ON BANACH SPACES WITHOUT
THE APPROXIMATION PROPERTY

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1. If $X$ is a Banach space of type $p$ and of cotype $q$, then every its $n$-dimensional subspace is $Cn^{1/p-1/q}$-complemented in $X$ (cf. [1]). Szankowski [2] has showed that if $T(X) = \sup \{ p : X$ of type $p \} \neq 2$ or $C(X) = \inf \{ q : X$ of cotype $q \} \neq 2$, then $X$ has a subspace without the approximation property. Thus, if each subspace of $X$ possesses the approximation property, then necessarily $T(X) = C(X) = 2$ and, therefore, all of finite dimensional subspaces in $X$ are ”well” complemented. Moreover, the space $X$ need not be Hilbertian (or isomorphic to a Hilbert space). The examples of such spaces were constructed by Johnson [3].

In connection with the examples of Szankowski and Johnson, the natural questions arises: 1) if $T(X) = C(X) = 2$, then is it true that every subspace in $X$ has the approximation property? 2) how ”well complemented” may be the finite dimensional subspaces of a space without the approximation property: in particular, does there exist a space $X$ without the approximation property, the constants $C$ and $A$ such that each $n$-dimensional subspace of $X$ is $C \log^A n$-complemented? This note is devoted to the answers to these questions (negative for the first one and positive for the second).

2. Our example is based entirely on the construction of Szankowski [2]. Let $I_n = \{ 2^n + 1, \ldots, 2^{n+1} \}$; $\{ e_k \}_{1}^{\infty}$ be the standard basis in $c_0$; $\{ e'_k \}_{1}^{\infty}$ be a sequence of the corresponding coordinate functionals. Further, let $z_i = e_{2i} - e_{2i+1} + e_{4i} + e_{4i+1} + e_{4i+2} + e_{4i+3}$ and $z'_i = 2^{-1}(e'_{2i} - e'_{2i+1})$. Denote by $W$ the linear span of the set $\{ z_i \}$ and put, for linear maps $U : W \to W$ and for $n = 1, 2, \ldots$, $\beta_n(U) = 2^{-n} \sum I_n \langle z'_i, Uz_i \rangle$.

In [2], there were constructed, for $n = 1, 2, \ldots$, the partitions $\Delta_n$ and $\nabla_n$ of the set $I_n$ and finite scalar sequences $y_j \in c_0 (j = 1, 2, \ldots)$ with the following properties:

a) $\beta_n(U) - \beta_{n-1}(U) = 2^{-n-1} \sum I_n \langle e'_i, y_i \rangle$;

b) if $B_1, B_2 \in \nabla_n$, then $\overline{B_1 \cup B_2} = m_n \geq C 2^{n/8}$; hence, $\overline{\nabla} = 2^n / m_n$;

c) if $A \in \Delta_n, B \in \nabla_n$, then $\overline{B \cap A} \leq 1, \overline{A} \geq C 2^{n/8}$;

d) if $B \in \nabla_n$, then $\sum_{i \in B} \varepsilon_i y_i = \sum_{j=1}^{10} (\pm \sum_{i \in A_j} \varepsilon_i e_i)$, where $\varepsilon_i = \pm 1$; $A_j (j = 1, \ldots, 10)$ are subsets of some elements from $\Delta_n - 1, \Delta_n$ and $\Delta_n + 1$ and, moreover, $\overline{A_j} = \overline{B}$.

It follows from a) and b) that

e) $\beta_n(U) - \beta_{n-1}(U) = 2^{-n-1} \sum B \in \nabla_n, 2^{-m_n} \sum \varepsilon \sum_{i \in B} \varepsilon_i e'_i \sum_{i \in B} \varepsilon_i y_i$, where $\sum \varepsilon$ denotes the summing over all collections of the signs $\varepsilon_i = \pm 1$ ($i \in B$).

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† Reinov O.I., *On Banach spaces without the approximation property*, Funkts. analiz i ego prilozhen. (1982), v. 16, vyp. 4, p. 84-85 (in Russian).
3. An example of a space without the approximation property which has good enough finite dimensional subspaces will be found among the subspaces of the space $Y = \left( \sum_{n} \left( \sum_{A \in \Delta_n} \frac{1}{p_n} \right)_{l^p_n} \right)_{l^2}$, where the numbers $p_n$ are defined from the relations

$1/p_n - 1/2 = n^{-1} \log_n n^{1+\varepsilon}$ ($\varepsilon > 0, a = \sqrt{2}$). It is clear that $T(Y) = C(Y) = 2$. Let $X_\varepsilon$ be the closure in $Y$ of the linear subspace $W$.

**Theorem.** The space $X_\varepsilon$ does not possess the approximation property. There exists a constant $C_\varepsilon > 0$ such that if $E$ is an $n$-dimensional subspace of $X_\varepsilon$, then 1) $d(E, l_n^2) \leq C_\varepsilon \log^{1+\varepsilon} n$ and 2) $E$ is $C_\varepsilon \log^{1+\varepsilon} n$-complemented in $Y$ (and, hence, in $X_\varepsilon$).

4. **Proof.** If $B \in \nabla_n$ then, for $\varepsilon_i = \pm 1$, we get from b), c) and d):

$$f) \left\| \sum_{i \in B} \varepsilon_i e_i' \right\|_{Y^*} \leq \left( \sum_{A \in \Delta_n} (A \cap B)_{p_n'/2} \right)^{1/p_n'} \leq \left\{ A \in \Delta_n : A \cap B \neq \emptyset \right\}^{1/p_n'} \leq B^{1/p_n'} = m_n^{1/p_n'},$$

$$g) \left\| \sum_{i \in B} \varepsilon_i y_i \right\|_Y \leq 10 \max_{1 \leq j \leq 10} \left\| \sum_{i \in A_j} \varepsilon_i e_i \right\| \leq 10 \max_{1 \leq j \leq 10} \left( \frac{A_j}{A} \right)^{1/2} \leq 10 m_n^{1/2}.$$

Thus, it follows from e)

$$h) \left| \beta_n(U) - \beta_{n-1}(U) \right| \leq 2^{-n-1} \nabla_n m_n^{1/p_n'} \max \left\{ \left\| U \left( \sum_{i \in B} \varepsilon_i y_i \right) \right\| : \varepsilon_i = \pm 1, B \in \nabla_n \right\}.$$

Set $F_n = \left\{ m_n^{-1/p_n} \sum_{i \in B} \varepsilon_i y_i : \varepsilon_i = \pm 1, B \in \nabla_n \right\}$. Then $\left| \beta_n(U) - \beta_{n-1}(U) \right| \leq 2^{-1} \max \left\{ \| Uf \| : f \in F_n \right\}$ and, moreover,

$$\alpha_n = \max \left\{ \| f \| : f \in F_n \right\} \leq m_n^{-1/p_n'} \cdot 10 m_n^{1/2} \leq 10(C2^{n/8})^{-n^{-1} \log n^{1+\varepsilon}} \leq \text{const} \cdot n^{-1-\varepsilon}.$$

So we are in conditions of Lemma 1 of [2] and, therefore, the space $X_\varepsilon$ does not have the approximation property.

For $T \in L(X, Z)$, let us put $\gamma_2(T) = \inf \{ \| A \| \| B \| : T = BA : X \to l^2 \to Z \}$. Let

$$Y_N = \left( \sum_{n \leq N} \left( \sum_{A \in \Delta_n} l_{1/n}^2 \right)_{l^p_n} \right)_{l^2} \quad \text{and} \quad Y^N = \left( \sum_{n > N} \left( \sum_{A \in \Delta_n} l_{1/n}^2 \right)_{l^p_n} \right)_{l^2}.$$

Let $P_N$ and $P^N$ be the natural projections from $Y$ onto $Y_N$ and $Y^N$ respectively. Since $\Delta_n \leq C2^{n/8}$, then

$$d \left( \left( \sum_{A \in \Delta_n} l_{1/n}^2 \right)_{l^p_n}, l_{2n}^2 \right) \leq (\Delta_n)^{1/p_n - 1/2} \leq C_1 n^{1+\varepsilon},$$
Whence, the Banach-Mazur distance from $Y_N$ to an euclidean space does not exceed $C_2 N^{1+\varepsilon}$.

Let now $E$ be an arbitrary $2^N$-dimensional subspace of $Y$; $E_N = P_{8N}(E)$ and $E^N = P^{8N}(E)$. Because of above arguments, there exists a projection $P_1$ from $Y_{8N}$ onto $E_N$, and moreover $\gamma_2(P_1) \leq C_3 N^{1+\varepsilon}$. On the other hand, $Y^{8N}$ is a space of cotype 2 and of type $p_{8N}$; therefore, there exists a projection $P_2$ from $Y^{8N}$ onto $E^N$ such that $\gamma_2(P_2) \leq C_4 (\dim E^N)^{1/p_{8N}-1/2} \leq C_5 N^{1+\varepsilon}$. Put $P_0 = P_1 P_{8N} + P_2 P^{8N}$ and $E_0 = P_0(Y)$. Then $P_0$ is a projection from $Y$ onto $E_0$, and $\gamma_2(P_0) \leq C_6 N^{1+\varepsilon}$. Hence, firstly, there exists a projection from $Y$ onto $E$ with the norm $\leq C_6 N^{1+\varepsilon}$ and, secondly, $d(E, l_2^{2N}) \leq C_6 N^{1+\varepsilon}$.

5. If we define $\{p_n\}$ from the relations $p_n^{-1} - 2^{-1} = n^{-1} \log_a n^{1+\varepsilon_n}$, where $\varepsilon_n \to 0$ and the series $\sum n^{-1-\varepsilon_n}$ is convergent, then the same arguments show that there exists a Banach space $X$ without the approximation property and such that if $\varepsilon > 0$ and $E$ is a $n$-dimensional subspace in $X$, then there is a projection $P$ from $X$ onto $E$ such that $\gamma_2(P) \leq C_7 \log^{1+\varepsilon} n$. In particular, one can take, for each $\delta > 0$, $n^{\varepsilon_n} = (\log n)(\log \log n) \ldots (\log^{1+\delta} \log \ldots \log n)$. Naturally, a question arises: does there exist a Banach space without the approximation property, all $n$-dimensional subspaces of which are $C \log n$-complemented? $C \log n$-euclidean?

References

1. Pisier G., *Estimations des distances à un espace euclidien et des constantes de projection des espace de Banach de dimensions finie*, Seminaire d'analyse fonctionnelle 1978–1979 (1979), exp. 10, 1–21.
2. A. Szankowski, *Subspaces without approximation property*, Israel J. Math. 30 (1978), 123–130.
3. Johnson W.B., *Banach spaces all of whose subspaces have the approximation property*, Seminaire d'analyse fonctionnelle 1979–1980 (1980), exp. 16, 1–11.

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