GLOBAL WELL-POSEDNESS, DISSIPATION AND BLOW UP FOR SEMILINEAR HEAT EQUATIONS IN GENERAL ENERGY SPACES

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Abstract. The purpose in this paper is to determine the global behavior of solutions to the initial-boundary value problems for the focusing energy-subcritical and critical semilinear heat equations by initial data at low energy level in various situations by a unified treatment.

1. Introduction

Let \( \Omega \) be an open set of \( \mathbb{R}^d \) with \( d \geq 1 \). We consider the Cauchy problem of the energy-subcritical semilinear evolution equation:

\[
\begin{aligned}
\partial_t u + Lu + u = |u|^{p-1}u & \quad \text{in } \mathbb{R}_+ \times \Omega, \\
|u(0)| = u_0 & \in H^1(\Omega)
\end{aligned}
\]  

(1.1)

with \( 1 < p < p^* \), and the Cauchy problem of the energy-critical semilinear evolution equation:

\[
\begin{aligned}
\partial_t u + Lu = |u|^{4\frac{d}{d-2}}u & \quad \text{in } \mathbb{R}_+ \times \Omega, \\
|u(0)| = u_0 & \in \dot{H}^1(\Omega)
\end{aligned}
\]  

(1.2)

for \( d \geq 3 \), where \( u_0 = u_0(x) \) is a given complex-valued function on \( \Omega \), \( u = u(t, x) \) is an unknown complex-valued function on \( \mathbb{R}_+ \times \Omega \), \( L \) is a suitable self-adjoint operator on \( L^2(\Omega) \), and \( p^* \) is the critical exponent given by

\[
p^* = \begin{cases} 
\frac{d+2}{d-2} & \text{if } d \geq 3, \\
\infty & \text{if } d = 1, 2.
\end{cases}
\]

Here \( H^1(L) \) and \( \dot{H}^1(L) \) are Sobolev spaces with norms

\[
\|f\|_{H^1(L)} := \|(I + L)^{\frac{d}{2}}f\|_{L^2(\Omega)} \quad \text{and} \quad \|f\|_{\dot{H}^1(L)} := \|L^{\frac{d}{2}}f\|_{L^2(\Omega)},
\]

respectively, where \( I \) is the identity operator on \( L^2(\Omega) \). For the details of definitions, we refer to Definition 1.2 below. For the sake of convenience we set \( \mathcal{E} = H^1(L) \) or \( \dot{H}^1(L) \), and choose \( \mathcal{E} = H^1(L) \) in the case (1.1) and \( \mathcal{E} = \dot{H}^1(L) \) in the case (1.2). The energy functional \( E_L : \mathcal{E} \to \mathbb{R} \) is defined by

\[
E_L(u) = \frac{1}{2}\|u\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}(\Omega)}^{p+1},
\]

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and in particular the energy is dissipated along solutions to (1.1) and (1.2):

$$\frac{d}{dt}E_L(u(t)) = -\int_{\Omega} |u_t(t)|^2 \, dx \leq 0.$$  \hfill (1.3)

The problems (1.1) and (1.2) correspond to the focusing case, while the problems with the nonlinearity term $-|u|^{p-1}u$ correspond to the defocusing case. In the defocusing case, it is proved that all solutions exist globally in time and decay to zero as $t \to \infty$ at least in the subcritical case (see Remark 2.3 below). On the other hand, the situation of the focusing case is completely different. In this case, the global behavior of solutions depends on initial data, that is, the solutions are global or blow up in finite time. Our purpose is to determine the global behavior of solutions by initial data, when the energy of $u_0$ is less than or equal to the mountain pass energy

$$l_L := \inf_{u \in E \setminus \{0\}} \max_{\lambda \geq 0} E_L(\lambda u).$$  \hfill (1.4)

For this purpose, let us introduce the Nehari functional and Nehari manifold:

$$J_L(\phi) := \frac{d}{d\lambda}E_L(\lambda \phi)|_{\lambda = 1} = \|\phi\|^2_E - \|\phi\|^{p+1}_{L^{p+1}(\Omega)},$$

$$N_L := \{\phi \in E \setminus \{0\} : J_L(\phi) = 0\}.$$  \hfill (1.5)

Then the functional $J_L$ is (formally) written as

$$J(u(t)) = \frac{1}{2} \frac{d}{dt}||u(t)||^2_{L^2(\Omega)}$$

for solutions $u$ to (1.1) or (1.2).

In the case when $L$ is the Laplace operator $-\Delta$ on $L^2(\mathbb{R}^d)$ or the Dirichlet Laplacian $-\Delta_D$ on $L^2(\Omega)$, there are many literatures on the global behavior of solutions of (1.1) and (1.2). For the focusing energy-subcritical semilinear heat equations

$$\partial_t u - \Delta u + u = |u|^{p-1}u \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d$$

and

$$\partial_t u - \Delta_D u = |u|^{p-1}u \quad \text{in } \mathbb{R}_+ \times \Omega$$

for bounded domains $\Omega$ of $\mathbb{R}^d$, the behavior of solutions has been well investigated (see [6, 8, 10, 13, 20, 26, 29]). For example, Tsutsumi [29] studied the first and second order semilinear differential equations in the abstract setting, and applied the results to semilinear heat equations and wave equations with low energy initial data, i.e., $E_L(u_0) < l_L$. In the high energy case, i.e., $E_L(u_0) \geq l_L$, the behavior of solutions to semilinear heat equations is studied by Gazzola and Weth [10] and Dickstein, Mizoguchi, Souplet and Weissler [6].

In terms of the focusing and energy-critical case, the pioneer works by Kenig and Merle [17, 18] are well known for semilinear Schrödinger equations and wave equations on $\mathbb{R}^d$ with $d = 3, 4, 5$ in the low energy case (see also Killip and Visan [19] for Schrödinger equations with $d \geq 6$). Recently, Gustafson and Roxanas proved a similar result for the semilinear heat equation

$$\partial_t u - \Delta u = |u|^{p-1}u \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d$$
with \( d = 4 \) (see \([12]\)). Furthermore, Collot, Merle and Raphael gave a complete classification of flow near the ground state solution in higher dimensional case. More precisely, they proved that one of the following always occurs: Global existence and asymptotical attraction by a soliton wave; global existence and dissipation; type I blow up (see \([5]\)). In the high energy case, Schweyer constructed type II blow up solutions in the case \( d = 4 \) (see \([27]\)).

However it seems that there are not the above results on the problems (1.1) and (1.2) when \( L \) is the Robin Laplacian or Schrödinger operators, etc. In this paper, we study the problems (1.1) and (1.2) at low energy level in these situations by a unified treatment. More precisely, in the subcritical case (1.1), we assume that the semigroup \( \{e^{-tL}\}_{t>0} \) generated by \( L \) satisfies the following \( L^2-L^q \)-estimates:

**Assumption A.** \( L \) is a self-adjoint operator on \( L^2(\Omega) \) such that \( \{e^{-tL}\}_{t>0} \) satisfies the following: For any \( 2 \leq q < p^* + 1 \), there exist two constants \( C > 0 \) and \( 0 \leq \omega < 1 \) such that

\[
\|e^{-tL}\|_{L^2(\Omega) \to L^q(\Omega)} \leq Ct^{-\frac{d}{2} \left(\frac{1}{2} - \frac{1}{q}\right)} e^{\omega t}, \quad t > 0.
\]  

(1.6)

In the critical case (1.2), we assume that the kernel of \( \{e^{-tL}\}_{t>0} \) satisfies the Gaussian upper estimate:

**Assumption B.** \( L \) is a non-negative and self-adjoint operator on \( L^2(\Omega) \) such that the kernel \( K_L(t; x, y) \) of \( \{e^{-tL}\}_{t>0} \) satisfies the following: There exist two constants \( c \) and \( C > 0 \) such that

\[
|K_L(t; x, y)| \leq Ct^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{ct}}, \quad t > 0, \quad \text{a.e.} \ x, y \in \Omega.
\]  

(1.7)

**Remark 1.1.** Assumption B is stronger than Assumption A.

Let us define the inhomogeneous Sobolev spaces \( H^1(L) \) under Assumption A and the homogeneous Sobolev space \( \dot{H}^1(L) \) under Assumption B.

**Definition 1.2.**

(i) Suppose that \( L \) satisfies Assumption A. Then the inhomogeneous Sobolev space \( H^1(L) \) is defined by

\[
H^1(L) := \{f \in L^2(\Omega) : \|f\|_{H^1(L)} < \infty\}
\]

with norm

\[
\|f\|_{H^1(L)} := \|(I + L)^{\frac{1}{2}} f\|_{L^2(\Omega)}.
\]

(ii) Suppose that \( L \) satisfies Assumption B. Then the homogeneous Sobolev space \( \dot{H}^1(L) \) is defined by

\[
\dot{H}^1(L) := \{f \in \mathcal{Z}'(L) : \|f\|_{\dot{H}^1(L)} < \infty\}
\]

with norm

\[
\|f\|_{\dot{H}^1(L)} := \|L^{\frac{1}{2}} f\|_{L^2(\Omega)},
\]

where \( \mathcal{Z}'(L) \) is the topological dual of \( \mathcal{Z}(L) \) defined by

\[
\mathcal{Z}(L) := \{f \in L^1(\Omega) \cap \mathcal{D}(L) : L^M f \in L^1(\Omega) \cap \mathcal{D}(L) \text{ for all } M \in \mathbb{Z}\},
\]

and \( \mathcal{D}(L) \) denotes the domain of \( L \).
Then $H^1(L)$ and $\dot{H}^1(L)$ are well defined and complete (see [15] and Appendix A). For these spaces, we have the Sobolev inequalities, which play a fundamental role in studying the problems (1.1) and (1.2).

**Proposition 1.3.** (i) Suppose that $L$ satisfies Assumption A. Then for any $1 < p < p^*$, there exists a constant $C > 0$ such that

$$\|f\|_{L^{p+1}(\Omega)} \leq C \|f\|_{H^1(L)} \quad (1.8)$$

for any $f \in H^1(L)$.

(ii) Let $d \geq 3$. Suppose that $L$ satisfies Assumption B. Then there exists a constant $C > 0$ such that

$$\|f\|_{L^{2d/(d-2)}(\Omega)} \leq C \|f\|_{\dot{H}^1(L)} \quad (1.9)$$

for any $f \in \dot{H}^1(L)$.

For the proof we refer to Appendix B. We note that the number $l_L$ in (1.4) is also characterized as

$$l_L = \inf_{f \in N_L} E_L(f) = \frac{p-1}{2(p+1)} S_{p+1}^{-\frac{2(p+1)}{p-1}}, \quad (1.10)$$

where $S_{p+1} = S_{p+1}(d, L)$ is the best constant of the Sobolev inequality (1.8) if $1 < p < p^*$, and (1.9) if $d \geq 3$ and $p = p^*$ (see Appendix C).

When $\Omega = \mathbb{R}^d$, the Laplace operator $-\Delta$ on $L^2(\mathbb{R}^d)$ satisfies Assumption B. In the rest of this section, let us give other major examples of $L$ satisfying Assumption A or B.

(a) **(The Schrödinger operator with the Dirichlet boundary condition)**

The Schrödinger operator $-\Delta_D + V$ with the Dirichlet boundary condition on an open set $\Omega$ of $\mathbb{R}^d$ with $d \geq 1$ satisfies Assumption A, where $V = V(x)$ is a real-valued measurable function on $\Omega$ such that the infimum of the spectrum of $-\Delta_D + V$ is strictly larger than $-1$, and

$$V = V_+ - V_-, \quad V_+ \geq 0, \quad V_+ \in L^1_{\text{loc}}(\Omega) \quad \text{and} \quad V_- \in K_d(\Omega)$$

(see, e.g., Propositions 2.1 and 3.1 in [16]). Here we say that $V_-$ belongs to the Kato class $K_d(\Omega)$ if

$$\begin{align*}
\limsup_{r \to 0} \int_{\Omega \cap \{|x-y|<r\}} \frac{V_-(y)}{|x-y|^{d-2}} dy &= 0 \quad \text{for } d \geq 3, \\
\limsup_{r \to 0} \int_{\Omega \cap \{|x-y|<r\}} \log(|x-y|^{-1}) V_-(y) dy &= 0 \quad \text{for } d = 2, \\
\sup_{x \in \Omega} \int_{\Omega \cap \{|x-y|<1\}} V_-(y) dy &< \infty \quad \text{for } d = 1
\end{align*}$$

(see Section A.2 in Simon [28]). It is readily seen that the potential $V_-(x) = 1/|x|^{\alpha}$ with $0 \leq \alpha < 2$ if $d \geq 2$ and $0 \leq \alpha < 1$ if $d = 1$ is included in $K_d(\Omega)$ (see Fukaya and Ohta [9]). It should be noted that the potential like $V_-(x) = 1/|x|^2$ as $|x| \to 0$ is excluded from $K_d(\Omega)$ (see Example (e) below).
In addition, if the negative part $V_-$ satisfies
\[
\begin{cases}
\sup_{x \in \Omega} \int_{\Omega} \frac{V_-(y)}{|x-y|^{d-2}} \, dy < \frac{\pi^{\frac{d}{2}}}{\Gamma(d/2 - 1)} & \text{if } d \geq 3, \\
V_- = 0 & \text{if } d = 1, 2,
\end{cases}
\]
then $-\Delta_D + V$ satisfies Assumption B (see Propositions 2.1 and 3.1 in [16]). In particular, $-\Delta_D$ (i.e., the case of $V = 0$) satisfies Assumption B.

(b) (The Neumann Laplacian) Let $\Omega$ be a domain of $\mathbb{R}^d$ having the extension property. Then the Laplace operator $-\Delta_N$ with the Neumann boundary condition on $\Omega$ satisfies Assumption A. Indeed, when $\Omega$ has the extension property, the following Sobolev inequality holds:
\[
\|f\|_{L^q(\Omega)} \leq C \|f\|_{H^1(\Omega)}, \quad f \in H^1(\Omega)
\]
for any $2 \leq q < p^* + 1$. Applying the above estimate to $f = e^{t\Delta_N}u_0$, we have the estimate (1.6).

(c) (The Robin Laplacian on an exterior domain) Let $d \geq 3$ and $\Omega$ be the exterior domain in $\mathbb{R}^d$ of a compact and connected set with Lipschitz boundary. We consider the Laplace operator $-\Delta_\sigma$ on $L^2(\Omega)$ associated with a quadratic form
\[
q_\sigma(f, g) = \int_{\Omega} \nabla f \cdot \nabla g \, dx + \int_{\partial \Omega} \sigma f g \, dS
\]
for any $f, g \in H^1(\Omega)$, where $\sigma$ is a function $\partial \Omega \to \mathbb{R}$ and $\partial \Omega$ denotes the boundary of $\Omega$. Note that $-\Delta_0$ (i.e., the case of $\sigma = 0$) is the Neumann Laplacian on $L^2(\Omega)$. Assume that $\sigma \in L^\infty(\partial \Omega)$. Then $-\Delta_\sigma$ satisfies Assumption B. This is a consequence of the following two estimates: By domination of semigroups, we have
\[
K_{-\Delta_\sigma}(t; x, y) \leq K_{-\Delta_0}(t; x, y), \quad t > 0, \quad \text{a.e. } x, y \in \Omega,
\]
and there exist two constants $c$ and $C > 0$ such that
\[
0 \leq K_{-\Delta_0}(t; x, y) \leq C t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{Ct}}, \quad t > 0, \quad \text{a.e. } x, y \in \Omega,
\]
(see Chen, Williams and Zhao [4]).

In the case $d = 2$, if $\Omega$ is the exterior domain in $\mathbb{R}^2$ of a compact and connected set with $C^2$-boundary and
\[
\operatorname{ess inf}_{x \in \partial \Omega} \sigma(x) > 0,
\]
then $-\Delta_\sigma$ satisfies Assumption B (see Section 2 in Kovařík and Mugnolo [22]).

In the case $d = 1$, let $\Omega = \mathbb{R}^+$ and $-\Delta_\sigma$ is the Laplace operator on $L^2(\mathbb{R}^+)$ associated with a quadratic form
\[
q_\sigma(f, g) = \int_0^\infty f' g' \, dx + \sigma f(0) g(0)
\]
for any $f, g \in H^1(\mathbb{R}^+)$, where $\sigma \geq 0$ is a constant. Then $-\Delta_\sigma$ satisfies Assumption B (see Section 4 in [22]).
(d) (The elliptic operator) Let $L$ be the self-adjoint operator associated with a quadratic form

$$q(f, g) = \int_{\mathbb{R}^d} \left\{ \sum_{k,j=1}^{d} a_{kj} D_k f D_j g + \sum_{k=1}^{d} (b_k D_k f + c_k f D_k g) + a_0 fg \right\} \, dx$$

for any $f, g \in H^1(\mathbb{R}^d)$, where $a_{kj}, b_k, c_k, a_0 \in L^\infty(\mathbb{R}^d)$ are real-valued functions for all $1 \leq j, k \leq d$, and the principle part is elliptic, i.e., there exists a constant $\eta > 0$ such that

$$\sum_{j,k=1}^{d} a_{kj}(x) \xi_j \xi_k \geq \eta |\xi|^2$$

for any $\xi \in \mathbb{C}$, a.e. $x \in \mathbb{R}^d$.

Then $L$ is self-adjoint on $L^2(\mathbb{R}^d)$ and satisfies the Gaussian upper estimate:

There exist three constants $c, C > 0$ and $\omega \geq 0$ such that

$$|K_L(t; x, y)| \leq C t^{-\frac{d}{2}} e^{\omega t} e^{-\frac{|x-y|^2}{ct}}, \quad t > 0, \quad \text{a.e. } x, y \in \Omega$$

(see, e.g., [25]). If $\omega < 1$, then $L$ satisfies Assumption A.

(e) (The Schrödinger operator with a negative inverse-square potential)

The Schrödinger operator $-\Delta - c/|x|^2$ on $\mathbb{R}^d$ with $d \geq 3$, where

$$0 < c \leq \frac{(d-2)^2}{4},$$

satisfies Assumption A, and not Assumption B (see Theorem 1.3 in Ioku and Ogawa [14]).

This paper is organized as follows. In Section 2 we state main results on the global behavior of solutions to (1.1) or (1.2). In Section 3 we provide the results on local well-posedness of (1.1) and (1.2). In Section 4 we show some lemmas on variational estimates. In Section 5 the proofs of main results will be given.

2. Statement of results

First we state the result on the subcritical case (1.1).

Definition 2.1. Let $T \in (0, \infty]$. A function $u : [0, T) \times \Omega \to \mathbb{C}$ is called a solution to (1.1) if $u \in C((0, T'); H^1(L)) \cap C((0, T'); D(L)) \cap C^1((0, T'); L^2(\Omega))$ for any $T' \in [0, T)$, and it satisfies the Duhamel formula

$$u(t) = e^{-t(I+L)} u_0 + \int_0^t e^{-(t-s)(I+L)} |u(s)|^{p-1} u(s) \, ds$$

for any $t \in [0, T)$. The time $T$ is said to be maximal if the solution cannot be extended beyond $[0, T)$, and we denote by $T_m$ the maximal time. We say that $u$ is a global solution if $T_m = +\infty$, and that $u$ blows up in finite time if $T_m < +\infty$.

We shall prove the following:

Theorem 2.2. Suppose that $L$ satisfies Assumption A. Let $u$ be a solution to the problem (1.1) with initial data $u_0 \in H^1(L)$ satisfying $E_L(u_0) \leq l_L$. Then the following assertions hold:
Remark 2.3. In the defocusing and energy-subcritical case, all solutions are global and decay to zero as \( t \to \infty \). Indeed, in this case,

\[
\|u(t)\|_{L^2(\Omega)}^2 + 2 \int_0^t J_L(u(s)) \, ds = \|u_0\|_{L^2(\Omega)}^2,
\]

for any \( t > 0 \). Since \( J_L(u(t)) \geq \|u(t)\|_{H^1(L)}^2 \) for any \( t > 0 \), we find that

\[
\sup_{t \geq 0} \|u(t)\|_{L^2(\Omega)}^2 + 2\|u\|_{L^2(\mathbb{R}_+; H^1(L))}^2 \leq \|u_0\|_{L^2(\Omega)}^2,
\]

which proves that \( u(t) \to 0 \) in \( H^1(L) \) as \( t \to \infty \).

Remark 2.4. When we consider the problem

\[
\begin{cases}
\partial_t u + Lu = |u|^{p-1}u & \text{in } \mathbb{R}_+ \times \Omega, \\
u(0) = u_0 \in H^1(L)
\end{cases}
\]

with \( 1 < p < p^* \), we can prove the same statements on the above problem as in Theorem 2.2 if the infimum of spectrum of \( L \) is strictly positive.

Next we state the result on the critical case (1.2).

Definition 2.5. Let \( T \in (0, \infty) \). A function \( u : [0, T] \times \Omega \to \mathbb{R} \) is called a solution to (1.2) if \( u \in C([0, T]; \dot{H}^1(L)) \cap C^1([0, T]; L^2(\Omega)) \cap L^{\frac{2(d+2)}{d-2}}((0, T') \times \Omega) \) for any \( T' \in [0, T) \), and it satisfies the Duhamel formula

\[
u(t) = e^{-tL}u_0 + \int_0^t e^{-(t-s)L}|u(s)|^{p-1}u(s) \, ds
\]

for any \( t \in [0, T) \).

We shall prove the following:

Theorem 2.6. Suppose that \( L \) satisfies Assumption B. Let \( u \) be a solution to the problem (1.2) with initial data \( u_0 \in \dot{H}^1(L) \) satisfying \( E_L(u_0) \leq l_L \). Then the following assertions hold:

(i) If \( J_L(u_0) \geq 0 \), then \( T_m = +\infty \) for small initial data \( u_0 \) in \( \dot{H}^1(L) \). Furthermore, if \( E_L(u_0) < l_L \) or \( J_L(u_0) > 0 \), then

\[
\lim_{t \to \infty} \|u(t)\|_{\dot{H}^1(L)} = 0.
\]

(ii) If \( J_L(u_0) < 0 \), then \( T_m < +\infty \).
Remark 2.7. In the case when $L$ is the Laplace operator $-\Delta$ on $L^2(\mathbb{R}^d)$ with $d = 3, 4$, we can remove the assumption on smallness of $\|u_0\|_{\dot{H}^1(-\Delta)}$ in (i) of Theorem 2.6 via concentration compactness plus rigidity (see [12]). Furthermore, the global behavior of flow near the ground state solution was completely classified in higher dimensional case $d \geq 7$. More precisely, one of the following always occurs: Global existence and asymptotical attraction by a soliton wave; global existence and dissipation; type I blow up (see [5] and references therein). For any $\varepsilon > 0$, there exists a radially symmetric initial data $u_0 \in H^1(\mathbb{R}^4)$ with $l_{-\Delta} < E_{-\Delta}(u_0) < l_{-\Delta} + \varepsilon$ such that the corresponding solution to (1.2) blows up in type II (see [27]).

Remark 2.8. (i) We can define another homogeneous Sobolev space as the completion of $C^\infty_0(\Omega)$ with respect to $\| \cdot \|_{\dot{H}^1(L)}$: $\dot{H}^1_0(L) := C^\infty_0(\Omega)/\{H^1(L)\}$. The space $\dot{H}^1_0(L)$ is often used in studying partial differential equations with Dirichlet boundary condition on a domain (see, e.g., [21]). We note that $\dot{H}^1_0(-\Delta) \subset \dot{H}^1(\mathbb{R}^d)$, and in the particular case when $L = -\Delta$ on $L^2(\mathbb{R}^d)$,

$$\dot{H}^1_0(-\Delta) = \dot{H}^1(-\Delta) \left( = \dot{H}^1(\mathbb{R}^d) \right).$$

(ii) When we adopt $\dot{H}^1_0(L)$ as the energy space, we can prove the same statements as in Theorem 2.6 under the following weaker assumption than Assumption B: For any $2 \leq q \leq r \leq \infty$ there exists a constant $C > 0$ such that

$$\|e^{-tL}\|_{L^q(\Omega) \to L^r(\Omega)} \leq C t^{-\frac{d}{2}\left(\frac{1}{q} - \frac{1}{r}\right)}, \quad t > 0. \tag{2.1}$$

Indeed, the Sobolev inequality

$$\|f\|_{L^\frac{d}{d-2}(\Omega)} \leq C \|f\|_{\dot{H}^1(L)}, \quad f \in \dot{H}^1_0(L) \tag{2.2}$$

is assured by the assumption (2.1) (see Theorem 6.4 in Chapter 6 from Ouhabaz [25]). Thanks to (2.2), we can obtain the statements in Theorem 2.6 for initial data $u_0 \in \dot{H}^1_0(L)$ in the same way as the case of $\dot{H}^1(L)$. So we may omit the details.

3. Local Theory

3.1. The subcritical case. In this subsection we state a result on local well-posedness for the problem (1.1). For this purpose, we prepare the following:

Lemma 3.1. Suppose that $L$ satisfies Assumption A. Then for any $2 \leq q < p^* + 1$, there exists a constant $C > 0$ such that

$$\|e^{-t(L+I)}\|_{L^q(\Omega) \to L^q(\Omega)} + \|e^{-t(I+L)}\|_{L^q'(\Omega) \to L^2(\Omega)} \leq C t^{-\frac{d}{2}\left(\frac{1}{q} - \frac{1}{q'}\right)} \quad \text{for any } t > 0,$$

where $1/q + 1/q' = 1$.

The estimate for the first term in the left hand side of (3.1) immediately follows from Assumption A, and the estimate for the second term is obtained by the duality argument.
Proposition 3.2. Suppose that $L$ satisfies Assumption A. Let $u_0 \in H^1(L)$. Then the following assertions hold:

(i) (Existence) There exists a maximal time $T_m > 0$, depending only on $\|u_0\|_{H^1(L)}$, such that there exists a solution $u$ to (1.1) on $[0, T_m)$ with $u(0) = u_0$ in the sense of Definition 2.1.

(ii) (Uniqueness) Let $T > 0$. If $u_1, u_2 \in L^\infty([0, T]; H^1(L))$ are solutions to (1.1) with $u_1(0) = u_2(0) = u_0$, then $u_1 = u_2$ on $[0, T]$.

(iii) (Continuous dependence on initial data) The function $T_m : H^1(L) \to (0, \infty]$ is lower semicontinuous. Furthermore, if $u_{0,n} \to u_0$ in $H^1(L)$ as $n \to \infty$ and $u_n$ is a solution to (1.1) with $u_n(0) = u_{0,n}$, then $u_n \to u$ in $L^\infty([0, T]; H^1(L))$ as $n \to \infty$ for any $0 < T < T_m$.

(iv) (Blow-up criterion) If $T_m < +\infty$, then $\lim_{t \to T_m} \|u(t)\|_{H^1(L)} = +\infty$.

(v) (Energy identity)

$$E_L(u(t)) + \int_0^t \int_\Omega |u_s|^2 \, dx \, dt = E_L(u_0), \quad 0 < t < T_m.$$ 

3.2. The critical case. In this subsection we state a result on the case of the problem (1.2). For this purpose, we prepare the following:

Lemma 3.3. Suppose that $L$ satisfies Assumption B. Then the following assertions hold:

(i) For any $1 \leq q \leq r \leq \infty$, there exists a constant $C > 0$ such that

$$\|e^{-tL}\|_{L^q(\Omega) \to L^r(\Omega)} \leq Ct^{-\frac{d}{2}\left(\frac{1}{q} - \frac{1}{r}\right)}, \quad t > 0.$$ 

(ii) Let $1 \leq q < r \leq \infty$ and

$$\frac{1}{\gamma} = \frac{d}{2} \left(\frac{1}{q} - \frac{1}{r}\right).$$

Then there exists a constant $C > 0$ such that

$$\|e^{-tL}f\|_{L^\gamma(\mathbb{R}^+; L^r(\Omega))} \leq C\|f\|_{L^q(\Omega)}$$

for any $f \in L^q(\Omega)$.

(iii) Let $1 \leq q_1 \leq q_2 \leq \infty$ and $1 < \gamma_1, \gamma_2 < \infty$ satisfying

$$\frac{1}{\gamma_2} = \frac{1}{\gamma_1} + \frac{d}{2} \left(\frac{1}{q_1} - \frac{1}{q_2}\right) - 1, \quad \frac{d}{2} \left(\frac{1}{q_1} - \frac{1}{q_2}\right) < 1.$$ 

Then there exists a constant $C > 0$ such that

$$\left\|\int_0^t e^{-(t-s)L}F(s) \, ds\right\|_{L^{\gamma_2}(\mathbb{R}^+; L^{q_2}(\Omega))} \leq C\|F\|_{L^{\gamma_1}(\mathbb{R}^+; L^{q_1}(\Omega))}.$$ 

(iv) There exists a constant $C > 0$ such that

$$\left\|\int_0^t e^{-(t-s)L}F(s) \, ds\right\|_{L^{\infty}(\mathbb{R}^+; H^1(L))} \leq C\|F\|_{L^2(\mathbb{R}^+; L^2(\Omega))}.$$
Proposition 3.4. \( (1.2) \):
in Lemma 3.3, we have the following result on local well-posedness for the problem details.

We define the space-time norm by

\[
\|u\|_{S(I)} := \|u\|_{L^{\frac{2(d+2)}{d-2}}(I \times \Omega)} = \left( \int_I \int_{\Omega} |u(t, x)|^{\frac{2(d+2)}{d-2}} \, dxdt \right)^{\frac{d-2}{2(d+2)}}.
\]
for an interval \( I \subset \mathbb{R}_+ \). By using the fixed point argument and Strichartz estimates in Lemma 3.3, we have the following result on local well-posedness for the problem \( (1.2) \):

**Proposition 3.4.** Suppose that \( L \) satisfies Assumption B. Let \( u_0 \in \dot{H}^1(L) \). Then the following assertions hold:

(i) (Existence) There exists a maximal time \( T_m > 0 \) such that there exists a solution \( u \) to \( (1.2) \) on \( [0, T_m) \) with \( u(0) = u_0 \) in the sense of Definition 2.3.

(ii) (Uniqueness) Let \( T > 0 \). If \( u_1, u_2 \in L^{\frac{2(d+2)}{d-2}}((0, T] \times \Omega) \) are solutions to \( (1.2) \) with \( u_1(0) = u_2(0) = u_0 \), then \( u_1 = u_2 \) on \( [0, T] \).

(iii) (Continuous dependence on initial data) The function \( T_m : \dot{H}^1(L) \to (0, \infty] \) is lower semicontinuous. Furthermore, if \( u_{0,n} \to u_0 \) in \( \dot{H}^1(L) \) as \( n \to \infty \) and \( \{u_n\} \subset \dot{H}^1(L) \) is a solution to \( (1.2) \) with \( u_n(0) = u_{0,n} \), then \( u_n \to u \) in \( L^\gamma([0, T]; \dot{H}^1(L)) \) as \( n \to \infty \) for any \( 1 \leq \gamma < \infty \) and \( 0 < T < T_m \).

(iv) (Blow-up criterion) identity If \( T_m = +\infty \), then \( \|u\|_{S((0, T_m))} = +\infty \).

(v) (Energy identity)

\[
E_L(u(t)) + \int_0^t \int_\Omega |u|^2 \, dxdt = E_L(u_0), \quad 0 < t < T_m.
\]

(vi) (Small data global existence) There exists \( \varepsilon_0 > 0 \) such that if \( \|e^{-tL}u_0\|_{S(\mathbb{R}_+)} < \varepsilon_0 \), then \( T_m = +\infty \) and

\[
\|u\|_{S(\mathbb{R}_+)} \leq 2\varepsilon_0.
\]

In particular, if \( \|u_0\|_{\dot{H}^1(L)} \) is sufficiently small, then \( \|e^{-tL}u_0\|_{S(\mathbb{R}_+)} < \varepsilon_0 \).

We note that the maximal time \( T_m \) depends on the profile of \( u_0 \), not only the size of \( u_0 \). For the proof we refer to Theorem 4.5.1 in Cazenave [2] for instance. Let us give two remarks on continuous dependence on initial data in the critical case.

**Remark 3.5.** The statement (i) of continuous dependence on initial data in Theorem 3.4 is weaker than that of (i) in Theorem 3.2. Indeed, \( u_n \to u \) in \( L^\gamma([0, T]; \dot{H}^1(L)) \) holds for any \( 1 \leq \gamma < \infty \), but possibly not for \( \gamma = \infty \) (see Remark 4.5.2 in [2]). However, in the defocusing case, there is also convergence for \( \gamma = \infty \) (see Remark 4.5.4 (iii) in [2]).
Proof. We consider the Cauchy problem:

\[ \partial_t v + Lv = |v|^{\frac{4}{d-2}}v + e \quad \text{in } (0, T) \times \Omega \]

with initial data \( v(0) = v_0 \in \dot{H}^1(L) \), where \( e = e(t, x) \) is a function on \( (0, T) \times \Omega \). Let \( M > 0 \). Assume that \( v \) satisfies

\[ \|v\|_{L^\infty((0,T);\dot{H}^1(L))} + \|v\|_{S((0,T))} \leq M. \]

Then there exist constants \( \delta = \delta(M) > 0 \) and \( C = C(M, \delta) > 0 \) such that the following assertion holds: If \( e \in L^2((0, T) \times \Omega) \) and \( u_0 \in \dot{H}^1(L) \) satisfy

\[ \|e\|_{L^2((0,T)\times\Omega)} \leq \delta, \quad \|u_0 - v_0\|_{\dot{H}^1(L)} \leq \delta, \]

then there exists a unique strong solution \( u \) to (1.2) on \( (0, T) \times \Omega \) with \( u(0) = u_0 \) such that

\[ \|u - v\|_{L^\infty((0,T);\dot{H}^1(L))} + \|u - v\|_{S((0,T))} \leq C. \]

Proof. We consider the Cauchy problem

\[
\begin{cases}
\partial_t w + Lw = |v + w|^{\frac{4}{d-2}}(v + w) - |v|^{\frac{4}{d-2}}v - e & \text{in } (0, T) \times \Omega, \\
w(0) = w_0 := u_0 - v_0 & \text{in } \dot{H}^1(L).
\end{cases}
\] (3.2)

Define the map \( \Phi \) by

\[ \Phi[w](t) := e^{-tL}w_0 + \int_0^t e^{-(t-s)L} \left( |v + w|^{\frac{4}{d-2}}(v + w) - |v|^{\frac{4}{d-2}}v - e \right) ds, \]

and the complete metric space \( X([0, T]) \) by

\[ X([0, T]) := \left\{ w \in L^\infty([0, T]; \dot{H}^1(L)) \cap L^{\frac{2(d+2)}{d-2}}((0, T) \times \Omega) : \|w\|_{X([0, T])} \leq K_0\|w_0\|_{\dot{H}^1(L)} \right\}, \]

\[ \|w\|_{X([0, T])} := \|w\|_{L^\infty([0, T]; \dot{H}^1(L))} + \|w\|_{S((0,T))}, \]

where \( K_0 \) is the constant such that

\[ K_0 := \sum_{n=1}^{n_0} (4C_1)^n, \]

and \( C_1 \) and \( n_0 \) are determined later. By Lemma 3.3 we estimate

\[ \|\Phi[w]\|_{X([0, T])} \leq C_1\|w_0\|_{\dot{H}^1(L)} + C_2 \left( \|v\|^{\frac{4}{d-2}}_{S((0,T))} + \|w\|^{\frac{4}{d-2}}_{S((0,T))} \right) \|w\|_{S((0,T))} + C_3\|e\|_{L^2([0,T] \times \Omega)} \]

Remark 3.6. We have the following statement: There exists a time \( T = T(u_0) > 0 \) such that if \( u_{0,n} \rightarrow u_0 \) in \( \dot{H}^1(L) \), then \( u_n \rightarrow u \) in \( L^\infty([0, T']; \dot{H}^1(L)) \) for any \( 0 < T' < T \) (see, e.g., Theorem 5.5 in Linares and Ponce [23]). We do not know whether this \( T \) coincides with the maximal time \( T_m \) or not.
for any \( w \in X([0, T]) \). Then, choosing \( \|w_0\|_{\dot{H}^1(L)} \) so small that
\[
C_2\|w\|_{\frac{4}{3}\dot{S}(0, T)} \leq C_2 (K_0C_1)^{\frac{4}{3}} \|w_0\|_{\dot{H}^1(L)} \leq C_1 \|w_0\|_{\dot{H}^1(L)},
\]
we have
\[
\|\Phi[w]\|_{X([0, T])} \leq 2C_1\|w_0\|_{\dot{H}^1(L)} + C_2\|v\|_{\frac{4}{3}\dot{S}(0, T)} \|w\|_{S(0, T)} + C_3\|e\|_{L^2([0, T) \times \Omega)}. \tag{3.3}
\]
Furthermore, splitting the interval \([0, T)\) into intervals \(\{[T_{n-1}, T_n]\}_{n=1}^{n_0}\) such that \(T_0 = 0, T_{n_0} = T\) and
\[
\|w\|_{S([T_{n-1}, T_n])} \leq \|w_0\|_{\dot{H}^1(L)};
\]
\[
C_2\|v\|_{\frac{4}{3}S(T_{n-1}, T_n)} \leq C_1;
\]
\[
C_3\|e\|_{L^2([T_{n-1}, T_n] \times \Omega)} \leq C_1 \|w_0\|_{\dot{H}^1(L)}
\]
for \(n = 1, \ldots, n_0\), we find from (3.3) that
\[
\|\Phi[w]\|_{X([T_0, T_1])} \leq 4C_1\|w_0\|_{\dot{H}^1(L)}. \tag{3.4}
\]
Similarly, we get
\[
\|\Phi[w]\|_{X([T_1, T_2])} \leq 4C_1\|w(T_1)\|_{\dot{H}^1(L)} \leq (4C_1)^2\|w_0\|_{\dot{H}^1(L)},
\]
where we used (3.4) in the last step. Repeating this argument, we obtain
\[
\|\Phi[w]\|_{X([T_{n-1}, T_n])} \leq (4C_1)^n\|w_0\|_{\dot{H}^1(L)}
\]
for \(n = 1, \ldots, n_0\). Hence
\[
\|\Phi[w]\|_{X([0, T])} \leq \sum_{n=1}^{n_0} (4C_1)^n\|w_0\|_{\dot{H}^1(L)} = K_0\|w_0\|_{\dot{H}^1(L)},
\]
which implies that \(\Phi\) is a mapping from \(X([0, T])\) into itself. In a similar argument, we can prove that \(\Phi\) is contractive from \(X([0, T])\) into itself. Hence it follows from the fixed point argument that there exists a unique solution \(w\) to (3.2), and \(u := v + w\) is the required solution to (1.2). Thus we conclude Proposition 3.7. \(\Box\)

As a corollary, we have the following:

**Corollary 3.8.** Suppose that \(L\) satisfies Assumption B. Let \(u_0, u_{0,n} \in \dot{H}^1(L)\) for \(n \in \mathbb{N}\), and let \(u\) and \(u_n\) be solutions to (1.2) with \(u(0) = u_0\) and \(u_n(0) = u_{0,n}\), respectively. If \(u_{0,n} \to u_0\) in \(\dot{H}^1(L)\) as \(n \to \infty\), then
\[
T_m(u_0) \leq \liminf_{n \to \infty} T_m(u_{0,n})
\]
and
\[
\lim_{n \to \infty} \|u_n(t) - u(t)\|_{\dot{H}^1(L)} = 0
\]
for any \(t \in [0, T_m(u_0))\).

For the details of proof, we refer to Remark 2.17 in [17].
4. Some lemmas on variational estimates

In this section we show some lemmas on the elementary variational inequalities. We recall \( E = H^1(L) \) or \( H^1(L) \), and choose \( E = H^1(L) \) in the case (1.1) and \( E = H^1(L) \) in the case (1.2). Let us introduce
\[
\mathcal{M}_L^+ := \{ f \in E : E_L(f) < l_L, J_L(f) \geq 0 \},
\]
\[
\mathcal{M}_L^- := \{ f \in E : E_L(f) < l_L, J_L(f) < 0 \}.
\]
In the following, we denote by \( u = u(t) \) the solution to (1.1) or (1.2) with initial data \( u_0 \). The following lemma states that \( \mathcal{M}_L^\pm \) are invariant under the semiflow associated to (1.1) or (1.2).

**Lemma 4.1.** If \( u_0 \in \mathcal{M}_L^\pm \), then \( u(t) \in \mathcal{M}_L^\pm \) for any \( t \in [0, T_m] \), where double-sign corresponds.

*Proof.* Let \( u_0 \in \mathcal{M}_L^+ \). Then \( u(t) \in \mathcal{M}_L^+ \cup \mathcal{M}_L^- \) for any \( t \in [0, T_m] \), since
\[
E_L(u(t)) \leq E_L(u_0) \tag{4.1}
\]
for any \( t \in [0, T_m] \) by (1.3). Suppose that there exists a time \( t_0 \in (0, T_m) \) such that \( u(t_0) \in \mathcal{M}_L^- \). Then, since \( J_L(u(t)) \) is continuous on \([0, T_m]\), there exists a time \( t_1 \in [0, t_0) \) such that \( J_L(u(t_1)) = 0 \). Hence we see from (1.10) and (4.1) that
\[
l_L \leq E_L(u(t_1)) \leq E_L(u_0).
\]
However this contradicts the assumption \( E_L(u_0) < l_L \). Thus \( u(t) \in \mathcal{M}_L^+ \) for any \( t \in [0, T_m] \). Similarly, the case of \( \mathcal{M}_L^- \) is also proved. The proof of Lemma 4.1 is finished.

**Lemma 4.2.** If \( u_0 \in \mathcal{M}_L^+ \), then there exists \( \delta > 0 \) such that
\[
J_L(u(t)) \geq \delta \| u(t) \|_E^2 \tag{4.2}
\]
for any \( t \in [0, T_m] \).

*Proof.* Since \( E_L(u_0) < l_L \), there exists \( \delta_0 > 0 \) such that
\[
E_L(u_0) \leq (1 - \delta_0)l_L \tag{4.3}
\]
Consider the function
\[
F(y) := \frac{1}{2} y - \frac{S_{p+1}}{p+1} y^{\frac{p+1}{2}}, \quad y \geq 0.
\]
It is readily seen that \( F'(y) = 0 \) if and only if \( y = y_C \), where
\[
y_C := \frac{S_{p+1}}{2(p+1)}.
\]
Then we see that
\[
F(y_C) = \frac{p - 1}{2(p+1)} S_{p+1}^{\frac{2(p+1)}{p+1}} = l_L \quad \text{and} \quad F''(y_C) < 0 \tag{4.4}
\]
Hence it follows from (1.3), (4.1), (4.3) and (4.4) that
\[
F(\| u(t) \|_E) \leq E_L(u(t)) \leq E_L(u_0) \leq (1 - \delta_0)l_L = (1 - \delta_0)F(y_C).
\]
for any \( t \in [0, T_m] \). Note that
\[
\| u(t) \|_E < y_C \quad \text{for any} \quad t \in [0, T_m], \tag{4.5}
\]
Lemma 4.3. If 

\[ \|u(t)\|_\varepsilon \leq \frac{2(p+1)}{p-1} E_L(u(t)) < \frac{2(p+1)}{p-1} l_L = S_{p+1}^{\frac{2(p+1)}{p-1}} = y_C. \]

Since \( F \) is strictly increasing on \((0, y_C)\), there exists \( \delta_1 > 0 \) such that

\[ \|u(t)\|_\varepsilon \leq (1-\delta_1)y_C \quad (4.6) \]

for any \( t \in [0, T_m) \). Next, we consider the function

\[ G(y) := y - S_{p+1}^{p+1} y^{\frac{p+1}{2}}. \]

Then \( G(y) = 0 \) if and only if \( y = 0 \) or \( y = y_C \). Furthermore, \( G'(0) = 1 \) and \( G'(y_C) = -(p-1)/2 \). Hence

\[ G(y) \geq C \min \{y, y_C - y\} \quad (4.7) \]

for any \( 0 < y < y_C \). Therefore, noting (4.5), and taking \( y = \|u(t)\|_\varepsilon \), we deduce from (4.6) and (4.7) that

\[ J_L(u(t)) \geq G(\|u(t)\|_\varepsilon) \geq C \min \{\|u(t)\|_\varepsilon, y_C - \|u(t)\|_\varepsilon\} \geq C\delta_1 \|u(t)\|_\varepsilon \]

for any \( t \in [0, T_m) \). Thus (4.2) is proved. The proof of Lemma 4.2 is complete. \( \square \)

Lemma 4.3. If \( u_0 \in \mathcal{M}_L \), then

\[ J_L(u(t)) < -(p+1)\{l_L - E_L(u(t))\} \]

for any \( t \in (0, T_m) \).

Proof. Let \( t \in [0, T_m) \) be fixed. By Lemma 4.1, we have

\[ J_L(u(t)) < 0 \quad \text{for any} \ t \in [0, T_m). \quad (4.8) \]

Define the function

\[ K(\lambda) := E_L(e^\lambda u(t)), \quad \lambda \in \mathbb{R}. \]

Then we calculate

\[ K'(\lambda) = e^{2\lambda}\|u(t)\|_\varepsilon^2 - e^{(p+1)\lambda}\|u(t)\|_{L^{p+1}(\Omega)}^{p+1}, \quad (4.9) \]

\[ K''(\lambda) = 2e^{2\lambda}\|u(t)\|_\varepsilon^2 - (p+1)e^{(p+1)\lambda}\|u(t)\|_{L^{p+1}(\Omega)}^{p+1}. \]

Hence

\[ K''(\lambda) - (p+1)K'(\lambda) = -(p-1)e^{2\lambda}\|u(t)\|_\varepsilon^2 < 0 \quad (4.10) \]

for any \( \lambda \in \mathbb{R} \), since \( p > 1 \). We note from (4.8) and (4.9) that \( K' \) is continuous in \( \lambda \) and

\[ K'(0) = J_L(u(t)) < 0, \quad K'(\lambda) > 0 \quad \text{for} \ -1 \leq \lambda \leq 0. \]

Then there exists \( \lambda_0 < 0 \) such that \( K'(\lambda_0) = 0 \), which implies that \( e^{\lambda_0}u(t) \in \mathcal{N}_L \) and \( K(\lambda_0) \geq l_L \). Integrating the inequality (4.10) for the interval \((\lambda_0, 0]\), we have

\[ K'(0) - K'(\lambda_0) < (p+1)(K(0) - K(\lambda_0)). \]

From the above, we obtain

\[ J_L(u(t)) = K'(0) - K'(\lambda_0) < (p+1)(K(0) - K(\lambda_0)) \leq (p+1)(E_L(u(t)) - l_L). \]

Thus we conclude Lemma 4.3. \( \square \)
5. Proof of Theorems 2.2 and 2.6

We may assume that $E_L(u_0) < l_L$ without loss of generality. Indeed, we consider the subcritical case (1.1). Let $E_L(u_0) = l_L$. When
\[ E_L(u(t)) = E_L(u_0) \text{ for any } t \in [0, T_m), \] (5.1)
the solution $u$ must be a solution to the stationary problem
\[
\begin{cases}
Lv + v = |v|^{p-1}v & \text{in } \Omega, \\
v \in H^1(L).
\end{cases}
\] (5.2)

Therefore, if (5.2) has a nontrivial solution, then $u \equiv u_0$, and hence, $T_m = +\infty$ and $J_L(u(t)) = 0$ for any $t \geq 0$. If (5.2) has no nontrivial solution, then (5.1) does not occur. On the other hand, when (5.1) does not hold, i.e., there exists a time $t_0 \in (0, T_m)$ such that $E_L(u(t_0)) < E_L(u_0)$, the problem is reduced to the case $E_L(u_0) < l_L$ by regarding $u(t_0)$ as initial data. The critical case (1.2) is similar. Hence, to prove Theorems 2.2 and 2.6, it is sufficient to consider the case of $E_L(u_0) < l_L$.

First we prove Theorem 2.2.

Proof of (i) in Theorem 2.2. Let $u_0 \in \mathcal{M}_L^+$. By the definitions of $E_L$ and $J_L$, we calculate
\[ J_L(u(t)) = -\frac{p-1}{2} \|u(t)\|^2_{H^1(L)} + (p+1)E_L(u(t)). \]

Since $J_L(u(t)) \geq 0$ for any $t \in [0, T_m)$ by Lemma 4.1, we have
\[ \|u(t)\|^2_{H^1(L)} \leq \frac{2(p+1)}{p-1} E_L(u(t)) \leq \frac{2(p+1)}{p-1} E_L(u_0) \]
for any $t \in [0, T_m)$. Then $T_m = +\infty$ by (iii) in Proposition 3.2. Furthermore, we have
\[ \|u(t)\|^2_{L^2(\Omega)} + 2 \int_0^t J_L(u(s)) \, ds = \|u_0\|^2_{L^2(\Omega)} \]
for any $t > 0$ by (1.5). Then we find from Lemma 4.2 that
\[ \sup_{t \geq 0} \|u(t)\|^2_{L^2(\Omega)} + 2\delta \|u\|^2_{L^2(\mathbb{R}_+; H^1(L))} \leq \|u_0\|^2_{L^2(\Omega)}, \]
which proves that
\[ \lim_{t \to \infty} \|u(t)\|_{H^1(L)} = 0. \]
Thus we conclude the assertion (i) in Theorem 2.2. \[ \square \]

The proof of (ii) is done by the argument of proof of Proposition 6.1 in [12]. For completeness, we give the proof.

Proof of (ii) in Theorem 2.2. Let $u_0 \in \mathcal{M}_L^-$. Define
\[ I(t) := \int_0^t \|u(s)\|^2_{L^2(\Omega)} \, ds + A, \quad t \in [0, T_m), \]
where $A > 0$, which is chosen later. Then
\[ I'(t) = \|u(t)\|^2_{L^2(\Omega)} \quad \text{and} \quad I''(t) = -2J_L(u(t)). \]
By Schwarz’ inequality, we estimate
\[ I'(t)^2 = \left( \|u_0\|_{L^2(\Omega)}^2 + 2 \text{Re} \int_0^t (u, u_s)_{L^2(\Omega)} \, ds \right)^2 \]
\[ \leq (1 + \epsilon^{-1}) \|u_0\|_{L^2(\Omega)}^2 + 4(1 + \epsilon) \left( \int_0^t (u, u_s)_{L^2(\Omega)} \, ds \right)^2 \]
\[ \leq (1 + \epsilon^{-1}) \|u_0\|_{L^2(\Omega)}^2 + 4(1 + \epsilon) \left( \int_0^t \|u(s)\|_{L^2(\Omega)}^2 \, ds \right) \left( \int_0^t \|u_s(s)\|_{L^2(\Omega)}^2 \, ds \right) \]
for any \( \epsilon > 0 \), where \((\cdot, \cdot)_{L^2(\Omega)}\) stands for the inner product of \(L^2(\Omega)\). Furthermore, it follows from Lemma 4.3 that
\[ I''(t) \geq 2(p + 1) \{ l_L - E_L(u(t)) \} \geq 2(p + 1) \left( l_L - E_L(u_0) + \int_0^t \|u_s(s)\|_{L^2(\Omega)}^2 \, ds \right). \]
Let \( \alpha > 0 \). By summarizing the above estimates, we have
\[ I''(t)I(t) - (1 + \alpha)I'(t)^2 \geq 2(p + 1) \left( l_L - E_L(u_0) + \int_0^t \|u_s(s)\|_{L^2(\Omega)}^2 \, ds + A \right) \]
\[ - 4(1 + \alpha)(1 + \epsilon) \left( \int_0^t \|u(s)\|_{L^2(\Omega)}^2 \, ds \right) \left( \int_0^t \|u_s(s)\|_{L^2(\Omega)}^2 \, ds \right) \]
\[ - (1 + \alpha)(1 + \epsilon^{-1}) \|u_0\|_{L^2(\Omega)}^2 \]
for any \( t \in (0, T_m) \) and \( \epsilon > 0 \). Noting that \( l_L - E_L(u_0) \) is a positive constant, and choosing \( \alpha, \epsilon \) sufficiently small and \( A \) sufficiently large, we can ensure that
\[ I''(t)I(t) - (1 + \alpha)I'(t)^2 > 0 \]
for any \( t \in (0, T_m) \). This is equivalent to
\[ \frac{d}{dt} \left( \frac{I'(t)}{I(t)^{\alpha+1}} \right) > 0, \]
which implies that
\[ \frac{I'(t)}{I(t)^{\alpha+1}} > \frac{I'(0)}{I(0)^{\alpha+1}} = \frac{\|u_0\|_{L^2(\Omega)}^2}{A^{\alpha+1}} =: a \]
for any \( t \in (0, T_m) \). Integrating the above inequality gives
\[ \frac{1}{\alpha} \left( \frac{1}{I(0)^{\alpha}} - \frac{1}{I(t)^{\alpha}} \right) > at. \]
Hence
\[ I(t)^\alpha > \frac{I(0)^\alpha}{1 - I(0)^\alpha at} \to +\infty \]
as \( t \to 1/(I(0)^\alpha at) =: \tilde{t} \ (< +\infty) \). This shows that
\[ \limsup_{t \to \tilde{t}} \|u(t)\|_{L^2(\Omega)} = +\infty. \] (5.3)
If \( T_m = +\infty \), then the solution \( u \) must satisfy
\[ u \in C([0, T]; L^2(\Omega)) \] for any \( T > 0 \).
by Proposition 3.2. However this contradicts (5.3). Thus we prove that $T_m < +\infty$.

The proof of (ii) in Theorem 2.2 is complete. □

Next we prove Theorem 2.6.

Proof of (i) in Theorem 2.6. Let $u_0 \in \dot{H}^1(L)$ satisfying $J_L(u_0) \geq 0$. The first part, i.e., $T_m = +\infty$, is already proved in (v) of Proposition 3.4. Hence it suffices to prove the latter part:

$$\lim_{t \to \infty} \|u(t)\|_{\dot{H}^1(L)} = 0.$$  

Let $\varepsilon > 0$ be fixed. The solution $u$ to (1.2) is written as

$$u(t) = e^{-tL}u_0 + \int_0^t e^{-(t-s)L}|u(s)|^{\frac{4}{d-2}}u(s) \, ds + \int_0^t e^{-(t-s)L}|u(s)|^{\frac{4}{d-2}}u(s) \, ds$$

$$:= I(t) + II(t) + III(t)$$

for $0 < \tau < t$. By density, there exists $v_\varepsilon \in L^1(\Omega) \cap L^2(\Omega)$ such that

$$\|L^{\frac{1}{2}}u_0 - v_\varepsilon\|_{L^2(\Omega)} < \frac{\varepsilon}{2}.$$  

Then it follows from (i) in Lemma 3.3 that

$$\|I(t)\|_{\dot{H}^1(L)} \leq \|L^{\frac{1}{2}}u_0 - v_\varepsilon\|_{L^2(\Omega)} + Ct^{-\frac{d}{4}}\|v_\varepsilon\|_{L^1(\Omega)}$$

$$\leq \frac{\varepsilon}{2} + Ct^{-\frac{d}{4}}\|v_\varepsilon\|_{L^1(\Omega)}$$

for any $t > 0$. Hence there exists a time $t_1 = t_1(\varepsilon) > 0$ such that

$$\|I(t)\|_{\dot{H}^1(L)} \leq \varepsilon \quad \text{for any } t > t_1. \quad (5.4)$$

As to the second term $II(t)$, we write

$$II(t) = e^{-(t-\tau)L} \int_0^\tau e^{-(\tau-s)L}|u(s)|^{\frac{4}{d-2}}u(s) \, ds.$$  

Since

$$w(\tau) := \int_0^\tau e^{-(\tau-s)L}|u(s)|^{\frac{4}{d-2}}u(s) \, ds \in \dot{H}^1(L)$$

for $0 < \tau < t$, we can apply the same argument as in $I(t)$ to $II(t)$, and hence, there exists a time $t_2 = t_2(\varepsilon) > 0$ such that

$$\|II(t)\|_{\dot{H}^1(L)} \leq \varepsilon \quad \text{for any } t > t_2. \quad (5.5)$$

As to the third term $III(t)$, we estimate

$$\|III(t)\|_{\dot{H}^1(L)} \leq C\|u\|_{S((\tau, t))}^{\frac{d+2}{4}}.$$  

Since $\|u\|_{S(\mathbb{R}^+)} < \infty$ by (iv) of Proposition 3.4, there exist $\tau_0 = \tau_0(\varepsilon) > 0$ such that

$$\|III(t)\|_{\dot{H}^1(L)} \leq C\|u\|_{S((\tau, t))}^{\frac{d+2}{4}} < \varepsilon \quad \text{for any } t > \tau > \tau_0. \quad (5.6)$$

By combining (5.4)–(5.6), we conclude that

$$\lim_{t \to +\infty} \|u(t)\|_{\dot{H}^1(L)} = 0.$$  

The proof of (i) in Theorem 2.6 is finished. □
Proof of (ii) in Theorem 2.6. Let \( u_0 \in M^{-L} \). Suppose that \( T_m = T_m(u_0) = +\infty \). By Proposition D.1 in Appendix D for any \( \varepsilon > 0 \) there exists a function \( v_0 \in H^1(L) \) such that
\[
\|u_0 - v_0\|_{\dot{H}^1(L)} < \varepsilon.
\]
Let \( v \) be a solution to (1.2) with \( v(0) = v_0 \). Choosing \( \varepsilon \) sufficiently small, we have
\[
E_L(v_0) < L \quad \text{and} \quad J_L(v_0) < 0.
\]
Hence we can apply the same argument as in the proof of (ii) in Theorem 2.2 to get
\[
\limsup_{t \to \tilde{t}^-} \|v(t)\|_{L^2(\Omega)} = +\infty
\]
for some \( 0 < \tilde{t} < \infty \). This implies that
\[
\liminf_{t \to \tilde{t}^-} J_L(v(t)) \to -\infty
\]
by (1.5), and hence,
\[
\limsup_{t \to \tilde{t}^-} \|v(t)\|_{\dot{H}^1(L)} = +\infty \quad (5.7)
\]
by the definition of \( J_L \) and Sobolev inequality (1.9). Furthermore, choosing \( \varepsilon \) sufficiently small, we find from Proposition 3.7 and (5.7) that
\[
\limsup_{t \to \tilde{t}^-} \|u(t)\|_{\dot{H}^1(L)} = +\infty.
\]
However this contradicts that
\[
u \in C([0, T]; \dot{H}^1(L)) \quad \text{for any} \quad T > 0
\]
by Proposition 3.4 since \( T_m = +\infty \). Thus we conclude that \( T_m < +\infty \). The proof of (ii) in Theorem 2.6 is complete.

\[\square\]

Appendix A. Definition of homogeneous Sobolev space

In this appendix we mention the definition of homogeneous Sobolev space \( \dot{H}^1(L) \). The definition is based on [15], in which the theory of homogeneous Besov spaces \( \dot{B}^{s}_{p,q}(-\Delta_D) \) generated by the Dirichlet Laplacian \( -\Delta_D \) is established on an arbitrary open set of \( \mathbb{R}^d \). The key points to define \( \dot{B}^{s}_{p,q}(-\Delta_D) \) are the following two facts:

(i) \( L^p \)-boundedness of spectral multiplier operators \( \phi(-\theta \Delta_D) \) for \( 1 \leq p \leq \infty \):
\[
\sup_{\theta > 0} \|\phi(-\theta \Delta_D)\|_{L^p(\Omega) \to L^p(\Omega)} < \infty, \quad (A.1)
\]
provided \( \phi \in C^\infty_0((0, \infty)) \);
(ii) zero is not an eigenvalue of \( -\Delta_D \).

If \( L \) is a non-negative and self-adjoint operator on \( L^2(\Omega) \) satisfying (i) and (ii), then the argument of [15] can be applied to \( L \), and hence, we can define the homogeneous Besov spaces \( \dot{B}^{s}_{p,q}(L) \) generated by \( L \) with norm
\[
\|f\|_{\dot{B}^{s}_{p,q}(L)} := \left\{ \sum_{j=-\infty}^{\infty} \left( 2^{sj} \|\phi_j(\sqrt{L})f\|_{L^p(\Omega)} \right)^q \right\}^{1/q}
\]
for \( s \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \), where \( \{ \phi_j \}_j \) is the Littlewood-Paley dyadic decomposition. Then the homogeneous Sobolev space \( \dot{H}^1(L) \) is defined by
\[
\dot{H}^1(L) := B^1_{2,2}(L). \tag{A.2}
\]
It is proved that \( \dot{H}^1(L) \) is complete (see Theorem 2.5 in [15]), and the definition \((A.2)\) is equivalent to (ii) in Definition 1.2. Therefore, in order to define \( \dot{H}^1(L) \), it is sufficient to show that \( L \) satisfies (i) and (ii) under Assumption B.

As to (i), the spectral multiplier theorem is already established for non-negative self-adjoint operators with Gaussian upper bound (1.7) (see Duong, Ouhabaz and Sikora [7] and also Bui, D’Ancona and Nicola [1]). From this theorem, we obtain \( L^p \)-boundedness \((A.1)\) for \( L \) under Assumption B.

As to (ii), we have the following:

**Proposition A.1.** Let \( L \) be a self-adjoint operator on \( L^2(\Omega) \) satisfying
\[
\|e^{-tL}f\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty \tag{A.3}
\]
for any \( f \in L^2(\Omega) \). Then zero is not an eigenvalue of \( L \).

**Proof.** Suppose that zero is an eigenvalue of \( L \), i.e., there exists a function \( f_0 \in \mathcal{D}(L) \setminus \{0\} \) such that \( Lf_0 = 0 \). Then
\[
\partial_t e^{-tL}f_0 = Le^{-tL}f_0 = e^{-tL}Lf_0 = 0
\]
for any \( t > 0 \), which implies that \( e^{-tL}f_0 \) is a constant in \( t > 0 \). Taking account of \((A.3)\), we have \( e^{-tL}f_0 = 0 \), and hence,
\[
(I + L)^{-1}f_0 = \int_0^\infty e^{-t}e^{-tL}f_0 \, dt = 0 \tag{A.4}
\]
almost everywhere in \( \Omega \). Since \(-1\) belongs to the resolvent set of \( L \), the operator \((I + L)^{-1}\) is injective from \( \mathcal{D}(L) \) to \( L^2(\Omega) \). Therefore we deduce from \((A.4)\) that \( f_0 = 0 \). However this contradicts that zero is an eigenvalue of \( L \). Thus we conclude that zero is not an eigenvalue of \( L \).  

Hence it follows from Proposition \((A.1)\) that zero is not an eigenvalue of \( L \) under Assumption B. Thus \( \dot{H}^1(L) \) is well defined and complete.

**Appendix B. Proof of Proposition 1.3**

We first prove the assertion (i). By using the formula
\[
(I + L)^{-1/2} = \frac{1}{2\sqrt{\pi}} \int_0^\infty t^{-1/2}e^{-t}e^{-tL} \, dt,
\]
we deduce from \((1.6)\) that
\[
\|(I + L)^{-1/2}g\|_{L^{p+1}(\Omega)} \leq C \int_0^\infty t^{-1/2}e^{-t}\|e^{-tL}g\|_{L^{p+1}(\Omega)} \, dt
\]
\[
\leq C\|g\|_{L^2(\Omega)} \int_0^\infty t^{-1/2}t^{-\frac{d}{2}(1/2 - 1/p + 1)}e^{-t} \, dt
\]
\[
\leq C\|g\|_{L^2(\Omega)}
\]
for any \( g \in L^2(\Omega) \), since \( 2 < p + 1 < 2d/(d - 2) \). Hence we conclude that
\[
\|f\|_{L^{p+1}(\Omega)} = \|(I + L)^{\frac{1}{2}}(I + L)^{\frac{1}{2}} f\|_{L^{p+1}(\Omega)} \leq C\|(I + L)^{\frac{1}{2}} f\|_{L^2(\Omega)} = C\|f\|_{H^1(L)}
\]
for any \( f \in H^1(L) \), which concludes the assertion (i).

As to the assertion (ii), under Assumption B, the same arguments as Propositions 3.2 and 3.3 in [15] allow us to obtain the following embedding relations:
\[
\dot{B}^1_{2,2}(L) \hookrightarrow \dot{B}^0_{2d/(d-2),2}(L), \quad \dot{B}^0_{2d/(d-2),2}(L) \hookrightarrow L^{\frac{2d}{d-2}}(\Omega).
\]
Therefore we conclude the assertion (ii) from the above embeddings and the definition (A.2) of \( \dot{H}^1(L) \). The proof of Proposition 1.3 is complete.

Appendix C. Proof of (1.10)

First we show that
\[
\frac{p - 1}{2(p + 1)} S_{p+1}^{2(p+1)} \leq l_L. \tag{C.1}
\]
Let \( f \in \mathcal{N}_L \). Then we write
\[
E_L(f) = \frac{p - 1}{2(p + 1)} \|f\|^2_{\dot{H}^1(L)}.
\]
By the Sobolev inequality (1.8) and (1.9), we estimate
\[
\|f\|^2_{\dot{H}^1(L)} = \|f\|^2_{L^{p+1}(\Omega)} \leq (S_{p+1}\|f\|_{\dot{H}^1(L)})^{p+1},
\]
which implies that
\[
S_{p+1}^{2(p+1)} \leq \|f\|^2_{\dot{H}^1(L)}.
\]
Hence, combining the estimates obtained now, we have
\[
\frac{p - 1}{2(p + 1)} S_{p+1}^{2(p+1)} \leq E_L(f),
\]
which shows (C.1) by taking the infimum of the right hand side over \( f \in \mathcal{N}_L \).

Next we show (1.10). Since \( S_{p+1}^{-1} = \inf \{ \|f\|_{\dot{H}^1(L)} : f \in \mathcal{E}, \|f\|_{L^{p+1}(\Omega)} = 1 \} \), it follows that there exists \( \{f_\varepsilon\}_{\varepsilon > 0} \subset \mathcal{E} \) such that
\[
\|f_\varepsilon\|_{\dot{H}^1(L)} < S_{p+1}^{-1} + \varepsilon \quad \text{and} \quad \|f_\varepsilon\|_{L^{p+1}(\Omega)} = 1. \tag{C.2}
\]
Furthermore, there exists \( \lambda_\varepsilon > 0 \) such that \( \lambda_\varepsilon f_\varepsilon \in \mathcal{N}_L \), and we put \( g_\varepsilon := \lambda_\varepsilon f_\varepsilon \) for \( \varepsilon > 0 \). Then, noting the equality in (C.2) and \( g_\varepsilon \in \mathcal{N}_L \), we write
\[
\|f_\varepsilon\|_{\dot{H}^1(L)} = \frac{\|g_\varepsilon\|_{\dot{H}^1(L)}}{\|g_\varepsilon\|_{L^{p+1}(\Omega)}} = \|g_\varepsilon\|_{L^{p+1}(\Omega)}^{\frac{p}{p+1}} = \left\{ \frac{2(p + 1)}{p - 1} E_L(g_\varepsilon) \right\}^{\frac{p}{2(p+1)}}.
\]
Hence, combining the inequality in (C.2) and the above equality, we have
\[
E_L(g_\varepsilon) < \frac{p - 1}{2(p + 1)}(S_{p+1}^{-1} + \varepsilon)^{\frac{2(p+1)}{p-1}} \tag{C.3}
\]
for any \( \varepsilon > 0 \). Suppose that the right hand side of (C.1) is strictly less than \( l_L \). Then it follows from (C.3) that
\[
E_L(g_\varepsilon) < l_L
\]
for sufficiently small \( \varepsilon > 0 \). This contradicts the definition of \( l_L \), since \( g_\varepsilon \in \mathcal{N}_L \). Thus we conclude (1.10).

**Appendix D.**

**Proposition D.1.** Let \( d \geq 3 \) and \( L \) satisfy Assumption B. Then \( H^1(L) \) is dense in \( \dot{H}^1(L) \).

**Proof.** We see from Proposition 3.4 in [15] that \( \dot{H}^1(L) \) is isomorphic to

\[
\left\{ f \in \mathcal{X}'(L) : \| f \|_{\dot{H}^1(L)} < \infty, \ f = \sum_{j=-\infty}^{\infty} \phi_j(\sqrt{L})f \text{ in } \mathcal{X}'(L) \right\},
\]

where \( \mathcal{X}'(L) \) is the dual space of \( \mathcal{X}(L) \) defined by

\[
\mathcal{X}(L) := \left\{ f \in L^1(\Omega) \cap D(L) : L^M f \in L^1(\Omega) \cap D(L) \text{ for all } M \in \mathbb{N} \right\}.
\]

Let \( f \in \dot{H}^1(L) \). It is readily seen from [D.1] that

\[
L^\frac{1}{2} f = \sum_{j=-\infty}^{\infty} \phi_j(\sqrt{L})L^\frac{1}{2} f \text{ in } L^2(\Omega).
\]

Then, putting

\[
f_N := \sum_{j=-N}^{N} \phi_j(\sqrt{L})f \in H^1(L), \ N \in \mathbb{N},
\]

we have

\[
f_N \to f \text{ in } \dot{H}^1(L) \text{ as } N \to \infty,
\]

which conclude that \( H^1(L) \) is dense in \( \dot{H}^1(L) \). \( \square \)

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