Derivation of an equation of pair correlation function from BBGKY hierarchy in a weakly coupled self-gravitating system

Anirban Bose
Serampore College, Serampore, Hooghly 712201, West Bengal, India

Received 26 November 2022 / Accepted 9 March 2023 / Published online 31 March 2023

© The Author(s), under exclusive licence to EDP Sciences, SIF and Springer-Verlag GmbH Germany, part of Springer Nature 2023

Abstract. An equation of pair correlation function has been derived from the first two members of BBGKY hierarchy in a weakly coupled inhomogeneous self-gravitating system in quasi-thermal equilibrium. This work may be useful to study the thermodynamic properties of the central region of a star cluster which is older than a few or more central relaxation time.

1 Introduction

Finite, bound self-gravitating systems have negative heat capacity. Thermodynamic properties [1–8] of such systems differ significantly from that of the normal laboratory systems and there is a question that if a self-gravitating system where the constituent particles are interacting with each other via long-range gravitational force, may at all attain the thermal equilibrium state. In fact, absolute thermal equilibrium is not possible. Since in contrast to the laboratory systems, there is no physical container to confine the system to a particular place. Therefore, evaporation of particles from the high energy tail of the distribution function does not allow the system to reach thermal equilibrium.

This confinement problem may be theoretically resolved by the introduction of an artificial boundary. Thermodynamic properties of such systems have been extensively investigated [9–11] by confining N point particles each with mass m in spherical containers. The systems are assumed to be thermally conducting. Therefore, the equilibrium state is expected to be the isothermal sphere. But that is not the actual fact. For both canonical and microcanonical ensembles, instabilities occur if the density contrast between the centre and edge of an isothermal gas exceeds certain values. For micro-canonical ensembles, this phenomenon is recognised as gravothermal catastrophe [9–11] and such system can increase entropy by settling down to a state which is not isothermal.

However, a self-gravitating system (globular cluster) whose relaxation time is less than it’s age, has a chance to reach a state which is close to thermal equilibrium.

At least, the central part of a cluster older than a few central relaxation time is close to thermal equilibrium [2]. These globular clusters are approximately spherical in shape and may be modelled by the N particle system confined in a spherical container.

There are different theoretical approaches [1,2,12–15] to study the self gravitating systems. Probably, the most common way is to explore the Fokker–Planck equation [1,2] of the concerned system.

There are other ways. For example, integrating the N particle Liouville equation over the phase space of (N − 1) particles, we obtain the first member of the BBGKY hierarchy. In the limit of N → ∞, it becomes the collision less Vlasov equation. This equation deals successfully with the self-gravitating system where the influence of encounters are neglected. Integrating the Liouville equation over the phase space of (N − 2) particles, we obtain the second member of BBGKY hierarchy. These two equations, in the limit of large but finite values of N, are capable of exploring the thermodynamics of systems with weak encounters [12,16].

In this article, we attempt to derive an equation of pair correlation function from the first two members of the BBGKY hierarchy for the inhomogeneous self-gravitating system in quasi thermal equilibrium. This method has been previously applied [17] to study the thermodynamic properties of the weakly correlated inhomogeneous plasma system.

2 Derivation of the equation of pair correlation in thermal equilibrium

We consider a self-gravitating system of N particles enclosed in a spherical container. We assume, for sim-
plicity, that all the particles have the same mass \( m \). This system is older than a few collisional relaxation time so that the distribution function is in thermal equilibrium. Under these assumptions, the first two members of the BBGKY hierarchy are written as

\[
\begin{align*}
\frac{df_1}{dt} + v_1 \cdot \frac{\partial f_1}{\partial x_1} &+ n_0 \int dX_2 a_{12} \left[ f_1(X_1)f_1(X_2) + g_{12} \right] = 0, \\
\frac{dg_{12}}{dt} + v_1 \cdot \frac{\partial g_{12}}{\partial x_1} + v_2 \cdot \frac{\partial g_{12}}{\partial x_2} &+ n_0 \int dX_3 f_1(X_3) \left( a_{13} \cdot \frac{\partial g_{12}}{\partial v_1} + a_{23} \cdot \frac{\partial g_{12}}{\partial v_2} \right) \\
&= - \left( a_{12} \cdot \frac{\partial}{\partial v_1} + a_{21} \cdot \frac{\partial}{\partial v_2} \right) \left[ f_1(X_1)f_1(X_2) + g_{12} \right] \\
&- n_0 \int dX_3 \left[ a_{13} \cdot \frac{\partial f_1(X_1)}{\partial v_1} g_{23} + a_{23} \cdot \frac{\partial f_1(X_2)}{\partial v_2} g_{13} \right],
\end{align*}
\]

where \( X = (x, v) \), \( f_1 \) is the single particle distribution, \( g_{12} \) is the pair correlation function and

\[
a_{ij} = - \frac{1}{m} \frac{\partial}{\partial x_i} \phi_{ij},
\]

where

\[
\phi_{ij} = \frac{Gm^2}{|x_i - x_j|}.
\]

The term \( a_{ij} \) denotes acceleration of the \( i \)th particle due to a force exerted by the \( j \)th particle and \( \phi_{ij} \) is the energy of interaction between them. Three particle correlation function is ignored for large but finite values of \( N \).

The system is considered to be in thermal equilibrium. Hence, the first terms of Eqs. (1) and (2) are ignored. The pair correlation function \((g_{12})\) is written as

\[
g_{12}(X_1, X_2) = f_1(X_1)f_1(X_2)\chi_{12}(x_1, x_2). \tag{3}
\]

Single particle distributions \((f_1(X_1)\) and \(f_1(X_2)\)) are functions of position and velocity. \( \chi_{12} \) is a symmetric function of \( x_1 \) and \( x_2 \). Using Eqs. (1) and (3), we obtain

\[
\begin{align*}
\frac{\partial g_{12}}{\partial x_1} &+ v_1 \cdot \frac{\partial f_1(X_2)}{\partial x_2} = f_1(X_1)\chi_{12} v_2 \cdot \frac{\partial f_1(X_2)}{\partial x_2} + f_1(X_1)f_1(X_2) v_2 \cdot \frac{\partial \chi_{12}}{\partial x_2}, \tag{4}
\\
v_2 \cdot \frac{\partial f_1(X_2)}{\partial x_2} &+ n_0 \int dX_3 a_{23} \cdot \frac{\partial}{\partial v_2} (f_1(X_2)f_1(X_3)) \chi_{23}.
\end{align*}
\]

and similar expressions for \( v_1 \cdot \partial g_{12}/\partial x_1 \). Using Eqs. (4) and (5) in Eq. (2), we obtain

\[
\begin{align*}
f_1(X_1)f_1(X_2) \left[ v_1 \cdot \frac{\partial \chi_{12}}{\partial x_1} + v_2 \cdot \frac{\partial \chi_{12}}{\partial x_2} \right] \\
- n_0 \int dX_3 \left[ f_1(X_2)\chi_{12} a_{13} \cdot \frac{\partial g_{13}}{\partial v_1} \right] \\
+ f_1(X_1)\chi_{21} a_{23} \cdot \frac{\partial g_{23}}{\partial v_2} \\
= - \left( a_{12} \cdot \frac{\partial}{\partial v_1} + a_{21} \cdot \frac{\partial}{\partial v_2} \right) (f_1(X_1)f_1(X_2) + g_{12}) \\
- n_0 \int dX_3 \left[ a_{13} \cdot \frac{\partial f_1(X_1)}{\partial v_1} g_{23} + a_{23} \cdot \frac{\partial f_1(X_2)}{\partial v_2} g_{13} \right].
\end{align*}
\]

The single particle distribution functions are written in the following form

\[
f_1(X_1) = f_M(v_1)F_1(x_1),
\]

where \( f_M \) is a Maxwellian distribution and \( F_1 \) is the space part.

Inserting \( g_{23} \) and \( g_{13} \), we obtain

\[
\begin{align*}
f_1(X_1)f_1(X_2) \left[ v_1 \cdot \frac{\partial \chi_{12}}{\partial x_1} + v_2 \cdot \frac{\partial \chi_{21}}{\partial x_2} \right] \\
- \frac{1}{k_B T} \left[ \frac{\partial \phi_{12}}{\partial x_1} \cdot v_1 + \frac{\partial \phi_{12}}{\partial x_2} \cdot v_2 \right] \\
\times f_1(X_1)f_1(X_2) (1 + \chi_{12}) + f_1(X_1)f_1(X_2) \frac{n_{om}}{k_B T} \\
\times f_1(X_1)f_1(X_2) \frac{n_{om}}{k_B T} \int dX_3 \left[ a_{13} \cdot v_1 f_1(X_3) \chi_{13} \chi_{23} + a_{23} \cdot v_2 f_1(X_3) \chi_{13} \right] \\
- f_1(X_1)f_1(X_2) \frac{n_{om}}{k_B T} \int dX_3 \left[ a_{13} \cdot v_1 f_1(X_3) \chi_{13} \chi_{23} + a_{23} \cdot v_2 f_1(X_3) \chi_{13} \right].
\end{align*}
\]

We can write,

\[
f_1(X_1)f_1(X_2)A \cdot v_1 + f_1(X_1)f_1(X_2)B \cdot v_2 = 0, \tag{8}
\]

where

\[
\begin{align*}
A &= \frac{\partial \chi_{12}}{\partial x_1} + \frac{1}{k_B T} \frac{\partial \phi_{12}}{\partial x_1} (1 + \chi_{12}) \\
+ \frac{n_{om}}{k_B T} \int dX_3 \frac{\partial \phi_{13}}{\partial x_1} f_1(X_3) \chi_{23} \\
- \frac{n_{om}}{k_B T} \int dX_3 \frac{\partial \phi_{13}}{\partial x_1} f_1(X_3) \chi_{13}, \tag{9}
\\
B &= \frac{\partial \chi_{21}}{\partial x_2} + \frac{1}{k_B T} \frac{\partial \phi_{21}}{\partial x_2} (1 + \chi_{21}) \\
+ \frac{n_{om}}{k_B T} \int dX_3 \frac{\partial \phi_{23}}{\partial x_2} f_1(X_3) \chi_{13} \\
- \frac{n_{om}}{k_B T} \int dX_3 \frac{\partial \phi_{23}}{\partial x_2} f_1(X_3) \chi_{23}. \tag{10}
\end{align*}
\]
For arbitrary and linearly independent $\mathbf{v}_1$ and $\mathbf{v}_2$, $A$ and $B$ both vanish. Therefore,

$$
\frac{\partial \chi_{12}}{\partial \mathbf{x}_1} + \frac{1}{k_B T} \frac{\partial \phi_{12}}{\partial \mathbf{x}_1} + \frac{1}{k_B T} \frac{\partial \phi_{12}}{\partial \mathbf{x}_1} \chi_{12} + \frac{n_0}{k_B T} \int d\mathbf{x}_3 \frac{\partial \phi_{13}}{\partial \mathbf{x}_1} f_1(\mathbf{x}_3) \chi_{23} - \frac{n_0 \chi_{12}}{k_B T} \int d\mathbf{x}_3 \frac{\partial \phi_{13}}{\partial \mathbf{x}_1} f_1(\mathbf{x}_3) \chi_{13} = 0. \quad (11)
$$

This is the equation of the pair correlation function, which is derived from the BBGKY hierarchy. This equation is applicable to both homogeneous and inhomogeneous systems and capable of describing both short and long range behaviours. To derive this complex equation, we have assumed that the three-particle correlation function is negligible. However, that does not mean that pair correlation ($\chi_{12}$) should always be very small. For example, we may bring the first two particles closer and take the remaining particle far away from them. In that case, the three-particle correlation function is negligible but the value of pair correlation ($\chi_{12}$) of the first two particles may not be very weak. Finally, we check the weak correlation limit ($\chi \ll 1$). We drop the third and last term of the equation to obtain

$$
\frac{\partial \chi_{12}}{\partial \mathbf{x}_1} + \frac{1}{k_B T} \frac{\partial \phi_{12}}{\partial \mathbf{x}_1} + \frac{n_0}{k_B T} \int d\mathbf{x}_3 \frac{\partial \phi_{13}}{\partial \mathbf{x}_1} f_1(\mathbf{x}_3) \chi_{23} = 0.
$$

Hence,

$$\chi_{12} = -\frac{1}{k_B T} \phi_{12} - \frac{n_0}{k_B T} \int d\mathbf{x}_3 \phi_{13} f_1(\mathbf{x}_3) \chi_{23}. \quad (13)$$

Performing the velocity integral

$$\chi_{12} = -\frac{1}{k_B T} \phi_{12} - \frac{1}{k_B T} \int d\mathbf{r}_3 \phi_{13} n_1(\mathbf{r}_3) \chi_{23}. \quad (14)$$

The pair correlation equation is obtained in the weak coupling limit. This equation is identical to what has been previously derived by Chavanis [18] from a BBGKY-like hierarchy for inhomogeneous systems which generalizes his former result [19] obtained for homogeneous systems with long-range interactions.

### 3 Results and discussions

In the previous section, we have derived an equation of pair correlation function of a weakly coupled self-gravitating system from BBGKY hierarchy.

To obtain the equation, we have assumed that the pair correlation function is the product of the single-particle distribution functions and a general function of positions of the pairing particles. This general function arises due to correlation.

The structure of this function depends on the nature of the system. For homogeneous cases, it is the sole function of $|\mathbf{x}_1 - \mathbf{x}_2|$. For inhomogeneous cases, the position of the pair is important. Hence, the pair correlation function can not be the sole function of $|\mathbf{x}_1 - \mathbf{x}_2|$ because it does not reflect where the pair is placed in the system. Therefore, it must be a symmetric function of $\mathbf{x}_1$ and $\mathbf{x}_2$ and should also reflect the inhomogeneity of the system.

We must choose a general form of the pair correlation function so that after inserting it in BBGKY equations, we should obtain an equation which is applicable to both homogeneous and inhomogeneous cases.

We may calculate the pair correlation function for homogeneous cases to check how does the equation work. Although self-gravitating system is intrinsically inhomogeneous, if the distance between the pairing particles is much less than the size of the system, we may consider it to be locally flat. Moreover, the homogeneous cases are relatively easy to tackle mathematically. Therefore, we apply Eq. (14) to calculate the pair correlation function in the homogeneous limit. We use the following transformations

$$\chi_{ij} = \int d\mathbf{k} \chi(\mathbf{k}) e^{i\mathbf{k} \cdot (\mathbf{x}_i - \mathbf{x}_j)},$$

$$\phi_{ij} = -\frac{Gm_i^2}{2\pi^2} \int \frac{d\mathbf{k}}{k^2} e^{i\mathbf{k} \cdot (\mathbf{x}_i - \mathbf{x}_j)},$$

in Eq. (14) and obtain

$$\chi(\mathbf{k}) = \frac{1}{2\pi^2 k_B T k^2 - k_j^2} Gm_i^2 \cos (k_j r), \quad (15)$$

where $k_j^2 = 4\pi n_0 Gm^2 / k_B T$. Therefore, the pair correlation function with collective effects is obtained as

$$\chi_{ij} = \frac{1}{k_B T} \frac{Gm_i^2}{r} \cos (k_j r), \quad (16)$$

where $r = |\mathbf{x}_1 - \mathbf{x}_j|$. Hence, the potential for interaction is

$$-\frac{Gm_i^2}{r} \cos (k_j r),$$

which is identical to what has been shown by Chavanis [20]. We could neglect the collective effects by switching off the last term of Eq. (14) to obtain

$$\chi_{ij} = \frac{1}{k_B T} \frac{Gm_i^2}{r}.$$  \quad (17)

It is interesting to note that in the limit $k_j r << 1$, the result with collective effects (Eq. 16) tends to be identical to the result without collective effects (Eq. 17).
It has a serious implication for self-gravitating system. Since, as described by Binney and Tremaine [2], equal octaves in impact parameter contribute equally to gravitational scattering, most of the contributions of the two body interactions are local in nature, i.e. the distance between the pairing particles is much less than Jean’s length. Therefore, under such circumstances, we could effectively use the result without collective effects in place of the result with collective effects. It may be the reason that the theory of Chandrasekhar [21–23], which did not consider the collective effects, gives a reasonably good description of the self-gravitating system. Obviously, there is scope to incorporate the collective effects in the system to make the calculation more rigorous. We may also expect the above discussion to be true for the inhomogeneous systems. After all, an inhomogeneous system appears to be homogeneous in the shorter length scale and the underlying physics of collective phenomenon does remain the same in both cases. If we increase the distance between the pairing particles, the collective effects should reflect in the expression of the pair correlation function. Intuitively, some symmetric function of positions, other than the function of |x1 − x2|, may appear in the expression. For example, it may be a function of (x1 + x2), which is the simplest function among the functions which are symmetric in positions of the particles. Finally, consideration of strong inhomogeneity needs numerical techniques which is beyond the scope of this work.

4 Conclusion

We have obtained an equation of pair correlation (Eq. 11) function from the first two members of BBGKY hierarchy for a weakly interacting inhomogeneous self-gravitating system confined by a finite spherical container in thermal equilibrium. This equation may describe both the short and long range behaviour of the pair correlation function. This complex equation (Eq. 11), in the weak coupling limit, reduces to a simpler equation (Eq. 14) which is identical to what has been derived by Chavanis from a BBGKY-like hierarchy for a weakly interacting inhomogeneous self-gravitating system confined by a finite spherical container [19–21,24]. However, in contrast to the homogeneous and locally flat cases, it is not the sole function of |x1 − x2|. Consequently, we are not bound to consider our system to be locally flat. In addition to that, the presence of third term of Eq. (11) confirms the presence of collective effects.

Acknowledgements I would like to acknowledge helpful discussions with Prof. S. Tremaine.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: Data sharing is not applicable to this article as no data sets were generated or analysed during the current study.]