Nesting and lifetime effects in the FFLO state of quasi-one-dimensional imbalanced Fermi gases

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Abstract
Motivated by the recent experimental realization of a candidate to the Fulde–Ferrell (FF) and the Larkin–Ovchinnikov (LO) states in one-dimensional (1D) atomic Fermi gases, we study the quantum phase transitions in these enigmatic, finite-momentum-paired superfluids. We focus on the FF state and investigate the effects of the induced interaction on the stability of the FFLO phase in homogeneous spin-imbalanced quasi-1D Fermi gases at zero temperature. When this is taken into account, we find a direct transition from the fully polarized to the FFLO state in agreement with exact solutions. Also, we consider the effect of a finite lifetime of the quasi-particle states in the normal-superfluid instability. In the limit of long lifetimes, the lifetime effect is irrelevant and the transition is directly from the fully polarized to the FFLO state. We show, however, that for sufficiently short lifetimes, there is a quantum critical point, at a finite value of the mismatch of the Fermi wave-vectors of the different quasi-particles, that we fully characterize. In this case, the transition is from the FFLO phase to a normal partially polarized state with increasing mismatch.

(Some figures may appear in colour only in the online journal)

1. Introduction
Recently, there has been increased interest in the theory of one-dimensional (1D) imbalanced Fermi systems, partly because of the relevance of these theories for the understanding of the Fulde and Ferrell\cite{fulde1964} and Larkin and Ovchinnikov\cite{larkin1965} (FFLO) phase. The FFLO is an exotic phase proposed approximately 40 years ago, where atoms of opposite momenta and spins form Cooper pairs with finite momentum. In spite of intense theoretical and experimental efforts, the FFLO phase remains elusive.

In three-dimensional (3D) systems, in the strongly interacting limit, experiments show\cite{kimchis2003, helmes2007, huai2005, fouquet2009, huai2010} that the gas phase separates with an unpolarized superfluid core surrounded by a polarized shell\cite{zhang2004, gornyi2004}, with no evidence for the FFLO phase\cite{fouquet2009}. However, in 1D imbalanced Fermi gases, the observed density profiles\cite{huai2010} agree quantitatively well with theories that exhibit the one dimension equivalent to FFLO correlations at low temperatures\cite{fouquet2009, huai2010, zhang2012}. These experimental measurements\cite{huai2010} of density profiles of a two-spin mixture of ultracold $^6$Li atoms trapped in an array of 1D tubes show that at finite spin imbalance, the system phase separates with an inverted phase profile as compared to the 3D case. In these 1D experiments, a partially polarized (PP) core was observed, surrounded by wings composed of either a completely paired or a fully polarized (FP) Fermi gas, depending on the degree of polarization.

This recent experimental observation of what can be seen as a strong candidate for FFLO-like correlations in one dimension has motivated theoretical investigations of possible mechanisms responsible for its stability. The increased stability of FFLO-like phases in one dimension can be understood as a nesting effect, where a single wave-vector connects all points on the Fermi surface, allowing all atoms on
the Fermi surface to participate in finite momentum pairing, while in 3D, only a small portion of these atoms is able to contribute. This 1D Fermi surface nesting enhancement of the instability of the normal to the FFLO state is analogous to the conventional charge density wave instability [15].

In the pairing mechanism, besides the particle–particle channel considered by Nozières and Schmitt-Rink (NSR) [16], there is a correction of the two-body pairing interactions, considered first by Gorkov and Melik-Barkhudarov (GMB) [17]. This correction accounts for the induced interactions which arise between atoms at the Fermi level due to the polarization of the medium. It has been shown that these induced interactions suppress the superfluid transition temperature by a factor of about 2.22 in 3D [18] and 2.72 in 2D [19, 20] spin-balanced Fermi gases, respectively, when compared with the mean-field (MF) results. The GMB correction was considered recently in various situations such as, for instance, in the spin-balanced Fermi gas in an optical lattice [21, 22], in a homogeneous three-component Fermi gas [23], and in the unitary limit of spin-balanced [24] and imbalanced 3D Fermi gases [25].

In this paper, we study the zero temperature ($T$) phase diagram of a quasi-1D imbalanced Fermi system as a function of the mismatch $h$ between their Fermi wave-vectors [26]. This is relevant for the nearly 1D Fermi gases we are interested in. We show that including induced interactions through a random phase approximation (RPA) is essential to correct the MF (naive) pairing fluctuation results, since they give rise to a finite critical field (or mismatch) separating an FP phase from the FFLO state, and in this way, reveal the presence of the nesting effect in spin-imbalanced quasi-1D (ideal) Fermi gases.

It is possible to conceive in actual physical systems mechanisms by which the quasi-particle states in the normal phase acquire a finite lifetime, for example, due to different types of unknown (or unrecognized) scattering mechanisms not included in the pairing interaction or due to an inhomogeneous distribution of the atoms in a trap. In condensed matter systems, for quasi-1D superconductors, disorder may have a more mundane origin, as defects or impurities [27]. We assume the existence of weak inter-tube interactions, so that the effects of localization are not so severe. We show here how this lifetime effect (LT) modifies the $T = 0$ phase diagram of the 1D gas. In the limit of short lifetimes of the quasi-particle states in one dimension, we find a quantum phase transition from the normal-to-inhomogeneous superfluid as the Fermi wave-vector mismatch is reduced from the normal phase. This $T = 0$ transition is continuous or second order. It is associated with a quantum critical point (QCP) at a critical value of the field (mismatch) $h_c$. We fully characterize this QCP obtaining its dynamic quantum critical exponent and universality class. On the other hand, for sufficiently long lifetimes (weak disorder), LTs turn out to be irrelevant and we recover the previous results of including only induced interactions. Our results imply that for sufficiently strong disorder, the region in the phase diagram where the FFLO phase exits is reduced without necessarily being destroyed, even in one dimension. The instability of the normal state that we consider is that for an FFLO superfluid state characterized by a single wave-vector $q$. This is the first zero temperature instability that occurs as the effective Zeeman field $h$ is reduced [26] from the normal phase.

The majority of the recent literature on 1D spin-imbalanced Fermi gases considers the simplest possible system that exhibits FFLO-type pairing, namely the Yang–Gaudin model or its lattice version, the Hubbard model with attractive interactions [28–32]. Both models are exactly solvable (or integrable) and their energy spectra and thermodynamical properties can be calculated exactly using the Bethe ansatz and numerical methods [28]. The Yang–Gaudin model does not include inter-tube couplings and eventually it will be necessary to consider these [30, 33] to fully describe the experiments.

The theoretical predictions and in particular the phase diagram obtained using the integrable Yang–Gaudin model [10, 11, 34, 35] agree very well with the experimentally observed density profiles and support the stability of the FFLO phase in one dimension. Here, we give a robust and transparent physical explanation of this stability as a consequence of nesting effects. We also show explicitly that the medium indubitably modifies the fermion–fermion interaction $g$, due to many-body effects, an effect which cannot be clearly seen in the exact approaches. We point out that our results including induced interactions are consistent with the exact results since, for long lifetimes, we find a direct transition from the FP to the FFLO state. This is not surprising since the physics of this problem is determined by the nesting effect which is present in both approaches, either explicitly or implicitly.

Our study is based on calculations of pair and density fluctuations, i.e., of the particle–particle (pair) and particle–hole susceptibilities, respectively, that certainly are present in a Fermi gas, in any dimension. In one dimension, these two physical quantities are of extreme importance, since both diverge, which are indications of some order in the system. As will be shown below, our main results do not depend on the particular Hamiltonian used to describe the attractive fermionic gas. Rather they arise from fundamental quantities, namely the 1D fermionic dispersion relations that can always be linearized close to the Fermi points, regardless of its precise nature, and the subsequent calculation of the particle–hole and particle–particle susceptibilities [36]. Thus, given that it is of fundamental importance to consider the Fermi surface properties at 1D or the particle–particle and particle–hole interactions near it, our results can be considered as model independent. In this sense, our approach is complementary to those previous studies based on the real-space Yang–Gaudin and Hubbard models.

2. Model Hamiltonian

To begin, let us consider a non-relativistic dilute (i.e., the particles interact through a short-range attractive interaction) 1D spin-polarized Fermi gas, described by the following single-channel model Hamiltonian:

$$
\mathcal{H} = \sum_{k, \alpha} \mu_{\alpha} n_{k\alpha} + \sum_{k} \epsilon_k b_{k\alpha}^\dagger b_{k\alpha} + g_{1D} \sum_{k \neq k'} a_{k\alpha}^\dagger b_{-k\alpha}^\dagger b_{-k'\alpha} a_{k'\alpha},
$$

(1)
where \( a_\alpha^\dagger \) and \( a_\alpha \) are the creation and annihilation operators of the \( a \) particles, respectively (and the same for the \( b \) particles) and \( \epsilon_\alpha^b \) are their dispersion relations, defined by \( \epsilon_\alpha^b = \xi_k - \mu_\alpha \), with \( \xi_k = h^2 k^2 / 2m \), and \( \mu_\alpha \) is the chemical potential of the non-interacting \( \alpha \)-species, \( \alpha = a, b \). A special case described by this Hamiltonian is that of the identical spin \( S = 1/2 \) particles in an external magnetic field \( h \). In this case, \( a \) and \( b \) correspond to the spin-up and spin-down bands with their degeneracy raised by the magnetic field. The dispersion relations are then given by \( \epsilon_\alpha^{ab} = \epsilon_k \mp \mu_\alpha \), where \( \epsilon_k = k \mp \xi_k \). In particular, we will approximate the dispersions around their Fermi energies by \( \epsilon_\alpha^{ab} = v_F (k - k_F) \mp h \), since the relevant states are near the Fermi momenta, \( v_F \) is the Fermi velocity. For future notation, we rewrite this equation as \( \epsilon_\alpha^{ab} = v_F (k - k_F^\alpha) \mp h \), where \( k_F^\alpha \equiv k_F \pm h/v_F \) are the Fermi wave-vectors of the different particles or bands. An important parameter in this study is the Fermi wave-vector mismatch, \( \delta k_F \equiv k_F^a - k_F^b = 2h/v_F \). To reflect an attractive (s-wave) interaction between particles \( a \) and \( b \), we take \( g_{1D} \equiv g < 0 \).

3. The ideal case: infinite lifetimes

In this section, we consider a quasi-1D idenbaled Fermi system as a function of the mismatch \( h \) between their chemical potentials in the ideal case, i.e., in the absence of LTs. Our aim is to investigate the quantum phase transition from the normal-to-inhomogeneous FFLO superfluid phase as the Fermi-wave-vector mismatch is reduced from the normal polarized state.

3.1. Ginzburg–Landau theory and the FFLO phase

Since in a homogeneous 1D system the quantum phase transition to the FFLO phase is continuous [11, 12, 15], we expand the action in fluctuations, \( \Delta k \), and obtain [37, 38]

\[
S_{eff} = \sum_{\vec{q}} \int_0^{\omega_0} d\omega_0 (\alpha(|\vec{q}|, \omega_0) \Delta \chi(\omega_0))| + O(|\Delta|),
\]

(2)

where \( \alpha(|\vec{q}|, \omega_0) = -\frac{\hbar^2}{2m} \chi_{pp}(\vec{q}, \omega_0) \) with \( \chi_{pp}(\vec{q}, \omega_0) \) being the external momentum of the particle–particle bubble diagram and \( \chi_{pp}(\vec{q}, \omega_0) \) the pair susceptibility,

\[
\chi_{pp}(\vec{q}, \omega_0) = \sum_{\vec{k}, \vec{k}'} \frac{1}{\epsilon_k - \epsilon_{\vec{k}'} - \omega - \omega_0} \bar{\Delta}^a \bar{\Delta}^b \bar{\Delta}^b \bar{\Delta}^a \chi_{pp}(\vec{q}, \omega_0)
\]

(3)

or

\[
\chi(|\vec{q}|, \omega_0) = \frac{m}{4\pi k_F} \int_0^{\omega_0} d\omega_0 \tanh \left( \frac{\omega_0}{2T} \right) \times \left[ \frac{1}{\omega_0 + h + \frac{(a - \omega_0)^2}{2}} + \frac{1}{\omega_0 + h - \frac{(a - \omega_0)^2}{2}} \right]
\]

(4)

where \( \omega_0 \) is an energy cut-off and \( a \equiv v_F q_0 / 2 \). In the zero temperature limit, \( \chi_{pp}(\vec{q}, \omega_0) \) is given by

\[
\text{Re} \chi(|\vec{q}|, \omega_0) = N(0) \left[ \ln \left( \frac{\omega_0}{h} \right) - \frac{1}{4} \ln \left( 1 - (\vec{q} + \omega_0)^2 \right) \right] - \frac{1}{4} \ln \left( 1 - (\vec{q} - \omega_0)^2 \right)
\]

(5)

where \( N(0) = \frac{\pi}{2k_F} \) is the density of states for both spins at the Fermi energy \( E_F = k_F^2 / 2m, \tilde{q} \equiv |q_0| / 2h, \tilde{\omega}_0 \equiv \omega_0 / 2h \). Here, \( v_F \) is the Fermi velocity and

\[
\text{Im} \chi(|\vec{q}|, \omega_0) = \left( \frac{\pi}{2} \right) \frac{\sqrt{\omega_0 - \omega - \frac{(a - \omega_0)^2}{2}}} {\sqrt{\omega_0 + h + \frac{(a - \omega_0)^2}{2}}} - \frac{1}{4} \ln \left( 1 - (\vec{q} + \omega_0)^2 \right)
\]

(6)

\[
\text{Im} \chi(|\vec{q}|, \omega_0) = \left( \frac{\pi}{2} \right) \frac{\sqrt{\omega_0 + h - \frac{(a - \omega_0)^2}{2}}} {\sqrt{\omega_0 + h + \frac{(a - \omega_0)^2}{2}}} - \frac{1}{4} \ln \left( 1 - (\vec{q} - \omega_0)^2 \right)
\]

(7)

The static pair susceptibility reads

\[
\chi_{pp}(\vec{q}, \omega_0) = N(0) \left[ \ln \left( \frac{2\omega_0}{\bar{q}_c} \right) - \frac{1}{2} \ln \left( 1 - \frac{\bar{q}_c^2}{\bar{q}_c^2} \right) \right]
\]

(8)

where \( \bar{q}_c = 2\omega_0 \exp(-1/N(0)|\vec{q}|) \) is the zero temperature BCS gap. This expression diverges for \( \bar{q}_c = \bar{q}_c = 1 \), i.e., for \( q_0 = \frac{2\hbar}{v_F} \), yielding \( h_c = \infty \). Since the two-species Fermi momenta can be written as \( k_F^h = k_F^b = \frac{h}{v_F} \), their difference is \( k_F^h - k_F^b = \frac{2\hbar}{v_F} \) and we find that at the critical mismatch the pair-wave-vector reads \( q_c = k_F^h - k_F^b \). Thus, the calculation above shows that FFLO types of correlations are so strong in quasi-1D systems with attractive interactions that differently from \( d = 2 \) [40] and \( d = 3 \) [41], in 1D, the FFLO phase persists even in the presence of an arbitrarily large magnetic field [39]. However, we show below that including induced interactions substantially modifies this scenario and there is a finite \( h_c \) beyond which FFLO correlations disappear. This field coincides with that at which the system becomes FP in agreement with the exact results [33].

3.2. Induced interaction in a spin-polarized Fermi gas

NSR have shown that as the superconducting transition temperature \( T_c \) is approached from above, Cooper pair fluctuations grow in amplitude, and the pair susceptibility (which measures the tendency of pairs to form in response to an external pair field) diverges. The pair fluctuation is expressed by the pair susceptibility of equation (3), and NSR showed that \( \alpha(|\vec{q}| = 0, \omega_0 = 0) = 0 \) is simply the Thouless condition for weak-coupling superconductivity [16].

Besides the pairing fluctuations that must be taken into account, as pointed out by NSR, to obtain the correct superfluid transition temperature \( T_c \) of the BEC–BCS crossover, there is another effect of particle–hole fluctuations that affects the superfluid state. Namely, there is a change in the coefficient \( \alpha(|\vec{q}|) \) due to screening of the interspecies (or induced) interaction, known as the GMB correction [17]. In the BEC side, the NSR fluctuation is dominant, while the GMB fluctuation becomes weaker towards the BEC side and vanishes in this region due to the disappearance of the Fermi surface [20].

In the original work by GMB [17], the induced interaction was obtained in the BCS limit by second-order perturbation [17, 18]. The diagram in figure 1 describes a scattering process in which the conservation of total momentum implies that
where $k' = |k'|$ is equal to the magnitude of $\tilde{p}_1 + \tilde{p}_3 = \tilde{p}_1 - \tilde{p}_4$, so $k' = \sqrt{(\tilde{p}_1 + \tilde{p}_3)(\tilde{p}_1 + \tilde{p}_3)} = \sqrt{\tilde{p}_1^2 + \tilde{p}_3^2 + 2\tilde{p}_1\tilde{p}_3} = \sqrt{\tilde{p}_1^2 + \tilde{p}_3^2 + 2(|\tilde{p}_1||\tilde{p}_3|)\cos\phi}$, where $\phi$ is the angle between $\tilde{p}_1$ and $\tilde{p}_3$. Since the scattering is in 1D, the only values of $\phi$ are 0 or $\pi$. Performing the calculation, we obtain the real function $f(x, h)$ in the form of a generalized Lindhard function given by

$$f(x, h) = \frac{1}{4x} \left[ \ln \frac{1 + xH^b} {1 + xH^a} + \ln \frac{1 + xH^a} {1 - xH^a} \right].$$  

where $x = \frac{k_{1b}}{k_{1f}}, \chi^b = \frac{k_{1b}^2}{2m}, k_{1f}^2 = \sqrt{2m\mu_{b,a}}$. Note that at $h = 0$, the well-known result for 1D balanced systems is recovered,

$$f(x) = \frac{1}{2x} \ln \frac{1 + x}{1 - x},$$  

for which $\chi_{ph}(k' = 0) = -N(0)$. The function $f(x, h)$ diverges for both $xH^aH^b = 1$ or for either $xH^a = 1$ or $xH^b = 1$. Solving, for example, $xH^a = 1$ for $k'$, we find

$$k'^b = k'^b_f + k'^b_b = 2k_f.$$

Thus, at $k'^b \equiv k'^b_f + k'^b_b = 2k_f$, the function $f(x, h)$ in equation (13) diverges (a similarly condition holds for $k'^a$). This corresponds to $\phi = 0$ and $|\tilde{p}_1| = |\tilde{p}_3| = k_F = \sqrt{2m\mu}$, meaning that both scattering particles are at the Fermi surface. Note that in metallic systems, the divergence of $\chi_{ph}(k')$ for a given value of $g$ is often related to an instability to a charge-ordered phase [21]. The result above shows that even in the presence of an external magnetic field, the particle–hole susceptibility diverges at the same value of $k$, as in the absence of the ‘field’ $h$, namely for $k' = 2k_f$.

Considering particle–hole fluctuations, we replace $g$ by $g_{\text{eff}}$ and the instability condition for the superconducting phase is given by

$$\alpha_{\text{eff}}(\bar{q}) = -1 - g_{\text{eff}}(k') \chi_{ph}(\bar{q}) = 0.$$  

It can be easily verified from this equation that the wave-vector for which this condition is first satisfied is still given by $\bar{q} = \bar{q}_c$, which gives a critical field $h_c = \infty$. An interesting and new possibility occurs when $k' = k'^b_f + k'^b_b = 2k_f = q_c = k'^b_b - k'^b_f$. However, this is possible if and only if $k'^b_b = 0$. Then, we conclude that the many-body effects brought about by the nesting wave-vector $k'$ which connects the two Fermi points $k'^b_b$ and $k'^b_f$ gives rise a new effective $b$ species Fermi surface, $k'^b_{\text{eff}} = 2k_f$, with $k'^b_f = 0$. Indeed, it can be seen from the divergence of the particle–hole susceptibility and equation (10) that the effective interaction is $g_{\text{eff}} = 0$. This situation $q_c = 2k_f$, as we just verified, corresponds to the FP gas. This FP gas, which is equivalent to empty the band of down spins and accommodates all in band $b$, such that $k'^b_b = 0$ and $k'^b_f = 2k_f$, is reached for a field $h_f = \mu = \frac{1}{2}v_F k_F$, as illustrated in figure 2. Then for $h \geq h_f$, the FP system can be considered as non-interacting and it remains normal for $h > h_f$. For $h < h_f$, the system enters the FFLO phase.

The spin polarization is defined as $P = \frac{n^b - n^a}{n^b + n^a}$, where $n^b,a$ are the number densities. Since in a 1D system, we have $k_F^b = \frac{\pi}{2} n^b$, the polarization can be written as

$$P = \frac{(\mu + h)^{1/2} - (\mu - h)^{1/2}}{(\mu + h)^{1/2} + (\mu - h)^{1/2}}.$$  

For $h_f = \mu$, we have $P = 1$ as expected.
As we mentioned in the introduction section, exact results are obtained for 1D imbalanced Fermi gases using the Bethe ansatz within the Gaudin–Yang model. See, for instance [10, 33], where the ground-state energy expression for a homogeneous system is given in terms of spectral functions, which in turn are solutions of two coupled integral equations. The main results presented in [10, 33] for a homogeneous 1D imbalanced Fermi gas with the fixed total density $n = n_1 + n_1$ and density difference $s = n_1 - n_1$, with $0 < s < n$, are as follows. For $s = 0$, the ground state of the system is a fully paired (BCS) state. For $s = n$, the system is the FFLO gas consisting of solely $\uparrow$ fermions. And finally, for any $0 < s < n$, the gas is PP and is a superfluid of the FFLO type. These are exactly the same results we have obtained as a manifestation of the nesting effect. We stress that since the nesting effect is an intrinsic and universal phenomenon in 1D Fermi systems [36, 33], it should be properly considered, as we did here.

We conclude this section with the result that in the ideal case, where the quasi-particles have an infinite lifetime, the FFLO phase will occur for all $h < h_f$ and for any strength of the attractive interaction. Here, $h_f$ is the field above which the system is FP. The quantum phase transition in this case is directly from the FP state to a phase with FFLO correlations [33]. The nature of the quantum phase transitions in the pure case of infinite lifetime has been investigated by Guan and Ho [35] and at least for the case they are driven by changes in the chemical potential they belong to the universality class of density-driven transitions with dynamic exponent $z = 2$ and $v = 1/2$ as in the case of the repulsive 1D Hubbard model [42, 43].

4. Lifetime effects

In this section, we study LTs in the phase diagram of 1D attractive imbalanced Fermi gases. In cold-atom systems, the trap to confine the atoms gives rise to an inhomogeneous atomic distribution, which can be described, for example, by a chemical potential, which depends on the distance from the centre of the trap. In this case, since translation invariance is broken, the momentum or wave-vector $k$ is not the good quantum number to describe the quasi-particle states in the trap. However, it is still convenient to use this representation, in which case, it is appropriate to introduce a finite lifetime to these states. In the quasi-1D organic superconductors [27], LTs arise from impurities or defects.

Taking into account the finite lifetime of the quasi-particle states in the momentum representation, the particle–particle dynamic susceptibility can be written as [44]

$$\chi(q, \omega_n) = \frac{N(0)}{4} \int_0^{\omega_n} dx \left( \frac{1}{x + h + v_F q/2 - \omega_0/2 + i\gamma/2} \right. \right.$$  

$$+ \left. \frac{1}{x + h - v_F q/2 + \omega_0/2 - i\gamma/2} \right) + \frac{v_F q/2}{-v_F q/2},$$  

(18)

where $\gamma = \tau^{-1}$ is the inverse of the lifetime of a quasi-particle $q$-state in the normal phase. This approach is formally similar to that used to investigate the effect of non-magnetic impurities in higher dimensional (3D and 2D) FFLO superconductors [45, 46]. In this case, the main interest was to obtain the reduction in the critical temperature of the superconductor. Here, we will concentrate on the zero temperature phase diagram of the 1D system and will be able to fully characterize the new QCP that arises due to the LTs.

Since we are interested in studying the effect of this finite lifetime on quantum criticality, we start calculating the real part of the static particle–particle susceptibility:

$$\text{Re} \chi(q, \omega_n = 0) = \frac{N(0)}{4} \int_0^{\omega_n} dx \left[ \frac{x + h + v_F q/2}{(x + h + v_F q/2)^2 + \gamma^2/4} \right. \right.$$  

$$+ \left. \frac{x + h - v_F q/2}{(x + h - v_F q/2)^2 + \gamma^2/4} \right].$$  

(19)

This can be easily integrated and for $\omega_n \tau \gg 1$, we obtain

$$\text{Re} \chi(q, \omega_0 = 0) = N(0) \frac{\alpha}{h} = \frac{N(0)}{4} \ln \left[ \left(1 + \tilde{q}^2\right) + \frac{\gamma^2}{4h^2} \right] \times \left[ \left(1 - \tilde{q}^2\right) + \frac{\gamma^2}{4h^2} \right].$$  

(20)

The condition for the divergence of the interacting pair susceptibility becomes

$$g_s N(0) \left[ \ln \frac{2h}{\Delta_0} + \frac{1}{4} \ln \left[ \left(1 + \tilde{q}^2\right) + \frac{\gamma^2}{4h^2} \right] \right] \left[ \left(1 - \tilde{q}^2\right) + \frac{\gamma^2}{4h^2} \right] = 0,$$  

where $g_s$ is an effective (renormalized by the GMB correction) coupling constant. This condition is first satisfied when the argument of the logarithmic is maximum, that is, for $\tilde{q}_c = \sqrt{1 - \gamma^2/4h^2}$. The critical field is given by $h_c = h(\tilde{q}_c)$, where

$$\frac{2h(\tilde{q})}{\Delta_0} = \left[ \left(1 + \tilde{q}^2\right) + \frac{\gamma^2}{4h^2} \right] \left[ \left(1 - \tilde{q}^2\right) + \frac{\gamma^2}{4h^2} \right]^{1/4}.$$  

(22)

Substituting for $\tilde{q}_c$, we finally obtain

$$h_c = h(\tilde{q}_c) = \Delta_0^2/4\gamma,$$  

(23)

implying that in quasi-1D imbalanced Fermi systems, for sufficiently short lifetimes of the quasi-particle states, the FFLO phase appears only below a critical mismatch $h < h_c = \Delta_0^2/4\gamma$ (see figure 3).

Taking into account the results of the previous section, we can summarize our results as follows.
respectively. When the induced interactions are considered, on the boundary of the BCS phase may be described as of the magnetic field for the different approximations used here (see Figure 4.

Besides, they are not dependent on the Hamiltonian given by density fluctuations that go beyond the MF approximation.

For completeness, it is interesting to apply the Thouless criterion to determine the boundary of the homogeneous BCS phase in the case of a small field or mismatch. For this, it is sufficient to take $q = 0$ in equations (8) and (22), which yields $h^0_{c,\text{MF}} = \Delta_0/2$ [10, 33], and $h^0_{c,\text{LT}} \approx (\Delta_0/2)(1/(1 - (\gamma/\Delta_0)^2))^{1/4}$, for the pure (MF) and disordered cases, respectively. When the induced interactions are considered, $h^0_{c,\text{GMB}} = \Delta_0/2$, where the value of $\Delta$ is reduced, compared to the bare case as $\Delta_0 = 2\omega_c\exp(-1/N(0))|g_{\text{eff}}|$, and $g_{\text{eff}} = g/(1 - gN(0))$ is the interaction corrected by the GMB correction. Thus, the ‘hierarchy’ of the critical fields on the boundary of the BCS phase may be described as $h^0_{c,\text{GMB}} < h^0_{c,\text{LT}} < h^0_{c,\text{MF}} = \Delta_0/2$.

A zero temperature phase diagram showing the different phases as the field is increased is shown in figure 4 for the different cases studied here.

5. Nature of the transition at $h_c < h_f$

From here on, we will consider the case $h_c < h_f$ and study the quantum phase transition from the normal PP state to the FFLO phase. The real part of the dynamic susceptibility is obtained as

$$\text{Re} \chi(q, \omega_0) = N(0) \frac{2\omega_c}{\hbar} \ln \left\{ \frac{N(0)}{8} \right\} \times \ln \left[ \frac{(1 + \bar{q} - \bar{\omega}_0)^2 + \gamma^2/4}{(1 - \bar{q} - \bar{\omega}_0)^2 + \gamma^2/4} \right]$$

$$- \frac{N(0)}{8} \ln \left[ \frac{(1 - \bar{q} + \bar{\omega}_0)^2 + \gamma^2/4}{(1 + \bar{q} + \bar{\omega}_0)^2 + \gamma^2/4} \right] \times \left[ (1 + \bar{q} + \bar{\omega}_0)^2 + \gamma^2/4h^2 \right].$$

Expanding close to $\bar{q} = \bar{q}_c$ and $\omega_0 = 0$, we obtain

$$1 - g_r \text{Re} \chi(q, \omega_0) \approx g_r N(0) \ln \left( \frac{\hbar g_c}{\hbar_c} \right) \left( \bar{q} - \bar{q}_c \right)^2$$

$$+ \left( \frac{\hbar g_c}{\gamma} \right)^2 \bar{\omega}_0^2.$$ (25)

The imaginary part of the dynamic susceptibility is given by

$$\text{Im} \chi(q, \omega_0) = \frac{N(0)}{4} \int_0^{\omega_0} dx \left[ \frac{\gamma/2}{(x - \omega_0^2)} \right]$$

$$+ \frac{\gamma/2}{(x - \omega_0^2 + \hbar + \v_{\text{FFLO}}/2)^2 + \gamma^2/4} + (\omega_0 \rightarrow -\omega_0),$$ (26)

which can be easily integrated to give

$$\text{Im} \chi(q, \omega_0) = -\frac{N(0)}{4} \left[ \tan^{-1} \left( \frac{2(\bar{q} - \bar{\omega}_0)}{\gamma (1 - (\bar{q} - \bar{\omega}_0)^2)} \right) \right]$$

$$- \tan^{-1} \left( \frac{2(\bar{q} + \bar{\omega}_0)}{\gamma (1 - (\bar{q} + \bar{\omega}_0)^2)} \right),$$ (27)

where we recall, $\bar{q} = \v_{\text{FFLO}}/2\hbar$, $\bar{\omega}_0 = \omega_0/2\hbar$ and we defined $\gamma = \gamma/2\hbar$.

Expanding close to $\omega_0 = 0$, $\bar{q} = \bar{q}_c$, we obtain

$$\text{Im} \chi(q, \omega_0) = N(0) \frac{\hbar}{\gamma} \omega_0. $$ (28)

Then in the limit $\omega_0 \rightarrow 0$, the frequency-dependent part of the imaginary susceptibility dominates over the real part.

As concerns its quantum critical behaviour, the zero temperature phase transition from the normal-to-inhomogeneous superconductor state of the homogeneous quasi-1D system in the presence of a finite lifetime of the quasi-particle states can be described by the following effective action at the Gaussian level [38]:

$$S_{\text{eff}} = \int dQ \int d\omega_0 [\delta + Q^2 + |\omega_0|] \Delta(Q, \omega_0)^2,$$ (29)

where $\delta = h - h_c$ and $Q = Q(q - q_c)$. The QCP associated with this phase transition has a dynamic exponent $z = 2$, such that, its effective dimension, $d_{\text{eff}} = d + z = 3$ [47, 42]. Consequently, we expect the superfluid transition in the 1D system in the

\[\text{Figures and equations from the original text are omitted for brevity.}\]
presence of a finite lifetime to be in the universality class of the 3D XY model [42] due to the two-component nature of the superfluid order parameter. Effects of temperature [48] can also be obtained from the knowledge of the critical exponents of the QCP. In this case, the finite-temperature critical line is obtained as \( T_c \propto |h - h_c|^z \), where \( v \approx 2/3 \) is the correlation length exponent of the 3D-XY model [49] and \( z = 2 \), as obtained previously. Since there is no long-range magnetic order in \( d = 1 \) at finite temperatures, this line in practice provides the temperature scale below which the FFLO correlations become important and this varies with the distance to the QCP. Since \( (\partial T_c/\partial h)_{h=h_c} = 0 \), the characteristic temperature \( T_c \) turns out to be very small near the critical field.

The identification of the universality class of the lifetime-induced QCP as being that of the 3D XY model also allows us to obtain the behaviour of the correlation function of the FFLO fluctuations at \( T = 0 \). Using that the exponent \( z \) for the order parameter correlation function of the 3D XY model takes the value \( z = 0.0381 \) [50], we find that at the QCP, the FFLO correlation function decays with distance \( r \), as \( G(r) \propto 1/r^{2+2z} \) [42], i.e., \( G(r) = 1/r^{1.038} \). This exponent turns out to be small or of the same order of that obtained numerically for these types of correlations using the Bethe ansatz (see [29] and [33]).

6. Discussion and conclusion

In spite of theoretical predictions and intensive experimental activity, the FFLO phase remains elusive. Motivated by experimental results in cold-atom systems and aiming to understand the reasons for the difficulty in observing this phase, we have carried out a detailed study of an FFLO phase in 1D systems, which provide the most favourable conditions for the appearance of this phenomenon. We have considered the effect of the induced interaction in an ideal gas to show how the result which predicts long-range FFLO correlations in the presence of an arbitrarily large magnetic field or mismatch is modified, leading to a finite critical field \( h_f \). In this field, a nesting condition is satisfied and long-range FFLO correlations set in. In accordance with the exact results, the system at \( h_f \) goes directly from a regime with strong FFLO correlations \( (h < h_f) \) to the FP normal phase \( (h > h_f) \).

In systems, with additional interactions not included in the pairing Hamiltonian, or in the presence of artificial disorder, a finite lifetime of the quasi-particle excitations may be considered. We have shown how the LT modifies the \( T = 0 \) phase diagram in one dimension. It gives rise to a new characteristic or a critical field \( h_c \), which depends on the \( h = 0 \) BCS gap and on the lifetime, \( \tau = 1/\gamma \) of the states (equation (23)). If a disorder is weak, such that, \( h_c > h_f \), then it is irrelevant and the transition with increasing field is from the FFLO phase to the FP system at \( h_f \). However, for a strong disorder \( (h_c < h_f) \), there is a new QCP in the system that we have fully characterized. In the case of increasing mismatch, the system goes from the FFLO phase to a normal PP phase at \( h = h_c \) and finally to an FP phase at \( h_f \). We have fully characterized the QCP at \( h_c \). In this case, the region of the phase diagram where the FFLO phase appears is reduced.

We have also applied the Thouless criterion to determine the boundary of the homogeneous BCS phase in the case of the small-field \( h^2 \) in the various approaches we considered namely MF and MF corrected by induced interactions (GMB), and MF considering LTs. We have found that the transition to the BCS phase occurs at \( h_{0,MF}^2 \approx h_{0,MF}^2 < h_{0,GMB}^2 \). We hope our results will stimulate further experiments to confirm unambiguously the existence of the FFLO phase in quasi-1D imbalanced Fermi gases.

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Note added in proof. After the completion of this work, we became aware of recent papers that investigated imbalanced fermionic superfluids in arrays of 1D tubes, allowing inter-tube tunneling [51, 52]. They found that the evolution of the physical properties between 1D and 3D (including the inverted phase profiles) can be well described at an MF level. It would be very interesting to consider both the effects of induced interactions and finite lifetime of the particles in the normal phase in the systems considered in the references above.

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