ON EXISTENCE OF PI-EXPONENTS OF UNITAL ALGEBRAS

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Abstract. We construct a family of unital non-associative algebras \( \{ T_\alpha \mid 2 < \alpha \in \mathbb{R} \} \) such that \( \exp(T_\alpha) = 2 \), whereas \( \alpha \leq \exp(T_\alpha) \leq \alpha + 1 \). In particular, it follows that ordinary PI-exponent of codimension growth of algebra \( T_\alpha \) does not exist for any \( \alpha > 2 \). This is the first example of a unital algebra whose PI-exponent does not exist.

1. Introduction. We consider numerical invariants associated with polynomial identities of algebras over a field of characteristic zero. Given an algebra \( A \), one can construct a sequence of non-negative integers \( \{ c_n(A) \}, n = 1, 2, \ldots \), called the codimensions of \( A \), which is an important numerical characteristic of identical relations of \( A \). In general, the sequence \( \{ c_n(A) \} \) grows faster than \( n! \). However, there is a wide class of algebras with exponentially bounded codimension growth. This class includes all associative PI-algebras [2], all finite-dimensional algebras [2], Kac-Moody algebras [12], infinite-dimensional simple Lie algebras of Cartan type [9], and many others. If the sequence \( \{ c_n(A) \} \) is exponentially bounded then the following natural question arises: does the limit

\[
\lim_{n \to \infty} \sqrt[n]{c_n(A)} \tag{1.1}
\]

exist and what are its possible values? In case of existence, the limit (1.1) is called the PI-exponent of \( A \), denoted as \( \exp(A) \). At the end of 1980’s, Amitsur conjectured that for any associative PI-algebra, the limit (1.1) exists and is a non-negative integer. Amitsur’s conjecture was confirmed in [5, 6]. Later, Amitsur’s conjecture was also confirmed for finite-dimensional Lie and Jordan algebras [4, 15]. Existence of \( \exp(A) \) was also proved for all finite-dimensional simple algebras [8] and many others.

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Nevertheless, the answer to Amitsur’s question in the general case is negative: a counterexample was presented in [14]. Namely, for any real \( \alpha > 1 \), an algebra \( R_\alpha \) was constructed such that the lower limit of \( \sqrt[\alpha]{c_n(A)} \) is equal to 1, whereas the upper limit is equal to \( \alpha \). It now looks natural to describe classes of algebras in which for any algebra \( A \), its PI-exponent \( \exp(A) \) exists. One of the candidates is the class of all finite-dimensional algebras. Another one is the class of so-called special Lie algebras. The next interesting class consists of unital algebras, it contains in particular, all algebras with an external unit. Given an algebra \( A \), we denote by \( A^f \) the algebra obtained from \( A \) by adjoining the external unit. There is a number of papers where the existence of \( \exp(A^f) \) has been proved, provided that \( \exp(A) \) exists [11, 16, 17]. Moreover, in all these cases, \( \exp(A^f) = \exp(A) + 1 \).

The main goal of the present paper is to construct a series of unital algebras such that \( \exp(A) \) does not exist, although the sequence \( \{c_n(A)\} \) is exponentially bounded (see Theorem 3.1 and Corollary 3.1 below). All details about polynomial identities and their numerical characteristics can be found in [1, 3, 7].

2. Definitions and preliminary structures. Let \( A \) be an algebra over a field \( F \) and let \( F\{X\} \) be a free \( F \)-algebra with an infinite set \( X \) of free generators. The set \( Id(A) \subset F\{X\} \) of all identities of \( A \) forms an ideal of \( F\{X\} \). Denote by \( P_n = P_n(x_1, \ldots, x_n) \) the subspace of \( F\{X\} \) of all multilinear polynomials on \( x_1, \ldots, x_n \in X \). Then \( P_n \cap Id(A) \) is actually the set of all multilinear identities of \( A \) of degree \( n \). An important numerical characteristic of \( Id(A) \) is the sequence of non-negative integers \( \{c_n(A)\}, n = 1, 2, \ldots \), where

\[
c_n(A) = \dim \frac{P_n}{P_n \cap Id(A)}.
\]

If the sequence \( \{c_n(A)\} \) is exponentially bounded, then the lower and the upper PI-exponents of \( A \), defined as follows

\[
\exp(A) = \liminf_{n \to \infty} \sqrt[n]{c_n(A)}, \quad \overline{\exp}(A) = \limsup_{n \to \infty} \sqrt[n]{c_n(A)},
\]

are well-defined. An existence of ordinary PI-exponent (1.1) is equivalent to the equality \( \exp(A) = \overline{\exp}(A) \).

In [14], an algebra \( R = R(\alpha) \) such that \( \exp(R) = 1, \overline{\exp}(R) = \alpha \), was constructed for any real \( \alpha > 0 \). Slightly modifying the construction from [14], we want to get for any real \( \alpha > 2 \), an algebra \( R_\alpha \) with \( \exp(R_\alpha)^2 = 2 \) and \( \alpha \leq \overline{\exp}(R_\alpha) \leq \alpha + 1 \).

Clearly, polynomial identities of \( A^f \) strongly depend on the identities of \( A \). In particular, we make the following observation. Note that if \( f = f(x_1, \ldots, x_n) \) is a multilinear polynomial from \( F\{X\} \) then \( f(1 + x_1, \ldots, 1 + x_n) \in F\{X\}^f \) is the sum

\[
f = \sum f_{i_1, \ldots, i_k}, \quad \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}, \quad 0 \leq k \leq n, \quad (2.1)
\]

where \( f_{i_1, \ldots, i_k} \) is a multilinear polynomial on \( x_{i_1}, \ldots, x_{i_k} \) obtained from \( f \) by replacing all \( x_j, j \neq i_1, \ldots, i_k \) with 1.

**Remark 2.1.** A multilinear polynomial \( f = f(x_1, \ldots, x_n) \) is an identity of \( A^f \) if and only if all of its components \( f_{i_1, \ldots, i_k} \) on the left hand side of (2.1) are identities of \( A \).

The next statement easily follows from Remark 2.1.
Remark 2.2. Suppose that an algebra $A$ satisfies all multilinear identities of an algebra $B$ of degree $\deg f = k \leq n$ for some fixed $n$. Then $A^k$ satisfies all identities of $B^k$ of degree $k \leq n$.

Using results of [13], we obtain the following inequalities.

Lemma 2.1. ([13, Theorem 2]) Let $A$ be an algebra with an exponentially bounded codimension growth. Then $\exp A^k \leq \exp A + 1$. □

Lemma 2.2. ([13, Theorem 3]) Let $A$ be an algebra with an exponentially bounded codimension growth satisfying the identity (2.2). Then $\exp A^k \geq \exp A + 1$. □

Given an integer $T \geq 2$, we define an infinite-dimensional algebra $B_T$ by its basis 
\[ \{a, b, z_1^i, \ldots, z_T^i | i = 1, 2, \ldots \} \]
and by the multiplication table
\[ z_j^i a = \begin{cases} z_j^{i+1} & \text{if } j \leq T - 1, \\ 0 & \text{if } j = T \end{cases} \]
for all $i \geq 1$ and
\[ z_T^i b = z_1^{i+1}, \quad i \geq 1. \]
All other products of basis elements are equal to zero. Clearly, algebra $B_T$ is right nilpotent of class 3, that is
\[ x_1(x_2 x_3) = 0 \quad (2.2) \]
is an identity of $B_T$. Due to (2.2), any nonzero product of elements of $B_T$ must be left-normed. Therefore we omit brackets in the left-normed products and write
\[ (y_1 y_2) y_3 = y_1 y_2 y_3 \quad \text{and} \quad (y_1 \cdots y_k) y_{k+1} = y_1 \cdots y_{k+1} \]
if $k \geq 3$.

We will use the following properties of algebra $B_T$.

Lemma 2.3. ([14, Lema 2.1]) Let $n \leq T$. Then $c_n(B_T) \leq 2n^3$. □

Lemma 2.4. ([14, Lema 2.2]) Let $n = kT + 1$. Then
\[ c_n(B_T) \geq k! = \left( \frac{N - 1}{T} \right)!. \]
□

Lemma 2.5. ([14, Lema 2.3]) Any multilinear identity $f = f(x_1, \ldots, x_n)$ of degree $n \leq T$ of algebra $B_T$ is an identity of $B_T^{+1}$. □

Let $F[\theta]$ be a polynomial ring over $F$ on one indeterminate $\theta$ and let $F[\theta]_0$ be its subring of all polynomials without free term. Denote by $Q_N$ the quotient algebra
\[ Q_N = \frac{F[\theta]_0}{(Q^{N+1})}, \]
where $(Q^{N+1})$ is an ideal of $F[\theta]$ generated by $Q^{N+1}$. Fix an infinite sequence of integers $T_1 < N_1 < T_2 < N_2 \ldots$ and consider the algebra
\[ R = B(T_1, N_1) \oplus B(T_2, N_2) \oplus \cdots, \quad (2.3) \]
where $B(T, N) = B_T \otimes Q_N$.

Let $R$ be an algebra of the type (2.3). Then the following lemma holds.

Lemma 2.6. For any $i \geq 1$, the following equalities hold:
(a) if $T_i \leq n \leq N_i$ then
\[ P_n \cap Id(R) = P_n \cap Id(B(T_i, N_i) \oplus B(T_{i+1}, N_{i+1})) = P_n \cap Id(B_{T_i} \oplus B_{T_{i+1}}); \]
(b) if $N_i < n \leq T_{i+1}$ then

$$P_n \cap Id(R) = P_n \cap Id(B(T_{i+1}, N_{i+1})) = P_n \cap (Id(B_{T_{i+1}})).$$

Proof. This follows immediately from the equality $B(T_i, N_i)^{N_i+1} = 0$ and from Lemma 2.5.

The following remark is obvious.

Remark 2.3. Let $R$ be an algebra of type (2.3). Then

$$Id(R^\sharp) = Id(B(T_1, N_1)^\sharp \oplus B(T_2, N_2)^\sharp \oplus \cdots).$$

\[\square\]

3. The main result.

Theorem 3.1. For any real $\alpha > 1$, there exists an algebra $R_\alpha$ with $\exp(R_\alpha) = 1$, $\exp(R_\alpha) = \alpha$ such that $\exp(R_\alpha^\sharp) = 2$ and $\alpha \leq \exp(R_\alpha^\sharp) \leq \alpha + 1$.

Proof. Note that

$$c_n(A) \leq n c_{n-1}(A)$$

for any algebra $A$ satisfying (2.2). We will construct $R_\alpha$ of type (2.3) by a special choice of the sequence $T_1, N_1, T_2, N_2, \ldots$, depending on $\alpha$. First, choose $T_1$ such that

$$2m^3 < \alpha^m$$

for all $m \geq T_1$. By Lemma 2.4, algebra $B_{T_1}$ has an overexponential codimension growth. Hence there exists $N_1 > T_1$ such that

$$c_n(B_{T_1}) < \alpha^n \quad \text{for all} \quad n \leq N_1 - 1 \quad \text{and} \quad c_{N_1}(B_{T_1}) \geq \alpha^{N_1}.$$  

Consider an arbitrary $n > N_1$. By Remark 2.1, we have

$$c_n(R^\sharp) \leq \sum_{k=0}^{n} \binom{n}{k} c_k(R) = \Sigma_1' + \Sigma_2',$$

where

$$\Sigma_1' = \sum_{k=0}^{N_1} \binom{n}{k} c_k(R), \quad \Sigma_1'' = \sum_{k=N_1+1}^{n} \binom{n}{k} c_k(R).$$

By Lemma 2.6, we have $\Sigma_1' + \Sigma_1'' \leq \Sigma_1 + \Sigma_2$, where

$$\Sigma_1 = \sum_{k=0}^{N_1} \binom{n}{k} c_k(B_{T_1}), \quad \Sigma_2 = \sum_{k=0}^{n} \binom{n}{k} c_k(B_{T_2}).$$

Then for any $T_2 > N_1$, an upper bound for $\Sigma_2$ is

$$\Sigma_2 \leq \sum_{k=0}^{n} \binom{n}{k} 2k^3 \leq 2n^3 \sum_{k=0}^{n} \binom{n}{k} = 2n^3 2^n,$$

which follows from (3.2), provided that $n \leq T_2$.

Let us find an upper bound for $\Sigma_1$ assuming that $n$ is sufficiently large. Clearly,

$$\Sigma_1 \leq N_1 \alpha^{N_1} \sum_{k=0}^{N_1} \binom{n}{k}$$

which follows from the choice of $N_1$, relation (3.1), and the equality $B(T_1, N_1)^n = 0$ for all $n \geq N_1 + 1$. Since $N_1 \alpha^{N_1}$ is a constant for fixed $N_1$, we only need to estimate the sum of binomial coefficients.
From the Stirling formula
\[ m! = \sqrt{2\pi m} \left(\frac{m}{e}\right)^m e^{-m+1} = m^{n+1} \cdot 0 < \theta_m < 1, \]
it follows that
\[ \binom{n}{k} \leq \sqrt{\frac{n}{k(n-k)}} \cdot \frac{n^n}{k^n(n-k)^{n-k}}. \tag{3.5} \]

Now we define the function \( \Phi : [0; 1] \to \mathbb{R} \) by setting
\[ \Phi(x) = \frac{1}{x}(1-x). \]

It is not difficult to show that \( \Phi \) increases on \([0; 1/2]\), \( \Phi(0) = 1 \), and \( \Phi(x) \leq 2 \) on \([0; 1]\). In terms of the function \( \Phi \) we rewrite (3.5) as
\[ \binom{n}{k} \leq \sqrt{\Phi(k/n)^n} \cdot \Phi(k/n)^n. \tag{3.6} \]

providing that \( n > 2N_1 \). Now (3.4) and (3.6) together with (3.3) imply
\[ \Sigma_1 \leq 2N_1 \alpha^{N_1}(N_1 + 1)\Phi\left(\frac{N_1}{n}\right)^n, \quad \Sigma_2 \leq 2n^{3/2}. \tag{3.7} \]

Now we take \( T_2 \) to be equal to the minimal \( n > 2N_1 \) satisfying (3.7). Note that for such \( T_2 \) we have
\[ c_n(R^2) < \left(2 + \frac{1}{2}\right)^n \]
for \( n = T_2 \).

As soon as \( T_2 \) is chosen, we can take \( N_2 \) as the minimal \( n \) such that \( c_m(B_{T_2}) \geq \alpha^n \). Then again, \( c_m(R) < m\alpha^m \) if \( m < N_2 \). Repeating this procedure, we can construct an infinite chain \( T_1 < N_1 < T_2 < N_2 \cdots \) such that
\[ c_n(R) < \alpha^n + 2n^3 \tag{3.8} \]
for all \( n \neq N_1, N_2, \ldots \),
\[ \alpha^n \leq c_n(R) < \alpha^n + n(\alpha^{n-1} + 2n^3) \tag{3.9} \]
for all \( n = N_1, N_2, \ldots \) and
\[ 2N_j(N_j + 1)\alpha^{N_j}\Phi\left(\frac{N_j}{T_{j+1}}\right)^{T_{j+1}} + 2T_{j+1}^3 \cdot 2^{T_{j+1}} < (2 + \frac{1}{2^j})^{T_{j+1}} \tag{3.10} \]
for all \( j = 1, 2, \ldots \).

Let us denote by \( R_\alpha \) the just constructed algebra \( R \) of type (2.3). Then (3.10) means that
\[ c_n(R^2_n) < \left(2 + \frac{1}{2^j}\right)^n \tag{3.11} \]
if \( n = T_{j+1}, j = 1, 2, \ldots \). It follows from inequality (3.11) that
\[ \exp(R^2_n) \leq 2. \tag{3.12} \]
On the other hand, since $R_{\alpha}$ is not nilpotent, it follows that
\[ \exp(R_{\alpha}) \geq 1. \] (3.13)
Since the PI-exponent of non-nilpotent algebra cannot be strictly less than 1, relations (3.12), (3.13) and Lemma 2.2 imply
\[ \exp(R_{\alpha}) = 1, \quad \exp(R_{\alpha}^{\sharp}) = 2. \]
Finally, relations (3.8), (3.9) imply the equality $\exp(R_{\alpha}) = \alpha$. Applying Lemma 2.1, we see that $\exp(R_{\alpha}^{\sharp}) \leq \alpha + 1$. The inequality $\alpha = \exp(R_{\alpha}) \leq \exp(R_{\alpha}^{\sharp})$ is obvious, since $R_{\alpha}$ is a subalgebra of $R_{\alpha}^{\sharp}$. Thus we have completed the proof of Theorem 3.1. □

As a consequence of Theorem 3.1 we get an infinite family of unital algebras of exponential codimension growth without ordinary PI-exponent.

Corollary 3.1. Let $\beta > 2$ be an arbitrary real number. Then the ordinary PI-exponent of unital algebra $R_{\beta}^{\sharp}$ from Theorem 3.1 does not exist. Moreover, $\exp(R_{\beta}^{\sharp}) = 2$, whereas $\beta \leq \exp(R_{\beta}^{\sharp}) \leq \beta + 1$. □

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