Uniform Decay Estimates for Solutions of a Class of Retarded Integral Inequalities

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Abstract. Some uniform decay estimates are established for solutions of the following type of retarded integral inequalities:

\[ y(t) \leq E(t, \tau) \| y_\tau \| + \int_{\tau}^{t} K_1(t, s) \| y_s \| ds + \int_{t}^{\infty} K_2(t, s) \| y_s \| ds + \rho, \quad t \geq \tau \geq 0. \]

As a simple example of application, the retarded scalar functional differential equation \( \dot{x} = -a(t)x + B(t, x_\tau) \) is considered, and the global asymptotic stability of the equation is proved under weaker conditions. Another example is the ODE system \( \dot{x} = F_0(t, x) + \sum_{i=1}^{m} F_i(t, x(t - r_i(t))) \) on \( \mathbb{R}^n \) with superlinear nonlinearities \( F_i (0 \leq i \leq m) \). The existence of a global pullback attractor of the system is established under appropriate dissipation conditions.

The third example for application concerns the study of the dynamics of the functional cocycle system \( \frac{du}{dp} + Au = F(\theta_p u_\theta) \) in a Banach space \( X \) with sublinear nonlinearity. In particular, the existence and uniqueness of a nonautonomous stationary solution \( \Gamma \) is obtained under the hyperbolicity assumption on operator \( A \) and some additional hypotheses, and the global asymptotic stability of \( \Gamma \) is also addressed.

Keywords: Retarded integral inequality, delay differential equation, asymptotic stability, exponential asymptotic stability, pullback attractor, nonautonomous stationary solution

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1 Introduction

Decay estimate of solutions is a fundamental problem in the qualitative analysis of evolution equations. In most cases this problem can be reduced to differential or integral inequalities. For non-retarded evolution equations, numerous inequalities are available to make the performance of decay estimate fruitful (see e.g. [2, 29, 42, 43]), among which is the remarkable Gronwall-Bellman inequality which was first proposed in Gronwall [14] and later extended to a more general form in Bellman [1]. In contrast, the situation in the case of retarded equations seems to be more complicated. Although there have appeared many nice retarded differential and integral inequalities in the literature (see e.g. [11, 12, 29, 32, 33, 34, 42, 58] and references cited therein), the existing ones are far from being adequate to provide easy-to-handle and efficient tools for studying the dynamics of this type of equations, and it is still a challenging task to derive decay estimates for their solutions, even if for the scalar functional differential equation \( \dot{x} = f(t, x, x_t) \). In fact, it is often the case that one has to fall his back on differential/integral inequalities without delay when dealing with retarded differential or integral equations, which makes the calculations in the argument much involved and restrictive.

In this paper we investigate the following type of retarded integral inequalities:

\[
y(t) \leq E(t, \tau) \|y_r\| + \int_{\tau}^{t} K_1(t, s) \|y_s\| ds
+ \int_{t}^{\infty} K_2(t, s) \|y_s\| ds + \rho, \quad \forall t \geq \tau \geq 0,
\]

where \( E, K_1 \) and \( K_2 \) are nonnegative measurable functions on \( Q := (\mathbb{R}^+)^2, \rho \geq 0 \) is a constant, \( \| \cdot \| \) denotes the usual sup-norm of the space \( \mathcal{C} := C([-r, 0]) \) for some given \( r \geq 0, y(t) \) is a nonnegative continuous function on \([−r, \infty) \) (called a solution of (1.1)), and \( y_t \) denotes the element in the space \( \mathcal{C} \),

\[
y_t(s) = y(t+s), \quad s \in [-r, 0].
\]

Our main purpose is to establish some uniform decay estimates for its solutions. Specifically, let \( E \) be a bounded function on \( Q \) satisfying

\[
\lim_{t \to +\infty} E(t+s, s) = 0 \text{ uniformly w.r.t. } s \in \mathbb{R}^+,
\]

and suppose

\[
\kappa(K_1, K_2) := \sup_{t \geq 0} \left( \int_{0}^{t} K_1(t, s) ds + \int_{t}^{\infty} K_2(t, s) ds \right) < \infty.
\]

Denote \( \mathcal{L}_r(E; K_1, K_2; \rho) \) the solution set of (1.1), i.e.,

\[
\mathcal{L}_r(E; K_1, K_2; \rho) = \{ y \in C([-r, \infty)) : y \geq 0 \text{ and satisfies (1.1)} \}.
\]

We show that the following theorem holds true.
Theorem 1.1 Let \( \vartheta = \sup_{t \geq s \geq 0} E(t, s) \), and write \( \kappa = \kappa(K_1, K_2) \).

1. If \( \kappa < 1 \) then for any \( R, \varepsilon > 0 \), there exists \( T > 0 \) such that
   \[
   \| y_t \| < \mu \rho + \varepsilon, \quad t > T \tag{1.5}
   \]
   for all bounded functions \( y \in \mathcal{L}_r(E; K_1, K_2; \rho) \) with \( \| y_0 \| \leq R \), where
   \[
   \mu = 1/(1 - \kappa). \tag{1.6}
   \]

2. If \( \kappa < 1/(1 + \vartheta) \) then there exist \( M, \lambda > 0 \) (independent of \( \rho \)) such that
   \[
   \| y_t \| \leq M \| y_0 \| e^{-\lambda t} + \gamma \rho, \quad t \geq 0 \tag{1.7}
   \]
   for all bounded functions \( y \in \mathcal{L}_r(E; K_1, K_2; \rho) \), where
   \[
   \gamma = (\mu + 1)/(1 - \kappa c), \quad c = \max(\vartheta/(1 - \kappa), 1). \tag{1.8}
   \]

Remark 1.2 If \( \kappa < 1/(1 + \vartheta) \) then one trivially verifies that \( \kappa c < 1 \).

The particular case where \( K_2 = 0 \) is of crucial importance in applications. In such a case we show that if \( \kappa = \kappa(K_1, 0) < 1 \) then any function \( y \in \mathcal{L}_r(E; K_1, 0; \rho) \) is automatically bounded. Hence the boundedness requirement on \( y \) in Theorem 1.1 can be removed. Consequently we have

Theorem 1.3 Let \( (K_1, K_2) = (K, 0) \), and let \( \vartheta, \kappa, \mu \) and \( \gamma \) be the constants as defined in Theorem 1.1. Then the following assertions hold.

1. If \( \kappa < 1 \) then for any \( R, \varepsilon > 0 \), there exists \( T > 0 \) such that
   \[
   \| y_t \| < \mu \rho + \varepsilon, \quad t > T \tag{1.9}
   \]
   for all \( y \in \mathcal{L}_r(E; K, 0; \rho) \) with \( \| y_0 \| \leq R \).

2. If \( \kappa < 1/(1 + \vartheta) \) then there exist \( M, \lambda > 0 \) such that for all \( y \in \mathcal{L}_r(E; K, 0; \rho) \),
   \[
   \| y_t \| \leq M \| y_0 \| e^{-\lambda t} + \gamma \rho, \quad t \geq 0. \tag{1.10}
   \]

Remark 1.4 The smallness requirement \( \kappa < 1 \) in the above theorems is optimal in some sense. This can be seen from the simple example of scalar equation:
   \[
   \dot{x} = -ax + bx(t - 1), \tag{1.11}
   \]
where \( a, b > 0 \) are constants, for which the assumption \( \kappa < 1 \) in Theorem 1.1 on the corresponding integral inequality \( 3.6 \) to guarantee the global asymptotic stability of the equation amounts to require that \( b < a \). On the other hand, if \( b > a \) then simple calculations show that \( 1.11 \) has a positive eigenvalue and hence is unstable; see e.g. Kuang [28, Chap. 3, Sect. 2].
Remark 1.5 It remains open whether the assumption \( \kappa < \frac{1}{1 + \vartheta} \) in Theorem 1.3 to guarantee global exponential decay for (1.1) can be further relaxed in the full generality of the theorem.

Theorem 1.1 can be seen as an extension of the following result in Hale [15] (see [15, Lemma 6.2]) which plays a fundamental role in constructing invariant manifolds of differential equations.

Proposition 1.6 [15] Suppose \( \alpha > 0, \gamma > 0, K, L, M \) are nonnegative constants and \( u \) is a nonnegative bounded continuous solution of the inequality

\[
 u(t) \leq Ke^{-\alpha t} + L \int_0^t e^{-\alpha(t-s)}u(s)ds + M \int_0^\infty e^{-\gamma s}u(t+s)ds, \quad t \geq 0. \tag{1.12}
\]

If \( \beta := L/\alpha + M/\gamma < 1 \) then

\[
 u(t) \leq (1 - \beta)^{-1} Ke^{-[\alpha-(1-\beta)^{-1}]L}t, \quad t \geq 0. \tag{1.13}
\]

Note that there is in fact an additional requirement in (1.13) to guarantee exponential decay of \( u \), that is, \( \alpha - (1 - \beta)^{-1}L > 0 \) or, equivalently,

\[
 L/\alpha + M/\gamma < 1 - L/\alpha. \tag{1.14}
\]

Let us say a little more about the special case \( M = 0 \), in which (1.14) reads as \( L/\alpha < 1/2 \). In such a case \( L/\alpha \) coincides with the constant \( \kappa \) in Theorem 1.3. Setting \( K = u_0 \) in (1.12), we see that the upper bound \( \vartheta \) of the decay functor \( E \) in (1.12) (corresponding to (1.1)) equals 1. Consequently the smallness requirement on \( \kappa \) in assertion (2) of Theorem 1.3 reduces to that \( \kappa = L/\alpha < 1/2 \).

On the other hand, if \( 1/2 \leq \kappa = L/\alpha < 1 \) then we can only infer from (1.13) that \( u \) has at most an exponential growth. However, Theorem 1.3 still assures that a function satisfying the corresponding integral inequality must approach 0 in a uniform manner with respect to initial data in bounded sets.

We also mention that our proof for Theorem 1.1 is significantly different not only from the one for Proposition 1.6 given in [15], but also from those in the literature for other types of differential or integral inequalities.

As a simple example of applications, we consider asymptotic stability of the scalar functional differential equation:

\[
 \dot{x} = -a(t)x + B(t, x_t), \quad t > 0 \tag{1.15}
\]

where \( a \in C(\mathbb{R}^+) \), and \( B \) is a continuous function on \( \mathbb{R}^+ \times C([-r, 0]) \) for some fixed \( r \geq 0 \) with \( |B(t, \phi)| \leq b(t)\|\phi\| \). Special cases of the equation were studied
in the literature by many authors. For instance, in an earlier work of Winston [54], the author considered the case where \( a(t) \) is nonnegative and \( b(t) \leq \theta a(t) \) for some \( \theta < 1 \). Using Razumikhin’s method the author proved the exponential asymptotic stability and the asymptotic stability of the equation under the assumption \( a(t) \geq \alpha > 0 \) and that \( a(t) \geq 0 \) with \( \int_0^\infty a(t)dt = +\infty \), respectively. Here we revisit this problem and allow \( a(t) \) to be a function which may change sign on \( \mathbb{R}^+ \). Assume \( \lim_{t \to +\infty} \int_s^{s+t} a(\tau) d\tau \to +\infty \) uniformly w.r.t \( s \in \mathbb{R}^+ \).

We show that (1.15) is globally asymptotically stable provided that

\[
\kappa := \sup_{t \geq 0} \int_0^t E(t,s)b(s)ds < 1,
\]

where \( E(t,s) = \exp\left( -\int_s^t a(\tau)d\tau \right) \). It is not difficulty to check that if \( a(t) \) is a nonnegative function with \( \int_0^\infty a(s)ds = +\infty \) and \( b(t) \leq \theta a(t) \) for some \( \theta < 1 \), then the above smallness requirement on \( \kappa \) is automatically fulfilled.

As another example of applications for our integral inequalities, we discuss the existence of pullback attractor for ODE system

\[
\dot{x} = F_0(t,x) + \sum_{i=1}^m F_i(t,x(t-r_i)), \quad x = x(t) \in \mathbb{R}^n, \quad (1.16)
\]

where \( F_i(t,x) \) (\( 0 \leq i \leq m \)) are continuous mappings from \( \mathbb{R} \times \mathbb{R}^n \) to \( \mathbb{R}^n \) which are locally Lipschitz in \( x \) in a uniform manner with respect to \( t \) on bounded intervals, and \( r_i : \mathbb{R} \to [0,r] \) (\( 1 \leq i \leq m \)) are measurable functions. The investigation of the dynamics of delayed differential equations in the framework of pullback attractor theory developed in [10, 24, 25] etc. was first initiated by Caraballo et al. [4]. In recent years there is an increasing interest on this topic for both retarded ODEs and PDEs; see e.g. [5, 6, 8, 9, 26, 36, 44, 53, 60]. However, we find that the existing works mainly focus on the case where the terms involving time lags have at most sublinear nonlinearities. Here we allow the nonlinearities \( F_i(t,x) \) (\( 0 \leq i \leq m \)) in (1.16) to be superlinear in space variable \( x \). Suppose

(F) there exist positive constants \( p > q \geq 1 \), \( \alpha_i > 0 \) (\( 0 \leq i \leq m \)), and nonnegative measurable functions \( \beta_i(t) \) (\( 0 \leq i \leq m \)) on \( \mathbb{R} \) such that

\[
(F_0(t,x),x) \leq -\alpha_0 |x|^{p+1} + \beta_0(t), \quad \forall x \in \mathbb{R}^n, \ t \in \mathbb{R},
\]

\[
|F_i(t,x)| \leq \alpha_i |x|^q + \beta_i(t), \quad \forall x \in \mathbb{R}^n, \ t \in \mathbb{R}.
\]

We show under some additional assumptions on \( \beta_i(t) \) (\( 0 \leq i \leq m \)) that system (1.16) is dissipative and has a global pullback attractor.
As our third example to illustrate applications of Theorems \ref{teo1} and \ref{teo3}, we finally consider the dynamics of retarded nonlinear evolution equations with sub-linear nonlinearities in the general setting of the cocycle system:

$$\frac{du}{dt} + Au = F(\theta_t p, u), \quad p \in \mathcal{H}$$  \hspace{1cm} (1.17)

in a Banach space $X$, where $A$ is a sectorial operator in $X$ with compact resolvent, $\mathcal{H}$ is a compact metric space, and $\theta_t$ is a dynamical system on $\mathcal{H}$. We will show under a hyperbolicity assumption on $A$ and some smallness requirements on the growth rate and the Lipschitz constant of $F(p, u)$ in $u$ that the system has a unique nonautonomous stationary solution $\Gamma$. The global asymptotic stability and exponential stability of $\Gamma$ will also be addressed.

This paper is organized as follows. Section 2 is devoted to the proofs of the main results, namely, Theorems \ref{teo1} and \ref{teo3}, and Section 3 consists of the two examples of ODE systems mentioned above. Section 4 is concerned with the dynamics of system (1.17). We will also talk about in this section how to put a differential equation with multiple variable delays and external forces into the general setting of (1.17).

2 Proofs of Theorems \ref{teo1} and \ref{teo3}

For convenience in statement, let us first introduce several classes of functions.

Denote $\mathcal{E}$ the family of bounded nonnegative measurable functions on $Q := (\mathbb{R}^+)^2$ satisfying (1.3), and let

$$\mathcal{H}_1 = \{ K \in \mathcal{M}^+(Q) : \int_0^t K(t, s)ds < \infty \text{ for all } t \geq 0 \},$$

$$\mathcal{H}_2 = \{ K \in \mathcal{M}^+(Q) : \int_t^\infty K(t, s)ds < \infty \text{ for all } t \geq 0 \},$$

where $\mathcal{M}^+(Q)$ is the family of nonnegative measurable functions on $Q$. Denote $\kappa(K_1, K_2)$ the constant defined in (1.4) for any $(K_1, K_2) \in \mathcal{H}_1 \times \mathcal{H}_2$.

Let $\mathcal{C}$ be the space $C([-r, 0])$ equipped with the usual sup-norm

$$\|\phi\| = \sup_{s \in [-r, 0]} |\phi(s)|, \quad \phi \in \mathcal{C}.$$

Given $y \in C([-r, T])$ ($T > 0$), one can assign a function $y_t$ from $[0, T)$ to $\mathcal{C}$ as follows: for each $t \in [0, T)$, $y_t$ is the element in $\mathcal{C}$ defined by (1.2). For convenience, $y_t$ will be referred to as the lift of $y$ in $\mathcal{C}$.
2.1 Proof of Theorem 1.1

Proof. (1) Assume $\kappa < 1$. We split the argument in this part into three steps. 

Step 1. We first show that for any bounded function $y \in \mathcal{L}_r(E; K_1, K_2; \rho)$,
\[
\|y_t\| \leq c\|y_0\| + \mu \rho, \quad t \geq 0, \tag{2.1}
\]
where $c$ is the constant defined in (1.8).

It can be assumed that there is $t > 0$ such that $y(t) > \|y_0\| + \mu \rho$; otherwise (2.1) readily holds true. Write
\[
\sup_{t \in \mathbb{R}^+}\|y_t\| = N_\varepsilon(\|y_0\| + \varepsilon) + \mu \rho
\]
for $\varepsilon > 0$. We show that $N_\varepsilon \leq c$ for all $\varepsilon > 0$, and the conclusion follows.

For each $\delta > 0$ sufficiently small, pick an $\eta > 0$ with $y(\eta) > \sup_{t \in \mathbb{R}^+}\|y_t\| - \delta$. Then by (1.1) we have
\[
N_\varepsilon(\|y_0\| + \varepsilon) + \mu \rho - \delta = \sup_{t \in \mathbb{R}^+}\|y_t\| - \delta < y(\eta)
\]
\[
\leq E(\eta, 0)\|y_0\| + \int_0^\eta K_1(\eta, s)\|y_s\| ds
\]
\[
+ \int_\eta^\infty K_2(\eta, s)\|y_s\| ds + \rho
\]
\[
\leq (\vartheta(\|y_0\| + \varepsilon) + \kappa (N_\varepsilon(\|y_0\| + \varepsilon) + \mu \rho) + \rho.
\]
Setting $\delta \to 0$ we obtain that
\[
N_\varepsilon(\|y_0\| + \varepsilon) + \mu \rho \leq \vartheta(\|y_0\| + \varepsilon) + \kappa (N_\varepsilon(\|y_0\| + \varepsilon) + \mu \rho) + \rho
\]
\[
= (\vartheta + \kappa N_\varepsilon)(\|y_0\| + \varepsilon) + (\kappa \mu + 1)\rho. \tag{2.2}
\]
The choice of $\mu$ implies that
\[
\kappa \mu + 1 = \mu. \tag{2.3}
\]
Hence (2.2) implies that
\[
N_\varepsilon(\|y_0\| + \varepsilon) \leq (\vartheta + \kappa N_\varepsilon)(\|y_0\| + \varepsilon).
\]
It follows that $N_\varepsilon \leq \vartheta/(1 - \kappa) \leq c$. This completes the proof of (2.1).

For $\sigma > 0$, if we set $\tilde{y}(t) = y(\sigma + t)$ and define
\[
\tilde{E}(t, s) = E(t + \sigma, s + \sigma), \quad \tilde{K}_i(t, s) = K_i(t + \sigma, s + \sigma) \quad (i = 1, 2)
\]
for $t, s \geq 0$, then one trivially checks that $\tilde{y} \in \mathcal{L}_r(\tilde{E}; \tilde{K}_1, \tilde{K}_2; \rho)$ with $\kappa(\tilde{K}_1, \tilde{K}_2) \leq \kappa(K_1, K_2) < 1$. Thus by (2.1) one also concludes that

$$
\|y_{t+\sigma}\| \leq c\|y_{\sigma}\| + \mu\rho, \quad t, \sigma \geq 0.
$$

(2.4)

**Step 2.** We verify that

$$
\limsup_{t \to +\infty} \|y_t\| \leq \mu\rho.
$$

(2.5)

Let us argue by contradiction and suppose

$$
\limsup_{t \to +\infty} \|y_t\| = \mu\rho + \delta
$$

for some $\delta > 0$. Take a monotone sequence $\tau_n \to +\infty$ such that $\lim_{n \to \infty} y(\tau_n) = \mu\rho + \delta$. For any $\varepsilon > 0$, take a $\tau > 0$ sufficiently large so that

$$
\|y_t\| < \mu\rho + \delta + \varepsilon, \quad t \geq \tau.
$$

Then for $\tau_n > \tau$, by (1.1) we deduce that

$$
y(\tau_n) \leq E(\tau_n, \tau)\|y_\tau\| + \int_{\tau}^{\tau_n} K_1(\tau_n, s)\|y_s\|ds + \int_{\tau_n}^{\infty} K_2(\tau_n, s)\|y_s\|ds + \rho
$$

$$
\leq E(\tau_n, \tau)\|y_\tau\| + \kappa(\mu\rho + \delta + \varepsilon + \rho), \quad t \geq 0.
$$

Setting $n \to \infty$ in the above inequality, it yields

$$
\mu\rho + \delta \leq \kappa(\mu\rho + \delta + \varepsilon + \rho).
$$

Since $\varepsilon$ is arbitrary, we conclude that

$$
\mu\rho + \delta \leq (\kappa\mu + 1)\rho + \kappa\delta.
$$

Therefore by (2.3) one has $\delta \leq \kappa\delta$, which leads to a contradiction.

**Step 3.** We show that assertion (1) holds true. Let $R > 0$. Denote

$$
\mathcal{B}_R = \{y \in \mathcal{L}_r(E; K_1, K_2; \rho) : y \text{ is bounded with } \|y_0\| \leq R\}.
$$

By (2.1) we see that $\mathcal{B}_R$ is uniformly bounded. Hence the envelope

$$
y^*(t) = \sup_{y \in \mathcal{B}_R} y(t)
$$

of the family $\mathcal{B}_R$ is well-defined and is a bounded nonnegative measurable function on $[-r, \infty)$. We infer from (1.1) that

$$
y(t) \leq E(t, \tau)\|y^*_\tau\| + \int_{\tau}^{t} K_1(t, s)\|y^*_s\|ds
$$

$$
+ \int_{t}^{\infty} K_2(t, s)\|y^*_s\|ds + \rho, \quad \forall \ t \geq \tau \geq 0
$$
for any $y \in \mathcal{B}_R$. Further taking supremum in the lefthand side of the above inequality with respect to $y \in \mathcal{B}_R$ it yields

$$y^*(t) \leq E(t, \tau)\|y^*_\| + \int_t^\tau K_1(t, s)\|y^*_s\|ds + \int_t^\infty K_2(t, s)\|y^*_s\|ds + \rho, \quad \forall t \geq \tau \geq 0.$$ (2.6)

The only difference between (1.1) and the above inequality (2.6) is that the function $y^*$ in (2.6) is not continuous. Note that we do not make use of any continuity requirement on $y$ in the arguments in Steps 1 and 2. Therefore all the arguments in the above two steps can be directly carried over to $y^*$ without any modifications except that the function $y$ therein is replaced by $y^*$. As a result, we deduce that $\limsup_{t \to \infty} y^*(t) \leq \mu \rho$. Hence for any $\varepsilon > 0$ there is a $T > 0$ such that

$$y^*(t) < \mu \rho + \varepsilon, \quad t > T,$$

from which assertion (1) immediately follows.

(2) Now we assume $\kappa < 1/(1 + \vartheta)$. To obtain the exponential decay estimate in (1.7), we first prove a temporary result: There exist $T, \lambda > 0$ such that if $\|y_0\| \leq N_0 + \gamma \rho$ with $N_0 > 0$, then

$$\|y_t\| \leq N_0 e^{-\lambda t} + \gamma \rho, \quad t \geq T.$$ (2.7)

For this purpose, we fix an appropriate number $\sigma \in (0, 1)$ (which will be further specified) and define

$$\eta = \min\{s \geq 0 : \|y_t\| \leq \sigma N_0 + \gamma \rho \text{ for all } t \geq s\}.$$

The key point is to estimate the upper bound of $\eta$.

Since $\gamma > \mu$ and $N_0 > 0$, by (2.5) it is clear that $\eta < +\infty$. We may assume $\eta > r$ (otherwise we are done). Then by continuity of $y$ one necessarily has

$$\|y_\eta\| = \sigma N_0 + \gamma \rho.$$

For simplicity, write $E(t, 0) := b(t)$. Given $t \in [\eta - r, \eta]$, by (1.1) we have

$$y(t) \leq b(t)\|y_0\| + \int_0^t K_1(t, s)\|y_s\|ds + \int_t^\infty K_2(t, s)\|y_s\|ds + \rho$$

$$\leq (\|b_\eta\| + \kappa c\|y_0\| + \kappa \mu \rho) + \rho$$

$$\leq (\|b_\eta\| + \kappa c)\|y_0\| + (\kappa \mu + 1)\rho$$

$$\leq (\|b_\eta\| + \kappa c)(N_0 + \gamma \rho) + \mu \rho.$$
Here we have used the fact that $\kappa\mu + 1 = \mu$ (see (2.3)). Therefore

$$\sigma N_0 + \gamma \rho = \|y_0\| = \max_{t \in [\eta - r, \eta]} y(t)$$

$$\leq (\|b_\eta\| + \kappa c) N_0 + ((\|b_\eta\| + \kappa c) \gamma + \mu) \rho.$$  \hfill (2.8)

Take a number $t_0 > 0$ such that

$$E(t + s, s) \gamma < 1, \quad \forall t > t_0, \ s \in \mathbb{R}^+.$$  \hfill (2.9)

If $\eta \leq t_0 + r$ then we are done. Thus we assume that $\eta > t_0 + r$. Then by the definition of $\gamma$ and (2.9) one deduces that

$$\gamma = \kappa c \gamma + \mu + 1 \geq (\|b_\eta\| + \kappa c) \gamma + \mu.$$  \hfill (2.10)

It follows by (2.8) that $\sigma N_0 \leq (\|b_\eta\| + \kappa c) N_0$. Hence

$$\|b_\eta\| \geq \sigma - \kappa c.$$  \hfill (2.10)

As $\kappa c < 1$ (see Remark 1.2), we can fix a number $\sigma \in (0, 1)$ so that $\sigma - \kappa c > 0$. Define a number

$$t_1 := \sup\{t \geq 0 : E(t + s, s) \geq (\sigma - \kappa c)/2 \text{ for some } s \in \mathbb{R}^+\}.$$  \hfill (2.11)

Since $E \in \mathcal{E}$, one easily sees that $t_1 < \infty$. By the definition of $t_1$ it is clear that

$$E(t + s, s) \leq (\sigma - \kappa c)/2, \quad t \geq t_1, \ s \in \mathbb{R}^+.$$  \hfill (2.10)

Then implies that $\eta \leq t_1 + r$.

Hence we conclude that $\eta \leq T := \max (t_0, t_1) + r$.

By far we have proved that if $\|y_0\| \leq N_0 + \gamma \rho$ ($N_0 > 0$) then

$$\|y_t\| \leq \sigma N_0 + \gamma \rho, \quad t \geq T.$$  \hfill (2.10)

Let $\tilde{y}(t) = y(t + T)$, and set

$$E(t, s) = E(t + T, s + T), \quad \tilde{K}_i(t, s) = K_i(t + T, s + T)$$

for $t, s \geq 0, \ i = 1, 2$. Then $\tilde{y} \in \mathcal{L}_r(E; \tilde{K}_1, \tilde{K}_2; \rho)$ with

$$\kappa(\tilde{K}_1, \tilde{K}_2) \leq \kappa(K_1, K_2) < 1/(1 + \vartheta).$$

Since $\|\tilde{y}_0\| \leq \sigma N_0 + \gamma \rho$, the same argument as above applies to show that

$$\|\tilde{y}_t\| \leq \sigma(\sigma N_0) + \gamma \rho, \quad t \geq T,$$
that is,
\[ \|y_t\| \leq \sigma^2 N_0 + \gamma \rho, \quad t \geq 2T. \]
(We emphasize that the independence of the numbers \( t_0 \) and \( t_1 \) upon \( s \in \mathbb{R}^+ \)
(see (2.9) and (2.11)) plays a key role in the argument.) Repeating the above
procedure we finally obtain that
\[ \|y_t\| \leq \sigma^n N_0 + \gamma \rho, \quad t \geq nT, \ n = 1, 2, \cdots. \] (2.12)
Setting \( \lambda = -(\ln \sigma)/2T \), one trivially verifies that
\[ \sigma^n \leq e^{-\lambda t}, \quad t \in [nT, (n+1)T] \]
for all \( n \geq 1 \). (2.7) then follows from (2.12).

We are now in a position to complete the proof of the theorem.

Note that (2.1) implies that if \( \|y_0\| = 0 \) then
\[ \|y_t\| \leq \mu \rho \leq \gamma \rho, \quad t \geq 0, \]
and hence the conclusion readily holds true. Thus in what follows we assume that
\( \|y_0\| > 0 \). Take \( N_0 = \|y_0\| \). Clearly \( N_0 > 0 \) and \( \|y_0\| = N_0 \leq N_0 + \gamma \rho \). Therefore
by (2.7) we have
\[ \|y_t\| \leq \|y_0\|e^{-\lambda t} + \gamma \rho, \quad t \geq T. \] (2.13)
On the other hand, by (2.4) we deduce that
\[ \|y_t\| \leq c\|y_0\| + \mu \rho \leq c\|y_0\| + \gamma \rho, \quad t \in [0, T]. \]
Set \( M = ce^{\lambda T} \). Then
\[ \|y_t\| \leq c\|y_0\| + \gamma \rho \leq Me^{-\lambda t}\|y_0\| + \gamma \rho, \quad t \in [0, T]. \]
Combining this with (2.13) we finally arrive at the estimate
\[ \|y_t\| \leq M\|y_0\|e^{-\lambda t} + \gamma \rho, \quad t \geq 0. \]
The proof of the theorem is complete. \( \Box \)

2.2 Proof of Theorem 1.3

**Proof.** The conclusions of Theorem 1.3 immediately follow from Theorem 1.1 as
long as Lemma 2.1 below is proved. \( \Box \)
Lemma 2.1 Let $E \in \mathcal{E}$, and $K_1 = K \in \mathcal{K}_1$. Suppose $\kappa := \kappa(K, 0) < 1$. Let $r, \rho \geq 0$, and let $y$ be a nonnegative continuous function on $[-r, T]$ $(0 < T \leq \infty)$ satisfying the integral inequality

$$y(t) \leq E(t, 0)\|y_0\| + \int_0^t K(t, s)\|y_s\|ds + \rho, \quad 0 \leq t < T. \quad (2.14)$$

Then

$$y(t) \leq (c + 1)(\|y_0\| + 1) + \mu \rho, \quad t \in [0, T), \quad (2.15)$$

where $\mu$ and $c$ are the constants defined in Theorem 1.1.

Proof. Suppose the contrary. There would exist $0 < \tau < T$ such that

$$y(\tau) = c'(\|y_0\| + 1) + \mu \rho, \quad y(t) \leq c'(\|y_0\| + 1) + \mu \rho \quad (t \in [0, \tau]),$$

where $c' = c + 1$. By (2.14) we see that

$$c'(\|y_0\| + 1) + \mu \rho = y(\tau) \leq E(\tau, 0)\|y_0\| + \int_0^\tau K_1(\tau, s)\|y_s\|ds + \rho \leq \vartheta(\|y_0\| + 1) + \kappa c'(\|y_0\| + 1) + \mu \rho + \rho \leq (\vartheta + \kappa c')(\|y_0\| + 1) + (\kappa \mu + 1)\rho. \quad (2.16)$$

By (2.3) we have $\kappa \mu + 1 = \mu$. Hence (2.16) implies

$$c'(\|y_0\| + 1) \leq (\vartheta + \kappa c')(\|y_0\| + 1),$$

that is, $c' \leq \vartheta + \kappa c'$. Therefore

$$c + 1 = c' \leq \vartheta/(1 - \kappa) \leq c,$$

a contradiction. \qed

3 Asymptotic Behavior of ODE Systems

This section consists of two examples of ODE systems illustrating possible applications of the integral inequalities given here. For the general theory of delay differential equations, one may consult the excellent books \[16\] \[28\] \[45\] \[55\].
3.1 Asymptotic stability of a scalar functional ODE

Our first example concerns the asymptotic stability of the scalar functional differential equation:

\[ \dot{x} = -a(t)x + B(t, x_t), \quad t > 0 \]  

where \( x_t \) is the lift of \( x = x(t) \) in \( C = C([-r, 0]) \) for some fixed \( r \geq 0 \), \( a \in C(\mathbb{R}^+) \), and \( B \) is a continuous function on \( \mathbb{R}^+ \times C \) satisfying that

\[ |B(t, \phi)| \leq b(t)\|\phi\|, \quad \phi \in C, \ t \geq 0 \]  

for some nonnegative function \( b \in C(\mathbb{R}^+) \). We assume that

(A1) \( \lim_{t \to \infty} \int_s^{s+t} a(\tau)d\tau \to +\infty \) uniformly with respect to \( s \in \mathbb{R}^+ \).

Let

\[ E(t, s) = \exp \left( -\int_s^t a(\tau)d\tau \right), \quad K(t, s) = E(t, s)b(s), \]

and set

\[ \vartheta = \sup_{t \geq s \geq 0} E(t, s), \quad \kappa = \sup_{t \geq 0} \int_0^t K(t, s)ds. \]

Denote \( x(t; \phi) \) the solution \( x(t) \) of (3.1) with initial value \( x_0 = \phi \in C \).

**Proposition 3.1** If \( \kappa < 1 \) then system (3.1) is globally uniformly asymptotically stable, i.e., for any \( R, \varepsilon > 0 \), there exists \( T > 0 \) such that

\[ |x(t; \phi)| < \varepsilon, \quad t > T \]

for all \( \phi \in C \) with \( \|\phi\| \leq R \).

If \( \kappa < 1/(1 + \vartheta) \) then (3.1) is globally exponentially asymptotically stable. Specifically, there exist \( M, \lambda > 0 \) such that

\[ |x(t; \phi)| \leq M\|\phi\|e^{-\lambda t}, \quad t \geq 0, \ \phi \in C. \]  

**Proof.** Let \( x(t) = x(t; \phi) \). Multiplying (3.1) with \( E(t, \tau)^{-1} = \exp \left( \int_\tau^t a(\eta)d\eta \right) \), we obtain that

\[ \frac{d}{dt} \left( E(t, \tau)^{-1}x \right) = E(t, \tau)^{-1}B(t, x_t). \]  

Given \( t \geq \tau \geq 0 \), integrating (3.4) in \( t \) between \( \tau \) and \( t \) it yields

\[ x(t) = E(t, \tau)x(\tau) + \int_\tau^t E(t, s)B(s, x_s)ds. \]
Here we have used the simple observation that $E(t,\tau)E(s,\tau)^{-1} = E(t,s)$. Let 
\[ y(t) = |x(t)|. \]
By (3.5) one deduces that
\[ y(t) \leq E(t,\tau)\|y\| + \int_t^\tau K(t,s)\|y_s\|ds, \quad \forall \tau \geq t \geq 0, \tag{3.6} \]
and the conclusions follow immediately from Theorems 1.3.

**Remark 3.2** If $a(t) \geq 0$ for $t \in \mathbb{R}$, then $\vartheta = 1$, and the hypothesis on $\kappa$ to guarantee (3.3) reduces to that $\kappa < \frac{1}{2}$.

In such a case one can also easily verify that $\kappa \leq \frac{1}{2}$ if $a(t)$ and $b(t)$ satisfy the following two hypotheses in Winston [54]:

1. $b(t) \leq \theta a(t)$ for some $\theta < 1$,
2. $\int_0^\infty a(t)dt = \infty$.

**Example 3.1.** Let $a(t)$ be a continuous periodic function with period $\omega > 0$. Denote $a^+(t)$ ($a^-(t)$) the positive (negative) part of $a(t)$ (hence $a(t) = a^+(t) - a^-(t)$). Let

\[ I = \int_0^\omega a(t)dt, \quad I^\pm = \int_0^\omega a^\pm(t)dt. \]

Clearly $I = I^+ - I^-$. We assume that $I^+ > I^-$. Then for $t \geq s \geq 0$, one trivially verifies that

\[
\int_s^t a^+(\tau)d\tau \geq mI^+ \geq ((t-s)/\omega - 1)I^+,
\]

\[
\int_s^t a^-(\tau)d\tau \leq (m+1)I^- \leq ((t-s)/\omega + 1)I^-,
\]

where $m = [(t-s)/\omega]$ is the integer part of $(t-s)/\omega$. Using these basic facts it can be easily shown via some simple calculations that

\[
E(t,s) = \exp\left(-\int_s^t a(\tau)d\tau\right) = \exp\left(-\int_s^t a^+(\tau)d\tau + \int_s^t a^-(\tau)d\tau\right) \leq e^{(I^+ + I^-)} \exp\left(-\frac{t-s}{\omega}\right). \tag{3.7}
\]

Thus we see that $E$ belongs to the family $\mathcal{E}$.

It is also easy to check that $\int_s^t a(\tau)d\tau \geq -I^- \quad (t \geq s \geq 0)$. It follows that

\[
E(t,s) = \exp\left(-\int_s^t a(\tau)d\tau\right) \leq e^{I^-}, \quad t \geq s \geq 0. \tag{3.8}
\]

Assume that the function $b(t)$ in (3.2) is bounded. Set $\beta = \sup_{t \geq 0} b(t)$. Then

\[
\int_0^t K(t,s)ds = \int_0^t E(t,s)b(s)ds \leq (by \ (3.7)) \leq \beta e^{(I^+ + I^-)}\omega/I \tag{3.9}
\]

for all $t \geq 0$. Thus by Theorem 3.1 we have
Proposition 3.3 Suppose $\beta < I/\omega e^{(I^+-I^-)}$. Then system (3.1) is globally asymptotically stable in the sense of Theorem 3.1.

If we further assume that $\beta < I/\omega e^{(I^++I^-)(1+e^{I^-})}$, then (3.1) is globally exponentially asymptotically stable.

**Proof.** Assume $\beta < I/\omega e^{(I^+-I^-)}$. Then by (3.9) we see that

$$\kappa := \sup_{t \geq 0} \int_0^t K(t,s) ds < 1.$$ 

By (3.8) we have $\vartheta := \sup_{t \geq s \geq 0} E(t,s) \leq e^{I^-}$. Thus if we assume $\beta < I/\omega e^{(I^++I^-)(1+e^{I^-})}$, then one trivially verifies that $\kappa < 1/(1+\vartheta)$.

Now the conclusion directly follows from Theorem 3.1. □

A concrete example as in Example 3.1 is the linear equation:

$$\dot{x} = -(\sin t + \varepsilon)x + \delta x(t-1), \quad t > 0,$$

where $0 < \varepsilon, \delta < 1$ are constants. Simple calculations show that

$$I^+ > 2 + \pi \varepsilon, \quad I^- < 2 - \pi \varepsilon, \quad I^+ + I^- < 4 + 2\pi \varepsilon.$$ 

It is easy to check that if $\delta < \varepsilon e^{-(4+2\pi \varepsilon)}$, then the first hypothesis in Proposition 3.3 is fulfilled. Hence the equation is globally asymptotically stable.

3.2 Pullback attractors of an ODE system with delays

As a second example, we consider in this part the existence of pullback attractors of the ODE system:

$$\dot{x} = F_0(t,x) + \sum_{i=1}^{m} F_i(t,x(t-r_i))), \quad x = x(t) \in \mathbb{R}^n$$

(3.11)

with superlinear nonlinearities $F_i(t,x) \ (0 \leq i \leq m)$.

Assume that $F_i(t,x) \ (0 \leq i \leq m)$ are continuous mappings from $\mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ which are locally Lipschitz in the space variable $x$ in a uniform manner with respect to $t$ on bounded intervals and satisfy the structure condition (F) given in Section 1, and $r_i : \mathbb{R} \rightarrow [0,r] \ (1 \leq i \leq m)$ are measurable functions.

Denote $C$ the space $C([-r,0],\mathbb{R}^n)$ equipped with the usual norm $\| \cdot \|$. By the hypotheses on $F_i$ and the delay functions $r_i$, it can be easily shown that the initial value problem of (3.11) is well-posed. Specifically, for each $\tau \in \mathbb{R}$ and $\phi \in C$ the system has a unique solution $x(t;\tau,\phi) := x(t)$ on a maximal existence interval $[\tau - r, T_\phi) \ (T_\phi > \tau)$ with

$$x(\tau + s) = \phi(s), \quad s \in [-r,0].$$
For convenience, we call the lift $x_t$ of $x(t)$ the solution curve of (3.11) in $C$ with initial value $x_\tau = \phi$, denoted hereafter by $x_t(\tau, \phi)$.

**Lemma 3.4** Suppose that there exist $M, N > 0$ such that

$$\sum_{i=0}^{m} \int_{s}^{t} \beta_i(\mu) d\mu \leq M(t - s) + N, \quad -\infty < s < t < \infty. \quad (3.12)$$

Then each solution $x(t; \tau, \phi)$ of (3.11) is globally defined for $t \geq \tau$. Furthermore, there exist $C, \lambda, \rho > 0$ such that

$$|x(t; \tau, \phi)| \leq C\|\phi\|e^{-\lambda(t-\tau)} + \rho, \quad \forall \phi \in C, \ t \geq \tau.$$

**Proof.** Let $x = x(t) := x(t; \tau, \phi)$ be a solution of (3.11) with maximal existence interval $[\tau - r, T_\phi]$. Set $\gamma := p(q - 1)/(p - q) + 1$. Taking the inner product of both sides of (3.11) with $|x|^{-1}x$, we find that

$$\frac{1}{\gamma + 1} \frac{d}{dt} |x|^{\gamma + 1} = |x|^{-1}(F_0(t, x), x) + |x|^{-1} \sum_{i=1}^{m} (F_i(t, x(t - r_i)), x) \leq (-\alpha_0 |x|^{\gamma + p} + \beta_0(t)) |x|^{-1} + \sum_{i=1}^{m} (\alpha_i |x|^p \|x_t\|^q + \beta_i(t)|x|^\gamma).$$

The classical Young’s inequality implies that

$$|x|^{\gamma} \|x_t\|^q \leq \varepsilon \|x_t\|^{\gamma+1} + C_\varepsilon |x|^{(\gamma+1)/(\gamma+1-q)}$$

for any $\varepsilon > 0$. Here and below $C_\varepsilon$ denotes a general constant depending upon $\varepsilon$. By the choice of $\gamma$ one easily verify that $\gamma(\gamma + 1)/((\gamma + 1) - q) < \gamma + p$. Hence using the Young’s inequality once again we deduce that

$$|x|^{\gamma} \|x_t\|^q \leq \varepsilon \|x_t\|^{\gamma+1} + \varepsilon |x|^{\gamma+p} + C_\varepsilon.$$

We also have

$$|x|^{\gamma-1}, |x|^{\gamma} \leq \varepsilon |x|^{\gamma+1} + C_\varepsilon.$$

Combining the above estimates together it gives

$$\frac{1}{\gamma + 1} \frac{d}{dt} |x|^{\gamma + 1} \leq - (\alpha_0 - \varepsilon \varepsilon) |x|^{\gamma+p} + \varepsilon \alpha \|x_t\|^{\gamma+1} + \varepsilon \beta(t)|x|^{\gamma+1} + C_\varepsilon(\beta(t) + 1), \quad (3.13)$$

where $\alpha = \sum_{i=1}^{m} \alpha_i, \beta(t) = \sum_{i=0}^{m} \beta_i(t)$. It can be assumed that $\varepsilon \alpha < \alpha_0$. Noticing that $s^{\gamma+1} \leq s^{\gamma+p} + 1$ for all $s \geq 0$, by (3.13) we find that

$$\frac{d}{dt} |x|^{\gamma + 1} \leq -a_\varepsilon(t) |x|^{\gamma+1} + \varepsilon(\gamma + 1) \alpha \|x_t\|^{\gamma+1} + C_\varepsilon(\beta(t) + 1), \quad (3.14)$$
where
\[ a_\varepsilon(t) = (\gamma + 1) (\alpha_0 - \varepsilon \alpha - \varepsilon \beta(t)) . \]

Let \( E_\varepsilon(t, s) = \exp \left( - \int_s^t a_\varepsilon(\mu) d\mu \right) \). Note that
\[ - \int_s^t a_\varepsilon(\mu) d\mu = -c_0 (t - s) + \varepsilon (\gamma + 1) \left( \alpha (t - s) + \int_s^t \beta(\mu) d\mu \right) , \tag{3.15} \]
where \( c_0 = (\gamma + 1) \alpha_0 \). Hereafter we assume \( \varepsilon (\gamma + 1) (\alpha + M) < c_0 / 2 \). Then by (3.12) and (3.15) it is trivial to see that
\[ \lim_{t \to +\infty} E_\varepsilon(t + s, s) = 0 \]
uniformly with respect to \( s \in \mathbb{R} \).

Now performing a similar argument as in the proof of Theorem 3.1 on (3.14), one can obtain that
\[ |x(t)|^{\gamma + 1} \leq E_\varepsilon(t, \eta)\|x_\eta\|^{\gamma + 1} + \int_\eta^t K_\varepsilon(t, s)\|x_s\|^{\gamma + 1} ds + C \varepsilon \int_\eta^t E_\varepsilon(t, s) \tilde{\beta}(s) ds \tag{3.16} \]
for any \( \tau \leq \eta < t < T_\phi \), where \( K_\varepsilon(t, s) = \varepsilon (\gamma + 1) \alpha E_\varepsilon(t, s) \), and \( \tilde{\beta}(t) = \beta(t) + 1 \).

We observe that
\[ \int_\eta^t E_\varepsilon(t, s) \tilde{\beta}(s) ds \leq (\text{by } (3.12)) \leq e^{\varepsilon (\gamma + 1) N} \int_\eta^t e^{-\lambda(t-s)} \tilde{\beta}(s) ds \]
\[ = e^{\varepsilon (\gamma + 1) N} \int_0^{(t-\eta)} e^{-\lambda s} \tilde{\beta}(t - s) ds \]
\[ \leq e^{\varepsilon (\gamma + 1) N} \int_0^\infty e^{-\lambda s} \tilde{\beta}(t - s) ds , \]
where \( \lambda = c_0 / 2 \). Note that
\[ \int_0^\infty e^{-\lambda s} \tilde{\beta}(t - s) ds = \sum_{k=0}^\infty \int_k^{k+1} e^{-\lambda s} \tilde{\beta}(t - s) ds = \sum_{k=0}^\infty e^{-\lambda k} \int_k^{k+1} \tilde{\beta}(t - s) ds \]
\[ \leq (\text{by } (3.12)) \leq (M + N + 1) \sum_{k=0}^\infty e^{-\lambda k} . \]

Therefore by (3.16)
\[ |x(t)|^{\gamma + 1} \leq E_\varepsilon(t, \eta)\|x_\eta\|^{\gamma + 1} + \int_\eta^t K_\varepsilon(t, s)\|x_s\|^{\gamma + 1} ds + C \varepsilon . \]

Using (3.12) one can easily verify that
\[ \lim_{\varepsilon \to 0} \left( \sup_{-\infty < \eta < t < \infty} \int_\eta^t E_\varepsilon(t, s) ds \right) = 1/c_0 = 1/(\gamma + 1)\alpha_0 , \]

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and consequently
\[
\lim_{\varepsilon \to 0} \left( \sup_{-\infty < \eta < \infty} \int_{\eta}^{t} K_\varepsilon(t, s) ds \right) = 0.
\]

We now fix an \( \varepsilon > 0 \) sufficiently small so that
\[
\kappa := \sup_{-\infty < \eta < \infty} \int_{\eta}^{t} K_\varepsilon(t, s) ds < 1/2.
\]

Then since \( \vartheta := \sup_{t \geq s} E_\varepsilon(t, s) = 1 \), by Lemma 2.1 one deduces that \( x(t) \) is bounded on \( [\tau - r, T_\phi] \). It follows that \( T_\phi = \infty \). The second conclusion in the theorem then immediately follows from Theorem 1.3. \( \square \)

Under the hypotheses of Lemma 3.4 we see that the solutions of (3.11) globally exist. This allows us to define a process \( \Phi(t, \tau) \) on \( C \) as follows:
\[
\Phi(t, \tau) \phi = x_t(\tau, \phi), \quad t \geq \tau > -\infty, \ \phi \in C,
\]
where \( x_t(\tau, \phi) \) is the solution curve of (3.11) in \( C \) with \( x_\tau(\tau, \phi) = \phi \) defined as above. \( \Phi \) possesses the following basic properties:

- \( \Phi(t, \tau) : C \to C \) is a continuous mapping for all \( t \geq \tau \);
- \( \Phi(\tau, \tau) = \text{id}_C \) for all \( \tau \in \mathbb{R} \), where \( \text{id}_C \) is the identity mapping on \( C \);
- \( \Phi(t, \tau) = \Phi(t, s) \Phi(s, \tau) \) for all \( t \geq s \geq \tau \).

For system (3.11), the estimate given in Lemma 3.4 is sufficient to guarantee the existence of a global pullback attractor; see [3, 4] etc. (The interested reader is referred to [7] etc. for the general theory of pullback attractors.) Hence we have

**Theorem 3.5** Assume the hypotheses in Lemma 3.4. Then the solution process \( \Phi \) of (3.11) possesses a (unique) global pullback attractor in \( C \). Specifically, there is a unique family \( \mathcal{A} = \{ A(t) \}_{t \in \mathbb{R}} \) of compact sets contained in the ball \( \overline{B}_\rho \) in \( C \) centered at 0 with radius \( \rho \) satisfying that

1. \( \Phi(t, \tau) A(\tau) = A(t) \) for all \( t \geq \tau \);
2. for any bounded set \( B \subset C \),
\[
\lim_{\tau \to -\infty} d_H(\Phi(t, \tau) B, A(t)) = 0
\]
for all \( t \in \mathbb{R} \), where \( d_H(\cdot, \cdot) \) denotes the Hausdorff semi-distance in \( C \),
\[
d_H(M, N) = \sup_{\phi \in M} \inf_{\psi \in N} \{ \| \phi - \psi \| : \psi \in N \}, \quad \forall M, N \subset C.
\]
4 On the Dynamics of Retarded Evolution Equations with Sublinear Nonlinearities

As our third example to illustrate the application of Theorems 1.1 and 1.3, we investigate the dynamics of abstract retarded functional differential equations with sublinear nonlinearities in the general setting of cocycle systems.

Let $\mathcal{H}$ be a compact metric space with metric $d(\cdot, \cdot)$. Assume that there has been given a dynamical system $\theta$ on $\mathcal{H}$, i.e., a continuous mapping $\theta : \mathbb{R} \times \mathcal{H} \to \mathcal{H}$ satisfying the group property: for all $p \in \mathcal{H}$ and $s, t \in \mathbb{R}$,

$$\theta(0, p) = p, \quad \theta(s + t, p) = \theta(s, \theta(t, p)).$$

As usual, we will rewrite $\theta(t, p) = \theta_t p$.

In what follows we always assume that $\mathcal{H}$ is minimal (with respect to $\theta$). This means that $\theta$ has no proper nonempty compact invariant subsets in $\mathcal{H}$.

Let $X$ be a real Banach space with norm $\| \cdot \|_0$, and let $A$ be a sectorial operator on $X$ with compact resolvent. Denote $X^s$ ($s \geq 0$) the fractional power of $X$ generated by $A$ with norm $\| \cdot \|_s$; see [18, Chap. 1] for details.

Let $0 \leq r < \infty$, and $\alpha \in [0, 1)$. Denote $C_\alpha = C([-r, 0], X^\alpha)$. $C_\alpha$ is equipped with the norm $\| \cdot \|_{C_\alpha}$ defined by

$$\| \phi \|_{C_\alpha} = \max_{[-r, 0]} \| \phi(s) \|_\alpha, \quad \phi \in C_\alpha.$$

Given a continuous function $u : [t_0 - r, T) \to X^\alpha$, denote by $u_t$ the lift of $u$ in $C_\alpha$,

$$u_t(s) = u(t + s), \quad s \in [-r, 0], \ t \geq t_0.$$

The retarded functional cocycle system we are concerned with is as follows:

$$\frac{du}{dt} + Au = F(\theta_t p, u_t), \quad p \in \mathcal{H}, \quad \text{(4.1)}$$

where $F$ is a continuous mapping from $\mathcal{H} \times C_\alpha$ to $X$. Later we will show how to put a nonlinear evolution equation like

$$\frac{du}{dt} + Au = f(u(t - r_1), \ldots, u(t - r_m)) + h(t)$$

into the abstract form of (4.1). For convenience in statement, $\mathcal{H}$ and $\theta$ are usually called the base space and the driving system of (4.1), respectively.

Hereafter we denote $B_R$ the ball in $C_\alpha$ centered at 0 with radius $R$,

$$B_R = \{ \phi \in C_\alpha : \| \phi \|_{C_\alpha} < R \}.$$

Assume that $F$ satisfies the following conditions:
(F1) \( F(p, \phi) \) is locally Lipschitz in \( \phi \) uniformly w.r.t \( p \in \mathcal{H} \), namely, for any \( R > 0 \), there exists \( L_F = L_F(R) > 0 \) such that
\[
\| F(p, \phi) - F(p, \phi') \|_0 \leq L_F \| \phi - \phi' \|_{C_\alpha}, \quad \forall \phi, \phi' \in B_R, \ p \in \mathcal{H}.
\]

(F2) There exist \( C_0, C_1 > 0 \) such that
\[
\| F(p, \phi) \|_0 \leq C_0 \| \phi \|_{C_\alpha} + C_1, \quad \forall (p, \phi) \in \mathcal{H} \times C_\alpha.
\]

Under the above assumptions, the same argument as in the proof of [47, Proposition 3.1] with minor modifications applies to show the existence and uniqueness of global mild solutions for (4.1). Specifically, for each initial data \( \phi \in C_\alpha := C([-r, 0], X^\alpha) \) and \( p \in \mathcal{H} \), there is a unique continuous function \( u : [-r, \infty) \to X^\alpha \) with \( u(t) = \phi(t) \) \((-r \leq t \leq 0)\) satisfying the integral equation
\[
u(t) = e^{-At}\phi(0) + \int_0^t e^{-A(t-s)} F(\theta_s p, u_s) ds, \quad t \geq 0.
\]

A solution of (4.1) clearly depends on \( p \). For convenience, given \( p \in \mathcal{H} \), we call a solution \( u \) of (4.1) a solution pertaining to \( p \). We will use the notation \( u(t; p, \phi) \) to denote the solution of (4.1) on \([-r, \infty)\) pertaining to \( p \) with initial value \( \phi \in C_\alpha \). The solutions of (4.1) generates a cocycle \( \Phi \) on \( C_\alpha \),
\[
\Phi(t, p) \phi = u_t, \quad t \geq 0, \ (p, \phi) \in \mathcal{H} \times C_\alpha,
\]
where \( u_t \) is the lift of the solution \( u(t) = u(t; p, \phi) \) in \( C_\alpha \).

Since \( \mathcal{H} \) is compact and \( A \) has compact resolvent, using a similar argument as in the proof of [48, Proposition 4.1], it can be shown that for each fixed \( t > r \), \( \Phi(t, p) \phi \) is compact as a mapping from \( \mathcal{H} \times C_\alpha \) to \( C_\alpha \). Making use of this basic fact one can easily verify that \( \Phi \) is asymptotically compact, that is, \( \Phi \) enjoys the following property:

(AC) For any sequences \( t_n \to +\infty \) and \( (p_n, \phi_n) \in \mathcal{H} \times C_\alpha \), if \( \bigcup_{n \geq 1} \Phi([0, t_n], p_n) \phi_n \) is bounded in \( C_\alpha \) then the sequence \( \Phi(t_n, p_n) \phi_n \) has a convergent subsequence.

4.1 Basic integral formulas on bounded solutions

Suppose \( A \) has a spectral decomposition \( \sigma(A) = \sigma^- \cup \sigma^+ \), where
\[
\text{Re } z \leq -\beta < 0 \ (z \in \sigma^-), \quad \text{Re } z \geq \beta > 0 \ (z \in \sigma^+) \quad (4.2)
\]
for some \( \beta > 0 \). Let \( X = X_1 \oplus X_2 \) be the corresponding direct sum decomposition of \( X \) with \( X_1 \) and \( X_2 \) being invariant subspaces of \( A \). Denote \( \Pi_i : X \to X_i \)
(i = 1, 2) the projection from $X$ to $X_i$, and write $A_i = A|_{X_i}$. By the basic knowledge on sectorial operators (see Henry [18, Chap. 1]), there exists $M \geq 1$ such that
\[
\|\Lambda^\alpha e^{-A_1 t}\| \leq Me^{\beta t}, \quad \|e^{-A_1 t}\| \leq Me^{\beta t}, \quad t \leq 0, \quad (4.3)
\]
\[
\|\Lambda^\alpha e^{-A_2 t}\Pi_2 \Lambda^{-\alpha}\| \leq Me^{-\beta t}, \quad \|\Lambda^\alpha e^{-A_2 t}\| \leq Mt^{-\alpha} e^{-\beta t}, \quad t > 0. \quad (4.4)
\]

The verification of the following basic integral formulas on bounded solutions are just slight modifications of the corresponding ones for that of equations without delays (see e.g. [20]), and hence is omitted.

**Lemma 4.1** (1) Let $u : [-r, +\infty) \to X^\alpha$ be a bounded continuous function. Then $u$ is a solution of (4.1) on $[-r, +\infty)$ pertaining to $p \in \mathcal{H}$ if and only if $u$ solves the integral equation
\[
u(t) = e^{-A_2 t} \Pi_2 u(0) + \int_0^t e^{-A_2 (t-s)} \Pi_2 F(\theta_s p, u_s) ds
- \int_t^\infty e^{-A_1 (t-s)} \Pi_1 F(\theta_s p, u_s) ds, \quad t \geq 0. \quad (4.5)
\]

(2) Let $u : \mathbb{R} \to X^\alpha$ be a bounded continuous function. Then $u$ is a complete solution of (4.1) pertaining to $p \in \mathcal{H}$ if and only if it solves the integral equation
\[
u(t) = \int_{-\infty}^t e^{-A_2 (t-s)} \Pi_2 F(\theta_s p, u_s) ds
- \int_t^\infty e^{-A_1 (t-s)} \Pi_1 F(\theta_s p, u_s) ds.
\]

### 4.2 Existence of bounded complete solutions

For nonlinear evolution equations, bounded complete solutions are of equal importance as stationary ones. This is because that the long-term dynamics of an equation is determined not only by the distribution of its stationary solutions, but also by that of all its bounded complete trajectories. In fact, for a nonautonomous evolution equation it may be of little sense to talk about stationary solutions in the usual terminology.

In this subsection we establish an existence result for bounded complete solutions of equation (4.1). For this purpose we need first to give some a priori estimates.

Let $C_0, C_1$ be the constants in $(F2)$, and set
\[
\kappa = \sup_{t \geq 0} \left( \int_0^t (t-s)^{-\alpha} e^{-\beta(t-s)} ds + \int_t^\infty e^{\beta(t-s)} ds \right). \quad (4.6)
\]
Lemma 4.2 Suppose $A$ has a spectral decomposition as in (4.2), and that

$$C_0 < 1/\kappa M. \quad (4.7)$$

Then for any $R, \varepsilon > 0$, there exists $T > 0$ such that for all bounded solutions $u(t) = u(t; p, \phi)$ of (4.1) with $\phi \in \overline{B}_R$,

$$\|u(t)\|_\alpha < \rho + \varepsilon, \quad t > T, \quad (4.8)$$

where $\rho = C_1 M (1 - \kappa C_0 M)^{-1} \int_0^\infty (1 + s^{-\alpha}) e^{-\beta s} ds$. Consequently

$$\sup_{t \in \mathbb{R}} \|\gamma(t)\|_\alpha \leq \rho \quad (4.9)$$

for all bounded complete solutions $\gamma(t)$ of (4.1).

Proof. (1) Let $u(t) = u(t; p, \phi)$ be a bounded solution of (4.1) on $[-r, \infty)$. For any $\tau \geq 0$, set $v(t) = u(t + \tau)$ ($t \geq 0$). Then $v$ is a bounded solution of (4.1) pertaining to $q = \theta_{\tau} p$. Hence we infer from Lemma 4.1 that

$$v(t) = e^{-A_2 t} v(0) + \int_0^t e^{-A_2 (t-s)} \Pi_2 F(\theta_s q, v_s) ds - \int_t^\infty e^{-A_1 (t-s)} \Pi_1 F(\theta_s q, v_s) ds, \quad t \geq 0. \quad (4.10)$$

Therefore by (4.3), (4.4) and (F2), we deduce that

$$\|v(t)\|_\alpha \leq M e^{-\beta t} \|v_0\|_{C_{\alpha}} + \int_0^t K_1(t, s) \|v_s\|_{C_{\alpha}} ds + \int_t^\infty K_2(t, s) \|v_s\|_{C_{\alpha}} ds + C_2, \quad t \geq 0,$$

where

$$K_1(t, s) = C_0 M (t-s)^{-\alpha} e^{-\beta (t-s)}, \quad K_2(t, s) = C_0 M e^{\beta (t-s)},$$

and $C_2 = C_1 M \int_0^\infty (1 + s^{-\alpha}) e^{-\beta s} ds$. That is, $u$ satisfies

$$\|u(t)\|_\alpha \leq M e^{-\beta (t-\tau)} \|u_\tau\|_{C_{\alpha}} + \int_\tau^t K_1(t, s) \|u_s\|_{C_{\alpha}} ds + \int_t^\infty K_2(t, s) \|u_s\|_{C_{\alpha}} ds + C_2, \quad t \geq \tau \geq 0.$$

Applying Theorem 1.1 one deduces that if $C_0$ satisfies (4.7) then for any $R, \varepsilon > 0$, there exists $T > 0$ such that (4.8) holds true for all $p \in \mathcal{H}$ and $\phi \in \overline{B}_R$.

(2) Let $\gamma(t)$ be a bounded complete solution of (4.1) pertaining to some $q \in \mathcal{H}$. Pick an $R > 0$ such that $\|\gamma(t)\|_\alpha < R$ for all $t \in \mathbb{R}$. Then for any $\varepsilon > 0$, there is $T > 0$ such that (4.8) holds for all $p \in \mathcal{H}$ and $\phi \in \overline{B}_R$. Taking $p = \theta_{-T} q$ and $\phi = \gamma(-T)$, one finds that

$$\|\gamma(0)\|_\alpha = \|u(T; p, \phi)\|_\alpha < \rho + \varepsilon.$$

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Since $\varepsilon$ is arbitrary, we conclude that $\|\gamma(0)\|_\alpha \leq \rho$.

In a similar manner it can be shown that $\|\gamma(t)\|_\alpha \leq \rho$ for all $t \in \mathbb{R}$. □

Thanks to Lemma 4.2 one can now show by very standard argument via the Conley index theory that equation (4.1) has a bounded complete solution $u$. Specifically, we have the following existence result.

**Theorem 4.3** Assume the hypotheses in Lemma 4.2. Then for any $p \in \mathcal{H}$, equation (4.1) has at least one bounded complete solution $u$ pertaining to $p$.

**Proof.** The estimate (4.9) allows us to prove by using the Conley index theory and some standard argument that (4.1) has at least one bounded complete solution $\gamma = \gamma(t)$ pertaining to some $p_0 \in \mathcal{H}$. The interested reader is referred to [52, Sect. 7] for details.

To show that for any $p \in \mathcal{H}$, equation (4.1) has at least one bounded complete solution $u$ pertaining to $p$, we consider the skew-product flow $\Pi$ on $\mathcal{X} = \mathcal{H} \times C_\alpha$ defined as below:

$$\Pi(t)(p, \phi) = (\theta_t p, \Phi(t, p) \phi), \quad (p, \phi) \in \mathcal{X}, \ t \geq 0. \quad (4.11)$$

The asymptotic compactness of $\Phi$ imply that $\Pi$ is asymptotically compact. Let $\varphi(t) = (\theta_t p_0, \gamma_t)$. Then $\varphi = \varphi(t)$ is a bounded complete trajectory of $\Pi$.

Let $\mathcal{S} = \omega(\varphi)$ be the $\omega$-limit set of $\varphi$,

$$\omega(\varphi) = \bigcap_{\tau \geq 0} \{ \varphi(t) : t \geq \tau \}. \quad (4.12)$$

By the basic knowledge in the dynamical systems theory we know that $\mathcal{S}$ is a nonempty compact invariant set of $\Pi$. Set $K = P_\mathcal{H} \mathcal{S}$, where $P_\mathcal{H} : \mathcal{X} \to \mathcal{H}$ is the projection. One can easily verify that $K$ is a nonempty compact invariant set of the driving system $\theta$. Hence due to the minimality hypothesis on $\mathcal{H}$ we deduce that $K = \mathcal{H}$. Consequently for each $p \in \mathcal{H}$, there is a $\phi \in C_\alpha$ such that $(p, \phi) \in \mathcal{S}$. Let $(\theta_t p, u_t)$ be a bounded complete trajectory of $\Pi$ in $\mathcal{S}$ through $(p, \phi)$. Set

$$u(t) = u_t(0), \quad t \in \mathbb{R}.$$

Then $u(t)$ is bounded complete solution of (4.1) pertaining to $p$. □

### 4.3 Existence of a nonautonomous stationary solution

Let us begin with the notion of a *nonautonomous stationary solution*. 
Definition 4.4 A nonautonomous stationary solution of (4.1) is a continuous mapping \( \Gamma \in C(\mathcal{H}, X^\alpha) \) such that \( \gamma_p(t) := \Gamma(\theta_t p) \) is a bounded complete solution of (4.1) pertaining to \( p \) for each \( p \in \mathcal{H} \).

Theorem 4.5 In addition to the hypotheses in Lemma 4.2, suppose that

\[
L_F(\rho + 1) < 1/\kappa M, \tag{4.13}
\]

where \( L_F(\rho + 1) \) is the local Lipschitz constant of \( F \) given in (F1), and \( \rho \) is the constant given in Lemma 4.2. Then equation (4.1) has a nonautonomous stationary solution \( \Gamma \in C(\mathcal{H}, X^\alpha) \). Moreover, for any \( R, \varepsilon > 0 \), there exists \( T > 0 \) such that for any bounded solution \( u(t) = u(t; p, \phi) \) with \( \phi \in \overline{B}_R \),

\[
\|u(t) - \Gamma(\theta_t p)\|_\alpha < \varepsilon, \quad t > T. \tag{4.14}
\]

Proof. We continue the argument in the proof of Theorem 4.3. Set

\[
\mathcal{S}[p] = \{ \phi : (p, \phi) \in \mathcal{S} \}, \quad p \in \mathcal{H},
\]

where \( \mathcal{S} \) is the \( \omega \)-limit set of \( \varphi \) given by (4.12). Since \( \mathcal{S} \) is compact, one can easily check that \( \mathcal{S}[p] \) is upper semicontinuous, i.e., given \( p \in \mathcal{H} \), for any \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( \mathcal{S}[q] \) is contained in the \( \varepsilon \)-neighborhood of \( \mathcal{S}[p] \) for all \( q \) with \( d(q, p) < \delta \). In what follows we show that \( \mathcal{S}[p] \) is a singleton. Consequently the upper semicontinuity of \( \mathcal{S}[p] \) reduces to the continuity of \( \mathcal{S}[p] \) in \( p \).

Let \( \phi_1, \phi_2 \in \mathcal{S}[p] \). As in the proof of Theorem 4.3 we deduce that \( \Phi \) has two bounded complete trajectories \( \gamma_i^t (i = 1, 2) \) in \( \mathcal{C}_\alpha \) pertaining to \( p \). We check that \( \gamma_t := \gamma_1^t - \gamma_2^t \equiv 0 \) for \( t \in \mathbb{R} \), or equivalently,

\[
\gamma(t) := \gamma^1(t) - \gamma^2(t) \equiv 0, \quad \text{where} \quad \gamma^i(t) = \gamma^i_t(0). \tag{4.15}
\]

It then follows that \( \mathcal{S}[p] \) is a singleton.

For \( \eta \in \mathbb{R} \), we write \( \varphi^i(t) = \gamma^i(t + \eta) \). Then \( \varphi^i(t) \) is a solution of (4.1) pertaining to \( q = \theta_\eta p \). By Lemma 4.1 we have

\[
\varphi^i(t) = e^{-A_2 t} \Pi_2 \varphi^i(0) + \int_0^t e^{-A_2(t-s)} \Pi_2 F(\theta_s q, \varphi^i_s) ds - \int_t^\infty e^{-A_2(t-s)} \Pi_1 F(\theta_s q, \varphi^i_s) ds, \quad t \geq 0.
\]

Let \( \varphi(t) := \varphi^1(t) - \varphi^2(t) \). Then

\[
\varphi(t) = e^{-A_2 t} \Pi_2 \varphi(0) + \int_0^t e^{-A_2(t-s)} \Pi_2 (F(\theta_s q, \varphi^1_s) - F(\theta_s q, \varphi^2_s)) ds - \int_t^\infty e^{-A_1(t-s)} \Pi_1 (F(\theta_s q, \varphi^1_s) - F(\theta_s q, \varphi^2_s)) ds, \quad t \geq 0.
\]
Thus by (F1) we deduce that
\[
\|\varphi(t)\|_\alpha \leq M e^{-\beta t} \|\varphi_0\|_\alpha + L_F M \int_0^t (t-s)^{-\alpha} e^{-\beta (t-s)} \|\varphi_s\|_\alpha \, ds + L_F M \int_t^\infty e^{-\beta (s-t)} \|\varphi_s\|_\alpha \, ds, \quad \forall t \geq 0,
\]
where \(L_F := L_F(\rho + 1)\) (recall that by Lemma 4.2, both \(\varphi^1\) and \(\varphi^2\) are contained in the ball \(\overline{B}_{\rho+1}\) in \(X^\alpha\) centered at 0 with radius \(R = \rho + 1\)). Since \(\varphi(t) = \gamma^1(t+\eta) - \gamma^2(t+\eta)\) and \(\eta \in \mathbb{R}\) can be taken arbitrary, (4.16) readily implies that the inequality holds true for all translations \(\varphi(\tau+\cdot)\) of \(\varphi\).

Now suppose \(L_F\) satisfies (4.13). Then one trivially verifies that all the requirements in Theorem 1.1 are fulfilled by \(y(t) := \|\varphi(t)\|_\alpha\). Therefore for any \(\varepsilon > 0\), there is a \(T > 0\) (independent of \(\eta\)) such that
\[
\|\varphi(t)\|_\alpha < \varepsilon, \quad t > T
\]
(4.17)
That is,
\[
\|\gamma(t+\eta)\|_\alpha < \varepsilon, \quad t > T, \ \eta \in \mathbb{R}.
\]
(4.18)
For any \(\tau \in \mathbb{R}\), setting \(t = T + 1\) and \(\eta = \tau - (T + 1)\) in (4.18) we find that \(\|\gamma(\tau)\|_\alpha < \varepsilon\). Since \(\varepsilon\) is arbitrary, one immediately concludes that \(\gamma(\tau) = 0\), which justifies the validity of (4.15).

Now let \(S[p] = \{\phi_p\}\). Then \(\phi_p\) is continuous in \(p\), and the invariance property of \(S\) implies that \(\phi_{\theta_p}\) is a complete trajectory of the cocycle \(\Phi\) in \(C_\alpha\) for each \(p \in \mathcal{H}\). Define
\[
\Gamma(p) = \phi_p(0), \quad p \in \mathcal{H}.
\]
Clearly \(\Gamma \in C(\mathcal{H}, X^\alpha)\). One easily sees that for each \(p \in \mathcal{H}\), \(\gamma_p(t) = \Gamma(\theta_t p) = \phi_{\theta_t p}(0)\) is a complete solution of (4.1) pertaining to \(p\). Hence \(\Gamma\) is a nonautonomous stationary solution of equation (4.1).

Let \(R > 0\). We infer from Lemma 4.2 that there exists \(T' > 0\) such that
\[
\|u(t)\|_\alpha < \rho + 1, \quad t \geq T'
\]
for all bounded solutions \(u(t) = u(t; p, \phi)\) of (4.1) with \(\phi \in \overline{B}_R\). Since we are interested in the asymptotic behavior of \(u(t)\) as \(t \to \infty\), it can be assumed that \(u(t) \in \overline{B}_{\rho+1}\) for all \(t \geq 0\). Then following a similar argument as the one leading to (4.17) with \(\varphi\) therein replaced by \(u(t) - \Gamma(\theta t p)\), one can easily verify the validity of (4.14). We omit the details. □
4.4 Global asymptotic stability of the nonautonomous stationary solution

Now we consider the particular case where \( \sigma(A) \) lies in the right half plane, namely, \( \sigma^- = \emptyset \). Using the constant variation formula it can be shown that

\[
\|u(t)\|_\alpha \leq Me^{-\beta(t-\tau)}\|u_\tau\|_C + \int_\tau^t K(t,s)\|u_s\|_C ds + C_2, \quad t \geq \tau \geq 0
\]

for any solution \( u \) of (4.1) on \([-r, \infty)\), where

\[K(t,s) = C_0 M (t-s)^{-\alpha} e^{-\beta(t-s)}, \quad C_2 = C_1 M \int_0^\infty s^{-\alpha} e^{-\beta s} ds.\]

(The calculations involved here are similar to those as in the proof of Lemma 4.2. We omit the details.) Let

\[
\kappa = \sup_{t \geq 0} \left( \int_0^t (t-s)^{-\alpha} e^{-\beta(t-s)} ds \right), \quad \rho = C_1 M (1 - \kappa C_0 M)^{-1} \int_0^\infty s^{-\alpha} e^{-\beta s} ds
\]

Applying Theorem 1.3 one deduces that if \( C_0 < 1/\kappa M \) then for any \( R, \varepsilon > 0 \), there exists \( T > 0 \) such that

\[
\|u(t)\|_\alpha < \rho + \varepsilon, \quad t > T
\]

for all \( p \in H \) and \( \phi \in \mathcal{B}_R \).

Combining (4.19) with Theorem 4.5 one obtains the following result.

**Theorem 4.6** Assume

\[
Re z \geq \beta > 0, \quad \forall z \in \sigma(A). \tag{4.20}
\]

Suppose \( C_0 < 1/\kappa M \) and that \( L_F(\rho + 1) < 1/\kappa M \). Then the equilibrium solution \( \Gamma \) given in Theorem 4.5 is globally uniformly asymptotically stable. Specifically, for any \( R, \varepsilon > 0 \), there exists \( T > 0 \) such that for any solution \( u(t) = u(t;p,\phi) \) of (4.1) with \( \phi \in \mathcal{B}_R \),

\[
\|u(t) - \Gamma(\theta_t p)\|_\alpha < \varepsilon, \quad t > T. \tag{4.21}
\]

Instead of assuming \( L_F(\rho + 1) < 1/\kappa M \), if we impose on \( F \) a stronger global Lipschitz continuity assumption, then it can be shown that \( \Gamma \) is globally exponentially asymptotically stable.

**Theorem 4.7** In addition to (4.20), assume that

\[
L := \sup_{R > 0} L_F(R) < 1/\kappa M(1 + M). \tag{4.22}
\]

Then there exist \( C, \lambda > 0 \) such that for any solution \( u(t) = u(t;p,\phi) \) of (4.1),

\[
\|u(t) - \Gamma(\theta_t p)\|_\alpha \leq Ce^{-\lambda t} \sup_{s \in [-r,0]} \|\phi(s) - \Gamma(\theta_s p)\|_\alpha, \quad t \geq 0.
\]
Proof. It is easy to verify that all the hypotheses in Theorem 4.6 are fulfilled. Hence the conclusions in Theorem 4.6 remain valid.

Let \( u(t) = u(t; p, \phi) \) be a solution of (4.1). Set \( \varphi(t) = u(t) - \Gamma(\theta t p) \). Using a parallel argument as in the proof of Lemma 4.2 (1), we can obtain that

\[
\| \varphi(t) \|_\alpha \leq Me^{-\beta(t-\tau)}\| \varphi_\tau \|_{C_\alpha} + \int_\tau^t K(t,s)\| \varphi_s \|_{C_\alpha} ds,
\]

where \( K(t,s) = ML(t-s)^{-\alpha}e^{-\beta(t-s)} \). If \( L \) satisfies (4.22) then the functions \( E(t,s) := Me^{-\lambda t}\| \varphi_0 \|_{C_\alpha} \) and \( K(t,s) \) satisfy the requirements in Theorem 1.3. Thus there exist constants \( C, \lambda > 0 \) independent of \( \varphi \) such that

\[
\| \varphi(t) \|_\alpha \leq Ce^{-\lambda t}\| \varphi_0 \|_{C_\alpha},
\]

\( t \geq 0 \).

The conclusion of the theorem then immediately follows. □

4.5 Nonlinear evolution equations with multiple delays

Let us now consider the nonlinear evolution equation

\[
\frac{du}{dt} + Au = f(u(t-r_1), \ldots, u(t-r_m)) + h(t)
\]

with multiple delays, where \( X \) and \( A \) are the same as in Subsection 4.1, \( f \) is a continuous mapping from \( (X^\alpha)^m \) to \( X \) for some \( \alpha \in [0,1) \), \( h \in C(\mathbb{R}, X) \), \( r_i \in C(\mathbb{R}, \mathbb{R}^+) \), and

\[
0 \leq r_i(t) \leq r < \infty, \quad 1 \leq i \leq m.
\]

It is well known that (4.23) covers a large number of concrete examples from applications. Our main goal here is to show how to put such an equation into the abstract form of (4.1), and therefore the general results given above can be directly carried over to the equation.

Since equation (4.23) is nonautonomous, one has to take into account the initial time when considering its initial value problem. Hence the initial value problem of the equation generally reads as

\[
\begin{align*}
\frac{du}{dt} + Au &= f(u(t-r_1), \ldots, u(t-r_m)) + h(t), \quad t \geq \tau, \\
u(\tau+s) &= \phi(s), \quad s \in [-r,0],
\end{align*}
\]

where \( \phi \in C_\alpha = C([-r,0], X^\alpha) \), and \( \tau \in \mathbb{R} \) is given arbitrary. Rewriting \( t - \tau \) as \( t \), one obtains an equivalent form of (4.23):

\[
\begin{align*}
\frac{dv}{dt} + Av &= f(v(t-\tilde{r}_1), \ldots, v(t-\tilde{r}_m)) + \tilde{h}(t), \quad t \geq 0, \\
v(s) &= \phi(s), \quad s \in [-r,0],
\end{align*}
\]
where \( v(t) = u(t + \tau) \), and
\[
\tilde{r}_i(t) = r_i(t + \tau), \quad \tilde{h}(t) = h(t + \tau).
\]

Denote \( \mathcal{Y} \) the space \( C(\mathbb{R})^m \times C(\mathbb{R}, X) \) equipped with the compact-open topology (under which a sequence \( p_n(t) \) in \( \mathcal{Y} \) is convergent if and only if it is uniformly convergent on any compact interval \( I \subset \mathbb{R} \)). Let \( \theta \) be the translation operator on \( \mathcal{Y} \),
\[
\theta_{\tau}p = p(\tau + \cdot), \quad \forall p \in \mathcal{Y}, \quad \tau \in \mathbb{R}.
\]

Set
\[
p^*(t) = (r_1(t), \ldots, r_m(t), h(t)),
\]
and assume that \( p^*(t) \) is translation compact in \( \mathcal{Y} \), i.e., the hull
\[
\mathcal{H} = \mathcal{H}[p^*] := \{ \theta_{\tau}p^* : \tau \in \mathbb{R} \}
\]
of \( p^* \) in \( \mathcal{Y} \) is a compact subset of \( \mathcal{Y} \).

We also assume that \( \mathcal{H} \) is minimal w.r.t \( \theta \). This requirement is naturally fulfilled when \( p^* \) is, say for instance, periodic, pseudo-periodic, or almost periodic.

Define a function \( F : \mathcal{H} \times C_\alpha \to X \) as
\[
F(p, \phi) = f(\phi(-p_1(0)), \ldots, \phi(-p_m(0))) + p_{m+1}(0)
\]
for any \( p = (p_1, \ldots, p_{m+1}) \in \mathcal{H} \). Observing that
\[
(r_1(t + \tau), \ldots, r_m(t + \tau), h(t + \tau)) = p^*(t + \tau) = (\theta_{t+\tau}p^*)(0),
\]
we can rewrite the righthand side of the equation in (4.25) as follows:
\[
f(v(t - \tilde{r}_1), \ldots, v(t - \tilde{r}_m)) + \tilde{h}(t)
\]
\[
= F(\theta_{t+\tau}p^*, v_t) = F(\theta_{t}p, v_t), \quad p = \theta_{\tau}p^*.
\]

Consequently (4.25) can be reformulated as
\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{dv}{dt} + Av = F(\theta_{t}p, v_t), \\
v_0 = \phi.
\end{array} \right. & \quad t \geq 0, \quad p \in \{ \theta_{\tau}p^* : \tau \in \mathbb{R} \},
\end{aligned}
\]
(4.28)

Since \( \{ p = \theta_{\tau}p^* : \tau \in \mathbb{R} \} \) is dense in \( \mathcal{H} \), for theoretical completeness we usually embed (4.28) into the following cocycle system:
\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{dv}{dt} + Av = F(\theta_{t}p, v_t), \\
v_0 = \phi.
\end{array} \right. & \quad t \geq 0, \quad p \in \mathcal{H},
\end{aligned}
\]
(4.29)

Now assume \( f \) satisfies the following conditions:
(f1) $f$ is locally Lipschitz, namely, for any $R > 0$, there exists $L_f = L_f(R) > 0$ such that for all $u, u' \in X^\alpha$ with $\|u\|_\alpha, \|u'\|_\alpha \leq R$,

$$\|f(u_1, \ldots, u_m) - f(u'_1, \ldots, u'_m)\|_0 \leq L_f(\|u_1 - u'_1\|_\alpha + \cdots + \|u_m - u'_m\|_\alpha).$$

(f2) There exist $C_0, C_1 > 0$ such that

$$\|f(u_1, \ldots, u_m)\|_0 \leq C_0(\|u_1\|_\alpha + \cdots + \|u_m\|_\alpha) + C_1, \quad \forall u_i \in X^\alpha.$$

Then one can trivially verify that the mapping $F$ defined by (4.27) satisfies hypotheses (F1) and (F2).

One can of course consider the more general case where $f = f(t, u_1, \ldots, u_m)$ depends on $t$. The only difference between such a case and the one we have considered above is that, one has to take into account the closure of the set of translations of the function $f$ in $t$ in appropriate functional spaces when constructing the base space $\mathcal{H}$. We omit the details.

**Remark 4.8** Note that if the function $p^*$ in (4.26) is periodic (resp. quasi-periodic, almost periodic), then $\theta_t p$ is periodic (resp. quasi-periodic, almost periodic) for any fixed $p \in \mathcal{H} := \mathcal{H}[p^*]$. Let $\Gamma$ be the equilibrium solution of (4.29) given in Theorems 4.6 and 4.7. Then since $\Gamma(q)$ is continuous in $q$, we deduce that $\gamma_p := \Gamma(\theta_t p)$ is periodic (resp. quasi-periodic, almost periodic) as well. Therefore these two theorems give the existence of asymptotically stable periodic (resp. pseudo periodic, almost periodic) solutions for equation (4.23).

The interested reader is referred to [16, 21, 22, 23, 31, 35, 37, 39, 40, 50, 51] etc. for some classical results and new trends on periodic solutions of delay differential equations, and to [19, 30, 38, 46, 55, 56, 57, 59] and references therein for typical results on almost periodic solutions.

### 4.6 Neural networks with multiple delays

As an concrete example, we consider the following reaction diffusion neural network system with multiple delays:

$$\begin{aligned}
\frac{\partial u_i}{\partial t} &= \text{div} \left( a_i(x) \nabla u_i \right) + \sum_{j=1}^n b_{ij} u_j + \\
&\quad + \sum_{j=1}^n T_{ij} g_j(x, u_j(x, t - r_{ij})) + J_i(x, t), \\
u_i(x, t) &= 0, \quad t \geq 0, \ x \in \partial \Omega, \quad i = 1, 2, \ldots, n.
\end{aligned}$$

(4.30)

Here $\Omega \subset \mathbb{R}^m$ is a bounded domain with a smooth boundary $\partial \Omega$, $a_i \in C^1(\overline{\Omega})$ and is positive everywhere on $\overline{\Omega}$, $b_{ij}$ and $T_{ij}$ are constant coefficients,

$$0 \leq r_{ij} \leq r < \infty, \quad 1 \leq i, j \leq n,$$

and
and $J_i(x, t)$ are bounded inputs. We refer the interested reader to [13, 49] etc. for a physical background of this type of systems.

Let $A_i$ be the elliptic operator given by

$$A_i u = -\sum_{k=1}^{m} \frac{\partial}{\partial x_k} \left( a_i(x) \frac{\partial u}{\partial x_k} \right)$$

associated with the corresponding boundary condition. It is a basic knowledge (see e.g. Henry [18, Chap.7]) that $A_i$ is a sectorial operator in $L^2(\Omega)$ with compact resolvent.

For notational simplicity, we use the same notation $g_j$ to denote the Nemytskii operator generated by the function $g_j(x, u, v)$, i.e.,

$$g_j(u, v)(x) = g_j(x, u, v) \quad (x \in \Omega), \quad u, v \in L^2(\Omega).$$

Let $J_i(t) = J_i(\cdot, t)$. Then (4.30) takes a slightly abstract form:

$$\frac{du_i}{dt} + A_i u_i = \sum_{j=1}^{n} b_{ij} u_j + \sum_{j=1}^{n} T_{ij} g_j(u_j, u_j(t - r_{ij})) + J_i(t), \quad 1 \leq i \leq n. \quad (4.31)$$

Set $H = (L^2(\Omega))^n$, and let $u = (u_1, \cdots, u_n)'$. Denote

$$Au = (A_1 u_1, \cdots, A_n u_n)', \quad u \in D(A) \subset H.$$ 

(It is clear that $A$ is a sectorial operator in $H$.) Let $C_0 = C([-r, 0], H)$, and define an operator $G : C_0 \rightarrow H^n = (L^2(\Omega))^{n \times n}$ as follows: $\forall \phi = (\phi_1, \cdots, \phi_n)' \in C_0,$

$$G(\phi) = (\psi_{ij})_{n \times n}, \quad \text{where } \psi_{ij} = g_j(\phi_j(0), \phi_j(-r_{ij})).$$

Let $T = (T_{ij})_{n \times n}$. Write $TG(\phi) = (TG(\phi)_{ij})_{n \times n}$, and define

$$F(\phi) = (F_1(\phi), F_2(\phi), \cdots, F_n(\phi))', \quad F_i(\phi) = TG(\phi)_{ii}.$$ 

Then (4.31) can be reformulated as

$$\frac{du}{dt} + Au = Bu + F(u_t) + J(t), \quad (4.32)$$

where $B = (b_{ij})_{n \times n}$, and $J = (J_1, \cdots, J_n)'$.

Since (4.32) is nonautonomous, generally the initial value problem reads

$$\begin{cases} \frac{dv}{dt} + Av = Bv + F(v_t) + J(t + \tau), \quad t \geq 0, \\ v_0 = \phi \in C_0, \end{cases} \quad (4.33)$$
where \( v(t) = u(t + \tau) \), and \( \tau \in \mathbb{R} \) denotes the initial time. We assume that \( J \) is translation compact in \( \mathcal{Y} \). Denote \( \mathcal{H} \) the hull \( \mathcal{H}[J] \) of the function \( J \) in \( \mathcal{Y} \). Then as in the previous subsection one can embed (4.33) into the cocycle system:

\[
\begin{aligned}
\left\{
\begin{array}{l}
\frac{dv}{dt} + (A - B)v = F(\theta_t p, v_t), \\
v_0 = \phi \in \mathcal{C}_0,
\end{array}
\right. \quad t \geq 0, \ p \in \mathcal{H},
\end{aligned}
\tag{4.34}
\]

where \( F(p, \phi) = F(\phi) + p(0), \quad p \in \mathcal{H}, \ \phi \in \mathcal{C}_0. \)

For simplicity, we always assume that \( g_j(x, u, v) \) are continuous and \textit{globally} Lipschitz in \((u, v)\) uniformly for \( x \in \Omega \), that is, there exists \( L_j > 0 \) such that

\[ |g_j(x, t_1, s_1) - g_j(x, t_2, s_2)| \leq L_j(|t_1 - t_2| + |s_1 - s_2|) \]

for all \( t_i, s_i \in \mathbb{R} \) and \( x \in \Omega \). Then for the Nemytskii operator \( g_j \) of the function \( g_j(x, u, v) \), we have

\[ \|g_j(u_1, v_1) - g_j(u_2, v_2)\|_{L^2(\Omega)} \leq L_j(\|u_1 - u_2\|_{L^2(\Omega)} + \|v_1 - v_2\|_{L^2(\Omega)}). \]

Further by some simple calculations it can be shown that

\[ \|F(p, \phi) - F(p, \phi')\| H \leq L\|\phi - \phi'\|_{\mathcal{C}_0} \]

with \( L = 2 \left( \sum_{i=1}^n \left( \sum_{j=1}^n |T_{ij}|L_j \right)^2 \right)^{1/2} \). This allows us to carry over all the results on the abstract evolution equation (4.1) to system (4.34). In particular, by Remark 4.8 we have the following theorem.

**Theorem 4.9** Suppose \( \Re(\sigma(A - B)) \geq \beta > 0 \), and that \( L < 1/M \int_0^t e^{-\beta(t-s)}ds \), where \( M \) is the constant appearing in (4.3) corresponding to operator \( A - B \). Let \( J(t) = (J_1(t), \ldots, J_n(t))' \) be a periodic (resp. quasi-periodic, almost periodic) function. Then system (4.30) has a unique periodic (resp. quasi-periodic, almost periodic) solution \( \gamma \) which is globally uniformly asymptotically stable.

If we further assume \( L < 1/M(1 + M) \int_0^t e^{-\beta(t-s)}ds \), then \( \gamma \) is globally exponentially asymptotically stable.

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