Milnor-Orr invariants from the Kontsevich invariant.

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Abstract

As nilpotent studies in knot theory, we focus on invariants of Milnor, Orr, and Kontsevich. We show that the Orr invariant of degree \( k \) is equivalent to the tree reduction of the Kontsevich invariant of degree \(< 2k\). Furthermore, we will see a close relation between the Orr invariant and the Milnor invariant, and discuss a method of computing these invariants.

Keywords

Knot, Milnor invariant, nilpotent group, Magnus expansion

1 Introduction

In [Mi1, Mi2], Milnor defines his \( \bar{\mu} \)-invariants of links, which extract numerical information from the lower central series of the link groups and the link longitudes. The \( \bar{\mu} \)-invariants have been studied from topological viewpoints (see [IO, CDM, MY] and references therein, and KN for a powerful computation). Furthermore, as a homotopical approach to \( \bar{\mu} \)-invariants, Orr [O] introduced an invariant of “based links”. This Orr invariant provides obstruction of slicing links in a nilpotent sense [O, IO]. However, since the invariant is defined as a homotopy 3-class of a homotopy group, there are few examples of the computation, and it is not clear whether the invariant is properly a generalization of the \( \bar{\mu} \)-invariants or not.

Meanwhile, in quantum topology, a standard way to study nilpotent information is to carefully observe the tree parts of the Kontsevich invariants or LMO functor; see, e.g., [HM, GL, Ma2]. For example, the first non-vanishing term of \( \bar{\mu} \)-invariants is equal to that of a tree reduction of the Kontsevich invariant [HM], with a relation to the Chen integral. Concerning the mapping class group, the Johnson-Morita homomorphism and Goldmann Lie algebra can be also nilpotently studied from the LMO functor (see [Ma1, Ma2]). Furthermore, such observations of tree parts sometimes approach the fundamental homology 3-class of 3-manifolds, together with relations to Massey products [GL].

In this paper, inspired by the works of Massuyeau [Ma1, Ma2], we show that the (Milnor-)Orr invariant can be recovered from the Kontsevich invariant. We should note that these invariants are appropriately graded, and that, given \( k \in \mathbb{N} \), the Orr invariant of degree \( k \) is defined for any based link \( L \) whose \( \bar{\mu} \)-invariants of degree \( \leq k \) are zero. The theorem is as follows:

**Theorem 1.1** (Corollary 3.3). *Given a based link \( L \) whose \( \bar{\mu} \)-invariants of degree \( \leq k \) vanish, the Orr invariant of \( L \) is equivalent to the tree-shaped reduction of the Kontsevich invariant of degree \(< 2k\).*

This theorem is a generalization of the above result [HM] (see Remark 5.1) and gives a topological interpretation of the tree reduction of degree \(< 2k\). Moreover, it is natural for us to ask...
what finite type invariants recover the Orr invariant; as a solution, we suggest a computation of the Orr invariant from HOMFLYPT polynomials (see §A), where this computation is based on [MY].

Furthermore, we will show the equivalence between the Orr invariant of degree $k$ and Milnor $\mu$-invariants of degree $< 2k$; see Theorem 2.3. Accordingly, while the Orr invariant is a homotopy 3-class, the 3-class turns out to be described by the link longitudes, as is implicitly pointed out in [Ma2, Ko]; see the figure below as a summary. Our result is analogous to the result [Ma1] concerning the mapping class group, which claims an equivalence between “the Morita homomorphism” of degree $k$ and “the total Johnson homomorphisms” of degree $< 2k$.

Thus, it can be hoped that and the tree parts of the LMO functor can be described in terms of homology 3-classes or algebraic topology.

| Milnor $\mu$-invariant of degree $< 2k$ | Theorem 2.3 | Orr invariant of degree $k$ | Corollary 2.4 | Tree-reduced Kontsevich invariant of degree $< 2k$ |
|----------------------------------------|-------------|----------------------------|---------------|-----------------------------------------------|
|                                        |             | Recover via HOMFLYPT       |               | [MY]                                          |

This paper is organized as follows. Section 2 reviews Milnor-Orr invariants and states Theorem 2.3, and Section 3 describes the relation to the Kontsevich invariant. Section 4 gives the proofs. Appendix A explains the computation from HOMFLYPT polynomials.

2 Theorem on Milnor-Orr invariant

The first part of this section reviews the Milnor-Orr invariant, and the second part states Theorem 2.3.

2.1 Review of Milnor invariant and Orr invariant

We will start by reviewing string links. Let $I$ be the interval $[0, 1]$, and fix $q \in \mathbb{N}$. A $(q$-component) string link is a smooth embedding of $q$ disjoint oriented arcs $A_1, \ldots, A_q$ in the 3-cube $I^3$, which satisfy the boundary condition $A_i = \{p_i, q_i\}$, where $p_i = (j/2q, 0, 0)$ and $q_i = (j/2q, 0, 1) \in I^3$. We define $SL(q)$ to be the set of string links of $q$-components. Given two string links $T$ and $T'$, we can define another string link, $T \cdot T' \subset [0, 1]^2 \times [0, 2] \cong I^3$, by connecting $q_i$ in $T$ and $p'_i$ in $T'$. If $T = \{A_i\}$ is a string link, then an oriented link $L = \{L_1, \ldots, L_q\}$ can be defined to be $L_i = A_i \cup a_i$, where $a_i$ is a semi-circle in $S^3 \setminus L$ connecting $p_i$ and $q_i$. We call the link the closure of $T$ and denote it by $\overline{T}$; see Figure 1.

Next, we will describe the groups used throughout this paper. Let $F$ be the free group with generator $x_1, \ldots, x_q$. For a group $G$, we define $G_1$ to be $G$ and $G_m$ to be the commutator

![Figure 1: A string link and the closure as a link in $S^3$.](image)
[G_{m-1},G]$ by induction. If $G$ is the free group $G$, then the projection $p_{m-1} : F/F_m \to F/F_{m-1}$ implies the central extension,
\[
0 \to F_{m-1}/F_m \to F/F_m \xrightarrow{p_{m-1}} F/F_{m-1} \to 0 \quad \text{(central extension)}
\]
The abelian kernel $F_{m-1}/F_m$ is known to be free with a finite basis.

Now let us explain the $m$-th leading terms of the Milnor invariant, according to [MH] [O] [KN]. We will suppose that the reader has elementary knowledge of knot theory, as can be found in [CDM] §1 and §12. Given a $q$-component link $L$ in the 3-sphere $S^3$ and $\ell \leq q$, we can uniquely define the (preferred) longitude $l_{\ell} \in \pi_1(S^3 \setminus L)$ of the $\ell$-th component. In addition, let $f_2 : \pi_1(S^3 \setminus L) \to F/F_2 = \mathbb{Z}^q$ be the abelianization $\text{Ab}$. Furthermore, for $k \in \mathbb{N}$, we assume

- **Assumption $\mathfrak{A}_k$.** There are homomorphisms $f_s : \pi_1(S^3 \setminus L) \to F/F_s$ for $s$ with $s \leq k$, which satisfy the commutative diagram,

$$
\begin{array}{c}
\pi_1(S^3 \setminus L) \\
\downarrow f_2 \\
F/F_2 \xrightarrow{p_2} F/F_3 \xrightarrow{p_3} \cdots \xrightarrow{p_{k-1}} F/F_k.
\end{array}
$$

Here, we should note that if there is another extension $f'_k$ instead of $f_k$, then $f_k$ equals $f'_k$ up to conjugacy, by centrality. Further, we should note the following proposition.

**Proposition 2.1 (MH).** Suppose Assumption $\mathfrak{A}_k$. Then, $f_k$ admits a lift $f_{k+1} : \pi_1(S^3 \setminus L) \to F/F_{k+1}$ if and only if all the Milnor invariants of length $< k$ vanish, i.e., $f_k(l_{\ell}) = 0$.

Thus, the map $f_k$ sends the preferred longitude $l_{\ell}$ to the center $F_{k-1}/F_k$. Then, the $q$-tuple,
\[
(f_k(l_1), \ldots, f_k(l_q)) \in (F_{k-1}/F_k)^q,
\]
is called the first non-vanishing Milnor $\mu$-invariant or the Milnor $\mu$-invariant of length $k - 1$. Proposition 2.1 implies that $\mu$-invariant is known to be a complete obstruction for lifting $f_m$. The paper [KN] gives an algorithm to describe $f_m$ explicitly, and a method of computing the Milnor invariants.

We further review the Orr invariant [O], where $L$ satisfies Assumption $\mathfrak{A}_{k+1}$, that is, all the $\mu$-invariants of length $\leq k$ are zero; $f_k(l_{\ell}) = 0$. Fix a homomorphism $\tau : F \to \pi_1(S^3 \setminus L)$ that sends each generator $x_\ell$ to some meridian $m_\ell$ of the $\ell$-th component for $\ell \leq \# L$. This $\tau$ is called a basing, and the pair $(L, \tau)$ is referred as to a based link. As examples that we will refer to later, given a string link $T$, the closure $\overline{T}$ has a canonical basing, where $\tau$ is obtained from choosing the loop circling $\{(j/2q,0,1)\}$. Furthermore, for a group homomorphism $f : G \to H$, we will write $f_* : K(G,1) \to K(H,1)$ for the induced map between Eilenberg-MacLane spaces. We define the space $K_k$ to be the mapping cone,
\[
K_k := \text{Cone}((f_k \circ \tau)_* : K(F,1) \to K(F/F_k,1)).
\]
Then, from the assumption $f_k(l_{\ell}) = 0$, $f_k$ gives rise to a continuous map $\rho_L : S^3 \to K_k$. It is reasonable to consider the homotopy 3-class,
\[
\theta_k(L, \tau) := [\rho_L] \in \pi_3(K_k),
\]
which we call the Orr invariant. The following is a list of known results on the invariant and 
\( \pi_3(K_k) \).

**Theorem 2.2 (\[O\] \[IO\]).** \( (I) \) Let \( N_h \in \mathbb{N} \) be the rank of \( H_2(F/F_h; \mathbb{Z}) = F_h/F_{h+1} \). The following are isomorphisms on \( \pi_3(K_k) \) and on \( H_3(K_k) \):

\[
\pi_3(K_k) \cong \bigoplus_{h=k}^{2k-1} \mathbb{Z}^{q^N_{h-N_{h+1}}}, \quad H_3(K_k; \mathbb{Z}) \cong H_3(K(F/F_k, 1); \mathbb{Z}) \cong \bigoplus_{h=k}^{2k-2} \mathbb{Z}^{q^N_{h-N_{h+1}}}.
\]

Furthermore, the Hurewicz homomorphism \( \mathcal{H} : \pi_3(K_k) \to H_3(K_k; \mathbb{Z}) \) is equal to the projection according to the direct sums on the right-hand sides.

\( (II) \) The lowest summand of \( \theta_k(L, \tau) \) is equivalent to the Milnor invariant of length \( k \); see [\(O\] or \[IO\] \( \S 10\) for details.

\( (III) \) For any element \( \kappa \in \pi_3(K_k) \), there exist a link \( L \) and a homomorphism \( g_k : \pi_1(S^3 \setminus L) \to F/F_k \) satisfying \( \theta_k(L, \tau) = \kappa \).

\( (IV) \) The Orr invariant has additivity with respect to “band connected sums”; see [\(O\] \( \S 3\)]. As a special case, for two string links \( T_1 \) and \( T_2 \) such that the closures \( \overline{T}_1 \) and \( \overline{T}_2 \) satisfy Assumption \( \mathcal{A}_{k+1} \), we have \( \theta(T_1 \cdot T_2, \tau_1) = \theta(T_1, \tau_1) + \theta(T_2, \tau_2) \).

\( (V) \) Let \( \iota_k : K_k \to K_{k+1} \) be the continuous map arising from the projection \( F/F_k \to F/F_{k+1} \). The Orr invariant has functoriality. To be precise, if \( L \) satisfies \( \mathcal{A}_{k+1} \), then the equality \( (\iota_k)_*(\theta_k(L, \tau)) = \theta_{k+1}(L, \tau) \) holds in \( \text{Im}(\iota_k)_* \cap \pi_3(K_{k+1}) \).

Next, we mention the homological reduction of the Orr invariant \( \theta_k(L) \) via the Hurewicz map \( \mathcal{H} : \pi_3(K_k) \to H_3(K_k; \mathbb{Z}) \). Note that the inclusion \( K(F/F_k, 1) \to K_k \) induces the isomorphism,

\[
P^{gr} : H_3(K(F/F_k, 1), \mathbb{Z}) \cong H_3(K_k; \mathbb{Z}),
\]

from the relative homology. To summarize, the value \( \mathcal{H}(\theta_k(L)) \) is the reduction of \( \theta_k(L) \) without the top summand \( \mathbb{Z}^{q^N_{2k-1-N_{2k}}} \). Moreover, by definition, this \( \mathcal{H}(\theta_k(L)) \) can be regarded as the pushforward of the fundamental 3-class \( [S^3 \setminus L, \partial(S^3 \setminus L)] \in H_3(S^3 \setminus L, \partial(S^3 \setminus L); \mathbb{Z}) \cong \mathbb{Z} \):

\[
P^{gr} \circ (f_k)_*(S^3 \setminus L, \partial(S^3 \setminus L)) = \mathcal{H}(\theta_k(L, \tau)) \in H_3(K_k; \mathbb{Z}). \tag{3}
\]

The author [\(N\os\] showed that the cohomology \( H^3(K(F/F_k, 1)) \) are generated by some Massey products; thus, the reduction [\(B\)] are characterized by some Massey products of \( S^3 \setminus L \).

### 2.2 Results: Orr invariant of higher invariants

In order to state the theorem, let us briefly review the Milnor \( \mu \)-invariant for string links (see \[HM\] \[KO\] \[L1\]). For a string link \( T \in SL(q) \), let \( y_j \in \pi_1([0, 1]^3 \setminus T) \) be an element arising from the loop circling \( \{(j/2q, 0, 1)\} \); Let \( G_m \) be the \( m \)-th nilpotent quotient of \( \pi_1([0, 1]^3 \setminus T) \). Then, as is shown \[M2\] \[L1\], the homomorphism \( F \to \pi_1([0, 1]^3 \setminus T) \) which sends \( x_i \) to \( y_i \) descends to an isomorphism between the \( m \)-th nilpotent quotients:

\[
\phi_* : F/F_m \cong \pi_1([0, 1]^3 \setminus T)/G_m, \quad \text{for any } m \in \mathbb{N}. \tag{4}
\]
Here, we should note that, if $T$ is a pure braid, this $\phi_*$ is the identity map. Furthermore, a framing of the $\ell$-th component of $T$ defines a parallel curve which determines an element, $\lambda_\ell \in \pi_1([0,1]^3 \setminus T)$. This $\lambda_\ell$ is referred to as the $\ell$-th longitude of $T$. We call the reduction $\phi_*^{-1}(\lambda_\ell) \in F/F_m$ the $\mu$-invariant of $T$ (of degree $\leq m$). Later, we will omit writing $\phi_*^{-1}$ for simplicity. We should notice, from the definitions, that the closure $\bar{T}$ of a string link $T$ satisfies $\mathfrak{A}_{k+1}$ if and only if $\lambda_j$ lies in $F_k$, and $\lambda_j = f_{k+1}(t_j) \in F_k/F_{k+1}$ modulo $F_{k+1}$.

Thus, for such a string link $T$, as in the paper [L1] of Levine, it is reasonable to consider the invariant, $\lambda_j$ modulo $F_{2k}$. Notice that $F_k/F_{2k}$ is abelian, since $[F_k,F_k] \subset F_{2k}$. Furthermore, as can be seen from [L1, Proposition 4], we can verify that the equality,

\[
[x_1,\lambda_1][x_2,\lambda_2] \cdots [x_q,\lambda_q] = 1 \in F,
\]

always holds. Thus, for $m \leq 2k$, the sum,

\[
\sum_{j=1}^q (x_j \otimes \lambda_j) \in \mathbb{Z}^q \otimes \mathbb{Z} F_k/F_m \text{ modulo } F_m,
\]

is contained in the kernel of the commuting operator,

\[
[\bullet, \bullet]_{k,m} : F/F_2 \otimes F_k/F_m \longrightarrow F_{k+1}/F_{m+1}; \quad x \otimes y \longmapsto xyx^{-1}y^{-1}.
\]

This operation will be used in many times. Now let us show the equivalence of Milnor and Orr invariants:

**Theorem 2.3** (See §3 for the proof). (I) There is a $\mathbb{Q}$-vector isomorphism,

\[
\Phi \circ \eta^{-1} : \mathbb{Q} \otimes \text{Ker}([\bullet, \bullet]_{k,2k-1}) \xrightarrow{\sim} H_3(F/F_k; \mathbb{Q}),
\]

such that the following holds for any string link $T$ satisfying $\mathfrak{A}_{k+1}$ of the closure $\bar{T}$:

\[
\Phi \circ \eta^{-1}((x_1 \otimes \lambda_1) + \cdots + (x_q \otimes \lambda_q)) = \mathfrak{f}_j \circ \theta_k(\bar{T}, \tau).
\]

(II) Furthermore, concerning the homotopy group $\pi_3(K_k)$, there is a bijection

\[
\bar{\Phi} : \mathbb{Q} \otimes \text{Ker}([\bullet, \bullet]_{k,2k}) \xrightarrow{\sim} \pi_3(K_k) \otimes \mathbb{Q},
\]

as an extension of $\Phi \circ \eta^{-1}$, such that a similar equality $\bar{\Phi}((x_1 \otimes \lambda_1) + \cdots + (x_q \otimes \lambda_q)) = \theta_k(\bar{T}, \tau)$ holds for any string link $T$ satisfying $\mathfrak{A}_{k+1}$ of the closure $\bar{T}$.

We conjecture that the bijection in (II) is an isomorphism. However, from Theorem 2.2 (III), we have the realizability result as a generalization of [L1, Proposition 5]:

**Corollary 2.4.** Let $(\alpha_1, \ldots, \alpha_q) \in F_k/F_{2k}$ satisfy $[x_1,\alpha_1] \cdots [x_q,\alpha_q] = 1 \in F/F_{2k}$. There exists a string link with $\bar{\mu}$-invariants $(\lambda_1, \ldots, \lambda_q) \in (F_k)^q$ such that $\lambda_j \equiv \alpha_j$ modulo $F_{2k}$.

### 3 As a tree reduction of the Kontsevich invariant

As described in the Introduction, we will relate the $\bar{\mu}$-invariants with the Kontsevich invariant.

**Notation.** Throughout this paper, the expression $O(n)$ will be used to denote terms of degree greater than or equal to $n$. 
3.1 A brief review of the Kontsevich invariants of (string) links

Let us start by briefly reviewing the definition of the $\mathbb{Q}$-vector space $\mathcal{A}(\uparrow^q)$, where a chord diagram (of $q$-components) is a union $\sqcup_{j=1}^q [0,1] \cup \Gamma$ such that $\Gamma$ is a uni-trivalent graph, whose univalent vertices lie in the interior of $\sqcup_{j=1}^q [0,1]$, and each component of $\Gamma$ is required to have a univalent vertex. It is customary to refer to the components of $\Gamma$ as dashed. Then, $\mathcal{A}(\uparrow^q)$ is defined by the $\mathbb{Q}$-vector space generated by all chord diagrams subject to the STU, AS, and IHX relations. The three relations are described in Figure 2. The space $\mathcal{A}(\uparrow^q)$ is graded by the degree, where the degree of a diagram is half the number of vertices of $\Gamma$. We denote the subspace of $\mathcal{A}(\uparrow^q)$ of degree $n$ by $\mathcal{A}_n(\uparrow^q)$. By abuse of notation, we will denote the graded completion of $\mathcal{A}(\uparrow^q)$ by $\mathcal{A}(\uparrow^q)$ as well. We define $\mathcal{A}^t(\uparrow^q)$ as the subspace of $\mathcal{A}(\uparrow^q)$ generated by the chord diagram such that all trivalent diagrams containing a non-simply connected dashed component are considered relations. Furthermore, the sticking connection of $\sqcup_{j=1}^q [0,1]$ and $\sqcup_{j=1}^q [0,1]$ gives rise to a ring structure of $\mathcal{A}(\uparrow^q)$. Moreover, there is a cocommutative multiplication $\Delta : \mathcal{A}(\uparrow^q) \to \mathcal{A}(\uparrow^q) \otimes \mathcal{A}(\uparrow^q)$ (see, e.g., [CDM, Chapter 4], for the details), and $\mathcal{A}(\uparrow^q)$ is made into a Hopf algebra.

There are several formulations of the Kontsevich invariant. In this paper, we employ the Kontsevich invariant defined for “$q$-tangles”, as in [HM]. Since we can choose an injection from the set of string links into the set of $q$-tangles which is invariant with respect to the composite of ($q$-)tangles, we can regard the Kontsevich invariant of $q$-tangles that of string links. (Throughout this paper, we fix such an injection). In this paper, we only consider some of the properties of the Kontsevich invariant of string links $T$. Thus, while we refer the reader to, e.g., [CDM, §§8–10], [Ma2, §6], and references therein for the definition of the Kontsevich invariant for $q$-tangles, here are the properties that we will use later.

(I) The invariant, $Z(T)$, is defined as an element of $\mathcal{A}(\uparrow^q)$.

(II) $Z(T)$ is multiplicative, i.e., $Z(T_1)Z(T_2) = Z(T_1 \cdot T_2)$ holds for two string links $T_1, T_2$.

(III) Every $Z(T)$ is group-like in $\mathcal{A}(\uparrow^q)$, i.e., $\Delta(Z(T)) = Z(T) \otimes Z(T)$.

(IV) (Doubling Formula) For $i \leq q$ and $T \in \mathcal{A}(\uparrow^q)$, one has that

$$\Delta_i(Z(T)) = Z(D_i(T)) \in \mathcal{A}(\uparrow^{q+1})$$

where $D_i(T)$ denotes the double of $T$ along the $i$-th component, $C_i$, and $\Delta_i$ of a diagram is the sum over all lifts of vertices on $C_i$, to vertices on the two components over $C_i$.

Recalling the tree subspace $\mathcal{A}^t(\uparrow^q)$, we take the projection $p^t : \mathcal{A}(q) \to \mathcal{A}^t(\uparrow^q)$. We denote by $Z^t_{<m}(T)$ the composite $p^t \circ Z(T)$ subject to $O(m)$. In other words, this $Z^t_{<m}(T)$ is a tree reduction of the Kontsevich invariant $Z_{<m}(T)$.

![Figure 2: The IHX, AS, and STU relations among chord diagrams.](image-url)
Furthermore, we need to use the notion of primitive subspaces. Here, \( m \in \mathcal{A}(\uparrow^q) \) is called primitive if \( \Delta(m) = 1 \otimes m + m \otimes 1 \). We denote by \( P^i(\uparrow^q) \) the subspace of primitive elements of \( A^i(\uparrow^q) \), and by \( P^i_h(\uparrow^q) \) the subspace of degree \( h \). As is known, \( P^i(q) \) is the graded subspace of \( A^i(\uparrow^q) \) generated by chord diagrams such that the the dashed graph \( \Gamma \) is simply connected. Furthermore, the rank of \( P^i_h(\uparrow^q) \) is known to be \( qN_h - N_{h+1} \); see (12) and Theorem 5.2.

### 3.2 Results

Before stating the theorem, we should mention the following easily proven lemma.

**Lemma 3.1.** (1) Fix \( k \in \mathbb{Z} \). Every elements \( a, b \) in \( \bigoplus_{h=k}^{2k-1} A_h(\uparrow^q) \) satisfy \( (1 + a) \cdot (1 + b) \equiv 1 + a + b + O(2k) \in \mathcal{A}(\uparrow^q) \).

(2) Let \( a \in \mathcal{A}(\uparrow^q) \) satisfy \( a = 1 + O(k) \), and \( \Delta(a) = a \otimes a \). If we decompose \( a = 1 + b + c \) such that \( b \in A_{2k}(\uparrow^q) \) and \( c \in O(2k) \), then \( b \) is primitive.

As is known [HM], if a string link \( T \) satisfies \( A_{k+1} \) of the closure \( \overline{T} \), \( Z^i(T) = 1 + O(k - 1) \); thus, we can see that \( Z^i(T)_{<2k} - 1 \in P^i_{<2k} \) from property (III). The theorem is as follows:

**Theorem 3.2.** There is a linear isomorphism \( R : \bigoplus_{j=k}^{2k-1} P^i_j(q) \rightarrow \text{Ker}([\bullet, \bullet]_{k, 2k} \otimes Q) \) such that the following holds for any string link \( T \) satisfying \( A_{k+1} \) of the closure \( \overline{T} \):

\[
R(Z^i_{<2k}(T) - 1) = x_1 \otimes \lambda_1 + \cdots + x_q \otimes \lambda_q.
\]

(6)

Since Theorem 2.3 implies that the left-hand side is equivalent to the Orr invariant, we have the following equivalence:

**Corollary 3.3.** For any string link \( T \) satisfying \( A_{k+1} \) of the closure \( \overline{T} \), the Orr invariant \( \theta_k(L, \tau) \) is equivalent to the tree reduction of the Kontsevich invariant of degree \( < 2k \).

**Example 3.4.** As an example, we give a computation of the Boromean rings \( 6_3^2 \). Let \( PB(q+1) \) denote the pure braid group on \( q + 1 \). Let \( \sigma_i \) be the geometric braid formed by crossing the \( i \)-th string over the \((i + 1)\)-th one. Consider the string link \( T \) presented by \( \sigma_i^{-1} \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \).

Then \( \overline{T} = 6_3^2 \). We can easily verify \( A_3 \) and the expressions of the longitudes as

\[
\lambda_i = x_{i+2}x_{i+1}x_i^{-1}x_{i+2}^{-1}x_{i+1}^{-1}x_i^{-1} = [x_{i+2}, x_{i+1}x_i^{-1}x_i^{-1}].
\]

where \( i \in \mathbb{Z}/3 \). Thus, the \( \mu \)-invariant forms \( \sum_{i=1}^3 x_i \otimes [x_{i+2}, x_{i+1}x_i^{-1}] \) modulo \( F_4 \). This value can be computed as something quantitative by using Magnus expansion \( \mathcal{M}_4 \) (see [12] for the definition of \( \mathcal{M}_m \)).

### 4 Proof of Theorem 3.2

As preparation, let us review the notion of group-like expansions and look at Example 12.

Let \( I_m \subset \mathbb{Q}[X_1, \ldots, X_q] \) be the both-sided ideal generated by polynomials of degree \( \geq m \). Consider the augmentation \( \varepsilon : \mathbb{Q}[X_1, \ldots, X_q] \rightarrow \mathbb{Q} \) with \( \varepsilon(X_i) = 1 \), and a coproduct defined by \( \Delta(X_i) = X_i \otimes 1 + X_i \otimes 1 \). Then, the involution \( S : \mathbb{Q}[X_1, \ldots, X_q] \rightarrow \mathbb{Q}[X_1, \ldots, X_q] \) which sends \( X_i \) to \(-X_i \) makes it into a Hopf algebra. A Magnus expansion (modulo \( O(m) \)) is a group homomorphism \( \theta : F \rightarrow (\mathbb{Q}[X_1, \ldots, X_q]/I_m)^\times \), that satisfies \( \theta(y) = 1 + [y] + O(2) \).
for any \( y \in Y \). Furthermore, a group-like expansion is a Magnus expansion \( \theta \) satisfying 
\[
\Delta(\theta(y)) = \theta(y) \otimes \theta(y) \quad \text{and} \quad \varepsilon(\theta(y)) = 1 \quad \text{for any} \quad y \in Y.
\]
For example, the homomorphism \( \mathcal{M}_m : F \to \mathbb{Q}\langle X_1, \ldots, X_q \rangle/\mathcal{I}_m \) which sends \( x_i \) to \( 1 + X_i \) is not a group-like expansion, but a Magnus expansion.

**Remark 4.1.** We should mention some of the properties of these expansions (see [Ka] Theorem 1.3)

1. Given another Magnus expansion \( \theta' \), there is a ring automorphism \( S_{\theta'} \) on \( \mathbb{Q}\langle X_1, \ldots, X_q \rangle/\mathcal{I}_m \) such that \( \theta_{\text{str}} = S_{\theta'} \circ \theta' \).

2. We have \( \theta(F_m) = 0 \), and \( \theta \) induces an injection \( \theta : F/F_m \to \mathbb{Q}\langle X_1, \ldots, X_q \rangle/\mathcal{I}_m \). In fact, since the above \( \mathcal{M}_m \) is injective, the injectivity inherits every \( \theta \), by (1). Furthermore, since the restriction on \( F_j/F_{j+1} \) of \( \mathcal{M}_{j+1} \) is additive by definition, that of \( \theta \) is also an additive map.

3. There is a Lie algebra isomorphism from \( \mathcal{L}/\mathcal{L}_{\geq m} \) to the subspace,
\[
\mathcal{P}(F/F_m) := \{ a \in \mathbb{Q}\langle X_1, \ldots, X_q \rangle/\mathcal{I}_m \mid \Delta(a) = a \otimes 1 + 1 \otimes a \} \tag{7}
\]
with the Lie bracket \( [a, b] = ab - ba \). The restricted image \( \mathcal{M}_m(F_m/F_{m+1}) \) is contained in this \( \mathcal{P}(F/F_m) \).

Next, the isomorphism \( \Theta \) below is related to another example of \( \theta \) arising from the Kontsevich invariant. For an index pair \( (i, j) \in \{1, \ldots, q+1\}^2 \), let \( t_{i,j} \in \mathcal{A}_1((\uparrow q+1)) \) be the Jacobi diagram with only one edge connecting the \( i \)-th strand to the \( j \)-th one. Namely,
\[
t_{i,j} = \begin{array}{cccc}
\ldots & \ldots & \ldots & \ldots \\
1 & \cdots & i & \cdots \\
 & & & j \cdots q + 1
\end{array}
\]
We denote \( \mathcal{A}^\leftarrow((\uparrow q, \ast)) \) by the subalgebra of \( \mathcal{A}((\uparrow q+1)) \) generated by \( t_{1,q+1}, t_{2,q+1}, \ldots, t_{q,q+1} \). Let \( FI \) denote the framing independence relation in \( \mathcal{A}^\leftarrow((\uparrow q, \ast)) \), where any Jacobi diagram with an isolated chord on the same interval is equal to 0. Then, as shown [Ma2, (6.2)], we can verify that the map,
\[
\mathbb{Q}\langle X_1, \ldots, X_q \rangle/\mathcal{I}_m \longrightarrow \mathcal{A}^\leftarrow((\uparrow q, \ast))/((\mathcal{A}^\leftarrow_{\geq m}((\uparrow q, \ast)) + FI), \tag{8}
\]
which sends \( X_i \) to \( t_{i,q+1} \) is a Hopf algebra isomorphism.

**Example 4.2** ([Ma2, Proposition 6.2]). We use notation on the pure braid group \( PB(q+1) \) in Example 3.4. Let \( \sigma_{jq+1} \in PB(q+1) \) be \( \sigma_q \sigma_{q-1} \cdots \sigma_{jq+1} \sigma_{jq+1} \cdots \sigma_j \); see Figure 3. As is well known, there is a semi-direct product decomposition \( PB(q+1) \cong F(q) \ltimes PB(q) \), where \( F(q) \) is the free group generated by \( \sigma_1, \ldots, \sigma_{q+1} \), and \( PB(q) \) is embedded into \( PB(q+1) \) via \( \beta \mapsto \beta \times \uparrow \). Thus, any element \( g \) of the free group \( F(q) \) can be regarded as a pure-braid \( PB(q+1) \subset SL(q+1) \). Therefore, we can define \( Z^\leftarrow(q) \in \mathcal{A}((\uparrow q+1)) \). As is shown [Ma2], \( Z^\leftarrow(g) \) lies in the subalgebra \( \mathcal{A}((\uparrow q, \ast)) \), and the composite,
\[
\theta^Z : F \stackrel{\mathbb{Z}}{\longrightarrow} \mathcal{A}((\uparrow q, \ast))/((\mathcal{A}^\leftarrow_{\geq m}((\uparrow q+1)) + FI) \mathcal{S}^{-1} \longrightarrow \mathbb{Q}\langle X_1, \ldots, X_q \rangle/\mathcal{I}_m, \tag{9}
\]
turns out to be a group-like expansion (Moreover, it was shown to be a “special expansion”).
Before turning back to Theorem 3.2 we should mention (10) from [HM Corollary 12.2]. For a pure braid \( \sigma \in P B(q) \), the \( \ell \)-th longitude, \( \lambda_\ell \in F(q) \), of \( \sigma \) is equal to \( (\sigma \times 1)^{-1} \beta_\ell D_\ell(\sigma) \beta_\ell^{-1} \) in \( P B(q+1) \). Here, \( \beta_\ell \) is a braid of the form \( \sigma_1 \sigma_2 \cdots \sigma_{\ell-1} \in B_n \). Thus, the \( \theta^Z(\lambda_\ell) \) can be computed as

\[
\theta^Z(\lambda_\ell) = Z^t(\sigma \times 1)^{-1} Z^t(\beta_\ell) \Delta_t(Z^t(\sigma)) Z^t(\beta_\ell)^{-1} \in A^\infty(\lambda^q, \ast).
\]  

Proof of Theorem 3.2. Inspired by (10), for \( \ell \leq q \), we set up a homomorphism,

\[
\Upsilon^j_\ell : P^j(\lambda^q) \longrightarrow A^\infty_\ell(\lambda^q, \ast)/FI \cong \mathbb{Q}[X_1, \ldots, X_q]/I_{j+1}; \ a \mapsto ((a+1)\times 1)^{-1} Z^t(\beta_\ell) \Delta_t(a+1) Z^t(\beta_\ell)^{-1}.
\]

Furthermore, we consider the linear homomorphism,

\[
\Upsilon^j : P^j(\lambda^q) \longrightarrow F/F_2 \otimes_{\mathbb{Z}} \mathbb{Q}[X_1, \ldots, X_q]/I_{j+1}; \ a \mapsto \sum_{\ell: \ 1 \leq \ell \leq q} x_\ell \otimes \Upsilon^j_\ell(a).
\]

We will show the injectivity of \( \Upsilon^j \) and that the image is equal to

\[
\{ \sum_{1 \leq \ell \leq q} x_\ell \otimes S_g a(\ell) \in F/F_2 \otimes (I_j/I_{j+1}) \mid \Delta(a_\ell) = a_\ell \otimes a_\ell, \ \sum_{1 \leq \ell \leq q} a_\ell X_\ell - X_\ell a_\ell = 0 \}.
\]  

(11)

From Remark 4.4 (3), this subspace (11) can be identified with the kernel of \( [\bullet, \bullet]_{j,j+1} : F/F_2 \otimes L_j/L_{j+1} \rightarrow L_{j+1}/L_{j+2} \). Thus, the rank of (11) is \( qN_j - N_{j+1} \). As is known (see [L1 HM]), if some \( (\alpha_1, \ldots, \alpha_q) \in F_j/F_{j+1} \) satisfies \( [x_1, \alpha_1] \cdots [x_q, \alpha_q] = 1 \in F/F_{j+1} \), then there exists a string link \( T_\alpha \) with \( \bar{\mu} \)-invariants \( (\lambda_1, \ldots, \lambda_q) \in (F_j)^q \) such that \( \lambda_j \equiv \alpha_j \mod F_{j+1} \). Therefore, by (10), the image of \( \Upsilon^j \) is generated by \( \bar{\Upsilon}^j(\lambda^q_\leq_j(T_\alpha) - 1) = \sum_{i=1}^q x_i \otimes \theta^Z(\lambda_i) \), where \( \alpha \) runs over a basis of \( \text{Ker}([\bullet, \bullet]_{j,j+1}) \). Since the rank of \( P^j(\lambda^q) \) is \( qN_j - N_{j+1} \), \( \Upsilon^j \) must be injective, and the image is (11), as required.

To complete the proof, we will construct the isomorphism \( R \) and show the equality (6). Consider

\[
\text{Id}_{\mathbb{Z}^q} \otimes \theta^Z : Z^t \otimes_{\mathbb{Z}} F_k/F_{2k} \rightarrow F/F_2 \otimes_{\mathbb{Z}} \mathbb{Q}(X_1, \ldots, X_q)/I_{2k},
\]

which is an injective homomorphism, by Remark 4.4 (2). The image of \( \text{Ker}([\bullet, \bullet]_{k,2k-1} \otimes \mathbb{Q}) \) is contained in (11). We define the isomorphism \( R : \text{Ker}([\bullet, \bullet]_{k,2k-1} \otimes \mathbb{Q}) \rightarrow \bigoplus_{j=k}^{2k-1} P^j_2(\lambda^q) \) as a linear extension of \( \text{Id}_{\mathbb{Z}^q} \otimes (\text{id}_{\mathbb{Z}^q} \otimes \theta^Z) \circ \bigoplus_{j=k}^{2k-1} P^j_2(\lambda^q)^{-1} \).

We now prove the equality (6) where the string link \( T \) is a pure braid. Noting that \( Z_{2k}(T) - 1 \) is primitive by Lemma 3.1, we compute \( \bar{\Upsilon}(\Upsilon^j(Z^t_{2k}(T) - 1)) \) as

\[
\sum_{\ell=1}^q x_\ell \otimes \prod_{j=k}^{2k-1} \Upsilon^j_\ell(\Upsilon^j_{\leq 2k}(T) - 1) = \sum_{\ell=1}^q x_\ell \otimes (\theta^Z(\lambda_\ell)) = \sum_{\ell=1}^q \theta^Z(x_\ell) \otimes (\theta^Z(\lambda_\ell))
\]

Figure 3: The pure braid \( \sigma_{j,q+1} \)
which is immediately deduced from from (10). By the injectivity of $\theta^Z$, this equality implies the desired (6).

Finally, we will prove (6) for any string link $T$ satisfying $\mathfrak{A}_{k+1}$ of the closure $\overline{T}$. As implicitly shown in the proofs of [HM] Propositions 10.6 and Theorem 6.1, there is a pure braid $\sigma$ such that the Milnor invariants of $T$ and $\sigma$ in $F_k/F_{2k}$ are equal. Thus, the invariant of $T \sigma^{-1}$ is zero. Thus, [HM] Theorem 6.1 immediately implies $Z^t(T \sigma^{-1}) = 1 + O(2k)$, which leading to $Z^t(T) = Z^t(\sigma)$ modulo $O(2k)$ by Lemma 5.1 (1). Hence, $R(Z^t(T) - 1) = R(Z^t(\sigma) - 1)$ is equal to $\sum_{i=\ell} x_i \otimes \lambda_\ell$ by the above paragraph, as desired. It completes the proof. 

5 Proof of Theorem 2.3

5.1 Review of the infinitesimal Morita-Milnor homomorphism, and tree reduction

As a preliminary to prove Theorem 2.3, we review some of the results in [Ma1, IO, Ko].

We will start by briefly reviewing the Lie algebra homology of $F/F_k$. Let $H$ be the $\mathbb{Q}$-vector space of rank $q$ with basis $X_1, \ldots, X_q$, i.e., $H = \text{Span}_\mathbb{Q}\langle X_1, \ldots, X_q \rangle$. Let $\mathfrak{L}$ be the free Lie algebra generated by $H$. This $\mathfrak{L}_{\geq k}$ is the subspace generated by the commutator of length $\geq k$. We have the quotient Lie algebra $\mathfrak{L}/\mathfrak{L}_{\geq k}$. Then, the Koszul complex of $\mathfrak{L}/\mathfrak{L}_{\geq k}$ is the exterior tensor algebra $\Lambda^*(\mathfrak{L}/\mathfrak{L}_{\geq k})$ with the boundary map $\partial_n : \Lambda^n(\mathfrak{L}/\mathfrak{L}_{\geq k}) \rightarrow \Lambda^{n-1}(\mathfrak{L}/\mathfrak{L}_{\geq k})$ given by

$$\partial_n(h_1 \wedge \cdots \wedge h_n) = \sum_{i<j} (-1)^{i+j}[h_i, h_j] \wedge h_1 \wedge \cdots \wedge \hat{h}_i \wedge \cdots \wedge \hat{h}_j \wedge \cdots \wedge h_n.$$  

Later, we will use the known isomorphism $H_n(\Lambda^*(\mathfrak{L}/\mathfrak{L}_{\geq k})) \cong H_n(F/F_k; \mathbb{Q})$, which is called Pickel’s isomorphism; see, e.g., [IO, Ma1, SW].

Next, let us review Jacobi diagrams. A Jacobi diagram is a uni-trivalent graph whose univalent vertices are labeled by one of $\{1, 2, \ldots, q\}$, where each trivalent vertex is oriented. Consider the graded $\mathbb{Q}$-vector space generated by Jacobi diagrams, where the degree of such a diagram is half the number of vertices. Let $\mathcal{J}(q)$ be the quotient space subject to the AS and IHX relations, and let $\mathcal{J}(q)$ be the subspace generated by Jacobi diagrams which are simply connected. As a diagrammatic analogy of the Poincaré-Birkhoff-Witt theorem, we can construct a graded vector isomorphism,

$$\chi_q : \mathcal{P}_n(\uparrow q) \cong \mathcal{J}_n(q).$$  

(12)

See, e.g., [CDM] §5.7 for details.

Remark 5.1. Theorem 3.2 implies a generalization of the main theorem 6.1 of [HM]. In fact, recalling from Theorem 2.2 (II) the $k$-th summand of $\theta_k(L, \tau)$ is the Milnor invariant of length $k$, the equivalence between the Milnor invariant and $\chi_q(Z^t_k(T) - 1)$ coincides with the theorem 6.1 of [HM] exactly.

Furthermore, we consider the subspace $\mathcal{J}_n(q, 0)$ of $\mathcal{J}_n(q + 1)$ generated by tree diagrams in which the label $q + 1$ occurs exactly once. Given such a diagram $J \in \mathcal{J}_n(q, 0)$, all of its univalent vertices apart from one labeled by $r$ defines $\text{comm}(J)$ in $F_n/F_{n+1}$ in a canonical way. For example,
Then, we can easily see that the following correspondence is an isomorphism,

\[ \text{comm} : \mathcal{J}_n(q, 0) \longrightarrow F_n/F_{n+1} \otimes \mathbb{Q}. \]

Next, we will review the infinitesimal Morita-Milnor homomorphism, \( M_k \), defined in \([\text{Ko}] \) §5.1 (cf. the infinitesimal Morita map in \([\text{Ma2}] \)). For this, a string link \( T \in SL(q) \) is of degree \( k \) if the closure \( \overline{T} \) satisfies Assumption \( \mathfrak{A}_{k+1} \). Let \( SL(q)_k \) be the subset consisting of string links of degree \( k \). Then, we have a filtration,

\[ SL(q) = SL(q)_0 \supset SL(q)_1 \supset \cdots \supset SL(q)_k \supset \cdots. \]

Further, we recall the \( \bar{\mu} \)-invariants \( \lambda_\ell \in F_k/F_{2k-1} \) and expand them as \( \lambda_\ell = \lambda_\ell^{(k)} + \cdots + \lambda_\ell^{(2k-2)} \) with \( \lambda_\ell^{(j)} \in F_j/F_{j+1} \) according to \( F_k/F_{2k-1} \cong \bigoplus_{j=k}^{2k-2} F_j/F_{j+1} \). Recall that, if \( L \in SL(q)_k \) is of degree \( k \), the \( \bar{\mu} \)-invariants \( \lambda_\ell \) are contained in \( F_k \). Then, identifying \( F_j/F_{j+1} \otimes \mathbb{Q} \) with \( \mathfrak{L}_j/\mathfrak{L}_{j+1} \) as a \( \mathbb{Q} \)-vector space, let us consider a 2-form,

\[ \sigma_L := \sum_{j=k}^{2k-2} \sum_{\ell=1}^q X_\ell \wedge \lambda_\ell^{(j)} \in \Lambda^2(\mathfrak{L}/\mathfrak{L}_{\geq 2k-1}). \quad (13) \]

As is shown in §5.1 of \([\text{Ko}] \), there exists \( t_L \in \Lambda^3(\mathfrak{L}/\mathfrak{L}_{\geq 2k-1}) \) satisfying \( \partial_3(t_L) = \sigma_L \). Since \( \lambda_\ell \in F_k \) by Assumption \( \mathfrak{A}_{k+1} \), the 2-form \( \sigma_L \) reduced in \( \Lambda^3(\mathfrak{L}/\mathfrak{L}_{\geq k}) \) is zero. Hence, \( t_L \) is a 3-cycle. It has been shown \([\text{Ko}] \) Lemma 5.1.2 that the homology 3-class \( \{t_L\} \in H_3(\mathfrak{L}/\mathfrak{L}_{\geq k}) \) is independent of the choice of \( t_L \). To summarize, we have a map,

\[ M_k : SL(q)_k \longrightarrow H_3(\mathfrak{L}/\mathfrak{L}_{\geq k}); \quad L \longmapsto \{t_L\}. \]

Kodani \([\text{Ko}] \) §5 showed that \( M_k \) is a monoid homomorphism and its kernel is \( SL(q)_{2k-1} \).

Now let us review the tree description of the third homology \([\text{Ma1}] \). Fix a tree diagram \( J \in \mathcal{J}_j^t(q) \). For each trivalent vertex \( r \) of \( J \), \( J \) is the union of the three tree diagrams rooted as \( r \). We denote the three tree diagrams by \( J_r^{(1)} \), \( J_r^{(2)} \), and \( J_r^{(3)} \), by clockwise rotation. Here, the numbering 1,2,3 is according to the cyclic ordering of \( r \). Furthermore, if we label a univalent \( r \) by \( q + 1 \), each \( J_r^{(j)} \) can be regarded as an element in \( \mathcal{J}^t(q, 0) \). Then, the fission map \( \phi : \mathcal{J}_j^t(q) \to \Lambda^3(\mathfrak{L}/\mathfrak{L}_{\geq k}) \) is defined by

\[ \phi(J) = \sum_{r:\text{trivalent vertex of } J} \text{comm}(J_r^{(3)}) \wedge \text{comm}(J_r^{(2)}) \wedge \text{comm}(J_r^{(1)}). \]

**Theorem 5.2 \([\text{Ma1}] \).** If \( k \leq j \leq 2k - 2 \), this \( \phi(J) \) is a 3-cycle. Moreover, the fission map gives rise to a linear isomorphism,

\[ \Phi : \bigoplus_{j=k}^{2k-2} \mathcal{J}_j^t(q) \sim H_3(\mathfrak{L}/\mathfrak{L}_{\geq k}). \]
Next, let us review the isomorphism (15) below. For a Jacobi diagram \( J^{(j)} \in J_j^t(q) \), we define \( \eta_j(J^{(j)}) \) to be the sum,

\[
\sum_{v:\text{univalent vertex of } J^{(j)}} [x_{\text{col}(v)}] \otimes \text{comm}(J^{(j)}_v) \in (F/F_2) \otimes \mathbb{Q}(\mathcal{L}/\mathcal{L}_{\geq j}),
\]

where \( \text{col}(v) \in \{1, \ldots, q\} \) is the label on \( v \), and \( J^{(j)}_v \in J_j^t(q,0) \) is the labelled tree obtained by replacing the label on \( v \) with \( q + 1 \). Taking the bracket \([\cdot, -] : F/F_2 \otimes \mathcal{L}_j \rightarrow \mathcal{L}_{j+1} \), which \((x, y)\) sends \( xyx^{-1}y^{-1} \), we can easily verify that the sum (14) lies in the kernel. Then, this \( \eta_j \) defines a linear homomorphism,

\[
\eta_j : J_j^t(q) \rightarrow \text{Ker}([\cdot, -] : F/F_2 \otimes \mathcal{L}_j/\mathcal{L}_{j+1} \rightarrow \mathcal{L}_{j+1}/\mathcal{L}_{j+2}),
\]

which is known to be an isomorphism; see [L2]. We denote by \( \eta \) the sum of the isomorphisms \( \oplus_{j=k}^{2k-2} \eta_j \) for short.

### 5.2 Proof of Theorem 2.3

Then, the following lemma relates to the 3-class \([t_L]\) from tree diagrams:

**Lemma 5.3.** For \( T \in SL(q)_k \), the composite \( \Phi \circ \eta^{-1}_{j=1} (\sum_{t=1}^q x_t \otimes \lambda_t) \) coincides with \( M_k(T) = [t_L] \).

**Proof.** Let \( J^{(j)} \in J_j^t(q) \) be \( (\eta)^{-1}_{j=1} (\sum_{i=1}^q x_i \otimes \lambda_i(j)) \). As is shown in [Ma1, Lemma 1.2], the composite \( \partial_3(\phi(J^{(j)})) \) is formed as

\[
\partial_3(\phi(J^{(j)})) = \sum_{v:\text{univalent vertex of } J^{(j)}} X_{\text{col}(v)} \wedge \text{comm}(J^{(j)}_v) \in \wedge^2(\mathcal{L}/\mathcal{L}_{\geq 2k}).
\]

Compared with (14), this \( \partial_3(\phi(J^{(j)})) \) is equal to \( \sigma_L \in \wedge^2(\mathcal{L}/\mathcal{L}_{\geq 2k}) \) by definition. Thus, letting \( t_L \) be \( \sum_{j=k}^{2k-2} \phi(J^{(j)}) \), we get a 3-chain \( t_L \) satisfying \( \partial_3(t_L) = \sigma_L \), leading to the desired coincidence. \( \square \)

We can prove Theorem 2.3 by showing that the 3-class \([t_L]\) is equal to the Orr invariant of degree \( k \). For this, we will have to prove the following lemma.

**Lemma 5.4.** From the identification \( H_3(\mathcal{L}/\mathcal{L}_{\geq k}) \cong H_3^L(F/F_k; \mathbb{Q}) \), the homomorphism \( M_k \) is equal to a map which sends the 3-class \( t_L \) to \( \tilde{\phi} \circ \theta_k(\overline{T}, \tau) \).

Before proving this lemma, we now prove Theorem 2.3.

**Proof of Theorem 2.3** The proof of (I) immediately follows from Lemmas 5.3 and 5.4.

Now let us show (II). Fix a basis \( \{b_s\} \) of \( H_3(F/F_k; \mathbb{Z}) \), where \( s \) ranges over \( 1 \leq s \leq \text{rk} H_3(F/F_k; \mathbb{Z}) \). Thanks to Theorem 2.2 (III), we can choose a string link \( L_s \) such that \( \tilde{\phi}(\theta_k(\overline{L_s}, \tau)) = b_s \) and the component of \( \theta_k(\overline{L_s}, \tau) \) in the kernel \( \text{Ker} \tilde{\phi} \) is zero. Then, from (1), the \( \tilde{\mu} \)-invariant of \( L_s \) in \( F_k/F_{2k} \) is \( \eta \circ \Phi^{-1}(b_s) + \sum_{j=1}^q x_j \otimes \lambda_{j,s}^{(2k)} \) for some \( \lambda_{j,s}^{(2k)} \in F_{2k-1}/F_{2k} \otimes \mathbb{Q} \). Recalling the isomorphism \( \text{Ker} \tilde{\phi} \otimes \mathbb{Q} \cong \mathbb{Q}[N_{2k-1} - N_{2k}] \), we fix the isomorphism \( \iota : \mathbb{Q}[N_{2k-1} - N_{2k}] \rightarrow \text{Ker}[\bullet, \bullet]_{2k-1,2k} \) and define the bijection,

\[
\overline{\Psi}_k : (H_3(F/F_k; \mathbb{Z}) \oplus \text{Ker} \tilde{\phi}) \otimes \mathbb{Q} \rightarrow \text{Ker}[\bullet, \bullet]_{k,2k},
\]
Proof of Lemma 5.4. First, we set up some complexes and chain maps. In what follows, we consider only complexes over \( \mathbb{Q} \) and omit writing the coefficients \( \mathbb{Q} \). Let \( (C^*_k(F/F_k), \partial) \) be the non-homogenous group complex of \( F/F_k \). Then, on the basis of Suslin and Wodzicki’s paper [SW], Massuyeau (see the proof of [Ma1, Proposition 4.3]) showed the natural existence of a chain map \( \kappa : \wedge^*(L/L_{\geq k}) \rightarrow C^*_k(F/F_k) \) that induces an isomorphism on the homology. Thus, it is enough for us to show that the 3-cycle \( \kappa(t_L) \) is equivalent to the pushforward \( (f_k)_* [S^3 \setminus \overline{T}, \partial(S^3 \setminus \overline{T})] \).

To do so, we can study the 3-cycle \( t_L \) from the viewpoint of the group complex. For \( \ell \leq q \), let \( K_\ell \) be the abelian subgroup of \( F \) generated by the meridian-longitude pair \( (m_\ell, l_\ell) \). Let us consider the commutative diagram,

\[
\begin{array}{ccc}
0 & \rightarrow & \oplus_{\ell=1}^q C^3_3(K_\ell) \\
& \downarrow{\varphi_3} & \downarrow{\varphi_3} \\
0 & \rightarrow & \oplus_{\ell=1}^q C^3_2(K_\ell)
\end{array}
\]

where the right-hand sides are defined as the cokernel of \( \iota_\star \). In the subcomplex \( C^3_2(K_\ell) \), the cross product \( m_\ell \times l_\ell \) is a 2-cycle that generates \( H^2_2(K_\ell) \cong \mathbb{Z} \). Let \( \tau_L \) be a 2-cycle \( \sum_{\ell=1}^q m_\ell \times l_\ell \in \oplus_{\ell=1}^q C^3_2(K_\ell) \). Accordingly, since \( H^2_2(F) = 0 \), we can choose a 3-cycle \( \eta_L \) in \( C^3_3(F, K_1 \cup \cdots \cup K_q) \) such that \( \delta_\star(\eta_L) = \tau_L \).
Next, we will examine the diagrams subject to $F_k$ and $F_{2k}$ with regard to their functoriality:

$$
\begin{array}{ccccccc}
0 & \oplus_{l=1}^{q} C_2^{gr}(K_l) & \rightarrow & C_2^{gr}(F/F_{2k-1}) & \rightarrow & C_2^{gr}(F/F_{2k-1}, \cup_t K_t) & 0 \\
& \downarrow \partial_2 & & \downarrow \partial_1 & & \downarrow \partial & \\
0 & \oplus_{l=1}^{q} C_2^{gr}(K_l) & \rightarrow & C_2^{gr}(F/F_{2k-1}) & \rightarrow & C_2^{gr}(F/F_{2k-1}, \cup_t K_t) & 0 \\
& \downarrow \partial_2 & & \downarrow \partial_1 & & \downarrow \partial & \\
0 & \oplus_{l=1}^{q} C_2^{gr}(K_l) & \rightarrow & C_2^{gr}(F/F_k) & \rightarrow & C_2^{gr}(F/F_k, \cup_t K_t) & 0 \\
& \downarrow \partial_2 & & \downarrow \partial_1 & & \downarrow \partial & \\
0 & \oplus_{l=1}^{q} C_2^{gr}(K_l) & \rightarrow & C_2^{gr}(F/F_k) & \rightarrow & C_2^{gr}(F/F_k, \cup_t K_t) & 0.
\end{array}
$$

Here, the horizontal arrows are exact, and the slanting ones are the maps induced from the projection $F/F_{2k-1} \rightarrow F/F_k$. Since the above quasi-isomorphism $\kappa$ was constructed from the projective resolution of the augmentation $\varepsilon : \mathbb{Q}[G] \rightarrow \mathbb{Q}$, this $\kappa$ replaces the wedge product $\wedge$ by the cross product $\times$. Therefore, recalling the isomorphism $\phi^*$ in (1), the 2-cycle $\phi_*^{-1}(\tau_L)$ modulo $F_{2k}$ is exactly equal to $\sigma_L$ in (13). Thus, the 3-cycle $t_L \in C_3^{gr}(F/F_{2k-1})$ satisfies $P^{gr}(t_L) = \phi_*^{-1}(\eta_L) \in H_3^{gr}(F/F_k)$.

Finally, we give a relation to the link complement $S^3 \setminus \overline{T}$. Let $E$ denote $S^3 \setminus \overline{T}$. Similarly, let us consider the long exact sequence on the cellular homology:

$$
\begin{align*}
0 & \rightarrow H_3^{cell}(E, \partial E; \mathbb{Q}) \xrightarrow{\delta} H_2^{cell}(\partial E; \mathbb{Q}) \rightarrow H_2^{cell}(E; \mathbb{Q}) \rightarrow H_2^{cell}(E, \partial E; \mathbb{Q}).
\end{align*}
$$

Here are some well-known facts from knot theory: The first term is $\mathbb{Q}$ generated by the fundamental 3-class $[E, \partial E]$, and the second is $\mathbb{Q}$ generated by the cross products $m_t \times \ell_t$. Furthermore, the sum $\sum_l m_l \times \ell_t$ is zero in $H_2^{cell}(E; \mathbb{Q})$ by relation (3). Thus, we have a cellular 3-chain $b_L \in C_3^{cell}(E, \partial E; \mathbb{Q})$ such that $(P_3)_*\{b_L\} = [E, \partial E]$ in the relative $C_3^{cell}(E, \partial E; \mathbb{Q})$ and $\partial_3(b_L) = \sum_{l=1}^{q} m_t \times \ell_t$. In particular, letting $I : [0,1]^3 \setminus T \rightarrow S^3 \setminus \overline{T}$ be the inclusion, we have $[E, \partial E] = I_*([E])$. Notice that $f_k \circ I_* \circ \pi_1([0,1]^3 \setminus T) \rightarrow F/F_k$ is equal to the reduction $F \rightarrow F/F_k$. Hence, noting $(f_k)_* \circ I_* = \phi_*^{-1}$, we obtain the computation:

$$(f_k)_*[E, \partial E] = (f_k)_* \circ I_*([E]) = \phi_*^{-1}(\eta_L) = P^{gr}(t_L) \in H_3^{gr}(F/F_k, \cup_t K_t).$$

Since $K_\ell$ modulo $F_k$ is isomorphic to $\mathbb{Z}$ from Assumption $\alpha_{k+1}$, $P^{gr}$ induces $H_3^{gr}(F/F_k) \cong H_3^{gr}(F/F_k, \cup_t K_t)$. Hence, $t_L$ is equivalent to the pushforward $(f_k)_*[E, \partial E]$, as desired.

## A HOMFLYPT polynomials and Orr invariants

Let $T$ be a string link and $(\overline{T}, \tau)$ be the associated based link. According to Theorem 2.3, the computation of the Orr invariant of $(\overline{T}, \tau)$ is equivalent to the $\bar{\mu}$-invariant of $T$ of degree $< 2k$. In general, it is hard to get a presentation of longitudes $\lambda_\ell$ as a word of $x_1, \ldots, x_q$ (However, if $T$ is a pure braid, we can easily get such a presentation). Furthermore, in quantum topology, it is natural to ask what finite type invariants recover the $\bar{\mu}$-invariants. As a solution, we will show that the result of Meilhan-Yasuhashi [MY] give a computation of the $\bar{\mu}$-invariants from HOMFLYPT polynomials, without having to write longitudes (Theorem A.1).
Furthermore, given a $q$-component link $L$, we define the oriented link $L_{i,j}$ with the order induced by the lexicographic order of the index $(i,j)$. This ordering defines a bijection $\{(i,j)| 1 \leq i \leq q, 1 \leq j \leq r_i \} \rightarrow \{1,\ldots,m\}$.

In addition, we define a sequence $D_I(T) \in \{1,\ldots,m\}^m$ without a repeated index as follows. First, we take a sequence of elements of $\{(i,j)| 1 \leq i \leq q, 1 \leq j \leq r_i \} \rightarrow \{1,\ldots,m\}$ by replacing each $i$ in $I$ with $(i,1),\ldots,(i,r_i)$ in this order. Next, we replace each term $(i,j)$ of this sequence with $\varphi((i,j))$.

In addition, given a subsequence $H < D(I)$, we define another link $D_I(T)_J$. Let $B_i$ be an oriented $2m$-gon, and denote by $p_j (j = 1,\ldots,m)$ a set of $m$ nonadjacent edges of $B_i$ according to the boundary orientation. Suppose that $B_i$ is embedded in $S^3$ such that $B_i \cap L = \cup_{j=1}^m p_j$ and such that each $p_j$ is contained in $L_i$ with opposite orientation. We call such a disk an $I$-fusion disk of $D_I(L)$. For any subsequence $J$ of $D(I)$, we define the oriented link $L_J$ as the closure of $\bigcup_{j \in \{J\}} (L_j \cap \partial B_i) \setminus \bigcup_{j \in \{J\}} (L_j \cap B_i)$, where $\{J\}$ is the subset of $\{1,\ldots,n\}$ of all indices appearing in the sequence $J$.

![Figure 4: The links $L_+$, $L_-$, and $L_0$.](image)

To describe the result, we should recall the HOMFLYPT polynomial and some of its properties. The HOMFLYPT polynomial $P(L; t, z) \in \mathbb{Z}[t^{\pm 1}, z^{\pm 1}]$ of an oriented link $L \subset S^3$ is defined formulas as follows:

(I) Concerning the unknot $U$ in $S^3$, the polynomial $P(U; t, z)$ is 1.

(II) The skein relation $t \cdot P(L_+; t, z) + t^{-1} \cdot P(L_-; t, z) = z \cdot P(L_0; t, z)$ holds, where $L_-$, $L_+$, and $L_0$ are links formed by crossing and smoothing changes on a local region of a link diagram, as indicated in Figure 4.
Consider the homomorphism $\mathcal{M}_m : F \to \mathbb{Z}[X_1, \ldots, X_q]/I_m$ defined by $\mathcal{M}_m(x_i) = 1 + X_i$. For a sequence in $I = i_1 \cdots i_m \in \{1, 2, \ldots, q\}^m$, $\mu_I(T)$ is defined by the coefficient of $X_{i_1} \cdots X_{i_{m-1}}$ in $\mathcal{M}_m(\lambda_{i_m})$ as in [Mi2, L1, O, MY].

**Theorem A.1** ([MY]). Let $T$ be a $q$-component string link which satisfies $\mathfrak{A}_{k+1}$. Assume $3 \leq m \leq 2k + 2$. Let $I$ be a sequence in $\{1, 2, \ldots, q\}^m$ of length $m$. For any $D_I$-fusion disk for $D_I(T)$, we have

$$\mu_I(T) = \frac{(-1)}{m!2^m} \sum_{J < D(I)} (-1)^{|J|} \log P_0(D_I(T)_{J}(m)).$$

The original statement [MY, Theorem 1.3] dealt with only links $S^3$ and takes the same formula modulo some integers. However, as can be seen in their proof, after the authors proved Theorem A.1 for string links before they proved the original statement. Thus, we do not need to give a detailed proof of Theorem A.1.

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