Third and Fourth Order Phase Transitions:
Exact Solution for the Ising Model on the Cayley Tree

Borko D. Stošić
Departamento de Estatística e Informática, Universidade Federal Rural de Pernambuco,
Rua Dom Manoel de Medeiros s/n, Dois Irmãos, 52171-900 Recife-PE, Brasil

Tatijana Stošić
Laboratory for Theoretical Physics, Institute for Nuclear Sciences,
Vinča, P.O. Box 522, YU-11001 Belgrade, Yugoslavia

Ivon P. Fittipaldi
Ministério da Ciência e Tecnologia, Esplanada dos Ministérios,
Bloco E, 2º andar, Sala 215, 70067-900 Brasília-DF, Brazil

(Dated: January 9, 2022)

An exact analytical derivation is presented, showing that the Ising model on the Cayley tree exhibits a line of third order phase transition points, between temperatures $T_2 = 2k_B^{-1}J\ln(\sqrt{2}+1)$ and $T_{BP} = k_B^{-1}J\ln(3)$, and a line of fourth order phase transitions between $T_{BP}$ and $\infty$, where $k_B$ is the Boltzmann constant, and $J$ is the nearest-neighbor interaction parameter.

PACS numbers: 05.50.+q, 64.60.Cn, 75.10.Hk

Ever since the pioneering works on phase transitions and critical phenomena, it has been clear that phase transitions of higher order are conceptually possible, but, to the best of our knowledge, up to date there has been no rigorous proof of existence of a single system exhibiting a phase transition of finite order higher than two.

One notable attempt in this direction is the work of Müller-Hartmann and Zittartz \[1\], interpreting the series expansion of the free energy of the Ising model on the Cayley tree in the limit of small field $H \rightarrow 0^+$, in terms of phase transitions of all even orders (from two to infinity). More precisely, their claim was that the susceptibility of order $2\ell$ (i.e. the second order phase transition was found to occur at the Bethe-Peierls temperature $T_{BP} = k_B^{-1}J\ln[(B + 1)/(B - 1)]$, representing a fourth order phase transition line.

We consider the nearest-neighbor Ising model with the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle nn \rangle} S_i S_j - H \sum_i S_i, \quad (1)$$

where $J$ is the coupling constant, $H$ is the external magnetic field, $S_i = \pm 1$ is the spin at site $i$, and $\langle nn \rangle$ denotes summation over the nearest-neighbor pairs. For simplicity, hereafter we consider only the tree with $B = 2$, while the analysis which follows may be directly generalized for arbitrary tree branching number. Following Eggarter \[2\], we further consider systems situated on a single $n$-generation branch of a Cayley tree, composed of two $(n-1)$-generation branches connected to a single initial site. Thus, the $n$-generation branch consists of $N_n = 2^{n+1} - 1$ spins, the 0-generation branch being a single spin. The exact recursion relations for the partial partition functions of any two consecutive generation branches are easily derived \[2\] to be

$$Z_{n+1}^\pm = y^{\pm 1} [x^{\pm 1} Z_n^+ + x^{\mp 1} Z_n^-]^2, \quad (2)$$

where $Z_n^+$ and $Z_n^-$ denote the partition functions restricted by fixing the initial spin (connecting the two $n$-generation branches) into the $\{+\}$ and $\{-\}$ position, respectively, and where we have used the notation $x \equiv \exp(\beta J)$ and $y \equiv \exp(\beta H)$, with $\beta = 1/k_B T$ denoting the reciprocal of the product of the Boltzmann constant $k_B$ and the temperature $T$. The usual approach of attempting to establish the field dependent expression for...
the partition function in the thermodynamic limit, then finding its field derivatives, and finally taking the limit \( H \to 0 \) yields only approximate solutions for the zero-field susceptibility. In an earlier work, current authors have derived an exact expression for the zero-field magnetization and susceptibility, by using the strategy of finding the recursion relations for the field derivatives of the partition function, taking the limit \( H \to 0 \), and only then performing the actual iterations to reach the thermodynamic limit. Here we extend this approach to find higher derivatives of the partition function.

Equation (2) can be formally differentiated with respect to field, to find the recursion relations for the field derivatives of the partition function. Up to the forth derivative we find the following recursion relations

\[
\frac{\partial Z_{n+1}^\pm}{\partial h} = y^{\pm 1} \left( \pm \Gamma_n^{0,\pm^2} + 2 \Gamma_n^{0,\pm^2} \Gamma_n^{1,\pm} \right), \quad (3)
\]

\[
\frac{\partial^2 Z_{n+1}^\pm}{\partial h^2} = y^{\pm 1} \left[ (\Gamma_n^{0,\pm^2})^2 + 4 \Gamma_n^{0,\pm^2} \Gamma_n^{1,\pm} + 2 (\Gamma_n^{1,\pm})^2 + 2 \Gamma_n^{0,\pm^2} \Gamma_n^{1,\pm} \right], \quad (4)
\]

\[
\frac{\partial^3 Z_{n+1}^\pm}{\partial h^3} = y^{\pm 1} \left[ (\Gamma_n^{0,\pm^2})^2 + 6 \Gamma_n^{0,\pm^2} \Gamma_n^{1,\pm} + 6 (\Gamma_n^{1,\pm})^2 \pm 6 \Gamma_n^{0,\pm^2} \Gamma_n^{2,\pm} + 6 \Gamma_n^{1,\pm} \Gamma_n^{1,\pm} + 2 \Gamma_n^{0,\pm^2} \Gamma_n^{3,\pm} \right], \quad (5)
\]

\[
\frac{\partial^4 Z_{n+1}^\pm}{\partial h^4} = y^{\pm 1} \left[ (\Gamma_n^{0,\pm^2})^2 + 8 \Gamma_n^{0,\pm^2} \Gamma_n^{1,\pm} + 12 (\Gamma_n^{1,\pm})^2 + 12 \Gamma_n^{0,\pm^2} \Gamma_n^{2,\pm} + 24 \Gamma_n^{1,\pm} \Gamma_n^{1,\pm} + 2 \Gamma_n^{0,\pm^2} \Gamma_n^{3,\pm} + 6 (\Gamma_n^{2,\pm})^2 + 8 \Gamma_n^{1,\pm} \Gamma_n^{1,\pm} + 2 \Gamma_n^{0,\pm^2} \Gamma_n^{4,\pm} \right], \quad (6)
\]

where we have used notation \( h \equiv \beta H \) and

\[
\Gamma_n^{1,\pm} = \frac{\partial^4 Z_n^+}{\partial h^4} - \frac{\partial^4 Z^-}{\partial h^4} \equiv 1.
\]

Starting from a single spin (0-th generation branch), for which we have \( \partial^k Z_0^\pm / \partial h^k = (\pm 1)^k y^{\pm 1} \), it is straightforward to show by mathematical induction, using (6), that for zero field \( (y = 1) \) the symmetry equations

\[
\frac{\partial^k Z_n^+}{\partial (\beta H)^k} \bigg|_{H=0} = (-1)^k \frac{\partial^k Z_n^-}{\partial (\beta H)^k} \bigg|_{H=0}, \quad (7)
\]

hold for a branch of arbitrary generation \( n \). It then follows from (2) that the moments \( S_n = \frac{1}{Z_n} \frac{\partial Z_n^+}{\partial h} \), \( T_n = \frac{1}{Z_n} \frac{\partial^2 Z_n^+}{\partial h^2} \), \( U_n = \frac{1}{Z_n} \frac{\partial^3 Z_n^+}{\partial h^3} \), and \( V_n = \frac{1}{Z_n} \frac{\partial^4 Z_n^+}{\partial h^4} \) satisfy the recursion relations

\[
S_{n+1} = 1 + 2 t S_n, \quad S_0 = 0, \quad (8)
\]

\[
T_{n+1} = 1 + 4 t S_n + 2 t^2 S_n^2 + 2 T_n, \quad T_0 = 1, \quad (9)
\]

\[
U_{n+1} = 1 + 6 t S_n + 6 t^2 S_n^2 + 6 T_n + 6 t S_n T_n + 2 t U_n, \quad U_0 = 1, \quad (10)
\]

\[
V_{n+1} = 1 + 8 t S_n + 12 t^2 S_n^2 + 12 T_n + 24 t S_n T_n + 8 t U_n + 24 t^2 S_n U_n + 2 V_n, \quad V_0 = 1, \quad (11)
\]

where \( t \equiv \tanh(\beta J) \). Relations (8) can be iterated (by summing the geometric series) to yield closed-form expressions for \( S_n \), \( T_n \), \( U_n \) and \( V_n \) for arbitrary tree generation \( n \), and we find

\[
S_n = \frac{2^{n+1} t^{n+1} - 1}{2 t - 1}, \quad (12)
\]

\[
T_n = \frac{4 t^4 (4 t^2)^n}{(2 t^2 - 1)(2 t - 1)^2} - \frac{2^{n+1} (t+1)^2}{2 t^2 - 1} + \frac{4 t^2 (2 t)^n}{(2 t - 1)^2} + \frac{2 t^2 - 1}{(2 t - 1)^2}, \quad (13)
\]

\[
U_n = \frac{24 t^5 (8 t^3)^n}{(2 t + 1)(2 t^2 - 1)(2 t - 1)^2} - \frac{12 (t+1)^2 t (4 t)^n}{(2 t - 1)(2 t^2 - 1)} + \frac{12 t^3 (4 t^3 + 2 t^2 - t - 2)(4 t^2)^n}{(2 t^2 - 1)(2 t - 1)} + \frac{(2 t)^n+1 (24 n t^4 - 30 n t^2 + 6 n + 40 t^4 - 16 t^3 - 60 t^2 - 2 t + 11)}{(2 t - 1)^2 (2 t + 1)} + \frac{62 n (t+1)^2}{(2 t - 1)(2 t^2 - 1)} \]

\[
- \frac{8 t^3 - 6 t^2 - 6 t + 5}{(2 t - 1)^4}, \quad (14)
\]
\[
\mathcal{V}_n = \frac{48 t^8 (12 t^2 - 5) (16 t^4)^n}{(2 t + 1) (8 t^4 - 1) (2 t^2 - 1)^2 (2 t - 1)^5} + \frac{12 (t + 1)^4 4^n}{(2 t^2 - 1)^2} - \frac{48 t^4 (t + 1)^2 (8 t^2)^n}{(2 t - 1)^2 (2 t^2 - 1)^2} + \frac{96 t^6 (8 t^4 + 16 t^3 - 6 t - 3) (8 t^3)^n}{(2 t^2 - 1) (2 t - 1)^5 (2 t + 1) (4 t^3 - 1)} - \frac{48 (t + 1)^2 t^2 (4 t)^n}{(2 t^2 - 1) (2 t - 1)^2} + \frac{8 t^4 (64 t^6 + 80 t^5 - 104 t^4 - 120 t^3 + 50 t^2 + 40 t - 1) (4 t^2)^n}{(2 t - 1)^5 (2 t + 1) (2 t^2 - 1)^2} + \frac{96 (t^2 - 1) t^4 (n + 1) (4 t^2)^n}{(2 t^2 - 1) (2 t - 1)^4} + \frac{8 (t^4 - 20 t^3 - 30 t^2 + 2 t + 7) t^3 (2 t)^n}{(2 t - 1)^5} + \frac{48 t^2 (t^2 - 1) (n + 1) (2 t)^n}{(2 t - 1)^4} - \frac{2^{n+2} (t + 1) (320 t^{12} - 784 t^{10} - 384 t^8 + 176 t^8 + 372 t^7 - 192 t^6 - 56 t^4 - 45 t^3 - 11 t^2 + 6 t + 4)}{(2 t^2 - 1)^2 (8 t^3 - 1) (2 t - 1)^2 (4 t^3 - 1)} + \frac{16 t^5 - 24 t^4 - 8 t^3 + 28 t^2 - 6 t - 5}{(2 t - 1)^5}\]

(12)

Before proceeding with the analysis of the above exact expressions, a word is due on symmetry breaking. It has been commonly accepted for this model that all odd derivatives of the free energy with respect to field are identically zero in zero field because of symmetry, and only even derivatives have been analyzed\[1, 6, 7\] (in fact, equations similar to (10) and (12) for the second and fourth derivatives have been derived in references\[4, 7\] by neglecting odd correlations). However, this is true for any lattice in strictly zero field, as every spin configuration has a mirror image (obtained by flipping all the spins) with exactly the same energy, and inverted sign of the configurational magnetization \(\sum_j S_j\). The usual procedure of breaking the symmetry by retaining an infinitesimal positive field while taking the thermodynamic limit, and only then taking the zero field limit, is not suitable in the present case because the derivatives of the free energy diverge in wide temperature regions (rather than just in a single critical point). Here it seems more appropriate to break the symmetry by applying an infinitesimally small field of infinitesimal range, which can be implemented by restricting a single spin into one of the two possible orientations, while considering all the possible orientations of all the other spins. In a recent work\[10\], the present authors have analyzed the effect of restricting a single spin in the \(+\) orientation on magnetization (first derivative of the free energy with respect to field), for the current system. It was found that symmetry is indeed broken by fixing an arbitrary (surface, bulk or central) spin. While without this restriction magnetization is identically zero in strictly zero field for arbitrary system size, it was shown that fixing any spin leads to magnetic ordering of extremely large systems, in a wide temperature range (even if magnetization does go to zero in the thermodynamic limit for all nonzero temperatures).

In the rest of this paper, we shall therefore adopt the strategy of breaking the symmetry by fixing a single (central) spin in the \(+\) (upward) orientation while taking the thermodynamic limit, and we shall henceforth use the term \(\ell\)-th order susceptibility for the quantities

\[
\chi_n^{(\ell)} \equiv \beta^{\ell - 1} \frac{1}{N_n} \left. \frac{\partial^\ell Z_n^+}{\partial (\beta H)^\ell} \right|_{H=0}
\]

(13)

for finite size systems, and

\[
\chi^{(\ell)} \equiv \lim_{n \to \infty} \chi_n^{(\ell)},
\]

(14)

for the thermodynamic limit.

The restricted magnetization, \(\langle m \rangle^+ \equiv \chi^{(1)}\) is found\[5, 8\] to be zero for all nonzero temperatures in the thermodynamic limit, (even if it retains nonzero values in a wide temperature range for systems far exceeding in number of particles the observable Universe\[10\]), while 2-nd order susceptibility \(\chi_n^{(2)}\) diverges below \(T_2 = 2k_B J \ln(\sqrt{2} + 1)\). It was also recently shown\[5\] that in the thermodynamic limit the divergence of \(\chi_n^{(2)}\) in the vicinity and at \(T_2\) is extremely weak, with critical exponent \(\gamma = 0\), where susceptibility of a finite tree \(\chi_n^{(2)}\) diverges proportionally to the three generation level \(n\), as \(n \to \infty\). In the rest of this paper, we analyze the third and fourth order susceptibility given by \(\chi_n^{(3)} \equiv \beta^2 U_n/N_n\) and \(\chi_n^{(4)} \equiv \beta^3 \nu_n/N_n\), respectively.

Equations\[10, 12\] are deceptive in the sense that at first glance they suggest divergence of second, third and fourth order susceptibilities for arbitrary tree generation level \(n\), at points \(2t^2 = 1\) and \(2t = 1\) (corresponding to \(T_2\) and \(T_{BP}\), respectively), while in addition, the fourth order susceptibility seems to diverge (irrespective of \(n\)) at \(4t^4 = 1\) and \(8t^4 = 1\) (the last corresponding to the Müller-Hartmann and Zittartz temperature \(T_4\)[1]). In fact, the exact expression corresponding to (12), but obtained by neglecting odd correlations\[5\], indeed does diverge at
TABLE I: Leading terms (for large $n$) of the series expansion of susceptibilities $\chi^{(2)}$, $\chi^{(3)}$, and $\chi^{(4)}$, around points $2t = 1$ and $2t^2 = 1$, demonstrating that expressions do not diverge at these points for any finite $n$ (see text for details).

| $t \to 1/2$ | $t \to 1/\sqrt{2}$ |
|-------------|---------------------|
| $\chi^{(2)}$ | $9/4 \ln 3 - n (\sqrt{2} + 3/2) \ln (\sqrt{2} - 1)$ |
| $\chi^{(3)}$ | $n (27/8) \ln^2 3 + n 2^{n/2} (21/\sqrt{2} + 15) \ln (\sqrt{2} - 1)^2$ |
| $\chi^{(4)}$ | $2^{4/2} (243/16) \ln^3 3 - n^2 2^{n} (2/1 + 18/\sqrt{2}) \ln (\sqrt{2} - 1)^{15}$ |

TABLE II: Leading terms of $\chi^{(4)}$ around $4t^3 = 1$ and $8t^4 = 1$.

| $t \to 2^{-\frac{1}{3}}$ | $2^n \ln \left(\frac{2^{\frac{1}{3}} + 1}{2^{\frac{1}{3}} - 1}\right)$ |
| $t \to 2^{-\frac{1}{4}}$ | $2^n \ln \left(\frac{2^{\frac{1}{4}} + 1}{2^{\frac{1}{4}} - 1}\right)$ |

$8t^4 = 1$. On the other hand, a more detailed analysis shows that the interplay between the individual terms cancels out these apparent singularities at the mentioned temperature points. In particular, upon series expansion of the exact expressions around these points (from either side), it is found that there is no divergence for any finite $n$. We give the leading terms of these expansions for large $n$ in Tabs. I and II.

Consequently, divergence of higher order susceptibilities in the thermodynamic limit in different temperature regions is caused only by diverging generation level $n$, and the finite size scaling is accomplished simply by the formula $\ln \chi^{(n)} / n$. In Fig. 1 we show the scaled curves of higher order derivatives, obtained using formulas (10-12), for several system sizes $n = 64, 128, 256$ and 1024. The dotted vertical lines indicate the points $T_2$ and $T_{BP}$, where the transition changes from second to third, and from third to fourth order, respectively.

It is seen from Fig. 1 that there are three different temperature regions, where each temperature represents a point of second, third, or fourth order phase transition. Between zero and $T_2$ the second derivative diverges (the first derivative being finite), and we have a second order phase transition line. Between $T_2$ and $T_{BP}$ the third derivative diverges (first and second derivatives being finite), representing a line of third order phase transitions, while the fourth order derivative diverges between $T_{BP}$ and $\infty$ (where all lower derivatives are finite), representing a fourth order phase transition line. The explicit expressions for the limiting curves, corresponding to the finite size system curves shown in Fig. 1, are obtained by taking the limit $\kappa^\ell = \lim_{n \to \infty} \ln \chi^{(n)} / n$, and the results obtained using formulas (10-12) for different temperature regions are systematized in Tab. III.

In conclusion, after decades of continuous interest, the Ising model on the Cayley tree continues to furnish new insights into critical phenomena. Being a highly unphysical system (with its infinite dimension and finite order of ramification), Cayley tree may turn out a unique structure where phase transitions of order higher than two indeed do exist, and the latter may prove to be of only academic interest. On the other hand, it is possible that current findings may turn out relevant for interpretation of experimental data on finite size branching structures, of the real observable physical world.

* Electronic address: borko@ufpe.br

[1] E. Müller-Hartmann and J. Zittartz, Phys. Rev. Lett. 33 (1974) 893.
[2] T.P. Eggarter, Phys. Rev. B 9 (1974) 2989.
[3] J. von Heimburg and H. Thomas, J. Phys. C 7 (1974) 3433.
[4] H. Matsuda, Prog. Theor. Phys. 51 (1974) 1053.
[5] T. Stošić, B.D. Stošić and I.P. Fittipaldi, J. Mag. Mag. Mater. 177-181 (1998) 185.
[6] T. Morita and T. Horiguchi, Prog. Theor. Phys. 54 (1975) 692.
[7] T. Morita and T. Horiguchi, J. Stat. Phys. 26 (1981) 665.
[8] R. Mélin, J.C. Anglès d’Auriac, P. Chandra and B. Doucet, J. Phys. A 29 (1996) 5773.
[9] T. Stošić, B.D. Stošić and I.P. Fittipaldi, Physica A 320 (2003) 443.
[10] T. Stošić, B.D. Stošić and I.P. Fittipaldi, cond-mat/0305581 to appear in Physica A (2005).