EXPONENTIAL STABILIZATION OF CASCADE ODE-REACTION-DIFFUSION PDE BY POINTWISE ACTUATION

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ABSTRACT. In this paper, we are concerned with the state feedback stabilization of ODE-PDE cascade systems governed by a linear ordinary differential equation and the $1-d$ reaction-diffusion equation posed on a bounded interval. In contrast to the previous works in the literature where the control acts at the boundary, the control for the entire system acts at an inside point of the PDE domain whereas the PDE acts in the linear ODE by a Neumann connection. We use the infinite dimensional backstepping design to convert system under consideration to an exponentially target system. By invertibility of the design and Lyapunov analysis, we prove the well posedness and exponential stability of such system.

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1. INTRODUCTION

Originally developed for finite dimensional control systems governed by ODE [8], the first extension of the backstepping method appeared in [3] and [9] for parabolic PDE. Later, in [12] and [10], the authors have introduced an invertible integral transformation that

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transforms the original parabolic PDE into an asymptotically stable one. Recently, the backstepping method is used to design a feedback control law for coupled PDE-ODE (see the textbook [6] and references therein). Many problems of state and output feedback stabilization for coupled ODE-Heat has been solved [13], [15], [5] and ODE- Wave [7], [14], to cite few. In all those works, the actuator acts at the left or the right boundary of the PDE domain by Dirichlet or Neumann actuation. In contrast to the previous works in the literature, in this paper, we propose a model where the controller acts at a point inside the domain of the PDE subsystem. More precisely, we consider an ODE-PDE cascade system governed by a linear ordinary differential equation and the $1-d$ reaction-diffusion equation posed on a bounded interval $(0,l)$ where the control of the entire system is located at a point $\xi \in (0,l)$ under a transmission conditions and the PDE acts in the ODE by a Neumann connection. We prove that for all $\xi \in (0,l)$, system is well posed and exponentially stable in the sense of the $H^1$-norm. In recent years, by using semi semigroups theory, a lot of researches have been devoted to the study of distributed plants with pointwise actuator. Surprisingly, the control properties of those systems are very different depending of the location of the actuator and the type of boundary conditions. (see [1] for the wave equation and [2] for the beam equation, to cite few.) In the other hand, by using the backstepping method, the exponential stability for reaction-diffusion equation [16] is proved if $\frac{q}{p} = \frac{k}{q}$ co-prime where $p$ odd. However, in [4], a feedback law has been proposed to achieve exponential stability for wave equation for all $\xi \in (0,l)$. The paper is organized as follows. In section 2, the problem is stated and the main result of this paper is summarized in Theorem 2.1. In section 3, the backstepping method is used to derive the state-feedback control law. Section 4 is devoted to the proof of the main Theorem 2.1. In section 5, we present the conclusion and the future work.
2. Problem Formulation and Main Result

Let $l > 0$, $\xi \in (0,l)$ and $\lambda > 0$, we consider the following cascade ODE-PDE system

\[
\begin{align*}
\dot{X}(t) &= AX(t) + B \frac{\partial u}{\partial x}(0,t), \quad t > 0, \\
\frac{\partial u}{\partial t}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,t) + \lambda u(x,t), \quad t > 0, \quad x \in (0,\xi) \cup (\xi,l), \\
u(\xi^-,t) &= u(\xi^+,t), \quad t > 0, \\
\frac{\partial u}{\partial t}(\xi^-,t) - \frac{\partial^2 u}{\partial x^2}(\xi^+,t) &= U(t), \quad t > 0, \\
X(0) &= X^0, \\
u(0,t) &= u(l,t) = 0, \quad t > 0, \\
u(x,0) &= u^0(x), \quad x \in (0,l).
\end{align*}
\]

(1)

where $X(t) \in \mathbb{R}^n$ is the state of the ODE subsystem, $u(x,t) \in \mathbb{R}$ is the state of the reaction-diffusion subsystem, $U(t) \in \mathbb{R}$ is the control input to the entire system acting in the interior point $x = \xi$ of the PDE domain $(0,l)$, and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$ such that the pair $(A,B)$ is controllable. When $U(t) = 0$, the PDE subsystem is equivalent to the following system

\[
\begin{align*}
\frac{\partial u}{\partial t}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,t) + \lambda u(x,t), \quad t > 0, \quad x \in (0,l) \\
u(0,t) &= u(l,t) = 0, \quad t > 0, \\
u(x,0) &= u^0(x), \quad x \in (0,l).
\end{align*}
\]

(2)

System (2) is unstable with arbitrarily many unstable eigenvalues for large $\lambda$. Thus the open-loop system (1) is unstable for large $\lambda$. The control objective is to exponentially stabilize system (1) around its zero equilibrium. Dividing the domain $[0,l]$ into the subdomains $[0,\xi]$ and $[\xi,l]$, we can reformulate system (1) as the following cascade ODE-transmission system

\[
\begin{align*}
\dot{X}(t) &= AX(t) + B \frac{\partial u}{\partial x}(0,t), \quad t > 0, \\
\frac{\partial u}{\partial t}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,t) + \lambda u_1(x,t), \quad t > 0, \quad x \in (0,\xi), \\
\frac{\partial u}{\partial t}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,t) + \lambda u_2(x,t), \quad t > 0, \quad x \in (\xi,l), \\
u_1(\xi^-,t) &= u_2(\xi^+,t), \\
\frac{\partial u}{\partial t}(\xi^-,t) - \frac{\partial^2 u}{\partial x^2}(\xi^+,t) &= U(t), \quad t > 0, \\
u_1(0,t) &= u_2(l,t) = 0, \quad t > 0, \\
X(0) &= X^0, \\
u_1(x,0) &= u_1^0(x), \quad x \in (0,\xi), \\
u_2(x,0) &= u_2^0(x), \quad x \in (\xi,l).
\end{align*}
\]

(3)
where
\[ u(x,t) = \begin{cases} 
    u_1(x,t), & x \in [0, \xi], \quad t \geq 0, \\
    u_2(x,t), & x \in [\xi, l], \quad t \geq 0. 
\end{cases} \]

The backstepping method is to use the transformations
\[ \begin{aligned}
   w_1(x,t) &= u_1(x,t) - \int_0^x k_1(x,y)u_1(y,t)dy + \varphi(x)X(t), \quad x \in [0, \xi], \quad t \geq 0, \\
   w_2(x,t) &= u_2(x,t) + \int_x^l k_2(x,y)u_2(y,t)dy, \quad x \in [\xi, l], \quad t \geq 0,
\end{aligned} \]

such that with the feedback law
\[ \begin{aligned}
   U(t) &= \left( k_1(\xi, \xi) - k_2(\xi, \xi) \right)u_1(\xi,t) + \int_0^\xi \frac{\partial k_1}{\partial x}(\xi,t)u_1(y,t)dy \\
   &+ \int_\xi^1 \frac{\partial k_2}{\partial x}(\xi,t)u_2(y,t)dy - \varphi'(\xi)X(t),
\end{aligned} \]

where the gain kernels \( k_1(x,y) \in \mathbb{R} \) and \( k_2(x,y) \in \mathbb{R} \) and the gain function \( \varphi(x)^T \in \mathbb{R}^n \) are appropriately chosen to transform system (3) into the following exponentially stable target system
\[ \begin{aligned}
   \dot{X}(t) &= (A + BK)X(t) + B\frac{\partial u_2}{\partial x}(0,t), \quad t > 0, \\
   \frac{\partial w_1}{\partial t}(x,t) &= \frac{\partial^2 w_1}{\partial x^2}(x,t), \quad t > 0, \quad x \in (0, \xi), \\
   \frac{\partial^2 w_2}{\partial x^2}(x,t) &= \frac{\partial^2 w_2}{\partial x^2}(x,t), \quad t > 0, \quad x \in (\xi, l), \\
   w_1(\xi,t) &= w_2(\xi,t), \\
   \frac{\partial w_1}{\partial x}(\xi,t) &= \frac{\partial w_2}{\partial x}(\xi,t), \quad t > 0, \\
   w_1(0,t) &= w_2(l,t) = 0, \quad t > 0, \\
   X(0) &= X^0, \\
   w_1(x,0) &= w_2^0(x), \quad x \in (0, \xi), \\
   w_2(x,0) &= w_2^0(x), \quad x \in (\xi, l),
\end{aligned} \]

where \( K \in \mathbb{R}^{1 \times n} \) is such that the matrix \( A + BK \) is Hurwitz. Once the transformations (4) and (5) (namely \( k_1(z,y) \), \( k_2(z,y) \) and \( \varphi(x)^T \)) are found, we use their invertibility and exponential stability of (7) to get that of the original plant (1) with the feedback law (6) acting at \( \xi \). Along this paper, the Euclidean norm of a vector \( X \) in \( \mathbb{R}^n \) and the \( L^2 \)-norm of a function \( u \) in \( L^2(a,b) \) are denoted by \( |X| \) and
\[ ||u|| = \left( \int_a^b u^2(x)dx \right)^{\frac{1}{2}}, \]
respectively. Let \( H = \mathbb{R}^n \times L^2(0, \xi) \times L^2(\xi, l) \) be the state space of the system (3). It is obvious that the vector space \( H \) equipped with its norm (8)

\[
\|(X, u_1, u_2)\|_H = \left( |X|^2 + \|u_1\|^2 + \|u_2\|^2 \right)^{\frac{1}{2}},
\]

is a Hilbert space. As far as that goes, we denote by \( Y = \mathbb{R}^n \times H^1_L(0, \xi) \times H^1_R(\xi, l) \) the dense subspace of \( H \) endowed with the norm (9)

\[
\|(X, u_1, u_2)\|_Y = \left( |X|^2 + \|u_1\|^2_{H^1(0, \xi)} + \|u_2\|^2_{H^1(\xi, l)} \right)^{\frac{1}{2}},
\]

where \( H^1_L(0, \xi) := \{ v \in H^1(0, \xi), v(0) = 0 \} \) and \( H^1_R(\xi, l) := \{ v \in H^1(\xi, l), v(l) = 0 \} \).

Now we are in position to establish the following main result.

**Theorem 2.1.** For any initial data \((X^0, u_1^0, u_2^0) \in Y\) satisfying the following compatibility conditions (9)

\[
u_1^0(\xi) - \int_0^\xi k_1(\xi, y)u_1^0(y)dy + \varphi(\xi)X^0 = u_2^0(\xi) + \int_\xi^l k_2(\xi, y)u_2^0(y)dy
\]

and

\[
(u_1^0)'(\xi) - k_1(\xi, \xi)u_1^0(\xi) - \int_0^\xi \frac{\partial k_1}{\partial x}(\xi, y)u_1^0(y)dy + \varphi'(\xi)X^0
= (u_2^0)'(\xi) - k_2(\xi, \xi)u_2^0(\xi) + \int_\xi^l \frac{\partial k_2}{\partial x}(\xi, y)u_2^0(y)dy;
\]

system (3) with the feedback law (2) has a unique classical solution in \( C([0, +\infty[, Y) \cap C^1([0, +\infty[, H]) \). Moreover, there exists \( C > 0 \) and \( d > 0 \) such that the solution satisfies (10)

\[
\|(X(t), u_1(t, \cdot), u_2(t, \cdot))\|_Y \leq Ce^{-dt}\|(X^0, u_1^0, u_2^0)\|_Y, \quad \forall t \geq 0.
\]

**Remark 2.1.** In particular, the compatibility condition (5) implies that \( w_1^0(\xi) = w_2^0(\xi) \).

Hence, by property of semi-groups, we get immediately

\[
w_1(t, \xi) = w_2(\xi, t), \quad \forall t \geq 0.
\]
3. Backstepping Design

We are now going to find the gain kernels \(k(x,y)\) and \(k_2(x,y)\) and the gain function \(\varphi(x)^T\).

From (3), (5) and (7), we get

\[
\frac{\partial w_1}{\partial t}(x,t) = \frac{\partial u_1}{\partial t}(x,t) - \int_0^x k_1(x,y) \frac{\partial u_1}{\partial t}(y,t) dy + \varphi(x)X(t)
\]

\[
= \frac{\partial^2 u_1}{\partial x^2}(x,t) + \lambda u_1(x,t) - \int_0^x k_1(x,y) \left( \frac{\partial^2 u_1}{\partial y^2}(y,t) + \lambda u_1(y,t) \right) dy + \varphi(x)X(t)
\]

\[
= \frac{\partial^2 u_1}{\partial x^2}(x,t) + \lambda u_1(x,t) - k_1(x,x) \frac{\partial u_1}{\partial x}(x,t) + k_1(x,0) \frac{\partial u_1}{\partial x}(0,t) + \frac{\partial k_1}{\partial y}(x,x)u_1(x,t)
\]

\[
(12) \quad - \int_0^x \left( \frac{\partial^2 k_1}{\partial y^2}(x,y) + \lambda k_1(x,y) \right) u_1(y,t) dy + \varphi(x)AX(t) + \varphi(x)B \frac{\partial u_1}{\partial x}(0,t),
\]

\[
\frac{\partial w_1}{\partial x}(x,t) = \frac{\partial u_1}{\partial x}(x,t) - k_1(x,x)u_1(x,t) - \int_0^x \frac{\partial k_1}{\partial x}(x,y)u_1(y,t) dy + \varphi'(x)X(t),
\]

and

\[
\frac{\partial^2 w_1}{\partial x^2}(z,t) = \frac{\partial^2 u_1}{\partial x^2}(x,t) - \frac{d}{dx}(k_1(x,x))u_1(x,t) - k_1(x,x) \frac{\partial u_1}{\partial x}(x,t) - \frac{\partial k_1}{\partial x}(x,x)u_1(x,t)
\]

\[
- \int_0^x \frac{\partial^2 k_1}{\partial y^2}(x,y)u_1(y,t) dy + \varphi''(x)X(t).
\]

Combining (12) and (14) gives

\[
0 = \frac{\partial w_1}{\partial t}(x,t) - \frac{\partial^2 w_1}{\partial x^2}(x,t)
\]

\[
= \left( \lambda + 2 \frac{d}{dx}(k_1(x,x)) \right) u_1(x,t) + \left( k_1(x,0) + \varphi(x)B \right) \frac{\partial u_1}{\partial x}(0,t) - (\varphi''(x) - \varphi(x)A)X(t)
\]

\[
(15) \quad + \int_0^x \left( \frac{\partial^2 k_1}{\partial y^2}(x,y) - \lambda k_1(x,y) \right) u_1(y,t) dy.
\]

In the same way, we can just get the following identity for \(w_2\)

\[
0 = \frac{\partial w_2}{\partial t}(x,t) - \frac{\partial^2 w_2}{\partial x^2}(x,t)
\]

\[
= \left( \lambda + 2 \frac{d}{dx}(k_2(x,x)) \right) u_2(x,t) + k_2(x,0) \frac{\partial u_2}{\partial x}(l,t)
\]

\[
+ \int_x^l \left( \frac{\partial^2 k_2}{\partial y^2}(x,y) + \lambda k_2(x,y) \right) u_2(y,t) dy.
\]

(16)

Moreover, setting \(x = 0\) in \(w_1(x,t)\), \(x = l\) in \(w_2(x,t)\) and \(\xi\) in both \(\frac{\partial w_1}{\partial x}(x,t)\) and \(\frac{\partial w_2}{\partial x}(x,t)\), and taking Remark 6.1 in mind, it follows that, if the gain function \(\varphi(x)^T\) defined in \([0,l]\),
the gain kernel \( q_1(x, y) \) defined in

\[
T_1 = \{ (x, y) \mid x \in [0, \xi], y \in [x, \xi] \}
\]

and the gain kernel \( q_2(x, y) \) defined in

\[
T_2 = \{ (x, y) \mid x \in [\xi, \ell], y \in [x, \ell] \}
\]

satisfy

\[
\begin{cases}
\phi''(x) - \phi(x)A = 0, \\
\phi(0) = 0, \\
\phi'(0) = -K,
\end{cases}
\]

\[
\begin{align*}
\frac{\partial^2 k_1}{\partial x^2}(x, y) - \frac{\partial^2 k_1}{\partial y^2}(x, y) &= \lambda k_1(x, y), \\
k_1(x, 0) &= -\phi(x)B, \\
k_1(x, x) &= \frac{\lambda}{2} x,
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial^2 k_2}{\partial x^2}(x, y) - \frac{\partial^2 k_2}{\partial y^2}(x, y) &= -\lambda k_2(x, y), \\
k_2(x, 0) &= 0, \\
k_2(x, x) &= \frac{\lambda}{2} (\ell - x),
\end{align*}
\]

respectively, then we obtain the target system (7) for every solution of the closed loop (3) with the feedback law (6). Obviously, the solution of the linear differential equation (19) is

\[
\phi(x) = (0, -K)e^{Mx},
\]

where \( M \) and \( E \) are the constant matrices

\[
M = \begin{pmatrix} 0 & A \\ I_n & 0 \end{pmatrix}, \quad E = \begin{pmatrix} I_n \\ 0 \end{pmatrix}.
\]

According to [11], the solution of (21) can be done explicitly. For (20), because of the presence of \( \phi(x) \) in the boundary, we can only prove the existence of \( k_1(x, y) \). To study (20), we first convert it into an integral equation. For this reason, introducing the change of variables

\[
\zeta = x + y, \quad \eta = x - y,
\]

and define

\[
G(\zeta, \eta) := q(x, y).
\]
Then, the function $G$ defined in the triangle $T_0 = \{ (\xi, \eta), \eta \in [0, l], \xi \in [\eta, 2l - \eta] \}$, satisfies

$$
\frac{\partial^2 G}{\partial \xi \partial \eta} (\xi, \eta) = \frac{\lambda}{4} G(\xi, \eta), \quad (\xi, \eta) \in T_0,
$$

(23)

$$
G(\eta, \eta) = -\varphi(\eta) B, \quad \eta \in [0, l],
$$

$$
G(\xi, 0) = -\frac{\lambda}{4} \xi, \quad \xi \in [0, 2l].
$$

Integrating the first equation of (23) with respect to $\eta$ from 0 to $\eta$ and using the boundary condition $G(\xi, 0)$, we get

$$
\frac{\partial G}{\partial \xi}(\xi, \eta) = -\frac{\lambda}{4} + \int_0^\eta \frac{\lambda}{4} G(\xi, s) ds.
$$

Next, integrating the above identity with respect to $\zeta$ over the interval $[\eta, \zeta]$ and using the boundary condition $G(\eta, \eta)$, it follows

$$
G(\zeta, \eta) = -\varphi(\eta) B - \frac{\lambda}{4} (\zeta - \eta) + \int_\eta^\zeta \int_0^\eta \frac{\lambda}{4} G(t, s) ds dt, \forall n \in \mathbb{N}.
$$

(24)

To achieve the existence of $G$ in $T_0$, we use the method of successive approximations. To this end, let us set

$$
G^0(\zeta, \eta) = 0,
$$

(25)

$$
G^{n+1}(\zeta, \eta) = -\varphi(\eta) B - \frac{\lambda}{4} (\zeta - \eta) + \int_\eta^\zeta \int_0^\eta \frac{\lambda}{4} G^n(t, s) ds dt, \forall n \in \mathbb{N},
$$

(26)

and denote the difference between two consecutive terms by

$$
\Delta G^n(\zeta, \eta) = G^{n+1}(\zeta, \eta) - G^n(\zeta, \eta).
$$

(27)

Then,

$$
\Delta G^{n+1}(\zeta, \eta) = \frac{\lambda}{4} \int_\eta^\zeta \int_0^\eta \Delta G^n(t, s) ds dt.
$$

(28)

We have

$$
\Delta G^0(\zeta, \eta) = -\frac{\lambda}{4} (\zeta - \eta) - \varphi(\eta) B.
$$

Since $\varphi$ is continuous on $[0, l]$, there exists $\mu > 0$ such that $|\varphi(\eta) B| \leq \mu, \forall \eta \in [0, l]$. Thus,

$$
|\Delta G^0(\zeta, \eta)| \leq \frac{\lambda}{4} (\zeta - \eta) + \mu.
$$

(29)

Using the fact that $0 \leq \eta \leq \zeta$, by an immediate mathematical induction, it can be shown that

$$
|\Delta G^n(\zeta, \eta)| \leq \left( \frac{\lambda}{4} \right)^{n+1} \frac{\zeta - \eta}{n!(n+1)!} \mu + \frac{\lambda}{4} \frac{\zeta^n \eta^n}{(n!)^2}.
$$
It then follows from the Weierstrass M-test that the series

\[ G(\zeta, \eta) = \lim_{n \to \infty} G_n(\zeta, \eta) = \sum_{n=0}^{\infty} \Delta G_n(\zeta, \eta) \]

converges absolutely and uniformly in \( T_0 \). Having proved the existence of \( G(\zeta, \eta) \) in \( T_0 \), that of \( k_1(x, y) \) in \( T \) follows immediately. For the kernel \( k_2(x, y) \), without loss of generality, we suppose that \( l - \xi \leq \xi \). Consider the change of coordinates

\[ s = l - x \ , \ t = l - y, \]

that maps the triangle \( T_2 \) into the triangle \( T_1 \), and define the function \( h(s, t) = -k_2(x, y) \).

Then, we get for the function \( h(s, t) \) the following PDE

\[
\begin{align*}
\frac{\partial^2 h}{\partial s^2}(s, t) - \frac{\partial^2 h}{\partial t^2}(s, t) &= \lambda h(s, t), \\
h(s, 0) &= 0, \\
h(s, s) &= \frac{\lambda}{2} s.
\end{align*}
\] (30)

According to [11], the solution of (30) in \( T_1 \) is

\[ h(s, t) = -\lambda \frac{I_1 \left( \sqrt{\lambda (s^2 - t^2)} \right)}{\sqrt{\lambda (s^2 - t^2)}}, \text{ if } s \neq t, \] (31)

where \( I_1 \) is the first order modified Bessel function of the first kind. Consequently, we obtained the solution of (31) in \( T_2 \) as follows

\[ k_2(x, y) = \frac{\lambda}{2} \frac{I_1 \left( \sqrt{\lambda ((l-x)^2 - (l-y)^2)} \right)}{\sqrt{\lambda ((l-x)^2 - (l-y)^2)}}, \text{ if } x \neq y. \] (32)

Let’s move to the proof of the main Theorem 2.1.

4. **Proof of Theorem 2.1**

4.1. **Well posedness of the initial plant** [3]. Let us define the map

\[ \Omega : Y \to Y \]

\[ (X, u_1, u_2) \to (X, w_1, w_2), \] (33)
where \( w_1 \) and \( w_2 \) satisfy (4) and (5), respectively. This linear map is well defined and bounded. Hence, there exists a positive constant \( c_1 \) such that

\[
\| \Omega(X, u_1, u_2) \|_Y \leq c_1 \| (X, u_1, u_2) \|_y, \quad \forall (X, u_1, u_2) \in Y.
\]

It is a straightforward that the transformation \( \Omega \) is an isomorphism, and

\[
\Omega^{-1} : \ Y \to Y
\]

\[
(X, w_1, w_2) \mapsto (X, u_1, u_2)
\]

has the following form

\[
X(t) = X(t)
\]

\[
u_1(x, t) = w_1(x, t) - \int_0^x w_1(y, t) g_1(x, y) dy + \psi(x)X(t), \quad x \in [0, \xi], \quad t \geq 0,
\]

\[
u_2(x, t) = w_2(x, t) + \int_x^t w_2(y, t) g_2(x, y) dy, \quad x \in [\xi, l], \quad t \geq 0.
\]

As is done in the study of the direct transformation \( \Omega \) and in the same way, on can prove the existence of the gain kernel \( g_1(x, y) \) and compute explicitly the gain kernel \( g_2(x, y) \) and the gain function \( \psi(x) \). Therefore, there exists a positive constant \( c_2 \) such that

\[
\| \Omega^{-1}(X, w_1, w_2) \|_Y \leq c_2 \| (X, w_1, w_2) \|_y, \quad \forall (X, w_1, w_2) \in Y.
\]

It is well known that if the initial condition \((w_1^0, w_2^0)\) of the target system \( \Box \) belongs to the subspace

\[
D = \left\{ (w_1, w_2) \in H^2(0, \xi) \cap H^1_l(0, \xi) \times H^2(\xi, l) \cap H^1_{\xi l}(\xi, l) \mid w_1(\xi) = w_2(\xi), \frac{\partial w_1}{\partial x}(\xi) = \frac{\partial w_2}{\partial x}(\xi) \right\}
\]

the function \( w \) defined as

\[
w(x, t) = \begin{cases} w_1(x, t), & x \in [0, \xi], \quad t \geq 0, \\ w_2(x, t), & x \in [\xi, l], \quad t \geq 0. \end{cases}
\]

belongs to \( H^2(0, l) \cap H^1_l(0, l) \) and satisfying the following boundary problem

\[
\begin{cases}
\frac{\partial w}{\partial x}(x, t) = \frac{\partial^2 w}{\partial t^2}(x, t), & t > 0, \quad x \in (0, l), \\
w(0, t) = w(l, t) = 0, & t > 0, \\
w(x, 0) = w^0(x) = \mathbb{I}_{(0, \xi)} w_1^0(x) + \mathbb{I}_{(\xi, l)} w_2^0(x), & x \in (0, l),
\end{cases}
\]
The existence, uniqueness and regularity of the solution of (40) follow by standard arguments of semi-groups theory. Furthermore, by the method of separation of variables, the system (40) is not only well posed but its solution is explicitly done as follows

\[ w(x,t) = \frac{2}{T} \sum_{k=1}^{\infty} e^{-\frac{k^2 \pi^2}{T^2} t} \sin \left( \frac{k \pi}{T} x \right) \int_{0}^{t} w(t-s) \sin \left( \frac{k \pi}{T} s \right) ds. \]

Thus, By Duhamel’s formula, the solution of the ODE in the target system (7) yields

\[ X(t) = e^{(A+BK)} X_0 + \int_{0}^{t} e^{(t-\tau)(A+BK)} B \frac{\partial w_1}{\partial x}(0, \tau) d\tau. \]

Since the isomorphism \( \Omega \) transforms system (3) to system (7), it follows that system (3) with the feedback law (6) is well posed. Hence, the regularity of the solution given by Theorem 2.1 holds true.

4.2. Exponential stability. Consider the Lyapunov function candidate

\[ V(t) = X(t)^T P X(t) + \frac{a}{2} \| w(.,t) \|^2 + \frac{b}{2} \left\| \frac{\partial w}{\partial x}(.,t) \right\|^2, \]

where \( a > 0 \) and \( b > 0 \) are two constants which we will specify later and the positive definite matrix \( P = P^T > 0 \) is the solution of the Lyapunov equation

\[ P(A+BK) + (A+BK)^T P = -Q, \]

for some positive definite matrix \( Q = Q^T > 0 \). From (43), it can be obtained that for all \( t \geq 0 \),

\[ \alpha_1 \| (X(t), w(.,t)) \|^2 \leq V(t) \leq \alpha_2 \| (X(t), w(.,t)) \|^2, \]

where

\[ \alpha_1 = \min \left( \lambda_{\min}(P), \frac{a}{2}, \frac{b}{2} \right), \quad \alpha_2 = \max \left( \lambda_{\max}(P), \frac{a}{2}, \frac{b}{2} \right) \]

and \( Z = \mathbb{R}^n \times H^1(0,1) \) equipped with its norm \( \| (X,w) \|^2_Z = \| X \|^2 + \| w \|^2_{H^1(0,1)} \). The derivative of \( V \) along the solutions of (40) is given by

\[ \dot{V}(t) = -X^T(t) Q X(t) + 2X(t)^T P B \frac{\partial w}{\partial x}(0,t) - a \left\| \frac{\partial w}{\partial x}(.,t) \right\|^2 - b \left\| \frac{\partial^2 w}{\partial x^2}(.,t) \right\|^2. \]

By Young’s inequality, we get

\[ 2X(t)^T P B \frac{\partial w}{\partial x}(0,t) \leq \frac{\lambda_{\min}(Q)}{2} \| X(t) \|^2 + \frac{2}{\lambda_{\min}(Q)} \| P B \|^2 \left( \frac{\partial w}{\partial x}(0,t) \right)^2, \]

and by Agmon’s inequality [15], it can be proved that the following inequality holds:

\[ -\left\| \frac{\partial^2 w}{\partial x^2}(.,t) \right\|^2 \leq \frac{1+l}{l} \| \frac{\partial w}{\partial x}(.,t) \|^2 - \left( \frac{\partial w}{\partial x}(0,t) \right)^2. \]
Thus,

\[ \dot{V}(t) \leq -\frac{\lambda_{\text{min}}(Q)}{2} |X(t)|^2 - \left( \frac{a}{2} - \frac{b(1+l)}{l} \right) \left\| \frac{\partial w}{\partial x} (.,t) \right\|^2 - \frac{a}{2} \left\| \frac{\partial w}{\partial x} (.,t) \right\|^2 - \left( b - \frac{2|PB|^2}{\lambda_{\text{min}}(Q)} \right) \left( \frac{\partial w}{\partial x} (0,t) \right)^2. \]

Since \( w(0,t) = 0 \), by Poincaré inequality, we get

\[ \|w(.,t)\|^2 \leq 4l^2 \left\| \frac{\partial w}{\partial x} (.,t) \right\|^2. \]

Now, if we choose

\[ b > \frac{2|PB|^2}{\lambda_{\text{min}}(Q)} \quad \text{and} \quad a > \frac{2b(1+l)}{l} + 2, \]

from estimation above, it yields

\[
\begin{align*}
\dot{V}(t) &\leq -\frac{\lambda_{\text{min}}(Q)}{2} |X(t)|^2 - \frac{a}{8l^2} \|w(.,t)\|^2 - \|\frac{\partial w}{\partial x} (.,t)\|^2 \\
&\leq -\delta V(t),
\end{align*}
\]

where

\[ \delta = \min \left( \frac{\lambda_{\text{min}}(Q)}{2\lambda_{\text{max}}(P)} \frac{1}{4l^2} \frac{2}{b} \right). \]

Therefore,

\[ V(t) \leq V(0)e^{-\delta t}, \forall t \geq 0. \]

Consequently, From (44) and (46), the following estimation

\[ \| (X(t), w(.,t)) \|_Z^2 \leq \alpha e^{-\delta t} \| (X^0, w^0) \|_Z^2 \]

holds for all \( t \geq 0 \), where \( \alpha = \frac{c_1}{c_2} \).

Ultimately, since \( \| (X, w_1, w_2) \|_Y = \| (X, w) \|_Z \), from (34), (39) and (47), it follows

\[ \| (X(t), u_1(.,t), u_2(.,t)) \|_Y \leq C e^{-dt} \| (X^0, u_1^0, u_2^0) \|_Y, \quad \forall t \geq 0, \]

where \( C = c_1c_2\sqrt{\alpha} \) and \( d = \frac{\delta}{2} \). Thus, the proof of Theorem 2.1 is complete.

5. Conclusion and future work

We have considered the exponential stabilization for a cascaded ODE-reaction-diffusion system (1) coupling at the left boundary by Neumann connection and a control acting at an internal point of the PDE subdomain. By the backstepping method, we have construct
a feedback law to achieve the result. However for a general coupled system where the cascade ODE-PDE system is coupled at an intermediate point $x_0 \in (0, l)$ as follows

\[
\begin{cases}
\dot{X}(t) = AX(t) + B \frac{\partial u}{\partial x}(x_0, t), & t > 0, \\
\frac{\partial u}{\partial t}(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t) + \lambda u(x, t), & t > 0, \quad x \in (0, \xi) \cup (\xi, l), \\
u(\xi -, t) = u(\xi +, t), & t > 0, \\
\frac{\partial u}{\partial x}(\xi -, t) - \frac{\partial u}{\partial x}(\xi +, t) = U(t), & t > 0, \\
\dot{X}(0) = X^0, \\
u(0, t) = u(l, t) = 0, & t > 0, \\
u(x, 0) = u^0(x), & x \in (0, l),
\end{cases}
\]

the stabilization controller design is not obvious, which needs to be considered in the future.

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