MODEL-BUILDING FOR FRACTIONAL SUPERSTRINGS

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ABSTRACT

We investigate heterotic string-type model-building for the recently-proposed fractional superstring theories. We concentrate on the cases with critical spacetime dimensions four and six, and find that a correspondence can be drawn between the new fractional superstring models and a special subset of the traditional heterotic string models. This allows us to generate the partition functions of the new models, and demonstrate that their number is indeed relatively limited. It also appears that these strings have uniquely natural compactifications to lower dimensions. In particular, the $D_c = 6$ fractional superstring has a natural interpretation in four-dimensional spacetime.

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1. INTRODUCTION

It is generally accepted that string theory is a realistic hope of unifying all forces and matter, and provides an attractive solution to the problem of reconciling quantum mechanics and general relativity. To date, the only known consistent string theories are the superstring theory and the closely-related heterotic string theory, yet one of the significant problems involved in superstring model-building in four spacetime dimensions is the existence of a multitude of classical solutions which these string theories allow. This lack of uniqueness arises because these theories have critical dimension $D = 10$; in order to have a sensible interpretation in four spacetime dimensions, one must therefore choose a compactification scheme for these six extra dimensions or represent these extra degrees of freedom in terms of arbitrarily chosen additional worldsheet fields. There are many ways in which this can be done, resulting in many millions of possible classical vacua. It is presumed that some dynamical mechanism or symmetry argument might be used to select the vacuum corresponding to the present-day physical world, but at present such approaches have not been successful.

Recently, however, a new approach to superstring model-building has been proposed. Rather than work within the superstring/heterotic framework, the fundamental idea is to construct a new type of string theory called the fractional superstring, in which four can appear as the critical spacetime dimension directly. Such fractional superstrings only serve as a more natural starting-point for...
but hopefully also lead to a smaller and thereby more compelling set of self-consistent classical vacua. The basic idea behind the fractional superstring is to replace the traditional right-moving worldsheet supersymmetry found in heterotic strings with a right-moving worldsheet fractional supersymmetry parametrized by an integer $K_R \geq 2$. Such a fractional supersymmetry relates worldsheet bosons not to fermions but rather to worldsheet parafermions, and one finds that the corresponding critical dimension of such a string theory is

$$D = \begin{cases} 
2 + \frac{16}{K_R} & \text{for } K_R \geq 2 \text{ (fractional supersymmetry)}, \\
26 & \text{for } K_R = 1 \text{ (no supersymmetry)}.
\end{cases}$$

The case $K_R = 1$ ($D = 26$) corresponds to the traditional bosonic string, and $K_R = 2$ ($D = 10$) corresponds to the traditional super- or heterotic string. The new theories are those for which $K_R > 2$, and we see that by choosing the cases $K_R = 8$ ($D = 4$) or $K_R = 4$ ($D = 6$), we can obtain significantly lower critical dimensions. For these cases we can therefore expect a smaller set of classical vacua and hopefully a more natural description of the physical world.

These fractional superstring theories are indeed natural extensions of the $K_R = 2$ superstring theories, and it is straightforward to relate the two in terms of their underlying worldsheet physics. For the traditional superstring theory, it is well-known that the underlying worldsheet structure is closely related to the $SU(2)_2$ Wess-Zumino-Witten (WZW) theory, the worldsheet superpartner of the spacetime coordinate $X^\mu$ is a Majorana fermion $\psi^\mu$, and this fermionic theory can be simply described by the WZW coset $SU(2)_2/U(1)$. Setting the remaining $U(1)$ boson to an appropriately chosen radius reproduces the full $SU(2)_2$ WZW theory, yet if we relax the radius, we can interpret this boson as the spacetime coordinate, and this decompactification procedure destroys the superconformal symmetry of the model, but its superconformal symmetry survives. The fractional superstring theory has a similar structure, and is related in precisely the same way to the parafermion theories for $K \geq 2$. The general coset theories $SU(2)_{K_R}/U(1)$ are indeed natural extensions of the parafermion theories, and we once again obtain the spacetime coordinate field $X^\mu$ by decompactifying the remaining $U(1)$ boson. The usual supercurrent for $K > 2$ is a new current whose conformal dimension (or equivalently, spin) is $(K + 4)/(K + 2)$; these new currents have fractional spin, and (as we shall see) transform $X^\mu$ to the fractional-spin field $\tilde{\epsilon}^\mu$, the energy operator in the parafermion theory. It is therefore natural to refer to this remaining worldsheet symmetry as a fractional superconformal symmetry, and to the strings based on these worldsheet symmetries as fractional superstrings. Because the corresponding fractional superconformal supercurrent is non-local on the worldsheet, the analysis for such a string theory is substantially more involved than for the simpler, local case, but as we shall see, concrete progress can indeed be made.

In this paper we shall assume the underlying consistency of the fractional superstring and concern ourselves primarily with the space of models that such fractional string theories allow; other important issues such as understanding the underlying Fock-space structure...
scattering amplitudes, examining the ghost system and developing a no-ghost theorem, are actively being pursued. Our approach, therefore, is to examine this space of models by studying the allowed partition functions that such models might have; in this way we are able to obtain a number of interesting results.

First, we demonstrate that the fractional superstring partition functions are straightforward generalizations of the traditional \((K_L, K_R) = (2, 2)\) superstring and \((1, 2)\) heterotic string partition functions, and we explicitly develop a general procedure for constructing modular-invariant partition functions for our new models which are consistent with \(N = 1\) spacetime supersymmetry.

Second, and perhaps more importantly, by focusing much of our attention on the \((K_L, K_R) = (1, 8)\) and \((1, 4)\) heterotic-type fractional superstring models, we find that we are able to make direct correspondences between these models and the traditional \((1, 2)\) models; these correspondences are possible because both theories are built from identical bosonic left-moving sectors. These correspondences afford us a means of generating what we believe to be valid \((1, K_R)\)-type fractional superstring models, and we present a number of concrete examples with critical spacetime dimensions four and six.

Third, our correspondences suggest that only traditional \((1, 2)\) models with a maximal number of spacetime supersymmetries can be related to fractional \((1, K_R)\) models (which were themselves constructed with \(N = 1\) spacetime supersymmetry). This result therefore severely constrains the space of fractional superstring models in \(D = 4\) and \(D = 6\), confirming our expectation that the number of allowed models is indeed relatively small.

Finally, we discuss an intriguing feature which these fractional superstring theories: quantum mechanics, locality, and Lorentz invariance together seem to intrinsically select certain “natural” dimensions in which these fractional superstring theories naturally live; that our theories themselves seem to induce such compactifications to achieve Lorentz invariance. These compactifications of fractional string theories, are not at all arbitrary, and the \(K = 4\) fractional superstring has a “natural” interpretation in four spacetime. It turns out that this compactification affords us with a means of building models containing chiral fermions in fundamental representations of relevant gauge groups.

All of our results therefore not only lend credence to the fractional superstring idea, but may also, we hope, serve as the first steps of a model-building program. In particular, the correspondences we develop arise from very general principles, and thus should generically be true for the \((K_L, K_R)\) fractional superstring theories in either their critical or “natural” dimensions.

Our goals in this paper are two-fold: not only do we present the new results discussed above, but we also aim to provide an introduction to the original fractional superstring idea than was given in Ref. [3]. Accordingly, this paper is somewhat lengthy; its organization is as follows. In Sect. II we provide a self-contained introduction to the fractional
how this approach forms a natural generalization of the traditional superstring approach. We then in Sect. III proceed to survey the algebraic forms we expect partition functions to have for a general $(K_L, K_R)$ string theory: for $K = 1, 2$ we present known models which will play a role in later sections, and for other values of $K$ we introduce the parafermionic string functions\textsuperscript{[9]} and discuss how they enter into the new total partition functions of $(K_L, K_R)$ models. We also demonstrate that spacetime supersymmetry can be incorporated for these models by choosing these string functions in certain linear combinations, and present a number of important new string-function identities. In Sect. IV we then turn our attention to the heterotic $(1, K_R)$ theories, ultimately deriving various “dictionaries” relating these models to the traditional $(1, 2)$ models. We illustrate the use of these dictionaries by obtaining a number of new fractional-superstring models in $D = 4$ and $D = 6$, and in Sect. V we discuss precisely which traditional models may be “translated” with these dictionaries. In this way we observe an expected truncation in the size of the space of fractional superstring models relative to that corresponding to traditional superstring models in $D < 10$, reflecting the fact that these new models are indeed in their critical dimensions. We close in Sect. VI with our discussion of various further issues in fractional superstring model-building, among them the creation of $(1, 4)$ models with chiral fermions and the necessity of compactifying or interpreting the $(1, 4)$ models in four spacetime dimensions. As we will see, these issues are intimately connected, and we expect the dictionaries we derive in Sect. IV to be easily generalizable to these cases as well. In Appendix A we gather together various definitions and properties of the parafermion characters (or string functions) which play an important role throughout our work, and in Appendix B we prove an assertion made in Sect. V.
2. FRACTIONAL SUPERSTRINGS

In this section we provide a self-contained introduction to the fractional superstring theory as a natural generalization of the traditional superstring and heterotic string theories. We also review, where necessary, some relevant features of the underlying $\mathbb{Z}_K$ parafermion theories, originally constructed by Zamolodchikov and Fateev.\(^{[5]}\)

As outlined in Sect. I, the basic idea behind the fractional superstring is to modify the worldsheet symmetry in such a manner as to obtain a correspondingly smaller critical spacetime dimension. In order to do this, let us begin by considering the general $SU(2)_K$ WZW theory.\(^{[4]}\) As is well-known, this theory consists of primary fields $\Phi^j_m(z)$ which can be organized into $SU(2)$ representations labelled by an integer $j$, where $0 \leq j \leq K/2$ and $|m| \leq j$ with $j-m \in \mathbb{Z}$ (for simplicity we are considering only the holomorphic components). Since $SU(2)$ always has a $U(1)$ subgroup which can be bosonized as a free boson $\varphi$ on a circle of radius $\sqrt{K}$, we can correspondingly factor these primary fields $\Phi^j_m$:

$$\Phi^j_m(z) = \phi^j_m(z) \exp \left\{ \frac{m}{\sqrt{K}} \varphi(z) \right\}. \quad (2.1)$$

Here $\varphi$ is the free $U(1)$ boson, and the $\phi^j_m(z)$ are the primary fields of the coset $SU(2)_K/U(1)$ theory. This coset theory is the well-known $\mathbb{Z}_K$ parafermion theory,\(^*\) and these fields $\phi^j_m$ are the corresponding parafermion fields with highest weights (or conformal dimensions):

$$h^j_m = \frac{j(j+1)}{K+2} - \frac{m^2}{K}$$

The fusion rules of these parafermion fields $\phi^j_m$ follow from those of the $SU(2)_K$ theory:

$$[\phi^j_{m_1}] \otimes [\phi^j_{m_2}] = \sum_{j=|j_1-j_2|}^r [\phi^j_{m_1+m_2}], \quad (2.2)$$

where $r \equiv \min(j_1+j_2, K-j_1-j_2)$ and where the sectors $[\phi^j_m]$ include the primary fields $\phi^j_m$ and their descendants. The characters for $[\phi^j_m]$ are $\eta_{c_j}^{2j}$, where $\eta$ is the Dedekind $\eta$-function and the $c_j^\ell_n$ are the so-called string functions;\(^{[9,6]}\) these functions will be discussed in more detail in Sect. III, and definitions and properties of these functions are collected in Appendix A.

Upon factorizing the primary fields $\Phi^j_m$ as in (2.1), one finds that the $SU(2)$ currents factorize as well:

$$J^+ = \sqrt{K} \psi^*_1 e^{i\varphi/\sqrt{K}}, \quad J^0 = \sqrt{K/2} i \partial \varphi, \quad J^- = \sqrt{K} \psi_1 e^{-i\varphi/\sqrt{K}}$$

where the parafermion currents $\psi_i \sim \phi^0_i \sim \phi^K_i/\sqrt{K}$ have conformal dimensions $i(K-i)/K$ in accordance with

\(^*\) It is important to realize that for fixed $K$, the $\mathbb{Z}_K$ parafermion theory can be realized in more than one way. Different $\mathbb{Z}_K$ models will have different field contents, where any $[\phi^j_m]$ can appear with multiplicity other than one. In general, the coupling constants will be different as well.


see that the parafermion stress-energy tensor \( T_{\text{para}}(z) \equiv T_{SU(2)_K}(z) - T_{\varphi}(z) \) and
the parafermion currents \( \psi_i, i = 1, 2, ..., K - 1 \) form a closed algebra, namely,
the \( \mathbb{Z}_K \) parafermion current algebra.

Note that \( \psi_1 \) acting on a field \( \phi^j_m \) increases the \( m \) quantum number by
one but does not change the \( SU(2) \) spin \( j \). Specifically, we can perform a
mode-expansion for \( \psi_1 \)
\[
\psi_1(z) = \sum_{n \in \mathbb{Z}} A_{d+n} z^{-1-d-n+1/K} \tag{2.5}
\]
where \( d \) is a fractional number and the conformal dimension of \( A_{d+n} \) is \(-(d+n)\); the \( A^\dagger \) mode-expansion for the \( \psi_1^\dagger \) field can be handled similarly. In (2.5), of
course, the value of \( d \) must be chosen appropriately for the particular parafermion field on which \( \psi_1(z) \) is to operate, e.g., for consistency we must choose a
\( \psi_1 \) moding with \( d = (2m+1)/K \) when operating on \( \phi^j_m \). We thus have
\[
A_{(2m+1)/K+n} : [\phi^j_m] \to [\phi^j_{m+1}] \tag{2.6}
\]
where \( n \) are integers. There is also another special field in the parafermion theory, namely the energy operator \( \epsilon \equiv \phi^1_0 \); operating on a field \( \phi^j_m \) with \( \epsilon \)
preserves the \( m \) quantum number but yields sectors with \( j \) quantum numbers
\( j+1 \), \( j \), and \( j-1 \). Specifically, performing a mode-expansion for \( \epsilon \) as in (2.5)
and choosing \( d \) as indicated below, we find the actions of its modes:
\[
\epsilon_{-2(j+1)/(K+2)+n} : [\phi^j_m] \to [\phi^j_{m+1}] \tag{2.7}
\]
\[
\epsilon_n : [\phi^j_m] \to [\phi^j_m] \tag{2.7}
\]
\[
\epsilon_{2j/(K+2)+n} : [\phi^j_m] \to [\phi^j_{m-1}] \tag{2.7}
\]
Of course, (2.6) and (2.7) are valid only when permitted by the fusion rule (2.3).

We thus see that the two fields \( \psi_1 = \phi^1_0 \) and \( \epsilon \) can form the entire set of fields \( \phi^j_m \), starting from any one of them.

The next step in the fractional superstring construction is to decompactify the free boson \( \varphi \), replacing it with a field \( X \) of infinite radius; this field \( X \) will be interpreted as a space-time coordinate. This decompactification preserves the underlying \( SU(2) \) WZW symmetry of our theory.

In fact, the symmetry remaining after the boson decompactification is larger than simple conformal symmetry. We can see this for general \( K \geq 2 \) in the following way. Let us construct the current\[^7\]
\[
\hat{J} \equiv \epsilon \partial X + :\epsilon\epsilon:\tag{2.8}
\]
where \( \epsilon(z) \) is the energy-operator field and where \( :\epsilon\epsilon:\ ) is a normal-ordered product (and is in fact a parafermion descendent of \( \epsilon \)).

For \( K \geq 2 \), of course, the field \( \epsilon \) has conformal dimension
\[
\Delta_\epsilon = \frac{2}{K+2}
\]
and it can be shown that the normal-ordered term \( :\epsilon\epsilon:\ \) has
\[
\Delta_{\epsilon\epsilon} = 1 + \Delta_\epsilon.
\] This is a non-trivial fact, implying...
: \epsilon \epsilon : appears in the \epsilon \epsilon operator product expansion as follows:

$$\epsilon(z)\epsilon(w) = \frac{1}{(z-w)^{2\Delta}} + \ldots + (z-w)^{1-\Delta} : \epsilon \epsilon : (w) + \ldots$$  \hspace{1cm} (2.10)

Thus, for \( K \geq 2 \) the current \( \hat{J} \) in (2.8) has conformal dimension (or equivalently, spin):

$$\Delta_j = 1 + \frac{2}{K+2} = \frac{K+4}{K+2}. \hspace{1cm} (2.11)$$

On the worldsheet this current \( \hat{J} \) forms a closed algebra \(^8\) with \( T(z) \), where \( T(z) \equiv T_X(z) + T_{\text{para}}(z) \) is the stress-energy tensor of the decompactified boson field \( X \) plus that of the \( Z_K \) parafermion theory. Thus we see that \( \hat{J} \) indeed generates an additional worldsheet symmetry which we refer to as a fractional worldsheet supersymmetry. Note that for \( K > 2 \) the dimensions \( \Delta_\epsilon \) and \( \Delta_j \) are not simple half-integers. Thus, our underlying worldsheet \( (\hat{J}, T) \) algebra is non-local, with Riemann cuts (rather than poles) appearing in the various OPEs. Note that this \( (\hat{J}, T) \) algebra is merely the simplest algebra that can be constructed. For other fractional-superstring applications, this \( (\hat{J}, T) \) algebra can indeed be extended to include additional currents.

In order to achieve a sensible interpretation for a \( D \)-dimensional spacetime, we associate the decompactified bosonic field \( X \) with a single spacetime coordinate and tensor together \( D \) copies of the \( (X, \phi^m) \) (or boson plus \( Z_K \) parafermion) theory. We therefore obtain the fractional supersymmetry current

$$J = \epsilon^\mu \partial X_\mu + : \epsilon^\mu \epsilon_\mu :$$  \hspace{1cm} (2.12)

where the Lorentz indices \( \mu = 0, 1, \ldots, D-1 \) are to be contracted with the Minkowski metric. A crucial issue, however, is to determine the dimension \( D \) for arbitrary \( K > 2 \). A formal derivation would proceed through the established path: since the central charge of each \( (X, \phi^m) \) theory is

$$c_0 = 1 + \frac{2K-2}{K+2} = \frac{4K}{K+2}$$

one would need only to determine the central charge contributed by the fractionally-superreparametrization ghosts, and then choose \( D \) so that \( Dc_0 \) cancels this quantity. Such a calculation is presently remains to be done. Another approach is to examine the Fock space of this theory and find a dimension \( D \) and a model which extra null states appear; such work is currently in progress. Another approach is to require that any fractional superstring model produced have a sensible phenomenology, e.g., a massless graviton in the case of a closed string theory, or equivalently a massless vector particle in the case of an open string theory. We emphasize that this is a requirement and not an assumption, for we are interested in constructing only those fractional string theories which contain gravity; other possibilities are, from this standpoint, phenomenologically unappealing. It turns out that this requirement is not difficult to implement.

For a general closed string theory, the graviton and the massless vector both arise from the same right-moving excitation state \( |V_R \rangle \); they differ only in that they are tensored with dissimilar left-moving states \( |V_L \rangle \). Therefore, our requirement simply becomes a requirement on the state \( |V_R \rangle \) – we must...
demand that this state exist (i.e., satisfy the physical-state conditions) and be massless. For a graviton or spacetime vector particle in a string theory with arbitrary \( K \geq 2 \), this state is

\[
|V\rangle_R = \zeta_\mu \epsilon_{-2/(K+2)}^\mu |p\rangle_R
\]

(2.14)

where \( |p\rangle_R \) is the right-moving vacuum state with momentum \( p \), \( \zeta_\mu \) is a polarization vector, and where \( \epsilon_{-2/(K+2)}^\mu \) is the lowest excitation mode of the parafermion field \( \epsilon \). Note that the moding of the \( \epsilon \) field follows from its conformal dimension. This state is indeed the analogue of what appears for the \( K = 2 \) case, in which a single lowest-mode excitation of a worldsheet Neveu-Schwarz fermion produces the needed state:

\[
|V\rangle_R = \zeta_\mu \epsilon_{-1/2}^\mu |p\rangle \text{ where } |p\rangle \text{ is the usual Neveu-Schwarz vacuum state and } \epsilon_{-1/2}^\mu = b_{-1/2}^\mu \text{ is the lowest Neveu-Schwarz creation operator. (For the bosonic string the } \epsilon \text{ field is absent, and we accordingly substitute the bosonic creation operator } a_{-1}^\mu \text{ for } b_{-1/2}^\mu \text{ or } \epsilon_{-2/(K+2)}^\mu \text{. The same argument then applies.) Thus, requiring the state (2.14) to be massless, we find in general that the vacuum state } |p\rangle \text{ in (2.14) must have vacuum energy } \text{VE} = -v = -2/(K + 2). \text{ This information can also be stated in terms of the fractional superstring character } \chi(q). \text{ In general the character has a } q\text{-expansion of the form}
\]

\[
\chi(q) = \sum_n a_n q^n
\]

(2.15)

where \( q = \exp(2\pi i \tau) \), \( \tau \) is the complex modular parameter of the torus, and where \( a_n \) is the number of propagating degrees of freedom at mass level \( M^2 = n \).

We thus see that the fractional superstring character

\[
\chi(q) = q^{-v} (1 + \ldots) = q^{-2/(K+2)} (1 + \ldots)
\]

where inside the parentheses all \( q \)-powers are non-negative integers.

The only remaining non-trivial physical-state condition on the state \( |V\rangle_R \) is

\[
J_{2/(K+2)} |V\rangle_R = 0
\]

(2.17)

where \( J_n \) are the modes of the fractional supercurrent; this can indeed be shown to yield the expected transversality constraint on the polarization vector \( \zeta_\mu \) (which is consistent with its interpretation as the polarization vector of the massless vector state \( |V\rangle_R \)). The state \( |V\rangle_R \) therefore has only \( D - 2 \) polarizations (or degrees of freedom), and from this it follows that at all mass levels of the physical spectrum only \( D - 2 \) transverse dimensions worth of polarization states are propagating degrees of freedom. Note that this latter assertion is not an additional assumption, but rather follows directly from modular invariance. We can see this as follows. Removal of the longitudinal and time-like components of the massless state \( |V\rangle_R \) implies the removal of the corresponding \( q^0 \) terms in the products of string functions which (as we will see in Sect. III) comprise the total fractional superstring partition functions. However, the string functions \( c^\ell_n \), \( c^\mu_n \), and \( T \) modular transformations, form an admissible representation of the modular group. Modular invariance therefore requires
string functions from the total fractional-superstring partition functions, which in turn implies that there is indeed a large string gauge symmetry whose gauge-fixing provides the physical conditions effectively striking out all states involving longitudinal and/or time-like modes from every mass level in the theory.

However, we now recall that conformal invariance requires the character 
\[ \chi(q) = q^{-c/24}(1 + ...) \]  

(2.18)

where \( c \) is the total effective conformal anomaly of the propagating degrees of freedom. Since each spacetime dimension contributes the central charge \( c_0 \) given in (2.13), and since we have determined that effectively only \( D-2 \) dimensions contribute to propagating fields, we have \( c = (D-2)c_0 \). Thus, comparing (2.16) and (2.18) and substituting (2.13), we find the result

\[ D = 2 + \frac{16}{K}, \quad K \geq 2. \]  

(2.19)

We see, then, that the critical dimension of the fractional superstring is a function of the level \( K \) of the \( SU(2)_K/U(1) \) coset (i.e., of the \( \mathbb{Z}_K \) parafermionic theory): for \( K = 2 \) we have \( D = 10 \), for \( K = 4 \) we have \( D = 6 \), and for \( K = 8 \) we obtain \( D = 4 \). For \( K = 1 \) we need only set \( v = 1 \) in (2.16) [as explained after (2.14)], whereupon the argument above yields \( D = 26 \). Thus, by appropriately choosing \( K \) and building the corresponding worldsheet fractional supersymmetry as discussed above, we can hope to obtain a series of new string theories with a variety of critical dimensions.

It is easy to check that the \( K = 1 \) and \( K = 2 \) string and superstring respectively. For \( K = 1 \), the \( SU(2)_1/U(1) \) coset is trivial, containing only the identity field \( X^\mu \) which they become), and hence for \( K = 1 \) the only worldsheet symmetry is conformal symmetry. This reproduces, of course, the worldsheet structure of the bosonic string. Similarly, for \( K = 2 \), the \( SU(2)_2/U(1) \) (i.e., the \( \mathbb{Z}_2 \) parafermion theory) is simply the Ising model: \( \phi^0_0 \) (the identity), \( \phi^1_0 = \phi^0_1 \) (which is of course the \( \epsilon \) field, or equivalently the parafermion current \( \psi_1 \), or equivalently the Majorana fermion \( \psi \)), \( \phi^{1/2}_1/2^{1/2} \) (the spin field \( \sigma \)), and \( \phi^{1/2}_1 - 1/2^{1/2} \) (the conjugate spin field \( \sigma^\dagger \)). However, excitations of the worldsheet spin fields are responsible for spacetime fermions, and indeed together these fields form a Ramond worldsheet fermion. The same is true for the identity and \( \psi \) fields: when appropriately mixed they form a Neveu-Schwarz worldsheet Majorana fermion theory, and the symmetry relating this theory to the bosonic \( X^\mu \) fields is an ordinary worldsheet supersymmetry. This then reproduces the traditional superstring, and the heterotic string is the left/right tensoring of a \( K = 1 \) and \( K = 2 \) theory respectively. It is therefore apparent that this fractional superstring language provides a natural framework in which to classify and uniformly handle all of the traditional string theories, and doing it also points the way to their non-trivial generalizations.
3. PARTITION FUNCTIONS FOR FRACTIONAL SUPERSTRINGS

Having thus presented the underlying basis for the fractional superstring theories, we turn our attention to the partition functions that these theories must have. We begin by reviewing the generic forms of these partition functions, and start with the known $K = 1$ and $K = 2$ cases to establish our notation and conventions. We will see, once again, that the $K > 2$ string theories have partition functions whose forms are straightforward extensions of those of the traditional cases. This will therefore permit us to write the partition functions for all of the $(K_L, K_R)$ theories we shall consider in a common language.

3.1. Traditional String Theories

The first case to consider, of course, is the pure bosonic string; since there is no left-moving or right-moving worldsheet supersymmetry, we may refer to this in our new fractional superstring language as the $(K_L, K_R) = (1, 1)$ case. As discussed in Sect. II, the critical spacetime dimension for this string theory is $D = 26$, and thus this theory contains 26 bosonic worldsheet fields $X^\mu$, each of which contributes to the total one-loop partition function a factor

$$\text{each boson} \implies \frac{1}{\sqrt{\tau_2} \eta \bar{\eta}}. \quad (3.1)$$

Here $\eta(\tau)$ is the well-known Dedekind $\eta$-function, and $\tau = \tau_1 + i\tau_2$ is the torus modular parameter. Note that this factor is explicitly real because the left- and right-moving components of each boson contribute equally. Since we have already seen that only $D - 2 = 24$ transverse directions are propagating (as is evident in a light-cone gauge approach), our total partition function for the $(1, 1)$ string is therefore of the form

$$Z_{(1,1)} = \tau_2^{-12} \frac{1}{|\eta|^48} = \tau_2^{-12} \Delta^{-2}, \quad (3.2)$$

where we have defined $\Delta \equiv \eta^{24}$. Note that the overall factor, which we will denote $k$, is in general given by

$$k = 1 - D/2$$

where $D$ is the number of spacetime dimensions in which the theory is formulated. Hence, for the $(1,1)$ string in $D = 26$ dimensions, we find agreement with (3.2). In the theory of modular functions, $k$ is known as the modular weight; the $\eta$-function transforms as a modular function of weight $1/2$, and $T : \tau \rightarrow \tau + 1$ as a modular function of weight $0$. Hence, the modular weight of the partition function $Z$ is $k = -12$. Modular-invariance for the entire partition function requires that the modular weight of its holomorphic factors equal that of its anti-holomorphic factors, and that this weight also equal the power of the overall $\tau_2$ factor. We can see that in (3.2) this is indeed the case.

The next string to consider is the traditional superstring: since this theory has a full superconformal worldsheet symmetry, in our new language this is simply the $(K_L, K_R) = (2, 2)$ case. As discussed in Sect. II, the critical spacetime dimension for this string theory is $D = 10$, and thus this theory contains 10 bosonic worldsheet fields $X^\mu$, each of which contributes to the total one-loop partition function a factor

$$\text{each boson} \implies \frac{1}{\sqrt{\eta \bar{\eta}}} \eta \bar{\eta}. \quad (3.3)$$

Here $\eta(\tau)$ is the well-known Dedekind $\eta$-function, and $\tau = \tau_1 + i\tau_2$ is the torus modular parameter. Note that this factor is explicitly real because the left- and right-moving components of each boson contribute equally. Since we have already seen that only $D - 2 = 8$ transverse directions are propagating (as is evident in a light-cone gauge approach), our total partition function for the $(2,2)$ string is therefore of the form

$$Z_{(2,2)} = \tau_2^{-24} \frac{1}{|\eta|^48} = \tau_2^{-24} \Delta^{-2}, \quad (3.4)$$

where we have defined $\Delta \equiv \eta^{24}$. Note that the overall factor, which we will denote $k$, is in general given by

$$k = 1 - D/2$$

where $D$ is the number of spacetime dimensions in which the theory is formulated. Hence, for the $(2,2)$ string in $D = 10$ dimensions, we find agreement with (3.4). In the theory of modular functions, $k$ is known as the modular weight; the $\eta$-function transforms as a modular function of weight $1/2$, and $T : \tau \rightarrow \tau + 1$ as a modular function of weight $0$. Hence, the modular weight of the partition function $Z$ is $k = -24$. Modular-invariance for the entire partition function requires that the modular weight of its holomorphic factors equal that of its anti-holomorphic factors, and that this weight also equal the power of the overall $\tau_2$ factor. We can see that in (3.4) this is indeed the case.
That its critical dimension is 10 follows from (2.19), or equivalently from the
traditional argument in which the conformal anomaly $c = -26 + 11 = -15$ from
the reparametrization ghosts and their superpartners must be cancelled by the
contributions from $D$ worldsheet boson/fermion pairs, each of which has total
central charge $c_0 = 1 + \frac{1}{2} = \frac{3}{2}$ [in accordance with (2.13) for $K = 2$]. In light-
cone gauge only the eight transverse field pairs propagate, and therefore this
theory has a total worldsheet field content consisting of eight real bosons (each
with a left-moving and right-moving component) and eight Majorana fermions
(or eight Majorana-Weyl left-movers and eight Majorana-Weyl right-movers).
These fermions, we recall, can be described in our new viewpoint as simply the
fields of the $SU(2)_2/U(1)$ Ising model, and we are free to group pairs of these
Majorana-Weyl fermions together to form single complex Weyl fermions. The
factor contributed by each of the eight bosons to the one-loop partition function
is given in (3.1), and the contribution of each complex Weyl fermion is

$$\text{each Weyl fermion} \implies \frac{\vartheta}{\eta}$$

(3.4)

where $\vartheta$ represents one of the well-known Jacobi $\vartheta$-functions defined in App-
pendix A.\* Since the contributions of right-moving fields are the complex-
conjugates of those of left-moving fields, the total partition function for the

\* Which particular $\vartheta$-function is appropriate depends on the boundary con-
ditions assigned to the fermion as it traverses the two cycles of the torus.
Only periodic or antiperiodic boundary conditions will yield the Jacobi $\vartheta$-functions, but if a fermion is chosen to have periodic/periodic boundary
conditions then its zero-modes cause the total partition function to vanish
identically.

\[ Z_{(2,2)} \sim \tau_2^{-4} \frac{1}{|\eta|^{16}} \sum \left| \frac{\vartheta}{\eta} \right|^8 \sim \tau_2^{-4} \Delta^{-1} \]

where $\vartheta^4$ indicates four (not necessarily identical) Jacobi $\vartheta$-function factors.
The summation in (3.5) is over fermion boundary conditions, as is needed in
order to achieve modular invariance. In this section we shall frequently use
the schematic notation in (3.5), employing the general forms of partition functions. Note that
transforms under the modular group with weightorphic and anti-holomorphic factors in (3.5) — a
that $k = -4$ for this theory. This result is once again agreement with our
spacetime dimension $D = 10$ according to (3.3).

We shall be most concerned in this paper with heterotic-type string theories.
The traditional heterotic string has a right-moving worldsheet supersymmetry
as in the superstring; hence the right-moving propagating field content
consist, as before, of eight bosons and eight Majorana-Weyl fermions, with
critical spacetime dimension $D = 10$. As above, this is simply the choice
for the right-moving sector. However, the left-moving sector must also have the eight transverse left-moving
fields, and thus in order to cancel the $c = 26$ contribution from the left-moving
reparametrization ghosts we must additionally have 32 Majorana-Weyl fermions
(or equivalently 16 complex Weyl fermions, or 16 compactified scalar bosons
$\phi$). Our traditional heterotic string is therefore

\[ (2,2) \text{ string therefore takes the form} \]

$$Z_{(2,2)} \sim \tau_2^{-4} \frac{1}{|\eta|^{16}} \sum \left| \frac{\vartheta}{\eta} \right|^8 \sim \tau_2^{-4} \Delta^{-1} \]

\[ \Delta \]

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partition function of the form

\[ D = 10 : \quad Z_{(1,2)} \sim \tau^2 \Delta^{-1} \sum_{\vartheta} \vartheta^{16} \vartheta^{-1} \]  \quad (3.6)

Note that we continue to have \( k = -4 \) from each factor in this partition function, in agreement with (3.3). As examples of (3.6) which will be relevant later, we can look at those known \((1,2)\) \(D = 10\) theories (or models) which have spacetime supersymmetry. As is well-known, there are only two such self-consistent models: these have gauge groups \(SO(32)\) and \(E_8 \otimes E_8\). Their partition functions respectively are as follows:

\[ D = 10, \ SO(32) : \quad Z = \left( \frac{1}{2} \right)^2 \tau^2 \Delta^{-1} K (\beta^4 + \gamma^4 + \delta^4) \]  \quad (3.7)

\[ D = 10, \ E_8 \otimes E_8 : \quad Z = \left( \frac{1}{2} \right)^3 \tau^2 \Delta^{-1} K (\beta^2 + \gamma^2 + \delta^2)^2 \]

where we have made the following definitions which we will use throughout:

\[ \beta \equiv \vartheta_2^{-4}, \quad \gamma \equiv \vartheta_3^{-4}, \quad \delta \equiv \vartheta_4^{-4}, \]
\[ J \equiv \gamma - \beta - \delta, \]
\[ K \equiv \Delta^{-1} \sum_{\vartheta} \vartheta^{16} \vartheta^{-1} \]  \quad (3.8)

Note that the partition functions (3.7) are indeed of the form (3.6), with the holomorphic parts factorizing into group characters to reflect the underlying group structure. The quantity \( J \) in (3.8) is of course the Jacobi factor, and the spacetime supersymmetry of these models (or equivalently, the vanishing of their partition functions) arises from the identity \( J = 0 \). It is in fact a further identity that

\[ \beta^4 + \gamma^4 + \delta^4 = \frac{1}{2} (\beta^2 + \gamma^2 + \delta^2)^2 \]

as a consequence of which the two partition functions (3.7) are equal even without use of the Jacobi identity \( J = 0 \).

It is straightforward to check the modular invariance of these partition functions (3.7). For any \( A \equiv (a \ b) \) in the modular group, the so-called “stroke” operator \([A] : f \to f[A]\) where

\[ (f[A])(\tau) \equiv (c\tau + d)^{-k} f(a\tau + b) \]  \quad (3.10)

(here \( k \) is the modular weight of \( f \)). It then follows from the transformation properties of the \( \eta \) and \( \vartheta \) functions (see Appendix A) that

\[ \beta[S] = -\delta, \quad \beta[T] = -\gamma, \quad \gamma[S] = -\gamma, \quad \gamma[T] = -\beta \]
\[ \delta[S] = -\beta, \quad \delta[T] = -\gamma \]
\[ K[S] = +K, \quad K[T] = +K \]

Thus, \( K \) is itself modular-invariant (i.e., invariant under \([S]\) and \([T]\)), and in each partition function the other factors involving \( \beta \), \( \gamma \), and \( \delta \) must be (and are) themselves modular-invariant as well.

It is also possible to construct self-consistent \((1,2)\) theories in dimensions \( D < 10 \); one needs simply, for example, to re-interpret the extra \( 10 - D \) bosonic degrees of freedom or in some other manner hide them from low-energy physics.
(e.g., through compactification). We stress, however, that all such methods do not alter the underlying critical dimension away from 10; these theories remain theories of the \((1, 2)\) variety. For example, one method of constructing \((1, 2)\) models in \(D = 4\) involves fermionization: each of the six extra left- or right-moving bosonic degrees of freedom can, for example, be represented in terms of two free worldsheet Majorana-Weyl fermions (or a single free Weyl fermion). \(^{[10]}\)

The propagating field content for this four-dimensional \((1, 2)\) theory therefore consists of two left-moving and two right-moving bosons, as well as 22 left-moving and 10 right-moving Weyl fermions. The total partition function for such a \(D = 4\) \((1, 2)\) theory hence takes the form

\[
D = 4: \quad Z_{(1,2)} \sim \tau^{-1} \Delta^{-1} \prod_{n} \vartheta^{2n-\frac{1}{2}}. \tag{3.12}
\]

Note that once again (3.3) is satisfied, with \(k = -1\) for \(D = 4\). As an example, we present the partition function of what is possibly the simplest such spacetime-supersymmetric model that can be constructed. Named Model \(M1\) in Ref. [10], it has gauge group \(SO(44)\), the largest possible in \(D = 4\); its (modular-invariant) partition function is

\[
D = 4, \quad SO(44): \quad Z = (\frac{1}{2})^2 \tau^{-1} K \left( |\beta| \overline{\beta} \gamma^5 + |\gamma| \overline{\gamma} \delta^5 + |\delta| \overline{\delta} \gamma^5 \right). \tag{3.13}
\]

Since the “maximal” gauge group allowed in \(D\) dimensions is \(SO(52 - 2D)\), we see that this model is the \(D = 4\) analogue of the \(D = 10\) \(SO(32)\) model in (3.7).

We note for future reference that this idea can in fact be generalized to obtain \((1, 2)\)-type theories in any spacetime dimension \(D \leq 10\). We in general obtain the partition function form

\[
\text{general } D: \quad Z_{(1,2)} \sim \tau_k \Delta^{-1} \prod_{n} \vartheta^{2n-\frac{1}{2}}. \tag{3.14}
\]

where \(k\) is given by (3.3) and

\[
\begin{align*}
   n &= 26 - D = 24, \\
   \pi &= 14 - D = 12.
\end{align*}
\]

Much of our work will be concerned with models \(k = -2, n = 20, \) and \(\pi = 8\). The “maximally symmetric” SUSY has gauge group \(SO(40)\); its partition function, along with those of other supersymmetric models we will be discussing, is as follows:

\[
\begin{align*}
   D = 6, \quad SO(40) : \quad Z &= (\frac{1}{2})^2 \tau^{-2} K, \\
   D = 6, \quad SO(24) \otimes E_8 : \quad Z &= (\frac{1}{2})^3 \tau^{-2} K, \\
\end{align*}
\]

\[
\begin{align*}
   D = 6, \quad SO(24) \otimes SO(16) : \quad Z &= (\frac{1}{2})^4 \tau^{-2} K, \\
      &\quad \times \left( \overline{\beta} \gamma^2 (\beta^3 + \gamma^3) \right).
\end{align*}
\]

We again observe the familiar factorization, with the same \(E_8\) factor above as in (3.7). It is simple, using (3.11), to see that these partition functions are indeed modular-invariant.
3.2. Fractional Superstring Theories

The partition functions for fractional superstrings can be determined in precisely the same manner as above. Since we know the critical dimension \(D\) for a given value of \(K\), we can readily deduce the forms that partition functions must take for general \((K_L, K_R)\) combinations. The factor contributed to the total partition function from each worldsheet boson is given, as before, by (3.1); recall that this factor includes the contributions from both holomorphic and anti-holomorphic (or left- and right-moving) components. The factor contributed by each worldsheet parafermion, however, is a generalization of (3.4); in general we have

\[
\text{each parafermion } \implies \eta c
\]

(3.17)

(the above is for left-moving parafermions; right-moving parafermions contribute the complex-conjugate). Here \(\eta\) is the usual Dedekind function, and \(c\) schematically represents one of the so-called parafermionic string functions. These functions are defined in Appendix A, but for our present purposes we need record only the following facts. These functions \(c^\ell_n\) may be defined in terms of \(q\)-expansions which depend on \(K\) as well as the two parameters \(\ell\) and \(n\), and one string function \(c^{2j}_{2m}\) may correspond to more than one parafermion field \(\phi^l_m\) (for example, the distinct fields \(\phi^l_m\) and \(\phi^l_{-m}\) both give rise to \(c^{2j}_{2m}\)). Each \(c^\ell_n\) is an eigenfunction under \(T: \tau \rightarrow \tau + 1\), and under \(S: \tau \rightarrow -1/\tau\) they mix forming a closed set; furthermore, they transform under the \(S\) and \(T\) transformations with the negative modular weight \(k = -\frac{1}{2}\).

As expected, these string functions (which are functions of \(K\)) should reduce to the traditional modular functions for the \(K = 1\) and \(K = 2\) cases, and this is indeed precisely what occurs. For the \(K = 1\) function \(c^0_0\), and since our \(SU(2)_1/U(1)\) theory contains \(\phi^0_0\), we quickly have

\[
K = 1: \quad \eta c^0_0 = 1 \implies c^0_0 = \eta.
\]

(3.18)

Thus, for \(K = 1\) the one string function \(c^0_0\) is related to the boson character \(\eta\).

Similarly, for \(K = 2\) there are precisely three string functions \(c^0_0, c^1_1, c^2_0\), and these are related to the three fermionic Jacobi \(\vartheta\)-functions as follows:

\[
K = 2: \quad \begin{cases} 
2(c_1^1)^2 = \vartheta_2/\eta^3 \\
(c_0^0 + c_2^0)^2 = \vartheta_3/\eta^3 \\
(c_0^0 - c_2^0)^2 = \vartheta_4/\eta^3.
\end{cases}
\]

(3.19)

For \(K > 2\), of course, the string functions involve more than just these simpler functions.

Therefore, looking first at the \((K_L, K_R) = (K, K)\) theories with \(K > 2\) (these are the generalizations of the usual Type II theories), we see that our worldsheet field content consists (in light-cone gauge) of \(D - 2 = 16/K\) coordinate bosons and \(16/K\) each of left- and right-moving parafermions, and we therefore obtain the form for the total \((K, K)\) partition function:

\[
Z_{(K, K)} \sim \tau_2^{-8/K} \sum c^\ell_n^2.
\]

(3.20)

Note that \(k \equiv -8/K = 1 - D/2\) in accordance with (3.3) and (2.19).
the η-functions have cancelled between the bosonic and parafermionic contributions. Thus, in our cases of interest we obtain:

\[ D = 6 : \quad Z_{(4,4)} \sim \tau_2^{-2} \sum \vartheta^4 e^4 \]
\[ D = 4 : \quad Z_{(8,8)} \sim \tau_2^{-1} \sum \vartheta^2 e^2 \quad (3.21) \]

Similarly, one can consider the analogues of heterotic string theories; these would be the \((K_L, K_R) = (1, K)\) situations. The right-moving sectors of these theories are precisely as in the \((K, K)\) cases, and hence the anti-holomorphic parts of their partition functions have the same form as in (3.20). Since their left-moving sectors must already contain the left-moving components of the coordinate bosons, we must augment their left-moving field contents to achieve conformal anomaly cancellation. The contribution of the bosons to the central charge is of course \(2 + 16/K\), and since this sector is a bosonic theory we must cancel the usual ghost contribution \(c = -26\). This requires additional matter fields with central charge

\[ \Delta c = 8 \left( \frac{3K - 2}{K} \right), \quad (3.22) \]

and we may choose these fields to be \(\Delta c\) complex Weyl worldsheet fermions [each of which has the partition-function contribution given in (3.4)]. Thus, for a general heterotic-type \((1, K)\) theory we expect a partition function of the form

\[ Z_{(1,K)} \sim \tau_2^{-8/K} \sum \vartheta^{16/K} \left( \frac{1}{\eta} \right)^{16/K} \left( \frac{\vartheta}{\eta} \right)^{8(3K-2)/K} \sim \tau_2^{-8/K} \sum \vartheta^{16/K} \Delta^{-1} \vartheta^{8(3K-2)/K}. \quad (3.23) \]

Note that since \(\Delta c = 8(3K - 2)/K = n\) [where \(n\) is the quantity defined in (3.15)], the holomorphic part of this partition function is as we obtained in (3.14) for general spacetime dimensions, as to be expected; these left-moving sectors are bosonic. The difference, however, is that the heterotic \((1, K)\) theories are conformal for all values of \(K\).

For completeness, we should also note that, for example, to consider \((1, K)\) theories in spacetime dimensions \(D < D_c\), one can choose to retain \(D - 2\) of the above bosons and replace the remaining \(D_c - D\) bosons with as many worldsheet Weyl fermions. It is clear that from such a sector the partition function contribution

\[ \left( \frac{1}{\eta} \right)^{D-2} (\eta)^{16/K} \left( \frac{\vartheta}{\eta} \right)^{D_c-D} \]

the first factor is the contribution of the \(D - 2\) bosons, the second is that of the original \(D_c - 2 = 16/K\) parafermions, the third is the contribution of the fermionized \(D_c - D\) bosons. Thus, for example, there are two ways to achieve a \(D = 4\) theory: one can consider a \((1, 8)\) theory in its critical dimension, or a \((1, 4)\) theory in which two dimensions are in some way compactified or fermionized. We shall discuss this latter possibility in Sect. VI.

In fact, use of these parafermionic string functions allows us to collect together in a simple way all of the partition function forms we have considered for the general \((K_L, K_R)\) theory in arbitrary dimension. For the general \((K_L, K_R)\) theory in arbitrary dimension, for a given value of \(K_R\), we have considered the two cases of traditional string theories as well as our new
simply

\[ Z_{(K_L, K_R)} = \tau_2^k \sum_i L_i^{(K_L)} R_i^{(K_R)} \]  

(3.25)

where the \( L_i^{(K_L)} \) are the contributions from the left-moving sectors and the \( R_i^{(K_R)} \) are from the right-movers. In the above formula \( k \) is always given by (3.3) (where \( D \) is the dimension of spacetime, not the critical dimension). If we define the critical dimensions \( D^R_c = 2 + 16/K_R \) and \( D^L_c = 2 + 16/K_L \) (or 26 if the corresponding \( K = 1 \)), then the general forms for the \( L_i \) and \( R_i \) can be given as follows:

\[ L_i^{(K_L)} \sim c^{D^L_c - 2} \vartheta^{D^L_c - D} \]
\[ R_i^{(K_R)} \sim c^{D^R_c - 2} \vartheta^{D^R_c - D} \]  

(3.26)

We can easily check the special cases \( K = 1 \) and \( K = 2 \). For a \((1,1)\) theory formulated in arbitrary \( D \leq D_c = 26 \), (3.18) yields \( L^{(1)} = R^{(1)} = \Delta^{-1} \vartheta^{26-D} \); for \( D = D_c \) this therefore reproduces (3.2), and for \( D < D_c \) this reproduces the holomorphic part of (3.14) for the \((1,2)\) theory. Similarly, for \( K_R = 2 \) in arbitrary dimension \( D \leq D_c \), (3.19) yields \( R^{(2)} = \Delta^{-1} \vartheta^{14-D} \), in accord with the anti-holomorphic part of (3.14). Thus, (3.25) and (3.26) are indeed the most general partition function forms for the traditional as well as the fractional string theories, brought together in a natural way through our use of the parafermionic string functions.

In order to construct sensible partition functions having the above forms, it is first necessary to find suitable combinations of the string functions which can replace the \( c^{D_c - 2} \) factors above. There are several requirements. First, we want linear combinations which are tachyon-free: this means that we must have \( a_n = 0 \) for all \( n < 0 \). Second, we must also require that there be a spectrum of massless particles: this means that for at least a subset of terms corresponding to a massless sector in the linear combination we must have \( a_0 \neq 0 \). Third, since we would like to use the resulting heterotic partition functions, we must demand modular-invariance for our total partition function. Finally, for a given \( K \), we must of course demand that our linear combinations each involve \( D_c - 2 \) powers of level-\( K \) string functions: for \( K \geq 2 \) this means that we require \( 16/K \) string-function factors, and for \( K = 1 \) we require 24.

It is clear that linear combinations satisfying all of the above constraints exist for the \( K = 1 \) and \( K = 2 \) cases; for \( K = 1 \), for example, we have

\[ A_1 = (c_0^2)^{24} = \Delta \]  

(3.28)

* This expression \( A_1 \) actually has \( a_n \neq 0 \) for \( n < 0 \), in accordance with the known result that the bosonic string contains on-shell tachyonic states.
and for \(K = 2\) we can choose:

\[
A_2 = 8(c_0^6)^2 c_0^2 - 8(c_1^1)^8 + 56(c_0^5)(c_0^2)^3 + 56(c_0^3)(c_0^3)^3 + 8c_0^6(c_0^2)^2 = \frac{1}{2} \Delta^{-1/2} J .
\]

(3.29)

We note that in each of these cases only one linear combination is necessary to achieve closure under \(S\) and \(T\), and we observe that \(A_2 = 0\) as a consequence of the Jacobi identity \(J = 0\). This is simply a reflection of the *spacetime* supersymmetry of the superstring. These two expressions have, of course, already been determined: \(Z_{(1,1)}\) in (3.2) is merely \(\tau_2^{-12}|A_1|^2\), and \(\tau_2^{-4}|A_2|^2\) is merely a special (spacetime-supersymmetric) case of \(Z_{(2,2)}\) in (3.5).

Similarly, it is possible to construct linear combinations satisfying all of the above constraints for the \(K = 4\) and \(K = 8\) cases as well.\(^3\) These are as follows.

For \(K = 4\) we have the two combinations:

\[
A_4 = 4(c_0^0 + c_1^1)^3(c_0^2) - 4(c_2^2)^4 + 32(c_0^4)(c_0^2)^3 - 4(c_0^2)^4 ,
\]

\[
B_4 = -4(c_2^2)^2(c_0^2)^2 + 8(c_0^6 + c_0^4)(c_0^2)(c_0^2)^2
\]

(3.30)

and for \(K = 8\) we have the three combinations:

\[
A_8 = 2(c_0^0 + c_0^8)(c_0^2) + 2(c_1^1)^2 + 8(c_4^4)(c_0^2) - 2(c_0^2)^2 ,
\]

\[
B_8 = 4(c_0^0 + c_0^8)(c_4^4) + 4(c_2^2 + c_0^6)(c_4^4) - 4(c_0^2c_4^4) ,
\]

\[
C_8 = 4(c_2^2 + c_0^6)(c_2^2 + c_0^8) - 4(c_2^2c_2^2) .
\]

(3.31)

[In fact, combinations satisfying these requirements exist for the \(K = 16\) (\(D = 3\)) case as well.\(^{11}\)] The transformation properties of these functions under \(S\) and \(T\) will be given in Sect. IV; in particular, \(A_4\) has a \(q\)-expansion of the form \(q^h(1 + \ldots)\), where the \(h\) exponents are non-negative integers and where \(h = 3/4\) for \(B_4\) and \(B_8\), and \(h = 3/4\) for \(C_8\). Thus, \(A_4\) and \(A_8\) particles can contribute to the these partition functions \(A_8\): indeed, the terms \((c_0^0)^{D_0 - 3}c_0^2\) can be interpreted as spacetime vector particles, and the terms \(-c_0^0\) correspond to massless spacetime fermions.

We will, therefore, take these combinations as the building blocks when constructing our fractional superstring partition functions, substituting them corresponding to massless spacetime fermions.

In fact, it turns out that there is one additional \(\delta\)-function identity \(J = \gamma - \beta - \delta = 0\), it can be parafemionic string-function expressions vanish.

\[
A_4 = B_4 = A_8 = B_8 = C_8 = 0.
\]

Thus, for \(K = 4\) we have the two new Jacobi-like identities, for \(K = 8\) we have the three new identities \(A_8\) that exist for the \(K = 16\) case as well.\(^{11}\) This result can then serve as the mechanism by which any \(S\) can be made consistent with spacetime supersymmetry. The transformation properties of these functions under...
these Jacobi-like factors, and the $K = 2$ full Jacobi identity $A_2 = 0$ appears merely as the $K = 2$ special case of the more general identities for general even integers $K$.

We emphasize that such $B$ and $C$ sectors, necessitated by closure under $[S]$ for the $K = 4$ and $K = 8$ theories, are completely new features arising only for the $K > 2$ fractional superstrings, and as such they are responsible for much of the new physics these strings contain. For example, not only do they lead (as we have seen) to multiple independent Jacobi identities, but they may also be responsible for self-induced compactifications of these strings. We will discuss this possibility in Sect. VI.

In fact, the Jacobi identity is not the only famous identity which generalizes to higher $K$. For $K = 2$, there is another well-known identity involving the $\eta$ and $\vartheta$-functions:

$$\vartheta_2 \vartheta_3 \vartheta_4 = 2 \eta^3,$$  \hspace{1cm} (3.33)

this identity relates the fermion characters $\vartheta_i$ to the boson character $\eta$, and hence we may refer to this as a bosonization identity. It turns out \cite{11} that (3.33) is only the $K = 2$ special case of another series of identities, each relating the more general $\mathbb{Z}_K$ parafermion characters $c_{\ell \ell}^\ell$ for a different $K \geq 2$ to the boson character $\eta$. In fact, there also exist various other series of identities whose $K = 2$ special cases are known and have well-understood physical interpretations; proofs of all of these identities, as well as methods for generating them for arbitrary $K$, are presented in Ref. \cite{11}. We therefore see that the string functions $c_{\ell \ell}^\ell$ provide a uniquely compelling language in which to generalize previously known string-theory results, providing us with new Jacobi-like and other identities whose physical interpretations are only beginning to be explored. Thus we see again (this time on the string-function level) that by leading directly to these string-function expressions and identities, the fractional superstring construction does indeed provide a natural means of generalizing the traditional string theories.
4. DICTIONARIES FOR MODEL-BUILDING

Having established that spacetime supersymmetry can be incorporated by using the expressions (3.30) and (3.31), we now turn our attention to the construction of actual supersymmetric \((K_L, K_R)\) models. We focus our attention primarily on the heterotic \((K_L, K_R) = (1, 4)\) and \((1, 8)\) cases, and construct a procedure for generating models in these classes. Our procedure involves “translating” or drawing correspondences between the \((1, K)\) models and known \((1, 2)\) models in \(D = 2 + 16/K\), and we construct “dictionaries” which enable these translations to take place for a given \(K\). We find that these dictionaries are intuitive and practical, and furthermore (as we will see in Sect. V) they yield substantially and understandably smaller spaces of \((1, K)\) models than \((1, 2)\) models in \(D < 10\). In particular, we will find that only those \((1, 2)\) models which have a maximal number of spacetime supersymmetries are translatable; these are the models with \(N = N_{\text{max}}\) SUSY, where

\[
N_{\text{max}} = \begin{cases} 
1 & \text{for } D = 10 \\
2 & \text{for } D = 6 \\
4 & \text{for } D = 4 \ . 
\end{cases} 
\] (4.1)

We should point out that throughout this and the next section we will be focusing our attention on those \((1, 2)\) models whose partition functions can be built from Jacobi \(\vartheta\)-functions, in accordance with the presentation in Sect. III. This is indeed a broad class of models, but it is not all-inclusive. However, the dictionaries we will be developing for these models rest on very general principles, and we expect this dictionary idea to be equally applicable to other methods of \((1, 2)\) model-construction as well.

4.1. MODULAR INVARIANCE

As discussed in Sect. III, models of the \((1, K)\) variety in their critical dimensions \(D = 2 + 16/K\) must have partition functions

\[
Z_{(1, K)} \sim \tau_2^k \sum \tau^{16/K} \Delta^{-1} \vartheta^{22} 
\]

and spacetime supersymmetry can be incorporated to take the values \(A_K, B_K\), and (for \(K = 8\)) \(C_K\). Let us change notation slightly, and refer to these quantities collectively as \(A_i(K)\), where \(i = 1, 2\) for \(K = 4\). Supersymmetric partition functions

\[
Z_{(1, K)} = \tau_2^k \sum_i A_i^K F_i^K 
\]

where the \(F_i^K\)’s take the forms

\[
F_i^{(4)} \sim \Delta^{-1} \sum \vartheta^{20} \quad \text{and} \quad F_i^{(8)} \sim \Delta^{-1} \sum \vartheta^{22} 
\]

Let us now determine the constraints we impose on these partition functions due to modular-invariance. It follows from the transformation properties of the \(\vartheta\) under the \([S]\)-transformations that the expressions \(A_i^K\) transform as

\[
A_i^K[S] = e^{4\pi i/K} \sum_j M_{ij} A_j^K 
\]
where the matrices $M_{(K)}$ are

$$M_{(4)} = \begin{pmatrix} 1/2 & 3 \\ 1/4 & -1/2 \end{pmatrix}, \quad M_{(8)} = \begin{pmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & -1 \\ 1/2 & -1/2 & 0 \end{pmatrix}, \quad (4.6)$$

and where the stroke operator $[S]$ was defined in (3.10). Demanding $[S]$-invariance of the total partition function $Z$ therefore yields

$$Z[S] = e^{-4\pi i/K} \tau_2^k \sum_{i,j} M^{ij} A^j F^i[S] \equiv \tau_2^k \sum_j A^j F^j, \quad (4.7)$$

or

$$F^i = e^{-4\pi i/K} \sum_j (M^t)^{ij} F^j, \quad (4.8)$$

where $M^t$ is the transpose of $M$. Since $M^2 = (M^t)^2 = 1$ for both cases $K = 4$ and 8, we can immediately solve (4.8) for $F^j[S]$, yielding the constraint

$$F^i[S] = e^{4\pi i/K} \sum_j (M^t)^{ij} F^j. \quad (4.9)$$

Similarly, under $[T]$ the $A$'s have the transformation

$$A^i_{(K)}[T] = \sum_j N^{ij}_{(K)} A^j_{(K)} \quad (4.10)$$

where the matrices $N_{(K)}$ are

$$N_{(4)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad N_{(8)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -i \end{pmatrix}. \quad (4.11)$$

Proceeding precisely as above, we find the additional constraints:

$$F^i[T] = \sum_j N^{ij} F^j, \quad (4.12)$$

which in this case imply simply that each $F^i$ must transform as does the corresponding $A^i$. This is also clear from the transformation under $T: \tau \mapsto \tau + 1$ means that in $Z = \tau_2^k \sum_m a_m q^m$ only if $m-n \in \mathbb{Z}$. Since each $A^i$ has a $q$-expansion ($...,q^m,...$) where inside the parentheses all powers of $q$ are integral, we see that the corresponding $F^i$ must take the same form with the same value of $h$. This is the content of (4.12).

We now give a general procedure for obtaining expressions $F^i$ of the forms (4.4) which satisfy both (4.9) and (4.12). Let us first examine the $K=8$ case, after which the $K=4$ case will be straightforward. In order to do this, we consider the space $F$ of polynomials in the three quantities $\{\vartheta_2^2, \vartheta_3, \vartheta_4\}$ (so that $f[T^4] = f$ for all $f \in F$), and establish four projection operators $P_\ell$ ($\ell = 0, ..., 3$) in this space. These operators $P_\ell$ are defined

$$P_\ell f = \frac{1}{4} \sum_{n=0}^3 \exp \left\{ -i\pi \frac{\ell n}{4} \right\} f[T^n] \quad (4.13)$$

where $f \in F$ is any polynomial in this space, and $\sum_\ell P_\ell = 1$ and $P_\ell P_\ell' = P_\ell \delta_{\ell\ell'}$. These operators $P_\ell$ when operating on any $f \in F$, $P_\ell$ selects out the...
by projection. The crucial element in $K$ which itself is $S$-invariant, and from which the individual $F$'s can be obtained by projection. The crucial element in $K = 8$, however, is the fact that this $f$ must satisfy $P_1 f = 0$; we cannot accommodate an $S$-invariant $f$ equalling 1/4 modulo unity in building the desired solution. Since there is no corresponding term $D_8$ with this spin, we have no difficulty as follows. Starting from any $f$, a first choice for the $f$'s can be obtained: $f^{(0)} = \frac{1}{2} \Delta^{-1}(1+S) f$. We then enforce $S$-invariance by modifying the guess: $f^{(1)} = (1 - P_1) f^{(0)}$. However, the $f^{(1)}$ is $S$-invariant, and we therefore re-apply the operator $P_0$, $f^{(2)} = P_0 f^{(1)}$. This process iterates until we have finally achieved a $f$ which is the common term $D_8$ and $(1 - P_1)$. We write this solution for $(4.6)$; of course if this iterative process fails to converge, we replace it with this solution. (In practice, however, with $f$ restricted to the space $F$ of polynomials to be that generated by $S$-invariant $f$, we can almost immediately.) Having thus found $f$, we handle this $f$ as follows. Choose an $f \in F$ of the form $\vartheta^{26-D}$, and calculate the quantity

$$
\vartheta = \Delta^{-1} \lim_{n \to \infty} \left[ \frac{1}{2} (1 - P_1)(1+S) \right]^n f .
$$

(4.16)

If $P_3 S P_3 f = 0$, then up to one common scale factor the corresponding solution for the $F$'s is:

$$
F^1 = P_0 S P_3 f , \quad F^2 = -P_2 S P_3 f , \quad F^3 = P_3 f .
$$

(4.17)

It is easy to see how this procedure works. The goal is to construct a set of $F$'s which are distinguished by their eigenvalues under $[T]$, and which furthermore are closed under $[S]$. This we achieve by constructing a quantity $f$ which itself is $S$-invariant, and from which the individual $F$'s can be obtained by projection. The crucial element in $K = 8$, however, is the fact that this $f$
\( f \in F \), we define
\[
f \equiv \frac{1}{2} \Delta^{-1} (1 + S) f \,.
\]
whereupon we quickly have the solutions (up to a common scale factor)
\[
F^1 = P_0 f \,, \quad F^2 = 4 P_3 S P_0 f \,.
\]

Note that there were two reasons this \( K = 4 \) case was significantly simpler than the \( K = 8 \) case. First, for \( K = 4 \) we must demand \( P_1 f = P_3 f = 0 \) instead of the more difficult \( K = 8 \) constraints \( P_1 f = 0 \), \( P_3 f \neq 0 \); the presence of two zero-constraints in \( K = 4 \) instead of only one allowed us to subsume them together into a restriction in the space \( F \). Second, the matrix \( M_{(4)} \) has no zero entries; hence the \( F \)’s can always be found by simple projections and only their relative normalizations need be determined. In \( K = 8 \), however, we must further assert \( P_3 S P_3 f = 0 \); as stated above, this occurs because \( M_{(8)}^{33} = 0 \).

Given these procedures for generating solutions for the \( F \)’s satisfying (4.9) and (4.12) in the \( K = 4 \) and \( K = 8 \) cases, it is easy to build modular-invariant partition functions of the proper forms. Let us first construct some examples for the \((1, 4)\) case. Taking \( f = 2 \beta^5 \) [where we remind the reader of the definitions in (3.8)], we find \( f = \Delta^{-1} (\beta^5 + \delta^5) \), whereupon (4.20) yields the results
\[
F^1 = \frac{1}{2} \Delta^{-1} (\gamma^5 + \delta^5) \,, \quad F^2 = \Delta^{-1} \left[ 2 \beta^5 + (\gamma^5 - \delta^5) \right] \,.
\]
Since the internal gauge symmetry for such models is determined, as usual, by the left-movers, we can quickly identify this solution as corresponding to gauge group \( SO(40) \), the largest allowed in six spacetime dimensions. For example, we start with \( f = 2 \beta^{11/2} \), we find (or \( \Delta^{-1} (\beta^{11/2} + \delta^{11/2}) \), which immediately leads to
\[
F^1 = \frac{1}{2} \Delta^{-1} (\gamma^{11/2} + \delta^{11/2}) \,, \quad F^2 = \frac{1}{2} \Delta^{-1} (\gamma^{11/2} - \delta^{11/2}) \,, \quad F^3 = \Delta^{-1} \beta^{11/2} \,.
\]

Note that \( P_3 S P_3 f = 0 \), so this solution is independent. This solution corresponds to \( SO(44) \), the largest allowed in six spacetime dimensions.
4.2. Dictionaries for Model-Construction

While thus far it has been quite straightforward to identify the gauge groups corresponding to our partition functions, we have been dealing only with the simplest of cases; furthermore, in principle almost any properly-chosen function $f$ can serve in generating solutions, and we require a way to discern which of all possible solutions correspond to bona-fide fractional superstring models. Toward this end we now develop a method for generating partition functions which we believe do precisely this in the heterotic $(1,K)$ cases, and for which the underlying physics is substantially more transparent. Our approach rests on two fundamental observations.

The first observation has to do with the existence of a model with maximal gauge symmetry for the $(1,2)$ heterotic string in $D$ dimensions. For any value $D \leq 10$, there is always a self-consistent $(1,2)$ model which can be formulated with gauge group $SO(52 - 2D)$ – such a model is, in a sense, the starting point in model-building, for all other models can be obtained from it by altering this known solution via orbifolding, e.g., by adding twists, altering the boundary conditions of worldsheet fields, adding new sectors, etc., all of which tends to break the gauge group and correspondingly add new terms to the partition function. We therefore assert that such a self-consistent model exists as well for the $(1,K)$ fractional superstring, and has gauge group $SO(52 - 2D) = SO(48 - 16/K)$; indeed, the validity of this assertion follows directly from the (assumed) self-consistency of the $K > 2$ fractional superstring right-moving sector and the near-decoupling of the gauge sector (the $K = 1$ left-moving sector), as will be discussed below. Since the unique partition functions corresponding to these [see (4.21) and (4.24)], it therefore follows that these two correspond to actual $(1,K)$ models.

Our second and more important observation is well-known for the traditional $(1,2)$ heterotic string theory: the left-moving sector carries with it all of the information concerning the internal gauge group and particle representations; the right-moving sector, on the other hand, carries with it the linkage to spacetime physics, Lorentz spin and statistics, and spacetime supersymmetry. One builds a model by choosing these respective sectors so that certain physical constraints are satisfied: one must maintain worldsheet (super)conformal invariance, modular invariance (which incorporates proper level-matching), spacetime Lorentz invariance, and physically sensible internal (GSO-like) projections (thereby incorporating proper spacetime spin-statistics). Of course, not all of these requirements are independent. Some of these requirements can clearly be placed on the left- and right-moving sectors separately: among these are worldsheet (super)conformal invariance, spacetime Lorentz invariance, and physically sensible projections. Modular invariance, on the other hand, constrains both sectors jointly. Thus, when building a model, one must satisfy essentially two kinds of constraints: those which involve the left- and right-moving sectors independently (guaranteeing that they are each
those (i.e., modular invariance and the implied level-matching) which insure that they are properly coupled or linked to each other.

In the case of the heterotic \((1, K)\) fractional superstring, we expect the same situation to prevail: we must determine those \(K = 1\) left-moving sectors which are themselves internally self-consistent, and then we must join them with our (assumed self-consistent) \(K > 2\) right-moving sector in such a way that modular-invariance is satisfied. Fortunately, the underlying physics of a \(K = 1\) left-moving sector is well-understood; for example, descriptions of it in terms of lattices, orbifolds, or Fock-space spectrum-generating formulae abound. In particular, it is well-known how to construct valid \((1, 2)\) models which satisfy all of the physical model-building constraints we have listed. Therefore, one might hope to be able to build valid \((1, K)\) models by first building valid \((1, 2)\) models, and then “replacing” their \(K = 2\) right-moving sectors with our new \(K > 2\) right-moving sectors in such a way that modular-invariance (the sole “linking” constraint) is not violated. Such a procedure would thereby guarantee, in the language of the previous subsection, a set of \(F^i\)'s which themselves are known to correspond to valid \(K = 1\) left-moving sectors.

It turns out that these arguments can be phrased directly in terms of a correspondence or “dictionary” between right-moving \(K = 2\) physics and \(K > 2\) physics, in the sense that they may be substituted for each other in this way when building models. At the level of the partition function (which has been the basis of our approach), this means that we are able to draw a correspondence between the respective \(\Theta\)-functions of these right-moving sectors. For the \(K > 2\) theory these \(\Theta\)-functions are simply the expressions we have been using to build our partition functions, and the \(\Theta\)-functions are the usual Jacobi \(\vartheta\)-functions. We thus construct a dictionary relating the expressions, in the sense that the two underlying sectors’ expressions couple in the same manner to left-moving physics and can be used interchangeably in building self-consistent \((1, K)\) type.

It is fairly straightforward to construct this dictionary between the \(\Theta\)-functions and the \(\vartheta\)-functions, for our first assumption [the validity of the \((1, K)\) solutions \((4.21)\) and \((4.24)\)] allows us to make this connection in the case of maximal gauge symmetry. Let us first concentrate on the case \(K = 4\). Recall from Sect. III that the supersymmetric \((1, 2)\) model in \(D = 6\) with the maximal gauge group \(SO(40)\) has the partition function

\[
\mathcal{Z}_{SO(40)} = (\frac{1}{2})^2 \tau_2^{-2} \mathcal{K} (\overline{\beta} \beta)^5
\]

(where \(\mathcal{K} \equiv \Delta^{-1} \Delta^{-1/2} \mathcal{J}\)). Repeatedly making use of the algebraic identity

\[
Aa + Bb = \frac{1}{2} (A + B) (a + b) + \frac{1}{2} (A - B) (a - b)
\]

it turns out that we can write

\[
\overline{\beta} \beta^5 + \overline{\gamma} \gamma^5 + \overline{\delta} \delta^5 = \frac{1}{2} (\overline{\gamma} + \overline{\delta}) (\gamma + \delta) + \frac{1}{2} (\overline{\gamma} - \overline{\delta}) (\gamma - \delta)
\]
We therefore have

\[ \mathcal{Z}_{SO(40)} = \frac{1}{4} \tau_2^{-2} \Delta^{-1} \overline{\Delta}^{-1/2} \mathcal{J} \left\{ \frac{1}{4} (\overline{\beta} + \overline{\gamma} - \overline{\delta}) [2 \beta^5 + (\gamma^5 - \delta^5)] 
+ \frac{1}{2} (\overline{\gamma} + \overline{\delta})(\gamma^5 + \delta^5) - \frac{1}{4} \mathcal{J} [2 \beta^5 - (\gamma^5 - \delta^5)] \right\} \]  

(4.28) for the (1, 2) maximally symmetric model in \( D = 6 \).

The next step involves properly interpreting some of the factors in (4.28). Since this is the partition function of a (1, 2) model, the critical dimension is 10, and reducing the spacetime dimension to 6 through fermionization (as discussed in Sect. III) yields worldsheet matter consisting of the 4 transverse bosonic coordinate fields, 16 Majorana-Weyl right-moving fermions, and 40 Majorana-Weyl left-moving fermions. This is why each term in (4.28) contains two anti-holomorphic powers of \( \overline{\beta}, \overline{\gamma}, \) or \( \overline{\delta} \) (recall that \( J = \gamma - \beta - \delta \)), and five holomorphic powers of \( \beta, \gamma, \) or \( \delta \). Of these 16 right-moving fermions, four are the superpartners of the bosonic coordinate fields (and hence carry spacetime Lorentz indices) while the remaining twelve are internal and carry only internal quantum numbers. However, of these twelve, four had previously been the superpartners of the (now fermionized) 10 – 6 coordinate bosons, and as such their degrees of freedom (in particular, their toroidal boundary conditions) must be chosen to be the same as those of the four fermions carrying Lorentz indices.

In Ref. [10], for example, this is the result of the so-called “triplet” constraint, which arises by demanding the periodicity or anti-periodicity of the worldsheet supercurrent on the worldsheet torus. Thus, the 16 right-moving Majorana-Weyl fermions split into two groups: the first eight (which include the four with spacetime Lorentz indices) must all have the same boundary conditions, and the remaining eight (all of which are internal) can be chosen independently. As we can see from the partition function, the factor \( \mathcal{J} \) of eight right-moving worldsheet fermions are present to produce the overall factor of \( \mathcal{J} \): this is indeed related to the worldsheet supersymmetry. The remaining eight right-moving fermions, however, are internal, and these are responsible for the remaining \( \overline{\delta} \) within the braces in (4.28) above.

This understanding is very important, for it enables us to interpret the \( J \) factor in the second line of (4.28). As we stated, the \( J \) in the first line expresses spacetime supersymmetry: the identity \( J = 0 \) represents the complete cancellation of spacetime bosonic states against spacetime fermionic states. The second factor of \( J \), however, arises from exclusively internal degrees of freedom, and thus for this term the identity \( J = 0 \) represents an internal GSO-like projection between particles of the same spacetime statistics (in fact, between the same particle states). Thus, the last term in (4.28) contains no physical states whatsoever, and may be legitimately dropped. We therefore obtain

\[ \mathcal{Z}_{SO(40)} = \frac{1}{4} \tau_2^{-2} \Delta^{-1} \overline{\Delta}^{-1/2} \mathcal{J} \times \mathcal{J} \times \left\{ \frac{1}{4} (\overline{\beta} + \overline{\gamma} - \overline{\delta}) [2 \beta^5 + (\gamma^5 - \delta^5)] \right\} \]  

(4.29) for this (1, 2) model, or equivalently

\[ \mathcal{Z}_{SO(40)} = \tau_2^{-2} \left\{ \frac{\Gamma_{K=2, D=6}}{R_{1}^{K=2, D=6}} L_{1}^{SO(40)} \right\} \]

for this (1, 2) model.
where the two factors arising from the right-moving $K = 2, D = 6$ theory are

\[
R_{1}^{K=2,D=6} \equiv \frac{1}{4} \Delta^{-1/2} J (\gamma + \delta)
\]

\[
R_{2}^{K=2,D=6} \equiv \frac{1}{16} \Delta^{-1/2} J (\beta + \gamma - \delta)
\]

and where the two factors arising from the left-moving $K = 1$ $SO(40)$ theory are:

\[
L_{1}^{SO(40)} \equiv \frac{1}{2} \Delta^{-1} (\gamma^5 + \delta^5)
\]

\[
L_{2}^{SO(40)} \equiv \Delta^{-1} (2\beta^5 + \gamma^5 - \delta^5)
\]

Now that we have rewritten (4.25) in the form (4.30), we can compare this result with the partition function for the $(1, 4)$ $SO(40)$ model. Recall that this latter partition function was found to be [see (4.3) and (4.21)]:

\[
Z = \tau^{-2} \left\{ \frac{1}{2} A_{4} \Delta^{-1} (\gamma^5 + \delta^5) + B_{4} \Delta^{-1} [2\beta^5 + (\gamma^5 - \delta^5)] \right\}
\]

\[
= \tau^{-2} \left\{ A_{1} L_{1}^{SO(40)} + B_{1} L_{2}^{SO(40)} \right\}
\]

(4.33)

note that one can verify the overall normalization of (4.33) by counting the numbers of low-lying (e.g., tachyonic or massless fermionic) degrees of freedom. It is clear that (4.30) and (4.33) have the same left-moving holomorphic pieces $L_{i}^{SO(40)}$, and therefore we can relate them to each other, i.e.,

\[
Z_{(1,2)} = Z_{(1,4)} \quad \text{for } SO(40)
\]

(4.34)

(where $\simeq$ indicates this relation or correspondence), if we make the following correspondences:

\[
A_{1}^{(4)} \equiv A_{4} \simeq R_{1}^{K=2,D=6}
\]

\[
A_{2}^{(4)} \equiv B_{4} \simeq R_{2}^{K=2,D=6}
\]

where the $R$'s are given in (4.31). This result, a “dictionary” relating $A_{4}$ and $B_{4}$ to the Jacobi factors (and thereby enabling us to interchange their corresponding $K = 2$ and $K = 4$ right-moving sectors at the partition-function level) in order to build models of the $(1, 4)$ variety. We remark that in principle it is just as straightforward to construct suitable for building $(2, 4)$ models; indeed, the expressions we have obtained can themselves be taken as the partition-function contributions from the self-consistent left-moving sectors of such theories.

Note that if we had not dropped the third term in the $(1, 2)$ partition function (4.28), we would have required a third string-function expression $C_{4}$ to relate to

\[
R_{3}^{K=2,D=6} \equiv \Delta^{-1/2} J (\gamma - \beta - \delta)
\]

Here we have explicitly indicated the origins of the two independent Jacobi factors, labelling with subscripts whether they arose in the $(1, 2)$ $D = 6$ theory from spacetime or internal degrees of freedom. However, such a string-function expression $C_{4}$ does not exist. It is indeed fortunate that the GSO-projection $J_{\text{int}} = 0$ enables us to avoid this unwanted term internally self-consistent manner, and thereby

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There are several things to note about this dictionary. First, we observe that this dictionary is self-consistent as a relation between modular functions, with each side of the relation transforming under the modular group with weight \( k = -2 \) [as appropriate for \( D = 6 \), according to (3.3)]. Furthermore, the expressions on the right side of (4.35) mix under the \([S]\)-transformation with the same matrix \( M(4) \) found for the \( A_i(4) \), and the same, of course, holds true for the \([T]\)-transformation and the \( N(4) \) matrix. Indeed, the right sides of (4.35) each have \( q \)-expansions of the forms \( q^h(1+...) \) where inside the parentheses all \( q \)-powers are integral and where \( h \)-values are equal \((\text{mod } 1)\) on both sides of each equation.

Additionally, this dictionary incorporates the spacetime supersymmetry of \( A_4 \) and \( B_4 \) in a natural way, allowing these expressions (which are themselves the “Jacobi identities” for \( K = 4 \)) to correspond to expressions proportional to the spacetime factor \( J \), the \( K = 2 \) Jacobi identity.

In fact, from this dictionary it is now clear that one cannot expect to factorize the string-function expressions \( A_i(4) \) into two pieces, one of which might vanish on its own. In the \((1,2)\) theory \([i.e., \text{the right side of (4.35)}]\), the \( J \) factor is the result of spacetime-related degrees of freedom, and the remaining factors linear in \( \beta, \gamma, \) and \( \delta \) in (4.35) are the contributions from the internal right-moving degrees of freedom. For the \( K = 4 \) theory, however, we are in the critical dimension, and hence all right-moving modes carry spacetime information and play a part in yielding spacetime supersymmetry. We therefore do not expect to be able to factorize the left sides of (4.35) correspondences between individual factors on both sides of each equation.

This dictionary allows us to easily generate fractional superstring models: we start with a known \((1,2)\) model compactified in some manner to six spacetime dimensions, and then make the “translation” given in (4.35) to the \( K = 4 \) right-moving sector. As examples, we can translate \((1,2)\) models whose partition functions \( SO(24) \otimes E_8 \) model is particularly simple to the \((1,4)\) case the analogue of (4.27) becomes

\[
\overline{\beta}^3 + \overline{\gamma}^3 + \overline{\delta}^3 = \frac{1}{4} (\overline{\beta} + \overline{\gamma} - \overline{\delta}) [2\beta^3 + (\gamma^3 - \delta^3)]
\]

where we have set \( J_{\text{internal}} = 0 \). Upon substituting this internal partition function in (3.16) and translating according to (4.35), we indeed find the partition function (4.3) with the \( F \)'s given in (4.21). This confirms that the solution in (4.21) with the \( F \)'s given corresponds to a valid \((1,4)\) model with the \( (\text{left-moving}) \) gauge group claimed. Similarly translating the \( SO(24) \otimes SO(16) \) partition function in (3.16), we obtain the solution quoted in (4.23), confirming the interpretation of that solution as corresponding to a valid model with the quoted gauge group.

We stress again, of course, that this dictionary derivation does not merely duplicate the results found earlier, for the general procedure presented in the previous subsection merely assures the creation of modular-invariant expressions \( Z \). It is the crucial fact that we can derive
our dictionary translation which guarantees their interpretation as the partition functions of actual models. To illustrate this fact, let us consider the creation of (1, 4) models having gauge groups $G$ of the form $G_{12} \otimes G_{8}$, where $G_r$ denotes a simple (and simply-laced) group of rank $r$. Recall that the procedure given in the previous subsection allows us to specify our possible (1, 4) functions $\mathcal{Z}$ in terms of the simpler $S$-invariant expression $f$. Since we expect a gauge group $G_r$ to reveal itself in $\mathcal{Z}$ through the presence of factors which are linear combinations of $\beta^p$, $\gamma^p$, and $\delta^p$ where $p = r/4$ [for example, the factor (or character) corresponding to $E_8$ is $(\beta^2 + \gamma^2 + \delta^2)$], we can survey many such $\mathcal{Z}$'s by building functions $f$ of the form $f^{(i,j)} = \Delta^{-1} Q_3^{(i)} Q_2^{(j)}$ where, for example,

$$Q_p^{(1)} = \gamma^p, \quad Q_p^{(2)} = \beta^p + \delta^p, \quad Q_p^{(3)} = \beta^p + \gamma^p + \delta^p, \quad Q_p^{(4)} = \beta^p - \gamma^p + \delta^p. \quad (4.38)$$

Some of these possibilities we have already examined: for example, we have confirmed that $f^{(3,1)}$ generates an $SO(24) \otimes SO(16)$ model, and that $f^{(1,3)}$, $f^{(2,3)}$, and $f^{(3,3)}$ each generate the $SO(24) \otimes E_8$ model. In fact, we have also seen that $f^{(1,1)}$ leads to the maximally-symmetric $SO(40)$ model (by construction), and $f^{(4,3)}$ turns out to be a null solution (all the $F$'s vanish). However, there are eight other distinct modular-invariant functions $\mathcal{Z}$ which can be constructed in this form [eight rather than ten because $f^{(4,1)}$, $f^{(4,2)}$, and $f^{(4,4)}$ each lead to the same $\mathcal{Z}$], and each of these might reasonably correspond to a valid (1, 4) model. For instance, $f^{(4,1)}$ leads to a modular-invariant function $\mathcal{Z}$ which is the translated version of

$${\mathcal{Z}} = \tau_2^{-2} K (\beta^2 - \gamma^2 + \delta^2) (\beta^3 - \gamma^3 + \delta^3). \quad (4.39)$$

However, there does not exist a (1, 2) model having this form. Our dictionary therefore enables us to conclude that there is no (1, 2) model which can be generated for this case.

In a similar manner, we can obtain the corresponding $z$-invariant expressions $f$ in the $(D = 4)$ model given in (3.13). It is a simple matter to rewrite

$$\mathcal{Z} = (\gamma^2 + \beta^2 + \delta^2)/(\gamma^2 + \beta^2 + \delta^2) \quad (3.13)$$

whereupon a quick comparison with the solution $z^{(3,1)}$ gives the dictionary:

$$A_{(8)}^1 \equiv A_8 = R_{(1)}^K$$
$$A_{(8)}^2 \equiv B_8 = R_{(2)}^K$$
$$A_{(8)}^3 \equiv C_8 = R_{(3)}^K$$

where the factors from the right-moving $K = 2$ theory are:

$$R_{(1)}^{K=2,D=4} \equiv \frac{1}{2} \Delta^{-1/2} J$$
$$R_{(2)}^{K=2,D=4} \equiv \frac{1}{2} \Delta^{-1/2} J$$
$$R_{(3)}^{K=2,D=4} \equiv \frac{1}{2} \Delta^{-1/2} J$$

Note that it is extremely fortunate that the solution $z^{(3,1)} = 0$, satisfying $P_3 S P_3 f = 0$, for if this solution had not
could not have been built. Once again, we observe that this dictionary is self-consistent, with both sides of (4.41) having modular weight \( k = -1 \) and mixing identically according the matrices \( M(8) \) and \( N(8) \) under the \([S]\)- and \([T]\)-transformations respectively. We remark that (2, 8) models can be built as well, simply by taking these expressions \( R_i^{K=2,D=4} \) as the valid representative partition-function contributions from self-consistent left-moving \( K = 2 \) sectors.

As in the \( K = 4 \) case, the \( J \) factors in (4.42) arise in the \((1, 2)\) theory from spacetime-related degrees of freedom; the remaining factors, on the other hand, arise from purely internal degrees of freedom. However, unlike the \( K = 4 \) case, it would have been impossible now to obtain a complete GSO projection with these internal factors, for the analogous combination \( \gamma^{3/2} - \beta^{3/2} - \delta^{3/2} \) is non-vanishing. Fortunately, the \( K = 8 \) theory provides three string-function expressions \( A_8, B_8, \) and \( C_8 \) to relate to our three \( R^{K=2,D=4} \) factors, so our \( K = 8 \) dictionary could nevertheless be constructed. It is indeed curious and fortuitous that the \( K = 4 \) and \( K = 8 \) theories provide exactly the needed number of independent string-function expressions with which to build supersymmetric fractional superstring partition functions.

5. MODEL TRANSLATABILITY

In the previous section we established our dictionaries and they enable us to confirm whether the possible functions \( Z(1,K) \) correspond to valid \((1,K)\) models. Of considerably more interest, however, is the question of examining those that \textit{do} correspond to models in this space of new \((1,K)\) models and determining the relevant features. This is, in a sense, the opposite issue, for now we must use our dictionaries to determine which of the \((1,2)\) models are in correspondence with valid \((1,K)\) models.

At first glance it may seem that our dictionaries for \( K = 4 \) and \( K = 8 \) allow any spacetime-supersymmetric \((1,2)\) model to be translated. We will find that not all \((1,2)\) models may be translated. We will find that only those possessing a \textit{maximal} number of spacetime supersymmetries are in correspondence with valid \((1,K)\) models.

At first glance it may seem that our dictionaries for \( K = 4 \) and \( K = 8 \) allow any spacetime-supersymmetric \((1,2)\) model to be translated. We will find that not all \((1,2)\) models may be translated. We will find that only those possessing a \textit{maximal} number of spacetime supersymmetries are in correspondence with valid \((1,K)\) models.
corresponding \((1,4)\) model:

\[
Z_{(1,4)} = \tau_2^{-2} \Delta^{-1} \left\{ A_4 (Y + Z) + B_4 [4X + 2(Y - Z)] \right\}.
\] (5.3)

This result will be modular-invariant provided (5.1) is modular-invariant. Note that the identity (5.2), which makes the necessary rewriting possible, is simply the statement that the linear combinations \(\bar{\tau} + \bar{\delta}, \bar{\beta} + \bar{\tau} - \bar{\delta}\), and \(\bar{\bar{J}} = \bar{\tau} - \bar{\beta} - \bar{\delta}\) span the three-dimensional space spanned by the combinations \(\bar{\beta}, \bar{\tau}\), and \(\bar{\delta}\) separately, and therefore setting \(\bar{J} = 0\) (the internal GSO projection) always leaves the remaining two left-moving factors \(F^1\) and \(F^2\) to multiply the right-moving factors \(\bar{A}_4\) and \(\bar{B}_4\) in the \((1,4)\) partition function. The same argument can be made for the \(K = 8\) case as well. It would therefore seem that the spaces of \((1,K)\) models are as large as the spaces of \((1,2)\) models compactified to the appropriate dimensions \(D = 2 + 16/K\), a conclusion which would suggest that the dictionary translation idea does not yield the expected substantial truncation in the sizes of the space of \((1,K)\) models (which are of course in their critical dimensions).

Fortunately, this is not the case, for there are two important reasons why valid supersymmetric \((1,2)\) models may fail to be translatable. First, they may fail to have partition functions of the needed general forms

\[
Z = \tau_2^{-2} \Delta^{-1} \bar{\Delta}^{-1/2} \bar{J} \left\{ \bar{\beta}^{(10-D)/4} X + \bar{\tau}^{(10-D)/4} Y + \bar{\delta}^{(10-D)/4} Z \right\};
\] (5.4)

indeed, we will see that the vast majority of known supersymmetric \((1,2)\) models do not have this form. Second, even though a given model may technically have a partition function of the form (5.4), its contributions to that partition function may not be of the correct forms. We will discuss each of these possibilities in turn.

The first possibility is that a given \((1,2)\) spacetime-supersymmetric model may fail to have a partition function of the form (5.4). While the assumed spacetime supersymmetry guarantees the presence of the factors \(\bar{J}\), and the \(\Delta\)-functions must always appear as in (5.4) for a heterotic \((1,2)\) string theory, we are not assured that our remaining right-moving factors \(\bar{\beta}^{(10-D)/4}, \bar{\tau}^{(10-D)/4}, \bar{\delta}^{(10-D)/4}\), or their linear combinations, will be of the correct forms. We will discuss each of these possibilities in turn.

Fortunately, this is not the case, for there are two important reasons why valid supersymmetric \((1,2)\) models may fail to be translatable. First, they may fail to have partition functions of the needed general forms
factors—and simultaneously avoid the unwanted ones—our underlying model must have a symmetry relating the internal (i.e., non-spacetime-related) right-moving worldsheet fermions so that they each uniformly produce the same contributions to $Z$. For example, these $10-D$ internal fermions must have the same toroidal boundary conditions in all sectors of the model, and be in all respects interchangeable. Such a worldsheet symmetry is not new: if such a symmetry appears amongst the left-moving worldsheet fermions, there will be corresponding massless spacetime vector particles transforming in the adjoint representation of this symmetry group. Thus, such a left-moving worldsheet symmetry can be interpreted as a spacetime gauge symmetry. What, however, are the spacetime consequences of such a right-moving worldsheet symmetry involving these $10-D$ worldsheet fermions?

Fortunately, such a worldsheet symmetry is easy to interpret: it is responsible for a multiplicity in the number of spacetime gravitinos, so that the larger the rank of the symmetry group, the larger the multiplicity. In fact, if the rank of this worldsheet symmetry group is $10-D$ (so that all internal right-moving fermions are involved, as is needed for translatability), then the number of gravitinos in the spectrum of the model is $N_{\text{max}}$, where $N_{\text{max}}$ is given in (4.1). The analysis needed for proving this assertion is not difficult, but varies greatly with the type of $(1,2)$ model-construction procedure we employ; in Appendix B we provide a proof using the free-fermion construction of Ref. [10]. We therefore conclude that only models with $N = N_{\text{max}}$ supersymmetry have partition functions of the form (5.4).

In the $K = 2$ case ($D = 10$), this of course gives $N_{\text{max}} = 1$ for $D = 10$, the only models which have partition functions of the form (5.4) are indeed those which are supersymmetric models,$^2$ and their partition functions are given in (3.7). However, this translatability requirement is much more restrictive for the majority of supersymmetric $(1,2)$ models in $D < 10$. For example, it has been found$^{[12]}$ that of $D = 4$ free-fermion models, there exist fewer than 1150 with $N_{\text{max}} = 4$susy; this is to be compared against over 32,000 models with $N_{\text{max}} = 1$susy$^{[12]}$ and a virtually limitless supply with no spacetime supersymmetry at all.$^{[13]}$ We see, therefore, that the number of translatable $(1,2)$ models in $D = 4$—and hence the number of $(1,8)$ models—is severely restricted. A similar restriction exists for the $K = 4$ case as well.

As mentioned earlier, there is also a second reason why a $(1,2)$ model may fail to be translatable, even if it does technically have a partition function of the algebraic form (5.4). Let us consider the spacetime and the algebraic form (5.4). Let us consider the separate contributions to the total partition function of a model from the spacetime bosonic and fermionic states respectively. Recall that in the spacetime Jacobi factor $J$, the term $(\gamma - \delta)$ is the contribution from spacetime bosons (the Neveu-Schwarz sector) and the term $\beta$ is the contribution from spacetime fermions (the Ramond sector). This statement assumes that we have carefully avoided cancelling spacetime bosonic and fermionic states in constructing the partition function, as will be discussed below.
holomorphic factors in the partition function represent purely internal degrees of freedom. Let us therefore separate these two pieces, and write the general bosonic and fermionic partition functions from an arbitrary model as follows:

\[
Z_b = \tau^2 k \Delta^{-1} \Delta^{-1/2} \left\{ (\gamma - \delta) V + W_b \right\}, \\
Z_f = \tau^2 k \Delta^{-1} \Delta^{-1/2} \left\{ \beta V + W_f \right\}.
\]

(5.5)

Here \(W_b\) is that part of \(Z_b\) whose spacetime anti-holomorphic factor is not proportional to \((\gamma - \delta)\). It is clear that the total partition function \(Z = Z_b - Z_f\) will be of the algebraic form (5.4) if \(W_b = W_f = W\) where \(W\) is arbitrary and if \(V\) is of the form of the term in braces in (5.4). However, translatability additionally requires that \(W\) vanish, regardless of its form. The reason for this is quite simple. If \(W \neq 0\), then allowing \(W_b\) and \(W_f\) to cancel in the difference \(Z_b - Z_f\) amounts to cancelling spacetime bosonic and fermionic states and is therefore inconsistent with our retention of the spacetime Jacobi factor \(\mathcal{J}\) in (5.4); indeed, writing \(W_b - W_f = 0\) is tantamount to simply writing \(\mathcal{J} = 0\) in (5.4). Another way of seeing this is to realize that while \(W_b\) and \(W_f\) might be algebraically equal, the \textit{spacetime} factor in \(W_b\) must include combinations of \(\gamma\) and \(\delta\) only (since it comes from spacetime bosonic degrees of freedom), while the spacetime factor in \(W_f\) must be simply \(\beta\). Such expressions \(W_b\) and \(W_f\) should therefore not be cancelled, so translatability requires that they not appear at all. Note that this additional constraint \(W = 0\) is not vacuous, for there exist many models for which the total \(Z\) is of the algebraic form (5.4) but for which \(W \neq 0\): these are models with partition functions of the form \(Z + W - W\). It turns out, however, that any \((1, 2)\) model is guaranteed to have \(W = 0\), so the \(N = N_{\text{max}}\) constraint ensures model translatability. (In fact, as noted earlier, so does the \(N = N_{\text{max}}\) constraint cease to be \textit{overly} restrictive again that \(N = N_{\text{max}}\) SUSY is the \(1, 2\) physics underlying translatability.

One might argue that this \(N = N_{\text{max}}\) translatability constraint is somewhat artificial: our dictionary was constructed by comparing the \(SO(48 - 16)/K\) model with the maximally symmetric \(SO(52 - 2D)\) \((1, K)\) model, and since this latter model always has an \(N = N_{\text{max}}\) right-moving sector, this \(N = N_{\text{max}}\) constraint was thereby “encoded” in the beginning. Indeed, one might claim that other dictionaries could be constructed for \(N < N_{\text{max}}\) cases simply by comparing, for example, our maximally symmetric \((1, K)\) model with a maximally symmetric \((N < N_{\text{max}})\) \((1, 2)\) model. However, such approaches ultimately fail to yield self-consistent dictionaries. At a mathematical level, this occurs because it is not possible to construct \(\theta\)-function expressions which could appear on the \(K = 1\) sides of such dictionaries: such expressions would have to be not only eigenfunctions of \(T\) but also closed under \(S\), and the \(N = N_{\text{max}}\) solutions we have constructed are the only ones possible. On a physical level, we can understand this result as follows. Unlike the \((1, 2)\) models in \(D < 10\), our \((1, K)\) models are in their critical dimensions, and therefore they lack “intrinsic” right-moving degrees of freedom. Thus, we expect that any dictionary relating a \(K = 2\) right-moving sector to a \(K > 2\) right-moving sector should
tra degrees of freedom in the $K = 2$ sector in order to introduce twists or (super)symmetry-breaking; rather, we expect these degrees of freedom to be “frozen out”, handled jointly as though they were one block. This is precisely how they appear in the dictionaries (4.35) and (4.41), and is the root of the previously encountered indistinguishability of (or symmetry relating) the internal right-moving fermions. Thus, it is indeed sensible that our $(1, K)$ models are the analogues of the $N = N_{\text{max}} (1, 2)$ models, for both are the unique models in which no internal right-moving degrees of freedom are available for (super-)symmetry breakings. Note that this argument does not tell us the degree of supersymmetry for the $(1, K)$ models themselves. However, both an examination of the individual bosonic and fermionic degrees of freedom and an overall counting of states indicate that the $(1, K)$ strings have $N = 1$ supersymmetry [as distinguished from the $(K, K)$ strings, which have $N = 2$ supersymmetry\(^\text{[3]}\)].

Finally, we should point out that it is also possible to build fractional superstring models which are not spacetime-supersymmetric. There are primarily two ways in which this can be done. First, it is possible to construct string-function expressions which are similar to $A_K$, $B_K$, and $C_K$ but which do not vanish; partition functions built with these expressions would then correspond only to non-supersymmetric models, and analogous dictionaries could be constructed (in the manner presented in Sect. IV) guaranteeing that such self-consistent $(1, K)$ models actually exist. Constructing such expressions is not difficult: of all the requirements listed after (3.27), we need only eliminate the tachyon-free constraint $a_n = 0$ for all $n < 0$. Note that removal of this require-

ment need not introduce spacetime tachyons into the $(1, K)$ model, for in general a partition function

$$Z = T^k \sum_{m,n} a_{mn} \tau^m q^n$$

will have physical tachyons only if a diagonal element $a_{nn}$ is non-zero for some $n < 0$. Indeed, it is easy to choose holomorphic $K = 1$ sectors for these models in such a way that $a_{nn} = 0$ for all $n < 0$.

A second way to build non-supersymmetric models with the supersymmetric expressions $A_K$, $B_K$, and $C_K$ but then separate them into their spacetime bosonic and fermionic sectors in a way that $A_K = A_K^{(b)} - A_K^{(f)}$, etc. For the $K = 2$ case, it is clear how to do this: the term $\beta$ represents the contributions from Ramond (i.e., spacetime-fermionic) sectors in the theory, and the remaining term $\gamma - \delta$ arises only from a Neveu-Schwarz (i.e., spacetime-bosonic) sector. For the $K > 2$ “para-Jacobi” identities $A_K = 0$, the situation is more complicated. A first approach might be to use the dictionaries developed in Sect. IV to relate our desired $K > 2$ bosonic and fermionic pieces to the known $K = 2$ pieces: we would simply split the spacetime Jacobi factor $J$ which appears in the $K = 4$ and $K = 8$ dictionaries, for example, the $K = 4$ string-function expressions $A_4^{(b)}$, $A_4^{(f)}$, and $B_4^{(b,f)}$ which would be consistent with the
\[ A_4^{(b)} = 4k \Delta^{-1/2} (\gamma - \delta)_{\text{st}} (\gamma + \delta)_{\text{int}} \]
\[ A_4^{(f)} = 4k \Delta^{-1/2} (\beta)_{\text{st}} (\gamma + \delta)_{\text{int}} \]
\[ B_4^{(b)} = k \Delta^{-1/2} (\gamma - \delta)_{\text{st}} (\beta + \gamma - \delta)_{\text{int}} \]
\[ B_4^{(f)} = k \Delta^{-1/2} (\beta)_{\text{st}} (\beta + \gamma - \delta)_{\text{int}} \]

where we have explicitly indicated with subscripts the separate $K = 1$ spacetime and internal factors. However, it is easy to see that this method is not appropriate, for we do not expect the cancellations occurring in the full expressions $A_K$ and $B_K$ to mirror the relatively simple cancellation occurring in the $K = 2$ case.

Indeed, it is easy to show that no string functions for the left sides of (5.7) can be found which transform under $[S]$ and $[T]$ as do the right sides. Instead, one can determine the separate bosonic and fermionic contributions to the expressions $A_K$, $B_K$ and $C_K$ by demanding that these individual contributions each have $q$-expansions $\sum_n a_n q^n$ in which all coefficients $a_n$ are non-negative, and have relatively simple closure relations under $[S]$. This approach, in fact, proves successful, and yields results consistent with our interpretation [discussed after (3.31)] that the terms $(c_0^0)^{D-3}(c_0^0)$ are bosonic and the terms $(c_{K/2}^{K/2})^{D-2}$ are fermionic. Thus, the construction of non-supersymmetric $(1, K)$ models is in principle no more difficult than that of the supersymmetric models we have already considered, and we expect our dictionary techniques to generalize to these cases as well [though of course not yielding dictionaries similar to (5.7)].

To summarize the results of this and the previous section, then, we have succeeded in developing a method by which the partition functions of valid supersymmetric $(1, K)$ models can be generated and their gauge groups identified. This was achieved by establishing a correspondence between $(1, K)$ and $(1, 2)$ models in $D = 2 + 16/K$ dimensions [Eqs. (4.35) and (4.41)] enabling one to “translate” from the understood $K = 2$ sector and the less-understood $K > 2$ sector. We found that only $(1, 2)$ models were translatable, and we were thereby able to estimate the sizes of the spaces of $(1, K)$ models.
6. FURTHER DISCUSSION AND REMARKS

In this concluding section we discuss two different extensions of our results: these are the questions of obtaining chiral fermions and achieving Lorentz invariance. We begin by investigating how chiral fermions – or more generally, chiral supermultiplets – might arise in the fractional superstring models we have been investigating. We discuss several different methods for obtaining such multiplets, one of which involves formulating or interpreting these models in dimensions less than their critical dimensions. While this might seem to spoil the original attraction of the fractional superstring approach, we find instead that such a compactification is not at all arbitrary (as it is for the traditional superstrings), but rather is required in order to achieve a self-consistent Lorentz-invariant interpretation. Indeed, we find that requiring Lorentz invariance seems to specify a “natural” dimension in which the theory must be formulated, thereby (in a unique manner) simultaneously offering a possible solution to the chiral fermion problem. We emphasize that such a “forced” compactification appears to be a feature wholly new to fractional superstrings. Furthermore, we find that the “natural” dimension for the $K = 4$ string appears to be $D = 4$, rendering the $K = 4$ fractional superstring the most likely candidate for achieving chiral particle representations in four-dimensional spacetime while maintaining Lorentz invariance.

We begin by investigating how we might obtain chiral massless spacetime supermultiplets (i.e., supermultiplets which transform in a complex representation of the gauge group) in our fractional superstrings. Recall from Sect. III that the massless particles appear only in the $A_K$ such as we have examined; indeed, for $K \geq 2$, the $(d_0)$ contains the contributions of the right-moving components of the massless particles $A^\mu$ while the $( \ell_{K/2}^{K/2}, D_{-2}^{D_{-2}})$ term contains the contributions of the left-moving components of the massless fermions $\psi^\alpha$ (where $\alpha$ is a spacetime Lorentz spinor index). For the heterotic $(1, K)$ theory in its critical dimension, the massless spacetime supermultiplet structure is in general

$$ (\psi^\alpha, A^\mu)_R \otimes \left( X^\nu \oplus \phi^{(h_1)} \oplus \ldots \right) $$

where $(\psi^\alpha, A^\mu)_R$ is the fermion/vector supermultiplet discussed above, and for the left-moving excitations we have indicated the various possible massless states: $X^\nu$ denotes the Lorentz-vector (gauge-singlet) state achieved by exciting the worldsheet left-moving component of the spacetime coordinate boson, and the $\phi^{(h_i)}$ denote the various remaining Lorentz-scalar states combined into representations (with highest weights $h_i$) of the relevant left-moving gauge group.

Within (6.1), the combination

$$ (\psi^\alpha, A^\mu)_R \otimes (X^\nu) $$

forms the usual $N = 1$ supergravity multiplet containing the spin-2 graviton $g^{\mu \nu}$, spin-1 antisymmetric tensor field $B^{\mu \nu}$, spin-0 dilaton $\phi$, spin-3/2 gravitino $\lambda^\nu$, and spin-1/2 fermion $\lambda^\alpha$. All of these states are of course gauge-singlets. The only other combination within (6.1) allows
is

\[(\psi^\alpha, A^\mu)_R \otimes (\phi^{(h_{\text{adj}})})_L\]  \hspace{1cm} (6.3)

where these left-moving states fill out the adjoint representation of the gauge group. This yields, of course, a spacetime supermultiplet consisting of spin-1 gauge bosons \(A^\mu_{\text{adj}}\) and their spin-1/2 fermionic superpartners \(\psi^\alpha_{\text{adj}}\). The crucial point, however, is that this supermultiplet structure therefore does not permit spacetime fermions which transform in any other (e.g., complex) representations of the gauge group.

In the usual superstring phenomenology, a chiral supermultiplet can be achieved by introducing an additional right-moving supermultiplet of the form \((\phi_i, \psi^\alpha)_R\) where the \(\phi_i, i = 1, ..., D - 2\), are a collection of Lorentz scalar fields. We then obtain, as in (6.1), the additional states

\[(\phi_i, \psi^\alpha)_R \otimes \left\{ X_\nu \oplus \phi^{(h_1)} \oplus \phi^{(h_2)} \oplus ... \right\}_L \hspace{1cm} (6.4)\]

The combination

\[(\phi_i, \psi^\alpha)_R \otimes (X^\nu)_L \hspace{1cm} (6.5)\]

yields additional spin-1 vector bosons \(\tilde{A}_\nu\), spin-1/2 fermions \(\psi^\alpha\), and spin-3/2 gravitinos \(\lambda^{\nu\alpha}\) (thereby producing an \(N > 1\) SUSY unless these extra gravitinos are GSO-projected out of the spectrum). However, since (6.4) is built with the right-moving supermultiplet \((\phi_i, \psi^\alpha)_R\), the level-matching constraints now also allow the additional combinations

\[(\phi_i, \psi^\alpha)_R \otimes (\phi^{(h_i)})_L\]

where the gauge-group representations are not restricted to the adjoint. These combinations (6.6) can yield chiral spacetime supermultiplets with the highest weights of complex representations containing chiral massless spacetime fermions transforming in the gauge-group fundamental representation.

How might these additional supermultiplets arise in our fractional superstring theories? At the partition function level, we can simply introduce new terms of the form \((\epsilon_0^0)^{Dc-3}(\epsilon_0^2)^2\) to be interpreted as containing contributions of massless scalar fields. Such \((\phi_i, \psi^\alpha)_R\) multiplets could then be accommodated simply by multiplying the overall \(K = 4\) or \(K = 8\) partition functions by an appropriate integer (so as to maintain modular invariance). At the level of the actual particle spectrum, however, it is not clear how such extra fields might arise. Extra fermions are not difficult to find, since we start with \(2^D\) degrees of freedom while the Dirac \(\gamma\)-matrix algebra, along with the Majorana/Weyl condition, reduces this to \(2^{D/2-1}\) degrees of freedom. Extra scalar fields, on the other hand, are more difficult to obtain as follows. For \(K > 2\) there also exist extended parafermion theories; these have the same central charge as our usual parafermion theories, but contain more than one \(\phi_0^1\) field. At first sight, these extended
Lorentz indices $\mu$ associated with them. However, recall that in the parafermion theories there are additional parafermion fields whose characters have not appeared in the partition functions: these are the half-integer spin fields. It may therefore be possible to interpret these extra $\phi_{0}^{1}$ fields as Lorentz scalar
composites of the half-integer spin fields – e.g., the composite $(\phi_{1/2}^{1})^{\mu}(\phi_{1/2}^{1/2})_{\mu}$ may contain the additional $\phi_{0}^{1}$ field in the extended parafermion theory. Forming these $\phi_{0}^{1}$ fields as composites may allow them to be interpreted as Lorentz scalars, much as the scalar composite:

$$\epsilon^{\mu}\epsilon_{\mu}$$

appears as one of the descendants of $\phi^{\mu}$ as in (2.10). These scalars would then supply the scalar fields needed for chiral supermultiplets. Of course, a detailed analysis is necessary to see if these scenarios are possible.

A more widely-known and established method of generating supermultiplets $(\phi, \psi^{\alpha})$ in string theory is through spacetime compactification. Since the $K=8$ string is already in $D_{c}=4$, let us focus on the $(1,4)$ heterotic string with $D_{c}=6$.

For this string, we can choose to compactify two space dimensions:

$$(\psi^{\alpha}, A^{\mu}) \rightarrow (\psi^{\alpha}, A^{\mu}) \oplus (\phi^{1}, \phi^{2}) ; \quad (6.7)$$

here the $A^{\mu}$, $\mu = 0, 1, \ldots, 5$ in the $D=6$ theory breaks down to $A^{\mu}$, $\mu = 0, 1, 2, 3$ in $D=4$, with the remaining fields reidentified as internal scalars: $A_{4} = \phi_{1}$ and $A_{5} = \phi_{2}$. Similarly, the four-component Weyl fermion in $D=6$ splits into two two-component fermions in $D=4$. Therefore, the degree-of-freedom count corresponding to (6.7) is

$$(4+4) \rightarrow (2+2) + (2+2) . \quad (6.8)$$

In the usual $(1,2)$ string compactified to four spacetime dimensions, it is precisely this mechanism which breaks the ten-dimensional Lorentz symmetry. The separate $\phi^{1}$ and $\phi^{2}$ can then couple to different left-moving sectors, allowing the possibility of chiral fermions in the fundamental representation.

There is, however, a crucial difference between the usual $(1,2)$ heterotic string and the compactification to four spacetime dimensions. In the usual superstring, one arbitrarily chooses the resulting dimension in which one wishes to formulate the $(1,2)$ theory, and there are neither dynamical nor symmetry reasons why four dimensions is chosen. For the $(1,4)$ string, we can argue that even though the critical dimension is $D_{c}=6$, the partition function itself indicates that there does not exist a six-dimensional Minkowskian spacetime interpretation consistent with Lorentz invariance, locality, and quantum mechanics. Indeed, we will see that the largest number of spacetime dimensions in which a consistent interpretation exists is $D=4$. We shall call this the natural dimension. Hence, the $(1,4)$ string appears unique in that its compactification to (or interpretation in) four spacetime dimensions is essentially induced by the intrinsic structure of the theory itself.

Let us see how this comes about. Recall that $c_{m}^{\ell_{n}} = c_{2j}^{2k}$ for $[\phi_{m}^{j}]$. Also recall that the parafermion $m$-quantum number is additive (modulo $K/2$).
ber is not. As we have already observed, the \((c_0^0)^3(c_0^2)^3\) term in \(A_4\) represents the contributions from spacetime bosons, and the \((c_2^2)^4\) term in \(A_4\) represents the contributions from spacetime fermions. Hence, as is argued in Ref. [11], terms with \(D - 2\) factors of \(c_{K/2}^\ell\) (with arbitrary values of \(\ell\)) correspond to spacetime fermions, while terms of the forms \((c_0^0)^{D-2}\) (with arbitrary values of \(\ell\)) correspond to spacetime bosons. Such identifications arise essentially from recognizing the worldsheet vacuum state corresponding to \((\varphi_0^0)^{D-2}\) as yielding spacetime bosons [it is the analogue of the \(K = 2\) Neveu-Schwarz vacuum \((\varphi_0^0)^8\)], and recognizing the worldsheet vacuum state corresponding to \((\varphi_\pm_{1,2}^K)^{D-2}\) as yielding spacetime fermions [it is the analogue of the \(K = 2\) Ramond vacuum \((\varphi_{1/2}^{1/2})^8\)]. Note that this identification is in fact consistent at all mass levels, since excitations upon these respective vacua occur only through the \(\epsilon = \varphi_0^1\) field, and the parafermion fusion rules show that applications of \(\varphi_0^1\) cannot change the \(m\)-quantum numbers of these respective vacua. Furthermore, since the \(m\)-quantum number is additive modulo \(K/2\), this identification indeed reproduces the desired Lorentz fusion rules \(B \otimes B = B, B \otimes F = F,\) and \(F \otimes F = B\), where \(B\) and \(F\) represent spacetime bosonic and fermionic fields. Thus, with this interpretation in \(D = D_c\) spacetime dimensions, we see that it is straightforward to determine the spacetime statistics of the particles contributing to each term in \(A_4\) and \(A_8\).

However, we now note that in \(B_4\) there exists a term of the form \((c_0^2)^2(c_0^2)^2\) which has only massive states \(\text{i.e., in a } q\text{-expansion of this term we find } q^h(1 + ... \text{ where } h > 0).\) Although this term may have more than one interpretation, the elementary particles giving rise to them either bosons or fermions in six spacetime dimensions must arise through \(c\)-excitations from either the six-dimensional vacuum state. It is, therefore, natural to interpret the contributions from either:

1. fermions in four spacetime dimensions [the \(c_0^2\) term]
2. bosons in four spacetime dimensions [the \(c_0^2\) term]

How might such internal factors arise? Let us recall the case of the usual superstring theory, where in \(D_c = 10\) spacetime dimensions the \(c_0^1\) field (corresponding to spacetime fermions) always carries a spacetime Lorentz index) and where the factor \(-(c_1^1)^8 = -\varphi_0^1/(16\Delta^{1/2})\).

However, compactifying a model to a dimension \(D < D_c\) with additional internal quantum numbers required; indeed, we have seen in Sect. III that in \(D < D_c\) flat spacetime dimensions, we obtain terms with string-function factors of the form \(c^{D-2}c^D\) from spacetime-associated degrees of freedom (\(c\)-excitations carrying a spacetime Lorentz index) and where the \(D\)-term represents additional internal degrees of freedom (\(\text{i.e., worldsheet indices)\). In the present case, upon compactification...
the form $(c^{ℓ}_{K/2})^{D-2}(c^{ℓ}_{0})^{D_c-D}$ where the second factor represents contributions from spacetime bosons (and is the $K > 2$ analogue of the contributions from the usual Neveu-Schwarz sectors of the theory). Remarkably, this is precisely what occurs in $B_4$. Apparently, six-dimensional Lorentz invariance for the $K = 4$ string appears to be explicitly broken.

Note that – at tree-level – the particles in the $A_4$ sector couple only to other fields in the $A_4$ sector; in particular, the tree-level scatterings of massless particles do not involve the $B_4$ sector. At the string loop level, however, fields in the $B_4$ sector couple to the massless particles in the $A_4$ sector, so we see that it is the string’s own quantum effects which render the theory in its critical dimension inconsistent with Lorentz invariance. Thus, in this sense, it is quantum mechanics and the requirement of Lorentz invariance which together necessitate or induce the compactification from the critical dimension (six) to the natural dimension (four). This mechanism is quite remarkable, and has no analogue in the traditional $K = 1, 2$ string theories (in which the critical dimensions are the natural dimensions). In particular, only for the $K > 2$ theories do such independent $B$-sectors appear; for $K = 2$ there is only one possible supersymmetric $A$ sector, corresponding to the one supersymmetry identity (Jacobi identity) for $K = 2$. It is the presence of a unique $B_K$ sector for $K > 2$, itself necessitated by modular invariance and closure under $[S]$, which induces this compactification.

The same argument applies to the $K = 8$ string (for which the critical dimension is four but the natural dimension is three), but with one additional feature. As before, the $A_8$ expression [Eq. (3.31)] is consistent with a $D = D_c = 4$ spacetime boson/fermion interpretation; rather, the form of this expression dictates a three-dimensional interpretation. Now, the third expression $C_8$ contains contributions only of the type $c^{ℓ}_{K/4} = c_2$. (Recall that only is $C_8$ therefore consistent with three-dimensional in fact Lorentz invariance in any number of dimensions only.) Above interpretation indicates that $C_8$ must correspond to fields with fractional spacetime Lorentz spins $1/4$ and $3/4$. However, such spin fields (i.e., anyons) are consistent with quantum mechanics and Lorentz invariance in three spacetime dimensions, the way in which self-consistency selects this dimensionality $K > 2$ strings seem to select naturally their proper spacetime dimensions are the dimensions in which consistent Lorentz-invariant interpretations can be achieved.

Note that if such speculations are indeed correct, then the $(1, 4)$ and $(1, 8)$ models we have constructed in this paper have continuous (and non-chiral) spectra, for when we interpret them in four and three spacetime dimensions respectively these string theories contain worldsheet bosons which remain uncompactified. However, we expect the “dictionary” approach we have developed in this paper to be easily generalizable to the compactified (and chiral) case as well.
In summary, then, although there may well exist other possible interpretations of these partition functions, we find the above speculations both tantalizing and intriguing. It is indeed fortuitous that the solution to what might have seemed a problem (i.e., finding an interpretation consistent with Lorentz invariance) also simultaneously provides a possible solution to the more phenomenological problem of obtaining models with chiral fermions. It is also strongly compelling that the quantum structure of the theory itself seems to dictate this solution. These are issues clearly worth investigating.

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APPENDIX A

In this appendix we introduce the string functions (or parafermion characters) \( c_n^\ell \), collecting together their definitions, modular transformation properties, special cases, and identities. The results quoted here are due mostly to Kac and Peterson.\(^9\)

The string functions are essentially the characters \( Z_{2m}^{2j} \) of the \( Z_K \) parafermion fields \( \phi_m^j \):

\[
Z_n^\ell = \eta c_n^\ell
\]

(A.1)

where \( \ell \equiv 2j, n \equiv 2m \), and where \( \eta \) is the Dedekind \( \eta \)-function:

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{3(n-1/6)^2/2}
\]

(A.2)

with \( q \equiv e^{2\pi i \tau} \). Since the \( Z_K \) parafermion theory can be represented as the coset theory \( SU(2)_K/U(1) \), the string functions can therefore be obtained by expanding the full \( SU(2)_K \) characters \( \chi(\tau, z) \) in a basis of \( U(1) \) characters \( \Theta_{n,K}(\tau, z)/\eta \):

\[
\chi(\tau, z) = \sum_{n=-\infty}^{\infty} Z_n^\ell(\tau) \frac{\Theta_{n,K}(\tau, z)}{\eta} = \sum_{n=-\ell}^{2K-\ell-1} c_n^\ell(\tau) \Theta_{n,K}(\tau, z).
\]

(A.3)

Here \( \Theta_{n,K}(\tau, z)/\eta \) are the characters of a \( U(1) \) boson compactified on a radius \( \sqrt{K} \). Since the \( SU(2)_K \) characters and the \( \Theta_{n,K} \) functions are well-known, explicit expressions for the string functions can be extracted. Eq. (A.3), then, can be taken as a definition of the string functions \( c_n^\ell \).

Expressions for the string functions \( c_n^\ell \) were due to Distler and Qiu:\(^{15}\)

\[
c_n^\ell(\tau) \equiv q^{h_n^\ell + [4(K+2)]^{-1}} \sum_{r,s=0}^{\infty} (-1)^{r+s} q^{\frac{r}{2} + 2s + 1} \times \left\{ q^{(j+m)+s(j-m)} - q^{K+1-2j+r+s} \right\}
\]

where \( \ell - n \in 2\mathbb{Z} \) and where the highest weight

\[
h_n^\ell = \frac{\ell(\ell+2)}{4(K+2)} - \frac{n^2}{4K}
\]

The string functions have the symmetries

\[
c_n^\ell = c_{-n}^{\ell + K} = c_{-n}^\ell
\]

as a consequence of which for any \( K \) we are free to choose a "basis" of string functions \( c_n^\ell \) where \( 0 \leq \ell \leq K \) and \( 0 \leq n \leq n_{\text{max}} \) and \( \ell - 2 \) otherwise.

The string functions are closed under the modular transformation \( \tau \to \tau + 1 \). In fact, under \( T : \tau \to \tau + 1 \) they transform as

\[
c_n^\ell(\tau + 1) = e^{2\pi is_n^\ell} c_n^\ell(\tau)
\]

where the phase \( s_n^\ell \) is given by

\[
s_n^\ell = h_n^\ell - \frac{1}{24} c_0 = h_n^\ell + \frac{1}{24}
\]

Here \( h_n^\ell \) is defined in (A.5), and \( c_0 \) is the central charge.
theory [given in (2.13)]. Under \( S : \tau \to -1/\tau \), the level-\( K \) string functions mix among themselves:

\[
\ell_n(-1/\tau) = \frac{1}{\sqrt{-i\tau}} \frac{1}{\sqrt{K(K+2)}} \sum_{\ell'=-K+1}^{K} \sum_{n'=0}^{K} b(\ell,n,\ell',n') c_{n'}^\ell,
\]

(A.9)

where the first square root indicates the branch with non-negative real part and where the mixing coefficients \( b(\ell,n,\ell',n') \) are

\[
b(\ell,n,\ell',n') \equiv \exp\left\{ \frac{in\eta\ell\ell'}{K} \right\} \sin\left\{ \frac{\pi(\ell+1)(\ell'+1)}{K+2} \right\}.
\]

(A.10)

From (A.7) and (A.9) it follows that the string functions have modular weight \( k = -\frac{1}{2} \).

Note that in the special \( K = 1 \) and \( K = 2 \) cases, the string functions can be expressed in terms of the Dedekind \( \eta \)-function (A.2) and the more familiar Jacobi \( \vartheta \)-functions; these relations are given in (3.18) and (3.19). In particular, these \( \vartheta \)-functions are

\[
\vartheta_2(\tau) \equiv 2q^{1/8} \prod_{n=1}^{\infty} (1+q^n)^2 (1-q^n)
\]

\[
\vartheta_3(\tau) \equiv \prod_{n=1}^{\infty} (1+q^{n-1/2})^2 (1-q^n)
\]

(A.11)

\[
\vartheta_4(\tau) \equiv \prod_{n=1}^{\infty} (1-q^{n-1/2})^2 (1-q^n),
\]

and under the \([S]\) and \([T]\) transformations [where the bracket indicates the stroke operator, defined in (3.10)], we find

\[
\vartheta_2[S] = \exp(7\pi i/4) \vartheta_4, \quad \vartheta_2[T] = \vartheta_4
\]

\[
\vartheta_3[S] = \exp(7\pi i/4) \vartheta_3, \quad \vartheta_3[T] = \vartheta_3
\]

\[
\vartheta_4[S] = \exp(7\pi i/4) \vartheta_2, \quad \vartheta_4[T] = \vartheta_2
\]

Note that (A.12) is indeed consistent with the string-function transformations (A.7) and (A.9) for \( K = 2 \); in particular, under \([T]\) \( c_0^0 + c_0^1 \) transforms as an eigenfunction while \( c_0^0 \) transforms into each other.

The \( S \)-transformation formula (A.9) assumes an analogously simple form for the \( K = 4 \) case. In this case there are seven distinct string functions, and under \([S]\) they can be block-diagonalized as follows:

\[
\left( \begin{array}{c}
c_0^0 + c_0^1 \\
c_0^2 \\
c_0^3 \\
c_1^2 \\
c_1^3
\end{array} \right)[S] = e^{i\pi/4} \mathbf{R}^T
\]

and

\[
\left( \begin{array}{c}
c_0^0 - c_0^4 \\
c_1^4 \\
c_1^3
\end{array} \right)[S] = e^{i\pi/4} \mathbf{R}^T
\]

where

\[
\mathbf{R}^+ = \frac{1}{\sqrt{24}} \begin{pmatrix} 2 & 4 & 4 & 4 \\ 2 & -2 & 4 & -2 \\ 1 & 2 & -2 & -2 \\ 2 & -2 & -4 & 2 \end{pmatrix}, \quad \mathbf{R}^- = \begin{pmatrix} 2 & 4 & 4 & 4 \\ 2 & -2 & 4 & -2 \\ 1 & 2 & -2 & -2 \\ 2 & -2 & -4 & 2 \end{pmatrix}^{-1}
\]
Note that \((R^+)^2 = (R^-)^2 = 1\), as required.

This mixing pattern seen for the \(K = 2\) and \(K = 4\) cases exists for other higher even values of \(K\) as well. For any such \(K\), we can define the linear combinations \(d^\pm_n \equiv c^\ell_n \pm c^K_n\) when \((\ell, n)\) are even; note that these \(d\)-functions will also be eigenfunctions of \([T]\) if \(K \in 4\mathbb{Z}\) because in these cases \(s^K_n = s^{K-\ell}_n \mod 1\). It then follows from (A.9) that the \(d^+\)-functions mix exclusively among themselves under \(S\) [as in (A.13)], and that the \(d^-\)-functions mix exclusively with themselves and with the odd \((\ell, n)\) string functions [as in (A.14)]. It is for this reason that (for \(K > 2\)) it is possible to construct modular-invariant expressions involving the string functions in such a manner that the odd string functions do not appear. Indeed, one can check that in all the string-function expressions which have appeared for the fractional superstring (such as \(A_K\), \(B_K\), and \(C_K\)), only the \(d^+\) combinations have played a role.

In this appendix we demonstrate, using the free-fermionic spin-structure construction of Ref. [10], that any \((1, 2)\) model with a partition function of the form

\[
Z = \tau_2^{-2} \Delta^{-1} \sum m^{-1/2} \mathcal{F} \left( \frac{\beta^{(10-D)/4}}{X + \gamma^{(10-D)/4}} \right)
\]

has an \(N = N_{\text{max}}\) spacetime supersymmetry. We assume the reader to be familiar with the model-construction procedure described in Ref. [10], and we use the same notation here.

It is clear that a model with partition function (B.1) is spacetime-supersymmetric; therefore, as discussed in Ref. [10], its set of spin-structure generating vectors \(W_i\) must include the two vectors

\[
W_0 \equiv \left[ (\frac{1}{\tau})^{12+2k} \right] \quad \text{and} \quad W_1 \equiv \left[ (0)^{-k} (0 \frac{1}{2} \frac{1}{2})^{4+k} \right]
\]

where \(k\), the modular weight, equals \(1 - D/2\). These vectors indicate the boundary conditions of the worldsheet fermions, and we see that the four primary conditions in \(W_1\) (i.e., the four whose components which together contribute the spacetime remaining \(2(4+k) = 10 - D\) right-moving components equal \(\frac{1}{2}\)) are the remaining factors of \(\beta^{(10-D)/4}, \gamma^{(10-D)/4}\), and \(\beta^{(10-D)/4}\).
In order for our model to have a partition function of the form (B.1), these remaining \((10 - D)\) worldsheet fermions must have the same toroidal boundary conditions in all sectors of the model which contribute to the partition function. In particular, this must be true separately for spacetime bosonic and fermionic sectors (i.e., those which have \(\alpha s = \frac{1}{2}, 0\) respectively in the notation of Ref. [10]). This can occur only if every other spin-structure generating vector \(W_i\) has equal components for these \(10 - D\) fermions (thereby giving rise to the maximal right-moving “gauge” symmetry discussed in Sect. V). Thus, any other spin-structure vectors \(W_i\) in the generating set must have a right-moving component of the form

\[
W_i^R = [(0)^{-k}(X)^{4+k}] \tag{B.3}
\]

where each factor \(X\) is independently either \((00)\) or \((0\frac{1}{2}2)\). In determining (B.3) we have had to satisfy the “triplet” constraint; we have also made use of our freedom to choose (without loss of generality) vectors which have their first components vanishing.

Let us now consider the constraints which must be satisfied by the gravitino states. These states all arise in the \(W_1\) sector, and have the charge vectors

\[
Q_{\text{grav}} \equiv N_{W_1} + W_1 - W_0 = [(\frac{1}{2})^{-k}(\frac{1}{2}00)^{4+k} | (0)^{24+k}] \tag{B.4}
\]

where in principle all \(\pm \frac{1}{2}\) combinations are allowed. The constraint due to the \(W_0\) vector, however, immediately projects out half of these states, allowing only those with charge vectors in which the product of non-zero components is positive. This by itself leaves us with eight distinct allowed states, and these states are precisely those which combine to form the gravitinos if and only if the other \(W\)-vectors in the generating set produce constraint equations removing some of these gravitinos from the physical spectrum. The \(W_1\)-vector, however, and since any additional \(W\)-vectors are of the forms (B.3), they each give rise to constraint equations of the form

\[
0 = f(k_{ij}) \tag{B.5}
\]

where \(f(k_{ij})\) is a function of the projection constants. Note, in particular, that (B.5) does not involve \((B.4)\); we see that either all or none of the \(N_{\text{max}}\) gravitinos remain in the spectrum: all gravitinos remain if the projection constants are chosen so that in each case \(f(k_{ij}) = 0\), and none remain otherwise. In order to have a partition function of the form (B.1), our model must have at least an \(N = 1\) spacetime supersymmetry.
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