3D morphology of a random field from its 2D cross-section

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We show that both aspect ratios of randomly oriented triaxial ellipsoids (representing isosurfaces of an isotropic 3D random field) can be determined from a single 2D cross-section of their sample using the probability distribution of the filamentarity $F$ of the structures seen in the cross-section ($F = 0$ for a circle and $F = 1$ for a line). The probability distribution of $F$ has a robust form with a sharp maximum and truncation that are sensitive to the ellipsoids’ aspect ratios. We show that the aspect ratios of triaxial ellipsoids with randomly distributed dimensions can still be recovered from the probability distribution of $F$. This method is applicable to many shape recognition and classification problems, here illustrated with neutral hydrogen density in the turbulent interstellar medium of the Milky Way. The gas distribution is shown to be filamentary with the mean axis ratio $1:2:20$.

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Introduction.—An efficient approach to quantify the morphology of a random field is based on the classification of its excursion sets [1], i.e., the interior of its isosurfaces. The isosurfaces of a differentiable random field at a sufficiently high level can be approximated by ellipsoids. Thus, we consider the statistical morphology of the cross-sections of a large number of randomly oriented structures (infinitely long cylinders or triaxial ellipsoids) representing the isosurfaces of an isotropic random field. The approach we develop allows us to retrieve the 3D morphology of the objects (i.e., their aspect ratios) from a single 2D cross-section of the system.

The recovery of 3D shapes from their 2D cross-sections or projections is a standard problem of tomography which can be solved if a sufficiently large number of such 2D images is available [e.g., 2]. Recovering the intrinsic morphology in 3D from a single 2D image, which is a common problem in astronomy and cosmology, is more difficult. Approaches based on the Fourier transform of a 2D projection are limited by strong restrictions [3]. A powerful method for morphological analysis of convex objects in 3D, based on a classification using Minkowski functionals, is provided by Blaschke diagrams [4], but we are not aware of any applications of this approach to the deprojection problem from 2D to 3D.

We employ the probability distribution of the filamentarity, a dimensionless morphological characteristic based on the Minkowski functionals of the 2D structures [5] [6]. Unsurprisingly, additional assumptions are required to obtain a 3D information from a single 2D cross-section, but these are not particularly restrictive in our approach. Here we only assume that the objects are oriented isotropically in 3D. We only consider 2D cross-sections in this paper; the statistical properties of 2D projections will be discussed elsewhere.

In 2D, the three Minkowski functionals of a closed contour are its enclosed area $S$, perimeter $P$, and the Euler characteristic [1], but it is convenient to define the dimensionless filamentarity:

$$ F = \frac{P^2 - 4\pi S}{P^2 + 4\pi S}. $$

By definition, $0 \leq F \leq 1$, with $F = 0$ for a circle, $F = (4 - \pi)/(4 + \pi) \approx 0.12$ for a square and $F = 1$ for a line segment (not necessarily straight). For an ellipse of the semi-major and semi-minor axes $a$ and $b$, the area and perimeter are

$$ S = \pi ab, \quad P \approx \pi (a + b) \frac{1 - 3\lambda^4/64}{1 - \lambda^2/4}, \quad \lambda = \frac{a - b}{a + b}, $$

with the accuracy better than 0.2% for $P$.

The filamentarity $F$ is only sensitive to the shape of a structure, but not to its size. Therefore, this diagnostic is especially useful as a quantifier of multi-scale, self-similar, intermittent structures, such as those in turbulent flows. Filamentarity has been used as a shapefinder in astrophysical applications [5] [6], but we are not aware of any earlier exploration of its statistical properties.

Ellipsoids, from filaments to pancakes.—To explore the statistical properties of a cross-section of an isosurface of a random field, we analyze a sample of the cross-sections of a single ellipsoid by isotropically oriented and positioned planes. The ellipsoid dimensions (principal axes) are referred to as the thickness, $T = \min(T, W, L)$, width $W = \text{med}(T, W, L)$, and length $L = \max(T, W, L)$. Triaxial ellipsoids have $T < W < L$, oblate and prolate spheroids have $T < W = L$ and $T < W < L$, respectively, and infinite elliptical cylinders are obtained for $L \rightarrow \infty$. We first fix the shape of the ellipsoid and only vary the cross-section’s orientation and position, and then consider randomly distributed aspect ratios $W/T$ and $L/T$. For the sample of the random cross-sections thus obtained, we compute the probability density function (PDF) of the filamentarity $P(F)$.

For an infinitely long cylinder, $P(F)$ is shown in Fig. [1] for various values of $W/T$. The PDF has a single sharp maximum at a certain $F = F_m$ that depends on $W/T$ remarkably simply, as shown in Fig. [2].
The filamentarity PDF has distinct asymptotic behaviors, also sensitive to $W/T$, exponential at $F \to 0$ and a power law at $F \to 1$. For small $F$ and $W/T > 3$, we have $P \approx a_1 \exp(b_1 F)$ with $a_1 = (8.7 \pm 0.4)(W/T)^{-2.34 \pm 0.06}$ and $b_1 = (5.7 \pm 0.3) + (0.05 \pm 0.02)W/T$. At $F \to 1$, we obtain $P \approx a_2 F^{b_2}$ with $a_2 \approx 0.093 \pm 0.002W/T + 0.06 \pm 0.02$ and $b_2 \approx -0.047 \pm 0.009W/T - 0.97 \pm 0.08$. We also considered infinite cylinders with a rectangular base. The results are very similar with the only significant difference that now $P = 0$ for $F < 0.12$ ($F \approx 0.12$ is the filamentarity of the square).

The effect of a finite length of the structure should be the strongest when the cross-section is nearly aligned with the structure’s major axis, i.e., at large $F$. This expectation is corroborated by Fig. 3 obtained for triaxial ellipsoids with fixed $L/T = 16$ and various widths, $1 \leq W/T \leq 16$. Unlike the case of an infinite cylinder shown in Fig. 1, the PDF of Fig. 3 is truncated: $P = 0$ at $F > F_t \approx 0.73$ (the filamentarity of an ellipse with the axis ratio 16). The truncation filamentarity $F_t$ provides a direct measure of $L/T$.

Infinite cylinders and triaxial ellipsoids have the same modal filamentarity $F_m$, so that Fig. 2 can be used to obtain $W/T$ from $F_m$ in either case. Remarkably, the dependence of $F_t$ on $L/T$ is identical to that of $F_m$ on $W/T$, and both aspect ratios can be recovered from the single curve of Fig. 2. The two most prominent aspects of the filamentarity PDF, its mode and truncation, obtained from a single 2D cross-section provide complete, easily accessible information on the shape of the structures in 3D: both $W/T$ and $L/T$ can be recovered from $F_m$ and $F_t$, respectively, using the curve shown in Fig. 2.

As a circular filament ($W/T = 1, L/T = 16$) gradually turns into a pancake ($W/T = L/T = 16$), $F_m$ increases. This, perhaps counterintuitive, result can easily be understood: a large fraction of the random cross-sections of the filament are nearly circular ($F \approx 0$), whereas the majority of the pancake’s cross-sections are elongated.

**Spheroids, from pancakes to spheres.**—For prolate spheroids ($W = T < L$), the PDF maximum is always at $F_m = 0$ (as one of the cross-sections is circular), so the left tail of $P(F)$ disappears (blue open circles, $W/T = 1$, in Fig. 3), but $F_t$ remains a diagnostic of $L/T$. For oblate spheroids ($T < W = L$), the mode and truncation filamentarities are the same, $F_m = F_t$, as shown with filled red circles in Fig. 3 and, separately, in Fig. 4. As an oblate spheroid (pancake) gradually turns into a sphere, the mode of the PDF shifts to the left, $F_m \to 0$. The value of $F_m = F_t$ immediately gives us the value of
Now, consider how the PDF of $P(F)$ changes when the ellipsoids have randomly distributed aspect ratios, conservatively choosing the uniform probability distribution (any other better localized distribution would make a weaker effect).

Figure 5 shows with filled red circles $P(F)$ for ellipsoids of $W/T = 8$ and $L/T$ uniformly distributed in the interval $L_1 \leq L \leq L_2$, where $L_1/T = 8$ and $L_2/T = 24$, i.e., $L/T \sim U(8, 24)$, with the mean value $\langle L/T \rangle = 16$ and the standard deviation $\sigma_{L/T} = 4.62$. The PDF maximum is at the same value of $F_m$ as for the fixed length $L/T = 16$ (blue line in Fig. 5 and yellow, in Fig. 3), but the right tail of the PDF becomes longer and has an inflection point at $F_i = 0.7$ corresponding to $\langle L/T \rangle = 16$ (provided $W < L_1$). Thus, for ellipsoids with a range of lengths, the inflection point of $P(F)$ occurs at exactly the value of $F$ where $P$ is truncated for ellipsoids whose relative length is equal to $\langle L/T \rangle$. The position of $F_i$, such that $P(F_i) = 0$, provides a direct measure of the maximum length $L_2/T$: $F_i \approx 0.8$ in Fig. 5 corresponding to $L_2/T \approx 24$ from Fig. 2. The extent of the right tail from the inflection point to $F_i$ is proportional to $\sigma_L$: $F_i - F_0 \approx 0.1 \sigma_{L/T}$. Thus, Fig. 2 can be used to determine all the dimension parameters of random-shape ellipsoids from the corresponding values of $F$.

The right tail of the PDF is slightly different when $W > L_1$ as the inflection point disappears [but log $P(F)$ still has an inflection point]. Prolate spheroids with randomly distributed length ($T = W < L_1$) produce a similar picture: the truncation point gives $L_2/T$, and the inflection point gives $\langle L/T \rangle$.

A uniform distribution of the width $W$ at fixed $T$ and $L$ of the ellipsoid has a stronger, but easily understandable, influence on the shape of the PDF. The peak, sensitive to $W/T$, becomes lower and broader (the green line in Fig. 6). Its width, $0.45 \leq F_m \leq 0.6$, reflects directly the range $6 \leq W/T \leq 10$, using Fig. 2.

Application to the galactic turbulence.—The interstellar medium (ISM) of spiral galaxies is involved in transonic turbulent motions [7, 8]. The distribution of neutral hydrogen, an important component of the diffuse ISM, is apparently dominated by filamentary structures which may be partially due to quasi-spherical shells and their parts viewed tangentially, and it remains unclear if the filaments are real or just an artifact of projection [9]. Numerical simulations of the ISM also show filamentary structures produced by compression [10, 11]. However, the morphology of the gas filaments has never been quantified in either observations or simulations.

Figure 6 shows the PDF of the filamentarity of the number density distribution of neutral hydrogen ($H_1$) in the Milky Way obtained from a recent GASS $H_1$ survey [12]. We have subtracted large-scale trends along
and across the gas layer to obtain gas density fluctuations \(n(r, \phi, z)\) with vanishing mean value, \(\langle n \rangle = 0\). We used the 2D cross-section of the isosurfaces of \(n\) at \(r = 16 \text{kpc}, 200^\circ \leq \phi \leq 237^\circ, \abs{z} \leq 4.5 \text{kpc}\), with \((r, \phi, z)\) the Galactocentric cylindrical coordinates [with the Sun at \((r, \phi, z) = (8.5 \text{kpc}, 180^\circ, 0)\)]. Different curves in Fig. 6 are obtained for \(n = \nu \sigma\), where \(1 \leq \nu \leq 5\) and \(\sigma = 0.01 \text{cm}^{-3}\) is the standards deviation of \(n\).

The PDFs of Fig 6 are remarkably similar to those of triaxial ellipsoids as they have a well pronounced maximum and a clear truncation that occurs at the same \(F\) for all the isosurface levels. The latter suggests a self-similar gas distribution. The form of the PDF indicates that the \(\text{H}\) \text{I} distribution is indeed filamentary in 3D (compare Figs 6 and 3). The locations of the maximum at \(F_m\) = 0.10–0.15 and truncation at \(F_t\) = 0.7–0.8 yield, using Fig. 2, the mean width of the filaments as \(W/T = 2–3\) and their largest length as \(L/T \approx 15–25\). A detailed discussion of the \(\text{H}\) \text{I} density fluctuations will be published elsewhere.

Conclusions: What can be learned from the filamentarity PDF?—The probability density function \(\mathcal{P}\) of the filamentarity \(F\) of the structures in a 2D cross-section of a 3D sample of isotropically oriented triaxial ellipsoids can provide rich information about the morphology of the ellipsoids that can be recovered using a unique and simple relation shown in Fig. 2.

If all the ellipsoids have the same axis ratios \(W/T\) and \(L/T\) (but not necessarily the same size):

- \(\mathcal{P}(F)\) has a single pronounced maximum at \(F = F_m\), and \(F_m\) is related uniquely and simply to \(W/T\) as shown in Fig. 2.
- For structures of finite length, \(\mathcal{P} = 0\) for \(F > F_t\), and \(F_t\) depends on \(L/T\) exactly as \(F_m\) depends on \(W/T\).
- Non-zero probability at \(F \rightarrow 0\) indicates that the ellipsoids have a circular cross-section.
- Non-zero probability at \(F \rightarrow 1\) indicates that the length of the ellipsoids is comparable to or exceeds the size of the volume sampled, so that they are effectively infinitely long. Otherwise, this can indicate insufficient resolution of the image.
- When \(F_m = F_t\), the structures are oblate spheroids. The closer \(F_m\) is to zero, the closer the objects are to spheres.
- When \(F_m = 0\), the structures are prolate spheroids.

We have also considered the effect of randomly distributed \(W/T\) and \(L/T\), i.e., random structures of variable shape. For the width and length uniformly distributed over the ranges \(W_1 < W < W_2\) and \(L_1 < L < L_2\), we have shown that:

- The maximum of \(\mathcal{P}\) at \(F = F_m\) becomes flatter and less pronounced, but its position depends on the mean value of \(W\) exactly as above. The width of the maximum is related to the range of \(W\).
- The abrupt truncation of \(\mathcal{P}\) at larger \(F\) is replaced by a range of \(F\) where \(\mathcal{P}\) decreases smoothly before being truncated. Now, \(\mathcal{P}(F)\) acquires an inflection point at \(F = F_t\) corresponding to the mean value of \(L/T\). The eventual truncation to \(\mathcal{P} = 0\) occurs at \(F = F_t\) that uniquely depends on \(L_2/T\), and \(F_i = F_t\) is proportional to \(L_2 - L_1\).

Altogether, the PDF of the filamentarity in a 2D cross-section provides remarkably rich and easily accessible information about the morphology of the 3D random structures which can be recovered from a single, simple dependence shown in Fig. 2. Anisotropic structure distributions produce a secondary maximum in \(\mathcal{P}(F)\), which can be used to characterize the anisotropy as discussed elsewhere.

We have demonstrated the efficiency of this approach to morphological analysis using density isosurfaces of the compressible, turbulent interstellar gas in the Milky Way. We have confirmed that the structures visible in a 2D cross-section of the isosurfaces are consistent with isotropically distributed filaments (rather than shells or other flattened objects) and estimated, for the first time, the relative width and length of the filamentary structures.

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