Decompositions of the Moonshine Module
with respect to subVOAs associated to codes over $\mathbb{Z}_{2k}$

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Abstract

In this paper, we give decompositions of the moonshine module $V^\natural$ with respect to subVOAs associated to extremal Type II codes over $\mathbb{Z}_{2k}$ for an integer $k \geq 2$. Those subVOAs are isomorphic to the tensor product of 24 copies of the charge conjugation orbifold VOA. Using such decompositions, we obtain some elements of type $4A$ ($k$ odd) and $2B$ ($k$ even) of the Monster simple group $\text{Aut}(V^\natural)$.

Introduction

The notation of a vertex operator algebra (VOA) is introduced in [Bo1, FLM]. One of the most interesting examples of a VOA is the moonshine module $V^\natural$ constructed in [FLM]. The automorphism group of $V^\natural$ is the Monster, the largest sporadic finite simple group.

The moonshine module $V^\natural$ has many subalgebras having good symmetry ([Mi, DLMN]). The decompositions of $V^\natural$ with respect to those subalgebras are computed in [DGH, KLY1, KLY2]. In particular, $V^\natural$ contains a subVOA isomorphic to the tensor product of 48 copies of $L(1/2,0)$, called a Virasoro frame in [DGH], and the decomposition of $V^\natural$ is computed.

Since $L(1/2,0)$ is isomorphic to $W_2$ algebra at $c = 1/2$, we should study the first member of the unitary series of $W_n$ algebras. For details of $W_n$ algebras, see [FKW] and references given there. In particular, $W_4$ algebra at $c = 1$ is realized as the fixed-point subspace $V_L^+$ of the lattice VOA $V_L$ corresponding to the rank one lattice $L = \mathbb{Z}\alpha$ with $\langle \alpha, \alpha \rangle = 6$ with respect to the $-1$ automorphism of the lattice. Its fusion rules have a nice symmetry according to [AR] and for each embedding of $V_L^+$ into $V^\natural$, we get a $4A$ element of the Monster [Ma].

In this paper, we consider more general case $L = \mathbb{Z}\alpha$ with $\langle \alpha, \alpha \rangle = 2k$ for an integer $k \geq 2$. Since the Leech lattice $\Lambda$ contains many elements of squared length $2k$ and $V^\natural$ contains $V_\Lambda^+$ as a subVOA, $V^\natural$ contains many copies of $V_L^+$. We consider a set of 24...
mutually orthogonal pairs of opposite vectors of \( \Lambda \) with squared length \( 2k \). We call such a set a \( 2k \)-frame of \( \Lambda \). The existence of \( 2k \)-frame of \( \Lambda \) is shown in [CR, GH]. By using a \( 2k \)-frame, we show that \( V^2 \) contains a subVOA isomorphic to the tensor product of 24 copies of \( V_L^+ \). Then we give the decomposition of \( V^2 \) as a \( (V_L^+)^{\otimes 24} \)-module. By using the fact that there exists the natural bijection between equivalence classes of \( 2k \)-frames and extremal Type II codes of length 24 over \( \mathbb{Z}_{2k} \), the decompositions of \( V^2 \) are described in terms of such codes. By the fusion rules of \( V_L^+ \), we obtain automorphisms of \( V^2 \) with respect to the decompositions. More precisely, we have \( 4A \) elements (\( k \) is odd) and \( 2B \) elements (\( k \) is even) of the Monster simple group. Moreover, we give new expressions of McKay-Thompson series for \( 4A \) elements and obtain formulas of modular functions.

Professor Ching Hung Lam studies the subject independently.

Throughout the paper, we will work over the field \( \mathbb{C} \) of complex numbers unless otherwise stated. We denote the set of integers by \( \mathbb{Z} \) and the ring of the integers modulo \( k \) by \( \mathbb{Z}_k \).

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1 Preliminaries

In this section, we recall or give some definitions and facts necessary in this paper.

1.1 Charge conjugation orbifold VOA \( V_L^+ \) and its irreducible modules

In this section, we give the charge conjugation orbifold VOA \( V_L^+ \) and its irreducible modules. The description of \( V_L^+ \) in this paper is slightly different from those given in [DN, AB] because it is useful to show the main theorem.

Let \( L = \mathbb{Z} \alpha \) be an even lattice of rank one with \( \langle \alpha, \alpha \rangle = 2k \). Set \( H = \mathbb{C} \otimes \mathbb{Z} L \) and we regard \( L \) as a subgroup of \( H \). We extend the form \( \langle \cdot, \cdot \rangle \) to a \( \mathbb{C} \)-bilinear form on \( H \). Let \( L^o = \{ h \in H \mid \langle h, \alpha \rangle \in \mathbb{Z} \} = L/2k \) be the dual lattice of \( L \). Let \( \hat{L}^o \) be the trivial extension of \( L^o \) by the order 2 cyclic group \( \langle -1 \rangle \). Form the induced \( \hat{L}^o \)-module \( \mathbb{C}\{L^o\} = \mathbb{C}[\hat{L}^o] \otimes_{\mathbb{C}[\pm 1]} \mathbb{C} \), where \( \mathbb{C}[\cdot] \) denotes the group algebra and \( -1 \) acts on \( \mathbb{C} \) as multiplication by \( -1 \). We choose a section \( L^o \to \hat{L}^o \), \( \beta \mapsto e^\beta \) such that \( e^\beta e^\gamma = (-1)^{\langle \beta, \gamma \rangle} e^{\beta + \gamma} \) for \( \beta, \gamma \in L \). Set \( V_{L^o} = S(H \otimes t^{-1} \mathbb{C}[t^{-1}]) \otimes \mathbb{C}\{L^o\} \). For \( \lambda + L \in L^o/L \), we set \( V_{\lambda+L} = S(H \otimes t^{-1} \mathbb{C}[t^{-1}]) \otimes \mathbb{C}\{\lambda + L\} \), where \( \lambda + L \) is regarded as a subset of \( L^o \) and
\( \mathbb{C}\{\lambda + L\} \) is the subspace of \( \mathbb{C}\{L^0\} \) spanned by \( \{e^\mu \mid \mu \in \lambda + L\} \). It is well known that \( V_L \) has a VOA structure and \( V_{\lambda + L} \) has a \( V_L \)-module structure for each \( \lambda + L \in L^0/L \) (cf. [FLM]). Note that all the irreducible \( V_L \)-modules are given by the set \( \{V_{\lambda + L} \mid \lambda + L \in L^0/L\} \) (cf. [D1]). For convenience, we denote \( h \otimes t^n \) by \( h(n) \) for \( h \in H \) and \( n \in \mathbb{Z} \).

Let \( \rho_L : L \to \mathbb{C}^\times \) be a group homomorphism such that \( \rho_L(2\mu) = (-1)^{(\mu, \mu)/2} \) for \( \mu \in L \). Let \( \theta_L \) be the automorphism of \( \hat{L} \) given by \( \theta_L(e^\mu) = \rho_L(2\mu)(e^\mu)^{-1} \) for \( \mu \in L \) and \( \theta_L(-1) = -1 \). Let \( \theta_{V_L} \) be the unique commutative algebra automorphism of \( V_L \) such that \( \theta_{V_L}(h(n) \otimes 1) = -h(n) \otimes 1 \) and \( \theta_{V_L}(1 \otimes e^\mu) = 1 \otimes \theta_L(e^\mu) \) for \( h \in H \) and \( \mu \in L \).

Let \( \lambda \) be a representative of a coset in \( L^0/L \). We will extend \( \theta_{V_L} \) on \( V_L^{o} \) as a \( V_L \)-module isomorphism. Let \( \theta_{V_L} : V_{\lambda + L} \to V_{-\lambda + L} \) be the \( V_L \)-module isomorphism such that \( \theta_{V_L}(e^\lambda) = e^{-\lambda} \). When \( \lambda \in L/2 \), the linear map \( V_{\lambda + L} \to V_{-\lambda + L} \), \( x \mapsto \rho_L(-2\lambda)x \) is an isomorphism of \( V_L \)-modules. Then we have the automorphism \( \theta_{V_L} \) of \( V_{\lambda + L} \) as a \( V_L \)-module such that \( \theta_{V_L}(e^\lambda) = \rho_L(2\lambda)e^{-\lambda} \). For each \( \theta_{V_L} \)-stable subspace \( M \) of \( V_L^{o} \), let \( M^\pm \) denote the \( \pm 1 \) eigenspaces of \( M \) with respect to \( \theta_{V_L} \) respectively. Note that \( V_L^+ \) is a subVOA of \( V_L \).

Let \( K_L = \{\theta_L(a)a^{-1} \mid a \in \hat{L}\} \) be the subgroup of \( \hat{L} \). Let \( T_0 \) and \( T_1 \) be irreducible \( \hat{L}/K_L \)-modules over \( \mathbb{C} \) such that \( e^\alpha \) acts by \( \rho_L(\alpha) \) and \( e^\alpha \) acts by \( -\rho_L(\alpha) \) respectively. Note that \( -1 \in \hat{L} \) acts by \( -1 \) on \( T_i \). Set \( V_L^{T_i} = S(H \otimes t^{-1/2}\mathbb{C}[t^{-1}]) \otimes T_i \). Then it has an irreducible \( \theta_{V_L} \)-twisted \( V_L \) module structure. Moreover, \( V_L^{T_0} \) and \( V_L^{T_1} \) give all the inequivalent irreducible \( \theta_{V_L} \)-twisted \( V_L \)-modules (cf. [D2]).

Let \( \theta_H \) be the unique commutative algebra automorphism of \( S(H \otimes t^{-1/2}\mathbb{C}[t^{-1}]) \) such that \( \theta_H(h(n)) = -h(n) \) for \( h \in H \) and \( n \in 1/2 + \mathbb{Z} \). We consider the linear automorphism of \( V_L^{T_i} \) given by \( \theta_H \otimes 1_{T_i} \), where \( 1_{T_i} \) is the identity operator of \( T_i \). By abuse of notation, we denote both automorphisms of \( V_L^{T_0} \) and \( V_L^{T_1} \) by \( \theta_{V_L} \). Since \( e^\alpha = \theta_{V_L}(e^\alpha) \) on \( T_i \), \( \theta_{V_L} \) is an automorphism as a \( V_L \)-module. We denote the \( \pm 1 \) eigenspaces of \( V_L^{T_i} \) with respect to \( \theta_{V_L} \) by \( V_L^{T_i, \pm} \) respectively. Then \( V_L^{T_i, \pm} \) becomes an irreducible \( V_L \)-module.

Note that \( V_L^+ \) and its irreducible modules defined above are isomorphic to those given in [DN]. By Theorem 5.13 of [DN], the set

\[
\{V_L^\pm, V_{\alpha/2+L}^\pm, V_{T_i, \pm}^\pm, V_{\omega/2k+L}^\pm \mid i = 0, 1, \ldots, k - 1\}
\]

gives all inequivalent irreducible \( V_L^+ \)-modules.

### 1.2 Moonshine module

Let us review the moonshine module \( V_2 \) from [FLM] for what we need in this paper.

Let \( \Lambda \) be the Leech lattice with the positive-definite \( \mathbb{Z} \)-bilinear form \( \langle \cdot, \cdot \rangle \). It is a unique positive-definite even unimodular lattice of rank 24 without roots.
Let \( \hat{\Lambda} \) be the central extension of \( \Lambda \) by the cyclic group \( \langle -1 \rangle \cong \mathbb{Z}_2 \):

\[
1 \to \langle -1 \rangle \to \hat{\Lambda} \to \Lambda \to 0
\]  

(1.1)

with the commutator map \( c_0(\beta, \gamma) = \langle \beta, \gamma \rangle + 2\mathbb{Z} \) for \( \beta, \gamma \in \Lambda \).

Let \( \theta_\Lambda \) be the automorphism of \( \hat{\Lambda} \) given by \( \theta_\Lambda(a) = a^{-1}(-1)^{(a, a)/2} \) for \( a \in \hat{\Lambda} \). We denote the center of \( \hat{\Lambda} \) by \( \text{Cent} \hat{\Lambda} \). Set \( K_\Lambda = \{ \theta_\Lambda(a)a^{-1} \mid a \in \hat{\Lambda} \} \subset \text{Cent} \hat{\Lambda} \). Then \( K_\Lambda \) is a normal subgroup of \( \hat{\Lambda} \); consider the group \( \hat{\Lambda}/K_\Lambda \). Note that the center of \( \hat{\Lambda}/K_\Lambda \) is \( \langle -K_\Lambda \rangle \cong \mathbb{Z}_2 \). By Theorem 5.5.1 of [FLM], we have the following proposition.

**Proposition 1.1.** ([FLM]) Let \( G = \hat{\Lambda}/K_\Lambda \) and let \( \chi \) be a character of \( \text{Cent} G = \langle -K_\Lambda \rangle \) such that \( \chi(-K_\Lambda) = -1 \). Let \( A \) be a maximal abelian subgroup of \( G \) and let \( \psi : A \to \mathbb{C}^\times \) be a character with \( \psi|_{\text{cent} \ G} = \chi \). Then \( T = \text{Ind}_{A}^{\hat{\Lambda}} \mathbb{C}_\psi \) is the unique irreducible \( G \)-module on which \( x \in \text{Cent} G \) acts by \( \chi(x) \mathbf{1}_T \), where \( \mathbb{C}_\psi \) is the one-dimensional \( A \)-module corresponding to \( \psi \). Moreover, \( T \cong \oplus \mathbb{C}_\varphi \), where \( \varphi \) ranges over the characters of \( A \) whose restriction to \( \text{Cent} G \) is \( \chi \), and \( \dim T = 2^{12} \).

Set the induced \( \hat{\Lambda} \)-module \( \mathbb{C}\{\Lambda\} = \mathbb{C}[\hat{\Lambda}] \otimes_{\mathbb{C}[\pm 1]} \mathbb{C} \), where \( -1 \) acts on \( \mathbb{C} \) as multiplication by \( -1 \). We extend \( \theta_\Lambda \) to \( \mathbb{C}[\hat{\Lambda}] \) linearly. Since \( \theta_\Lambda \) fixes \( -1 \), we view the automorphism \( \theta_\Lambda \) as an automorphism of \( \mathbb{C}\{\Lambda\} \).

Set \( \mathfrak{h} = \mathbb{C} \otimes_\mathbb{Z} \Lambda \). We regard \( \Lambda \) as a subgroup of \( \mathfrak{h} \). We extend \( \langle \cdot, \cdot \rangle \) to a \( \mathbb{C} \)-bilinear form on \( \mathfrak{h} \). Let \( T \) be the irreducible \( \hat{\Lambda}/K_\Lambda \)-module given in Proposition 1.1.1. Set \( V_\Lambda = S(\mathfrak{h} \otimes t^{-1/2}\mathbb{C}[t^{-1}]) \otimes \mathbb{C}\{\Lambda\} \) and \( V_\Lambda^T = S(\mathfrak{h} \otimes t^{-1/2}\mathbb{C}[t^{-1}]) \otimes T \). For convenience, we also denote \( h \otimes t^n \) by \( h(n) \) for \( h \in \mathfrak{h} \) and \( n \in \mathbb{Z}/2 \). Let \( \theta_{V_\Lambda} \) be the unique commutative algebra automorphism of \( V_\Lambda \) such that \( \theta_{V_\Lambda}(h(n) \otimes 1) = -h(n) \otimes 1 \) and \( \theta_{V_\Lambda}(1 \otimes b) = 1 \otimes \theta_\Lambda(b) \), where \( h \in \mathfrak{h} \), \( n \in \mathbb{Z}_{<0} \) and \( b \in \mathbb{C}\{\hat{\Lambda}\} \). Let \( \theta_\mathfrak{h} \) be the unique commutative algebra automorphism of \( S(\mathfrak{h} \otimes t^{-1/2}\mathbb{C}[t^{-1}]) \) such that \( \theta_\mathfrak{h}(h(n)) = -h(n) \) for \( h \in \mathfrak{h} \), \( n \in 1/2 + \mathbb{Z}_{<0} \). Then \( \theta_{V_\Lambda^T} = \theta_\mathfrak{h} \otimes (-1)_T \) is an automorphism of \( V_\Lambda^T \), where \( (-1)_T \) is the identity operator on \( T \). Thus we have \( V^z = V_\Lambda^+ \oplus V_\Lambda^{T+} \), where \( V_\Lambda^+ \) is the \( \theta_{V_\Lambda} \)-fixed-point subspace of \( V_\Lambda \) and \( V_\Lambda^{T+} \) is the \( \theta_{V_\Lambda^T} \)-fixed-point subspace of \( V_\Lambda^T \).

### 1.3 2k-frames and codes over \( \mathbb{Z}_{2k} \)

In this subsection, we give some terminology on a code over \( \mathbb{Z}_{2k} \) (cf. [DHS]).

A (linear) code \( C \) of length \( n \) over \( \mathbb{Z}_{2k} \) is a \( \mathbb{Z}_{2k} \)-submodule of \( \mathbb{Z}_{2k}^n \). We denote the image of \( x \in \mathbb{Z} \) with respect to the canonical map \( \mathbb{Z} \to \mathbb{Z}_{2k} \) by \( \bar{x} \). We fix an ordered basis of \( \mathbb{Z}_{2k}^n \) and denote \( i \)-th element of this basis by \( (\bar{0}, \ldots, \bar{0}, \bar{1}, \bar{0}, \ldots, \bar{0}) \), where \( \bar{1} \) only appears in the \( i \)-th position.
An element of $C$ is called a *codeword*. The Euclidean weight $\text{Ewt}(\cdot)$ on $\mathbb{Z}_{2k}^n$ is given by $\text{Ewt}(c) = \sum_{i=1}^{2k} m_i^2$, where $c = (\bar{m}_1, \ldots, \bar{m}_n) \in \mathbb{Z}_{2k}^n$ and $-k < m_i \leq k$ for $i = 1, \ldots, n$. We define the inner product of $x$ and $y$ in $\mathbb{Z}_{2k}^n$ by $\langle x, y \rangle = \sum_{i=1}^{n} \bar{x}_i \bar{y}_i$, where $x = (\bar{x}_1, \ldots, \bar{x}_n)$ and $y = (\bar{y}_1, \ldots, \bar{y}_n)$. The *dual code* $C^\perp$ of $C$ is defined as $C^\perp = \{ x \in \mathbb{Z}_{2k}^n \mid \langle x, y \rangle = 0 \text{ for all } y \in C \}$. $C$ is *self-orthogonal* if $C \subset C^\perp$ and $C$ is *self-dual* if $C = C^\perp$. We define a *Type II* code over $\mathbb{Z}_{2k}^k$ to be a self-dual code with all codewords having Euclidean weight divisible by $4k$.

It is well known that $\tilde{S}_n = \mathbb{Z}_2 \wr S_n = \text{Aut}(\mathbb{Z}_{2k}^n)$, where $S_n$ is the symmetric group. $\tilde{S}_n$ acts on $\mathbb{Z}_{2k}^n$ by the permutation of the coordinate positions and the change of the signs of some positions of $\mathbb{Z}_{2k}^n$. Two codes $C_0$ and $C_1$ over $\mathbb{Z}_{2k}$ are called *equivalent* if they both have length $n$ and if there exists $\sigma \in \tilde{S}_n$ such that $C_0 = \sigma(C_1)$.

**Definition 1.2.** A *2k-frame* of a lattice of rank $n$ is a set of $n$ mutually orthogonal pairs of opposite vectors of squared length $2k$.

We denote the group of isometries of a lattice $\Lambda$ by $\text{Aut}(\Lambda)$. Since $\text{Aut}(\Lambda)$ preserves the inner product, $\text{Aut}(\Lambda)$ acts on the set of $2k$-frames of $\Lambda$. We say that $2k$-frames $S_0$, $S_1$ of $\Lambda$ are *equivalent* if there exists $\tau \in \text{Aut}(\Lambda)$ such that $S_0 = \tau(S_1)$.

We define an *extremal* Type II code of length 24 over $\mathbb{Z}_{2k}$ to be a Type II code with minimum Euclidean weight $8k$.

**Proposition 1.3.** [Ch, GH] For any positive integer $k$, there exists an extremal Type II code of length 24 over $\mathbb{Z}_{2k}$.

Using the fact that the Leech lattice $\Lambda$ is the unique positive definite unimodular lattice in dimension 24 without roots, it is easy to see that for $k \geq 2$, equivalence classes of $2k$-frames of $\Lambda$ are the same as equivalence classes of extremal Type II codes of length 24 over $\mathbb{Z}_{2k}$. More precisely, for a $2k$-frame $S$ of $\Lambda$, we have an extremal Type II code $C = \Lambda/N \subset N^\circ/N \cong \mathbb{Z}_{2k}^{24}$, where $N$ is the sublattice of $\Lambda$ generated by $S$ and $N^\circ$ is the dual lattice of $N$, and for an extremal Type II code $C$ of length 24 over $\mathbb{Z}_{2k}$, the lattice constructed by the generalized Construction A with $C$ is the Leech lattice and contains a $2k$-frame.

By direct calculation, we have the following proposition.

**Proposition 1.4.** Let $S$ be a $2k$-frame of the Leech lattice $\Lambda$ and let $N$ be the sublattice of $\Lambda$ generated by $S$. Let $C = \Lambda/N$ be a code over $\mathbb{Z}_{2k}$ and let $C_2 = (\Lambda \cap (N/2))/N$ be a binary code. Set $m = \dim C_2$ over $\mathbb{Z}_2$.

1. $|\Lambda/N| = (2k)^{12}$. 


(2) \( m \geq 12 \).

(3) \( N/(2\Lambda \cap N) \) is an elementary abelian 2-group with order \( 2^{24-m} \).

(4) If \( k \) is odd, then \( C_2 \) is a Type II code.

Remark 1.5. Let \( S \) be a \( 2k \)-frame of \( \Lambda \) and \( C \) be an extremal Type II code corresponding to \( S \). Let \( N \) be the sublattice of \( \Lambda \) generated by \( S \). Then it is easy to see that \( C_2 = (\Lambda \cap (N/2))/N \cong \{(c_1, \ldots, c_{24}) \in C \mid c_i = 0 \pmod{k} \text{ for all } i\} \), where we regard the both codes as binary codes.

2 Decompositions of \( V^\Lambda \) and \( \mathbb{Z}_{2k} \) codes

In this section, for an integer \( k \geq 2 \), we give the decomposition of \( V^2 \) as a \( (V_L^+)^{\otimes 24} \)-module associated with an extremal Type II code over \( \mathbb{Z}_{2k} \). It is easy to see the embedding of \( V_L \) into \( V^\Lambda \). But the 24 tensor product of the involution of \( V_L \) given in [DN] is not the same involution \( \theta_{V_\Lambda} \) of \( V_\Lambda \). By using the definition of \( V_L \) given in Section 1.1, we will clarify the problem.

For convenience, we use the following notation. For \( c = (\bar{c}_1, \ldots, \bar{c}_{24}) \in \mathbb{Z}_{2k}^{24} \), we set

\[
M(c) = \bigotimes_{i=1}^{24} V_{\bar{c}_i/2k+L}.
\]

Note that \( M(c) \cong M(\sigma(c)) \) as a \( (V_L^+)^{\otimes 24} \)-module for \( c \in \mathbb{Z}_{2k}^{24} \), where \( \sigma \) is an operator which changes signs of some positions.

In particular, for \( c = (\bar{c}_1, \ldots, \bar{c}_{24}) \in \mathbb{Z}_2^{24} \), we set

\[
M^+(c) = \bigoplus_{\substack{e_i \in \{\pm\} \atop \prod e_i = +}} \bigotimes_{i=1}^{24} V_{\bar{c}_i/2k+L}^{e_i},
\]

\[
M^-(c) = \bigoplus_{\substack{e_i \in \{\pm\} \atop \prod e_i = -}} \bigotimes_{i=1}^{24} V_{L}^{T_{c_i} e_i},
\]

where \( \{\pm\} \cong \{-\} \cong \mathbb{Z}_2 \).

Remark 2.1. Let \( c = (c_1, \ldots, c_{24}) \) be an element of \( \mathbb{Z}_2^{24} \).

(1) \( M^+(c) \) is the fixed-point subspace of \( \otimes V_{c_i/2k+L} \) with respect to \( \theta_{V_L}^{\otimes 24} \).

(2) \( M^-(c) \) is the \( -1 \) eigenspace of \( \otimes V_{L}^{T_{c_i} e_i} \) with respect to \( \theta_{V_L}^{\otimes 24} \).
2.1 Main results

The following is our main theorem.

Theorem 2.2. For $k \geq 2$, let $C$ be an extremal Type II code of length 24 over $\mathbb{Z}_{2k}$ and set $C_2 = \{(c_1, \ldots, c_{24}) \in C \mid c_i = 0 \pmod{k} \text{ for all } i\}$, which is binary code. Set $m = \dim C_2$ over $\mathbb{Z}_2$. Then there exists an embedding of $(V_L^+)^{\otimes 24}$ into $V_A^+$ as a subVOA such that $V_V^+ = V_A^+ \oplus V_A^{T,+}$ decomposes into $(V_L^+)^{\otimes 24}$-modules as the following:

1. \[ V_A^+ \cong \bigoplus_{c \in C_2} M(c)^+ \oplus \frac{1}{2} \bigoplus_{c \in C \setminus C_2} M(c). \]

2. \[ V_A^{T,+} \cong \bigoplus_{c \in C_2^*} 2^{m-12} M^T(c)^-. \]

Remark 2.3. This decomposition is uniquely determined by the extremal Type II code of length 24 over $\mathbb{Z}_{2k}$, up to the action of $\tilde{S}_n = \mathbb{Z}_2 \wr S_n$.

The rest of this section is devoted to the proof of the theorem.

2.2 Key lemma

In this subsection, we will give the key lemma for Theorem 2.2. It is an extension of Theorem D.6 of [DGH]. More precisely, we consider the case of a dual lattice. We will use the lemma to identify $\theta_{V_L^+}$ with $\theta_{V_A^+}$ on $V_A$.

Let $U$ be an even integral lattice and let $V_U$ be the lattice VOA associated to $U$. By [D1], $V_U^\circ$ is a $V_U$-module. We choose a section $U^\circ \to \tilde{U}^\circ$, $x \mapsto e^x$ such that the 2-cocycle with respect to it is $\mathbb{Z}$-bilinear.

Definition 2.4. A lift of $-1$ of $U^\circ$ is an automorphism $\theta$ of $V_U^\circ$ as a $V_U$-module such that for all $x \in U^\circ$, there is a scalar $c_x$ so that $\theta : e^x \mapsto c_x e^{-x}$.

Set $A_U = \text{Hom}(U, \mathbb{C}^\times)$ and set $A_U^\circ = \{ f : U^\circ \to \mathbb{C}^\times \mid f(x + y) = f(x)f(y) \text{ for } x \in U, y \in U^\circ, f|_U \in A_U\}$. For $f \in A_U^\circ$, we set the automorphism $\tilde{f}$ of $V_U^\circ$ as a $V_U$-module by $\tilde{f} : y \otimes e^x \mapsto f(x)y \otimes e^x$, and we set $\tilde{A}_U^\circ = \{ \tilde{f} \mid f \in A_U^\circ\}$.

By Theorem D.6 of [DGH], we have the following proposition.
Proposition 2.5. For lifts $f$ and $g$ of $-1$ of $U^\circ$, there exists an element $s \in \tilde{A}_{U^\circ}$ such that $s^{-1}f|_{V_U} = g|_{V_U}$.

Moreover, we consider the case of $V_{U^\circ}$.

Lemma 2.6. Let $f, g$ be lifts of $-1$ of $U^\circ$ such that $f = g$ on $V_U$. Let $S$ be a set of representatives of $(U^\circ \cap U/2)/U$. Then $f$ and $g$ are conjugate by an element of $\tilde{A}_{U^\circ}$ whose restriction on $V_U$ is the identity map if and only if $f(e^x) = g(e^x)$ for $x \in S$.

Proof. In order to simplify the proof, we assume $g(e^x) = e^{-x}$ for $x \in U^\circ$. Let $\tilde{S}$ be a set of representatives of $U^\circ/U$ containing $S$.

First, we assume $f(e^x) = g(e^x)$ for $x \in \tilde{S}$. By a conjugation of $\tilde{A}_{U^\circ}$, we assume $f(e^x) = e^{-x}$ for $x \in \tilde{S}$. Namely, if $f(e^x) = c_x e^{-x}$ for $x \in S$, we set $s \in A_{U^\circ}$ such that $s(x) = c_x^{-1/2}$ for $x \in \tilde{S} \setminus S$ and $s(y) = 1$ for $y \in U \cup S$. It is easy to see that $\tilde{s}^{-1} f \tilde{s}(e^x) = e^{-x}$ for $x \in \tilde{S}$. Note that $V_{U^\circ} = \oplus_{\lambda \in \tilde{S}} V_{\lambda^+ U}$, and for $\lambda \in \tilde{S}$, $V_{\lambda^+ U}$ is generated by $e^\lambda$ as a $V_U$-module. Since $f$ is a $V_U$-module automorphism and 2-cocycle is a Z-bilinear map, we have $f(e^x) = e^{-x}$ for $x \in U^\circ$ by the direct calculation.

Next, we assume $\tilde{s}^{-1} f \tilde{s} = g$ for $\tilde{s} \in \tilde{A}_{U^\circ}$ such that $\tilde{s}|_{V_U}$ is the identity map. For $x \in S$, we set $f(e^x) = c_x e^{-x}$, where $c_x \in \mathbb{C}^\times$. Then we have $\tilde{s}^{-1} f \tilde{s}(e^x) = s(x)c_x \tilde{s}^{-1}(e^{-2x} e^x) = s(-2x)^{-1}c_x e^{-x} = s(-2x)g(e^x)$ for $x \in S$. Since $-2x \in U$ and $s = 1$ on $U$, we have $f(e^x) = g(e^x)$ for $x \in \tilde{S}$. $\square$

2.3 Decomposition of the untwisted space $V^+_\Lambda$

In this subsection, we give the decomposition of $V^+_\Lambda$.

Let $F = (x_1, \ldots, x_{24})$ be the $2k$-frame of $\Lambda$ corresponding to the code $C$ and let $N$ be the sublattice of $\Lambda$ generated by $F$. Since $L = \mathbb{Z} \alpha$ is an even lattice of rank one with $\langle \alpha, \alpha \rangle = 2k$, we have $N \cong L^{(24)}$. From [D1], it follows that $V_\Lambda$ is decomposed as

$$V_\Lambda = \bigoplus_{\lambda + N \in \Lambda/N} V_{\lambda + N}$$

(2.1)

as a $V_N$-module.

It is well known that

$$V_{\lambda + N} \cong \bigotimes_{i=1}^{24} V_{c_i/2k + L} = M(c)$$

(2.2)

as a $(V^+_L)^{\otimes 24}$-module, where $\lambda + N = \sum_{i=1}^{24} c_i x_i/2k + N$ and $c = (\bar{c}_1, \ldots, \bar{c}_{24})$ is a codeword of $C = \Lambda/N$. 

8
Now, we have two involutions of $V_{\Lambda}$ as a $V_N$-module. One is $f = \theta_{V_{\Lambda}}$ given in Section 1.2 and the other one is $g = \theta_{V_L}^{\otimes 24}$. Note that $\theta_{V_L}$ is the automorphism of $V_L$ as a $V_L$-module given in Section 1.1. We will apply Proposition 2.5 and Lemma 2.6 to $V_{\Lambda} \subset V_{N^2}$, and identify $f$ and $g$. Since we consider $V_{\Lambda}$ in this section, we use the notation $A_{\Lambda}$ instead of $A_{N^2}$. By [FLM], we can choose a $\mathbb{Z}$-bilinear 2-cocycle $\varepsilon : \Lambda \times \Lambda \to \mathbb{Z}_2$ such that $\varepsilon(x, x) = \langle x, x \rangle / 2$.

By Proposition 2.5 there exists $\tilde{s} \in \tilde{A}_{\Lambda}$ such that $\tilde{s}^{-1}f\tilde{s} = g$ on $V_N$. Therefore we identify $g$ with $f$ on $V_N$, and we have the inclusion of subVOAs $(V_L^+)_{\otimes 24} \subset V_{\Lambda}^+ \subset V^2$.

**Remark 2.7.** Let $V_{L,R}^+$ and $V_{R}^3$ be the real forms of $V_L^+$ and $V^3$ as constructed in [FLM]. By [FLM], those have the positive-definite symmetric bilinear forms. By the above inclusion, we have the embedding $V_{L,R}^+ \subset V_{R}^3$. In this embedding, the positive-definite symmetric bilinear form of the subVOA is the restriction of that of $V_{R}^3$.

Next, in order to apply to lemma 2.6, we have to check the hypothesis of the lemma. Let $s$ be the element of $A_{\Lambda}$ corresponding to $\tilde{s}$. For $x \in \Lambda \cap N/2$, we have $\tilde{s}^{-1}f\tilde{s}(e^x) = s(-2x)e^{-x}$ and $g(e^x) = \rho_N(2x)e^{-x}$, where $\rho_N = \oplus \rho_{2z_i}$. Since $\rho_N$ and $s$ are linear on $N$, we can choose $\rho_N$ and the set $S$ of representatives of $(\Lambda \cap N/2)/N$ such that $\tilde{s}^{-1}f\tilde{s} = g$ on $\tilde{S}$. Therefore we can use lemma 2.6, and we have $f$ and $g$ are conjugate by $A_{\Lambda}$. So, we identify $f$ with $g$ on $V_{\Lambda}$.

We consider the $\theta_{V_L}^{\otimes 24}$-fixed-point subspace of $V_{\Lambda}$. We obtain the following lemma (cf. [KLY1]).

**Lemma 2.8.** Let $c$ be a codeword of $C = \Lambda/N$.

1. If $c \in C_2 = (\Lambda \cap N/2)/N$, then $M(c)$ is $\theta_{V_L}^{\otimes 24}$-invariant, and $M(c)^+$ is the $\theta_{V_L}^{\otimes 24}$-fixed-point subspace.

2. If $c \in C \setminus C_2$, then $M(c) \oplus M(-c)$ is $\theta_{V_L}^{\otimes 24}$-invariant, and the $\theta_{V_L}^{\otimes 24}$-fixed-point subspace $(M(c) \oplus M(-c))^+$ is isomorphic to $M(c)$ as a $(V_L^+)_{\otimes 24}$-module.

By Lemma 2.8, (2.1) and (2.2), we obtain Theorem 2.2 (i).

### 2.4 Decomposition of the twisted space $V_{\Lambda}^{T,+}$

In this subsection, we give the decomposition of $V_{\Lambda}^{T,+}$.

Set $K(N) = K_\Lambda \cap \hat{N}$ and $K(2N) = K_\Lambda \cap 2\hat{N}$. Since $T$ is a $\hat{A}/K_\Lambda$-module, $T$ is a $\hat{N}K_\Lambda/K_\Lambda$-module. Note that $-1 \in \hat{\Lambda}$ acts on $T$ as multiplication by $-1$. Let $M$ be a maximal abelian subgroup of $\hat{\Lambda}/K_\Lambda$ such that $M \supset \hat{N}K_\Lambda/K_\Lambda$. Note that $|M| =
2^{13}. By Proposition 1.1, $T$ decomposes into the direct sum of all the irreducible $M$-modules on which $-K_A$ acts by $-1$. Therefore $T$ decomposes into the direct sum of irreducible $\tilde{N}K_A/K_A$-modules on which $-K_A$ acts by $-1$. By Proposition 1.4 (3), we have $|\tilde{N}K_A/K_A| = 2|N/(N \cap 2\Lambda)| = 2^{25} - m$. Then any multiplicities of the irreducible $\tilde{N}K_A/K_A$-module is $2^{m-12}$. Since $\tilde{N}K_A/K_A \cong \hat{N}/K(N)$, $T$ is a $\hat{N}/K(N)$-module. Thus we obtain

$$T \cong \bigoplus_{\psi \in \text{Irr}(\hat{N}/K(N))} 2^{m-12}T\psi,$$

(2.3)

where $\text{Irr}(\hat{N}/K(N))$ is the set of the characters for $\hat{N}/K(N)$ and $T\psi$ is the one-dimensional module $\mathbb{C}$ with a character $\psi$.

By using the canonical map $\pi: \tilde{N}/K(2N) \to \hat{N}/K(N)$, we view $T$ as a $\tilde{N}/K(2N)$-module. Note that the kernel of $\pi$ is $K(N)/K(2N)$. Therefore, as a $\tilde{N}/K(2N)$-module, $T$ is decomposed

$$T \cong \bigoplus_{\chi \in \text{Irr}(\tilde{N}/K(2N))} m\chi T\chi,$$

(2.4)

where $m\chi$ is the multiplicity. By (2.3), we have $m\chi \in \{0, 2^{m-12}\}$. By (2.4), $V^T_A$ has a decomposition,

$$V^T_A \cong \bigoplus_{\chi \in \text{Irr}(\tilde{N}/K(2N))} m\chi V^T_N\chi,$$

(2.5)

where $V^T_N\chi$ is the $\theta_{\chi}$-twisted $V_N$-module corresponding to the $\hat{N}/K(2N)$-module $T\chi$.

As in Section 2.3, we fix the ordered basis of $N$ as $(x_1, \ldots, x_{24})$. Let $L_i = \mathbb{Z}x_i$ be a sublattice of $N$ and set $K(L_i) = K_A \cap \hat{L}_i$. For $\chi \in \text{Irr}(\tilde{N}/K(2N))$, we set the character of $\hat{L}_i/K(L_i)$ $\chi_i(e^{x_i}) = \chi(e^{x_i})$. For a character $\chi = \oplus_{i=1}^{24} \chi_i$ such that $\chi(-K(2N)) = -1$, we set $P(\chi) = \langle c_1, \ldots, c_{24} \rangle \in \mathbb{Z}_2^{24}$, where $\chi_i(s(e^{x_i})) = \rho_i(x_i)(-1)^{c_i}$. For $x = \sum d_i x_i \in N$, we set $Q(x) = \langle \tilde{d}_1, \ldots, \tilde{d}_{24} \rangle \in \mathbb{Z}_2^{24}$.

Then we have

$$\chi(\tilde{s}(e^x)) = \rho(\sum d_i x_i)(-1)^{\sum c_i d_i} = \rho(x)(-1)^{\langle P(\chi), Q(x) \rangle}.$$ 

Now, let us determine the multiplicities $\{m\chi\}$. Let $\chi$ be an element of $\text{Irr}(\tilde{N}/K(2N))$ such that $\chi(-K(2N)) = -1$. Set $n = 2^{m-12}$. Then $m\chi = n$ if and only if there exists $\tilde{\chi} \in \text{Irr}(\hat{N}/K(N))$ such that $\chi = \tilde{\chi} \circ \pi$, namely, $\chi(x) = 1$ for any $x \in \text{Ker} \pi = K(N)/K(2N)$. Since $\theta_{\Lambda}(e^x)(e^x)^{-1} = e^{-x}(-1)^{e(x,-x)} e^{-x} = e^{-2x} = \rho_N(2x) s(-2x) e^{-2x}$ for $x \in \Lambda \cap N/2$, we
have $K(N) = \{ \theta_V(a)a^{-1} \mid a \in \Lambda \cap N/2 \} = \{ \rho_N(-\beta)\tilde{s}(e^\beta) \mid \beta \in 2\Lambda \cap N \}$. Therefore $\chi(x) = 1$ for any $x \in K(N)/K(2N)$ if and only if $\langle P(\chi), Q(x) \rangle = 0$ for any $x \in K(N)/K(2N)$.

Since $K(N)/K(2N) \cong (N \cap 2\Lambda)/2N \cong (\Lambda \cap (N/2))/N = C_2$, $m_\chi = n$ if and only if $P(\chi) \in C_2^\perp$. By (2.5), we have an isomorphism

$$V^\chi \cong 2^{m-12} \bigoplus_{\chi \in \text{Irr}(\hat{N}/K(2N))} V^\chi_{P(\chi)}.$$  \hspace{1cm} (2.6)

Next, we fix a character $\chi$ and consider the space $V^\chi_{P(\chi)}$. If $P(\chi) = (c_1, \ldots, c_{24})$, then $T^\chi = \otimes T^\chi_{c_i}$. Thus, we get an isomorphism $V^\chi_{P(\chi)} \cong \otimes V^\chi_{T^\chi_{c_i}}$. Note that we have $\theta_{V^\chi} = -\theta_{V^\chi_{24}}$, because $\theta_{V^\chi}$ acts by $-1$ on $T^\chi$ and $\theta_{V^\chi_{24}}$ acts by $1$ on $T^\chi_{c_i}$. Therefore, for $\chi \in \text{Irr}(\hat{N}/K(2N))$ with $\chi(-K(2N)) = -1$, we have an isomorphism of $(V^\chi_{24})^{\otimes 24}$-modules

$$V^\chi_{P(\chi)} \cong M^T(P(\chi))^{-}.$$  \hspace{1cm} (2.7)

By (2.6) and (2.7), we get Theorem 2.2 (ii).

### 3 Character and automorphisms of the moonshine module

In this section, we give the character of $V^\sharp$ and give some automorphisms of $V^\sharp$.

#### 3.1 Character of the moonshine module

We recall the character of a VOA. Let $V = \bigoplus_{n=0}^{\infty} V_n$ be a VOA and the character of $V$ is given by $\text{ch}(V) = \sum_{n=0}^{\infty} \dim(V_n)q^n$.

In order to give the character of $V^\sharp$, we consider the symmetrized weight enumerator of a code over $\mathbb{Z}_{2^k}$. The symmetrized weight enumerator of a code $C$ over $\mathbb{Z}_{2^k}$ is defined as:

$$\text{swe}_C(p_0, \ldots, p_k) = \sum_{c \in C} \prod_{i=0}^{k} p_i^{n_i(c)},$$

where $n_i(c)$ denotes the number of $j$ such that $c_j = \pm i$. Note that the symmetrized weight enumerators of equivalence codes are same. For $0 \leq i \leq k$, we set

$$a_i = \text{ch}V_{\alpha/2^{k+i+L}} = \frac{1}{\phi(q)} \sum_{j \in \mathbb{Z}} q^{k(j+i)/2},$$

$$b = (\text{ch}V^+_L - \text{ch}V^-_L) = \frac{1}{\phi(q)} \sum_{j \in \mathbb{Z}} (-1)^j q^{j^2},$$

11
where \( \phi(q) = \prod_{n \geq 1}(1 - q^n) \). Note that for \( c \in \mathbb{C}_2 \setminus \{0\} \), we have \( \text{ch}(M(c)^+) = \text{ch}(M(c))/2 \), and \( \text{ch}(M(0)^+) = b^{24} + \text{ch}(M(0))/2 \).

By Theorem \( \ref{thm:2.2} \), we obtain the following corollary.

**Corollary 3.1.** Let \( C \) be an extremal Type II code of length 24 over \( \mathbb{Z}_{2k} \). Then we have

\[
\text{ch}(V^\ast) = q(J(q) - 744) = \frac{1}{2} \text{swe}_C(a_0, a_1, \ldots, a_k) + \frac{1}{2} b^{24} + \phi(q)^{24}/(\phi(q^{1/2})^{24} - \phi(q^{12})^{18}).
\]

**Remark 3.2.** It is easy to see that \( \text{swe}_C(a_0, a_1, \ldots, a_k) = \Theta_{\Lambda}(q)/(\phi(q)^{24}) \), where \( \Theta_{\Lambda}(q) \) is the theta function associated with \( \Lambda \). Since \( b = \phi(q)/\phi(q^2) \), we obtain

\[
\text{ch}(V^\ast) = \frac{1}{2} \left( \Theta_{\Lambda}(q) + \phi(q)^{24}/(\phi(q^{1/2})^{24}) \right) + \phi(q)^{24}/(\phi(q^{1/2})^{24} - \phi(q^{12})^{18}).
\]

This equation is given in Remark 10.5.8 of [FLM].

### 3.2 Automorphisms of the moonshine module

Let \( \Gamma \) be the set of inequivalent irreducible \( V_L^+ \)-modules. If \( k \) is odd then we define the map \( \mu_k : \Gamma \to \mathbb{C}^\times \) setting by

\[
\mu_k(W) = \begin{cases} 
1 & \text{if } W \in \{V_L^+, V_{j\alpha/k+L}^+ | 1 \leq j \leq (k-1)/2\}; \\
-1 & \text{if } W \in \{V_{\alpha/2+L}, V_{(2j-1)\alpha/2k+L}^+ | 1 \leq j \leq (k-1)/2\}; \\
i & \text{if } W \in \{V_{T_0, \pm}^+\}, \\
-i & \text{if } W \in \{V_{T_1, \pm}^+\},
\end{cases}
\]

and if \( k \) is even then we defined the map \( \mu_k : \Gamma \to \mathbb{C}^\times \) setting by

\[
\mu_k(W) = \begin{cases} 
1 & \text{if } W \in \{V_L^+, V_{\alpha/2+L}, V_{j\alpha/2k+L}^+ | 1 \leq j \leq (k-1)\}; \\
-1 & \text{if } W \in \{V_{T_0, \pm}^+, V_{T_1, \pm}^+\}.
\end{cases}
\]

The fusion algebra of \( V_L^+ \) is the vector space \( U = \bigoplus_{W \in \Gamma} \mathbb{C}W \) equipped products \( \times \) given by fusion rules, where we regard \( W \) as a formal element. The definition of fusion rules is given in [DL]. An automorphism of the fusion algebra \( U \) is a linear automorphism \( g \) such that \( g(A \times B) = g(A) \times g(B) \) for \( A, B \in U \). By the fusion rules of \( V_L^+ \) determined in [AL], we have the following proposition (cf. [Ma]).

**Proposition 3.3.** The linear map of the fusion algebra of \( V_L^+ \), \( W \mapsto \mu_k(W)W \), is an automorphism of the fusion algebra of \( V_L^+ \).
Suppose we are given a decomposition

\[ V^2 = \bigoplus_{w_i \in \Gamma} m_{W_1, \ldots, W_{24}} W_1 \otimes \cdots \otimes W_{24} \]

as a \((V_L^+)\otimes^{24}\)-module, where \(m_{W_1, \ldots, W_{24}}\) is the multiplicity. For each \(i \in \{1, \ldots, 24\}\), we define a linear automorphism \(\sigma_i\) of \(V^2\) by

\[ \sigma_i(x) = \mu_k(W_i)x \]

for \(x \in \bigotimes_{j=1}^{24} W_j\). By Proposition 3.3, we have the following proposition (cf. [Mi]).

**Proposition 3.4.** \(\sigma_i\) is a VOA automorphism of \(V^2\).

*Proof.* By the definition of \(\mu_k\), \(\sigma_i\) fixes any element of \((V_L^+)\otimes^{24}\). In particular, \(\sigma_i\) fixes the Virasoro element and the vacuum vector. Note that the fusion rules of tensor products of modules are the tensor products of the fusion rules of those modules.

Let \(\Gamma_{24} = \{\otimes_{i=1}^{24} W_i \mid W_i \in \Gamma\}\). Let \(\mu_{k,i} : \Gamma_{24} \to \mathbb{C}^\times\) be a map such that \(\mu_{k,i}(W) = \mu_k(W_i)\) for \(W = \otimes W_j \in \Gamma_{24}\). Let \(W_1, W_2\) be elements of \(\Gamma_{24}\). By the definition of fusion rules, we have \(Y(v,z)w = (W_1 \times W_2)[z, z^{-1}]\) for \(v \in W_1, w \in W_2\), where \(Y(\cdot, \cdot)\) is the vertex operator of \(V^2\) and \(\times\) is fusion rules for \(V_L^+\). Therefore we have \(Y(\sigma_i(v), z)\sigma_i(w) = \mu_{k,i}(W_1)\mu_{k,i}(W_2)Y(v, z)w = \mu_{k,i}(W_1 \times W_2)Y(v, z)w = \sigma_i(Y(v, z)w)\).

By Theorem 2.2, we have the following proposition.

**Proposition 3.5.** Suppose \(k\) is odd. In decompositions of \(V^2\) given by Theorem 2.2, \(\sigma_i\) is a 4A element of the Monster. In fact, \(\sigma_i \in 2^{1+24}_+ \subset 2^{1+24}_+\). Conway_4 \subset Monster, where 2^{1+24}_+. Conway_4 is a non-split extension of Conway_4 (largest simple Conway group) by the extra-special group 2^{1+24}_+.

*Proof.* By Theorem 2.2 and Proposition 3.3, \(\sigma_i^2\) acts by 1 on \(V_A^+\) and acts by \(-1\) on \(V_A^{T^+}\). By [FLM], the centralizer of \(\sigma_i^2\) in the Monster simple group is 2^{1+24}_+. Conway_4. Since \(\sigma_i\) commutes with \(\sigma_i^2\), we have \(\sigma_i \in 2^{1+24}_+\). Conway_4. Since \(\sigma_i\) preserves the space \(\mathbb{C}e^\beta\) for any \(\beta \in \hat{A}\), we have \(\sigma_i \in 2^{1+24}_+\). Conway_4. It is well known that there are the four types of Monster elements 1A, 2A, 2B and 4A contained in 2^{1+24}_+. Since \(\sigma_i\) is an order 4 element, \(\sigma_i\) is a 4A element of the Monster.

**Remark 3.6.** Suppose \(k\) is even. In decompositions of \(V^2\) given by Theorem 2.2, \(\sigma_i\) is a 2B element of the Monster. More precisely, \(\sigma_i\) acts by 1 on \(V_A\) and acts by \(-1\) on \(V_A^T\), namely, \(\sigma_i = z\) given in (10.4.48) of [FLM].
3.3 McKay-Thompson series for 4A elements of the Monster

In this section, we assume that $k$ is odd. Then we give the McKay-Thompson series for 4A elements $\sigma_i$ given in Section 3.2. The expressions of it are different from [CN], and we obtain formulas of modular functions.

We recall the McKay-Thompson series. Let $g$ be an element of the Monster simple group and let $V^\natural = \sum_{n=0}^{\infty} V^\natural_n$ be the moonshine module. Then the McKay-Thompson series for $g$ is given by $T_g(q) = q^{-1}(\sum_{n=0}^{\infty} (\text{Tr}_{g|V^\natural_n})q^n)$, where $\text{Tr}_{g|V^\natural_n}$ is the trace of the action of $g$ on $V_n$.

Let $\tau_i: \mathbb{Z}_{2^{k}}^{24} \rightarrow \mathbb{Z}_2$ be the composite map of the projection map $\mathbb{Z}_{2^{k}}^{24} \rightarrow \mathbb{Z}_{2^k}$ with respect to $i$-th elements and the canonical map $\mathbb{Z}_{2^k} \rightarrow \mathbb{Z}_2$. Note that for $c \in C$, the automorphism $\sigma_i$ acts by the multiplication $(-1)^{\tau_i(c)}$ on $M(c)$.

**Lemma 3.7.** Let $\sigma_i$ be the automorphism of $V^\natural$ given in Section 3.2. In the decomposition given in Theorem 2.1, we have

1. $\sum_{n=0}^{\infty} (\text{Tr}_{\sigma_i|(V^\natural_+)_{n}})q^n = \frac{1}{2} \sum_{c \in C} (-1)^{\tau_i(c)}\text{ch}(M(c)) + b^{24}$,

where $b$ is given in Section 3.1.

2. $\sum_{n=0}^{\infty} (\text{Tr}_{\sigma_i|(V^\natural_{T^+})_{n}})q^n = 0$.

**Proof.** By direct calculation, we have (i). Since $C_2$ contains the all 1 element, the numbers of elements of $C_2$ whose $i$-th coordinate is 0, and whose $i$-th coordinate is 1 are equal. By the definitions of $V^T_L$, we have $\text{ch}(V^T_{L_0}\pm) = \text{ch}(V^T_{L_1}\pm)$ respectively. Therefore we have (ii). \qed

**Corollary 3.8.** Let $C$ be an extremal Type II code over $\mathbb{Z}_{2^k}$. The McKay-Thompson series for the 4A element $\sigma_i$ is given by

$$T_{4A}(q) = q^{-1}\left\{\frac{1}{2} \sum_{c \in C} (-1)^{\tau_i(c)}\text{ch}(M(c)) + b^{24}\right\}.$$ 

**Remark 3.9.** In [Bo2, CN], all the McKay-Thompson series are computed. So we have the following formulas of modular functions.

$$\frac{\eta(q)^{48}}{\eta(q)^{24}\eta(q^4)^{24}} - 24 = q^{-1}\frac{1}{2} \sum_{c \in C} (-1)^{\tau_i(c)}\text{ch}(M(c)) + \frac{\eta(q)^{24}}{\eta(q^2)^{24}},$$

where the Dedekind $\eta$-function $\eta(q) = q^{1/24} \prod_{n \geq 1}(1 - q^n)$. 

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14
References

[Ab] T. Abe, Fusion Rules for the Charge Conjugation Orbifold, *J. Algebra* **242** (2001), 624-655.

[Bo1] R. E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Nat'l. Acad. Sci. U.S.A.* **83** (1986), 3068-3071.

[Bo2] R. E. Borcherds, Monstrous moonshine and monstrous Lie superalgebras, *Invent. Math* **109** (1992), 405-444.

[Ch] R. Chapman, Double circulant constructions of the Leech lattice, *J. Aust. Math. Soc. Ser. A* **69** (2000), 287-297.

[CN] J.H. Conway and S.P. Norton, Monstrous moonshine, *Bull. London Math. Soc.* **11** (1979), 308-339.

[D1] C. Dong, Vertex algebras associated with even lattices, *J. Algebra*, **161** (1993), 245-265.

[D2] C. Dong, Twisted modules for vertex algebras associated with even lattice, *J. Algebra* **165** (1994), 91-112.

[DGH] C. Dong, R. L. Griess Jr., and G. Höhn, Framed vertex operator algebras, codes and moonshine module, *Comm. Math. Phys.* **193** (1998), 407-448.

[DL] C. Dong and J. Lepowsky, Generalized vertex operator algebras and relative vertex operators, Progress in Math. Vol. 112, Birkhäuser, Boston, 1993.

[DLMN] C. Dong, H. Li, G. Mason, and S. P. Norton, Associative subalgebras of the Griess algebra and related topics, “Proceedings of the Conference on the Monster and Lie algebra at Ohio State University” (J. Ferrar and K. Harada, Eds), de Gruyter, Berlin, New York, 1996.

[DHS] S. T. Dougherty, M. Harada and P. Solé, Self-dual codes over rings and the Chinese remainder theorem. *J. Math. Hokkaido Univ.* **28** (1999), 253-283.

[DN] C. Dong and K. Nagatomo, Representations of Vertex operator algebra $V^+_L$ for rank one lattice $L$, *Comm. Math. Phys.* **202** (1999), 169-195.

[FKW] E. Frenkel, V. Kac and M. Wakimoto, Characters and fusion rules for $W$-algebras via quantized Drinfeld-Sokolov reduction, *Comm. Math. Phys.* **147** (1992), 295-328.
[FLM] I. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, Pure and Appl. Math. Vol.134, Academic Press, Boston, (1988).

[GH] T. A. Gulliver, M. Harada, Orthogonal Frames in the Leech Lattice and a Type II code over $\mathbb{Z}_{22}$, J. Combin. Theory. Ser. A 95 (2001), 185-188.

[KLY1] M. Kitazume, C.H. Lam and H. Yamada, Decomposition of the Moonshine Vertex Operator Algebra as Virasoro Modules, J. Algebra 226 (2000), 893-919.

[KLY2] M. Kitazume, C.H. Lam and H. Yamada, Moonshine Vertex Operator Algebra as $L(\frac{1}{2},0) \otimes L(\frac{7}{10},0) \otimes L(\frac{4}{5},0) \otimes L(1,0)$-Modules, to appear.

[Ma] A. Matsuo, Norton’s Trace Formulae for the Griess Algebra of a Vertex Operator Algebra with Larger Symmetry, Comm. Math. Phys. 224 (2001), 565-591.

[Mi] M. Miyamoto, Griess Algebras and Conformal Vectors in Vertex Operator Algebras, J. Algebra 179 (1996), 523-548.