Surprises with Angular Momentum

Paul K. Townsend

DAMTP, University of Cambridge,
Centre for Mathematical Sciences, Wilberforce Road,
Cambridge CB3 0WA, UK

ABSTRACT

The physics of angular momentum in even space dimensions can be surprisingly counter-intuitive. Three such suprises, all associated with the properties of supersymmetric rotating objects, are examined: (i) 5D black holes, (ii) Dyonic instantons and (iii) Supertubes.

* To appear in proceedings of TH-2002, The International Conference on Theoretical Physics, Paris, UNESCO, July 2002.
1 Prelude

In his book *Surprises in Theoretical Physics*, Peierls examines various occasions on which his research led to a surprising conclusion although, as he says, *there would have been no surprise if one had really understood the problem from the start*. Such surprises are significant in that they expose flaws in one’s physical intuition and thus serve to refine it. In the same spirit, this article recounts three surprising results of the author’s co-investigations into the properties of supersymmetric rotating objects. In each case one could imagine a similar surprise arising for non-supersymmetric objects, so supersymmetry just provides a convenient, and simplifying, context; the surprises are chiefly due to unexpected features of angular momentum, a fact that might itself be considered a meta-surprise given the central role of angular momentum in physics. However, intuition for angular momentum is usually acquired from a study of objects rotating in three-dimensional space, whereas the cases reviewed here involve rotation in either four space dimensions or, in the last case to be considered, two space dimensions. Although the context of each of the three surprises is quite different, there are a number of points of contact that make a comparative review seem worthwhile. I thank the organisers of TH-2002 for allowing me the opportunity to present such a review, and I congratulate the Mayor of Paris for *Paris Plage*.

2 Supersymmetric Rotating Black Holes

The first of our three surprises arose from a study of supersymmetric black hole solutions of 5D supergravity. Supersymmetric black holes are special cases of stationary black holes. A stationary (asymptotically flat) black hole spacetime admits a Killing vector field (KVF) that is timelike near spatial infinity, and unique up to normalization. However, there may be interior regions outside the horizon, called ‘ergoregions’, within which the KVF is spacelike; in fact, an event horizon with a non-zero angular velocity necessarily lies within an ergoregion. Supersymmetric spacetimes cannot have ergoregions, however, because supersymmetry implies that the KVF can be expressed in terms of a Killing spinor field, and this expression allows the KVF to be timelike or null but not spacelike. It follows that the *event horizon of a supersymmetric black hole must be non-rotating*. This general observation, which applies in any spacetime dimension, should be kept in mind in what follows.

The angular momentum \( J \) of any D-dimensional asymptotically-flat spacetime can be expressed as the surface integral

\[
J = \frac{1}{16\pi G_D} \int_{\infty} dS_{\mu\nu} D^\mu m^\nu
\]

where \( G_D \) is the D-dimensional Newton constant, \( D \) the standard covariant derivative, \( m \) a suitably normalized spacelike KVF with closed orbits and \( dS \) the dual of the \((D-2)\)-surface element at spatial infinity, which can be viewed as the boundary at infinity of a spacelike \((D-1)\)-surface \( \Sigma \). In the case of a black hole spacetime one may choose \( \Sigma \) to intersect the event horizon on a \((D-2)\)-surface \( H \), in which case the expression above may be re-expressed in the form

\[
J = J_H + J_\Sigma
\]

where \( J_H \) is a surface integral over \( H \) with the same integrand as in (1) and \( J_\Sigma \) is a ‘bulk’ integral over the region of \( \Sigma \) outside the horizon.

For solutions of 4D Einstein-Maxwell theory, and hence of pure \( N = 2 \) 4D supergravity, the integral \( J_\Sigma \) vanishes and so \( J = J_H \). In other words, the angular momentum is due to the black hole itself. This seems reasonable given that non-zero \( J \) implies a rotating
horizon by a theorem of Wald (which states that a stationary black hole with a non-rotating horizon is static [3]). But a rotating horizon is incompatible with supersymmetry so any 4D supersymmetric black hole must be static.

The situation in 5D is more subtle. Firstly, the 5D Einstein-Maxwell theory admits a possible 'FFA' Chern-Simons (CS) term, which is present in the pure 5D supergravity theory with a particular coefficient [4]. A supersymmetric black hole solution of this Einstein-Maxwell-CS theory must again have a non-rotating horizon but this no longer implies that $J = 0$; there is no 5D analogue of Wald’s theorem because the bulk Maxwell field may now carry angular momentum. In fact, a stationary supersymmetric black hole solution of 5D supergravity with non-zero $J$ exists. It was first found in a slightly different context by Breckenridge et al. [5] and is usually called the ‘BMPV’ black hole. It was later shown to be a 1/2 supersymmetric solution of 5D matter-coupled supergravity [6, 7].

In the form found in [2], as a solution of the pure minimal 5D supergravity, the metric is

$$ds^2 = \left(1 + \frac{\mu}{r^2}\right)^{-2} \left(dt + \frac{j \sigma_3}{2r^2}\right)^2 + \left(1 + \frac{\mu}{r^2}\right) \left(dr^2 + r^2 d\Omega_3^2\right)$$

(3)

where $d\Omega_3^2$ is the $(SU(2)_L \times SU(2)_R)$-invariant metric on $S^3 \cong SU(2)$ and $\sigma_3$ is one of the three left-invariant forms on $SU(2)$ satisfying $d\sigma_3 = \sigma_1 \wedge \sigma_2$ and cyclic permutations. The parameters $\mu$ and $j$ are related to the total mass $M$ and total angular momentum $J$ as follows:

$$\mu = \frac{4MG_5}{3\pi}, \quad j = -\frac{2JG_5}{\pi}.$$  

(4)

The singularity at $r = 0$ is just a coordinate singularity at a degenerate non-rotating event horizon provided that $j^2 < \mu^3$.

Although the horizon has zero angular velocity, it is affected by the rotation; the horizon is a 3-sphere, topologically, but geometrically it is a squashed 3-sphere, with a squashing parameter proportional to $J$. As $j^2 \to \mu^3$ the squashed 3-sphere degenerates, and for $j^2 > \mu^3$ there are closed timelike curves through every point [2]. For this reason we restrict $j$ as above (and refer the interested reader to [8, 9] for details of the ‘over-rotating’ case).

Because the angular velocity of the horizon vanishes, one might expect to find that $J_H = 0$ and hence that $J_\Sigma = J$. However, and this is the promised surprise, a calculation shows that

$$J_\Sigma = \left[1 + \frac{1}{2} \left(1 - \frac{j^2}{\mu^3}\right)\right] J > J.$$  

(6)

This implies not only that $J_H$ is non-zero but also that it is negative, as a direct computation confirms! What this means is that a negative fraction of the total angular momentum is stored in the Maxwell field behind the horizon.

Of course, given that there can be a contribution to the total angular momentum of a charged black hole from the Maxwell field outside the horizon there is no good reason to suppose that there is no similar bulk contribution from inside the horizon, and once this has been appreciated it is not difficult to see why the fraction should be negative: given a positive bulk contribution to the angular momentum, one would expect frame dragging effects to cause the horizon to rotate unless these effects are counterbalanced by the frame dragging effects due to a negative contribution to the angular momentum in the fields behind the horizon [3]. Because the horizon of a supersymmetric black hole cannot rotate, $J_\Sigma > J$ should be expected. So why was it a surprise? The answer is presumably that there is a clash with intuition derived from approaches to black hole physics such as the membrane paradigm [10, 11] in which physical properties of the black hole relevant to an exterior observer are expressed entirely in terms of the horizon and its exterior spacetime. Rotating 5D black holes appear to present an interesting challenge to this paradigm.
3 Interlude: symmetries and angular momentum: I

Before considering the next surprise it will be useful to consider the effects, or expected effects, of rotation on rotational symmetry. In four space dimensions the angular momentum 2-form $L$ has two skew eigenvalues $J \pm J'$, where $J$ and $J'$ are the quantum numbers associated to the rotation group $Spin(4) \cong SU(2)_R \times SU(2)_L$. One expects $spin(4)$ to be broken to $U(1)$ for generic $(J, J')$, but to $U(1)_R \times SU(2)_L$ when $J' = 0$, in which case $L$ is self-dual, and to $SU(2)_R \times U(1)_L$ when $J = 0$, in which case $L$ is anti-self-dual.

The supersymmetric rotating 5D black hole has $J' = 0$, which is why there is only a single rotation parameter $J$; the $U(1)_R \times SU(2)_L$ symmetry is evident from the metric (3), and this is also the isometry group of the squashed 3-sphere.

The supersymmetry generators of minimal 5D supersymmetry transform as the $(2, 1) \oplus (1, 2)$ representation of $SU(2)_R \times SU(2)_L$. Half the generators, call them $Q_L$, are singlets of $SU(2)_R$ and the other half, $Q_R$, are singlets of $SU(2)_L$. A half-supersymmetric configuration on which $Q_L$ acts trivially will preserve $SU(2)_L$ in which case the rotational symmetry group must be unbroken or broken to $U(1)_R \times SU(2)_L$. In the case of the 5D black hole the former possibility applies when $J = 0$ and the latter when $J \neq 0$. More generally, we deduce that preservation of 1/2 supersymmetry implies a self-dual or anti-self-dual angular momentum 2-form.

4 Dyonic Instantons

The instanton solution of Euclidean 4D Yang-Mills theory has an alternative interpretation as a static soliton of the 5D Yang-Mills theory. As a solution of 5D Super-Yang-Mills (SYM) theory these ‘instanton-solitons’ preserve 1/2 of the supersymmetry of the gauge theory vacuum. Let us concentrate on the minimal 5D SYM theory with gauge group $SU(2)$, for which the bosonic field content consists of the $SU(2)$ triplet of gauge potential 1-forms $A^a$ ($a = 1, 2, 3$) and a single scalar (Higgs) triplet $\phi^a$. Supersymmetry does not permit a potential for the Higgs field so its expectation value is arbitrary. The vacua are thus parametrized by the constant 3-vector

$$\langle \phi^a \rangle = v^a. \quad (7)$$

For vanishing Higgs field, and hence $v = 0$, a Yang-Mills instanton is a (marginally) stable supersymmetric soliton solution of the 5D SYM theory with unbroken $SU(2)$. A class of multi-soliton solutions, with arbitrary instanton number $I$, is given by the ’t Hooft ansatz

$$A^a_i = \tilde{\eta}^a_{ij} \partial_j \log H \quad (i = 1, 2, 3, 4) \quad (8)$$

where $\tilde{\eta}^a$ is the triplet of anti-self-dual complex structures on $\mathbb{E}^4$ and $H$ is a harmonic function on $\mathbb{E}^4$ with point singularities such that $H \rightarrow 1$ as $r \rightarrow \infty$, where $r$ is the radial distance from the origin of $\mathbb{E}^4$. The simplest possibility,

$$H = 1 + \frac{\rho^2}{r^2}, \quad (9)$$

yields the one-instanton ($I = 1$) solution of ‘size’ $\rho$.

When $v \neq 0$ then $SU(2)$ is spontaneously broken to $U(1)$ and the instanton-soliton is destabilized; a simple scaling argument shows that the energy is reduced if $\rho$ is reduced, so the soliton will implode to a singular instanton configuration with $\rho = 0$. However, the addition of an electric $U(1)$ charge can stabilize the soliton at some equilibrium radius, at which supersymmetry is again partially preserved [12]. Specifically, if we set

$$A^a_0 = \phi^a = v^a H^{-1} \quad (10)$$
then one again has a supersymmetric configuration. It is $1/2$ supersymmetric as a solution of the minimal SYM theory discussed here but $1/4$ supersymmetric as a solution of the maximally-supersymmetric 5D SYM theory, as is suggested by the formula

$$M = 4\pi^2 |I| + |vq|$$

(11)

for the mass of a dyonic instanton.

The special case of

$$H = 1 + \frac{\rho}{r^2}$$

(12)
yields the one dyonic instanton ($I = 1$) of size $\rho$. A computation of the $U(1)$ electric charge $q$ of this solution shows that $q \sim v/\rho^2$; equivalently

$$\rho \sim \sqrt{q/v}.$$  
(13)

The energy density takes the form

$$\mathcal{E} = v^4 f(vr)$$

(14)

for some function $f$. As long as

$$vq < 16\pi^2$$

(15)

then the energy density takes its maximum on some 3-sphere centred on the origin in the limit of large $vq$ the radius of this 3-sphere is of order $\rho$.

Note that the energy density of this $I = 1$ solution is hyper-spherically-symmetric, and the 3-sphere around which the energy density is distributed at large $vq$ is a round 3-sphere, not a squashed one. The rotational $Spin(4) \cong SU(2)_L \times SU(2)_R$ symmetry is broken, to either $SU(2)_L$ or $SU(2)_R$, by the Yang-Mills-Higgs field configuration itself. However, these fields are also acted on by an ‘isospin’ group $SU(2)_I$, and the diagonal subgroup $SU(2)_D$ of either $SU(2)_R \times SU(2)_I$ or $SU(2)_L \times SU(2)_I$ survives. Thus the Yang-Mills-Higgs fields of the dyonic instanton preserve either $SU(2)_L \times SU(2)_D$ or $SU(2)_D \times SU(2)_R$. This is interpreted as the rotational invariance group of gauge-invariant quantities such as the energy density. Compare this state of affairs to that of the BPS monopole of 4D $N=2$ SYM theory. In that case the rotation group is $SU(2)$ and the group $SU(2)_I \times SU(2)_I$ is broken to the diagonal $SU(2)_D$ by the one-monopole solution, which is therefore spherically symmetric. The one dyonic instanton solution is hyper-spherically symmetric for essentially the same reason.

The spherical symmetry of the BPS monopole indicates that it carries no angular momentum, because angular momentum would break the spherical symmetry. One might similarly expect the hyper-spherically symmetric dyonic instanton to carry no angular momentum but, and this is our second surprise, a computation yields

$$L_{ij} = -q \hat{v} \cdot \bar{\eta}_{ij}$$

(17)

where $\hat{v}$ is the unit 3-vector with components $v^a/v$. Thus, a self-dual dyonic instanton has an anti-self-dual angular momentum 2-form proportional to $q$, and this is true even for the hyper-spherically symmetric dyonic instanton solution of $\mathbb{R}^4$.

The realization that the one dyonic instanton must carry angular momentum, despite its hyper-spherical symmetry emerged from a computation of its gravitational field $\mathbb{R}^4$. Surprisingly, this turned out to be stationary rather than static, and a subsequent calculation of the angular momentum of the flat space dyonic instanton yielded the above
formula. Should this result not have been anticipated from the start? Essentially, the reason that the BPS monopole has no angular momentum is that there is nothing in the monopole ansatz that could produce a non-zero result; the angular momentum is zero because there is nothing else that it could be. Applied to the dyonic instanton ansatz, the same argument shows only that \( L \propto \vec{v} \cdot \vec{\eta} \), so the real question is why one should expect the constant of proportionality to vanish. It vanishes when \( q = 0 \) because the instanton is genuinely static, but when \( q \neq 0 \) we have electric fields and a configuration with electric fields is not ‘genuinely’ static (for a reason to be explained below) so the formula should not really have been a surprise.

Nevertheless, I still find that the most common reaction to the statement \textit{spherical symmetry in four space dimensions does not imply vanishing angular momentum} is surprise. To mitigate the surprise I usually point out that circular symmetry in two space dimensions is obviously compatible with non-zero angular momentum. Pursuing this point will lead to our third surprise.

5 Interlude: symmetries and angular momentum: II

A supersymmetric field configuration of a supersymmetric field theory is one that is unchanged by the action of some linear combination \( Q \) of supersymmetry charges. This means that \( Q^2 \) acts trivially too, but the action of \( Q^2 \) on any \textit{gauge-invariant} field is equivalent to the action of \( H \), which generates time translations. It follows that a \textit{field configuration can be supersymmetric only if all gauge-invariant quantities are time-independent}; in particular, this means that the energy density must be time-independent. It is important to appreciate that it is possible for some \textit{gauge-dependent} field of a supersymmetric field configuration to be time-dependent, and this is why there can exist supersymmetric field configurations with non-zero angular momentum. For example, a non-zero electric field \( \vec{E} \) can have this effect because \( \vec{E} = \dot{\vec{A}} \) in an \( A_0 = 0 \) gauge, and in this gauge \( \vec{A} \) is time-dependent if \( \vec{E} \) is non-zero.

Although the energy density of a supersymmetric field configuration must be time-independent, the possibility of a non-zero angular momentum indicates that there is motion nevertheless. Consider a circular planar loop of elastic string; this may rotate about the axis of the plane, and the associated angular momentum will (if it has a sufficiently large magnitude) support the string, against the force exerted by its tension, at some equilibrium radius. A rotating configuration of this kind would not be incompatible with supersymmetry because the circular symmetry ensures that the energy density of the string loop is time-independent. This illustrates the point that angular momentum is obviously compatible with circular symmetry in two space dimensions.

It is even more obvious that circular \textit{asymmetry} is compatible with non-zero angular momentum but in this case it might seem unlikely that the energy density could be time-independent (as would be required by supersymmetry); any rotating ‘bump’ on the string would clearly imply a time-dependent energy density.

6 Supertubes

The supertube, as originally considered [14], is a kind of string theory realization of the spinning string loop supported by angular momentum. I say ‘kind of’ because a relativistic string that is described by a Nambu-Goto action cannot support momentum along the string and hence cannot rotate if it is circularly symmetric. However, IIA superstring

\footnote{In the case of gravitational theories this argument needs modification because \( Q \) is defined only as an integral at infinity, but the end conclusion is similar: \textit{a spacetime can be supersymmetric only if it is stationary.}}
theory has membrane solitons that appear as D2-branes, and a cylindrical D2-brane can be supported against collapse by angular momentum in a plane orthogonal to the axis of the cylinder. This is possible because the D2-brane action is not of Nambu-Goto type but rather of Dirac-Born-Infeld type and the angular momentum can be generated by the Born-Infeld (BI) electric and magnetic fields. Specifically, the Lagrangian density is

\[ \mathcal{L} = -\sqrt{-\det(g + F)} \]  

where \( g \) is the induced metric on the 3D worldvolume and \( F \) is the worldvolume BI field strength 2-form. In principle we have a membrane in \( E^9 \) (since the spacetime is 10-dimensional) but we may choose to consider a membrane in \( E^3 \subset E^9 \). A membrane of cylindrical topology can be parameterized by worldspace coordinates \((z, \sigma) \in \mathbb{R} \times S^1 \). In a physical gauge adapted to this topology, the geometry of a static membrane is determined by a single function \( R(z, \sigma) \) which gives the radial position in the plane orthogonal to the axis of the cylinder as a function of position on the brane. The BI magnetic field \( B \) is a worldspace scalar. The BI electric field is a worldspace 2-vector but, as we wish to generate an angular momentum in the plane orthogonal to the axis of the cylinder, we will choose this 2-vector to be parallel to the axis of the cylinder. Thus, the BI 2-form is

\[ F = E(z, \sigma) \, dt \wedge dz + B(z, \sigma) \, dz \wedge d\sigma. \]  

The Lagrangian density now reduces to

\[ \mathcal{L} = -\sqrt{R^2 + R^2_{\sigma} (1 - E^2)} + B^2 + R^2 R^2_{\sigma} \]  

where \( R_{\sigma} = \partial_{\sigma} R \) and \( R_z = \partial_z R \).

We have assumed that the D2-brane has cylindrical topology. If we further assume that it has cylindrical geometry then we must set \( R_{\sigma} = 0 \) and \( R_z = 0 \); the radial function \( R \) thus becomes a real variable. The BI fields \( E \) and \( B \) similarly reduce to real variables if we assume cylindrical symmetry, and the Lagrangian density becomes a function of three variables

\[ \mathcal{L}(E, B, R) = -\sqrt{R^2(1 - E^2)} + B^2. \]  

Introducing the electric displacement

\[ D = \frac{\partial \mathcal{L}}{\partial E}, \]  

the Hamiltonian density \( \mathcal{H} = DE - \mathcal{L} \) is

\[ \mathcal{H}(D, B, R) = R^{-1} \sqrt{(D^2 + R^2)(B^2 + R^2)}. \]  

This is equivalent to

\[ \mathcal{H}^2 = (D \pm B)^2 + \left( \frac{BD}{R} \mp 1 \right)^2. \]  

From this formula we see that the energy is minimised for given \( B \) and \( D \) when

\[ R = |BD|. \]  

This is therefore the equilibrium value of the cylinder radius. The equilibrium energy is

\[ \mathcal{H}_{\text{min}} = |D| + |B|. \]  

This energy formula is typical of 1/4 supersymmetric configurations, and a calculation confirms that the D2-brane configuration just describes preserves 1/4 supersymmetry, hence the name ‘supertube’.
As we go round the circle parametrized by $\sigma$, the tangent planes to the tube at a point with coordinates $(z, \sigma)$ are rotated by an angle in the plane orthogonal to the axis of the cylinder. Under normal circumstances a configuration of this type would not preserve supersymmetry because the D2 constraint on the supersymmetry parameter associated with one tangent plane would be incompatible with the constraint associated with any of the other tangent planes. As an extreme example consider two tangent planes at diametrically opposite points on the circle: if we declare the constraint associated with one to be the D2-constraint then the constraint associated with the other is the anti-D2-constraint. Thus, the supertube is effectively a supersymmetric configuration that includes both D2-branes and anti-D2-branes! There are various related ways to understand how this is possible. Note that the relation between $E$ and $D$ is

$$E = \frac{D}{R} \sqrt{\frac{B^2 + R^2}{D^2 + R^2}}$$

and that this yields $E = 1$ when $R = |BD|$. An electric field has the effect of reducing the D2-brane tension, and increasing $E$ to its ‘critical’ value $E = 1$ would reduce the tension to zero if the magnetic field were zero; this can be seen from the fact that $\mathcal{L} = -|B|$ for $E = 1$. This explains why the supertube has no energy associated to the D2-brane tension: its energy comes entirely from the electric and magnetic fields, which can be interpreted as ‘dissolved’ strings and D0-branes, respectively. The energy from the D2-brane tension has been cancelled by the binding energy released as the strings and D0-branes are dissolved by the D2-brane. Given that the D2-brane energy has been cancelled, it is perhaps not so surprising to discover that the D2-brane constraint is also absent, and hence that D2-branes can co-exist with anti-D2-branes without breaking supersymmetry. In any case, this is what happens and I refer to [14, 15] for a much more complete discussion of this point.

While the supersymmetry of the supertube might appear surprising, this feature was not discovered accidentally and, in any case, is not a surprise specifically related to angular momentum. Following the initial supertube paper [14], a matrix model version of it was introduced by Bak and Lee [16]. A subsequent paper by Bak and Karch [17] found a more general solution of the matrix model describing an elliptical supertube, which included a plane parallel D2/anti-D2 pair as a limiting case. The fact that a circular cross-section could be deformed to an ellipse was certainly a surprise to me because it was hard to understand how any shape other than a circle could be consistent with both rotation and the time-independent energy profile required by supersymmetry (the limiting parallel brane/anti-brane case, in which angular momentum is replaced by linear momentum, seemed much less problematical). And what was so special about an ellipse? Was this some artefact of the matrix model approach? The possibility of a non-circular cross-section had been considered, and rejected, in [14], but what was actually shown there is that the cross-section must be circular if $R_z \neq 0$, whereas the supertube has $R_z = 0$. Let us return to (20) and set $R_z = 0$ but keep the variables $(E, B, R)$ as functions of $\sigma$. The same steps as before now lead to

$$H^2 = (D \pm B)^2 + \left(\frac{BD}{\sqrt{R^2 + R_z^2}} \mp 1\right)^2$$

As $D$ and $B$ are now functions of $\sigma$ we should minimise $H$ for fixed average electric displacement $\bar{D}$ and average magnetic field $\bar{B}$, these quantities being proportional to the IIA string charge and D0-brane charge per unit length, respectively. The result of this minimization procedure is that suggested by the above formula; the energy is minimised when

$$\sqrt{R^2 + R_z^2} = |BD|,$$
with no restriction on $R(\sigma)$, and the energy at this minimum is $|\vec{D}| + |\vec{B}|$. This implies preservation of 1/4 supersymmetry, as a direct computation confirms \[13\]. Note that the cross-sectional curve described by $R(\sigma)$ need not even be closed; the net D2-brane charge is non-zero for an open curve, which means that the supersymmetry algebra will include a D2-brane term in addition to the IIA string and D0 terms that are present for a closed curve. While this may be another surprise it is not one related to angular momentum, so I refer to \[13\] for details of its resolution.

One way to understand how an arbitrary tubular cross-section can be compatible with supersymmetry is to note that the supertube is TST-dual to a wave of arbitrary profile on a IIA string \[15\] because it has been appreciated for a long time that any uni-directional wave on a string preserves supersymmetry \[16\]. However, this explanation provides little in the way of intuition that might help us understand the original problem: how can a rotating tubular D2-brane with a non-circular cross section have a time-independent profile? There is a simple explanation, which can be found in \[15\]. I shall explain it here in terms of a closely related undergraduate mechanics problem that I heard about from Brandon Carter, whose earlier paper with Martin \[20\] describes a similar phenomenon in the context of superconducting cosmic strings.

Consider an elastic hosepipe through which superfluid flows at variable velocity $v(s)$, where $s$ parametrizes the curve $C$ described by the hosepipe. If $C$ is not straight then the motion of the fluid will produce a centrifugal force proportional to $v^2(s)/r(s)$ where $r(s)$ is the radius of curvature of $C$ at position $s$. If we are free to choose the function $v(s)$ then we may choose it such that the centrifugal force produced by the fluid motion exactly balances, for all $s$, the centripetal force due to the hosepipe tension. Given $C$, and a parametrization of it, the function $v(s)$ will be determined by this local force balance condition. We may choose $C$ to be closed, in which case the hosepipe forms a closed loop of arbitrary shape that is prevented from collapse by the angular momentum generated by the circulating superfluid. A crucial feature of this hosepipe loop is that its shape does not rotate, so the energy density profile is time-independent, despite the non-zero angular momentum. This shows that time-independence does not imply circular symmetry.

7 Epilogue

I have discussed three surprises involving angular momentum in even space dimension arising from various research projects on rotating supersymmetric objects undertaken over the past few years with Eduardo Eyras, Jerome Gauntlett, David Mateos, Robert Myers, Selena Ng and Marija Zamaklar. Although each surprise arose in its own distinct context, this article was motivated by the idea that there is something to be learnt by considering them together. Some of the connections between the three surprises have been discussed above, but there are others. For example, the supertube of circular cross-section is special in that it maximizes the angular momentum for given energy. This angular momentum upper bound arises in the context of the supergravity solution sourced by a supertube as a condition for the absence of global causality violations due to closed time-like curves \[21\], which is precisely the origin of the upper bound on the angular momentum of a supersymmetric rotating 5D black hole. Also, the fact that the shape of a supertube does not rotate despite the non-zero angular momentum is reminiscent of the fact that the horizon of the ‘rotating’ 5D black hole does not rotate despite its non-zero angular momentum; of course, the underlying reason here is supersymmetry, but one wonders whether the mechanisms might not also be related.

There are also several further connections between dyonic instantons and supertubes. The dyonic instanton of 5D gauge theory has a 3D sigma-model analogue known as a Q-lump \[22\]: this is a charged sigma-model lump that expands to a loop for large charge.
in the same way that the dyonic instanton expands to a 3-sphere \[23\]. In fact, it was this analogy that led to the realization that there should exist a tubular supersymmetric D2-brane supported by angular momentum; as explained in \[14\], the supertube can be viewed as an effective worldvolume description of the sigma-model Q-lump. Given this, it is natural to wonder whether the dyonic instanton has a similar effective realization as a tubular D(3+p)-brane with a 3-sphere cross section (the 5D SYM/Higgs theory has a D4-brane realization, and the dyonic instanton then acquires a string-theory interpretation \[24\] but this is quite different from the effective brane description being suggested here).

Finally, there is the question, which I leave unanswered, whether the special features of angular momentum in two and four space dimensions extend to higher even dimensions. For example, is non-zero angular momentum compatible with $SO(2n)$ symmetry in $2n$ space dimensions for $n > 2$?

References

[1] R. Peierls, *Surprises in Theoretical Physics*, Princeton University Press (1979), Princeton University Press (1991).

[2] J.P. Gauntlett, R. Myers and P.K. Townsend, *Black holes of D=5 supergravity*, Class.Quant.Grav.16:1-21,1999

[3] R. Wald, *The first law of black hole mechanics: vol 1*, in *Directions in general relativity*, eds. B.L. Hu, M.P. Ryan, Jr., C.V. Vishveshwara and T.A. Jacobson, pp 358-366, Cambridge University Press (1993); gr-qc/9305022

[4] E. Cremmer, *Supergravities in five dimensions*, in *Superspace and supergravity*, eds. S.W. Hawking and M. Rocek, pp 267-282, Cambridge University Press (1981)

[5] J.C. Breckenridge, R.C. Myers, A.W. Peet and C. Vafa, *D-branes and spinning black holes*, Phys.Lett.B391:93-98,1997

[6] R. Kallosh, A. Rajaraman and W.K. Wong, *Supersymmetric rotating black holes and attractors*, Phys.Rev.D55:3246-3249,1997

[7] A.H. Chamseddine and W.A. Sabra, *Metrics admitting Killing spinors in five dimensions*, Phys.Lett.B426:36-42,1998

[8] G.W. Gibbons and C.A.R. Herdeiro, *Supersymmetric rotating black holes and causality violation*, Class.Quant.Grav.16:3619-3652,1999

[9] C.A.R. Herdeiro, *Special properties of five-dimensional BPS rotating black holes*, Nucl.Phys.B582:363-392,2000

[10] T. Damour, *Black hole eddy currents*, Phys.Rev.D18:3598-3604,1978

[11] *Black Holes: the membrane paradigm*, eds. K.S. Thorne, R.H. Price and D.A. Macdonald, Yale University Press (1986).

[12] N.D. Lambert and D. Tong, *Dyonic instantons in five-dimensional gauge theories*, Phys.Lett.B462:89-94,1999

[13] E. Eyras, P.K. Townsend and M. Zamaklar, *The heterotic dyonic instanton*, JHEP 0105:046,2001

[14] D. Mateos and P.K. Townsend, *Supertubes*, Phys.Rev.Lett.87:011602,2001
[15] D. Mateos, S.K.L. Ng and P.K. Townsend, Tachyons, supertubes and brane/anti-brane systems, JHEP 0203:016, 2002

[16] D. Bak and K. Lee, Noncommutative supersymmetric tubes, Phys.Lett.B509:168-174, 2001

[17] D. Bak and A. Karch, Supersymmetric brane anti-brane configurations, Nucl.Phys.B626:165-182, 2002

[18] D. Mateos, S.K.L. Ng and P.K. Townsend, Supercurves, Phys.Lett.B538:366-374, 2002

[19] A. Dabholkar, J.P. Gauntlett, J.A. Harvey and D. Waldram, Strings as solitons and black holes as strings, Nucl.Phys.B474:85-121, 1996

[20] B. Carter and X. Martin, Dynamic instability criterion for circular string loops, Annals Phys.227:151-171, 1993

[21] R. Emparan, D. Mateos and P.K. Townsend, Supergravity supertubes, JHEP 0107:011, 2001

[22] R. Leese, Q-Jumps and their interactions, Nucl.Phys.B366:283-314, 1991; E.R.C. Abraham nonlinear sigma models and their Q-lump solutions, Phys.Lett.B278:291-296, 1992

[23] E.R.C. Abraham and P.K. Townsend, More on Q-kinks: a (1+1)-dimensional analogue of dyons, Phys.Lett.B295:225-232, 1992

[24] M. Zamaklar, Geometry of the non-abelian DBI dyonic instanton, Phys.Lett.B493:411-420, 2000