An error-based active disturbance rejection control with memory structure

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Abstract
The paper studies the control problem for nonlinear uncertain systems with the situation that only the current reference signal is available. By constructing a memory structure to save the previous reference signals, a novel error-based active disturbance rejection control with an approximation for the second-order derivative of reference signal is proposed. The transient performance of the proposed method is rigorously studied, which implies the high consistence of the closed-loop system. More importantly, to attain the satisfactory tracking performance, the necessary condition for nominal control input gain is quantitatively investigated. Furthermore, the superiority of the proposed method is illuminated by contrastively evaluating the sizes of the total disturbance and its derivative. The proposed method can alleviate the burden of the estimation and compensation for total disturbance. Finally, the experiment for a manipulator platform shows the effectiveness of the proposed method.

Keywords
Active disturbance rejection control, extended state observer, error-based, memory structure

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Introduction
Uncertainties, including external disturbances, unmodeled nonlinear dynamics and parametric perturbations, are ubiquitous in industrial process. In control science and technology, it is a centering issue to design a controller featured with strong capability of handling uncertainties. Among various substantially developed control methods, uncertainty estimation and compensation-based methods show the powerful capability of tackling uncertainties and the simplicity of practical implementation, including but not limited to disturbance observer–based control,¹⁻³ uncertainty and disturbance estimator–based control,⁴ unknown input observer–based control,⁵,⁶ and active disturbance rejection control (ADRC).⁷,⁸

ADRC is innovatively proposed by Jingqing Han,⁷ which is featured with two-degree-of-freedom (2-DOF) design. More specifically, the extended state observer (ESO) is proposed to actively estimate the total disturbance and the derivatives of controlled output. Based on the online estimation from ESO, the controller composed of compensation for the total disturbance and stabilizing of the nominal system is constructed. Due to the effectiveness 2-DOF design, ADRC has been successfully applied to many practical systems, including aerospace systems,⁹⁻¹¹ motion control systems,¹²⁻¹⁵ energy systems,¹⁶⁻¹⁹ and robotic systems.²⁰ In engineering practice, the closed-loop performance of ADRC design is featured with strong robustness to uncertainties and simplicity in implementation. Moreover, the theoretical analysis of ADRC has been substantially established by a series of studies.²¹⁻²⁵ The studies by Guo and Zhao²¹ and Zhao and Guo²² rigorously prove the convergence of nonlinear ESO and nonlinear ADRC design. For discontinuous disturbances and unknown nonlinearity, the literature²³ theoretically investigates that ADRC has strong capability of tackling various uncertainties. The studies²⁴,²⁵ demonstrate that ADRC can deal with mismatched uncertainties via the compensation for the total disturbance. Based on the successful development of ADRC in both practical and theoretical aspects, ADRC has drawn amounts of attentions from both industrial and academic communities.
Motivated by different practical problems, several effective modified ADRC designs are proposed. To reduce the effect of measurement noise, the combination of ADRC design and Kalman filter algorithm is successfully applied to gasoline engine systems. To deal with the input and output delays in control systems, some efficient modifications of ADRC are proposed, including Smith predictor–based ADRC, matched delay design in ESO, and predictor observer–based ADRC.

In the conventional design of ADRC, the reference signal and its derivatives are assumed to be known before control design. However, in numerous practical systems, only the reference signal at the current time is available, since there is a decision loop calculating the reference signal instantly beyond the low-level control loop. Motivated by this problem, an error-based ADRC design is innovatively proposed and successfully applied in DC–DC buck converter. Moreover, the literature shows that error-based ADRC has strong capability of handling harmonic disturbances. However, the existing stability analysis for error-based ADRC is based on the assumption that the initial estimation error is sufficiently small, which might not be satisfied in practice. It is in urgent need to establish the rigorous stability theory for error-based ADRC. Furthermore, it is a significant issue to develop a more effective error-based ADRC for practical process.

In this paper, by establishing a memory structure for the previous reference signals, a novel error-based ADRC with an approximation for the second-order derivative of reference signal is proposed. Furthermore, the theoretical analyses, including the transient performance, the necessary condition for nominal control input gain, and the superiority of the proposed memory structure, are comprehensively presented. The experiment of a manipulator platform illustrates the effectiveness of the proposed ADRC. The main contributions of the paper are shown as follows.

1. For the situation that only the current reference signal is available, a novel error-based ADRC with a memory structure to approximate the second-order derivative of reference signal is proposed. The proposed approximation value for the second-order derivative of reference signal is continuous, which can reduce the bump of control input.

2. With general assumptions for uncertainties and initial values, the transient performance of the proposed ADRC based closed-loop system is rigorously studied. More importantly, the necessary condition for the nominal control input gain is quantitatively analyzed.

3. Compared with the existing error-based ADRC, the sizes of total disturbance and its derivatives in the proposed ADRC are proved to be smaller, which alleviates the estimating burden of ESO.

The rest of the paper has the following organization. In “Problem formulation” section, the problem formulation is presented. The error-based ADRC with memory structure is proposed in “Error-based ADRC with memory structure” section. The theoretical analysis for the proposed ADRC is given in “Theoretical analysis” section. “Experimental verification” section shows the experimental results. The conclusion is presented in “Conclusion” section.

Problem formulation

Consider the following second-order nonlinear uncertain systems

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= bu(t) + f(x_1(t), x_2(t), t), \\
y(t) &= x_1(t),
\end{align*}
\]  

(1)

where \(x_1(t) \in R\) and \(x_2(t) \in R\) are the system states, \(u(t) \in R\) is the control input, \(y(t) \in R\) is the measured output to be controlled, and the unknown nonzero constant \(b \in R\) represents the control input gain whose nominal value is known and denoted as \(\bar{b}\). By lumping external disturbances and unmodeled dynamics together, the function \(f(\cdot)\) represents various uncertainties, which satisfies the following assumption:

**Assumption 1.** The function \(f(\cdot)\) is continuously differentiable, and there exists a continuous function \(\psi_f\) such that

\[
\sup_{t \geq 0} \left\{ |f|, \left| \frac{\partial f}{\partial x_1} \right|, \left| \frac{\partial f}{\partial x_2} \right|, \left| \frac{\partial f}{\partial u} \right| \right\} \leq \psi_f(x_1, x_2)
\]  

(2)

for \(\forall [x_1, x_2]^T \in R^2\).

**Remark 1.** Assumption 1 implies that the uncertainty \(f(\cdot)\) and its partial derivatives are bounded if the system states are bounded.

The system (1) can model a vast of practical systems, including flight systems, robotic systems, and vehicle systems. Furthermore, the control objective of the system (1) is to design the control input \(u(t)\) such that the measured output \(y(t)\) can track the reference signal \(r(t)\), which satisfies the following assumption:

**Assumption 2.** There exists a positive \(\eta_r\) such that the reference signal and its derivatives satisfy \(\sup_{t > 0} |r^{(i)}(t)| \leq \eta_r\) for \(i \geq 0\).

**Remark 2.** Assumption 2 implies that the reference signal and its derivatives are bounded.

The reference signal is often generated by a decision-making scheme in practical systems, such as robotic systems, flight systems, and energy systems. As a result, only the instant reference signal is available for the low-level control loop. Moreover, the derivatives of
the reference signal are definitely unknown. However, in the conventional ADRC, the derivatives of reference signal are assumed to be known. It is significant to properly modify ADRC scheme based on the limited information of reference signal.

Based on the reference signal at the current time, a modified ADRC for a large scope of nonlinear uncertainty signal are assumed to be known. It is significant that the system (1) can be applied to systems with mismatched uncertainty. Therefore, the control design for the system (1) is the core of the total disturbance. Moreover, only \( e_1(t) \) and \( r(t) \) at the current time are available for control design.

In the practical implementation of the existing error-based ADRC, only the sampled reference signal \( r(kT_s) \) is utilized for \( t \in [kT_s, (k + 1)T_s] \), where \( T_s > 0 \) is the sampling time and the integer \( k \geq 0 \) is the count of sampling time.

In the paper, an approximation for \( r^{(2)}(t) \) is designed as follows

\[
\hat{r}_{dd}(t) = \begin{cases} 
0, & t \in [0, 2T_s), \\
\frac{r(2T_s) - 2r(T_s) + r(0)}{T_s^2} (t - 2T_s), & t \in [2T_s, 3T_s), \\
\frac{r(T_s) - 2r(k-2T_s) + r(k-3T_s)}{T_s^2} + (t-kT_s), & t \in [kT_s, (k + 1)T_s), \quad k \geq 3 \end{cases}
\]  

(5)

The detailed analysis for the approximation (5) is presented in Lemma 1 in the appendix. From the perspective of implementation, the main difference with the existing error-based ADRC is that an additional memory structure for saving \( r(k-1)T_s \), \( r(k-2)T_s \), and \( r(k-3)T_s \) is required to achieve the approximation (5).

**Remark 3.** From the reference Chen et al., the integrator chain form, that is, the system (1), is the core of uncertain systems which might contain mismatched uncertainties. Therefore, the control design for the system (1) can be applied to systems with mismatched uncertainties.

**Error-based ADRC with memory structure**

Due to the measured output and reference signal at current time, the current tracking error

\[
e_1(t) \triangleq y(t) - r(t)
\]  

(3)
is available. Combined with the system (1) and the nominal control input term \( bu(t) \), the dynamics of tracking error is deduced as follows

\[
\begin{align*}
\dot{e}_1(t) &= e_2(t), \\
\dot{e}_2(t) &= bu(t) + f(e_1 + r, e_2 + r^{(1)}(t), (b - \hat{b})u(t) - r^{(2)}(t))
\end{align*}
\]  

(4)

where \( e_2(t) \in \mathbb{R} \) is the derivative of tracking error. Moreover, only \( e_1(t) \) and \( r(t) \) at the current time are available for control design.

With the approximated value \( \hat{r}_{dd}(5) \), the dynamics of the tracking error (4) can be reformulated as

\[
\begin{align*}
\dot{e}_1(t) &= e_2(t) + f_{\hat{r}_{dd}}(t) + f_1(e_1(t), e_2(t), u(t), t) \\
\dot{e}_2(t) &= bu(t) - \hat{r}_{dd}(t) + f_1(e_1(t), e_2(t), u(t), t) - r^{(2)}(t)
\end{align*}
\]  

(6)

where the total disturbance \( f_1 \in \mathbb{R} \) has the following form

\[
\begin{align*}
f_1(e_1(t), e_2(t), u(t), t) &= \hat{r}_{dd}(t) - r^{(2)}(t) + f(e_1(t) + r(t), e_2(t) + r^{(1)}(t), (b - \hat{b})u(t))
\end{align*}
\]  

(7)

From the above analysis, the tracking problem of the system (1) is transformed into the stabilization problem of the error system (6). Next, ADRC design is presented.

To online estimate the total disturbance, the following ESO is presented

\[
\begin{align*}
\dot{\hat{e}}_1(t) &= \hat{e}_2(t) + \beta_1(e_1(t) - \hat{e}_1(t)), \\
\dot{\hat{e}}_2(t) &= bu(t) - \hat{r}_{dd}(t) + \hat{f}_1(t) + \beta_2(e_1(t) - \hat{e}_1(t)), \\
\dot{\hat{f}}_1(t) &= \beta_3(e_1(t) - \hat{e}_1(t))
\end{align*}
\]  

(8)

where \( \hat{e}_1(t) \in \mathbb{R} \) is the estimation for \( e_1(t) \), \( \hat{e}_2(t) \in \mathbb{R} \) is the estimation for \( e_2(t) \) and \( \hat{f}_1(t) \in \mathbb{R} \) is the estimation for the total disturbance \( f_1 \). Since \( e_1(t) \) is measurable, the initial value \( \hat{e}_1(0) \) can be naturally designed as \( \hat{e}_1(0) = e_1(0) \). The parameters of ESO \( \beta_1, \beta_2 \) and \( \beta_3 \) are chosen such that the polynomial \( f_{\text{ESO}}(s) = s^3 + \beta_1 s^2 + \beta_2 s + \beta_3 \) is stable. A popular design of ESO’s parameters is setting all solutions of the equation \( f_1(s) = 0 \) as \(-\omega_o\), where \( \omega_o > 0 \) is defined as the bandwidth of ESO, that is

\[
\beta_1 = 3\omega_o, \quad \beta_2 = 3\omega_o^2, \quad \beta_3 = \omega_o^3
\]  

(9)

Based on the approximation (5) and the estimation from the ESO (8), the ADRC input is designed as follows

\[
u(t) = \begin{cases} 
0, & 0 \leq t \leq T_s \ 
-\hat{f}_1(t) - k_1 e_1(t) - k_2 \hat{e}_2(t) + \hat{r}_{dd}(t) / \hat{b}, & t > T_s
\end{cases}
\]  

(10)
where the feedback gains $k_1 \in R$ and $k_2 \in R$ are chosen such that the polynomial $f_{cz}(s) = s^2 + k_2 s + k_1$ is stable, that is, $k_1 > 0$ and $k_2 > 0$. Moreover, $t_u$ is the time after which the peaking of the ESO (8) ends. Since it is hard to obtain the minimal $t_u$, inspired by the reference Xue and Huang, the following explicit design for $t_u$ is presented

$$
t_u = \frac{2||P_1||}{\max\{\ln(\omega_o|e_2(0) - \hat{e}_2(0)|), 0\}}$$  \hspace{1cm} (11)

where the positive definite matrix $P_1$ satisfies

$$A^T P_1 + P_1A = -I, \quad A = \begin{bmatrix} -3 & 1 & 0 \\ -3 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$  \hspace{1cm} (12)

**Remark 5.** If the initial value satisfies that $|e_2(0) - \hat{e}_2(0)| \leq 1/\omega_o$, the definition of $t_u$ (11) implies that $t_u = 0$. Moreover, for $|e_2(0) - \hat{e}_2(0)| > 1/\omega_o$, since the equation $\lim_{\omega_o \to -} t_u = 0$ is satisfied, $t_u$ can be sufficiently small by tuning the bandwidth of ESO $\omega_o$.

**Remark 6.** In the existing error-based ADRC, the term $-r^2(t) + f(e_1 + r, e_2 + r^1(t)) + (b - b_2)u(t)$ is regarded as the total disturbance. In this paper, by approximating the second-order derivative of the reference signal based on additional memory structure, the total disturbance is shown in (7), which can effectively alleviate the estimating burden of ESO.

**Remark 7.** Although the paper considers the control for the two-dimensional uncertain system (1), the same design methodology can be used for n-dimensional uncertain systems.

Finally, the error-based ADRC with memory structure, that is, (3), (5), and (8)–(11), is proposed, whose control block diagram is shown in Figure 1. In the next section, the properties of the proposed ADRC are comprehensively investigated, including transient performance, necessary condition for nominal control input gain and superiority of memory design.

### Theoretical analysis

#### Transient performance

In this subsection, by comparing the ideal trajectory and the actual trajectory of the output, the transient tracking performance of closed-loop system is rigorously described. Meanwhile, the estimating performance of ESO is investigated.

Consider the following ideal trajectory for the output $y(t)$

$$\begin{align*}
    x_1^*(t) &= x_1(t), \\
    x_2^*(t) &= r^2(t) - k_1(x_1^*(t) - r(t)) - k_2(x_2^*(t) - r^1(t)), \\
    y^*(t) &= x_2^*(t)
\end{align*}$$  \hspace{1cm} (13)

where $[x_1^*(t) x_2^*(t)]^T \in R^2$ is the state vector for the ideal system (13), $y^*(t) \in R$ is the ideal output, and the initial condition satisfies that $x_1^*(0) = x_1(0)$ and $x_2^*(0) = x_2(0)$.

By calculating the error dynamics between the state vector $[x_1^* x_2^*]^T$ and $[r^* r]^T$, it can be verified that the ideal trajectory $y^*(t)$ can exponentially converge to the reference signal $r(t)$ with the desired speed by choosing the feedback gain $k_1$ and $k_2$.

Compared with the ideal trajectory $y^*(t)$, the transient tracking performance of the closed-loop system is illuminated by the following theorem.

**Theorem 1.** Consider the system (1) with Assumptions 1–2 and the proposed ADRC (3), (5), and (8)–(11). Assume that

$$\frac{b}{b_2} \in (0, 9)$$  \hspace{1cm} (14)

![Figure 1. Control block diagram for error-based ADRC with memory structure.](image-url)
Then, there exist positives \( \omega^* \) and \( \eta_i^*(i = 1, 2, 3) \) dependent on \((x_1(0), x_2(0), \dot{e}_2(0), \dot{f}(0), \psi_i, b, \bar{b}, k_1, k_2, \eta_i)\) such that

\[
\sup_{t \in [0, \infty)} |y(t) - y^*(t)| \leq \eta_1^* \max \left\{ \frac{\ln \omega_0}{\omega_0}, \frac{1}{\omega_0} \right\} \tag{15}
\]

\[
\left\| \begin{bmatrix} e_1(t) \\ e_2(t) \\ f(t) \end{bmatrix} - \begin{bmatrix} \hat{e}_1(t) \\ \hat{e}_2(t) \\ \hat{f}(t) \end{bmatrix} \right\| \leq \eta_2^* \left( \frac{1}{\omega_0} + e^{-\eta_1^* \omega_0 t} \right), \quad t \geq t_u \tag{16}
\]

for all \( \omega_0 \geq \omega^* \).

Theorem 1 ensures the stability of the proposed ADRC based closed-loop system. More specifically, the upper bounds of both the tracking and the estimating errors are explicitly shown. From (15), the upper bounds of tracking errors between the actual and ideal trajectories can be tuned sufficiently small by the ESO’s bandwidth \( \omega_0 \) despite various uncertainties, which implies the highly consistent performance of the closed-loop system. Moreover, as shown in (16), the estimating error of the ESO (8) can exponentially decreases, and finally stay in a sufficiently small region by designing a sufficiently large \( \omega_0 \). The proof of Theorem 1 is presented in the appendix.

**Remark 8.** The tolerable range of uncertain control input gain is shown as (14). Compared with the existing results\(^{22} \) that \( b/h \in (0, 3) \), the presented condition (14) makes improvement. Moreover, the analysis in the next subsection demonstrates that (14) is almost necessary for the well-designed ADRC.

**Necessary condition for nominal control transient gain**

In this subsection, to ensure the satisfied transient performance of the proposed ADRC, that is, (3), (5), and (8)–(11), the necessary condition for nominal control input gain is studied.

From Theorem 1, the satisfied transient performance of the proposed ADRC is described by the equation

\[
\sup_{t \in [0, \infty)} |y(t) - y^*(t)| \leq \varepsilon \tag{17}
\]

for some small positive \( \varepsilon \). Based on the satisfied transient performance (17), the definition of well-designed ADRC is presented as follows.

**Definition 1.** For the system (1) and the proposed ADRC, that is, (3), (5), and (8)–(11), if for any given positive \( \varepsilon \), the equation (17) is satisfied for some \( \omega_0 \), then the proposed ADRC is defined to be well designed.

Due to Definition 1, if the error between the ideal trajectory and the actual trajectory can be arbitrarily small by designing the bandwidth of ESO \( \omega_0 \), then the proposed ADRC is named to be well designed.

To achieve well-designed property of the proposed ADRC, the following theorem investigates the necessary condition for the nominal control input gain \( b \).

**Theorem 2.** Consider the system (1) with Assumptions 1–2 and the proposed ADRC (3), (5), and (8)–(11). If the proposed ADRC is well designed for any given \((x_1(0), x_2(0), \dot{e}_2(0), \dot{f}(0), \psi_i, b, \bar{b}, k_1, k_2, \eta_i)\), then

\[
\frac{b}{\bar{b}} \in (0, 9) \tag{18}
\]

From Theorem 2, the necessary condition for the nominal control input gain \( b \) is explicitly presented as (18). The condition (18) provides the fundamental requirement for the control systems, that is, the range of uncertain control input gain. More importantly, based on (14) in Theorem 1, the capability of the proposed ADRC to handle uncertainties of control input is illustrated. With the combination of Theorem 1 and Theorem 2, the range of uncertain control input gain for the proposed ADRC being well designed is comprehensively studied. The proof of Theorem 2 is presented in the appendix.

**Remark 9.** The necessary condition for the range of uncertain control input gain is quantitatively presented as Theorem 2, which has not been revealed in the existing studies.

**Superiority of memory design**

In this subsection, the superiority of the proposed approximation based on memory structure is demonstrated by the comparison with the existing error-based ADRC.

To essentially investigate the main difference between the proposed ADRC and the existing error-based ADRC,\(^{31,32} \) the results in this subsection are under the condition that

\[
t > \max\{t_u, 3T_s\}, \quad b = \bar{b}, \quad f = f(t) \tag{19}
\]

To distinguish the variables from those of the proposed ADRC, the symbol \( \bar{a} \) is utilized for the variable \( a \) in the control system with the existing error-based ADRC. Therefore, the dynamics of tracking error for existing error-based ADRC is

\[
\begin{aligned}
\dot{\bar{e}}_1(t) &= \bar{e}_2(t), \\
\dot{\bar{e}}_2(t) &= \bar{b}u(t) + \bar{f}(t)
\end{aligned} \tag{20}
\]

where \( \bar{e}_1(t) \triangleq \bar{y}(t) - \bar{r}(t) \) is the tracking error, \( \bar{e}_2(t) \in R \) is the derivative of tracking error, \( \bar{u}(t) \in R \) is the control input and the function

\[
\bar{f}(t) = -\rho^2(t) + f(t) \tag{21}
\]

represents the total disturbance. Then, the existing error-based ADRC for the system (1) is presented as follows.
where \( \hat{e}_1(t) \in R \) is the estimation for \( e_1(t) \), \( \hat{e}_2(t) \in R \) is the estimation for \( e_2(t) \), and \( \hat{f}_j(t) \in R \) is the estimation for the total disturbance \( f_j \). The parameters \( \beta_1, \beta_2 \) and \( \beta_j \) have the same design with (9).

Based on the condition (19) and the notation \( u_0 = u - \hat{r}_{ad} / b \), the proposed ADRC based closed-loop system can be reformulated as

\[
\begin{align*}
\dot{e}_1(t) &= e_2(t), \\
\dot{e}_2(t) &= h_0(t) + f_j(t), \\
\dot{e}_1(t) &= \hat{e}_2(t) + \beta_1(e_1(t) - \hat{e}_1(t)), \\
\dot{e}_2(t) &= h_0(t) + f_j(t) + \beta_2(e_1(t) - \hat{e}_1(t)), \\
\dot{f}_j(t) &= \beta_3(e_1(t) - \hat{e}_1(t)), \\
\dot{u}_0(t) &= -\frac{f_j(t) - k_1 e_1(t) - k_2 \hat{e}_2(t)}{b}
\end{align*}
\]

where

\[
f_j(t) = \hat{r}_{ad}(t) - r^{(2)}(t) + f(t)
\]

With the comparison of (20)–(24), the structures of the closed-loop systems for these two ADRC designs are exactly the same. However, the formulas of the total disturbances, that is, (21) and (24), are different. For ADRC design, the norms of the total disturbance and its derivative are supposed to be as small as possible, which can alleviate the burdens of estimating and compensating for the total disturbance.

Then, the sizes of the total disturbance and its derivative for these two ADRC designs are compared, as shown in the following theorem.

**Theorem 3.** Consider the system (1) with Assumptions 1–2, the proposed ADRC (3), (5), and (8)–(11), and the existing error-based ADRC (22). Assume that the condition (19) is satisfied. For any given positive constants \( \varepsilon_2 \in (0, 0.5) \) and \( \eta_1 \), the following statements hold.

(S1) If \( |r^{(2)}(t)| \leq \eta_1 \) and \( |f(t)| \leq \varepsilon_2 |r^{(2)}(t)| \) for \( e_2(t) \in [0, \varepsilon_2] \) and \( t > \max \{ t_0, 3T_s \} \), then there exists \( T^*_1 > 0 \) such that

\[
|f(t)| < \tilde{|f|}(t), \forall t > \max \{ t_0, 3T_s \}, \forall T_s \in (0, T^*_1)
\]

(S2) If \( |r^{(2)}(t)| \leq \eta_1 \) and \( \tilde{|f|}(t) \leq \varepsilon_2 |r^{(2)}(t)| \) for \( e_2(t) \in [0, \varepsilon_2] \) and \( t > \max \{ t_0, 3T_s \} \), then there exists \( T^*_1 > 0 \) such that

\[
|\tilde{f}(t)| < \tilde{|f|}(t), \forall t > \max \{ t_0, 3T_s \}, \forall T_s \in (0, T^*_1)
\]

Theorem 3 demonstrates that the norms of the total disturbance and its derivative in the proposed ADRC are definitely smaller, if the derivatives of reference signal play a dominant role. From the statement (S1), the inequality \( |f(t)| \leq \varepsilon_2 |r^{(2)}(t)| (e_2 \in (0, 0.5)) \) implies that \( r^{(2)} \) is more dominant than \( f \). With the dominance of \( r^{(2)} \), the norm of the total disturbance in the proposed ADRC is smaller than those in the existing error-based ADRC. The similar result of the derivative of total disturbance can be obtained, as shown in the statement (S2). Hence, for the control systems whose reference signal has drastic variation, the proposed ADRC can attain the preferable closed-loop performance. The proof of Theorem 3 is presented in the appendix.

**Remark 10.** For the situation that \( r^{(2)}(t) = 0 \) for \( t \) belonging to a interval \( D \), it can be verified from \( |f(t)| \leq \varepsilon_2 |r^{(2)}(t)| \) and (5) that \( f_1 = f_2 = 0 \) for \( t \in D \) with the suitably small \( T_s \). For the case that \( r^{(2)}(t) = 0 \), the similar deduction can be obtained. To simplify the description of Theorem 3, the paper considers the case that \( r^{(2)}(t) \) and \( r^{(3)}(t) \) are nonzero.

**Experimental verification**

In this section, the experiment of a manipulator platform is presented.

The setup of the manipulator platform is shown in Figure 2. The reference signal is artificially generated by the freely operated manipulator. With the measured angular position of the controlled manipulator and the current reference signal, the master computer calculates the control input by the designed control algorithm. Based on a series of devices, including the permanent magnet synchronous motor and the gear reducer, the produced input signal eventually affects the controlled manipulator. To verify the capability of uncertainty rejection, the additional digital uncertainty is further produced by the master computer.

The dynamics of the manipulator platform can be formulated as

\[
\begin{align*}
\dot{x}_1(t) &= \dot{x}_2(t), \\
\dot{x}_2(t) &= \frac{2}{J} u(t) + f_{mp}(x_1, x_2, u, t)
\end{align*}
\]

where \( x_1(t) \in R \) is the measured angular position of the controlled manipulator, \( x_2(t) \in R \) is the angular velocity, the control input \( u(t) \in R \) is the voltage of the electric motor, \( J \in R \) is the nominal moment of inertia, \( c_0 \in R \) is the coefficient of driving torque, and \( f_{mp}(\cdot) \in R \) represents the total disturbance, including external disturbances, parametric perturbations, and unmodeled nonlinear dynamics.

The control objective is to design the voltage \( u(t) \) such that the angular position \( x_1(t) \) can track the reference signal \( r(t) \). Hence, the tracking error is denoted as \( e_1(t) = x_1(t) - r(t) \), whose unit is rad. Moreover, only the reference signal \( r(t) \) at the current time is available, which is artificially generated by the freely operated manipulator. The control objective of the system (27) can represent several practical control systems such as remote control of robotics and tracking problem of radar servo systems.
The system parameters are $c_u = 115.5 (N \cdot m/V)$ and $J = 0.128 (kg \cdot m^2)$. The sampling time is 0.001 s. Moreover, the following three cases of uncertainties are considered in the experiment:

Case1: $f_{mp}(x_1, x_2, u, t) = f_0(x_1, x_2, u, t)$,
Case2: $f_{mp}(x_1, x_2, u, t) = f_0(x_1, x_2, u, t) + \frac{c_u}{J} (x_1 + 0.1u)$,
Case3: $f_{mp}(x_1, x_2, u, t) = f_0(x_1, x_2, u, t) + \frac{c_u}{J} (x_1 + x_2' + 0.1e^{x_1} + 0.3 \sin(t))$,

where $f_0$ presents the original nonlinear uncertainty of the manipulator platform. In Cases 2–3, the additional digital external disturbances and unmodeled dynamics are presented.

Then, the experimental results of the existing error-based ADRC (22) and the proposed ADRC (3), (5), and (8)–(11) are comparatively presented. To make the comparison be fair, the same bandwidth of ESO and feedback gains is selected:

$$\omega_o = 100, \quad k_1 = 900, \quad k_2 = 60$$ \hfill (28)

The experimental results are shown in Figures 3 and 4. Moreover, the performance indicators are presented in Table 1, including integral of absolute error (IAE), integral of time absolute error (ITAE), and integral of squared error (ISE).
From Figure 3, the controlled angular positions and tracking errors for various uncertainties are shown. Figure 3 shows that these two ADRC designs have consistent tracking performance despite various uncertainties. Moreover, the higher tracking accuracy of the proposed ADRC is visually demonstrated by Figure 3, especially during the time that the reference signal drastically changes. Figure 4 presents the control inputs for various uncertainties, which changes smoothly.

Furthermore, from Table 1, the indicators of tracking errors have at least 70% reduction by the proposed ADRC, which illustrates the superiority of the proposed ADRC.

### Conclusion

The paper studies the control problem for nonlinear uncertain systems with the situation that only the current reference signal is available. An error-based ADRC with memory structure is novelly proposed. By constructing the memory structure to save the previous reference signals, an approximation for the second-order derivative of the reference signal is proposed. Then, with the estimation for the total disturbance by ESO, the control input composed of the compensations for the total disturbance and the second-order derivative of reference signal is designed. The properties of the proposed ADRC are comprehensively studied, including transient performance, necessary condition for nominal control input gain, and superiority of the memory structure. The analysis of transient performance implies the high consistence of the closed-loop system. More importantly, the necessary condition for nominal control input gain is quantitatively analyzed, which illustrates the fundamental requirement for uncertain control input gain. Furthermore, the sizes of the total disturbance and its derivative are proved to be smaller, which can alleviate the burdens of estimating and compensating for total disturbance. Finally, the experiment for a manipulator platform shows the effectiveness of the proposed method.

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**Appendix**

**Lemma 1**

Consider the reference signal $r(t)$ with Assumption 2 and the approximation (5), then the following statements hold:

1. $\dot{\hat{r}}_{dd}(t)$ is continuous for $t \geq 0$ and differentiable in each time interval $(kT_s,(k+1)T_s)$ ($k \geq 0$).

2. The approximation errors have the following bounds

   $$\sup_{t > \delta T_s} |\dot{\hat{r}}_{dd}(t) - r(t)| \leq 3\eta T_s + O(T_s^2)$$

   $$\sup_{t > \delta T_s} |\ddot{\hat{r}}_{dd}(t) - r(t)| \leq \frac{7}{2} \eta T_s + O(T_s^2)$$

3. There exists a positive constant $M_r$ dependent on $T_s$ such that

   $$\sup_{t > 0} \{ |\dot{\hat{r}}_{dd}(t)|, |\ddot{\hat{r}}_{dd}(t)| \} < M_r$$
Proof of Lemma 1. According to Assumption 2, the Taylor expansions for \( r((k-1)T_s) \), \( r((k-2)T_s) \) and \( r((k-3)T_s) \) at \( t = kT_s \) are shown as follows

\[
\begin{align*}
\{ r((k-1)T_s) &= r(kT_s) - r'(kT_s)T_s + \frac{r''(kT_s)}{2}T_s^2 - \frac{r'''(kT_s)}{6}T_s^3 + O(T_s^4), \\
\{ r((k-2)T_s) &= r(kT_s) - r'(kT_s)T_s + \frac{r''(kT_s)}{2}T_s^2 - \frac{r'''(kT_s)}{3}T_s^3 + O(T_s^4), \\
\{ r((k-3)T_s) &= r(kT_s) - 3r(kT_s)T_s + 3r'(kT_s)T_s^2 - 3r''(kT_s)T_s^3 + O(T_s^4)
\end{align*}
\]

(32)

where the term \( m = O(T_s^2) \) means that there exists a constant \( M_1 \) such that \( |m| \leq M_1 T_s^2 \). Hence, the following equations are satisfied

\[
\begin{align*}
\frac{r(kT_s) - 2r((k-1)T_s) + r((k-2)T_s)}{T_s^2} &= r'(kT_s) + \frac{3}{2}r'''(kT_s)T_s + O(T_s^2) \\
\frac{r(kT_s) - 3r((k-1)T_s) + 3r((k-2)T_s) - r((k-3)T_s)}{T_s^3} &= r''(kT_s) + \frac{3}{2}r'''(kT_s)T_s + O(T_s^2)
\end{align*}
\]

(33)

(34)

The proof of the statement (L2). With (33), the error between \( \hat{r}_{ad}(k+1)T_s \) and \( r^{(2)}(kT_s) \) for \( k \geq 3 \) satisfies that

\[
|\hat{r}_{ad}(k+1)T_s - r^{(2)}(kT_s)| \leq \eta_s T_s + O(T_s^2)
\]

From (5), \( \hat{r}_{ad}(t) \) can be calculated as shown in the following form

\[
\hat{r}_{ad}(t) = \frac{(r(kT_s) - 3r((k-1)T_s) + 3r((k-2)T_s) - r((k-3)T_s))}{(T_s^2)}, \quad t \in (kT_s, (k+1)T_s), \quad k \geq 3
\]

(37)

Based on (34) and (37), the error between \( \hat{r}_{ad}(t) \) and \( r^{(3)}(kT_s) \) for \( k \geq 3 \) has the following bound

\[
\sup_{t \in (kT_s, (k+1)T_s)} |\hat{r}_{ad}(t) - r^{(3)}(kT_s)| \leq \frac{3}{2} \eta_s T_s + O(T_s^2)
\]

(38)

Moreover, by Taylor expansion, \( r^{(2)}(t) \) and \( r^{(3)}(t) \) have the following expression

\[
r^{(2)}(t) = \frac{r(kT_s) + r^{(3)}(kT_s)(t-kT_s) + O((t-kT_s)^2)}{T_s^2}
\]

(39)

\[
r^{(3)}(t) = \frac{r^{(3)}(kT_s) + r^{(4)}(kT_s)(t-kT_s) + O((t-kT_s)^2)}{T_s^3}
\]

(40)

With the combination of (36)–(40), the bounds of \( |\hat{r}_{ad}(k+1)T_s - r^{(2)}(t)| \) and \( |\hat{r}_{ad}(kT_s) - r^{(2)}(t)| \) for \( t \in (kT_s, (k+1)T_s) \) \((k \geq 3)\) can be calculated as follows

\[
|\hat{r}_{ad}(k+1)T_s - r^{(2)}(t)| \leq |\hat{r}_{ad}(k+1)T_s - r^{(2)}(kT_s)| + |r^{(2)}(kT_s) - r^{(2)}(t)| \\
\leq 2\eta_s T_s + O(T_s^2), \quad \forall t \in [kT_s, (k+1)T_s]
\]

(41)

\[
|\hat{r}_{ad}(kT_s) - r^{(2)}(t)| \leq |\hat{r}_{ad}(kT_s) - r^{(2)}(kT_s)| + |r^{(2)}(kT_s) - r^{(2)}(t)| \\
\leq 2\eta_s T_s + O(T_s^2), \quad \forall t \in [kT_s, (k+1)T_s]
\]

(42)

From (41)–(42), the error between \( \hat{r}_{ad} \) and \( r^{(2)} \) has the following bound

\[
|\hat{r}_{ad}(t) - r^{(2)}(t)| \\
\leq \max(|\hat{r}_{ad}(k+1)T_s - r^{(2)}(t)|, |\hat{r}_{ad}(kT_s) - r^{(2)}(t)|) \\
\leq 3\eta_s T_s + O(T_s^2), \quad \forall t \in [kT_s, (k+1)T_s], \forall k \geq 3
\]

(43)

Similar with (41)–(42), the equations (38) and (40) declare that for \( \forall k \geq 3 \)

\[
|\hat{r}_{ad}(t) - r^{(3)}(t)| \leq |\hat{r}_{ad}(t) - r^{(3)}(kT_s)| \\
+ |r^{(3)}(kT_s) - r^{(3)}(t)| \leq \frac{5}{2} \eta_s T_s + O(T_s^2)
\]

(44)

(\forall t \in (kT_s, (k+1)T_s))
which further implies that
\[
\lim_{t \to kT_s} \hat{r}_{dd}(t) - r^{(3)}(kT_s) \leq \frac{5}{2} \eta_s T_s + O(T_s^2)
\]
for \( k \geq 3 \). Owing to (40), the bound of \( \lim_{t \to kT_s} (\hat{r}_{dd}(t) - r^{(3)}(kT_s)) \) satisfies that
\[
\left| \lim_{t \to kT_s} \hat{r}_{dd}(t) - r^{(3)}(kT_s) \right| \\
\leq \frac{1}{2} \left( |r^{(2)}(kT_s)| + |r^{(3)}((k - 1)T_s)| \right) + \frac{7}{2} \eta_s T_s + O(T_s^2),
\]
\( \forall k \geq 4 \)
(46)

With the combination of (44)–(46), the bound of the error \( \left| \hat{r}_{dd}(t) - r^{(3)}(t) \right| \) for \( t \in [kT_s, (k + 1)T_s] \) \( (k \geq 4) \) is deduced as follows
\[
\sup_{t \in [kT_s, (k + 1)T_s]} \left| \hat{r}_{dd}(t) - r^{(3)}(t) \right| \leq \frac{3}{2} \eta_s T_s + O(T_s^2)
\]
(47)

According to (43), (44), and (47), the statement (L2) is proved.

The proof of the statement (L3). Based on the results in the statement (L2), we only need to analyze the bounds of \( \hat{r}_{dL} \) and \( \hat{r}_{dd} \) for \( t \in [0, 3T_s] \).

It can be directly deduced from (5) that
\[
\sup_{t \in [0, 2T_s]} \left| \hat{r}_{dL}(t) \right| = 0, \quad \sup_{t \in [0, 2T_s]} \left| \hat{r}_{dd}(t) \right| = 0
\]
(48)

With the help of (33), the bounds of \( \hat{r}_{dL} \) and \( \hat{r}_{dd} \) are shown as follows
\[
\sup_{t \in [2T_s, 3T_s]} \left| \hat{r}_{dL}(t) \right| \leq \frac{3}{2} \eta_s T_s + O(T_s^2)
\]
\[
\sup_{t \in [2T_s, 3T_s]} \left| \hat{r}_{dd}(t) \right| \leq \frac{3}{2} \eta_s T_s + O(T_s^2)
\]
Hence, by setting
\[
M_r = \frac{\eta_s}{T_s} + 2 \eta_s + O(T_s) + O(T_s^2)
\]
(49)

(31) is proved.

Proof of theorem 1: define.
\[
E(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}, \quad \dot{E}(t) = \begin{bmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \end{bmatrix}
\]
\[
T_1^{-1} \begin{bmatrix} e_1(t) - \hat{e}_1(t) \\ e_2(t) - \hat{e}_2(t) \end{bmatrix}
\]
where \( T_1 = \begin{bmatrix} \omega_r^{-2} & 0 & 0 \\ 0 & \omega_r^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \). Combined with (7), the control input (10) can be rewritten as
\[
U(t) = \begin{cases} 0, & 0 \leq t < T_u, \\ \frac{-f(\cdot) - K^T E(t) + K^T T_1 \dot{E}(t) + r^{(2)}(t)}{b}, & t \geq T_u \end{cases}
\]
(50)

where \( K = [k_1 \ k_2]^T \) and \( K_c = [0 \ 0 \ k_1 \ k_2]^T \). Moreover, since
\[
\dot{u}(t) = \frac{-f_1 - k_1 \hat{e}_1 - k_2 \hat{e}_2 + \hat{r}_{dd}}{b} \\
- \frac{\beta_3 (e_1 - \hat{e}_1) - k_1 \hat{e}_2}{b} + \frac{k_2 (k_1 e_1 - k_2 \hat{e}_2 + \beta_2 (e_1 - \hat{e}_1)) + \hat{r}_{dd}}{b}
\]
\[
- \frac{-\omega_0 \dot{e}_1 + k_1 k_2 \hat{e}_1 + (k_2^2 - k_1) \hat{e}_2}{b} + \frac{-3 k_2 \hat{e}_2 - k_2 \hat{e}_2 / \omega_0 + \hat{r}_{dd}}{b}, \quad t \geq T_u
\]
(51)

the closed-loop system (1), (3), (5), and (8)–(11) is reformulated as
\[
\begin{cases}
\dot{E} = AE + B \Gamma_0 (E, t), \\
\dot{E} = \omega_0 A_1 \dot{e}_1 + B_1 \Gamma_1 (E, t),
\end{cases}
\]
(52)
\[
\begin{cases}
\dot{E} = A_2 \dot{E} + B \Gamma_2 (E, \dot{E}, \dot{\xi}, t), \\
\dot{E} = \omega_0 A_2 \dot{\xi} + B_2 \Gamma_2 (E, \dot{E}, \dot{\xi}, t),
\end{cases}
\]
(53)

where \( A_1 \) is defined in (12) and
\[
A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ B_2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & -k_1 \\ -k_1 & -k_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_K = \begin{bmatrix} 0 \\ -k_1 \\ -k_2 \end{bmatrix}
\]
\[
\Gamma_0 = f(e_1 + r, e_2 + r^{(1)}, t) - r^{(2)}, \Gamma_1 = \frac{\partial f}{\partial r} + \frac{\partial f}{\partial (e_1 + r)} e_2 + \frac{\partial f}{\partial (e_2 + r^{(1)})} (f - r^{(2)}) + \frac{\partial f}{\partial \dot{r}_{dd}} (r^{(3)}), \Gamma_2 = \frac{\partial f}{\partial r} + \frac{\partial f}{\partial (e_1 + r)} e_2 + \frac{\partial f}{\partial (e_2 + r^{(1)})} (f - r^{(2)}) + \frac{\partial f}{\partial \dot{r}_{dd}} (r^{(3)}) + \frac{b - \dot{b}}{b} (e_1 k_2 \dot{e}_2 + (k_2^2 - k_1) e_2) + \frac{b - \dot{b}}{b} (-3 k_2 \dot{e}_2 - k_2 \dot{e}_2 / \omega_0 + \hat{r}_{dd})
\]
(54)

First, similar with (12), the properties of \( A_K \) and \( A_2 \) will be analyzed.

The characteristic polynomial of \( A_2 \) is
\[
f_{A_2}(s) = s^3 + 3 s^2 + 3 s + (b / \dot{b})
\]
(55)

Since (14) holds, it can be verified from Routh Criterion that the polynomial \( f_{A_2}(s) \) is stable, which implies that \( A_2 \) is Hurwitz. Moreover, \( A_K \) is Hurwitz due to the stability of \( f_{Cl}(s) \). Therefore, there exist positive definite matrices \( P_K \) and \( P_2 \) such that
\[
P_K^T P_K + P_K A_K = -I, \quad P_2^T P_2 + P_2 A_2 = -I
\]
(55)
Next, the bounds of $\Gamma_i (i = 0, 1, 2)$ are analyzed. Define the function
\[
\Psi_f (a) \triangleq \sup_{||x_1, x_2|| \leq a} \psi_f (x_1, x_2)
\]
which is non-decreasing with respect to the variable $a$. Let $p_r$, $p_t$, and $\omega_0^*$ be any positives. Based on Assumption 1 and Lemma 1, for any $E \in [E^i, E^j] \leq p_r$, $x \in [E^i, E^j] \leq p_t$, and $\omega_0^* \in [\omega_0^*, \omega_0^*]$, $\Gamma_i (i = 0, 1, 2)$ has the following bound
\[
|\Gamma_0| \leq \pi_0 (p_r), \ |\Gamma_1| \leq \pi_1 (p_r), \ |\Gamma_2| \leq \pi_2 (p_r, p_t, \omega_0^*)
\]
where the positives $\pi_0 (p_r) \triangleq \Psi_f (p_r + \sqrt{2} \eta_r) + \eta_r$ and $\pi_1 (p_r) \triangleq \Psi_f (p_r + \sqrt{2} \eta_r) (1 + p_r + \eta_r + \Psi_f (p_r + \sqrt{2} \eta_r))$ are non-decreasing with respect to the variable $p_r$, and the positive $\pi_2 (p_r, p_t, \omega_0^*) \triangleq \Psi_f (p_r + \sqrt{2} \eta_r) (1 + p_r + \eta_r + \Psi_f (p_r + \sqrt{2} \eta_r)) + M_r + \eta_r$ is non-decreasing with respect to $\omega_0^*$. Moreover, it can be deduced from (50) that the control input has the following bound
\[
|u(t)| \leq 4e (p_r, p_t, \omega_0^*), \quad \forall t \geq t_0
\]
where $4e (p_r, p_t, \omega_0^*) = (\Psi_f (p_r + \sqrt{2} \eta_r) + \eta_r + \|K\| p_r + \|K\| \|T (\omega_0^*)\| p_t) / b$

Based on (52)–(57), the results (15)–(16) can be proved by the same process in the reference Xue and Huang.\(^{23}\)

1. The bounds of the tracking and estimation errors in the time sequence $[0, t_0]$ are analyzed.
2. For the time sequence $[t_0, \infty)$, the invariant set of the tracking and estimation errors is calculated.
3. Based on the invariant set of the tracking and estimation errors, the detailed trajectories of the tracking and estimation errors are studied.

The rest of the proof is omitted to save more room for the other significant results.

**Proof of theorem 2.** The proof is based on reduction to absurdity. Assume that the condition (18) is not satisfied, and for any given $(x_1(0), x_2(0), \hat{x}_1(0), \hat{x}_2(0), \bar{f}(0), \psi_f, b, b_1, k_1, k_2, \eta_r)$ and given positive $\varepsilon$, there exists $\omega_0^*$ such that (17) holds.

With $b / b > 0$, the contradiction to (17) can be easily proved by analyzing the closed-loop system for $f = 0$, $r = 0$ and $t \geq t_0$. Thus, the rest of the proof is based on $b / b > 0$.

Let $r(t) = 0$ and $f = \bar{f}(t)$. Denote
\[
\begin{bmatrix}
\dot{e}_1(t) \\
\dot{e}_2(t)
\end{bmatrix} = \begin{bmatrix}
x_1(t) - x_1^*(t) \\
\dot{x}_2(t) - \dot{x}_2^*(t)
\end{bmatrix}, \quad \begin{bmatrix}
\dot{\hat{e}}_1(t) \\
\dot{\hat{e}}_2(t)
\end{bmatrix} = \begin{bmatrix}
\dot{e}_1(t) - \dot{\hat{e}}_1(t) \\
\dot{e}_2(t) - \dot{\hat{e}}_2(t)
\end{bmatrix}
\]

Combined with (13), (52), and (53), the dynamics of variables (58) is
\[
\begin{cases}
\dot{e}_1(t) = A_1 e_1(t) + B_2 \hat{e}_1(t), \quad 0 \leq t < t_u \\
\dot{\hat{e}}_2(t) = A_2 e_2(t) + B_2 \hat{\hat{e}}_2(t), \quad t \geq t_u
\end{cases}
\]

where
\[
A_1 = \begin{bmatrix}
-3\omega_0^* & 1 \\
-3\omega_0^* & 0
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
-3\omega_0^* & 0 \\
-3\omega_0^* & 1
\end{bmatrix}
\]

According to the condition that there exists $\omega_0^*$ such that (17) holds for any given $\varepsilon$, it can be deduced from the dynamics of $[e_1^T, e_2^T]^T$ in (60) that the following inequality holds for some positive $\varepsilon_1$ satisfying $\lim_{t \to 0} \varepsilon_1 = 0$
\[
\sup_{t \geq t_u} \{ |\varepsilon_2(t)|, |k_2 \dot{\hat{e}}_2(t) + \dot{\hat{e}}_3(t)| \} \leq \varepsilon_1
\]

Since $\bar{f}(t)$ can be arbitrarily chosen in any bounded set, the necessary condition for (62) is that the matrix $A_2$ is Hurwitz.

Then, with Routh Criterion, the necessary and sufficient condition for $A_2$ being Hurwitz is that
\[
\begin{cases}
\omega_0^* > 0, \\
(9 - \frac{b}{b}) \omega_0^* + \frac{3(b - \frac{b}{b})k_2 \omega_0^*}{b} > 0,
\end{cases}
\]

Since
\[
(9 - \frac{b}{b}) < 0, \quad -\frac{3(b - \frac{b}{b})k_2 \omega_0^*}{b} < 0
\]
there exist positives $M_{\omega_0}$ and $\overline{M}_{\omega_0}$ dependent on $(\bar{b}, b, k_2)$ such that

\[
\left(\frac{9 - \bar{b}}{b}\right) \omega_0^3 + \frac{3(\bar{b} - b)k_2\omega_0^2}{b} < 0, \quad \forall \omega_0 \geq M_{\omega_0} \quad (65)
\]

\[
b\omega_0^3 + \frac{3(\bar{b} - b)k_2\omega_0^2}{b} - \frac{3(b - \bar{b})k_2\omega_0^2}{b} < 0, \quad \forall 0 < \omega_0 \leq \overline{M}_{\omega_0} \quad (66)
\]

Therefore, the stability of the matrix $\tilde{A}_2$ implies that $\omega_0 \in [M_{\omega_0}, \overline{M}_{\omega_0}]$. Next, we will provide an initial condition $[x_2(0) \ \dot{e}_3(0)]^T$ and an uncertainty $f$, which lead to a contradiction to (62).

With the definition of $t_u$ (11), the initial values $x_2(0)$ and $\dot{e}_2(0)$ can be chosen such that $t_u > M_{t_u}$ for some positive constant $M_{t_u}$ and $\forall \omega_0 \in [M_{\omega_0}, \overline{M}_{\omega_0}]$. Based on (61), the following equation

\[
\dot{\Gamma}_0 > M_{\Gamma} > 0, \quad 0 \leq t \leq t_u \quad (67)
\]

can be ensured for any positive constant $M_{\Gamma}$ by selecting the uncertainty $f$. By designing $M_{\Gamma}$ being suitably large, the dynamics (59) illustrates that $|\dot{e}_3(M_{t_u})| > \epsilon$ which contradicts to (17). Thus, $b/\bar{b} \in (0, 9)$ is proved.

**Proof of theorem 3.** From (29) in Lemma 1, there exists $T^* > 0$ such that

\[
\sup_{t \geq T^*} |\tilde{r}_{\dot{e}_2}(t) - \tilde{r}^{(2)}(t)| \leq \frac{1 - \frac{2}{\epsilon}}{2} \tilde{\eta}_e, \forall T \in (0, T^*)
\]

Combined with the conditions in the statement (S1), the upper bound of $|f(t)|$ can be expressed as follows

\[
|f(t)| \leq |\tilde{r}_{\dot{e}_2}(t) - \tilde{r}^{(2)}(t)| + |f(t)| \leq \frac{1 - \frac{2}{\epsilon}}{2} \tilde{\eta}_e + \bar{\epsilon}_e |r^{(2)}(t)|
\]

for $t > \max\{t_u, 3T_s\}$ and $T \in (0, T^*)$. On the contrary, the lower bound of $|\tilde{f}(t)|$ satisfies

\[
|\tilde{f}(t)| \geq |\tilde{r}^{(2)}(t)| - |f(t)| \geq (1 - \tilde{\epsilon}_e) |r^{(2)}(t)|
\]

\[
\geq \bar{\epsilon}_e |r^{(2)}(t)| + (1 - 2\tilde{\epsilon}_e) \tilde{\eta}_e
\]

for $t > \max\{t_u, 3T_s\}$. With the combination of (68)–(69), we have

\[
|f(t)| < |\tilde{f}(t)|, \forall t > \max\{t_u, 3T_s\}, \forall T \in (0, T^*) \quad (70)
\]

which completes the proof of the statement (S1).

With the similar deduction, the proof of the statement (S2) can be obtained, which is omitted here.