CONTACT STRUCTURE ON 2-STEP NILPOTENT LIE GROUPS

BABAK HASANZADEYE SEYEDI

Abstract. In this paper we study contact structure on 2-step nilpotent Lie groups. We consider properties of normal subgroups and center of Lie groups while cosymplectic and Sasakian structure defined on Lie group.

1. Introduction

Let $\tilde{M}$ be an odd dimensional Riemannian manifold with a Riemannian metric $g$ and Riemannian connection $\tilde{\nabla}$. Denote by $TM$ the Lie algebra of vector fields on $\tilde{M}$. Then $\tilde{M}$ is said to be an almost contact metric manifold if there exist on $\tilde{M}$ a tensor $\phi$ of type $(1,1)$, a vector field $\xi$ called structure vector field and $\eta$, the dual 1-form of $\xi$ satisfying the following

\begin{equation}
\phi^2X = -X + \eta(X)\xi, \quad g(X, \xi) = \eta(X)
\end{equation}

\begin{equation}
\eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0
\end{equation}

\begin{equation}
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)
\end{equation}

For any $X, Y \in T\tilde{M}$. In this case

\begin{equation}
g(\phi X, Y) = -g(X, \phi Y)
\end{equation}

Now, let $M$ be a submanifold immersed in $\tilde{M}$. A normal almost contact manifold is called a cosymplectic manifold if

\begin{equation}
(\tilde{\nabla}_X \phi)(Y) = 0, \quad \tilde{\nabla}_X \xi = 0
\end{equation}

Theorem 1.1 An almost contact metric structure $(\phi, \xi, \eta, g)$ is Sasakian if and only if

\begin{equation}
(\nabla \phi)Y = g(X, Y)\xi - \eta(Y)X
\end{equation}

The Riemannian metric induced on $M$ is denoted by the same symbol $g$. Let $TM$ and $T^\perp M$ be the Lie algebras of vector fields tangential and normal to $M$ respectively, and $\nabla$ be the induced Levi-Civita connection on $M$, then the Gauss and Weingarten formulas are given by

\begin{equation}
\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)
\end{equation}

\begin{equation}
\tilde{\nabla}_X V = -A_X V + \nabla^\perp_X V
\end{equation}

1991 Mathematics Subject Classification. Primary 53B40, Secondary 53C60.

Key words and phrases. contact structure, Lie group, nonsingular, skew adjoint.
for any $X, Y \in TM$ and $V \in T^\perp M$. Where $\nabla^\perp$ is the connection on the normal bundle $T^\perp M$, $h$ is the second fundamental form and $A_V$ is the Weingarten map associated with $V$ as

$$g(A_V X, Y) = g(h(X, Y), V)$$

(1.9)

For any $x \in M$ and $X \in T_x M$, we write

$$\phi X = \Psi X + \Gamma X$$

(1.10)

where $\Psi X \in T_x M$ and $\Gamma X \in T^\perp_x M$. Similarly, for $V \in T^\perp_x M$, we have

$$\phi V = \psi V + \gamma V$$

(1.11)

where $\psi V$ (resp. $\gamma V$) is the tangential component (resp. normal component) of $\phi V$. From (1.4) and (1.9), it is easy to observe that for each $x \in M$, and $X, Y \in T_x M$

$$g(\Psi X, Y) = -g(X, \Psi Y)$$

(1.12)

and therefore $g(\Psi^2 X, Y) = g(X, \Psi^2 Y)$ which implies that the endomorphism $\Psi^2 = Q$ is self adjoint. Moreover, it can be seen that the eigenvalues of $Q$ belong to $[-1, 0]$ and that each non-vanishing eigenvalue of $Q$ has even multiplicity. We define $\nabla \Psi, \nabla Q$ and $\nabla N$ by

$$((\nabla^\perp \Psi) Y = \nabla_X \Psi Y - \Psi \nabla_X Y$$

(1.13)

$$((\nabla_X Q) Y = \nabla_X Q Y - Q \nabla_X Y$$

(1.14)

$$((\nabla_X N) Y = \nabla^\perp_X N Y - N \nabla_X Y$$

(1.15)

for any $X, Y \in TM$.

2. PRELIMINARIES

**Definition 1.1.** A Lie group $G$ is a smooth manifold with group structure such that

$$((i) G \times G \mapsto G (ii) G \mapsto G$$

$$(x, y) \mapsto xy \quad x \mapsto x^{-1}$$

(2.1)

Are smooth.  

**Definition 1.2.** Let $G$ is a Lie group and $a \in G$, thus

$$L_a : G \mapsto G$$

$$x \mapsto ax$$

(2.2)

is called left translation with $a$. Also map

$$R_a : G \mapsto G$$

$$x \mapsto xa$$

(2.3)

Is called right translation with $a$. Let $G$ is a Lie group. Vector field $X$ on $G$ is left invariant if

$$L_a(X) = X \quad \forall a \in G$$

(2.4)

And that is right invariant if

$$R_a(X) = X \quad \forall a \in G$$

(2.5)
Difinition 1.3. A Lie group $H$ of a Lie group $G$ is a subgroup which is also a submanifold.

Difinition 1.4. Here $F = \mathbb{R}$ or $\mathbb{C}$. A Lie algebra over $F$ is pair $(\mathfrak{g}, [\cdot, \cdot])$, where $\mathfrak{g}$ is a vector space over $F$ and

$$
[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}
$$

is an $F$-bilinear map satisfying the following properties

$$
[X, Y] = -[Y, X]
$$

(2.6)

is the Jacobi identity. In this paper for any $X, Y \in \mathfrak{g}$ we have

$$
[X, Y] = \nabla_X Y - \nabla_Y X
$$

(2.9)

A Lie subalgebra of a Lie algebra is a vector space that is closed under bracket.

Theorem 1.1. Let $G$ is a Lie group and $\mathfrak{g}$ is a set of left invariant vector field on $G$. We have

(i) $\mathfrak{g}$ is a vector space and map

$$
E : \mathfrak{g} \to T_e G
$$

$$
X \mapsto X_e
$$

is a linear isomorphism and therefore $\dim \mathfrak{g} = \dim T_e G = \dim G$. (e is identity element)

(ii) Left invariant vectro fields necessity are differentiable.

(iii) $(\mathfrak{g}, [\cdot, \cdot])$ is Lie algebra. [6]

Difinition 5.1. Lie algebra made of left invariant vector field on Lie group $G$ is called Lie algebras $\mathfrak{g}$ of G. This Lie algebra is isomorphism with $T_e G$, and we have

$$
[X_e, Y_e] = [X, Y]_e
$$

(2.10)

X and Y are unique left invariant vector field.

Theorem 2.1. Let $G$ be a Lie group. [6]

(a) If $H$ is a Lie subgroup of $G$, then $\mathfrak{h} \simeq T_e H \subset T_e G \simeq \mathfrak{g}$ $g$ is a Lie subalgebra.

(b) If $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra, there exists a unique connected Lie subgroup $H \subset G$ with Lie algebra $\mathfrak{h}$.

For each nonzero vector field $X \in \mathfrak{g}$, the angle $\theta(X)$; $0 \leq \theta(X) \leq \frac{\pi}{2}$; between $\phi X$ and $\mathfrak{g}$ is called the Wirtinger angle of $X$. If the Wirtinger angle $\theta$ is a constant its called slant angle of $\mathfrak{g}$. Let $H \subset G$ is a Lie subgroup and $\mathfrak{h}$ is their Lie algebra, thus $\mathfrak{h}$ is Lie subalgebra of $\mathfrak{g}$ and if $\mathfrak{h}$ is a slant Lie subalgebra, $H$ is called slant Lie subgroup.

If $(G,\mathfrak{g})$ be a Lie group equipted by Riemannian metric, ad is skew adjoint if for $X, Y, Z \in \mathfrak{g}$, if

$$
g(adX(Y), Z) = -g(Y, adX(Z))
$$

If $X \in Z(\mathfrak{g})$ and $Y, z \in \mathfrak{g}$ we have

$$
X\langle Y, Z \rangle = \langle \nabla_{Y+Z} X, Y \rangle
$$

Theorem 2.3. Let $\mathfrak{g}$ be a left invariant metric on a connected Lie group $G$. This metric will also be right invariant if and only if $ad(X)$ is skew-adjoint for every
\[ X \in \mathfrak{g}. \] 

**Definition 3.1.** A nilpotent Lie group is a Lie group \( G \) which is connected and whose Lie algebras is nilpotent Lie algebra \( \mathfrak{g} \), that is, its Lie algebra have a sequence of ideals of \( \mathfrak{g} \) by \[ \mathfrak{g}^0 = \mathfrak{g}, \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1], \ldots, \mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}]. \] \( \mathfrak{g} \) is called nilpotent if \( \mathfrak{g}^n = 0 \) for some \( n \).

**Proposition 3.1.** \[\text{Let } \mathfrak{g} \text{ be a Lie algebra.} \]

(i) If \( \mathfrak{g} \) is a nilpotent, then so are all subalgebras and homomorphic images of \( \mathfrak{g} \).

(ii) If \( \mathfrak{g} \) is nilpotent, then so is \( \mathfrak{g} \).

(iii) If \( \mathfrak{g} \) is nilpotent and nonzero, then \( \mathfrak{Z}(\mathfrak{g}) \neq 0 \).

**Definition 3.2.** A finite dimensional Lie algebra \( \mathfrak{g} \) is 2-step nilpotent if \( \mathfrak{g} \) is not abelian and \( [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0 \). A Lie group \( G \) is 2-step nilpotent if its Lie algebra \( \mathfrak{g} \) is 2-step nilpotent. In the other word A Lie algebra \( \mathfrak{g} \) is 2-step nilpotent if \( [\mathfrak{g}, \mathfrak{g}] \) is non zero and Lies in the center of \( \mathfrak{g} \).

We may identify an element of \( \mathfrak{g} \) with a left invariant vector field on \( G \) since \( T_eG \) may be identified with \( \mathfrak{g} \). If \( X, Y \) are left invariant vector field on \( G \), then \( \nabla_X Y \) is left invariant also. for \( X, Y \in Z^\perp(\mathfrak{g}) \) we have following formula: \[ \nabla_X Y = \frac{1}{2}[X, Y] \]

**Definition 3.3.** A 2-step nilpotent Lie algebra \( \mathfrak{g} \) is nonsingular if \( \text{ad}X : \mathfrak{g} \to \mathfrak{Z}(\mathfrak{g}) \) is surjective for each \( X \in Z^\perp(\mathfrak{g}) \). A 2-step nilpotent Lie group \( G \) is nonsingular if its Lie algebra \( \mathfrak{g} \) is nonsingular.

Let \( G \) denote a simply connected, 2-step nilpotent Lie group with a left invariant metric \( \langle \cdot, \cdot \rangle \) and let \( \mathfrak{g} \) denote the Lie algebra of \( G \). Write \( \mathfrak{g} = \mathfrak{Z}(\mathfrak{g}) \oplus Z^\perp(\mathfrak{g}) \) where \( Z^\perp(\mathfrak{g}) \) its orthogonal complement of center \( \mathfrak{Z}(\mathfrak{g}) \).

### 3. Cosymplectic and Sasakian

In this section we study cosymplectic structure on 2-step nilpotent Lie groups. Let \( G \) is a 2-step nilpotent Lie group with cosymplectic structure, from (1.5) for any \( X, Y \in \mathfrak{g} \) we have

\[
[\phi X, Y] = [X, \phi Y] = \phi[X, Y]
\]

and

\[
[X, \xi] = 0
\]

Let \( G \) be a 2-step nilpotent nonsingular cosymplectic Lie group. if ad is skew adjoint for any \( X, Y \in \mathfrak{g} \) we have

\[
\eta([X, Y]) = g([X, Y], \xi) = -g([X, \xi], Y)
\]

From() we conclude \( \eta(\mathfrak{Z}(\mathfrak{g})) = 0 \), thus \( \xi \) is normal to \( \mathfrak{Z}(\mathfrak{g}) \) and \( \xi \in Z^\perp(\mathfrak{g}) \), therefore \( \mathfrak{Z}(\mathfrak{g}) \) is integral Lie subgroup and

\[
\phi^2([X, Y]) = [Y, X]
\]
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Young Researchers and Elit Club Islamic Azad University Tabriz Branch, Iran
E-mail address: babakmath777@gmail.com