COFINAL MORPHISM OF POLYNOMIAL MONADS AND DOUBLE DELOOPING

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Abstract. Using the theory of internal algebras classifiers developed by Batanin and Berger, we construct a morphism of polynomial monads which we prove is homotopically cofinal. We then describe how this result constitutes the main conceptual argument for a categorical direct double delooping proof of the Turchin-Dwyer-Hess theorem concerning the explicit double delooping of spaces of long knots.

Contents

1. Introduction 1
2. Homotopy theory for polynomial monads 3
3. Cofinality result 6
4. Turchin-Dwyer-Hess delooping theorem 13
References 25

1. Introduction

In [3], Batanin and the author extended some classical results of the homotopy theory of small categories and presheaves to polynomial monads and their algebras. A notion of homotopically cofinal morphism between polynomial monad was given, which extends the notion of homotopy left cofinal functors [12]. In this present work, we will construct a homotopically cofinal morphism of polynomial monads and use it to give a direct double delooping proof of the Turchin-Dwyer-Hess theorem [10, 18]. Recall that a non-symmetric operad $O$ is multiplicative if it is equipped with an operadic map from the terminal non-symmetric operad $Ass$ to $O$. Such a map endows the collection $(O_n)_{n \geq 0}$ with a structure of cosimplicial object [18], which we will write $O^\bullet$. The Turchin-Dwyer-Hess theorem can be stated as follows:

Theorem 1.1. If $O$ is a simplicial multiplicative operad such that $O_0$ and $O_1$ are contractible, then there is a weak equivalence between simplicial sets

$$\Omega^2 Map_{\text{QP}(Ass, u^*(O))} \sim \text{Tot}(O^\bullet),$$

where $\Omega$ is the loop space functor, $Map_{\text{QP}(-,-)}$ is the homotopy mapping space in the category of simplicial non-symmetric operads, $u^*$ is the forgetful functor from multiplicative to non-symmetric operads and $\text{Tot}(-)$ is the homotopy totalization.

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This result is remarkable especially because of an earlier result from Sinha [10] which connects the space of long knots modulo immersions [10] with the totalization of the Kontsevich operad. Due to its importance in algebraic geometry, many proofs of the Turchin-Dwyer-Hess theorem, and generalisation to higher dimensions, exist in the literature [3, 6, 8, 9, 10, 18]. However, we believe that our techniques can provide a new insight and a conceptual explanation of the result which can be generalised to higher dimensions, as we develop hereafter.

This present work is structured as follows. We will start by recalling the notions of polynomial monads, internal algebra classifiers and homotopically cofinal maps in Section 2. Our presentation is strongly inspired from [3]. In Section 3 we will construct a polynomial monad IBimod whose algebras are quintuples \((A, B, C, D, E)\) where \(A\) and \(B\) are non-symmetric operads, \(C\) and \(D\) are pointed \(A\)-\(B\)-bimodules and \(E\) is a pointed infinitesimal \(C\)-\(D\)-bimodule. This data can be organised in the following diagram:

\[
\begin{array}{c}
E \\
\downarrow \\
C \\
\downarrow \\
A
\end{array}
\quad \begin{array}{c}
\downarrow \\
D \\
\downarrow \\
B
\end{array}
\quad \text{pointed infinitesimal } C - D\text{-bimodule}
\]

\[
\begin{array}{c}
E \\
\downarrow \\
C \\
\downarrow \\
A
\end{array}
\quad \begin{array}{c}
\downarrow \\
D \\
\downarrow \\
B
\end{array}
\quad \text{pointed } A - B\text{-bimodules}
\]

\[
\begin{array}{c}
A \\
\downarrow \\
B
\end{array}
\quad \text{non-symmetric operads}
\]

The notions of pointed bimodules and pointed infinitesimal bimodules are introduced in Subsection 3.1. The reason why \(E\) is called infinitesimal is that it coincides with the classical notion of infinitesimal bimodules (called weak bimodules in [18]) when \(A = B = C = D\). It is not hard to see how this construction can be generalised to higher dimensions: we would have a tower of pairs of elements as in the previous diagram, where each pair are some kind of bimodules over the pair below, and with a single element sitting on top. We will also construct a polynomial NOp\(_\Phi\) whose algebras are commutative diagrams

\[
\begin{array}{c}
E \\
\downarrow \\
C \\
\downarrow \\
A
\end{array}
\quad \begin{array}{c}
\downarrow \\
D \\
\downarrow \\
B
\end{array}
\quad \text{of non-symmetric operads. The non-symmetric operad } E \text{ should be seen as sitting on top of a “circle” of non-symmetric operads. The objective of Section 4 is to prove Theorem 3.8 which states that there is a homotopically cofinal morphism of polynomial monads from IBimod\(_2\) to NOp\(_\Phi\). In Section 4 we will show how this cofinality result can be applied to the Turchin-Dwyer-Hess theorem. We will construct circles } S^1 \text{ in any simplicial model category } M \text{ and show that, if } M \text{ is left proper, computing the double loop space of mapping spaces in } M \text{ reduces to studying mapping spaces in the comma category } S^1 / M. \text{ Because } S^1 \text{ is not a set theoretical object but is homotopical by nature, we can not directly apply our cofinality result. However, we can look at the categories of algebras of our previously}
Cofinal morphism of polynomial monads and double delooping

定义 2.2 (2, 11, 13)。一个 (有限)多项式 P 是在 Set 中的图，形式为

\[
\begin{array}{cccc}
J & \xleftarrow{s} & E & \xrightarrow{p} & B & \xrightarrow{t} & I \\
\end{array}
\]

其中 \( p^{-1}(b) \) 对于任何 \( b \in B \) 是有限的。

每个多项式 \( P \) 生成一个称为多项式映射的函数，将 functor categories

\[
P : Set^J \to Set^I
\]

这被定义为复合函数

\[
Set^J \xrightarrow{s^*} Set^E \xrightarrow{p^*} Set^B \xrightarrow{t} Set^I
\]

我们考虑集 \( J, E, B, I \) 作为离散 category，\( s^* \) 是限制映射，而 \( p^* \) 和 \( t^* \) 分别是右 Kan 和左 Kan 延伸。具体地，函数 \( P \) 由公式

\[
P(X)_i = \prod_{b \in t^{-1}(i)} \prod_{e \in p^{-1}(b)} X_{s(e)}.
\]

一个 cartesian morphism 是两个多项式映射从 \( Set^J \) 到 \( Set^I \) 之间的自然映射，使得每个 naturality square 是一个 fibred category。我们将证明映射空间在有限纤维 bimodule 和 pointed infinitesimal bimodule 类别是弱等价的。

2. Homotopy theory for polynomial monads

2.1. Polynomial monads and their algebras.

定义 2.3。一个 (有限)多项式 monad 是在 2-category Poly 中的 monad。

一个多项式 monad 的 morphism 给出一个相互交换的图

\[
\begin{array}{cccc}
J & \xleftarrow{v} & D & \xrightarrow{q} & C & \xrightarrow{u} & J \\
\phi \downarrow & & & & & & \phi \\
I & \xleftarrow{s} & E & \xrightarrow{p} & B & \xrightarrow{t} & I \\
\psi \downarrow & & & & & & \psi \\
\end{array}
\]

在 Set 中。有限多项式 monads 形成一个 2-category Poly。
where the middle square is a pullback, such that

\[
\begin{array}{c}
D \xrightarrow{\phi \circ v} C \\
\downarrow \phi \circ u \\
I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I
\end{array}
\]

gives a morphism of monads.

For a polynomial monad \( T \) given by

\[
I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I
\]

we will call \( I \) the set of \textit{colours} of \( T \), \( B \) the set of \textit{operations}, \( E \) the set of \textit{marked operations}, the map \( t \) \textit{target} and the map \( s \) \textit{source}. The map \( p \) will be called the \textit{middle map} of \( T \).

Examples of polynomial monads are small categories, the free monoid monad and the free monad NOp for non-symmetric operads [3]. Let us describe NOp again, as it is an important example for us. For an integer \( n \geq 0 \), we write \([n] = \{1, \ldots, n\}\), \([0] \) being the empty set. Recall that a \textit{non-symmetric operad} \( A \) in a symmetric monoidal category \((E, \otimes, I)\) is given by

- an object \( A_n \) in \( E \) for all integers \( n \geq 0 \)
- a morphism \( \epsilon : I \to A_1 \) called \textit{unit}
- for any order-preserving map \( f : [m] \to [n] \), morphisms

\[
\mu_f : A_n \otimes A_f \to A_m
\]

called \textit{multiplication}, where we use the notation of operadic categories [4]

\[
A_f = \bigotimes_{k \in [n]} A_{f^{-1}(k)}
\]

such that the usual associativity and unity conditions are satisfied.

\textbf{Example 2.4.} The polynomial for NOp is given by

\[
\begin{array}{c}
\mathbb{N} \xleftarrow{s} PTr^* \xrightarrow{p} PTr \xrightarrow{t} \mathbb{N}
\end{array}
\]

where \( PTr \) and \( PTr^* \) are the sets of isomorphism classes of planar trees and planar trees with a marked vertex respectively. The middle map forgets the marked point, the source map is given by the number of incoming edges for the marked point and the target map associates to a tree its number of leaves. The multiplication in this monad is generated by insertion of a tree inside a marked point.

Let \( \mathcal{E} \) be a cocomplete symmetric monoidal category and \( T \) be a polynomial functor. One can construct a functor \( T^\mathcal{E} : \mathcal{E}^J \to \mathcal{E}^I \) given by a formula similar to \[1\]

\[
T^\mathcal{E}(X)_i = \bigotimes_{b \in \text{bet}^{-1}(i)} \otimes_{c \in \text{exp}^{-1}(b)} X_{s(c)}.
\]

If \( I = J \) and \( T \) was given the structure of a polynomial monad then \( T^\mathcal{E} \) would acquire a structure of monad on \( \mathcal{E}^I \).

\textbf{Definition 2.5.} Let \( \mathcal{E} \) be a cocomplete symmetric monoidal category and \( T \) a polynomial monad. The category of algebras of \( T \) in \( \mathcal{E} \) is the category of algebras of the monad \( T^\mathcal{E} \).
If $\mathcal{E}$ is the category $\text{Cat}$ of small categories (resp. the category $\text{SSet}$ of simplicial sets), we will call an algebra of $T$ in $\mathcal{E}$ a categorical (resp. simplicial) $T$-algebra respectively. Explicitly, an algebra $A$ in $\mathcal{E}$ of a polynomial monad $T$ is given by a collection $A_i \in \mathcal{E}$, $i \in I$, equipped with the following structure maps:

$$m_{(b,\sigma)} : A_{s(\sigma(1))} \otimes \ldots \otimes A_{s(\sigma(k))} \to A_t(b)$$

for each $b \in B$ and each bijection $\sigma : \{1,\ldots,k\} \to p^{-1}(b)$, which satisfy some appropriate associativity, unity and equivariance conditions [2].

2.6. **Internal algebras classifiers and homotopically cofinal maps.** The category of categorical $T$-algebras, which we will write $\text{Alg}_T(\text{Cat})$, is naturally a 2-category since it is isomorphic to the category of internal categories in the category of $T$-algebras in $\text{Set}$. The terminal internal category has a unique structure of $T$-algebra in $\text{Set}$ for any polynomial monad $T$; the latter promotes it to a terminal categorical $T$-algebra. Let us recall the notions of internal algebras [1, 2]:

**Definition 2.7.** Let $T$ be a polynomial monad and $A$ be a categorical $T$-algebra. An internal $T$-algebra in $A$ is a lax morphism of categorical $T$-algebras from the terminal categorical $T$-algebra to $A$. Internal $T$-algebras in $A$ and $T$-natural transformations form a category $\text{Int}_T(A)$ and this construction extends to a 2-functor

$$\text{Int}_T : \text{Alg}_T(\text{Cat}) \to \text{Cat}.$$ 

Given a morphism of polynomial monads $f : S \to T$ we have a restriction 2-functor $f^* : \text{Alg}_T(\text{Cat}) \to \text{Alg}_S(\text{Cat})$. Definition [2] has the following generalisation:

**Definition 2.8.** Let $f : S \to T$ be a morphism of polynomial monads and $A$ be a categorical $T$-algebra. An internal $S$-algebra in $A$ is a lax morphism of categorical $S$-algebras from the terminal categorical $S$-algebra to $f^*(A)$. Internal $S$-algebras in $A$ and $S$-natural transformations form a category $\text{Int}_S(A)$ and this construction extends to a 2-functor

$$\text{Int}_S : \text{Alg}_T(\text{Cat}) \to \text{Cat}.$$ 

(3) 

For any morphism of polynomial monads $f : S \to T$ one can associate a categorical $T$-algebra with certain universal property [1, 2]. Namely, this is the object representing the 2-functor [3]. This categorical $T$-algebra is called the classifier of internal $S$-algebras inside categorical $T$-algebras and is denoted $T^S$. In particular, if $f = \text{Id}$ the $T$-algebra $T^T$ is called absolute classifier of $T$.

It was proved in [1, 2] that the absolute classifier of $T$ is given by an internal categorical object in the category of $T$-algebras in $\text{Set}$:

$$\mathcal{F}_T(1) \longrightarrow \mathcal{F}_T(T1) \longrightarrow \mathcal{F}_T(T^21)$$

where 1 is the terminal $I$-collection of sets, $\mathcal{F}_T$ is the free $T$-algebra functor and the source (resp. target) map of this internal categorical object is given by the multiplication of the monad (resp. the unique map to the terminal object). The other structure maps are described in [3].

There is an analogous formula in the non absolute case. Namely, given a morphism of polynomial monads $f : S \to T$ as in [2] we have the following commutative
square of adjunctions:

\[
\begin{array}{ccc}
\text{Alg}_S & \xrightarrow{f^*} & \text{Alg}_T \\
\downarrow f & \downarrow f & \downarrow f \\
\text{Set}^J & \xrightarrow{\phi^*} & \text{Set}^I
\end{array}
\]

Here \(\phi^*\) is the restriction functor induced by \(\phi : J \to I\) and \(\phi_! : \text{Set}^I \to \text{Set}^J\) is its left adjoint given by coproducts over fibres of \(\phi\). The categorical \(T\)-algebra \(T^S\) is given then by an internal categorical object similar to the absolute case:

\[
\mathcal{F}_T(\phi_(S1)) \xrightarrow{\sim} \mathcal{F}_T(\phi_(S21))
\]

where 1 is now the terminal \(J\)-collection.

For a polynomial monad \(T\), the category of simplicial \(T\)-algebras is a simplicial model category, with the model structure transferred from the projective model structure on the category of collections of simplicial sets \([2]\). Also note that, if \(A\) is a categorical \(T\)-algebra, then we get a simplicial \(T\)-algebra, which we will write \(N(A)\), by taking the nerve of each category of the underlying collection. We have the following lemma \([3, \text{Corollary 5.2}]\).

**Lemma 2.9.** Let \(f : S \to T\) be a morphism of polynomial monads. Let \(f^*\) be the restriction functor from simplicial \(T\)-algebras to simplicial \(S\)-algebras and \(f_!\) its left adjoint. There is a weak equivalence

\[
\mathbb{L}f_!(1) \sim N(T^S).
\]

where \(\mathbb{L}f_!\) is the left derived functor of \(f_!\).

This motivates the following definition, which is central in the homotopy theory for polynomial monads \([3]\).

**Definition 2.10.** A cartesian map \(f : S \to T\) between polynomial monads is called homotopically cofinal if \(N(T^S)\) is contractible.

### 3. Cofinality result

#### 3.1. Pointed infinitesimal bimodules over operadic bimodules.

Recall that, if \(A\) and \(B\) are two non-symmetric operads in a symmetric monoidal category \((\mathcal{E}, \otimes, I)\), an \(A - B\)-bimodule \(C\) is given by

- a collection \(C_n\) of objects in \(\mathcal{E}\) for all integers \(n \geq 0\),
- for any order-preserving map \(f : [m] \to [n]\), morphisms
  \[
  \lambda_f : A_n \otimes C_f \to C_m
  \]
  and
  \[
  \rho_f : C_n \otimes B_f \to C_m
  \]
  such that the usual associativity and unity conditions are satisfied.

For a composite \([m] \xrightarrow{i} [n] \xrightarrow{g} [p]\) of order-preserving maps with \(h = gf\) and \(i \in [p]\), we will write \(f_i : h^{-1}(i) \to g^{-1}(i)\) for the induced map between fibres.
Definition 3.2. Let $A$ and $B$ be two non-symmetric operads in a symmetric monoidal category $(E,\otimes,1)$. Let $C$ and $D$ be two $A-B$-bimodules. An infinitesimal $C-D$-bimodule $E$ is given by

- a collection of objects $E_n$ in $E$ for $n \geq 0$
- for any order-preserving map $f : [m] \to [n]$ and $k \in [n]$, a map

\[ \phi^k_f : A_n \otimes E^k_f \to E_m \]

where we write

\[ E^k_f = \bigotimes_{j \in [n]} C_{f^{-1}(j)} \otimes E_{f^{-1}(k)} \otimes \bigotimes_{j \in [n]} D_{f^{-1}(j)} \]

- for any order-preserving map $f : [m] \to [n]$, a map

\[ \psi_f : E_n \otimes B_f \to E_m \]

satisfying the usual unit and associativity axioms, that is

- the composites

\[ E_n \simeq I \otimes E_n \xrightarrow{\tau \otimes 1} A_1 \otimes E_n \xrightarrow{\phi^1_n} E_n, \]

where $\tau : [n] \to [1]$ is the unique map, and

\[ E_n \simeq E_n \otimes \bigotimes_{i \in [n]} I \xrightarrow{1 \otimes \psi_{id}} E_n \otimes \bigotimes_{i \in [n]} B_i \xrightarrow{\psi_{id}} E_n, \]

where $id : [n] \to [n]$ is the identity map, are the identity on $E_n$

- for any composite $[m] \xrightarrow{f} [n] \xrightarrow{g} [p]$ of order-preserving maps with $h = gf$, $k \in [n]$ and $l \in [p]$ with $g(k) = l$, the following diagram commutes

\[ \xymatrix{ A_p \otimes A_g \otimes E^k_f \ar@{=>}[rr]^{\simeq} \ar[dr]_{\mu^g \otimes 1} & & A_p \otimes \bigotimes_{i \in [p]} (A_{g^{-1}(i)} \otimes X_i) \ar[dl]^{1 \otimes \psi_{id}} \ar@{=>}[dd]^\theta_i \ar[dr]_{\phi^h_l} & & A_p \otimes E^l_h \ar[dl]_{\phi^h_l} \\
A_n \otimes E^k_f & & E_m & & \}

where

\[ X_i = \begin{cases} 
C_{f_i} & \text{if } i < l \\
E^k_{f_i} & \text{if } i = l \\
D_{f_i} & \text{if } i > l 
\end{cases} \quad \text{and} \quad \theta_i = \begin{cases} 
\lambda^C_{f_i} & \text{if } i < l \\
\phi^h_{f_i} & \text{if } i = l \\
\lambda^D_{f_i} & \text{if } i > l 
\end{cases} \]
Let \( m \in \text{the classical sense.} \) Let \( f \in E \) and Markl operads [14]. First assume that

The proof is inspired from the proof of equivalence between May operads

\[ A \to \text{the category of infinitesimal bimodules over } A \]

\[ \text{category of infinitesimal symmetric operad} \]

Lemma 3.3.

Note that there is a classical notion of infinitesimal bimodule \( E \) is an \( E \)-bimodule over itself and the \( E \)-module structure on \( E \) is given by

\[ \frac{E_{g^{-1}(i)}}{E_{g^{-1}(i)}} \]

\[ \text{for } i < l \]

\[ \frac{E_{g^{-1}(i)}}{E_{g^{-1}(i)}} \]

\[ \text{for } i = l \]

\[ \frac{D_{g^{-1}(i)}}{D_{g^{-1}(i)}} \]

\[ \text{for } i > l \]

\[ Y_i = \begin{cases} 
C_{g^{-1}(i)} & \text{if } i < l \\
E_{g^{-1}(i)} & \text{if } i = l \\
D_{g^{-1}(i)} & \text{if } i > l 
\end{cases} \]

\[ \nu_i = \begin{cases} 
\rho^C_{f_i} & \text{if } i < l \\
\psi_{f_i} & \text{if } i = l \\
\rho^D_{f_i} & \text{if } i > l 
\end{cases} \]

- for any composite \([m] \xrightarrow{f} [n] \xrightarrow{g} [p]\) of order-preserving maps with \( h = gf \), the following diagram commutes

\[ A_p \otimes E_g B_f \xrightarrow{\sim} A_p \otimes \otimes_{i \in [p]} (Y_i \otimes B_{f_i}) \]

\[ \phi_g \otimes 1 \downarrow \quad 1 \otimes \otimes_{i \in [p]} \nu_i \]

\[ E_n \otimes B_f \quad E_m \]

\[ \psi_f \downarrow \phi_h \]

\[ \psi_f \]

where

Note that there is a classical notion of infinitesimal bimodule \( E \) over a non-symmetric operad \( A \) in \( E \). It is given by

- a collection of objects \( E_n \) in \( E \) for \( n \geq 0 \)
- for \( 1 \leq k \leq n \) and \( m \geq 0 \), maps

\[ i : A_n \otimes E_m \to E_{m+n-1} \]

- for \( 1 \leq k \leq n \) and \( m \geq 0 \), maps

\[ * : E_n \otimes A_m \to E_{m+n-1} \]

satisfying unity, associativity and compatibility axioms [14].

**Lemma 3.3.** Let \( A \) be a non-symmetric operad. \( A \) is a bimodule over itself and the category of infinitesimal \( A \)-\( A \)-bimodules in the sense of Definition 3.2 is equivalent to the category of infinitesimal bimodules over \( A \) in the classical sense.

**Proof.** The proof is inspired from the proof of equivalence between May operads and Markl operads [14]. First assume that \( E \) is an infinitesimal bimodule over \( A \) in the classical sense. Let \( f : [m] \to [n] \) be an order-preserving map and \( k \in [n] \). Let \( m_k = |f^{-1}(k)| \) and \( \tilde{f} : [m-m_k+1] \to [n] \) be the order-preserving map such that \( \tilde{f}^{-1}(j) = f^{-1}(j) \) for \( j \in [n] \) such that \( j \neq k \) and \( \tilde{f}^{-1}(k) = [1] \). Remark that, using

\[ m \]
the structure of the non-symmetric operad \( A \), we have the composite
\[
A_n \otimes \bigotimes_{j \in [n]} A_{|f^{-1}(j)|} \to A_n \otimes A_f^{\mu_T} \to A_{m-m_k+1}
\]
where the first map is obtained using the unit of \( A \).
Let \( \gamma \) be this composite. This allows us to define \( \phi^k_f \) as the composite
\[
A_0 \otimes E_f^k \to A_n \otimes \bigotimes_{j \in [n]} A_{|f^{-1}(j)|} \otimes E_m \to A_n \otimes \bigotimes_{j \in [n]} A_{|f^{-1}(j)|} \otimes E_m \to \cdots \to E_{m-m_n+1} \otimes A_m \to E_m
\]
Observe that, by iterating \( n \) times the right action, we can also construct \( \psi_f \) as the composite
\[
E_n \otimes \bigotimes_{j=1}^n A_{m_j} \to E_{n+m_{n-1}} \otimes \bigotimes_{j=2}^n A_{m_j} \to E_{m-m_n+1} \otimes A_m \to E_m
\]
where \( m_j = |f^{-1}(j)| \) for \( j \in [n] \). We get a functor from the category of infinitesimal bimodules over \( A \) to the category of infinitesimal \( A - A \)-bimodules.
In the other direction, assume that \( E \) is an infinitesimal \( A - A \)-bimodule in the sense of Definition 3.2. Let \( 1 \leq k \leq n \) and \( m \geq 0 \). Let \( f : [m+n-1] \to [n] \) be the order-preserving map such that \( f^{-1}(j) = [1] \) for \( j \in [n] \) with \( j \neq k \) and \( f^{-1}(k) = [m] \).
Then the left action \( \bullet_k \) is obtained as the following composite
\[
A_n \otimes E_m \to A_n \otimes E_f^k \to E_{m+n-1}
\]
where the first map is obtained using \( n-1 \) times the unit of \( A \).
Similarly, the right action \( *_k \) is obtained as the following composite
\[
E_n \otimes A_m \to E_n \otimes A_f \to E_{m+n-1}
\]
where the first map is obtained using \( n-1 \) times the unit of \( A \).
We get a functor from the category of infinitesimal \( A - A \)-bimodules to the category of infinitesimal bimodules over \( A \).
It is easy to check that the two functors we have constructed are inverse of each other. This concludes the proof.

In the rest of this subsection, we will give a notion of pointedness for bimodules and infinitesimal bimodules.

**Definition 3.4.** Let \( A \) and \( B \) be two non-symmetric operads in \( \mathcal{E} \), and \( C \) an \( A - B \)-bimodule. We say that \( C \) is pointed if it is equipped with a map \( e : A \to C \).

**Remark 3.5.** Equivalently, an \( A - B \)-bimodule \( C \) is pointed if it equipped with a map \( A \to C \) of left \( A \)-modules and a map \( B \to C \) of right \( B \)-modules such that the following diagram commutes
\[
\begin{array}{ccc}
eq A & \\
\downarrow & \\
B & \to C
\end{array}
\]
Note that for \( A \) and \( B \) two non-symmetric operads in a symmetric monoidal category \( \mathcal{E} \) and an \( A - B \)-bimodule \( C \), there is an obvious notion of infinitesimal left \( C \)-module. It is given by a collection of objects \( E_n \) in \( \mathcal{E} \) for \( n \geq 0 \), together with,
for any order-preserving map \(f : [m] \to [n]\), a map \(\phi^f_n\) as in (4) and a map \(\psi_f\) as in (5) satisfying the same axioms as in Definition 3.2. Similarly, one has a notion of infinitesimal right \(C\)-module. Also note that \(C\) is an infinitesimal \(C - C\)-bimodule, and so in particular, an infinitesimal left and right \(C\)-module.

**Definition 3.6.** Let \(A\) and \(B\) be two non-symmetric operads and \(C\) and \(D\) two pointed \(A - B\)-bimodules. An infinitesimal \(C - D\)-bimodule \(E\) is pointed if it is equipped with a map \(C \to E\) of infinitesimal left \(C\)-modules and a map \(D \to E\) of infinitesimal right \(D\)-modules such that, for all \(h : [n] \to [l]\) and \(i, j \in [l]\) with \(i + 1 = j\), the following diagram commutes

\[
A_l \otimes \bigotimes_{k \in [l]} C_{[h^{-1}(k)]} \otimes \bigotimes_{k \in [l]} D_{[h^{-1}(k)]} \to A_l \otimes E^i_h \\
\downarrow \quad \quad \downarrow \\
A_l \otimes E^i_h \to E_n
\]

and the following diagram commutes in the category of collections

\[
\begin{array}{ccc}
B & & \\
& C \to E & \to D \\
A & & \\
\end{array}
\]

where the maps from \(A\) and \(B\) are given by Remark 3.5.

3.7. **Description of the morphism of polynomial monads.** Let \(\text{IBimod}_{\odot}\) be the polynomial monad for quintuples \((A, B, C, D, E)\), where \(A\) and \(B\) are non-symmetric operads, \(C\) and \(D\) are pointed \(A - B\)-bimodules and \(E\) is a pointed infinitesimal \(C - D\)-bimodule, and \(\text{NOP}_\Phi\) be the polynomial monad for commutative diagrams

(6)

\[
\begin{array}{ccc}
B & & \\
& C \to E & \to D \\
A & & \\
\end{array}
\]

of non-symmetric operads.

The objective of this section is to prove the following theorem.

**Theorem 3.8.** There is a homotopically cofinal morphism of polynomial monads

(7)  \(f : \text{IBimod}_{\odot} \to \text{NOP}_\Phi\)

First, let us give an explicit description of \(\text{NOP}_\Phi\). In order to do this, we define a partial order on the set \(\{A, B, C, D, E\}\).
**Definition 3.9.** For any \( l_1, l_2 \in \{A, B, C, D, E\} \), we write \( l_1 \leq l_2 \) if there is an arrow from \( l_1 \) to \( l_2 \) in the diagram. Explicitly, we have the following relations:

\[
A \leq C \leq E \quad B \leq C \leq E \\
A \leq D \leq E \quad B \leq D \leq E
\]

Let \( \mathbb{N} \) be the set of non-negative integers. The polynomial monad \( \text{NOP}_\Phi \) is given by the polynomial

\[
\{A, B, C, D, E\} \times \mathbb{N} \longrightarrow \text{PTr}_5^* \longrightarrow \text{PTr}_5 \longrightarrow \{A, B, C, D, E\} \times \mathbb{N}
\]

where \( \text{PTr}_5 \) is the set of isomorphism classes of planar rooted tree where each vertex \( v \) has a label \( l_v \in \{A, B, C, D, E\} \). Each tree itself also has a label \( l \in \{A, B, C, D, E\} \), called target label, satisfying \( l_v \leq l \) for each vertex \( v \) of the tree. \( \text{PTr}_5^* \) is the set of elements of \( \text{PTr}_5 \) with one vertex marked. The source map produces a label corresponding to the label of the marked vertex, and a number equal to the number of incoming edges of this vertex. The target map returns the target label and the number of leaves. The multiplication is by insertion of trees inside the vertices of another tree. The condition of insertion is that the number of leaves and the target label of each inserted tree must correspond to the number of incoming edges and the label of the vertex where it is inserted.

Now we will describe in more detail the polynomial monad \( \text{IBimod}_{\cdot} \).

**Definition 3.10.** We say that the labels \( l_v \in \{A, B, C, D, E\} \) of the vertices \( v \) of an isomorphism class of planar trees lie on a line and a point if, for a representant of the isomorphism class, it is possible to draw a line and a point on this line such that

- all the vertices below the line have label \( A \)
- all the vertices above the line have label \( B \)
- all the vertices on the line and to the left of the point have label \( C \)
- all the vertices on the line and to the right of the point have label \( D \)
- if a vertex lies on the point, it has label \( E \)

The polynomial monad \( \text{IBimod}_{\cdot} \) is given by the polynomial

\[
\{A, B, C, D, E\} \times \mathbb{N} \longrightarrow \text{IPT}_5^* \longrightarrow \text{IPT}_5 \longrightarrow \{A, B, C, D, E\} \times \mathbb{N}
\]

where \( \text{IPT}_5 \) is the subset of \( \text{PTr}_5 \) of isomorphisms classes of planar trees whose labels lie on a line and a point. The rest of the description of the polynomial monad \( \text{IBimod}_{\cdot} \) is straightforward, as it is completely similar to the description of the polynomial monad \( \text{NOP}_\Phi \).

The map \( f: \text{IBimod}_{\cdot} \rightarrow \text{NOP}_\Phi \) is given by the identity on colours and inclusion on the other sets.
3.11. Description of the induced classifier. Let us describe the classifier associated to the map \[7\]. It is given by a collection of categories indexed by the set of colours of \(\text{NOp}_\Phi\), that is \(\{A, B, C, D, E\} \times \mathbb{N}\). Let \((l, n)\) be an element of this set. The category indexed by \((l, n)\) has the following description.

The objects are isomorphism classes of planar tree with \(n\) leaves, where each vertex \(v\) has a label \(l_v \in \{A, B, C, D, E\}\) satisfying \(l_v \leq l\). We will describe morphisms as nested trees. It is an isomorphism class of planar tree \(T\) where each vertex itself contains an isomorphism class of planar tree. The number of incoming edges of each vertex must be equal to the number of leaves of the tree inside it. The tree \(T\) has \(n\) leaves and each vertex \(v\) has a label \(l_v \in \{A, B, C, D, E\}\) satisfying \(l_v \leq l\). Each vertex \(w\) of each tree inside a vertex \(v\) also has a label \(l_w\) such that \(l_w \leq l_v\). Finally and importantly, the vertices of a tree inside a vertex must satisfy the condition of Definition 3.10. Here is an example of nested tree.

![Nested Tree Example](image)

The source of a nested tree is obtained by inserting each tree into the vertex it decorates and the target is obtained by forgetting the trees inside each vertex. For example, the nested tree of the previous picture represents the following morphism.

![Nested Tree Morphism](image)

3.12. Proof of the cofinality result. In order to prove Theorem 3.8, we need the Cisinski lemma about smooth functors. For a functor \(F : \mathcal{X} \to \mathcal{Y}\) between categories and \(y \in \mathcal{Y}\), we will write \(F_y\) for the fibre of \(F\) over \(y\), that is the full subcategory of objects \(x \in \mathcal{X}\) such that \(F(x) = y\).

**Definition 3.13.** A functor \(F : \mathcal{X} \to \mathcal{Y}\) is smooth if, for all \(y \in \mathcal{Y}\), the canonical functor

\[F_y \to y/F\]
induces a weak equivalence between nerves.

Dually, a functor $F : \mathcal{X} \to \mathcal{Y}$ is proper if, for all $y \in \mathcal{Y}$, the canonical functor $F_y \to F/y$ induces a weak equivalence between nerves.

Let us state the Cisinski lemma [7, Proposition 5.3.4].

**Lemma 3.14.** A functor $F : \mathcal{X} \to \mathcal{Y}$ is smooth if and only if for all maps $f_1 : y_0 \to y_1$ in $\mathcal{Y}$ and objects $x_1$ in $\mathcal{X}$ such that $F(x_1) = y_1$, the nerve of the lifting category of $f_1$ over $x_1$, whose objects are arrows $f : x \to x_1$ such that $F(f) = f_1$ and morphisms are commutative triangles

$$
\begin{array}{ccc}
x & \xrightarrow{g} & x' \\
\downarrow{f} & & \downarrow{f'} \\
\quad x_1 \quad & & \quad x_1 \\
\end{array}
$$

with $g$ a morphism in $F_{y_0}$, is contractible.

*There is a dual characterisation for proper functors.*

Now we will construct a functor $F : \mathcal{X} \to \mathcal{Y}$ which we will prove is smooth. We have the following commutative square of polynomial monads

$$
\begin{array}{ccc}
\text{IBimod} & \xrightarrow{u_f} & \text{NOP} \\
\downarrow{f} & & \downarrow{u} \\
\text{NOP} & \to & \text{NOP} \\
\end{array}
$$

where $u$ is the morphism of polynomial monads given by the projections. As it was proven in [3, Proposition 4.7], this commutative square induces a strict map of algebras

(8) $\text{NOP}^{\text{IBimod}} \to u^* (\text{NOP}^{\text{NOP}})$

To simplify the notations, let us assume that a colour of $\text{NOP}_\Phi$ is fixed and let $F : \mathcal{X} \to \mathcal{Y}$ be the underlying functor of the map $\Phi$ between the categories indexed by this colour. This functor sends a labelled tree to the same tree, but without labels.

**Lemma 3.15.** The functor $F : \mathcal{X} \to \mathcal{Y}$ is smooth.

First, let us prove the following technical lemma.

**Lemma 3.16.** Let $f_1 : y_0 \to y_1$ be a map in $\mathcal{Y}$ and $x_1$ be an object in $\mathcal{X}$ such that $F(x_1) = y_1$. If $x_1$ is a corolla, that is a tree with exactly one vertex, then the nerve of the lifting category $\mathcal{X}(x_1, f_1)$ of $f_1$ over $x_1$ is contractible.

**Proof.** Note that, since $\mathcal{X}$ and $\mathcal{Y}$ are posets, $\mathcal{X}(x_1, f_1)$ can be seen as the full subcategory of $\mathcal{X}$, whose objects are the objects $x \in F_{y_0}$ such that there is an arrow $x \to x_1$. Also note that, to the pair $(x_1, f_1)$, one can associate a pair $(l, T)$, where $l \in \{A, B, C, D, E\}$ is the label of the unique vertex of $x_1$ and $T$ is the tree given by the object $y_0 \in \mathcal{Y}$. This association is a one-to-one correspondence, so that we can define $\chi(l, T) = \mathcal{X}(x_1, f_1)$. 
For example, if the label $l$ is $E$ and $T$ is the following tree

```
  A
 /\  |
 / \ |
A C A
```

then $\chi(l, T)$ is represented in the following picture

```
A A
 /\  |
 / \ |
A C A
```

We will prove that the nerve of $\chi(l, T)$ is contractible by induction on the number of vertices of $T$. If $T$ has no vertices, that is the free living edge, then $\chi(l, T)$ is the terminal category and its nerve is contractible. Now assume that $T$ has at least one vertex. Let $v$ be the root vertex and $k$ be the number of incoming edges of $v$. Let $T_1, \ldots, T_k$ be the subtrees above the edges of $v$.

If the label $l$ is $A$ or $B$ then the result is trivial. Otherwise, let $\chi_A(l, T)$ be the full subcategory of $\chi(l, T)$ of trees where the root vertex is labelled with $A$. Let $\chi_B(l, T)$ be the full subcategory of $\chi(l, T)$ of trees where all the vertices other than the root vertex are labelled with $B$. The union of $\chi_A(l, T)$ and $\chi_B(l, T)$ gives the whole category $\chi(l, T)$. If we can prove that the nerves of $\chi_A(l, T)$, $\chi_B(l, T)$ and the intersection of these two subcategories are contractible, then we can conclude that the nerve of their union is also contractible, which is the desired result. The intersection of $\chi_A(l, T)$ and $\chi_B(l, T)$ is the terminal category. The subcategory $\chi_B(l, T)$ is equivalent to the category $\chi(l, v)$, where $v$ is the corolla with one vertex and $k$ leaves, which has obviously a contractible nerve.

Now, let us consider $\chi_A(l, T)$. If the label $l$ is $C$ or $D$, then this category is equivalent to the product of categories $\prod_{i=1}^k \chi(l, T_i)$, which has contractible nerve by induction. If the label $l$ is $E$, we need to do an induction on $k$. Let $\chi_C(E, T)$ be the full subcategory of $\chi_A(E, T)$ of trees where the vertices of $T_1$ have label $A$, $B$ or $C$. Let $\chi_D(E, T)$ be the full subcategory of $\chi_A(E, T)$ of trees where the vertices of $T_2, \ldots, T_k$ have label $A$, $B$ or $D$. Finally, let us write $T'$ for the tree obtained by removing the first edge above the root vertex $v$ and the tree $T_1$. The union of $\chi_C(E, T)$ and $\chi_D(E, T)$ gives the subcategory $\chi_A(E, T)$. As before, we want to prove that the nerves of $\chi_C(E, T)$, $\chi_D(E, T)$ and the intersection of these two subcategories
are contractible. The subcategory $\chi_C(E,T)$ is equivalent to $\chi(C,T_1) \times \chi_A(E,T')$, the subcategory $\chi_D(E,T)$ is equivalent to $\chi(E,T_1) \times \chi_A(D,T')$ and their intersection is equivalent to $\chi(C,T_1) \times \chi_A(D,T')$. By induction, all three have contractible nerve. □

Proof of Lemma 3.15. Let $f_1 : y_0 \to y_1$ be a map in $\mathcal{Y}$ and $x_1$ be an object in $\mathcal{X}$ such that $F(x_1) = y_1$. According to the Cisinski lemma 3.14, we get the desired result if we can prove that the nerve of the lifting category $\mathcal{X}(x_1,f_1)$ of $f_1$ over $x_1$ is contractible. Remark that $f_1$ can be described by maps $f^{(v)}_1 : y^{(v)}_0 \to y^{(v)}_1$ for $v \in V$, where $V$ is the set of vertices of $y_1$ and $y^{(v)}_1$ is always the corolla. For example, the morphism

\[
\begin{array}{ccc}
\vdots & \rightarrow & \vdots \\
\end{array}
\]

We also have objects $x^{(v)}_1$ for each $v \in V$ which decompose $x_1$, just like $y^{(v)}_1$ decompose $y_1$. The category $\mathcal{X}(x_1,f_1)$ is equivalent to the product of the categories $\mathcal{X}(x^{(v)}_1,f^{(v)}_1)$ over $v \in V$, which has contractible nerve thanks to Lemma 3.16. □

Proof of Theorem 3.8. The functor $F : \mathcal{X} \to \mathcal{Y}$ is smooth and its fibres have contractible nerve, since they have a terminal object. Using Quillen Theorem A, we deduce that $F$ induces a weak equivalence between nerve. Again, the nerve of $\mathcal{Y}$ is contractible since this category has a terminal object. So the nerve of $\mathcal{X}$ is contractible, which concludes the proof. □

4. Turchin-Dwyer-Hess delooping theorem

4.1. Delooping theorem for a left proper model category. Let us define, by induction on $n$, spheres $S^n$ and disks $D^n$ in a general model category.

Definition 4.2. Let $M$ be a model category. We will write $S^{-1}$ (resp. $D^{-1}$) for the initial (resp. terminal) object in $M$. Note that there is a unique map $S^{-1} \to D^{-1}$.

Assume we have defined $S^{n-1}$ and $D^{n-1}$ for $n \geq 0$, together with a map $S^{n-1} \to D^{n-1}$. We can factorise this map as follows

\[
\begin{array}{ccc}
S^{n-1} & \rightarrow & D^{n-1} \\
\downarrow & & \downarrow \\
D^n & \sim & \end{array}
\]
where the first map is a cofibration and the second map is a weak equivalence. We define $S^n$ as the pushout

$$
\begin{array}{c}
S^{n-1} \\
\downarrow \\
D^n
\end{array} \rightarrow _\sim \begin{array}{c}
D^n \\
\downarrow \\
S^n
\end{array}
$$

This comes with a map $S^n \rightarrow D^n$ induced by the identity on $D^n$.

For $n \geq 0$, note that the triangle allows us to construct a composite map

$$
S^{n-1} \rightarrow D^n \rightarrow D^{n-1} \rightarrow \ldots \rightarrow D^0
$$

which we will denote by $g_n$. This induces a Quillen adjunction between comma categories

$$
\begin{array}{c}
S_{n-1}/M \\
\downarrow \\
D^0/M
\end{array} \rightarrow _{(g_n)} \begin{array}{c}
D^n \\
\downarrow \\
S^n
\end{array}
$$

where $g_n^*$ is given by precomposition and $(g_n)$ by pushout respectively.

We also have a composite

$$
D^0 \rightarrow S^0 \rightarrow S^1 \rightarrow \ldots \rightarrow S^n
$$

where the first map is one of the two maps to the coproduct and the other maps are given by the commutative square. Finally, recall that a model category is left proper if weak equivalences are preserved by pushouts along cofibrations.

**Lemma 4.3.** Let $n \geq 0$ and $M$ be a left proper model category. Then $L(g_n)(1)$, that is the value of the left derived functor of $(g_n)$ on the terminal object, is weakly equivalent to the object in $D^0/M$ given by the map.

**Proof.** We look at the following diagram

$$
\begin{array}{c}
S^{n-1} \\
\downarrow \\
D^n
\end{array} \rightarrow _{g_n} \begin{array}{c}
D^n \\
\downarrow \\
P^n
\end{array}
$$

where the square on the left is the pushout and the square on the right is also a pushout. Then the outer rectangle is also a pushout. Remark that $D^n \rightarrow D^0$ is a weak equivalence and the middle vertical map is a cofibration. Since $M$ is left proper, this proves that $S^n \rightarrow P^n$ is a weak equivalence.

For a simplicial model category $M$ and $X, Y \in M$, we will write $SSet_M(X, Y)$ for the simplicial hom. Note that if $M$ is a simplicial category, then for $e \in M$, the comma category $e/M$ is also a simplicial model category. Finally, for an object of $e/M$ given by $x \in M$ and a morphism $\alpha : e \rightarrow x$, we will write just $x$ when there is no ambiguity for the map $\alpha$.

**Lemma 4.4.** Let $M$ be a simplicial model category and $n \geq 0$. For all fibrant $X \in D^0/M$, we have a weak equivalence

$$
\Omega^n SSet_{D^0/M} (S^0, X) \sim SSet_{D^0/M} (S^n, X)
$$
Proof: For \( n \geq 1 \), the pushout \([10]\) becomes a pushout in \( D^0/M \) using maps such as in \([12]\). Since all maps are cofibrations, this pushout is also a homotopy pushout. By applying the functor \( SSet_{D^0/M}(\cdot, X) \) to this homotopy pushout \([10]\) we get a homotopy pullback

\[
\begin{array}{ccc}
SSet_{D^0/M}(S^k, X) & \longrightarrow & SSet_{D^0/M}(D^k, X) \\
\downarrow & & \downarrow \\
SSet_{D^0/M}(D^k, X) & \longrightarrow & SSet_{D^0/M}(S^{k-1}, X)
\end{array}
\] (13)

Observe that the map \( D^k \to D^0 \) is a weak equivalence between cofibrant objects. Since \( X \) is fibrant, as is well-known \([12]\, Corollary 9.3.3]\), \( D^k \to D^0 \) induces a weak equivalence

\[
SSet_{D^0/M}(D^0, X) \simeq SSet_{D^0/M}(D^k, X)
\]

Since \( SSet_{D^0/M}(D^0, X) \) is contractible as \( D^0 \) is the initial object in \( D^0/M \), we deduce that \( SSet_{D^0/M}(D^k, X) \) is also contractible. We can therefore deduce from the homotopy pullback \([13]\) that there is a weak equivalence

\[
\Omega SSet_{D^0/M}(S^{k-1}, X) \sim SSet_{D^0/M}(S^k, X)
\]

We get the desired result by iterating this delooping. \( \square \)

For a simplicial model category \( M \), we will write \( Map_M(\cdot, \cdot) \) for the homotopy mapping spaces in \( M \). Recall that, for \( X, Y \in M \), \( Map_M(X,Y) \) can be computed as \( SSet_M(X^c, Y^f) \) where \( X^c \) and \( Y^f \) are cofibrant and fibrant replacements of \( X \) and \( Y \) respectively \([12]\).

**Theorem 4.5.** Let \( M \) be a left proper simplicial model category and \( n \geq 0 \). For all \( X \in D^0/M \), we have a weak equivalence

\[
\Omega^n Map_M(D^0, g^*_n(X)) \sim Map_{S^n/M}(D^n, g^*_n(X))
\]

Proof. Since \( g^*_n \) preserves weak equivalences, we can assume without loss of generality that \( X \) is fibrant. By adjunction and using Lemma 4.3 we get a weak equivalence

\[
Map_{S^n/M}(D^n, g^*_n(X)) \sim SSet_{D^0/M}(S^n, X)
\]

We get the conclusion using Lemma 4.4. \( \square \)

### 4.6. Left cofinal Quillen functors.

**Definition 4.7.** A left Quillen functor \( G : B \to C \) is **cofinal** if \( LG(1) \) is contractible, where \( LG \) is the left derived functor of \( G \) and \( 1 \) is the terminal object in \( B \).

**Remark 4.8.** We deduce from Lemma 2.9 that a morphism of polynomial monads \( f : S \to T \) is homotopically cofinal if and only if \( f_1 \) is a left cofinal Quillen functor.

Let \( \text{Bimod}_{\cdot,\cdot} \) be the polynomial monad for quadruples \((A, B, C, D)\) where \( A \) and \( B \) are non-symmetric operads and \( C \) and \( D \) are pointed \( A-B \)-bimodules. Let \( \mathcal{B} \) be the category of simplicial algebras of \( \text{Bimod}_{\cdot,\cdot} \) and \( \Phi : \mathcal{B}^{op} \to \text{CAT} \) be the functor which sends a quadruple \((A, B, C, D)\) to the category of pointed infinitesimal \( C-D \)-bimodules. Recall that the Grothendieck construction \( \int \Phi \) is the category whose objects are pairs \((b, x)\) where \( b \in \mathcal{B} \) and \( x \in \Phi(b) \) and morphisms \((b, x) \to (c, y)\) are pairs \((f, \gamma)\) where \( f : b \to c \) is a morphism in \( \mathcal{B} \) and \( \gamma : x \to \Phi(f)(y) \) is a morphism in
\[ \Phi(b) \]. There is a categorical equivalence between \( f \Phi \) and the category of simplicial algebras of the polynomial monad \( \text{IBimod}_{\cdot} \); defined in Subsection 3.7.

**Lemma 4.9.** For any \( b \in \mathcal{B} \), \( \Phi(b) \) has a model structure.

*Proof.* First note that \( \Phi \) is homotopically structured \( [5] \), that is, \( \mathcal{B} \) is equipped with two classes of morphisms called horizontal weak equivalences and fibrations, and for each \( b \in \mathcal{B} \), the category \( \Phi(b) \) is also equipped with classes of weak equivalences and fibrations called vertical. Also, \( \int \Phi \) is complete, cocomplete and admits a global model structure, that is, where a morphism \( (f, \gamma) \) in \( \int \Phi \) is a weak equivalence (resp. fibration) if \( f \) is a horizontal and \( \gamma \) is a vertical weak equivalence (resp. fibration). Finally, for all \( b \in \mathcal{B} \), \( \Phi(b) \) is complete and cocomplete, and for all morphisms \( f : b \to c \) in \( \mathcal{B} \), the functor \( \Phi(f) \) preserves weak equivalences, fibrations and terminal objects. Batanin and White \( [5] \) proved that, with such assumptions, \( \Phi(b) \) admits a model structure for all \( b \in \mathcal{B} \). \( \square \)

We will write \( \text{NOp} \) for the category of simplicial non-symmetric operads.

**Lemma 4.10.** For \( b \in \mathcal{B} \) a cofibrant replacement of the terminal object, there is a left cofinal Quillen functor \( \Phi(b) \to S^1 / \text{NOp} \), where \( S^1 \) is a circle constructed as in Definition 4.2.

*Proof.* Let \( \text{NOp}_\triangle \) be the polynomial monad for diagrams

\[
\begin{array}{ccc}
B & \xleftarrow{C} & D \\
\xleftarrow{A} & & \\
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\end{array}
\]

of non-symmetric operads. There is a morphism of polynomial monads

\[
g : \text{Bimod}_{\cdot} \to \text{NOp}_\triangle
\]

which is given by restriction on colours of the map

\[
f : \text{IBimod}_{\cdot} \to \text{NOp}_\Phi
\]

of Theorem 3.8. Since \( f \) is homotopically cofinal, so is \( g \). Let \( \mathcal{C} \) be the category of simplicial algebras of \( \text{NOp}_\triangle \) and \( \Psi : \mathcal{C}^\text{op} \to \text{CAT} \) be the functor which sends a diagram \( c \) to the comma category \( \text{colim}(c)/\text{NOp} \). There is a categorical equivalence between \( f \Psi \) and the category of simplicial algebras of the polynomial monad \( \text{NOp}_\Phi \). For \( (c,y) \in f \Psi \), the restriction functor induced by \( f \) is given by

\[
f^*(c,y) = (g^*(c), W_c(y)),
\]

where \( W_c : \Psi(c) \to \Phi g^*(c) \) is the obvious forgetful functor. Let \( \eta \) be the unit of the adjunction induced by \( g \) and, for \( b \in \mathcal{B} \), let \( H_b : \Phi(b) \to \Psi g(b) \) be the left adjoint of \( \Phi(\eta_b)W_{g(b)} \). By adjunction, for all \( (b,x) \in f \Phi \), we have

\[
f_!(b,x) = (g!(b), H_b(x)).
\]

From Remark 4.8 we deduce that \( f_! \) and \( g_! \) are left cofinal Quillen functors. If \( b \in \mathcal{B} \) is a cofibrant replacement of the terminal object, then \( H_b \) is left cofinal. The diagram \( g!(b) \) forms a cofibrant replacement of the terminal object in \( \mathcal{C} \), so its colimit is a circle \( S^1 \) in \( \text{NOp} \), as constructed in Definition 4.2. This means
that the categories $\Psi g(b)$ and $S^1/\text{NOp}$ are equivalent, and $H_b$ gives us the desired functor.

4.11. Fibration sequence theorem.

Lemma 4.12. Let $M$ be a model category. Let $x \to a \to b$ be a composite in $M$. Assume $x$ cofibrant and the map $a \to b$ is a cofibration. The following square is a homotopy pushout in $M$

(14)

\[
\begin{array}{ccc}
    a \cup x & \longrightarrow & a \\
    \downarrow & & \downarrow \\
    b \cup x & \longrightarrow & b
\end{array}
\]

Proof. Let $a^c$ be an object in $M$ together with a factorisation

\[
\begin{array}{ccc}
    a \cup x & \longrightarrow & a \\
    \downarrow & & \downarrow \\
    a^c & \longrightarrow & \leftarrow
\end{array}
\]

Let us consider the following commutative diagram

\[
\begin{array}{ccc}
    a & \longrightarrow & a \cup x \\
    \downarrow i & & \downarrow ii \\
    b & \longrightarrow & b \cup x \\
    \downarrow & & \downarrow \\
    & & p
\end{array}
\]

where $p$ is defined as the pushout of the square $ii$.

We want to prove that the composite $b \to p$ is a weak equivalence. It is actually a trivial cofibration. To see this, observe first that the composite $a \to a^c$ is a trivial cofibration. Indeed, it is a weak equivalence by the two-out-of-three property and it is also a cofibration since $x$ is cofibrant. Moreover, since the square $i$ is a pushout, the big rectangle is also a pushout. We get the conclusion using the fact that trivial cofibrations are stable under pushout.

Since the square $ii$ is a pushout and all the maps in this square are cofibrations, it is also a homotopy pushout. Since all the objects in the square $ii$ are weakly equivalent to the objects in the square (14), this last square is also a homotopy pushout.

Theorem 4.13. Let $e \in M$ be cofibrant. Then for all maps $r \to s$ in $e/M$, there is a fibration sequence

\[
\text{Map}_{e/M}(r,s) \to \text{Map}_M(V(r),V(s)) \to \text{Map}_M(e,V(s)),
\]

where $V : e/M \to M$ is the forgetful functor.

Proof. Since $V$ preserves weak equivalences, we can assume without loss of generality that $r$ is cofibrant and $s$ is fibrant in $e/M$. By definition of being cofibrant in $e/M$, there is a cofibration $e \to V(r)$. Since $e$ is cofibrant, this means that $V(r)$ is also cofibrant in $M$. Let $G$ be the left adjoint of $V$. Then

\[VG(x) = x \cup e.\]
Let $\iota$ be the initial object in $e/M$, given by the identity map on $e$. We have the following commutative square in $e/M$:

\[
\begin{array}{ccc}
G(e) & \rightarrow & \iota \\
\downarrow & & \downarrow \\
GV(r) & \rightarrow & r
\end{array}
\]

Observe that a square in $e/M$ is a homotopy pushout if and only if $V$ applied to this square gives a homotopy pushout. But when we apply $V$ to the square $15$, we get the square

\[
\begin{array}{ccc}
e \cup e & \rightarrow & e \\
\downarrow & & \downarrow \\
V(r) \cup e & \rightarrow & V(r)
\end{array}
\]

which is a homotopy pushout by Lemma 4.12. This proves that the square $15$ is a homotopy pushout. Applying $\text{SSet}_{e/M}(-,s)$ to $15$ gives us the homotopy pullback

\[
\begin{array}{ccc}
\text{SSet}_{e/M}(r,s) & \rightarrow & \text{SSet}_{e/M}(\iota,s) \\
\downarrow & & \downarrow \\
\text{SSet}_{e/M}(GV(r),s) & \rightarrow & \text{SSet}_{e/M}(G(e),s)
\end{array}
\]

By adjunction, the bottom map is equivalent to the map

$\text{SSet}_M(V(r),V(s)) \rightarrow \text{SSet}_M(e,V(s))$

Since $\iota$ is the initial object in $e/M$, $\text{SSet}_{e/M}(\iota,s)$ is contractible. This concludes the proof.

For a functor $U : B \rightarrow M$ and $e \in M$, we write $e/U$ for the slice construction, whose objects are pairs $(b, \alpha)$, with $b \in B$ and $\alpha : e \rightarrow U(b)$ a morphism in $M$.

**Theorem 4.14.** Let $U : B \rightarrow M$ be a right Quillen functor and $e \in M$ cofibrant. Then for all maps $r \rightarrow s$ in $e/U$, there is a fibration sequence

$\text{Map}_{e/U}(r,s) \rightarrow \text{Map}_B(V(r),V(s)) \rightarrow \text{Map}_M(e,UV(s))$

where $V : e/U \rightarrow B$ is the forgetful functor.

**Proof.** Let $F : M \rightarrow B$ be the left adjoint of $U$. Since $F(e)$ is cofibrant in $B$, we can apply Lemma 4.13 to get, for all maps $r \rightarrow s$ in $F(e)/B$, the fibration sequence

$\text{Map}_{F(e)/B}(r,s) \rightarrow \text{Map}_B(V(r),V(s)) \rightarrow \text{Map}_B(F(e),V(s))$

We get the conclusion by adjunction and from the fact that the categories $F(e)/B$ and $e/U$ are equivalent.

**4.15. Tame polynomial monads.** The notions of this subsection are taken from [2].

**Definition 4.16.** Let $T$ be a finitary monad on a cocomplete category $C$. We denote by $T+1$ the finitary monad on $C \times C$ given by

$(T + 1)(X,Y) = (T(X),Y)$

$(T + 1)(\phi,\psi) = (T(\phi),\psi)$
with evident multiplication and unit.

If \( T \) is a polynomial monad generated by the polynomial

\[
I \leftarrow E \xrightarrow{p} B \xrightarrow{t} I
\]

then \( T + 1 \) is a polynomial monad on \( \text{Set}^I \times \text{Set}^I = \text{Set}^{I \times I} \) generated by

\[
I \cup I \xleftarrow{\text{sum}} E \cup I \xrightarrow{\text{prod}} B \cup I \xrightarrow{\text{term}} I \cup I
\]

Moreover, there is a morphism of polynomial monads

\[
\begin{array}{cccc}
I \cup I & \xleftarrow{\text{sum}} & E \cup I & \xrightarrow{\text{prod}} B \cup I & \xrightarrow{\text{term}} I \cup I \\
\downarrow \downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \downarrow \\
I & \xleftarrow{s} & E & \xrightarrow{p} & B & \xrightarrow{t} & I
\end{array}
\]

where \( \nabla_I \) is the codiagonal and \( \phi \) (resp. \( \psi \)) is the identity on \( B \) (resp. \( E \)) and the unit of \( T \) on \( I \).

**Definition 4.17.** A polynomial monad \( T \) is tame if the classifier \( T^{T+1} \) is a coproduct of categories with terminal object.

**Lemma 4.18.** The polynomial monads \( \text{NOp} \) of Example 2.4 is tame.

**Sketch of the proof.** The tameness of the polynomial monad \( \text{NOp} \) is proved in [2, Section 9.2]. The proof consists in an explicit description of the classifier \( \text{NOp}^{NOp+1} \). The objects are given by isomorphism classes of planar rooted trees with vertices coloured by \( X \) and \( K \) and morphisms contract edges between vertices labelled with \( X \), or add a unary vertex labelled with \( X \). A typical terminal object in a connected component is a planar rooted tree with vertices coloured by \( X \) and \( K \) such that adjacent vertices have different colours, and such that vertices incident to the root or to the leaves are coloured with \( X \).

According to [2, Theorem 8.1], the category of simplicial algebras over a tame polynomial monad is left proper, so we have the following corollary.

**Corollary 4.19.** The category of simplicial non-symmetric operads \( \text{NOp} \) is left proper.

4.20. **Pointed infinitesimal bimodules as a comma category.** Let \( B \) be the category and \( \Phi : B^{\text{op}} \to \text{CAT} \) be the functor defined in Subsection 4.6. Let \( \Phi^\circ : B^{\text{op}} \to \text{CAT} \) be the functor which sends a quadruple \( (A, B, C, D) \) to the category of infinitesimal \( C-D \)-bimodules.

**Lemma 4.21.** For all \( b \in B \), there is an object \( \alpha \in \Phi^\circ(b) \) such that \( \Phi(b) \) is equivalent to the comma category \( \alpha/\Phi^\circ(b) \).

**Proof.** This is obvious from the definition of pointed infinitesimal bimodules. Since it exists, \( \alpha \) must be given by the image of the initial object through the forgetful functor \( V : \Phi(b) \to \Phi^\circ(b) \).

**Definition 4.22.** Let \( (E, \otimes, e) \) be a symmetric monoidal category and \( (A_1, A_2) \) be a pair of objects in \( E \) where \( A_1 \) is a monoid. Let \( \mathcal{SP}(A_1, A_2) \) be the category of pairs of objects \( (E_0, E_1) \) in \( E \), where \( E_0 \) is a left module over \( A_1 \), equipped with a cospan

\[
A_2 \otimes A_1 \otimes E_0 \to E_1 \leftarrow A_2 \otimes E_0 \otimes A_1.
\]
Recall that $\mathcal{B}$ was defined as the category of simplicial algebras over the polynomial monad $\text{Bimod}_{\cdot}$, so there is a particular object in this category given by the nerve of the absolute classifier of the polynomial monad. We will write $b = (A, B, C, D)$ for this element and consider $\alpha$ associated to it, given by Lemma 4.21. Let $F$ be the left adjoint of the forgetful functor $U : \Phi(b) \to \mathcal{S}(A_1, A_2)$.

**Lemma 4.23.** There is a weak equivalence $\alpha \sim LF(1)$, where $1$ is the terminal object in $\mathcal{S}(A_1, A_2)$.

**Proof.** There is an obvious morphism of polynomial monads

(16) $\text{Bimod}_{\cdot} \to \text{IBimod}_{\cdot}$

given by inclusion of sets. The nerve of the classifier induced by this map gives us in particular the initial object of $\Phi(b)$. Let $\text{BimodSP}_{\cdot}$ be the polynomial monad for quintuples $(A, B, C, D, E)$ where $A$ and $B$ are non-symmetric operads, $C$ and $D$ are pointed $A - B$-bimodules and $E = (E_0, E_1)$ is an object of $\mathcal{S}(A_1, A_2)$. Let $\text{IBimod}^o_{\cdot}$ be the polynomial monad for quintuple $(A, B, C, D, E)$ where $A, B, C$ and $D$ are as before and $E$ is an infinitesimal $C - D$-bimodule. Again, there is an obvious morphism of polynomial monads

(17) $\text{BimodSP}_{\cdot} \to \text{IBimod}^o_{\cdot}$

given by inclusion of sets. The nerve of the classifier induced by this map gives us in particular an element in $\Phi^o(b)$, which is weakly equivalent to $LF(1)$ according to Lemma 2.9. We have to prove that the nerves of the classifiers induced by the maps (16) and (17) are weakly equivalent.

We will define two new morphisms of polynomial monads very similar to the ones defined above. Let $\text{Bimod}_{\cdot}$ be the polynomial monad whose algebras are the same as the algebras of $\text{Bimod}_{\cdot}$, but we force the non-symmetric operad $A$ to be the terminal non-symmetric operad $\text{Ass}$. That is, the polynomial monad for quadruples $(A, B, C, D)$ where $A = \text{Ass}$, $B$ is any non-symmetric operad and $C$ and $D$ are pointed $A - B$-bimodules. In a totally similar way, one can define the polynomial monads $\text{IBimod}^o_{\cdot}$, $\text{BimodSP}_{\cdot}$, and $\text{IBimod}^o_{\cdot}$. We have two more morphisms of polynomial monads

(18) $\text{Bimod}_{\cdot} \to \text{IBimod}_{\cdot}$

and

(19) $\text{BimodSP}_{\cdot} \to \text{IBimod}^o_{\cdot}$

To simplify the notations, let us fix an integer $n \geq 0$ and let us write $\mathcal{X}$, $\mathcal{X}'$, $\mathcal{Y}$ and $\mathcal{Y}'$ for the underlying categories indexed by the label $E$ and the integer $n$ of the classifiers induced by the morphisms (17), (19), (16) and (18) respectively. Let us describe these categories more explicitly. The objects for each category are isomorphisms classes of planar trees with $n$ leaves, where each vertex is decorated with a label in $\{ A, B, C, D, E \}$, satisfying the condition of Definition 3.10. We have the following extra conditions. For the categories $\mathcal{X}$ and $\mathcal{X}'$, the trees must contain exactly one vertex labelled with $E$, and this vertex must have $0$ or $1$ incoming edges, while for the categories $\mathcal{Y}$ and $\mathcal{Y}'$ there should be no vertices labelled with $E$. Moreover, the categories $\mathcal{X}'$ and $\mathcal{Y}'$ also have the extra condition that there is exactly one vertex, the root vertex, with label $A$. The morphisms for each category can be described by nested trees, as in Subsection 3.11. The trees nested inside vertices have a labelling
which also satisfies the condition of Definition \[\text{3.10}\]. For the categories \(\mathcal{X}\) and \(\mathcal{X}'\), the tree nested inside the vertex with label \(E\) has exactly one vertex with label \(E\), and all other vertices have label \(A\). For the categories \(\mathcal{X}'\) and \(\mathcal{Y}'\), if the source of a nested tree has several vertices with label \(A\), then the edges between them are automatically contracted. This means that we have, for example in \(\mathcal{X}'\) morphisms such as in the following picture:

\[
\begin{array}{c}
A \xrightarrow{C} B
\end{array}
\]

Let us now construct a diagram of functors

\[
\mathcal{X} \xrightarrow{F} \mathcal{X}' \xrightarrow{G} \mathcal{Y}' \xleftarrow{H} \mathcal{Y}
\]

The functors \(F\) and \(H\) automatically contract all the edges between vertices with label \(A\) if there is at least one vertex with label \(A\). If there is no such vertex, then an unary vertex with label \(A\) is added to the root vertex. The functor \(G: \mathcal{X}' \rightarrow \mathcal{Y}'\) acts on a tree as follows. If the unique vertex with label \(E\) has no incoming edges, then it removes this vertex and the edge below it. If the unique vertex with label \(E\) has an incoming edge, then it just removes this vertex. Such functors on objects can obviously be extended on morphisms.

First, note that the fibres of each functor have contractible nerve. Indeed, the fibres of \(F\) and \(H\) have a terminal object. The fibres of \(G\) consist in zigzags of morphisms. For example, the fibre over the object

\[
\begin{array}{c}
B \xrightarrow{A} B
\end{array}
\]

is given by the following subcategory

\[
\begin{array}{c}
E \xrightarrow{A} B \xrightarrow{B} E \xrightarrow{A} B \xrightarrow{B} E \xrightarrow{A}
\end{array}
\]

Let us now prove that each functor is smooth, which will conclude the proof. For the functor \(F\), let us take an object \(x_1\) in \(\mathcal{X}\) and a map \(f_1: y_0 \rightarrow y_1\) in \(\mathcal{X}'\) such that \(F(x_1) = y_1\). The lifting category of \(f_1\) over \(x_1\) has a terminal object, which can be construct in the following way. From the tree \(x_1\), one can make a cut just below the line given by Definition \[\text{3.10}\] to get a tree \(T\) whose vertices are labelled with \(A\) and all the other vertices have been removed. It is easy to see that there are trees \(T_1, \ldots, T_k\), with \(k\) the number of leaves of \(T\), such that the tree obtained by attaching these trees to each leaf of \(T\) gives an object in the fibre of \(y_0\). Moreover, one can take \(T_1, \ldots, T_k\) with at most one vertex labelled with \(A\), which is not unary. Then the obtained tree is the terminal object of the lifting category. Similarly, \(H\) is also a smooth functor. For the functor \(G\), let us take an object \(x_1\) in \(\mathcal{X}'\) and a map \(f_1: y_0 \rightarrow y_1\) in \(\mathcal{Y}'\) such that \(G(x_1) = y_1\). The lifting category of \(f_1\) over \(x_1\) is the terminal category if the unique vertex labelled with \(E\) in \(x_1\) has 0 incoming edges and consists in a cospan if it has 1 incoming edge.

Let \(e\) be a cofibrant replacement of the terminal object in \(\mathcal{S}\mathcal{P}(A_1, A_2)\).
Lemma 4.24. There is a Quillen equivalence between $e/U$ and $\Phi(b)$.

Proof. The category $e/U$ is equivalent to the category $F(e)/\Phi^0(b)$ and, according to Lemma [4.21], the category $\Phi(b)$ is equivalent to the category $\alpha/\Phi^0(b)$.

Let us prove that $\Phi^0(b)$ is left proper. The polynomial monad $IBimod^2$, defined in the proof of Lemma [4.22] is tame, and the proof of tameness works just like the proof of Lemma [4.18]. The category of simplicial algebras of this polynomial monad, which is therefore left proper, is equivalent to the Grothendieck construction $\int \Phi^0$. We deduce from the model structure that the fibre $\Phi^0(b)$ must also be left proper.

According to Lemma [4.23] there is a weak equivalence between $F(e)$ and $\alpha$. We get the conclusion from the fact that a weak equivalence in a left proper model category induces a Quillen equivalence between the comma categories $[15,\text{Proposition 2.7}].$

Lemma 4.25. Let $x \to y$ be a map in $\Phi(b)$. If $y_0$ and $y_1$ are contractible, then there is a weak equivalence

$$\text{Map}_{\Phi(b)}(x,y) \to \text{Map}_{\Phi^0(b)}(V(x),V(y)),$$

where $V : \Phi(b) \to \Phi^0(b)$ is the forgetful functor.

Proof. Let $x \to y$ be a map in $e/U$. Applying Theorem [4.14], we get a fibration sequence

$$\text{Map}_{e/U}(x,y) \to \text{Map}_{\Phi^0(b)}(V(x),V(y)) \to \text{Map}_{S\mathcal{P}(A_1,A_2)}(e,UV(y))$$

If $y_0$ and $y_1$ are contractible, then the last mapping space in this fibration sequence is also contractible, which means that the first map is a weak equivalence. We get the conclusion from Lemma [4.24].

4.26. Direct double delooping proof.

Proof of Theorem [1.1] According to Corollary [4.19] the category of simplicial non-symmetric operads $NOp$ is left proper, so we can apply Theorem [4.5] to get a weak equivalence

$$\Omega^2\text{Map}_{NOp}(D^0,g_0^*(y)) \to \text{Map}_{S^{1}/NOp}(D^2,g_2^*(y))$$

where disks and spheres are constructed as in Definition [4.2] and $g_n : S^{n-1} \to D^0$ is the map [4]. Let $b \in \mathcal{B}$ as in Subsection [4.20]. Let $H_b : \Phi(b) \to S^{1}/NOp$ be the left cofinal Quillen functor given by Lemma [4.10] and $W_b$ its right adjoint. By adjunction, for $x \in \Phi(b)$ cofibrant and $y \in S^{1}/NOp$ fibrant, there is an isomorphism

$$\text{Map}_{S^{1}/NOp}(H_b(x),y) \to \text{Map}_{\Phi(b)}(x,W_b(y))$$

The assumption that $y$ is fibrant can be dropped since $W_b$ preserves weak equivalences. Also, if $x$ is a cofibrant replacement of the terminal object, $H_b(x)$ is contractible since $H_b$ is left cofinal. Let $1$ be the terminal object in $\mathcal{B}$ and again in $\Phi^0(1)$, abusing notations. The unique map $b \to 1$ is a weak equivalence, so it induces a Quillen equivalence between the categories of simplicial presheaves $\Phi^0(b)$ and $\Phi^0(1)$. Therefore, putting together the weak equivalences $[22,23]$ and $[21]$ we get, for a multiplicative operad $\mathcal{O}$ such that $\mathcal{O}_0$ and $\mathcal{O}_1$ are contractible, a weak equivalence

$$\Omega^2\text{Map}_{NOp}(Ass,u^*(\mathcal{O})) \sim \text{Map}_{\Phi^0(1)}(1,f^*(\mathcal{O}))$$
According to Lemma 3.3, $\Phi^\circ(1)$ is equivalent to the category of infinitesimal $Ass$-bimodules in the classical sense, which is in turn equivalent to the category cosimplicial objects [17, Lemma 4.2]. This concludes the proof. □

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References

[1] Michael Batanin. The Eckmann–Hilton argument and higher operads. Advances in Mathematics, 217(1):324–385, 2008.
[2] Michael Batanin and Clemens Berger. Homotopy theory for algebras over polynomial monads. Theory and Applications of Categories, 32(6):148–253, 2017.
[3] Michael Batanin and Florian De Leger. Polynomial monads and delooping of mapping spaces. Journal of Noncommutative Geometry, 13(4):1521++, 2019.
[4] Michael Batanin and Martin Markl. Operadic categories and duoidal deligne’s conjecture. Advances in Mathematics, 285:1630–1687, 2015.
[5] Michael Batanin and David White. Model structures on operads and algebras from a global perspective. In preparation.
[6] Pedro Boavida de Brito and Michael Weiss. Spaces of smooth embeddings and configuration categories. Journal of Topology, 11(1):65–143, 2018.
[7] Denis-Charles Cisinski. Les préfaisceaux comme modèles des types d’homotopie. Société mathématique de France, 2006.
[8] Julien Ducoulombier. Delooping derived mapping spaces of bimodules over an operad. Journal of Homotopy and Related Structures, 14(2):411–453, 2019.
[9] Julien Ducoulombier and Victor Turchin. Delooping the functor calculus tower. arXiv preprint arXiv:1708.02203, 2017.
[10] William Dwyer and Kathryn Hess. Long knots and maps between operads. Geometry & Topology, 16(2):919–955, 2012.
[11] Nicola Gambino and Joachim Kock. Polynomial functors and polynomial monads. Mathematical Proceedings of the Cambridge Philosophical Society, 154(1):153–192, 2013.
[12] Philip Hirschhorn. Model categories and their localizations. Number 99. American Mathematical Society, 2009.
[13] Joachim Kock, André Joyal, Michael Batanin, and Jean-François Mascari. Polynomial functors and opetopes. Advances in Mathematics, 224(6):2690–2737, 2010.
[14] Martin Markl. Operads and props. Handbook of algebra, 5:87–140, 2008.
[15] Charles Rezk. Every homotopy theory of simplicial algebras admits a proper model. Topology and its Applications, 119(1):65–94, 2002.
[16] Dev Sinha. Operads and knot spaces. Journal of the American Mathematical Society, 19(2):461–486, 2006.
[17] Victor Turchin. Hodge-type decomposition in the homology of long knots. Journal of Topology, 3(3):487–534, 2010.
[18] Victor Turchin. Delooping totalization of a multiplicative operad. Journal of Homotopy and Related Structures, 9(2):349–418, 2014.
[19] Mark Weber. Internal algebra classifiers as codescent objects of crossed internal categories. Theory and Applications of Categories, 30(50):1713–1792, 2015.

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