RATE OF CONVERGENCE OF THE KAC-LIKE PARTICLE SYSTEM FOR HARD POTENTIALS

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Abstract. In this talk, we consider the Kac stochastic particle system associated to the spatially homogeneous Boltzmann equation for true hard potentials. We establish a rate of propagation of chaos of the particle system to the unique solution of the Boltzmann equation. We use a probabilistic coupling method and give, under suitable assumptions on the initial condition, a rate of convergence of the empirical measure of the particle system to the solution of the Boltzmann equation for this singular interaction.

1. Introduction and main results

1.1. The Boltzmann equation. We consider a 3-dimensional spatially homogeneous Boltzmann equation, which depicts the density $f(t,v)$ of particles in a gas, moving with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$. The density $f_t(v)$ solves

\begin{equation}
\partial_t f_t(v) = \frac{1}{2} \int_{\mathbb{R}^3} dv_* \int_{S^2} d\sigma B(|v-v_*|, \theta) [f_t(v')f_t(v'_*) - f_t(v)f_t(v*)],
\end{equation}

where

\begin{equation}
v' = v'(v, v_*, \sigma) = \frac{v + v_*}{2} + \frac{|v-v_*|}{2} \sigma, \quad v'_* = v'_*(v, v_*, \sigma) = \frac{v + v_*}{2} - \frac{|v-v_*|}{2} \sigma
\end{equation}

and $\theta$ is the deviation angle defined by $\cos \theta = \frac{(v-v_*)}{|v-v_*|} \cdot \sigma$. The collision kernel $B(|v-v_*|, \theta) \geq 0$ depends on the type of interaction between particles. It only depends on $|v-v_*|$ and on the cosine of the deviation angle $\theta$. Conservation of mass, momentum and kinetic energy hold for reasonable solutions and we may assume without loss of generality that $\int_{\mathbb{R}^3} f_0(v)dv = 1$.

1.2. Assumptions. We will assume that there is a measurable function $\beta : (0, \pi] \to \mathbb{R}_+$ such that

\begin{equation}
B(|v-v_*|, \theta) \sin \theta = |v-v_*|^{\gamma} \beta(\theta), \quad \forall \theta \in [\pi/2, \pi], \quad \beta(\theta) = 0
\end{equation}

with

\begin{equation}
\exists \gamma \in [0, 1], \forall z \geq 0, \quad \Phi(z) = z^\gamma,
\end{equation}

and either

\begin{equation}
\forall \theta \in (0, \pi/2), \quad \beta(\theta) = 1
\end{equation}

or

\begin{equation}
\exists \nu \in (0, 1), \exists 0 < c_0 < c_1, \forall \theta \in (0, \pi/2), \quad c_0 \theta^{-1-\nu} \leq \beta(\theta) \leq c_1 \theta^{-1-\nu},
\end{equation}

The propagation of exponential moments requires the following additional condition

\begin{equation}
\beta(\theta) = b(\cos \theta) \quad \text{with } b \text{ non-decreasing, convex and } C^1 \text{ on } [0, 1).
\end{equation}
We now introduce, for $\theta \in \mathbb{R}$, the set of bounded globally Lipschitz functions $\phi : \mathbb{R}^3 \to \mathbb{R}$. When $f \in \mathcal{P}(\mathbb{R}^3)$ has a density, we also denote this density by $f$. For $q > 0$, we set

$$\mathcal{P}_q(\mathbb{R}^3) = \{ f \in \mathcal{P}(\mathbb{R}^3) : m_q(f) < \infty \} \quad \text{with} \quad m_q(f) := \int_{\mathbb{R}^3} |v|^q f(dv).$$

We now introduce, for $\theta \in (0, \pi/2)$ and $z \in [0, \infty)$,

$$H(\theta) = \int_0^{\pi/2} \beta(x)dx \quad \text{and} \quad G(z) = H^{-1}(z).$$

Under (1.6), it is clear that $H$ is a continuous decreasing function valued in $[0, \infty)$, so it has an inverse function $G : [0, \infty) \to (0, \pi/2)$ defined by $G(H(\theta)) = \theta$ and $H(G(z)) = z$. Furthermore, it is easy to verify that there exist some constants $0 < c_2 < c_3$ such that for all $z > 0$,

$$c_2(1 + z)^{-1/\nu} \leq G(z) \leq c_3(1 + z)^{-1/\nu},$$

and we know from [21, Lemma 1.1] that there exists a constant $c_4 > 0$ such that for all $x, y \in \mathbb{R}_+$,

$$\int_0^\infty (G(z/x) - G(z/y))^2dz \leq c_4 \frac{(x-y)^2}{x+y}.$$

Let us now introduce the Wasserstein distance with quadratic cost on $\mathcal{P}_2(\mathbb{R}^3)$. For $g, \tilde{g} \in \mathcal{P}_2(\mathbb{R}^3)$, let $\mathcal{H}(g, \tilde{g})$ be the set of probability measures on $\mathbb{R}^3 \times \mathbb{R}^3$ with first marginal $g$ and second marginal $\tilde{g}$. We then set

$$W_2(g, \tilde{g}) = \inf \left\{ \left( \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - \tilde{v}|^2 R(dv, d\tilde{v}) \right)^{1/2}, R \in \mathcal{H}(g, \tilde{g}) \right\}.$$

For more details on this distance, one can see [43, Chapter 2].

1.4. Weak solutions. We now introduce a suitable spherical parameterization of $S^2$ as in [24]. For each $X \in \mathbb{R}^3$, we consider vectors $I(X), J(X) \in \mathbb{R}^3$ such that $(\frac{X}{|X|}, \frac{I(X)}{|I(X)|}, \frac{J(X)}{|J(X)|})$ is a direct orthonormal basis of $\mathbb{R}^3$. Then for $X, v, v_* \in \mathbb{R}^3$, for $\theta \in (0, \pi/2)$ and $\varphi \in [0, 2\pi)$, we set

$$\Gamma(X, \varphi) := (\cos \varphi)I(X) + (\sin \varphi)J(X),$$

$$a(v, v_*, \theta, \varphi) := -\frac{1 - \cos \theta}{2} (v - v_*) + \frac{\sin \theta}{2} \Gamma(v - v_*),$$

$$v'(v, v_*, \theta, \varphi) := v + a(v, v_*, \theta, \varphi),$$

then we write $\sigma \in S^2$ as $\sigma = \frac{v - v_*}{|v - v_*|} \cos \theta + \frac{I(v - v_*)}{|I(v - v_*)|} \sin \theta \cos \varphi + \frac{J(v - v_*)}{|J(v - v_*)|} \sin \theta \sin \varphi$, and observe at once that $\Gamma(X, \varphi)$ is orthogonal to $X$ and has the same norm as $X$, from which it is easy to check that

$$|a(v, v_*, \theta, \varphi)| = \sqrt{\frac{1 - \cos \theta}{2}} |v - v_*|.$$

Let us define, classically, weak solutions to (1.1).
Definition 1.1. Assume (1.3), (1.4) and (1.5) or (1.6). A family \((f_t)_{t \geq 0} \in C([0, \infty), \mathcal{P}_2(\mathbb{R}^3))\) is called a weak solution to (1.1) if it preserves momentum and energy, i.e.

\[
\forall t \geq 0, \quad \int_{\mathbb{R}^3} v f_t (dv) = \int_{\mathbb{R}^3} v f_0 (dv) \quad \text{and} \quad \int_{\mathbb{R}^3} \lvert v \rvert^2 f_t (dv) = \int_{\mathbb{R}^3} \lvert v \rvert^2 f_0 (dv)
\]

and if for any \(\phi : \mathbb{R}^3 \to \mathbb{R}\) bounded and Lipschitz-continuous, any \(t \in [0, T]\),

\[
\int_{\mathbb{R}^3} \phi(v) f_t (dv) = \int_{\mathbb{R}^3} \phi(v) f_0 (dv) + \int_0^t \int_{\mathbb{R}^3} A \phi(v, v_*) f_s (dv_*) f_s (dv) ds
\]

where

\[
A \phi(v, v_*) = |v - v_*|^\gamma \int_0^{\pi/2} \beta(\theta) d\theta \int_0^{2\pi} d\varphi \left[ \phi(v + a(v, v_*, \theta, \varphi)) - \phi(v) \right].
\]

Noting that \(\lvert a(v, v_*, \theta, \varphi) \rvert \leq C \theta \lvert v - v_* \rvert\) and that \(\int_0^{\pi/2} \beta(\theta) d\theta\), we easily get \(\lvert A \phi(v, v_*) \rvert \leq C_\phi \lvert v - v_* \rvert^{1+\gamma} \leq C_\phi (1 + \lvert v - v_* \rvert^2)\), so that everything makes sense in (1.14).

We next rewrite the collision operator in a way that makes disappear the velocity-dependence \(\lvert v - v_* \rvert^\gamma\) in the rate. Such a trick was already used in [24] and [21].

1.5. The particle system. Let us now recall the Kac particle system introduced by [30]. It is the \((\mathbb{R}^3)^N\)-valued Markov process with infinitesimal generator with infinitesimal generator \(\mathcal{L}_N\) defined as follows: for any bounded Lipschitz function \(\phi : (\mathbb{R}^3)^N \to \mathbb{R}\) sufficiently regular and \(v = (v_1, \ldots, v_N) \in (\mathbb{R}^3)^N\), by

\[
\mathcal{L}_N \phi(v) = \frac{1}{2(N-1)} \sum_{i \neq j} \int_{S^2} [\phi(v + (v'(v_i, v_j, \sigma) - v_i) e_i + (v'(v_i, v_j, \sigma) - v_j) e_j) - \phi(v)] B(|v_i - v_j|, \sigma) d\sigma.
\]

where \(ve_i = (0, \ldots, 0, i, 0, \ldots, 0) \in (\mathbb{R}^3)^N\) with \(v\) at the \(i\)-th place for \(v \in \mathbb{R}^3\).

In other words, the system contains \(N\) particles with velocities \(v = (v_1, \ldots, v_N)\). Each pair of particles (with velocities \((v_i, v_j)\)), interact, for each \(\sigma \in S^2\), at rate \(B(|v_i - v_j|, \sigma)/N\). Then one changes the velocity \(v_i\) to \(v'(v_i, v_j, \sigma)\) given by (1.5) and \(v_j\) changes to \(v'(v_i, v_j, \sigma)\).

Let us now classically rewrite the collision operator by making disappear the velocity-dependence \(\lvert v - v_* \rvert^\gamma\) in the rate using a substitution.

Lemma 1.2. Assume (1.3), (1.4) and (1.5) or (1.6). Recalling (1.8) and (1.11), define, for \(z \in (0, \infty), \varphi \in [0, 2\pi)\), \(v, v_* \in \mathbb{R}^3\) and \(K \in [1, \infty)\),

\[
c(v, v_*, z, \varphi) := a[v, v_*, G(z/|v - v_*|^\gamma), \varphi] \quad \text{and} \quad c_K(v, v_*, z, \varphi) := c(v, v_*, z, \varphi) 1_{\{z \leq K\}}.
\]

For any bounded Lipschitz \(\phi : \mathbb{R}^3 \to \mathbb{R}\), any \(v, v_* \in \mathbb{R}^3\)

\[
A \phi(v, v_*) = \int_0^\infty dz \int_0^{2\pi} d\varphi \left( \phi[v + c_K(v, v_*, z, \varphi)] - \phi[v] \right).
\]

The fact that \(\int_0^\pi \beta(\theta) d\theta = \infty\) (i.e. \(\beta\) is non cutoff) means that there are infinitely many jumps with a very small deviation angle. It is thus impossible to simulate it directly. For this reason, we will study a truncated version of Kac’s particle system

\[
\mathcal{L}_{N,K} \phi(v) = \frac{1}{2(N-1)} \sum_{i \neq j} \int_0^\infty dz \int_0^{2\pi} d\varphi \left[ \phi(v + c_K(v_i, v_j, z, \varphi)e_i + c_K(v_j, v_i, z, \varphi)e_j) - \phi(v) \right].
\]
Proof. To get (1.17), start from (1.15) and use the substitution \( \theta = G(z/|v - v_*|) \) or equivalently \( H(\theta) = z/|v - v_*|^\gamma \), which implies \( |v - v_*|^\gamma \beta(\theta) d\theta = dz \). The expressions (1.18) are checked similarly.

\[ \square \]

1.6. Well-posedness. Let \( \mathcal{P}_k(\mathbb{R}^3) \) be the set of all probability measures \( f \) on \( \mathbb{R}^3 \) such that \( \int_{\mathbb{R}^3} |v|^k f(dv) < \infty \). We first recall known well-posedness results for the Boltzmann equation, as well as some properties of solutions we will need. A precise definition of weak solutions is stated in the next section.

**Theorem 1.3.** Assume \((1.5), (1.6), (1.7)\) or \((1.8)\). Let \( f_0 \in \mathcal{P}_2(\mathbb{R}^3) \).

For \( \gamma \in (0, 1], \) assume additionally \((1.7)\) and that

\[ (1.19) \quad \exists \ p \in (\gamma, 2), \ \int_{\mathbb{R}^3} e^{\gamma \|v\|^p} f_0(dv) < \infty. \]

There is a unique weak solution \((f_t)_{t \geq 0} \in C([0, \infty), \mathcal{P}_2(\mathbb{R}^3))\) to (1.1) such that

\[ (1.20) \quad \forall \ q \in (0, p), \ \sup_{[0, \infty)} \int_{\mathbb{R}^3} e^{\gamma \|v\|^q} f_t(dv) < \infty. \]

Concerning well-posedness, see [26, 16] for hard potentials and [5, 37, 31, 17, 32] for hard spheres. The propagation of exponential moments for hard potentials and hard spheres, initiated by Bobylev [7], is checked in [26, 32]. Finally, the existence of a density for \( f_1 \) has been proved in [19] (under (1.3) and when \( f_0 \) is not a Dirac mass and belongs to \( \mathcal{P}_4(\mathbb{R}^3) \), in [57] (under (1.5) when \( f_0 \) has a density) and is very classical by monotonicity of the entropy when \( f_0 \) has a finite entropy, see e.g. Arkeryd [4].

**Proposition 1.4.** Assume \((1.5), (1.6)\) and \((1.7)\) or \((1.8)\). Let \( f_0 \in \mathcal{P}_2(\mathbb{R}^3) \) and a number of particles \( N \geq 1 \) be fixed. Let \((V_0^i)_{i=1,...,N} \) be i.i.d. with common law \( f_0 \).

(i) For each cutoff parameter \( K \in [1, \infty) \), there exists a unique (in law) Markov process \((V_{i,N,K}^i)_{i=1,...,N,t \geq 0} \) with values in \((\mathbb{R}^3)^N\), starting from \((V_0^i)_{i=1,...,N} \) and with generator \( \mathcal{L}_{N,K} \) defined, for all bounded measurable \( \phi : (\mathbb{R}^3)^N \rightarrow \mathbb{R} \) and any \( v = (v_1, \ldots, v_N) \in \mathbb{R}^3 \), by

\[
\mathcal{L}_{N,K} \phi(v) = \frac{1}{2(N-1)} \sum_{i \neq j} \int_{\mathbb{R}^2} [\phi(v) + (v_i(v_i, v_j, \sigma) - v_i) e_i + (v_j(v_i, v_j, \sigma) - v_j) e_j] - \phi(v)]
\]

\[ B(|v_i - v_j|, \theta) \mathbb{1}_{\{\theta \geq G(K/|v_i - v_j|)\}} d\sigma, \]

with \( G \) defined by (1.3) and, for \( h \in \mathbb{R}^3 \), \( h e_i = (0, \ldots, h, \ldots, 0) \in (\mathbb{R}^3)^N \) with \( h \) at the \( i \)-th place.

(ii) There exists a unique (in law) Markov process \((V_{i,\infty}^i)_{i=1,...,N,t \geq 0} \) with values in \((\mathbb{R}^3)^N\), starting from \((V_0^i)_{i=1,...,N} \) and with generator \( \mathcal{L}_N \) defined, for all Lipschitz bounded function \( \phi : (\mathbb{R}^3)^N \rightarrow \mathbb{R} \) and any \( v = (v_1, \ldots, v_N) \in \mathbb{R}^3 \).

The generator \( \mathcal{L}_{N,K} \) uniquely defines a strong Markov process with values in \((\mathbb{R}^3)^N\). This comes from the fact that the corresponding jump rate is finite and constant: for any configuration \( v = (v_1, \ldots, v_N) \in (\mathbb{R}^3)^N \), it holds that \( N^{-1} \sum_{i \neq j} \int_{\mathbb{R}^2} B(|v_i - v_j|, \theta) \mathbb{1}_{\{\theta \geq G(K/|v_i - v_j|)\}} d\sigma = 2\pi(N-1)K \).

Indeed, for any \( z \in [0, \infty) \), we have \( \int_{\mathbb{R}^2} B(x, \theta) \mathbb{1}_{\{\theta \geq G(K/|x'|)\}} d\sigma = 2\pi K \), which is easily checked recalling that \( B(x, \theta) = x^\gamma \beta(\theta) \) and the definition of \( G \).
1.7. Main result. Our study concerns both the particle systems with and without cutoff. It is worth to notice that for true Hard spheres molecules and hard potentials, \( \nu \in (0, 1/2) \) so that \( 1 - 2/\nu \leq -3 \) and the contribution of the cut-off approximation vanishes rapidly in the limit \( K \to \infty \).

**Theorem 1.5.** Let \( B \) be a collision kernel satisfying (1.3), (1.4) and (1.5) or (1.9) and let \( f_0 \in \mathcal{P}_2(\mathbb{R}^3) \) not be a Dirac mass. If \( \gamma > 0 \), assume additionally (1.7) and (1.19). Consider the unique weak solution \((f_t)_{t \geq 0}\) to (1.1) defined in Theorem 1.3 and, for each \( N \geq 1, K \in [1, \infty] \), the unique Markov process \((V_t^{i,N,K})_{i=1,\ldots,N,t \geq 0}\) defined in Proposition 1.4. Let \( \mu_t^{N,K} := N^{-1} \sum_i \delta_{V_t^{i,N,K}}. \)

(i) Hard potentials. Assume that \( \gamma \in (0, 1) \) and (1.7). For all \( \varepsilon \in (0, 1) \), all \( T \geq 0 \), there is \( C_{\varepsilon,T} \) such that for all \( N \geq 1, K \in [1, \infty] \),

\[
\sup_{[0,T]} \mathbb{E}[W_2^2(\mu_t^{N,K}, f_t)] \leq C_{\varepsilon,T}(N^{-\frac{1}{2}} + K^{1-2/\nu})^{1-\varepsilon}.
\]

(ii) Hard spheres. Assume finally that \( \gamma = 1 \), (1.5) and that \( f_0 \) has a density. For all \( \varepsilon \in (0, 1) \), all \( T \geq 0 \), all \( q \in (1, p) \), there is \( C_{\varepsilon,q,T} \) such that for all \( N \geq 1, K \in [1, \infty] \),

\[
\sup_{[0,T]} \mathbb{E}[W_2^2(\mu_t^{N,K}, f_t)] \leq C_{\varepsilon,q,T}(N^{-1/3+\varepsilon} + e^{-K^q})e^{C_{\varepsilon,q,T}K}.
\]

We excluded the case where \( f_0 \) is a Dirac mass because we need that \( f_t \) has a density and because if \( f_0 = \delta_{v_0} \), then the unique solution to (1.1) is given by \( f_t = \delta_{v_0} \) and the Markov process of Proposition 1.4 is nothing but \( V_t^{i,N,K} = (v_0, \ldots, v_0) \) (for any value of \( K \in [1, \infty] \)), so that \( \mu_t^{N,K} = \delta_{v_0} \) and thus \( W_2(f_t, \mu_t^{N,K}) = 0 \).

2. Main computations of the paper

2.1. Accurate version of Tanaka’s trick. As was already noted by Tanaka [11], it is not possible to choose \( I \) in such a way that \( X \to I(X) \) is continuous. However, he found a way to overcome this difficulty, see also [23] Lemma 2.6. Here we need the following accurate version of Tanaka’s trick.

**Lemma 2.1.** Recall (1.11). There are some measurable functions \( \varphi_0, \varphi_1 : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, 2\pi) \), such that for all \( X, Y \in \mathbb{R}^3 \), all \( \varphi \in [0, 2\pi) \),

\[
\Gamma(X, \varphi) \cdot \Gamma(Y, \varphi + \varphi_0(X, Y)) = X \cdot Y \cos^2(\varphi + \varphi_1(X, Y)) + |X||Y| \sin^2(\varphi + \varphi_1(X, Y)),
\]

\[
|\Gamma(X, \varphi) - \Gamma(Y, \varphi + \varphi_0)|^2 \leq |\Gamma(X, \varphi)|^2 + |\Gamma(Y, \varphi + \varphi_0)|^2 - 2\Gamma(X, \varphi) \cdot \Gamma(Y, \varphi + \varphi_0)
\]

\[
= |X|^2 + |Y|^2 - 2(X \cdot Y \cos^2(\varphi + \varphi_1) + |X||Y| \sin^2(\varphi + \varphi_1))
\]

\[
\leq |X|^2 + |Y|^2 - 2(X \cdot Y = |X - Y|^2.
\]

We next check the first claim. Let thus \( X \) and \( Y \) be fixed. Observe that \( \Gamma(X, \varphi) \) goes (at constant speed) all over the circle \( C_X \) with radius \( |X| \) lying in the plane orthogonal to \( X \). Let \( i_X \in C_X \) and \( i_Y \in C_Y \) such that \( X, Y, i_X, i_Y \) belong to the same plane and \( i_X \cdot i_Y = X \cdot Y \) (there are exactly two possible choices for the couple \((i_X, i_Y)\) if \( X \) and \( Y \) are not collinear, infinitely many otherwise).

Consider \( \varphi_X \) and \( \varphi_Y \) such that \( i_X := \Gamma(X, \varphi_X) \) and \( i_Y := \Gamma(Y, \varphi_Y) \). Define \( j_X := \Gamma(X, \varphi_X + \pi/2) \) and \( j_Y := \Gamma(Y, \varphi_Y + \pi/2) \). Then \( j_X \) and \( j_Y \) are collinear (because both are orthogonal to the plane containing \( X, Y, i_X, i_Y \), satisfy \( j_X \cdot j_Y = |j_X||j_Y| = |X||Y| \) and \( i_X \cdot j_Y = i_Y \cdot j_X = 0 \). Next, observe that \( \Gamma(X, \varphi + \varphi_X) = i_X \cos \varphi + j_X \sin \varphi \) while \( \Gamma(Y, \varphi + \varphi_Y) = i_Y \cos \varphi + j_Y \sin \varphi \). Consequently,
\[ 
\Gamma(X, \varphi + \varphi_X) \cdot \Gamma(Y, \varphi + \varphi_Y) = i_X \cdot i_Y \cos^2 \varphi + j_X \cdot j_Y \sin^2 \varphi = X \cdot Y \cos^2 \varphi + |X||Y| \sin^2 \varphi. 
\]

The conclusion follows: choose \( \varphi_0 := \varphi_Y - \varphi_X \) and \( \varphi_1 := -\varphi_X \) (all this modulo \( 2\pi \)). \qed

The following estimate is our central argument.

**Lemma 2.2.** Recall that \( G \) was defined in \( [1.10] \) and that the deviation functions \( c \) and \( c_K \) were defined in \( [1.16] \). For any \( v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^3 \), any \( K \in [1, \infty) \),

\[
\int_0^\infty \int_0^{2\pi} \left( |v + c(v, v_*, z, \varphi) - \tilde{v} - c_K(\tilde{v}, \tilde{v}_*, z, \varphi + \varphi_0(v - v_*, \tilde{v} - \tilde{v}_*))|^2 - |v - \tilde{v}|^2 \right) d\varphi dz \\
\leq A_1^K(v, v_*, \tilde{v}, \tilde{v}_*) + A_2^K(v, v_*, \tilde{v}, \tilde{v}_*) + A_3^K(v, v_*, \tilde{v}, \tilde{v}_*),
\]

where, setting \( \Phi_K(x) = \pi \int_0^K (1 - \cos G(z/x^\gamma)) dz, \)

\[
A_1^K(v, v_*, \tilde{v}, \tilde{v}_*) = 2|v - v_*|^2 |\tilde{v} - \tilde{v}_*| \int_0^K \left[ G(z/|v - v_*|) - G(z/|\tilde{v} - \tilde{v}_*|) \right]^2 dz,
\]

\[
A_2^K(v, v_*, \tilde{v}, \tilde{v}_*) = - \left[ (v - \tilde{v}) + (v_* - \tilde{v}_*) \right] \cdot [G(v - v_*) - (v - \tilde{v}_*)] \Phi_K(|v - v_*|),
\]

\[
A_3^K(v, v_*, \tilde{v}, \tilde{v}_*) = |(v - v_*)^2 + 2(v - \tilde{v})(v_* - \tilde{v}_*)| \Phi_K(|v - v_*|).
\]

**Proof.** We need to shorten notation. We write \( x = |v - v_*|, \tilde{x} = |\tilde{v} - \tilde{v}_*|, \varphi_0 = \varphi_0(v - v_*, \tilde{v} - \tilde{v}_*), \)

\( c = c(v, v_*, z, \varphi), \tilde{c} = \tilde{c}(\tilde{v}, \tilde{v}_*, z, \varphi + \varphi_0) \) and \( \tilde{c}_K = c_K(\tilde{v}, \tilde{v}_*, z, \varphi + \varphi_0) = \tilde{c}_K \{ z \leq K \}. \) We start with

\[
\Delta_K := \int_0^\infty \int_0^{2\pi} \left( |v + c - \tilde{c} - \tilde{c}_K|^2 - |v - \tilde{v}|^2 \right) d\varphi dz \\
= \int_0^K \int_0^{2\pi} \left( |c|^2 + |\tilde{c}|^2 - 2c \cdot \tilde{c} + 2(v - \tilde{v}) \cdot (c - \tilde{c}) \right) d\varphi dz.
\]

First, it holds that \( |c|^2 = | - (1 - \cos G(z/x^\gamma))(v - v_*) + (\sin G(z/x^\gamma)) \Gamma(v - v_*, \varphi)|^2/4 = (1 - \cos G(z/x^\gamma))|v - v_*|^2/2. \) We used that by definition, see \( [1.11] \). \( \Gamma(v - v_*, \varphi) \) has the same norm as \( v - v_* \) and is orthogonal to \( v - v_* \) and that \( (1 - \cos \theta)^2 + (\sin \theta)^2 = 2 - 2 \cos \theta. \) Consequently, we have

\[
\int_0^K \int_0^{2\pi} |c|^2 d\varphi dz = \pi |v - v_*|^2 \int_0^K (1 - \cos G(z/x^\gamma)) dz = x^2 \Phi_K(x).
\]

Similarly, we also have \( \int_0^K \int_0^{2\pi} |\tilde{c}|^2 d\varphi dz = \tilde{x}^2 \Phi_K(\tilde{x}). \)

Next, using that \( c = -(1 - \cos G(z/x^\gamma))(v - v_*)/2 + (\sin G(z/x^\gamma)) \Gamma(v - v_*, \varphi)/2 \) and that \( \int_0^{2\pi} \Gamma(v - v_*, \varphi)d\varphi = 0, \)

\[
\int_0^K \int_0^{2\pi} c d\varphi dz = -(v - v_*) \pi \int_0^K (1 - \cos G(z/x^\gamma)) dz = -(v - v_*) \Phi_K(x).
\]

By the same way, \( \int_0^K \int_0^{2\pi} \tilde{c} d\varphi dz = -(\tilde{v} - \tilde{v}_*) \Phi_K(\tilde{x}). \)

Finally, \( c \cdot \tilde{c} = [(1 - \cos G(z/x^\gamma))(v - v_*) - (\sin G(z/x^\gamma)) \Gamma(v - v_*, \varphi)] \cdot [(1 - \cos G(z/x^\gamma))(\tilde{v} - \tilde{v}_*) - (\sin G(z/x^\gamma)) \Gamma(\tilde{v} - \tilde{v}_*, \varphi + \varphi_0)]/4. \) Since \( \int_0^{2\pi} \Gamma(v - v_*, \varphi)d\varphi = \int_0^{2\pi} \Gamma(\tilde{v} - \tilde{v}_*, \varphi + \varphi_0)d\varphi = 0, \)

we get

\[
\int_0^{2\pi} c \cdot \tilde{c} d\varphi = \frac{\pi}{2} \left( (1 - \cos G(z/x^\gamma))(1 - \cos G(z/x^\gamma))(v - v_*) \cdot (\tilde{v} - \tilde{v}_*) \right) \\
+ \frac{1}{4} (\sin G(z/x^\gamma)) (\sin G(z/x^\gamma)) \int_0^{2\pi} \Gamma(v - v_*, \varphi) \cdot \Gamma(\tilde{v} - \tilde{v}_*, \varphi + \varphi_0)d\varphi.
\]
Recalling Lemma 2.1 and using that $\int_0^{2\pi} \cos^2(\varphi + \varphi_1)d\varphi = \int_0^{2\pi} \sin^2(\varphi + \varphi_1)d\varphi = \pi$, we obtain

$$\int_0^{2\pi} c \cdot \hat{c} d\varphi = \frac{\pi}{2}(1 - \cos G(z/x^\gamma))(1 - \cos G(z/x^\gamma))(v - v_\ast) \cdot (\hat{v} - \hat{v}_\ast)$$

$$+ \frac{\pi}{4}(\sin G(z/x^\gamma))(\sin G(z/x^\gamma))[(v - v_\ast) \cdot (\hat{v} - \hat{v}_\ast) + |v - v_\ast||\hat{v} - \hat{v}_\ast|].$$

But $G$ takes values in $(0, \pi/2)$, so that, since $|v - v_\ast||\hat{v} - \hat{v}_\ast| \geq (v - v_\ast) \cdot (\hat{v} - \hat{v}_\ast)$,

$$\int_0^{2\pi} c \cdot \hat{c} d\varphi$$

$$\geq \frac{\pi}{2}(1 - \cos G(z/x^\gamma))(1 - \cos G(z/x^\gamma)) + (\sin G(z/x^\gamma))(\sin G(z/x^\gamma))(v - v_\ast) \cdot (\hat{v} - \hat{v}_\ast)$$

$$- \frac{\pi}{2}(1 - \cos G(z/x^\gamma) - G(z/x^\gamma))(v - v_\ast) \cdot (\hat{v} - \hat{v}_\ast).$$

Using that $\pi(1 - \cos \theta) \leq 2\theta^2$, we thus get

$$\int_0^K \int_0^{2\pi} c \cdot \hat{c} d\varphi dz \geq (v - v_\ast) \cdot (\hat{v} - \hat{v}_\ast) \frac{\Phi_K(x) + \Phi_K(\hat{x})}{2} - x\hat{x} \int_0^K (G(z/x^\gamma) - G(z/x^\gamma))^2 dz.$$

All in all, we find

$$\Delta_K \leq x^2 \Phi_K(x) + \hat{x}^2 \Phi_K(\hat{x}) - (v - v_\ast) \cdot (\hat{v} - \hat{v}_\ast)[\Phi_K(x) + \Phi_K(\hat{x})]$$

$$+ 2(v - \hat{v}) \cdot ((\hat{v} - \hat{v}_\ast)\Phi_K(\hat{x}) - (v - v_\ast)\Phi_K(x))$$

$$+ 2x\hat{x} \int_0^K (G(z/x^\gamma) - G(z/x^\gamma))^2 dz.$$

Recalling that $x = |v - v_\ast|$, $\hat{x} = |\hat{v} - \hat{v}_\ast|$, we realize that the third line is nothing but $A^K_1(v, v_\ast, \hat{v}, \hat{v}_\ast)$ while the fourth one is bounded from above by $A^K_3(v, v_\ast, \hat{v}, \hat{v}_\ast)$. To conclude, it suffices to note that the sum of the terms on the two first lines equals

$$= (v - v_\ast) \cdot [(v - v_\ast) - (\hat{v} - \hat{v}_\ast) - 2(v - \hat{v})]\Phi_K(x)$$

$$+ (v - v_\ast) \cdot [(\hat{v} - \hat{v}_\ast) - (v - v_\ast) + 2(v - \hat{v})]\Phi_K(\hat{x})$$

$$= - (v - v_\ast) \cdot ((v - \hat{v}) + (v - v_\ast))\Phi_K(x) + (\hat{v} - \hat{v}_\ast) \cdot ((v - \hat{v}) + (v - v_\ast))\Phi_K(\hat{x})$$

which is $A^K_2(v, v_\ast, \hat{v}, \hat{v}_\ast)$ as desired. \[\square\]

Next, we study each term found in the previous inequality.

The case of hard potentials is much more complicated. The following result gives a possible and useful upper bound on the $A^K_3$ functions.

**Lemma 2.3.** Assume the notation of (1.3) with $\gamma \in (0, 1)$, and adopt the notation of Lemma 2.2.

(i) For all $q > 0$, there is $C_q > 0$ such that for all $M \geq 1$, all $K \in [1, \infty)$, all $v, v_\ast, \hat{v}, \hat{v}_\ast \in \mathbb{R}^3$,

$$A^K_1(v, v_\ast, \hat{v}, \hat{v}_\ast) \leq M(|v - \hat{v}|^2 + |v_\ast - \hat{v}_\ast|^2) + C_q e^{-M^{\gamma/\gamma}} e^{C_q(|v|^q + |v_\ast|^q)}.$$

(ii) There is $C > 0$ such that for all $K \in [1, \infty)$, all $v, v_\ast, \hat{v}, \hat{v}_\ast \in \mathbb{R}^3$ and all $z_\ast \in \mathbb{R}^3$,

$$A^K_2(v, v_\ast, \hat{v}, \hat{v}_\ast) - A^K_2(v, z_\ast, \hat{v}, \hat{v}_\ast) \leq C \left[|v - \hat{v}|^2 + |v_\ast - \hat{v}_\ast|^2 + |v_\ast - z_\ast|^2(1 + |v| + |v_\ast| + |z_\ast|)^{2\gamma/(1 - \gamma)} \right].$$
(iii) There is $C > 0$ such that for all $K \in [1, \infty)$, all $v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^3$,

$$A^K_3 (v, v_*, \tilde{v}, \tilde{v}_*) \leq C (1 + |v|^{4\gamma/\nu + 2} + |v_*|^{4\gamma/\nu + 2} + |\tilde{v}|^2 + |\tilde{v}_*|^2) K^{1 - 2/\nu}.$$  

Proof. Using (1.11) and that $|x^\gamma - y^\gamma| \leq 2|x - y|/(x^{1 - \gamma} + y^{1 - \gamma})$, we get

$$A^K_3 (v, v_*, \tilde{v}, \tilde{v}_*) \leq 2c_4 |v - v_*| |\tilde{v} - \tilde{v}_*| \frac{|(v - v_*)^\gamma - |\tilde{v} - \tilde{v}_*|^\gamma|}{|v - v_*|^{\gamma} + |\tilde{v} - \tilde{v}_*|^{\gamma}} \leq 8c_4 (|v - v_*| \wedge |\tilde{v} - \tilde{v}_*|)^{1 - \gamma} (|v - v_*| - |\tilde{v} - \tilde{v}_*|)^2.$$  

Now for any $M \geq 1$, this is bounded from above by

$$\frac{M}{2} (|v - v_*| - |\tilde{v} - \tilde{v}_*|)^2 + 8c_4 (|v - v_*| \vee |\tilde{v} - \tilde{v}_*|)^{2 + \gamma} \frac{M}{8c_4} (\frac{|v - v_*| \wedge |\tilde{v} - \tilde{v}_*|}{|v - v_*|^{\gamma} + |\tilde{v} - \tilde{v}_*|^{\gamma}} \geq M^{1 - 2/\nu})$$  

$$\leq \frac{M}{2} (|v - \tilde{v}| + |v_* - \tilde{v}_*|)^2 + 8c_4 \left[16c_4 |v - v_*| \wedge |\tilde{v} - \tilde{v}_*|\right]^{\gamma + 1} \frac{M}{8c_4} (\frac{|v - v_*| \wedge |\tilde{v} - \tilde{v}_*|}{|v - v_*|^{\gamma} + |\tilde{v} - \tilde{v}_*|^{\gamma}} \geq M^{1 - 2/\nu})$$  

$$\leq M (|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) + 8c_4 (16c_4 |v - v_*| \wedge |\tilde{v} - \tilde{v}_*|)^{\gamma + 1} \frac{M}{8c_4} (\frac{|v - v_*| \wedge |\tilde{v} - \tilde{v}_*|}{|v - v_*|^{\gamma} + |\tilde{v} - \tilde{v}_*|^{\gamma}} \geq M^{1 - 2/\nu})$$  

Fix now $q > 0$ and observe that

$$x^{\gamma + \frac{\nu}{\gamma - \nu}} \frac{M}{8c_4} \leq e^{\frac{\gamma}{\gamma - \nu}} e^{(16c_4)^{\gamma/\nu} x^q} \leq C_0 e^{-M^{q/\nu}} e^{2(16c_4)^{\gamma/\nu} x^q}.$$  

Point (i) follows.

Point (ii) is quite delicate. First, there is $C$ such that for all $K \in [1, \infty)$, all $x, y > 0$,

$$|X \Phi_K(x) - Y \Phi_K(y)| \leq C |x^\gamma - y^\gamma|$$

Indeed, it is enough to prove that for $\Gamma_K(x) = \int_0^K (1 - \cos G(z/x)) dz$, $\Gamma_K(0) = 0$ and $|\Gamma_K''(x)| \leq C$. But $\Gamma_K(x) = x \int_0^{K/x} (1 - \cos G(z)) dz \leq x \int_0^{K/x} G^2(z) dz$, so that $\Gamma_K(0) = 0$ and $|\Gamma_K''(x)| \leq \int_0^\infty (1 - \cos G(z)) dz + x(K/x)^2 (1 - \cos G(K/x)) \leq \int_0^\infty G^2(z) dz + (K/x) G^2(K/x)$, which is uniformly bounded by $M$. Consequently, for all $X, Y \in \mathbb{R}^3$,

$$|X \Phi_K(\{X\}) - Y \Phi_K(\{Y\})| \leq C |X - Y|(|X^\gamma + |Y^\gamma|) + C(|X| + |Y|)|X^\gamma - |Y^\gamma|.$$  

Using again that $|x^\gamma - y^\gamma| \leq 2|x - y|/(x^{1 - \gamma} + y^{1 - \gamma})$, we easily conclude that

$$|X \Phi_K(\{X\}) - Y \Phi_K(\{Y\})| \leq C |X - Y|(|X^\gamma + |Y^\gamma|).$$  

Now we write

$$\Delta^K_3 := A^K_3 (v, v_*, \tilde{v}, \tilde{v}_*) - A^K_3 (z_*, \tilde{v}, \tilde{v}_*)$$

$$= - [(v - \tilde{v}) + (v_* - \tilde{v}_*)] \cdot [(v - v_*) \Phi_K(\{v - v_*\}) - (\tilde{v} - \tilde{v}_*) \Phi_K(\{\tilde{v} - \tilde{v}_*\})]$$

$$+ [(v - \tilde{v}) + (z_* - \tilde{v}_*)] \cdot [(v - z_*) \Phi_K(\{v - z_*\}) - (\tilde{v} - \tilde{v}_*) \Phi_K(\{\tilde{v} - \tilde{v}_*\})]$$

$$= [(v - \tilde{v}) + (v_* - \tilde{v}_*)] \cdot [(v - v_*) \Phi_K(\{v - v_*\}) - (v - z_*) \Phi_K(\{v - z_*\})]$$

$$+ (z_* - v_*) \cdot [(v - z_*) \Phi_K(\{v - z_*\}) - (\tilde{v} - \tilde{v}_*) \Phi_K(\{\tilde{v} - \tilde{v}_*\})].$$  

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The first term is clearly bounded by $C(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2 + |v_* - z_*|^2(1 + |v| + |v_*| + |z_*|)^2\gamma)$ which fits the statement, since $2\gamma \leq 2\gamma/(1 - \gamma)$. We next bound the second term by

$$
C|z_* - v_*|^2(1 + |v| + |z_*| + |v_*|)^\gamma \\
+ C|z_* - v_*|^2(|v - \tilde{v}| + |v_* - \tilde{v}_*|)^\gamma \\
+ C|z_* - v_*||v - \tilde{v}| + |v_* - \tilde{v}_*|(|v - z_*| + |v - v_*|)^\gamma \\
+ C|z_* - v_*|(|v - \tilde{v}| + |v_* - \tilde{v}_*|)^{1+\gamma},
$$

Using that $x^2 y^\gamma \leq x^{4/(2 - \gamma)} + y^2$ (for the second line), that $xyz \leq (xz\gamma)^2 + y^2$ (for the third line) and that $xy^{1+\gamma} \leq x^{2/(1-\gamma)} + y^2$, we obtain the upper-bound

$$
C|z_* - v_*|^2(1 + |v| + |z_*| + |v_*|)^\gamma \\
+ C(|v - \tilde{v}| + |v_* - \tilde{v}_*|)^2 + |v_* - v_*|^4/(2 - \gamma) \\
+ C(|v - \tilde{v}| + |v_* - \tilde{v}_*|)^2 + |z_* - v_*|^2(|v - z_*| + |v - v_*|)^\gamma \\
+ C(|v - \tilde{v}| + |v_* - \tilde{v}_*|)^2 + |z_* - v_*|^2/(1-\gamma),
$$

which is bounded by

$$
C(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) + C|z_* - v_*|^2\left\{(1 + |v| + |z_*| + |v_*|)^\gamma + |z_* - v_*|^{4/(2 - \gamma) - 2} \\
+ |v - z_*| + |v - v_*|)^{2\gamma} + |z_* - v_*|^{2/(1-\gamma) - 2}\right\}.
$$

One easily concludes, using that $\max\{\gamma, 4/(2 - \gamma) - 2, 2\gamma, 2/(1 - \gamma) - 2\} = 2\gamma/(1 - \gamma)$.

We finally check point (iii). Using (1.9), we deduce that $1 - \cos(G(z/x\gamma)) \leq C^2(z/x\gamma) \leq C(z/x\gamma)^{-2/\nu}$, whence $\Psi_K(x) \leq Cx^{2\gamma/\nu} \int_K z^{-2/\nu}dz = Cx^{2\gamma/\nu} K^{1-2/\nu}$. Thus

$$
A^K(v, v_*, \tilde{v}, \tilde{v}_*) \leq C(|v - v_*|^2 + |v - v_*|\tilde{v} - \tilde{v}_*||v - v_*|^{2\gamma/\nu} K^{1-2/\nu},
$$

from which we easily conclude, using that $|\tilde{v} - \tilde{v}_*||v - v_*|^{1+2\gamma/\nu} \leq |\tilde{v} - \tilde{v}_*|^2 + |v - v_*|^{2+4\gamma/\nu}$.

We conclude with the hard spheres case.

**Lemma 2.4.** Assume $[1.3], [1.4]$ with $\gamma = 1, [1.5]$ and adopt the notation of Lemma 2.2. 

(i) For all $q > 0$, there is $C_q > 0$ such that for all $M \geq 1$, all $K \in [1, \infty)$, all $v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^3$,

$$
A^K(v, v_*, \tilde{v}, \tilde{v}_*) \leq M(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) + C_q K(\tilde{v}^\nu + |v_*|) e^{-M^q} e^{C_q(|v|^\nu + |v_*|^\nu)}.\]

(ii) For all $q > 0$, there is $C_q > 0$ such that for all $M \geq 1$, all $K \in [1, \infty)$, all $v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^3$,

$$
A^K(v, v_*, \tilde{v}, \tilde{v}_*) - A^K(v, v_*, \tilde{v}_*) \leq M(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) + C|v_* - z_*|^2(1 + |v| + |v_*| + |z_*|)^2 \\
+ C_q(1 + |\tilde{v}| + |\tilde{v}_*|) K e^{-M^q} e^{C_q(|v|^\nu + |v_*|^\nu + |z_*|^\nu)}.
$$

\[\square\]
(iii) For all \( q > 0 \), there is \( C_q > 0 \) such that for all \( K \in [1, \infty) \), all \( v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^d \),

\[
A^{K}_q(v, v_*, \tilde{v}, \tilde{v}_*) \leq C_q (1 + |\tilde{v}|) e^{-K^q} e^{C_q(|v|^q + |v_*|^q + |\tilde{v}|^q)}.
\]

Proof. On the one hand, (10.10) implies

\[
A^K_1(v, v_*, \tilde{v}, \tilde{v}_*) \leq 2c_4 |v - v_*| |\tilde{v} - \tilde{v}_*| \left( \frac{|v - v_*| - |\tilde{v} - \tilde{v}_*|}{|v - v_*| + |\tilde{v} - \tilde{v}_*|} \right)^2 
\leq 4c_4 \left( |v - v_*| \wedge |\tilde{v} - \tilde{v}_*| \right) (|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2).
\]

On the other hand, since \( G \) takes values in \((0, \pi/2)\), we obviously have

\[
A^K_1(v, v_*, \tilde{v}, \tilde{v}_*) \leq \frac{\pi^2}{2} K |v - v_*| |\tilde{v} - \tilde{v}_*|.
\]

Consequently, we may write

\[
A^K_1(v, v_*, \tilde{v}, \tilde{v}_*) \leq M (|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) + \frac{\pi^2}{2} K |v - v_*| |\tilde{v} - \tilde{v}_*| \mathbb{1}_{\{4c_4 |v - v_*| \wedge |\tilde{v} - \tilde{v}_*| \geq M\}}.
\]

Point (i) easily follows, using that \( |v - v_*| \mathbb{1}_{\{4c_4 |v - v_*| \wedge |\tilde{v} - \tilde{v}_*| \geq M\}} \leq |v - v_*| \mathbb{1}_{\{4c_4 |v - v_*| \geq M\}} \leq |v - v_*| e^{-M' e^{2(4c_4 |v - v_*|)} \gamma} \leq C_q e^{-M' e^{2(4c_4 |v - v_*|)} \gamma} \leq C_q e^{-M' e^{2(4c_4 |v - v_*|)} \gamma} \).

Using all the computations of the proof of Lemma 2.3(ii) except the one that makes appear the power \( 2/(1 - \gamma) \), we see that for \( \Delta^K_2 := A^K_2(v, v_*, \tilde{v}, \tilde{v}_*) - A^K_2(v, z, z, \tilde{v}, \tilde{v}_*) \)

\[
\Delta^K_2 \leq C |v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2 + |v_* - z_*|^2 (1 + |v| + |v_*| + |z_*|)^2 + |z_* - v_*| (|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) 
\leq C (1 + |z_* - v_*|) (|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) + C |v_* - z_*|^2 (1 + |v| + |v_*| + |z_*|)^2.
\]

On the other hand, starting from (2.3) and using that \( \phi_K(x) \leq \pi K \), we realize that

\[
\Delta^K_2 \leq C K (1 + |\tilde{v}| + |\tilde{v}_*|) (1 + |v|^2 + |v_*|^2 + |z_*|^2).
\]

Hence we can write, for any \( M > 1 \),

\[
\Delta^K_2 \leq M (|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) + C |v_* - z_*|^2 (1 + |v| + |v_*| + |z_*|)^2 
+ C K (1 + |\tilde{v}| + |\tilde{v}_*|) (1 + |v|^2 + |v_*|^2 + |z_*|^2) \mathbb{1}_{\{C (1 + |v_*| + |z_*|) \geq M\}}.
\]

But \( (1 + |v|^2 + |v_*|^2 + |z_*|^2) \mathbb{1}_{\{C (1 + |v_*| + |z_*|) \geq M\}} \leq (1 + |v| + |v_*| + |z_*|)^2 e^{-M' e^{C (1 + |v_*| + |z_*|)}} \mathbb{1}_{\{C (1 + |v_*| + |z_*|) \geq M\}} \leq (1 + |v| + |v_*| + |z_*|)^2 e^{-M' e^{C (1 + |v| + |v_*| + |z_*|)}} \mathbb{1}_{\{C (1 + |v_*| + |z_*|) \geq M\}} \).

Finally, we observe that \( \Psi_K(x) \leq \pi \int_K^\infty G^2(z/x) dz \). But here, \( G(z) = (\pi/2 - z)_+ \) whence \( \Psi_K(x) \leq (\pi^4/24) x \mathbb{1}_{\{x \geq 2K/\pi\}} \leq 5x \mathbb{1}_{\{x \geq 2K/\pi\}} \). Thus for any \( q > 0 \), \( \Psi_K(x) \leq 5x e^{-K^q} e^{2q x^q} \), so that

\[
A^K_2(v, v_*, \tilde{v}, \tilde{v}_*) \leq C (1 + |\tilde{v}|) (1 + |v|^2 + |v_*|^2) e^{-K^q} |v - v_*| e^{2q |v - v_*|^q} 
\leq C_q (1 + |\tilde{v}|) e^{-K^q} e^{C_q (|v|^q + |v_*|^q)}
\]

as desired. \( \square \)

3. CONVERGENCE OF THE PARTICLE SYSTEM WITH CUTOFF

The idea of this proof are followed from [12] and [25]. To build a suitable coupling between the particle system and the solution to (1.1), we need to introduce the (stochastic) paths associated to (1.11). To do so, we follow the ideas of Tanaka [40, 41] and make use of two probability spaces. The main one is an abstract \((\Omega, \mathcal{F}, \Pr)\), on which the random objects are defined when nothing is precised. But we will also need an auxiliary one, \([0,1]\) endowed with its Borel \(\sigma\)-field and its
Lebesgue measure. In order to avoid confusion, a random variable defined on this latter probability space will be called an $\alpha$-random variable, expectation on $[0, 1]$ will be denoted by $E_{\alpha}$, etc.

3.1. A SDE for the particle system.

**Proposition 3.1.** Assume $\{1, 2\}$, $\{2, 2\}$, $\{1, 3\}$ or $\{1, 4\}$ and let $f_{0} \in \mathcal{P}_{2}(\mathbb{R}^{3})$, $N \geq 1$ and $K \in [1, \infty)$. Consider a family $(V_{0}^{i})_{i=1,\ldots,N}$ of i.i.d. $f_{0}$-distributed random variables and an independent family $(O_{i,j}^{N}(ds, dz, d\varphi))_{1 \leq i < j \leq N}$ of Poisson measures on $[0, \infty) \times \{1, \ldots, N\} \times [0, \infty) \times [0, 2\pi)$ with intensity measures $\frac{1}{2\pi}dsdzd\varphi$. For $1 \leq j < i \leq N$, we put $O_{i,j}^{N}(ds, dz, d\varphi) = O_{j,i}^{N}(ds, dz, d\varphi)$. We also set $O_{i,j}^{N}(ds, dz, d\varphi) = 0$ for $i = 1, \ldots, N$. There exists a unique (càdlàg and adapted) strong solution to

$$(3.1) \quad V_{t}^{i,K} = V_{0}^{i} + \sum_{j=1}^{N} \int_{0}^{t} \int_{0}^{\infty} \int_{0}^{2\pi} c_{K}(V_{s}^{i,K}, V_{s}^{j,K}, z, \varphi)O_{i,j}^{N}(ds, dz, d\varphi), \quad i = 1, \ldots, N.$$ 

Furthermore, $(V_{t}^{i,K})_{i=1,\ldots,N,t \geq 0}$ is Markov with generator $\mathcal{L}_{N,K}$. And the system is conservative: a.s. for all $t \geq 0$, it holds that $\sum_{i=1}^{N} V_{t}^{i,K} = \sum_{i=1}^{N} V_{0}^{i,K}$, $\sum_{i=1}^{N} V_{t}^{i,K}^{2} = \sum_{i=1}^{N} V_{0}^{i,K}^{2}$.

**Proof.** By (3.1), we have

$$\sum_{i=1}^{N} V_{t}^{i,K} = \sum_{i=1}^{N} V_{0}^{i} + \sum_{i=1}^{N} \int_{0}^{t} \int_{0}^{\infty} \int_{0}^{2\pi} c_{K}(V_{s}^{i,K}, V_{s}^{j,K}, z, \varphi)O_{i,j}^{N}(ds, dz, d\varphi)$$

Since $O_{i,j}^{N}(ds, dz, d\varphi) = O_{j,i}^{N}(ds, dz, d\varphi)$ and $c(V_{s}^{i,K}, V_{s}^{j,K}, z, \varphi) = -c(V_{s}^{j,K}, V_{s}^{i,K}, z, \varphi)$. We conclude that

$$\sum_{i=1}^{N} V_{t}^{i,K} = \sum_{i=1}^{N} V_{0}^{i,K}.$$ 

We get $\sum_{i=1}^{N} V_{t}^{i,K} = \sum_{i=1}^{N} V_{0}^{i,K}$. Next, we apply the Itô’s formula,

$$\sum_{i=1}^{N} V_{t}^{i,K}^{2} = \sum_{i=1}^{N} V_{0}^{i}^{2} + \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{0}^{t} \int_{0}^{\infty} \int_{0}^{2\pi} (c_{K}(V_{s}^{i,K}, V_{s}^{j,K}, z, \varphi) + c_{K}(V_{s}^{j,K}, V_{s}^{i,K}, z, \varphi))O_{i,j}^{N}(ds, dz, d\varphi)$$

Recall the definition of $c_{K}(v, v^{*}, z, \varphi)$. Since $(v - v^{*})$ and $\Gamma(v, v^{*}, \varphi)$ are orthogonal, we get for $i \neq j$,

$$c_{K}(V_{s}^{i,K}, V_{s}^{j,K}, z, \varphi) + c_{K}(V_{s}^{j,K}, V_{s}^{i,K}, z, \varphi) = 2c_{K}(V_{s}^{i,K}, V_{s}^{j,K}, z, \varphi)$$

which implies

$$\sum_{i=1}^{N} V_{t}^{i,K}^{2} = \sum_{i=1}^{N} V_{0}^{i}^{2}.$$ 

This completes the proof of the proposition.
where we put $\theta = G(z/V_{s_1,N,K} - V_{s_2,N,K}(\gamma))$.

Finally, we conclude that $\sum_{i=1}^N |V_{t,i,N,K}|^2 = \sum_{i=1}^N |V_0,i,N,K|^2$. □

For $\phi : (\mathbb{R}^3)^N \to \mathbb{R}$ sufficiently regular and $v = (v_1, \ldots, v_N) \in (\mathbb{R}^3)^N$, by

$$L_{N,K} \phi(v) = \frac{1}{2(N-1)} \sum_{i \neq j} \int_0^\infty dz \int_0^{2\pi} d\varphi \left[ \phi(v + cK(v_i, v_j, z, \varphi)e_i + cK(v_j, v_i, z, \varphi)e_j) - \phi(v) \right]$$

For $h \in \mathbb{R}^3$, we note $he_i = (0, \ldots, 0, h, 0, \ldots, 0) \in (\mathbb{R}^3)^N$ with $h$ at the $i$-th place.

3.2. The coupling. Here we explain how we couple our particle system with a family of i.i.d. Boltzmann processes.

Here we write another version of the [12] Lemma 5.

**Proposition 3.2.** For all $t \geq 0$, $N \geq 2$, $w \in (\mathbb{R}^3)^N$, for $i, j = 1, \ldots, N$, there exists an $\mathbb{R}^3$-valued function $\Pi_{t,i,j}(w, \alpha)$, measurable in $(t, w, \alpha) \in [0, \infty) \times (\mathbb{R}^3)^N \times [0, 1]$ with the following property:

(i) If $Y$ is any exchangeable random vector in $(\mathbb{R}^3)^N$ and for any $j \neq i$, then the $\alpha$ law of $\Pi_{t,i,j}(Y, \alpha)$ is $f_i$, which is equivalent to for any bounded measurable function $\phi$,

$$E\left[ \int_0^1 \phi(\Pi_{t,i,j}(Y, \alpha)) d\alpha \right] = \int_{\mathbb{R}^3} \phi(u) f_i(du).$$

(ii) For any $w \in (\mathbb{R}^3)^N$, $0 \leq s$, $i = 1, \ldots, N$, we have $\int_0^1 \frac{1}{N-1} \sum_{j \neq i} |\Pi_{s,i,j}(w, \alpha) - w_j|^2 d\alpha = W_s^2(\tilde{w}^i, f_s)$ where $\tilde{w}^i = \frac{1}{N-1} \sum_{j \neq i} w_j$.

Here is the coupling we propose.

**Lemma 3.3.** Assume [13], [14], [15] or [16]. Let $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$. Assume additionally [17] and [18] if $\gamma \in (0, 1]$. Let $(f_t)_{t \geq 0}$ be the unique weak solution to [11] and assume that $f_t$ has a density for all $t > 0$ (see Theorem [13]). Consider $N \geq 2$ and $K \in [1, \infty)$ fixed. Let $(V_0^j)_{i=1, \ldots, N}$ be i.i.d. with common law $f_0$ and let $(M_{ij}(ds, d\alpha, dz, d\varphi))_{1 \leq i < j \leq N}$ be an i.i.d. family of Poisson measures on $[0, \infty) \times [0, 1] \times (0, \infty) \times [0, 2\pi]$ with intensity measures $\frac{1}{\pi}dsd\alpha dzd\varphi$, independent of $(V_0^j)_{i=1, \ldots, N}$. For $1 \leq j < i \leq N$, we put $M_{ij}(ds, d\alpha, dz, d\varphi) = M_{ji}(ds, d\alpha, dz, d\varphi)$. We also set $M_{i,i}(ds, d\alpha, dz, d\varphi) = 0$ for $i = 1, \ldots, N$. We then consider the following system

(i) The following SDE’s, for $i = 1, \ldots, N$, define $N$ copies of the Boltzmann process:

$$W_t^i = V_0^i + \sum_{j=1}^N \int_0^t \int_0^\infty \int_0^{2\pi} c(W_{s-, i}^j, \Pi_{s-, i,j}^j(W_{s-, i}^j, \alpha), z, \varphi) M_{ij}(ds, d\alpha, dz, d\varphi).$$

In particular, for each $t \geq 0$, $(W_t^j)_{i=1, \ldots, N}$ are with common law $f_t$.

(ii) Next, we consider the system of SDE’s, for $i = 1, \ldots, N$,

$$V_{t,i}^{i,K} = V_0^i + \sum_{j=1}^N \int_0^t \int_0^\infty \int_0^{2\pi} c_K(V_{s-, i}^j, V_{s-, i,j}^j, z, \varphi, \varphi_{i,j,i,s}) M_{ij}(ds, d\alpha, dz, d\varphi),$$

where we used the notation $V_{s-, i}^{i,K} = (V_{s-, 1}^{1,K}, \ldots, V_{s-, N}^{N,K}) \in (\mathbb{R}^3)^N$, $W_{s-, i}^j = (W_{s-, 1}^j, \ldots, W_{s-, N}^j) \in (\mathbb{R}^3)^N$ and where we have set $\varphi_{i,j,i,s} := \varphi(\Pi_{s-, i,j}^j(W_{s-, i}^j, \alpha), V_{s-, i,j}^j, V_{s-, i,j}^{i,K} - V_{s-, i,j}^{j,K})$ for simplicity. This system of SDEs has a unique solution, and this solution is a Markov process with generator $L_{N,K}$ and initial condition $(V_0^j)_{i=1, \ldots, N}$.

(iii) The family $((W_t^1, V_t^{1,K}), t \geq 0, \ldots, (W_t^N, V_t^{N,K}), t \geq 0)$ is exchangeable.
Proof. In order to prove point (i), we need to rewrite $W_i^{i,N}$.
We define a family of random measure $(Q_i^N(ds, d\xi, dz, d\varphi))_{1 \leq i \leq N}$ on $[0, \infty) \times [0, N] \times [0, \infty) \times [0, 2\pi)$. For any measurable set $A_1 \subseteq [0, \infty), A_2 \subseteq [0, N], A_3 \subseteq [0, \infty), A_4 \subseteq [0, 2\pi)$.

$$Q_i^N(A_1, A_2, A_3, A_4) = \sum_{j=1}^N M_{i,j}^N(A_1, (A_2 \cap (j - 1, j]) - j, A_3, A_4)$$

We can show that the family of $\{Q_i^N(ds, d\xi, dz, d\varphi)\}_{i=1,...,N}$ are Poisson measures on $[0, \infty) \times [0, N] \times [0, \infty) \times [0, 2\pi)$ with intensity measures $dsdzd\varphi$, independent of $(V_0^i)_{i=1,...,N}$. Then we can rewrite:

$$W_i^{i,N} = V_0^i + \int_0^t \int_0^\infty \int_{\mathbb{R}^3} \int_{0}^{2\pi} c(W_i^{i,N,k}, \Pi_s(W_i^{i,N}, \xi), z, \varphi + \varphi_{i,s})Q_i^N(ds, d\xi, dz, d\varphi)$$

where $\Pi_s(W_i^{i,N}, \xi) = \Pi_s(W_i^{i,N}, \xi - [\xi])$ as well as $\varphi_{i,s} = \varphi_{i,(\xi - [\xi]), s}$. Next, we define $Q_i^{N^*}(ds, dv, dz, d\varphi)$ to be the point measure on $[0, \infty) \times \mathbb{R}^3 \times [0, \infty) \times [0, 2\pi)$ with atoms $(t, \Pi_s^{i,N}(W_s^{i,N}, \epsilon), z, \varphi + \varphi_{i,s} - c)$, which means: for any measurable set $B \subseteq [0, \infty) \times \mathbb{R}^3 \times [0, \infty) \times [0, 2\pi)$.

$$Q_i^{N^*}(B) := Q_i^N\left(\{(s, \xi, z, \varphi)| (s, \Pi_s(W_s^{i,N}, \xi), z, \varphi + \varphi_{i,s} - c) \in B\}\right)$$

We finally have the expression: for $i = 1, ..., k$,

$$W_i^{i,N} = V_0^i + \int_0^t \int_0^\infty \int_{\mathbb{R}^3} \int_{0}^{2\pi} c(W_i^{i,N}, v, z, \varphi)Q_i^{N^*}(ds, dv, dz, d\varphi)$$

In order to prove $\{W_i^{i,N}\}_{i=1,..,N}$, it is sufficient to prove that $\{Q_i^{N^*}(ds, dv, dz, d\varphi)\}$ are Poisson measures with the same intensity $tdtdzd\varphi f_s(ds)$. For any bounded, measurable, positive function $g : \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+ \times [0, 2\pi) \to \mathbb{R}$, we put $G_i = \int_0^t \int_{\mathbb{R}^3} \int_{0}^{2\pi} g(s, v, z, \varphi)Q_i^{N,N^*}(ds, dv, dz, d\varphi)$

$$\mathbb{E}\left[\exp(-G_i)\right] = 1 + \mathbb{E}\left[\exp\left(-\int_0^t \int_{\mathbb{R}^3} \int_{0}^{2\pi} g(s, \Pi_s(W_s^{i,N}, \xi), z, \varphi + \varphi_{i,s})Q_i^N(ds, d\xi, dz, d\varphi)\right)\right]$$

$$= 1 + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^3} \int_{0}^{2\pi} \exp\left(-G_s - g(s, \Pi_s(W_s^{i,N}, \xi), z, \varphi + \varphi_{i,s})\right) - \exp(-G_s)dsd\xi dz d\varphi\right]$$

$$= 1 + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^3} \int_{0}^{2\pi} \exp(-G_s)\int_{0}^{\infty} \exp\left(-g(s, \Pi_s(W_s^{i,N}, \xi), z, \varphi)\right) - 1\right)d\varphi d\xi dz ds$$

$$= 1 + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^3} \int_{0}^{2\pi} \exp(-G_s)\int_{0}^{\infty} \int_{\mathbb{R}^3} (e^{-g(s,v,z,\varphi)} - 1)f_s(ds)dz d\varphi\right]$$

$$= 1 + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^3} \int_{0}^{2\pi} \exp(-G_s)\int_{0}^{\infty} \int_{\mathbb{R}^3} (e^{-g(s,v,z,\varphi)} - 1)dz d\varphi f_s(ds)\right]$$

We put $\tau_t = \int_{0}^{\infty} \int_{\mathbb{R}^3} (e^{-g(t,v,z,\varphi)} - 1)dz d\varphi f_s(ds), r_t = \mathbb{E}\left[\exp(-G_t)\right]$, then we have the following integration equation

$$r_t = 1 + \int_0^t r_{t-s} ds.$$
This implies for fix \(i = 1, \ldots, N, Q_t^{N,i}(ds, dv, dz, d\varphi)\) to be the point measure on \([0, \infty) \times \mathbb{R}^3 \times [0, \infty) \times [0, 2\pi]\) with intensity \(dsf_s(v)dzd\varphi\). This finishes the proof of (i).

Next, we are going to explain why \((V_t^{i,N,K})_{i=1,\ldots,N,t\geq 0}\) is Markov with generator \(L_{N,K}\). We define:

\[
V_t^{i,N,K} := \sum_{i=1}^N V_t^{i,N,K}e_i.
\]

First of all, observe that we actually deal with finite Poisson measures, since \(c_K\) vanishes for \(z \geq K\). Thus, strong existence and uniqueness for (3.1) is trivial: it suffices to work recursively on the instants of jumps (which are discrete) of the family \((M_t^{N,i}(ds, dz, d\varphi))_{i,j=1,\ldots,N}\). Consequently, \((V_t^{i,N,K})_{i=1,\ldots,N,t\geq 0}\) is a Markov process, since it solves a well-posed time-homogeneous SDE. By Itô’s formula, we have:

\[
\phi(V_t^{N,K}) = \phi(V_0^N) + \frac{1}{2} \sum_{i \neq j} \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} \left[ \phi(V_{s-}^{N,K} + c_K(V_{s-}^{i,N,K}, V_{s-}^{j,N,K}, z, \varphi)e_i \
+ c_K(V_{s-}^{j,N,K}, V_{s-}^{i,N,K}, z, \varphi)e_j \right] - \phi(V_{s-}^{N,K}) \right] M_{ij}^N(ds, d\alpha, dz, d\varphi)
\]

Take expectation each side, we get

\[
\mathbb{E}[\phi(V_t^{N,K})] = \mathbb{E}[\phi(V_0^N)] + \frac{1}{2(N-1)} \sum_{i \neq j} \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} \mathbb{E}\left[ \phi(V_{s-}^{N,K} + c_K(V_{s-}^{i,N,K}, V_{s-}^{j,N,K}, z, \varphi)e_i \
+ c_K(V_{s-}^{j,N,K}, V_{s-}^{i,N,K}, z, \varphi)e_j \right] - \phi(V_{s-}^{N,K}) \right] dsd\alpha dzd\varphi.
\]

Since

\[
\lim_{s \to t} \int_0^1 \int_0^\infty \int_0^{2\pi} \mathbb{E}\left[ \phi(V_{s-}^{N,K} + c_K(V_{s-}^{i,N,K}, V_{s-}^{j,N,K}, z, \varphi)e_i \
+ c_K(V_{s-}^{j,N,K}, V_{s-}^{i,N,K}, z, \varphi)e_j \right] - \phi(V_{s-}^{N,K}) \right] dsd\alpha dzd\varphi
\]

\[
= \int_0^\infty \int_0^{2\pi} \mathbb{E}\left[ \lim_{s \to t} \phi(V_{0}^{N} + c_K(V_{0}^{i}, V_{0}^{j}, z, \varphi)e_i + c_K(V_{0}^{j}, V_{0}^{i}, z, \varphi)e_j) - \phi(V_{0}^{N}) \right].
\]

We put \(\mathbb{E}_v\) is the conditional expectation under the condition \(V_0^N = v\).

\[
\lim_{t \to 0} \frac{\mathbb{E}_v[\phi(V_t^{i,N,K})] - \mathbb{E}_v[\phi(V_0^N)]}{t} = \frac{1}{2(N-1)} \sum_{i \neq j} \int_0^\infty \int_0^{2\pi} \mathbb{E}_v \left[ \phi(v + c_K(v_i, v_j, z, \varphi)e_i + c_K(v_j, v_i, z, \varphi)e_j) - \phi(v) \right]
\]

This allows us to deduce point (ii).

Point (iii) follows from the exchangeability of the family \((V_0^i)_{i=1,\ldots,N}, (M_{ij})_{i,j=1,\ldots,N}\) are independent and from uniqueness (in law). 

\[\square\]
3.3. Estimate of the Wasserstein distance. We now define

$$
\varepsilon^N_t(f) := \mathbb{E}\left[ W^2(f, \frac{1}{N-1} \sum_{i=2}^N \delta_W) \right]
$$

where $W^i_t$ were defined in Lemma 3.3. By proposition 4.2 and exchangeability, we have

$$
\mathbb{E}\left[ \int_0^1 \frac{1}{N-1} \sum_{j \neq i}^N |\Pi_i^{t,j}(W_t, \alpha)) - W^j_t|^2 d\alpha \right] = \varepsilon^N_t(f_t)
$$

We also denote that

$$
\mu^N_{W_t} := \mathbb{E}\left[ W^2(f, \frac{1}{N} \sum_{i=1}^N \delta_W) \right].
$$

We can now prove our main result in the case with cutoff.

**Proposition 3.4.** Let $B$ be a collision kernel satisfying (1.3), (1.5), and (1.6) and let $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ not be a Dirac mass. If $\gamma > 0$, assume additionally (1.7) and (1.19). Consider the unique weak solution $(f_t)_{t \geq 0}$ to (1.1) defined in Theorem 1.3 and, for each $N \geq 1$, $K \in [1, \infty]$, the unique Markov process $(V^{i, K})_{i=1, \ldots, N, t \geq 0}$ defined in Proposition 4.4.

(i) Hard potentials. Assume that $\gamma \in (0, 1)$ and (1.6). For all $\varepsilon \in (0, 1)$, all $T \geq 0$, there is a constant $C_{\varepsilon, T}$ such that for all $N \geq 1, \alpha K \in [1, \infty)$,

$$
\sup_{[0, T]} \mathbb{E}[W^2(\mu^N_{V^{i, K}}, f_t)] \leq C_{\varepsilon, T} \left( \sup_{[0, T]} \varepsilon^N_t(f_t) + K^{1-2/\kappa} \right)^{1-\varepsilon} + \frac{C}{N}.
$$

(ii) Hard spheres. Assume finally that $\gamma = 1$, (1.5), and that $f_0$ has a density. For all $\varepsilon \in (0, 1)$, all $T \geq 0$, all $q \in (1, p)$, there is a constant $C_{\varepsilon, q, T}$ such that for all $N \geq 1$, all $K \in [1, \infty)$,

$$
\sup_{[0, T]} \mathbb{E}[W^2(\mu^N_{V^{i, K}}, f_t)] \leq C_{\varepsilon, q, T} \left( \sup_{[0, T]} \varepsilon^N_t(f_t) + e^{-K^q} \right) e^{C_{\varepsilon, q, T} K} + \frac{C}{N}.
$$

Before giving the proof of Proposition 3.4, we need the following simple Lemma.

**Lemma 3.5.** Recall that $\varepsilon^N_t(f_t)$ and $\mu^N_{W_t}$ were defined in (3.2) and (3.3), then we have

$$
\mathbb{E}[W^2(\mu^N_{W_t}, f_t)] \leq \frac{N-1}{N} \varepsilon^N_t(f_t) + \frac{C}{N}
$$

**Proof.** We recall the well-known fact that for $f, f', g, g' \in \mathcal{P}_2(\mathbb{R}^3)$ and $\lambda \in (0, 1)$, it holds that $W^2(\lambda f + (1-\lambda)g, \lambda f' + (1-\lambda)g') \leq \lambda W^2(f, f') + (1-\lambda)W^2(g, g')$. Indeed, consider $X \sim f$ and $X' \sim f'$ such that $\mathbb{E}[|X - X'|^2] = W^2(f, f'), Y \sim g$ and $Y' \sim g'$ such that $\mathbb{E}[|Y - Y'|^2] = W^2(g, g')$, and $U \sim \text{Bernoulli}(\lambda)$, with $(X, X'), (Y, Y'), U$ independent. Then $Z := UX + (1 - U)Y \sim \lambda f + (1-\lambda)g$, $Z' := UX' + (1 - U)Y' \sim \lambda f' + (1-\lambda)g'$, and one easily verifies that $\mathbb{E}[|Z - Z'|^2] = \lambda \mathbb{E}[|X - X'|^2] + (1-\lambda)\mathbb{E}[|Y - Y'|^2] = \lambda W^2(f, f') + (1-\lambda)W^2(g, g')$. Then we have

$$
\mathbb{E}[W^2(\mu^N_{W_t}, f_t)] \leq \mathbb{E}\left[ W^2\left( \frac{\delta_{W^i_t}}{N} + \frac{N-1}{N} \frac{1}{N-1} \sum_{i=2}^N \delta_{W^i_t}, f_t \right) \right] 
$$

$$
= \frac{1}{N} \mathbb{E}[W^2(\delta_{W^i_t}, f_t)] + \frac{N-1}{N} \mathbb{E}\left[ W^2\left( \frac{1}{N-1} \sum_{i=2}^N \delta_{W^i_t}, f_t \right) \right] \leq \frac{C}{N} + \frac{N-1}{N} \varepsilon^N_t(f_t)
$$
The last inequality is given by
\[ E \left[ |W^2_t(\delta_{W^1_t}, f_t)| \right] \leq 2E[|W^1_t|^2] + 2 \int_{\mathbb{R}^3} |v|^2 f_t(\text{dv}) = 4 \int_{\mathbb{R}^3} |v|^2 f_t(\text{dv}) < \infty. \]

Proof of Proposition 3.4 (i) when \( K \in [1, \infty) \). We thus assume (1.3), (1.4) with \( \gamma \in (0, 1) \) and (1.6). We consider \( f_0 \in \mathcal{P}_2(\mathbb{R}^3) \) satisfying (1.19) for some \( p \in (\gamma, 2) \) and fix \( q \in (\gamma, p) \) for the rest of the proof. We also assume that \( f_0 \) is not a Dirac mass, so that \( f_t \) has a density for all \( t > 0 \). We fix \( N \geq 1 \) and \( K \in [1, \infty) \) and consider the processes introduced in Lemma 3.3.

**Step 1.** A direct application of the Itô calculus for jump processes shows that
\[
E[|W^1_t - V^1_t K|^2] = \frac{1}{N - 1} \sum_{j=2}^{N} \int_0^t \int_1^t \int_0^1 \int_0^{2\pi} E\left[ |W^1_s - V^1_s K + \Delta^{1,j}(s, \alpha, z, \varphi)|^2 - |W^1_s - V^1_s K|^2 \right] d\alpha dz ds,
\]
where
\[
\Delta^{1,j}(s, \alpha, z, \varphi) = c(W^1_s, \Pi^{1,j}(W_{s-}, \alpha), z, \varphi + \varphi_{1,j,\alpha,s}) - c_K(V^1_s K, V^j_s K, z, \varphi).
\]
Using Lemma 2.2 we thus obtain
\[
E[|W^1_t - V^1_t K|^2] \leq \int_0^t \left[ B^K_1(s) + B^K_2(s) + B^K_3(s) \right] ds,
\]
where for \( i = 1, 2, 3 \)
\[
B^K_i(s) := \frac{1}{N - 1} \int_0^1 \left[ \sum_{j=2}^{N} A^K_i(W^1_s, \Pi^{1,j}(W_{s-}, \alpha), V^{1,K}_s, V^{j,K}_s) \right] d\alpha.
\]

**Step 2.** Using Lemma 2.3 (i), we see that for all \( M \geq 1 \) (recall that \( q \in (\gamma, p) \) is fixed).
\[
B^K_1(s) \leq \frac{M}{N - 1} \sum_{j=2}^{N} \int_0^1 E\left[ |W^1_s - W^1_s|^2 + |V^{j,K}_s - \Pi^{1,j}(W_{s-}, \alpha)|^2 \right] d\alpha
\]
\[
+ C e^{-M^{\gamma/\gamma}} \sum_{j=2}^{N} E\left[ \int_0^1 \exp(C(|W^1_s|^q + |\Pi^{1,j}(W_{s-}, \alpha)|^q)) d\alpha \right]
\]
\[
\leq \frac{M}{N - 1} \sum_{j=2}^{N} \int_0^1 E\left[ |W^1_s - V^{1,K}_s|^2 + |V^{j,K}_s - \Pi^{1,j}(W_{s-}, \alpha)|^2 \right] d\alpha + C e^{-M^{\gamma/\gamma}}.
\]
To get the last inequality, we used that the \( \alpha \) law of \( \Pi^{1,j}(W_{s-}, \cdot) \) is \( f_s \) and the distribution of \( W^1_s \) is also \( f_s \). Hence for any \( j = 2, \ldots, N \)
\[
(3.6) \quad E\left[ \int_0^1 \exp(C(|W^1_s|^q + |\Pi^{1,j}(W_{s-}, \alpha)|^q)) d\alpha \right] \leq E\left[ \int_0^1 \exp(2C|W^1_s|^q) d\alpha \right] \leq \frac{1}{2} E \left[ \int_0^1 \exp(2C|\Pi^{1,j}(W_{s-}, \alpha)|^q) d\alpha \right]^{\frac{1}{2}}
\]
\[
= \left( \int_{\mathbb{R}^3} e^{2C|w|^q} f_s(\text{dw}) \right)^2 < \infty
\]
by (1.20).
Step 3. Roughly speaking, $B^K_2$ should not be far to be zero for symmetry reasons. We claim that $B^K_2$ would be zero if $\Pi^{1,j}_s(W_{s-}, \alpha)$ was replaced by $W^1_s$. More precisely, we check here that

$$
\hat{B}^K_2(s) := \frac{1}{N-1} \sum_{j=2}^N \int_0^1 \mathbb{E} \left[ A^K_2(W^1_s, W^j_s, V_s^1, K, V_s^{j,K}) \right] d\alpha = 0.
$$

We simply have

$$
\tilde{B}^K_2(s) = \mathbb{E} \left[ A^K_2(W^1_s, W^2_s, V_s^1, K, V_s^{2,K}) \right]
$$

by exchangeability. Finally, we write, using again exchangeability,

$$
\hat{B}^K_2(s) = \frac{1}{2} \mathbb{E} \left[ A^K_2(W^1_s, W^j_s, V_s^1, K, V_s^{j,K}) \right] + \frac{1}{2} \mathbb{E} \left[ A^K_2(W^2_s, W^1_s, V_s^2, K, V_s^{1,K}) \right].
$$

This is zero by symmetry of $A^K_2$: it holds that $A^K_2(v, v_s, \tilde{\nu}_s) + A^K_2(v, v_s, \tilde{\nu}_s) = 0$.

Step 4. By Step 3, we thus have

$$
\hat{B}^K_2(s) = \frac{1}{N-1} \sum_{j=2}^N \int_0^1 \mathbb{E} \left[ A^K_2(W^1_s, \Pi^{1,j}_s(W_{s-}, \alpha), V_s^1, K, V_s^{j,K}) - A^K_2(W^1_s, W^j_s, V_s^1, K, V_s^{j,K}) \right] d\alpha.
$$

Consequently, Lemma 2.3 (ii) implies

$$
\hat{B}^K_2(s) \leq \frac{C}{N-1} \sum_{j=2}^N \int_0^1 \mathbb{E} \left[ |W^1_s - V_s^{1,K}|^2 + |\Pi^{1,j}_s(W_{s-}, \alpha) - V_s^{j,K}|^2 
+ |\Pi^{1,j}_s(W_{s-}, \alpha) - W^j_s|^2 (1 + |W^1_s| + |\Pi^{1,j}_s(W_{s-}, \alpha)| + |W^j_s|)^{2\gamma/(1-\gamma)} \right] d\alpha.
$$

Step 5. Finally, we use Lemma 2.3 (iii) to obtain

$$
\hat{B}^K_2(s) \leq \frac{CK^{1-2/\nu}}{N-1} \sum_{j=2}^N \int_0^1 \mathbb{E} \left[ 1 + |W^1_s|^{4\gamma/\nu+2} + |\Pi^{1,j}_s(W_{s-}, \alpha)|^{4\gamma/\nu+2} + |V_s^{1,K}|^2 + |V_s^{j,K}|^2 \right] d\alpha
= CK^{1-2/\nu} \int_0^1 \mathbb{E} \left[ 1 + |W^1_s|^{4\gamma/\nu+2} + |\Pi^{1,2}_s(W_{s-}, \alpha)|^{4\gamma/\nu+2} + |V_s^{1,K}|^2 + |V_s^{2,K}|^2 \right] d\alpha.
$$

Since $W^1_s \sim f_s$, we deduce from (1.20) that $\mathbb{E}[|W^1_s|^{\gamma/\nu+2}] = \int_{\mathbb{R}^3} |v|^{\gamma/\nu+2} f_s(dv) \leq C$. By Proposition 3.2 (i), we also have $\int_0^1 \mathbb{E}[|\Pi^{1,j}_s(W_{s-}, \alpha)|^{\gamma/\nu+2}] d\alpha = \int_{\mathbb{R}^3} |v|^{\gamma/\nu+2} f_s(dv) \leq C$. Proposition 3.3 shows that $\mathbb{E}[|V_s^{1,K}|^2] = \int_{\mathbb{R}^3} |v|^2 f_0(dv)$.

Step 6. We set $u_s := \mathbb{E} [W^1_t - V_s^{1,K,2}]$. Using the previous steps, we see that for all $M \geq 1$,

$$
u_s \leq Ce^{-M^{\gamma/\nu}} + CM^{1-2/\nu} + (M + C) \int_0^t \left[ u_s + \frac{1}{N-1} \sum_{j=2}^N \int_0^1 \mathbb{E} \left[ |\Pi^{1,j}_s(W_{s-}, \alpha) - W^j_s|^2 (1 + |W^1_s| + |\Pi^{1,j}_s(W_{s-}, \alpha)| + |W^j_s|)^{2\gamma/(1-\gamma)} \right] d\alpha \right] ds
+ \frac{C}{(N-1)} \sum_{j=2}^N \int_0^1 \int_0^1 \mathbb{E} \left[ |\Pi^{1,j}_s(W_{s-}, \alpha) - W^j_s|^2 (1 + |W^1_s| + |\Pi^{1,j}_s(W_{s-}, \alpha)| + |W^j_s|)^{2\gamma/(1-\gamma)} \right] d\alpha ds.$$
We now write,

\begin{equation}
\int_0^1 \mathbb{E} \left[ |V_s^{j,K} - \Pi_s^{1,j}(W_{s-}, \alpha)|^2 \right] d\alpha \leq 2 \int_0^1 \mathbb{E} \left[ |V_s^{j,K} - W_s^{j}|^2 \right] d\alpha + 2 \int_0^1 \mathbb{E} \left[ |W_s^{j} - \Pi_s^{1,j}(W_{s-}, \alpha)|^2 \right] d\alpha
\end{equation}

Then,

\begin{align*}
&\frac{1}{N-1} \sum_{j=2}^N \int_0^1 \mathbb{E} \left[ |V_s^{j,K} - \Pi_s^{1,j}(W_{s-}, \alpha)|^2 \right] d\alpha \\
&\leq \frac{2}{N-1} \sum_{j=2}^N \int_0^1 \mathbb{E} \left[ |V_s^{j,K} - W_s^{j}|^2 \right] d\alpha + \frac{2}{N-1} \sum_{j=2}^N \int_0^1 \mathbb{E} \left[ |W_s^{j} - \Pi_s^{1,j}(W_{s-}, \alpha)|^2 \right] d\alpha \\
&= 2u_s + 2\varepsilon^N(f_t)
\end{align*}

by exchangeability and Proposition 3.2. We deduce that

\begin{equation}
\frac{1}{N-1} \sum_{j=2}^N \int_0^1 \mathbb{E} \left[ |V_s^{j,K} - \Pi_s^{1,j}(W_{s-}, \alpha)|^2 \right] d\alpha \leq 2\varepsilon^N(f_t) + 2u_s.
\end{equation}

Next, a simple computation shows that for all \( \varepsilon \in (0, 1) \),

\begin{equation}
\int_0^1 \mathbb{E} \left[ \frac{1}{N-1} \sum_{j=2}^N |\Pi_s^{1,j}(W_{s-}, \alpha) - W_s^{j}|^2 (1 + |W_s^{j}| + |\Pi_s^{1,j}(W_{s-}, \alpha)| + |W_s^{j}|^{2\gamma/(1-\gamma)}) \right] d\alpha
\end{equation}

\begin{align*}
&\leq \int_0^1 \mathbb{E} \left[ \frac{1}{N-1} \sum_{j=2}^N |\Pi_s^{1,j}(W_{s-}, \alpha) - W_s^{j}|^{2-\varepsilon} (1 + |W_s^{j}| + |\Pi_s^{1,j}(W_{s-}, \alpha)| + |W_s^{j}|^{2\gamma/(1-\gamma)+\varepsilon}) \right] d\alpha \\
&\leq \left( \int_0^1 \mathbb{E} \left[ \frac{1}{N-1} \sum_{j=2}^N |\Pi_s^{1,j}(W_{s-}, \alpha) - W_s^{j}|^2 \right] d\alpha \right)^{\frac{2-\varepsilon}{2}} \\
&\quad \times \left( \int_0^1 \mathbb{E} \left[ \frac{1}{N-1} \sum_{j=2}^N (1 + |W_s^{j}| + |\Pi_s^{1,j}(W_{s-}, \alpha)| + |W_s^{j}|^{2\gamma/(1-\gamma)+\varepsilon}) \right] d\alpha \right)^{\frac{\varepsilon}{2}} \\
&= \left( \int_0^1 \mathbb{E} \left[ \frac{1}{N-1} \sum_{j=2}^N |\Pi_s^{1,j}(W_{s-}, \alpha) - W_s^{j}|^2 \right] d\alpha \right)^{\frac{2-\varepsilon}{2}} \\
&\quad \times \left( \frac{1}{N-1} \sum_{j=2}^N \int_0^1 \mathbb{E} \left[ (1 + |W_s^{j}| + |\Pi_s^{1,j}(W_{s-}, \alpha)| + |W_s^{j}|^{2\gamma/(1-\gamma)+\varepsilon}) \right] d\alpha \right)^{\frac{\varepsilon}{2}} \\
&\leq C_s (\varepsilon^N(f_t))^{\frac{2-\varepsilon}{2}}.
\end{align*}
For the last inequality, we used Proposition [3.2] the fact that by
\[ E \left[ |W_s^2|^{\frac{4\gamma}{N-\gamma}} \right] = E \left[ |W_t^2|^{\frac{4\gamma}{N-\gamma}} \right] = \int_0^1 E \left[ \Pi^{1-j} (W_{s-}\alpha)^{\frac{4\gamma}{N-\gamma}} \right] d\alpha \]
\[ = \int_{\mathbb{R}^3} \|v\|^{\frac{4\gamma}{N-\gamma}} + 2f_s(dv) \leq C \varepsilon \]

We end up with: for all \( \varepsilon \in (0,1) \), all \( M \geq 1 \),
\[ u_t \leq C \varepsilon T e^{-M^\nu/\gamma} + C t K^{1-2/\nu} + 3(M + C) \int_0^t \left[ u_s + \varepsilon N^\alpha (f_s) \right] ds \]
\[ + C \varepsilon \int_0^t \left( \varepsilon N^\alpha (f_s) \right)^{1-\varepsilon/2} ds. \]

Recall (3.2), because \( W_1^T, \ldots, W_N^T \) share the same law as \( f_t \)-distributed. Since \( \varepsilon N^\alpha (f_t) \leq 2 \int_{\mathbb{R}^3} |v|^2 f_t(dv) \), since \( M \geq 1 \) and \( K \in [1, \infty) \), we get
\[ u_t \leq C \varepsilon \left( T e^{-M^\nu/\gamma} + M t \delta_{N,K,T}^{-1/\varepsilon/2} + M \int_0^t u_s ds \right). \]

where we have set
\[ \delta_{N,K,t} := K^{1-2/\nu} + \sup_{[0,t]} \varepsilon N^\alpha (f_s). \]

Hence by Grönwall’s lemma,
\[ \sup_{[0,T]} u_t \leq C \varepsilon T e^{-M^\nu/\gamma} + M \delta_{N,K,T}^{-1/\varepsilon/2} e^{C \varepsilon T}, \]
this holding for any value of \( M \geq 1 \). We easily conclude that
\[ \sup_{[0,T]} u_t \leq C \varepsilon T \delta_{N,K,T}^{-1/\varepsilon}, \]
by choosing \( M = 1 \) if \( \delta_{N,K,T} \geq 1/\varepsilon \) and \( M = \| \log \delta_{N,K,T} \|^{\gamma/\nu} \) otherwise, which gives
\[ \sup_{[0,T]} u_t \leq C \varepsilon \left( T \delta_{N,K,T} + \delta_{N,K,T}^{-1/\varepsilon} \| \log \delta_{N,K,T} \|^{\gamma/\nu} \right) e^{C \varepsilon T} \delta_{N,K,T}^{-1/\varepsilon}, \]
the last inequality following from the fact that \( \gamma/\nu < 1 \).

Final step. We now recall that \( \mu_t^N = \mu_t^{N,K} \) and write
\[ E[\mathcal{W}_2^2(\mu_t^N, f_t)] \leq 2E[\mathcal{W}_2^2(\mu_t^{N,K}, \mu_t^{N,W})] + 2E[\mathcal{W}_2^2(\mu_t^{N,K}, f_t)]. \]
But \( E[\mathcal{W}_2^2(\mu_t^{N,K}, \mu_t^{N,W})] \leq E[N^{-1} \sum_1^N |V_t^{1,K} - W_t^1|^2] = E[|V_t^{1,K} - W_t^1|^2] = u_t \) by exchangeability, and we have already seen that \( E[\mathcal{W}_2^2(\mu_t^{N,K}, f_t)] \leq N^{-1} \varepsilon N^\alpha (f_t) + C \varepsilon \) from Lemma 3.6. Consequently, for all \( \varepsilon \in (0,1) \), all \( t \in [0,T] \),
\[ E[\mathcal{W}_2^2(\mu_t^N, f_t)] \leq C \varepsilon T \delta_{N,K,T}^{-1/\varepsilon} + 2\varepsilon N^\alpha (f_t) + \frac{C \varepsilon}{N} \leq C \varepsilon T \left( K^{1-2/\nu} + \sup_{[0,T]} \varepsilon N^\alpha (f_s) \right)^{1-\varepsilon} + \frac{C \varepsilon}{N}, \]
and this proves point (i).

We conclude with hard spheres.

Proof of Proposition 3.4 (ii). We thus assume (1.3), (1.4) with \( \gamma = 1 \) and (1.5). We consider \( f_0 \in \mathcal{P}_2(\mathbb{R}^3) \) satisfying (1.19) for some \( p \in (\gamma, 2) \) and fix \( q \in (\gamma, p) \) for the rest of the proof. We also
assume that $f_0$ has a density, so that $f_t$ has a density for all $t > 0$. We fix $N \geq 1$ and $K \in [1, \infty)$ and consider the processes introduced in Lemma 3.3.

**Step 1.** Exactly as in the case of hard potentials, we find that

$$
E[|W^1_t - V^{1,K}_t|^2] \leq \int_0^t |B^K_i(s) + B^K_j(s) + B^K_k(s)|ds,
$$

where $B^K_i(s) := \frac{1}{\sqrt{N}} \sum_{j=2}^N \mathbb{E} \left[ \sum_{j=2}^N A^K_i(W^1_s, \Pi^{1,j}(W_{s-}, \alpha), V^{1,K}_s, V^{j,K}_s) \right] da$ for $i = 1, 2, 3$.

**Steps 2, 3, 4, 5, 6.** Following the case of hard potentials, using Lemma 2.4 instead of Lemma 2.3, we deduce that for all $M > 1$,

$$
\sum_{i=1}^3 B^K_i(s) \leq 2M \int_0^1 \mathbb{E} \left[ |W^1_s - V^{1,K}_s|^2 + \frac{1}{N-1} \sum_{j=2}^N |\Pi^{1,j}(W_{s-}, \alpha) - V^{j,K}_s|^2 \right] da
$$

$$
+ \frac{C(Ke^{-M^q} + e^{-K^q})}{N-1} \sum_{j=2}^N \int_0^1 \mathbb{E} \left[ (1 + |V^{1,K}_s| + |V^{j,K}_s|) \times e^{C(|W^1_s|^q + |\Pi^{1,j}(W_{s-}, \alpha)|^q + |W^{j}_s|^q)} \right] da
$$

$$
+ \frac{C}{N-1} \sum_{j=2}^N \int_0^1 \mathbb{E} \left[ |\Pi^{1,j}(W_{s-}, \alpha) - W^{j}_s|^2 \times (1 + |W^1_s| + |\Pi^{1,j}(W_{s-}, \alpha)| + |W^{j}_s|)^2 \right] da
$$

Proceeding as in (3.10), we deduce that the last line is bounded, for all $\varepsilon \in (0, 1)$, by

$$
C\varepsilon \left( \varepsilon_s^N(f_s) \right)^{2+\varepsilon/2}
$$

using (3.3), the first term is bounded by

$$
4M\varepsilon^N_s(f_t) + 6Mu_s.
$$

Using finally the Cauchy-Schwarz inequality, that, thanks to Proposition 3.2 (i) and by exchangeability, $E[\int_0^1 |\Pi^{1,j}(W_{s-}, \alpha)|^2 da] = \int_{\mathbb{R}^3} |v|^2 f_s(dv)$ which is uniformly bounded of $s$ and $K$. And (1.20), we easily bound the second line by $C(Ke^{-M^q} + e^{-K^q})$ (recall that $W^1_s \sim f_s$, and that, by Lemma 3.3 (b), $E[\int_0^1 e^{C(|W^1_s|^q + |\Pi^{1,j}(W_{s-}, \alpha)|^q + |W^{j}_s|^q)} da] = \int_{\mathbb{R}^3} e^{Cau} f_s(du)$).

Recalling that $\delta_{N,t} := \sup_{[0,t]} \varepsilon^N_s(f_s)$, we thus have, for any $M > 1$, any $\varepsilon \in (0, 1)$,

$$
u_t \leq 6M \int_0^t u_s ds + C t (Ke^{-M^q} + e^{-K^q}) + C\varepsilon t \delta_{N,t}^{1-\varepsilon/2}.
$$

Thus by Grönwall’s Lemma,

$$
u_t \leq C\varepsilon t (Ke^{-M^q} + e^{-K^q} + \delta_{N,t}^{1-\varepsilon/2}) e^{6Mt}.
$$

Choosing $M = 2K$ and using that $Ke^{-(2K)^q} \leq Ke^{-K^q}$, we deduce that

$$
\sup_{[0,T]} u_t \leq C\varepsilon T (e^{-K^q} + \delta_{N,t}^{1-\varepsilon/2}) e^{12KT} = C\varepsilon T (e^{-K^q} + (\sup_{[0,T]} \varepsilon^N_s(f_s))^{1-\varepsilon/2}) e^{12K T}.
$$

**Final step.** We conclude as usual, using that $E[|W^2_{2\mu_s^N, K}(f_t)|] \leq 2\varepsilon^N_s(f_t) + 2u_t + \frac{C}{N}$ to obtain 3.3. \qed
3.4. Decoupling. In this part, we are going to prove the following proposition:

**Proposition 3.6.** Let $B$ be a collision kernel satisfying (1.3), (1.4) and (1.5) or (1.6) and let $f_0 \in P_2(\mathbb{R}^3)$ not be a Dirac mass. If $0 < \gamma \leq 1$, assume additionally (1.7) and (1.19). Consider the unique weak solution $(f_t)_{t \geq 0}$ to (1.1) defined in Theorem 1.3 and, for each $N \geq 1$, we have there exist some constant $C > 0$, such that, for any $0 < \varepsilon < 1$, there exist $C_{\varepsilon} > 0$

$$\varepsilon_i^N(f_t) \leq C_{\varepsilon}(\frac{Ct}{N^\varepsilon})^{1-\varepsilon}.$$

**Lemma 3.7.** Recall that $G$ was defined in (1.8) and that the deviation functions $c$ and $c_K$ were defined in (1.16). Assume (1.3), (1.4), for any $v, v, \tilde{v} \in \mathbb{R}^3$, any $K \in [1, \infty)$, we have

(i) for $\gamma \in (0, 1]$

$$\int_0^\infty \int_0^{2\pi} \left( |v + c(v, v, z, \varphi) - \tilde{v}|^2 - |v - \tilde{v}|^2 \right) d\varphi dz \leq C(1 + |v|^{2+2\gamma} + |\tilde{v}|^2 + |v_\ast|^{2+2\gamma})$$

(ii) We set $\Phi(x) = \pi \int_0^\infty (1 - \cos G(z/x^\gamma)) dz$

$$\int_0^\infty \int_0^{2\pi} \left( |v + c(v, v, z, \varphi) - \tilde{v} - c(\tilde{v}, v, z, \varphi + \varphi_0(v - v_\ast, \tilde{v} - v_\ast))|^2 - |v - \tilde{v}|^2 \right) d\varphi dz$$

$$\leq A_1(v, v, \tilde{v}, v_\ast) + A_2(v, v, \tilde{v}, v_\ast),$$

where

$$A_1(v, v_\ast, \tilde{v}, v_\ast) = 2|v - v_\ast| |\tilde{v} - v_\ast| \int_0^\infty |G(z/|v - v_\ast|^\gamma) - G(z/|\tilde{v} - v_\ast|^\gamma)|^2 dz,$$

$$A_2(v, v_\ast, \tilde{v}, v_\ast) = -(v - \tilde{v}) \cdot [(v - v_\ast) \Phi(|v - v_\ast|) - (\tilde{v} - v_\ast) \Phi(|\tilde{v} - v_\ast|)].$$

**Proof.** In lemma 2.2 we choose $\tilde{v}_\ast = v_\ast$, we have

$$\int_0^\infty \int_0^{2\pi} \left( |v + c(v, v, z, \varphi) - \tilde{v}|^2 - |v - \tilde{v}|^2 \right) d\varphi dz$$

$$= \int_0^\infty \int_0^{2\pi} \left( |c(v, v_\ast, z, \varphi)|^2 + 2(v - \tilde{v}) \cdot c(v, v, z, \varphi) \right) d\varphi dz$$

Since

$$\int_0^\infty \int_0^{2\pi} c(v, v_\ast, z, \varphi)^2 d\varphi dz = \pi |v - v_\ast|^2 \int_0^\infty (1 - \cos G(z/|v - v_\ast|^\gamma)) dz \leq C|v - v_\ast|^{2+\gamma}$$

and

$$\left| \int_0^\infty \int_0^{2\pi} c(v, v_\ast, z, \varphi) d\varphi dz \right| = \left| (v - v_\ast) \pi \int_0^\infty (1 - \cos G(z/|v - v_\ast|^\gamma)) dz \right| \leq C|v - v_\ast|^{1+\gamma}.$$

We conclude (i).

(ii) It is nothing but Lemma 2.2 with $v_\ast = \tilde{v}_\ast$ and $K = \infty$. We finished the proof. 

**Lemma 3.8.** Recall $A_1$ in Lemma 3.7, we have: for any $v, v_\ast, \tilde{v} \in \mathbb{R}^3$, any $M \in [1, \infty)$ and any $q > \gamma$ we have for $0 < \gamma \leq 1,$

$$A_1(v, v_\ast, \tilde{v}, v_\ast) + A_2(v, v_\ast, \tilde{v}, v_\ast) \leq CM|v - \tilde{v}|^2 + C_q e^{-M^\gamma} e^{C_q(|v|^\gamma + |\tilde{v}|^\gamma + |v_\ast|^\gamma)}.$$
Proof. For \( 0 < \gamma < 1 \), using \([1.10]\) and that \(|x^\gamma - y^\gamma| \leq 2|x - y|/(x^{1-\gamma} + y^{1-\gamma})\), we get
\[
A_1(v, v_*, \bar{v}, v_*) \leq 2c_4 |v - v_*||\bar{v} - v_*| |v - v_*| \gamma + |\bar{v} - v_*| \gamma
\leq 8c_4 \frac{|v - v_*| \gamma + |\bar{v} - v_*| \gamma}{|v - v_*| \gamma + |\bar{v} - v_*| \gamma} (|v - v_*| - |\bar{v} - v_*|)^2.
\]
For any \( M \geq 1 \), we have
\[
A_1(v, v_*, \bar{v}, v_*) \leq 8c_4 M |v - \bar{v}|^2 + 8c_4 (|v - v_*| \gamma + |\bar{v} - v_*| \gamma)^2 M \{ (\frac{|v - v_*| \gamma + |\bar{v} - v_*| \gamma}{|v - v_*| \gamma + |\bar{v} - v_*| \gamma}) \geq M \}
\leq CM |v - \bar{v}|^2 + 8c_4 \left( \frac{1}{M} |v - v_*| \gamma + |\bar{v} - v_*| \gamma \right)^2 M \{ (\frac{|v - v_*| \gamma + |\bar{v} - v_*| \gamma}{|v - v_*| \gamma + |\bar{v} - v_*| \gamma}) \geq M \}
\leq CM |v - \bar{v}|^2 + C_q e^{-M^2} C_4 (|v|^\gamma + |v_*|^\gamma).
\]
For \( \gamma = 1 \)
\[
A_1(v, v_*, \bar{v}, v_*) \leq 2c_4 |v - v_*||\bar{v} - v_*| (|v - v_*| - |\bar{v} - v_*|)^2
\leq 8c_4 (|v - v_*| \gamma + |\bar{v} - v_*| \gamma)^2 |v - \bar{v}|^2
\leq CM |v - \bar{v}|^2 + C_q e^{-M^2} C_4 (|v|^\gamma + |v_*|^\gamma)
\]
Recall that \( \Phi(x) = \pi \int_0^\infty (1 - \cos(G(z/x^\gamma)))dz \), it’s not hard to see that there is \( C \) such that for \( 0 < \gamma \leq 1 \)
\[
\Phi(x) \leq C x^\gamma, \quad |\Phi(x) - \Phi(y)| \leq C|x^\gamma - y^\gamma|.
\]
Hence, for all \( X, Y \in \mathbb{R}^3 \),
\[
|X \Phi(|X|) - Y \Phi(|Y|)| \leq C |X - Y||X|^\gamma + C|Y|||X|^\gamma - |Y|^\gamma|.
\]
Symmetrically, we have
\[
|X \Phi(|X|) - Y \Phi(|Y|)| \leq C |X - Y||Y|^\gamma + C|X|||X|^\gamma - |Y|^\gamma|.
\]
So,
\[
|X \Phi(|X|) - Y \Phi(|Y|)| \leq C |X - Y|||X|^\gamma + |Y|^\gamma| + C(|X| + |Y|)||X|^\gamma - |Y|^\gamma|.
\]
We know that \(|x^\gamma - y^\gamma| \leq 2|x - y|/(x^{1-\gamma} + y^{1-\gamma})\) for \( x > 0, y > 0 \), then
\[
|X \Phi(|X|) - Y \Phi(|Y|)| \leq C |X - Y|||X|^\gamma + |Y|^\gamma|.
\]
We now look at
\[
A_2(v, v_*, \bar{v}, v_*) \leq C|v - \bar{v}|^2 (|v - v_*|^\gamma + |\bar{v} - v_*|^\gamma)
\]
Since
\[
|v - \bar{v}|^2 (|v - v_*|^\gamma + |\bar{v} - v_*|^\gamma) \leq M |v - \bar{v}|^2 (|v - v_*|^\gamma + |\bar{v} - v_*|^\gamma)
+ |v - \bar{v}|^2 (|v - v_*|^\gamma + |\bar{v} - v_*|^\gamma) \{ (|v - v_*|^\gamma + |\bar{v} - v_*|^\gamma) \geq M \}
\leq CM |v - \bar{v}|^2 + C_q e^{-M^2} C_4 (|v|^\gamma + |v_*|^\gamma)
\]
This finishes the proof.

We also denote measure $\mu^N_w := N^{-1} \sum_{i=1}^N \delta_{w_i}$.

Given $k \in \{1, ..., N\}$ fixed, we are going to construct $k$ independent cutoff nonlinear processes $\tilde{W}^1, ..., \tilde{W}^k$ such that $\mathbb{E}[\|\tilde{W}^i - W^i\|^2]$ is small, for $i = 1, ..., k$. We recall that the Poisson random measures $(M_{ij}(ds, da, dz, d\varphi))_{1 \leq i < j \leq N}$ are independent, that $M_{ij}(ds, da, dz, d\varphi) = M_{ji}(ds, da, dz, d\varphi)$, and we introduce a new family $(\tilde{M}_{ij}(ds, da, dz, d\varphi))_{1 \leq i < j \leq N}$ of independent family of Poisson measures on $[0, \infty) \times [0, 1] \times [0, \infty) \times [0, 2\pi]$ with intensity measures $\frac{1}{N} dsdzd\varphi$ (also independent of everything else). And $\tilde{M}_{ii}(ds, da, dz, d\varphi) = 0$ for $i = 1, ..., N$. We now introduce for $i = 1, ..., k$,

$$N_{ij}(dt, da, dz, d\varphi) = \tilde{M}_{ij}(dt, da, dz, d\varphi) \mathbb{1}_{1 \leq j \leq k} + M_{ij}(dt, da, dz, d\varphi) \mathbb{1}_{j > k}$$

which are independent Poisson measures, with intensity $\frac{1}{N} dsdzd\varphi$. We thus know that there is a solution $\tilde{W}^i$ starting from $V_0^i$ solving the stochastic equation

$$\tilde{W}^i_t = V_0^i + \sum_{j=1}^N \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c(W^i_{s-}, \Pi_{s-}^i(W_{s-}, \alpha), z, \varphi, \tilde{\varphi}_{i,j,\alpha,s-}) N_{ij}(ds, da, dz, d\varphi),$$

where $\tilde{\varphi}_{i,j,\alpha,s-} := \varphi_0(W^i_{s-} - \Pi^i_{s-}(W_{s-}, \alpha), \tilde{W}^i_{s-} - \Pi^j_{s-}(W_{s-}, \alpha))$.

**Proposition 3.9.** For fixed $k \geq 1$, the family $\{(\tilde{W}^i_t)_{t \geq 0}\}_{i=1,...,k}$ are i.i.d Boltzmann processes with the same law $(f_t)_{t \geq 0}$.

Since the proof of Proposition 3.9 is almost the same as that of Lemma 3.3 (i), we put it in the appendix.

At the end of the part, we can give the proof of Proposition 3.6.

**Proof.** In order to prove ....... For $i = 1, ..., k$,

$$W^i_t - \tilde{W}^i_t = \sum_{j=1}^k \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} c_{s}^{ij} M_{ij}(ds, da, dz, d\varphi)$$

$$- \sum_{j=1}^k \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} \tilde{c}_{s}^{ij} \tilde{M}_{ij}(ds, da, dz, d\varphi)$$

$$+ \sum_{j=k+1}^N \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} (c_{s}^{ij} - \tilde{c}_{s}^{ij}) M_{ij}(ds, da, dz, d\varphi).$$

where

$$c_{s}^{ij} = c(W^i_{s-}, \Pi_{s-}^i(W_{s-}, \alpha), z, \varphi), \quad \tilde{c}_{s}^{ij} = c(\tilde{W}^i_{s-}, \Pi_{s-}^i(W_{s-}, \alpha), z, \varphi + \tilde{\varphi}_{i,j,\alpha,s-}).$$

By Itô's formula, we have

$$\mathbb{E}[|W^i_t - \tilde{W}^i_t|^2] = J_1^t + J_2^t + J_3^t$$
where

\[ J_1^t = \frac{1}{N-1} \sum_{j=1}^{k} \int_0^t \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \mathbb{E} \left[ |W^i_j - \tilde{W}^i_j + c^i_j|^2 - |W^i_j - \tilde{W}^i_j|^2 \right] ds dz d\phi \]

\[ J_2^t = \frac{1}{N-1} \sum_{j=1}^{k} \int_0^t \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \mathbb{E} \left[ |W^i_j - \tilde{W}^i_j + \tilde{c}^i_j|^2 - |W^i_j - \tilde{W}^i_j|^2 \right] ds dz d\phi \]

\[ J_3^t = \frac{1}{N-1} \sum_{j=K+1}^{N} \mathbb{E} \int_0^t \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \left[ |W^i_j - \tilde{W}^i_j + c^i_j - \tilde{c}^i_j|^2 - |W^i_j - \tilde{W}^i_j|^2 \right] ds dz d\phi \]

\( J_1^t \) and \( J_2^t \) are easy to get; by Lemma 3.7 (i), we have:

\[ J_1^t \leq \frac{C}{N-1} \sum_{j=1}^{k} \int_0^t \int_0^1 \mathbb{E} \left[ |W^i_j, N, |2 + |W^i_j, N, |^2 + |\Pi^{i,j}(W^N_j, \alpha)|^2 + 1 \right] ds d\alpha \]

\[ \leq \frac{C t^k}{N-1} \int_0^t \left[ \int_{\mathbb{R}^3} u^2 f_s(du) + 1 \right] ds \leq \frac{C t^k}{N-1}. \]

By the same way, we also have \( J_2^t \leq \frac{C t^k}{N-1} \). Next, we are going to analysis \( J_3^t \). By Lemma 3.7, we have:

\[ J_3^t \leq \int_0^t B^i(s) ds, \]

where

\[ B^i(s) = \frac{1}{N-1} \int_0^1 \mathbb{E} \left[ \sum_{j \neq i}^N (A_1 + A_2)(W^i_N, \Pi^{i,j}(W^N_j, \alpha), \tilde{W}^i_j, N, k, \Pi^{i,j}(W^N_j, \alpha)) \right] ds. \]

From Lemma 3.8 (i), we get

\[ B^i(s) \leq \frac{C}{N-1} \sum_{j \neq i}^N \int_0^1 \mathbb{E} \left[ |W^i_j, N, |^2 \right] ds + \frac{C e^{-M q}}{N-1} \sum_{j \neq i}^N \int_0^1 \mathbb{E} \left[ \exp \left( C_q (1 + |W^i_j, N, |^q + |\Pi^{i,j}(W^N_j, \alpha)|^q + |\tilde{W}^i_j, N, k, |^q) \right) \right] ds \]

\[ \leq C M \mathbb{E} \left[ W^i_j, N, - \tilde{W}^i_j, N, k, |^2 \right] + C e^{-M q} \]

the last 2 step is since we have already mentioned a lot of times,

\[ \int_0^1 \mathbb{E} \left[ \exp \left( C_q (1 + |W^i_j, N, |^q + |\Pi^{i,j}(W^N_j, \alpha)|^q + |\tilde{W}^i_j, N, k, |^q) \right) \right] ds \leq C \int_{\mathbb{R}^3} \exp (8 C_q u^q) f_s(du) < \infty. \]

Over all, we have

\[ \mathbb{E} \left[ |\tilde{W}^i_j, N, k, - W^i_j, N, |^2 \right] \leq \frac{C t^k}{N} + C M \int_0^t \mathbb{E} \left[ |\tilde{W}^i_j, N, k, - W^i_j, N, K, |^2 \right] ds + C e^{-M q}. \]

By Gronwall’s inequality, we have

\[ \mathbb{E} \left[ |\tilde{W}^i_j, N, k, - W^i_j, N, |^2 \right] \leq C e^{C M \left( \frac{t^k}{N} + e^{-C_t M q} \right)}. \]
By choosing \( M = (-\log \frac{t_k}{N})^{\frac{1}{2}} \), we conclude \( \mathbb{E}[|\tilde{W}_{i}^{t,N,k} - W_{i}^{t,N}|^2] \leq C(-\log \frac{t_k}{N})^{\frac{1}{2}} \frac{t_k}{N} \). Thus, we easily conclude that for all \( 0 < \varepsilon < 1 \)

\[
\mathbb{E}[|\tilde{W}_{i}^{t,N,k} - W_{i}^{t,N}|^2] \leq C_\varepsilon \left( \frac{t}{N} \right)^{1-\varepsilon},
\]

the last inequality following from the fact that \( \gamma/q < 1 \).
Hence, by choosing \( k = N^{-\frac{2}{3}} \) for any \( 0 < \varepsilon < 1 \),

\[
\varepsilon_i^N (f_t) \leq \sup_{i=1,...,k} 2\mathbb{E}[|\tilde{W}_{i}^{t,N} - W_{i}^{t,N}|^2] + \mathbb{E} \left[ W_2^2 \left( \mu^N_{\tilde{W}_{i}^{t,N,k}, f_t} \right) \right]
\leq C_\varepsilon \left( \frac{t}{N^2} \right)^{1-\varepsilon}
\]

\( \Box \)

3.5. **Proof of Theorem 3.5** From Proposition 3.4 and 3.6 we already know for \( 0 < \gamma < 1 \), for any \( 0 < \varepsilon < 1 \), there exist \( C_\varepsilon > 0 \) such that

\[
\sup_{[0,T]} \mathbb{E}[W_2^2(\mu_t^{N,K}, f_t)] \leq C_\varepsilon T \left( \sup_{[0,T]} \varepsilon_i^{N} (f_t) + K^{1-2/\nu} \right)^{1-\varepsilon} + C \frac{1}{N} \leq C_\varepsilon T \left( \frac{t}{N^2} + K^{1-2/\nu} \right)^{1-2\varepsilon}
\]

which finish the proof of point (i). Point (ii) is exactly the same, it is also by Proposition 3.4 and 3.6 we conclude the result.

4. **Extension to the particle system without cutoff**

It remains to check that the particle system without cutoff is well-posed and that we can pass to the limit as \( K \to \infty \) in the convergence estimates. We will need the following rough computations.

**Lemma 4.1.** Assume [1.3], [1.4] and [1.5] or [1.6]. Adopt the notation of Lemma 2.2. There are \( C > 0 \), \( \kappa > 0 \) and \( \delta > 0 \) (depending on \( \gamma, \nu \)) such that for all \( K \in [1, \infty) \), all \( v, v_s, \tilde{v}, \tilde{v}_s \in \mathbb{R}^3 \),

\[
\sum_{i=1}^{3} A^K_i (v, v_s, \tilde{v}, \tilde{v}_s) \leq C(1 + |v| + |v_s| + |\tilde{v}| + |\tilde{v}_s|)^{\kappa}(|v - \tilde{v}|^2 + |v_s - \tilde{v}_s|^2 + K^{-\delta}).
\]

**Proof.** Concerning \( A^K_1 \), we start from (2.11) (this is valid for all \( \gamma \in [0,1] \)) and we deduce that

\[
A^K_1 (v, v_s, \tilde{v}, \tilde{v}_s) \leq 8c_4 (|v - \tilde{v}| \wedge |v_s - \tilde{v}_s|)^\gamma(|v - \tilde{v}| + |v_s - \tilde{v}_s|)^2
\leq C(1 + |v| + |v_s| + |\tilde{v}| + |\tilde{v}_s|)^{\gamma}(|v - \tilde{v}|^2 + |v_s - \tilde{v}_s|^2).
\]

We then make use of (2.22) (also valid for all \( \gamma \in [0,1] \)) to write

\[
A^K_2 (v, v_s, \tilde{v}, \tilde{v}_s) \leq C(|v - \tilde{v}| + |v_s - \tilde{v}_s|)^2(|v - \tilde{v}|^\gamma + |v_s - \tilde{v}_s|)^\gamma
\leq C(1 + |v| + |v_s| + |\tilde{v}| + |\tilde{v}_s|)^{(\gamma+1)}(|v - \tilde{v}|^2 + |v_s - \tilde{v}_s|^2).
\]

For \( A^K_3 \), we separate two cases. Under hypothesis [1.9], we immediately deduce from (2.4) that

\[
A^K_3 (v, v_s, \tilde{v}, \tilde{v}_s) \leq C(1 + |v| + |v_s| + |\tilde{v}| + |\tilde{v}_s|)^{2+2\gamma/\nu} K^{1-2/\nu}.
\]
Under hypothesis (1.5), we have seen (when $\gamma = 1$, at the end of the proof of Lemma 2.4) that $\Psi_K(x) \leq 5x^3 \mathbb{I}_{\{x \geq K/2\}}$, whence $\Psi_K(x) \leq 10x^{2\gamma}/K$ and thus

$$A_x^K(v, v_*, \tilde{v}, \tilde{v}_*) \leq C( |v - v_*| v_0 + |\tilde{v} - \tilde{v}_*|)^{2+2\gamma} K^{-1} \leq C (1 + |v| + |v_*| + |\tilde{v}| + |\tilde{v}_*|)^{2+2\gamma} K^{-1}.$$  

The conclusion follows, choosing $\kappa = 2 + 2\gamma/\nu$ and $\delta = 2/\nu - 1$ under (1.6) and $\kappa = 2 + 2\gamma$ and $\delta = 1$ under (1.5).

Now we can give the

Proof of Proposition 1.4-(ii). We only sketch the proof, since it is quite standard. In the whole proof of Proposition 1.4-(ii), consider the solution to

$$\Psi_K(v, v_*, \tilde{v}, \tilde{v}_*) \leq C( |v - v_*| v_0 + |\tilde{v} - \tilde{v}_*|)^{2+2\gamma} K^{-1} \leq C (1 + |v| + |v_*| + |\tilde{v}| + |\tilde{v}_*|)^{2+2\gamma} K^{-1}.$$  

The existence of a solution (in law) to (4.1) is easily checked, using martingale problems (tightness and consistency), by passing to the limit in (3.1). The main estimates to be used are that, uniformly in $K \in [1, \infty)$ if it solves

$$(4.1) \quad V_t^{1,N,K} = V_0 + \int_0^t \int \int \int c \left( V_s^{1,N,K}, V_s^{2,N,K}, z, \varphi \right) O_1^N ds, dj, dz, d\varphi, \quad i = 1, \ldots, N$$

for some i.i.d. Poisson measures $O_1^N (ds, dz, d\varphi)_{i=1,\ldots,N}$ on $[0, \infty) \times \{1, \ldots, N\} \times [0, \infty) \times [0, 2\pi)$ with intensity measures $ds \left( N^{-1} \sum_{i=1}^N \delta_k (dj) \right) dz d\varphi$.

Step 2. The existence of a solution (in law) to (4.1) is easily checked, using martingale problems methods (tightness and consistency), by passing to the limit in (3.1). The main estimates to be used are that, uniformly in $K \in [1, \infty)$ (and in $N \geq 1$ but this is not the point here),

$$\mathbb{E}[|V_t^{1,N,K}|^2] = \int_{\mathbb{R}^3} |v|^2 f_0 (dv) \quad \text{and} \quad \mathbb{E} \left[ \sup_{[0,T]} |V_t^{1,N,K}| \right] \leq C_T$$

for all $T > 0$. This second estimate is immediately deduced from the first one and the fact that

$$\int_0^\infty \int_{-\infty}^{\infty} c (v, v_*, z, \varphi) \, dz d\varphi \mid v_0 \geq 0 \quad \text{and} \quad C |v_*|^{1+\gamma} \leq C (1 + |v| + |v_*|)^{2+2\gamma}.$$

The tightness is easily checked by using Aldous’s criterion (1.6).

Step 3. Uniqueness (in law) for (4.1) is more difficult. Consider a (cadlag and adapted) solution $(V_t^{1,N,K})_{i=1,\ldots,N,t \geq 0}$ to (4.1). For $K \in [1, \infty)$, consider the solution to

$$V_t^{1,N,K} = V_0 + \int_0^t \int \int \int c_K \left( V_s^{1,N,K}, V_s^{2,N,K}, z, \varphi + \varphi_{s,i,j} \right) O_1^N ds, dj, dz, d\varphi, \quad i = 1, \ldots, N$$

where $\varphi_{s,i,j} := \varphi_0 \left( V_{s-j,N,K} - V_{s-j,N,K} \right) \left( V_{s-i,N,K} - V_{s-i,N,K} \right)$. Such a solution obviously exists and is unique, because the involved Poisson measures are finite (recall that $c_K (v, v_*, z, \varphi) = 0$ for $z \geq K$).

Furthermore, this solution $(V_t^{i,N,K})_{i=1,\ldots,N,t \geq 0}$ is a Markov process with generator $\mathcal{L}_{N,K}$ starting from $(V_0^{i,N})_{i=1,\ldots,N}$ (because the only difference with (4.1) is the presence of $\varphi_{s,i,j}$ which does not change the law of the particle system, see Lemma 3.3 (ii) for a similar claim). Hence Proposition 1.4-(i) implies that the law of $(V_t^{i,N,K})_{i=1,\ldots,N,t \geq 0}$ is uniquely determined.

We next introduce $\tau_{N,K,A} = \inf \{ t \geq 0 : \exists i \in \{1, \ldots, N\}, |V_t^{i,N,K}| + |V_t^{i,N,K}| \geq A \}$. Using, on the one hand, the fact that $(V_t^{i,N,K})_{i=1,\ldots,N,t \geq 0}$ is a.s. cadlag (and thus locally bounded) and, on the other hand, the (uniform in $K$) estimate established in Step 2, one easily gets convinced that

$$(4.2) \quad \forall T > 0, \quad \lim_{A \to \infty} \sup_{K \geq 1} \Pr [ \tau_{N,K,A} \leq T ] = 0.$$
Next, a simple computation shows that
\[
\mathbb{E} |V_{t\wedge T_N,K,A}^{1,N,\infty} - V_{t\wedge T_N,K,A}^{1,N,K}|^2 \leq \frac{1}{N} \sum_{j=1}^{N} \mathbb{E} \left[ \int_{0}^{t\wedge T_N,K,A} \int_{0}^{\infty} \left( |V_{s^-}^{1,N,\infty} - V_{s^-}^{1,N,K} + \Delta_{s,j}^{1,N,K}(z,\varphi)|^2 - |V_{s^-}^{1,N,\infty} - V_{s^-}^{1,N,K}|^2 \right) d\varphi dz \right]
\]
where
\[
\Delta_{s,j}^{1,N,K}(z,\varphi) := c(V_{s^-}^{1,N,\infty}, V_{s^-}^{1,N,K}, z,\varphi) - c_K(V_{s^-}^{1,N,K}, z,\varphi + \varphi_{s,i,j}).
\]
Using Lemmas 2.2 and 4.1 and the fact that all the velocities are bounded by \(A\) until \(\tau_{N,K,A}\), we easily deduce that
\[
\mathbb{E} |V_{t\wedge T_N,K,A}^{1,N,\infty} - V_{t\wedge T_N,K,A}^{1,N,K}|^2 \leq C(1 + A)^{\kappa} K^{-\delta} + C(1 + A)^{\kappa} \int_{0}^{t} \mathbb{E} |V_{s\wedge T_N,K,A}^{1,N,\infty} - V_{s\wedge T_N,K,A}^{1,N,K}|^2 ds
\]
by exchangeability. We now use the Grönwall lemma and then deduce that for any \(A > 0\),
\[
(4.3) \quad \lim_{K \to \infty} \sup_{[0,T]} \mathbb{E} |V_{t\wedge T_N,K,A}^{1,N,\infty} - V_{t\wedge T_N,K,A}^{1,N,K}|^2 = 0.
\]
Gathering (4.2) and (4.3), we easily conclude that for all \(t \geq 0\), \(V_{t}^{1,N,K}\) tends in probability to \(V_{t}^{1,N,\infty}\) as \(K \to \infty\). Thus for any finite family \(0 \leq t_1 \leq \cdots \leq t_l\), \((V_{t_j}^{1,N,K})_{i=1,\ldots,N,j=1,\ldots,l}\) goes in probability to \((V_{t}^{1,N,\infty})_{i=1,\ldots,N,j=1,\ldots,l}\), of which the law is thus uniquely determined. This is classically sufficient to characterize the whole law of the process \((V_{t}^{1,N,\infty})_{i=1,\ldots,N,t \geq 0}\)

**Conclusion.** We thus have the existence of a unique Markov process \((V_{t}^{1,N,\infty})_{i=1,\ldots,N,t \geq 0}\) with generator \(\mathcal{L}_N\) starting from \((V_{0}^{1,N})_{i=1,\ldots,N}\), and it holds that for each \(t \geq 0\), each \(N \geq 2\), \((V_{t}^{1,N,\infty})_{i=1,\ldots,N}\) is the limit in law, as \(K \to \infty\), of \((V_{t}^{1,N,K})_{i=1,\ldots,N}\).

To conclude, we will need the following lemma.

**Lemma 4.2.** Let \(N \geq 2\) be fixed. Let \((X_{t}^{i,N,K})_{i=1,\ldots,N}\) be a sequence of \((\mathbb{R}^3)^N\)-valued random variable going in law, as \(K \to \infty\), to some \((\mathbb{R}^3)^N\)-valued random variable \((X_{t}^{i,N})_{i=1,\ldots,N}\). Consider the associated empirical measures \(\nu_{N,K} := N^{-1} \sum_{i=1}^{N} \delta_{X_{t}^{i,N,K}}\) and \(\nu_{N} := N^{-1} \sum_{i=1}^{N} \delta_{X_{t}^{i,N}}\). Then for any \(g \in \mathcal{P}(\mathbb{R}^3)\),
\[
\mathbb{E} \left[ W_2^2 (\nu_{N,K}, g) \right] \leq \liminf_{K \to \infty} \mathbb{E} \left[ W_2^2 (\nu_{N,K}, g) \right].
\]

**Proof.** First observe that the map \((x_1, \ldots, x_N) \mapsto W_2(N^{-1} \sum_{i=1}^{N} \delta_{x_i}, g)\) is continuous on \((\mathbb{R}^3)^N\).
Indeed, it suffices to use the triangular inequality for \(W_2\) and the easy estimate

Consequently, \(W_2^2 (\nu_{N,K}, g)\) goes in law to \(W_2^2 (\nu_{N}, g)\). Thus for any \(A > 1\), we have
\[
\mathbb{E} \left[ W_2^2 (\nu_{N,K}, g) \wedge A \right] = \lim_{K \to \infty} \mathbb{E} \left[ W_2^2 (\nu_{N,K}, g) \wedge A \right] \leq \liminf_{K \to \infty} \mathbb{E} \left[ W_2^2 (\nu_{N,K}, g) \right].
\]
It then suffices to let \(A\) increase to infinity and to use the monotonic convergence theorem.

This allows us to conclude the proof of our main results.

**Proof of Theorem 4.1 (i)-(ii) when \(K = \infty\).** Recall that (4.1) have already been established when \(K \in [1, \infty)\). Since \((V_{t}^{1,N,\infty})_{i=1,\ldots,N}\) is the limit (in law) of \((V_{t}^{1,N,K})_{i=1,\ldots,N}\) as \(K \to \infty\) for each
for any bounded measurable function $\phi$ where $w$ since $i, \ldots, N$ and the remaining coordinates in positions 2 or $\pi$ mapping, we define $R$ associated to $g$ and $\delta e_i \in \{i : e_i = e_1\}$.

for all $t \geq 0$, $e \in (\mathbb{R}^3)^n$ and any Borel set $B \subseteq \mathbb{R}^3$. We claim that $G$ is a probability kernel from $\mathbb{R}^+ \times (\mathbb{R}^3)^N$ into $\mathbb{R}^3$. With the kernel $G^K$ defined above we can associate a measurable mapping $g^K : \mathbb{R} \times (\mathbb{R}^3)^n \times [0, 1] \mapsto \mathbb{R}^3$ such that $g^K(t, \varepsilon, \alpha)$ has distribution $G(t, \varepsilon, \cdot)$ whenever $\alpha$ is a uniform random variable in $[0, 1]$. Now, given $N \geq 1$, $i \neq j$, for $w \in (\mathbb{R}^3)^N$, we now put

$$
\Pi^{i,j}(w, \alpha) = g(t, w^{i,j}, \alpha)
$$

where $w^{i,j}$ denotes the vector $w$ with its $i$ coordinate removed, the $j$ coordinate in the first position, and the remaining coordinates in positions 2, $\ldots$, $N$ in increasing order. By definition of $\Pi_i^{i,j}(w, \alpha)$, for any bounded measurable function $\phi$ from $\mathbb{R}^3$ to $\mathbb{R}$, we have:

$$
\mathbb{E}\left[\int_0^1 \phi(\Pi_i^{i,j}(Y, \alpha)) \, d\alpha\right] = \mathbb{E}\left[\int_0^1 \phi(g(t, Y^{i,j}, \alpha)) \, d\alpha\right] = \mathbb{E}\left[\int_{\mathbb{R}^3} \phi(u) \frac{\pi_{t,Y}(du \times \{Y_j\})}{\pi_{t,Y}(\mathbb{R}^3 \times \{Y_j\})}\right]
$$

$$
= \frac{1}{N-1} \sum_{j \neq i} \int_{\mathbb{R}^3} \phi(u) \frac{\pi_{t,Y}(du \times \{Y_j\})}{\pi_{t,Y}(\mathbb{R}^3 \times \{Y_j\})} = \int_{\mathbb{R}^3} \phi(u) f_t(du),
$$

Since $\pi_{t,Y}$ is an optimal transference plan between $f_t$ and $g^3$, we then have

$$
\frac{1}{N-1} \sum_{j \neq i} \pi_{t,Y}(du \times \{Y_j\}) = \pi_{t,Y}(\mathbb{R}^3 \times \{Y_j\}) = f_t(du).
$$

Since

$$
\int_{\mathbb{R}^3} \phi(u) f_t(du) = \int_{\mathbb{R}^3} \phi(u) \pi_{t,Y}(du \times \mathbb{R}^3) = \sum_{j \neq i} \int_{\mathbb{R}^3} \phi(u) \pi_{t,Y}(du \times \{Y_j\})
$$

$$
= \frac{1}{N-1} \sum_{j \neq i} \int_{\mathbb{R}^3} \phi(u) G^K(t, Y^{i,j}, du) = \frac{1}{N-1} \sum_{j \neq i} \int_0^1 \phi(G^K(t, Y^{i,j}, \alpha)) d\alpha
$$

$$
= \frac{1}{N-1} \sum_{j \neq i} \int_0^1 \phi(\Pi_i^{i,j}(Y, \alpha)) d\alpha.
$$
From the exchangeability of $\mathbf{Y}$, it is clear that the last expression has the same distribution, for all $j \neq i$. Thus, its expected value must be the same for all $j \neq i$. Hence

$$
\mathbb{E}[\int_0^1 \phi(\Pi^j_t(\mathbf{Y}, \alpha)) \, d\alpha] = \frac{1}{N-1} \mathbb{E}\left[\sum_{l \neq i}^N \int_0^1 \phi(\Pi^j_t(\mathbf{Y}, \alpha)) \, d\alpha\right] = \int_{\mathbb{R}^3} \phi(u) f_t(du).
$$

This finished the proof of (i).

For every fixed measurable set $B \subseteq \mathbb{R}^3$, we have:

$$
\int_0^1 \mathbb{I}_B(\Pi^j_t(w, \alpha)) \, d\alpha = \int_0^1 \mathbb{I}_B(g(t, w^i, j, \alpha)) \, d\alpha = \frac{(N-1)\pi^K_t(B, \{w_j\})}{\mathbb{P}\{l : l \neq i, w_l = w_j\}}
$$

This implies that

$$
\int_0^1 |\Pi^j_t(w, \alpha)| - w_j|^2 \, d\alpha = \frac{N-1}{\mathbb{P}\{l : l \neq i, w_l = w_j\}} \int_{\mathbb{R}^3} (u - w_j)^2 \pi^K_t(du, \{w_j\})
$$

Further, we have

$$
\int_0^1 \frac{1}{N-1} \sum_{j \neq i} |\Pi^j_t(w, \alpha)| - w_j|^2 \, d\alpha = \sum_{j \neq i} \int_{\mathbb{R}^3} (u - w_j)^2 \frac{\pi^K_t(du, \{w_j\})}{\mathbb{P}\{l : l \neq i, w_l = w_j\}}
$$

which completes the proof of (ii).

5.2. Proof of Proposition 3.9. We set a family of random measure $\{\tilde{Q}^{N,k}_i(ds, d\xi, dz, d\varphi)\}_{1 \leq k \leq k}$ on $[0, \infty) \times [0, N] \times [0, \infty) \times [0, 2\pi)$. For any measurable set $A_1 \subseteq [0, \infty)$, $A_2 \subseteq [0, N]$, $A_3 \subseteq [0, \infty)$, $A_4 \subseteq [0, 2\pi)$.

$$
\tilde{Q}^{N,k}_i(A_1, A_2, A_3, A_4) = \sum_{j=1}^N \left[ M_i^N(A_1, (A_2 \cap (j-1, j))) - j, A_3, A_4) \mathbb{I}_{1 \leq j \leq k}
$$

$$
\quad + M_i^N(A_1, (A_2 \cap (j-1, j))) - j, A_3, A_4) \mathbb{I}_{j \geq k} \right]
$$

We can show that the family of Poisson measures $\{\tilde{Q}^{N,k}_i(ds, d\xi, dz, d\varphi)\}_{i=1,...,k}$ are i.i.d on $[0, \infty) \times [0, N] \times [0, \infty) \times [0, 2\pi]$ with intensity measures $ds d\xi dz d\varphi$, independent of $(V^i_0)_{i=1,...,N}$. Then we can rewrite:

$$
\tilde{W}^{i,N}_t = V_0^i + \int_0^t \int_0^N \int_0^\infty \int_0^{2\pi} c(\tilde{W}^{i,N,k}_s, \Pi_s^i(\mathbf{W}^N_{s^-}, \xi), z, \varphi + \varphi_{i, \xi, s}) \tilde{Q}^{N,k}_i(ds, d\xi, dz, d\varphi).
$$

where $\Pi^i_s(\mathbf{W}^N_{s^-}, \xi) = \Pi^i_s(\mathbf{W}^N_{s^-}, \xi - \{\xi\})$ as well as $\varphi_{i, \xi, s} = \varphi_{i, \xi, -\{\xi\}, s}$. Next, we define $\tilde{Q}^{N,k_s}_i(ds, dv, dz, d\varphi)$ to be the point measure on $[0, \infty) \times \mathbb{R}^3 \times [0, \infty) \times [0, 2\pi)$ with the atoms $(t, \Pi^i_t(\mathbf{W}^N_{s^-}, \alpha), z, \varphi + \varphi_{i, \xi, s})$, which means: for any measurable set $B \subseteq [0, \infty) \times \mathbb{R}^3 \times [0, \infty) \times [0, 2\pi)$.

$$
\tilde{Q}^{N,k}_i(B) := \tilde{Q}^{N,k}_i\left(\{(s, \xi, z, \varphi) | (s, \Pi^i_s(\mathbf{W}^N_{s^-}, \xi), z, \varphi + \varphi_{i, \xi, s}) \in B\}\right).
$$

We finally have the expression: for $i = 1, ..., k$,

$$
\tilde{W}^{i,N}_t = V_0^i + \int_0^t \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} c(\tilde{W}^{i,N,k}_s, v, z, \varphi) \tilde{Q}^{N,k}_i(ds, dv, dz, d\varphi).
$$
We can see from [12], the family \( \{ Q_i^{N,k}\}_{i=1,...,k} \) are i.i.d with the same intensity \( dt dz d\varphi f_i(dv) \).

For any bounded, measurable, positive function \( g: \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R} \times [0,2\pi) \rightarrow \mathbb{R} \), we put \( G_i = \int_0^t \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} g(s, v, z, \phi) Q_i^{N,k}(ds, dv, dz, d\varphi) \)

\[
E \left[ \exp(-G_i) \right] = 1 + E \left[ \exp \left( - \int_0^t \int_0^N \int_0^\infty \int_0^{2\pi} g(s, \Pi_s^{N}(W_i^{N}, \xi), z, \varphi + \tilde{\phi}_i, \xi, s_{-}) Q_i^{N,k}(ds, d\xi, dz, d\varphi) \right) \right]
\]

\[
= 1 + E \left[ \int_0^t \int_0^N \int_0^\infty \int_0^{2\pi} \exp \left( -G_s - g(s, \Pi_s^{N}(W_i^{N}, \xi), z, \varphi + \tilde{\phi}_i, \xi, s_{-}) \right) \exp(-G_s) ds dz d\varphi \right]
\]

\[
= 1 + E \left[ \int_0^t \int_0^N \int_0^\infty \int_0^{2\pi} \exp(-G_s) \left( \int_{\mathbb{R}^3} \left( e^{-g(s,v,z,\varphi)} - 1 \right) f_i^K(dv) \right) ds dz d\varphi \right]
\]

We put \( \tau_i = \int_0^\infty \int_0^{2\pi} \int_{\mathbb{R}^3} \left( e^{-g(t,v,z,\varphi)} - 1 \right) dz d\varphi f_i^K(dv) \), \( r_i = E \left[ \exp(-G_i) \right] \)

Then we have the following integration equation

\[
r_i = 1 + \int_0^t r_s \tau_s ds.
\]

We have

\[
E \left[ \exp(-G_i) \right] = r_i = \exp \left( \int_0^t \tau_s ds \right) = \exp \left( \int_0^t \int_0^\infty \int_0^{2\pi} \int_{\mathbb{R}^3} \left( e^{-g(s,v,z,\varphi)} - 1 \right) ds d\varphi dz df_i^K(dv) \right).
\]

This implies for fix \( i = 1,...,k \), \( \tilde{Q}_i^{N,k}(ds, dv, dz, d\varphi) \) to be the point measure on \( [0,\infty) \times \mathbb{R}^3 \times [0,\infty) \times [0,2\pi) \) with intensity \( ds dz d\varphi \). Next, we are going to prove the independence.

We define

\[
G_1^i = \int_0^t \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} g^1(s, v, z, \phi) \tilde{Q}_1^{N,k}(ds, dv, dz, d\varphi)
\]

\[
G_2^i = \int_0^t \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} g^2(s, v, z, \phi) \tilde{Q}_2^{N,k}(ds, dv, dz, d\varphi).
\]

We also define

\[
\tau_1^i = \int_0^\infty \int_0^{2\pi} \int_{\mathbb{R}^3} \left( e^{-g^1(t,v,z,\varphi)} - 1 \right) dz d\varphi f_i^K(dv),
\]

\[
\tau_2^i = \int_0^\infty \int_0^{2\pi} \int_{\mathbb{R}^3} \left( e^{-g^2(t,v,z,\varphi)} - 1 \right) dz d\varphi f_i^K(dv).
\]

The same method in the proof of intensity, we get:

\[
E[\exp(-G_1^i - G_2^i)] = 1 + E \left[ \int_0^t e^{G_1^i - G_2^i} (\tau_1^i + \tau_2^i) ds \right].
\]

We get \( E[\exp(-G_1^i - G_2^i)] = \exp(\int_0^t \tau_1^i + \tau_2^i ds) = \exp(\int_0^t \tau_1^i ds) \exp(\int_0^t \tau_2^i ds) \), which implies the independence of \( \{ \tilde{Q}_i^{N,k}(ds, dz, d\varphi) \}_{i=1,...,k} \). It is clear that \( \{ \tilde{W}_i^{N,k} \}_{i \geq 0} \) depends only on \( V_0^i \) and the random measure \( \tilde{Q}_i^{N,k} \). This finished the proof.
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