Oscillation and variation inequalities for the multilinear singular integrals related to Lipschitz functions

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Abstract

The main purpose of this paper is to establish the weighted \(L^p, L^q\) inequalities of the oscillation and variation operators for the multilinear Calderón-Zygmund singular integral with a Lipschitz function.

MSC: 42B25; 42B20; 47G10

Keywords: oscillation operator; variation operator; multilinear operator; Lipschitz function

1 Introduction and results

Let \(K\) be a kernel on \(\mathbb{R} \times \mathbb{R} \setminus \{(x, x) : x \in \mathbb{R}\}\). Suppose that there exist two constants \(\delta\) and \(C\) such that

\[
|K(x, y)| \leq \frac{C}{|x-y|} \quad \text{for } x \neq y;
\]

\[
|K(x, y) - K(x', y)| \leq \frac{C|x-x'|^\delta}{|x-y|^{1+\delta}} \quad \text{for } |x-y| \geq 2|x-x'|;
\]

\[
|K(x, y) - K(x, y')| \leq \frac{C|y-y'|^\delta}{|x-y|^{1+\delta}} \quad \text{for } |x-y| \geq 2|y-y'|.
\]

We consider the family of operators \(T = \{T_\epsilon\}_{\epsilon>0}\) given by

\[
T_\epsilon f(x) = \int_{|x-y|>\epsilon} K(x, y)f(y) \, dy.
\]

A common method of measuring the speed of convergence of the family \(T_\epsilon\) is to consider the square functions

\[
\left( \sum_{i=1}^\infty |T_{\epsilon_i}f - T_{\epsilon_j}f|^2 \right)^{1/2},
\]

where \(\epsilon_i\) is a monotonically decreasing sequence which approaches 0. For convenience, other expressions have also been considered. Let \(\{t_i\}\) be a fixed sequence which decreases
to zero. Following [1], the oscillation operator is defined as
\[
O(Tf)(x) = \left( \sum_{i=1}^{\infty} \sup_{t_i, t_{i+1} \leq x} |T_{t_i+\epsilon}f(x) - T_{t_i}f(x)|^2 \right)^{1/2}
\]
and the \( \rho \)-variation operator is defined as
\[
V_\rho(Tf)(x) = \sup_{\epsilon_i \downarrow 0} \left( \sum_{i=1}^{\infty} |T_{t_i+\epsilon}f(x) - T_{t_i}f(x)|^\rho \right)^{1/\rho},
\]
where the sup is taken over all sequences of real number \( \{\epsilon_i\} \) decreasing to zero.

The oscillation and variation for some families of operators have been studied by many authors on probability, ergodic theory, and harmonic analysis; see [2–4]. Recently, some authors [5–8] researched the weighted estimates of the oscillation and variation operators for the commutators of singular integrals.

Let \( m \) be a positive integer, let \( b \) be a function on \( \mathbb{R} \), and let \( R_{m+1}(b; x, y) \) be the \( m+1 \)th Taylor series remainder of \( b \) at \( x \) expander about \( y \), i.e.
\[
R_{m+1}(b; x, y) = b(x) - \sum_{\alpha \leq m} \frac{1}{\alpha!} b^{(\alpha)}(y)(x-y)^\alpha.
\]

We consider the family of operators \( T^b = \{T^b_\epsilon\}_{\epsilon>0} \), where \( T^b_\epsilon \) are the multilinear singular integral operators of \( T_\epsilon \),
\[
T^b f(x) = \int_{|x-y|<\epsilon} \frac{R_{m+1}(b; x, y)}{|x-y|^m} K(x, y)f(y)\,dy.
\]

Note that when \( m = 0 \), \( T^b_\epsilon \) is just the commutator of \( T_\epsilon \) and \( b \), which is denoted by \( T_{\epsilon,b} \), that is to say
\[
T_{\epsilon,b} f(x) = \int_{|x-y|<\epsilon} (b(x) - b(y))K(x, y)f(y)\,dy.
\]

However, when \( m > 0 \), \( T^b_\epsilon \) is a non-trivial generation of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [9–13]).

A locally integrable function \( b \) is said to be in Lipschitz space \( \text{Lip}_\beta(\mathbb{R}) \) if
\[
\|b\|_{\text{Lip}_\beta} = \sup_I \frac{1}{|I|^{1+\beta}} \int_I |b(x) - b_I|\,dx < \infty,
\]
where
\[
b_I = \frac{1}{|I|} \int_I b(x)\,dx.
\]

In this paper, we will study the boundedness of oscillation and variation operators for the family of the multilinear singular integral related to a Lipschitz function defined by (1.5) in weighted Lebesgue space. Our main results are as follows.
Theorem 1.1 Suppose that $K(x,y)$ satisfies (1.1)-(1.3), $b^{(m)} \in \dot{A}_p$, $0 < \beta \leq \delta < 1$, where $\delta$ is the same as in (1.2). Let $\rho > 2$, $T = \{T_{\epsilon} \}_{\epsilon > 0}$ and $T^b = \{T^b_{\epsilon} \}_{\epsilon > 0}$ be given by (1.4) and (1.5), respectively. If $O(T)$ and $V_p(T)$ are bounded on $L^p(\mathbb{R}, dx)$ for some $1 < p_0 < \infty$, then, for any $1 < p < 1/\beta$ with $1/q = 1/p - \beta$, $\omega \in A_{p,q}(\mathbb{R})$, $O(T^b)$ and $V_p(T^b)$ are bounded from $L^p(\mathbb{R}, \omega^p dx)$ into $L^q(\mathbb{R}, \omega^q dx)$.

Corollary 1.1 Suppose that $K(x,y)$ satisfies (1.1)-(1.3), $b \in \dot{A}_p$, $0 < \beta \leq \delta < 1$, where $\delta$ is the same as in (1.2). Let $\rho > 2$, $T = \{T_{\epsilon} \}_{\epsilon > 0}$ and $T_b = \{T_b_{\epsilon} \}_{\epsilon > 0}$ be given by (1.4) and (1.6), respectively. If $O(T)$ and $V_p(T)$ are bounded on $L^p(\mathbb{R}, dx)$ for some $1 < p_0 < \infty$, then, for any $1 < p < 1/\beta$ with $1/q = 1/p - \beta$, $\omega \in A_{p,q}(\mathbb{R})$, $O(T_b)$ and $V_p(T_b)$ are bounded from $L^p(\mathbb{R}, \omega^p dx)$ into $L^q(\mathbb{R}, \omega^q dx)$.

In this paper, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant $C$, independent of all important parameters, such that $A \leq CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2 Some preliminaries

2.1 Weight

A weight $\omega$ is a nonnegative, locally integrable function on $\mathbb{R}$. The classical weight theories were introduced by Muckenhoupt and Wheeden in [14] and [15].

A weight $\omega$ is said to belong to the Muckenhoup class $A_p(\mathbb{R})$ for $1 < p < \infty$, if there exists a constant $C$ such that

$$
\left( \frac{1}{|I|} \int_I \omega(x) \, dx \right)^{p-1} \leq C
$$

for every interval $I$. The class $A_1(\mathbb{R})$ is defined by replacing the above inequality with

$$
\frac{1}{|I|} \int_I \omega(x) \, dx \leq \text{ess inf}_{x \in I} \omega(x) \quad \text{for every ball } I \subset \mathbb{R}.
$$

When $p = \infty$, we define $A_\infty(\mathbb{R}) = \bigcup_{1 \leq p < \infty} A_p(\mathbb{R})$.

A weight $\omega(x)$ is said to belong to the class $A_{p,q}(\mathbb{R})$, $1 < p < q < \infty$, if

$$
\left( \frac{1}{|I|} \int_I \omega(x)^q \, dx \right)^{1/q} \left( \frac{1}{|I|} \int_I \omega(x)^{-p'} \, dx \right)^{1/p'} \leq C.
$$

It is well known that if $\omega \in A_{p,q}(\mathbb{R})$, then $\omega^q \in A_\infty(\mathbb{R})$.

2.2 Function of Lip$_\beta$($\mathbb{R}$)

The function of Lip$_\beta$($\mathbb{R}$) has the following important properties.

Lemma 2.1 Let $b \in \text{Lip}_\beta(\mathbb{R})$. Then

1. $1 \leq p < \infty$

$$
\sup_I \frac{1}{|I|^\beta} \left( \frac{1}{|I|} \int_I |b(x) - b_I|^p \, dx \right)^{1/p} \leq C \|b\|_{\dot{A}_p};
$$
for any $I_1 \subset I_2$,
\[
\frac{1}{|I_2|} \int_{I_2} |b(y) - b_{I_1}| \, dy \lesssim \frac{|I_2|}{|I_1|} |I_2|^\beta \|b\|_{\dot{B}^\beta}.
\]

2.3 Maximal function
We recall the definition of Hardy-Littlewood maximal operator and fractional maximal operator. The Hardy-Littlewood maximal operator is defined by
\[
M(f)(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)| \, dy.
\]
The fractional maximal function is defined as
\[
M_{\beta,r}(f)(x) = \sup_{I \ni x} \left( \frac{1}{|I|^{1-r\beta}} \int_I |f(y)|^r \, dy \right)^{1/r}
\]
for $1 \leq r < \infty$. In order to simplify the notation, we set $M_{\beta}(f)(x) = M_{\beta,1}(f)(x)$.

Lemma 2.2 Let $1 < p < \infty$ and $\omega \in A_\infty(\mathbb{R})$. Then
\[
\|Mf\|_{L^p(\omega)} \lesssim \|M^\#f\|_{L^p(\omega)}
\]
for all $f$ such that the left hand side is finite.

Lemma 2.3 Suppose $0 < \beta < 1$, $1 \leq r < 1/p - 1/q = 1/p - \beta$. If $\omega \in A_{p,q}(\mathbb{R})$, then
\[
\|M_{\beta,r}f\|_{L^q(\omega)} \lesssim \|f\|_{L^p(\omega^\beta)}.\]

2.4 Taylor series remainder
The following lemma gives an estimate on Taylor series remainder.

Lemma 2.4 [10] Let $b$ be a function on $\mathbb{R}$ and $b^{(m)} \in L^s(\mathbb{R})$ for any $s > 1$. Then
\[
|R_m(b; x, y)| \lesssim |x - y|^m \left( \frac{1}{|I_x^s|} \int_{I_x^s} |b^{(m)}(z)|^s \, dz \right)^{1/s},
\]
where $I_x^s$ is the interval $(x - 5|x - y|, x + 5|x - y|)$.

2.5 Oscillation and variation operators
We consider the operator
\[
O'(Tf)(x) = \left( \sum_{j=1}^{\infty} \sup_{b_1 \subset b_2 : c_1 < c_2} |T_{c_2}f(x) - T_{c_1}f(x)|^2 \right)^{1/2}.
\]
It is easy to check that
\[
O'(Tf) \approx O(Tf).
\]
Following [4], we denote by $E$ the mixed norm Banach space of two variable function $h$ defined on $\mathbb{R} \times \mathbb{N}$ such that

$$\|h\|_E \equiv \left( \sum_i \left( \sup_s |h(s,i)| \right)^2 \right)^{1/2} < \infty.$$ 

Given $T = \{T_\epsilon\}_{\epsilon > 0}$, where $T_\epsilon$ defined as (1.4), for a fixed decreasing sequence $\{t_i\}$ with $t_i \downarrow 0$, let $J_i = (t_{i+1}, t_i]$ and define the $E$-valued operator $\mathcal{U}(T): f \rightarrow \mathcal{U}(T)f$ by

$$\mathcal{U}(T)f(x) = \left\{ T_{t_{i+1}}f(x) - T_{t_i}f(x) \right\}_{x \in J_i, i \in \mathbb{N}} = \left\{ \int_{[t_{i+1} < |x-y| < t_i]} K(x,y)f(y) \, dy \right\}_{x \in J_i, i \in \mathbb{N}}.$$ 

Then

$$O'(T)f(x) = \|\mathcal{U}(T)f(x)\|_E = \left\| \left\{ T_{t_{i+1}}f(x) - T_{t_i}f(x) \right\}_{x \in J_i, i \in \mathbb{N}} \right\|_E = \left\| \left\{ \int_{[t_{i+1} < |x-y| < t_i]} K(x,y)f(y) \, dy \right\}_{x \in J_i, i \in \mathbb{N}} \right\|_E.$$ 

On the other hand, let $\Theta = \{\beta : \beta = \{\epsilon_i\}, \epsilon_i \in \mathbb{R}, \epsilon_i \downarrow 0\}$. We denote by $F_\rho$ the mixed norm space of two variable functions $g(i, \beta)$ such that

$$\|g\|_{F_\rho} \equiv \sup_\beta \left( \sum_i |g(i, \beta)|^\rho \right)^{1/\rho}.$$ 

We also consider the $F_\rho$-valued operator $\mathcal{V}(T): f \rightarrow \mathcal{V}(T)f$ given by

$$\mathcal{V}(T)f(x) = \left\{ T_{t_{i+1}}f(x) - T_{t_i}f(x) \right\}_{\beta = \{\epsilon_i\} \in \Theta}.$$ 

Then

$$\mathcal{V}_\rho(T)f(x) = \|\mathcal{V}(T)f(x)\|_{F_\rho}.$$ 

Next, let $B$ be a Banach space and $\varphi$ be a $B$-valued function, we define the sharp maximal operator as follows:

$$\varphi^\sharp(x) = \sup_{x \in I} \frac{1}{|I|} \int_I \|\varphi(y) - \frac{1}{|I|} \int_I \varphi(z) \, dz \|_B \, dy \approx \sup_{x \in I} \frac{1}{|I|} \int_I \|\varphi(y) - c\|_B \, dy.$$ 

Then

$$M^\sharp(O'(T)f) \leq 2(\mathcal{U}(T)f)^\sharp(x)$$ 

and

$$M^\sharp(\mathcal{V}_\rho(T)f) \leq 2(\mathcal{V}(T)f)^\sharp(x).$$

Finally, let us recall some results about oscillation and variation operators.
Lemma 2.5 ([5]) Suppose that \(K(x,y)\) satisfies (1.1)-(1.3), \(\rho > 2\). Let \(T = \{T_\varepsilon\}_{\varepsilon > 0}\) be given by (1.4). If \(O(T)\) and \(V_\rho(T)\) are bounded on \(L^{p_0}(\mathbb{R})\) for some \(1 < p_0 < \infty\), then, for any \(1 < p < \infty\), \(\omega \in A_p(\mathbb{R})\),

\[
\|O'(Tf)\|_{L^p(\omega)} \leq \|O(Tf)\|_{L^p(\omega)} \lesssim \|f\|_{L^p(\omega)}
\]

and

\[
\|V_\rho(Tf)\|_{L^p(\omega)} \lesssim \|f\|_{L^p(\omega)}.
\]

3 The proof of main results

Note that if \(\omega \in A_p(\mathbb{R})\), then \(\omega^\# \in A_\infty(\mathbb{R})\). By Lemma 2.2 and Lemma 2.3, we only need to prove

\[
M^r(\|O'(T^b)f\|)(x) \lesssim \|b(m)\|_{L^p} \left( M_{p,r}(f)(x) + M_p(f)(x) \right)
\]

and

\[
M^r(\|V_\rho(T^b)f\|)(x) \lesssim \|b(m)\|_{L^p} \left( M_{p,r}(f)(x) + M_p(f)(x) \right)
\]

hold for any \(1 < r < \infty\).

We will prove only inequality (3.1), since (3.2) can be obtained by a similar argument.

Fix \(f\) and \(x_0\) with an interval \(I = (x_0 - l, x_0 + l)\). Write \(f = f_1 + f_2 = f \chi_{[l,\infty)} + f \chi_{[-\infty,-l]}\), and let

\[
C_l = \left\{ \int_{[t_1 < |x_0 - y| < s]} \frac{\rho_{m+1}(b;x_0,y)}{|x_0 - y|^m} K(x_0,y) \right\} \chi_{l,j} = \mathcal{U}(\rho^b) f_2(x_0).
\]

Then

\[
\mathcal{U}(\rho^b)f(x) = \int_{[t_1 < |x-y| < s]} \frac{\rho_{m+1}(b;x,y)}{|x-y|^m} K(x,y) f(y) \chi_{l,j} dx.
\]

Therefore

\[
\frac{1}{|I|} \int_I \left\| \mathcal{U}(\rho^b)f(x) - C_l \right\|_E dx \leq \frac{1}{|I|} \int_I \left\| \mathcal{U}(\rho^b) \left( \frac{\rho_{m+1}(b;x_0,y)}{|x_0 - y|^m} f_1 \right) \right\|_E dx + \frac{1}{|I|} \int_I \left\| \mathcal{U}(\rho^b)f_2(x) - \mathcal{U}(\rho^b)f_2(x_0) \right\|_E dx
\]

\[
= M_1 + M_2.
\]

For \(x \in I\), \(k = 0, -1, -2, \ldots\), let \(E_k = \{ y : 2^{-k} \cdot 6l \leq |y-x| < 2^{k+1} \cdot 6l \}\), let \(I_k = \{ y : |y-x| < 2^k \cdot 6l \}\), and let \(b_k(z) = b(z) - \frac{1}{m(b(m))} \rho_{m+1}(b;x_0,y)\). By [10] we have \(R_{m+1}(b;x,y) = R_{m+1}(b_k;x,y)\) for any \(y \in E_k\).
By Lemma 2.5, we know $O'(T)$ is bounded on $L^u(\mathbb{R})$ for $u > 1$. Then, using Hölder's inequality, we deduce

\[ M_1 \lesssim \left( \frac{1}{|I|} \int |O(J)| \left( \frac{R_{m+1}(b_\beta x, \cdot)}{|x-y|^m} f \right)^u \, dx \right)^{1/u} \]

\[ \lesssim \left( \frac{1}{|I|} \int_{|y-x|>0} |R_{m+1}(b_\beta y, y)| \left( \frac{R_{m+1}(b_\beta y)}{|y-y'|^m} f(y) \right)^u \, dy \right)^{1/u} \]

\[ = \left( \frac{1}{|I|} \sum_{k=\infty}^0 \int_{E_k} \left| \frac{R_{m+1}(b_\beta y, y)}{|y-y'|^m} f(y) \right|^u \, dy \right)^{1/u} \]

\[ \lesssim \left( \frac{1}{|I|} \sum_{k=\infty}^0 \int_{E_k} \left| \left( \frac{R_{m+1}(b_\beta y, y)}{|y-y'|^m} - \frac{1}{m!} \frac{(y-\cdot)^m b_k(m)(y)}{|y-y'|^m} f(y) \right) \right|^u \, dy \right)^{1/u} \]

\[ + \left( \frac{1}{|I|} \sum_{k=\infty}^0 \int_{E_k} \frac{1}{m!} \frac{(y-\cdot)^m b_k(m)(y)}{|y-y'|^m} f(y) \, dy \right)^{1/u} \]

\[ = M_{11} + M_{12}. \]

By Lemma 2.4 and Lemma 2.1,

\[ |R_m(b_\beta x, y)| \lesssim |x-y|^m \left( \frac{1}{|I|^2} \int_{I_k} |b_k^{(m)}(z)|^s \, dz \right)^{1/s} \]

\[ \lesssim |x-y|^m \left( \frac{1}{2^k \cdot 30t} \int_{|y-x|>2^k \cdot 30t} |b^{(m)}(y) - (b^{(m)})(y)|^s \, dz \right)^{1/s} \]

\[ \lesssim |x-y|^m (2^k t)^\beta \| b^{(m)} \|_{\lambda_\beta}. \]

Then

\[ M_{11} \lesssim \| b^{(m)} \|_{\lambda_\beta} p^{\beta} \left( \frac{1}{|I|} \sum_{k=\infty}^0 2^k \frac{1}{|I|} \int_{E_k} |f(y)|^u \, dy \right)^{1/u} \]

\[ \lesssim \| b^{(m)} \|_{\lambda_\beta} p^{\beta} \left( \frac{1}{|I|} \sum_{k=\infty}^0 \int_{E_k} |f(y)|^u \, dy \right)^{1/u} \]

\[ \lesssim \| b^{(m)} \|_{\lambda_\beta} p^{\beta} \left( \frac{1}{|I|} \int_{I_k} |f(y)|^u \, dy \right)^{1/u} \]

\[ \lesssim \| b^{(m)} \|_{\lambda_\beta} p^{\beta} \left( \frac{1}{|I|} \int_{I_k} |f(y)|^r \, dy \right)^{1/r} \]

\[ \lesssim \| b^{(m)} \|_{\lambda_\beta} M_{\beta,r}(f)(x_0). \]
We now estimate \( M_{21} \). For \( x \in I \), we have

\[
\| U(T^b)f_2(x) - U(T^b)f_2(x_0) \|_E
\]

\[
= \left\| \int_{\{y: |y-y_0| < 2^k \cdot 4l \}} \left( R_{m+1}(b;x,y) \frac{K(x,y)}{|x-y|^m} - R_{m+1}(b_0;x_0,y) \frac{K(x_0,y)}{|x_0-y|^m} \right) dy \right\|_E
\]

\[
\leq \left\| \int_{\{y: |y-y_0| < 2^k \cdot 4l \}} \left( R_{m+1}(b;x,y) - R_{m+1}(b_0;x_0,y) \frac{K(x_0,y)}{|x_0-y|^m} \right) K(x,y) dy \right\|_E
\]

\[
= N_1 + N_2.
\]

For \( k = 0, 1, 2, \ldots \), let \( F_k = \{ y: 2^k \cdot 4l \leq |y-x_0| < 2^{k+1} \cdot 4l \} \), let \( F_k = \{ y: |y-x_0| < 2^k \cdot 4l \} \), let \( \tilde{F}_k = \{ y: |y-x_0| < 2^k \cdot 4l \} \), and let \( \tilde{b}_k(z) = b(z) - \frac{1}{m} R_{m+1}(b;x_0,y)z^m \). Note that

\[
R_{m+1}(b;x,y) = R_{m+1}(b_0;x_0,y) + \int_{F_k} \left( R_{m+1}(\tilde{b}_k;x,y) - R_{m+1}(\tilde{b}_k;x_0,y) \right) K(x,y) dy
\]

\[
= \frac{1}{|x-y|^m} \left( R_{m+1}(\tilde{b}_k;x,y) - R_{m+1}(\tilde{b}_k;x_0,y) \right) K(x,y)
\]

\[
+ \frac{1}{m} \int_{F_k} \left( R_{m+1}(\tilde{b}_k;x_0,y) \left( \frac{1}{|x-y|^m} - \frac{1}{|x_0-y|^m} \right) K(x,y)
\]

\[
- \frac{1}{m} \int_{F_k} \left( R_{m+1}(\tilde{b}_k;x_0,y) \left( \frac{(x-y)^m}{|x-y|^m} - \frac{(x_0-y)^m}{|x_0-y|^m} \right) K(x,y)
\]

\[
+ \frac{R_{m+1}(\tilde{b}_k;x_0,y)}{|x_0-y|^m} \left( K(x,y) - K(x_0,y) \right).
\]
By Minkowski’s inequalities and \[ \|\chi_{[t_{i+1};[x-y];i]}\|_{E} \leq 1, \] we obtain
\[
N_1 \leq \int_{\mathbb{R}} \left\| \chi_{[t_{i+1};[x-y];i]} \right\|_{E} d y \times \left| R_{m+1}(\tilde{b}_k; x, y) \right| K(x, y) - \frac{R_{m+1}(\tilde{b}_k; x, y)}{|x-y|^m} K(x, y) \right| f_2(y) dy \\
\leq \sum_{k=0}^{\infty} \int_{F_k} \frac{1}{|x-y|^m} \left| R_m(\tilde{b}_k; x, y) - R_m(\tilde{b}_k; x_0, y) \right| K(x, y) \right| f_2(y) dy \\
+ \sum_{k=0}^{\infty} \int_{F_k} \frac{1}{|x-y|^m} \left| \frac{1}{|x-y|^m} - \frac{1}{|x_0-y|^m} \right| K(x, y) \right| f_2(y) dy \\
+ \sum_{k=0}^{\infty} \int_{F_k} \frac{1}{|x-y|^m} \left| \frac{(x-y)^m}{|x-y|^m} - \frac{(x_0-y)^m}{|x_0-y|^m} \right| K(x, y) \right| f_2(y) dy \\
+ \sum_{k=0}^{\infty} \int_{F_k} \frac{1}{|x-y|^m} \left| K(x, y) - K(x_0, y) \right| f_2(y) dy \\
= N_{11} + N_{12} + N_{13} + N_{14}. 
\]

From the mean value theorem, there exists \( \eta \in I \) such that
\[
R_m(\tilde{b}_k; x, y) - R_m(\tilde{b}_k; x_0, y) = (x - x_0)R_{m-1}(\tilde{b}_k; \eta, y). 
\]

For \( \eta, x \in I, y \in F_k \), we have \( |y - x| \approx |y - x| \approx |y - \eta| \) and \( 5|y - \eta| \approx 5|y - x_0| \leq 2^{k+1} \cdot 20l. \)

By Lemma 2.4 and Lemma 2.1 we get
\[
\left| R_{m-1}(\tilde{b}_k; \eta, y) \right| \leq |\eta - y|^{m-1} \left( \frac{1}{|\eta|} \int_{\eta} |b^{(m)}(z)|^\beta \, dz \right)^{1/\beta} \leq |x - y|^{m-1} \left( \frac{1}{2^{k+1} \cdot 20l} \int_{|z - x_0| < 2^{k+1} \cdot 20l} |b^{(m)}(z) - (b^{(m)})_k| \, dz \right)^{1/\beta} \leq \| b^{(m)} \|_{\beta} (2^k l)^\beta |x - y|^{m-1}. 
\]

Then
\[
\left| R_m(\tilde{b}_k; x, y) - R_m(\tilde{b}_k; x_0, y) \right| \leq \| b^{(m)} \|_{\beta} (2^k l)^\beta |x - x_0| |x - y|^{m-1}. 
\]

Since \( |K(x, y)| \leq C|x_0 - y|^{-1} \),
\[
N_{11} \leq \| b^{(m)} \|_{\beta} \sum_{k=0}^{\infty} (2^k l)^\beta \int_{2^k 4l \leq |x-y| < 2^{k+1} 4l} \frac{l}{(2^k \cdot 4l)^2} |f(y)| \, dy \\
\leq \| b^{(m)} \|_{\beta} \sum_{k=0}^{\infty} \frac{1}{2^k - 2^k l} \int_{|x-y| < 2^{k+1} 4l} |f(y)| \, dy \\
\leq \| b^{(m)} \|_{\beta} M_\beta(f)(x_0). 
\]
For $N_{12}$, since $x \in I, y \in F_k$, 

$$|R_m(\tilde{b}_k; x, y)| \lesssim |x - y|^m \left( \frac{1}{|F_k|} \int_{F_k} |\tilde{b}_k^{(m)}(z)|^2 \, dz \right)^{1/s} \lesssim \|b^{(m)}\|_{\dot{\mathcal{H}}_{\beta}} (2^k l)^{\beta} |x - y|^m$$

and

$$\left| \frac{1}{|x - y|^m} - \frac{1}{|x_0 - y|^m} \right| \lesssim \frac{|x - x_0|}{|x - y|^{m+1}}.$$ 

Thus

$$N_{12} \lesssim \|b^{(m)}\|_{\dot{\mathcal{H}}_{\beta}} \sum_{k=0}^{\infty} (2^k l)^{\beta} \int_{2^k \cdot 4l \leq |x_0 - y| < 2^{k+1} \cdot 4l} \frac{l}{(2^k \cdot 4l)^2} |f(y)| \, dy \lesssim \|b^{(m)}\|_{\dot{\mathcal{H}}_{\beta}} M_{\beta}(f)(x_0).$$

As for $N_{13}$, due to

$$\frac{|x - y|^m}{|x - y|^m} - \frac{(x_0 - y)^m}{|x_0 - y|^m} \lesssim \frac{|x - x_0|}{|x - y|},$$

and noting $\tilde{b}_k^{(m)}(y) = b^{(m)}(y) - (b^{(m)})_{\tilde{I}_k}$, we have

$$N_{13} \lesssim \sum_{k=0}^{\infty} \int_{F_k} |b^{(m)}(y) - (b^{(m)})_{\tilde{I}_k}| \frac{|x - x_0|}{|x_0 - y|^2} |f(y)| \, dy$$

$$\lesssim \sum_{k=0}^{\infty} \frac{1}{2^k} \left( \frac{1}{2^k \cdot 4l} \int_{|x_0 - y| < 2^k \cdot 4l} |b^{(m)}(y) - (b^{(m)})_{\tilde{I}_k}|^2 |f(y)| \, dy \right)^{1/2}$$

$$\lesssim \sum_{k=0}^{\infty} \frac{1}{2^k} \left( \frac{1}{2^k \cdot 4l} \int_{|x_0 - y| < 2^k \cdot 4l} |f(y)| \, dy \right)^{1/2}$$

$$\times \left( \frac{1}{2^k \cdot 4l} \int_{|x_0 - y| < 2^k \cdot 4l} |b^{(m)}(y) - (b^{(m)})_{\tilde{I}_k}|^{r'} \, dy \right)^{1/r'}$$

$$\lesssim \|b^{(m)}\|_{\dot{\mathcal{H}}_{\beta}} M_{r, \beta}(f)(x_0) \sum_{k=0}^{\infty} \frac{1}{2^k} \lesssim \|b^{(m)}\|_{\dot{\mathcal{H}}_{\beta}} M_{\beta, r}(f)(x_0).$$

Notice

$$|\frac{\mathcal{R}_m(\tilde{b}_k; x_0, y)}{2^k l^{\beta}}| \lesssim \left| \frac{\mathcal{R}_m(\tilde{b}_k; x_0, y)}{2^k l^{\beta}} \right| + \frac{1}{m!} \left| \tilde{b}_k^{(m)}(y)(x_0 - y)^m \right|$$

$$\lesssim \|b^{(m)}\|_{\dot{\mathcal{H}}_{\beta}} (2^k l)^{\beta} |x_0 - y|^m + |b^{(m)}(y) - (b^{(m)})_{\tilde{I}_k}| |x_0 - y|^m$$

and by (1.2),

$$|K(x, y) - K(x_0, y)| \lesssim \frac{|x - x_0|^\beta}{|x - y|^{1+\beta}}.$$

Similar to the estimates for $N_{11}$, we have

$$\sum_{k=0}^{\infty} \int_{F_k} \left| \frac{\mathcal{R}_m(\tilde{b}_k; x_0, y)}{2^k l^{\beta}} \right| \frac{|x - x_0|^\beta}{|x_0 - y|^{1+\beta}} |f(y)| \, dy \lesssim \|b^{(m)}\|_{\dot{\mathcal{H}}_{\beta}} M_{\beta}(f)(x_0).$$
Similar to the estimates for $N_{13}$, we have

$$\sum_{k=0}^{\infty} \int_{F_k} |\tilde{b}^{(m)}_k(y)(x_0 - y)^m| \frac{|x - x_0|^t}{|x_0 - y|^t} |f(y)| \, dy \lesssim \|b^{(m)}_n\|_{\lambda, \beta} \mathcal{M}_{\beta, r}(f)(x_0).$$

Then

$$N_{14} \lesssim \|b^{(m)}_n\|_{\lambda, \beta} \left( \mathcal{M}_{\beta}(f)(x_0) + \mathcal{M}_{\beta, r}(f)(x_0) \right).$$

Finally, let us estimate $N_2$. Notice that the integral

$$\int_R \left( x_{[t_0, t_1]}(y) - x_{[1, t_1]}(y) \right) \frac{R_{m+1}(b; x_0, y)}{|x_0 - y|^m} K(x_0, y)f_2(y) \, dy$$

will be non-zero in the following cases:

(i) $t_{i+1} < |y| < s$ and $|x_0 - y| \leq t_{i+1}$;

(ii) $t_{i+1} < |y| < s$ and $|x_0 - y| \geq s$;

(iii) $t_{i+1} < |x_0 - y| < s$ and $|y| \leq t_{i+1}$;

(iv) $t_{i+1} < |x_0 - y| < s$ and $|y| \geq s$.

In case (i) we have $t_{i+1} < |x - y| \leq |x_0 - x| + |x_0 - y| < l + t_{i+1}$ as $|x_0 - x| < l$. Similarly, in case (iii) we have $t_{i+1} < |x_0 - y| < l + t_{i+1}$ as $|x_0 - y| < l$. In case (ii) we have $s < |x_0 - y| < l + s$ and in case (iv) we have $s < |x - y| < l + s$. By (1.1) and taking $1 < \varepsilon < r$, we have

$$\int_R \left( x_{[t_0, t_1]}(y) - x_{[1, t_1]}(y) \right) \frac{R_{m+1}(b; x_0, y)}{|x_0 - y|^m} K(x_0, y)f_2(y) \, dy$$

$$\lesssim \int_R x_{[t_0, t_1]}(y) x_{[1, t_1]}(y) \frac{R_{m+1}(b; x_0, y)}{|x_0 - y|^m} \frac{|f_2(y)|}{|x_0 - y|} \, dy$$

$$+ \int_R x_{[1, t_1]}(y) x_{[1, t_1]}(y) \frac{R_{m+1}(b; x_0, y)}{|x_0 - y|^m} \frac{|f_2(y)|}{|x_0 - y|} \, dy$$

$$+ \int_R x_{[1, t_1]}(y) x_{[1, t_1]}(y) \frac{R_{m+1}(b; x_0, y)}{|x_0 - y|^m} \frac{|f_2(y)|}{|x_0 - y|} \, dy$$

$$\lesssim \lambda^{t/\varepsilon} \left( \int_R x_{[t_0, t_1]}(y) \frac{R_{m+1}(b; x_0, y)}{|x_0 - y|^m} \frac{|f_2(y)|}{|x_0 - y|} \, dy \right)^{1/t}$$

$$+ \lambda^{t/\varepsilon} \left( \int_R x_{[t_0, t_1]}(y) \frac{R_{m+1}(b; x_0, y)}{|x_0 - y|^m} \frac{|f_2(y)|}{|x_0 - y|} \, dy \right)^{1/t}.$$

Then

$$N_2 \lesssim \lambda^{t/\varepsilon} \left\{ \left( \int_R x_{[t_0, t_1]}(y) \frac{R_{m+1}(b; x_0, y)}{|x_0 - y|^m} \frac{|f_2(y)|}{|x_0 - y|} \, dy \right)^{1/t} \right\}_{s \in E, i \in \mathbb{N}}$$

$$+ \lambda^{t/\varepsilon} \left\{ \left( \int_R x_{[t_0, t_1]}(y) \frac{R_{m+1}(b; x_0, y)}{|x_0 - y|^m} \frac{|f_2(y)|}{|x_0 - y|} \, dy \right)^{1/t} \right\}_{s \in E, i \in \mathbb{N}}$$

$$= N_{21} + N_{22}.$$
Notice

\[ |R_{m+1}(\tilde{g}; x_0, y)| \lesssim \|b^{(m)}\|_{\lambda_\beta}(2^k t)^{\beta} |x_0 - y|^m + |b^{(m)}(y) - (b^{(m)})_k| |x_0 - y|^m. \]

Choosing \( 1 < r < p \) with \( t = \sqrt{r} \), we have

\[ N_{21} \lesssim \| \frac{R_{m+1}(b; x_0, y)}{|x_0 - y|^m} \|_{t/r}^{1/t} \left( \int_{|x_1| < |x - y|} \left( \frac{|f(y)|^t}{|x_0 - y|^t} \right) dy \right)^{2/t} \]

\[ \lesssim \| \frac{R_{m+1}(b; x_0, y)}{|x_0 - y|^m} \|_{t/r}^{1/t} \left( \int_{|x_1| < |x - y|} \left( \frac{|f_2(y)|^t}{|x_0 - y|^t} \right) dy \right)^{1/t} \]

\[ \lesssim \| \frac{R_{m+1}(b; x_0, y)}{|x_0 - y|^m} \|_{t/r}^{1/t} \left( \int_{|x_1| < |x - y|} \left( \frac{|f(y)|^t}{|x_0 - y|^t} \right) dy \right)^{1/t} \]

\[ \lesssim \| b^{(m)} \|_{\lambda_\beta} \| \frac{R_{m+1}(b; x_0, y)}{|x_0 - y|^m} \|_{t/r}^{1/t} \left( \int_{|x_1| < |x - y|} \left( \frac{|f(y)|^t}{|x_0 - y|^t} \right) dy \right)^{1/t} \]

But

\[ \| \frac{R_{m+1}(b; x_0, y)}{|x_0 - y|^m} \|_{t/r}^{1/t} \left( \int_{|x_1| < |x - y|} \left( \frac{|f(y)|^t}{|x_0 - y|^t} \right) dy \right)^{1/t} \]

\[ \lesssim \| \frac{R_{m+1}(b; x_0, y)}{|x_0 - y|^m} \|_{t/r}^{1/t} \left( \int_{|x_1| < |x - y| < 2^{k+1} 4l} \left( \frac{1}{2^k} \frac{|f(y)|^t}{|x_0 - y|^t} \right) dy \right)^{1/t} \]

\[ \lesssim \left( \sum_{k=1}^{\infty} \frac{1}{2^{k(t-1)}} \left( \frac{2^k}{2^k \cdot 5l} \int_{|x_1| < |x - y| < 2^k 5l} \left( \frac{|f(y)|^t}{|x_0 - y|^t} \right) dy \right)^{1/t} \]

\[ \lesssim \left( \sum_{k=1}^{\infty} \frac{1}{2^{k(t-1)}} \left( \frac{2^k}{2^k \cdot 5l} \int_{|x_1| < |x - y| < 2^k 5l} \left( \frac{|f(y)|^2}{|x_0 - y|^t} \right) dy \right)^{1/t} \]

\[ \lesssim \left( \sum_{k=1}^{\infty} \frac{1}{2^{k(t-1)}} \right)^{1/t} M_{\beta, 2\gamma}(x_0) \lesssim M_{\beta, 2\gamma}(f)(x_0) \]

and

\[ \| \frac{R_{m+1}(b; x_0, y)}{|x_0 - y|^m} \|_{t/r}^{1/t} \left( \int_{|x_1| < |x - y|} \left( \frac{|f(y)|^t}{|x_0 - y|^t} \right) dy \right)^{1/t} \]

\[ \lesssim \left( \sum_{k=1}^{\infty} \frac{1}{2^{k(t-1)}} \frac{1}{2^k \cdot 4l} \int_{|x_1| < |x - y| < 2^k 4l} \left( \frac{|f(y)|^t}{|x_0 - y|^t} \right) dy \right)^{1/t} \]
\begin{align*}
&\lesssim \left( \sum_{k=0}^{\infty} \frac{1}{2^k(t-1)} \left( \frac{1}{2^k \cdot 4^l} \int_{|x-y|<2^k \cdot 4^l} |f(y)|^2 \, dy \right)^{1/t} \right)
\times \left( \frac{1}{2^k \cdot 4^l} \int_{|x-y|<2^k \cdot 4^l} \left| b^{(m)}(y) - \left( b^{(m)} \right)_{2^k \cdot 4^l} \right|^{1/t} \right)^{1/t} \\
&\lesssim \| b^{(m)} \|_{\ell_p} \left( \sum_{k=0}^{\infty} \frac{1}{2^k(t-1)} \left( \frac{2^k \cdot 4^l}{2^k \cdot 4^l} \int_{|x-y|<2^k \cdot 4^l} |f(y)|^2 \, dy \right)^{1/t} \right)^{1/t} \\
&\lesssim \| b^{(m)} \|_{\ell_p} M_{\beta, r}(f)(x_0) \left( \sum_{k=0}^{\infty} \frac{1}{2^k(t-1)} \right)^{1/t} \\
&\lesssim \| b^{(m)} \|_{\ell_p} M_{\beta, r}(f)(x_0).
\end{align*}

Therefore

\[ N_{21} \lesssim \| b^{(m)} \|_{\ell_p} M_{\beta, r}(f)(x_0). \]

Similarly,

\[ N_{22} \lesssim \| b^{(m)} \|_{\ell_p} M_{\beta, r}(f)(x_0). \]

This completes the proof of (3.1). Hence, Theorem 1.1 is proved.

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
The authors completed the paper together. They also read and approved the final manuscript.

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Publisher’s Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 12 October 2017 Accepted: 9 November 2017 Published online: 25 November 2017

References
1. Campbell, JT, Jones, RL, Reinhold, K, Wendl, M: Oscillation and variation for the Hilbert transform. Duke Math. J. 105, 59-83 (2000)
2. Akcoglu, M, Jones, RL, Schwartz, P: Variation in probability, ergodic theory and analysis. Ill. J. Math. 42, 154-177 (1998)
3. Crescimbeni, R, Martin-Reyes, FL, Torrea, AL, Torrea, JL: The $\varrho$-variation of the Hermitian Riesz transform. Acta Math. Sin. Engl. Ser. 26, 1827-1838 (2010)
4. Gillespie, TA, Torrea, JL: Dimension free estimates for the oscillation of Riesz transforms. Isr. J. Math. 141, 125-144 (2004)
5. Liu, F, Wu, HX: A criterion on oscillation and variation for the commutators of singular integral operators. Forum Math. 27, 77-97 (2015)
6. Zhang, J, Wu, HX: Oscillation and variation inequalities for singular integrals and commutators on weighted Morrey spaces. Front. Math. China 11, 423-447 (2016)
7. Zhang, J, Wu, HX: Oscillation and variation inequalities for the commutators of singular integrals with Lipschitz functions. J. Inequal. Appl. 2015, 214, 21 pp. (2015)
8. Zhang, J, Wu, HX: Weighted oscillation and variation inequalities for singular integrals and commutators satisfying Hormander type condition. Acta Math. Sin. 33, 1397-1420 (2017)
9. Cohen, J: A sharp estimate for a multilinear singular integral on $\mathbb{R}^n$. Indiana Univ. Math. J. 30, 693-702 (1981)
10. Cohen, J, Gosselin, J: A $BMO$ estimate for multilinear singular integral operators. Ill. J. Math. 30, 445-465 (1986)
11. Ding, Y, Lu, SZ: Weighted boundedness for a class rough multilinear operators. Acta Math. Sin. 17, 517-526 (2001)
12. Lu, SZ, Wu, HX, Zhang, P: Multilinear singular integral with rough kernel. Acta Math. Sin. 19, 51-62 (2003)
13. Chen, WG: A Besov estimate for multilinear singular integrals. Acta Math. Sin. 16, 613-626 (2000)
14. Muckenhoupt, B: Weighted norm inequalities for the Hardy maximal function. Trans. Am. Math. Soc. 165, 207-226 (1972)
15. Muckenhoupt, B, Wheeden, RL: Weighted norm inequalities for singular and fractional integrals. Trans. Am. Math. Soc. 161, 249-258 (1971)