Theoretical confirmation of Feynman’s Hypothesis on the creation of circular vortices in superfluid helium

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The changes observed in the topology of superfluid helium vortices have intrigued people for some time now [1]. These vortices either extend from wall to wall, however tangled they may be in between, or else can be roughly circular and freely move around the superfluid [1]. Some time ago, Richard Feynman postulated that two oppositely polarized line vortices could, if they cross at two points, reconnect so as to create a circular vortex that snaps off and subsequently lives a life of its own [2]. This is often simply postulated in numerical experiments, e. g. [1], [3]. That an opposite line vortex pair solution of the Nonlinear Schrödinger equation (NLS) for HeII is unstable has been demonstrated theoretically [4]. Reconnection at a point has been obtained numerically [5]. What remained to prove was that known, stationary, double vortex line solutions can thus reproduce one of the one parameter solitonic family of solutions for circular vortices as found for NLS [6]. In other words, can this reconnection really lead to a full confirmation of Feynman’s hypothesis? Our first step was a similar changeover calculation for a limiting case, unfortunately such that the vortex configuration was degenerate [7], [8]. However, surprisingly complete dynamic changeover from cylindrical to spherical symmetry of the soliton was observed in [7]. This augured well for the present effort.

An imperfect Bose condensate, such as HeII, can (also imperfectly) be described by a single particle wavefunction \( \psi(x, t) \) of \( N \) bosons of mass \( m \) that obeys the Nonlinear Schrödinger equation (NLS), according to Gross, Pitaevski and Ginzburg:

\[
\frac{i\hbar}{\partial t} \psi = -\frac{\hbar^2}{2m}\nabla^2 \psi + W_0|\psi|^2. \tag{1}
\]

Here \( W_0 \) characterizes the potential between bosons. Opposite vortex pair solutions, as well as those describing circular vortices are known [6]. Each solution has a unique velocity perpendicular to the vortex plane. However, to answer the crucial question of whether dynamics can lead from the former to the latter kind, we must resort to numerics. All theory tells us is that the double line vortex configuration is unstable [4].

Before giving the results of our simulations, we wish to point out that a preliminary idea of the problem can be gained from the linear equation, \( W_0 = 0 \). At a vortex, \( \psi = 0 \), so the cubic term not contributing locally might not be too surprising. However, the extent of global similarities with solutions to equation (1) may be more so. The preliminary results will help us appreciate just what the role of the nonlinear, \( W_0 > 0 \), term is in the act of reconnection.

One can easily check by substitution that equation (1), \( W_0 = 0 \), is solved by

\[
\psi = \text{const} \left[ a^2 - x^2 + ib \left( z(t) + \frac{ht}{mb} \right) \right] e^{ik_z z - i\hbar k_z^2 t/(2m)}, \quad z(t) = z - \frac{h k_z}{m} t, \quad b > 0. \tag{2}
\]

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Vortices are situated where both $\text{Re}\psi$ and $\text{Im}\psi$ are zero. They constitute two oppositely polarized lines along $y$ at $x = \pm a$ and move together at velocity $U_z = \hbar (k_z - b^{-1})/m$. There is no correlation between the separation, $2a$, and the uniform velocity $U_z$, which can in fact have either sense.

A second solution to equation (1), $W_0 = 0$, is given by [9]:

$$\psi = \text{const} \left[ R^2 - x^2 - y^2 + i d \left( z(t) + \frac{2\hbar}{md} t \right) \right] e^{i k_z z - i \hbar k_z^2 t/(2m)}. \quad (3)$$

A circular vortex at $x^2 + y^2 = R^2$ is now moving up $z$ at velocity $U_z = \hbar (k_z - 2d^{-1})/m$. Again, there is no connection between $R$ and the uniform velocity, or even with its sense.

Jones and Roberts found both a class of stationary, double line vortex solutions to (1), $W_0 > 0$, as well as circular ones. Correlations between $a$, $R$ and corresponding $U_z$ were given in tables. Otherwise, the similarities between their solutions with the above are at first surprising, especially if we choose the velocities in (2) and (3) such as to mimic those of Jones and Roberts. The role of the nonlinear term would then ostensibly be limited to ensuring that $|\psi|$ tend to a uniform value in the far field. However, there is a less obvious difference. Even if we perturbed (2) such that two vortices touched at a point, say by adding $a \cos(k_y y)$ to $x$ initially, a circular vortex would not be produced at any $t > 0$.

Further calculations will be compared with the solutions of Jones & Roberts. Therefore we cast equation (1) in dimensionless form such that we can use their tables (here $E$ is the average energy level per unit mass of a boson):

$$\psi \to e^{-imEt/\hbar}\psi, \quad x \to \sqrt{\frac{\hbar}{2Em}}x, \quad t \to \frac{\hbar}{2mE}t, \quad (4)$$

so finally $\psi \to \sqrt{\frac{\hbar E}{W_0}}\psi$. (Linear models will match the temporal dependence if $k_z = \sqrt{2Em}/\hbar$.)

Now

$$2i\frac{\partial \psi}{\partial t} = -\nabla^2 \psi - \psi(1 - |\psi|^2). \quad (5)$$

If we write $\psi = \rho^{1/2}e^{iS}$, then $\rho$ and $\mathbf{v} = \nabla S$ have a fluid interpretation. The variables $\rho$ and $\mathbf{v}$ satisfy the usual continuity equation, but due to the nonlinear term, the Newtonian equation has a rather strange pressure tensor [6]. This may explain the possibility of reconnection. Importantly, if we encircle a $\psi = 0$ line once, $S$ must increase by $\pm 2\pi n$ so that $\psi$ is single valued. This was the case for (2) and (3), where $n = 1$ (unless $a = 0$ in (2), in which case $n = 0$). At infinity, $|\psi| \to 1$ and this must be included in our initial conditions describing the pair of line vortices.

As initial condition, we took a vortex configuration in which the separation and velocity were lifted from Jones and Roberts, Table 2. Thus

$$\psi(t = 0) = \frac{r_1r_2}{\sqrt{r_1^2 + b^2}\sqrt{r_2^2 + b^2}} e^{i(\theta_1 + \theta_2)}, \quad (6)$$

where

$$r_1 = (1 - 2U^2)(x + a)^2 + z^2, \quad r_2 = (1 - 2U^2)(x - a)^2 + z^2, \quad \tan \theta_1,2 = \frac{z}{\sqrt{1 - 2U^2}(a \pm x)}.$$
We repeated the calculation for different initial conditions, always obtaining viable circular vortex solutions, see Fig. 2. Thus, Feynman’s hypothesis is confirmed, completing the tentative steps of [5] and [7]. Of course, this confirmation is only as conclusive as is the NLS model for a Bose gas. With the above reservation, the experimentally found abundance of circular vortices in superfluid HeII is now explained theoretically. The proximity at two points of two opposite line vortices in so tangled a web is quite commonplace [1]. The generation of vortex rings due to the helical instability of a vortex line is also of primary interest in superconductivity theory [11], [12]. Perhaps our experience could be useful there, though unfortunately the equations are more complicated (in the Ginzburg–Landau model, the vector potential \( \mathbf{A} \) appears in an extension of (1). An additional vector equation relates \( \mathbf{A} \) and \( \psi \)).

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FIGURE CAPTIONS

Figure 1. Three stages in a Feynman transformation of two perturbed line vortices into a circular vortex in HeII, as follows from the NLS equation. Densities on the axes are zero.

Figure 2. Our numerically obtained circular vortices (circles) as compared to those of Jones and Roberts (continuous line) in $R, U$ space.
