Calculation of the Meniscus Shape Formed under Gravitational Force by Solving the Young–Laplace Differential Equation Using the Bézier Curve Method

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ABSTRACT: This work presents a method to calculate the meniscus shape by solving the differential equation based on the Young–Laplace equation. More specifically, the differential equation is solved by applying the cubic Bézier curve. A complicated nonlinear differential equation is solved using the Bézier control points and the least-squares method while maintaining computational simplicity. The results show all of the expected features of the meniscus under the gravitational force. A brief discussion is also made on the effect of the errors on the results. The method is further validated by its agreement with the numerical solutions reported in the existing literature.

INTRODUCTION

The calculation of the meniscus shape is actively researched because of its importance in surface and interfacial science. To solve the problem, the Young–Laplace equation

$$\Delta p = \sigma \left( \frac{1}{R_1} + \frac{1}{R_2} \right),$$

where $$\Delta p$$ is the pressure difference between both sides of the meniscus, $$\sigma$$ is the surface tension of the liquid, and $$R_1$$ and $$R_2$$ are two radii of curvature, is often used. Considering the meniscus in a cylindrical capillary as an example, when $$R$$, the radius of the capillary, is significantly smaller than the capillary length of water,

$$\sqrt{\frac{\Delta p}{\rho g}} = 2.713 \times 10^{-3} \text{ m},$$

where $$\rho$$ and $$g$$ are the density of water and gravity constant, respectively, and the effect of gravity on the meniscus shape of water is negligible, so the meniscus can be accurately approximated with a spherical surface. In this case, the equation

$$\Delta p = \frac{2\sigma \cos \theta}{R}$$

can be used in the meniscus calculation, where $$\theta$$ is the contact angle. However, when $$R$$ is significantly larger than the capillary length, the gravitational force is more impactful, flattening the meniscus in the center, which makes the spherical approximation inaccurate. Thus, providing a more accurate description of the shape of the meniscus under the effect of gravity has been an area of active research.

There exist a number of other works discussing this topic. Eslami and Elliot evaluated the depth of capillary menisci by considering the three-phase contact line as the initial point for the integration of the differential equations they developed. Bullard and Garboczi minimized the free energy to calculate the meniscus shape in a cylindrical capillary, as well as the capillary rise between two closely placed parallel planar surfaces. Lobanov calculated the shape and volume of the meniscus at the upper horizontal plane of a specimen, at the lower horizontal plane of a specimen, and at the surface of a cylindrical specimen. Biery and Oblak modified the Young–Laplace equation to a differential equation in which the two radii of curvature, $$R_1$$ and $$R_2$$, are shown individually. Using this equation, they numerically calculated the meniscus shape and validated the calculation by experiments. Soligno, Dijkstra, and van Roij proposed a numerical method to calculate the meniscus shape between vertical and inclined walls and curved surfaces by minimizing the thermodynamic potential of the system.

Note that all of the preceding approaches use special techniques and tedious numerical integration to solve the differential equation.

The Bézier curve is a parametric function, consisting of one polynomial function for each dimension. Therefore, like polynomials, Bézier curves can be of any degree and express
complicated shapes as the degree increases. This work uses the cubic Bézier curve because it can approximate a wide variety of curves while still having only eight parameters. The cubic Bézier curve is determined by four points: \( P_0, P_1, P_2, \) and \( P_3 \). The curve begins at \( P_0 \), then moves in the directions \( P_1 \) and \( P_2 \) without necessarily touching them, and then ends at \( P_3 \). Bézier curves are typically applied in the fields of computer graphics, animation, and robotics, but they are also useful in science, engineering, and technology.

This work uses the Bézier curve method, which approximates the solution to any differential equation by changing the parameters of the Bézier curve to minimize the difference between the two sides of the differential equation. According to Venkataraman, the Bézier curve is advantageous when solving differential equations for four main reasons: its approximations are accurate, its formulation is simple, the differential equations can be handled in their original forms, and standard optimization techniques can be applied.\(^5\)

This work attempts to calculate, first, the meniscus shape in a cylindrical capillary under the force of gravity using the Bézier curve to solve the Young–Laplace equation. This is the first work, known to the authors, that uses the Bézier curve in the calculation of the meniscus shape. The cylindrical capillary is the simplest case; however, this method can be applied in the same way for more complicated systems such as the meniscus between two inclined walls, two vertical walls with different contact angles, on a horizontal plate, and so forth. Thus, its broader applicability is shown by solving the meniscus shape formed when a plate is immersed vertically in water.

### METHOD FOR THE MENISCUS FORMED IN A CYLINDRICAL CAPILLARY

Figure 1 schematically depicts the meniscus formed in a cylindrical capillary.

![Schematic representation of the meniscus in a cylindrical capillary.](Image)

Figure 1. Schematic representation of the meniscus in a cylindrical capillary.

By considering a force balance on a small section of the meniscus surface, the differential eq 1 is derived based on the Young–Laplace equation.\(^5\) The same equation can be derived by minimizing the Helmholtz energy of the system.\(^9\)

\[
\left(1 + \left(\frac{dh}{dr}\right)^2\right)^{3/2} + r\left(1 + \left(\frac{dh}{dr}\right)^2\right)^{1/2} = \frac{h}{\sqrt{r^2 + h^2}} \tag{1}
\]

where \( r \) and \( h \) are the radial distance from the center of the cylinder and the longitudinal distance from a reference point, respectively (see Figure 1). Note that the first and the second term of eq 1 correspond to the reciprocal of \( R_1 \) and \( R_2 \); the principal radii involved in the Young–Laplace equation. Note also that an equilibrium state is considered in this approach where the pinning force does not distort the meniscus.\(^10\)

Let \( l = \sqrt{\frac{r^2}{g}} \) be the capillary length. Then, eq 1 can be rewritten in a dimensionless form as

\[
\frac{dy}{dx} = \frac{y}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2}} + \frac{dy}{dx}\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} - y = 0 \tag{2}
\]

where \( x = \frac{r}{l} \) and \( y = \frac{h}{l} \).

The nonlinear second-order differential eq 2 can be solved with the boundary conditions

\[
\frac{dy}{dx} = 0 \text{ at } x = 0 \tag{3}
\]

and

\[
\frac{dy}{dx} = \cot \theta \text{ at } x = x_3 \tag{4}
\]

where \( x_3 = \frac{r}{l} \), \( R \) is the radius of the cylindrical capillary, and \( \theta \) is the contact angle.

### CUBIC BÉZIER CURVE

The function for the cubic Bézier curve and first and second derivatives are

\[
B(t) = (1 - t)^3P_0 + 3(1 - t)^2tP_1 + 3(1 - t)t^2P_2 + t^3P_3 \tag{5}
\]

\[
B'(t) = 3(1 - t)^2(P_1 - P_0) + 6(1 - t)t(P_2 - P_1) + 3t^2(P_3 - P_2) \tag{6}
\]

\[
B''(t) = 6(1 - t)(P_2 - 2P_1 + P_0) + 6t(P_3 - 2P_2 + P_1) \tag{7}
\]

In terms of \((x, y)\) using rectangular coordinates,

\[
x(t) = (1 - t)^3x_0 + 3(1 - t)^2tx_1 + 3(1 - t)t^2x_2 + t^3x_3 \tag{8}
\]

\[
y(t) = (1 - t)^3y_0 + 3(1 - t)^2ty_1 + 3(1 - t)t^2y_2 + t^3y_3 \tag{9}
\]

\[
x'(t) = \frac{dx}{dt} = 3(1 - t)^2(x_1 - x_0) + 6(1 - t)t(x_2 - x_1) + 3t^2(x_3 - x_2) \tag{10}
\]

\[
y'(t) = \frac{dy}{dt} = 3(1 - t)^2(y_1 - y_0) + 6(1 - t)t(y_2 - y_1) + 3t^2(y_3 - y_2) \tag{11}
\]
Furthermore, the derivatives \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \) are

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \tag{14}
\]

\[
\frac{d^2y}{dx^2} = \left( \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right)^2 \left( \frac{\frac{d^2y}{dt^2}}{\frac{dx^2}{dt^2}} \right) - \left( \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right) \left( \frac{\frac{d^2x}{dt^2}}{\frac{dx^2}{dt^2}} \right) \left( \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right) \tag{15}
\]

Thus, for a given set of parameters \( (x_0, x_1, x_2, x_3, y_0, y_1, y_2, \text{ and } y_3) \), we can obtain \( x, y, \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \) via eqs 8–15. Using these equations, the differential equation is solved as follows.

At \( t = 0 \) and 1 (the initial and end point of the Bézier curve), \( x = x_0 \) and \( x_3 \), respectively, from eq 8. Now the values of \( x_0 \) and \( x_3 \) are fixed to 0 and \( \frac{R}{l} \), respectively, which correspond to the center and the edge (wall) of the \( x \) axis (see Figure 1), while \( x_1 \) and \( x_2 \) are kept free to change.

At \( t = 0 \) (which means \( x = 0 \)), \( \frac{dy}{dx} = \frac{y_1 - y_0}{x_1 - x_0} \) from eqs 10, 11, and 14. Then, using the boundary condition (3), \( y_1 = y_0 \). Thus, \( y_1 \) is bound to \( y_0 \) while \( y_0 \) is kept free to change.

Similarly, at \( t = 1 \) (which means \( x = x_3 \)), \( \frac{dy}{dx} = \frac{y_3 - y_2}{x_3 - x_2} \) from eqs 10, 11, and 14. Then, using the boundary condition (4), \( y_3 = y_2 + \cot \theta \ (x_3 - x_2) \). Thus, \( y_3 \) is bound to \( y_2 \), \( x_0 \), and \( x_3 \), among which \( y_2 \) and \( x_3 \) are free to change.

Thus, the four remaining parameters \( x_0, x_2, y_0 \) and \( y_2 \) are kept free for optimization, and our task is now to find those parameters that can make the left side of eq 2, called hereafter difference, as close to zero as possible for many chosen values of \( t \).

The optimization is carried out using the following algorithm.

1. Read \( x_0 = 0 \) and \( x_3 = \frac{R}{l} \)
2. Guess \( x_1, x_2, y_0 \) and \( y_2 \)
3. Calculate Difference = \( \frac{\frac{dy}{dx}}{\frac{dx}{dt}} \left( \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right)^2 \left( \frac{\frac{d^2y}{dt^2}}{\frac{dx^2}{dt^2}} \right) - \left( \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right) \left( \frac{\frac{d^2x}{dt^2}}{\frac{dx^2}{dt^2}} \right) \left( \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right) \) by using equations (8)–(15)
4. for \( t = 0.01, 0.1, 0.2, \ldots, 0.9, 0.99, 1.0 \)
5. Calculate \( \frac{\Sigma |\text{Difference}|}{\Sigma y} < \varepsilon \)
6. No
7. Yes
8. Record \( x \) and \( y \) at each \( t \)

It should be noted that \( \frac{\Sigma |\text{Difference}|}{\Sigma y} \) was minimized using Excel (Data, Solver). \( t = 0 \) was excluded from optimization.
because $\frac{\Sigma \text{differential}_y}{\Sigma y}$ is undefined at $t = 0$ ($\frac{\Sigma \text{differential}_y}{\Sigma y}$ was not used since the ratio $\frac{\text{differential}_y}{y}$ becomes unreasonably large when $y$ is close to zero).

**RESULTS AND DISCUSSION FOR THE MENISCUS FORMED IN A CYLINDRICAL CAPILLARY**

Figure 2 shows $y$ versus $x$ while $x_3$ is fixed to 2 and the contact angle changes from 10 to 80°. $x$ and $y$ need to be multiplied by $l$ to produce the actual meniscus. Thus, $x_3 = 2$ corresponds to the capillary radius of 5.42 mm. The meniscus approaches the contact angle at the capillary wall ($x = 2$) and becomes flat by the center of the capillary ($x = 0$), as made necessary by the boundary conditions. When the contact angle is as large as 80°, the meniscus is nearly flat for the entire range of $x$; however, when it is smaller, $y$ increases more steeply near the capillary wall. This is because the capillary wall experiences stronger adhesion (measurable by the method developed by Tadmor et al.\textsuperscript{11}) with the liquid when the contact angle is smaller, lifting the center of the meniscus as well.

Figure 3 shows $y$ versus $x$ for different contact angles when $x_3$ is fixed to 10. Note that this $x_3$ corresponds to a capillary radius of 27.1 mm. $y$ mainly just increases when $x > 6$. As the contact angle decreases, the flat region gets smaller, and $y$ increases more steeply near the capillary wall.

Figure 4 shows $y$ versus $x$ while $\theta$ is fixed at 20° and $x_3$ changes from 2 to 10. The meniscus increases steeply when it approaches the capillary wall for all values of $x_3$ because the meniscus experiences adhesion strongly when the contact angle is as low as 20°. As $x_3$ increases, a larger portion of the meniscus becomes flat because the surface tension is no longer able to overcome the gravitational force near the center of the capillary. When $x_3$ is as small as 2, the effect of surface tension is powerful everywhere, including the center of the capillary, so $y$ is also lifted up everywhere.

In the optimization process, $\frac{\Sigma \text{differential}_y}{\Sigma y}$ was minimized, since, when $\frac{\Sigma \text{differential}_y}{\Sigma y}$ is smaller, the approximation of the solution to differential eq 2 is more accurate.

Table 1 shows the change of the minimum $\frac{\Sigma \text{differential}_y}{\Sigma y}$ with $\theta$ while $x_3$ is fixed to 10, when using the optimized parameters for the Bézier curve. The table shows that the minimum $\frac{\Sigma \text{differential}_y}{\Sigma y}$ increases when $\theta$ decreases.
As mentioned earlier, the meniscus should approach a spherical surface when the capillary radius is less than the capillary length $l$. This was examined by reducing $x_3$ to 0.5 (half of the capillary length) while maintaining $\theta$ at 20°. Figure 6 shows the spherical surface (top) and the Bézier curve (bottom) for $x_3 = 0.5$ and $\theta = 20°$. The Bézier curve is very close to the spherical one in this case.

These results obtained by the Bézier curve method were further compared with the data provided by Eslami and Elliot's numerical calculation of the meniscus shape based on the differential equation that they had derived, which was essentially the same as eq 1. Figure 7 shows that the methods agree on the meniscus depth while $\theta$ goes from 25 to 75° and $R$ is fixed to 10 mm $(x_3 = 3.686)$. Figure 8 shows the meniscus depths provided by both methods when $R$ goes from 5 to 40 mm $(x_3 = 1.843$ to 14.74) and $\theta$ is fixed to 25°. In this case, the methods mainly agree; however, Eslami and Elliot’s data have a local maximum at $R = 10$ mm, decreasing afterward before leveling off, while the Bézier curve method shows a continuous increase. When $R$ increases, it makes the most sense for the meniscus depth to increase continuously as well because the center of the capillary will be further from the walls, leading to a more significant difference between its height and the height of contact point. Thus, the Bézier curve solution actually makes more sense here.

The agreement of the data resulting from the two different methods of calculation further validates the results of the Bézier curve method, even when $R$ is much larger than the capillary length (i.e., $x_3$ is much larger than unity).

### Meniscus Formed Between a Plate and Horizontal Water Surface When the Plate Is Immersed Vertically in Water

As mentioned earlier, the Bézier curve method can be applied not only to the meniscus formed in a cylindrical capillary but also to calculate many other meniscus shapes. As one of those examples, the meniscus formed between a plate and horizontal water surface when the plate is immersed vertically in water is

![Graph](https://example.com/graph.png)

**Figure 5.** $y$ versus $x$ with the change in $\frac{\Sigma y_{\text{difference}}}{\Sigma y}$ $(x_3$ and $\theta$ are fixed at 10 and 20°; differential eq 2 was solved with boundary conditions (3) and (4). $x$ is the normalized radial distance from the center of the cylinder, and $y$ is the normalized height).
calculated. At the point where water touches the plate, the contact angle is $\theta$ (Fig. 9).

Bullard and Garboczi derived the differential equation 16 for the meniscus formed between two plates immersed vertically in liquid.

$$\frac{d^2 y}{dx^2} \left(1 + \frac{dy}{dx}\right)^{3/2} - y = 0$$

Figure 6. Comparison of the spherical meniscus and the meniscus generated by the Bézier curve method for $x_1 = 0.5$ and $\theta = 20^\circ$. (Differential eq 2 was solved with boundary conditions (3) and (4). $x$ is the normalized radial distance from the center of the cylinder, and $y$ is the normalized height).

Figure 7. Comparison of the meniscus depth calculated by the Bézier curve method with the data presented by Eslami and Elliott in Figure 11 of their paper (for different $\theta$s at $R = 10$ mm).

Figure 8. Comparison of the meniscus depth calculated by the Bézier curve method with the data presented by Eslami and Elliott in Figure 11 of their paper (for different $R_s$ at $\theta = 25^\circ$).
Only one principal radius of curvature, $R_2$, is considered in eq 16 because the other one, perpendicular to the meniscus line, is infinity.

The above equation is applicable for the case presently considered, with $y$ and $x$ being the normalized height, $h_l$, and the normalized distance, $d_l$, respectively. The boundary conditions are also slightly modified to

$$\frac{dy}{dx} = -\cot \theta \text{ at } x = 0$$ \hspace{1cm} (17)

and

$$\frac{dy}{dx} = 0 \text{ at } x = x_3 \text{ where } x_3 \text{ is very large}$$ \hspace{1cm} (18)

The results of the calculation using the Bézier curve method are given in Figure 10 for different contact angles at $x_3 = 10$, which is 10 times as large as the capillary length $l$ and considered large enough.

In the figure, the meniscus is flat when the contact angle is 90° and moves up as the contact angle decreases due to the increased adhesion of water to the flat plate.

The data are further validated by examining their agreement with the following Newman equation for the meniscus height shown as $h'$ in Figure 9. The latter equation is based on the rigorous solution of the Young–Laplace equation.

$$\sin \theta = 1 - \frac{\rho g}{2\sigma} (h')^2$$ \hspace{1cm} (19)

After rearranging

$$\frac{h'}{l} = \sqrt{1 - \sin \theta}$$ \hspace{1cm} (20)

Figure 11 compares the $h'$ versus $\theta$ curves obtained by the Bézier curve method and the Newman equation. The agreement is good enough to validate the Bézier curve method.

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**ABOUT SURFACE EVOLVER**

It should be noted that, in principle, there are two ways by which one may determine the meniscus shape. The first is to establish a point-by-point balance between surface tension and gravitational force, which leads to the differential equations such as eqs 2 and 16. These equations are solved either analytically or numerically under certain boundary conditions. The Bézier curve method belongs to this approach as a method to solve the differential equations approximately.

On the other hand, the meniscus shape can also be obtained by minimizing the total energy, including the potential energy term and surface energy term, of the system. The method is called “Surface Evolver” and was written initially by Brakke for the purpose of studying surface-tension-defined shapes. Starting from an initial surface, the program evolves it toward the minimum energy by interacting with the user. The advantage of this method is that it can draw the shape of complex geometries. Using the shape of meniscus wetting perpendicular plane surfaces as an example, Racz et al. said that it would be more desirable to choose the analytical method, and the same would be true for other classical solutions. The evolver’s capability is demonstrated best in geometries of greater complexity. Liu et al. used the Surface Solver simulation for the meniscus shape and compared the results with the analytical solution in the planar model and the numerical solution in the axisymmetric model. van Honschoten et al. obtained the shape of a liquid bridge formed between two solid surfaces by Surface Evolver. This shape represents that of the liquid bridge formed between the hydrophilic sample and AFM tip by capillary condensation.

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![Figure 9](image-url)

**Figure 9.** Schematic representation of the meniscus formed between a plate and horizontal water surface when the plate is immersed vertically in water.

![Figure 10](image-url)

**Figure 10.** Effect of $\theta$ on the $y$ versus $x$ curve while $x_3$ is fixed to 10. (Differential eq 16 was solved with boundary conditions (17) and (18). $x$ and $y$ are the normalized distance from the plate and normalized height, respectively).
CONCLUSIONS

The following conclusions can be drawn from this work:

1. The Bézier curve method could be successfully applied to solve the differential equation and calculate the meniscus shape in a cylindrical capillary under the effect of gravitational force.

2. The Bézier curve method was also applicable to solve the differential equation and calculate the meniscus formed between a plate and horizontal water surface when the plate is immersed vertically in water.

3. The meniscus shape produced by this method could satisfy all expectations: it is flat in the capillary center, approaches the contact angle at the capillary wall, and has a spherical shape when the capillary radius is small.

4. These results agreed with the rigorous solutions of other works.

5. The error of the approximation by the Bézier curve increased as the capillary radius increased and the contact angle decreased.

6. The error defined by $\sum \text{differential} / \sum \text{values}$ reached 13% when the capillary radius was as high as 10 times of the capillary length and the contact angle was as low as 10° in this work.

7. The Bézier parameter converged smoothly to its best-fit values, even when the $\sum \text{differential} / \sum \text{values}$ value remained relatively high.

8. The Bézier curve method uses straightforward optimization techniques and avoids special techniques and complicated numerical calculations. But for the meniscus calculation of complicated geometries, Surface Solver could be more powerful.

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Notes

The authors declare no competing financial interest.

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