Finsleroid-regular space. Landsberg-to-Berwald implication

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Abstract

By performing required evaluations, we show that in the Finsleroid-regular space the Landsberg-space condition just degenerates to the Berwald-space condition (at any dimension number $N \geq 2$). Simple and clear expository representations are obtained. Due comparisons with the Finsleroid-Finsler space are indicated.

Keywords: Finsler metrics, spray coefficients, curvature tensors.
1. Description of new conclusions

The Finsler geometry theory can mentally be divided into two great parts: the development of the general theory to search for tensorial and geometrical implications of a general Finsler metric function “$F$” and the investigation of possible results of specifying the $F$ in an attractive particular way [1,2]. Obviously, it is the latter way that one is to follow when hoping to develop genius and handy applications.

Below, we deal with the Finsler space notion specified by the condition that the basic Finslerian metric function, to be denoted by $K(x, y)$, is constructed functionally from the set $\{g(x), b_i(x), a_{ij}(x), y\}$, where $g(x)$ is a scalar, $b_i(x)$ is an involved vector field, and $a_{ij}(x)$ is a (positive-definite) Riemannian metric tensor; $y$ stands for the tangent vectors supported by a point $x$ of the underlined manifold. Denoting by $c = ||b|| \equiv ||b||_{\text{Riemannian}}$ the respective Riemannian norm value of the input 1-form $b = b_i(x)y^i$ and assuming the range $0 < c < 1$, we construct the particular function $K(x, y)$ which occurs being globally regular. The entailed positive-definite Finsler space will be denoted by $\mathcal{F}R_{g;c}^{PD}$.

The extrapolation $\mathcal{F}R_{g;c=1}^{PD} = \mathcal{F}F_{g}^{PD}$ (1.3) takes place, where $\mathcal{F}F_{g}^{PD}$ is the Finsleroid-Finsler space which was constructed and developed in [3-6] under the assumption $||b|| = 1$.

This scalar $c(x)$ proves to play the role of the regularization factor. Indeed, in the spaces $\mathcal{F}R_{g;c}^{PD}$ and $\mathcal{F}F_{g}^{PD}$ the metric function $K$ is constructed such that $K$ involves the square-root variable $q(x, y) = \sqrt{S^2 - b^2}$ (see (A.4)). Differentiating various tensors of the space $\mathcal{F}R_{g;c}^{PD}$ as well as the space $\mathcal{F}F_{g}^{PD}$ gives rise, therefore, to appearance of degrees of the fraction $1/q$.

If $||b|| = 1$, we have $q = 0$ when $y = \pm b$. As far as $0 < c < 1$, we have $q \neq 0$ whenever $y \neq 0$ (because of the inequality (A.5)), so that the fraction $1/q$ does not produce any singularities on $TM \setminus 0$.

We use

REGULARITY DEFINITION. The Finsler space $\mathcal{F}R_{g;c}^{PD}$ under consideration is regular in the following sense: globally over all the slit tangent bundle $TM \setminus 0$, the Finsler metric function $K(x, y)$ of the space is smooth of the class $C^\infty$ regarding both the arguments $x$ and $y$, and also the entailed Finsler metric tensor $g_{ij}(x, y)$ is positive-definite: $\det(g_{ij}) > 0$.

The $\mathcal{F}F_{g}^{PD}$–space is smooth of the class $C^2$, and not of the class $C^3$, on all of the slit tangent bundle $TM \setminus 0$. The $\mathcal{F}F_{g}^{PD}$–space is smooth of the class $C^\infty$ on all of the $b$-slit tangent bundle

$$T_bM := TM \setminus 0 \setminus b \setminus -b$$

(1.4) (obtained by deleting out in $TM \setminus 0$ all the directions which point along, or oppose, the directions given rise to by the 1-form $b$).

The Finsleroid-Finsler space $\mathcal{F}F_{g}^{PD}$ developed involves an attractive realization of the Landsberg condition over the $b$-slit tangent bundle (see [3-6]). The realization cannot
be extrapolated to the \( b \)-section of the tangent bundle \( TM \), because on the section the smoothness of the space \( \mathcal{FF}^{PD}_g \) degenerates to but the \( C^2 \)-level.

It is impossible to lift realization of the Landsberg condition from the space \( \mathcal{FF}^{PD}_g \) to the space \( \mathcal{FR}^{PD}_{g;c} \). Indeed, the following theorem is valid.

**Landsberg-to-Berwald Theorem.** In the Finsleroid-regular space \( \mathcal{FR}^{PD}_{g;c} \) the Landsberg-space condition entails the Berwald-space condition.

To verify this theorem, it is sufficient to pay a due attention to the factor \((1 - c^2)\) which enters the right-hand part of the formula (A.32) which precedes the explicit (and simple) expression (A.33) for the contraction \( A^iA_i \) (see more detail in Note placed in the end of Appendix A).

In Appendix A, the explicit form of the Finsleroid-regular metric function \( K \) is presented, the space \( \mathcal{FR}^{PD}_{g;c} \) is rigorously defined, and entailed representations of various key tensors are given. The knowledge of the associated spray coefficients \( G^i \), as given by the explicit representation (A.37) derived, can open up various convenient possibilities to evaluate and study associated geodesic equations, connection coefficients, as well as curvature tensors. Amazingly, the coefficients (A.37) provide us readily with the Berwald space of the regular type: the Berwald case would imply \( g = \text{const} \) and \( \nabla_i b_j = 0 \) (see (A.35) and (A.36)), and also \( c = \text{const} \). In the two-dimensional case, however, the space degenerates to the locally Minkowskian space (whenever \( g \neq 0 \)). In the dimensions \( N \geq 3 \) the obtainable \( \mathcal{FR}^{PD}_{g;c} \)-Berwald spaces can be \"neither Riemannian nor locally Minkowskian.\" The Berwald case of the Finsler space is attractive because of its simplicity.

In Appendix B, we investigate the derivatives of the spray coefficients in the particular case (B.1). Calculations involved are simple, showing the validity of the following theorem.

**Particular Theorem.** Given a scalar \( k = k(x) \). If the conditions \( g = \text{const} \) and \( \nabla_i b_j = kr_{ij} \) are fulfilled, then

\[
\dot{A}_{knj} = (1 - c^2)k (m_1 A_{knj} + m_2 A_k A_n A_j). \tag{1.5}
\]

In (1.5), \( m_1 \) and \( m_2 \) are two scalars, which form is indicated explicitly in (B.20) of Appendix B. Remarkably, \( m_1 \), as well as \( m_2 \), doesn’t vanish identically when \( g \neq 0 \). Therefore, if \( g \neq 0 \) and \( 0 < c < 1 \), the right-hand part in (1.5) can vanish identically only in the case \( k = 0 \) which is the Berwald case.

We shall use the notation

\[
\mathcal{D} = \frac{1}{K} y^j D_j \tag{1.6}
\]

with \( D_j \) standing for the \( h \)-covariant (horizontal) Finslerian derivative (see [2]); the tensor \( \dot{A}_{knj} \) is identical to that used in [2], namely,

\[
\dot{A}_{knj} := \mathcal{D} A_{knj}. \tag{1.7}
\]

The condition \( \nabla_i b_j = kr_{ij} \), when taken in conjunction with \( g = \text{const} \), realizes the Landsberg space, that is, \( \dot{A}_{knj} = 0 \), in the Finsleroid-Finsler space \( \mathcal{FF}^{PD}_g \) (see [3-6]). Lifting the condition to the Finsleroid-regular space \( \mathcal{FR}^{PD}_{g;c} \) results in the representation (1.5) which (beautifully?) extends the Landsberg condition \( \dot{A}_{knj} = 0 \).
The occurrence of the regularizing factor \((1 - c^2)\) in the right-hand parts of the formulae (A.32) and (1.5) presents an astonishingly simple and explicit illustration to the above Landsberg-to-Berwald Theorem.

Appendix A: Involved \(\mathcal{FR}_{x;c}^P\)-notions

Let \(M\) be an \(N\)-dimensional \(C^\infty\) differentiable manifold, \(T_x M\) denote the tangent space to \(M\) at a point \(x \in M\), and \(y \in T_x M \setminus 0\) mean tangent vectors. Suppose we are given on \(M\) a Riemannian metric \(S = S(x,y)\). Denote by \(\mathcal{R}_N = (M, S)\) the obtained \(N\)-dimensional Riemannian space. Let us also assume that the manifold \(M\) admits a non–vanishing 1-form \(b = b(x,y)\), denote by 

\[
c = ||b|| \equiv ||b||_{\text{Riemannian}} \tag{A.1}
\]

the respective Riemannian norm value. Assuming

\[
0 < c < 1, \tag{A.2}
\]

we get

\[
S^2 - b^2 > 0 \tag{A.3}
\]

and may conveniently use the variable

\[
q := \sqrt{S^2 - b^2}. \tag{A.4}
\]

Obviously, the inequality

\[
q^2 \geq \frac{1 - c^2}{c^2} b^2 \tag{A.5}
\]

is valid.

With respect to natural local coordinates in the space \(\mathcal{R}_N\) we have the local representations

\[
\sqrt{a^{ij}(x)b_i(x)b_j(x)} = c(x) \tag{A.6}
\]

and

\[
b = b_i(x)y^i, \quad S = \sqrt{a_{ij}(x)y^iy^j}. \tag{A.7}
\]

The reciprocity \(a^{in}a_{nj} = \delta^i_j\) is assumed, where \(\delta^i_j\) stands for the Kronecker symbol. The covariant index of the vector \(b_i\) will be raised by means of the Riemannian rule \(b^i = a^{ij}b_j\), which inverse reads \(b_i = a_{ij}b^j\). We also introduce the tensor

\[
r_{ij}(x) := a_{ij}(x) - b_i(x)b_j(x) \tag{A.8}
\]

to have the representation

\[
q = \sqrt{r_{ij}(x)y^iy^j}. \tag{A.9}
\]

We choose the Finsler space notion specified by the condition that the Finslerian metric function \(K(x,y)\) be of the functional dependence

\[
K(x,y) = \Phi(g(x), b_i(x), a_{ij}(x), y), \tag{A.10}
\]

where \(g(x)\) is a scalar (on the background manifold \(M\)), subjected to ranging

\[
-2 < g(x) < 2, \tag{A.11}
\]
and apply the convenient notation
\[ h(x) = \sqrt{1 - \frac{1}{4}(g(x))^2}, \quad G(x) = \frac{g(x)}{h(x)}. \] (A.12)

We introduce the characteristic quadratic form
\[ B(x, y) := b^2 + gqb + q^2 \equiv \frac{1}{2} \left( (b + g)^2 + (b + g - q)^2 \right), \] (A.13)
where \( g_+ = (1/2)g + h \) and \( g_- = (1/2)g - h \). The discriminant \( D_B \) of the quadratic form \( B \) is negative:
\[ D_B = -4h^2 < 0. \] (A.14)

Therefore, the quadratic form \( B \) is positively definite. In the limit \( g \to 0 \), the definition (A.13) degenerates to the quadratic form of the input Riemannian metric tensor: \( B|_{g=0} = b^2 + q^2 \equiv S^2 \). Also, \( \eta B|_{\eta=1} = c^2 \), where \( \eta = 1/(1 + gc\sqrt{1 - c^2}) \). It can readily be verified that on the definition range (A.11) of the \( g \) we have \( \eta > 0 \).

Under these conditions, we set forth the following definition.

**Definition.** The scalar function \( K(x, y) \) given by the formulas
\[ K(x, y) = \sqrt{B(x, y)} J(x, y) \] (A.15)
and
\[ J(x, y) = e^{-\frac{1}{2}G(x)f(x,y)}, \] (A.16)
where
\[ f = -\arctan \frac{G}{2} + \arctan \frac{L}{hb}, \quad \text{if} \quad b \geq 0, \] (A.17)
and
\[ f = \pi - \arctan \frac{G}{2} + \arctan \frac{L}{hb}, \quad \text{if} \quad b \leq 0, \] (A.18)
with
\[ L = q + \frac{g}{2}b, \] (A.19)
is called the Finsleroid-regular metric function.

The function \( L \) obeys the identity
\[ L^2 + h^2b^2 = B. \] (A.20)

**Definition.** The arisen space
\[ \mathcal{FR}_{g,c}^{FD} := \{ \mathcal{R}_N; b_i(x); g(x); K(x, y) \} \] (A.21)
is called the Finsleroid-regular space.

**Definition.** The space \( \mathcal{R}_N \) entering the above definition is called the associated Riemannian space.
The associated Riemannian metric tensor \( a_{ij} \) has the meaning
\[
a_{ij} = g_{ij} \bigg |_{g=0}. \quad (A.22)
\]

Definition. Within any tangent space \( T_xM \), the Finsleroid-regular metric function \( K(x, y) \) produces the regular Finsleroid
\[
\mathcal{F}R_{gc\{x\}}^{PD} := \{ y \in \mathcal{F}R_{gc\{x\}}^{PD} : y \in T_xM, K(x, y) \leq 1 \}. \quad (A.23)
\]

Definition. The regular Finsleroid Indicatrix \( I_{g; c}\{x\} \subset T_xM \) is the boundary of the regular Finsleroid, that is,
\[
I_{g; c}\{x\} := \{ y \in I_{g; c}\{x\} : y \in T_xM, K(x, y) = 1 \}. \quad (A.24)
\]

Definition. The scalar \( g(x) \) is called the Finsleroid charge. The 1-form \( b = b_i(x) y^i \) is called the Finsleroid–axis 1-form.

We shall meet the function
\[
\nu := q + (1 - c^2)gb \quad (A.25)
\]
for which
\[
\nu > 0 \text{ when } |g| < 2. \quad (A.26)
\]
Indeed, if \( gb > 0 \), then the right-hand part of (A.25) is positive. When \( gb < 0 \), we may note that at any fixed \( c \) and \( b \) the minimal value of \( q \) equals \( \sqrt{1 - c^2} |b|/c \) (see (A.5)), arriving again at (A.26).

Under these conditions, we can explicitly extract from the function \( K \) the distinguished Finslerian tensors, and first of all the covariant tangent vector \( \hat{y} = \{y_i\} \) from
\[
y_i := (1/2) \partial K^2 / \partial y^i, \quad (A.27)
\]
where \( u_i = a_{ij} y^j \). After that, we can find the Finslerian metric tensor \( \{g_{ij}\} \) together with the contravariant tensor \( \{g^{ij}\} \) defined by the reciprocity conditions \( g_{ij} g^{jk} = \delta^k_i \), and the angular metric tensor \( \{h_{ij}\} \), by making use of the following conventional Finslerian rules in succession:
\[
g_{ij} := \frac{1}{2} \frac{\partial^2 K^2}{\partial y^i \partial y^j} = \frac{\partial y_i}{\partial y^j}, \quad h_{ij} := g_{ij} - y_i y_j \frac{1}{K^2},
\]
thereafter the Cartan tensor
\[
A_{ijk} := \frac{K}{2} \frac{\partial g_{ij}}{\partial y^k} \quad (A.28)
\]
and the contraction
\[
A_i := g^{jk} A_{ijk} = K \frac{\partial \ln \left( \sqrt{\det(g_{mn})} \right)}{\partial y^i} \quad (A.29)
\]
can readily be evaluated.

It can straightforwardly be verified that
\[
\det(g_{ij}) = \nu \left( \frac{K^2}{B} \right)^N \det(a_{ij}) > 0 \quad (A.30)
\]
with the function $\nu$ given by (A.25) [7], and

$$A_i = \frac{Kg}{2qB} \left( \frac{q^2 b_i - b v_i}{X} \right), \quad (A.31)$$

where the function $X$ is given by

$$\frac{1}{X} = N + (1 - c^2) \frac{B}{qv}. \quad (A.32)$$

Contracting yields the formula

$$A^i A_i = \frac{g^2}{4} \frac{1}{X^2} \left( N + 1 - \frac{1}{X} \right) \quad (A.33)$$

and evaluating the Cartan tensor results in the lucid representation

$$A_{ijk} = X \left[ A_i h_{jk} + A_j h_{ik} + A_k h_{ij} - \left( N + 1 - \frac{1}{X} \right) \frac{1}{A_h A^h} A_i A_j A_k \right]. \quad (A.34)$$

We use the Riemannian covariant derivative

$$\nabla_i b_j := \partial_i b_j - b_k a_{ij}, \quad (A.35)$$

where

$$a_{ij} := \frac{1}{2} a^{kn} (\partial_j a_{ni} + \partial_i a_{nj} - \partial_n a_{ji}) \quad (A.36)$$

are the Christoffel symbols given rise to by the associated Riemannian metric $S$.

Attentive direct calculations of the induced spray coefficients $G^i = \gamma^i_{nm} y^n y^m$, where $\gamma^i_{nm}$ denote the associated Finslerian Christoffel symbols, can be used to arrive at the following result.

Theorem 1. In the Finsleroid-regular space $\mathcal{F}R^P_D$ the spray coefficients $G^i$ can explicitly be written in the form

$$G^i = \frac{g}{\nu} \left( y^j b^h \nabla_j b^h + gq b^j f_j \right) v^i - gq f^i + a^i_{nm} y^n y^m. \quad (A.37)$$

We use the notation

$$v^i = y^i - b b^i \quad (A.38)$$

and

$$f_j = f_{jn} y^n, \quad f^i = f^i_{n} y^n, \quad f^i_n = a^{ik} f_{kn}, \quad f_{mn} = \nabla_m b_n - \nabla_n b_m \equiv \frac{\partial b_n}{\partial x^m} - \frac{\partial b_m}{\partial x^n}. \quad (A.39)$$

where $\nabla$ means the covariant derivative in terms of the associated Riemannian space $R_N = (M, S)$ (see (A.35)); $a^i_{nm}$ stands for the Riemannian Christoffel symbols (A.36) constructed from the input Riemannian metric tensor $a_{ij}(x)$; the coefficients $E^i$ involving the gradients $g_h = \partial g / \partial x^h$ of the Finsleroid charge can be taken as

$$E^i = \bar{M} (yg-y^i) + K \frac{4q^2}{gB} (yg) X A^i - \frac{1}{2} \bar{M} K^2 g_h g^{ih}, \quad (A.40)$$
where \((yg) = g h y h\), \(X\) is the function given in (A.32), and the function \(\bar{M}\) is defined by the equality \(\partial K/\partial g = (1/2)\bar{M} K\). It can readily be verified that

\[
b_h E^h = b\bar{M} (yg) + \left[\frac{q^2}{B} (yg) - \frac{1}{2} \bar{M} \left(q(bg) + g[b(bg) - (yg)]\right)\right] \frac{1}{\nu} \left(c^2 S^2 - b^2\right) - \frac{1}{2} \bar{M} (bg) b^2, \tag{A.41}\]

where \((bg) = b h g h\).

Contracting (A.37) by \(b_i\) yields

\[
b_i \left(\tilde{G}^i - a^{i mn} y^m y^n\right) = \frac{q}{\nu} (ys) + g q \sigma \left(1 - c^2\right) b - g q \sigma + b_i E^i, \tag{A.42}\]

or

\[
b_k \left(\tilde{G}^k - a^{k mn} y^m y^n\right) = \frac{q}{\nu} (ys) \left(1 - c^2\right) b - \frac{g q^2}{\nu} \sigma + b_k E^k, \tag{A.43}\]

where \((ys) = y^h y^k \nabla_h b_k\) and

\[
\sigma = b_k f^k = b^k y^n f_{kn}.\]

Let us apply the operator \(\mathcal{D}\) (defined in (1.6)) to the input 1-form \(b\):

\[
\mathcal{D} b = y^k \partial_k b - G^k b_k = y^k y^m \nabla_k b_m - (G^k - a^{k mn} y^m y^n) b_k, \tag{A.44}\]

that is,

\[
\mathcal{D} b = (ys) - (G^k - a^{k mn} y^m y^n) b_k. \tag{A.45}\]

Taking into account (A.42) and denoting \(\dot{b} = \mathcal{D} b\), we obtain simply

\[
\dot{b} = \frac{q}{\nu} (ys) + \frac{g q^2}{\nu} \sigma - b_k E^k, \tag{A.46}\]

With the help of the coefficients

\[
G^k_n = \frac{\partial G^k}{\partial y^n}, \tag{A.47}\]

we can consider also the contracted derivative

\[
\mathcal{D} b_n = y^h \partial_h b_n - \frac{1}{2} G^k_n b_k = y^h \nabla_h b_n - \left(\frac{1}{2} G^k_n - a^{k nh} y^h\right) b_k, \tag{A.48}\]

or

\[
\mathcal{D} b_n = y^h \nabla_h b_n - \frac{1}{2} \frac{\partial}{\partial y^n} \left(b_k \left(G^k - a^{k mn} y^m y^n\right)\right). \tag{A.49}\]

This, by virtue of (A.43), can be written as

\[
\mathcal{D} b_n = \frac{1}{2} y^h \left(\nabla_h b_n - \nabla_n b_h\right) + \frac{1}{2} \frac{\partial \mathcal{D} b}{\partial y^n}. \tag{A.50}\]

In evaluations, it is convenient to use the derivative value

\[
\nu_k = \frac{v_k}{q} + (1 - c^2) g b_k \equiv \frac{\partial \nu}{\partial y^k}. \tag{A.51}\]
If the input 1-form $b = y^i b_i$ is exact:

$$f_{mn} = 0,$$

(A.50)

then the above representation (A.37) reduces to read merely

$$G^i = \frac{g^i}{\nu}(ys)v^i + E^i + a^i_{nm}y^ny^m,$$

(A.51)

entailing

$$G^i_k = gU_k v^i + \frac{1}{\nu}g(y^i s^k) + E^i + 2a^i_{km}y^m \equiv \frac{\partial G^i}{\partial y^k},$$

(A.52)

with

$$U_k = -\frac{1}{\nu^2}v_k(y^i s^k) + \frac{2}{\nu}r_k,$$

(A.53)

where $(ys) = y^h s_h$ and $s_k = y^k \nabla_h b_k$; $E^i_k = \partial E^i / \partial y^k$ and $r^i_k = a^{ih} r_{hk}$. Differentiating (A.53) yields the coefficients $U_{km} = \partial U_{k} / \partial y^m$ given by the formula

$$U_{km} = 2\frac{1}{\nu^2}v_k v_m(y^i s^m) - \frac{2}{\nu^2}(v_k s_m + v_m s_k) - \frac{1}{\nu^2}q_{km}(ys) + \frac{2}{\nu}\nabla_m b_k,$$

(A.54)

where

$$q_{km} = r_{km} - \frac{1}{\nu^2}v_k v_m, \quad v_k = u_k - b b_k \equiv a_{kj} v^j, \quad u_k = a_{kj} y^j.$$

(A.55)

From (A.52) we find that the coefficients $G^i_{km} = \partial G^i_k / \partial y^m$ are given by the representation

$$G^i_{km} = gU_k v^i + gU_m v^i_k + gU_{km} v^i + E^i_{km} + 2a^i_{km},$$

(A.56)

where $E^i_{km} = \partial E^i_k / \partial y^m$. By contracting we obtain

$$b_i G^i_{km} = (1 - c^2) \left( gU_k b_i + gU_i b_k + gU_{km} b \right) + b_i E^i_{km} + 2b_i a^i_{km},$$

(A.57)

and

$$u_i G^i_{km} = \left( gU_k v_m + gU_m v_k + gU_{km} q^2 \right) + u_i E^i_{km} + 2u_i a^i_{km}.$$

(A.58)

The following theorem is valid.

**Theorem 2.** If the Finsleroid-regular space $\mathcal{FR}^{PD}_{\nu^2}$ is the Berwald space and is not the locally-Minkowskian space, then $g = const$.

To verify the theorem, it is worth noting that in the Berwald case the covariant derivative $D_{n}A_{ijk}$ of the Cartan tensor vanishes identically (see [1,2]). The implication $D_{n}A_{ijk} = 0 \implies D_{n}(A^l A_i) = 0$ is obviously valid in any Finsler space. Therefore, in the Berwald case of dimension $N \geq 3$ the representations (A.33) and (A.34) just entail $g = const$. In the two-dimensional case the representation (A.34) reduces to

$$A_{ijk} = I \alpha_i \alpha_j \alpha_k \quad \text{at} \quad N = 2$$

(A.59)

with the normalized vector $\alpha_i = A_i / \sqrt{A^h A_h}$, and with the main scalar $I$ given by

$$I = \sqrt{A^h A_h}.$$
It is well-known \[1,2\] that the two-dimensional Finsler space is the non-Minkowskian Berwald space if and only if the main scalar is independent of the argument \(y\). However, from (A.32) and (A.33) it just follows that the identical vanishing of the derivative \(\partial (A^h A_h)/\partial y^n\) entails \(g = 0\), that is, the Riemannian space. Thus, the theorem is valid in any dimension \(N \geq 2\).

Also, the following theorem is valid.

**Theorem 3.** The Finsleroid-regular space \(\mathcal{FR}_{g;c}^{PD}\) is the Berwald space if and only the conditions 
\[
g = \text{const} \quad \text{and} \quad \nabla_m b_n = 0
\]  hold.

The vanishing \(\nabla_m b_n = 0\) can well be interpreted geometrically by phrasing that the 1-form \(b\) is parallel (in the sense of the associated Riemannian space).

When the coefficients \((1/2)\partial^2 G^i/\partial y^m \partial y^n\) are independent of the variable \(y\), one says that the Finsler space is the Berwald space (see \[1,2\]). The sufficiency of the conditions (A.61) is obvious from the representation (A.37), reducing the coefficients \((1/2)\partial^2 G^i/\partial y^m \partial y^n\) to the Riemannian Christoffel symbols:

\[
G^i = a^i_{nm} y^n y^m \quad \text{in the Berwald case.} \tag{A.62}
\]

To verify the necessity, we apply Theorem 2 to conclude \(g = \text{const}\), which in turn yields \(\dot{E}^i = 0\) in the representation (A.37) of \(G^i\), after which it is easy to see that the Berwald space arises if only \(\nabla_m b_n = 0\), as far as the value of \(g\) is kept differing from zero (the choice \(g = 0\) would reduce the Finsler spaces under consideration to Riemannian spaces).

The above theorem yields an attractive example of the regular Berwald space.

Comparing the conditions (A.61) with the representation (A.44) of \(\dot{b}\) yields the following elegant theorem.

**Notable Theorem.** In the dimensions \(N \geq 3\), the Finsleroid-regular space \(\mathcal{FR}_{g;c}^{PD}\) is the non-Riemannian Berwald space if and only if
\[
\dot{b} = 0 \quad \text{and} \quad g = \text{const} \neq 0. \tag{A.63}
\]

From (A.44) it is obvious that (A.61) entails \(\dot{b} = 0\). The opposed implication that the conditions \(\dot{b} = 0\) and \(g = \text{const} \neq 0\) entail \(\nabla_m b_n = 0\) can straightforwardly and readily be arrived at on the basis of the representation (A.44).

**NOTE.** The Landsberg-space condition means the requirement that the (identical) vanishing \(\mathcal{DA}_{ijk} = 0\) hold. Obviously, \(\mathcal{DA}_{ijk} = 0 \implies \mathcal{D}(A^i A_i) = 0\). In dimensions \(N \geq 3\), the Cartan tensor representation (A.34) communicates us that \(\mathcal{DA}_{ijk} = 0\) would entail \(\mathcal{DX} = 0\). When \(\mathcal{D}(A^i A_i) = 0\) and \(\mathcal{DX} = 0\) are applied to the representation (A.33) of \(A^i A_i\), we conclude \(\mathcal{DG} = 0\). Since \(g\) is a function of \(x\), the last vanishing entails \(g = \text{const}\). In the two-dimensional case, the Cartan tensor representation (A.34) reduces to the representation (A.59)-(A.60) which doesn’t involve the function \(X\), which circumstance does not make possible to conclude \(\mathcal{DX} = 0\) in a direct way. However, the basic definition of \(K\), and hence the derivative of \(K\) with respect to \(g\), involves the function arctan\((L/hb)\) (see (A.17)). Obviously, this trigonometric function cannot be cancelled by polynomials constructed from the variable set \(\{b, q\}\). At the same time,
simple evaluations show that the function enters the quantity $\mathcal{D}X$ through the term $(1 - e^2)f(x, y)(\arctan(L/hb))\mathcal{D}g$ with some function $f$ which doesn’t vanish identically. The identical vanishing of $\mathcal{D}g$ [which is equal to $(yg) = y^i\partial g/\partial x^i$] means $g = \text{const}$. Therefore, the Landsberg-space condition entails $g = \text{const}$ and $\mathcal{D}X = 0$ in any case of the dimension $N \geq 2$. Attentive consideration can be applied to conclude after direct calculations that whenever $g = \text{const}$ the identical vanishing $\mathcal{D}X = 0$ is possible when either $g = 0$ (which is the Riemannian case) or $\nabla_m b_n = 0$ (which is the Berwald case). In this way the Landsberg-to-Berwald Theorem set forth in Section 1 is getting valid.

More detail of calculations involved in the space $\mathcal{F}\mathcal{R}_{g;c}^{P,D}$ can be found in [7].

**Appendix B: Evaluation of $A_{kmj}$ in particular case**

Below we evaluate the particular case

$$g = \text{const} \quad \text{and} \quad \nabla_i b_j = k r_{ij}, \quad k = k(x), \quad (B.1)$$

obtaining the following representations from (A.53)-(A.55):

$$U_k = -k q^2 v_k + k \frac{2}{v} v_k,$$

$$G^i_k = g U_k v^i + g k q^2 r^i_k + 2 a^i_{km} y^m,$$

and

$$U_{km} = 2 k q^2 v_k v_m - k \frac{2}{v^2} (v_k v_m + v_m v_k) - k q \frac{2}{\nu} \eta_{km} + k \frac{2}{v} r_{mk}. \quad (B.2)$$

Let us find the tensor $U_{kmj} = \partial U_{km}/\partial y^j$:

$$U_{kmj} = -6 k q^2 v_k v_m v_j + 4 k \frac{1}{\nu^2} (v_k v_m v_j + v_j v_m v_k + v_k v_j v_m) + 2 k q \frac{2}{\nu^2} (v_m \eta_{kj} + v_k \eta_{mj} + v_j \eta_{km})$$

$$+ 2 k \frac{q}{\nu^2} (v_m \eta_{kj} + v_k \eta_{mj} + v_j \eta_{km}) - 2 k \frac{1}{\nu^2} (v_m r_{kj} + v_k r_{mj} + v_j r_{km}) - k \frac{1}{q \nu^2} (\eta_{km} v_j + \eta_{jm} v_k + \eta_{kj} v_m),$$

or

$$U_{kmj} = -6 k q^2 v_k v_m v_j + 4 k \frac{1}{\nu^2} (v_k v_m v_j + v_j v_m v_k + v_k v_j v_m) + 2 k q \frac{2}{\nu^2} (v_m \eta_{kj} + v_k \eta_{mj} + v_j \eta_{km})$$

$$- 2 k \frac{1}{q \nu^2} (v_m v_k v_j + v_k v_m v_j + v_j v_k v_m) - k \frac{1}{q \nu^2} (\eta_{km} v_j + \eta_{jm} v_k + \eta_{kj} v_m). \quad (B.3)$$

After that, we can evaluate the coefficients $G^i_{kmj} = \partial G^i_{km}/\partial y^j$ with the help of (A.56), which yields the representation

$$G^i_{kmj} = g U_{kji} r^i_m + g U_{mji} r^i_k + g U_{kmj} v^i, \quad (B.4)$$

from which we can find the contraction

$$u_i G^i_{kmj} = g U_{kji} v_m + g U_{mji} v_k + g U_{kmj} v_j + g U_{kmj} q^2. \quad (B.5)$$
It is convenient to use the vector
\[ e_k = \frac{b}{q^2}v_k - b_k, \] (B.6)

having
\[ \nu_k = \frac{1}{b}(q\nu_k + \nu b_k) \] (B.7)
on the basis of (A.49).

We obtain readily
\[ U_k = kq^2\nu - q\nu_k + k\frac{q^2}{\nu^2}b_k, \] (B.8)
together with
\[ U_{km} = \frac{2\nu - q}{\nu^2}\eta_{km} + 2k(1 - c^2)^2\frac{g^2q^2}{\nu^3}e_k e_m \] (B.9)
which entails
\[ U_{kmj} = -\frac{1}{q^2}(v_k U_{mj} + v_m U_{kj} + v_j U_{km}) \]
\[ + 2k(1 - c^2)^2\frac{g^2b}{\nu^3}(\eta_{kj} e_m + \eta_{mj} e_k + \eta_{mk} e_j) + 6k(1 - c^2)^3\frac{g^3q^2}{\nu^4}e_k e_m e_j \] (B.10)
and
\[ G^i_{kmj} = gU_{kj}\eta^i_m + gU_{mj}\eta^i_k + gU_{km}\eta^i_j + g \left[ 2k(1 - c^2)^2\frac{g^2b}{\nu^3}(\eta_{kj} e_m + \eta_{mj} e_k + \eta_{mk} e_j) + 6k(1 - c^2)^3\frac{g^3q^2}{\nu^4}e_k e_m e_j \right] v^i, \] (B.11)
where \( \eta^i_j = a^i h_{ij}. \) Noting that
\[ b_i v^i = (1 - c^2)b, \quad b_i \eta^i_j = -(1 - c^2)e_j, \] (B.12)
and
\[ u_i \eta^i_j = 0, \quad u_i v^i = q^2, \] (B.13)
and taking into account the representation \( y_i = (u_i + gqb_i)K^2/B \) (which is valid in the space \( \mathcal{F\mathcal{R}}^{PP}_{\gamma;e} \) under study (see (A.27)), we can readily conclude that
\[ y_i G^i_{kmj} = -(1 - c^2)k^2\frac{g^2q^2}{\nu^2}(\eta_{kj} e_m + \eta_{mj} e_k + \eta_{mk} e_j) \frac{K^2}{B}. \] (B.14)

Indeed, we have
\[ (u_i + gqb_i)G^i_{kmj} = \]
\[ gq^2 \left[ 2k(1 - c^2)^2\frac{g^2b}{\nu^3}(\eta_{kj} e_m + \eta_{mj} e_k + \eta_{mk} e_j) + 6k(1 - c^2)^3\frac{g^3q^2}{\nu^4}e_k e_m e_j \right] \]
\[ -(1 - c^2)g^2 q(U_{kj} e_m + U_{mj} e_k + U_{km} e_j) \]
\[ + g^2qb(1 - c^2) \left[ 2k(1 - c^2)^2 \frac{g^2b}{\nu^3} (\eta_{kj} \epsilon_m + \eta_{mj} \epsilon_k + \eta_{mk} \epsilon_j) + 6k(1 - c^2)^3 \frac{g^3q^2}{\nu^4} \epsilon_k \epsilon_m \epsilon_j \right] \]

\[ = gqk \left[ 2(1 - c^2)^2 \frac{g^2b}{\nu^2} (\eta_{kj} \epsilon_m + \eta_{mj} \epsilon_k + \eta_{mk} \epsilon_j) + 6(1 - c^2)^3 \frac{g^3q^2}{\nu^3} \epsilon_k \epsilon_m \epsilon_j \right] \]

\[-g^2kq(1 - c^2) \left[ \left( \frac{2\nu - q}{\nu^2} \eta_{kj} + 2(1 - c^2)^2 \frac{g^2q^2}{\nu^3} \epsilon_k \epsilon_j \right) \epsilon_m + \left( \frac{2\nu - q}{\nu^2} \eta_{mj} + 2(1 - c^2)^2 \frac{g^2q^2}{\nu^3} \epsilon_m \epsilon_j \right) \epsilon_k \right. \]

\[+ \left( \frac{2\nu - q}{\nu^2} \eta_{km} + 2(1 - c^2)^2 \frac{g^2q^2}{\nu^3} \epsilon_k \epsilon_m \right) \epsilon_j \right] \]

\[= 2gkq(1 - c^2)(\nu - q) \frac{g}{\nu^2} (\eta_{kj} \epsilon_m + \eta_{mj} \epsilon_k + \eta_{mk} \epsilon_j) \]

which shows that (B.14) is valid.

Let us also verify that (B.3) entails (B.10):

\[ U_{kmj} + \frac{1}{q^2} (v_k U_{mj} + v_m U_{kj} + v_j U_{km}) \]

\[= \frac{2}{q^2} \left[ \nu - q \right] \left( v_j \eta_{km} + v_k \eta_{jm} + v_m \eta_{kj} \right) \]

\[+ 2k(1 - c^2)^2 \frac{g^2}{\nu^3} (\epsilon_k \epsilon_m v_j + \epsilon_k \epsilon_j v_m + \epsilon_j \epsilon_m v_k) \]

\[-6k \frac{q^2}{\nu^4 b^4} \left[ q^3 \epsilon_k \epsilon_m \epsilon_j + q^2 \nu (\epsilon_k \epsilon_m b_j + \epsilon_k \epsilon_j b_m + \epsilon_j \epsilon_m b_k) + q \nu^2 (\epsilon_k b_m \epsilon_j + b_k \epsilon_j \epsilon_m + \epsilon_j b_m \epsilon_k) + \nu^3 b_k b_m b_j \right] \]

\[+ 4k \frac{q^2}{b^3 \nu^3} \left[ q^2 \epsilon_k \epsilon_m + q \nu (\epsilon_k b_m + \epsilon_m b_k) + \nu^2 b_k b_m \right] \epsilon_j + \left[ q^2 \epsilon_k \epsilon_m + q \nu (\epsilon_k b_m + \epsilon_m b_k) + \nu^2 b_k b_m \right] \epsilon_j \]

\[+ 4k \frac{q^2}{b^3 \nu^3} \left[ q^2 \epsilon_k \epsilon_j + q \nu (\epsilon_k b_j + \epsilon_j b_k) + \nu^2 b_k b_j \right] \epsilon_m + \left[ q^2 \epsilon_k \epsilon_j + q \nu (\epsilon_k b_j + \epsilon_j b_k) + \nu^2 b_k b_j \right] \epsilon_m \]
\[+4k \frac{q^2}{b^3 \nu^3} \left[ q^2 e_j e_m + q \nu (e_j b_m + e_m b_j) + \nu^2 b_j b_m ] e_k + [q^2 e_j e_m + q \nu (e_j b_m + e_m b_j) + \nu^2 b_j b_m ] b_k \right]
\]

\[+2k \frac{q - \nu}{\nu^3} (\nu_m \eta_{kj} + \nu_k \eta_{mj} + \nu_j \eta_{km})\]

\[-2k \frac{q^2}{b^3 \nu^2} \left[ q[e_k e_m + e_k b_m + e_m b_k + b_k b_m ] e_j + \nu[e_k e_m + e_k b_m + e_m b_k + b_k b_m ] b_j \right]\]

\[-2k \frac{q^2}{b^3 \nu^2} \left[ q[e_k b_j + e_j b_k + b_k b_j] e_m + \nu[e_k b_j + e_j b_k + b_k b_j] b_m \right]\]

\[-2k \frac{q^2}{b^3 \nu^2} \left[ q[e_j e_m + e_j b_m + e_m b_j + b_j b_m] e_k + \nu[e_j e_m + e_j b_m + e_m b_j + b_j b_m] b_k \right], \quad (B.15)\]

therefore (B.10) is issued.

The representation (B.14) can be written as

\[y_i G^{i}_{kmj} = -(1 - \rho^2) k \frac{g^2 q^2}{B^2} (h_{kj} e_m + h_{mj} e_k + h_{mk} e_j) + (1 - \rho^2) k \frac{3g^2 q^4}{B b^2} e_k e_m e_j \frac{K^2}{B}, \quad (B.16)\]

where we have used the following representation of the angular metric tensor:

\[h_{ij} = \frac{K^2}{B} \left( \eta_{ij} + \frac{q^2}{B} e_i e_j \right) \quad (B.17)\]

(see (A.51) in [7]).

We can use the equality

\[A_i = -\frac{Kgq}{2B} \frac{1}{X} e_i \quad (B.18)\]

(compare (A.31) with (B.6)) and take into account the representation (A.34) for the Cartan tensor, which turns (B.16) to

\[y_i G^{i}_{kmj} = -4(1 - \rho^2) k (m_1 A_{kmj} + m_2 A_k A_m A_j), \quad (B.19)\]

where

\[m_1 = -\frac{gqB}{2K\nu^2}, \quad m_2 = -m_1 X \left( N + 1 - \frac{1}{X} \right) \frac{1}{A_h A^h}. \quad (B.20)\]

The representation (B.19) tells us that

\[\dot{A}_{kmj} = (1 - \rho^2) k (m_1 A_{kmj} + m_2 A_k A_m A_j). \quad (B.21)\]
We have applied the known formula

$$\dot{A}_{kmj} = -\frac{1}{4} y_i G^i_{\ kmj}$$

(see p. 67 in [2]).

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