PROPERTIES OF THE CONNECTION ASSOCIATED WITH PLANAR WEBS AND APPLICATIONS

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Abstract. We give various results and applications using the connection \((E, \nabla)\) associated with a \(d\)-web in \(\mathcal{H}_{04}\). More precisely, we exhibit fundamental \(\mathcal{O}^*\)-invariants of the web related to the differential equation of first order which presents the web. They cast some new lights on the connection and its construction, both conceptually and effectively. We describe 4 and 5webs from this point of view and show for instance that the connection gives account for the linearizability conditions of the web. Moreover, we get characterization of maximal rank webs such as exceptional 5webs, and 4webs via a new proof of the Poincaré theorem in terms of differential systems. We establish the trace formula related to the determinant bundle \((\det E, \det \nabla)\) and the extracted 3webs. Furthermore, the theorem of determination of the rank is proved to give an explicit criterion for measuring the rank of a web.

1. Introduction

Planar webs geometry is dedicated to the study of classes of first order ordinary differential equations

\[ F(x, y, y') := a_0(x, y) \cdot (y')^d + a_1(x, y) \cdot (y')^{d-1} + \cdots + a_d(x, y) = 0 \]

with coefficients in the ring \(\mathcal{O} = \mathbb{C}\{x, y\}\) of convergent power series in two variables, up to an element in the group \(\mathcal{O}^*\) of invertible in \(\mathcal{O}\). Particularly, we are interested in the study of specific relations between the solutions of these equations, called the abelian relations.

These differential equations of degree \(d\) admit \(d\) solutions out of the singular locus given by their \(y'\)-resultant. They are called the leaves of the web.

A basic problem in web geometry is to count the number of linearly independent abelian relations, which is the rank of the web, and to determine the webs of maximal rank. The connection \((E, \nabla)\) we deal with here answers to this last problem, but not only, as we will prove it in this paper. Questions about the simultaneous linearizability of the leaves of the web also appear naturally and we will show how they relate to the first ones.

According to the classical definition of webs, the leaves of a planar \(d\)-web can equivalently be given by \(d\) foliations defined by the level sets \(F_i(x, y) = \text{constant}\) for \(1 \leq i \leq d\) where the \(F_i\) are in \(\mathcal{O}\) in general position, with \(F_i(0) = 0\). An abelian relation is then a relation of the form

\[ \sum_{i=1}^{d} g_i(F_i)dF_i = 0 \]

where the \(g_i\) are analytic in one variable.

The pioneers of web geometry are W. Blaschke and G. Bol in the thirties (B-B). Then the subject get another raise thanks to S. S. Chern and P. A. Griffiths in the...
seventies (C-G). Nowadays, the subject is still up to date with the works of M. A. Akivis, V. V. Goldberg and V. V. Lychagin (See for instance Go-L), I. Nakai, E. Ghys, D. Cerveau and A. Hénaut, more recently D. Lehmann and V. Cavalier (C-L), L. Pirio and J.-M. Trépreau (P or P-T), G. Robert, J. V. Pereira and D. Marin (M2P).

Thus, the subject has drawn links with many subjects such as foliations, $D$-modules, polylogarithms, differential equations and Cartan-Spencer techniques for instance and could meet now some topics like differential algebra (see for instance R-S-1) on singular solutions of planar webs or R-S-2 on Darboux polynomials), differential Galois theory and meromorphic connections (see H-07). But web geometry has also some applications in applied mathematics. Non linear optic geometry (see for instance the article of J.-L. Joly, G. Métivier and J. Rauch in W) but also economy use web geometry where the question of finding the rank of the web is crucial.

This paper aims to give new conceptual and effective results by using the connection associated with webs. Let us describe the content of this paper. We first introduce shortly the tools used here by giving a description of the abelian relations of a web via the data of the first order differential equation which presents the web. The connection is also described.

We then introduce the associated polynomials in theorem 3.1 which lead us to the fundamental $O^*$-invariants of the web. They are closely linked with the geometry of the differential equation. Those $O^*$-invariants will give significative results in association with the connection. But first, we show how they relate to linear and algebraic webs and how they can be linked with abelian relations. We then reduce their number by showing that they can only be given by the data of a particular polynomial and a fundamental 1-form. These will be at the centre of this study.

We then fully construct the connection associated with 4 and 5-webs. We show that it gives account of the linearizability conditions (theorem 4.1) and makes it clearer that if the Poincaré theorem is true for 4-webs, as we prove it in theorem 4.2, it is not the same for 5-webs. By the way, the obstruction is given in this case (theorem 5.2). Hence, we give a new conceptual description of exceptional 5-webs (proposition 5.1).

The trace formula is then proved (theorem 6.1). For this, we need a few propositions which state the link between the connections associated with extracted webs and the connection of the initial web. They focus on the nature of the abelian relations. Then the proof is completed.

We end with the theorem of determination of the rank (theorem 7.1). It gives explicitly the locally free $O$-modules of finite type deduced from the local system of the abelian relations. This fibre bundle is thus generated by the abelian relations of the web. The particular form of the connection allows to embody this $O$-modules so that it is now possible to compute the rank without searching the abelian relations, as it was done before in general. Then an exemple is given.

2. Objects and tools

References for this introduction of our objects and tools are the founder books B-B or B of W. Blaschke and G. Bol, the article of S. S. Chern C and the book W. Unless further specifications, the notations adopted for the sequel of this article are fixed in this part.

Definition 2.1. A non singular planar $d$-web $W(d)$ is defined by the family of the solutions of an ordinary differential equation of the first order

$$F(x, y, y') := a_0(x, y) \cdot (y')^d + a_1(x, y) \cdot (y')^{d-1} + \cdots + a_d(x, y) = 0$$
which is a \( y' \)-polynomial with coefficients in \( \mathcal{O} \), in a neighborhood of a point where the \( y' \)-resultant \( R(x, y) \) of \( F \) and \( \partial_y(F) \) is non zero.

We say that a \( d \)-web is \textit{presented} by such a differential equation. In the sequel, a \( d \)-web will always be presented by such a differential equations, denoted \( F = 0 \), if it is not otherwise specified. An \( \mathcal{O}^* \) invariant of a web \( \mathcal{W}(d) \) only depends on the class of differential equations, modulo an invertible in \( \mathcal{O} \) and not on a particular equation in this class. We could easily justify this definition of invariance by saying that the natural condition we could expect from our objects is that as \( F = 0 \) and \( \rho F = 0 \) present the same web if \( \rho \) is invertible in \( \mathcal{O} \), they do not depend on the presentation of the web.

A theorem of Cauchy asserts that in a neighborhood of a point \((x_0, y_0)\) such that \( R(x_0, y_0) \neq 0 \), the equation admits \( d \) integral curves, which are the leaves of the \( d \)-web. Close to this point, the polynomial \( F(x, y, p) \) admits \( d \) distinct roots which are the slopes of the leaves, usually noted \( p_i(x, y) \) for \( 1 \leq i \leq d \). We will denote by \( \mathcal{W}(F_1, F_2, \ldots, F_d) \) a \( d \)-web given by the level functions \( F_1, F_2, \ldots, F_d \), as it was defined in the introduction. Thus, if we set \( p_i := -\partial_x(F_i)/\partial_y(F_i) \), then \( \mathcal{W}(F_1, F_2, \ldots, F_d) \) is presented by the differential equation \( \prod_{i=1}^d (y' - p_i) = 0 \).

We will use here the word web instead of non singular web, which is more convenient, since we will always be in the neighborhood of a point where the web is defined, that is to say where the \( y' \)-resultant of \( F \) is non zero.

**Definition 2.2.** The \( \mathbb{C} \)-vector space defined by

\[
\mathcal{A}(d) = \left\{ (g_1(F_1), \ldots, g_d(F_d)) \in \mathcal{O}^d \text{ with } g_i \in \mathbb{C} \{t\} \text{ and } \sum_{i=1}^d g_i(F_i) dF_i = 0 \right\}
\]

is called the space of abelian relations of the \( d \)-web \( \mathcal{W}(d) \)

We have the following theorem:

**Theorem 2.1** (Bol 1932, Blaschke 1933). The dimension of \( \mathcal{A}(d) \) is finite and called the rank of the web. It is an \( \mathcal{O}^* \)-invariant of the web and more, we have the following optimal inequality:

\[
\text{rk} \ \mathcal{W}(d) := \dim_\mathbb{C} \mathcal{A}(d) \leq \frac{1}{2} (d - 1)(d - 2).
\]

Note that in the sequel, \( d \) will always be an integer greater or equal to 3, since abelian relations are trivial otherwise. We denote by \( \pi_d \) the integer \( \frac{1}{2} (d - 1)(d - 2) \).

Taking an example, consider the 3-web \( \mathcal{H} = \mathcal{W}(x, y, x + y) \), given by the level sets \( x = \text{constant}, y = \text{constant} \) and \( x + y = \text{constant} \). The rank of \( \mathcal{H} \) is maximal, equal to 1 since \( dx + dy - d(x + y) = 0 \) is a non trivial abelian relation.

For a \( 3 \)-web \( \mathcal{W}(F_1, F_2, F_3) \), we know with the work of Blaschke that there exists a differential 1-form \( \gamma \) with coefficients in \( \mathcal{O} \) such that \( d\omega_i = \gamma \wedge \omega_i \) where \( \omega_i = \rho_i dF_i \), with \( \rho_i \in \mathcal{O}^* \) chosen such that the normalization \( \omega_1 + \omega_2 + \omega_3 = 0 \) holds. The 2-form \( d\gamma \) is, unlike \( \gamma \), an \( \mathcal{O}^* \)-invariant of the web called the Blaschke curvature of \( \mathcal{W}(3) \). We then have the following equivalence \( \text{rk} \ \mathcal{W}(3) = 1 \ if \ and \ only \ if \ d\gamma = 0 \).

Our interest is specially turned on natural classes of webs. For instance, one can consider the class of linear webs, whose leaves are germs of straight lines. Linearizable webs are those for which there exists a change of coordinates which transforms the web to a linear one. Among them, one can consider algebraic webs. Given a reduced algebraic curve \( C \) in \( \mathbb{P}^2 \) of degree \( d \), a straight line cuts generically \( C \) in \( d \) points. By duality, we get a \( d \)-web whose leaves are tangent to the dual curve of \( C \), provided that \( C \) does not contain straight lines. Webs constructed thanks to an algebraic curve are algebraic ones and algebrizable webs are those for which there exists a change of coordinates which transform them in algebraic ones. One can see
that Abel’s theorem states that the rank of algebraic webs is maximal. Thanks to an Abel-inverse type theorem (cf. [G-02]), we get the following:

**Theorem 2.2** (Lie-Darboux-Griffiths). A linear d-web \( L(d) \) admitting an abelian relation whose terms are all different from zero, is algebraic. In particular, a linearizable d-web with maximal rank is algebrizable.

A web of maximal rank is not always linearizable if \( d \) is greater or equal to 5. Such a web will be called exceptional. Several exceptional webs have been discovered in the last years by Robert and Pirio-Trépreau for instance (see [P] and [P-T-T]), with the consequence to tone down the range of the word exceptional but among them, one remains so. It is the first example of such a web, given by G. Bol in 1936. The leaves of this 5-web at a point is the only one to be also hexagonal: all its extracted 3-webs are of maximal rank. One can refer to the article of P. A. Griffiths [G-02] for a prospective survey of the link between polylogarithms and exceptional webs.

With the setting adopted here, the \( \mathbb{C} \)-vector space \( A(d) \) of the abelian relations of a web is described in [H-04] as follow: let \( S = \{ F(x, y, p) = 0 \} \) be the surface defined in \( \mathbb{C}^3 \) by \( F \) with the projection \( \pi \) on the \( (x, y) \) plane. There exists a \( \mathbb{C} \)-isomorphism between \( A(d) \) and the space of 1-forms

\[
a_F = \{ \omega = (b_3 \cdot p^{d-3} + b_4 \cdot p^{d-4} + \cdots + b_d) \cdot \frac{dy - pdx}{\partial_p(F)} \in \pi_*(\Omega^1_S), \ b_i \in \mathcal{O} \text{ and } d\omega = 0 \}\]

where \( (\Omega^*_S, d) \) is the usual de Rham complex, with \( \Omega^*_S = \Omega^*_{\mathbb{C}^3} / (dF \wedge \Omega^*_{\mathbb{C}^3} - 1, F.\Omega^*_{\mathbb{C}^3}) \).

Such a 1-form \( \omega = r \cdot \frac{dy - pdx}{\partial_p(F)} \) is in \( \Omega^1_S \) if and only if there exists elements \( r_p \) in \( \mathcal{O}[p] \) of degree less or equal to \( d - 1 \) in \( p \), and \( t := t_2 \cdot p^{d-2} + \cdots + t_d \) with coefficients in \( \mathcal{O} \) such that

\[
r.(\partial_x(F) + p.\partial_y(F)) + r_p.\partial_p(F) = (\partial_x(r) + p.\partial_y(r) + \partial_p(r_p) - t).F
\]

One then gets with the previous relation that

\[
d\omega = t \cdot \frac{dy - pdx}{\partial_p(F)}
\]

So, the form is closed if and only if \( t = 0 \) that is to say that there exists \( r_p \) in \( \mathcal{O}[p] \) of degree at most \( d - 1 \) such that

(\( \ast \)) \quad \( r.(\partial_x(F) + p.\partial_y(F)) + r_p.\partial_p(F) = (\partial_x(r) + p.\partial_y(r) + \partial_p(r_p)).F \)

holds. It is also the same to say that the \( b_i \) are analytic solutions of the following homogeneous differential systems \( M(d) \), where the left members are the coefficients \( t_2, \ldots, t_d \) of the polynomial \( t \), equal to zero in this case:

\[
M(d) \begin{cases}
\partial_x(b_3) + A_{1,1} \cdot b_3 + \cdots + A_{1,d-2} \cdot b_d &= 0 \\
\partial_x(b_{d-1}) + \partial_y(b_d) + A_{2,1} \cdot b_3 + \cdots + A_{2,d-2} \cdot b_d &= 0 \\
\vdots \\
\partial_x(b_d) + \partial_y(b_4) + A_{d-2,1} \cdot b_3 + \cdots + A_{d-2,d-2} \cdot b_d &= 0 \\
\partial_y(b_3) + A_{d-1,1} \cdot b_3 + \cdots + A_{d-1,d-2} \cdot b_d &= 0
\end{cases}
\]

The coefficients \( A_{i,j} \) are in \( \mathcal{O}[1/\Delta] \), where \( \Delta \) is the \( y' \)-discriminant of \( F \). So the space of abelian relations of the web is identified with the space of solutions of the differential system \( M(d) \). By the nature of this system, it is actually a local system, and so for the space of abelian relations, seen as a sheaf over \( \mathbb{C}^2 - \{ \Delta = 0 \} \).
The exterior differential on $S$ induces a linear differential operator
\[ \rho : \mathcal{O}^{d-2} \rightarrow \mathcal{O}^{d-1} \]
with the $t_i$ being given by the $d-1$ equations of $\mathcal{M}(d)$.

The study of this system leads us to consider, with the notations defined in [BC53G], the jet space $J_k(\mathcal{O}^{d-2})$ of order $k$ over $\mathcal{O}^{d-2}$ and $j_k : \mathcal{O}^{d-2} \rightarrow J_k(\mathcal{O}^{d-2})$ the natural derivation map. From this differential operator, one gets the $\mathcal{O}$-morphism $p_0 : J_1(\mathcal{O}^{d-2}) \rightarrow \mathcal{O}^{d-1}$ and its successive prolongations $p_k$. Let $R_k$ be the kernel of $p_k$. The Spencer complex associated with the prolongations $p_k$ is then given by
\[ 0 \rightarrow \text{Sol } \mathcal{M}(d) \xrightarrow{j_{k+1}} R_k \xrightarrow{D_1} \mathcal{O}^1 \otimes_{\mathcal{O}} R_{k-1} \xrightarrow{D_2} \mathcal{O}^2 \otimes_{\mathcal{O}} R_{k-2} \rightarrow 0. \]

Using properties of these objects which will be detailed in the section 3 for 4webs, the main result of [H-04] proves the existence of a $\mathbb{C}$-vector bundle $E := R_{d-3}$ included in $J_{d-2}(\mathcal{O}^{d-2})$ of rank $\pi_d$ on $(\mathbb{C}^2, 0)$ which admits a connection
\[ \nabla : E \rightarrow \mathcal{O}^1 \otimes_{\mathcal{O}} E \]
such that the space $\text{Ker } \nabla$ of its horizontal sections is isomorphic to $\mathcal{M}(d)$. Moreover, the curvature of $(E, \nabla)$ takes its values in $\mathcal{O}^2 \otimes_{\mathcal{O}} g \subset \mathcal{O}^2 \otimes_{\mathcal{O}} E$ where $g$ is a free $\mathcal{O}$-module of rank one over $(\mathbb{C}^2, 0)$.

Note that this connection is a meromorphic connection, with poles on the $y'$-discriminant of $F$.

In the case of 3-webs, we get a natural normalization of the webs by considering special 1-forms on the surface $S$, so that we can prove that the curvature of the connection thus constructed is exactly the Blaschke curvature of the 3-web. So the curvature extends for $d$-webs the Blaschke curvature of 3-webs. In the sequel, referring to the Blaschke curvature of a 3-web will mean that we consider the curvature of the connection associated to a 3-web.

3. Associated polynomials

3.1. Introduction. We are looking for $\mathcal{O}^*$-invariants both linked to the differential equation and the abelian relations of the web presented by this equation. For instance, this will potentially allows us to join together conditions of ranking and linearizability. Still, we want to get a minimal system of $\mathcal{O}^*$-invariants which will describe the web as completely as possible. These $\mathcal{O}^*$-invariants will be deduced from the so called associated polynomials, whose existence is given by the following theorem.

**Theorem 3.1** (Associated polynomials). Let $W(d)$ be a d-web presented by a differential equation $F = 0$. There exists two polynomials $U$ and $V$ in $p$ respectively of degree $d-2$ and $d-1$ whose coefficients admit poles on $R = \text{Result}(F, \partial_y(F))$, such that the following equalities hold:
\[ \partial_z(F) + p \partial_y(F) = U \cdot F + V \cdot \partial_p(F) \]
Moreover such an expression is unique: if two polynomials $\bar{U}$ and $\bar{V}$ in $p$ respectively of degree $d-2$ and $d-1$ satisfy $(\varphi)$, then $U = \bar{U}$ and $V = \bar{V}$.

**Proof.** Such polynomials $U := u_2 \cdot p^{d-2} + \ldots + u_d$ and $V := v_1 \cdot p^{d-1} + \ldots + v_d$ must satisfy a system $S(\varphi)$, deduced from $(\varphi)$. If $R$ stands for the Sylvester square matrix of order $2d - 1$ whose determinant is $R$, the system is given by
\[ S(\varphi) = R^{\dagger} \begin{pmatrix} u_2, & \ldots & u_d, & v_1, & \ldots & v_d \end{pmatrix} = t(0, \ldots, 0, \partial_y(a_0), \partial_x(a_0) + \partial_y(a_1), \ldots, \partial_x(a_{d-1}) + \partial_y(a_i), \ldots, \partial_x(a_d)) \]
We then get $(\varphi)$ and its uniqueness via Cramer's rule. \qed
In this article we fix the following notations:

\[ U = u_2 \cdot p^{d-2} + \ldots + u_d \quad \text{and} \quad V := v_1 \cdot p^{d-1} + \ldots + v_d. \]

**Corollary 3.2.** For all \( 1 \leq i \leq d - 3 \), there exists two polynomials \( U_i \) and \( V_i \) respectively of degree \( d - 2 \) and \( d - 1 \) whose coefficients admit poles on \( R \), such that the following equality hold:
\[
(\circ_i) \quad p^i \cdot (\partial_x(F) + p\partial_y(F)) = U_i \cdot F + V_i \cdot \partial_p(F)
\]
Moreover such an expression is unique.

The proof of this corollary is the same as the previous one. For \( 0 \leq i \leq d - 3 \), the couples of polynomials

\[
U_i := u_i^1 \cdot p^{d-2} + \ldots + u_i^d 
\quad \text{and} \quad
V_i := v_i^1 \cdot p^{d-1} + \ldots + v_i^d
\]
are called the associated polynomials of \( F \) of order \( i \), the associated polynomials of order 0 being the couple \((U, V)\).

### 3.2. Associated polynomials as web’s \( O^\ast \)-invariants. The associated polynomials are linked to the web thanks to the following properties:

**Proposition 3.1.** Let \( W(d) \) be a \( d \)-web presented by \( F = 0 \) and let \((U_i^g, V_i^g)\) be the associated polynomials of order \( i \). If \((U_i^g, V_i^g)\) are the associated polynomial of the equation \( g \cdot F = 0 \), where \( g \) is invertible in \( \mathcal{O} \), we then have the following relations:

\[
\begin{align*}
U_i^g &= U_i^F + \frac{1}{g} \cdot p^i \cdot (\partial_x(g) + p\partial_y(g)) \\
V_i^g &= V_i^F
\end{align*}
\]

**Proof.** We need to write the relation we get from theorem 3.1:

\[
p^i \cdot (\partial_x(g \cdot F) + p\partial_y(g \cdot F)) = U_i^g \cdot g \cdot F + V_i^g \cdot \partial_p(g \cdot F), \]
that is to say

\[
p^i \cdot (\partial_x(g) + p\partial_y(g))F + p^i \cdot (\partial_x(F) + p\partial_y(F)) = gU_i^g \cdot F + V_i^g \cdot g\partial_p(F). \]

The equalities result from the uniqueness obtained in theorem 3.1 and its corollary.

So, the polynomials \( V_i \) are \( O^\ast \)-invariants of the web, as the following coefficients of \( U_i \): \( u_i^2, \ldots, u_d - d - 1, u_d - d, \ldots, u_d \) where the hat means that the coefficients are omitted.

Moreover, for \( 0 \leq k \leq d - 2 \), the \((d - 3)(d - 2)\) differences

\[
\begin{align*}
u_i^{d-1} - u_i^d & \quad \text{for} \quad 2 \leq i \leq d - 2 \\
u_i^{d-j+1} - u_i^{d-j} & \quad \text{for} \quad 3 \leq j \leq d - 1
\end{align*}
\]
and the \( d - 2 \) forms \( d(u_d^{i-1}dx + u_d^{i-1}dy) \) are \( O^\ast \)-invariants of the web.

### 3.3. Linear and algebraic webs. We will begin this section with results concerning the linearizability of the webs.

Let \( W(d) \) be a planar \( d \)-web whose slopes of the leaves are denoted for \( 1 \leq i \leq d \) by \( p_i \in \mathcal{O} \). There exists a unique polynomial of degree less or equal to \( d - 1 \)

\[ P_{W(d)} := l_1 \cdot p^{d-1} + l_2 \cdot p^{d-2} + \ldots + l_d \]
whose coefficients are in \( \mathcal{O} \), such that the following equality holds for all \( i \):

\[ X_i(p_i) := \partial_x(p_i) + p_i\partial_y(p_i) = P_{W(d)}(x, y, p_i(x, y)). \]

Then, the graphs of the leaves are solutions of the equation \( y'' = P_{W(d)}(x, y, y') \).

The following properties of \( P_{W(d)} \) are due to Hénaut:

**Properties 1.**

(1) The web \( W(d) \) is linear if and only if \( X_i(p_i) = 0 \) for all \( i \), if and only if \( P_{W(d)}(x) = 0 \) ;
(2) For $d \geq 4$, the web $W(d)$ is linearizable if and only if $\deg(P_{W(d)}) \leq 3$ and $(l_d,l_{d-1},l_{d-2},l_{d-3})$ is a solution of the nonlinear differential system:

$$
\begin{align*}
L_1 &= -\partial_x(\partial_y(l_{d-2}) - 2\partial_y(l_{d-1}) - l_{d-1}(\partial_x(l_{d-2}) - 2\partial_y(l_{d-1})) - 3\partial_y^2(l_d) \\
&= -3\partial_y(l_{d-2}l_d) + 3\partial_x(l_{d-2}l_{d-3}) + 3l_d\partial_x(l_{d-3}) = 0 \\
L_2 &= \partial_y(\partial_x(l_{d-2}) - \partial_y(l_{d-1})) - l_{d-2}(2\partial_x(l_{d-2}) - \partial_y(l_{d-1})) - 3\partial_y^2(l_{d-3}) \\
&= +3\partial_y(l_{d-1}l_{d-3}) - 3\partial_y(l_{d-2}l_{d-3}) - 3l_d^2\partial_y(l_d) = 0.
\end{align*}
$$

The polynomial $P_{W(d)}$ is hence called the linearization polynomial. It appears to be an important $O^*$-invariant of the web, as we will see. One can compare this property with webs linearization results of Akivis, Goldberg and Lychagin (cf. [Gd]).

**Proposition 3.2.** For $0 \leq k \leq d - 3$ and all $1 \leq i \leq d$, we have the following equality: $V_k(x,y,p_i) = -(p_i)^k X_i(p_i)$. In particular, $P_{W(d)} = -V$

**Proof.** Since $F = \prod_i^d(p - p_i)$ presents the web, we have the following equalities:

$$
p^k(\partial_x(F) + p\partial_y(F)) = p^k\sum_{i=1}^d -\partial_x(p_i) + p\partial_y(p_i)) \prod_{j=1,j\neq i}^d(p - p_j)
$$

and

$$
p^k(\partial_x(F) + p\partial_y(F)) = U_k.F + V_k\sum_{i=1}^d \prod_{j=1,j\neq i}^d(p - p_j).\quad \text{Let } p = p_i, \text{ we have then } V_k(x,y,p_i) = -(p_i)^k X_i(p_i).
$$

By the uniqueness of $P_{W(d)}$, the second equality follows.

As a consequence of the properties of linear webs given in the beginning of this subsection, we have the following corollary:

**Corollary 3.3 (Linear webs).** A $d$-web is linear if and only there exists an integer $0 \leq k \leq d - 3$ such that $V_k = 0$.

These are geometric properties, since they are expressed with $O^*$-invariants of the web $W(d)$.

It can be checked that an algebraic web is presented by an equation

$$
F(x,y,y') = g \cdot P(y - y'x,y') = 0
$$

where $P \in \mathbb{C}[s,t]$ is an affine equation of the reduced algebraic curve which defines the web, and $g$ is invertible in $O$. As linear webs, the polynomials $V_i$ are equal to zero, and we get an additional condition for such a web, given in the following theorem:

**Proposition 3.3 (Algebraic webs).** Let $W(d)$ be a $d$-web presented by a differential equation $F(x,y,y') = 0$ and let $(U,V)$ be its associated polynomials. We have the following equivalences:

i) $W(d)$ is algebraic;

ii) There exists $\phi$ in $O$ such that $V = 0$ and $U = \partial_y(\phi)p + \partial_x(\phi)$;

**Proof.** If $W(d)$ is algebraic, an equation $F(x,y,p) = e^\phi P(y - px, p)$ presents the web, where $\phi$ is in $O$ and $P$ in $\mathbb{C}[s,t]$. So $\partial_x(P(y - px, p)) + p\partial_y(P(y - px, p)) = 0$ and, according to theorem 3.1 we get the properties ii).

Conversely, let $G(x,y,p) = e^{-\phi}F(x,y,p)$. Since $V(F) = 0$ by hypothesis, then $V(G) = 0$. Moreover, we have $U(G) = U(F) + e^\phi(\partial_x(e^{-\phi}) + p\partial_y(e^{-\phi})) = 0$ and so $\partial_x(G) + p\partial_y(G) = 0$. The polynomials $y - px$ and $p$ are linearly independent solutions of the previous equation. By Frobenius theorem, there exists an analytic function $\gamma$ in $O$ such that $G(x,y,p) = \gamma(y - px, p)$. In fact, $\gamma$ is a polynomial in $p$ of degree $d$ since we can show that for its partial derivatives of order greater than $d$ are equal to zero. The second equivalence is a consequence of the preceding one.

This proposition is closely related to the Lie-Darboux-Griffiths theorem. The condition that the web is linear is expressed by the cancelation of $V$. The maximal...
rank condition can not be yet interpreted as the condition on \( U \), but we will see that it is the case (cf. proposition 3.5).

**Remark 3.1.** The associated polynomials \( (U, V) \) allow us to find specific singular solutions of the differential equation \( F = 0 \). Moreover, in the case where such a singular solution exist, the connection does not admit poles on the locus of this solution. These results were obtained by J. Sebag and the author in [R-S] by crossing the classical results on singular solutions given by G. Darboux, and those of J. F. Ritt and E. R. Kolchin in differential algebra, around the \( y' \)-resultant of \( F \) which is omnipresent in our constructions.

### 3.4. Link with the abelian relations.

The following lemma will make a bridge between those associated polynomials and the abelian relation of the web.

**Lemma 3.4.** With the previous notations, let \( r := r(x, y, p) = b_3 \cdot p^{d-3} + \ldots + b_d \) with coefficients in \( \mathcal{O} \). The polynomials \( U_r = b_3 \cdot U_{d-3} + \ldots + b_d \cdot U \) and \( V_r = b_3 \cdot V_{d-3} + \ldots + b_d \cdot V \) are such that \( r \cdot (\partial_x(F) + p\partial_y(F)) = U_r \cdot F + V_r \cdot \partial_y(F) \).

Then, a 1-form \( \omega = (b_3 \cdot p^{d-3} + b_4 \cdot p^{d-4} + \ldots + b_d) \cdot \frac{dy - pdx}{\partial_y(F)} \in \pi_s(\Omega^1_S) \) belongs to \( a_F \) if and only if \( r \) satisfies the equation \( U_r + \partial_y(V_r) = \partial_x(r) + p\partial_y(r) \).

Indeed, the first part is a direct consequence of the uniqueness property in theorem 3.1 and its corollary. The relation (*) given in section 2 gives the second part, again using the uniqueness property.

This allows us to express the system \( \mathcal{M}(d) \) in terms of our associated polynomials.

After a short computation we get the following expression for \( \mathcal{M}(d) \):

\[
\begin{align*}
\partial_x(b_d) - (u_d^{d-3} + v_{d-1}^{d-3})b_3 - \ldots - (u_{d-1})b_d &= 0 \\
\vdots \\
\partial_x(b_{d+1-i}) + \partial_y(b_{d+2-i}) - \ldots - (u_{d+1-i} + i \cdot v_{d-i-1})b_j \ldots &= 0 \\
\vdots \\
\partial_y(b_3) - (u_2^{d-3} + (d-1)v_1^{d-3})b_3 - \ldots - (u_2 + (d-1)v_1)b_d &= 0
\end{align*}
\]

**Proposition 3.4.** Let \( \alpha = A_1 dx + A_2 dy := A_{1,d-2} dx + A_{2,d-2} dy \). We have the following equality:

\[
\alpha = \left(-\frac{\partial_x(a_0)}{a_0} - \frac{\partial_y(a_1)}{a_0} + \sum_{i=1}^{d-1} v_i \sum_{k=1}^{d} p_k^{d-1-i} \right) dx + \left(-\frac{\partial_y(a_0)}{a_0} + \sum_{i=1}^{d-2} v_i \sum_{k=1}^{d} p_k^{d-2-i} \right) dy.
\]

The proof is a direct computation, using the writing of \( \alpha \) with our \( \mathcal{O}^* \)-invariants:

\[
\alpha = -(u_d + v_{d-1})dx - (u_{d-1} + (d-2)v_{d-2})dy.
\]

The form \( \alpha \) will be called the fundamental 1-form. Its differential is an \( \mathcal{O}^* \)-invariant of the web with proposition 3.1. This form will play an important part in the sequel, justifying the distinction we make.

For \( d = 3 \), the system \( \mathcal{M}(3) \) is given by the coefficients \( A_{11} = A_1 \) and \( A_{21} = A_2 \). The associated connection is a 1-form, which, in a suitable basis is the fundamental 1-form \( \alpha = A_1 dx + A_2 dy \). It has been shown in [H-04] that its curvature is then the Blaschke curvature \( d\gamma \) of the 3-web, which emphasizes on the importance of \( \alpha \).

So, in the case of a 3-web the differential of the fundamental form is the Blaschke curvature.

We are now looking for a minimal set of \( \mathcal{O}^* \)-invariants. The next theorem gives us such a minimal set, which will play a central part in this article.

**Theorem 3.5.** The system \( \mathcal{M}(d) \) and consequently, the connection, can only be written thanks to \( \alpha \) and \( V \). Moreover, \( dx \) and \( V \) are \( \mathcal{O}^* \)-invariants of the web.
Proof. First we can deduce all the polynomials \( V_k \) from the polynomial \( V \) and \( F \). Let us remark that for all \( 1 \leq k \leq d-3 \), we have the following polynomial equalities:

\[
(V_k - pV_{k-1})(x, y, p) = -\frac{v_k^{k-1}}{a_0}F(x, y, p) \quad \text{and} \quad U_k - pU_{k-1} = \frac{v_k^{k-1}}{a_0}\partial_p(F)(x, y, p).
\]

Indeed, since the polynomials \( (V_k - pV_{k-1})(x, y, p) \) of degree \( d \) admits the \( d \) slopes of the web \( p \in \mathcal{O} \) as solutions, we get the first equality. The second one is deduced from the uniqueness in theorem \( \text{3.1} \). So, if \( V = -P_{\mathcal{W}(d)} \) is known, we know all the others polynomials \( V_k \).

It remains to show that \( U \) can be deduced from \( V \). Considering the system \( (\diamond) \) in theorem \( \text{3.1} \) and taking \( V \) as a parameter, the special form of the Sylvester determinant gives a triangular system in the \( a_i \) and \( V \) with \( U \) as unknown. One can check that using the Newton’s relations between the coefficients and the roots of our equation \( F \), the data of \( \alpha \) and \( V \) allows us to compute all the coefficients of the system \( \mathcal{M}(d) \). \( \square \)

We will see in the next section the usefulness of this theorem, and the special form of the coefficients of \( \mathcal{M}(d) \) expressed with \( \alpha \) and \( V \). But first, to emphasize on the fundamental form, let us consider the linear case. We said that the form \( \alpha \) is linked to the abelian relations. Precisely, we have the following proposition which completes the result given in proposition \( \text{3.3} \).

**Proposition 3.5.** The following properties are equivalent:

i) \( \mathcal{W}(d) \) is algebraic;

ii) \( \mathcal{W}(d) \) is linear and the fundamental form satisfies \( d\alpha = \partial_p^2(\frac{\alpha}{a_0}) = 0 \).

Proof. The coefficients of the system \( \mathcal{M}(d) \) for a linear \( d \)-web is the following, since the \( V_k \) are zero:

\[
\begin{pmatrix}
0 & \ldots & 0 & 0 & -u_d \\
0 & \ldots & 0 & -u_{d-1} & -u_{d-1} \\
0 & \ldots & \ldots & -u_{d-2} & 0 \\
\vdots & \ddots & \vdots & \ddots & \ddots \\
-u_{d-3} & 0 & 0 & \ldots & 0 \\
u_2 & 0 & 0 & \ldots & \\
\end{pmatrix}
\]

But writing the \( u_j^i \) in terms of the \( v_j^i \) shows that, since all the \( v_j^i = 0 \):

\[
u_d = u_{d-1} = \ldots = u_{d-3} = \frac{\partial_x(a_0)}{a_0} + \frac{\partial_y(a_1)}{a_0} \quad \text{and} \quad u_{d-1} = u_{d-2} = \ldots = u_{d-3} = \frac{\partial_y(a_0)}{a_0}.
\]

According to proposition \( \text{3.3} \) the linear web is algebraic if and only if the following system admits a solution:

\[
\begin{cases}
\partial_y(\phi) = \frac{\partial_y(a_0)}{a_0} \\
\partial_x(\phi) = \frac{\partial_x(a_0)}{a_0} + \frac{\partial_y(a_1)}{a_0}
\end{cases}
\]

The integrability condition is then \( \partial_p^2(\frac{\alpha}{a_0}) = 0 \), that is to say \( d\alpha = 0 \), which proves the proposition. \( \square \)

4. Geometric study of the 4-webs associated connection

The general settings given in the preceding sections will find a direct application in this part. We give here an explicit computation of the connection associated with a 4-web. Not only that it gives a computation tool, it will allow us to find some more properties of the connection.
4.1. Construction. We adopt here the notations of the book [BC3G] and the methods developed in the article [H-04]. In the case of 4-webs, one can define the operator

\[ p_0 : J_1(O^2) \longrightarrow \mathcal{O}^3 \]

\[
\begin{pmatrix}
  z_3, p_3, q_3 \\
  z_4, p_4, q_4
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
p_4 + A_{11}z_3 + A_{12}z_4 \\
p_3 + q_4 + A_{21}z_3 + A_{22}z_4 \\
g_3 + A_{31}z_3 + A_{32}z_4
\end{pmatrix}.
\]

Then we get an exact and commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & g_0 & \longrightarrow & S_1(O^2) & \xrightarrow{\sigma_0} & \mathcal{O}^3 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & R_0 & \longrightarrow & J_1(O^2) & \xrightarrow{p_0} & \mathcal{O}^3 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & R_{-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
\]

where the upper line concerns symbols of \( p_0 \).

Explicitly, we have: \( \sigma_0(p_3, q_3) = (p_4, p_3 + q_4, q_3) \) whose kernel \( g_0 \) is then isomorphic to \( O \).

The first prolongation \( p_1 : J_2(O^2) \longrightarrow J_1(O^3) \) of \( p_0 \) is then given by

\[
p_1 \begin{pmatrix}
z_3, p_3, q_3, r_3, s_3, t_3 \\
z_4, p_4, q_4, r_3, s_4, t_4
\end{pmatrix} = \begin{pmatrix}
p_4 + A_{11}z_3 + A_{12}z_4 \\
p_3 + q_4 + A_{21}z_3 + A_{22}z_4 \\
g_3 + A_{31}z_3 + A_{32}z_4
\end{pmatrix}.
\]

which leads us to another commutative and exact diagram:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & g_1 = 0 & \longrightarrow & S_2(O^2) & \xrightarrow{\sigma_1} & S_1(O^3) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & R_1 & \longrightarrow & J_2(O^2) & \xrightarrow{p_1} & J_1(O^3) & \longrightarrow & 0. \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & R_0 & \longrightarrow & J_1(O^2) & \xrightarrow{p_0} & \mathcal{O}^3 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{O}^3 & \longrightarrow & \mathcal{O}^3 & \longrightarrow & 0 \\
\end{array}
\]

Thus \( \pi_0 \) is an isomorphism. The first Spencer complex is given by

\[
0 \longrightarrow \text{Sol } \mathcal{M}(4) \xrightarrow{J_2} E \xrightarrow{D} \Omega^1 \otimes_O R_0 \xrightarrow{D} \Omega^2 \otimes_O R_{-1} \longrightarrow 0
\]
where $E$ stands for the kernel of $p_1$. We deduce from this the following commutative diagram, whose lines are exact and whose columns are exact in $R_k$, and $j_2$ and $j_3$ are monomorphisms:

\[
\begin{array}{c}
0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 & \rightarrow & \text{Sol } M(4) & \rightarrow & \text{Sol } M(4) & \rightarrow & 0 \\
\downarrow j_3 & & \downarrow j_2 & & \downarrow & & \downarrow \\
0 & \rightarrow & R_2 & \rightarrow & E = R_1 & \rightarrow & R_1 \\
\downarrow \sqrt{\nabla} & & \downarrow D & & \downarrow & & \\
0 & \rightarrow & \Omega^1 \otimes \mathcal{O} E & \rightarrow & \Omega^1 \otimes \mathcal{O} R_0 & \rightarrow & 0 \\
\downarrow & & \downarrow D & & \downarrow & & \\
0 & \rightarrow & \Omega^2 \otimes \mathcal{O} g_0 & \rightarrow & \Omega^2 \otimes \mathcal{O} R_0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & \\
\end{array}
\]

One can check that the kernel $E$ of $p_1$ is a $\mathbb{C}$-vector bundle of rank 3. The main result of this construction is that, using properties of $D$, the map $\nabla = \pi_0^{-1} \circ D : E \rightarrow \Omega^1 \otimes \mathcal{O} E$ is a connection on $E$, and its kernel is isomorphic to $\text{Sol } M(4)$.

**Remark 4.1.** Since $\text{Ker } \nabla$ is isomorphic to $\text{Sol } M(4)$ which is a local system, the kernel of $\text{Ker } \nabla$ is also a local system, even if the connection is not integrable.

One can pick up an adapted basis of $E = \text{Ker } (p_1)$. We can choose such a basis $e_1, e_2, e_3$ this way, so that the curvature takes its values in $\Omega^2 \otimes \mathcal{O} g_0$:

\[
e_1 = \begin{pmatrix} 0 & -1 & 0 & A_{21} + A_{12} & A_{31} & -A_{32} \\ 0 & 0 & 1 & A_{11} & -A_{12} & -A_{22} - A_{31} \end{pmatrix}
\]

\[
e_2 = \begin{pmatrix} -1 & A_{21} & A_{31} - \partial_y(A_{11}) + \partial_x(A_{21}) - A_{21}^2 & -A_{21}^2 - A_{11}(A_{22} - A_{31}) \\ 0 & A_{11} & 0 & -A_{11}(A_{21} + A_{12}) + \partial_x(A_{11}) - A_{31}A_{21} - A_{32}A_{11} + \partial_x(A_{31}) & -A_{32} + \partial_y(A_{31}) \\ & -A_{11}A_{31} + \partial_y(A_{11}) & -\partial_x(A_{31}) + \partial_y(A_{21}) & -A_{32} - A_{32}A_{11} & \end{pmatrix}
\]

\[
e_3 = \begin{pmatrix} 0 & 0 & -A_{32} & -\partial_y(A_{12}) - \partial_x(A_{22}) - A_{11}A_{32} \\ 1 & -A_{12} & -A_{22} & A_{12}^2 - \partial_x(A_{12}) & A_{32}(A_{31} + A_{22}) - \partial_y(A_{31}) \\ & A_{32}(A_{31} + A_{22}) - \partial_y(A_{12}) & A_{12}A_{22} - A_{12} - A_{22} + A_{22}^2 + \partial_x(A_{31}) & \end{pmatrix}
\]

Since $j_2(f) = (f, \partial_x f, \partial_y f, \partial_x^2 f, \partial_x \partial_y f, \partial_y^2 f)$, the exactness in $E$ of the Spencer complex allows us to compute $D(e_1), D(e_2)$ and $D(e_3)$ in $\Omega^1 \otimes \mathcal{O} R_0$, that must be composed with $\pi_0$ to get the value of $\nabla(e_i)$. Thus, we get the connection matrix $\gamma$ of $\nabla$ in this basis:

\[
\gamma = \begin{pmatrix} A_{12} dx + A_{31} dy & \xi_1 & \xi_2 \\ -dx & A_{21} dx + A_{31} dy & -A_{32} dy \\ -dy & -A_{11} dx & A_{12} dx + A_{22} dy \end{pmatrix}
\]

where $\xi_1$ and $\xi_2$ are expression in the $A_{ij}$ which will be detailed later. The matrix of the system $M(4)$ written with the fundamental 1-form $\alpha = A_1 dx + A_2 dy$ and $V$ is

\[
(A_{ij}) = \begin{pmatrix} -v_4 & A_1 \\ A_1 - v_3 & A_2 \\ A_2 - v_2 & v_1 \end{pmatrix}
\]
Then we have the following expression of the connection matrix:

\[
\gamma = \begin{pmatrix}
A_1 dx + (A_2 - v_2) dy \\
-dx & (A_1 - v_3) dx + (A_2 - v_2) dy \\
-dy & v_4 dx & -v_1 dy
\end{pmatrix}
\]

where \( \xi_1 = (\partial_y(v_4) + v_4 v_2) dx + (v_1 v_4 + \partial_x(A_2 - v_2) - \partial_y(A_1 - v_3)) dy \) and
\( \xi_2 = (v_4 v_1 - (\partial_x(A_2) - \partial_y(A_1))) dx + (v_1 v_3 - \partial_x(v_1)) dy \).

Its curvature is:

\[
d\gamma + \gamma \wedge \gamma = \begin{pmatrix}
k_1 & k_2 & k_3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
dx \wedge dy
\]

where \( k_1 = d(\text{tr} \: \gamma) \), \( k_2 \) and \( k_3 \) are computable but will be omitted here (cf. [R2] for details).

4.2. Interpretation. We will give several applications of the writing of the connection in terms of the 1-form \( \alpha \) and the polynomial \( V \). Since we have the equality \( V = -P_{W(d)} \) established in proposition [3.2] the linearizability conditions for the web will be seen in the connection as we can see in the following theorem.

**Theorem 4.1.** Up to a change of basis, the curvature matrix associated with a 4-web is:

\[
K = \begin{pmatrix}
k_1 & \partial_x(k_1) + L_1 & \partial_y(k_1) + L_2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
dx \wedge dy
\]

where \( L_1 \) and \( L_2 \) are defined in properties [7]. Thus, the curvature is an \( O^* \)-invariant of the web.

**Proof.** Using the identification \( V = -P_{W(d)} \), and the expression of the curvature in terms of \( V \), one can compute that

\[
k_2 = \frac{1}{3}(\partial_x(k_1) + v_3 k_1 + L_1) \quad \text{and} \quad k_3 = \frac{1}{3}(\partial_y(k_1) - v_2 k_1 + L_2).
\]

The change of basis defined by the matrix

\[
P = \begin{pmatrix}
\frac{1}{3} & -\frac{1}{3} v_3 & \frac{1}{3} v_2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

gives the needed expression. \( \square \)

Now we can give a new proof of a Poincaré theorem, using only differential systems tools:

**Theorem 4.2** (Poincaré, 1901). A 4-web of maximal rank is linearizable.

**Proof.** Since the web is of maximal rank, the curvature is zero, thus, \( k_1 = k_2 = k_3 = 0 \). Hence \( L_1 = L_2 = 0 \) and since the degree of \( P_{W(4)} \) is 3, the web is linearizable by the properties [4]. \( \square \)

5. The case of 5-webs

In the case of 5-webs (and higher), computations are more complicated, but the method is still the same and we can get similar results than in the case of 4-webs.
5.1. The connection associated with 5-webs. Let $\mathcal{W}(5)$ be a planar 5-web, presented by a differential equation $F = 0$ of degree 5. The matrix $(A_{ij})$ of the system $\mathcal{M}(5)$ is

$$(A_{ij}) = \begin{pmatrix}
\frac{a_2}{a_0} v_1 & -v_5 & A_1 \\
-2v_5 + \frac{a_4}{a_0} v_1 & A_1 - v_4 & A_2 \\
A_1 - 2v_4 + \frac{a_4}{a_0} v_1 & A_2 - v_3 & 2v_2 - \frac{a_4}{a_0} v_1 \\
A_2 - 2v_3 + \frac{a_2}{a_0} v_2 & v_2 - \frac{a_2}{a_0} v_1 & v_1
\end{pmatrix}$$

where we have written $P_{\mathcal{W}(5)} = -v_1 p^4 - v_2 p^3 - v_3 p^2 - v_4 p - v_5$ and the fundamental form $\alpha = A_1 \, dx + A_2 \, dy$. The construction of the connection associated with the web gives us a $\mathbb{C}$-vector bundle $(E, \nabla)$ of rank 6 and a choice of an adapted basis of $E$.

We can write the trace $k_1$ of the curvature matrix:

$$k_1 = 6(\partial_x(A_2) - \partial_y(A_1)) + 4\partial_y(v_4) - 8\partial_x(v_3) + 3\partial_x(v_1 \frac{a_2}{a_0}) - \partial_y(v_1 \frac{a_2}{a_0}),$$

which is still an $\mathcal{O}^*$-invariant of the web. With the notations of the properties of 4-webs, we have by computation the following theorem, the analogous of theorem 4.1 for 5-webs:

**Theorem 5.1.** The curvature is given in a suitable basis by the matrix:

$$K = \begin{pmatrix}
k_1 & \tilde{k}_2 + < v_1 >_2 & \tilde{k}_3 + < v_1 >_3 & \tilde{k}_4 + < v_1 >_4 & \tilde{k}_5 + < v_1 >_5 & \tilde{k}_6 + < v_1 >_6 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \, dx \wedge dy$$

where

$$\tilde{k}_2 = \partial_x(k_1) + \frac{5}{2} L_1, \quad \tilde{k}_3 = \partial_y(k_1) + \frac{5}{2} L_2, \quad \tilde{k}_4 = \partial_x^2(k_1) + 4\partial_x(L_1) + \frac{v_4}{2} L_1 - \frac{3}{2} v_5 L_2,$$

$$\tilde{k}_5 = \partial_x \partial_y(k_1) - 2\partial_x(L_2) + 2\partial_y(L_1) + \frac{v_3}{2} L_1 - \frac{v_4}{2} L_2 \quad \tilde{k}_6 = \partial_y^2(k_1) + 4\partial_y(L_2) + \frac{3}{2} v_2 L_1 - \frac{v_3}{2} L_2,$$

and where $< v_1 >_i$ belongs to the differential ideal generated by $v_1$ so that if $v_1 = 0$, then $< v_1 >_i = 0$. Moreover, the curvature is an $\mathcal{O}^*$-invariant of the web.

This result is quite similar to the case of 4-webs. Nevertheless, the existing difference makes clearer the fact that Poincaré theorem is not valid for 5-webs. Instead, we see in the following theorem that $v_1$ is an obstruction for a maximal rank web to be linearizable:

**Theorem 5.2.** Let $\mathcal{W}(5)$ be a 5-web of maximal rank. Then $\mathcal{W}(5)$ is linearizable if and only if $\deg P_{\mathcal{W}(5)} \leq 3$.

**Proof.** If $\deg (P_{\mathcal{W}(5)}) \leq 3$, then $v_1 = 0$ and the curvature is

$$K = \begin{pmatrix}
k_1 & \tilde{k}_2 & \tilde{k}_3 & \tilde{k}_4 & \tilde{k}_5 & \tilde{k}_6 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \, dx \wedge dy$$

Since the web is of maximal rank, the curvature is equal to zero. So $k_1 = 0$ and then $L_1$ and $L_2$ too. The properties gives us the conclusion. The other implication is also a direct consequence of the same properties. □
This theorem was first proved in [H-94], but we give here a new proof only using the connection.

5.2. Revisiting exceptional 5-webs. An exceptional web is a web of maximal rank which is not algebrizable, or with the theorem of Lie-Darboux-Griffiths, it is the same to say that it is not linearizable. The Poincaré theorem for 4-webs says that this configuration is only possible for \( d \geq 5 \).

Our expression of the connection offers a new approach of exceptional webs. We do not need to exhibit abelian relations to determine whether a web is exceptional or not. Moreover, all exceptional 5-webs can be described thanks to a differential system we give now:

**Proposition 5.1.** Let \( \mathcal{W}(5) \) be a planar 5-web. We have the following equivalence:

i) \( \mathcal{W}(5) \) is exceptional;

ii) The following explicit conditions are satisfied:

\[
\begin{cases}
  k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = 0 \\
  v_1 \neq 0 \in O
\end{cases}
\]

This system can be seen as systems in the coefficients \( a_i \) of the differential equation which presents the web. Hence, the study of exceptional 5-webs could be lead theoretically through the study of this system. But clearly, its complexity draws the limits of such an approach.

6. **The trace formula**

The trace formula links the trace of the curvature associated with a \( d \)-web \( \mathcal{W}(d) \), namely \( k_1 \), to the Blaschke curvatures \( d \gamma_k \) of the 3-webs extracted from \( \mathcal{W}(d) \), \((3)\) in number:

**Theorem 6.1** (Trace formula). Let \( \mathcal{W}(d) \) a planar \( d \)-web. With the previous notations,

\[
\text{tr } K = k_1 = \sum_{k=1}^{\frac{d}{3}} d \gamma_k.
\]

This formula was first demonstrated for 4, 5 and 6 webs in the author thesis [R2]. In this continuity, we will give here a full proof of this theorem in subsection 6.2.

The formula has various interpretations and consequences. Still in the author thesis, we made a construction of a poly-hexagon that generalizes the one of Thomassen for 3-webs, based on this trace formula. It gives also a simple but very useful criteria for searching exceptional webs.

One must notice that in 1938, Pantazi ([Pa]) gave a construction to determine the maximal rank webs which lead him to the introduction of \( \pi_d \) expressions whose annulation gives the conditions for a web to be of maximal rank. Mihaileanu ([Mi]), following Pantazi, identified one of this coefficient to be the sum of the Blaschke curvatures of extracted 3-webs. Our results were stated independently from them, since they were ignored until Luc Pirio digs them in his thesis. These results offer obvious similarity with the previous statements, even if the construction is not the same and Pantazi and Mihaileanu did not give a truly proof of there results. But the links between both approaches can be found in [H2R], unifying the trace formulas. Note also that a general proof of the trace formula which use another, but equivalent, formalism will be given in the same paper.

Here, the trace formula will always refer to the relation satisfied by the trace of the curvature introduced here.

In order to produce our proof for a \( d \)-web, we must first exhibit the relations that link, say the coefficients of the connection associated with the extracted 3-webs and
the coefficients of the $d$-web itself. We then could calculate the sum of the Blaschke curvatures in terms of the coefficients of the connection of the $d$-web. This is our first step. Then we need to give a general expression of the trace $k_1$ of the curvature, and compare the two expressions which will be the second step.

Before we prove the trace formula, let us make a remark. We let $W(d)$ be a $d$-web and its presentation $F(x, y, y') = 0$. Out of the singular locus, the slopes of the web will be denoted by $p_i \in \mathcal{O}$, for $1 \leq i \leq d$. Given such a $d$-web, we want to know whether an abelian relation comes from extracted webs of $W(d)$.

Let $p_k$ be one of the slope of the web, and $W_k(d-1)$ be the extracted $d-1$ web of $W(d)$ obtained by forgetting the slope $p_k$. We want to know the $\mathcal{O}$*-invariants associated with this web thanks to those of the $d$-web.

On can show ([H-03]) that the isomorphism $T$ between $\mathcal{A}(d)$ and $\mathcal{A}_F$ associates to an abelian relation $(g_i(F_i))_{1 \leq i \leq d}$ the 1-form on $S$:

$$T((g_i(F_i))_{1 \leq i \leq d}) = r(x, y, p) \frac{dy - p \, dx}{\partial_p(F)} \text{ where } r = F \left( \sum_{i=1}^{d} g_i(F_i) \partial_y(F_i) \right).$$

**Proposition 6.1.** An abelian relation of $W(d)$ is an abelian relation of the web $W_k(d-1)$ if and only if $r(x, y, p_j) = 0$ i.e. $r = (p - p_k) r_k$ where $r_k$ is a polynomial in $p$ of degree $d - 1$.

**Proof.** An abelian relation $(g_i(F_i))_{1 \leq i \leq d}$ of $W(d)$ is an abelian relation of $W_k(d-1)$ if and only if $g_k(F_k) = 0$. Via the isomorphism $T$ this is a necessary and sufficient condition for $r$ to admit $p_k$ as a root. \qed

### 6.1. Extracted webs

Let $p_k$ be one of the slope of the web, and $W_k(d-1)$ be the extracted $d-1$ web of $W(d)$ obtained by forgetting the slope $p_k$. It is presented by $F_k(x, y, p) = 0$ with the relation

$$F(x, y, p) = (p - p_k(x, y)) F_k(x, y, p).$$

We want to compute the coefficients of the system $M(d-1)$ associated with the web $W_k(d-1)$ with the ones of $M(d)$, associated with $W(d)$. Let $S$ be the surface of $\mathbb{C}^3$ defined by $S = \{ F(x, y, p) = 0 \}$ and $S_k$ the surface defined by $S_k = \{ F_k(x, y, p) = 0 \}$ and let $r_k(x, y, p)$ be a polynomial in $p$ with coefficients in $\mathcal{O}$ of degree $d - 4$. The canonical monomorphism $i : S_k \to S$ induces the morphism

$$i^* : \Omega^1_{S_k} = \Omega^1_{\mathbb{C}^3} / (df, F \Omega^1_{\mathbb{C}^3}) \to \Omega^1_S = \Omega^1_{\mathbb{C}^3} / (df, F_k \Omega^1_{\mathbb{C}^3})$$

where $\omega_k = r_k(x, y, p) \frac{dy - p \, dx}{\partial_p(F_k)}$.

Indeed, we have

$$\partial_p(F) \partial_p(F_k) (p - p_k) + F_k \text{ where } F_k = 0 \text{ on } S_k$$

by definition of pull back. By the same way, we show that for a polynomial $t$ in $\mathcal{O}[p]$, we have

$$i^* (d x \wedge dy) = t \frac{dx \wedge dy}{(p - p_k) \partial_p(F_k)}.$$

As reminded in section 2, given a 1-form $\sigma$ on $S$, there exists a polynomial $t_\sigma$ of degree $d - 1$ in $\mathcal{O}[p]$ such that the differential can be written $d \sigma = t_\sigma \frac{dx \wedge dy}{\partial_p(F)}$. We thus have the following proposition:

**Proposition 6.2.** Let $\omega_k$ be a 1-form on $S_k$ and $t_{\omega_k}$ the polynomial of degree $d - 1$ such that $d \omega_k = t_{\omega_k} \frac{dx \wedge dy}{\partial_p(F_k)}$. Let $\omega = (p - p_k) \omega_k$ be the corresponding 1-form on $S$ by $i^*$. Then we have

$$d \omega = (p - p_k) \cdot t_{\omega_k} \frac{dx \wedge dy}{\partial_p(F)}.$$
Proof. Indeed, we have the following equalities: \[ d(i^*(\omega)) = d\omega_k = t_{\omega_k} \frac{dx \wedge dy}{\partial p_i(F_k)} \] and
\[ i^*(d\omega) = i^*(t_{\omega_k} \frac{dx \wedge dy}{\partial p_i(F)}) = t_{\omega_k} \frac{dx \wedge dy}{(p - p_k)\partial p_i(F)}. \] Since the differential and the pull back commutes \( d(i^*(\omega)) = i^*(d\omega) \), so \( t_{\omega_k} = t_{\omega_k}(p - p_k). \)

Notice that this is another proof of proposition 6.1 by taking \( t_{\omega_k} = 0 \).

If we let \( r_k(x,y,p) = b_3 p^{d-4} + \ldots + b_{d-1} \), the differential on \( \omega_k \) of \( r_k \) induces a system denoted as follow, where the \( \partial_i \)'s are the coefficients of \( t_{\omega_k} \):

\[
\begin{align*}
\partial_x(b_{d-1}) & + A^{k}_{1,1} \cdot b_3 + \ldots + A^{k}_{1,d-3} \cdot b_{d-1} = t_{d-1} \\
\partial_x(b_{d-2}) + \partial_y(b_{d-1}) & + A^{k}_{2,1} \cdot b_3 + \ldots + A^{k}_{2,d-3} \cdot b_{d-1} = t_{d-2} \\
& \vdots \\
\partial_x(b_3) + \partial_y(b_4) & + A^{k}_{d-3,1} \cdot b_3 + \ldots + A^{k}_{d-3,d-3} \cdot b_{d-1} = t_3 \\
\partial_y(b_3) & + A^{k}_{d-2,1} \cdot b_3 + \ldots + A^{k}_{d-2,d-3} \cdot b_{d-1} = t_2 \\
\end{align*}
\]

So the corresponding system for the differential on \( S \) is then

\[
\begin{align*}
\partial_x(-p_k \cdot b_{d-1}) + A_{1,1} \cdot b_3 + \ldots + A_{1,d-2} \cdot (-p_k \cdot b_{d-1}) & = -p_k \cdot t_{d-1} \\
\partial_x(b_{d-1} - p_k \cdot b_{d-2}) + \partial_y(-p_k \cdot b_{d-1}) + A_{2,1} \cdot b_3 + \ldots + A_{2,d-2} \cdot (-p_k \cdot b_{d-1}) & = t_{d-1} - p_k \cdot t_{d-2} \\
& \vdots \\
\partial_x(b_3) + \partial_y(-p_k \cdot b_3 + p_k \cdot b_{d-1}) + A_{d-2,1} \cdot b_3 + \ldots + A_{d-2,d-2} \cdot (-p_k \cdot b_{d-1}) & = t_3 - p_k \cdot t_2 \\
\partial_y(b_3) + A_{d-1,1} \cdot b_3 + \ldots + A_{d-1,d-2} \cdot (-p_k \cdot b_{d-1}) & = t_2 \\
\end{align*}
\]

By a combination of the preceding systems, but omitting the explicit expressions, we give the following claim:

**Theorem 6.2.** With the previous development, we get an explicit writing of the coefficients \( A_{ij}^k \) of the extracted web thanks to the \( A_{ij} \) and the forgotten slope.

**Example 6.1.** For a 4-web, we have the following relations:

\[
\begin{align*}
A^k_2 & = A_2 - v_2 - v_1 p_k \\
A^k_1 & = -\partial_y(p_k) + A_1 - v_3 - v_2 p_k - v_1 p_k^2 \\
V(p_k) & = \partial_x(p_k) + p_k \partial_y(p_k)
\end{align*}
\]

In the linear case, where \( A_{11} = A_{32} = 0, A_{12} = A_{21} = 1, \) and \( A_{22} = A_{33} = A_2 \), we get \( A_2 = A_2^k \) and \( A_1 = A_1^k - \partial_y(p_k) \), which means that the trace formula is directly checked, since the Blaschke curvatures are the 2-forms \( \partial_x(A^k_2) - \partial_y(A^k_1) dx \wedge dy \).

6.2. **Proof of the trace formula.** The expression of the coefficients \( (A^k_1, A^k_2) \) of all the extracted 3-webs of a \( d \)-web allows us after a long computation to give the expression of the sum of the Blaschke curvatures in terms of the coefficients \( A_{ij} \) associated with the \( d \)-web. This can be done using heavily the Newton relations between the roots and the coefficients of the equation presenting the web.

The result of this sum is a quite simple expression, which needs to be the trace of the curvature matrix in order to prove the trace formula.

**Proposition 6.3.** The trace of the connection, in a suitable basis is given by:

\[
\text{tr} \, \gamma = \sum_{q=1}^{d-2} A_{d-q-1,q} dx + A_{d-q-1,q} dy + \sum_{q=2}^{d-2} A_{d-q-1,q} dx + A_{d-q+1,q-1} dy
\]

Again the proof is a computation. Since the differential of this trace is the trace of the curvature matrix \( \text{tr} \, K \), it does not depend on the choice of the basis. We then compute more easily this trace by choosing a suitable basis for this.

Moreover, we do not need to have a general expression of the basis, since the construction of the adapted basis constrain the vectors of the basis to have a particular form which is sufficient to compute the trace. This trace is, as expected, the sum of the Blaschke curvatures computed before.

We give now an interpretation of the trace formula in terms of determinant:
Theorem 6.3. Let \( \mathcal{W}(d) \) be a planar \( d \)-web and let \( (L_k, \nabla_k) \) be the line bundles associated with extracted 3-webs of \( \mathcal{W}(d) \) for \( 1 \leq k \leq \left( \frac{d}{3} \right) \). We have an isomorphism of line bundles with connection:

\[
(\det E, \det \nabla) \cong \bigotimes_{k=1}^{\left( \frac{d}{3} \right)} L_k, \bigotimes_{k=1}^{\left( \frac{d}{3} \right)} \nabla_k,
\]

Proof. The isomorphism denoted \( \tau \) is defined by its action on the basis. To the basis \( e_1 \wedge \ldots \wedge e_{\pi d} \) of \( \det E \) which is the wedge product of the vector of the basis of \( E \), we associate the basis \( \bigotimes_{k=1}^{\left( \frac{d}{3} \right)} e_k \) of \( \bigotimes_{k=1}^{\left( \frac{d}{3} \right)} L_k \) where \( e_k \) is a basis of \( L_k \). The isomorphism commutes with the connections since the equality

\[
\bigotimes_{k} \gamma_k \circ \tau \left( \bigwedge_{k} e_k \right) = \tau \circ \det \nabla \left( \bigwedge_{k} e_k \right)
\]

is a consequence, regarding matrices, of the trace formula

\[
\text{tr} \gamma \otimes \left( \bigotimes_{k} e_k \right) = \left( \sum_{k=1}^{\left( \frac{d}{3} \right)} \gamma_k \right) \otimes \left( \bigotimes_{k} e_k \right).
\]

\( \square \)

7. Determination of the rank

7.1. The main result. The determination of the rank of a \( d \)-web was not effective since Blaschke introduced the subject, except for \( d = 3 \) as we have seen. In fact, the main way used before to get the rank was to compute all the abelian relations.

The following theorem gives, not only a determination of the rank, but also an explicit locally free \( \mathcal{O} \)-module of finite type whose rank is the rank of the web. It embodies the abelian relations, since it is generated by them. The existence of such a \( \mathcal{O} \)-modules is theoretically given, but the fact that it can be embodied explicitly is not clear in general. We give here a complete proof of this result first announced in our note [R1]:

Theorem 7.1 (Determination of the rank). Let \( \mathcal{W}(d) \) a non singular planar \( d \)-web. There exists a \( \mathbb{C} \)-vector bundle \( \mathcal{K} \) of rank \( \text{rk} \mathcal{W}(d) \), which is the kernel of an explicit endomorphism of \( \mathcal{O}^{\pi d} \) such that

\[
\mathcal{K} = \mathcal{O} \otimes_{\mathbb{C}} \text{Ker} \nabla.
\]

So if \( (k_{m\ell}) \) denotes the matrix of this endomorphism, the rank of the web is given by:

\[
\text{rk} \mathcal{W}(d) = \text{corank} (k_{m\ell}).
\]

Before we prove this theorem, we give the construction of the matrix \( (k_{m\ell}) \). The horizontal sections \( f = (f_1, f_2, \ldots, f_{\pi d}) \in E^\nabla := \text{Ker} \nabla \) of \( \nabla \) are identified to the abelian relations of the web by construction of the connection, and satisfy the differential system \( df + \gamma f = 0 \) where \( \gamma = \gamma_x dx + \gamma_y dy \) is the connection matrix in a suitable basis. The integrability condition is then given by the only one relation

\[
(1) \quad k \cdot f = k_1 f_1 + k_2 f_2 + \cdots + k_{\pi d} f_{\pi d} = 0
\]

where the \( k_i \) are the coefficients of the curvature matrix. We consider then the \( \pi_d \) equations obtained from the derivation until the order \( d - 3 \) of (1) where we substitute the derivative of \( f \) thanks to \( df = -\gamma f \). We get then a square matrix \( (k_{m\ell}) \) of order \( \pi_d \) whose first line is the first line of the curvature matrix:
Example 7.1. For \( d = 4 \), we have the explicit expression of the coefficients of the matrix \((k_{mi})\):

\[
\begin{pmatrix}
  k_1 & k_2 & \ldots & k_{\pi_d} \\
  k_{21} & k_{22} & \ldots & k_{2,\pi_d} \\
  \vdots & \vdots & \ddots & \vdots \\
  k_{\pi_d,1} & k_{\pi_d,2} & \ldots & k_{\pi_d,\pi_d}
\end{pmatrix}
\]

Proof. We will largely use the fact that \( E^\nabla \) is a local system. Let \( \mathcal{K} := \text{Ker}(k_{mi}) \) the \( \mathcal{O} \)-module of finite type defined by our matrix. We have the inclusion

\[
\mathcal{O} \otimes_C E^\nabla \subseteq \mathcal{K}
\]

by construction. The converse will be true by Nakayama’s lemma if we show the following

\[
\mathcal{K} = \mathcal{O} \otimes_C E^\nabla + m \cdot \mathcal{K}
\]

where \( m \) is the maximal ideal of \( \mathcal{O} \). Let \((g_1, \ldots, g_r)\) a system of generators of \( \mathcal{K} \) such that the \( g_i(0) \) are linearly independent and whose existence is again given by Nakayama’s lemma.

Let \( g \in \mathcal{K} \). One can write that \( g = \sum_{1 \leq i \leq r} \lambda_ig_i \) with \( \lambda_i \in \mathcal{O} \). If \( g(0) = 0 \), then \( g(0) = \sum_{1 \leq i \leq r} \lambda_i(0)g_i(0) = 0 \) and so \( \lambda_i(0) = 0 \) since the \( g_i(0) \) are linearly independent. This gives \( g \in m\mathcal{K} \) and the needed equality.

If \( g(0) \neq 0 \), we will construct an analytic function \( f \) in \( E^\nabla \) such that \( f(0) = g(0) \), where \( g(0) \) stands for the initial conditions in the classical Cauchy Theorem.

Indeed, Cauchy theorem in one variable gives us a unique function \( \sigma(y) \) such that \( \sigma(0) = g(0) \) and \( \partial_y(\sigma) + \gamma\sigma = 0 \). Again, Cauchy theorem gives us a unique function \( f(x, y) \) such that \( f(0, y) = \sigma(y) \) and \( \partial_x(f) + \gamma_xf = 0 \). We have the following two equalities:

\[
\begin{aligned}
\partial_y(f(0, y)) + \gamma_yf(0, y) &= 0 \quad (4) \\
\partial_x(f(x, y)) + \gamma_xf(x, y) &= 0 \quad (5)
\end{aligned}
\]

Let \( \tau = \partial_y(f) + \gamma_yf \). The equation (4) gives us that \( \tau(0, y) = 0 \), and using (5), we have \( \partial_x(\tau) + \gamma_x\tau = k \cdot f \).

Since \( g \in \mathcal{K} \) and \( f(0) = g(0) \), we have \( k(0) \cdot f(0) = 0 \), and all the successive derivatives of \( k \cdot f \) taken at 0 are zero, again by (4), (5) and the equality \( \mathcal{K}(0) \cdot g(0) = 0 \). So we have \( k \cdot f = 0 \), and then, \( \partial_x(\tau) + \gamma_x\tau = 0 \) and \( \tau(0, y) = 0 \). The uniqueness theorem of Cauchy gives again \( \tau = \partial_y(f) + \gamma_yf = 0 \) and so \( f \) belongs to \( E^\nabla \) since we have

\[
\begin{aligned}
\partial_y(f) + \gamma_yf &= 0 \\
\partial_x(f) + \gamma_xf &= 0.
\end{aligned}
\]

The inclusion (2) gives \( g - f \in \mathcal{K} \) and \((g - f)(0) = 0 \). But we have seen that if \( h \in \mathcal{K} \) is such that \( h(0) = 0 \), the system of generators of \( \mathcal{K} \) chosen allows us to show that \( h \in m\mathcal{K} \). So \( g - f \in m\mathcal{K} \), which shows the needed equality. \( \square \)

As a corollary, we precise the fact that a web is, in general, of rank equal to 0:

**Corollary 7.2.** A d-web is of rank greater or equal to 1 if and only if \( \det(k_{mi}) = 0 \)

**Example 7.1.** For \( d = 4 \), we have the explicit expression of the coefficients of the matrix \((k_{mi})\):

\[
\begin{align*}
 k_{21} &= \partial_x(k_1) - A_1k_1 + k_2 & k_{22} &= \partial_x(k_2) - (\partial_y(v_4) + v_4v_2)k_1 - (A_1 - v_3)k_2 - v_4k_3 \\
 k_{23} &= \partial_x(k_3) - (v_1v_4 - k_2)k_1 - A_1k_3 & k_{31} &= \partial_y(k_1) - (A_2 - v_2)k_1 + k_3 \\
 k_{32} &= \partial_y(k_2) - (v_1v_4 + k_1)k_1 - (A_2 - v_2)k_2 & k_{33} &= \partial_y(k_3) - (v_1v_3 - \partial_x(v_1))k_1 + v_1k_2 - A_2k_3
\end{align*}
\]
7.2. Linear case for 4-webs.

**Proposition 7.1.** The rank of a linear 4-web is not 2.

Indeed, let \( \mathcal{L}(4) \) be a linear 4-web. The Lie-Darboux-Griffiths says that if such a web admits a complete abelian relation, \( i.e. (g_i(F_i))_{1 \leq i \leq 4} \) where \( g_i \) is non zero for all \( i \), then the web is of maximal rank.

If \( \mathcal{L}(4) \) is of rank 2 and so does not have any complete abelian relations, this web admits two independent abelian relations coming from (different) extracted 3-webs. In fact, there exists a linear combination of these two abelian relations which is then complete, in contradiction with the hypothesis. Thus, a linear 4 web can not be of rank 2.

We can also prove this using our preceding results: the cancelation of all the minors of order 1 and 2 implies that \( k_1 = 0 \) which is impossible since this implies for a linear web to be of maximal rank.

7.3. Bol’s theorem for 4-webs. As a nice application of the trace formula and the theorem of determination of the rank, we can prove the Bol’s theorem for 4-webs. But before, we have this rigidity type proposition:

**Proposition 7.2.** A 4-web of rank at least 2 with \( k_1 = 0 \) is of maximal rank.

**Proof.** A computation show that if \( k_1 = 0 \) and all the minors of order 1 and 2 of \( k_{ml} \) are equal to zero, then the matrix of curvature is zero. Hence the web is of maximal rank. \( \square \)

**Theorem 7.3.** Let \( \mathcal{H}(4) \) be a hexagonal 4-web (i.e. the Blaschke curvatures of all extracted 3-webs are zero). Then \( \mathcal{H}(4) \) is of maximal rank.

**Proof.** Since the web is hexagonal, the trace formula shows that \( k_1 = 0 \) and there exists at least two abelian relations of 3-webs linearly independent. So the rank is at least 2 and the preceding proposition applies. \( \square \)

8. Example

Let \( \mathcal{W}(4) \) be the 4-web presented by \( p^4 + y^2p^2 - yp = 0 \). Its discriminant is \( \Delta = -y^4(27 + 4y^4) \) and

\[
P_{\mathcal{W}(4)} = -12\frac{p^3}{27 + 4y^4} + \frac{(9 + 4y^4)p^2}{y(27 + 4y^4)} - 8\frac{y^2p}{27 + 4y^4}.
\]

The fundamental forms is \( \alpha = -2y(9 + 4y^4)\frac{dy}{y(27 + 4y^4)} \). We can compute the connexion matrix:

\[
\gamma = \begin{pmatrix}
-\frac{(9 + 4y^4)dy}{y(27 + 4y^4)} & -16y\frac{(-27 + 4y^4)dy}{y(27 + 4y^4)} & 96y\frac{dy}{y(27 + 4y^4)}
\end{pmatrix}
\]

and the curvature matrix:

\[
K = \begin{pmatrix}
-16y\frac{(-27 + 4y^4)}{(27 + 4y^4)^2} & -128y\frac{(-27 + 4y^4)}{(27 + 4y^4)^3} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
The rank of this web is 2 since the matrix \((k_{ml})\) is the following:

\[
(k_{ml}) = \begin{pmatrix}
-16 \frac{y(-27+4y)}{(27+4y)^3} & -128 \frac{y^2(-27+4y)}{(27+4y)^3} & 0 \\
-128 \frac{y(-27+4y)}{(27+4y)^3} & -1024 \frac{y^2(-27+4y)}{(27+4y)^3} & 0 \\
64 \frac{243-306y^4+8y^8}{(27+4y)^3} & 512 \frac{y(243-306y^4+8y^8)}{(27+4y)^3} & 0
\end{pmatrix}
\]

whose determinant and order 2 minors are all equal to zero.

References

[B] W. Blaschke, Einfuhrung in die Geometrie der Waben, Birkhauser, Basel, 1955.

[B-B] W. Blaschke und G. Bol, Geometrie der Gewebe, Springer, Berlin, 1938.

[BC3G] R.L. Bryant, S.S. Chern, R.B. Gardner, H.L. Goldschmidt and P.A. Griffiths, Exterior differential systems, Springer, Berlin, 1991.

[C] S.S. Chern, Web Geometry, Bull. Amer. Math. Soc. 6 (1982), 1-8.

[C-G] S.S. Chern and P.A. Griffiths, Abel’s Theorem and Webs, Jahressber. Deutsch. Math.-Verein. 80 (1978), 13-110 and Corrections and Addenda to Our Paper: Abel’s Theorem and Webs, Jahressber. Deutsch. Math.-Verein. 83 (1981), 78-83.

[C-L] V. Cavalier et D. Lehmann, Introduction à une étude globale des tissus sur une surface holomorphe, To appear.

[G-02] P.A. Griffiths, The legacy of Abel in algebraic geometry, in: Uludag, Olav Arnfinn (ed.), The legacy of Niels Henrik Abel. Papers from the Abel bicentennial conference, University of Oslo, Oslo, Norway, June 3-8, 2002. Springer, Berlin (2004), 179-205.

[G-76] P.A. Griffiths, Variations on a Theorem of Abel, Invent. Math. 35 (1976), 321-390.

[Go] V.V. Goldberg, 4-webs in the plane and their linearizability Acta. Appl. Math. 80 (2004), 35-55.

[Go-L] V.V. Goldberg and V.V. Lychagin, On linearisation of planar three-webs and Blaschke’s conjecture, C.R. Acad. Sci. Paris, Ser. I 341 (2005), 169-173.

[H2R] A. Hénaut, O. Ripoll et G. Robert, Formule de la trace pour la connexion d’un tissu du plan, in preparation.

[H-07] A. Hénaut, Planar web geometry through abelian relations and singularities, in: Inspired by Chern, A memorial volume in honor of a great mathematician, World Scientifi, Sci. Publishing co., River Edge, NJ, 2006.

[H-04] A. Hénaut, On planar web geometry through abelian relations and connections, Ann. of Math. 159 (2004), 425-445.

[H-94] A. Hénaut, Caractérisation des tissus de \(\mathbb{C}^2\) dont le rang est maximal et qui sont linéarisables, Compositio Math. 94 (1994), 247-268.

[M2P] D. Marín, J.V. Pereira and L. Pirio, On planar webs with infinitesimal automorphisms, in: Inspired by Chern, A memorial volume in honor of a great mathematician, World Scientific, Sci. Publishing co., River Edge, NJ, 2006.

[Mih] M.N. Mihaileanu, Sur les tissus plans de première espèce, Bull. Math. Soc. Roum. Sci. 43 (1941), 23-26.

[N] I. Nakai, Curvature of curvilinear 4-webs and pencils of one forms: Variation on a theorem of Poincaré, Mayrhofer and Reidemeister, Comment. Math. Helv. 73 (1998), 177-205.

[P] L. Pirio, Équations fonctionnelles abéliennes et géométrie des tissus, Thèse de doctorat, Université Paris VI, décembre 2004.

[P-T] L. Pirio and J.-M. Trépreau, Tissus plans exceptionnels et fonction thêta, Annales de l’Institut Fourier, 55 no. 7 (2005), 2209-2237.

[Pa] A. Pantazi, Sur la détermination du rang d’un tissu plan, C.R. Acad. Sci. Roumanie 4 (1938), 108-111.

[R1] O. Ripoll, Détermination du rang des tissus du plan et autres invariants géométriques, C.R. Acad. Sci. Paris, Ser. I 341 (2005), 247-252.

[R2] O. Ripoll, Géométrie des tissus du plan et équations différentielles, Thèse de doctorat, Université Bordeaux I, décembre 2005, disponible sur http://tel.archives-ouvertes.fr/tel-00011928.

[R-S-1] O. Ripoll et J. Sebag, Solutions singulières des tissus polynomiaux du plan, J. of Algebra 310, (2007), 351-370.

[R-S-2] O. Ripoll et J. Sebag, Tissus du plan et polynomes de Darboux, (submitted).

[W] J. Grifone and É. Salem (Eds), Web Theory and Related Topics, World Scientific, Sci. Publishing co., River Edge, NJ, 2001.

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