Quasiparticles, flat bands and the melting of hydrodynamic matter

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The concept of quasiparticles—long-lived low-energy particle-like excitations—has become a cornerstone of condensed quantum matter, where it explains a variety of emergent many-body phenomena such as superfluidity and superconductivity. Here we use quasiparticles to explain the collective behaviour of a classical system of hydrodynamically interacting particles in two dimensions. In the disordered phase of this matter, measurements reveal a subpopulation of long-lived particle pairs. Modelling and simulation of the ordered crystalline phase identify the pairs as quasiparticles, emerging at the Dirac cones of the spectrum. The quasiparticles stimulate supersonic pairing avalanches, bringing about the melting of the crystal. In hexagonal crystals, where the intrinsic three-fold symmetry of the hydrodynamic interaction matches that of the crystal, the spectrum forms a flat band dense with ultra-slow, low-frequency phonons whose collective interactions induce a much sharper melting transition. Altogether, these findings demonstrate the usefulness of concepts from quantum matter theory in understanding many-body physics in classical dissipative settings.

Hydrodynamic pairing

We examine ensembles of polystyrene colloids of diameter \( \ell = 2R = 7–20 \, \mu m \), confined between the parallel plates of a thin microfluidic channel of height 10–30 \( \mu m \). The height is designed to be slightly larger than the particles’ diameter, such that their horizontal \((x, y)\) positions cannot overlap, rendering the system effectively 2D, with an areal density of the particles of \( \rho \approx 0.3–7\% \) (Fig. 1a, Methods and Supplementary Video 1). A steady flow of water drives the particles in the \( x \) direction. Slowed down by friction and viscous shear forces at the floor and ceiling of the channel, the particles move at a velocity of \( u = 100–400 \, \mu m \, s^{-1} \) relative to the water and thus experience a drag force \( \gamma u \) with a drag coefficient \( \gamma \). To make the particles flow, the momentum loss must be compensated by constantly pumping...
The hydrodynamic force $f(r)$ exerted on a particle by another particle at a distance $r = (r, \theta)$, where $\theta$ is the angle with respect to the flow direction, has a magnitude decaying as the distance squared, $r^{-2}$, and is oriented at twice the angle, $2\theta$ (Fig. 1a and Methods). This two-fold symmetry implies that the dipolar force is invariant under parity, $f(-r) = f(r)$. Thus, the hydrodynamic forces that a pair of particles exert momentum through the pressure gradient along the channel. The Reynolds and Péclet numbers were $Re \approx 10^{-4}$ to $10^{-3}$ and $Pe \approx 10^3$ to $10^4$, allowing us to safely disregard inertial and thermal forces.

As the driven particles are slower than the surrounding water, they perturb the streamlines. In the quasi-2D geometry of our setup, the perturbations are known to induce long-range dipolar interactions. The hydrodynamic forces that a pair of particles exert on each other can be described by the dipole-dipole interaction, which is given by $f(r) \propto r^{-2}$. This force is invariant under parity, meaning that the force is the same for a particle pair at distance $r$ as for the pair at distance $-r$.

The hydrodynamic force $f(r)$ can be decomposed into two components: a direct force and a reaction force. The direct force is the force that a particle exerts on another particle, while the reaction force is the force that the second particle exerts on the first. The reaction force is equal and opposite to the direct force, but it is not zero. This is because the flow is not perfect, and there are always small disturbances in the flow that create a reaction force.

In our experiment, we used a channel with height $10 \mu m$ and particles of diameter $7 \mu m$ with an areal density of $\rho = 5.0\%$. The velocity of the particles was approximately $200 \mu m \cdot s^{-1}$ in the $x$-direction. The particles move at similar velocities, as depicted in the schematic (right). The measured pair velocity $u_p$ is shown in the figure with the direction $\hat{u}_p \approx (\cos^2 \theta, \sin^2 \theta)$ (left) and the magnitude $||u_p|| \approx r^{-2}$ (right). Solid lines are theoretical predictions.

The distribution of the velocity with respect to the centre of mass (in units of $u$) of all the particles (gold) and in the pairs (blue, 7.6% of all particles). The lifetime of pairs (in units of $R/u$) as a function of the pair size $r/R$. The data are presented as mean ± s.e.m., shown as error bars with whiskers, for a sample size of $n = 2 \times 10^6$ to $7.3 \times 10^5$ pairs measured in each experiment. Solid lines are fits to sums of two exponentials.

The hydrodynamic force $f(r)$ is a function of the distance $r$ between two particles. As the distance between the particles increases, the magnitude of the force decreases as $r^{-2}$. This is because the force is proportional to the gradient of the pressure, which decreases with distance.

The effective primitive cell is shown as a dashed black square. The spectrum with all the hydrodynamic interactions, with typical pairing modes depicted. The top of the spectrum shows the frequency bands of a square lattice with nearest-neighbour interactions, with four Dirac points (blue) and four vHSs inside the BZ (orange). The effective primitive cell is shown as a dashed black square. The spectrum shows the density of states $g(\omega)$ with strong selection of the pairing modes at the Dirac points.
on each other are equal, and isolated pairs should therefore be stable. A pair oriented at an angle $\theta$ moves at a velocity $u_p \approx u(R/r)^2 (\cos^2 \theta, \sin^2 \theta)$, as verified in the experiment (Fig. 1b). Note that pairs can be stable only thanks to the dissipative nature of the forces. Momentum-conserving forces, in contrast, would be anti-symmetric, $f(-r) = -f(r)$, and thereby destabilize the pairs. From the parity symmetry of the hydrodynamic dipoles originate all physical phenomena described in this article.

Due to the inverse square decay of the hydrodynamic force, intra-pair forces are typically much stronger than interactions with the surrounding particles, and one would expect to see weakly interacting metastable pairs. Analysis of particle trajectories verified this prediction: a significant fraction of the particles, typically about 5–20%, traverse in pairs, geometrically defined as couples of particles much closer to each other than to the next nearest neighbour—by a factor of ~3.5—such that their interactions with other particles are at least ten-fold weaker (Fig. 1c). The pairs move considerably faster than the whole population (relative to the centre of mass of the population).

These weakly interacting couples persist through typical lifetimes -(10–20)R/u, until they approach other particles (Fig. 1d and Supplementary Video 1). To exclude the possibility that the system is substantially affected by non-hydrodynamic interactions, such as van der Waals or electrostatic forces, we compared the measurements with simulations of particle ensembles with purely hydrodynamic interaction (and hard-core repulsion; Methods), which exhibited similar velocity and lifetime distributions (Extended Data Fig. 1).

**Emergence of quasiparticles in hydrodynamic crystals**

The emergence of pairs observed in the disordered phase hints that these might be elementary particle-like excitations in the system. To explore this possibility, we consider a driven hydrodynamic crystal made of identical particles. A method of generating large hydrodynamic lattices of hundreds to thousands of particles is yet to be developed (though densely packed or spatially structured microfluidic crystals can be produced by various techniques). Thus, we investigate the ordered crystalline phase using the analytic model.
and computer simulations tested against the experiment in the disordered phase.

At steady state, the viscous drag force experienced by each particle in the crystal is counterbalanced by the driving force, \( \mathbf{F} \), and the crystal flows uniformly at a velocity \( \mathbf{u} = \mathbf{F}/\gamma \) (Methods). The long-range hydrodynamic forces excite collective modes in the lattice. Expanding the dynamical equation in small deviations around the steady-state motion, we find that these normal modes are plane waves of wavevector \( \mathbf{k} \), with a polarization \( \mathbf{\varphi}_k \) and frequency \( \omega_k \), which are the eigenvector and eigenvalue of a Schrödinger-like equation, \( \mathcal{H}_k \mathbf{\varphi}_k = \omega_k \mathbf{\varphi}_k \) (ref. 39). The hydrodynamic ‘Hamiltonian’ is

\[
\mathcal{H}_k = \mathbf{\Omega}_k \cdot \mathbf{\sigma} ,
\]

where \( \mathbf{\sigma} = (\sigma_x, \sigma_y) \) are Pauli matrices and \( \mathbf{\Omega}_k = (\Omega_x, \Omega_y) \) are Fourier sums of the hydrodynamic interactions over the steady-state lattice positions \( \mathbf{r}_j = (x_j, y_j) \).

\[
\mathbf{\Omega}_k \equiv \begin{bmatrix} \Omega_x \\ \Omega_y \end{bmatrix} \approx \sum_j \text{e}^{i \mathbf{k} \cdot \mathbf{r}_j} \begin{bmatrix} \cos 3\theta_j \\ \sin 3\theta_j \end{bmatrix} ,
\]

The Hermitian operator \( \mathcal{H}_k \) exhibits two purely real eigenfrequency bands, \( \omega_k = \pm |\mathbf{\Omega}_k| = \pm (\Omega^2_x + \Omega^2_y)^{1/2} \), corresponding to marginally stable phonon modes that propagate without any damping (Methods)\(^{40-42}\).

Notably, while the dipolar force shows two-fold symmetry (2\( \theta \) in Fig. 1b), the rotational symmetry of \( \mathcal{H}_k \) is three-fold (3\( \theta \) in equation (2)). This is because \( \mathcal{H}_k \) is the momentum-space ‘spring constant’, linking the stress and the strain in the hydrodynamic crystal (Methods). Thus, since the force is dipolar \( \mathbf{f} = \text{e}^{i |\mathbf{r}|^2} \), the spring constant \( \mathcal{H}_k \) is ‘tripolar’, \( \mathcal{H}_k \propto \sum_j (\mathbf{\nabla}_j \mathbf{f}) \text{e}^{i |\mathbf{r}|} \propto \sum_j (\text{e}^{i |\mathbf{r}| |\mathbf{r}|^2}) \text{e}^{i |\mathbf{r}|} \) (equation (2)), which has interesting implications for the collective modes, as discussed in the following.

It is instructive to consider first a simple case: a square crystal with lattice constant \( a \) in which particles interact only with their nearest neighbours and the resulting frequency bands are

\[ \omega_k \approx \pm \sqrt{2 - 2 \cos 2k_x a - 2 \cos 2k_y a} \]  

Two distinctive features of the spectrum are: (1) four Dirac points (X points, \( \mathbf{k} = \mathbf{k}_x \)), where the positive and negative bands meet, forming a double cone, and (2) four corresponding vHSs that occur at saddle points within the Brillouin zone (BZ) \( \mathbf{k} = \frac{1}{2} \mathbf{k}_x \) where the density of states diverges logarithmically, \( \rho(\omega) \sim \log(\omega - \omega_0) \) (Fig. 1f)\(^{19}\). Both features are hallmarks of quasiparticle spectra.

The periodicity of the spectrum allows one to define a smaller effective BZ (dashed black square in Fig. 1e, top). This zone would be a primitive cell in a crystal with a doubled lattice constant, \( 2a \), another indication of pairing and quasiparticles. One can see the pairing mode as an optical phonon, with neighbouring particles moving in opposite directions, although owing to the parity symmetry, the pairing modes have zero frequency, unlike standard optical phonons. Taking into account all hydrodynamic interactions masks the pairing symmetry of the nearest-neighbour spectrum but preserves the topology of its critical points (Fig. 1e, bottom), as verified in a simulation (Extended Data Fig. 2).

The excitations at the Dirac points are pairing modes that generate lines of pairs (Fig. 1e). The Dirac cone describes long-wavelength acoustic modes of the pair lines. Due to the parity symmetry, the forces on particles in each pair are equal, and the pairing modes, \( \mathbf{k} = \mathbf{k}_x \), are therefore marginally stable also when their amplitude is finite. In the nearest-neighbour spectrum, the Dirac and the acoustic cones are identical in shape, indicating the equal sound velocity of pairing modes and standard phonons (that is, the points \( \Gamma, X \) and \( M \) are identical). With all interactions taken into account, the cones are flattened towards the centre, slowing down the propagation of modes in the \( \Gamma-X \) direction.

For their purely real frequencies, the phonons excited in the linear dynamics (equations (1) and (2)) are marginally stable. Hence, any instability or damping can only stem from non-linear coupling of the phonons. To examine this possibility, we followed the progression of the power spectral density (PSD) in a simulation starting with white noise (Fig. 1g). The evolution of the PSD indicates strong selection of low-frequency excitations, presumably due to multi-phonon scattering events\(^{30}\), with sharp peaks at the Dirac pairing modes.

**Pair-induced melting**

The observation of pairing phenomena in both disordered and ordered phases puts forward a possible role of these excitations in the emergence of disorder. To examine this hypothesis, we performed numerical simulations, starting from an ordered square crystal (with little white noise) and following the progression of its structure and

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**Fig. 3** | Quasiparticle avalanche. Top: a simulation starting from a perfect square lattice doped with an isolated pair quasiparticle, positioned at the right-centre (Supplementary Video 3). Middle: the pair is propagating to the left while exciting an avalanche of pairs in a trailing Mach cone. Bottom: collisions among the excited pairs induce melting. White arrows denote velocity, and the distance to the nearest neighbour (that is, the pair length) is colour coded between 2\( R \) and 0.9\( a \). Supplementary Video 3 follows the progression.
The emergence of motions is driven by anharmonic terms in the equations of motion, beyond the linear Schrödinger equation (equation (1)). The emergence of Dirac peaks in $S(k)$ and quasiparticle avalanches reveals pairing as the mechanism inducing the non-equilibrium melting transition.

**Flat bands and monkey saddles in hexagonal crystals**

Hexagonal crystals are unique as the only class of 2D Bravais lattices whose symmetry matches the intrinsic three-fold symmetry of the hydrodynamic interaction (equation (2)), bringing about a qualitatively different pathway to disorder. The first hint comes from observing the nearest-neighbour spectrum of the hexagonal crystal, $\omega_k \approx \pm |k_x| \cos 3\phi$, which exhibits an extraordinary pattern of critical points (Fig. 4a): (1) a single vHS is positioned exactly at the $k = 0$ centre of the BZ (Γ point) and (2) the Dirac points (M) extend into a web of zero-frequency lines, connecting the vHSSs. These Dirac cones are flattened into ‘wedges’ (or ‘canyons’) stretched along the $\Gamma–M$ direction.

This vHS of the hexagonal crystal is a ‘monkey saddle’ composed of six standard vHSSs (orange) and six Dirac cones (blue). The band structure and corresponding density of states, showing power-law divergence in the nearly flat band, $g(\omega) \propto \omega^{-1/3}$. The PSD in a simulation starting from a white noise perturbation shows selection of low-frequency modes in the flat band around $k = 0$.

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**Fig. 4 | Flat bands and monkey saddles in hexagonal crystals.** a, Top: the frequency bands $\omega_k$ of a hexagonal lattice with nearest-neighbour interactions, in the first BZ. The six Dirac points (blue) extend into 1D zero-frequency lines (solid green) connected to a ‘monkey saddle’ vHS at the centre (orange point), where the band is nearly flat, $\omega_k \approx \pm k^3 |\cos 3\phi|$. Bottom: the spectrum including all interactions with six standard vHSSs (orange) and six Dirac cones (blue). b, The band structure and corresponding density of states, showing power-law divergence in the nearly flat band, $g(\omega) \propto \omega^{-1/3}$. c, The PSD in a simulation starting from a white noise perturbation shows selection of low-frequency modes in the flat band around $k = 0$. 

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**Table**

| Parameter       | Value          |
|-----------------|----------------|
| $\omega_k$      | $\pm |k_x| \cos 3\phi$ |
| $g(\omega)$     | $\propto \omega^{-1/3}$ |
With all long-range interactions included, the symmetry of the double monkey saddle is broken, as it splits into six canonical vHSs, and the Dirac cones regain their standard shape. Nevertheless, the band remains relatively shallow in the Γ–M direction (Fig. 4a, b), as verified in the simulation (Extended Data Fig. 2). The evolution of the power spectral density (PSD) in a simulation starting with a white noise exhibits strong amplification of slow excitations in the flat band around the monkey-saddle vHS (Fig. 4c).

Following the progression of a hexagonal hydrodynamic lattice, we see a melting transition governed by the flat band. The dominant modes that appear in the structure factor \( S(k) \) at the time of the melting transition are long-wavelength excitations sitting in the monkey saddle around each Bragg peak (Fig. 5 and Supplementary Video 4). This flat band spectrum is amplified as the system approaches the melting transition, as manifested in the widening peaks (most notably at \( t \approx 10\tau \)).

As in the square lattice, a radial modulation emerges and eventually becomes the structure factor of the disordered phase.

To further examine the role of the flat band, we follow the evolution of a hexagonal crystal doped with a single pair (Fig. 6a and Supplementary Video 5). Unlike the quasiparticle avalanche in the square crystal, here the pair remains wobbling around its original position, surrounded by a sea of excited flat-band phonons, for an extended period of \(-20\tau\). The quasiparticle stays put due to the ultra-slow group velocity in the flat band, \( \partial k_\omega \approx 0 \) (Fig. 4a, b). Then, many pairs rapidly emerge, presumably via multi-phonon collisions, inducing swift melting of the crystal.

The more collective nature of the transition is also shown in a sharp change in the slope of exponential growth of the disorder parameter, the mean squared deviation (MSD) from the crystal positions (Fig. 6b). In comparison, the MSD of the square lattice grows continuously and super-exponentially. The square crystal MSD curves overlap when scaled by \( \tau \propto a^3 \). In contrast, the hexagonal MSD curves overlap when normalized by a timescale of \(-a^{7/2}\) (Extended Data Fig. 4), another manifestation of the dissimilar nature of these two melting transitions.

**Discussion and outlook**

The present findings demonstrate that quantum matter concepts—quasiparticles, van Hove singularities and flat bands—provide insight
excitations in the flat band become extremely slow, leading to strong correlations and collective modes, which give rise to a sharper melting transition. Flat bands were recently found in bilayer graphene twisted at a specific magic angle\(^{43}\) or buckled\(^{44}\). We observed similar divergences in the density of excitations of a driven hydrodynamic system, raising the possibility that other emergent many-body phenomena of 2D electronic systems may be revealed in classical dissipative settings. As for future directions, the present findings propose that quantum matter notions can be widely useful for examining emergent many-body phenomena, particularly non-equilibrium phase transitions, in a variety of classical dissipative systems ranging from soft matter, driven\(^{14}\) and active\(^{15}\) alike, to complex plasma\(^{16}\), reaction–diffusion\(^{17}\), chemotaxis\(^{18}\), enzymes\(^{19}\) and ecology\(^{20}\).

**Online content**

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Methods

Experiment

Setup. We investigated the collective behaviour of hydrodynamically interacting particles in a quasi-2D flow. The particles were polystyrene microspheres of diameter 7–20 μm (Sigma-Aldrich). These microspheres were driven in water in microfluidic channels of height 10–30 μm. The typical roughness of the microscope cover-slips used as the floor and ceiling of the channel is ≤30 nm, not more 0.04–0.1% of the channel height. The roughness induces quenched disorder, which locally perturbs the pressure and velocity fields by a similar magnitude and may generate pairs in a hydrodynamic crystal. A stronger noise source is due to the polydispersity of the polystyrene beads, whose standard deviation is 0.2–0.3 μm.

The flow rate in the channel was controlled via a pressure pump (Flui gent LineU). The motion of particles was recorded digitally at a rate of 100–2,000 frames s⁻¹ with a high-speed camera (Phantom V-1120) and tracked using MATLAB software. To follow the motion of the ensemble, the camera is fixed on a translation stage, moving at the average speed of particles. The areal density of particles in the field of view was 0.3–7%. One end of the channel was kept open to ensure that the particle flow remained steady and to avoid pressure build-up in the channel.

The Reynolds and Péclet numbers for the flow were in the ranges of Re = 10⁻⁴ to 10⁻³ and Pe = 10³ to 10⁴, such that one can consider a low-Reynolds Stokes flow while disregarding any thermal and inertial effects. The typical velocities of the fluid and of the particles were in the range of 10⁻³ to 10⁻¹ μm s⁻¹. The velocities of particles were calculated from the tracked trajectories. The particles are slower than the surrounding fluid (by about 50%) due to friction with the solid boundaries of the channel. This relative motion of the particles induces drag forces on the particles and perturbations in the flow field.

Tracking dynamics and pair statistics. Pairs are defined geometrically as couples of particles that are much closer to each other than to other particles. Thus, the forces they exert on each other are much stronger than interactions with other particles, and they can be seen as compound, weakly interacting particles. Specifically, a particle belongs to a pair if the distance to the nearest neighbour (that is, the other particle in the pair) rij is shorter by a factor of 3–4 than the distance to the next nearest particle rj. In the analysis of the experiment and simulations, we take 3Sr ≥ rj. Due to the 1/r² dependence, this ensures that the intra-pair forces are at least ten-fold stronger than the interactions with other particles. The lifetime of a pair is computed from the auto-correlation function of the distance rij(t). The lifetime (measured in units of R/u) is defined when the correlation decreases by a factor of 25%.

Model

Hydrodynamic forces. Consider particles of size ℓ = 2R moving in the x–y plane of a thin 2D fluid layer between a rigid floor and ceiling (a Hele–Shaw cell) at a velocity u relative to the fluid. In this quasi-2D setting, the narrow dimension is z (perpendicular to the page in Fig. 1a). The viscous drag on each particle is ju, where j is the friction coefficient. All particles are driven by a force F along the x-axis, which compensates for the viscous friction. In the experiment, F is the friction of the particles with the floor and ceiling. The particles’ motion with respect to the fluid induces dipolar perturbations with a velocity field decaying as −u(R/r)², and these dipolar flow fields give rise to collective hydrodynamic interactions. The hydrodynamic force F exerted by the i-th particle on the j-th particle is (in x, y components)\(^{20}\)

\[
F = \begin{pmatrix} f_x(r_{ij}) \\ f_y(r_{ij}) \end{pmatrix} = \begin{pmatrix} \alpha \cdot (R/r_{ij})^2 \cos 2\theta_{ij} \\ \sin 2\theta_{ij} \end{pmatrix},
\]

where the positions of the dipoles are \(r_i\) and \(r_j = r_i - r_j\) are the distance vectors. In polar coordinates, \(r_{ij} = (r_{ij}, \theta_{ij})\), where \(r_{ij} = |r_{ij}|\) and \(\theta_{ij}\) is the angle. The geometric factor \(\alpha = O(1)\) depends on the shape of the particles (for example, discs or spheres). Equation (4) implies that \(F = F_i\), since the angles obey \(\theta_{ij} = \pi + \theta_{ij}\). This implies that an isolated pair moves at a uniform velocity \(u_{ij} = f(r_{ij})/y = a_u (R/r_{ij})^2 (\cos 2\theta_{ij} \sin 2\theta_{ij})\) (Fig. 1b), where in the experiment \(a_u = 0.3–0.5\).

We see that the hydrodynamic forces violate Newton’s third law of momentum conservation. While the microscopic molecular forces in the fluid obey Newton’s law, the hydrodynamic interactions are effective macroscopic forces that do not conserve momentum. This is because the viscous flow is an inherently open system, an effective representation of energy and momentum transfer from hydrodynamic degrees of freedom to microscopic ones. For a detailed treatise on the physics of 2D microfluidic ensembles, see refs. 41, 42.

Equations of motion. The hydrodynamically interacting particle ensembles exhibit complicated chaotic dynamics in the fully disordered phase and non-linear mode coupling in the ordered, crystalline phase. To compute their trajectories, one could in principle solve the underlying Stokes equations consistently with the moving boundaries of the particles, although this is in general a rather cumbersome procedure. One possible approach is to expand the hydrodynamic interactions as a multipole series over ‘hydrodynamic image charges induced by the particles’ solid boundaries’ 41,42. The procedure is very similar to electrostatics since the same Laplace equation solves the Hele–Shaw flow potential. For example, if we consider two finite particles of size ℓ, then the first image in the interaction will be a dipole −(ℓ/2), followed by an infinite series of multiple reflections, (ℓ/2), (ℓ/2), ... Thus, in an ensemble of particles, one in principle needs to sum over all possible multiple scattering paths among all the particles. Fortunately, since the system is always dilute (with areal density ≤5%), we can neglect all the higher terms and take only the first reflections. This is an excellent approximation, validated in numerous studies43,44,46,48–52,54. For a detailed discussion of the hydrodynamic images sum and their convergence, see refs. 41, 42.

Thus, we can use the following method, relying on two well-established approximations: (1) unless the particles almost touch each other, their induced velocity perturbations are approximated by the isolated dipole field (equation (4)) and (2) the total hydrodynamic force acting on a particle can be simply computed as the sum over the pairwise interactions with all other particles. In the low-Reynolds regime, inertia is negligible, so the drag force is balanced by the driving force and the hydrodynamic interactions. The resulting system of N coupled equations of motion is

\[
y\dot{r}_i = F + \sum_{j \neq i} f(r_{ij}) = F + \mathbf{f}_i(r_i).
\]

Dynamic equation of moving crystals. At steady state, the lattice interactions in equation (5) vanish by symmetry, \(f = \sum_{j \neq i} f = 0\), and the lattice moves uniformly at a velocity \(u = F/y\), relative to the surrounding fluid. Expansion of the equations of motion in small deviations of the lattice positions around the steady-state positions, \(\delta r = r - r_o\) yields a linear dynamic equation

\[
\dot{\delta r} = H \delta r.
\]
where $\delta r$ is a $2N$ vector of the $N$ particle deviations. The tensor $H_{ij}^{\text{eff}} = \delta_{ij} \omega_{ij}$ is a generalized spring constant that multiplies the deviation $\delta r$ to give the hydrodynamic force (where $\omega$ and $\beta$ are the coordinates $x$ and $y$). $H$ is a $2N \times 2N$ matrix composed of $2 \times 2$ blocks $H^p$ that account for hydrodynamic interactions between the $i$th and $j$th particles,

$$H^p = H(r_i - r_j) = \frac{\partial F_i}{\partial r_j} = 2 \sin \theta_y \cos \theta_y$$

The diagonal terms ensure zero sums, $H^p = -\sum_{j \neq i} F_i$. The three-fold symmetry of $H$ stems from its definition as the derivative of the force, $H = \delta H/\delta r$, whose symmetry is two-fold. Note that $H$ is a translation invariant ($H(\mathbf{r})$ is a function of only $\mathbf{r} - x$) and anti-symmetric, $H^{t} = -H^{p}$. In a crystal, the diagonal elements vanish due to reflection symmetry, $H^t = 0$. Thus, the $H$ of a crystal is skew Hermitian with $N$ purely imaginary eigenvalues, representing $N$ phononic modes.

Hereafter, we measure the physical quantities by the relevant scales of the crystal. Distances are measured in units of $a$, the typical distance between the particles (and wavevectors in $1/a$). In a lattice, $a$ is the lattice constant. Times are measured in units of $\tau$, the timescale of the hydrodynamic interaction, $\tau \equiv a^2/|a^2|$. The time it takes a perturbation to traverse a distance $a$ (frequencies are measured in $1/\tau$).

**Momentum space.** The equations of motion are then expanded in plane waves with 2D wavevectors $\mathbf{k} = (k_x, k_y)$, such that the deviation $\delta r$ of each particle from its mechanical equilibrium position $r_i$ is

$$\delta r_i(t) = \psi_{k_i}(t)e^{i\omega_k t} = \psi_{k_i}(e^{i(k \cdot r - \omega_k t)})$$

Here, $\psi_{k_i}(t) = \psi_{k_i}e^{i\omega_k t}$ is a 2D polarization vector in $k$ space. $\psi_{k_i}(t)$ evolves according to a Schrödinger-like equation

$$i \frac{\partial}{\partial t} \psi_{k_i}(t) = \mathcal{H}_k \psi_{k_i}(t)$$

with an eigenvector $\psi_{k_i}$ and eigenfrequency $\omega_k$ obeying equation (1):

$$\mathcal{H}_k \psi_{k_i} = \omega_k \psi_{k_i}$$

The ‘Hamiltonian’ $\mathcal{H}_k$ is a $2 \times 2$ matrix, which is a Fourier transform of $H^p = H(r_i - r_j)$:

$$\mathcal{H}_k = i \sum_{j=0}^{1} \mathbf{H}(\mathbf{r}_j - \mathbf{r}_i) e^{i(k \cdot r - \omega_k t)}$$

where we multiplied by the imaginary unit for convenience, such that $\mathcal{H}_k$ becomes Hermitian. There are $N$ operators $\mathcal{H}_k$ (one for each $k$) with $N$ real phonon eigenfrequencies $\omega_k$. Owing to the translation symmetry of both the crystal and $H^p$, $\mathcal{H}_k$ is also translation invariant. $\mathcal{H}_k$ can be expressed in terms of Pauli matrices as

$$\mathcal{H}_k = \Omega_{k} \sigma_x + \Omega_y \sigma_z$$

(7)

The contributions of the long-range hydrodynamic interaction to $\mathcal{H}_k$ are Fourier sums:

$$\mathcal{H}_k = \left[ \frac{\Omega_x}{\Omega_y} \right] = \sum_{j=0}^{1} \frac{2}{\rho_{j0}} \left[ \cos \theta_x \sin \theta_y \right] \sin(k \cdot r)$$

(8)

where $\mathbf{r}_j = r_j(\cos \theta_j, \sin \theta_j)$ are the distance vectors of the steady-state lattice positions from an arbitrary origin point. Due to the crystal’s parity symmetry, $\Omega_x$ and $\Omega_y$ are always real. Since $\Omega_x$ and $\Omega_y$ are odd functions of $\mathbf{k}$, $\mathcal{H}_k$ is also odd under parity: $\mathcal{H}_k = -\mathcal{H}_k$. One can represent the Hamiltonian in the basis of left and right circularly polarized unit vectors, $\sigma = [\sigma_x, \sigma_y]$ as $\mathcal{H}_k = \Omega_k \cdot \sigma = \Omega_x \sigma_x + \Omega_y \sigma_y$ (equation (1)).

**Spectra.** The eigenfrequencies $\omega_k$ are found by solving the secular equation corresponding to equation (7). There are two eigenfrequency bands, $\omega_k = \pm |\mathcal{H}_k| = \pm (\Omega_x^2 + \Omega_y^2)^{1/2}$, and the corresponding polarization eigenvectors are

$$\psi_{k}^+ = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos \frac{\omega_k}{2} \\ \sin \frac{\omega_k}{2} \end{bmatrix}$$

$$\psi_{k}^- = \frac{1}{\sqrt{2}} \begin{bmatrix} -\sin \frac{\omega_k}{2} \\ \cos \frac{\omega_k}{2} \end{bmatrix}$$

where the angle $\alpha = \arctan(\Omega_x / \Omega_y)$. In the circular basis, the eigenvectors are

$$\psi_{k}^+ = \frac{1}{\sqrt{2}} \begin{bmatrix} \pm e^{i\alpha} \frac{\omega_k}{2} \\ e^{i\alpha} \frac{\omega_k}{2} \end{bmatrix}$$

**Nearest-neighbour approximation.** For the sake of simplicity, we consider the case where the particles interact only with their nearest neighbours. For a square crystal of lattice constant $a$, one finds by taking the sum in equation (8) over nearest neighbours the bands

$$\omega_k = \pm 4 \sqrt{\sin^2 k_x a + \sin^2 k_y a}$$

$$= \pm 2 \sqrt{2 - 2 \cos 2k_x a - 2 \cos 2k_y a}$$

(9)

In the first BZ, the inner vHSs are located at the saddle points, $k = (\pm \pi / a, 0)$, and $(0, \pm \pi / a)$. The Dirac points are the midpoints of the Brillouin zone edges, $k = (\pi / a, 0)$, and $(0, \pi / a)$. Likewise, for the hexagonal crystal, the bands are

$$\omega_k = \pm 8 |\sin \frac{1}{2} k_x a \sin \frac{1}{2} k_y a - \cos \frac{1}{2} k_x a - \cos \frac{1}{2} k_y a|$$

(10)

From equation (10), we see that the hexagonal bands exhibit six $\omega = 0$ lines linking the Dirac cones and the vHS into a hexagonal web of flat bands.

**Dirac points.** Dirac points occur at wavevectors $k_0$ for which the hydrodynamic interaction vanishes, $\Omega_x = \Omega_y = 0$. Equation (8) implies that this happens when $\sin(k \cdot r) = 0$, that is, for $k$ that are halves of the reciprocal-space base vectors, $\frac{1}{4} b_1, \frac{1}{4} b_2$, and their combinations, $k_0 = \frac{1}{4} b_1 + \frac{1}{4} b_2$ (where $b_1, b_2 \in \{0, 1, 2\}$). In the hexagonal lattice, the six Dirac points are $(\beta_1, \beta_2) = (0, \pm 1), (\pm 1, 0), (\pm 1, \pm 1)$. Note that these are the M midpoints of the BZ edges and not the K corners as in graphene.

The expansion of equation (8) around the Dirac point is linear in $\delta k = k - k_0$, the wavevector of the quasiparticles, $\Omega_x = \Omega_y = \Omega_x \delta k$ and $\Omega_y = \Omega_y \delta k$, where the gradients are the sums

$$\Omega_x \delta k = \left[ \begin{array}{c} \cos \theta_x \\ \sin \theta_x \end{array} \right]$$

$$\Omega_y \delta k = \left[ \begin{array}{c} \sin \theta_y \\ -\cos \theta_y \end{array} \right]$$

(11)

Here, the $\delta_{i}$ are the indices of the lattice positions, $\mathbf{r}_i = \mathbf{a}_i + \mathbf{a}_j$, with the basis vectors, $\mathbf{a}_i$ and $\mathbf{a}_j$. The gradients at the Dirac point are orthogonal, $\Omega_x \Omega_y = \Omega_y \Omega_x = 0$, and the resulting cone is therefore elliptic (Fig. 1c,f). $\omega_k = (\mathbf{c}_x^1 + \mathbf{c}_y^2) \delta k^2 + (\mathbf{c}_x^3 + \mathbf{c}_y^4) \delta k^2$.

**Density of state and its van Hove singularities.** The density of states $g(\omega)$ is calculated by numerical summation of the integral

$$g(\omega) = \left( \frac{a}{2\pi} \right)^2 \int d^2k \delta(\omega - \omega(k))$$

(11)

where $\delta(\omega)$ is the Dirac delta function. The vHSs are located at saddle points, where $\Omega_x \omega = 0$ and the Gaussian curvature of the band, that is,
the determinant of the Hessian, is negative, det(\(\partial^2(x,y,\omega)\)) < 0. In 2D systems, the density of states diverges logarithmically at the vHS: 
\[ g(\omega) \sim \log |\omega - \omega_0|, \]

**Monkey saddles.** At higher-order multi-critical vHSSs, such as those found in the hexagonal crystal, the Gaussian curvature also vanishes. The bands around this vHS are two interlacing ‘monkey saddles’.

\[ n = \frac{\pi}{3}\text{ }k^3 \cos 3\phi \]

\[ \text{To calculate } g(\omega), \text{ we integrate along an isofrequency line. Around the monkey saddle, the norm of the is } r = (3/2)k^2. \]

\[ \text{We integrate in one sextant, } \pi/6 \leq \phi \leq \pi/6, \text{ and multiply by six. Along the isofrequency line, } k_y(\omega) = (2\omega/3\cos 3\phi)^{1/3}; \text{ the density of states in equation (11) becomes} \]

\[ g(\omega) = \frac{1}{2\pi} \int \frac{dk_y}{\Gamma(k_\omega)} \frac{1}{\Gamma(3/2)} \times \omega^{-3/2}, \quad (12) \]

where \(\Gamma(x)\) is the gamma function.

**Simulation**

As explained in Model section, we solve the coarse-grained equations of motion (equation (5)) for an ensemble of dynamically interacting particles in a quasi-2D geometry with hard-core repulsion. The corresponding system of 2N coupled ordinary differential equations (for the \(x\) and \(y\) coordinates of each of the \(N\) particles) are

\[ \dot{x}_i = au \epsilon \sum_{\substack{j \neq i \geq \phantom{1} \\	ext{all}}} \frac{x_j^2 - y_j^2}{(x_j^2 + y_j^2)}; \quad \dot{y}_i = au \epsilon \sum_{\substack{j \neq i \geq \phantom{1} \\	ext{all}}} \frac{2x_j y_j}{(x_j^2 + y_j^2)}. \quad (13) \]

Here, the positions of the dipoles are \(r_i = (x_i, y_i)\), and their velocities are \(\dot{r}_i = (\dot{x}_i, \dot{y}_i)\), which are the distance vectors. Using the natural space and time scales, \(\epsilon = m / N\) and \(\tau_0 = \ell / \epsilon\), the equations become dimensionless and invariant to the choice of \(m\). Hard boundary conditions are imposed at the surface of particles to prevent any overlap.

Equations (13) are numerically integrated, in short enough time steps, \(\Delta t \leq 0.1\tau_0\), yielding the trajectories of the particles. Typically, periodic boundary conditions are applied to avoid boundary effects, but we also examined a finite system. Simulations are then repeated for a range of initial conditions, crystal or random arrangements, with a wide range of aerial density of particles, \(\rho = 0.3-10\%\). In crystal simulations, a small level of white noise is usually imposed in the initial conditions, \(\langle \delta r \rangle \leq 10^{-6} a\). Even without these initial imperfections, rounding errors in the simulation provide perturbations that eventually destabilize the crystalline state.

**Pairing, structure factor and correlation function.** The structure factor \(S(k)\) is computed by transforming each frame of particle configuration into a high-resolution image, where each particle is represented by a small circle of diameter \(0.2\text{ }\AA\) to avoid the effects of the form factor. Then, the squared modulus of the Fourier transform of the image yields \(S(k)\). At the beginning of the simulation, only the Bragg peaks corresponding to the perfect crystal are apparent. For example, the peaks of the square lattice are at \(k = (\pi/a)(2m, 2n)\), for all integers \(m\) and \(n\). As the dynamics progress, the amplitude of the Bragg peaks decreases, and other patterns emerge, most notably peaks at the Dirac cones and flat bands. In the square lattice, the Dirac points are at \(k = (\pi/a)(m, n)\), where \(m\) is odd and \(n\) is even, or vice versa. These correspond to the radial positions \(k = |k| \sin S(\Delta)\) (Fig. 2):

\[ k = \frac{\pi}{4} \times \{1, \sqrt{5}, 3, \sqrt{13}, 7, \sqrt{29}, \sqrt{57}, \sqrt{41}, \sqrt{45}, 7, \ldots\}. \]

After the crystal is completely melted, all these peaks vanish, and \(S(k)\) is dominated by the disordered state, exhibiting an annular peak around \(k = 2\pi/\ell\) (the position of the peak depends on the particle density). In this regime, the angle-averaged structure factor \(S(k)\) is a Fourier transform of the radial pair correlation function, \(g(r)\). To calculate the radial pair correlation function \(g(r)\), we count the number of particles, \(d(r)\), within an annular region \(2\pi r dr\) around each particle in the ensemble, where periodic boundary conditions are employed to avoid finite size effects. Then, the pair correlation function is computed as a double average, over all particles in the system in multiple simulations, which is normalized by the number of particles in an uncorrelated system, \(g(r) = \langle d(r) \rangle / (2\pi r dr)\).

**Dispersion relation and PSD.** The positions of all particles are recorded at each time step. Then, the deviations from crystal positions or the perturbations of the distances between nearest neighbours are calculated in the square and hexagonal geometries. The data are Fourier transformed in time and space, yielding two three-dimensional arrays (2D in \(k\) and one-dimensional (1D) in \(\omega\)) for perturbations in the \(x\) and \(y\) directions, \(X_\omega\) and \(Y_\omega\). The calculation is carried out until the perturbation is too large and the original order of the particles is lost. The dispersion \(\omega_\delta\) and PSD are obtained by combining peaks of \(|X_\omega|\) and \(|Y_\omega|\) at each point in the \(k\) plane.

**Data availability**

Data supporting the figures within this paper are deposited at [https://doi.org/10.5281/zenodo.7295031](https://doi.org/10.5281/zenodo.7295031).

**Code availability**

The code used for the analysis of the experiment, analytic modelling and simulations in this study is available from the corresponding authors upon reasonable request.

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**Author contributions**

I.S. performed the experiment, analysed the measurements and ran simulations. H.K.P. and T.T. designed and supervised the research. T.T. conceived the study of quasiparticles and flat bands in hydrodynamic systems and developed the physical theory. H.K.P., I.S. and T.T. conducted the research and wrote and revised the manuscript.

**Competing interests**

The authors declare no competing interests.

**Additional information**

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**Extended Data Fig. 1 | Comparison of experiment and simulations.**  A. The pair velocity $\mathbf{u}_p$ in simulations, showing the direction $\hat{\mathbf{u}}_p \sim (\cos 2\theta, \sin 2\theta)$ (Left) and the magnitude $||\mathbf{u}_p|| \sim r^{-2}$ (Right), with a geometric factor $\alpha = 0.34$ estimated from the experiment. Solid lines are the theoretical predictions (as in the experiment, Fig. 1b). B. Distribution of velocity w.r.t. center of mass (in units of $\mathbf{u}$) of all particles (gold) and in the pairs (blue) in simulations, for areal densities $\rho = 1.8\%$ (left), and $5.0\%$ (right) (as in the experiment, Fig. 1c). C. Lifetime of pairs (in $R/u$ units) as a function of pair size $r/R$ for for areal densities $\rho = 0.9\%, 1.3\%, 1.8\%$ in simulations (as in the experiment, Fig. 1d). Data are presented as means ± SEM, shown as error bars with whiskers, for a sample size $n = 2 \times 10^5 - 7.3 \times 10^5$ of pairs measured in each simulation (as in Fig. 1d). Solid lines are fits to a sum of two exponentials.
Extended Data Fig. 2 | The spectrum computed from simulations. The spectrum $\omega^\pm_k = -\omega^-_k$ computed in simulations of square (left), and hexagonal (right) lattices (Methods). Compare to theoretical spectra (compare to Figs. 1e,f and 4a). Both lattices include $51 \times 51$ particles.
Extended Data Fig. 3 | Pair correlation function. The pair correlation function $g(r)$ computed in simulations of square (left) and hexagonal (right) lattices (Methods). Times are measured in units of $\tau = a^2/\langle \ell^2 \rangle$. Both lattices include $51 \times 51$ particles.
Extended Data Fig. 4 | Scaling in the melting transition. Progression of the mean squared deviation (MSD) in square and hexagonal crystals for $a/R = 5, 6, 8$, where time is normalized by $(R/u)(a/R)^{7/2} = \tau(a/R)^{1/2}$. 