PATH IDEALS OF WEIGHTED GRAPHS

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Abstract. We introduce and study the weighted $r$-path ideal of a weighted graph $G_\omega$, which is a common generalization of Conca and De Negri’s $r$-path ideal for unweighted graphs and Paulsen and Sather-Wagstaff’s edge ideal of the weighted graph. Over a field, we explicitly describe primary decompositions of these ideals, and we characterize Cohen-Macaulayness of these ideals for trees (with arbitrary $r$) and complete graphs (for $r = 2$).

Introduction

Assumption. Throughout this paper, let $G$ be a (finite, simple) graph with vertex set $V = V(G) = \{v_1, \ldots, v_n\}$ of cardinality $n \geq 1$ and edge set $E(G) = E$. Let $A$ be a non-zero commutative ring, and set $S = A[X_1, \ldots, X_n]$ unless otherwise specified. Fix an integer $r \in \mathbb{N} = \{1, 2, \ldots\}$.

Commutative algebra and combinatorics have a rich history of fruitful interactions. In this paper, we focus on the connections between commutative algebra and graph theory. For our purposes, this begins with Villarreal’s notion \cite{16,17} of the edge ideal associated to the graph $G$, which is the ideal $I(G)$ in $S$ “generated by the edges of $G$”. Much research has been done on the relations between the combinatorial properties of $G$ and the algebraic properties of $I(G)$; see, e.g., \cite{3,4,5,6,8,9,10,13,14,15}. For instance, it is straightforward to show that, when $A$ is a field, an irredundant primary decomposition of the ideal $I(G)$ is determined by “vertex covers” of the graph $G$. Thus, given decomposition information about $I(G)$, one can deduce combinatorial information about $G$, and vice versa.

Recently, this construction has been generalized in two different directions relevant to our work. First, Conca and De Negri \cite{2} introduce the $r$-path ideal of $G$, when $G$ is a tree. This is the ideal $I_r(G)$ of $S$ “generated by the paths in $G$ of length $r$”. This recovers Villarreal’s edge ideal as the special case $I_1(G) = I(G)$. See also \cite{1} for useful properties of this construction, including a characterization of the Cohen-Macaulay property of $I_r(G)$.

Next, Paulsen and Sather-Wagstaff \cite{11} introduce the edge ideal of a weighted graph $G_\omega$, i.e., a graph $G$ equipped with a function $\omega: E \to \mathbb{N}$ that assigns to each edge $e$ of $G$ a weight $\omega(e)$. The edge ideal $I(G_\omega)$ in this case is generated by the weighted edges of $G_\omega$. In particular, if $1: E \to \mathbb{N}$ is the constant function $1(e) = 1$, then $I(G_1) = I(G)$. See Section \cite{1} for foundational material about weighted graphs.

In the current paper, we introduce and study a common generalization of these two constructions, the weighted $r$-path ideal associated to $G_\omega$. This is the ideal

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$I_r(G_\omega)$ of $S$ that is “generated by the weighted paths of length $r$ of $G$”.

$$I_r(G_\omega) = \left( X_{e_1}^{r_1} \cdots X_{e_r}^{r_r} \mid \begin{array}{l}
v_i \cdots v_{i+1} \text{ is a path in } G \text{ with } e_i = \omega(v_i,v_{i+1}), \\
e_i = \max\{\omega(v_{i-1},v_i),\omega(v_i,v_{i+1})\} \text{ for } 1 < j \leq r \end{array} \right) S$$

As before, this recovers the previous constructions as special cases with $I_r(G_1) = I_r(G)$ and $I_r(G_\omega) = I(G_\omega)$.

We investigate foundational properties of $I_r(G_\omega)$ in Section 2. In particular, the following decomposition result is proved in Theorem 2.7.

**Theorem A.** Given a weighted graph $G_\omega$, one has

$$I_r(G_\omega) = \bigcap_{(W,\sigma)} P(W,\sigma) = \bigcap_{(W,\sigma) \text{ min}} P(W,\sigma)$$

where the first intersection is taken over all weighted $r$-path vertex covers of $G_\omega$, and the second intersection is taken over all minimal weighted $r$-path vertex covers of $G_\omega$. Moreover, the second intersection is irredundant.

(See Section 11 for definitions of terms like “weighted $r$-path vertex cover”.) When $A$ is a field, this result yields a primary decomposition of $I_r(G_\omega)$.

In Section 3, we turn our attention to Cohen-Macaulayness of $I_r(G_\omega)$ when the underlying graph $G$ is a tree. The main result of this section is the following, which is proved in Theorem 3.11.

**Theorem B.** Assume that $G_\omega$ is a weighted tree and that $A$ is a field. Then the following conditions are equivalent:

(i) $I_r(G_\omega)$ is Cohen-Macaulay;

(ii) $I_r(G_\omega)$ is m-unmixed; and

(iii) there is a weighted tree $\Gamma_\mu$ and an $r$-path suspension $H_\lambda$ of $\Gamma_\mu$ such that $H_\lambda$ is obtained by pruning a sequence of $r$-pathless leaves from $G_\omega$ and for all $v_i v_j \in E(\Gamma_\mu)$ we have $\omega(v_i v_j) \leq \min\{\omega(v_i v_{i+1}),\omega(v_{i+1} v_{i+2}),\ldots,\omega(v_{j-1} v_j)\}$.

Note that this shows that Cohen-Macaulayness of path ideals of weighted trees is independent of the characteristic of $A$.

Section 4 is devoted to Cohen-Macaulayness of $I_r(K_n^\omega)$, where $G = K_n$ is complete, i.e., an $n$-clique. Note that it is straightforward to show that the edge ideal $I_1(K_n^\omega)$ is always Cohen-Macaulay, since it is unmixed of dimension 1. On the other hand, the case of $I_r(G_\omega)$ with $r \geq 2$ is more complicated. We deal with the case $r = 2$, the proof of which takes up most of Section 4; see Theorems 4.7 and 4.12.

**Theorem C.** Assume that $n \geq 3$, and let $K_n^\omega$ be a weighted $n$-clique. Assume that $A$ is a field. Then the ideal $I_2(K_n^\omega)$ is Cohen-Macaulay if and only if every induced weighted sub-$3$-clique $K_3^\omega$ of $K_n^\omega$ has $I_2(K_3^\omega)$ Cohen-Macaulay.

As in Theorem 4, this shows that the Cohen-Macaulay property is characteristic-independent for cliques. Unlike Theorem 1, though, it does not say that Cohen-Macaulayness is equivalent to unmixedness. See Example 4.10 for a weighted 4-clique that is unmixed but not Cohen-Macaulay.

Finally, we note that in Sections 4 and 2, we deal with a more general situation than the one described in this introduction. It uses the following. 
Notation. Throughout this paper, $G_\omega$ is a weighted graph. Let $\mathcal{P}_2(\mathbb{N})$ denote the set of subsets $U \subseteq \mathbb{N}$ such that $|U| \leq 2$. Fix a function $f: \mathcal{P}_2(\mathbb{N}) \rightarrow \mathbb{N}$, and write $f\{a, b\}$ in place of $f(\{a, b\})$. For instance, $f$ may be max, min, gcd, or lcm.

1. Weighted Graphs and Weighted $r$-Path Vertex Covers

In this section, we develop the graph theory used in the rest of the paper, beginning with the unweighted situation.

Definition 1.1. An $r$-path in $G$ is a sequence $v_{i_1} \ldots v_{i_{r+1}}$ of distinct vertices in $G$ such that the pair $v_{i_j}v_{i_{j+1}}$ is an edge in $G$ for $j = 1, \ldots, r$.

Example 1.2. Let $G$ be the following tree

```
  v_1 -- v_2 -- v_3 -- v_4 -- v_5
      ^    |    |    |    |
      v_6  v_7  v_8  v_9  v_{10}
```

and consider the case $r = 3$. Then $G$ has four distinct 3-paths, namely $v_1v_2v_3v_4$, $v_1v_2v_3v_6$, $v_2v_3v_4v_5$, and $v_6v_3v_4v_5$.

The next notion is key for Theorem A and the rest of the paper.

Definition 1.3. An $r$-path vertex cover of $G$ is a subset $W \subseteq V$ such that for any path $v_{i_1} \ldots v_{i_{r+1}}$ of length $r$ in $G$ we have $v_{i_j} \in W$ for some $j$. In this case, we write that $v_{i_j}$ “covers” the path.

An $r$-path vertex cover of $G$ is minimal if it is minimal with respect to containment, that is, it does not properly contain another $r$-path vertex cover of $G$.

For instance, consider the tree $G$ from Example 1.2 with $r = 3$. Then the singleton $\{v_3\}$ is a 3-path vertex cover, since each 3-path in $G$ is covered by $v_3$. We represent this diagrammatically, as follows.

```
  v_1 -- v_2 -- v_3
      ^    |    |
      v_6  v_7  v_8
```

Moreover, this is a minimal 3-path vertex cover of $G$ since $\emptyset$ is not a 3-path vertex cover. On the other hand, no other singleton is a 3-path vertex cover. (For instance, the vertex $v_1$ does not cover the path $v_6v_3v_4v_5$.) However, the set $\{v_1, v_3\}$ is another minimal 3-path vertex cover of $G$.

For graphs represented diagrammatically, we use the diagram for a visual representation of the weight function $\omega$ by decorating each edge $v_iv_j$ with the weight $\omega(v_iv_j)$, as follows.

Example 1.4. A particular weight function $\omega$ on the tree $G$ from Example 1.2 is represented in the following diagram.

```
  v_1  v_2  v_3  v_4  v_5
  2    1    2    2    \\
  3    |
  v_6
```

For instance, this means that $\omega(v_3v_6) = 3$. 


As one may expect, the following definition provides a combinatorial description of decompositions of ideals constructed from $G_\omega$. See Section 2

**Definition 1.5.** Set $\Lambda = \{(W, \sigma) \mid W \subseteq V \text{ and } \sigma : W \to \mathbb{N}\}$. For each $(W, \sigma) \in \Lambda$, we set $|\{(W, \sigma)\}| = |W|$.

An $f$-weighted $r$-path vertex cover of a weighted graph $G_\omega$ is an ordered pair $(W, \sigma) \in \Lambda$ such that for every path $v_{i_1} \ldots v_{i_r+1}$ of length $r$ in $G$, there exists an index $j$ such that $v_{i_j} \in W$ and one of the following holds:

1. if $j = 1$, then $\sigma(v_{i_1}) \leq \omega(v_{i_1}v_{i_2})$;
2. if $j = r + 1$, then $\sigma(v_{i_j}) \leq \omega(v_{i_r}v_{i_{r+1}})$; or
3. if $1 < j < r$, then $\sigma(v_{i_j}) \leq f(\omega(v_{i_{j-1}}v_{i_j}), \omega(v_{i_j}v_{i_{j+1}}))$.

(In particular, when $(W, \sigma)$ is an $f$-weighted $r$-path vertex cover of $G_\omega$, the set $W$ is an $r$-path vertex cover of the unweighted graph $G$.) The number $\sigma(v_{i_j})$ is the weight of $v_{i_j}$. When $v_{i_j}$ satisfies one of the above conditions, we write that it covers the path $v_{i_1} \ldots v_{i_{r+1}}$. When $f = \text{max}$, we write that $(W, \sigma)$ is a weighted $r$-path vertex cover of $G_\omega$.

We represent $f$-weighted $r$-path vertex covers algebraically and diagrammatically, as follows.

**Example 1.6.** Consider the weighted tree $G_\omega$ from Example 1.4 with $r = 3$ and with $f = \text{max}$. The set $\{v_3\}$ is a 3-path vertex cover of $G$, and the function $\sigma : \{v_3\} \to \mathbb{N}$ given by $\sigma(v_3) = 2$ yields a weighted 3-path vertex cover of $G_\omega$. We represent this algebraically and diagrammatically, by decorating the vertex $v_3$ with the weight $\sigma(v_3) = 2$, as follows.

\[
(W, \sigma) = \{v_3^2\} \quad \text{and} \quad \begin{array}{c}
\begin{array}{c}
\ 2 \\
3 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
v_1 \ \\ v_2 \ \\ v_3^2 \ \\ v_4 \ \\ v_5 \ \\ v_6
\end{array}
\end{array}
\]

By definition, a function $\sigma' : \{v_3\} \to \mathbb{N}$ yields a weighted 3-path vertex cover of $G_\omega$ if and only if $\sigma'(v_3) \leq 2$. Similarly, a decorated set $\{v_1^{d_1}, v_5^{d_5}\}$ describes a weighted 3-path vertex cover of $G_\omega$ if and only if $d_1, d_5 \leq 2$.

**Definition 1.7.** Given $(W, \sigma), (W', \sigma') \in \Lambda$, we write $(W', \sigma') \leq (W, \sigma)$ if $W' \subseteq W$ and for all $v_i \in W'$ we have $\sigma(v_i) \leq \sigma'(v_i)$. Naturally, we write $(W', \sigma') < (W, \sigma)$ whenever we have $(W', \sigma') \leq (W, \sigma)$ and $(W', \sigma') \neq (W, \sigma)$. An $f$-weighted $r$-path vertex cover $(W, \sigma)$ is minimal if it is minimal with respect to this ordering, that is, if there does not exist another $f$-weighted $r$-path vertex cover $(W', \sigma')$ such that $(W', \sigma') < (W, \sigma)$.

**Example 1.8.** Consider the weighted tree $G_\omega$ from Example 1.4 with $r = 3$ and with $f = \text{max}$. The decorated sets $\{v_3^2\}$ and $\{v_1^2, v_5^2\}$ are minimal weighted 3-path vertex covers of $G_\omega$.

**Example 1.9.** Given an $r$-path vertex cover $W$ of $G$, it is straightforward to show that the constant function $\sigma : W \to \mathbb{N}$ with $\sigma(v) = 1$ provides an $f$-weighted $r$-path vertex cover $(W, \sigma)$.

The next two results are for use in the proof of Theorem A.
Lemma 1.10. Assume that for all \( j \in \mathbb{N} \) we have an \( f \)-weighted \( r \)-path vertex cover \((W_j, \sigma_j) = \{v_{i_1}^{a_1}, \ldots, v_{i_p}^{a_p}, v_{i_{p+1}}^{b_j}\}\) of \( G_\omega \). If the sequence \( \{b_1, b_2, \ldots\} \) is unbounded, then \((W, \sigma) = \{v_{i_1}^{a_1}, \ldots, v_{i_p}^{a_p}\}\) is also an \( f \)-weighted \( r \)-path vertex cover of \( G_\omega \).

Proof. By assumption, there exists an index \( j \) such that \( b_j \) is greater than each of the following numbers: \( \omega(v_p) \) for each edge \( v_p \) in \( G \), and \( f(\omega(v_jv_j), \omega(v_jv_k)) \) for each 2-path \( v_jv_jv_k \) in \( G \). It follows that the weighted vertex \( v_{i_{p+1}}^{b_j} \) does not cover any \( f \)-weighted path in \( G_\omega \). Since \((W_j, \sigma_j)\) is an \( f \)-weighted \( r \)-path vertex cover of \( G_\omega \), it follows that \((W, \sigma)\) is an \( f \)-weighted \( r \)-path vertex cover of \( G_\omega \). \( \square \)

Lemma 1.11. For every \( f \)-weighted \( r \)-path vertex cover \((W, \sigma)\) of \( G_\omega \) there is a minimal \( f \)-weighted \( r \)-path vertex cover \((W'', \sigma'')\) of \( G_\omega \) with \((W'', \sigma'') \leq (W, \sigma)\).

Proof. If \((W, \sigma)\) is a minimal \( f \)-weighted \( r \)-path vertex cover then we are done. If \((W, \sigma)\) is not minimal, then either there is a \( v \in W \) that can be removed or for some \( v \in W \) the function \( \sigma(v) \) can be increased. In the first case, remove vertices from \( W \) until the removal of one more vertex creates a path without a vertex to cover it. This process must terminate in finitely many steps because \( W \) is finite. Let us denote our new \( f \)-weighted \( r \)-path vertex cover as \((W', \sigma')\). If no vertices are removed, then \((W, \sigma) = (W', \sigma')\).

Lemma 1.10 shows that each vertex \( v_i \in W' \) has a bound beyond which one cannot increase the weight on \( v_i \) without losing the \( f \)-weighted \( r \)-path vertex covering property, assuming the weights on the other vertices are held constant. In sequence, increase the weight of each vertex to such a bound. Denote the new ordered pair \((W'', \sigma'')\). Then, by construction, \((W'', \sigma'')\) is a minimal \( f \)-weighted \( r \)-path vertex cover such that \((W'', \sigma'') \leq (W, \sigma)\), and we are done. \( \square \)

The next result uses \( f = \max \).

Lemma 1.12. Every minimal weighted \( r \)-path vertex cover of \( G_\omega \) has cardinality at most \( n - 1 \).

Proof. In the case \( n \leq r \), the graph \( G \) has no \( r \)-paths, so the empty set describes the unique minimal weighted \( r \)-path vertex cover of \( G_\omega \). This has cardinality 0 < \( n \), as desired. Thus, for the remainder of the proof, we assume that \( n > r \).

Let \((W, \sigma)\) be a weighted \( r \)-path vertex cover of \( G_\omega \). We show that, if \(|W| = n\), then \((W, \sigma)\) is not minimal.

Assume that \(|W| = n\), and write \((W, \sigma) = \{v_1^{e_1}, \ldots, v_n^{e_n}\}\). Reorder the \( v_i \) if necessary to assume that \( e_1 \leq e_2 \leq \cdots \leq e_n \). We show that \( v_n^{e_n} \) is superfluous in the vertex cover.

Suppose by way of contradiction that \( v_n^{e_n} \) cannot be removed from \((W, \sigma)\). This implies that one of the \( r \)-paths \( p \) in \( G \) can only be covered by the weighted vertex \( v_n^{e_n} \). In particular, \( p \) must pass through \( v_n \), so assume that \( p \) uses the vertices \( v_1, \ldots, v_{i_1}, v_{i_1} \) with \( i_1, \ldots, i_r < n \).

As a special case, assume that \( p \) has the following form.

```
\begin{array}{c}
\text{\( e_1 \)} \\
\text{\( v_1 \)} \\
\cdots \\
\text{\( e_{i_r-1} \)} \\
\text{\( v_{i_r-1} \)} \\
\text{\( e_{i_r} \)} \\
\text{\( v_{i_r} \)} \\
\text{\( e_{i_r} \)} \\
\text{\( v_{i_r} \)} \\
\end{array}
```

By assumption, the weighted vertices \( v_{i_r-1}^{e_{i_r-1}} \) and \( v_{i_r}^{e_{i_r}} \) do not cover this path, so we have \( e_{i_r-1} > a \) and \( e_{i_r} > b \). Also, the weighted vertex \( v_n^{e_n} \) does cover this path, so we have \( e_n \leq a < e_{i_r-1} \leq e_n \) or \( e_n \leq b < e_{i_r} \leq e_n \), a contradiction.
The general case where \( v_n \) is not an endpoint of \( p \) is handled similarly. The remaining case where \( v_n \) is an endpoint of \( p \) is similar, but easier. \( \square \)

**Definition 1.13.** A weighted graph \( G_\omega \) is \( r \)-path unmixed with respect to \( f \) if all minimal \( f \)-weighted \( r \)-path vertex covers have the same cardinality; \( G_\omega \) is \( r \)-path mixed with respect to \( f \) if it is not \( r \)-path unmixed. We write that the unweighted graph \( G \) is "\( r \)-path (un)mixed" when the trivially weighted graph (with \( \omega(e) = 1 \) for all \( e \in E \)) is so.

2. Weighted Path Ideals and their Decompositions

In this section, we introduce and study weighted path ideals. In particular, we prove Theorem A from the introduction here.

**Definition 2.1.** The \( f \)-weighted \( r \)-path ideal associated to \( G_\omega \) is the ideal \( I_{r,f}(G_\omega) \) of \( S \) that is "generated by the weighted \( r \)-paths in \( G_\omega \)."

\[
I_{r,f}(G_\omega) = \left( X_{e_{i_1}}^{e_{i_1}} \cdots X_{e_{i_r+1}}^{e_{i_r+1}} \mid v_{i_1} \cdots v_{i_r+1} \text{ is a path in } G \text{ with } e_{i_j} = \omega(v_{i_{j-1}}v_{i_j}), \right. \\
\left. e_{i_j} = f \{ \omega(v_{i_{j-1}}v_{i_j}), \omega(v_{i_j}v_{i_{j+1}}) \} \text{ for } 1 < j \leq r, \right. \\
\left. \text{and } e_{i_{r+1}} = \omega(v_{i_r}v_{i_{r+1}}) \right) S
\]

See Remark 2.4 for some justification for this definition.

**Example 2.2.** Consider the weighted tree \( G_\omega \) from Example 1.4 with \( r = 3 \) and with \( f = \max \). The 3-path \( v_1v_2v_3v_6 \) provides one generator of \( I_{3, \max}(G_\omega) \), namely

\[
X_1^{\omega(v_1v_2)} X_2^{\max(\omega(v_1v_2), \omega(v_2v_3))} X_3^{\max(\omega(v_2v_3), \omega(v_3v_6))} X_6^{\omega(v_3v_6)} = X_1^2 X_2^2 X_3^3 X_6^3.
\]

From the remaining 3-paths, we find that

\[
I_{3, \max}(G_\omega) = (X_1^2 X_2^2 X_3^3 X_6^3, X_1^2 X_3^2 X_4^2 X_6^2, X_1^2 X_4^2 X_5^3, X_3^2 X_4^2 X_6^3) S.
\]

**Remark 2.3.** In the case \( r = 1 \), the ideal \( I_{1, f}(G_\omega) \) is the "weighted edge ideal" of \([1] \). (Note that this is independent of the choice of \( f \).) When \( \omega(e) = 1 \) for all \( e \in E \) and \( f = \max \), we recover the "path ideal" \( I_r(G) \) of \([1, 2] \). Also, the special case \( f = \max \) yields the ideal \( I_r(G_\omega) \) from the introduction.

**Remark 2.4.** Our definition of \( I_{r,f}(G_\omega) \) probably deserves some justification. Our purpose is to have this definition satisfy the conclusions of Remark 2.3. In order to recover the path ideal of \([1, 2] \), the generators should correspond to the \( r \)-paths in \( G \). To recover the weighted edge ideal of \([1] \) in the case \( r = 1 \), the generator corresponding to a path \( \zeta = v_{i_1} \cdots v_{i_{r+1}} \) should be of the form \( X_{e_{i_1}}^{e_{i_1}} \cdots X_{e_{i_{r+1}}}^{e_{i_{r+1}}} \), where the exponent \( e_{i_j} \) depends on the weights of the edges in \( \zeta \) that are adjacent to the vertex \( v_{i_j} \). For the endpoints \( v_{i_1} \) and \( v_{i_{r+1}} \), it seems reasonable to simply use the weight of the only relevant edges, namely, \( \omega(v_{i_{j-1}}v_{i_j}) \) and \( \omega(v_{i_j}v_{i_{j+1}}) \). However, when \( 1 < j \leq r \), the value of \( e_{i_j} \) should depend on both weights \( \omega(v_{i_{j-1}}v_{i_j}) \) and \( \omega(v_{i_j}v_{i_{j+1}}) \). We entertained several ideas about the "best" way to combine these two weights to define \( e_{i_j} \), including max, min, gcd, and lcm.

Theorem B.4 shows that, from the point of view of decomposing \( I_{r,f}(G_\omega) \) (e.g., computing a primary decomposition of \( I_{r,f}(G_\omega) \), determining unmixedness, etc. when \( A \) is a field) there is no "best" choice for \( f \). In other words, every choice for \( f \) yields an ideal that we can explicitly decompose. (In principle, this explains our choice of condition (3) in Definition 1.5. While this condition may seem a little strange, it is the exact condition that works for our decomposition result.) On the other hand, our results on Cohen-Macaulayness in Sections 3 and 4 indicate that
the choice \( f = \max \) is somewhat nicer than others we considered, in that it seems more difficult to characterize Cohen-Macaulayness of \( I_{r,f}(G_\omega) \) when \( f \neq \max \).

In the next definition, recall the notation \( \Lambda \) from \ref{01}.

**Definition 2.5.** For all \((W, \sigma) \in \Lambda\) we write \( P_{(W,\sigma)} = (X_{i}^{\sigma(v_i)} | v_i \in W)S.\)

One advantage for the algebraic notation from Example \ref{13} for elements of \( \Lambda \), is that it explicitly provides generators for the ideal \( P_{(W,\sigma)} \). For instance, with \((W, \sigma) = \{v_1^2, v_3^2\}\), we have

\[
P_{(W,\sigma)} = P_{\{v_1^2, v_3^2\}} = (X_1^2, X_3^2)S.
\]

**Remark 2.6.** It is straightforward to show that the ideals in \( S \) of the form \( P_{(W,\sigma)} \) are precisely the indecomposable elements of the set of monomial ideals of \( S \). In other words, a monomial ideal \( I \) of \( S \) is of the form \( P_{(W,\sigma)} \) if and only if it satisfies the following: for all monomial ideals \( J_1, J_2 \) such that \( I = J_1 \cap J_2 \), one has \( I = J_i \) for some \( j \in \{1,2\} \). (In the language of \ref{12}, these are the “m-irreducible” monomial ideals of \( S \).) When the coefficient ring \( A \) is a field, the ideal \( P_{(W,\sigma)} \) is primary with \( \text{rad}(P_{(W,\sigma)}) = (X_i \mid v_i \in W)S \). Hence, when we are working over a field, Theorem \ref{27} below gives an irredundant primary decomposition of \( I_{r,f}(G_\omega) \). In general, this is the “m-irreducible decomposition” of \ref{12}.

It is straightforward to show that every monomial ideal \( I \) of \( S \) admits a unique irredundant m-irreducible decomposition \( I = P_{(W_i,\sigma_i)} \cap \cdots \cap P_{(W_j,\sigma_j)} \); uniqueness here is up to reordering of the ideals in the decomposition, and “irredundant” means that no ideal in this decomposition is contained in any other ideal in the decomposition. We write that \( I \) is \( m \)-unmixed provided that all the \( W_i \) in this decomposition have the same cardinality. We write that \( I \) is \( m \)-mixed provided that it is not \( m \)-unmixed. When we are working over a field, these are equivalent to \( I \) being unmixed or mixed, respectively.

The next result contains Theorem \ref{A} from the introduction.

**Theorem 2.7.** (a) Given \((W, \sigma) \in \Lambda\), one has \( I_{r,f}(G_\omega) \subseteq P_{(W,\sigma)} \) if and only if \((W, \sigma) \) is an \( f \)-weighted \( r \)-path vertex cover of \( G_\omega \).

(b) One has decompositions

\[
I_{r,f}(G_\omega) = \bigcap_{(W,\sigma)} P_{(W,\sigma)} = \bigcup_{(W,\sigma)} P_{(W,\sigma)}
\]

where the first intersection is taken over all \( f \)-weighted \( r \)-path vertex covers of \( G_\omega \), and the second intersection is taken over all minimal \( f \)-weighted \( r \)-path vertex covers of \( G_\omega \). Moreover, the second intersection is irredundant.

**Proof.** First assume that \((W, \sigma) \) is an \( f \)-weighted \( r \)-path vertex cover of \( G_\omega \), and let \( v_{i_1} \cdots v_{i_{r+1}} \) be an \( r \)-path in \( G \). By definition, there exists a \( j \in \{1, \ldots, r+1\} \) such that \( v_{i_j} \in W \) and one of the following holds:

- \( j = 1 \): we have \( \sigma(v_{i_1}) \leq \omega(v_{i_1}v_{i_2}) = e_{i_1} \);
- \( j = r+1 \): we have \( \sigma(v_{i_{r+1}}) \leq \omega(v_{i_{r+1}}v_{i_{r+1}}) = e_{i_{r+1}} \); or
- \( 1 < j < r \): we have \( \sigma(v_{i_j}) \leq f(\omega(v_{i_{j-1}}v_{i_j}), \omega(v_{i_j}v_{i_{j+1}})) = e_{i_j} \).

In each case we have \( v_{i_j} \in W \) and \( \sigma(v_{i_j}) \leq e_{i_j} \). Thus, \( X_{i_j}^{\sigma(v_{i_j})} \) divides \( X_{i_j}^{e_{i_j}} \), and hence the generator \( X_{i_1}^{e_{i_1}} \cdots X_{i_{r+1}}^{e_{i_{r+1}}} \) of \( I_{r,f}(G_\omega) \) is in \( P_{(W,\sigma)} \). Since this is true for each \( r \)-path in \( G \), we conclude that \( I_{r,f}(G_\omega) \subseteq P_{(W,\sigma)} \).
Conversely, assume that $I_{r,f}(G_\omega) \subseteq P_{(W,\sigma)}$ and let $v_1, \ldots, v_{i+1}$ be an $r$-path in $G$. By assumption we have $X_{v_1}^{e_{v_1}} \cdots X_{v_{r+1}}^{e_{v_{r+1}}} \in I_{r,f}(G_\omega) \subseteq P_{(W,\sigma)} = (X^{\sigma(v_i)}_i | v_i \in W)$. Hence there exists an $i$ such that $v_i \in W$ and the associated generator $X^{\sigma(v_i)}_i$ divides $X_{v_1}^{e_{v_1}} \cdots X_{v_{r+1}}^{e_{v_{r+1}}}$. Since $\sigma(v_i) \geq 1$, there exists a $j$ such that $i_j = i$ and $\sigma(v_i) \leq e_{i_j}$. That is, there exists a $j$ such that $v_{i_j} = v_i \in W$ and $\sigma(v_{i_j}) \leq e_{i_j}$. Since this is true for each $r$-path in $G$, we conclude that $(W,\sigma)$ is an $f$-weighted $r$-path vertex cover of $G_\omega$.

This follows from Lemma 2.11 and part (iii), as in [11] Theorem 3.5. \[\square\]

**Corollary 2.8.** We have $\text{depth}(S/I_{r,f}(G_\omega)) \geq 1$.

**Proof.** Lemma 2.12 and Theorem 2.7 imply that the maximal ideal $(X_1, \ldots, X_n)S$ is not associated to $I_{r,\max}(K^n)$, hence the desired conclusion. \[\square\]

**Remark 2.9.** Remark 2.6 and Theorem 2.7 imply that $I_{r,f}(G_\omega)$ is $m$-unmixed if and only if $G_\omega$ is $r$-path unmixed. In particular, the $r$-path ideal $I_r(G)$ of [12] is $m$-unmixed if and only if the unweighted graph $G$ is $r$-path unmixed.

**Example 2.10.** Consider the weighted tree $G_\omega$ from Example 1.4 with $r = 3$ and with $f = \text{max}$. The ideal $I_{3,\max}(G_\omega)$, computed in Example 2.2, decomposes irredundantly as follows:

$$I_{3,\max}(G_\omega) = (X^2_3)^\omega S \cap (X^2_1, X^2_2) \cap (X^2_1, X^2_3) \cap (X^2_2, X^2_3) \cap (X^2_2, X^2_3) \cap (X^2_3, X^2_3) \cap (X^2_3, X^2_3) \cap (X^2_2, X^2_3) \cap (X^2_2, X^2_3) \cap (X^2_2, X^2_3) \cap (X^2_3, X^2_3) \cap (X^2_3, X^2_3).$$

If one computes this algebraically (as we did), one can identify all of the minimal weighted $r$-path vertex covers of $G_\omega$. (For instance, the minimal weighted $r$-path vertex covers $\{v_3^2\}$ and $\{v_1^2, v_2^2\}$ from Example 1.8 are visible via the ideals $(X^2_3)^\omega$ and $(X^2_1, X^2_3)$ in the decomposition.) On the other hand, if one is combinatorially inclined, one can first identify all of the minimal $f$-weighted $r$-path vertex covers of $G_\omega$, and then obtain the decomposition from Theorem 2.7.

The next lemma is for use in the proof of Theorem 13.

**Lemma 2.11.** If $I_{r,f}(G_\omega)$ is $m$-unmixed, then $I_r(G)$ is also $m$-unmixed.

**Proof.** Assume that $I_{r,f}(G_\omega)$ is $m$-unmixed. Then there exists an integer $k$ such that every minimal $f$-weighted $r$-path vertex cover $(W,\sigma)$ of $G_\omega$ has $|W| = k$. Let $W'$ be a minimal $r$-path vertex cover of $G$. We show that $|W'| = k$.

As we observed in Example 1.9 the constant function $\sigma' : W' \to \mathbb{N}$ given by $\sigma'(v_i) = 1$ yields an $f$-weighted $r$-path vertex cover $(W',\sigma')$ of $G_\omega$. Lemma 2.11 implies that there exists a minimal $f$-weighted $r$-path vertex cover $(W'',\sigma'')$ of $G_\omega$ such that $(W'',\sigma'') \subseteq (W',\sigma')$. By assumption, we have $|W''| = k$. By the minimality of $W'$, we have $W'' = W'$, hence $|W'| = |W''| = k$. \[\square\]

We conclude this section with two lemmas used in the proof of Theorem 13.

**Lemma 2.12.** Let $G'_\omega$ denote the weighted subgraph of $G$ induced by $V \setminus \{v_n\}$. Set $S' = A[X_1, \ldots, X_{n-1}]$. Then the natural isomorphism $S/(X_n)S \cong S'/I_{r,\max}(G'_\omega)$ induces an isomorphism $S/(I_{r,\max}(G_\omega) + (X_n)S) \cong S'/I_{r,\max}(G'_\omega)$. 

Proof. Let \( \tau : S/(X_n)S \to S'/I_{r,\max}(G'_{\omega}) \) denote the composition of the natural maps \( S/(X_n)S \xrightarrow{=\pi} S' \to S'/I_{r,\max}(G'_{\omega}) \). To show that \( \tau \) induces a well-defined epimorphism \( \pi : S/(I_{r,\max}(G_{\omega})+(X_n)S) \to S'/I_{r,\max}(G'_{\omega}) \), it suffices to show that each generator of \( I_{r,\max}(G_{\omega})(S/(X_n)S) \) is in \( \text{Ker}(\tau) \). Note that the generators of \( I_{r,\max}(G_{\omega})(S/(X_n)S) \) correspond to the \( r \)-paths in \( G \) that do not pass through \( v_n \). That is, they correspond to the \( r \)-paths in \( G' \). Since \( \omega'(e) = \omega(e) \) for each edge in \( G' \), it follows that the generators of \( I_{r,\max}(G_{\omega})(S/(X_n)S) \) and \( I_{r,\max}(G'_{\omega}) \) corresponding to such a path are equal. This gives the desired result about \( \text{Ker}(\tau) \).

A similar argument shows that \( \text{Ker}(\tau) = I_{r,\max}(G_{\omega})(S/(X_n)S) \), so the induced map \( \pi \) is an isomorphism. \( \square \)

Lemma 2.13. The ideal \( I_{r,f}(G_{\omega}) \) can be written as

\[
I_{r,f}(G_{\omega}) = \sum I_{r,f}(G'_{\omega})S
\]

where the sum is taken over all weighted subgraphs \( G'_{\omega} \) of \( G_{\omega} \) induced by \( r+1 \) vertices. (If \( G'_{\omega} \) is induced by \( v_{i_1}, \ldots, v_{i_{r+1}} \) with \( i_1 < \cdots < i_{r+1} \), then we consider \( I_{r,f}(G'_{\omega}) \) in the polynomial subring \( A[X_{i_1}, \ldots, X_{i_{r+1}}] \subseteq S \).

Proof. For the containment \( \supseteq \), note that each generator \( g \) of \( I_{r,f}(G'_{\omega})S \) is determined by an \( r \)-path in \( G' \), which is an \( r \)-path in \( G \) with the same weights; hence \( g \) is also a generator of \( I_{r,f}(G_{\omega}) \). For the reverse containment, note that each generator \( h \) of \( I_{r,f}(G_{\omega}) \) comes from an \( r \)-path in \( G_{\omega} \), and this \( r \)-path lives in a (unique) induced weighted subgraph \( G'_{\omega'} \) of \( G_{\omega} \) on \( r+1 \) vertices; thus, \( h \) is also a generator of \( I_{r,f}(G'_{\omega'})S \). \( \square \)

3. COHEN-MACAULAY WEIGHTED TREES

Assumption. Throughout this section, \( A \) is a field.

The point of this section is to prove Theorem 3 from the introduction characterizing Cohen-Macaulayness of trees in the context of weighted path ideals for the function \( f = \max \).

Definition 3.1. Assume that \( v_1 \) is a vertex of degree 1 in \( G \) that is not a part of any \( r \)-path in \( G \). We write that \( v_1 \) is an \( r \)-pathless leaf of \( G_{\omega} \). Let \( H_\lambda \) be the weighted subgraph of \( G_{\omega} \) induced by the vertex subset \( V \setminus \{ v_1 \} \). We write that \( H_\lambda \) is obtained by pruning an \( r \)-pathless leaf from \( G_{\omega} \). A weighted subgraph \( \Gamma_\mu \) of \( G_{\omega} \) is obtained by pruning a sequence of \( r \)-pathless leaves from \( G_{\omega} \) if there exists a sequence of weighted graphs \( G_{\omega} = G^{(0)}_{\omega(0)}, G^{(1)}_{\omega(1)}, \ldots, G^{(l)}_{\omega(l)} = \Gamma_\mu \) such that each \( G^{(i+1)}_{\omega(i+1)} \) is obtained by pruning an \( r \)-pathless leaf from \( G^{(i)}_{\omega(i)} \).

Example 3.2. In the weighted tree \( G_{\omega} \) from Example 2, the vertex \( v_6 \) is a 4-pathless leaf. Pruning this leaf yields the following weighted path \( H_\lambda \).

\[
\begin{array}{cccccc}
v_1 & 2 & v_2 & 1 & v_3 & 2 & v_4 & 2 & v_5
\end{array}
\]

Next, we state some consequences of the existence of an \( r \)-pathless leaf in \( G_{\omega} \).

Lemma 3.3. Let \( H_\lambda \) be a weighted graph obtained by pruning a single \( r \)-pathless leaf \( v_i \) from \( G_{\omega} \).

(a) The set of \( r \)-paths in \( G \) is the same as the set of \( r \)-paths in \( H \).
(b) Assume that \((W, \sigma)\) is an \(f\)-weighted \(r\)-path vertex cover of \(G_\omega\) such that \(v_1 \in W\). Set \(W' = W \setminus \{v_i\}\) and \(\sigma' = \sigma|_{W'}\). Then \((W', \sigma')\) is an \(f\)-weighted \(r\)-path vertex cover of \(G_\omega\).

(c) The minimal \(f\)-weighted \(r\)-path vertex covers of \(G_\omega\) are the same as the minimal \(f\)-weighted \(r\)-path vertex covers of \(H_\lambda\), so \(G_\omega\) is \(r\)-path unmixed with respect to \(f\) if and only if \(H_\lambda\) is so.

Proof. (a) This follows by definition of \(H\) since no \(r\)-paths in \(G\) pass through \(v_i\).

(b) Since no \(r\)-paths pass through \(v_i\), this vertex does not cover any \(r\)-paths, so it can be removed.

(c) Combining parts (a) and (b), we conclude that the \(f\)-weighted \(r\)-path vertex covers of \(H_\lambda\) are exactly the \(f\)-weighted \(r\)-path vertex covers \((W, \sigma)\) of \(G_\omega\) such that \(v_i \not\in W\). The desired conclusion about minimal elements now follows. □

The next definition is key for Theorem B.

Definition 3.4. Let \(\Gamma_\mu\) be a weighted graph. The \(r\)-path suspension of the unweighted graph \(\Gamma\) is the graph obtained by adding a new path of length \(r\) to each vertex of \(\Gamma\). The new \(r\)-paths are called \(r\)-whiskers. A weighted graph \(H_\lambda\) is a weighted \(r\)-path suspension of \(\Gamma_\mu\) provided that the unweighted graph \(H\) is an \(r\)-path suspension of \(\Gamma\).

Example 3.5. The weighted tree \(G_\omega\) from Example 1.4 is a weighted 1-path suspension of the following weighted path.

\[
v_2 \overset{1}{\rightarrow} v_3 \overset{2}{\rightarrow} v_4
\]

Examples of weighted \(r\)-path suspensions of \(G_\omega\) itself are given by the following, where the edges of \(G\) are drawn double for emphasis.

\[
\begin{array}{cccccc}
y_{1,1} & y_{2,1} & y_{3,1} & y_{4,1} & y_{5,1} \\
4 & 3 & 3 & 4 & 2 \\
v_1 \overset{2}{\rightarrow} v_2 \overset{1}{\rightarrow} v_3 \overset{2}{\rightarrow} v_4 \overset{2}{\rightarrow} v_5
\end{array}
\quad r = 1
\]

\[
\begin{array}{cccc}
y_6,1 & \\
3 & \\
v_6 \overset{2}{\rightarrow} v_1
\end{array}
\]

\[
\begin{array}{cccccc}
y_{1,2} & y_{2,2} & y_{3,2} & y_{4,2} & y_{5,2} \\
3 & 3 & 5 & 4 & 2 \\
y_{1,1} & y_{2,1} & y_{3,1} & y_{4,1} & y_{5,1} \\
4 & 3 & 3 & 4 & 2 \\
v_1 \overset{2}{\rightarrow} v_2 \overset{1}{\rightarrow} v_3 \overset{2}{\rightarrow} v_4 \overset{2}{\rightarrow} v_5
\end{array}
\quad r = 2
\]

\[
\begin{array}{cccc}
y_6,2 & y_6,1 & \\
3 & 200 & \\
v_6 \overset{3}{\rightarrow} v_1
\end{array}
\]

Remark 3.6. A weighted graph \(H_\lambda\) is an \(r\)-path suspension of another weighted graph \(\Gamma_\mu\) if and only if \(H\) has a sequence of pair-wise disjoint paths \(p_1, p_2, \ldots, p_\beta\) of
The polynomial ring $\tilde{\mathcal{P}}$ is partially in $\Gamma$; paths that start in a $I$ in another polynomial ring $T$ such that $T/I$ is Cohen-Macaulay. Since $T/J$ is Cohen-Macaulay, in particular, $T/J$ is Cohen-Macaulay. Thus $I$ is the ideal in a new polynomial ring $\widetilde{S}$.

**Proposition 3.7.** Let $H_\mu$ be an $r$-path suspension of the weighted graph $\Gamma_\mu$, with notation as in Remark 3.6 such that for all $v_i,v_j \in E(\Gamma)$ we have $\omega(v_i,v_j) \leq \min\{\omega(v_i,v_1),\omega(v_1,v_j)\}$. Then $I_{r,\text{max}}(H_\lambda)$ is Cohen-Macaulay.

**Proof.** As in the proof of [11, Lemma 5.3], we polarize the ideal $I := I_{r,\text{max}}(H_\lambda)$ to obtain a new ideal $\tilde{I}$ in a new polynomial ring $\widetilde{S}$. We then show that $\tilde{I}$ is the polarization of another monomial ideal $J$ in another polynomial ring $T$ such that $T/J$ is Cohen-Macaulay.

In preparation, we set some notation

$$a_i := \omega(v_iy_i)$$
$$a_{i,j} := \max\{\omega(v_iy_j),\omega(y_iy_j)\}$$
$$a_{i,r} := \omega(y_iy_{r-1})$$
$$t_{i,j} = \omega(y_{i-1}y_j)$$
$$b_{p,q,r} = \max\{(p+1),\omega(v_pv_q)v_r\}$$
$$c_{i,j} = \omega(v_iy_j)$$

The polynomial ring $\widetilde{S}$ has coefficients in $A$ with the following list of variables.

$$X_{1,1}, \ldots, X_{1,a_1}, Y_{1,1,1}, \ldots, Y_{1,a_1,1}, Y_{1,2,1}, \ldots, Y_{1,a_1,2}, \ldots, Y_{1,r,1}, \ldots, Y_{1,r,a_1,1}, Y_{2,1}, \ldots, Y_{2,a_2}, Y_{2,2,1}, \ldots, Y_{2,a_2,2}, \ldots, Y_{2,r,1} \ldots Y_{2,r,a_2}, \ldots, Y_{\beta,1}, \ldots, Y_{\beta,a_\beta}, Y_{\beta,1,1}, \ldots, Y_{\beta,a_\beta,1}, Y_{\beta,2,1}, \ldots, Y_{\beta,a_\beta,2}, \ldots, Y_{\beta,r,1} \ldots Y_{\beta,r,a_\beta,1}.$$
this generator for $I$, we obtain the next generator for $\bar{I}$.

\[ X_{i_1,1} \cdots X_{i_1,c_{1,2}} X_{i_2,1} \cdots X_{i_2,b_{i_2,1},i_2,i_3} \cdots X_{i_p,1} \cdots X_{i_p,b_{i_p-1,i_p},j} \]

\[ \cdot X_{j,1} \cdots X_{j,a_j} Y_{j,1,1} \cdots Y_{j,1,a_j,1} \cdots Y_{j,k-1,1} \cdots Y_{j,k-1,a_{j,k-1},1} \cdots Y_{j,k,t_{j,k}} \] (3.7.2)

Observe that the assumption $\omega(v_i v_j) \leq \min(\omega(v_i y_{j,1}), \omega(v_j y_{j,1}))$ for all $v_i v_j \in E(\Gamma)$ implies that we have $c_{i_1,2} \leq a_{i_1}$. Similarly, we have $b_{i_1,i_2,i_3} \leq a_{i_2}$ and the inequality $t_{j,k} \leq a_{j,k}$ is by construction. Thus, the generator (3.7.2) is in $S$.

Next, consider an $r$-path $y_{j,q} \cdots y_{j,1} v_{j} v_{m_1} \cdots v_{m_t} y_{i_1} \cdots y_{i_p}$ that starts in an $r$-whisker, runs through part of $\Gamma$, and ends in another $r$-whisker. Note that we have $l \geq 0$ and $q + l + p + 1 = r$. The generator in $I$ associated to this type of path is the following.

\[ Y_{j,q}^r Y_{j,q}^{a_{j,q}-1} \cdots Y_{j,1}^{a_{j,1}} X_j^{a_{j,m_1}} X_j^{b_{j,m_1,m_2}} \cdots X_j^{b_{j,m_1,m_2,\ldots,1}} X_{i_1,1}^{a_{i_1,1}} \cdots Y_{i,p-1,1}^{a_{i,p-1,1}} Y_{i,p,1}^{a_{i,p,1}} (3.7.3) \]

When we polarize this generator we obtain the next generator for $\bar{I}$.

\[ Y_{j,q}^r Y_{j,q}^{a_{j,q}-1} \cdots Y_{j,1}^{a_{j,1}} X_j^{a_{j,m_1}} X_j^{b_{j,m_1,m_2}} \cdots X_j^{b_{j,m_1,m_2,\ldots,1}} X_{i_1,1}^{a_{i_1,1}} \cdots Y_{i,p-1,1}^{a_{i,p-1,1}} Y_{i,p,1}^{a_{i,p,1}} (3.7.3) \]

For the last type of generator, consider an $r$-path $v_{i_1} \cdots v_{i_r+1}$ entirely in $\Gamma$. The generator in $I$ associated to this path is the following.

\[ X_{i_1,1}^c X_{i_2,1}^{b_{i_1,i_2,i_3}} \cdots X_{i_r,1}^{b_{i_r-1,1},r-1,r+1} X_{i_{r+1,1}}^c (3.7.4) \]

When we polarize this generator we obtain the next generator for $\bar{I}$.

\[ X_{i_1,1}^c X_{i_2,1}^{b_{i_1,i_2,i_3}} \cdots X_{i_r,1}^{b_{i_r-1,1},r-1,r+1} X_{i_{r+1,1}}^c (3.7.4) \]

Set $T = A[X_{i_1,1}, \ldots, X_{i_r,1}]$, and let $J$ be the monomial ideal of $T$ with the following generators. For each $r$-whisker $v_{i_1} y_{i_1} \cdots y_{i_r}$, include the following generator.

\[ X_{i_1,1}^{a_{i_1,1}} (3.7.5) \]

For each $r$-path $v_{i_1} v_{i_2} \cdots v_{i_p} v_{j} v_{j,1} \cdots y_{j,k}$ that starts in $\Gamma$ and ends in an $r$-whisker, include the next generator.

\[ X_{i_1,1}^c X_{i_2,1}^{b_{i_1,i_2,i_3}} \cdots X_{i_p,1}^{b_{i_p-1,1},p-1,j} X_{j,1}^{a_{j,1} + \cdots + a_{j,k-1} + t_{j,k}} (3.7.6) \]

For each $r$-path $y_{j,q} \cdots y_{j,1} v_{j} v_{m_1} \cdots v_{m_t} y_{i_1} \cdots y_{i_p}$ that starts in an $r$-whisker, runs through part of $\Gamma$, and ends in another $r$-whisker, include the next generator.

\[ X_{j,1}^{a_{j,q-1} + \cdots + a_{j,1} + a_{j,k-1}} X_{j,m_1,1}^{b_{j,m_1,m_2}} \cdots X_{j,m_1,m_2,\ldots,1}^{b_{j,m_1,m_2,\ldots,1}} X_{i_1,1}^{a_{i_1,1} + \cdots + a_{i,p-1} + t_{i,p}} (3.7.7) \]

For each $r$-path $v_{i_1} \cdots v_{i_r+1}$ entirely in $\Gamma$, include the next generator.

\[ X_{i_1,1}^c X_{i_2,1}^{b_{i_1,i_2,i_3}} \cdots X_{i_r+1,1}^{c_{i_r,1},r+1} (3.7.8) \]

It is straightforward to show that the polarization of $J$ is exactly $\bar{I}$: for $n = 1, 2, 3, 4$, the polarization of the generator $(3.7.4)$ of $J$ is exactly the generator $(3.7.4)$ of $I$. Since $J$ contains a power of each of the variables in $T$, namely $(3.7.5)$, we conclude that $T/J$ is artinian. Thus, the first paragraph of this proof implies that $S/I$ is Cohen-Macaulay. \qed
Example 3.8. For the weighted graph $G_\omega$ in Example 1.4, Proposition 3.7 shows that $I_{r,\max}(G_\omega)$ is Cohen-Macaulay, and similarly for $I_{2,\max}(G_\omega')$ in Example 3.5. See also Examples 3.12 and 3.13.

Note that the ideals $I_{r,\max}(G_\omega)$ and $I_{r,\max}(H_\lambda)$ in the next result live in different polynomial rings.

Lemma 3.9. Let $H_\lambda$ be a weighted graph obtained by pruning a sequence of $r$-pathless leaves from $G_\omega$.

(a) The ideals $I_{r,\max}(G_\omega)$ and $I_{r,\max}(H_\lambda)$ have the same generators.
(b) The ideal $I_{r,\max}(G_\omega)$ is $m$-unmixed if and only if $I_{r,\max}(H_\lambda)$ is so.
(c) The ideal $I_{r,\max}(G_\omega)$ is Cohen-Macaulay if and only if $I_{r,\max}(H_\lambda)$ is so.

Proof. Arguing by induction on the number of $r$-pathless leaves being pruned from $G_\omega$, we assume that $H_\lambda$ is obtained by pruning a single $r$-pathless leaf $v_i$ from $G_\omega$.

(a) By Lemma 3.3(a), the set of $r$-paths in $G$ is the same as the set of $r$-paths in $H$, and $\lambda(e) = \omega(e)$ for each edge $e \in E(H) \subseteq E(G)$. The claim about the generators now follows directly.

(b) This follows from Theorem 2.7(b) and Lemma 3.3(c).

(c) Part (a) implies that $(S'/I_{r,\max}(H_\lambda))[X] \cong S/I_{r,\max}(G_\omega)$, where $S' := A[X_1, \ldots, X_{r-1}, X_{r+1}, \ldots, X_n]$. It follows that $S/I_{r,\max}(G_\omega)$ is Cohen-Macaulay if and only if $S'/I_{r,\max}(H_\lambda)$ is Cohen-Macaulay, as desired.

The next result compares directly to Theorem 3.3 from the introduction, though it does not assume that $G$ is a tree.

Proposition 3.10. Assume that $H_\lambda$ is obtained by pruning a sequence of $r$-pathless leaves from $G_\omega$ and that $H_\lambda$ is an $r$-path suspension of a weighted graph $\Gamma_\mu$. With notation as in Remark 3.6, the following conditions are equivalent:

(i) $I_{r,\max}(G_\omega)$ is Cohen-Macaulay;
(ii) $I_{r,\max}(G_\omega)$ is $m$-unmixed; and
(iii) for all $v_i v_j \in E(\Gamma_\mu)$ we have $\omega(v_i v_j) \leq \min\{\omega(v_i y_{i,1}), \omega(v_j y_{j,1})\}$.

Proof. The case $r = 1$ is handled in [11] Theorem 5.7, so we assume that $r \geq 2$ for the remainder of the proof. The implication (ii)$\implies$(iii) always holds.

Assume that $I_{r,\max}(G_\omega)$ is $m$-unmixed. It follows from Lemma 3.9(b) that $I_{r,\max}(H_\lambda)$ is also unmixed. From an analysis of the $r$-paths of $H$ as in the proof of Proposition 3.4, it is straightforward to show that $V(\Gamma_\mu)$ is a minimal $r$-path vertex cover of $H$. (It covers all the paths, and the $r$-whiskers show that it is minimal.) Let $\tau: V(\Gamma_\mu) \to \mathbb{N}$ be the constant function $\tau(v_i) = 1$. Lemma 4.11 implies that there is a minimal weighted $r$-path vertex cover $(W'',\sigma'')$ of $H_\lambda$ such that $(W'',\sigma'') \leq (V(\Gamma_\mu),\tau)$. The minimality of $V(\Gamma_\mu)$ implies that $W'' = V(\Gamma_\mu)$, so $(V(\Gamma_\mu),\sigma'')$ is a minimal weighted $r$-path vertex cover of $H_\lambda$. The unmixedness condition implies that every minimal weighted $r$-path vertex cover of $H_\lambda$ has cardinality $|V(\Gamma_\mu)|$.

We proceed by contradiction. Suppose that there is an edge $v_i v_j \in E(\Gamma_\mu)$ such that $\omega(v_i v_j) > \min\{\omega(v_i y_{i,1}), \omega(v_j y_{j,1})\}$. We produce a contradiction by showing that there exists a minimal weighted $r$-path vertex cover $(W,\sigma)$ of $H_\lambda$ such that $|W| > |V(\Gamma_\mu)|$. Assume by symmetry that

$$a := \omega(v_i y_{i,1}) = \min\{\omega(v_i y_{i,1}), \omega(v_j y_{j,1})\} < \omega(v_i v_j) =: b.$$
Set $c = \omega(v_j y_j, 1)$ and $a' := \omega(y_{i,r-1} y_i, r)$ and $c' := \omega(y_{j,r-1} y_{j,r})$. The following digram (where the column represents $\Gamma$, and the rows represent the $r$-whiskers in $H$) is our guide for constructing an approximation of $(W, \sigma)$.

\[
\begin{array}{cccccc}
  & v_i & \cdots & v_j & \cdots & v_k \\
  & b & c & \cdots & c & d \\
  & a & y_i, 1 & \cdots & y_j, 1 & \cdots & y_k, 1 \\
  & & y_i & \cdots & y_j & \cdots & y_k \\
  & & & & & & \\
\end{array}
\]

Set $W = \{ v_k | k \neq j \} \cup \{ y_{i,r}, y_{j,r} \}$ and define $\sigma : W \to \mathbb{N}$ by

$$
\sigma(v_k) = \begin{cases} 
1 & \text{if } k \neq i \\
b & \text{if } k = i 
\end{cases}
$$

$$
\sigma(y_{i,r}) = a'
$$

$$
\sigma(y_{j,r}) = c'.
$$

It is straightforward to show that $(W, \sigma)$ is a weighted $r$-path vertex cover of $H_\lambda$.

Lemma 1.11 provides a minimal weighted $r$-path vertex cover $(W', \sigma')$ of $G_\omega$ such that $(W', \sigma') \leq (W, \sigma)$.

We claim that $W' = W$. (This then yields the promised contradiction, completing the proof of this implication.) To this end, first note that we have $W' \subseteq W$, by assumption. So, we need to show that $W' \supseteq W$. We cannot remove the vertex $y_{j,r}$ from $W$, since that would leave the $r$-path $v_j y_{j,1} \cdots y_{j,r}$ uncovered. Thus, we have $y_{j,r} \in W'$. Similarly, for $k \neq i, j$ the vertex $v_k$ cannot be removed, so $v_k \in W'$. If we remove the vertex $v_i$, the $r$-path $v_j v_i v_{i,1} \cdots v_{i,r-1}$ is not covered, so $v_i \in W'$. Since $\sigma(v_i) = b > a$, the vertex $v_i$ does not cover the $r$-path $v_j y_{i,1} \cdots y_{i,r}$. It follows that the vertex $y_{i,r}$ cannot be removed. Thus, we have $y_{i,r} \in W'$, and it follows that $W' = W$, as claimed.

Assuming condition (i), Proposition 3.7 implies that $I_{r,\text{max}}(H_\lambda)$ is Cohen-Macaulay, so Lemma 3.9(c) implies that $I_{r,\text{max}}(G_\omega)$ is as well.

The next result contains Theorem B from the introduction.

**Theorem 3.11.** Assume that $G_\omega$ is a weighted tree. Then the following conditions are equivalent:

(i) $I_{r,\text{max}}(G_\omega)$ is Cohen-Macaulay;

(ii) $I_{r,\text{max}}(G_\omega)$ is $m$- unmixed; and

(iii) there is a weighted tree $\Gamma_\mu$ and an $r$-path suspension $H_\lambda$ of $\Gamma_\mu$ such that $H_\lambda$ is obtained by pruning a sequence of $r$-pathless leaves from $G_\omega$ and for all $v_i v_j \in E(\Gamma_\mu)$ we have $\omega(v_i v_j) \leq \min \{ \omega(v_i y_{i,1}), \omega(v_j y_{j,1}) \}$.

When $G_\omega$ satisfies the above equivalent conditions, the graph $H$ can be constructed by pruning $r$-pathless leaves from $G$ until no more $r$-pathless leaves remain.
Proof: The implications (i) $\implies$ (ii) $\implies$ (iii) are from Proposition 3.10. For the implication (i) $\implies$ (ii), assume that $I_{r, \max}(G_\omega)$ is m-unmixed. Since $G$ is finite, prune a sequence of $r$-pathless leaves from $G_\omega$ to obtain a weighted subgraph $H_\lambda$ that has no $r$-pathless leaves. Lemma 3.8 implies that $I_{r, \max}(H_\lambda)$ is m-unmixed, so Lemma 2.11 implies that $I_r(H)$ is m-unmixed. Thus, $H$ is an $r$-path suspension of a tree $\Gamma$ by Theorem 3.8 and Remark 3.9. Finally, Proposition 3.10 implies that $\omega(v_iv_j) \leq \min\{\omega(v_1y_{i,1}), \omega(v_jy_{j,1})\}$ for all $v_iv_j \in E(\Gamma_\mu)$. \[\square\]

Example 3.12. Consider the weighted graph $G_\omega$ in Example 1.2. Then $I_{r, \max}(G_\omega)$ is Cohen-Macaulay if and only if $r \neq 2, 3$, as follows. Example 3.8 deals with the case $r = 1$.

For $r > 5$, the ideal $I_{r, \max}(G_\omega)$ is trivially Cohen-Macaulay since $G$ has no $r$-paths. (One can also deduce this from Lemma 3.9 since every leaf is $r$-pathless.)

This graph has a single 4-path, so $S/I_{t, \max}(G_\omega)$ is a hypersurface, hence Cohen-Macaulay. One can also deduce this from Theorem 3.11 by pruning the 4-pathless leaf $v_0$ to obtain the weighted 4-path $H_\lambda$ in Example 3.2. Since $H_\lambda$ is a 4-path suspension of the trivial graph $v_1$, the desired conclusion follows from Theorem 3.11.

For $r = 2, 3$, the ideal $I_{r, \max}(G_\omega)$ is not Cohen-Macaulay by Theorem 3.11. To see this, observe that $G$ does not have any $r$-pathless leaves and is not an $r$-suspension for $r = 2, 3$.

Example 3.13. Arguing as in Example 3.12, we have the following for the weighted graphs $G_\omega'$ and $G_\omega''$ of Example 3.5. The ideal $I_{r, \max}(G_\omega')$ is Cohen-Macaulay if and only if $r \geq 6$, and $I_{r, \max}(G_\omega'')$ is Cohen-Macaulay if and only if $r \neq 1, 3, 4, 5, 6, 7$.

4. Cohen-Macaulay Weighted Complete Graphs when $r = 2$

Assumption. Throughout this section, $K_\omega^n$ is a weighted $n$-clique, and $A$ is a field.

In this section, we prove Theorem C from the introduction characterizing Cohen-Macaulayness of $n$-cliques in the context of weighted path ideals for the function $f = \max$ with $r = 2$. We begin with two results about arbitrary $f$ and $r$. Note that the assumption $r < n$ causes no loss of information since, when $r \geq n$, we have $I_{r, f}(G_\omega) = 0$.

Lemma 4.1. If $(W, \sigma)$ is an $f$-weighted $r$-path vertex cover for $K_\omega^n$ where $r < n$, then $|W| \geq n - r$.

Proof. Suppose that $|W| < n - r$ and assume that $v_1, \ldots, v_{r+1} \notin W$. Then the path $v_1 \ldots v_{r+1}$ in $K_\omega^n$ is not covered by $(W, \sigma)$. \[\square\]

Lemma 4.2. Assume that $r < n$, and consider an arbitrary subset $W \subseteq V$ with $|W| = n - r$. Then there is a function $\sigma'' : W \rightarrow \mathbb{N}$ such that $(W, \sigma'')$ is a minimal $f$-weighted $r$-path vertex cover for $K_\omega^n$.

Proof. Using the inclusion-exclusion principal, it is straightforward to show that $W$ is an $r$-path vertex cover of $K_\omega^n$. The trivial weight $\sigma : W \rightarrow \mathbb{N}$ with $\sigma(v) = 1$ for all $v \in W$ makes $(W, \sigma)$ into an $f$-weighted $r$-path vertex cover of $K_\omega^n$. Lemma 1.11 yields a minimal $f$-weighted $r$-path vertex cover $(W'', \sigma'')$ of $G_\omega$ such that $(W'', \sigma'') \subseteq (W, \sigma)$. Lemma 1.11 shows that $|W''| \geq n - r = |W|$. Since $W'' \subseteq W$, we must have $W = W''$, as desired. \[\square\]

For the remainder of this section, we focus on the case $f = \max$. 

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Proposition 4.3. If $r \leq n$, then $\dim(S/I_{r,\max}(K_n^\omega)) = r$.

Proof. If $r = n$, then $I_{r,\max}(K_n^\omega) = 0$ and therefore $\dim(S/I_{r,\max}(K_n^\omega)) = \dim(S) = n = r$, as claimed. Assume for the rest of the proof that $r < n$. Lemma 4.1 implies that for every weighted $r$-path vertex cover $(W, \sigma)$ we have $|W| \geq n - r$. Furthermore, Lemma 4.2 implies that there is a minimal weighted $r$-path vertex cover $(W, \sigma)$ with $|W| = n - r$. Thus, the desired conclusion follows from Theorem 2.7(b). \qed

For the rest of the section, we focus on the case $r = 2$.

The next result characterizes the weighted 3-cliques $K_3^\omega$ such that $I_{2,\max}(K_3^\omega)$ is Cohen-Macaulay. Note that these cliques are key for the characterization of Cohen-Macaulayness of larger $n$-cliques in Theorem C. Also, smaller $n$-cliques are very small trees that always give Cohen-Macaulay ideals; argue as in Example 3.12.

Proposition 4.4. Consider a weighted 3-clique $K_3^\omega$, which we assume by symmetry to be of the following form

```
   u __ a __ v
   |    \    |
   \  b  /  \\
   w
```

with weights $a, b, c$ such that $a \leq b \leq c$. Then the following conditions are equivalent:

(i) The ideal $I_{2,\max}(K_3^\omega)$ is Cohen-Macaulay;
(ii) The ideal $I_{2,\max}(K_3^\omega)$ is unmixed; and
(iii) We have $a = b$, that is, $a = b \leq c$.

Proof. First, we note that

$$I_{2,\max}(K_3^\omega) = (X^a Y^b Z^b, X^c Y^b Z^c, X^c Y^a Z^c)S = (X^a Y^b Z^b, X^c Y^a Z^c)S.$$  

(4.4.1)

The implication (i) $\implies$ (ii) is standard.

(ii) $\implies$ (iii) We argue by contrapositive. Assume that $a < b$. If $a < b = c$, then it is straightforward to show that the weighted 2-path ideal decomposes irredudantly as follows.

$$I_{2,\max}(K_3^\omega) = (X^a Y^b Z^b, X^c Y^a Z^c)S = (X^a)S \cap (Y^b)S \cap (Z^b)S \cap (X^c, Y^b)S \cap (Y^b, Z^c)S.$$  

In particular, this ideal is mixed. When $a < b < c$, the weighted 2-path ideal is also mixed because of the following irredundant decomposition.

$$I_{2,\max}(K_3^\omega) = (X^a Y^b Z^b, X^c Y^a Z^c)S = (X^a)S \cap (Y^b)S \cap (Z^b)S \cap (X^c, Y^b)S \cap (Y^b, Z^c)S.$$  

(iii) $\implies$ (i) If $a = b$, then we have

$$I_{2,\max}(K_3^\omega) = (X^a Y^a Z^a)S.$$  

which is generated by a regular element and is therefore Cohen-Macaulay. \qed

Remark 4.5. The first display in the proof of Proposition 4.4 shows that the generating sequence used to define $I_{r,f}(G_\omega)$ can be redundant, i.e., non-minimal.

Our next result uses the following information about colon ideals.
Remark 4.6. Let $I$ be a monomial ideal in $S$, that is an ideal of $S$ generated by a list $g_1, \ldots, g_t$ of monomials in the variables $X_1, \ldots, X_n$. Given another monomial $h \in S$, it is straightforward to show that the colon ideal $(I : S h)$ is generated by the following list of monomials: $g_1 / \gcd(g_1, h), \ldots, g_t / \gcd(g_t, h)$.

The next result contains one implication of Theorem C from the introduction. Note that the 2-path Cohen-Macaulay weighted 3-cliques are characterized in Proposition 4.4.

**Theorem 4.7.** Let $n \geq 3$. Assume that every induced weighted sub-3-clique $K_\omega$ of $K^n_\omega$ has $I_{2, \text{max}}(K^3_\omega)$ Cohen-Macaulay. Then $I_{2, \text{max}}(K^n_\omega)$ is also Cohen-Macaulay.

**Proof.** Set $I := I_{2, \text{max}}(K^n_\omega)$. Note that our hypothesis on the induced weighted sub-3-cliques of $K^n_\omega$ imply that $I$ is generated by the following set of monomials.

$$\{X_i^{a_{i,j,k}} X_j^{a_{i,j,k}} X_k^{a_{i,j,k}} : i < j < k \text{ and } a_{i,j,k} = \min(\omega(e_i e_j), \omega(e_i e_k), \omega(e_j e_k))\}$$

Indeed, this follows from Lemma 2.13 and the description of $I_{2, \text{max}}(K^3_\omega)$ from equation (4.4.1) in the proof of Proposition 4.4. In particular, the generators of $I$ are determined by the induced weighted sub-3-cliques of $K^n_\omega$.

We proceed by induction on $n$. The base case $n = 3$ is trivial.

For the inductive step, assume that $n \geq 4$ and the following: for every weighted $(n - 1)$-clique $K^{n-1}_\omega$, if every induced weighted sub-3-clique $K^{n-1}_{\mu'}$ of $K^{n-1}_\omega$ has $I_{2, \text{max}}(K^3_{\mu'})$ Cohen-Macaulay, then $I_{2, \text{max}}(K^{n-1}_\omega)$ is also Cohen-Macaulay. Let $R := S/I_{2, \text{max}}(K^n_\omega)$ and $a := \min(\omega(v_i v_j))$ over all $i$ and $j$. Assume by symmetry that $\omega(v_1 v_2) = a$. Let $K^{n-1}_\omega$ denote the weighted sub-clique of $K^n_\omega$ induced by $V \setminus \{v_1\}$. Set $S' = A[X_2, \ldots, X_n]$. Lemma 2.12 implies that $R' := R/(X_1) \cong S'/I_{2, \text{max}}(K^{n-1}_\omega)$. Since $K^{n-1}_\omega$ has the same condition on the induced weighted sub-3-cliques, $R'$ is Cohen-Macaulay by the inductive hypothesis. Note that Proposition 4.4 says that $\dim(R') = 2$. We consider the following short exact sequence.

$$0 \to X^n_1 R \to R \to R/X^n_1 R \to 0 \quad (4.7.1)$$

Since $a$ is the smallest edge weight on $K^n_\omega$, we have $R/X^n_1 R \cong R[T_1]/(T^n_1)$, which is Cohen-Macaulay of dimension 2. As $\dim(R) = 2$, in order to show that $R$ is Cohen-Macaulay, it suffices to show that $\text{depth}(R) \geq 2$. Applying the Depth Lemma to the sequence (4.7.1), we see that it suffices to show that $\text{depth}_S(X^n_1 R) = 2$.

Case 1: Assume that $\omega(v_i v_j) = a$ for all $i = 2, \ldots, n$.

Claim 1: $(I : S X^n_1) = (X^n_1 X^n_2) = (1 < i < j \leq n) S$. For the containment $\supseteq$, let $1 < i < j \leq n$. Our assumptions on $a$ imply that the generator of $I$ corresponding to the sub-clique induced by $v_1, v_i, v_j$ is $X^n_1 X^n_2 X^n_3$. It follows that the element $X^n_1 X^n_3$ is in $(I : S X^n_1)$, as desired. For the reverse containment, note that the generators for $I$ are of the form $X^n_1 X^n_2 X^n_3$ such that $p < q < r$ and $\omega : S$. The corresponding generator of $(I : S X^n_1)$ when $p = 1$ is $X^{n-\alpha} X^n_1 X^n_3 \in (X^n_1 X^n_3)$. When $p \neq 1$ we have $X^{\alpha} X^n_1 X^n_3 \in (X^n_1 X^n_3)$. Therefore the claim holds.

Also, we have

$$X^n_1 R \cong R/\text{Ann}_R(X^n_1) \cong S/(I : S X^n_1) \cong (S'/I_{1, \text{max}}(K^{n-1}_\omega))[X_1]$$

where the graph $K^{n-1}_\omega$ has constant weight $a$ on each edge; this is by Claim 1. The proof of (11) Proposition 5.2 shows that $S'/I_{1, \text{max}}(K^{n-1}_\omega)$ is Cohen-Macaulay of dimension 1. Therefore $X^n_1 R \cong (S'/I_{1, \text{max}}(K^{n-1}_\omega))[X_1]$ is Cohen-Macaulay of dimension 2.
Case 2: Assume that $\omega(v_1v_2) = a < \omega(v_1v_i)$ for some $i > 2$. This assumption implies that there exists a subset $W \subseteq V$ such that $v_1, v_i \in W$ and for each $v_j, v_k \in W$ we have $\omega(v_jv_k) > a$. By the finiteness of the graph $K^n$, there exists a maximal such set $W$. Note that $|W| \geq 2$.

Claim 2: for all $v_p \in V \setminus W$ and all $v_j \in W$, we have $\omega(v_jv_p) = a$. Suppose by way of contradiction that $\omega(v_jv_p) > a$. Let $v_k \in W$ such that $v_k \neq v_j$. By assumption, we have $\omega(v_jv_k) > a$ and $\omega(v_jv_p) > a$. Let $K^n_\omega$ be the weighted sub-3-clique of $K^n$ induced by $v_j, v_k, v_p$. By assumption, the ideal $I_{2,\text{max}}(K^n_\omega)$ is Cohen-Macaulay, so Proposition 4.4 implies that either $\omega(v_kv_p) \geq \omega(v_jv_p) > a$ or $\omega(v_kv_p) \geq \omega(v_jv_k) > a$. Since $v_k$ was chosen arbitrarily, the set $W \cup \{v_p\}$ satisfies the condition for $W$, contradicting the maximality of $W$.

Let $\lambda$ be a new weight on $K^n$ such that

$$\lambda(v_\alpha v_\beta) = \begin{cases} \omega(v_\alpha v_\beta) & \text{if } v_\alpha, v_\beta \in W \\ a & \text{if } v_\alpha \notin W \text{ or } v_\beta \notin W. \end{cases}$$

Observe that this implies for $v_j, v_k \in W$ and $v_p, v_q \notin W$ we have

$$\lambda(v_jv_k) = \omega(v_jv_k)$$
$$\lambda(v_jv_p) = a = \omega(v_jv_p)$$
$$\lambda(v_pv_q)$$ may be different from $\omega(v_pv_q)$.

Hence the graph $K^n_\lambda$ satisfies the induced weighted sub-3-clique assumption. (The four types of induced weighted sub-3-cliques are displayed next, with $v_j, v_k, v_l \in W$ and $v_p, v_q, v_r \notin W$.)

Since $\omega(v_1v_2) = a$, we have $v_2 \notin W$. Thus $\lambda(v_2v_l) = a$ for all $l \neq 2$. Hence the ideal $J := I_{2,\text{max}}(K^n_\lambda)$ is Cohen-Macaulay by Case 1. Note that the condition $\omega(e) \geq \lambda(e)$ for each edge $e$ implies that $I \subseteq J$.

Claim 3: We have the equality $(I :_S X^n_i) = (J :_S X^n_i)$. The containment $\subseteq$ follows from the fact that $I \subseteq J$. For the reverse containment, recall that the generators for the ideals $I$ and $J$ are determined by the induced sub-3-cliques of $K^n$. For the first three sub-3-cliques displayed above, the corresponding generators of $I$ and $J$ are the same. Therefore, the generators in the colon ideals produced by these generators are the same; see Remark 4.4. In the case of the fourth induced sub-3-clique, the associated generator for $J$ is $X^n_pX^n_qX^n_r$. Since $p, q, r \neq 1$, the associated generator for $(J :_S X^n_i)$ is $X^n_pX^n_qX^n_r \in (X^n_pX^n_q)S \subseteq (I :_S X^n_i)$; the last containment is explained as follows. The existence of distinct elements $v_p, v_q, v_r \in$
\( V \setminus W \) provides a sub-3-clique induced by \( v_1, v_p, v_q \), which is of the third type, with corresponding generator for the colon ideals being \( X_p^a X_q^a \). This establishes Claim 3.

Lastly, Case 1 shows that \( \text{depth}_S((X_1^a) S/J) = 2 \). Claim 3 implies that

\[
(X_1^a) S/J \cong S/(J : S X_1^a) \cong (X_1^a) S/I = X_1^a R.
\]

Therefore \( \text{depth}_S(X_1^a R) = 2 \), as desired. \( \square \)

The converse of Theorem 4.7 is more complicated. We break the proof into (hopefully) manageable pieces, culminating in Theorem 4.12.

**Proposition 4.8.** Let \( n \geq 3 \) and assume that \( K_n^\omega \) contains an induced weighted sub-3-clique of the form

\[
v_i \quad \xrightarrow{a} \quad v_j \quad \xleftarrow{c} \quad v_k \quad \xrightarrow{b} \quad v_l
\]

with weights \( a, b, \) and \( c \) such that \( a < b < c \). Then \( I_{2, \text{max}}(K_n^\omega) \) is mixed. In particular, \( I_{2, \text{max}}(K_n^\omega) \) is not Cohen-Macaulay.

**Proof.** By symmetry, assume without loss of generality that \( i = 1, j = 2, \) and \( k = 3 \). By Theorem 2.7(b), it suffices to exhibit two minimal weighted 2-path vertex covers for \( K_n^\omega \) whose cardinalities are not equal. Since \( \dim(S/I_{2, \text{max}}(K_n^\omega)) = 2 \) by Proposition 4.3, we know that \( K_n^\omega \) has a minimal weighted 2-path vertex cover of size \( n - 2 \). Thus, it suffices to find a minimal weighted 2-path vertex cover of size \( n - 1 \).

Consider the weighted set \( \{v_2^b, v_3^c, v_1^a, \ldots, v_n^1\} \). In light of the assumptions on \( a, b, \) and \( c \), it is straightforward to show that this is a weighted 2-path vertex cover for \( K_n^\omega \). We show that it gives rise to a minimal one of the form \( \{v_2^b, v_3^c, v_4^a, \ldots, v_n^1\} \).

Since \( c > b \), the weighted path

\[
v_2^b \quad \xrightarrow{b} \quad v_3^c \quad \xleftarrow{a} \quad v_1^a
\]

is covered only by the weighted vertex \( v_2^b \). If the weight \( b \) on this vertex were increased, then this weighted path would no longer be covered. Thus, the vertex \( v_2 \) cannot be removed from the cover, and its weight cannot be increased. Similarly, the weighted path \( v_3 v_1 v_2 \) shows that the vertex \( v_3 \) cannot be removed from the cover, and its weight cannot be increased. Lastly, for \( j \geq 4 \) the weighted path \( v_2 v_1 v_j \) is only covered by \( v_j^1 \). Thus, the vertex \( v_j \) cannot be removed from the cover; however, its weight can be increased. \( \square \)

**Remark 4.9.** The weighted 2-path vertex cover \( \{v_2^b, v_3^c, v_1^a, \ldots, v_n^1\} \) in the previous proof is not incredibly mysterious. Indeed, the induced weighted sub-3-clique

\[
v_1 \quad \xrightarrow{a} \quad v_2 \quad \xleftarrow{c} \quad v_3 \quad \xrightarrow{b} \quad v_4
\]

has \( \{v_2^b, v_3^c\} \) as a minimal weighted 2-path vertex cover. (This can be checked readily as in the previous proof. Alternately, it follows from the proof of Proposition 4.4; see the discussion in Example 2.10.) The given cover for \( K_n^\omega \) is built from this one.
When $a < b = c$, one might guess that the vertex cover $\{v^a_1, v^b_2, v^1_3, \ldots, v^n_1\}$ can be used to show that $I_{2,\text{max}}(K^a_n)$ is mixed in this case as well. However, the next example shows that this is not the case.

**Example 4.10.** Consider the following weighted 4-clique.

It is straightforward to show that we have the following.

$$I_{2,\text{max}}(K^4_4) = (X_1 X_2 X_3 X_4, X_1 X_2 X_3 X_4, X_1 X_2 X_3 X_4, X_1 X_2 X_3 X_4)$$

The decomposition here shows that $I_{2,\text{max}}(K^4_4)$ is unmixed. However, Theorem 4.12 below shows that it is not Cohen-Macaulay because the weighted sub-3-clique induced by $v_1, v_2, v_3$ is not Cohen-Macaulay; see Proposition 4.4.

**Proposition 4.11.** Assume that $I_{2,\text{max}}(K^a_n)$ is unmixed, and that $K^a_n$ has an induced weighted sub-3-clique $K^b_3$, such that $I_{2,\text{max}}(K^b_3)$ is not Cohen-Macaulay. Then $K^a_n$ has an induced weighted sub-4-clique of the form

such that $a < b$.

**Proof.** Without loss of generality, assume that the non-Cohen-Macaulay induced weighted sub-3-clique is on the vertices $v_1, v_2, v_3$ as follows

with $a < b$. Note that it must have this form by Propositions 4.2 and 4.8 because of our unmixedness assumption. Assume without loss of generality that $b$ is maximal among all weights occurring in a non-Cohen-Macaulay induced sub-3-clique.

It is readily shown that the set $\{v^a_1, v^b_2, v^1_3, \ldots, v^n_1\}$ is a weighted 2-path vertex cover. As in the proof of Proposition 4.8, the path $v_3 v_1 v_2$ shows that the vertex $v^b_1$ cannot be removed from this cover, and its weight cannot be increased. Similarly, the path $v_1 v_2 v_3$ shows that the vertex $v^b_2$ cannot be removed from this cover, and
its weight cannot be increased. Because of our unmixedness assumption, Theorem 2.7(b) and Proposition 4.3 imply that every minimal weighted 2-path vertex cover of $K_n^\omega$ has cardinality $n - 2$. Since the given cover has size $n - 1$, one of the vertices $v_4$ through $v_n$ can be removed to create a weighted 2-path vertex cover. Reorder the vertices if necessary so that $v_4$ is the vertex that can be removed. Lemma 1.11 shows that this gives rise to a minimal weighted 2-path vertex cover of the form $\{v_1^b, v_2^b, v_5^c, \ldots, v_n^r\}$.

Label the induced weighted subgraph with vertices $v_1, v_2, v_3, v_4$ as follows.

Since $a < b$, the path $v_1v_2v_4$ must be covered by $v_2^b$. Thus $b \leq d$. Similarly, the vertex $v_1^a$ must cover the path $v_2v_1v_4$, so $b \leq c$. Thus, we have $a < b \leq c, d$, so the weighted sub-3-clique induced by $v_1, v_2, v_3$ is not Cohen-Macaulay. Proposition 4.8 implies that $c = d$, and the maximality of $b$ implies that $c \leq b$, that is $c = b$. Thus, the above sub-4-clique has the desired form. □

The next result contains the remainder of Theorem C from the introduction.

**Theorem 4.12.** Assume that $K_n^\omega$ contains at least one induced weighted sub-3-clique $K_3^\omega$, such that $I_{2, \text{max}}(K_3^\omega)$ is not Cohen-Macaulay. Then $I_{2, \text{max}}(K_n^\omega)$ is not Cohen-Macaulay.

**Proof.** If $I := I_{2, \text{max}}(K_n^\omega)$ is mixed, then we are done. So, we assume that $I$ is unmixed. Theorem 2.7(b) and Proposition 4.3 imply that every minimal weighted 2-path vertex cover of $K_n^\omega$ has cardinality $n - 2$. Also, Lemma 1.11 shows that every subset of $V$ of cardinality $n - 2$ occurs as a minimal weighted 2-path vertex cover.

Every induced weighted sub-3-clique of $K_n^\omega$ has the form

\[
\begin{align*}
v_1 & \rightarrow v_j \\
& \downarrow b \\
v_k & \rightarrow v_1
\end{align*}
\]

with $a \leq b \leq c$. By assumption, $K_n^\omega$ contains at least one such sub-clique with $a < b \leq c$; see Proposition 4.4. Furthermore, Proposition 4.8 implies that every such sub-clique has $a < b = c$. 
Using Proposition 4.11 and reordering the vertices if necessary, we obtain an induced weighted subgraph of the following form

\[ v_1 \quad a \quad b \quad v_2 \quad b \quad v_3 \quad c \quad v_4 \]

with \( a < b \).

Using Theorem 2.7(b) we have a minimal m-irreducible decomposition

\[ I = \bigcap (X_1^{\alpha_1}, X_2^{\alpha_2}, \ldots, X_{k_n-3}^{\alpha_{k_n-3}})S \]  

(4.12.2)

where the intersection is taken over all minimal weighted 2-path vertex covers \( \{v_1^1, v_2^2, \ldots, v_{k_n-2}^n\} \) of \( K_n^0 \). We set

\[ I_1 := \bigcap (X_1^{\alpha_1}, X_2^{\alpha_2}, \ldots, X_{k_n-3}^{\alpha_{k_n-3}})S \]  

(4.12.3)

where the intersection is taken over all minimal weighted 2-path vertex covers of \( K_n^0 \) that contain the vertex \( v_1 \). Next, set

\[ I_* := \bigcap_{j_i \neq 1} (X_1^{\beta_1}, X_2^{\beta_2}, \ldots, X_{j_n-2}^{\beta_{j_n-2}})S \]  

(4.12.4)

where the intersection is taken over all minimal weighted 2-path vertex covers that do not contain the vertex \( v_1 \). By definition, this yields \( I = I_1 \cap I_* \). Moreover, the first paragraph of this proof implies that each of these intersections is taken over a non-empty index set.

Note that the irredundancy of the intersection in (4.12.2) implies that the two subsequence intersections are also irredundant. It follows that the maximal ideal \( m = (X_1, \ldots, X_n)S \) is not associated to \( I_1 \) and is not associated to \( I_* \). Thus, we have \( 1 \leq \text{depth}(S/I_1) \leq \dim(S/I_1) = 2 \) and \( 1 \leq \text{depth}(S/I_*) \leq \dim(S/I_*) = 2 \). Since we have \( \dim(S/I) = 2 \), it remains to show that \( \text{depth}(S/I) = 1 \).

Consider the short exact sequence

\[ 0 \to S/I \to S/I_1 \oplus S/I_* \to S/(I_1 + I_*) \to 0. \]

By the Depth Lemma (or a routine long-exact-sequence argument), in order to show that \( \text{depth}(S/I) = 1 \), it suffices to show that \( \text{depth}(S/(I_1 + I_*)) = 0 \), that is, that \( m \) is associated to \( I_1 + I_* \).

From the decompositions (4.12.3) and (4.12.4), we have

\[ I_1 + I_* = \bigcap_{j_i \neq 1} \left[ (X_1^{\alpha_1}, X_2^{\alpha_2}, \ldots, X_{k_n-3}^{\alpha_{k_n-3}})S + (X_1^{\beta_1}, X_2^{\beta_2}, \ldots, X_{j_n-2}^{\beta_{j_n-2}})S \right] \]  

(4.12.5)

where the first intersection is taken over all minimal weighted 2-path vertex covers that contain the vertex \( v_1 \), and the second intersection is taken over all minimal weighted 2-path vertex covers that do not contain the vertex \( v_1 \); see, e.g., [7, Lemma 2.7]. Note that this is an m-irreducible decomposition, though it may be redundant. We need to show that there is an ideal in this intersection of the form \((X_1^{\delta_1}, X_2^{\delta_2}, \ldots, X_n^{\delta_n})S\) that is irredundant in the intersection.
Given the sub-clique $\{1, 12, 1\}$, it is straightforward to show that there are minimal weighted 2-path vertex covers of $K_n^\omega$ of the form \{$v_1^b, v_2^b, v_5^{\alpha_5}, \ldots, v_n^{\alpha_n}$\} and \{$v_3^b, v_4^b, v_5^{\beta_5}, \ldots, v_n^{\beta_n}$\}. In particular, the ideal $P_1 := (X_1^b, X_2^b, X_5^{\alpha_5}, \ldots, X_n^{\alpha_n})S$ occurs in the decomposition (4.12.3), and the ideal $P_* := (X_1^b, X_4^b, X_5^{\beta_5}, \ldots, X_n^{\beta_n})S$ occurs in the decomposition (4.12.4). Thus, the ideal

$$P_1 + P_* = (X_1^b, X_2^b, X_3^b, X_4^b, X_5^{\gamma_5}, X_6^{\gamma_6}, \ldots, X_n^{\gamma_n})S$$

is in the intersection (4.12.5), where $\gamma_i = \min\{\alpha_i, \beta_i\}$.

Let $Q_1$ be an ideal occurring in the intersection (4.12.3), and let $Q_*$ be an ideal occurring in the intersection (4.12.4). Suppose that

$$(X_{i_1}^{\xi_1}, X_{i_2}^{\xi_2}, \ldots, X_{i_g}^{\xi_g})S = Q_1 + Q_* \subseteq P_1 + P_*$$

with $g \leq n - 1$. (4.12.6)

Claim 1: we have $Q_* = (X_{\eta_3}^{\eta_3}, X_{\eta_4}^{\eta_4}, X_{\eta_5}^{\eta_5}, X_{\eta_6}^{\eta_6}, \ldots, X_n^{\eta_n})S$ for some $\eta_3, \ldots, \eta_n$. By assumption, we have $Q_* = (X_{j_1}^{\eta_1}, X_{j_2}^{\eta_2}, \ldots, X_{j_{n-2}}^{\eta_{n-2}})S$ with $j_i > 1$ for $i = 1, \ldots, n - 2$. It suffices to show that $j_i \neq 2$ for all $i$. Suppose that $j_i = 2$ for some $i$. Given the conditions on the generators of $Q_*$, there must be an index $k \neq 1$ such that $j_i \neq k$ for all $i$. Then $v_2^* \mu$ must cover the path $v_1 v_2 v_k$. This implies that $\eta_2 \leq \alpha$. On the other hand, since

$$X_{j_i}^{\eta_i} \in Q_1 + Q_* \subseteq P_1 + P_* = (X_1^b, X_2^b, X_3^b, X_4^b, X_5^{\gamma_5}, X_6^{\gamma_6}, \ldots, X_n^{\gamma_n})S$$

we have $\eta_2 \geq b > a \geq \eta_2$, a contradiction. This establishes Claim 1.

Claim 2: we have $Q_1 = (X_{\mu_1}^{\mu_1}, X_{m_{m-3}}^{\mu_{m-3}}, \ldots, X_{m_{m-3}}^{\mu_{m-3}})S$ for some $\mu_1, \mu_{m-3}, \ldots, \mu_{m_{m-3}}$ with $m_i > 2$ for all $i$. By assumption, we have $Q_1 = (X_1^{\mu_1}, X_{m_1}^{\mu_1}, \ldots, X_{m_{m-3}}^{\mu_{m_{m-3}}})S$ with $m_i \geq 2$. From the equality in (4.12.6), we have

$$\{t_1, \ldots, t_g \} = \{3, \ldots, n\} \cup \{1, m_1, \ldots, m_{m-3}\}.$$  

Since $g \leq n - 1$, the inclusion-exclusion principle implies that

$$\left|\{3, \ldots, n\} \cap \{1, m_1, \ldots, m_{m-3}\}\right| \geq n - 3.$$  

Since $1 \notin \{3, \ldots, n\}$ it follows that $m_1, \ldots, m_{m-3} \in \{3, \ldots, n\}$, that is, that $m_i > 2$ for all $i$. This establishes Claim 2.

Claim 2 says that $X_2$ does not appear to any power in the list of generators of $Q_1$. Given the form and number of the generators of $Q_1$, it follows that there is another variable, say $X_p$ with $p \geq 3$, that has no power occurring in this list. By assumption, the set $\{v_1^{\mu_1}, v_{m_1}^{\mu_1}, \ldots, v_{m_{m-3}}^{\mu_{m-3}}\}$ is a minimal weighted 2-path vertex cover of $K_n^\omega$. It follows that the path $v_1 v_2 v_p$ is covered by $v_1^{\mu_1}$, which implies that $\mu_1 \leq a$. However, we have $X_1^{\mu_1} \in Q_1 + Q_* \subseteq P_1 + P_*$: as in the proof of Claim 1, this implies that $\mu_1 \geq b > a \geq \mu_1$, contradiction. We conclude that the supposition (4.12.6) is impossible.

From this, we deduce that the only way one can have $Q_1 + Q_* \subseteq P_1 + P_*$ is with

$$Q_1 + Q_* = (X_1^{\delta_1}, X_2^{\delta_2}, \ldots, X_n^{\delta_n})S$$

for some $\delta_i$. It follows that at least one ideal of this form is irredundant in the intersection (4.12.5), as desired. \qed

We end with a question motivated by the results of this section.
Questions 4.13. Is there a similar characterization of the Cohen-Macaulayness of $I_{r,\max}(K^n_\omega)$ when $r \geq 3$? For instance, must the ideal $I_{r,\max}(K^n_\omega)$ be Cohen-Macaulay if and only if every induced weighted sub-$(r + 1)$-clique $K^{r+1}_\omega$ of $K^n_\omega$ has $I_{r,\max}(K^{r+1}_\omega)$ Cohen-Macaulay?

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