Meson mass spectrum from relativistic equations in configuration space

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A method is described for solving relativistic quasi-potential equations in configuration space. The Blankenbecler-Sugar-Logunov-Tavkhelidze and an equal-time equation, both relativistic covariant two-body equations containing the full Dirac structure of positive and negative energy states, are studied in detail. These equations are solved for a system of two constituent quarks interacting through a potential consisting of a one-gluon exchange part with running coupling constant plus a linear confining potential which is mostly scalar and partly vector, and the spectrum of all light and heavy mesons is calculated.

I. INTRODUCTION

It is rather surprising how well the non-relativistic constituent quark model does describe the masses of the known mesons and baryons [1]. However, non-relativistic models with or without relativistic corrections can be criticized easily of having an inaccurate treatment of the quark dynamics when high momentum processes or light quarks are considered. It is therefore interesting to study relativistic covariant extensions of the constituent quark model, which are still likely to reproduce the hadron spectroscopy with similar agreement as the older models, but have a more reliable dynamical framework at extreme momenta. In a previous paper [2] we studied the pi and rho meson in such an extension, which was based on a covariant equal-time approximation to the Bethe-Salpeter equation. It was shown that the known high momentum behavior of the electromagnetic form factors could be reproduced within this model.

In this present work we investigate two different relativistic covariant approximations to the Bethe-Salpeter equation, namely the equal-time approximation [3], and the Blankenbecler-Sugar-Logunov-Tavkhelidze approximation [4] and calculate the meson mass spectrum. The ingredients of the calculations are constituent quarks interacting through a instantaneous flavor-independent phenomenological quark-antiquark potential. The potential consists of a Coulomb-like one-gluon-exchange part with running coupling constant and of a linear plus constant confining part which is a mixture of scalar and vector character. For the vector part of the potential two different gauges, Feynman and Coulomb, are examined.

The Bethe-Salpeter equation (BSE) is the relativistic covariant generalization of the Lippmann-Schwinger equation for a two-quark bound state and can be written as

\[ S^{-1}(P,p)\psi(P,p) = -\int \frac{d^3p'}{i(2\pi)^3} V(P,p-p') \psi(P,p'), \]  

(1.1)

where \( P \) and \( p \) are the total and relative momenta. Two new features are introduced as compared to the non-relativistic equation. The first one is that the bound state wave function \( \psi(P,p) \) and the potential \( V(P,p-p') \) now also depend on the fourth component of the relative momentum. The second is that for each quark the number of degrees of freedom is doubled due to the introduction of the negative energy states. These extensions make the numerical work necessary for solving the BSE highly non trivial. In order to reduce the complexity one may remove all or some of the negative energy states, or one may eliminate the dependence on the fourth component of the relative momentum. The Lorentz invariance and the structure of the negative energy states can be kept if this component is eliminated.

There have been various suggestions how to approximate the relative energy dependence of the BSE [1], most of which are based on a prescription for modifying the two body propagator in such a way that the relative energy is forced to some fixed value. This modification of the propagator is equivalent to a modification of the potential and the resulting wave equations are commonly called quasi-potential equations. Here we study two of them. The first one is the Blankenbecler-Sugar-Logunov-Tavkhelidze (BSLT) approximation, originally proposed by Logunov and Tavkhelidze, and Blankenbecler and Sugar [4], and extended to unequal masses by Cooper and Jennings [5]. The second is a two-body Dirac equation due to Wallace and Mandelzweig [3], which may be viewed as an extension of the equal-time approximation of Salpeter [7]. These quasi-potential equations are normally formulated in momentum...
space because many expressions in the equations depend on functions like $\sqrt{p^2 + m^2}$. However, the confining potential in momentum space is highly singular at zero relative momentum $q$. From $V(q=0) = \int dq^3 V(x)$ and the monotonical increasement of the confining potential $V(x)$ one easily finds that $V(q)$ diverges stronger than $q^{-3}$ for small momenta. Special care must be given when solving such equations \[^{[12]}\] This work uses a different approach. Using well-chosen representations of the quasi-potential equations we found that they can be Fourier transformed easily to configuration space, thus encompassing all problems associated with the confining potential.

This paper is organized as follows. In the next section we discuss the derivation of the BSLT and ET equations and how they can be Fourier transformed from momentum space to configuration space. The partial wave projection of the equations is given in appendix A. In section III we describe the interaction and its implications for the short distance behavior of the corresponding solutions. The presence of the Coulomb-like potential makes the wave function singular at zero relative distance. This sets an upper bound for fixed coupling constants. The short distance behavior of the solutions of the resulting integro-differential equations. Also we study the parameter dependence and global behavior of relativistic effects in the meson mass spectrum. Section V discusses the Regge-trajectories which follow from the mass spectra of the BSLT and ET equations. The slopes of these trajectories are found to be dependent on the degree of admixture of vector character in the confining potential. If it becomes too large the potential becomes repulsive at long distances in some channels making the solutions of the wave equation unbound. This gives an upper bound for the degree of the vector character in the confining potential. In section VI the full meson spectrum is discussed as obtained from the various quasi-potential equations and in the final section some concluding remarks are made.

\section{II. THE QUASI-POTENTIAL EQUATIONS}

For the full BSE it is of no importance how the total and relative momenta are defined. However, the prescriptions for the quasi-potential approximation depend on the definition of the relative momentum. Let us follow Wightman and Garding \[^{[12]}\] and define the total and relative momenta $P$ and $p$ for a two particle system by

$$P = p_1 + p_2, \quad p_1 = \alpha(s)P + p, \quad p_2 = \beta(s)P - p,$$

and

$$\alpha(s) = \frac{s + m_1^2 - m_2^2}{2s}, \quad \beta(s) = \frac{s - m_1^2 + m_2^2}{2s}, \quad (2.1)$$

where $s = -P^2 = P_0^2 - P^2$, and $p_1$ and $p_2$ denote the momenta of the two particles. In the center of mass system, where $P = (M, 0)$, they read

$$p_1 = \frac{(M^2 + m_1^2 - m_2^2)}{2M} + p_0, p = (E_1 + p_0, p), \quad (2.2)$$

$$p_2 = \frac{(M^2 - m_1^2 + m_2^2)}{2M} - p_0, -p = (E_2 - p_0, -p). \quad (2.3)$$

This definition of $p$ has the advantage that in the limit of one infinitely heavy particle, say $m_2 \to \infty$, that $E_1 \to M - m_2$. Now $M - m_2$ is precisely the total energy associated with the light particle, so this implies that the relative energy $p_0$ goes to zero. It turns out that this last property is essential for the quasi-potential approximations to have the correct one-body limit, i.e., that they reduce the one-body Dirac equation in the limit of one infinitely heavy particle.

\section{A. Blankenbecler-Sugar-Logunov-Tavkhelidze equation}

The propagator of the Bethe-Salpeter equation is

$$S(p_1, p_2) = (p_1 - m_1)^{-1}(p_2 - m_2)^{-1}$$

$$= (p_1 + m_1)(p_2 + m_2) S^{scalar}(p_1, p_2). \quad (2.5)$$
The pole-structure of this propagator is contained in the scalar part of the propagator $S^{\text{scalar}}(p_1, p_2)$. In the approximation first proposed by Logunov and Tavkhelidze, and Blankenbeler and Sugar [4], it is assumed that the pole-structure of the scalar part may be approximated by means of a dispersion relation according to

$$S^{\text{scalar}} \rightarrow S^{\text{BSLT}}_{\text{scalar}} = 2\pi i \int_{(m_1 + m_2)^2}^\infty ds' \frac{f(s', s)}{s' - s} \delta^+ \left( \left[ \alpha(s')P' + p \right]^2 + m_1^2 \right) \delta^+ \left( \left[ \beta(s')P' - p \right]^2 + m_2^2 \right),$$

where $P' = \sqrt{s'/s}P$, and $f(s', s)$ is any function that satisfies $f(s, s) = 1$. Performing the integral gives in the cm system

$$S^{\text{BSLT}}_{\text{scalar}} = f((\omega_1 + \omega_2)^2, s) \frac{\omega_1 + \omega_2}{(\omega_1 + \omega_2)^2 - s} \frac{2s}{\omega_1 + \omega_2} \delta(p_0),$$

with $\omega_i = \sqrt{p_i^2 + m_i^2}$. Taking for $f(s', s)$ the simple form proposed by Cooper and Jennings [5], which yields the proper one-body limit in the case that one of the masses goes to infinity, one finds

$$S^{\text{scalar}}_{\text{BSLT}} = \frac{1}{s - (\omega_1 + \omega_2)^2} \frac{1}{s - (\omega_1 - \omega_2)^2} \frac{2s}{\omega_1 + \omega_2} \delta(p_0),$$

and

$$S^{\text{BSLT}}(P, p) = \langle \hat{p}^+_1 + m_1 \rangle \langle \hat{p}^+_2 + m_2 \rangle S^{\text{scalar}}_{\text{BSLT}}(P, p).$$

The $\hat{p}_1$ and $\hat{p}_2$ follow from $p_1$ and $p_2$ by putting the relative energy to zero, thus $\hat{p}_1 = (E_1, p)$ and $\hat{p}_2 = (E_2, -p)$. Note that this propagator may be cast in the simple forms

$$S^{\text{BSLT}} = -\frac{4s}{[s - (\omega_1 + \omega_2)^2][s - (\omega_1 - \omega_2)^2]} \frac{\delta(p_0)}{2(\omega_1 + \omega_2)} \langle \hat{p}_1 + m_1 \rangle \langle \hat{p}_2 + m_2 \rangle,$$

$$= \frac{\delta(p_0)}{2(\omega_1 + \omega_2)} \frac{\hat{p}_1 + m_1}{\hat{p}_2 - m_2},$$

$$= \frac{\delta(p_0)}{2(\omega_1 + \omega_2)} \frac{\hat{p}_2 + m_2}{\hat{p}_1 - m_1}.$$

It is instructive to project the BSLT propagator upon positive and negative energy states. Using

$$\hat{p}_i + m_i = (\omega_i + p_{0i} - i\varepsilon) \Lambda^+_i(p_i) - (\omega_i - p_{0i} - i\varepsilon) \Lambda^-_i(p_i),$$

with

$$\Lambda^+_i(p_i) = \frac{\rho_i(-\gamma^{(i)} \cdot p_i + m_i) + \omega^{\lambda^{(i)} \gamma^{(i)}}_{0,0}}{2\omega_i},$$

and introducing $\Lambda^{\rho_1, \rho_2} = \Lambda^{\rho_1}_1(p) \Lambda^{\rho_2}_2(-p)$ gives

$$S^{\text{BSLT}} = \frac{\delta(p_0)}{2(\omega_1 + \omega_2)} \frac{1}{G} \left[ (\omega_1 + E_1)(\omega_2 + E_2)\Lambda^{+-} - (\omega_1 + E_1)(\omega_2 - E_2)\Lambda^{+-} 
- (\omega_1 - E_1)(\omega_2 + E_2)\Lambda^{+-} + (\omega_1 - E_1)(\omega_2 - E_2)\Lambda^{+-} \right],$$

where $G = \omega_1^2 - E_1^2 = \omega_2^2 - E_2^2$. This form clearly shows that the BSLT propagator is time-reversal invariant. It satisfies the condition that it does not change form if one interchanges simultaneously the energy labels $\rho_1 \rightarrow -\rho_1$ and the particle energies $E_i \rightarrow -E_i$. If the potential is also time-reversal invariant then the BSLT equation has the property that if a solution exists with mass $M$ then there will also be a solution at $-M$, which is identical to the solution at $M$ but with positive energy states changed to negative energy states and vice versa. These solutions at $-M$ correspond of course to the antiparticles of the solutions at $M$.

Since we want to solve the BSLT equation $S^{\text{BSLT}}_1 \psi = -V\psi$ in configuration space we have to Fourier transform it. In order to bring this equation in a form which may be easily transformed we multiply it by
\[ D = \frac{\tilde{p}_1 + \tilde{p}_2 + m_1 + m_2}{2(\omega_1 + \omega_2)} \]  

(2.16) which, using eq. (2.11) and (2.12), gives

\[ (\tilde{p}_1 + \tilde{p}_2 - m_1 - m_2)\psi(P, p) = \frac{1}{2(\omega_1 + \omega_2)}(\tilde{p}_1 + \tilde{p}_2 + m_1 + m_2) \int \frac{dp'}{(2\pi)^3} V(p' - p)\psi(P, p'). \]  

(2.17) In this form it is most clearly visible that in the limit that one of the particles masses goes to infinity, say \(m_2 \to \infty\), the BSLT equation reduces to the Dirac equation

\[ (\tilde{p}_1 - m_1)\psi(p_1) = \int \frac{dp'_1}{(2\pi)^3} V(p'_1 - p_1)\psi(p'_1). \]  

(2.18) The Fourier transform of the BSLT equation now reads

\[ \int \frac{dx'}{2(m_1 + m_2)} Z_{BSLT}(x' - x) \]

\[ \times \left[ i\gamma^{(1)} \cdot \nabla + E_1\gamma^{(1)} - i\gamma^{(2)} \cdot \nabla + E_2\gamma^{(2)} - m_1 - m_2 \right] \psi(P, x) = \]

\[ \int \frac{dx'}{2(m_1 + m_2)} \frac{Z_{BSLT}(x' - x)}{2(m_1 + m_2)} \]

\[ \times \left[ i\gamma^{(1)} \cdot \nabla + E_1\gamma^{(1)} - i\gamma^{(2)} \cdot \nabla + E_2\gamma^{(2)} + m_1 + m_2 \right] V(x')\psi(P, x'). \]  

(2.19) Due to the relativistic phase space factor in the BSLT propagator, a non-locality occurs in the relativistic equation, which is contained in the function

\[ Z_{BSLT}(R) = \int \frac{dp}{(2\pi)^3} \frac{m_1 + m_2}{\omega_1 + \omega_2} e^{i p \cdot R}. \]  

(2.20) This integral can be found by rewriting

\[ \frac{m_1 + m_2}{\omega_1 + \omega_2} = \frac{1}{m_1 - m_2} \left( \frac{m_1^2}{\omega_1} \nabla_R^2 - \frac{m_2^2}{\omega_2} \nabla_R^2 \right), \]  

(2.21) and using the Fourier transform of \(m_i/\omega_i\)

\[ Z_i(R) = \int \frac{dp}{(2\pi)^3} e^{i p \cdot R} \frac{m_i}{\omega_i} = \frac{m_i^2}{2\pi^2 R} K_1(m_i R). \]  

(2.22) \(K_1\) is the modified Bessel function of the second kind of order one. The non-locality becomes

\[ Z_{BSLT}(R) = \frac{1}{2\pi^2 R^2} \frac{1}{m_1 - m_2} \]

\[ \times \left[ -m_1^2 \left\{ K_0(m_1 R) + \frac{2}{m_1 R} K_1(m_1 R) \right\} + m_2^2 \left\{ K_0(m_2 R) + \frac{2}{m_2 R} K_1(m_2 R) \right\} \right]. \]  

(2.23) which is invariant under the interchange \(m_1 \leftrightarrow m_2\). Both \(Z_{BSLT}(R)\) and \(Z_i(R)\) behave like \(1/R^2\) for short distances. If the two quarks have equal masses then the non-localities become identical, \(Z_{BSLT}(R) = Z_1(R) = Z_2(R)\). In order to be solved, the BSLT equation is projected on eight basis states with definite parity and total angular momentum \(J\). These basis states, the partial wave analysis and the resulting eight coupled integro-differential equations are discussed in appendix A.1. As example, the reader may look at Eq. (B2) to see the explicit BSLT equation for the pion.

**B. Equal-time approximation**

In the equal-time (ET) approximation the assumption is made that the \(q\bar{q}\)-potential \(V\) does not depend on the relative energy in the cm system. This simplifies the BSE considerably. One may then introduce the equal-time wave
function $\psi(p) = 1/(2\pi i) \int dp_0 \psi(p_0, p)$, or $\psi(x) = \psi(x_0 = 0, x)$, and integrate the BSE on both sides over $p_0$ to obtain an equation for the ET wave function:

$$\psi(p) = - \left[ \int \frac{dp_0}{2\pi i} S(p_1, p_2) \right] \int \frac{dp}{(2\pi)^3} V(p - p') \psi(p').$$

(2.24)

For the sake of clarity the derivations in this section are formulated in the cm frame and therefore the ET approach does not show a manifestly relativistic covariant form. Yet it can be formulated in fully manifestly relativistic covariant way, see Ref. [3].

The ET equation as its stands in Eq. (2.24) does not reduce to the Dirac equation if one of the particles masses becomes infinitely heavy and so it does not have the correct one-body limit. However, as has been pointed out by Wallace and Mandelzweig [3], the ET approximation can be improved considerably by adding a propagator $S_{\text{cross}}$ to the integral between square brackets, which represents the propagation of the two particles within the crossed box diagram. With this extra term the ET propagator exhibits the one-body limit, and furthermore the contributions of the crossed diagrams are approximately taken into account. The propagator is now taken as

$$S_{\text{ET}} = \int \frac{dp_0}{2\pi i} [S(p_1, p_2) + S_{\text{cross}}(p_1, p_2)],$$

(2.25)

where in the cm system

$$S_{\text{cross}}(p_1, p_2) = S(p_1, p_2^{\text{cross}}), \quad p_2^{\text{cross}} = (E_2 + p_0, -p).$$

(2.26)

The only difference between $p_2$ and $p_2^{\text{cross}}$ is the sign in front of the relative energy. The origin of the different signs for $p_0$ can easily be understood from the different directions of flow of relative energy in the uncrossed and crossed diagrams. A detailed justification for this crossed box contribution based on the eikonal approximation has been given in ref. [3]. The integral over $p_0$ can be evaluated easily by using Eq. (2.13) and expressing the propagators in positive and negative energy projections. This gives

$$S_{\text{ET}} = \frac{\Lambda^{++}}{\omega_1 + \omega_2 - E_1 - E_2} - \frac{\Lambda^{-+}}{\omega_1 + \omega_2 + E_1 - E_2} - \frac{\Lambda^{-+}}{\omega_1 + \omega_2 - E_1 + E_2} + \frac{\Lambda^{++}}{\omega_1 + \omega_2 + E_1 + E_2}.$$

(2.27)

The terms containing $\Lambda^{++}$ and $\Lambda^{-+}$ are well-known and stem from the $p_0$-integration over $S$. They were first used by Salpeter [7]. The other two terms result from the $p_0$-integration over $S_{\text{cross}}$. This form shows that the ET propagator is time-reversal invariant, similarly as the BSLT propagator. The most important difference between the BSLT and ET approximation is the degree to which the propagation of negative energy states is suppressed as compared to the propagation of the positive energy states. For particles of equal mass ($\omega_1 = \omega_2 = \omega$, and $E_1 = E_2 = E$), the ratio of the propagation for positive and negative states is for the ET case

$$\frac{S_{\text{ET}}^{++}}{S_{\text{ET}}} = \frac{\omega + E}{\omega - E},$$

(2.28)

whereas Eq. (2.15) gives for the BSLT case

$$\frac{S_{\text{BSLT}}^{++}}{S_{\text{BSLT}}} = \left( \frac{\omega + E}{\omega - E} \right)^2.$$

(2.29)

Therefore we can expect a larger admixture of negative states in the ET solutions than in the BSLT solutions. Note that this ratio is always positive for the BSLT propagator whereas it can be negative for the ET propagator. For the full BSE it is always positive at $p_0 = 0$.

Writing out the projection operators $\Lambda^{p_1 p_2}$ gives

$$S_{\text{ET}}^{-1} = \frac{1}{\omega_1} \left[ -\gamma^{(1)} \cdot p + m_1 \right] \left[ \gamma^{(2)} \cdot p - \gamma^{(2)}_0 E_2 + m_2 \right]$$

$$+ \frac{1}{\omega_2} \left[ -\gamma^{(1)} \cdot p - \gamma^{(1)}_0 E_1 + m_1 \right] \left[ \gamma^{(2)} \cdot p + m_2 \right]$$

(2.30)
The ET bound state wave equation $S_{ET}^{-\frac{1}{2}}\psi = -V\psi$ may now be transformed easily to configuration space and becomes
\[
\int dx' \left[ \frac{Z_1(x - x')}{m_1} S_1(x') + \frac{Z_2(x - x')}{m_2} S_2(x') \right] \psi(x') = -V(x)\psi(x),
\]
with
\[
S_1(x) = \left[ i\gamma_\mu (1) \cdot \nabla + m_1 \right] \left[ -i\gamma_\mu (2) \cdot \nabla - \gamma_\mu^2 E_2 + m_2 \right],
\]
\[
S_2(x) = \left[ i\gamma_\mu (1) \cdot \nabla - \gamma_\mu^2 E_1 + m_1 \right] \left[ -i\gamma_\mu (2) \cdot \nabla + m_2 \right],
\]
and the non-localities $Z_i$ have already been given in Eq. (2.22). The ET equation can be solved after projecting it on basis states with definite parity and total angular momentum $J$. The resulting eight coupled integro-differential equations can be found in appendix \[A2\].

### III. INTERACTION

It is commonly accepted that the interaction between the two quarks consists of a short-range part describing the one-gluon-exchange (OGE) potential and a infinitely rising long-range part responsible for the confinement of the quarks \[13\]. We use
\[
\alpha \rightarrow \frac{\alpha(x)}{x} \gamma_\mu (1) \cdot \gamma_\mu (2) + (\kappa x + c) \left[ (1 - \varepsilon)\gamma_\mu (1) \cdot \gamma_\mu (2) + \varepsilon \gamma_\mu^2 \right].
\]
(The color factor 4/3 of the expectation value of the OGE in the meson color wave function has been absorbed in the definition of $\alpha$.) The OGE potential is a pure vector interaction. The confining potential is commonly believed to be purely scalar. However, we choose a confining potential which is mainly scalar, but it can have a fraction $\varepsilon$ of vector confinement. Asymptotic freedom requires that for short distances the OGE coupling constant decreases logarithmically as $\alpha(x) \sim \alpha_0 / \ln(x_0 / x)$ where $\alpha_0 = 8\pi/(33 - 2n_F)$ and $x_0 = e^{-\gamma}/\Lambda_{QCD}$ \[13\]. Richardson \[14\] has given an elegant prescription for a running coupling constant which for small distances reproduces the correct asymptotic freedom and which gives for large distances a linearly rising potential. We do not use this prescription since it can not specify the vector and scalar character of the potential. At large distances the confining interaction dominates and the exact behavior of $\alpha(x)$ becomes of little importance. If one assumes that the coupling constant grows to some saturation value $\alpha_{sat}$ then a smooth interpolation between the short and long range is given by \[13\]
\[
\alpha(x) = \alpha_0 \ln^{-1} \left[ \frac{x_0}{x} \exp(-\mu x) + \exp \left( \frac{\alpha_0}{\alpha_{sat}} \right) \right].
\]
Here the typical range of the running coupling regime is controlled by the parameter $\mu$; the maximum range is found at $\mu = 0$. A longer range can be obtained by interpolating
\[
\alpha(x) = \alpha_0 \left[ \ln \left( \frac{x_0 + x}{x} \right) + \frac{\alpha_0}{\alpha_{sat}} \right]^{-1}.
\]
We use $\Lambda_{QCD} = 0.2$ GeV and $n_F = 3$. The dependence on $\Lambda_{QCD}$ and $n_F$, which is not large, can be compensated for by modifying $\mu$ and $\alpha_{sat}$.

In the ET the solutions are independent of the choice of gauge for the vector part of the potential. This independence depends essentially on the presence of crossed diagrams in the interaction kernel \[16\]. However, in the BSLT and ET approximations the relative energy is fixed, so crossed diagrams do not occur and the gauge-independence of these quasi-potential equations must be broken. In order to estimate the gauge-dependence of our results we study two gauges. The first one is the Feynman gauge as used in Eq. (3.1). The other is the Coulomb, transverse or radiation gauge which is obtained by replacing
\[
\gamma_\mu (1) \cdot \gamma_\mu (2) \rightarrow \gamma_\mu (1) \cdot \gamma_\mu (2) + \frac{1}{2} \left[ \gamma^0 (1) \cdot \gamma^0 (2) - \frac{(\gamma^0 (1) \cdot x)(\gamma^0 (2) \cdot x)}{x^2} \right]
\]
in Eq. (3.1). Appendix \[A3\] gives the partial wave projections of these potentials.
The presence of the $1/x$ term in the potential has important consequences for the short distance behavior of the wave functions. This can be understood from the following simple picture. Consider the probability of the meson decaying through the annihilation of the quark and the antiquark. This process is proportional to the wave function at zero relative distance, or equivalently, to the wave function integrated over all relative momenta. Using the wave equation the integral over the wave function can be expressed as a $\int dp\psi(p) = -\int dpS(p) \int dqV(p-q)\psi(q)$, that is, a loop integral over the propagator $S$ and potential $V$ plus additional corrections. This is illustrated diagrammatically in Fig. 4. Let us focus on the ultra-violet (uv) behavior of the momentum integration over the triangle diagram. It is divergent for most dynamical models due to the OGE potential. For example, in the BSE the two-fermion propagator and the OGE interaction both fall off as $p^{-2}$ for large $p$ (we neglect for the moment the additional $\ln p^2$ behavior of the running coupling constant). So for large relative momenta the loop integral takes the uv-divergent form $\int d^4p p^{-4}$. Similarly, in the one-body Dirac equation and in the two-body quasi-potential equations the propagators behave as $p^{-1}$ and the potential as $p^{-2}$ what also leads to a uv-divergent loop $\int d^4p p^{-3}$. Divergences also occur in the light-cone formalism where the propagator and potential go as $p_x^{-2}$ and the two quark spinors as $p_x^2$ resulting in $\int d^2p_\perp p_x^{-2}$. This suggests that the solutions of these wave equations are singular at the origin. Indeed, at short distances the Dirac wave functions 28 and the Bethe-Salpeter wave functions [19, 20] behave like $\psi(x) \sim x^\gamma$, $-1 < \gamma < 0$, where $\gamma$ is a decreasing function of the coupling constant $\alpha$ of the interaction. At some maximum value of $\alpha$ the exponent $\gamma$ becomes smaller than $-1$ and physical acceptable solutions do no longer exist. These features are also shared by the BSLT and ET equations.

The extent to which these singularities actually appear in the physical wave functions is weakened, if the fixed coupling constant is replaced by a running coupling constant such that $\alpha(p) \sim \ln^{-1} p^2$ for $p \to \infty$. The triangle loop, however, is still uv divergent but the short distance behavior is like $\psi(x) \sim \ln^2 |x|$. Furthermore, the singular behavior can be removed by renormalizing the wave function by means of some cut-off scale. For example, this has been done in the light-cone calculations of Ref. [17]. Yet, the mathematical implications are important. Seemingly no attention has been given to the asymptotic behavior in most work on relativistic quark-quark dynamics such as in Refs. [21, 22]. In appendix B we analyze in detail the singular behavior of the wave functions of the $J^P = 0^-$ states (e.g. the pion) in the BSLT equation with a fixed coupling constant. We have already discussed the short distance behavior for the ET equation in Ref. 3.

IV. PARAMETER DEPENDENCE

In this section we describe the calculational procedure to solve the quasi-potential equations and we present some results on the parameter dependence of the spectrum of $J^{PC} = 0^{-+}$ states. For convenience we refer to these states as the $^1S_0$ states since their wave functions contain mostly $^1S_0$ components. Similarly, other $J^{PC}$ states are also named by their main $^{2S+1}L_J$ components. Numerically stable solutions are obtained by taking explicitly into account the singularity in the bound state wave equation at $x = 0$ due to the presence of the OGE term. In appendix B the behavior of the wave function of the $^1S_0$ state is analyzed in detail for small $x$. We find that the most singular component of it behaves as $\psi \sim x^{2\gamma}$, $-1 < \gamma < 0$, and $\gamma$ given by Eqs. [37], [39], [110] or [112]. Let us introduce a function $f_{\gamma}(x)$ which for small arguments behaves as $x^{2\gamma}$ and for large arguments becomes unity. By substituting $\psi(x) = f_{\gamma}(x)\varphi(x)$ we get an equation for $\varphi(x)$ which is regular at $x = 0$. For the non $^1S_0$ states, no special precautions have to be taken, because the wave function for the coupling constant strengths considered in this work vanishes sufficiently fast in the origin. Numerically accurate solutions are found by using the same $x^{2\gamma}$ factor as for $^1S_0$ channel. The same applies when we have a running coupling constant, since the equations in this case are less singular than those with a fixed coupling constant.

The partial wave projection of the BSLT and ET equations leads to the coupled set of differential-integral equations [13] and [20] for $n$ ($3 \leq n \leq 8$) components $\psi_i(x)$, which are expanded as $\psi_i(x) = f_i(x) \sum_{j=1}^k c_{ij} S_j(x)$. The $S_j(x)$ are cubic Hermite spline functions (see e.g. Ref. [23]). The $\psi_i(x)$ are cut off at some maximum value $x_{\text{max}}$ and we impose $\psi_i(x_{\text{max}}) = \psi'_i(x_{\text{max}}) = 0$. Near $x = 0$ $\psi_i(x)$ is forced to be of order $x^{2\gamma}$ or higher. By evaluating the equations [13] or [20] at $k$ fixed points $x_1, ..., x_k$ one obtains a set of $k \times n$ linear equations for the $k \times n$ spline coefficients $c_{ij}$, which only admits non-trivial solutions at the bound state energies. It should be noted that due to the multiplication of the BSLT equation by the operator $D$ of Eq. [10] a continuum of additional solutions have been introduced in Eq. [12] for $M > m_1 + m_2$, which are not present in the original BSLT equation. The cut-off makes these continuum unphysical solutions into a discrete set, thus rendering it possible to isolate and reject them. This is illustrated in Figs. 2, and b which show two $^1S_0$ solutions of the BSLT equation with $m = 0.2$ GeV, $\kappa = 0.2$ GeV$^2$, $\epsilon = \alpha = c = 0$. The wave function shown in 2a is the ground state found at $M = 1.23$ GeV; the solution shown in 2b found at $M = 1.30$ GeV can easily be identified as an outgoing wave. As a result the latter solution should be rejected. The masses found for the physical bound states are insensitive to variations of $x_{\text{max}}$, whereas the masses of the continuum
solutions do depend on \(x_{\text{max}}\). Typical values that we used are \(x_{\text{max}} = 2\) fm and \(k = 30\) for the heavy quark systems up to \(x_{\text{max}} = 5\) fm and \(k = 120\) for the pion system. For the latter system a large number of splines is needed to obtain good accuracy since its wave function can have a long oscillating tail. As an overall check on the method the configuration space program was also used to calculate the spectrum for a non-confining potential containing scalar and vector exchanges. The resulting masses were verified by solving the BSLT equation in momentum space \([24]\); agreement was found within 0.1%.

Let us now consider some important differences between the non-relativistic and the relativistic equations. In the conventional non-relativistic models as discussed for example in Ref. \([1]\) the hyperfine interaction resulting from the OGE contribution is singular near the origin and as result it has to be regularized by introducing a phenomenological cutoff. On the other hand, our relativistic equations are well defined, at least up to a critical coupling constant of the OGE interaction. In this case a natural cutoff essentially occurs due to relativity, where the scale is given by the mass of the constituent quarks. The singular behavior of the wave function at \(x = 0\) induced by the Coulomb-like interaction in the relativistic equations considerably modifies the meson mass if the the coupling constant \(\alpha\) is close to its maximum value. Fig. 3 shows the masses of a mostly non-relativistic \(^1\!S_0\) system \((m = 5\) GeV and \(\kappa = 0.2\) GeV\(^2\)) as a function of the coupling constant. For small \(\alpha\) the spectrum agrees well with the Schrödinger result. However, if the limit \(\alpha \to \alpha_{\text{max}}\) is taken, the relativistic results differ considerably; \(dM(\alpha)/d\alpha \to -\infty\) and \(M(\alpha) \to M_0\), where generally \(M_0\) is not equal to zero.

Furthermore, for the non-relativistic Schrödinger-like equations an additional constant \(c\) in the potential causes a shift in the meson mass spectrum. This does not happen in the relativistic case. Because the quasi-potential equations are time-reversal invariant each solution for a meson at some positive mass is accompanied by its anti-meson solution at the same negative mass. If the constant \(c\) is added to the potential and we let \(c\) grow to negative values, the absolute mass of both solutions decrease until both mesons become massless. At this \(c\) the solutions coincide and for more negative \(c\) no bound state can be found except for higher excitations. This can clearly be seen from Fig. 6 where the \(^1\!S_0\) spectrum is plotted as a function of \(c\) for the BSLT and ET equations.

The replacement of the non-relativistic kinetic energy \(p^2/2m\) by the relativistic expression \(\sqrt{p^2 + m^2}\) and the introduction of the negative energy states greatly reduces the spacings between the ground state solution and the successive excitations for small quark masses. Fig. 5 shows the calculated \(^1\!S_0\) mass spectrum as a function of the quark mass with the confinement taken such that \(\kappa/m^2\) is fixed. This confinement gives for the Schrödinger equation constant binding energies. The figure illustrates nicely that for large masses the non-relativistic results are obtained, whereas for small masses the level density becomes considerably higher.

Finally, another aspect which is absent in non-relativistic models are the solutions describing a heavy quark and a light quark-hole. Their presence is a consequence of the requirement that the one-body Dirac equation is obtained in the limit that one particle becomes very heavy, and they can be interpreted in an analogous way as the negative energy states in the single fermion Dirac hole theory. Fig. 4 shows the \(^1\!S_0\) mass spectrum for fixed mass of the first quark and various masses of the second quark. At equal masses the hole-quark states are not present, but as \(m_2\) grows they emerge from the zero total mass axis, and at large \(m_2\) two spectra symmetrical around \(M = \pm m_2\) are generated.

V. LONG DISTANCE BEHAVIOR AND REGGE-TRAJECTORIES

Experimentally it is found that the masses of the light mesons lie on a linear Regge-trajectory, that is, for large angular momenta \(J\) the squares of the masses \(M\) of the mesons are proportional to their angular momenta

\[
M^2 = \beta J + c,
\]

where the Regge-slope \(\beta \simeq 1.2\) GeV\(^2\). The value of the slope depends almost exclusively on the confinement strength and can be used to fix it. Figs. 7 and 8 show the Regge behavior as obtained from the numerical solutions of the BSLT and ET equations using \(m_q = 0.25\) GeV, \(\alpha = 0, c = -1.0\) GeV and a linear scalar confining potential with \(\kappa = 0.33\) GeV\(^2\). The figures indeed show a linear relation between \(M^2\) and \(J\). However, the Regge-slope is considerably smaller than predicted by non-relativistic models with energy operator \(\sqrt{p^2 + m^2}\), where \(\beta = 8\kappa\); these models give a good description of the \(c\bar{c}\) and \(b\bar{b}\) systems when \(\kappa \simeq 0.18\) GeV\(^2\) (see e.g. \([3]\)). It turns out that a small admixture of vector confinement \(\varepsilon\), as in Eq. \((\varepsilon\)), greatly affects the Regge-slopes. Figs. 6 and 8 also show the Regge behavior for a fraction \(\varepsilon = 0.15\) of vector confinement in the Feynman gauge and in the Coulomb gauge. All trajectories are increased except for the spin-singlet trajectory in the ET model in the Feynman gauge which is decreased. Table 1 summarizes the resulting Regge-slopes.

A vector admixture in the confining interaction introduces a spin dependence through the presence of the \(|++\rangle\), \(|+-\rangle\) and \(|--\rangle\) components. Because of this when too much vector confinement is chosen, \(\varepsilon > 0.2\), some light mesons are no
longer bound, i.e. certain channels may become deconfined. This can be used to set an upper bound on the degree of allowed vector admixture. Consider the strength of the confining potential in the $^1J_f^3$ channel in the Feynman gauge. From Eq. (A22) we see that the projection of $\gamma_{\mu}^{(1)}\gamma_{\mu}^{(2)}$ in this channel is $-4$, so the linear part of the $q\bar{q}$-interaction in this channel is

$$\kappa x \left[ (1-\varepsilon)\gamma_{\mu}^{(1)}\gamma_{\mu}^{(2)} + \varepsilon\gamma_{\mu}^{(1)}\gamma_{\mu}^{(2)} \right] \frac{1}{x} \kappa x \left[ (1-\varepsilon) - 4\varepsilon \right].$$

Hence if $\varepsilon > 0.2$ the potential becomes repulsive for large distances; the interaction tends to $-\infty$ for $x \to \infty$, which is clearly not physically admissable. This suggests that the $P = (-)^J$ mesons of the light quark system, of which the wave functions contain the $^1J_f^3$ channel and have $|++\rangle$ components of size comparable to the $|++\rangle$ components, do not have a bound state solution if $\varepsilon > 0.2$. This is confirmed by the numerical solutions. In the Coulomb gauge a slightly larger value is allowed, namely $\varepsilon = 0.25$. In order to avoid these problems we demand that the admixture of the vector interaction in the confining potential does not reach or exceed these values.

It is interesting to note that also in the one-body Dirac equation, which can be used to describe a system of a very heavy and a light quark a similar problem arise. Such a framework has recently been used to study the spectrum of the $D$ and $B$ mesons [2]. A pure scalar confinement interaction was used. The repulsion exercised by the vector confinement on the negative components gives also in this case an upper limit on the allowed degree of vector admixture in the confining potential. Consider the Dirac equation

$$(m - i \not{\partial})\psi(x) = -\kappa x [(1-\varepsilon) + \varepsilon\gamma_0]\psi(x).$$

Projection upon states of definite $J$ gives

$$
\begin{pmatrix}
  m - E & D_{J+3/2} \\
  D_{-J+1/2} & m + E
\end{pmatrix}
\begin{pmatrix}
  \psi(x)
\end{pmatrix} = -\kappa x
\begin{pmatrix}
  1 & 0 \\
  0 & 1 - 2\varepsilon
\end{pmatrix}
\begin{pmatrix}
  \psi(x)
\end{pmatrix},
$$

and $D_{J}$ as defined in Eq. (A3). Eliminating the lower component of $\psi(x)$ in favor of the upper component of $\psi_1(x) = F(x)/x$ and letting $x \to \infty$ leads to

$$\left\{ -\frac{\partial^2}{\partial x^2} + \frac{1}{x} \frac{\partial}{\partial x} + \frac{1}{x^2}(J-3/2)(J+1/2) \right\}(1 - 2\varepsilon)\kappa x^2 F(x) = 0.$$  

For $\varepsilon > 0.5$ the potential term becomes negative and the solutions for $F(x)$ are no longer bound.

VI. MESON MASS SPECTRUM

In Table 1 the parameters are listed that we used to calculate the meson mass spectrum. We did not include the light isoscalar mesons ($\eta$, $f$, $h$). The masses of these mesons are expected to be modified by the process of $q\bar{q}$ annihilation and creation, making the transitions $u\bar{u} \leftrightarrow d\bar{d} \leftrightarrow s\bar{s}$ likely.

We determine the strength of the confinement from the condition that the experimental Regge-slope $\beta \simeq 1.2$ GeV$^2$ is found. If the confinement is taken purely scalar then a confinement is needed which is much stronger than the strength known from non-relativistic models to give a good description of the $c\bar{c}$ and $b\bar{b}$ systems. Therefore the maximum of allowed vector admixture is chosen which gives a maximum Regge-slope at a fixed confinement strength. So in the BSLT model we take $\varepsilon = 0.2$ and $\varepsilon = 0.25$ in the Feynman gauge and in the Coulomb gauge, respectively. With these values the experimental Regge-slopes are found at $\kappa \simeq 0.33$ GeV$^2$. Although this strong confinement makes it difficult to get a perfect fit for the heavy quark systems we need it in order to get an acceptable light meson description. As can be seen from Table 2 and Figs. 3 and 4 an increase of vector confinement in the Feynman gauge in the ET model gives an increase in the Regge-slopes at the spin-triplet states but a decrease at the spin-singlet states. It is therefore impossible in the ET model in the Feynman gauge to obtain simultaneously a fair description of the heavy mesons and of the Regge-slopes of the light mesons, so for the fit of the meson mass spectrum we only used the ET model in the Coulomb gauge. For this last case we took the value $\varepsilon = 0.2$ which is slightly less than the maximum $\varepsilon = 0.25$; above $\varepsilon \simeq 0.2$ the spatial extension of the $^1J_f^3$ components becomes more than $\sim 10$ fm which is unphysically large.

For the OGE part of the potential the choice of the saturation value $\alpha_{sat}$ of the running coupling constant is not important since it only plays a role at large distances were the confining part dominates. Fair fits can be found with $\alpha_{sat} = 0.6$ as well with $\alpha_{sat} = 1.0$. We choose the commonly accepted value $\alpha_{sat} = 0.8$. The typical range of the running coupling constant regime is much more important. In the Coulomb gauge the maximum fixed coupling
constants allowed in the BSLT and in ET model are respectively $\alpha_{max} = 8/(3\pi) \simeq 0.85$ and $\alpha_{max} = 4/(3\pi) \simeq 0.42$ [see Eqs. (B11) and (B12)]. If the running coupling constant regime is small and the saturation value is approximately equal or larger than the maximum allowed fixed coupling constant, $\alpha_{sat} \geq \alpha_{max}$, then at medium small distances the wave function exhibits the singular behavior of the fixed coupling constant equations. Especially, the spectrum shows much lower energies for the $^1S_0$ states than the non-relativistic spectrum. Since $\alpha_{max}$ is rather low in the ET model this effect occurs rather strongly in the ET spectrum, and one must choose a much larger running coupling regime for the ET equations than for the BSLT equations. So we use for the BSLT model the coupling $\alpha_I$ of Eq. (3.2) and for the ET model the coupling $\alpha_{II}$ of Eq. (5.2). Yet, even with this choice the $^1S_0$ states are considerably lower in the ET fit than in the BSLT fits.

The masses of the u and d quarks are taken equal and chosen together with the constant $c$ to give a fair description of the light non-strange mesons. Next the masses of the s, c, and b quarks are fitted to the $^3S_1$ states $K^*$, $J/\psi$ and $\Upsilon$, except for the c mass in the ET model where a correct effect occurs rather strongly in the ET spectrum, and one must choose a much larger running coupling regime than for the BSLT equations. So we use for the ET model a coupling $\alpha_I$ of Eq. (3.2) and for the BSLT model the coupling $\alpha$ for the ET equations than for the BSLT equations. Therefore the spectrum shows much lower energies for the $^1S_0$ states than the non-relativistic spectrum. Since $\alpha_{max}$ is rather low in the ET model this effect occurs rather strongly in the ET spectrum, and one must choose a much larger running coupling regime for the ET equations than for the BSLT equations. So we use for the BSLT model the coupling $\alpha_I$ of Eq. (3.2) and for the ET model the coupling $\alpha_{II}$ of Eq. (5.2). Yet, even with this choice the $^1S_0$ states are considerably lower in the ET fit than in the BSLT fits.

Table III presents the resulting mass spectra of the BSLT approach in both the Feynman and in the Coulomb gauge and of the ET approach in the Coulomb gauge, together with the known experimental mass spectrum. As can be seen from these numbers the BSLT approach in the Feynman gauge gives the best description, but the differences between the BSLT results obtained from the Feynman gauge and those from the Coulomb gauge are minor. The ET model does not give a very good spectrum.

Let us discuss the spectra in more detail. The ground states of the $d\bar{d}$ and $d\bar{s}$ systems, the $\pi$ and $K$ mesons, are considerably lighter than the BSLT fits predict. The light masses of these mesons are commonly explained within in the framework of broken chiral symmetry where these mesons correspond to almost massless Goldstone bosons. Since the quasi-potential equations do not incorporate the chiral symmetry they give too heavy $\pi$ and $K$. The correct mass of the $\pi$ in the ET is a coincidence and due to the too singular behavior of the $^1S_0$ states in the ET model.

The description of the $^1P_1$-states in the ET model is not good. According to the Breit-reduction of the potential, the mass of this state should be equal to the center of gravity of the $^3P$ states up to order $1/m^2$. Clearly none of the $^1P_1$-states in the ET model satisfy this condition, in contrast to the BSLT model where this condition is more or less satisfied. Furthermore the splittings between the various $^3S_1$ and $^3D_1$ states, especially in the $d\bar{d}$ and $d\bar{s}$ system, are much too small in the ET model. We conclude that the ET model does not give the correct finite structure. At this point we would like to note that within a constituent quark model with flavor independent potential it is not possible to have simultaneously a correct description of the $P$-states in the $d\bar{d}$ and $d\bar{s}$ systems. The only difference between these systems is that one of the constituent quarks is a little heavier, so it can be expected that the levels of the $d\bar{s}$ system are raised a little as compared to the levels of the $d\bar{d}$ system, with a little smaller fine-structure splitting. But the observed spacings between the $P$-states of the $d\bar{d}$ and $d\bar{s}$ mesons do not follow this pattern.

Considerable deviations between the experimental masses and the calculated masses appear in the excited $P$-states of the $b\bar{b}$ system. This is due to the large confinement strength $\kappa$ that was chosen in order to have proper Regge-slopes for the light mesons. We have searched for various simple prescriptions to improve on the shape of the potential in such a way that the masses of the excited $P$-states of the bottomonium system show better agreement. The large confining potential needed to get the proper Regge-slope for the light mesons and the weaker confinement needed for the heavy excited $P$-states in bottomonium clearly suggest that maybe a better overall description can be found if one takes a smaller $\kappa$ at short distances than at large distances. We examined this possibility by modifying the potential at short distances according to

$$V_{conf}(x) \rightarrow \tilde{V}_{conf}(x) = \begin{cases} \kappa'x + c' + Ax^2 & x < x_0, \\ \kappa x + c & x > x_0, \end{cases}$$

with the two regions smoothly matching, $\kappa' < \kappa$, $A > 0$ and various character (vector or scalar) for $\tilde{V}_{conf} - V_{conf}$. We found that it was indeed possible to get the right Regge-slopes for the light mesons and simultaneously the $b\bar{b}$ excitations at the correct levels. But this also increased the fine-structure splittings in the non-strange $P$-states up to values as 0.3 GeV for the difference $M(a_2) - M(a_1)$ which is known to be only 0.051 GeV. Furthermore, the decrease of the splitting $M(2^3P_1) - M(1^3S_1)$ in $b\bar{b}$ causes almost a similar reduction of the shift $M(1^3P_1) - M(1^3S_1)$ in $\bar{c}c$ which is undesirable. These two differences depend more or less in the same way on the potential since the excited levels $2^3P_1$ of $b\bar{b}$ and $1^3P_1$ of $\bar{c}c$ both have approximately the same spatial extension.

VII. CONCLUDING REMARKS

We studied two approximations to the Bethe-Salpeter equation for the two quark system, the Blankenbecler-Sugar-Logunov-Tavkhelidze (BSLT) equation and an equal-time (ET) equation. In these quasi-potential approximations
the relative energy dependence of the wave function was eliminated by assuming a simplified form for the two-body propagator. The full Dirac structure of positive and negative energy states was kept. Both two-body propagators reduce to the one-body Dirac propagator if one of the particles is taken infinitely heavy. We applied these equations to a system of two constituent quarks interacting through a phenomenological potential which consisted of two parts. The first part was a Coulomb-like one-gluon exchange (OGE) part, the second part was a linear confining potential which was taken mostly scalar-like and partly vector-like. For the vector part we studied both the Feynman and the Coulomb gauge. Since the confining potential is highly singular in momentum space we transformed the quasi-potential equations to configuration space.

It was shown that for a fixed coupling constant for the OGE potential a maximum value exists —depending on the model and gauge— above which the ground state solutions no longer exist. Also the admixture of vector character in the confining potential has a maximum value (20% and 25% in the Feynman and Coulomb gauge respectively) above which some mesons become unbound. The latter could be explained from the repulsion between positive and negative energy states. We found linear Regge-trajectories for both models and gauges, but their Regge-slopes were much smaller than predicted by models with only positive energy states.

Using the BSLT and ET equations we calculated the full known meson mass spectrum of all light and heavy mesons. As compared to the non-relativistic model predictions given the limited number of parameters used here, the fit can be considered satisfactory. We believe that our prediction of the spectrum is of comparable or better quality than other relativistic studies based on quasi-potential and Dirac equations. We found only a small gauge dependence in the BSLT spectrum. For the ET equation we only used the Coulomb gauge since it was impossible to get satisfying Regge-slopes in the Feynman gauge. The fine-structure of the spectrum, such as the spacings between the $P$-states and between the $^3S_1$ and $^3D_1$ states, as calculated from the ET equations did not follow the experimental spacings nor did it follow predictions of the Breit-reduction of the potential. The BSLT fine-structure showed more or less agreement with these. However, from the agreement that was found between the calculated meson mass spectrum and the experimental spectrum we conclude that for a calculation able to reproduce all meson masses within $\sim 0.03$ GeV one cannot suffice with a constituent quark model and a phenomenological flavor independent potential. Specifically, one would like to incorporate chiral symmetry in order to reproduce the correct $\pi$ and $K$ mass and some flavor dependence in the confinement strength in order to reduce this strength in the heavy quark systems.

The wave functions found in these calculations can in principle be used to perform relativistic covariant calculations on interactions between mesons and other particles. Especially at high momenta significant deviations from the non-relativistic calculations can be expected; one example has been discussed in Ref. [2].

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APPENDIX A: PARTIAL WAVE PROJECTION

In this appendix we describe the projection of the potential and the BSLT and ET equations on sets of basis states with definite parity and total angular momentum. The equations and basis states are formulated for a two quark system; they can easily be transformed to a quark-antiquark system by performing a charge-conjugation on one of the quarks. The complete meson wave function is written as a combination of the states $|JLSJ_z\rangle \otimes |\rho_1^{(1)} \rho_2^{(2)}\rangle \otimes |\chi_{color}\rangle \otimes |\chi_{flavor}\rangle$. The color and flavor parts of the wave function are well-known and can be found e.g. in Ref. [1]. For the $\rho$-spin basis states we do not use the energy eigenstates projected out by the $\Delta^{\rho_1 \rho_2}$, but instead the four eigenstates of $\gamma_0^{(1)} \gamma_0^{(2)}$, corresponding to the four combinations of upper and lower components of the quark spinors. The energy eigenstates have the advantage that the propagator is simply diagonal on this basis as is shown by Eqs. (2.13) and (2.27). However, on this basis the matrix elements of the potential are complicated and many in number. In principle it is possible to transform them all to configuration space, but the numerical work involved is huge. On the other hand, on the basis of the eigenstates of $\gamma_0^{(1)} \gamma_0^{(2)}$ the matrix elements of the potential are almost trivial, whereas the projection of the propagator on this set gives only a limited number of functions to be transformed. For the BSLT propagator there is only one such function namely $Z_{BSLT}$ (2.23), whereas for the ET propagator two functions $Z_1$ and $Z_2$ (2.22) are needed.
1. Blankenbecler-Sugar-Logunov-Tavkhelidze equation

We decompose the kinetic part of the BSLT propagator into pieces acting in the subspaces of the spin and rho-spin:

\[ \tilde{p}_1 + \tilde{p}_2 = i \gamma^{(1)} \cdot \nabla + E_{1(0)}(1) - i \gamma^{(2)} \cdot \nabla + E_{2(0)}^{(2)} \]

\[ = -\rho_2^{(1)} \sigma^{(1)} \cdot \nabla + E_1 \rho_1^{(1)} + \rho_2^{(2)} \sigma^{(2)} \cdot \nabla + E_2 \rho_3 \]

(A1)

For a given total spin \( J \) there are four basis states for the spin subspace. In the spectroscopic notation \( 2S+1L_J \) they are \( 1^1J_J, 1^3J_J, 1^3(J-1)_J, 1^3(J+1)_J \). After some Clebsch-Gordan algebra one finds that on this basis

\[ \sigma^{(1)} \cdot \nabla = \begin{pmatrix} 0 & 0 & c_1 D_{J+1} & -c_2 D_{J+2} \\ 0 & 0 & -c_2 D_{J+1} & c_1 D_{J+2} \\ -c_2 D_{J-1} & c_1 D_{J-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]

(A2)

where

\[ D_{\ell} = \frac{\partial}{\partial x} + \frac{\ell}{x}, \quad c_1 = \sqrt{\frac{J}{2J+1}}, \quad c_2 = \sqrt{\frac{J+1}{2J+1}}. \]

(A3)

The matrix of \( \sigma^{(2)} \cdot \nabla \) is identical, except for the first row and first column — these correspond to the antisymmetric spin singlet state — which change sign. Let us combine the four eigenstates of \( \gamma^{(1)}_0, \gamma^{(2)}_0 \) into the rho-spin combinations

\[ s = \frac{+1-1}{\sqrt{2}}, \quad a = \frac{+1+1}{\sqrt{2}}, \quad e = \frac{-1+1}{\sqrt{2}}, \quad o = \frac{-1-1}{\sqrt{2}}. \]

(A4)

Phase factors \( i \) between the eigenstates are used according to

\[ |++\rangle = -i |p^{(1)}_1 \rangle = -i |p^{(2)}_1 \rangle, \quad -|--\rangle = |p^{(1)}_1 \rangle, \quad --\rangle = -|--\rangle. \]

(A5)

This ensures that the final equations will be real. Following Kubis [30] and Gamme [31] we combine the spin and rho-spin bases to form the basis set \( \mathcal{B} \) of eight states with unnatural parity \( P = (-)^{J+1} \)

\[ \mathcal{B} = \{ 1^1J_J, 1^3J_J, 1^3J_J, 1^3J_J, 1^3J_J, 1^3J_J, 1^3J_J, 1^3J_J \}, \]

(A6)

and the basis set \( \mathcal{B}^* \) of eight states with natural parity \( P = (-)^{J} \)

\[ \mathcal{B}^* = \{ 1^1J_J, 1^3J_J, 1^3J_J, 1^3J_J, 1^3J_J, 1^3J_J, 1^3J_J, 1^3J_J \}. \]

(A7)

These basis states for two quarks can be related to the often used Dirac matrix set for a quark and an anti-quark [32] by performing a charge conjugation on one of the spinors. The relation is shown in Table IV. Let us expand the expression for \( \tilde{p}_1 + \tilde{p}_2 \) on the set \( \mathcal{B} \) of unnatural parities and denote the resulting \( 8 \times 8 \) matrix by \( \mathbf{D}(x) \). This gives

\[ \mathbf{D} = \begin{pmatrix} 0 & E & 0 & 0 & -c_1 D_{J+1} & 0 & 0 & c_2 D_{J+2} \\ E & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 & 0 & 0 \\ -c_1 D_{J+1} & 0 & 0 & 0 & -c_2 D_{J+1} & 0 & 0 & c_1 D_{J+2} \\ 0 & 0 & -c_2 D_{J+1} & 0 & 0 & 0 & 0 & 0 \\ c_2 D_{J-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c_1 D_{J-1} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \]

(A8)

with \( 2E = E_1 + E_2 = M \) and \( 2\Delta = E_1 - E_2 = (m_1^2 - m_2^2)/M \). The matrix \( \mathbf{D}^* \) of \( \tilde{p}_1 + \tilde{p}_2 \) for the set \( \mathcal{B}^* \) of natural parity states can be obtained from \( \mathbf{D} \) by interchanging \( E \leftrightarrow \Delta \). The complete partial wave projected BSLT equation is the following set of eight coupled integro-differential equations

\[ [\mathbf{D}(x) - M] \psi(x) = \frac{1}{2(m_1 + m_2)} \int dx' x'^2 Z_{BSLT}(x, x') [\mathbf{D}(x') + M] V(x') \psi(x'). \]

(A9)
The number of coupled channels becomes less if the quarks have equal masses. In that case $\Delta = 0$ and the matrix $D$ for the unnatural parity $P = (-)^{J+1}$ states falls apart into two $4 \times 4$ parts describing the charge-parity $C = (-)^J$ states and the $C = (-)^{J+1}$ states

\[
\{1J^e, J^e_1, 3(J-1)^e, 3(J+1)^e\} \quad \text{and} \quad \{3J^e, J^e_1, 3(J-1)^e, 3(J+1)^e\},
\]

respectively. If the masses are equal the set of natural parity $P = (-)^J$ states can be split into the $C = (-)^J$ states and the $C = (-)^{J+1}$ states

\[
\{1J^e, 3J^e_1, 3(J-1)^e, 3(J+1)^e\} \quad \text{and} \quad \{1J^e_1, 3J^e\},
\]

respectively. The latter $C = (-)^{J+1}$ set contains no $|++\rangle$ components and therefore it has no non-relativistic analog. However since Eq. (A9) is independent of $E$ for this case, no bound states exist in this sector.

2. Equal-time approximation

As with the BSLT propagator, the parts $S_1(x)$ and $S_2(x)$ in Eqs. (2.32) and (2.33) of the ET propagator can be projected on the partial wave bases. One finds for the set $B$ (A6) of unnatural parity states

\[
S_1 = m_1m_2 I +
\]

\[
\begin{pmatrix}
\delta^2_j & -m_1 E_2 & 0 & 0 & c_1 A D_{-J+1} & c_1 E_2 D_{-J+1} & -c_2 A D_{J+2} & -c_2 E_2 D_{J+2}

-m_1 E_2 & -\delta^2_j & 0 & 0 & -c_1 E_2 D_{-J+1} & c_1 B D_{-J+1} & c_2 E_2 D_{J+2} & -c_2 B D_{J+2}

0 & 0 & -\delta^2_j & -m_1 E_2 & c_2 B D_{-J+1} & -c_2 E_2 D_{J+1} & c_2 A D_{J+1} & c_2 E_2 D_{J+2}

0 & 0 & 0 & -m_1 E_2 & c_1 B D_{J+1} & c_1 E_2 D_{J+1} & c_1 B D_{J+1} & c_1 E_2 D_{J+2}

c_1 E_2 D_{J+1} & -c_1 E_2 D_{J+1} & c_1 E_2 D_{J+1} & m_1 E_2 & 1/2J+1 \delta^2_j & 0 & 2c_1 c_2 H_2
nc_1 B D_{J+1} & c_2 B D_{J+1} & c_1 E_2 D_{J+1} & -2c_1 c_2 H_1 & 0 & 1/2J+1 \delta^2_j & m_1 E_2

-c_2 A D_{-J} & c_2 E_2 D_{-J} & c_1 B D_{-J} & c_1 E_2 D_{-J} & 0 & 2c_1 c_2 H_1
\end{pmatrix}
\]

with $D_\ell$, $c_1$, $c_2$ as defined in (A3), $A = m_1 + m_2$, $B = m_1 - m_2$, and

\[
H_1 = \frac{\partial^2}{\partial x^2} - \frac{2J+1}{x} \frac{\partial}{\partial x} - \frac{-J^2 + 1}{2x^2},
\]

\[
H_2 = \frac{\partial^2}{\partial x^2} + \frac{2J+3}{x} \frac{\partial}{\partial x} - \frac{J^2 - 2J}{2x^2},
\]

\[
\partial^2_\ell = \frac{\partial^2}{\partial x^2} + \frac{2 \partial}{x} \frac{\partial}{\partial x} - \frac{\ell(\ell + 1)}{2x^2}.
\]

The projection $S_1^*$ of $S_1$ for the set $B^*$ (A7) of natural parity states is given by

\[
S_1^* = T S_1 T, \quad T = \text{diag}(1, -1, 1, -1, 1, -1, 1).
\]

The projection $S_2$ of $S_2$ for the set $B$ can be expressed as

\[
S_2(m_1, m_2, E_1, E_2) = C S_1(m_2, m_1, E_2, E_1) C,
\]

with diagonal charge-parity matrix $C = \text{diag}(1, 1, -1, 1, -1, 1, -1, -1)$. And finally,
with charge-parity matrix $C^* = TC = \text{diag}(1, -1, -1, 1, 1, 1, 1, 1)$. The complete partial wave projected ET equation becomes the following set of eight coupled integro-differential equations

$$\int dx' x'^2 \left[ \frac{1}{m_1} Z_1(x, x') S_1(x') + \frac{1}{m_2} Z_2(x, x') S_2(x') \right] \psi(x') = -V(x) \psi(x). \quad (A20)$$

The diagonal matrices $Z_i$ have as $j$-th diagonal element the projection of $Z_i$ on the orbital angular momentum $\ell_j$ of the $j$-th basis state,

$$Z_i^{(j)}(x, x') = 2\pi \int_{-1}^{1} d\gamma P_\ell(\gamma) Z_i(\sqrt{x^2 + x'^2 - 2xx'\gamma}). \quad (A21)$$

### 3. Potential

The vector structure of the $q\bar{q}$-interaction gives

$$\gamma_\mu^{(1)} \gamma_\mu^{(2)} = \rho_3^{(1)} \rho_3^{(2)} + \rho_2^{(1)} \rho_2^{(2)} \sigma^{(1)} \cdot \sigma^{(2)} = \begin{pmatrix} 1 + f & 0 & 0 & 0 \\ 0 & 1 - f & 0 & 0 \\ 0 & 0 & -1 + f & 0 \\ 0 & 0 & 0 & -1 - f \end{pmatrix} \quad (A22)$$

on the basis \(A4\). Here $f = -3$ for spin singlet states and $f = 1$ for spin triplet states. Choosing the Coulomb gauge amounts to replacing

$$\sigma^{(1)} \cdot \sigma^{(2)} \rightarrow \frac{1}{2} \left( \sigma^{(1)} \cdot \sigma^{(2)} + \frac{(\sigma^{(1)} \cdot x)(\sigma^{(2)} \cdot x)}{x^2} \right). \quad (A23)$$

Thus in the Coulomb gauge the spin-spin interaction is reduced. The difference with the Feynman gauge can be expressed as

$$\frac{1}{2} \left[ \gamma^{(1)} \cdot \gamma^{(2)} - \left( \gamma^{(1)} \cdot x \right) \left( \gamma^{(2)} \cdot x \right) \frac{1}{x^2} \right] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_1^2 & 0 & c_1 c_2 & 0 & 0 \\ 0 & 0 & 0 & c_1 c_2 & 0 & -c_1 c_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1 c_2 & 0 & c_1 c_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c_1 c_2 & 0 & c_2^2 \end{pmatrix} \quad (A24)$$

both for the bases \(A8\) and \(A7\). The constants $c_i$ have been given in Eq. \(A3\).

### APPENDIX B: SHORT DISTANCE BEHAVIOR OF $^1S_0$ STATES

Here we discuss in detail the singular behavior of the wave function of the $J^P = 0^-$ states (e.g. pion), to which we refer to as $^1S_0$ states for convenience because their wave functions contain mostly $^1S_0$ components. At short distances, or equivalently at high momenta, the masses of the two quarks can be neglected, so the short distance behavior for the unequal quark mass case is identical to the equal quark mass case. Thus in this appendix only the equal mass case needs to be considered. Since the short distance behavior for the ET equation has already been discussed in Ref. \(3\) we concentrate on the BSLT approximation. Since $J = 0$ the $^3J - 1_j^J$ state is removed from the basis in \(A11\) and the wave function is decomposed as

$$\psi(x, \Omega) = \frac{1}{x} \left[ \psi_1(x) |^1S_0^+ > + \psi_2(x) |^1S_0^- > + \psi_3(x) |^3P_0^+ > \right]. \quad (B1)$$

In this basis the BSLT equation \(A9\) reduces to
\[
\begin{pmatrix}
-m & E & D_1 \\
E & -m & 0 \\
D_{-1} & 0 & -m
\end{pmatrix}
\begin{pmatrix}
\psi_1(x) \\
\psi_2(x) \\
\psi_3(x)
\end{pmatrix}
= \frac{1}{4m} \int_0^\infty dx' \, xx' \begin{pmatrix}
Z^{(0)}(x, x') & 0 & 0 \\
0 & Z^{(0)}(x, x') & 0 \\
0 & 0 & Z^{(1)}(x, x')
\end{pmatrix}
\begin{pmatrix}
\psi_1(x') \\
\psi_2(x') \\
\psi_3(x')
\end{pmatrix}.
\]

In this formula \( E = M/2 \) and \( V_V(x) \) and \( V_S(x) \) denote the vector and scalar part of the \( q\bar{q} \) potential. The Feynman gauge is assumed. We first consider the case of a fixed coupling constant \( \gamma \). Assuming that \( \psi_i(x) = c_i x^{\gamma_i} \) for \( x \to 0 \) one can reduce in leading order in \( x \) the BSLT equation to

\[
\begin{cases}
-m c_1 x^{\gamma_1} + E c_2 x^{\gamma_2} + c_3 (\gamma_3 + 1) x^{\gamma_3 - 1} & = -\frac{\alpha}{4} \left[ -2mc_1 x^{\gamma_1} I^{(0)}(\gamma_1) + 4E c_2 x^{\gamma_2} I^{(0)}(\gamma_2) \right] \\
E c_1 x^{\gamma_1} - m c_2 x^{\gamma_2} & = -\frac{\alpha}{4} \left[ -2E c_1 x^{\gamma_1} I^{(0)}(\gamma_1) + 4m c_2 x^{\gamma_2} I^{(0)}(\gamma_2) \right] \\
c_1 (\gamma_1 - 1) x^{\gamma_1 - 1} - m c_3 x^{\gamma_3} & = -\frac{\alpha}{4} \left[ -2c_1 (\gamma_1 - 2) x^{\gamma_1 - 1} I^{(1)}(\gamma_1 - 1) \right],
\end{cases}
\]

where the \( I^{(\ell)} \) follow from

\[
\frac{1}{m} \int_0^\infty dx' \, xx' \, Z^{(\ell)}(x, x') \, x^{\gamma - 1} = x^{\gamma} I^{(\ell)}(\gamma),
\]

which is well-defined if \( x \to 0 \) and \( |\gamma| < 1 \). Assuming that Eq. (B3) does not solely determine the energy eigenvalue \( E \), we may exclude the case that \( \gamma_1 = \gamma_2 \). As a result we find that the exponents satisfy \( \gamma_2 < \gamma_1, \gamma_3 = \gamma_2 + 1 \) and \( \alpha^{-1} = I^{(0)}(\gamma_2) \). Taking the explicit form of

\[
Z^{(0)}(x, x') = \frac{m}{\pi x x'} \left[ K_0(m|x - x'|) - K_0(m|x + x'|) \right]
\]

we get

\[
I^{(0)}(\gamma) = \frac{1}{\pi} \int_0^1 \frac{d\eta}{\eta} \ln \left( \frac{1 + \eta}{1 - \eta} \right) (\eta^\gamma + \eta^{-\gamma}) = \frac{1}{\gamma} \tan \frac{\pi \gamma}{2}.
\]

Explicit expressions for general \( I^{(\ell)} \) are given in Ref. [8]. Hence

\[
\alpha = \frac{1}{I^{(0)}(\gamma_2)} \leq \frac{2}{\pi}.
\]

This may be compared with the asymptotic behavior of the BSE [19], where

\[
\gamma_2 = \gamma_3 - 1 = \sqrt{1 - \frac{4\alpha}{\pi}} \quad \to \quad \alpha \leq \frac{\pi}{4} \quad \text{(BSE)}
\]

\[
\gamma_1 = \gamma_4 = \sqrt{5 - 2 \sqrt{4 + \frac{\alpha}{\pi} \left( 1 + \frac{\alpha}{\pi} \right)}}. \quad \text{(BSE)}
\]

Here \( \gamma_4 \) is the exponent of the \( ^3P_0 \) component which does not decouple in the BSE. Note that the dependence of \( \gamma_2 \) on \( \alpha \) is of first order identical for the BSLT equation as for the BSE.

The ET uv behavior can be found in a similar way and was already discussed in Ref. [2]. Here again \( \gamma_2 < \gamma_1, \gamma_3 = \gamma_2 + 1 \) and

\[
\alpha = \frac{1}{2} \gamma_2 (1 - \gamma_2) I^{(0)}(1 - \gamma_2) \leq \frac{1}{\pi} \quad \text{(ET)}
\]
If the Feynman gauge is replaced by the Coulomb gauge then according to Eq. (A24) we have to replace the diagonal matrix \((-2, 4, 0)\) in Eq. (B2) by the diagonal matrix \((-1, 3, -1)\) and one gets for the BSLT and ET equations

\[
\alpha = \frac{4}{37(\gamma_2)} \leq \frac{8}{3\pi}, \quad \text{(BSLT)} \tag{B11}
\]

\[
\alpha = \frac{2}{3} \gamma_2 (1 - \gamma_2) I^{(0)} (1 - \gamma_2) \leq \frac{4}{3\pi}, \quad \text{(ET)} \tag{B12}
\]

So the short distance behavior is less singular in the Coulomb gauge than in the Feynman gauge. This can be understood from the suppression of the spin-spin interaction in the Coulomb gauge. The same singular behavior has been found by Murota \cite{8} for the Salpeter equation, which is obtained from the ET equation by removing the \(|+\rangle\) and \(|-\rangle\) components.

If the short distance attraction of the Coulomb-like potential is weakened by taking a running coupling constant \(\alpha(x) \sim \ln^{-1}(x)\) then the singular short distance behavior of the \(\psi_i\) is also weakened to \(\psi(x) = c_i |\ln(x)|^{\gamma_i}\). However, to construct numerically the solutions to the quasi-potentials equations it is not necessary to know the expressions for the \(\gamma_i\). The equations can be solved by assuming the singular behavior associated with a fixed coupling constant which is equal to the running coupling constant at a typical small distance.

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\end{array}\]
FIG. 1. The meson wave function at zero relative distance written as a loop integral over the quark (quasi-) propagators and the potential, with additional corrections. If this loop integral is divergent, then the wave function is divergent at $x = 0$.

FIG. 2. A physical bound state $^1S_0$ solution (a) at $M = 1.23$ GeV and an unphysical continuum $^1S_0$ solution (b) at $M = 1.30$ GeV obtained from the BSLT equation, with $m = 0.2$ GeV, $\kappa = 0.2$ GeV$^2$, $\alpha = c = \epsilon = 0$.

FIG. 3. The masses of the ground state and the first few excitations of the $^1S_0$ systems are shown as a function of the fixed coupling constant $\alpha$ (calculated with $m = 5$ GeV, $\kappa = 0.2$ GeV$^2$, $\epsilon = \alpha = 0$). The solid lines are for BSLT, broken lines for ET, and dotted for Schrödinger.

FIG. 4. The masses of the ground state and the first few excitations of the $^1S_0$ system are shown as a function of the constant term $c$ in the $q\bar{q}$ potential (calculated with $m = 0.2$ GeV, $\kappa = 0.2$ GeV$^2$, $\epsilon = \alpha = 0$). States with negative mass represent anti-mesons. The solid lines are for BSLT, broken lines for ET.

FIG. 5. The masses of the ground state and the first few excitations of the $^1S_0$ systems are shown as a function of the quark mass (calculated with $c = \epsilon = \alpha = 0$), but with the ratio $\kappa/m^2$ fixed. Under this condition the Schrödinger levels are independent of $m$; they are indicated by the arrows on the right of the figure. The solid lines are for BSLT, broken lines for ET.

FIG. 6. The masses of the ground state and the first few excitations of the $^1S_0$ systems in the BSLT equation are shown as a function of the mass $m_1$ of the first quark (calculated $m_2 = 0.2$ GeV, $\kappa = 0.2$ GeV$^2$, $c = \epsilon = \alpha = 0$).

FIG. 7. Regge-slopes in the BSLT model. The squared meson masses as a function of the total angular momentum $J$ of the unexcited mesons (calculated with $m = 0.25$ GeV, $\alpha = 0$, $c = -1.0$ GeV and $\kappa = 0.33$ GeV$^2$). Solid lines show the masses for a pure scalar confinement ($\epsilon = 0$), broken lines for mixture of vector and mainly scalar confinement in the Feynman gauge ($\epsilon = 0.15$), and dotted lines for the same mixture using the Coulomb gauge. The lines labeled $\pi$ represent the $\pi$ family [$P = (-)^{J+1}$ and $C = (-)^{J^2}$], and similarly for the $\rho$ family [$P = C = (-)^{J+1}$] and the $a_1$ family [$P = C = (-)^J$]. The corresponding Regge-slopes are listed in Table I.

FIG. 8. Identical to Fig. 7, but now for the ET equation.

| TABLE I. Slope $\beta$ (in GeV$^2$) of the Regge-trajectories shown in Fig. 7 and Fig. 8. Listed values follow from a fit from $J = 2$ to $J = 6$. |
|-------------------------------|------------------|------------------|
| family | $\epsilon = 0$ | $\epsilon = 0.15$ | $\epsilon = 0.15$ |
|        | Feynman | Coulomb | Feynman | Coulomb |
|-------|---------|---------|---------|---------|
| BSLT | $\pi$ | 0.69 | 1.20 | 1.06 |
|       | $\rho$ | 0.69 | 1.13 | 1.01 |
|       | $a_1$ | 0.66 | 1.12 | 1.01 |
| ET   | $\pi$ | 0.67 | 0.47 | 0.64 |
|       | $\rho$ | 0.69 | 1.04 | 0.99 |
|       | $a_1$ | 0.65 | 0.92 | 0.83 |
| Parameter | BSLT Feynman gauge | BSLT Coulomb gauge | ET Coulomb gauge |
|-----------|-------------------|-------------------|-----------------|
| $m_{u,d}$ | 0.200             | 0.250             | 0.250           |
| $m_s$     | 0.404             | 0.447             | 0.390           |
| $m_c$     | 1.715             | 1.779             | 1.719           |
| $m_b$     | 5.121             | 5.199             | 5.096           |
| $\kappa$ | 0.33              | 0.33              | 0.33            |
| $c$       | -0.8              | -1.0              | -1.0            |
| $\varepsilon$ | 0.2      | 0.25              | 0.2             |
| $\alpha_{sat}$ | 0.8      | 0.8               | 0.8             |
| running type | $\alpha_f, \mu = 1$ | $\alpha_f, \mu = 1$ | $\alpha_{II}$ |
TABLE III. The meson mass spectrum (in GeV). The experimental data are from the particle data group [26]. Numbers between brackets need confirmation. Only experimental errors larger than 0.01 GeV are given. The column labeled \( N^{25+L_J} \) displays the quantum numbers of the main component of the wave function. Underlined values have been fitted. The most satisfying results are found from the BSLT using the Feynman gauge.

| quark content | meson | \( J^{PC} \) | \( N^{2S+1L_J} \) | Observed mass | BSLT | BSLT | ET |
|---------------|-------|-------------|----------------|--------------|------|------|----|
| \( u, d, d \) | \( \pi \) | 0\(^+\) | \( ^1S_0 \) | 0.135 | 0.439 | 0.521 | 0.134 |
| \( \pi' \) | 0\(^-\) | 2\(^+\) | \( ^1S_0 \) | 1.30±0.10 | 1.441 | 1.424 | 1.361 |
| \( \pi'' \) | 0\(^-\) | 3\(^+\) | \( ^1S_0 \) | (1.77±0.03) | 2.246 | 2.193 | 1.978 |
| \( \rho \) | 1\(^-\) | \( ^3S_1 \) | 0.768 | 0.798 | 0.796 | 0.838 |
| \( \rho' \) | 1\(^-\) | \( ^3D_1 \) | 1.47±0.03 | 1.454 | 1.470 | 1.613 |
| \( \rho'' \) | 1\(^-\) | \( ^3S_1 \) | 1.70±0.02 | 1.653 | 1.594 | 1.649 |
| \( \rho''' \) | 1\(^-\) | \( ^3D_1 \) | (2.14±0.03) | 2.185 | 2.180 | 2.230 |
| \( b_1 \) | 1\(^+\) | \( ^1P_1 \) | 1.23 | 1.091 | 1.136 | 1.196 |
| \( a_0 \) | 0\(^+\) | \( ^3P_0 \) | 0.983 | 0.993 | 1.001 | 1.186 |
| \( a_1 \) | 1\(^+\) | \( ^3P_1 \) | 1.26±0.03 | 1.126 | 1.142 | 1.249 |
| \( a_2 \) | 2\(^+\) | \( ^3P_2 \) | 1.318 | 1.297 | 1.277 | 1.311 |
| \( \pi_2 \) | 2\(^+\) | \( ^1D_2 \) | 1.67±0.02 | 1.524 | 1.552 | 1.569 |
| \( b_3 \) | 3\(^+\) | \( ^1F_3 \) | 1.886 | 1.896 | 1.820 |
| \( \pi_4 \) | 4\(^-\) | \( ^1G_4 \) | 2.205 | 2.198 | 2.011 |
| \( b_5 \) | 5\(^-\) | \( ^1H_5 \) | 2.494 | 2.471 | 2.164 |
| \( \pi_6 \) | 6\(^-\) | \( ^1I_6 \) | 2.761 | 2.722 | 2.236 |
| \( \rho_4 \) | 3\(^-\) | \( ^3D_3 \) | 1.69 | 1.689 | 1.654 | 1.660 |
| \( a_4 \) | 4\(^+\) | \( ^3F_4 \) | (2.04±0.03) | 2.021 | 1.973 | 1.946 |
| \( \rho_5 \) | 5\(^-\) | \( ^3G_5 \) | (2.35) | 2.314 | 2.253 | 2.191 |
| \( a_6 \) | 6\(^+\) | \( ^3H_6 \) | (2.45±0.13) | 2.583 | 2.507 | 2.407 |
| \( \rho_2 \) | 2\(^-\) | \( ^3D_2 \) | 1.547 | 1.562 | 1.624 |
| \( a_3 \) | 3\(^+\) | \( ^3F_3 \) | (2.08±0.04) | 1.897 | 1.903 | 1.904 |
| \( \rho_4 \) | 4\(^-\) | \( ^3G_4 \) | 2.202 | 2.199 | 2.133 |
| \( a_5 \) | 5\(^+\) | \( ^3H_5 \) | 2.479 | 2.465 | 2.330 |
| \( \rho_6 \) | 6\(^-\) | \( ^3I_6 \) | 2.732 | 2.709 | 2.503 |
| \( d\bar{s} \) | \( K \) | 0\(^-\) | \( ^1S_0 \) | 0.498 | 0.593 | 0.660 | 0.350 |
| \( K' \) | 0\(^-\) | \( ^1S_0 \) | (1.46) | 1.560 | 1.530 | 1.457 |
| \( K'' \) | 0\(^-\) | \( ^1S_0 \) | (1.83) | 2.326 | 2.268 | 2.069 |
| \( K^* \) | 1\(^-\) | \( ^3S_1 \) | 0.896 | 0.896 | 0.896 | 0.910 |
| \( K^{*-} \) | 1\(^-\) | \( ^3D_1 \) | 1.41 | 1.584 | 1.599 | 1.692 |
| \( K^{*-*} \) | 1\(^-\) | \( ^3S_1 \) | 1.71±0.02 | 1.727 | 1.680 | 1.700 |
| \( K_1 \) | 1\(^+\) | \( ^1P_1 \) | 1.27 | 1.232 | 1.257 | 1.282 |
| \( K_0 \) | 0\(^+\) | \( ^3P_0 \) | 1.43 | 1.113 | 1.129 | 1.262 |
| \( K_1^* \) | 1\(^+\) | \( ^3P_1 \) | 1.40 | 1.261 | 1.268 | 1.333 |
| \( K_2 \) | 2\(^+\) | \( ^3P_2 \) | 1.425 | 1.389 | 1.373 | 1.376 |
| \( K_2 \) | 2\(^-\) | \( ^3D_2 \) | 1.77 | 1.655 | 1.669 | 1.672 |
| \( K_3 \) | 3\(^+\) | \( ^1F_3 \) | (2.32±0.02) | 2.005 | 2.008 | 1.943 |
| \( K_4 \) | 4\(^-\) | \( ^1G_4 \) | (2.49±0.02) | 2.331 | 2.304 | 2.153 |
| \( K_5 \) | 5\(^+\) | \( ^1H_5 \) | 2.590 | 2.571 | 2.326 |
| \( K_3^* \) | 3\(^-\) | \( ^3D_3 \) | 1.77 | 1.776 | 1.749 | 1.726 |
| \( K_4^* \) | 4\(^+\) | \( ^3F_4 \) | 2.05 | 2.107 | 2.069 | 2.018 |
| \( K_5^* \) | 5\(^-\) | \( ^3G_5 \) | (2.38±0.03) | 2.399 | 2.350 | 2.271 |
| \( u\bar{c} \) | \( D \) | 0\(^-\) | \( ^1S_0 \) | 1.864 | 1.868 | 1.912 | 1.855 |
| \( D^* \) | 1\(^-\) | \( ^3S_1 \) | 2.007 | 2.015 | 2.032 | 2.058 |
|    | 1$^+$ | 1$^+$ | 2.42 | 2.388 | 2.404 | 2.423 |
|----|-------|-------|------|-------|-------|-------|
| $D_0$ | 0$^+$ | 1$^3P_0$ | 2.321 | 2.327 | 2.632 |
| $D_1$ | 1$^+$ | 1$^3P_1$ | 2.415 | 2.420 | 2.494 |
| $D_2$ | 2$^+$ | 1$^3P_2$ | 2.459 | 2.458 | 2.461 | 2.474 |
| $s^c$ | $D_s$ | 0$^-$ | 1$^3S_0$ | 1.969 | 1.952 | 1.983 | 1.939 |
| | $D_s^*$ | 1$^-$ | 1$^3S_1$ | (2.110) | 2.104 | 2.110 | 2.115 |
| | $D_{s1}$ | 1$^+$ | 1$^3P_1$ | 2.537 | 2.500 | 2.509 | 2.491 |
| | $D_{s0}$ | 0$^+$ | 1$^3P_1$ | 2.427 | 2.419 | 2.510 |
| | $D_{s1}$ | 1$^+$ | 1$^3P_1$ | 2.516 | 2.502 | 2.550 |
| | $D_{s2}$ | 2$^+$ | 1$^3P_2$ | (2.564) | 2.569 | 2.559 | 2.547 |
| $c^c$ | $\eta_c$ | 0$^-$ | 1$^3S_0$ | 2.980 | 2.969 | 2.988 | 2.883 |
| | $\eta_c^*$ | 0$^+$ | 2$^3S_0$ | (3.59) | 3.742 | 3.713 | 3.680 |
| $J/\psi$ | 1$^-$ | 1$^3S_1$ | 3.097 | 3.096 | 3.098 | 3.097 |
| $\psi'$ | 1$^-$ | 2$^3S_1$ | 3.686 | 3.810 | 3.779 | 3.789 |
| $\psi''$ | 1$^-$ | 1$^3P_1$ | 3.770 | 3.873 | 3.854 | 3.872 |
| $\psi'''$ | 1$^-$ | 3$^3S_1$ | 4.04 | 4.370 | 4.325 | 4.308 |
| $\psi^{iv}$ | 1$^-$ | 2$^3P_1$ | 4.16±0.02 | 4.409 | 4.372 | 4.363 |
| $\psi''''$ | 1$^-$ | 3$^3P_1$ | 4.42 | 4.860 | 4.805 | 4.744 |
| $h_{c1}$ | 1$^+$ | 1$^3P_1$ | 3.526 | 3.517 | 3.513 | 3.486 |
| $\chi_{c0}$ | 0$^+$ | 1$^3P_0$ | 3.415 | 3.461 | 3.442 | 3.486 |
| $\chi_{c1}$ | 1$^+$ | 1$^3P_1$ | 3.511 | 3.526 | 3.510 | 3.521 |
| $\chi_{c2}$ | 2$^+$ | 1$^3P_2$ | 3.556 | 3.572 | 3.559 | 3.553 |
| $d^b$ | $B_s$ | 0$^-$ | 1$^3S_0$ | 5.279 | 5.302 | 5.331 | 5.349 |
| | $B_s^*$ | 1$^-$ | 1$^3S_1$ | 5.324 | 5.360 | 5.383 | 5.391 |
| | $B_1$ | 1$^+$ | 1$^3P_1$ | 5.741 | 5.756 | 5.741 |
| | $B_0^*$ | 0$^+$ | 1$^3P_0$ | 5.714 | 5.717 | 5.800 |
| | $B_1'$ | 1$^+$ | 1$^3P_1$ | 5.760 | 5.766 | 5.817 |
| | $B_2^*$ | 2$^+$ | 1$^3P_2$ | 5.770 | 5.781 | 5.760 |
| $s^b$ | $B_s$ | 0$^-$ | 1$^3S_0$ | (5.38±0.03) | 5.371 | 5.383 | 5.390 |
| | $B_s^*$ | 1$^-$ | 1$^3S_1$ | (5.43±0.03) | 5.434 | 5.443 | 5.441 |
| | $B_{s1}$ | 1$^+$ | 1$^3P_1$ | 5.839 | 5.841 | 5.805 |
| | $B_{s0}^*$ | 0$^+$ | 1$^3P_0$ | 5.802 | 5.789 | 5.848 |
| | $B_{s1}'$ | 1$^+$ | 1$^3P_1$ | 5.846 | 5.838 | 5.865 |
| | $B_{s2}^*$ | 2$^+$ | 1$^3P_2$ | 5.869 | 5.866 | 5.827 |
| $c^b$ | $B_c$ | 0$^-$ | 1$^3S_0$ | 6.260 | 6.260 | 6.228 |
| | $B_c^*$ | 1$^-$ | 1$^3S_1$ | 6.331 | 6.329 | 6.336 |
| | $B_{c1}$ | 1$^+$ | 1$^3P_1$ | 6.760 | 6.754 | 6.719 |
| | $B_{c0}^*$ | 0$^+$ | 1$^3P_0$ | 6.724 | 6.702 | 6.724 |
| | $B_{c1}'$ | 1$^+$ | 1$^3P_1$ | 6.767 | 6.745 | 6.742 |
| | $B_{c2}^*$ | 2$^+$ | 1$^3P_2$ | 6.794 | 6.781 | 6.758 |
| $b^b$ | $\eta_b$ | 0$^-$ | 1$^3S_0$ | 9.401 | 9.402 | 9.350 |
| | $\eta_b^*$ | 0$^+$ | 2$^3S_0$ | 10.067 | 10.047 | 9.980 |
| $\Upsilon$ | 1$^-$ | 1$^3S_1$ | 9.460 | 9.460 | 9.459 | 9.460 |
| $\Upsilon'$ | 1$^-$ | 2$^3S_1$ | 10.023 | 10.099 | 10.081 | 10.039 |
| $\Upsilon''$ | 1$^-$ | 1$^3P_1$ | 10.355 | 10.206 | 10.187 | 10.138 |
| $\Upsilon'''$ | 1$^-$ | 3$^3S_1$ | 10.58 | 10.556 | 10.532 | 10.467 |
| $\Upsilon^{iv}$ | 1$^-$ | 2$^3P_1$ | 10.87 | 10.629 | 10.603 | 10.533 |
| | $\Upsilon^v$ | 1$^-$ | 4$^3S_1$ | 11.02 | 10.943 | 10.911 | 10.827 |
| $h_{s1}$ | 1$^+$ | 1$^3P_1$ | 9.881 | 9.879 | 9.823 |
| $\chi_{s0}$ | 0$^+$ | 1$^3P_0$ | 9.860 | 9.862 | 9.843 | 9.831 |
\[ \begin{array}{cccccc}
\chi_{b1} & 1^{++} & 1^3 P_1 & 9.892 & 9.890 & 9.876 & 9.845 \\
\chi_{b2} & 2^{++} & 1^3 P_2 & 9.913 & 9.911 & 9.901 & 9.863 \\
h_0' & 1^{+-} & 2^1 P_1 & 10.383 & 10.363 & 10.288 \\
\chi_{b1} & 0^{++} & 2^3 P_0 & 10.232 & 10.363 & 10.335 & 10.289 \\
\chi_{b2} & 1^{++} & 2^3 P_1 & 10.255 & 10.384 & 10.360 & 10.301 \\
\chi_{b2} & 2^{++} & 2^3 P_2 & 10.268 & 10.400 & 10.379 & 10.316 \\
\end{array} \]

*Ref. [2].*

**TABLE IV.** 16-component vector qq-basis states and corresponding 4 \times 4 matrix q\bar{q}-basis states. The angular dependence of the wave functions of the q\bar{q}-triplet states must determine to which qq-triplet state they correspond. For a correct normalization all matrix states should be multiplied by 1/2; furthermore, states proportional to \( P \) or \( p \) need an extra factor 1\( /M \) and 1\( /|p| \), respectively. The correspondence is only valid in the cm system and \( p_0 = 0 \) is assumed.

| \text{qq-state} | \text{q\bar{q}-state} |
|-----------------|----------------------|
| \( J = J_0 \)   | \( -\gamma_5 P \)     |
| \( J = 1/2 \)   | \( \gamma_5 \)        |
| \( J = 3/2 \)   | \(-1 \)               |
| \( J = 5/2 \)   | \( P \)               |
| \( J = 7/2 \)   | \( -\gamma_5 P_0 \)   |
| \( J = 9/2 \)   | \( i\sigma_{\mu\nu}P^\mu p^\nu \) |
| \( J = 11/2 \)  | \(-\gamma_5 \sigma_{\mu\nu}P^\mu p^\nu \) |
| \( J = 13/2 \)  | \( \gamma_5 \)       |

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