A Polynomial Time Algorithm for Learning Halfspaces with Tsybakov Noise

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October 6, 2020

We study the problem of PAC learning homogeneous halfspaces in the presence of Tsybakov noise. In the Tsybakov noise model, the label of every sample is independently flipped with an adversarially controlled probability that can be arbitrarily close to $1/2$ for a fraction of the samples. We give the first polynomial-time algorithm for this fundamental learning problem. Our algorithm learns the true halfspace within any desired accuracy $\epsilon$ and succeeds under a broad family of well-behaved distributions including log-concave distributions. Prior to our work, the only previous algorithm for this problem required quasi-polynomial runtime in $1/\epsilon$.

Our algorithm employs a recently developed reduction \cite{DKTZ20b} from learning to certifying the non-optimality of a candidate halfspace. This prior work developed a quasi-polynomial time certificate algorithm based on polynomial regression. The main technical contribution of the current paper is the first polynomial-time certificate algorithm. Starting from a non-trivial warm-start, our algorithm performs a novel “win-win” iterative process which, at each step, either finds a valid certificate or improves the angle between the current halfspace and the true one. Our warm-start algorithm for isotropic log-concave distributions involves a number of analytic tools that may be of broader interest. These include a new efficient method for reweighting the distribution in order to recenter it and a novel characterization of the spectrum of the degree-2 Chow parameters.

*Supported by NSF Award CCF-1652862 (CAREER), a Sloan Research Fellowship, and a DARPA Learning with Less Labels (LwLL) grant.
†Supported by NSF Award CCF-1553288 (CAREER) and a Sloan Research Fellowship.
‡Supported in part by NSF Award CCF-1652862 (CAREER) and a DARPA Learning with Less Labels (LwLL) grant.
1 Introduction

The main result of this paper is the first polynomial-time algorithm for learning halfspaces in the presence of Tsybakov noise under a broad family of distributions. Before we explain our contributions in detail, we provide some context and motivation for this work.

1.1 Background

Learning in the presence of noise is a central challenge in machine learning. In this paper, we study the (supervised) binary classification setting, where the goal is to learn a Boolean function from random labeled examples with noisy labels. In more detail, we focus on the problem of learning homogeneous halfspaces in Valiant’s PAC learning model [Val84] when the labels have been corrupted by Tsybakov noise [Tsy04].

A (homogeneous) halfspace is any function \( h_w : \mathbb{R}^d \rightarrow \{\pm 1\} \) of the form \( h_w(x) = \text{sign}(\langle w, x \rangle) \), where the vector \( w \in \mathbb{R}^d \) is called the weight vector of \( h_w \) and \( \text{sign} : \mathbb{R} \rightarrow \{\pm 1\} \) is defined by \( \text{sign}(t) = 1 \) if \( t \geq 0 \) and \( \text{sign}(t) = -1 \) otherwise. Halfspaces (or Linear Threshold Functions) are arguably the most fundamental and extensively studied concept class in the learning theory and machine learning literature, starting with early work in the 1950s and 60s [Ros58, Nov62, MP68] and leading to fundamental and practically important techniques [Vap98, FS97].

Halfspaces are known to be efficiently learnable without noise, i.e., when the labels are consistent with a halfspace, see, e.g., [MT94]. In the presence of noisy labels, the picture is more muddled. In the agnostic model [Hau92, KSS94] (when a constant fraction of the labels can be adversarially chosen), learning halfspaces is computationally hard [GR06, FGKP06, Dan16], even under the Gaussian distribution [DKZ20, GGK20]. This motivates the study of “benign” noise models, where positive results may be possible. The most basic such model, known as Random Classification Noise (RCN) [ALS8], prescribes that each label is flipped independently with probability exactly \( \eta < 1/2 \). In the RCN model, halfspaces are known to be learnable in polynomial time [BPKV96].

The uniform noise assumption in the RCN model is commonly accepted to be unrealistic. To address this issue, various natural noise models have been proposed and studied, capturing a number of realistic noise sources. The two most prominent such models are, in order of increasing difficulty, the Massart (or bounded) noise model [MN06], and the Tsybakov noise model [Tsy04]. In the Massart model, each label is flipped independently with probability at most \( \eta < 1/2 \), but the flipping probability can depend on the example. The Tsybakov noise condition prescribes that the label of each example is independently flipped with some probability which is controlled by an adversary but is not uniformly bounded by a constant less than 1/2. In particular, the Tsybakov condition allows the flipping probabilities to be arbitrarily close to 1/2 for a fraction of the examples. More formally, we have the following definition:

**Definition 1.1** (PAC Learning with Tsybakov Noise). Let \( \mathcal{C} \) be a concept class of Boolean-valued functions over \( X = \mathbb{R}^d \), \( \mathcal{F} \) be a family of distributions on \( X \), \( 0 < \epsilon < 1 \) be the error parameter, and \( 0 \leq \alpha < 1 \), \( A > 0 \) be parameters of the noise model.

Let \( f \) be an unknown target function in \( \mathcal{C} \). A **Tsybakov example oracle**, \( \text{EX}^{\text{Tsyb}}(f, \mathcal{F}) \), works as follows: Each time \( \text{EX}^{\text{Tsyb}}(f, \mathcal{F}) \) is invoked, it returns a labeled example \( (x, y) \), such that: (a) \( x \sim \mathcal{D}_x \), where \( \mathcal{D}_x \) is a fixed distribution in \( \mathcal{F} \), and (b) \( y = f(x) \) with probability \( 1 - \eta(x) \) and \( y = -f(x) \) with probability \( \eta(x) \). Here \( \eta(x) \) is an unknown function that satisfies the \( (\alpha, A) \)-Tsybakov noise condition. That is, for any \( 0 < t \leq 1/2 \), \( \eta(x) \) satisfies \( \Pr_{x \sim \mathcal{D}_x}[\eta(x) \geq 1/2 - t] \leq A t^\alpha \).

Let \( \mathcal{D} \) denote the joint distribution on \( (x, y) \) generated by the above oracle. A learning algorithm is given i.i.d. samples from \( \mathcal{D} \) and its goal is to output a hypothesis function \( h : X \rightarrow \{\pm 1\} \) such that with high probability \( h \) is \( \epsilon \)-close to \( f \), i.e., it holds \( \Pr_{x \sim \mathcal{D}_x}[h(x) \neq f(x)] \leq \epsilon \).
The Tsybakov noise model was proposed in [MT99], then refined in [Tsy04], and subsequently studied in a number of works, see, e.g., [Tsy04] [BBL05] [BJM06] [BBT07] [Han11] [HY15]. All these prior works address information-theoretic aspects of the model, i.e., do not provide computationally efficient algorithms in high dimensions. In fact, until very recently, no non-trivial algorithm was known in the Tsybakov model for any non-trivial concept class, even under Gaussian marginals.

The only algorithmic result we are aware of in this model is the prior work by a subset of the authors [DKTZ20b], which gave a quasi-polynomial time algorithm for learning homogeneous halfspaces under a family of well-behaved distributions (including log-concave distributions).

It is easy to see that the Tsybakov model becomes more challenging as the parameter $\alpha$ in Definition 1.1 decreases. In particular, it is well-known that poly($d,1/\epsilon^{1/\alpha}$) samples are necessary (and sufficient) to learn halfspaces in this model. That is, an exponential dependence in $1/\alpha$ is information-theoretically required for any algorithm that solves this problem.

We note that the error guarantee of Definition 1.1 requires that the learning algorithm identifies the true function in the agnostic model is known to require time $\Omega(1/\epsilon)$ for halfspaces under Gaussian marginals [KKMS08] [DKZ20] [GGK20]. On the positive side, [ABL17] [Dan15] [DKS18] [DKTZ20c] gave poly($d/\epsilon$) time algorithms for agnostically learning halfspaces under log-concave marginals. These algorithms have error of $O(\text{OPT}) + \epsilon$, which is significantly weaker as explained in Remark 1.2.

Remark 1.2 (Identifiability versus Misclassification Error). Definition 1.1 requires that the learning algorithm identifies the true function $f \in C$ within arbitrary accuracy $\epsilon$. A related commonly used loss function is the misclassification error, i.e., the probability $\Pr_{(x,y) \sim D}[h(x) \neq y]$. We note that having an efficient algorithm with misclassification error OPT + $\epsilon$, for all $\epsilon > 0$, where OPT = inf$_{g \in C} \Pr_{(x,y) \sim D}[g(x) \neq y]$, is equivalent to having an efficient algorithm with the guarantee of Definition 1.1. We emphasize however that there is a major qualitative difference between achieving misclassification error of OPT + $\epsilon$ and achieving error $c \cdot \text{OPT} + \epsilon$, for a constant $c > 1$. The latter guarantee only allows us to approximate $f$ within error $\Omega(\text{OPT})$.

Obtaining error OPT + $\epsilon$ in the agnostic model is known to require time $d^{\text{poly}(1/\epsilon)}$ for halfspaces under Gaussian marginals [KKMS08] [DKZ20] [GGK20]. On the positive side, [ABL17] [Dan15] [DKS18] [DKTZ20c] gave poly($d/\epsilon$) time algorithms for agnostically learning halfspaces under log-concave marginals. These algorithms have error of $O(\text{OPT}) + \epsilon$, which is significantly weaker as explained in Remark 1.2.

1.2 Our Contributions

The existence of a computationally efficient learning algorithm in the presence of Tsybakov noise for any natural concept class and under any distributional assumptions has been a long-standing open problem in learning theory. In this work, we make significant progress in this direction by essentially resolving the complexity of learning halfspaces in this model.

In this section, we formally state our contributions. We start by defining the distribution family for which our algorithms succeed.

Definition 1.3 (Well-Behaved Distributions). For $L, R, U > 0$ and $k \in \mathbb{Z}_+$, a distribution $D_x$ on $\mathbb{R}^d$ is called $(k, L, R, U)$-well-behaved if for any projection $(D_x)_V$ of $D_x$ on a $k$-dimensional subspace $V$ of $\mathbb{R}^d$, the corresponding pdf $\gamma_V$ on $V$ satisfies the following properties: (i) $\gamma_V(x) \geq L$, for all $x \in V$ with $\|x\|_2 \leq R$ (anti-anti-concentration), and (ii) $\gamma_V(x) \leq U$ for all $x \in V$ (anti-concentration). If, additionally, there exists $\beta \geq 1$ such that, for any $t > 0$ and unit vector $w \in \mathbb{R}^d$, we have that $\Pr_{x \sim D_x}[\langle w, x \rangle \geq t] \leq \exp(1 - t/\beta)$ (sub-exponential concentration), we call $D_x$ $(k, L, R, U, \beta)$-well-behaved.
We focus on the case that the marginal distribution $D_x$ on the examples is well-behaved for some values of the relevant parameters. Definition 1.3 specifies the concentration and anti-concentration conditions on the low-dimensional projections of the data distribution that are required for our learning algorithm. Throughout this paper, we will take $k = 3$, i.e., we only require 3-dimensional projections to have such properties.

Interestingly, the class of well-behaved distributions is quite broad. In particular, it is easy to show that the broad class of isotropic log-concave distributions is well-behaved for $L,R,U,\beta$ being universal constants. Moreover, as Definition 1.3 does not require a specific functional form for the underlying density function, it encompasses a much more general set of distributions.

Since the complexity of our algorithm depends (polynomially) on $1/L,1/R,U,\beta$, we state here a simplified version of our main result for the case that these parameters are bounded by a universal constant. To simplify the relevant theorem statements, we will sometimes say that a distribution $\mathcal{D}$ of labeled examples in $\mathbb{R}^d \times \{\pm 1\}$ is well-behaved to mean that its marginal distribution $D_x$ is well-behaved. We show:

**Theorem 1.4** (Learning Tsybakov Halfspaces under Well-Behaved Distributions). Let $\mathcal{D}$ be a well-behaved isotropic distribution on $\mathbb{R}^d \times \{\pm 1\}$ that satisfies the $(\alpha,A)$-Tsybakov noise condition with respect to an unknown halfspace $f(x) = \text{sign}((w^*,x))$. There exists an algorithm that draws $N = O_{A,\alpha}(d/e)^{O(1/\alpha)}$ samples from $\mathcal{D}$, runs in $\text{poly}(N,d)$ time, and computes a vector $\hat{w}$ such that, with high probability we have that $\text{err}_{\mathcal{D}}^{\mathcal{D}_x}(h_{\hat{w}},f) \leq \epsilon$.

See Theorem 5.1 for a more detailed statement.

For the class of log-concave distributions, we give a significantly more efficient algorithm:

**Theorem 1.5** (Learning Tsybakov Halfspaces under Log-concave Distributions). Let $\mathcal{D}$ be a distribution on $\mathbb{R}^d \times \{\pm 1\}$ that satisfies the $(\alpha,A)$-Tsybakov noise condition with respect to an unknown halfspace $f(x) = \text{sign}((w^*,x))$ and is such that $D_x$ is isotropic log-concave. There exists an algorithm that draws $N = \text{poly}(d)O(A/e)^{O(1/\alpha^2)}$ samples from $\mathcal{D}$, runs in $\text{poly}(N,d)$ time, and computes a vector $\hat{w}$ such that, with high probability, we have that $\text{err}_{\mathcal{D}_x}^{\mathcal{D}}(h_{\hat{w}},f) \leq \epsilon$.

See Theorem 5.2 for a more detailed statement. Since the sample complexity of the problem is $\text{poly}(d,1/e^{1/\alpha})$, the algorithm of Theorem 1.5 is qualitatively close to best possible.

### 1.3 Overview of Techniques

Here we give an intuitive summary of our techniques in tandem with a comparison to the most relevant prior work. A more detailed technical discussion is provided in the proceeding sections.

Our learning algorithms employ the certificate-based framework of [DKTZ20b]. At a high-level, this framework allows us to efficiently reduce the problem of finding a near-optimal halfspace $h_{\hat{w}}(x) = \text{sign}((\hat{w},x))$ to the (easier) problem of certifying whether a candidate halfspace $h_w(x) = \text{sign}((w,x))$ is “far” from the optimal halfspace $f(x) = \text{sign}((w^*,x))$. The idea is to use a certificate algorithm (as a black-box) and combine it with an online convex optimization routine. Roughly speaking, starting from an initial guess $w_0$ for $w^*$, a judicious combination of these two ingredients allows us to efficiently compute a near-optimal halfspace $\hat{w}$, i.e., one that the certifying algorithm cannot reject. We note that a similar approach has been used in [CKMY20] for converting nonproper learners to proper learners in the Massart noise model.

With the aforementioned approach as the starting point, the learning problem reduces to that of designing an efficient certifying algorithm. In recent work [DKTZ20b], the authors developed a certifying algorithm for Tsybakov halfspaces based on high-dimensional polynomial regression.
This method leads to a certifying algorithm with sample complexity and runtime $d^{\text{polylog}(1/\epsilon)}$, i.e., a quasi-polynomial upper bound. As we will explain in Section 3.1, the [DKTZ20b] approach is inherently limited to quasi-polynomial time and new ideas are needed to obtain a polynomial time algorithm. The main contribution of this paper is the design of a polynomial-time certificate algorithm for Tsybakov halfspaces under well-behaved distributions.

The key idea to design a certificate in the Tsybakov noise model is the following simple but crucial observation: If $w^*$ is the normal vector to true halfspace, then for any non-negative function $T(x)$, it holds that $E_{(x,y)\sim D}[T(x)y \langle w^*, x \rangle] \geq 0$. On the other hand, for any $w \neq w^*$ there exists a non-negative function $T(x)$ such that $E_{(x,y)\sim D}[T(x)y \langle w, x \rangle] < 0$. In other words, there exists a reweighting of the space that makes the expectation of $y \langle w, x \rangle$ negative (Fact 3.1). Note that we can always use as $T(x)$ the indicator of the disagreement region between the candidate halfspace $h_w(x)$ and the optimal halfspace $f(x) = h_{w^*}(x)$. Of course, since optimizing over the space of non-negative functions is intractable, we need to restrict our search space to a “simple” parametric family of functions. In [DKTZ20b], squares of low-degree polynomials were used, which led to a quasi-polynomial upper bound.

In this work, we consider certifying functions of the form:

$$ T(x) = \frac{1}{\langle w, x \rangle} \mathbb{1}\left\{ \sigma_1 \leq \langle w, x \rangle \leq \sigma_2, -t_1 \leq \left\langle v, \text{proj}_{w^\perp} x \right\rangle \leq -t_2 \right\} $$

that are parameterized by a vector $v$ and scalar thresholds $\sigma_1, \sigma_2, t_1, t_2 > 0$. Here $\text{proj}_{w^\perp}$ denotes the orthogonal projection on the subspace orthogonal to $w$. It will be important for our approach that functions of this form are specified by $O(d)$ parameters.

Of course, it may not be a priori clear why functions of this form can be used as certifying functions in our setting. The intuition behind choosing functions of this simple form is given in Section 3.1. In particular, in Claim 3.4, we show that for any incorrect guess $w$ there exists a certifying vector $v$ that makes the expectation $E_{(x,y)\sim D}[T(x)y \langle w, x \rangle]$ negative. In fact, the vector $v = \frac{\text{proj}_{w^\perp} w^*}{\|\text{proj}_{w^\perp} w^*\|_2} := (w^*)^{1-w}$ suffices for this purpose.

The key challenge is in finding such a certifying vector $v$ algorithmically. We note that our algorithm in general does not find $(w^*)^{1-w}$. But it does find a vector $v$ with similar behavior, in the sense of making the $E_{(x,y)\sim D}[T(x)y \langle w, x \rangle]$ sufficiently negative. To achieve this goal, we take a two-step approach: The first step involves computing an initialization vector $v_0$ that has non-trivial correlation with $(w^*)^{1-w}$. In our second step, we give a perceptron-like update rule that iteratively improves the initial guess until it converges to a certifying vector $v$. While this algorithm is relatively simple, its correctness relies on a win-win analysis (Lemma 3.12) whose proof is quite elaborate. In more detail, we show that for any non-certifying vector $v$ that is sufficiently correlated with $(w^*)^{1-w}$, we can efficiently compute a direction that improves its correlation to $(w^*)^{1-w}$. We then argue (Lemma 3.17) that by choosing an appropriate step size this iteration converges to a certifying vector within a small number of steps.

A subtle point is that the aforementioned analysis does not take place in the initial space, where the underlying distribution is well-behaved and the labels are Tsybakov homogeneous halfspaces, but in a transformed space. The transformed space is obtained by restricting our points in a band and then performing an appropriate “perspective” projection on the subspace orthogonal to $w$ (Section 3.2). Fortunately, we are able to show (Proposition 3.6) that this transformation preserves the structure of the problem: The transformed distribution remains well-behaved (albeit with somewhat worse parameters) and satisfies the Tsybakov noise condition (again with somewhat worse parameters) with respect to a potentially biased halfspace. In fact, this consideration motivated our use of the perspective projection in the definition of $T(x)$. 

4
It remains to argue how to compute an initialization vector $v_0$ that acts as a warm-start for our algorithm. Naturally, the sample complexity and runtime of our certificate algorithm depend on the quality of the initialization. The simplest way to initialize is by using a random unit vector. With random initialization, we achieve initial correlation roughly $1/\sqrt{d}$, which leads to a certifying algorithm with complexity $(d/\epsilon)^{O(1/\alpha)}$ (Theorem 3.3). This simple initialization suffices to obtain Theorem 1.4 for the general class of well-behaved distributions.

To obtain our faster algorithm for log-concave marginals (Theorem 1.5), we use the exact same approach described above starting from a better initialization. Our algorithm to obtain a better starting vector leverages additional structural properties of log-concave distributions. Our initialization algorithm runs in poly($d$) time (independent of $1/\alpha$) and computes a unit vector whose correlation with $(w^\star)^\perp$ is $\Omega(\epsilon^{1/\alpha})$ (Theorem 4.2).

Specifically, our initialization algorithm works as follows:

1. It starts by conditioning on a random sufficiently narrow band around the current candidate $w$ and projecting the samples on the subspace $w^\perp$.

2. It transforms the resulting distribution to ensure that it is isotropic log-concave through rescaling and rejection sampling.

3. It then computes the degree-2 Chow parameters and uses them to construct a low-dimensional subspace $V$ inside which $(w^\star)^\perp$ has sufficiently large projection. This subspace $V$ is the span of the degree-1 Chow vector and the large eigenvectors of the degree-2 Chow matrix.

4. Finally, the algorithm outputs a uniformly random vector in $V$ that can be shown to have the desired correlation with $(w^\star)^\perp$.

The resulting distribution after the initial conditioning in Step 1 is still log-concave and approximately satisfies the Tsybakov noise condition with respect to a near-origin centered halfspace orthogonal to $w$. However, the distribution may no longer be zero-centered and may contain a tiny amount of non-Tsybakov noise — in the sense that we may end with points $x$ having $\eta(x) > 1/2$. As we can control the total non-Tsybakov noise, the latter is not a significant issue. We address the former issue by reweighting the distribution to make it isotropic. We do this by applying rejection sampling with probability $\min(1, \exp(-\langle x, r \rangle))$, for some vector $r$ that we compute via SGD (so that the resulting mean is near-zero) and then rescaling by the inverse covariance matrix.

After the first two steps, our goal is to find any vector with non-trivial correlation $(w^\star)^\perp$ given that the underlying distribution is isotropic log-concave. We show that the labels $y$ must correlate with some degree-2 polynomial in $\langle (w^\star)^\perp, x \rangle$ (Lemma 4.9). Our algorithm crucially exploits this property, along with recently established “thin shell” estimates [LV17] for log-concave distributions, to show that a large part of this correlation is explained by the vector of degree-1 Chow parameters and the top few eigenvectors of the degree-2 Chow matrix (Lemma 4.10). This implies that the subspace $V$ spanned by those vectors contains a non-trivial part of $(w^\star)^\perp$, and thus a random vector from $V$ has non-trivial correlation with $(w^\star)^\perp$ with constant probability.

1.4 Related Work

Recent work by a subset of the authors [DKTZ20b] gave the first non-trivial algorithm for learning homogeneous halfspaces with Tsybakov noise under a family of “well-behaved” distributions. The notion of well-behaved distributions in that work is somewhat different than ours, but also contains log-concave distributions. The sample complexity and runtime of the [DKTZ20b] algorithm is $d^{\text{Polylog}(1/\epsilon)}$ and the quasi-polynomial upper bound is tight for their techniques.
Finally, we denote by \( w \), i.e., \( w \) to denote the subspace spanned by vectors orthogonal to \( U \) the unit sphere. We will denote by \( \text{proj}_A \) a distribution \( f,h \) between two hypotheses \( x,y \) and \( E \) an event. We use small boldface characters for vectors. For a vector \( x \) on \( \mathbb{R}^d \) and \( i \in [d] \), \( x_i \) denotes the \( i \)-th coordinate of \( x \), and \( \|x\|_2 = (\sum_{i=1}^{d} x_i^2)^{1/2} \) denotes the \( \ell_2 \)-norm of \( x \). We will use \((x,y)\) for the inner product of \( x,y \in \mathbb{R}^d \) and \( \theta(x,y) \) for the angle between \( x,y \). We will use \( 1_A \) to denote the characteristic function of the set \( A \), i.e., \( 1_A(x) = 1 \) if \( x \in A \) and \( 1_A(x) = 0 \) if \( x \notin A \).

Let \( e_i \) be the \( i \)-th standard basis vector in \( \mathbb{R}^d \). For \( d \in \mathbb{N} \), let \( S^{d-1} \overset{\text{def}}{=} \{x \in \mathbb{R}^d : \|x\|_2 = 1\} \) be the unit sphere. We will denote by \( \text{proj}_U(x) \) the projection of \( x \) onto the subspace \( U \subset \mathbb{R}^d \). For a subspace \( U \subset \mathbb{R}^d \), let \( U^\perp \) be the orthogonal complement of \( U \). For a vector \( w \in \mathbb{R}^d \), we use \( w^\perp \) to denote the subspace spanned by vectors orthogonal to \( w \), i.e., \( w^\perp = \{u \in \mathbb{R}^d : (w,u) = 0\} \). Finally, we denote by \( w^{+v} \) the projection of the vector \( w \) on the subspace \( v^\perp \) after normalization, i.e., \( w^{+v} = \frac{w-(w,v)v}{\|w-(w,v)v\|_2} \).

We use \( E[X] \) for the expectation of the random variable \( X \) and \( \Pr[E] \) for the probability of the event \( E \).

We study the binary classification setting where labeled examples \((x,y)\) are drawn i.i.d. from a distribution \( D \) on \( \mathbb{R}^d \times \{\pm 1\} \). We denote by \( D_x \) the marginal of \( D \) on \( x \). The zero-one error between two hypotheses \( f,h \) (with respect to \( D_x \)) is \( \text{err}_{D_x}^0(f,h) \overset{\text{def}}{=} \Pr_{x \sim D_x}[f(x) \neq h(x)] \).

The Tsybakov noise model lies in between the Massart model [Slo88, MN06] and the agnostic model [Han92, KSS04]. During the past five years, substantial algorithmic progress has been made on learning with Massart noise in both the distribution-specific setting [ABHU15, ABHZ16, ZLC17, YZ17, ZSA20, DKTZ20a] and the distribution-free PAC model [DKT19, CKMY20]. The algorithmic techniques in these prior works are known to inherently fail for the more challenging Tsybakov noise model, and new ideas are needed for this more general setting.

Learning in the agnostic model is known to be computationally hard, even under well-behaved marginals. Specifically, recent work [DKZ20, GGK20] proved Statistical Query lower bounds of \( d^{\text{poly}(1/\epsilon)} \) for agnostically learning halfspaces to error \( \text{OPT} + \epsilon \) under Gaussian marginals. This lower bound bound is qualitatively matched by the \( L_1 \) regression algorithm [KKMS08]. A related line of work [KLS09, ABL17, Dan15, DKS18, DKTZ20b] gave efficient algorithms for agnostically learning halfspaces under log-concave marginals. While these algorithms run in \( \text{poly}(d/\epsilon) \) time, they achieve a “semi-agnostic” error guarantee of \( O(\text{OPT}) + \epsilon \), instead of \( \text{OPT} + \epsilon \). As already mentioned in Remark 1.2 this guarantee is significantly weaker and cannot be used to approximate the true function within any desired accuracy.

This work is part of the broader direction of designing robust learning algorithms for a range of statistical models with respect to natural and challenging noise models. A line of work [KLS09, ABL17, DKK±16, LRV16, DKK+17, DKK+18, DKS18, KKM18, DKS19, DKK+19] has given efficient robust learners for a range of settings in the presence of adversarial corruptions. See [DK19] for a recent survey on the topic.

### 1.5 Structure of This Paper

After the required preliminaries in Section 2, in Section 3 we give our certifying algorithm for the class of well-behaved distributions. In Section 4, we give our more efficient certifying algorithm for log-concave distributions. Finally, in Section 5 we review the certificate framework and put everything together to prove our main results.

### 2 Preliminaries

For \( n \in \mathbb{Z}_+ \), let \( [n] \overset{\text{def}}{=} \{1,\ldots,n\} \). We will use small boldface characters for vectors. For \( x \in \mathbb{R}^d \) and \( i \in [d] \), \( x_i \) denotes the \( i \)-th coordinate of \( x \), and \( \|x\|_2 = (\sum_{i=1}^{d} x_i^2)^{1/2} \) denotes the \( \ell_2 \)-norm of \( x \). We will use \((x,y)\) for the inner product of \( x,y \in \mathbb{R}^d \) and \( \theta(x,y) \) for the angle between \( x,y \). We will use \( 1_A \) to denote the characteristic function of the set \( A \), i.e., \( 1_A(x) = 1 \) if \( x \in A \) and \( 1_A(x) = 0 \) if \( x \notin A \).

Let \( e_i \) be the \( i \)-th standard basis vector in \( \mathbb{R}^d \). For \( d \in \mathbb{N} \), let \( S^{d-1} \overset{\text{def}}{=} \{x \in \mathbb{R}^d : \|x\|_2 = 1\} \) be the unit sphere. We will denote by \( \text{proj}_U(x) \) the projection of \( x \) onto the subspace \( U \subset \mathbb{R}^d \). For a subspace \( U \subset \mathbb{R}^d \), let \( U^\perp \) be the orthogonal complement of \( U \). For a vector \( w \in \mathbb{R}^d \), we use \( w^\perp \) to denote the subspace spanned by vectors orthogonal to \( w \), i.e., \( w^\perp = \{u \in \mathbb{R}^d : (w,u) = 0\} \). Finally, we denote by \( w^{+v} \) the projection of the vector \( w \) on the subspace \( v^\perp \) after normalization, i.e., \( w^{+v} = \frac{w-(w,v)v}{\|w-(w,v)v\|_2} \).

We use \( E[X] \) for the expectation of the random variable \( X \) and \( \Pr[E] \) for the probability of the event \( E \).

We study the binary classification setting where labeled examples \((x,y)\) are drawn i.i.d. from a distribution \( D \) on \( \mathbb{R}^d \times \{\pm 1\} \). We denote by \( D_x \) the marginal of \( D \) on \( x \). The zero-one error between two hypotheses \( f,h \) (with respect to \( D_x \)) is \( \text{err}_{D_x}^0(f,h) \overset{\text{def}}{=} \Pr_{x \sim D_x}[f(x) \neq h(x)] \).
3 Efficiently Certifying Non-Optimality

In this section, we give an efficient algorithm that can certify whether a candidate weight vector \( w \) defines a halfspace \( h_w(x) = \text{sign}(\langle w, x \rangle) \) that is far from the optimal halfspace \( f(x) = \text{sign}(\langle w^*, x \rangle) \).

Before we formally describe and analyze our algorithm, we provide some intuition.

**Background: Certifying Non-Optimality.** Our approach relies on the following simple but powerful idea, introduced in [DKT20]: If a candidate weight vector \( w \) defines a halfspace \( h_w(x) = \text{sign}(\langle w, x \rangle) \) that differs from the target halfspace \( f(x) = \text{sign}(\langle w^*, x \rangle) \), there exists a certifying function of its non-optimality. In more detail, there exists a reweighting of the space that makes the expectation of \( y \langle w, x \rangle \) negative. This intuition is captured in Fact 3.1 stated below. We note that the only assumption required for this to hold is that the underlying distribution on examples assigns positive mass to the symmetric difference of any two distinct halfspaces.

**Fact 3.1 (Certifying Function).** Let \( D \) be a distribution on \( \mathbb{R}^d \times \{ \pm 1 \} \) such that: (a) For any pair of distinct unit vectors \( v, u \in \mathbb{R}^d \), we have that \( \mathbf{Pr}_{x \sim D}[h_v(x) \neq h_u(x)] > 0 \). (b) \( D \) satisfies the Tsybakov noise condition with optimal classifier \( f(x) = \text{sign}(\langle w^*, x \rangle) \). Then we have:

1. For any \( T : \mathbb{R}^d \mapsto \mathbb{R}_+ \), we have that \( E_{(x,y) \sim D}[T(x) y \langle w^*, x \rangle] \geq 0 \).

2. For any non-zero vector \( w \in \mathbb{R}^d \) such that \( \theta(w, w^*) > 0 \), there exists a function \( T : \mathbb{R}^d \mapsto \mathbb{R}_+ \) satisfying \( E_{(x,y) \sim D}[T(x) y \langle w, x \rangle] < 0 \).

**Proof.** For the first statement, note that

\[
E_{(x,y) \sim D}[T(x) y \langle w^*, x \rangle] = E_{x \sim D}[T(x) \langle w^*, x \rangle | (1 - \eta(x))] - E_{x \sim D}[T(x) \langle w^*, x \rangle | \eta(x)] = E_{x \sim D}[T(x) \langle w^*, x \rangle | (1 - 2\eta(x))] \geq 0,
\]

where we used the fact that \( \eta(x) \leq 1/2 \) and \( T(x) \geq 0 \).

For the second statement, let \( w \neq 0 \) and \( \theta(w, w^*) > 0 \). By picking as a certifying function \( T \) the indicator function of the disagreement region between \( f \) and \( h_w \), i.e., \( T(x) = 1 \{ h_w(x) \neq f(x) \} \), we have that

\[
E_{(x,y) \sim D}[T(x) y \langle w, x \rangle] = -E_{x \sim D}[T(x) \langle w, x \rangle | (1 - 2\eta(x))].
\]

We claim that \( E_{x \sim D}[T(x) \langle w, x \rangle | (1 - 2\eta(x))] > 0 \), which proves the second statement. To see this, we use our assumption that the symmetric difference between any pair of distinct homogeneous halfspaces has positive probability mass. First, we note that from the Tsybakov condition (for any choice of parameters) we have that \( \mathbf{Pr}_{x \sim D}[\eta(x) = 1/2] = 0 \). So, it suffices to show that \( E_{x \sim D}[T(x) \langle w, x \rangle] > 0 \).

Let \( w' \) be a non-zero vector such that the hyperplane \( \{ x : \langle w', x \rangle = 0 \} \) is contained in the disagreement region \( \{ x : h_w(x) \neq f(x) \} \) and \( \theta(w, w'), \theta(w^*, w') > 0 \). This implies that \( \{ x : h_w(x) \neq f(x) \} \supseteq \{ x : h_{w'}(x) \neq f(x) \} \) and \( \mathbf{Pr}_{x \sim D}[h_w(x) \neq f(x)] > 0 \). Note that \( \langle w, x \rangle > 0 \) for all \( x \) with \( h_{w'}(x) \neq f(x) \). Therefore, we get that

\[
E_{x \sim D}[T(x) \langle w, x \rangle] \geq E_{x \sim D}[1 \{ h_{w'}(x) \neq f(x) \} \langle w, x \rangle] > 0.
\]

This completes the proof of Fact 3.1.
Main Result of this Section. Fact 3.1 shows that a certifying function exists. However, in general, finding such a function is information-theoretically and computationally hard. By leveraging our distributional assumptions, we show that a certifying function of a specific simple form exists and can be computed in polynomial time.

For the rest of this section, we work with distributions that are \((3,L,R,\beta)\)-well-behaved. These distributions satisfy the same properties as those in Definition 1.3 except the anti-concentration condition. (The anti-concentration condition is only required at the end of our analysis in Section 5 to deduce that small angle between two halfspaces implies small 0-1 error.)

Definition 3.2. For \(L, R > 0, \beta \geq 1\), and \(k \in \mathbb{Z}_+\), a distribution \(D_x\) on \(\mathbb{R}^d\) is called \((k, L, R, \beta)\)-well-behaved if the following conditions hold: (i) For any projection \((D_x)_V\) of \(D_x\) on a \(k\)-dimensional subspace \(V\) of \(\mathbb{R}^d\), the corresponding pdf \(\gamma_V\) on \(V\) satisfies \(\gamma_V(x) \geq L\), for all \(x \in V\) with \(\|x\|_2 \leq R\) (anti-anti-concentration). (ii) For any \(t > 0\) and unit vector \(w \in \mathbb{R}^d\), we have that 

\[
\mathbb{P}_{x \sim D_x}[\langle w, x \rangle \geq t] \leq \exp(1 - t/\beta) \quad \text{(sub-exponential concentration)}.
\]

Specifically, we have:

Theorem 3.3 (Efficiently Certifying Non-Optimality). Let \(D\) be a \((3, L, R, \beta)\)-well-behaved isotropic distribution on \(\mathbb{R}^d \times \{\pm 1\}\) that satisfies the \((\alpha, A)\)-Tsybakov noise condition with respect to an unknown halfspace \(f(x) = \text{sign}(\langle w^*, x \rangle)\). Let \(w\) be a unit vector with \(\theta(w, w^*) \geq \theta\), where \(\theta \in (0, \pi]\). There is an algorithm that, given as input \(w, \theta\), and \(N = ((A/\theta)(LR)^{\alpha}(d/\theta)^{\alpha/2})\log(1/\delta)\) samples from \(D\), it runs in \(\text{poly}(N, d)\) time, and with probability at least \(1 - \delta\) returns a certifying function \(T_w : \mathbb{R}^d \rightarrow \mathbb{R}_+\) such that

\[
\mathbb{E}_{(x,y) \sim D}[T_w(x) \gamma(w, x)] \leq \frac{1}{\beta} \left(\frac{LR \theta}{A d}\right)^{O(1/\alpha)}.
\]

3.1 Intuition and Roadmap of the Proof

In this subsection, we give an intuitive proof overview of Theorem 3.3 along with pointers to the corresponding subsections where the proof of each component appears. First, we discuss the specific form of the certifying function that we compute. The proof of Fact 3.1 shows that a valid choice for the certifying function would be the characteristic function of the disagreement region between the candidate hypothesis \(w\) and the optimal halfspace \(w^*\), i.e., \(T_w(x) = \mathbb{I}\{\text{sign}(\langle w, x \rangle) \neq \text{sign}(\langle w^*, x \rangle)\}\). Unfortunately, we do not know \(w^*\) (this is the vector we are trying to approximate!), and therefore it is unclear how to algorithmically use this certifying function.

Our goal is to judiciously define a parameterized family of “simple” certifying functions and optimize over this family to find one that acts similarly to the indicator of the disagreement region. A natural attempt to construct a certifying function for a guess \(w\) would be to focus on a small “band” around the candidate halfspace \(w\). This idea bears some similarity with the technique of “localization”, an approach going back to [BBM05], which has previously seen success for the problem of efficiently learning homogeneous halfspaces with Massart noise [ABHU15, ABHZ16, ZSA20, DKTZ20a]. Unfortunately, this idea is inherently insufficient to provide us with a certifying function for the following reason: Even an arbitrarily thin band around \(w\) will assign more probability mass on points that do not belong in the disagreement region, and therefore the expectation \(\mathbb{E}_{(x,y) \sim D}[\mathbb{I}\{\sigma_1 \leq \langle w, x \rangle \leq \sigma_2\} \gamma(w, x)]\) will be positive. See Figure 1 for an illustration.

Intuitively, we need a way to boost the contribution of the disagreement region. One way to achieve this is by constructing a smooth reweighting of the space. In particular, we can look in the
direction of the projection of $w^*$ on the orthogonal complement of $w$, i.e., the vector

$$(w^*)^\perp = \frac{\text{proj}_{w^\perp}(w^*)}{\|\text{proj}_{w^\perp}(w^*)\|_2},$$

that lies in the 2-dimensional subspace spanned by $w$ and $w^*$; see Figure 1. Notice that the disagreement region is a subset of the points that have negative inner product with $(w^*)^\perp$. Therefore, a candidate reweighting can be obtained by using a polynomial $p((w^*)^\perp, x)$ of moderately large degree that will boost the points that lie in the disagreement region. This was the approach used in the recent work [DKTZ20b]. Since $(w^*)^\perp$ is unknown, one needs to formulate a convex program (SDP) over the space of all $d$-variate polynomials of sufficiently large degree $k$ implying that the corresponding SDP has $d^{O(k)}$ variables. Unfortunately, it is not hard to show that the required degree cannot be smaller than $\Omega(\log(1/\epsilon))$. Therefore, this approach can only give a $d^{\Omega(\log(1/\epsilon))}$, i.e., quasi-polynomial, certificate algorithm.

In this work, we instead use a hard threshold function together with a band to isolate (a non-trivial subset of) the disagreement region. In more detail, we consider a function of the form $1\{1_{\{\sigma_1 \leq \langle w, x \rangle \leq \sigma_2\}} 1_{\{\langle v, x \rangle < t\}} \langle w, x \rangle\}$ for some scalar threshold $t$; see Figure 1. Since $(w^*)^\perp$ is unknown, we need to find a certifying vector $v$ that is perpendicular to $w$, i.e., $v \in w^\perp$ and acts similarly to $(w^*)^\perp$. This leads us to the following non-convex optimization problem

$$\min_{t \in \mathbb{R}, v \in w^\perp} \mathbb{E}_{(x,y) \sim \mathcal{D}} [1\{\sigma_1 \leq \langle w, x \rangle \leq \sigma_2\} 1\{\langle v, x \rangle < t\} \langle w, x \rangle].$$

Thus far, we have succeeded in reducing the number of parameters that we want to compute down to $O(d)$, but now we are faced with a non-convex optimization problem. Our main result is an efficient algorithm that computes a certifying vector $v$ and a threshold $t$ that does not necessarily minimize the above non-convex objective, but still suffice to make the corresponding expectation sufficiently negative.

We now describe the main steps we use to compute the certifying vector $v$. The first obstacle we need to overcome is that, for $v \in w^\perp$, the corresponding instance fails to satisfy the Tsybakov noise condition. In particular, when we project the datapoints on $w^\perp$, the region close to the boundary of
the optimal halfspace becomes “fuzzy” even without noise: Points with different labels are mapped to the same point of \( \mathbf{w}^\perp \), see Figure 2a. We bypass this difficulty by using a \textit{perspective projection} to map the datapoints onto \( \mathbf{w}^\perp \). For non-zero vectors \( \mathbf{w}, \mathbf{x} \in \mathbb{R}^d \), the perspective projection of \( \mathbf{x} \) on \( \mathbf{w} \) is defined as follows:

\[
\pi_{\mathbf{w}}(\mathbf{x}) \equiv \text{proj}_{\mathbf{w}^\perp} \frac{\mathbf{x}}{\langle \mathbf{w}, \mathbf{x} \rangle}.
\]  

(2)

Notice that without noise the perspective projection keeps the dataset linearly separable (see Figure 2b), which means that after we perform this projection the label noise of the resulting instance will again satisfy the Tsybakov noise condition. In addition, we show that this transformation will preserve the crucial distributional properties (concentration, anti-anti-concentration) of the underlying marginal distribution \( \mathcal{D}_x \). For a detailed discussion and analysis of this data transformation, see Subsection 3.2.

Given this setup, the certificate that our algorithm will compute for a candidate weight vector \( \mathbf{w} \in \mathbb{R}^d \) is a function of the form

\[
T_w(\mathbf{x}) = \frac{1}{\langle \mathbf{w}, \mathbf{x} \rangle} \mathbb{I} \{ \sigma_1 \leq \langle \mathbf{w}, \mathbf{x} \rangle \leq \sigma_2, -t_1 \leq \langle \mathbf{v}, \pi_{\mathbf{w}}(\mathbf{x}) \rangle \leq -t_2 \} =: \frac{\psi(\mathbf{x})}{\langle \mathbf{w}, \mathbf{x} \rangle},
\]  

for some vector \( \mathbf{v} \in \mathbb{R}^d \) and scalars \( \sigma_1, \sigma_2, t_1, t_2 > 0 \). For an illustration, in Figure 2b we plot the set of the indicator function \( \psi(\mathbf{x}) \) which is a (high-dimensional) trapezoid.

It is not difficult to verify that by choosing \( \mathbf{v} = (\mathbf{w}^*)_{\mathbf{w}} \) and appropriately picking \( \sigma_1, \sigma_2, t_1, t_2 \), the corresponding certificate function \( T_w \) resembles the indicator function of the disagreement region and certifies the \textit{non-optimality} of the candidate halfspace \( \mathbf{w} \). In the following claim, we prove that for any non-optimal halfspace there exists a certifying function of the above form.

\textbf{Claim 3.4.} \( \) Let \( \mathcal{D} \) be a \((3, L, R, \beta)\)-well-behaved isotropic distribution on \( \mathbb{R}^d \times \{ \pm 1 \} \) that satisfies the \((\alpha, A)\)-Tsybakov noise condition with respect to an unknown halfspace \( f(\mathbf{x}) = \text{sign}(\langle \mathbf{w}^*, \mathbf{x} \rangle) \). Fix any non-zero vector \( \mathbf{w} \) such that \( \theta(\mathbf{w}, \mathbf{w}^*) > 0 \). Then, by setting \( \mathbf{v} = (\mathbf{w}^*)_{\mathbf{w}} \) in the definition (3) of \( T_w(\mathbf{x}) \), there exist \( \sigma_1, \sigma_2, t_1, t_2 > 0 \) such that \( \mathbb{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[T_w(\mathbf{x}) y \langle \mathbf{w}, \mathbf{x} \rangle] < 0 \).
We note here that the proof of Claim 3.4 is sketched below for the sake of intuition and is not required for the subsequent analysis.

Proof Sketch. Setting \( v = (w^*)^⊥ \) in [3], we have

\[
E_{(x,y) \sim D} [T_w(x) y \langle w, x \rangle] = E_{(x,y) \sim D} [\psi(x) y] = E_{(x,y) \sim D} [\psi(x) (1 - 2\eta(x)) \text{sign}(\langle w^*, x \rangle)].
\]

We will show that by appropriate choices of \( \sigma_1, \sigma_2, t_1, t_2 \) the indicator \( \psi(x) \) above corresponds to a subset of the disagreement region \( \{ x : \text{sign}(\langle w, x \rangle) \neq \text{sign}(\langle w^*, x \rangle) \} \). See Figure 3 for an illustration. More precisely, since the distribution satisfies an anti-anti-concentration property, we can choose \( \sigma_1, \sigma_2 = \Theta(R) \), so that inside the band \( \{ \sigma_1 \leq \langle w, x \rangle \leq \sigma_2 \} \) there is non-zero probability mass. In particular, by setting \( \sigma_1 = \rho R/2 \) and \( \sigma_2 = \rho R/\sqrt{2} \), for some \( \rho \in (0,1] \), we have that the band has mass roughly \( \Omega(\rho R^3) \). For these choices of \( \sigma_1 \) and \( \sigma_2 \), we can pick \( t_1 = \Theta(R/\rho) \) and guarantee that the slope of the corresponding line in the two-dimensional subspace is sufficiently small, so that we get a trapezoid whose intersection with the aforementioned horizontal band is large (see Figure 3). It remains to tune the parameter \( t_2 \). Since \( \theta = \theta(w, w^*) \) is known, we may pick \( t_2 = \Theta(R \tan \theta/\rho) \) in order to make sure that the trapezoid is a subset of the disagreement region between \( w^* \) and \( w \).

![Figure 3](image-url)

Figure 3: The function \( \psi(x) \) for \( v = (w^*)^⊥ \) defined in [3] and appropriate scalars \( \sigma_1, \sigma_2, t_1, t_2 \) is the indicator of a subset of the disagreement region \( \{ x : \text{sign}(\langle w, x \rangle) \neq \text{sign}(\langle w^*, x \rangle) \} \).

From the above proof, it is clear that one does not really need to optimize the scalars \( \sigma_1, \sigma_2, t_1 \). Their values can be chosen according to the parameters of the underlying well-behaved distribution. Our optimization problem will be with respect to the vector \( v \) and the threshold \( t_2 \). However, optimizing the expectation of the certifying function \( T_w \) of Equation (3) is still a non-convex problem. Given a candidate certifying vector \( v_0 \) that has non-trivial correlation with \( (w^*)^⊥ \), our main structural result is a win-win statement showing that either there exists a threshold \( t_2 \) that, together with \( v_0 \), makes the corresponding expectation of \( T_w \) sufficiently negative, or a perceptron-like update rule will improve the correlation between \( (w^*)^⊥ \) and \( w \). In particular, we show that after roughly \( \text{poly}(d/\epsilon) \) updates the correlation between the guess \( v \) and \( (w^*)^⊥ \) will be sufficiently large so that there exists some threshold \( t_2 \) that makes \( v \) a certifying vector. Having such a vector \( v \), it is easy to optimize over all possible thresholds and find a value for \( t_2 \) that works. For the formal statement of this claim and its proof, see Subsection 3.3 and Proposition 3.11.
3.2 Data Transformation

In this subsection, we show that we can simplify the problem of searching for a certifying vector $v$ in $T_w(x)$ defined in Equation (3) by projecting the samples to an appropriate $(d-1)$-dimensional subspace via the perspective projection (2). The main proposition of this subsection (Proposition 3.6) shows that this operation in some sense preserves the structure of the problem. In more detail, the transformed distribution remains well-behaved and satisfies the Tsybakov noise condition (albeit with somewhat worse parameters).

The transformation we perform is as follows:

1. We first condition on the band $B = \{x : \langle x, w \rangle \in [\sigma_1, \sigma_2]\}$, for some positive parameters $\sigma_1, \sigma_2$.

2. We then perform the perspective projection on the samples, $\pi_w(\cdot)$, defined in Equation (2).

To facilitate the proceeding formal description, we introduce the following definition.

**Definition 3.5 (Transformed Distribution).** Let $D$ be a distribution on $\mathbb{R}^d \times \{\pm 1\}$, $B \subseteq \mathbb{R}^d$ and $(x, y) \sim D$.

- We use $D_B$ to denote $D$ conditioned on $x$ being in the set $B$.
- Let $q : \mathbb{R}^d \mapsto \mathbb{R}^d$. We denote by $D_q$ the distribution of the random variable $(q(x), y)$.

With the above notation, $D_B$ is the distribution obtained by first conditioning on $B$ and then applying the transformation $q(\cdot)$ to $D_B$.

With Definition 3.5 in place, the distribution obtained from $D$ after we condition on the band $B$ is $D_B$, and the distribution obtained from $D_B$ after we perform the perspective projection is $D_B^{\pi_w}$. We can now state the main proposition of this subsection.

**Proposition 3.6 (Properties of $D_B^{\pi_w}$).** Let $D$ be a $(3, L, R, \beta)$-well-behaved isotropic distribution on $\mathbb{R}^d \times \{\pm 1\}$ that satisfies the $(\alpha, A)$-Tsybakov noise condition with respect to an unknown halfspace $f(x) = \text{sign}(\langle w^*, x \rangle)$. Fix any unit vector $w$ such that $\theta(w, w^*) = \theta$, and let $B = \{x : \langle x, w \rangle \in [\rho R/2, \rho R/\sqrt{2}]\}$, for some $\rho \in (0, 1]$. Then, for some $c = (LR)^{O(1)}$, the following conditions hold:

1. The distribution $D_B^{\pi_w}$ on $\mathbb{R}^d \times \{\pm 1\}$ is $(2, \rho^3, \frac{1}{\rho}, \frac{\rho}{\sqrt{2}} \log \frac{1}{\rho})$-well-behaved.

2. The distribution $D_B^{\pi_w}$ satisfies the $(\alpha, \frac{1}{\rho})$-Tsybakov noise condition with optimal classifier $\text{sign}(\langle (w^*)^{-1}w, x \rangle + 1/\tan \theta)$.

The rest of this subsection is devoted to the proof of Proposition 3.6. Before we proceed with the proof, we express the problem of finding a certifying vector $v$ satisfying (3) in the transformed domain. Indeed, it is not hard to see that after we condition on $B$ and perform the perspective projection $\pi_w$, our goal is to find a vector $v$ and scalars $t_1, t_2 > 0$ such that

$$E_{(z, y) \sim D_B^{\pi_w}}[I\{-t_1 \leq \langle v, z \rangle \leq -t_2\} y] < 0.$$  

(4)

More formally, we have the following simple lemma showing that if we find a certifying vector $v$ and parameters $t_1, t_2$ in the transformed instance $D_B^{\pi_w}$ satisfying Equation (4), the same vector and parameters will be a certificate with respect to the initial well-behaved distribution $D$. The relevant expectation remains negative but is slightly closer to zero.
Lemma 3.7. Let $D$ be a $(3, L, R, \beta)$-well-behaved distribution on $\mathbb{R}^d$ and let $B = \{x : \langle x, w \rangle \in [\rho R/2, \rho R/\sqrt{2}]\}$, for some $\rho \in (0, 1)$. Let $w \in \mathbb{R}^d$ be a unit vector and let $v \in w^\perp$, $t_1, t_2 > 0$ be such that $E_{(x,y) \sim D} \mathbb{P}_{B} \mathbb{I}\{t_1 \leq \langle v, z \rangle \leq t_2\} y < -C$, for some $C > 0$. Then we have that $E_{(x,y) \sim D} [T_w(x) y \langle w, x \rangle] = -\Omega(CLR^3 \rho)$.

Proof. It holds

$$E_{(z,y) \sim D_B} \mathbb{P}_{\pi_B} \mathbb{I}\{t_1 \leq \langle v, z \rangle \leq t_2\} y = E_{(x,y) \sim D_B} \mathbb{I}\{t_1 \leq \langle v, \pi_w(x) \rangle \leq t_2\} y = \frac{1}{\Pr_D[B]} E_{(x,y) \sim D} [T_w(x) \langle w, x \rangle y].$$

Using the anti-anti concentration property of $D_x$, we can bound $\Pr_D[B]$ from below. Observe that since the lower bound $L$ on the 3-dimensional marginal density holds inside a ball of radius $R$, to bound the above probability from below, we can multiply $L$ by the volume of the intersection of $B$ with the ball of radius $R$. Using the formula for the volume of spherical segments, we obtain $\Pr_D[B] = \Omega(LR^3 \rho)$. This completes the proof.

Proof of Proposition 3.6. Our goal is to compute a certificate of the form (3). As we already discussed, if we had chosen to simply project the points on the subspace $w^+$, we would have obtained an instance that is not linearly separable — even if the noise rate $\eta(x)$ was identically zero. By first conditioning on the set $B = \{x : \langle x, w \rangle \in [\sigma_1, \sigma_2]\}$, where $\sigma_1, \sigma_2 > 0$, and then performing the perspective projection $\pi_w$, we keep the dataset linearly separable (with respect to the noiseless distribution, i.e., for $\eta(x) = 0$), albeit by a biased linear classifier.

We have the following lemma.

Lemma 3.8. Let $D$ be a distribution on $\mathbb{R}^d \times \{\pm 1\}$ such that for $(x, y) \sim D$ we have that $y = \text{sign}(\langle w^*, x \rangle)$. Let $w$ be any unit vector such that $\theta(w, w^*) = \theta \in (0, \pi)$. For $(z, y) \sim D_B$ it holds $y = \text{sign}(\langle (w^*)^\perp, z \rangle + \frac{1}{\tan \theta})$, i.e., the transformed distribution is linearly separable by a biased hyperplane.

Proof. Observe that $w^* = \lambda_1 (w^*)^\perp + \lambda_2 w$, where $\lambda_1 > 0$. We then have

$$\text{sign}(\langle w^*, x \rangle) = \text{sign} \left( \lambda_1 \left( \langle w^* \rangle^\perp, x \right) + \lambda_2 \langle w, x \rangle \right) = \text{sign} \left( \lambda_1 \langle w, x \rangle \left( \frac{\langle (w^*)^\perp, x \rangle}{\langle w, x \rangle} + \frac{\lambda_2}{\lambda_1} \right) \right)$$

$$= \text{sign} \left( \langle (w^*)^\perp, \pi_w(x) \rangle + \frac{\lambda_2}{\lambda_1} \right),$$

where to get the last equality we use the fact that $\lambda_1$ and $\langle w, x \rangle$ are both positive given that we conditioned on the band $B$. Observe that if the angle between $w$ and $w^*$ is $\theta$, then $\lambda_1 = \sin \theta$ and $\lambda_2 = \cos \theta$. This completes the proof.

We next show that conditioning on the band $B$ will not make the Tsybakov noise condition substantially worse.

Lemma 3.9. Let $D$ be a $(3, L, R, \beta)$-well-behaved isotropic distribution on $\mathbb{R}^d \times \{\pm 1\}$ that satisfies the $(\alpha, A)$-Tsybakov noise condition with respect to an unknown halfspace $f(x) = \text{sign}(\langle w^*, x \rangle)$. Let $B = \{x : \langle x, w \rangle \in [\rho R/2, \rho R/\sqrt{2}]\}$, for some $\rho \in (0, 1)$. Then $D_B$ satisfies the Tsybakov noise condition with parameters $(\alpha, O(A/(R^3 L \rho)))$ and optimal linear classifier $w^*$. 

13
Proof. We have that \( \Pr_{x \sim D_x}[1 - 2\eta(x) > t|x \in B] \leq \Pr_{x \sim D_x}[1 - 2\eta(x) > t]/\Pr_{x \sim D_x}[B] \). From the proof of Lemma 3.7, we have seen that we can use the anti-anti-concentration property of \( D_x \) to bound \( \Pr_{x \sim D_x}[B] \) from below. Specifically, we have \( \Pr_{x \sim D_x}[B] \geq \Omega(LR^3\rho) \). Therefore, \( D_B \) satisfies the Tsybakov noise condition with parameters \( (\alpha, O(A/(R^3\rho L)) \).

Finally, we show that the transformation of Equation (2) also preserves the anti-anti-concentration and concentration properties of the marginal distribution \( D_x \).

**Lemma 3.10.** Let \( D \) be a \((3, L, R, \beta)\)-well-behaved distribution. Fix any unit vector \( w \) and let \( B = \{ x : \langle x, w \rangle \in [\rho R/2, \rho R/\sqrt{2}] \} \), for some \( \rho \in (0, 1) \). Then the transformed distribution \( D_B^{\pi_w} \) is \((2, \Omega(L^3R^3), 1/\rho, O(\beta/(R\rho)\log(1/(LR\rho)))\)-well-behaved.

**Proof.** Let \( \gamma(x) : \mathbb{R}^d \rightarrow \mathbb{R}^+ \) be the probability density function of \( D_x \) and \( B = \{ x : \rho R/2 \leq \langle x, w \rangle \leq \rho R/\sqrt{2} \} \). Note that the conditional distribution \( (D_x)_B \) of the random vector \( x \sim D_x \) on the band \( B \) has density \( \gamma_B(x) = \mathbb{1}_B(x)\gamma(x)/\int_B \gamma(x)dx \). Since the transformation \( \pi_w(\cdot) \) is not injective, we consider the transformation \( \phi(x) = (\langle x, w \rangle, \pi_w(x)) \) and observe that \( \phi(x) : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is injective. Denote by \( U \) the random variable corresponding to the image of \( x, x \sim (D_x)_B \), under \( \phi \). Without loss of generality, we may assume that \( w = e_1 \). By computing the Jacobian of the above one-to-one transformation, we get that the density function of the random vector \( U \) is given by \( \gamma_U(u) = |u_1|^{d-1}\gamma_B(u_1(1, u_2, \ldots, u_d)) \). We can marginalize out the “dummy” variable \( u_1 \) to obtain the density function \( g \) of \( z \sim (D_x)_B^{\pi_w} \), i.e.,

\[
g(z) = \int_{-\infty}^{\infty} |u_1|^{d-1}\gamma_B(u_1(1, u)) du_1.
\]

Let \( V \) be any 2-dimensional subspace of \( \mathbb{R}_+^4 \). Without loss of generality, we may assume that \( V = \text{span}(e_2, e_3) \). Denote \( z_{[3, d-1]} = (z_3, \ldots, z_{d-1}) \), \( U = \text{span}(e_1, e_2, e_3) \), and \( U^\perp = \text{span}(e_4, \ldots, e_d) \). The marginal density of \( z \sim (D_x)_B^{\pi_w} \) on \( V \) is then given by

\[
g_V(z_1, z_2) = \int_{U^\perp} \int_{-\infty}^{\infty} |u_1|^{d-1}\gamma_B(u_1(1, z)) du_1 dz_{[3, d-1]}
\]

\[
= \int_{-\infty}^{\infty} |u_1|^{d-1} \int_{U^\perp} \gamma_B(u_1(1, z)) dz_{[3, d-1]} du_1
\]

\[
= \int_{\gamma_B(x)dx}^{\rho R/\sqrt{7}} |u_1|^{d-1} \int_{U^\perp} \gamma_B(u_1(1, z)) dz_{[3, d-1]} du_1
\]

\[
= \int_{\gamma_B(x)dx}^{\rho R/2} |u_1|^{d-1} \int_{U^\perp} \gamma_B(u_1(1, z)) dz_{[3, d-1]} du_1
\]

where to get the third equality we used the definition of the conditional density on \( B \) and the fact that the set \( B \) only depends on the first coordinate. The last equality follows by a change of variables. Since \( D_x \) is \((3, L, R, \beta)\)-well-behaved, we have that if \( u_1^2(1 + z_2^2 + z_3^2) \leq R^2 \) we have that \( \gamma_U(u_1(1, z_1, z_2)) \geq L \). Therefore, using the fact that \( u_1^2 \leq R^2/2 \), \( u_1 \leq 2/\rho^2 - 1 \) it holds \( \gamma_U(u_1(1, z_1, z_2)) \geq L \). Observe that since \( \rho \leq 1 \), we can get the slightly looser bound \( z_2^2 + z_3^2 \leq 1/\rho^2 \). Note that \( \int_B \gamma(x)dx \leq 1 \) and also \( \int_{\rho R/\sqrt{7}}^{\rho R/2} |u_1|^2 du_1 = \Omega(\rho^3 R^3) \). Combining these bounds, we obtain that \( g_V(z_1, z_2) \geq \Omega(L\rho^3 R^3) \).

It remains to prove that the transformed distribution still has exponentially decaying tails. In the proof of Lemma 3.9, we have already argued that the probability mass of \( B \) is bounded below by \( C_B = \Omega(LR^3\rho) \). Therefore, the distribution \( (D_x)_B \) obtained after conditioning has exponential
concentration with parameter \( \beta(1 - \log C_B) \). After we perform the perspective projection (Equation 22) to obtain \((D_\infty)_B^w\), the concentration parameter becomes \(2\beta(1 - \log C_B)/(\rho R)\), since we divide each coordinate of \( x \) by a quantity that is bounded from below by \( R\rho/2 \). This completes the proof of Lemma 3.10.

Proposition 3.6 follows by combining Lemmas 3.8, 3.9, and 3.10.

### 3.3 Efficient Certificate Computation Given Initialization

In this subsection, we give our main algorithm for computing a non-optimality certificate in the transformed instance, i.e., a vector \( v \) and parameters \( t_1, t_2 > 0 \) satisfying Equation (4). Recall that after the perspective projection transformation of Subsection 3.2, we now have sample access to i.i.d. labeled examples \((x, y)\) from a well-behaved distribution \( D \) on \( \mathbb{R}^d \times \{\pm 1\} \) satisfying the Tsybakov noise condition (albeit with somewhat worse parameters) with the optimal classifier being a non-homogeneous halfspace (see Proposition 3.6).

Our certificate algorithm in this subsection assumes the existence of an initialization vector, i.e., a vector that has non-trivial correlation with \((w^*)^\perp\). The easiest way to find such a vector is by picking a uniformly random unit vector. A random initialization suffices for the guarantees of this subsection (and in particular for Theorem 3.3). We note that for the family of log-concave distributions, we can leverage additional structure to design a fairly sophisticated initialization algorithm that in turn leads to a faster certificate algorithm (see Section 4).

The main algorithmic result of this section is an efficient algorithm to compute a certifying vector satisfying Equation (4). Note that we are essentially working in \((d - 1)\) dimensions, since we have already projected the examples to the subspace \( w^\perp \). As shown in Proposition 3.6, the transformed distribution \( D_B^w \) is still well-behaved and follows the Tsybakov noise condition, but with somewhat worse parameters than the initial distribution \( D \).

To avoid clutter in the relevant expressions, we overload the notation and use \( D \) instead of \( D_B^w \) in the rest of this section. Moreover, we use the notation \((L, R, \beta)\) and \((\alpha, A)\) to denote the well-behaved distribution’s parameters and the Tsybakov noise parameters. The actual parameters of \( D_B^w \) (quantified in Proposition 3.6) are used in the proof of Theorem 3.3. To simplify notation, we will henceforth denote by \( v^* \) the vector \((w^*)^\perp\). We show:

**Proposition 3.11.** Let \( D \) be a \((2, L, R, \beta)\)-well-behaved distribution on \( \mathbb{R}^d \times \{\pm 1\} \) satisfying the \((\alpha, A)\)-Tsybakov noise condition with respect to an unknown halfspace \( f(x) = \text{sign}(\langle v^*, x \rangle + b) \). Let \( v_0 \in \mathbb{R}^d \) be a unit vector such that \( \langle v_0, v^* \rangle \geq 4b/R \). There is an algorithm (Algorithm 7) with the following performance guarantee: Given \( v_0 \) and \( N = d \frac{\beta^2 R^2}{\log(1/\delta)} \left( \frac{A}{RL} \right)^{O(1/\alpha)} \) samples from \( D \), the algorithm runs in \( \text{poly}(N, d) \) time, and with probability at least \( 1 - \delta \) returns a unit vector \( v \in \mathbb{R}^d \) and a scalar \( t \in \mathbb{R}_+ \) such that

\[
\mathbb{E}_{(x, y) \sim D} \left[ 1 - R \leq \langle v, x \rangle \leq -t \right] \mathbb{P}[y] \leq -\frac{b}{R\beta} \left( \frac{RL}{A} \right)^{O(1/\alpha)}.
\]

Algorithm 7 employs a “perceptron-like” update rule that in polynomially many rounds succeeds in improving the angle between the initial guess \( v_0 \) and the target vector \((w^*)^\perp = v^* \). While the algorithm is relatively simple, its proof of correctness relies on a novel structural result (Lemma 3.12) whose proof is the main technical contribution of this section. Roughly speaking, our structural result establishes the following win-win statement: Given a vector whose correlation with \( v^* \) is non-trivial, either this vector is already a certifying vector (see Item 1 of Lemma 3.12 and Lemma 3.7) or the update step will improve the angle with \( v^* \) (Item 2 of Lemma 3.12).
In more detail, starting with a vector \( v_0 \) that has non-trivial correlation with \( v^* \), we consider the following update rule
\[
v^{(t+1)} = v^{(t)} + \lambda g,
\]
where \( \lambda > 0 \) is an appropriately chosen step size and
\[
g = \mathbb{E}_{(x,y) \sim \mathcal{D}} \left[ I \{ -R \leq \langle v^{(t)}, x \rangle \leq -R/2 \} \right. y \left. \text{proj}_{(v^{(t)})^{ot}}(x) \right],
\]
where \( \text{proj}_{(v^{(t)})^{ot}}(x) \) is the projection of \( x \) to the subspace \( (v^{(t)})^{ot} \). In Lemma 3.17, we show that if \( v^{(t)} \) is not a certifying vector, i.e., it does not satisfy Item 1 of Lemma 3.17, then there exists an appropriately small step size \( \lambda \) that improves the correlation with \( v^* \) after the update. This is guaranteed by Item 2 of Lemma 3.17, which shows that \( g \) has positive correlation with \( (v^*)^{ot} \) (the normalized projection of \( v^* \) onto \( v^\bot \)), and thus will turn \( v^{(t)} \) towards the direction of \( v^* \) decreasing the angle between them.

**Algorithm 1** Computing a Certificate Given Initialization

1. **procedure** COMPUTE-CERTIFICATE((\( L, R, \beta \)), (\( A, \alpha \)), \( \delta, v_0, \hat{D} \))
2. **Input:** Empirical distribution \( \hat{D} \) of a \( (2, L, R, \beta) \)-well-behaved distribution that satisfies the \( (\alpha, A) \)-Tsypbakov noise condition, initialization vector \( v_0 \), confidence probability \( \delta \).
3. **Output:** A certifying vector \( v \) and positive scalars \( t_1, t_2 \) that satisfy (1).
4. \( v^{(0)} \leftarrow v_0 \)
5. \( T \leftarrow \text{poly}(1/L, 1/R, A)^{1/\alpha} \cdot \text{poly}(1/b, 1/\beta) \)
6. \( \lambda \leftarrow \frac{1}{T} \cdot \text{poly}(L, R, 1/A)^{1/\alpha}; \ c \leftarrow \frac{b}{R^2} \cdot \text{poly}(L, R, 1/A)^{1/\alpha} \)
7. **for** \( t = 1, \ldots, T \) **do**
8. \( B_t = \{ x : -R \leq \langle v^{(t-1)}, x \rangle \leq -t' \} \)
9. **if** there exists \( t_0 \in (R/2, R] \) such that \( \mathbb{E}_{(x,y) \sim \hat{D}} \left[ I_{B_{t_0}}(x) \right] \leq -c \)
10. **return** \( v^{(t-1)}, R, t_0 \)
11. \( \hat{g}^{(t)} \leftarrow \mathbb{E}_{(x,y) \sim \hat{D}} \left[ I_{B_t} \right. y \left. \text{proj}_{(v^{(t-1)})^{ot}}(x) \right] \)
12. \( v^{(t)} \leftarrow v^{(t-1)} + \frac{\lambda \hat{g}^{(t)}}{\|v^{(t-1)} + \lambda \hat{g}^{(t)}\|_2} \)

Figure 4: In the subspace \( w^\bot \), the certifying function is simply an indicator \( I \{ -R \leq \langle v, x \rangle \leq -t_0 \} \), for some \( t_0 > 0 \). See also Equation (4). This is shown in Figure 4(a). The blue regions in Figure 4(a) (resp. Figure 4(b)) have negative contribution to the value of \( I_1^t \) (resp. \( I_2 \)), while the red regions have positive contribution.
We are now ready to state and prove our win-win structural result:

**Lemma 3.12** (Win-Win Result). Let $\mathcal{D}$ be a $(2, L, R, \beta)$-well-behaved distribution on $\mathbb{R}^d \times \{\pm 1\}$ that satisfies the $(\alpha, A)$-Tsypakov noise condition with respect to $f(x) = \text{sign}(\langle v^*, x \rangle + b)$, and $v \in \mathbb{R}^d$ be a unit vector with $\langle v, v^* \rangle \geq 4b/R$. Consider the band $B^t = \{x : -R \leq \langle v, x \rangle \leq -t\}$ for $t \in [R/2, R]$ and define $g = E_{(x,y) \sim \mathcal{D}} [I_{B^t/(2)}(x) y \text{proj}_v(x)]$. For some $c = (RL/A)^{O(1/\alpha)}$, one of the following statements is satisfied:

1. There exists $t_0 \in (R/2, R]$, such that $E_{(x,y) \sim \mathcal{D}} [I_{B^{t_0}}(x) y] \leq -c^2 \frac{b}{R^2}$.

2. It holds $\langle g, v^* \rangle \geq c^2 \frac{\pi b}{4\beta}$.

Moreover, the first condition always holds if $\theta(v, v^*) \leq b/c/\beta$.

**Proof.** Since $v$ and $v^*$ span a 2-dimensional subspace, we can assume without loss of generality that $v = e_2$ and $v^* = (-\sin \theta, \cos \theta)$. Our analysis will consider the following regions: $B'_1 = \{x \in B^t : f(x) = +1\}$, $B'_2 = \{x \in B^t : f(x) = -1 \text{ and } \langle \text{proj}_v x, v^* \rangle \geq 0\}$, and $B'_3 = \{x \in B^t : f(x) = -1 \text{ and } \langle \text{proj}_v x, v^* \rangle < 0\}$. See Figures 4a, 4b for an illustration.

For notation convenience, we will also denote $(v^*)^\perp = \text{proj}_v(v^*)/\|\text{proj}_v(v^*)\|$2 and $\zeta(x) = 1 - 2\eta(x)$.

Given the above notation, we can rewrite the two quantities appearing in Items 1, 2 of Lemma 3.12 as follows:

$$I'_1 = \mathbb{E}_{(x,y) \sim \mathcal{D}} [I_{B^t}(x)] = \mathbb{E}_{x \sim \mathcal{D}_x} [(I_{B'_1}(x) - I_{B'_2}(x)) \zeta(x)] - \mathbb{E}_{x \sim \mathcal{D}_x} [(I_{B'_3}(x) \zeta(x))].$$

$$I_2 = \langle g, (v^*)^\perp \rangle = \langle \mathbb{E}_{(x,y) \sim \mathcal{D}} [I_{B^t/(2)}(x)y|x], (v^*)^\perp \rangle$$

$$= \mathbb{E}_{x \sim \mathcal{D}_x} [(I_{B'_1/(2)}(x) - I_{B'_2/(2)}(x)) \zeta(x)|x_1] + \mathbb{E}_{x \sim \mathcal{D}_x} [I_{B'_3/(2)}(x) \zeta(x)|x_1].$$

Since $v^* = (-\sin \theta, \cos \theta)$, the quantity $\langle g, v^* \rangle$ (that appears in Item 2 of Lemma 3.12) is equal to $\sin(\theta)I_2$. We work with the normalized $(v^*)^\perp$ in order to simplify notation.

Before we go into the details of the proof, we give a high-level description of the main steps with pointers to the relevant claims. Note that the quantity $I'_1$ corresponds to the value of the certifying function (in the subspace $w^\perp$) when we use $v$ as certifying vector and $t_1 = -R, t_2 = t$ as thresholds. See Equation 41. When $I'_1$ is small (see Item 1 of the lemma), we have a certifying function. On the other hand, $\sin(\theta)I_2$ corresponds to the inner product of the update $g$ and the optimal vector $v^*$. Item 2 of the lemma states that this quantity is large, which means that if we update according to $g$ we shall improve the correlation with $v^*$.

**Heuristic Argument.** Since the formal proof is somewhat technical, we start with a useful (but inaccurate) heuristic argument. If we ignore the presence of $|x_1|$ in $I_{2,1}$ and $I_{2,2}$, we see from Figure 4a that if the contribution of region $B^{R/2}_2$ is sufficiently large compared to the positive contribution of $B^{R/2}_1$ (red region in Figure 4a), then $I_1$ will be negative in total. That is, Item 1 is true. On the other hand, if the contribution of $B^{R/2}_2$ is not very large, then when we add the contribution of $B_3$ (red region in Figure 4b) overall, $I_2$ will be positive and Item 2 now holds.
Notice that in this setting we could take the threshold $t$ in the definition of $I_1^t$ to simply be $R/2$, i.e., use the entire band in our certificate.

Unfortunately, in the actual proof, we need to deal with the term $|x_1|$ in the expectations of $I_2$ that makes the previous argument invalid. Using the Mean Value Theorem (Fact 3.16), we show that there exists a threshold $t \in [-R, -R/2]$ that makes $I_1^t$ sufficiently negative. This is done in Claim 3.15.

We can now proceed with the formal proof. We will require several technical claims. First, we bound $I_{1,2}^{R/2}$ and $I_{2,2}$ from below using the fact that our distribution is well-behaved. We require the following claim in order to show that the expressions in Item 1 (resp. Item 2) of our lemma are not simply negative (resp. positive), but have a non-trivial gap instead. The proof of the claim relies on two important observations. First, the fact that the distribution is well-behaved means that the contribution of region $B_3$ would be sufficiently large if we ignore the noise function $\zeta(x)$ in the expectations. Second, we use the fact that the Tsybakov noise rate $\zeta(x) = 1 - 2\eta(x)$ cannot reduce the contribution of a region by a lot.

Claim 3.13. We have that $I_{1,2}^{R/2}$ and $I_{2,2}$ are bounded from below by some $c = (RL/A)^{O(1/\alpha)}$.

The proof of Claim 3.13 can be found in Appendix A.

Now we show that if the angle between the optimal vector and the current one is small, then $I_0^c$ would be sufficiently large if we ignore the noise function $\zeta(x)$ in the expectations. Second, we use the fact that the Tsybakov noise rate $\zeta(x) = 1 - 2\eta(x)$ cannot reduce the contribution of a region by a lot.

Claim 3.14. If $\theta(v, v^*) \leq bc/(4\beta)$, then $I_1^{R/2} \leq -c/4$.

The proof of Claim 3.14 can be found in Appendix A.

Our next claim shows that when Item 2 does not hold, then Item 1 always does. Having proved Claim 3.14, we may also assume that $\theta(v, v^*) \geq bc/(4\beta)$. Observe that, in this case, if $I_2 \geq c/2$, we have

$$I_2 \geq c/2 = c \sin \theta/(2 \sin \theta) \geq \pi c^2 b/(4 \beta \sin \theta),$$

where we used the fact that $\sin(\theta) \geq 2\theta/\pi$ for all $\theta \in [0, \pi/2]$ and the fact that $\theta \geq bc/(4\beta)$. Therefore, to complete the proof, we need to show the following claim proving that when $I_2 \leq c/2$, Item 1 of the lemma is always true.

Claim 3.15. If $\theta = \theta(v, v^*) \geq bc/(4\beta)$ and $I_2 \leq c/2$, there exists $t_0 \in (-R, -R/2]$ such that $I_{1,1}^{t_0} \leq -bc^2/(16R\beta)$.

Proof. Given the lower bounds on $I_{2,2}$ and $I_{1,2}^{R/2}$, we distinguish two cases. Assume that $I_2 \leq c/2$. This implies, from Claim 3.13, that $I_{2,1} \leq -c/2$. We show that in this case there exists a $t_0$ such that $I_{1,1}^{t_0} \leq -bc^2/(16R\beta)$. To show this, we are going to use the following variant of the standard Mean Value Theorem (MVT) for integrals.

Fact 3.16 (Second Integral MVT). Let $G : \mathbb{R} \mapsto \mathbb{R}_+$ be a non-negative, non-increasing, continuous function. There exists $s \in (a, b]$ such that $\int_a^b G(t) F(t) dt = G(a) \int_a^s F(t) dt$.

Let $\xi(x_2) = x_2/\tan \theta + b/\sin \theta$ be the first coordinate of a point $(x_1, x_2)$ that lies on the halfspace defined by $f$, where $f(x) = \text{sign}((v^*, x) + b)$ (see Figure 4b). We have

$$I_{1,1}^t = \int_{-R}^{-t} \left( \int_{-\infty}^{\xi(x_2)} \zeta(x_1, x_2) \gamma(x_1, x_2) dx_1 - \int_{\xi(x_2)}^{0} \zeta(x_1, x_2) \gamma(x_1, x_2) dx_1 \right) dx_2 = \int_{-R}^{-t} g(x_2) dx_2,$$
where \( g(x_2) = \int_{-\infty}^{\xi(x_2)} \zeta(x_1, x_2) \gamma(x_1, x_2) \, dx_1 - \int_{-\xi(x_2)}^{0} \zeta(x_1, x_2) \gamma(x_1, x_2) \, dx_1 \). Moreover,

\[
I_{2,1} = \int_{-R/2}^{-R} \left( \int_{-\xi(x_2)}^{\xi(x_2)} \zeta(x_1, x_2) \gamma(x_1, x_2) \, dx_1 - \int_{-\xi(x_2)}^{0} \zeta(x_1, x_2) \gamma(x_1, x_2) \, dx_1 \right) \, dx_2
\]

\[
\geq \int_{-R/2}^{-R} |\xi(x_2)| g(x_2) \, dx_2 = |\xi(-R)| \int_{-R}^{-t_0} g(x_2) \, dx_2 = |\xi(-R)| I_{1,1}^0,
\]

for some \( t_0 \in (-R, -R/2) \). Observe that the inequality above follows by replacing \( |x_1| \) with its lower bound \( |\xi(x_2)| \) in the first integral and by its upper bound \( |\xi(x_2)| \) in the second.

We now observe that \( |\xi(x_2)| = x_2 / \tan \theta - b / \sin \theta \), where to remove the absolute value we used the assumption that \( \cos \theta \geq 4b/R \). Therefore, \( |\xi(x_2)| \) is a decreasing and non-negative function of \( x_2 \). Using the Mean Value Theorem, Fact 3.16, we obtain

\[
I_{2,1} \geq \int_{-R/2}^{-R} |\xi(x_2)| g(x_2) \, dx_2 = |\xi(-R)| \int_{-R}^{-t_0} g(x_2) \, dx_2 = |\xi(-R)| I_{1,1}^0.
\]

Thus,

\[
I_{1,1}^0 \leq I_{2,1} / |\xi(-R)| \leq c \sin \theta / (2R) \leq c \beta^2 / (16R \beta),
\]

where we used that \( \theta \geq c / (4\beta) \). This completes the proof of Claim 3.15.

Putting together the above claims, Lemma 3.12 follows.

In the next lemma, we show that if Item 2 of Lemma 3.12 is satisfied, then an update step decreases the angle between the current vector \( \mathbf{v} \) and the optimal vector \( \mathbf{v}^* \).

**Lemma 3.17** (Correlation Improvement). For unit vectors \( \mathbf{v}^*, \mathbf{v} \in \mathbb{R}^d \), let \( \mathbf{g} \in \mathbb{R}^d \) such that \( \langle \mathbf{g}, \mathbf{v}^* \rangle \geq \frac{c}{\beta} \), \( \langle \mathbf{g}, \mathbf{v} \rangle = 0 \), and \( \| \mathbf{g} \|_2 \leq \beta \), with \( c > 0 \) and \( \beta \geq 1 \). Then, for \( \mathbf{v}' = \frac{\mathbf{v} + \lambda \mathbf{g}}{\| \mathbf{v} + \lambda \mathbf{g} \|_2} \), with \( \lambda = \frac{c}{2\beta^2} \), we have that \( \langle \mathbf{v}', \mathbf{v}^* \rangle \geq \langle \mathbf{v}, \mathbf{v}^* \rangle + \lambda \beta^2/2 \).

**Proof.** We will show that \( \langle \mathbf{v}', \mathbf{v}^* \rangle = \cos \theta' \geq \cos \theta + \lambda \beta^2/2 \), where \( \cos \theta = \langle \mathbf{v}, \mathbf{v}^* \rangle \). We have that

\[
\| \mathbf{v} + \lambda \mathbf{g} \|_2 = \sqrt{1 + \lambda^2 \| \mathbf{g} \|_2^2 + 2\lambda \langle \mathbf{g}, \mathbf{v} \rangle} \leq 1 + \lambda^2 \| \mathbf{g} \|_2^2,
\]

where we used that \( \sqrt{1 + a} \leq 1 + a/2 \). Using the update rule, we have

\[
\langle \mathbf{v}', \mathbf{v}^* \rangle = \left( \langle \mathbf{v}', (\mathbf{v}^*)^{\perp} \rangle \sin \theta + \langle \mathbf{v}', \mathbf{v} \rangle \cos \theta \right) \sin \theta + \frac{\lambda \langle \mathbf{g}, (\mathbf{v}^*)^{\perp} \rangle}{\| \mathbf{v} + \lambda \mathbf{g} \|_2} \sin \theta + \frac{\langle \mathbf{v} + \lambda \mathbf{g}, \mathbf{v} \rangle}{\| \mathbf{v} + \lambda \mathbf{g} \|_2} \cos \theta.
\]

Now using Equation (3), we get

\[
\langle \mathbf{v}', \mathbf{v}^* \rangle \geq \frac{\lambda}{1 + \lambda^2 \| \mathbf{g} \|_2^2} \sin \theta + \frac{\cos \theta}{1 + \lambda^2 \| \mathbf{g} \|_2^2} = \cos \theta + \frac{\lambda}{1 + \lambda^2 \| \mathbf{g} \|_2^2} \sin \theta + \frac{-\lambda^2 \| \mathbf{g} \|_2^2 \cos \theta}{1 + \lambda^2 \| \mathbf{g} \|_2^2}.
\]

Then, using that \( \langle \mathbf{g}, \mathbf{v}^{*} \rangle = \langle \mathbf{g}, (\mathbf{v}^{*})^{\perp} \sin \theta \rangle \), we have that \( \langle \mathbf{g}, (\mathbf{v}^{*})^{\perp} \rangle \geq \frac{c}{\beta \sin \theta} \), thus

\[
\langle \mathbf{v}', \mathbf{v}^{*} \rangle \geq \cos \theta + \frac{\lambda c / \beta - \lambda^2 \| \mathbf{g} \|_2^2}{1 + \lambda^2 \| \mathbf{g} \|_2^2} \geq \cos \theta + \frac{\lambda c / \beta - \lambda^2 \beta^2}{1 + \lambda^2 \| \mathbf{g} \|_2^2} = \cos \theta + \frac{\lambda c / \beta}{2(1 + \lambda^2 \| \mathbf{g} \|_2^2)},
\]

where in the first inequality we used that \( \| \mathbf{g} \|_2 \leq \beta \) and in the second that for \( \lambda = c/(2\beta^3) \) it holds \( c/\beta - \lambda \beta^2 \geq c/(2\beta) \). Finally, we have that

\[
\cos \theta' = \langle \mathbf{v}', \mathbf{v}^{*} \rangle \geq \cos \theta + \frac{\lambda c / \beta}{2(1 + \lambda^2 \| \mathbf{g} \|_2^2)} \geq \cos \theta + \frac{1}{4} \lambda c / \beta = \cos \theta + \frac{1}{2} \lambda^2 \beta^2.
\]

This completes the proof.
To analyze the sample complexity of Algorithm 1, we require the following simple lemma, which bounds the sample complexity of estimating the update function and testing the current candidate certificate. The simple proof can be found in Appendix A.

Lemma 3.18 (Estimating \( g \)). Let \( \mathcal{D} \) be a \((2, L, R, \beta)\)-well-behaved distribution. Given \( N = O(d\beta^2/\epsilon^2) \log(d/\delta) \) i.i.d samples \((x^{(i)}, y^{(i)})\) from \( \mathcal{D} \), the estimator \( \hat{g} = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{B_{R/2}}(x^{(i)}) y^{(i)} x^{(i)} \) satisfies the following with probability at least \( 1 - \delta \):

- \( \|\hat{g} - g\|_2 \leq \epsilon \), where \( g = \mathbb{E}_{(x, y) \sim \mathcal{D}}[\mathbb{1}_{B_{R/2}}(x) y x] \), and

- \( \|\hat{g}\|_2 \leq \epsilon \beta + \epsilon \).

Before we proceed with the proof of Proposition 3.11, we show that we can efficiently check for the certificate in Line 9 of Algorithm 1 with high probability.

Lemma 3.19. Let \( \hat{\mathcal{D}}_N \) be the empirical distribution obtained from \( \mathcal{D} \) with \( N = O(\log(1/\delta)/\epsilon^2) \) samples. Then, with probability \( 1 - \delta \), for every \( t \in \mathbb{R}_+ \), \( |\mathbb{E}_{(x, y) \sim \mathcal{D}}[\mathbb{1}_{B_t^*}(x)] - \mathbb{E}_{(x, y) \sim \hat{\mathcal{D}}_N}[\mathbb{1}_{B_t^*}(x)]| \leq \epsilon \).

The proof of Lemma 3.19 can be found in Appendix A. We are now ready to prove Proposition 3.11.

Proof of Proposition 3.11. Consider the \( k \)-th iteration of Algorithm 1. Let \( \hat{g}^{(k)} = \mathbb{E}_{(x, y) \sim \mathcal{D}}[\mathbb{1}_{B_{R/2}^k}(x) y x] \), where \( B_{R/2}^k(x) = \{x : -R \leq \langle x, \hat{g}^{(k)} \rangle \leq -R/2\} \) and \( G := \sqrt{b}(RL/A)^{O(1/\alpha)} \). Moreover, let \( \hat{g}^{(k)} = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{B_{R/2}^k}(x^{(i)}) y^{(i)} x^{(i)} \) and note that from Lemma 3.18 we have that given \( N = O \left( d\beta^2/G^4 \log(1/(L R)) \log(dT/\delta) \right) \) samples, for every iteration \( k \), it holds that \( \|\hat{g}^{(k)} - g^{(k)}\|_2 \leq G^2/(16\beta) \) and \( \|\hat{g}^{(k)}\|_2 \leq \epsilon \beta + G^2/(16\beta) \leq 3\beta \), with probability \( 1 - \delta/T \).

We first show that if Condition 1 of Lemma 3.12 is satisfied, then Algorithm 1 terminates at Line 10 returning a certifying vector. The only issue is that we have access to the empirical distribution \( \hat{\mathcal{D}}_N \) instead of \( \mathcal{D} \). From Lemma 3.19, we have that the empirical expectation of Line 9 is sufficiently close to the true expectation that appears in Condition 1 of Lemma 3.12, thus it is going to find it.

We now analyze the case when Condition 1 of Lemma 3.12 is not true. From Lemma 3.12, we immediately get that since Condition 1 is not satisfied, Condition 2 is true. Then, using the update rule \( v^{(k+1)} = \frac{v^{(k)} + \lambda \hat{g}^{(k)}}{\|v^{(k)} + \lambda \hat{g}^{(k)}\|_2} \) with \( \lambda = G^2/(64\beta^3) \), we have \( \hat{g}^{(k)} = \text{proj}_{v^{(k)} + \hat{g}^{(k)}}(\hat{g}^{(k)}) \) (here \( \hat{g}^{(k)} \) is the \( \hat{g}^{(k)} \) with the component on the direction \( v^{(k)} \) removed). Note that this procedure only decreases the norm of \( \hat{g} \) (by the Pythagorean theorem). Then, from Lemma 3.17, we have \( \langle v^{(k+1)}, v^* \rangle \geq \langle v^{(k)}, v^* \rangle + G^4/\beta^4 \).

The update rule is repeated for at most \( O(\beta^4/G^4) \) iterations. From Lemma 3.12, we have that a certificate exists if the angle with the optimal vector is sufficiently small. Putting everything together, our total sample complexity is \( N = \tilde{O} \left( \frac{d\delta^4}{\epsilon^4 C^2} \right) \log(1/\delta) \). It is also clear that the runtime is \( \text{poly}(N, d) \), which completes the proof.

3.4 Proof of Theorem 3.3

To prove Theorem 3.3, we will use the iterative algorithm developed in Proposition 3.11 initialized with a uniformly random unit vector \( v_0 \). It is easy to show that such a random vector will have non-trivial correlation with \( v^* \).

Fact 3.20 (see, e.g., Remark 3.2.5 of Ver18). Let \( v \) be a unit vector in \( \mathbb{R}^d \). For a random unit vector \( u \in \mathbb{R}^d \), with constant probability, it holds \( |\langle v, u \rangle| = \Omega(1/\sqrt{d}) \).
We now present the proof of Theorem 3.3 putting together the machinery developed in the previous subsections.

**Proof of Theorem 3.3.** As explained in Section 3.1, we are looking for a certificate function $T_w(x)$ of the form given in Equation (3). As argued in Section 3.2, the search for such a certificate function can be simplified by projecting the samples to a $(d-1)$-dimensional subspace via the perspective projection.

From Proposition 3.6, choosing $\rho = O(\sqrt{d})$, there is a $c = (LR)^{O(1)}$ such that the resulting distribution $\mathcal{D}_B^w$ is $(2, c\theta/\sqrt{d}, \sqrt{d}/\theta, \beta\sqrt{d}/(c\theta)\log(\sqrt{d}/\theta))$-well-behaved and satisfies the $(\alpha, Ad^{1/2}/(c\theta))$-Tsybakov noise condition.

From Fact 3.20, a random unit vector $v \in \mathbb{R}^{d-1}$ with constant probability satisfies $\langle v, (w^*)^\perp \rangle = \Omega(1/\sqrt{d})$. We call this event $\mathcal{E}$.

From Proposition 3.11, conditioning on the event $\mathcal{E}$ and using $\frac{\sigma^2}{\theta^2}(\frac{A}{\sqrt{LR}})^{O(1/\alpha)} \log(1/\delta)$ samples, with probability $1-\delta$, we get a $(v', R, t_0)$ such that

$$E_{(x,y) \sim \mathcal{D}_B^w} \left[ -R \leq \langle v', x \rangle \leq -t_0 \right] \leq - \left( \theta LR/(Ad) \right)^{O(1/\alpha)}/\beta .$$

By inverting the transformation (Lemma 3.7), we get that

$$E_{(x,y) \sim \mathcal{D}} \left[ T_w(x) \langle x, w \rangle \right] \leq - \left( \theta LR/(Ad) \right)^{O(1/\alpha)}/\beta .$$

Overall, we conclude that with constant probability Algorithm 1 returns a valid certificate. Repeating the process $k = O(\log(1/\delta))$ times, we can boost the probability to $1-\delta$. The total number of samples for finding and testing these candidate certificates until we find a correct one with probability at least $1-\delta$ is $N = \left( \frac{dA}{\theta LR} \right)^{O(1/\alpha)} \log(1/\delta)$. It is also clear that the runtime is $\text{poly}(N, d)$, which completes the proof. \qed

## 4 More Efficient Certificate for Log-Concave Distributions

In this section, we present a more efficient certificate algorithm for the important special case of isotropic log-concave distributions. To achieve this, we use Algorithm 1 from the previous section starting from a significantly better initialization vector. To obtain such an initialization, we leverage the structure of log-concave distributions. The main result of this section is the following theorem.

**Theorem 4.1** (Certificate for Log-concave Distributions). Let $\mathcal{D}$ be a distribution on $\mathbb{R}^d \times \{ \pm 1 \}$ that satisfies the $(\alpha, A)$-Tsybakov noise condition with respect to an unknown halfspace $f(x) = \text{sign}(\langle w^*, x \rangle)$ and is such that $\mathcal{D}_x$ is isotropic log-concave. Let $w$ be a unit vector that satisfies $\theta(w, w^*) \geq \theta$, where $\theta \in (0, \pi]$. There is an algorithm that, given as input $w$, $\theta$, and $N = \text{poly}(d) \cdot \left( \frac{A}{\theta LR} \right)^{O(1/\alpha^2)} \log(1/\delta)$ samples from $\mathcal{D}$, it runs in $\text{poly}(d, N)$ time, and with probability at least $1-\delta$ returns a certifying function $T_w : \mathbb{R}^d \mapsto \mathbb{R}_+$ such that

$$E_{(x,y) \sim \mathcal{D}} \left[ T_w(x) \langle w, x \rangle \right] \leq - \left( \frac{\theta}{A} \right)^{O(1/\alpha^2)} .$$

(10)

In other words, we give an algorithm whose sample complexity and running time as a function of $d$ is a fixed degree polynomial, independent of the noise parameters.

To establish Theorem 4.1, we apply Algorithm 1 starting from a better initialization vector. The main technical contribution of this section is an efficient algorithm to obtain such a vector for log-concave marginals.
Theorem 4.2 (Efficient Initialization for Log-Concave Distributions). Let $\mathcal{D}$ be a distribution on $\mathbb{R}^d \times \{\pm 1\}$ that satisfies the $(\alpha, A)$-Tsybakov noise condition with respect to an unknown halfspace $f(x) = \text{sign}((w^*, x))$ and is such that $\mathcal{D}_x$ is isotropic log-concave. There exists an algorithm that, given an $\epsilon > 0$, a unit vector $w$ such that $\|w^* - w\|_2 = \Theta(\epsilon)$, and $N = \text{poly}(d) \cdot (A/(\alpha \epsilon))^{O(1/\alpha)}$ samples from $\mathcal{D}$, it runs in $\text{poly}(d, N)$ time, and with constant probability returns a unit vector $v$ such that $\langle v, (w^*)^\perp w \rangle \geq (\alpha \epsilon/A)^{O(1/\alpha)}$, where $(w^*)^\perp w$ is the component of $w^*$ perpendicular to $w$.

4.1 Intuition and Roadmap of the Proof

Here we sketch the proof of Theorem 4.2 and point to the relevant lemmas in the formal argument (Section 4.2). Given a weight vector $w$ of unit length, our goal is to find a unit vector $v$ that has non-trivial correlation with $(w^*)^\perp w$, i.e., such that $\langle (w^*)^\perp w, v \rangle$ is roughly $\epsilon^{1/\alpha}$, where $w^*$ is the optimal halfspace. Our first step is to condition on a thin band around the current candidate $w$ (similarly to Section 3, see Figure 1). When the size of the band approaches 0, we get an instance whose separating hyperplane is perpendicular to $(w^*)^\perp w$ and has much larger Tsybakov noise rate. After that, we would like (similarly to Section 3) to project the points on the subspace $(w^*)^\perp w$. Instead of having a zero length band, we will instead take a very thin band. We have already seen in Section 3 that we can apply a perspective transformation in order to project the points on $(w^*)^\perp w$ and obtain an instance that satisfies the Tsybakov noise condition (with somewhat worse parameters). Unfortunately, for the current setting of log-concave distributions, we cannot use the perspective projection, as it does not preserve the log-concavity of the underlying distribution. On the other hand, we know that log-concavity is preserved when we condition on convex sets (such as the thin band we consider here) and when we perform orthogonal projections.

As we have seen (see Figure 2a), an orthogonal projection will create a “fuzzy” region with arbitrary sign. However, we can control the probability of this “fuzzy” region by taking a sufficiently thin random band. In particular, instead of Tsybakov noise, we will end up with the following noise condition: For some small $\xi > 0$, with probability $2/3$ the noise $\eta(x)$ is bounded above by $1/2 - \xi$, and with probability roughly $\xi^{\Theta(1)}$ we have $\eta(x) > 1/2$ (this corresponds to the probability of the “fuzzy” region). For the proof of this statement and detailed discussion on how the random band results in this above noise guarantee, see Lemma 4.11.

From this point on, we will be working in the subspace $w^\perp$ and assume that the distribution satisfies the aforementioned noise condition. As we have discussed, the marginal distribution on the examples remains log-concave and it is not hard to make its covariance be close to the identity. However, conditioning on the thin slice may result in a distribution with large mean, even though originally the distribution was centered. This is a non-trivial technical issue. We cannot simply translate the distribution to be origin-centered, as this would result in a potentially very biased optimal halfspace. Our proof crucially relies on the assumption of having a distribution that is nearly centered and at the same time for the optimal halfspace to have small bias. We overcome this obstacle in Step 1 below.

Our approach is as follows:

1. First, we show that there is an efficient rejection sampling procedure that preserves log-concavity and gives us a distribution that is nearly isotropic (see Definition 4.3). For the algorithm and its detailed proof of correctness, see Algorithm 3 and Lemma 4.14.

2. Then we show the following statement: Under the following assumptions

   (i) the $x$-marginal is nearly isotropic,
Lemma 4.4

low-degree multivariate polynomials under log-concave distributions.

be a polynomial of degree at most \( n \) (\cite{LV07} and Lemma 7 of \cite{KLT09}).

\( \langle q, 0 \rangle \)

We will require the following standard anti-concentration result for Useful Technical Tools.

with large eigenvalues. Our plan is to return a random unit vector of the subspace degree-1 Chow parameters (this is a single vector) and the eigenvectors of the degree-2 Chow matrix \( \langle w, y \rangle \) in order for this random vector to have non-trivial correlation with \( (w^*)^\perp \). In particular, we show that a \( \gamma \)-stationary point \( r \) of \( F(r) \) will make the above norm of the expectation roughly \( O(\gamma) \) (Claim 4.15). Therefore, in time \( \text{poly}(d/\gamma) \), we find a reweighting of the initial distribution whose mean is close to \( 0 \). Given this point \( r \), we then perform rejection sampling: We draw \( x \) from the initial distribution \( D \) and accept it with probability \( \max(1, \exp(-\langle r, x \rangle)) \), i.e., we "shrink" the distribution along the direction \( r \).

We now explain how to handle the setting that the distribution is approximately log-concave (Step 2 above). After we make our distribution nearly isotropic, we compute the degree-2 Chow parameters of the distribution, i.e., the vector \( E_{(x,y) \sim D}[yx] \) and the matrix \( E_{(x,y) \sim D}[yx(x^T I)] \). We show that there exists a degree-2 polynomial \( p(\langle (w^*)^\perp y, x \rangle) \) that correlates non-trivially with the labels \( y \) (Lemma 4.9). This means that \( (w^*)^\perp \) correlates reasonably with the degree-2 Chow parameters. In particular, \( (w^*)^\perp \) has a non-trivial projection on the subspace \( V \) spanned by the degree-1 Chow parameters (this is a single vector) and the eigenvectors of the degree-2 Chow matrix with large eigenvalues. Our plan is to return a random unit vector of the subspace \( V \). However, in order for this random vector to have non-trivial correlation with \( (w^*)^\perp \), we also need to show that the dimension of \( V \) is not very large.

The last part of our argument shows that \( V \) has reasonably small dimension. To prove this, we first show that the dimension of \( V \) can be bounded above by the variance of the projection of \( D \) onto \( V \), \( \text{Var}_{x \sim D}[\|x\|_2^2] \). Then we make essential use of a recent "thin-shell" result about log-concave measures that bounds from above \( \text{Var}_{x \sim D}[\|x\|_2^2] \), see Lemma 4.6 and Lemma 4.10.

4.2 Proof of Theorem 4.2

The proof of Theorem 4.2 requires a number of intermediate results. As already mentioned, our initialization algorithm works by restricting \( D \) to a narrow band perpendicular to \( w \). Unfortunately, this restriction will be log-concave but will no longer be isotropic, even in the directions perpendicular to \( w \). However, it will be close in the following sense.

Definition 4.3 \((\alpha, \beta)\)-isotropic distribution. \( \)We say that a distribution \( D \) is \((\alpha, \beta)\)-isotropic, if for every unit vector \( u \in \mathbb{R}^d \), it holds \( \|E_{x \sim D}[\langle x, u \rangle]\| \leq \alpha \) and \( 1/\beta \leq E_{x \sim D}[\langle x, u \rangle^2] \leq \beta \).

Useful Technical Tools. We will require the following standard anti-concentration result for low-degree multivariate polynomials under log-concave distributions.

Lemma 4.4 (Theorem 8 of \cite{CWW01}). Let \( D \) be a log-concave distribution on \( \mathbb{R}^d \) and \( p: \mathbb{R}^d \rightarrow \mathbb{R} \) a polynomial of degree at most \( n \). Then there is an absolute constant \( C > 0 \) such that for any \( 0 < q < \infty \) and \( t \in \mathbb{R}_+ \), it holds \( \Pr\_{x \sim D}[|p(x)| \leq t] \leq Ct^{1/n} E_{x \sim D}[|p(x)|^{q/n}]^{1/q} \).

The following statement is well-known. (It follows for example by combining Theorem 5.14 of \cite{LV07} and Lemma 7 of \cite{KLT09}.)
**Fact 4.5.** Let $z$ be an isotropic log-concave distribution on $\mathbb{R}^d$ and let $\gamma(\cdot)$ be its density function.

There exists a constant $c_d > 0$ such that:

1. For any $z$ with $\|z\|_2 \leq c_d$, we have that $\gamma(z) \geq c_d$.

2. For any $z$, we have that $\gamma(z) \leq 1/c_d \exp(-1/c_d \|z\|_2)$.

Our proof makes essential use of the following “thin-shell” estimate bounding the variance of the norm of any isotropic log-concave random vector.

**Lemma 4.6 (Corollary 13 of [LV17]).** Let $D$ be any isotropic log-concave distribution on $\mathbb{R}^d$. We have that $\text{Var}_{x \sim D}[\|x\|_2^2] \leq d^{3/2}$.

In particular, it is important for our analysis that the above bound is sub-quadratic in $d$.

Finally, we will require the following simple lemma bounding the sample complexity of approximating the degree-2 Chow parameters of a halfspace under isotropic log-concave distributions.

**Lemma 4.7.** Let $D$ be an isotropic log-concave distribution on $\mathbb{R}^d$ and $\mathcal{D}_N$ be the empirical distribution obtained from $D$ with $N = \text{poly}(d/\epsilon)$ samples. Then, with high constant probability, we have $\left\| E_{(x,y) \sim D}[yx] - E_{(x,y) \sim \mathcal{D}_N}[yx] \right\|_2 \leq \epsilon$ and $\left\| E_{(x,y) \sim D}[y(xx^\top - I)] - E_{(x,y) \sim \mathcal{D}_N}[y(xx^\top - I)] \right\|_F \leq \epsilon$.

The proof of this lemma can be found in Appendix B.

We now have the necessary tools to proceed with our proof. We start by showing how we can find a vector $v$ with non-trivial correlation with $(w^*)^\top w$ if the marginal distribution is (approximately) isotropic. Since in general this will not hold, we will then need to reduce to the isotropic case.

**Proposition 4.8.** Let $D$ be a distribution on $\mathbb{R}^d \times \{-1,1\}$ such that $D_x$ is $(\alpha, \beta)$-isotropic log-concave. Let $f(x) = \text{sign}((v^*, x) - \theta)$ be such that $\text{Pr}_{(x,y) \sim D}[y \neq f(x)|x] = \eta(x)$, where for some $\xi > 0$ we have that $\text{Pr}_{x \sim D_x}[^3\eta(x)] < 1/2 - \xi]$ $\geq 2/3$ and $\text{Pr}_{x \sim D_x}[^3\eta(x) > 1/2] \leq \xi'$, where $\xi'$ is a constant degree polynomial in $d$. Then, as long as $|\alpha| + |\theta|$ is less than a sufficiently small constant multiple of $1/\text{poly}(1/\xi)$, there exists an algorithm with sample complexity and runtime $\text{poly}(d/\xi)$ that with constant probability returns a unit vector $v \in \mathbb{R}^d$ such that $(v, v^*) > \text{poly}(\xi)$.

**Proof.** For clarity of the analysis, we begin by presenting our algorithm for the case that $D_x$ is exactly isotropic log-concave. We then show how the algorithm and its analysis can be modified for the approximate log-concave setting.

Our algorithm is fairly simple. We compute high-precision estimates $T_1$ and $T_2$ of the vector $T_1 := E_{(x,y) \sim D}[yx]$ and the matrix $T_2 := E_{(x,y) \sim D}[y(xx^\top - I)]$ respectively. This can be easily done by taking $\text{poly}(d/\epsilon)$ samples from $D$ and using the empirical estimates (see Lemma 4.7). We then define $V$ to be the subspace spanned by $T_1$ and the eigenvectors of $T_2$ whose eigenvalue has absolute value at least $2\xi'$, for $\xi'$ some sufficiently large constant power of $\xi$. The algorithm returns a uniform random unit vector $v$ from $V$.

It is clear that the above algorithm has polynomial sample complexity and runtime. We need to show that with constant probability it holds that $(v, v^*) > \text{poly}(\xi)$. The desired statement will follow by following the following two claims:

1. The size of the projection of $v^*$ onto $V$ is at least $\text{poly}(\xi)$.

2. The dimension of $V$ is at most $\text{poly}(1/\xi)$.

\[^1]\text{It is not difficult to verify that } \xi' = \Theta(\xi^3) \text{ suffices.}
The desired result then follows by noting that the median value of \(|\langle v^*, v \rangle|\) is on the order of \(\|\text{proj}_V(v^*)\|_2 / \sqrt{\dim(V)}\), and observing that the sign of the inner product is independent of its size.

To establish the first claim, we prove the following lemma for isotropic log-concave distributions.

**Lemma 4.9.** Let \(\mathcal{D}_x\) be isotropic log-concave. There exists a degree-2 polynomial \(p : \mathbb{R} \to \mathbb{R}\) such that \(E_{x \sim \mathcal{D}_x}[p(\langle v^*, x \rangle)] = 0\), \(E_{x \sim \mathcal{D}_x}[p(\langle v^*, x \rangle)^2] = 1\), and \(E_{(x,y) \sim \mathcal{D}}[yp(\langle v^*, x \rangle)] = \Omega(\xi)\).

**Proof.** We consider the polynomial

\[
q(x) = (x - \theta)(x + 1/\theta) = x^2 + (1/\theta - \theta)x - 1
\]

and we set \(p(x) = q(x)/\sqrt{E_{x \sim \mathcal{D}_x}[q(\langle v^*, x \rangle)^2]}\). It is easy to see that \(E_{x \sim \mathcal{D}_x}[p(\langle v^*, x \rangle)] = 0\) and \(E_{x \sim \mathcal{D}_x}[p(\langle v^*, x \rangle)^2] = 1\). To show that \(E_{(x,y) \sim \mathcal{D}}[yp(\langle v^*, x \rangle)] = \Omega(\xi)\), we note that

\[
E_{(x,y) \sim \mathcal{D}}[yp(\langle v^*, x \rangle)] = E_{x \sim \mathcal{D}_x}[(1-2\eta(x))f(x)p(\langle v^*, x \rangle)].
\]

We observe that if \(|\langle v^*, x \rangle| \leq 1/|\theta|\), then \(\text{sign}(p(\langle v^*, x \rangle)) = f(x)\), where \(f(x) = \text{sign}(\langle v^*, x \rangle - \theta)\). Thus, unless \(|\langle v^*, x \rangle| > 1/|\theta|\) or \(\eta(x) > 1/2\) (which happens with probability at most \(\xi'\), a sufficiently high power of \(\xi\)), we have that \((1-2\eta(x))f(x)p(\langle v^*, x \rangle) \geq 0\) except with probability at most \(\xi'\).

Let \(I(x)\) denote the indicator of the event \((1-2\eta(x))f(x)p(\langle v^*, x \rangle) < 0\). We have that

\[
E_{(x,y) \sim \mathcal{D}}[yp(\langle v^*, x \rangle)] = E_{x \sim \mathcal{D}_x}[(1-2\eta(x))p(\langle v^*, x \rangle)] - 2E_{x \sim \mathcal{D}_x}[(1-2\eta(x))p(\langle v^*, x \rangle)I(x)]
\]

\[
\geq E_{x \sim \mathcal{D}_x}[(1-2\eta(x))p(\langle v^*, x \rangle)] - 2\sqrt{E_{x \sim \mathcal{D}_x}[I^2(x)]E_{x \sim \mathcal{D}_x}[p(\langle v^*, x \rangle)^2]}
\]

\[
\geq E_{x \sim \mathcal{D}_x}[(1-2\eta(x))p(\langle v^*, x \rangle)] - 2\sqrt{\xi}.
\]

Recall that by assumption there is at least a 2/3 probability that \((1-2\eta(x)) \geq \xi\).

By anti-concentration of Gaussian polynomials, Lemma 4.4, applied for \(q = 4\) and \(n = 2\), we have that \(\text{Pr}_{x \sim \mathcal{D}_x}[|p(\langle v^*, x \rangle)| \leq t] = O(\sqrt{t})\). Thus, for small enough \(t\), we have that \(|p(\langle v^*, x \rangle)| = \Omega(1)\) with probability at least 2/3. Therefore, with probability at least 1/3 both statements hold. Since \(|1-2\eta(x)||p(\langle v^*, x \rangle)| \geq 0\) for all \(x\), we have that \(E_{x \sim \mathcal{D}_x}[(1-2\eta(x)||p(\langle v^*, x \rangle)|| = \Omega(\xi)\). This completes our proof. \(\square\)

Given Lemma 4.9, it is not hard to see that \(p(\langle v^*, x \rangle) = a(\langle v^*, x \rangle) + b((\langle v^*, x \rangle)^2 - 1)\) for some real numbers \(a\) and \(b\) with \(|a| + |b| = \Theta(1)\). We note that there is another way to compute \(E_{(x,y) \sim \mathcal{D}}[yp(\langle v^*, x \rangle)]\) relating it to \(T_1^1\) and \(T_2^1\). In particular, we can write

\[
E_{(x,y) \sim \mathcal{D}}[yp(\langle v^*, x \rangle)] = a E_{(x,y) \sim \mathcal{D}}[y(\langle v^*, x \rangle)] + b E_{(x,y) \sim \mathcal{D}}[y((\langle v^*, x \rangle)^2 - 1)]
\]

\[
= a \langle v^*, T_1^1 \rangle + b \langle v^*, T_2^1 \rangle \rangle + b \langle (v^*)^T (xx^T - I)v^* \rangle
\]

\[
= a \langle v^*, T_1^1 \rangle + b (v^*)^T T_2^1 v^*.
\]

Thus, Lemma 4.9 implies that either \(|\langle v^*, T_1^1 \rangle| = \Omega(\xi)| or \(|(v^*)^T T_2^1 v^*| = \Omega(\xi)|\).

Assuming that \(T_1^1\) and \(T_2^1\) estimate \(T_1\) and \(T_2\) to error less than this quantity, i.e., \(O(\xi)\), the above implies that either \(\langle v^*, T_1^1 \rangle = \Omega(\xi)| or \((v^*)^T T_2^1 v^* = \Omega(\xi)\). In the former case, we have that \(\|\text{proj}_V(v^*)\|_2 \geq |\langle v^*, T_1^1 \rangle| = \Omega(\xi)|. In the latter case, we note that since \(V\) contains the span of
all eigenvectors of $T'_2$ with eigenvalue having absolute value at least $\zeta$, it holds that $|\langle v^* \rangle^T T'_2 v^*| \leq \zeta + \|T'_2\|_2 \|\text{proj}_v(v^*)\|_2$. This will imply that in this case as well we have that $\|\text{proj}_v(v^*)\|_2 = \Omega(\xi)$, if $\|T'_2\|_2$ is $O(1)$. To show this, we note that for any unit vector $v$, we have

$$v^T T_2 v = \mathbb{E}_{(x,y) \sim D} \left[ y(v^T (xx^T - I) v) \right] = \mathbb{E}_{(x,y) \sim D} \left[ y(\langle v, x \rangle^2 - 1) \right] \leq \sqrt{\mathbb{E}_{(x,y) \sim D} \left[ y^2 \right] \mathbb{E}_{x \sim D_x} \left[ (\langle v, x \rangle^2 - 1)^2 \right]} = O(1).$$

This completes the proof that the projection of $v^*$ onto $V$ has size at least $\text{poly}(\xi)$.

It remains to show that the dimension of $V$ is at most $\text{poly}(\xi)$. We prove the following lemma:

**Lemma 4.10.** We have that $\dim(V) = O(\zeta^{-4}).$

**Proof.** Let $V_+$ denote the subspace spanned by the eigenvectors of $T'_2$ with eigenvalue at least $\zeta$. Let $V_-$ denote the subspace spanned by eigenvectors of eigenvalue at most $-\zeta$. Clearly $\dim(V) \leq \dim(V_+) + \dim(V_-) + 1$. We will show that $\dim(V_+) = O(\zeta^{-4})$ and the bound on $\dim(V_-)$ will follow symmetrically.

Let $m = \dim(V_+)$ and let $P$ be the projection matrix that maps a vector $z$ onto $V_+$. Since $T'_2$ is sufficiently close to $T_2$, the restriction of $T'_2$ to $V_+$ will have all of its eigenvalues at least $\zeta/2$. Therefore, it holds that

$$m\zeta/2 \leq \text{tr}(PT_2) = \mathbb{E}_{(x,y) \sim D} \left[ y \text{tr}(P(xx^T - I)) \right] = \mathbb{E}_{(x,y) \sim D} \left[ y\|Px\|_2^2 - m \right] \leq \sqrt{\mathbb{E}_{(x,y) \sim D} \left[ y^2 \right] \mathbb{E}_{x \sim D_x} \left[ (\|Px\|_2^2 - m)^2 \right]} = \sqrt{\text{Var}[\|Px\|_2^2]}.$$

In other words, we have that

$$m^2\zeta^2 \leq 4\text{Var}[\|Px\|_2^2].$$

To conclude the proof, observe that $Px$ is a log-concave distribution in $m$ dimensions, since projections preserve log-concavity. From Lemma 4.10 we have that $\text{Var}[\|Px\|_2^2] = O(m^{3/2})$ and together with the above, we obtain that $m = O(\zeta^{-4}).$ This completes our proof.

Thus far, we have shown the desired claim if the distribution is in isotropic position, $\theta = O(1/\log(1/\xi))$, and we have access to sufficiently accurate approximations $T'_1, T'_2$ to the degree-2 Chow parameters with accuracy $\zeta/2$. To handle the case that the distribution $D$ is $(\alpha, \beta)$-isotropic, we can let $z \sim D'_z$, where $z = \text{Cov}[x]^{-1/2} (x - E_{x \sim D}[x])$, be an isotropic log-concave distribution. We need to show that if we have good approximations of $E_{x \sim D}[x]$ and $\text{Cov}[x]$, we can compute $O(\zeta)$-approximations to $T_1$ and $T_2$ for $z$ (i.e., $E_{(z,y) \sim D}[yz]$ and $E_{(z,y) \sim D}[yz(z^T - I)]$).

By taking $\text{poly}(d/\zeta)$ samples, we can compute $\widehat{M}$ and $\widehat{\text{Cov}}$ such that $\|\widehat{M} - E_{x \sim D}[x]\|_2 \leq \zeta/16$ and $\|\widehat{\text{Cov}}[x]\|_2 \leq \zeta/16$. Let $\widehat{z} = \widehat{M}^{-1/2}(x - \widehat{m})$. Then we have that $\|\widehat{z} - z\|_2 \leq \zeta/4$. Thus, we obtain that $\|E_{(z,y) \sim D}[yz] - E_{(z,y) \sim D}[yz]\|_2 \leq \|E_{(z,y) \sim D}[z - \widehat{z}]\|_2 \leq \zeta/4$ and similarly that $\|E_{(z,y) \sim D}[yz(z^T - I)] - E_{(z,y) \sim D}[yz(z^T - I)]\|_2 \leq \zeta/4$. By approximating the degree-2 Chow parameters $T_1, T_2$ to accuracy $\zeta/4$, we obtain overall error $\zeta/2$.

We note that $(z, y)$ satisfies our assumptions for the function

$$f'(x) = \text{sign} \left( (v*)^T \text{Cov}[x]^{1/2}x - (\theta - \langle v^*, E_{x \sim D}[x] \rangle) \right).$$

From our assumptions, we have that $|\theta - \langle v^*, E_{x \sim D}[x] \rangle| = O(1/\log(1/\xi))$. Using the aforementioned argument for $z$, this allows us to compute a $v$ so that with constant probability $v^T \text{Cov}[x]^{1/2}v^* \geq \text{poly}(\xi)$, or $\langle \text{Cov}[x]^{1/2}v, v^* \rangle \geq \text{poly}(\xi)$. This completes the proof.
Thus far, we have dealt with the case that the mean of our log-concave distribution is sufficiently close to zero. As already mentioned, this property will not hold in general after projection. The following important lemma shows that by conditioning on a random thin band before projecting onto $w^\perp$, we obtain a log-concave distribution whose mean has small distance from the origin. Moreover, we show that the noise condition of the instance after we perform this transformation satisfies the assumptions of Proposition [4.8]. We note that this is the step that crucially relies on picking a random thin band.

**Lemma 4.11 (Properties of Transformed Instance).** Let $D$ be a distribution on $\mathbb{R}^d \times \{\pm 1\}$ that satisfies the $(\alpha, A)$-Tsybakov noise condition with respect to an unknown halfspace $f(x) = \text{sign}(\langle w^*, x \rangle)$ and is such that $D_x$ is isotropic log-concave. Fix $\epsilon > 0$ and unit vector $w$ such that $\theta(w, w^*) = \Theta(\epsilon)$. Let $s$ be a sufficiently small multiple of $\frac{\epsilon}{2}$. Set $\xi = (\Theta(s/A))^{1/\alpha}$ and $s' = \Theta(\xi^3 s \epsilon)$. Pick $x_0$ uniformly at random from $[s, 2s]$ and define the random band $B_{x_0} = \{x \in \mathbb{R}^d : \langle x, w \rangle \in [x_0, x_0 + s']\}$.

Define the distribution $D^\perp = D_{B_{x_0}}^{\text{proj}_w}$, the classifier $f^\perp(x^\perp) = \text{sign}(x_0 / \tan \theta + \langle x^\perp, (w^*)^\perp w \rangle)$, and the noise function $\eta^\perp(x^\perp) = \mathbb{P}_{(z,y) \sim D^\perp} [y \neq f^\perp(z) | z = x^\perp]$.

Then $D^\perp$ is an $(O(1), O(1))$-isotropic log-concave distribution and, with probability at least 99%, satisfies the following noise condition:

$$\mathbb{P}_{x^\perp \sim D^\perp_x} [\eta^\perp(x^\perp) \leq 1/2 - \xi] \geq 2/3 \quad \text{and} \quad \mathbb{P}_{x^\perp \sim D^\perp_x} [\eta^\perp(x^\perp) \geq 1/2] \leq \xi^3.$$

**Proof.** We first calculate how far the distribution $D_{B_{x_0}}^{\text{proj}_w}$ is from being isotropic. Since our final goal is to have a distribution whose mean is arbitrarily close to 0, we need to bound the distance from 0 of the mean of the distribution obtained after we condition on $B$ and project onto $w^\perp$. The following claim shows that the mean and covariance of $D_{B_{x_0}}^{\text{proj}_w}$ differ from these of the initial distribution $D$ only by constant factors (additive for the mean and multiplicative for the covariance).

**Claim 4.12.** $D_{B_{x_0}}^{\text{proj}_w}$ is $(O(1), O(1))$-isotropic.

The proof of Claim 4.12 relies on Fact 4.5 and is given in Appendix B.

It remains to prove how the noise condition changes via the transformation. In our argument, we are going to repeatedly use the following anti-concentration, and anti-anti-concentration properties of log-concave distributions that follow directly from Fact 4.5. In particular, for every interval $[a, b]$, we have that:

1. $\mathbb{P}_{x \sim D_x} [(x, v) \in [a, b]] = O(b - a)$ (anti-concentration).
2. If $|a|, |b|$ are smaller than some absolute constant (see Fact 4.5), then it also holds that $\mathbb{P}_{x \sim D_x} [(x, v) \in [a, b]] = \Omega(b - a)$ (anti-anti-concentration).

Using the condition $\theta(w, w^*) = \Theta(\epsilon)$, we can assume that $w^* = \lambda_1 w + \lambda_2 (w^*)^\perp w$, where $\lambda_1 = \cos \theta$ and $\lambda_2 = \sin \theta$. It holds $|\lambda_1| = 1 - \Theta(\epsilon)$ and $\lambda_2 = \Theta(\epsilon)$. Next we set $x = (x_w, x^\perp)$, where $x_w = \langle w, x \rangle$ and $x^\perp$ is the projection of $x$ on the subspace $w^\perp$.

For some $\zeta \in (0, 1)$, set $\xi = (\zeta s/A)^{1/\alpha}$.

Following important lemma shows that by conditioning on a random thin band before projecting onto $w^\perp$, we obtain a log-concave distribution whose mean has small distance from the origin. Moreover, formally we will show that

$$\mathbb{P}_{(x^\perp, y) \sim D^\perp} [\eta^\perp(x^\perp) \geq 1/2] \leq \xi^3.$$  

---

\footnote{We need $s$ to be smaller than the absolute constant of Fact 4.5 for dimension $d = 2$.}
Notice that in this part of the proof the randomness of \( x_0 \) is not important and we are able to establish a stronger claim that holds for every band \( B_{x_0} \). Conditioned on \( x \in B_{x_0} \), i.e., \( x_w \in [x_0, x_0 + s'] \), it holds that

\[
\langle w^*, x \rangle = \lambda_1 x_w + \lambda_2 \langle (w^*)^\perp w, x^\perp \rangle = \lambda_1 x_0 + \lambda_2 \langle (w^*)^\perp w, x^\perp \rangle + \rho s',
\]

for some \( \rho \in [-1, 1] \) (recall that \( |\lambda_1| \leq 1 \)). Notice that when \( |\lambda_1 x_0 + \lambda_2 \langle (w^*)^\perp w, x^\perp \rangle | > s' \), \( f^\perp(x^\perp) \) is equal to the sign of \( \langle w^*, x \rangle \) (recall that \( \lambda_2 > 0 \)), and therefore we are outside of the fuzzy region, see Figure 2a. Thus, we need to bound the probability of the event \( |\lambda_1 x_0 + \lambda_2 \langle (w^*)^\perp w, x^\perp \rangle | \leq s' \), or equivalently \( \langle (w^*)^\perp w, x^\perp \rangle \in [-\lambda_1 x_0 - s', -\lambda_1 x_0 + s'] =: I_{x_0}^s \). We have that

\[
\Pr_{x \sim D_x}[\langle (w^*)^\perp w, x^\perp \rangle \in I_{x_0}^s] = \frac{\Pr_{x \sim D_x}[\langle (w^*)^\perp w, x \rangle \in I_{x_0}^s]}{\Pr_{x \sim D_x}[x \in B_{x_0}]} = O(s'/(\lambda_2 s)) \leq \xi^3,
\]

where \( \Pr \) to bound the numerator we used the anti-concentration property of \( D \), Property 1 for the interval \( I_{x_0}^s \) of length \( s' \), and to bound the denominator we used the anti-anti-concentration, Property 2. The last inequality holds because we have that \( \lambda_2 = \Theta(\epsilon) \) and also, from the assumptions of the lemma, we have \( s' = \Theta(\xi^3 s \epsilon) \). This proves (11).

Now we deal with the case where \( |\lambda_1 x_0 + \lambda_2 \langle (w^*)^\perp w, x^\perp \rangle | \leq s' \), i.e., we are in the non-fuzzy region of Figure 2a. This is where the randomness of \( x_0 \) helps us control the probability that the noise is close to 1/2. Recall that,

\[
\eta^\perp(x^\perp) = \Pr_{(x^\perp, y) \sim D^\perp}[y \neq \text{sign}(\langle (w^*)^\perp w, x^\perp \rangle + x_0)] = \int_{x_0}^{x_0 + s'} \eta(x_w, x^\perp) \gamma(x_w | x^\perp) dx_w,
\]

where \( \gamma(x_w | x^\perp) \) is the density of \( D_{B_{x_0}} \) conditioned on \( x^\perp \), that is \( \gamma(x_w | x^\perp) = \gamma(x_w, x^\perp) / \int \gamma(x_w, x^\perp) dx_w \), and \( \gamma \) is the density of the \( x \)-marginal of \( D_{B_{x_0}} \). Note that, from Lemma 3.9 it follows that

\[
\Pr[\eta(x) \geq 1/2 - t | x_w \in [s, 2s + s']] = O(\xi^3 t^9).
\]

Therefore, \( \Pr[\eta(x) > 1/2 - 2\xi | x_w \in [s, 2s + s']] \leq \zeta, \) and it remains to prove that \( \Pr[\eta^\perp(x^\perp) > 1/2 - \xi] \) is at most a small constant multiple of \( \zeta \) with high constant probability.

To prove this, let \( M(x) \) be the indicator of the event \( \eta(x) > 1/2 - 2\xi \) and consider the random variable \( Y = \int_{x_0}^{x_0 + s'} M(x_w, x^\perp) \gamma(x_w | x^\perp) dx_w \). Observe that the randomness of \( Y \) is over the randomly chosen \( x_0 \) and \( x^\perp \). We will first show that the probability that the noise function \( \eta^\perp(x^\perp) \) exceeds \( 1/2 - \xi \) can be bounded above by the probability that the random variable \( Y \) exceeds \( 1/2 \), that is

\[
\Pr_{x^\perp \sim D^\perp, x_0}[\eta^\perp(x^\perp) > 1/2 - \xi] \leq \Pr_{x^\perp \sim D^\perp, x_0}[Y \geq 1/2].
\]

(12)

In fact, we show a stronger statement than Equation (12) that holds for any fixed \( x_0 \in [s, 2s] \). To see this, let \( \eta'(x) = 1/2 - 2\xi(1 - M(x)) \) and notice that \( \eta'(x) \geq \eta(x) \) for every \( x \). Then, it holds

\[
1/2 - \xi < \eta^\perp(x^\perp) = \int_{x_0}^{x_0 + s'} \eta(x_w, x^\perp) \gamma(x_w | x^\perp) dx_w \leq \int_{x_0}^{x_0 + s'} \eta'(x_w, x^\perp) \gamma(x_w | x^\perp) dx_w = 1/2 - 2\xi + 2\xi \int_{x_0}^{x_0 + s'} M(x_w, x^\perp) \gamma(x_w | x^\perp) dx_w = Y \geq 1/2,
\]

which is equivalent to \( \int_{x_0}^{x_0 + s'} M(x_w, x^\perp) \gamma(x_w | x^\perp) dx_w = Y \geq 1/2 \).

28
Our next step is to bound from above the probability of the event \( Y \geq 1/2 \). For convenience, let \( \phi(x) \) be the density of the initial isotropic log-concave marginal \( D_x \). Thus, we have \( \gamma(x, x^\perp) = \phi(x, x^\perp)/Pr_D[B_{x_0}] \). Moreover, set \( Q = \min_{x \in [s, 2s]} Pr_D[B_{x_0}] \) and recall that from Properties 1\( \text{ and } 2 \) we have that for any \( x_0 \in [s, 2s] \) it holds \( Pr_D[B_{x_0}] = \Theta(s') \), and thus \( Q = \Theta(s') \). We can bound from above the expectation of \( Y \), i.e.,

\[
E[Y] = \int_s^{2s} \frac{1}{s} \int_{s_0}^{x_0+s'} \int_{w^d} M(x, x^\perp) \phi(x, x^\perp) \, dx^\perp \, dx_w \, dx_0 \\
\leq \frac{1}{sQ} \int_s^{2s} \int_{s_0}^{x_0+s'} \int_{w^d} M(x, x^\perp) \phi(x, x^\perp) \, dx^\perp \, dx_w \, dx_0 \\
\leq \frac{s'}{sQ} \int_s^{2s+s'} \int_{w^d} M(x, x^\perp) \phi(x, x^\perp) \, dx^\perp \, dx_w \\
\leq \frac{s'}{sQ} \int_{s}^{s} \int_{w^d} M(x, x^\perp) \phi(x, x^\perp) \, dx^\perp \, dx_w \\
\leq \frac{s'}{sQ} \int_{s}^{s} \int_{w^d} M(x, x^\perp) \phi(x, x^\perp) \, dx^\perp \, dx_w \\
\leq \frac{s'}{sQ} Pr_{D_x [x_0 \in [s, 2s+s']]} \Pr[\eta(x) > 1/2 - 2\xi | x_0 \in [s, 2s+s']] \leq \frac{s'}{sQ} \xi = O(\xi) ,
\]

where to get the third inequality we used the fact that for any non-negative function \( g(t) \) it holds

\[
\int_s^{2s} \int_{s}^{u+s'} g(t) dt du = \int_0^{s'} \int_{s+u}^{2s+u} g(t) dt du \leq \int_0^{s'} \int_{s}^{2s+s'} g(t) dt du = s' \int_{s}^{2s+s'} g(t) dt .
\]

The final inequality follows from Properties 1\( \text{ and } 2 \) By Markov’s inequality, we obtain \( Pr[Y \geq 1/2] = O(\xi) \). Therefore, combining this bound with Equation 12, we obtain the probability that \( \eta(x^\perp) > 1/2 - \xi \) is at most

\[
Pr_{x^\perp \sim D_x, x_0}[\eta^\perp(x^\perp) > 1/2 - \xi] \leq Pr_{x^\perp \sim D_x, x_0}[Y \geq 1/2] = O(\xi) .
\]

So, choosing \( \xi \) to be a sufficiently small absolute constant, we get that \( Pr_{(x^\perp, y)} \sim D^\perp [\eta^\perp(x^\perp) \geq 1/2] \leq \xi^3 \) and \( Pr_{x^\perp} [\eta^\perp(x^\perp) > 1/2 - \xi] \leq 1/3 \) with probability at least 99\%. This completes the proof.

We next show how to efficiently decrease the mean of a nearly identity covariance log-concave distribution and make it arbitrary close to zero. We achieve this by further conditioning. In particular, we show that we can efficiently find a reweighting of the conditional distribution on \( x^\perp \) such that it is approximately mean zero isotropic. The high-level idea to achieve this is, for some vector \( r \), to run rejection sampling, where \( x \) is kept with probability \( \min(1, \exp(-\langle r, x \rangle)) \). The problem is then to find \( r \). We do this by finding an approximate stationary point of an appropriately defined constrained non-convex optimization problem.

We will use the following standard fact about the convergence of projected stochastic gradient descent (PSGD) to stationary points of smooth non-convex functions. Consider the constrained optimization setting of minimizing a (differentiable) function \( F \) in the set \( \mathcal{X} \). In this setting, a point \( x \) is called \( \epsilon \)-stationary, \( \epsilon > 0 \), if for all \( u \in \mathcal{X} \) it holds \( \langle \nabla F(x), u - x \rangle \geq -\epsilon \| u - x \|_2 \). Note that if \( x \in \text{int}(\mathcal{X}) \), i.e., \( x \) is not on the boundary of \( \mathcal{X} \), this inequality is equivalent to \( \| \nabla F(x) \|_2 \leq \epsilon \).

**Fact 4.13** (see, e.g., [GLZ16], Corollary 4 and Equations (4.23) and (4.25)). Let \( D \) be a distribution supported on \( \mathbb{R}^d \). Let \( F : \mathcal{X} \mapsto \mathbb{R} \) be an L-smooth differentiable function on a compact convex set \( \mathcal{X} \subset \mathbb{R}^d \) with diameter \( D \). Let \( g : \mathcal{X} \times \mathbb{R}^d \mapsto \mathbb{R}^d \) be such that \( E_{x \sim D}[g(r, x)] = \nabla F(r) \) and \( E_{x \sim D}[\|g(r, x)\|_2^2] \leq \sigma^2 \), for some \( \sigma > 0 \). Then randomized projected SGD uses \( T = O(\sigma^3 D^2 L^2 / \epsilon^2) \) samples from \( D \), runs \( \text{poly}(T, d) \) time, and returns a point \( r' \) such that with probability at least 2/3, \( r' \) is an \( \epsilon \)-stationary point of \( F \).
We show the following:

**Lemma 4.14.** Let $D$ be an isotropic log-concave distribution on $\mathbb{R}^d$. Let $w \in \mathbb{R}^d$ be a unit vector and let $B = \{x \in \mathbb{R}^d : \langle w, x \rangle \in [a, b]\}$ for $a, b > 0$ smaller than some universal absolute constant. There exists an algorithm that, given $\gamma > 0$ and $\text{poly}(d/\gamma)$ independent samples from $D_{B}^{\text{proj}w,\perp}$, runs in sample polynomial time, and returns a vector $r$ such that if $z$ is obtained from $D_{B}^{\text{proj}w,\perp}$ by rejection sampling, where a sample $x$ is accepted with probability $\min(1, e^{-\langle r, x \rangle})$, then:

- A sample is rejected with probability $p$, where $p \in (0, 1)$ is an absolute constant.
- The distribution of $z$ is $(\gamma, O(1))$-isotropic log-concave.

**Proof.** For notational convenience, let $D' = D_{B}^{\text{proj}w,\perp}$. First, we note that for $u$ any unit vector perpendicular to $w$ and any $r$ perpendicular to $w$, we can apply Fact 4.5 to the projection of $x$ onto the subspace spanned by $u, w$ and $r$.

We denote by $c$ the constant $c_3$ from Fact 4.5. As a result, we have that the distribution of $\langle u, x \rangle$ will have constant probability density in a neighborhood of 0 and will have exponential tails. Furthermore, this will still hold after rejection sampling with probability $\min(1, e^{-\langle r, x' \rangle})$. This implies that no matter what $r$ is chosen, $z$ will be approximately isotropic. Moreover, $z$ will be log-concave automatically, because the rejection sampling multiplies the pdf by a log-concave function. Furthermore, the probability of a sample being accepted will be at least $\Pr_{x' \sim D'}[\langle r, x' \rangle \leq 0]$, which is at least $c^4$.

It remains to prove the second condition of the lemma. We let $R$ be a sufficiently large constant and apply projected SGD to find an approximate stationary point of the non-convex function $F(r) := \|g(r)\|^2_2$, where $g(r) := E_{x' \sim D'}[x' \min\{1, \exp(-\langle r, x' \rangle\})]$, in the feasible set $\{r \in \mathbb{R}^d : \|r\|_2 \leq R\}$. Note that $g(r)$ is the mean of the distribution of $z$.

We will need the following claim about the approximate stationary points of $F(r)$.

**Claim 4.15.** Any interior point of the feasible region, i.e., a point $r$ such that $\|r\|_2 < R$, has $\|\grad F(r)\|_2 = \Omega(\|g(r)\|_2)$. Moreover, $F$ has no stationary points on the boundary, i.e., on the set $\{r \in \mathbb{R}^d : \|r\|_2 = R\}$.

**Proof.** We show that the Jacobian matrix of $g$ is negative definite. In particular, for any vector $u \neq 0$, we have

$$
\langle u, \grad g(r)u \rangle = \left\langle u, -E_{x' \sim D'}[x'(x')^\top 1\{\langle r, x' \rangle \geq 0\} \exp(-\langle r, x' \rangle)]u \right\rangle \\
= -E_{x' \sim D'}[1\{\langle r, x' \rangle \geq 0\} \exp(-\langle r, x' \rangle) \langle u, x' \rangle^2] \\
= -E_{x' \sim D'}[1\{\langle r, x' \rangle \geq 0\} \exp(-\langle r, x' \rangle) \left\langle \frac{u}{\|u\|_2}, x' \right\rangle^2] \|u\|^2_2 \\
\leq -\frac{c^2}{4} e^{-c} E_{x' \sim D'}[1\{c \geq \langle r, x' \rangle \geq 0\} 1\left\{\left\langle \frac{u}{\|u\|_2}, x' \right\rangle \geq c/2\right\}] \|u\|^2_2 \\
\leq -\frac{c^5}{24} e^{-c} \|u\|^2_2,
$$

where we used Fact 4.5 which gives $\langle u, \grad g(r)u \rangle = -O(\|u\|^2_2)$. Observe that the gradient of $F$ at $r$ is $\grad F(r) = 2 \grad g(r)g(r)$, where $\grad g(r)$ is the Jacobian of $g$ at point $r$, thus $\|\grad F(r)\|_2 \geq \langle u, \grad g(r)g(r) \rangle / \|u\|_2$ for any vector $u$. Setting $u = g(r)$, we have that $\|\grad F(r)\|_2 = \Omega(\|g(r)\|_2)$.  

30
It remains to prove that there is no stationary point on the boundary. That is, for a point \( r \) with \( \| r \|_2 = R \), the gradient of \( F \) at \( r \) is a negative multiple of \( r \). It is easy to see that, using Fact 4.5 for \( R \) at least a sufficiently large constant, \( \langle g(r), r \rangle < 0 \). So, if the gradient of \( F \) at \( r \) is a negative multiple of \( r \), we have that

\[
0 < \langle g(r), \nabla F(r) \rangle = 2 \langle g(r), \text{Jac}(g(r))g(r) \rangle < 0 ,
\]

which is a contradiction. \( \square \)

As a result, an internal stationary point of \( F \) must have \( \| g(r) \|_2 \) close to 0, which would imply that the conditional distribution \( z \) with that \( r \) has mean less than \( \gamma \). In the following claim, we prove that \( F(r) \) is smooth with respect the Euclidean norm. See Appendix B for the proof.

**Claim 4.16.** The function \( F(r) \) is \( L \)-smooth, for some \( L = \text{poly}(d) \).

Thus, by Fact 4.13, running Stochastic Gradient Descent for \( T = \text{poly}(d/\gamma) \), we obtain a \( \gamma \)-stationary point, assuming we have an unbiased estimator for the gradient of \( F \). Note that by taking two independent samples \( x^{(1)} \) and \( x^{(2)} \) from \( D' \) and setting \( \hat{g}(r, x) = x \min(1, \exp(-\langle r, x \rangle)) \), the quantity \( 2 \text{Jac}(\hat{g}(r, x^{(1)}))\hat{g}(r, x^{(2)}) \) is an unbiased estimator for \( \nabla F(r) \). This completes our proof. \( \square \)

**Algorithm 2** Computing a Good Initialization Vector

1. **procedure** WarmStart((\( A, \alpha \), \( \epsilon, w, D \))
2. **Input:** Samples from an \( O(\gamma, O(1)) \)-isotropic log-concave distribution that satisfies the \((\alpha, A)\)-Tsybakov noise condition, and a unit vector \( w \) such that \( \theta(w, w^*) = \Theta(\epsilon) \).
3. **Output:** A vector \( v \) such that \( \langle v, (w^*)^\perp w \rangle \geq (\alpha \epsilon / A)^{(O(1/\alpha))} \).
4. \( s \leftarrow \Theta(\alpha \epsilon / \log(A \log(A) / (\alpha \epsilon))) \), \( \xi \leftarrow (\Theta(A / s))^{1/\alpha} \), \( s' = \Theta(\xi^3 s \epsilon) \)
5. \( N \leftarrow \text{poly}(d) \cdot (A / (\alpha \epsilon))^{O(1/\alpha)} \)
6. Let \( x_0 \) be a uniform random number on \([s, 2s]\).
7. Let \( D' \) denote \( D \) conditioned on \( \langle w, x \rangle \in [x_0, x_0 + s'] \) and projected onto \( w^\perp \).
8. \( \hat{D} \leftarrow \text{MAKEISOTROPIC}(D', 1 / \log(1 / \xi), N) \)
9. \( \bar{x} \leftarrow E_{x \sim \hat{D}}[x] \); \( \bar{X} \leftarrow E_{x \sim \hat{D}}[xx^\top] \)
10. Normalize all samples in \( \hat{D} \) with \( \bar{x} \) and \( \bar{X} \)
11. \( T_1' \leftarrow E_{(x, y) \sim \hat{D}}[yx] \) and \( T_2' \leftarrow E_{(x, y) \sim \hat{D}}[yx^\top - I] \)
12. Let \( V \) be the subspace spanned by \( T_1' \) and the eigenvectors of \( T_2' \) whose eigenvalues have absolute value at least \( \xi \).
13. **return** a random vector in \( V \).
Proposition 4.8 to obtain a vector with non-trivial correlation. For the general case, we show how to apply Proposition 4.8 to the distribution on \( (\mathbf{w}, \mathbf{x}) \) produced according to Lemma 4.14 with \( \mathbf{x} \) from \( \mathcal{D} \) and then projected onto \( \mathbf{w}^\perp \).

Proof of Theorem 4.1. Let \( \mathbf{w}^\perp \) be a uniform random number in \([s, 2s]\). Consider the conditional distribution on the random band \( B_{x_0} = \{ (\mathbf{w}, \mathbf{x}) \in [x_0, x_0 + s'] \} \) and projected onto \( \mathbf{w}^\perp \), i.e., \( \mathcal{D}^\perp \). We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. Using the condition \( \theta(\mathbf{w}, \mathbf{w}^*) = \Theta(\epsilon) \), we can assume that \( \mathbf{w}^* = \lambda_1 \mathbf{w} + \lambda_2 (\mathbf{w}^*)^\perp \), where \( |\lambda_1| = 1 - \Theta(\epsilon) \), and \( \lambda_2 = \Theta(\epsilon) \). If \( \langle \mathbf{w}, \mathbf{w}^* \rangle = 0 \), then we can directly apply Proposition 4.8 to obtain a vector with non-trivial correlation. For the general case, we show how we can construct a distribution that satisfies the conditions of Proposition 4.8.

Let \( s \) be a sufficiently small multiple of \( \alpha \epsilon / \log(A \log(A)/(\alpha \epsilon)) \), \( \xi = (\Theta(s/A))^{1/\alpha} \), and let \( s' = \xi^3 s \epsilon \). Finally, let \( x_0 \) be a uniform random number in \([s, 2s]\). Consider the conditional distribution on the random band \( B_{x_0} = \{ (\mathbf{w}, \mathbf{x}) \in [x_0, x_0 + s'] \} \) and projected onto \( \mathbf{w}^\perp \).

Set \( \mathbf{x}^\perp = \text{proj}_{\mathbf{w}^\perp} \mathbf{x} \), \( f^\perp(\mathbf{x}^\perp) = \text{sign} \left( \langle \mathbf{x}^\perp, (\mathbf{w}^*)^\perp \mathbf{w} \rangle \right) \), and \( \eta^\perp(\mathbf{x}^\perp) = \Pr_{(\mathbf{x}^\perp, y) \sim \mathcal{D}^\perp} [y \neq f^\perp(\mathbf{z}) \mid \mathbf{z} = \mathbf{x}^\perp] \). Using Lemma 4.11 we get that \( \mathcal{D}^\perp \) is \((O(1), O(1))\)-isotropic and with high probability it holds \( \Pr_{\mathbf{x}^\perp \sim \mathcal{D}^\perp} [\eta^\perp(\mathbf{x}^\perp) \leq 1/2 - \xi] \geq 2/3 \) and \( \Pr_{\mathbf{x}^\perp \sim \mathcal{D}^\perp} [\eta^\perp(\mathbf{x}^\perp) \geq 1/2] \leq \xi^3 \).

At this point, we have that \( \mathcal{D}^\perp \) is approximately isotropic, but may be relatively far from mean 0 (the mean can be at constant distance from the origin, whereas we need it to be roughly \( 1/\log(1/\xi) \)). To overcome this issue, we apply Lemma 4.14. We define \( \mathcal{D} \) to be the distribution of \( \mathbf{z} \) that is produced according to Lemma 4.14 with \( \gamma \) a small multiple of \( 1/\log(1/\xi) \), and consider the distribution on \( \mathbf{z} \) and \( y \). Notice that \( y \) is a noisy version of \( f^\perp(\mathbf{x}) \) (with noise rate \( \eta^\perp(\mathbf{x}^\perp) \)), because rejection sampling does not increase the noise rate. Moreover, the mean of \( \langle \mathbf{z}, (\mathbf{w}^*)^\perp \mathbf{w} \rangle \) is at most \( \gamma + O(s/\epsilon) \), which is a sufficiently small multiple of \( 1/\log(1/\xi) \). This means that we can apply Proposition 4.8 to the distribution on \( (\mathbf{z}, y) \), yielding our final result.

4.3 Proof of Theorem 4.1

Using Theorem 4.2, we can prove Theorem 4.1. The proof is similar to the proof of Theorem 3.3, but we additionally need to guess how far the current guess \( \mathbf{w} \) is from \( \mathbf{w}^* \).

Proof of Theorem 4.2. First, we guess a value \( \epsilon \) such that \( \| \mathbf{w} - \mathbf{w}^* \|_2 = \Theta(\epsilon) \), where \( \epsilon = \Theta(\theta) \). From Proposition 3.6 for \( \rho = O(\theta(\alpha \epsilon/A)^{O(1/\alpha)}) \), the distribution \( \mathcal{D}^\perp \) is \((2, \Omega(\rho), 1/\rho, O(1/\rho \log(1/\rho)))\)-well-behaved and satisfies the \((\alpha, O(\rho))\)-Tsybakov noise condition, where we used (from Fact 4.5)
that the values $L, R$ are absolute constants. Using Theorem 4.2, a random unit vector $v \in \mathbb{R}^d$ with constant probability $\delta_1$ satisfies $\langle v, (w^*)^{-w} \rangle \geq \frac{(\alpha e/A)}{O(1/\alpha)}$. We call this event $\mathcal{E}$. Conditioning on the event $\mathcal{E}$, from Proposition 3.11, using $\frac{\beta^4}{\sqrt{\theta}} (\frac{A}{\alpha^2})^{O(1/\alpha^2)} \log (1/\delta)$ samples, with probability $1 - \delta$, we get a $(v', R, t_0)$ such that

$$
E_{(x,y) \sim D_{R}} \mathbb{P}[-R \leq \langle v', x \rangle \leq -t_0] y \leq - \left( \theta \alpha / A \right)^{O(1/\alpha^2)} / \beta .
$$

Using Lemma 3.7, we get that

$$
E_{(x,y) \sim D} \mathbb{P}[T_w(x) \langle x, w \rangle y \leq - \left( \theta \alpha / A \right)^{O(1/\alpha^2)} / \beta .
$$

Conditioning on the event $\mathcal{E}^c$, where $\mathcal{E}^c$ is the complement of $\mathcal{E}$, Algorithm 1 either returns a certificate or returns nothing. Thus, by taking $k = O(\log (1/\delta))$ random vectors, we get that the probability that event $\mathcal{E}^c$ happens is at most $(1 - \delta_1)^k \leq e^{-\delta_1 k}$. Thus, by taking $O(\log 1/\delta)$ random vectors and running Algorithm 1 with confidence $\delta / \log (1/\delta)$, we get a certificate with probability $1 - 2\delta$. Moreover, the number of samples needed to construct the empirical distribution is $(\frac{A}{\alpha^2}) \log (1/\delta)$. Finally, to guess the value of $\epsilon$, it suffices to run the algorithm for the values $\theta, 2\theta, \ldots, 1$ which will increase the complexity by a $\log (1/\theta)$ factor. This completes the proof of Theorem 4.1.

5 Learning a Near-Optimal Halfspace via Online Convex Optimization

In this section we present a black-box approach that uses our certificate algorithms from the previous sections to learn halfspaces in the presence of Tsybakov noise. In more detail, we provide a generic result showing that one can apply a certificate oracle in a black-box manner combined with online gradient descent to learn the unknown halfspace. We note that an essentially identical approach, with slightly different formalism, was given in [DKTZ20b].

Using the aforementioned approach, we establish the two main algorithmic results of this paper.

**Theorem 5.1** (Learning Tsybakov Halfspaces under Well-Behaved Distributions). Let $\mathcal{D}$ be a $(3, L, R, U, \beta)$-well-behaved isotropic distribution on $\mathbb{R}^d \times \{\pm 1\}$ that satisfies the $(\alpha, A)$-Tsybakov noise condition with respect to an unknown halfspace $f(x) = \text{sign}((w^*, x))$. There exists an algorithm that draws $N = \beta^4 (\frac{d U A}{\alpha})^{O(1/\alpha^2)} \log (1/\delta)$ samples from $\mathcal{D}$, runs in $\text{poly}(N, d)$ time, and computes a vector $\hat{w}$ such that, with probability $1 - \delta$, we have $\text{err}_{0-1}^{\mathcal{D}}(h_{\hat{w}}, f) \leq \epsilon$.

For the important special case of log-concave distributions on examples, we give a more efficient learning algorithm.

**Theorem 5.2** (Learning Tsybakov Halfspaces under Log-concave Distributions). Let $\mathcal{D}$ be a distribution on $\mathbb{R}^d \times \{\pm 1\}$ that satisfies the $(\alpha, A)$-Tsybakov noise condition with respect to an unknown halfspace $f(x) = \text{sign}((w^*, x))$ and is such that $\mathcal{D}_x$ is isotropic log-concave. There exists an algorithm that draws $N = \text{poly}(d) \cdot (\frac{1}{\epsilon})^{O(1/\alpha^2)} \log (1/\delta)$ samples from $\mathcal{D}$, runs in $\text{poly}(N, d)$ time, and computes a vector $\hat{w}$ such that, with probability $1 - \delta$, we have $\text{err}_{0-1}^{\mathcal{D}}(h_{\hat{w}}, f) \leq \epsilon$.

To formally describe the approach of this section, we require the notion of a certificate oracle. A certificate oracle is an algorithm that, given a candidate weight vector $w$ and an accuracy
parameter $\rho > 0$, it returns a certifying function $T(x)$. Recall that a certifying function is a non-negative function that satisfies $\mathbb{E}_{(x,y) \sim D}[T(x)y(x,w)] \leq -\rho$ for some $\rho > 0$. We have already described how to efficiently implement such an oracle in Section 3.

**Definition 5.3** (Certificate Oracle). Let $D$ be a distribution on $\mathbb{R}^d \times \{\pm 1\}$ that satisfies the $(\alpha, A)$-Tsybakov noise condition with respect to an unknown halfspace $f(x) = \text{sign}(\langle w^*, x \rangle)$. For a decreasing function $\rho(\cdot) : \mathbb{R} \to \mathbb{R}_+$, we define $C(w, \theta, \delta)$ to be the following $\rho$-certificate oracle: For any unit vector $w$ and $\theta > 0$, if $\theta(w, w^*) \geq \theta$, then a call to $C(w, \theta, \delta)$, with probability at least $1 - \delta$, returns a function $T(x)$, with $\|T\|_{\infty} \leq 1$ such that

$$\mathbb{E}_{(x,y) \sim D}[T(x)y(x,w)] \leq -\rho(\theta),$$

and with probability at most $\delta$ returns “FAIL”.

**Remark 5.4.** We note that the above oracle provides a “one-sided” guarantee in the following sense. When the candidate vector $w$ satisfies $\theta(w, w^*) \geq \theta$, the oracle is required to return a certifying function $T$ with high probability. But it may also return such a function when $\theta(w, w^*) \leq \theta$. In other words, the oracle is not required to output “FAIL” with high probability when $w$ is nearly parallel to $w^*$. We show that an one-sided oracle of non-optimality suffices for our purposes.

**Remark 5.5.** By Fact 3.1, the optimal halfspace $w^*$ satisfies $\mathbb{E}_{(x,y) \sim D}[T(x)y(x,w^*)] \geq 0$ for any non-negative function $T$. Therefore, as $w$ approaches $w^*$, we have that

$$\lim_{\theta(w,w^*) \to 0} \inf_{T : \|T\|_{\infty} \leq 1} \mathbb{E}_{(x,y) \sim D}[T(x)y(x,w)] = 0,$$

where $\|T\|_{\infty}$ is the $\ell_\infty$ norm for functions, i.e., $\|T\|_{\infty} = \sup_{x \in \mathbb{R}^d} |T(x)|$. That is, $\lim_{\theta \to 0} \rho(\theta) = 0$ and it is natural that the non-negative function $\rho(\theta)$ is a decreasing function of the (lower bound on the) angle between $w$ and $w^*$. Intuitively, the closer $w$ is to $w^*$, the harder it is to find a certifying function $T$ that makes $\mathbb{E}_{(x,y) \sim D}[T(x)y(x,w)]$ sufficiently negative. Moreover, if our goal is to estimate the vector $w^*$ within angle $\epsilon$, we can always give the oracle this worst-case target angle, i.e., $\theta = \epsilon$. Finally, notice that when the distribution $D$ is isotropic, we have $\rho(\theta) \leq 1$, as follows from $\|T\|_{\infty} \leq 1$ and the Cauchy-Schwarz inequality.

Given a certificate oracle, the following result shows we can efficiently approximate the optimal halfspace using projected online gradient descent.

**Proposition 5.6** (Certificate-Based Optimization). Let $D$ be a $(3, L, R, \beta)$-well-behaved isotropic distribution on $\mathbb{R}^d \times \{\pm 1\}$ that satisfies the $(\alpha, A)$-Tsybakov noise condition with respect to an unknown halfspace $f(x) = \text{sign}(\langle w^*, x \rangle)$, and let $C$ be a $\rho$-certificate oracle. There exists an algorithm that makes at most $T = \frac{1}{\rho(\epsilon)} \frac{1}{\alpha} \left( \frac{d}{L^d} \right)^{O(1/\alpha)}$ calls to $C(\cdot)$, draws $N = d \frac{T^2}{\rho^2(\epsilon)} \log \left( \frac{dT}{\delta \rho(\epsilon)} \right)$ samples from $D$, runs in time $\text{poly}(T, N, d)$, and computes a weight vector $\hat{w}$ such that with probability $1 - \delta$ we have that $\theta(\hat{w}, w^*) \leq \epsilon$.

The algorithm establishing Proposition 5.6 is given in pseudocode in Algorithm 4. In the remaining part of this section, we provide a proof sketch of Proposition 5.6. The full argument is given in Appendix C.

**Proof Sketch.** The main idea of the algorithm is to provide a sequence of adaptively chosen convex loss functions to an Online Convex Optimization algorithm, for example Online Gradient Descent.
(OGD). In more detail, we construct these loss functions using our certificate oracle $C$. At round $t$, we call the certificate oracle to obtain a certifying function $T(x)$ and set

$$\ell_t(w) = -\left\langle E_{(x,y) \sim D} [(T(x) + \lambda)y x], w \right\rangle,$$

where $\lambda > 0$ acts similarly to a regularizer. The term $\lambda \langle E_{(x,y) \sim D} [y x], w \rangle$ prevents the trivial vector $w = 0$ from being a valid solution (in the sense of one that minimizes regret, see also the full proof in Appendix C).

The crucial property of the above sequence of loss functions is that they are positive and bounded away from 0 when $\rho(\epsilon)$, given the guarantee of our certificate oracle from Definition 5.3 for $\theta = \epsilon$ and assuming that the regularizer $\lambda$ is sufficiently small.

We then provide this convex loss function to the OGD algorithm that updates the guess according to the gradient of $\ell_t(w)$. Our analysis follows from the regret guarantee of OGD. Since we provide convex (and in particular linear) loss functions to OGD, we know the average regret will converge to 0 as $T \to \infty$ with a convergence rate roughly $O(1/\sqrt{T})$. This means that the oracle can only succeed in returning certifying functions for a bounded number of rounds, since every time the oracle succeeds, OGD suffers loss of at least $\rho(\epsilon)$. Therefore, after roughly $1/\rho(\epsilon)^2$ rounds the regret will be so small that for at least one round the certificate oracle must have failed. Our algorithm then stops and returns the halfspace of that iteration. Even though our certificate is “one-sided”, we know that the probability that it failed with $\theta(w, w^*)$ being larger than $\epsilon$ is very small, which implies that we have indeed found a vector $w$ very close to $w^*$.

\begin{algorithm}
\caption{Learning Halfspaces with Tsybakov Noise using a $\rho$-certificate oracle $C$}
1: procedure ALG($\epsilon, \delta, D, C$) \Comment{$\epsilon$: accuracy, $\delta$: confidence}
2: Input: $D$ is a $(3, L, R, \beta)$-well-behaved distribution that satisfies the $(\alpha, A)$-Tsybakov noise condition, and $C$ is a $\rho$-certificate oracle.
3: Output: A vector $\hat{w}$ such that $\text{err}_{D,x}^2(h_{\hat{w}}, f) \leq \epsilon$ with probability at least $1 - \delta$.
4: $w^{(0)} \leftarrow e_1$
5: $T \leftarrow \frac{1}{\rho(\epsilon)^2 \alpha} \left( \frac{A}{\pi T} \right)^{O(1/\alpha)}$
6: Draw $N = \tilde{O} \left( d \cdot \frac{T^{\frac{2}{2 + \beta}}}{\rho^2(\epsilon)} \log \left( \frac{1}{\delta} \right) \right)$ samples from $D$ to form the empirical distribution $\hat{D}$
7: for $t = 1, \ldots, T$ do
8: \hspace{1em} $\eta_t \leftarrow 1/(\sqrt{T} + \rho(\epsilon))$
9: if $w^{(t-1)} = 0$ then
10: \hspace{2em} $\hat{\ell}_t(w) \leftarrow \left\langle w, -E_{(x,y) \sim \hat{D}} \left[ \frac{\rho(\epsilon)}{2} y x \right] \right\rangle$
11: \hspace{2em} $w^{(t)} \leftarrow \Pi_B \left( w^{(t-1)} - \eta_t \nabla w \hat{\ell}_t \left( w^{(t-1)} \right) \right)$
12: else
13: \hspace{2em} $\text{ANS} \leftarrow C(w^{(t-1)}/\|w^{(t-1)}\|_2, \epsilon, \delta/T)$
14: if $\text{ANS} = \text{FAIL}$ then
15: \hspace{3em} return $w^{(t-1)}$
16: $T_{w^{(t)}}(x) \leftarrow \text{ANS}$
17: \hspace{2em} $\hat{\ell}_t(w) \leftarrow \left\langle w, -E_{(x,y) \sim \hat{D}} \left[ \left( T_{w^{(t)}}(x) + \frac{\rho(\epsilon)}{2} \right) y x \right] \right\rangle$
18: \hspace{2em} $w^{(t)} \leftarrow \Pi_B \left( w^{(t-1)} - \eta_t \nabla w \hat{\ell}_t \left( w^{(t-1)} \right) \right)$ \Comment{\(B = \{x \in \mathbb{R}^d : \|x\|_2 \leq 1\}$}
\end{algorithm}
Given Proposition 5.6, it is straightforward to prove our main results. Here we give the proof for the case of log-concave densities and provide a similar argument for well-behaved distributions in Appendix C.

**Proof of Theorem 5.2.** First, we require a $\rho$-certificate oracle for log-concave distributions. The algorithm of Theorem 4.1 returns a function $T_w$ such that $\mathbb{E}_{(x,y) \sim D}[T_w(x)y \langle w, x \rangle] \leq -\left(\frac{\theta}{A}\right)^{O(1/\alpha^2)}$.

From the definition of $T_w$ (i.e., Equation (3)), it is clear that $\|T_w\|_{\infty} \leq \frac{1}{\min_{x \in B}|\langle w, x \rangle|} \leq \left(\frac{\log A}{\alpha \theta}\right)^{O(1/\alpha)}$, where $B$ is the band from Equation (3). Note that the function $T_w/\|T_w\|_{\infty}$ satisfies the conditions of the $\rho$-certificate oracle. Thus, by scaling the output of the algorithm of Theorem 4.1, we obtain a $(\theta\alpha/A)^{O(1/\alpha^2)}$-certificate oracle. From Proposition 5.6 this gives us an algorithm that returns a vector $\hat{w}$ such that $\theta(\hat{w}, w^*) \leq \frac{\epsilon}{\log^2(1/\epsilon)}$ with probability $1 - \delta$. Using the fact that for log-concave distributions $\text{err}_{0,1}(h_{\hat{w}}, f) \leq O\left(\log^2(1/\epsilon)\theta(\hat{w}, w^*)\right) + \epsilon$ (Claim C.5) the result follows.

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A  Omitted Proofs from Section 3

A.1  Proof of Claim 3.13

Proof of Claim 3.13  To bound from below the expectation $I_{2,2}$, we use the fact that the distribution is $(2, L, R, \beta)$-well-behaved. For $I_{1,2}$, we have

$$I_{2,2} = \mathbb{E}_{x \sim D_x} \left[ \mathbb{I}_{B_2^{R/2}(x) \cap \mathcal{C}(x)} x_1 \right] = \int_{B_2^{R/2}} x_1 \gamma(x) dx$$

$$= \int_0^{R/\sqrt{2}} \int_{R/(2\sqrt{2})} x_1 \gamma(x_1, x_2) dx_1 x_2$$

$$\geq \frac{R}{2\sqrt{2}} \int_{R/2}^{R/\sqrt{2}} \int_{R/(2\sqrt{2})} \gamma(x_1, x_2) dx_1 x_2$$

$$\geq \frac{R}{2\sqrt{2}} C^A \left( \int_{R/(2\sqrt{2})} \gamma(x_1, x_2) dx_1 x_2 \right)^{1/\alpha}$$

$$\geq \frac{R}{4} C^A \left( \frac{R^2 L}{4} \right)^{1/\alpha},$$

where we used Lemma A.7 and we bound from below the integral by a smaller square region, i.e., $[R/2, R/\sqrt{2}] \times [R/(2\sqrt{2}), R/\sqrt{2}]$. For $I_{2,2}$, we have

$$I_{1,2} = \mathbb{E}_{x \sim D_x} \left[ \mathbb{I}_{B_2^{R/2}(x) \cap \mathcal{C}(x)} x_1 \right] = \int_{B_2^{R/2}} \gamma(x) dx$$

$$= \int_0^{R/\sqrt{2}} \int_{R/(2\sqrt{2})} \gamma(x_1, x_2) dx_1 x_2$$

$$\geq C^A \left( \int_{B_2^{R/2}} \gamma(x) dx \right)^{1/\alpha}$$

$$\geq C^A \left( \int_0^{R/\sqrt{2}} \int_{R/(2\sqrt{2})} \gamma(x_1, x_2) dx_1 x_2 \right)^{1/\alpha}$$

$$\geq C^A \left( \frac{R^2 L}{4} \right)^{1/\alpha},$$

where we used Lemma A.7. Thus,

$$I_{1,2} \geq C^A \left( \frac{R^2 L}{4} \right)^{1/\alpha} = (RL/A)^{O(1/\alpha)} \quad \text{and} \quad I_{2,2} \geq (RL/A)^{O(1/\alpha)} .$$

This completes the proof of Claim 3.13 \hfill \Box

A.2  Proof of Claim 3.14

Proof of Claim 3.14  Recall that $\xi(x_2) = x_2 / tan \theta + b / sin \theta$. We have that

$$I_{1}^{R/2} \leq \mathbb{E}_{x \sim D_x} \left[ \mathbb{I}_{B_1^{R/2}(x) \cap \mathcal{C}(x)} \right] - \Gamma/2 .$$

We can bound from below the first term as follows

$$\mathbb{E}_{x \sim D_x} \left[ \mathbb{I}_{B_1^{R/2}(x) \cap \mathcal{C}(x)} \right] \leq \int_{-\infty}^{R} \int_{-\infty}^{\xi(x_2)} \gamma(x_1, x_2) dx_1 dx_2 \leq \int_{-R}^{R} \int_{-\infty}^{\xi(-R)} \gamma(x_1, x_2) dx_1 dx_2 \leq Pr[x_2 > |\xi(-R)|] \leq \exp(1 - |\xi(-R)|/\beta) .$$
Note that $|\xi(-R)| = (R \cos \theta - b)/\sin \theta \geq 3b/\sin \theta$, thus using the assumption $\theta < b\Gamma/(4\beta)$, we obtain $\exp(1-|\xi(-R)|/\beta) \leq \Gamma/4$, and therefore $I_1^{R/2} \leq -\Gamma/4$, completing the proof of Claim 3.14.

\[ \square \]

### A.3 Proof of Lemma 3.18

We start with a useful fact about the sub-exponential random variables.

**Fact A.1** (see, e.g., Corollary of Proposition 2.7.1 in [Ver18]). Let $X$ be sub-exponential random variable with tail parameter $\beta$. For any function $f : \mathbb{R} \to \mathbb{R}$, the random variable $Xf(X) - \mathbb{E}[Xf(X)]$ is zero mean sub-exponential with tail parameter $O(\beta \sup |f|)$.

Using Fact A.1 we can bound from above the sample complexity needed to construct $\hat{D}$.

**Proof of Lemma 3.18** Let $\hat{g} = \frac{1}{N} \sum_{i=1}^N 1_{B_{R/2}}(x^{(i)}) y^{(i)} x^{(i)}$. For any $u \in \mathbb{R}^d$, we have that
\[
|\langle u, g \rangle| \leq \mathbb{E}_{x \sim D_x} |\langle u, x \rangle| = \int_0^\infty \Pr_{x \sim D_x} |\langle u, x \rangle| \geq t dt \leq \int_0^\infty \exp(1-t/\beta) dt = e\beta ,
\]
thus $\|g\|_2 \leq e\beta$. Next we prove that the random variable $X = 1_{B_{R/2}}(x)y - g$ is zero-mean with sub-exponential tails. First, we clearly have that $\mathbb{E}[X] = 0$. Using Fact A.1 it follows that $X$ is sub-exponential with tail parameter $\beta' = O(\beta)$. We will now use the following Bernstein-type inequality.

**Fact A.2.** Let $X_1, X_2, \ldots, X_N$ be independent zero-mean sub-exponential random variables with tail parameter $\beta \geq 1$. There exists an absolute constant $c > 0$ such that for every $\epsilon > 0$ we have
\[
\Pr \left[ \left| \sum_{i=1}^N X_i \right| \geq \epsilon N \right] \leq 2 \exp \left( -cN\epsilon^2/\beta^2 \right).
\]

Using Fact A.2 we have that for every $1 \leq j \leq d$ it holds
\[
\Pr \left[ |\hat{g}_j - g_j| \geq \epsilon/\sqrt{d} \right] \leq 2 \exp \left( -cN\epsilon^2/(d\beta^2) \right).
\]

Thus, taking $N = O\left((d\beta^2/\epsilon^2)\log(d/\delta)\right)$, we get that $\|\hat{g} - g\|_2 \leq \epsilon$ with probability $1 - \delta$. For the second statement, using the triangle inequality and Equation (13) the result follows.

\[ \square \]

### A.4 Proof of Lemma 3.19

The proof requires a couple of known probabilistic facts. The first one is the bounded-difference inequality.

**Fact A.3** (see, e.g., Theorem 2.2 of [DL01]). Let $X_1, \ldots, X_d \in \mathcal{X}$ be independent random variables and let $f : \mathcal{X}^d \to \mathbb{R}$. Let $c_1, \ldots, c_d$ satisfy
\[
\sup_{x_1, \ldots, x_d, x_i'} \left| f(x_1, \ldots, x_i, \ldots, x_d) - f(x_1, \ldots, x_i', \ldots, x_d) \right| \leq c_i
\]
for $i \in [d]$. Then we have that $\Pr \left[ f(X) - \mathbb{E}[f(X)] \geq t \right] \leq \exp \left( -2t^2/\sum_{i=1}^d c_i^2 \right)$.

We additionally require the symmetrization of the empirical distribution.
Fact A.4 (see, e.g., Exercise 8.3.24 of [Ver18]). Let $F$ be a class of measurable real-valued functions. Let $X_1, \ldots, X_N$ be $N$ i.i.d. samples from a distribution $D$. Then

$$
\mathbb{E} \left[ \sup_{f \in F} \left| \frac{1}{N} \sum_{i=1}^{N} f(X_i) - \mathbb{E}[f(X)] \right| \right] \leq 2 \mathbb{E} \left[ \sup_{f \in F} \left| \frac{1}{N} \sum_{i=1}^{N} \epsilon_i f(X_i) \right| \right],
$$

where the $\epsilon_i$'s are independent Rademacher random variables.

The last fact we need connects the symmetrization with the VC dimension.

Definition A.5 (VC dimension). A collection of sets $F$ is said to shatter a set $S$ if for all $S' \subseteq S$, there is an $F \in F$ so that $F \cap S = S'$. The VC dimension of $F$, denoted $\text{VC}(F)$, is the largest $n$ for which there exists an $S$ with $|S| = n$ such that $F$ shatters $S$.

We note that a collection of sets $F$ over a ground set is equivalent to a class of Boolean-valued functions on the same ground set. With this terminology, we have the following fact.

Fact A.6 (VC Inequality, see, e.g., [DL01] or Theorem 8.3.3 in [Ver18]). Let $F$ be a class of Boolean-valued functions with $\text{VC}(F) \geq 1$. Let $X_1, \ldots, X_N$ be $N$ i.i.d. samples from a distribution $D$. Then

$$
\mathbb{E} \left[ \sup_{f \in F} \left| \frac{1}{N} \sum_{i=1}^{N} \epsilon_i f(X_i) \right| \right] \leq C \sqrt{\text{VC}(F)/N},
$$

where $C > 0$ is an absolute constant and the $\epsilon_i$'s are independent Rademacher random variables.

We are ready to bound the sample complexity required to check if Algorithm 1 finds a certificate.

Proof of Lemma 3.19. The proof is a simple application of the VC inequality. In more detail, we first use the bounded-difference inequality and then, using the symmetrization, we can apply the VC inequality to obtain the desired result.

For $N = O(\log(1/\delta)/\epsilon^2)$, we apply Fact A.4 for the function

$$
f((X_1,Y_1), \ldots, (X_N,Y_N)) = \sup_{t \in \mathbb{R}_+} \left| \mathbb{E}_{(x,y) \sim D} [\mathbb{1}_B(x) y] - \frac{1}{N} \sum_{i=1}^{N} [\mathbb{1}_B(X_i) Y_i] \right|,
$$

noting that $c_i = 2/N$ for all $i \leq N$. Therefore, with probability at least $1 - \delta$, we have that

$$
\sup_{t \in \mathbb{R}_+} \left| \mathbb{E}_{(x,y) \sim D} [\mathbb{1}_B(x) y] - \frac{1}{N} \sum_{i=1}^{N} [\mathbb{1}_B(X_i) Y_i] \right| \leq \mathbb{E} \left[ \sup_{t \in \mathbb{R}_+} \left| \mathbb{E}_{(x,y) \sim D} [\mathbb{1}_B(x) y] - \frac{1}{N} \sum_{i=1}^{N} [\mathbb{1}_B(X_i) Y_i] \right| \right] + \epsilon.
$$

Then, by Fact A.4, we have that

$$
\mathbb{E} \left[ \sup_{t \in \mathbb{R}_+} \left| \mathbb{E}_{(x,y) \sim D} [\mathbb{1}_B(x) y] - \frac{1}{N} \sum_{i=1}^{N} [\mathbb{1}_B(X_i) Y_i] \right| \right] \leq 2 \mathbb{E} \left[ \sup_{t \in \mathbb{R}_+} \left| \frac{1}{N} \sum_{i=1}^{N} \epsilon_i Y_i \mathbb{1}_B(X_i) \right| \right] = 2 \mathbb{E} \left[ \sup_{t \in \mathbb{R}_+} \frac{1}{N} \sum_{i=1}^{N} \epsilon_i \mathbb{1}_B(X_i) \right],
$$

where the last inequality follows from the fact that $Y_i \epsilon_i$ and $\epsilon_i$ have the same distribution (because $\epsilon_i$ and $Y_i$ are independent). Finally, using the fact that the class of indicators of the form $\mathbb{1}\{x \leq t\}$ has VC dimension 1, Fact A.6 implies that

$$
\mathbb{E} \left[ \sup_{t \in \mathbb{R}_+} \frac{1}{N} \sum_{i=1}^{N} \epsilon_i \mathbb{1}_B(X_i) \right] = O(\sqrt{1/N}) = O(\epsilon).
$$

Putting everything together completes the proof. □
A.5 Useful Technical Lemma

We are going to use the following simple fact about Tsybakov noise that shows that large probability regions will also have large integral even if we weight the integral with the noise function $1 - 2\eta(x) > 0$. Notice that larger noise $\eta(x)$ makes $1 - 2\eta(x)$ closer to 0, and therefore tends to reduce the probability mass of the regions where $\eta(x)$ is large. A similar lemma can be found in [Tsy04].

**Lemma A.7.** Let $D$ be a distribution on $\mathbb{R}^d \times \{\pm 1\}$ that satisfies the $(\alpha, A)$-Tsybakov noise condition. Then for every measurable set $S \subseteq \mathbb{R}^d$ it holds $E_{x \sim D_x}[\mathbb{I}_S(x)(1-2\eta(x))] \geq C^A_\alpha (E_{x \sim D_x}[\mathbb{I}_S(x)])^{\frac{1}{\alpha}}$, where $C^A_\alpha = \alpha \left(\frac{1}{\alpha}^A\right)^{\frac{1}{\alpha}}$

See [DKTZ20b] for the simple proof.

B Omitted Proofs from Section 4

B.1 Proof of Lemma 4.7

**Proof of Lemma 4.7** For the first condition, the lemma follows from Lemma 3.18. For the second condition, let $X = E_{(x,y) \sim D}[y(xx^\top - I)]$ and $\tilde{X} = E_{(x,y) \sim D}[y(xx^\top - I)]$. We are going to bound the variance, so we can apply Chebyshev’s inequality. For $0 < i, j \leq d$, we have

$$\text{Var}_{(x,y) \sim D}[\tilde{X}_{ij}] = \frac{1}{N} \text{Var}_{(x,y) \sim D}[X_{ij}] \leq \frac{1}{N} \frac{E_{(x,y) \sim D}[X_{ij}^2]}{E_{(x,y) \sim D}[Y^2(x,x_j - 1)^2]}$$

$$\leq \frac{2}{N} \left(\frac{E_{x \sim D_x}[x_i^2x_j^2]}{1} + 1\right) \leq \frac{2}{N} \left(\frac{\sqrt{E_{x \sim D_x}[x_i]}E_{x \sim D_x}[x_j]}{1} + 1\right) = O(1/N),$$

where the last inequality follows from the fact that the marginals of a log-concave density have sub-exponential tails. Thus, from Chebyshev’s inequality, for $0 < i, j \leq d$, we have that

$$\text{Pr}_{(x,y) \sim D}[|\tilde{X}_{ij} - X_{ij}| \geq \epsilon/d] = O\left(\frac{d^2}{\epsilon^2N}\right).$$

Choosing $N = O(d^4/\epsilon^2)$, we have that $\|X - \tilde{X}\|_F \leq \epsilon$ with high constant probability. This completes the proof. □

B.2 Proof of Claim 4.12

**Proof of Claim 4.12** For notational convenience, let $D^\perp = D^\perp_{B_{x_0}}$. Fix any unit vector $u \in \mathbf{w}$. Without loss of generality, we may assume that $\mathbf{w} = e_1$ and $u = e_2$. Denote by $\gamma(x_1, x_2)$ the marginal density of $D$ on the first two coordinates. We have that

$$E_{x \sim D^\perp} [\|x^\top u\|] = \frac{1}{\text{Pr}_D[B_{x_0}]} \int |x_2| \mathbb{I}\{x_0 \leq x_1 \leq x_0 + s'\} \gamma(x_1, x_2)dx_1dx_2.$$

From Fact 4.5, we have that $\gamma(x_1, x_2) \leq (1/c) \exp(-|x_2|/c)$, for some absolute constant $c > 0$. Therefore,

$$\frac{1}{\text{Pr}_D[B_{x_0}]} \int_{-\infty}^\infty \int_{x_0}^\infty |x_2| \gamma(x_1, x_2)dx_1dx_2 \leq \frac{s'}{c\text{Pr}_D[B_{x_0}]} \int_{-\infty}^\infty |x_2|e^{-|x_2|/c}dx_2 = O(1),$$

43
where we used that $x_0, x_0 + s'$ are sufficiently small and it holds $\Pr_D[B_{x_0}] = \Theta(s')$, see Fact 4.15.

We next bound the covariance. Pick a unit vector $u \in w^\perp$. Without loss of generality, we may assume that $u = e_2$. Let $\theta = e_2^T E_{x \sim D} [x]$ be the projection of the mean of $D^\perp$ on the direction $e_2$. To bound the maximum and minimum eigenvalues of the covariance matrix of $D^\perp$, we need to bound from above and below the following expectation:

$$E_{x \sim D^\perp}[(x_2 - \theta)^2] = \frac{1}{\Pr_D[B_{x_0}]} \int_{-\infty}^{\infty} \int_{x_0}^{x_0+2s'} (x_2 - \theta)^2 \gamma(x_1, x_2) dx_1 dx_2.$$

We first bound it from below. Using again Fact 4.13, we know that, for the same absolute constant $c$ as above, it holds that $\gamma(x_1, x_2) \geq c$ for points with distance smaller than $c$ from the origin. Therefore,

$$\frac{1}{\Pr_D[B_{x_0}]} \int_{-\infty}^{\infty} \int_{x_0}^{x_0+2s'} (x_2 - \theta)^2 \gamma(x_1, x_2) dx_1 dx_2 \geq \frac{c}{\Pr_D[B_{x_0}]} \int_{-c/\sqrt{2}}^{c/\sqrt{2}} (x_2 - \theta)^2 dx_2 \int_{x_0}^{x_0+2s'} dx_1 = \Omega(1),$$

where we used again the fact that $\Pr_D[B_{x_0}] = \Theta(s')$ and also picked the worst case $\theta$ to minimize the above expression, i.e., $\theta = 0$. We next bound the covariance eigenvalues from above. Using again the fact that $\gamma(x_1, x_2) \leq c \exp(-c|x_2|)$ for some absolute constant $c > 0$, we compute

$$\frac{1}{\Pr_D[B_{x_0}]} \int_{-\infty}^{\infty} \int_{x_0}^{x_0+2s'} (x_2 - \theta)^2 \gamma(x_1, x_2) dx_1 dx_2 \leq \frac{1}{c \Pr_D[B_{x_0}]} \int_{-\infty}^{\infty} \int_{x_0}^{x_0+2s'} (x_2 - \theta)^2 e^{-|x_2|/c} dx_1 dx_2 = O(1),$$

where we used the fact that $\theta = O(1)$, as already shown above, and that $\Pr_D[B_{x_0}] = \Theta(s')$. This completes the proof.

### B.3 Proof of Claim 4.16

**Proof of Claim 4.16.** To prove that $F$ is $L$-smooth, we need to show that $\sup_{\|r\|_2 \leq R} \|\nabla^2 F(r)\|_2 \leq L$, for some $L > 0$. We have

$$G(r) := \nabla F(r) = -2 \mathbb{E}_{x \sim D_x} [xx^T \mathbb{1}\{|r, x| \geq 0\}] \mathbb{E}_{x \sim D_x} [x \min(1, e^{-\langle r, x \rangle})]$$

$$= -2 \mathbb{E}_{x \sim D_x} [xx^T g_1(x^T r)] \mathbb{E}_{x \sim D_x} [g_2(x^T r)],$$

where $g_1(t) = \mathbb{1}\{|t| \geq 0\} e^{-t}$ and $g_2(t) = \min(1, e^{-t})$. Using the product rule, we obtain that the derivative of $G(r)$ at $r$, $DG|r$, is the following linear function from $\mathbb{R}^d$ to $\mathbb{R}^d$:

$$DG|r h = -2 \mathbb{E}_{x \sim D_x} [xx^T g_1'(x^T r) x^T h] \mathbb{E}_{x \sim D_x} [g_2(x^T r)] - 2 \mathbb{E}_{x \sim D_x} [xx^T g_1(x^T r)] \mathbb{E}_{x \sim D_x} [g_2'(x^T r)x^T h],$$

where $g_1'(t) = \delta(t) e^{-t} - \mathbb{1}\{|t| \geq 0\} e^{-t}$ (here by $\delta$ we denote the Dirac delta function), and $g_2'(t) = -\mathbb{1}\{|t| \geq 0\} e^{-t}$. To show that $F$ is smooth, we need to bound the operator norm of $DG|r$, i.e.,

$$\sup_{h: \|h\|_2 = 1} \|DG|r h\|_2.$$

Using the triangle and Cauchy-Schwarz inequalities, we can bound the first term as follows:

$$\left\| \mathbb{E}_{x \sim D_x} [xx^T g_1'(x^T r) x^T h] \mathbb{E}_{x \sim D_x} [g_2(x^T r)] \right\|_2$$

$$\leq \left\| \mathbb{E}_{x \sim D_x} [xx^T g_1'(x^T r) x^T h] \right\|_2 \left\| \mathbb{E}_{x \sim D_x} [g_2(x^T r)] \right\|_2$$

$$\leq \left\| \mathbb{E}_{x \sim D_x} [xx^T x^T h \delta(x^T r) e^{-x^T r}] \right\|_2 + \left\| \mathbb{E}_{x \sim D_x} [xx^T x^T h] \right\|_2 \left\| \mathbb{E}_{x \sim D_x} [x] \right\|_2.$$
We will first handle the term $\|E_{x \sim D_x}[xx^T x^T \delta(x^T r)]\|_2$. To simplify notation, we may set without loss of generality $r = e_1$. We have

$$E_{x \sim D_x}[xx^T x^T \delta(x^T r)] = E_{x \sim D_x}[x'(x')^T (x')^T \delta(x^T r)]$$

where $D'_x$ is the distribution $D_x$ conditioned on $x_1 = 0$, and $\gamma(0)$ is the one-dimensional p.d.f. at point 0 (which is bounded by a universal constant for log-concave distributions). Note that $D'_x$ is still log-concave.

Since $D_x$ is $(O(1), O(1))$-isotropic, it holds

$$\|E_{x' \sim D'_x}[x'(x')^T (x')^T \delta(x^T r)]\|_2 \leq E_{x' \sim D'_x}[\|x'(x')^T (x')^T \delta(x^T r)\|_2] \leq E_{x' \sim D'_x}[\|x'\|_2^2] \leq \text{poly}(d),$$

where we used that $\|AB\|_2 \leq \|A\|_2 \|B\|_2$, and that $\|h\|_2 = 1$. Similarly, $\|E_{x \sim D_x}[xx^T x^T h]\|_2 \leq \text{poly}(d)$. Finally,

$$\|E_{x \sim D_x}[x]\|_2 \leq E_{x \sim D_x}[\|x\|_2] \leq \text{poly}(d).$$

Putting everything together, we get that $L = \text{poly}(d)$, which completes the proof. □

### C Omitted Proofs from Section 5

#### C.1 Proof of Proposition 5.6

We will require the following standard regret bound from online convex optimization.

**Lemma C.1** (see, e.g., Theorem 3.1 of [Haz16]). Let $V \subseteq \mathbb{R}^n$ be a non-empty closed convex set with diameter $K$. Let $\ell_1, \ldots, \ell_T$ be a sequence of $T$ convex functions $V \mapsto \mathbb{R}$ differentiable in open sets containing $V$, and let $G = \max_{t \in [T]} \|\nabla \ell_t\|_2$. Pick any $w^{(1)} \in V$ and set $\eta_t = \frac{K}{G \sqrt{T}}$ for $t \in [T]$. Then, for all $u \in V$, we have that $\sum_{t=1}^T (\ell_t(w^{(t)}) - \ell_t(u)) \leq \frac{3}{2} G K \sqrt{T}$.

For the set $B$, i.e., the unit ball with respect the $\|\cdot\|_2$, the diameter $K$ equals to 2. We will show that the optimal vector $w^*$ and our current candidate vector $w^{(t)}$ have a separation in the value of $\ell_t$. Since we do not have access to $\ell_t$ precisely, we need a function $\hat{\ell}_t$, which is close to $\ell_t$ with high probability. The following simple lemma gives us an efficient way to compute an approximation $\hat{\ell}_t$ of $\ell_t$.

**Lemma C.2** (Estimating the function $\ell_t$). Let $D$ be a $(3, L, R, \beta)$-well-behaved distribution and $T_w(x)$ be the non-negative function given by a $p$-certificate oracle. Then after drawing $O(d \beta^2/\epsilon^2 \log(d/\delta))$ samples from $D$, with probability at least $1 - \delta$, the empirical distribution $\hat{D}$ satisfies the following conditions:

- $\left| E_{(x, y) \sim \hat{D}}[(T_w(x) + \frac{\epsilon}{2}) y \langle u, x \rangle] - E_{(x, y) \sim \hat{D}}[(T_w(x) + \frac{\epsilon}{2}) y \langle u, x \rangle] \right| \leq \epsilon$, for any $u \in B$.

- $\left\| E_{(x, y) \sim \hat{D}}[(T_w(x) + \frac{\epsilon}{2}) y x] \right\|_2 \leq 1 + \frac{\epsilon}{2} + \epsilon$.

**Proof.** The proof of this lemma is similar to the proof of Lemma 3.18. Let $\hat{g} = E_{(x, y) \sim \hat{D}}[(T_w(x) + \frac{\epsilon}{2}) y x]$ and $g = E_{(x, y) \sim \hat{D}}[(T_w(x) + \frac{\epsilon}{2}) y x]$. For any unit vector $u$, we have

$$| \langle u, g \rangle | \leq E_{x \sim D_x}[\|T_w(x)\| \langle u, x \rangle] + \frac{\epsilon}{2} E_{x \sim D_x}[\|u, x\|] \leq 1 + \frac{\epsilon}{2},$$

for all $u \in B$.
where we used that \(|T(x)| \leq 1\) and that the distribution \(D_x\) is in isotropic position. Moreover, from Fact A.1, the random variable \(X = (T_w(x) + \frac{\nu}{2})y - g\) is sub-exponential with tail bound \(\beta' = O(\beta)\). Thus, the rest of proof follows as in Lemma 3.18. 

The last item we need to proceed with our main proof is to establish that when the oracle \(C\) in Step 13 of Algorithm 4 returns a function \(T_{w(t)}\), then there exists a function \(\ell_t\) for which our current candidate vector \(w(t)\) and the optimal vector \(w^*\) are not close.

**Lemma C.3** (Error of \(\ell_t\)). Let \(w(t) \in B\) and \(w^*\) be the optimal weight vector. For \(g_t(x) = -(T_{w(t)}(x) + \frac{\nu}{2})\) and \(\ell_t(w) = E_{(x,y) \sim D}[\langle g_t(x) y, w \rangle]\), where \(T_{w(t)}(x)\) is the function given by a \(p\)-certificate oracle, we have that

\[
\ell_t(w^*) \leq -\rho \alpha \left( \frac{RL}{A} \right)^{O(1/\alpha)} \quad \text{and} \quad \ell_t(w(t)) \geq \|w(t)\|_2 \frac{\rho}{2}
\]

**Proof.** Without loss of generality, let \(w^* = e_1\). From Fact A.1 and the definition of \(\eta(x)\), we have that for every \(t \in [T]\), it holds \(\ell_t(w^*) \leq -\lambda E_{x \sim D_x}[\|w^*\|_2 (1 - 2\eta(x))]\). To bound from above this expectation, we use the \((3, L, R, \beta)\)-bound properties. We have that

\[
E_{x \sim D_x}[\langle w^*, x \rangle (1 - 2\eta(x))] \geq \frac{R}{4} C^A \left( \frac{R^3 L}{2} \right)^{1/\alpha} \gamma,
\]

where in the last inequality we used Lemma A.7. Therefore, \(\ell_t(w^*) \leq -\frac{\rho}{2} \frac{R}{4} C^A \left( \frac{R^3 L}{2} \right)^{1/\alpha} \). Then we bound from below \(\ell_t(w(t))\) as follows

\[
\ell_t(w(t)) = -E_{(x,y) \sim D}[\langle T_{w(t)}(x) + \lambda \rangle \langle w(t), x \rangle y] \geq \|w(t)\|_2 \|\rho - E_{x \sim D_x}[\frac{\rho}{2} \langle w(t), x \rangle y]\|
\]

\[
\geq \|w(t)\|_2 \|\rho - \frac{\rho}{2} \sqrt{E_{x \sim D_x}[\langle w(t), x \rangle^2]} \| \geq \|w(t)\|_2 \frac{\rho}{2},
\]

where we used the Cauchy-Schwarz inequality and the fact that \(x\) is in isotropic position.

We are ready to prove Proposition 5.6.

**Proof of Proposition 5.6** Let \(G = \alpha \left( \frac{RL}{A} \right)^{O(1/\alpha)}\). Assume, in order to reach a contradiction, that for all steps \(t \in [T]\) it holds that \(\theta(w(t), w^*) \geq \epsilon\). For each step \(t\), let \(T_{w(t)}(x)\) be the non-negative function output by the oracle \(C(w(t), \epsilon, \delta/T)\). Note that

\[
E_{(x,y) \sim D}[T_{w(t)}(x) y \langle w(t), x \rangle] \leq -\|w(t)\|_2 \frac{\rho}{2}.
\]

Let \(\hat{\ell}_t(w)\) be the empirical estimator of \(\ell_t(w) = E[\hat{\ell}_t(w)] = -E_{(x,y) \sim D}[\langle (T_{w(t)}(x) + \frac{\nu}{2}) y, w \rangle]\). Using Lemma C.2 for \(N = O \left( \frac{d^2}{\rho^2 G^2} \log \left( \frac{T}{\delta} \right) \right)\) samples, we have that \(E \left[ |\hat{\ell}_t(w(t)) - \ell_t(w(t))| \right] \leq \frac{\|w(t)\|_2}{\sqrt{T}} \frac{\rho}{2}\) and \(Pr \left[ |\hat{\ell}_t(w^*) - \ell_t(w^*)| \right] \leq \frac{\|w(t)\|_2}{\sqrt{T}} \frac{\rho}{2}\).

From Lemma C.3, for every step \(t\), we have that \(\ell_t(w(t)) \geq \frac{1}{2} \|w(t)\|_2 \rho \geq 0\) and \(\ell_t(w^*) \leq -\rho G\), thus, with probability at least \(1 - \frac{\delta}{T}\), \(\hat{\ell}_t(w^*) \geq -\frac{1}{2} G\rho\) and \(\hat{\ell}_t(w^*) \leq -\frac{3}{4} G\rho \). Using Lemma C.1 we get

\[
\frac{1}{T} \sum_{t=1}^{T} \left( \hat{\ell}_t(w(t)) - \hat{\ell}_t(w^*) \right) \leq 1 + \frac{\|w(t)\|_2}{\sqrt{T}} \frac{\rho}{2}.
\]

46
By the union bound, it follows that with probability at least $1 - \delta$, we have that

\[
\frac{1}{2} G_\rho \leq \frac{1}{T} \sum_{t=1}^{T} \left( \hat{\ell}_t \left( \mathbf{w}^{(t)} \right) - \hat{\ell}_t (\mathbf{w}^*) \right) \leq \frac{4}{\sqrt{T}},
\]

which leads to a contradiction for $T = \frac{16}{(\rho \delta)^2}$.

Thus, either there exists $t \in [T]$ such that $\theta \left( \mathbf{w}^{(t)}, \mathbf{w}^* \right) < \epsilon$, which the algorithm returns in Step 15, or the oracle $\mathcal{C}$ did not provide a correct certificate, which happens with probability at most $\delta$. Moreover, the algorithm calls the certificate $T$ times and the number of samples needed to construct the empirical distribution $\hat{D}$ is

\[
O(T N) = \frac{d \beta^2}{\rho^2} \log \left( \frac{1}{\rho \delta} \right) \left( \frac{A}{RL} \right)^{O(1/\alpha)}.
\]

This completes the proof. \qed

Using Proposition \ref{prop:cert_oracle} and our certificate algorithms, we obtain the following parameter estimation result for halfspaces with Tsybakov noise.

**Theorem C.4 (Parameter Estimation of Tsybakov Halfspaces Under Well-Behaved Distributions).** Let $\mathcal{D}$ be a $(3, L, R, \beta)$-well-behaved isotropic distribution on $\mathbb{R}^d \times \{\pm 1\}$ that satisfies the $(\alpha, A)$-Tsybakov noise condition with respect to an unknown halfspace $f(x) = \text{sign}(\langle \mathbf{w}^*, x \rangle)$. There exists an algorithm that draws $N = \beta^4 \left( \frac{dA}{RL} \right)^{O(1/\alpha)} \log \left( \frac{1}{\delta} \right)$ samples from $\mathcal{D}$, runs in $\text{poly}(N, d)$ time, and computes a vector $\hat{\mathbf{w}}$ such that with probability $1 - \delta$ we have $\theta(\hat{\mathbf{w}}, \mathbf{w}^*) \leq \epsilon$.

We note here that Theorem C.4 does not require the “U bounded” condition of the underlying distribution on examples that is required in our Theorem 5.1. Recall that this condition corresponds to an anti-concentration property of the data distribution. With this additional property, Theorem \ref{thm:parameter_estimation} follows easily from Theorem C.4 since it allows us to translate the small angle guarantee of Theorem C.4 to the zero-one loss.

**Proof of Theorem C.4.** We start by noting how to obtain a $\rho$-certificate oracle for $(3, L, R, \beta)$-well-behaved distributions. The algorithm of Theorem \ref{thm:oracle} returns a function $T_w$ such that $E_{(x,y) \sim \mathcal{D}} \left[ T_w(x) \langle y, \mathbf{w}, x \rangle \right] \leq -\frac{1}{\beta} \left( \theta LR / (\rho dA) \right)^{O(1/\alpha)}$. By definition, the function $T_w$ (i.e., Equation \ref{eq:certificate}) is bounded, namely $\|T_w\|_\infty \leq \frac{1}{\min_{x \in B} \langle \mathbf{w}, x \rangle} \leq O \left( \frac{1}{\gamma} \right)$, where $B$ is the band from Equation \ref{eq:certificate}. Therefore, the function $T_w / \|T_w\|_\infty$ satisfies the conditions of a $\rho$-certificate oracle. Thus, by scaling the output of the algorithm of Theorem \ref{thm:parameter_estimation}, we obtain a $\frac{1}{\gamma} \left( \theta LR / (dA) \right)^{O(1/\alpha)}$-certificate oracle. From Proposition \ref{prop:cert_oracle} this gives us an algorithm that returns a vector $\hat{\mathbf{w}}$ such that $\theta(\hat{\mathbf{w}}, \mathbf{w}^*) \leq \epsilon$ with probability $1 - \delta$. \qed

To prove Theorem \ref{thm:parameter_estimation} we need the following claim for $(3, L, R, U, \beta)$-well-behaved distributions.

**Claim C.5 (see, e.g., Claim 2.1 of DKTZ20a).** Let $\mathcal{D}_x$ be an $(3, L, R, U, \beta)$-well-behaved distribution on $\mathbb{R}^d$. Then, for any $0 < \epsilon \leq 1$, we have that $\text{err}_{0-1}(h_u, h_v) \leq U \beta^2 \log^2 (2/\epsilon) \cdot \theta(v, u) + \epsilon$.

**Proof of Theorem 5.1.** Running Algorithm 4 for $\epsilon' = \frac{\epsilon}{2U \beta^2 \log^2 (2/\epsilon)}$, by Theorem C.4, Algorithm 4 outputs a $\hat{\mathbf{w}}$ such that $\theta(\hat{\mathbf{w}}, \mathbf{w}^*) \leq \frac{\epsilon'}{2U \beta^2 \log^2 (1/\epsilon)}$, then from Claim C.5 we have $\text{err}_{0-1}(h_{\hat{\mathbf{w}}}, f) \leq \epsilon$. \qed