Original Research Article

SOME BLOW-UP PROPERTIES OF A SEMILINEAR HEAT EQUATION

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ABSTRACT

In this paper, we consider some blow-up properties of a semilinear heat equation, where the nonlinear term is of exponential type, subject to the zero Dirichlet boundary conditions, defined in a ball in $\mathbb{R}^{n}$. Firstly, we study the blow-up set showing that the blow-up can only occur at a single point. Secondly, the upper blow-up rate estimate is derived.

Keywords: Blow-up solution, semilinear heat equation, Dirichlet boundary conditions, pointwise estimate, radially symmetric.

1. INTRODUCTION

In this paper, we study the following parabolic problem:

$$\begin{align*}
u_t &= \Delta u + e^{\alpha u}, & (x, t) \in B_R \times (0, T), \\
u(x, t) &= 0, & (x, t) \in \partial B_R \times (0, T), \\
u(x, 0) &= u_0(x), & x \in B_R,
\end{align*}$$

where $p > 1$; $\lambda > 0$; $B_R$ is a ball in $\mathbb{R}^{n}$; $u_0$ is smooth, nonzero, nonnegative, nonincreasing radially symmetric function, moreover, it is further required to be vanishing on $\partial B_R$ and satisfying the following condition:

$$\Delta u_0(x) + \lambda e^{\alpha u_0(x)} \geq 0, \quad x \in B_R,$$

(2)

The blow-up phenomena in parabolic problems, defined on bounded domains, have been studied by many authors, see for instance [1-7].

One of the most studied cases, is the semilinear heat equation:

$$\begin{align*}
u_t &= \Delta u + f(u), & \text{in} \ B_R \times (0, T),
\end{align*}$$

(3)

In [1] it has been shown by Kaplan that, if the convex source terms $f = f(u)$ satisfies the condition

$$\int_0^\infty \frac{du}{f(u)} < \infty, \quad U \geq 1,$$

(4)

then diffusion cannot prevent blow-up, when the initial state is large enough.

Later, in [7], Friedman and McLeod have studied equation (3) with zero Dirichlet conditions, under fairly general assumptions on $u_0$ (nonincreasing radial function, vanishing on $\partial B_R$). They have considered the two special cases:

$$\begin{align*}
u_t &= \Delta u + u^p, & p > 1,
\end{align*}$$

(5)

$$u_t = \Delta u + e^{\alpha u}$$

(6)

For equation (5), they showed that for any $\alpha > 2/(p - 1)$, the upper pointwise estimate takes the following form

$$u(x, t) \leq C|x|^{-\alpha}, \quad x \in B_R \setminus \{0\} \times (0, T),$$

which indicates that the only possible blow-up point is $x = 0$. Moreover, under an additional assumption of monotonicity in time (2), the corresponding lower estimate on the blow-up can be established (see[6]) as follows

$$u(x, T) \geq C|x|^{-2/(p-1)}, \quad x \in B_R \setminus \{0\},$$

for some $R^* \leq R, C > 0$.

On the other hand, it has been shown in [4] that the upper (lower) blow-up rate estimates take the following form

$$c(T - t)^{-1/(p-1)} \leq u(0, t) \leq C(T - t)^{-1/(p-1)}, \quad t \in (0, T).$$

For the second case, (6), it has been proved that the point $x = 0$ is the only possible blow-up point, and that due to the upper pointwise estimate, which takes the form:

$$\int_0^\infty \frac{du}{f(u)} < \infty, \quad U \geq 1,$$

then the upper blow-up rate estimate takes the form:

$$\log C - \log(T - t) \leq u(0, t) \leq \log C - \log(T - t), \quad t \in (0, T).$$

The aim of paper, is to show that the results of Friedman and McLeod hold true for problem (1). In other words, we prove that $x = 0$ is the only possible blow-up point for this problem. Moreover, we derive the upper blow-up rate estimate for problem (1). The rest of
this paper is organized as follows: In section two, we discuss the local existence and blow-up with stating some properties of classical solutions of problem (1). In section three, we study the blow-up set. In section four, we derive the upper blow-up rate estimate. In section five, we state some conclusions.

2. Preliminaries

Since \( f(u) = \lambda e^{\alpha u} \) is \( C^1((0, \infty)) \) function, the existence and uniqueness of local classical solutions to problem (1) are well known, see [8, 9]. On the other hand, since the function \( f \) is convex on \((0, \infty)\) and satisfies the condition (4), it follows that the solution of problem (1) blow up in finite time for large initial function.

The next lemma shows some properties of the solutions of problem (1), for more details see [2]. We denote for simplicity \( u(r, t) = u(x, t) \), where \( r = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \).

**Lemma 2.1** Let \( u \) be a classical solution of (1). Then

1. \( u(x, t) \) is positive and radial,
2. \( u_r \leq 0 \) in \((0, R) \times (0, T)\). Moreover, \( u_r \leq 0 \) in \((0, R) \times (0, T)\).
3. \( u \geq 0 \), \((x, t) \in B_R \times (0, T)\).

3. Blow-up Set

This section considers the upper pointwise estimate to the solutions of problem (1), which shows that the blow-up cannot occur if \( x \) is not equal zero. In order to prove that, we need first to recall the following lemma, which has been proved by Friedman and McLeod in [7].

**Lemma 3.1** Let \( u \) be a blow-up solution of the zero Dirichlet problem of (3), and \( u_0 \) should be nonzero, nonincreasing radial function vanishing on \( \partial B_R \). Also suppose that

\[
u_{ur}(r) \leq -\delta r, \quad \text{for} \quad 0 < r \leq R, \quad \text{where} \quad \delta > 0.
\]

Then there exists a positive constant \( \delta > 0 \) such that

\[
u(0, t) \leq \log C - \frac{1}{p} \log(T - t), \quad t \in (0, T).
\]

**Proof.** Let \( F(u) = \lambda e^{u^p} \), \( 0 < \delta < 1 \).

It is clear that \( F \) satisfies (8). The next aim is to show that the inequality (9) holds.

A direct calculation shows

\[
u'F(u) - f(u)F'(u) = \lambda^2 p u e^{p-1\varepsilon(1+\delta)u^p} - \lambda^2 x p u e^{(1+\delta)u^p} = \lambda^2 p u e^{p-1\varepsilon(1+\delta)u^p}[1 - \delta].
\]

On the other hand, we have

\[
u F(u) = 2\lambda^2 p x u e^{2\varepsilon x u^p}.
\]

From (10) and (11) it is clear that (9) holds true provided \( \varepsilon, \delta \) are small enough.

Thus \( J = \int_{\nu \leq 0} \log C \left( u \right) d\nu \leq 0 \), \((r, t) \in (0, R) \times (0, T)\).

It follows that:

\[
u_{ur} + \delta r \geq 0\]

Set \( G(s) = \int_s^\infty \frac{ds}{e^{s\nu_{ur}}} \).

It is clear that

\[
u_{ur} + \delta r = \int_0^\infty \frac{ds}{e^{s\nu_{ur}}} \frac{dG(s)}{ds} = - \frac{dG(s)}{ds} \frac{dG(s)}{ds} \frac{dG(s)}{ds} = - \frac{dG(s)}{ds} \frac{dG(s)}{ds} \frac{dG(s)}{ds}.
\]

Thus, by (12), we obtain

\[
u_{ur}(r) = 0 = \lambda r.
\]

Now, by integrate the last equation from 0 to \( r \), we obtain

\[
u_{ur}(r) = \nu_{ur}(0) = \frac{1}{2} \lambda r^2.
\]

It follows that

\[
u_{ur}(r) = \frac{1}{2} \lambda r^2 = \frac{1}{2} \nu_{ur}^2.
\]

If for some \( r > 0 \), \( u(r, t) \to \infty \), as \( t \to T \), then \( G(u(r, t)) \to 0 \), as \( t \to T \), a contradiction to (13).

4. Blow-up Rate Estimate

The following theorem considers the upper bounds of the blow-up rate for problem (1).

**Theorem 3.1** Let \( u \) be a solution of (1), which blows up in a finite time \( T \). Also suppose that \( u_0 \) satisfies (7). Then there exists a positive constant \( C \) such that

\[
u(0, t) \leq \log C - \frac{1}{p} \log(T - t), \quad t \in (0, T).
\]

**Proof.** Define the function \( F \) as follows,

\[
u(x, t) = u_t - \alpha f(u), \quad (x, t) \in B_R \times (0, T),
\]

where \( f(u) = \lambda e^{u^p} \), \( \alpha > 0 \).

A direct calculation shows

\[
u_t - \Delta u = u_t - \alpha f(u), \quad (x, t) \in B_R \times (0, T),
\]

where \( f' \) is continuous, so that \( f'(u) \) is bounded in \( B_R \times [0, t] \), for \( t < T \).

By Lemma 2.1, \( u(x, t) > 0 \), in \( B_R \times (0, T) \), and since \( u \) blows up at \( x = 0 \), it follows that there exist \( k > 0 \), \( \epsilon \in (0, R) \), \( r \in (0, T) \) such that

\[
u_t(x, t) \geq k, \quad (x, t) \in B_r \times [0, t).
\]
Also, we can find $\alpha > 0$ such that $u_t(x,t) \geq \alpha f(u(x,t))$, for $x \in B_e$.

Thus $F(x,t) \geq 0$ for $x \in B_e$.

By Theorem 3.1, $u$ blows up at only $x = 0$, so that there exists $C_0 > 0$ such that

$$f(u(x,t)) \leq C_0 < \infty, \quad \text{in} \quad \partial B_e \times (0,T).$$

If we choose $\alpha$ is small enough such that $k \geq aC_0$, then we can get

$$F(x,t) \geq 0, \quad (x,t) \in \partial B_e \times [\tau,T).$$

By (15), (16), (17) and maximum principle [10], it follows that

$$F(x,t) \geq 0, \quad (x,t) \in B_e \times (\tau,T).$$

Thus

$$u_t(0,t) \geq a\lambda e^{\alpha(0,t)}, \quad \text{for} \quad \tau \leq t < T.$$  \hfill (18)

Since $u$ is increasing in time and blows at $T$, there exist $\tau^* \leq \tau$ such that

$$u(0,t) \geq p^{1/(p-1)} \quad \text{for} \quad \tau^* \leq t < T,$$

Provided that $\tau$ is closed enough to $T$, which leads to

$$e^{\alpha(0,t)} \geq e^{pu(0,t)}, \quad \tau^* \leq t < T.$$  \hfill (19)

From (18) and (19), it follows that

$$u_t(0,t) \geq a\lambda e^{\alpha(0,t)}, \quad \text{for} \quad \tau \leq t < T.$$  \hfill (20)

By integrating (20) from $t$ to $T$, we obtain

$$\int_t^T u_t(0,t) e^{-pu(0,t)} \geq a\lambda (T-t).$$

Thus

$$-\frac{1}{p} e^{-pu(0,t)} T^\tau \geq a\lambda (T-t).$$  \hfill (21)

Since $u(0,t) \rightarrow \infty, \quad e^{-pu(0,t)} \rightarrow 0 \quad \text{as} \quad t \rightarrow T$, it follows that (21) becomes

$$\frac{1}{e^{pu(0,t)}} \geq p\lambda (T-t).$$

Thus

$$e^{pu(0,t)}(T-t) \leq C^*, \quad C^* = 1/(p\alpha\lambda), \quad t \in [\tau,T)$$

Therefore, there exist a positive constant $C$ such that

$$u(0,t) \leq \log C - \frac{1}{p} \log(T-t), \quad t \in (0,T).$$

\section{5. Conclusion}

In this paper, we have studied the blow-up set and derived the upper blow-up rate estimate for problem (1). The main conclusion is that the blow-up in this problem can only occur at a single point. Moreover, the upper blow-up rate estimate is dependednt on $p$.

\section{Conflict of Interests}

The authors declare that there is no conflict of interest related to the publication of this article.

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