Estimating earthquake-induced tsunami inundation probabilities without sampling

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SUMMARY

Given a distribution of earthquake-induced seafloor elevations, we present a method to compute the probability of the resulting tsunamis reaching a certain size on shore. Instead of sampling, the proposed method relies on optimization to compute the most likely fault slips that result in a seafloor deformation inducing a large tsunami wave. We model tsunamis induced by bathymetry change using the shallow water equations on an idealized slice through the sea. The earthquake slip model is based on a sum of multivariate log-normal distributions, and follows the Gutenberg-Richter law for moment magnitudes 7–9. For a model problem inspired by the Tohoku-Oki 2011 earthquake and tsunami, we quantify annual probabilities of differently sized tsunami waves. Our method also identifies the most effective tsunami mechanisms. These mechanisms have smoothly varying fault slip patches that lead to an expansive but moderately large bathymetry change. The resulting tsunami waves are compressed as they approach shore and reach close-to-vertical leading wave edge close to shore.

Key words: tsunami hazard, probability estimation, shallow water equations, PDE-constrained optimization, adjoint equations.

1 INTRODUCTION

Among all potential sources, megathrust earthquakes are likely to cause the most extreme tsunami hazards for coastal regions (Grezio et al., 2017). Recent work has focused on defining megathrust tsunami source scenarios for simulations to evaluate a quantity of interest (QoI) such as the maximal wave height (Gao et al., 2018). Since earthquake fault slip patterns are unpredictable, a probabilistic study of these QoIs is needed. In (LeVeque et al., 2016; Williamson et al., 2020), the authors construct realistic tsunami source distributions and use simulations based on samples from that distribution to obtain hazard curves for maximum water depth, i.e., the annual probabilities of the water depth exceeding certain values. Typically, estimating small probabilities requires a large number of Monte Carlo samples due to the small probability to find samples corresponding to probability tails. Importance sampling (IS) can be an improvement, but requires a properly chosen proposal density and, typically, still a large number of samples (Liu, 2001). In this paper, we present a sampling-free method for computing annual probabilities of tsunamis on shore using ideas from optimization and applied probability. The approach uses a probabilistic model of earthquake fault slips, adapting ideas from (Gao et al., 2018; LeVeque et al., 2016), and simulates tsunami waves using the one-dimensional shallow water equations. Our target is not an online tsunami warning system, but offline hazard estimation that is capable of accurately computing very small probabilities. The proposed method is most efficient for scalar QoIs, e.g., to compute tsunami size probabilities in a single location on shore.

This paper builds on the approach proposed in (Dematteis et al., 2019) for extreme event probability estimation and extends our recent work (Tong et al., 2021) on tsunami prediction in multiple directions. First, it considers fault slip events that take into account the physics constraints such as slip orientation and magnitude and whose distribution fits the Gutenberg-Richter law for earthquake moments between 7 and 9. To obtain a realistic fault slip model distribution, we use a sum of weighted multivariate log-normals. Additionally, we study the most effective fault mechanisms for obtaining a tsunami of a certain height on shore. These optimizers are a side product of the proposed method which uses a sequence of optimization problems for probability estimation.

The proposed approach for probability estimation is not specific to tsunamis. Rather, it is applicable to a wide class of problems involving complex systems (e.g., governed by partial differential equations) with high-dimensional parameters. The method assumes some regularity of the functions involved and that certain

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optimization problems have a unique solution. Additionally, it requires access to derivatives of the map from parameters to the QoI, which typically can be computed efficiently using adjoint methods. The tsunami problem discussed in this paper assumes slips are of the buried rupture type, but the method can be also applied to other types (splay-faulting, trench-breaching) by choosing different slip distribution means.

2 SAMPLING-FREE ESTIMATION OF SMALL PROBABILITIES

We briefly present our approach to estimate probabilities of outputs (QoIs) arising from complex systems that depend on potentially high-dimensional uncertainties. We summarize the method for multivariate Gaussian parameter distributions. For the tsunami problem discussed in the next section, the approach is adapted for a sum of log-normal distributions.

We assume a random variable vector \( \theta \in \mathbb{R}^n \) with \( \theta \sim N(\mu, C) \), i.e., \( \theta \) follows a multivariate Gaussian distribution with mean \( \mu \in \mathbb{R}^n \) and a symmetric, positive definite covariance matrix \( C \in \mathbb{R}^{n \times n} \). The negative log-probability density function for this distribution is \( I(\theta) := \frac{1}{2} (\theta - \mu)^T C^{-1} (\theta - \mu) \). This quadratic is, at the same time, the so-called rate function in large deviation theory, which allows generalization of the approach to non-Gaussian distributions (Dematteis et al. 2019). However, for Gaussians, log-normals and their sums as considered here, it is sufficient to consider \( I(\cdot) \). The parameter vectors \( \theta \) are the inputs into the parameter-to-QoI map \( F \) discussed next.

We assume a sufficiently regular, possibly complicated map \( F : \mathbb{R}^n \rightarrow \mathbb{R} \), which maps the parameter vector to a scalar QoI \( F(\theta) \in \mathbb{R} \). We are interested in approximating the probability

\[
P(z) := P(F(\theta) \geq z),
\]

where \( z \) is a large value and thus \( P(z) \ll 1 \). While such a probability can be estimated using Monte-Carlo sampling, the performance of these samplers typically degrades for large \( z \) and thus small probabilities \( P(z) \), since random samples are unlikely to lead to large QoIs. Using geometric arguments and results from probability, in particular large deviation theory (LDT), shows that the following optimization problem plays an important role for the estimation of \( P(z) \):

\[
\theta^* (z) = \arg\min_{F(\theta) = z} I(\theta).
\]

The LDT-minimizer \( \theta^* (z) \) is the most likely point in the set \( \Omega(z) = \{ \theta : F(\theta) \geq z \} \), i.e., the random parameters that correspond to a QoI of size \( z \) or larger. Under reasonable assumptions, in particular the uniqueness of the solution of \( \theta^* \) (for details see Tong et al. 2020; Dematteis et al. 2019), one can show that for large \( z \), the probability measure is concentrated around the optimizer \( \theta^* \). Moreover, the probability can be well approximated using local derivative information of \( F \) and \( I \) at \( \theta^* \). Namely, provided sufficient smoothness of \( F \), we can approximate the nonlinear equation \( F(\theta^*) = z \) using a Taylor expansion about \( \theta^* \) truncated after the quadratic term:

\[
F^{SC}(\theta) := F(\theta^*) + \langle \nabla F(\theta^*), \theta - \theta^* \rangle + \frac{1}{2} (\theta - \theta^*)^T \nabla^2 F(\theta^*)(\theta - \theta^*),
\]

Replacing the boundary of the extreme QoI set, i.e., \( \{ \theta : F(\theta) = z \} \) with the quadratic approximation of the boundary, i.e., \( \{ \theta : F^{SC}(\theta) = z \} \) allows to compute the integration analytically, resulting in an estimate of the form \( P(z) \approx D_0(z)e^{-I(\theta^*)} \), where the prefactor \( D_0(z) \) can be computed using local curvature information of \( F \) at \( \theta^* \). Here, randomized linear algebra methods can be used to only compute the eigenvalues that are important for \( D_0(z) \). The number of such eigenvalues depends on the geometry of \( F^{SO}(\theta) \) and the underlying multivariate Gaussian, and is typically small. See the SOM for details.

3 EARTHQUAKE-INDUCED TSUNAMIS

Next, we present the tsunami inundation problem. We describe how we model the distribution of uncertain fault slips corresponding to the random parameter \( \theta \) and the forward model \( F \), which involves the solution of the shallow water equation. We use a model setup sketched in Figure S2 in the SOM, which is motivated by the Tohoku-Oki 2011 earthquake and tsunami. In Figure 1 we show snapshots of tsunami waves traveling towards shore, which is located on the left side of the domain.

3.1 Modeling bathymetry change as random field

We model earthquake-induced bathymetry change as a random variable to account for its uncertainty. More precisely, we model fault slips along patches in the subduction fault using a multivariate log-normal distribution. Random fault slips are propagated to the seafloor bathymetry change using the Okada model (Okada 1985), which assumes a linear elastic crust. The use of a log-normal distribution ensures that fault slips have a uniform sign as is realistic when stress in the overriding plate is released through slip along the fault. We assume that slip at the ith patch has the form \( s_i = \exp(\theta_i) \), where \( \theta_i \) is a component of a multivariate Gaussian distribution for the random vector \( \theta \). The choice of a mean \( \mu \) and covariance \( \mathcal{C} \) for the log-normal slip vector \( s \) corresponds to slip events leading by earthquakes around a certain Gutenberg-Richter moment magnitude \( M_w \). Details can be found in the SOM. The framework summarized in section 3 uses the multivariate Gaussian variable \( \theta \) underlying the log-normal variable \( s \).

As detailed in the SOM, we use four mean slips and covariances for earthquakes of moment magnitude \( M_w = 7.5, 8, 8.5, 9 \). These mean slips, as well as random draws from the four log-normal distributions are shown in Figure 2A. Note that this distribution assumes that the faults are of buried type, i.e., there is no slip close to the seafloor. By changing the means, other scenarios such as trench-breaking earthquakes can be modeled (Gao et al. 2018). The bathymetry changes induced by mean and random fault slips are shown in Figure 2B.

3.2 Governing shallow water equations

We model tsunami waves using the one-dimensional shallow water equations, describing the conservation of mass and momentum for points \( (x, t) \) in space and times \( t \in [0, T] \), \( T > 0 \). These nonlinear hyperbolic equations, written in terms of the water height \( h \) and the momentum variable \( v \) are:

\[
h_t + vh_x = 0, \quad (4a)
\]

\[
v_t + \left( \frac{v^2}{h} + \frac{gh^2}{2} \right)_x + ghB_x = 0. \quad (4b)
\]

Here, \( v := hu \) with \( u \) being the velocity, \( g \) is the gravitational constant, and \( B \) the earthquake-induced bathymetry change, which
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Figure 1. Snapshots of tsunami waves generated by the optimizer with \( z = 4 \) (dashed line in Figure 4B) at different times \( t \). The seafloor deformation generates waves in both directions, but we focus on the waves traveling towards the shore on the left. The region where we measure average tsunami height at time \( t_{\text{max}} \) is shaded in red. Note that the tsunami wave is compressed as it travels towards shore, its magnitude increases and its leading wave edge steepens.

![Figure 1](image1)

Figure 1. Snapshots of tsunami waves generated by the optimizer with \( z = 4 \) (dashed line in Figure 4B) at different times \( t \). The seafloor deformation generates waves in both directions, but we focus on the waves traveling towards the shore on the left. The region where we measure average tsunami height at time \( t_{\text{max}} \) is shaded in red. Note that the tsunami wave is compressed as it travels towards shore, its magnitude increases and its leading wave edge steepens.

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...
Comparison of the annual probability of tsunami waves on shore of size \( z \). Shown in (A) are random slip patterns (solid and dashed lines) from the log-normal distributions whose means (faded solid lines) generate earthquake slip samples for each of the four earthquake magnitudes. Using a tsunami model problem and a realistic distribution of random earthquake slips, we study the probability of large tsunamis occurring close to shore. We use an earthquake slip distribution that reflects a realistic distribution of magnitude 7–9 earthquakes on the Gutenberg-Richter scale. We show that tsunami height probabilities on shore can be estimated accurately by solving a series of optimization problems and verify these probabilities by comparison with direct but more costly Monte Carlo sampling. These optimization problems identify the most likely and thus most effective earthquake mechanisms that result in tsunamis of a certain size. We find that the most effective earthquake mechanisms have smoothly varying slip patches. These lead to waves with a steepening gradient as the crest travels towards shore, typically resulting in an approximate shock right around the region close to shore. This indicates that the nonlinearity in the shallow water equations, which are used to model tsunami waves, plays an important role for the generation of tsunami waves. We find that tsunamis with average wave height towards shore. The wave trough and crest initially stretches over about 80km. As approaching shore, the wave is compressed to a few kilometers. The wave’s crest leading edge steepens towards a shock as it approaches the observation region close to shore. We postulate that reaching a shocked state further away from shore would be a less efficient mechanism and, hence, is not found by the LDT-optimization.

To study this further, in Figure 5, we compare snapshots of waves computed by the shallow water equations and their linearization, both driven by the same bathymetry change profiles shown in Figure 4B. First, it can be seen that the differences between the nonlinear and linearized equations are more substantial for larger initial conditions, i.e., larger \( z \). That is, the nonlinearity is more important for larger bathymetry changes, i.e., when the linearization around the rest state is a poorer approximation to the nonlinear problem. It can also be seen that waves from the linearized equations are generally higher and less steep, while the waves from the shallow water equations have steep leading edge crests, which are caused by the nonlinearity in the equations. Since shocks (or, correspondingly, the steep regions in our numerical simulations in combination with the artificial viscosity) introduce dissipation at the wave crests, in Figure 5 the snapshots for the linearized system are slightly higher. While the linearized problem uses the same amount of artificial viscosity, it plays little role as the wave’s slopes are significantly smaller. This again illustrates the importance of the nonlinearity in the equations for the dynamics and thus the tsunami probabilities on shore.

5 DISCUSSION

Using a tsunami model problem and a realistic distribution of random earthquake slips, we study the probability of large tsunamis occurring close to shore. We use an earthquake slip distribution that reflects a realistic distribution of magnitude 7–9 earthquakes on the Gutenberg-Richter scale. We show that tsunami height probabilities on shore can be estimated accurately by solving a series of optimization problems and verify these probabilities by comparison with direct but more costly Monte Carlo sampling. These optimization problems identify the most likely and thus most effective mechanism that result in tsunamis of a certain size. We find that the most effective earthquake mechanisms have smoothly varying slip patches. These lead to waves with a steepening gradient as the crest travels towards shore, typically resulting in an approximate shock right around the region close to shore. This indicates that the nonlinearity in the shallow water equations, which are used to model tsunami waves, plays an important role for the generation of tsunami waves. We find that tsunamis with average wave height
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Figure 4. Shown in (A) are the slips corresponding to the LDT-optimizers that contributes most to the overall probability. These probabilities are indicated by the solid colored spheres in Figure 3. The dominant contributions for $z = 1$ is from the log-normal distribution with mean $M_w = 7.5$, for $z = 2$ from the distribution with mean $M_w = 8.5$ and for $z = 3, 4, 5$ from the distribution with mean $M_w = 9$. Shown in the middle are the corresponding vertical seafloor deformations. Shown in (C) are the tsunami waves close to shore generated by the seafloor deformations in (B). Snapshots of the waves are shown when $t_i$, i.e., the average wave height is largest in the measurement interval (red shaded area).

Figure 5. Comparison of waves on shore modeled by the shallow water equations and their linearization about the rest state. The waves are initialized by the bathymetry change corresponding to the optimizers from Figure 4. Snapshots show the waves when their average is largest in the observation interval. The gray waves, shown for reference, are the same as in Figure 3C. The waves shown in color are obtained by solving the linearized equations using the same initializations.

of $z \geq 3$ correspond to slips from the log-normal distribution with mean corresponding to the $M_w = 9$ earthquake.

Note that in Figure 4 the probability curve bends upwards. That is, tsunami size probabilities do not decay exponentially with earthquake moment magnitude, i.e., the Gutenberg-Richter law (see also Figure S3 in the SOM). Thus, model predicts a higher rate of large tsunamis than of large earthquakes. The onset of a similar behavior can also be seen in Williamson et al. [2020].

A limitation of the presented results is that they are based on a one-dimensional tsunami model and use one-dimensional slip patches to model random bathymetry changes. The identical approach applies to two-dimensional shallow water models and fault slips, but would require access to efficiently computed gradients of the parameter-to-QoI map [7]. Further, while the proposed optimization-based technique provides an approximation to the true probability, it can be combined with importance sampling to find the exact probabilities [Tong et al., 2021]. Finally, note that the optimization-based method employed here loses its computational advantage over Monte Carlo sampling if one aims at computing hazard probabilities at many different points and for many different probability thresholds $z$.

DATA AVAILABILITY

The data underlying this article are available in the article and in its online supplementary material or will be made available on request.

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Estimating earthquake-induced tsunami inundation probabilities without sampling: Supplemental Material

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S1 LDT-based method for probability estimation

As discussed in section 2, the most important step for probability estimation using the LDT approach is to solve the optimization problem (2); see also (Dematteis et al., 2019, 2018). After obtaining the optimizer $\theta^\star(z)$, we can integrate the probability measure of the set bounded by the second-order Taylor expansion (3) to obtain an approximation of the probability (1), as shown in Figure S1. For computing the prefactor $D_0(z)$ in the estimate $P(z) \approx D_0(z)e^{-I(\theta^\star)}$, we use local derivative information of $F$ and $I$ at $\theta^\star(z)$. The resulting expression for $D_0(z)$ is given by

$$D_0(z) := \frac{(2\pi)^{-\frac{1}{2}}}{\sqrt{2I(\theta^\star)}} \prod_{i=1}^{n-1} \left[ 1 - \frac{\|\nabla I(\theta^\star)\|}{\|\nabla F(\theta^\star)\|} \lambda_i \left( P_{P^C} C^{1/2} \nabla^2 F(\theta^\star) C^{1/2} P_{P^C} \right) \right]^{-\frac{1}{2}}.$$  

Here, for brevity, we write $\theta^\star$ instead of $\theta^\star(z)$. The operator $P_{P^C}$ is the projection onto the orthogonal space of the gradient $\nabla F(\theta^\star)$. Moreover, $\lambda_i(\cdot)$ represents the $i$th eigenvalue of a matrix. This estimation is equivalent to the Second Order Reliability Method (SORM) in engineering (Rackwitz, 2001). However, we use an formulation lending itself to higher dimensions, since it only requires the application of the second derivative matrix $\nabla^2 F(\theta^\star)$ to vectors, rather than building this matrix explicitly. We use finite differences of gradients to approximate these Hessian-vector products. Such gradients are computed using adjoints, as summarized in section S3. The estimation then uses randomized linear algebra methods to compute the dominating eigenvalues efficiently (Halko et al., 2011); details can be found in (Tong et al., 2021).

Figure S1: Two-dimensional illustration of the second-order approximation of the set $\Omega(z)$ for given $z$. These approximations exploit properties of the minimizer $\theta^\star$, the normal direction $n^\star := \nabla F(\theta^\star)/\|\nabla F(\theta^\star)\| = \nabla I(\theta^\star)/\|\nabla I(\theta^\star)\|$ and the curvature of $\partial \Omega(z)$ at $\theta^\star$.  

\begin{itemize}
  \item Level sets of $I(\theta)$
  \item Normal direction $n^\star$
  \item Second-order approx.
  \item $\Omega(z)$
\end{itemize}
Figure S2: Problem setup inspired by Tohoku-Oki 2011 earthquake/tsunami. Bathymetry changes (area in purple) are modeled as resulting from 20 randomly slipping patches in the fault region (in green, with end points (120km, −19.64km) and (180km, −9.75km)), using the Okada model. The tsunami event QoI is the average wave height in the interval [34km, 35km] close to shore (shaded in red), where the water depth at rest is 20m.

S2 Tsunami problem and random bathymetry setup

S2.1 Model setup for earthquake-induced tsunamis

Inspired by the 2011 Tohoku-Oki earthquake and tsunami (Fujiwara et al., 2011), we model the geometry of a slice through the sea as shown in Figure S2. The geometry represents a two-dimensional slice with a bathymetry that models the continental shelf and the pacific ocean to the east of Japan. The fault location and dip angle are taken from (Zhan et al., 2012). We generate random slips as discussed in the next section.

S2.2 Setup for random slips

As discussed in section 3.1, the slips \( s = \exp(\theta) \) follow log-normal distributions, where \( \theta \) is a multivariate Gaussian random parameter with mean \( \mu \) and covariance matrix \( C \). The mean \( \mu \) and covariance matrix \( C \) for \( \theta \) are given in terms of \( \hat{\mu} \) and \( \hat{C} \) as follows:

\[
C_{ij} = \log\left(\frac{\hat{C}_{ij}}{\hat{\mu}_i \hat{\mu}_j} + 1\right), \\
\mu_i = \log(\hat{\mu}_i) - \frac{1}{2} C_{ii}.
\]  

(S.1)

We next show how we choose the mean \( \hat{\mu} \) and covariance \( \hat{C} \) such that samples of the distribution correspond to earthquakes with a certain magnitude. We use four pairs of \((\hat{\mu}, \hat{C})\) and compute a weighted sum of multivariate log-normal distributions such that the corresponding earthquakes follow the Gutenberg-Richter scale. The mean \( \hat{\mu} \) is defined as the multiple of the taper proposed in (Gao et al., 2018),

\[
\hat{\mu}_i := S_{M_w} \tau((x_t - x_i)/(x_t - x_b)),
\]  

(S.2)

where \( S_{M_w} \) is a constant multiple determined by the moment magnitude discussed later. The value \( x_i \) is the horizontal location of the center in the \( i \)th sub-fault, and \( x_t \) and \( x_b \) are the horizontal location of the top and the bottom of the fault, respectively. The function \( \tau \) is the taper from (Gao et al., 2018):

\[
\tau(x') = \delta(x')(1 + \sin(\pi \delta(x')^b)),
\]  

(S.3)

\[
\delta(x') = \begin{cases} 
\frac{6}{q^3}(x')^2 \left(\frac{q}{2} - \frac{x'}{3}\right), & 0 \leq x' \leq q, \\
\frac{6}{(1-q)^3}(1-x')^2 \left(\frac{1-q}{2} - \frac{1-x'}{3}\right), & q \leq x' \leq 1.
\end{cases}
\]  

(S.4)
We use the values $b = 0.25$, $q = 0.65$. The covariance is defined similarly as in (LeVeque et al., 2016), namely as 

$$
\hat{C}_{ij} = (3/4)^2 \hat{\mu}_i \hat{\mu}_j \exp(-|x_i - x_j|/(0.4|x_1 - x_b|)).
$$

(S.5)

The only thing left is to choose $S_{M_w}$ such that $\hat{\mu}$ produce to an earthquake with moment magnitude $M_w$, by solving the relation

$$
M_w = \frac{2}{3} (\log_{10} M_0 - 9.05),
$$

(S.6)

where the seismic moment $M_0 = A \times$ (average slip) $\times$ (rigidity). Following (Murotani et al., 2008), the area $A$ is defined as

$$
A = 1.48 \cdot 10^{-10+9.05 \cdot 2/3+6+M_w} [m^2].
$$

(S.7)

The average overall slip is computed as the average of (S.2). The rigidity is $35 \times 10^9 N/m^2$, as suggested in (Hashima et al., 2016).

S2.3 Deciding weights for each distribution

The previous sections detail the probability distribution for $\theta$ with a fixed mean corresponding to moment magnitude $M_w$. We denote this multivariate Gaussian distribution as $\pi_{M_w}$. For this setup, we apply the methods discussed in section 2 to estimate $P_{\pi_{M_w}}(F(\theta) \geq z)$, for the probability of observing an average wave height of $z$ or higher from random slips from the distribution with moment magnitude $M_w$ mean. To obtain the annual assessment of this probability $P_{an}(z)$, i.e., the annual probability of wave higher than $z$, we use a weighted sum similar to (Williamson et al., 2020)

$$
P_{an}(z) = \sum_{M_w \in \{7.5, 8, 8.5, 9\}} w_{M_w} P_{\pi_{M_w}}(F(\theta) \geq z),
$$

(S.8)

where the weight $w_{M_w} = 10^{6.456-M_w}$ is the annual probability of a moment magnitude $M_w$ earthquake following the Gutenberg-Richter (GR) law. The Gutenberg-Richter (GR) law describes the return period of certain magnitude earthquakes. We choose 350 years as the return period for the Tohoku area from the study in (Kagan & Jackson, 2013), so the annual probability of occurring earthquakes with moment magnitude larger than $M_w$ is $10^{6.456-M_w}$. Using these weights and the distribution $\pi_{M_w}$, we compute the annual probability for earthquakes with magnitude larger than $M_w$ when samples are from $\pi_{M_w}$. Their sum fits the annual probability curve $10^{6.456-M_w}$ from the GR law, as shown in Figure S3.

S3 Numerical and optimization methods

Our approach for probability estimation requires to solve a sequence of optimization problems of the form (2) with the parameter-to-event map $F$ specified above. Since $F$ involves solution of the shallow water equations for given $B$ (and thus $\theta$), this is a PDE-constrained optimization problem (Borzi & Schulz, 2011; Hinze et al., 2009; De Los Reyes, 2015). For solving the one-dimensional shallow water equations, we use the discontinuous Galerkin finite element method (DG-FEM) with linear interpolating polynomials and a global Lax-Friedrichs flux to discretize the equations in space, and the strong stability-preserving second-order Runge-Kutta (SSP-RK2) method to discretize the equations in time (Hesthaven & Warburton, 2007). We use adjoint methods to efficiently compute gradients for this optimization problem, and a descent algorithm that approximates second-order derivative information using the BFGS method (Nocedal & Wright, 2006). After finding the optimizer, we use finite differences of gradients to approximate the application of Hessians to vectors as required to find the dominating curvature directions; for details we refer to (Tong et al., 2021). Numerical values used in the discretization the problem is summarized in Table S1.
Figure S3: Annual probabilities of earthquakes magnitude larger than $M_w$ caused by samples generated from $\pi_{M_w}$ whose means correspond to moment magnitude $M_w = 7.5, 8, 8.5, 9$. The sum of the four probability distributions provides the annual probabilities of observing an earthquake with magnitude 7–9. We compare it with the probability distribution $10^{6.456–M}$ from the Gutenberg-Richter law. It can be seen that our model fits the distribution of earthquakes for magnitude 7–9.

Table S1: Numerical values used for problem setup and discretization.

| Description                        | Value |
|------------------------------------|-------|
| Number of elements using in DG-FEM | 5000  |
| Final Time $T$                     | 1800  |
| Time step size                     | 0.0714|
| Smoothing parameter $\gamma$       | 0.03  |

As we have seen, solutions of nonlinear hyperbolic equations such as the shallow water equations can develop steep slopes or shocks, which play an important role for the dynamics of the system. These phenomena are challenging for numerical simulations. To prevent infinite slopes, in our simulations we use artificial viscosity, which decreases upon mesh refinement, thus retaining important aspects of the dynamics, details can be found in [Tong et al., 2021].

S4 List of symbols

$\mathbb{R}^n$ Space of $n$-dimensional real vector
$\mathbb{R}^{n \times n}$ Space of real matrix of size $n \times n$
$\| \cdot \|$ Euclidean norm
$\langle \cdot, \cdot \rangle$ Inner product induced by Euclidean norm
$\exp(\cdot)$ Exponential function
$\pi$ Pi, mathematical constant
$\sum$ Summation
$\prod$ Product
$\nabla$ Gradient/first-order derivative
\[ \nabla^2 \] Hessian/second-order derivative
\[ \approx \] Asymptotically equal, the ratio of RHS and LHS goes to one in limit
\[ \bar{f} \] Average integral
\[ \mathbb{P}(\cdot) \] Probability measure
\[ \theta \] \( n \)-dimensional Multivariate Gaussian random parameter
\[ N(\mu, C) \] Multivariate Gaussian distribution with mean \( \mu \) and covariance \( C \)
\[ \mu \] Mean of multivariate Gaussian random parameter
\[ C \] Covariance matrix of multivariate Gaussian random parameter
\[ I(\cdot) \] Rate function
\[ F(\cdot) \] Parameter-to-Quantity of interest (QoI) map
\[ P(z) \] Probability function at threshold \( z \)
\[ z \] Threshold/maximal average wave height in tsunami problem
\[ \theta^*(z) \] LDT optimizer for size \( z \)
\[ \Omega(z) \] Extreme event set, containing all \( \theta \) such that \( F(\theta) \geq z \)
\[ F^{SO}(\cdot) \] Second-order Taylor expansion of \( F \)
\[ P^{SO}(z) \] Approximation of probability \( P(z) \) based on second-order Taylor expansion
\[ D_0(z) \] Prefactor for approximation \( P^{SO}(z) \)
\[ \theta_i \] Component of multivariate Gaussian \( \theta \)
\[ s \] Random slip vector, multivariate log-normal random parameter
\[ s_i \] Component of multivariate log-normal \( s \)
\[ \hat{\mu} \] Mean of multivariate log-normal distribution
\[ \hat{C} \] Covariance of multivariate log-normal distribution
\[ M_w \] Moment magnitude of earthquake
\[ x \] Horizontal location
\[ t \] Time
\[ T \] Final Time
\[ h \] Water height
\[ v \] Momentum for a unit of water
\[ u \] Velocity
\[ g \] Gravitational constant
\[ B \] Changed bathymetry, negative depend from the sea level to the sea floor
\[ v_x \] Spatial derivative of \( v \)
\[ h_t \] Time derivative of \( h \)
\[ B_0 \] Initial bathymetry before earthquake
\[ [c, d] \] Horizontal interval for observing tsunami wave near shore
\[ \gamma \] Smoothing parameter
\[ G(\cdot) \] Smooth approximation of maximal average wave height
\[ \lambda_i(\cdot) \] The \( i \)-th eigenvalue, ordered by magnitude
\[ P_{Pr} \] Projection operator onto the orthogonal space for the gradient \( \nabla F(\theta^*) \)
\[ n^* \] Normal direction of \( I \) and \( F \) at optimizer \( \theta^* \)
\[ \partial \Omega(z) \] Boundary of set \( \Omega(z) \)
\[ S_{M_w} \] Scaling constant determined by magnitude
\[ \tau(\cdot) \] Taper function for mean slip
\[ \delta(\cdot) \] Piecewise function for defining taper function
\[ b \] A broadness parameter
\[ q \] A skewness parameter
\[ x' \] The ratio of the distance from the upper limit of the rupture zone to the width of the rupture zone
\[ x_t \] Horizontal location of the top of the fault
\[ x_b \] Horizontal location of the bottom of the fault
\[ x_i \] Horizontal location of the center of the \( i \)-th sub-fault

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$M_0$ Seismic moment
$A$ Fault area
$\pi_{M_w}$ Multivariate Gaussian distribution whose mean correspond to magnitude $M_w$ earthquake
$P_{\text{an}}(z)$ Annual probability of maximal average wave higher than $z$
$w_{M_w}$ Weight for each distribution, computed from the annual frequency
$P_{\pi_{M_w}}(\cdot)$ Probability measure under the distribution $\pi_{M_w}$

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