ON THE MEAN RADIUS OF QUASICONFORMAL MAPPINGS

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Abstract. We study the mean radius growth function for quasiconformal mappings. We give a new sub-class of quasiconformal mappings in $\mathbb{R}^n$, for $n \geq 2$, called bounded integrable parameterization mappings, or BIP maps for short. These have the property that the restriction of the Zorich transform to each slice has uniformly bounded derivative in $L^{n/(n-1)}$. For BIP maps, the logarithmic transform of the mean radius function is bi-Lipschitz. We then apply our result to BIP maps with simple infinitesimal spaces to show that the asymptotic representation is indeed quasiconformal by showing that its Zorich transform is a bi-Lipschitz map.

1. Introduction

The growth of quasiconformal mappings is a topic that has been well studied. If we set

$$L_f(x_0, r) = \max_{|x-x_0|=r} |f(x) - f(x_0)|$$

and

$$\ell_f(x_0, r) = \min_{|x-x_0|=r} |f(x) - f(x_0)|,$$

then it is well-known that for quasiconformal mappings we have

$$\limsup_{r \to 0} \frac{L_f(x_0, r)}{\ell_f(x_0, r)} \leq H$$

for some $H \geq 1$. On the other hand, for a fixed $x_0$, the (real) functions $L_f$ and $\ell_f$ are bounded above and below via Hölder-type inequalities. The local distortion result given by [9, Theorem III.4.7] shows that if $f$ is $K$-quasiconformal then there exist positive constants $A, B$ and $r_0 > 0$ so that

$$Ar^{K/(n-1)} \leq \ell_f(x_0, r) \leq L_f(x_0, r) \leq Br^{K/(1-n)}$$

for $0 < r < r_0$.

Another quantity that is comparable to $L_f$ and $\ell_f$ on small scales is the mean radius function that, as far as the authors are aware, was introduced into the study of quasiregular mappings by Gutlyanskii et al in [5]. We recall this notion here. Let $n \geq 2$, suppose that $U \subset \mathbb{R}^n$ is a domain and $f : U \to \mathbb{R}^n$ is a non-constant quasiregular mapping. Let $x_0 \in U$ and $r_0 = \text{dist}(x_0, \partial U)$. For $0 < r < r_0$, the mean radius of the image of the ball $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ under $f$ is defined by

$$\rho_f(x_0, r) = \left(\frac{\text{vol}_n f(B(x_0, r))}{\Omega_n}\right)^{1/n},$$

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where \( \text{vol}_n E \) denotes the \( n \)-dimensional Lebesgue measure of \( E \) and \( \Omega_n \) denotes the volume of the unit ball \( \mathbb{B}^n = B(0, 1) \subset \mathbb{R}^n \).

Since
\[
\Omega_n \ell_f(x_0, r) \leq \text{vol}_n f(B(x_0, r)) \leq \Omega_n L_f(x_0, r),
\]

it is clear that
\[
\ell_f(x_0, r) \leq \rho_f(x_0, r) \leq L_f(x_0, r).
\]

Henceforth, we will assume that \( x_0 = 0 \). This makes no difference to our discussion.

The mean radius was used by Gutlyanskii et al. [3] to construct generalized derivatives of quasiregular mappings, a topic which has been studied more recently in papers of the first named author and Wallis [4] and the authors [3]. In this paper, we will study the regularity properties of \( \rho_f \) itself. It is clear from the definition that \( \rho_f \) is increasing and, moreover, it was shown in the proof of [4, Lemma 3.1] that \( \rho_f \) is continuous as a consequence of the Lusin (N) condition. It follows from a standard result in analysis that, as a continuous monotonic real function, \( \rho_f \) is differentiable almost everywhere.

It will be convenient for our results to use the logarithmic transform of \( \rho_f \). Suppose \( \rho_f : (0, r_0) \to (0, \infty) \). Then the logarithmic transform of \( \rho_f \) is defined by the function \( \tilde{\rho}_f : (-\infty, \ln r_0) \to \mathbb{R} \) given by
\[
\tilde{\rho}_f(t) = \ln \rho_f(e^t).
\]

We will typically use the \( r \)-variable for real functions, and the \( t \) variable for logarithmic transforms.

As a generalization of the logarithmic transform to quasiregular maps, we will also use the Zorich transform introduced by the authors in [3]. This will be described in further detail below, but for a quasiconformal map \( f \) fixing \( 0 \), this is defined via a particular Zorich map \( \mathcal{Z} \) and the functional equation \( \mathcal{Z} \circ f = f \circ \mathcal{Z} \). The domain of definition of \( \mathcal{Z} \) is a half-beam \( B_M = Q \times (-\infty, M) \), where \( Q \) is a cuboid in \( \mathbb{R}^{n-1} \) of the form \([0, 1]^{n-2} \times [0, 2] \).

To illustrate why the logarithmic transform is a natural object to study in the growth of quasiconformal mappings, consider the following example.

**Example 1.1.** Let \( f : \mathbb{C} \to \mathbb{C} \) be a radial quasiconformal mapping of the form \( f(re^{i\theta}) = h(r)e^{i\theta} \) in polar coordinates, where \( h : [0, \infty) \to [0, \infty) \) is a strictly increasing homeomorphism satisfying \( h(0) = 0 \). Then via a direct computation we see that its complex dilatation is
\[
\mu_f = \frac{f_r + \frac{i}{r} f_\theta}{f_r - \frac{i}{r} f_\theta} = \frac{r h'(r)}{h(r)} - 1, \quad \frac{r h'(r)}{h(r)} + 1.
\]

wherever \( h'(r) \) exists. Since \( h \) is an increasing homeomorphism, it is differentiable almost everywhere, and so \( \mu_f \) has this expression for almost every \( r \) value. Letting \( \tilde{h} \) be the logarithmic transform of \( h \), and as
\[
\tilde{h}'(t) = \frac{e^t h'(e^t)}{h(e^t)},
\]

it is then clear that for \( f \) to be quasiconformal, it is necessary that there exist constants \( 0 < C_1 < C_2 < \infty \) so that
\[
C_1 \leq \tilde{h}'(t) \leq C_2
\]
for all \( t \) where \( \tilde{h}' \) is defined.
Radial maps are the nicest class of mappings from our viewpoint, since \( \ell_f, L_f \) and \( \rho_f \) will all coincide. In general, we expect other factors to play a role in the behaviour of \( \rho_f \). To that end, we build towards the key definition of our paper.

Let \( n \geq 2 \). For vectors \( v_1, \ldots, v_{n-1} \) in \( \mathbb{R}^n \), form the \( n \) by \( n \) matrix \( A \) where the top row consists of the unit vectors \( e_1, \ldots, e_n \) in \( \mathbb{R}^n \), and the \( i \)th row of \( A \) consists of entries from the vector \( v_{i-1} \), for \( 2 \leq i \leq n \). Then denote by \( \Pi(v_1, \ldots, v_{n-1}) \) the vector in \( \mathbb{R}^n \) given by \( \det A \).

Returning to the Zorich transform, \( \tilde{f} \) maps each slice \( Q \times \{ t \} \), for \(-\infty < t < M \), onto a hypersurface in the beam \( B \). The map \( \tilde{f} \) provides a parameterization for each image hypersurface through \( Q \). Recalling that

\[
\bar{Q} = \{(x_1, \ldots, x_{n-1}) : 0 \leq x_1, \ldots, x_{n-2} \leq 1, 0 \leq x_{n-1} \leq 2\},
\]

for \(-\infty < t < M \), write \( \gamma : Q \to \tilde{f}(Q \times \{ t \}) \) so that

\[
\gamma(x_1, \ldots, x_{n-1}) = \tilde{f}(x_1, \ldots, x_{n-1}, t).
\]

If \( \gamma(Q) \) has a well-defined \((n-1)\)-dimensional volume in \( B \) (noting that this is not necessarily the case for an arbitrary quasiconformal Zorich transform \( \tilde{f} \)), then it is given by

\[
\text{vol}_{n-1} \gamma(Q) = \int_Q \left\| \Pi \left( \frac{\partial \gamma}{\partial x_1}, \ldots, \frac{\partial \gamma}{\partial x_{n-1}} \right) \right\|_2 dV,
\]

where \( dV \) denotes the volume element on \( Q \), and \( \| \cdot \|_2 \) denotes the usual 2-norm.

**Definition 1.2.** Let \( n \geq 2 \), let \( e^M > 0 \) and let \( f : B(0, e^M) \to \mathbb{R}^n \) be quasiconformal with \( f(0) = 0 \). We say that \( f \) is a *bounded integrable parameterization map*, or is a *BIP map*, if there exists \( P > 0 \) such that for every \( t < M \), the parameterization \( \gamma : Q \to \tilde{f}(Q \times \{ t \}) \) via \( \tilde{f} \) satisfies

\[
\int_Q \left\| \Pi \left( \frac{\partial \gamma}{\partial x_1}, \ldots, \frac{\partial \gamma}{\partial x_{n-1}} \right) \right\|_2^{n/(n-1)} dV \leq P.
\]

Since \( n/(n-1) > 1 \), it follows that the BIP condition implies the images of the various slices \( \tilde{f}(Q \times \{ t \}) \) have finite \((n-1)\)-dimensional volume, but it also implies stronger regularity of the parameterizations.

For our main result, we show that BIP mappings have improved regularity for \( \tilde{\rho_f} \) than just almost everywhere differentiability.

**Theorem 1.3.** Let \( n \geq 2 \), \( K \geq 1 \), \( e^M > 0 \) and let \( f : B(0, e^M) \to \mathbb{R}^n \) be a BIP \( K \)-quasiconformal map with \( f(0) = 0 \). Then there exists \( L \) depending only on \( n \), \( K \) and \( P \) such that \( \tilde{\rho_f} \) is \( L \)-bi-Lipschitz on \((\infty, M)\).

It follows from Theorem 1.3 that

\[
\frac{1}{L} \leq \tilde{\rho_f}'(t) \leq L
\]

for every \( t \in (\infty, M) \) at which \( \tilde{\rho_f} \) is differentiable, recalling Example 1.1. As will be evident from the proof of Theorem 1.3, the implied lower bound in the bi-Lipschitz behaviour of \( \tilde{\rho_f} \) always holds. The requirement of BIP maps is needed for the upper bound.

Theorem 1.3 may be applied in neighbourhoods of a fixed point of a quasiregular mapping, provided that the local index \( i(x_0, f) = 1 \), that is, that \( f \) is injective in a neighbourhood of \( x_0 \). If the fixed point of \( f \) is geometrically attracting, then by
definition there exist \( c > 1 \) and \( r_1 > 0 \) such that \( |f(x)| < c|x| \) for \( |x| < r_1 \). It is worth pointing out that although \( \rho_f(r) < cr \), it does not follow that \( \tilde{\rho}_f(t) < 1 \): consider the simple example \( f(x) = x/2 \) for which \( \tilde{\rho}_f(t) = t - \ln 2 \).

The idea behind the proof of Theorem 1.3 is as follows. We first need to show that the Zorich transform \( \tilde{f} \) is not just quasiconformal in \( B_M \), but in fact quasisymmetric. This allows us to compare quantities of the form \( (\tilde{\rho}_f(t_0 + t) - \tilde{\rho}_f(t_0))/t \) to \( n \)-dimensional volumes of images of slices \( Q \times [t_0, t] \) under \( \tilde{f} \). Subdividing these slices into small cuboids and using the quasisymmetry of \( \tilde{f} \) allows us to compare the volume of the image of a cuboid to its diameter raised to the \( n \)th power. Summing over all such cuboids and a judicious use of an \( \ell^p \) inequality yields the lower bound. The upper bound arises by using the sum over all cuboids as an approximation for the expression in Definition 1.2. The BIP condition is then needed to provide the upper bound.

We record the fact that the BIP condition is necessary for Theorem 1.3 in the following proposition.

**Proposition 1.4.** Suppose that \( e^M > 0 \) and let \( f : B(0, e^M) \to \mathbb{R}^2 \) be quasiconformal, fix 0 and so that for some \( 0 < e^0 < e^M \), the image of the circle centred at 0 of radius \( e^0 \) under \( f \) is a non-rectifiable curve. Then \( \tilde{\rho}_f \) is not bi-Lipschitz at \( t_0 \).

It is well-known that the image of a circle under a quasiconformal map need not be rectifiable, for example, it could be a snowflake curve in \( \mathbb{R}^2 \) or a snowball in \( \mathbb{R}^3 \). We refer to [2] for snowballs and references in the literature addressing snowflake curves.

As an application of Theorem 1.3 we turn to infinitesimal spaces. Recall from [5] that a generalized derivative of a quasiregular map \( f : U \to \mathbb{R}^n \) at \( x_0 \in U \) is defined to be any local uniform limit of

\[
\frac{f(x_0 + r_k x) - f(x_0)}{\rho_f(r_k)}
\]

as \( r_k \to 0 \). Of course, not every such sequence need have a limit, but the quasiregular version of Montel’s Theorem implies there will be a subsequence along which there is local uniform convergence. See [5, p.103] for a discussion of this point. The collection of generalized derivatives of \( f \) at \( x_0 \) is called the infinitesimal space \( T(x_0, f) \).

In the special case where \( T(x_0, f) \) consists of only one mapping \( g \), then \( f \) is called simple at \( x_0 \). For example, if \( f \) is differentiable at \( x_0 \), then \( f \) is simple at \( x_0 \). If \( f \) is simple at \( x_0 \), then \( f \) has an asymptotic representation analogous to a first degree Taylor polynomial approximation of an analytic function. As usual, we suppose \( x_0 = f(x_0) = 0 \). Then [5, Proposition 4.7] states that as \( x \to 0 \), we have

\[
f(x) \sim D(x) := \rho_f(|x|)g(x/|x|),
\]

where \( p(x) \sim q(x) \) as \( x \to 0 \) means

\[
|p(x) - q(x)| = o(|p(x)| + |q(x)|).
\]

The map \( D \) is called the asymptotic representative of \( f \) at (in this case) 0. Our second main result concerns the asymptotic representative.

**Theorem 1.5.** Let \( n \geq 2 \), \( U \subset \mathbb{R}^n \) be a domain and suppose that \( f : U \to \mathbb{R}^n \) is a quasiregular map, \( 0 \in U \), \( i(0, f) = 1 \) and \( f \) is simple at 0 and \( f \) is a BIP
quasiconformal map on some neighbourhood of 0. Then there exists $M \in \mathbb{R}$ such that $D$ is bi-Lipschitz in $B_M$. Moreover, $D$ is quasiconformal in $B(x_0, e^M)$.

The simplicity assumption on $f$ implies via [5, Proposition 4.7] that $\rho_f$ is asymptotically $d$-homogeneous, that is, there exists $d > 0$ such that for any $s > 0$,

$$\rho_f(st) \sim s^d \rho_f(t)$$

as $t \to 0$. It should be noted that this conclusion does not imply that $\rho_f$ itself is $d$-homogeneous; one can check that if $d > 0$ then the two dimensional radial map \( f \) given in polar coordinates by

$$(r, \theta) \mapsto \left( \frac{r^d}{\log(1/r)}, \theta \right)$$

is quasiconformal, but $\rho_f$ is not $d$-homogeneous.

The assumption that $f$ is BIP in a neighbourhood of 0 is necessary in Theorem 1.5. We illustrate this by means of an example.

Example 1.6. We work in dimension two. There exists a quasiconformal map $h$ from the square $[-1,1]^2$ to itself which is the identity on the boundary and so that $h$ maps $[-1,1] \times \{0\}$ onto a non-rectifiable curve. Next, for $M < 0$, let $B_M = [0,2] \times (-\infty,M)$. For $d \in \mathbb{Z}$ and $d < \min\{M,-2\}$, consider the square $S_d$ centred at $z_d = (1,d)$ with sides parallel to the coordinate axes and of side-length $|d|^{-1}$. We define $\tilde{f}$ to be the identity outside $S_d$ in $B_M$. Let $A_d$ be the linear map

$$A_d(z) = \frac{z}{2|d|} + z_d$$

and then for $z \in S_d$, we define $\tilde{f}(z) = A_d(h(A_d^{-1}(z)))$.

There exists a quasiconformal map $f : B(0, e^M) \to \mathbb{R}^2$ whose Zorich transform is $\tilde{f}$. By construction, for $r > 0$, $f$ is not the identity in $B(0, r)$ on the union of the sets $Z(S_d)$, where $d < \log r$. For such $d$, the diameter of $Z(S_d)$ is bounded above by

$$e^{d+1/2|d|} - e^{-d-1/2|d|} < 2 \sinh \left( \frac{1}{2 \ln r} \right) = o(1)$$

as $r \to 0$. We conclude that $f(x) \sim x$ as $x \to 0$ and so $f$ has a simple infinitesimal space consisting of the identity. However, $\tilde{\rho}_f$ is not bi-Lipschitz on any neighbourhood of 0 by the argument in Proposition 1.4 from which it follows that $D(x) = \rho_f(|x|)|x|/|x|$ is not quasiconformal.

As a final remark, in [1], it was implicitly assumed that the map $D$ is quasiconformal. Since our results above show that this is not always the case, Theorem 1.5 can be viewed as providing an extra assumption necessary for the results from that paper to hold.

2. Preliminaries

Throughout, $B(x_0, r)$ denotes the ball centred at $x_0 \in \mathbb{R}^n$ of radius $r > 0$. Its boundary is the sphere $S(x_0, r)$ or, if $x_0 = 0$, we may simply write $S(r)$. If $x \in \mathbb{R}^n$, then we write $|x|_n \in \mathbb{R}$ for the $n$th coordinate.

We refer to standard references such as [9] for the definition and basic properties of quasiregular and quasiconformal mappings in $\mathbb{R}^n$. In particular, if $f$ is quasiregular, then $K_0(f), K_I(f)$ and $K(f)$ refer to the outer dilatation, inner dilatation and
maximal dilatation, respectively. Bounded length distortion maps, BLD for short, are a sub-class of quasiregular maps for which the finite length curves are mapped to curves of finite length, with uniform control on the length distortion. BLD maps are locally bi-Lipschitz. In some sense, BIP maps provide a generalization of BLD maps.

We will use the following result.

**Theorem 2.1 (Theorem 1.1, [2]).** Let \( U \subset \mathbb{R}^n \) be a domain for \( n \geq 2 \) and let \( f : U \to \mathbb{R}^n \) be a non-constant quasiregular map. If \( x \in U \), there exist \( C > 1 \) and \( r_0 > 0 \) such that for all \( 0 < T \leq 1 \) and \( r \in (0, r_0) \)

\[
T^{-\mu} \leq \frac{L_f(x,r)}{\ell_f(x,T)} \leq C^2 T^{-\nu}
\]

and

\[
\frac{T^{-\mu}}{C^2} \leq \frac{\ell_f(x,r)}{L_f(x,T)} \leq T^{-\nu},
\]

where \( \mu = (i(x,f)/K_I(f))^{1/(n-1)} \), \( \nu = (K_O(f)i(x,f))^{1/(n-1)} \) and \( C \) depends only on \( n \), \( K_O(f) \) and the local index \( i(x,f) \).

2.1. The Zorich transform. The class of Zorich mappings provides a well-known quasiregular generalization of the exponential function in the plane. These maps are strongly automorphic with respect to a discrete group \( G \) of isometries. This means that if \( Z \) is a Zorich map, then \( Z(g(x)) = Z(x) \) for all \( g \in G \) and all \( x \in \mathbb{R}^n \). 

For each \( n \geq 2 \), we will fix one particular Zorich map, which we denote simply by \( Z \), with the following properties.

(i) For each \( t \in \mathbb{R} \), \( Z \) maps the hyperplane \( H_t = \{ x \in \mathbb{R}^n : [x]_n = t \} \) onto the sphere \( S(e^t) \).

(ii) On each hyperplane \( H_t \), \( Z \) is locally bi-Lipschitz with isometric distortion scaling with \( e^t \). In fact,

\[
Z(x_1, \ldots, x_{n-1}, t) = e^t Z(z_1, \ldots, x_{n-1}, 0).
\]

(iii) The group \( G \) is generated by a translation subgroup of rank \( n-1 \) and a finite group of rotations about the origin such that each \( g \in G \) preserves the \( n \)th coordinate.

(iv) A fundamental set \( B \) for the action of \( G \) is given by a beam \( Q \times \mathbb{R} \), where the closure of \( Q \) is a cuboid in \( \mathbb{R}^{n-1} \) of the form \( [0,1]^{n-2} \times [0,2] \) and the sides of this beam are identified via the group \( G \).

For an explicit formula for such a Zorich map, together with its associated group \( G \), we refer to, for example, [3, Section 3]. If \( f \) is quasiconformal in a neighbourhood \( U \) of the origin in \( \mathbb{R}^n \) and \( f(0) = 0 \), then we may consider the Zorich transform \( \tilde{f} : Z^{-1}(U) \cap B \to B \) of \( f \). This is defined via the relation

\[
f \circ Z = Z \circ \tilde{f}.
\]

We often just assume the domain of \( \tilde{f} \) is

\[
B_M := \{ x \in B : [x]_n < M \},
\]

in the same way that we may assume that the domain of \( f \) is \( \{ x : |x| < e^M \} \).

We will use the fact that \( B_M \) can be made into a metric space via the quotient of the Euclidean metric under the group \( G \) and consider the appropriate topological
notions of balls, convergence and so on in this metric space. In particular, \( Z^{-1} : B_M \to \mathbb{R}^n \) is locally bi-Lipschitz with this metric in the domain and the Euclidean metric in the range.

2.2. Quasisymmetric maps. A standard reference for quasisymmetric maps is Heinonen’s book [6]. A map \( f : X \to Y \) between metric spaces is called quasisymmetric if there exists a homeomorphism \( \eta : [0, \infty) \to [0, \infty) \) such that for all triples of points \( x, a, b \in X \), we have

\[
\frac{d_Y(f(x), f(a))}{d_Y(f(x), f(b))} \leq \eta \left( \frac{d_X(x, a)}{d_X(x, b)} \right).
\]

If we wish to specify the function \( \eta \), then \( f \) is said to be \( \eta \)-quasisymmetric. A map \( f : X \to Y \) is called weakly quasisymmetric if there exists a constant \( H \geq 1 \) such that

\[ d_X(x, a) \leq d_X(x, b) \]

implies

\[ d_Y(f(x), f(a)) \leq H d_Y(f(x), f(b)) \]

for all triples of points \( x, a, b \in X \). While in general weakly quasisymmetric maps need not be quasisymmetric, they are in connected doubling spaces, see [6, Theorem 10.19].

Quasisymmetry can be viewed as a global version of quasiconformality. In particular, a quasisymmetric map \( f : U \to V \) between domains in \( \mathbb{R}^n \) is quasiconformal. Conversely, the egg-yolk principle states that a \( K \)-quasiconformal map \( f : U \to V \) is \( \eta \)-quasisymmetric on each ball \( B(x_0, \text{dist}(x_0, \partial U)/2) \), with \( \eta \) depending only on \( n \) and \( K \), see [6, Theorem 11.14].

2.3. \( l^p \) estimates. It is well-known that if \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N \) and \( x_i \geq 0 \) for \( i = 1, \ldots, N \), then for \( 1 \leq p < q \leq \infty \) we have

\[ ||x||_q \leq ||x||_p \leq N^{1/p-1/q} ||x||_q, \]

where \( ||x||_p \) denotes the usual \( p \)-norm

\[ ||x||_p = \left( \sum_{i=1}^{N} x_i^p \right)^{1/p}. \]

In particular, for \( n \geq 2 \), if \( p = n - 1 \) and \( q = n \), then

\[ \sum_{i=1}^{N} x_i^n \geq N^{1/(1-n)} \left( \sum_{i=1}^{N} x_i^{n-1} \right)^{n/(n-1)}. \]

3. Quasisymmetry

We will need the following result that shows the Zorich transform of a quasiconformal map is quasisymmetric. This is a non-trivial result, as \( Z^{-1} \) itself is not quasisymmetric on any punctured neighbourhood of the origin.

**Proposition 3.1.** Let \( n \geq 2 \), \( K > 1 \) and \( r_0 > 0 \). If \( f : B(0, r_0) \to \mathbb{R}^n \) is a \( K \)-quasiconformal map with \( f(0) = 0 \), then there exists \( M < \ln r_0 \) such that \( \tilde{f} \) is quasisymmetric in \( B_M \).
Proof. First, for any $M$, as $B_M$ is a connected doubling space, \cite[Theorem 10.19]{6} implies that it is sufficient to show that $\tilde{f}$ is weakly quasisymmetric. Suppose for a contradiction that $\tilde{f}$ is not weakly quasisymmetric. Then there exists a sequence of triples of points $x_k, a_k, b_k$ in $B_M$ such that

\begin{equation}
|x_k - a_k| \leq |x_k - b_k|, \quad \text{but} \quad |\tilde{f}(x_k) - \tilde{f}(a_k)| > k|\tilde{f}(x_k) - \tilde{f}(b_k)|.
\end{equation}

We will deal with large scales and small scales separately.

First, if $|\tilde{f}(x_k) - \tilde{f}(b_k)|$ does not converge to 0 then, passing to a subsequence if necessary, we have $|\tilde{f}(x_k) - \tilde{f}(a_k)| \to \infty$. This implies that $|[\tilde{f}(x_k)]_n - [\tilde{f}(a_k)]_n| \to \infty$. Now set $t_k = |x_k - b_k|$, so $t_k \to \infty$ by (3.1). Without loss of generality, suppose $[a_k]_n \leq [x_k]_n \leq [b_k]_n$. Then we have

\begin{equation}
[b_k]_n \leq [x_k]_n + t_k, \quad [a_k]_n \geq [x_k]_n - t_k.
\end{equation}

Next, set $s_k = [b_k]_n - [x_k]_n$. As $t_k^2 \leq s_k^2 + (\text{diam } Q)^2$ by Pythagoras and $(\text{diam } Q)^2 = n + 2$, we have

\begin{equation}
\sqrt{1 - \frac{(\text{diam } Q)^2}{t_k^2}} \leq \frac{s_k}{t_k} \leq 1
\end{equation}

and in particular, $s_k/t_k \to 1$ as $k \to \infty$.

Next, set $y_k = Z(x_k)$, $p_k = Z(a_k)$ and $q_k = Z(b_k)$. Using Theorem \cite[2.1]{2.1} and (3.2), we have

\begin{align*}
|\tilde{f}(x_k) - \tilde{f}(a_k)| &= \left| \log \frac{\tilde{f}(y_k)}{\tilde{f}(p_k)} \right| \\
&\leq \log \frac{L_f(0, e^{[x_k]_n})}{L_f(0, e^{[b_k]_n})} \\
&\leq \log \frac{L_f(0, e^{-t_k e^{[x_k]_n}})}{L_f(0, e^{-t_k e^{[b_k]_n}})} \\
&\leq \log \left( C^2(e^{-t_k} - 1) \right),
\end{align*}

recalling that $\nu = K_O(f)^{1/(n-1)}$ and $C$ only depends on $n$ and $K_O(f)$ since $f$ is quasiconformal. On the other hand, by Theorem \cite[2.1]{2.1} and the definition of $s_k$ we have

\begin{align*}
|\tilde{f}(b_k) - \tilde{f}(x_k)| &= \left| \log \frac{\tilde{f}(q_k)}{\tilde{f}(y_k)} \right| \\
&\geq \log \frac{L_f(0, e^{[b_k]_n})}{L_f(0, e^{[x_k]_n})} \\
&= \log \frac{L_f(0, e^{-s_k e^{[b_k]_n}})}{L_f(0, e^{-s_k e^{[b_k]_n}})} \\
&\geq \log \left( \frac{e^{-s_k} - 1}{C^2} \right),
\end{align*}

where $\mu = K_f(f)^{1/(1-n)}$. Hence by (3.3) we have

\begin{align*}
\frac{|\tilde{f}(b_k) - \tilde{f}(x_k)|}{|\tilde{f}(x_k) - \tilde{f}(a_k)|} \geq \frac{-2 \log C + \mu s_k}{2 \log C + \nu t_k} \to \frac{\mu}{\nu} = (K_O(f)K_f(f))^{1/(1-n)}
\end{align*}

as $k \to \infty$. This provides a contradiction to (3.1).
Turning now to the small scales case, assume that $|\tilde{f}(x_k) - \tilde{f}(b_k)| \to 0$ and hence $t_k = |x_k - b_k| \to 0$. As $f$ is $K$-quasiconformal, $\tilde{f}$ is $K'$-quasiconformal, where $K' \leq K[K(\mathbb{Z})]^2$. We may define a sequence of $K'$-quasiconformal maps $g_k : \mathbb{R}^n \to \mathbb{R}^n$ via

$$g_k(x) = \frac{\tilde{f}(x_k + |x_k - b_k|x) - \tilde{f}(x_k)}{L\tilde{f}(x_k, |x_k - b_k|)}.$$  

By construction, $\sup_{|x|=1} |g_k(x)| = 1$ for all $k$. Since the family $\{g_k\}$ is normal by the quasiregular version of Montel’s Theorem (see [8, Theorem 4]), it follows that there exists $\delta > 0$ such that

$$\inf_{|x|=1} |g_k(x)| > \delta$$

for all $k$.

However, as $|x_k - a_k| \leq |x_k - b_k|$ and $|\tilde{f}(x_k) - \tilde{f}(a_k)| > k|\tilde{f}(x_k) - \tilde{f}(b_k)|$, it follows that, with $y_k = (b_k - x_k)/|b_k - x_k|$, we have

$$|g_k(y_k)| = \frac{|f(b_k) - \tilde{f}(x_k)|}{L\tilde{f}(x_k, |x_k - b_k|)} \leq \frac{|\tilde{f}(a_k) - \tilde{f}(x_k)|}{kL\tilde{f}(x_k, |x_k - b_k|)} \leq \frac{|f(x_k - a_k)|}{kL\tilde{f}(|x_k - b_k|)} \to 0$$

as $k \to \infty$. This is a contradiction to (3.4).

We conclude that $\tilde{f}$ is weakly quasisymmetric, and hence quasisymmetric, on $B_M$. \hfill \Box

We will assume henceforth that $\tilde{f}$ is defined in $B_M$, and identify $B_M$ with $Q \times (-\infty, M)$. Given $t_0 < M - 1$ and $t < 1$, consider the slice

$$S_t = Q \times [t_0, t_0 + t] \subset B_M.$$

We subdivide $S_t$ into boxes in the following way. Subdivide $Q$ into $(n - 1)$-dimensional cubes $T_i$ of side length $(1/t)^{-1}$ with edges parallel to the unit vectors $e_1, \ldots, e_{n-1}$. Finally, we set

$$P_i = T_i \times [t_0, t_0 + t]$$

for $i = 1, \ldots, N$.

We collect important geometric information about the quasisymmetric images of $P_i$ in the following lemma. Recall that we denote by $\text{vol}_p$ the $p$-dimensional volume.

**Lemma 3.2.** Let $\tilde{f} : B_M \to B$ be $\eta$-quasisymmetric and let $t \in (0, 1/2)$. Then the number of boxes in the subdivision of $S_t$ is $N$, where $N/t^{1-n} \to 2$ as $t \to 0$. Moreover, there exists a constant $C_1 > 1$ depending only on $n$ and $\eta$ so that

$$\frac{1}{C_1} \leq \frac{\text{vol}_n \tilde{f}(P_i)}{(\text{diam} \tilde{f}(P_i))^n} \leq C_1,$$

for each box $P_i$ constructed as above.
Proof. The process of subdividing $Q$ into $(n-1)$-dimensional cubes of side length $([1/t])^{-1}$ yields $N = O(t^{1-n})$ of them as

$$\text{(3.6)} \quad t \leq ([1/t])^{-1} = \frac{t}{1-t} \leq 2t,$$

$\text{vol}_{n-1} Q = 2$ by construction and $\text{vol}_{n-1} Q = N([1/t])^{1-n}$. Note that as $t \to 0$, $([1/t])^{-1}/t \to 1$ and so $N/t^{(1-n)} \to 2$.

Next, let $x_0$ be the centroid of $P_i$. By (3.6) we have

$$\text{(3.7)} \quad \frac{t}{2} \leq \text{dist}(x_0, \partial P_i).$$

Let $y, z$ be any two points on $\partial P_i$. Then since $\tilde{f}$ is $\eta$-quasisymmetric, we have

$$\frac{|\tilde{f}(y) - \tilde{f}(z)|}{|\tilde{f}(y) - \tilde{f}(x_0)|} \leq \eta \left( \frac{|y - z|}{|y - x_0|} \right).$$

Choosing $y, z$ so that $|\tilde{f}(y) - \tilde{f}(z)|$ realizes the diameter of $\tilde{f}(P_i)$, we obtain via (3.7) that

$$\text{(3.8)} \quad \text{diam} \tilde{f}(P_i) \leq \text{dist}(\tilde{f}(x_0), \partial \tilde{f}(P_i)) \cdot \eta \left( \frac{\sqrt{n}}{2} \right) = \text{dist}(\tilde{f}(x_0), \partial \tilde{f}(P_i)) \cdot \eta(2\sqrt{n}).$$

Since

$$B(\tilde{f}(x_0), \text{dist}(\tilde{f}(x_0), \partial \tilde{f}(P_i))) \subset \tilde{f}(P_i) \subset B(\tilde{f}(x_0), \text{diam} \tilde{f}(P_i)),$$

it follows that

$$\Omega_n \left( \text{dist}(\tilde{f}(x_0), \partial \tilde{f}(P_i)) \right)^n \leq |\tilde{f}(P_i)| \leq \Omega_n \left( \text{diam} \tilde{f}(P_i) \right)^n.$$

Combining this with (3.8), we obtain

$$\frac{\Omega_n}{\eta(2\sqrt{n})} \leq \frac{|\tilde{f}(P_i)|}{(\text{diam} \tilde{f}(P_i))^n} \leq \Omega_n,$$

as required. \hfill \Box

**Definition 3.3.** Given a box $P_i$ constructed as above, denote by $\nu(\tilde{f}(P_i))$ the minimum distance between images under $\tilde{f}$ of opposite pairs of faces of $P_i$.

**Lemma 3.4.** Let $\tilde{f} : B_M \to B$ be $\eta$-quasisymmetric and let $t \in (0, 1/2)$. There exists a constant $C_2 > 1$ depending only on $n$ and $\eta$ such that if $P_i$ is a box constructed as in (3.8) with parameter $t$, then

$$1 \leq \frac{\text{diam}(\tilde{f}(P_i))}{\nu(\tilde{f}(P_i))} \leq C_2.$$

Proof. The lower bound here is trivial. Given $P_i$, find $x, y \in P_i$ which realize the minimum in the definition of $\nu(\tilde{f}(P_i))$. Necessarily, $x$ and $y$ must lie in opposite faces of the boundary of $P_i$. Applying the quasisymmetry of $\tilde{f}$ twice, for any $p, q \in P_i$, we have

$$\frac{|\tilde{f}(p) - \tilde{f}(q)|}{|\tilde{f}(x) - \tilde{f}(y)|} \leq \frac{|\tilde{f}(p) - \tilde{f}(q)|}{|\tilde{f}(x) - \tilde{f}(q)|} \cdot \frac{|\tilde{f}(x) - \tilde{f}(q)|}{|\tilde{f}(x) - \tilde{f}(y)|} \leq \eta \left( \frac{|p - q|}{|x - q|} \right) \eta \left( \frac{|x - q|}{|x - y|} \right).$$
We choose \( p, q \) so that their images realize \( \text{diam} \tilde{f}(P_i) \) and, without loss of generality, we may assume that \( |x - q| \geq t/2 \) (if not, switch the roles of \( x \) and \( y \)). From (3.6), we have \( |x - q| \leq 2t\sqrt{n}, |p - q| \leq 2t\sqrt{n} \) and \( |x - y| \geq t \), from which it follows that
\[
\frac{\tilde{f}(p) - \tilde{f}(q)}{|f(x) - f(y)|} \leq \eta(4\sqrt{n})\eta(2\sqrt{n}),
\]
completing the proof. □

**Lemma 3.5.** Let \( \tilde{f} : B_M \to B \) be \( \eta \)-quasisymmetric and let \( t \in (0, 1/2) \). There exists a constant \( C_3 \) depending only on \( n \) and \( \eta \) such that if \( P_i \) is a box constructed as in (3.3) with parameter \( t \), and \( Q_i \) denotes the base of the box, that is, \( Q_i = P_i \cap (Q \times \{ t_0 \}) \), then
\[
\frac{\nu(\tilde{f}(P_i))^{n-1}}{\text{vol}_{n-1} \tilde{f}(Q_i)} \leq C_3.
\]

**Proof.** Given \( Q_i \), let \( x_0 \) be its centroid. As \( Q_i \) is an \((n - 1)\)-dimensional cube of side length \((|1/t|)^{-1}\), it follows from (3.6) that
\[
\frac{t}{2} \leq \text{dist}(x_0, \partial Q_i) \leq t\sqrt{n - 1}.
\]
Here, we denote by \( \partial Q_i \) the \((n - 2)\)-dimensional faces on the edge of \( Q_i \). Let \( y, z \in \partial Q_i \) be such that \( \tilde{f}(y) \) and \( \tilde{f}(z) \) realize the minimum distance between images under \( \tilde{f} \) of opposite pairs of faces of \( \partial Q_i \). Then \( |\tilde{f}(y) - \tilde{f}(z)| \geq \nu(\tilde{f}(P_i)) \).

By the quasisymmetry of \( \tilde{f} \) and (3.3), it follows that
\[
\frac{\nu(\tilde{f}(P_i))}{|\tilde{f}(x_0) - \tilde{f}(y)|} \leq \frac{|\tilde{f}(y) - \tilde{f}(z)|}{|\tilde{f}(x_0) - \tilde{f}(y)|} \leq \eta \left( \frac{|y - z|}{|x_0 - y|} \right) \leq \eta \left( \frac{2t\sqrt{n - 1}}{t/2} \right) = \eta(4\sqrt{n - 1}).
\]
We conclude that
\[
\frac{\nu(\tilde{f}(P_i))}{\text{vol}_{n-1} \tilde{f}(Q_i)} \leq \eta(4\sqrt{n - 1}) \text{dist}(\tilde{f}(x_0), \partial Q_i).
\]
Thus for \( s = \frac{\nu(\tilde{f}(P_i))}{\eta(4\sqrt{n - 1})} \), the ball \( B(\tilde{f}(x_0), s) \) does not meet \( \tilde{f}(\partial Q_i) \). It follows that
\[
\frac{\text{vol}_{n-1} \tilde{f}(Q_i)}{s^{n-1}} \geq \frac{\Omega_{n-1} \nu(\tilde{f}(P_i))^{n-1}}{\eta(4\sqrt{n - 1})^{n-1}},
\]
from which the lemma follows. □

4. VOLUME COMPARISON

**Lemma 4.1.** Let \( \tilde{f} : B_M \to B \) be \( \eta \)-quasisymmetric and let \( t \in (0, 1/2) \). Let \( S_t = Q \times [t_0, t_0 + t] \) and denote by \( V_t \) the volume \( \text{vol}_n S_t \). Then there exist \( C_4 > 1 \), depending only on \( n \) and \( \eta \), and \( t_1 > 0 \) so that for \( t \in (0, t_1) \) we have
\[
\frac{1}{C_4} \leq \frac{\tilde{\rho_f}(t_0 + t) - \tilde{\rho_f}(t_0)}{V_t} \leq C_4.
\]

**Proof.** By the definition of \( \tilde{\rho_f} \), we have
\[
\tilde{\rho_f}(t_0 + t) - \tilde{\rho_f}(t_0) = \log \rho_f(e^{t_0+t}) - \log \rho_f(e^{t_0}) = \log \left( \frac{\rho_f(e^{t_0+t})}{\rho_f(e^{t_0})} \right).
\]
Interpreting \( \rho_f(s) \) as the normalized volume \( \text{vol}_n f(B(0,s))/\Omega_n \), (4.1) yields

\[
\tilde{\rho}_f(t_0 + t) - \tilde{\rho}_f(t_0) = \log \left( \frac{\text{vol}_n f(B(0, e^{t_0} e^t))}{\text{vol}_n f(B(0, e^{t_0}))} \right) = O \left( \frac{\text{vol}_n f(B(0, e^{t_0} e^t)) - \text{vol}_n f(B(0, e^{t_0}))}{\text{vol}_n f(B(0, e^{t_0}))} \right)
\]

(4.2)
as \( t \to 0 \). Denoting by \( A_t \) the annulus \( \{ x : e^{t_0} \leq |x| \leq e^{t_0} e^t \} \), we see that the numerator on the right hand side of (4.2) is \( \text{vol}_n f(A_t) \). Moreover, \( V_t = \text{vol}_n Z^{-1}(f(A_t)) \).

Now, \( Z^{-1} \) is locally bi-Lipschitz away from the origin, but we want an estimate for \( Z^{-1} \) that will work for all small \( t_0 \). To that end, we observe that on a punctured neighbourhood of the origin, we have

\[
Z^{-1}(x) = Z^{-1} \left( \frac{x}{\ell_f(e^{t_0})} \right) - \left[ \ln \frac{1}{\ell_f(e^{t_0})} \right] e_n.
\]

Since \( f \) is quasiconformal, it has finite linear distortion at 0, that is, there exist \( r_0 > 0 \) and a constant \( H \geq 1 \) such that

\[
\frac{L_f(r)}{\ell_f(r)} \leq H
\]

for \( 0 < r < r_0 \). Hence if \( e^{t_0} < r_0 \), we may conclude that the image of \( f(A_t) \) under the map \( x/\ell_f(e^{t_0}) \) is contained in the ring \( R_H = \{ x : 1 \leq |x| \leq H \} \). Since this set is compact in \( \mathbb{R}^n \setminus \{0\} \), \( Z^{-1} \) is \( \alpha \)-bi-Lipschitz on \( R_H \), for some \( \alpha \geq 1 \). Finally, the map \( x - \left[ \ln \frac{1}{\ell_f(e^{t_0})} \right] e_n \) is a translation, and hence just an isometry. Putting this together, we obtain

\[
\text{vol}_n f(A_t) \leq V_t \leq \alpha^n \text{vol}_n f(A_t).
\]

By (4.2) and (4.3) we have

\[
\tilde{\rho}_f(t_0 + t) - \tilde{\rho}_f(t_0) = O \left( \frac{\text{vol}_n f(A_t)}{\text{vol}_n f(B(0, e^{t_0}))} \right) = O \left( \frac{\ell_f(e^{t_0})^n V_t}{\text{vol}_n f(B(0, e^{t_0}))} \right) = O(V_t),
\]

where we use the fact that \( \ell_f(e^{t_0})^n \) is comparable to \( \text{vol}_n f(B(0, e^{t_0})) \) for all small enough \( e^{t_0} \). This completes the proof. \( \square \)

5. Proof of Theorem 1.3

Assuming \( \tilde{\rho}_f \) is locally \( L \)-bi-Lipschitz at every \( t \in (-\infty, M) \), for some \( L \) depending on \( n, K \) and \( P \), it follows that \( \tilde{\rho}_f \) is globally \( L \)-bi-Lipschitz on \( (-\infty, M) \). Hence it is enough to show that \( \tilde{\rho}_f \) is locally \( L \)-bi-Lipschitz. Since the quasisymmetry function \( \eta \) of \( f \) depends only on the maximal dilatation \( K \) of \( f \) (and the maximal dilatation of the fixed map \( Z \)), at any place below where a constant depends on \( \eta \), it depends on \( K \).

In the following subsections, we will show that bounds exist for

\[
\frac{\tilde{\rho}_f(t_0 + t) - \tilde{\rho}_f(t_0)}{t}
\]
as \( t \to 0 \) for \( t > 0 \). The arguments below can be easily modified to take into account the case where \( t < 0 \), and so we focus solely on the case when \( t > 0 \).

5.1. The lower bound. Suppose \( \tilde{f} \) satisfies the hypotheses of Theorem 1.3; \( \epsilon > 0 \) is small and for some \( t_0 < M^{-1} \), some \( 0 < t_2 < 1/2 \) and all \( 0 < t < t_2 \), we have

\[
\frac{\tilde{\rho}(t_0 + t) - \tilde{\rho}(t_0)}{t} < \epsilon.
\]

Our goal here is to show that if \( \epsilon \) is too small, we obtain a contradiction.

Set \( S_i = Q \times [t_0, t_0 + t] \) and let \( V_t = \text{vol}_n \tilde{f}(S_i) \). By Lemma 4.1 it follows that for \( 0 < t < \min\{t_1, t_2\} \) we have

\[
(5.1) \quad V_t \leq C_4 \epsilon t.
\]

Next, cover \( Q \times \{t_0\} \) by \((n-1)\)-dimensional cubes of side length \( \left(\lceil 1/t \rceil^{-1}\right) \) and form the \( n \)-dimensional boxes \( P_i \), for \( i = 1, \ldots, N \). Since the boxes \( P_i \) cover \( S_i \) with overlaps only on their boundaries, we have

\[
V_t = \sum_{i=1}^{N} \text{vol}_n \tilde{f}(P_i).
\]

By Lemma 3.2 this yields

\[
V_t \geq C_1^{-1} \sum_{i=1}^{N} \left( \text{diam} \tilde{f}(P_i) \right)^n.
\]

Applying (2.1) we obtain

\[
(5.2) \quad V_t \geq C_1^{-1} N^{1/(1-n)} \left( \sum_{i=1}^{N} \left( \text{diam} \tilde{f}(P_i) \right)^{n-1} \right)^{n/(n-1)}.
\]

Next, let \( \mathcal{P} : B \to Q \) denote the orthogonal projection from the beam \( B \) onto \( Q \). It is clear that

\[
\text{diam} \mathcal{P}(\tilde{f}(P_i)) \leq \text{diam} \tilde{f}(P_i)
\]

and hence that

\[
(5.3) \quad \text{vol}_{n-1} \mathcal{P}(\tilde{f}(P_i)) \leq \frac{\Omega_{n-1}(\text{diam} \tilde{f}(P_i))^{n-1}}{2^{n-1}}.
\]

Now, we must have

\[
Q \subset \bigcup_{i=1}^{N} \mathcal{P}(\tilde{f}(P_i)),
\]

for otherwise \( \partial f(B(0, e^{t_0})) \) would not separate 0 and \( \infty \). Hence

\[
(5.4) \quad \sum_{i=1}^{N} \text{vol}_{n-1} \mathcal{P}(\tilde{f}(P_i)) \geq \text{vol}_{n-1} Q = 2.
\]

Combining (5.2), (5.3) and (5.4), we obtain

\[
(5.5) \quad V_t \geq C_1^{-1} N^{1/(1-n)} \left( 2^n \Omega_{n-1}^{-1} \right)^{n/(n-1)} \geq C_5 N^{1/(1-n)},
\]

where \( C_5 \) depends only on \( n \) and \( \eta \). Since \( N/t^{1-n} \to 2 \) as \( t \to 0 \), by (5.1) and (5.5) we have

\[
C_4 \epsilon t \geq 2^{1/(1-n)} C_5 t,
\]

which yields a contradiction if \( \epsilon \) is small enough.
5.2. The upper bound. Suppose \( \tilde{f} \) satisfies the hypotheses of Theorem 1.3, \( \epsilon > 0 \) is small and for some \( t_0 < M - 1 \), some \( 0 < t_2 < 1/2 \) and all \( 0 < t < t_2 \), we have
\[
\frac{\tilde{\rho}_f(t_0 + t) - \tilde{\rho}_f(t_0)}{t} > \frac{1}{\epsilon}.
\]

Our goal here is to show that if \( \epsilon \) is too small, we obtain a contradiction.

As with the lower bound, set \( S_t = Q \times [t_0, t_0 + t] \) and let \( V_t = \text{vol}_n \tilde{f}(S_t) \). By Lemma 4.1 it follows that for \( 0 < t < \min\{t_1, t_2\} \) we have
\[
V_t \geq \frac{t}{C_3 \epsilon}.
\]

Next, cover \( Q \times \{t_0\} \) by \((n - 1)\)-dimensional cubes of side length \(((1/t)^{-1})^{-1} \) and form the \( n \)-dimensional boxes \( P_i \), for \( i = 1, \ldots, N \) with base \( Q_i \). Since the boxes \( P_i \) cover \( S_t \) with overlaps only on their boundaries, we have
\[
V_t = \sum_{i=1}^N \text{vol}_n \tilde{f}(P_i).
\]

By Lemma 3.2 this yields
\[
V_t \leq C_1 \sum_{i=1}^N \left( \text{diam} \tilde{f}(P_i) \right)^n.
\]

Then by Lemma 3.3 we obtain
\[
V_t \leq C_1 C_2^n \sum_{i=1}^N \left( \nu \tilde{f}(P_i) \right)^n.
\]

Now applying Lemma 3.5 we see that
\[
V_t \leq C_1 C_2^n C_3^{-n/(n-1)} \sum_{i=1}^N \left( \text{vol}_{n-1} \tilde{f}(Q_i) \right)^{n/(n-1)}.
\]

Next, we use the fact that \( f \) is a BIP map. Setting
\[
\gamma_t(x_1, \ldots, x_{n-1}) = \tilde{f}(x_1, \ldots, x_{n-1}, t),
\]
then for all \( t < M \),
\[
\int_Q \left\| \Pi \left( \frac{\partial \gamma_t}{\partial x_1}, \ldots, \frac{\partial \gamma_t}{\partial x_{n-1}} \right) \right\|_{L^2}^{n/(n-1)} \, dV \leq P.
\]

As a shorthand, for \( y_i \in Q \) and \( t < M \), we write \( \Pi(y_i) \) for the vector
\[
\Pi \left( \frac{\partial \gamma_t}{\partial x_1}(y_i), \ldots, \frac{\partial \gamma_t}{\partial x_{n-1}}(y_i) \right).
\]

We may suppose that \( t \) is small enough that the partition of \( Q \times \{t_0\} \) given by \( Q_1, \ldots, Q_N \) yields
\[
\text{vol}_{n-1} \tilde{f}(Q_i) \leq 2\| \Pi(y_i) \|_2 \text{vol}_{n-1}(Q_i) \leq 2^{n-1} \| \Pi(y_i) \|_2 t^{n-1},
\]
where \( y_i \in Q_i \) is chosen so that all the partial derivatives of \( \gamma_0 \) exist at \( y_i \), and the last inequality follows from (3.6). We may also suppose that \( t \) is chosen small enough that
\[
\sum_{i=1}^N \| \Pi(y_i) \|_2^{n/(n-1)} t^{n-1} \leq 2 \int_Q \left\| \Pi \left( \frac{\partial \gamma_t}{\partial x_1}, \ldots, \frac{\partial \gamma_t}{\partial x_{n-1}} \right) \right\|_{L^2}^{n/(n-1)} \, dV.
\]
It now follows from (5.7), (5.8) and (5.9) that

\[ V_t \leq C_1 C_2^n C_3^{n/(n-1)} \sum_{i=1}^{N} \left( \text{vol}_{n-1} \tilde{f}(Q_i) \right)^{n/(n-1)} \]

\[ \leq 2^n C_1 C_2^n C_3^{n/(n-1)} \sum_{i=1}^{N} \left( \| \Pi(y_i) \|_2^{n/(n-1)} t^{n-1} \right) \]

\[ \leq 2^{n+1} C_1 C_2^n C_3^{n/(n-1)} \left( \int_Q \left\| \Pi \left( \frac{\partial \gamma_t}{\partial x_1}, \ldots, \frac{\partial \gamma_t}{\partial x_{n-1}} \right) \right\|_2^{n/(n-1)} dV \right) t \]

\[ \leq C_6 P t, \]

where \( C_6 \) depends only on \( n \) and \( K \). Combining this with (5.6), we have

\[ \frac{t}{C_4 \epsilon} \leq C_6 P t, \]

which yields a contradiction for \( \epsilon \) small enough.

We emphasize that in both the upper and lower bounds the contradiction is obtained by finding \( \epsilon \) to be too small relative to constants that depend only on \( n \), \( K \) and \( P \). Hence \( \tilde{\rho_f} \) is locally \( L \)-bi-Lipschitz for some \( L \) depending only on \( n \), \( K \) and \( P \), which completes the proof of Theorem 1.3.

6. Non-rectifiable images

Proof of Proposition 1.4 Here we show in dimension two that if the image of a cross-section of the beam is a non-rectifiable curve, then \( \tilde{\rho_f} \) is not bi-Lipschitz. Recall that \( \tilde{f} \) is defined in \([0, 2] \times (-\infty, M)\), so suppose that \( \gamma(x) := \tilde{f}(x, t_0) \) parameterizes a non-rectifiable curve. In particular, its Hausdorff 1-measure is infinite.

Using the notation established above, we have by Lemma 3.2 that

\[ V_t = \sum_{i=1}^{N} \text{vol}_{n-1} \tilde{f}(P_i) \geq \frac{1}{C_1} \sum_{i=1}^{N} \left( \text{diam} \tilde{f}(P_i) \right)^2. \]

Since \( \gamma \) has infinite Hausdorff 1-measure, it follows that given \( \epsilon > 0 \) we can find \( t_1 > 0 \) so that if \( 0 < t < t_1 \), we have

\[ \sum_{i=1}^{N} \text{diam} \tilde{f}(P_i) > \frac{1}{\epsilon}. \]

By (2.1) and (3.6) we then have

\[ V_t \geq \frac{1}{C_1 N} \left( \sum_{i=1}^{N} \text{diam} \tilde{f}(P_i) \right)^2 > \frac{t}{C_1 \epsilon} \]

for \( 0 < t < t_1 \). An application of Lemma 4.1 then shows that

\[ \frac{\tilde{\rho_f}(t_0 + t) - \tilde{\rho_f}(t_0)}{t} > \frac{1}{C_1 C_4 \epsilon} \]

for \( 0 < t < t_1 \). Choosing \( \epsilon \) small enough shows that \( \tilde{\rho_f} \) is not bi-Lipschitz.

\( \square \)
7. Simple infinitesimal spaces

Proof of Theorem 1.3. Recall the asymptotic representative

\[ D(x) = \rho_f(x)g(x/|x|), \]

where \( g \) is the unique element in \( T(x_0, f) \). By [5 Proposition 4.18],

\[ g(x) = |x|^d \rho_f(x) \]

for some \( d > 0 \), and where \( g : S^{n-1} \to \mathbb{R}^n \) is BLD. In our case, \( g \) is bijective onto its image, as \( i(x_0, f) = 1 \), and so \( g|_{S^{n-1}} \) is bi-Lipschitz. Since \( g \) is quasiconformal, fixes 0 and preserves the measure of the unit ball, there exists \( C = C(K) \geq 1 \) so that \( g(S^{n-1}) \) is contained in the ring \( R = \{ x \in \mathbb{R}^n : 1/C \leq |x| \leq C \} \). As \( Z \) is bi-Lipschitz on \( Q \times \{ 0 \} \) and \( Z^{-1} \) is bi-Lipschitz on \( R \), and in particular on \( g(S^{n-1}) \), it follows that

\[ \tilde{g} : Q \times \{ 0 \} \to B \]

is bi-Lipschitz. Again letting \( \mathcal{P} : B \to Q \) be the orthogonal projection, we see from (7.1) that \( \tilde{g} : B \to B \) is given by

\[ \tilde{g}(x) = \tilde{g}(\mathcal{P}(x)) + (0, \ldots, 0, d|x|_n). \]

Since the restriction of \( \tilde{g} \) to each slice \( Q \times \{ t \} \) is bi-Lipschitz, and since \( \tilde{g} \) is quasiconformal, it follows from the bounded linear distortion that \( \tilde{g} \) is itself bi-Lipschitz.

Now, the Zorich transform of \( D \) is given by

\[ \tilde{D}(x) = \tilde{g}(\mathcal{P}(x)) + (0, \ldots, 0, \tilde{\rho}_f([x]_n)). \]

By Theorem 1.3, \( \tilde{\rho}_f \) is bi-Lipschitz on \( (-\infty, M) \). For every \( t < M \), it follows that there exists \( t' \in \mathbb{R} \) such that

\[ \tilde{D}|_{Q \times \{ t \}} = \tilde{g}|_{Q \times \{ t' \}}. \]

Writing \( h(t) = \tilde{\rho}_f(t)/d \), the map \( \tilde{g}^{-1} \circ \tilde{D} \) is nothing other than

\[ (x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_{n-1}, h(x_n)). \]

Since \( h \) is bi-Lipschitz on \( (-\infty, M) \), it follows that \( \tilde{g}^{-1} \circ \tilde{D} \) is bi-Lipschitz on \( B_M \). Since \( \tilde{g} \) is bi-Lipschitz, we conclude that \( \tilde{D} \) is also bi-Lipschitz and thus that \( D \) itself is quasiconformal.

\[ \square \]

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