Minimal Linear Codes over Finite Fields

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Abstract

As a special class of linear codes, minimal linear codes have important applications in secret sharing and secure two-party computation. Constructing minimal linear codes with new and desirable parameters has been an interesting research topic in coding theory and cryptography. Ashikhmin and Barg showed that $w_{\text{min}}/w_{\text{max}} > (q-1)/q$ is a sufficient condition for a linear code over the finite field $\text{GF}(q)$ to be minimal, where $q$ is a prime power, $w_{\text{min}}$ and $w_{\text{max}}$ denote the minimum and maximum nonzero weights in the code, respectively. The first objective of this paper is to present a sufficient and necessary condition for linear codes over finite fields to be minimal. The second objective of this paper is to construct an infinite family of ternary minimal linear codes satisfying $w_{\text{min}}/w_{\text{max}} \leq 2/3$. To the best of our knowledge, this is the first infinite family of nonbinary minimal linear codes violating Ashikhmin and Barg’s condition.

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1. Introduction

Let $q$ be a prime power and $\text{GF}(q)$ denote the finite field with $q$ elements. An $[n, k, d]$ code $C$ over $\text{GF}(q)$ is a $k$-dimensional subspace of $\text{GF}(q)^n$ with minimum (Hamming) distance $d$. Let $A_i$ denote the number of codewords with Hamming weight $i$ in a code $C$ of length $n$. The weight enumerator of $C$ is defined by $1 + A_1z + A_2z^2 + \cdots + A_nz^n$. The sequence $(1, A_1, A_2, \cdots, A_n)$ is called the weight distribution of the code $C$.

The support of a vector $v = (v_1, v_2, \ldots, v_n) \in \text{GF}(q)^n$, denoted by $\text{Suppt}(v)$, is defined by

$$\text{Suppt}(v) = \{1 \leq i \leq n : v_i \neq 0\}.$$ 

The vector $v$ is called the characteristic vector or the incidence vector of the set $\text{Suppt}(v)$. A vector $u \in \text{GF}(q)^n$ covers another vector $v \in \text{GF}(q)^n$ if $\text{Suppt}(u)$ contains $\text{Suppt}(v)$. We write
\( v \preceq u \) if \( v \) is covered by \( u \), and \( v \prec u \) if \( \text{Supp}(v) \) is a proper subset of \( \text{Supp}(u) \). A codeword \( u \) in a linear code \( C \) is said to be **minimal** if \( u \) covers only the codeword \( au \) for all \( a \in \GF(q) \), but no other codewords in \( C \). A linear code \( C \) is said to be **minimal** if every codeword in \( C \) is minimal.

Minimal linear codes have interesting applications in secret sharing \([3, 21, 24]\) and secure two-party computation \([3, 22]\), and could be decoded with a minimum distance decoding method \([1]\). Searching for minimal linear codes has been an interesting research topic in coding theory and cryptography. The following sufficient condition for a linear code to be minimal is due to Ashikhmin and Barg \([1]\).

**Lemma 1** (Ashikhmin-Barg). A linear code \( C \) over \( \GF(q) \) is minimal if

\[
\frac{w_{\min}}{w_{\max}} > \frac{q - 1}{q},
\]

where \( w_{\min} \) and \( w_{\max} \) denote the minimum and maximum nonzero Hamming weights in the code \( C \), respectively.

With the help of Lemma 1, a number of families of minimal linear codes with \( w_{\min}/w_{\max} > (q - 1)/q \) have been reported in the literature (see, \([5, 10, 14, 24]\), for example). Sporadic examples in \([9]\) show that Ashikhmin-Barg’s condition is not necessary for linear codes to be minimal. However, no infinite family of minimal linear codes with \( w_{\min}/w_{\max} \leq (q - 1)/q \) was found until the breakthrough in \([7]\), where an infinite family of such binary codes was discovered. Inspired by the work in \([7]\), the authors of the present paper gave a further study of binary minimal linear codes \([12]\). Specifically, a necessary and sufficient condition for binary linear codes to be minimal was derived in \([12]\). With this new condition, three infinite families of minimal binary linear codes with \( w_{\min}/w_{\max} \leq 1/2 \) were obtained from a general construction in \([12]\).

The first objective of this paper is to present a sufficient and necessary condition for linear codes over finite fields to be minimal, which generalizes the result about the binary case given in \([12]\). The second objective of this paper is to construct an infinite family of ternary minimal linear codes with \( w_{\min}/w_{\max} < 2/3 \). To the best of our knowledge, this is the first infinite family of nonbinary minimal linear codes violating the Ashikhmin-Barg condition.

The rest of this paper is organized as follows. In Section 2, we recall basic properties of Krawtchouk polynomials which will be needed in the sequel. In Section 3, we present a new sufficient and necessary condition for linear codes over finite fields to be minimal. In Section 4, we use the Walsh spectrum of generalized Boolean functions to characterize when ternary linear codes from a general construction are minimal. We then propose a family of ternary minimal codes violating the Ashikhmin-Barg condition with this characterization. Finally, we conclude this paper and make concluding comments in Section 5.

### 2. Krawtchouk polynomials and their properties

Krawtchouk polynomials were introduced by Lloyd in 1957 \([19]\) and have wide applications in coding theory \([3, 15, 16]\), cryptography \([6]\), and combinatorics \([17]\). In this section, we only give a short introduction to Krawtchouk polynomials with their essential properties. For more information, the reader is referred to \([3, 15, 17, 19]\).
Let $m$ be a positive integer, $q$ a positive integer and $x$ a variable taking nonnegative values. The Krawtchouk polynomial (of degree $t$ and with parameters $q$ and $m$) is defined by

$$K_t(x,m) = \sum_{j=0}^{t} (-1)^j (q-1)^{t-j} \binom{x}{j} \binom{m-x}{t-j}.$$  

Accordingly, the Lloyd polynomial $\Psi_k(x,m)$ (of degree $k$ and with parameters $q$ and $m$) is given by

$$\Psi_k(x,m) = \sum_{i=0}^{k} K_i(x,m). \quad (1)$$

The following results will be useful in the sequel.

**Lemma 2.** [3, Lemma 3.2.1] For $x, m \geq 1$, $\Psi_k(x,m) = K_k(x-1, m-1)$.

**Lemma 3.** Let symbols and notation be as before. Then the following holds:

1. $K_t(0, m) = (q-1)^t \binom{m}{t}$.
2. $K_t(1, m) = (q-1)^t \binom{m-1}{t-1} - (q-1)^{t-1} \binom{m-1}{t-1}$.
3. $K_t(m, m) = (-1)^t \binom{m}{t}$.
4. [3, Lemma 3.1.1] For any integers $0 \leq x, t \leq m$,
   
   $$(q-1)(m-x)K_t(x+1, m) - (x+(q-1)(m-x)-qt)K_t(x, m) + xK_t(x-1, m) = 0.$$  

5. [17, Equation (21)] For any integers $0 \leq x, t \leq m$,
   
   $$(q-1)^t \binom{m}{x} K_t(x, m) = (q-1)^t \binom{m}{t} K_t(t, m).$$

**Lemma 4.** [17, Equation (6)] For $x, t \in \{0, 1, 2, \cdots, m\}$,

$K_t(x, m) \leq K_t(0, m)$.

**Lemma 5.** For $x, t \in \{0, 1, 2, \cdots, m\}$,

$$|K_t(x, m)| \leq (q-1)^t \binom{m}{t}.$$  

**Proof.** For any given $x, t \in \{0, 1, 2, \cdots, m\}$, we have

$$|K_t(x, m)| = \left| \sum_{j=0}^{t} (-1)^j (q-1)^{t-j} \binom{x}{j} \binom{m-x}{t-j} \right| \leq \sum_{j=0}^{t} \left| (-1)^j (q-1)^{t-j} \binom{x}{j} \binom{m-x}{t-j} \right|.$$
\[ \leq (q - 1)^t \sum_{j=0}^{t} \binom{x}{j} \binom{m-x}{t-j} \]
\[ = (q - 1)^t \binom{m}{t}, \]

where the last equality followed from the Vandermonde convolution formula. This completes the proof of this lemma.

The following result follows directly from Lemmas 2 and 5.

**Corollary 6.** For any integers \( 1 \leq x \leq m \) and \( 1 \leq k \leq m - 1 \), we have

\[ |\Psi_k(x,m)| \leq (q - 1)^k \binom{m-1}{k}. \]

We remark that the upper bound for \( |\Psi_k(x,m)| \) in Corollary 6 is tight since

\[ \Psi_k(1,m) = K_k(0,m-1) = (q - 1)^k \binom{m-1}{k}. \]

The next result will be employed to calculate the Hamming weights of the proposed linear codes in Section 4.

**Lemma 7.** [3, Lemma 4.2.1] Let \( u \in \mathbb{Z}_q^m \) with Hamming weight \( \text{wt}(u) = i \). Then

\[ \sum_{v \in \mathbb{Z}_q^m} \delta_{\text{wt}(v)=i} 1_{u \cap v} = K_i(i,m), \]

where \( \zeta_q \) denotes the \( q \)-th primitive root of complex unity, and the inner product \( u \cdot v \) in \( \mathbb{Z}_q^m \) is defined by \( u \cdot v = u_1 v_1 + \cdots + u_m v_m \).

3. A sufficient and necessary condition for \( q \)-ary linear codes to be minimal

In this section, we shall present a sufficient and necessary condition for linear codes over \( \text{GF}(q) \) to be minimal. From now on, we always assume that \( q \) is a prime power. Let \( \text{GF}(q)^* = \text{GF}(q) \setminus \{0\} \) and \( \text{GF}(q)^{m*} = \text{GF}(q)^* \setminus \{0\} \). For any \( a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_n) \in \text{GF}(q)^n \), define \( a \cap b \) to be the vector \( (f_1, f_2, \ldots, f_n) \) as

\[ f_i = \begin{cases} a_i, & \text{if } a_i = b_i \in \text{GF}(q)^*, \\ 0, & \text{otherwise}, \end{cases} \]

For example, for \( a = (1, 0, 2, 1, 0), b = (2, 1, 2, 0, 0) \in \text{GF}(3)^5 \), we have

\[ a \cap b = (0, 0, 2, 0, 0). \]

The following three lemmas will be needed in the sequel.
Lemma 8. For any $a, b \in \mathbb{GF}(q)^n$, $b \preceq a$ if and only if
\[ \sum_{c \in \mathbb{GF}(q)^n} (ca \cap b) = b \tag{2} \]
if and only if
\[ \sum_{c \in \mathbb{GF}(q)^n} wt(ca \cap b) = wt(b). \tag{3} \]

Proof. Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be any two vectors in $\mathbb{GF}(q)^n$. We first prove that $b \preceq a$ if and only if \(2\) holds. Assume that $b \preceq a$, then for any $1 \leq i \leq n$, $b_i \neq 0$ implies $a_i \neq 0$ and there is one and only one $c \in \mathbb{GF}(q)^n$ such that $b_i = ca_i$. This together with the definition of $a \cap b$ leads to (2). On the other hand, suppose that (2) holds, then for any $1 \leq i \leq n$, these exists some $c \in \mathbb{GF}(q)^n$ such that $b_i = ca_i$. Therefore $b_i \neq 0$ implies $a_i \neq 0$ and further implies $b \preceq a$. The conclusion that (2) holds if and only if (3) holds follows directly from the fact that
\[ \text{Supp}(c_1a \cap b) \cap \text{Supp}(c_2a \cap b) = \emptyset \]
for any distinct pair $(c_1, c_2) \in \mathbb{GF}(q)^n \times \mathbb{GF}(q)^n$.

Lemma 9. For any $a, b \in \mathbb{GF}(q)^n$,
\[ (q-1)(wt(a) + wt(b)) = \sum_{c \in \mathbb{GF}(q)^n} wt(a + cb) + q \sum_{c \in \mathbb{GF}(q)^n} wt(ca \cap b). \tag{4} \]

Proof. For any $c \in \mathbb{GF}(q)^n$, with a direct verification, we have
\[ wt(a) + wt(b) = wt(a + cb) + \sum_{y \in \mathbb{GF}(q)^n} wt(ya \cap b) + wt\left(\frac{1}{c}a \cap b\right). \]
It then follows that
\[ (q-1)(wt(a) + wt(b)) = \sum_{c \in \mathbb{GF}(q)^n} wt(a + cb) + (q-1) \sum_{y \in \mathbb{GF}(q)^n} wt(yc \cap b) + \sum_{c \in \mathbb{GF}(q)^n} wt(ca \cap b) \]
\[ = \sum_{c \in \mathbb{GF}(q)^n} wt(a + cb) + q \sum_{c \in \mathbb{GF}(q)^n} wt(ca \cap b). \]
This comoltes the proof of this lemma.

Lemma 10. For any $a, b \in \mathbb{GF}(q)^n$, $b \preceq a$ if and only if
\[ \sum_{c \in \mathbb{GF}(q)^n} wt(a + cb) = (q-1)wt(a) - wt(b). \]

Proof. The conclusion follows from Lemmas 8 and 9 by plugging Equation 3 into Equation 4.

We are now in a position to present the sufficient and necessary condition for linear codes over $\mathbb{GF}(q)$ to be minimal.
**Theorem 11.** Let $C \subseteq GF(q)^n$ be a linear code. Then it is minimal if and only if

$$\sum_{c \in GF(q)^*} \text{wt}(a + cb) \neq (q - 1)\text{wt}(a) - \text{wt}(b)$$

for any $GF(q)$-linearly independent codewords $a, b \in C$.

**Proof.** The conclusion follows from the definition of minimal codes and Lemma 10. \qed

**Remark 1.** When $q = 2$, according to Theorem 11, $C$ is minimal if and only if

$$\text{wt}(a + b) \neq \text{wt}(a) - \text{wt}(b)$$

for any two distinct codewords $a, b \in C$. This sufficient and necessary condition was derived in [12]. Therefore, the result in Theorem 11 generalizes the one in [12] since it works for any prime power $q$.

By Theorem 11 the minimality of $C$ is completely determined by the weights of its codewords. In particular, if $C$ is a $q$-ary linear code with only two weights, by Theorem 11, we can judge the minimality of it as follows.

**Corollary 12.** Let $C \subseteq GF(q)^n$ be a two-weight $q$-ary linear code with nonzero weights $w_1$ and $w_2$, where $0 < w_1 < w_2 < n$. Then $C$ is minimal, provided that

$$jw_1 \neq (j - 1)w_2$$

for any integer $j$ with $2 \leq j \leq q$.

**Proof.** Suppose that $C$ is not minimal. By Theorem 11 there exists a pair of $GF(q)$-linearly independent codewords $a, b \in C$ such that

$$\text{wt}(b) + \sum_{c \in GF(q)^*} \text{wt}(a + cb) = (q - 1)\text{wt}(a). \quad (5)$$

Note that $\text{wt}(a + cb) > 0$ for any $c \in GF(q)^*$, as $\text{wt}(a) > 0$ and $\text{wt}(b) > 0$. Thus we have $\text{wt}(a) = w_2$. Consider the multiset

$$\{\text{wt}(a + cb) : c \in GF(q)^*\} \cup \{\text{wt}(b)\}.$$

Assume that the multiplicity of $w_1$ in this multiset is $j$ and the multiplicity of $w_2$ is $q - j$. Then we have $2 \leq j \leq q$. It follows from (5) that

$$jw_1 = (j - 1)w_2,$$

This completes the proof. \qed

4. A family of minimal ternary linear codes violating the Ashikhmin-Barg condition

In this section, we present a family of minimal ternary linear codes violating the Ashikhmin-Barg condition with a general construction. The idea is similar to the one in our construction for the binary case [12].
4.1. A general construction of ternary linear codes

Throughout this section, we always assume that \( f(x) \) is a function from \( \text{GF}(3)^m \) to \( \text{GF}(3) \) such that \( f(0) = 0 \) but \( f(b) \neq 0 \) for at least one \( b \in \text{GF}(3)^m \). Recall that the Walsh transform of \( f \) is given as

\[
\hat{f}(w) = \sum_{x \in \text{GF}(3)^m} \zeta_3^{f(x) - w \cdot x}, \quad w \in \text{GF}(3)^m,
\]

where \( \zeta_3 \) is a primitive 3-th complex root of unity, and \( w \cdot x \) denotes the standard inner product of \( w \) and \( x \). Using such function \( f \), we define a linear code by

\[
C_f = \{(uf(x) + v \cdot x)_{x \in \text{GF}(3)^m} : u \in \text{GF}(3), \ v \in \text{GF}(3)^m \}.
\]

The construction above is general in the sense that it works for any function \( f \) from \( \text{GF}(3)^m \) to \( \text{GF}(3) \) with \( f(0) = 0 \). The following result shows that the weight distribution of \( C_f \) could be determined by the Walsh spectrum of \( f \).

**Theorem 13.** Assume that \( f(x) \neq w \cdot x \) for any \( w \in \text{GF}(3)^m \). The linear code \( C_f \) in (6) has length \( 3^m - 1 \) and dimension \( m + 1 \). In addition, the weight distribution of \( C_f \) is given by the following multiset union:

\[
\{ 2 \left( 3^m - 1 - \frac{\Re (\hat{f}(v))}{3} \right) : u \in \text{GF}(3)^*, v \in \text{GF}(3)^m \} \cup \{3^m - 3^m - 1 : u = 0, v \in (\text{GF}(3)^m)^* \} \cup \{0\}.
\]

Herein and hereafter, \( \Re(x) \) denotes the real part of the complex number \( x \).

**Proof.** In terms of exponential sums, the Hamming weight \( \omega_t(c) \) of any codeword \( c = (uf(x) + v \cdot x)_{x \in \text{GF}(3)^m \setminus \{0\}} \) of \( C_f \) in (6) can be calculated as

\[
\omega_t(c) = \sharp \{ x \in \text{GF}(3)^m \setminus \{0\} : uf(x) + v \cdot x \neq 0 \} \\
= (3^m - 1) - \frac{1}{3} \sum_{y \in \text{GF}(3)} \sum_{x \in \text{GF}(3)^m \setminus \{0\}} \sum_{\zeta_3} \zeta_3^{uf(x) + v \cdot x} \\
= (3^m - 1) - \frac{1}{3} \sum_{y \in \text{GF}(3)^*} \sum_{x \in \text{GF}(3)^m} \sum_{\zeta_3} \zeta_3^{uf(x) + v \cdot x} \\
= 3^m - 3^m - 1 - \frac{1}{3} \left( \sum_{x \in \text{GF}(3)^m} \sum_{\zeta_3} \zeta_3^{uf(x) + v \cdot x} + \sum_{x \in \text{GF}(3)^m} \sum_{\zeta_3} \zeta_3^{-uf(x) - v \cdot x} \right) \\
= 3^m - 3^m - 1 - \frac{2}{3} \Re \left( \sum_{x \in \text{GF}(3)^m} \sum_{\zeta_3} \zeta_3^{uf(x) + v \cdot x} \right).
\]

We discuss the value of \( \omega_t(c) \) by considering the following cases.

- If \( u = 0 \) and \( v = 0 \), then \( \omega_t(c) = 0 \).
- If \( u = 0 \) and \( v \neq 0 \), then \( \omega_t(c) = 3^m - 3^m - 1 \).
- If \( u = 1 \) and \( v \neq 0 \), then \( \omega_t(c) = 3^m - 3^m - 1 - \frac{2}{3} \Re(\hat{f}(-v)) \).
The weight distribution in (7) then follows from the discussions above. Note that the dimension of the code \( C_f \) is \( m+1 \) if and only if
\[
\text{Re}(\tilde{f}(w)) \neq 3^m \text{ for any } w \in \text{GF}(3)^m,
\]
where
\[
\text{Re}(\tilde{f}(w)) = \text{Re} \left( \sum_{x \in \text{GF}(3)^m} \zeta_3^{f(x)-w \cdot x} \right) = \sum_{x \in \text{GF}(3)^m} \text{Re} \left( \zeta_3^{f(x)-w \cdot x} \right).
\]
Hence \( \text{Re}(\tilde{f}(w)) = 3^m \) if and only if \( f(x) = w \cdot x \) for all \( x \in \text{GF}(3)^m \). This together with the hypothesis that \( f(x) \neq w \cdot x \) for any \( w \in \text{GF}(3)^m \) means that the dimension of \( C_f \) is \( m+1 \).

4.2. When are these codes minimal?

Now, a natural question is when the linear code \( C_f \) defined in (6) is minimal. The following gives a sufficient and necessary condition for \( C_f \) to be minimal in terms of the Walsh spectrum of \( f \).

**Theorem 14.** Let \( C_f \) be the ternary code of Theorem 7. Assume that \( f(x) \neq v \cdot x \) for any \( v \in \text{GF}(3)^m \). Then \( C_f \) is a minimal \( [3^m-1, m+1] \) code if and only if
\[
\text{Re}(\tilde{f}(w_1)) + \text{Re}(\tilde{f}(w_2)) = 2\text{Re}(\tilde{f}(w_3)) \neq 3^m
\]
and
\[
\text{Re}(\tilde{f}(w_1)) + \text{Re}(\tilde{f}(w_2)) + \text{Re}(\tilde{f}(w_3)) \neq 3^m
\]
for any pairwise distinct vectors \( w_1, w_2 \) and \( w_3 \) in \( \text{GF}(3)^m \) satisfying \( w_1 + w_2 + w_3 = 0 \).

**Proof.** We define the following linear code
\[
S_m = \{ (v \cdot x)_{x \in \text{GF}(3)^m \setminus \{0\}} : v \in \text{GF}(3)^m \}.
\]
This code is a ternary code with parameters \( [3^m-1, m, 3^m-3^m-1] \) and the only nonzero Hamming weight \( 3^m-3^m-1 \).

Assume that \( f(x) \neq v \cdot x \) for any \( v \in \text{GF}(3)^m \). Let \( f = (f(x))_{x \in \text{GF}(3)^m \setminus \{0\}} \). By definition, every codeword \( a \in C_f \) can be expressed as
\[
a = u_a f + s_a,
\]
where \( u_a \in \{0, 1, -1\} \) and \( s_a = (v_a \cdot x)_{x \in \text{GF}(3)^m \setminus \{0\}} \in S_m \) for some \( v_a \in \text{GF}(3)^m \). We next consider the coverage of codewords in \( C_f \) by distinguishing the following cases.

Case I: Let \( a = s_a \) and \( b = s_b \) be two linearly independent codewords in \( S_m \). Then one cannot cover the other as the one-weight code \( S_m \) is obviously minimal.

Case II: Let \( a = f + s_a \) and \( b = f + s_b \), where \( a, b \) are linearly independent. This implies that
\[
a \pm b \neq 0, \text{ i.e., } s_a \neq s_b \text{ and } s_a + s_b \neq f.
\]
By assumption, \( s_a + s_b \neq f \) always holds. Hence \( a, b \) are linearly independent if and only if \( s_a \neq s_b \). Since

\[
s_a = (v_a \cdot x)_{x \in GF(3)^m \backslash \{0\}}, \quad s_b = (v_b \cdot x)_{x \in GF(3)^m \backslash \{0\}},
\]

we further deduce that \( a, b \) are linearly independent if and only if \( v_a \neq v_b \). Suppose that \( b \preceq a \). It then follows from Lemma 10 and the proof of Theorem 13 that

\[
b \preceq a \iff \mathsf{wt}(a + b) + \mathsf{wt}(a - b) = 2\mathsf{wt}(a) - \mathsf{wt}(b)
\]

\[
\iff \left( 3^m - 3^{m-1} - \frac{2}{3} \text{Re}(\hat{f}(v_a + v_b)) \right) + (3^m - 3^{m-1})
\]

\[
= 2 \left( 3^m - 3^{m-1} - \frac{2}{3} \text{Re}(\hat{f}(-v_a)) \right) - \left( 3^m - 3^{m-1} - \frac{2}{3} \text{Re}(\hat{f}(-v_b)) \right)
\]

\[
\iff \text{Re}(\hat{f}(v_a + v_b)) - 2\text{Re}(\hat{f}(-v_a)) + \text{Re}(\hat{f}(-v_b)) = 3^m.
\]

Similarly, we have

\[
a \preceq b \iff \text{Re}(\hat{f}(v_a + v_b)) - 2\text{Re}(\hat{f}(-v_b)) + \text{Re}(\hat{f}(-v_a)) = 3^m.
\]

Note that \( v_a + v_b, -v_a, -v_b \) are pairwise distinct as \( v_a \neq v_b \). In addition,

\[
v_a + v_b - v_a - v_b = 0.
\]

Case III: Let \( a = -f + s_a \) and \( b = -f + s_b \), where \( a, b \) are linearly independent. Similarly as in Case II, we deduce that \( v_a \neq v_b \) and

\[
b \preceq a \iff \text{Re}(\hat{f}(-v_a - v_b)) - 2\text{Re}(\hat{f}(v_a)) + \text{Re}(\hat{f}(v_b)) = 3^m
\]

and

\[
a \preceq b \iff \text{Re}(\hat{f}(-v_a - v_b)) - 2\text{Re}(\hat{f}(v_b)) + \text{Re}(\hat{f}(v_a)) = 3^m.
\]

Note that \( -v_a - v_b, v_a, v_b \) are pairwise distinct as \( v_a \neq v_b \). In addition,

\[
-v_a - v_b + v_a + v_b = 0.
\]

Case IV: Let \( a = s_a \) and \( b = f + s_b \) with \( a \) being nonzero. Then \( a, b \) are linearly independent because of the assumption. Similarly as in Case II, we deduce that

\[
b \preceq a \iff \text{Re}(\hat{f}(-v_a - v_b)) + \text{Re}(\hat{f}(v_a - v_b)) + \text{Re}(\hat{f}(v_b)) = 3^m
\]

and

\[
a \preceq b \iff \text{Re}(\hat{f}(-v_a - v_b)) + \text{Re}(\hat{f}(v_a - v_b)) - 2\text{Re}(\hat{f}(-v_b)) = 3^m.
\]

Note that \( -v_a - v_b, v_a - v_b, -v_b \) are pairwise distinct as \( v_a \neq 0 \). In addition,

\[
-v_a - v_b + v_a - v_b = 0.
\]

Case V: Let \( a = s_a \) and \( b = -f + s_b \) with \( a \) being nonzero. Then \( a, b \) are linearly independent because of the assumption. Similarly as in Case II, we deduce that

\[
b \preceq a \iff \text{Re}(\hat{f}(v_a + v_b)) + \text{Re}(\hat{f}(v_b - v_a)) + \text{Re}(\hat{f}(v_b)) = 3^m
\]
and
\[ b \leq a \iff \Re(\hat{f}(v_a + v_b)) + \Re(\hat{f}(v_b - v_a)) - 2\Re(\hat{f}(v_b)) = 3^m. \]

Note that \( v_a + v_b, v_b - v_a, v_b \) are pairwise distinct as \( v_a \neq 0 \). In addition,
\[ v_a + v_b + v_b - v_a + v_b = 0. \]

Case VI: Let \( a = t + s_a \) and \( b = -t + s_b \) with \( s_a \neq -s_b \). Then \( a, b \) are linearly independent because of the assumption. Similarly as in Case II, we deduce that
\[ b \leq a \iff \Re(\hat{f}(v_a - v_b)) + \Re(\hat{f}(v_b)) - 2\Re(\hat{f}(v_a)) = 3^m \]
and
\[ a \leq b \iff \Re(\hat{f}(v_a - v_b)) + \Re(\hat{f}(-v_a)) - 2\Re(\hat{f}(v_b)) = 3^m. \]

Note that \( v_a - v_b, v_b, -v_a \) are pairwise distinct as \( v_a \neq -v_b \). In addition,
\[ v_a - v_b + v_b - v_a = 0. \]

Combining the discussions above, we complete the proof.

4.3. A family of minimal ternary linear codes with \( w_{\min}/w_{\max} \leq 2/3 \)

In this section, we present a family of minimal ternary linear codes with \( w_{\min}/w_{\max} \leq 2/3 \) with the help of Theorem 14. Before doing this, we recall that when \( q = 3 \), the Lloyd polynomial in (1) becomes
\[ \Psi_k(x, m) = \sum_{i=0}^{k} K_i(x, m) = \sum_{i=0}^{k} \sum_{j=0}^{t} (-1)^i 2^{i-j} \binom{m-x}{i-j}. \] (8)

For a positive integer \( k \) with \( 1 \leq k \leq m \), let \( S(m, k) \) denote the set of vectors in \( \text{GF}(3)^m \setminus \{0\} \) with Hamming weight at most \( k \). It is clear that
\[ |S(m, k)| = \sum_{j=1}^{k} 2^j \binom{m}{j}. \]

Define a function \( g_{(m, k)} \) from \( \text{GF}(3)^m \) to \( \text{GF}(3) \) as
\[ g_{(m, k)}(x) = \begin{cases} 1 & \text{if } x \in S(m, k), \\ 0 & \text{otherwise.} \end{cases} \] (9)

Using \( g_{(m, k)} \) to replace the function \( f \) in (6), we automatically obtain a ternary linear code \( C_{g_{(m, k)}} \). The parameters and weight distribution of \( C_{g_{(m, k)}} \) are given as follows.

Theorem 15. The ternary code \( C_{g_{(m, k)}} \) has length \( 3^m - 1 \), dimension \( m + 1 \), and the weight distribution in Table 7, where \( \Psi_k(x, m) \) is the Lloyd polynomial given by (8).

Proof. From the definition of \( g_{(m, k)} \) in Equation (9),
\[ \tilde{g}_{(m, k)}(w) = \sum_{x \in \text{GF}(3)^m} g_{(m, k)}(x) \cdot w^x. \]
\[
\sum_{x \in S(m,k)} \zeta_3^{1-w \cdot x} + \sum_{x \in \text{GF}(3)^m \setminus S(m,k)} \zeta_3^{-w \cdot x}
\]

\[
= \sum_{x \in \text{GF}(3)^m} \zeta_3^{-w \cdot x} + (\zeta_3 - 1) \sum_{x \in S(m,k)} \zeta_3^{-w \cdot x}
\]

for \(w \in \text{GF}(3)^m\). If \(w = 0\), then

\[
\hat{g}_{(m,k)}(0) = 3^m + (\zeta_3 - 1) \sum_{j=1}^{k} 2^j \binom{m}{j}
\]

and

\[
\text{Re} \left( \hat{g}_{(m,k)}(0) \right) = 3^m - \frac{3}{2} \sum_{j=1}^{k} 2^j \binom{m}{j}.
\]

If \(w \neq 0\) with \(\text{wt}(w) = i\), then by Lemma \[7\] we have

\[
\hat{g}_{(m,k)}(w) = (\zeta_3 - 1) \sum_{j=1}^{k} K_i(i,m) = (\zeta_3 - 1)(\Psi_k(i,m) - 1)
\]

for \(q = 3\). Thus

\[
\text{Re} \left( \hat{g}_{(m,k)}(w) \right) = -\frac{3}{2}(\Psi_k(i,m) - 1).
\]

Then the weight distribution follows from the proof of Theorem \[13\]. \(\square\)

**Remark 2.** It is noticed that the Lloyd polynomial \(\Psi_k(x,m)\) may take the same value for different values of \(x\). Therefore, the set \(\mathcal{C}_{g_{(m,k)}}\) in Theorem \[15\] is a ternary linear code with at most \(m + 2\) weights. For example, when \(k = 2\) and \(m = 5\),

\[
\Psi_2(x,5) = \frac{9x^2}{2} - \frac{63x}{2} + 51.
\]

It is easy to see that \(\Psi_2(x,5) = 6\) for \(x \in \{2,5\}\), and \(\Psi_2(x,5) = -3\) for \(x \in \{3,4\}\). Thus the set \(\mathcal{C}_{g_{(7,2)}}\) in Theorem \[15\] is a five-weight (instead of seven-weight) linear code. Similarly, when \(k = 2\) and \(m = 7\), it is easily verified that the Lloyd polynomial \(\Psi_2(x,7)\) takes different values when \(x\) runs from 1 to 7. Therefore, the set \(\mathcal{C}_{g_{(7,2)}}\) in Theorem \[15\] is a nine-weight linear code.

| Table 1: Weight distribution |
|-----------------------------|
| Weight \(w\) | No. of codewords \(A_w\) |
|----------------|-----------------|
| 0 | 1 |
| \(3^m - 3^{m-1} + \Psi_k(i,m) - 1\) | \(2^{i+1} \binom{m}{i}\) for \(1 \leq i \leq m\) |
| \(\sum_{j=1}^{k} 2^j \binom{m}{j}\) | 2 |
| \(3^m - 3^{m-1}\) | 3^m - 1 |
Corollary 16. Let $m, k$ be integers with $m \geq 5$ and $2 \leq k \leq \lfloor \frac{m-1}{2} \rfloor$. Then the linear code $C_{G(m,k)}$ in Theorem 15 has parameters

$$\left[ 3^m - 1, m + 1, \sum_{j=1}^{k} 2^j \binom{m}{j} \right].$$

Furthermore, $w_{\text{min}}/w_{\text{max}} \leq 2/3$ if and only if

$$3 \sum_{j=1}^{k} 2^j \binom{m}{j} \leq 2(3^m - 3^{m-1}) + 2^{k+1} \binom{m-1}{k} - 2.$$

Proof. By Table 1, we denote all the nonzero weights in $C_{G(m,k)}$ as

$$\begin{cases}
w(i) = 3^m - 3^{m-1} + \Psi_k(i,m) - 1, & 1 \leq i \leq m, \\
w' = \sum_{j=0}^{k} 2^j \binom{m}{j}, \\
w'' = 3^m - 3^{m-1}.
\end{cases}$$

For $1 \leq i \leq m$, by Corollary 6 we deduce that

$$w(i) = 3^m - 3^{m-1} + \Psi_k(i,m) - 1$$

$$\geq 3^m - 3^{m-1} - 2^k \binom{m-1}{k} - 1$$

$$= 2 \times 3^{m-1} - 2^k \binom{m-1}{k} - 1$$

$$= 2 \times (2+1)^{m-1} - 2^k \binom{m-1}{k} - 1$$

$$= 2 \times \sum_{j=0}^{m-1} 2^j \binom{m-1}{j} - 2^k \binom{m-1}{k} - 1. \quad (10)$$

Note that

$$w' = \sum_{j=1}^{k} 2^j \binom{m}{j} = \sum_{j=1}^{k} 2^j \binom{m-1}{j} + \sum_{j=1}^{k} 2^j \binom{m-1}{j-1}. \quad (11)$$

Since $2 \leq k \leq \lfloor \frac{m-1}{2} \rfloor$, we have

$$2 \times \sum_{j=0}^{m-1} 2^j \binom{m-1}{j} - 2^k \binom{m-1}{k} - 1$$

$$= \sum_{j=0}^{m-1} 2^j \binom{m-1}{j} + \sum_{j=0}^{m-1} 2^j \binom{m-1}{j} - 2^k \binom{m-1}{k} - 1$$

$$> \sum_{j=0}^{k} 2^j \binom{m-1}{j} + \sum_{j=1}^{m-1} 2^j \binom{m-1}{j} - 2^k \binom{m-1}{k} - 1$$

$$= \sum_{j=1}^{k} 2^j \binom{m-1}{j} + \sum_{j=1}^{m-1} 2^j \binom{m-1}{j} + \sum_{j=k+1}^{m-1} 2^j \binom{m-1}{j}$$
\[ \sum_{j=1}^{k} 2^j \binom{m-1}{j} + \sum_{j=1}^{k-1} 2^j \binom{m-1}{j} + 2^{k+1} \binom{m-1}{k+1} > \sum_{j=1}^{k} 2^j \binom{m-1}{j} + \sum_{j=1}^{k-1} 2^j \binom{m-1}{j} + 2^k \binom{m-1}{k-1} \]
\[ = \sum_{j=1}^{k} 2^j \binom{m-1}{j} + \sum_{j=1}^{k} 2^j \binom{m-1}{j}. \]

It then follows from Equations 10 and 11 that
\[ w(i) > w' \text{ for all } 1 \leq i \leq m. \]

From the discussions above, we also have
\[ w'' = 3^m - 3^{m-1} > w'. \]

Hence, the minimum Hamming weight of \( C_{g(m,k)} \) is given by \( w_{\text{min}} = w' \). According to Corollary 6, the maximum Hamming weight of \( C_{g(m,k)} \) is given by
\[ w_{\text{max}} = w(1) = 3^m - 3^{m-1} + \Psi_k(1,m) - 1 = 3^m - 3^{m-1} + 2^k \binom{m-1}{k} - 1. \]

This completes the proof. □

The following lemma will be used to prove the minimality of the linear code in Theorem 15.

**Lemma 17.** Let \( m,k \) be integers with \( m \geq 5 \) and \( 2 \leq k \leq \lfloor \frac{m-1}{2} \rfloor \). Then we have
\[ \sum_{j=1}^{k} \left( \binom{m}{j} \right) \neq -2 \left( \Psi_k(i,m) - 1 \right) \text{ for all } 1 \leq i \leq m. \]

**Proof.** Note that the inequality
\[ \sum_{j=1}^{k} \left( \binom{m}{j} \right) \neq -2 \left( \Psi_k(i,m) - 1 \right) \text{ for all } 1 \leq i \leq m \]
is equivalent to
\[ -\sum_{j=1}^{k} 2^{j-1} \binom{m}{j} - \Psi_k(i,m) \neq -1 \text{ for all } 1 \leq i \leq m. \]

By Lemma 2, it is sufficient to prove that
\[ -\sum_{j=1}^{k} 2^{j-1} \binom{m}{j} - K_k(i-1,m-1) \neq -1 \text{ for all } 1 \leq i \leq m. \] (12)

Note that
\[ -\sum_{j=1}^{k} 2^{j-1} \binom{m}{j} - K_k(i-1,m-1) \]
This means that (12) holds provided that
\[-2^{k-1} \binom{m-1}{k} - K_k(i-1, m-1) < -1 \text{ for all } 1 \leq i \leq m.\]

In fact, by the Vandermonde convolution formula and the definition of the Krawthouk polynomial, we have
\[-2^{k-1} \binom{m-1}{k} - K_k(i-1, m-1) = -\sum_{j=0}^{k-1} \binom{i-1}{j} \binom{m-i}{k-j} - \sum_{j=0}^{k-1} (-1)^j 2^{k-j} \binom{i-1}{j} \binom{m-i}{k-j} < -1,\]
where we used the fact that \(2 \leq k \leq \lfloor \frac{m-1}{2} \rfloor\) and \(m \geq 5\). This completes the proof of this lemma.

The following result shows that the linear code in Theorem 15 is minimal and violates the Ashikhmin-Barg condition in many cases.

**Theorem 18.** Let \(m, k\) be integers with \(m \geq 5\) and \(2 \leq k \leq \lfloor \frac{m-1}{2} \rfloor\). Then the linear code \(C_{\mathcal{R}(m,k)}\) is minimal and has parameters
\[
\left[3^m - 1, m + 1, \sum_{j=1}^{k} 2^{j} \binom{m}{j}\right].
\]
Furthermore, \(w_{\min}/w_{\max} \leq 2/3\) if and only if
\[
3 \sum_{j=1}^{k} 2^{j} \binom{m}{j} \leq 2(3^m - 3^{m-1}) + 2^{k+1} \binom{m-1}{k} - 2.
\]

**Proof.** According to Corollary 16, we only need to prove that \(C_{\mathcal{R}(m,k)}\) is minimal. From the proof of Theorem 15, we have
\[
\Re \left( \hat{g}_{(m,k)}(w) \right) = \begin{cases} 
3^m - \frac{3}{2} \sum_{j=1}^{k} 2^{j} \binom{m}{j} & \text{if } w = 0, \\
-\frac{3}{2} \left( \Psi_k(i, m) - 1 \right) & \text{if } \exists t(w) = i > 0.
\end{cases}
\]
(13)

Theorem 14 implies that \(C_{\mathcal{R}(m,k)}\) is minimal if and only if
\[
\Re(\hat{g}_{(m,k)}(w_1)) + \Re(\hat{g}_{(m,k)}(w_2)) - 2\Re(\hat{g}_{(m,k)}(w_3)) \neq 3^m
\]
(14)
\[ \text{Re}(\hat{g}(m,k)(w_1)) + \text{Re}(\hat{g}(m,k)(w_2)) + \text{Re}(\hat{g}(m,k)(w_3)) \neq 3^m \]  \hspace{1cm} (15)

for any pairwise distinct vectors \( w_1, w_2, w_3 \in \text{GF}(3)^m \) satisfying \( w_1 + w_2 + w_3 = 0 \). We distinguish between the following two cases to show that (14) and (15) hold for the claimed vectors.

Case 1: Assume that one of \( w_1, w_2, w_3 \) is 0.

We firstly consider Inequality (15). Without loss of generality, we assume that \( w_1 = 0 \) and then \( w_2 = -w_3 \neq 0 \), where \( \text{wt}(w_2) = \text{wt}(w_3) = i \) and \( 1 \leq i \leq m \). Then by Equation (13), Inequality (15) is equivalent to

\[
\sum_{j=1}^{k} 2^j \binom{m}{j} \neq -2(\Psi_k(i,m) - 1) \text{ for all } 1 \leq i \leq m,
\]

which holds by Lemma 17. Next we consider Inequality (14) in two cases.

1. If \( w_3 = 0 \), then \( \text{wt}(w_1) = \text{wt}(w_2) = i \) with \( 1 \leq i \leq m \). Then by Equation (13), Inequality (14) is equivalent to

\[
\sum_{j=0}^{k} 2^j \binom{m}{j} - \Psi_k(i,m) \neq 3^m \text{ for all } 1 \leq i \leq m.
\]  \hspace{1cm} (16)

Due to Corollary 6 for \( q = 3 \), we deduce that

\[
\left| \sum_{j=0}^{k} 2^j \binom{m}{j} - \Psi_k(i,m) \right| \leq \sum_{j=0}^{k} 2^j \binom{m}{j} + |\Psi_k(i,m)| \leq \sum_{j=0}^{k} 2^j \binom{m}{j} + 2^k \binom{m-1}{k} < \sum_{j=0}^{k} 2^j \binom{m}{j} + 2^k \binom{m-1}{k+1} = \sum_{j=0}^{k} 2^j \binom{m}{j} + 2^{k+1} \binom{m}{k+1} < \sum_{j=0}^{m} 2^j \binom{m}{j} = 3^m.
\]

Thus Inequality (16) holds and then Inequality (14) holds.

2. If one of \( \text{wt}(w_1) \) and \( \text{wt}(w_2) \) is 0, we assume that \( w_1 = 0 \) without loss of generality. Then \( \text{wt}(w_2) = \text{wt}(w_3) = i \) with \( 1 \leq i \leq m \). Then by Equation (13), Inequality (14) is equivalent to

\[
\sum_{j=0}^{k} 2^j \binom{m}{j} \neq \Psi_k(i,m) \text{ for all } 1 \leq i \leq m.
\]  \hspace{1cm} (17)
Due to Corollary 6 for \( q = 3 \), we have
\[
\Psi_k(i, m) \leq 2^k \binom{m-1}{k} < 2^k \binom{m}{k} < \sum_{j=0}^{k} 2^j \binom{m}{j}.
\]
Thus Inequality (17) holds and then Inequality (14) holds.

In this case, (14) and (15) follow from the discussion above.

**Case 2: Assume that all \( w_1, w_2, w_3 \) are nonzero.**

Due to Corollary 6 for \( q = 3 \) and Equation (13), we derive that
\[
\left| \Re(\hat{g}(m, k)(w_1)) + \Re(\hat{g}(m, k)(w_2)) - 2\Re(\hat{g}(m, k)(w_3)) \right| \leq 6 \times 2^k \binom{m-1}{k} + 6
\]
and
\[
\left| \Re(\hat{g}(m, k)(w_1)) + \Re(\hat{g}(m, k)(w_2)) + \Re(\hat{g}(m, k)(w_3)) \right| \leq \frac{9}{2} \times 2^k \binom{m-1}{k} + \frac{9}{2}.
\]
To show that Inequalities (14) and (15) hold, it is sufficient to show that
\[
6 \times 2^k \binom{m-1}{k} + 6 < 3^m,
\]
which is equivalent to
\[
2^{k+1} \binom{m-1}{k} + 2 < 3^{m-1}.
\]
Since \( 2 \leq k \leq \left\lfloor \frac{m-1}{2} \right\rfloor \), we have
\[
2^{k+1} \binom{m-1}{k} + 2 = 2^k \binom{m-1}{k} + 2^k \binom{m-1}{k} + 2 < 2^k \binom{m-1}{k} + 2^{k+1} \binom{m-1}{k+1} + 2 < \sum_{j=0}^{m-1} 2^j \binom{m-1}{j} = 3^{m-1}.
\]
Thus Inequalities (14) and (15) hold in this case.

Summarizing the discussion above completes the proof of this theorem.

As a corollary of Theorem 18, one can easily derive the following.

**Corollary 19.** Let \( k = 2, m \geq 5 \). Then \( C_{g(m, 2)} \) in Theorem 18 is a minimal code with parameters
\[
\left[ 3^m - 1, m + 1, \sum_{j=1}^{2} 2^j \binom{m}{j} \right].
\]
Furthermore, \( w_{\text{min}}/w_{\text{max}} < 2/3 \).
Corollary 19 demonstrates that the codes in Theorem 18 contain an infinite family of ternary minimal linear codes with $w_{\min}/w_{\max} \leq 2/3$. We conclude this section with two examples computed by Magma, which are consistent with our theoretical results.

**Example 1.** The set $C_{g(5,2)}$ in Theorem 18 is a minimal code with parameters $[242, 6, 50]$ and weight enumerator

$$1 + 2z^{50} + 320z^{158} + 242z^{162} + 144z^{167} + 20z^{185}$$

where $w_{\min}/w_{\max} = 50/185 < 2/3$.

**Example 2.** The set $C_{g(7,2)}$ in Theorem 18 is a minimal code with parameters $[2186, 8, 98]$ and weight enumerator

$$1 + 2z^{98} + 1344z^{1451} + 1120z^{1454} + 896z^{1457} + 2186z^{1458} + 560z^{1466} + 256z^{1472} + 168z^{1487} + 28z^{1517}$$

where $w_{\min}/w_{\max} = 98/1517 < 2/3$.

### 5. Concluding remarks

In this paper, we derived a necessary and sufficient condition for linear codes over finite fields to be minimal. This condition generalizes the construction for the binary case given in [12]. It enabled us to obtain a family of ternary minimal linear codes violating the Ashikhmin-Barg condition. This is the first infinite family of nonbinary minimal linear codes violating the Ashikhmin-Barg condition. It would be interesting to construct more such infinite families of nonbinary minimal linear codes. The reader is cordially invited to join this adventure.

Finally, we mention that the construction of the ternary code in (6) can be generalised to obtain $q$-ary codes, and the function $g_{m,k}(x)$ of (9) is also a function from $\text{GF}(q)$ to $\text{GF}(q)$ when the set $S(m, k)$ is defined to be the set of all vectors in $\text{GF}(q)^m \setminus \{0\}$ with Hamming weight at most $k$. In this case, the code $C_{g(m,k)}$ is a linear code over $\text{GF}(q)$. Experimental data show that the code $C_{g(m,k)}$ over $\text{GF}(q)$ could be minimal under certain conditions. However, it seems very hard to work out similar results for the general case $q$. We were able to handle only the ternary case in this paper.

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