in terms of the phase space variables, one gets

\[
\mathcal{H}_c = \pi_\alpha^* \pi_\alpha - i A_0 \left( \pi_\alpha^* \dot{z}_\alpha - \pi_\alpha \dot{z}_\alpha^* + z_\alpha \frac{\delta \mathcal{L}_H}{\delta \dot{z}_\alpha} - z_\alpha^* \frac{\delta \mathcal{L}_H}{\delta \dot{z}_\alpha^*} \right) - \left( \frac{\delta \mathcal{L}_H}{\delta \dot{z}_\alpha} + \frac{\delta \mathcal{L}_H}{\delta \dot{z}_\alpha^*} \right) + \frac{\delta \mathcal{L}_H}{\delta \dot{z}_\alpha} \frac{\delta \mathcal{L}_H}{\delta \dot{z}_\alpha^*} + |D_i Z|^2 + \lambda (Z^\dagger Z - 1)
\]

(4.9)

Preservation of the primary constraints (4.4-4.6) in time yield the following set of secondary constraints

\[
A_i + \frac{i}{2 (Z^\dagger Z)} Z^\dagger \partial_i Z \approx 0
\]

(4.10)

\[
i \left( \pi_\alpha^* \dot{z}_\alpha - \pi_\alpha \dot{z}_\alpha^* + z_\alpha \frac{\delta \mathcal{L}_H}{\delta \dot{z}_\alpha} - z_\alpha^* \frac{\delta \mathcal{L}_H}{\delta \dot{z}_\alpha^*} \right) \approx 0
\]

(4.11)
respectively. From the constraint (4.12), a new tertiary constraint
\[ \pi_\alpha^* z_\alpha^* + \pi_\alpha z_\alpha \approx 0 \] (4.13)
is obtained. The constraint (4.11) can be simplified further using (4.13) to yield
\[ i(\pi_\alpha^* z_\alpha^* - \pi_\alpha z_\alpha + 2\theta \epsilon^{ij} \partial_i Z^\dagger \partial_j Z) \approx 0 \] (4.14)
Finally, by demanding the preservation of (4.14) in time, one more constraint
\[ \pi_\alpha^* \pi_\alpha + (D_i D_i Z)^\dagger Z - \lambda + \theta - \text{dependent terms} \approx 0 \] (4.15)
is obtained, where the last \( \theta \)-dependent terms are independent of \( \lambda \). It can be checked that there exist no further constraints.

At this stage it is necessary to classify the total set of constraints (4.4-4.6, 4.10, 4.12-4.15) into first class and second class constraints. A comparison with the \( S^3 \) model discussed in the previous section shows that the constraints (4.6, 4.12 and 4.13) are similar to the constraints (3.4, 3.5 and 3.6) respectively. In addition here we have the constraints (4.4, 4.5 and 4.14). The constraint (4.15) corresponds to the condition \( \lambda = 0 \) (3.12) of the previous section. The appearance of (4.4 and 4.5) is in keeping with the fact that we have introduced a background gauge field \( A_\mu \) to gauge the \( U(1) \) subgroup of the global \( SU(2) \) group of the \( S^3 \) model. The other nontrivial constraint (4.14) is therefore expected to be the Gauss constraint. Besides, (4.14) is obtained by preserving the constraint (4.5) in time, just as in Maxwell electrodynamics [10]. It can be easily seen that (4.4, 4.10), (4.6,4.15), and (4.12,4.13) form pairs of second class constraints. Only the constraint (4.14) is first class, leaving apart the trivial constraint (4.5). The first two pairs are ‘strongly’ implemented by the (DB)
\[ \{A_i(x), \pi^j(y)\} = 0 \] (4.16)
\[ \{\lambda(x), \pi_\lambda(y)\} = 0 \] (4.17)
These are the additional DB’s that we get in this model other than those in (3.9) obtained from the last pair (4.12, 4.13).

The constraint (4.14) when simplified further using (4.13) gives
\[ G(x) \equiv i(2\pi_\alpha(x) z_\alpha(x) - 2\theta \epsilon^{ij} \partial_i Z^\dagger(x) \partial_j Z(x)) \approx 0 \] (4.18)
which can be shown using (3.9) to generate a \( U(1) \) gauge transformation
\[ \delta z_\alpha(x) = \int d^2 y f(y) \{z_\alpha(x), G(y)\} = i f(x) z_\alpha(x) \] (4.19)
Therefore (4.18) can be identified with the Gauss constraint. The transformation property of
the momenta variables $\pi_\alpha$ can be obtained either through the DB (3.9), or else by using
the basic transformation properties of the fundamental fields $z_\alpha$ (4.19) to get

$$\delta\pi_\alpha(x) = \int d^2y f(y)\{\pi_\alpha(x), G(y)\} = i[f(x)\pi_\alpha(x) - 2\theta\epsilon_{ij}\partial_i f(D_jz_\alpha^*)] \quad (4.20)$$

Note that the Gauss constraint (4.18) was absent in the $S^3$ model where the $\lambda = 0$
condition was put in by hand. In contrast, here the constraint (4.15) can be obtained by preserving
the Gauss constraint in time. The exact form of $\lambda$ is inconsequential for the DB (4.17) and the
Hamiltonian since $\lambda$ being a Lagrangian multiplier, enforces the ‘strongly’ valid second class
constraint (4.12).

The final form of the total Hamiltonian is given by

$$\mathcal{H} = \pi_\alpha^*\pi_\alpha - \left(\pi_\alpha \frac{\delta L_H}{\delta z_\alpha^*} + \pi_\alpha^* \frac{\delta L_H}{\delta z_\alpha}\right) + \frac{\delta L_H}{\delta z_\alpha^*} \frac{\delta L_H}{\delta z_\alpha} + |D_1Z|^2 - A_0G \quad (4.21)$$

Note that since (4.10) and (4.12) are second class constraints, and therefore are ‘strongly’ valid,
one can simplify (4.10) to write

$$A_i = -iZ^\dagger \partial_i Z \quad (4.22)$$

$A_i$ therefore ceases to be an independent degree of freedom. Note also, that the Gauss constraint
(4.18) can be expressed alternatively using (4.2) and (4.3) as

$$Z^\dagger(D_0Z) - (D_0Z)^\dagger Z \approx 0 \quad (4.23)$$

which can be solved for $A_0$ to yield

$$A_0 = -iZ^\dagger \partial_0 Z \quad (4.24)$$

However, (4.22) is not a constraint equation as it involves a time derivative, unlike (4.24). Nevertheless, it could sometimes be useful to write (4.22 and 4.24) compactly as

$$A_\mu = -iZ^\dagger \partial_\mu Z \quad (4.25)$$

which is just the expression (2.25b) obtained by the geometrical approach discussed in section
2. The consistency of the whole approach is apparent since the $CP^1 \simeq S^2$ manifold admits a
unique $U(1)$ connection up to a gauge transformation. This follows from the fact that the first
derRham cohomology group vanishes for the $CP^1$ manifold $\left(H^1(CP^1) = 0\right)$ [15].

After having studied the $CP^1$ model coupled to the Hopf term, let us now consider the case
when the Hopf term does not exist, i.e., if $\theta = 0$ in (4.1a). The Lagrangian for the pure $CP^1$
model is given by

$$\mathcal{L} = (D_\mu Z)^\dagger(D^\mu Z) - \lambda(Z^\dagger Z - 1) \quad (4.26)$$
In this case the symplectic structure can easily be shown to remain essentially the same as in the case with the presence of the Hopf term. However, the canonically conjugate momenta variables (i.e., the counterparts of (4.2 and 4.3)) are different, and are given by

\[ \pi_\alpha = (D_0 Z)_\alpha^* \]  
\[ \pi_\alpha^* = (D_0 Z)_\alpha \]  

The other momenta variables remain the same as (4.4 - 6). The secondary and the tertiary constraints following from the Legendre transformed Hamiltonian

\[ H_c = \pi_\alpha^* \pi_\alpha - i A_0 (\pi_\alpha^* z_\alpha^* - \pi_\alpha z_\alpha) + |D_i Z|^2 + \lambda (Z^\dagger Z - 1) \]  

which do not undergo any change are given by (4.10, 4.12 and 4.13). However, the Gauss constraint (i.e., the counterpart of (4.14 or 4.18) now becomes

\[ G(x) \equiv i \left( \pi_\alpha^* (x) z_\alpha^* (x) - \pi_\alpha (x) z_\alpha (x) \right) = 2i \pi_\alpha (x) z_\alpha (x) \approx 0 \]  

The preservation of the Gauss constraint in time yields

\[ \pi_\alpha^* \pi_\alpha + (D_i D_i Z)^\dagger Z - \lambda \approx 0 \]  

The final symplectic structure is given by the DB’s (3.9, 4.16 and 4.17) and therefore undergoes no change. The total Hamiltonian obtained from (4.29, and 4.30) is given by

\[ H = \pi_\alpha^* \pi_\alpha + |D_i Z|^2 - A_0 G \]  

Finally, the expression for the gauge field \( A_\mu \) (4.25) in terms of the matter fields \( Z \) holds together with the ‘strongly’ valid relation (4.22) for the spatial components. Thus it is clearly evident that the presence or absence of the Hopf term has no effect on the symplectic structure.

4.2 Angular momentum in the \( CP^1 \) model

Here we shall construct various spacetime symmetry generators of the model (4.1) obtained from both the Noether’s prescription, and the symmetric expression of the energy momentum (EM) tensor. The latter can be obtained by functionally differentiating the action with respect to the spacetime metric. We shall focus particularly on the angular momentum since our goal is to look for any fractional spin generated by the presence of the Hopf term. At this stage it needs to be mentioned that fractional spin was found to be induced by the Hopf term in the equivalent nonlinear sigma model (NLSM) by Bowick et al [9]. However, it is important to note that in [9] a gauge had to be fixed right at the beginning in order to express the gauge field \( A_\mu \).
in terms of the current $j_\mu$ (see (2.43)). On the other hand, in the $CP^1$ version, no gauge fixing is necessary, and one can perform a gauge independent Hamiltonian analysis.

The symmetric expression for the energy momentum (EM) tensor is given by

$$T^s_{\mu\nu} = (D_\mu Z)^\dagger(D_\nu Z) + (D_\nu Z)^\dagger(D_\mu Z) - g_{\mu\nu}(D_\rho Z)^\dagger(D^\rho Z)$$  \hspace{1cm} (4.33)

Note that the $\theta$- dependent term $L_H$ does not contribute to this expression since it is a topological term which is independent of the metric. It follows that the symmetric expression for the Hamiltonian is given by

$$H^s = \int d^2x[2(D_0 Z)^\dagger(D_0 Z) - (D_\mu Z)^\dagger(D^\mu Z)]$$  \hspace{1cm} (4.34)

which can be rewritten using (4.8 and 4.12) as

$$H^s = \int d^2x\left(\mathcal{H}_c - iA_0[(D_0 Z)^\dagger Z - Z^\dagger(D_0 Z)]\right)$$  \hspace{1cm} (4.35)

It can be checked that (4.35) generates the appropriate time translation. The last expression reduces to

$$H^s \approx \int d^2x\mathcal{H}_c$$  \hspace{1cm} (4.36)

on the Gauss constraint surface.

Let us now consider the momentum generator obtained from (4.33) given by

$$P^s_i = \int d^2xT^s_{0i} = \int d^2x[(D_0 Z)^\dagger(D_i Z) + (D_i Z)^\dagger(D_0 Z)]$$  \hspace{1cm} (4.37)

This expression can be simplified further to get

$$P^s_i = P^N_i + 2i\theta\epsilon^{ijk}\int d^2x[A_j\partial_i Z^\dagger\partial_j Z - A_i\partial_j Z^\dagger\partial_j Z - A_j\partial_k Z^\dagger\partial_i Z] - \int d^2xA_i(x)G(x)$$  \hspace{1cm} (4.38)

where

$$P^N_i = \int d^2xP^N_{0i} = \int d^2x[\pi_\alpha\partial_i z_\alpha + \pi^*_\alpha\partial_i z^*_\alpha]$$  \hspace{1cm} (4.39)

is the expression of momentum obtained from the Noether theorem. The presence of the second $\theta$ -dependent term is a reflection of the fact that the canonically conjugate momentum variables $\pi_\alpha$ (4.2) and $\pi^*_\alpha$ (4.3) get a $\theta$ -dependent contribution from the Hopf term, over the corresponding variables (4.27) and (4.28) in the pure $CP^1$ case. Now using the fact that in two spatial dimensions one can write $\partial_i A_j - \partial_j A_i = \epsilon_{ij}B$ ($B$ being the magnetic field), it can be shown that the $\theta$—dependent term in (4.38) vanishes. However because of the presence of the last term involving the Gauss constraint $G(x)$ in (4.38), $P^s_i$ fails to generate appropriate translations, i.e.,

$$\{z_\alpha(x), P^s_i\} = D_i z_\alpha$$  \hspace{1cm} (4.40)
in contrast to $P^N$ (4.39) which by construction generates the appropriate translation

$$\{z_\alpha(x), P^N_i\} = \partial_i z_\alpha$$

(4.41)

However, note that one has the liberty to modify the EM tensor (4.33) by an appropriate linear combination of first class constraint(s) (here only (4.18)) with tensor valued coefficients $u_{\mu\nu}$

$$\tilde{T}^s_{\mu\nu} = T^s_{\mu\nu} + u_{\mu\nu}G$$

(4.42)

By looking at the expression (4.38) it is clear that with the choice

$$u_{0i} = A_i$$

(4.43)

one gets

$$\tilde{T}^s_{0i} = T^N_{0i}$$

(4.44)

and correspondingly

$$\tilde{P}^s_i \equiv \int d^2x \tilde{T}^s_{0i} = P^N_i$$

(4.45)

which generates the appropriate translation $\{z_\alpha, \tilde{P}^s_i\} = \partial_i z_\alpha$ just as (4.41), and can therefore be identified as momentum. Note that this way of obtaining the modified expression $\tilde{T}^s_{0i}$ from $T^s_{0i}$ is tantamount to simplifying $T^s_{0i}$ on the Gauss constraint surface.

Nextly, one can write down the two expressions of angular momentum as

$$J^s = \int d^2x \epsilon_{mj} x_m \tilde{T}^s_{0j}$$

$$J^N = \int d^2x \epsilon_{mj} x_m T^N_{0j}$$

(4.46)

Note that the Noether expression $J^N$ corresponds only to the orbital angular momentum since we are dealing with scalar fields. Apart from the fact that they generate appropriate rotation, one can easily see in view of (4.44) that these two expressions match exactly. Thus

$$J^s = J^N$$

(4.47)

Usually, $J^s$ is regarded as the physical angular momentum as it is obtained from the symmetric expression of the EM tensor which is gauge invariant by construction. On the other hand $J^N$ is usually found to be gauge invariant only on the Gauss constraint surface, and that too only under those gauge transformations that reduce to identity at infinity [7]. In various models involving the Chern-Simons term [4-6], as well as the NLSM model coupled to the Hopf term [9], fractional spin had been revealed by essentially computing the difference between $J^s$ and $J^N$. Since, in the present case $J^s$ matches exactly with the orbital angular momentum $J^N$, we conclude that the system (4.1) does not exhibit the existence of fractional spin in spite of
the presence of the Hopf term. This should not be very surprising considering the fact that the Hopf term is a total divergence. However, since this result is purely classical, one cannot rule out the emergence of fractional spin at the quantum level if Dirac quantization of the model is carried out.

### 4.3 Fractional spin in a radiation-gauge-fixed Hopf-$CP^1$ model

The result of no fractional spin obtained in the last subsection is in sharp contradiction to the scenario of NLSM [9] where the expression of angular momentum is modified by the presence of an extra part corresponding to fractional spin emanating from the Hopf term. The result of fractional spin in [9] is not a typical quantum effect since it can be obtained in a classical analysis itself which can then be extended to the quantum level. Wilczek and Zee [8] had also considered the same model, but instead of a Hamiltonian analysis, they considered a slow adiabatic rotation of the system by an angle of $2\pi$. They found the wave function to acquire an additional phase on this rotation which provided the fractional spin for the system. It needs to be stressed that the latter way of obtaining fractional spin is a purely quantum effect. In this paper we have carried out our analysis in the Dirac Hamiltonian framework, as in [9]. In fact, ours is a $CP^1$ version of [9]. We have confined our analysis to the classical level since transition to the quantum level by elevating the field dependent DB’s (3.9) to quantum commutators is problematic [10] because of operator ordering ambiguities. It is therefore unexpected to disagree with [9].

In what follows we shall show that the Lagrangian considered in [9] is basically inequivalent to the one (4.1) used by us. In [8,9] the Lagrangian is

$$\mathcal{L} = \frac{1}{4}(\partial_\mu M_\alpha)(\partial^\mu M_\alpha) - \theta j_\mu A_\mu - \lambda(M_\alpha M_\alpha - 1)$$  \hspace{1cm} (4.48)

which is the NLSM coupled to the Hopf term (2.32). At this stage the identity

$$\int d^2x A_0(x) j_0(x) = -\int d^2x A_i(x) j_i(x)$$  \hspace{1cm} (4.49)

which is valid in the radiation gauge, is used inside the action [9] to reduce the Hopf term in the Lagrangian (4.48) to get

$$\mathcal{L} = \frac{1}{4}(\partial_\mu M_\alpha)^2 + 2\theta j_i(x) A_i(x) - \lambda(M_\alpha M_\alpha - 1)$$  \hspace{1cm} (4.50)

It needs to be noted here that the derivation of the identity (4.46) requires the inversion of (2.30) to express the gauge field $A_\mu$ in terms of the current $j_\mu$ using the radiation gauge condition. However, the spatial components of (2.30) and (4.24) by virtue of being relations involving time derivatives (not constraint equations) when expressed in terms of the matter fields
$M_a$, are likely to lead to discrepancies in the dynamical structure of any model if substituted into the original Lagrangian. Besides, in this case the $j_i A_i$ term is no longer a total divergence unlike the pure Hopf term ($\sim j_{\mu} A_{\mu}$) considered in the last section. Hence, it is improper to hold the original Lagrangian (4.48) consequent for any result which is obtained after the substitution of (4.49) into it. In order to clarify this point, let us consider the gauge variant $CP^1$ version of the Lagrangian (4.50) given by

$$L = |D_\mu Z|^2 + \frac{\theta}{\pi} \epsilon^{\mu\nu\lambda} \partial_\nu Z^\dagger \partial_\lambda Z Z^\dagger \partial_\lambda Z - \lambda (Z^\dagger Z - 1) \quad (4.51)$$

where the use of (2.25) and (2.30) has been made. (4.51) can also be obtained by making use of the identity (4.49) which in terms of the $Z$ fields looks as

$$\int d^2 x Z^\dagger \nabla \cdot Z^\dagger = \int d^2 x Z^\dagger \nabla \times Z = \int d^2 x Z^\dagger \nabla \times [\nabla \cdot Z^\dagger \dot{Z} - \dot{Z}^\dagger \nabla \cdot Z] \quad (4.52)$$

and is clearly an identity involving time derivatives but not a constraint equation.

Note that (4.51) has certain similarities with (4.1), and can be expressed in the form of (4.1a) with $L_0$ given by (4.1b). $L_H$ is now simplified in the radiation gauge to

$$L_H = \frac{\theta}{\pi} \epsilon^{\mu\nu\lambda} \partial_\nu Z^\dagger \partial_\lambda Z Z^\dagger \partial_\lambda Z \quad (4.53)$$

Nevertheless, (4.53) is first order in time derivative, so that (4.7) is still valid. The canonically conjugate momenta corresponding to $z_\alpha$ and $z^{\star}_\alpha$ are now changed to

$$\pi_\alpha = (D_0 z_\alpha)^* + \frac{\theta}{\pi} \epsilon^{ij} Z^\dagger \partial_i z_j z^{\star}_\alpha \quad (4.54)$$

$$\pi^{\star}_\alpha = (D_0 z_\alpha) - \frac{\theta}{\pi} \epsilon^{ij} Z^\dagger \partial_i z_j z_\alpha \quad (4.55)$$

The rest of the momenta given in (4.4-4.6) remain the same.

It can be checked through the Hamiltonian analysis that the set of constraints and the symplectic structure given by the DB’s (3.9, 4.16 and 4.17) remain the same as that of the model (4.1), except for the Gauss constraint which is now changed to that of the pure $CP^1$ model (4.30). This indicates that the Gauss constraint affects the $U(1)$ gauge transformation on the $Z$ fields as before (4.19). The Lagrangian (4.51) being gauge variant, does not possess this symmetry. But this is not a serious problem, as (3.9),(4.16) and (4.17) do not represent the final symplectic structure of the model. They will undergo further modification when we implement strongly the Gauss constraint along with the radiation gauge condition. With that the model actually ceases to be a gauge theory as we are left with no first class constraint. The exact form of this final set of DB is however not needed for our purpose, as our interest is the angular momentum operator.
The symmetric expression for the EM tensor (4.33) for the model (4.1) undergoes no change in this case since $L_H(4.53)$ is still metric independent. The expression for linear momentum is given by (4.37), and thus the angular momentum is

$$J^s = \int d^2x \epsilon_{ij} x_i [(D_0 Z)^\dagger (D_j Z) + (D_j Z)^\dagger (D_0 Z)]$$  \hspace{1cm} (4.56)$$

In terms of the phase space variables, and after some simplification using the strongly valid Gauss constraint (4.30), $J^s$ reduces to

$$J^s = J^N + \frac{i\theta}{\pi} \epsilon^{pq} \int d^2x \epsilon_{ij} x_i A_p [(D_j Z)^\dagger \partial_q Z - \partial_q Z^\dagger (D_j Z)]$$  \hspace{1cm} (4.57)$$

where $J^N$ is the Noether expression of angular momentum. It can be easily checked that both $J^s$ and $J^N$ generate appropriate spatial rotation with respect to the DBs (3.9,4.16 and 4.17). Clearly they will continue to do so with respect to the modified brackets as well, as the radiation gauge condition preserves rotational symmetry[5].

The $\theta$- dependent term in (4.57) can be expressed in terms of the topological charge density $j_0$ (2.30) by using the ‘strong’ constraint (4.12), as

$$J^s = J^N + \theta N^2$$  \hspace{1cm} (4.61)$$

where $N$ is the soliton number and is given by (2.28). Since $J^N$ is just the orbital angular momentum, one concludes that the model described by (4.51) exhibits fractional spin. Let us emphasize, once again, that the model (4.51) is basically inequivalent to the model (4.1) which does not display any fractional spin. Nevertheless, it is possible to construct a variant of the model (4.48) by making use of the identity (4.49), which exhibits fractional spin as was shown in [9]. In this way the possibility of having fractional spin in $2 + 1$ dimensions as discussed in section 2, is realized.
5 Introducing the Chern-Simons term

In this section we shall investigate the modifications of the symplectic structure, if any, due to the addition of dynamical terms for the gauge fields in the form of a Chern-Simons (CS) term and compare the analysis with the corresponding nonrelativistic case \[12\]. In \[12\] the Gauss constraint was found to be modified in such a manner that the model ceases to be a $CP^1$ model. In this light, it would be interesting to study the effect of the CS term on the relativistic model, as well.

The model is

$$\mathcal{L} = (D_\mu Z)^\dagger (D^\mu Z) + \theta e^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - \lambda (Z^\dagger Z - 1) \quad (5.1)$$

where $D_\mu = \partial_\mu - i A_\mu$ is the covariant derivative operator. The canonically conjugate momenta variables are given by

$$\pi_\alpha = \frac{\delta \mathcal{L}}{\delta \dot{z}_\alpha} = (D_0 z)_\alpha^* \quad (5.2)$$

$$\pi_\alpha^* = \frac{\delta \mathcal{L}}{\delta \dot{z}^*_\alpha} = (D_0 z)_\alpha \quad (5.3)$$

$$\pi^i = \frac{\delta \mathcal{L}}{\delta A_i} = -\theta e^{ij} A_j \quad (5.4)$$

$$\pi^0 = \frac{\delta \mathcal{L}}{\delta A_0} = 0 \quad (5.5)$$

$$\pi_\lambda = \frac{\delta \mathcal{L}}{\delta \lambda} = 0 \quad (5.6)$$

Note that except for (5.4), all the other relations (5.2-5.6) appear in case of the pure $CP^1$ model (4.26) also. (5.4) represents a pair of second class constraints coming from the CS sector which is first order in time derivative. The relevant DB’s which implement (5.4) ‘strongly’, can either be determined by using the Dirac scheme, or else can be almost read off by using the symplectic method of Faddeev and Jackiw [20], to obtain

$$\{A_i(x), A_j(y)\} = \frac{\epsilon_{ij\lambda}}{2\theta} \delta(x - y) \quad (5.7)$$

The other two constraints (5.5 and 5.6) represent the only two primary constraints of the model.

The Legendre transformed Hamiltonian is given by

$$\mathcal{H}_c = \pi_\alpha \dot{z}_\alpha + \pi_\alpha^* \dot{z}^*_\alpha + \pi^0 \dot{A}_0 + \pi^j \dot{A}_j + \pi_\lambda \dot{\lambda} - \mathcal{L}$$

$$= \pi_\alpha^* \pi_\alpha - i A_0 (\pi_\alpha^* \pi_\alpha^* - \pi_\alpha^* \pi_\alpha) - 2 \theta A_0 e^{ij} \partial_i A_j + |D_i Z|^2 + \lambda (Z^\dagger Z - 1) \quad (5.8)$$
Preservation of the primary constraints (5.5 and 5.6) yield the following secondary constraints
\[ G(x) \equiv i(\pi^*_\alpha(x)z^*_\alpha(x) - \pi_\alpha(x)z_\alpha(x)) + 2\theta e^{ij}\partial_iA_j(x) \approx 0 \quad (5.9) \]
and
\[ C(x) \equiv Z^\dagger(x)Z(x) - 1 \approx 0 \quad (5.10) \]
The expressions of \( \pi_\alpha (5.2) \) and \( \pi^*_\alpha (5.3) \) are the same as that of the pure \( CP^1 \) model. Consequently, the constraint
\[ \pi^*_\alpha z^*_\alpha + \pi_\alpha z_\alpha \approx 0 \quad (5.11) \]
is present here as well. Just as in the \( S^3 \) model (3.1) the constraints (5.10) and (5.11) form a second class pair, and the symplectic structure is given by the DB (3.9) and (5.7).

It can be easily seen that that (5.9) generates the \( U(1) \) gauge transformation. For example
\[ \delta z^\alpha(x) = \int d^2 y f(y)\{z^\alpha(x), G(y)\} = -if(x)z^\alpha(x) \quad (5.12a) \]
\[ \delta A^i(x) = \int d^2 y f(y)\{A^i(x), G(y)\} = -\partial^i f(x) \quad (5.12b) \]
Thus (5.9) can be identified with the Gauss constraint. Note at this stage that the gauge field \( A_i \) is an independent degree of freedom now. A relation like \( A_i = -iZ^\dagger\partial_iZ \) (4.22) is absent in this case, and therefore, the gauge field has nothing to do with the monopole connection on \( CP^1 \). Consequently, the first term \((D_\mu Z)^\dagger(D^\mu Z)\) does not yield the nonlinear sigma model. Recall that the equivalence of the \( CP^1 \) and the NLSM (2.25a) hinges on two relations, (i) \( Z^\dagger Z = 1 \) (5.10), and (ii) \( A_\mu = -iZ^\dagger\partial_\mu Z \) (2.25b). So just like its nonrelativistic counterpart where the relation \( Z^\dagger Z = 1 \) is itself altered [12], here too the model ceases to be the \( CP^1 \) model, albeit for a different reason as (5.10) still holds.

Let us now consider the momentum and the angular momentum generator. The symmetric expression of the EM tensor is identical to that of the model with the Hopf term (4.1). Similarly, here also \( P^s_i \equiv \int d^2 x T^s_0i \) fails to generate appropriate translations ((4.40) holding true again). Therefore, we modify \( T^s_0i \) by making the same choice as (4.43) to get
\[ \tilde{T}^s_0i = \pi_\alpha \partial^i z^\alpha + \pi^*_\alpha \partial^i z^*_\alpha - 2\theta e^{kl}A_i(\partial_k A_l) \quad (5.13) \]
so that the integrated expression
\[ \tilde{P}^s_i \equiv \int d^2 x \tilde{T}^s_0i \equiv \int d^2 x[\pi_\alpha \partial^i z^\alpha + \pi^*_\alpha \partial^i z^*_\alpha - 2\theta e^{kl}A_i(\partial_k A_l)] \quad (5.14) \]
can now be identified as the momentum generator giving correct translation. But unlike the case of the Hopf term, this expression of momentum does not match with the one obtained through the Noether prescription
\[ P^N_i = \int d^2 x T^N_0i \equiv \int d^2 x[\pi_\alpha \partial^i z^\alpha + \pi^*_\alpha \partial^i z^*_\alpha + \pi^i \partial_j A_j] \]
\[ J^s = \int d^2x [\pi_\alpha \partial_i z_\alpha + \pi^*_\alpha \partial_i z^*_\alpha + \theta \epsilon_{kl} A_i (\partial_i A_k)] \tag{5.15} \]

by using the 'strong' equality (5.4). \( P_i^N \) generates appropriate translations as well, just as in (4.41).

The difference between the Hopf and the CS terms is more striking in the case of the angular momentum \( J \). The expression obtained through the symmetric EM tensor is given as,

\[ J^s = \int d^2x \epsilon^{ij} x_i \tilde{T}^s_{0j} \tag{5.16} \]

whereas the one using Noether’s prescription is given by

\[ J^N = \int d^2x [\epsilon^{ij} x_i T^N_{0j} + \pi^i \Sigma^{12}_{ij} A^j] \tag{5.17a} \]

where,

\[ \Sigma^{12}_{ij} = (\delta^1_i \delta^2_j - \delta^1_j \delta^2_i) \tag{5.17b} \]

The presence of an additional \( \Sigma \)-dependent piece is due to the presence of an independent dynamical variable \( A_i \) which transforms as a vector under spatial rotation. Note that such a term was absent in the case of the model involving Hopf term (4.46), as in that case, the background gauge field \( A_i \) ceased to be an independent degree of freedom because of the strong equality (4.22) relating the gauge fields \( A_i \) to the \( Z \) fields.

The expression (5.17a) which can be simplified as

\[ J^N = \int d^2x [\epsilon^{ij} x_i T^N_{0j} + \theta A_j A^j] \tag{5.18} \]

can be shown to generate appropriate spatial rotation:

\[ \{ Z(x), J^N \} = \epsilon_{ij} x_i \partial_j Z(x) \tag{5.19a} \]

\[ \{ A_k(x), J^N \} = \epsilon_{ij} x_i \partial_j A_k(x) + \epsilon_{ki} A_i \tag{5.19b} \]

An identical set of equation holds for \( J^s \) (5.16) also, indicating that \( J^s \) too generates appropriate rotation. However \( J^s \) and \( J^N \) are not identical—they differ by a nontrivial boundary term \( J_b \) given as,

\[ J_b = J^s - J^N = \theta \int d^2x \partial_i [x_j A^j A^i - x^i A_j A^j] \tag{5.20} \]

Precisely the same term appears in the context of other models involving CS term [6,7]. Some of its important properties have already been studied in [6,7], namely that \( J_b \) is gauge invariant only under those gauge transformation, which reduce to identity asymptotically. Also that the CS gauge field do not fall off to zero asymptotically fast enough in any of the standard gauges, as can be seen by looking at the Gauss constraint (5.9). So \( J_b \) evaluated in two different gauges
having different asymptotic behaviour is likely to yield different results. Proceeding as in [7], we can therefore evaluate this in a rotationally symmetric gauge like radiation gauge to get

\[ J_b = \frac{Q^2}{8\pi\theta} \quad (5.21) \]

where

\[ Q = i \int d^2x (\pi_\alpha(x)z_\alpha(x) - \pi^*_\alpha(x)z^{*\alpha}_\alpha(x)) \quad (5.22) \]

can be interpreted as the total charge of the system. This is because the integrand corresponds to the zeroth component of the Euler-Lagrange equation of motion of the gauge field \( A_i \).

\[ j^\mu = \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda \quad (5.23) \]

where,

\[ j_\mu = i[(D_\mu Z)^\dagger Z - Z^\dagger(D_\mu Z)] \quad (5.24) \]

The total charge \( Q \) (5.22) can be expressed on the Gauss constraint surface (5.9) as

\[ Q \approx 2\theta \int d^2x \epsilon^{ij}\partial_i A_j \quad (5.25) \]

so that this represents the total magnetic flux for the CS gauge field. Since \( A_j \) is no longer equal to \(-iZ^\dagger\partial_j Z\), the integrand in (5.25) cannot be identified with the topological density any more. A non-zero value of \( J_b \) indicates that unlike the Hopf term, the CS term imparts an "fractional" spin term right at the classical level.

Here we would like to clarify that the fractional spin in the context of CS term is conceptually somewhat different from the corresponding case of Hopf term we have considered earlier. This is because the Noether’s expression for the angular momentum in Hopf case consists solely of the orbital angular momentum unlike the CS case (5.17, 5.18) which consists of an additional spin term apart from the orbital one. The term “fractional” in the CS case just indicates the “anomalous” term we get over and above that of the Noether’s expression. Also note that the fractional spin (4.61) in the model involving radiation gauge fixed Hopf term (4.51) is given in terms of the soliton number \( N \), in contrast to the case here (5.21), where it is given in terms of the total charge \( Q \) (5.22), which in turn is the reflection of the fact that the CS gauge field \( A_i \) has an independent existence now and has nothing to do with the monopole connection on \( CP^1 \), as we have mentioned earlier.

## 6 Conclusions

In this paper we have first investigated some effects on the symplectic structure due to gauging and introducing the Hopf term to the \( CP^1 \) model. To begin with, we considered the line
element on group $SU(2)$ and its coset space $CP^1 \sim S^2$ and following [13], constructed the $S^3$ model and the $CP^1$ model associated with these respective line elements. The former enjoys only a global $SU(2)$ symmetry. The $CP^1$ model can be obtained from this by gauging the $U(1)$ subgroup. We found that the symplectic structure given by the set of DB’s in the $S^3$ model remains unaffected by this. The only difference being the appearance of a first class (Gauss) constraint generating $U(1)$ gauge transformation. The $CP^1$ model coupled to the Hopf term also has identical symplectic structure except that the structure of the Gauss constraint is modified by an additional $\theta$-dependent term.

Although a “background” gauge field was introduced in the $CP^1$ model, this however gets related to the $CP^1$ fields ‘strongly’ so that it effectively becomes a pull back onto the spacetime of the Dirac magnetic monopole connection on $CP^1 \sim S^2$. This gauge field therefore ceases to be an independent degree of freedom. This is in contrast to the case where the gauge field is given dynamics through the Chern-Simons term. The gauge field here is an independent degree of freedom, and is not related with the monopole connection of $CP^1$. The symplectic structure of this model also remains essentially the same, except that the independent CS gauge fields have an additional Faddeev-Jackiw bracket between themselves.

The fact that the CS gauge field has an independent existence has its bearing on the Noether expression of angular momentum in the form of an additional spin term apart from the usual orbital piece. This is in contrast to the case of the Hopf term where the Noether expression of angular momentum consists of the orbital part only, arising from the presence of scalar fields. We have shown that the angular momentum obtained from the symmetric expression of the EM tensor agrees with that obtained from the Noether prescription for the $CP^1$ model coupled to the Hopf term. This indicates that no fractional spin is imparted at the classical level by the Hopf term, a result that appears to be in disagreement with that obtained in [9]. One can attribute this discrepancy to the fact that an identity involving time derivatives, when expressed in terms of the $CP^1$ variables, is used to simplify the Hopf term in the radiation gauge. This is because relations involving time derivatives are not constraint equations and is liable to alter the dynamical content of the theory when substituted in the original Lagrangian. To corroborate, we have also carried out a similar analysis for the radiation gauge fixed Hopf term coupled to the $CP^1$ model to reveal fractional spin. This indicates the possibility of having fractional spin, arising from the nontrivial fundamental group of the configuration space (2.31), can be realised by (4.50) [9] or its $CP^1$ version (4.51), where radiation gauge condition has been incorporated right at the beginning to construct the model itself.

For the model involving the CS term, one finds that in contrast to the Hopf case, the two expressions of angular momentum obtained through the symmetric EM tensor and the Noether prescription differ by a nontrivial boundary term. This boundary term can be evaluated using the radiation gauge condition to yield the standard anomalous spin at the classical level itself. The difference between this result and the result obtained in the radiation gauge fixed Hopf model is that in the former case fractional spin is given by the total charge of the system,
whereas in the latter case it is given by the soliton number $\mathcal{N}$.

It would be interesting to quantize the $\mathbb{C}P^1$-Hopf theory in the Dirac scheme, instead of the reduced phase space scheme, to see if any fractional spin emerges as a pure quantum effect. Further, it would be also interesting to study if the analysis can be generalized for an arbitrary compact semi-simple Lie group $G$ and its coset $G/H$ to see whether the model obtained by gauging the subgroup $H$ has any effect on the original symplectic structure.

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