The duality of Weyl and linear extension of Kostka matrices

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Abstract. Application of the Robinson-Schensted algorithm to the basis of magnetic configurations of the one-dimensional Heisenberg magnet with an arbitrary spin gives an efficient way for a classification of the irreducible basis of the Weyl duality. The plactic monoid is shown to be an adequate tool for describing this irreducible basis in a way consistent with the Schensted insertion procedure, i.e. the creation of a new single-particle state (a letter of the single-node spin) in already constructed Young and Weyl tableaux. Schensted insertion is interpreted in terms of Gelfand triangles - combinatoric analogues of Weyl tableaux with exposed occupation numbers, consistent with canonical chains of subgroups of both the symmetric and the unitary group. A transition matrix between these two bases should exist due to the linear structure of the Hilbert space. This matrix can be looked at as the linear extension of the famous Kostka matrix. We show how to obtain this matrix and give an interpretation of its elements as coefficients of certain wave packet with exactly defined symmetry.

1. Introduction

The one-dimensional Heisenberg magnet is a paradigmatic example of use of the Weyl duality [1] between the actions of the symmetric group \( \Sigma_N \) - the group of all permutations of nodes of a magnet, and the unitary group \( U(n) \) of the highest quantum symmetry of each constituent part of the magnet - the node with the spin \( s \). The irreducible basis of the Weyl duality is given by the combinatoric algorithm of Robinson and Schensted (RS) [2, 3] in terms of a pair \( (P,Q) \) of Weyl and Young (semi)standard tableaux. In this paper we aim to show an explicit construction of this irreducible basis. It can be interpreted as a generalization of famous Kostka matrices \( \{K_{\lambda,\mu}\} \) at the level of bases, spanned on an orbit of action of the symmetric group with the weight \( \mu \), corresponding to an \( \Sigma \)-irrep of the shape \( \lambda \).

Within the paper, we use interchangeably semistandard tableaux and Gelfand patterns. The former are perhaps better adapted to RS algorithm of insertion of a letter into a tableau, whereas the latter are more convenient in physical interpretation as collections of appropriate occupation numbers.

2. Heisenberg magnetic ring

We consider a one dimensional Heisenberg magnet consisting of \( N \) nodes with single-node spin \( s \). A state of the magnet is given by ascribing a projection of the single node spin to each node
of a magnet. Mathematically, this is formulated in terms of two sets:

\[ \tilde{N} = \{ j = 1, 2, ..., N \} \quad \text{and} \quad \tilde{n} = \{ i = 1, 2, ..., n \}, \quad n = 2s + 1, \]

the set of \( N \) nodes of the crystal - the alphabet of nodes, and the set of the single node states - the alphabet of spins, respectively. The state of the whole magnet is a mapping \( f : \tilde{N} \rightarrow \tilde{n} \) and can be presented as

\[ |f\rangle = |i_1, i_2, ..., i_N \rangle \equiv |f(1), f(2), ..., f(N) \rangle \equiv |f, f', f'' \rangle, \]

We call this state a magnetic configuration and treat “it” as a word of the length \( N \) in the alphabet of spins.

The set \( \tilde{n}^{N} = \{ f : \tilde{N} \rightarrow \tilde{n} \} \) of all magnetic configurations spans the Hilbert space

\[ \mathcal{H} = \text{lc} \mathbb{C} \tilde{n}^{\tilde{N}}, \]

with the scalar product \( \langle f | f' \rangle = \delta_{f f'}, \quad f, f' \in \tilde{n}^{\tilde{N}}. \)

The dynamics of the Heisenberg magnet is defined by the Hamiltonian - a hermitian operator \( \hat{H} \in \text{End} \mathcal{H} \). In general a setting, this Hamiltonian is constructed from generators \( e_{ii'}, i, i' \in \tilde{n} \) of the Lie algebra of the unitary group \( U(n) \). For example, for the case \( n = 3 \) (or \( s = 1 \)) for the isotropic XXX model we have

\[ H = (J/4) \sum_{j \in \tilde{N}} \left( \hat{s}_j \cdot \hat{s}_{j+1} - (\hat{s}_j \cdot \hat{s}_{j+1})^2 \right), \]

where \( \hat{s}_j = (\hat{s}_j^x, \hat{s}_j^y, \hat{s}_j^z) \) is the vector operator for the spin \( s = 1 \).

3. The Duality of Weyl

We recall that the space \( \mathcal{H} \) is a scene of two dual actions, \( A : \Sigma_N \times \mathcal{H} \rightarrow \mathcal{H} \) and \( B : U(n) \times \mathcal{H} \rightarrow \mathcal{H} \). The most important quantum -mechanical observation is that the two actions mutually commute, that is,

\[ [A(\sigma), B(u)] = 0, \quad \sigma \in \Sigma_N, \quad u \in U(n), \]

despite the fact that both dual groups are, for \( N > 2, n > 1 \), highly noncommutative. It implies compatibility of appropriate quantities related to both groups in the spirit of the Heisenberg uncertainty principle: these quantities ”can be measured simultaneously”. A maximal system of such compatible (commuting) observables is realized in an irreducible basis in the space \( \mathcal{H} \), adapted to the symmetry of both dual groups. We proceed to describe this scheme in more detail.

We introduce the set of partitions of the integer \( N \) with no more than \( n \) parts

\[ D_W(N, n) = \{ \lambda \vdash N, \quad |\lambda| \leq n \} \]

where \( |\lambda| \) denotes the number of non zero parts of the partition \( \lambda \) of \( N \). Each element \( \lambda \) of this set of partitions serves as the label of the irreducible representation (irrep) \( \Delta^\lambda \) of the symmetric group \( \Sigma_N \) and, at the same time, as the label of the irrep \( D^\lambda \) of the unitary group \( U(n) \).

The corresponding decompositions of both dual actions into irreps read

\[ A = \sum_{\lambda \in D_W(N, n)} m(A, \Delta^\lambda) \Delta^\lambda \quad \text{and} \quad B = \sum_{\lambda \in D_W(N, n)} m(B, D^\lambda) D^\lambda, \]
where appropriate multiplicities, on the strength of duality of Weyl [1], satisfy relations

\[ m(A, \Delta^\lambda) = \dim D^\lambda, \quad m(B, D^\lambda) = \dim \Delta^\lambda. \]  

(8)

In this way, the duality of Weyl decomposes the whole space \( \mathcal{H} \) of quantum states of the composite system into sectors \( \mathcal{H}^\lambda \),

\[ \mathcal{H} = \sum_{\lambda \in D_W(N,n)} \oplus \mathcal{H}^\lambda \]  

such that \( A|_{\mathcal{H}^\lambda} = (\dim D^\lambda) \Delta^\lambda \), and \( B|_{\mathcal{H}^\lambda} = (\dim \Delta^\lambda) D^\lambda \).  

(9)

Thus each sector \( \mathcal{H}^\lambda \) carries \( m(A, \lambda) = \dim D^\lambda \) copies of \( \Delta^\lambda \)-irreducible subspaces, and, at the same time, \( m(B, \lambda) = \dim \Delta^\lambda \) copies of \( D^\lambda \)-irreducible ones.

The duality of Weyl admits therefore the irreducible basis of the form

\[ b_{irr} = \{ |\lambda t y \rangle | \lambda \in D_W(N,n), \ t \in \tilde{D}^\lambda, \ y \in \tilde{\Delta}^\lambda \}, \]  

(10)

where \( \tilde{D}^\lambda \) and \( \tilde{\Delta}^\lambda \) are some standard bases for the irrep \( D^\lambda \) and \( \Delta^\lambda \), respectively. It is convenient to take standard bases in the form \( \tilde{\Delta}^\lambda = SYT(\lambda), \ \tilde{D}^\lambda = WT(\lambda, \tilde{n}) \), where \( SYT(\lambda) \) denotes the set of all standard Young tableaux of the shape \( \lambda \) in the alphabet \( \tilde{N} \) of nodes, and \( WT(\lambda, \tilde{n}) \) is the set of all Weyl tableaux of the shape \( \lambda \) in the alphabet \( \tilde{n} \) of spins. In this way, the duality of Weyl provides a complete classification of basis states in the space \( \mathcal{H} \).

4. The Robinson-Schensted combinatoric bijection

Now we describe a way of labelling scheme of Weyl duality, namely the RS algorithm. Let us take the set of words

\[ \tilde{n}^* = \bigcup_{N=0}^{\infty} \tilde{n}^\tilde{N}, \]  

(11)

i.e. the free monoid [4] with juxtaposition of words as monomial multiplication and the empty word \( \emptyset \) as the unit. Physically, the free monoid is the disjoint union of sets \( \tilde{n}^\tilde{N} \) of all magnetic configurations for the rings \( \tilde{N}, \ N = 0, 1, \ldots \), with \( N = 0 \) corresponding to the empty set. Let

\[ b^* = \bigcup_{N=0}^{\infty} b_{irr}(N) \]  

(12)

be the disjoint union of irreducible bases of the duality of Weyl for the rings \( \tilde{N}, \ N = 0, 1, \ldots \), with \( b_{irr}(0) = \emptyset \). The Robinson-Schensted algorithm [2, 3] defines a bijection \( RS : \tilde{n}^* \rightarrow b^* \) between magnetic configurations and appropriate irreducible bases of the duality of Weyl, by putting

\[ RS(f) = (P(f), Q(f)), \ f \in \tilde{n}^*, \]  

(13)

where \( P(f) \) and \( Q(f) \) denotes the Weyl and Young tableau, respectively.

The construction of these tableaux is given by the following rules [5, 11]:

- The magnetic configuration \( f \in \tilde{n}^\tilde{N} \) can be presented in two-line notation: \( f = (\begin{array}{cccc} i_1 & \ldots & i_N \\ j_1 & \ldots & j_N \end{array}) \) where the upper row contains the labels of nodes and the lower one the corresponding single node spin states.

- Next we construct a sequence of pairs of tableaux: \( (\emptyset, \emptyset) = (P_0, Q_0), (P_1, Q_1), (P_2, Q_2), \ldots, (P_N, Q_N) = (P, Q) \) where one has to insert letters \( i_1, i_2, \ldots, i_N \) into the \( P \)'s and the numbers \( 1, 2, \ldots, N \) into the \( Q \)'s so that \( sh P_k = sh Q_k \) for all \( k \) (\( sh \) denotes the shape of a tableau).

In order to insert the letter \( x \) into the intermediate tableau \( P \), we proceed as follows:
set $r:=\text{the first row of } P$

while $x$ is less than some element of row $r$ do

let $y$ be the smallest element of $r$ greater than $x$ and replace $y$ by $x$ in row $r$

set $x:=y$ and $r:=\text{the next row down}$

now $x$ is greater than every element of $r$, so place $x$ at the end of row $r$ and stop

Performing the RS algorithm on all magnetic configurations we obtain the complete set of labels of the basis of duality of Weyl.

5. The decomposition of Kostka at the level of bases

Consider the action $A$ of the symmetric group $\Sigma_N$ as a purely permutational representation $A : \Sigma_N \times \tilde{n} \tilde{N} \rightarrow \tilde{n} \tilde{N}$. This action decomposes the set $\tilde{n} \tilde{N}$ of all magnetic configurations of the ring into orbits

$$\mathcal{O}_\mu = \{ f \circ \sigma^{-1} | \sigma \in \Sigma_N \}$$

labelled by weights. The weight is a composition $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$, $\sum_{i} \mu_i = N$, where $\mu_i = |\{ i_j = i \; | \; j \in \tilde{N} \}|$, $i \in \tilde{n}$ is the occupation number for the single-node state $i \in \tilde{n}$, for any $f \in \mathcal{O}_\mu$. The restriction

$$A|_{\mathcal{O}_\mu} \equiv \mathcal{R}^{\Sigma_N: \Sigma_\mu}$$

of the action $A$ to the orbit $\mathcal{O}_\mu$ is a transitive representation of $\Sigma_N$, with the stabiliser being the Young subgroup $\Sigma_\mu = \Sigma_{\mu_1} \times \Sigma_{\mu_2} \times \cdots \times \Sigma_{\mu_N}$.

Let us consider next a (non-empty) intersection $lC\mathcal{O}_\mu \cap \mathcal{H}_\lambda$, $\lambda \trianglerighteq \mu$, of the carrier space of the action $\mathcal{R}_\mu$ with the sector $\mathcal{H}_\lambda$ of the Weyl duality. $\lambda \trianglerighteq \mu$, means that $\lambda$ is greater than or equal to $\mu$ in the dominance order. This intersection carries some copies of the irrep $\Delta^\lambda$ of the symmetric group $\Sigma_N$, according to the Kostka decomposition

$$\mathcal{R}^{\Sigma_N: \Sigma_\mu} = \sum_{\lambda \trianglerighteq \mu} K_{\lambda \mu} \Delta^\lambda,$$

with the multiplicities $K_{\lambda \mu}$ being Kostka numbers. The Kostka decomposition can be specified at the level of bases as

$$| \mu \lambda t y \rangle = \sum_{f \in \mathcal{O}_\mu} \begin{bmatrix} \mu & \lambda & t \end{bmatrix} \begin{bmatrix} f \end{bmatrix},$$

where the symbol in the rectangular brackets forms a unitary matrix in the space $lC\mathcal{O}_\mu$, which transforms the initial basis $\mathcal{O}_\mu$ of magnetic configurations to the irreducible basis of the Weyl duality, $t$ denotes the Weyl tableau, and $y$ the Young tableau.

In the sequel we identify the Weyl and Young tableaux with combinatorially equivalent Gelfand patterns along $[6, 7]$. The elements of this matrix are calculated on the basis of a ladder construction following the combinatorial growth of a Weyl tableau $t$ by attachement of consecutive nodes $j = 1, 2, \ldots, N$ to already constructed state of $j - 1$ nodes $[10, 12]$. Thus we have

$$\begin{bmatrix} \mu & \lambda & t \end{bmatrix} = \sum_{\{t_{12} \; t_{123} \cdots \; t_{1..N-1} \; t \}} \begin{bmatrix} \{1\} & \{1\} & \lambda_{12} & \{1\} & \lambda_{123} & \cdots & \lambda_{1..N-1} & \{1\} & \lambda \end{bmatrix} \begin{bmatrix} f(1) \; f(2) \; t_{12} \; f(3) \; t_{123} \; \cdots \; t_{1..N-1} \; f(N) \; t \end{bmatrix},$$

where the sum is over all possible chains of Gelfand patterns $\{t_{12} \; t_{123} \cdots \; t_{1..N-1} \; t \}$, adjusted to the configuration $f$ and successive steps of RS algorithm along the configuration $f' = RS^{-1}(\lambda t y)$. 

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In more detail, a chain \( \{t_{12} t_{123} \cdots t_{1..N-1} t\} \) is constructed recursively, namely: \( t_{1.j-1} \) is formed from \( t_{1,j-1} \) by adding (in all possible ways) a letter \( f(j) \) of magnetic configuration to \( t_{1,j-1} \) such that: the n-th row of the Gelfand pattern \( t_{1,j} \) is equal to \( \lambda_{1..j} \) and the standardness of Gelfand pattern has to be conserved.

Note that adding a letter \( a \) into a Weyl tableau implies that the appropriate entries in all rows \( j, (a \leq j \leq n) \) of the corresponding Gelfand pattern increase by one. Procedure of insertion of a letter \( a \) yields that the row \( \lambda_{1..j} \) of Gelfand pattern is given by:

\[
\lambda_{1..j} f(j) \rightarrow \lambda_{1..j} + f(j)
\]

what means that this row can be changed in various ways, depending on the choice of \( l \). The symbol

\[
\begin{bmatrix}
\lambda_{1..j-1} & 1 \\
t_{1..j-1} & f(j)
\end{bmatrix}
\]

in (18) denotes the coupling coefficient (Wigner-Clebsch-Gordan coefficient) at the \( j \)-th stage of growth, corresponding to the Littlewood-Richardson decomposition

\[
D^{\lambda_{1..j-1}} \otimes D^{(1)} = \sum_{\lambda_{1..j}} \oplus D^{\lambda_{1..j}}
\]

for the unitary group \( U(n) \). The bottom row of the coupling coefficient (19) exhibits the standard bases corresponding to irreps presented in the first row. Intermediate irrep \( \lambda_{1..j} \) is defined in terms of quantum numbers \( \lambda t y \) by means of the Robinson-Schensted correspondence, whereas the corresponding bases \( t_{1..j} \), \( j \in N \) \( \lambda_{1..N} = \lambda, t_{1..N} = t \) are defined by the insertion procedure, adjusted to the shapes from RS correspondence.

The Wigner-Clebsch-Gordan coefficient (19) can be calculated using a technique called \textit{Pattern Calculus} [7, 8, 9] by interpreting it in terms of fundamental tensor operators. A fundamental tensor operator \( t_{i_1 i_2} \) (double Gelfand pattern) is defines by the second column (\( \begin{bmatrix} \{i\} & f(j) \end{bmatrix} \)) of (19), by assigning: \( i_1 = f(j), i_2 = \) the row of \( \lambda_{1..j} \) in which the new box is added at the \( j \)-th step. Then we have

\[
\begin{bmatrix}
\lambda_{1..j-1} & 1 \\
t_{1..j-1} & f(j)
\end{bmatrix} = \langle t_{1..j} | t_{i_1 i_2} | t_{1..j-1} \rangle,
\]

this allow us to calculate this coefficient. Examples of matrix (17) are given in Table 1.

6. Conclusions
We have shown that the Robinson-Schensted algorithm plays two roles in the context of the Heisenberg magnet. Firstly, it serves for labelling the irreducible basis of the Weyl duality, so that \( |\lambda t y\rangle = |RS(f)\rangle \). Secondly, it provides a complete information for evaluation of elements \( \langle f|RS(f)\rangle \) of the Kostka matrix at the level of bases. This allow us to construct a wave packet with exactly defined symmetry.
Table 1. Kostka matrix at a level of bases for $N = 3$, $n = 2$ ($S = 1/2$), and $N = 3$, $n = 3$ ($S = 1$).

| $t$ | $a$ $a$ $b$ | $t$ | $a$ $b$ $c$ |
|-----|--------------|-----|--------------|
| $y$ | $1$ $2$ $3$  | $y$ | $1$ $2$ $3$  |

| $f$  | $aab$ | $0$ | $abc$ | $bac$ | $0$ | $2$ | $1/6$ | $1/2$ | $2$ | $-1/12$ | $-2/12$ | $1/6$ |
|------|-------|-----|-------|-------|-----|-----|-------|-------|-----|---------|---------|------|
|      | $1/\sqrt{3}$ | $0$ | $1/\sqrt{3}$ | $1/\sqrt{3}$ | $2/\sqrt{6}$ | $1/\sqrt{3}$ | $1/\sqrt{3}$ | $1/\sqrt{3}$ | $2/\sqrt{6}$ | $1/\sqrt{3}$ | $1/\sqrt{3}$ | $1/\sqrt{3}$ | $1/\sqrt{3}$ | $1/\sqrt{3}$ | $1/\sqrt{3}$ | $1/\sqrt{3}$ | $1/\sqrt{3}$ | $1/\sqrt{3}$ | $1/\sqrt{3}$ |
|      | $1/\sqrt{3}$ | $2/\sqrt{6}$ | $1/\sqrt{3}$ | $2/\sqrt{6}$ | $1/\sqrt{3}$ | $2/\sqrt{6}$ | $1/\sqrt{3}$ | $2/\sqrt{6}$ | $1/\sqrt{3}$ | $2/\sqrt{6}$ | $1/\sqrt{3}$ | $2/\sqrt{6}$ | $1/\sqrt{3}$ | $2/\sqrt{6}$ | $1/\sqrt{3}$ | $2/\sqrt{6}$ | $1/\sqrt{3}$ | $2/\sqrt{6}$ | $1/\sqrt{3}$ | $2/\sqrt{6}$ |

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