The principle of least action in the space of Kähler potentials

László Lempert ∗
Department of Mathematics
Purdue University
West Lafayette, IN 47907-2067, USA

Abstract

Given a compact Kähler manifold, the space \( \mathcal{H} \) of its (relative) Kähler potentials is an infinite dimensional Fréchet manifold, on which Mabuchi and Semmes have introduced a natural connection \( \nabla \). We study certain Lagrangians on \( T\mathcal{H} \), in particular Finsler metrics, that are parallel with respect to the connection. We show that geodesics of \( \nabla \) are paths of least action, and prove a certain convexity property of the least action. This generalizes earlier results of Calabi, Chen, and Darvas.

1 Introduction

Let \((X, \omega)\) be an \( n \) dimensional, connected, compact Kähler manifold and

\[
\mathcal{H} = \mathcal{H}_\omega = \{ u \in C^\infty(X) : \omega + dd^c u = \omega_u > 0 \}
\]

its space of relative Kähler potentials. Here \( C^\infty(X) \) refers to the Fréchet space of real valued smooth functions on \( X \), and \( dd^c = i(\overline{\partial} - \partial) / 2 \), so that \( dd^c = i\partial\overline{\partial} \). The space \( \mathcal{H} \), as an open subset of a Fréchet space, inherits a Fréchet manifold structure, whose tangent bundle has a canonical trivialization \( T\mathcal{H} \approx \mathcal{H} \times C^\infty(X) \). Mabuchi and Semmes [M, S] independently and with different motivations have introduced a torsion free connection \( \nabla \) on \( T\mathcal{H} \). Mabuchi, as a tool to study special Kähler metrics, defined a Riemannian metric on \( \mathcal{H} \) and obtained \( \nabla \) as the Levi–Civita connection of the metric. Somewhat later Semmes found the connection in search for a geometric interpretation of interpolation of Banach spaces and of a certain homogeneous complex Monge–Ampère equation associated with interpolation. He also determined all Riemannian metrics compatible with the connection: they are linear combinations of Mabuchi’s metric and the square of a linear form.

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One way to explain $\nabla$ is through its parallel transport. We will use dot $\dot{}$ to denote derivative of a function of one real variable, and $\text{grad}_v$ to refer to gradient of a function $X \to \mathbb{R}$ with respect to the Kähler metric of $\omega_v$. Let $u : [a, b] \to \mathcal{H}$ be a smooth path. By integrating the time dependent vector field $(-1/2) \text{grad}_u(t) \dot{u}(t)$ on $X$ we obtain a smooth family of diffeomorphisms $\varphi(t) : X \to X$. In fact $\varphi(t) : (X, \omega_{u(0)}) \to (X, \omega_{u(t)})$ is symplectomorphic. The parallel translate of $\xi \in T_{u(t)}\mathcal{H} \approx C^\infty(X)$ to $u(0)$ along the path $u$ is then

$$\xi \circ \varphi(t) \in C^\infty(X) \approx T_{u(0)}\mathcal{H}. \label{1.1}$$

Understanding the geodesics of this connection was already marked in $[M, S]$ as an interesting and potentially important problem, and Donaldson’s subsequent work $[Do]$ gave further impetus to study them. By now the boundary value problem for geodesics is well understood. On the one hand Darvas, Hu, Vivas, and myself $[D1, DL, Hu, LV]$ proved that points in $\mathcal{H}$ cannot always be connected by a geodesic, not even if they are close to each other. On the other hand work by Berman–Demailly, Berndtsson, Blocki, Chen, Chen–Feldman–Hu, Chu–Tosatti–Weinkove, Darvas, and He $[BD, Be1, Bl1, Bl2, C, CFH, CTW, D2, D3, He]$ gave that the geodesic equation extends to various enlargements of $\mathcal{H}$, and in these enlargements any pair of points, or at least nearby points, can be connected by solutions of the extended geodesic equation, weak geodesics. It follows from Chen’s work that in those enlargements to which Mabuchi’s metric extends, weak geodesics minimize length. In $[D2]$ Darvas generalized Mabuchi’s metric to certain Orlicz type Finsler metrics on $\mathcal{H}$, determined the metric completions of $\mathcal{H}$ under these metrics, and again found that weak geodesics in these completions minimize length. In a slight overstatement the length minimizing paths are independent of which of Darvas’s metric we use to compute length. This was surprising at first sight.

But in fact in geometry one encounters other similar phenomena. In a normed vector space straight line segments minimize length no matter what norm is chosen. There is also the analogy between $\mathcal{H}$ and the space $\mathcal{Q}$ of positive definite quadratic forms on $\mathbb{R}^k$. $\mathcal{Q}$ has a natural torsion free connection that turns it in a symmetric space $\approx \text{GL}^+_k(\mathbb{R})/\text{SO}_k(\mathbb{R})$; and for all parallel Finsler metrics—i.e. those that are invariant under $\text{GL}^+_k(\mathbb{R})$—the shortest paths are the same: subarcs of left translates of certain one parameter subgroups in $\text{GL}^+_k(\mathbb{R})$, projected to $\text{GL}^+_k(\mathbb{R})/\text{SO}_k(\mathbb{R})$.

Now $\mathcal{H}$ with Mabuchi’s connection is itself a symmetric space $[Do, M, S]$, at least according some definitions of a symmetric space (while it is not according to some others, $[L3]$). Although there is no group acting transitively on $\mathcal{H}$, the holonomy groupoid $\Gamma$ of $(\mathcal{H}, \nabla)$ acts on $T\mathcal{H}$. Thus $\Gamma = \bigcup_{u,v \in \mathcal{H}} \Gamma_{uv}$, where $\Gamma_{uv}$ consists of linear isomorphisms $T_u\mathcal{H} \to T_v\mathcal{H}$ that arise as parallel transport along piecewise smooth paths from $u$ to $v$. Concatenation of parallel transports defines an operation $\Gamma_{uw} \times \Gamma_{vw} \to \Gamma_{uw}$ that turns $\Gamma$ in a grupoid. That a Finsler metric or a function $L : T\mathcal{H} \to \mathbb{R}$ is parallel means it is invariant under $\Gamma$.

*This follows from $[L2]$. Even though Theorem 1.2 there is formulated for isometries of Mabuchi’s metric, the proof, verbatim, gives that if $\omega_u$ is analytic while $\omega_v$ is not, then no diffeomorphism of $\mathcal{H}$ can preserve $\nabla$ and map $u$ to $v$. 
Darvas’s metrics are parallel. They are defined in terms of integrals

\[ \int_X \chi(\xi)\omega_u^\alpha, \quad u \in \mathcal{H}, \quad \xi \in T_u\mathcal{H} \approx C^\infty(X), \]

with a fixed “Young weight” \( \chi : \mathbb{R} \to [0, \infty] \), and are invariant under parallel transport simply because in the formula (1.1) for parallel transport, \( \varphi(t) \) satisfies \( \varphi(t)^*\omega_{u(t)} = \omega_{u(0)} \). But there are many parallel Finsler metrics on \( T\mathcal{H} \) beyond those considered in [D2]. The simplest is, for given \( 0 < \alpha < 1 \),

\[ p(\xi) = \sup \left\{ \int_E |\xi|\omega_u^\alpha / \left( \int_E \omega_u^\alpha \right)^\alpha : E \subset X \text{ is measurable} \right\}, \]

\( \xi \in T_u\mathcal{H} \). This is known as weak \( L^q \)-norm or \( L^q, \infty \) Lorentz norm, \( q = 1/\alpha \).

Our thesis is that the proper generality of Darvas’s results on his metrics is parallel, or holonomy invariant, Finsler metrics and more generally, fiberwise convex functions \( T\mathcal{H} \to \mathbb{R} \), “Lagrangians”. In this paper and in a sequel we will show that many of his results generalize to this framework. Most of the time we will consider Lagrangians on \( T\mathcal{H} \) that extend to the space of bounded \( \omega \)-plurisubharmonic functions. We denote by \( B(X) \) the Banach space of bounded Borel functions \( \xi : X \to \mathbb{R} \) with the norm \( ||\xi|| = \sup |\xi| \); the Lagrangians of interest extend to \( (B(X) \cap \text{PSH}(\omega)) \times B(X) \). (The more common space \( L^\infty(X) \) is a quotient of \( B(X) \), but we have little use for it in this paper.) A generalization of holonomy invariance can be defined for such functions.

Our results pertain to invariant Lagrangians that are convex in the \( B(X) \) variable and have a certain continuity property, that we call strong continuity.

**Theorem 1.1** (=Theorem 8.1, Principle of least action). If \( v : [0, T] \to B(X) \cap \text{PSH}(\omega) \) is a weak geodesic, and \( C^1 \) as a map into the Banach space \( B(X) \), then it minimizes action

\[ \int_0^T L(\dot{u}(t))dt \]

among all piecewise \( C^1 \) paths \( u : [0, T] \to B(X) \cap \text{PSH}(\omega) \) with \( u(0) = v(0), u(T) = v(T) \).

We do not know whether weak geodesy of \( v \) already implies it is \( C^1 \), but from Chen’s work [C] we do know that a weak geodesic with endpoints in \( \mathcal{H} \) is \( C^1 \). The theorem can be proved for weak geodesics rather less regular than \( C^1 \), but we still do not know if all weak geodesics have this relaxed regularity.

The second result is about how the least action varies as one moves along weak geodesics; it is a manifestation of seminegative curvature. Fix \( T > 0 \). If \( w, w' \in B(X) \cap \text{PSH}(\omega) \), the least action \( L_T(w, w') \) between them is the infimum of the actions \( \int_0^T L(\dot{u}(t))dt \) over all piecewise \( C^1 \) paths \( u : [0, T] \to B(X) \cap \text{PSH}(\omega) \) connecting \( w \) with \( w' \). It is not obvious, but by Lemma 9.3, \( L_T(w, w') \) is finite.

**Theorem 1.2** (=Theorem 9.1). If \( u, v : [a, b] \to B(X) \cap \text{PSH}(\omega) \) are weak geodesics, then the function \( L_T(u, v) : [a, b] \to \mathbb{R} \) is convex.
The tools of this paper are Chen’s work on $\varepsilon$-geodesics, rudiments of Guedj–Zeriahi’s pluripotential theory, and our results on invariant convex functions on $C^\infty(X)$ \cite{C, GZ1, GZ2, L4}. Even if the details are different, overall we will be able to follow the strategy of Calabi, Chen, and Darvas \cite{C, CC, D2, D3} here, and also in a sequel devoted to least action in spaces of $\omega$-plurisubharmonic functions larger than $B(X) \cap \text{PSH}(\omega)$. Once basic properties of our Lagrangians are established, the greater generality occasionally results in less computation in the proofs for the following reason. Say, for a holonomy invariant Finsler metric $p : TH \to [0, \infty)$, there is a family $\mathcal{F} \subset TH \approx H \times C^\infty(X)$ such that

\begin{equation}
 p(\xi) = \sup \left\{ \int_X f \omega^n_u : f \in \mathcal{F} \cap TuH \right\}, \quad \xi \in TuH,
\end{equation}

and the integrals in (1.3), linear in $\xi$, can be easier to manipulate than the nonlinear integrals in (1.2).

It appears that the greatest generality in which action can be defined by an integral is the space of bounded $\omega$–plurisubharmonic functions. Nonetheless, action can be defined for any path in $\text{PSH}(\omega)$ as a limit of integrals. Whether this action is finite or $\pm \infty$ of course depends on the path and on the Lagrangian. We plan to address this and related questions in a sequel to this paper.

Lagrangians even beyond Finsler metrics are not new to the subject. Chen’s $\varepsilon$-geodesics are trajectories of a Lagrangian $L : TH \to \mathbb{R}$ (albeit not holonomy invariant), with kinetic energy term the square of Mabuchi’s metric and potential energy a multiple of $V(u) = -\int_X u \omega^n$. Functions on $H$ that its geometry motivates, and that are used in existence problems in Kähler geometry, are also not new. Aubin’s functional $I : H \to \mathbb{R}$ \cite[p.146]{Au},

\[ I(u) = \int_X u(\omega^n - \omega^n_u) \]

is a constant multiple of the total geodesic curvature of the line segment $[0, 1] \ni t \mapsto tu \in H$, measured in Darvas’s $L^1$ Finsler metric. Monge–Ampère energy also arises from the geometry of $H$. It is a convex function on $H$, for example in the sense that its restrictions to geodesics of $\nabla$ are convex; but its negative is also convex and, up to scaling and adding a constant, it is the only continuous function that has this property.

We hope that a geometrical approach to functions on $H$ and on related spaces, in the spirit of this paper, will be of use in analytical problems on Kähler manifolds.

Contents. Section 2 is about basic properties of holonomy invariant convex Lagrangians $TH \to \mathbb{R}$. Section 3 is about a subclass of Lagrangians that have an extra continuity property, which makes it possible to extend them to a larger vector bundle. Most of the results in these sections are direct consequences of results in \cite{L4}. Section 4 reviews the notion of weak and $\varepsilon$–geodesics, and $\varepsilon$–Jacobi fields. Section 5 introduces the action and formulates Theorems 1.1 and 1.2 in precise forms. It also gives a road map to their proofs, which occupy the rest of the paper, sections 6–9.

In this paper we freely use basic notions of infinite dimensional analysis and geometry. There are many sources the reader can consult on these matters, one of them \cite{L1}, written with an eye on the space $H$ of Kähler potentials.

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2 Lagrangians

The central objects of this paper are continuous functions $L : T\mathcal{H} \to \mathbb{R}$ that are convex on each tangent space $T_u\mathcal{H}$ and have a certain invariance property; as well as the associated action functional $\mathcal{L}(u) = \int_u^b L(\dot{u}(t))dt = \int_u^b L \circ \dot{u}$ for brevity. In this section and in the next we record basic facts about such functions which follow more or less directly from [L4], that dealt with the action on $C^\infty(X)$ of Hamiltonian diffeomorphisms of $(X, \omega)$ and with invariant convex functions on $C^\infty(X)$. As explained in the Introduction, for $L$ the invariance property in question is invariance under the holonomy groupoid $\Gamma$ of $(\mathcal{H}, \nabla)$. Thus, if $\xi_1 \in T_{u(1)}\mathcal{H}$ is the parallel translate of $\xi_0 \in T_{u(0)}\mathcal{H}$ along a piecewise smooth path $u : [0, 1] \to \mathcal{H}$, then $L(\xi_0) = L(\xi_1)$. This property in fact implies a much stronger and more primitive notion of invariance.

Definition 2.1. Given two measure spaces $(X, \mu)$ and $(Y, \nu)$, we say that measurable functions $\xi : X \to \mathbb{R}$ and $\eta : Y \to \mathbb{R}$ are equidistributed, or are strict rearrangements of each other, if $\mu(\xi^{-1}B) = \nu(\eta^{-1}B)$ for every Borel set $B \subset \mathbb{R}$.

In finite measure spaces this is equivalent to requiring $\mu(\xi > t) = \nu(\eta > t)$ for all $t \in \mathbb{R}$.

Back to our Kähler manifold $(X, \omega)$, if $u \in \mathcal{H}$ we let $\mu_u$ denote the measure induced by $\omega_u^n$. Given measurable $\xi, \eta : X \to \mathbb{R}$ we will write

\begin{equation}
(\xi, u) \sim (\eta, v) \text{ if } \xi, \eta \text{ are equidistributed as functions on } (X, \mu_u), (Y, \mu_v).
\end{equation}

When smooth $\xi, \eta$ are viewed as tangent vectors in $T_u\mathcal{H}$, $T_v\mathcal{H}$, we will just write $\xi \sim \eta$.

Theorem 2.2. A function $L : T\mathcal{H} \to \mathbb{R}$, continuous and convex on each fiber $T_u\mathcal{H}$, is invariant under the holonomy groupoid $\Gamma$ if and only if it is invariant under strict rearrangements: $L(\xi) = L(\eta)$ when $\xi \sim \eta$.

For the proof we need to understand the holonomy groups $\Gamma_{uv}$. (1.1) shows that in general, elements of $\Gamma_{uv}$, isomorphisms $T_u\mathcal{H} \to T_v\mathcal{H}$, are pullbacks by certain symplectomorphisms $\varphi : (X, \omega_u) \to (X, \omega_v)$. Let us write $G$ for those symplectomorphisms that induce elements of $\Gamma_{00}$. Thus $G$ is a subgroup of the Fréchet–Lie group Diff $X$ of diffeomorphisms of $X$.

Lemma 2.3. The closure of $G$ in Diff $X$ contains all Hamiltonian diffeomorphisms of $(X, \omega)$.

Recall that Hamiltonian diffeomorphisms are time–1 maps of time dependent Hamiltonian vector fields $\text{sgrad } \zeta_t$, i.e., vector fields that are symplectic gradients with $\zeta_t \in C^\infty(X, \omega)$ a smooth family, $t \in [0, 1]$.

Proof. Let $\mathfrak{g}$ (the “Lie algebra” of $G$) consist of smooth vector fields $V$ on $X$ for which there is a smooth map $\varphi : [0, 1] \to G \subset \text{Diff } X$ such that $\varphi(0) = \text{id}_X$ and $\dot{\varphi}(0) = d\varphi(t)/dt|_{t=0} = V$. This is a vector subspace of the space of all vector fields: for example, if $\varphi, \psi$ realize vector fields $V, W \in \mathfrak{g}$, then $\varphi(t) \circ \psi(t)$ realizes $V + W$. In [S] pp. 512-513] Semmes essentially proved that $\mathfrak{g}$ contains all Hamiltonian vector fields. Essentially only, because the proof of his Lemma 4.1 is given only in Sobolev
spaces, not in $C^\infty(X)$. At any rate, we will need a slightly stronger statement, to wit: If $\zeta : [a, b] \to C^\infty(X)$ is smooth, then there is a smooth family

\begin{equation}
[a, b] \times [0, 1] \ni (s, t) \mapsto \varphi_t^s \in G \subset \text{Diff} X
\end{equation}

such that $\varphi_0^s = \text{id}_X$ and $\partial_t \varphi_t^s|_{t=0} = \text{sgrad} \zeta(s)$ for all $s$.

To verify this, recall Semmes’ construction in [S, top of p. 512] that, given $\xi, \eta \in C^\infty(X)$, shows that the Poisson bracket $\{\xi, \eta\} \in C^\infty(X)$, determined by $\omega$, has symplectic gradient in $\mathfrak{g}$. The same construction works with a parameter appended. Thus, if $\xi, \eta : [a, b] \to C^\infty(X)$ are smooth, there is a smooth family $\varphi_t^s \in G$ as in (2.2), $\varphi_0^s = \text{id}_X$ and $\partial_t \varphi_t^s|_{t=0} = \text{sgrad}\{\xi(s), \eta(s)\}$. But any smooth $\zeta : [a, b] \to C^\infty(X)$ such that $\int_X \zeta(s) \omega^n = 0$ can be written

\begin{equation}
\zeta(s) = \sum_{j=1}^m \{\xi_j(s), \eta_j(s)\}, \quad m = 4n + 1,
\end{equation}

with suitable smooth $\xi_j, \eta_j : [a, b] \to C^\infty(X)$. In fact $\xi_j$ can be chosen constant, and arbitrary as long as $\xi_j(s) \equiv \xi_j$ embed $X$ into $\mathbb{R}^m$. The statement, without $s$–dependence, corresponds to [S, Lemma 4.1], but was already proved in [AG]. Atkin and Grabowski’s proof is easily modified to provide (2.3). The proof of [AG] (5.2) Theorem| depends on [AG] (2.6) Proposition, the $s$–dependent version of which says that if $\xi_j \in C^\infty(X), j = 1, \ldots, m$, embed $X$ into $\mathbb{R}^m$, then any smooth smooth family $\psi^s$ of smooth $k$–forms on $X$, $s \in [a, b]$, can be written

$$\psi^s = \sum_{i_1, i_2, \ldots} f_{i_1 \ldots i_k}(s) d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_k}$$

with $f_{i_1 \ldots i_k} : [a, b] \to C^\infty(X)$ smooth. This is proved by an obvious cohomology vanishing as in [AG]. Another ingredient of the proof of [AG] (5.2) Theorem| on p. 325 there, in $s$–dependent version says that given a smooth family $\alpha^s$ of exact smooth forms on $X$, there is a smooth family $\beta^s$ of smooth forms such that $d\beta^s = \alpha^s$. One way to prove this is by Hodge theory, which gives that the unique solution $\beta^s$ of $d\beta^s = \alpha^s$ that is orthogonal to $\text{Ker} d$ depends smoothly on $s$. The rest of the proof in [AG] manipulates identities, and changes not if a parameter $s$ is appended. Thus (2.3) is proved.

We can now construct $\varphi_t^s \in G$ as in (2.2). First, subtracting from $\zeta$ a smooth function $c : [a, b] \to \mathbb{R}$ we obtain $\zeta^t : [a, b] \to C^\infty(X)$ with $\int_X \zeta^t(s) \omega^n = 0$. We find $\xi_j, \eta_j$ as in (2.3), corresponding to $\zeta^t$ rather than $\zeta$, and then smooth maps $(s, t) \mapsto \varphi^s_{jt}$ in $G$ such that $\varphi^s_{jt} = \text{id}_X$ and $\partial_t \varphi^s_{jt} = \{\xi_j(s), \eta_j(s)\}$ at $t = 0$. Since $\text{sgrad} \zeta = \text{sgrad} \zeta^t$, the diffeomorphisms

$$\varphi^t_t = \varphi^s_t \circ \varphi^s_{2t} \circ \cdots \circ \varphi^s_{mt}$$

have $t$–derivative $\text{sgrad} \zeta(s)$ at $t = 0$.

After these preparations we are ready to prove the lemma. Suppose $\varphi^1 : (X, \omega) \to (X, \omega)$ is a Hamiltonian diffeomorphism. This means it can be included in the flow $\varphi^s$ of Hamiltonian vector fields $V^s = \text{sgrad} \zeta(s)$.

\begin{equation}
\partial_s \varphi^s = V^s(\varphi^s), \quad 0 \leq s \leq 1, \quad \varphi^0 = \text{id}_X.
\end{equation}
Here $\zeta : [0, 1] \to C^\infty(X)$ is smooth. The $\varphi^s_k$ constructed above for this $\zeta$ can be used as integrators in a 1–step scheme to approximate the solution of the initial value problem (2.4). General theory gives that

$$\varphi^{(k-1)/k}_1 \circ \varphi^{(k-2)/k}_1 \circ \cdots \circ \varphi^0_1 \to \varphi^1 \quad \text{in } C^\infty(X)$$

as $k \to \infty$.

(Details are as follows. Smoothly embed $X$ in some $\mathbb{R}^m$ and with $p \in \mathbb{N}$, view $\varphi^s$ as an element of the Banach space $B = C^p(X) \times \cdots \times C^p(X)$, $m$ copies of $C^p(X)$. Extend $\varphi^s : X \to X$ to a smooth family of maps $\psi^s : \mathbb{R}^m \to \mathbb{R}^m$ and extend $V^s$ to a vector field on $\mathbb{R}^m$ by $V^s = \partial \psi^s|_{t=0}$. The error analysis of e.g. [HNW] p.160, Theorem 3.4, or more directly [An] Theorem 4.1, Corollary 4.2, gives that

$$\psi^{(k-1)/k}_1 \circ \psi^{(k-2)/k}_1 \circ \cdots \circ \psi^0_1 \to \varphi^1 \quad \text{in } B$$

as $k \to \infty$. Both [HNW] [An] work in finite dimensional Banach spaces, the latter in $\mathbb{C}^m$, but the same reasoning proves the result in any Banach space.)

Since the left hand side of (2.6) is $\varphi^{(k-1)/k}_1 \circ \cdots \circ \varphi^0_1 \in G$, we proved that $\varphi^1$ is indeed in the closure of $G$.

*Proof of Theorem 2.2.* That invariance under strict rearrangements implies holonomy invariance follows since parallel transport is realized by composition with a symplectomorphism, and such compositions send functions to their strict rearrangements. The converse implication depends on Lemma 2.3. This implies that $L(\xi) = L(\xi \circ \varphi)$ if $\xi \in T_0\mathcal{H}$ and $\varphi \in \text{Diff}(X, \omega)$ is Hamiltonian. By [LA, Theorem 1.2], $L|_{T_0\mathcal{H}}$ is therefore invariant under strict rearrangements. To complete the proof, take $\xi \in T_0\mathcal{H}$, $\eta \in T_0\mathcal{H}$ such that $\xi \sim \eta$. Parallel translate $\xi, \eta$ to $\xi', \eta' \in T_0\mathcal{H}$ along arbitrary smooth paths. Then $\xi' \sim \xi \sim \eta \sim \eta'$, whence $L(\xi) = L(\xi') = L(\eta') = L(\eta)$.

In what follows, a fiberwise continuous and convex function $L : T\mathcal{H} \to \mathbb{R}$ that is invariant under strict rearrangements will be called an invariant convex Lagrangian. The chief device to analyze their finer properties is the following representation theorem. We write $B(X)$ or $B(X, \mu)$—when a Borel measure $\mu$ on $X$ plays a role—for the Banach space of bounded Borel functions on $X$, with the supremum norm $\| \|$.

**Theorem 2.4.** Given an invariant convex Lagrangian $L : T\mathcal{H} \to \mathbb{R}$, there are families $A_u \subset \mathbb{R} \times B(X)$, $u \in \mathcal{H}$ such that for $\xi \in T_0\mathcal{H} \approx C^\infty(X)$

$$L(\xi) = \sup_{(a,f) \in A_u} a + \int_X f \omega^u.$$

$A_u$ can be chosen in $\mathbb{R} \times C^\infty(X)$, and have the property that whenever $(a,f) \in A_u$ and $\varphi : (X, \omega_u) \to (X, \omega_u)$ is a symplectomorphism, then $(a, f \circ \varphi) \in A_v$. Alternatively, $A_u$ can be chosen to be strict rearrangement invariant: if $f \in B(X, \mu_u)$ and $g \in B(X, \mu_v)$ are equidistributed, and $(a, f) \in A_u$, then $(a, g) \in A_v$.

If $L$ is also positively homogeneous, $(L(c\xi) = L(\xi)$ whenever $c \in (0, \infty))$, then in addition $A_u$ can be chosen in $\{0\} \times C^\infty(X)$, respectively, in $\{0\} \times B(X)$. 

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Proof. Most of the proof was done in \[L4\]. Lemma 2.1 there produces \( \mathcal{A}_0 \subset \mathbb{R} \times C^\infty(X) \) that satisfies (2.7) when \( u = 0 \). If we adjoin to \( \mathcal{A}_0 \) all pairs \((a, f \circ \varphi)\) with \((a, f) \in \mathcal{A}_0 \) and \( \varphi : (X, \omega) \to (X, \omega) \) a symplectomorphism, because of the invariance of \( L \) the supremum in (2.7) is not going to change (for \( u = 0 \)). So we can assume that \( \mathcal{A}_0 \) already is invariant under symplectomorphisms. We then define \( \mathcal{A}_u \) to consist of pairs \((a, f \circ \psi)\) with \((a, f) \in \mathcal{A}_0 \) and \( \psi : (X, \omega_u) \to (X, \omega) \) a symplectomorphism. This will do, since if \( \xi \in T_u \mathcal{H} \), with the above \( \psi \)

\[
\sup_{(a, g) \in \mathcal{A}_u} a + \int_X g \xi \omega_v^n = \sup_{(a, g) \in \mathcal{A}_u} a + \int_X (g \circ \psi^{-1})(\xi \circ \psi^{-1}) \omega_v^n \\
= \sup_{(a, f) \in \mathcal{A}_0} a + \int_X (\xi \circ \psi^{-1}) f \omega_v^n = \sup_{(a, f) \in \mathcal{A}_0} a + \int_X (\xi \circ \psi^{-1}) L(\xi) = L(\xi).
\]

Alternatively, we can modify the above \( \mathcal{A}_u \) to \( \mathcal{A}'_u \) consisting of all \((a, g) \in \mathbb{R} \times B(X)\) for which there is \((a, f) \in \mathcal{A}_0 \) such that \((f, \mu_0) \sim (g, \mu_u)\). This will not change the supremum in (2.7), with \( \mathcal{A}'_u \) now, either. It suffices to check this for \( u = 0 \). By a variant of a lemma of Katok, \[L4\], Lemma 3.2], if \((f, \mu_0) \sim (g, \mu_u)\) then there is a sequence \( \varphi_k : (X, \omega) \to (X, \omega) \) of symplectomorphisms such that \( \int_X |g - f \circ \varphi_k| \omega^n \to 0 \). Therefore

\[
\int_X g \xi \omega_v^n = \lim_{k \to \infty} \int_X (f \circ \varphi_k) \xi \omega_v^n = \lim_{k \to \infty} \int_X (\xi \circ \varphi^{-1}_k) f \omega_v^n, \text{ and so}
\]

\[
a + \int_X g \xi \omega_v^n \leq \lim_{k \to \infty} L(\xi \circ \varphi^{-1}_k) = L(\xi).
\]

Thus replacing \( \mathcal{A}_u \) with \( \mathcal{A}'_u \), (2.7) will still hold, and \( \mathcal{A}'_u \) is now strict rearrangement invariant.

Finally, if \( L \) is positively homogeneous, the statement of the theorem follows in the same way from the corresponding part of \[L4\] Lemma 2.1.

The Lagrangians in this section were required to be continuous on the fibers of \( T \mathcal{H} \). But, coupled with invariance, this implies continuity on \( T \mathcal{H} \):

**Theorem 2.5.** An invariant convex Lagrangian \( L : T \mathcal{H} \to \mathbb{R} \) is a continuous function on the Fréchet manifold \( T \mathcal{H} \).

Proof. Suppose \( u, u_k \in \mathcal{H} \), \( \xi \in T_u \mathcal{H} \), \( \xi_k \in T_{u_k} \mathcal{H} \), and \( \xi_k \to \xi \). This simply means that as elements of \( C^\infty(X) \), \( u_k \to u \) and \( \xi_k \to \xi \). Parallel translate \( \xi_k \) to \( \eta_k \in T_u \mathcal{H} \) along the straight line segment \( t \mapsto tu + (1 - t)u_k \). This is done by integrating the time dependent vector field \( (1/2) \text{grad}_{tu + (1-t)u_k} (u_k - u) \) on \( X \), for \( 0 \leq t \leq 1 \). If the time–1 map is \( \psi_k : X \to X \), then \( \eta_k = \xi_k \circ \psi_k \). Since \( \psi_k \to \id_X \) in the \( C^\infty \) topology, \( \eta_k \to \xi \) in \( C^\infty(X) \approx T_u \mathcal{H} \), as \( k \to \infty \). Hence \( \lim_k L(\xi_k) = \lim_k L(\eta_k) = L(\xi) \), as claimed.

### 3 Extending Lagrangians

As said in the Introduction, weak geodesics tend not to stay in the space \( \mathcal{H} \). Therefore, even in order to formulate a principle of least action we need to evaluate the action
of a Lagragian along paths in spaces larger than \( \mathcal{H} \). In this section we will extend certain invariant convex Lagrangians \( T\mathcal{H} \to \mathbb{R} \) to a larger Banach bundle and describe properties of the extended Lagrangians.

We start by recalling definitions. Let \( Y \) be a complex manifold and \( \Omega \) a smooth real \((1,1)\) form on it, \( d\Omega = 0 \). A function \( u : Y \to [-\infty, \infty) \) is \( \Omega \)-plurisubharmonic if \( \rho + u \) is plurisubharmonic whenever \( \rho \) is a local potential of \( \Omega \), i.e., \( \Omega = dd^c \rho \). We use the convention that \( \equiv -\infty \) is not plurisubharmonic, and write \( \text{PSH}(\Omega) \) for the set of \( \Omega \)-plurisubharmonic functions. Back to our Kähler manifold \((X,\omega)\), we denote by \( E(\omega) \) the class of \( u \in \text{PSH}(\omega) \) with full Monge–Ampère mass, see [GZ1]. This class contains all bounded \( \omega \)-plurisubharmonic functions. The Monge–Ampère measure on \( X \), corresponding to \( \omega^n u \), will again be denoted \( \mu_u \). This is a Borel measure on \( X \), its crucial property is \( \mu_u(X) = \int_X \omega^n \). We endow \( E(\omega) \) with the discrete topology, and let

\[
(3.1) \quad T^\infty \mathcal{E}(\omega) = \mathcal{E}(\omega) \times B(X),
\]

a trivial Banach bundle with fibers the bounded Borel functions on \( X \). Corresponding to usage in the subject we will not distinguish between elements \( \xi \in T^\infty \mathcal{E}(\omega) \) and their representation \( \xi \in B(X) \) in the trivialization (3.1). The embedding \( C^\infty(X) \hookrightarrow B(X) \) induces an embedding \( \mathcal{H} \hookrightarrow T^\infty \mathcal{E}(\omega) \) of vector bundles, continuous if \( \mathcal{H} \) is considered with the discrete topology.

**Definition 3.1.** Suppose \( u \in \mathcal{E}(\omega) \) and \( V \subset B(X, \mu_u) \) is a vector subspace. We say that a function \( p : V \to \mathbb{R} \) is strongly continuous if \( p(\xi_k) \) converges whenever \( \xi_k \in V \) is a uniformly bounded sequence that converges \( \mu_u \) almost everywhere.

In this case \( \lim_k p(\xi_k) \) depends only on \( \xi = \lim_k \xi_k \), since another sequence \( \eta_k \to \xi \) can be combined with \( \xi_k \) into one sequence.

**Theorem 3.2.** Any invariant convex Lagrangian \( L : T\mathcal{H} \to \mathbb{R} \) that is strongly continuous on the fibers \( T_u \mathcal{H} \) has a unique extension to \( \overline{L} : T^\infty \mathcal{E}(\omega) \to \mathbb{R} \) that is strict rearrangement invariant, and strongly continuous on the fibers \( T^\infty \mathcal{E}(\omega) \). This extension is fiberwise convex.

For example, Darvas’s metrics in [D2], coming from a finite Young weight, cf. (1.2), are strongly continuous on the fibers. —The proof of the theorem will use the notion of decreasing rearrangement of measurable functions \( \eta : (Y, \nu) \to \mathbb{R} \) on a measure space. This is a decreasing, upper semicontinuous function \( \eta^* : [0, \nu(Y)] \to \mathbb{R} \), equidistributed with \( \eta \). Thus \( \nu(s \leq \eta \leq t) \) is equal to the length of the maximal interval on which \( s \leq \eta^* \leq t \). The requirement of upper semicontinuity for the decreasing function \( \eta^* \) translates to left continuity, which differs from the more usual right continuity requirement, but the difference is of no consequence. In our setting

\[
(3.2) \quad \nu(\eta \geq \eta^*(s)) = s,
\]

and more generally,

\[
(3.3) \quad \nu(\eta \geq t) \leq \tau \text{ implies } \eta^*(\tau) \leq t, \quad \nu(\eta \geq t) \geq \tau \text{ implies } \eta^*(\tau) \geq t,
\]
Proof of Theorem 3.2. By [L4] Theorem 5.2 \( L|T_0 \mathcal{H} : C^\infty(X) \to \mathbb{R} \) has a unique strongly continuous and strict rearrangement invariant extension \( q : B(X) \to \mathbb{R} \); this extension is convex, and if uniformly bounded \( \xi_k \in B(X) \) converge to \( \xi \in B(X) \) \( \mu_0 \)-a.e., then \( q(\xi_k) \to q(\xi) \). From \( q \) we obtain \( T \) by taking a \( \xi \in T^\infty_0 \mathcal{E}(\omega) \simeq B(X, \mu_0) \), finding an \( \eta \in B(X, \mu_0) \) equidistributed with it, and letting \( T(\xi) = q(\eta) \). Such an \( \eta \) can be found in the form \( \eta = \xi^* \circ \theta \), where \( \theta : (X, \mu_0) \to [0, \mu_0(X)] \) is a measure preserving map (the target endowed with Lebesgue measure), see e.g. [L4] Lemma 5.5. This is clearly the only strict rearrangement invariant way to extend \( q : T^\infty_0 \mathcal{E}(\omega) \to \mathbb{R} \) to \( T : T^\infty \mathcal{E}(\omega) \to \mathbb{R} \). It is immediate that \( T \) thus constructed has the properties claimed in the theorem.

Further down we will not distinguish between an invariant convex Lagrangian \( T \mathcal{H} \to \mathbb{R} \) that is strongly continuous on the fibers and its extension \( T^\infty \mathcal{E}(\omega) \to \mathbb{R} \) provided by Theorem 3.2, and will just refer to a strongly continuous, invariant, convex Lagrangian \( T : T^\infty \mathcal{E}(\omega) \to \mathbb{R} \).

Lemma 3.3. A strongly continuous, invariant, convex Lagrangian \( L : T^\infty \mathcal{E}(\omega) \to \mathbb{R} \) is equi–Lipschitz continuous on bounded subsets of the fibers \( T_u \mathcal{E}(\omega) \) in the sense that given \( R \in (0, \infty) \) there is an \( A \in (0, \infty) \) such that for \( u \in \mathcal{E}(\omega) \) and \( \xi, \eta \in T^\infty_u \mathcal{E}(\omega) \)

\[
(3.4) \quad \text{if } \|\xi\|, \|\eta\| < R \text{ then } |L(\xi) - L(\eta)| \leq A\|\xi - \eta\|.
\]

Proof. According to [L4] Theorem 5.4 (3.4) holds when \( u = 0 \). The same \( A \) will work for any \( u \), for with a measure preserving \( \theta : (X, \mu_0) \to [0, \mu_0(X)] \) as in the proof of Theorem 3.2 and \( \xi, \eta \in T^\infty_u \mathcal{E}(\omega) \)

\[
|L(\xi) - L(\eta)| = |L(\xi^* \circ \theta) - L(\eta^* \circ \theta)| \leq A \sup |\xi^* \circ \theta - \eta^* \circ \theta| \leq A\|\xi - \eta\|.
\]

Although we have endowed \( \mathcal{E}(\omega) \) with the discrete topology, we will need a continuity property of Lagrangians \( T^\infty \mathcal{E}(\omega) \to \mathbb{R} \) stronger than fiberwise. This will involve the notion of Monge–Ampère capacity cap of subsets of \( X \). Recall that a function \( \xi : X \to \mathbb{R} \) is quasicontinuous if for every \( \varepsilon > 0 \) there is an open \( G \subset X \) of capacity \( (G) < \varepsilon \) such that \( \xi|X \setminus G \) is continuous; and a sequence of functions \( \xi_j : X \to \mathbb{R} \) converges to \( \xi : X \to \mathbb{R} \) in capacity if \( \lim_{j \to \infty} \text{cap}(|\xi_j - \xi| > \delta) = 0 \) for every \( \delta > 0 \). In particular, a uniformly convergent sequence converges in capacity.

Lemma 3.4. Let \( L : T^\infty \mathcal{E}(\omega) \to \mathbb{R} \) be strongly continuous, invariant, and convex. Suppose \( u_k \in \mathcal{E}(\omega) \) either decrease, or uniformly converge, to a bounded \( u \in \mathcal{E}(\omega) \) as \( k \to \infty \), and \( \xi_k \in T^\infty_u \mathcal{E}(\omega) \simeq B(X) \) converge in capacity to \( \xi \in T^\infty_u \mathcal{E}(\omega) \simeq B(X) \). If \( \xi \) is quasicontinuous, then \( \lim_k L(\xi_k) = L(\xi) \).

Proof. Upon adding a constant to the \( u_k \) and scaling \( u, u_k, \omega, \) and \( L \), we can arrange that \( 0 \leq u, u_k \leq 1 \). Suppose first the \( u_k \) decrease. The point of the proof is to show that the decreasing rearrangements \( \xi^*_k \) of \( \xi_k \in B(X, \mu_{u_k}) \) converge to the rearrangement \( \xi^* \) of \( \xi \in B(X, \mu_0) \), away from a countable subset of \([0, \mu_0(X)]\). Define decreasing functions \( f, g \) on \([0, \mu_0(X)]\)

\[
f = \liminf_k \xi^*_k \leq \limsup_k \xi^*_k = g.
\]
Let $s \in (0, \mu_0(X))$, $S = \xi^*(s)$, and $\varepsilon > 0$. With $A_k = \{|\xi_k - \xi| \geq \varepsilon\}$, $k \in \mathbb{N}$,

$$\{\xi_k \geq S + \varepsilon\} \cup A_k \quad \text{and} \quad \{\xi > S - \varepsilon\} \cup \{\xi_k \geq S - 2\varepsilon\} \cup A_k.$$ 

For $j \in \mathbb{N}$ define continuous functions $F_j, G_j : \mathbb{R} \to [0, 1]

$$F_j(t) = \begin{cases} 0 \text{ if } t \leq S - 1/j \\ 1 \text{ if } t \geq S \\ \text{linear in between,} \end{cases} \quad \text{and} \quad G_j(t) = \begin{cases} 0 \text{ if } t \leq S - \varepsilon \\ 1 \text{ if } t \geq S - \varepsilon + 1/j \\ \text{linear in between.} \end{cases}$$

Note that $F_j$ decreases, $G_j$ decreases with increasing $j$. We can estimate

$$\mu_{u_k}(\xi_k \geq S + \varepsilon) \leq \mu_{u_k}(\xi \geq S) + \mu_{u_k}(A_k) \leq \int_X F_j \circ \xi \, d\mu_{u_k} + \text{cap}(A_k)$$

$$\mu_{u_k}(\xi_k \geq S - 2\varepsilon) \geq \mu_{u_k}(\xi > S - \varepsilon) - \mu_{u_k}(A_k) \geq \int_X G_j \circ \xi \, d\mu_{u_k} - \text{cap}(A_k).$$

Since $F_j \circ \xi, G_j \circ \xi$ are quasicontinuous, by [GZ2] Theorem 4.2.6

$$\lim_{k \to \infty} \int_X F_j \circ \xi \, d\mu_{u_k} = \int_X F_j \circ \xi \, d\mu_u, \quad \lim_{k \to \infty} \int_X G_j \circ \xi \, d\mu_{u_k} = \int_X G_j \circ \xi \, d\mu_u.$$ 

Therefore, letting first $k \to \infty$ in (3.5), then $j \to \infty$, and using the monotone convergence theorem as well,

$$\limsup_{k \to \infty} \mu_{u_k}(\xi_k \geq S + \varepsilon) \leq \mu_u(\xi \geq S)$$

$$\liminf_{k \to \infty} \mu_{u_k}(\xi_k \geq S - 2\varepsilon) \geq \mu_u(\xi > S - \varepsilon) \geq \mu_u(\xi \geq S).$$

Now $\mu_u(\xi \geq S) = s$, see (3.2). Hence, given $\sigma < s < \rho$, for sufficiently large $k$

$$\mu_{u_k}(\xi_k \geq \xi^*(s) + \varepsilon) < \rho = \mu_u(\xi \geq \xi^*(\rho)),$$

$$\mu_u(\xi \geq \xi^*(\sigma)) = \sigma < \mu_{u_k}(\xi_k \geq \xi^*(s) - 2\varepsilon).$$

We apply (3.3) with $\nu = \mu_{u_k}, \mu_u, t = \xi^*(s) + \varepsilon, \xi^*(s) - 2\varepsilon$, and $\tau = \rho, \sigma$ to conclude $\xi_k^*(\rho) \leq \xi^*(s) + \varepsilon$ and $\xi_k^*(\sigma) \geq \xi^*(s) - 2\varepsilon$. In the limit $k \to \infty$

$$g(\rho) \leq \xi^*(s) + \varepsilon \quad \text{and} \quad f(\sigma) \geq \xi^*(s) - 2\varepsilon, \text{ for all } \sigma < s < \rho.$$ 

If $f, g$ are continuous at $s$—which occurs apart from countably many values—, $g(s) \leq \xi^*(s) \leq f(s) \leq g(s)$ follows, i.e., $\lim_k \xi_k^*(s) = \xi^*(s)$ as claimed.

It is now easy to finish the proof. With a measure preserving $\theta : (X, \mu_0) \to [0, \mu_0(X)]$, as in the proof of Theorem 3.2, $\xi_k \in B(X, \mu_{u_k})$ and $\xi_k^* \circ \theta \in B(X, \mu_0)$ are equidistributed, and $\xi_k^* \circ \theta \to \xi^* \circ \theta \mu_0$–almost everywhere. Hence by Theorem 3.2

$$\lim_k L(\xi_k) = \lim_k L(\xi_k^* \circ \theta) = L(\xi^* \circ \theta) = L(\xi).$$

We are done if $u_k$ are known to decrease.

Now suppose that $u_k$ converge uniformly. It suffices to prove that a subsequence of $L(\xi_k)$ converges to $L(\xi)$, and for this reason we can assume that $\|u_k - u_{k-1}\| < 2^{-k}$ for $k = 2, 3, \ldots$. Then the sequence $v_k = u_k + 2^{-k}$ decreases to $u$, and $\mu_{u_k} = \mu_{v_k}$. We can view $\xi_k \in T_{u_k} \mathcal{E}(\omega) \approx B(X, \mu_{u_k})$ as elements $\xi_k^* \in T_{v_k} \mathcal{E}(\omega) \approx B(X, \mu_{v_k})$, which are strict rearrangements of $\xi_k$. Hence $L(\xi_k) = L(\xi_k^*) \to L(\xi)$ by the first part of the proof.
4 Weak geodesics, \(\varepsilon\)-geodesics, Jacobi fields

If \(a < b\) are real numbers, we let

\[
S_{ab} = \{ s \in \mathbb{C} : a < \text{Re} s < b \},
\]

and denote by \(\pi\) the projection \(S_{ab} \times X \to X\).

Following Berndtsson and Darvas [Be1, pp. 156-157] we make the following definition.

**Definition 4.1.** A path \(u : [a, b] \to \text{PSH}(\omega)\) is a subgeodesic if the function \(U : S_{ab} \times X \to [-\infty, \infty)\) given by \(U(s, x) = u(\text{Re}(s))(x)\) is \(\pi^*\omega\)-plurisubharmonic.

If \(u_a, u_b \in \text{PSH}(\omega)\), the weak geodesic determined by \((a, b) \to \text{PSH}(\omega)\),

\[
(4.1) \quad u = \sup \{ v | v : (a, b) \to \text{PSH}(\omega) \text{ is subgeodesic}, \lim_{t \to a} v \leq u_a, \lim_{t \to b} v \leq u_b \}. \]

The limits are understood pointwise on \(X\); they exist because \(\pi^*\omega\)-plurisubharmonicity implies that for each \(x \in X\) the function \(v(\cdot)(x)\) is convex. It is possible that \((4.1)\) gives \(u \equiv -\infty\), not valued in \(\text{PSH}(\omega)\); but otherwise the weak geodesic is indeed a path in \(\text{PSH}(\omega)\) and is itself a subgeodesic [D4, section 3.1]. Darvas points out that in general the term “connecting” weak geodesic is misleading, as \(\lim_a u\) may have little to do with \(u_a\). But, if \(u_a, u_b\) are bounded, Berndtsson proves by a simple argument that the weak geodesic indeed connects, \(\lim_a u = u_a, \lim_b u = u_b\), uniformly on \(X\), [Be1, pp. 156-157]. If \(c < d\), the weak geodesic \(u' : (c, d) \to \text{PSH}(\omega)\) between \(u_a\) and \(u_b\) is \(u\), composed with an affine reparametrization, because affine reparametrizations of subgeodesics yield subgeodesics.

In what follows we will only deal with weak geodesics \(u\) determined by bounded \(u_a, u_b\). Such a \(u\) is a Lipschitz map into \(B(X)\), and we will refer to its continuous extension to the closed interval \([a, b]\) as a weak geodesic, too.

In (1.1) we defined Mabuchi’s connection on \(\mathcal{T}\mathcal{H}\) through its parallel transport. A more direct definition takes a smooth path \(u : [a, b] \to \mathcal{H}\) and a smooth vector field \(\xi : [a, b] \to \mathcal{T}\mathcal{H}, \xi(t) \in T_{u(t)}\mathcal{H}\), along it; the covariant derivative of \(\xi\) along \(u\) is then the vector field \(\nabla_t \xi\) given by

\[
(4.2) \quad \nabla_t \xi(t) = \dot{\xi}(t) - \frac{1}{2} (d_X \dot{u}(t), d_X \xi(t))|_{u(t)} \in C^\infty(X) \approx T_{u(t)}\mathcal{H}. \]

Here \(d_X\) is differential on \(X\), for fixed \(t\), and \((\ , \ )|_{u(t)}\) is inner product on \(T^*X\) induced by the Kähler metric of \(\omega_{u(t)}\). In (4.2) the left hand side is to be computed for \(\xi\) a section of \(\mathcal{T}\mathcal{H}\) along \(u\); on the right \(\xi\) stands for the representation of this section in the canonical trivialization \(\mathcal{T}\mathcal{H} \approx \mathcal{H} \times \mathcal{C}^\infty(X)\), so for a function \([a, b] \to \mathcal{C}^\infty(X)\); and the equality of the two sides again uses the trivialization of \(\mathcal{T}\mathcal{H}\).

Geodesics \(u : [a, b] \to \mathcal{H}\) of \(\nabla\) satisfy \(\nabla_t \dot{u}(t) = 0\). Chen, however, had the idea that the geometry of \(\mathcal{H}\) can be better accessed through \(\varepsilon\)-geodesics. Define a vector field \(F\) on \(\mathcal{H}\) by

\[
(4.3) \quad F(v)\omega^n_v = \omega^n, \quad v \in \mathcal{H}, \quad F(v) \in \mathcal{C}^\infty(X) \approx T_v\mathcal{H}. \]
If \( \varepsilon > 0 \), an \( \varepsilon \)-geodesic is a solution \( u : [a, b] \to \mathcal{H} \) of
\[
\nabla_t \dot{u}(t) = \varepsilon F(u(t)), \quad t \in [a, b].
\]

In what follows, we will just write \( d \) for \( d_X \). Chen proves [B1, B2]

**Theorem 4.2.** Given \( a < b \) and two potentials \( u_a, u_b \in \mathcal{H} \), (4.4) has a unique \( C^2 \)
solution \( u = u^\varepsilon : [a, b] \to \mathcal{H} \) satisfying \( u(a) = u_a, \ u(b) = u_b \). The solution \( u \) is smooth, and as an element of \( C^\infty([a, b] \to \mathcal{H}) \), it depends smoothly on \( u_a, u_b \) (and \( a, b, \varepsilon \)). Finally, if \( u_a, u_b \) are in a fixed compact subset of \( \mathcal{H} \), the forms \( d\varepsilon^u(t), d\dot{u}(t) \) and \( \ddot{u}(t) \) are uniformly bounded on \( X \) for \( 0 < \varepsilon < \varepsilon_0 \) and \( a \leq t \leq b \).

It follows by the Arzelà–Ascoli theorem and a maximum principle for the Monge–Ampère operator that for fixed \( u_a, u_b \) the uniform limit \( u = \lim_{\varepsilon \to 0} u^\varepsilon \) exists. The limit \( u \) maps into the space
\[
\mathcal{H}^{1 \mathbf{T}} = \{ w \in C(X) \cap \text{PSH(}\omega\text{)} : \text{the current } d\varepsilon^w \text{ is bounded}\},
\]
and the currents \( d\varepsilon^u(t), d\dot{u}(t), \ddot{u}(t) \) are represented by uniformly bounded forms. Also, \( u \) is the weak geodesic in the sense of Definition 4.1 to connect \( u_a, u_b \).

Consider an \( \varepsilon \)-geodesic \( u : [a, b] \to \mathcal{H} \).

**Definition 4.3.** A vector field \( \xi : [a, b] \to T\mathcal{H} \) along \( u \) is an \( \varepsilon \)-Jacobi field if there are an interval \( I \) containing \( 0 \in \mathbb{R} \) and a smooth family \( I \ni s \mapsto u^s \), each \( u^s : [a, b] \to \mathcal{H} \) an \( \varepsilon \)-geodesic such that \( u^0 = u \) and \( \xi = \partial_s u^s|_{s=0} \).

**Lemma 4.4.** (a) If \( I \subset \mathbb{R} \) is an interval and \( v : I \to \mathcal{H} \) a smooth path, then the covariant derivative of \( F \) along \( v \) satisfies
\[
\omega^n_{v(s)} \nabla_s F(v(s)) = -nd(F(v(s))d\varepsilon^v(s)) \wedge \omega^{n-1}_{v(s)},
\]
(b) If \( \xi : [a, b] \to T\mathcal{H} \) is an \( \varepsilon \)-Jacobi field along an \( \varepsilon \)-geodesic \( u : [a, b] \to \mathcal{H} \), then
\[
\omega^n_{u(t)} \nabla^2 \xi(t) = \frac{1}{4} \{ \{ \dot{u}(t), \xi(t) \}, \dot{u}(t) \} \omega^n_{u(t)} - \varepsilon nd(F(u(t))d\varepsilon^\xi(t)) \wedge \omega^{n-1}_{u(t)},
\]
where \( \{ , \} \) is Poisson bracket on \( T_{u(t)}\mathcal{H} \approx C^\infty(X) \) for the symplectic form \( \omega_{u(t)} \).

Calabi and Chen [CC] Section 2.3] derive an equivalent equation for \( \varepsilon \)-Jacobi fields.

**Proof.** (a) We will apply (4.2) with \( \xi = F \circ v \). Differentiating \( F(v(s))\omega^n_{v(s)} = \omega^n \) with respect to \( s \) gives
\[
\omega^n_{v(s)} \partial_s F(v(s)) = -F(v(s))\partial_s(\omega + d\varepsilon^v(s))^n = -nF(v(s))d\varepsilon^v(s) \wedge \omega^{n-1}_{v(s)}.
\]
At the same time
\[
(d\dot{v}(s), dF(v(s)))_{v(s)} \omega^n_{v(s)} = 2ndF(v(s)) \wedge d\varepsilon^v(s) \wedge \omega^{n-1}_{v(s)},
\]
see e.g. [B2] p.103. Substituting into (4.2) now gives (4.5).
(b) Let \( u^s : [a, b] \to \mathcal{H} \) be a smooth family of \( \varepsilon \)-geodesics such that \( \xi = \partial_s u^s|_{s=0} \), and set \( U(s, t) = u^s(t) \). As Mabuchi’s connection is torsion free, \( \nabla_s \partial_t U = \nabla_t \partial_s U \). The curvature of \( \nabla \), evaluated on \( \partial_s U(s, t), \partial_t U(s, t) \in T_{U(s, t)} \mathcal{H} \) is an endomorphism of \( T_{U(s, t)} \mathcal{H} \) that acts on a vector field \( \eta(s, t) \) by

\[
R(\partial_s U, \partial_t U)\eta = (\nabla_s \nabla_t - \nabla_t \nabla_s)\eta = \{\{\partial_s U, \partial_t U\}, \eta\}/4,
\]

see [M Theorem 4.3]. (Mabuchi’s formula does not contain the factor 1/4, due to different conventions.)

We apply \( \nabla_s \) to the \( \varepsilon \)-geodesic equation \( \nabla_t \partial_t U(s, t) = \varepsilon F(U(s, t)) \), to obtain at

\[ s = 0 \]

\[
\varepsilon \nabla_s F(U) = \nabla_s \nabla_t \partial_t U = R(\partial_s U, \partial_t U)\partial_t U + \nabla_t \nabla_s \partial_t U = (1/4)\{\{\partial_s U, \partial_t U\}, \partial_t U\} + \nabla_t \nabla_t \partial_t U = (1/4)\{\{\xi, \dot{u}\}, \ddot{u}\} + \nabla_t^2 \xi.
\]

Combining (4.5) with this, (4.6) follows.

## 5 The action

Consider an invariant convex Lagrangian \( T\mathcal{H} \to \mathbb{R} \). If \( u : [a, b] \to \mathcal{H} \) is a piecewise \( C^1 \) path, its action is

\[
\mathcal{L}(u) = \int_a^b L(\dot{u}(t))dt.
\]

Depending on the nature of \( L \), this can represent length or energy of a path, but in general it is neither. No matter what \( L \), the integral (5.1) is that of a piecewise continuous function by Theorem 2.5, so that it exists as a Riemann integral.

For the purposes of this paper we must consider action for paths beyond \( \mathcal{H} \). The material developed in section 3 allows to define action for paths in the space \( B(X) \cap \text{PSH}(\omega) \subset \mathcal{E}(\omega) \) of continuous \( \omega \)-plurisubharmonic functions. This is a subset of the Banach space \( B(X) \) and, viewing maps into it as maps into \( B(X) \), we can talk about various regularity classes of such maps. If \( a < b \) are real, the following is easy to check.

**Lemma 5.1.** A map \( u : [a, b] \to B(X) \) is continuous if and only if the functions

\[
(5.2) \quad u(\cdot)(x), \quad \text{for } x \in X
\]

are equicontinuous, and it is \( C^k \) for \( k = 1, 2, \ldots \) if and only if the functions in (5.2) are \( k \) times differentiable, and the \( k \)’th derivatives are also equicontinuous.

According to Theorem 3.2, an invariant convex Lagrangian \( T\mathcal{H} \to \mathbb{R} \) that is strongly continuous on the fibers determines a strongly continuous, invariant, convex Lagrangian \( L : T^\infty \mathcal{E}(\omega) \to \mathbb{R} \). Suppose \( u : [a, b] \to B(X) \cap \text{PSH}(\omega) \) is a \( C^1 \) path. Since \( \omega \)-plurisubharmonic functions are quasicontinuous [BT Theorem 3.5], [GZ2 Corollary 9.12], the difference quotients \( (u(t) - u(s))/(t - s) \) are quasicontinuous and so are their uniform limits \( \dot{u}(t) \). Hence by Lemma 3.4 \( L \circ \dot{u} : [a, b] \to \mathbb{R} \) is continuous. Clearly if
\( u \) is just piecewise \( C^1 \), the integral in (5.1) still exists as the integral of a piecewise continuous function, and defines action \( \mathcal{L}(u) \).

If \( w, w' \in B(X) \cap \text{PSH}(\omega) \) and \( T \in (0, \infty) \), we define the least action, or just action, \( \mathcal{L}_T(w, w') \) between \( w, w' \) as
\[
\mathcal{L}_T(w, w') = \inf_u \mathcal{L}(u),
\]
the infimum taken over all piecewise \( C^1 \) paths \( u : [0, T] \to B(X) \cap \text{PSH}(\omega) \) such that \( u(0) = w, \ u(T) = w' \). Note that \( w, w' \) can be connected by a smooth path, e.g. \( u(t) = (1 - t/T)w + (t/T)w' \) connects. We will see that \( \mathcal{L}_T(w, w') > -\infty \) (Lemma 9.3).

Instead of \([0, T]\) if we minimize over paths \([a, a + T] \to B(X) \cap \text{PSH}(\omega)\), the infimum in (5.3) does not change. However, in general \( \mathcal{L}_T(w, w') \) will depend on \( T \); it will not if \( L \) is positively homogeneous. In general
\[
\mathcal{L}_T(w, w') + \mathcal{L}_S(w', w'') \geq \mathcal{L}_{T+S}(w, w'')
\]
follows by concatenating paths. Of course, \( \mathcal{L}_T(w, w') = \mathcal{L}_T(w', w) \) should be expected only if \( L \) is even, \( L(-\xi) = L(\xi) \).

In our two main results below, \( L : T^\infty \mathcal{E}(\omega) \to \mathbb{R} \) is a strongly continuous, invariant, convex Lagrangian.

**Theorem 5.2** (Principle of least action). If a \( C^1 \) path \( v : [0, T] \to B(X) \cap \text{PSH}(\omega) \) is a weak geodesic, then \( \mathcal{L}(v) = \mathcal{L}_T(\xi(v(0)), v(T)) \).

**Theorem 5.3.** If \( u, v : [a, b] \to B(X) \cap \text{PSH}(\omega) \) are weak geodesics, then for \( S > 0 \) the function
\[
[a, b] \ni t \mapsto \mathcal{L}_S(u(t), v(t)) \in \mathbb{R}
\]
is convex.

If \( L \) is absolutely homogeneous, \( L(c \xi) = |c|L(\xi) \), and vanishes only on zero vectors, then action \( \mathcal{L}_S \) is distance measured in a Finsler metric and is independent of \( S \); the statement of Theorem 5.3 is an indication of seminegative curvature.

The proofs will take the rest of the paper. First we prove them in approximate versions, with \( \varepsilon \)-geodesics in \( \mathcal{H} \) replacing weak geodesics. The approximate versions depend on two facts. First, that \( L \) is convex along \( \varepsilon \)-Jacobi fields; second, as a consequence, for some Lagrangians a triangle inequality holds for triangles in \( \mathcal{H} \) with two sides \( \varepsilon \)-geodesics (Theorem 6.1, Lemma 7.2). It is a technical point but noteworthy that the approximate results contain no error term, no \( O(\varepsilon) \). By letting \( \varepsilon \to 0 \) we obtain a principle of least action in \( \mathcal{H}^{11} \) (Corollary 7.4). Approximating weak geodesics in \( B(X) \cap \text{PSH}(\omega) \) by weak geodesics in \( \mathcal{H}^{11} \) we obtain the same in \( B(X) \cap \text{PSH}(\omega) \) in section 8. Theorem 5.3 is proved by the same approximation scheme in section 9.

### 6 Divergence of \( \varepsilon \)-Jacobi fields

In this section we stay in \( \mathcal{H} \), and consider invariant convex Lagrangians \( L : T\mathcal{H} \to \mathbb{R} \), strongly continuous or not. Recall that given \( \varepsilon > 0 \), an \( \varepsilon \)-geodesic \( u : [a, b] \to \mathcal{H} \) satisfies the equation
\[
\nabla_t u(t) = \varepsilon F(u(t)),
\]

```
where the vector field $F : \mathcal{H} \to T\mathcal{H}$ is defined by $F(v)\omega^n = \omega^n$. Infinitesimal variations of $\varepsilon$-geodesics are $\varepsilon$-Jacobi fields. If $\xi : [a, b] \to T\mathcal{H}$ is an $\varepsilon$-Jacobi field along an $\varepsilon$-geodesic $u : [a, b] \to T\mathcal{H}$, by Lemma 4.4(b)

$$\omega^n u(t) \nabla^2 \xi(t) = \frac{1}{4}\{\{\dot{u}(t), \xi(t)\}, \dot{u}(t)\}\omega^n u(t) - \varepsilon d(\alpha(u(t)) d\xi(t)) \wedge \omega^{n-1}_u(t).$$

All our subsequent results rest on the following theorem.

**Theorem 6.1.** If $\xi : [a, b] \to T\mathcal{H}$ is an $\varepsilon$-Jacobi field along an $\varepsilon$-geodesic $u : [a, b] \to \mathcal{H}$, then $L \circ \xi$ is a convex function on $[a, b]$.

This will be derived from a special case.

**Lemma 6.2.** Given $u_0 \in \mathcal{H}$ and $f_0 \in B(X)$, Theorem 6.1 holds for the Lagrangian

$$L(\eta) = \sup_{(f, v) \sim (f_0, u_0)} \int_X f \eta d\mu_v, \quad \eta \in T_v \mathcal{H},$$

cf. (2.1).

To prove Lemma 6.2 we need some preparation. If $Y$ is any set, we say that functions $g, h : Y \to \mathbb{R}$ are similarly ordered if $(g(x) - g(y))(h(x) - h(y)) \geq 0$ for all $x, y \in Y$. Equivalently, $g(x) < g(y)$ should imply $h(x) \leq h(y)$. The relation is not transitive, any function is similarly ordered as a constant.

**Lemma 6.3.** Let $Y$ be a smooth manifold and $a_{ij}$ smooth functions, $V_i$ smooth vector fields on it, $i, j = 1, \ldots, k$. Assume the matrix $(a_{ij})$ is symmetric and positive semidefinite everywhere. If $g \in C^\infty(Y)$ and a locally integrable $h : Y \to \mathbb{R}$ are similarly ordered, then the current $Q = \sum_{i,j} a_{ij}(V_i g)(V_j h) \geq 0$.

**Proof.** Assume first that there is a smooth increasing $H : \mathbb{R} \to \mathbb{R}$ such that $h = H \circ g$. Then $Q = H'(g) \sum a_{ij}(V_i g)(V_j g) \geq 0$. The same follows if $H$ is any increasing function, by writing it as $\lim_{p} \lim_{q} H_{pq}$ (pointwise limit), with locally uniformly bounded smooth increasing $H_{pq}$.

Now consider general $g, h$. Let $I$ denote the range of $g$, and for $t \in I$ define

$$m(t) = \inf\{h(x) : g(x) = t\}, \quad M(t) = \sup\{h(x) : g(x) = t\}.$$

If $g(x) = t < g(y) = \tau$, then $h(x) \leq h(y)$, which means that

$$m(t) \leq M(t) \leq m(\tau) \leq M(\tau) \quad \text{when } t \leq \tau.$$

In particular, $m$ and $M$ are increasing functions, and coincide on $\text{int} T$ wherever one of them is continuous, that is, apart from a countable set $T \subset I$. On $g^{-1}(I \setminus T)$ we have $h = m \circ g$. If $t$ is a regular value of $g$, then $g^{-1}(t)$ has measure 0. Hence on the regular set of $g$ the functions $h$ and $m \circ g$ agree a.e., and the induced currents simply agree there. By what we already proved, $Q \geq 0$ on the set where $dg \neq 0$. We still need to understand what happens on the critical set $C = (dg = 0)$.

Let $\chi : [0, \infty) \to [0, 1]$ be a smooth function, $\chi(t) = 0$ if $t \leq 1$, $\chi(t) = 1$ if $t \geq 2$. Endow $Y$ with a Riemannian metric and denote by $\text{dist}(\cdot, C)$ distance to $C$; this is a
Lipschitz function with Lipschitz constant 1. For $s > 0$, the function $\chi(s \text{ dist}(\cdot, C))$ has Lipschitz constant $O(s)$; it vanishes in the $1/s$ neighborhood of $C$ and equals 1 outside the $2/s$ neighborhood. Let $\rho_s \in C^\infty(Y)$ have the same properties. To prove the lemma we need to show that if $\theta \geq 0$ is a compactly supported smooth volume form on $Y$, then

$$0 \leq \int_Y Q\theta = -\int_Y h \sum_{i,j} \mathcal{L}_j(\theta a_{ij} V_i g),$$

where $\mathcal{L}_j$ stands for Lie derivative along $V_j$.

The inequality holds if $\theta$ is replaced by $\theta \rho_s$, because $Q \geq 0$ in a neighborhood of $\text{supp} \, \theta \rho_s$. The point is that the functions $\mathcal{L}_j(\theta \rho_s a_{ij} V_i g)$ are uniformly bounded and tend to $\mathcal{L}_j(\theta a_{ij} V_i g)$ pointwise as $s \to \infty$. The latter is obvious; the former is verified by applying Leibniz rule to the products, and checking each term. The only term that needs speaking for is $\theta a_{ij}(V_i g)(V_j \rho_s)$. But since $|V_i g| |V_j \rho_s|$ attains its maximum on $\{y \in Y : 1/s \leq \text{dist}(y, C) \leq 2/s\}$, this maximum is $O(1/s)O(s) = O(1)$. Therefore by dominated convergence

$$\int_Y Q\theta = -\lim_{s \to \infty} \int_Y h \sum_{i,j} \mathcal{L}_j(\theta \rho_s a_{ij} V_i g) = \lim_{s \to \infty} \int_Y Q\theta \rho_s \geq 0.$$

**Proof of Lemma 6.2.** The plan is to construct for every $t_0 \in (a, b)$ a family $f(t) \in B(X)$ such that $(f(t), u(t)) \sim (f_0, u_0)$ and $A(t) = \int_X f(t) \xi(t) d\mu_{u(t)} \leq L(\xi(t))$ satisfies

$$A(t_0) = L(\xi(t_0)), \quad A(t_0) \geq 0.$$

To simplify notation we can assume $t_0 = 0$. At the price of replacing $f_0$ by $f_1$ such that $(f_0, u_0) \sim (f_1, u(0))$, we can assume $u(0) = u_0$. Further to simplify we can arrange that $f = f_0$ realizes

$$\sup_{(f,u(0)) \sim (f_0,u(0))} \int_X f(0) d\mu_{u(0)};$$

this is possible simply because the supremum is attained, see e.g., [L4, Lemma 6.2]. The same lemma says that there is a maximizing $f$ that is similarly ordered as $\xi(0)$, and accordingly we will work with $f_0$ similarly ordered as $\xi(0)$.

For a moment suppose $u : [a, b] \to \mathcal{H}$ is an arbitrary smooth path, and parallel transport $T_{u(0)} \mathcal{H} \to T_{u(t)} \mathcal{H}$ along $u$ is given by pull back by a symplectomorphism $\varphi(t) : (X,\omega_{u(t)}) \to (X,\omega_{u(0)})$. Suppose $\eta : [a, b] \to T_{u(0)} \mathcal{H}$ is smooth; then $t \mapsto \eta(t) \circ \varphi(t)$ defines a vector field along $u$. Parallel transport intertwines differentiation and covariant differentiation:

$$\nabla_t (\eta(t) \circ \varphi(t)) = \dot{\eta}(t) \circ \varphi(t) \quad \text{and} \quad \nabla_t^\sharp (\eta(t) \circ \varphi(t)) = \dot{\eta}(t) \circ \varphi(t).$$

When $u$ is an $\varepsilon$–geodesic and $\xi$ an $\varepsilon$–Jacobi field along it, as in the lemma, choose $\eta$ so that $\eta(t) \circ \varphi(t) = \xi(t)$. By (6.2), at $t = 0$,

$$\dot{\eta}(0) \omega^n_{u(0)} = (1/4) \{ \dot{u}(0), \eta(0) \}, \quad \dot{\eta}(0) \omega^n_{u(0)} - \varepsilon nd(F(u(0)) d\ell \eta(0)) \wedge \omega^n_{u(0)}.$$

With $f(t) = f_0 \circ \varphi(t)$ we let

$$A(t) = \int_X f(t) \xi(t) \omega^n_{u(t)} = \int_X f_0 \eta(t) \omega^n_{u(0)},$$

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then \( \bar{A}(t) = \int_X f_0 \eta(t) \omega^n_{u(0)} \). In view of (6.4)
\[
\bar{A}(0) = \frac{1}{4} \int_X f_0 \{\{\dot{u}(0), \eta(0)\}, \dot{u}(0)\} \omega^n_{u(0)} + \varepsilon n \int_X f_0 d(F(u(0)) d\xi(0)) \wedge \omega^{n-1}_{u(0)}
\]
\[
= \frac{1}{4} \int_X \{\dot{u}(0), f_0\} \{\dot{u}(0), \eta(0)\} \omega^n_{u(0)} + \varepsilon n \int_X F(u(0)) d\eta_0 \wedge d\eta(0) \wedge \omega^{n-1}_{u(0)}.
\]
In the last line \( \{\dot{u}(0), f_0\} \) and \( d\eta_0 \) are currents. By Lemma 6.3 the first integrand in this last line is \( \geq 0 \), since \( f_0 \) and \( \eta(0) = \xi(0) \) are similarly ordered; and so is, for the same reason, \( 2n d\eta_0 \wedge d\eta(0) \wedge \omega^{n-1}_{u(0)} = (d\eta_0, d\eta(0))_u(0) \omega^n_{u(0)} \), cf. [B2] p.103.

To summarize, we have shown that for every \( t_0 \in (a, b) \) there is a function \( A \in C^\infty[a, b] \) such that
\[
A(t) \leq L(\xi(t)), \quad \text{with equality when } t = t_0, \quad \text{and } \bar{A}(t_0) \geq 0.
\]
By a standard argument this implies that \( L \circ \xi \) is convex. First one notes that if \( p > 0 \) and \( q \in \mathbb{R} \), the function \( L(\xi(t)) + pt^2 + qt \) cannot have a local maximum at any \( t_0 \in (a, b) \), because with the \( A \) we have constructed \( A(t) + pt^2 + qt \) has no local maximum at \( t_0 \). It follows that on any subinterval \( \alpha, \beta \subset [a, b] \), \( L(\xi(t)) + pt^2 + qt \) attains its maximum at the endpoints, whence \( L(\xi(t)) + pt^2 \) is convex. Letting \( p \to 0 \) we see that \( L \circ \xi \) itself is also convex.

**Proof of Theorem 6.1.** Clearly, Lemma 6.2 implies that if \( a \in \mathbb{R} \), \( g \in B(X) \), and
\[
L_{a,g}(\eta) = a + \sup_{(f,v) \sim (g,u_0)} \int_X f \eta d\mu_v, \quad \eta \in T_v \mathcal{H},
\]
then \( L_{a,g} \circ \xi \) is convex for any \( \varepsilon \)-Jacobi field. Since by Theorem 2.4 a general invariant convex Lagrangian is the supremum of a family of such \( L_{a,g} \), the theorem follows.

### 7 Least action in \( \mathcal{H} \) and \( \mathcal{H}^{1\dagger} \)

In this section unless otherwise indicated the Lagrangian \( L \) is strongly continuous, invariant, and convex on \( T^\infty \mathcal{E}(\omega) \). We will compare the actions along weak geodesics in \( \mathcal{H}^{1\dagger} \) and along general paths in \( \mathcal{H} \).

**Theorem 7.1.** Consider a piecewise \( C^1 \) path \( u : [0, T] \to \mathcal{H} \) and a weak geodesic \( v : [0, T] \to \mathcal{H}^{1\dagger} \). If \( u(0) = v(0) \) and \( u(T) = v(T) \), then
\[
\frac{1}{T} \int_0^T L \circ \dot{u} \geq L(\dot{v}(0)).
\]

First we prove a variant.

**Lemma 7.2.** Suppose an invariant convex Lagrangian \( L : T\mathcal{H} \to \mathbb{R} \) is positively homogeneous, \( L(c\xi) = cL(\xi) \) if \( c > 0 \). Consider a triangle in \( \mathcal{H} \) formed by a piecewise \( C^1 \) path \( u : [a, b] \to \mathcal{H} \) and \( \varepsilon \)-geodesics \( v_a, v_b : [0, T] \to \mathcal{H} \); so that \( v_a(0) = v_b(0) \) and \( v_a(T) = u(a), v_b(T) = u(b) \). Then
\[
\frac{1}{T} \int_a^b L \circ \dot{u} \geq L(\dot{v}_b(0)) - L(\dot{v}_a(0)).
\]
Note that positive homogeneity implies the triangle inequality $L(\xi+\eta) \leq L(\xi)+L(\eta)$ for $w \in \mathcal{H}$ and $\xi, \eta \in T_w \mathcal{H}$.

**Proof.** Because of the additive nature of (7.2), we can assume $u$ is $C^1$, not only piecewise, and then by simple approximation that it is even $C^\infty$. For each $s \in [a, b]$ let $U(s, \cdot) : [0, T] \to \mathcal{H}$ denote the $\varepsilon$-geodesic connecting $v_a(0) = v_b(0)$ with $u(s)$. According to Theorem 4.2, that is, by Chen’s work, there is a unique such geodesic, and $U \in C^\infty([a, b] \times [0, T])$. Thus $\xi^s = \partial_s U(s, \cdot)$ is an $\varepsilon$-Jacobi field and $\xi^s(0) = 0$. By Theorem 6.1 $L \circ \xi^s$ is convex on $[0, T]$. Using $\partial_t$ (and later, dot) to denote right derivative, therefore

$$L(\xi^s(T)) \geq L(\xi^s(0)) + T\partial_t|_{t=0}L(\xi^s(t)).$$

By homogeneity, the first term on the right is 0. To compute the second, let $u(t) \in T_{U(0,0)} \mathcal{H}$ denote the parallel translate of $\xi^s(t) \in T_{U(s,t)} \mathcal{H}$ along $U(s, \cdot)$. Thus

$$\lim_{t \to 0} \frac{1}{T} \int_a^b L(\partial_s u(s))ds = \frac{1}{T} \int_a^b L(\xi^s(T))ds \geq \int_a^b L(\partial_s \partial_t|_{t=0}U(s,t))ds$$

$$\geq L\left(\int_a^b \partial_s \partial_t|_{t=0}U(s,t)ds\right) = L(\dot{v}_b(0) - \dot{v}_a(0)) \geq L(\dot{v}_b(0)) - L(\dot{v}_a(0)),$$

as claimed.

**Proof of Theorem 7.1.** For $\varepsilon > 0$ let $v^\varepsilon : [0, T] \to \mathcal{H}$ denote the $\varepsilon$-geodesic connecting $u(0)$ and $u(T)$. Again by Chen [C], see also Błocki [B11, B12], $v^\varepsilon \to v$ in such a way that $\dot{v}^\varepsilon(0) \to \dot{v}(0)$ in $T_{v(0)}^{\infty} \mathcal{E}(\omega)$ as $\varepsilon \to 0$. Suppose first that $L$ is positively homogeneous, and apply Lemma 7.2 with $[a, b] = [0, T]$, $v_a \equiv u(0)$, $v_b = v^\varepsilon$. We obtain

$$\frac{1}{T} \int_0^T L \circ \dot{u} \geq L(\dot{v}(0)).$$

Hence (7.1) follows by letting $\varepsilon \to 0$.

It is clear that (7.1) follows even if $L$ is not positively homogeneous but $L$ plus a constant is. Since a general $L$ is the supremum of Lagrangians of form positively homogeneous plus constant, see Theorem 2.4, (7.1) holds in complete generality.

**Lemma 7.3.** If $v : [0, T] \to \mathcal{H}^{11}$ is a weak geodesic, then $L \circ \dot{v}$ is constant. Hence

$$L(\dot{v}(0)) = \frac{1}{T} \int_0^T L \circ \dot{v}.$$

This can be seen as an instance of Noether’s theorem on conserved quantities, albeit in an unusual setting.
Proof. Berndtsson [Be2, Proposition 2.2] discovered that $\dot{v}(t) \in B(X, \mu_v(t))$ are equidistributed for all $t$, although he worked with integral Kähler classes [ω] only. At any rate, [D2, Lemma 4.6] implies the general result. Since $L$ is invariant, the lemma follows.

Together with Theorem 7.1 this almost proves the principle of least action in $\mathcal{H}^{1\bar{1}}$:

Corollary 7.4. If $u : [0, T] \rightarrow \mathcal{H}$ is a piecewise $C^1$ path and $v : [0, T] \rightarrow \mathcal{H}^{1\bar{1}}$ is a weak geodesic between the same endpoints, then $\int_0^T L \circ \dot{u} \geq \int_0^T L \circ \dot{v}$.

8 Least action in $B(X) \cap \text{PSH}(\omega)$

Here we will extend Corollary 7.4 to $u, v$ taking values in $B(X) \cap \text{PSH}(\omega)$ (Theorem 5.2). In this section $L : T^\infty \mathcal{E}(\omega) \rightarrow \mathbb{R}$ is assumed to be strongly continuous, invariant, and convex.

Theorem 8.1. Suppose $u, v : [0, T] \rightarrow B(X) \cap \text{PSH}(\omega)$ have the same endpoints: $u(0) = v(0)$, $u(T) = v(T)$. If $u$ is piecewise $C^1$ and $v$ is a $C^1$ weak geodesic, then $\int_0^T L \circ \dot{u} \geq \int_0^T L \circ \dot{v}$.

This will be derived from Corollary 7.4 by approximation.

Lemma 8.2. Suppose $u : [0, T] \rightarrow B(X) \cap \text{PSH}(\omega)$ is a piecewise $C^1$ path, and $w_j, w_j' \in \mathcal{H}$ decrease to $u(0)$, respectively, $u(T)$, as $j \rightarrow \infty$. Then there are a sequence $J \subset \mathbb{N}$ and for $j \in J$ piecewise linear $u_j : [0, T] \rightarrow \mathcal{H}$ such that $u_j(0) = w_j$, $u_j(T) = w'_j$, and $\int_0^T L \circ \dot{u}_j \rightarrow \int_0^T L \circ \dot{u}$ as $J \ni j \rightarrow \infty$.

As said, at points where $u, u_j$ are not differentiable, $\dot{u}, \dot{u}_j$ mean right derivatives.

Proof. Choose $t_0 = 0 < t_1 < \cdots < t_p = T$ so that $u$ is $C^1$ on each $[t_{i-1}, t_i]$. Suppose that $u$ is even linear on $[t_{i-1}, t_i]$. In this case $J$ will be all of $\mathbb{N}$. A simple special case of regularization, see [Dal, DP] and especially [LK], provides $z_{ij} \in \mathcal{H}$ such that $z_{ij}$ decreases to $u(t_i)$ as $j \rightarrow \infty$ for $i = 0, \ldots, p$. We take $z_0 = w_j$, and $z_{pj} = w'_j$, and arrange that the $z_{ij}$ are uniformly bounded. Linearly interpolating on $[t_{i-1}, t_i]$ between $z_{i-1,j}$ and $z_{ij}$ we obtain the functions $u_j$ sought. Indeed, $u_j(t)$ decreases to $u(t)$, and

$$ \dot{u}_j(t) = \frac{z_{ij} - z_{i-1,j}}{t_i - t_{i-1}} \in T_{u_j(t)} \mathcal{H}, \quad \text{when } t \in [t_{i-1}, t_i], $$

are uniformly bounded and tend to $\dot{u}(t)$ as $j \rightarrow \infty$. Since $\omega$–plurisubharmonic functions are quasicontinuous [GZ2 Corollary 9.12], so are the difference quotients $\dot{u}(t)$. According to [GZ2 Proposition 9.11] this implies convergence in capacity, and so $\lim_j L(\dot{u}_j(t)) = L(\dot{u}(t))$ by Lemma 3.4. Since $\dot{u}_j(t)$ are uniformly bounded, so are $L(\dot{u}_j(t))$ by equi–Lipschitz continuity, Lemma 3.3. The dominated convergence theorem gives therefore $\lim_j \int_0^T L \circ \dot{u}_j = \int_0^T L \circ \dot{u}$.

For general $u$, partition each $[t_{i-1}, t_i]$ into $k$ equal parts. Construct $v_k : [0, T] \rightarrow B(X) \cap \text{PSH}(\omega)$ that agrees with $u$ at each partition point, and is linear in between. Then $v_k \rightarrow u$ and $\dot{v}_k \rightarrow \dot{u}$ uniformly, because $\dot{u}$ is uniformly continuous on $[t_{i-1}, t_i]$. Hence $L \circ \dot{v}_k \rightarrow L \circ \dot{u}$ by Lemma 3.4 and, again by dominated convergence, $\int_0^T L \circ \dot{v}_k$
\[ \int_0^T L \circ \dot{u}. \] By what we have already proved, for each \( k \) we can find \( j = j_k > j_{k-1} \) and piecewise linear \( u_j : [0, T] \to \mathcal{H} \) such that \( u_j(0) = w_j, u_j(T) = w'_j, \) and
\[
\left| \int_0^T L \circ \dot{u}_j - \int_0^T L \circ \dot{v}_k \right| < \frac{1}{k}.
\]
Thus \( J = \{ j_1, j_2, \ldots \} \) will do.

**Lemma 8.3.** Let \( v, v_j : [a, b] \to \text{PSH}(\omega) \) be weak geodesics. If \( v_j(t) \) decreases to \( v(t) \) when \( t = a, b, \) then \( v_j(t) \) decreases to \( v(t) \) for all \( t \in [a, b] \).

This is \( [D4, \text{Proposition 3.15}] \).—There is one more ingredient that goes into the proof of Theorem 8.1.

**Lemma 8.4.** Consider a weak geodesic \( v : [0, T] \to B(X) \cap \text{PSH}(\omega) \). If it is right differentiable at \( t \in [0, T) \), then the right derivative \( \dot{v}(t) \) is quasicontinuous. Moreover, \( L \circ \dot{v} \) is constant on the subset \( D \subset (0, T) \) where \( v \) is differentiable. Finally, if \( v_j : [0, T] \to H^{11} \) are weak geodesics that decrease to \( v \), then \( L \circ \dot{v}_j \to L \circ \dot{v} \) on \( D \).

**Proof.** As said, plurisubharmonic functions are quasicontinuous, hence so are the difference quotients \( (v(t + s) - v(t))/s \), and their uniform limit, \( \dot{v}(t) \). Next we turn to the last statement, that we reduce to Lemma 3.4. First we show that \( \dot{\dot{v}}_j(t) \to \dot{\dot{v}}(t) \) in \( B(X) \) if \( t \in D \). Let \( t \in D \) and \( \varepsilon > 0 \). There is an \( s > 0 \) such that
\[
\left\| \dot{v}(t) - \frac{v(t \pm s) - v(t)}{\pm s} \right\| < \varepsilon,
\]
and so there is a \( j_0 \) such that for \( j > j_0 \)
\[
\left\| \dot{v}(t) - \frac{v_j(t \pm s) - v_j(t)}{\pm s} \right\| < \varepsilon.
\]
Convexity implies
\[
\frac{v_j(t - s) - v_j(t)}{-s} \leq \dot{v}_j(t) \leq \frac{v_j(t + s) - v_j(t)}{s},
\]
whence \( \|\dot{v}_j(t) - \dot{v}(t)\| < \varepsilon \).

Given that \( v_j(t) \) decreases to \( v(t) \), that \( \dot{v}_j(t) \to \dot{v}(t) \) in \( B(X) \), and that \( \dot{v}(t) \) is quasicontinuous, \( t \in D \), Lemma 3.4 implies \( L \circ \dot{v}_j \to L \circ \dot{v} \) on \( D \).

To prove the second statement, construct \( w_j, w'_j \in \mathcal{H} \) that decrease to \( v(0), v(T) \), and let \( v_j : [0, T] \to H^{11} \) be the weak geodesic that joins them. By Lemma 8.3 \( v_j \) decreases to \( v \) and by Lemma 7.3 \( L \circ \dot{v}_j \) is constant. According to what we just proved, \( L \circ \dot{v}_j \to L \circ \dot{v} \) on \( D \), and \( L \circ \dot{v} \) must be constant there.

In particular, if \( v : [a, b] \to B(X) \cap \text{PSH}(\omega) \) is a weak geodesic of class \( C^1 \), then \( L \circ \dot{v} \) is constant on \( (a, b) \). Using this with different Lagrangians one can show that in fact \( \dot{v}(t) \in B(X, \mu_{v(t)}) \) are equidistributed for \( a < t < b \). Darvas points out in \( [D3, \text{p. 1305}] \) that for general weak geodesics even in \( C(X) \cap \text{PSH}(\omega) \) this is no longer true for \( t = a, b \). The big question is whether it is true for \( a < t < b \) and a general weak
geodesic in $\mathcal{H}$ that the left and right derivatives $\lim_{s \to 0^\pm} (v(t + s)(x) - v(t)(x))/s$, computed pointwise on $X$, are equidistributed. If so, at least for almost every $t$, the Principle of least action could be extended to include all weak geodesics in $\mathcal{H}$.

Proof of Theorem 8.1. Construct $w_j, w'_j \in \mathcal{H}$ decreasing to $u(0), u(T)$, and let $u_j : [0, T] \to \mathcal{H}$, $j \in J$, be as in Lemma 8.2. Let $v_j : [0, T] \to \mathcal{H}$ be the weak geodesic connecting $w_j$ and $w'_j$, $j \in J$. By Corollary 7.4

\[
\int_0^T L \circ \dot{u}_j \geq \int_0^T L \circ \dot{v}_j. \tag{8.1}
\]

The integral on the left tends to $\int_0^T L \circ \dot{u}$ as $j \to \infty$. The integrand on the right is constant for each $j$, and on $(0, T)$ converges uniformly to $L \circ \dot{v}$ by Lemma 8.4. Hence $\lim_{j \to \infty} \int_0^T L \circ \dot{v}_j = \int_0^T L \circ \dot{v}$ and letting $j \to \infty$ in (8.1) we obtain the theorem.

9 Convexity of the action

In this section the Lagrangian $L : T^\infty \mathcal{E}_r(\omega) \to \mathbb{R}$ is strongly continuous, invariant, and convex. We first investigate the least action, cf. (5.3), between two $\varepsilon$-geodesics, and then by letting $\varepsilon \to 0$ we prove Theorem 5.3, which was:

Theorem 9.1. If $u, v : [a, b] \to B(X) \cap \text{PSH}(\omega)$ are weak geodesics, then for any $S \in (0, \infty)$ the function $\mathcal{L}_S(u, v) : [a, b] \to \mathbb{R}$ is convex.

The $\varepsilon$-variant is as follows:

Lemma 9.2. If $u, v : [a, b] \to \mathcal{H}$ are $\varepsilon$-geodesics, then for any $S \in (0, \infty)$ the function $\mathcal{L}_S(u, v) : [a, b] \to \mathbb{R}$ is convex.

Proof. Let $a \leq \alpha < \beta \leq b$. Suppose $U : [0, S] \times [\alpha, \beta] \to \mathcal{H}$ is a smooth map such that $U(s, \cdot)$ is an $\varepsilon$-geodesic for all $s$, and $U(0, \cdot) = u$, $U(S, \cdot) = v$. Hence $\xi^s = \partial_s U(s, \cdot)$ is an $\varepsilon$-Jacobi field, $0 \leq s \leq S$, and by Theorem 6.1 $L \circ \xi^s$ is convex. Therefore, with $0 \leq \lambda \leq 1$ and $t_\lambda = (1 - \lambda)\alpha + \lambda \beta$

\[
\mathcal{L}(U(\cdot, t_\lambda)) = \int_0^S L(\xi^s(t_\lambda)) ds \leq (1 - \lambda) \int_0^S L(\xi^s(\alpha)) ds + \lambda \int_0^S L(\xi^s(\beta)) ds. \tag{9.1}
\]

Fix $\delta > 0$. Given $u, v$, we can choose $U$ (uniquely) so that both $w^\delta_\alpha = U(\cdot, \alpha)$ and $w^\delta_\beta = U(\cdot, \beta)$ are $\delta$-geodesics. From (9.1)

\[
\mathcal{L}(u(t_\lambda), v(t_\lambda)) \leq \mathcal{L}(U(\cdot, t_\lambda)) \leq (1 - \lambda) \int_0^S L \circ \dot{w}^\delta_\alpha + \lambda \int_0^S L \circ \dot{w}^\delta_\beta. \tag{9.2}
\]

Now $\lim_{\delta \to 0} w^\delta_\alpha = w_\alpha$ and $\lim_{\delta \to 0} w^\delta_\beta = w_\beta$ are the weak geodesics in $\mathcal{H}$ connecting $u(\alpha), v(\alpha)$, respectively, $u(\beta), v(\beta)$; and, as explained in section 4,

\[
w^\delta_\alpha \to w_\alpha, \quad w^\delta_\beta \to w_\beta, \quad \dot{w}^\delta_\alpha \to \dot{w}_\alpha, \quad \dot{w}^\delta_\beta \to \dot{w}_\beta
\]
uniformly as $\delta \to 0$. Thus by Lemma 3.4 and Theorem 8.1
\[
\lim_{\delta \to 0} \int_0^S L \circ \dot{w}_\delta^\delta = \int_0^S L \circ \dot{w}_\alpha = \mathcal{L}(u(\alpha), v(\alpha)),
\]
and similarly for the other integral in (9.2). Hence letting $\delta \to 0$ in (9.2) gives
\[
\mathcal{L}(u(t_\lambda), v(t_\lambda)) \leq (1 - \lambda)\mathcal{L}(u(\alpha), v(\alpha)) + \lambda\mathcal{L}(u(\beta), v(\beta)),
\]
what was to be proved.

**Lemma 9.3.** If $w, w' \in B(X) \cap \text{PSH}(\omega)$ and $T > 0$, then $\mathcal{L}_T(w, w')$ is finite. If $w_j, w'_j \in C(X) \cap \text{PSH}(\omega)$ decrease, or converge uniformly, to $w$, resp. $w'$, then

\[
\mathcal{L}_T(w_j, w'_j) \to \mathcal{L}_T(w, w') \quad \text{as } j \to \infty.
\]

We do not know if (9.3) holds when $w_j, w'_j \in B(X) \cap \text{PSH}(\omega)$.

**Proof.** We will prove for decreasing sequences $w_j, w'_j$; the case of uniformly convergent sequences can be reduced to decreasing sequences in a standard way, as in Lemma 3.4.

Invagination implies that $L$ is constant on the zero section of $T^\infty \mathcal{E}(\omega)$. Since adding a constant to $L$ will not affect the validity of the lemma, we will assume $L$ vanishes on the zero section. Let us start with (9.3). It suffices to prove it along a subsequence $j = j_k$.

Assume first that $w_j, w'_j \in \mathcal{H}$. Let $u : [0, T] \to B(X) \cap \text{PSH}(\omega)$ be piecewise $C^1$ connecting $w$ and $w'$. At the price of passing to a subsequence, by Lemma 8.2 there are $u_j : [0, T] \to \mathcal{H}$ piecewise $C^1$ such that $u_j(0) = w_j$, $u_j(T) = w'_j$, and
\[
\int_0^T L \circ \dot{u}_j \to \int_0^T L \circ \dot{u}.
\]

Therefore
\[
\mathcal{L}(u) = \lim_{j \to \infty} \mathcal{L}(u_j) \geq \limsup_{j \to \infty} \mathcal{L}_T(w_j, w'_j).
\]

Passing to the infimum over all paths $u$ connecting $w, w'$,

\[
\mathcal{L}_T(w, w') \geq \limsup_{j \to \infty} \mathcal{L}_T(w_j, w'_j).
\]

Let $v_j : [0, T] \to \mathcal{H}^{12}$ be the weak geodesics connecting $w_j$ and $w'_j$.

We take a pause in the proof of (9.3) and show how (9.4) implies $\mathcal{L}_T(w, w') > -\infty$.

Fix numbers $m, M$ so that for all $j$

\[
m \leq \inf_X w_j, \inf_X w'_j, \quad M \geq \sup_X w_j, \sup_X w'_j.
\]

By convexity $\dot{v}_j(0) \leq (v_j(T) - v_j(0))/T \leq (M - m)/T$. Furthermore, $u(t) = w_j + (m - M)t/T$ is a subgeodesic, $u(0) = w_j$, $u(T) \leq w'_j$. Hence $u(t) \leq v_j(t)$ for all $t$, and

\[
\dot{v}_j(0) \geq \lim_{t \to 0} \frac{u(t) - v(0)}{t} \geq \frac{m - M}{T}.
\]

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Thus \( \|\dot{v}_j(0)\| \leq (M - m)/T \). Since \( L \) is equi–Lipschitz on bounded subsets of the fibers (Lemma 3.3), using Lemma 7.3 as well, \( \mathcal{L}_T(w_j, w'_j) = \mathcal{L}(v_j) = TL(\dot{v}_j(0)) \) is a bounded sequence, and (9.4) implies \( \mathcal{L}_T(w, w') > -\infty \).

We return to the proof of (9.3); we need to estimate \( \mathcal{L}_T(w, w') \) from above. For fixed \( \delta > 0 \) there are infinitely many \( k \) with

\[
(9.5) \quad \liminf_{j \to \infty} \mathcal{L}_T(w_j, w'_j) \geq \mathcal{L}_T(w_k, w'_k) - \delta = TL(\dot{v}_k(0)) - \delta.
\]

If \( 0 < \varepsilon < T/2 \), define

\[
v_j^\varepsilon(t) = \begin{cases} tw_j/\varepsilon + (\varepsilon - t)w/\varepsilon & \text{if } 0 \leq t < \varepsilon \\ v_j \left( t - \frac{\varepsilon}{T - 2\varepsilon} \right) & \text{if } \varepsilon \leq t < T - \varepsilon \\ (T - t)w_j'/\varepsilon + (t + \varepsilon - T)w'/\varepsilon & \text{if } T - \varepsilon \leq t \leq T.
\end{cases}
\]

The piecewise \( C^1 \) paths \( v_j^\varepsilon : [0, T] \to B(X) \cap \text{PSH}(\omega) \) connect \( w \) and \( w' \), hence

\[
(9.6) \quad \mathcal{L}_T(w, w') \leq \mathcal{L}(v_j^\varepsilon) = \left( \int_0^\varepsilon + \int_{\varepsilon}^{T - \varepsilon} + \int_{T - \varepsilon}^T \right) L \circ \dot{v}_j^\varepsilon.
\]

The middle integral on the right is

\[
\int_{\varepsilon}^{T - \varepsilon} L \circ \dot{v}_j^\varepsilon = (T - 2\varepsilon)L(\frac{T\dot{v}_j(0)}{T - 2\varepsilon}).
\]

As we saw, the \( \dot{v}_j(0) \) are uniformly bounded. By the equi–Lipschitz property of \( L \) an \( \varepsilon \in (0, 1) \) can be chosen so that for all \( j \)

\[
(9.7) \quad \int_{\varepsilon}^{T - \varepsilon} L \circ \dot{v}_j^\varepsilon \leq TL(\dot{v}_j(0)) + \delta.
\]

When \( 0 \leq t \leq \varepsilon \), we have \( \dot{v}_j^\varepsilon(t) = (w_j - w)/\varepsilon \in T_{v_j^\varepsilon(t)}^\infty \mathcal{E}(\omega) \). Again by the equi–Lipschitz property, if \( j \) is sufficiently large, \( |L(\dot{v}_j^\varepsilon(t))| < \delta \); and similarly for \( T - \varepsilon \leq t < T \). Putting this and (9.6), (9.7) together,

\[
\mathcal{L}_T(w, w') \leq 3\delta + TL(\dot{v}_j(0))
\]

if \( j \) is sufficiently large. Choosing \( j \) from among the \( k \) in (9.5) therefore yields

\[
\mathcal{L}_T(w, w') \leq 4\delta + \liminf_{j \to \infty} \mathcal{L}_T(w_j, w'_j).
\]

This being true for all \( \delta > 0 \), (9.3) follows in view of (9.4).

So far we dealt with \( w_j, w'_j \in \mathcal{H} \). If \( w_j, w'_j \in C(X) \cap \text{PSH}(\omega) \) only, upon adding constants to them we can arrange that \( w_j < w_{j-1} \) and \( w'_j < w'_{j-1} \) everywhere. We will express this by saying that \( w_j, w'_j \) strictly decrease. We construct recursively \( z_j > w_j, z'_j > w'_j \) in \( \mathcal{H} \) that strictly decrease to \( w, w' \) and satisfy \( |\mathcal{L}_T(z_j, z'_j) - \mathcal{L}_T(w_j, w'_j)| < 1/j \) as follows. Suppose we already have \( z_{j-1}, z'_{j-1} \). Construct sequences \( y_i < z_{j-1}, y'_i < z'_{j-1} \) (\( i \in \mathbb{N} \) in \( \mathcal{H} \) that decrease to \( w_j, w'_j \). By what we have already proved, \( |\mathcal{L}_T(y_i, y'_i) - \mathcal{L}_T(w_j, w'_j)| < 1/j \) for some \( i \), and we let \( z_j = y_i, z'_j = y'_i \) with that \( i \). Thus

\[
\mathcal{L}_T(w, w') = \lim_j \mathcal{L}_T(z_j, z'_j) = \lim_j \mathcal{L}_T(w_j, w'_j),
\]

as claimed.
Proof of Theorem 9.1. Assume first that $u, v$ are weak geodesics in $H^1$ with endpoints in $H$, and connect $u(a), u(b)$, respectively, $v(a), v(b)$ by $\varepsilon$–geodesics $u^\varepsilon, v^\varepsilon$. By Chen’s theorem $u^\varepsilon \to u$ and $v^\varepsilon \to v$ uniformly as $\varepsilon \to 0$. Hence by Lemma 9.3, $L_S(u^\varepsilon, v^\varepsilon) \to L_S(u, v)$, and so the latter, as the limit of convex functions (Lemma 9.2) is itself convex.

Second, consider general $u, v$. Choose $w_j, w'_j \in H$ decreasing to $u(a), u(b)$ and $z_j, z'_j \in H$ decreasing to $v(a), v(b)$. Join $w_j, w'_j$ by weak geodesics $u_j : [a, b] \to H^1$ and $z_j, z'_j$ by weak geodesics $v_j : [a, b] \to H^1$. By Lemma 8.3 $u_j, v_j$ decrease to $u, v$, hence by Lemma 9.3 the convex functions $L_S(u_j, v_j)$ converge to $L_S(u, v)$. It follows that the latter is also convex.

References

[An] Erik Andersén, Volume–preserving automorphisms of $\mathbb{C}^n$, Complex Variables Theory Appl. 14 (1990) 223–235.

[AG] Christopher Atkin, Janusz Grabowski, Homomorphisms of the Lie algebras associated with a symplectic manifold, Compositio Math. 76 (1990) 315–349.

[Au] Thierry Aubin, Réduction du cas positif de l’équation de Monge–Ampère sur les variétés kählériennes compactes à la démonstration d’une inégalité, J. Funct. Anal. 57 (1984) 143–153.

[BT] Eric Bedford, Alan Taylor, A new capacity for plurisubharmonic functions. Acta Math. 149 (1982) 1–40.

[BD] Robert Berman, Jean–Pierre Demailly, Regularity of plurisubharmonic envelopes in big cohomology classes. Perspectives in analysis, geometry, and topology, 39–66, Progr. Math. 296, Birkhäuser/Springer, New York, 2012.

[Be1] Bo Berndtsson, A Brunn–Minkowski type inequality for Fano manifolds and some uniqueness theorems in Kähler geometry, Invent. Math. 200 (2015) 149–200.

[Be2] Bo Berndtsson, Probability measures associated to geodesics in the space of Kähler metrics. Algebraic and analytic microlocal analysis, 395–419, Springer, Cham, 2018.

[BI1] Zbigniew Błocki, On geodesics in the space of Kähler metrics, Advances in geometric analysis, 3–19, Int. Press, Somerville, MA, 2012.

[BI2] Zbigniew Błocki, The complex Monge–Ampère equation in Kähler geometry. Pluripotential theory, 95–41, Lecture Notes in Math. 2075, Springer, Heidelberg, 2013.

[BK] Zbigniew Błocki, Sławomir Kołodziej, On regularization of plurisubharmonic functions on manifolds, Proc. Amer. Math. Soc. 135 (2007) 2089–2093.

[CC] Eugenio Calabi, Xiuxiong Chen, The space of Kähler metrics II, J. Diff. Geom. 61 (2002) 173–193.
[C] Xiuxiong Chen, The space of Kähler metrics, J. Diff. Geom. 56 (2000) 189–234.

[CFH] Xiuxiong Chen, Mikhail Feldman, Jingchen Hu, Geodesic convexity of small neighborhood in the space of Kähler potentials, J. Funct. Anal. 279 (2020)

[CTW] Jianchun Chu, Valentino Tosatti, Ben Weinkove, $C^{1,1}$ regularity of geodesics in the space of Kähler metrics, Ann. PDE 3 (2017) paper No. 15.

[D1] Tamás Darvas, Morse theory and geodesics in the space of Kähler metrics, Proc. Amer. Math. Soc. 142 (2014) 2775–2782.

[D2] Tamás Darvas, The Mabuchi geometry of finite energy classes, Adv. Math. 285 (2015) 182–219.

[D3] Tamás Darvas, The Mabuchi completion of the space of Kähler potentials, Amer. J. Math. 139 (2017) 1275–1313.

[D4] Tamás Darvas, Geometric pluripotential theory on Kähler manifolds. Advances in complex geometry, 1–104, Contemp. Math. 735, Amer. Math. Soc., Providence, RI, 2019.

[DL] Tamás Darvas, László Lempert, Weak geodesics in the space of Kähler metrics, Math. Res. Lett. 19 (2012) 1127–1135.

[De] Jean-Pierre Demailly, Regularization of closed positive currents and intersection theory, J. Alg. Geom. 1 (1992) 361–409.

[DP] Jean-Pierre Demailly, Mihai Păun, Numerical characterization of the Kähler cone of a compact Kähler manifold, Ann. of Math. (2) 159 (2004) 1247–1274.

[Do] Simon Donaldson, Symmetric spaces, Kähler geometry and Hamiltonian dynamics. Northern California Symplectic Geometry Seminar 13–33, Amer. Math. Soc. Transl. Ser. 2, 196, Amer, Math Soc. Providence, RI, 1999.

[GZ1] Vincent Guedj, Ahmed Zeriahi, The weighted Monge–Ampère energy of quasiplusharisharmonic functions, J. Funct. Anal. 250 (2007) 442–482.

[GZ2] Vincent Guedj, Ahmed Zeriahi, Degenerate complex Monge–Ampère equations, EMS Tracts in Mathematics, 26. European Mathematical Society, Zürich, 2017.

[HNW] Ernst Hairer, Syvert Nørsett, Gerhard Wanner, Solving ordinary differential equations. I. Nonstiff problems, Springer–Verlag, Berlin, 1987.

[He] Weiyong He, On the space of Kähler potentials, Comm. Pure Appl. Math. 68 (2015) 332–343.

[Hu] Jingchen Hu, An obstacle for higher regularity of geodesics in the space of Kähler potentials, Internat. Math. Research Notices, to appear; arXiv:1808.03498

[K] Sławomir Kołodziej, The complex Monge-Ampère equation. Acta Math. 180 (1998) 69–117.
[L1] László Lempert, Riemannian geometry in infinite dimensional spaces. arXiv: 1702.04896

[L2] László Lempert, Isometries in spaces of Kähler potentials, Ann. Polon. Math. 123 (2019) 423–458.

[L3] László Lempert, On complex Legendre duality, J. Geom. Anal 30 (2020) 2581–2592.

[L4] László Lempert, On the adjoint action of the group of symplectic diffeomorphisms, arXiv:2009.06729.

[LV] László Lempert, Liz Vivas, Geodesics in the space of Kähler metrics, Duke Math. J. 162 (2013) 1369–1381.

[M] Toshiki Mabuchi, Some symplectic geometry on compact Kähler manifolds I, Osaka J. Math. 24 (1987) 227–252.

[S] Stephen Semmes, Complex Monge–Ampère and symplectic manifolds, Amer. J. Math. 114 (1992) 495–550.