ON HILBERT COEFFICIENTS AND SEQUENTIALLY GENERALIZED COHEN-MACAULAY MODULES

NGUYEN TU CUONG, NGUYEN TUAN LONG, AND HOANG LE TRUONG

Abstract. This paper shows that if $R$ is a homomorphic image of a Cohen-Macaulay local ring, then $R$-module $M$ is sequentially generalized Cohen-Macaulay if and only if the difference between Hilbert coefficients and arithmetic degrees for all distinguished parameter ideals of $M$ are bounded.

1. Introduction

Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring, where $\mathfrak{m}$ is the maximal ideal. Let $M$ be a finitely generated $R$-module of dimension $d$. For an $\mathfrak{m}$-primary ideal $I$ of $R$, it is well-known that there are integers $\{e_i(I; M)\}_{i=0}^d$, called the Hilbert coefficients of $M$ with respect to $I$, such that

$$\ell_R(M/I^{n+1}M) = e_0(I; M) \left(\frac{n+d}{d}\right) - e_1(I; M) \left(\frac{n+d-1}{d-1}\right) + \cdots + (-1)^d e_d(I; M)$$

for all $n \gg 0$. Here $\ell_R(N)$ denotes the length of an $R$-module $N$. In particular, the leading coefficient $e_0(I; M)$ is said to be the multiplicity of $M$ with respect to $I$ and $e_1(I; M)$ is called by Vasconcelos (27) the Chern coefficient of $M$ with respect to $I$. In 2008, Vasconcelos posed the Vanishing Conjecture: $M$ is Cohen-Macaulay if and only if $e_1(q; M) = 0$ for some parameter ideal $q$ of $M$. It is shown that the relation between Cohen-Macaulayness and the Chern number of parameter ideals is quite surprising. Motivated by some profound results of [5, 20] and also by the fact that this is true for $M$ is unmixed as shown in [10], it was asked whether the characterization of many classes of non-unmixed rings such as Buchsbaum rings, generalized Cohen-Macaulay rings, sequentially Cohen-Macaulay rings in terms of the Hilbert coefficients and other invariants of $M$ (see [11, 6, 22, 23, 18, 24]). The aim of our paper is to continue this research direction.

To state the results of this paper, first of all let us fix our notation and terminology. First, a filtration $D : M = D_0 \supset D_1 \supset \ldots \supset D_t = W$ of $R$-submodules of $M$ is called the dimension filtration of $M$, if for all $1 \leq i \leq \ell$, $D_i$ is the largest submodule of $D_{i-1}$ with $\dim_R D_i < \dim_R D_{i-1}$, where $\dim_R(0) = -\infty$ for convention. We say that $M$ is sequentially (generalized) Cohen-Macaulay, if $C_i = D_i/D_{i+1}$ is (generalized) Cohen-Macaulay for all $0 \leq i \leq \ell - 1$. A system of parameters $\underline{x} = x_1, x_2, \ldots, x_d$ of $M$ is said to be distinguished, if $(x_j \mid d_i < j \leq d)D_i = (0)$ for all $0 \leq i \leq \ell$, where $d_i = \dim_R D_i$ ([19 Definition 2.5]). A parameter

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ideal \( q \) of \( M \) is called \textit{distinguished}, if there exists a distinguished system \( x_1, x_2, \ldots, x_d \) of parameters of \( M \) such that \( q = (x_1, x_2, \ldots, x_d) \). For each \( i = 1, \ldots, s \), set

\[
\Lambda_i(M) = \{(-1)^j e_i(q, M) - \text{adeg}_i(q; M) \mid q \text{ is a distinguished parameter ideal of } M\},
\]

where \( \text{adeg}_i(I; M) = \sum_{p \in \text{Ass}(M), \dim R/p = i} \ell_{R_p}(H^0_{pR_p}(M_p))e_0(I; R/p) \) is the \( i \)-th arithmetic degree of \( M \) with respect to \( I \) (see [1], [25], [26]). Then we have the following results as in Table 1. A symbol \( (\text{\textbf{*}}) \) requires that the module \( M \) be unmixed.

| \( \Lambda_1(M) \subseteq (\infty, 0] \) | \( M \) | 5 |
| \( 0 \in \Lambda_1(M), (\text{\textbf{*}}) \) | \( M \) is Cohen-Macaulay | 10 |
| \( |\Lambda_1(M)| < \infty, (\text{\textbf{*}}) \) | \( M \) is generalized Cohen-Macaulay | 10 [11] |
| \( 0 \in \Lambda_i(M) \text{ for all } i = 1, \ldots, d \) | \( M \) is sequentially Cohen-Macaulay | 5 |

Table 1. Properties of a finitely generated module \( M \) are carried by the behavior of the specific set. A symbol \( (\text{\textbf{*}}) \) requires that the module \( M \) be unmixed.

This paper aims to extend these results in the sequentially generalized Cohen-Macaulay case. The answer is affirmative, which we are eager to report in the present writing.

\textbf{Theorem 1.1} \textbf{(Theorem 4.1).} Assume that \( R \) is a homomorphic image of a Cohen-Macaulay local ring. Then the following statements are equivalent.

i) \( M \) sequentially generalized Cohen-Macaulay.

ii) The set \( \Lambda_i(M) \) is finite for all \( 1 \leq i \leq d \).

The paper is divided into four sections. The next section presents some preliminaries. In Section 3, we prove that if \( M \) is a sequentially generalized Cohen-Macaulay module, then the set \( \Lambda_i(M) \) is finite for all \( i = 1, \ldots, d \). A characterization of sequentially generalized Cohen-Macaulay modules by the finiteness of \( \Lambda_i(M) \) will be shown in the last section.

2. Preliminaries

In what follows, throughout this paper, let \( (R, \mathfrak{m}, k) \) be a Noetherian local ring, where \( \mathfrak{m} \) is the maximal ideal and \( k = R/\mathfrak{m} \) is the infinite residue field of \( R \). Suppose that \( R \) is a homomorphic image of a Cohen-Macaulay local ring. Let \( M \) be a finitely generated \( R \)-module of dimension \( d \).

\textbf{Definition 2.1} \textbf{([7], [2], [19])}. \( i \) We say that a finite filtration of submodules of \( M \)

\[
\mathcal{F}: M = M_0 \supset M_1 \supset \ldots \supset M_s
\]

satisfies the dimension condition if \( \dim M_i > \dim M_{i+1} \), for all \( i = 0, \ldots, s-1 \) and we say this case that the filtration \( \mathcal{F} \) has the length \( s \).

\( ii \) A filtration

\[
\mathcal{D}: M = D_0 \supset D_1 \supset \ldots \supset D_t = W
\]
of submodules of $M$ is said to be the dimension filtration if $D_i$ is the largest submodule of $D_{i-1}$ with $\dim D_i < \dim D_{i-1}$ for all $i = 1, \ldots, t$. Note that the dimension filtration always exists uniquely (see [7]).

**Notation 2.2.**
- $t$: the length of the dimension filtration of $M$,
- $\mathcal{D} = \{D_i\}_{i=0}^t$: the dimension filtration of $M$,
- $d_i = \dim D_i$ for all $i = 0, \ldots, t$,
- $\mathcal{F} = \{M_i\}_{i=0}^t$: a filtration of submodules of $M$ of length $s$ satisfying the dimension condition,
- $\mathcal{F}/x\mathcal{F} = \{(M_i + xM)/xM\}_{i=0}^k$: a filtration of submodules of $M/xM$, where $x$ is a parameter element of $M$ and $k = \{s - 1$ if $\dim (M_{s-1}) = 1,$
  $s$ otherwise,
- $\mathcal{F}(M) = \{\mathcal{F} = \{M_i\}_{i=0}^t | \ell(D_i/M_i) < \infty$ for all $i = 0, \ldots, t\}$.

**Definition 2.3.**

1) A system of parameters $x_1, \ldots, x_d$ of $M$ is called a distinguished system of parameters with respect to $\mathcal{F}$ if $(x_{\dim M_i+1}, \ldots, x_d)M_i = 0$ for all $i = 1, \ldots, s$. A distinguished system of parameters of $M$ with respect to $\mathcal{D}$ is simply called a distinguished system of parameters of $M$. An ideal $q$ is said to be a distinguished parameter ideal of $M$ with respect to $\mathcal{F}$ if it is generated by a distinguished system of parameters of $M$ with respect to $\mathcal{F}$. A distinguished parameter ideal of $M$ with respect to $\mathcal{D}$ is simply called a distinguished parameter ideal of $M$ ([19]).

2) A system of parameters $x_1, \ldots, x_d$ is called a good system of parameters of $M$ if $(x_{d+1}, \ldots, x_d)M \cap D_i = 0$ for all $i = 1, \ldots, t$. An ideal $q$ is said to be a good parameter ideal of $M$ if it is generated by a good system of parameters of $M$ (see [2]).

Recall that there always exists distinguished systems of parameters of $M$ with respect to $\mathcal{F}$ (see [19 Lemma 2.6]). Note that, if $x_1, x_2, \ldots, x_d$ is a distinguished system of parameters of $M$ with respect to $\mathcal{F}$, then $x_2, \ldots, x_d$ is also a distinguished system of parameters of $M/x_1M$ with respect to $\mathcal{F}/x_1\mathcal{F}$.

Clearly, a good system of parameters is a distinguished system of parameters.

**Definition 2.4 ([17, 22, 26]).** Let $I$ be an $m$-primary ideal of $R$. The $i$-th arithmetic degree of $M$ with respect to $I$ is defined by

$$\text{adeg}_i(I; M) = \sum_{p \in \text{Ass}(M), \dim R/p = j} \ell(H^0_{pR_y}(M_p))c_0(I; R/p).$$

The following result is an immediate consequence of Proposition 3.2 in [5].

**Lemma 2.5** (c.f. [5 Proposition 3.2]). Let $I$ be an $m$-primary ideal of $R$ and $\mathcal{F} \in \mathcal{F}(M)$. Then

$$\text{adeg}_i(I; M) = \begin{cases} \ell(H^0_{m}(M)) & \text{if } i = 0, \\ c_0(I; M_i) & \text{if } d_i = j \text{ for some } i = 0, \ldots, t - 1, \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 2.6.** Let $\mathcal{F} \in \mathcal{F}(M)$ and $q$ a parameter ideal of $M$. Suppose that there exists a filter regular element $x \in q$ of $M$ such that $\mathcal{F}/xF \in \mathcal{F}(M/xM)$. We have

$$\text{adeg}_j(q; M/xM) = \text{adeg}_{j+1}(q; M),$$

where $j = 0, \ldots, t - 1$, $t = \dim M$, $m = \dim R$, and $\text{adeg}_i(q; M)$ is the arithmetic degree of $M$ with respect to $q$. For $q = 0$, i.e., $x = 0$, we have $\text{adeg}_i(0; M) = 0$ for all $i = 0, \ldots, t$.
for all \( j \geq 1 \). Moreover, if \( q := (x := x_1, x_2, \ldots, x_d) \) is a distinguished parameter ideal of \( M \) with respect to \( F \), then \( \text{adeg}_0(q; M/xM) \geq \text{adeg}_1(q; M) \).

**Proof.** Since \( F/xF \in F(M/xM) \), we have \( D/xM \in F(M/xM) \). Moreover, the length of \( D/xM \) is \( t - 1 \) if \( d_{t-1} = 1 \) and \( t \) otherwise. Since \( x \) is a filter regular element of \( M \), \( x \) is a regular element of \( M/D_i \) for all \( i = 1, \ldots, t \). Thus we have \( xM \cap D_i = xD_i \). By Lemma 2.5 and \( D/xM \in F(M/xM) \), we have

\[
\text{adeg}_{d_i-1}(q; M/xM) = e_0(q; (D_i + xM)/xM) = e_0(q; D_i) = \text{adeg}_d(q; M)
\]

and \( \text{adeg}_j(q; M) = 0 = \text{adeg}_{j-1}(q; M/xM) \) for all \( i = 0, \ldots, t - 1 \) and \( 2 \leq d_i < j < d_{i-1} \). Hence

\[
\text{adeg}_j(q; M/xM) = \text{adeg}_{j+1}(q; M),
\]

for all \( j \geq 1 \).

Now, if \( d_{t-1} > 1 \), then \( \text{adeg}_1(q; M) = 0 \leq \text{adeg}_0(q; M) \). So we can assume that \( d_{t-1} = 1 \). Since \( q = (x, x_2, \ldots, x_d) \) is a distinguished ideal of \( M \) with respect to \( F \in F(M) \), by Lemma 2.5 we get

\[
\text{adeg}_1(q; M) = e_0(q; M_{t-1}) = e_0(x; M_{t-1}) = e_0(x; D_{t-1}) \leq \ell(D_{t-1}/xD_{t-1}).
\]

Since \( H_0^0(M/xM) \supseteq (D_{t-1} + xM)/xM \cong D_{t-1}/(D_{t-1} \cap xM) = D_{t-1}/xD_{t-1} \), we have

\[
\text{adeg}_1(q; M) \leq \ell(D_{t-1}/xD_{t-1}) \leq \ell(H_0^0(M/xM)) = \text{adeg}_0(q; M/xM),
\]

as required. \( \square \)

The following lemma was proved by [3] Lemma 3.3, Lemma 3.4].

**Lemma 2.7.** The following statements are true.

(i) Assume that \( d \geq 2 \). Let \( x \) be a superficial element of \( M \) for a parameter ideal \( q \) of \( M \). Then

\[
e_i(q; M/xM) = e_i(q; M/xM)
\]

for all \( i = 0, \ldots, d - 2 \) and \( (-1)^{d-1}e_{d-1}(q; M) = (-1)^{d-1}e_{d-1}(q; M/xM) + \ell(0 :_M x) \).

(ii) Let \( N \) be a submodule of \( M \) with \( \dim N = s < d \) and \( I \) an \( m \)-primary ideal of \( R \). Then

\[
e_j(I; M) = \begin{cases} e_j(I; M/N), & \text{if } 0 \leq j \leq d - s - 1, \\ e_{d-s}(I; M/N) + (-1)^{d-s}e_0(I, N) & \text{if } j = d - s. \end{cases}
\]

3. The finiteness of the set \( \mathcal{P}_D(M) \)

Recall that the definition of a sequentially Cohen-Macaulay module was introduced first by LT. Nhan and the first author ([7]).

**Definition 3.1.** A filtration of submodules \( F = \{ M_i \}_{i=0}^t \) of \( M \) is called a **generalized Cohen-Macaulay filtration** if \( F \in F(M) \) and \( M_{i-1}/M_i \) are generalized Cohen-Macaulay modules for all \( i = 1, \ldots, t - 1 \). A module \( M \) is called **sequentially generalized Cohen-Macaulay** if it has a generalized Cohen-Macaulay filtration. In particular, \( F(M) \) is the set of all generalized Cohen-Macaulay filtrations of \( M \). If \( M \) is a sequentially Cohen-Macaulay module, then \( M \) is a sequentially generalized Cohen-Macaulay module.
Now, the function
\[
H_{I,M}^{ad}(n) = \ell(M/I^{n+1}M) - \sum_{i=0}^{d} \text{adeq}_i(I; M) \binom{n+i}{i}
\]
is called an adjusted Hilbert-Samuel function of \( M \) with respect to \( I \). It is well known that \( \ell(M/I^{n+1}M) \) becomes a polynomial for large enough \( n > 0 \). So the function \( H_{I,M}^{ad}(n) \) becomes a polynomial \( P_{q,M}^{ad}(n) \). Such polynomial is called adjusted Hilbert-Samuel polynomial and of the form
\[
P_{I,M}^{ad}(n) = \sum_{i=1}^{d} \left( (-1)^i e_i(I; M) - \text{adeq}_{d-i}(I; M) \right) \binom{n+d-i}{d-i}.
\]
These integers \( a_i(I; M) = (-1)^i e_i(I; M) - \text{adeq}_{d-i}(I; M) \) are called adjusted Hilbert coefficients of \( M \) with respect to \( I \) for all \( i = 1, \ldots, d \). We denote by \( \mathcal{P}_F(M) \) the set of all adjusted Hilbert-Samuel polynomials \( P_{q,M}^{ad}(n) \), where \( q \) runs over the set of all distinguished parameter ideals of \( M \) with respect to \( F \).

Recall that a system \( x_1, \ldots, x_m \) in \( R \) is said to be \( d \)-sequence on \( M \) (see [13], [21]) if
\[
(x_1, \ldots, x_i)M : x_i x_k = (x_1, \ldots, x_i-1)M : x_k
\]
for all \( i = 1, \ldots, m \) and \( k \geq i \). The sequence \( \mathbf{e} \) is said to be \( \text{dd-sequence} \) on \( M \) (see [4]) if \( x_1^{n_1}, \ldots, x_s^{n_s} \) is a \( d \)-sequence on \( M \) and \( x_1^{n_1}, \ldots, x_i^{n_i} \) is a \( d \)-sequence on \( M/(x_1^{n_1+1}, \ldots, x_i^{n_i})M \) for all positive integers \( n_1, \ldots, n_s \) and all \( i = 1, \ldots, s - 1 \). According to D. T. Cuong and the first author, if the parameter ideal \( q \) is generated by a \( \text{dd-sequence} \) on \( M \), the adjusted Hilbert coefficients are described as follows.

**Lemma 3.2** ([3] Theorem 6.2). Let \( M \) be a sequentially generalized Cohen-Macaulay module and \( q = (x_1, \ldots, x_d) \) a system of parameters of \( M \). Assume that \( x_1, \ldots, x_d \) is a \( d \)-sequence on \( M \). Then the adjusted Hilbert coefficients are of the form
\[
a_d - d_k(q; M) = \sum_{j=1}^{d_k} \frac{(d_k - 1)}{j - 1} \ell(H^1_m(M/D_k))
\]
for \( k = 0, \ldots, t - 1 \),
\[
a_d - i(q; M) = \sum_{j=1}^{i} \frac{(i - 1)}{j - 1} \ell(H^1_m(M/D_k))
\]
for \( d_k < i < d_{k-1} \) and \( a_d(q; M) = 0 \).

Now with the above notations, we have the main result in this section.

**Theorem 3.3.** Let \( M \) be a sequentially generalized Cohen-Macaulay module and \( F \in \mathcal{F}(M) \). Then the set \( \mathcal{P}_F(M) \) of adjusted Hilbert-Samuel polynomials is finite.

**Setting 3.4.** In this section, from now on, we assume that \( M \) is a sequentially generalized Cohen-Macaulay module. Set \( W = H^0_m(M), M = M/W, \text{ and } \overline{N} = (N + W)/W \) for all submodules \( N \) of \( M \). Let \( F \in \mathcal{F}(M), q := (x := x_1, x_2, \ldots, x_d) \) be a distinguished parameter ideal of \( M \) with respect to \( F \).

Set
\[
I(M) = \sup\{\ell(M/qM) - e_0(q; M) \mid q \text{ is a parameter ideal of } M\}.
\]
and \( I(\mathcal{F}, M) = \sum_{i=0}^{t-1} I(M_i/M_{i+1}) + \ell(M_i) \).

**Fact 3.5.** With this notation, we have

i) The filtration of submodules \( \mathcal{F} := \{ M_i \}_{i=0}^t \) of \( M \) is a generalized Cohen-Macaulay filtration of \( M \) and

\[
I(\mathcal{F}, M) = I(\mathcal{F}, M) + \ell(W)
\]

([6] Lemma 6).

ii) The module \( M/xM \) is sequentially generalized Cohen-Macaulay and \( \mathcal{F}/x\mathcal{F} \subseteq \mathcal{F}(M/xM) \) ([6], Corollary 2]). Moreover, we have

\[
I(\mathcal{F}/x\mathcal{F}, M/xM) \leq I(\mathcal{F}, M)
\]

([6] Lemma 4).

Now, let \( S = \bigoplus_{n \geq 0} S_n \) be a standard Noetherian graded ring and \( E = \bigoplus_{n \in \mathbb{Z}} E_n \) a finitely generated graded \( S \)-module. The *Castelnuovo-Mumford regularity* \( \text{reg}(E) \) of \( E \) is defined by

\[
\text{reg}(E) = \sup \{ n + i \mid [H^i_{S_+}(E)]_n \neq 0, i \geq 0 \}
\]

and simply called *regularity*, where \( S_+ = \bigoplus_{n \geq 0} S_n \). Let \( N \) be a finitely generated \( R \)-module and \( Q \) a parameter ideal of \( N \). We always denote the associate graded module of \( N \) with respect to \( Q \) by \( G_Q(N) \), i.e. \( G_Q(N) = \bigoplus_{n \geq 0} Q^n N/Q^{n+1} N \). With this notation, we have

**Fact 3.6.**

i) There is a constant \( C = C_\mathcal{F} \) such that

\[
\text{reg}(G_Q(M)) \leq C = (3I(\mathcal{F}, M))^d - 2I(\mathcal{F}, M),
\]

for all distinguished parameter ideals \( q \) of \( M \) with respect to \( \mathcal{F} \) ([6] Theorem 4]).

ii) We have \( H^d_{q,M}(n) \geq 0 \) for all

\[
n \geq \text{reg}(G_q(M)) + \left( \frac{\text{reg}(G_q(M)) + d - 1}{d - 1} \right) I(\mathcal{F}, M) + d,
\]

([15] Theorem 4.4]).

The following result give an upper bound for the adjusted Hilbert-Samuel function.

**Lemma 3.7.** We have

\[
H^d_{q,M}(n) \leq \sum_{i=0}^{t-1} \binom{n + d_i - 1}{d_i - 1} I(M_i/M_{i+1}) + \ell(M_i) - \ell(W),
\]

for all \( n \geq 0 \),

**Proof.** Note that from the following exact sequence

\[
0 \to (q^{n+1} M \cap M_1)/q^{n+1} M_1 \to M_1/q^{n+1} M_1 \to M/q^{n+1} M \to M/q^{n+1} M + M_1 \to 0,
\]

we get \( \ell(M/q^{n+1} M) \leq \ell(M/q^{n+1} M + M_1) + \ell(M_1/q^{n+1} M_1) \).
Now we argue by the induction on the length $t$ of the dimension filtration of $M$. The case $t = 1$, it follows from $qM_1 = 0$ and Lemma 1.1 in [10] that we have
\[
\ell(M/q^{n+1}M) \leq \ell(M/q^{n+1}M + M_1) + \ell(M_1)
\]
\[
\leq \binom{n + d}{d} e_0(q; M) + \binom{n + d - 1}{d - 1} I(M/M_1) + \ell(M_1),
\]
for all $n \geq 0$.

Now, assume that $t > 1$ and that our assertion holds true for $t - 1$. By the inductive hypothesis, we have
\[
\ell(M/q^{n+1}M) \leq \ell(M/q^{n+1}M + M_1) + \ell(M_1/q^{n+1}M_1)
\]
\[
\leq \binom{n + d}{d} e_0(q; M_0) + \binom{n + d - 1}{d - 1} I(M/M_1) + \ell(M_1/q^{n+1}M_1)
\]
\[
\leq \sum_{i=0}^{t-1} \binom{n + d_i}{d_i} e_0(q; M_i) + \sum_{i=0}^{t-1} \binom{n + d_i - 1}{d_i - 1} I(M_i/M_{i+1}) + \ell(M_i),
\]
for all $n \geq 0$. By Lemma 3.5 we have
\[
H_{q,M}^{rd}(n) \leq \sum_{i=0}^{t-1} \binom{n + d_i - 1}{d_i - 1} I(M_i/M_{i+1}) + \ell(M_i) - \ell(W),
\]
for all $n \geq 0$, as requested. \qed

**Theorem 3.8.** Let $C = C_\mathcal{F} = (3I(\mathcal{F}, M))^{d_1} - 2I(\mathcal{F}, M)$ as in Fact 3.6 i). Then we have

1. $|e_1(q; M) + \text{adeg}_{d-1}(q; M)| \leq I(M/M_1)$.
2. $|(-1)^i e_i(q; M) - \text{adeg}_{d-i}(q; M)| \leq 2^{i-1} ((C + 1)^{d-1}I(\mathcal{F}, M) + d + C + 2)^{i-1} I(\mathcal{F}, M)$ for all $i = 2, \ldots, d - 1$.
3. $|e_d(q; M)| \leq 2^{d-1} ((C + 1)^{d-1}I(\mathcal{F}, M) + d + C + 2)^{d-1} I(\mathcal{F}, M)$.

**Proof.** i) By Fact 3.6 ii), we have $P_{q,M}^{rd}(n) = H_{q,M}^{rd}(n) \geq 0$ for all $n \geq 0$. Thus by Lemma 3.7 we have
\[
0 \leq P_{q,M}^{rd}(n) = \binom{n + d - 1}{d - 1}((-1)^{d-1}e_1(q; M) - \text{adeg}_{d-1}(q; M)) + \text{lower terms}
\]
\[
\leq \binom{n + d - 1}{d - 1} I(M/M_1) + \text{lower terms}.
\]
Therefore we have
\[
0 \leq (-1)e_1(q; M) - \text{adeg}_{d-1}(q; M) \leq I(M/M_1),
\]
as requested.

ii) and iii). Now we proceed by induction on $d$ to show that ii) and iii). Set $r = \text{reg}(G_q(M))$. The case $d = 2$, we have
\[
e_2(q; M) = H_{q,M}^{rd}(n) + \ell(H_{q,M}^{rd}(M)) + (e_1(q; M) + \text{adeg}_1(q; M))\binom{n + 1}{1}
\]
for all $n \geq r + 1$. Thus by Lemma 3.7 and Theorem 3.6 we have
\[
|e_2(q; M)| \leq \sum_{i=0}^{t} \binom{n + d_i - 1}{d_i - 1} I(M_i/M_{i+1}) + I(M/M_1)(n + 1)
\]
\[
\leq 2(n + 1)^{r-1} I(\mathcal{F}, M).
\]
for all \( n \geq r + \binom{r+1}{1} I(\mathcal{F}, M) + d \).

Now choose \( n = (C + 1)^{2-1} I(\mathcal{F}, M) + d + C + 1 \). Then \( n \geq r + \binom{r+1}{1} I(\mathcal{F}, M) + d \) by Fact \( \text{3.4} \) i).

Therefore we have
\[
|e_2(q; M)| \leq 2^{2-1} \left( (C + 1)^{2-1} I(\mathcal{F}, M) + d + C + 1 \right) I(\mathcal{F}, M)
= 2^{d-1} \left( (C + 1)^{d-1} I(\mathcal{F}, M) + d + C + 2 \right)^{d-1} I(\mathcal{F}, M),
\]
as required.

Now suppose that \( d > 2 \) and that our assertion holds true for \( d - 1 \). We have by Lemma \( \text{2.7} \) ii) that
\[
e_i(q; M) = e_i(q; M/xM)
\]
for all \( i = 0, \ldots, d - 1 \) and \( (-1)^d e_d(q; M) = (-1)^d e_d(q; M/xM) + \ell(W) \). Moreover, by Fact \( \text{3.5} \) i) we have
\[
I(\mathcal{F}, M/xM) + \ell(W) = I(\mathcal{F}, M).
\]
Consequently, \( C_\mathcal{F} \geq C_\mathcal{F} \). Therefore we can assume \( W = 0 \). Recall that \( x = x_1, \ldots, x_d \) is a distinguished system of parameters of \( M \) with respect to \( \mathcal{F} \) such that \( q = (x) \) and \( x = x_1 \) is a superficial element of \( M \) for \( q \). Hence \( x \) is a regular element of \( M \) since \( W = 0 \). So by Lemma \( \text{2.7} \) i),
\[
e_i(q; M) = e_i(q; M/xM)
\]
for all \( i = 0, \ldots, d - 1 \). On the other hand, it follows from Lemma \( \text{2.6} \) and Fact \( \text{3.5} \) ii) that we have
\[
\text{adeg}_{d-1}(q; M) = \text{adeg}_{d-2-i}(q; M/xM)
\]
for all \( i = 0, \ldots, d - 2 \) and
\[
I(\mathcal{F}/x\mathcal{F}, M/xM) \leq I(\mathcal{F}, M).
\]
Set \( C_x = C_{\mathcal{F}/x\mathcal{F}} \). It follows that
\[
C_x = C_{\mathcal{F}/x\mathcal{F}} \leq C_\mathcal{F} = C.
\]
Note that \( x_2, \ldots, x_d \) is a distinguished system of parameters of sequentially generalized Cohen-Macaulay \( M/xM \) with respect to \( \mathcal{F}/x\mathcal{F} \in \mathcal{F}(M/xM) \). Therefore it follows from the inductive hypothesis and (1) – (4) that we have
\[
|(-1)^i e_i(q; M) - \text{adeg}_{d-1}(q; M)|
= |(-1)^i e_i(q; M/xM) - \text{adeg}_{d-1-i}(q; M/xM)|
\leq 2^{i-1} \left( (C_2 + 1)^{d-2} I(\mathcal{F}/x\mathcal{F}, M/xM) + (d - 1) + C_x + 2 \right)^{i-1} I(\mathcal{F}/x\mathcal{F}, M/xM)
\leq 2^{i-1} \left( (C + 1)^{d-1} I(\mathcal{F}, M) + d + C + 2 + I(\mathcal{F}, M) \right)^{i-1} I(\mathcal{F}, M)
\]
for all \( i = 2, \ldots, d - 2 \).

Fact, if \( d_{d-1} > 1 \) then \( \text{adeg}_1(q; M) = 0 \) by Lemma \( \text{2.5} \). So
\[
|(-1)^{d-1} e_{d-1}(q; M) - \text{adeg}_1(q; M)|
= |(-1)^{d-1} e_{d-1}(q; M/xM)|
\leq 2^{(d-1)-1} \left( (C_2 + 1)^{d-2} I(\mathcal{F}/x\mathcal{F}, M/xM) + (d - 1) + C_x + 2 \right)^{(d-1)-1} I(\mathcal{F}/x\mathcal{F}, M/xM)
\leq 2^{d-1} \left( (C + 1)^{d-1} I(\mathcal{F}, M) + d + C + 2 \right)^{d-1} I(\mathcal{F}, M).
\]
The last inequality is followed by (3) and (4).
Now we can assume that $d_i = 1$. Set $\tilde{M} = M/D_{i-1}$ and $\tilde{N} = (N+D_{i-1})/D_{i-1}$ for all submodules $N$ of $M$. Then $\tilde{M}$ is sequentially generalized Cohen-Macaulay, $\tilde{F} = \{\tilde{M}_i\}_{i=0}^{t-1} \in \mathcal{F}(\tilde{M})$ and $q$ is also a distinguished parameter ideal of $\tilde{M}$ with respect to $\tilde{F}$. Note that $x_1, \ldots, x_d$ is a distinguished system of parameters of sequentially generalized Cohen-Macaulay $\tilde{M}$ with respect to $\tilde{F}$. Using similar above arguments, it follows from $H^0_{\mathfrak{m}}(\tilde{M}) = 0$ that we show that results 1) – 4) for module $\tilde{M}$.

By Fact 3.6 i), we have

$$\text{reg}(G_q(\tilde{M})) \leq \tilde{C} := (3I(\tilde{F}, \tilde{M}))^d - 2I(\tilde{F}, \tilde{M}).$$

Since $(M_i \cap D_{t-1})/(M_{i+1} \cap D_{t-1}) = (M_{i+1} + M_i \cap D_{t-1})/M_{i+1}$ is a submodule of the module $D_{t-1}/M_{t-1}$ of finite length for all $i = 0, \ldots, t - 2$, we have

$$I(\mathcal{F}, M) = \sum_{i=0}^{t-2} I(M_i/M_{i+1}) + I(M_{t-2}/M_t) + l(M_t)
\geq \sum_{i=0}^{t-2} (I(M_i/M_{i+1}) - l((M_i \cap D_{t-1})/(M_{i+1} \cap D_{t-1}))) + I(M_{t-1})
= \sum_{i=0}^{t-2} I((M_i + D_{t-1})/(M_{i+1} + D_{t-1})) + I(M_{t-1})
= I(\mathcal{F}, \tilde{M}) + I(M_{t-1}).$$

This implies that $\tilde{C} \leq C$. We have by Lemma 2.7 ii) and Lemma 2.8 that

$$(-1)^{d-1}e_{d-1}(q, M) - \text{adeg}_1(q, M) = (-1)^{d-1}e_{d-1}(q, \tilde{M}).$$

Therefore it follows from the inductive hypothesis and similar results 1) – 4) for module $\tilde{M}$ that we have

$$\left| (-1)^{d-1}e_{d-1}(q; M) - \text{adeg}_1(q; M) \right| = \left| (-1)^{d-1}e_{d-1}(q; \tilde{M}) \right| = \left| (-1)^{d-1}e_{d-1}(q; \tilde{M}/x\tilde{M}) \right| \leq 2^{d-2} \left( (\tilde{C}_x + 1)^{d-1}I(\tilde{F}/x\tilde{F}, \tilde{M}/x\tilde{M}) + d - 1 + \tilde{C}_x + 2 \right)^{d-1} I(\tilde{F}/x\tilde{F}, \tilde{M}/x\tilde{M})
\leq 2^{d-1} \left( (\tilde{C} + 1)^{d-1}I(\tilde{F}, \tilde{M}) + d + \tilde{C} + 2 \right)^{d-1} I(\tilde{F}, \tilde{M})
\leq 2^{d-1} \left( (C + 1)^{d-1}I(\mathcal{F}, M) + d + C + 2 \right)^{d-1} I(\mathcal{F}, M)$$

where $\tilde{C}_x = C_{\tilde{F}/x\tilde{F}}$.

Now, we have

$$(-1)^d e_d(q; M) = H_{q, d}^d(n) - \sum_{i=1}^{d-1} \left( (-1)^i e_i(q; M) - \text{adeg}_{d-i}(q; M) \right) \binom{n + d - i}{d - i}.$$
Theorem 4.1. For all $n \geq r + 1$. Furthermore, for all $n \geq r + \binom{r+1}{1} I(\mathcal{F}, M) + d$, by Lemma 3.7 and Fact 3.6 we have

$$| e_d(q; M) | \leq t-1 \sum_{i=0}^{d-1} \left( \frac{n + d_i - 1}{d_i - 1} \right) I(M_i/M_{i+1}) + | (-1)^i e_i(q; M) - \text{adeg}_{d-i}(q; M) | \left( \frac{n + d - 1}{d - 1} \right)$$

$$+ \sum_{i=2}^{d-1} \left( (-1)^i e_i(q; M) - \text{adeg}_{d-i}(q; M) \right) \left( \frac{n + d - 1}{d - i} \right)$$

$$\leq \sum_{i=0}^{d-1} (n + 1)^{d-1} I(M_i/M_{i+1}) + (n + 1)^{d-1} I(\mathcal{F}, M)$$

$$+ \sum_{i=2}^{d-1} 2^{d-i} ((C + 1)^{d-1} I(\mathcal{F}, M) + d + C + 2)^{i-1} I(\mathcal{F}, M)(n + 1)^{d-i},$$

where the last inequality is followed by $i)$, $ii)$. Choose $n = (C + 1)^{d-1} I(\mathcal{F}, M) + d + C + 1$ and note that $n \geq r + \binom{r+d-1}{d-1} I(\mathcal{F}, M) + d$. Then we have

$$| e_d(q; M) | \leq 2 ((C + 1)^{d-1} I(\mathcal{F}, M) + d + C + 2)^{d-1} I(\mathcal{F}, M)$$

$$+ \sum_{i=2}^{d-1} 2^{d-1} ((C + 1)^{d-1} I(\mathcal{F}, M) + d + C + 2)^{d-1} I(\mathcal{F}, M)$$

$$= 2^{d-1} ((C + 1)^{d-1} I(\mathcal{F}, M) + d + C + 2)^{d-1} I(\mathcal{F}, M).$$

as required. □

Now, we are in a position to prove the main theorem in this section.

Proof of Theorem 3.3. This is now immediately seen from Theorem 3.8 □

We close this section with the following example, which shows that the condition $\mathcal{F} \in \mathcal{F}(M)$ in Theorem 3.3 and 3.8 cannot be omitted whenever $M$ is sequentially Cohen-Macaulay.

Example 3.9. Let $R = k[[X, Y, Z]]$ be the formal power series ring over a field $k$. Let $M = k[[X, Y, Z]] \oplus (k[[X, Y, Z]]/(Z^2))$ be $R$-module. For an integer $m \geq 1$, set $q_m = (X^m, Y^m, Z)$. Then we have the following.

i) $M$ is sequentially Cohen-Macaulay of dimension 2.

ii) $q$ is a parameter ideal of $M$ but $q$ is not a distinguished parameter ideal with respect to $F \in \mathcal{F}(M)$.

iii) $H^{ad}_{q_m, M}(n) = -m^2 \binom{n+1}{1}$ for all $m, n > 1$. Hence the set $\Lambda_1(M)$ is finite but the set $\mathcal{P}_\mathcal{F}(M)$ and $\Lambda_2(M)$ is infinite.

4. Characterization of the finiteness of the set $\mathcal{P}_\mathcal{D}(M)$

In this section, we give a characterization of sequentially generalized Cohen-Macaulay modules. In particular, we have

Theorem 4.1. Let $R$ be a homomorphic image of a Cohen-Macaulay local ring. Then the following statements are equivalent:

i) $M$ is sequentially generalized Cohen-Macaulay.

ii) The set $\mathcal{P}_\mathcal{F}(M)$ is finite for all $\mathcal{F} \in \mathcal{F}(M)$.

iii) The set $\mathcal{P}_\mathcal{F}(M)$ is finite for some $\mathcal{F} \in \mathcal{F}(M)$. 
iv) The set $\mathcal{P}_D(M)$ is finite.

**Setting 4.2.** In this section, from now on, we assume that $R$ is a homomorphic image of a Cohen-Macaulay local ring. Let $\mathcal{F} = \{M_i\}_{i=0}^t \in \mathcal{F}(M)$ and $\underline{a} = x_1, \ldots, x_d$ be a distinguished system of parameters of $M$.

**Notation 4.3.** \textbullet{} Set $\underline{q} := (\underline{x} := x_1, x_2, \ldots, x_d)$ and $q_i = (x_1, \ldots, x_i)$ for all $i = 1, \ldots, d$. Let $\underline{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$ and $\underline{d} = (d_{i,1}, \ldots, d_{i,d})$. We set

$$\mathcal{N}(\underline{a}; M) = \{n \in \mathbb{N}^d | \underline{d} = (d_{i,1}, \ldots, d_{i,d}) \text{ is a distinguished system of parameters of } M/n_i^M \text{ for all } 0 \leq i \leq d - 1\}.$$

\textbullet{} Let $d_{i+1} < k \leq d_i$ for some $i = 0, \ldots, t - 1$. Then $\mathcal{F}/q_k \mathcal{F} := \{(M_i + q_kM)/q_kM\}_{i=0}^t$ is a filtration of submodules of $M$ satisfying the dimension condition. The length $s$ of a filtration of submodules $\mathcal{F}/q_k \mathcal{F}$ of $M$ is $i$, if $k = d_i$ and $i + 1$, otherwise. Furthermore, $x_k+1, \ldots, x_d$ is a distinguished system of parameters of $M/q_kM$ with respect to $\mathcal{F}/q_k \mathcal{F}$. Stipulating $x_0 = 0$ and $\mathcal{F}_0 = \mathcal{F}$.

**Lemma 4.4.** Suppose that $d \geq 2$ and $\underline{a}$ is a distinguished system of parameters of $M$ such that $\mathcal{N}(\underline{a}; M) \neq \emptyset$. Then we have

i) $\underline{d}$ is a $d$-sequence on $M$ for all $\underline{n} \in \mathcal{N}(\underline{a}; M)$.

ii) $(n_1, n_2m_2, \ldots, n_dm_d) \in \mathcal{N}(\underline{a}; M)$ for all $(m_2, \ldots, m_d) \in \mathcal{N}(\underline{a}; M)/x_1^M$.

iii) If $M$ is a sequentially generalized Cohen-Macaulay module, then there is an $\underline{n} \in \mathcal{N}(\underline{a}; M)$ such that $\underline{d}$ is a distinguished $d$-sequence on $M$.

**Proof.** i). Let $\underline{n} = (n_1, \ldots, n_d) \in \mathcal{N}(\underline{a}; M)$. Then $\underline{d}_{i-1}^{n_i}$ is a distinguished system of parameters of $M/q_{i-1}^M$. Thus $(\underline{d}_{i-1}^{n_i})H_0^M(M/q_{i-1}^M) = 0$. Therefore $\underline{d}$ is a filter regular $M$-sequence. Moreover,

$$q_{i-1}^M : x_1^{n_i} = H_m^M(M/q_{i-1}^M) = \bigcup_{n=1}^\infty (q_{i-1}^M : (\underline{d}^n))$$

for all $i = 1, \ldots, d$. So $\underline{d}$ is a $d$-sequence by [21] Theorem 1.1(vii).

ii). This is immediately seen from the definition of the distinguished system of parameters.

iii). By the Artin-Rees Lemma, there are positive integers $m_1, \ldots, m_d$ such that $x_1^m_1, \ldots, x_d^m_d$ is a good system of parameters of $M$. We have by [3] Theorem 3.8, Corollary 3.9 that there are positive integers $k_1, \ldots, k_d$ such that $x_1^{k_1}, \ldots, x_d^{k_d}$ is a $d$-sequence on $M$, where $n_i = m_ik_i$ for all $i = 1, \ldots, d$. We have $(n_1, \ldots, n_d) \in \mathcal{N}(\underline{a}; M)$ by [3] Corollary 3.7 and the definition of $d$-sequence. \hfill $\square$

**Lemma 4.5.** There is a distinguished system of parameters $\underline{a} = x_1, \ldots, x_d$ of $M$ such that $\mathcal{F}/q_i \mathcal{F} \subseteq \mathcal{F}(M/q_iM)$ for all $\mathcal{F} \in \mathcal{F}(M)$ and $i = 1, \ldots, d$. Moreover, $\mathcal{N}(\underline{a}; M) \neq \emptyset$.

**Proof.** The first assertion is now immediately seen from Lemma 2.7 in [3].

Now we will prove that $\mathcal{N}(\underline{a}; M) \neq \emptyset$ by induction on $d$. The case $d = 1$ is obvious. Assume that $d > 1$. Then $x_2, \ldots, x_d$ is a distinguished system of parameters of $M/x_1M$ with respect to $D/x_1D$. Since $D/x_1D \in \mathcal{F}(M/x_1M)$, there are positive integers $n_2, \ldots, n_d$ such that $x_2^{n_2}, \ldots, x_d^{n_d}$ is a distinguished system of parameters of $M/x_1M$. By the inductive hypothesis, we have $(m_2, \ldots, m_d) \in \mathcal{N}(x_2^{m_2}, \ldots, x_d^{m_d}; M/x_1M)$ for some $m_i$. Therefore we have $(1, n_2m_2, \ldots, n_dm_d) \in \mathcal{N}(\underline{a}; M)$, as requested. \hfill $\square$
Lemma 4.6. Assume that $d \geq 2$ and $\mathcal{F} = \{M_i\}_{i=0}^{t} \in \mathcal{F}(M)$. Then, $H^1_m(M/M_{i+1})$ are of finite length for $i = 0, 1, \ldots, t - 1$ and $d_i \geq 2$.

Proof. We first see that $\text{Ass}_R(M_i/M_{i+1}) = \text{Ass}_R(M_i/M_{i+1}) \cup \{m\}$. It follows by [12, Lemm 3.1] that $H^1_m(M_i/M_{i+1})$ are of finite length for all $i = 0, \ldots, t - 1$ and $d_i \geq 2$. Now we will prove by induction on $i$. If $i = 0$, clearly that $H^1_m(M/M_1)$ is of finite length. Assume that $i > 0$ and that assertion holds true for $i - 1$. Now the short exact sequence

$$0 \rightarrow M_i/M_{i+1} \rightarrow M/M_{i+1} \rightarrow M/M_i \rightarrow 0$$

induces the long exact sequence

$$\ldots \rightarrow H^1_m(M_i/M_{i+1}) \rightarrow H^1_m(M/M_{i+1}) \rightarrow H^1_m(M/M_i) \rightarrow \ldots.$$  

If $d_i \geq 2$ then $d_{i-1} > d_i \geq 2$. Thus $H^1_m(M/M_i)$ is of finite length by induction. So $H^1_m(M/M_{i+1})$ is of finite length. $\square$

Theorem 4.7. Assume that $d \geq 2$. Suppose that there is an integer $C$ and a distinguished parameter ideal $q = (x_1, \ldots, x_d)$ of $M$ as in Lemma 4.6 such that

$$|a_i(q^\infty M)| \leq C$$

for all $n \in \mathcal{N}(q; M)$ and $i = 0, \ldots, d - 1$. Then we have

$$m^C H^1_m(M/D_{k+1}) = 0$$

for all $j = 1, \ldots, d_k - 1$, $d_k \geq 2$ and $k = 0, \ldots, t - 1$.

Proof. Let $n \in \mathcal{N}(q; M)$. We have by Lemma 2.7(ii) and Lemma 2.3(i) that

$$e_i(\hat{\mathfrak{m}}^n M) = e_i(\hat{\mathfrak{m}}^n M/W),$$

$$\text{adeg}_{d-i}(\hat{\mathfrak{m}}^n M) = \text{adeg}_{d-i}(\hat{\mathfrak{m}}^n M/W)$$

for all $i = 0, \ldots, d - 1$, where $W = H^1_m(M)$. Thus we can assume that $W = 0$.

Now we argue by the induction on the dimension $d$ of $M$. In the case $d = 2$, $M/D_1$ is a generalized Cohen-Macaulay module by Lemma 4.6. Therefore $M$ is sequentially generalized Cohen-Macaulay. By Lemma 4.4(iii), there is a $(n_1, n_2) \in \mathcal{N}(q; M)$ such that $x_1^{n_1}, x_2^{n_2}$ is a $d$-sequence on $M$. Therefore, by Lemma 3.2, we have

$$\ell(H^1_m(M/D_1)) = |(-1)e_1(x_1^{n_1}, x_2^{n_2}; M) - \text{adeg}_1(x_1^{n_1}, x_2^{n_2}; M)| \leq C.$$  

Hence $m^C H^1_m(M/D_1) = 0$.

Fact, suppose that $d > 2$ and our assertion holds true for $d - 1$. Since $\mathfrak{m}' = (m_1, n_2m_2, \ldots, n_dm_d) \in \mathcal{N}(q; M)$ for all $\mathfrak{m} = (m_2, \ldots, m_d) \in \mathcal{N}(q^\infty M/x_1^{n_1} M)$ by Lemma 4.4(ii), $x_2^{n_1} = x_1^{n_1}, x_2^{n_2m_2}, \ldots, x_d^{n_dm_d}$ is a $d$-sequence on $M$ because of Lemma 4.4(i). Set $y = x_1^{n_1}$ and $\mathfrak{m} = x_2^{n_2}, \ldots, x_d^{n_d}$. Therefore $y$ is a
superficial of $M$ for ideals $(y, y^n)$ for all $m \in \mathcal{N}(y; M/yM)$. It follows by Lemma 2.7(i) and Lemma 2.6 that
\[
\left| (-1)^i e_i(y^n, M/yM) - \text{adeg}_{d-i}(y^n, M/yM) \right| = \left| (-1)^i e_i(y, y^n, M) - \text{adeg}_{d-i}(y, y^n, M) \right| \leq C,
\]
for all $i = 0, \ldots, d - 2$ and for all $m \in \mathcal{N}(y; M/yM)$.

Now set $\overline{M} = M/yM$ and and $\overline{N} = (N + yM)/yM$ for all submodules $N$ of $M$. Let $\mathcal{D} = \{ \mathcal{D}_i \}_{i=0}^3$ be the dimension filtration of $\overline{M}$, where $s$ is $t$ if $d_{t-1} > 1$ and $t - 1$, otherwise. By the inductive hypothesis, we have
\[
m^C H^j_m(\overline{M}/\mathcal{D}_{i+1}^t) = 0
\]
for all $d_i \geq 3$ and $j = 1, \ldots, d_i - 2$. Since $\mathcal{D}/\mathcal{D} \in \mathcal{F}(M/yM)$ by Lemma 2.6, we have $\ell(\mathcal{D}_{i+1}/\mathcal{D}_i) < \infty$ for all $i = 0, \ldots, s - 1$. Therefore we get $H^j_m(\overline{M}/\mathcal{D}_{i+1}^t) \cong H^j_m(\overline{M}/\mathcal{D}_{i+1}^s)$ for all $j \geq 1$ and $i = 0, \ldots, s - 1$. This implies that
\[
m^C H^j_m(M/(yM + \mathcal{D}_{i+1}^t)) = 0
\]
for all $d_i \geq 3$ and $j = 1, \ldots, d_i - 2$. Since $y$ is $M/\mathcal{D}_{i+1}$-regular for all $i = 0, \ldots, t - 1$, it follows from short exact sequences
\[
0 \rightarrow \frac{M}{\mathcal{D}_{i+1}} \rightarrow \frac{M}{\mathcal{D}_{i+1}^t} \rightarrow \frac{M}{\mathcal{D}_{i+1}^t + y^n M} \rightarrow 0
\]
that long sequences
\[
\ldots \rightarrow H^j_m(\frac{M}{\mathcal{D}_{i+1}^t + y^n M}) \rightarrow H^j_m(\frac{M}{\mathcal{D}_{i+1}^s}) \rightarrow H^j_m(\frac{M}{\mathcal{D}_{i+1}^t}) \rightarrow \ldots
\]
are exact for all $n \geq 1$. Thus we have
\[
m^C(0 : H^j_m(M/\mathcal{D}_{i+1}^t) y^n) = 0
\]
for all $j = 2, \ldots, d_i - 1$, $d_i \geq 3$ and $n \geq 1$. Note that $n$ and $C$ are independent of each other. Hence
\[
m^C H^j_m(M/\mathcal{D}_{i+1}^t) = 0
\]
for all $j = 2, \ldots, d_i - 1$ and $d_i \geq 3$. Moreover, $\ell(H^j_m(M/\mathcal{D}_{i+1}^t)) < \infty$ by Lemma 4.6 for all $d_i \geq 2$. So $M$ is a sequentially generalized Cohen-Macaulay module by [3, Proposition 3.5]. Therefore there is a $n = (n_1, \ldots, n_{d}) \in \mathcal{N}(x; M)$ such that $x^n$ is a dd-sequence on $M$. By Lemma 3.2, we have $\ell(H^j_m(M/\mathcal{D}_{i+1}^t)) \leq C$ for all $d_i \geq 2$. The proof is completed.

Proof of Theorem 4.1. (i) $\Rightarrow$ (ii) is followed by Theorem 3.3.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (iv) is followed by $\mathcal{P}_D(M) \subseteq \mathcal{P}_F(M)$ for all $F \in \mathcal{F}(M)$.

(iv) $\Rightarrow$ (i). Let $x = x_1, \ldots, x_d$ be a distinguished system of parameters of $M$ as in Lemma 4.5. Since $\mathcal{P}_D(M)$ is finite, so is $\{ P^d_D M(n) \mid n \in \mathcal{N}(x; M) \}$. We have by Theorem 4.7 and [3, Proposition 3.5] that $M$ is sequentially generalized Cohen-Macaulay, as required. $\Box$
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Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet Road, 10307 Hanoi, Vietnam

Email address: ntculong@math.ac.vn

National Economics University, 207 Giai Phong Road, Hanoi, Vietnam

Email address: ntlong01@gmail.com

Institute of Mathematics, VAST, 18 Hoang Quoc Viet Road, 10307 Hanoi, Vietnam

Thang Long Institute of Mathematics and Applied Sciences, Hanoi, Vietnam

Email address: hltruong@math.ac.vn, truonghoangle@gmail.com