THE LOWEST HIDDEN CHARMED TETRAQUARK STATE FROM QCD SUM RULES

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Abstract

In this article, we study the $SS$ type scalar tetraquark state $cq\bar{c}\bar{q}$ in details with the QCD sum rules by calculating the contributions of the vacuum condensates up to dimension-10 in the operator product expansion, and obtain the value $M_{Zc} = (3.82^{+0.08}_{-0.08})$ GeV, which is the lowest mass for the hidden charmed tetraquark states from the QCD sum rules. Furthermore, we calculate the hadronic coupling constants $G_{Zc,0,8}$ and $G_{Zc,DD}$ with the three-point QCD sum rules, then study the strong decays $Z_c \rightarrow \eta, \pi, DD$, and observe that the total width $\Gamma_{Z_c} \approx 21$ MeV. The present predictions can be confronted with the experimental data in the futures at the BESIII, LHCb and Belle-II.

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1 Introduction

The scattering amplitude for one-gluon exchange in an $SU(N_c)$ gauge theory is proportional to

$$t^a_{ki}t^a_{lj} = -\frac{N_c + 1}{4N_c}(\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl}) + \frac{N_c - 1}{4N_c}(\delta_{jk}\delta_{il} + \delta_{ik}\delta_{jl}),$$

(1)

where the $t^a$ is the generator of the gauge group, and the $i, j$ and $k, l$ are the color indexes of the two quarks in the incoming and outgoing channels respectively. For $N_c = 3$, the negative sign in front of the antisymmetric antitriplet indicates the interaction is attractive and favors the formation of the diquark states in the color antitriplet, while the positive sign in front of the symmetric sextet indicates the interaction is repulsive and disfavors the formation of the diquark states in the color sextet [1].

The antitriplet diquark states have five Dirac tensor structures, scalar $C_{\gamma 5}$, pseudoscalar $C$, vector $C_{\gamma \mu \gamma 5}$, axial vector $C_{\gamma \mu}$ and tensor $C_{\sigma \mu \nu}$. The structures $C_{\gamma 5}$, $C_{\gamma \mu}$ and $C_{\sigma \mu \nu}$ are symmetric, the structures $C_{\gamma 5}$, $C$ and $C_{\gamma \mu \gamma 5}$ are antisymmetric. The attractive interactions of one-gluon exchange favor formation of the diquarks in color antitriplet $3_c$, flavor antitriplet $3_f$ and spin singlet $1_s$ (or flavor sextet $6_f$ and spin triplet $3_c$) [2,3], so the favored configurations are the scalar and axial-vector diquark states. The scalar ($S$) and axial-vector ($A$) heavy-light diquark states have almost degenerate masses from the QCD sum rules [4,5]. In Refs. [6,7], we take the $C_{\gamma 5} - C_{\gamma 5}$, $C_{\gamma \mu} - C_{\gamma \mu}$, $C_{\gamma \mu \gamma 5} - C_{\gamma \mu \gamma 5}$ type interpolating currents to study the masses of the scalar diquark states in a systematic way using the QCD sum rules, and observe that the $SS$ and $A\bar{A}$ type scalar diquark states have almost degenerate masses, about 4.36 GeV, which is much larger than that from the phenomenological models [8,9,10].

In Ref. [8], Ebert, Faustov and Galkin calculate the masses of the excited heavy tetraquarks with hidden charm within the relativistic diquark-antidiquark picture based on the quasipotential approach, and obtain the values $M_{J=0} = 3.852$ GeV and $3.812$ GeV for the $A\bar{A}$ and $SS$ type scalar tetraquark states $cq\bar{c}\bar{q}$, respectively. While L. Maiani et al obtain the values $M_{J=0} = 3.832$ GeV and $3.723$ GeV for the $A\bar{A}$ and $SS$ type scalar tetraquark states $cq\bar{c}\bar{q}$ respectively in the type-I diquark-antidiquark model [9], and $M_{J=0} = 4.000$ GeV and $3.770$ GeV for the $A\bar{A}$ and $SS$ type scalar tetraquark states $cq\bar{c}\bar{q}$ respectively in the type-II diquark-antidiquark model [10]. In those model-dependent studies, the masses of the $A\bar{A}$-type scalar tetraquark states are larger than that of the $SS$ type scalar tetraquark states.

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In Refs. [11, 12, 13, 14, 15, 16], we explore the energy scale dependence of the hidden charmed (bottom) tetraquark states and molecular states in details for the first time, and suggest a formula

$$\mu = \sqrt{M_{X/Y/Z}^2 - (2M_Q)^2},$$

(2)

with the effective heavy $Q$-quark mass $M_Q$ to determine the energy scales of the QCD spectral densities in the QCD sum rules, which works well. According to the formula, the energy scale $\mu = 1$ GeV taken in Refs. [6, 7] is too low to result in robust predictions.

In Ref. [13], we choose the $C\gamma_\mu - C\gamma_\nu$ type interpolating currents to study the $A\bar{A}$-type scalar, axial-vector and tensor tetraquark states in details with the QCD sum rules. The predicted masses of the axial-vector and tensor tetraquark states favor assigning the $Z_c(4020)$ and $Z_c(4025)$ as the $J^{PC} = 1^{+-}$ or $2^{++}$ diquark-antidiquark type tetraquark states. While there are no experimental candidates to match the predicted mass of the scalar tetraquark state $M_{J=0} = (3.85^{+0.15}_{-0.05})$ GeV. The value is consistent with the prediction $M_{J=0} = 3.852$ GeV based on the quasipotential approach [8], while the upper bound reaches the prediction $M_{J=0} = 4.000$ GeV based on the type-II diquark-antidiquark model [10]. According to Refs. [8, 9, 10], the $SS$-type scalar tetraquark states have smaller masses than that of the corresponding $A\bar{A}$-type scalar tetraquark states. It is interesting to see whether or not such conclusion survives when confronted with the QCD sum rules. In Refs. [11, 12, 13], we observe that the masses of the $S\bar{A}$ or $A\bar{S}$ type axial-vector tetraquark states are larger than that of the $A\bar{A}$ type scalar tetraquark states. So the $SS$ scalar tetraquark state maybe the lowest tetraquark state.

In this article, we study the scalar $SS$-type hidden charmed tetraquark state (thereafter we will denote it as $Z_c$) by calculating the contributions of the vacuum condensates up to dimension-10, and try to obtain the lowest mass based on the QCD sum rules. Furthermore, we calculate the hadronic coupling constants $G_{Z_c,\eta,\pi}$ and $G_{Z_c,DD}$ with the three-point QCD sum rules, then study the strong decays $Z_c \rightarrow \eta, \pi, DD$.

The article is arranged as follows: we derive the QCD sum rules for the mass and pole residue of the scalar tetraquark state $Z_c$ and for the hadronic coupling constants $G_{Z_c,\eta,\pi}$ and $G_{Z_c,DD}$ in section 2; in section 3, we present the numerical results and discussions; section 4 is reserved for our conclusion.

## 2 QCD sum rules for the scalar tetraquark state

In the following, we write down the two-point correlation function $\Pi(p)$ in the QCD sum rules,

$$\Pi(p) = i \int d^4xe^{ip\cdot x} \langle 0|T \{J(x)J^\dagger(0)\}|0\rangle,$$

(3)

$$J(x) = \epsilon^{ijk}\epsilon^{lmn}u^j(x)C\gamma_5\epsilon^k(x)d^m(x)\gamma_5\bar{c}^n(x),$$

(4)

where the $i, j, k, l, m, n$ are color indexes, the $C$ is the charge conjugation matrix.

At the hadronic side, we can insert a complete set of intermediate hadronic states with the same quantum numbers as the current operator $J(x)$ into the correlation function $\Pi(p)$ to obtain the hadronic representation [17, 18, 19]. After isolating the ground state contribution of the scalar tetraquark state, we get the following result,

$$\Pi(p) = \frac{\lambda_{Z_c}^2}{M_{Z_c}^2 - p^2} + \cdots,$$

(5)

where the pole residue $\lambda_{Z_c}$ is defined by $\langle 0|J(0)|Z(p)\rangle = \lambda_{Z_c}$. 

2
In the following, we briefly outline the operator product expansion for the correlation function $\Pi(p)$ in perturbative QCD. We contract the $u$, $d$ and $c$ quark fields in the correlation function $\Pi(p)$ with Wick theorem, and obtain the result:

$$
\Pi(p) = i\epsilon^{ijk} \epsilon^{lmn} \epsilon^{i'j'k'} \epsilon^{l'm'n'} \int d^4x e^{ipx} \left[ \gamma_5 C \langle \pi(x) \rangle \right] \left[ \gamma_5 C \langle \pi(T(x)) \rangle \right],
$$

where the $U_{ij}(x)$, $D_{ij}(x)$ and $C_{ij}(x)$ are the full $u$, $d$ and $c$ quark propagators respectively (the $U_{ij}(x)$ and $D_{ij}(x)$ can be written as $S_{ij}(x)$ for simplicity),

$$
S_{ij}(x) = \frac{i}{2\pi^2} \int d^4k e^{-ikx} \left\{ \frac{\delta_{ij}}{12} - \frac{i\delta_{ij}x^2(\bar{q}q, \sigma Gq)}{192} - \frac{ig_s G^a_{\alpha} t_{ij} \sigma_{\alpha\beta}(k + m_c) + (k + m_c) \sigma_{\alpha\beta}}{32\pi^2 x^2} - \frac{i\delta_{ij}x^2 \bar{q}q^2(\bar{q}q)^2}{7776} \right\}
$$

$$
C_{ij}(x) = \frac{i}{(2\pi)^4} \int d^4k e^{-ikx} \left\{ \frac{\delta_{ij}}{12} - \frac{g_s G^a_{\alpha} t_{ij} \sigma_{\alpha\beta}(k + m_c) + (k + m_c) \sigma_{\alpha\beta}}{4(k^2 - m_c^2)^2} \right\}
$$

and $f^{\alpha\beta\mu\nu} = (k + m_c) \gamma^\alpha(k + m_c) \gamma^\beta(k + m_c)$, $f^{\alpha\beta} = (k + m_c) \gamma^\alpha(k + m_c) \gamma^\beta(k + m_c)$,

Once the analytical expression is obtained, we can take the quark-hadron duality below the continuum threshold $s_0$ and perform Borel transform with respect to the variable $P^2 = -p^2$ to obtain the following QCD sum rule:

$$
\lambda_{Z_c}^2 \exp \left( -\frac{M_{Z_c}^2}{T^2} \right) = \int_{m_c^2}^{s_0} ds \rho(s) \exp \left( -\frac{s}{T^2} \right),
$$

where

$$
\rho(s) = \rho_0(s) + \rho_1(s) + \rho_2(s) + \rho_3(s) + \rho_4(s) + \rho_5(s) + \rho_6(s) + \rho_7(s) + \rho_8(s) + \rho_{10}(s),
$$

$$
\rho_0(s) = \frac{1}{512\pi^2} \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \int_{z_1}^{1-y} dz \int_{z_1}^{1-y} dz (1 - y - z)^3 (s - m_c^2)^2 (7s_2^2 - 6m_c^2 + m_c^2),
$$

$$
\rho_3(s) = -\frac{m_c(\bar{q}q)}{16\pi^4} \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz (y + z)(1 - y - z) (s - m_c^2) (2s - m_c^2),
$$

$$
\rho_4(s) = -\frac{m_c^2}{384\pi^4} \frac{\alpha_s G}{\pi} \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \left\{ \frac{z}{y^2} + \frac{y}{z^2} \right\} (1 - y - z)^3 \left( 2s - m_c^2 + \frac{m_c^2}{6} \delta(s - m_c^2) \right) + \frac{1}{512\pi^4} \frac{\alpha_s G}{\pi} \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz (y + z)(1 - y - z) (10s_2^2 - 12s m_c^2 + 3m_c^4),
$$

3
\[
\rho_5(s) = \frac{m_c \langle \bar{q} q, \sigma G q \rangle}{64\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{y+z}{y} \right) (3s - 2m_c^2) \\
- \frac{m_c \langle q \bar{q}, \sigma G \rangle}{64\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{y+z}{y} \right) (1-y-z) (3s - 2m_c^2),
\]

\[
\rho_6(s) = \frac{m_c^2 \langle \bar{q} q \rangle^2}{12\pi^2} \int_{y_i}^{y_f} dy + \frac{g_s^2 \langle \bar{q} q \rangle^2}{108\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz yz \left\{ 2s - m_c^2 + \frac{m_c^4}{6} \delta (s - m_c^2) \right\} \\
- \frac{g_s^2 \langle \bar{q} q \rangle^2}{512\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz (1-y-z) \left\{ \frac{z}{y} + \left( \frac{1}{y^2} + \frac{1}{z^2} \right) \right\} \\
- \frac{m_c^2 [2 + m_c^2 \delta (s - m_c^2)]}{3888\pi^4} + (y+z) \left[ 12 (2s - m_c^2) + 2m_c^4 \delta (s - m_c^2) \right],
\]

\[
\rho_7(s) = \frac{m_c^3 \langle \bar{q} q \rangle}{288\pi^2} \left\{ \frac{\alpha_s \sigma G}{\pi} \right\} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{y+z}{y^2} + \frac{z}{y^2} + \frac{1}{y^2} + \frac{1}{z^2} \right) (1-y-z) \\
\left( \frac{1 + m_c^2}{T^2} \right) \delta (s - m_c^2) \\
- \frac{m_c^3 \langle \bar{q} q \rangle}{96\pi^2} \left\{ \frac{\alpha_s \sigma G}{\pi} \right\} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{y+z}{y^2} + \frac{z}{y^2} \right) (1-y-z) \left\{ 2 + m_c^2 \delta (s - m_c^2) \right\} \\
- \frac{m_c^3 \langle \bar{q} q \rangle}{96\pi^2} \left\{ \frac{\alpha_s \sigma G}{\pi} \right\} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left\{ 2 + m_c^2 \delta (s - m_c^2) \right\} \\
- \frac{m_c^3 \langle \bar{q} q \rangle}{576\pi^2} \left\{ \frac{\alpha_s \sigma G}{\pi} \right\} \int_{y_i}^{y_f} dy \left\{ 2 + m_c^2 \delta (s - m_c^2) \right\},
\]

\[
\rho_8(s) = -\frac{m_c^2 \langle \bar{q} q, \sigma G q \rangle}{24\pi^2} \int_0^1 dy \left( 1 + \frac{m_c^2}{T^2} \right) \delta (s - m_c^2) \\
+ \frac{m_c^2 \langle \bar{q} q, \sigma G q \rangle}{48\pi^2} \int_0^1 dy \frac{1}{y(1-y)} \delta (s - m_c^2),
\]

\[
\rho_{10}(s) = \frac{m_c^2 \langle \bar{q} g, \sigma G q \rangle^2}{192\pi^2 T^6} \int_0^1 dy \tilde{m}_c^4 \delta (s - m_c^2) \\
- \frac{m_c^4 \langle \bar{q} q \rangle^2}{216 T^4} \left\{ \frac{\alpha_s \sigma G}{\pi} \right\} \int_0^1 dy \left\{ \frac{1}{y^3} + \frac{1}{(1-y)^3} \right\} \delta (s - m_c^2) \\
+ \frac{m_c^4 \langle \bar{q} q \rangle^2}{72 T^2} \left\{ \frac{\alpha_s \sigma G}{\pi} \right\} \int_0^1 dy \left\{ \frac{1}{y^2} + \frac{1}{(1-y)^2} \right\} \delta (s - m_c^2) \\
- \frac{m_c^4 \langle \bar{q} g, \sigma G q \rangle^2}{192\pi^2 T^4} \int_0^1 dy \frac{1}{y(1-y)} \tilde{m}_c^2 \delta (s - m_c^2) \\
+ \frac{m_c^4 \langle \bar{q} g, \sigma G q \rangle^2}{384\pi^2 T^2} \int_0^1 dy \frac{1}{y(1-y)} \delta (s - m_c^2) \\
+ \frac{m_c^4 \langle \bar{q} g, \sigma G q \rangle^2}{216 T^6} \left\{ \frac{\alpha_s \sigma G}{\pi} \right\} \int_0^1 dy \tilde{m}_c^4 \delta (s - m_c^2),
\]
the subscripts 0, 3, 4, 5, 6, 7, 8, 10 denote the dimensions of the vacuum condensates, \( y_i = \frac{1 + \sqrt{1 - 4m_i^2/s}}{2}, \) \( y_i = \frac{1 - \sqrt{1 - 4m_i^2/s}}{2}, \) \( z_i = \frac{x^m_i}{y_i - m_i^2}, \) \( m_c^2 = \frac{m^2}{y_i(1 - y_i)} \), \( f_{y_i}^{1-\gamma} dy \to \int_0^1 dy, \) \( f_{z_i}^{1-\gamma} dz \to \int_0^1 y dz \) when the \( \delta \) functions \( \delta (s - m_i^2) \) and \( \delta (s - m_c^2) \) appear. We take into account the vacuum condensates which are vacuum expectations of the operators of the orders \( O(\alpha_s^k) \) with \( k \leq 1 \) consistently.

Differentiate Eq.(9) with respect to \( \frac{1}{M} \), then eliminate the pole residues \( \lambda_+ \), we obtain the QCD sum rule for the mass of the scalar tetraquark state,

\[
M_Z^2 = \frac{\int_{m^2}^{M^2} ds \frac{d}{d(1/T^2)} \rho(s) \exp \left( -\frac{1}{T^2} \right)}{\int_{m^2}^{M^2} ds \frac{dp(s) \exp \left( -\frac{1}{T^2} \right)}}. \tag{19}
\]

In the following, we perform Fierz re-arrangement to the current \( J \) both in the color and Dirac-spinor spaces to obtain the result,

\[
J = \frac{1}{4} \left\{ -\bar{c}c d\gamma_5 \bar{c}d\gamma_5 u - \bar{c}c \gamma^{\mu\nu} d\sigma_{\mu\nu} u - \bar{c}c \gamma^{\mu\nu} \gamma_5 d\sigma_{\mu\nu} u + \frac{1}{2} \bar{c}c \sigma_{\mu\nu} d\sigma_{\mu\nu} u \right. \\
\left. + \bar{c}c d\gamma_5 \bar{c}d\gamma_5 u + \bar{c}c \gamma^{\mu\nu} d\sigma_{\mu\nu} u + \bar{c}c \gamma^{\mu\nu} \gamma_5 d\sigma_{\mu\nu} u - \frac{1}{2} \bar{c}c \sigma_{\mu\nu} d\sigma_{\mu\nu} c \right\} , \tag{20}
\]

the components couple to the meson pairs \( \pi c \bar{c} \), \( \eta \pi \), \( J/\psi \rho \), \( \chi_{c1} \pi \), \( \chi_{c1} a_1 \), \( h_c h_1 \), \( D_0(2400)D_0(2400) \), \( (D \bar{D}) \), \( (D^* \bar{D}^*) \), \( (D_1(2420)D_1(2420)) \), \( (D_1(2430)D_1(2430)) \), respectively. The strong decays

\[
Z^+_c(0^{++}) \to \chi_{c0} \bar{c}c, \eta_c \pi \pi, J/\psi \rho \pi, \chi_{c1} \pi, \chi_{c1} a_1, h_c h_1, D_0(2400)D_0(2400), (D \bar{D}) \), \( (D^* \bar{D}^*) \), \( (D_1(2420)D_1(2420)) \), \( (D_1(2430)D_1(2430)) \), \tag{21}
\]

are Okubo-Zweig-Iizuka super-allowed, if they are kinematically allowed. The diquark-antidiquark type tetraquark state can be taken as a special superposition of a series of meson-meson pairs, and embodies the net effects. The decays to its components (meson-meson pairs) are Okubo-Zweig-Iizuka super-allowed, but the re-arrangements in the color-space are non-trivial \[20, 21\].

The numerical analysis indicates that the ground state mass of the \( SS \)-type scalar tetraquark state is about 3.82 GeV, the strong decays

\[
Z_c^+(0^{++}) \to \eta_c \pi \pi, \chi_{c1} \pi \pi, (D \bar{D}) \to \chi_{c1} \pi \pi \tag{22}
\]

are kinematically allowed. The decay \( Z_c^+(0^{++}) \to \chi_{c1} \pi \pi \) takes place through relative P-wave and is kinematically suppressed.

Now we write down the three-point correlation functions \( \Pi_1(p, q) \) and \( \Pi_2(p, q) \) to study the strong decays \( Z_c^+(0^{++}) \to \eta_c \pi \pi, (D \bar{D}) \) \[20, 21\],

\[
\Pi_1(p, q) = i^2 \int d^4 x d^4 y e^{i p \cdot x} e^{i q \cdot y} \langle 0 \mid \{ J_{\eta_c}(x) J_{\pi}(y) J(0) \} \mid 0 \rangle ,
\]

\[
\Pi_2(p, q) = i^2 \int d^4 x d^4 y e^{i p \cdot x} e^{i q \cdot y} \langle 0 \mid \{ J_{D_-}(x) J_{D_0}(y) J(0) \} \mid 0 \rangle , \tag{23}
\]

where the currents

\[
J_{\eta_c}(x) = \bar{c}(x)i\gamma_5 c(x) , \\
J_\pi(y) = \bar{u}(y)i\gamma_5 d(y) , \\
J_{D_-}(x) = \bar{c}(x)i\gamma_5 d(x) , \\
J_{D_0}(y) = \bar{u}(y)i\gamma_5 c(y) , \tag{24}
\]
interpolate the mesons $\eta_c$, $\pi$, $D^-$, $D^0$, respectively.

We insert a complete set of intermediate hadronic states with the same quantum numbers as the current operators into the three-point correlation functions $\Pi_1(p,q)$ and $\Pi_2(p,q)$ and isolate the ground state contributions to obtain the following results,

$$\Pi_1(p,q) = f_\pi M_\pi^2 f_{\eta_c} M_{\eta_c}^2 \lambda Z_c G_{Z_c,\eta_c,\pi} \frac{-q \cdot p}{2(m_u + m_d) m_c} (M_{\eta_c}^2 - p'^2)(M_{\eta_c}^2 - p^2)(M_{\eta_c}^2 - q'^2) + \cdots,$$

$$\Pi_2(p,q) = f_D^2 M_D^2 \lambda Z_c G_{Z_c,DD} \frac{-q \cdot p}{(m_c + m_q)^2} (M_{Z_c}^2 - p'^2)(M_{Z_c}^2 - p^2)(M_{Z_c}^2 - q^2) + \cdots,$$

where $p' = p + q$, the $f_D$, $f_{\eta_c}$, and $f_\pi$ are the decay constants of the mesons $D$, $\eta_c$, and $\pi$, respectively, the $G_{Z_c,\eta_c,\pi}$ and $G_{Z_c,DD}$ are the hadronic coupling constants. In the following, we write down the definitions,

$$\langle 0 \mid J_{\eta_c}(0) \mid \eta_c(p) \rangle = \frac{f_{\eta_c} M_{\eta_c}^2}{2m_c},$$

$$\langle 0 \mid J_\pi(0) \mid \pi(q) \rangle = \frac{f_\pi M_\pi^2}{m_u + m_d},$$

$$\langle 0 \mid J_D(0) \mid D(p/q) \rangle = \frac{f_D M_D^2}{m_c + m_q},$$

$$\langle \eta_c(p) | \pi(q) | Z_c(p') \rangle = -i q \cdot p G_{Z_c,\eta_c,\pi}(q^2),$$

$$\langle D(p) | D(q) | Z_c(p') \rangle = -i q \cdot p G_{Z_c,DD}(q^2).$$

We carry out the operator product expansion and take into account the color connected Feynman diagrams \cite{20, 21}, and obtain the following results,

$$\Pi_1(p,q) = -\frac{m_c (\bar{q} g_s \sigma G q)}{32 \pi^2 q^2} \int_0^1 dx \frac{q \cdot p}{m_c^2 - x(1-x)p^2} + \cdots.$$

Figure 1: The connected Feynman diagram contributes to the correlation function $\Pi_1(p,q)$, where the dashed and solid lines denote the heavy quark and light quark lines, respectively. Other diagrams obtained by interchanging of the heavy quark lines or light quark lines are implied.

Figure 2: The connected Feynman diagram contributes to the correlation function $\Pi_2(p,q)$, where the dashed and solid lines denote the heavy quark and light quark lines, respectively. Other diagrams obtained by interchanging of the heavy quark lines and (or) light quark lines are implied.
\[
\Pi_2(p, q) = -\frac{m_c\langle \bar{q}q, \sigma Gq \rangle}{64\pi^2} \frac{q \cdot p}{q^2 - m_c^2} \int_0^1 dx \frac{1 + x}{m_c^2 - (1 - x)p^2}
- \frac{m_c\langle \bar{q}q, \sigma Gq \rangle}{64\pi^2} \frac{q \cdot p}{m_c^2 - p^2} \int_0^1 dx \frac{2 - x}{xq^2 - m_c^2} + \cdots .
\]

(29)

In Fig.1 and Fig.2, we draw the connected Feynman diagrams contribute to the correlation functions \(\Pi_1(p, q)\) and \(\Pi_2(p, q)\), respectively. The \(\Pi_1(p, q)\) and \(\Pi_2(p, q)\) can be expanded in terms of the \(\cos \theta\), \(\Pi_{1/2}(p, q) = \Pi^0(p^2, q^2) + \Pi^1(p^2, q^2) \cos \theta + \Pi^2(p^2, q^2) \cos^2 \theta + \cdots\), at the QCD side, where \(\theta\) is the included angle of the Euclidean momenta \(p\) and \(q\), i.e. \(\cos \theta = p \cdot q / \sqrt{q^2p^2}\). There exists only one term \((\Pi^1(p^2, q^2) \cos \theta)\) for the \(\Pi_1(p, q)\), while there exist two terms \((\Pi^0(p^2, q^2)\) and \(\Pi^1(p^2, q^2) \cos \theta)\) for the \(\Pi_2(p, q)\). At the phenomenological side, the hadronic coupling constants \(G_{SPP'}(p, q)\) have the possible forms \(G_{SPP'}^\theta\), \(G_{SPP'}^\omega\), \(\theta\), \(G_{SPP'}^\rho\) \(\rho\), \(\cdots\), where the \(S\) denotes the scalar mesons, the \(P\) and \(P'\) denote the pseudoscalar mesons. In the present case, it is better to choose the form \(G_{SPP'}^\theta\) and take the replacement \(2p \cdot q = p^2 - p'^2 - q^2\), then set \(p^2 = p'^2\) and perform the Borel transform with respect to the variable \(p^2 = -p'^2\), as the \(p\), \(q\) and \(p'\) are not independent variables, the \(\cos \theta\) cannot be replaced.

Once the analytical expressions of the correlation functions \(\Pi_1(p, q)\) and \(\Pi_2(p, q)\) at both the QCD side and hadron side are obtained, we perform the Borel transform with respect to the variable \(p^2 = -p'^2\) by setting \(p^2 = p'^2\), then take the quark-hadron duality and obtain the following QCD sum rules,

\[
\frac{f_\pi M_\pi^2 f_\eta M_\eta^2 \lambda_{GZ, GZ, \eta, \pi}}{2(m_u + m_d)m_c(M_c^2 - \eta^2)} \left\{ \exp\left(-\frac{M_c^2}{T^2}\right) - \exp\left(-\frac{\eta^2}{T^2}\right) \right\} + C_{Z, \eta, \pi} \exp\left(-\frac{s_0}{T^2}\right)
- \frac{m_c\langle \bar{q}q, \sigma Gq \rangle}{32\pi^2} \frac{Q^2 + M_c^2}{Q^2} \int_0^1 dx \frac{1}{x(1 - x)} \exp\left(-\frac{m_c^2}{x(1 - x)T^2}\right),
\]

(30)

\[
\frac{f_D^2 M_\pi^4 \lambda_{GZ, GZ, DD}}{(m_c + m_d)^2(m_c^2 - M_D^2)} \left\{ \exp\left(-\frac{M_D^2}{T^2}\right) - \exp\left(-\frac{\eta^2}{T^2}\right) \right\} + C_{Z, DD} \exp\left(-\frac{s_0}{T^2}\right)
- \frac{m_c\langle \bar{q}q, \sigma Gq \rangle}{64\pi^2} \frac{(Q^2 + M_D^2)}{Q^2} \int_0^1 dx \left\{ \frac{1}{Q^2 + m_c^2} \frac{1 + x}{(1 - x)} \exp\left(-\frac{m_c^2}{Q^2 + m_c^2 (1 - x)T^2}\right) + \frac{2 - x}{Q^2 + m_c^2} \exp\left(-\frac{m_c^2}{Q^2 + m_c^2 T^2}\right) \right\},
\]

(31)

where the \(s_0\) is the continuum threshold parameter for the \(Z_c\), and the \(C_{Z, \eta, \pi}\) and \(C_{Z, DD}\) are unknown parameters introduced to take into account the single-pole contributions associated with pole-continuum transitions. In numerical analysis, we will denote the right sides of Eqs. (30-31) as \(F_1(Q^2)\) and \(F_2(Q^2)\) respectively. In the three-point QCD sum rules, the single-pole contributions are not suppressed if a single Borel transform is taken.

3 Numerical results and discussions

The vacuum condensates are taken to be the standard values \(\langle \bar{q}q \rangle = -(0.24\pm0.01 \text{ GeV})^3\), \(\langle \bar{q}g \sigma Gq \rangle = m_0^2\langle \bar{q}q \rangle\), \(m_0^2 = (0.8 \pm 0.1) \text{ GeV}^2\), \(\langle m_{\sigma G G} \rangle = (0.33 \text{ GeV})^4\) at the energy scale \(\mu = 1 \text{ GeV}\) [17, 18, 19, 22, 23]. The quark condensate and mixed quark condensate evolve with the renormalization group equation, \(\langle \bar{q}q \rangle(\mu) = \langle \bar{q}q \rangle(\bar{Q}) \left[ \frac{\alpha_s(Q)}{\alpha_s(\mu)} \right]^{\frac{3}{2}}\) and \(\langle \bar{q}g \sigma Gq \rangle(\mu) = \langle \bar{q}g \sigma Gq \rangle(\bar{Q}) \left[ \frac{\alpha_s(Q)}{\alpha_s(\mu)} \right]^{\frac{3}{2}}\).
The hadronic input parameters are taken as $M_\pi = 0.13957$ GeV, $f_\pi = 0.130$ GeV, $M_{D^0} = 1.8695$ GeV, $M_{D^+} = 1.86491$ GeV, $f_D = 0.208$ GeV, $M_{\eta_c} = 2.9837$ GeV, $f_{\eta_c} = 0.350$ GeV \cite{24,25,26}.

We take the values $m_u(\mu = 1\text{GeV}) = m_d(\mu = 1\text{GeV}) = m_q(\mu = 1\text{GeV}) = 0.006$ GeV from the Gell-Mann-Oakes-Renner relation, and choose the $\overline{MS}$ mass $m_c(m_c) = (1.275 \pm 0.025)$ GeV from the Particle Data Group \cite{24}, and take into account the energy-scale dependence of the $\overline{MS}$ masses from the renormalization group equation,

$$m_q(\mu) = m_q(1\text{GeV}) \left[ \frac{\alpha_s(\mu)}{\alpha_s(1\text{GeV})} \right]^{\frac{1}{2}},$$

$$m_c(\mu) = m_c(m_c) \left[ \frac{\alpha_s(\mu)}{\alpha_s(m_c)} \right]^{\frac{1}{2}},$$

$$\alpha_s(\mu) = \frac{1}{b_0 t} \left[ 1 - \frac{b_1 \log t}{b_0} + \frac{b_2 (\log^2 t - \log t - 1) + b_0 b_2}{b_0^2} \right],$$

where $t = \log \frac{\mu^2}{\Lambda^2}$, $b_0 = \frac{33 - 2n_f}{12\pi}$, $b_1 = \frac{153 - 19n_f}{24\pi^2}$, $b_2 = \frac{2857 - 5033n_f + 339\pi^2}{128\pi^4}$, $\Lambda = 213$ MeV, 296 MeV and 339 MeV for the flavors $n_f = 5, 4$ and 3, respectively \cite{24}.

Now we study the mass and pole residue of the $S$ type scalar tetraquark state. We impose the two criteria (pole dominance and convergence of the operator product expansion) on the hidden charmed tetraquark state to choose the Borel parameter $T^2$ and threshold parameter $s_0$.

In the heavy quark limit, the $c$ (and $b$) quark can be taken as a static well potential, which binds the light quark $q'$ to form a diquark in the color antitriplet channel or binds the light antiquark $\bar{q}$ to form a meson in the color singlet channel (or a meson-like state in the color octet channel). Then the heavy tetraquark states are characterized by the effective heavy quark masses $M_Q$ (or constituent quark masses) and the virtuality $V = \sqrt{M_{X/Y/Z}^2 - (2M_Q)^2}$ (or bound energy not as robust). It is natural to take the energy scale $\mu = V$, the formula works well for the $X(3872)$, $Z_c(3900)$, $Z_c(4020)$, $Z_c(4025)$, $Z(4250)$, $Y(4360)$, $Z(4430)$, $Y(4630)$, $Y(4660)$, $Z_b(10610)$ and $Z_b(10650)$ in the scenario of tetraquark states \cite{11,12,13,14}. The relation

$$M_{X/Y/Z}^2 = (2M_Q)^2 + \mu^2,$$

with the value $M_Q = 1.8$ GeV determined in previous works \cite{11,12,13,14} puts a strong constraint on the masses of the possible tetraquark states.

The mass gaps between the ground states and the first radial excited states are usually taken as $(0.4 - 0.6)$ GeV, for example, the $Z(4430)$ is tentatively assigned as the first radial excitation of the $Z_c(3900)$ according to the analogous decays,

$$\begin{align*}
Z_c(3900)^{\pm} & \rightarrow J/\psi \pi^{\pm}, \\
Z(4430)^{\pm} & \rightarrow \psi' \pi^{\pm},
\end{align*}$$

and the mass differences $M_{Z(4430)} - M_{Z_c(3900)} = 576$ MeV, $M_{\psi'} - M_{J/\psi} = 589$ MeV \cite{10,27,28}. The relation

$$\sqrt{s_0} = M_{X/Y/Z} + (0.4 - 0.6) \text{ GeV},$$

puts another strong constraint on the masses of the possible tetraquark states.

In calculations, we observe that

$$\begin{align*}
\mu & \uparrow \ M_Z \downarrow, \\
\mu & \downarrow \ M_Z \uparrow,
\end{align*}$$

(36)
from the QCD sum rule in Eq.(19). While Eq.(33) indicates that
\[ \mu \uparrow \quad M_Z \uparrow, \]
\[ \mu \downarrow \quad M_Z \downarrow. \] (37)

There must be a compromise, which leads to the optimal energy scale \( \mu \), mass \( M_Z \) and threshold parameter \( s_0 \).

In Fig.3, the contribution of the pole term is plotted with variations of the threshold parameter \( s_0 \) and Borel parameter \( T^2 \) at the energy scale \( \mu = 1.3 \text{ GeV} \). From the figure, we can see that the value \( \sqrt{s_0} \leq 4.1 \text{ GeV} \) is too small to satisfy the pole dominance condition and result in reasonable Borel window.

In Fig.4, the contributions of different terms in the operator product expansion are plotted with variations of the Borel parameter \( T^2 \) for the threshold parameter \( \sqrt{s_0} = 4.3 \text{ GeV} \) at the energy scale \( \mu = 1.3 \text{ GeV} \). From the figure, we can see that the \( \sqrt{s_0} \leq 4.1 \text{ GeV} \) is too small to satisfy the pole dominance condition and result in reasonable Borel window.

In this article, we take the Borel parameter \( T^2 = (2.2 - 2.6) \text{ GeV}^2 \), the continuum threshold parameter \( \sqrt{s_0} = (4.2 - 4.4) \text{ GeV} \) and the energy scale \( \mu = 1.3 \text{ GeV} \), the pole dominance is well satisfied. The Borel parameter, continuum threshold parameter and the pole contribution are shown explicitly in Table 1. The two criteria (pole dominance and convergence of the operator product expansion) of the QCD sum rules are fully satisfied, furthermore, the relations in Eq.(33) and Eq.(35) are also satisfied.

Taking into account all uncertainties of the input parameters, finally we obtain the values of
Figure 4: The contributions of different terms in the operator product expansion with variations of the Borel parameter $T^2$, where the 0, 3, 4, 5, 6, 7, 8, 10 denote the dimensions of the vacuum condensates.

![Graph showing the contributions of different terms](image)

Table 1: The Borel parameter, continuum threshold parameter, pole contribution, mass and pole residue of the scalar tetraquark state.

| $J^{PC}$ | $T^2$(GeV$^2$) | $\sqrt{s_0}$(GeV) | pole | $M_Z$(GeV) | $\lambda_Z$ |
|----------|----------------|-------------------|------|------------|------------|
| $0^{++}$ | $2.2 - 2.6$    | $4.3 \pm 0.1$    | $(49 - 74)\%$ | $3.82^{+0.08}_{-0.08}$ | $1.79^{+0.29}_{-0.24} \times 10^{-2}$GeV$^3$ |

Figure 5: The mass with variations of the Borel parameter $T^2$.
the mass and pole residue of the $S\bar{S}$ type scalar tetraquark state, which are shown explicitly in Figs.5-6 and Table 1.

The central value of the present prediction $M_{Z_c} = (3.82^{+0.08}_{-0.08})$ GeV for the $S\bar{S}$ type scalar tetraquark state is smaller than that of the $A\bar{A}$ type scalar tetraquark state $M_{J=0} = (3.85^{+0.15}_{-0.09})$ GeV obtained in Ref.\[14\]. The predictions based on the QCD sum rules are consistent with the values $M_{J=0} = 3.852$ GeV and $3.812$ GeV for the $A\bar{A}$ and $S\bar{S}$ type scalar tetraquark states $cq\bar{c}q$ respectively from the quasipotential approach \[8\].

Now we take the mass $M_{Z_c}$ and pole residue $\lambda_{Z_c}$ as basic input parameters to study the hadronic coupling constants $G_{Z_c\eta \pi}$ and $G_{Z_cDD}$, and take the same threshold parameter and Borel parameter as in the QCD sum rule for the mass and pole residue. In calculations, we choose the unknown parameters as $C_{Z_c\eta \pi} = 0.0009$ GeV$^6$ and $C_{Z_cDD} = 0.0004$ GeV$^6$ to obtain stable QCD sum rules with variations of the Borel parameter $T^2$ at the Borel windows $T^2 = (2.2 - 2.6)$ GeV$^2$; the left side and right side of the QCD sum rules coincide. In fact, it is not necessary to choose the same Borel parameters both in the two-point and three-point QCD sum rules. If we take larger Borel parameter, say $T^2 = (2.5 - 3.0)$ GeV$^2$ instead of $T^2 = (2.2 - 2.6)$ GeV$^2$, we should alter the unknown parameters $C_{Z_c\eta \pi}$ and $C_{Z_cDD}$ slightly, then obtain stable QCD sum rules, the resulting values of the hadronic coupling constants change slightly.

Based on Eqs.\(30-31\), we can study the $Q^2$ dependence of the right side of the QCD sum rules,

$$F_1(Q^2) \propto \frac{Q^2 + M_\pi^2}{Q^2} \approx 1,$$

(38)

at the region of large (or intermediate) $Q^2$ due to the tiny mass of the $\pi$, while the $F_2(Q^2)$ has no such simple $Q^2$ dependence due to the heavy quark mass $m_c$ and heavy meson mass $M_D$. In the limit $Q^2 \to \infty$,

$$F_2(Q^2) = -\frac{m_c\langle \bar{q}q, \sigma Gq \rangle}{64\pi^4} \int_0^1 dx \left\{ \frac{1 + x}{1 - x} \exp \left( -\frac{m_c^2}{(1 - x)T^2} \right) + \frac{2 - x}{x} \exp \left( -\frac{m_c^2}{T^2} \right) \right\},$$

(39)

which is independence on $Q^2$. In Fig.7, we plot the central values of the $F_2(Q^2)$ with variations of the $Q^2$ at the range $Q^2 = (1 - 5)$ GeV$^2$ for the Borel parameters $T^2 = 2.2$ GeV$^2$, 2.4 GeV$^2$. 

Figure 6: The pole residue with variations of the Borel parameter $T^2$. 

The central value of the present prediction $M_{Z_c} = (3.82^{+0.08}_{-0.08})$ GeV for the $S\bar{S}$ type scalar tetraquark state is smaller than that of the $A\bar{A}$ type scalar tetraquark state $M_{J=0} = (3.85^{+0.15}_{-0.09})$ GeV obtained in Ref.\[14\]. The predictions based on the QCD sum rules are consistent with the values $M_{J=0} = 3.852$ GeV and $3.812$ GeV for the $A\bar{A}$ and $S\bar{S}$ type scalar tetraquark states $cq\bar{c}q$ respectively from the quasipotential approach \[8\].
Figure 7: The central values of the $F_2(Q^2)$ with variations of the $Q^2$.

Figure 8: The central values of the hadronic coupling constants with variations of the $Q^2$, where the $A$ and $B$ denote the $G_{ZcDD}(Q^2)$ and $G_{Zc\eta\pi}(Q^2)$, respectively.
and 2.6 GeV$^2$, respectively. From the figure, we can see that the $Q^2$ dependence of the $F_2(Q^2)$ is rather mild and can be neglected approximately. The left sides of the QCD sum rules in Eqs.(30-31) have no explicit $Q^2$ dependence, the $Q^2$ dependence is embodied in the right sides of the QCD sum rules ($F_1(Q^2)$ and $F_2(Q^2)$), so the hadronic coupling constants $G_{Z,\eta_\pi}$ and $G_{Z,DD}$ are independent on the $Q^2$ in the limit $Q^2 \to \infty$, the conclusion survives even for much smaller $Q^2$, say $Q^2 = (1 - 5)$ GeV$^2$ according to Eq.(38) and Fig.7. The central values of the $G_{Z,\eta_\pi}(Q^2)$ and $G_{Z,DD}(Q^2)$ can be fitted to the following constant forms,

$$ G_{Z,\eta_\pi}(Q^2) = 0.43 \text{ GeV}^{-1}, $$

$$ G_{Z,DD}(Q^2) = 1.06 \text{ GeV}^{-1}, $$

at the region $Q^2 = (1 - 5)$ GeV$^2$; the uncertainties of the $G_{Z,\eta_\pi}$ and $G_{Z,DD}$ are about 25% and 18%, respectively. We plot the central values of the hadronic coupling constants $G_{Z,DD}(Q^2)$ and $G_{Z,\eta_\pi}(Q^2)$ with variations of the $Q^2$ at the region $Q^2 = (1 - 5)$ GeV$^2$ for the Borel parameter $T^2 = 2.4 \text{ GeV}^2$ in Fig.8. From the figure, we can see that the fitted functions in Eq.(40) are satisfactory. We extend the coupling constants to the physical regions without difficulty, and calculate the partial decay widths,

$$ \Gamma_{Z,\eta_\pi} = \frac{G_{Z,\eta_\pi}^2(M_{Z,\eta_\pi}^2 - M_{\eta_\pi}^2 - M_{\pi}^2)^2 p_{\eta_\pi}}{32\pi M_{Z,\eta_\pi}^2} = (3.0 \pm 1.5) \text{ MeV}, $$

$$ \Gamma_{Z,DD} = \frac{G_{Z,DD}^2(M_{Z,DD}^2 - M_{DD}^2)^2 p_{DD}}{32\pi M_{Z,DD}^2} = (17.9 \pm 6.4) \text{ MeV}, $$

where

$$ p_{\eta_\pi} = \frac{\sqrt{[M_{Z,\eta_\pi}^2 - (M_{\eta_\pi} + M_\pi)^2] [M_{Z,\eta_\pi}^2 - (M_{\eta_\pi} - M_\pi)^2]}}{2M_{Z,\eta_\pi}}, $$

$$ p_{DD} = \frac{\sqrt{[M_{Z,DD}^2 - (M_{DD} + M_{DD})^2] [M_{Z,DD}^2 - (M_{DD} - M_{DD})^2]}}{2M_{Z,DD}}. $$

The total width $\Gamma_{Z,\pi}$ of the $Z_c(3820)$ can be approximated by $\Gamma_{Z,\eta_\pi} + \Gamma_{Z,DD}$, the numerical value is about $(20.9 \pm 6.6) \text{ MeV}$. The radiative decay widths can be estimated by assuming vector meson dominance, for example, $\Gamma_{Z^+_c \to \rho^+} \propto \alpha |\Gamma_{Z^+_c \to J/\psi^* \rho^+}|$ for the radiative decays $Z^+_c(3820) \to J/\psi^* \rho^+ \to \gamma \rho^+$, the partial decay widths are of the order $O(\text{KeV})$ due to the factor $\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$. The strong decays $Z^+_c(3820) \to J/\psi \rho^+$ are kinematically forbidden, the values of the $\Gamma_{Z^+_c \to J/\psi^* \rho^+}$ are complex, so we take $|\Gamma_{Z^+_c \to J/\psi^* \rho^+}|$. The contributions of the radiative decays to the total width $\Gamma_{Z_c}$ are small and can be neglected.

### 4 Conclusion

In this article, we calculate the contributions of the vacuum condensates up to dimension-10 in the operator product expansion, study the $SS$ type scalar tetraquark state $cq\bar{c}q$ in details with the QCD sum rules. In calculations, we search for the optimal Borel parameter and threshold parameter to satisfy the energy scale formula $M_{Z}^2 = (2M_c)^2 + \mu^2$ and the experiential threshold formula $\sqrt{s_0} = M_Z + (0.4 - 0.6) \text{ GeV}$, where the $\mu$ is the energy scale of the QCD spectral density, and obtain the values $M_{Z_c} = (3.82^{+0.08}_{-0.06}) \text{ GeV}$ and $\lambda_{Z_c} = (1.79^{+0.29}_{-0.24}) \times 10^{-3} \text{ GeV}^3$. The central value of the mass of the $SS$ type scalar tetraquark state is smaller than that of the $AA$ type scalar tetraquark state, the $SS$ type scalar tetraquark state $cq\bar{c}q$ may be the lowest hidden charmed tetraquark state. Furthermore, we calculate the hadronic coupling constants $G_{Z,\eta_\pi}$ and $G_{Z,DD}$ with the three-point QCD sum rules by taking into account the color-connected diagrams, then
study the strong decays $Z_c \rightarrow \eta_c \pi$, $DD$, and observe that the total width $\Gamma_{Z_c} \approx 21$ MeV. The present predictions can be confronted with the experimental data in the futures at the BESIII, LHCb and Belle-II.

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References

[1] M. Huang, Int. J. Mod. Phys. E14 (2005) 675.
[2] A. De Rujula, H. Georgi and S. L. Glashow, Phys. Rev. D12 (1975) 147.
[3] T. DeGrand, R. L. Jaffe, K. Johnson and J. E. Kiskis, Phys. Rev. D12 (1975) 2060.
[4] Z. G. Wang, Eur. Phys. J. C71 (2011) 1524.
[5] R. T. Kleiv, T. G. Steele and A. Zhang, Phys. Rev. D87 (2013) 125018.
[6] Z. G. Wang, Phys. Rev. D79 (2009) 094027.
[7] Z. G. Wang, Eur. Phys. J. C67 (2010) 411.
[8] D. Ebert, R. N. Faustov and V. O. Galkin, Eur. Phys. J. C58 (2008) 399.
[9] L. Maiani, F. Piccinini, A. D. Polosa and V. Riquer, Phys. Rev. D71 (2005) 014028.
[10] L. Maiani, F. Piccinini, A. D. Polosa and V. Riquer, Phys. Rev. D89 (2014) 114010.
[11] Z. G. Wang and T. Huang, Phys. Rev. D89 (2014) 054019.
[12] Z. G. Wang, Eur. Phys. J. C74 (2014) 2874.
[13] Z. G. Wang and T. Huang, Nucl. Phys. A930 (2014) 63.
[14] Z. G. Wang, arXiv:1312.1537.
[15] Z. G. Wang and T. Huang, Eur. Phys. J. C74 (2014) 2891.
[16] Z. G. Wang, Eur. Phys. J. C74 (2014) 2963.
[17] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B147 (1979) 385.
[18] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B147 (1979) 448.
[19] L. J. Reinders, H. Rubinstein and S. Yazaki, Phys. Rept. 127 (1985) 1.
[20] F. S. Navarra and M. Nielsen, Phys. Lett. B639 (2006) 272.
[21] J. M. Dias, F. S. Navarra, M. Nielsen and C. M. Zanetti, Phys. Rev. D88 (2013) 016004.
[22] P. Colangelo and A. Khodjamirian, hep-ph/0010175.
[23] B. L. Ioffe, Prog. Part. Nucl. Phys. 56 (2006) 232.
[24] J. Beringer et al, Phys. Rev. D86 (2012) 010001.
[25] Z. G. Wang, JHEP 1310 (2013) 208.

[26] V. A. Novikov, L. B. Okun, M. A. Shifman, A. I. Vainshtein, M. B. Voloshin and V. I. Zakharov, Phys. Rept. 41 (1978) 1.

[27] M. Nielsen and F. S. Navarra, Mod. Phys. Lett. A29 (2014) 1430005.

[28] Z. G. Wang, arXiv:1405.3581