Longitudinal wave band gaps in metamaterial-based elastic rods containing multi-degree-of-freedom resonators

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Abstract. Wave propagation and vibration transmission in metamaterial-based elastic rods containing periodically attached multi-degree-of-freedom spring–mass resonators are investigated. A methodology based on a combination of the spectral element (SE) method and the Bloch theorem is developed, yielding an explicit formulation for the complex band structure calculation. The effects of resonator parameters on the band gap behavior are investigated by employing the attenuation constant surface plots, which display information on the location, the width and the attenuation performance of all band gaps. It is found that Bragg-type and resonance-type gaps co-exist in these systems. In some special situations, exact coupling between Bragg and resonance gaps occurs, giving rise to super-wide coupled gaps. The advantage of multi-degree-of-freedom resonators in achieving multiband and/or broadband gaps in metamaterial-based rods is demonstrated. Band gap formation mechanisms are further examined by analytical and physical models, providing explicit formulae to locate the band edge frequencies of all the band gaps.

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In the last two decades, artificial periodic composite materials known as phononic crystals (PCs), consisting of a periodic array of acoustic scatterers embedded in a matrix material, have received considerable attention [1, 2]. One of the most attractive characteristics of PCs is their wave filtering behavior, which guarantees some frequency ranges known as band gaps or stop bands, within which the harmonic acoustic/elastic waves cannot propagate freely.

The most commonly studied band gaps in the area of PCs are of Bragg type, appearing around frequencies governed by the Bragg condition \( L = n(\lambda/2) \) \((n = 1, 2, 3, \ldots)\), where \( L \) is the lattice constant of the periodic system and \( \lambda \) is the wavelength of the waves in the matrix material. Thus, Bragg gaps are not practical for insulating audio-frequency sound in the low frequency range, since the dimension of the structure tends to be too large. In contrast, the locally resonant (LR) mechanism recently proposed by Liu et al provides a new solution to this problem [3]. They fabricated a class of three-dimensional (3D) PCs comprising a cubic array of coated spheres immersed in an epoxy matrix. Resonance-type band gaps were obtained in a frequency range two orders of magnitude lower than that given by the Bragg limit. This pioneering work has triggered further exciting investigations in this field [4–10]. More recently, the development of LR PCs has been extended to a newly emerging field: acoustic metamaterials (AMs) [11–13]. AMs are generally regarded as materials with manmade microstructures that possess unusual physical behavior such as negative effective mass and/or modulus [14, 15], negative refraction [16], acoustic cloaking [17–20], etc. Theoretical and experimental studies on several types of LR PCs and AMs have shown the effects of negative effective acoustic properties [12, 21–24].

The construction of LR PCs and AMs is always based on the idea of introducing localized resonant structures into matrix materials. Such a design strategy can be implemented in the context of vibration control engineering, where a number of structural elastic waveguides such as rods, beams, plates, shells, etc are widely used. Low-frequency resonance gaps can be achieved in these structural waveguides by mounting local resonators periodically. The concept

1. Introduction

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The construction of LR PCs and AMs is always based on the idea of introducing localized resonant structures into matrix materials. Such a design strategy can be implemented in the context of vibration control engineering, where a number of structural elastic waveguides such as rods, beams, plates, shells, etc are widely used. Low-frequency resonance gaps can be achieved in these structural waveguides by mounting local resonators periodically. The concept
has recently been validated theoretically and experimentally for rods [25] and beams [7, 26], which demonstrate considerable vibration attenuation within resonance band gaps. However, the attached resonators involved in these works are restricted to single-degree-of-freedom (SDOF) system; thus only one tunable resonance gap can be achieved. In addition, the effects of resonator parameters on the band gap behavior, including the position, width and attenuation performance of band gaps, as well as the associated band gap formation mechanisms, have not yet been fully realized so far. Furthermore, recent studies on similar systems show that Bragg-type gaps also exist in these systems due to the systematic periodicity [27, 28], and their interplay with resonance gaps produces some unusual physics. This aspect has received little attention in previous efforts on LR rods [25, 29].

In this paper, we consider more general LR metamaterial-based rod systems consisting of uniform rods with periodically attached multi-degree-of-freedom (MDOF) spring–mass resonators. The main purpose is to develop and expand the properties of metamaterial-based rods, and to clarify some underlying physics. It should be mentioned that the band gap characteristics of periodic rods have also attracted considerable interest in the field of mechanical engineering [30], with the focus placed on Bragg-type gaps induced by material periodicity. In contrast, the present work will explore both Bragg and resonance gaps to achieve multiband and broadband vibration reduction in rods. Parametric influence on the band gap behavior, as well as the band gap formation mechanisms, will be examined in detail.

The paper is organized as follows. Following the introduction, section 2 concerns the derivation of formulations for the calculation of band gaps in infinite systems and vibration transmission through finite structures. Section 3 is devoted to a study of the band gap behavior and band gap formation mechanisms. Finally, section 4 concludes the paper.

2. The model and the method

2.1. Wave propagation in an infinite periodic system

Consider an LR metamaterial-based rod system consisting of an infinite length uniform hollow rod with periodic internally attached identical MDOF resonators, as sketched in figure 1. The spacing of the resonators is L. Each resonator consists of a series of springs $k_i$ and masses $m_i$ ($i = 1, 2, 3, \ldots, N$).

The equation of motion of the resonator can be written as

$$ (K - \omega^2 M)q = F, $$

(1)

Figure 1. An infinite metamaterial-based rod with a periodic array of MDOF spring–mass resonators.
where the stiffness and mass matrices and the vectors of degrees of freedom and forces are

\[ K = \begin{bmatrix} k_1 & k_{1r} \\ k_{1r} & k_{rr} \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 \\ 0 & m_{rr} \end{bmatrix}, \quad q = \begin{bmatrix} q_0 \\ q_r \end{bmatrix}, \quad F = \begin{bmatrix} F_0 \\ F_r \end{bmatrix}. \quad (2) \]

In the above

\[ k_{1r} = k_1T_1r = \begin{bmatrix} -k_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad k_{rr} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \cdots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & \cdots & \vdots \\ 0 & -k_3 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & -k_N & -k_N \end{bmatrix}, \quad m_{rr} = \begin{bmatrix} m_1 & 0 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & m_{N-1} & 0 \\ 0 & 0 & \cdots & 0 & m_N \end{bmatrix}, \quad q_r = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{N-1} \\ q_N \end{bmatrix}, \quad F_r = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{N-1} \\ F_N \end{bmatrix}. \quad (3) \]

\( q_0 \) is the displacement of the attachment point, \( q_i \) \( (i = 1, 2, 3, \ldots, N) \) is the displacement of the \( i \)th mass, \( F_0 \) is the reaction force from the rod at the attachment point and \( F_i \) \( (i = 1, 2, 3, \ldots, N) \) is the external force acting on the \( i \)th mass. If no external forces act on the masses so that \( F_r = 0 \), the resonator displacement vector \( q_r \) in (1) can be condensed. Therefore, (1) can be reduced to

\[ F_0 = D_0q_0, \quad (4) \]

where \( D_0 \) is the dynamic stiffness of the resonator at the attachment point, which can be expressed as

\[ D_0 = k_1 - k_{1r}(k_{rr} - \omega^2 m_{rr})^{-1}k_{1r}. \quad (5) \]

Explicit expressions for \( D_0 \) for some particular cases of resonators are presented in appendix A. It should be noted that if an additional lumped mass, \( m_0 \), is mounted at the attachment point, the expression for \( D_0 \) should be modified by \( D_0' = -\omega^2 m_0 + D_0 \) to include the dynamic effects of the lumped mass.

The longitudinal vibration of the finite rod within a unit cell can be modeled using the so-called SE method, which gives the SE matrix [31, 32]

\[ D_{\text{rod}} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = EA\beta \begin{bmatrix} \cot(\beta L) & -\csc(\beta L) \\ -\csc(\beta L) & \cot(\beta L) \end{bmatrix}, \quad (6) \]

where \( \beta = \omega/c_0 = \omega/(E/\rho)^{1/2} \) is the wavenumber of the longitudinal waves in the rod, \( c_0 \) is the wave speed, \( E \) and \( \rho \) are the Young’s modulus and density of the rod material, and \( A \) is the cross-sectional area of the rod.

Therefore, the equation of motion for the entire unit cell of the infinite periodic system can be written as

\[ Du_c = f_c. \quad (7) \]
where
\[ \mathbf{u}_c = \begin{pmatrix} u_L \\ u_R \end{pmatrix}, \quad \mathbf{f}_c = \begin{pmatrix} f_L \\ f_R \end{pmatrix}, \]
\[ \mathbf{D} = \begin{bmatrix} D_{LL} & D_{LR} \\ D_{RL} & D_{RR} \end{bmatrix} = \begin{bmatrix} D_{11} + D_0 & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = E A \beta \begin{bmatrix} \cot(\beta L) + D_0/E A \beta & -\csc(\beta L) \\ -\csc(\beta L) & \cot(\beta L) \end{bmatrix}. \quad (8) \]

In (7) and (8), \( \mathbf{D} \) is the dynamic stiffness matrix of the entire unit cell including the dynamic effects of the local resonator (represented by \( D_0 \)). \( \mathbf{u}_c \) and \( \mathbf{f}_c \) are, respectively, the displacement vector and loading force vector of the boundary of the unit cell, and the subscripts ‘L’ and ‘R’ denote the left and right sides of the unit cell.

The Bloch theorem for wave propagation in an infinite periodic system states that
\[ u_R = e^{\mu} u_L, \quad \text{(9a)} \]
\[ f_R = -e^{\mu} f_L, \quad \text{(9b)} \]
where \( \mu \) is the so-called propagation constant [33–36]. The real part of \( \mu \) is known as the ‘attenuation constant’ [33–36], and the imaginary part as the ‘phase constant’. They, respectively, represent the amplitude decay and phase difference of wave motion between two adjacent periodic unit cells. It should be mentioned that the Bloch wave vector (also known as the Bloch wavenumber), \( q \), which is commonly used in studies of PCs, is related to the propagation constant by \( \mu = -iq \). Thus the imaginary part of the Bloch wave vector, \( \text{Im}(q) \), which quantifies wave attenuation per unit length, is related to the attenuation constant by \( \text{Re}(\mu) = \text{Im}(q) \times L \).

Inserting (9a) into the first row of (7) leads to the following relationship between \( f_L \) and \( u_L \):
\[ f_L = (D_{LL} + e^{\mu} D_{LR}) u_L. \quad \text{(10)} \]

Combining (9b) with the second row of (7) yields the well-known quadratic eigenvalue problem for \( e^{\mu} \):
\[ [D_{RL} + (D_{LL} + D_{RR}) e^{\mu} + D_{LR} e^{2\mu}] u_L = 0. \quad \text{(11)} \]

A solution exists for this equation only if
\[ D_{RL} + (D_{LL} + D_{RR}) e^{\mu} + D_{LR} e^{2\mu} = 0. \quad \text{(12)} \]

Substituting the elements of the dynamic stiffness matrix \( \mathbf{D} \) into (12) gives
\[ \cosh(\pm \mu) = \cos(\beta L) + \frac{\tilde{D}}{2} \sin(\beta L), \quad \text{(13)} \]
where \( \tilde{D} = D_0/(E A \beta) \) is the nondimensional dynamic stiffness of the resonators. Equation (13) provides a straightforward way of calculating the propagation constants and thus the complex band structure of the infinite periodic system.

2.2. Vibration transmission in a finite periodic structure

Consider now a finite periodic structure with \( N \) unit cells, as sketched in figure 2. The objective is to calculate the dynamic stiffness matrix across this structure, which gives the relationship between the forces and displacements at each end. Such a problem has recently been treated
systematically \cite{35, 37} for general 1D periodic structures. The methods developed in these investigations are based on a post-processing of the unit-cell dynamic stiffness matrix and the introduction of a wave basis.

The method developed in \cite{35} starts from a reformulation of the eigenvalue problem (11). The reformulated eigenvalue problem becomes

\[
\begin{pmatrix}
D_{RL} & D_{RR} \\
0 & 1
\end{pmatrix} - e^{\mu} \begin{pmatrix}
-D_{LL} & -D_{LR} \\
1 & 0
\end{pmatrix} \begin{bmatrix}
u_L \\
u_R
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \end{bmatrix}.
\]

The solution of (14) yields two sets of eigenvalues associated with displacement eigenvectors of \(u_L\), which can be denoted by \((e^{-\mu}, \phi_u^+)\) and \((e^{\mu}, \phi_u^-)\), representing a pair of positive-going and negative-going characteristic waves. The associated force eigenvectors \(\phi_f^+\) and \(\phi_f^-\) are related to the displacement eigenvectors through (10), i.e.

\[
\phi_f^+ = (D_{LL} + e^{-\mu} D_{LR})\phi_u^+, \quad \phi_f^- = (D_{LL} + e^{\mu} D_{LR})\phi_u^-.
\]

The values of \(\mu\) in (15) are defined such that they have positive real parts or, if \(\text{Re}\{\mu\} = 0\), the power is positive, i.e. \(\text{Re}\{i\omega \phi_H^T \phi_f\} > 0\), where the superscript ‘H’ indicates complex conjugate. Such a definition states that if the wave is traveling in the positive \(x\)-direction, then its amplitude should be decaying, or that if its amplitude remains constant, then there is time-averaged power transmission in the positive \(x\)-direction. The displacement and force at a junction between two unit cells of the periodic system can be given by the sum of positive- and negative-going waves with amplitudes \(w^+\) and \(w^-\), i.e.

\[
u = \phi_u^+ w^+ + \phi_u^- w^-,
\]

\[f = \phi_f^+ w^+ + \phi_f^- w^-.
\]

Specifically, the motion at the two ends of the finite periodic structure illustrated in figure 2 can be expressed by

\[
u_L^1 = \phi_u^+ w_1^+ + \phi_u^- w_1^-,
\]

\[
u_R^N = \phi_u^+ w_2^+ + \phi_u^- w_2^-.
\]

The wave amplitudes at the two ends are related by

\[
w_1^+ = (e^{-\mu}) w_1^+,
\]

\[
w_1^- = (e^{-\mu}) w_2^-.
\]

Substituting (18) into (17) gives

\[
u_L^1 = \phi_u^+ (e^{-\mu}) w_1^+ + \phi_u^- (e^{-\mu}) w_2^-,
\]

\[
u_R^N = \phi_u^+ (e^{-\mu}) w_1^+ + \phi_u^- w_2^-.
\]

This can be expressed in matrix form

\[
\begin{bmatrix}
u_L^1 \\
u_R^N
\end{bmatrix} = \begin{bmatrix}
\phi_u^+ & \phi_u^- (e^{-\mu})^N \\
\phi_u^+ (e^{-\mu})^N & \phi_u^-
\end{bmatrix} \begin{bmatrix} w_1^+ \\
w_2^-
\end{bmatrix}.
\]
In a similar way, one can obtain the associated force vector at the ends expressed in matrix form
\[
\begin{bmatrix}
f_1^L \\
f_1^R
\end{bmatrix} =
\begin{bmatrix}
\phi_f^+ & \phi_f^-(e^{-\mu}N) \\
-\phi_f^+(e^{-\mu}N) & -\phi_f^-
\end{bmatrix}
\begin{bmatrix}
w_1^+ \\
w_2^-
\end{bmatrix}.
\]
(21)

Combining (20) and (21) yields
\[
\begin{bmatrix}
f_1^L \\
f_1^R
\end{bmatrix} = D_N
\begin{bmatrix}
u_1^L \\
u_2^R
\end{bmatrix},
\]
(22)
where \(D_N\) is the dynamic stiffness matrix of the entire finite structure, given by
\[
D_N =
\begin{bmatrix}
D_{N11} & D_{N12} \\
D_{N21} & D_{N22}
\end{bmatrix} =
\begin{bmatrix}
\phi_f^+ & \phi_f^-(e^{-\mu}N) \\
-\phi_f^+(e^{-\mu}N) & -\phi_f^-
\end{bmatrix}
\begin{bmatrix}
\phi_u^+ & \phi_u^-(e^{-\mu}N) \\
\phi_u^+(e^{-\mu}N) & \phi_u^-
\end{bmatrix}^{-1}.
\]
(23)

Therefore, the vibration transmission function (transmittance) of the finite system can be derived from (22). Assume that the finite structure is excited by a harmonic force \(f_1^L\) at its left end and no external force is applied to its right end, i.e. \(f_1^R = 0\); then the vibration transmittance \(T = |u_1^R/u_1^L|\) of the finite periodic structure is given by
\[
T = |D_{N22}^{-1}D_{N21}|.
\]
(24)

3. Results and discussion

3.1. Illustrative examples

The purpose of this subsection is twofold: (i) to validate the methodology presented in the previous section and (ii) to illustrate the band gap characteristics of some simple metamaterial-based rod systems.

In the first example, we consider a system consisting of a uniform rod with periodically attached lumped masses and SDOF resonators, which has been investigated theoretically and experimentally in [25]. The results predicted by the present SE method are compared with those obtained in [25], where the classical transfer matrix (TM) method is employed. The parameters used in the simulation are the same as in [25], i.e. \(L = 0.05\) m, \(A = 50 \times 10^{-6}\) m\(^2\), \(E = 1.5 \times 10^{10}\) Pa, \(\rho = 1200\) kg m\(^{-3}\), \(m_0 = 0.016\) kg, \(m_1 = 0.0476\) kg and \(k_1 = 5.12 \times 10^6\) N m\(^{-1}\). The resonance frequency of the SDOF resonator is calculated to be \(f_1 = (1/2\pi) \times (k_1/m_1)^{1/2} = 1650\) Hz.

For comparison, the TM formulations are presented in appendix B, where the TM \(T\) of a unit cell of the periodic system is derived directly from the dynamic stiffness matrix \(D\) (see (8)) of the unit cell. The relationship between the matrices \(T\) and \(D\) is shown in (B.6). Under the TM framework, the propagation constants/complex band structure of an infinite periodic system are calculated by (B.8), while the vibration transmittance of a finite periodic structure is predicted by (B.11). We may notice that when dealing with an infinite periodic system there is little difference between the TM method and the SE approach presented in this paper (see section 2.1). However, a vast difference can be found in the treatment of a finite periodic structure. In the TM formulation, an entire TM \(T_N\), which is determined by a multiple multiplication of the unit-cell TM \(T\), is adopted for the calculation of vibration transmittance. Attention should be paid to the term \(D_0^N\) that will arise in \(T_N\) due to the matrix multiplication (see (B.10)). The resonant nature of the resonators implies that the dynamic stiffness, \(D_0\), is very
large around the resonance frequency, and thus the term $D_0^N$ can be extremely large in a frequency range around the resonance frequency if $N$ is big enough. As a consequence, numerical ill-conditioning may occur in the computational processes of the TM method when a large number of unit cells are involved in the finite structure. In contrast, the SE formulation presented in this paper (see section 2.2) is well conditioned because the dynamic stiffness matrix $D_N$ of the entire finite periodic structure, as presented in (23), is obtained by knowledge of the eigenvalues, $e^{-\mu}$, and the eigenvectors, $\phi^+_f$, $\phi^-_f$, $\phi^+_u$ and $\phi^-_u$. Note that the definition after (15) suggests that $\text{Re}(\mu) \geq 0$. Thus, the numerical computation can be easily performed even if $N$ is very large.

The numerical results are shown in figure 3, where the left panel depicts the propagation constants/complex band structure of the infinite system, and the right panel illustrates the vibration transmittance of the finite system. It can be seen that the results calculated by the SE method and the TM method agree well with each other. The comparative results confirm the validity of the methodology presented in this work. In figure 3, a typical asymmetric resonance gap can be observed with sharp attenuation at the resonance frequency (i.e. $f_1 = 1650$ Hz) of the SDOF resonator. Such observations are further verified by the experimental results of [25]. It should be mentioned that symmetric resonance gaps can also be achieved in this system by tuning the resonance frequency to some higher frequency ranges [28]. In addition, Bragg-type gaps, which have not received attention in [25], also exist in this system, and their coupling with the resonance gap will give rise to super-wide gaps [28], which may be of great interest in the applications of broadband vibration control. These aspects will be discussed in the next subsection.

In the second example, we consider the case of 2DOF resonator attachments. In the calculation, the parameters are chosen the same as used in the first example except that the SDOF resonator is now replaced by a 2DOF resonator (characterized by $k_1$, $k_2$, $m_1$ and $m_2$) with equivalent total mass (i.e. $m_1 + m_2 = 0.0476$ kg). The first and the second resonance frequencies of the 2DOF resonator are, respectively, tuned to $f_1 = 1650$ Hz and $f_2 = 3500$ Hz.

The resulting band gap properties are illustrated by the solid line in figure 4. It can be seen that two resonance gaps are achieved around the resonance frequencies. The results are

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**Figure 3.** Band gap characteristics of a metamaterial-based rod with a periodic array of SDOF resonators. (a) Propagation constants/complex band structure of the infinite system and (b) vibration transmittance of a finite system with eight unit cells.
also compared with two cases of SDOF resonator attachments (with the same total mass as the 2DOF resonators). The dotted line shows the band gap behavior corresponding to the first case of SDOF resonators (with the resonance frequency tuned to \( f_1 = 1650 \) Hz), while the dashed line depicts the second case (with the resonance frequency tuned to \( f_2 = 3500 \) Hz). It is shown that 2DOF resonators have an advantage over SDOF resonators in achieving two band gaps with great attenuation performance at two desired frequencies, although the band-gap width corresponding to the SDOF case is much broader.

3.2. Parametric influence study

In the previous subsection, we only consider several special examples with specifically chosen system parameters. However, in the practical design, it will be of great interest to know the effects of designing parameters on the band gap behavior, including the location, the width and the attenuation performance of all the band gaps. This is the topic of this subsection. Generally, the band gap behavior can be well represented by the attenuation constant, which can be mathematically described by a multivariable function:

\[
\text{Re}(\mu) = f(E, A, \rho, L, \omega, m_0, k_1, m_1, k_2, m_2, \ldots) = \text{Re}\{\text{acosh}[\cos(\beta L) + (\bar{D}/2) \sin(\beta L)]\}.
\] (25)

In what follows, we focus our attention on the effects of resonator parameters. The rod parameters are chosen the same as those used in the previous examples and it is assumed that \( m_0 = 0 \) and \( L = 0.05 \) m. In addition, we assume that the mass ratio of the resonators, which is defined as \( \gamma = (m_1 + m_2 + \ldots)/\rho AL \), has a constrained maximum value, which is always true in the practical design. The mass ratio of resonators will be fixed at \( \gamma = 0.5 \) in the following numerical examples.

First, we consider the simplest case, i.e. the metamaterial-based rod with a periodic array of SDOF resonators. The attenuation constant function can be written as \( \text{Re}(\mu) = f(\omega, k_1, m_1) \).
Figure 5. ACS of a metamaterial-based rod containing a periodic array of SDOF resonators. (a) 3D surface view, (b) planform view and (c) transformed planform view in the $\Omega-\Omega_1$ plane. The ridge observed in the figure shows that $K_1$ has a quadratic dependence on $\Omega$, while the dependence of $\Omega_1$ on $\Omega$ is linear.

Since the mass ratio $\gamma$ is fixed, the rod properties and the lattice constant are given, so that the mass $m_1$ is determined (i.e. $m_1 = \gamma \rho AL$). Hence, the attenuation constant can be described by a bi-variable function:

$$\text{Re}(\mu) = f(\omega, k_1).$$  \hspace{1cm} (26)

It should be noted that this could also be expressed in terms of other physical parameters, e.g. $\text{Re}(\mu) = f(\omega, \omega_1)$, where the relation $k_1 = m_1 \omega_1^2$ is employed. Using (26), the influence of the spring stiffness $k_1$ on the band gap behavior can be illustrated clearly by plotting the bi-variable function $f(\omega, k_1)$ in the $\omega-k_1$ plane, which leads to a so-called ‘attenuation constant surface (ACS)’ as introduced in [28]. For convenience, the spring stiffness can be nondimensionalized as $K_1 = k_1/(E A/L)$, and a nondimensional frequency is now defined as

$$\Omega = \frac{\beta L}{\pi} = \frac{L}{\pi} \sqrt{\frac{\rho}{E \omega}}.$$  \hspace{1cm} (27)

This is equal to the ratio of the resonator interval to half the wavelength of wave motion in the rod. Therefore, the nondimensional resonance frequency of the resonator can be written as

$$\Omega_1 = (L/\pi) \sqrt{\rho/E} \sqrt{k_1/m_1}.$$ 

In figure 5, different forms of ACS corresponding to the metamaterial-based rod with SDOF resonators are illustrated. Figure 5(a) presents a direct 3D description of the ACS, whereas figure 5(b) displays the planform view of the ACS. The colored region in figure 5(b) represents the band gap range with nonzero attenuation constants. It can be seen that there exist three band gaps in the frequency range considered (i.e. $0 < \Omega < 3$). The band gap with a sharp maximum attenuation constant should be classified as a resonance gap since its band edges can be tuned to any frequency range by varying the spring stiffness, while other band gaps can be categorized as Bragg-type gaps because one of their band edges is governed by the Bragg condition $L = n(\lambda/2)$ ($n = 1, 2, 3, \ldots$), which well explains those integral nondimensional band edge frequencies (i.e. $\Omega = 1, 2, 3, \ldots$). In figure 5(b), it is interesting to note that, under some special situations, i.e. $K_1 = K_a, K_b, \ldots$, the resonance gap and the nearest Bragg gap merge, with the bandwidth of the pass band between them becoming zero. Such a phenomenon is called the band gap coupling phenomenon, which gives rise to a coupled super-wide gap
behaving like a general Bragg gap with a smooth attenuation profile. Such a phenomenon has been examined in detail in [28] for a simple LR string system.

As illustrated in figure 5(b), the ACS depicted in the $\Omega-K_1$ plane provides a straightforward guideline for the parameter design. However, to extract more physical information from the ACS, it would be better to transform the ACS from the $\Omega-K_1$ plane into the $\Omega-\Omega_1$ plane. The transformed ACS is shown in figure 5(c). Some important features can be found in the new plot. Firstly, the maximum sharp attenuation within a resonance gap occurs at the resonance frequencies represented by the line $\Omega = \Omega_1$. Secondly, the band gap coupling phenomenon arises when the resonance frequency is tuned to $\Omega_1 = n$ ($n = 1, 2, 3, \ldots$), which yields a design formula for the spring stiffness:

$$k_1 = \frac{EA}{L} \frac{m_1}{\rho A L} n^2 \pi^2.$$  

(28)

This can also be expressed in a nondimensional form, i.e. $K_1 = \gamma n^2 \pi^2$, which exactly predicts that $K_a = \gamma \pi^2$ and $K_b = 4 \gamma \pi^2$.

In order to illustrate the unique attenuation performance of a coupled band gap, a numerical example is presented for the case $K_1 = K_a$. The attenuation constant of the infinite system and the vibration transmittance of a finite periodic structure are, respectively, shown in figures 6(a) and (b) by solid lines. To make a direct comparison, the results associated with a conventional periodic structure consisting of the same rod with periodic rigidly mounted lumped masses (with the mass ratio $\gamma = 0.5$) are also depicted in figure 6. We focus our attention on the first band gap. It is found that the bandwidth of the coupled band gap induced by the present carefully designed SDOF resonators is much broader than that of the conventional counterpart (i.e. the first Bragg gap illustrated by the dashed line in figure 6). Furthermore, the coupled gap starts at a lower frequency range, and the corresponding vibration attenuation is much more significant than that resulting from the conventionally engineered Bragg gap. Such a comparison indicates...
that the band gap coupling phenomenon can be utilized as a new technique for the broadband control of longitudinal vibrations and wave propagation in elastic rods.

At this step, we explore the second case, i.e. the metamaterial-based rod with a periodic array of 2DOF resonators. The attenuation constant function can therefore be expressed by $\text{Re}(\mu) = f(\omega, m_1, m_2, k_1, k_2)$. As assumed previously, the mass ratio is fixed at $\gamma = 0.5$, which determines the total mass $m_1 + m_2 = \gamma \rho AL$. Now we also assume that our purpose is to achieve two resonance band gaps covering two disturbing frequencies $\omega_1$ and $\omega_2$. Hence, the two resonance frequencies of the 2DOF resonators are supposed to be tuned to the frequencies $\omega_1$ and $\omega_2$. These assumptions imply that if one of the resonator parameters (i.e. $k_1$, $k_2$, $m_1$, $m_2$) is selected, the others are all determined by the knowledge of $m_1 + m_2$, $\omega_1$ and $\omega_2$. For example, if the value of $k_1$ is assigned, the other three parameters can be, respectively, given by

$$m_1 = \frac{k_1^2}{ak_1 - bc}, \quad m_2 = c - m_1, \quad k_2 = \frac{bm_1m_2}{k_1},$$

where

$$a = \omega_1^2 + \omega_2^2, \quad b = \omega_1^2\omega_2^2, \quad c = m_1 + m_2.$$

It should be mentioned that the mass constraint is implied in nature such that $0 < m_1 < c$. Therefore, the assignment of $k_1$ must be restricted within the region $(k_{1,\text{min}}, k_{1,\text{max}})$, where

$$k_{1,\text{min}} = \frac{a - \sqrt{(a^2 - 4b)}}{2} c, \quad k_{1,\text{max}} = \frac{a + \sqrt{(a^2 - 4b)}}{2} c.$$

As a consequence, the parametric influence analysis can also be performed by employing a bi-variable function:

$$\text{Re}(\mu) = f(\omega, k_1)|_{k_1 \in (k_{1,\text{min}}, k_{1,\text{max}})}.$$  

Figure 7 displays the ACSs corresponding to two examples of metamaterial-based rods with 2DOF resonators. In the first example, as shown in figure 7(a), the two resonance frequencies of the resonator are both tuned to be well below the first Bragg condition. Therefore,
two low-frequency resonance gaps around $\Omega_1$ and $\Omega_2$ are achieved, respectively. According to the ACS plot, the width of the two band gaps can be adjusted by varying the parameter $k_1$. The lower and upper limits (i.e. $K_1 = K_{1,\text{min}}$ and $K_{1,\text{max}}$), respectively, represent the case of the SDOF resonator with its single resonance frequency being tuned to $\Omega_1$ and $\Omega_2$.

In the second example, as presented in figure 7(b), one resonance frequency is set below the first Bragg condition, while the other is tuned precisely to the first Bragg condition (i.e. $\Omega_2 = 1$). It can be seen that the band gap coupling phenomenon occurs, giving rise to a super-wide resonance–Bragg coupled band gap. In order to illustrate this behavior more clearly, the associated attenuation constant curve for an infinite system and the vibration transmittance of a finite sample are, respectively, depicted in figures 8(a) and (b) by solid lines. For a comparison, the results associated with the case of SDOF resonator attachments (with the same mass ratio) are also presented, as depicted by the dotted lines in figure 8. The resonance frequency of the SDOF resonators is tuned to $\Omega_1 = 0.5$. It is found that, by employing the band gap coupling behavior, using 2DOF resonators to replace SDOF resonators can dramatically improve the band width and attenuation performance of the second gap around the Bragg condition, but has little influence on the first resonance gap (i.e. the gap around $\Omega_1$). This clearly indicates the advantage of 2DOF resonators, or more generally, MDOF resonators, over SDOF resonators in the control of multiband or broadband vibrations in rods utilizing band gap behavior.

### 3.3. Band gap formation mechanisms

This subsection is aimed at providing mathematical and physical descriptions of band edge frequencies. The outputs are expected to facilitate the understanding of band gap formation mechanisms, as well as to simplify the designing of band gaps in LR metamaterial-based rods at an initial stage. A detailed investigation on this topic for a simple LR string system can be
found in [28]. In what follows, only the main steps of formulation will be presented for the system considered here, since much of the derivation is similar to that proposed in [28].

For the system considered, free waves can propagate provided \( \mu \) is pure imaginary, which requires \(-1 \leq \cosh(\mu) \leq 1\). Therefore the edge frequencies of the band gaps are given by

\[
\cosh(\mu) = \pm 1. \tag{33}
\]

By inserting (13) and performing some manipulations we can obtain two groups of edge frequencies. These can be denoted as mode-A and mode-B frequencies, respectively. The mode-A frequencies are given by

\[
\sin \frac{\beta L}{2} = 0 \tag{34}
\]

or

\[
\tilde{D} = -2 \cot \frac{\beta L}{2}. \tag{35}
\]

Similarly, the mode-B frequencies are determined by

\[
\cos \frac{\beta L}{2} = 0 \tag{36}
\]

or

\[
\tilde{D} = 2 \tan \frac{\beta L}{2}. \tag{37}
\]

By analyzing the free vibrations of a symmetric unit cell of the infinite system with fixed–fixed or free–free end conditions, as sketched in figure 9, one can find that the mode-A frequencies can be interpreted by the natural frequencies of the unit cell with fixed–fixed ends, while the mode-B edge frequencies can be explained by those of the unit cell with free–free ends [28].

Note that the solutions of (34) and (36) simply represent the Bragg conditions

\[
\Omega = \frac{\beta L}{\pi} = n \quad (n = 1, 2, 3, \ldots). \tag{38}
\]

These frequencies actually represent the natural frequencies corresponding to the antisymmetric modes of the unit cells. It should be noted that such frequencies are identical to the natural frequencies associated with the antisymmetric modes of the bare unit-cell rod (i.e. the rod without attachments). For instance, consider a fixed–fixed bare rod. Its natural frequencies are given by \( \beta_n L = n\pi \) \((n = 1, 2, 3, \ldots)\), and the normal modes (or mode shapes) are described by a spatial function \( Y_n(x) = \sin \beta_n x \) \((0 \leq x \leq L)\) [38]. It is noted that the modes for \( n = 1, 3, 5, \ldots \) give symmetric motion, while antisymmetric modes result from \( n = 2, 4, 6, \ldots \) [38]. Moreover, it is evident that the even-order natural frequencies, i.e. the frequencies for \( n = 2, 4, 6, \ldots \), are identical to the solutions of (34), and they represent the even-order Bragg conditions (i.e. \( \Omega = 2, 4, 6, \ldots \)) presented in (38). For a free–free bare rod,
its natural frequencies are also given by $\beta_n L = n\pi$ \((n = 1, 2, 3, \ldots)\), but the normal modes are governed by the function $Y_n(x) = \cos \beta_n x$ \((0 \leq x \leq L)\) \([38]\). This implies that the antisymmetric modes are associated with \(n = 1, 3, 5, \ldots\). We further note that the corresponding natural frequencies, given by $\beta_n L = n\pi$ \((n = 1, 3, 5, \ldots)\), are indeed the same as the solutions of \((36)\), and they represent the odd-order Bragg conditions (i.e. $\Omega = 1, 3, 5, \ldots$) shown in \((38)\).

The band edge frequencies determined by \((34)\) and \((36)\), as have been known as the natural frequencies corresponding to the antisymmetric modes of the unit cells, have no dependence on the resonator parameters. This is because the attachment point at the midspan of the unit cell becomes a node of such antisymmetric modes; thus the attached resonator does not affect the motion of the unit-cell rod. However, the other band edge frequencies, as governed by \((35)\) and \((37)\), do relate to the resonator parameters, since they represent the natural frequencies relating to the symmetric modes of the unit cells, in which the motion of the resonator and that of the unit-cell rod are coupled with each other.

We reconsider the planform plots of ACS shown in figures 5 and 7. The boundary curves separating the colored and white regions in these figures actually represent the map of band edge frequencies; thus they must be governed by \((34)–(37)\). The solutions to \((34)\) and \((36)\) are straightforward, which respectively represent the even (i.e. $\Omega = 2, 4, 6, \ldots$) and odd (i.e. $\Omega = 1, 3, 5, \ldots$) order Bragg frequencies. By substituting the expression for $\bar{D}$, \((35)\) and \((37)\) can be further described by explicit relations between $k_1$ and $\omega$. For example, for SDOF resonators, the expression for $\bar{D}$ can be written as

$$\bar{D} = \frac{1}{EA\beta} \frac{-\omega^2 k_1 m_1}{k_1 - \omega^2 m_1}. \quad (39)$$

Substituting this into \((35)\) yields

$$k_1 = \left[ \frac{1}{\omega^2 m_1} - \frac{1}{2EA\beta \cot(\beta L/2)} \right]^{-1}. \quad (40)$$

Combining \((40)\) with the solutions to \((34)\) and employing nondimensional parameters leads to the mode-A band edge frequency equations:

$$K_1 = \left[ \frac{1}{\gamma \pi^2 \Omega^2} - \frac{1}{2\pi \Omega \cot(\pi \Omega/2)} \right]^{-1}. \quad (41a, b)$$

$$\Omega = 2, 4, 6, \ldots$$

Similarly, we can obtain the mode-B band edge frequency equations:

$$K_1 = \left[ \frac{1}{\gamma \pi^2 \Omega^2} + \frac{1}{2\pi \Omega \tan(\pi \Omega/2)} \right]^{-1}. \quad (42a, b)$$

$$\Omega = 1, 3, 5, \ldots$$

By substituting the relation $K_1 = \gamma \pi^2 \Omega^2$, \((41)\) and \((42)\) can be respectively reformulated as

$$\Omega = 2, 4, 6, \ldots$$

$$\left\{ \begin{array}{l}
\Omega = \left[ \frac{1}{\Omega^2} - \frac{\gamma \pi}{2 \Omega \cot(\pi \Omega/2)} \right]^{-1/2}, \\
\Omega = 2, 4, 6, \ldots
\end{array} \right. \quad (43a, b)$$
Figure 10. Band edge frequency curves for the case of SDOF resonators, plotted in the $\Omega-K_1$ plane (a) and in the $\Omega-\Omega_1$ plane (b). The parameters used in the calculation are the same as those in figure 5.

and

$$\begin{align*}
\Omega_1 &= \left[ \frac{1}{\Omega^2} + \frac{\gamma \pi}{2 \Omega \tan(\pi \Omega/2)} \right]^{-1/2}, \\
\Omega &= 1, 3, 5, \ldots .
\end{align*}$$

(44a, b)

For the case of 2DOF resonators, we can derive the band edge frequency equations in a similar way. The associated mode-A and mode-B equations are, respectively, given by

$$\begin{align*}
K_1 &= -2\Omega_1 \pi \cot \frac{\Omega_1 \pi}{2} \left( 1 - \frac{\Omega_1^2 + \Omega_2^2}{\Omega^2} + \frac{\Omega_1^2 \Omega_2^2}{\Omega^4} \right) + \gamma \frac{\pi^2 \Omega_1^2 \Omega_2^2}{\Omega^2}, \\
\Omega &= 2, 4, 6, \ldots
\end{align*}$$

(45a, b)

and

$$\begin{align*}
K_1 &= 2\Omega_1 \pi \tan \frac{\Omega_1 \pi}{2} \left( 1 - \frac{\Omega_1^2 + \Omega_2^2}{\Omega^2} + \frac{\Omega_1^2 \Omega_2^2}{\Omega^4} \right) + \gamma \frac{\pi^2 \Omega_1^2 \Omega_2^2}{\Omega^2} .
\end{align*}$$

(46a, b)

In order to verify the above derivations, numerical examples for the cases of SDOF resonators and 2DOF resonators are, respectively, presented in figures 10 and 11. As expected, the band edge frequency curves predicted analytically by (41)–(46) agree well with those boundary curves shown in figures 5 and 7. In addition, the physical significance of all the corresponding band edge frequencies is now clarified. It is demonstrated that each band gap is bounded by a mode-A and mode-B frequency, respectively. Generally, the band edges of Bragg gaps are, respectively, governed by an antisymmetric mode and a symmetric mode, whereas those of resonance gaps are both determined by symmetric modes. Furthermore, it can be seen in figure 10 that the band gap coupling phenomenon is due to the overlap of an antisymmetric mode and a symmetric mode.
Figure 11. Band edge frequency curves for the case of 2DOF resonators. (a) $\Omega_1 = 0.2$, $\Omega_2 = 0.3$; (b) $\Omega_1 = 0.5$, $\Omega_2 = 1$. The other system parameters used in the calculation are the same as those in figure 7.

4. Conclusions

In summary, this paper dealt with wave propagation and vibration transmission in LR metamaterial-based rod systems consisting of uniform rods with a periodic array of MDOF resonators. An analytical methodology based on a combination of the SE method and the periodic structure theory was developed to predict the propagation constants and thus the complex band structure of infinite periodic systems, as well as the vibration transmittance of a finite periodic structure. The validity of this methodology was confirmed by a comparison with the traditional TM method.

Numerical results showed that both Bragg-type and resonance-type band gaps exist in metamaterial-based rod systems. It was found that multiple desired resonance gaps can be achieved in metamaterial-based rods containing MDOF resonators. We demonstrated that the bandwidth and the attenuation performance of gaps around Bragg conditions can be dramatically improved by employing the band gap coupling behavior between resonance and Bragg gaps. The effects of resonator parameters on the band gap behavior, including the location, the width and the attenuation performance of all the band gaps, were studied by plotting the attenuation constant surfaces. In order to understand the band gap behavior of metamaterial-based rod systems, both mathematical derivations and physical models were provided to explain the band gap formation mechanisms. Explicit formulae were further derived to predict all the band edge frequencies in an exact manner without the need to calculate band structures. The results presented in this paper will facilitate the design and understanding of band gap behavior in metamaterial-based elastic rods containing arrays of local resonators.

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Appendix A. Explicit expression for $D_0$

(a) SDOF resonator:
\[ D_{\text{sdof}} = -\frac{\omega^2 k_1 m_1}{k_1 - \omega^2 m_1}. \]  
(A.1)

(b) 2DOF resonator:
\[ D_{\text{2dof}} = \frac{k_1 m_1 m_2 \omega^4 - k_1 k_2 (m_1 + m_2) \omega^2}{m_1 m_2 \omega^4 - (m_1 k_2 + m_2 (k_1 + k_2)) \omega^2 + k_1 k_2}. \]  
(A.2)

(c) 3DOF resonator:
\[ D_{\text{3dof}} = k_1 + \frac{k_1^2 k_2^2 m_2^2 - k_1 k_2 [k_3 (m_1 + m_2) + k_1 k_2 m_3] + k_3^2 (m_1 + m_2 + m_3)}{a_1 \omega^6 + a_2 \omega^4 + a_3 \omega^2 + a_4}, \]  
(A.3)
where
\[ a_1 = m_1 m_2 m_3, \quad a_2 = -k_2 m_3 - k_2 m_3 (m_1 + m_2) - k_3 m_1 (m_2 + m_3), \]
\[ a_3 = k_2 k_3 (m_1 + m_2 + m_3) + k_1 [k_2 m_3 + k_3 (m_2 + m_3)], \quad a_4 = -k_1 k_2 k_3. \]  
(A.4)

Appendix B. The transfer matrix method

From (7), we obtain the following relation:
\[ \begin{bmatrix} u_R \\ f_R \end{bmatrix}_{n+1} = U \begin{bmatrix} u_L \\ f_L \end{bmatrix}_{n+1} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} u_L \\ f_L \end{bmatrix}_{n+1}, \]  
(B.1)
where
\[ U_{11} = -D_{LR}^{-1} D_{LL}, \quad U_{12} = D_{LR}^{-1}, \]
\[ U_{21} = D_{RL} - D_{RR} D_{LR}^{-1} D_{LL}, \quad U_{22} = D_{RR} D_{LR}^{-1}. \]  
(B.2)

The subscript ‘$n + 1$’ in (B.1) denotes the ($n + 1$)th unit cell. Imposing continuity and compatibility at the left boundary of the unit cell yields
\[ \begin{bmatrix} u_L \\ f_L \end{bmatrix}_{n+1} = \begin{bmatrix} u_R \\ -f_R \end{bmatrix}_{n} = J \begin{bmatrix} u_R \\ f_R \end{bmatrix}_{n}, \]  
(B.3)
where
\[ J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \]  
(B.4)

Combining (B.1) and (B.3) gives
\[ \begin{bmatrix} u_R \\ f_R \end{bmatrix}_{n+1} = T \begin{bmatrix} u_R \\ f_R \end{bmatrix}_{n}, \]  
(B.5)
where $T = \mathbf{U} \cdot \mathbf{J}$ is referred to as the TM for the unit cell, and $\{u_R, f_R\}^T$ is considered as a state vector. The explicit expression for the TM $T$ is given by

$$T = \begin{bmatrix} -D_{LR}^{-1}D_{LL} & -D_{LR}^{-1} \\ D_{RL} - D_{RR}D_{LR}^{-1}D_{LL} & -D_{RR}D_{LR}^{-1} \end{bmatrix} = \begin{bmatrix} \cos(\beta L) + (D_0/EA\beta) \sin(\beta L) & (EA\beta)^{-1} \sin(\beta L) \\ -EA\beta \sin(\beta L) + D_0 \cos(\beta L) & \cos(\beta L) \end{bmatrix}. \quad (B.6)$$

For an infinite periodic system, the Bloch theorem guarantees that $\{u_R, f_R\}_{n+1} = e^{\mu} \{u_R, f_R\}_n$. Subtracting (B.7) from (B.5) yields the following eigenvalue problem:

$$T \begin{bmatrix} u_R \\ f_R \end{bmatrix}_n = e^{\mu} \begin{bmatrix} u_R \\ f_R \end{bmatrix}_n. \quad (B.8)$$

Such an eigenvalue problem can be solved numerically. And further, we may derive an explicit solution such as that presented in (13) by using the equation $|T - e^{\mu}I| = 0$.

As to a finite periodic system with $N$ unit cells, the state vectors at the beginning and the end of the finite structure are related by using (B.5), i.e.

$$\begin{bmatrix} u_R \\ f_R \end{bmatrix}_N = T_N \begin{bmatrix} u_R \\ f_R \end{bmatrix}_0 = T_N \begin{bmatrix} u_L \\ -f_L \end{bmatrix}, \quad (B.9)$$

where

$$T_N = (T)^N = \begin{bmatrix} T_{N11} & T_{N12} \\ T_{N21} & T_{N22} \end{bmatrix}. \quad (B.10)$$

The matrix $T_N$ is referred to as the entire TM for the finite periodic structure. Therefore, the vibration transmittance of the finite periodic structure is given by

$$T = |u_R^N/u_L^1| = |T_{N11} - T_{N12}T_{N22}^{-1}T_{N21}|. \quad (B.11)$$

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