Shadowing in a linear skew product over Bernoulli shift

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Abstract

We investigate the probability of the event that a finite random pseudotrajectory can be effectively shadowed by an exact trajectory. The main result of the work states that this probability tends to one as the length of a pseudotrajectory tends to infinity for an arbitrary continuous linear skew product over Bernoulli shift. The Cramer’s large deviation theorem is used in the proof.

Introduction

The theory of shadowing of pseudotrajectories is a well-developed area of the fundamental theory of dynamical systems. There are a lot of results that show the connection between the shadowing property and structural stability [1], [2]. At the same time, the results of various numerical experiments show that finite pseudotrajectories can be shadowed effectively even for systems that are not structurally stable [3], [4]. The purpose of this research is to study the stochastic setting of the problem mentioned above. We introduce a natural way of generating a random pseudotrajectory and estimate the probability of a random pseudotrajectory of a finite length to be shadowed by an exact one.

It was shown in the papers [2], [5], [6] that the parameters of the shadowing property for finite pseudotrajectories can be obtained using the asymptotics of the growth rate of the solution of the corresponding inhomogeneous linear system. The problem that we study in this paper can be considered a simplified case of a linear inhomogeneous system, obtained from a nonuniformly hyperbolic system with one-dimensional center direction and nonzero Lyapunov exponents, which fact lets us hope that similar methods to those shown in this work can be used in a more general setting. The result we obtain can be considered a generalisation of the one shown in [7], where a special case of a linear skew product over the Bernoulli shift is studied.

The proof we present uses the Cramér’s large deviations theorem, which is known to be true for dynamical systems with exponential mixing [8]. This fact also tells us that our approach might be used for establishing the shadowing property in a more general setting.

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1 Statement of the theorem

Consider the metric space $\Sigma = \{0, 1\}$ with $\text{dist}_\Sigma(0, 1) = 1$ and probability measure $\mu(\{0\}) = \frac{1}{2}$. Let us denote the space $\Sigma^\mathbb{Z}$ with standard topology and probability measure $\nu$ that arises from the product structure by $X$. Note that the topology on $X$ is generated by the metric $\text{dist}_X(\{w(j)\}_{j \in \mathbb{Z}}, \{\tilde{w}(j)\}_{j \in \mathbb{Z}}) = \frac{1}{2^k}$, where $k = \min\{|j| : w(j) \neq \tilde{w}(j)\}$.

On the space $X$ let us define the shift map $T: X \to X$;

$$T(\{w(j)\}_{j \in \mathbb{Z}}) = w(j+1).$$

Let us consider a continuous function $\lambda: X \to \mathbb{R}^+$ such that $\mathbb{E}_\nu(\log(\lambda)) \neq 0$.

We also define the space $Q$ by the equality $Q = X \times \mathbb{R}$. We consider the measure $m = \nu \times \text{Leb}$ and maximum metric:

$$\text{dist}((w, x), (\tilde{w}, \tilde{x})) = \max(\text{dist}_X(w, \tilde{w}), |x - \tilde{x}|).$$

Now we consider the map $f: Q \to Q$ that is given by the following formula:

$$f(w, x) = (T(w), \lambda(w)x).$$

This map is a simple example of a nonuniformly hyperbolic map, and the condition (1) is analogous to having nonzero Lyapunov exponents.

We call the sequence of points $\{y_k\}_{k=a}^b$ a $d$-pseudotrajectory for the map $f: Q \to Q$ if

$$\text{dist}(y_{k+1}, f(y_k)) < d, \ k \in \{a, \ldots, b-1\}.$$ 

We say that trajectory $\{x_k\}_{k=a}^b$ $\varepsilon$-shadows the pseudotrajectory $\{y_k\}$ if

$$\text{dist}(y_k, x_k) < \varepsilon, \ k \in \{a, \ldots, b\}.$$ 

Let us denote by $B(q, r)$ the open ball of radius $r$ in the space $Q$ centered at the point $q$. For every $q \in Q$, every radius $d > 0$, and every natural $N \in \mathbb{N}$ we denote by $\Omega_{q, d, N}$ the set of all $d$-pseudotrajectories of the length $N$ that start at the point $q$. In our model the next element $y_{k+1}$ is chosen in the ball $B(f(q_k), d)$ at random with respect to the measure $m$. Now we have a Markov chain on $Q$, which gives the set $\Omega_{q, d, N}$ a certain probabilistic measure $\mathbb{P}$.

For a positive number $\varepsilon > 0$ let us denote by $p(q, d, N, \varepsilon)$ the probability that a pseudotrajectory from $\Omega_{q, d, N}$ chosen at random with respect to $\mathbb{P}$ can be $\varepsilon$-shadowed by an exact trajectory.

The proof of the following lemma can be found in [7] (Lemma 1).

**Lemma 1.** Consider two points $q = (w, x)$ and $\tilde{q} = (w, 0)$. For arbitrary positive numbers $d, \varepsilon > 0, N \in \mathbb{N}$ the following equality holds:

$$p(q, d, N, \varepsilon) = p(\tilde{q}, d, N, \varepsilon).$$
For positive numbers \( d, \varepsilon > 0 \), \( N \in \mathbb{N} \) let us define
\[
p(d, N, \varepsilon) = \int_{w \in X} p((w, 0), d, N, \varepsilon) d\nu.
\]

Now we can state the main theorem of this paper:

**Theorem 1.** If a continuous function \( \lambda : X \to \mathbb{R}_+ \) satisfies the property (1), then there exist positive \( \varepsilon_0 > 0 \) and \( \gamma > 0 \) such that for every \( 0 < \varepsilon < \varepsilon_0 \) the following equality holds:
\[
limit_{N \to \infty} p\left(\frac{\varepsilon}{N^\gamma}, N, \varepsilon\right) = 1.
\] (2)

**Remark 1.** This result generalises the Theorem 2 in [7], where the function \( \lambda \) could only depend on the coordinate \( w(0) \).

## 2 Proof of theorem [1]

We will need some auxiliary lemmas to prove the theorem. First of all, let us consider the sequences \( (\lambda_j)_{j=0}^N \) and \( (r_j)_{j=1}^N \) such that \( |r_j| < 1 \). We introduce following notations:
\[
A_j = \sum_{p=0}^{j-1} \log(\lambda_p),
\]
\[
z_0 = 0, \quad z_{j+1} = z_j + \frac{r_j}{e^{A_j+1}}.
\]
and define
\[
B(p, q) = \frac{e^{A_p+A_q}}{e^{A_p}+e^{A_q}} |z_p - z_q|;
\]
\[
F((A_j)_{j=0}^{N-1}, (r_j)_{j=1}^N) = \max_{0 \leq p < q \leq N} B(p, q).
\]

The following lemma is a reformulation of lemma 2 in [7].

**Lemma 2.** For every sequence \( (x_j)_{j=0}^N \), given by
\[
x_0 = 0, \quad x_{j+1} = \lambda_j x_j + r_{j+1},
\]
There exist an \( y_0 \) such that
\[
y_{j+1} = \lambda_j y_j,
\]
implies
\[
|x_j - y_j| < F((A_j)_{j=0}^{N-1}, (r_j)_{j=1}^N), \quad \forall j \in \{0, \ldots, N\}.
\]

As far as the shadowing properties for \( f \) and \( f^{-1} \) are equivalent we assume that
\[
a = \mathbb{E}_\nu(\log(\lambda)) < 0.
\]
Lemma 3. Consider the sequence \((w_j)_{j=0}^{N-1}\) that is a \(d\)-pseudotrajectory of \(T\) in the space \(X\) taken at random with respect to our Markov chain. Then we can define \(w = w((w_j)_{j=0}^N) \in X\) such that

\[
\text{dist}(w_j, T^j(w)) < 2d,
\]

and \(w\) has the distribution equal to \(\nu\).

Proof. Let us consider natural \(n\) such that \(\frac{1}{2^{n+1}} < d \leq \frac{1}{2^n}\). Define the map \(w((w_j)_{j=0}^N)\) as follows:

\[
w(j) = \begin{cases} w_0^{(j)}, & \text{for } j < n \text{ and } j \geq n + N, \\ w_k^{(n)}, & \text{for } j = n + k, \text{ where } 0 < k < N. \end{cases}
\]

It is clear that \(\text{dist}(w_j, T^j(w)) \leq \frac{1}{2^n} < 2d\). Now we need to show that the distribution of \(w\) equals to \(\nu\), which fact is clear because \(P(w_j = 0 | w_i = x, i < j) = P(w_j = 1 | w_i = x, i < j) = \frac{1}{2}\).

Note that since \(\lambda\) is continuous there exist a natural number \(t \in \mathbb{N}\) such that \(w_j = \tilde{w}_j\) with \(j \in \{-t, \ldots, t\}\) implies

\[
|\log(\lambda(w)) - \log(\lambda(\tilde{w}))| < -\frac{a}{2}.
\]

Consider \(\Lambda - \) the set of all continuous functions \(\xi : X \to \mathbb{R}_+\) such that \(\xi(w)\) depends only on \(w^{(j)}, j \in \{-t, \ldots, t\}, \xi(w) \geq \min_X \{\lambda\} > 0\) and \(|\log(\xi(w)) - \log(\lambda(w))| \leq -\frac{a}{2}\). It is easy to see that \(\Lambda\) is a nonempty compact set in \(\mathbb{R}^{2t+1}\). Note that \(\mathbb{E}\log(\xi)\) is a continuous function on \(\Lambda\) and \(\inf_{\xi \in \Lambda} \mathbb{E}\log(\xi) \leq a \leq \sup_{\xi \in \Lambda} \mathbb{E}\log(\xi)\). Now it follows that there exists a function \(\tilde{\lambda} \in \Lambda\) such that

\[
|\log(\lambda(w)) - \log(\tilde{\lambda}(w))| < -\frac{a}{2}, \quad \forall w \in X
\]

and

\[
\mathbb{E}\log(\tilde{\lambda}) = \mathbb{E}\log(\lambda) = a.
\]

**Lemma 4.** There exist constants \(k, C\) such that if we take \(w\) as a random point of \(X\) and \(A_j\) given by formulas

\[
A_j = \sum_{p=0}^{j-1} \log(\tilde{\lambda}(T^p(w))),
\]

then

\[
P \left( \left| \frac{A_j}{j} - a \right| \geq \varepsilon \right) < C e^{-k\varepsilon^2 j} \quad \forall \varepsilon > 0, j \in \mathbb{N}.
\]

Proof. Random variables \(\tilde{\lambda}(w)\) and \(\tilde{\lambda}(T^{2t+1}w)\) are independent since \(\tilde{\lambda}\) depends only on every coordinates with indeces \(j \in \{-t, \ldots, t\}\), and tuples \((w_j^{(j)})_{j=-t}^{t}\) and
Let us consider the partial sums that are given by the following formula:

\[ A_j^{(q)} = \sum_{p=0}^{\lfloor j/2t+1 \rfloor} \log(\tilde{\lambda}(T^p(w))) \text{, where } q \in \{0, \ldots, 2t\}. \]

Note that for all \( q \) the values of \( A_j^{(q)} \) are sums of i. i. d. random variables, hence a large deviation principle is applicable \[9, \text{Chapter 12, \S 2}\], so there are constants \( C_0, k_0 > 0 \) such that:

\[ P \left( \left| A_j^{(q)} - a \right| \geq \varepsilon \right) < C_0 e^{-k_0 \varepsilon^2 |\frac{j-q}{2t+1}|} \forall \varepsilon > 0, j \in \mathbb{N}. \]

Also observe that

\[ P \left( \left| A_j^{(q)} - a \right| \geq \varepsilon \right) \leq \sum_{q=0}^{2t} P \left( \left| A_j^{(q)} - a \right| \geq \varepsilon \right), \]

hence if we define constants \( C = (2t+1)C_0, k = \frac{k_0}{4t+2} \), the inequality (4) will hold.

Now we will prove analogous large deviation probability estimation along a random pseudotrajectory.

**Lemma 5.** There exist constants \( C, k > 0 \) such that if \( d < \frac{1}{2t+1} \), \( (w_j)_{j \in \mathbb{N}} \) is a \( d \)-pseudotrajectory for \( T \) chosen at random with respect to our Markov chain, and \( \tilde{A}_j \) are given by the following formula:

\[ \tilde{A}_j = \sum_{p=0}^{j-1} \log(\tilde{\lambda}(w_p)), \]

then

\[ P \left( \left| \tilde{A}_j - a \right| > \varepsilon \right) < C e^{-k \varepsilon^2 j} \forall \varepsilon > 0, j \in \mathbb{N}. \] (5)

**Proof.** Let us fix \( j \in \mathbb{N} \) and consider the point \( w = w((w_p)_{p=0}^j) \) that is defined in lemma (3). The inequality \( \text{dist}(T^p(w), w_p) < 2d < \frac{1}{2t} \) implies that \( \tilde{\lambda}(T^p(w)) = \tilde{\lambda}(w_p) \) and using lemma (4) we obtain needed result. \( \Box \)

From now on we fix constants \( k \) and \( C \) such that the inequalities (4) and (5) hold

\[ \gamma > -\frac{1}{2ka} + 1 \] (6)
Now we are ready to prove the theorem \[1\] Let \((w_j, x_j)_{j=0}^{N-1}\) be a random \(d\)-pseudotrajectory with \(d < \frac{1}{2^{r+1}}\). We introduce the following notations:

\[
A_j = \sum_{p=0}^{j-1} \log(\lambda(w_p)); \\
r_j = \frac{x_j - \lambda(w_{j-1})x_{j-1}}{d}.
\]

We will start with an estimation for the following probability:

\[
S_1 = \{ \text{there exists such } y_0 \text{ that } y_j = \lambda(w_{j-1})y_{j-1} \text{ implies } |x_j - y_j| \leq dN^\gamma \}
\]

It is clear that \(|r_j| < 1\) and according to lemma \[2\]

\[
\mathbb{P}(S_1) = \mathbb{P}(F((A_j)_{j=0}^{N-1}, (r_j)_{j=1}^{N}) \leq N^\gamma).
\]

It is easy to see that

\[
e^{A_q} |z_q - z_p| \leq \sum_{j=p}^{q} e^{-(A_j - A_q)},
\]

\[
e^{A_p} + e^{A_q} \leq 1.
\]

Now, using previously mentioned inequalities and the stationarity of our process we obtain:

\[
1 - \mathbb{P}(S_1) = \mathbb{P}(F((A_j)_{j=0}^{N-1}, (r_j)_{j=1}^{N}) > N^\gamma) \leq
\]

\[
\leq \mathbb{P} \left( \exists 0 \leq p < q \leq N : \sum_{j=p}^{q} e^{-(A_j - A_q)} > N^\gamma \right) \leq
\]

\[
\leq N \mathbb{P} \left( \exists n \leq N : \sum_{j=n}^{N} e^{-(A_j - A_N)} > N^\gamma \right) \leq
\]

\[
\leq N \mathbb{P} \left( \sum_{j=0}^{N} e^{-(A_j - A_N)} > N^\gamma \right) \leq
\]

\[
\leq N \sum_{j=0}^{N} \mathbb{P}(e^{A_N - A_j} > N^\gamma - 1) = N \sum_{j=0}^{N} \mathbb{P}(e^{A_j} > N^\gamma - 1) \leq
\]

\[
\leq N \sum_{j=0}^{N} \mathbb{P}(A_j > (\gamma - 1) \log(N)) \leq
\]

\[
\leq N \sum_{j=0}^{N} \mathbb{P} \left( \left| \frac{A_j}{j} - a \right| > \frac{(\gamma - 1) \log(N)}{j} - a \right).
\]

In what follows we will need another piece of notation:

\[
\tilde{A}_j = \sum_{p=0}^{j-1} \log(\tilde{\lambda}(w_p)).
\]
Applying (3) we can see that

\[ \left| \frac{A_j}{j} - \tilde{A}_j \right| < \frac{a}{2} \]

We can continue our sequence of estimations:

\[
1 - \mathbb{P}(S_1) \leq N \sum_{j=0}^{N} \mathbb{P}\left( \left| \frac{A_j}{j} - a \right| > \left( \frac{\gamma - 1}{j} \log(N) \right) - a \right) \leq \\
N \sum_{j=0}^{N} \mathbb{P}\left( \left| \frac{\tilde{A}_j}{j} - a \right| > \left( \frac{\gamma - 1}{j} \log(N) \right) - a \right),
\]

Finally, applying lemma 4 we obtain

\[
1 - \mathbb{P}(S_1) \leq N \sum_{j=0}^{N} C e^{-k \left( \frac{(\gamma - 1) \log(N)}{j} - \frac{a}{2} \right)^2} \leq \\
\leq CN \left[ \sum_{j=0}^{(\log(N))^2} e^{4k \left( \frac{(\gamma - 1) \log(N)}{j} - \frac{a}{2} \right)^2} + \sum_{j>(\log(N))^2} e^{-k \left( \frac{a}{2} \right)^2} \right] \leq \\
\leq CN \left[ \sum_{j=0}^{(\log(N))^2} N^{2k(\gamma - 1)a} + \sum_{j>(\log(N))^2} N^{-k \frac{a^2}{2} \log(N)} \right] \leq \\
\leq CN^{1+2k(\gamma - 1)a} \left( (\log(N))^2 + N^{1-2k(\gamma - 1)a-k \frac{a}{2} \log(N)} \right).
\]

Now it is easy to see that (6) implies

\[ \mathbb{P}(S_1) \to N \to \infty 1. \]

For the next step we assume that \( S_1 \) holds. We fix \( y_0 \) mentioned above and introduce \( z_0 = y_0 \). Moreover, we denote \( w = w \left( (w_j)_{j=0}^{N} \right) \) from lemma 3. It is important to remember that the distribution of \( w \) equals to \( \nu \). We use the following notation:

\[
y_{j+1} = \lambda(w_j)y_j, \quad (7) \\
z_{j+1} = \lambda(T^j(w)z_j), \quad (8) \\
A'_j = \sum_{p=0}^{j-1} \log(\lambda(T^p(w))). \quad (9)
\]
Now we estimate the following probability (we use \(d < 1\) in the first inequality):

\[
P(S_2) = P \left( \max_{j \in \{0, \ldots, N-1\}} |z_j - y_j| > dN^\gamma \right) \leq \leq P \left( \exists n \in \{0, \ldots, N-1\} : |e^{A_n} - e^{A_n'}| > N^\gamma \right) \leq \leq P \left( \exists n \in \{0, \ldots, N-1\} : A_n > \gamma \log(N) \right) + P \left( \exists n \in \{0, \ldots, N-1\} : A_n' > \gamma \log(N) \right) \leq \sum_{n=0}^{N} P \left( \frac{|A_n - a|}{n} > \frac{\gamma \log(N)}{n} - a \right) + \sum_{n=0}^{N} P \left( \frac{|A_n' - a|}{n} > \frac{\gamma \log(N)}{n} - a \right).
\]

Now we need to estimate the two summands from previous line. For the first one we can repeat the calculations that were used to estimate \(P(S_1)\) and obtain that the first summand vanishes as \(N\) goes to infinity. To estimate the second summand we argue in a similar way. First we introduce the variables

\[
A_n'' = \sum_{p=0}^{j-1} \log(\tilde{\lambda}(T^p(w))).
\]

Due to the lemma 4 the inequality (4) holds for them. We also note that due to the inequality (3) the following holds:

\[
\left| \frac{A_j'}{j} - \frac{A_j''}{j} \right| < -\frac{a}{2},
\]

and hence

\[
\sum_{n=0}^{N} P \left( \frac{|A_n' - a|}{n} > \frac{\gamma \log(N)}{n} - a \right) \leq \sum_{n=0}^{N} P \left( \frac{|A_n'' - a|}{n} > \frac{\gamma \log(N)}{n} - a \right),
\]

and required equality

\[
\lim_{N \to \infty} \sum_{n=0}^{N} P \left( \frac{|A_n'' - a|}{n} > \frac{\gamma \log(N)}{n} - \frac{a}{2} \right) = 0
\]

can be obtained in a way that is analogous to the one described above. We obtain

\[
P(S_2) \to_{N \to \infty} 0.
\]

To finish the proof we take \(d = \frac{\varepsilon}{2N^\gamma}\), and obtain that for a random pseudotrajectory \((w_j, x_j)_{j=0}^{N-1}\) the sequence \((y_j)_{j=0}^{N-1}\) given by the equation (7) will satisfy

\[
|y_j - x_j| < dN^\gamma
\]

\(8\)
with probability not less than \( P(S_1) \). We also know that with probability not less than \( 1 - P(S_2) \) there exist an exact trajectory \((T^j(w), z_j)_{j=0}^{N-1}\) given by (8) and satisfying the inequalities

\[
|z_j - y_j| < dN^\gamma.
\]

We have obtained that the trajectory \((T^j(w), z_j)_{j=0}^{N-1}\) \(\varepsilon\)-shadows the pseudotrajectory \((w_j, x_j)_{j=0}^{N-1}\) (since \(2dN^\gamma = \varepsilon\)) with probability not less than \( 1 - (1 - P(S_1) + P(S_2)) = P(S_1) - P(S_2) \to_{N\to\infty} 1 \). To be precise, now we have proved that

\[
\lim_{N\to\infty} p\left(\frac{\varepsilon}{2N^\gamma}, N, \varepsilon\right) = 1,
\]

which differs from the statement of the theorem (1) by the factor \(1/2\) in the first argument, but we can easily get rid of it by using in our proof \(\gamma'\) that lies between \(\gamma\) and \(1 - \frac{1}{2ka}\). Since the inequality

\[
\frac{\varepsilon}{N^\gamma} < \frac{\varepsilon}{2N^\gamma}
\]

holds starting from some \(N \in \mathbb{N}\) we still obtain inequality (2) for any \(\gamma < 1 - \frac{1}{2ka}\), which fact finishes the proof.

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