FAILURE OF 0-1 LAW FOR SPARSE RANDOM GRAPH
IN STRONG LOGICS

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Dedicated to Yuri Gurevich on the Occasion of his 75th Birthday

Abstract. Let \( \alpha \in (0, 1) \) be irrational and \( G_n = G_{n, 1/n^\alpha} \) be the random graph on \([n]\) with edge probability \( 1/n^\alpha \); we know that it satisfies the 0-1 law for first order logic. We deal with the failure of the 0-1 law for stronger logics: \( \mathcal{L}_{\infty, k} \), \( k \) large enough and the LFP, least fix point logic.
§ 0(A). The Question.

Let $G_{n,p}$ be the random graph with set of nodes $[n] = \{1, \ldots, n\}$, each edge of probability $p \in [0,1]$ chosen independently, see $\exists_2$ below. On 0-1 laws (and random graphs) see the book of Spencer [Spe01] or Alon-Spencer [AS08], in particular on the behaviour of the random graph $G_{n,1/n^\alpha}$ for $\alpha \in (0,1)$ irrational. On finite model theory see Flum-Ebbinghaus [EF06], e.g. on the logic $\mathbb{L}_{\infty, \kappa}$ (see §1) and on LFP (least fixed point$^1$) logic. A characteristic example of what can be expressed by it is “in the graph $G$ there is a path from the node $x$ to node $y$”; this is closed to what we use. We know that $G_{n,p}$, i.e. $p$ constant satisfies the 0-1 law for first order logic (proved independently by Fagin [Fag76] and Glebskii-et-al [GKLT69]). This holds also for many stronger logics like $\mathbb{L}_{\infty, \kappa}$ and LFP logic. If $\alpha \in (0,1)$ is irrational, the 0-1 law holds for $G_{n,(1/n^\alpha)}$ and first order logic, see e.g. [AS08].

The question we address is whether this holds also for stronger logics as above. Though our main aim is to address the problem for the case of graphs, the proof seems more transparent when we have two random graph relations (we make them directed graphs just for extra transparency). So here we shall deal with two cases A and B. In Case A, the usual graph, we have to show that there are (just first order) formulas $\varphi_\ell(x,y)$ for $\ell = 1,2$ with some special properties (actually also $\varphi_0$), see Claim 1.2. For Case B, those formulas are $R_\ell(x,y)$, $\ell = 1,2$, the two directed graph relations. Note that (for Case B), the satisfaction of the cases of the $R_\ell$ are decided directly by the drawing and so are independent, whereas for Case A there are (small) dependencies for different pairs, so the probability estimates are more complicated.

In the case of constant probability $p \in (0,1)$, the 0-1 law is strong: it is obtained by proving elimination of quantifier and it works also for stronger logics: $\mathbb{L}_{\infty, \kappa}$ and LFP logic. If $\alpha \in (0,1)$ is irrational, the 0-1 law holds for $G_{n,(1/n^\alpha)}$ and first order logic, see e.g. [AS08].

The question we address is whether this holds also for stronger logics as above. Though our main aim is to address the problem for the case of graphs, the proof seems more transparent when we have two random graph relations (we make them directed graphs just for extra transparency). So here we shall deal with two cases A and B. In Case A, the usual graph, we have to show that there are (just first order) formulas $\varphi_\ell(x,y)$ for $\ell = 1,2$ with some special properties (actually also $\varphi_0$), see Claim 1.2. For Case B, those formulas are $R_\ell(x,y)$, $\ell = 1,2$, the two directed graph relations. Note that (for Case B), the satisfaction of the cases of the $R_\ell$ are decided directly by the drawing and so are independent, whereas for Case A there are (small) dependencies for different pairs, so the probability estimates are more complicated.

In the case of constant probability $p \in (0,1)$, the 0-1 law is strong: it is obtained by proving elimination of quantifier and it works also for stronger logics: $\mathbb{L}_{\infty, \kappa}$ and LFP logic. Another worthwhile case is:

$\exists_1$ $G_{n,1/n^\alpha}$ where $\alpha \in (0,1)$; so $p_n = 1/n^\alpha$.

Again the edges are drawn independently but the probability depends on $n$.

The 0-1 law holds if $\alpha$ is irrational, but we have elimination of quantifiers only up to (Boolean combinations of) existential formulas. Do we have 0-1 law also for those stronger logics? We shall show that by proving that for some so called scheme of interpretation $\varphi$, for random enough $G_n, \varphi$ interpret number theory up to $m_n$, where $m_n$ is not too small, e.g. $m_n \geq \log_2 \log_3(n)$.

A somewhat related problem asks whether for some logic the 0-1 law holds for $G_{n,p}$ (e.g. $p = 1/2$) but does not have the elimination of quantifiers, see [Sh:1077].

We now try to informally describe the proof, naturally concentrating on case B. Fix reals $\alpha_1 < \alpha_2$ from $(0, 1/2)$ for transparency, so $\alpha = (\alpha_1, \alpha_2)$ letting $\alpha(\ell) = \alpha_\ell$;

$\exists_2$ let the random digraph $G_{n,\alpha} = ([n], R_1, R_2) = ([n], R_1^{G_{n,\alpha}}, R_2^{G_{n,\alpha}})$ with $R_1, R_2$ irreflexive 2-place relations drawn as follows:

$^1$There are some variants, but those are immaterial for our perspective
(a) for each $a \neq b$, we draw a truth value for $R_2(a,b)$ with probability $\frac{1}{n^2}$ for yes

(b) for each $a \neq b$, we draw a truth value for $R_1(a,b)$ with probability $\frac{1}{n^{\alpha_1}}$ for yes

(c) those drawings are independent.

Now for random enough digraph $G = G_n = G_{n,0} = ([n], R_1, R_2)$ and node $a \in G$; we try to define the set $S_k = S_{G,a,k}$ of nodes of $G$ not from $\cup \{ S_m : m < k \}$ by induction on $k$ as follows:

For $k = 0$ let $S_k = \{ a \}$. Assume $S_0, \ldots, S_k$ has been chosen, and we shall choose $S_{k+1}$.

• For $\ell = 1, 2$ we ask: is there an $R_{\ell}$-edge $(a, b)$ with $a \in S_k$ and $b$ not from $\cup \{ S_m : m \leq k \}$?

If the answer is no for both $\ell = 1, 2$ we stop and let $\text{height}(a, G) = k$. If the answer is yes for $\ell = 1$, we let $S_{k+1}$ be the set of $b$ such that for some $a$ the pair $(a, b)$ is as above for $\ell = 1$. If the answer is no for $\ell = 1$ but yes for $\ell = 2$ we define $S_{k+1}$ similarly using $\ell = 2$.

Let the height of $G$ be $\max \{ \text{height}(a, G) : a \in G \}$.

Now we can prove that for every random enough $G_n$, for $a \in G_n$ or easier - for most $a \in G_n$, for every not too large $k$ we have:

$$S_{G,a,k} \text{ is on the one hand not empty and on the other hand with } \leq n^{2\alpha_2} \text{ members.}$$

This is proved by drawing the edges not all at once but in $k$ stages. In stage $m \leq k$ we already can compute $S_{G_{n,a,0}, \ldots, S_{G_{n,a,m}}}$ and we have already drawn all the $R_1$-edges and $R_2$-edges having first node in $S_{G_{n,a,0}} \cup \cdots \cup S_{G_{n,a,m-1}}$: that is for every such pair $(a, b)$ we draw the truth values of $R_1(a, b), R_2(a, b)$. For $m = 0$ this is clear. So arriving to $m$ we can draw the edges having the first node in $S_m$ and not dealt with earlier, and hence can compute $S_{m+1}$.

The point is that in the question $\exists$ above, if the answer is yes for $\ell = 1$ then the number of nodes in $S_{m+1}$ will be small, almost surely smaller than in $S_m$ because its expected value is $|S_m| \cdot |[n] - \cup_{\ell \leq m} S_{\ell}| \cdot \frac{1}{n^{\alpha_1}} \leq n^{1+2\alpha_2-1-\alpha_1} = n^{2\alpha_2-\alpha_1}$ and the drawings are independent so except for an event of very small probability this is what will occur. Further, if for $\ell = 1$ the answer is no but for $\ell = 2$ the answer is yes then almost surely $S_m$ is smaller than a number near $n^{\alpha_1}$ but it is known that the $R_2$-valency of any node of $G_n$ is near to $n^{\alpha_2}$. Of course, the “almost surely” is such that the probability that at least one undesirable event mentioned above occurs is negligible.

So the desired inequality holds.

By a similar argument, if we stop at $k$ then there is no $R_2$-edge from $S_k$ into $[n] \setminus (S_0 \cup \ldots S_k)$ so the expected value is $\geq |S_k| \cdot (n - \sum_{\ell \leq k} (S_k)) \cdot \frac{1}{n^{1-\alpha_2}}$ hence in $S_0 \cup \cdots \cup S_k$ there are many nodes, e.g. at least near $n/2$ by a crude argument. As each $S_m$ is not too large necessarily the height of $G_n$ is large.

The next step is to express in our logic the relation $\{(a_1, b_1, a_2, b_2) :$ for some $k_1, k_2$ we have $b_1 \in S_{G_n, a_1, k_1}, b_2 \in S_{G_n, a_2, k_2} \text{ and } k_1 \leq k_2 \}$. 

By this we can interpret a linear order with height($G_n$) members. Again using the relevant logic this suffice to interpret number theory up to this height. Working more we can define a linear order with $n$ elements, so can essentially find a formula “saying” $n$ is even (or odd).

For random graphs we have to work harder: instead of having two relations we have two formulas; one of the complications is that their satisfaction for the relevant pairs are not fully independent.

In [Sh:1096] we shall deal with the strong failure of the 0-1 for Case A, i.e. $G_{n,p}$, (e.g. can “express” $n$ is even) and also intend to deal with the $\alpha$ rational case. The irrationality can be replaced by discarding few exceptions.

We thank the referee for helping to improve the presentation.

§ 0(B). **History.**

The history is non-trivial having non-trivial opaque points. I have a clear memory of the events but vague on the exact statements and more so on the proof (and a concise entry in my (private F-list, [Sh:F159])) that in January 1996, in a Conference in DIMACS, Monica McArthur gave a lecture claiming that the graph $G_{n,\alpha}$ satisfies the 0-1 law not only for first order logic (by Shelah-Spencer [ShSp:304]) but also for a stronger logic. Joel Spencer said this can be contradicted in a way he outlined. I thought on this and saw further things (explain to her saying more and) wrote them in a letter to Monica and Joel. I understood that it was agreed that Monica would write a paper with us saying more but eventually she left academia.

As the referee found out, MacArthur’s claim in [?] (DIMACS): failure of the law in $L_{\infty,\omega}$, but refers the proof to a paper in preparation with Spencer that never appeared. She claims also that there is 0-1 law for $L_{\infty,k}$ if $k = \lfloor 1/\alpha \rfloor$, referring again to the paper in preparation. The later claim is not contradicted by the results of this paper. Lynch [Lyn97], refers also to a joint paper with McArthur and Spencer that never appeared proving that for the TC (= transitive closure) logic satisfies the 0-1 law.

Having sent Joel (in 2011) an earlier version of this paper, his recollection of talking to Monica was that “we hadn’t really gotten a handle on the situation”.

Discussing with Simi Haber, (December 2011) this question arised again, trying to recollect it was not clear to me what was the logic (inductive logic? $L_{\infty,k}$?) Looking at it again, I saw a proof for the logic $L_{\infty,k}$. No trace of the letter or the notes mentioned above were found. The only tangible evidence, is in an entry [Sh:F159] from my F-list. Joel declined a suggestion that Haber, he and I will deal with it, and eventually also Haber left.

The notes on §1 are from January 2012; for §2 from Sept. 4, 2012; revised in Nov/Dec. 2014 and expanded March 2015, June, 2015.

The intention was that it would appear in the Yurifest, commemorating Yuri Gurevich’s 75th birthday, but it was not in a final version in time, so only a short version (with the abstract and §(0A)) appear in the Yurifest volume, [Sh:1061].

§ 0(C). **Preliminaries.**
Notation 0.1. 1) \( n \in \mathbb{N} \setminus \{0\} \) will be used for \( G_n \in K_n \) random enough”.
2) \( G, H \) denote graphs and \( M, N \) denote more general structures = models.
3) \( a, b, c, d, e \) denote nodes of graphs or elements of structures.
4) \( m, k, \ell \) denote natural numbers.
5) \( \tau \) denotes a vocabulary, \( M \) a model with vocabulary \( \tau = \tau_M \) (see 0.1(9),(10) below).
6) \( \mathcal{L} \) denotes a logic, \( \mathcal{L} \) is first order logic, so \( \mathcal{L}(\tau) \) is first order language (= set of formulas) for the vocabulary \( \tau \). \( \mathcal{L}(\tau) \) is the language for the logic \( \mathcal{L} \) and the vocabulary \( \tau \).
7) \( \mathcal{L}_{\text{LFP}} \) is the least fix point logic, short hand \( \text{LFP} \).
8) \( \mathcal{L}_\tau \) denotes the vocabulary of graphs, but we may write \( \mathcal{L}(\text{graph}) \) or \( \mathcal{L}(\text{graph}) \) instead of \( \mathcal{L}(\tau_\text{gr}) \), \( \mathcal{L}(\tau_\text{gr}) \). So \( \tau_\text{gr} \) consists of one two-place predicate \( R \), (below always interpreted as a symmetric irreflexive relation).
9) Let \( \tau_\text{gr} \) denote the vocabulary of graphs, but we may write \( \mathcal{L}(\text{graph}) \) or \( \mathcal{L}(\text{graph}) \) instead of \( \mathcal{L}(\tau_\text{gr}) \), \( \mathcal{L}(\tau_\text{gr}) \). So \( \tau_\text{gr} \) consists of one two-place predicate \( R \), (below always interpreted as a symmetric irreflexive relation).
10) Let \( \tau_{\text{dg}} \) denote the vocabularies of bound directed graphs, so it consists of two two-place predicates, below always interpreted as irreflexive relations. Let \( \tau_{\text{N}} \) be the vocabulary of number theory, see 0.2(1).
11) We define the function \( \log_* \) from \( \mathbb{R}_{\geq 0} \) to \( \mathbb{N} \) by:
\[
\log_+(x) = \begin{cases} 0 & \text{if } x < 2 \\ \log_+(\log_2(x)) + 1 & \text{if } x \geq 2 \end{cases}
\]
12) \( |u| \) is the cardinality = the number of elements of a set \( u \).

Explanation 0.2. 1) Above recall that the vocabulary of the model = the structure \( \mathbb{N} \), number theory, is the set of symbols \( \{0, 1, +, \times, <\} \) where \( 0, 1 \) are individual constants (interpreted in \( \mathbb{N} \) as the corresponding elements) and \( +, \times \) are two-place function symbols interpreted as \( +^\mathbb{N}, \times^\mathbb{N} \) the two-place functions of addition and multiplication, and \( < \) is a two-place predicate (= relation symbol) interpreted as \( <^\mathbb{N} \), the usual order on \( \mathbb{N} \).
2) In general
\( (A) \) a vocabulary is a set of predicates, individual constants and function symbols each with a given arity = number of places; individual constants (like 0,1 above) are considered as 0-place function symbols
(\( B \) a \( \tau \)-model or a \( \tau \)-structure \( M \) consists of:
\( (a) \) its universe, \( |M| \), a non-empty set of elements so \( |M| \) is their number
(\( b \) if \( P \in \tau \) is an \( n \)-place predicate, \( P^M \) is a set of \( n \)-tuples of members of \( M \)
(\( c \) if \( F \in T \) is an \( n \)-place function symbol then \( F^M \) is an \( n \)-place function from \( |M| \) to \( |M| \).
Definition 0.3. Let $\tau$ be a finite vocabulary, for simplicity with predicates only or we just consider a function as a relation. Here we use $\tau_{gr}, \tau_{ds}$ only except when we interpret.

1) We say $\bar{\varphi}$ is in a $(\tau_{*}, \tau)$-scheme of interpretation when:
   (a) $\bar{\varphi} = \langle \varphi_{R}(\bar{x}, n_{\tau}(R)) : R \in \tau_{*} \cup \{=\}\rangle$ where $n_{\tau}(R)$ is the arity (= number of places) of $R$
   (b) $\varphi_{R} \in \mathbb{L}(\tau)$
   (c) $\varphi_{=}(x_{0}, x_{1})$ is always an equivalence relation on $\{y : \varphi(y, y)\}$; if $\varphi_{=}$ is $x_{0} = x_{1}$ then we may omit it.

2) For a $\tau$-model $M$ (here a graph or diagram) and $\bar{\varphi}$ as above, let $N = N_{M, \bar{\varphi}}$ be the following structure:
   (a) $|N|$ the set of elements of $N$, is $\{a/\varphi_{=}(M) : a \in M \text{ and } M \models \varphi_{=}(a, a)\}$; note that $\varphi(M)$ is an equivalence relation on $\{a : M \models \varphi_{=}(a, a)\}$
   (b) if $R \in \tau$ has arity $m$ then $R_{N}$, the interpretation of $r$ is $\{a_{\ell}/\varphi_{=}(M) : \ell < m : M \models \bigwedge_{\ell<m} \varphi_{=}(a_{\ell}, a_{\ell}) \land \varphi_{R}(a_{0}, \ldots, a_{m-1}) \text{ so } a_{0}, \ldots, a_{m-1} \in M\}$.

Recall that here “for every random enough $G_{n}$” is a central notion.

Definition 0.4. 1) A 0-1 context consists of:
   (a) a vocabulary $\tau$, here just the one of graphs or double directed graphs, see 0.1(5),(9),(10)
   (b) for each $n, K_{n}$ is a set of $\tau$-models with set of elements = nodes $[n]$, in our case graphs or double directed graphs
   (c) a distribution $\mu_{n}$ on $K_{n}$, i.e. $\mu_{n} : K_{n} \rightarrow [0, 1]_{\mathbb{R}}$ satisfying $\Sigma\{\mu_{n}(G) : G \in K_{n}\} = 1$
   (d) the random structure is called $G_{n} = G_{\mu_{n}}$ and we tend to speak on $G_{\mu_{n}}$ rather than on the context.

2) For a given 0-1 context, let “for every random enough $G_{n}$ we have $G_{n} \models \psi$, i.e. $G$ satisfies $\psi$” and “if $G_{n}$ is random enough then $\psi$”, etc. means that the sequence $\langle \text{Prob}(G_{n} \models \psi) : n \in \mathbb{N}\rangle$ converge to 1; of course, $\text{Prob}(G_{n} \models \psi) = \Sigma\{\mu_{n}(G) : G \in K_{n} \text{ and } G \models \psi\}$.

3) For $\bar{p} = \langle p_{n} = p(n) : n \rangle$ a sequence of probabilities, $G_{n, \bar{p}}$ is the case $K_{n} = \text{graphs on } |n|$ and we draw the edges independently
   (a) with probability $p$ when $\bar{p}$ is constantly $p$, e.g. $1/2$, and
   (b) with probability $p(n)$ or $p_{n}$ when $p$ is a function from $\mathbb{N}$ to $[0, 1]_{\mathbb{R}}$.

Below we add the second context because for it the proof is more transparent.

Context 0.5. 1) Case A:
   (a) $a \in (0, 1)$ is irrational
   (b) $p_{n} = 1/n^{\alpha}$.
2) Case B:
\[ \bar{\alpha}^* = (\alpha_1^*, \alpha_2^*) \] where \( \alpha_1^*, \alpha_2^* \in (0, 1/4) \) are irrational numbers, (natural to add linearly independent over \( \mathbb{Q} \)) hence \( 0 < \alpha_1^* < \alpha_2^* < \alpha_2^* + \alpha_2^* < 1/2 \) and let \( \alpha_0^* = \alpha_1^* \).

**Definition 0.6.** For Case A:
1) Let \( K^1 := \bigcup_{n} K_n^1 \) where we let \( K_n^1 \) be the set of graphs \( G \) on \([n] = \{1, \ldots, n\} \) so \( R^G \subseteq \{\{i, j\} : i \neq j \in [n]\} \).
2) For \( \alpha \in (0, 1)_{\mathbb{R}} \) let \( G_n = G_{n, \alpha} \) be the random graph on \([n]\) with the probability of an edge being \( 1/n^\alpha \) and the drawing of the edges being independent.
3) Let \( \mu_n = \mu_{n, \alpha} \) be the corresponding distribution on \( K_n^1 \), so \( \mu_n : K_n^1 \rightarrow [0, 1]_{\mathbb{R}} \) and \( 1 = \sum \{\mu_n(M) : M \in K_n\} \), in fact, \( \mu_n(G) = (1/n^\alpha)|R^G| \times (1 - 1/n^\alpha)^{2-|R^G|} \).

**Convention 0.7.** Writing \( K_n \) means we intend \( K_n^1 \) or \( K_n^2 \) (see below), similarly \( G_n \) is \( G_{n, \alpha} \) if Case A and \( G_{n, \bar{\alpha}} \) if Case B and similarly \( K \) is \( K^1 \) or \( K^2 \).

The more transparent related case is the following

**Definition 0.8.** On Case B, for \( G_{n, \bar{\alpha}} \):
1) Recall \( \tau_{\bar{\alpha}} \) is the vocabulary \( \{R_1, R_2\} \) intended to be two directed graph relations.
2) Let \( K^2 = \bigcup_{n} K_n^2 \) where we let \( K_n^2 = \{G : G = ([n], R^G_1, R^G_2) \text{ satisfying } ([n], R^G_1, R^G_2) \text{ is a directed graph for } \ell = 1, 2; \text{ we may write } R_\ell \text{ instead of } R^G_\ell \text{ when } G \text{ is clear from the context} \} \). We assume\(^2\) irreflexivity, i.e. \( (a, a) \notin R^G_1 \) but allow \( (a, b), (b, a) \in R^G_2 \).
3) For reals \( \alpha_1 < \alpha_2 \) from \( (0, \frac{1}{2})_{\mathbb{R}}, \text{ say from } 0.5(2) \) so \( \bar{\alpha} = (\alpha_1, \alpha_2) \) let \( \alpha(\ell) = \alpha_\ell \); let the random model \( G_{n, \bar{\alpha}} = ([n], R_1, R_2) = ([n], R_1^{G_{n, \bar{\alpha}}}, R_2^{G_{n, \bar{\alpha}}}) \) with \( R_1, R_2 \) irreflexive relations be drawn as follows:
   
   - (a) for each \( a \neq b \), we draw a truth value for \( R_2(a, b) \) with probability \( \frac{1}{n^{\alpha_2}} \) for yes
   - (b) for each \( a \neq b \), we draw a truth value for \( R_1(a, b) \) with probability \( \frac{1}{n^{\alpha_1}} \) for yes
   - (c) those drawings are independent.

4) We define the distribution \( \mu_{n, \bar{\alpha}} \) as follows:
   
   - (a) \( \mu_n = \mu_{n, \bar{\alpha}} = \mu_{n, \alpha_1, \alpha_2} \) is the following distributions on \( K_n^2 \):
     - \( \mu_n(G) = \mu_{n, \alpha_1, \alpha_2}([n], R^G_1), \mu_{n, \alpha_1, \alpha_2}([n], R^G_2) \)
     - \( \mu_{n, \alpha_1, \alpha_2}([n], R^G_1) = \left( \frac{1}{n^{\alpha_1}} \right)^{|R^G_1|} \times (1 - \frac{1}{n})^{n(n-1)-|R^G_1|} \)
   - (b) \( G_n = G_{n, \bar{\alpha}} = G_{n, \alpha_1, \alpha_2} \) denote a random enough \( G \in K_n^2 \) for \( \mu_{n, \bar{\alpha}} \) so \( n \) is large enough.

**Observation 0.9.** For random enough (recalling 0.4(2)) \( G_n = G_{n, \bar{\alpha}} = G_{n, \alpha_1, \alpha_2} \):
   
   - (a) For \( a \in [n], \text{ the expected value of the } R_2\text{-valency of } a, \text{ that is, } |\{b : aR^G_2b\}| \) is \( (n - 1) \cdot \frac{1}{n^{1-\alpha_2}} \sim n^{\alpha_2(2)} \);
   - (b) for every random enough \( G_{n, \bar{\alpha}} \) for every \( a \in [n] \) this number is close enough to \( n^{\alpha_2(1)} \), e.g.

\(^2\text{We may change the definition of } K^2_n \text{ by requiring } R^G_1 \cap R^G_2 = \emptyset, \text{ this makes little difference. We could further demand } R_\ell \text{ is asymmetric, i.e. } (a, b) \in R^G_\ell \Rightarrow (b, a) \notin R^G_\ell, \text{ again this makes little difference.} \)
\( \bullet_2 \) for some \( \varepsilon \in (0, \alpha_1)_R \), the probability of the difference being \( \geq n^{\alpha(1)(1-\varepsilon)} \) for at least one \( a \in [n] \), goes to zero with \( n \).

(c) the expected number of \( R_1 \)-edges is \( n(n-1)/n^{(1+\alpha_1)} \sim n^{1-\alpha_1} \) hence the expected value of \( |\{a : aR_1b \text{ for some } b\}| \) is close to it.

(d) for every random enough \( G_{n,\bar{\alpha}} \) the two numbers in (c) are close enough to \( n^{1-\alpha_1} \) (similarly to (b)).

Remark 0.10. 1) For \( K^2 \), this is a parallel of Claim 1.2 for \( K^1 \).

2) Note that the Clause (a) does not imply clause (b) in 0.9 because a priori the variance may be too large.
§ 1. ON THE LOGIC $L_{\infty,k}$

As the proof for $L_{\infty,k}$ is simpler and more transparent than for LFP, we shall explain it.

First, we try to define and then explain the logic $L_{\infty,k}$ for $k$ a finite number.

For a vocabulary $\tau$, we define the set $L_{\infty,k}(\tau)$ of formulas as the closure of the set of atomic formulas under some operation similarly to first order logic, but:

- we restrict ourselves to formulas having $<k$ free variables
- we allow arbitrary conjunctions and disjunctions (that is even infinite ones)
- as in first order logic we allow negation $\neg \varphi$ and existential quantifier (on one variable) $\exists x \varphi(x,\bar{y})$.

So any formula in $L_{\infty,k}$ not just have only $<k$ free variables but also every sub-formula has.

It may be helpful to recall the standard game which express equivalence. Recall (0.3(1)) that for transparency we assume the vocabulary below has only predicates and is finite.

$\oplus$ we say $\mathcal{F}$ is an $(M_1, M_2)$ - $L_{\infty,k}$-equivalence witness when for some vocabulary $\tau$ with predicates only

(a) $M_1, M_2$ are $\tau$-models
(b) $\mathcal{F}$ is a non-empty set of partial isomorphisms from $M_1$ to $M_2$
(c) if $f \in \mathcal{F}$ then $|\text{dom}(f)| < k$
(d) if $f \in \mathcal{F}, A \subseteq \text{dom}(f), |A| + 1 < k, \iota \in \{1, 2\}$ and $a_\iota \in M_\iota$ then there is $g$ such that $g \in \mathcal{F}, f|A \subseteq g$ and $\iota = 1 \Rightarrow a_\iota \in \text{dom}(g)$ and $\iota = 2 \Rightarrow a_\iota \in \text{rang}(g)$.

Now $\oplus_1$ for $M_1, M_2$ as in (a) of $\oplus$ above, the following are equivalent:

(a) $M_1, M_2$ are $L_{\infty,k}$-equivalent, i.e. for every sentence $\psi \in L_{\infty,k}(\tau)$, i.e. a formula with no free variables, $M_1 \models \psi \iff M_2 \models \psi$
(b) there is an $(M_1, M_2)$ - $L_{\infty,k}$-equivalence witness $\mathcal{F}$, i.e. as in $\oplus$.

Also $\oplus_2$ for $M_1, M_2, \mathcal{F}$ as in $\oplus$ above we have

(c) if $k < k,a_0, \ldots, a_{k-1} \in M_1$ and $g \in \mathcal{F}$ and $\{a_\ell : \ell < k\} \subseteq \text{dom}(g)$ then for every formula $\varphi(x_0, \ldots, x_{k-1}) \in L_{\infty,k}(\tau)$ we have $M_1 \models \varphi[a_0, \ldots, a_{k-1}] \iff M_2 \models \varphi[g(a_0), \ldots, g(a_{k-1})]$.

* * *

Having explained the logic, how can we prove for it the failure of the 0-1 law? Consider Case B where we have two kinds of edges, $R_1$ and $R_2$. Consider $\eta$ a sequence from $^{k}\{1, 2\}$, see 0.1(11) and $a \neq b$. There may be $(\eta, 0, k)$-pre-paths from

$^3$As we consider only finite models, countable conjunctions and injunctions are enough.
a to b in G, see Definition 1.6, i.e. \( a = a_0, a_1, \ldots, a_k = b \) such that \( (a_\ell, a_{\ell+1}) \) is an \( R_{\eta}(\ell) \)-edge for \( \ell < k \).

Now depending on \( \eta \) there may be many such pre-paths or few. If \( \eta \) is constantly 2 and \( k > 1/\alpha^*_2 \) then there are many such pre-paths - as fixing \( a \) in \( G_{n, \bar{a}} \), the expected number of \( b \)'s for which there is pre-\((\eta, 0, k)\)-paths from \( a \) to \( b \) is 1 for \( k = 0 \), is \( \approx n^{\alpha^*_2} \) for \( k = 1 \) is \( \approx n^{2\alpha^*_2} \) for \( k = 2 \), etc., so for \( k > 1/\alpha^*_2 \) it is every \( b \in G_n \); not helpful. If \( \eta \) is constantly 1, there are few such pre-paths and they are all short, even \( \leq k \) for any random enough \( G_n \), when \( 1 < \alpha^*_1 k \), not helpful.

But we may choose a “Goldilock’s” \( \eta \), that is, such that for every initial segment of \( \eta \) the expected number is not too large and not too small. This means that for some \( a \) for every \( k' \leq k \) for some \( b \) there is such a pre-path but not too many. We need more so that we can define by a formula from \( L_{\infty,k} \) the set \( S_{G_{n, \bar{a}}, k'} := \{ b \text{; there is such pre-path from } a \text{ to } b \text{ of length } k' \} \text{ but not a shorter pre-path} \} \text{ and it is } \neq \emptyset ; \text{ moreover we can define the natural order on the set} \{ S_{G_{n, \bar{a}}, k'} : k \leq n \} \text{. Fact} \ 1.4 \text{ below indicates what kind of } \eta \text{'s we need, and we use it proving 1.8; however in later sections, because we have to estimate the probabilities, we shall use only a closely related definition.}

**Hypothesis 1.1.** 1) Case A of 0.5 holds or Case B there holds.

2)

(a) for case A: \( \alpha^*_1, \varphi_\ell(x, y), n^*_2 \) for \( \ell = 0, 1, 2 \) will be as in Claim 1.2 below

(b) for case B: \( \alpha^*_1, \alpha^*_2 \) are as in 0.5 and \( \varphi_\ell(x, y) = R_\ell(x, y) \) for \( \ell = 1, 2 \) and let
\[ \alpha^*_0 = \alpha^*_1, \varphi_0(x, y) = \varphi_1(x, y) \]

(c) let \( \bar{\varphi} = (\varphi_\ell(x, y) : \ell = 0, 1, 2) \).

**Claim 1.2.** Assume we are in Case A. There are \( \alpha^*_1, \varphi_\ell(x, y) \) and \( \gamma^*_\ell \) for \( \ell = 0, 1, 2 \) such that:

(a) \( 0 < \alpha^*_1 < \alpha^*_0 < \alpha^*_2 < \alpha^*_2 + \alpha^*_2 \) are reals with \( \alpha^*_1 \in (0, 1/4)_\mathbb{R} \) and \( \gamma^*_\ell \in \mathbb{R}_{>0} \)

(b) \( \varphi_\ell(x, y) \) are first order formulas (in the vocabulary of graphs) even existential positive formulas such that \( \varphi_\ell(x, y) \vdash x \neq y \) for random enough \( G_{n, \bar{a}} \)

(c) if \( G_{n, \bar{a}} \) is random enough then for every \( a \in G_{n, \bar{a}} \) the set \( \varphi_2(G_{n, \bar{a}}, a) \) has \( \approx \gamma^*_\ell n^{\alpha^*_2} \) elements, i.e. for some \( \varepsilon \in (0, 1)_\mathbb{R} \). if \( G_{n, \bar{a}} \) is random enough, then for every \( a \in [n] \), the number of members of \( \varphi_1(G_{n, \bar{a}}, a) \) belongs to the interval \( (\gamma^*_\ell n^{\alpha^*_2} - n^{\alpha^*_2(1-\varepsilon)}, \gamma^*_\ell n^{\alpha^*_2} + n^{\alpha^*_2(1-\varepsilon)}) \)

(d) if \( \ell = 0, 1 \) and \( G_{n, \bar{a}} \) is random enough then \( \{ a \in [n] : \varphi_\ell(G_{n, \bar{a}}, a) \neq \emptyset \} \) has \( \approx \gamma^*_\ell n^{\alpha^*_2} \) members.

**Remark 1.3.** We shall use not just the statements but also the proofs of 1.2, 1.4.

**Proof.** Also here we shall use freely the analysis of \( G_{n, \alpha} \) for \( \alpha \in (0, 1)_\mathbb{R} \) irrational (see, e.g. [AS08]).

Let \( m^*_2, n^*_2 \) be such that:

(a) \( n^*_2 \) is large enough

(b) \( m^*_2 \leq \left( \frac{\alpha^*_2}{2} \right) \)

(c) \( \alpha^*_2 := (n^*_2 - 1) - \alpha m^*_2 \) is positive but, e.g. \( < \frac{1}{17} \).
As $\alpha \in (0, 1)_\mathbb{R}$ is irrational we can find such $m^*_1, n^*_1$. Let $H^*_2$ be a random enough graph on $[n^*_2]$ with $m^*_2$ edges such that $(1, 2) \notin R^{H^*_2}$. (Note that this “random enough” is just used for the existence proof).

We choose $n^*_1, m^*_1, H^*_1$ similarly except that $-\alpha^*_1 := n^*_1 - 1 - \alpha m^*_1$ is negative with value close enough, e.g. to $-\alpha^*_2/3$. Lastly, we choose $n^*_0, m^*_0, H^*_0$ similarly except that $-\alpha^*_0 = n^*_0 - 1 - \alpha m^*_0$ is negative and $\alpha^*_0 \in (\alpha^*_1, \alpha^*_2)$.

For $\ell = 1, 2$ let $\varphi_\ell(x, y) = (3 \ldots x_1 \ldots)_{i \in [n^*_1]}(x = x_1 \wedge y = x_2 \wedge \{x_i R x_j : i, j \in [m^*_1]\} \wedge \{x_i \neq x_j : i \neq j \in [n^*_1]\})$.

Now check clauses (a)-(d). Clearly $\alpha^*_2, \alpha^*_1$ satisfy clause (a) and $\varphi_1, \varphi_2$ are as in clause (b).

For $\ell = 1, 2$, let $\gamma^*_\ell = 1$. So for any $n$ large enough compared to $n^*_1, n^*_2$ and $a_1 \neq a_2 \in [n]$, the set $\mathcal{F} := \{f : f$ is a one-to-one function from $[n^*_2]$ to $[n]$ such that $f(1) = a_1, f(2) = a_2\}$ has $\prod_{i < n^*_2 - 2} (n - 2 - i) \sim n^{n^*_2 - 2}$ members.

For each $f \in \mathcal{F}$ the probability of the event $\mathcal{E}_f = \{f$ maps every edge of $H^*_2$ to an edge of $G_{n, \alpha}\}$ is $(\frac{1}{n^*_2})^{m^*_2}$ so the expected value of $\{f \in \mathcal{F} : \mathcal{E}_f$ occurs $\}$ is $\approx n^{n^*_2 - \alpha m^*_2} = \frac{1}{n^{\alpha m^*_2}}$. Clearly as in 0.9 the expected value is as required in clause (c) and by the well known analysis of $G_{n, \alpha}$ (see, e.g. [AS08]), clause (c) holds and see more in §4.

Claim (d) is proved similarly. \hfill $\square_{1.2}$

**Fact 1.4.** There is a sequence $\eta \in \mathbb{N}\{1, 2\}$ such that: for every $n > 0$,

$\gamma_n = (\eta(n))^{-1}\{2\} 2n^2 - (\eta(n))^{-1}\{1\} |\alpha^*_1|$ belongs$^4$ to $[\alpha^*_2 - \alpha^*_1, \alpha^*_2 + \alpha^*_1]_\mathbb{R}$.

**Proof.** We choose $\eta(n)$ by induction on $n$. Let $\eta(n)$ be 2 if $\gamma_n \leq \alpha^*_2$, e.g. $n = 0$ and $\eta(n)$ be 1 if $\gamma_n > \alpha^*_2$.

Easily $\eta$ is as required. \hfill $\square_{1.4}$

**Claim 1.5.**  1) If $\eta$ is as in 1.4 then for any $m$ and every random enough $G_n$, there is an $(\eta, m)$-path in $G_n$, see below.

2) Moreover, also there is an $(\eta, \varepsilon([\log(n)])\)-path and even an $(\eta, \lfloor n^\varepsilon \rfloor)$-path for appropriate $\varepsilon \in \mathbb{R}_{>0}$.

**Proof.** As in [AS08] on $G_{n, 1/n^\varepsilon}$ and see more in §3. \hfill $\square_{1.5}$

**Definition 1.6.** 1) A sequence $\bar{a} = (a_\ell : \ell \in [m_1, m_2])$ of nodes, that is, of members of $G \in K_n$ is called a pre-$(\nu, m_1, m_2)$-path, if $m_1 = 0$ we may omit it, when:

(a) $\nu$ is a sequence of length $\geq m_2$ and $i < \ell(\nu) \Rightarrow \nu(i) \in \{1, 2\}$

(b) if $\ell \in \{m_1, m_1 + 1, \ldots, m_2 - 1\}$ then $G \models \varphi_{\ell}(\nu, a_\ell, a_{\ell+1})$.

2) Above we say “$(\nu, m_1, m_2)$-path” when in addition:

(c) if $m_1 \leq \ell_1 < \ell_2 \leq m_2$ and $\langle a'_\ell : \ell \in [m_1, m_2]\rangle$ is a pre-$(\nu, m_1, \ell_2)$-path then $a'_m = a_{m_1} \wedge a'_t = a_{\ell_2} \Rightarrow a'_{\ell_1} = a_{\ell_1}$

(d) if $m_1 \leq \ell_1 < \ell_2 \leq m_2$ then $a_{\ell_1} \neq a_{\ell_2}$.

3) We say “$\bar{a}$ is a (pre)-$(\nu, m_1, m_2)$-path from $a$ to $b$" when in addition $a_m, = a \wedge a_m = b$. 

$^4$Can and will use also other intervals and similar sequences.
Remark 1.7. 1) Note that if $\langle a_\ell : \ell \leq m \rangle$ is a pre-$(\nu, m)$-path, it is possible that 
$\ell_1 + 1 < \ell_2 \leq m$ and $a_{\ell_1} = a_{\ell_2}$. For a $(\nu, m)$-path this is impossible.
2) In 1.6(2)(c), really the case $\ell_2 = m_2$ suffice.
3) We use the “log($n$)” Case in 1.5(2), but having log(log($n$)) and even much less has no real affect on the proof.

Conclusion 1.8. Let $k \geq \max\{n_0^*, n_1^*, n_2^*\}$ then $G_n$ fails the 0-1 law for $L_{\infty, k}$.

Remark 1.9. 1) Note that if $\langle a_\ell : \ell \leq m \rangle$ is a pre-$(\nu, m)$-path, it is possible that 
$\ell_1 + 1 < \ell_2 \leq m$ and $a_{\ell_1} = a_{\ell_2}$. For a $(\nu, m)$-path this is impossible.
2) In 1.6(2)(c), really the case $\ell_2 = m_2$ suffice.
3) We use the “log($n$)” case in 1.5(2), but having log(log($n$)) and even much less has no real affect on the proof.

Note that we rely on 1.5(2) but we prove more in §3.

Proof. For a finite graph $G$ and $\eta$ as in 1.4 or any $\eta \in \mathbb{N}\{1, 2\}$ let length$_\eta(G)$ be the 
maximal $m$ such that there is an $(\eta, m)$-path in $G$.

Now consider the statement 
$\oplus$ there is a sentence $\psi_m = \psi_{\eta, m} \in L_{\infty, k}(\tau_{et})$ such that for a finite graph 
$G, G \models \psi_m$ iff there is an $(\eta, m)$-path in $G$.

Why $\oplus$ is enough? Because then we let 
$\psi = \bigvee\{(\psi_m \land \neg \psi_{m+1}) : m \geq 10 \text{ and } (\log_m(m) \text{ is even})\}$

where log$_m(m)$ is essentially the inverse of the tower function, see 0.1(3). Note that using 1.4, 1.5(2), of course, we should be able to say much more.

Why $\oplus$ is true? First, we define the formula $\psi_{m_1, m_2}(x, y)$ for $m_1 \leq m_2$ by 
induction on $m_2 - m_1$ as follows:

$(*)_1$ if $m_1 = m_2$ it is $x = y$
$(*)_2$ if $m_1 < m_2$ it is $(\exists x_1)[\varphi_{\eta(m_1)}(x, x_1) \land \psi_{m_1, m_2}(x_1, y)]$.

So clearly

$(*)_3$ if $G \in K_n$ and $a, b \in [n]$ then $G \models \psi_{m_1, m_2}(a, b)$ iff there is a pre-$(\eta, m_1, m_2)$- 
path from $a$ to $b$.

Second, we define $\psi'_{m_2}(x, y)$ as $\psi_{0, m_2}(x, y) \land \bigwedge_{\ell_1 < \ell_2 \leq m_2} \neg(\exists z'_1, z'_2)(z'_1 \neq z'_2 \land \psi_0, \ell_1, \ell_2)(x, z'_1) \land 
\psi_{0, \ell_1}(x, z'_2) \land \psi_{\ell_1, \ell_2}(z'_2, z_2) \land \psi_{\ell_2, m_2}(z_2, y)]$.

This just formalizes 1.6(2)(c) so

$(*)_4$ $G \models \psi'_{m_2}(a, b)$ iff there is an $(\eta, m_2)$-path from $a$ to $b$.

As said above this is enough. Note that complicating the sentence we may weaken the demand on $G_n$. $\Box_{1.8}$
§ 2. The LFP Logic

In this section we try to interpret an initial segment of number theory in a random enough $G \in K_n$, i.e. with set of nodes $[n]$. In Definition 2.2 for $G \in K_n$ and $a \in G$ we define a model $M_{G,a}$. Now in $M \in M_{G,a}$, the equivalence classes of $E^M$ represent natural numbers. Concentrating on Case B, starting with $\{a_\ast\}$ as the first equivalence class, usually its set of $R_2$-neighbors will be the second equivalence class. Generally, if for an equivalence class $a/E^M$ we let the next one be the set $\text{suc}(a/E^M) = \{b \in G : R_2(a',b) \text{ for some } a' \in a/E^M\}$, then we expect that $\text{suc}(A/E^M)$ has $|a/E^M| \cdot n^2$ members. So if we continue in this way, shortly we get the equivalence classes cover essentially all the nodes of $G$. Hence we try to sometimes use the $R_1$-neighbors instead of the $R_2$-neighbors, but when? For $L_{\infty,k}(\tau_\ast)$ we can decide a priori, e.g. use $\eta$ as in 1.4 and the proof of 1.8 so that the expected number will be small. But for LFP logic this is not clear, so we just say: use the $R_1$-neighbors if there is at least one and the $R_2$-neighbors otherwise, so this is close to what is done in 1.4, 1.5, 1.8 but not the same.

For case A we use $\varphi_\ell$-neighbors instead of $R_\ell$-neighbors for $\ell = 1, 2$ except that the question on existence is for $\varphi_0$-neighbors.

How do we from equivalence relations and the successor relation re-construct the initial segment of number theory? This is exactly the power of definition by induction.

Naturally we need just

Let $\text{height}(G)$ be the maximal number of $E^M$-equivalence classes for $M \in M_{G,a}, a \in G$, we have:

(*) for every $m$, for every random enough $G_n, m \leq \text{height}(G)$, moreover letting $f : \mathbb{N} \to \mathbb{N}$ be $f(n) = \log(n)$ for every random enough $G_n, f(n) \leq \text{height}(G_n)$.

For failure of 0-1 laws, $\boxplus$ is enough, but we may wish to prove a stronger version, say finding a sentence $\psi$ which for every random enough $G_n, G_n$ satisfies $\psi \boxplus \alpha$ is even.

We intend to return to it elsewhere; but for now note that for a set $A \subseteq G$ we can define $(R$ is $R_2$ for Case B, $\varphi_2$ for Case A) $c_{\ell,G_n}(A) = \{b : b \in A \setminus \{a\} \text{ for every random enough } G_n \}$ but for no $\alpha \in G \setminus \{a\}$ do we have $(\forall x \in A)[R(x,\alpha) \equiv R(x,b)]$. Now from a definition of a linear order on $A$ we can derive one on $c_{\ell,G_n}(A)$. We can replace $R$ by any formula $\varphi(x,y)$. Now in our context, if we know that, with parameters, we define such $A$ of size $\approx n^\epsilon$ for appropriate $\epsilon$ then we can define a linear order on $[n]$; why there is such $A$? because if $M \in M_{G,a}$, and $k$ is not too large then there is $b \in M, \text{lev}(b, M) = k$ such that there is a unique maximal $\prec_2$-path from $a_\ast$ to $b$.

For $L_{\infty,k}$ this is much easier.

Context 2.1. (A) or (B):

(A) (case A of 0.5) the vocabulary $\tau_\ast$ is $\tau_{gr}$, the one for the class of graphs, $\varphi_\ell(x,y), \ell = 0, 1, 2$ are as in 1.2 so $\in \mathbb{L}(\tau_\ast)$ and $\tilde{\alpha}^\ast = (\alpha_0^\ast, \alpha_1^\ast, \alpha_2^\ast)$, is as there, $G = G_n = G_{n,a}, K_n = K_n^\ast$ as in Definition 1.4,

(B) (case B of 0.5) $\tau_\ast = \tau_{gr} = \{R_1, R_2\}, K_n = K_n^\ast$ and $\tilde{\alpha}^\ast = (\alpha_0^\ast, \alpha_2^\ast)$ are as in 0.5(2) and $G_n = G_{n,a}$ and $\varphi_\ell(x,y) = xR_\ell y$ for $\ell = 1, 2$, with $G_n, K_n^\ast$ as in Definition 0.8 and let $\alpha_0^\ast = \alpha_1^\ast, \varphi_0(x,y) = \varphi_1(x,y)$.
Definition 2.2. For $G \in K_n$ and $a_\ast \in G$ we define $M = M(G,a_\ast) = M_{G,a_\ast}$ as the set of $\tau_1$-structures of $M$ such that (the vocabulary $\tau_1$ is defined implicitly):

(A) (a) the universe of $M$ is $P^M \subseteq [n]$
(b) $c^M_\ast = a_\ast$, so $c_\ast$ is an individual constant from $\tau_1$
(c) $E^M$ is an equivalence relation on $M$
(d) $<_1$ is a linear order on $P^M/E^M$, i.e.
(\alpha) $a_1E^M a_2 \land b_1E^M b_2 \land a_1 <^M_1 b_1 \Rightarrow a_2 <^M_1 b_2$
(\beta) for every $a,b \in P^M$ exactly one of the following holds: $a <^M_1 b$, $b <^M_1 a$ and $aE^M b$
(e) $<_2^M$ is a partial order included in $<_1^M$
(f) (a) $a/E^M$ is a singleton and $a_\ast$ is $<_2^M$-minimal, i.e. $b \in M \backslash \{a_\ast\} \Rightarrow a <^M_2 b$
(\beta) if $a <^M_1 b_1 <^M_1 c$ and $a <^M_2 c$ then for some $b' \in b/E^M$ we have $a <^M_2 b' <^M_2 c$
(\gamma) if $a,b \in M$ and $b/E^M$ is the immediate successor of $a/E^M$ (by $<_1^M$) then for some $a' \in a/E^M$, we have $a' <^M_2 b$
(g) if $b \in M$ is a $<_2^M$-immediate successor of $a \in M$ (i.e. $a <^M_2 b$ and $\neg(\exists y)(a <^M_2 y <^M_2 b)$, equivalently, $\neg(\exists y)(a <^M_1 y <^M_1 b)$) then for some $\iota \in \{1,2\}$ we have $G \models \varphi_\iota[a,b]$
(h) $P_0, P_1, P_2 = P_+^\iota, P_3 = P_\iota, P_4 = P_\iota^\setminus$ are predicates of $\tau_1$ with 1,1,3,3,2 places respectively such that using the definitions in clauses (B)(a),(b),(c) below, $P^M_\iota$ are defined in clauses (B)(d) below

(B) (a) for $a \in M$, $\text{lev}(a,M)$ is the maximal $k$ such that there are $a_0 <^M_1 a_1^M < \ldots <^M_1 a_k = a$; so necessarily $a_0 = a_\ast$
(b) $\text{height}(M) = \max\{\text{lev}(a,M) : a \in M\}$
(c) for $k < \text{height}(M)$ let $\iota = \iota(k,M) \in \{1,2\}$ be such that if $b$ is a $<_2^M$-immediate successor of $a$ and $k = \text{lev}(a,M)$ then $G \models \varphi_\iota[a,b]$, in the unlikely case both $\iota = 1$ and $\iota = 2$ are as required we use $\iota = 1$

(d) (a) $P^M_0 = \{a_\ast\}$
(\beta) $P^M_1 = \{a \in M : \text{lev}(a,M) = 1\}$
(\gamma) $P^M_2 = \{(a,b,c) : a,b,c \in M \text{ and } \mathbb{N} \models \text{"lev}(a,M) + \text{lev}(b,M) = \text{lev}(c,M)\}\}$
(\delta) $P^M_3 = \{(a,b,c) : a,b,c \in M \text{ and } \mathbb{N} \models \text{"lev}(a,M) \times \text{lev}(b,M) = \text{lev}(c,M)\}\}$
(c) $P^M_4 = \{(a,b) : N \models \text{"lev}(a,M) < \text{lev}(a,N)\}\}$.

Definition 2.3. 1) Let $\iota \in \{1,2\}$.
We say $N$ is the $\iota$-successor of $M$ in $M_{G,a}$ when for some $k$

(a) $M, N \in M_{G,a}$ so $G \in K_n$ for some $n$
(b) $M \subseteq N$ as models so $M = N \upharpoonright |M|$, recalling $|M|$ is the set of elements of $M$
(c) $k = \text{height}(M)$ and $\text{height}(N) = k + 1$
(d) \( b \in N \setminus M \) iff \( \text{lev}(b, N) = k + 1 \) iff \( b \in G \setminus M \) and for some \( a \in M \) we have \( \text{lev}(a, M) = k \) and \( G \models \varphi_i[a, b] \).

2) We may omit \( \iota \) above when \( \iota = 1 \) iff (*) holds where:

(*) there is \( e \in M \) such that \( \text{lev}(e, M) = k = \text{height}(G) \) and for some \( b \in G \setminus M \) we have \( G \models \varphi_0[b, c] \); yes not \( \varphi_1 \) but for case B there is no difference.

3) For a sentence \( \psi \) in the vocabulary \( \tau_1 \cup \tau_\ast \), (see 2.1, 2.2), for \( M, N \in M_{G,a} \), we say \( N \) is the \( \psi \)-successor of \( M \) when for some \( \iota \in \{ 1, 2 \} \), \( N \) is the \( \iota \)-successor of \( M \) and \( (G, M) \models \psi \iff (\iota = 1) \).

4) For \( G \in K_n, a \in G \) and \( M \in M_{G,a} \), we define \( N_{M,a} \) as the following structure \( N \) with the vocabulary \( \tau \) of number theory:

(a) set of elements \( \{ a/E^M : a \in M \} \)
(b) \( 0^N = a_0/E^M = P_0^N \)
(c) \( 1^N = P_1^M \)
(d) if \( a_\ell = a_\ell/E^M \in N, a_\ell \in M \) for \( \ell = 1, 2, 3 \) then
   \( (a) \ \text{N} \models a_1 + a_2 = a_3 \) iff \( (a_1, a_2, a_3) \in P_2^M \)
   \( (b) \ \text{N} \models a_1 \times a_2 = a_3 \) iff \( (a_1, a_2, a_3) \in (a) \)
   \( (c) \ \text{N} \models a_1 < a_2 \) iff \( (a_1, a_2) \in P_1^M \).

Claim 2.4. 1) If \( \iota, G, M, a \) are as in 2.3(1) then there is at most one \( \iota \)-successor \( N \) of \( M \) in \( M_{G,a} \).

1A) For some \( \psi_\ast \in L(\tau_1 \cup \tau_\ast) \), being a \( \psi_\ast \)-successor is equivalent to being a successor.

2) For a given \( G \in K_n, a \in G, M \in M_{G,a} \) and \( \psi \in L(\tau_1 \cup \tau_\ast) \) there is at most one \( \psi \)-successor \( N \) of \( M \) in \( M_{G,a} \).

3) For \( G \in K_n \) and \( a \in G \) there is one and only one sequence \( \langle M_k = M_{k,a} : k \leq k_{G,a} \rangle \) such that:

(a) \( M_k \in M_{G,a} \)
(b) \( M_0 \) has universe \{a\}
(c) \( M_{k+1} \) is the successor of \( M_k \) in \( M_{G,a} \), recall 2.3(2)
(d) if \( k = k_{G,a} \) then there is no \( N \in M_{G,a} \) which is the successor of \( M_k \) in \( M_{G,a} \).

3A) Above \( N_{M_{k,a}} \) is isomorphic to \( \mathbb{N} \{ 0, \ldots, k \} \). Also for every sentence \( \psi \in L \) or even \( \in L_{\text{LFP}} \) in the vocabulary of number theory there is a sentence \( \varphi \in L_{\text{LFP}}(\tau_\ast) \) such that \( N_{M_{k,a}} \models \psi \iff M_{k,a} \models \varphi \); of course, \( \varphi \) depends on \( \psi \) but not on \( G, a \) (and \( k \)).

4) In the LFP logic for \( \tau_\ast \), we can find a sequence \( \varphi \) of formulas with \( \psi \) variable \( x_0, \ldots, y \) such that: for any \( G \in K_n \) and \( a \in G \), the sequence \( \varphi \) substituting \( y \) by a defines \( M = M_{G,a} \) which is \( M_{k,a} \) for \( k = k_{G,a} \) from part (3).

4A) For \( \psi \) as in 2.3(3), i.e. \( \psi \in L(\tau_1 \cup \tau_\ast) \), a sentence, the parallel of 2.4(3),(4),(5) holds for "\( \psi \)-successor", (so we should write \( M_{G,a,\psi} \) instead \( M_{G,a} \)).

\(^5\)No real harm to demand here (and in 2.2) "unique"

\(^6\)The variable \( y \) stands for the parameters \( a \); instead we may define in 2.3 one model \( M_k \) coding all \( M_{a,\ell} \in M_{G,a} \) for \( \ell \leq k, a \in G \).
5) Letting \( \text{height}(G) = \max\{\text{height}(M_{G,a}) : a \in G\} \), in LFP logic there is \( \varphi_*(x) \) such that \( G \models \varphi_*(a) \) iff \( a \in G \) and \( \text{height}(M_{G,a}) = \text{height}(G) \) iff for every \( a_1 \in G \), \( \text{height}(M_{G,a_1}) \leq \text{height}(M_{G,a}) \).

6) For any sentence \( \psi \) in the vocabulary of number theory (in first order or LFP logic) there is a sentence \( \phi \) in induction logic for \( \tau_\ast \) (recalling 2.1) such that for any \( G, G \models \phi \) iff \( \text{height}(M_{G,a}) = \text{height}(G) \) iff for every \( a_1 \in G \), \( \text{height}(M_{G,a_1}) \leq \text{height}(M_{G,a}) \).

Proof. 1) Read the Definition 2.3(1).

1A) Read 2.3(2).

2) Read 2.3(3) and recall part (1).

3) We choose \( M_k \) and prove its uniqueness by induction on \( k \) till we are stuck. Recalling 2.4(3A) we are done.

3A) Easy.

4),4A) Should be clear.

5) We can express by induction when “\( \text{lev}(b_1, M_{G,a_1}) \leq \text{lev}(b_2, M_{G,a_2}) \)”.

6) Should be clear but we elaborate.

Recall the formula \( \varphi_*(x) \in L_{\text{LFP}}(\tau_\ast) \) from 2.4(5). By the choice of \( \varphi_\ast \) necessarily for some \( a_\ast, G_n \models \varphi_\ast[a_\ast] \) (as in a finite non-empty set of natural numbers there is a maximal member) so \( \text{height}(a_\ast, G_n) = \text{height}(G_n) \).

Now for a given \( \psi \), let \( \varphi \in L_{\text{LFP}}(\tau_\ast) \) say: for some (equivalently every) \( a \in G_n \) such that \( G_n \models \varphi(a) \), the model \( N_{G_n,a_n} \) defined in 2.3(4), which is isomorphic to \( N|\{0, \ldots, \text{height}(a, M_{G_n,a_n})\} \), see 2.4(3A), satisfies \( \psi \).

\[ \square \]

Conclusion 2.5. We have “\( G_n \) fail the 0-1 law for the LFP logic, moreover for some \( \varphi \in L_{\text{ind}}(\tau_\ast) \) we have \( \text{Prob}(G_n \models \varphi) \) has \( \lim \sup = 1 \) and \( \lim \inf = 0 \)”.

Proof. Should be clear by the above, in particular 2.4(6), see 3.3, 3.4(2) for details on the probabilistic estimate needed for 1.5(2) on which we rely. But we elaborate.

Note that just the following is not sufficient:

\[ (*)_1 \] some \( \bar{\varphi}, m \) satisfies

\( (a) \) \( \bar{\varphi} \) an interpretation scheme, see 0.3

\( (b) \) \( m \) is a function from the class of finite graphs to \( \mathbb{N} \), depending on the isomorphism type only

\( (c) \) for every \( m \) for every random enough \( G_n, m(G_n) \geq m \)

\( (d) \) for random enough \( G_n, \bar{\varphi} \) defines an isomorphic copy of \( \mathbb{N}|\{0, \ldots, m(G_n)\} \).

However it is enough if we add, e.g.

\[ (*)_2 \] \( m(G_n) \geq \log_2(\log_2(n)) \).

Why it is enough? Let \( \psi \) be a first order sentence in the vocabulary such that \( \mathbb{N}|\{0, \ldots, k\} \models \varphi \iff \log_2(k) \text{ belong to } \{10n + \ell : \ell = \{0, 1, 2, 3, 4\} \text{ and } n \in \mathbb{N}\} \).

Now use the interpretation \( \bar{\varphi} \), i.e. we use 2.4(6) in our case.

Why \( (*)_1 + (*)_2 \) holds: By 3.3, 3.4(1) and 2.4(3A).
§ 3. Revisiting induction

As discussed in §2, we need to prove that for random enough \( G_n \), \( \text{height}(G_n) \) is large enough, equivalently, for some \( a \in G_n \) (we shall prove that even for most), \( \text{height}(a, G_n) \) is large enough. For this a more detailed specific statement is proved - see (**) in the proof of 3.3. That is, we prove that for most \( a \in G_n \) (for random enough \( G_n \)): on the one hand \( M_{G,a} = k \) is not too large, and, on the other hand, is not empty; and for Case A, even not too small. The computation naturally depends on what \( \eta_{G,a} \) is, see 3.2(3), this is a delicate point.

For Case B, things are simpler. For each \( k \) we ask if there is an \( R_1 \)-edge out of \( M_{G,a} = k \) to \( G \backslash M_{G,a,k} \). If there is, clearly \( M_{G,a} = k + 1 \) will be quite small but not empty. If not, then necessarily \( M_{G,a} = k \) has \( \leq n^{a_2 - \xi} \) members hence the number of \( R_2 \)-neighbors of members of \( M_{G,a} = k \) cannot be too large (well \( < n^{a_2 n^{a_2}} \)) so we are done.

Case A seems harder, so we simplify considering only small enough \( k \), see 3.4, hence we can consider all possible \( \eta \)'s of length \( k \), that is, summing the probability of the “undesirable” events on all of them; so if each has small enough probability, even the unions of all those events has small enough probability. Now we divide the \( \eta \)'s to those which are “reasonable candidates to be \( \eta_{G,a} \)” and those which are not. For the former \( \eta \)'s, for almost all \( a \in G_n \) there is a pre-(\( \eta, 0, k_a \))-path starting with \( a \). So it is enough to prove that for almost all \( a \in G_n \), \( \eta_{G,a} |k^*| \) is one of those former \( \eta \)'s where \( k_a \) is the relevant large enough height, e.g. \( \leq [\log_2(\log_2(n))] \). For this it is enough to prove that the other \( \eta \)'s cannot occur and this is what we do.

We in this section fulfill promises from §2 (and §1).

Context 3.1. As in 2.1.

Below we shall use

**Definition 3.2.** 1) For \( G \in K_n \) we define \( M_k(a,G) = M_{G,a,k} \) by induction as in 2.4(3) for \( \psi = \psi_0 \) from 2.4(1A) and also \( k = k_{G,a} \) as there and height \( (G) \) as in 2.4(6).

2) Let \( M_{G,a,k} = M_{G,a,k} \cup \{ M_{G,a,m} : m < k \} \) and similarly \( M_{G,a,k} < k \).

3) Let \( \eta = \eta_{G,a} \) be the following sequence of length \( k_{G,a} \); if \( \ell = k_{G,a} \), then \( \eta(\ell) = \eta(\ell, M_{G,a}) \in \{1,2\} \) from Definition 2.2(B)(c).

**Claim 3.3.** For small enough \( \varepsilon \in (0,1)_R \), for random enough \( G_n \), for some \( a \in [n], k_{G,a} \geq k^* = \lceil n^\varepsilon \rceil \) in case B and \( k_{G,a} \geq \lceil \log(\log(n)) \rceil \) in case A.

**Remark 3.4.** It would be nice to use an \( \eta \in \{1,2\} \) defined similarly to 1.4, say such that \( \gamma_n \in [a_0^*, a_2^* + a_2^*] \), but this is not clear. In case B, in the proof the problem is that the \( \gamma_n \)'s from 1.4 may be very near to \( a_0^* \). Also the parallel problem for case A is that the answer to the question asked there is near the critical stage, so we are not almost sure about the answer.

**Proof.** For case A, we presently prove it, e.g. for \( k^* = \lceil \log_2(\log_2 n) \rceil \) and for Case B, \( \lceil n^\varepsilon \rceil \), an overkill, but this suffices for the failure of the 0-1 law. We intend to fill the general case elsewhere. Actually for any \( \varepsilon \in (0,1)_R \) we can get \( k^* = \lceil n^{1-\varepsilon} \rceil \).

Let \( \zeta \in (0,1)_R \) be small enough and \( k^* \) be as above.

Clearly it is enough to prove:
(⋆) if $a \in [n]$ and $k < k^*$ then the probability that at least one of the following
(i) $a_k^i (ii) a_{2.1}^i (iii) a_{k}$ fails (assuming $\ell < k \Rightarrow (i)_{a, \ell} \wedge (ii)_{a, \ell}$ is small enough;
$< \frac{1}{k \log(n)}$ suffices, being $< \frac{1}{kn^i}$ for each $i$ for large enough $n$ is natural)

$i_k = k \leq k_{G, a}^i$,

$a_k G, a = k$ has $\leq n^{\alpha_1^* + \alpha_2^*}$ elements

$a_{k} G, a, k$ (noting that $(i) a_k$ implies $M_{a, \ell, k} \neq \emptyset$) has

$\geq n^{\alpha_0^* - \xi}$ elements $^7$ except when $k = 0$, not need for Case B.

Why does (⋆) hold?

Case 1 Case B of the Context 2.1 and Definition 0.8

We are given $n \geq 1$ and $a \in [n]$; we draw the edges in $k$ stages so by induction
on $k$. For $k = 0$ draw the edges starting with $a$ (of both kinds, an overkill), i.e. for
$\iota \in \{1, 2\}$ the truth value of $R_{\iota}(a, b)$ for every $b \in [n] \setminus \{a\}$, hence we can compute
$M_{\iota}(a, G)$.

The induction hypothesis on stage $k$ is that $|M_{G, a, i} : i \leq k|$ have been computed
and we have drawn the truth value of $R_{\iota}(c, b)$ for $b \in \bigcup \{M_{G, a, i} : i < k\}$ and
c $\in [n] \setminus \{b\}$. If $k < k_0$, we now draw the edges $R_{\iota}(b, c)$ for $b \in M_{G, a, = k}$ and any
$c \neq b$; actually the $c \in M_{G, i, k}$ are irrelevant and so we can compute $M_{G, a, k+1}$.

Now we ask: if $(i) \alpha + (ii) a_m$ holds for $m \leq k$ what is the probability that
$(i) a_{k+1} (ii) a_{k+1}$? (recalling $(iii) a_{k+1}$ is irrelevant), i.e., is it small enough? This
is easy and as required.

In details, we ask

Question: Are there $c \in [n] \setminus M_{G, a, k}$ and $b \in M_{G, a, = k}$ such that $(b, c) \in R_{1}^G$?

First note

(⋆) if $|M_{G, a, = k}| \geq n^{\alpha_0^* - \xi}$ then the probability that the answer is no is $< 1/2^n$.

[Why? We have $M_{G, a, = k} \times ([n] \setminus M_{G, a, k})$ independent drawings so their number is

$\geq n^{\alpha_0^* - \xi} n/2$, each with probability $\frac{1}{n^{\alpha_1^* + 1}}$ of success and $(1 + \alpha_0^* - \zeta) - (1 + \alpha_1^*) =

\alpha_0^* - \zeta - \alpha_1^* > 0$ so the probability of the no answer is $(1 - \frac{1}{n^{\alpha_1^* + 1}})^{n^{(1 + \alpha_1^*)^*}} \sim

1/e(n^{\alpha_0^* - \alpha_1^*})$, clearly more than enough.]

By (⋆) it suffices to deal with the following two possibilities.

Possibility 1: The answer is yes.

In this case $M_{G, a, k+1}$ is well defined and $\iota(k, M_{G, a, k}) = 1, M_{G, a, = k+1} = \{c : c \in

G \setminus M_{G, a, k}$ and $(b, c) \in R_{1}^G$ for some $b \in M_{G, a, = k}\}$. Now for each $c \in [n] \setminus M_{G, a, k}$ and

$b \in M_{G, a, = k}$ the probability of $(c, b) \in R_{1}^G$ is $\frac{1}{n^{\alpha_0^* + \alpha_1^*}}$ hence by the independence of
the drawing, recalling $|M_{G, a, = k}| \leq n^{\alpha_0^* + \alpha_2^*}$ the probability of $|M_{G, a, = k+1}| \geq n^{\alpha_0^* + \alpha_2^*}$
is negligible, e.g. $< 2^n$ so can be ignored. Also by the possibility we are in, $M_{G, a, = k+1} \neq \emptyset$.

Possibility 2: The answer is no and $|M_{G, a, = k}| \leq n^{\alpha_0^* - \zeta}$.

This is easy, too, recalling that almost surely for every $a' \in G$ the number of
$R_{2}$-neighbors in the interval $[n^{\alpha_0^*} - n^{\alpha_0^* (1 - \zeta)}, n^{\alpha_0^*} + n^{\alpha_0^* (1 - \zeta)}]$ and so the probability
that $M_{G, a, = k+1}$ is too large is negligible.

Case 2: Case A of the context 2.1

$^7$We can use $\geq n^{\alpha_0^* - \alpha_0^* - \zeta}$
Here it helps to use "\(\varphi_0, \varphi_1\) are distinct".

Now it suffices to prove:

\((*)_1\) for random enough \(G_n\), for every \(a \in M\), the following has negligible probability of failure: (i) \(a, b, c\), (ii) \(a, k\), (iii) \(a, k\).

Note that for this it seems more transparent to\(^9\) assume \(k < \log_2(\log_2(n))\) and to translate \((*)_1\) to statement on paths.

\((*)_2\) For \(k \leq k_*\) let

(a) \(\Omega = \{\eta \in \{1, 2\}\} \text{ such that } \alpha(\eta) := |\eta^{-1}\{2\}| \cdot \alpha_2^{\eta} - |\eta^{-1}\{1\}| \cdot \alpha_1^{\eta}\) belongs to the interval \([0, \alpha_2^{\eta} + \alpha_1^{\eta} + \zeta]\)

(b) \(\Omega_k\) be the set of \(\eta \in \{1, 2\}\) such that for every \(\ell < n\) the sequence \(\eta(\ell) = (\eta(0), \ldots, \eta(\ell - 1))\) belongs to \(\Omega = \ell\)

(c) \(\Delta_k\) be the set of \(\eta \in \{1, 2\}\) such that \(\eta \in \Omega = k\) and even \(\eta \in \Omega = k\) but \(\alpha(\eta) \leq \alpha_2^{\eta} - \zeta\)

\((*)_3\) recalling that \(\zeta \in (0, 1)_n\) is small enough, for any random enough \(G_n\), for every \(a \in M\) the following has probability \(\leq 1/n^k\):

- for some \(k \leq |\log(\log(n))|\) and \(\eta \in \{1, 2\}\) at least one of the following holds
  - (a) \(\eta \in \Omega_k\) but there is no pre-\((\eta, 0, k)\)-path in \(G_n\) starting with a
  - (b) \(\eta \in \Delta_k\) but there is a pre-\((\eta, 0, k + 1)\)-path in \(G_n\) from \(a\) to some \(b \in G_n \setminus \{a\}\) such that \(G_n \models (\exists x)\varphi_0(b, x)\).

Otherwise the proof is as in the earlier case. \(\square\)

**Claim 3.5.** Let \(G_n = G_{n, \tilde{a}}\), i.e. we are in Case B. For some \((\tau_n, \tau_{dn})\)-scheme \(\tilde{\psi}\), for every random enough \(G_n, \tilde{a}, \psi\) defines in \(G_n\) a structure isomorphic to \(\mathbb{N}, <\).

**Proof.** We make a minor change in Definition 2.3(1),(d).

Clause \((d)^{\uparrow}\): We require the \(a \in M\) is unique.

This makes no real difference above because the probability of the occurrence if even one "\(b\) with two predecessors" is small and we just need that there is one; this makes \(M_{G, a, k}\) smaller but not empty.

We start as in the proof of 3.3, for \(k_* = \|\eta k\|\), we use Case 1 but for stage \(k\) we draw the truth values of \(\mathbb{R}_c(b, c)\) only when \(b \in M_{G, a, = k}\) and \(c \in [n]\) but \(c \notin M_{G, a, 0} \cup \ldots \cup M_{G, a, k - 1}\).

So there is \(b \in M_{G, a, = k}\) and there is a unique sequence \(\langle a_\ell : \ell \leq k_*\rangle, a_0 = a, a_{k_*} = b\) and \(\langle a_\ell, a_{\ell + 1}\rangle \in R^G_{a, \ell + 1}\) where \(\ell\) is such that \(M_{G, a, \ell + 1}\) is the \(\iota\)-successor of \(M_{G, a, \ell}\) and so there are formulas \(\psi_2 \in \mathbb{L}_{\text{LFP}}(\tau_{d_k})\) such that not depending on the pair \((a, n), \psi_2(G, a, b) = \{\langle a_\ell, a_i\rangle : \ell \leq i \leq k_*\}\).

Now

- the probability of the following event is negligible \((c < 2)\): for some \(d_1 \neq d_2 \in [n] \setminus M_{G, a, k}\) for every \(c\) if \(\psi_2(c, c, a, b)\) then \(\langle d_1, c\rangle \in R^G_2 \iff \langle d_2, c\rangle \in R^G_2\).

Ignoring this event, the following formulas defines a linear order on \([n] \setminus M_{G, a, k}\):
$\psi_3(d_1, d_2, a, b)$ say: for some $c$ we have $\psi_2(c, c, a, b) \land R_2(d_2, c) \land \neg R_2(d_1, c)$ and for any $c_1$ if $\psi_2(c_1, c, a, b)$ then $R_2(d_1, c') \leftrightarrow R_2(d_2, c')$.

So $(x, y, a, b)$ defines a linear order on $[n] \setminus MG,a,k$ which has $\geq n - \lfloor n^{\varepsilon} \rfloor$ elements.

We get using the same trick $\psi_4 \in \mathbb{L}_{\text{LFP}}(\tau_{dn})$ and $\psi_4(x, y, a, b)$ defines a linear order on $[n]$. Now the formulas $\psi_0, \ldots, \psi_4$ do not depend on $n$. Also for some $\psi_5 \in \mathbb{L}_{\text{LFP}}(\tau_{dg})$

- $G \models \psi_5(a, b)$ iff $\psi_4(\cdot, \cdot, a, b)$ defines a linear order (on $G$)
  and for some $\psi_0 \in \mathbb{L}_{\text{LFP}}(\tau_{dg})$
    - $G \models \psi_0[c_1, a_1, b_1, c_2, a_2, b_2]$ iff:
      - $(a_1, b_1), (a_2, b_2) \in \varphi_5(G)$
      - $(b) \ |\{ c : G \models \psi_4[c, c_1, a_2, b_2] \}| = |\{ c : G \models \psi_4[c, c_2, a_2, b_2] \}|.$

So the interpretation should be clear. \( \square_{3.5} \)

**Conclusion 3.6.** (Case B) For some $\psi \in \mathbb{L}_{\text{LFP}}(\tau_{dg})$ for every random enough $G_n = G_{n, \alpha}$ we have:

- $G_n \models \psi$ iff $n$ is even.
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