In this paper we establish the $L^p$-boundedness properties of the variation operators associated with the heat semigroup, Riesz transforms and commutator between Riesz transforms and multiplication by $BMO(\mathbb{R}^n)$-functions in the Schrödinger setting.

1. Introduction and main results

We consider the time independent Schrödinger operator on $\mathbb{R}^n$, $n \geq 3$, defined by

$$\mathcal{L} = -\Delta + V,$$

where the potential $V$ is nonzero, nonnegative and belongs, for some $q \geq n/2$, to the reverse Hölder class $B_q$, that is, there exists $C > 0$ such that

$$\left( \frac{1}{|B|} \int_B V(x)^q \, dx \right)^{1/q} \leq C \int_B V(x) \, dx,$$

for every ball $B$ in $\mathbb{R}^n$. Since any nonnegative polynomial belongs to $B_q$ for every $1 < q < \infty$, the Hermite operator $-\Delta + |x|^2$ falls under our considerations.

Harmonic analysis associated with the operator $\mathcal{L}$ has been studied by several authors in the last decade. Most of them had, as starting point, the paper of Shen [23]. This author investigated $L^p$-boundedness properties of the Riesz transforms formally defined in the $L^p$-setting by

$$R_L^\tau f(x) = \nabla \mathcal{L}^{\tau/2},$$

where as usual $\nabla$ denotes the gradient operator and the negative square root $\mathcal{L}^{\tau/2}$ of $\mathcal{L}$ is given by the functional calculus as follows

$$\mathcal{L}^{\tau/2} f(x) = \int_{\mathbb{R}^n} K(x,y) f(y) \, dy,$$

being

$$K(x,y) = -\frac{1}{2\pi} \int_\mathbb{R} (-i\tau)^{-1/2} \Gamma(x,y,\tau) \, d\tau.$$

Here, for every $\tau \in \mathbb{R}$, $\Gamma(x,y,\tau)$, $x,y \in \mathbb{R}^n$, represents the fundamental solution for the operator $\mathcal{L} + i\tau$.

Bongioanni, Harboure and Salinas [3] studied the behavior in $L^p$ spaces of the commutator operator $[R_L^\tau, b]$ defined by

$$[R_L^\tau, b] f = b R_L^\tau(f) - R_L^\tau(bf),$$

where $b$ is in an appropriate class containing the space $BMO$ of bounded mean oscillation functions.

The heat semigroup $\{W_t^\mathcal{L}\}_{t>0}$ generated by the operator $-\mathcal{L}$ can be written on $L^2(\mathbb{R}^n)$ as the following integral operator

$$W_t^\mathcal{L}(f)(x) = \int_{\mathbb{R}^n} W_t^\mathcal{L}(x,y) f(y) \, dy, \quad f \in L^2(\mathbb{R}^n).$$

The semigroup $\{W_t^\mathcal{L}\}_{t>0}$ is $C_0$ in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, but it is not Markovian. The main properties of the kernel $W_t^\mathcal{L}(x,y)$, $x,y \in \mathbb{R}^n$, $t > 0$, can be encountered in [11].

Other operators associated with the Schrödinger operator $\mathcal{L}$ have been studied on $L^p(\mathbb{R}^n)$ and on other kind of function spaces also in [1], [2], [11], [12], [16], [26], and [29].
The variation operators were introduced in ergodic theory ([19]) in order to measure the speed of convergence. Suppose that \( \{T_t\}_{t \geq 0} \) is an uniparametric system of linear operators bounded in \( L^p(\mathbb{R}^n) \), for some \( 1 \leq p < \infty \), such that \( \lim_{t \to 0^+} T_t f \) exists in \( L^p(\mathbb{R}^n) \).

If \( p > 2 \), the variation operator \( V_p(T_t) \) is given by

\[
V_p(T_t)(f)(x) = \sup_{\{t_j\}_{j \in \mathbb{N}} > 0} \left( \sum_{j=0}^{\infty} |T_{t_j} f(x) - T_{t_{j+1}} f(x)|^p \right)^{1/p}, \quad f \in L^p(\mathbb{R}^n).
\]

Here the supremum is taken over all the real decreasing sequences \( \{t_j\}_{j \in \mathbb{N}} \) that converge to zero. By \( E_p \) we denote the space that includes all the functions \( w : (0, \infty) \to \mathbb{R} \), such that

\[
\|w\|_{E_p} = \sup_{\{t_j\}_{j \in \mathbb{N}} > 0} \left( \sum_{j=0}^{\infty} |w(t_j) - w(t_{j+1})|^p \right)^{1/p} < \infty.
\]

\( \| \cdot \|_{E_p} \) is a seminorm on \( E_p \). It can be written

\[
V_p(T_t)(f) = \|T_t f\|_{E_p}.
\]

Variation operators for \( C_0 \)-semigroups of operators and singular integrals have been analyzed on \( L^p \)-spaces by Campbell et al. ([6], [7] and [19]). More recently, in [9] and [18] the authors have studied variation operators on \( L^p \)-spaces for semigroups and Riesz transforms in the Ornstein-Uhlenbeck and Hermite settings.

As it was mentioned, for every \( 1 \leq p < \infty \), the semigroup \( \{W_t^{x'}\}_{t > 0} \) is \( C_0 \) in \( L^p(\mathbb{R}^n) \), that is, for every \( f \in L^p(\mathbb{R}^n) \), \( W_t^{x'}(f) \to f \), as \( t \to 0^+ \), in \( L^p(\mathbb{R}^n) \). The \( L^p \)-boundedness properties of the oscillation and variation operators for \( \{W_t^{x'}\}_{t > 0} \) are established in the following.

**Theorem 1.1.** Let \( p > 2 \). Then, the variation operator \( V_p(W_t^{x'}) \) is bounded from \( L^p(\mathbb{R}^n) \) into itself, for every \( 1 < p < \infty \), and from \( L^1(\mathbb{R}^n) \) into \( L^{1,\infty}(\mathbb{R}^n) \).

Note that, since \( \{W_t^{x'}\}_{t > 0} \) is not Markovian, none part of Theorem 1.1 can be deduced from [20] Theorem 3.3.

According to standard ideas, Shen in [23] actually defined (although he did not write it in this way), for every \( \ell = 1, \ldots, n \), the \( \ell \)-th Riesz transform in the \( \mathcal{L} \)-context by

\[
R^{x'}_{\ell} (f)(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y| > \varepsilon} R^{x'}_{\ell} (x, y) f(y) dy, \text{ a.e. } x \in \mathbb{R}^n,
\]

provided that \( f \in L^p(\mathbb{R}^n) \) and either

(i) \( 1 \leq p < \infty \) and \( V \in B_p \); or

(ii) \( 1 \leq p < p_0 = \frac{\ell}{\ell - n}, \) and \( V \in B_q, n/2 \leq q < n. \)

Here, for every \( x, y \in \mathbb{R}^n, x \neq y, \)

\[
R^{x'}_{\ell} (x, y) = -\frac{1}{2\pi} \int_{\mathbb{R}} (-i\tau)^{-1/2} \frac{\partial}{\partial x_{\ell}} \Gamma(x, y, \tau) d\tau.
\]

In the sequel we complete Shen’s result proving that actually the limit in (3) exists and the Riesz transform \( R^{x'} = (R^{x'}_1, \ldots, R^{x'}_n) \) can be represented by (11) on \( C_{c}^{\infty}(\mathbb{R}^n) \), the space of \( C^\infty \)-functions in \( \mathbb{R}^n \) that have compact support.

**Proposition 1.1.** Let \( \ell = 1, \ldots, n. \) Suppose that one of the following two conditions holds:

(i) \( f \in L^p(\mathbb{R}^n), 1 \leq p < \infty, \) and \( V \in B_p \);

(ii) \( f \in L^p(\mathbb{R}^n), 1 \leq p < p_0, \) where \( \frac{1}{p_0} = \frac{1}{q} - \frac{1}{n}, \) and \( V \in B_q, n/2 \leq q < n. \)

Then, there exists the following limit

\[
\lim_{\varepsilon \to 0^+} \int_{|x-y| > \varepsilon} R^{x'}_{\ell} (x, y) f(y) dy, \text{ a.e. } x \in \mathbb{R}^n.
\]

Moreover, if \( f \in C_{c}^{\infty}(\mathbb{R}^n), \mathcal{L}^{-1/2} f, \) as defined in (2), admits partial derivative with respect to \( x_{\ell} \) almost everywhere in \( \mathbb{R}^n \) and

\[
\frac{\partial}{\partial x_{\ell}} \mathcal{L}^{-1/2} f(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y| > \varepsilon} R^{x'}_{\ell} (x, y) f(y) dy, \text{ a.e. } x \in \mathbb{R}^n.
\]
For every $\varepsilon > 0$, the $\varepsilon$-truncation $R^\varepsilon(f)$ of $Rf$ is defined as usual by
\[
R^\varepsilon(f)(x) = \int_{|x-y|>\varepsilon} Rf(x,y)dy, \quad \ell = 1,\ldots,n.
\]

The behavior on $L^p$ spaces of the variation operators associated with the family of truncations \{\(R^\varepsilon\)\} \(\varepsilon>0\), \(\ell = 1,\ldots,n\), is contained in the following result.

**Theorem 1.2.** Let \(\ell = 1,\ldots,n\). Assume that \(\rho > 2\) and that \(V \in B_q\), with \(q \geq n/2\), and \(1 < p < p_0\), where \(\frac{1}{p_0} = \left(\frac{1}{p} - \frac{1}{n}\right)_+\). Then, the variation operator \(V^\ell R^\varepsilon\) is bounded from \(L^p(\rn)\) into itself. Moreover, \(V^\ell R^\varepsilon\) is bounded from \(L^1(\rn)\) into \(L^{1,\infty}(\rn)\).

By $BMO(\rn)$ we denote the usual John-Nirenberg space. A locally integrable function \(b\) on \(\rn\) is in $BMO(\rn)$ if and only if there exists \(C > 0\) such that
\[
\frac{1}{|B|} \int_B |b(x) - b_B| dx \leq C,
\]
for every ball \(B\) in \(\rn\). Here \(b_B = \frac{1}{|B|} \int_B b(x) dx\), where \(B\) is a ball in \(\rn\). For \(f \in BMO(\rn)\) we define
\[
\|f\|_{BMO(\rn)} = \sup_B \frac{1}{|B|} \int_B |b(x) - b_B| dx,
\]
where the supremum is taken over all the balls \(B\) in \(\rn\).

For every \(V \in B_{n/2}\), we consider the function $\gamma$ defined by
\[
\gamma(x) = \sup \left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \leq 1 \right\}, \quad x \in \rn.
\]
Under our assumptions it is not hard to see that \(0 < \gamma(x) < \infty\), for all \(x \in \rn\). This function $\gamma$ was introduced in [22] when the potential \(V\) satisfies that
\[
\max_{x \in B} V(x) \leq C \frac{1}{|B|} \int_B V(y) dy,
\]
for every ball \(B\) in \(\rn\), to study the Neumann problem for the operator $\mathcal{L}$ in the region above a Lipschitz graph. The main properties of $\gamma$ were showed in [23, Section 1] (see also [22]). Here, the function $\gamma$ plays a crucial role.

In [23] the space $BMO_\theta(\gamma)$, \(\theta > 0\), was defined as follows. Let \(\theta > 0\). A locally integrable function \(b\) in \(\rn\) is in $BMO_\theta(\gamma)$ provided that
\[
\frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy \leq C \left(1 + \frac{r}{\gamma(x)}\right)^\theta,
\]
for all \(x \in \rn\) and \(r > 0\). We denote for \(b \in BMO_\theta(\gamma)\)
\[
\|b\|_{BMO_\theta(\gamma)} = \sup_{x \in \rn, \ r > 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy \left(1 + \frac{r}{\gamma(x)}\right)^{-\theta}.
\]
Note that $BMO(\rn) = BMO_0(\gamma) \subset BMO_\theta(\gamma) \subset BMO_\theta(\gamma)$, when \(0 \leq \theta \leq \theta'\). We set $BMO_\infty(\gamma) = \bigcup_{\theta > 0} BMO_\theta(\gamma)$. As it is pointed out in [23], $BMO_\infty(\gamma)$ is in general larger than $BMO(\rn)$.

For \(b \in BMO_\infty(\gamma)\) and \(\ell = 1,\ldots,n\), the commutator operator $C^\varepsilon_{b,\ell}$ is defined by
\[
C^\varepsilon_{b,\ell}(f) = bR^\varepsilon_{\ell}(f) - R^\varepsilon_{\ell}(bf), \quad f \in C^\infty_c(\rn).
\]
Note that $bf \in L^1(\rn)$, for every $f \in C^\infty_c(\rn)$ and $b \in BMO_\infty(\gamma)$.

In [23 Theorem 1] it was shown that, for every $b \in BMO_\infty(\gamma)$ and \(\ell = 1,\ldots,n\), the operator $C^\varepsilon_{b,\ell}$ is bounded from $L^p(\rn)$ into itself, provided that \(1 < p < p_0\), where \(\frac{1}{p_0} = \left(\frac{1}{p} - \frac{1}{n}\right)_+\) and $V \in B_q$, \(q \geq n/2\).

In the next result we obtain a pointwise representation of the commutator operator by a principal value integral.
Proposition 1.2. Let $\ell = 1, \ldots, n$. If $b \in BMO_{\infty}(\gamma)$, $V \in B_q$, with $q \geq n/2$, and $f \in L^p(\mathbb{R}^n)$, where $1 < p < p_0$ and $\frac{1}{p_0} = \left(\frac{1}{q} - \frac{1}{n}\right)_+$, then

$$C_{b,\ell}^{L}(f)(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y| < \varepsilon} (b(x) - b(y)) R_{\ell}^{V}(x,y) f(y) dy, \ a.e. \ x \in \mathbb{R}^n.$$ 

For every $b \in BMO_{\infty}(\gamma)$, $\varepsilon > 0$, and $\ell = 1, \ldots, n$, we define the $\varepsilon$-truncation $C_{b,\ell}^{L,\varepsilon}$ of $C_{b,\ell}^{L}$ by

$$C_{b,\ell}^{L,\varepsilon}(f)(x) = \int_{|x-y| < \varepsilon} (b(x) - b(y)) R_{\ell}^{V}(x,y) f(y) dy, \ x \in \mathbb{R}^n.$$ 

The $L^p$-boundedness properties of the variation operators associated with the family of truncations $\{C_{b,\ell}^{L,\varepsilon}\}_{\varepsilon > 0}$ are contained in the following.

Theorem 1.3. Let $\ell = 1, \ldots, n$ and $b \in BMO_{\infty}(\gamma)$. Assume that $V \in B_q$, with $q \geq n/2$, and $1 < p < p_0$, where $\frac{1}{p_0} = \left(\frac{1}{q} - \frac{1}{n}\right)_+$. Then, if $p > 2$, the variation operator $V_p(C_{b,\ell}^{L,\varepsilon})$ is bounded from $L^p(\mathbb{R}^n)$ into itself.

In [23, p. 516] it was proved that if $V$ is a nonnegative polynomial, then $V \in B_q$, for every $1 < q < \infty$. Then, as special cases of our results appear the corresponding ones to the Hermite operator $H = -\Delta + |x|^2$ ([9] and [10]).

This paper is organized as follows. In Section 2 we describe a general procedure that we shall use to prove our main results and we present the $L^p$-properties of the variation operators associated with the classical ($V \equiv 0$) heat semigroup $\{W_t\}_{t>0}$, Riesz transforms $R_1$ and their commutators $C_{b,\ell}$, $\ell = 1, \ldots, n$, that will be very useful to our purposes. The proof of Theorem 1.1 is carried out in Section 3. We present proofs of Proposition 1.4 and Theorem 1.2 in Section 4. Finally, in Section 5 we give proofs for Proposition 1.2 and Theorem 1.3.

Throughout this paper by $c$ and $C$ we will always denote positive constants that can change in each occurrence. If $1 < p < \infty$, by $p'$ we represent the exponent conjugated of $p$, that is, $p' = \frac{p}{p-1}$.

2. Procedure and auxiliary results

In order to establish boundedness properties for harmonic analysis operators (semigroup, maximal operators, Riesz transforms, Littlewood-Paley functions,...) in the Schrödinger setting it is usual to exploit that $\mathcal{L}$ is actually a nice perturbation of the Laplacian operator $-\Delta$. We now describe a general procedure to analyze harmonic operators associated with the Schrödinger operator. Suppose that $T$ is a $\mathcal{L}$-harmonic analysis operator and that $\mathcal{F}$ is the corresponding $\Delta$-harmonic operator. According to the function $\gamma$ described above, we split $\mathbb{R}^n \times \mathbb{R}^n$ in two parts as follows

$$A = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |x-y| < \gamma(x)\},$$

and

$$B = (\mathbb{R}^n \times \mathbb{R}^n) \setminus A.$$ 

The sets $A$ and $B$ are usually called local and global region associated with $\mathcal{L}$, respectively. The local part of the operator $T$ is defined by

$$T_{loc}(f)(x) = T(f\chi_B(x,\gamma(x)))(x), \ x \in \mathbb{R}^n.$$ 

In a similar way we consider the operator

$$\mathcal{F}_{loc}(f)(x) = \mathcal{F}(f\chi_B(x,\gamma(x)))(x), \ x \in \mathbb{R}^n.$$ 

Then, we decompose the operator $T$ through

$$T = (T_{loc} - \mathcal{F}_{loc}) + \mathcal{F}_{loc} + (T - T_{loc}).$$ 

It is clear that $(T - T_{loc})(f)(x) = T(f\chi_{\mathbb{R}^n \setminus B(x,\gamma(x))})(x)$. Since the set $\{(x,y) : x \in \mathbb{R}^n, y \in \mathbb{R}^n \setminus B(x,\gamma(x))\}$ is sufficiently far away from the diagonal (usual line of singularities) $\{(x,y) : x \in \mathbb{R}^n\}$, the operator $T - T_{loc}$ will be controlled by a positive and $L^p$-bounded operator. We said that $\mathcal{L}$ is actually a nice perturbation of the Laplacian operator $-\Delta$. That niceness leads to the operators $T$ and $\mathcal{F}$ to have the same singularity in the local region. Then, cancelation of singularities in $T_{loc} - \mathcal{F}_{loc}$ takes place and $T_{loc} - \mathcal{F}_{loc}$ is controlled by a positive and $L^p$-bounded operator for the given range of $p$. In this way $L^p$-boundedness of $T$ is reduced to the corresponding property for
the operator $\mathcal{H}_\text{loc}$. Finally, $L^p$-boundedness properties of the operator $\mathcal{H}_\text{loc}$ rely on well known properties for the classical harmonic operator $\mathcal{T}$.

This procedure has been used in [23] to establish $L^p$-boundedness properties for $\mathcal{L}$-Riesz transform.

We will employ this comparative method to describe the behavior in $L^p$-spaces of the variation operators for the heat semigroup $\{W_t^\gamma\}_{t>0}$ generated by $-\mathcal{L}$. Riesz transforms and commutators of Riesz transforms with the multiplication by $\text{BMO}_\infty(\gamma)$-functions in the Schrödinger setting. Following this pattern we will need to know $L^p$-boundedness properties of the variation operators associated with the classical heat semigroup, Riesz transforms and commutators between Riesz transforms and multiplication by $\text{BMO}(\mathbb{R}^n)$-functions.

In [20] Theorem 3.3 it was established that if $\{T_t\}_{t>0}$ is a symmetric diffusion semigroup (in the sense of [24] p. 65) then the variation operator $V_p(T_t)$, with $p > 2$, is bounded from $L^p(\mathbb{R}^n)$ into itself for every $1 < p < \infty$. This result applies to the symmetric diffusion semigroup $\{W_t\}_{t>0}$ generated by the Euclidean Laplacian $\Delta$. Recently, Crescimbeni, Macías, Menárguez, Torrea and Viviani [9], by using vector valued Calderón-Zygmund theory, have proved that the operators $V_p(W_t)$ map $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$, for each $p > 2$. These results are contained in the following.

**Theorem 2.1.** ([20] Theorem 3.3 and [9] Theorem 1.1) Let $p > 2$. Then, the variation operator $V_p(W_t)$ is bounded from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.

For every $\ell = 1, \ldots, n$, the $\ell$-th Riesz transform $R_\ell$ is defined by

$$R_\ell f(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{x_\ell - y_\ell}{|x-y|^{n+1}} f(y) \, dy, \quad \text{a.e. } x \in \mathbb{R}^n,$$

for each $f \in L^p(\mathbb{R}^n), 1 \leq p < \infty$. For each $\varepsilon > 0$ and $\ell = 1, \ldots, n$, the $\varepsilon$-truncation of $R_\ell$ is given by

$$R_\ell^\varepsilon(f)(x) = \int_{|x-y|>\varepsilon} \frac{x_\ell - y_\ell}{|x-y|^{n+1}} f(y) \, dy, \quad x \in \mathbb{R}^n.$$

We denote the kernel function of $R_\ell$ by $R_\ell(z) = \frac{x_\ell - y_\ell}{|x-y|^{n+1}}, z = (z_1, \ldots, z_n) \in \mathbb{R}^n \setminus \{0\}$. The variation operators for $R_\ell^\varepsilon, \ell = 1, \ldots, n$, were investigated in [6] and [7] where the following results were proved.

**Theorem 2.2.** ([6] Theorems 1.1 and 1.2] and [7] Corollary 1.4). Let $\ell = 1, \ldots, n$. Assume that $p > 2$. Then, the variation operator $V_p(R_\ell^\varepsilon)$ is bounded from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.

Let us mention that by using transfer techniques, the above results for $R_\ell^\varepsilon$ have been established by means of dimension free $L^p(\mathbb{R}^n), |x|^\alpha \, dx$ norm inequalities, for every $1 < p < \infty$ and $-1 < \alpha < p - 1$, for variation operators of the Riesz transform $R_\ell, \ell = 1, \ldots, n$.

Next let $b \in \text{BMO}(\mathbb{R}^n)$. It is well known that, for every $\ell = 1, \ldots, n$, the commutator operator $C_{b,\ell}$ defined by

$$C_{b,\ell}(f) = bR_\ell(f) - R_\ell(bf),$$

is bounded from $L^p(\mathbb{R}^n)$ into itself, and for each $f \in L^p(\mathbb{R}^n), 1 < p < \infty$,

$$C_{b,\ell} f(x) = \lim_{\varepsilon \to 0^+} C_{b,\ell}^{\varepsilon}(f)(x), \quad \text{a.e. } x \in \mathbb{R}^n,$$

where

$$C_{b,\ell}^{\varepsilon}(f)(x) = \int_{|x-y|>\varepsilon} (b(x) - b(y)) R_\ell(x-y) f(y) \, dy, x \in \mathbb{R}^n,$$

([S] Theorem 1). $L^p$-boundedness properties for the variation operators associated with $C_{b,\ell}, \ell = 1, \ldots, n$, are stated in the following. To our knowledge the result is new, so we provide a proof.

**Theorem 2.3.** Let $b \in \text{BMO}(\mathbb{R}^n)$ and $\ell = 1, 2, \ldots, n$. Assume that $p > 2$. Then, the variation operator $V_p(C_{b,\ell}^{\varepsilon})$ is bounded from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$. 

Proof. Let $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$. Inspired in the ideas developed in [14], our goal is to estimate the sharp maximal function

$$(V_p(C_{\ell,\rho}^e(f))(f))#(x) = \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |V_p(C_{\ell,\rho}^e(f))(y) - c_{B(x, r)}|dy, \quad x \in \mathbb{R}^n,$$

where, for every $x \in \mathbb{R}^n$ and $r > 0$, $c_{B(x, r)}$ is a constant that will be specified later.

Let $x_0 \in \mathbb{R}^n$ and $r_0 > 0$ and denote by $B = B(x_0, r_0)$. We decompose $f = f_1 + f_2$, where $f_1 = f_{\chi_{AB}}$ and $f_2 = f_{\chi_{(4B)^c}}$, and we write

$$C_{\ell,\rho}^e f = (b - b_B)R_\ell^e(f) - R_\ell^e((b - b_B)f_1) - R_\ell^e((b - b_B)f_2) = A_\ell(f) + A_\ell^2(f) + A_\ell^3(f), \quad \epsilon > 0.$$

We have that

$$\int_{(4B)^c} \frac{|b(y) - b_B|}{|x - y|^n} |f(y)|dy = \sum_{k=2}^{\infty} \int_{k^{2k+1}r_0 < |y - x_0| \leq 2k^{2k+1}r_0} \frac{|b(y) - b_B|}{|x - y|^n} |f(y)|dy$$

$$\leq C \sum_{k=2}^{\infty} \left( \int_{k^{2k+1}r_0 < |y - x_0| \leq 2k^{2k+1}r_0} \frac{|b(y) - b_B|^{p'}}{|x - y|^n} dy \right)^{1/p'} \left( \int_{k^{2k+1}r_0 < |y - x_0| \leq 2k^{2k+1}r_0} \frac{|f(y)|^{p}}{|x - y|^n} dy \right)^{1/p}$$

$$\leq C \sum_{k=2}^{\infty} \left( \frac{1}{(2k^{2k+1}r_0)^n} \int_{|y - x_0| \leq 2k^{2k+1}r_0} |b(y) - b_B|^{p'} dy \right)^{1/p'} \frac{1}{(2k^{2k+1}r_0)^{n/p}} ||f||_{L^p(\mathbb{R}^n)}$$

$$\leq C ||b||_{BMO(\mathbb{R}^n)} ||f||_{L^p(\mathbb{R}^n)} \sum_{k=2}^{\infty} \frac{k}{(2k^{2k+1}r_0)^{n/p}}, \quad x \in B.$$

Then, we deduce that

$$V_p(R_\ell^e)((b - b_B)f_2)(x)$$

$$= \sup_{(\epsilon_j)_{\epsilon \in \mathbb{N}^n}} \left( \int_{|y - x_0| < \epsilon_j} |R_\ell^e(x - y)(b(y) - b_B)f(y)\chi_{(4B)^c}(y)dy|^{p/2} \right)^{1/p}$$

$$\leq \sup_{(\epsilon_j)_{\epsilon \in \mathbb{N}^n}} \left( \int_{|y - x_0| < \epsilon_j} \frac{|b(y) - b_B||f(y)|}{|x - y|^n} \chi_{(4B)^c}(y)dy \right)^{1/p}$$

$$\leq \int_{(4B)^c} \frac{|b(y) - b_B||f(y)|}{|x - y|^n} dy \leq C r_0^{-n/p} ||b||_{BMO(\mathbb{R}^n)} ||f||_{L^p(\mathbb{R}^n)}, \quad x \in B.$$

We denote $c_B = -V_p(R_\ell^e)((b - b_B)f_2)(x_0)$. It is clear that $c_B = V_p(A_\ell^3(f))(x_0)$.

We have

$$(5) \quad \frac{1}{|B|} \int_B |V_p(C_{\ell,\rho}^e(f))(x) - c_B|dx$$

$$= \frac{1}{|B|} \int_B \left| ||A_\ell(f)(x) + A_\ell^2(f)(x) + A_\ell^3(f)(x)||_{E_p} - ||A_\ell^3(f)(x_0)||_{E_p} \right|dx$$

$$\leq \frac{1}{|B|} \int_B ||A_\ell(f)(x)||_{E_p} dx + \frac{1}{|B|} \int_B ||A_\ell^2(f)(x)||_{E_p} dx$$

$$+ \frac{1}{|B|} \int_B ||A_\ell^3(f)(x) - A_\ell^3(f)(x_0)||_{E_p} dx.$$

We analyze each term. First, we obtain

$$\frac{1}{|B|} \int_B ||A_\ell^3(f)(x)||_{E_p} dx = \frac{1}{|B|} \int_B |b(x) - b_B|V_p(R_\ell^e)(f)(x)dx$$

$$\leq \left( \frac{1}{|B|} \int_B |b(x) - b_B|^{r'} dx \right)^{1/r'} \left( \frac{1}{|B|} \int_B \left( V_p(R_\ell^e)(f)(x) \right)^r dx \right)^{1/r}$$

$$\leq C ||b||_{BMO(\mathbb{R}^n)} M_r(V_p(R_\ell^e)(f))(z), \quad z \in B.$$
Here $1 \leq r < \infty$ and $\mathcal{M}_r$ is the $r$-maximal function defined by
\[
\mathcal{M}_r(g)(x) = \sup_{x \in B} \left( \frac{1}{|B|} \int_B |g(y)|^r \, dy \right)^{1/r}, \quad x \in \mathbb{R}^n,
\]
for every measurable function $g$ on $\mathbb{R}^n$.

On the other hand, according to [7, Theorem A], we have that
\[
\frac{1}{|B|} \int_B ||A_2^s(f)(x)||_{E_r} \, dx \leq C \left( \frac{1}{|B|} \int_B |V_p(R^s_\rho)(b-b_B)f(x)|^\beta \, dx \right)^{1/\beta}
\]
\[
\leq C \left( \frac{1}{|B|} \int_{4B} |b(x) - b_B|^\beta |f(x)|^\beta \, dx \right)^{1/\beta}
\]
\[
\leq C \left( \frac{1}{|B|} \int_{4B} |b(x) - b_B|^{s/3} \, dx \right)^{1/(s/3)} \left( \frac{1}{|B|} \int_{4B} |f(x)|^{3\beta} \, dx \right)^{1/(3\beta)},
\]
where $1 < s, \beta < \infty$. Then, for every $1 < r < \infty$, we have
\[
\frac{1}{|B|} \int_B ||A_2^s(f)(x)||_{E_r} \, dx \leq C||b||_{\text{BMO}(\mathbb{R}^n)} \mathcal{M}_r(f)(z), \quad z \in B.
\]
In order to study $||A_2^s(f)(x) - A_2^s(f)(x_0)||_{E_r}$, we use a procedure developed in [14]. We have that
\[
||R^s_\rho((b-b_B)f_2(x)) - R^s_\rho((b-b_B)f_2(x_0))||_{E_r} \leq H_1(x) + H_2(x), \quad x \in B,
\]
where, for every $x \in B$,
\[
H_1(x) = \left\| \int_{|x-y| > \varepsilon} (R\varepsilon(x-y) - R\varepsilon(x_0-y))(b(y) - b_B)f_2(y) \, dy \right\|_{E_r},
\]
and
\[
H_2(x) = \left\| \int_{\mathbb{R}^n} (\chi_{\{|x-y| > \varepsilon\}}(y) - \chi_{\{|x_0-y| > \varepsilon\}}(y)) R\varepsilon(x_0-y)(b(y) - b_B)f_2(y) \, dy \right\|_{E_r}.
\]
By using Minkowski inequality and well known properties of the function $R\varepsilon$ we get
\[
H_1(x) \leq \int_{\mathbb{R}^n} |R\varepsilon(x-y) - R\varepsilon(x_0-y)||b(y) - b_B||f(y)||\chi_{\{4B\}^c}(y) \, dy 
\]
\[
\leq C \sum_{k=2}^{\infty} \int_{2^k \rho_0 \leq |x| - |x_0| \leq 2^{k+1} \rho_0} \frac{|x - x_0|}{|x_0 - y|^n} |b(y) - b_B||f(y)| \, dy 
\]
\[
\leq C \sum_{k=2}^{\infty} \frac{1}{2^k (2^k \rho_0)^n} \int_{2^{k+1} B} |b(y) - b_B||f(y)| \, dy, \quad x \in B.
\]
To analyze $H_2$ we split the integral appearing in the norm in four terms as follows. Let $\{\varepsilon_j\}_{j \in \mathbb{N}}$ be a real decreasing sequence that converges to zero. We have
\[
\int_{\mathbb{R}^n} \chi_{\{|x-y| < \varepsilon_j\}}(y) - \chi_{\{|x_0-y| < \varepsilon_j\}}(y) ||R\varepsilon(x_0-y)||b(y) - b_B||f_2(y)| \, dy 
\]
\[
\leq C \left( \int_{\mathbb{R}^n} \chi_{\{|x-y| < \varepsilon_j\} \cap \{x_0-y \geq \rho_0\}}(y) \chi_{\{|x-y| < \varepsilon_j\}}(y) \frac{1}{|x_0 - y|^n} |b(y) - b_B||f_2(y)| \, dy 
\]
\[
+ \int_{\mathbb{R}^n} \chi_{\{|x-y| < \varepsilon_j\} \cap \{x_0-y < \rho_0\}}(y) \chi_{\{|x-y| < \varepsilon_j\}}(y) \frac{1}{|x_0 - y|^n} |b(y) - b_B||f_2(y)| \, dy 
\]
\[
+ \int_{\mathbb{R}^n} \chi_{\{|x-y| < \varepsilon_j\} \cap \{x_0-y \geq \rho_0\}}(y) \chi_{\{|x-y| < \varepsilon_j\}}(y) \frac{1}{|x_0 - y|^n} |b(y) - b_B||f_2(y)| \, dy 
\]
\[
+ \int_{\mathbb{R}^n} \chi_{\{|x-y| < \varepsilon_j\} \cap \{x_0-y < \rho_0\}}(y) \chi_{\{|x-y| < \varepsilon_j\}}(y) \frac{1}{|x_0 - y|^n} |b(y) - b_B||f_2(y)| \, dy 
\]
\[
= C(H^j_{2,1}(x) + H^j_{2,2}(x) + H^j_{2,3}(x) + H^j_{2,4}(x)), \quad x \in B \text{ and } j \in \mathbb{N}.
\]
We observe that \( \frac{1}{j} |x - y| \geq |x_0 - y| \geq \frac{1}{j} |x - y| \), when \( y \notin 4B \) and \( x \in B \). Moreover, if \( x \in B \), then \( H^j_{2,m}(x) = 0 \), for \( m = 1, 3 \), when \( j \in \mathbb{N} \) and \( r_0 \geq \varepsilon_{j+1} \) and \( H^j_{2,m}(x) = 0 \), when \( m = 2, 4 \), \( j \in \mathbb{N} \) and \( r_0 \geq \varepsilon_j \). For every \( j \in \mathbb{N} \), Hölder inequality leads to

\[
H^j_{2,1}(x) \leq C \left( \int_{\mathbb{R}^n} \chi_{[x_j+1<[x-y]<\varepsilon_j]}(y) \frac{1}{|x-y|^n} (|y| - bB||f_2(y)||x)^s dy \right)^{\frac{1}{s}} v_j \frac{1}{v_j}, \quad x \in B,
\]

\[
H^j_{2,2}(x) \leq C \left( \int_{\mathbb{R}^n} \chi_{(x_j+\varepsilon_j, x_j+\varepsilon_j)}(y) \frac{1}{|x-y|^n} (|y| - bB||f_2(y)||x)^s dy \right)^{\frac{1}{s}} v_j \frac{1}{v_j}, \quad x \in B,
\]

\[
H^j_{2,3}(x) \leq C \left( \int_{\mathbb{R}^n} \chi_{[x_{j-1}+<[x-y]<\varepsilon_j]}(y) \frac{1}{|x_0-y|^n} (|y| - bB||f_2(y)||x)^s dy \right)^{\frac{1}{s}} v_j \frac{1}{v_j}, \quad x \in B,
\]

and

\[
H^j_{2,4}(x) \leq C \left( \int_{\mathbb{R}^n} \chi_{(x_j+\varepsilon_j, x_j+\varepsilon_j)}(y) \frac{1}{|x_0-y|^n} (|y| - bB||f_2(y)||x)^s dy \right)^{\frac{1}{s}} v_j \frac{1}{v_j}, \quad x \in B.
\]

Here we take \( 1 < s < \rho \), and \( v_j = (\varepsilon_j + r_0)^n - \varepsilon_j^n \), \( j \in \mathbb{N} \). Note that \( v_j \leq C(\max\{r_0, \varepsilon_j\})^{n+1}r_0 \), \( j \in \mathbb{N} \), for a certain \( C > 0 \) that does not depend on \( j \).

We define the set \( \mathcal{A} = \{ j \in \mathbb{N} : r_0 < \varepsilon_j \} \). We have that

\[
H^j_{2,1}(x) \leq C \left( \int_{\mathbb{R}^n} \chi_{[x_j+1<[x-y]<\varepsilon_j]}(y) \frac{1}{|x-y|^n} (|y| - bB||f_2(y)||x)^s dy \right)^{\frac{1}{s}} v_j \frac{1}{v_j}, \quad x \in B \text{ and } j + 1 \in \mathcal{A},
\]

for every \( x \in B \) and \( j + 1 \in \mathcal{A} \). In a similar way we can see that

\[
H^j_{2,2}(x) \leq C r_0^{1/s} \left( \int_{\mathbb{R}^n} \chi_{[x_j+1<[x-y]<\varepsilon_j]}(y) \frac{1}{|x-y|^n} (|y| - bB||f_2(y)||x)^s dy \right)^{\frac{1}{s}} v_j \frac{1}{v_j}, \quad x \in B \text{ and } j \in \mathcal{A},
\]

\[
H^j_{2,3}(x) \leq C r_0^{1/s} \left( \int_{\mathbb{R}^n} \chi_{[x_{j-1}+<[x-y]<\varepsilon_j]}(y) \frac{1}{|x_0-y|^n} (|y| - bB||f_2(y)||x)^s dy \right)^{\frac{1}{s}} v_j \frac{1}{v_j}, \quad x \in B \text{ and } j + 1 \in \mathcal{A},
\]

and

\[
H^j_{2,4}(x) \leq C r_0^{1/s} \left( \int_{\mathbb{R}^n} \chi_{[x_j+1<[x-y]<\varepsilon_j]}(y) \frac{1}{|x_0-y|^n} (|y| - bB||f_2(y)||x)^s dy \right)^{\frac{1}{s}} v_j \frac{1}{v_j}, \quad x \in B \text{ and } j \in \mathcal{A}.
\]

Hence, we get by using Minkowski’s inequality

\[
\left( \sum_{j=0}^{\infty} |H^j_{2,1}(x)|^\rho \right)^{1/\rho} \leq C \left( \sum_{j+1 \in \mathcal{A}} \sum_{j \in \mathcal{A}} |H^j_{2,1}(x)|^\rho \right)^{1/\rho}
\]

\[
\leq C \left( \sum_{j=0}^{\infty} \left( \int_{\mathbb{R}^n} \chi_{[x_j+1<[x-y]<\varepsilon_j]}(y) \frac{1}{|x-y|^n} (|y| - bB||f_2(y)||x)^s dy \right)^{\frac{\rho}{s}} r_0^{\rho/s} \right)^{1/\rho}
\]

\[
\leq C \left( \int_{\mathbb{R}^n} \frac{(|y| - bB||f_2(y)||x)^s}{|x-y|^n} dy \right)^{1/s} r_0^{1/s}
\]

\[
\left( \sum_{k=2}^{\infty} \frac{1}{(2^k r_0)^n} \right) \left( \int_{|x_0-y|<2^{k+1}r_0} (|y| - bB||f(y)||x)^s dy \right)^{1/s} \frac{1}{2^{k(s-1)}} \right)^{1/s}, \quad x \in B.
\]

In a similar way we get

\[
\left( \sum_{j=0}^{\infty} |H^j_{2,3}(x)|^\rho \right)^{1/\rho}
\]

\[
\leq C \left( \sum_{k=1}^{\infty} \frac{1}{(2^k r_0)^n} \int_{|x_0-y|<2^{k+1}r_0} (|y| - bB||f(y)||x)^s dy \right)^{1/s} \frac{1}{2^{k(s-1)}} \right)^{1/s}, \quad x \in B.
\]
By combining (3), (9), (11), and (12) it follows that
\[
||A^*_B f(x) - A^*_B f(x_0)||_{E_p} = ||R\epsilon ((b - b_B)f_2(x) - R\epsilon ((b - b_B)f_2)(x_0)||_{E_p}
\leq C \left( \sum_{k=1}^{\infty} \frac{2^{-k}}{(2k^{1/2})^n} \int_{|x_0 - y| < 2^k r_0} |b(y) - b_B||f(y)|dy 
+ \left( \sum_{k=1}^{\infty} \frac{2^{-k(s-1)}}{(2k^{1/2})^n} \int_{|x_0 - y| < 2^k r_0} |(b(y) - b_B)f(y)|^s dy \right)^{1/s} \right),
\]
for \( x \in B \). Then, Hölder's inequality implies that
\[
||A^*_B (f)(x) - A^*_B (f)(x_0)||_{E_p} \leq C \left( \sum_{k=1}^{\infty} \frac{1}{2^k} \left( \frac{1}{(2k^{1/2})^n} \int_{|x_0 - y| < 2^k r_0} |b(y) - b_B|^{t'} dy \right)^{1/t'} 
\times \left( \frac{1}{(2k^{1/2})^n} \int_{|x_0 - y| < 2^k r_0} |f(y)|^{r'} dy \right)^{1/r} 
+ \left( \sum_{k=1}^{\infty} \frac{1}{2^{k(s-1)}} \left( \frac{1}{(2k^{1/2})^n} \int_{|x_0 - y| < 2^k r_0} |b(y) - b_B|^{t''} dy \right)^{1/t''} 
\times \left( \frac{1}{(2k^{1/2})^n} \int_{|x_0 - y| < 2^k r_0} |f(y)|^{r''} dy \right)^{1/r} \right) \right)^{1/s}
\leq C \left( \sum_{k=1}^{\infty} \frac{k}{2^{k}} ||b||_{BMO(\mathbb{R}^n)} \mathcal{M}_t (f)(z) 
+ \left( \sum_{k=1}^{\infty} \frac{k}{2^{k(s-1)}} \right)^{1/s} ||b||_{BMO(\mathbb{R}^n)} \mathcal{M}_{t'} (f)(z) \right)
\leq C ||b||_{BMO(\mathbb{R}^n)} (\mathcal{M}_t (f)(z) + \mathcal{M}_{t'} (f)(z)), \quad x, z \in B,
\]
where \( 1 < t, r < \infty \).

From (13) it follows
\[
\frac{1}{|B|} \int_B ||A^*_B f(x) - A^*_B f(x_0)||_{E_p} dx \leq C ||b||_{BMO(\mathbb{R}^n)} \mathcal{M}_t (f)(z), \quad z \in B,
\]
where \( 1 < r < \infty \).

By combining (5), (6), (7) and (14) we conclude that
\[
(V_p(C^\gamma_{b,t})(f))^{\#}(x) \leq C ||b||_{BMO(\mathbb{R}^n)} (\mathcal{M}_t (f)(x) + \mathcal{M}_{t'} (V_p(R^*_\gamma)(f))(x)), \quad x \in B,
\]
where \( 1 < r < p \).

Since \( \mathcal{M}_t \) is bounded from \( L^p(\mathbb{R}^n) \) into itself provided that \( 1 < r < p < \infty \), [7] Theorem A and [25] Corollary 1, p. 154 allow us to conclude that \( V_p(C^\gamma_{b,t}) \) is bounded from \( L^p(\mathbb{R}^n) \) into itself.

In [25] Dachun Yang, Dongyong Yang and Yuan Zhou introduced localized Riesz transform associated with the classical Laplacian and Schrödinger operators. Here, we consider localized commutator operators with the classical Riesz transforms.

Let \( b \in BMO_0(\theta), \theta > 0 \). We define, for every \( \varepsilon > 0 \), the local truncation \( C^\varepsilon_{b,t} f \) of \( f \in L^p(\mathbb{R}^n) \), \( 1 < p < \infty \), by
\[
C^\varepsilon_{b,t} f(x) = \int_{|x - y| < \varepsilon(x)} (b(x) - b(y)) R\epsilon(x - y) f(y) dy, \quad x \in \mathbb{R}^n.
\]

In order to define the local commutator \( C^\varepsilon_{b,t} \) on \( L^p(\mathbb{R}^n) \), \( 1 < p < \infty \), we show the \( L^p \) boundedness properties for the maximal operator \( C^\varepsilon_{b,t} \) defined by
\[
C^\varepsilon_{b,t} f(x) = \sup_{\varepsilon > 0} |C^\varepsilon_{b,t} f(x)|, \quad x \in \mathbb{R}^n.
\]
Previously we state an auxiliary result that will be also useful in other sections of this paper. According to [11 Proposition 5] we choose a sequence \( \{ x_k \}_{k=1}^{\infty} \subset \mathbb{R}^n \), such that if \( Q_k = B(x_k, \gamma(x_k)) \), \( k \in \mathbb{N} \), the following properties hold:

(i) \( \bigcup_{k=1}^{\infty} Q_k = \mathbb{R}^n \);

(ii) For every \( m \in \mathbb{N} \) there exist \( C, \beta > 0 \) such that, for every \( k \in \mathbb{N} \),

\[
\text{card}\{l \in \mathbb{N} : 2^m Q_l \cap 2^m Q_k \neq \emptyset \} \leq C 2^m \beta.
\]

The sequence \( \{ Q_k \}_{k \in \mathbb{N}} \) of balls will appear in different occasions throughout this paper.

**Lemma 2.1.** Let \( b \in BMO_{\theta}(\gamma) \), \( \theta \geq 0 \), and \( M > 0 \). We define the operator \( T_{b,M} \) by

\[
T_{b,M}(f) = \frac{1}{\gamma(x)^n} \int_{|x-y| \leq M \gamma(x)} |b(x) - b(y)| f(y) dy, \ x \in \mathbb{R}^n.
\]

Then, for every \( 1 < p < \infty \), \( T_{b,M} \) is bounded from \( L^p(\mathbb{R}^n) \) into itself. Moreover,

\[
\| T_{b,M}(f) \|_{L^p(\mathbb{R}^n)} \leq C (1 + MA)^{n + \theta} \|b\|_{BMO_{\theta}(\gamma)} \|f\|_{L^p(\mathbb{R}^n)}, \ f \in L^p(\mathbb{R}^n).
\]

Here, \( A, C > 0 \) and \( \theta > 0 \) do not depend on \( M \).

**Proof.** Let \( 1 < p < \infty \) and \( f \in L^p(\mathbb{R}^n) \). We have that

\[
\| T_{b,M}(f) \|_{L^p(\mathbb{R}^n)} \leq C \left( \sum_{k=1}^{\infty} \int_{Q_k} \| T_{b,M}(f)(x) \|^p dx \right)^{1/p}.
\]

Since \( \gamma(x) \sim \gamma(x_k) \), for every \( x \in Q_k \), \( k \in \mathbb{N} \), by using Hölder and Minkowski inequalities and [3 Proposition 3], it follows that, for certain \( A > 0 \) and \( \theta' > \theta \),

\[
\begin{align*}
\int_{Q_k} \| T_{b,M}(f)(x) \|^p dx &\leq C \int_{Q_k} \left( \frac{1}{\gamma(x)^n} \int_{|x-y| \leq (1+MA)\gamma(x_k)} |b(x) - b(y)| |f(y)| dy \right)^p dx \\
&\leq C \int_{Q_k} \frac{1}{\gamma(x)^n p} \int_{|x-y| \leq (1+MA)\gamma(x_k)} |f(y)|^p dy \int_{|x-y| \leq (1+MA)\gamma(x_k)} |b(x) - b(y)|^p dy^p \\
&\leq C \int_{Q_k} \frac{1}{\gamma(x)^n p} \int_{|x-y| \leq (1+MA)\gamma(x_k)} |f(y)|^p dy \times \left( \int_{|x-y| \leq (1+MA)\gamma(x_k)} |b(x) - b_Qk|^p dy^p \right)^{p/p'} + \left( \int_{|x-y| \leq (1+MA)\gamma(x_k)} |b(y) - b_Qk|^p dy^p \right)^{p/p'} dx \\
&\leq C \int_{Q_k} \frac{1}{\gamma(x)^n p} \int_{|x-y| \leq (1+MA)\gamma(x_k)} |f(y)|^p dy \times \left( \int_{|x-y| \leq (1+MA)\gamma(x_k)} |b(x) - b_Qk|^p dy^p + \int_{|x-y| \leq (1+MA)\gamma(x_k)} |b(y) - b_Qk|^p dy^p \right) dx \\
&\leq C \int_{Q_k} \frac{1}{\gamma(x)^n p} \int_{|x-y| \leq (1+MA)\gamma(x_k)} |f(y)|^p dy \times \left( \|b\|_{BMO_{\theta}(\gamma)}^p + \|b\|_{BMO_{\theta}(\gamma)}^p (1 + MA)^{\theta'} \right)^{p/p'} dx \\
&\leq C \int_{Q_k} \frac{1}{\gamma(x)^n p} \int_{|x-y| \leq (1+MA)\gamma(x_k)} |f(y)|^p dy \times \|b\|_{BMO_{\theta}(\gamma)}^p (1 + MA)^{(n + \theta)p}, \ k \in \mathbb{N},
\end{align*}
\]

where \( Q_k = B(x_k, (1 + MA)\gamma(x_k)) \), \( k \in \mathbb{N} \).

Hence, by taking into account the properties of \( \{ Q_k \}_{k \in \mathbb{N}} \), we get

\[
\| T_{b,M}(f) \|_{L^p(\mathbb{R}^n)} \leq C \left( \sum_{k=1}^{\infty} \int_{|x-y| \leq (1+MA)\gamma(x_k)} |f(y)|^p dy \|b\|_{BMO_{\theta}(\gamma)}^p (1 + MA)^{(n + \theta)p} \right)^{1/p} \\
\leq C (1 + MA)^{n + \theta} \|b\|_{BMO_{\theta}(\gamma)} \|f\|_{L^p(\mathbb{R}^n)}.
\]

\( \square \)

Our next step is to advance on the boundedness of the corresponding local operators. The case of heat semigroup and Riesz transforms are not very difficult and the will be done while proving Theorems [11 and 12 in Sections 3 and 4, respectively. As far the commutator, it involves an additional difficulty. According to our procedure we start with the commutator \( C_{b,f} \) with \( b \in BMO_{\theta}(\gamma) \) and we reduce the problem to the local classical commutator but now \( b \) is not necessarily in \( BMO \) in the classical sense. In the next lemma and proposition we show how to overcome this problem.
Lemma 2.2. Let $\theta \geq 0$ and $L > 0$. There exists $C > 0$ such that, for every $k \in \mathbb{N}$ and $b \in BMO_b(\gamma)$, we can find a function $b_k \in BMO(\mathbb{R}^n)$ for which $b_k = b$ on $LQ_k = B(x_k, L\gamma(x_k))$ and $\|b_k\|_{BMO(\mathbb{R}^n)} \leq C\|b\|_{BMO_b(\gamma)}$.

Proof. Let $k \in \mathbb{N}$ and $b \in BMO_b(\gamma)$. We define $b_k = b\chi_{LQ_k}$. Assume that $z_0 \in \mathbb{R}^n$ and $r_0 > 0$ such that $B(z_0, r_0) \subset LQ_k$. It is clear that $|z_0 - x_k| \leq L\gamma(x_k)$. Then, for a certain $A > 0$ that does not depend on $k$, $\gamma(x_k) \leq A\gamma(z_0)$.

We have that

$$\frac{1}{|B(z_0, r_0)|} \int_{B(z_0, r_0)} |b(x) - b_{B(z_0, r_0)}| dx \leq \|b\|_{BMO(\mathbb{R}^n)} \left(1 + \frac{r_0}{\gamma(z_0)} \right)^\theta \leq C\|b\|_{BMO_b(\gamma)},$$

where $C > 0$ depends on $L$ and $\theta$ but does not depend on $k$. Hence, $b \in BMO(LQ_k)$ and $\|b\|_{BMO(LQ_k)} \leq C\|b\|_{BMO_b(\gamma)}$.

We define $w_k = \exp(b_k)$. It is well known that $w_k \in A_2(LQ_k)$, where $A_2(LQ_k)$ denotes the Muckenhoupt class of weights. Moreover, the $A_2$-characteristic $[w_k]_{A_2(LQ_k)}$ satisfies that $[w_k]_{A_2(LQ_k)} \leq C\|b\|_{BMO(LQ_k)}$, where $C > 0$ does not depend on $k$. According to [3 Lemma 4] (see also [17]) there exists $\bar{w}_k \in A_2(\mathbb{R}^n)$ such that $\bar{w}_k = w_k$ in $LQ_k$, and $[\bar{w}_k, A_2(\mathbb{R}^n)] \leq C[w_k, A_2(LQ_k)]$, $C > 0$ being independent of $k$. Then, there exists $b_k \in BMO(\mathbb{R}^n)$ satisfying that $\bar{w}_k = \exp(b_k)$ and $\|b_k\|_{BMO(\mathbb{R}^n)} \leq C[\bar{w}_k, A_2(\mathbb{R}^n)]$, where $C > 0$ does not depend on $k$.

We conclude that $b_k = b$ on $LQ_k$ and $\|b_k\|_{BMO(\mathbb{R}^n)} \leq C\|b\|_{BMO_b(\gamma)}$, for a certain $C > 0$ independent of $k$. \hfill $\Box$

Proposition 2.1. Let $b \in BMO_\infty(\gamma)$ and $\ell = 1, \ldots, n$. Then, the maximal operator $C_{b,\ell}^{w,loc}$ is bounded from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$.

Proof. Assume that $b \in BMO_b(\gamma)$, with $\theta \geq 0$. Fix $k \in \mathbb{N}$. According to [23 Lemma 1.4] $\gamma(x) \sim \gamma(x_k)$, for every $x \in Q_k$. We choose $L > 0$ independent of $k$ such that, for any $x \in Q_k$, $B(x, \gamma(x)) \subset Q_k$, where $Q_k = B(x_k, L\gamma(x_k))$. We can write

$$\left| \int_{\gamma(x) \leq |x - y| \leq \gamma(x_k)} (b(x) - b(y)) R_\ell(x - y) f(y) dy \right|$$

$$\leq \left| \int_{\gamma(x) \leq |x - y| \leq \gamma(x)} (b(x) - b(y)) R_\ell(x - y) f(y) dy \right| - \int_{y \in \mathbb{R}^n, |x - y| \geq \varepsilon} (b(x) - b(y)) R_\ell(x - y) f(y) dy$$

$$+ \left| \int_{y \in \mathbb{R}^n, |x - y| \geq \varepsilon} (b(x) - b(y)) R_\ell(x - y) f(y) dy \right|$$

$$= J_{1,k}(x, \varepsilon) + J_{2,k}(x, \varepsilon), \quad x \in Q_k, \quad \varepsilon > 0.$$

Let us analyze $J_{i,k}(x, \varepsilon)$, for $x \in Q_k$ and $\varepsilon > 0$, $i = 1, 2$. Observe that, since $\gamma(x) \sim \gamma(x_k)$, for every $y \in Q_k$, we can find $C, M > 0$ that do not depend on $k \in \mathbb{N}$ such that

$$J_{1,k}(x, \varepsilon) \leq C \int_{Q_k \setminus B(x, \gamma(x_k))} \frac{|b(x) - b(y)|}{|x - y|^\theta} f(y) dy$$

$$\leq C \int_{\gamma(x) \leq |x - y| \leq M\gamma(x)} \frac{|b(x) - b(y)|}{|x - y|^\theta} f(y) \chi_{Q_k}(y) dy$$

$$\leq C \int_{Q_k} \frac{|b(x) - b(y)|}{|x - y|^\theta} f(y) \chi_{Q_k}(y) dy, \quad x \in Q_k \quad \varepsilon > 0.$$

From Lemma 2.1, we have that

$$\int_{Q_k} \sup_{\varepsilon > 0} J_{1,k}(x, \varepsilon) \leq C \int_{\mathbb{R}^n} |T_{b,M}(f\chi_{Q_k})(x)|^p dx$$

$$\leq C\|b\|_{BMO(\gamma)}^p \int_{Q_k} |f(y)|^p dy, \quad k \in \mathbb{N},$$

with a constant independent of $k$.

On the other hand, we have that, for $x \in Q_k$ and $\varepsilon > 0$,

$$J_{2,k}(x, \varepsilon) = \left| \int_{|x - y| \geq \varepsilon} (b_k(x) - b_k(y)) R_\ell(x - y) f(y) dy \right|,$$
where \( b_k = b \chi_{Q_k} \). According to Lemma 2.2, there exists a function \( \tilde{b}_k \in BMO(\mathbb{R}^n) \) such that \( \tilde{b}_k = b \), on \( Q_k \), and \( \| \tilde{b}_k \|_{BMO(\mathbb{R}^n)} \leq C\|b\|_{BMO(\mathbb{R}^n)} \), where \( C > 0 \) does not depend on \( k \in \mathbb{N} \). It follows that

\[
\sup_{\varepsilon > 0} J_{2,k}(x, \varepsilon) = C_{b,\ell}^*(f \chi_{Q_k})(x), \ x \in Q_k.
\]

Hence, we get

\[
\int_{Q_k} \left| \sup_{\varepsilon > 0} J_{2,k}(x, \varepsilon) \right|^p \, dx \leq C \int_{\mathbb{R}^n} \left| C_{b,\ell}^*(f \chi_{Q_k})(x) \right|^p \, dx
\]

\[
\leq C\|b\|_{BMO(\mathbb{R}^n)} \int_{Q_k} |f(y)|^p \, dy
\]

\[\leq C\|b\|_{BMO(\mathbb{R}^n)} \int_{Q_k} |f(y)|^p \, dy, \ k \in \mathbb{N}. \tag{17}\]

By combining (15), (16) and (17) and using the properties of the sequence \( \{Q_k\}_{k \in \mathbb{N}} \), we conclude, for every \( 1 < p < \infty \),

\[
\|C_{b,\ell}^*(f)\|_{L^p(\mathbb{R}^n)} \leq \left( \sum_{k=1}^{\infty} \int_{Q_k} |C_{b,\ell}^*(f)(x)|^p \, dx \right)^{1/p}
\]

\[
\leq C\|b\|_{BMO(\mathbb{R}^n)} \left( \sum_{k=1}^{\infty} \int_{Q_k} |f(y)|^p \, dy \right)^{1/p}
\]

\[
\leq C\|b\|_{BMO(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}, \ f \in L^p(\mathbb{R}^n).
\]

\(\square\)

Suppose now that \( \ell = 1, \ldots, n \), \( f \in C_c^\infty(\mathbb{R}^n) \) and \( b \in BMO(\mathbb{R}^n) \). Then, \( b f \in L^1(\mathbb{R}^n) \). Moreover, we can write, for \( \varepsilon > 0 \) small enough,

\[
C_{b,\ell}^*(f)(x) = C_{b,\ell}^c(f)(x) - \int_{|x-y| \geq \varepsilon} (b(x) - b(y)) R_\ell(x-y)f(y) \, dy, \ x \in \mathbb{R}^n.
\]

Hence, there exists the limit

\[
\lim_{\varepsilon \to 0^+} C_{b,\ell}^c(f)(x), \ x \in \mathbb{R}^n.
\]

By using standard arguments, from Proposition 2.1, we deduce that, for every \( f \in L^p(\mathbb{R}^n) \), \( 1 < p < \infty \), there exists the limit

\[
\lim_{\varepsilon \to 0^+} C_{b,\ell}^c(f)(x), \ a.e. \ x \in \mathbb{R}^n.
\]

We define, for every \( f \in L^p(\mathbb{R}^n) \), \( 1 < p < \infty \),

\[
C_{b,\ell}^c(f)(x) = \lim_{\varepsilon \to 0^+} C_{b,\ell}^c(f)(x), \ a.e. \ x \in \mathbb{R}^n.
\]

The behavior in \( L^p(\mathbb{R}^n) \) of the variation operators for the family of truncations \( \{C_{b,\ell}^c\}_{\varepsilon > 0} \) associated with the local commutator operator \( C_{b,\ell}^c \) is established in the following.

**Theorem 2.4.** Let \( b \in BMO(\mathbb{R}^n) \) and \( \ell = 1, \ldots, n \). Assume that \( p > 2 \). Then, the variation operator \( V_{\rho}(C_{b,\ell}^c) \) is bounded from \( L^p(\mathbb{R}^n) \) into itself, for every \( 1 < p < \infty \).

**Proof.** Suppose that \( f \in L^p(\mathbb{R}^n) \) with \( 1 < p < \infty \). Let \( k \in \mathbb{N} \). As in the proof of Proposition 2.1, we define \( Q_k = B(x_k, L \gamma(x_k)) \), where \( L > 0 \) is such that \( B(x, \gamma(x)) \subset Q_k \), for every \( x \in Q_k \). Moreover, \( L \) does not depend on \( k \).
Thus the proof is finished.

Observe that

\[ V_p(W_f^Z)(f) \leq V_p(W_{f,\text{loc}}^Z) \leq V_p(W_{f,\text{loc}}^Z) + V_p(W_{f,\text{loc}}^Z) + V_p(W_f^Z - W_{f,\text{loc}}^Z)(f). \]
Assume that \( \{t_j\}_{j \in \mathbb{N}} \) is a real decreasing sequence that converges to zero. We can write
\[
\left( \sum_{j=0}^{\infty} |W_{t_j,\text{loc}}(f)(x) - W_{t_{j+1},\text{loc}}(f)(x)|^{r} \right)^{\frac{1}{r}} 
\leq \left( \sum_{j=0}^{\infty} |W_{t_j}(f)(x) - W_{t_{j+1}}(f)(x)|^{r} \right)^{\frac{1}{r}}
\]
\[
+ \left( \sum_{j=0}^{\infty} \left| \int_{|x-y|>\gamma(x)} (W_{t_j}(x,y) - W_{t_{j+1}}(x,y)) f(y)dy \right|^{r} \right)^{\frac{1}{r}}
\]
\[
\leq V_p(W_t)(f)(x) + \sup_{\varepsilon>0} \left\| \int_{|x-y|>\varepsilon} W_t(x,y)f(y)dy \right\|_{E_p}, \ x \in \mathbb{R}^n,
\]
where the space \( E_p \) is defined in Section 1.

Then,
\[
V_p(W_{t,\text{loc}})(f)(x) \leq V_p(W_t)(f) + \sup_{\varepsilon>0} \left\| \int_{|x-y|>\varepsilon} W_t(x,y)f(y)dy \right\|_{E_p}, \ x \in \mathbb{R}^n.
\]

We consider the operator defined by
\[
T: \ L^2(\mathbb{R}^n) \to L^2_{E_p}(\mathbb{R}^n)
\]
\[
f \mapsto Tf(x) = \int_{\mathbb{R}^n} W_t(x,y)f(y)dy.
\]

According to [20] Theorem 3.3 \( T \) is bounded from \( L^2(\mathbb{R}^n) \) into \( L^2_{E_p}(\mathbb{R}^n) \). Moreover, \( T \) is a Calderón-Zygmund operator associated with the \( E_p \)-valued kernel
\[
K(x,y;t) = W_t(x,y), \ x,y \in \mathbb{R}^n, \ t > 0,
\]
that satisfies the following properties (see [9]):
\[
(1) \ |K(x,y;\cdot)|_{E_p} \leq \frac{C}{|x-y|^n}, \ x,y \in \mathbb{R}^n, \ x \neq y,
\]
\[
(2) \ \left\| \frac{\partial}{\partial x} K(x,y;\cdot) \right\|_{E_p} + \left\| \frac{\partial}{\partial y} K(x,y;\cdot) \right\|_{E_p} \leq \frac{C}{|x-y|^{n+1}}, \ x,y \in \mathbb{R}^n, \ x \neq y.
\]

Then, by proceeding as in the proof of [25] Proposition 2, p. 34 and Corollary 2, p. 36], we prove that the maximal operator \( T^* \) defined by
\[
T^* f(x) = \sup_{\varepsilon>0} \left\| \int_{|x-y|>\varepsilon} W_t(x,y)f(y)dy \right\|_{E_p}
\]
is bounded from \( L^p(\mathbb{R}^n) \) into itself, for every \( 1 < p < \infty \), and from \( L^1(\mathbb{R}^n) \) into \( L^{1,\infty}(\mathbb{R}^n) \). By combining this fact with [20] Theorem 3.3 we conclude that the operator \( V_p(W_{t,\text{loc}}) \) is bounded from \( L^p(\mathbb{R}^n) \) into itself, for every \( 1 < p < \infty \), and from \( L^1(\mathbb{R}^n) \) into \( L^{1,\infty}(\mathbb{R}^n) \).

We consider now the variation operator defined by
\[
V_p(W^Z_{t^Z} - W^Z_{t,\text{loc}})(f)(x) = \sup_{\{t_j\}_{j \in \mathbb{N}} > 0} \left( \sum_{j=0}^{\infty} \left| \int_{|x-y|>\gamma(x)} (W^Z_{t_j}(x,y) - W^Z_{t_{j+1}}(x,y)) f(y)dy \right|^{r} \right)^{\frac{1}{r}}.
\]
Assume that \( \{ t_j \} \in \mathbb{N} \) is a real decreasing sequence that converges to zero. We can write

\[
(19) \quad \left( \sum_{j=0}^{\infty} \int_{|x-y| > \gamma(x)} \left| W_{t_j}^x(x,y) - W_{t_{j+1}}^x(x,y) \right| f(y)dy \right)^{\frac{1}{p}}
\]

\[
\leq \sum_{j=0}^{\infty} \int_{|x-y| > \gamma(x)} |f(y)| \int_{t_{j+1}}^{t_j} |\frac{\partial}{\partial t} W_{t}^x(x,y)| dt dy
\]

\[
= \int_{|x-y| > \gamma(x)} |f(y)| \int_{0}^{\infty} |\frac{\partial}{\partial t} W_{t}^x(x,y)| dt dy, \quad x \in \mathbb{R}^n.
\]

According to \( [11] (2.7) \), for every \( N \in \mathbb{N} \) there exist \( c, C > 0 \) such that

\[
(20) \quad \left| \frac{\partial}{\partial t} W_{t}^x(x,y) \right| \leq C \left( 1 + \frac{t}{\gamma(x)^2} + \frac{t}{\gamma(y)^2} \right)^{-N} e^{-|x-y|^2 t}, \quad x, y \in \mathbb{R}^n, \quad t > 0.
\]

Estimation (20) allows us to obtain

\[
(21) \quad \int_{|x-y| > \gamma(x)} \int_{0}^{\infty} \left| \frac{\partial}{\partial t} W_{t}^x(x,y) \right| dt |f(y)| dy
\]

\[
\leq C \int_{|x-y| > \gamma(x)} |f(y)| \int_{0}^{\infty} \frac{1}{t^{2+2n}} \left( 1 + \frac{t}{\gamma(x)^2} \right)^{-n-2} e^{-|x-y|^2 t} dt dy
\]

\[
\leq C \gamma(x)^n \int_{|x-y| > \gamma(x)} |f(y)| \int_{0}^{\infty} \frac{1}{u^{2+2n}} \left( 1 + u^{2+n} \right)^{\frac{n+2}{2}} u^{n-1} e^{-\frac{|x-y|^2}{|x-y|^2 u^2}} du dy
\]

\[
\leq C \gamma(x)^n \int_{|x-y| > \gamma(x)} |f(y)| \left( \frac{\gamma(x)}{|x-y|} \right)^{n+2} dy
\]

\[
= C \sum_{k=0}^{\infty} \frac{1}{2^{2k}(2^k \gamma(x))^n} \int_{|x-y| \leq 2k+1\gamma(x)} |f(y)| |dy| \leq C \mathcal{M}(f)(x), \quad x \in \mathbb{R}^n.
\]

Here \( \mathcal{M} = \mathcal{M}_1 \) denotes the Hardy-Littlewood maximal function.

Also we have that

\[
(22) \quad \int_{|x-y| > \gamma(x)} \int_{0}^{\gamma(x)^2} \left| \frac{\partial}{\partial t} W_{t}^x(x,y) \right| dt |f(y)| dy
\]

\[
\leq C \int_{0}^{\gamma(x)^2} \int_{|x-y| > \gamma(x)} \frac{1}{t^{2+2n}} e^{-\frac{|x-y|^2}{t}} |f(y)| dy dt
\]

\[
\leq C \int_{0}^{\gamma(x)^2} \frac{e^{-\frac{|x-y|^2}{t}}}{t^{2+2n}} \int_{\mathbb{R}^n} 1 e^{-\frac{|x-y|^2}{t}} |f(y)| dy dt
\]

\[
\leq C \sup_{t > 0} \frac{1}{t^{2+2n}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{t}} |f(y)| dy \leq C \mathcal{M}(f)(x), \quad x \in \mathbb{R}^n.
\]

From (19), (21) and (22) we conclude that

\[
V_p \left( W_{t}^x - W_{t, \text{loc}}^x \right)(f)(x) \leq C \mathcal{M}(f)(x), \quad x \in \mathbb{R}^n.
\]
Hence, the operator \( V_\rho (W_t \to W_{t,\text{loc}}) \) is bounded from \( L^p(\mathbb{R}^n) \) into itself, for every \( 1 < p < \infty \), and from \( L^1(\mathbb{R}^n) \) into \( L^{1,\infty}(\mathbb{R}^n) \). We now analyze the operator

\[
V_\rho (W_{t,\text{loc}} - W_{t,\text{loc}})(f) = \sup_{\{t_j\}_{j\in\mathbb{N}} > 0} \left( \sum_{j=0}^{\infty} \int_{|x-y| < \gamma(x)} \left( W_{t_j}^\rho (x, y) - W_{t_j}(x, y) \right) f(y) dy \right)^{\frac{1}{\rho}}, \quad x \in \mathbb{R}^n.
\]

(23) \[
\begin{align*}
&\left( \sum_{j=0}^{\infty} \int_{|x-y| < \gamma(x)} \left( W_{t_j}^\rho (x, y) - W_{t_j}(x, y) - \left( W_{t_{j+1}}^\rho (x, y) - W_{t_{j+1}}(x, y) \right) \right) f(y) dy \right)^{\frac{1}{\rho}} \\
&= \left( \sum_{j=0}^{\infty} \int_{|x-y| < \gamma(x)} f(y) \int_{t_j}^{t_{j+1}} \frac{\partial}{\partial t} \left( W_t^\rho (x, y) - W_t(x, y) \right) dt dy \right)^{\frac{1}{\rho}} \\
&\leq \sum_{j=0}^{\infty} \int_{|x-y| < \gamma(x)} \left| f(y) \right| \int_{t_j}^{t_{j+1}} \frac{\partial}{\partial t} \left( W_t^\rho (x, y) - W_t(x, y) \right) dt dy \\
&= \int_{|x-y| < \gamma(x)} \left| f(y) \right| \int_{0}^{\gamma(x)} \frac{\partial}{\partial t} \left( W_t^\rho (x, y) - W_t(x, y) \right) dt dy \\
&= I_1(f)(x) + I_2(f)(x), \quad x \in \mathbb{R}^n.
\end{align*}
\]

(24) \[
I_2(f)(x) \leq C \int_{|x-y| < \gamma(x)} \left| f(y) \right| \int_{|x-y|^2 / \frac{t+1}{\rho}}^{\infty} e^{-\frac{|x-y|^2}{t+1}} dt dy \\
\leq C \int_{|x-y| < \gamma(x)} \left| f(y) \right| \int_{0}^{\gamma(x)^2} \frac{dt}{t^{\frac{1}{\rho}} + 1} dy \\
\leq \frac{C}{\gamma(x)^{\rho}} \int_{|x-y| < \gamma(x)} \left| f(y) \right| dy \leq C \mathcal{M}(f)(x), \quad x \in \mathbb{R}^n.
\]

Note firstly that, according to (20) and the known estimates for \( \frac{\partial}{\partial t} W_t \), for certain \( C, c > 0 \), we get

\[
I_2(f)(x) \leq C \int_{|x-y| < \gamma(x)} \left| f(y) \right| \int_{|x-y|^2 / \frac{t+1}{\rho}}^{\infty} e^{-\frac{|x-y|^2}{t+1}} dt dy \\
\leq C \int_{|x-y| < \gamma(x)} \left| f(y) \right| \int_{0}^{\gamma(x)^2} \frac{dt}{t^{\frac{1}{\rho}} + 1} dy \\
\leq \frac{C}{\gamma(x)^{\rho}} \int_{|x-y| < \gamma(x)} \left| f(y) \right| dy \leq C \mathcal{M}(f)(x), \quad x \in \mathbb{R}^n.
\]

On the other hand, the perturbation formula (11) (5.25)) leads to

\[
\frac{\partial}{\partial t} \left( W_t(x, y) - W_t^\rho (x, y) \right) = \int_{\mathbb{R}^n} V(z) W_{t,\gamma}^\rho (x, z) W_{t,\gamma}^\rho (z, y) dz \\
+ \int_{0}^{t} \int_{\mathbb{R}^n} V(z) \frac{\partial}{\partial t} W_{t-s}(x, z) W_{t-s}^\rho (z, y) dz ds \\
+ \int_{0}^{t} \int_{\mathbb{R}^n} W_{t-s}(x, z) V(z) \frac{\partial}{\partial s} W_{t-s}^\rho (z, y) dz ds \\
= K_1(x, y, t) + K_2(x, y, t) + K_3(x, y, t), \quad x, y \in \mathbb{R}^n \text{ and } t > 0.
\]

Hence \( I_1(f) = T_1(f) + T_2(f) + T_3(f) \), where, for \( m = 1, 2, 3 \),

\[
T_m f(x) = \int_{|x-y| < \gamma(x)} \left| f(y) \right| \int_{0}^{\gamma(x)^2} \left| K_m(x, y, t) \right| dt dy, \quad x \in \mathbb{R}^n.
\]
By [11] (2.2) and (2.8), we obtain

\[
\int_0^{\gamma(x)^2} |K_1(x, y, t)| dt \leq C \int_0^{\gamma(x)^2} \int_{\mathbb{R}^n} V(z) \frac{1}{t^{\frac{n}{2}+1}} e^{-\frac{|z-y|^2}{t}} e^{-\frac{|z-y|^2}{t}} dz dt
\]

\[
\leq C \int_0^{\gamma(x)^2} \frac{1}{t^{\frac{n}{2}+1}} e^{-\frac{|y|^2}{t}} \int_{\mathbb{R}^n} V(z) dz dt
\]

\[
\leq C \int_0^{\gamma(x)^2} \frac{1}{t^{\frac{n}{2}+1}} e^{-\frac{|y|^2}{t}} \left( \frac{\sqrt{t}}{\gamma(x)} \right)^{\delta} dt,
\]

for a certain \( \delta > 0 \). Then,

\[
|T_1(f)(x)| \leq C \int_{|y-y'|<\gamma(x)} |f(y)| \int_0^{\gamma(x)^2} \frac{1}{t^{\frac{n}{2}+1}} e^{-\frac{|y-y'|^2}{t}} \left( \frac{\sqrt{t}}{\gamma(x)} \right)^{\delta} dt dy
\]

\[
\leq C \int_0^{\gamma(x)^2} \frac{t^{\frac{n}{2}+1}}{\gamma(x)^{\delta}} \int_{\mathbb{R}^n} |f(y)| e^{-\frac{|y-y'|^2}{t}} dy dt
\]

(25)

\[
\leq C \sup_{t>0} \frac{1}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n} |f(y)| e^{-\frac{|y-y'|^2}{t}} dy \leq C \mathcal{M}(f)(x), \quad x \in \mathbb{R}^n.
\]

Also, since \( \frac{t}{4} < t - s < t \) provided that \( 0 < s < \frac{t}{4} \), [11] (2.2) and (2.8)] imply that, for some \( 0 < c < \frac{1}{4} \),

\[
\int_0^{\gamma(x)^2} |K_2(x, y, t)| dt \leq C \int_0^{\gamma(x)^2} \int_0^{\frac{t}{4}} \int_{\mathbb{R}^n} V(z) \frac{1}{(t-s)^{\frac{n}{2}+1}} e^{-\frac{|z-y|^2}{t-s}} e^{-\frac{|z-y|^2}{s}} dz ds dt
\]

\[
\leq C \int_0^{\gamma(x)^2} \frac{1}{(t-s)^{\frac{n}{2}+1}} \int_{\mathbb{R}^n} V(z) e^{-\frac{|z-y|^2}{t-s}} e^{-\frac{|z-y|^2}{s}} dz ds dt
\]

\[
\leq C \int_0^{\gamma(x)^2} \frac{1}{(t-s)^{\frac{n}{2}+1}} \int_{\mathbb{R}^n} e^{-\frac{|z-y|^2}{t-s}} \int_{\mathbb{R}^n} e^{-\frac{|z-y|^2}{s}} V(z) dz ds dt
\]

\[
\leq C \int_0^{\gamma(x)^2} \frac{1}{(t-s)^{\frac{n}{2}+1}} \int_{\mathbb{R}^n} e^{-\frac{|z-y|^2}{t-s}} \int_{\gamma(y)^2} s^{\frac{n}{2}+\frac{1}{2}} e^{-\frac{|z-y|^2}{s}} dz ds dt
\]

\[
\leq C \int_0^{\gamma(x)^2} \frac{1}{(t-s)^{\frac{n}{2}+1}} e^{-\frac{|y|^2}{t-s}} \int_{\gamma(y)^2} s^{\frac{n}{2}+\frac{1}{2}} e^{-\frac{|y|^2}{s}} dy |x-y|^2 dt, \quad x, y \in \mathbb{R}^n, \quad |x-y| \leq \gamma(x).
\]

We have taken into account that \( \gamma(x) \sim \gamma(y) \), provided that \( |x-y| \leq \gamma(x) \). Then,

\[
|T_2f(x)| \leq C \frac{\gamma(x)}{\gamma(x)^{\delta}} \int_{|y-y'|<\gamma(x)} |f(y)| \int_0^{\gamma(x)^2} e^{-\frac{\delta}{t} |y-y'|^2} dt dy
\]

\[
\leq C \frac{\gamma(x)}{\gamma(x)^{\delta}} \int_0^{\gamma(x)^2} \frac{dt}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{\delta}{t} |y-y'|^2} |f(y)| dy dt
\]

(26)

\[
\leq C \sup_{t>0} \frac{1}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{\delta}{t} |y-y'|^2} |f(y)| dy \leq C \mathcal{M}(f)(x), \quad x \in \mathbb{R}^n.
\]

By proceeding in a similar way and using [11] (2.7) we see that

(27)

\[
|T_3f(x)| \leq C \mathcal{M}(f)(x), \quad x \in \mathbb{R}^n.
\]

From (25), (26) and (27), we get

(28)

\[
I_1(f)(x) \leq C \mathcal{M}(f)(x), \quad x \in \mathbb{R}^n.
\]

Inequalities (24) and (25) imply that

\[
V_p(W_{t,loc}^Z - W_{t,loc})(f)(x) \leq C \mathcal{M}(f)(x), \quad x \in \mathbb{R}^n.
\]

Hence, the operator \( V_p(W_{t,loc}^Z - W_{t,loc}) \) is bounded from \( L^p(\mathbb{R}^n) \) into itself, for every \( 1 < p < \infty \), and from \( L^1(\mathbb{R}^n) \) into \( L^{1,\infty}(\mathbb{R}^n) \).

Finally, by (18) we conclude that \( V_p(W_{t,loc}^Z) \) is a bounded operator from \( L^p(\mathbb{R}^n) \) into itself, for every \( 1 < p < \infty \), and from \( L^1(\mathbb{R}^n) \) into \( L^{1,\infty}(\mathbb{R}^n) \).
4. Variation operators associated with Riesz transform $R^\mathcal{L}_\ell$

In this section we prove Proposition 1.1 and Theorem 1.2. As it was mentioned in Section 1, for every $\ell = 1, \cdots, n$, the $\ell$-th Riesz transform associated with $\mathcal{L}$ is defined formally by

$$R^\mathcal{L}_\ell = \frac{\partial}{\partial x_\ell} \mathcal{L}^{-\frac{1}{2}}.$$  

Here $\mathcal{L}^{-\frac{1}{2}}$ denotes the negative square root of the operator $\mathcal{L}$ given by

$$\mathcal{L}^{-\frac{1}{2}} f(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^n} \int_{-\infty}^{+\infty} (i\tau)^{-\frac{1}{2}} \Gamma(x, y, \tau) d\tau f(y) dy,$$

where $\Gamma(x, y, \tau)$ represents the fundamental solution for the operator $\mathcal{L} + i\tau$, with $\tau \in \mathbb{R}$ (see [23, §2]).

We recall that, for every $\ell = 1, \cdots, n$, the function $R^\mathcal{L}_\ell (x, y)$ is defined by

$$R^\mathcal{L}_\ell (x, y) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} (i\tau)^{-\frac{1}{2}} \frac{\partial}{\partial x_\ell} \Gamma(x, y, \tau) d\tau, \quad x, y \in \mathbb{R}^n, \ x \neq y.$$  

The following estimates for the kernels $R^\mathcal{L}_\ell$, $\ell = 1, \cdots, n$, were established in [23, Sections 5 and 6] (see also [2] Lemma 1)) and will be very useful in the sequel.

**Lemma 4.1.** Let $\ell = 1, \cdots, n$ and $V \in B_q$.

(i) Assume that $q > n$. Then, for every $k \in \mathbb{N}$ there exists $C > 0$ such that

$$|R^\mathcal{L}_\ell (x, y)| \leq C \frac{1}{(1 + |x-y|/\gamma(x))^k} \frac{1}{|x-y|^n}.$$  

Moreover, we have

$$|R^\mathcal{L}_\ell (x, y) - R_\ell (x-y)| \leq C \frac{1}{|x-y|^n} \left( \frac{|x-y|}{\gamma(x)} \right)^{2-n/q}, \quad 0 < |x-y| < \gamma(x).$$

(ii) Suppose that $n/2 < q < n$. Then, for every $k \in \mathbb{N}$ there exists $C > 0$ such that

$$|R^\mathcal{L}_\ell (x, y)| \leq C \frac{1}{(1 + |x-y|/\gamma(y))^k} \frac{1}{|x-y|^{n-1}} \times \left( \frac{1}{|x-y|} + \int_{B(x,|x-y|/4)} \frac{V(z)}{|z-x|^{n-1}} dz \right).$$

Also, we have

$$|R^\mathcal{L}_\ell (x, y) - R_\ell (x-y)| \leq C \frac{1}{|x-y|^{n-1}} \times \left( \frac{1}{|x-y|} \left( \frac{|x-y|}{\gamma(x)} \right)^{2-n/q} + \int_{B(x,|x-y|/4)} \frac{V(z)}{|z-x|^{n-1}} dz \right), \quad 0 < |x-y| < \gamma(x).$$

**Remark 4.1.** Note that, according to [13], if $V \in B_n$, there exists $\varepsilon > 0$ such that $V \in B_{n+\varepsilon}$. Then, the estimates in Lemma 4.1, (i), can be applied to $V \in B_n$ by taking $q = n + \varepsilon$, where $\varepsilon > 0$ is small enough and it depends on $V$.

We consider, for every $\ell = 1, \cdots, n$ and $\varepsilon > 0$,

$$R^\mathcal{L}_{\ell, \varepsilon} (f)(x) = \int_{|x-y| > \varepsilon} R^\mathcal{L}_\ell (x, y) f(y) dy, \quad x \in \mathbb{R}^n.$$  

According to Lemma 4.1 it is not hard to see that, for every $\ell = 1, \cdots, n$, $\varepsilon > 0$, $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, the integral defining $R^\mathcal{L}_{\ell, \varepsilon} (f)(x)$ is absolutely convergent for each $x \in \mathbb{R}^n$.

Before proving Proposition 1.1 we establish the $L^p$-boundedness properties of the maximal operator $R^\mathcal{L}_{\ell, \ast}$ defined by

$$R^\mathcal{L}_{\ell, \ast} (f) = \sup_{\varepsilon > 0} |R^\mathcal{L}_{\ell, \varepsilon} (f)|.$$  

**Proposition 4.1.** Let $\ell = 1, \cdots, n$ and $V \in B_q$, $q \geq n/2$. Then, if $1 < p < p_0$, where $\frac{1}{p_0} = \left( \frac{1}{q} - \frac{n}{2} \right)_+$, $R^\mathcal{L}_{\ell, \ast}$ is bounded from $L^p(\mathbb{R}^n)$ into itself. Moreover, $R^\mathcal{L}_{\ell, \ast}$ is bounded from $L^1(\mathbb{R}^n)$ into $L^{1, \infty}(\mathbb{R}^n)$. 
By Proposition 4.1, in order to show Proposition 1.1 it is sufficient in the spirit of the general procedure described at the beginning of this section, as follows

\[ R_{\ell,\tau}^{L^p}(f)(x) \leq \int_{|x-y|<\gamma(x)} |R_{\ell,\tau}^{L^p}(x,y) - R_{\ell}(x-y)||f(y)|dy \\
+ \int_{|x-y|>\gamma(x)} |R_{\ell,\tau}^{L^p}(x,y)||f(y)|dy + \sup_{\varepsilon>0} \int_{|x-y|<\gamma(x)} R_{\ell}(x-y)f(y)dy \]

= \tau_1(|f|)(x) + \tau_2(|f|)(x) + \tau_3(f)(x), \ x \in \mathbb{R}^n.

It is clear that

\[ \tau_3(f)(x) \leq \sup_{\varepsilon>0} \int_{|x-y|>\varepsilon} R_{\ell}(x-y)f(y)dy, \ x \in \mathbb{R}^n. \]

Then, from well known results we infer that \( \tau_3 \) is bounded from \( L^r(\mathbb{R}^n) \) into itself when \( 1 < r < \infty \), and from \( L^1(\mathbb{R}^n) \) into \( L^{1,\infty}(\mathbb{R}^n) \).

For a certain \( M > 1 \), \( \frac{1}{M} \leq \frac{\gamma(x)}{\gamma(y)} \leq M \), when \( |x-y| \leq \gamma(y) \). Moreover, if \( |x-y| > \gamma(x) \), \( \gamma(x)/M \). Indeed, it is sufficient to observe that

\[ \{ y \in \mathbb{R}^n : |x-y| > \gamma(x) \} \subset \{ y \in \mathbb{R}^n : |x-y| < \gamma(y) \ and \ |x-y| > \gamma(x) \} \]

\[ \cup \ y \in \mathbb{R}^n : |x-y| > \gamma(y) \ and \ |x-y| > \gamma(x) \} \}

We denote by \( \tau_j^* \) the adjoint operator of \( \tau_j \), \( j = 1, 2 \). We have that

\[ \tau_1^*|g||g| = \int_{|x-y|<\gamma(x)} |R_{\ell,\tau}^{L^p}(x,y) - R_{\ell}(x-y)||g(x)||dx \]

\[ \leq \int_{|x-y|<M\gamma(y)} |R_{\ell,\tau}^{L^p}(x,y) - R_{\ell}(x-y)||g(x)||dx, \ y \in \mathbb{R}^n, \]

and

\[ \tau_2^*|g||g| = \int_{|x-y|\geq\gamma(x)} |R_{\ell,\tau}^{L^p}(x,y)||g(x)||dx \]

\[ \leq \int_{|x-y|\geq\gamma(y)/M} |R_{\ell,\tau}^{L^p}(x,y)||g(x)||dx, \ y \in \mathbb{R}^n. \]

According to [23, (5.9)] and the proof of Lemma 5.8 and the arguments in the proof of [23, Lemma 5.7] we obtain that \( \tau_j^* \), \( i = 1, 2 \), are bounded from \( L^r(\mathbb{R}^n) \) into itself when \( p_0 < r \leq \infty \). Then, \( \tau_i \), \( i = 1, 2 \), are bounded from \( L^r(\mathbb{R}^n) \) into itself when \( 1 \leq r < p_0 \).

By combining the above properties we conclude that \( R_{\ell,\tau}^{L^p} \) is a bounded operator from \( L^p(\mathbb{R}^n) \) into itself, provided that \( 1 < p < p_0 \), and from \( L^1(\mathbb{R}^n) \) to \( L^{1,\infty}(\mathbb{R}^n) \).

Proof of Proposition [1.1] By Proposition 4.1 in order to show Proposition 1.1 it is sufficient to see [3] for every \( \phi \in C_c^\infty(\mathbb{R}^n) \).

Suppose that \( \phi \in C_c^\infty(\mathbb{R}^n) \). By \( \Gamma_0(x,y,\tau) \) we denote the fundamental solution of the operator \( -\Delta + i\tau \) in \( \mathbb{R}^n \), with \( \tau \in \mathbb{R} \). We are going to see that, for every \( \ell = 1, \ldots, n, \mathcal{L}^{-\frac{\ell}{2}}\phi = (-\Delta)^{-\frac{\ell}{2}}\phi \)

admits derivative with respect to \( x_\ell \), for almost all \( \mathbb{R}^n \), and that a.e. \( x \in \mathbb{R}^n \),

\[ \frac{\partial}{\partial x_\ell} \left( \mathcal{L}^{-\frac{\ell}{2}}\phi(x) - (-\Delta)^{-\frac{\ell}{2}}\phi(x) \right) = -\frac{1}{2\pi} \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{+\infty} \frac{\partial}{\partial x_\ell} \left( \Gamma(x,y,\tau) - \Gamma_0(x,y,\tau) \right) (i\tau)^{-\frac{\ell}{2}} \tau dy. \]

To simplify the notation we consider \( \ell = 1 \). Also we assume that \( q > n/2 \) ([13]). According to [23, Lemma 4.5], for every \( k \in \mathbb{N} \) there exists \( C_k > 0 \) such that

\[ |\Gamma(x,y,\tau) - \Gamma_0(x,y,\tau)| \leq \frac{C_k|x-y|^{2-n}}{(1 + |\tau|^2|x-y|)^k} \left( \frac{|x-y|}{\gamma(x)} \right)^{2-n} \]
provided that \( \tau \in \mathbb{R}, x, y \in \mathbb{R}^n, |x - y| \leq \gamma(x) \). Moreover, by [23, Theorem 2.7], for every \( k \in \mathbb{N} \) there exists \( C_k > 0 \) such that

\[
|\Gamma(x, y, \tau)| \leq \frac{C_k|x-y|^{2-n}}{(1 + |\tau|^{\frac{n}{2}})^k (1 + \frac{|x-y|}{\tau|\tau|})}, \quad x, y \in \mathbb{R}^n, \quad \tau \in \mathbb{R},
\]

and

\[
|\Gamma_0(x, y, \tau)| \leq \frac{C_k|x-y|^{2-n}}{(1 + |\tau|^{\frac{n}{2}})^k}, \quad x, y \in \mathbb{R}^n, \quad \tau \in \mathbb{R}.
\]

We will use repeatedly without reference the following equality

\[
\int_{-\infty}^{\infty} \frac{1}{|\tau|^{\frac{n}{2}} (1 + |x-y||\tau|^{\frac{n}{2}})^k} d\tau = \frac{C_k}{|x-y|}, \quad x, y \in \mathbb{R}^n, \quad x \neq y \text{ and } k \geq 1.
\]

From (35) and (36) we deduce that

\[
\int_{-\infty}^{\infty} \frac{1}{|\tau|^{\frac{n}{2}} (1 + |x-y||\tau|^{\frac{n}{2}})^k} d\tau = \frac{C_k}{|x-y|}, \quad x, y \in \mathbb{R}^n, \quad x \neq y \text{ and } k \geq 1.
\]

Then, the function \( F = \mathcal{L}^{-\frac{n}{2}} \phi - (-\Delta)^{-\frac{n}{2}} \phi \) defines in \( \mathbb{R}^n \) a distribution \( S_F \) by

\[
(S_F, \psi) = \int_{\mathbb{R}^n} F(y) \psi(y) dy, \quad \psi \in C_c^\infty(\mathbb{R}^n).
\]

We have that, by writing \( \mathcal{F} = (x_2, \ldots, x_n) \) and considering \( \psi \in C_c^\infty(\mathbb{R}^n) \),

\[
\frac{\partial}{\partial x_1} S_F, \psi \) = - \int_{\mathbb{R}^n} F(x) \frac{\partial}{\partial x_1} \psi(x) dx
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{\infty} (-i\tau)^{-\frac{n}{2}} (\Gamma(x, y, \tau) - \Gamma_0(x, y, \tau)) d\tau dy \frac{\partial}{\partial x_1} \psi(x) dx
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{\infty} (-i\tau)^{-\frac{n}{2}} \int_{-\infty}^{\infty} (\Gamma(x, y, \tau) - \Gamma_0(x, y, \tau)) \frac{\partial}{\partial x_1} \psi(x) dx d\tau dy d\mathcal{F}.
\]

We now apply partial integration in the following way. Since the integral is absolutely convergent (see (37)) it follows that

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{\infty} (-i\tau)^{-\frac{n}{2}} \int_{-\infty}^{\infty} (\Gamma(x, y, \tau) - \Gamma_0(x, y, \tau)) \frac{\partial}{\partial x_1} \psi(x) dx d\tau dy d\mathcal{F}
\]

\[
= \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{\infty} (-i\tau)^{-\frac{n}{2}} \left( \int_{-\infty}^{y_1 - \epsilon} + \int_{y_1 + \epsilon}^{\infty} \right) (\Gamma(x, y, \tau) - \Gamma_0(x, y, \tau)) \frac{\partial}{\partial x_1} \psi(x) dx d\tau dy d\mathcal{F}
\]

\[
= \lim_{\epsilon \to 0^+} \left( - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{\infty} (-i\tau)^{-\frac{n}{2}} \left( \int_{-\infty}^{y_1 - \epsilon} + \int_{y_1 + \epsilon}^{\infty} \right) \frac{\partial}{\partial x_1} (\Gamma(x, y, \tau) - \Gamma_0(x, y, \tau)) \psi(x) dx d\tau dy d\mathcal{F} 
\]

\[
+ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{\infty} (-i\tau)^{-\frac{n}{2}} \left( (\Gamma(y_1 - \epsilon, \mathcal{F}, y, \tau) - \Gamma_0(y_1 - \epsilon, \mathcal{F}, y, \tau)) \psi(y_1 - \epsilon, \mathcal{F}) 
\]

\[
- (\Gamma(y_1 + \epsilon, \mathcal{F}, y, \tau) - \Gamma_0(y_1 + \epsilon, \mathcal{F}, y, \tau)) \psi(y_1 + \epsilon, \mathcal{F}) \right) d\tau dy d\mathcal{F}.
\]
We have that
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{-\infty}^{+\infty} |\phi(y)||\psi(x)||\tau|^{-1/2} \left| \frac{\partial}{\partial x_1} (\Gamma(x, y, \tau) - \Gamma_0(x, y, \tau)) \right| drdxdy < \infty.
\]
Indeed, by [23, Theorem 2.7] (see also the proof of [23, Lemma 5.7]), for every \( k \in \mathbb{N} \), there exists \( C_k > 0 \) such that, for every \( x, y \in \mathbb{R}^n \) and \( \tau \in \mathbb{R} \),
\[
\left| \frac{\partial}{\partial x_1} \Gamma(x, y, \tau) \right| \leq C_k \frac{|x-y|^{2-n}}{(1+|\tau|^{1/2}|x-y|)^k} \left( \int_{|z-x|<|x-y|} V(z) + \frac{1}{|x-y|} \right).
\]
Moreover, for every \( k \in \mathbb{N} \), there exists \( C_k > 0 \) such that
\[
\left| \frac{\partial}{\partial x_1} \Gamma_0(x, y, \tau) \right| \leq C_k \frac{|x-y|^{1-n}}{(1+|\tau|^{1/2}|x-y|)^k}, \quad x, y \in \mathbb{R}^n, \ \tau \in \mathbb{R}.
\]
By [23, p. 541], for every \( k \in \mathbb{N} \), there exists \( C_k > 0 \) such that
\[
\left| \frac{\partial}{\partial x_1} \Gamma(x, y, \tau) - \frac{\partial}{\partial x_1} \Gamma_0(x, y, \tau) \right| \leq C_k \frac{|x-y|^{2-n}}{(1+|\tau|^{1/2}|x-y|)^k} \left( \int_{|z-x|<|x-y|} V(z) + \frac{1}{|x-y|} \right), \quad |x-y| < \gamma(y), \ \tau \in \mathbb{R},
\]
where \( q > n/2 \).
Assume firstly that \( V \in B_q \). Then \( V \in B_q \) for some \( q > n \) (13). According to [2, Lemma 1], we deduce, from (11) and (13), that for every \( k \in \mathbb{N} \), there exists \( C_k > 0 \) such that
\[
\left| \frac{\partial}{\partial x_1} \Gamma(x, y, \tau) \right| \leq C_k \frac{|x-y|^{1-n}}{(1+|\tau|^{1/2}|x-y|)^k} \left( \int_{|z-x|<|x-y|} V(z) + \frac{1}{|x-y|} \right), \quad x, y \in \mathbb{R}^n, \ \tau \in \mathbb{R},
\]
and, for every \( \tau \in \mathbb{R} \) and \( |x-y| < \gamma(y) \),
\[
\left| \frac{\partial}{\partial x_1} \Gamma(x, y, \tau) - \frac{\partial}{\partial x_1} \Gamma_0(x, y, \tau) \right| \leq C_k \frac{|x-y|^{2-n}}{(1+|\tau|^{1/2}|x-y|)^k} \left( \int_{|z-x|<|x-y|} V(z) + \frac{1}{|x-y|} \right).
\]
According to [23, Lemma 1.4], \( \gamma \) and \( 1/\gamma \) are bounded on any compact subset of \( \mathbb{R}^n \). Since \( \phi \) and \( \psi \) have compact support, there exists \( A > 0 \) such that \( |x-y| \leq A\gamma(y), x \in supp \psi \) and \( y \in supp \phi \). Then, by using (12), (14) and (15) we get
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{-\infty}^{+\infty} |\phi(y)||\psi(x)||\tau|^{-1/2} \left| \frac{\partial}{\partial x_1} (\Gamma(x, y, \tau) - \Gamma_0(x, y, \tau)) \right| drdxdy \leq C \int_{supp \psi} |\psi(x)| \int_{supp \phi} |\phi(y)| \int_{-\infty}^{+\infty} \frac{|x-y|^{3-n-n/q}}{|\tau|^{1/2}(1+|\tau|^{1/2}|x-y|)^2} d\tau dy dx \leq C \int_{supp \psi} \int_{supp \phi} \frac{1}{|x-y|^{n+q}} dy dx < \infty.
\]
We consider now \( \frac{2}{q} < q < n \). We recall that the 1-th Euclidean fractional integral is bounded from \( L^q(\mathbb{R}^n) \) into \( L^p(\mathbb{R}^n) \), when \( \frac{1}{p} = \frac{1}{q} - \frac{1}{n} \) (25). Since \( q > n/2 \) in order to establish (10) we only have to see that
\[
\int_{\mathbb{R}^n} |\phi(y)||\psi(x)| \int_{|x-y| \leq \gamma(y)} \int_{-\infty}^{+\infty} \frac{|x-y|^{2-n}}{|\tau|^{1/2}(1+|\tau|^{1/2}|x-y|)^2} d\tau dy dx \leq C \int_{supp \psi} \int_{supp \phi} \frac{1}{|x-y|^{n-1}} d\tau dy dx \leq C \int_{supp \phi} \left( \int_{supp \psi} \frac{1}{|x-y|^{p(n-1)}} d\tau \right)^{1/p_0} \left( \int_{\mathbb{R}^n} |\Gamma_0(x, y) V(x)|^{p_0} dx \right)^{1/p_0} dy \leq C \int_{supp \phi} \left( \int_{supp \psi} \frac{1}{|x-y|^{p(n-1)}} d\tau \right)^{1/p_0} \left( \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} V(z) dz \right|^{q} dx \right)^{1/q} dy < \infty,
\]
because $V \in L_{\text{loc}}^p(\mathbb{R}^n)$, and $(n-1)(p'_0 - 1) < 1$ when $q > n/2$.

Hence, (40) holds and we can write

$$
\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{+\infty} \left( -i\tau \right)^{-\frac{n}{2}} \left( \int_{-\infty}^{y_1+\varepsilon} + \int_{y_1+\varepsilon}^{+\infty} \right) \frac{\partial}{\partial x_1} \left( \Gamma(x, y, \tau) - \Gamma_0(x, y, \tau) \right) \psi(x) dx_1 d\tau d\sigma d\tau = 
\int_{\mathbb{R}^n} \psi(x) \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{+\infty} \left( -i\tau \right)^{-\frac{n}{2}} \frac{\partial}{\partial x_1} \left( \Gamma(x, y, \tau) - \Gamma_0(x, y, \tau) \right) d\tau dy dx.
$$

Our next task is to see that $\lim_{\varepsilon \to 0^+} I(\varepsilon) = 0$, where

$$
I(\varepsilon) = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{+\infty} \left( -i\tau \right)^{-\frac{n}{2}} \left( \Gamma(y_1 - \varepsilon, \bar{x}, y, \tau) - \Gamma_0(y_1 - \varepsilon, \bar{x}, y, \tau) \right) \psi(y_1 - \varepsilon, \bar{x}) d\tau dy d\tau.
$$

We have that

$$
\left( \Gamma(y_1 - \varepsilon, \bar{x}, y, \tau) - \Gamma_0(y_1 - \varepsilon, \bar{x}, y, \tau) \right) \psi(y_1 - \varepsilon, \bar{x}) = (\Gamma(y_1 - \varepsilon, \bar{x}, y, \tau) - \Gamma(y_1 + \varepsilon, \bar{x}, y, \tau) - \Gamma_0(y_1 + \varepsilon, \bar{x}, y, \tau)) \psi(y_1 - \varepsilon, \bar{x})
$$

and

$$
\left( \Gamma(y_1 + \varepsilon, \bar{x}, y, \tau) - \Gamma_0(y_1 + \varepsilon, \bar{x}, y, \tau) \right) \psi(y_1 - \varepsilon, \bar{x}) = J_1(\bar{x}, y_1 + \varepsilon, \bar{x}, y, \tau) + J_2(\bar{x}, y, \varepsilon, \bar{x}, y, \tau),
$$

where $\bar{x} \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}^n$, $0 < \varepsilon < |\bar{x} - \bar{y}|/15$ and $\tau \in \mathbb{R}$.

According to [23] p. 535, if $\tau \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}^n$, $0 < \varepsilon < |\bar{x} - \bar{y}|/15$ and $\tau \in \mathbb{R}$,

$$
|\Gamma(y_1 - \varepsilon, \bar{x}, y, \tau) - \Gamma(y_1 + \varepsilon, \bar{x}, y, \tau)| \leq C \frac{\varepsilon^\delta}{(1 + |\tau|^{1/2}|\bar{x} - \bar{y}|)^{1/2}},
$$

and

$$
|\Gamma_0(y_1 - \varepsilon, \bar{x}, y, \tau) - \Gamma_0(y_1 + \varepsilon, \bar{x}, y, \tau)| \leq C \frac{\varepsilon^\delta}{(1 + |\tau|^{1/2}|\bar{x} - \bar{y}|)^{3/2}},
$$

for a certain $\delta > 0$ that depends on $q$.

By using (43), (54) and (55), we can deduce that

$$
\lim_{\eta \to 0^+} \int_{\mathbb{R}^n} |\phi(y)| \int_{|\bar{x} - \bar{y}| < \eta} \int_{-\infty}^{+\infty} |\tau|^{-1/2} |J_m(\bar{x}, y, \varepsilon, \tau)| d\tau d\sigma d\tau = 0, \quad m = 1, 2,
$$

uniformly in $\varepsilon \in (0, 1)$.

Indeed, let $\varepsilon \in (0, 1)$. According to (55) and (56), the mean value theorem leads to

$$
|J_2(\bar{x}, y, \varepsilon, \tau)| \leq C \varepsilon \left( \varepsilon + |\bar{x} - \bar{y}| \right)^{2-n}, \quad \bar{x}, \bar{y} \in \mathbb{R}^{n-1} \text{ and } \tau \in \mathbb{R}.
$$

Then, we have

$$
\int_{\mathbb{R}^n} |\phi(y)| \int_{|\bar{x} - \bar{y}| < \eta} \int_{-\infty}^{+\infty} |\tau|^{-1/2} |J_2(\bar{x}, y, \varepsilon, \tau)| d\tau d\sigma d\tau
\leq C \int_{\mathbb{R}^n} |\phi(y)| \int_{|\bar{x} - \bar{y}| < \eta} \int_{-\infty}^{+\infty} |\tau|^{-1/2} \varepsilon \left( \varepsilon + |\bar{x} - \bar{y}| \right)^{2-n} (1 + |\tau|^{1/2}(\varepsilon + |\bar{x} - \bar{y}|)^{1/2})^3 d\tau d\sigma d\tau
\leq C \int_{\mathbb{R}^n} |\phi(y)| \int_{|\bar{x} - \bar{y}| < \eta} \varepsilon (\varepsilon + |\bar{x} - \bar{y}|)^{-n} d\tau d\sigma d\tau
\leq C \int_{\mathbb{R}^n} |\phi(y)| \int_{|\bar{x} - \bar{y}| < \eta} |\bar{x} - \bar{y}|^{-n} d\tau d\sigma d\tau, \quad \eta > 0,
$$

where $C > 0$ does not depend on $\varepsilon$. Hence,

$$
\lim_{\eta \to 0^+} \int_{\mathbb{R}^n} |\phi(y)| \int_{|\bar{x} - \bar{y}| < \eta} \int_{-\infty}^{+\infty} |\tau|^{-1/2} |J_2(\bar{x}, y, \varepsilon, \tau)| d\tau d\sigma d\tau = 0,
$$

uniformly in $\varepsilon \in (0, 1)$. 

By using (13), since \( \text{supp} \, \phi \) is compact, we get, for a certain \( a > 0 \), that if \( \varepsilon \in (0, 1) \)
\[
\int_{\mathbb{R}^n} |\phi(y)| \int_{|\vec{x}| < \eta} \int_{-\infty}^{\infty} |\tau|^{-1/2} |J_1(\vec{x}, y, \varepsilon, \tau)| d\tau dy \\
\leq C \int_{\mathbb{R}^n} |\phi(y)| \int_{|\vec{x}| < \eta} \int_{-\infty}^{\infty} |\tau|^{-1/2} \int_{y_1 - \varepsilon}^{y_1 + \varepsilon} \frac{\partial}{\partial u} \left( \Gamma(u, \vec{x}, y, \tau) - \Gamma_0(u, \vec{x}, y, \tau) \right) du d\tau dy \\
\leq C \int_{\mathbb{R}^n} |\phi(y)| \int_{|\vec{x}| < \eta} \int_{y_1 - \varepsilon}^{y_1 + \varepsilon} \int_{-\infty}^{\infty} |\tau|^{-1/2} \frac{\partial}{\partial u} \left( \Gamma(u, \vec{x}, y, \tau) - \Gamma_0(u, \vec{x}, y, \tau) \right) d\tau du d\tau dy \\
\leq C \int_{\mathbb{R}^n} |\phi(y)| \int_{|\vec{x}| < \eta} \int_{y_1 - \varepsilon}^{a} \int_{-\infty}^{\infty} |\tau|^{-1/2} \frac{\partial}{\partial u} \left( \Gamma(u, \vec{x}, y, \tau) - \Gamma_0(u, \vec{x}, y, \tau) \right) d\tau du d\tau dy, \; \eta > 0.
\]

It was showed in (10) that, for every \( K \) compact subset of \( \mathbb{R}^n \), we have that
\[
\int_{K} \int_{K} \int_{-\infty}^{\infty} |\tau|^{-1/2} \frac{\partial}{\partial u} \left( \Gamma(u, \vec{x}, y, \tau) - \Gamma_0(u, \vec{x}, y, \tau) \right) d\tau du dy < \infty.
\]

Then, it follows that
\[
\lim_{\eta \to 0^+} \int_{\mathbb{R}^n} |\phi(y)| \int_{|\vec{x}| < \eta} \int_{-\infty}^{a} \int_{-\infty}^{\infty} |\tau|^{-1/2} |J_1(\vec{x}, y, \varepsilon, \tau)| d\tau du dy = 0,
\]
uiformly in \( \varepsilon \in (0, 1) \).

Therefore, in order to achieve our purpose it is sufficient to see that, for every \( \eta > 0 \),
\[
\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} \int_{|\vec{x}| > \eta} |\phi(y)| \int_{-\infty}^{\infty} |\tau|^{-1/2} |J_m(\vec{x}, y, \varepsilon, \tau)| d\tau du dy = 0, \; m = 1, 2.
\]

Assume that \( \eta > 0 \). By (16) and (17), we get, for certain \( A_1, A_2 > 0 \),
\[
\int_{\mathbb{R}^n} \int_{|\vec{x}| > \eta} |\phi(y)| \int_{-\infty}^{\infty} |\tau|^{-1/2} |J_1(\vec{x}, y, \varepsilon, \tau)| d\tau dy \\
\leq C \epsilon \delta \int_{B(0, A_1)} \int_{\eta < |\vec{x}| < A_2} \int_{-\infty}^{\infty} |\tau|^{-1/2} |\vec{x} - \vec{y}|^{2-n-\delta} d\tau d\vec{y} d\vec{x} \\
\leq C \epsilon \delta \int_{B(0, A_1)} \int_{\eta < |\vec{x}| < A_2} |\vec{x} - \vec{y}|^{-1-n-\delta} d\vec{y} d\vec{x}, \; 0 < \epsilon < \eta/15.
\]

Hence
\[
\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} \int_{|\vec{x}| > \eta} |\phi(y)| \int_{-\infty}^{\infty} |\tau|^{-1/2} |J_1(\vec{x}, y, \varepsilon, \tau)| d\tau dy = 0.
\]

Finally, in order to see that
\[
\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} \int_{|\vec{x}| > \eta} |\phi(y)| \int_{-\infty}^{\infty} |\tau|^{-1/2} |J_2(\vec{x}, y, \varepsilon, \tau)| d\tau dy = 0,
\]
we use the mean value theorem, (35) and (36).

Thus we have proved that \( \lim_{\varepsilon \to 0^+} I(\varepsilon) = 0 \).

Hence, coming back to (39) we obtain
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{\infty} (-i\tau) \frac{1}{2} \int_{-\infty}^{\infty} \left( \Gamma(x, y, \tau) - \Gamma_0(x, y, \tau) \right) \frac{\partial}{\partial x_1} \psi(x) dx_1 d\tau dy \\
= - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(y) \psi(x) \int_{-\infty}^{\infty} (-i\tau) \frac{1}{2} \frac{\partial}{\partial x_1} \left( \Gamma(x, y, \tau) - \Gamma_0(x, y, \tau) \right) d\tau dy dx,
\]
and from (38) it follows that
\[
\langle \frac{\partial}{\partial x_1} S_T \psi \rangle = - \frac{1}{2\pi} \int_{\mathbb{R}^n} \psi(x) \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{\infty} (-i\tau) \frac{1}{2} \frac{\partial}{\partial x_1} \left( \Gamma(x, y, \tau) - \Gamma_0(x, y, \tau) \right) d\tau dy dx.
\]
Therefore we have proved that the distributional derivative \( \frac{\partial}{\partial x_1} S_F \) of \( S_F \) is

\[
\frac{\partial}{\partial x_1} S_F = -\frac{1}{2\pi} \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{+\infty} (-i\tau)^{-\frac{1}{2}} \frac{\partial}{\partial x_1} (\Gamma(x, y, \tau) - \Gamma_0(x, y, \tau)) \, d\tau \, dy.
\]

Moreover, the above argument shows that the right hand side in the last equality defines a locally integrable function in \( \mathbb{R}^n \).

Now, by invoking now [21, §5, Theorem V, part (2)] we can conclude that

\[
\mathcal{L}^{-\frac{1}{2}} \phi - (-\Delta)^{-\frac{1}{2}} \phi
\]

admits classical derivative with respect to \( x_1 \) for almost all \( x \in \mathbb{R}^n \), and

\[
\frac{\partial}{\partial x_1} \left( \mathcal{L}^{-\frac{1}{2}} \phi(x) - (-\Delta)^{-\frac{1}{2}} \phi(x) \right) = -\frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \phi(y) \int_{-\infty}^{+\infty} (-i\tau)^{-\frac{1}{2}} \frac{\partial}{\partial x_1} (\Gamma(x, y, \tau) - \Gamma_0(x, y, \tau)) \, d\tau \, dy,
\]
a.e. \( x \in \mathbb{R}^n \). Moreover, the last integral is absolutely convergent. Then,

\[
\frac{\partial}{\partial x_1} \left( \mathcal{L}^{-\frac{1}{2}} \phi(x) - (-\Delta)^{-\frac{1}{2}} \phi(x) \right) = -\frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \phi(y) \int_{-\infty}^{+\infty} (-i\tau)^{-\frac{1}{2}} \frac{\partial}{\partial x_1} (\Gamma(x, y, \tau) - \Gamma_0(x, y, \tau)) \, d\tau \, dy,
\]
a.e. \( x \in \mathbb{R}^n \), and we obtain that

\[
\frac{\partial}{\partial x_1} \mathcal{L}^{-\frac{1}{2}} \phi(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \phi(y) \mathcal{L}_{x_1}^{\mathcal{L}}(x, y) \, dy,
\]
a.e. \( x \in \mathbb{R}^n \), since, as it is well known,

\[
\frac{\partial}{\partial x_1} (-\Delta)^{-\frac{1}{2}} \phi(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \phi(y) \int_{-\infty}^{+\infty} (-i\tau)^{-\frac{1}{2}} \frac{\partial}{\partial x_1} \Gamma_0(x, y, \tau) \, d\tau \, dy,
\]
a.e. \( x \in \mathbb{R}^n \).

We now prove \( L^p \)-boundedness properties for the variation operators associated with the Riesz transforms \( R_{\ell}^{\mathcal{L}} \).

**Proof of Theorem 1.2.** As in the proof of Proposition 4.1 it is enough to assume \( n/2 < q < n \).

We consider the operators

\[
R_{\ell,\text{loc}}^{\mathcal{L}}(f)(x) = P.V. \int_{|x-y|<\gamma(x)} R_{\ell}^{\mathcal{L}}(x, y) f(y) \, dy
\]

and

\[
R_{\ell,\text{loc}}(f)(x) = P.V. \int_{|x-y|<\gamma(x)} R_{\ell}(x-y) f(y) \, dy,
\]

where \( f \in L^p(\mathbb{R}^n) \) for a suitable \( 1 \leq p < \infty \).
Suppose that \( \{ \eta_j \}_{j \in \mathbb{N}} \) is a real decreasing sequence that converges to zero. Following the general procedure we may write
\[
\left( \sum_{j=0}^{\infty} \left| R_{\ell}^{x_j}(f)(x) - R_{\ell}^{x_{j+1}}(f)(x) \right|^p \right)^{\frac{1}{p}}
\]
\[
\leq \left( \sum_{j=0}^{\infty} \int \left| \eta_{j+1} < |x-y| < \eta_j, |x-y| < \gamma(x) \right. \left( R_{\ell}^{x_j}(x,y) - R_{\ell}(x-y) \right) f(y) dy \right)^{\frac{1}{p}}
\]
\[
+ \left( \sum_{j=0}^{\infty} \int \left| \eta_{j+1} < |x-y| < \eta_j, |x-y| > \gamma(x) \right. \left. \right| f(y) dy \right)^{\frac{1}{p}}
\]
\[
+ \left( \sum_{j=0}^{\infty} \int \left| \eta_{j+1} < |x-y| < \eta_j, |x-y| > \gamma(x) \right. \left. \right| f(y) dy \right)^{\frac{1}{p}}
\]
\[
\leq \left( \sum_{j=0}^{\infty} \int \left| R_{\ell}^{x_j}(x,y) - R_{\ell}(x-y) \right| |f(y)| dy \right)^{\frac{1}{p}}
\]
\[
+ \left( \sum_{j=0}^{\infty} \int \left| R_{\ell}^{x_j}(x,y) \right| |f(y)| dy \right)^{\frac{1}{p}}
\]
\[
+ \left( \sum_{j=0}^{\infty} \int \left| R_{\ell}(x-y) \right| |f(y)| dy \right)^{\frac{1}{p}}
\]
\[
\leq \int_{|x-y| < \gamma(x)} |R_{\ell}^{x_j}(x,y) - R_{\ell}(x-y)| |f(y)| dy + \int_{|x-y| > \gamma(x)} |R_{\ell}^{x_j}(x,y)| |f(y)| dy
\]
\[
+ \sup_{(t_j)_{j \in \mathbb{N}}} \left( \sum_{j=0}^{\infty} \int_{t_{j+1} < |x-y| < t_j} R_{\ell}(x-y) f(y) dy \right)^{\frac{1}{p}}
\].

Hence, we get
\[
V_\rho(R_{\ell}^{x_j})(f)(x) \leq \int_{|x-y| < \gamma(x)} |R_{\ell}^{x_j}(x,y) - R_{\ell}(x-y)| |f(y)| dy
\]
\[
+ \int_{|x-y| > \gamma(x)} |R_{\ell}^{x_j}(x,y)| |f(y)| dy + V_\rho(R_{\ell}^{x_j})(f)(x)
\]
\[
\overset{(48)}{=} \tau_1(\|f\|)(x) + \tau_2(\|f\|)(x) + V_\rho(R_{\ell}^{x_j})(f)(x), \quad x \in \mathbb{R}^n.
\]

Note that the operators \( \tau_1 \) and \( \tau_2 \) are the same ones that appeared in the proof of Proposition 4.1. Then, as it was proved there, \( \tau_1 \) and \( \tau_2 \) are bounded from \( L^r(\mathbb{R}^n) \) into itself provided that \( 1 \leq r < p_0 \).

By [13] and [7, Theorem A and Corollary 1.4] we conclude the desired \( L^p \)-boundedness properties for \( V_\rho(R_{\ell}^{x_j}) \).

\[ \square \]

5. Variation operators associated with commutators \( C_0^{x_j} \)

Proposition [12] and Theorem [14] are proved in this section. Assume that \( b \in \text{BMO}_\theta(\gamma) \), for some \( \theta \geq 0 \). Let us remind that, for every \( \ell = 1, \ldots, n \), the commutator operator \( C_0^{x_j} \) for the Riesz transform \( R_{\ell}^{x_j} \) is given by
\[
C_0^{x_j} f = b R_{\ell}^{x_j} f - R_{\ell}^{x_j}(bf), \quad f \in C^{\infty}_c(\mathbb{R}^n).
\]

Some \( L^p \)-boundedness results for these operators were established in [3, Theorem 1].

From Proposition [11] we deduce that, for every \( f \in C^{\infty}_c(\mathbb{R}^n) \),
\[
C_0^{x_j}(f)(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y| > \varepsilon} (b(x) - b(y)) R_{\ell}^{x_j}(x,y) f(y) dy, \quad x \in \mathbb{R}^n.
\]
In Proposition 1.2 that is proved in the following, we extend this property for every \( f \in L^p(\mathbb{R}^n) \), \( 1 < p < p_0 \), where \( \frac{1}{p_0} = \left( \frac{1}{q} - \frac{1}{n} \right)_+ \) and \( V \in B_q, q \geq n/2 \).

**Proof of Proposition 1.2** It is enough to prove that the maximal operator \( C_{b,\ell}^{Z, *}(f) \) defined by

\[
C_{b,\ell}^{Z, *}(f)(x) = \sup_{\varepsilon > 0} \left| \int_{|x - y| > \varepsilon} (b(x) - b(y)) R_{\ell}^{Z}(x, y) f(y) dy \right|, \quad x \in \mathbb{R}^n,
\]

is bounded from \( L^p(\mathbb{R}^n) \) into itself when \( V \) and \( p \) satisfy the conditions specified in this proposition.

As in the proof of Theorem 1.2 it suffices to take care of the case \( n/2 < q < n \). Suppose that \( f \in L^p(\mathbb{R}^n) \), where \( 1 < p < p_0 \).

Let us consider the local operators

\[
C_{b,\ell}^{Z, *}(f)(x) = \sup_{\varepsilon > 0} \left| \int_{|x - y| < \varepsilon} (b(x) - b(y)) R_{\ell}^{Z}(x, y) f(y) dy \right|, \quad x \in \mathbb{R}^n,
\]

and

\[
C_{b,\ell}^{*, \text{loc}}(f)(x) = \sup_{\varepsilon > 0} \left| \int_{|x - y| < \varepsilon} (b(x) - b(y)) R_{\ell}(x, y) f(y) dy \right|, \quad x \in \mathbb{R}^n.
\]

With this notation we have that

\[
C_{b,\ell}^{Z, *}(f)(x) = C_{b,\ell}^{*, \text{loc}}(f)(x) - C_{b,\ell}^{*, \text{loc}}(f)(x) - C_{b,\ell}^{Z, *}(f)(x) + C_{b,\ell}^{*, \text{loc}}(f)(x)
\]

\[
= T_1(|f|)(x) + T_2(|f|)(x) + C_{b,\ell}^{*, \text{loc}}(f)(x), \quad x \in \mathbb{R}^n.
\]

According to Proposition 2.1, the operator \( C_{b,\ell}^{*, \text{loc}} \) is bounded from \( L^r(\mathbb{R}^n) \) into itself, for every \( 1 < r < \infty \).

We now analyze the \( L^p \)-boundedness properties for the operators \( T_1 \) and \( T_2 \) studying the behavior of their adjoints \( T_1^* \) and \( T_2^* \).

The operator \( T_1^* \) adjoint of \( T_1 \) is defined by

\[
T_1^*(g)(y) = \int_{|x - y| < \gamma(y)} |b(y) - b(x)| R_{\ell}^{Z}(x, y) - R_{\ell}(x, y) g(x) dx.
\]

According to (33), since \( \gamma(x) \sim \gamma(y) \) when \( |x - y| \leq \gamma(x) \) there exists \( A > 0 \) for which

\[
|T_1^*(g)(y)| \leq C \left( \int_{|x - y| < A \gamma(y)} \frac{1}{|x - y|^q} \left( \frac{|x - y|}{\gamma(y)} \right)^{2-n/q} |b(y) - b(x)| |g(x)| dx \right.
\]

\[
+ \int_{|x - y| < A \gamma(y)} \frac{1}{|x - y|^n} \int_{B(x, |x - y|/4)} V(z) |z - x|^{n-1} |b(y) - b(x)| |g(x)| dx \right)
\]

\[
= C(T_{1,1}^* |g|)(y) + T_{1,2}^* |g|(y), \quad y \in \mathbb{R}^n.
\]

We have that

\[
T_{1,1}^* |g|(y) \leq C \sum_{j=0}^{\infty} 2^{-j\delta} \int_{2^{-j-1}A \gamma(y) \leq |x - y| \leq 2^{-j}A \gamma(y)} |b(x) - b(y)| |g(x)| dx, \quad y \in \mathbb{R}^n,
\]

where \( \delta = 2 - \frac{n}{q} > 0 \).

Then,

\[
\|T_{1,1}^* |g|\|_{L^p(\mathbb{R}^n)} \leq C \sum_{j=0}^{\infty} \|T_{1,j}^* |f|\|_{L^p(\mathbb{R}^n)},
\]

(51)
where, for every \( j \in \mathbb{N} \),

\[
T_{1,1,j}(g)(y) = \frac{2^{-j\delta}}{(2^{-j\gamma}(y))^n} \int_{|x-y| \leq 2^{-j}A_1(y)} |b(x) - b(y)||g(x)|dx, \quad y \in \mathbb{R}^n.
\]

To deal with these operators we consider the covering \( \{Q_k\}_{k \in \mathbb{N}} \) as given in Section 2. We know that there exists \( C > 0 \) such that \( \gamma(y) \leq C_\gamma(x_k) \), for every \( y \in 2^kQ_k \). We choose the smaller \( L \in \mathbb{N} \) such that \( AC + 1 \leq 2^L \). It is not hard to see that we can find \( M \in \mathbb{N} \) such that, for every \( k,j \in \mathbb{N} \), there exist \( N_j \in \mathbb{N} \) and \( x^i_{k,j} \), \( i = 1, \ldots, N_j \), such that, by denoting \( Q^i_{k,j} = B(x^i_{k,j}, 2^{-j}(x^i_{k,j})) \), \( i = 1, \ldots, N_j \), the following properties hold:

(i) \( Q_k \subset \bigcup_{i=1}^{N_j} Q^i_{k,j} \subset 2Q_k \);

(ii) \( \text{card} \{ l \in \mathbb{N} : 2^LQ^i_{k,j} \cap 2^LQ^i_{k,j} \neq \emptyset \} \leq M, i = 1, \ldots, N_j \).

Clearly, we have that \( B(y, 2^{-j}A_1(y)) \subset 2^LQ^i_{k,j} = Q^j_{k,j} \), when \( y \in Q^i_{k,j} \), \( j,k \in \mathbb{N} \) and \( i = 1, \ldots, N_j \).

We can write, for every \( j \in \mathbb{N} \),

\[
\int_{\mathbb{R}^n} |T_{1,1,j}(g)(y)|^{p'} dy \leq C \sum_{k=0}^{\infty} \sum_{j=0}^{N_j} \int_{Q^i_{k,j}} \left( \frac{2^{-j\delta}}{(2^{-j\gamma}(y))^n} \int_{|x-y| \leq 2^{-j}A_1(y)} |g(x)||b(x) - b(y)|dx \right)^{p'} dy \\
\leq C \sum_{k=0}^{\infty} \sum_{j=0}^{N_j} \int_{Q^i_{k,j}} \left( \frac{2^{-j\delta}}{(2^{-j\gamma}(x_k))^n} \int_{|x-y| \leq 2^{-j}A_1(y)} |g(x)||b(x) - b(y)|dx \right)^{p'} dy \\
\leq C \left( \sum_{j=0}^{\infty} \sum_{k=0}^{N_j} \frac{2^{-j\delta p'}}{(2^{-j\gamma}(x_k))^{np'}} \int_{Q^i_{k,j}} |b(y) - b_{Q^i_{k,j}}|^{p'} \left( \int_{Q^i_{k,j}} |g(x)|dx \right)^{p'} dy \right. \\
\left. + \sum_{j=0}^{\infty} \sum_{k=0}^{N_j} \frac{2^{-j\delta p'}}{(2^{-j\gamma}(x_k))^{np'}} \int_{Q^i_{k,j}} \left( \int_{Q^i_{k,j}} |b(x) - b_{Q^i_{k,j}}||g(x)|dx \right)^{p'} dy \right).
\]

Each summand is estimated separately. For the first one, since \( \gamma(x^i_{k,j}) \sim \gamma(x_k) \), by \([3]\) Proposition 3], we have that

\[
\sum_{j=0}^{\infty} \left( \sum_{k=0}^{\infty} \sum_{j=0}^{N_j} \frac{2^{-j\delta p'}}{(2^{-j\gamma}(x_k))^{np'}} \int_{Q^i_{k,j}} |b(y) - b_{Q^i_{k,j}}|^{p'} \left( \int_{Q^i_{k,j}} |g(x)|dx \right)^{p'} dy \right)^{1/p'} \\
\leq C \sum_{j=0}^{\infty} \frac{2^{-j\delta}}{(2^{-j\gamma}(x_k))^{np'}} \int_{Q^i_{k,j}} |b(y) - b_{Q^i_{k,j}}|^{p'} dy \\
\times \left( \int_{Q^i_{k,j}} |g(x)|^{p'} dx (2^{-j\gamma}(x_k))^{np'/p} \right)^{1/p'} \\
\leq C \sum_{j=0}^{\infty} \frac{2^{-j\delta}}{(2^{-j\gamma}(x_k))^{np'}} \left( \int_{Q^i_{k,j}} |b(y) - b_{Q^i_{k,j}}|^{p'} dy \int_{Q^i_{k,j}} |g(x)|^{p'} dx \right)^{1/p'} \\
\leq C ||b||_{\text{BMO}_\gamma} \sum_{j=0}^{\infty} \frac{2^{-j\delta}}{(2^{-j\gamma}(x_k))^{np'}} \int_{2^{j+1}Q_k} |g(x)|^{p'} dx \\
\leq C ||b||_{\text{BMO}_\gamma} ||g||_{L^{p'}(\mathbb{R}^n)}.
\]
Also, by using again (3), Proposition 3,

\[
\sum_{j=0}^{\infty} \left( \sum_{k=0}^{\infty} \sum_{i=1}^{N_j} \frac{2^{-j\beta p'}}{2^{-j-\gamma}(x_k)^{n\beta - n}} \int_{Q_{k,j}} \left( \int_{Q_{k,j}} |b(x) - b_{Q_{k,j}}||g(x)|dx \right)^{p'} \frac{dy}{dy'} \right)^{1/p'} \\
\leq C \sum_{j=0}^{\infty} 2^{-j\delta} \left( \sum_{k=0}^{\infty} \sum_{i=1}^{N_j} \frac{1}{2^{-j-\gamma}(x_k)^{n\beta - n}} \int_{Q_{k,j}} |b(x) - b_{Q_{k,j}}|^p dx \int_{Q_{k,j}} |g(x)|^{p'} dx \right)^{1/p'} \\
\leq C ||b||_{BMO(\gamma)} \sum_{j=0}^{\infty} 2^{-j\delta} \left( \sum_{k=0}^{\infty} \sum_{i=1}^{N_j} \int_{Q_{k,j}} |g(x)|^p dx \right)^{1/p'} \\
\leq C ||b||_{BMO(\gamma)} ||g||_{L^{p'}(\mathbb{R}^n)}. 
\]

By combining (51), (52), (53) and (54), we obtain

\[
||T_{1,2}^*(y)||_{L^{p'}(\mathbb{R}^n)} \leq C ||b||_{BMO(\gamma)} ||g||_{L^{p'}(\mathbb{R}^n)}. 
\]

On the other hand, we have

\[
|T_{1,2}^*(g)(y)| \leq \sum_{j=0}^{\infty} \int_{2^{-j-1}A(\gamma)} |b(y)| \int_{2^{-j-1}A(\gamma)} |g(x)| \frac{|b(y) - b(x)|}{|x-y|^{n-1}} \frac{V(z)}{|z-x|^{n-1}} \frac{dz}{dz} \\
\leq C \sum_{j=0}^{\infty} \frac{1}{2^{-j-\gamma}(y)^n-1} \int_{|x-y| \leq 2^{-j-1}A(\gamma)} \int_{B(y,2^{-j+1}A(\gamma))} \frac{V(z)}{|z-x|^{n-1}} |b(y) - b(x)||g(x)| dx. 
\]

Since \(\frac{1}{\theta} = \frac{1}{q} - \frac{1}{p}\), by using (2, Lemma 1), we get

\[
|I_1(\chi_{B(y,2^{-j+1}A(\gamma))} V)||_{L^{p_0}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \left| \int_{B(y,2^{-j+1}A(\gamma))} \frac{V(z)}{|z-x|^{n-1}} dz \right|^{p_0} dx \right)^{1/p_0} \\
\leq C \left( \int_{B(y,2^{-j+1}A(\gamma))} |V(z)|^{q_0} dz \right)^{1/q} \\
\leq C \left( 2^{-j\gamma}(y) \right)^{n(-1+1/q)} B(y,2^{-j+1}A(\gamma)) \frac{V(z)}{|z-x|^{n-1}} dz \\
\leq C \left( 2^{-j\gamma}(y) \right)^{n(-1+1/q)} 2^{-j(2-n/q)} \left( 2^{-j\gamma}(y) \right)^{n-2} \\
\leq C \gamma(y)^{2-n/q}, \quad y \in \mathbb{R}^n. 
\]

Then, Hölder inequality implies that

\[
|T_{1,2}^*(g)(y)| \leq C \sum_{j=0}^{\infty} \frac{1}{2^{-j\gamma}(y)^{n-1}} \left( \int_{|x-y| \leq 2^{-j-1}A(\gamma)} |b(y) - b(x)||g(x)|^{p_0} dx \right)^{1/p_0} \\
\times ||I_1(\chi_{B(y,2^{-j+1}A(\gamma))} V)||_{L^{p_0}(\mathbb{R}^n)} \\
\leq C \sum_{j=0}^{\infty} \frac{1}{2^{-j\gamma}(y)^{n-1}} \left( \int_{|x-y| \leq 2^{-j-1}A(\gamma)} |b(y) - b(x)||g(x)|^{p_0} dx \right)^{1/p_0} \gamma(y)^{n/q-2}, \quad y \in \mathbb{R}^n. 
\]

We can write,

\[
||T_{1,2}^*(g)||_{L^{p'}(\mathbb{R}^n)} \leq C \sum_{j=0}^{\infty} ||T_{1,2,j}^*(g)||_{L^{p'}(\mathbb{R}^n)}, 
\]

where, for every \(j \in \mathbb{N}\),

\[
T_{1,2,j}^*(g)(y) = \frac{\gamma(y)^{n/q-2}}{(2^{-j+\gamma}(y))^{n-1}} \left( \int_{|x-y| \leq 2^{-j-1}A(\gamma)} |b(x) - b(y)||g(x)|^{p_0} dx \right)^{1/p_0}, \quad y \in \mathbb{R}^n. 
\]
Then,

\[
\int_{\mathbb{R}^n} |T_{1,2,j}(g)(y)|^{p'} \, dy \leq C \left( \sum_{k=0}^{\infty} \sum_{i=1}^{N_j} \frac{2^{-j \delta p'}}{(2^{-j}\gamma(x_k))^{(n-n/q+1)p'}} \int_{Q_{k,j}^i} |b(y) - b_{Q_{k,j}^i}|^{p'} \left( \int_{Q_{k,j}^i} |g(x)|^{p''} \, dx \right)^{p'/p''} \right)^{1/p'}
\]

(58)

\[
+ \sum_{k=0}^{\infty} \sum_{i=1}^{N_j} \frac{2^{-j \delta p'}}{(2^{-j}\gamma(x_k))^{(n-n/q+1)p'}} \int_{Q_{k,j}^i} \left( \int_{Q_{k,j}^i} (|b(y) - b_{Q_{k,j}^i}|^{p''}) \, dy \right)^{p'/p''} \right)^{1/p'}
\]

where \( \delta = 2 - n/q \). Here \( Q_k, k \in \mathbb{N} \), and \( Q_{k,j}^i, j = 1, \ldots, N_j, k, j \in \mathbb{N} \), are the same balls that we considered above.

Also, since \( p' > p_0 \), we get

\[
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=1}^{N_j} \frac{2^{-j \delta p'}}{(2^{-j}\gamma(x_k))^{(n-q+1)p'}} \int_{Q_{k,j}^i} |b(y) - b_{Q_{k,j}^i}|^{p'} \left( \int_{Q_{k,j}^i} |g(x)|^{p''} \, dx \right)^{p'/p''} \right)^{1/p'}
\]

(59)

\[
\leq C \sum_{j=0}^{\infty} 2^{-j \delta} \left( \sum_{k=0}^{\infty} \sum_{i=1}^{N_j} \frac{1}{(2^{-j}\gamma(x_k))^{(n-q+1)p'}} \int_{Q_{k,j}^i} |b(y) - b_{Q_{k,j}^i}|^{p'} \, dy \right)^{1/p'}
\]

\[
\times \left( \int_{Q_{k,j}^i} |g(x)|^{p''} \, dx \right)^{1/p''}
\]

\[
\leq C ||b||_{BMO_{n}(\gamma)} \sum_{j=0}^{\infty} 2^{-j \delta} \left( \sum_{k=0}^{\infty} \sum_{i=1}^{N_j} \int_{Q_{k,j}^i} |g(x)|^{p''} \, dx \right)^{1/p''}
\]

\[
\leq C ||b||_{BMO_{n}(\gamma)} ||g||_{L^{p'}(\mathbb{R}^n)},
\]

and

\[
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=1}^{N_j} \frac{2^{-j \delta p'}}{(2^{-j}\gamma(x_k))^{(n-q+1)p'}} \int_{Q_{k,j}^i} \left( \int_{Q_{k,j}^i} (|b(x) - b_{Q_{k,j}^i}|^{p''}) \, dx \right)^{p'/p''} \right)^{1/p'}
\]

(60)

\[
\leq C \sum_{j=0}^{\infty} 2^{-j \delta} \left( \sum_{k=0}^{\infty} \sum_{i=1}^{N_j} \frac{1}{(2^{-j}\gamma(x_k))^{(n-q+1)p'-n}} \right)^{1/p'}
\]

\[
\times \left( \int_{Q_{k,j}^i} |b(x) - b_{Q_{k,j}^i}|^{p''} / (p'-p_0) \, dx \right)^{(p'-p_0)/p''} \left( \int_{Q_{k,j}^i} |g(x)|^{p''} \, dx \right)^{1/p'}
\]

\[
\leq C ||b||_{BMO_{n}(\gamma)} \sum_{j=0}^{\infty} 2^{-j \delta} \left( \sum_{k=0}^{\infty} \sum_{i=1}^{N_j} \int_{Q_{k,j}^i} |g(x)|^{p''} \, dx \right)^{1/p''}
\]

\[
\leq C ||b||_{BMO_{n}(\gamma)} ||g||_{L^{p'}(\mathbb{R}^n)},
\]

From (57), (58), (59) and (60) we conclude that

\[
||T_{1,2}(g)||_{L^{p'}(\mathbb{R}^n)} \leq C ||b||_{BMO_{n}(\gamma)} ||g||_{L^{p'}(\mathbb{R}^n)}, \quad g \in L^{p'}(\mathbb{R}^n).
\]

By invoking (50), (55), and (61), it follows that

\[
||T_1 g||_{L^{p'}(\mathbb{R}^n)} \leq C ||b||_{BMO_{n}(\gamma)} ||g||_{L^{p'}(\mathbb{R}^n)}, \quad g \in L^{p'}(\mathbb{R}^n).
\]

Then, \( T_1 \) is a bounded operator from \( L^{p}(\mathbb{R}^n) \) into itself.

The operator \( T_2 \) adjoint of \( T_2 \) is defined by

\[
T_2(g)(y) = \int_{|x-y| \geq \gamma(x)} |b(x) - b(y)||R_{x}^{x'}(x, y)| g(x) \, dx, \quad y \in \mathbb{R}^n.
\]
According to (32), since for a certain \( A > 0 \), \(|x - y| \geq A\gamma(y)\), when \(|x - y| \geq \gamma(x)\), we have that

\[
|T_2^*(g)(y)| \leq C_0 \left( \int_{|x-y| \geq A\gamma(y)} \frac{1}{|x-y|^n} \frac{1}{1 + \frac{|x-y|}{\gamma(y)}} |b(x) - b(y)||g(x)|dx \right)
+ \int_{|x-y| \geq A\gamma(y)} \frac{1}{|x-y|^{n-1}} \frac{1}{1 + \frac{|x-y|}{\gamma(y)}} \int_{B(x, |x-y|/\gamma(y))} V(z) \left| \frac{1}{|z-x|^{n-1}} dz \right| dx
\]

(62) \quad = C_0(T_{2,1}^*(-g)(y) + T_{2,2}^*(-g)(y)), \quad y \in \mathbb{R}^n.

Here \( \alpha > 0 \) will be sufficiently large and it will be fixed later.

By choosing \( \alpha > \theta' + 1 \), where \( \theta' \) is the one appearing in Lemma 2.1 we can prove that

\[
|T_{2,1}^*(-g)(y)| \leq C \sum_{m=0}^{\infty} \int_{2m\gamma(x) < |x-y| \leq 2^{m+1}\gamma(x)} \frac{|b(x) - b(y)||g(x)|}{|x-y|^n} \left( 1 + \frac{|x-y|}{\gamma(y)} \right)^{-(\theta' + 1)} \left| g(x) \right| dx
\]

(63) \quad \leq C \frac{1}{2m(\theta' + n + 1)m} \left| T_{b,2m+1}^A(|g|)(y) \right|, \quad y \in \mathbb{R}^n,

where the operators \( T_{b,M} \) are the ones introduced in Lemma 2.1.

Then, by using Lemma 2.1 we get

\[
\|T_{2,1}^*(-g)\|_{L^p'((\mathbb{R}^n))} \leq \sum_{m=0}^{\infty} \frac{1}{2m(\theta' + n + 1)m} \left| T_{b,2m+1}^A(|g|) \right|_{L^p'((\mathbb{R}^n))}
\]

On the other hand, since \( \gamma(y) \sim \gamma(x_k), y \in Q_k \) and \( k \in \mathbb{N} \), we have, for a certain \( \beta > 0 \),

\[
\|T_{2,2}^*(-g)\|_{L^p'((\mathbb{R}^n))} \leq C \sum_{k=0}^{\infty} \int_{Q_k} \left( \int_{|x-y| \geq \beta\gamma(x_k)} \frac{|b(x) - b(y)||g(x)|}{|x-y|^{n-1}} \right.
\]

\[
\times \left( 1 + \frac{|x-y|}{\gamma(x_k)} \right)^{\theta'} \int_{B(x, |x-y|/\gamma(x_k))} V(z) \left| \frac{1}{|z-x|^{n-1}} dz \right| dx
\]

(64) \quad = \sum_{k=0}^{\infty} \int_{Q_k} J_k(y) \rho dy.

Let \( k \in \mathbb{N} \). It follows that

\[
J_k(y) \leq C \sum_{j=0}^{\infty} \int_{2^j \beta_1(x_k) < |x-y| \leq 2^{j+1} \beta_1(x_k)} \frac{|b(x) - b(y)||g(x)|}{(2\beta_1(x_k))^{n-1}} \left| \frac{1}{2^{\alpha}} \int_{B(x, \beta_2 + 2^{j+1} \gamma(x_k))} V(z)dz \left| \frac{1}{z-x|^{n-1}} dz \right| \right| dx
\]

(65) \quad \leq C \sum_{j=0}^{\infty} \frac{1}{2^{j\alpha}(2\beta_1(x_k)^{n-1})} \left( \int_{Q^c_j} (|b(x) - b(y)||g(x)|)^{\rho_0} dx \right)^{1/\rho_0}

\times \left( \int_{\mathbb{R}^n} \int_{Q^c_j} V(z)^{\rho_0} dz \frac{1}{|z-x|^{n-1}} dx \right)^{1/\rho_0}, \quad y \in Q_k,

where \( \rho_0 \) is such that \( \frac{1}{\rho_0} = \frac{1}{\rho} - \frac{1}{n} \), and for every \( j \in \mathbb{N} \) we consider \( Q^c_j = B(x_k, c2^j\gamma(x_k)) \), where \( c > 0 \) is independent of \( j, k \in \mathbb{N} \) and such that \( B(y, \beta_2 2^{j+1} \gamma(x_k)) \subset Q^c_j, y \in Q_k \).

Since the 1-th Euclidean fractional integral is bounded from \( L^q((\mathbb{R}^n)) \) into \( L^p((\mathbb{R}^n)) \) we deduce

\[
J_k(y) \leq C \sum_{j=0}^{\infty} \frac{2^{-j\alpha}}{(2\beta_1(x_k)^{n-1})} \left( \int_{Q^c_j} (|b(x) - b(y)||g(x)|)^{\rho_0} dx \right)^{1/\rho_0} \left( \int_{Q^c_j} V(z)^{\rho_0} dz \right)^{1/\rho_0}, \quad y \in Q_k.
\]
Moreover, by using the doubling property of $V$ it follows that, for some $\mu > 0$,
\[
\left( \int_{Q_k^*} V(z)^{\theta'} \, dz \right)^{1/\theta'} \leq C (2^j \gamma(x_k))^{-n/\theta'} 2^{j\mu} \int_{Q_k} V(z) \, dz \leq C (2^j \gamma(x_k))^{-n/\theta'} \gamma(x_k)^{n-2}, \quad y \in Q_k,
\]
where in the last inequality we just use the definition of $\gamma$.

Then,
\[
J_k(y) \leq C \sum_{j=0}^{\infty} \frac{2^{j(\mu - \alpha - n + 1 - n/q)}}{\gamma(x_k)^{1+n/q'}} \left( \int_{Q_k^*} |b(x) - b(y)|^{\nu} \, dx \right)^{1/\nu} \left( \int_{Q_k^*} |g(x)|^{\nu'} \, dx \right)^{1/\nu'}
\]
\[
\leq C \sum_{j=0}^{\infty} \frac{2^{j(\mu - \alpha - n + 1 - n/q)}}{\gamma(x_k)^{1+n/q'}} \left( \int_{Q_k^*} |b(x) - b_{Q_k}|^{\nu} \, dx \right)^{1/\nu} + \left( \int_{Q_k^*} |b(y) - b_{Q_k}|^{\nu} \, dx \right)^{1/\nu} \left( \int_{Q_k^*} |g(x)|^{\nu'} \, dx \right)^{1/\nu'}
\]
\[
\leq C \sum_{j=0}^{\infty} \gamma(x_k)^{-n/\theta'} 2^{-2j(n-2+\alpha - \mu + n/\theta')} (j+1)^{1/\nu'} \left( \int_{Q_k^*} |b(x)|^{\nu} \, dx \right)^{1/\nu} \left( \int_{Q_k^*} |g(x)|^{\nu'} \, dx \right)^{1/\nu'}
\]
\[
\times (j+1)^{2j\theta'} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{-j(n-2+\alpha - \mu + \theta' + n/\theta')} (j+1)^{1/\nu'} \left( \int_{Q_k^*} |g(x)|^{\nu'} \, dx \right)^{1/\nu'}
\]
\[
\leq C \sum_{k=0}^{\infty} \left( \sum_{j=0}^{\infty} 2^{-j(n-2+\alpha - \mu + \theta' + n/\theta')} (j+1)^{1/\nu'} \left( \int_{Q_k^*} |g(x)|^{\nu'} \, dx \right)^{1/\nu'} \right)^{1/\nu'}
\]
\[
\leq C \sum_{k=0}^{\infty} \left( \sum_{j=0}^{\infty} 2^{-j(n-2+\alpha - \mu + \theta' + n/\theta')} (j+1)^{1/\nu'} \left( \int_{Q_k^*} |g(x)|^{\nu'} \, dx \right)^{1/\nu'} \right)^{1/\nu'}
\]
\[
\leq C \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} 2^{-j(n-2+\alpha - \mu + \theta' + n/\theta')} (j+1)^{1/\nu'} \left( \int_{Q_k^*} |g(x)|^{\nu'} \, dx \right)^{1/\nu'}
\]
\[
\leq C \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} 2^{-j(n-2+\alpha - \mu + \theta' + n/\theta')} (j+1)^{1/\nu'} \left( \int_{Q_k^*} |g(x)|^{\nu'} \, dx \right)^{1/\nu'}
\]
\[
\leq C \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} 2^{-j(n-2+\alpha - \mu + \theta' + n/\theta')} (j+1)^{1/\nu'} \left( \int_{Q_k^*} |g(x)|^{\nu'} \, dx \right)^{1/\nu'}
\]
\[
\leq C \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} 2^{-j(n-2+\alpha - \mu + \theta' + n/\theta')} (j+1)^{1/\nu'} \left( \int_{Q_k^*} |g(x)|^{\nu'} \, dx \right)^{1/\nu'}
\]
\[
\leq C \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} 2^{-j(n-2+\alpha - \mu + \theta' + n/\theta')} (j+1)^{1/\nu'} \left( \int_{Q_k^*} |g(x)|^{\nu'} \, dx \right)^{1/\nu'}
\]
\[
\leq C \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} 2^{-j(n-2+\alpha - \mu + \theta' + n/\theta')} (j+1)^{1/\nu'} \left( \int_{Q_k^*} |g(x)|^{\nu'} \, dx \right)^{1/\nu'}
\]

Thus, we prove that
\[
||T_{2,2}(g)||_{L^{\theta'}(\mathbb{R}^n)} \leq C ||g||_{L^{\nu'}(\mathbb{R}^n)}, \quad g \in L^{\nu'}(\mathbb{R}^n).
\]

By combining (62), (63) and (65) we obtain that
\[
||T_2^*(g)||_{L^{\nu'}(\mathbb{R}^n)} \leq C ||g||_{L^{\nu'}(\mathbb{R}^n)}, \quad g \in L^{\nu'}(\mathbb{R}^n).
\]

Hence, the operator $T_2$ is bounded from $L^{\nu'}(\mathbb{R}^n)$ into itself.

Finally, the $L^{\nu'}$-boundedness of $T_1$ and $T_2$ allows us to conclude that the operator $C_{b,\ell}^{\nu',\nu}$ is bounded from $L^{\nu'}(\mathbb{R}^n)$ into itself.
Proof of Theorem 1.3. Assume that $\rho > 2$. We consider the operators

$$C_{b, \ell}^{\varphi, \text{loc}}(f)(x) = \lim_{\varepsilon \to 0^+} \int_{x-y<\gamma(x)} (b(x) - b(y))(R_\ell^\varphi (x, y) - R_\ell(x-y))f(y)dy,$$

and

$$C_{b, \ell}^{\text{loc}}(f)(x) = \lim_{\varepsilon \to 0^+} \int_{x-y<\gamma(x)} (b(x) - b(y))R_\ell(x-y)f(y)dy,$$

and we define the truncations $C_{b, \ell}^{\varphi, \text{loc}}(f)$ and $C_{b, \ell}^{\text{loc}}(f)$, $\varepsilon > 0$, in the usual way.

If $\{\varepsilon_j\}_{j \in \mathbb{N}}$ is a real decreasing sequence that converges to zero, we can write

$$\left(\sum_{j=0}^{\infty} |C_{b, \ell}^{\varphi, \varepsilon_j}(f)(x) - C_{b, \ell}^{\varphi, \varepsilon_{j+1}}(f)(x)|^p\right)^{1/p} \leq \left(\sum_{j=0}^{\infty} \int_{\varepsilon_{j+1}<|x-y|<\varepsilon_j, |x-y|<\gamma(x)} (b(x) - b(y))(R_\ell^\varphi (x, y) - R_\ell(x-y))f(y)dy \right)^{1/p}$$

$$+ \left(\sum_{j=0}^{\infty} \int_{\varepsilon_{j+1}<|x-y|<\varepsilon_j, |x-y|\geq\gamma(x)} (b(x) - b(y))R_\ell^\varphi (x, y)f(y)dy \right)^{1/p}$$

$$+ \left(\sum_{j=0}^{\infty} \int_{\varepsilon_{j+1}<|x-y|<\varepsilon_j, |x-y|<\gamma(x)} (b(x) - b(y))R_\ell(x-y)f(y)dy \right)^{1/p}$$

$$\leq \int_{|x-y|<\gamma(x)} |b(x) - b(y)||R_\ell^\varphi (x, y) - R_\ell(x-y)||f(y)||dy$$

$$+ \int_{|x-y|\geq\gamma(x)} |b(x) - b(y)||R_\ell^\varphi (x, y)||f(y)||dy + V_\rho(C_{b, \ell}^{\varphi, \text{loc}})(f)(x)$$

Hence,

$$V_\rho(C_{b, \ell}^{\varphi, \varepsilon})(f) \leq T_1(f) + T_2(f) + V_\rho(C_{b, \ell}^{\varphi, \text{loc}})(f),$$

where the operator $T_1$ and $T_2$ are the ones defined in the proof of Proposition 1.2.

According to the $L^p$-boundedness properties of the operators $T_1$ and $T_2$ (see the proof of Proposition 1.2 and Proposition 2.1) we conclude that the variation operator $V_\rho(C_{b, \ell}^{\varphi, \varepsilon})$ is bounded from $L^p(\mathbb{R}^n)$ into itself.

\[ \square \]

Remark 5.1. In [23] (see also [1] and [2]) it is considered, for every $\ell = 1, \cdots, n$, the adjoint $(R_\ell^\varphi)^*$ of $R_\ell^\varphi$, when $V \in B_q$ with $\frac{n}{q} < q < n$. By proceeding as in the previous results of this section we can prove the following properties.

Assume that $V \in B_q$, with $\frac{n}{q} < q$, $\ell = 1, 2, \cdots, n$, and $p_0 < p < \infty$, where $\frac{1}{p_0} = \left(\frac{1}{q} - \frac{1}{n}\right)$. For every $f \in L^p(\mathbb{R}^n)$ there exists the limit

$$\lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} R_\ell^\varphi (y, x)f(y)dy, \quad a.e. \ x \in \mathbb{R}^n,$$

and defining the operator $R_\ell^\varphi$ on $L^p(\mathbb{R}^n)$ as

$$R_\ell^\varphi f(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} R_\ell^\varphi (y, x)f(y)dy, \quad a.e. \ x \in \mathbb{R}^n,$$

$R_\ell^\varphi$ is bounded from $L^p(\mathbb{R}^n)$ into itself.

Moreover, if $p > 2$ the variation operator $V_\rho(R_\ell^\varphi)$ is bounded from $L^p(\mathbb{R}^n)$ into itself. Suppose that $b \in BMO_b(\gamma)$. For every $f \in L^p(\mathbb{R}^n)$, there exists the limit

$$\lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} (b(x) - b(y))R_\ell^\varphi (y, x)f(y)dy, \quad a.e. \ x \in \mathbb{R}^n.$$
and the operator $\mathcal{G}_{b,\ell}^p$ defined on $L^p(\mathbb{R}^n)$ by

$$
\mathcal{G}_{b,\ell}^p(f)(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{(b(x)-b(y))R_{\ell}^p(y,x)f(y)dy}{|x-y|^n}, \quad \text{a.e. } x \in \mathbb{R}^n,
$$

is bounded from $L^p(\mathbb{R}^n)$ into itself.

Moreover, if $p > 2$ the variation operator $V_p(\mathcal{G}_{b,\ell}^p)$ is bounded from $L^p(\mathbb{R}^n)$ into itself.

**Remark 5.2.** The fluctuations of a family $\{T_t\}_{t>0}$ of operators when $t \to 0^+$ also can be analyzed by using oscillation operators (see, for instance, [5] and [19]). If $\{t_j\}_{j \in \mathbb{N}}$ is a real decreasing sequence that converges to zero, the oscillation operator $O(T_t; \{t_j\}_{j \in \mathbb{N}})$ is defined by

$$
O(T_t; \{t_j\}_{j \in \mathbb{N}})(f)(x) = \left( \sum_{j=0}^{\infty} \sup_{t_{j+1} \leq \varepsilon_j < t_j} \left| T_{\varepsilon_j}f(x) - T_{\varepsilon_j+1}f(x) \right|^2 \right)^{1/2}, \quad f \in L^p(\mathbb{R}^n).
$$

$L^p$-boundedness properties for the oscillation operators associated with the heat semigroup, Riesz transforms and commutators with the Riesz transforms in the Schrödinger setting can be established by using the procedures developed in this paper.

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