The Quantum Spectrum of the Conserved Charges in Affine Toda Theories

Max R. Niedermaier

Max-Planck-Institut für Physik
-Werner Heisenberg Institut -
Föhringer Ring 6
80805 Munich (Fed. Rep. Germany)

Abstract

The exact eigenvalues of the infinite set of conserved charges on the multi-particle states in affine Toda theories are determined. This is done by constructing a free field realization of the Zamolodchikov-Faddeev algebra in which the conserved charges are realized as derivative operators. The resulting eigenvalues are renormalization group (RG) invariant, have the correct classical limit and pass checks in first order perturbation theory. For \( n = 1 \) one recovers the (RG invariant form of the) quantum masses of Destri and DeVega.
1. Introduction

The infinite set of local conserved charges in involution is a key structure in an integrable relativistic field theory. The simultaneous eigenstates are the multiparticle states of the theory and the knowledge of their spectrum is a major step towards the linearization of the dynamics. In the classical case the linearized dynamics can in principle be constructed by means of the inverse scattering transform. The transition to the quantum regime rests on the construction of the renormalized monodromy matrix satisfying a Yang Baxter equation. This however has only been achieved in rare cases due to technical, and perhaps fundamental obstructions to the applicability of the method. This motivates the search for alternative techniques.

The recent progress towards establishing such an alternative has two major sources. The first is the improved understanding of the form factor equations in terms of infinite dimensional symmetry algebras \[27, 28\]. The second is a group of ideas collectively referred to as ‘perturbed conformal field theories’ \[1, 26\]. A central result there is that the local conserved charges of an integrable massive QFT can already be identified and constructed at the level of the conformal field theory (CFT) describing its UV behaviour \[1, 11, 12\]. The next step should consist in addressing the diagonalization problem for these charges in the context of perturbed CFTs and, in particular, to find expressions for the eigenvalues. In the form factor approach knowledge of the eigenvalues of the conserved charges provides part of the additional dynamical input required to set up the correspondence between solutions of the form factor equations and local operators. It is the purpose of the present paper to deduce exact expressions for these eigenvalues on the multiparticle states in affine Toda theories.

Affine Toda theories are relativistic models of \( r \) interacting scalar fields which generalize the Sinh/Sine Gordon model. They are associated with the simple root system \( \alpha_0, \ldots, \alpha_r \) of some affine Lie algebra \( \hat{g} \). The Lagrangian in 2-dim Minkowski space is given by

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi - \frac{m^2}{\beta^2} \sum_{j=0}^{r} a_j \left( e^{\beta \alpha_j \cdot \phi} - 1 \right),
\]

where \( a_j \) are the labels of the Dynkin diagram (\[15\], p 54). The fields \( \phi^a(t, x), \ a = 1, \ldots, r \) are considered as the components of a real field with values in the Cartan subalgebra \( h \) of \( g \). The coupling constant \( \beta \) is real and \( m \) fixes the (bare) mass scale. At the classical level the theories (1.1) are known to be integrable and an infinite set of local conserved currents can be constructed by various techniques \[4, 5, 6\]. The application of the inverse
scattering method however meets certain obstructions. Nevertheless, the principle aims of the method can still be achieved by deriving a system of GLM-type equations whose solutions are given in terms of generalized tau functions\[7\]. This leads to a parametrization and construction of generic classical solutions in terms of ‘scattering data’ which linearize the dynamics. The classical spectral problem then consists in the derivation of ‘trace identities’. This means that one aims to find explicit expressions for the local conserved charges in terms of the scattering data i.e.

\[ I^{(n)}[\text{solution}] = I^{(n)}(\text{scattering data}) , \quad n \in E \subset \mathbb{N} . \]

In section 4 we will recall such a trace identity \[7\] to check the classical limit of our quantum result.

In the quantum theory, the natural analogue of a trace identity is the sequence of eigenvalues of the conserved charges on the asymptotic multi-particle states. In section 5 we will use a vertex operator construction to obtain an exact formula for these eigenvalues in the real coupling affine Toda theories. Since the scattering operator commutes with all the conserved charges \([S , I^{(n)}] = 0\), one expects that the bootstrap S-matrix (once known) also carries information about their spectrum. Indeed, the intertwining concept is just that of a Zamolodchikov-Faddeev (ZF) algebra. We call a ZF algebra \(Z(S)\) associated with \(S\) an associative algebra with generators \(Z_a(\theta)\), \(\theta \in \mathbb{C}\), \(I^{(n)}\), \(n \in E\) and \(K\) subject to the relations

\[
Z_a(\theta_a) Z_b(\theta_b) = S_{ab}(\theta_a - \theta_b) Z_b(\theta_b) Z_a(\theta_a) \]

\[
[I^{(n)} , Z_a(\theta)] = e^{-n\theta} I^{(n)}(a) Z_a(\theta) , \quad a, b = 1, \ldots, r , \quad n \in E \]

\[
[K , Z_a(\theta)] = \frac{d}{d\theta} Z_a(\theta) . \tag{1.2}
\]

\(K\) is the generator of Lorentz boosts and \(I^{(n)}(a)\) is interpreted as the eigenvalue of \(I^{(n)}\) on an asymptotic 1-particle state of type \(a\). Usually, a ZF algebra is supposed to act on the space of scattering states of the theory \(\Sigma_{in/out}\). These are Fock spaces but the relation to the fundamental fields of the theory is elusive in general. It is therefore more useful to construct realizations \(\rho : Z(S) \longrightarrow \pi\) of (1.2) on some auxiliary Fock space \(\pi\), on which the \(Z_a(\theta)\)’s act as generalized vertex operators. This, of course, can be done in many ways and one will choose the realization according to purpose. In section 5 we will show that there exists a realization \(\rho_I\) in which the conserved charges are realized just as derivative operators

\[
\rho_I(I^{(n)}) = \frac{\partial}{\partial x_n} , \quad n \in E . \tag{1.3}
\]
Moreover this realization is shown to be essentially uniquely determined by (1.3). Once
the realization has been constructed, the exact eigenvalues of the conserved charges can
be obtained from the second of the relations (1.2). Since this is a novel technique, we
have included a number of independent checks on the result. In particular, it turns out that

- the eigenvalues obtained are renormalization group (RG) invariant.
- they have the correct classical limit and pass checks in 1st order perturbation theory.
- for \( n = 1 \) one recovers the (RG invariant form of the) exact quantum masses found
  by Destri and DeVega \[2\].

The paper is organized as follows. In section 3 we give a quantum field theoretical defini-
tion of the eigenvalues via a LSZ reduction formula, relating them to the Greens functions
of the conserved densities. From this one can show that the eigenvalues are RG invariant
and that their functional forms (as a function of the coupling constant \( \beta \)) are governed
by universal, scheme independent functions \( K_n(\beta) \). In section 4 we will determine the
classical limit of \( K_n(\beta) \) and calculate some first order quantum corrections. Finally, in
section 5 the exact expression for \( K_n(\beta) \) is obtained as indicated. We conclude with some
remarks on the diagonalization problem of the conserved charges on the Verma modules,
which should be governed by ‘generalized Kac determinants’.

2. Real coupling affine Toda theories

Here we prepare some basic results on real coupling affine Toda theories (AT). Our kinem-
tical conventions are as follows: \((t, x) = (x^0, x^1)\) are coordinates on 2-dimensional
Minkowski space \( \mathbb{R}^{1,1} \) with metric \( \eta = \text{diag}(1, -1) \). Lightcone coordinates are intro-
duced by \( x^\pm = (x^0 \pm x^1)/\sqrt{2} \). Elements \( \Lambda \) of the restricted Lorentz group \( SO(1,1) \) are
parametrized by the rapidity \( \theta \in \mathbb{R} \) via \( \Lambda(\theta) = e^{\theta K} \), where \( K \) is the generator of the
Lie algebra. The \( 2^{n_+ + n_-} \) components of a Lorentz tensor of type \((n_+, n_-)\) transform
as \((\Lambda t)_{+\ldots,+\ldots,-\ldots,-\ldots} = e^{n_+ \theta} e^{-n_- \theta} t_{+\ldots,+\ldots,-\ldots,-\ldots} \) in lightcone coordinates. Indices are raised and
lowered according to \( t^{\ldots+} = t_{\ldots-} \) and vice versa. If \( P_\pm \) are the lightcone momenta, the
Poincaré algebra takes the form \([K, P_\pm] = \pm P_\pm \). On test functions one has the usual
realization in terms of differential operators \( iK = x^+ \partial_+ - x^- \partial_- \), \( iP_\pm = \partial_\pm \).
We will consider the case of simply laced Lie algebras $g$, and mainly the $A$-series. It is convenient to use a complex basis $T_1, \ldots, T_r$ for the Cartan subalgebra $h$, satisfying $(T_a, T_b) = \delta_{a,b}$, $T^\dagger_a = T_{\bar{a}}$. Here $(\cdot, \cdot)$ is the non-degenerate bilinear form on $h$ and $\bar{a}$ is the charge conjugate of $a$. The expansion coefficients $\Phi_a = (T^\dagger_a, \Phi)$ are then complex fields $\Phi^* \bar{a} = \Phi_{\bar{a}}$. Let $\alpha_1, \ldots, \alpha_r$ be the simple roots of $g$ and $\alpha_0 = -\theta$ the negative highest root. The inner products $(\alpha_j, \Phi)$ appearing in the interaction Lagrangian take the form

$$(\alpha_j, \Phi) = \sum_a (\gamma_j)^a \Phi_a =: \gamma_j \cdot \Phi,$$

where $(\gamma_j)^a$ can be interpreted as the components of complex simple roots. They are related to the Frobenius-Perron eigenvector $q_a^{(1)}$ of the Cartan matrix (c.f. appendix A). The Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} \sum_a \partial_\mu \Phi_a \partial^\mu \Phi_{\bar{a}} - \sum_{N \geq 0} \frac{1}{N!} V_N[\Phi]. \quad (2.1)$$

The vertices are

$$V_N = \sum_{a_1, \ldots, a_N} \lambda_{a_1 \ldots a_N} : \Phi_{a_1} \ldots \Phi_{a_N} :,$$

where $: :$ denotes some normal ordering and

$$\lambda_{a_1 \ldots a_N} = m^2 \beta^{2(N-2)} \sum_{a_1, \ldots, a_N} a_j (\gamma)_{a_1} \ldots (\gamma)_{a_N}, \quad (2.2)$$

if $a_j$ are the Dynkin labels ([13], p. 54). In particular, $V_0 = h$ (Coxeter number), $V_1 = 0$ and from

$$V_2 = 2hm^2 \sum_a q_a^{(1)} q_{\bar{a}}^{(1)} \Phi_a \Phi_{\bar{a}},$$

one reads off the classical masses $m_a = m \sqrt{2h q_a^{(1)}}$. For $g = A_r$ one finds explicitly,

$$T_a = \frac{1}{2 \sqrt{r + 1}} \sin \frac{a \pi}{r+1} \sum_{j=1}^r (\omega^{aj} - 1) \alpha_j,$$

$$T^\dagger_a = T_{\bar{a}}, \quad \bar{a} = r + 1 - a, \quad \text{Tr}(T_a T_b) = \delta_{a,\bar{b}},$$

$$(\gamma_j)^a = \frac{2}{\sqrt{r + 1}} \sin \frac{a \pi}{r+1} \omega^{aj}, \quad m_a = 2m \sin \frac{a \pi}{r+1}. \quad (2.3)$$
2.1. Tadpole function and renormalization group

Next we prepare the set-up for the renormalized perturbation theory. The free propagator is

\[
\Delta_{ab}(x|m_a) = \delta_{a,\bar{b}} \langle 0 | T \Phi_a(x) \Phi_b(0) | 0 \rangle = \delta_{a,\bar{b}} \int \frac{d^2 p}{(2\pi)^2} \frac{e^{ip \cdot x}}{p^2 - m_a^2 + i\epsilon},
\]

(2.4)

For spacelike distances \( x^2 < 0 \) the evaluation results in a Bessel function, from which one can read off the asymptotics

\[
\Delta_{ab}(x|m_a) = \delta_{a,\bar{b}} \frac{1}{2\pi} K_0(m_a | x|)
\]

\[
= -\delta_{a,\bar{b}} \frac{1}{2\pi} \left( \ln \frac{1}{2} m_a | x| + \gamma \right) + o(|x|), \quad |x| \to 0,
\]

(2.5)

where \( |x| = \sqrt{-x \cdot x} \) and \( \gamma \) is the Euler constant. Since the models have a mass gap, UV regularization is sufficient and is done by means of a momentum cutoff \( \Lambda \). For example, one can use \( \Delta_{ab}(x|m_a, \Lambda) = \Delta_{ab}(x|m_a) - \Delta_{ab}(x|\Lambda) \) or \( \Delta_{ab}(x|m_a, \Lambda) = \delta_{ab} \int_{-\Lambda}^{\Lambda} \frac{dk}{4\pi} \frac{e^{-ikx}}{\sqrt{k^2 + m_a^2}} \) as the regularized free propagator.

Renormalization is done by normal ordering w.r.t. a normal ordering mass \( M \). Let \( F \) be the vector space of local field operators and define \( : M : \) to be a linear map satisfying

\[
\exp (\lambda \gamma \cdot \Phi) : M = \exp \left( \lambda \gamma \cdot \Phi - \frac{1}{2} \lambda^2 \langle (\gamma \cdot \Phi)^2 \rangle_M \right),
\]

(2.6)

where \( \lambda \in \mathbb{C} \) and \( \langle (\gamma \cdot \Phi)^2 \rangle_M = \gamma_a \gamma_b \Delta_{ab}(0|\Lambda, \Lambda) \) is the contraction function. It is easy to see that (2.6) specifies \( : M \) completely. In particular, monomials of the form \( \Phi_{a_1} \ldots \Phi_{a_N} : M \) get mapped onto generalized Hermite polynomials in \( \Phi_{a_1}, \ldots, \Phi_{a_N} \). Let \( \mathcal{O} \) be a local operator and let \( \langle \cdot \rangle_m \) be the expectation value evaluated by means of Wicks theorem and the free regularized propagator \( \Delta_{ab}(x|m_a, \Lambda) \). (We make use of the fact that for simply laced ATs all – classical and quantum corrected – masses are proportional to a single mass scale \( m \), which is used to label \( \langle \cdot \rangle_m \).) The construction then is designed s.t. all Greens functions

\[
G_{a_1 \ldots a_n}^{\mathcal{O}}(y_1, \ldots, y_n; x) := \langle T \Phi_{a_1}(y_1) \ldots \Phi_{a_n}(y_n) : \mathcal{O}(x) : M \rangle_m
\]

are finite as \( \Lambda \to \infty \). Even after removal of the cutoff the theory contains two mass scales \( m \) and \( M \). One can also consider \( m \) and the dimensionless ratio \( M/m \) as parameters, and put the latter equal to any function \( f(\beta) \) of the coupling constant. We will refer
to a choice of the pair \((m, M/m = f(\beta))\) as a choice of the renormalization scheme. Interaction Lagrangians which satisfy

\[
m_1^2 \sum_{j=0}^{r} : e^{\beta \gamma_j \cdot \Phi} :_{M_1} = m_2^2 \sum_{j=0}^{r} : e^{\beta \gamma_j \cdot \Phi} :_{M_2}
\]

define equivalent renormalized perturbation theories. From (2.6) one checks that this holds iff

\[
m_1(M_1)^{\beta^2/4\pi} = m_2(M_2)^{\beta^2/4\pi}.
\]  

(2.7)

The relation (2.7) defines the normal ordering renormalization group (RG). All physical quantities should be RG invariant, i.e. should be annihilated by the differential operator

\[
M \frac{\partial}{\partial M} - \frac{\beta^2}{4\pi} \frac{\partial}{\partial m}.
\]

A major result about the renormalized perturbation theory is that the dependence on \(M/m\) in all Greens functions enters only through a universal function \(T(M/m, \beta)\), which does not depend on the Greens function considered. Its exponential is defined through the relation

\[
e^T = \frac{\beta^2}{m^2 \hbar} \langle : V[\Phi](x) :M \rangle_m,
\]

where \(V[\Phi]\) is the interaction Lagrangian. The ‘tadpole function’ \(T\) itself can be interpreted (and calculated) as the sum of all connected vacuum diagrams. To lowest order

\[
T \left( \frac{M}{m}, \beta \right) = \frac{\beta^2}{m^2 \hbar} \sum_{a=1}^{r} \frac{m_a^2}{4\pi} \ln \left( \frac{M}{m_a} \right) + o(\beta^4).
\]  

(2.8)

From the previous remarks it follows then that different renormalization schemes will affect Greens functions only through different choices for the function

\[
T_f(\beta) := T \left( \frac{M}{m} = f(\beta), \beta \right).
\]

For example, one can choose a scheme for which the lowest order contribution (2.8) (the ‘fundamental tadpole’) vanishes: Define

\[
\ln \xi := \frac{1}{2\hbar} \sum_{a} \left( \frac{m_a}{m} \right)^2 \ln \frac{m_a}{m}.
\]

From \(\sum_a m_a^2 = 2\hbar m^2\) one checks

\[
T \left( \frac{M}{m}, \beta \right) = \frac{\beta^2}{2\pi} \ln \frac{M}{m \xi} + o(\beta^4),
\]
so that the choice $M/m = \xi$ kills the fundamental tadpole. Another preferred scheme
is to choose $M/m = \hat{f}(\beta)$ by the implicit function theorem s.t. $T(\hat{f}(\beta), \beta)$ vanishes
identically (for a certain range in $\beta$). Of course, in practice the function $T(M/m, \beta)$ –
and hence the defining equation for $\hat{f}$ – will not be known to all orders. Theoretically,
however, the scheme $(m, \hat{f})$ is distinguished in that the Greens functions $\hat{G}$ in this scheme
are obtained by deleting all tadpole diagrams from the set of diagrams defining the same
Greens function $G$ in a generic scheme. Moreover, once $\hat{G}$ is known, the dependence on
the tadpoles – and hence on $M/m$ – can be restored through the formula [2]

$$G = \exp \left( T \frac{m^2}{\partial} \right) \hat{G}.$$  

(2.9)

The fact that this ‘tadpole dressing’ can be described in closed form is a consequence of
the exponential nature of the interaction. For the same reason also the $M/m$ – dependence
of $T(M/m, \beta)$ can be found explicitly[2]

$$T \left( \frac{M}{m}, \beta \right) = \frac{\beta^2/2\pi}{1 + \beta^2/4\pi} \left[ \ln \frac{M}{\xi m} + \frac{2\pi}{\beta^2} \hat{T}(\beta) \right] ,$$  

(2.10)

where $\hat{T}(\beta)$ is a function of the coupling constant alone. An important consequence of
(2.10) is that the combination $m^2 e^T$ is RG invariant i.e.

$$m_1^2 e^{T(M_1/m_1, \beta)} = m_2^2 e^{T(M_2/m_2, \beta)} ,$$  

(2.11)

if $(m_1, M_1)$ and $(m_2, M_2)$ are related by (2.7). In fact, the combination $m^2 e^T$ essentially
defines the physical mass scale and hence better should be RG invariant.

In preparation of section 4 let us show in detail that the physical masses and the
wave function renormalizations are RG invariant. Beyond the tree level the fields $\Phi_a$ may
develop a non-zero vacuum expectation value $\lambda_a$. (For $g = A_r$ the $\lambda_a$’s vanish to all orders
due to $Z_{r+1}$-invariance). Thus define the wave function renormalization by

$$\langle 0 | \Phi_a(x) - \lambda_a | b(p) \rangle_m = \sqrt{Z_a} \delta_{a,b} e^{-ip \cdot x} ,$$
$$\langle 0 | \Phi_a(x) - \lambda_a | 0 \rangle_m = 0 .$$

The shift $\Phi_a \rightarrow \Phi_a - \lambda_a$, when inserted into the Lagrangian (1.1) affects only the 1-point
functions, and is irrelevant for the stability of the classical mass ratios under quantum
corrections. This stability also allows one to choose the matrix of wave function renor-
malizations diagonal. The exact propagator becomes

$$G_{ab}(p^2) = \frac{(Z_a Z_b)^{-1/2}}{(p^2 - m_2^2)\delta_{ab} - \Sigma_{ab}(p^2)} ,$$  

(2.12)

7
where $\Sigma_{ab}(p^2)$ is the $\Phi_a\Phi_b$ self energy i.e. $-i\Sigma_{ab}(p^2)$ is the sum of all 1 PI graphs with external legs $a, b$. The physical mass is defined as the position of the pole in (2.12). On dimensional grounds $G_{ab}(p^2)$ is of the form

$$G_{ab}(p^2) = \frac{1}{m^2} g_{ab} \left( \frac{p^2}{m^2}, \frac{M^2}{m^2}, \beta \right).$$

On the other hand the tadpole-free part $\hat{G}_{ab}(p^2)$ does not depend on $M$, so that

$$\hat{G}_{ab}(p^2) = \frac{1}{m^2} \hat{g}_{ab} \left( \frac{p^2}{m^2}, \beta \right).$$

From the tadpole dressing formula (2.9) one finds the relation

$$G_{ab}(p^2) = e^{-T} \hat{G}_{ab}(e^{-T} p^2). \quad (2.13)$$

Let $\hat{m}_a^2$ denote the position of the pole in $\hat{G}_{aa}(p^2)$. The stability of the mass ratios under quantum corrections implies

$$\hat{m}_a = K(\beta) m \sqrt{2h q_a^{(1)}}, \quad (2.14)$$

for some function $K(\beta)$, which on dimensional grounds is a function of the coupling constant alone. The pole in $G_{ab}(p^2)$ is at $p^2 = (m_a^2)_{phys}$,

$$(m_a^2)_{phys} = \hat{m}_a^2 e^T, \quad (2.15)$$

which by (2.11) is indeed RG invariant. To proceed, solve (2.12) for the self energy i.e.

$$\Sigma_{ab}(p^2) = [p^2 - m_a^2] \delta_{\bar{a} \bar{b}} - (Z_a Z_b)^{-1/2} [G(p^2)^{-1}]_{ab}. \quad (2.16)$$

From (2.9) one obtains the tadpole dressing relation for $\Sigma_{ab}$,

$$\Sigma_{ab}(p^2) = m_a^2 \delta_{\bar{a} \bar{b}} (e^T - 1) + e^T \hat{\Sigma}_{ab}(p^2 e^{-T}).$$

Together,

$$Z_a^{-1} = 1 - \frac{\partial \Sigma_{aa}(p^2)}{\partial p^2} \bigg|_{p^2 = (m_a^2)_{phys}} = 1 - e^T \frac{\partial}{\partial p^2} \hat{\Sigma}_{aa}(p^2 e^{-T}) \bigg|_{p^2 = (m_a^2)_{phys}}. \quad (2.16)$$
From the definition of the tadpole-free parts, and on dimensional grounds the quantity 

\[ e^{T} \frac{\partial}{\partial p^2} \hat{\Sigma}_{ab}(p^2 e^{-T}) \]

is of the form \( S_a \left( \frac{e^{-T}p^2}{m} \right) \) for some functions \( S_a : \mathbb{R}^2 \to \mathbb{C} \). Thus, when evaluated at \( p^2 = (n_a^2)_{\text{phys}} \), the \((M, m)\)-dependence drops out, leaving a scheme-independent function of \( \beta \) alone. Guided by similar results in the Sine-Gordon model \[24\], the exact expressions for \( Z_a(\beta) \) were found in \[2\].

### 2.2. Bootstrap S-matrix and minimal form factor

As a consequence of the presence of higher order conservation laws the S-matrix of an integrable theory factorizes into a product of two particle ones and the set of in- and outgoing particle momenta is preserved separately with only the assignment to particles permuted. The S-matrix element \( S_{ab}(\theta) \) for the scattering of particles \( a \) and \( b \) is meromorphic in \( \theta = \theta_a - \theta_b \) with period \( 2\pi i \). The physical sheet corresponds to \( 0 \leq \text{Im} \theta < \pi \). On very general grounds \( S_{ab}(\theta) \) is subject to a number of functional equations, collectively referred to as ‘bootstrap equations’\[17\]. For a theory without genuine particle degeneracies (i.e without multiplets transforming under a continuous symmetry) these take the simple form

\[
S_{ab}(\theta) = S_{ba}(\theta) = S_{ab}(-\theta)^{-1} = S_{ab}^*(-\theta^*)
\]  
\[
S_{ab}(i\pi - \theta) = S_{ab}(\theta)
\]  
\[
S_{dc}(\theta + i\eta(a)) S_{db}(\theta + i\eta(b)) S_{dc}(\theta + i\eta(c)) = 1.
\]

The first equation expresses hermitian analyticity and (formal) unitarity. The second equation implements crossing invariance and the third one is the bootstrap equation proper. The charge conjugation operation \( a \to \bar{a} \) is defined in appendix B. \( \eta(l) \), \( l = a, b, c \) are the imaginary rapidities of the particles \( a, b, c \). They are related to the conventional fusing angles \( U_{bc}^a = \pi - \bar{U}_{bc}^a \) etc. by

\[
\eta(a) = -\frac{1}{3}(\bar{U}_{ac}^b - \bar{U}_{ba}^c)
\]
\[
\eta(b) = -\frac{1}{3}((\bar{U}_{ba}^c - \bar{U}_{cb}^a - 2\pi)
\]
\[
\eta(c) = -\frac{1}{3}((\bar{U}_{cb}^a - \bar{U}_{ac}^b + 2\pi),
\]

where \( \bar{U}_{bc}^a + \bar{U}_{ac}^b + \bar{U}_{ab}^c = \pi \). Using \((2.17.a,b)\) and suitably redefining \( \theta \) yields the bootstrap equation in the usual form \( S_{dc}(\theta) = S_{da}(\theta - i\bar{U}_{ac}^b) S_{db}(\theta + i\bar{U}_{bc}^a) \).
For every simple Lie algebra $g$ there exists a minimal parameter free solution to these equations\cite{13}. Using the notation of appendix A it is written as \cite{22}

\[ S_{ab}^{\text{min}}(\theta) = \prod_{p=1}^{h} \left( 2p + \frac{c(a) - c(b)}{2} \right)^{\lambda_{a} \Omega^{-p \gamma_{b}}} \theta, \quad (2.19) \]

where

\[ (\mu)_{a} := \frac{\sinh(\frac{\theta}{2} + \frac{i\pi \mu}{2h})}{\sinh(\frac{\theta}{2} - \frac{i\pi \mu}{2h})}. \quad (2.20) \]

If $g$ is simply laced (2.19) in addition also has the required meromorphy. We henceforth take $g$ to be simply laced. The S-matrix for real coupling affine Toda theories is of the form $S_{ab}(\theta) = f_{ab}(\theta) S_{ab}^{\text{min}}(\theta)$, where $f_{ab}(\theta)$ carries the dependence on the coupling constant s.t. $S_{ab}(\theta)|_{\beta=0} = 1$. Further requirements on $f$ are that it should not introduce further poles into the physical strip for $\beta > 0$ (as the particle content of the theory is already encoded in $S_{ab}^{\text{min}}(\theta)$) and that the signs of the residues of poles of $S_{ab}^{\text{min}}(\theta)$ should be unchanged after multiplication by $f$. An expression consistent with these requirements as well as various checks in perturbation theory is

\[ S_{ab}(\theta) = \prod_{p=1}^{h} \left[ \frac{\left( 2p + \frac{c(a) - c(b)}{2} \right)_{\theta}}{\left( 2p + \frac{c(a) - c(b)}{2} + B \right)_{\theta}} \right]^{(\lambda_{a} \Omega^{-p \gamma_{b}})} \cdot (2.21) \]

The parameter $B$ is not fixed by the bootstrap equations but all results known are consistent with

\[ B = \frac{\beta^{2}/2\pi}{1 + \beta^{2}/4\pi} \quad (2.22) \]

Thus, $B$ plays the role of an ‘effective coupling’ and contains non-perturbative information. The fact that (2.19), (2.21) are indeed solutions to (2.17) will be seen below.

For a local operator $\mathcal{O}(t, x)$ the form factor on an asymptotic $N$-particle state is defined by

\[ F_{a_{n}...a_{1}}^{\mathcal{O}}(\theta_{i} - \theta_{j}) = \langle \Omega | \mathcal{O}(0) | a_{n}(\theta_{n}), \ldots, a_{1}(\theta_{1}) \rangle, \quad (2.23) \]

where $|\Omega\rangle$ is the physical vacuum. From the Wightman axioms one can argue that these form factors are subject to a number of functional equations which conversely are then taken to define the quantities (2.23) axiomatically. For a given bootstrap S-matrix these
equations have the form of a generalized Riemann Hilbert problem and the reconstruction of the correlation functions from them is known as the form factor bootstrap program\cite{24, 25}. Similar monodromy problems are known in the context of CFT and quantum groups, which has been the source of recent progress\cite{27, 28, 29}. For a diagonal S-matrix the defining equations for the 2-particle form factor are

\[ F_{ab}^O(\theta) = S_{ab}(\theta) F_{ba}^O(-\theta) \]
\[ F_{ab}^O(i\pi - \theta) = e^{2\pi i \omega(\Phi, O)} F_{ba}^O(i\pi + \theta), \] (2.24)

where \( \omega(\Phi, O) \) is the relative locality index of \( \Phi \) and \( O \) (defined via the monodromy of their operator product). Moreover \( F_{ab}^O(\theta) \) is required to be meromorphic on the physical strip \( 0 \leq \text{Im} \theta \leq \pi \) with possible poles and zeros only on the imaginary axis and \( F_{ab}^O(\theta) = o(e^{e|\theta|}) \) for \( |\text{Re} \theta| \to \infty \). The solution to (2.24) is then uniquely determined by the position of the poles and zeros up to a normalization constant. It can be written as

\[ F_{ab}^O(\theta) = K_{ab}^O(\theta) F_{ab}^{\text{min}}(\theta), \] (2.25)

where \( F_{ab}^{\text{min}}(\theta) \) is analytic in \( 0 \leq \text{Im} \theta \leq 2\pi \) and is normalized as \( F_{ab}^{\text{min}}(i\pi) = 1 \). The pole factor \( K_{ab}^O(\theta) \) carries the dependence on the local operator and has trivial monodromy \( K(\theta) = K(-\theta) = K(2\pi i + \theta) \).

Suppose that the S-matrix allows for an integral representation of the form

\[ S_{ab}(\theta) = e^{i\delta_{ab}(\theta)}, \quad \delta_{ab}(\theta) = \int_0^\infty \frac{dt}{t} h_{ab}(t) \sin \frac{h \theta}{\pi} t, \] (2.26)

where \( \delta_{ab}(\theta) \) is the scattering phase and \( h_{ab}(t) = h_{ba}(t) \) is real. The minimal form factor is then given by\cite{24}

\[ F_{ab}^{\text{min}}(\theta) = e^{f_{ab}(\theta)}, \quad f_{ab}(\theta) = \int_0^\infty \frac{dt}{t} h_{ab}(t) \frac{\sin^2(i\pi - \theta) h t}{\text{sh} h t} \] (2.27)

with real and imaginary parts

\[ \text{Re} f_{ab}(\theta) = \int_0^\infty \frac{dt}{t} \frac{h_{ab}(t)}{\text{sh} h t} \left( 1 - \text{ch} h t \cos \frac{h \theta}{\pi} t \right) \]
\[ \text{Im} f_{ab}(\theta) = \frac{1}{2} \delta_{ab}(\theta). \] (2.28)

For the S-matrices (2.19), (2.21) the relation

\[ (\mu)_a = -\exp \left\{ -2i \int_0^\infty \frac{dt}{t} \frac{\text{sh}(\mu - h)t}{\text{sh} h t} \sin \frac{h \theta}{\pi} t \right\}, \quad |\text{Im} \theta| < \frac{\pi \mu}{h} \] (2.29)
leads to integral representations of the form (2.26), (2.27). For (2.21) one finds
\[
h_{ab}(t) = \frac{2h \sin \frac{tB}{2} \sh \frac{t}{2}(2 - B)}{\sh(ht) \sh t} \left( q_a(t) q_b(t) + q_b(t) q_a(t) \right),
\]  
(2.30)
and a similar expression for (2.19). The functions \(q_a(t), a = 1, \ldots, r\) are \(2\pi i\)-periodic functions in \(t\) defined by
\[
q_a(n) = iq_a^{(n)}, \quad n \in E,
\]
where \(q_a^{(n)}\) are the eigenvectors (A.5) of the Cartan matrix. The integral representations (2.26), (2.27) are also convenient to derive power series expansions in \(e^{\pm n\theta}\) by complex contour deformation. For the scattering phase one finds
\[
\delta_{ab}(\theta) = \pm \delta_{ab}^{<\theta>(\theta)} = \sum_{n \in E} \frac{1}{n} D_n q_a^{(n)} q_b^{(n)} e^{\pm n\theta}, \quad \text{Re} \ \theta <> 0
\]  
(2.31)
The notations are \(\theta_n = \pi n/h\) and
\[
D_n = \frac{4h \sin \frac{\theta_n}{2} B \sin \frac{\theta_n}{2}(2 - B)}{\sin \theta_n} = \beta^2 n \left( 1 - \frac{\beta^2}{4\pi n} \cot \theta_n \right) + o(\beta^6),
\]  
(2.32)
where for later use we displayed the semi classical limit. As a check one can also obtain (2.32) directly from (2.21) using the formulae of appendix A. Similar expansions exist for the function elements of \(\text{Re} f_{ab}(\theta)\). Due to the second order pole in \(\sh ht\) the coefficients will also carry a linear \(\theta\) dependence. We shall not need there explicit form.

Remark: i. The corresponding expression (2.31) for the minimal S-matrix is obtained by replacing \(D_n\) by \(-2h \cot \theta_n\). In addition, there is also a zero mode contribution to \(\delta_{ab}^{<\theta>(\theta)}\) with value \(\pm \pi (1_{ab} - 2(a^{-1})_{ab})\), where 1 denotes the unit matrix and \(a^{-1}\) is the inverse of the Cartan matrix. This implies \(-\frac{1}{2\pi i} \left[ \ln S_{ab}^{\min}(\theta) \right]_{\theta=-\infty} = 2a^{-1} - 1\), which has been obtained previously(e.g. [22]). The absence of a zero mode in \(\delta_{ab}^{<\theta>(\theta)}\), of course, is required for \(S_{ab}(\theta)|_{\beta=0} = 1\).

ii. Given the additional information that the series expansion (2.31) indeed defines a meromorphic function with the correct pole structure[23], one can also easily verify that it provides a solution to the bootstrap equations (2.17). From \(q_{a}^{(n)} = (-)^{n+1} q_{a}^{(n)}\) one has by analytic continuation
\[
\delta_{ab}(i\pi - \theta) = \delta_{ab}(\theta) = -\delta_{ab}(-\theta),
\]  
(2.33)
which implements equations (2.17.a,b) for the S-matrix. In terms of the scattering phase the bootstrap eqn. is equivalent to
\[
\sum_{l=a,b,c} \delta_{dl}(\theta + i\eta(l)) = 0,
\]  
(2.34)
which follows from (A.13).
3. Definition of the quantum spectrum

We will consider ATs in $\partial_+\,$-lightcone dynamics. Such a lightcone quantization simplifies the structure of the conserved currents, but also introduces some complications. See [8, 7] for a detailed discussion. Being complicated composite operators, the construction of the quantum conserved currents is a difficult problem and except for some sample calculations [19] has not been successful in earlier attempts. The major result in lightcone dynamics is that the construction of the quantum conserved currents can be mapped onto a conformal field theory (CFT) problem. In the CFT context, the existence of infinitely many quantum conserved currents – and hence the integrability of the quantum ATs – can be proved [11, 12]. The construction can also be rephrased in algorithmic terms and, at least in principle allows one to explicitly construct the quantum conserved currents. To specify the structure of the quantum conserved currents it therefore suffices to describe the mapping onto (and back from) the CFT problem.

3.1. Construction of the quantum conserved currents

Recall that in lightcone dynamics the interaction lagrangian plays the role of the hamiltonian. For ATs thus

$$H[\Phi] = \int dx^- V[\Phi] = \frac{m^2}{\beta^2} \sum_{j=0}^{\infty} \int dx^- \left[ : e^{\beta \gamma_j \cdot \Phi} :_M - 1 \right]$$

(3.1)

is the normal ordered lightcone hamiltonian. By definition it is RG invariant. We wish to construct quantum versions of the local conserved charges $I_\pm^{(n)}$, $n \in E$ present in the classical theory. A quantum conserved charge is defined to be a functional $I[\Phi]$ commuting with the Hamiltonian (3.1). A local conserved charge $I[\Phi] = \int dx^- J[\Phi]$ is supposed to arise by integration from some density $J[\Phi]$, which is a normal ordered differential polynomial in $\partial_- \Phi$ and $e^{\beta \gamma_j \cdot \Phi}$. In terms of the densities $V[\Phi]$ and $J[\Phi]$ the condition reads

$$[V(x), J(y)] = \delta(x - y) \partial Q^{(1)}(y) + \sum_{k \geq 2} \delta^{(k)}(x - y) Q^{(k)}(y),$$

(3.2)

for some local operators $Q^{(k)}$, $k \geq 1$. Here the commutator is taken at equal ‘$x^+$-time’ and $x = x^-$, $y = y^-$, etc.. In the classical theory the equation (3.2) (with the commutator replaced by Poisson brackets) is known to possess two infinite sequences of solutions $J_\pm^{(n)}[\phi]$, one pair for each affine exponent $n \in E$ (including multiplicities). For
the purposes here it suffices to consider the densities $J^{(n)}[\partial_- \Phi] := J_{-}^{(n)}[\partial_- \Phi]$, which are differential polynomials in $\partial_- \Phi$. For such differential polynomials the commutator (3.2) in the quantum theory is equivalent to an operator product expansion of the form

$$V(x) J(y) = \frac{1}{x - y + i\epsilon} \partial Q^{(1)}(y) + \sum_{k \geq 2} \frac{\bar{Q}^{(k)}(y)}{(x - y + i\epsilon)^k}.$$  

From the distributional formula $(x + i\epsilon)^{-k} - (x - i\epsilon)^{-k} = \frac{2\pi i(-1)^k}{(k-1)!} \delta^{(k-1)}(x)$ one recovers the commutator (3.2) (with $Q^{(k)}$ related to $\bar{Q}^{(l)}$, $l \geq k$). Usually, the calculation of an operator product expansion in a massive QFT is not an attractive task. In the case at hand, however, we can take advantage of the fact that the operator $V[\Phi]$ is RG invariant. In particular, we can choose the scheme $(m, \hat{f})$ defined in section 2.1 to find solutions to (3.2). In this scheme

$$1 = e^T = \frac{\beta^2}{h m^2} \langle : V[\Phi] : M \rangle_m.$$ 

Further, the quantum masses coincide with the classical masses $\hat{m}_a = m_a$, and the exact propagator coincides with the free propagator. Thus, one can simply use the short distance behaviour of the free propagator (2.5) to calculate the singular part of the OPE. Moreover, since (for $J = J^{(n)}[\partial_- \Phi]$) all contractions in (3.2) involve at least one derivative field, even the dependence on the classical masses drops out. The singular part of the resulting OPE will be form-identical to the one obtained by using the contraction function

$$\Phi_a(x) \Phi_b(y) = -\delta_{ab} \frac{1}{2\pi} \ln(x + i\epsilon).$$

But this is (up to a normalization) just the contraction function that one would use for free massless fields in a Minkowski space CFT. Hence if one is able to find solutions to (3.2) in the CFT, the normal ordered operators $J[\partial_- \Phi]$ will also have an interpretation in the massive QFT: They define quantum conserved densities $J[\partial_- \Phi]$ of the ATs in the particular renormalization scheme $(m, \hat{f}(\beta))$. It remains to solve the problem in the CFT context. The result [11, 12] is that to each classical conserved density $(J^{(n)}[\partial_- \Phi])_{class}$, $n \in E$, there exists a unique normal ordered quantum operator $J^{(n)}[\partial_- \Phi]$ solving (3.2). In the CFT this can be shown for all affine Lie algebras. Since in non-simply laced AT’s the stability of the mass ratios under quantum corrections ceases to hold, the preferred scheme $(m, \hat{f}(\beta))$ does not exist in these cases. Therefore, for non-simply laced Lie algebras, the CFT solutions to (3.2) presumably have no direct significance in AT’s. In principle (returning to the simply laced cases) one can also use the defining relation (3.2) to explicitly construct the conserved densities $J^{(n)}[\partial_- \Phi]$ by fixing the coefficients in an in an appropriate Ansatz.
The coefficients turn out to be constrained by overdetermined linear systems (of rapidly increasing size in $n$). The existence theorem quoted guarantees that these overdetermined linear systems can always be solved, but doing it explicitly soon becomes inconvenient. A more economical way to construct the conserved charges explicitly is to exploit the relation to $W$-algebras because then part of the constraints are built in\cite{10, 11}. For later use we quote some examples for the $A$-series. We normalize the $J^{(n)}$’s according to the form in which they appear in the (quantum) Miura transformation. That is

$$J^{(n)}[\partial \Phi] = (-\beta)^{n+1}s_{n+1}[(h_j, \partial \Phi)] + \ldots , \quad 1 \leq n \leq r,$$

where $h_0, \ldots, h_r$ are the weights of the $r+1$-dimensional fundamental representation and $s_n$ is the totally symmetric polynomial of order $n$ in $r+1$ variables. The normalization of the higher quantum conserved densities is fixed with reference to their classical counterparts (c.f. section 4.1). Using the transformation

$$(h_j, \Phi) = \frac{i}{\sqrt{r+1}} \sum_{a=1}^{r} \omega^{-a/2} \omega^a \Phi_a,$$

one finds

$$\beta^{-2} J^{(1)} = \frac{1}{2} \sum_a \partial \Phi_a \partial \Phi_a,$$

$$\beta^{-3} J^{(2)} = -i \frac{r\sqrt{r+1}}{18} \sum_{a+b+c=0} \omega^{(a+b+c)/2} \partial \Phi_a \partial \Phi_b \partial \Phi_c$$

$$+ \frac{i}{2} \left( \frac{1}{\beta} + \frac{\beta}{4\pi} \right) \sum_a \cot \frac{\pi a}{r+1} \partial^2 \Phi_a \partial \Phi_a$$

Similarly $J^{(n)}$, $1 \leq n \leq r$ can be obtained from the Miura transformation and the higher conserved densities are then calculated by means of the procedure outlined. In general they will not be known explicitly. Remarkably, at least the quadratic part $J^{(n,2)}$ of the $J^{(n)}$’s can be given in closed form

$$\beta^{-(n+1)} J^{(n,2)} = \beta^{-n+1} \left( 1 + \frac{\beta^2}{4\pi} + o(\beta^4) \right) \sum_a c^{(n)}_a \partial^n \Phi_a \partial \Phi_a,$$

$$c^{(n)}_a = i^{n-1} \frac{\sin \frac{an\pi}{r+1}}{\left( 2 \sin \frac{\pi}{r+1} \right)^n}.$$
3.2. The current algebra

The results of the last subsection allow one to deduce some results on the structure of generalized current algebras in AT’s. In 2-dim. Minkowski space consider a current algebra in $\partial_+{-}$lightcone dynamics of the following form.

$$[P^{(i)}(f), P^{(j)}(g)] = \sum_{\{k: \Delta_{ijk} \geq 1\}} C^{ij}_{k} P^{(k)}(p^{\Delta_{i} \Delta_{j}}_{\Delta_{k}}(f, g)) + D^{ij} \omega^{\Delta_{i}}(f, g).$$

(3.8)

The commutator is taken at equal $x^+$-time and the range of the summation is defined in terms of $\Delta_{ijk} = \Delta_{i} + \Delta_{j} - \Delta_{k}$, where $\Delta_{i}$ is the Lorentz spin of the field $P^{(i)}$. Here $P^{(i)}(x) := P^{(i)}(0, x)$, $x = x^-$ denotes some normal ordered differential polynomial of the fundamental fields. The structure constants $C^{ij}_{k}$, $D^{ij}$ are dynamical parameters of the theory. The operators are smeared via

$$P^{(i)}(f) := \int dx P^{(i)}(x) f(x).$$

$f$, $g$ and $p^{\Delta_{i} \Delta_{j}}_{\Delta_{k}}(f, g)$ are smooth test functions, the latter being given by

$$p^{\Delta_{i} \Delta_{j}}_{\Delta_{k}}(f, g) = \frac{1}{(d-1)! (2\Delta_{k} + d-2)!} \sum_{r=0}^{d-1} (-)^r \binom{d-1}{r} c_r f^{(d-1-r)}(x) g^{(r)}.$$

(3.9)

where $d = \Delta_{ijk}$, $f^{(n)} = (-i\partial_{-})^n f$ and

$$c_r = \frac{(2\Delta_{j} - 2 - r)!(2\Delta_{i} - d - 1 + r)!}{(2\Delta_{j} - d - 1)! (2\Delta_{i} - d - 1)!}.$$

The cocycle is

$$\omega^{\Delta_{k}}(f, g) = \frac{1}{(2\Delta_{k} - 1)!} \int dx - f^{(2\Delta_{k} - 1)} g.$$

In particular, let $P_{-} = 2I_{-}^{(1)}$ and $K$ denote the generators of infinitesimal translations and Lorentz boosts, respectively. Then

$$[P_{-}, P^{(k)}(x)] = -i\partial_{-} P^{(k)}$$

$$[K, P^{(k)}(x)] = (-ix\partial + \Delta_{k}) P^{(k)}.$$

(3.10)

We claim that
(3.8) is the most general current algebra possible in a 2-dim. QFT in lightcone dynamics, which is compatible with Poincare' invariance. Only the ‘structure constants’ $C^i_k$, $D^{ij}$ are dynamical parameters of the theory.

The algebra in (3.8) is isomorphic to a vertex operator algebra or meromorphic CFT (in the sense of [14]) with the same structure constants. The isomorphism can be constructed explicitly.

For the proof we refer to [13, 8]. To illustrate the point of the construction consider a scalar QFT with non-derivative interaction. The $T_{--}$ component of the energy momentum tensor is then given by $\partial_-\Phi\cdot\partial_-\Phi$ and the operators $L_s = \frac{1}{2}\int dx^- (x^-)^s+1 \partial_-\Phi\cdot\partial_-\Phi$, $s = 0, \pm 1$ generate an $so(1,2)$ algebra. By adding suitable total derivative terms one can achieve that $P^{(k)}[\partial_-\Phi]$, and hence a suitable basis of normal ordered products thereof, transforms covariantly under this $so(1,2)$ algebra. This is due to the fact that in lightcone coordinates Lorentz boosts $x^+ \to e^{\theta}x^+$, $x^- \to e^{-\theta}x^-$ can be viewed as scale transformations $x^\pm \to \rho x^\pm$ with separate scale factors for each lightcone sector. For either one of the lightcone sectors (but not for both simultaneously) the action of the Poincare algebra on the equivalence classes of the field algebra modulo total derivatives therefore extends to an action of the (chiral part of the) finite conformal group $SO(1,2)$. One can then use CFT techniques to determine the $SO(1,2)$-covariant form of the commutator $[P^{(k)}(f), P^{(l)}(g)]$.

In particular, the mCFT definition of the structure constants $C^i_k$, $D^{ij}$ implies that, if one of the fields is the Virasoro generator, only the structure constant $C^{L_0}_{pk}$ is non-vanishing and equals 2 in standard normalizations. This guarantees the consistency of the general formula (3.8) with (3.10) and gives the correspondence $P_- \to L_{-1}$, $K \to L_0$ to mCFT generators. To check this, note that

$$p^{2j}_j(f,g) = \frac{1}{2}[(\Delta_j - 1) f^{(1)} g - g^{(1)} f] .$$

Choosing $f = 1, x$, respectively one recovers the equations (3.10).

Let us also briefly describe the mapping onto a mCFT in general. For a current of Lorentz spin $\Delta$ in (3.8), the image in the mCFT is given by

$$P(x) \rightarrow P(z) = \sum_{n \in \mathbb{Z}} P_n z^{-n-\Delta} ,$$

where on the r.h.s. $P(z)$ is a quasiprimary field of weight $\Delta$. The modes are defined by

$$P(x) =: \left( \frac{2}{1 + x^2} \right)^\Delta \sum_{n} P_n \left( \frac{1 + ix}{1 - ix} \right)^{-n} .$$

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In particular the integral $\int dx P(x)$ is mapped onto a mCFT quantity lying in the vacuum preserving subalgebra $P_n$, $|n| \leq \Delta - 1$ of the mCFT.

$$\int dx P(x) \rightarrow 4\pi 2^{-\Delta_i} \sum_{|n| \leq \Delta_i - 1} \left( \frac{2\Delta_i - 2}{|n + \Delta_i - 1|} \right) P_n .$$

(3.13)

By means of a further transformation one can then achieve that $\int P$ is mapped onto $P_{\pm(\Delta - 1)}$ or even onto a zero mode $\tilde{P}_0$ for some modified field $\tilde{P}(z)$. See [13] for more details.

These results hold for any normal ordered differential polynomial in the fundamental fields. In particular, the conserved densities $J^{(n)} = J^{(n)}[\partial_- \Phi]$ are of that type. We conclude that the conserved densities $J^{(n)}$ together with all their normal ordered products form a generalized current algebra of the form (3.8). The $J^{(n)}$’s alone will in general not form a closed algebra by themselves. The existence of the conserved charges is equivalent to the vanishing of a certain subset of the structure constants. From (3.8), (3.9) one checks

$$[I^{(n)}, I^{(m)}] = 0 \quad \text{iff} \quad C_{k}^{mn} = 0 \quad \forall k : \Delta_{mnk} = 1 ,$$

(3.14)

if $n, m$ refer to the densities $J^{(n)}$, $J^{(m)}$ and $k$ refers to any composite field $P^{(k)}$ appearing on the r.h.s. of (3.8). Since the structure constants in (3.8) coincide with that in the mCFT, this confirms again that the existence of the conserved charges is decided on the CFT level.

3.3. Definition of the quantum spectrum

Consider now the eigenvalues of the conserved charges $I^{(n)}_{\pm}$ on multiparticle states. Let $|a_N(\theta_N), \ldots, a_1(\theta_1)\rangle$ denote an asymptotic multiparticle state, where $\theta_i$ is the rapidity of a particle of type $a_i$. The action of the charges $I^{(n)}_{\pm}$ on these states is expected to be of the form

$$I^{(n)}_{-}|a_N(\theta_N), \ldots, a_1(\theta_1)\rangle = \sum_{k=1}^{N} I^{(n)}_{-}(a_k) e^{-(\Delta_{n}-1)\theta_k}|a_N(\theta_N), \ldots, a_1(\theta_1)\rangle$$

$$I^{(n)}_{+}|a_N(\theta_N), \ldots, a_1(\theta_1)\rangle = \sum_{k=1}^{N} I^{(n)}_{+}(a_k) e^{-(\Delta_{n}-3)\theta_k}|a_N(\theta_N), \ldots, a_1(\theta_1)\rangle$$

(3.15)

Such common eigenstates exist because of $[I^{(n)}_{+}, I^{(n)}_{-}] = 0$. The factorization of the eigenvalues into a sum of single particle eigenvalues is a consequence of the locality of the
charges and the approximate independence of the localized asymptotic wavepackets. We anticipate also that the single particle eigenvalues \( I_{\pm}^{(n)}(a) \) coincide up to a sign

\[
I^{(n)}(a) := I_{-}^{(n)}(a) = -I_{+}^{(n)}(a) .
\] (3.16)

This is a consequence of the current conservation equation in momentum space

\[
p_+ J^{(n)}_+ (p_+, p_-) + p_- J^{(n)}_+ (p_+, p_-) = 0,
\]

which holds on the operator level and a-fortiori relates the matrix elements of \( J^{(n)}_+ (p_+, p_-) \). On the other hand the eigenvalues \( I_{\pm}^{(n)}(a) \) can be calculated from on-shell single particle matrix elements (c.f. below)

\[
I_{\pm}^{(n)}(a) \sim \langle a(p)|J_{\pm}^{(n)}(p_+, 0)|a(p) \rangle |_{p^2 = (m_a^2)_{phys}}.
\]

This implies \( p_+ I_{-}^{(n)}(a) + p_- I_{+}^{(n)}(a) = 0 \) at \( p^2 = (m_a^2)_{phys} \) and hence (3.16) for rapidity \( \theta = 0 \).

Therefore it suffices to consider the \( I_-^{(n)} \) charge and we simplify the notation by putting \( J^{(n)} = J_-^{(n)} \), \( I^{(n)} = I_-^{(n)} \) etc. Consider then in detail the matrix elements of \( I_{-}^{(n)} \) between single particle states of type \( a \). We use the standard normalization transcribed to rapidity variables

\[
\langle a(p_1)|a(p_2') \rangle = 2\pi p_0 \delta(p_1 - p_2') ; \quad \langle a(\theta)|a(\theta') \rangle = 4\pi \delta(\theta - \theta') .
\]

By definition

\[
4\pi \delta(\theta - \theta') e^{-\eta \theta} I^{(n)}(a) = \langle a(\theta)|I^{(n)}|a(\theta') \rangle = \langle a(\theta)| \int dx^- J^{(n)} [\partial_\eta \Phi]|a(\theta') \rangle .
\] (3.17)

On the other hand for any local operator \( O(x) \) an LSZ reduction yields the relation

\[
\langle a(p)|O(x)|a(q) \rangle = \lim_{p^2 = (m_a^2)_{phys}, q^2 = (m_b^2)_{phys}} \left( \frac{-i}{\sqrt{Z_a}} \right) \left( \frac{-i}{\sqrt{Z_b}} \right) \times
\]

\[
\times [p^2 - (m_a^2)_{phys}][q^2 - (m_b^2)_{phys}] G_{ab}^{\eta}(-p, q) .
\] (3.18)

In the case at hand \( J^{(n)}(0, x^-) \) is integrated along \( x^- \) and the relevant momentum space Greens function is defined by

\[
2\pi \delta(p_- + q_-) G_{a\bar{a}}^{J^{(n)}(p, q)} := \int d^2y_1 d^2y_2 e^{ipy_1} e^{iqy_2} G_{a\bar{a}}^{J^{(n)}}(y_1, y_2) ,
\] (3.19)

where \( p = (p_+, p_-) \), \( q = (p_+, q_-) \) and

\[
i G_{a\bar{a}}^{J^{(n)}}(y_1, y_2) = \langle \Omega| \int dx^- J^{(n)}(0, x^+) \Phi_a(y_1)\Phi_{\bar{a}}(y_2)|\Omega \rangle .
\]
Since only connected diagrams contribute, the r.h.s. before integration will depend on the differences $x - y_1$, $x - y_2$ only. The $dx^-$ integration thus will result in a factor $2\pi\delta(p_+ + q_-)$, which has already been extracted in the definition of $G_{\alpha\dot{\alpha}}^{J^{(n)}}(p, q)$. The same factor will appear on the r.h.s. of (3.17) when applying the LSZ formula (3.18) to $\int dx^- J^{(n)}$. Together

$$I^{(n)}(a) = -(Z_a Z_{\dot{a}})^{-1/2} \lim_{p^2 \to (m_{\alpha}^2)_{\text{phys}}} \frac{e^{n\theta}}{2p_-} [p^2 - (m_{\alpha}^2)_{\text{phys}}]^2 G_{\alpha}^{(n)}(p),$$  \hspace{1cm} (3.20)

where $G_{\alpha}^{(n)}(p) = G_{\alpha\dot{\alpha}}^{J^{(n)}}(p_+, -p_-, p_+, p_-)$. The equation (3.20) gives a perturbative definition of the eigenvalue $I^{(n)}(a)$. As for the SG model in the super-renormalizable regime, one expects the perturbation theory to have non-zero radius of convergence, so that (3.20) amounts to an exact, unambiguous definition. As a check note that the Feynman diagrams contributing to $G_{\alpha}^{(n)}(p)$ will produce a factor $(p_-)^{n+1} \times [p^2 - (m_{\alpha}^2)_{\text{phys}}]^{-2} \times (\text{dim.less quantity})$. This means for the eigenvalue $I^{(n)}(a) \sim m^n \times (\text{dim.less quantity})$. In detail, we claim that $I^{(n)}(a)$ has the following form

$$I^{(n)}(a) = K_n(\beta) \left( \frac{m^2 e^T}{2\beta^2} \right)^{n/2} q_{\alpha}^{(n)}.$$  \hspace{1cm} (3.21)

Here $K_n(\beta)$ is an unknown function of the coupling constant to be determined (from which we extracted a factor $(2\beta^2)^{-n/2}$ for later convenience). Further, $m^2 e^T$ is the RG invariant combination (2.11) and $q_{\alpha}^{(n)} = q_{\alpha}^{(n+h)}$ are the components of the $r$-th ($n = r \mod h$) eigenvector of the Cartan matrix.

To prove (3.21), first recall that $G_{\alpha}^{(n)}(p)$ is the product of $(p_-)^{n+1} \times [p^2 - (m_{\alpha}^2)_{\text{phys}}]^{-2}$ and a dimensionless quantity composed of Lorentz scalars. On dimensional grounds one thus knows

$$\frac{e^{n\theta}}{2p_-} [p^2 - (m_{\alpha}^2)_{\text{phys}}]^2 G_{\alpha}^{(n)}(p) = m^n g_{\alpha}^{(n)} \left( \frac{p^2}{m^2}, \frac{p^2}{M^2}, \beta \right),$$

for some function $g_{\alpha}^{(n)} : \mathbb{R}^3 \to \mathbb{C}$. Similarly one defines its tadpole-free counterpart $\tilde{g}_{\alpha}^{(n)}(p^2/m^2, \beta)$. As usual, both are related by tadpole dressing

$$m^n g_{\alpha}^{(n)}(p^2/m^2, \beta) = \exp \left( T m^2 \frac{\partial}{\partial m^2} \right) \left[ m^n \tilde{g}_{\alpha}^{(n)}(p^2/m^2, \beta) \right]$$

$$= \left( e^T m^2 \right)^{n/2} \tilde{g}_{\alpha}^{(n)}(e^{-T} p^2/m^2, \beta).$$  \hspace{1cm} (3.22)
This has to be evaluated at \( p^2 = (m_a^2)_{\text{phys}} \). Since \( (m_a^2)_{\text{phys}} \sim m^2 e^T \), the \((M, m)\) dependence in \( \tilde{g}^{(n)} \) drops out, leaving a scheme independent function \(-k_a^{(n)}\) of the coupling alone. Together

\[
I^{(n)}(a) = \left[(m_a^2)_{\text{phys}}\right]^{n/2} (Z_a Z_{\bar{a}})^{-1/2} k_a^{(n)}(\beta).
\] (3.23)

The wave function renormalizations \( Z_a(\beta) \) have already been seen to be scheme independent. The form (3.23) of the eigenvalues therefore implies in particular that they are scheme independent physical quantities. The form of \( I^{(n)}(a) \) is further constrained by the ‘conserved charge bootstrap’[4]. This is to say that the eigenvalue equations (3.15) have to be compatible with the formation of bound states. In the notation of appendix A, this enforces the condition

\[
\sum_{l=a,b,c} e^{\pm \eta(l)} I^{(n)}(l) = 0,
\] (3.24)

where \( \eta(l) \) is defined in (2.18). This requires \( I^{(n)}(a) \sim q_a^{(n)} \) i.e. in (3.23) \( (q_a^{(1)})^n k_a^{(n)} \sim q_a^{(n)} \).

We conclude that \( I^{(n)}(a) \) has the form (3.21), which is what we wanted to show.

We note some consequences of the relation (3.21). First, \( I^{(n)}(a) \) is composed of scheme independent quantities, and hence is itself scheme independent. Still, both factors have a slightly different status. \( K_n(\beta) \) is a universal, scheme independent function of \( \beta \), to be determined. The second factor \( (m^2 e^T)^{n/2} \), although numerically constant under a RG transformation, will be represented by different functions of \( \beta \) in different schemes. However, once \( m^2 e^T \) is known in one scheme as a function of \( \beta \), it is known in any other: Suppose two renormalization schemes \( M_1/m_1 = f_1(\beta) \) and \( M_2/m_2 = f_2(\beta) \) to be given. Then

\[
\frac{m_1}{m_2} = \left(\frac{f_2(\beta)}{f_1(\beta)}\right)^{B/2}, \quad B = \frac{\beta^2/2\pi}{1 + \beta^2/4\pi}.
\]

Suppose further that \( e^{T(f_1(\beta),\beta)} =: T_1(\beta) \) is known explicitly as a function of \( \beta \). By RG invariance \( m_1^2 T_1(\beta) = m_2^2 T_2(\beta) \), where \( T_2(\beta) := e^{T(f_2(\beta),\beta)} \). Thus also

\[
T_2(\beta) = \left(\frac{m_1}{m_2}\right)^2 T_1(\beta) = \left(\frac{f_2(\beta)}{f_1(\beta)}\right)^B T_1(\beta)
\] (3.25)

is known explicitly. For the eigenvalues this implies that if \( I^{(n)}(a) \) is known explicitly as a function of some bare mass scale \( m_1 \) and the coupling constant, it is also known
explicitely as a function of any other bare mass scale \( m_2 \) and the coupling constant. It remains to determine the unknown functions \( K_n(\beta) \).

4. Perturbative evaluation of \( K_n(\beta) \)

In this section we restrict attention to the \( A_{r}^{(1)} \) series.

4.1. The classical limit

The classical limit corresponds to the tree level diagrams in (3.20). At tree level \( Z_a = 1 = T \) and since only connected diagrams enter, only the quadratic part of \( J^{(n)} \) will contribute. Modulo total derivatives \( J^{(n)} \) is invariant under the action of the dihedral group generated by \( \Phi_a \rightarrow \omega^a \Phi_a, \Phi_a \rightarrow \Phi_a \). For the quadratic part \( J^{(n,2)} \) this implies

\[
J^{(n,2)} = \beta^{-(n+1)} \sum_a c^{(n)}_a \partial^n \Phi_a \partial \Phi_a ,
\]

for some coefficients \( c^{(n)}_a \) satisfying \( c^{(n)}_a = (-)^{n+1} c^{(n)}_{\bar{a}} \). Inserting into (3.19) gives

\[
G_a^{(n)}(p) = 2(-i)^{n+1} \beta^{-n+1} c^{(n)}_a p^{n+1} \frac{1}{(p^2 - m^2 a)^2} ,
\]

where \( m_a = 2m \sin \frac{\alpha}{r+1} \) are the classical masses. From \( p_+ = m_a e^{-\theta}/\sqrt{2} \) one obtains

\[
I^{(n)}(a) = (-i)^{n-1} \left( \frac{m_a}{\sqrt{2}} \right)^n \beta^{-n+1} c^{(n)}_a .
\]

(4.2)

It remains to determine the coefficients \( c^{(n)}_a \). From the defining relations (4.1), (3.2) this is likely to be formidable. From (4.2) it is also clear that the calculation of the \( c^{(n)}_a \)'s essentially amounts to a direct calculation of the eigenvalues \( I^{(n)}(a) \). Fortunately, in the classical model the latter is possible by means of a trace identity, which gives an expression for the conserved charges in terms of the scattering data of a given solution

\[
I^{(n)}[\text{solution}] = I^{(n)}(\text{scattering data}) , \quad n \in E \subset \mathbb{N} .
\]

(4.3)

The solutions relevant here should have the following features
− The solutions should be singular solitary wave solutions i.e. solutions with discrete scattering data, which describe the propagation of point-like singularities in the initial data.

− The classical masses should coincide with the ones obtained from the quadratic part of the Lagrangian (1.1).

− Their trace identity (4.3) should coincide with the leading order term in $\beta$ of the trace identity for the breather solutions in the imaginary coupling model.

These requirements parallel the situation in the Sinh-Gordon model\[9\]. Since the trace identity for the breathers in the imaginary coupling AT’s is known\[7\], the last condition is the most convenient one to obtain the result. By construction, the result will then also meet the second requirement, and a closer inspection of the corresponding solutions (defined in terms of their $\tau$-functions) shows that they indeed describe the propagation of pointlike singularities in the initial data. We omit the details. Thus, the trace identity for the breather solutions in the imaginary coupling model is needed. Such breather solutions are parametrized by their type $a \in \{1, \ldots, r\}$ and a triplet of action-angle variables. The triplet of action variables is $((\theta, \rho, j)$, where $\theta$ is the rapidity, and $\rho$ and $j \in \{1, \ldots, a\}$ are a continuous and discrete excitation number, respectively. Conjugate to them is a triplet of angle variables which does not enter the trace identity. Explicitly, the trace identity at zero rapidity for a breather of type $a$ and action variables $((\theta = 0, \rho, j)$ is\[4\]

$$I^{(n)}[\text{breath}_a] = -\frac{r + 1}{n} (i\beta)^{-n+1} \left( \frac{m}{\sqrt{2}} \right)^n \times$$

$$\times \sin \left( \frac{\beta^2 n \rho}{r + 1} + \frac{\pi n}{r + 1} (j - 1) \right) \sin \frac{an\pi}{r + 1}.$$ (4.4)

For $n = 1$ the leading term in $\beta$ (after changing $i\beta$ to $\beta$) should reproduce the classical masses in (2.3). This fixes the parameters $\rho = 1, j = 1$ i.e. the singular solitary wave solution of type $a$ in the real coupling model is in correspondence with the breather solution of type $a$ and excitation numbers $(\rho, j) = (1, 1)$ in the imaginary coupling model. With these parameters fixed the procedure is repeated for generic $n$. Changing $i\beta$ to $\beta$ in (4.4) and extracting the leading order term in $\beta$ yields the trace identity at zero rapidity

$$I^{(n)}[\text{ssw}_a] = \beta^{-n+1} \left( \frac{m}{\sqrt{2}} \right)^n \sin \frac{an\pi}{r + 1},$$ (4.5)

\*We use slightly different conventions as in [8, 7]. The classical mass of a solution is $m[\text{solution}] = 2\sqrt{2}I^{(1)}[\text{solution}]$. 24
for the singular solitary wave solution $ssw_a$ of type $a$ in the real coupling model. Equation
(4.5) gives the classical eigenvalues searched for
\[ I^{(n)}(a) = I^{(n)}[ssw_a]. \] (4.6)
In particular, comparing with (4.2) one can read off the coefficients
\[ c^{(n)}_a = \frac{\sin \frac{an\pi}{r+1}}{2 \sin \frac{a\pi}{r+1} \pi}. \] (4.7)
defined by (4.1). For $n = 1, 2$ this is also confirmed by direct calculation (c.f. (3.6)). On
the other hand comparing (4.6) with the generic form (3.21) one can read off the classical
limit of the unknown function $K_n(\beta)$.
\[ (K_n(\beta))_{\text{class}} = \sqrt{\frac{h}{2\beta}}. \] (4.8)

4.2. First order perturbation theory

To do explicit calculations in PT it is convenient to rescale the fields $\Phi_a \to \beta^{-1}\Phi_a$. The coupling constant $\beta^2$ will then serve as a loop counting parameter. All vertices $\lambda_{a_1...a_N}$ are proportional to $\beta^{-2}$ and the propagator carries an extra factor of $\beta^2$. The classical conserved densities are then $\beta^2$-independent and the quantum corrections in the scheme $(m, \hat{f})$ are a polynomial in $\beta^2$. We shall be interested in the 1 loop corrections to $K_n(\beta)$ characterizing the eigenvalues (3.21). This amounts to calculating the 1 loop corrections to the Greens function $G_a^{(n)}(p)$ appearing in (3.20). It is convenient to pick the renormalization scheme $(m, M/m = \xi)$ in which the fundamental tadpole (2.8) vanishes. In this case only the parts $J^{(n,3)} + J^{(n,2)}$ of the conserved densities (i.e. those of power 2 and 3 in $\partial \Phi_a$) are needed for the 1 loop corrections. Notice that to lowest order the schemes $(m, \hat{f})$ and $m, M/m = \xi$ coincide, so that one can use the mCFT-form of the conserved densities as input for the calculation. The Feynman diagrams contributing to $G_a^{(n)}(p)$ are listed in Fig. 1. Diagrams of type $A$ and $B$ arise from the classical part of $J^{(n,3)}$ and $J^{(n,2)}$, respectively. Diagrams of type $C$ involve contributions from the wave function renormalization and from the first quantum corrections of $J^{(n,2)}$. The $\otimes$ indicates the operator insertion at zero momentum. The wave function renormalization constants are known exactly [2]. To $o(\beta^4)$ they read
\[ Z_a(\beta) = 1 - \frac{\beta^2(h-2)}{4\pi h} + \frac{\beta^2(h-2a)}{4h^2} \cot \frac{a\pi}{h} + o(\beta^4), \] (4.9)
where \( h = r+1 \). Moreover from (4.1), (4.7) also \( J^{(n,2)} \) is known to \( o(\beta^4) \). The contributions from diagrams of type \( C \) can therefore directly be evaluated. For type \( A \)-diagrams one needs the explicit form of \( J^{(n,3)} \), which is not known for generic \( n \).

Fig.1: Feynman diagrams for \( o(\beta^2) \) contribution to \( G_a^{(n)}(p) \)

In the following we will consider \( n = 1, 2 \), where the densities have been listed in (3.6). This is sufficient for our purposes because in section 5 an exact expression for \( K_a(\beta) \) will be derived, for which the 1-loop PT serves only as a check. In particular, \( n = 1 \) is a good check on the consistency of the set-up, because \( I^{(1)}(a) \) should reproduce the known first order corrections to the quantum masses. The case \( n = 2 \) then provides a non-trivial check on the \( n \)-dependence. The results obtained are consistent with the expression

\[
I^{(n)}(a) = I^{(n)}(a)_{class} \left( 1 - \frac{\beta^2}{8(r+1)} n \cot \frac{a\pi n}{r+1} + o(\beta^4) \right) .
\]  

(4.10)

Consider first the case \( n=1 \). The eigenvalue should reproduce the quantum masses via

\[
2\sqrt{2}I^{(1)}(a) = (m_a)_{phys} ,
\]  

(4.11)

where \((m_a)_{phys}\) now are the physical masses evaluated in the scheme \((m, M/m = \xi)\). To lowest order (see the last ref. in [18])

\[
(m_a)_{phys} = m_a \left( 1 - \frac{\beta^2}{8(r+1)} \cot \frac{a\pi}{r+1} + o(\beta^4) \right) .
\]  

(4.12)

On the other hand calculating \( I^{(1)}(a) \) from (3.20), (3.19) results in

\[
2\sqrt{2}I^{(1)}(a) = (m_a)_{phys} Z_a^{-1} \left[ 1 + \frac{1}{ip^2} B^{(1)}(a) \right] ,
\]

\[
B^{(1)}(a) = \sum_{bc} \lambda_{abc} \lambda_{abc}^* Q_2(m_b, m_c | m_a) ,
\]
where $Q_2$ is defined and evaluated in appendix B. Performing the sum one finds

$$
\frac{1}{ip^2} B^{(1)}(a) = \frac{\beta^2}{4\pi h} \left( 2 - h + \frac{4\pi(h - 2a)}{h} \cot \frac{a\pi}{h} \right) + o(\beta^4),
$$

which precisely cancels against the $o(\beta^2)$ contribution coming from $Z^{-1}_a$. Thus, to $o(\beta^2)$ one has indeed (4.10).

For $n = 2$ we have verified the relation (4.10) for $r = 2$ and 3.† We illustrate the calculation for $r = 2$; the case $r = 3$ is done similarly. From (3.6) the conserved density is (after the rescaling $\Phi_a \to \beta^{-1} \Phi_a$)

$$
J^{(2)} = \mp \frac{i}{3\sqrt{3}} (\Phi_1^3 - \Phi_2^3) \pm \frac{i}{2\sqrt{3}} \left( 1 + \frac{\beta^2}{4\pi} \right) (\Phi''_1 \Phi'_2 - \Phi'_1 \Phi''_2),
$$

(4.13)

where $\Phi'_a = \partial \Phi_a$ etc. The 3-point vertex is $V_3[\Phi] = \frac{3m^2}{\beta^2} (\Phi_1^3 + \Phi_2^3)$. From (3.20) one obtains

$$
I^{(2)}(1) = \frac{1}{2\beta} Z^{-1}_1 (m^2)_{phys} \left[ \frac{1}{2\sqrt{3}} \left( 1 + \frac{\beta^2}{4\pi} \right) \right.
\left. - \frac{\beta^2}{(2\pi)^2 i(p_-)^3} \left( a A^{(2)}(1) + b B^{(2)}(1) \right) \right],
$$

(4.14)

where the momentum $p_-$ is on mass-shell $p^2 = (m^2)_{phys}$. The integrals

$$
A^{(2)}(1) = m^2 \int d^2k \frac{k_-(p_--k_-)p_-}{(k^2 - 3m^2)[(k-p)^2 - 3m^2]}
$$

$$
B^{(2)}(1) = m^4 \int d^2k \frac{k^2}{(k^2 - 3m^2)^2[(k-p)^2 - 3m^2]}
$$

are $m$-independent, and $a, b$ are the corresponding symmetry factors. From (4.13) and the form of $V_3[\Phi]$ one finds $a = 0, b = 3\sqrt{3}/2$ and the evaluation of (B.1) gives

$$
B^{(2)}(1) = m^4 Q_3(3m^2, 3m^2 | 3m^2) = \frac{i\pi(p_-)^3}{27} \left[ 4 - \frac{7\pi}{3\sqrt{3}} \right].
$$

Inserting into (4.14) one obtains

$$
I^{(2)}(1) = \left( I^{(2)}(1) \right)_{class} \left( 1 + \frac{\beta^2}{12\sqrt{3}} \right) = -I^{(2)}(2),
$$

(4.15)

consistent with (4.10).

†The $r = 2$ case was also done in [20]. The results do not coincide because in [20] the eigenvalues were defined w.r.t. the bare mass scale. In this case $I^{(1)}(a)$ does not reproduce the quantum masses.
5. Vertex operator construction of $K_n(\beta)$

5.1. Realizations of the ZF algebra

In order to obtain an exact expression for $K_n(\beta)$ we use the bootstrap S-matrix as an additional input. Since the scattering operator commutes with all the conserved charges $[S, I^{(n)}] = 0$, one expects the S-matrix also to carry information about their spectrum. Indeed, the intertwining concept is just that of a Zamolodchikov-Faddeev (ZF) operator.

If $Z_a(\theta)$ denotes the ZF operator creating an asymptotic 1-particle state $|a(\theta)\rangle$ of type $a$ and rapidity $\theta$, one has by definition

$$Z_a(\theta_a) Z_b(\theta_b) = S_{ab}(\theta_a - \theta_b) Z_b(\theta_b) Z_a(\theta_a)$$  \hspace{1cm} (5.1a)

$$[I^{(n)}, Z_a(\theta)] = e^{-n\theta} I^{(n)}(a) Z_a(\theta)$$, \hspace{1cm} (5.1b)

$$[K, Z_a(\theta)] = \frac{d}{d\theta} Z_a(\theta)$$, \hspace{1cm} (5.1c)

where $I^{(n)}(a)$ is the eigenvalue of $I^{(n)}$ on $|a(\theta)\rangle = Z_a(\theta)|\Omega\rangle$ and $K$ is the generator of Lorentz boosts. We will call an associative algebra with generators $Z_a(\theta)$, $\theta \in \mathbb{C}$, $I^{(n)}$, $n \in E$ and $K$ subject to the relations (5.1) (and possibly others) a ZF algebra $Z(S)$ associated with $S$. Usually, a ZF algebra is supposed to act on the space of scattering states of the theory $\Sigma_{in/out}$. These are Fock spaces but the relation to the fundamental fields of the theory is elusive in general. It is therefore more useful to construct realizations

$$\rho : Z(S) \rightarrow \pi , \hspace{1cm} \rho(AB) = \rho(A) \rho(B) ,$$

of (5.1) on some auxiliary Fock space $\pi$, on which the $Z_a(\theta)$’s act as generalized vertex operators. This, of course, can be done in many ways and one will choose the realization according to purpose. The following two realizations are of particular interest.

1. The realization $\rho_I$ adapted to the conserved charges. In this case $\pi = \mathbb{C}[x_n, \ n \in E]$ and the realization is defined by

$$\rho_I(I^{(n)}) = \frac{\partial}{\partial x_n}, \ n \in E .$$  \hspace{1cm} (5.2)

We will show below that, supplemented by the invariance under some involution $\omega$, the relation (5.2) fixes the realization almost uniquely. In particular, the 2-point function coincides with the phase of the minimal form factor (2.27)

$$\langle 0| \rho_I(Z_a)(\theta_1) \rho_I(Z_b)(\theta_2)|0\rangle = e^{i \epsilon_{ab}(\theta_1 - \theta_2)} .$$  \hspace{1cm} (5.3)

Once the realization is known, the eigenvalues $I^{(n)}(a)$ can be obtained from the relation (5.1b).
The realization $\rho_F$ adapted to the form factor equations. In this case one has some freedom in the choice of the Fock space, but usually again the Fock space of a single free boson will be sufficient i.e. $\pi = \mathbb{C}[y_n, \ n \in E]$. The main requirement now is that the two point function $G_{ab}(\theta_1 - \theta_2) := \langle 0|\rho_F(Z_a)(\theta_1)\rho_F(Z_b)(\theta_2)|0 \rangle$ satisfies the conditions:

- $G_{ab}(\theta)$ is analytic for $\text{Im} \ \theta \leq 0$, except for a simple pole at $\theta = -i\pi$.
- $G_{ab}(\theta)$ is bounded in the lower half plane, $G_{ab}(\theta) = O(1)$, $\theta \to \infty$, $\text{Im} \ \theta \leq 0$.

Again this fixes the realization essentially uniquely. Suppose now the realization to be given and let $\Lambda$ be any linear operator on $\pi$ satisfying

$$\Lambda Z_a(\theta) = e^{2\pi i l} Z_a(\theta) \Lambda$$

$$e^{\theta K} \Lambda e^{-\theta K} = e^{\theta s} \Lambda,$$  \hspace{1cm} (5.4)

for some $l \in \mathbb{R}$, $s \in \mathbb{Z}$. According to [31], eqn. (3.9) the function

$$F_{a_n \ldots a_1}^\Lambda (\theta_1, \ldots, \theta_n) = \text{Tr}_\pi \left( e^{2\pi i \rho_F(K)} \Lambda \rho_F(Z_{a_n}(\theta_n) \ldots Z_{a_1}(\theta_1)) \right)$$  \hspace{1cm} (5.5)

then formally satisfies the form factor equations.

Both realizations are designed to solve some aspect of the full problem (non-perturbative construction of the QFT) but fail to incorporate others. The construction (2) does not specify which of the solutions (5.4) correspond to the form factors of some given local operator $O$. If one interprets $l$ and $s$ in (5.4) as the locality index (relative to the fundamental field) and the spin of $O$, respectively, any solution of (5.4) can apparently be taken to represent $\rho_F(O)$. Clearly some additional dynamical input is needed to determine the realizations $\rho_F(O)$ of local operators.

Conversely, in the realization $\rho_I$ one fixes the image of at least an infinite subset of local operators, namely the conserved charges, via (5.2). This essentially fixes the realization and allows one to determine the eigenvalues of the conserved charges. On the other hand the two point function (5.3) will in general fail to have simple analyticity properties in $\theta_1 - \theta_2$ and the operators $\rho_I(Z_a)$ will not be useful to obtain solutions of the form factor equations. We expect that important technical progress can be made by understanding the relation between the realizations (1) and (2) (and possibly others). In particular, one should find $\rho_F(I^{(n)})$.

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*It is worth emphasizing that $G_{ab}(\theta)$ is not supposed to satisfy $G_{ab}(\theta + 2\pi i) = e^{2\pi i} G_{ab}(-\theta)$ for some $l \in \mathbb{R}$. 

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5.2. Construction of $\rho_I$

Recall that the quantization has been done w.r.t. the $\partial_-$-lightcone dynamics. In particular the conserved charges $I^{(n)}$ and their proposed realization $\rho_I$ refer to the choice of the $\partial_-$-dynamics. Of course one could also have chosen the $\partial_+$-lightcone dynamics and the result for the eigenvalues should be the same. In other words, if we define $\omega$ to be the involution which maps the theory quantized w.r.t. the $\partial_-$-dynamics onto that quantized w.r.t. the $\partial_+$-dynamics, one should have

$$\omega(I^{(n)}(a)) = I^{(n)}(a) .$$  \hspace{1cm} (5.6)

On the imaginary axis in rapidity space the action of $\omega$ should correspond to $e^{-i\theta} \rightarrow c^2 e^{i\theta}$, where the real constant $c$ is related to the overall mass scale. For the rapidity factors $e^{-in\theta}$ accompanying $I^{(n)}(a)$ one sees that changing the sign of $\theta$ formally amounts to the same as changing the sign of $n$. We can thus introduce an infinite dimensional Heisenberg algebra

$$[a_{-m}, a_n] = i k \delta_{m,n} , \quad m, n \in E , \hspace{1cm} (5.7)$$

and associate the positive modes with the charges $I^{(n)}$ and the negative modes with $\omega I^{(n)}$. Consistency then requires

$$\omega(a_m) = -c^{2m} a_{-m} , \quad \omega([a_{-m}, a_n]) = [\omega(a_{-m}), \omega(a_n)]$$
$$\omega(a_{-m}) = -c^{-2m} a_m , \quad \omega(z) = z^* , \quad z \in \mathbb{C} . \hspace{1cm} (5.8)$$

i.e. $\omega$ is a linear anti-involution. Let $\pi$ denote the Fock space $\mathbb{C}[a_{-n} , n \in E]$ with vacuum $|0\rangle$. On $\pi$ there exists a unique bilinear form $\langle \ , \ \rangle$ s.t. $\omega$ is contravariant w.r.t. it, i.e. $\langle A, B \rangle = \langle 0, \omega(A) B \rangle$. In particular $\langle a_{-1}, a_{-1} \rangle = -i c^{-2} k$. The defining relation (5.1.b) then becomes\footnote{To simplify the notation we will write $Z_a(\theta)$ instead of $\rho_I(Z_a(\theta))$ when no confusion is possible. For real $\theta$ these are unbounded operators on $\pi$. The following formulae can be justified e.g. through analytic continuation from complex values of $\theta$.}

$$[a_n , Z_a(\theta)] = e^{-in\theta} I^{(n)}(a) Z_a(\theta) , \quad n \in E . \hspace{1cm} (5.9)$$

If we require that the realization $\rho_I$ is $\omega$-invariant,

$$\omega(Z_a(\theta)) = Z_a(\theta) , \hspace{1cm} (5.10)$$
this implies that

\[ [a_n, Z_a(\theta)] = -c^{2n} e^{in\theta} I^{(n)}(a) Z_a(\theta), \quad n \in E. \] (5.11)

Slightly extending a well-known Lemma ([15], Lemma 14.5), we infer that a linear operator on (the formal completion of) \( \pi \) satisfying the commutation relations (5.9), (5.10) is a generalized vertex operator i.e. of the form

\[ \rho_I(Z_a(\theta)) = : e^{\Upsilon_a(\theta)} : , \]

where \( : \) : denotes normal ordering w.r.t. the Heisenberg algebra (5.7). The coefficients \( d_n(a) \) are to be determined s.t. the equations (5.1.a), (5.10) holds. From \( \omega \Upsilon_a(\theta) = \Upsilon_a(\theta) \) one finds

\[ d^*_n(a) = -d_n(a) c^{2n}, \quad n \in E. \] (5.13)

To implement the relation (5.1.a) first notice that the compatibility of (5.1) with the S-matrix bootstrap equations (2.17) imposes the consistency consitions

\[ Z_a(i\pi - \theta) = c_a Z_a(-\theta)^{-1} \]

\[ Z_a(\theta + i\eta(a)) Z_b(\theta + i\eta(b)) Z_c(\theta + i\eta(c)) = c_{abc}, \] (5.14)

where \( c_a \) and \( c_{abc} \) are constants. We may assume these constants to equal 1. In terms of the fields \( \Upsilon_a(\theta) \) the conditions (5.14) then become

\[ \Upsilon_a(i\pi - \theta) = \Upsilon_a(-\theta), \]

\[ \sum_{l=a,b,c} \Upsilon_l(\theta + i\eta(l)) = 0, \]

Using the results of appendix A, these equations enforce \( d_n(a) = d_nq_a^{(n)}, n \in \pm E \) for some \( a \)-independent constants \( d_n \). Consider now

\[ Z_a(\theta_1) Z_b(\theta_2) = : Z_a(\theta_1) Z_b(\theta_2) : \exp \left( i k \sum_{n \in E} d_n(a) d_{-n}(b) e^{in(\theta_1 - \theta_2)} \right). \] (5.15)

The monodromy of \( Z_a(\theta_1) Z_b(\theta_2) \) will therefore be given by the meromorphic continuation of the function \( \exp \left( 2i k \sum_{n \in E} d_n(a) d_{-n}(b) e^{in(\theta_1 - \theta_2)} \right) \). Comparing with the expansion (2.31) of the scattering phase one reads off the condition

\[ \frac{1}{n} D_n q_a^{(n)} q_b^{(n)} = 2k d_n(a) d_{-n}(b) = 2k d_n d_{-n} q_a^{(n)} q_b^{(-n)}, \quad n \in E. \] (5.16)
Since $q_b^{(-n)} = -q_b^{(n)}$ this implies $D_n = -2nk d_n d_{-n} = -D_{-n}$, so that by (5.13)

$$d_n = i\epsilon_n c^{-n} \sqrt{\frac{D_n}{2nk}},$$

(5.17)

where $\epsilon_n$ is a sign to be determined later. In summary we have obtained a 1-parameter family of realizations of (5.1) from the conditions (5.2), (5.10)

$$\rho_I(Z_a)(\theta) = e^{\gamma_a(\theta)},$$

$$\Upsilon_a(\theta) = i \sum_{n \in \pm E} \epsilon_n c^{-n} \sqrt{\frac{D_n}{2nk}} q_a^{(n)} a_n e^{in\theta}, \quad a = 1, \ldots, r.$$  

(5.18)

Combining (5.15) and (5.17) one sees that this realization has the feature (5.3). The Lorentz boost operator is realized as

$$\rho_I(K) = -\frac{1}{k} \sum_{n \in E} n a_{-n} a_n$$

(5.19)

and is $\omega$-invariant. Having constructed the realization we can now use the the defining relation (5.1.b) – or equivalently (5.9), (5.11) – to find the eigenvalues $I^{(a)}(a)$. Equating the expressions obtained from (5.9) and (5.11) fixes the sign $\epsilon_n = (-)^{\text{sign}(n)+1} = -\epsilon_{-n}$. (i.e. $q_a^{(n)}$ in (5.18) gets replaced by $q_a^{(-n)}$.) Comparison with the generic form (3.21) then gives

$$I^{(n)}(a) = c^n \sqrt{\frac{k D_n}{2n}} q_a^{(n)} = K_n(\beta) \left( \frac{m^2 e^T}{2\beta^2} \right)^{n/2} q_a^{(n)}.$$  

(5.20)

This fixes

$$c = \left( \frac{m^2 e^T}{2\beta^2} \right)^{1/2}, \quad K_n(\beta) = \sqrt{\frac{k D_n}{2n}}.$$  

(5.21)

Further, since $D_n$ is known, the unknown function $K_n(\beta)$ has been determined up to the constant $k$, appearing in the commutation relations (5.7). Expanding $D_n$ and matching against the perturbative result (4.10) one sees that $k$ actually has to be a function of $\beta$ of the form

$$k(\beta) = h \left( 1 + \frac{\beta^2}{4\pi} + o(\beta^4) \right).$$  

(5.22)
Since \( k(\beta) \) is \( n \)-independent, the higher order terms in (5.22) can be found by comparison with the exact quantum masses of Destri and DeVega\[^4\]. When written in a RG invariant form, these are

\[
(m_a)_{\text{phys}} = 2\sqrt{2} I^{(1)}(a) = \left( \frac{m^2 e^T h D_1}{\pi B} \right)^{1/2} q_\alpha^{(1)}, \tag{5.23}
\]

so that \( k(\beta) = h \beta^2 / 2\pi B \), i.e. (5.22) is in fact exact.

In summary we have obtained the following result: There exists a 1-parameter family of \( \omega \)-invariant realizations of the ZF algebra s.t. \( \rho_I(I^{(n)}) = \frac{\partial}{\partial x_n} = a_n \). The free parameter \( c \) corresponds to the ambiguity in setting the physical mass scale through a (renormalization group invariant) combination of the bare mass and the normal ordering mass and is given explicitly in (5.21). The parameter \( c \) also sets the scale for the involution \( \omega(a_n) = -c^{2n} a_{-n} \) and the commutation relations are \([a_{-m}, a_n] = i\hbar(1 + \beta^2/4\pi)\delta_{m,n}\). On the (formal completion of the) corresponding Fock space representation the asymptotic multi-particle states are realized as generalized coherent states

\[
\rho_I(|a_N(\theta_N), \ldots, a_1(\theta)\rangle) = \rho_I(Z_{a_N}(\theta_N) \ldots \rho_I(Z_{a_1})(\theta_1)|0\rangle).
\]

These are simultaneous eigenstates of the conserved charges \( I^{(n)}, n \in E \) and their eigenvalues decompose into a sum of eigenvalues \( I^{(n)}(a) \) on the single particle states. For these eigenvalues we have the exact result

\[
I^{(n)}(a) = \left( \frac{m^2 e^T}{2\beta^2} \right)^{n/2} \beta^{n/2} \left[ \frac{\hbar \sin \frac{\pi n}{2k} B \sin \frac{\pi n}{2h}(2 - B)}{4\pi n B \sin \frac{\pi n}{h}} \right]^{1/2} q_\alpha^{(n)}, \quad n \in E. \tag{5.24}
\]

Here \( \hbar \) is the Coxeter number, \( B \) is the effective coupling (2.22) and \( q_\alpha^{(n)} \) are the eigenvectors of the Cartan matrix. The tadpole function \( T \) enters through the RG invariant combination \( m^2 e^T \).

### 6. Conclusions

The equation (5.24) solves the diagonalization problem of the conserved charges \( I^{(n)} \) in a QFT context, using the bootstrap S-matrix as an input. Conversely, combining equation
(5.24) with (2.31) the scattering phase can also be re-expressed in terms of the eigenvalues of the conserved charges via
\[
\delta_{ab}(\theta) = \frac{4\pi B}{h\beta^2} \sum_{n \in E} c^{-2n} I^{(n)}(a) I^{(n)}(b) e^{n\theta}, \quad \text{Re} \, \theta < 0. \tag{6.1}
\]
This means that an independent justification of the eigenvalues (5.24) would also provide a dynamical justification of the conjectured bootstrap S-matrix. Since UV finite expressions for the conserved charges can be constructed directly from CFT techniques, this amounts to addressing the diagonalization problem on suitable representation spaces of the associated chiral algebra. For the real coupling models the Fock space of rank\((g)\) free bosons should be appropriate.

Generally one is lead to consider the diagonalization problem of the conserved charges on the Verma modules of \(W(g)\). This amounts to the calculation of generalized Kac determinants in the following sense. Let \(V(\lambda)\) be a Verma module of \(W(g)\), where \(\lambda\) are the parameters of the highest weight state. Let \(V_N\) denote the subspace of degree \(N\) w.r.t. the \(L_0\)-grading (\(L_0\) being the zero mode of the Virasoro subalgebra) with basis \(v_\alpha, 1 \leq \alpha \leq \dim V_N =: K\). If \(\langle \cdot, \cdot \rangle\) denotes the usual contravariant hermitian form on \(V_N\), the Kac determinant formula (see [16] and references therein) gives an expression for
\[
\det (\langle v_\alpha, L_0 v_\beta \rangle)_{1 \leq \alpha, \beta \leq K} \tag{6.2}
\]
as a function of \(\lambda\) and the central charge. The insertion of \(L_0\) of course just gives rise to an overall factor and is usually omitted. From the viewpoint of \(W\)-algebras however the conserved charges \(I^{(n)}\) arise as an infinite dimensional abelian subalgebra of \(W(g)\). Moreover one can construct a basis of \(W(g)\) for which \(I^{(1)} = L_0\) (rather than \(I^{(1)} = L_{-1}\)) so that the charges \(I^{(n)}\) preserve the \(L_0\) graduation of the Verma modules \([L_0, I^{(n)}] = 0, n \in E\) [13]. From this viewpoint, the choice of the lowest conserved charge \(I^{(1)} = L_0\) in (6.2) is not preferred and one can study ‘generalized Kac determinants’ of the form
\[
\det (\langle v_\alpha, I^{(n)} v_\beta \rangle)_{1 \leq \alpha, \beta \leq K, n \in E}. \tag{6.3}
\]
Finding an explicit expression for (6.3) would solve the diagonalization problem of the conserved charges on Verma modules.

For the ATs one should study similar determinants on the Hilbert space of the CFT characterizing its UV behaviour. Let \(v_{\alpha, \bar{\alpha}}, 1 \leq \alpha \leq K, 1 \leq \bar{\alpha} \leq \overline{K}\) be a basis of the subspace of degree \(N\) in the CFT (w.r.t. the \(L_0 + \overline{L}_0\) graduation) and consider
\[
\det (\langle v_{\alpha \bar{\alpha}}, I^{(n)} v_{\beta \bar{\beta}} \rangle)_{1 \leq \alpha, \beta \leq K, 1 \leq \bar{\alpha}, \bar{\beta} \leq \overline{K}}. \tag{6.4}
\]
On general grounds we expect that the large $N$ asymptotics of these determinants is governed by factors whose zeros are proportional to sums of factors $K^n I^{(n)}(a)$ for some constant $K$.

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Appendix

A. Orbits of the bicolored Coxeter element

Let $g$ be a simple simply laced Lie algebra. The simple root system of $g$ admits a bicoloring s.t. roots of different colors are orthogonal. Let $\Omega_+$ denote the product of the Weyl reflections in the ‘white’ simple roots $\alpha_a$ endowed with the color value $c(a) = +1$ and $\Omega_-$ the product of the Weyl reflections in the ‘black’ simple roots endowed with the color value $c(a) = -1$. Then $\Omega = \Omega_-\Omega_+$ is a preferred Coxeter element of $g$. In particular, its eigenvectors are labeled by the exponents $\{s_1, \ldots, s_r\}$ of $g$

$$\Omega e^{(s)} = e^{\frac{2\pi i s}{h}} e^{(s)} , \quad e^{(s)} \cdot e^{(t)} = \delta_{s+t,h} ,$$

(A.1)

where $s \in \{s_1, \ldots, s_r\}$ and $h$ is the Coxeter number. The components of some real $\lambda \in h^*$ are given by

$$\lambda = \sum_s \lambda^{(h-s)} e^{(s)} , \quad \lambda^{(s)} = (e^{(s)}, \lambda) , \quad \lambda^{(h-s)} = (\lambda^{(s)})^* .$$

(A.2)

It suffices to know the components of the simple roots and the fundamental weights. The result is (see e.g. [22])

$$\begin{align*}
(e^{(s)}, \lambda_a) &= \frac{i}{\sqrt{2 \sin \theta_s}} e^{-i\theta_s \frac{1+c(a)}{2}} q_a^{(s)}, \\
(e^{(s)}, \alpha_a) &= -c(a) \sqrt{2} e^{i\theta_s \frac{1-c(a)}{2}} q_a^{(s)},
\end{align*}$$

(A.3)

*The following results hold with minor modifications also for non simply laced algebras. The status of the associated bootstrap S-matrices is however not clear yet.
where $\theta_s = \pi s/h$ and $q_a^{(s)} = c(a)q_a^{(h-s)}$ is the normalized eigenvector of the Cartan matrix defined by
\[
\sum_a a_{ab} q_b^{(s)} = 2(1 - \cos \theta_s) q_a^{(s)}, \quad q^{(s)} \cdot q^{(t)} = \delta_{s,t} .
\] (A.4)

This allows to evaluate inner products of the form
\[
(\lambda, \Omega^{-p} \mu) = \sum_s \omega^{ps} \lambda^{(h-s)} \mu^{(s)}
\] (A.5)

In particular
\[
(\lambda, \Omega^{-p} \mu) = \sum_s q_a^{(s)} q_b^{(s)} e^{i\theta_s(2p+c(a,b))} d_s = \sum_s q_a^{(s)} q_b^{(s)} \tilde{d}_s,
\] (A.6)

where $\gamma_a = c(a)\alpha_a$, $c(a, b) = (c(a) - c(b))/2$ and the symmetric form of (B7) displays that the r.h.s. of (B6) is real. Set
\[
A_{\lambda,\mu}(\theta) = \prod_{p=1}^{h} \left(1 - e^{i\pi(2p+c(a,b))} e^{\theta}\right)^{(\lambda,\Omega^{-p} \mu)} .
\] (A.8)

Then
\[
\ln A_{\lambda,\mu}(\theta) = -\sum_{n \in E} \frac{\hbar}{n} d_n q_a^{(n)} q_b^{(n)} e^{n\theta} ,
\] (A.9)

with the subcases (B7).

**Remark:** $A_{\gamma_a,\gamma_b}$ is the 2-soliton interaction constant. For $g = A_r$ it reduces to
\[
A_{\gamma_a,\gamma_b} = \frac{\sinh \left( \frac{\theta}{2} + \frac{i\pi(a-b)}{2(r+1)} \right) \sinh \left( \frac{\theta}{2} - \frac{i\pi(a-b)}{2(r+1)} \right)}{\sinh \left( \frac{\theta}{2} + \frac{i\pi(a+b)}{2(r+1)} \right) \sinh \left( \frac{\theta}{2} - \frac{i\pi(a+b)}{2(r+1)} \right)}
\] (A.10)

The quantity $A_{\gamma_a,\gamma_b}$ enters the bootstrap S-matrices and $A_{\lambda_a,\lambda_b}$ has been used in [21] for the construction of vertex operators.
Let $\Omega = \Omega_- \Omega_+ = r_{i_r} \ldots r_{i_1}$ denote a reduced expression for the bicolored Coxeter element. Set
\[
\Delta_+^\Omega = \{ \alpha \in \Delta_+ | \Omega(\alpha) < 0 \},
\]
which by $|\Delta_+^\Omega| = l(\Omega) = r$ has $r$ elements. An explicit enumeration is
\[
\Delta_+^\Omega = \{ \alpha_{i_r}, r_{i_r} \alpha_{i_{r-1}}, \ldots, r_{i_2} r_{i_1} \alpha_{i_1} \} = \{ (1 - \Omega^{-1}) \lambda_a, 1 \leq a \leq r \}.
\]
(A.11)

Consistent with $|\Delta| = 2|\Delta_+^\Omega| = 2^r h = \dim g - r$ the root system decomposes into $r$ disjoint orbits of $\Omega$ each of which is $h$-dimensional. Let $\Omega_a = Z_h \cdot (1 - \Omega^{-1}) \lambda_a$ denote the orbit of the $a$-th element in (B11) under the action of the cyclic group $Z_h$ generated by $\Omega$. Then $\Delta = \Omega_1 \oplus \ldots \oplus \Omega_r$. The orbits $\Omega_a$ have the following properties:

i. $\gamma_a = c(a) \alpha_a \in \Omega_a$.

ii. If $\Omega_a$ is an orbit, so is $-\Omega_a$ and hence has to coincide with some $\Omega_{\bar{a}}$, where '¯' denotes an idempotent permutation of $\{1, \ldots, r\}$. Explicitly
\[
\gamma_{\bar{a}} = -\Omega \frac{b + c(a) - c(\bar{a})}{4} \gamma_a, \quad c(\bar{a}) c(a) = (-)^h.
\]

iii. For $(a, b, c) \in \{1, \ldots, r\}^3$ the equivalent equations
\[
\begin{align*}
\sum_{l=a,b,c} \Omega^\zeta(l) \gamma_l &= 0 \\
\sum_{l=a,b,c} \Omega^{-\zeta(l)} \lambda_l &= 0
\end{align*}
\]
(A.12)
are called ‘fusing equations’ for the process $a, b \rightarrow \bar{c}$. The triplet $(\zeta(a), \zeta(b), \zeta(c)) \in \mathbb{Z}^3/\mathbb{Z}$ is called a solution. For given $(a, b, c) \in \{1, \ldots, r\}^3$ (B12) has either none or two independent solutions. If $(\zeta(a), \zeta(b), \zeta(c))$ is one solution then $(\zeta'(a), \zeta'(b), \zeta'(c))$, $\zeta'(l) = -\zeta(l) + (c(l) - 1)/2$ is the second. The projections of (B12) onto the eigenstates of $\Omega$ are given by
\[
\sum_{l=a,b,c} e^{\pm i \eta(l)} q_{l}^{(s)} = 0,
\]
respectively, where $\eta(l) = -\frac{\pi}{h}(2 \zeta(l) + \frac{1-c(l)}{2})$. Since $\eta'(l) = \frac{\pi}{h}(2 \zeta'(l) + \frac{1-c(l)}{2}) = -\eta(l)$ the two eqn. s (B13) correspond to the two solutions of the fusing equations. Charge conjugation corresponds to $\eta(\bar{a}) = \eta(a) - \pi$. 

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iv. Let $T_a$ denote a basis of the Cartan subalgebra satisfying $T_a^\dagger = T_\bar{a}$, $\text{Tr}(T_a T_b) = \delta_{ab}$ and let $E = \sum_{i=1}^r e_i + e_{-\theta}$, $F = \sum_{i=1}^s \bar{n}_i f_i + e_{\theta}$ (with $\theta$ the highest root and $\bar{n}_i$ the dual Kac labels) denote the standard regular elements of $g$. The 3-point coupling $C_{abc} = \text{Tr}([T_a, E], [T_b, F], T_c)$ is non-zero iff the fusing equation for $(a, b, c)$ has a nontrivial solution\[22, 23\].

B. Some Integrals

Here we spell out some details needed for the perturbation theory calculation in section 4.2. The required integrals are

$$Q_n(a^2, b^2|p^2) = \int d^2k [\frac{(k^-)^n}{[k^2 - a^2 + i\epsilon][k^2 - b^2 + i\epsilon]}] = \int dk^+dk^- [\frac{(k^-)^n}{[2k^+k^- - a^2 + i\epsilon][2(k^+ - p^+)(k^- + p^-) - b^2 + i\epsilon]}].$$

Closing the contour in the lower half $k^+$ plane gives

$$Q_n(a^2, b^2|p^2) = -i\pi \left[ K_{n+1}(0) - K_{n+1}(p^-) - p^- K_n(0) + p^- K_n(p^-) \right], \quad (B.1)$$

where

$$K_n = \int_{-\infty}^{\infty} dk \frac{k^n}{2p^+k^2 + (b^2 - a^2 - p^2)k + a^2p^-}.$$ 

In particular

$$Q_2(a^2, b^2|p^2) = \frac{-i\pi(p^-)^2}{p^4\Delta} \left[ p^2(a^2 - b^2) + p^4 + \left( (b^2 - a^2 - p^2) - p^2 a^2(p^2 + b^2 - a^2) \right) Q \right] - \frac{i\pi(p^-)^2}{2p^4} \ln \frac{a^2}{b^2},$$

$$\Delta = -p^4 + 2p^2(a^2 + b^2) - (a^2 - b^2)^2,$$

$$Q = -\frac{2}{\sqrt{\Delta}} \left[ \arctan \left( \frac{-p^2 + b^2 - a^2}{\sqrt{\Delta}} \right) - \arctan \left( \frac{p^2 + b^2 - a^2}{\sqrt{\Delta}} \right) \right]. \quad (B.2)$$
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