A MATRIX RODRIGUES FORMULA FOR CLASSICAL
ORTHOGONAL POLYNOMIALS IN TWO VARIABLES

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Abstract. Classical orthogonal polynomials in one variable can be characterized as the only orthogonal polynomials satisfying a Rodrigues formula. In this paper, using the second kind Kronecker power of a matrix, a Rodrigues formula is introduced for classical orthogonal polynomials in two variables.

1. Introduction

One of the most important characterizations for classical orthogonal polynomials in one variable (Hermite, Laguerre, Jacobi and Bessel) is the so-called Rodrigues formula (see, for instance, [3]).

Using this kind of formula we can write the $n$–th classical orthogonal polynomial in terms of a $n$–th order derivative. In fact, if we denote by $\{P_n\}_n$ a classical family of orthogonal polynomials in one variable, then

$$P_n(x) = \frac{k_n}{\omega(x)} \frac{d^n}{dx^n} (\phi(x)^n \omega(x)), \quad n = 0, 1, 2, \ldots,$$

where $k_n$ is a constant, $\phi(x)$ is a polynomial of degree less than or equal to 2, independent of $n$, and $\omega(x)$ is an integrable function in a appropriate support set.

If $\deg \phi = 0$, Hermite polynomials appear, up to a linear change in the variable. If $\deg \phi = 1$, Laguerre polynomials are obtained, and if $\deg \phi = 2$, we can deduce two families of polynomials, Jacobi polynomials when $\phi(x)$ has two simple roots, and Bessel polynomials when $\phi(x)$ has a double root.

Formula (1) is called Rodrigues formula, honoring B. O. Rodrigues who established the formula in 1814 for Legendre polynomials.

Orthogonal polynomials in two variables which are solutions of partial differential equations were systematically studied by H. L. Krall and I. M. Sheffer ([14]), in 1967. They defined classical orthogonal polynomials in two variables as the sequences of orthogonal polynomials $\{P_{h,k}\}_{h,k \geq 0}$ such that every polynomial $P_{h,k}$, with $h + k = n$, satisfies the second order PDE

$$L[w] = a w_{xx} + 2b w_{xy} + c w_{yy} + d w_x + e w_y = \lambda_n w,$$

where $a(x, y) = ax^2 + d_1 x + e_1 y + f_1$; $b(x, y) = axy + d_2 x + e_2 y + f_2$; $c(x, y) = ay^2 + d_3 x + e_3 y + f_3$; $d(x, y) = gx + h_1$; $e(x, y) = gy + h_2$, and $\lambda_n = a(n-1) + gn$.

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The special shape of the polynomials involved in the above equation is a direct consequence of the fact that every orthogonal polynomial of total degree \( n \) must satisfy the same PDE. Krall and Sheffer showed that, up to a linear change in the variables, there are nine different sets of orthogonal polynomials satisfying such type of PDE.

The first reference to a Rodrigues formula for classical orthogonal polynomials in two variables appears in the classical text by P. Appell and J. Kampé de Fériet [1]. Later, P. K. Suetin [16], and Y. J. Kim, K. H. Kwon and J. K. Lee [10] consider an analogue of the Rodrigues formula for Krall and Sheffer classical orthogonal polynomials in two variables. In fact, for \( n \) a positive integer, they define

\[
P_{n-i,i}(x, y) = \frac{1}{\omega} \partial_x^{n-i} \partial_y^i (p^{n-i} q^i \omega),
\]

where \( w(x, y) \) is a weight function over a simply connected domain, and a symmetry factor of \( L \), the linear differential operator defined in (2), and \( p(x, y), q(x, y) \) are polynomials related with the polynomial coefficients in (2). Then, under some additional hypothesis, (3) defines an algebraic polynomial in two variables orthogonal to all polynomials of lower degree (see example 3, in Section 6).

The above Rodrigues formula runs only for classical orthogonal polynomials associated with a positive definite moment functional, since it needs a weight function. Nevertheless, H. L. Krall and I. M. Sheffer founded classical orthogonal polynomials in two variables associated with a non positive definite moment functional which has a symmetry factor (see L. L. Littlejohn [15]), but not a Rodrigues formula like (3) [10].

On the other hand, tensor product of two classical orthogonal polynomials in one variable, defined by

\[
P_{h,k}(x, y) = R_h(x)S_k(y), \quad h, k \geq 0,
\]

where \( \{R_h\}_{h \geq 0} \) and \( \{S_k\}_{k \geq 0} \) are Hermite, Laguerre, Jacobi or Bessel polynomials, satisfies a Rodrigues formula as (3). In fact, \( P_{h,k}(x, y) \) can be written as a product of the respective Rodrigues formulas

\[
P_{h,k}(x, y) = \frac{1}{\omega_1} \partial_x^h (\phi_1^h \omega_1) \frac{1}{\omega_2} \partial_y^k (\phi_2^k \omega_2), \quad h, k \geq 0.
\]

However, tensor products of classical orthogonal polynomials in one variable are not classical according to the Krall and Sheffer definition since they do not satisfy equation (2), except for Hermite and Laguerre polynomials.

Recently, the authors (see [5, 6, 7, 8]) extended the concept of classical orthogonal polynomials in two variables to a wider framework, which, of course, includes the Krall and Sheffer definition and tensor products of classical orthogonal polynomials in one variable.

The vector representation for orthogonal polynomials introduced in [12, 13], and developed in [17] is the key to introduce the concept of classical orthogonal polynomials in two variables. Let \( \{P_n\}_n \) denote a weak orthogonal polynomial sequence (see Section 2), it will be called classical (in an extended sense) if there exist non singular matrices \( \Lambda_n \in M_{n+1}(\mathbb{R}) \), such that,

\[
L[P_n] \equiv \text{div} (\Phi \nabla P_n^f) + \tilde{\Psi}^f \nabla P_n^f = P_n^f \Lambda_n,
\]
where
\[
\Phi = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad \tilde{\Psi} = \begin{pmatrix} d - a_x - b_y \\ e - b_x - c_y \end{pmatrix},
\]
and \(a, b, c\) are polynomials in two variables of total degree less than or equal to 2, and \(d, e\) are polynomials in two variables of total degree less than or equal to 1, and \(\text{div}\) and \(\nabla\) denote the usual divergence and gradient operators in two variables. Observe that the left hand side of (4) generalizes the left hand side of the Krall and Sheffer PDE (2), without any restrictions on the polynomial coefficients. Moreover, this new definition also include the tensor product of classical orthogonal polynomials in one variable. In the Krall and Sheffer case the matrices \(\Lambda_n\) are scalar matrices, and in the tensor product case, they are diagonal non–singular matrices.

In this paper, we will obtain a matrix Rodrigues type formula for classical orthogonal polynomials in extended sense. Denoting by \(\Phi^{(n)}\) the second kind Kronecker power of the matrix \(\Phi\) (see Bellman [2]), we will show, under some hypothesis, that the expression,
\[
Q^t_n = \frac{1}{\omega} \text{div}^{(n)}(\Phi^{(n)}\omega), \quad n \geq 0,
\]
provides a classical WOPS, where \(\omega(x, y)\) is a symmetry factor of the PDE (4), and \(\text{div}^{(n)}\) is a \(n\)–th order differential operator.

This formula generalizes in a natural way the Rodrigues formula proved in [16], and [10], and the Rodrigues formula for tensor product of classical orthogonal polynomials in one variable.

Moreover, using our results, we will deduce a matrix Rodrigues formula for classical orthogonal polynomials associated with a non positive definite moment functional whose PDE has a symmetry factor (see example 6, in Section 6). The structure of the paper is as follows. In Section 2 we collect the necessary basic tools. Section 3 and 4, are devoted to introduce classical orthogonal polynomials in two variables and symmetry factors associated with the partial differential equation (4). The matrix Rodrigues formula (5), as well as some examples are studied in Section 6, and finally, the proof of the main result is given in the last section.

2. Orthogonal polynomials in two variables

First, we introduce some notations. Let \(\mathcal{P}\) denote the linear space of real polynomials in two variables, and \(\mathcal{P}_n\) the subspace of polynomials of total degree not greater than \(n\).

Let \(\mathcal{M}_{h \times k}(\mathbb{R})\) and \(\mathcal{M}_{h \times k}(\mathcal{P})\) denote the linear spaces of \(h \times k\) real and polynomial matrices, respectively. When \(h = k\), the second index will be omitted.

Let \(A\) be a matrix, we denote by \(A^t\) its transpose, and by \(\det(A)\) its determinant. As usual, we say that \(A\) is non–singular if \(\det(A) \neq 0\). Furthermore, we introduce \(I_h\) as the identity matrix of dimension \(h\).

Moreover, we define the degree of a matrix of polynomials \(A \in \mathcal{M}_{h \times k}(\mathcal{P})\), as
\[
\deg A = \max\{\deg a_{i,j}(x, y), 1 \leq i \leq h, 1 \leq j \leq k\} \geq 0,
\]
where \(a_{i,j}(x, y)\) denotes the \((i, j)\)–entry of \(A\).
Before discussing our approach, we briefly give some general properties and tools about bivariate orthogonal polynomials. For an exhaustive description of this and another related subjects see, for instance, [4, 9, 10, 11, 12, 13, 16, 17].

Let \( \{ \mu_{h,k} \}_{h,k \geq 0} \) be a double indexed sequence of real numbers, and let \( u : \mathcal{P} \to \mathbb{R} \) be a functional defined by means of the moments \( \mu_{h,k} = \langle u, x^h y^k \rangle, \ h, k = 0, 1, 2, \ldots, \) and extended by linearity. Then, we will say that \( u \) is a moment functional.

Some elementary properties about moment functionals acting over polynomial matrices \( A \in \mathcal{M}_{h \times k} \) and \( B \in \mathcal{M}_{h \times l} \) are given by (see [4, 9, 10, 17]),

\[
\begin{align*}
(1) \quad & \langle u, A \rangle = \langle (u, a_{i,j}) \rangle_{i,j=1}^{h,k} \in \mathcal{M}_{h \times l}(\mathbb{R}), \text{ where } A = (a_{i,j})_{i,j=1}^{h,k}, \\
(2) \quad & \langle Au, B \rangle = \langle u, A^t B \rangle.
\end{align*}
\]

We say that a polynomial \( p(x, y) \in \mathcal{P}_n \) is orthogonal with respect to \( u \) if

\[
\langle u, pq \rangle = 0, \quad \forall q \in \mathcal{P}, \quad \text{deg } q < \text{deg } p.
\]

Then, we can define

\[
\mathcal{V}_n = \{ p \in \mathcal{P}_n / \langle u, pq \rangle = 0, \forall q \in \mathcal{P}_{n-1} \}.
\]

A moment functional \( u \) is called quasi definite if \( \dim \mathcal{V}_n = n + 1 \).

**Definition 2.1 ([9]).** A polynomial system (PS) is a vector sequence \( \{ \mathcal{P}_n \}_{n \geq 0} \) such that

\[
\mathcal{P}_n = \{ P_{n,0}, P_{n-1,1}, \ldots, P_{0,n} \} \in \mathcal{M}(n+1) \times 1(\mathcal{P}_n),
\]

where \( \{ P_{n,0}, P_{n-1,1}, \ldots, P_{0,n} \} \) are polynomials of total degree \( n \) independent modulus \( \mathcal{P}_{n-1} \).

Observe that a PS is a sequence of vectors whose dimension and total degree are increasing: \( \mathcal{P}_0 \) is a constant, \( \mathcal{P}_1 \) is a column vector of dimension 2 of bivariate polynomials of total degree 1, \( \mathcal{P}_2 \) is a column vector of dimension 3 whose elements are bivariate polynomials of total degree 2, and so on.

**Definition 2.2 ([9]).** We will say that a PS \( \{ \mathcal{P}_n \}_{n \geq 0} \) is a weak orthogonal polynomial system (WOPS) with respect to a moment functional \( u \) if

\[
\begin{align*}
\langle u, \mathcal{P}_n \mathcal{P}_m^t \rangle &= 0, \quad n \neq m, \\
\langle u, \mathcal{P}_n \mathcal{P}_n^t \rangle &= H_n, \quad n = 0, 1, 2, \ldots
\end{align*}
\]

where \( H_n \in \mathcal{M}_{n+1}(\mathbb{R}) \) is a non–singular matrix.

In the particular case where \( H_n \) is a diagonal matrix, we will say that the WOPS \( \{ \mathcal{P}_n \}_{n \geq 0} \) is an orthonormal polynomial system (OPS). Moreover, if \( H_n = I_{n+1} \), we call \( \{ \mathcal{P}_n \}_{n \geq 0} \) an orthonormal polynomial system. A moment functional \( u \) is quasi definite if and only if there exists a WOPS with respect to \( u \) ([9]).

In addition, a WOPS is called a monic WOPS if every polynomial contains only one monic term of higher degree, that is,

\[
P_{h,k}(x, y) = x^h y^k + R(x, y), \quad h + k = n,
\]

where \( R(x, y) \in \mathcal{P}_{n-1} \). And finally, we have that for a quasi definite moment functional \( u \), there exists a unique monic WOPS associated with \( u \).

In this paper, we will need some differentiation tools. In fact, we will use the gradient operator \( \nabla \), and the divergence operator \( \text{div} \), defined as usual. The extension
of this operators for matrices is introduced in [3, 6, 7, 8]. Let \( A, B_0, B_1 \in \mathcal{M}_{h \times h}(\mathcal{P}) \) be polynomial matrices. We define

\[
\nabla A = \begin{pmatrix} \partial_x A \\ \partial_y A \end{pmatrix} \in \mathcal{M}_{2h \times k}(\mathcal{P}), \quad \text{div} \begin{pmatrix} B_0 \\ B_1 \end{pmatrix} = \partial_x B_0 + \partial_y B_1 \in \mathcal{M}_{h \times k}(\mathcal{P}).
\]

We extend these definitions for \( n \geq 1 \). In fact, writing \( \nabla^{(1)} = \nabla \), and \( \text{div}^{(1)} = \text{div} \), if we denote \( \mathcal{D}^i_k = (^{n}_{i}) \partial_x^{n-i} \partial_y^i, \ i = 0, 1, \ldots, n \), we can introduce the differential operators \( \nabla^{(n)} \) and \( \text{div}^{(n)} \) by means of

\[
\nabla^{(n)} A = (\mathcal{D}_0^n A, \mathcal{D}_1^n A, \ldots, \mathcal{D}_n^n A)^t \in \mathcal{M}_{((n+1)h) \times k}(\mathcal{P}),
\]

\[
\text{div}^{(n)} (B_0, B_1, \ldots, B_n)^t = \sum_{i=0}^{n} \mathcal{D}_i^n B_i \in \mathcal{M}_{h \times k}(\mathcal{P}),
\]

where \( A, B_0, B_1, \ldots, B_n \in \mathcal{M}_{h \times k}(\mathcal{P}) \) are polynomial matrices. In addition, we establish \( \nabla^{(0)} A = A \), and \( \text{div}^{(0)} A = A \).

The previous definitions can be translated to the linear space of moment functionals using duality. For \( n \geq 0 \), we define the \( n \)-th distributional gradient operator and the \( n \)-th distributional divergence operator acting over moment functionals in the following way

\[
\langle \nabla^{(n)} u, \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{pmatrix} \rangle = (-1)^n \langle u, \text{div}^{(n)} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{pmatrix} \rangle = (-1)^n \sum_{i=0}^{n} \langle u, \mathcal{D}_i^n p_i \rangle,
\]

\[
\langle \text{div}^{(n)} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{pmatrix}, p \rangle = (-1)^n \langle \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{pmatrix}, \nabla^{(n)} p \rangle = (-1)^n \sum_{i=0}^{n} \langle u_i, \mathcal{D}_i^n p \rangle.
\]

3. Classical orthogonal polynomials in two variables

In [3, 6, 7, 8], the authors extended the concept of classical bivariate orthogonal polynomials. In fact, we define classical orthogonal polynomial starting from a matrix partial differential equation with matrix coefficients, a direct generalization of the partial differential equation studied by H. L. Krall and I. M. Sheffer in [14], and P. K. Suetin in [16].

Let \( a(x, y), b(x, y), c(x, y) \) be polynomials in two variables of total degree less than or equal to 2, and let \( d(x, y), e(x, y) \) be polynomials in two variables of total degree less than or equal to 1. Define the partial differential operator \( L \) acting over \( \mathcal{P} \) by means of

\[
L[p] = a \partial_{xx} p + 2b \partial_{xy} p + c \partial_{yy} p + d \partial_x p + e \partial_y p.
\]

The operator \( L[\cdot] \) preserves the degree of the polynomials, that is, \( L[\mathcal{P}_n] \subset \mathcal{P}_n \).

Observe that \( L \) is the left hand side of the partial differential equation (2) studied in [14] and [16], without restrictions on the polynomial coefficients.

Define

\[
\Phi = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathcal{M}_{2}(\mathcal{P}_2), \quad \Psi = \begin{pmatrix} d \\ e \end{pmatrix} \in \mathcal{M}_{2 \times 1}(\mathcal{P}_1).
\]
Then, we can write

\[ L[p] = \text{div} (\Phi \nabla p) + \tilde{\Psi}^t \nabla p, \]

where \( \tilde{\Psi} = \Psi - (\text{div} \Phi)^t \).

**Definition 3.1.** Let \( u \) be a quasi definite moment functional, and let \( \{ P_n \}_{n \geq 0} \) be the monic WOPS with respect to \( u \). Then, we say that \( u \) is **classical** (or \( \{ P_n \}_{n \geq 0} \) is a **classical WOPS**), if there exist non–singular matrices \( \Lambda_n \in M_{n+1}(\mathbb{R}) \), such that, for \( n \geq 1 \),

\[ L[P_n] = \text{div} (\Phi \nabla P_n) + \tilde{\Psi}^t \nabla P_n = P_n \Lambda_n, \]

and

\[ \text{det} \langle u, \Phi \rangle \neq 0. \]

Differential equation (9) can be write also for \( n = 0 \), taking \( \Lambda_0 = 0 \).

Observe that, for \( n = 1 \), then \( \nabla P_1 = I_2 \), and equation (9) can be written as

\[ \text{div} \Phi + \tilde{\Psi} = P_1 \Lambda_1 = \Psi. \]

The non–singular character of \( \Lambda_1 \) implies that \( d \) and \( e \) are independent polynomials of exact degree 1.

In [14] and [16], every orthogonal polynomial of total degree \( n \) satisfies the same partial differential equation, and therefore \( \Lambda_n = \lambda_n I_{n+1} \), is a scalar matrix.

Let \( L^* \) be the **formal Lagrange adjoint** of \( L \), defined by means of

\[ L^*[u] = \text{div} (\Phi \nabla u) - \text{div} (\Psi u), \]

then it satisfies \( \langle L^*[u], p \rangle = \langle u, L[p] \rangle \), \( \forall p \in \mathcal{P} \).

Observe that, if \( u \) is classical, then \( L^*[u] = 0 \). In fact, for any \( n \geq 0 \),

\[ \langle L^*[u], P_n \rangle = \langle u, L[P_n] \rangle = \langle u, P_n \rangle \Lambda_n = 0, \]

since \( \Lambda_0 = 0 \), and \( \langle u, P_0 \rangle = 0 \), for \( n \geq 1 \), using the orthogonality of the polynomials.

In [9], the authors obtained several characterizations for bivariate classical orthogonal polynomials. In particular, it is shown that, under some mild regularity conditions, a quasi definite moment functional \( u \) is classical if and only if \( u \) satisfies the **matrix Pearson–equation**

\[ \text{div} (\Phi u) = \Psi^t u, \]

where \( \Phi \) and \( \Psi \) are the same polynomial matrices defined in (8). Equivalently, \( u \) is classical if and only if

\[ \Phi \nabla u = \tilde{\Psi} u. \]

### 4. The Symmetry Factor

As L. L. Littlejohn did in [15], in order to obtain a Rodrigues formula, we consider symmetry factors for the differential operator \( L[\cdot] \).

**Definition 4.1** ([15]). We say that \( L[\cdot] \) is symmetric if \( L[\cdot] = L^*[\cdot] \). \( L[\cdot] \) is symmetrizable if there exists a nontrivial function \( \omega(x,y) \) such that it is \( C^2 \) in some open set, and \( \omega L[\cdot] \) is symmetric. In this case, \( \omega \) is called a symmetry factor for \( L[\cdot] \).
Proposition 4.1. Let $\omega$ be a nontrivial function. Then, $\omega$ is a symmetry factor for $L[\cdot]$ if and only if $\omega$ satisfies the matrix Pearson–type equation \([13]\).

$$\Phi \nabla \omega = \Psi \omega.$$  

Proof. From equation \([13]\), we get

$$\omega L[\cdot] = \omega [\text{div}(\Phi \nabla \omega)] =$$

$$= \omega [\text{div}(\Phi \nabla P^t_n)] =$$

$$= \text{div} [\omega (\Phi \nabla P^t_n) - (\nabla \omega)^t \Phi \nabla P^t_n + \omega \tilde{\Psi}^t \nabla P^t_n] =$$

$$= \text{div} [\omega (\Phi \nabla P^t_n) - (\Phi \nabla \omega - \tilde{\Psi} \omega)^t \nabla P^t_n],$$

and the result follows using \([14]\). \qed

Observe that \([13]\) is equivalent to

$$\text{div} (\Phi \omega) = \Psi^t \omega.$$  

The explicit expression for equations \([15]\) and \([16]\) are

$$\begin{cases}
(a \omega)_x + (b \omega)_y = d \omega, \\
(b \omega)_x + (c \omega)_y = e \omega,
\end{cases} \iff \begin{cases}
a \omega_x + b \omega_y = (d - a_x - b_y) \omega, \\
b \omega_x + c \omega_y = (e - b_x - c_y) \omega.
\end{cases}$$

Obviously, the nontrivial solutions of the above system of partial differential equations give us the symmetry factors for $L[\cdot]$.

Proposition 4.2 \([13]\). Suppose that $ac - b^2 \neq 0$. Then, the differential operator $L[\cdot]$ is symmetrizable if and only if

$$\frac{\partial}{\partial y} \left[ \frac{c(d - a_x - b_y) - b(e - b_x - c_y)}{ac - b^2} \right] = \frac{\partial}{\partial x} \left[ \frac{a(e - b_x - c_y) - b(d - a_x - b_y)}{ac - b^2} \right].$$

In \([10]\), the authors shows that in the Krall and Sheffer case, the existence of a symmetry factor is a necessary condition for the existence of an OPS solution of the PDE \([9]\). A quite similar proof shows that the result is also true in the general case.

Proposition 4.3. If the differential equation \([12]\) has an OPS as solutions, and $ac - b^2 \neq 0$, then $L[\cdot]$ must be symmetrizable.

5. The second kind Kronecker power of a matrix

The second kind Kronecker power is defined in \([2]\) p. 236, for a square matrix $A = (a_{i,j})$ of dimension 2. Let $z = At$ be the linear transformation defined by $A$, that is,

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

$$\Rightarrow \begin{cases}
z_1 = a_{0,0} t_1 + a_{0,1} t_2, \\
z_2 = a_{1,0} t_1 + a_{1,1} t_2.
\end{cases}$$

Each of the $(n + 1)$ homogeneous products $z_1^{n-i}z_2^i$, $i = 0, 1, \ldots, n$, is transformed under \([18]\) into a linear combination of the $(n + 1)$ homogeneous products $t_1^{n-i}t_2^i$, $i = 0, 1, \ldots, n$. Then, the square matrix of order $(n + 1)$ specifying these linear
transformation is known as the second kind n–th Kronecker power of A, which we
denote by A^{(n)}.

Throughout, we will adopt the convention A^{(0)} ≡ 1. Observe that A^{(1)} = A.
Moreover, we can check that

\[ A^{(2)} = \begin{pmatrix}
    a_{0,0}^2 & 2a_{0,0}a_{0,1} & a_{0,1}^2 \\
    a_{0,0}a_{1,0} & (a_{0,0}a_{1,1} + a_{0,1}a_{1,0}) & 2a_{0,1}a_{1,1} \\
    a_{1,0}^2 & 2a_{1,0}a_{1,1} & a_{1,1}^2
\end{pmatrix}. \]

It is possible to give the explicit expression for the entries of A^{(n)}, n ≥ 0. In fact,
if we denote A^{(n)} = (a_{i,j}^{(n)})_{i,j=0}^{n}, then, a direct calculation shows that

\[ a_{i,j}^{(n)} = \sum_{k=0}^{j} \binom{n-i}{k} \binom{i}{j-k} a_{0,0}^{n-i-k} a_{0,1}^{i-j+k} a_{1,1}^{j-k}, \quad 0 ≤ i, j ≤ n, \tag{19} \]

where, as usual, \( \binom{m}{l} = 0 \), if \( m < l \).

From the above explicit expression we can obtain recurrence formulas for the
second kind Kronecker power of a matrix A.

**Lemma 5.1.** There exist two recurrence formulas for A^{(n)}, n ≥ 1. In fact, for
\( 0 ≤ j ≤ n \), we get

**Recurrence I:**

\[
\begin{align*}
    a_{i,j}^{(n)} &= a_{0,0} a_{i,j}^{(n-1)} + a_{0,1} a_{i,j-1}^{(n-1)}, \quad 0 ≤ i ≤ n - 1, \\
    a_{n,j}^{(n)} &= a_{1,0} a_{n-1,j} + a_{1,1} a_{n-1,j-1},
\end{align*}
\]

**Recurrence II:**

\[
\begin{align*}
    a_{0,i}^{(n)} &= a_{0,0} a_{0,i}^{(n-1)} + a_{0,1} a_{0,i-1}^{(n-1)}, \\
    a_{1,i}^{(n)} &= a_{1,0} a_{1,i-1}^{(n-1)} + a_{1,1} a_{i-1,j-1}, \quad 1 ≤ i ≤ n.
\end{align*}
\]

In a matrix form, we can express the above formulas as follows

**Recurrence I:**

\[
A^{(n)} = \begin{pmatrix}
    a_{0,0} & \cdots & 0 \\
    \vdots & \ddots & \vdots \\
    0 & \cdots & a_{1,0}
\end{pmatrix} A^{(n-1)} L_{n-1}^{0} + \begin{pmatrix}
    a_{0,1} & \cdots & 0 \\
    \vdots & \ddots & \vdots \\
    0 & \cdots & a_{1,1}
\end{pmatrix} A^{(n-1)} L_{n-1}^{1}.
\]

**Recurrence II:**

\[
A^{(n)} = \begin{pmatrix}
    a_{0,0} & \cdots & 0 \\
    a_{1,0} & \cdots & 0 \\
    0 & \cdots & a_{1,0}
\end{pmatrix} A^{(n-1)} L_{n-1}^{0} + \begin{pmatrix}
    a_{0,1} & \cdots & 0 \\
    a_{1,1} & \cdots & 0 \\
    0 & \cdots & a_{1,1}
\end{pmatrix} A^{(n-1)} L_{n-1}^{1},
\]

where \( L_{n-1}^{k} \), \( k = 0, 1 \), are \( n \times (n + 1) \) matrices defined as

\[
L_{n-1}^{0} = \begin{pmatrix}
    1 & \cdots & 0 \\
    \vdots & \ddots & \vdots \\
    0 & \cdots & 1
\end{pmatrix}, \quad \text{and} \quad L_{n-1}^{1} = \begin{pmatrix}
    0 & 1 & \cdots \\
    \vdots & \ddots & \vdots \\
    0 & \cdots & 1
\end{pmatrix}.
\]
6. The matrix Rodrigues–type formula

In this Section, we shall assume that the differential operator $L[\cdot]$ is symmetrizable, and we denote by $\omega$ a nontrivial symmetry factor of $L[\cdot]$. Our main result is stated in the following theorem.

**Theorem 6.1.** Let $u$ be a classical moment functional, and let $L[\cdot]$ be the matrix partial differential equation associated with $u$. Let $\omega$ be a nontrivial symmetry factor of $L[\cdot]$, and let us assume that there exist polynomial matrices $\Psi_0, \Psi_1 \in \mathcal{M}_2(\mathcal{P}_1)$, such that

$$
(a \Phi)_x + (b \Phi)_y = \Phi \Psi_0,
$$

$$
(b \Phi)_x + (c \Phi)_y = \Phi \Psi_1.
$$

Then, for $n \geq 0$, the expression

$$
Q_n = \frac{1}{\omega} \text{div}^{(n)}(\Phi^{(n)} \omega), \quad n \geq 0,
$$

provides a $1 \times (n + 1)$ polynomial vector of degree $n$, such that

$$
\langle u, Q_n p \rangle = 0, \quad \forall p \in \mathcal{P}_{n-1}.
$$

Moreover, if $\{Q_n\}_n$ is a PS, then it is a WOPS associated with $u$.

As we will show later, we can establish sufficient conditions in order to obtain WOPS from the matrix Rodrigues formula.

**Corollary 6.2.** In the hypothesis of Theorem 6.1, if $\Phi$ is a diagonal matrix, then $\{Q_n\}_n$ is a WOPS associated with $u$.

**Corollary 6.3.** In the hypothesis of Theorem 6.1, if the $(n + 1)$ square matrix

$$
\langle u, \Phi^{(n)} \rangle,
$$

is non–singular, $n \geq 0$, then $\{Q_n\}_n$ is a WOPS associated with $u$.

Next, we will study some particular cases. Moreover, we will deduce that the Rodrigues–type formula given by P. K. Suetin ([16]) and revisited in Y. J. Kim et al. ([13]) is a particular case of the matrix Rodrigues–type formula (22).

**Example 1: The diagonal case**

Let us assume that $\Phi$ is diagonal, i.e., $b \equiv 0$. In this case, the matrix Pearson–type equation (21) reduces to

$$
(a \omega)_x = d \omega, \quad (c \omega)_y = e \omega,
$$

and the $2 \times 2$ matrices $\Psi_k = (\psi_{i,j}^k)_{i,j=0}^1, k = 0, 1$, in (21) are polynomials of degree less than or equal to 1 satisfying

$$
(a \Phi)_x = \Phi \Psi_0, \quad (c \Phi)_y = \Phi \Psi_1.
$$

This condition holds if we take $\psi_{0,0}^1 = 0$, $\psi_{0,0}^0 = 2 a_x$, $\psi_{1,1}^1 = 2 c_y$, and $\psi_{1,1}^0$, such that

$$
a_x c + c c_x = c \psi_{1,1}^0, \quad a_y c + a c_y = a \psi_{0,0}^1.
$$

In this way, if, for instance, the expressions

$$
\frac{a c_x}{c}, \quad \text{and} \quad \frac{a y c}{a},
$$

is rotationally valid.
are polynomials of total degree less than or equal to 1, then (21) holds. This is possible, for example, when \( a \) and \( c \) are equal up to a multiplicative constant factor, or \( a_y = c_x = 0 \), that is, the polynomial \( a \) only depends on \( x \), and \( c \) only depends on \( y \). This last situation corresponds to the case where the moment functional is a tensor product of univariate moment functionals.

Moreover, the second kind \( n \)-th order Kronecker power of \( \Phi \), is again a diagonal matrix whose elements are given by

\[
\phi_{i,i}^{(n)} = a^{n-i} c^i, \quad i = 0, 1, \ldots, n,
\]

and the Rodrigues formula gives

\[
Q_n = (Q_{n,0}, Q_{n-1,1}, \ldots, Q_{0,n})^t,
\]

where

\[
Q_{n-i,i}(x,y) = \frac{1}{\omega} \binom{n}{i} \frac{\partial_x^{n-i}}{\partial_x^i} (a^{n-i} c^i \omega), \quad 0 \leq i \leq n.
\]

In addition, from Corollary 6.2, we deduce that \( \{Q_n\}_n \) is a WOPS associated to \( u \).

**Example 2: Tensor product of classical orthogonal polynomials in one variable**

Tensor product of two families of classical orthogonal polynomials in one variable (Hermite, Laguerre, Jacobi or Bessel), \( \{R_h\}_{h \geq 0} \) and \( \{S_k\}_{k \geq 0} \) is defined by means of

\[
P_{h,k}(x,y) = R_h(x) S_k(y), \quad h, k \geq 0.
\]

These families of two-dimensional polynomials are classical according to our definition (see [6]). The symmetry factor for the partial differential equation (9) is given by

\[
\omega(x,y) = \omega_1(x) \omega_2(y),
\]

where \( \omega_1(x) \) and \( \omega_2(y) \) are symmetry factors of the differential equations for the polynomials \( \{R_h(x)\}_{h \geq 0} \) and \( \{S_k(y)\}_{k \geq 0} \), respectively. This is a particular case of Example 1, since

\[
\Phi = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \quad \Psi = \begin{pmatrix} d \\ e \end{pmatrix},
\]

and

\[
(a \omega_1)_x = d \omega_1, \quad (c \omega_2)_y = e \omega_2,
\]

are the respective Pearson equations for the univariate polynomials. Observe that \( a_y = d_y = 0 \), that is, \( a \) and \( d \) only depend on \( x \); and \( c_x = e_x = 0 \), that is, \( c \) and \( e \) only depend on \( y \). Then, (21) holds and the Rodrigues formula gives

\[
Q_n = (Q_{n,0}, Q_{n-1,1}, \ldots, Q_{0,n})^t,
\]

where

\[
Q_{n-i,i}(x,y) = \binom{n}{i} \frac{1}{\omega_1} \frac{\partial_x^{n-i}}{\partial_x^i} (a^{n-i} \omega_1) \frac{1}{\omega_2} \frac{\partial_y^i}{\partial_y^i} (c^i \omega_2), \quad 0 \leq i \leq n.
\]

Observe that this expression provides \( Q_{n-i,i}(x,y) \) as the product of the one dimensional Rodrigues formulas for \( \{R_h\}_{h \geq 0} \) and \( \{S_k\}_{k \geq 0} \) (see, for example, [3]), up to a constant factor.

General tensor product of univariate classical orthogonal polynomials is not included in the Krall and Sheffer ([14]), and Suetin ([16]) classifications for the bivariate classical orthogonal polynomials, except for Hermite and Laguerre families.
Example 3: The Suetin–Rodrigues formula

Under several conditions, P. K. Suetin [16] and Y. J. Kim et al [10] proved a Rodrigues formula for some classical orthogonal polynomials in two variables in the Krall and Sheffer sense. We will show that this Rodrigues formula is again a particular case of the matrix Rodrigues formula given in (22).

By elementary algebraic manipulations, equation (17) can be reduced to the form

\[
\begin{pmatrix}
\alpha & 0 \\
0 & \alpha
\end{pmatrix}
\begin{pmatrix}
\omega_x \\
\omega_y
\end{pmatrix}
= \begin{pmatrix}
\beta \\
\gamma
\end{pmatrix}
\omega
\Rightarrow
\begin{cases}
\alpha \omega_x = \beta \omega, \\
\alpha \omega_y = \gamma \omega,
\end{cases}
\]

where \(\alpha = \det \Phi, \beta = c(d - ax - by) - b(e - bx - cy),\) and \(\gamma = -b(d - ax - by) + a(e - bx - cy).\)

In the Suetin case, the special elections for the polynomials \(a, b, c, d\) and \(e,\) makes \(\text{deg}(\alpha) \leq 3, \text{deg}(\beta) \leq 2,\) and \(\text{deg}(\gamma) \leq 2.\) Then, if we assume

\[a = a_1 a_2, \quad b = a_1 b_1 c_1, \quad c = c_1 c_2,\]

where \(a_1 \neq 0, c_1 \neq 0.\) Then

\[\alpha = a_1 c_1 \alpha_0, \quad \beta = c_1 \beta_0, \quad \gamma = a_1 \gamma_0,\]

and \(\omega\) satisfies the following system of partial differential equations

\[p \omega_x = \beta_0 \omega, \quad q \omega_y = \gamma_0 \omega,\]

where \(p = a_1 \alpha_0,\) and \(q = c_1 \alpha_0.\)

In [10] and [17], the authors prove that if \((a_1)_y = (c_1)_x = 0, \text{deg}(p), \text{deg}(q) \leq 2,\) and \(\text{deg}(\beta_0), \text{deg}(\gamma_0) \leq 1,\) then the sequence \(\{P_n\}_n,\) with

\[P_n = (P_{n,0}, P_{n-1,1}, \ldots, P_{0,n})^t,\]

defined by means of

\[P_{n-i,i}(x,y) = \frac{1}{\omega} \partial_z^{n-i} \partial_y^i (p^{n-i} q^i \omega),\]

provides a WOPS satisfying the partial differential equation (2).

This is again a particular case of Example 1, since Suetin hypothesis \((a_1)_y = (c_1)_x = 0\) implies (21). Matrix Rodrigues formula (22) provides, for \(n \geq 0,\) a WOPS

\[Q_n = (Q_{n,0}, Q_{n-1,1}, \ldots, Q_{0,n})^t,\]

where

\[Q_{n-i,i}(x,y) = \frac{1}{\omega} \binom{n}{i} \partial_z^{n-i} \partial_y^i (p^{n-i} q^i \omega), \quad 0 \leq i \leq n.\]

Observe that the above expression coincides with Suetin–Rodrigues formula up to the binomial coefficients.

Example 4: Orthogonal polynomials on the unit ball

Classical orthogonal polynomials on the unit ball, \(B_2 = \{(x,y) : x^2 + y^2 \leq 1\},\) are associated with the weight function (symmetry factor)

\[\omega(x,y) = (1 - x^2 - y^2)^{\mu - 1/2}, \quad \mu > -1/2.\]

In this case, the matrices \(\Phi\) and \(\Psi\) are given by

\[\Phi = \begin{pmatrix}
\frac{x^2 - 1}{x} & x \\
x & \frac{y^2 - 1}{y}
\end{pmatrix}, \quad \Psi = \begin{pmatrix}
(2\mu + 2)x \\
(2\mu + 2)y
\end{pmatrix},\]
and the matrix Pearson–type equation for the symmetry factor $\omega$ is given by

$$(x^2 - 1 \ x \ y \ y^2 - 1) \begin{pmatrix} \omega_x \\ \omega_y \end{pmatrix} = \begin{pmatrix} (2\mu - 1)x \\ (2\mu - 1)y \end{pmatrix} \omega.$$ 

We can check that condition (21) is not true, and matrix Rodrigues formula does not provide a WOPS. However, we can transform the above matrix Pearson–type equation and obtain

$$(1 - x^2 - y^2 \ 0 \ 0 \ 1 - x^2 - y^2) \begin{pmatrix} \omega_x \\ \omega_y \end{pmatrix} = \begin{pmatrix} -(2\mu - 1)x \\ -(2\mu - 1)y \end{pmatrix} \omega.$$ 

Equation (25) is a Pearson–type equation for the symmetry factor $\omega$, it is diagonal, and it satisfies condition (24). Then, matrix Rodrigues formula provides a WOPS defined as follows

$$Q_n^i = (Q_{n,0}, Q_{n-1,1}, \ldots, Q_{0,n})^t,$$

where

$$Q_{n-i,i}(x, y) = \frac{1}{\omega} \binom{n}{i} \partial_x^{n-i} \partial_y^i ((1 - x^2 - y^2)^n \omega), \quad 0 \leq i \leq n,$$

which coincides with the Rodrigues formula for the orthogonal polynomials on the unit ball as described in [4, 10, 16].

**Example 5: Orthogonal polynomials on the simplex (Appell polynomials)**

The weight function associated with the classical orthogonal polynomials on the simplex, $T = \{(x, y) : x, y \geq 0, 1 - x - y \geq 0\}$, is defined by

$$\omega(x, y) = x^\alpha y^\beta (1 - x - y)^\gamma, \quad \alpha, \beta, \gamma > -1.$$ 

Here, the matrices $\Phi$ and $\Psi$ are given by

$$\Phi = \begin{pmatrix} x(x-1) & x \ y \ y(y-1) \end{pmatrix}, \quad \Psi = \begin{pmatrix} (\alpha + \beta + \gamma + 3)x - (\alpha + 1) \\ (\alpha + \beta + \gamma + 3)y - (\beta + 1) \end{pmatrix}. $$

Observe that condition (21) holds, and therefore, the matrix Rodrigues formula for triangle polynomials provides a WOPS. Since the obtained polynomials are monic, they coincide with the so–called second kind Appell polynomials (see [16]).

Y. J. Kim et al (see [10]) proved that $\omega$ satisfies also a diagonal matrix Pearson–type equation

$$(x(1-x-y) \ 0 \ 0 \ y(1-x-y)) \begin{pmatrix} \omega_x \\ \omega_y \end{pmatrix} = \begin{pmatrix} (\alpha - x - y - \gamma x) \\ (\beta(1-x-y) - \gamma y \end{pmatrix} \omega.$$ 

In this way, condition (21) holds, and we are again in the situation described in Example 1. Then, we get

$$Q_n^i = (Q_{n,0}, Q_{n-1,1}, \ldots, Q_{0,n})^t,$$

defined from the matrix Rodrigues formula

$$(27) \quad Q_{n-i,i}(x, y) = \frac{1}{\omega} \binom{n}{i} \partial_x^{n-i} \partial_y^i ((1 - x - y)^n \omega), \quad 0 \leq i \leq n,$$

is a WOPS relative to $\omega$.

Rodrigues formula for the circle and the triangle polynomials, (26) and (27) respectively, coincide with the classical expressions for these polynomials which appear in the literature (see, for instance, [1, 4, 10, 11, 16]).
Example 6: The most intriguing case (sic, L. L. Littlejohn, [15])

Krall and Sheffer ([14]) showed that the differential equation

\[ L[p] \equiv 3y p_{xx} + 2p_{xy} - xp_x - yp_y = -np. \]

has an OPS as solutions. In this case, the matrices \( \Phi \) and \( \Psi \) are given by

\[ \Phi = \begin{pmatrix} 3y & 1 \\ 1 & 0 \end{pmatrix}, \quad \Psi = \begin{pmatrix} -x \\ -y \end{pmatrix}. \]

A symmetry factor for the partial differential equation is \( \omega(x, y) = \exp(y^3 - xy) \) ([13]). Observe that condition (21) is satisfied, and then we can obtain a matrix Rodrigues formula for these polynomials. In particular, we get

\[ \begin{align*}
Q_0^1 &= 1,
Q_1^1 &= (-x, -y),
Q_2^1 &= (x^2 - 6y, 2xy - 2, y^2),
Q_3^1 &= (-x^3 + 18xy - 12, -3x^2y + 18y^2 + 6x, -3xy^2 + 6y, -y^3).
\end{align*} \]

We must remark that Rodrigues–type formula for these polynomials can not be obtained using the Suetin tools (see [10]). Moreover, the symmetry factor \( \omega \) is not a weight function. These polynomials has attracted considerable attention ([9, 10, 14, 15, 16]) since this is the simplest case of classical orthogonal polynomials in two variables orthogonal with respect to a non positive definite moment functional.

7. Proof of the Main Result

In this section, we will prove the matrix Rodrigues formula, introduced in Theorem 6.1 in several steps that we will organize in a series of lemmas.

Lemma 7.1. Let \( A = (a_{i,j})_{i,j=0}^{1} \) be a \( 2 \times 2 \) polynomial matrix. Assume that, there exist polynomial matrices \( \Psi_k = (\psi_{i,j}^k)_{i,j=0}^{1}, \) such that

\[ (a_{k,0} A_x) + (a_{k,1} A_y) = A \Psi_k, \quad k = 0, 1. \]

Then, for \( n \geq 1, \) we have

\[ (a_{k,0} A^{(n)})_x + (a_{k,1} A^{(n)})_y = A^{(n)} \Psi^n_k, \quad k = 0, 1, \]

where \( \Psi^n_k = (\psi_{i,j}^{n,k})_{i,j=0}^{n} \in \mathcal{M}_{n+1}(P), \) \( k = 0, 1, \) are three–diagonal matrices with

\[ \begin{align*}
\psi_{n-1,j}^{n,k} &= (n + 1 - j) \psi_{0,1}^{n,k},
\psi_{n,j}^{n,k} &= (n - j) \psi_{0,0}^{n,k} + j \psi_{1,1}^{n,k} - (n - 1) \psi_{0,0}^{n,k} + (a_{k,0})_x + (a_{k,1})_y, 
\psi_{j+1,j}^{n,k} &= (j + 1) \psi_{1,0}^{n,k},
\end{align*} \]

1 \( \leq j \leq n, \)

0 \( \leq j \leq n, \)

0 \( \leq j \leq n - 1. \)

Moreover, \( \deg \Psi^n_k \leq \deg A - 1, \quad k = 0, 1, \quad n \geq 1. \)

Proof. The Lemma follows by induction on \( n. \) For \( n = 1, \) the result holds using \( A^{(1)} = A, \) and \( \Psi_0^1 = \Psi_1^1, \ k = 0, 1. \)

Let \( 0 \leq i \leq n - 1, \) and \( 0 \leq j \leq n, \) and assume that the result holds for \( n - 1. \) First, we compute the left hand side of (29) using the induction hypothesis, and Recurrence I of Lemma 6.1.
\[
\begin{align*}
(a_0, a_{i,j})_x + (a_1, a_{i,j})_y &= \\
&= \left( a_0, \left[ a_0 a_{i,j}^{(n-1)} + a_1 a_{i,j}^{(n-1)} \right] \right)_x + \left( a_1, \left[ a_0 a_{i,j}^{(n-1)} + a_1 a_{i,j}^{(n-1)} \right] \right)_y \\
&= a_0 \left[ (a_0 a_{i,j}^{(n-1)})_x + (a_0, a_{i,j}^{(n-1)})_y \right] + \left[ (a_0, a_{i,j}^{(n-1)})_x a_0 + (a_0, a_{i,j}^{(n-1)})_y a_1 \right] a_{i,j}^{(n-1)} \\
&+ a_1 \left[ (a_0 a_{i,j}^{(n-1)})_x + (a_1 a_{i,j}^{(n-1)})_y \right] + \left[ (a_1, a_{i,j}^{(n-1)})_x a_0 + (a_1, a_{i,j}^{(n-1)})_y a_1 \right] a_{i,j}^{(n-1)} \\
&= a_0 \sum_{l=j-2}^{j+1} a_{i,l}^{(n-1)} \psi_{i,j}^{n-1,0} + \left\{ a_0, 0, (a_0, a_{i,j}^{(n-1)})_x + (a_1, a_{i,j}^{(n-1)})_y \right\} a_{i,j}^{(n-1)} \\
&+ a_1 \sum_{l=j-2}^{j+1} a_{i,l}^{(n-1)} \psi_{i,j}^{n-1,0} + \left\{ a_0, 0, (a_0, a_{i,j}^{(n-1)})_x + (a_1, a_{i,j}^{(n-1)})_y \right\} a_{i,j}^{(n-1)}.
\end{align*}
\]

Now, replace the recurrence relations for \( \psi_{i,j}^{n-1,0} \), and Recurrence I for the elements \( a_{i,j}^{(n)} \) to obtain

\[
\begin{align*}
(a_0, a_{i,j})_x + (a_1, a_{i,j})_y &= a_{i,j}^{(n-1)} \psi_{i,j}^{n-1,0} + a_{i,j}^{(n-1)} \psi_{i,j}^{n-1,0} + a_{i,j}^{(n-1)} \psi_{i,j}^{n-1,0}.
\end{align*}
\]

The rest of the cases follows in a similar way. \( \square \)

**Lemma 7.2.** Assume that the matrix polynomial \( \Phi \) satisfies [24]. Let \( A_n \in M_{(n+1) \times (n+1)}(\mathbb{P}) \) be an arbitrary polynomial matrix. Then,

\[
\text{div}^{(n)}(\Phi^{(n)} A_n \omega) = \text{div}^{(n-1)}(\Phi^{(n-1)} A_{n-1} \omega),
\]

where \( A_{n-1} \in M_{n \times (n+1)}(\mathbb{P}) \) is a polynomial matrix satisfying

\[
\deg A_{n-1} \leq \deg A_n + 1.
\]

**Proof.** From the definition of \( \text{div}^{(n)} \) in [24], we can easily deduce

\[
\text{div}^{(n)}(\Phi^{(n)} A_n \omega) = \text{div}^{(n-1)} \left[ \partial_x(L_{n-1}^{0} \Phi^{(n)} A_n \omega) + \partial_y(L_{n-1}^{1} \Phi^{(n)} A_n \omega) \right].
\]

Now, if we apply recurrence formulas in Lemma 5.1 we get

\[
\begin{align*}
\partial_x(L_{n-1}^{0} \Phi^{(n)} A_n \omega) + \partial_y(L_{n-1}^{1} \Phi^{(n)} A_n \omega) &= \left[ \left( a \Phi^{(n-1)} \right)_x + \left( b \Phi^{(n-1)} \right)_y \right] L_{n-1}^{0} A_n \omega \\
&+ \left[ \left( b \Phi^{(n-1)} \right)_x + \left( c \Phi^{(n-1)} \right)_y \right] L_{n-1}^{1} A_n \omega \\
&+ \Phi^{(n-1)} \left[ \left( a L_{n-1}^{0} + b L_{n-1}^{1} \right)(A_n)_x + \left( b L_{n-1}^{0} + c L_{n-1}^{1} \right)(A_n)_y \right] \omega \\
&+ \Phi^{(n-1)} \left[ L_{n-1}^{0} A_n(a \omega_x + b \omega_y) + L_{n-1}^{1} A_n(b \omega_x + c \omega_y) \right].
\end{align*}
\]
Condition (21), Lemma 7.3, and equation (15) give
\[
\text{div}^{(n)}(\Phi^{(n)} A_n \omega) = \text{div}^{(n-1)}(\Phi^{(n-1)} A_{n-1} \omega),
\]
where
\[
A_{n-1} = \left\{ \Psi_0^{n-1} + (d - a_x - b_y) I_n L_{n-1}^0 + [\Psi_1^{n-1} + (e - b_x - c_y) I_n L_{n-1}^0] A_n + (a L_{n-1}^0 + b L_{n-1}^1) (A_n x) + (b L_{n-1}^0 + c L_{n-1}^1) (A_n y) \right\}.
\]
Observe that the degree condition can be deduced from the explicit expression for \( A_{n-1} \).

Now, taking \( A_n = I_{n+1} \), and applying induction on \( n \), we get

**Lemma 7.3.** In the hypothesis of Lemma 7.3, for any \( n \geq 0 \),
\[
Q_n^t := \frac{1}{\omega} \text{div}^{(n)}(\Phi^{(n)} \omega),
\]
is a \( 1 \times (n + 1) \) polynomial matrix of degree less than or equal to \( n \).

Using the same technique as Lemma 7.3, we can write \( Q_n^t \) in terms of the moment functional \( u \).

**Lemma 7.4.** Let \( u \) be a classical moment functional, and let \( Q \) be the matrix partial differential equation associated with \( u \). Assume that condition (21) holds. Then, for \( n \geq 0 \), \( Q_n \) defined in Lemma 7.3 satisfies
(i) \( Q_n^t u = \text{div}^{(n)}(\Phi^{(n)} u) \), \( n \geq 0 \),
(ii) \( \langle u, Q_n X_m^t \rangle = 0 \), \( 0 \leq m \leq n - 1 \), where \( X_m^t = (x^{m-i} y^i)_{i=0}^m \),
(iii) \( \{Q_n^t\}_n \) is a \( 1 \times (n + 1) \) vector of polynomials of degree less than or equal to \( n \). In fact, if we denote
\[
Q_n^t = (Q_{n,0}, Q_{n-1,1}, \ldots, Q_{0,n}),
\]
then \( Q_{n-i,i} \) has exact degree \( n \) or \( Q_{n-i,i} \equiv 0 \), for \( 0 \leq i \leq n \).

**Proof.** (i) follows from a similar reasoning as used in Lemma 7.3.

In order to prove (ii), we compute
\[
\langle u, Q_n X_m^t \rangle = \langle Q_n^t u, X_m^t \rangle = \langle \text{div}^{(n)}(\Phi^{(n)} u), X_m^t \rangle = (-1)^n \langle \Phi^{(n)} u, \nabla^{(n)} X_m^t \rangle = 0, \quad 0 \leq m \leq n - 1.
\]
As a consequence, \( \langle u, Q_n p(x,y) \rangle = 0 \), for any \( p(x,y) \in P_{n-1} \).

From the above property, we will prove (iii). Let \( n \geq 1 \), and suppose that there exists \( 0 \leq i \leq n \), such that \( \text{deg} Q_{n-i,i} < n \). Then,
\[
\langle u, Q_n Q_{n-i,i} \rangle = 0,
\]
and therefore \( \langle u, Q_{n-i,i}^2 \rangle = 0 \). Since \( u \) is a quasi definite moment functional, we deduce \( Q_{n-i,i} \equiv 0 \).

To end the Section, we will prove Corollaries 6.2 and 6.3.

**Proof of Corollary 6.2**
Proof. If $\Phi$ is a diagonal matrix, 
$$Q_n^l = (Q_{n,0}, Q_{n-1,1}, \ldots, Q_{0,n}),$$
where
$$Q_{n-k,k}(x, y) u = \binom{n}{k} \partial_x^{n-k} \partial_y^k (a^{n-k} c^k u), \quad 0 \leq k \leq n.$$ 

From the previous Lemma, $Q_{n-k,k}$ has exact degree $n$ or $Q_{n-k,k} \equiv 0$, for $0 \leq k \leq n$. Assume that $Q_{n-k,k} \equiv 0$, then $\partial_x^{n-k} \partial_y^k (a^{n-k} c^k u) = 0$, and therefore $a = 0$ or $c = 0$ which contradicts (10). Moreover,
$$\langle u, Q_{n-k,k}(x, y) x^{n-l} y^l \rangle = (-1)^n \binom{n}{k} \langle a^{n-k} c^k u, \partial_x^{n-k} \partial_y^k (x^{n-l} y^l) \rangle = 0,$$
if $k \neq l$. In the case $k = l$, we have
$$\langle u, Q_{n-k,k}(x, y) x^{n-k} y^k \rangle \neq 0,$$

since $Q_{n-k,k}$ has exact degree $n$.

Finally, we are going to show that the polynomials $\{Q_{n-k,k}, 0 \leq k \leq n\}$ are independent modulo $\mathcal{P}_{n-1}$. Let $\lambda_0, \lambda_1, \ldots, \lambda_n$ be constants such that
$$q(x, y) = \sum_{k=0}^{n} \lambda_k Q_{n-k,k}(x, y),$$
is a polynomial of degree less than or equal to $n - 1$. Since
$$\langle u, q(x, y)p(x, y) \rangle = 0,$$
for any $p \in \mathcal{P}_{n-1}$, we have $\langle u, q(x, y)^2 \rangle = 0$, and therefore $q(x, y) \equiv 0$.

Then, we get
$$0 = \langle u, q(x, y) x^{n-i} y^i \rangle = \sum_{k=0}^{n} \lambda_k \langle Q_{n-k,k}(x, y) u, x^{n-i} y^i \rangle$$
$$= \lambda_i \langle Q_{n-i,i}(x, y) u, x^{n-i} y^i \rangle,$$
so $\lambda_i = 0, 0 \leq i \leq n$. \qed

Proof of Corollary 6.8

Proof. Observe that
$$\langle u, Q_n X_n^l \rangle = \langle Q_n^l u, X_n^l \rangle = \langle \text{div}^{(n)}(\Phi^{(n)} u), X_n^l \rangle =$$
$$= (-1)^n \langle \Phi^{(n)} u, \nabla^{(n)} X_n^l \rangle = (-1)^n n! \langle u, (\Phi^{(n)})^l \rangle,$$
and then, the result follows. \qed

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