Dispersionful analogues of Benney’s equations and $N$-wave systems.

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Abstract

We recall Krichever’s construction of additional flows to Benney’s hierarchy, attached to poles at finite distance of the Lax operator. Then we construct a “dispersionful” analogue of this hierarchy, in which the role of poles at finite distance is played by Miura fields. We connect this hierarchy with $N$-wave systems, and prove several facts about the latter (Lax representation, Chern-Simons-type Lagrangian, connection with Liouville equation, $\tau$-functions).

1 Introduction.

It was realized recently that topological field theories (TFT) are closely connected with integrable hierarchies. The starting point of this connection were the Witten conjectures (proven by Kontsevich) that the free energy of 2-dimensional (2D) gravity is the logarithm of the KdV $\tau$-function, specified by the string equation.

Later, Dijkgraaf, E. and H. Verlinde, and Witten ([6]) showed that classes of 2D TFT’s are classified by solutions of a system of nonlinear partial differential equations bearing on some function (primary free energy of the theory). This system is abbreviated WDVV; various classes of solutions to it were found by Dubrovin and Krichever ([7], [12]), using the integrable systems arising from problems of isomonodromic deformations. One of their results is that a class of WDVV solutions is provided by $\tau$-functions for semiclassical limits of Hamiltonian integrable systems.

$\tau$-functions of semiclassical (or dispersionless) limits of Lax-type integrable hierarchies were studied in the papers of Krichever ([12]) and Takasaki and Takebe ([22]), and generalized $W$-constraints for these systems were computed. These results inspired the study of dispersionless hierarchies and their relations with other solvable systems, in particular of hydrodynamic type ([8]). Among the long-known dispersionless systems of equations were the Benney equations, introduced in [2], which model long waves with a free boundary. They were studied from the Hamiltonian point of view by B.
Kupershmidt and Y. Manin ([15]) and their relations with the Kadomtsev-Petviashvili system were established in [17].

I. Krichever, in his talk at the 1994 Alushta conference, introduced a dispersionless-type hierarchy, that may be viewed as a multiple poles analogue of the Benney equations (he also introduced an elliptic generalisation of these systems). The Lax operator of this system is a rational function on $\mathbb{C}P^1$, with singularities at infinity and at some points at finite distance; in addition to the usual KP flows, corresponding to the pole at infinity, there are flows attached to the points at finite distance. Krichever posed the problem to find dispersionful analogues to these systems.

We solve this problem in the case where all poles at finite distance, are simple. The resulting systems are expressed in Lax form; we call them Krichever systems. The Lax operator takes the form $AB^{-1}$, $A$ and $B$ differential operators. We use the idea that the evolution operators attached to poles at finite distance are classical analogues of “quantum” Lax operators, written using Miura fields of $B$. We show that the resulting vector fields commute pairwise. We also study the (bi-)Hamiltonian properties of this system. Our results can be considered as a generalization of results of [5], [3], [20], [1].

In the last part of the paper, we study the relation of the Krichever systems with the $N$-wave systems ([19]; they also appear in [7], in the context of WDVV equations). We show that the solutions to the $N$-wave equations can be constructed from solutions to the degree 1 flows of the Krichever hierarchy. We also show that the $N$-wave system can be expressed as a commutation condition on certain integro-differential operators; we give a Chern-Simons-like Lagrangian for the $N$-wave system; we show a relation with a system of “connected” Liouville equations (i.e. fulfilled for each pair of indices); we construct $\tau$-functions for the $N$-wave systems and connect them with $\tau$-functions for KP systems. The question of integration of the Krichever systems remains open; we make some observations in section 4, indicating that it should be treated using methods of [14]. We hope to return to this question elsewhere.

## 2 Extended Benney systems.

The phase space of these systems consists of the set of smooth maps from $S^1 = \mathbb{R}/\mathbb{Z}$ to the space of rational functions on $\mathbb{C}P^1$ of the form

$$E(p) = p^n + \sum_{k=0}^{n-1} a_k p^k + \sum_{\alpha=1}^{N} \sum_{i=1}^{i_{\alpha}} \frac{a_{\alpha,i}}{(p - p_{\alpha})^i}, \quad (n, i_{\alpha} \geq 0).$$

(1)

Viewing $E(p)$ as the symbol of a pseudodifferential operator, we can define following [18] a bracket operation on the space of maps from $S^1$ to all rational functions on $\mathbb{C}P^1$, by degeneration of the Lie bracket. We denote it by $\{ , \}$. So $\{ A(p, x), B(p, x) \} = \frac{\partial A}{\partial p} \frac{\partial B}{\partial x} - \frac{\partial A}{\partial x} \frac{\partial B}{\partial p}$.
Let us consider then the rational functions $E_{\infty,i}(p), E_{\alpha,i}(p), i \geq 1$, associated to $E(p)$, given by $E_{\infty,i}(p) = (E(p)^{i/n})_{+\infty}, E_{\alpha,i}(p) = (E(p)^{i/i_\alpha})_{+\alpha}$. Here the powers of $E(p)$ are understood as formal expansions near the points $\infty$ and $p_\alpha$ respectively, and we set as usual

$$\sum_{s \leq s_0} p^s u_s = \sum_{0 \leq s \leq s_0} p^s u_s,$$

and

$$\sum_{s \leq s_0} \frac{u_s}{(p - p_\alpha)^s} = \sum_{1 \leq s \leq s_0} \frac{u_s}{(p - p_\alpha)^s}.$$

Define then flows $\partial_{\infty,i}, \partial_{\alpha,i}$, by

$$\partial_{\nu,i}(E(p)) = \{E_{\nu,i}(p), E(p)\}, \quad \nu = 1, ..., N, \infty, \quad i \geq 0 \tag{2}$$

It can be shown that this defines vector fields on the phase space. (For example, the immobility of the pole at $\infty$ is connected to the fact that the leading coefficient of $E(p)$ is $p^n$, whereas the non-constantness of $x \mapsto a_{\alpha,i_\alpha}(x)$ induces the motion of $p_\alpha(x)$, i.e. its non constantness as a function of time, $x$ being fixed.) Moreover, these vector fields commute. Conserved quantities are in addition to the usual KP conserved quantities $H_{\infty,i}$

$$H_{\infty,i} = \text{tr}_{\infty}(E(p)^{i/n}) = \int_{S^1} \text{res}_{\infty}(E(p))^{\frac{1}{n}+1} dx,$$

$$H_{\alpha,i} = \text{tr}_{p_\alpha}(E(p)^{i/i_\alpha}) = \int_{S^1} \text{res}_{p_\alpha}(E(p))^{\frac{1}{i_\alpha}+1} dx, \tag{3}$$

where in the last formula $E(p)$ should be expanded in formal series near $p_\alpha$.

These systems were introduced by I. Krichever in his talk at the 1994 Alushta conference ([13]). They are a generalization of the Benney systems (where all poles in expression (1) are simple), introduced in [2].

**Remark.** It is possible to define, on the dispersionless phase space, a compatible family of Poisson structures attached to the poles at finite distance, in the same fashion as the degenerations of the Adler-Gelfand-Dickey structures are attached to the pole at infinity. But it seems difficult to extend this construction in the dispersionful situation.

### 3 Dispersionful analogues.

We will construct analogues of the systems of section 2, in the case where all $i_\alpha$’s are equal to 1. In that case, we replace $p - p_\alpha$ by some operator $\partial + \phi_\alpha = \exp(\varphi_\alpha)\partial \exp(-\varphi_\alpha)$, and $a_\alpha(p - p_\alpha)^{-1}$ by the operator $a_\alpha \exp(\varphi_\alpha)\partial^{-1} \exp(-\varphi_\alpha)$; so that the analogue of $E(p)$
should be some operator of the form
\[ \mathcal{L} = \partial^n + \sum_{k=0}^{n-1} p_k \partial^k + \sum_{i=1}^{N} \varphi_i \partial^{-1} \psi_i, \] (4)

\( p_k, \varphi_i, \psi_i \) are some functions. So the phase space will be
\[ \mathcal{P} = \{ (P, \varphi_i, \psi_i)_{i=1,...,N}, P = \partial^n + \sum_{k=0}^{n-1} p_k \partial^k, p_k, \phi_i, \psi_i \in C^\infty(\mathbb{R}) \}. \]

We have a mapping
\[ \mathcal{P} \to \mathcal{L}_n = \{ \partial^n + \sum_{k \leq n-1} u_k \partial^k, u_k \in C^\infty(\mathbb{R}) \} \]
\[ \mathcal{P} \] admits a \( GL_N(\mathbb{C}) \)-action by
\[ g \cdot (P, \varphi_i, \psi_i) = (P, (g^{-1})_{ji} \varphi_j, g_{ij} \psi_j), \]
and the fibres of the mapping \( \mathcal{P} \to \mathcal{L}_n \) are generically the orbits of the action of \( GL_N(\mathbb{C}) \). Note also that the part of \( \mathcal{P} \) formed of the \( (P, \varphi_i, \psi_i) \) s.t. the Wronskian \( W(\varphi_i) \) (respectively, \( W(\psi_i) \)) is invertible, maps to the subset of \( \mathcal{L}_n \) formed of the operators \( AB^{-1} \) (respectively, \( B'^{-1}A' \)), with \( A, A' \in \{ \partial^{n+N} + \sum_{0 \leq i < n+N} a_i \partial^i \}, B, B' \in \{ \partial^N + \sum_{0 \leq i < N} b_i \partial^i \}; \) it is enough to take for \( B^* \) (respectively, for \( B' \)) the operator with kernel \( \oplus_{i=1}^N C \psi_i \), (respectively, \( \oplus_{i=1}^N C \varphi_i \)); the involution \( B \mapsto B^* \) is defined below.

We now show:

**Proposition 1**
\[ \mathcal{L}^p = (\mathcal{L}^p)' + \sum_{i=1}^{N} \sum_{k=0}^{p} [\mathcal{L}^k \varphi_i] \partial^{-1} [\mathcal{L}^*(p-k-1) \psi_i], \] (5)
or equivalently
\[ (1 - \lambda \mathcal{L})^{-1} = (1 - \lambda \mathcal{L})_+^{-1} + \sum_{i=1}^{N} [(1 - \lambda \mathcal{L})^{-1} \varphi_i] \lambda \partial^{-1} [(1 - \lambda \mathcal{L}^*)^{-1} \psi_i]. \] (6)

Here \( \mathcal{L}^* \) denotes the involution \( (\partial^m + \sum_{i<m} a_i \partial^i)^* = (-\partial)^m + \sum_{i<m} (-\partial)^i a_i \), we set \( (\sum_{i\leq m} x_i \partial^i)_+ = \sum_{0 \leq i \leq m} x_i \partial^i \) for any smooth \( x_i \)'s, \( [L \varphi] \) denotes the result of the action of \( L \) on the function \( \varphi \), with the convention
\[ [(\alpha \partial^{-1} \beta) \varphi](x) = \alpha(x) \int_0^x \beta \varphi; \]
in (6), \( \lambda \) is a formal parameter.
so we define the flows

\[ \partial \]
which can be translated into

\[ L^{p+1} = LL^p = \sum_{i=1}^{N} \sum_{k=0}^{p} [[Lk[\varphi_i]]^{-1}[L^{(p-k)}psi_i] + \sum_{i=1}^{N} \varphi_i^{-1}[L^{*p}\psi_i] + \text{diff. operator}, \]

this follows from the identity

\[ MN = \sum_{j} [Mc_j][\varphi_j]^{-1}d_j + \sum_{i} a_i \varphi_i^{-1}[N^*b_i] + \text{diff. operator}, \]

for \( \mathcal{M} = \mathcal{M}_+ + \sum a_i \varphi_i^{-1}b_i \) and \( \mathcal{N} = \mathcal{N}_+ + \sum c_j \varphi_j^{-1}d_j \); for a proof of this identity cf. [9].

We define now the following flows on \( L \):

\[ \partial_{i,s}L = \sum_{k=0}^{s-1} [L^k\varphi_i]\partial^{-1}[L^{(s-k)}\psi_i], \mathcal{L}, \]

(7a)

\[ \partial_{\infty,s}L = [(L^{s/n})_+, \mathcal{L}], \]

(7b)

These flows are defined on the phase space \( \mathcal{P} \) because of the following identities:

\[ \partial_{i,s}L = \sum_{k=0}^{s-1} \sum_{j=0}^{N} [L^k\varphi_i]\partial^{-1}[L^{(s-k)}\psi_i]\partial^{-1}\psi_j + \sum_{k=0}^{s-1} \sum_{j=0}^{N} [L^k\varphi_i]\partial^{-1}[L^{(s-k)}\psi_i] - \sum_{k=0}^{s-1} \sum_{j=0}^{N} \varphi_i\partial^{-1}([L^k\varphi_i]\psi_j[L^{(s-k)}\psi_i] - \sum_{k=0}^{s-1} \sum_{j=0}^{N} [L^k\varphi_i]\partial^{-1}([L^{(s-k)}\psi_i] + \text{diff. operator}, \]

\[ \partial_{\infty,s}L = \sum_{i=0}^{N} [(L^{s/n})_+\varphi_i]\partial^{-1}\psi_i - \sum_{i=0}^{N} \varphi_i\partial^{-1}([L^{s/n}_+\psi_i] + \text{diff. operator}; \]

so we define the flows \( \partial_{i,s} \) on \( \mathcal{P} \) by

\[ \partial_{i,s}\varphi_j = \sum_{k=0}^{s-1} [L^k\varphi_i]\partial^{-1}([L^{(s-k)}\psi_i]\varphi_j) - [L^s\varphi_i]\delta_{ij}, \]

\[ \partial_{i,s}\psi_j = \sum_{k=0}^{s-1} [L^k\psi_i]\partial^{-1}([L^{(s-k)}\varphi_i]\psi_j) + [L^s\psi_i]\delta_{ij}, \]

(8)

which can be translated into

\[ \partial_{i,\lambda}\varphi_j = [\mathcal{L}_i(\lambda)\varphi_j] - [(1 - \lambda \mathcal{L})^{-1}\varphi_j] \delta_{ij}, \]

\[ \partial_{i,\lambda}\psi_j = -[\mathcal{L}_i(\lambda)^s\psi_j] + [(1 - \lambda \mathcal{L}^s)^{-1}\psi_j] \delta_{ij}, \]

(9)

\[ \partial_{i,\lambda}P = [\mathcal{L}_i(\lambda), \mathcal{L}]_+(9bis), \]

where \( \partial_{i,\lambda} = \sum_{s\geq 0} \lambda^s \partial_{i,s} \), and

\[ \mathcal{L}_i(\lambda) = [(1 - \lambda \mathcal{L})^{-1}\varphi_i] \lambda \partial^{-1}[(1 - \lambda \mathcal{L}^s)^{-1}\psi_i] \]

(10)
is the generating function for the flows (7), so that (7a) holds. Similarly, we define the flows \( \partial_{\infty,s} P \) on \( \mathcal{P} \) by

\[
\partial_{\infty,s} \varphi_i = [(\mathcal{L}^{s/n})_+ \varphi_i], \quad \partial_{\infty,s} \psi_i = -[(\mathcal{L}^{s/n})_+ \psi_i], \quad \partial_{\infty,s} P = [(\mathcal{L}^{s/n})_+, \mathcal{L}]_+ \tag{11}
\]

so that (7b) holds. (We could also provide generating formulae \( \partial_{\infty,\lambda} P = [[(1 - \lambda \mathcal{L}^{1/n})^{-1})_+, \mathcal{L}]_+ \), etc.)

Let us prove now:

**Proposition 2** The flows \( \partial_i, i = 1, \ldots, N, \infty, s \geq 0 \), defined by equations (8), (9bis), (11) commute pairwise.

**Proof.** Let us first assume the zero-curvature conditions

\[
[\partial_i \lambda - \mathcal{L}_i(\lambda), \partial_j \mu - \mathcal{L}_j(\mu)] = [\partial_i \lambda - \mathcal{L}_i(\lambda), \partial_{\infty,s} - (\mathcal{L}^s)_+] = 0 \tag{12}
\]

are proved. (The relations \( [\partial_{\infty,s} - (\mathcal{L}^{s/n})_+], \partial_{\infty,t} - (\mathcal{L}^{t/n})_+ = 0 \) are known classically.) They imply the analogous relations, with \( \partial_{i\lambda} - \mathcal{L}_i(\lambda) \) replaced by \( \partial_{i\lambda} - \mathcal{L}_i(\lambda) + \delta_{ik}(1 - \lambda \mathcal{L})^{-1} \) and \( \partial_{j\mu} - \mathcal{L}_j(\mu) \) replaced by \( \partial_{j\mu} - \mathcal{L}_j(\mu) + \delta_{jk}(1 - \mu \mathcal{L})^{-1} \), due to (7b) and to \( \partial_{i\lambda} \mathcal{L} = [\mathcal{L}_i(\lambda), \mathcal{L}] \), and this shows that the vector fields commute on the variables \( \varphi_i, \psi_j \). Since on the other hand, they imply that they commute on \( \mathcal{L} \), they will commute on the phase space \( \mathcal{P} \).

Let us turn to the proof of (12).

\[
\partial_{i\lambda} \mathcal{L}_j(\mu) = [\mathcal{L}_i(\lambda)(1 - \mu \mathcal{L})^{-1} \varphi_j - \delta_{ij}(1 - \mu \mathcal{L})^{-1} (1 - \lambda \mathcal{L})^{-1} \varphi_j] \mu \partial^{-1}[(1 - \mu \mathcal{L}^*)^{-1} \psi_j]
\]

\[+ [(1 - \mu \mathcal{L})^{-1} \varphi_j] \mu \partial^{-1}[-\mathcal{L}_i(\lambda)^*(1 - \lambda \mathcal{L}^*)^{-1} \psi_j + \delta_{ij}(1 - \mu \mathcal{L}^*)^{-1} (1 - \lambda \mathcal{L}^*)^{-1} \psi_j].
\]

The terms in \( \delta_{ij} \) give \( \delta_{ij} \left[ \frac{1}{\lambda - \mu} \varphi_j \right] \partial^{-1} \left[ \frac{1}{1 - \lambda \mathcal{L}} \psi_j \right] \) and \( \delta_{ij} \left[ \frac{1}{\mu - \lambda} \varphi_j \right] \partial^{-1} \left[ \frac{1}{1 - \mu \mathcal{L}} \psi_j \right] \). So

\[
\partial_{i\lambda} \mathcal{L}_j(\mu) - \partial_{j\mu} \mathcal{L}_i(\lambda) = \sum_r \left[ \mathcal{L}_i(\lambda) \alpha^{(r)}_j \partial^{-1} \beta^{(r)}_i - \alpha^{(r)}_j \partial^{-1} \beta^{(r)}_i \mathcal{L}_i(\lambda)^* \beta^{(r)}_j \right]
\]

\[= \sum_s \left[ \mathcal{L}_j(\mu) \alpha^{(s)}_i \partial^{-1} \beta^{(s)}_j - \alpha^{(s)}_i \partial^{-1} \mathcal{L}_j(\mu)^* \beta^{(s)}_i \right],
\]

if \( \mathcal{L}_i(\lambda) \) and \( \mathcal{L}_j(\mu) \) are written \( \mathcal{L}_i(\lambda) = \sum_x \alpha^{(x)}_i \partial^{-1} \beta^{(x)}_i \) and \( \mathcal{L}_j(\mu) = \sum_x \alpha^{(x)}_j \partial^{-1} \beta^{(x)}_j \), and this coincides with \( \mathcal{L}_i(\lambda), \mathcal{L}_j(\mu) \) (applying again [3], lemma 2 and the fact that the operators have no differential part).

\[
\partial_{i\lambda} (\mathcal{L}^s_+) = [\mathcal{L}_i(\lambda), \mathcal{L}^s_+]_+ \Rightarrow \partial_{i\lambda} (\mathcal{L}^s_+) = [\mathcal{L}_i(\lambda), \mathcal{L}^s_+]_+ \Rightarrow \partial_{i\lambda} (\mathcal{L}^s_+) = [(\mathcal{L}^s_+) (1 - \lambda \mathcal{L})^{-1} \varphi_i] \lambda \partial^{-1} [(1 - \lambda \mathcal{L}^*)^{-1} \psi_i] - [(1 - \lambda \mathcal{L}^*)^{-1} \varphi_i] \lambda \partial^{-1} [(\mathcal{L}^s_+) (1 - \lambda \mathcal{L}^*)^{-1} \psi_i],
\]

so the second part of (12) follows from [3], lemma 2.

\[\square\]

* recall that for a formal series \( f(\lambda, \mu) \in C[[\lambda, \mu]] \), s.t. \( f(\lambda, \lambda) = 0 \), we can define the ratio \( \frac{f(\lambda, \mu)}{\lambda - \mu} \in C[[\lambda, \mu]] \); the expressions in this computation are understood in this sense.
4 Poisson structures.

We are going to define a Poisson structure on \( \mathcal{P} \), such that the natural mapping \( \mathcal{P} \to \mathcal{L}_n \) is Poisson, \( \mathcal{L}_n \) being endowed with the second Adler-Gelfand-Dickey structure (AGD2). The cotangent space to \( \mathcal{P} \) at \((P, \varphi_i, \psi_i)\) is defined as \( IOP/IOP_{\leq -n} \oplus DOP/\{Q \in DOP | Q \varphi_i = Q^* \psi_i = 0, i = 1, \ldots, N\} \), where \( DOP = \{\sum_{0 \leq i \leq n} a_i \partial^i, a_i \in C^\infty(\mathbb{R})\}, i_0 < \infty\) and \( IOP = \sum_{i < 0} a_i \partial^i, a_i \in C^\infty(\mathbb{R})\}\); the pairing with the tangent space being given by

\[
\langle (X, Q), (\delta P, \delta \varphi_i, \delta \psi_i) \rangle = \text{tr}(X + Q)(\delta P + \sum_{i=1}^N \delta \varphi_i \partial^{-1} \psi_i + \varphi_i \partial^{-1} \delta \psi_i)
\]

\[
= \text{tr}X \delta P + \sum_{i=1}^n \delta \varphi_i [Q^* \psi_i] + \delta \psi_i [Q \varphi_i].
\]

The Hamiltonian vector field on \( \mathcal{L}_n \) corresponding to the covector \((X, Q)\) is \( V_{(X,Q)}(\mathcal{L}) = (\mathcal{L}(X + Q))_+ \mathcal{L} - \mathcal{L}((X + Q)\mathcal{L})_+\); this formula can be realized defining

\[
V_{(X,Q)}(\varphi_i) = [(PX + LQ)_+ \varphi_i] = [(\mathcal{L}(X + Q))_+ \varphi_i],
\]

\[
V_{(X,Q)}(\psi_i) = -[((XP)^* + (QL)^*)_+ \psi_i] = -[\mathcal{L}(X + Q)^*_+ \psi_i],
\]

\[
V_{(X,Q)}(P) = [(\mathcal{L}(X + Q))_+ \mathcal{L} - \mathcal{L}((X + Q)\mathcal{L})_+]_+.
\]

The corresponding first structure is given by

\[
V^{(1)}_{(X,Q)}(\varphi_i) = [(X+Q)_+ \varphi_i], V^{(1)}_{(X,Q)}(\psi_i) = -[(X+Q)^*_+ \psi_i], V^{(1)}_{(X,Q)}(P) = [\mathcal{L}, X+Q]_+ + [Q, \mathcal{L}]
\]

Let us show:

**Proposition 3** Formulæ (13) and (14) define compatible Poisson structures on \( \mathcal{P} \), such that the map \( \mathcal{P} \to \mathcal{L}_n \) is Poisson with \( \mathcal{L}_n \) endowed with the second (resp. first) Adler-Gelfand-Dickey structures.

**Proof.** Let \( \tilde{\mathcal{P}} \) be the subset of \( \mathcal{P} \) consisting of the \((P, \varphi_i, \psi_i)\) such that the Wronskians \( W(\varphi_i) \) and \( W(\psi_i) \) are invertible. We have a sequence of maps \( \tilde{\mathcal{P}} \to \tilde{\mathcal{L}}_{n+N} \times C^\infty(\mathbb{R})^N \to \tilde{\mathcal{L}}_{n+N} \times \tilde{\mathcal{L}}_N \to \mathcal{L}_n\); the first map is defined by \((P, \varphi_i, \psi_i) \mapsto (A, \frac{\varphi_i}{\psi_i}, \psi_1(\frac{\psi_1}{\psi_i})', \ldots)\); it is a bijection between open subsets of the spaces. The second map is the product of the identity and the Miura mapping. So the composition of the two first maps is \((P, \varphi_i, \psi_i) \mapsto (A, B)\); the inverse of the first map is \((A, b_1, \ldots, b_N) \mapsto (P, \varphi_i, \psi_1(x) = \int_0^x b_i \int_0^y b_j dy, \ldots)\); it is equal to \((\partial + b_1)(\partial + b_N)\). The last map is \((A, B) \mapsto AB^{-1}\). If \( \tilde{\mathcal{L}}_{n+N} \) and \( \tilde{\mathcal{L}}_N \) are endowed respectively with AGD2 and its opposite, the last map is Poisson; this follows from [4], add. remark, or from the fact that each \( \tilde{\mathcal{L}}_p \) is a union
of symplectic leaves of the group $\mathcal{L}$ of [10], and that the map $S \times S' \to G$, $(g, h) \mapsto gh^{-1}$ in any Poisson-Lie group $G$ is Poisson, $S$ and $S'$ symplectic leaves of $G$ endowed with the induced structure and its opposite. If now $\mathcal{L}_{n+N} \times C^\infty(\mathbb{R})^N$ is endowed with the product of AGD2 and the free field structures, the second map is again Poisson, by the Kupershmidt-Wilson theorem ([16]). We wish now to show that the first map is Poisson, if $\mathcal{P}$ is endowed with the structure (13).

Under the first map, the correspondence of tangent spaces is given by $\delta \mathcal{L} - \delta AB^{-1} - AB^{-1} \delta BB^{-1}$, and for cotangent spaces it is $X_A = (B^{-1}(X+Q))_-$, $X_B = -((X+Q)\mathcal{L})_-$. On the other hand, we know from [21], [11] that the action of the Hamiltonian vector field corresponding to $X_B$ on solutions of $B^*$ is given by $V_{X_B}(\psi_i) = -(X_B^*B^*)_+\psi_i$. But $(BX_B)_+ = -((X+Q)\mathcal{L})_+ + B(B^{-1}(X+Q)\mathcal{L})_+$ so since $B^*\psi_i = 0$, $V_{X_B}(\psi_i)$ coincides with the vector field of (13).

To obtain the other parts of (13), we write the Hamiltonian vector field $V_{(X,Q)}$ on $\mathcal{L}$; it contains $2N$ terms of the form $a \partial^{-1}b$, namely the $\varphi_i \partial^{-1}V_{(X,Q)}(\psi_i)$ and the $V_{(X,Q)}(\varphi_i) \partial^{-1}\psi_i$, and we identified the first ones with what is in (13). It means that $\sum_{i=1}^N (V_{(X,Q)}(\varphi_i) - \text{formula for it in (13)}) \partial^{-1}\psi_i$ is a differential operator, and this implies (13a), (13b) and (13c) are deduced directly. The Poisson and compatibility of the first structure, given by (14) and the second one are obtained as usual by consideration of the map $(P, \varphi_i, \psi_i) \mapsto (P + \lambda, \varphi_i, \psi_i)$.

\textbf{Remarks.} 1) It is plausible that the flows $\partial_{\lambda}$ preserve the Poisson structures (13) and (14) on $\mathcal{P}$. The most natural expressions for Hamiltonians governing them would be $\int [1 - \lambda \mathcal{L}]^{-1}\varphi_i \psi_i$ but they are not defined on our phase space.

2) Stationary solutions problem. This is the problem of finding solutions to the stationary flow equation $[\mathcal{L}, \sum_i k \alpha_{i,k} \mathcal{L}_{i,k} + \sum_i \alpha_{i,k} \mathcal{L}^i_+] = 0$ (where $\mathcal{L}_i(\lambda) = \sum_k \lambda^k \mathcal{L}_{i,k}$). We define a spectral curve for this problem in the following way. Consider a commuting pair $\mathcal{A} = \sum_i \alpha_i \partial^{-1} \beta_i = A^{-1}B$ and $\mathcal{B} = P + \sum_j \gamma_j \partial^{-1} \delta_j = C^{-1}D$, $A, \ldots, D, P$ differential operators with leading term $\partial^n$, and coefficients in $C^\infty(\mathbb{R})$ (we assume that $\deg B > \deg A$). For any $\lambda \in \mathbb{C}$, let $E_\lambda$ be the kernel of $B - \lambda A$. Then $\dim E_\lambda = \deg B$ for all $\lambda$. Consider the map $j : E_\lambda \to \text{Ker}A$, $\psi \mapsto \mathcal{A}\psi - \lambda \psi$; let $V_\lambda$ be its kernel; so $V_\lambda \neq 0$. $V_\lambda$ is stable under the action of $\mathcal{B}$; let $\mu_i(\lambda)$ be the list of the eigenvalues of its action, and $\psi_{\lambda,\mu_i(\lambda)}$ be a common eigenvector. (Here $\mathcal{A}$ and $\mathcal{B}$ act in the way explained in prop. 1.)

The system

$$(B - \lambda A)\psi_{\lambda,\mu_i(\lambda)} = 0, \quad (D - \mu_i(\lambda)C)\psi_{\lambda,\mu_i(\lambda)} = 0$$

is satisfied; we can apply to it the classical method ([4]) of successive Euclidean divisions to find an algebraic relation between $\lambda$ and the $\mu_i(\lambda)$, and a vector bundle over the curve it defines. To go further in the integration of this system, we would need to impose additional constraints, following the method of [14].
Example. Consider the case $L = \psi_1 \partial^{-1} \psi_2 - \psi_2 \partial^{-1} \psi_1 = (\partial^2 + q)^{-1}$, $\partial^2 + q$ the operator with solutions $\psi_1$ and $\psi_2$ (we assume $W(\psi_1, \psi_2) = 1$). We have then a flow $\partial_t L = [\psi_1 \partial^{-1} \psi_2, L]$. Writing $\psi_1 = e^\varphi$, $\psi_2(x) = e^{\varphi(x)} \int_0^x e^{-2\varphi(x)}$, we find $\partial_t \varphi(x) = (\int_0^x e^{2\varphi}) (\int_0^x e^{-2\varphi})$.

5 $N$-wave equations.

Let $(\beta_{ij}(x))_{1 \leq i,j \leq N}$ be a system of smooth functions of $x = (x_1, ..., x_N)$ where $x \in \mathbb{R}^N$. Let us consider the $N$-wave system of equations

$$
\frac{\partial \beta_{jk}}{\partial x_i} = \beta_{ji} \beta_{ik}, \text{ if } i, j, k \text{ are all different, and } \left( \sum_{\alpha=1}^N \frac{\partial}{\partial x_\alpha} \right) \beta_{ij} = 0, \text{ for all } i \neq j \quad (15)
$$

Let now $\tilde{\beta}_{1i}$, $\tilde{\beta}_{i1}$ ($i = 2, ..., N$) be smooth functions on $\mathbb{R}$; let $L$ be the Lax operator

$$
L = \partial + \sum_{2 \leq i \leq N} \tilde{\beta}_{1i} \partial^{-1} \tilde{\beta}_{i1}.
$$

Proposition 4 Define $\tilde{\beta}_{ij}(x) = \int_0^x \tilde{\beta}_{1i} \tilde{\beta}_{i1}$, for $i, j$ distinct and $> 1$. Then the functions $\tilde{\beta}_{ij}$ satisfy equations (15), $\partial / \partial x_i$ being replaced by the the Krichever flow (7a), for $s = 1$ and index $i$, and $\partial / \partial x_1$ by $\partial / \partial x$. Moreover, with these definitions of $\tilde{\beta}_{ij}$ for $i \neq j > 1$, the system (15) is equivalent to the zero-curvature conditions

$$
[\partial_i - \tilde{\beta}_{1i} \partial^{-1} \tilde{\beta}_{i1}, \partial_j - \tilde{\beta}_{1j} \partial^{-1} \tilde{\beta}_{j1}] = 0, \text{ for } i \neq j.
$$

Proof. The fact that the $\tilde{\beta}_{ij}$ obey the first part of (15) is clear; for the second part, it follows from the form of $L$. The last part is obtained directly. \qed

We now show how equations closely connected to the system (15) can be obtained from a Chern-Simons-type Lagrangian. Let us set $N = 3$ in what follows. Consider the following functional, depending on functions $(\beta_{ij})_{i,j=1,2,3}$

$$
S(\beta_{ij}) = \int_{\mathbb{R}^3} A dA + \frac{2}{3} A^3,
$$

where $A$ is the matrix-valued one-form on $\mathbb{R}^3$ with coefficients $A_{ij} = \beta_{ij} dx_i$. The associated Euler-Lagrange equations are as usual $dA + [A, A] = 0$; this can be written

$$
\frac{\partial \beta_{jk}}{\partial x_i} = \beta_{ji} \beta_{ik} \text{ for } i \neq j
$$

Any solution to these equations satisfies $\partial_i \partial_k \log \beta_{ik} = \beta_{kl} \beta_{ik}$ and $\partial_k \partial_l \log \beta_{kl} = \beta_{lk} \beta_{kl}$, hence the function $\phi_{kl} = \log \beta_{kl} \beta_{ik}$ satisfies the Liouville equation $\partial_k \partial_l \phi_{kl} = 2 e^{\phi_{kl}}$.\[9]
Finally, let us make some comments of $\tau$-functions for the system (15). For any solution to (15), we have a function $\log \tau$ such that \[ \beta_{ik}\beta_{ki} = \partial_{x_i}\partial_{x_k}\log \tau. \] This $\tau$-function is connected with the KP one as follows: consider the KP operator $\partial_{y_i} - \partial_{x_i}^2 - u_i$, ($y_i$ are additional KP times), with $u_i = 2 \sum_{k=1,\ldots,N, k \neq i} \beta_{ik}\beta_{ki}$, then $u_i = 2 \sum_{k \neq i} \partial_{x_i}\partial_{x_k}\log \tau = -2\partial_{x_i}^2\log \tau$, in accordance with the usual formulae of the KP theory.

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