On lower bounds of the solutions of some simple reaction-diffusion equations

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Abstract

The mean-field reaction-diffusion equations of the diffusive pair-annihilation and triplett-annihilation processes are considered. A direct lower bound on the time-dependent mean particle-density is derived. The results are applied to the mean-field theory of the diffusive pair-contact process.

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1 Introduction

The interplay of reaction kinetics with particle diffusion has since a long time been a topic of intensive research. Here we shall consider the following reaction-diffusion equations

\[
\partial_t a(t, r) = \Delta a(t, r) - \lambda a(t, r)^2
\]

and

\[
\partial_t b(t, r) = \Delta b(t, r) - \mu b(t, r)^3
\]

where \( t \in \mathbb{R}_+ \) and \( r \in \mathbb{R}^n \) are time and space coordinates, \( \Delta \) is the Laplacian and \( \lambda, \mu > 0 \) are constant reaction rates. It is well-known that eq. (1.1) provides a mean-field description of the pair-annihilation process \( A + A \to \emptyset \) together with single-particle diffusion and similarly, (1.2) describes triplet annihilation \( A + A + A \to \emptyset \) (we have rescaled the diffusion-constant to one). Still, the derivation of the long-time behaviour of solutions of such non-linear partial differential equations is not completely trivial. It is convenient to consider eq. (1.1) inside a spatial domain \( \Omega \subset \mathbb{R}^n \) and to define the mean densities

\[
\bar{\pi} = \bar{\pi}(t) = \frac{1}{|\Omega|} \int_{\Omega} \, \mathrm{d}r \, a(t, r) \quad ; \quad \bar{\nu} = \bar{\nu}(t) = \frac{1}{|\Omega|} \int_{\Omega} \, \mathrm{d}r \, b(t, r)
\]

where \(|\Omega|\) denotes the volume of \( \Omega \).

The most simplistic treatment of (1.1) simply suppresses the diffusion term, which leads to \( \partial_t \bar{\pi}_s = -\lambda \bar{\pi}_s^2 \) and the solution \( \bar{\pi}_s(t) = \frac{a_0}{1 + a_0 \lambda t} \). Then, it may be asked to what extent this drastic simplification may be justified. This, and more generally the long-time behaviour of the space-time-dependent particle density \( a(t, r) \), has received a lot of attention. For example, \textit{a priori} estimates such as the strong maximum principle may be invoked to obtain bounds \( v_- (t) \leq a(t, r) \leq v_+ (t) \) where \( v_{\pm}(t) \) satisfy the simplistic equation \( \partial_t v_{\pm} = -\lambda v_{\pm}^2 \) with the initial conditions \( v_-(0) \leq a(0, r) \leq v_+(0) \) \textit{[12] p. 94}. The latter conditions might however be difficult to meet for very ‘rough’ initial data \( a(0, r) \). For domains with a finite (and ‘small’) volume \(|\Omega|\), one may define an invariant region \( \Sigma \) and define \( \sigma := \eta - M \), where \( \eta \) is the principal eigenvalue of \( -\Delta \) on \( \Omega \) and \( M = \max_{\Sigma} |\nabla \bar{\pi}_s| \). If \( \sigma > 0 \), then it can be shown that \( a(t, r) \) converges exponentially fast with a characteristic time \( 1/\sigma \) towards the solution \( \bar{\pi}_s(t) \) \textit{[12] p. 223}. However, the implied exponential approach need not hold any longer in spatially infinite regions. Then methods based on a scaling \textit{ansatz} of the form \( a(t, r) = t^{-\alpha/2} f(r t^{-1/2}) \) permit to extract the long-time asymptotics of the solution from phenomenological scaling \textit{[3]} or by using rigorous renormalization-group arguments \textit{[3]}. This kind of argument can also be extended to systems of reaction-diffusion equations and in particular the reaction fronts in the two-component system \( kA + kB \to \emptyset \) with initial conditions such that there is a reaction front between \( \text{A-rich} \) and \( \text{B-rich} \) regions. In \( n = 1 \) dimension, it was shown rigorously for \( k = 1 \) \textit{[10]} and \( k \geq 4 \) \textit{[2]} that the particle densities and the reaction fronts satisfy a multiscaling behaviour and that the convergence towards the scaling solutions is controlled by algebraically (and not exponentially) small corrections in \( t \). On the diffusive scale, when \( ||r||/\sqrt{t} \gg 1 \), the problem essentially reduces to \( \partial_t a = \Delta a - \alpha^k \) \textit{[2]}. Reaction-diffusion systems of the form \( \partial_t a = \Delta a - \nu a - f(a) + g \) with \( a|_{t=0} = a_0 \), \( a|_{\partial \Omega} = 0 \), \( \nu > 0 \) and suitable assumptions on \( f \) are reviewed in \textit{[13]}. The long-time behaviour of the unique solution (roughly, on the Sobolev space \( W^{2,q}(\Omega) \) where \( g \in L^q(\Omega) \) with \( q > \max(2, n/2) \)) is described in terms of an attractor whose complexity can be analysed through its Kolmogorov entropy in great detail.

Here we wish to present a simple direct estimate on the mean densities \( \bar{\pi}(t) \) and \( \bar{\nu}(t) \). We have (see section 2 for notations)
Theorem: (i) Let \( a \in C^1(\mathbb{R}_+; W^{2,2}(\Omega)) \) be an (almost) everywhere non-negative solution of \( (1.1) \). Let in addition be \( \nabla a = 0 \) on the boundary \( \partial \Omega \). Then there is a positive and \( |\Omega| \)-independent constant \( \lambda' \) such that the mean density \( \bar{a} \) satisfies
\[
-\lambda' \bar{a}(t)^2 \leq \partial_t \bar{a}(t) \leq -\lambda \bar{a}(t)^2 \quad \text{(1.4)}
\]
(ii) If \( b \in C^1(\mathbb{R}_+; W^{2,3}(\Omega)) \) is an (almost) everywhere non-negative solution of \( (1.2) \) such that \( \nabla b|_{\partial \Omega} = 0 \), there is a positive and \( |\Omega| \)-independent constant \( \mu' \) such that
\[
-\mu' \bar{b}(t)^3 \leq \partial_t \bar{b}(t) \leq -\mu \bar{b}(t)^3 \quad \text{(1.5)}
\]
For \( n \leq 3 \), the condition \( b \in C^1(\mathbb{R}_+; W^{2,2}(\Omega) \cap L^3(\Omega)) \) is sufficient.

The Sobolev space \( W^{2,p}(\Omega) \) may be too restrictive for unbounded domains. In section 2 we define a generalized space \( \widetilde{W}^{2,p}(\Omega) \) for which the theorem still holds and which includes spatially homogeneous initial states.

The upper bound in \( (1.4) \) has been known since a long time \([1]\). In particular, \( (1.4,1.5) \) imply for \( t \) sufficiently large (i.e. \( a_0 \lambda' t > 1 \) and \( 2b_0^2 \mu' t > 1 \), respectively, where \( a_0 > 0 \) and \( b_0 > 0 \) are the initial mean densities) the bounds
\[
\frac{1}{2\lambda'} \leq t \bar{a}(t) \leq \frac{1}{\lambda}; \quad \frac{1}{2\sqrt{\mu'}} \leq t^{1/2} \bar{b}(t) \leq \frac{1}{\sqrt{2\mu}} \quad \text{(1.6)}
\]
It is admissible to take the infinite-volume limit \( |\Omega| \to \infty \). Eq. \( (1.6) \) is in full agreement with the results established earlier by different means \([2, 3, 5, 7, 12]\). For similar upper bounds on \( ||a||_p, ||b||_p \) (with \( 1 \leq p \leq \infty \)) which explicitly depend on the initial data, see \([7]\). We point out that our derivation makes no explicit reference to the initial conditions (beyond the requirement \( a(t, r) \geq 0, b(t, r) \geq 0 \)) and neither a scaling ansatz is needed. The bounds \( (1.6) \) reproduce the expected mean-field scaling \( \bar{a}(t) \sim t^{-1} \). For the mathematically oriented reader we recall that in low dimensions \( n < 2 \), the description of the diffusive pair-annihilation processes through a more microscopic approach such as a master equation (where fluctuations are taken into account) leads to a different long-time behaviour \( \bar{a}_{\text{micro}}(t) \sim t^{-n/2} \), which has also been observed experimentally for \( n = 1 \), see \([8]\) for a recent review. This manifests once again the character of equations such as \( (1.1) \) as mean-field approximations.

The approach of \( a(t, r) \) towards the mean density can be described as follows.

Corollary: Under the same conditions as in the theorem, there is a constant \( K' \leq 1 \) such that for times satisfying the conditions used in eq. \( (1.6) \)
\[
\frac{1}{|\Omega|} \int_{\Omega} d\mathbf{r} \ (a(t, r) - \bar{a}(t))^2 \leq \frac{K'}{\lambda^2} \cdot t^{-2}
\]
\[
\frac{1}{|\Omega|} \int_{\Omega} d\mathbf{r} \ (b(t, r) - \bar{b}(t))^2 \leq \frac{K'}{2\mu} \cdot t^{-1}
\]

As an application, we consider the pair-contact process \( 2A \to \emptyset, 2A \to 3A \) with single-particle diffusion (PCPD). We consider a domain \( \Omega \subset \mathbb{R}^n \) with the boundary condition \( \nabla a|_{\partial \Omega} = 0 \). The mean-field reaction-diffusion equation is
\[
\partial_t a(t, r) = D \Delta a(t, r) + \lambda a(t, r)^2 - \mu a(t, r)^3
\]

(1.8)
with constants $\lambda \in \mathbb{R}$ and $\mu > 0$. If the diffusion constant $D = 0$, $a = a(t)$ evolves for $\lambda > 0$ towards a steady-state density $a_\infty = \lambda / \mu$, while $a_\infty = 0$ for $\lambda \leq 0$. It is known that

$$a(t) - a_\infty \sim \begin{cases} O \left( e^{-t/\tau} \right) & ; \text{if } \lambda > 0 \\ t^{-1/2} & ; \text{if } \lambda = 0 \\ t^{-1} & ; \text{if } \lambda < 0 \end{cases}$$

(1.9)
as $t \to \infty$ and where $\tau > 0$ is a known constant. On the other hand, for a non-vanishing diffusion constant, we scale to $D = 1$ and have for the mean density

$$\lambda \overline{a}(t)^2 - \mu' \overline{a}(t)^3 \leq \partial_t \overline{a}(t) \leq \lambda' \overline{a}(t)^2 - \mu \overline{a}(t)^3$$

(1.10)

for $\lambda \geq 0$ and

$$- |\lambda| \overline{a}(t)^2 - \mu' \overline{a}(t)^3 \leq \partial_t \overline{a}(t) \leq -|\lambda| \overline{a}(t)^2 - \mu \overline{a}(t)^3$$

(1.11)

for $\lambda \leq 0$, respectively and we can now let $|\Omega| \to \infty$, if we so desire. For $\lambda > 0$, there is an active steady-state with density $\lambda / \mu' \leq a_\infty \leq \lambda' / \mu$ but if $\lambda \leq 0$, one has $a_\infty = 0$. Furthermore, eq. (1.9) can be taken over. This proves the existence of a continuous steady-state transition at $\lambda_c = 0$ of the mean-field equation (1.8). Finally, the approach of $a(t, r)$ towards the mean density is according to

$$\frac{1}{|\Omega|} \int_\Omega \text{d}r \ (a(t, r) - \overline{a}(t))^2 \lesssim \begin{cases} O \left( e^{-2t/\tau'} \right) & ; \text{if } \lambda > 0 \\ t^{-1} & ; \text{if } \lambda = 0 \\ t^{-2} & ; \text{if } \lambda < 0 \end{cases}$$

(1.12)

In $n = 1$ dimension, fluctuation effects create a very rich behaviour of the PCPD which is under active investigation, see [9] for a review.

In section 2, we recall some inequalities which are needed in the proofs. The upper bounds in (1.4,1.5) are derived in section 3 and in section 4, the lower bounds are obtained.

## 2 Mathematical background

We recall here some standard notations and some inequalities which will be needed in establishing the lower bound in (1.4,1.5). We shall work with the $p$-norms, for $1 \leq p < \infty$

$$||u||_p := \left( \int_\Omega |u(r)|^p \right)^{1/p}$$

(2.1)

Denote by $L^p(\Omega)$ the space of (equivalence classes of) functions with $||u||_p$ finite. Here and in the following $\Omega \subset \mathbb{R}^n$. Furthermore, if $\Omega$ has a boundary, it is assumed to be sufficiently smooth throughout. The space $L^\infty(\Omega)$ is defined with respect to the supremum norm $||u||_\infty = \text{ess sup}_{\Omega} |u(r)|$. For derivatives, we use the multiindex notation $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and where $|\alpha| := \alpha_1 + \ldots + \alpha_n$. Then derivatives are denoted by

$$\partial^\alpha u = \frac{\partial^{(|\alpha|)} u}{(\partial r_1)^{\alpha_1} \ldots (\partial r_n)^{\alpha_n}}$$

(2.2)

These derivatives can be taken to be weak (distributional) derivatives. The Sobolev space is

$$W^{k,p}(\Omega) := \{ u \in L^p(\Omega), \partial^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq k \}$$

(2.3)

with its norm $||u||_{k,p} = \sum_{|\alpha| \leq k} ||\partial^\alpha u||_p$. Finally, $C^1(\mathbb{R}_+; W^{k,p}(\Omega))$ is the space of functions which are continuously differentiable with respect to $t$ for all times $0 \leq t < \infty$ and whose values at any given $t$ are in $W^{k,p}(\Omega)$. 

After these preparations, we can state some known results which we need later. The first one is the Gagliardo-Nirenberg inequality, see \[6, 11\].

**Lemma 1.** For functions \( u \in W^{k,p}(\Omega) \cap L^q(\Omega) \) with \( 1 \leq q \leq \infty \) and for any integer \( 0 \leq j < k \), there is a constant \( C > 0 \) such that
\[
||\partial^j u||_r \leq C ||u||_q^{1-\theta} ||\partial^k u||_p^\theta
\]
where
\[
\frac{1}{r} - \frac{j}{n} = \frac{1}{q} - \theta \left( \frac{1}{p} - \frac{k}{n} \right)
\]
and if \( 1 \leq p < n/(k-j) \), then \( j/k \leq \theta \leq 1 \). On the other hand, if \( 1 \leq p = n/(k-j) \), only \( j/k \leq \theta < 1 \) is allowed.

We shall need two special cases of this. For \( n \geq 2 \), we set \( p = r = 2 \), \( k = q = 1 \). Then \( j = 0 \) and \( \theta = n/(2+n) \in [\frac{1}{2}, 1] \). We get (here and in the following, we suppress the integration variable and write \( \int_\Omega = \int_\Omega dr \))
\[
\int_\Omega |u|^2 \leq C_1 \left( \int_\Omega |u|^{4/(2+n)} \right)^{4/(2+n)} \left( \int_\Omega |\partial u|^2 \right)^{n/(2+n)} \tag{2.6}
\]
Eq. (2.6) does not hold if \( n < 2 \). For \( 1 \leq n \leq 2 \), we set \( r = 2 \), \( p = q = k = 1 \). Then \( j = 0 \), \( \theta = n/2 \in [\frac{1}{2}, 1] \) and
\[
\int_\Omega |u|^2 \leq C_2 \left( \int_\Omega |u|^{2-n} \right)^{2-n} \left( \int_\Omega |\partial u|^n \right) \tag{2.7}
\]
where the constants \( C_{1,2} \) equal \( C^2 \) from Lemma 1. It can be checked from dimensional analysis that \( C_1 \) and \( C_2 \) are independent of \( |\Omega| \).

Next, we quote an inequality due to Nirenberg, see [6]. Let \( B_R(0) \) be the ball of radius \( R \) around the origin.

**Lemma 2.** For \( u \in W^2(\Omega) \), \( p \geq 1 \), there exists a constant \( \varepsilon_0 = \varepsilon_0(p, \Omega) \) such that for any \( \varepsilon \) with \( 0 < \varepsilon < \varepsilon_0 \), there is a positive constant \( c = c(p, \Omega) \) such that
\[
\int_\Omega d r |\partial u|^p \leq \frac{c}{\varepsilon} \int_\Omega d r |u|^p + \varepsilon \sum_{|\alpha| = 2} \int_\Omega d r |\partial^\alpha u|^p \tag{2.8}
\]
Finally, we quote Poincaré’s inequality, see [11].

**Lemma 3.** For any \( u \in W^{1,p}(B_R(0)) \) with \( 1 < p < \infty \), there is a positive constant \( C_p^{(p)} \) such that
\[
\int_{B_R(0)} d r |u - \bar{u}|^p \leq C_p^{(p)} |B_R(0)|^{p/n} \int_{B_R(0)} d r |\partial u|^p \tag{2.9}
\]
and the mean value \( \bar{u} \) is defined in analogy with (1.9).

It is sometimes desirable to consider spaces which are less restrictive than the spaces \( L^p(\Omega) \). We define \( \tilde{L}^p(\Omega) \) as the space of (equivalence classes of) functions such that \( m_p(u) := |\Omega|^{-1} ||u||_p^p \) is finite. For unbounded domains (e.g. \( \Omega = \mathbb{R}^n \)) a limit procedure must be used in the definition of \( m_p(u) \). We also set
\[
\tilde{W}^{k,p}(\Omega) := \left\{ u \in \tilde{L}^p(\Omega), \partial^\alpha u \in \tilde{L}^p(\Omega) \; \text{for all} \; |\alpha| \leq k \right\} \tag{2.10}
\]
As an example, consider the function \( f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x) = f_0 \neq 0 \). While \( f \in \tilde{L}^p(\mathbb{R}) \), since \( m_p(f) = ||f_0||^p \), clearly \( f \not\in L^p(\mathbb{R}) \). Lemmas 2 and 3 readily extend to the space \( \tilde{W}^{k,p}(\Omega) \).
3 The upper bound

We briefly recall the proof of the upper bound in (1.4), following [1]. The mean density satisfies

\[ \partial_t \bar{a} = \frac{1}{|\Omega|} \int_\Omega \Delta a - \frac{\lambda}{|\Omega|} \int_\Omega a^2 \]

\[ = \frac{1}{|\Omega|} \int_{\partial\Omega} \mathbf{d}\sigma \cdot \nabla a - \frac{\lambda}{|\Omega|} \int_\Omega a^2 \]  

(3.1)

where \( \sigma \) is a normal vector to the boundary \( \partial\Omega \). The first term describes the flux of particles through the boundary and vanishes either in the limit of large volumes \( |\Omega| \to \infty \) or else if the boundary condition \( \nabla a|_{\partial\Omega} = 0 \) is imposed. Then \( \partial_t \bar{a} = -\lambda \bar{a}^2 \leq -\lambda \bar{a}^2 \) by the Cauchy-Schwarz inequality.

A similar argument works for triplett annihilation. By Hölder’s inequality, \( \partial_t \bar{b} = -\mu \bar{b}^3 \leq -\mu \bar{b}^3 \).

4 The lower bound

In order to obtain the lower bound in (1.4), we recall from section 3 that \( \partial_t \bar{a} = -\lambda \bar{a}^2 \). The right-hand side is now estimated through the Gagliardo-Nirenberg inequality. We have to distinguish the cases \( n \geq 2 \) and \( n \leq 2 \) and obtain from eqs. (2.6) and (2.7)

\[ \partial_t \bar{a} \geq \begin{cases} 
-\lambda C_1 |\Omega|^{-1} \left( \int_\Omega a \right)^{4/(2+n)} \left( \int_\Omega |\nabla a|^2 \right)^{n/(2+n)} ; & \text{if } n \geq 2 \\
-\lambda C_2 |\Omega|^{-1} \left( \int_\Omega a \right)^{2-n} \left( \int_\Omega |\nabla a|^2 \right)^{n/2} ; & \text{if } n \leq 2
\end{cases} \]

(4.1)

where for \( n \leq 2 \) the Cauchy-Schwarz inequality was used again. Next, we need an upper estimate for \( \int_\Omega |\nabla a|^2 \), which is provided by the following

**Proposition:** For \( u \in W^{2,p}(\Omega) \) there is a constant \( c > 0 \) and an \( \varepsilon_* \) such that \( 0 < \varepsilon_* < \infty \) and that for all \( \varepsilon < \varepsilon_* \) one has

\[ \int_\Omega |\nabla u|^p \leq \frac{2c}{\varepsilon} \int_\Omega |u|^p \]  

(4.2)

**Proof:** If \( \sum_{|\alpha|=2} \int_\Omega |\partial^\alpha u|^p = 0 \), the proposition holds true trivially, because of (2.8). We can thus suppose that \( \sum_{|\alpha|=2} \int_\Omega |\partial^\alpha u|^p > 0 \). Next, the function \( f(x) := A/x + Bx \), where \( A, B \) are positive constants, has an absolute minimum at \( x_* = \sqrt{A/B} \). For \( x < x_* \), the first term dominates over the second and \( f(x) < 2A/x \) for all \( x < x_* \). We apply this to the inequality (2.8) of Lemma 2. The right-hand side is minimal if \( \varepsilon = \varepsilon_* \), where

\[ \varepsilon_* = \min \left( \left( \frac{c \int_\Omega |u|^p}{\sum_{|\alpha|=2} \int_\Omega |\partial^\alpha u|^p} \right)^{1/2} , \varepsilon_0 \right) \]

with the \( \varepsilon_0 \) of Lemma 2. Then the assertion follows. q.e.d.

The extension to \( \tilde{W}^{2,p}(\Omega) \) is immediate.

Therefore, setting \( p = 2 \) and appealing to dimensional analysis, there is a positive constant \( K > 0 \) such that
\[
\int_{\Omega} |\nabla a|^2 \leq K|\Omega|^{-2/n} \int_{\Omega} a^2
\]
\[
= K|\Omega|^{-2/n} \int_{\Omega} [(a - \overline{a})^2 + 2\overline{a} (a - \overline{a}) + \overline{a}^2]
\]
\[
= K|\Omega|^{1-2/n} \overline{a}^2 + K|\Omega|^{-2/n} \int_{\Omega} (a - \overline{a})^2
\]
\[
\leq 2K|\Omega|^{1-2/n} \overline{a}^2
\]
(4.3)

From the eqs. (4.1) it follows

\[
\partial_t \overline{a} \geq \begin{cases} 
-\lambda C_1 (2K)^{n/(2+n)} \overline{a}^2 & \text{if } n \geq 2 \\
-\lambda C_2 (2K)^{n/2} \overline{a}^2 & \text{if } n \leq 2 
\end{cases}
\]
(4.4)

This is exactly the form asserted in the theorem and we can identify the effective reaction rate

\[
\lambda' := \begin{cases} 
\lambda C_1 (2K)^{n/(2+n)} & \text{if } n \geq 2 \\
\lambda C_2 (2K)^{n/2} & \text{if } n \leq 2 
\end{cases}
\]
(4.5)

Remark: the last bound in (4.3) might be improved by restricting \( \Omega \) to a ball around the origin and applying the Poincaré inequality. If \( C_p^2 K < 1/2 \), one gets \( \int_{\Omega} |\nabla a|^2 < K/(1 - KC_p^2)|\Omega|^{-1/2} \overline{a}^2 \). For \( n = 1 \), the bounds might be further sharpened with the help of the inequality

\[
\int_{\Omega} |\partial u|^p \leq c \varepsilon^{-\mu(p)} \left( \int_{\Omega} |u|^p \right)^p + \varepsilon \int_{\Omega} |\partial^2 u|^p
\]
(4.6)

where \( \mu(p) = -(p - 3 + 1/p)/p \) and which can be proven for \( p \geq 1 \) through a slight generalization of the proof of the inequality (2.8) of Lemma 2. Since \( \mu(p) > 0 \) if \( 1 \leq p < p_c = (3 + \sqrt{5})/2 \), the method of the proposition can be used for example in the \( p = 1 \) and \( p = 2 \) cases.

The lower bound in (4.3) is proved similarly. First, we set \( k = q = 1 \) and \( r = 3 \) in Lemma 1. Then \( j = 0 \). For \( 1 \leq n \leq 3/2 \), we set \( p = 1 \), find \( \theta = 2n/3 \in [2/3, 1] \) and

\[
\int_{\Omega} |u|^3 \leq \Gamma_1 \left( \int_{\Omega} |u| \right)^{3-2n} \left( \int_{\Omega} |\partial u| \right)^{2n}
\]
(4.7)

Next, for \( 3/2 \leq n \leq 3 \), we set \( p = 3/2 \), find \( \theta = 2n/(n+3) \in [2/3, 1] \) and

\[
\int_{\Omega} |u|^3 \leq \Gamma_2 \left( \int_{\Omega} |u| \right)^{3(3-n)/(n+3)} \left( \int_{\Omega} |\partial u|^{3/2} \right)^{4n/(n+3)}
\]
(4.8)

Finally, for \( n \geq 3 \), we set \( p = 3 \), find \( \theta = 2n/(2n+3) \in [2/3, 1] \) and

\[
\int_{\Omega} |u|^3 \leq \Gamma_3 \left( \int_{\Omega} |u| \right)^{9/(2n+3)} \left( \int_{\Omega} |\partial u|^3 \right)^{2n/(2n+3)}
\]
(4.9)

Dimensional analysis shows that \( \Gamma_{1,2,3} \) (which stand for \( C^3 \) of Lemma 1) are independent of \( |\Omega| \). For the further analysis of (4.9), the proposition above states that there is a positive constant \( K_3 \) such that
\[
\int_\Omega |\nabla b|^3 \leq K_3 |\Omega|^{-3/n} \int_\Omega b^3 \\
= K_3 |\Omega|^{-3/n} \int_\Omega \left[ b^3 + 3\overline{\beta}(b - \overline{\beta})^2 + (b - \overline{\beta})^3 \right] \\
\leq K_3 |\Omega|^{-3/n} \left[ |\Omega|\overline{\beta}^3 + 3\overline{\beta} \int_\Omega |b - \overline{\beta}|^2 + \int_\Omega |b - \overline{\beta}|^3 \right] \\
\leq 5K_3 |\Omega|^{1-3/n}\overline{\beta}^3
\]  
(4.10)

Now, from \( \partial_t \overline{\beta} = -\mu \overline{\beta}^3 \), we obtain, using first eqns. (4.8,4.9), then the Cauchy-Schwarz inequality and the following consequence of Hölder’s inequality \( \int_\Omega |\nabla b|^3/2 \leq |\Omega|^{1/4} (\int_\Omega |\nabla b|^2)^{3/4} \) and finally eqns. (4.8,4.10)

\[
\partial_t \overline{\beta} \geq \begin{cases} 
-\mu \Gamma_1 |\Omega|^{-1} (\int_\Omega b)^{3-2n} (\int_\Omega |\nabla b|^2)^{2n} & \text{if } 1 \leq n \leq 3/2 \\
-\mu \Gamma_2 |\Omega|^{-1} (\int_\Omega b)^{3(3-n)/(3+n)} (\int_\Omega |\nabla b|^3/2)^{4n/(3+n)} & \text{if } 3/2 \leq n \leq 3 \\
-\mu \Gamma_3 |\Omega|^{-1} (\int_\Omega b)^{9/(2n+3)} (\int_\Omega |\nabla b|^3)^{2n/(2n+3)} & \text{if } 3 \leq n 
\end{cases}
\]

\[
\geq \begin{cases} 
-\mu \Gamma_1 (2K)^n \overline{\beta}^3 & \text{if } 1 \leq n \leq 3/2 \\
-\mu \Gamma_2 (2K)^{3n/(n+3)} \overline{\beta}^3 & \text{if } 3/2 \leq n \leq 3 \\
-\mu \Gamma_3 (5K_3)^{2n/(2n+3)} \overline{\beta}^3 & \text{if } 3 \leq n 
\end{cases}
\]  
(4.11)

which is the form asserted in (1.5). We identify the effective reaction rate

\[
\mu' := \begin{cases} 
\mu \Gamma_1 (2K)^n & \text{if } 1 \leq n \leq 3/2 \\
\mu \Gamma_2 (2K)^{3n/(n+3)} & \text{if } 3/2 \leq n \leq 3 \\
\mu \Gamma_3 (5K_3)^{2n/(2n+3)} & \text{if } 3 \leq n 
\end{cases}
\]  
(4.12)

This completes the proof of the theorem.

The bounds (1.6) are established by direct integration, since \( \partial_t \overline{\beta}(t) \geq -\lambda \overline{\beta}(t)^2 \) implies \( \overline{\beta}(t) \geq a_0/(1 + a_0 \lambda t) \) from which the assertion is immediate, under the stated condition on \( t \) and for \( a_0 > 0 \). The bounds for \( \overline{\beta}(t) \) follow similarly.

We now prove the corollary. First, \( |\Omega|^{-1} \int_\Omega (a - \overline{\beta})^2 \leq |\Omega|^{-1} \int_\Omega \overline{\beta}^2 \leq \overline{\beta}^2 \leq (\lambda t)^{-2} \). Next, we try to improve this bound by letting \( \Omega = B_R(0) \) be a ball around the origin and apply Poincaré’s inequality

\[
\frac{1}{|\Omega|} \int_\Omega (a - \overline{\beta})^2 \leq C_{\text{P}}^{(2)} |\Omega|^{-1+2/n} \int_\Omega |\nabla a|^2 \leq 2KC_{\text{P}}^{(2)} \overline{\beta}^2
\]

and therefore \( K' = \min(1, 2KC_{\text{P}}^{(2)}) \). The estimates for \( b \) are obtained similarly.

\[\text{q.e.d.}\]

Remark: for the PCPD with \( D = 1 \), eqns. (1.10,1.11) are obtained by applying the results of the theorem separately to the two terms on the right-hand side of eqn. (1.8). Eq. (1.12) is obtained by replacing \( a \mapsto a - a_\infty \) in the corollary and then using (1.9). Let \( a_t(r) := a(t, r) \). For sufficiently long times, \( a_t \in \overline{W}^{2,3}(\mathbb{R}^n) \), but if \( \lambda > 0 \), \( a_t \notin W^{2,3}(\mathbb{R}^n) \).
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