An asymptotic cell category for cyclic groups
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Abstract. — In his theory of unipotent characters of finite groups of Lie type, Lusztig constructed modular categories from two-sided cells in Weyl groups. Broué, Malle and Michel have extended parts of Lusztig’s theory to complex reflection groups. This includes generalizations of the corresponding fusion algebras, although the presence of negative structure constants prevents them from arising from modular categories. We give here the first construction of braided pivotal monoidal categories associated with non-real reflection groups (later reinterpreted by Lacabanne as super modular categories). They are associated with cyclic groups, and their fusion algebras are those constructed by Malle.

Broué, Malle and Michel [BMM] have constructed a combinatorial version of Lusztig’s theory of unipotent characters of finite groups of Lie type for certain complex reflection groups (“spetsial groups”). The case of spetsial imprimitive complex reflection groups has been considered by Malle in [Ma]: Malle defines a combinatorial set which generalizes the one defined by Lusztig to parametrize unipotent characters of the associated finite reductive group when \( W \) is a Weyl group. Malle generalizes also the partition of this set into Lusztig families. To each family, he associates a \( \mathbb{Z} \)-fusion datum: a \( \mathbb{Z} \)-fusion datum is a structure similar to a usual fusion datum (which we will
call a $\mathbb{Z}_d$-fusion datum) except that the structure constants of the associated fusion ring might be negative.

It is a classical problem to find a tensor category with suitable extra-structure (pivot, twist) corresponding to a given $\mathbb{Z}_d$-fusion datum. The aim of this paper is to provide an ad hoc categorification of the $\mathbb{Z}$-fusion datum associated with the non-trivial family of the cyclic complex reflection group of order $d$: it is provided by a quotient category of the representation category of the Drinfeld double of the Taft algebra of dimension $d^2$ (the Taft algebra is the positive part of a Borel subalgebra of quantum $\mathfrak{sl}_2$ at a $d$-th root of unity).

The constructions of this paper have recently been extended by Lacabanne [La1], [La2] to some families of the complex reflection groups $G(d, 1, n)$. His construction involves super-categories instead of triangulated categories, as suggested by Etingof. In the particular case studied in our paper, Lacabanne’s construction gives a reinterpretation as well as some clarifications of our results.

In the first section, we recall some basic properties of the Taft algebra and its Drinfeld double. The second section is devoted to recalling some of the structure of its category of representations: simple modules, blocks and structure of projective modules. We summarize some elementary facts on the tensor structure in the third section: invertible objects and tensor product of simple objects by the defining two-dimensional representation. The fourth section provides generators and characters of the Grothendieck group of $D(B)$, a commutative ring.

Our original work starts in the fifth section, with the study of the stable module category of $D(B)$ and a further quotient and the determination of their Grothendieck rings. In the sixth section, we define and study a pivotal structure on $D(B)$-mod and determine characters of its Grothendieck group associated to left or right traces. This gives rise to positive and negative $S$ and $T$-matrices. We proceed similarly for the small quotient triangulated category. These are our fusion data.

We recall in the final section the construction of Malle’s fusion datum associated with cyclic groups and we check that it coincides with the fusion data we defined in the previous section.

In Appendix A, we provide a description of $S$-matrices in the setting of pivotal tensor categories.

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1. The Drinfeld double of the Taft algebra

From now on, $\otimes$ will denote the tensor product $\otimes_{\mathbb{C}}$. We fix a natural number $d \geq 2$ as well as a primitive $d$-th root of unity $\zeta \in \mathbb{C}^\times$. We denote by $\mu_d = \langle \zeta \rangle$ the group of $d$-th roots of unity.

Given $n \geq 1$ a natural number and $\xi \in \mathbb{C}$, we set

$$(n)\xi = 1 + \xi + \cdots + \xi^{n-1}$$

and

$$(n)!\xi = \prod_{i=1}^{n}(i)\xi.$$  

We also set $(0)!\xi = 1$.

1.A. The Taft algebra. — We denote by $B$ the $\mathbb{C}$-algebra defined by the following presentation:

- Generators: $K$, $E$.
- Relations:
  \[
  \begin{align*}
  K^d &= 1, \\
  E^d &= 0, \\
  KE &= \zeta EK.
  \end{align*}
  \]

It follows from [Ka, Proposition IX.6.1] that:

- $(\Delta)$ There exists a unique morphism of algebras $\Delta : B \to B \otimes B$ such that
  \[
  \Delta(K) = K \otimes K \quad \text{and} \quad \Delta(E) = (1 \otimes E) + (E \otimes K).
  \]

- $(\varepsilon)$ There exists a unique morphism of algebras $\varepsilon : B \to \mathbb{C}$ such that
  \[
  \varepsilon(K) = 1 \quad \text{and} \quad \varepsilon(E) = 0.
  \]

- $(S)$ There exists a unique anti-automorphism $S$ of $B$ such that
  \[
  S(K) = K^{-1} \quad \text{and} \quad S(E) = -EK^{-1}.
  \]

With $\Delta$ as a coproduct, $\varepsilon$ as a counit and $S$ as an (invertible) antipode, $B$ becomes a Hopf algebra, called the Taft algebra [EGNO, Example 5.5.6]. It is easily checked that

\[
B = \bigoplus_{i,j=0}^{d-1} \mathbb{C}K^iE^j = \bigoplus_{i,j=0}^{d-1} \mathbb{C}E^jK^i.
\]
1.B. Dual algebra. — Let $K^*$ and $E^*$ denote the elements of $B^*$ such that

$$K^*(E^i K^j) = \delta_{i,0} \zeta_j^j \quad \text{and} \quad E^*(E^i K^j) = \delta_{i,1}.$$ 

Recall that $B^*$ is naturally a Hopf algebra [Ka, Proposition III.3.3] and it follows from [Ka, Lemma IX.6.3] that

$$E^*(E^i K^j) = \delta_{i,1}.$$ 

We deduce easily that $(E^i K^j)_{0 \leq i, j \leq d-1}$ is a $\mathbb{C}$-basis of $B^*$.

We will give explicit formulas for the coproduct, the counit and the antipode in the next subsection. We will in fact use the Hopf algebra $(B^*)^{\text{cop}}$, which is the Hopf algebra whose underlying space is $B^*$, whose product is the same as in $B^*$ and whose coproduct is opposite to the one in $B^*$.

1.C. Drinfeld double. — We denote by $D(B)$ the Drinfeld quantum double of $B$, as defined for instance in [Ka, Definition IX.4.1] or [EGNO, Definition 7.14.1]. Recall that $D(B)$ contains $B$ and $(B^*)^{\text{cop}}$ as Hopf subalgebras and that the multiplication induces an isomorphism of vector spaces $(B^*)^{\text{cop}} \otimes B \rightarrow D(B)$. A presentation of $D(B)$, with generators $E, E^*, K, K^*$ is given for instance in [Ka, Proposition IX.6.4]. We shall slightly modify it by setting $z = K^{-1} K$ and $F = \zeta E^* K^{-1}$.

Then [Ka, Proposition IX.6.4] can we rewritten as follows:

**Proposition 1.3.** — The $\mathbb{C}$-algebra $D(B)$ admits the following presentation:

- Generators: $E, F, K, z$;
  $$
  \begin{aligned}
  K^d &= z^d = 1, \\
  E^d &= F^d = 0, \\
  [z, E] &= [z, F] = [z, K] = 0, \\
  KE &= \zeta E K, \\
  KF &= \zeta^{-1} F K, \\
  [E, F] &= K - z K^{-1}.
  \end{aligned}
  $$

The next corollary follows from an easy induction argument:

**Corollary 1.4.** — If $i \geq 1$, then

$$[E, F^i] = (i)_{\zeta} F^{i-1}(\zeta^{1-i} K - z K^{-1})$$

and

$$[F, E^i] = (i)_{\zeta} E^{i-1}(\zeta^{1-i} z K^{-1} - K).$$
The algebra $D(B)$ is endowed with a structure of Hopf algebra, where the comultiplication, the counit and the antipode are still denoted by $\Delta$, $\varepsilon$ and $S$ respectively (as they extend the corresponding objects for $B$). We have [Ka, Proposition IX.6.2]:

\begin{align}
\Delta(K) &= K \otimes K, \\
\Delta(z) &= z \otimes z, \\
\Delta(E) &= (1 \otimes E) + (E \otimes K), \\
\Delta(F) &= (F \otimes 1) + (z K^{-1} \otimes F), \\
S(K) &= K^{-1}, \\
S(z) &= z^{-1}, \\
S(E) &= -EK^{-1}, \\
S(F) &= -z^{-1} F K z^{-1},
\end{align}

(1.5)

\begin{align}
\varepsilon(K) &= \varepsilon(z) = 1 \quad \text{and} \quad \varepsilon(E) = \varepsilon(F) = 0.
\end{align}

(1.6)

1.D. Morphisms to $\mathbb{C}$. — Given $\xi \in \mu_d$, we denote by $\varepsilon_{\xi} : D(B) \to \mathbb{C}$ the unique morphism of algebras such that

$\varepsilon_{\xi}(K) = \xi, \quad \varepsilon_{\xi}(z) = \xi^2 \quad \text{and} \quad \varepsilon_{\xi}(E) = \varepsilon_{\xi}(F) = 0.$

It is easily checked that the $\varepsilon_{\xi}$’s are the only morphisms of algebras $D(B) \to \mathbb{C}$. Note that $\varepsilon_1 = \varepsilon$ is the counit.

1.E. Group-like elements. — It follows from (1.5) that $K$ and $z$ are group-like, so that $K^j z^i$ is group-like for all $i, j \in \mathbb{Z}$. The converse also holds (and is certainly already well-known).

Lemma 1.7. — If $g \in D(B)$ is group-like, then there exist $i, j \in \mathbb{Z}$ such that $g = K^j z^i$.

Proof. — Let $g \in D(B)$ be a group-like element. Let us write

$g = \sum_{i,j,k,l=0}^{d-1} \alpha_{i,j,k,l} K^j z^i E^k F^l.$

We denote by $(k_0, l_0)$ the biggest pair (for the lexicographic order) such that there exist $i, j \in \{0, 1, \ldots, d-1\}$ such that $\alpha_{i,j,k_0,l_0} \neq 0$. The coefficient of $K^j z^i E^{k_0} F^{l_0} \otimes K^j z^i E^{k_0} F^{l_0}$ in $g \otimes g$ is equal to $\alpha_{i,j,k_0,l_0}^2$, so it is different from 0.

But, if we compute the coefficient of $K^j z^i E^{k_0} F^{l_0} \otimes K^j z^i E^{k_0} F^{l_0}$ in

$g \otimes g = \Delta(g) = \sum_{i,j,k,l=0}^{d-1} \alpha_{i,j,k,l} \Delta(K)^i \Delta(z)^i \Delta(E)^k \Delta(F)^l$
using the formulas (1.5), we see that it is equal to 0 if \((k_0, l_0) \neq (0, 0)\). Therefore \((k_0, l_0) = (0, 0)\), and so \(g\) belongs to the linear span of the family \((K^i z^j)_{i, j \in \mathbb{Z}}\). Now the result follows from the linear independence of group-like elements. \(\square\)

1.F. Braiding. — For \(0 \leq i, j \leq d - 1\), we set

\[ \beta_{i,j} = \frac{E^i}{d \cdot (i)! \zeta} \sum_{k=0}^{d-1} \zeta^{-k(i+j)} K^k. \]

It follows from (1.2) that \((\beta_{i,j})_{0 \leq i, j \leq d-1}\) is a dual basis to \((E^i K^j)_{0 \leq i, j \leq d-1}\).

We set now

\[ R = \sum_{i,j=0}^{d-1} E^i K^j \otimes \beta_{i,j} \in D(B) \otimes D(B). \]

Note that \(R\) is a universal \(R\)-matrix for \(D(B)\) and it endows \(D(B)\) with a structure of braided Hopf algebra [Ka, Theorem IX.4.4]). Using our generators \(E, F, K, z\), we have:

\[ R = \frac{1}{d} \sum_{i,j,k=0}^{d-1} \zeta^{(i-k)(i+j)-(i+i)/2} (i)! \zeta E^i K^j \otimes z^{-k} F^i K^k. \]

1.G. Twist. — Let us define

\[ \tau : D(B) \otimes D(B) \rightarrow D(B) \otimes D(B) \]

\[ a \otimes b \mapsto b \otimes a. \]

Following [Ka, §VIII.4], we set

\[ u = \sum_{i,j=0}^{d-1} S(\beta_{i,j}) E^i K^j \in D(B). \]

Recall that \(u\) is called the Drinfeld element of \(D(B)\). It satisfies several properties (see for instance [Ka, Proposition VIII.4.5]). For instance, \(u\) is invertible and we will recall only three equalities:

\[ \varepsilon(u) = 1, \quad \Delta(u) = (\tau(R)R)^{-1}(u \otimes u) \quad \text{and} \quad S^2(b) = ubu^{-1} \]

for all \(b \in D(B)\). A straightforward computation shows that

\[ S^2(b) = KbK^{-1} \]

for all \(b \in D(B)\). We now set

\[ \theta = K^{-1} u. \]

The following proposition is a consequence of (1.9) and (1.10).
Proposition 1.11. — The element $\theta$ is central and invertible in $D(B)$ and satisfies $\epsilon(\theta) = 1$ and $\Delta(\theta) = (\tau(R)R)^{-1}(\theta \otimes \theta)$.

Let us give a formula for $\theta$:

\begin{equation}
\theta = \frac{1}{d} \sum_{i,j,k=0}^{d-1} (-1)^i \zeta^{(i-k)(i+j)-i} z^{k-l} E^i E^l K^{i+j-k-1}.
\end{equation}

Corollary 1.13. — We have $S(\theta) = z \theta$.

Proof. — Let $g = S(\theta)\theta^{-1}$. Since $\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta$ and $(S \otimes S)(R) = R$ (see for instance [Ka, Theorems III.3.4 and VIII.2.4]), it follows from Proposition 1.11 that $g$ is central and group-like. Hence, by Lemma 1.7, there exists $l \in \mathbb{Z}$ such that $S(\theta) = \theta z^l$. So, by (1.12), we have

\begin{equation}
S(\theta)E^{d-1} = \theta z^l E^{d-1} = \frac{1}{d} \sum_{j,k \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-jk} z^{k+l} K^{j-k-1} E^{d-1}.
\end{equation}

Let us now compute $S(\theta)E^{d-1}$ by using directly (1.12). We get

\begin{align*}
S(\theta)E^{d-1} &= E^{d-1} S(\theta) \\
&= \frac{1}{d} \sum_{j,k \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-jk} E^{d-1} z^{-k} K^{1+k-j} \\
&= \frac{1}{d} \sum_{j,k \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-jk} \zeta^{1-k-j} z^{-k} K^{1+k-j} E^{d-1} \\
&= \frac{1}{d} \sum_{j,k \in \mathbb{Z}/d\mathbb{Z}} \zeta^{[1-j][1+k]} z^{-k} K^{1+k-j} E^{d-1}.
\end{align*}

So, if we set $j' = 1 - j$ and $k' = -1 - k$, we get

\begin{equation}
S(\theta)E^{d-1} = \frac{1}{d} \sum_{j',k' \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-j'k'} z^{k'+1} K^{j'-k'-1} E^{d-1}.
\end{equation}

Comparing with (1.12), we get that $z^l = z$. \hfill $\Box$
2. $D(B)$-modules

Most of the result of this section are due to Chen [Ch1] or Erdmann, Green, Snashall and Taillefer [EGST1], [EGST2]. By a $D(B)$-module, we mean a finite dimensional left $D(B)$-module. We denote by $D(B)$-mod the category of (finite dimensional left) $D(B)$-modules. Given $\alpha_1, \ldots, \alpha_{l-1} \in \mathbb{C}$, we set

$$J^+(\alpha_1, \ldots, \alpha_{l-1}) = \begin{pmatrix} 0 & \alpha_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \alpha_{l-1} \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

and

$$J^-((\alpha_1, \ldots, \alpha_{l-1}) = \tau J^+(\alpha_1, \ldots, \alpha_{l-1}).$$

Given $M$ a $D(B)$-module and $b \in D(B)$, we denote by $b|_M$ the endomorphism of $M$ induced by $b$. For instance, $E|_M$ and $F|_M$ are nilpotent and $K|_M$ and $z|_M$ are semisimple.

2.A. Simple modules. — Given $1 \leq l \leq d$ and $p \in \mathbb{Z}/d\mathbb{Z}$, we denote by $M_{l,p}$ the $D(B)$-module with $\mathbb{C}$-basis $\mathcal{M}((l,p)) = \{e_i^{(l,p)}\}_{1 \leq i \leq l}$ where the action of $z$, $K$, $E$ and $F$ in the basis $\mathcal{M}((l,p))$ is given by the following matrices:

$$z|_{M_{l,p}} = \zeta^{2p+l-1} \text{Id}_{M_{l,p}},$$

$$K|_{M_{l,p}} = \zeta^p \text{diag}(\zeta^{l-1}, \zeta^{l-2}, \ldots, \zeta, 1),$$

$$E|_{M_{l,p}} = \zeta^p J^+(1) \big( \zeta^{l-1} - 1, (2) \zeta^{l-2} - 1, \ldots, (l-1) \zeta - 1 \big),$$

$$F|_{M_{l,p}} = J^-((1, \ldots, 1)).$$

It is readily checked from the relations given in Proposition 1.3 that this defines a $D(B)$-module of dimension $l$. The next result is proved in [Ch1, Theorem 2.5].

**Theorem 2.1 (Chen).** — The map

$$\begin{align*}
\{1, 2, \ldots, d\} \times \mathbb{Z}/d\mathbb{Z} & \longrightarrow \text{Irr}(D(B)) \\
(l, p) & \longmapsto M_{l,p}
\end{align*}$$

is bijective.
2.B. Blocks. — We put $\Lambda(d) = \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$, a set in canonical bijection with \{1, 2, \ldots, d\} \times \mathbb{Z}/d\mathbb{Z}, which parametrizes the simple $D(B)$-modules. Given $\lambda \in \Lambda(d)$, we denote by $M_{\lambda}$ the corresponding simple $D(B)$-module. We also set $(\mathbb{Z}/d\mathbb{Z})^d = (\mathbb{Z}/d\mathbb{Z}) \setminus \{0\}$ and $\Lambda^0(d) = (\mathbb{Z}/d\mathbb{Z})^d \times \mathbb{Z}/d\mathbb{Z}$. Finally, let $\Lambda^0(d) = \{0\} \times \mathbb{Z}/d\mathbb{Z}$ be the complement of $\Lambda^0(d)$ in $\Lambda(d)$.

Define
$$\iota : \Lambda(d) \longrightarrow \Lambda(d) \quad (l, p) \longmapsto (-l, p+l).$$

We have $\iota^2 = \text{Id}_{\Lambda(d)}$ and $\Lambda^0(d)$ is the set of fixed points of $\iota$. Given $\mathcal{L}$ a $\iota$-stable subset of $\Lambda(d)$, we denote by $[\mathcal{L}/\iota]$ a set of representatives of $\iota$-orbits in $\mathcal{L}$.

The next result is proved in [EGST1, Theorem 2.26].

**Theorem 2.2 (Erdmann-Green-Snashall-Taillefer).** — Let $\lambda, \lambda' \in \Lambda(d)$. Then $M_{\lambda}$ and $M_{\lambda'}$ belong to the same block of $D(B)$ if and only if $\lambda$ and $\lambda'$ are in the same $\iota$-orbit.

We have constructed in §1.G a central element, namely $\theta$. Note that

$$(2.3) \quad \text{The element } \theta \text{ acts on } M_{l,p} \text{ by multiplication by } \zeta^{(p-1)(l+p-1)}. $$

**Proof.** — It is sufficient to compute the action of $\theta$ on the vector $e_1^{l,p}$. Note that $E_i e_1^{l,p} = 0$ as soon as $i \geq 1$. Therefore, for computing $\theta e_1^{l,p}$ using the formula (1.12), only the terms corresponding to $i = 0$ remain. Consequently,

$$\omega_{l,p}(\theta) = \frac{1}{d} \sum_{j,k=0}^{d-1} \zeta^{-jk} \zeta^{(2p+l-1)k} \zeta^{(p+l-1)(j-k-1)} = \frac{\zeta^{l+p-1} \sum_{k=0}^{d-1} \zeta^{pk} \left( \sum_{j=0}^{d-1} \zeta^{(p+l-1-k)j} \right)}{d}$$

The term inside the big parenthesis is equal to $d$ if $p+l-1-k \equiv 0 \mod d$, and is equal to 0 otherwise. The result follows. \qed
2.C. Projective modules. — Given \( \lambda \in \Lambda(d) \), we denote by \( P_\lambda \) a projective cover of \( M_\lambda \). The next result is proved in [EGST1, Corollary 2.25].

**Theorem 2.4 (Erdmann-Green-Snashall-Taillefer).** — Let \( \lambda \in \Lambda(d) \).

(a) If \( \lambda \in \Lambda^d(d) \), then \( \dim_C(P_\lambda) = 2d \), \( \text{Rad}^3(P_\lambda) = 0 \) and the Loewy structure of \( P_\lambda \) is given by:

\[
\begin{align*}
P_\lambda / \text{Rad}(P_\lambda) & \cong M_\lambda \\
\text{Rad}(P_\lambda)/\text{Rad}^2(P_\lambda) & \cong M_{\lambda(1)} \oplus M_{\lambda(2)} \\
\text{Rad}^2(P_\lambda) & \cong M_\lambda
\end{align*}
\]

(b) \( P_{d,p} = M_{d,p} \) has dimension \( d \).

3. Tensor structure

We mainly refer here to the work of Erdmann, Green, Snashall and Taillefer [EGST1], [EGST2]. Since \( D(B) \) is a finite dimensional Hopf algebra, the category \( D(B)\text{-mod} \) inherits a structure of a tensor category. We will compute here some tensor products between simple modules. We will denote by \( M_i \) the simple module \( M_{1,0} \).

3.A. Invertible modules. — We denote by \( V_\xi = \mathbb{C}v_\xi \) the one-dimensional \( D(B)\text{-module} \) associated with the morphism \( \varepsilon_\xi : D(B) \to \mathbb{C} \) defined in §1.D:

\[
b v_\xi = \varepsilon_\xi(b)v_\xi
\]

for all \( b \in D(B) \). We have

\[(3.1) \quad V_\xi \cong M_{1,p}.
\]

An immediate computation using the comultiplication \( \Delta \) shows that

\[(3.2) \quad M_{1,p} \otimes V_{\xi q} \cong V_{\xi q} \otimes M_{1,p} \cong M_{1,p+q}
\]

as \( D(B)\text{-modules} \). The \( V_\xi \)'s are (up to isomorphism) the only invertible objects in the tensor category \( D(B)\text{-mod} \).
3.B. Tensor product with $M_2$. — We set $e_i = e^{(2,0)}_i$ for $i \in \{1, 2\}$, so that $(e_1, e_2)$ is the standard basis of $M_2$. The next result is a particular case of [EGST1, Theorem 4.1].

**Theorem 3.3 (Erdmann-Green-Snashall-Tailléfer).** — Let $(l, p)$ be an element of $\{1, 2, \ldots, d\} \times \mathbb{Z}/d\mathbb{Z}$.

(a) If $l \leq d - 1$, then $M_2 \otimes M_{l,p} \simeq M_{l+1,p} \oplus M_{l-1,p+1}$.

(b) $M_2 \otimes M_{d,p} \simeq P_{d-1,p}$.

4. Grothendieck rings

We denote by $\text{Gr}(D(B))$ the Grothendieck ring of the category of (left) $D(B)$-modules.

4.A. Structure. — Since $D(B)$ is a braided Hopf algebra (with universal $R$-matrix $R$),

\[(4.1) \quad \text{the ring } \text{Gr}(D(B)) \text{ is commutative.}\]

Given $M$ a $D(B)$-module, we denote by $[M]$ the class of $M$ in $\text{Gr}(D(B))$. We set

\[m_\lambda = [M_\lambda], \quad m_l = [M_{l,0}] \quad \text{and} \quad v_\zeta = [V_\zeta] \in \text{Gr}(D(B)),\]

Recall that $v_\zeta^p = m_{1,p}$. It follows from (3.2) and Theorem 3.3 that

\[(4.2) \quad v_\zeta^q m_{l,p} = m_{l,p+q} \quad \text{and} \quad m_2 m_{l,p} = \begin{cases} m_{l+1,p} + m_{l-1,p+1} & \text{if } l \leq d - 1, \\ 2(m_{d-1,p} + m_{1,p-1}) & \text{if } l = d. \end{cases}\]

**Proposition 4.3.** — The Grothendieck ring $\text{Gr}(D(B))$ is generated by $v_\zeta$ and $m_2$.

*Proof.* — We will prove by induction on $l$ that $m_{l,p} \in \mathbb{Z}[v_\zeta, m_2]$. Since $m_{1,p} = (v_\zeta)^p$, this is true for $l = 1$. Since $m_{2,p} = (v_\zeta)^p m_2$, this is also true for $l = 2$. Now the induction proceeds easily by using (4.2).
4.B. Some characters. — If $b \in D(B)$ is group-like, then the map
\[
\begin{array}{ccc}
Gr(D(B)) & \longrightarrow & \mathbb{C} \\
[M] & \longrightarrow & Tr(b|_M)
\end{array}
\]
is a morphism of rings. Here, $Tr$ denotes the usual trace (not the quantum trace) of an endomorphism of a finite dimensional vector space. Recall from Lemma 1.7 that the only group-like elements of $D(B)$ are the $K^i z^j$, where $(i, j) \in \Lambda(d)$. We set
\[
\chi_{i,j}: Gr(D(B)) \longrightarrow \mathbb{C}
\]
mapsto $\langle Tr(K^i z^j | M) \rangle$.

An easy computation yields
\[
\chi_{i,j}(m_{i,p}) = \zeta p i + (2p + l - 1) j \cdot (1)_{\zeta}.
\]

Note that the $\chi_{i,j}$’s are not necessarily distinct:

**Lemma 4.5.** — Let $\lambda$ and $\lambda'$ be two elements of $\Lambda(d)$. Then $\chi_{\lambda} = \chi_{\lambda'}$ if and only if $\lambda$ and $\lambda'$ are in the same $\iota$-orbit.

**Proof.** — Let us write $\lambda = (i, j)$ and $\lambda' = (i', j')$. The “if” part follows directly from (4.4). Conversely, assume that $\chi_{i,j} = \chi_{i',j'}$. By applying these two characters to $v_\zeta$ and $m_2$, we get:
\[
\begin{cases}
\zeta^{i+2j} = \zeta^{i'+2j'}, \\
\zeta^j(1 + \zeta') = \zeta^{i'}(1 + \zeta')
\end{cases}
\]
It means that the pairs $(\zeta^j, \zeta^{i+j})$ and $(\zeta^{i'}, \zeta^{i'+j'})$ have the same sum and the same product, so $(\zeta^j, \zeta^{i+j}) = (\zeta^{i'}, \zeta^{i'+j'})$ or $(\zeta^j, \zeta^{i+j}) = (\zeta^{i'+j'}, \zeta^{j'})$. In other words, $(i', j') = (i, j)$ or $(i', j') = \iota(i, j)$, as expected.

5. Triangulated categories

5.A. Stable category. — As $B$ is a Hopf algebra, it is selfinjective, i.e., $B$ is an injective $B$-module. Recall that the stable category $B$-stab of $B$ is the additive category quotient of $B$-mod by the full subcategory $B$-proj of projective modules. Since $B$ is selfinjective, the category $B$-stab has a natural triangulated structure. Similarly, the category $D(B)$-stab is triangulated. Note that a $B$-module (resp. a $D(B)$-module) is projective if and only if it is injective. Since the tensor product of a projective $D(B)$-module by any $D(B)$-module is still
We denote by $\text{Gr}(D(B))$ a further quotient. In particular, its Grothendieck group (as a triangulated category), which will be denoted by $\text{Gr}^{\text{st}}(D(B))$, is a ring and the natural map

$$\text{Gr}(D(B)) \longrightarrow \text{Gr}^{\text{st}}(D(B))$$

$$m \longrightarrow m^{\text{st}}$$

is a morphism of rings. Given $M$ a $D(B)$-module, we denote by $[M]_{\text{st}}$ its class in $\text{Gr}^{\text{st}}(D(B))$.

It follows from Theorem 2.4 that

$$m^{\text{st}}_{d,p} = 0 \quad \text{and} \quad 2(m^{\text{st}}_{l,p} + m^{\text{st}}_{d-l,p}) = 0$$

if $l \leq d - 1$.

5.B. A further quotient. — We denote by $D(B)$-$\text{proj}_{B}$ the full subcategory of $D(B)$-$\text{mod}$ whose objects are the $D(B)$-modules $M$ such that $\text{Res}^{D(B)}_{B} M$ is a projective $B$-module. Since $D(B)$ is a free $B$-module (of rank $d^2$), $D(B)$-$\text{proj}$ is a full subcategory of $D(B)$-$\text{proj}_{B}$. We denote by $D(B)$-$\text{stab}_{B}$ the additive quotient of the category $D(B)$-$\text{mod}$ by the full subcategory $D(B)$-$\text{proj}_{B}$: it is also the quotient of $D(B)$-$\text{stab}$ by the image of $D(B)$-$\text{proj}_{B}$ in $D(B)$-$\text{stab}$.

Lemma 5.2. — The image of $D(B)$-$\text{proj}_{B}$ in $D(B)$-$\text{stab}$ is a thick triangulated subcategory. In particular, $D(B)$-$\text{stab}_{B}$ is triangulated.

Proof. — Given $M$ a $D(B)$-module, we denote by $\pi_{M} : P(M) \rightarrow M$ (resp. $i_{M} : M \hookrightarrow I(M)$) a projective cover (resp. injective hull) of $M$. We need to prove the following facts:

(a) If $M$ belongs to $D(B)$-$\text{proj}_{B}$, then $\text{Ker}(\pi_{M})$ and $I(M)/\text{Im}(i_{M})$ also belong to $D(B)$-$\text{proj}_{B}$.

(b) If $M \oplus N$ belongs to $D(B)$-$\text{proj}_{B}$, then $M$ and $N$ also belong to $D(B)$-$\text{proj}_{B}$.

(c) If $M$ and $N$ belong to $D(B)$-$\text{proj}_{B}$ and $f : M \rightarrow N$ is a morphism of $D(B)$-modules, then the cone of $f$ also belong to $D(B)$-$\text{proj}_{B}$.

(a) Assume that $M$ belongs to $D(B)$-$\text{proj}_{B}$. Since it is a projective $B$-module, there exists a morphism of $B$-modules $f : M \rightarrow P(M)$ such that $\pi_{M} \circ f = \text{Id}_{M}$. In particular, $P(M) \simeq \text{Ker}(\pi_{M}) \oplus M$, as a $B$-module. So $\text{Ker}(\pi_{M})$ is a projective $B$-module.

On the other hand, $I(M)$ is a projective $D(B)$-module since $D(B)$ so it is a projective $B$-module and so it is an injective $B$-module. So, again, $I(M) \simeq M \oplus I(M)/\text{Im}(i_{M})$, so $I(M)/\text{Im}(i_{M})$ is a projective $B$-module. This proves (a).
(b) is obvious.

(c) Let $M$ and $N$ belong to $(B)\text{-proj}_B$ and $f : M \to N$ be a morphism of $(B)\text{-modules}$. Let $\Delta_f : M \to I(M) \oplus N$, $m \mapsto (i_M(m), f(m))$. Then the cone of $f$ is isomorphic in $(B)\text{-stab}$ to $(I(M) \oplus N)/\text{Im}(\Delta_f)$. But $\Delta_f$ is injective, $M$ is an injective $B$-module and so $I(M) \simeq M \oplus (I(M) \oplus N)/\text{Im}(\Delta_f)$ as a $B$-module, which shows that $(I(M) \oplus N)/\text{Im}(\Delta_f)$ is a projective $B$-module. \[\square\]

We denote by $Gr^{st}_B(D(B))$ the Grothendieck group of $(B)\text{-stab}_B$, viewed as a triangulated category. If $M$ belongs to $(B)\text{-proj}_B$ and $N$ is any $(B)\text{-module}$, then $M \otimes N$ and $N \otimes M$ are projective $B$-modules [EGNO, Proposition 4.2.12], so $(B)\text{-stab}_B$ inherits a structure of monoidal category, compatible with the triangulated structure. This endows $Gr^{st}_B(D(B))$ with a ring structure. The natural map $Gr(D(B)) \to Gr^{st}_B(D(B))$ will be denoted by $m \mapsto m^{st}_B$: it is a surjective morphism of rings that factors through $Gr^{st}_B(D(B))$.

If $\lambda \in \Lambda^0(d)$, then it follows from [EGST2, Property 1.4] that there exists a $(B)\text{-module}$ $P^B_{\lambda}$ which is projective as a $B$-module and such that there is an exact sequence

$$0 \to M_{\iota(\lambda)} \to P^B_{\lambda} \to M_{\lambda} \to 0.$$  

It then follows that

$$(5.3) \quad m^{st}_\lambda + m^{st}_{\iota(\lambda)} = 0.$$  

Also, we still have

$$(5.4) \quad m^{st}_{d, \mu} = 0.$$  

The next theorem follows from (5.3), (5.4) and Proposition 4.3.

**Theorem 5.5.** — The ring $Gr^{st}_B(D(B))$ is generated by $v^{st}_x$ and $m^{st}_x$. Moreover,

$$Gr^{st}_B(D(B)) = \bigoplus_{\lambda \in \Lambda^0(d)/\iota} \mathbb{Z}m^{st}_\lambda$$

and $Gr^{st}_B(D(B))$ is a free $\mathbb{Z}$-module of rank $d(d - 1)/2$.

Recall that Lemma 4.5 shows that, through the $\chi_i, j$’s, only $d(d + 1)/2$ different characters of the ring $Gr(D(B))$ have been defined. It is not clear if $\mathbb{C}Gr(D(B))$ is semisimple in general but, for $d = 2$, it can be checked that it is semisimple (of dimension 4), so that there is a fourth character $Gr(D(B)) \to \mathbb{C}$ which is not obtained through the $\chi_i, j$’s.

Now, a character $\chi : Gr(D(B)) \to \mathbb{C}$ factors through $Gr^{st}_B(D(B))$ if and only if its kernel contains the $m_{\lambda, \delta}$’s (where $\lambda$ runs over $\Lambda^0(d)$) and the $m_{\lambda, \iota(\lambda)}$’s (where $\lambda$ runs over $\Lambda^0(d)$). This implies the following result.
Theorem 5.6. — The character \( \chi_\lambda : \text{Gr}(D(B)) \to \mathbb{C} \) factors through \( \text{Gr}^{\text{st}}(D(B)) \) if and only if \( \lambda \in \Lambda^d(d) \). So the \( \{ \chi_\lambda \}_{\lambda \in \Lambda^d(d)} \) are all the characters of \( \text{Gr}^{\text{st}}(D(B)) \) and the \( \mathbb{C} \)-algebra \( \mathbb{C}\text{Gr}^{\text{st}}(D(B)) \) is semisimple.

5.C. Complements. — Given \( \mathcal{C} \) a monoidal category, we denote by \( Z(\mathcal{C}) \) its Drinfeld center (see [Ka, §XIII.4]) and we denote by \( \text{For}_\mathcal{C} : Z(\mathcal{C}) \to \mathcal{C} \) the forgetful functor.

There is an equivalence between \( Z(B\text{-mod}) \) and \( D(B)\text{-mod} \) such that the forgetful functor becomes the restriction functor \( \text{Res}_B^{D(B)} \). The canonical functors between these categories will be denoted by \( \text{can}_{\text{st}}^{D(B)} : D(B)\text{-mod} \to D(B)\text{-stab} \) and \( \text{can}_{\text{st}}^B : B\text{-mod} \to B\text{-stab} \). The functor

\[
\text{can}_{\text{st}}^B \circ \text{Res}_B^{D(B)} : D(B)\text{-mod} \to B\text{-stab}
\]

factors through \( Z(B\text{-stab}) \) (this triangulated category needs to be defined in a homotopic setting, for example that of stable \( \infty \)-categories). We obtain a commutative diagram of functors

\[
\begin{array}{c}
D(B)\text{-mod} \xrightarrow{\text{Res}_B^{D(B)}} B\text{-mod} \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
Z(B\text{-stab}) \xrightarrow{\text{For}_{B\text{-stab}}} B\text{-stab}.
\end{array}
\]

Since any \( D(B) \)-module that is projective as a \( B \)-module is sent to the zero object of \( Z(B\text{-stab}) \) through \( \mathcal{F} \), the functor \( \mathcal{F} \) factors through \( D(B)\text{-proj}_B \) and we obtain a commutative diagram of functors

\[
\begin{array}{c}
D(B)\text{-stab}_B \xrightarrow{\text{can}_{\text{stab}}^{D(B)}} D(B)\text{-mod} \xrightarrow{\text{Res}_B^{D(B)}} B\text{-mod} \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
Z(B\text{-stab}) \xrightarrow{\text{For}_{B\text{-stab}}} B\text{-stab}.
\end{array}
\]

Question. Is \( \mathcal{F} : D(B)\text{-stab}_B \to Z(B\text{-stab}) \) an equivalence of categories?
6. Fusion datum

6.A. Quantum traces. — The element \( R \in D(B) \otimes D(B) \) defined in §1.F is a universal \( R \)-matrix which endows \( D(B) \) with a structure of braided Hopf algebra. The category \( D(B)\text{-mod} \) is braided as follows: given \( M \) and \( N \) two \( D(B) \)-modules, the braiding \( c_{M,N} : M \otimes N \to N \otimes M \) is given by

\[
c_{M,N}(m \otimes n) = \tau(R)(n \otimes m).
\]

Recall that \( \tau : D(B) \otimes D(B) \to D(B) \otimes D(B) \) is given by \( \tau(a \otimes b) = b \otimes a \). In particular,

\[
(6.1) \quad c_{N,M}c_{M,N} : M \otimes N \to M \otimes N \text{ is given by the action of } \tau(R)R.
\]

Given \( i \in \mathbb{Z} \), we have \( S^2(b) = (z^{-i}K)b(z^{-i}K)^{-1} \) for all \( b \in D(B) \) and \( z^{-i}K \) is group-like, so the algebra \( D(B) \) is pivotal with pivot \( z^{-i}K \). This endows the tensor category \( D(B)\text{-mod} \) with a structure of pivotal category (see Appendix A) whose associated traces \( \text{Tr}^i \) and \( \text{Tr}^i_- \) are given as follows: given \( M \) a \( D(B) \)-module and \( f \in \text{End}_{D(B)}(M) \), we have

\[
\text{Tr}^i(f) = \text{Tr}(z^{-i}Kf) \quad \text{and} \quad \text{Tr}^i_-(f) = \text{Tr}(fK^{-1}z^i).
\]

Recall that \( \text{Tr} \) denotes the “classical” trace for endomorphisms of a finite dimensional vector space. So the pivotal structure depends on the choice of \( i \) (modulo \( d \)). The corresponding twist is \( \theta_i = z^i \theta \), which endows \( D(B)\text{-mod} \) with a structure of balanced braided category (depending on \( i \)).

Hypothesis and notation. From now on, and until the end of this paper, we assume that the Hopf algebra \( D(B) \) is endowed with the pivotal structure whose pivot is \( z^{-i}K \). The structure of balanced braided category is given by \( \theta_i = z \theta \) and the associated quantum traces \( \text{Tr}^i_\pm \) are denoted by \( \text{Tr}^i \).

Given \( M \) a \( D(B) \)-module, we set \( \dim_+ (M) = \text{Tr}_+ (M) \).

We define

\[
\dim(D(B)) = \sum_{M \in \text{Irr} D(B)} \dim_+ (M) \dim_- (M).
\]

We have

\[
(6.2) \quad \dim(D(B)) = \frac{2d^2}{(1-\zeta)(1-\zeta^{-1})}.
\]

This follows easily from the fact that

\[
(6.3) \quad \dim_+ M_{l,p} = \zeta^{1-l-p}(I) \zeta \quad \text{and} \quad \dim_- M_{l,p} = \zeta^{p+l-1}(I) \zeta^{-1} = \zeta^p(I) \zeta.
\]
6.B. Characters of \( \text{Gr}(D(B)) \) via the pivotal structure. — As in Appendix A, these structures (braiding, pivot) allow to define characters of \( \text{Gr}(D(B)) \) associated with simple modules (or bricks). Given \( \lambda \in \Lambda(d) \), we set

\[
\begin{align*}
    s_{M_\lambda}^+: \quad \text{Gr}(D(B)) & \rightarrow \mathbb{C} \\
    [M] & \rightarrow (\text{Id}_{M_\lambda} \otimes \text{Tr}_+^M)(c_{M,M_\lambda}c_{M_\lambda,M})
\end{align*}
\]
and

\[
\begin{align*}
    s_{M_\lambda}^-: \quad \text{Gr}(D(B)) & \rightarrow \mathbb{C} \\
    [M] & \rightarrow (\text{Tr}_+^M \otimes \text{Id}_{M_\lambda})(c_{M,M_\lambda}c_{M_\lambda,M})
\end{align*}
\]

These are morphism of rings (see Proposition A.4). The main result of this section is the following.

**Theorem 6.4.** — Given \( \lambda \in \Lambda(d) \), we have

\[
s_{M_\lambda}^+ = \chi_\lambda \quad \text{and} \quad s_{M_{(0,1)}}^- = \chi_{(0,1) - \lambda}.
\]

**Proof.** — Write \( \lambda = (l, p) \) and

\[
T_{i,j,k} = \frac{\varphi(i-k)(i+j)-(i+1)/2}{(i)!z}.
\]

We have

\[
\tau(R) = \frac{1}{d^2} \sum_{i', j', k', k''=0}^{d-1} T_{i,j,k} T_{i',j',k'} \varphi(i-k') (z^{-k'} E^{i'} K^{k'+j}) \otimes (z^{-k} E^i K^{i+k}).
\]

We need to compute the endomorphism of \( M_{l,p} \) given by

\[
(\text{Id}_{M_{l,p}} \otimes \text{Tr}_+^M)(\tau(R) |_{M_{l,p} \otimes M}).
\]

Since \( M_{l,p} \) is simple, this endomorphism is the multiplication by a scalar \( \sigma \), and so it is sufficient to compute the action on \( e_1^{(l,p)} \in M_{l,p} \). Therefore, all the terms (in the big sum giving \( \tau(R) \)) corresponding to \( i \neq 0 \) disappear (because \( E e_1^{(l,p)} = 0 \)). Also, since we are only interested in the coefficient on \( e_1^{(l,p)} \) of the result (because the coefficients on other vectors will be zero), all the terms corresponding to \( i' \neq 0 \) also disappear. Therefore,

\[
\sigma = \frac{1}{d^2} \sum_{j',k',k'' \in \mathbb{Z}/d \mathbb{Z}} \varphi^{-k-j-k'} (z^{2p+1-i'-1} E^{i'} K^{i'+j} \text{Tr}(z^{-1} K z^j K^{i'+k})).
\]

So it remains to compute the element

\[
b = \frac{1}{d^2} \sum_{j',k',k'' \in \mathbb{Z}/d \mathbb{Z}} \varphi^{-k-j-k'} (z^{2p+1-i'-1} E^{i'} K^{i'+j} z^{-k-1} K^{i'+j+1})
\]
of $D(B)$. Since
\[ b = \frac{1}{d^2} \sum_{j',k \in \mathbb{Z}/d\mathbb{Z}} \left( \sum_{k' \in \mathbb{Z}/d\mathbb{Z}} \zeta^{l(p+l-1-k) + k'(-p-j')} z^{-k-1} K_j' k+1, \right) \]
only the terms corresponding to $k = l + p - 1$ and $j' = -p$ remain, hence
\[ b = z^{l-p} K_l. \]
So $s_{M_l}^k = \chi_{-l} = \chi_{-\lambda}$, as expected.

The other formula is obtained via a similar computation.

We denote by $S^\pm = (S^\pm_{\lambda,\lambda'})_{\lambda,\lambda' \in \Lambda(d)}$ the square matrix defined by
\[ S^\pm_{\lambda,\lambda'} = \text{Tr}_4 (c_{M_{\lambda'}, M_{\lambda}} \circ c_{M_{\lambda}, M_{\lambda'}}). \]
Similary, we define $T^\pm$ to be the diagonal matrix (whose rows and columns are indexed by $\lambda \in \Lambda(d)$) and whose $\lambda$-entry is
\[ T^\pm_{\lambda} = \omega_{\lambda} (\vartheta_1^{\pm 1}). \]
Let us first give a formula for $S^\pm_{\lambda,\lambda'}$ and $T^\pm_{\lambda}$.

**Corollary 6.5.** — Let $(l, p), (l', p') \in \Lambda(d)$. We have
\[ S^+_{(l, p), (l', p')} = \frac{\zeta}{1 - \zeta} \zeta^{l'l' - l - p - p' - 2pp'} (1 - \zeta^{ll'}), \quad T^+_{(l, p)} = \zeta^{l'} \zeta^{l - l'}, \]
\[ S^-_{(l, p), (l', p')} = \frac{\zeta^{2p+l} + 2p' + l' - 1}{1 - \zeta} \zeta^{l'l' - l - p - p' - 2pp'} (1 - \zeta^{ll'}) \quad \text{and} \quad T^-_{(l, p)} = \zeta^{p + p'}. \]

**Proof.** — This follows immediately from formulas (2.3), (4.4), (6.3) and Theorem 6.4.
6.C. Fusion datum associated with $D(B)$-stab$_g$. — Let $\mathcal{E}$ denote a set of representatives of $\iota$-orbits in $\{1, 2, \ldots, d - 1\} \times \mathbb{Z}/d\mathbb{Z}$. We define

$$\dim^{st}g(D(B)) = \sum_{(l, p) \in \mathcal{E}} \dim_{-}(M_{l, p}) \dim_{+}(M_{l, p}).$$

This can be understood in terms of super-categories, as explained recently by Lacabanne $[\text{La1}]$. We have

$$\dim^{st}g(D(B)) = \frac{1}{2} \dim(D(B)) = \frac{d^2}{(1 - \zeta)(1 - \zeta^{-1})}. $$

So $\dim^{st}g(D(B))$ is a positive real number and we denote by $\sqrt{\dim^{st}g(D(B))}$ its positive square root. Since $1 - \zeta^{-1} = -\zeta^{-1}(1 - \zeta)$, there exists a unique square root $\sqrt{-\zeta}$ of $-\zeta$ such that

$$\sqrt{\dim^{st}g(D(B))} = \frac{d \sqrt{-\zeta}}{1 - \zeta}.$$ 

We denote by $S^{st}(\lambda, \lambda') = \left(\sum_{\iota(\lambda, \lambda')} S^{st}_{\lambda, \lambda'}(l, p), \iota(\lambda, \lambda') = \left(\sum_{\iota(\lambda', \lambda)} S^{st}_{\lambda', \lambda}(l', p'), \iota(\lambda, \lambda') = \left(\sum_{\iota(\lambda', \lambda)} S^{st}_{\lambda, \lambda'}(l, p), \iota(\lambda', \lambda) = \left(\sum_{\iota(\lambda, \lambda')} S^{st}_{\lambda', \lambda}(l', p') \right) \right) \right).$$. 

We denote by $T^{st}(\lambda, \lambda') = \left(\sum_{\iota(\lambda, \lambda')} T^{st}_{\lambda, \lambda'}(l, p), \iota(\lambda, \lambda') = \left(\sum_{\iota(\lambda', \lambda)} T^{st}_{\lambda', \lambda}(l', p'), \iota(\lambda, \lambda') = \left(\sum_{\iota(\lambda, \lambda')} T^{st}_{\lambda', \lambda}(l, p), \iota(\lambda', \lambda) = \left(\sum_{\iota(\lambda, \lambda')} T^{st}_{\lambda, \lambda'}(l', p') \right) \right) \right).$$. 

The root of unity $\sqrt{-\zeta}$ appearing in this formula has been interpreted in terms of super-categories by Lacabanne $[\text{La1}]$: it is due to the fact that our category is not spherical. Finally, note that

$$\sqrt{-\zeta} = \frac{d \sqrt{-\zeta}}{1 - \zeta}.$$ 

We denote by $S^{st}(\lambda, \lambda') = \left(\sum_{\iota(\lambda, \lambda')} S^{st}_{\lambda, \lambda'}(l, p), \iota(\lambda, \lambda') = \left(\sum_{\iota(\lambda', \lambda)} S^{st}_{\lambda', \lambda}(l', p'), \iota(\lambda, \lambda') = \left(\sum_{\iota(\lambda, \lambda')} S^{st}_{\lambda, \lambda'}(l, p), \iota(\lambda', \lambda) = \left(\sum_{\iota(\lambda, \lambda')} S^{st}_{\lambda', \lambda}(l', p') \right) \right) \right).$$. 

7. Comparison with Malle $\mathbb{Z}$-fusion datum

We refer to $[\text{Ma}]$ and $[\text{Cu}]$ for most of the material of this section. We denote by $\mathcal{E}(d)$ the set of pairs $(i, j)$ of integers with $0 \leq i < j \leq d - 1$. 


7.A. Set-up. — Let \( Y = \{0, 1, \ldots, d\} \) and let \( \pi : Y \rightarrow \{0, 1\} \) be the map defined by

\[
\pi(i) = \begin{cases} 
1 & \text{if } i \in \{0, 1\}, \\
0 & \text{if } i \geq 2.
\end{cases}
\]

We denote by \( \Psi(Y, \pi) \) the set of maps \( f : Y \rightarrow \{0, 1, \ldots, d-1\} \) such that \( f \) is strictly increasing on \( \pi^{-1}(0) = \{2, 3, \ldots, d\} \) and strictly increasing on \( \pi^{-1}(1) = \{0, 1\} \). Since \( f \) is injective on \( \{2, 3, \ldots, d\} \), there exists a unique element \( k(f) \in \{0, 1, \ldots, d-1\} \) which does not belong to \( f([2, 3, \ldots, d]) \). Note that, since \( f \) is strictly increasing on \( \{2, 3, \ldots, d\} \), the element \( k(f) \) determines the restriction of \( f \) to \( \{2, 3, \ldots, d\} \). So the map

\[
\Psi(Y, \pi) \rightarrow \mathcal{J}(d) \times \{0, 1, \ldots, d-1\}
\]

\[
f \mapsto (f(0), f(1), k(f))
\]

is bijective. For \( f \in \Psi(Y, \pi) \), we set

\[
\varepsilon(f) = (-1)\det(\mathcal{Y}) = \prod_{0 \leq i < j \leq d-1} (\zeta^j - \zeta^i).
\]

We put by \( V = \bigoplus_{i=0}^{d-1} C v_i \) and we denote by \( \mathcal{Y} \) the square matrix \((\zeta^{ij})_{0 \leq i, j \leq d-1}\), which will be viewed as an automorphism of \( V \). Note that \( \mathcal{Y} \) is the character stable of the cyclic group \( \mu_d \). We set \( \delta(d) = \det(\mathcal{Y}) = \prod_{0 \leq i < j \leq d-1} (\zeta^j - \zeta^i) \).

Recall that \( \delta(d)^2 = (-1)^{\frac{(d-1)(d-2)}{2} d^d} \).

Given \( f \in \Psi(Y, \pi) \), let

\[
v_f = (v_f(0) \wedge v_f(1)) \otimes (v_f(2) \wedge v_f(3) \wedge \cdots \wedge v_f(d)) \in \bigwedge^2 V \otimes \bigwedge^{d-1} V.
\]

Note that \((v_f)_f \in \Psi(Y, \pi)\) is a C-basis of \( \bigwedge^2 V \otimes \bigwedge^{d-1} V \). Given \( f' \in \Psi(Y, \pi) \), we put

\[
(\bigwedge^2 \mathcal{Y} \otimes \bigwedge^{d-1} \mathcal{Y})(v_{f'}) = \sum_{f \in \Psi(Y, \pi)} S_{f', f} v_f.
\]

In other words, \((S_{f', f})_{f', f \in \Psi(Y, \pi)}\) is the matrix of the automorphism \( \bigwedge^2 \mathcal{Y} \otimes \bigwedge^{d-1} \mathcal{Y} \) of \( \bigwedge^2 V \otimes \bigwedge^{d-1} V \) in the basis \((v_f)_{f \in \Psi(Y, \pi)}\).

**Lemma 7.2.** — Let \( f, f' \in \Psi(Y, \pi) \). We define

\[
i = f(0), \quad j = f(1), \quad k = k(f),
\]

\[
i' = f'(0), \quad j' = f'(1), \quad k' = k(f').
\]

We have

\[
S_{f, f'} = (-1)^{k + k'} \frac{\delta(d)}{d} \zeta^{-kk'} (\zeta^{ij + ji'} - \zeta^{ij' + ji}).
\]
Proof. — The computation of the action of $\bigwedge^2 \mathcal{F}$ is easy, and gives the term $\zeta^{(i+j)i'} - \zeta^{(i+j)i}$, $i, j, i' \in \mathbb{Z}$. It remains to show that the determinant of the matrix $\mathcal{F}(k, k')$ obtained from $\mathcal{F}$ by removing the $k$-th row and the $k'$-th column is equal to $(-1)^{k+k'} \zeta^{-kk'} \delta(d)/d$. For this, let $\mathcal{F}'(k)$ denote the matrix whose $k$-th row is equal to $(1, t, t^2, \ldots, t^{d-1})$ (where $t$ is an indeterminate) and whose other rows coincide with those of $\mathcal{F}$. Then $(-1)^{k+k'} \det(\mathcal{F}'(k), k')$ is equal to the coefficient of $t^{k'}$ in the polynomial $\det(\mathcal{F}'(k))$. This is a Vandermonde determinant and

$$
\det(\mathcal{F}'(k)) = \prod_{0 \leq i < j \leq d-1 \atop i \neq k, j \neq k} (\zeta^i - \zeta^j) \cdot \prod_{i=0}^{d-1} (t - \zeta^i) \cdot \prod_{i=i+1}^{d-1} (\zeta^i - t)
$$

$$
= \delta(d) \prod_{i=0}^{d-1} (t - \zeta^i).
$$

Since

$$
\prod_{i=0}^{d-1} (t - \zeta^i) = \frac{t^d - 1}{t - \zeta} = \sum_{i=0}^{d-1} \zeta^{(d-1-i)k},
$$

we have

$$
\det(\mathcal{F}(k, k')) = (-1)^{k+k'} \delta(d) \frac{\zeta^{(d-1-k)k}}{\zeta^{(d-1)k}} = (-1)^{k+k'} \frac{\delta(d)}{d} \zeta^{-kk'},
$$

as desired.

\[ \square \]

7.B. Malle $\mathbb{Z}$-fusion datum. — Let

$$
\Psi^\#(Y, \pi) = \{ f \in \Psi(Y, \pi) | \sum_{y \in Y} f(y) \equiv \frac{d(d-1)}{2} \mod d \}.
$$

Given $f \in \Psi^\#(Y, \pi)$, we define

$$
\text{Fr}(f) = \zeta_{\pi^\#}^{d(1-d^2)} \prod_{y \in Y} \zeta_{\pi^\#}^{-d(f(y)^2 + df(y))},
$$

where $\zeta_{\pi}$ is a primitive $(12d)$-th root of unity such that $\zeta_{\pi}^{12} = \zeta$.

We denote by $T$ diagonal matrix (whose rows and column are indexed by $\Psi^\#(Y, \pi)$) equal to $\text{diag}(\text{Fr}(f))_{f \in \Psi^\#(Y, \pi)}$. We denote by $S = (S_{f,g})_{f,g \in \Psi^\#(Y, \pi)}$ the square matrix defined by

$$
S_{f,g} = \frac{(-1)^{d-1}}{\delta(d)} \zeta(f)\zeta(g) S_{f,g}.
$$

Note that $S_{f,f_{01}} \neq 0$ for all $f \in \Psi^\#(Y, \pi)$ (see Lemma 7.2).
**Proposition 7.3 (Malle [Ma], Cuntz [Cu]).** — With the previous notation, we have:

(a) \( S^4 = (ST)^3 = [S^2, T] = 1 \).
(b) \( T^S = S \) and \( T^S T^S = 1 \).
(c) For all \( f, g, h \in \Psi^\pi(Y, \pi) \), the number

\[
N^h_{f, g} = \sum_{i \in \Psi^\pi(Y, \pi)} \frac{S_{i,f} S_{i,g} S_{i,h}}{S_{i,f_0,1}}
\]

belongs to \( \mathbb{Z} \).

The pair \((S, T)\) is called the **Malle \( Z \)-fusion datum**.

**7.C. Comparison.** — We wish to compare the \( Z \)-fusion datum \((S, T)\) with the ones obtained from the tensor categories \( D(B)\)-mod and \( D(B)\)-stab. For this, we will use the bijection (7.1) to characterize elements of \( \Psi^\pi(Y, \pi) \). Given \( k \in \mathbb{Z} \), we denote by \( k^\text{res} \) the unique element in \( \{0, 1, \ldots, d-1\} \) such that \( k \equiv k^\text{res} \mod d \).

**Lemma 7.4.** — Let \( f \in \Psi^\pi(Y, \pi) \). Then \( f \in \Psi^\pi(Y, \pi) \) if and only if \( k(f) = (f(0) + f(1))^\text{res} \). Consequently, the map

\[
\Psi^\pi(Y, \pi) \to \mathcal{E}(d)
\]

\[
f \mapsto (f(0), f(1))
\]

is bijective.

**Proof.** — We have

\[
\sum_{y \in Y} f(y) = f(0) + f(1) + \frac{d(d-1)}{2} - k(f)
\]

and the result follows. \( \Box \)

Given \((i, j) \in \mathcal{E}(d)\), we denote by \( f_{i,j} \) the unique element of \( \Psi^\pi(Y, \pi) \) such that \( f_{i,j}(0) = i \) and \( f_{i,j}(1) = j \). We have

(7.5) \( Fr(f_{i,j}) = \zeta^{ij} \)

and, if \((i, j), (i', j') \in \Lambda(d)\), then

(7.6) \[
S_{f_{i,j}, f_{i', j'}} = \frac{(-1)^{(i+j)^\text{res} + (i'+j')^\text{res}}}{d} \epsilon(f_{i,j}) \epsilon(f_{i', j'}) (\zeta^{ij' + j'} - \zeta^{ij + j'}). 
\]
Proof. — The second equality follows immediately from Lemmas 7.2 and 7.4. Let us prove the first one. By definition, $\text{Fr}(f_{i,j}) = \xi^a$, where

$$a = d(1 - d^2) - 6 \sum_{y \in Y} (f_{i,j}(y)^2 + d f_{i,j}(y)).$$

The construction of $f_{i,j}$ shows that

$$a = d(1 - d^2) - 6(i^2 + d i) - 6(j^2 + d j) - 6 \sum_{k=0}^{d-1} (k^2 + d k) + 6((i + j)^{\text{res}})^2 + d(i + j)^{\text{res}}.$$

Write $i + j = (i + j)^{\text{res}} + \eta d$, with $\eta \in \{0, 1\}$. Then $\eta^2 = \eta$ and so

$$(i + j)^2 + d(i + j) = ((i + j)^{\text{res}})^2 + d(i + j)^{\text{res}} + 2\eta d(i + j) + 2\eta d^2 \equiv ((i + j)^{\text{res}})^2 + d(i + j)^{\text{res}} \mod 2d.$$

Therefore,

$$a \equiv 12i j + d(1 - d^2) - 6 \sum_{k=0}^{d-1} (k^2 + d k) \mod 12d \equiv 12i j \mod 12d.$$

So $\text{Fr}(f_{i,j}) = \xi^{12i j} = \xi^{i j}$. $lacksquare$

We define

$$\varphi : \mathcal{E}(d) \longrightarrow \Lambda^6(d),$$

$$(i, j) \longmapsto (j - i, i).$$

Note that $\varphi(\mathcal{E}(d))$ is a set of representatives of $t$-orbits in $\Lambda^6(d)$. We set

$$\tilde{\varphi}(i, j) = \begin{cases} 
\varphi(i, j) & \text{if } (-1)^{i+j} \varepsilon(f_{i,j}) = 1, \\
\delta(\varphi(i, j)) & \text{if } (-1)^{i+j} \varepsilon(f_{i,j}) = -1.
\end{cases}$$

Then $\tilde{\varphi}(\mathcal{E}(d))$ is also a set of representatives of $t$-orbits in $\Lambda^6(d)$ and the pairs of matrices $(\Sigma^{\text{stab}}_t, \Sigma^{\text{stab}}_t)$ and $(S, T)$ are related by the following equality (which follows immediately from Corollary 6.5 and formulas (6.7), (7.5) and (7.6)):

$$\Sigma_{f_{i,j}, f_{i',j'}} = \sqrt{-\xi} \Sigma_{\tilde{\varphi}(i,j), \tilde{\varphi}(i',j')} \quad \text{and} \quad T_{f_{i,j}} = T_{\tilde{\varphi}(i,j)},$$

Therefore, up to the change of $\xi$ into $\xi^{-1}$, we obtain our main result.

**Theorem 7.8.** — Malle $Z$-fusion datum $(S, T)$ can be categorified by the monoidal category $D(B)$-stab$_B$, endowed with the pivotal structure induced by the pivot $\xi^{-1}K$ and the balanced structure induced by $\varepsilon \Theta$.
A. Appendix. Reminders on S-matrices

We follow closely [EGNO, Chapters 4 and 8].

Let *C* be a tensor category over *C*, as defined in [EGNO, Definition 4.1.1]:

*C* is a locally finite *C*-linear rigid monoidal category (whose unit object is denoted by 1) such that the bifunctor \(\otimes: \mathcal{C}\times\mathcal{C}\to\mathcal{C}\) is *C*-bilinear on morphisms and End\(_{\mathcal{C}}(1) = \mathbb{C}\). If \(X\) is an object in \(\mathcal{C}\), its left (respectively right) dual is denoted by \(X^*\) (respectively \(*X\) and we denote by

\[
\text{coev}_X: 1 \to X \otimes X^* \quad \text{and} \quad \text{ev}_X: X^* \otimes X \to 1
\]

the coevaluation and evaluation morphisms respectively.

We assume that \(\mathcal{C}\) is braided, namely that it is endowed with a bifunctorial family of isomorphisms \(c_{X,Y}: X \otimes Y \to Y \otimes X\) such that

\[
\begin{align*}
(A.1) \quad & c_{X,Y \otimes Y'} = (\text{Id}_Y \otimes c_{X,Y'}) \circ (c_{X,Y} \otimes \text{Id}_{Y'}) \\
(A.2) \quad & c_{X \otimes X', Y} = (c_{X,Y} \otimes \text{Id}_{X'}) \circ (\text{Id}_X \otimes c_{X', Y}).
\end{align*}
\]

for all objects \(X, X', Y\) and \(Y'\) in \(\mathcal{C}\) (we have omitted the associativity constraints).

Finally, we also assume that \(\mathcal{C}\) is pivotal [EGNO, Definition 4.7.8], i.e. that it is equipped with a family of functorial isomorphisms \(a_X: X \to X^{**}\) (for \(X\) running over the objects of \(\mathcal{C}\)) such that \(a_{X \otimes Y} = a_X \otimes a_Y\). Given \(f \in \text{End}_{\mathcal{C}}(X)\), the pivotal structure allows to define two traces:

\[
\begin{align*}
\text{Tr}_+(f) &= \text{ev}_{X^*} \circ (a_X f \otimes \text{Id}_{X^*}) \circ \text{coev}_X \in \text{End}_{\mathcal{C}}(1) = \mathbb{C} \\
\text{Tr}_-(f) &= \text{ev}_X \circ (\text{Id}_X \otimes f a_X^{-1}) \circ \text{coev}_{X^*} \in \text{End}_{\mathcal{C}}(1) = \mathbb{C}.
\end{align*}
\]

We will sometimes write \(\text{Tr}_X(f)\) or \(\text{Tr}_X^Y(f)\) for \(\text{Tr}_+(f)\) and \(\text{Tr}_-(f)\). We define two dimensions

\[
\begin{align*}
\dim_+(X) &= \text{Tr}_+(\text{Id}_X) \quad \text{and} \quad \dim_-(X) = \text{Tr}_-(\text{Id}_X).
\end{align*}
\]

To summarize, we will work under the following hypothesis:

---

**Hypothesis and notation.** We fix in this section a braided pivotal tensor category \(\mathcal{C}\) as above. We denote by Gr(\(\mathcal{C}\)) its Grothendieck ring. Given \(X\) is an object in \(\mathcal{C}\), we denote by \([X]\) its class in Gr(\(\mathcal{C}\)). The set of isomorphism classes of simple objects in \(\mathcal{C}\) will be denoted by Irr(\(\mathcal{C}\)). If \(X \in \text{Irr}(\mathcal{C})\) and \(Y\) is any object in \(\mathcal{C}\), we denote by \([Y:X]\) the multiplicity of \(X\) in a Jordan-Hölder series of \(Y\).
Given $X, Y$ two objects in $\mathcal{C}$, we set

$$s^+_X, Y = (\text{Id}_X \otimes \text{Tr}_Y)(c_{Y, X} c_{X, Y}) \in \text{End}_{\mathcal{C}}(X).$$

and

$$s^-_X, Y = (\text{Tr}_Y \otimes \text{Id}_X)(c_{Y, X} c_{X, Y}) \in \text{End}_{\mathcal{C}}(X).$$

These induce two morphisms of abelian groups

$$s^+_X : \text{Gr}(\mathcal{C}) \to \text{End}_{\mathcal{C}}(X) \quad \text{and} \quad s^-_X : \text{Gr}(\mathcal{C}) \to \text{End}_{\mathcal{C}}(X).$$

**Definition A.3.** — An object $X$ in $\mathcal{C}$ is called a **brick** if $\text{End}_{\mathcal{C}}(X) = \mathbb{C}$. For instance, a simple object is a brick (and $1$ is also a brick, but $1$ is simple in a tensor category [EGNO, Theorem 4.3.1]). Note also that a brick is indecomposable. So if $\mathcal{C}$ is moreover semisimple, then an object is a brick if and only if it is simple.

If $X$ is a brick, then we will view $s^+_X$ and $s^-_X$ as elements of $\mathbb{C} = \text{End}_{\mathcal{C}}(X)$.

**Proposition A.4.** — If $X$ is a brick, then $s^+_X : \text{Gr}(\mathcal{C}) \to \mathbb{C}$ and $s^-_X : \text{Gr}(\mathcal{C}) \to \mathbb{C}$ are morphisms of rings.

**Proof.** — Assume that $X$ is a brick. We will only prove the result for $s^+_X$, which amounts to show that

$$(*) \quad s^+_X \circ Y = s^+_X s^+_X \circ Y.$$  

First, note that the following equality

$$c_{Y \circ Y, X} c_{X, Y \circ Y} = (c_{Y, X} \circ \text{Id}_Y) \circ (\text{Id}_Y \otimes c_{Y, X} c_{X, Y}) \circ (c_{X, Y} \circ \text{Id}_Y).$$

holds by (A.1) and (A.2). Taking $\text{Id}_X \otimes \text{Id}_Y \otimes \text{Tr}_Y$ on the right-hand side, one gets $s^+_X c_{Y, X} c_{X, Y} \in \text{End}_{\mathcal{C}}(X \otimes Y)$ (because $X$ is a brick). Applying now $\text{Id}_X \otimes \text{Tr}_Y$, one get $s^+_X \circ Y s^+_X \circ Y$. Since

$$(\text{Id}_X \otimes \text{Tr}_Y) \circ (\text{Id}_X \otimes \text{Id}_Y \otimes \text{Tr}_Y) = \text{Id}_X \otimes \text{Tr}_Y \circ Y,$$

this proves $(*)$.  

**Proposition A.5.** — Let $X$ be a brick and let $X'$ be a subquotient of $X$ which is also a brick. Then

$$s^+_X = s^+_X \quad \text{and} \quad s^-_X = s^-_X.$$
Proof. — Indeed, the endomorphism \((\text{Id}_X \otimes \text{Tr}_{+}^Y)(c_{Y,X} c_{X,Y})\) of \(X\) is multiplication by a scalar, and this scalar can be computed on any non-trivial subquotient of \(X\).

Corollary A.6. — Let \(X\) and \(X'\) be two bricks in \(\mathcal{C}\) belonging to the same block. Then

\[ s_X^+ = s_{X'}^+ \quad \text{and} \quad s_X^- = s_{X'}^- . \]

Proof. — By Proposition A.5, we may assume that \(X\) and \(X'\) are simple. We may also assume that \(X\) is not isomorphic to \(X'\) and that \(\text{Ext}^1_{\mathcal{C}}(X, X') = 0\). Let \(X \in \mathcal{C}\) such that there exists a non-split exact sequence

\[ 0 \rightarrow X' \rightarrow X \rightarrow X' \rightarrow 0 .\]

Since \(X \not\cong X'\), we have \(\text{End}_{\mathcal{C}}(X) = \mathbb{C}\), hence \(X\) is a brick. It follows from Proposition A.5 that

\[ s_X^+ = s_{X'}^+ = s_X^- = s_{X'}^- , \]

as desired.

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