Quantum $f$-divergences in von Neumann algebras I.
Standard $f$-divergences

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Dedicated to the memory of Dénes Petz

Abstract

We make a systematic study of standard $f$-divergences in general von Neumann algebras. An important ingredient of our study is to extend Kosaki’s variational expression of the relative entropy to an arbitrary standard $f$-divergence, from which most of the important properties of standard $f$-divergences follow immediately. In a similar manner we give a comprehensive exposition on the Rényi divergence in von Neumann algebra. Some results on relative hamiltonians formerly studied by Araki and Donald are improved as a by-product.

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1 Introduction

The notion of quantum divergences is among the most significant ones in quantum information theory, with various applications, in particular, to defining important quantum quantities to discriminate between states of a quantum system. A quantum system is mathematically described, in most cases, by an operator algebra $A$ on a Hilbert space (either finite-dimensional or infinite-dimensional), and a quantum divergence is generally given as a function $S(\rho||\sigma)$ of two states (or more generally, two positive linear functionals) $\rho$ and $\sigma$ on $A$. Among various quantum divergences, the most notable is the relative entropy having a long history as the quantum version of the Kullback-Leibler divergence in classical theory. Indeed, the relative entropy $D(\rho||\sigma)$ was first introduced in 1962 by Umegaki for normal states $\rho, \sigma$ on a semifinite von Neumann algebra $M$ as follows:

$$D(\rho||\sigma) := \begin{cases} \tau(d_\rho(\log d_\rho - \log d_\sigma)) & \text{if } s(\rho) \leq s(\sigma), \\ +\infty & \text{otherwise}, \end{cases} \quad (1.1)$$

where $\tau$ is a semifinite trace on $M$, $d_\rho$ is the density operator of $\rho$ with respect to $\tau$ and $s(\rho)$ is the support projection of $\rho$. Later in 1970’s Araki extended Umegaki’s relative
entropy, by introducing the relative modular operator \( \Delta_{\rho,\sigma} \) for normal states \( \rho, \sigma \), to general von Neumann algebras as

\[
D(\rho\|\sigma) := \begin{cases} 
-\langle \xi_\rho, (\log \Delta_{\sigma,\rho}) \xi_\rho \rangle + \langle \xi_\sigma, (\Delta_{\rho,\sigma} \log \Delta_{\rho,\sigma}) \xi_\sigma \rangle & \text{if } s(\rho) \leq s(\sigma), \\
+\infty & \text{otherwise},
\end{cases}
\]

where \( \xi_\rho \) is the vector representative of \( \rho \) in the standard representation of \( M \) (see Section 2.1 below). A remarkable progress on the relative entropy was made when Kosaki \[32\] gave a variational expression of \( D(\rho\|\sigma) \) and showed that all important properties of \( D(\rho\|\sigma) \) immediately follow from the expression.

A more general form of quantum divergences was considered by Kosaki \[30\] to generalize the Wigner-Yanase-Dyson-Lieb concavity, and later discussed in more detail by Petz \[43, 44\] with name quasi-entropy. The reader can refer to \[22\] for details on the relative entropy and quasi-entropies. The standard \( f \)-divergences \( S_f(\rho\|\sigma) \) studied in, e.g., \[23, 22\] in the finite-dimensional case are a special case of quasi-entropies but a natural class of quantum divergences generalizing the classical \( f \)-divergences. A significant property satisfied by quantum divergences mentioned above is the monotonicity property, that is, the inequality

\[
S(\rho \circ \Phi \| \sigma \circ \Phi) \leq S(\rho \| \sigma)
\]

under positive linear maps \( \Phi : \mathcal{B} \to \mathcal{A} \) between operator algebras assumed to be unit-preserving and completely positive (or more weakly a Schwarz map). An important issue in connection with this property is to prove the reversibility of \( \Phi \), i.e., the existence of a recovery map \( \Psi : \mathcal{A} \to \mathcal{B} \) satisfying \( \rho \circ \Phi \circ \Psi = \rho \) and \( \sigma \circ \Phi \circ \Psi = \sigma \) under the equality condition in (1.3). In the special case where \( \mathcal{B} \) is a subalgebra of \( \mathcal{A} \) and \( \Phi \) is the injection, the reversibility of \( \Phi \) on \( \{\rho, \sigma\} \) is called the sufficiency of \( \mathcal{B} \) for \( \{\rho, \sigma\} \).

This line of research in von Neumann algebras was initiated by Petz, e.g., \[45, 46, 29\] for the relative entropy and the transition probability (i.e., the Rényi divergence with parameter 1/2). The extension of the reversibility to more general standard \( f \)-divergence, though in the finite-dimensional case, has been done in \[23, 26, 22\].

Our aim in this paper is to propose a new approach to the theory of standard \( f \)-divergences in general von Neumann algebras. For this, in Section 2 we first give the definition and some basic properties of the standard \( f \)-divergence \( S_f(\rho\|\sigma) \) of normal positive linear functionals \( \rho, \sigma \) on a von Neumann algebra, when \( f \) is a general convex function on \( (0, \infty) \). The idea of the definition is essentially the same as (1.2), based on the relative modular operator \( \Delta_{\rho,\sigma} \), but without any assumption on the boundary values of \( f(t) \) at \( t \) zero and infinity. For further discussions we assume that \( f \) is an operator convex function on \( (0, +\infty) \). In Section 3 we give a variational expression of \( S_f(\rho\|\sigma) \) by utilizing the integral expression of \( f \) and modifying Kosaki’s expression \[32\] of the relative entropy. Our variational expression is a bit different from Kosaki’s one even in the case of the relative entropy. Next in Section 4, we present various properties of \( S_f(\rho\|\sigma) \) such as monotonicity property in (1.3), joint lower semicontinuity, joint convexity, etc. as straightforward consequences from the variational expression. In this way, we can study the standard \( f \)-divergence in von Neumann algebras along a very streamlined track, which is a special feature of our presentation.

The quantum Rényi divergence \( D_\alpha(\rho\|\sigma) \) with parameter \( \alpha \in [0, +\infty) \setminus \{1\} \) is of quite use in quantum information as a quantum version of the classical Rényi divergence. In the finite-dimensional (or the matrix) case, the Rényi divergence is defined by

\[
D_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \frac{Q_\alpha(\rho\|\sigma)}{\text{Tr} \rho},
\]
where $Q_\alpha(\rho \| \sigma) := \text{Tr} (\rho^\alpha \sigma^{1-\alpha})$ that is essentially the standard $f$-divergence with $f(t) = t^\alpha$. So $D_\alpha$ is indeed a variant of standard $f$-divergences. In these years it has also been widely known that another type of quantum Rényi divergence, called the sandwiched Rényi divergence and denoted by $\tilde{D}_\alpha(\rho \| \sigma)$, is equally useful in quantum information, in particular, in quantum state discrimination, see \cite{85} for example. The definition of $\tilde{D}_\alpha$ is similar to $D_\alpha$ by replacing $Q_\alpha(\rho \| \sigma)$ with $\tilde{Q}_\alpha(\rho \| \sigma) := \text{Tr} \left((\sigma^{(1-\alpha)/2\alpha}\rho\sigma^{(1-\alpha)/2\alpha})^\alpha\right)$, though $\tilde{Q}_\alpha$ is no longer in the class of standard $f$-divergences. Moreover, the so-called $\alpha$-$z$-Rényi divergence introduced in \cite{8} is a two-parameter common generalization of $D_\alpha$ and $\tilde{D}_\alpha$. Motivated by the current situation of quantum Rényi divergences, the authors in \cite{10, 26, 27} have recently extended the sandwiched version $\tilde{D}_\alpha$ to the von Neumann algebra setting. Indeed, in these papers, the quantity $\tilde{Q}_\alpha$ is defined in von Neumann algebras by using Araki and Masuda’s $L^\alpha$-spaces and Haagerup’s $L^\alpha$-spaces \cite{10, 57}. Although those papers contain some discussions on $D_\alpha$ as well, it seems that the expositions on $D_\alpha$ there are not comprehensive. Thus, in Section 5 we present a thorough exposition on the Rényi divergence in von Neumann algebras, while it is more or less specialization of the results of Section 4.

This paper has two appendices. In Appendix A we give a brief survey on Haagerup’s $L^\alpha$-spaces and the description of the Rényi divergence in terms of them. In Appendix B we revisit the former results in \cite{2, 15} on relative hamiltonians and their relation to the relative entropy, and improve them based on Haagerup’s $L^\alpha$-spaces and the fact that $D = \lim_{\alpha \rightarrow 1} D_\alpha$.

## 2 Definition of standard $f$-divergences

### 2.1 Relative modular operators

Let $M$ be a general von Neumann algebra, and $M^+_s$ be the positive cone of the predual $M_*$ consisting of normal positive linear functionals on $M$. Throughout the paper, we consider $M$ in its standard form $(M, \mathcal{H}, J, \mathcal{P})$, that is, $M$ is represented on a Hilbert space $\mathcal{H}$ with a conjugate-linear involution $J$ and a self-dual cone $\mathcal{P}$ called the natural cone, for which the following hold:

1. $JMJ = M'$,
2. $JxJ = x^*$, $x \in M \cap M'$ (the center of $M$),
3. $J\xi = \xi$, $\xi \in \mathcal{P}$,
4. $xJx\mathcal{P} \subset \mathcal{P}$, $x \in M$.

Recall \cite{17} that any von Neumann algebra has a standard form, which is unique in the sense that if $(M, \mathcal{H}, J, \mathcal{P})$ and $(\tilde{M}, \tilde{\mathcal{H}}, \tilde{J}, \tilde{\mathcal{P}})$ are two standard forms and $\Phi : M \rightarrow \tilde{M}$ is a $*$-isomorphism, then there is a unique unitary $u : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$ such that $\Phi(x) = uxu^*$ for $x \in M$, $\tilde{J} = uJu^*$ and $\tilde{\mathcal{P}} = u\mathcal{P}$. See \cite{17} (also \cite{1, 12}) for more details on the standard form.

Every $\sigma \in M^+_s$ has a unique vector representative $\xi_\sigma \in \mathcal{P}$ so that $\sigma(x) = \langle \xi_\sigma, x\xi_\sigma \rangle$, $x \in M$. We have the support projections $s_M(\sigma) \in M$ and $s_{M'}(\sigma) \in M'$ of $\sigma$, that is, $s_M(\sigma)$ is the orthogonal projection onto $M\xi_\sigma$ and $s_{M'}(\sigma)$ is that onto $M'\xi_\sigma$. Note that $s_{M'}(\sigma) = Js_M(\sigma)J$.

For each $\rho, \sigma \in M^+_s$ the operators $S_{\rho, \sigma}$ and $F_{\rho, \sigma}$ are defined by

\[
S_{\rho, \sigma}(x\xi_\sigma + \eta) := s_M(\sigma)x^*\rho \xi_\sigma, \quad x \in M, \quad \eta \in (1 - s_M(\sigma))\mathcal{H},
\]
\[
F_{\rho, \sigma}(x'\xi_\sigma + \zeta) := s_M(\sigma)x'^*\rho \xi_\sigma, \quad x' \in M', \quad \zeta \in (1 - s_M(\sigma))\mathcal{H}.
\]
Then $S_{\rho,\sigma}$ and $F_{\rho,\sigma}$ are closable conjugate-linear operators such that $S_{\rho,\sigma}^* = \overline{F}_{\rho,\sigma}$, see [6, Lemma 2.2]. The relative modular operator $\Delta_{\rho,\sigma}$ introduced in [6] is

$$\Delta_{\rho,\sigma} := S_{\rho,\sigma}^* S_{\rho,\sigma},$$

and the polar decomposition of $\overline{S}_{\rho,\sigma}$ is given as

$$\overline{S}_{\rho,\sigma} = J\Delta_{\rho,\sigma}^{1/2}.$$  \hspace{1cm} (2.1)

Recall that the support projection of $\Delta_{\rho,\sigma}$ is $s_M(\rho)s_M'(\sigma)$. We write the spectral decomposition of $\Delta_{\rho,\sigma}$ as

$$\Delta_{\rho,\sigma} = \int_{(0,+\infty)} t dE_{\rho,\sigma}(t).$$  \hspace{1cm} (2.2)

When $\rho = \sigma$, $\Delta_{\sigma,\sigma}$ is the modular operator $\Delta_\sigma$ for $\sigma$. Note that $\xi_\sigma$ is an eigenvector of $\Delta_\sigma$ with eigenvalue 1, so that $\Delta_\sigma \xi_\sigma = \xi_\sigma$.

### 2.2 Standard $f$-divergence $S_f(\rho\|\sigma)$

Let $f : (0, +\infty) \to \mathbb{R}$ be a convex function. Then the limits

$$f(0^+) := \lim_{t \searrow 0} f(t), \quad f'(+\infty) := \lim_{t \to +\infty} \frac{f(t)}{t}$$

exist in $(-\infty, +\infty]$. Below we will understand the expression $bf(a/b)$ for $a = 0$ or $b = 0$ in the following way:

$$bf(0/b) := f(0^+) b \quad \text{for } b \geq 0, \quad 0f(a/0) := \lim_{t \searrow 0} tf(a/t) = f'(+\infty) a \quad \text{for } a > 0,$$  \hspace{1cm} (2.3)

where we use the convention that $(+\infty)0 := 0$ and $(+\infty)c := +\infty$ for $c > 0$. In particular, we fix $0f(0/0) = 0$.

The next definition is a specialization of the quasi-entropy [30, 43] with modifications.

**Definition 2.1.** For each $\rho, \sigma \in M_+^*$, with the spectral decomposition in (2.2), we define the self-adjoint operator $f(\Delta_{\rho,\sigma})$ on $s_M(\rho)s_M'(\sigma)\mathcal{H}$ by

$$f(\Delta_{\rho,\sigma}) := \int_{(0, +\infty)} f(t) dE_{\rho,\sigma}(t),$$  \hspace{1cm} (2.4)

and define

$$\langle \xi_\sigma, f(\Delta_{\rho,\sigma})\xi_\sigma \rangle := \int_{(0, +\infty)} f(t) d\|E_{\rho,\sigma}(t)\xi_\sigma\|^2.$$  \hspace{1cm} (2.5)

We then introduce the standard $f$-divergence $S_f(\rho\|\sigma)$ of $\rho, \sigma$ by

$$S_f(\rho\|\sigma) := \langle \xi_\sigma, f(\Delta_{\rho,\sigma})\xi_\sigma \rangle + f(0^+)\sigma(1 - s_M(\rho)) + f'(+\infty)\rho(1 - s_M(\sigma)).$$  \hspace{1cm} (2.6)

Note that the integral in (2.4) is on $(0, +\infty)$ instead of $[0, +\infty)$. The left-hand expression of (2.5) should be understood, to be precise, in the sense of a lower-bounded form (see [49]), which equals the integral in the right-hand side. We first give a lemma to justify the above definition.
Lemma 2.2. For every $\rho, \sigma \in M_+^+$, $S_f(\rho\|\sigma)$ is well defined with values in $(-\infty, +\infty]$.

Proof. From the convexity of $f$, there are $a, b \in \mathbb{R}$ such that $f(t) \geq a + bt$ for all $t \in (0, +\infty)$. We have

$$
\int_{(0, +\infty)} f(t) d\|E_{\rho,\sigma}(t)\xi_{\sigma}\|^2 \geq \int_{(0, +\infty)} (a + bt) d\|E_{\rho,\sigma}(t)\xi_{\sigma}\|^2
$$

$$
= a\|s_M(\rho)s_M(\sigma)\xi_{\sigma}\|^2 + b\|\Delta_{\rho,\sigma}^{1/2}\xi_{\sigma}\|^2
$$

$$
= a\|s_M(\rho)\xi_{\sigma}\|^2 + b\|s_M(\sigma)\xi_{\sigma}\|^2
$$

$$
= a\sigma(s_M(\rho)) + b\rho(s_M(\sigma)) > -\infty,
$$

since $J\Delta_{\rho,\sigma}^{1/2}\xi_{\sigma} = s_M(\sigma)\xi_{\rho}$ by (2.1). Hence $S_f(\rho\|\sigma) \in (-\infty, +\infty]$.

By the above proof and (2.6) we also see that

$$
S_{a+bt}(\rho\|\sigma) = a\sigma(1) + b\rho(1). \quad (2.7)
$$

The following are some basic properties of $S_f(\rho\|\sigma)$:

**Proposition 2.3.** Let $\rho, \sigma \in M_+^+$.

1. If $\Phi : \tilde{M} \to M$ is a *-isomorphism between von Neumann algebras, then

$$
S_f(\rho \circ \Phi \| \sigma \circ \Phi) = S_f(\rho\|\sigma).
$$

2. In the case $\rho = 0$ or $\sigma = 0$ or $\rho = \sigma$,

$$
S_f(0\|\sigma) = f(0^+)\sigma(1), \quad S_f(\rho\|0) = f'(0)\rho(1), \quad S_f(\sigma\|\sigma) = f(0)\sigma(1).
$$

3. Homogeneity: For every $\lambda \in [0, +\infty)$,

$$
S_f(\lambda \rho \| \lambda \sigma) = \lambda S_f(\rho\|\sigma). \quad (2.8)
$$

4. Additivity: Let $M = M_1 \oplus M_2$ be the direct sum of von Neumann algebras $M_1$ and $M_2$. If $\rho_i, \sigma_i \in (M_i)_+^+$ for $i = 1, 2$, then

$$
S_f(\rho_1 \oplus \rho_2\|\sigma_1 \oplus \sigma_2) = S_f(\rho_1\|\sigma_1) + S_f(\rho_2\|\sigma_2).
$$

Proof. (1) is clear from the uniqueness of the standard form stated in Section 2.1.

(2) is seen directly from definition (2.6).

(3) For $\lambda > 0$, since $\xi_{\lambda \sigma} = \sqrt{\lambda}\xi_{\sigma}$ and $\Delta_{\lambda \rho,\lambda \sigma} = \Delta_{\rho,\sigma}$, equality (2.8) follows. For $\lambda = 0$ both sides are zero from (2).

(4) Note that the standard form of $M$ is the direct sum of the standard forms of $M_i$, $i = 1, 2$. For $\rho := \rho_1 \oplus \rho_2$ and $\sigma := \sigma_1 \oplus \sigma_2$ we have $\xi_{\rho} = \xi_{\rho_1} \oplus \xi_{\rho_2}$, $\xi_{\sigma} = \xi_{\sigma_1} \oplus \xi_{\sigma_2}$ and $\Delta_{\rho,\sigma} = \Delta_{\rho_1,\sigma_1} \oplus \Delta_{\rho_2,\sigma_2}$, from which the result immediately follows.

The additivity in (4) above will be improved in Section 4 (see Corollary 4.3 (2)).

In our definition of $S_f(\rho\|\sigma)$ the parameter function $f$ is a convex function on $(0, +\infty)$, not on $[0, +\infty)$. This is reasonable as the next proposition shows that $S_f(\rho\|\sigma)$ is symmetric between $\rho$ and $\sigma$ under exchanging $f$ with its transpose $\widetilde{f}$ defined by

$$
\widetilde{f}(t) := tf(t^{-1}), \quad t \in (0, +\infty).
$$
Proposition 2.4. For every $\rho, \sigma \in M^+_1$,

$$S_f(\rho\|\sigma) = S_f(\sigma\|\rho).$$

Proof. Since $f(0^+) = f'(\pm\infty)$ and $\tilde{f}'(\pm\infty) = f(0^+)$, it suffices to prove that

$$\langle \xi_\sigma, f(\Delta_{\rho,\sigma})\xi_\sigma \rangle = \langle \xi_\rho, \tilde{f}(\Delta_{\sigma,\rho})\xi_\rho \rangle. \quad (2.9)$$

Recall [6, Theorem 2.4] that

$$\Delta_{\rho,\sigma} = J\Delta_{\sigma,\rho}^{-1}J$$

together with

$$s_M(\rho)s_M'(\sigma) = Js_M(\rho)Js_M(\sigma)J = Js_M(\sigma)s_M'(\rho)J.$$  

Hence, since $J\Delta_{\sigma,\rho}^{1/2}\xi_\rho = s_M(\rho)\xi_\sigma$, one has

$$\|E_{\rho,\sigma}((0, t))\xi_\rho\|^2 = \|JE_{\rho,\sigma}((t^{-1}, +\infty))J\Delta_{\sigma,\rho}^{1/2}\xi_\rho\|^2 = \|E_{\rho,\sigma}((t^{-1}, +\infty))\Delta_{\sigma,\rho}^{1/2}\xi_\rho\|^2$$

$$= \int (t^{-1}, +\infty) s d\|E_{\sigma,\rho}(s)\xi_\rho\|^2 = \int (0, t) s^{-1} d\|E_{\sigma,\rho}(s^{-1})\xi_\rho\|^2,$$

so that $d\|E_{\rho,\sigma}(t)\xi_\rho\|^2 = t^{-1}d\|E_{\sigma,\rho}(t^{-1})\xi_\rho\|^2$ on $(0, +\infty)$. Therefore,

$$\int (0, +\infty) f(t) d\|E_{\rho,\sigma}(t)\xi_\rho\|^2 = \int (0, +\infty) f(t)t^{-1} d\|E_{\sigma,\rho}(t^{-1})\xi_\rho\|^2$$

$$= \int (0, +\infty) tf(t^{-1}) d\|E_{\sigma,\rho}(t)\xi_\rho\|^2,$$

which means (2.9).

Example 2.5. Let $M$ be an abelian von Neumann algebra such that $M \cong L^\infty(\Omega, \mu)$, where $(\Omega, A, \mu)$ is a $\sigma$-finite measure space. Let $\rho, \sigma \in M^+_1$, which correspond to $\phi, \psi \in L^1(\Omega, \mu)^+$ so that $\rho(x) = \int_\Omega x\phi d\mu$, $\sigma(x) = \int_\Omega x\psi d\mu$ for $x \in L^\infty(\Omega, \mu)$. The standard form of $L^\infty(\Omega, \mu)$ is $(L^\infty(\Omega, \mu), L^2(\Omega, \mu), \xi \mapsto \xi L^2(\Omega, \mu)^+)$, where $x \in L^\infty(\Omega, \mu)$ is represented on $L^2(\Omega, \mu)$ as the multiplication operator $\xi \mapsto x\xi$ for $\xi \in L^2(\Omega, \mu)$. It is straightforward to find that $\Delta_{\rho,\phi}$ is the multiplication of $1_{\{\phi>0\}}(\phi/\psi)$, which is the Radon-Nikodym derivative $d\rho/d\sigma$ (restricted on the support of $\sigma$) in the classical sense. We then have

$$S_f(\rho\|\sigma) = \int_{\{\phi>0\}\cap\{\psi>0\}} \psi f(\phi/\psi) d\mu + f(0^+) \int_{\{\phi=0\}} \psi d\mu + f'(\pm\infty) \int_{\{\psi=0\}} \phi d\mu,$$

which equals the classical $f$-divergence $S_f(\phi/\psi) = \int_\Omega \psi f(\phi/\psi) d\mu$ under the convention that $\psi(\omega)f(0/\psi(\omega)) = f(0^+)(\psi(\omega))$ for $\psi(\omega) \geq 0$ and $0f(\phi(\omega)/0) = \lim_{t\searrow 0} tf(\phi(\omega)/t) = f'(\pm\infty)\phi(\omega)$ for $\phi(\omega) > 0$.

Example 2.6. Let $M = B(\mathcal{H})$, i.e., a factor of type I, where $\mathcal{H}$ is an arbitrary Hilbert space. The standard form of $B(\mathcal{H})$ is $(B(\mathcal{H}), C_2(\mathcal{H}), J = *, C_2(\mathcal{H})_+)$, where $C_2(\mathcal{H})$ is the Hilbert-Schmidt class with the Hilbert-Schmidt inner product, $C_2(\mathcal{H})_+$ is the set of positive operators in $C_2(\mathcal{H})$, and the representation of $M = B(\mathcal{H})$ on $C_2(\mathcal{H})$ is the left multiplication. Then $M'$ is the right multiplication of $M = B(\mathcal{H})$ on $C_2(\mathcal{H})$. For $\rho, \sigma \in B(\mathcal{H})^+_1$ we have the vector representatives $D^1/\rho, D^1/\sigma \in C_2(\mathcal{H})_+$, where $D^\rho$ and $D^\sigma$ are the positive-trace-class operators such that $\rho(X) = \text{Tr} D^\rho X$ and $\sigma(X) = \text{Tr} D^\sigma X$ for $X \in B(\mathcal{H})$. Let

$$D^\rho = \sum_{a \in \text{spec } D^\rho, a>0} aP_a, \quad D^\sigma = \sum_{b \in \text{spec } D^\sigma, b>0} bQ_b.$$
be the spectral decompositions of $D_{\rho}, D_{\sigma}$, where $\sum_{a>0}$ and $\sum_{b>0}$ are finite or countable sums, and $P_a$ and $Q_b$ are finite-dimensional orthogonal projections. Then the relative modular operators $\Delta_{\rho,\sigma}$ on $C_2(\mathcal{H})$ is given as

$$\Delta_{\rho,\sigma} = L_{D_{\rho}} R_{D_{\sigma}^{-1}} = \sum_{a>0, b>0} ab^{-1} L_{P_a} R_{Q_b},$$

(2.10)

where $L_{[-]}$ and $R_{[-]}$ denote the left and the right multiplications and $D_{\sigma}^{-1}$ is the generalized inverse of $D_{\sigma}$. The proof of this is easy as follows:

If $\Phi \in P_a \mathcal{H}$ and $\Omega \in Q_b \mathcal{H}$ for $a, b > 0$, then

$$S_{\rho,\sigma}(|\Phi\rangle\langle\Omega|) = S_{\rho,\sigma}(b^{-1/2}|\Phi\rangle\langle\Omega| D_{\sigma}^{1/2}) = b^{-1/2} s_{M}(\sigma)|\Omega\rangle\langle\Phi| D_{\rho}^{1/2}$$

$$= a^{1/2} b^{-1/2} |\Omega\rangle\langle\Phi|,$$

$$F_{\rho,\sigma}(|\Omega\rangle\langle\Phi|) = F_{\rho,\sigma}(b^{-1/2} D_{\sigma}^{1/2} |\Omega\rangle\langle\Phi|) = b^{-1/2} s_{M}(\sigma) D_{\rho}^{1/2} |\Omega\rangle\langle\Phi|$$

$$= a^{1/2} b^{-1/2} |\Omega\rangle\langle\Phi| s_{M}(\sigma) = a^{1/2} b^{-1/2} |\Omega\rangle\langle\Phi|,$$

which imply that

$$\Delta_{\rho,\sigma}(|\Phi\rangle\langle\Omega|) = ab^{-1} |\Phi\rangle\langle\Omega|.$$  

Since the range of $L_{P_a} R_{Q_b}$ is the span of $|\Phi\rangle\langle\Omega|$ for $\Phi \in P_a \mathcal{H}$ and $\Omega \in Q_b \mathcal{H}$,

$$\sum_{a,b>0, ab^{-1}=c} L_{P_a} R_{Q_b}$$

is the spectral projection of $\Delta_{\rho,\sigma}$ corresponding to the eigenvalue $c > 0$.

Moreover, it follows from (2.10) that the definition of $S_f(\rho||\sigma)$ in (2.6) is rewritten as

$$S_f(\rho||\sigma) = \sum_{a>0, b>0} b f(ab^{-1}) \text{Tr} P_a Q_b + f(0^+).$$

$$\text{Tr} (I - D_{\rho}^{1/2}) D_{\sigma} + f'(+\infty).$$

which coincides with an expression in [22, Proposition 3.2] when $\dim \mathcal{H} < +\infty$.

**Remark 2.7.** Let $f : [0, +\infty) \to \mathbb{R}$ be a continuous function. For $\rho, \sigma \in M_+^*$ and $k \in M$, the quasi-entropy $S_f^k(\rho||\sigma)$ was introduced in [43] by

$$S_f^k(\rho||\sigma) := \langle k\xi_{\sigma}, f(\Delta_{\rho,\sigma}) k\xi_{\sigma} \rangle = \int_{[0, +\infty)} f(t) \text{d}\|E_{\rho,\sigma}(t)k\xi_{\sigma}\|^2$$

$$= \int_{[0, +\infty)} f(t) \text{d}\|E_{\rho,\sigma}(t)k\xi_{\sigma}\|^2 + f(0^+)(k\xi_{\sigma}, (1 - s_{M}(\rho)s_{M'}(\sigma))k\xi_{\sigma}).$$

(2.11)

Comparing (2.11) with (2.6) and (2.8) we note that

$$S_f(\rho||\sigma) = S_f^{k=1}(\rho||\sigma) + f'(+\infty)\rho(1 - s_{M}(\sigma)).$$

In particular, when $M = B(\mathcal{H})$ with $\dim \mathcal{H} < +\infty$ and $f(0^+) < +\infty$, the quasi-entropy (2.11) with $k = 1$ has finite values for all $\rho, \sigma$, which is improper as a standard $f$-divergence. For example, when $f(t) := t \log t$ so that $f(0^+) = 0$ and $f'(+\infty) = +\infty$, one can easily check that for $\rho = \text{Tr} (D_{\rho} \cdot)$ and $\sigma = \text{Tr} (D_{\sigma} \cdot)$ with density operators $D_{\rho}, D_{\sigma}$, expression (2.11) with $k = 1$ is

$$\text{Tr} D_{\rho}(\log D_{\rho} - \log^+ D_{\sigma}),$$

where $\log^+ t := \log t$ ($t > 0$), $\log^+ 0 := 0$. On the other hand, $S_{t\log t}(\rho||\sigma)$ in (2.6) coincides with the usual relative entropy

$$D(\rho||\sigma) := \begin{cases} \text{Tr} D_{\rho}(\log D_{\rho} - \log D_{\sigma}), & s_{M}(\rho) \leq s_{M}(\sigma); \\ +\infty, & s_{M}(\rho) \notin s_{M}(\sigma). \end{cases}$$

7
3 Variational expression of standard $f$-divergences

In this section we extend the variational expression of the relative entropy given in [32] to standard $f$-divergences. The extended expression will be quite useful in the next section to verify various properties of standard $f$-divergences.

Throughout this and the next sections, we assume that a function $f : (0, +\infty) \to \mathbb{R}$ is operator convex, i.e., the operator inequality

$$f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B), \quad 0 \leq \lambda \leq 1$$

holds for every invertible $A, B \in \mathcal{B}(\mathcal{H})^+$ of any $\mathcal{H}$. Also, a function $h : (0, +\infty) \to \mathbb{R}$ is said to be operator monotone if $A \preceq B \implies h(A) \leq h(B)$ for every invertible $A, B \in \mathcal{B}(\mathcal{H})^+$ of any $\mathcal{H}$. It is well-known that an operator monotone function $h$ on $(0, +\infty)$ is automatically operator concave (i.e., $-h$ is operator convex). For general theory on operator monotone and operator convex functions, see, e.g., [9, 21].

Recall [34] (see also [16, Theorem 5.1] for a more general form) that the operator convex function $f$ has an integral expression

$$f(t) = a + b(t - 1) + c(t - 1)^2 + \int_{[0, +\infty)} \frac{(t - 1)^2}{t + s} \, d\mu(s), \quad t \in (0, +\infty), \quad (3.1)$$

where $a, b \in \mathbb{R}, c \geq 0$ and $\mu$ is a positive measure on $[0, +\infty)$ with

$$\int_{[0, +\infty)} \frac{1}{1 + s} \, d\mu(s) < +\infty, \quad (3.2)$$

and moreover $a, b, c$ and $\mu$ are uniquely determined. Letting $d := \mu(\{0\}) \geq 0$ we also write

$$f(t) = a + b(t - 1) + c(t - 1)^2 + d \frac{(t - 1)^2}{t} + \int_{(0, +\infty)} \frac{(t - 1)^2}{t + s} \, d\mu(s), \quad t \in (0, +\infty). \quad (3.3)$$

One can easily verify that

$$f(0^+) = a - b + c + (+\infty)d + \int_{(0, +\infty)} s^{-1} \, d\mu(s), \quad (3.4)$$

$$f'(+) = b + (+\infty)c + d + \int_{(0, +\infty)} \, d\mu(s). \quad (3.5)$$

For each $n \in \mathbb{N}$ we define

$$f_n(t) := a + b(t - 1) + c \frac{n(t - 1)^2}{t + n} + d \frac{(t - 1)^2}{t + (1/n)}$$

$$+ \int_{[1/n, n]} \frac{(t - 1)^2}{t + s} \, d\mu(s), \quad t \in (0, +\infty). \quad (3.6)$$

We then have

**Lemma 3.1.** For each $n \in \mathbb{N}$, $f_n$ is operator convex on $(0, +\infty)$, $f_n(0^+) < +\infty$, $f_n'(+) < +\infty$ and $f_n(0^+) \nearrow f(0^+)$, $f_n'(+) \nearrow f'(+)$, $f_n(t) \nearrow f(t)$ as $n \to \infty$ for all $t \in (0, +\infty)$. 


Proof. By definition (3.6) it is immediate to see that \( f \) is an operator convex function on \((0, +\infty)\) and

\[
 f_n(0^+) = a - b + c + nd + \int_{[1/n, n]} s^{-1} d\mu(s),
\]

(3.7)

\[
 f'_n(+\infty) = b + nc + d + \int_{[1/n, n]} d\mu(s).
\]

(3.8)

It follows from (3.2) that (3.7) and (3.8) are finite, which increase, by the monotone convergence theorem, to (3.4) and (3.5), respectively. Moreover, for any \( t \in (0, +\infty) \), since

\[
 \frac{n(t - 1)^2}{t + n} \nearrow (t - 1)^2, \quad \frac{(t - 1)^2}{t + (1/n)} \nearrow \frac{(t - 1)^2}{t} \quad \text{as} \quad n \nearrow \infty,
\]

we have \( f_n(t) \nearrow f(t) \) from the monotone convergence theorem again.

\[ \square \]

Lemma 3.2. For every \( \rho, \sigma \in M^+_* \),

\[
 S_{f_n}(\rho\|\sigma) \nearrow S_f(\rho\|\sigma) \quad \text{as} \quad n \nearrow \infty.
\]

Proof. By Definition 2.1,

\[
 S_{f_n}(\rho\|\sigma) = f_n(0^+)\sigma(1 - s(\rho)) + f'_n(+\infty)\rho(1 - s(\sigma)) + \int_{(0, +\infty)} f_n(t) d\|E_{\rho, \sigma}(t)\xi\|^2.
\]

By Lemma 3.1 and the monotone convergence theorem, \( S_{f_n}(\rho\|\sigma) \) increases to \( S_f(\rho\|\sigma) \) as \( n \nearrow \infty \).

\[ \square \]

Lemma 3.3. For each \( n \in \mathbb{N} \) define an operator monotone function \( h_n \) on \([0, +\infty)\) by

\[
 h_n(t) = \int_{(0, +\infty)} \frac{t(1 + s)}{t + s} d\nu_n(s), \quad t \in [0, +\infty),
\]

(3.9)

where \( \nu_n \) is a finite positive measure supported on \([1/n, n] \) given by

\[
 d\nu_n(s) := c(1 + n)\delta_n + d(1 + n)\delta_{1/n} + 1_{[1/n, n]}(s) \frac{1 + s}{s} d\mu(s)
\]

(3.10)

with the point masses \( \delta_n \) at \( n \) and \( \delta_{1/n} \) at \( 1/n \). Then \( f_n \) defined in (3.6) is written as

\[
 f_n(t) = f_n(0^+) + f'_n(+\infty)t - h_n(t), \quad t \in (0, +\infty).
\]

(3.11)

Proof. Compute

\[
 \frac{n(t - 1)^2}{t + n} = 1 + nt - (1 + n)\frac{t(1 + n)}{t + n},
\]

\[
 \frac{(t - 1)^2}{t + (1/n)} = n + t - (1 + n)\frac{t(1 + 1/n)}{t + 1/n},
\]

\[
 \frac{(t - 1)^2}{t + s} = \frac{1}{s} + t - \frac{1 + s}{s} \cdot \frac{t(1 + s)}{t + s}.
\]

Inserting these into definition (3.6) one can write

\[
 f_n(t) = \left( a - b + c + nd + \int_{[1/n, n]} s^{-1} d\mu(s) \right)
\]
\[
\begin{align*}
&\left(b + nc + d + \int_{[1/n,n]} d\mu(s)\right)t \\
&- c(1 + n)\frac{t(1 + n)}{t + n} - d(1 + n)\frac{t(1 + \frac{1}{n})}{t + \frac{1}{n}} \\
&- \int_{[1/n,n]} \frac{1 + s}{s} \cdot \frac{t(1 + s)}{t + s} d\mu(s) \\
&= f_n(0^+) + f_n'(+\infty)t - h_n(t)
\end{align*}
\]

thanks to (3.7) and (3.8).

Now, let \(L\) be a subspace of \(M\) containing 1, and assume that \(L\) is dense in \(M\) with respect to the strong* operator topology. Since \(h_n(0) = h_n'(+\infty) = 0\), the next lemma follows from [32, Theorem 2.2].

**Lemma 3.4.** Let \(h_n\) be given in (3.5). Then for any \(\rho, \sigma \in M^*_n\),
\[
\int_{(0, +\infty)} h_n(t) d\|E_{\rho, \sigma}(t)\xi_\sigma\|^2 = \inf_{x(\cdot)} \left\{ \int_{[1/n,n]} \left\{ \sigma((1 - x(s))^*(1 - x(s))) + s^{-1} \rho(x(s)x(s)^*) \right\} (1 + s) d\nu_n(s), \right\}
\]
where the infimum is taken over all \(L\)-valued (finitely many values) step functions \(x(\cdot)\) on \((0, +\infty)\).

**Theorem 3.5.** Let \(f\) be an operator convex function on \((0, +\infty)\). For each \(n \in \mathbb{N}\) let \(f_n(0^+)\), \(f_n'(+\infty)\) and \(\nu_n\) be given in (3.7), (3.8) and (3.11), respectively. Then for every \(\rho, \sigma \in M^*_n\),
\[
S_f(\rho\|\sigma) = \sup_{n \in \mathbb{N}} \sup_{x(\cdot)} \left\{ f_n(0^+)\sigma(1) + f_n'(+\infty)\rho(1) \\
- \int_{[1/n,n]} \left\{ \sigma((1 - x(s))^*(1 - x(s))) + s^{-1} \rho(x(s)x(s)^*) \right\} (1 + s) d\nu_n(s) \right\}, \tag{3.12}
\]
where the supremum over \(x(\cdot)\) is taken over all \(L\)-valued step functions as the infimum in Lemma 3.4.

**Proof.** By (3.11) and (2.7) we have
\[
S_{f_n}(\rho\|\sigma) = f_n(0^+)\sigma(1) + f_n'(+\infty)\rho(1) + S_{-h_n}(\rho\|\sigma) \\
= f_n(0^+)\sigma(1) + f_n'(+\infty)\rho(1) - \int_{(0, +\infty)} h_n(t) d\|E_{\rho, \sigma}(t)\xi_\sigma\|^2.
\]

By Lemma 3.4 we hence have
\[
S_{f_n}(\rho\|\sigma) = \sup_{x(\cdot)} \left\{ f_n(0^+)\sigma(1) + f_n'(+\infty)\rho(1) \\
- \int_{[1/n,n]} \left\{ \sigma((1 - x(s))^*(1 - x(s))) + s^{-1} \rho(x(s)x(s)^*) \right\} (1 + s) d\nu_n(s) \right\}.
\]
The result follows by taking \(\sup_n\) of both sides of the above and using Lemma 3.2. \(\square\)
Example 3.6. Consider \( f(t) = -\log t \), whose integral expression in (3.1) is
\[
-\log t = -(t - 1) + \int_{(0, +\infty)} \frac{(t - 1)^2}{(t + s)(1 + s)^2} \, ds.
\]
Hence, in this case,
\[
a = c = d = 0, \quad b = -1, \quad d\mu(s) = \frac{1}{(1 + s)^2} \, ds,
\]
\[
f(0^+) = +\infty, \quad f'(+) = 0.
\]
Moreover, compute
\[
f_n(0^+) = 1 + \int_{1/n}^{n} \frac{1}{s(1 + s)^2} \, ds = \frac{2}{n + 1} + \log n,
\]
\[
f'_n(+) = -1 + \int_{1/n}^{n} \frac{1}{(1 + s)^2} \, ds = -\frac{2}{n + 1},
\]
\[
d\nu_n(s) = 1_{[1/n, n]}(s) \frac{1}{s(1 + s)} \, ds.
\]
For every \( \rho, \sigma \in M^*_+ \) the relative entropy is
\[
D(\sigma || \rho) = S_{\log}(\sigma || \rho) = S_{-\log}(\rho || \sigma),
\]
for which one can write expression (3.12) as
\[
D(\sigma || \rho) = \sup_{n \in \mathbb{N}} \sup_{x(\cdot)} \left[ \sigma(1) \log n + (\sigma(1) - \rho(1)) \frac{2}{n + 1} \right.
\]
\[
- \int_{[1/n, n]} \left\{ \sigma((1 - x(s))\star(1 - x(s))) + s^{-1}\rho(x(s)x(s)^*) \right\} s^{-1} \, ds \right].
\]
This expression is similar to but a bit different from the variational expression
\[
D(\sigma || \rho) = \sup_{n \in \mathbb{N}} \sup_{x(\cdot)} \left[ \sigma(1) \log n \right.
\]
\[
- \int_{[1/n, +\infty)} \left\{ \sigma((1 - x(s))\star(1 - x(s))) + s^{-1}\rho(x(s)x(s)^*) \right\} s^{-1} \, ds \right]
\]
in [32] Theorem 3.2].

Remark 3.7. The variational expression in (3.12) with the cut-off interval \([1/n, n]\) is natural when we consider \( S_f(\rho || \sigma) \) for general operator convex functions \( f \) on \((0, +\infty)\) with no assumption on the boundary values \( f(0^+) \) and \( f'(+) \). This is more explicitly justified by the fact that our expression is well behaved under taking the transpose \( \tilde{f}(t) = tf(t^{-1}) \). Indeed, for \( f \) given in (3.3), the integral expression of \( \tilde{f} \) is
\[
\tilde{f}(t) = \tilde{a} + \tilde{b}(t - 1) + \tilde{c}(t - 1)^2 + \tilde{d} \frac{(t - 1)^2}{t} + \int_{(0, +\infty)} \frac{(t - 1)^2}{t + s} \, d\tilde{\mu}(s)
\]
where
\[
\tilde{a} = a, \quad \tilde{b} = a - b, \quad \tilde{c} = d, \quad \tilde{d} = c, \quad d\tilde{\mu}(s) = s \, d\mu(s^{-1}).
\]
Hence one can easily find that \( \bar{f}_n(0^+) = f'_n(+\infty) \), \( \bar{f}_n'(+\infty) = f_n(0^+) \), \( d\nu_n(s) = d\nu_n(s^{-1}) \), and the expression inside the bracket \([\cdots]\) of (3.12) for \( S_f(\rho||\sigma) \) is

\[
\bar{f}_n(0^+)\sigma(1) + \bar{f}_n'(+\infty)\rho(1) - \int _{[1/n,n]} \{ \sigma((1-x(s))^*(1-x(s)))+s^{-1}\rho(x(s)x(s)^*) \}(1+s)
d\nu_n(s) = f_n(0^+)\rho(1) + f'_n(+\infty)\sigma(1) - \int _{[1/n,n]} \{ \rho((1-y(s))^*(1-y(s)))+s^{-1}\sigma(y(s)y(s)^*) \}(1+s)
d\nu_n(s),
\]

where \( y(s) := 1 - x(s)^{-1} \). In this way, the variational expression in (3.12) enjoys complete invariance under exchanging \((f, \rho, \sigma)\) with \((\bar{f}, \sigma, \rho)\).

### 4 Properties of standard \( f \)-divergences

As in [32] where the relative entropy was treated, most of the important properties of standard \( f \)-divergences can immediately be verified from the variational expression in Theorem 3.5.

**Theorem 4.1.** Let \( f \) be an operator convex function on \((0, +\infty)\). Let \( \rho, \sigma, \rho_i, \sigma_i \in M^+_\sigma \) for \( i = 1, 2 \).

(i) Joint lower semicontinuity: The map \((\rho, \sigma) \in M^+_\sigma \times M^+_\sigma \mapsto S_f(\rho||\sigma) \in (-\infty, +\infty) \) is jointly lower semicontinuous in the \( \sigma(M, M) \)-topology.

(ii) Joint convexity: The map in (i) is jointly convex and jointly subadditive, i.e., for every \( \rho_i, \sigma_i \in M^+_\sigma \), \( 1 \leq i \leq k \),

\[
S_f \left( \sum_{i=1}^k \rho_i \right) \left( \sum_{i=1}^k \sigma_i \right) \leq \sum_{i=1}^k S_f(\rho_i||\sigma_i).
\]

(iii) If \( f(0^+) \leq 0 \) and \( \sigma_1 \preceq \sigma_2 \), then \( S_f(\rho||\sigma_1) \geq S_f(\rho||\sigma_2) \). Also, if \( f'(+\infty) \leq 0 \) and \( \rho_1 \preceq \rho_2 \), then \( S_f(\rho_1||\sigma) \geq S_f(\rho_2||\sigma) \).

(iv) Monotonicity: Let \( N \) be another von Neumann algebra and \( \Phi : N \to M \) be a unital positive linear map that is normal (i.e., if \( \{x_\alpha\} \) is an increasing net in \( M_+ \) with \( x_\alpha \nearrow x \in M_+ \), then \( \Phi(x_\alpha) \nearrow \Phi(x) \)) and is a Schwarz map (i.e., \( \Phi(x^*x) \geq \Phi(x)^*\Phi(x) \) for all \( x \in N \)). Then

\[
S_f(\rho \circ \Phi||\sigma \circ \Phi) \leq S_f(\rho||\sigma) \tag{4.1}
\]

In particular, if \( N \) is a unital von Neumann subalgebra of \( M \), then

\[
S_f(\rho|_N||\sigma|_N) \leq S_f(\rho||\sigma) \tag{4.2}
\]

(v) Martingale convergence: If \( \{M_\alpha\} \) is an increasing net of unital von Neumann subalgebras of \( M \) such that \((\bigcup_\alpha M_\alpha)'' = M\), then

\[
S_f(\rho|M_\alpha||\sigma|M_\alpha) \nearrow S_f(\rho||\sigma).
\]
Proof. To prove (i)–(iv), we apply expression \((3.12)\) with \(L = M\). Since \((1 + s)dv_n(s)\) is a finite positive measure supported on \([1/n, n]\) (see \((3.10)\)), it is clear that the function of \((\rho, \sigma)\) inside the bracket \([\cdots]\) in \((3.12)\) is linear and continuous in the \((M_\ast, M)\)-topology, so (i) and (ii) are shown. Here note that joint convexity and joint subadditivity in (ii) are equivalent due to the homogeneity property in \((2.8)\).

Assume that \(f(0^+) \leq 0\); then \(f_n(0^+) \leq f(0^+) \leq 0\) for all \(n \in \mathbb{N}\). Hence the first assertion of (iii) is obvious by expression \((3.12)\), and the second assertion is similar.

To prove (iv), note first that \(\rho \circ \Phi, \sigma \circ \Phi \in N^+_\ast\) since \(\Phi\) is a normal positive linear map. For any \(N\)-valued step function \(x(\cdot)\) on \((0, +\infty)\), let \(y(s) := \Phi(x(s))\), which is an \(M\)-valued step function. Since \(\Phi\) is a unital Schwarz map, one has

\[
\sigma((1 - y(s))\tau(1 - y(s))) \leq \sigma \circ \Phi((1 - x(s))\tau(1 - x(s))),
\]

\[
\rho(y(s)y(s)^\ast) \leq \rho \circ \Phi(x(s)x(s)^\ast),
\]

which implies that the bracket \([\cdots]\) in \((3.12)\) for \(\rho \circ \Phi, \sigma \circ \Phi\) and \(x(\cdot)\) is dominated by \(S_f(\rho||\sigma)\). Hence inequality \((4.1)\) follows. When \(N\) is a unital von Neumann subalgebra, applying \((4.1)\) to the injection \(\Phi : N \to M\) gives \((4.2)\).

To prove (v), apply \((3.12)\) with \(L = \bigcup_\alpha M_\alpha\). When we restrict \(x(\cdot)\) in \((3.12)\) to \(M_\alpha\)-valued step functions, we have the expression of \(S_f(\rho|_{M_\alpha})||\sigma|_{M_\alpha})\). This shows that \(S_f(\rho|_{M_\alpha})||\sigma|_{M_\alpha})\) is increasing and \(S_f(\rho|_{M_\alpha})||\sigma|_{M_\alpha}) \leq S_f(\rho||\sigma)\). (This also follows from monotonicity in \((4.2)\)). Hence it remains to show that \(S_f(\rho||\sigma) \leq \sup_\alpha S_f(\rho|_{M_\alpha})||\sigma|_{M_\alpha})\). For any \(c < S_f(\rho||\sigma)\) we choose an \(n \in \mathbb{N}\) and an \(L\)-valued step function \(x(\cdot)\) such that

\[
c < f_n(0^+)\sigma(1) + f_n(\infty)^\ast\rho(1)
\]

\[
- \int_{[1/n,n]} \{\sigma((1 - x(s))\tau(1 - x(s))) + s^{-1}\rho(x(s)x(s)^\ast)\}(1 + s) dv_n(s).
\]

Since \(x(\cdot)\) is \(M_\alpha\)-valued for some \(\alpha\), we have \(c < S_f(\rho|_{M_\alpha})||\sigma|_{M_\alpha})\), implying the desired conclusion.

The next corollary shows that \(S_f(\rho||\sigma)\) is strictly positive in some typical situation.

**Corollary 4.2.** Let \(\rho, \sigma \in M^+_\ast\).

1. **The Peierls-Bogoliubov inequality holds:**

\[
S_f(\rho||\sigma) \geq \sigma(1)f(\rho(1)/\sigma(1)).
\]

2. **Assume that \(f\) is non-linear and \(\rho, \sigma \neq 0\). Then equality holds in \((4.3)\) if and only if \(\rho = (\rho(1)/\sigma(1))\sigma\).**

**Proof.** (1) When \(N := C1\) in \((4.2)\), inequality \((4.3)\) arises. If \(\rho = k\sigma\) with a constant \(k > 0\), then we have \(\Delta_{\rho, \sigma} = k\Delta_\sigma\), \(\Delta_\sigma\) being the modular operator for \(\sigma\), and hence \(d\|E_{\rho, \sigma}(t)\xi_\sigma\|^2 = (\sigma(1))d\delta_1(t)\), giving \(S_f(\rho||\sigma) = f(k)\sigma(1)\). Conversely, assume that \(\rho, \sigma \neq 0\) and equality holds in \((4.3)\). Further, assume that \(f\) is non-linear. Since \(f\) is operator convex on \((0, +\infty)\), it is strictly convex there. For every projection \(e \in M\), applying \((4.2)\) to \(N := Ce + Ce^\perp\) (where \(e^\perp := 1 - e\)) gives

\[
S_f(\rho||\sigma) \geq S_f(\rho|_N||\sigma|_N) = \sigma(e)f(\rho(e)/\sigma(e)) + \sigma(e^\perp)f(\rho(e^\perp)/\sigma(e^\perp)).
\]
From this and (2.3) for $\rho|_N$ and $\sigma|_N$ one has
$$\sigma(1)f(\rho(1)/\sigma(1)) = \sigma(e)f(\rho(e)/\sigma(e)) + \sigma(e^\perp)f(\rho(e^\perp)/\sigma(e^\perp)).$$

By Lemma 4.3 below one has $f(2.3)$ with convention (2.3). Let
$$f(\rho(1)/\sigma(1))$$
for $\rho = (\rho(1)/\sigma(1))\sigma$, showing that $\rho = (\rho(1)/\sigma(1))\sigma$.

(2) is immediately seen from (1). \qed

The next elementary lemma has been used in the above, whose proof is given for completeness, since we find no suitable reference.

**Lemma 4.3.** Let $f : (0, +\infty) \to \mathbb{R}$ is a strictly convex function (not necessarily operator convex). Let $a_i, b_i \in [0, +\infty)$ for $i = 1, 2$ be such that $a_1 + a_2 > 0$ and $b_1 + b_2 > 0$. If
$$(b_1 + b_2)f\left(\frac{a_1 + a_2}{b_1 + b_2}\right) = b_1f(a_1/b_1) + b_2f(a_2/b_2)$$
with convention (2.3), then $(a_1, a_2) = k(b_1, b_2)$ for some $k > 0$.

**Proof.** We may consider the following four cases separately.

**Case** $a_i, b_i > 0$ for $i = 1, 2$. Since
$$(b_1 + b_2)f\left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right) = b_1f(a_1/b_1) + b_2f(a_2/b_2),$$
the strict convexity of $f$ implies that $a_1/b_1 = a_2/b_2$.

**Case** $a_1 = 0$ and $b_1, b_2 > 0$. The assumption means that
$$(b_1 + b_2)f\left(\frac{a_2}{b_1 + b_2}\right) = b_1f(0/b_1) + b_2f(a_2/b_2) = b_1f(0^+) + b_2f(a_2/b_2),$$
which implies that $f(0^+) < +\infty$. Hence $f$ extends to a strictly convex function on $[0, +\infty)$, and the above equality gives $a_2/b_2 = 0$, which is impossible since $a_1 + a_2 > 0$.

**Case** $a_1, a_2 > 0$ and $b_1 = 0$. This case reduces to the previous case if we replace $f$ with its transpose $\tilde{f}$.

**Case** $a_1 = b_1 = 0$ or $a_2 = b_2 = 0$. The assertion trivially holds in this case.

**Case** $a_1 = b_2 = 0$. The assumption means that
$$b_1f(a_2/b_1) = b_1f(0/b_1) + 0f(a_2/0) = f(0^+)b_1 + f'(\infty)a_2,$$
which implies that $f(0^+) < +\infty$ and $f'(\infty) < +\infty$. Then it is easy to find that $f(t) < f(0^+) + f'(\infty)t$ for all $t > 0$, which contradicts the above equality. \qed

For $\sigma \in M_+^+$ and a projection $e \in M$, we write $e\sigma e$ for the restriction of $\sigma$ to the reduced von Neumann algebra $eMe$.

**Corollary 4.4.** (1) If $e \in M$ is a projection such that $s_M(\rho), s_M(\sigma) \leq e$, then
$$S_f(\rho\|\sigma) = S_f(e\rho e\|e\sigma e).$$
(2) If \( \rho_i, \sigma_i \in M^+_* \), \( i = 1, 2 \), and \( s_M(\rho_1) \lor s_M(\sigma_1) \perp s_M(\rho_2) \lor s_M(\sigma_2) \), then
\[
S_f(\rho_1 + \rho_2\|\sigma_1 + \sigma_2) = S_f(\rho_1\|\sigma_1) + S_f(\rho_2\|\sigma_2).
\]

(3) If \( \omega_1, \omega_2 \in M^+_* \) and \( S_f(\omega_1\|\omega_2) < +\infty \), then for every \( \rho, \sigma \in M^+_* \),
\[
S_f(\rho\|\sigma) = \lim_{\varepsilon \searrow 0} S_f(\rho + \varepsilon\omega_1\|\sigma + \varepsilon\omega_2).
\]

In particular, for every \( \rho, \sigma, \omega \in M^+_* \),
\[
S_f(\rho\|\sigma) = \lim_{\varepsilon \searrow 0} S_f(\rho + \varepsilon\omega\|\sigma + \varepsilon\omega).
\]

**Proof.** (1) By monotonicity \( 1.2 \) we have \( S_f(e\rho e\|e\sigma e) \leq S_f(\rho\|\sigma) \). For any \( M \)-valued step function \( x(\cdot) \) on \( (0, +\infty) \), let \( y(\cdot) := ex(\cdot)e \), which is an \( eMe \)-valued step function. Since
\[
(e\sigma e)((e - y(s))e - y(s))) = \sigma(e(1 - x(s))e(1 - x(s))e) \leq \sigma(1 - x(s))e(1 - x(s))e),
\]
the bracket \([\cdots]\) in \( 3.12 \) for \( x(\cdot) \) is dominated by \( S_f(e\rho e\|e\sigma e) \). Hence equality \( 1.3 \) follows.

(2) Let \( e := s_M(\rho_1) \lor s_M(\rho_2) \) and so \( s_M(\rho_2) \lor s_M(\sigma_2) \leq e^\perp \). Let \( \rho := \rho_1 + \rho_2, \sigma := \sigma_1 + \sigma_2 \), and \( N := eMe \oplus e^\perp Me^\perp \). Since \( \rho|_N = e\rho_1 e \oplus e^\perp \rho_2 e^\perp \) and \( \sigma|_N = e\sigma_1 e \oplus e^\perp \sigma_2 e^\perp \), from monotonicity \( 1.2 \) and Proposition \( 2.3(4) \) one has
\[
S_f(\rho\|\sigma) \geq S_f(e\rho_1 e\|e\sigma_1 e) + S_f(e^\perp \rho_2 e^\perp\|e^\perp \sigma_2 e^\perp) = S_f(\rho_1\|\sigma_1) + S_f(\rho_2\|\sigma_2),
\]
where we have used (1) for the last equality. On the other hand, consider the map
\[
\Phi : M \rightarrow eMe \oplus e^\perp Me^\perp, \quad \Phi(x) := exe + e^\perp x e^\perp,
\]
which is unital and completely positive (hence a Schwarz map). Since \( \rho = (e\rho_1 e \oplus e^\perp \rho_2 e^\perp) \circ \Phi \) and \( \sigma = (e\sigma_1 e \oplus e^\perp \sigma_2 e^\perp) \circ \Phi \), from monotonicity \( 1.1 \) and Proposition \( 2.3(4) \) one has
\[
S_f(\rho\|\sigma) \leq S_f(\rho_1\|\sigma_1) + S_f(\rho_2\|\sigma_2).
\]
Hence equality \( 4.3 \) is shown.

(3) From joint subadditivity in Theorem \( 4.1(\text{ii}) \) and homogeneity \( 2.9 \) one has
\[
S_f(\rho + \varepsilon\omega_1\|\sigma + \varepsilon\omega_2) \leq S_f(\rho\|\sigma) + \varepsilon S_f(\omega_1\|\omega_2) \rightarrow S_f(\rho\|\sigma)
\]
as \( \varepsilon \searrow 0 \). On the other hand, from lower semicontinuity in Theorem \( 4.1(\text{i}) \) one has
\[
S_f(\rho\|\sigma) \leq \liminf_{\varepsilon \searrow 0} S_f(\rho + \varepsilon\omega_1\|\sigma + \varepsilon\omega_2),
\]
showing the asserted convergence. \( \square \)

The additivity in Corollary \( 4.4(\text{2}) \) improves that in Proposition \( 2.3(4) \); yet we have used the latter in the above proof of the former. When \( M \) is \( \sigma \)-finite so that a faithful \( \omega \in M^+_* \) exists, we can sometimes reduce arguments on \( S_f(\rho\|\sigma) \) to the case of faithful \( \rho, \sigma \in M^+_* \) by using the convergence property in \( 1.6 \).

The next continuity property is not included in the martingale convergence in Theorem \( 4.1 \) since \( eMe \) is not a unital von Neumann subalgebra of \( M \).
Theorem 4.5. Let \( \{e_\alpha\} \) be an increasing net of projections in \( M \) such that \( e_\alpha \not\nearrow 1 \). Then for every \( \rho, \sigma \in M_+^* \),
\[
\lim_{\alpha} S_f(e_\alpha \rho e_\alpha \| e_\alpha \sigma e_\alpha) = S_f(\rho \| \sigma).
\]

Proof. By replacing \( f \) with \( f(t) - (a + bt) \) and noting \((2.7)\), we may assume that \( f(t) \geq 0 \) for all \( t \in (0, +\infty) \). Let \( M_\alpha := e_\alpha M e_\alpha + \mathbb{C}(1 - e_\alpha) \); then \( \{M_\alpha\} \) is an increasing net of von Neumann subalgebras with \( (\bigcup_\alpha M_\alpha)^\prime = M \). Hence the martingale convergence in Theorem 4.1 and Corollary 4.4(2) imply that
\[
S_f(\rho \| \sigma) = \lim_{\alpha} \left[ S_f(e_\alpha \rho e_\alpha \| e_\alpha \sigma e_\alpha) + \sigma(1 - e_\alpha) f\left(\frac{\rho(1 - e_\alpha)}{\sigma(1 - e_\alpha)}\right)\right],
\]
increasingly in \( \alpha \). First, assume that \( S_f(\rho \| \sigma) = +\infty \) and prove that
\[
\lim_{\alpha} S_f(e_\alpha \rho e_\alpha \| e_\alpha \sigma e_\alpha) = +\infty.
\]
If \( \limsup_\alpha \sigma(1 - e_\alpha) f(\rho(1 - e_\alpha)/\sigma(1 - e_\alpha)) < +\infty \), then \((4.8)\) clearly follows from \((4.7)\). Assume that \( \limsup_\alpha \sigma(1 - e_\alpha) f(\rho(1 - e_\alpha)/\sigma(1 - e_\alpha)) = +\infty \). Then for any \( K > 0 \) choose an \( \alpha_0 \) such that \( \sigma(1 - e_{\alpha_0}) f(\rho(1 - e_{\alpha_0})/\sigma(1 - e_{\alpha_0})) > K \). Since
\[
\rho(e_\alpha - e_{\alpha_0}) \not\nearrow \rho(1 - e_{\alpha_0}), \quad \sigma(e_\alpha - e_{\alpha_0}) \not\nearrow \sigma(1 - \alpha_0) \quad \text{as} \quad \alpha_0 \leq \alpha \to "\infty", \]
we easily see that
\[
\sigma(e_\alpha - e_{\alpha_0}) f\left(\frac{\rho(e_\alpha - e_{\alpha_0})}{\sigma(e_\alpha - e_{\alpha_0})}\right) \to \sigma(1 - e_{\alpha_0}) f\left(\frac{\rho(1 - e_{\alpha_0})}{\sigma(1 - e_{\alpha_0})}\right) > K.
\]
By monotonicity of \( S_f \) and the assumption \( f \geq 0 \) we have for \( \alpha \geq \alpha_0 \)
\[
S_f(e_\alpha \rho e_\alpha \| e_\alpha \sigma e_\alpha) \geq \sigma(e_\alpha - e_{\alpha_0}) f\left(\frac{\rho(e_\alpha - e_{\alpha_0})}{\sigma(e_\alpha - e_{\alpha_0})}\right) + \sigma(e_{\alpha_0}) f\left(\frac{\rho(e_{\alpha_0})}{\sigma(e_{\alpha_0})}\right)
\]
\[
\geq \sigma(e_\alpha - e_{\alpha_0}) f\left(\frac{\rho(e_\alpha - e_{\alpha_0})}{\sigma(e_\alpha - e_{\alpha_0})}\right),
\]
which is \( > K \) for all sufficiently large \( \alpha \geq \alpha_0 \). Hence \((4.8)\) follows.

Next, assume that \( S_f(\rho \| \sigma) < +\infty \), and prove that \( \lim_\alpha S_f(e_\alpha \rho e_\alpha \| e_\alpha \sigma e_\alpha) = S_f(\rho \| \sigma) \). To do this, by \((4.7)\) we may prove that \( \lim_\alpha \sigma(1 - e_\alpha) f(\rho(1 - e_\alpha)/\sigma(1 - e_\alpha)) = 0 \). Assume on the contrary that \( \limsup_\alpha \sigma(1 - e_\alpha) f(\rho(1 - e_\alpha)/\sigma(1 - e_\alpha)) > \varepsilon > 0 \) for some \( \varepsilon > 0 \) (here recall that \( f \geq 0 \) was assumed). Choose an \( \alpha_1 \) such that \( \sigma(1 - e_{\alpha_1}) f(\rho(1 - e_{\alpha_1})/\sigma(1 - e_{\alpha_1})) > \varepsilon \). Then we can choose a \( \beta_1 > \alpha_1 \) such that
\[
\sigma(e_{\beta_1} - e_{\alpha_1}) f\left(\frac{\rho(e_{\beta_1} - e_{\alpha_1})}{\sigma(e_{\beta_1} - e_{\alpha_1})}\right) > \varepsilon.
\]
Next, choose an \( \alpha_1 > \beta_2 \) such that \( \sigma(1 - e_{\alpha_2}) f(\rho(1 - e_{\alpha_2})/\sigma(1 - e_{\alpha_2})) > \varepsilon \), and a \( \beta_2 > \alpha_2 \) such that
\[
\sigma(e_{\beta_2} - e_{\alpha_2}) f\left(\frac{\rho(e_{\beta_2} - e_{\alpha_2})}{\sigma(e_{\beta_2} - e_{\alpha_2})}\right) > \varepsilon.
\]
Repeating the above argument we have \( \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots \) in such a way that
\[
\sigma(e_{\beta_k} - e_{\alpha_k}) f\left(\frac{\rho(e_{\beta_k} - e_{\alpha_k})}{\sigma(e_{\beta_k} - e_{\alpha_k})}\right) > \varepsilon.
\]
for all \( k \in \mathbb{N} \). Let \( e_{\alpha_k} \not\rightarrow e_\infty \) and \( e_0 := 1 - e_\infty \), and consider a unital abelian von Neumann subalgebra of \( M \)
\[
\bigoplus_{k=1}^\infty C(e_{\beta_k} - e_{\alpha_k}) \oplus \bigoplus_{k=1}^\infty C(e_{\alpha_k} - e_{\beta_k-1}) \oplus Ce_0,
\]
where \( e_{\beta_0} := 0 \). By monotonicity of \( S_f \) and Example 2.5 together with \( f \geq 0 \), we have
\[
S_f(\rho\|\sigma) \geq \sum_{k=1}^\infty \sigma(e_{\beta_k} - e_{\alpha_k})f\left(\frac{\rho(e_{\beta_k} - e_{\alpha_k})}{\sigma(e_{\beta_k} - e_{\alpha_k})}\right) + \sigma(e_0)f\left(\frac{\rho(e_0)}{\sigma(e_0)}\right) = +\infty,
\]
which contradicts the assumption \( S_f(\rho\|\sigma) < +\infty \).

When \( f \geq 0 \) in Theorem 4.5 from the monotonicity of \( S_f \) we see that \( S_f(e_\alpha \rho e_\alpha\| e_\alpha \sigma e_\alpha) \) is increasing as \( e_\alpha \not\rightarrow 1 \). But this is not the case unless \( f \geq 0 \).

**Remark 4.6.** When \( M = B(H) \) with \( \dim H = \infty \), according to Theorem 4.5, one can define the relative entropy \( D(\rho\|\sigma) \) for trace-class operators \( \rho, \sigma \geq 0 \) as
\[
D(\rho\|\sigma) = \lim_\alpha D(E_\alpha \rho E_\alpha\| E_\alpha \sigma E_\alpha),
\]
where \( \{E_\alpha\} \) is an increasing net of finite rank projections with \( E_\alpha \not\rightarrow I \). But it seems that there is no simpler proof other than that of Theorem 4.5 for the existence of the limit in (4.9) and its independence of the choice of \( \{E_\alpha\} \).

## 5 Rényi divergences

We define the notion of Rényi divergences \( D_\alpha(\rho\|\sigma) \) for \( \alpha \geq 0 \) in the general von Neumann algebra setting.

**Definition 5.1.** Let \( \rho, \sigma \in M_+^* \). Since \( \xi_\sigma \in \mathcal{D}(\Delta_{\rho,\sigma}^{1/2}) \), the domain of \( \Delta_{\rho,\sigma}^{1/2} \), note that \( \xi_\sigma \in \mathcal{D}(\Delta_{\rho,\sigma}^{\alpha/2}) \) for any \( \alpha \in [0, 1] \). We define the quantities \( Q_\alpha(\rho\|\sigma) \) for \( \alpha \geq 0 \) as follows: When \( 0 \leq \alpha < 1 \),
\[
Q_\alpha(\rho\|\sigma) := \|\Delta_{\rho,\sigma}^{\alpha/2} \xi_\sigma\|^2 \in [0, +\infty),
\]
and when \( \alpha > 1 \),
\[
Q_\alpha(\rho\|\sigma) := \begin{cases} \|\Delta_{\rho,\sigma}^{\alpha/2} \xi_\sigma\|^2 & \text{if } s_M(\rho) \leq s_M(\sigma) \text{ and } \xi_\sigma \in \mathcal{D}(\Delta_{\rho,\sigma}^{\alpha/2}), \\ +\infty & \text{otherwise}. \end{cases}
\]

Moreover, when \( \alpha = 1 \), define \( Q_1(\rho\|\sigma) := \rho(1) \). Then for every \( \rho, \sigma \in M_+^* \) with \( \rho \neq 0 \) and for each \( \alpha \in [0, +\infty) \setminus \{1\} \), the \( \alpha \)-Rényi divergence \( D_\alpha(\rho\|\sigma) \) is defined by
\[
D_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \frac{Q_\alpha(\rho\|\sigma)}{\rho(1)}. \tag{5.3}
\]

In particular, note that \( Q_0(\alpha\|\sigma) = \sigma(s_M(\rho)) \) and \( D_0(\rho\|\sigma) = -\log[\sigma(s_M(\rho))/\rho(1)] \). The next lemma shows that \( Q_\alpha(\rho\|\sigma) \) is essentially a standard \( f \)-divergence and so \( D_\alpha(\rho\|\sigma) \) is a variant of standard \( f \)-divergences.
Lemma 5.2. Define convex functions $f_\alpha$ on $[0, +\infty)$ by

$$f_\alpha(t) := \begin{cases} t^\alpha & \text{if } \alpha \geq 1, \\ -t^\alpha & \text{if } 0 < \alpha < 1. \end{cases}$$

Then for every $\rho, \sigma \in M^+_\ast$, $Q_\alpha(\rho\|\sigma)$ is given as

$$Q_\alpha(\rho\|\sigma) = \begin{cases} S_{f_\alpha}(\rho\|\sigma) & \text{if } \alpha \geq 1, \\ -S_{f_\alpha}(\rho\|\sigma) & \text{if } 0 < \alpha < 1. \end{cases} \quad (5.4)$$

Moreover,

$$Q_\alpha(\rho\|\sigma) = \begin{cases} \int_{(0, +\infty)} t^\alpha d_2 \|E_{\rho,\sigma}(t)\xi_\sigma\|^2 & \text{if } 0 \leq \alpha < 1 \text{ or } s_M(\rho) \leq s_M(\sigma), \\ +\infty & \text{if } \alpha > 1 \text{ and } s_M(\rho) \nless s_M(\sigma). \end{cases} \quad (5.5)$$

Proof. When $0 < \alpha < 1$, since $f_\alpha(0) = f'_\alpha(+\infty) = 0$, we have by (5.1)

$$S_{f_\alpha}(\rho\|\sigma) = \int_{(0, +\infty)} (-t^\alpha) d_2 \|E_{\rho,\sigma}(t)\xi_\sigma\|^2 = -Q_\alpha(\rho\|\sigma).$$

When $\alpha = 1$, (2.7) gives $S_{f_1}(\rho\|\sigma) = \rho(1) = Q_1(\rho\|\sigma)$. When $\alpha > 1$, since $f_\alpha(0) = 0$ and $f'_\alpha(+\infty) = +\infty$,

$$S_{f_\alpha}(\rho\|\sigma) = \int_{(0, +\infty)} t^\alpha d_2 \|E_{\rho,\sigma}(t)\xi_\sigma\|^2 + (+\infty)\rho(1 - s_M(\sigma)).$$

Note that $\rho(1 - s_M(\sigma)) = 0 \iff s_M(\rho) \leq s_M(\sigma)$, and $\int_{(0, +\infty)} t^\alpha d_2 \|E_{\rho,\sigma}(t)\xi_\sigma\|^2 < +\infty \iff \xi_\sigma \in D(\Delta^{\alpha/2}_\rho, \sigma)$. Hence by (5.2) we see that $S_{f_\alpha}(\rho\|\sigma) = Q_\alpha(\rho\|\sigma)$. Moreover, (5.5) immediately follows from the above argument, where the case $\alpha = 0$ is obvious. □

Some properties of $Q_\alpha$ and $D_\alpha$ are found in, e.g., [35, 37, 20, 36] though mostly in the finite-dimensional situation. More comprehensive summary of them are given in the next proposition, mainly based on Theorem 4.1. Although (3) and (4) when $\alpha \in [0, 2]$ have been shown in [10], we give their proofs as well for convenience of the reader.

Proposition 5.3. Let $\rho, \sigma \in M^+_\ast$ with $\rho \neq 0$.

1. If $s_M(\rho) \perp s_M(\sigma)$, then $D_\alpha(\rho\|\sigma) = +\infty$ for all $\alpha \in [0, +\infty) \setminus \{1\}$.

2. If $s_M(\rho) \nparallel s_M(\sigma)$, then $Q_\alpha(\rho\|\sigma) > 0$ for all $\alpha \geq 0$ and the function $\alpha \in [0, +\infty) \mapsto \log Q_\alpha(\rho\|\sigma)$ is convex.

3. The limit $D_1(\rho\|\sigma) := \lim_{\alpha \searrow 1} D_\alpha(\rho\|\sigma)$ exists and

$$D_1(\rho\|\sigma) = \frac{D(\rho\|\sigma)}{\rho(1)} \quad (5.6)$$

where $D(\rho\|\sigma)$ is the relative entropy. Moreover, if $D_\alpha(\rho\|\sigma) < +\infty$ for some $\alpha > 1$, then $\lim_{\alpha \searrow 1} D_\alpha(\rho\|\sigma) = D_1(\rho\|\sigma)$.

4. The function $\alpha \in [0, +\infty) \mapsto D_\alpha(\rho\|\sigma)$ is monotone increasing.
(5) Assume that $0 < \alpha < 1$. We have
\[ Q_\alpha(\rho||\sigma) = Q_{1-\alpha}(\rho||\sigma), \]  
and whenever $\rho, \sigma \neq 0$,
\[ \frac{1}{\alpha} D_\alpha(\rho||\sigma) = \frac{1}{1-\alpha} D_{1-\alpha}(\sigma||\rho) + \frac{1}{\alpha(1-\alpha)} \log \frac{\rho(1)}{\sigma(1)}. \]  
Hence, if $\rho(1) = \sigma(1)$, then $\lim_{\alpha \searrow 0} \frac{1}{\alpha} D_\alpha(\rho||\sigma) = D_1(\sigma||\rho)$.

(6) Joint lower semicontinuity: For every $\alpha \in [0, 2]$, the map $(\rho, \sigma) \in (M_+^+ \setminus \{0\}) \times M_+^+ \mapsto D_\alpha(\rho||\sigma)$ is jointly lower semicontinuous in the $\sigma(M_+, M)$-topology.

(7) The map $(\rho, \sigma) \in M_+^+ \times M_+^+ \mapsto Q_\alpha(\rho||\sigma)$ is jointly concave and jointly superadditive for $0 \leq \alpha \leq 1$, and jointly convex and jointly subadditive for $1 \leq \alpha \leq 2$. Hence, when $0 \leq \alpha \leq 1$, $D_\alpha(\rho||\sigma)$ is jointly convex on $\{(\rho, \sigma) \in M_+^+ \times M_+^+ : \rho(1) = c\}$ for any fixed $c > 0$.

(8) Let $\rho_i, \sigma_i \in M_+^+$ for $i = 1, 2$. If $0 \leq \alpha < 1$, $\rho_1 \leq \rho_2$ and $\sigma_1 \leq \sigma_2$, then $Q_\alpha(\rho_1||\sigma_1) \leq Q_\alpha(\rho_2||\sigma_2)$. If $1 \leq \alpha \leq 2$ and $\sigma_1 \leq \sigma_2$, then $Q_\alpha(\rho||\sigma_1) \geq Q_\alpha(\rho||\sigma_2)$. If $\sigma_1 \leq \sigma_2$, then $D_\alpha(\rho||\sigma_1) \geq D_\alpha(\rho||\sigma_2)$ for all $\alpha \in [0, 2]$.

(9) Monotonicity: For each $\alpha \in [0, 2]$, $D_\alpha(\rho||\sigma)$ is monotone under unital normal Schwarz maps, i.e.,
\[ D_\alpha(\rho \circ \Phi||\sigma \circ \Phi) \leq D_\alpha(\rho||\sigma) \]  
for any unital normal Schwarz map $\Phi : N \to M$ as in Theorem 4.1(iv).

(10) Strict positivity: Let $\alpha \in (0, +\infty)$ and $\rho, \sigma \neq 0$. The inequality
\[ D_\alpha(\rho||\sigma) \geq \log \frac{\rho(1)}{\sigma(1)} \]  
holds, and equality holds in (5.10) if and only if $\rho = (\rho(1)/\sigma(1))\sigma$. If $\rho(1) = \sigma(1)$, then $D_\alpha(\rho||\sigma) \geq 0$ and $D_\alpha(\rho||\sigma) = 0 \iff \rho = \sigma$.

Proof. Write $F(\alpha) := \int_{(0, +\infty)} t^\alpha d\mu(t)$ for $\alpha > 0$, where $d\mu(t) := d\|E_{\rho, \sigma}(t)\xi_\sigma\|^2$ for $t \in (0, +\infty)$. Note that $F(1) = \rho(s_M(\sigma))$. By (5.5) we note that

(A) if $s_M(\rho) \leq s_M(\sigma)$ then $Q_\alpha(\rho||\sigma) = F(\alpha)$ for all $\alpha \geq 0$,

(B) if $s_M(\rho) \not\leq s_M(\sigma)$ then
\[ Q_\alpha(\rho||\sigma) = \begin{cases} F(\alpha) & \text{for } 0 \leq \alpha < 1, \\ \rho(1) > F(1) & \text{for } \alpha = 1, \\ +\infty & \text{for } \alpha > 1. \end{cases} \]

(1) If $s_M(\rho) \perp s_M(\sigma)$, i.e., $F(1) = \rho(s_M(\sigma)) = 0$, then we have $\mu = 0$ so that $F(\alpha) = 0$ for all $\alpha \in [0, +\infty)$. Hence the conclusion follows from (B).
(2) If \( s_M(\rho) \not\perp s_M(\sigma) \), then we have \( \mu \neq 0 \) so that \( F(\alpha) > 0 \) for all \( \alpha \in [0, +\infty) \). Now by (A) and (B) we may show that \( \log F(\alpha) \) is convex on \( [0, +\infty) \). Let \( \alpha_1, \alpha_2 \in (0, +\infty) \) and \( 0 < \lambda < 1 \). Hölder’s inequality implies that

\[
\int t^{(1-\lambda)\alpha_1 + \lambda \alpha_2} \, d\mu(t) \leq \left[ \int t^{\alpha_1} \, d\mu(t) \right]^{1-\lambda} \left[ \int t^{\alpha_2} \, d\mu(t) \right]^\lambda,
\]

which shows the convexity of \( \log F(\alpha) \).

(3) First, assume that \( s_M(\rho) \not\leq s_M(\sigma) \). As \( 0 < \alpha \not\searrow 1 \), since \( t^\alpha \not\nearrow t \) for \( t \geq 1 \), by the monotone convergence theorem, we have \( F(\alpha) \to F(1) < \rho(1) \). Hence

\[
D_\alpha(\rho\|\sigma) = \frac{\log F(\alpha) - \log \rho(1)}{\alpha - 1} \to +\infty = \frac{D(\rho\|\sigma)}{\rho(1)}.
\]

Second, assume that \( s_M(\rho) \leq s_M(\sigma) \), i.e., \( F(1) = \rho(1) \). For any \( t \in (0, +\infty) \), since \( \alpha \in (0, +\infty) \mapsto t^\alpha \) is convex, we see that as \( 0 < \alpha \not\searrow 1 \),

\[ t - 1 \leq \frac{t^\alpha - t}{\alpha - 1} \not\nearrow t \log t, \quad t \in (0, +\infty), \]

so that the monotone convergence theorem gives

\[
\frac{F(\alpha) - F(1)}{\alpha - 1} = \int \frac{t^\alpha - t}{\alpha - 1} \, d\mu(t) \not\nearrow \int t \log t \, d\mu(t) = S_t \log_t(\rho\|\sigma) = D(\rho\|\sigma). \tag{5.11}
\]

This implies that as \( 0 < \alpha \not\searrow 1 \),

\[ F(\alpha) = \rho(1) + D(\rho\|\sigma)(\alpha - 1) + o(1 - \alpha) \]

so that

\[ \log \frac{F(\alpha)}{\rho(1)} = \frac{D(\rho\|\sigma)(\alpha - 1) + o(1 - \alpha)}{\rho(1)}. \]

Therefore, \( D_\alpha(\rho\|\sigma) \to D(\rho\|\sigma)/\rho(1) \).

Next, assume that \( D_\alpha(\rho\|\sigma) < +\infty \), i.e., \( s_M(\rho) \leq s_M(\sigma) \) and \( \int t^{\alpha_0} \, d\mu(t) < +\infty \) for some \( \alpha_0 > 1 \). As \( \alpha_0 \geq \alpha \not\searrow 1 \), since

\[ \frac{t^{\alpha_0} - 1}{\alpha_0 - 1} \geq \frac{t^\alpha - t}{\alpha - 1} \not\nearrow t \log t, \quad t \in (0, +\infty), \]

the Lebesgue convergence theorem gives, as in (5.11),

\[ \frac{F(\alpha) - F(1)}{\alpha - 1} \not\nearrow D(\rho\|\sigma), \]

and hence the latter assertion is shown similarly to the above.

(4) When \( s_M(\rho) \perp s_M(\sigma) \), this is obvious from (1). Otherwise, this immediately follows from convexity of \( \alpha \mapsto \log Q_\alpha(\rho\|\sigma) \) in (2) (and from definition of \( D_\alpha, Q_1 \) and \( D_1 \)).

(5) Let \( 0 < \alpha < 1 \). Since \( \bar{f}_\alpha = f_{1-\alpha} \), Proposition 2.4 with (5.4) gives \( Q_\alpha(\rho\|\sigma) = Q_{1-\alpha}(\sigma\|\rho) \) and hence for \( \sigma \neq 0 \),

\[ D_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \frac{Q_{1-\alpha}(\sigma\|\rho)}{\rho(1)} = \frac{\alpha}{1 - \alpha} D_{1-\alpha}(\sigma\|\rho) + \frac{1}{1 - \alpha} \log \frac{\rho(1)}{\sigma(1)}, \]
implying \([5.8]\). From this and (3) the second assertion follows.

(6) One can consider \(\log\) as a continuous increasing function from \([0, +\infty]\) to \([-\infty, +\infty]\). We see from (A) and (B) above that for every \(\rho, \sigma \in M^+_*\), \(Q_\alpha(\rho|\sigma)\) is in \([0, +\infty]\) for \(0 \leq \alpha < 1\) and in \((0, +\infty)\) for \(\alpha > 1\). Hence \(D_\alpha(\rho|\sigma)\) is in \((-\infty, +\infty)\) for any \(\alpha \in [0, +\infty) \setminus \{1\}\). Since \(f_\alpha\) is operator convex on \((0, +\infty)\) if \(0 \leq \alpha \leq 2\), by \([5.4]\) and Theorem \([4.1](i)\) the map \((\rho, \alpha) \in M^+_* \times M^+_* \mapsto Q_\alpha(\rho|\sigma)\) is upper semicontinuous for \(0 \leq \alpha < 1\) and lower semicontinuous for \(1 < \alpha \leq 2\) in the \((M^+_*, M)\)-topology. Hence \((\rho, \sigma) \in (M^+_* \setminus \{0\}) \times M^+_* \mapsto D_\alpha(\rho|\sigma)\) is lower semicontinuous for any \(\alpha \in [0, 2] \setminus \{1\}\). For \(\alpha = 1\), the result reduces to the case of the relative entropy due to \([5.6]\).

(7) The first part is a consequence of Theorem \([4.1](ii)\) in view of \([5.4]\). Then the second is clear since a non-negative concave function is log-concave.

(8) The results for \(Q_\alpha\) follow from Theorem \([4.1](iii)\) and \([5.4]\). This gives the assertion on \(D_\alpha\) for \(\alpha \in [0, 2] \setminus \{1\}\). The case of \(D_1\) follows from (3) or it is a well-known fact of \(D\).

(9) follows from Theorem \([4.1](iv)\) in view of \([5.4]\) and \([5.6]\) for \(\alpha = 1\).

(10) When \(\alpha \in [0, 2] \setminus \{1\}\), inequality \([5.10]\) is a special case of \([5.9]\) for \(N = C1\), since for scalars \(\rho(1)\) and \(\sigma(1)\),

\[
D_\alpha(\rho(1)|\sigma(1)) = \frac{1}{\alpha - 1} \log \frac{\rho(1)^\alpha \sigma(1)^{1-\alpha}}{\rho(1)} = \log \frac{\rho(1)}{\sigma(1)}.
\]

By (4) the inequality holds for \(\alpha > 2\) as well. If \(\rho = k\sigma\) with \(k = \rho(1)/\sigma(1)\), then \(\Delta_{\rho,\sigma} \xi_\sigma = k\Delta_\sigma \xi_\sigma = k\xi_\sigma\) and hence \(Q_\alpha(\rho|\sigma) = k^{\alpha}\sigma(1)\). Therefore, \(D_\alpha(\rho|\sigma) = \log(\rho(1)/\sigma(1))\) for all \(\alpha > 0\) (including \(\alpha = 1\)). Conversely, if equality holds for some \(\alpha > 0\), then by (4) the same holds for some \(\alpha \in (0, 1)\). This means that equality \([4.3]\) holds for \(f = f_\alpha\), so that \(\rho = (\rho(1)/\sigma(1))\) follows from Corollary \([4.2](1)\). Finally, the second part of (10) is clear from the first.

\[\square\]

**Remark 5.4.** (1) In Proposition \([5.3](3)\), the assumption that \(D_\alpha(\rho|\sigma) < +\infty\) for some \(\alpha > 1\) is essential to have \(\lim_{\alpha \rightarrow 1+} D_\alpha(\rho|\sigma) = D_1(\rho|\sigma)\). Indeed, it is not difficult to find commuting density operators \(\rho = \sum_{i=1}^\infty a_i |e_i\rangle\langle e_i|\) and \(\sigma = \sum_{i=1}^\infty b_i |e_i\rangle\langle e_i|\) on \(\mathcal{H}\) such that

\[
D(\rho|\sigma) = \sum_{i=1}^\infty a_i \log \frac{a_i}{b_i} < +\infty, \quad D_\alpha(\rho|\sigma) = \sum_{i=1}^\infty a_i^\alpha b_i^{1-\alpha} = +\infty \quad \text{for all } \alpha > 1.
\]

(2) The convexity of \(Q_\alpha\) for \(1 \leq \alpha \leq 2\) in Proposition \([5.3](7)\) cannot extend to \(\alpha > 2\) even in the finite-dimensional case and in separate arguments. This implies that the monotonicity property of \(D_\alpha\) in (9) fails to hold for \(\alpha > 2\), because the monotonicity of \(Q_\alpha\) under unital completely positive maps yields its joint convexity. Also, the monotone decreasing of \(\sigma \mapsto Q_\alpha(\rho|\sigma)\) for \(1 \leq \alpha \leq 2\) in (8) cannot extend to \(\alpha > 2\). But it seems possible that the joint lower semicontinuity of \(D_\alpha\) as in (6) (or in the norm topology) is true for \(\alpha > 2\) as well (this is easily verified in the finite-dimensional case).

(3) In Proposition \([5.3](7)\), due to division by \(\rho(1)\) in definition \([5.3]\), the map \(\rho \mapsto D_\alpha(\rho|\sigma)\) with \(\sigma \in M^+_*\) fixed cannot be convex on the whole \(M^+_*\). However, in the finite-dimensional case it was shown \([37]\, Theorem II.1]\) that \(\sigma \mapsto D_\alpha(\rho|\sigma)\) with \(\rho \in M^+_* \setminus \{0\}\) fixed is convex on \(M^+_*\) for any \(\alpha \in [0, 2]\). It is natural to expect that this extends to the general von Neumann algebra case.

We end the main body of the paper with a remark on relations of \(D_\alpha(\rho|\sigma)\) with other Rényi type divergences from recent papers \([10, 26]\).
Remark 5.5. For every $\rho, \sigma \in M^+_*$, in view of Proposition 5.3 (4) one can define

$$D_\infty(\rho\|\sigma) := \lim_{\alpha \to +\infty} D_\alpha(\rho\|\sigma).$$

The max-relative entropy introduced in [13] is

$$D_{\text{max}}(\rho\|\sigma) := \log \inf \{t > 0 : \rho \leq t\sigma\},$$

where $\inf \emptyset = +\infty$ as usual. The sandwiched Rényi divergence $\tilde{D}_\alpha(\rho\|\sigma)$ [40, 57] has recently been extended to the von Neumann algebra setting by Berta, Scholz and Tomamichel [10] and Jenčová [26, 27]. From [10, 26] we remark that for every $\rho, \sigma \in M^+_*$,

(a) $\tilde{D}_\alpha(\rho\|\sigma) \leq D_\alpha(\rho\|\sigma)$ for $\alpha > 1$,
(b) $\lim_{\alpha \to +\infty} \tilde{D}_\alpha(\rho\|\sigma) = D_{\text{max}}(\rho\|\sigma)$,
(c) $D_2(\rho\|\sigma) \leq D_{\text{max}}(\rho\|\sigma) \leq D_\infty(\rho\|\sigma)$.

6 Closing remarks

In this paper we present a systematic treatment of standard $f$-divergences in the setting of general von Neumann algebras and general operator convex functions $f$ on $(0, +\infty)$. The main theorem is the variational expression of an arbitrary standard $f$-divergence $S_f(\rho\|\sigma)$. We also present a comprehensive account on the quantum Rényi divergence in von Neumann algebras on the basis of theory of standard $f$-divergences. There are some other important quantum divergences; in particular, the maximal $f$-divergence (discussed in [22] in the finite-dimensional case) is worth studying. The most significant problem related to the standard $f$-divergence and other quantum divergences is the reversibility via them, as explained in the Introduction. These should be our forthcoming research topics.

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A Rényi divergences in terms of Haagerup’s $L^p$-spaces

A.1 Haagerup’s $L^p$-spaces and Connes’ Radon-Nikodym cocycles

We first give, for the convenience of the reader, a brief survey on the Haagerup $L^p$-spaces (see [54] for details). Let us take a faithful normal semifinite weight $\varphi_0$ on $M$ and denote by $N$ the crossed product $M \rtimes_{\varphi_0^\tau} \mathbb{R}$ of $M$ by the modular automorphism group $\sigma_t^{\varphi_0} = \Delta^{it}_{\varphi_0} \cdot \Delta^{-it}_{\varphi_0}$, $t \in \mathbb{R}$. Let $\theta_s, s \in \mathbb{R}$, be the dual action of $N$ so that $\tau \circ \theta_s = e^{-s}\tau$, $s \in \mathbb{R}$, where $\tau$ is the canonical trace on $N$. Let $\tilde{N}$ denote the space of $\tau$-measurable operators [41, 54] affiliated with $N$. For $0 < p \leq \infty$ Haagerup’s $L^p$-space $L^p(M)$ [19] is defined by

$$L^p(M) = \{x \in \tilde{N} : \theta_s(x) = e^{-s/p}x, s \in \mathbb{R}\}.$$
In particular, \( L^\infty(M) = M \). Let \( L^p(M)_+ = L^p(M) \cap \tilde{N}_+ \) where \( \tilde{N}_+ \) is the positive part of \( \tilde{N} \). Then \( M_+ \) is canonically order-isomorphic to \( L^1(M) \) by a linear bijection \( \psi \in M_+ \mapsto h_\psi \in L^1(M) \), so that the positive linear functional \( \text{tr} \) on \( L^1(M) \) is defined by \( \text{tr}(h_\psi) = \psi(1) \), \( \psi \in M_+ \).

For \( 0 < p < \infty \) the \( L^p \)-(quasi-)norm \( \|x\|_p \) of \( x \in L^p(M) \) is given by \( \|x\|_p = \text{tr}(|x|^p)^{1/p} \). Also \( \| \cdot \|_\infty \) denotes the operator norm \( \| \cdot \| \) on \( M \). When \( 1 \leq p < \infty \), \( L^p(M) \) is a Banach space with the norm \( \| \cdot \|_p \) and whose dual Banach space is \( L^q(M) \) where \( 1/p + 1/q = 1 \) by the duality

\[
(x,y) \in L^p(M) \times L^q(M) \mapsto \text{tr}(xy) (= \text{tr}(yx)).
\]

In particular, \( L^2(M) \) is a Hilbert space with the inner product \( \langle x,y \rangle = \text{tr}(xy) \) (\( = \text{tr}(yx^*) \)). Then \( (M,L^2(M),J = \ast, L^2(M)_+) \) becomes a standard form of \( M \) where \( M \) is represented on \( L^2(M) \) by the left multiplication. By the uniqueness (up to unitary equivalence) of a standard form we can always choose \( (M,L^2(M),*,L^2(M)_+) \) as a standard form of \( M \). This standard form has the advantage of enjoying the Haagerup \( L^p \)-space technique. Each \( \omega \in M_+^\ast \) is represented as

\[
\omega(x) = \text{tr}(x h_\omega) = \langle h_\omega^{1/2}, xh_\omega^{1/2} \rangle, \quad x \in M,
\]

with the vector representative \( h_\omega^{1/2} \in L^2(M)_+ \). Note that \( s_M(\omega) \) and \( s_{M^\ast}(\omega) \) are the left and right multiplications, respectively, on \( L^2(M) \) of the support projection \( s(\omega) \in M \).

Note that \( L^p(M) \) is independent (up to isomorphism) of the choice of \( \varphi_0 \) and that when \( M \) is semifinite, \( L^p(M) \) coincides with the \( L^p \)-space in the sense of [14 50].

Next let us briefly recall the definition of Connes’ Radon-Nikodym cocycles (see [51 §3], [53 §VIII.3]). For each \( \varphi, \omega \in M_+^\ast \) the balanced functional \( \theta = \theta(\varphi,\omega) \) on \( M_2^\ast(M) = M \otimes M_2^\ast(\mathbb{C}) \) is given by

\[
\theta \left( \sum_{i,j=1}^2 x_{ij} \otimes e_{ij} \right) = \varphi(x_{11}) + \omega(x_{22}), \quad x_{ij} \in M,
\]

where \( e_{ij} \) \((i,j = 1,2)\) are the matrix units of the \( 2 \times 2 \) matrix algebra \( M_2^\ast(\mathbb{C}) \). Note that the support projection of \( \theta \) is \( s(\theta) = s(\varphi) \otimes e_{11} + s(\omega) \otimes e_{22} \) and \( s(\omega)s(\varphi) \otimes e_{21} \in s(\theta)M_2^\ast(M)s(\theta) \). Then Connes’ Radon-Nikodym cocycle \( [D\omega : D\varphi]_t = (s(\omega)s(\varphi)) \) is defined by

\[
\sigma^\theta_t(s(\omega)s(\varphi) \otimes e_{21}) = [D\omega : D\varphi]_t \otimes e_{21}, \quad t \in \mathbb{R},
\]

where \( \sigma^\theta_t \) is the modular automorphism group defined on \( s(\theta)M_2^\ast(M)s(\theta) \). When \( M \) is represented on \( L^2(M) \) as above, we have (see [31])

\[
[D\omega : D\varphi]_t = h^\omega h^\varphi_{-it}, \quad t \in \mathbb{R}. \tag{A.1}
\]

### A.2 Lemmas

For later use we state the following two lemmas, while it seems that they are known to specialists in the subject matter. The first lemma generalizes [11 Theorem 3] and [12 Lemma 3.13]. These lemmas can be shown by [11 9.24] and [52 §9.24] together with a usual argument in analytic function theory (see also [33]).

**Lemma A.1.** For each \( \varphi, \omega \in M_+^\ast \) and \( \delta > 0 \) the following conditions are equivalent:

1. \( h^\varphi_\omega \leq \mu h^\delta_\varphi \), i.e., \( \mu h^\delta_\varphi - h^\delta_\varphi \in L^\delta(M)_+ \) for some \( \mu > 0 \);
(ii) \( s(\omega) \leq s(\varphi) \) and \([D\omega : D\varphi]_t\) extends to a weakly continuous \((M\text{-valued})\) function \([D\omega : D\varphi]_z\) on the strip \(-\delta/2 \leq \text{Im}\ z \leq 0\) which is analytic in the interior.

If the above conditions hold, then \(\| [D\omega : D\varphi]_z \| \leq \mu^{1/2} \) and \([D\omega : D\varphi]_z\) is strongly continuous on \(-\delta/2 \leq \text{Im}\ z \leq 0\), and

\[
h_{\omega}^{p/2} = [D\omega : D\varphi]_z -_{\text{ip}} h_{\varphi}^{p/2}, \quad 0 < p \leq \delta.
\]

**Lemma A.2.** For each \(\varphi, \omega \in M_{s}^\ast\) and \(\delta > 0\) the following conditions are equivalent:

(i) \(\mu^{-1} h_{\omega}^\delta \leq h_{\varphi}^\delta \leq \mu h_{\varphi}^\delta\) for some \(\mu > 0\);

(ii) \( s(\omega) = s(\varphi) \) and \([D\omega : D\varphi]_t\) extends to a weakly continuous \((M\text{-valued})\) function \([D\omega : D\varphi]_z\) on the strip \(-\delta/2 \leq \text{Im}\ z \leq \delta/2\) which is analytic in the interior.

If the above conditions hold, then \(\| [D\omega : D\varphi]_z \| \leq \mu^{1/2}\),

\[
[D\omega : D\varphi]_z = [D\varphi : D\omega]_z = [D\omega : D\varphi]_z^{-1},
\]

and \([D\omega : D\varphi]_z\) is strongly* continuous on \(-\delta/2 \leq \text{Im}\ z \leq \delta/2\).

For every \(\rho, \sigma \in M_{s}^\ast\) and \(x \in M\) recall the following well-known identity

\[
\Delta^p_{\rho, \sigma}(xh_{\sigma}^{1/2}) = h_{\rho}^p x h_{\sigma}^{1/2-p}, \quad 0 \leq p \leq 1/2,
\] (A.2)

with the convention that \(h_{\rho}^0 = s(\rho), h_{\sigma}^0 = s(\sigma)\) and \(\Delta^0_{\rho, \sigma} = s(\rho).J.s(\sigma)J\). Indeed, this is seen from the uniqueness of analytic continuation of \(\Delta^i_{\rho, \sigma}(xh_{\sigma}^{1/2}) = h_{\rho}^i x h_{\sigma}^{1/2-i}\) for \(t \in \mathbb{R}\) (see [31]).

Another lemma we need is the following:

**Lemma A.3.** For every \(\rho, \sigma \in M_{s}^\ast\) and \(0 \leq p \leq 1/2\), the domain of \(\Delta^p_{\rho, \sigma}\) is

\[
\mathcal{D}(\Delta^p_{\rho, \sigma}) = \{ \xi \in L^2(M) : h_{\rho}^p \xi s(\sigma) = \eta h_{\sigma}^p \text{ for some } \eta \in L^2(M) \}.
\]

Moreover, if \(\xi, \eta \in L^2(M)\) are given as in (A.3), then

\[
\Delta^p_{\rho, \sigma}\xi = \Delta^p_{\rho, \sigma}(\xi s(\sigma)) = \eta s(\sigma).
\] (A.4)

**Proof.** Assume first that \(\xi, \eta \in L^2(M)\) are given with \(h_{\rho}^p \xi s(\sigma) = \eta h_{\sigma}^p\) and so \(s(\sigma)\xi^* h_{\rho}^p = h_{\sigma}^p \eta^*\). For every \(x \in M\) we have by (A.2)

\[
\langle \xi, \Delta^p_{\rho, \sigma}(xh_{\sigma}^{1/2}) \rangle = \langle \xi, h_{\rho}^p xh_{\sigma}^{1/2-p} \rangle = \text{tr}(s(\sigma)\xi^* h_{\rho}^p xh_{\sigma}^{1/2-p})
\]

\[
= \text{tr}(h_{\rho}^p \eta^* xh_{\sigma}^{1/2-p}) = \text{tr}(s(\sigma)\eta^* xh_{\sigma}^{1/2}) = \langle \eta s(\sigma), xh_{\sigma}^{1/2} \rangle.
\]

This equality immediately extends to

\[
\langle \xi, \Delta^p_{\rho, \sigma}\zeta \rangle = \langle \eta s(\sigma), \zeta \rangle, \quad \zeta \in Mh_{\sigma}^{1/2} + L^2(M)(1-s(\sigma)).
\]

Since \(Mh_{\sigma}^{1/2} + L^2(M)(1-s(\sigma))\) is a core of \(\Delta^p_{\rho, \sigma}\), it is also a core of \(\Delta^p_{\rho, \sigma}\) (since \(0 < p \leq 1/2\)) by [H] Lemma 4. Hence we find that \(\xi \in \mathcal{D}(\Delta^p_{\rho, \sigma})\) and (A.4) holds.

Conversely, assume that \(\xi \in \mathcal{D}(\Delta^p_{\rho, \sigma})\) and \(\Delta^p_{\rho, \sigma}\xi = \eta \in L^2(M), \) so \(\Delta^p_{\rho, \sigma}(\xi s(\sigma)) = \eta\) and \(\eta s(\sigma) = \eta\). Since \(Mh_{\sigma}^{1/2} + L^2(M)(1-s(\sigma))\) is a core of \(\Delta^p_{\rho, \sigma}\), there exists a sequence \(\{x_n\}\) in \(M\) such that

\[
\| x_n h_{\sigma}^{1/2} - \xi s(\sigma) \| \rightarrow 0, \quad \| \Delta^p_{\rho, \sigma}(x_n h_{\sigma}^{1/2}) - \eta \| \rightarrow 0.
\]
Let \( \eta_n := \Delta_{\rho,\sigma}^p(x_n h_\sigma^{1/2}) \); then \( \eta_n = h_\rho^p x_n h_\sigma^{1/2 - p} \) by (A.2). We hence have

\[
\eta_n h_\sigma^p = h_\rho^p x_n h_\sigma^{1/2}. \tag{A.5}
\]

By Hölder’s inequality \ref{4},

\[
||\eta_n h_\sigma^p - \eta h_\sigma^p||_{\frac{2p}{p+2}} \leq ||\eta_n - \eta||_2 ||h_\sigma^p||_p \to 0, \tag{A.6}
\]

\[
||h_\rho^p x_n h_\sigma^{1/2} - h_\rho^p \xi s(\sigma)||_{\frac{2p}{p+2}} \leq ||h_\rho^p||_p ||x_n h_\sigma^{1/2} - \xi s(\sigma)||_2 \to 0. \tag{A.7}
\]

Combining (A.5)–(A.7) yields \( h_\rho^p \xi s(\sigma) = \eta h_\sigma^p \). Thus (A.3) follows. \( \square \)

### A.3 Description of Rényi divergences

The following provides a useful description of the Rényi divergence \( D_\alpha(\rho \| \sigma) \) in terms of \( h_\rho, h_\sigma \in L^1(M)_+ \).

**Proposition A.4.** Let \( \rho, \sigma \in M^+_\ast \).

1. When \( 0 \leq \alpha < 1 \),

\[
Q_\alpha(\rho \| \sigma) = \text{tr}(h_\rho^\alpha h_\sigma^{1-\alpha}). \tag{A.8}
\]

2. When \( s(\rho) \leq s(\sigma) \) and \( 1 < \alpha \leq 2 \), the following conditions are equivalent:

(i) \( h_\sigma^{1/2} \in \mathcal{D}(\Delta_{\rho,\sigma}^{\alpha/2}) \);

(ii) \( h_\rho^{1/2} \in \mathcal{D}(\Delta_{\rho,\sigma}^{(\alpha-1)/2}) \);

(iii) there exists an \( \eta \in L^2(M)s(\sigma) \) such that \( h_\rho^{\alpha/2} = \eta h_\sigma^{(\alpha-1)/2} \).

If the above conditions hold, then \( \eta \) in (iii) is unique and \( Q_\alpha(\rho \| \sigma) = ||\eta||^2_2 \).

**Proof.** (1) For \( 0 < \alpha < 1 \) we have by (A.2)

\[
Q_\alpha(\rho \| \sigma) = ||\Delta_{\rho,\sigma}^{\alpha/2} h_\sigma^{1/2}||^2 = ||h_\rho^{1/2} h_\sigma^{(1-\alpha)/2}||^2 = \text{tr}(h_\rho^{(1-\alpha)/2} h_\sigma^{(1-\alpha)/2}) = \text{tr}(h_\rho^\alpha h_\sigma^{1-\alpha}).
\]

(2) Assume that \( s(\rho) \leq s(\sigma) \) and \( 1 < \alpha \leq 2 \). Since \( h_\sigma^{1/2} \in \mathcal{D}(\Delta_{\rho,\sigma}^{\alpha/2}) \) and \( \Delta_{\rho,\sigma}^{1/2} h_\sigma^{1/2} = h_\rho^{1/2} s(\sigma) = h_\rho^{1/2} \), it follows that (i) \( \iff \) (ii) and in this case \( \Delta_{\rho,\sigma}^{\alpha/2} h_\sigma^{1/2} = \Delta_{\rho,\sigma}^{(\alpha-1)/2} h_\rho^{1/2} \) (see \[52\] Theorem 9.20 for details). Hence Lemma A.3 with \( p = (\alpha - 1)/2 \) implies that (ii) \( \iff \) (iii) and in this case \( Q_\alpha(\rho \| \sigma) = ||\eta||^2_2 \). The uniqueness of \( \eta \) in (iii) is obvious. \( \square \)

**Remark A.5.** In the above (2), if \( h_\rho^{\alpha/2} = \eta h_\sigma^{(\alpha-1)/2} \) with \( \eta \in L^2(M)s(\sigma) \), then we may write \( \eta = h_\rho^{\alpha/2} h_\sigma^{(1-\alpha)/2} \) in a formal sense, so that

\[
Q_\alpha(\rho \| \sigma) = ||h_\rho^{\alpha/2} h_\sigma^{(1-\alpha)/2}||^2 = \text{tr}(h_\rho^{\alpha/2} h_\sigma^{1-\alpha} h_\rho^{\alpha/2}) = \text{tr}(h_\rho^\alpha h_\sigma^{1-\alpha}),
\]

which is the same expression as in (1). This is in the same form as the quantity \( Q_\alpha \) in the matrix case if we consider \( \text{tr} \) as the usual trace and \( h_\rho, h_\sigma \) as the density matrices.

The next proposition gives a strengthening of Proposition A.3 (8).

**Proposition A.6.** Let \( \rho_i, \sigma_i \in M^+_\ast \) for \( i = 1, 2 \), and \( \mu > 0 \).
(1) Assume that $0 \leq \alpha < 1$. Then $h_{\sigma_1}^{1-\alpha} \leq \mu h_{\sigma_2}^{1-\alpha}$ if and only if $Q_\alpha(\rho\|\sigma_1) \leq \mu Q_\alpha(\rho\|\sigma_2)$ for all $\rho \in M_0^+$.

(2) Assume that $1 < \alpha \leq 2$. If $h_{\rho_1}^\alpha \leq \mu h_{\rho_2}^\alpha$, then $Q_\alpha(\rho_1\|\sigma) \leq \mu Q_\alpha(\rho_2\|\sigma)$ for all $\sigma \in M_+^\delta$.

Proof. (1) Let $0 \leq \alpha < 1$. Recall \cite[Proposition II.33]{54} that for any $b \in L^{1/(1-\alpha)}(M)$, $b \geq 0$ if and only if $tr(ab) \geq 0$ for all $a \in L^{1/(\alpha)}(M)_+$. Hence the assertion is immediate from (A.8).

(2) Let $1 < \alpha \leq 2$. Assume that $h_{\rho_1}^\alpha \leq \mu h_{\rho_2}^\alpha$. To prove the asserted inequality, we may assume that $Q_\alpha(\rho_2\|\sigma) < +\infty$ so that $s(\rho_2) \leq s(\sigma)$ and $h_{\sigma_2}^{1/2} \in D(\Delta_{\rho_2,\sigma}^{\alpha/2})$. By Proposition A.3 (2) there exists an $\eta \in L^2(M)s(\sigma)$ such that $h_{\rho_2}^{1/2} = \eta h_{\sigma_2}^{(\alpha-1)/2}$. By Lemma A.1 one has $a := [D\rho_1 : D\rho_2]_{-\alpha/2} \in M$, for which $h_{\rho_2}^{1/2} = ah_{\rho_2}^{\alpha/2}$ and $\|a\| \leq \mu^{1/2}$. Therefore, $h_{\rho_1}^{1/2} = a\eta h_{\sigma_2}^{(\alpha-1)/2}$, so Proposition A.3 (2) gives $Q_\alpha(\rho_1\|\sigma) = \|a\eta\|^2_2 \leq \mu \|\eta\|^2_2 = \mu Q_\alpha(\rho_2\|\sigma)$.

Remark A.7. Anna Jenčová \cite{28} informed the author that she could prove Lemma A.3 for every $p \geq 0$ by using analyticity of $z \mapsto h_z^p$ in $Re z > 0$ \cite[Lemma II.18]{54} and a convergence argument. Then the equivalence of (i)–(iii) in Proposition A.3 (2) is true for all $\alpha > 1$, thus Propositions A.6 (2) holds for all $\alpha > 1$.

B Relative Hamiltonians in Terms of Haagerup’s $L^p$-Spaces

The theory of relative Hamiltonians in the general von Neumann algebra setting was formerly developed by Araki \cite{2} and Donald \cite{15} in close relation to the relative entropy. Although the topic is not strongly related to the main body of this paper, it is worthwhile to consider that in a more general framework in terms of Haagerup’s $L^p$-spaces, as a sequel of Appendix A.

B.1 Survey on Relative Hamiltonians

Let $(M, \mathcal{H}, J, \mathcal{P})$ be a standard form of $M$ and $\varphi \in M_+^\delta$ be faithful so that $\varphi = \langle \Phi, \cdot \Phi \rangle$ with the cyclic and separating vector $\Phi \in \mathcal{P}$. For each $h \in M_{sa}$ Araki \cite{2} defined the perturbed vector $\Phi^h$ by

$$\Phi^h := \sum_{n=0}^{\infty} \int_0^{1/2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_n-1} dt_n \Delta_{\varphi}^{t_n} h \Delta_{\varphi}^{t_{n-1}-t_n} h \cdots \Delta_{\varphi}^{t_1-t_2} h \Phi,$$  \hspace{1cm} (B.1)

where $\Phi$ is in the domain of $\Delta_{\varphi}^{z_1} h \Delta_{\varphi}^{z_2} h \cdots \Delta_{\varphi}^{z_n} h$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ with $Re \ z \in \{(s_1, \ldots, s_n) \in \mathbb{R}^n : s_1, \ldots, s_n \geq 0, s_1 + \cdots + s_n \leq 1/2\}$ and the right-hand side of (B.1) absolutely converges. Then $\Phi^h$ is also a cyclic and separating vector in $\mathcal{P}$. It is known \cite{3} that if $\Phi$ is in the domain of $e^{1/2}(\log \Delta_{\varphi} + h)$ and $\Phi^h = e^{1/2}(\log \Delta_{\varphi} + h) \Phi$. A version of the Trotter product formula was given in \cite[Remarks 1, 2]{3} as follows:

$$\Phi^h = \lim_{n \to \infty} (\Delta_{\varphi}^{h/2n} e^{h/2n})^n \Phi = \lim_{n \to \infty} (e^{h/2n} \Delta_{\varphi}^{h/2n})^n \Phi$$ (strong limit).
The perturbed functional \( \varphi^h \) is defined by \( \varphi^h = \langle \Phi^h, \cdot \rangle \) independently of the choice of a standard form. In fact, we have (see \[2\] Proposition 4.3 and the proof of \[15\] Theorem 6)

\[
[D\varphi^h : D\varphi]_t = \sum_{n=0}^\infty t^n \int_0^t \int_0^{t_1} \cdots dt_{n-1} \int_0^{t_n} dt_{n} \varphi_{\alpha}(h) \cdots \varphi_{\alpha}(h). \tag{B.2}
\]

When \( \omega = \varphi^h \) with \( h \in M_{sa} \), \(-h\) is called a \textit{relative hamiltonian} of \( \omega \) relative to \( \varphi \).

Let \( \mathcal{S}(M) \) denote the set of normal states on \( M \). If \( h : \mathcal{S}(M) \to (-\infty, +\infty) \) is a weakly lower semicontinuous affine map whose range is lower bounded, then \( h \) is called an \textit{extended-valued lower-bounded operator} affiliated with \( M \). We denote by \( M_{ext} \) the set of all such extended-valued operators. Note \[15\] Proposition 2.13 (also \[18\] Theorem 1.5) that for each \( h \in M_{ext} \) there exist a projection \( p \in M \) and a spectral resolution \( \{ e_{\lambda} : s \leq \lambda < \infty \} \) in \( M \) with \( s \in \mathbb{R} \) and \( e_{\infty} = 1 - p \) such that \( h \) is represented as

\[
h(\rho) = \int_s^\infty \lambda d\rho(e_\lambda) + \infty \rho(p), \quad \rho \in \mathcal{S}(M),
\]

and that \( h \in M_{ext} \) if and only if there exists an increasing net \( \{ h_\alpha \} \) in \( M_{sa} \) such that \( h(\rho) = \sup_{\alpha} \rho(h_\alpha) \) for \( \rho \in \mathcal{S}(M) \). Obviously, \( h \in M_{ext} \) can extend to a positively homogeneous map \( h : M_{sa}^+ \to (-\infty, +\infty] \). We further set \( -M_{ext} := \{ -h : h \in M_{ext} \} \). Namely, \( h \in -M_{ext} \) is represented as

\[
h(\rho) = \int_{-\infty}^r \lambda d\rho(e_\lambda) + (-\infty) \rho(p), \quad \rho \in M_{sa}^+,
\]

where \( \{ e_{\lambda} : -\infty < \lambda \leq r \} \) is a spectral resolution in \( M \) with \( r \in \mathbb{R} \) and \( e_r = 1 - p \). In this case we write \( h \leq r \). For each \( \varphi \in M_{sa}^+ \) and \( h \in M_{ext} \), define \( c(\varphi, h) \in (-\infty, +\infty] \) by

\[
c(\varphi, h) := \inf \{ h(\rho) + D(\rho \| \varphi) : \rho \in \mathcal{S}(M) \}.
\]

Donald \[15\] Theorem 3.1 proved that if \( c(\varphi, h) < +\infty \), that is, \( h(\rho) + D(\rho \| \varphi) < +\infty \) for some \( \rho \in \mathcal{S}(M) \), then there exists a unique \( \omega \in \mathcal{S}(M) \) such that \( h(\omega) + D(\omega \| \varphi) = c(\varphi, h) \). We denote this \( \omega \) by \([\varphi^h]\); this was denoted in \[15\] by \( \varphi^h \) but we prefer to use notation consistently with Araki’s one.

Note \[47\] Proposition 1 (also \[48\] Appendix) that when \( \varphi \in M_{sa}^+ \) is faithful and \( h \in M_{sa} \), \([\varphi^h]\) coincides with \( \varphi^{-h} \) up to a normalization constant; more precisely, \( \varphi^{-h} = e^{-c(\varphi, -h)}[\varphi^{-h}] \) and \( \varphi^{-h}(1) = e^{-c(\varphi, -h)} \). So, for any \( \varphi \in M_{sa}^+ \) and \( h \in -M_{ext} \) such that \( c(\varphi, h) < +\infty \) we can define the perturbed functional \( \varphi^h \) by

\[
\varphi^h := e^{-c(\varphi, -h)}[\varphi^{-h}]
\]

and call \(-h\) a \textit{relative hamiltonian} of \( \omega = \varphi^h \) relative to \( \varphi \). But when \( c(\varphi, -h) = +\infty \) let \( \varphi^h := 0 \) as convention. Moreover, it is worthwhile to recall Petz’ variational expression of the relative entropy \[47\] Theorem 9; namely, if \( \varphi \) is faithful and \( \omega \in \mathcal{S}(M) \), then

\[
D(\omega \| \varphi) = \sup \{ \omega(h) - \log \varphi^h(1) : h \in M_{sa} \}. \tag{B.3}
\]

(Even when \( \varphi \) is not faithful, \[48\] remains valid with \( \omega(s(\varphi)hs(\varphi)) \) in place of \( \omega(h) \).)

**B.2 Theorems**

We here prove the next theorems as to the existence of relative hamiltonian, generalizing \[2\] Theorem 6.3 and \[15\] Theorem 4.3, respectively, in the framework of the standard form \((M, L^2(M), \ast, L^2(M)_{\ast})\).
Theorem B.1. If $\varphi, \omega \in M^+_s$ and $\nu h^\delta_{\varphi} \leq h^\delta_\omega \leq \mu h^\delta_{\varphi}$ for some $\delta, \mu, \nu > 0$, then there exists an $h \in M_{sa}$ such that $\omega = \varphi^h$ and $\delta^{-1} \log \nu \leq h \leq \delta^{-1} \log \mu$.

Theorem B.2. If $\varphi, \omega \in M^+_s$ and $h^\delta_\omega \leq \mu h^\delta_{\varphi}$ for some $\delta, \mu > 0$, then there exists an $h \in -M_{ext}$ such that $\omega = \varphi^h$, $h \leq \delta^{-1} \log \mu$, and
\[
D(\rho\|\omega) = -h(\rho) + D(\rho\|\varphi), \quad \rho \in M^+_s.
\]

To prove the theorems, we first give the following:

Lemma B.3. If $\varphi, \omega \in M^+_s$ and $h^\delta_\omega \leq \mu h^\delta_{\varphi}$ for some $\delta, \mu > 0$, then
\[
D(\rho\|\omega) \geq D(\rho\|\varphi) - \rho(1)\frac{\log \mu}{\delta}, \quad \rho \in M^+_s.
\]

Proof. Since both sides of the asserted inequality are zero if $\rho = 0$, we may assume that $\rho \neq 0$. By assumption we have $h^{1-\alpha}_{\varphi} \leq \mu^{(1-\alpha)/\delta} h^{1-\alpha}_{\omega}$ for any $\alpha \in (0, 1)$ with $1 - \alpha \leq \delta$. Then Proposition A.6(1) implies that $Q_\alpha(\rho\|\omega) \leq \mu^{(1-\alpha)/\delta} Q_\alpha(\rho\|\varphi)$ so that
\[
\log \frac{Q_\alpha(\rho\|\omega)}{\rho(1)} \leq \log \frac{Q_\alpha(\rho\|\varphi)}{\rho(1)} + \frac{1 - \alpha}{\delta} \log \mu.
\]
Therefore, $D_\alpha(\rho\|\omega) \geq D_\alpha(\rho\|\varphi) - \delta^{-1} \log \mu$. Letting $\alpha \nearrow 1$ gives the desired inequality due to Proposition A.3(3).

Proof of Theorem B.1. Let $\varphi, \omega \in M^+_s$ satisfy the assumption of the theorem. We may suppose that $\varphi$ is faithful (hence so is $\omega$). By Lemma A.2 and [52, §9.24] one can define $h \in M$ by $h = -i \frac{d}{dt}[D\omega : D\varphi]_{|t=0}$. Then since $[D\omega : D\varphi]_t$ is a $\sigma^*_t$-unitary cocycle (see [51 [53]), one has $h \in M_{sa}$ and
\[
\frac{d}{dt}[D\omega : D\varphi]_t = [D\omega : D\varphi]_t \sigma^*_t(ih), \quad t \in \mathbb{R},
\]
so that $[D\omega : D\varphi]_t = [D\varphi^h : D\varphi]_t$ for all $t \in \mathbb{R}$ by [4, Theorem 1] and (B.2). Therefore $\omega = \varphi^h$. For every $\rho \in M^+_s$ one has by [6, Theorem 3.10]
\[
D(\rho\|\omega) = -\rho(h) + D(\rho\|\varphi) \tag{B.5}
\]
and by Lemma B.3
\[
D(\rho\|\omega) \geq D(\rho\|\varphi) - \rho(1)\frac{\log \mu}{\delta}, \tag{B.6}
\]
\[
D(\rho\|\varphi) \geq D(\rho\|\omega) + \rho(1)\frac{\log \nu}{\delta}. \tag{B.7}
\]
When $D(\rho\|\varphi) < \infty$, estimates (B.5)–(B.7) imply that
\[
\rho(1)\frac{\log \nu}{\delta} \leq \rho(h) \leq \rho(1)\frac{\log \mu}{\delta}.
\]
But the set of $\rho \in M^+_s$ with $D(\rho\|\varphi) < \infty$ is dense in $M^+_s$ by the faithfulness of $\varphi$. Hence $\delta^{-1} \log \nu \leq h \leq \delta^{-1} \log \mu$.

Proof of Theorem B.2. Assume that $\varphi, \omega \in M^+_s$ and $h^\delta_\omega \leq \mu h^\delta_{\varphi}$ with $\delta, \mu > 0$. We may suppose that $\varphi$ is faithful and $\omega$ is nonzero (the case $\omega = 0$ is trivial). For each $\varepsilon > 0$ define
\( \omega(\varepsilon) \in M^+_* \) by \( h_{\omega(\varepsilon)} = (h_{\omega} + \varepsilon h_{\varphi})^{1/\delta} \). Since \( \varepsilon h_{\varphi} \leq h_{\omega(\varepsilon)} \leq (\mu + \varepsilon)h_{\varphi}^{\delta} \), Theorem B.1 implies that there exists an \( h(\varepsilon) \in M_{sa} \) such that \( \omega(\varepsilon) = \varphi^{h(\varepsilon)} \) and \( h(\varepsilon) \leq \delta^{-1} \log(\mu + \varepsilon) \). When \( 0 < \varepsilon < \varepsilon' \), it follows from Lemma B.3 that \( D(\rho\|\omega(\varepsilon)) \geq D(\rho\|\omega(\varepsilon')) \) for all \( \rho \in M^+_* \). Since by [6, Theorem 3.10]

\[
D(\rho\|\omega(\varepsilon)) = -\rho(h(\varepsilon)) + D(\rho\|\varphi), \quad \rho \in M^+_*,
\]

one has \(-\rho(h(\varepsilon)) \geq -\rho(h(\varepsilon'))\) for all \( \rho \in M^+_* \) with \( D(\rho\|\varphi) < \infty \), so that \(-h(\varepsilon) \geq -h(\varepsilon')\) as in the proof of Theorem B.1. So \( h \in -M_{ext} \) can be defined by

\[
-h(\rho) = \sup_{\varepsilon>0} \rho(-h(\varepsilon)) = \lim_{\varepsilon \searrow 0} \rho(-h(\varepsilon)), \quad \rho \in M^+_*.
\]

Then one has \( h \leq \delta^{-1} \log \mu \) as the limit of \( h(\varepsilon) \leq \delta^{-1} \log(\mu + \varepsilon) \).

Next, let us prove (B.4). Note [54, Lemma II.16 and Corollary II.17] that the norm topology on \( L^p(M) (\subset N) \) coincides with the measure topology (induced by the canonical trace on \( N \)). We see by [55] that \( h_{\omega(\varepsilon)} \to h_{\omega} \) as \( \varepsilon \searrow 0 \) in the measure topology. Hence \( ||\omega(\varepsilon) - \omega|| \to 0 \), so that we have for every \( \rho \in M^+_* \)

\[
D(\rho\|\omega) \leq \liminf_{\varepsilon \searrow 0} D(\rho\|\omega(\varepsilon))
\]

by the lower semicontinuity of relative entropy [6, Theorem 3.7(1)]. On the other hand, Lemma B.3 shows that \( D(\rho\|\omega) \geq D(\rho\|\omega(\varepsilon)) \) for all \( \varepsilon > 0 \) because \( h_{\omega} \leq h_{\omega(\varepsilon)} \). Therefore,

\[
D(\rho\|\omega) = \lim_{\varepsilon \searrow 0} D(\rho\|\omega(\varepsilon)).
\]

Now (B.4) is immediate by taking the limit of (B.8). Finally, (B.4) implies that for every \( \rho \in \mathcal{G}(M) \)

\[
-h(\rho) + D(\rho\|\varphi) = D(\rho\|\omega) \geq -\log \omega(1)
\]

and in particular, when \( \rho = \omega/\omega(1) \),

\[
-h(\rho) + D(\rho\|\varphi) = D(\omega/\omega(1)||\omega) = -\log \omega(1).
\]

Hence \( c(\varphi, -h) = -\log \omega(1) < \infty \) and \( \omega/\omega(1) = [\varphi^{-h}] \), so that \( \omega = e^{-c(\varphi, -h)}[\varphi^{-h}] = \varphi^h \), completing the proof. \( \square \)

**Remark B.4.** Assume that \( \varphi \in M^+_* \) be faithful and \( \omega = \varphi^h \) for some \( h \in M_{sa} \). Note [2, Proposition 4.6] that \(-h\) is a unique relative hamiltonian of \( \omega \) relative to \( \varphi \). This is seen also from [6, Theorem 3.10] or more explicitly from [6, (4.28)]. Moreover, according to [24, Theorem III.1] and its remarks we have

\[
h_{\omega}^{1/2} = \exp\left( \frac{\log h_{\varphi} + h}{2} \right) = \lim_{n \to \infty} (h_{\varphi}^{1/2n} e^{h/2n})^n \quad (\text{weak limit})
\]

and hence

\[
h_{\omega} = \exp(\log h_{\varphi} + h). \quad (B.9)
\]

(In fact, the results of [24] in the spatial \( L^p \)-spaces can be automatically transformed into those in the Haagerup \( L^p \)-spaces due to [54, Theorem IV.12].) Since \( D(\omega\|\varphi) = \omega(h) \), (B.9) yields the formula

\[
D(\omega\|\varphi) = \text{tr}(h_{\omega}(\log h_{\omega} - \log h_{\varphi})), \quad (B.10)
\]

which has a complete resemblance to Umegaki’s relative entropy in the semifinite case (see [12]). However, when \( \omega = \varphi^h \) with a relative hamiltonian \(-h \in M_{ext} \) unbounded from above, it seems problematic to determine whether formulas (B.9) and (B.10) remain to make sense (see [39] for a related discussion).
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