Non-Hermitian Topological Anderson Insulators

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Non-Hermitian systems can exhibit exotic topological and localization properties. Here we elucidate the non-Hermitian effects on disordered topological systems by studying a nonreciprocal disordered Su-Schrieffer-Heeger model. We show that the non-Hermiticity can enhance the topological phase against disorders by increasing energy gaps. Moreover, we uncover a topological phase which emerges only under both moderate non-Hermiticity and disorders, and is characterized by localized insulating bulk states with a disorder-averaged winding number and zero-energy edge modes. Such topological phases induced by the combination of non-Hermiticity and disorders are dubbed non-Hermitian topological Anderson insulators. We also find that the system has unique non-monotonous localization behaviour and the topological transition is accompanied by an Anderson transition.

Topological states of matter have been widely explored in condensed-matter materials [1–5] and various engineered systems, which include ultracold atoms [6–8], photonic lattices [9, 10], mechanic systems [11], classic electronic circuits [12–15], and superconducting circuits [16–20]. One hallmark of topological insulators is the robustness of nontrivial boundary states against certain types of weak disorders, since the topological band gap (topological invariants) preserves under these perturbations [1–3]. However, the band gap closes for sufficiently strong disorders and the system becomes trivial as all states are localized according to the Anderson localization [21]. Surprisingly, there is a topological phase driven from a trivial phase by disorders, known as topological Anderson insulator (TAI) [22]. The TAI was first predicted in two-dimensional (2D) quantum wells and then shown to exhibit in a wide range of systems [22–30], such as Su-Schrieffer-Heeger (SSH) chains [31]. Recently, the TAI has been observed in one-dimensional (1D) cold atomic wires and 2D photonic waveguide arrays [32, 33].

On the other hand, recent advances in non-Hermitian physics show that non-Hermitian systems have many intriguing features and applications [34–37]. Particularly, growing efforts have been made to reveal topological properties in non-Hermitian systems [38–69], which include new topological invariants [65], the non-Hermitian skin effect [48], the revised bulk-edge correspondence [47–52], and gain-and-loss induced topological phases [56]. In addition, non-Hermitian systems can exhibit unique localization properties in the presence of disorders [70–77]. Notably, the topological phases have been studied in 1D non-Hermitian (generalized) Aubry-André-Harper model [78–83], which describes topological quasicrystals and can be mapped to a 2D quantum Hall system in the Hermitian case [84–86]. The topological non-Hermitian quasicrystals were predicted [82, 83] and the topological nature of the Anderson transition in the systems was revealed [80, 81]. However, the interplay among topology, disorder and non-Hermiticity can induce rich physical phenomena that have been rarely explored, in particular, it is still unclear whether the TAI phase can exhibit in non-Hermitian systems.

In this Letter, we elucidate the non-Hermitian effects on disordered topological systems and uncover a different TAI phase. We construct an SSH model with nonreciprocal and disordered hopping terms and propose a real-space winding number to characterize its topology. Our main results are as follows: (i) We show that the non-Hermiticity can enhance the topological phase and make it more robust against disorders. This effect can be interpreted by the increase of the effective energy gaps due to the non-Hermiticity. (ii) We reveal a topological phase which emerges only under both moderate non-Hermiticity and disorders, and is characterized by localized insulating bulk states with a nontrivial winding number and two zero-energy edge modes. Such topological phases induced by the combination of non-Hermiticity and disorders are thus dubbed non-Hermitian topological Anderson insulators (NHTAI). (iii) We find that the system has non-monotonous localization behaviour with respect to disorder strengths due to its interplay with the skin effect. This behaviour is unique for non-Hermitian systems. We also show that the topological transition is accompanied by an Anderson transition. The model and the predicted NHTAI can be realized in some artificial systems and can be generalized to higher dimensions.

Model and topological invariant. — Let us begin by con-
considering the SSH model with nonreciprocal and disordered hoppings, which is depicted in Fig. 1. The tight-binding model with two-site unit cell reads

$$H = \sum_j (m_j a_j^\dagger b_j + h.c.) + t_j^{(r)} a_{j+1}^\dagger b_j + t_j^{(l)} b_j a_{j+1},$$  \hspace{1cm} (1)

where \(a_j^\dagger (b_j^\dagger)\) creates a particle on the sublattice A (B) in the \(j\)th lattice cell, and \(a_j \) (\(b_j\)) is the corresponding annihilation operator. Here \(m_j\) denotes the \(j\)-dependent (Hermitian) intracell hopping energy, and \(t_j^{(r,l)}\) characterize the non-Hermitian intercell hoppings. This Hamiltonian has the chiral symmetry as \(H\) satisfies \(\Gamma^{-1} H \Gamma = -H\), where the chiral operator is \(\Gamma = \sigma_z \otimes I\) with the Pauli matrix \(\sigma_z\) referring to the sublattice.

In contrast to the site-potential disorder, the pure tunneling disorder is crucial for preserving the chiral symmetry. In particular, we consider the hopping terms as 

$$m_j = t + W_1 \omega_j, \quad t_j^{(l)} = t' + W_2 \omega'_j, \quad t_j^{(r)} = t_j^{(l)} + t' \gamma.$$  \hspace{1cm} (2)

Here \(t\) and \(t'\) are the characteristic intracell and intercell tunneling energies, \(\omega_j\) and \(\omega'_j\) are independent random real numbers chosen uniformly in the range \([-1, 1]\). \(W_1\) and \(W_2\) are the corresponding disorder strengths, and \(\gamma\) denotes the non-Hermiticity. Hereafter we set \(t = 1\) as the energy unit and focus on the case \(W_2 = 0\).

Similar as the topological phase in the SSH model with \(t' > t\), this non-Hermitian disordered SSH model is topologically characterized by zero-energy edge modes and the corresponding winding number [see Eq. (4)]. Note that the system recovers to the Hermitian disordered SSH chain when \(\gamma = 0\) [26, 27, 32] and to the non-Hermitian clean chain when \(W_1 = W_2 = 0\) [47–50], respectively. In the clean limit, the topological invariant of the nonreciprocal SSH model can be a non-Bloch winding number in complex momentum space [48] or a dual open-bulk winding number in real space [49].

We now generalize the open-bulk winding number to our non-Hermitian disordered SSH model. Given a disorder configuration denoted by \(s\), we diagonalize the Hamiltonian (1) under open boundary conditions (OBCs) with two chiral-symmetric parts: \(H_s |nR_x\rangle_s = \pm E_{n,s} |nR_x\rangle_s\) with \(|nR_+\rangle_s = \Gamma |nR_-\rangle_s\). In the biorthonormal basis, the corresponding left eigenstates \(|nL_\pm\rangle\) orthonormal to the right eigenstates (i.e., \(s' \langle n' L_\eta | n Rospel \rangle_s = \delta_{n,n'} \delta_{\eta,\eta'}\)) can be taken from the columns of \((T^{-1}_s)\) by writing \(H_s = T_s \Lambda_s T_s^{-1}\) with \(\Lambda_s\) diagonal. The homotopically equivalent flat band version of the Hamiltonian \(H_s\) under OBCs is the open-boundary \(Q_s\) matrix, which is given by 

$$Q_s = \sum_n \langle nR_+ \rangle_s |nL_+\rangle_s - \langle nR_- \rangle_s |nL_-\rangle_s.$$  \hspace{1cm} (3)

Here the summing takes over the eigenstates in the bulk continuum spectrum without the discrete edge modes. The winding number in real space is then defined as [49]

$$\nu_s = \frac{1}{2L'} Tr'(\Gamma Q_s(Q_s, X)),$$  \hspace{1cm} (4)

where \(X\) is the coordinate operator of the lattice cell, \(L = L' + 2l\) is the chain length with three intervals of lengths \(l, L', l\), and \(Tr'\) denotes the trace over the middle interval of length \(L'\) (with \(l\) sufficiently large to avoid boundary effects). This real-space winding number does not require the translation invariance and thus serves well for disordered systems. Here \(L \to \infty\) is assumed to quantize \(\nu_s\) to an integer [27, 87]. With many disorder configurations, we can define the disorder-averaged winding number

$$\nu = \frac{1}{N_s} \sum_{s=1}^{N_s} \nu_s,$$  \hspace{1cm} (5)

where a modest sample number \(N_s\) suffices in practice. For \(\gamma = 0\), the topological invariants in Eqs. (3) and (4) reduce to those in Hermitian systems [27, 32, 87–90], where the boundary condition is irrelevant. However, Eq. (3) correctly corresponds to the topological edge modes only under OBCs, due to the unconventional bulk-edge correspondence in the non-Hermitian cases [47–52].

**Enhancing topological phase by non-Hermiticity.**—We first consider the effects of non-Hermiticity on the topological phase of the open SSH chain \((t' > t)\) with length \(L\) under disorders. Figure 2(a) shows the calculated disorder-averaged winding number \(\nu\) as a function of the non-Hermiticity \(\gamma\) and disorder strengths \(W_1 = W\) and \(W_2 = 0\). We find that when the non-Hermiticity increases from \(\gamma = 0\) to 3.5, the topological phase with \(\nu \approx 1\) pre-

![FIG. 2. (Color online) (a) Disorder-averaged winding number \(\nu\) with varying non-Hermiticity \(\gamma\) and disorder strength \(W\); and (b) fixed \(W = 3, 3.5, 4\). (c)(d) Four middle disorder-averaged energy eigenvalues \(E_{n,s}\) (real and imaginary parts) under OBCs, as a function of \(W\) with \(\gamma = 1\). From bottom to up are \(n = L/2 - 1, L/2, L/2 + 1, L/2 + 2\), respectively. (e) The energy gap \(E_\gamma\) in the clean limit as a function of \(\gamma\) calculated in simulations (hollow dots) and in Eq. (5) (solid line). The other parameters are \(t = 1, t' = 1.2, L = 5l = 100, W_1 = W, W_2 = 0\) and \(N_s = 200\).](image-url)
serves from a region $W \lesssim 2$ to a larger one $W \lesssim 5$. Figure 2(b) shows that increasing $\gamma$ for fixed $W = 3, 3.5, 4$ leads to the increase of $\nu$ to nearly unit. We rewrite the eigenequation of the open chain $H_n|\psi_n\rangle = E_n|x\rangle$ with wave functions $|\psi_n\rangle = [\psi_n^1, \psi_n^2, ..., \psi_n^L]^T$ and eigenenergies $E_n$, where $x$ denotes the index of the lattice site. Then the disorder-averaged eigenenergies is given by $E_n = \frac{1}{N_L}\sum_{s=1}^{N_L} E_n(x)$. Figures 2(c,d) show the real and imaginary parts of $E_n$ for four center eigenstates as a function of $W$, respectively. The energy spectrum of this open-boundary chain is purely real and two zero-energy edge modes exhibit in the topological phase.

To understand the numerical results, we take a similarity transformation [48]: $\tilde{H} = S^{-1} H S$ with the diagonal matrix $S = \text{diag}(1, 1, r, r^2, ..., r^{L/2-1}, r^{L/2-1})$, the eigenequation is equivalent to $H_n|\tilde{\psi}_n\rangle = E_n(x)|\tilde{\psi}_n\rangle$ with $|\tilde{\psi}_n\rangle = S^{-1}|\psi_n\rangle$. Let $r = \sqrt{1 + \gamma}$ for $\gamma > -1$, then $\tilde{H}$ becomes the Hermitian disordered SSH model with intracell and intercell hoppings $\tilde{m}_i = m_i$ and $\tilde{t}' = t'\sqrt{1 + \gamma}$ for $W_2 = 0$. This transformation indicates that all eigenenergies of the open-chain Hamiltonian are real, as shown in Figs. 2(c,d). The transformation also accumulates the wave functions of bulk states to one boundary, which is the non-Hermitian skin effect [48] and will be discussed later. We can define the energy gap $E_g = |E_{L/2+1} - E_{L/2}|$ in the clean limit. After the similarity transformation to the Hermitian SSH chain, the energy gap can approximately obtained as (with minor boundary effects)

$$E_g \approx 2|\tilde{t}' - \tilde{t}| = 2|t'\sqrt{1 + \gamma} - t|.$$  \hspace{1cm} (5)

The numerical results of $E_g$ for $L = 100$ are consistent with Eq. (5), as shown in Fig. 2(e). Thus, the enhancement of the topological phase in this non-Hermitian disordered SSH chain can be interpreted as the increase of the effective energy gap by non-Hermiticity.

**NHTAI from non-Hermiticity and disorder.**—We proceed to consider the effects of combined non-Hermiticity and disorders on an initially trivial phase in the SSH chain. To do this, we set $t > t'$ and numerically calculate the disorder-averaged winding number $\nu$ as a function of the non-Hermiticity $\gamma$ and disorder strengths $W_1 = W$ and $W_2 = 0$, with the results for $t' = 0.7t$ and $L = 400$ shown in Fig. 3(a). In the Hermitian and clean limit $\gamma = W = 0$, the system is in the topologically trivial phase with $\nu \approx 0$. Interestingly, we find $\nu \approx 1$ in a region with moderate non-Hermiticity and disorder strength in Fig. 3(a). Actually, $\nu$ can approach to unit in this region by increasing the lattice size, with an example as a function of $W$ with $\gamma = 0.6$ shown in Fig. 3(c). The four middle disorder-averaged eigenenergies of the open chain of $L = 100$ are plotted in Fig. 3(d), which shows that two zero-energy edge modes inside a small gap between upper and lower eigenstates corresponding to the topological phase. Notably, here the small gap essentially vanish when $L \to \infty$ (confirmed by the finite-size scaling) at moderate disorder strength ($W \gtrsim 1$ in this case). Similar to the Hermitian disordered chiral wires, it is excepted that the energy gap is replaced by a mobility gap and the band insulator of the clean system is replaced by an Anderson insulator, with the topology carried by localized bulk states [27]. Our results indicate a topological insulator induced by the combination of moderate non-Hermiticity and disorders, which is dubbed NHTAI as a non-Hermitian extension of the TAI [22-27, 32].

To reveal the connection between the NHTAI and the TAI, we perform the similarity transformation and map the non-Hermitian open SSH chain to the Hermitian one with $\tilde{t}' = t'\sqrt{1 + \gamma} \in [0.49, 1.22]$ for $t' = 0.7$ and $\gamma$ in Fig. 3(a). We calculate the corresponding winding number with the Hermitian Hamiltonian, for which only the right eigenstates (instead of the biorthonormal basis) are used in the calculations [27, 32]. The results for an open chain of length $L = 400$ are shown in Fig. 3(b), which indicates the $\tilde{t}'$-$W$ region of the TAI for $0.7 < \tilde{t}' < 0.97$, which corresponds well to the $\gamma$-$W$ region of the NHTAI for $0 < \gamma < 0.95$. Thus, NHTAIs can be topologically connected to TAs through the similarity transformation with the same energy spectrum under OBCs. However, due to the non-Hermitian skin effect [48], NHTAIs have unique bulk-edge correspondence and localization properties under OBCs as we will discuss below.

**Localization properties.**—We now study the localization properties of the system. First, in Figs. 4(a,b,c), we show the density distribution of the center $L/2$-th eigen-
state with $L = 100$ under OBCs or PBCs for $\gamma = 0.6$ and $W = 0, 1, 2, 3$ in a disorder configuration, respectively. In the clean limit [Fig. 4(a)], this eigenstate is a bulk state, which is extended under PBCs but pinned to the right edge (because $t_j^{(r)} > t_j^{(l)}$) of the lattice under OPC, as a manifestation of the non-Hermitian skin effect [48]. For moderate disorder strength, this eigenstate becomes a zero-energy mode localized at the right edge of the system that is in the NHTAI phase, with the disorder-averaged density distribution of the edge state in the inset of Fig. 4(b). For large disorder strength [Fig. 4(c)], it turns to a localized state in the bulk and the skin effect is broken. The same results are for the $(L/2+1)$-th eigenstate due to the chiral symmetry. Then, we calculate the disorder-averaged inverse participation ratio for the $n$-th eigenstate $\tilde{I}_n$ and its averaging over all eigenstates $\tilde{I}$, which are given by

$$I_n = \frac{1}{N_s} \sum_{s=1}^{N_s} \sum_{x=1}^{L} |\psi_{n,x}^{(s)}|^4, \quad \tilde{I} = \frac{1}{L} \sum_{n=1}^{L} I_n, \quad (6)$$

respectively. We find that $I_n > 1/L$ for all the bulk states with $E_n \neq 0$ when $W > 0$ under OPCs or OBCs, which implies that the entire energy spectrum (excluded $E = 0$) of the chiral wire are localized immediately after the disorder is turned on [27]. The results for $\tilde{I}$ and $\tilde{I}_{L/2}$ with $L = 400$ are shown in Fig. 4(d). Under OBCs, $\tilde{I}_{L/2} \gg 1/L$ for all $W$ shows that the $L/2$-th eigenstate always localized in this case; and $\tilde{I}_{L/2}$ with a maximum value in the NHTAI phase implies that the zero-energy edge modes are more localized. The global localization index $\tilde{I}$ increasing rapidly as a function of $W$ under PBCs also shows that the bulk states are localized in the presence of disorders. Under OPCs, $\tilde{I}$ has non-monotonous behaviour when $W \lesssim 1$, which is due to the interplay of Anderson localization and the skin effect. When $W \gtrsim 1$, $\tilde{I}$ for OBCs takes the value almost the same for that of PBCs, which indicates that the skin effect of bulk states is destroyed by disorders of strength $W \gtrsim 1$.

Following Ref. [27], we derive an analytical formula for the localization length $\Lambda$ of our model (with $W_2 = 0$) at energy $E = 0$ under OBCs. We obtain (see the SM)

$$\Lambda^{-1} = \ln \left[ \frac{2e\gamma}{2W|\pi\gamma + 2|} \right], \quad (7)$$

where $e$ is the natural constant. Figure. 4(e) shows the critical points with $\Lambda^{-1} \to 0$ in the $\gamma-W$ plane, where the localization length of zero-energy states diverges. The delocalized critical points match with the topological transition points in Fig. 3(a), indicating that the topological transition is accompanied by an Anderson localization-delocalization transition. The results demonstrate that the non-Hermitian topological numbers can be carried by localized bulk states and disorders can drive a localized non-Hermitian topological phase through a delocalized point, similar as those in Hermitian chiral chains [27].

Discussion and conclusion.—Now we discuss the realization of the non-Hermitian disordered SSH model in some artificial systems. The first feasible system is ultracold atoms [6–8]. The TAI has been realized in an atomic SSH wire with controllable disordered hoppings [32], where the effective nonreciprocal hopping terms can be engineered by a collective one-body loss [53, 91]. Another system is photonic crystals [9, 10], where the TAI phase and tunable non-Hermiticity have been experimentally realized [33, 92–95]. The realization of nonreciprocal hoppings in optics was suggested [96, 97]. The third possible system is topological electronic circuits [12–15], which have been experimentally implemented to study the Hermitian and nonreciprocal SSH chains [98–100]. One can add tunable hopping disorders in the nonreciprocal topoclectric circuit [100] to realize our model. In view that non-Hermiticity and disorders have been engineered in these artificial systems, the studied model with the NHTAI is realizable in current experiments. The robust or disorder-induced topological edge modes can be detected, and it would be interesting to measure the topological numbers.

We make several remarks on the model. First, the energy spectrum can be generally complex when $W_2 \neq 0$ and the similarity transformation is inapplicable in this case. However, the calculated winding numbers remain when $W_2$ is relatively small, and the results shown in Figs. 2 and 3 preserve if $W_2 \lesssim W_1/4$ with $W_1 = W$ (see the SM). Thus, the non-Hermiticity-enhanced topological phase and the NHTAI exhibit even without the similar-
In summary, we have studied the topological and localization properties of the SSH model with nonreciprocal and disordered hoppings. We have revealed the enhancement of the topological phase by the non-Hermiticity and the NHTAI induced by the combination of non-Hermiticity and disorders. The non-monotonous localization behaviour and the topological nature of the Anderson transition have been demonstrated. Moreover, the predicted NHTAI can be experimentally realized with some artificial systems, such as ultracold atoms, photonic crystals, and electronic circuits.

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Supplemental Material for Non-Hermitian Topological Anderson Insulators

Localization length at zero energy

For the Hermitian disordered chiral SSH chain, the localization length $\Lambda$ at energy $E = 0$ can be analytically obtained [27]. Indeed, the Schrödinger equation of the Hermitian SSH model $H \phi = 0$ reads: $t_j \phi_{j-a,\alpha} + \bar{m}_j \phi_{j,\alpha} = 0$, where $\alpha = \pm 1$ represents A and B sublattice, respectively. The solution is given by

$$\phi_{N,\alpha} = \prod_{j=1}^{N} \left( \frac{t_j}{\bar{m}_j} \right)^{1/2} \phi_{0,\alpha},$$

(8)

where the unit cell is labeled by $j = 0, 1, \ldots, N$. The inverse of the localization length is given by (in the thermodynamic limit disregarding the boundaries) [27]

$$\Lambda^{-1} = \max_{\alpha = \pm 1} \left[ - \lim_{N \to \infty} \frac{1}{N} \ln |\phi_{N,\alpha}| \right] = |\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \ln |t_j - |\phi_j||, \tag{9}$$

We consider the disordered hopping parameters: $\bar{m}_j = \bar{t} + W_1 \omega_j$ and $\bar{t}_j = \bar{t}' + W_2 \omega_j'$, where $\bar{t}$ and $\bar{t}'$ are the characteristic intracell and intercell tunneling energies, $\omega_j$ and $\omega_j'$ are independent random real numbers chosen uniformly from the range $[-1, 1]$, $W_1$ and $W_2$ are the corresponding disorder strengths, and $\bar{t} = 1$ is set as the energy unit. Note that the notations used here are slightly different from those in Ref. [27].

According to Birkhoff’s ergodic theorem, one can use the ensemble average to evaluate the last expression in Eq. (9), which is then given by

$$\Lambda^{-1} = \int_{-1}^{1} d\omega \int_{-1}^{1} d\omega' \left( \ln |t' + W_2 \omega| - \ln |t + W_1 \omega'|| \right) \cdot \tag{10}$$

The integrations can be calculated explicitly, and the arguments of the logarithms can become negative in the regime of large $W$. One can obtain

$$\Lambda^{-1} = \ln \left[ \frac{2\bar{t}' + 2W_2 \sqrt{\bar{t}' + \frac{1}{2} \left| 2 - 2W_1 \sqrt{\bar{t}' - \frac{1}{2}} \right|}}{2\bar{t}' - 2W_2 \sqrt{\bar{t}' - \frac{1}{2} \left| 2 + 2W_1 \sqrt{\bar{t}' - \frac{1}{2}} \right|}} \right]. \tag{11}$$

By using the similarity transformation $\tilde{H} = S^{-1}HS$ on our non-Hermitian disordered SSH Hamiltonian $H$ with $W_2 = 0$ under OBCs, one can transform the Hermitian disordered SSH Hamiltonian $\tilde{H}$. After the transformation, we can obtain the solution of the Schrödinger equation of $H \phi = 0$ with $\phi = S \phi$, which replaces the solution in Eq. (8) with the form:

$$\phi_{N,\alpha} = \prod_{j=1}^{N} \left( \frac{t_j}{m_j} \right)^{1/2} \phi_{0,\alpha}, \tag{12}$$

FIG. 5. (Color online) $\Lambda^{-1}$ as a function of $\gamma$ and $W$. The parameters are $t = 1$, $t' = 1.2$, $W_1 = W$, $W_2 = 0$.

With the mapping, one has the hopping parameters $t_j = t'r + W_1 \omega_j = t'\sqrt{1 + \gamma} + W_1 \omega_j$ and $m_j = m_j = 1$ for $t = t = 1$ and $W_2 = 0$. Thus, by substituting $t' = t'\sqrt{1 + \gamma}$, $W_1 = W$ and $W_2 = 0$ into Eq. (11), we obtain the inverse of the localization length of zero-energy states in our non-Hermitian disordered SSH open chain

$$\Lambda^{-1} = \ln \left[ \frac{2\bar{t}' + 2W_2 \sqrt{1 + \gamma} |2 - 2W_1 | \sqrt{\bar{t}' - \frac{1}{2}}}{|2 + 2W_1 | \sqrt{1 + \gamma} + \frac{1}{2}} \right]. \tag{13}$$

where $e$ is the natural constant. Figure 5 shows the critical points with $\Lambda^{-1} \to 0$ in the $\gamma$-$W$ plane, where the delocalized critical points match with the topological transition points in Fig. 2(a). This result, as well as Fig. 4(e) and Fig. 3(a) in the text, demonstrates that the topological transition is accompanied by an Anderson localization-delocalization transition in our non-Hermitian disordered SSH model.

Results for the case of $W_2 \neq 0$

When the intracell hopping disorder strength $W_2 \neq 0$, one cannot find a similarity transformation of the non-Hermitian disordered SSH chain under OBCs for every disorder configuration (especially in the strong disorder case). In this case, the (disorder-averaged) energy spectrum can be generally complex. Figures 6 (a,b) show the complex energy spectrum for $W_1 = 4W_2 = W$ as a function of $W$ in a disorder configuration. The corresponding disorder-averaged winding number $\nu$ as a function of $W$ and $\gamma$ is shown in Fig. 6 (c), which is close to the result of $W_2 = 0$ shown in Fig. 2 (c). Thus, the non-Hermitian enhancement of the topological phase can still exhibit when $W_2 \neq 0$. For comparison, we plot the winding number as a function of $W$ and $\bar{t}'$ in Fig. 6 (d), corresponding to the Hermitian open SSH chain with $\bar{t}' = t'\sqrt{1 + \gamma}$ after the similarity transformation when $W_2 = 0$, where the topological regime matches well with those in Figs. 2 (a) and 6 (c).
FIG. 6. (Color online) (a,b) The energy spectrum $E_n$ (real and imaginary parts) under OBCs as a function of $W$ with fixed $\gamma = 1$ in a disorder configuration with strengths $W_1 = 4W_2 = W$. (c) Disorder-averaged winding number $\nu$ as a function of $W$ ($W_2 = W/4$) and $\gamma$; (d) $\nu$ as a function of $W$ ($W_2 = 0$) and $\tilde{t}'$. Other parameters are $t = 1$, $t' = 1.2$, $L = 5l = 100$, and $N_s = 200$.

FIG. 7. (Color online) (a,b) Disorder-averaged winding number $\nu$ as a function of $W$ and $\gamma$ for $L = 100$. Other parameters are $t = 1$, $t' = 0.7$, $l = 0.2L$, and $N_s = 200$.

In Figs. 7(a,b), we show the disorder-averaged winding number $\nu$ as a function of $W$ and $\gamma$ for $W_1 = 4W_2 = W$ and $W_1 = 2W_2 = W$, respectively. Although we here adopt a lattice of length $L = 100$ smaller than that of $L = 400$ in Fig. 3, the topological regime with the NHTAI can be roughly seen. Tuning on the disorder of $W_1$ up to $W_1 = W/4$, the topological regime almost preserves, and the NHTAI phase remains. However, the topological regime will be reduced when $W_1$ becomes larger, such as the case of $W_1 = W/2$ in Fig. 7(b) with a narrow parameter regime for the NHTAI phase.