RADIATIVE POINCARE TYPE EON AND ITS FOLLOWER

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Abstract. We consider two consecutive eons \( \hat{M} \) and \( \hat{M} \) from Penrose’s Conformal Cyclic Cosmology and study how the matter content of the past eon \( (\hat{M}) \) determines the matter content of the present eon \( (\hat{M}) \) by means of the reciprocity hypothesis.

We assume that the only matter content in the final stages of the past eon is a spherical wave described by Einstein’s equations with the pure radiation energy momentum tensor
\[
\hat{T}^{ij} = \hat{\Phi} K^i K^j, \quad \hat{g}_{ij} K^i K^j = 0,
\]
and with cosmological constant \( \hat{\Lambda} \). We solve these Einstein’s equations associating to \( \hat{M} \) the metric \( \hat{g} = t^{-2} \left( -dt^2 + h(t) \right) \), which is a Lorentzian analog of the Poincaré-Einstein metric known from the theory of conformal invariants. The solution is obtained under the assumption that the 3-dimensional conformal structure \([h]\) on the \( I^+ \) of \( \hat{M} \) is flat, that the metric \( \hat{g} \) admits a power series expansion in the time variable \( t \), and that \( h(0) \in [h] \). Such solution depends on one real arbitrary function of the radial variable \( r \).

Applying the reciprocal hypothesis, \( \hat{g} \to \hat{g} = t^4 \hat{g} \), we show that the new eon \((\hat{M}, \hat{g})\) created from the one containing a single spherical wave, is filled at its initial state with three types of radiation: (i) the damped spherical wave which continues its life from the previous eon, (ii) the in-going spherical wave obtained as a result of a collision of the wave from the past eon with the Bang hypersurface and (3) randomly scattered waves that could be interpreted as perfect fluid with the energy density \( \hat{\rho} \) and the isotropic pressure \( \hat{p} \) such that \( \hat{p} = \frac{1}{3} \hat{\rho} \). The metric \( \hat{g} \) solves the Einstein’s equations without cosmological constant and with the energy-momentum tensor
\[
\hat{T}^{ij} = \hat{\Phi} K^i K^j + \hat{\Psi} L^i L^j + (\hat{\rho} + \hat{p}) \hat{u}^i \hat{u}^j + \hat{p} \hat{g}^{ij},
\]
in which \( \hat{u}^i \hat{u}^j \hat{g}_{ij} = -1, \hat{g}_{ij} L^i L^j = 0 \) and \( L^i K^j \hat{g}_{ij} = -2 \).

1. The setting

In this short note we show a model of a bandage region of two consecutive eons from the Penrose’s Conformal Cyclic Cosmology (CCC) [3], which have the following properties:\(^1\)

- The common three-surface \( \Sigma \) of \( \mathcal{I}^+ \) of the past eon and the Big Bang of the present eon is equipped with a conformal class \([h_0]\) of signature \((+,+,+)\) which has vanishing Cotton tensor, i.e. \([h_0]\) is conformally flat; in the following we will chose \( h_0 \) to be the flat representative of the conformal class \([h_0]\);

\(^1\)We closely follow Paul Tod’s setting, notation and terminology as presented in [5].

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• The Poincaré-type extension \((\hat{M}, \hat{g})\), with \(\hat{M} = ]0, \epsilon[ \times \Sigma\), of the conformal
three-manifold \((\Sigma, [h_0])\) has the Lorentzian metric \([1]\):
\[
\hat{g} = t^{-2}(-dt^2 + h_t);
\]
Here \(t \in ]0, \epsilon[\) is a coordinate along the extension \(\mathbb{R}_+\) of \(\Sigma\) in \(\hat{M}\),
and \(h_t = h(t, x)\) is a \(t\)-dependent 1-parameter family of metrics on \(\Sigma\) such that \(h(0, x) = h_0 \in [h_0]\); here \(x\) is a point in \(\Sigma, x \in \Sigma\);
• The Poincaré-type metric \(\hat{g}\) in \(\hat{M}\) satisfies the pure radiation Einstein’s
equations with cosmological constant \(\Lambda\):
\[
\hat{R}^{ij} = \hat{\Lambda}\hat{g}^{ij} + \hat{\Phi}K^iK^j;
\]
Here \(K^i\) is an expanding null vector field without shear and without twist
on \(M\); in particular we have \(\hat{g}_{ij}K^iK^j = 0\);
• The Lorentzian four-metric
\[
g = -dt^2 + h_t
\]
conformal to the Poincaré-type metric \(\hat{g}\) in \(\hat{M}\) is naturally extended to \(M = \hat{M} \cup \Sigma \cup \hat{M}\),
which is a bundle \(\Sigma \to M \xrightarrow{\pi} I\) over the interval \(I = ]-\epsilon, \epsilon[ \subset \mathbb{R}\)
parameterized by \(t\), with the following preimages of \(I\): \(\pi^{-1}(t > 0) = \hat{M}\),
\(\pi^{-1}(t = 0) = \Sigma\), and \(\pi^{-1}(t < 0) = \hat{M}\);
• The metric \(g\) is used to define a Lorentzian metric \(\hat{g}\) in \(\hat{M}\), which is
\[
\hat{g} = t^2(-dt^2 + h_t), \quad \text{for} \quad t < 0;
\]
• Note that for \(t > 0\) we have \(\hat{g} = \hat{\Omega}^2g\) and that for \(t < 0\) we have \(\hat{g} = \hat{\Omega}^2g\)
with \(\hat{\Omega} = -\hat{\Omega}^{-1} = t\).

One of the aims of this note is to identify the above four-manifold \(M\), equipped
with the three Lorentzian metrics \(\hat{g}, g\) and \(\hat{g}\), with the bandage region \([5]\) of the
Penrose’s cyclic Universe \([5]\) in which the past eon ends as filled with only one spherical wave, propagating along the null vector \(K^i\). Forcing the Poincaré-type
expansion metric \(\hat{g}\) to satisfy the Einstein’s equations \([1, 2]\) is the first step to achieve
this aim. Another aim is to see how the wave contained in the past eon will change
into a matter content at the beginning of the present eon, by means of Penrose’s
reciprocal hypothesis, stating that the three metrics \(\hat{g}, g\) and \(\hat{g}\) in the bandage region
should be related via: \(\hat{g} = \Omega^{-2}g\) and \(\hat{g} = \hat{\Omega}^2g\).

As we show below, under further simplification assumptions, the explicit form
of the Poincaré-type metric \(\hat{g}\) can be easily found up to an arbitrarily prescribed
accuracy, and as a byproduct one gets a remarkably pleasant consequences of the so
obtained \(\hat{g}\) for \(\hat{g}\), and in particular for the matter content of the spacetime \((\hat{M}, \hat{g})\),
which is interpreted as the beginning of the present eon.

2. The Ansatz and the Model for the Past Eon

In the theory of conformal invariants as presented by Fefferman and Graham in
\([1]\), given a conformal class \([h]\) on \(\Sigma\), one obtains the system of conformal invariants
of \([h]\) in terms of the (pseudo)Riemannian invariants of a certain (pseudo)Riemannian
metric \(\hat{g}\). This metric is naturally associated with the conformal class \([h]\) via

\[\text{We do not specify what kind of wave it is: it may consists of the incoherent superposition of waves representing directed massless radiation with random phases and polarizations, but the same propagation direction } K^i.\]
\[ \hat{g} = t^{-2}(\varepsilon dt^2 + h_t), \] where \( h_t \) is a 1-parameter family of metrics on \( \Sigma \), such that \( h_0 = h \) and \( h \) is a representative of \([h]\). The metric \( \hat{g} \) is defined for \( t > 0 \) and the value \( \varepsilon = 1 \) is chosen. To encode the conformal properties of \([h]\), this metric is demanded to be unique. This is done by the requirements that \( \hat{g} \) is Einstein, \( \hat{Ric}(\hat{g}) = \hat{\Lambda} \hat{g} \), and that \( h_t \) is real analytic and symmetric in \( t \).

We present here a milder version of this construction, applied to the conformally flat Riemannian structure \((\Sigma, [h])\) on a three-dimensional \( \Sigma \), to obtain the metric \( \hat{g} \) of the past eon with a desirable physical properties. In our ‘milder version’ we do as follows:

(a) We replace \( \varepsilon = 1 \) by \( \varepsilon = -1 \) - to have the Lorentzian signature of \( \hat{g} \);
(b) We replace the Einstein condition \( \hat{Ric}(\hat{g}) = \hat{\Lambda} \hat{g} \) by the Einstein equation (1.2) - to have the past eon filled by a spherical wave;
(c) And we drop the condition that \( h_t \) is symmetric in the variable \( t \) - to have more flexibility on the matter content of the present eon \( \hat{M} \).

Since in our milder-than-Fefferman-Graham-setting we do not have uniqueness theorems as in [1] the outcome past eon metric \( \hat{g} \) is not rigidly constrained. Thus, instead of working with the most general form of the the 1-parameter family of metrics \( h_t \) we make a physically motivated ansatz for them, hoping that it is compatible with the Einstein’s equations (1.2).

We start with a conformal class \([h_0]\) represented by the flat 3-dimensional metric

\[ h_0 = \frac{2r^2dzd\bar{z}}{(1 + \frac{z^2}{2})^2} + dr^2. \]

Then as \( h_t \) we take the spherically symmetric 1-parameter family

\[ h_t = \frac{2r^2(1 + \nu(t,r))dzd\bar{z}}{(1 + \frac{z^2}{2})^2} + (1 + \mu(t,r))dr^2, \]

where the unknown function \( \nu = \nu(t,r) \) and \( \mu = \mu(t,r) \) are both real analytic in the variable \( t \) and such that:

\[ \nu(0,r) = 0 \quad \text{and} \quad \mu(0,r) = 0. \]

This obviously satisfies \( h_{t=0} = h_0 \) and because of the analyticity assumption we have

\[ \nu(t,r) = \sum_{i=1}^{\infty} a_i(t) t^i \quad \text{and} \quad \mu(t,r) = \sum_{i=1}^{\infty} b_i(t) t^i, \]

with a set of differentiable functions \( a_i = a_i(r) \) and \( b_i = b_i(r) \) depending on the \( r \) variable only.

This leads to the following ansatz for the pre-Poincaré-type metric \( \hat{g} \) in \( \hat{M} \):

\[ \hat{g} = t^{-2} \left( -dt^2 + \frac{2r^2(1 + \sum_{i=1}^{\infty} a_i(t) t^i)dzd\bar{z}}{(1 + \frac{z^2}{2})^2} + (1 + \sum_{i=1}^{\infty} b_i(t) t^i)dr^2 \right). \]

Our (pre)past eon manifold \( \hat{M} \) is parameterized by \( t > 0, r > 0 \) and \( z \in \mathbb{C} \cup \{\infty\} \).

We now consider the following null vector field \( K \) on \( \hat{M} \):

\[ K = \partial_t + \left( 1 + \sum_{i=1}^{\infty} b_i(t) t^i \right)^{-\frac{1}{2}} \partial_r. \]

It is tangent to a congruence of null geodesics without shear and twist, which represents light rays emanating from the source at the surface \( r = 0 \). We require that
the Poincaré-type metric (2.1) satisfies the Einstein equations (1.2) with this null vector field $K$ and some functions $\hat{\Phi}$ and $\hat{\Lambda}$. We have the following theorem.

**Theorem 1.**

If the metric

\[
\hat{g} = t^{-2}(-dt^2 + h_t) = 
\]

\[
t^{-2} \left( -dt^2 + \frac{2r^2 \left( 1 + \frac{r}{2} \right) }{(1 + \frac{r}{2})^2} d\bar{z} \bar{z} + (1 + \mu(t, r))dr^2 \right) = 
\]

\[
t^{-2} \left( -dt^2 + \frac{2r^2 \left( 1 + \sum_{i=1}^{\infty} a_i(t) t_i \right) }{(1 + \frac{r}{2})^2} d\bar{z} \bar{z} + (1 + \sum_{i=1}^{\infty} b_i(t) t_i) dr^2 \right) 
\]

satisfies Einstein’s equations

\[
\hat{E}_{ij} := \hat{R}_{ij} - \hat{\Lambda} \hat{g}_{ij} - \hat{\Phi} \hat{K}_i \hat{K}_j = 0 
\]

with

\[
K = K^i \partial_i = \partial_t + \left( 1 + \sum_{i=1}^{\infty} b_i(t) t_i \right) \frac{1}{2} \partial_r, \quad \hat{K}_i = \hat{g}_{ij} \hat{K}_j, 
\]

then we have:

- The coefficients $a_1(r)$, $a_2(r)$ $b_1(r)$ and $b_2(r)$ identically vanish, $a_1(r) = a_2(r) = b_1(r) = b_2(r) = 0$, and the power series expansion of $h_t$ starts at the $t^3$ terms, $h_t = t^3 \chi(r) + O(t^4)$.
- The metric $\hat{g}$, or what is the same, the power series expansions $\nu(t, r) = \sum_{i=1}^{\infty} a_i(r) t^i$ and $\mu(t, r) = \sum_{i=1}^{\infty} b_i(r) t^i$, are totally determined up to infinite order by an arbitrary differentiable function $f = f(r)$.
- More precisely, the Einstein equations $\hat{E}_{ij} = \mathcal{O}(t^{k+1})$ solved up to an order $k$, together with an arbitrary differentiable function $f = f(r)$, uniquely determine $\nu(t, r)$ and $\mu(t, r)$ up to the order $(k + 2)$.
- In the lowest order the solution reads:

\[
\nu = \frac{f}{r^3} t^3 + \mathcal{O}(t^4) \quad \text{and} \quad \mu = -\frac{2f}{r^3} t^3 + \mathcal{O}(t^4);
\]

The energy function $\hat{\Phi}$ and the cosmological constant $\hat{\Lambda}$ are:

\[
\hat{\Phi} = 3 \frac{f'}{r^3} t^6 + \mathcal{O}(t^7) \quad \text{and} \quad \hat{\Lambda} = 3 + \mathcal{O}(t^4);
\]

the Weyl tensor of the solution is

\[
\hat{W}^i{}_j{}^{kl} = \mathcal{O}(t).
\]

In particular, the Weyl tensor $\hat{W}^i{}_j{}^{kl}$ vanishes at $t = 0$ and $\hat{\Lambda} > 0$ there.

With the use of computers we calculated this solution up to the order $k = 10$, finding explicitly $\nu = \sum_{k=3}^{10} a_k t^k$ and $\mu = \sum_{k=3}^{10} b_k t^k$. The formulas are compact...
enough up to $k = 8$ and up to the order $k = 8$ they read:

$$
\nu(t, r) = f^4 \frac{t^4}{r^4} - \frac{3}{4} f^2 \frac{t^4}{r^4} + \frac{1}{10} (-2 r f' + 3 r^2 f') \frac{t^4}{r^4} +
\frac{1}{34} (3 f^2 - 3 r f' + 3 r^2 f'' - 2 r^3 f^{(3)}) \frac{t^4}{r^4} +
\frac{r}{280} (1 - 24 f' - 105 f f' + 24 r f'' - 24 r^2 f^{(3)} + 5 r^3 f^{(4)}) \frac{t^4}{r^4} -
\frac{r}{960} (60 f' + 288 f f' - 150 r f'' - 60 r f'' - 216 r f f'' + 30 r^2 f^{(3)} - 10 r^3 f^{(4)} + 3 r^4 f^{(5)}) \frac{t^4}{r^4} +
\mathcal{O}(\left(\frac{t}{r}\right)^9)
$$

$$
\mu(t, r) = -2 f^4 \frac{t^4}{r^4} + \frac{3}{4} f^2 \frac{t^4}{r^4} - \frac{5}{4} f'' \frac{t^4}{r^4} + \frac{1}{24} (39 f^2 + r^3 f^{(3)}) \frac{t^4}{r^4} - \frac{r}{280} (390 f f' + 2 r^3 f^{(4)}) \frac{t^4}{r^4} +
\frac{r}{960} (-18 f f' + 300 r f'' + 378 r f f'' + r^4 f^{(5)}) \frac{t^4}{r^4} + \mathcal{O}(\left(\frac{t}{r}\right)^9).
$$

For a solution up to this order we find that:

$$
\Phi = 3 r^3 f'' \frac{t^4}{r^4} + 3 r^3 (f' - r f'') \frac{t^4}{r^4} + \frac{3}{2} \frac{r^3}{2 r} (2 f' - 2 r f'' + r^2 f^{(3)}) \frac{t^4}{r^4} +
\frac{r^3}{2 r} (6 f' + 6 f'' - 6 r f'' + 3 r^2 f^{(3)} - r^3 f^{(4)}) \frac{t^4}{r^4} +
\frac{r^3}{2 r} (24 f' + 60 f'' - 12 r f'' - 24 r f'' - 30 r f'' + 12 r^2 f^{(3)} - 4 r^3 f^{(4)} + r^4 f^{(5)}) \frac{t^4}{r^4} +
\frac{r^3}{2 r} (120 f' + 522 f'' - 177 r f'' - 120 r f'' + 378 r f f'' + 93 r^2 f' f'' + 60 r^2 f^{(3)} + 92 r^2 f^{(3)} + 20 r^3 f^{(4)} + 5 r^4 f^{(5)} - r^5 f^{(6)}) \frac{t^4}{r^4} +
\mathcal{O}(\left(\frac{t}{r}\right)^{12}),
$$

$$
\hat{\Lambda} = 3 + \mathcal{O}(t^9).
$$

I have no patience to type the Weyl tensor components up to higher order. It is enough to say that up to the 4th order in $t$, modulo a nonzero constant tensor $C^i_{jkl}$, it is equal to:

$$
\hat{W}^i_{jkl} = \left(\frac{f}{r^2} \frac{t}{r} - \frac{f'}{r} \frac{t^2}{r^2} + \frac{f''}{2} \frac{t^3}{r^3}\right) C^i_{jkl} + \mathcal{O}\left(\frac{t}{r}\right)^4.
$$

Of course, for the positivity of the energy density $\hat{\Phi}$ close to the surface $\mathcal{I}^+$ of $M$ we need

$$f' > 0.$$

**Remark 2.1.** We were unable to find a recurrence relation for the functions $a_i(r)$ and $b_i(r)$ for arbitrary $i > 10$. We nevertheless claim that such relations do exist and that the corresponding power series are convergent. The reason for these claims is that our solution for $\hat{g}$ is a pure radiation Einstein metric with cosmological constant, which have a sherafree expanding but notwisting congruence of null geodesics which is tangent to the wave propagation vector $K^i$. All such solutions of Einstein’s equations are known. They belong to the Robinson-Trautman class of solutions described e.g. in Chapter 28.4 of Ref. [2]. Our solution is the spherically symmetric solution from this class [6], and can be written in terms of the Robinson-Trautman coordinates [4] as:

$$
(2.5) \quad \hat{g} = \frac{2 v^2 d\zeta d\bar{\zeta}}{(1 + \frac{1}{2} \zeta \bar{\zeta})^2} - 2 dv (dv + \left(1 - \frac{2m(u)}{v} - \frac{1}{2} \Lambda v^2\right) du).
$$

The trouble is that, because of the appearance of the free function $m = m(u)$ in (2.5), there is no an easy way of getting the explicit coordinate transformation from the Robinson-Trautman null coordinates $(\zeta, \bar{\zeta}, v, u)$ to our coordinates $(z, \bar{z}, r, t)$. Such transformation would bring $\hat{g}$ as in (2.5) to ours $\hat{g}$ from (2.1) in which all the
coefficients $a_i$ and $b_i$ are determined up to infinite order. Anyhow, knowing this transformation or not, the geometric features of our solution with all $a_i$s and $b_i$s determined for $i \to \infty$, show that our $\hat{g}$ must be identified with the Vaidya solution (2.5). Thus not only the coefficients $a_i$ and $b_i$ in our solution are determined up to infinite order, but also the power series defining our $\hat{g}$ converges to $\hat{g}$ given by (2.5).

**Corollary 2.2.** The Poincaré-type metric (2.2) can be interpreted as the ending stage of the evolution of the past eon in Penrose’s CCC. The eon has a positive cosmological constant $\hat{\Lambda} \simeq 3$, which is filled with a spherically symmetric pure radiation moving along the null congruence generated by the vector field $K$.

3. Using reciprocity for the model of the present eon

Now, following the Penrose-Tod reciprocal hypothesis procedure, we summarize the properties of the spacetime $\hat{\mathcal{M}}$ equipped with the metric $\hat{\mathcal{g}}$ obtained from $\hat{g}$ as in Theorem 1, by the reciprocal change $\hat{\Omega} \to -\hat{\Omega}^{-1} = \Omega$. In other words, we are now interested in the properties of the metric $\mathcal{g} = t^{-2} \hat{g}$. We have the following theorem.

**Theorem 2.**
Assume that the metric $\hat{g}$ as in (2.2) satisfies the Einstein equations (2.3)-(2.4), $\hat{E}_{ij} = 0$. Then, the reciprocal metric

$$
\mathcal{g} = t^2 \left( -dt^2 + \frac{2r^2 (1 + \nu(t, r)) dz d\bar{z}}{(1 + \frac{\nu}{2})^2} + (1 + \mu(t, r)) dr^2 \right) = 
$$

satisfies the Einstein equations

$$
\mathcal{E}_{ij} = \mathcal{K}_{ij} - \mathcal{P}_{ij} (\mathcal{F} + \mathcal{P} \eta_{ij} - \frac{1}{2} (\mathcal{F} - \mathcal{P}) \mathcal{u}_i \mathcal{u}_j = 0.
$$

Here $\mathcal{K}_i$ and $\mathcal{L}_i$ are the null 1-forms corresponding to the pair of outgoing-ingoing null vector fields

$$
K = K^i \partial_i = \partial_t + \left( 1 + \sum_{i=1}^{\infty} b_i(r) t^i \right) ^{\frac{1}{2}} \partial_r \quad \text{and} \quad L = L^i \partial_i = \partial_t - \left( 1 + \sum_{i=1}^{\infty} b_i(r) t^i \right) ^{\frac{1}{2}} \partial_r,
$$

via $\hat{K}_i = \hat{g}_{ij} K^j$ and $\hat{L} = \hat{g}_{ij} L^j$, and the 1-form vector field $\mathcal{u}_i$ corresponds to the future oriented timelike unit vector field

$$
\mathcal{u} = \hat{u}^{-1} \partial_t = -t^{-1} \partial_t,
$$

via $\mathcal{u}_i = \hat{g}_{ij} \hat{u}^j$.

Before giving the explicit formulas for the power expansions of functions $\hat{\Phi}$, $\hat{\Psi}$, $\hat{\rho}$ and $\hat{\phi}$ appearing in this theorem, we make the following remark.

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$^3$Note that now $t < 0$ (!)
Remark 3.1. The Einstein equations (3.1) are equations with an energy momentum tensor consisting of radiation propagating with spherical fronts outward (along \(K\)) and inward (along \(L\)); it also consists of a perfect fluid comoving with the present eon’s cosmological time \(T = \int f \, dt\). Each front of the spherical wave present in the past eon that reached the \(t = 0\) surface in the present eon produces (i) a spherical outward wave with energy density \(\Phi\) going along \(K\) out of this sphere, (ii) a spherical inward wave with energy density \(\Psi\) going along \(L\) towards the center of this sphere, and (iii) a portion of a perfect fluid with energy density \(\dot{\rho}\) and isotropic pressure \(\ddot{p}\).

For the solutions \(\nu(t, r), \mu(t, r)\) of the past eon’s Einstein’s equations (2.3)-(2.4), which were given in terms of the power series expansions as \(\nu(t, r) = \sum_{i=3}^{k+2} a_i(r)t^i + O(t^{k+3})\) and \(\mu(t, r) = \sum_{i=3}^{k+2} b_i(r)t^i + O(t^{k+3})\) in Theorem 1, the formulae for the power series expansions of the energy densities \(\Phi, \Psi, \dot{\rho}\) and the pressure \(\ddot{p}\) are as follows:

\[
\Phi = -\frac{9f}{r^3}t^{-3} + \frac{9f'}{r^3}t^{-2} + \frac{1}{2r^4}(8f' - 11rf'')t + \frac{3}{4r^5}(5f' - 5rf'' + 3r^2f^{(3)}) + \frac{9}{40r^6}(16f' + 5ff' - 16rf'' + 8r^2f^{(3)} - 3r^3f^{(4)})t + \frac{1}{120r^7}(420f' + 1068ff' - 30rf^2 - 420rf'' - 384rf f'' + 210rf^{(3)} - 70r^3f^{(4)} + 19r^4f^{(5)})t^2 + \cdots + O(t^{k-3}),
\]

\[
\Psi = -\frac{9f}{r^3}t^{-3} + \frac{6f'}{r^3}t^{-2} + \frac{1}{2r^4}(2f' - 5rf'')t^{-1} + \frac{3}{4r^5}(f' - r f'' + r^2f^{(3)}) + \frac{1}{40r^6}(24f' - 75ff' - 24rf'' + 12r^2f^{(3)} - 7r^3f^{(4)})t + \frac{1}{60r^7}(30f' + 39ff' + 75r f^2 - 30rf'' + 33rf f'' + 15r^2f^{(3)} - 5r^3f^{(4)} + 2r^4f^{(5)})t^2 + \cdots + O(t^{k-3}),
\]

\[
\dot{\rho} = 3t^{-4} + \frac{18f}{r^3}t^{-1} - \frac{18f'}{r^3} + \frac{-6f' + 9rf''}{r^4}t - \frac{3}{4r^5}(9f^2 + 3rf' - 3r^2f'' + 2r^3f^{(3)})t^2 + \frac{3}{20r^6}(-24f' + 105ff' + 24rf'' - 12r^2f^{(3)} + 5r^3f^{(4)})t^3 - \frac{1}{20r^7}(60f' + 96ff' + 120rf^2 - 60rf'' + 72rf f'' + 30rf^{(3)} - 10r^3f^{(4)} + 3r^4f^{(5)})t^4 + \cdots + O(t^{k-1}),
\]

\[
\ddot{p} = t^{-4} + \frac{6f}{r^3}t^{-1} + \frac{1}{r^2}(2f' - rf'')t + \frac{1}{2r^4}(18f^2 + 3rf' - 3r^2f'' + r^3f^{(3)})t^2 - \frac{3}{20r^6}(-8f' + 45ff' + 8rf'' - 4r^2f^{(3)} + r^3f^{(4)})t^3 + \frac{1}{30r^7}(30f' + 57ff' + 45rf^2 - 30rf'' + 39rf f'' + 15r^2f^{(3)} - 5r^3f^{(4)} + r^4f^{(5)})t^4 + \cdots + O(t^{k-1}).
\]
In these formulas all the dotted terms are explicitly determined in terms of $f$ and its derivatives (I was lazy, and typed only the terms adapted to the choice $k = 6$ in Theorem 1).

The following remarks are in order:

**Remark 3.2.**

- Note that since in $\tilde{M}$ the time $t < 0$, then the requirement that the energy densities are positive near the Big Bang hypersurface $t = 0$ implies that
  
  \[ f > 0 \]

  in addition to $f' > 0$, which was the requirement we got from the past con. Indeed, the leading terms in $\Phi$ and $\Psi$ are $\Phi = \Psi = -\frac{3}{2} f t^{-3}$, hence $\Phi$ and $\Psi$ are both positive in the regime $t \to 0^-$ provided that $f > 0$. Note also that $f > 0$ and $f' > 0$ are the only conditions needed for the positivity of energy densities, as the leading term in $\tilde{\rho}$ is $\tilde{\rho} \simeq 3 t^{-4}$, and is positive regardless of the sign of $t$.

- Remarkably the leading terms in $\tilde{\rho}$ and $\tilde{p}$, i.e. the terms with negative powers in $t$, are proportional to each other with the numerical factor three. We have
  
  \[ \tilde{p} = \frac{1}{3} \tilde{\rho} + O(t^0). \]

  This means that immediately after the Bang, apart from the matter content of two spherical ingoing and outgoing waves in the new con, there is also a scattered radiation there, described by the perfect fluid with $\tilde{p} = \frac{1}{3} \tilde{\rho}$.

- So what the Penrose-Tod scenario does to the new con out of a single spherical wave in the past con, is it splits this wave into three portions of radiation: the two spherical waves, one which is a dumped continuation from the previous con, the other that is focusing in the new con, as it encountered a mirror at the Bang surface, and in addition a lump of scattered radiation described by the statistical physics.

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