ON DYNAMICS OF THE SIERPIŃSKI CARPET

JAN P. BOROŃSKI AND PIOTR OPROCHA

Abstract. We prove that the Sierpiński curve admits a homeomorphism with strong mixing properties. We also prove that the constructed example does not have Bowen’s specification property.

1. Introduction

The aim of this note is to exhibit a homeomorphism of the Sierpiński curve (known as the planar universal curve or Sierpiński carpet) with some strong mixing properties. In 1993, Aarts and Oversteegen proved that the Sierpiński curve admits a transitive homeomorphism [1], answering a question of Gottschalk. They also showed that it does not admit a minimal one. Earlier, in 1991 Kato proved that the Sierpiński curve, does not admit expansive homeomorphisms [11]. In [2] the authors proved that the Sierpiński curve admits a homeomorphism with positive entropy. They also showed that it admits a minimal group action (by [1] it cannot be done using single homeomorphism). There has been quite a lot of interest in dynamical properties of the planar universal curve, also due to its occurrence as Julia sets of various complex maps (see e.g. [7]). Nonetheless, we were unable to find any examples in the literature that would explicitly show homeomorphisms of the Sierpiński curve with chaocity beyond Devaney chaos. The writing of the note was also motivated by some recent questions. During the Workshop on Dynamical Systems and Continuum Theory, at University of Vienna, in June of 2015 the following question was raised.

Question 1.1. Suppose a 1-dimensional continuum $X$ admits a mixing homeomorphism. Must $X$ be $\frac{1}{n}$-indecomposable for some $n$?

Recall that a continuum $X$ is $\frac{1}{n}$-indecomposable, if given $n$ mutually disjoint subcontinua of $X$ at least one of them must have empty interior in $X$. Note that the Sierpiński curve is $\frac{1}{n}$-indecomposable for no $n \in \mathbb{N}$. This is because it is locally connected, so every point has an arbitrarily small connected neighborhood. Our example is quite simple, however it relies on many nontrivial facts from topology and ergodic theory. In principle, the general strategy is very similar to the one in [1], but the starting point is a bit different. We start with an Anosov torus diffeomorphism, which allows us to say much more about the dynamics of the constructed map.

Theorem 1.2. The Sierpiński curve $S$ admits a homeomorphism $H: S \to S$ such that:

1. $H$ has a fully supported measure $\mu$, such that $(H, \mu)$ is Bernoulli,
2. $H$ has a dense set of periodic points,
(3) $H$ does not have specification property.

Since every Bernoulli measure is strongly mixing, and $\mu$ in Theorem 1.2 is fully supported, we immediately obtain the following result, answering Question 1.1 in the negative.

**Corollary 1.3.** The Sierpiński curve $S$ admits a topologically mixing homeomorphism with dense set of periodic points.

By the arguments in the proof of Theorem 1.2, it seems very likely that Aarts-Oversteegen technique [1] which we employ here, will never lead to a map with specification property. This motivates the following natural question.

**Question 1.4.** Does the Sierpiński curve admit a homeomorphism with the specification property?

### 2. Preliminaries

By a dynamical system $(X, T)$ we mean a compact metric space $(X, d)$ with a continuous map $T: X \to X$.

We identify $\mathbb{T}^2$ with the quotient space $\mathbb{R}^2/\mathbb{Z}^2$ and denote $K^2 = [-1/2, 1/2]^2$.

A Sierpiński curve is any set homeomorphic to $S^2 \setminus \bigcup_{i=1}^{\infty} \text{int } D_i$ where

1. Each $D_i$ is a disc and $D_i \cap D_j = \emptyset$ for $i \neq j$.
2. $\{D_i\}_{i=1}^{\infty}$ is a null sequence, i.e. the diameters of $D_i$ tend to zero, as $i \to \infty$.
3. $\bigcup_{i=1}^{\infty} D_i$ is dense in $S^2$.

Whyburn [16] proved that Sierpiński curve does not depend on the choice of the sequence of discs $\{D_i\}_{i=1}^{\infty}$, that is any two Sierpiński curves are homeomorphic.

#### 2.1. Topological notions of mixing.

A dynamical system $(X, T)$ is topologically mixing if for any two nonempty open sets $U, V$ there is an $N > 0$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$. There are many different extensions of the above property to characterize stronger mixing in the system. From the point of view of our work the following two are very important. It is not hard to see that they imply topological mixing.

In his seminal paper [3] Bowen introduced an important, strong version of mixing, called (periodic) specification property. Let $T: X \to X$ be a continuous onto map. Following Bowen, we say that $(X, T)$ has the specification property if for any $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that for any integer $s \geq 2$, any $s$ points $y_1, \ldots, y_s \in X$, and any sequence $0 = j_1 \leq j_2 < k_2 \leq \cdots < j_s \leq k_s$ of $2s$ integers with $j_{m+1} - k_m \geq N$ for $m = 1, \ldots, s - 1$, there is a point $x \in X$ such that, for each positive integer $m \leq s$ and all integers $i$ with $j_m \leq i \leq k_m$, the following two conditions hold: $d(T^i(x), T^i(y_m)) < \varepsilon$. If, in addition, we can always select $x$ as a periodic point such that $T^{k_m-j_m+N}(x) = x$ then $(X, T)$ has the periodic specification property. Note that the problem of characterizing the relations between various types of mixing for maps in specified classes of one-dimensional continua is of high interest (e.g. see [9] and references therein).

#### 2.2. Invariant measures.

Let $X$ be a compact metric space with metric $d$ and let $M(X)$ be the space of Borel probability measures on $X$ equipped with the Lévy-Prokhorov metric $\rho$ defined by

$$\rho(\mu, \nu) = \inf\{\varepsilon: \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ for all Borel subsets } A \subset X\},$$
where $A^\delta = \{x : \text{dist}(x, A) < \delta\}$. The topology induced by $\rho$ coincides with the weak*-topology on $M(X)$. It is also well known that $(M(X), \rho)$ is a compact metric space. For a dynamical system $(X, T)$ we denote by $M_T(X)$ the set of all $T$-invariant measures. For more details on Lévy-Prokhorov metric and weak*-topology the reader is referred to [10], and basic properties related to invariant measures (ergodicity, strong mixing, Bernoulli shift) can be found in [15].

2.3. Quasi-Hyperbolic Toral Automorphisms. Let $A$ be a $2 \times 2$ matrix with integer entries such that $|\det A| = 1$. Then $A^{-1}$ also has integer entries, and so $A$ induces a homeomorphism of the 2-dimensional torus $F: \mathbb{T}^2 \to \mathbb{T}^2$ by $F(x) = Ax(mod 1)$, e.g. see [4] for more details. Since $|\det A| = 1$, every toral automorphism preserves Lebesgue measure. It is known that the periodic points of an ergodic toral automorphism are exactly those with rational coordinates (see [6 Proposition 24.7]). It was first proved by Adler and Weiss for $\mathbb{T}^2$ and then extended by Katznelson to each $\mathbb{T}^n$, that if toral automorphism is ergodic with respect to the Lebesgue measure, then it is measure-theoretically conjugate to a Bernoulli shift (e.g. see [6 Theorem 24.6]). Following Lind [12] we say that $F$ is quasi-hyperbolic if $A$ does not have roots of unity as eigenvalues. In dimension 2 every quasi-hyperbolic automorphism must be hyperbolic, that is, it does not have eigenvalues on the unit circle, and has strong specification property [12].

2.4. Branched Covering from $\mathbb{T}^2$ to $S^2$. Represent $\mathbb{T}^2$ as $\mathbb{R}^2/\mathbb{Z}^2$. Take a quotient of $\mathbb{T}^2$ by the relation $J$, that identifies $(x, y)$ with $(-x, -y)$. $J$ induces a branched covering map $\pi: \mathbb{T}^2 \to S^2$ (see e.g. [15 p. 140]), which is 2-to-1 except at four branch points in $\mathbb{T}^2$ given by $\mathcal{C} = \{(0, 0), (1/2, 0), (0, 1/2), (1/2, 1/2)\}$. Since the relation $J$ is preserved by any toral automorphism, for every toral automorphism $F$ we obtain a factor map $G: S^2 \to S^2$ such that $G \circ \pi = \pi \circ F$. Note that if $x \notin \mathcal{C} \cup F^{-1}(\mathcal{C})$ then there is an open neighborhood $U$ of $x$ such that $U$ has at most one element of any equivalence class of the relation $J$ and the same holds for $F(U)$. Then on $U$ the factor map $\pi$ is a local isometry.

3. Proof of Theorem 1.2

Start with “Arnold’s cat map” $F: \mathbb{T}^2 \to \mathbb{T}^2$ on the torus given by

\[
F(x, y) = (2x + y, x + y)(\text{mod } 1).
\]

Clearly $F$ is hyperbolic with eigenvalues $\lambda_1 = \frac{3 + \sqrt{5}}{2}$ and $\lambda_2 = \frac{-\sqrt{5} - 1}{2}$, hence it has periodic specification property. There is also an invariant foliation of $\mathbb{T}^2$ by lines $\mathcal{L}$, that is $F(\mathcal{L}) = \mathcal{L}$. Denote the Lebesgue measure on $\mathbb{T}^2$ by $\lambda$. $F$ has a dense set of periodic points and $(\mathbb{T}^2, F, \lambda)$ is measure-theoretically conjugate to Bernoulli shift. Let $\pi: \mathbb{T}^2 \to S^2$ be the quotient map from Section 2.4 and let $G$ be the induced homeomorphism of $S^2$. Since $G$ is a factor of $F$, $(S^2, G, \mu)$ is Bernoulli with respect to a fully supported measure $\mu$ which is a push-forward of $\lambda$ by $\pi$ (see [15 Theorem 4.29(ii)]). Let $\mathcal{G} = \pi(\mathcal{L})$. Clearly $\mathcal{G}$ is an invariant foliation of $S^2$; i.e. $G(\mathcal{G}) = \mathcal{G}$.

Let $\mathcal{O} = \{O_n : n \in \mathbb{N}\}$ be a dense family of periodic orbits of $G$. We assume that each $O_n \cap \pi(\mathcal{C}) = \emptyset$ and that $\text{Per}(G) \setminus \mathcal{O}$ is dense in $S^2$.

We will modify $G$ inductively, blowing up consecutive periodic orbit. Take $O_1 \in \mathcal{O}$, and assume that it has period $p_1$, say $O_1 = \{c, G(c), \ldots, G^{p_1-1}(c)\}$. Since $\pi^{-1}(O_1) \cap \mathcal{C} = \emptyset$, there are open discs $D_0, \ldots, D_{p_1-1}$ such that $\pi(D_i) \cap \pi(D_j) = \emptyset$.
for $i \neq j$ and $\pi$ is 1-1 on each $D_i$. Let $U_i = \pi(D_i)$. Then we have a natural foliation of each $U_i$ (induced locally from $D_i$) by $G^i(x)\cap G$, such that if $L \subset U_i$ is a sufficiently short line emerging from $G^i(c)$ then $G(L) \cap U_{i+1}$ is contained in the corresponding line. In other words, we have a continuous foliation of a neighborhood of each point $G^i(c)$ into lines, and $G$ preserves these lines.

Let $\mathcal{F}_i = \{t^i_\theta : \theta \in [0, 2\pi]\}$ be a foliation of a neighborhood $U_i$ of $G^i(c)$ mentioned above by the lines emanating from $G^i(c)$. We remove $O_1$ and compactify each $c\cup U_i \setminus \{G^i(c)\}$ by a topological copy $S^1_i$ of the unit circle $S^1$ adding, for a fixed $i$, a point $\theta_i \in S^1_i$ compactifying the leaf $t^i_\theta$. That way we obtain a $p_1$-punctured sphere $S_1$. We may easily extend $G$ to a map $H_1 : S_1 \to S_1$ by setting $H_1(\theta_1) = \theta_j'$, where $G(t^i_\theta) = t^i_{\theta'}$, $j = i + 1(\text{mod } p_1)$, and $\theta_j' \in S^1_j$. Clearly $H_1$ defined that way is invertible with a continuous inverse, so $H_1$ is a homeomorphism of $S_1$. Observe that the dynamics of all other points under $H_1$ in $S_1$ is exactly the same as on $S^2$ for $G$. Hence we can repeat this procedure, puncturing $S_2$ and replacing the periodic orbit $O_2$ by a periodic sequence of circles. Inductively, we obtain a sequence of spheres with $\sum_{n=1}^\infty p_i$-holes $S_n$, homeomorphisms $H_n : S_n \to S_n$ and factor maps $\pi_n : S_n \to S_{n-1}$ which collapse newly introduced circles into points of $O_{n-1}$, where $S_0 = S^2$ and $H_0 = G$. In other words, $\pi_n$ reverts the modification made in step $n$. Clearly, each $\pi_n$ is a continuous onto map and $\pi_n \circ H_{n+1} = H_n \circ \pi_n$.

Embed each $S_n$ in $S^2$ in a natural way, and extend $\pi_n$ to a map $\eta_n : S^2 \to S^2$ in the following way. If $D$ is an open disc bounded by $S_n$ in $S^2$ then we fix any $y \in \partial D \cap S_n$ and define $\eta_n(x) = y$ for every $x \in D$. Since $\pi_n$ collapses the circle $\partial D$ to a point, each $\eta_n$ is an almost homeomorphism; i.e. it can be obtained as a uniform sequence of homeomorphism. Denote by $S_\infty$ the inverse limit of spaces $S_n$ with bonding maps $\pi_n$ and by $Q_\infty$ the inverse limit of spheres $S^2$ with $\eta_n$ as bonding maps; i.e.

$$S_\infty = \{(z_0, z_1, \ldots) : \pi_n(z_n) = z_{n-1}\},$$

$$Q_\infty = \{(z_0, z_1, \ldots) : \eta_n(z_n) = z_{n-1}\}.$$

Since each $\eta_n$ is an almost homeomorphism, a result of Brown [5, Theorem 4] implies that $Q_\infty$ is homeomorphic to $S^2$. Observe that if we fix any $c \in O_n$ then the set $B$ of all inverse sequences in $Q_\infty$ with $c$ on the first coordinate is homeomorphic to a disc. Simply, after dropping $n$ first coordinates we see that $B$ is an inverse limit of a disk $D$ with the identity as a unique bounding map. But $\bigcup_{n=1}^\infty O_n = S^2$, hence $S_\infty$ satisfies conditions (S1)-(S3) and so is a Sierpinski curve. Observe that if we put $H = H_1 \times H_2 \times \ldots \times H_n \times \ldots$ then $H(S_\infty) = S_\infty$, hence $H$ is a homeomorphism of the Sierpiński curve. Let $M = S^2 \setminus \bigcup_{n=1}^\infty O_n$ and $M_\infty = \{z_0, z_1, \ldots \in S_\infty : z_0 \in M\}$ be the set of all inverse sequences in $S_\infty$ with the first coordinate in $M$. It follows directly from the construction that if $z \in M_\infty$ then $z = (x, x, x, \ldots)$ for some $x \in M$ and $H(z) = (G(x), G(x), \ldots)$. Since periodic orbits of $G$ in $M$ are dense in $S^2$, it is not hard to see that $H$ has a dense set of periodic points. The set $M_\infty$ is Borel, so for any Borel set $U \in S_\infty$ we can view $U \cap M_\infty$ as a Borel subset of $M$ (by projection onto the first coordinate) and so we obtain a well defined $H$-invariant Borel probability measure $\nu(U) = \mu(U \cap M_\infty)$. The measure $\mu$ is ergodic, so we have $\mu(S^2 \setminus M_\infty) = 0$, hence also $\nu(S_\infty \setminus M_\infty) = 0$, and so $\nu$ and $\mu$ are isomorphic, in particular $(S_\infty, H, \nu)$ is measure-theoretically conjugate to a Bernoulli shift. Take any open set $U$ in $S_\infty$. We claim that $\nu(U) > 0$. Indeed, the basic open sets in $S_\infty$ are given by $U_\infty = (\eta_1 \circ \ldots \circ \eta_{n-1}(U_i), \ldots, \eta_{n-1}(U_i), U_i, \eta_i^{-1}(U_i), \ldots)$, where $U_i \subseteq S_i$
is open, for some \(i \in \mathbb{N}\) (see e.g. Theorem 3 on p.79 in \[8\]). Since \(S_t\) is a sphere with a finite number of holes, the Lebesgue measure of \(U_t\) in \(S_t\) is positive and 
\[\nu(U_\infty) = \nu(U_\infty \cap M_\infty) = \mu(U_t \cap M),\] therefore \(U_\infty\) has positive product measure. This shows that \(\nu\) has full support, which completes the proof of Theorem \[12, 11\].

It remains to prove (3). Assume on the contrary that \(H\) has the specification property. Since the specification property is preserved under higher iterations, \(H^{p_1}\) has the specification property, where \(p_1\) is the period of \(O_1\). For simplicity of notation replace \(H\) by \(H^{p_1}\) and \(A\) by \(A^{p_1}\). By [14, Theorem 2.1] the specification property implies that for every invariant measure \(\mu \in M_T(S_\infty)\) there exists a sequence of ergodic measures such that \(\mu_n \to \mu\), when \(n \to \infty\), in \(\text{Lévy-Prohorov metric.}\) Since we blew up a hyperbolic periodic point \(c\) in \(O_1\) in the first step of our construction, after passing from \(A\) to the coordinates giving its Jordan form, we have locally a phase portrait (for \(G\) and \(H\)) as on Figure 1. Let us start with the following observation. Consider the hyperbolic linear map \(f(x, y) = (ax, by)\) where \(0 < a < 1 < b\) and \(ab = 1\). Let \(D = [-\varepsilon, \varepsilon]^2\) for some small \(\varepsilon > 0\). Now let \(z = (p, q) \in D\) with \(|p| \geq |q|\). Next assume that the trajectory of \(z\) is not fully contained in \(D\). Then there exists a minimal \(m \geq 1\) such that \(a^m |p| \geq b^m |q|\) and \(a^{m+1}|p| < b^{m+1}|q|\). Observe that \(b^{2m}|q| < \varepsilon\) as otherwise \(\varepsilon \leq b^{2m}|q| = a^{-m}b^m|q| \leq |p|\) and so \((p, q) \notin D\) which is a contradiction. Now, let \(v\) be a compactification of the leaf representing the stable direction for hyperbolic point \(c\) and take a small neighborhood \(U\) of \(v\). Let \(U' = \pi(U)\) where \(\pi\) is the natural factor map \(\pi: (S_\infty, H) \to (\mathbb{S}^2, G).\) If \(U\) is sufficiently small, then \(\pi(U) \subset D\) and furthermore, if \((p, q) \in \pi(U)\) then \(|p| \geq |q|\).

Fix any periodic point \(u \in S_\infty\), say of period \(s\), and consider the invariant measure \(\hat{\mu} = (1 - \alpha)\delta_c + \alpha \sum_{\ell=0}^{s-1} \delta_{H^i(u)}\) with a small \(\alpha\), say \(\alpha < \frac{1}{49}\). Assume also that \(\pi(u) \notin D\). Take \(\delta < 2\varepsilon\) such that \(\text{dist}(c, \{u, H(u), \ldots, H^{s-1}(u)\}) > 3\varepsilon\). Let \(\delta\) be small enough, so that there exists an open set \(c \in V\) such that \(B(V, 2\delta) \subset U\) and \(\pi(B(u, 2\delta)) \cap D = \emptyset\). Denote \(W = B(u, 2\delta)\). There exists an ergodic measure \(\hat{\nu}\) such that \(\rho(\hat{\nu}, \hat{\mu}) < \delta\). This implies that \(\hat{\nu}(U) \geq \hat{\nu}(V^\delta) \geq \hat{\mu}(V) - \delta > 4/5\) and \(\hat{\nu}(W) \geq \hat{\mu}(U^\delta) - \delta > 0\). By the Birkhoff ergodic theorem there exists \(x \in S_\infty\) such that \(\lim_{n \to \infty} \frac{1}{n} \{ j < n : H^j(x) \in U \} = \nu(U)\) and \(\lim_{n \to \infty} \frac{1}{n} \{ j < n : H^j(x) \in W \} = \hat{\nu}(W)\). Since \(\hat{\nu}(W) > 0\) there exists an increasing sequence \(k_i\) such that \(H^{k_i}(x) \in W\). Let us estimate the number of iterations \(k_i \leq j < k_{i+1}\) such that \(H^j(x) \in U\). Observe that if \(\pi(H^{k_i}(x)) \notin D\) and \(\pi(H^{k_{i+1}}(x)) \notin D\) then, by the earlier analysis, we see that no more than half of iterations \(H^j(x)\) for \(j = k_i + 1, \ldots, k_{i+1}\) can visit \(U\). This implies that \(\lim_{i \to \infty} \frac{1}{k_i} |\{ j : H^j(x) \in U \}| \leq 1/2\). By the choice of \(x\) we
obtain that $\hat{\nu}(U) = 1/2 < 4/5$ which is a contradiction. This shows that $(S_\infty, H)$ does not have the specification property, completing the proof.

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