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TITS ENDOMORPHISMS AND BUILDINGS OF TYPE $F_4$

by Tom DE MEDTS, Yoav SEGEV & Richard M. WEISS

Abstract. — The fixed point building of a polarity of a Moufang quadrangle of type $F_4$ is a Moufang set, as is the fixed point building of a semi-linear automorphism of order 2 of a Moufang octagon that stabilizes at least two panels of one type but none of the other. We show that these two classes of Moufang sets are, in fact, the same, that each member of this class can be constructed as the fixed point building of a group of order 4 acting on a building of type $F_4$ and that the group generated by all the root groups of any one of these Moufang sets is simple.

Résumé. — L’immeuble de points fixes d’une polarité d’un quadrangle de Moufang de type $F_4$ est un ensemble de Moufang. Il en va de même pour l’immeuble de points fixes d’un automorphisme semi-linéaire d’ordre 2 d’un octogone de Moufang qui stabilise au moins deux cloisons d’un type mais aucun de l’autre. Nous montrons que ces deux classes d’ensembles de Moufang sont en fait identiques, que chaque membre de cette classe peut être construit comme l’immeuble de points fixes d’un groupe d’ordre 4 agissant sur un immeuble de type $F_4$, et que pour chacun de ces ensembles de Moufang, le groupe engendré par tous les sous-groupes radiciels est un groupe simple.

1. Introduction

The notion of a building was introduced by J. Tits in order to give a uniform geometric/combinatorial description of the groups of rational points of an isotropic absolutely simple group. The buildings that arise in this context are spherical. In [19], Tits classified irreducible spherical buildings of rank at least 3 and this classification was extended to the rank 2 case in [23] under the assumption that the building is Moufang (which is automatic when the rank is at least 3).

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The result of this classification is that most Moufang buildings (as defined in Definition 2.5) are the spherical buildings associated with isotropic absolutely simple algebraic groups. The exceptions are buildings determined by algebraic data involving infinite dimensional structures, defective quadratic or pseudo-quadratic forms, inseparable field extension and/or the square root of a Frobenius endomorphism. Among these exceptions are the mixed buildings of type $B_2$, $G_2$ and $F_4$, the Moufang quadrangles of type $F_4$ and the Moufang octagons.

The classification results in [19] and [23] are proved by coordinatizing with an appropriate algebraic structure. These methods do not reveal the connection between the automorphism group of a Moufang building and an associated split group which is the central concern in the theory of Galois descent. In [11], this shortcoming is remedied with a theory of descent for buildings. This theory provides, in particular, a combinatorial interpretation of the Tits indices which appear in [18].

In Definition 2.10, we define the notion of a descent group of a building $\Delta$ and in Definition 3.9, we define the notion of a Galois subgroup of $\text{Aut}(\Delta)$ in the case that $\Delta$ is Moufang. One of the fundamental results in the theory of Galois descent in buildings in [11] says that the set of residues fixed by a descent group $\Gamma$ has, in a canonical way, the structure of a building, which we call the fixed point building of $\Gamma$ (see Theorem 2.18). This result applies to arbitrary descent groups acting on arbitrary buildings. A second fundamental result (proved in [15]) says that if $\Delta$ is Moufang and $\Gamma$ is a Galois subgroup of $\text{Aut}(\Delta)$ acting with finite orbits on $\Delta$ and stabilizing at least one proper residue of $\Delta$, then $\Gamma$ is a descent group.

A third fundamental result says that if the fixed point building of a descent group of a Moufang building $\Delta$ has rank 1, then the fixed point building inherits from the Moufang condition on $\Delta$ the structure of a Moufang set. Moufang sets, which were first introduced in [21, 4.4], are a class of 2-transitive permutation groups. The notion of a Moufang set is closely related to the notion of a split $BN$-pair of rank 1. All known Moufang sets which are proper (i.e. not sharply 2-transitive) arise as fixed point buildings of Moufang buildings. For a survey of recent results in the study of Moufang sets, see [4].

Polarities (i.e. non-type-preserving automorphisms of order 2) of Moufang buildings of type $B_2$, $G_2$ and $F_4$ are a second source of descent groups. In the case that the building is pseudo-split (as defined in [11, 28.16]), polarities give rise to the Suzuki and Ree groups. In [20], Tits characterized
Moufang octagons as the fixed point building of an arbitrary polarity of a building of type $F_4$.

The Moufang quadrangles of type $F_4$ (a class of non-pseudo-split Moufang buildings of type $B_2$) were discovered in the course of classifying Moufang buildings of rank 2 in [23]. Subsequently, it was shown (in [12]) that these quadrangles can be constructed as the fixed point building of a type-preserving Galois involution acting on a building of type $F_4$. (It was due to this result that the designation “of type $F_4$” was chosen in [23].)

Among all non-pseudo-split Moufang buildings of type $B_2$, $G_2$ or $F_4$, the Moufang quadrangles of type $F_4$ are the only ones that can have a polarity. If a Moufang quadrangle $\Xi$ of type $F_4$ has a polarity $\rho$, then the centralizer of $\rho$ in $\text{Aut}(\Xi)$ induces a Moufang set on the set $\Xi^{(\rho)}$ of chambers fixed by $\rho$. The first examples of such Moufang sets were constructed in [13]. It is also possible for a Moufang octagon $\Omega$ to have a type-preserving Galois involution $\kappa$ such that the centralizer of $\kappa$ in $\text{Aut}(\Omega)$ induces a Moufang set on the set $\Omega^{(\kappa)}$ of panels fixed by $\kappa$. In [13], it was conjectured that these two classes of Moufang sets are the same. Our main goal here is to prove this conjecture.

In the course of verifying this conjecture, we show that each of these Moufang sets is, in fact, the fixed point building of an elementary abelian subgroup $\Gamma$ of order 4 of the automorphism group of a building $\Delta$ of type $F_4$. The group $\Gamma$ is not type-preserving. For this reason, we say (in Definition 4.5) that these Moufang sets are of outer $F_4$-type. The fixed point building of each of the two polarities in $\Gamma$ is a Moufang octagon (but the two octagons are not, in general, isomorphic to each other) and the fixed point building of the third involution in $\Gamma$ is a Moufang quadrangle of type $F_4$. This lattice of fixed point buildings is all the more interesting in light of the fact that there is no obvious connection between an arbitrary Moufang quadrangle of type $F_4$ and an arbitrary Moufang octagon except that they both “descend from” a building of type $F_4$ in characteristic 2.

The root groups of a Moufang set of outer $F_4$-type are nilpotent of class 3 as are the root groups in the Ree groups of type $G_2$. In every other known proper Moufang set, the root groups are either abelian or nilpotent of class 2.

In Sections 15–19, we show that the group generated by all the root groups of a Moufang set of outer $F_4$-type is simple and describe a number of other properties of these Moufang sets.

The basic invariant of a Moufang quadrangle $\Xi$ of type $F_4$ is a pair of quadratic forms $q$ and $\hat{q}$ of type $F_4$ (as defined in Definition 5.3 below), one
defined over a field $K$ of characteristic 2 and the other over a field $F$ such that $F^2 \subset K \subset F$. The quadratic forms $q$ and $\hat{q}$ are anisotropic but have a defect of dimension $\dim_{K^2} F$, respectively, $\dim_{F} K$. (We do not make any assumptions about these dimensions; in particular, either one or both can be infinite.)

Let $V$ denote the vector space over $K$ on which $q$ is defined. We show that if the Moufang quadrangle $\Xi$ has a polarity $\rho$, then the quadratic forms $q$ and $\hat{q}$ are similar to each other and there is a Tits endomorphism $\theta$ of $K$ (i.e. an endomorphism whose square is the Frobenius endomorphism) with image $F$ and a non-associative algebra structure on $V$ with respect to which the quadratic form $q$ satisfies the identity
\[(1.1) \quad q(uv) = q(u)q(v)^\theta\]
for all $u, v \in V$ (see Propositions 6.23 and 7.9). Thus $q$ is multiplicative “with a twist” (cf. [7], for instance).

We call the non-associative algebras that arise in this way polarity algebras. In Section 7, we describe polarity algebras in terms of a system of axioms and deduce from these axioms equation (1.1) along with a number of other intriguing identities. In Theorem 19.1, we use some of these identities to show that $q$ is, up to similarity, an invariant not only of the quadrangle $\Xi$, but also of the Moufang set $\Xi^{(\rho)}$. See also [17].

Tits endomorphisms and their extensions play a central role in the study of polarities. The first thorough study of this connection is in [22]; in fact, it was the influence of this paper which led to the common attribution of these endomorphisms to Tits. See, in particular, Section 8.

**Convention 1.2.** — If $x$ and $y$ are two elements of a group, we set $x^y = y^{-1}xy$ and
\[[x, y] = x^{-1}y^{-1}xy = (y^{-1})^x y = x^{-1}x^y\]
(as in [23]). As a consequence of these conventions, the following two identities
\[(1.3) \quad [xy, z] = [x, z]^y \cdot [y, z] \quad \text{and} \quad [x, yz] = [x, z] \cdot [x, y]^z\]
hold.

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2. Fixed Point Buildings and Moufang Sets

Before we can give precise formulations of our main results in Section 4, we need to introduce some basic notions.

**Definition 2.1.** — Let $E$ be a field of positive characteristic $p$. We will denote the Frobenius endomorphism $x \mapsto x^p$ of $E$ by $\text{Frob}_E$. A Tits endomorphism of $E$ is an endomorphism of $E$ whose square is the Frobenius endomorphism. An octagonal set is a pair $(E, \theta)$, where $E$ is a field of characteristic 2 and $\theta$ is a Tits endomorphism of $E$.

**Definition 2.2.** — Let $\Delta$ be a building. An involution of $\Delta$ is an automorphism of order 2. A polarity of $\Delta$ is an involution which is not type-preserving. (We will only use this term when $\Delta$ is of type $B_2$, $F_4$ or $G_2$.)

**Notation 2.3.** — Let $(E, \theta)$ be an octagonal set. We denote by $O(E, \theta)$ the Moufang octagon defined in [23, 16.9] and we denote by $F_4(E, \theta)$ the building of type $F_4$ called $F_4(E, \theta)$ in [26, 30.15].

**Definition 2.4.** — A Moufang set is a pair $(X, \{U_x\}_{x \in X})$, where $X$ is a set of cardinality at least 3, $U_x$ is a subgroup of $\text{Sym}(X)$ fixing $x$ and acting sharply transitively on $X \setminus \{x\}$ for all $x \in X$ and $g^{-1}U_xg = U_{(x)g}$ for all $x \in X$ and all $g \in G^\dagger := \langle U_x \mid x \in X \rangle$. The subgroups $U_x$ are called the root groups of the Moufang set. A Moufang set is proper if the group $G^\dagger$ does not act sharply 2-transitively on $X$.

**Definition 2.5.** — As in [24, 11.2], we call a building Moufang if it is spherical, irreducible and of rank at least 2 and for each root $\alpha$, the root group $U_\alpha$ (as defined in [24, 11.1]) acts transitively on the set of apartments containing $\alpha$. (There are more general notions of a Moufang building, but they are not relevant in this paper.)

**Proposition 2.6.** — Let $\kappa$ be an involution acting on a spherical building $\Delta$. Then there exists an apartment of $\Delta$ stabilized by $\kappa$.

**Proof.** — This holds by [11, 25.15].

**Definition 2.7.** — Let $\Delta$ be a building and let $\Gamma$ be a subgroup of $\text{Aut}(\Delta)$. A $\Gamma$-residue is a residue of $\Delta$ stabilized by $\Gamma$. A $\Gamma$-chamber is a $\Gamma$-residue which is minimal with respect to inclusion. A $\Gamma$-panel is a $\Gamma$-residue $P$ such that for some $\Gamma$-chamber $C$, $P$ is minimal in the set of all $\Gamma$-residues containing $C$.

**Definition 2.8.** — Let $\Delta$ and $\Gamma$ be as in Definition 2.7. A residue of $\Delta$ is proper if it is different from $\Delta$ itself. (In particular, chambers are proper.
residues.) The group $\Gamma$ is anisotropic if $\Gamma$ stabilizes no proper residues of $\Delta$, and $\Gamma$ is isotropic if this is not the case. Thus $\Gamma$ is isotropic if and only if there exist $\Gamma$-panels. An automorphism $\xi$ of $\Delta$ is isotropic (or anisotropic) if $\langle \xi \rangle$ is isotropic (or anisotropic).

Notation 2.9. — Let $\Delta$ be a building and let $\Gamma$ be an isotropic subgroup of $\text{Aut}(\Delta)$. We denote by $\Delta^\Gamma$ the graph with vertex set the set of all $\Gamma$-chambers, where two $\Gamma$-chambers are joined by an edge of $\Delta^\Gamma$ if and only if there is a $\Gamma$-panel containing them both.

Definition 2.10. — Let $\Delta$ be a building. A descent group of $\Delta$ is an isotropic subgroup $\Gamma$ of $\text{Aut}(\Delta)$ such that each $\Gamma$-panel contains at least three $\Gamma$-chambers.

Proposition 2.11. — Suppose that a building $\Delta$ is Moufang as defined in Definition 2.5. Let $R$ be a residue of $\Delta$, let $\Sigma$ be an apartment containing chambers of $R$ and let $U_R$ denote the subgroup generated by the root groups $U_\alpha$ for all roots $\alpha$ of $\Sigma$ containing $R \cap \Sigma$. Then $U_R$ is independent of the choice of $\Sigma$.

Proof. — This holds by [11, 24.17]

Definition 2.12. — Let $R$ and $\Delta$ be as in Proposition 2.11. The group $U_R$ is called the unipotent radical of $R$.

Proposition 2.13. — Let $R$, $\Delta$ and $U_R$ be as in Definition 2.12. Then $U_R$ acts sharply transitively on the residues of $\Delta$ opposite $R$

Proof. — This holds by [11, 24.21].

Definition 2.14. — Let $\Pi$ be a Coxeter diagram and let $(W,S)$ be the corresponding Coxeter system. Thus $S$ is both a distinguished set of generators of the Coxeter group $W$ and the vertex set of $\Pi$. Let $J$ be a subset of $S$ such that the subdiagram $\Pi_J$ spanned by $J$ is spherical and let $w_J$ denote the longest element of the Coxeter system $(W_J,J)$, where $W_J$ denotes the subgroup of $W$ generated by $J$. By [24, 5.11], the map $s \mapsto w_J s w_J$ is an automorphism of $\Pi$. We denote this automorphism by $\text{op}_J$. This map is called the opposite map of the diagram $\Pi_J$ (and ought, in fact, to be denoted by $\text{op}_{\Pi_J}$). This map stabilizes every connected component of $\Pi_J$ and acts non-trivially on a given connected component if and only if it is isomorphic to the Coxeter diagram $A_n$ for some $n \geq 2$, to $D_n$ for some odd $n \geq 5$, to $E_6$ or to $I_2(n)$ for some odd $n \geq 5$. In particular, its order is at most 2.
Definition 2.15. — A Tits index is a triple \((\Pi, \Theta, A)\) where \(\Pi\) is a Coxeter diagram, \(\Theta\) is a subgroup of \(\text{Aut}(\Pi)\) and \(A\) is a \(\Theta\)-invariant subset of the vertex set \(S\) of \(\Pi\) such that for each \(s \in S \setminus A\), the subdiagram of \(\Pi\) spanned by \(A \cup \Theta(s)\) is spherical and \(A\) is invariant under the opposite map \(\text{op}_{A \cup \Theta(s)}\) defined in Definition 2.14. Here \(\Theta(s)\) denotes the \(\Theta\)-orbit containing \(s\).

Definition 2.16. — Let \(T = (\Pi, \Theta, A)\) be a Tits index. For each \(s \in S \setminus A\), let \(\tilde{s} = w_A w_{A \cup \Theta(s)}\), where \(w_J\) for \(J = A\) and \(J = A \cup \Theta(s)\) is as in Definition 2.14. Thus there is one element \(\tilde{s}\) for each \(\Theta\)-orbit in \(S \setminus A\). Let \(\tilde{S}\) be the set of all these elements \(\tilde{s}\). By [11, 20.32], \((\tilde{W}, \tilde{S})\) is a Coxeter system. Let \(\tilde{\Pi}\) be the corresponding Coxeter diagram. We call \(\Pi\) the absolute Coxeter diagram of \(T\) and \(\tilde{\Pi}\) the relative Coxeter diagram of \(T\). An algorithm for calculating the relative Coxeter diagram of a Tits index is described in [23, 42.3.5(c)].

Example 2.17. — Let \(T = (\Pi, \Theta, A)\), where \(\Pi\) is the Coxeter diagram \(F_4\). If \(\Theta = \text{Aut}(\Pi)\) and \(A = \emptyset\), then \(T\) is a Tits index with relative Coxeter diagram is \(I_2(8)\). If \(\Theta\) is trivial and \(A = \{2, 3\}\) with respect to the standard numbering of the vertex set of \(\Pi\), then \(T\) is a Tits index with relative Coxeter diagram \(B_2\).

Theorem 2.18. — Let \(\Gamma\) be a descent group of a building \(\Delta\). Let \(\Pi\) be the Coxeter diagram of \(\Delta\), let \(S\) denote the vertex set of \(\Pi\) and let \(\Theta\) denote the subgroup of \(\text{Aut}(\Pi)\) induced by \(\Gamma\). Then the following hold:

1. The graph \(\Delta^\Gamma\) defined in Notation 2.9 is a building with respect to a canonical coloring of its edges.
2. All \(\Gamma\)-chambers are residues of \(\Delta\) of the same type \(A \subset S\) and the rank of \(\Delta^\Gamma\) is the number of \(\Theta\)-orbits in \(S\) disjoint from \(A\).
3. If \(A\) is spherical, then the triple \(T := (\Pi, \Theta, A)\) is a Tits index and \(\Delta^\Gamma\) is a building of type \(\tilde{\Pi}\), where \(\tilde{\Pi}\) is the relative Coxeter diagram of \(T\).
4. If \(\Delta\) is Moufang and the rank of \(\Delta^\Gamma\) is at least 2, then \(\Delta^\Gamma\) is also Moufang.
5. Suppose that \(\Delta\) is Moufang and that the rank of \(\Delta^\Gamma\) is 1 and let \(X\) be the set of all \(\Gamma\)-chambers, For each \(C \in X\), let \(\tilde{U}_C\) denote the subgroup of \(\text{Sym}(X)\) induced by the centralizer \(C_{U_C}(\Gamma)\) of \(\Gamma\) in the unipotent radical \(U_C\). Then

\[ (X, \{\tilde{U}_C \mid C \in X\}) \]

is a Moufang set.
Proof. — Assertions (1) and (2) hold by [11, 22.20(v) and (viii)], assertion (3) holds by [11, 22.20(iv) and (viii)] and the remaining two assertions hold by [11, 24.31]. □

Definition 2.19. — The building \( \Delta^\Gamma \) in 2.18(1) is called the fixed point building of \( \Gamma \). The rank of \( \Delta^\Gamma \) is called the relative rank of \( \Gamma \). If the relative rank of \( \Gamma \) is 1, we interpret \( \Delta^\Gamma \) to mean the Moufang set described in 2.18(5).

3. Polarities and Galois subgroups

In this section we describe two ways in which descent groups arise.

Proposition 3.1. — Let \( \Delta \) be a Moufang building of type \( \Pi \) where \( \Pi \) is the Coxeter diagram \( B_2 \), \( G_2 \) or \( F_4 \). Suppose that \( \sigma \) is a polarity of \( \Delta \) as defined in Definition 2.2 and let \( \Gamma = \langle \sigma \rangle \). Then \( \Gamma \) is a descent group of \( \Delta \) and \( \Gamma \)-chambers are chambers of \( \Delta \). If \( \Pi = B_2 \) or \( G_2 \), the fixed point building \( \Delta^\Gamma \) is a Moufang set and if \( \Pi = F_4 \), the fixed point building \( \Delta^\Gamma \) is a Moufang octagon.

Proof. — By Proposition 2.6, we can choose an apartment \( \Sigma \) stabilized by \( \sigma \). By Definition 2.14, the automorphism of \( \Sigma \) sending each chamber to its unique opposite is color-preserving. By [11, 25.17], therefore, there exists a \( \Gamma \)-residue \( R \) containing chambers of \( \Sigma \). We can assume that \( R \) is minimal with respect to containment.

Since \( \Gamma \) is not type-preserving, the type \( J \) of \( R \) is \( \Theta \)-invariant, where \( \Theta \) is the automorphism group of \( \Pi \). Suppose that \( J \neq \emptyset \). Since \( J \) is \( \Theta \)-invariant, we must have \( \Pi = F_4 \) and \( J \) is either \( \{1, 4\} \) or \( \{2, 3\} \) (with respect to the standard numbering of the vertex set of the Coxeter diagram \( F_4 \)). Thus \( R \cap \Sigma \) is a thin building of type \( A_1 \times A_1 \) or \( B_2 \). By Definition 2.14, the map which sends each chamber of \( R \cap \Sigma \) to its unique opposite is again color-preserving. Another application of [11, 25.17] thus implies that \( \sigma \) stabilizes a proper residue of \( R \). This contradicts the choice of \( R \). With this contradiction, we conclude that there are chambers of \( \Sigma \) fixed by \( \sigma \).

Let \( c \) be a chamber of \( \Sigma \) fixed by \( \sigma \) and let \( d \) be the unique chamber of \( \Sigma \) opposite \( c \). Since \( \sigma \) stabilizes \( \Sigma \) and \( c \), it fixes \( d \) as well. Suppose that \( \Pi = B_2 \) or \( G_2 \). Among the roots of \( \Sigma \) containing \( c \), there are two such that \( c \) is at maximal distance from the root that is opposite in \( \Sigma \). We call these two roots \( \alpha \) and \( \beta \). They are interchanged by \( \sigma \) and by [23, 5.5 and 5.6], we have \( U_\alpha \cap U_\beta = 1 \) and \([U_\alpha, U_\beta] = 1 \). Suppose, instead, that \( \Pi = F_4 \). Let
c_1 denote the unique chamber of \( \Sigma \) that is 1-adjacent to \( c \), let \( \alpha \) denote the unique root of \( \Sigma \) containing \( c \) but not \( c_1 \) and let \( \beta = \alpha^\sigma \). By [24, 11.28(i) and (iii)] with \( \{i, j\} = \{1, 4\} \), we have \( U_\alpha \cap U_\beta = 1 \) and \( [U_\alpha, U_\beta] = 1 \) also in this case. We now return to the assumption that \( \Pi \) is in any one of the three cases \( B_2, G_2 \) or \( F_4 \) and let \( u \) be a non-trivial element of \( U_\alpha \). Both \( U_\alpha \) and \( U_\beta \) are contained in the unipotent radical \( U_c \) and hence \( b := uu^\sigma \in U_c \). Since \( U_\alpha \cap U_\beta = 1 \), we have \( b \neq 1 \) and since \([U_\alpha, U_\beta] = 1\), we have \( b^\sigma = b \). Thus by Proposition 2.13, \( d \neq d^b \). We conclude that \( c, d \) and \( d^b \) are distinct chambers fixed by \( \Gamma \).

Suppose that \( \Pi \) is \( B_2 \) or \( G_2 \). Since the type of a \( \Gamma \)-residue is \( \Theta \)-invariant, \( \Delta \) itself is the unique \( \Gamma \)-panel. Since there at least three \( \Gamma \)-chambers, \( \Gamma \) is a descent group. By Theorem 2.18, the Tits index of \( \Gamma \) is \((\Pi, \Theta, \emptyset)\) and \( \Delta^\Gamma \) (interpreted as in Definition 2.19) is a Moufang set.

Suppose now that \( \Pi = F_4 \). In this case, we let \( R_{ij} \) be the unique \( \{i, j\}\)-residue containing the chamber \( c \) and we let \( U_{ij} \) denote the group induced on \( R_{ij} \) by the unipotent radical \( U_c \) for \( \{i, j\} = \{1, 4\} \) and \( \{2, 3\} \). Then \( R_{14} \) and \( R_{23} \) are the two \( \Gamma \)-panels containing \( c \). Choose \( ij = 14 \) or \( 23 \) and let \( d_{ij} \) be the unique chamber of \( R_{ij} \cap \Sigma \) opposite \( c \). Just as above, we can choose a non-trivial element \( b_{ij} \) in \( U_{ij} \) centralized by \( \sigma \). By [11, 24.8(iii) and 24.33], \( U_{ij} \) acts sharply transitively on the set of chambers of \( R_{ij} \) opposite \( c \). Thus \( c, d_{ij} \) and \( d_{ij}^b \) are distinct \( \Gamma \)-chambers contained in \( R_{ij} \). We conclude that both \( \Gamma \)-panels \( R_{14} \) and \( R_{23} \) contain at least three \( \Gamma \)-chambers. By [11, 22.37], therefore, \( \Gamma \) is a descent group of \( \Delta \). By 2.18(3), therefore, the fixed point building \( \Delta^\Gamma \) is of type \( \tilde{\Pi} \), where \( \tilde{\Pi} \) is the relative Coxeter diagram of the Tits index \((\Pi, \Theta, \emptyset)\). By Examples 2.17, the relative Coxeter diagram of this Tits index is \( I_2(8) \). By 2.18(4), we conclude that \( \Delta^\Gamma \) is a Moufang octagon.

Notation 3.2. — Suppose that \( \Delta \) is Moufang. Let \( G^\circ \) denote the group of all type-preserving automorphisms of \( \Delta \), let \( G = \text{Aut}(\Delta) \) if \( \Delta \) is simply laced and let \( G = G^\circ \) if \( \Delta \) is not simply laced. (Thus if \( \Delta \) and \( \rho \) are as in Proposition 3.1, then \( \rho \notin G \).) Let \( G^\dagger \) denote the subgroup of \( \text{Aut}(\Delta) \) generated by all the root groups of \( \Delta \). Root groups are type-preserving, so \( G^\dagger \subset G^\circ \).

Notation 3.3. — Let \( \Pi \) be a Coxeter diagram, let \( \Delta \) be a building of type \( \Pi \) (as defined in [24, 7.1]) and suppose that \( \Delta \) is Moufang as defined in Definition 2.5. Let \( c \) be a chamber of \( \Delta \) and let \( \Sigma \) be an apartment containing \( c \). Let \( B_\Pi \) be the set of ordered pairs of \( \{i, j\} \) such that \( \{i, j\} \) is an edge of \( \Pi \). For each \( \{i, j\} \in B_\Pi \), let \( R_{ij} \) be the unique \( \{i, j\}\)-residue of \( \Delta \) containing \( c \) and let \( \Omega_{ij} \) be the root group sequence of \( R_{ij} \) based at
as defined in [11, 3.1–3.3]. The first term of $R_{ij}$ acts non-trivially on the $i$-panel of $R_{ij}$ containing $c$.

**Remark 3.4.** — Let $(i, j) \in B_{\Pi}$. Interchanging $i$ and $j$ if necessary, there exists an isomorphism from $\Omega_{ij}$ to one of the root group sequences described in [23, 16.1–16.9] (by the classification of Moufang polygons). We say that an element $(i, j)$ of $B_{\Pi}$ is *standard* if there is such an isomorphism.

**Notation 3.5.** — Let $F$ be a field of characteristic $p > 0$ and let $E/F$ be an extension such that $E^p \subset F$. By identifying $E$ with $E^p$ via Frob$_E$, we can regard $F$ as an extension of $E$. We can recover the extension $E/F$ from the extension $F/E$ by the same trick. We describe this situation by saying simply that we have a pair of extensions $\{E/F, F/E\}$. Let $\text{Aut}(E, F)$ be the set of all elements of $\text{Aut}(E)$ stabilizing $F$. This group is canonically isomorphic to the group $\text{Aut}(F, E)$ of all elements of $\text{Aut}(F)$ stabilizing $E$.

**Notation 3.6.** — Suppose that $\Delta$ is Moufang. By [11, 28.8], the building $\Delta$ has either a field of definition $F$ or a pair $\{E/F, F/E\}$ of defining extensions as in Notation 3.5. In the first case, we set $A = \text{Aut}(F)$ and in the second case, we let $A$ denote the group $\text{Aut}(E, F)$ defined in Notation 3.5.

**Remark 3.7.** — If the building $\Delta$ has a pair $\{E/F, F/E\}$ of defining extensions rather than a field of definition $F$, then $\Delta \cong B_{2}^P(\Lambda)$ for some indifferent set $\Lambda$, $\Delta \cong B_{2}^F(\Lambda)$ for some quadratic space $\Lambda$ of type $F_4$, $\Delta \cong G_2(\Lambda)$ for some inseparable hexagonal system $\Lambda$ or $\Delta \cong F_4(\Lambda)$ for some inseparable composition algebra $\Lambda$. (See [26, 30.15 and 30.23] for the definition of these terms.) By [23, 35.9, 35.12 and 35.13], the pair $\{E/F, F/E\}$ is an invariant of $\Delta$ in all these cases even though neither $E$ nor $F$ is an invariant. (By [23, 35.6-35.8, 35.10, 35.11 and 35.14], the field of definition $F$ is an invariant of $\Delta$ in every other case.) Thus the group $A$ is an invariant of $\Delta$ in every case.)

**Notation 3.8.** — Suppose that $\Delta$ is Moufang, let $G$ be as in Notation 3.2, let $\Sigma, c, B_{\Pi}$ and $\Omega_{ij}$ be as in Notation 3.3 and let $A$ be as in Notation 3.6. Let $(s, t)$ be a standard element of $B_{\Pi}$ as defined in Remark 3.4 and let $\varphi$ be an isomorphism from $\Omega_{st}$ to $\Theta$, where $\Theta$ is one of the root group sequences described in [23, 16.1–16.9]. For each $g \in G$ acting trivially on $\Sigma$, let $g_{st}$ denote the automorphism of $\Omega_{st}$ induced by $g$ and let $g_{st}^*$ denote the automorphism $\varphi^{-1}g_{st}\varphi$ of $\Theta$. By [11, (29.22) and 29.23–29.25], there exists a unique homomorphism $\psi$ from $G$ to $A$ such that the following hold:

1. $G^\dagger$ is contained in the kernel of $\psi$.
2. For each $g \in G$ acting trivially on $\Sigma$, $\psi(g)$ is equal to the element called $\lambda_\Omega(h)$ in [11, 29.5], where $\Omega = \Theta$ and $h = g_{st}^*$.
(3) \([15, 4.7(iii)]\) holds for all non-type-preserving elements \(g \in G\).

(We are only interested here in applying \(\psi\) to type-preserving elements, so we do not take the trouble to state (3) more precisely.) A homomorphism \(\psi : G \rightarrow A\) satisfying conditions (1)–(3) for some choice of \((s, t)\) and \(\varphi\) is called a Galois map of \(\Delta\). If \(\psi\) and \(\psi_1\) are two Galois maps of \(\Delta\), then there is an inner automorphism \(\iota\) of \(A\) such that \(\psi_1 = \psi \cdot \iota\) (by \([11, 29.25]\)). Thus, in particular, all Galois maps of \(\Delta\) have the same kernel.

**Definition 3.9.** — A Galois subgroup of \(\text{Aut}(\Delta)\) is a subgroup \(\Gamma\) of the group \(G\) defined in Notation 3.2 whose intersection with the kernel of a Galois map of \(\Delta\) is trivial. Since two Galois maps differ by an inner automorphism of the group \(A\), this notion is independent of the choice of the Galois map.

**Theorem 3.10.** — Suppose that \(\Delta\) is Moufang and that \(\Gamma\) is an isotropic Galois subgroup of \(\text{Aut}(\Delta)\) that acts on the set of chambers of \(\Delta\) with finite orbits. Then \(\Gamma\) is a descent group of \(\Delta\).

**Proof.** — This is \([15, 12.2(ii)]\). \(\square\)

**Notation 3.11.** — Let \(\Delta\) be Moufang and let \(\psi\) be a Galois map of \(\Delta\). A Galois involution \(g\) of \(\Delta\) is an element of order 2 in \(\text{Aut}(\Delta)\) such that \(\langle g \rangle\) is a Galois subgroup. A \(\chi\)-involution of \(\Delta\) for some \(\chi \in A\) is a Galois involution \(g\) such that \(\chi = \psi(g)\).

**Proposition 3.12.** — Let \(\Delta\) be a building of type \(F_4\) and suppose that \(\Gamma\) is a type-preserving Galois subgroup of \(\text{Aut}(\Delta)\) acting on the set of chambers of \(\Delta\) with finite orbits such that \(\Gamma\)-chambers are residues of type \(\{2, 3\}\) with respect to the standard numbering of the vertex set of the Coxeter diagram \(F_4\). Then \(\Gamma\) is a descent group and \(\Delta^\Gamma\) is a Moufang quadrangle of type \(F_4\).

**Proof.** — By Theorem 3.10, \(\Gamma\) is a descent group. Let \(T\) denote the Tits index \((\Pi, \Theta, A)\), where \(\Pi\) is the Coxeter diagram \(F_4\), \(\Theta\) is the trivial subgroup of \(\text{Aut}(\Pi)\) and \(A = \{2, 3\}\). By Example 2.17, the relative Coxeter diagram of this index is \(B_2\). By 2.18(3), the fixed point building \(\Delta^\Gamma\) is of type \(B_2\). By 2.18(4), we conclude that \(\Delta^\Gamma\) is a Moufang quadrangle. By 11.11(2) below, \(\Delta^\Gamma\) is, in fact, a Moufang quadrangle of type \(F_4\). \(\square\)

**Proposition 3.13.** — Let \(\Omega = \mathcal{O}(E, \theta)\) for some octagonal set \((E, \theta)\) and let \(\Gamma\) be a Galois subgroup of \(\text{Aut}(\Omega)\) acting on the set of chambers of \(\Delta\) with finite orbits and fixing panels of one type but none of the other. Then \(\Gamma\) is a descent group and \(\Omega^\Gamma\) is a Moufang set.

**Proof.** — This holds by 2.18(5) and Theorem 3.10. \(\square\)
4. Main Results

We can now state our main results. The proofs of Theorems 4.1 and 4.2 are in Section 14.

**Theorem 4.1.** — Let $\Xi$ be a Moufang quadrangle of type $F_4$ with a polarity $\rho$. Then there exists an octagonal set $(E, \theta)$ and an automorphism $\chi$ of the field $E$ of order 2 that commutes with the Tits endomorphism $\theta$ such that the following hold:

1. The building $\Delta := F_4(E, \theta)$ possesses a type-preserving $\chi$-involution $\xi$ and a polarity $\sigma$ such that $\Gamma := \langle \xi, \sigma \rangle \subset \text{Aut}(\Delta)$ is a descent group of order 4.
2. There exists an isomorphism from $\Xi$ to the fixed point building $\Delta^{(\xi)}$ which carries the polarity $\rho$ to the restriction of $\sigma$ to $\Delta^{(\xi)}$.
3. The fixed point buildings $\Delta^{(\sigma)}$ and $\Delta^{(\sigma \xi)}$ are Moufang octagons, one isomorphic to $\mathcal{O}(E, \theta)$ and the other to $\mathcal{O}(E, \theta \chi)$.
4. The Moufang sets $\Delta^\Gamma$, $(\Delta^{(\xi)})^{(\sigma)}$, $(\Delta^{(\sigma)})^{(\xi)}$ and $(\Delta^{(\sigma \xi)})^{(\xi)}$ are canonically isomorphic.
5. The restriction of $\xi$ to each of the two octagons in (3) is a $\chi$-involution which fixes panels of one type and none of the other.

**Theorem 4.2.** — Let $(E, \theta)$ be an octagonal set, let $\chi$ be an automorphism of $E$ of order 2, let $\Omega = \mathcal{O}(E, \theta)$, let $\Delta = F_4(E, \theta)$ and suppose that there exists a $\chi$-involution $\kappa$ of $\Omega$ which fixes panels of one type but not of the other type. Then there is a type-preserving $\chi$-involution $\xi$ of $\Delta$ and a polarity $\sigma$ of $\Delta$ such that the following hold:

1. $\Gamma := \langle \xi, \sigma \rangle \subset \text{Aut}(\Delta)$ is a descent group of $\Delta$ of order 4.
2. There is an isomorphism from $\Omega$ to the fixed point building $\Delta^{(\sigma)}$ which carries $\kappa$ to the restriction of $\xi$ to $\Delta^{(\sigma)}$.
3. $\langle \xi \rangle$-chambers are residues of type $\{2, 3\}$ with respect to the standard numbering of the vertex set of the Coxeter diagram $F_4$, the fixed point building $\Xi := \Delta^{(\xi)}$ is a Moufang quadrangle of type $F_4$ and the polarity $\sigma$ of $\Delta$ induces a polarity $\rho$ on $\Xi$.
4. The Moufang sets $\Delta^\Gamma$, $(\Delta^{(\xi)})^{(\sigma)}$, $(\Delta^{(\sigma)})^{(\xi)}$ and $(\Delta^{(\sigma \xi)})^{(\xi)}$ are canonically isomorphic.
5. The automorphism $\chi$ commutes with the Tits endomorphism $\theta$ and the fixed point building $\Delta^{(\xi \sigma)}$ is isomorphic to $\mathcal{O}(E, \theta \chi)$. 
Corollary 4.3. — The class of Moufang sets of the form $\Xi^{(\rho)}$, where $\Xi$ and $\rho$ are as in Theorem 4.1, coincides with the class of Moufang sets of the form $\Omega^{(\kappa)}$, where $\Omega$ and $\kappa$ are as in Theorem 4.2.

Proof. — Let $\Xi$, $\rho$, $\Delta$ and $\Gamma$ be as in Theorem 4.1. By 4.1(4), $\Xi^{(\rho)} \cong \Delta^\Gamma$ and hence by 4.1(5), $\Xi^{(\rho)}$ is isomorphic to a Moufang set of the form $\Omega^{(\kappa)}$, where $\Omega$ and $\kappa$ are as in Theorem 4.2. Suppose, conversely, that $\Omega$, $\kappa$, $\Delta$ and $\Gamma$ are as in 4.2. By 4.2(4), $\Omega^{(\kappa)} \cong \Delta^\Gamma$. By 4.2(3), it follows that $\Omega^{(\kappa)}$ is isomorphic to a Moufang set of the form $\Xi^{(\rho)}$, where $\Xi$ and $\rho$ are as in Theorem 4.1. $\square$

Remark 4.4. — In both Theorems 4.1 and 4.2, we are using the notion of a $\chi$-involution (defined in 3.11) with respect to Galois maps $\psi_\Delta$, $\psi_\Omega$ and $\psi_\Omega^\xi$ as in Propositions 13.4 and 13.8(4).

Definition 4.5. — We call a Moufang set of the form $\Delta^\Gamma$, where $\Delta$ and $\Gamma$ are as in Theorem 4.1 (or 4.2) a Moufang set of outer $F_4$-type (to distinguish them from other Moufang sets which arise as the fixed point buildings of type-preserving descent groups of a building of type $F_4$; see [5]).

Remark 4.6. — With the conventions described in [11, 34.2],

\[ \bullet \quad \bullet \quad \bullet \]

is the Tits index of $\langle \xi \rangle$,

\[ \bullet \quad \bullet \quad \bullet \]

is the Tits index of $\langle \sigma \rangle$ and $\langle \sigma \xi \rangle$ and

\[ \bullet \quad \bullet \quad \bullet \]

is the Tits index of $\Gamma$, where $\Gamma = \langle \xi, \sigma \rangle$ is as in Theorem 4.1 (or 4.2).

The following is proved in Section 18.

Theorem 4.7. — The group generated by all the root groups of a Moufang set of outer $F_4$-type is simple.

5. Moufang Quadrangles of Type $F_4$

The Moufang quadrangles of type $F_4$ were first described in [23, Chapter 14 and 16.7]; see also [2], [3] and [25]. In [11, Chapter 16], it is shown that these quadrangles arise in the study of pseudo-reductive quotients of parahoric subgroups of groups of absolute type $E_6$, $E_7$ and $E_8$. 
Definition 5.1. — Let $K$ be a field of characteristic 2 and let $F$ be a subfield such that $K^2 \subset F$. We set $t \ast s = \text{Frob}_K(t)s = t^2s$ for all $t \in K$ and all $s \in F$ and $q_{F/K}(s) := s$ for all $s \in F$. The map $\ast$ makes $F$ into a vector space over $K$ on which the map $q_{F/K}$ is a quadratic form. We write $[F]_K$ to refer to $F$ considered as a vector space over $K$ with respect to $\ast$.

Notation 5.2. — An $F_4$-datum is a 4-tuple $S = (E/K, F, \alpha, \beta)$, where $E/K$ is a separable quadratic extension with $\text{char}(K) = 2$, $F$ is a subfield of $K$ containing $K^2$, $\alpha$ is a non-zero element of $F$, and $\beta$ is a non-zero element of $K$, such that the quadratic form on $V_S := E \oplus E \oplus [F]_K$ given by
\[
(a, b, s) \mapsto \beta^{-1}(N(a) + \alpha N(b)) + q_{F/K}(s) = \beta^{-1}(N(a) + \alpha N(b)) + s
\]
is anisotropic, where $N = N_{E/K}$ is the norm of the extension $E/K$ and $q_{F/K}$ is as in Definition 5.1. We denote this quadratic form by $q_S$.

Definition 5.3. — A quadratic form $q$ over a field $K$ is of type $F_4$ if $q$ is similar to $q_S$ for some $F_4$-datum $(E/K, F, \alpha, \beta)$. If this case, we will say that $S$ is a standard decomposition of $q$ and that $E$ is a splitting field of $q$. We say that a quadratic space $(K, V, q)$ is of type $F_4$ if the quadratic form $q$ is of type $F_4$.

Definition 5.4. — Let $S = (E/K, F, \alpha, \beta)$ be an $F_4$-datum, let $D$ denote the composite field $FE^2$ and let $[K]_F$ denote $K$ regarded as a vector space over its subfield $F$ in the standard way. By [23, 14.8], the quadratic form $\tilde{q}_S$ on $\hat{V}_S := D \oplus D \oplus [K]_F$ over $F$ given by
\[
(x, y, t) \mapsto \alpha(N(x) + \beta^2 N(y)) + t^2
\]
is a quadratic form of type $F_4$ with standard decomposition
\[
(D/F, K^2, \beta^2, \alpha^{-1}).
\]

Notation 5.6. — Let $S = (E/K, F, \alpha, \beta)$ be an $F_4$-datum, let $q := q_S$ and $V = V_S$ be as in Notation 5.2 and let $\hat{V} = \hat{V}_S$ and $\hat{q} := \hat{q}_S$ be as in Definition 5.4. Let $f$ and $\hat{f}$ denote the bilinear forms $\partial q$ and $\partial \hat{q}$ associated with $q$ and $\hat{q}$ and let $x \mapsto \bar{x}$ be the non-trivial element of $\text{Gal}(E/K)$.

We now introduce the Moufang quadrangle that corresponds to this data.

Notation 5.7. — Let $S$, $\hat{V}$, $V$, etc. be as in Notation 5.6, let
\[
U_+ := U_1U_2U_3U_4
\]
be the group defined in terms of the isomorphisms $x_i : V \to U_i$ for $i = 2$ and 4 and $x_i : \hat{V} \to U_i$ for $i = 1$ and 3 the following commutator relations
taken from [23, 16.7]):
\[
[x_1(x, y, t), x_3(x', y', t')] = x_2(0, 0, \alpha(x\bar{x}' + x'\bar{x} + \beta^2(y\bar{y}' + y'\bar{y}))), \\
x_2(u, v, s), x_4(u', v', s') = x_3(0, 0, \beta^{-1}(u\bar{u}' + u'\bar{u} + \alpha(v\bar{v}' + v'\bar{v}))), \\
[x_1(x, y, t), x_4(u, v, s)] = x_2(tu + \alpha(xv + \beta y\bar{v}), tv + xu + \beta y\bar{u}, \\
t^2s + s\alpha(x\bar{x} + \beta^2y\bar{y}) \\
\quad + \alpha(u^2x\bar{y} + u^2\bar{x}y + \alpha(v^2x\bar{y} + v^2\bar{x}y))) \\
x_3(sx + \bar{u}^2y + \alpha v^2\bar{y}, sy + \beta^{-2}(u^2x + \alpha v^2\bar{x}), \\
st + t\beta^{-1}(u\bar{u} + \alpha v\bar{v}) \\
\quad + \alpha(\beta^{-1}(xuv + \bar{x}\bar{u}v) + y\bar{u}v + \bar{y}uv))
\]
for all \((u, v, s), (u', v', s') \in V\) and all \((x, y, t), (x', y', t') \in \bar{V}\) and
\[
[U_1, U_2] = [U_2, U_3] = [U_3, U_4] = 1.
\]
The group \(U_+\), its subgroups \(U_1, \ldots, U_4\) and the isomorphisms \(x_1, \ldots, x_4\) depend only on the \(F_4\)-datum \(S\). By [23, 16.7 and 32.11],
\[
(U_+, U_1, U_2, U_3, U_4)
\]
is a root group sequence. Let
\[
Q(S)
\]
denote the Moufang quadrangle, \(\Sigma\) the apartment of \(Q(S)\) and \(c\) the chamber of \(\Sigma\) obtained by applying [23, 8.3] to this root group sequence. There is a canonical identification of \(U_1, \ldots, U_4\) with the root groups of \(\Xi := Q(S)\) associated with the four roots of \(\Sigma\) containing \(c\) and we always identify \(U_+\) with the subgroup of \(\text{Aut}(\Xi)\) generated by these four root groups.

**Definition 5.8.** — A Moufang quadrangle of type \(F_4\) is a Moufang quadrangle isomorphic to \(Q(S)\) (as defined in Notation 5.7) for some \(F_4\)-datum \(S\).

**Notation 5.9.** — Let \(S = (E/K, F, \alpha, \beta)\) and \(\Omega := (U_+, U_1, \ldots, U_4)\) be as in Notation 5.7. By [23, 28.45], there is an anti-isomorphism (as defined in [23, 8.9]) from \(\Omega\) to the root group sequence obtained by applying Notation 5.7 to the \(F_4\)-datum \((D/F, K^2, \beta^2, \alpha^{-1})\) in Definition 5.4.

**Notation 5.10.** — Let \(S\) and \((K, V, q)\) be as in Notation 5.6 and let
\[
\Omega = (U_+, U_1, \ldots, U_4)
\]
and \(\Xi = Q(S)\) be as in Notation 5.7. The quadrangle \(\Xi\) is called \(Q_F(K, V, q)\) in [23, 16.7] and \(B_2^F(K, V, q)\) in [26, 30.15]. Suppose that \(S'\) is any other
$F_4$-datum and let $\Omega' = (U_+, U'_1, \ldots, U'_4)$ be the root group sequence obtained by applying Notation 5.7 to $S'$. Then by [23, 35.12], there is a type-preserving isomorphism from $\Omega$ to $\Omega'$ if and only if the quadratic form $q_{S'}$ is similar to $q$, where $q_{S'}$ is the quadratic form obtained by applying Notation 5.2 to $S'$. In particular, $\Xi \cong Q(S')$ for every standard decomposition $S'$ of $q$ (as defined in Definition 5.3).

**Remark 5.11.** — Let $\Xi = Q(S)$ for some $F_4$-datum $S = (E/K, F, \alpha, \beta)$. By [11, 28.4] (see Notation 3.6), $\{K/F, F/K\}$ is the pair of defining extensions of $\Xi$. By [23, 35.12], it is an invariant of $\Xi$.

**Remark 5.12.** — Let $s_0 \in F^*$ and let $S_1 = (E/K, F, \alpha, \beta/s_0)$. Then $S_1$ is an $F_4$-datum, $V_{S_1} = V, \hat{V}_{S_1} = \hat{V}$ and the maps $x_1(x, y, t) \mapsto x_1(x, s_0 y, t)$, $x_3(x, y, t) \mapsto x_3(x/s_0, y, t/s_0)$ and $x_i(u, v, s) \mapsto x_i(u/s_0, v/s_0, s/s_0)$ for $i = 2$ and 4 extend to an isomorphism from $Q(S)$ to $Q(S_1)$ (by [23, 7.5]). Thus by reparametrizing $U_+$, we can replace the element $x_4(0, 0, s_0)$ by $x_4(0, 0, 1)$ without changing the element $x_1(0, 0, 1)$.

**Notation 5.13.** — We define two maps, one from $V \times \hat{V}$ to $V$ and the other from $\hat{V} \times V$ to $\hat{V}$, both denoted either by $\cdot$ or juxtaposition, so that

\[(5.14) \quad [x_1(\hat{v}), x_4(v)] = x_2(v \cdot \hat{v}) x_3(\hat{v} \cdot v)\]

in $U_+$ for all $(\hat{v}, v) \in \hat{V} \times V$. (Note that we will also denote scalar multiplication by $\cdot$ or juxtaposition, but this should not cause any confusion.)

**Remark 5.15.** — Let $(K, V, q)$ and $f$ be as in Notation 5.6 and let $d, e$ be linearly independent elements of $V$ such that $f(d, e) = 1$. Then $q(d)x^2 + x + q(e) = q(xd + e) \neq 0$ for all $x \in K$ since $q$ is anisotropic, and the restriction of $q/q(d)$ to $\langle e, d \rangle$ is isometric to the norm of the quadratic extension $L/K$, where $L$ is the splitting field of the polynomial $q(d)x^2 + x + q(e)$ over $K$. Note that $L$ is also the splitting field of the polynomial $x^2 + x + q(d)q(e)$ over $K$.

**Theorem 5.16.** — Let $(K, V, q)$, $\hat{V}$ and $f$ be as in Notation 5.6 and let $(U_+, U_1, \ldots, U_4)$ and $x_1, \ldots, x_4$ be as in Notation 5.7. Let $d, e$ be two elements of $V$ and let $\xi$ be an element of $\hat{V}$ such that $f(d, e) = 1$ and $f(d, e\xi) = 0$. Let $\alpha_0 = f(d\xi, e\xi), \beta_0 = q(d)^{-1}$, let $L$ be the splitting field of the polynomial $p(x) = q(d)x^2 + x + q(e)$ over $K$ and let $\omega$ be a root of $p(x)$ in $L$. Then the following hold:

1. $S_0 := (L/K, F, \alpha_0, \beta_0)$ is a standard decomposition of $q$. 

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(2) There exists an isometry \( \pi \) from \((K, V, q)\) to \((K, V_{S_0}, q_{S_0})\) sending the elements \( d, e, d\xi \) and \( e\xi \) to \((1, 0, 0), (\omega, 0, 0), (0, 1, 0)\) and \((0, \omega, 0)\), respectively, and \((0, 0, s)\) to \((0, 0, s)\) for all \( s \in F \).

(3) There exists an isometry \( \hat{\pi} \) from \((F, \hat{V}, \hat{q})\) to \((F, \hat{V}_{S_0}, \hat{q}_{S_0})\) sending the elements \( \xi, \xi e \cdot d^{-1}, \xi e \cdot d^{-1} \) and \( \xi d^{-1} \) to \((1, 0, 0), (\omega^2, 0, 0), (0, 1, 0)\) and \((0, \omega^2, 0)\), respectively, and \((0, 0, t)\) to \((0, 0, t)\) for all \( t \in K \).

(4) Let \((\tilde{U}_+, \tilde{U}_1, \ldots, \tilde{U}_4)\) and \(\tilde{x}_1, \ldots, \tilde{x}_4\) be the root group sequence and the isomorphisms obtained by applying Notation 5.7 to \(S_0\). Then there is an isomorphism from \(\tilde{U}_+\) to \(\tilde{U}_4\) extending the maps \(x_i(v) \mapsto \tilde{x}_i(\pi(v))\) for \(i = 1\) and \(3\) and \(x_i(v) \mapsto \tilde{x}_i(\hat{\pi}(v))\) for \(i = 2\) and \(4\).

**Proof.** — This is proved in [3, 8.98-8.106]. See, in particular, the equations at the top of page 77 in [3]. \( \square \)

**Notation 5.17.** — Let \([s]_K = (0, 0, s) \in V\) for each \(s \in F\) and \([t]_F = (0, 0, t) \in \hat{V}\) for each \(t \in K\). Thus \(t[s]_K = [t^2s]_K\) and \(s[t]_F = [st]_F\) for all \(s \in F\) and all \(t \in K\).

**Proposition 5.18.** — The following identities hold for all \(t \in K\), all \(s \in F\), all \(u, v, w \in V\) and all \(\hat{u}, \hat{v}, \hat{w} \in \hat{V}\):

(F0) \(x \mapsto x\hat{w}\) and \(\hat{x} \mapsto \hat{x}w\) are linear maps from \(V\) to \(V\) and from \(\hat{V}\) to \(\hat{V}\).

(F1) \(v[t]_F = tv\).

(F2) \(\hat{v}[s]_K = s\hat{v}\).

(F3) \(v \cdot \hat{w} = v\hat{w} \cdot [s]_F\).

(F4) \(\hat{v} \cdot tw = \hat{v}w \cdot [t^2]_K\).

(F5) \([t]_F v = [tq(v)]_F\).

(F6) \([s]_K \hat{v} = [s\hat{q}(\hat{v})]_K\).

(F7) \(vw\hat{w} = v \cdot [\hat{q}(\hat{w})]_F\).

(F8) \(\hat{v}wv = \hat{v} \cdot [q(w)^2]_K\).

(F9) \(v \cdot \hat{w}v = q(v)v\hat{w}\).

(F10) \(\hat{v} \cdot \hat{w} = \hat{q}(\hat{v})\hat{w}\).

(F11) \(w(\hat{u} + \hat{v}) = \hat{w}u + \hat{w}v + [\hat{f}(\hat{w}, u) + \hat{v}]_K\).

(F12) \(\hat{w}(u + v) = \hat{w}u + \hat{w}v + [f(u\hat{w}, v)]_F\).

**Proof.** — Comparing [23, 16.7] with (5.14), we can write the products \(\hat{v} \cdot v\) and \(v \cdot \hat{v}\) in terms of the functions given in [23, 14.15–14.16]. The identities (F0)–(F12) can then be verified with the help of the identities in [23, 14.18]. \( \square \)
The identities (F0)–(F12) are the axioms of a radical quadrangular system as defined in [3, Appendix A.3.2]. These axioms can be used to characterize Moufang quadrangles of type $F_4$ (defined in Definition 5.8 above); see [3, §8.5] and [23, Chapter 28] for details.

**Notation 5.19.** Using Notation 5.13, we can re-write the commutator relations in Notation 5.7 as follows:

\[
\begin{align*}
[x_1(\hat{v}), x_3(\hat{u})] & = x_2([\hat{f}(\hat{v}, \hat{u})]_K) \\
[x_2(v), x_4(u)] & = x_3([f(v, u)]_F) \\
[x_1(\hat{v}), x_4(v)] & = x_2(v\hat{v})x_3(\hat{v}v)
\end{align*}
\]

for all $u, v \in V$ and all $\hat{u}, \hat{v} \in \hat{V}$ as well as $[U_1, U_2] = [U_2, U_3] = [U_3, U_4] = 1$.

**6. The Polarity $\rho$**

We continue with all the notation of the previous section.

**HYPOTHESIS 6.1.** We suppose now that the Moufang quadrangle $\Xi = Q(S)$ introduced in Notation 5.7 has a polarity $\rho$.

**Remark 6.2.** By Proposition 2.6, the polarity $\rho$ fixes an apartment. Since $\rho$ is a non-type-preserving involution and apartments are circuits of length 8, $\rho$ fixes two opposite chambers of this apartment. Since $\text{Aut}(\Xi)$ acts transitively on incident pairs of apartments and chambers (by [24, 11.12]), we can assume that $\rho$ fixes the apartment $\Sigma$ and the chamber $c$ in Notation 5.7. This means that

\[
(6.3) \quad \rho U_i \rho = U_{5-i}
\]

for all $i \in [1, 4]$.

**Notation 6.4.** Let $\hat{\varphi}, \hat{\varphi}_1 : \hat{V} \to V$ and $\varphi, \varphi_1 : V \to \hat{V}$ be the unique additive bijections such that

\[
\begin{align*}
\rho(x_1(\hat{v})) & = x_4(\hat{\varphi}(\hat{v})) \\
\rho(x_2(v)) & = x_3(\varphi_1(v)) \\
\rho(x_3(\hat{v})) & = x_2(\hat{\varphi}_1(\hat{v})) \\
\rho(x_4(v)) & = x_1(\varphi(v))
\end{align*}
\]

for all $v \in V$ and all $\hat{v} \in \hat{V}$. By Remark 5.12, we can assume that $\varphi([1]_K) = [1]_F$. Since $\rho$ is of order 2, we have

$\hat{\varphi} = \varphi^{-1}$ and $\hat{\varphi}_1 = \varphi_1^{-1}$.
LEMMA 6.5. — The following hold:

(1) \( \varphi = \varphi_1. \)

(2) \( \varphi([\hat{q}(\varphi(v))]_K) = [q(v)]_F \) for all \( v \in V. \)

Proof. — Let \( v \in V. \) By (5.14), (F1) and (F5), we have

\[
x_1([1]_F), x_4(v) = x_2(v)x_3([q(v)]_F).
\]

Applying \( \rho \), we obtain

\[
x_1(\varphi(v)), x_4([1]_K) = x_2(\varphi_1^{-1}([q(v)]_F))x_3(\varphi_1(v)).
\]

By (5.14), (F2) and (F6), on the other hand,

\[
x_1(\varphi(v)), x_4([1]_K) = x_2([\hat{q}(\varphi(v))]_K)x_3(\varphi(v)).
\]

Therefore \( \varphi(v) = \varphi_1(v) \) and \( \varphi_1^{-1}([q(v)]_F) = [\hat{q}(\varphi(v))]_K. \)

\( \square \)

LEMMA 6.6. — \( \varphi^{-1}([f(u,v)]_F) = [\hat{f}(\varphi(u), \varphi(v))]_K \) for all \( u, v \in V \) and \( \varphi([\hat{f}(\hat{u}, \hat{v})]_K) = [f(\varphi^{-1}(\hat{u}), \varphi^{-1}(\hat{v})]_F \)

to \( \hat{u}, \hat{v} \in \hat{V}. \) In particular, \( \varphi([F]_K) = [K]_F \) and \( \varphi^{-1}([K]_F) = [F]_K. \)

Proof. — This follows from 6.5 (2).

\( \square \)

PROPOSITION 6.7. — The following hold for all \( v \in V \) and all \( \hat{v} \in \hat{V}:

(1) \( \varphi(v\hat{v}) = \varphi(v)\varphi^{-1}(\hat{v}). \)

(2) \( \varphi^{-1}(\hat{v}v) = \varphi^{-1}(\hat{v})\varphi(v). \)

Proof. — Applying \( \rho \) to the identity (5.14), we obtain both claims by 6.5 (1).

\( \square \)

Notation 6.8. — By Lemma 6.6, there exists a unique additive bijection \( \theta \) from \( K \) to \( F \) such that \( \varphi^{-1}([t]_F) = [t^\theta]_K. \) Note that it means that

\[
\hat{f}(\varphi(u), \varphi(v)) = f(u, v)^\theta
\]

for all \( u, v \in V. \)

PROPOSITION 6.10. — The following hold:

(1) The map \( \theta \) defined in Notation 6.8 is a Tits endomorphism of \( K. \)

(2) \( \varphi(tv) = t^\theta\varphi(v) \) for all \( v \in V \) and all \( t \in K. \)

(3) \( q(v)^\theta = \hat{q}(\varphi(v)) \) for all \( v \in V. \)

(4) \( u \cdot \varphi(tv) = t^\theta \cdot u\varphi(v) \) for all \( u, v \in V \) and all \( t \in K. \)

Proof. — Choose \( t \in K \) and \( s \in F. \) By Notation 5.17 and (F2), we have \([t]_F \cdot [s]_K = [st]_F. \) Applying \( \varphi^{-1} \), we obtain \([t^\theta]_K \cdot [s^{\theta^{-1}}]_F = [(st)^\theta]_K. \)
By 5.17 and (F1), on the other hand, we have $[t^θ]_K \cdot [s^{θ^{-1}}]_F = [(s^{θ^{-1}})^2 t^θ]_K$. Therefore

\[(6.11)\] 
\[(ts)^θ = (s^{θ^{-1}})^2 t^θ.\]

Setting $t = 1$ in (6.11), we obtain

\[(6.12)\] 
\[s^θ = (s^{θ^{-1}})^2.\]

Substituting this back into (6.11), we then have

\[(6.13)\] 
\[(ts)^θ = s^θ t^θ.\]

Let $x = s^{θ^{-1}}$. Substituting $x^θ$ for $s$ in (6.12), we conclude that

\[(6.14)\] 
\[(x^θ)^θ = x^2.\]

Thus

\[(6.15)\] 
\[(x^2)^θ = (x^{θ^2})^θ = (x^θ)^2 = (x^θ)^2\]

for all $x \in K$. Choose $u \in K$. Then $u^2$ and $t^2$ are in $F$, so

\[(t^2 u^2)^θ = (t^θ)^2 (u^2)^θ\]

by (6.13). By (6.15), it follows that $θ$ is multiplicative. By (6.14), therefore, $θ$ is a Tits endomorphism. Thus (1) holds.

Now let $v \in V$ and $t \in K$. Then

\[\varphi(tv) = \varphi(v[t]_F)\]
\[= \varphi(v)\varphi^{-1}([t]_F)\]
\[= \varphi(v)[t^θ]_K = t^θ \varphi(v)\]

by (F1), so (2) holds, and

\[[q(v)^θ]_K = \varphi^{-1}[q(v)]_F\]
\[= [\hat{q}(\varphi(v))]_K\]

by 6.5(2), so (3) holds. Now choose another $u \in V$. Then

\[u \varphi(tv) = u \cdot t^θ \varphi(v)\]
\[= u \varphi(v) \cdot [t^θ]_F\]
\[= t^θ \cdot u \varphi(v)\]

by (F3), so (4) holds.

\[\square\]
Notation 6.16. — Let \( x_1, \ldots, x_4 \) be as in Notation 5.7. We replace \( x_i \) by \( x_i \cdot \varphi \) for \( i = 1 \) and 3. After these replacements, we have \( x_i: V \to U_i \) for all \( i \in [1, 4] \),
\[
[x_1(u), x_3(v)] = x_2([\hat{f}(\varphi(u), \varphi(v))]_K)
\]
and
\[
[x_2(u), x_4(v)] = x_3(\varphi^{-1}([f(u, v)]_F)) = x_3([\hat{f}(\varphi(u), \varphi(v))]_K)
\]
for all \( u, v \in V \) by Lemma 6.6,
\[
[x_1(u), x_4(v)] = x_2(v \varphi(u))x_3(\varphi^{-1}(\varphi(u)v)) = x_2(v \varphi(u))x_3(u \varphi(v))
\]
for all \( u, v \in V \) by Proposition 6.7 and
\[
(6.17) \quad x_i(u)^\theta = x_{5-i}(u)
\]
for all \( u \in V \) and for all \( i \in [1, 4] \). We define a product on \( V \) by
\[
(6.18) \quad uv = u \varphi(v)
\]
and a symmetric map \( g: V \times V \to [F]_K \subset V \) by
\[
(6.19) \quad g(u, v) = [\hat{f}(\varphi(u), \varphi(v))]_K
\]
for all \( u, v \in V \). With this notation, we have
\[
(6.20) \begin{align*}
[x_1(u), x_3(v)] &= x_2(g(u, v)) \\
[x_2(u), x_4(v)] &= x_3(g(u, v)) \\
[x_1(u), x_4(v)] &= x_2(\varphi(v)u)x_3(\varphi(v)u)
\end{align*}
\]
for all \( u, v \in V \) as well as \([U_1, U_2] = [U_2, U_3] = [U_3, U_4] = 1\).

Notation 6.21. — From now on, we set \([t] = [t^\theta]_K\) for all \( t \in K \). Thus
\[
[K] := \{[t] \mid t \in K\} = [F]_K
\]
is a vector space over \( K \) with scalar multiplication given by \( a[t] = [a^\theta t] \) for all \( a, t \in K \), and
\[
(6.22) \quad g(u, v) = [f(u, v)^\theta]_K = [f(u, v)]_K
\]
for all \( u, v \in V \) by (6.9) and (6.19).
**Proposition 6.23.** — Let the multiplication on $V = E \oplus E \oplus [K]$ and the map $g$ be as in (6.18) and (6.22). Then the following hold for all $u, v \in V$ and all $t \in K$:

(R1) The map $x \mapsto xv$ from $V$ to itself is $K$-linear.
(R2) $v[t] = tv$.
(R3) $u \cdot tv = t^\theta \cdot uv$.
(R4) $[t]v = [tq(v)]$.
(R5) $uv \cdot v = q(v)^\theta \cdot u$.
(R6) $v \cdot uv = q(v) \cdot vu$.
(R7) $u(v + w) = uv + uw + g(vu, w)$.

**Proof.** — Just for this proof, we will denote by $*$ both the map from $V \times \hat{V}$ to $V$ and the map from $\hat{V} \times V$ to $V$ defined in Notation 5.13 to distinguish them from the multiplication on $V$ defined in (6.18). Thus, in particular, $wu = u \ast \varphi(v)$ for all $u, v \in V$.

Let $u, v, w \in V$ and $t \in K$. The assertion (R1) is just a special case of (F0). We have

$$v[t] = v[t]^\theta_K = v \ast [t]_F = tv$$

by (F1). Thus (R2) holds. The assertion (R3) follows from 6.10(iv). To see that (R4) holds, we observe that

$$[t]v = [t^\theta]_K \ast \varphi(v)$$

$$= [t^\theta \hat{q}(\varphi(v))]_K$$

$$= [t^\theta q(v)^\theta]_K$$

$$= [tq(v)].$$

Next we have

$$uv \cdot v = u \ast [\hat{q}(\varphi(v))]_F$$

$$= u \ast [q(v)^\theta]_F$$

$$= u[q(v)^\theta]$$

$$= q(v)^\theta \cdot u$$

by (R2),

so (R5) holds, and

$$v \cdot uv = v \ast \varphi(u \ast \varphi(v))$$

$$= v \ast (\varphi(u) \ast v)$$

$$= q(v) \cdot vu$$

by (F9),
so (R6) holds. Finally, we have
\[ u(v + w) = u * \varphi(v + w) \]
\[ = uv + uw + \left[ \hat{f}(\varphi(v) * u, \varphi(w)) \right]_K \]
by (F11)
\[ = uv + uw + \left[ \hat{f}(\varphi(vu), \varphi(w)) \right]_K \]
by 6.7 (2)
\[ = uv + uw + g(vu, w) \]
by (6.19).

Thus (R7) holds. □

7. Polarity Algebras

In this section we introduce polarity algebras and prove a series of identities. Some (for instance Proposition 7.9) we have included only because they are compelling, not because we will apply them later on.

**Definition 7.1.** A polarity algebra is a 6-tuple \((K, V, q, \theta, t \mapsto [t], \cdot)\), where \(K\) is a field of characteristic 2, \((K, V, q)\) is an anisotropic quadratic space such that the bilinear form \(f := \partial q\) is not identically zero, \(\theta\) is a Tits endomorphism of \(K\), \(t \mapsto [t]\) is a \(K\)-linear embedding of the \(K\)-vector space \([K] = [K^\theta]_K\) defined in Definition 5.1 and Notation 6.21 into the radical of \(f\), so
\[
\hat{a}[t] = [a^\theta t]
\]
for all \(a, t \in K\), and \((u, v) \mapsto u \cdot v\) is a multiplication on \(V\) (which we often denote by juxtaposition), satisfying the conditions (R1)–(R7) in Proposition 6.23 with
\[
g(u, v) = [f(u, v)]
\]
for all \(u, v \in V\) in (R7).

Throughout this section we assume that \((K, V, q, \theta, t \mapsto [t], \cdot)\) is a polarity algebra. We let \(f\) and \(g\) be as in Definition 7.1 and we set
\[
v^{-1} = q(v)^{-1}v
\]
for all non-zero \(v \in V\). Since \(q\) is anisotropic, this is allowed.

**Remark 7.3.** Let \(K\) and \(f\) be as in Definition 7.1. An anisotropic form over a finite field is either 1-dimensional or similar to the norm of a quadratic extension. Since \(f\) is not identically zero and its radical is non-trivial, we conclude that \(K\) is infinite.
Proposition 7.4. — The following hold for all \( u, v, w \in V \) and all \( t \in K \):

1. \( g(u, uw) = 0 \).
2. \( g(u, vw) = g(uw, v) \).
3. \( g(u, v)w = g(q(w)u, v) \).
4. \( tg(u, v) = g(t^\theta u, v) \).

Proof. — Assertions (1) and (2) follow immediately from (R7) and assertion (3) follows immediately from (R4). We have \( tg(u, v) = t[f(u, v)] = [t^\theta f(u, v)] = g(t^\theta u, v) \) for all \( u, v \in V \) and all \( t \in K \), so also assertion (4) holds. \( \square \)

Proposition 7.5. — \( v \cdot wu = f(u, vw)u + f(u, v)uw + q(u)vw \) for all \( u, v, w \in V \).

Proof. — Let \( t \in K \). By (R6), we have

\[
(7.6) \quad (u + tv) \cdot (w(u + tv)) \\
= q(u + tv)(u + tv)w \\
= (q(u) + tf(u, v) + t^2 q(v))(u + tv)w \\
= q(u)uw + t(q(u)vw + f(u, v)uw) \\
\quad + t^2(q(v)uw + f(u, v)vw) + t^3 q(v)vw.
\]

By (R7) and (R3), on the other hand, we have

\[
(7.7) \quad (u + tv) \cdot (w(u + tv)) \\
= (u + tv)(wu + t^\theta vw + g(u, twv)) \\
= u(wu + t^\theta vw + g(u, twv)) + tw(wu + t^\theta vw + g(u, twv)) \\
= u(wu) + u \cdot t^\theta vw + ug(u, twv) + g(wu \cdot u, t^\theta vw) \\
\quad + tw \cdot wu + t^3 v \cdot vw + tv \cdot g(u, twv) + g(wu, t^\theta vw \cdot tv) \\
= q(u)uw + t^2 u \cdot vw + ug(u, twv) + g(wu \cdot u, t^\theta vw) \\
\quad + tv \cdot wu + t^3 q(v)vw + tv \cdot g(u, twv) + g(wu, t^\theta vw \cdot tv).
\]

Thus the sum of the expressions (7.6) and (7.7) is zero. By (R2) and 7.4(4), this sum lies in \( K[t] \) and the coefficient of \( t \) is

\( q(u)vw + f(u, v)uw + f(u, vw)u + v \cdot wu. \)

To verify this, we need only observe that \( ug(u, twv) = u[f(u, twv)] = f(u, twv)u \) by (R2) and \( g(wu \cdot u, vw) = q(u)g(w, vw) = 0 \) by (R5).
and 7.4(1). It follows from Remark 7.3 and [23, 2.26] that the coefficient of each power of $t$ in this sum is zero. □

**Proposition 7.8.** — $uv \cdot w + uw \cdot v = g(v, f(v, wu)v) + f(v, w)^\theta u$ for all $u, v, w \in V$.

**Proof.** — By (R5), we have

$$u(v + w) \cdot (v + w) = q(v + w)^\theta u$$

$$= (q(v) + f(v, w) + q(w))^\theta u$$

$$= q(v)^\theta u + f(v, w)^\theta u + q(w)^\theta u.$$  

By (R5), (R6), (R7) and 7.4(3), on the other hand, we have

$$u(v + w) \cdot (v + w) = (uv + uw + q(v, wu))(v + w)$$

$$= uv(v + w) + uw(v + w) + g(v, wu)(v + w)$$

$$= uv \cdot v + uv \cdot w + g(v \cdot w, w)$$

$$+ uw \cdot v + uw \cdot w + g(v, w \cdot uw)$$

$$+ g(q(v)v, wu) + g(q(w)v, wu) + g(f(v, w)v, wu)$$

$$= q(v)^\theta u + uv \cdot w + g(q(v)wu, w)$$

$$\begin{array}{c}
\quad 1 \\
\quad 2 \\
\end{array}$$

$$+ uw \cdot v + q(w)^\theta u + g(v, q(w)wu)$$

$$\begin{array}{c}
\quad 1 \\
\quad 2 \\
\end{array}$$

$$+ g(q(v)v, wu) + g(q(w)v, wu) + g(v, f(v, wu)w).$$

Note that

$$g(f(v, w)v, wu) = [f(v, w)f(v, wu)]$$

$$= [f(v, f(v, wu)w)] = g(v, f(v, wu)w).$$

Therefore $f(v, w)^\theta u = uv \cdot w + uw \cdot v + g(v, f(v, wu)w).$ □

The following observation says that the quadratic form $q$ is multiplicative “with a twist.”

**Proposition 7.9.** — $q(uv) = q(u)q(v)^\theta$ for all $u, v \in V$.

**Proof.** — Choose $u, w \in V$ and recall that $f(V, [K]) = 0$. Setting $v = [1]$ in Proposition 7.5, we obtain

$$[1] \cdot wu = f(u, [1]w)u + f(u, [1]uw + q(u)[1]w.$$  

By (R3) and (R4), therefore, $[q(wu)] = q(u)[q(w)] = [q(w)q(u)^\theta].$ □
Proposition 7.10. — \( q([t]) = t^\theta \) for all \( t \in K \).

Proof. — If \( t \in K \), then \([q([t])] = [1][t] = t[1] = [t^\theta] \) by (R2) and (R4).

Proposition 7.11. — The following hold for all \( u,v,w \in K \):

1. \( f(uv, uw) = f(v, wu)^\theta + q(u)f(v, w)^\theta \)
2. \( f(uw, wv) = q(v)^\theta f(u, w) \).
3. \( (uv)^{-1} = u^{-1}v^{-1} \) if \( u,v \neq 0 \).

Proof. — Let \( u,v,w \in K \). We have
\[
q(u(v + w)) = q(u)q(v + w)^\theta
= q(u)(q(v) + q(w) + f(v, w))^\theta
= q(uv) + q(uw) + q(u)f(v, w)^\theta
\]
by Proposition 7.9, whereas
\[
q(u(v + w)) = q(uv + uw + q(v, wu))
= q(uv) + q(uw) + q([f(v, wu)]) + f(uw, uw)
= q(uv) + q(uw) + f(v, wu)^\theta + f(uv, uw)
\]
by (R7)
Thus (1) holds.

We have
\[
q((u + w)v) = q(u + w)q(v)^\theta
= (q(u) + q(w) + f(u, w))q(v)^\theta
= q(uv) + q(wv) + q(v)^\theta f(u, w),
\]
by Proposition 7.9, whereas
\[
q((u + w)v) = q(uv) + q(wv) + q(wv) + f(uv, wv)
\]
by (R1). Thus (2) holds. Finally, we have
\[
(uv)^{-1} = q(uv)^{-1}uv
= q(u)^{-1}q(v)^{-\theta}uv = q(u)^{-1}u \cdot q(v)^{-1}v = u^{-1}v^{-1}
\]
by (R3) and Proposition 7.9. Thus (3) holds. □

Proposition 7.12. — The following hold for all \( u,v,w \in K \):

1. \( u^{-1} \cdot vu = uv \) if \( u \neq 0 \).
2. \( uv \cdot v^{-1} = u \) if \( v \neq 0 \).

Proof. — These identities follow immediately from (R3), (R5) and (R6). □
Remark 7.13. — Let $v \in V^*$. By (R1) and 7.12(2), the map $x \mapsto xv$ is an automorphism of $V$. By 7.9, this map is a similitude of $q$ with similarity factor $q(v)^\theta$.

Proposition 7.14. — The following hold for all $u, v, z, w \in V$:

1. $g(uv \cdot w, zv) = f(w, v)g(uv, z) + q(v)g(uw, z)$.
2. $g(uv \cdot w, uz) = f(w, v)z + f(wu, v)z + f(wu, z)v + f(w, z)wu$.
3. $f(uv, zw) + f(uw, zv) = f(u, z)f(v, w)^\theta$.
4. $uv \cdot w = q(vw)u$.
5. $uv \cdot w = w(u \cdot vz) + g(uw, z)$ by Prop. 7.5.
6. $uv \cdot w = (u \cdot vz) \cdot v$.
7. $(vv)^{-1} \cdot vv = v$ if $v \neq 0$.
8. $u^{-1} \cdot u^{-1} u = u$ if $u \neq 0$.

Proof. — On the one hand,

\begin{align*}
w \cdot (u + z)v &= f(w(u + z), v)\\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\ Quad
whereas
\[ w \cdot u(v + z) = f(wu, v + z)(v + z) + f(w, v + z)(vu + zu) + q(v + z)wu \]
\[ = (f(wu, v)v + f(w, v)vu + q(v)wu) \]
\[ + (f(wu, z)z + f(w, z)zu + q(z)wu) \]
\[ + f(wu, v)z + f(wu, z)v + f(w, v)zu \]
\[ + f(w, z)vu + f(v, z)wu \]
\[ = w \cdot uv + w \cdot uz + f(wu, v)z + f(wu, z)v + f(w, v)zu \]
\[ + f(w, z)vu + f(v, z)wu. \]

by several applications of Proposition 7.5. Thus (2) holds.

Using 7.4(2) and Proposition 7.8, we obtain
\[ uv \cdot w + uw \cdot v = g(v, f(v, wu)w) + f(v, w)^\theta u \]
and
\[ f(uv, zw) + f(uw, zv) = f(uv \cdot w, z) + f(uw \cdot v, z) \]
\[ = f(uv \cdot w + uw \cdot v, z) \]
\[ = f(f(v, w)^\theta u, z) = f(u, z)f(v, w)^\theta. \]

Thus (3) holds.

We have
\[ uv \cdot vu = f(u, uv \cdot v)u + f(u, uv)uv + q(u)uv \cdot v \]
by Proposition 7.5 and
\[ uv \cdot v = uv \cdot q(v)v^{-1} = q(v)^\theta u \]
by 7.12(2) and (R2). Hence
\[ uv \cdot vu = q(u)q(v)^\theta v = q(uv)u \]
by 7.4(1), Proposition 7.9 and 7.12(2). Thus (4) holds.

By Propositions 7.5 and 7.9 and (R5), we have
\[ uv \cdot vw = f(uv \cdot v, w)w + f(uv, w)uv + q(w)uv \cdot v \]
\[ = f(u, w)q(v)^\theta w + f(uv, w)uv + q(wv)u. \]

Thus (5) holds.

Using 7.8, we have
\[ (P \overline{u} U \overline{v} V + (u \cdot vw) \cdot v = g(U, f(U, V P) V) + f(U, V)^\theta P \]
\[ = g(v, f(v, vv \cdot u)vw) + f(v, vw)^\theta u = 0. \]
Thus (6) holds.

By (4), we have \( vv \cdot vv = q(vv) \cdot v \). Hence
\[
v = q(vv)^{-1} vv \cdot vv = (vv)^{-1}(vv)
\]
and thus (7) holds.

Using (R3), Proposition 7.9 and (4), finally, we have
\[
uv^{-1}uv = q(u)^{-1}q(u)\theta uu \cdot uu = q(uu)^{-1}q(uu)u = u.
\]
Thus (8) holds.

\[\square\]

Proposition 7.15. — For each nonzero \( a \in V \), there exists \( b \in V \) such that \( a = b^{-1}b \).

Proof. — This holds by 7.14(7).

\[\square\]

8. Tits Endomorphisms

We begin this section by proving some elementary properties of arbitrary Tits endomorphisms in Propositions 8.2 and 8.3.

Definition 8.1. — Let \((K, \theta)\) be an octagonal set as defined in Definition 2.1. We will call an element of \( K \) a Tits trace (with respect to \( \theta \)) if it is of the form \( x\theta + x \) for some \( x \in K \).

Proposition 8.2. — Let \((K, \theta)\) be an octagonal set. Then the following hold:

1. The map \( x \mapsto x\theta + x \) from \( K \) to itself is additive.
2. If \( u^\theta = u \) for some \( u \in K \), then \( u = 0 \) or \( 1 \).
3. If \( u^\theta + u = v^\theta + v \) for \( u, v \in K \), then either \( u = v \) or \( u = v + 1 \).
4. If \( z^\theta \) is a Tits trace, then so is \( z \).
5. \( u^2 + u \) is a Tits trace for every \( u \in K \).
6. \( 1 \) is not a Tits trace.

Proof. — Since \( \theta \) is additive, (1) holds. Suppose that \( u^\theta = u \) for some \( u \in K \), then \( u^2 = u \) and hence \( u = 0 \) or \( 1 \). Thus (2) holds, and (3) follows from (1) and (2). Suppose that \( z^\theta = u^\theta + u \) for some \( u \in K \). Let \( v = u + z \). Then \( v^\theta = u \) and hence \( z = v^\theta + v \). Thus (4) holds. Applying the map \( x \mapsto x^\theta + x \) twice to an element \( u \in K \) yields \( u^2 + u \). Thus (5) holds. Suppose, finally, that \( u^\theta + u = 1 \) for some \( u \in K \). Then \( u^2 = (u^\theta)^\theta = (u + 1)^\theta = u^\theta + 1 = u \) and hence \( u = 0 \) or \( 1 \). Thus (6) holds. \[\square\]
Proposition 8.3. — Let $\theta$ be a Tits endomorphism of a field $K$, let $\delta \in K$, let $L$ be the splitting field over $K$ of the polynomial $x^2 + x + \delta$ and let $\chi$ be the non-trivial element of $\text{Gal}(L/K)$. Then the following hold:

1. $\theta$ extends to a Tits endomorphism of $L$ if and only if $\delta$ is a Tits trace.

2. If $\theta$ extends to a Tits endomorphism of $L$, then there are exactly two extensions $\theta_1$ and $\theta_2$, both commute with $\chi$ and $\chi = \theta_2^{-1} \cdot \theta_1$.

Proof. — Let $\gamma$ be a root of $x^2 + x + \delta$ in $L$. Suppose that $\delta = \lambda^\theta + \lambda$ for some $\lambda \in K$. We extend $\theta$ to an endomorphism $\theta_1$ of $L$ by setting $\gamma^{\theta_1} = \gamma + \lambda$ and 

$$(a + b\gamma)^{\theta_1} = a^\theta + b^\theta \gamma^{\theta_1}$$

for all $a, b \in K$. We have

$$\gamma + \lambda + \lambda^\theta = \gamma + \delta = \gamma^2.$$ 

Hence $\theta_1^2 = \text{Frob}_L$. Thus $\theta_1$ is a Tits endomorphism of $L$.

Suppose, conversely, that $\theta$ extends to a Tits endomorphism $\theta_1$ of $L$. Then $\gamma^{\theta_1} = a + b\gamma$ for some $a, b \in K$. Therefore

$$\gamma + \delta = \gamma^2 = a^\theta + b^\theta \gamma^{\theta_1} = a^\theta + b^\theta (a + b\gamma) = a^\theta + ab^\theta + b^{\theta+1} \gamma,$$

so $\delta = a^\theta + ab^\theta$ and $b^{\theta+1} = 1$. Hence $b = b^{\theta^2-1} = (b^{\theta+1})^{\theta-1} = 1$ and therefore

$$\delta = a^\theta + a,$$

so $\delta$ is a Tits trace and hence (1) holds. Furthermore

$$\gamma^{\chi^{\theta_1}} = \gamma^{\theta_1} + 1 = a + 1 + \gamma = a + \gamma^\chi = \gamma^{\theta_1 \chi},$$

so $\theta_1$ commutes with $\chi$ and thus the product $\theta_2 := \chi \theta_1$ is a second Tits endomorphism of $L$ extending $\theta$. By 8.2(3) and (8.4), there are no others.

We now go back to assuming that $S, K, V, q, f$ and $\Xi = Q(S)$ are as in Notations 5.6 and 5.7, that $\rho, \varphi$ and $\theta$ are as is as in Hypothesis 6.1 and Notations 6.4 and 6.8, and that the product $uv$ on $V$ is as in (6.18). Our goal for the rest of this section is to prove Proposition 8.5.
Proposition 8.5. — Let $d$ be a non-zero element of $V$. Then there exists an element $e \in V$ such that $f(d, e) = 1$, $f(d, ed) = 0$, $f(dd, ed) = q(d)^\theta$

as well as

$$q(d)^\theta \cdot de = f(de, ed) \cdot dd + q(d) \cdot ed$$

and $q(d)q(e) = f(de, ed) + f(de, ed)^\theta$. In particular, $q(d)q(e)$ is a Tits trace.

Proof. — By [2, Theorem 2.1(i)], there exists $e \in V$ such that

$$f(d, e) = 1$$

and $f(d, ed) = 0$. By 7.4(2), it follows that

$$f(dd, e) = 0.$$  

By 7.4(1), we have

$$f(d, dd) = 0$$

and by 7.4(2), we have

$$f(dd, de) = f(de, ed)^\theta + q(d)f(d, e)^\theta = q(d)$$

and by 7.14(2) and (8.8), we have

$$f(dd, ed) = q(d)^\theta \cdot f(d, e) = q(d)^\theta.$$  

Thus (8.6) holds.

Choose $\lambda \in K$. The image of the map $g$ is the radical of $f$. By (R1), (R3) and (R7), therefore, we have

$$f((e + \lambda \cdot dd)(e + \lambda \cdot dd), d) = f(ee, d) + \lambda f(dd \cdot e, d) + \lambda^\theta f(e \cdot dd, d) + \lambda^{\theta+1} f(dd \cdot dd, d).$$

We observe that $f(dd \cdot e, d) = f(dd, de) = q(d)$ by 7.4(2) and (8.12) as well as

$$f(e \cdot dd, d) = f(e, d \cdot dd) \quad \text{by 7.4 (2)}$$

$$= q(d)f(e, dd) \quad \text{by (R6)}$$

$$= 0 \quad \text{by (8.9)},$$

and

$$f(dd \cdot dd, d) = f(q(dd) \cdot d, d) = q(dd)f(d, d) = 0.$$
by 7.14(4). It follows that
\[ f((e + \lambda \cdot dd)(e + \lambda \cdot dd), d) = f(ee, d) + \lambda q(d). \]

Furthermore,
\[ f(d, e + \lambda \cdot dd) = f(d, e) = 1 \]
by (8.8) and (8.10) and \( f(d, (e + \lambda \cdot dd)d) = f(d, ed) + \lambda f(d, dd \cdot d) \) by (R1),
so, in fact,
\[ f(d, (e + \lambda \cdot dd)d) = 0 \]
by (8.8) and (8.11). Thus if we replace \( e \) by \( e + \lambda \cdot dd \) and \( \lambda \) by \( f(ee, d)/q(d) \), we can assume that \( f(ee, d) = 0 \) while (8.8) and therefore also (8.6) remain valid. Hence also
\[ f(e, de) = 0 \]
by 7.4(2). By 7.11(1), it follows that
\[ f(ed, ee) = f(e, de)^\theta + q(e)f(d, e) = q(e). \]

By 7.4(1) and (8.13), we have \( de \in \langle d, e \rangle \perp \) (where \( \langle d, e \rangle \perp \) denotes the subspace orthogonal to \( \langle d, e \rangle \) with respect to the bilinear form \( f \)). Setting \( \xi = d \) in [2, Theorem 2.1], we obtain \( \langle d, e \rangle \perp = \langle dd, ed \rangle + [K] \). Hence
\[ de = \kappa \cdot dd + \mu \cdot ed + [t] \]
for some \( \kappa, \mu, t \in K \). Therefore
\[ f(dd, de) = f(dd, \kappa \cdot dd + \mu \cdot ed) = \mu \cdot f(dd, ed) = \mu \cdot q(d)^\theta \]
and
\[ f(de, ed) = f(\kappa \cdot dd + \mu \cdot ed, ed) = \kappa \cdot f(dd, ed) = \kappa \cdot q(d)^\theta \]
by (8.6), so
\[ \mu = q(d)^{1-\theta} \]
by (8.12) and
\[ \kappa = f(de, ed)/q(d)^\theta. \]

Substituting (8.16) but not (8.17) in (8.15) and applying \( q \), we obtain
\[ q(de) = q(d)q(e)^\theta = \kappa^2 q(d)^{\theta+1} + q(d)^{2-2\theta} \cdot q(e)q(d)^\theta \]
\[ + t^\theta + \kappa q(d)^{1-\theta}f(dd, ed) \]
by Propositions 7.9 and 7.10. Multiplying by \( q(d)^{\theta-1} \) and applying (8.6), we then obtain
\[ q(d)^\theta q(e)^\theta = \kappa^2 q(d)^{2\theta} + \kappa q(d)^\theta + q(d)q(e) + t^\theta q(d)^{\theta-1}. \]
Thus \( \kappa q(d)^\theta \) is a root of the polynomial
\[
\tag{8.18} p(x) := x^2 + x + q(d)q(e) + q(d)^\theta q(e)^\theta + t^\theta q(d)^{\theta - 1}.
\]
By 8.17, \( \kappa q(d)^\theta = f(de, ed) \) and by 7.14(3) and (8.8), we have
\[
\tag{8.19} f(de, ed) + f(dd, ee) = f(d, e)f(e, d)^\theta = 1.
\]
We conclude that \( f(de, ed) \) and \( f(dd, ee) \) are the two roots of the polynomial \( p \).

Next we observe that
\[
q(e)^\theta = q(e)^\theta f(d, e) \quad \text{by (8.8)}
\]
\[
= f(de, ee) \quad \text{by 7.11(2)}
\]
\[
= \kappa \cdot f(dd, ee) + \mu \cdot q(e) \quad \text{by (8.14) and (8.15)}
\]
\[
= (f(de, ed) \cdot f(dd, ee) + q(d)q(e)) / q(d)^\theta \quad \text{by (8.16) and (8.17)},
\]
so
\[
f(de, ed) \cdot f(dd, ee) = q(d)q(e) + q(d)^\theta q(e)^\theta.
\]
Thus by (8.19), \( f(de, ed) \) is a root of the polynomial
\[
x^2 + x + q(d)q(e) + q(d)^\theta q(e)^\theta.
\]
Since \( f(de, ed) \) is also a root of the polynomial \( p(x) \) defined in (8.18), we conclude that \( t = 0 \). Hence
\[
q(d)^\theta \cdot de = q(d)^\theta (\kappa \cdot dd + \mu \cdot ed) \quad \text{by (8.15)}
\]
\[
= f(de, ed) \cdot dd + q(d) \cdot ed \quad \text{by (8.16) and (8.17)}.
\]
Thus (8.7) holds.

Let \( s = f(de, ed) \). We multiply (8.7) on the left by \( d \). Applying (R3) and (R7), we obtain
\[
\tag{8.20} q(d)^2 d \cdot de = s^\theta d \cdot dd + q(d)^\theta d \cdot ed + g(s \cdot (dd \cdot d), q(d) \cdot ed).
\]
By Proposition 7.5, (8.8) and (8.9), we have
\[
d \cdot de = f(e, dd)e + f(e, d)ed + q(e)dd = ed + q(e)dd.
\]
By (R6), we have \( d \cdot dd = q(d)dd \) and \( d \cdot ed = q(d)de \). We also have
\[
g(dd \cdot d, ed) = g(q(d)^\theta d, ed) = g(q(d)^\theta dd, e) = 0
\]
by (R5) and (8.9). Hence we can rewrite (8.20) as
\[
q(d)^2 \cdot ed + q(d)^2 q(e) \cdot dd = s^\theta q(d) \cdot dd + q(d)^{\theta + 1} \cdot de.
\]
Dividing by \( q(d) \) and rearranging terms, we obtain
\[
q(d)^\theta \cdot de = (s^\theta + q(d)q(e)) \cdot dd + q(d) \cdot ed.
\]
Comparing this equation with (8.7), we conclude that
\[ s^\theta + q(d)q(e) = s. \]
In other words, \( q(d)q(e) = f(de, ed) + f(de, ed)^\theta. \)

\[\square\]

9. Polar Triples

We continue to assume that \( S, K, F, V, q, f, \Xi = Q(S), \Sigma, c \) and \( x_1, \ldots, x_4 \) are as in Notations 5.6 and 5.7, that \( \rho, \phi \) and \( \theta \) are as is as in Hypothesis 6.1 and Notations 6.4 and 6.8, and that the product \( uv \) on \( V \) is as in (6.18). Thus \( F = K^\theta, \theta \) is a Tits endomorphism of \( K \) by 6.10(1) and \( \Xi = Q_x(K, V, q) = B_2^x(K, V, q) \) by Notation 5.10. The main results of this section are Theorem 9.12 and Corollary 9.26.

**Proposition 9.1.** — Let \( d \) and \( e \) as in Proposition 8.5 and let \( \xi = \phi(d) \), so that \( f(d, e) = 1 \) and \( f(d, e\xi) = 0 \). Let
\[ S_0 := (L/K, K^\theta, \alpha_0, \beta_0) \]
be the standard decomposition of \( q \) obtained by applying Theorem 5.16 to the triple \( d, e \) and \( \xi \). Then \( \alpha_0 = \beta_0^{-\theta} \) and \( \theta \) has an extension to a Tits endomorphism of \( L \).

**Proof.** — By Remark 5.15, the field \( L \) is the splitting field of the polynomial
\[ x^2 + x + q(d)q(e) \]
over \( K \). Hence 8.3(1) and the last assertion in Proposition 8.5, \( \theta \) has an extension to a Tits endomorphism of \( L \). By Theorem 5.16, we have \( \alpha_0 = f(d\xi, e\xi) \) and \( \beta_0 = q(d)^{-1} \). Thus by (8.6), \( \alpha_0 = q(d)^\theta = \beta_0^{-\theta}. \)

**Hypothesis 9.2.** — We assume from now on that \( d \) and \( e \) are as in Proposition 8.5 and that \( \xi = \phi(d) \) and that the standard decomposition \( S \) in Notation 5.6 is the standard decomposition \( S_0 \) in Proposition 9.1. Thus \( E/K \) is now the extension called \( L/K \) and \( \alpha \) and \( \beta \) are now the constants called \( \alpha_0 \) and \( \beta_0 \) in Proposition 9.1, the Tits endomorphism \( \theta \) has an extension to the field \( E \) and \( \beta^\theta = \alpha^{-1} \). The group \( U_+ \) is unchanged by this assumption, but we assume that \( V, \hat{V} \) and the isomorphisms \( x_1, \ldots, x_4 \) are as in Notation 5.7 with respect to the new \( S \).
Notation 9.3. — By 8.3(2), \( \theta \) has exactly two extensions to \( E \). We denote these extensions by \( \theta_1 \) and \( \theta_2 \). Both commute with the non-trivial element \( \chi \) of \( \text{Gal}(E/K) \) and \( \theta_2 = \chi \theta_1 \). Let \( \bar{x} = x^\chi \) for all \( x \in E \).

Remark 9.4. — Let \( \theta_1, \theta_2 \) and \( \chi \) be as in Notation 9.3, let \( \gamma \in E \) be a root of

\[ x^2 + x + q(d)q(e) \]

and let \( i = 1 \) or \( 2 \). Since \( \chi \) commutes with \( \theta_i \), we have \( \gamma^{\theta_i} \notin K \). Thus \( E = K(\gamma^{\theta_i}) \) and hence

\[ D = E^2 F = F(\gamma^2) = K(\gamma^{\theta_i})^{\theta_i} = E^{\theta_i}. \]

**Proposition 9.5.** — For either \( i = 1 \) or \( i = 2 \), the map \( \varphi \) from \( V = E \oplus E \oplus [F]_K \) to \( \hat{V} = D \oplus D \oplus [K]_F \) is given by

\[ \varphi(a, b, s) = (a^{\theta_i}, \beta^{-2}b^{\theta_i}, s^{\theta^{-1}}) \]

for all \( (a, b, s) \in V \).

**Proof.** — Let \( \eta = \varphi(e) \). Then by (R5) and (8.7), we have

\[ d \eta \cdot \xi \in \langle d, e \rangle. \]

Applying \( \varphi \), we obtain

\[ \xi e \cdot d \in \langle \xi, \eta \rangle \]

by 6.7(1). By (6.9) and 7.4(2), we have

\[ \hat{f}(\xi e \cdot d, \xi) = \hat{f}(\varphi(d \eta \cdot \xi), \varphi(d)) = f(d \eta \cdot \xi, d) = f(d \eta, d \xi)^{\theta}. \]

By (8.12), it follows that \( \hat{f}(\xi e \cdot d, \xi) \neq 0 \). Thus \( \xi e \cdot d \) and \( \xi \) are linearly independent. By (9.6), therefore,

\[ \eta \in \langle \xi, \xi e \cdot d \rangle. \]

By 6.7(1) again, we have \( \varphi(e \xi) = \eta d \) and \( \varphi(d \xi) = \xi d \). Hence by (R1), (R5) and (9.7),

\[ \varphi(e \xi) \in \langle \xi d, \xi e \rangle. \]

By 6.10(2), \( \varphi \) is an isomorphism of vector spaces. Thus

\[ \varphi(\langle d, e \rangle) = \langle \xi, \xi e \cdot d \rangle \]

and

\[ \varphi(\langle d \xi, e \xi \rangle) = \langle \xi d, \xi e \rangle. \]
Let \( \gamma, \chi, \theta_1 \) and \( \theta_2 \) be as in Remark 9.4 and let \( N(x) = x \cdot x^\chi \) and \( T(x) = x + x^\chi \) for all \( x \in E \). We set \( \omega = \beta \gamma \). Thus
\[
\hat{q}(\xi) = \hat{q}(\varphi(d)) = q(d)^\theta = \beta^{-\theta} = \alpha
\]
and \( \omega \) is a root in \( E \) of \( q(d)x^2 + x + q(e) \). By Theorem 5.16, we can assume that \( d = (1,0,0), e = (\omega,0,0), d\xi = (0,1,0) \) and \( e\xi = (0,\omega,0) \) in \( V \) and \( \xi = (1,0,0), \xi e d^{-1} = (\omega^2,0,0), \xi d^{-1} = (0,1,0) \) and \( \beta^2 \xi e = (0,\omega^2,0) \) in \( V \). Thus \( \varphi(d) = \xi = (1,0,0) \).

By Notation 6.8, (9.8) and (9.9), there exist maps \( \varphi_1 \) and \( \varphi_2 \) from \( E \) to \( D \) such that
\[
\varphi(a,b,s) = (\varphi_1(a), \varphi_2(b), s^{\theta^{-1}})
\]
for all \( (a,b,s) \in V \). By 6.10(2), \( \varphi_1 \) and \( \varphi_2 \) are \( \theta \)-linear and since \( \varphi(d) = \xi \), we have \( \varphi_1(1) = 1 \). Furthermore, \( f(\xi, \eta) = f(\varphi(d), \varphi(e)) = f(d,e)^\theta = 1 \) and \( \hat{q}(\eta) = \hat{q}(\varphi(e)) = q(e)^\theta \). By (5.5), therefore, \( T(\varphi_1(\omega)) = \alpha^{-1} \) and \( N(\varphi_1(\omega)) = \alpha^{-1} q(e)^\theta \). Thus \( \varphi_1(\omega) \) is a root of \( q(d)^\theta x^2 + x + q(e)^\theta \). Hence we can choose \( i \in \{1,2\} \) such that
\[
\varphi_1(\omega) = \omega^{\theta_i}.
\]
Since \( \varphi_1(1) = 1 \) and \( \varphi_1 \) is \( \theta \)-linear, it follows that
\[
\varphi_1 = \theta_i.
\]

Let \( v \in E \). By [23, 16.7], we have
\[
[x_1(1,0,0), x_4(0,v,0)]_2 = x_2(\alpha v, 0, 0)
\]
and
\[
[x_4(1,0,0), x_1(0, \varphi_2(v), 0)]_3 = x_3(\beta^{-2} \varphi_2(v), 0, 0).
\]
Applying \( \rho \) to the first of these equations, we obtain
\[
[x_4(1,0,0), x_1(0, \varphi_2(v), 0)]_3 = x_3(\varphi_1(\alpha v), 0, 0).
\]
Hence
\[
\varphi_2(x) = \varphi_1(\alpha x) = \beta^{-2} \varphi_1(x) = \beta^{-2} x^{\theta_i}.
\]
Thus by (9.10) and (9.11), we have
\[
\varphi(a,b,s) = (a^{\theta_i}, \beta^{-2} b^{\theta_i}, s^{\theta^{-1}})
\]
for all \( (a,b,s) \in V \). \( \square \)
We summarize our results as follows:

**Theorem 9.12.** — Let $\Xi = B_2^F(K, V, q)$ for some quadratic space $(K, V, q)$ of type $F_4$ and suppose that $\rho$ is a polarity of $\Xi$. Then there exists a standard decomposition $S = (E/K, F, \alpha, \beta)$ of $q$ and a Tits endomorphism $\theta$ of $E$ such that the following hold:

1. $F = K^\theta$.
2. $\alpha = \beta - \theta$.
3. $\Xi$ can be identified with $Q(S)$ in such a way that $\rho$ stabilizes $c$ and $\Sigma$ and $x_i(u, v, s)^\rho = x_{5-i}(u^\theta, \beta^{-2}v^\theta, s^{\theta^{-1}})$ for $i = 2$ and $4$ and all $(u, v, s) \in V$, where $\Sigma, c, V$ and $x_i$ for $i \in [1, 4]$ are as in Notation 5.7 applied to $S$.

**Proof.** — By Definition 5.3, we can choose a standard decomposition $S$ of $q$ and by Notation 5.10, $\Xi \cong Q(S)$. Let $\Sigma$ and $c$ be as in Notation 5.7 applied to $S$. By Remark 6.2, there exists an isomorphism $\xi_S$ from $\Xi$ with $Q(S)$ such that $\xi_S^{-1} \rho \xi_S$ stabilizes $c$ and $\Sigma$. By Proposition 9.1, the standard decomposition $S$ and a Tits endomorphism $\theta$ of $E$ can be chosen so that (1) and (2) hold. If we identify $\Xi$ with $Q(S)$ via $\xi_S$ and replace $\rho$ by $\xi_S^{-1} \rho \xi_S$ for his choice of $S$, then (3) holds by Proposition 9.5.

**Remark 9.13.** — In Example 10.1 we give an example of $\Xi, \rho, S = (E/K, F, \alpha, \beta)$ and $\theta$ satisfying the conditions (1) and (2) in Theorem 9.12 and a splitting field $\tilde{E}$ of $q_S$ (as defined in Definition 5.3) such that the restriction of $\theta$ to $K$ does not have an extension to a Tits endomorphism of $\tilde{E}$. See also Proposition 10.4. In Example 10.12, we give an example of a Moufang quadrangle of type $F_4$ that has non-type-preserving automorphisms but no polarity.

**Notation 9.14.** — Let $\Xi, \rho, S = (E/K, F, \alpha, \beta), \theta, \Sigma, c$ and the identification of $\Xi$ with $Q(S)$ be as in Theorem 9.12, let

$$\quad (U_+, U_1, \ldots, U_4)$$

and $x_1, \ldots, x_4$ be as in Notation 5.7 applied to $S$, let $x \mapsto \bar{x}$ be as in Notation 9.3 and let $\iota$ denote the map $(a, b, s) \mapsto (a, b, s^{\theta^{-1}})$ from $V_S = E \oplus E \oplus [F]_K$ to $E \oplus E \oplus [K]$, where $[K]$ is as in Notation 6.21. We identify $V = V_S$ with its image under $\iota$ and we reparametrize $U_+$ by replacing $x_i$ by $\varphi \cdot x_i$ for $i = 1$ and $3$ as in Notation 6.16 and then replacing $x_i$ by
\(\iota \cdot x_i\) for all \(i \in [1, 4]\). Thus \(x_i\) is an isomorphism from the additive group of \(V = E \oplus E \oplus [K]\) to \(U_i\) for all \(i \in [1, 4]\) and the following identities hold:

\[
[x_1(a, b, r), x_3(a', b', r')] = x_2(0, 0, \beta^{-1}(aa' + \alpha(bb' + \bar{bb}'))),
\]

\[
[x_2(u, v, s), x_4(u', v', s')] = x_3(0, 0, \beta^{-1}(u\bar{u}' + \alpha(vv' + \bar{v}v'))),
\]

\[
[x_1(a, b, r), x_4(u, v, s)] = x_2(ru + \alpha(\bar{a}\theta v + \beta^{-1}b\theta \bar{v}), rv + \alpha u + \beta^{-1}b\theta \bar{u},
\]

\[
\quad r\theta s + \beta^{-1}s(a\bar{a} + \alpha bb)
\]

\[
\quad + \alpha \beta^{-1}(u\theta a\bar{b} + \bar{a}\theta ab + \beta^{-1}(v\theta a\bar{b} + \bar{a}\theta ab)))
\]

\[
\cdot x_3(sa + \alpha(u\theta b + \beta^{-1}v\theta \bar{b}), sb + u\theta a + \beta^{-1}v\theta \bar{a},
\]

\[
\quad s\theta r + \beta^{-1}r(u\bar{u} + \alpha vv)
\]

\[
\quad + \alpha \beta^{-1}(a\theta uv + a\theta v\bar{u} + \beta^{-1}(b\theta u\bar{v} + \bar{b}\theta uv)))
\]

for all \((a, b, r), (a', b', r'), (u, v, s), (u', v', s') \in V\),

\[
[U_1, U_2] = [U_2, U_3] = [U_3, U_4] = 1
\]

and

\[
x_i(v) = x_{5-i}(v)
\]

for all \(i \in [1, 4]\) and all \(v \in V\).

**Proposition 9.16.** — Let \(\Xi, S = (E/K, F, \alpha, \beta), \theta, \rho\) and the identification of \(\Xi\) with \(Q(S)\) be as in Theorem 9.12, let \(\cdot\) be the multiplication on \(V\) defined in (6.18), let \(x \mapsto \bar{x}\) be as in Notation 9.3 and let \(V\) be identified with \(E \oplus E \oplus [K]\) as in Notation 9.14. Then

\[
(a, b, r) \cdot (u, v, s) = (sa + \alpha(u\theta b + \beta^{-1}v\theta \bar{b}), sb + u\theta a + \beta^{-1}v\theta \bar{a},
\]

\[
\quad s\theta r + \beta^{-1}r(u\bar{u} + \alpha vv)
\]

\[
\quad + \alpha \beta^{-1}(a\theta uv + a\theta v\bar{u} + \beta^{-1}(b\theta u\bar{v} + \bar{b}\theta uv)))
\]

for all \((a, b, r), (u, v, s) \in V\).

**Proof.** — This holds by (6.20) and Notation 9.14. \(\Box\)

**Notation 9.17.** — Let \(\Xi, (K, V, q), \theta\) and \(S = (E/K, F, \alpha, \beta)\) and the identification of \(\Xi\) with \(Q(S)\) be as in Theorem 9.12, so \(F = K^\theta\) and \(\alpha = \beta^{-\theta}\). We identify \(V\) with \(E \oplus E \oplus [K]\) as in Notation 9.14, so that

\[
q(u, v, t) = \beta^{-1}(N(u) + \alpha N(v)) + t^\theta
\]

for all \((u, v, t) \in V\). Let \([t] = (0, 0, t)\) for all \(t \in K\) and let \(\cdot\) be the multiplication on \(V\) given by the formula in Proposition 9.16. By Proposition 6.23,
$(K, V, q, \theta, t \mapsto [t], \cdot)$ is a polarity algebra. We denote this polarity algebra by $A = A(E/K, \theta, \beta)$.

**Definition 9.19.** — A polar triple is a triple $(E/K, \theta, \beta)$, where $E/K$ is a separable quadratic extension in characteristic $2$, $\theta$ is a Tits endomorphism of $E$ such that $F := K^\theta \subset K$ and $\beta$ is an element of $K$ such that the quadratic form on $E \oplus E \oplus [K]$ given by (9.18) is anisotropic, where $[K]$ is as defined in Notation 6.21.

In the next result, we show that every polarity algebra is of the form $A(E/K, \theta, \beta)$ for some polar triple $(E/K, \theta, \beta)$ as defined in Definition 9.19. See also Theorem 19.1.

**Theorem 9.20.** — Let $P = (K, V, q, \theta, t \mapsto [t], \cdot)$ be a polarity algebra as defined in Definition 7.1. Then $q$ is a quadratic form of type $F_4$ and there exists:

1. a standard decomposition $S = (E/K, F, \alpha, \beta)$ of $q$ such that $\alpha = \beta^{-\theta}$ and $F = K^\theta$,
2. an extension of $\theta$ to a Tits endomorphism of $E$ and
3. an identification of $V$ with $E \oplus E \oplus [K]$ with respect to which $t \mapsto [t]$ is the map $t \mapsto (0, 0, t), \cdot$ is given by the formula in Proposition 9.16 and

$$q(u, v, t) = \beta^{-1}(N(u) + \beta^\theta N(v)) + t^\theta$$

for all $(u, v, t) \in E \oplus E \oplus [K]$, where $N$ is the norm of the extension $E/K$.

Proof. — Let $F := K^\theta$, and let $\hat{V}$ be the set consisting of the symbols $\hat{v}$ for all $v \in V$, i.e. the map $v \mapsto \hat{v}$ is a bijection from $V$ to $\hat{V}$. We make $\hat{V}$ into an $F$-vector space by defining

$$s \cdot \hat{v} := \hat{s^{-1}v}$$

for all $s \in F$ and all $\hat{v} \in \hat{V}$, or equivalently,

$$t^\theta \cdot \hat{v} := \hat{tv}$$

for all $t \in K$ and all $v \in V$. The map $\hat{q} : \hat{V} \to F$ given by

$$\hat{q}(\hat{v}) := q(v)^\theta$$

for all $\hat{v} \in \hat{V}$ is a quadratic form over $F$. For each $t \in K$, we define

$$[t]_F := [\hat{t}] \in [\hat{K}] \subset \hat{V},$$

and for each $s \in F$, we define

$$[s]_K := [s^{-1}] \in [K] \subset V.$$
Next we define maps from $V \times \hat{V}$ to $V$ and from $\hat{V} \times V$ to $\hat{V}$ (both denoted by juxtaposition) by
\begin{equation}
(9.25) \quad v\hat{w} = vw \quad \text{and} \quad \hat{v}w = \hat{vw}
\end{equation}
for all $v, w \in V$, where the multiplication on the right hand side of both equations is the multiplication of the polarity algebra. We claim that these data satisfy the axioms (F0)–(F12) of 5.18. To illustrate this, we will prove (F2), (F4) and (F7) and leave the verification of the other axioms to the reader.

So let $v, w \in V, s \in F,$ and $t \in K$; then using (9.24), (9.25), (R2) and (9.21), we obtain
\begin{equation}
\hat{v}[s]_K = \hat{v}[s^{\theta^{-1}}] = \hat{v}[s^{\theta^{-1}}] = \hat{s}^{\theta^{-1}}v = s\hat{v}.
\end{equation}
Thus (F2) holds. Next, by (9.25), (R3), (9.21) again and (F2), we obtain
\begin{equation}
\hat{v} \cdot tw = \hat{v} \cdot tw = (t^{\theta})^{\theta} \cdot \hat{vw} = (t^{\theta})^{\theta} \cdot \hat{vw} = t^{2}\hat{vw} = \hat{vw} \cdot [t^2]_K.
\end{equation}
Thus (F4) holds. By (9.25), (R5), (9.22) and (F1), we obtain
\begin{equation}
v\hat{w} \cdot \hat{w} = vw \cdot w = q(w)^{\theta} \cdot v = q(\hat{v}) \cdot v = v \cdot [q(\hat{v})]_F.
\end{equation}
Thus (F7) holds. This (together with the proof of the remaining identities) shows that $V$ and $\hat{V}$, together with the maps we just defined, form a radical quadrangular system as defined in [3, Appendix A.3.2]. We denote this quadrangular system by $\Theta$.

Let $\Omega = (U_+, U_1, \ldots, U_4)$ and $x_1, \ldots, x_4$ be as in Notation 5.19. By [3, Chapter 4], $\Omega$ is a root group sequence and thus $\Omega$ determines a unique Moufang quadrangle $\Xi$ by [23, 7.5 and 8.5]. It follows from (9.22), (9.23), (9.24) and (9.25) that there is a unique anti-automorphism $\rho$ of $\Omega$ extending the maps $x_i(\hat{v}) \mapsto x_{5-i}(v)$ for $i = 1$ and 3, and $x_i(v) \mapsto x_{5-i}(\hat{v})$ for $i = 2$ and 4. The square $\rho^2$ centralizes $U_+$. Thus $\rho$ induces a polarity of the Moufang quadrangle $\Xi$ (by [23, 7.5]).

By Definition 7.1, $\partial f$ is not identically zero. By Notation 5.19, therefore, $[U_2, U_4] \neq 1$. By [23, 17.4], $[U_2, U_4] \neq 1$ and the existence of an anti-automorphism of $\Omega$ imply that $\Xi$ is a quadrangle of type $F_4$. In other words, $\Xi$ is isomorphic to the quadrangle $Q_F(\hat{\Lambda}) = B_2^F(\hat{\Lambda})$ obtained by applying [23, 16.7] to some quadratic space $\Lambda = (\hat{K}, \hat{V}, \hat{q})$ of type $F_4$. Let $Y_1 = C_{U_2}(U_3)$, let $Y_3 = C_{U_3}(U_1)$ and let $Y_+ = Y_1U_2Y_3U_4$. By (F5) and Notation 5.19, $Y_1 = x_i([K]_F)$ for $i = 1$ and 3, $Y_+$ is a subgroup of $U_+$ and ($Y_+, Y_1U_2Y_3U_4$) is a root group sequence isomorphic to the root group sequence $Q_Q(K, V, q)$ obtained by applying [23, 16.3] to $(K, V, q)$. By [23,
16.7], on the other hand, \((Y_+, Y_1, U_2, Y_3, U_4)\) is a root group sequence isomorphic to \(Q_{\tilde{\Lambda}}\). By [23, 35.8], it follows that \(K \cong \tilde{K}\) and \(q\) is similar to \(\tilde{q}\). Thus \(q\) is of type \(F_4\) and \(\Xi \cong B_2^F(K, V, q)\) (by [23, 35.12]). We conclude that the quadrangular system \(\Theta\) is an extension of the quadrangular system associated with \((K, V, q)\); see the beginning of [3, Chapter 8] for the definition of these terms. This is exactly the situation investigated in [3, §8.5] (and [23, Chapter 28]). By [3, Theorem 8.107], there exists a standard decomposition \(S\) of \(q\) and an isomorphism \(\xi\) from \(\Omega\) to the root group sequence \(\Omega_S\) obtained by applying Notation 5.7 to \(S\) extending the maps 
\[x_i([t]_F) \mapsto x_i(0, 0, t)\] for \(i = 1\) and \(3\) and 
\[x_i([s]_K) \mapsto x_i(0, 0, s)\] for \(i = 2\) and \(4\). We now replace \(\rho\) by the unique automorphism of \(Q(S)\) obtained by applying \([23, 7.5]\) to the automorphism \(\xi^{-1}\rho\xi\) of \(\Omega_S\). By (9.23) and (9.24), we have 
\[x_i(0, 0, t^\rho) = x_i(0, 0, t^{\theta})\] for all \(t \in K\). Thus \(\theta\) is as in Notation 6.8. By Proposition 9.1, we conclude that (1)–(3) hold.

**Corollary 9.26.** — Every polarity algebra is of the form \(A(E/K, \theta, \beta)\) for some polar triple \((E/K, \theta, \beta)\) as defined in Definition 9.19.

**Proof.** — This holds by Notation 9.17 and Theorem 9.20.

## 10. Two Examples

In this section, we give two examples illustrating the results of the previous section; see Remark 9.13.

**Example 10.1.** — Let \(K = \mathbb{F}_2(\alpha, \beta)\) be a purely transcendental extension of the field \(\mathbb{F}_2\), let \(E\) be the splitting field of the polynomial 
\[p(x) = x^2 + x + \alpha + \beta^2\]
over \(K\), let \(\gamma \in E\) be a root of \(p(x)\), let \(\theta\) denote the unique Tits endomorphism of \(K\) that maps \(\beta\) to \(\alpha\) and let \(F = K^\theta\). By 8.3(1), \(\theta\) has an extension to a Tits endomorphism of \(E\). We leave it to the reader to check that \(S := (E/K, F, \alpha^{-1}, \beta)\) is an \(F_4\)-datum as defined in Notation 5.2. Let 
\[q = q_S\]
be the quadratic form of type \(F_4\) on \(E \oplus E \oplus [F]_K\) as defined in Notation 5.2. We claim that \(\beta\) is not a Tits trace of \(K\) (with respect to \(\theta\)). To show this, we assume that 
\[
\beta = g(\alpha, \beta) + g(\alpha, \beta)\theta = g(\alpha, \beta) + g(\beta^2, \alpha)
\]
for some rational function \(g(\alpha, \beta) \in K\). Let \(k = \text{deg}_\alpha(g)\) and \(m = \text{deg}_\beta(g)\). (If \(g = g_1/g_2\) for polynomials \(g_1\) and \(g_2\) in \(\mathbb{F}_2[\alpha, \beta]\) and \(u = \alpha\) or \(\beta\), then \(\text{deg}_u(g) = \text{deg}_u(g_1) - \text{deg}_u(g_2)\).) We have
\[(10.2) \quad 1 = \text{deg}_\beta^2(\beta) = \text{deg}_\beta^2(\text{deg}_\beta^2(\beta)) \leq \max(2k, m)\]
Applying Theorem 5.16 with $\theta$ and the choice of trace and let $\rho$ and let $\alpha$ and let $\phi$.

and

\[(10.3) \quad 1 = \max(2k, m) \text{ if } m \neq 2k\]

and

\[0 = \deg_\alpha(\beta) = \deg_\beta(g(\alpha, \beta) + g(\beta_2, \alpha)) \leq \max(k, m)\]

and $0 = \max(k, m)$ if $k \neq m$. By (10.2), we have $0 \neq \max(k, m)$. Thus $k = m \neq 0$. Hence $m \neq 2k$ and $\max(2k, m) \geq 2$, which is impossible by (10.3). We conclude that $\beta$ is not a Tits trace in $K$ as claimed. By 8.2(4), it follows that also $\beta_2$ is not a Tits trace in $K$. Let $L$ be the splitting field of the polynomial

\[p_1(x) = x^2 + x + \beta^2\]

over $K$. By 8.3(1), $\theta$ does not have an extension to a Tits endomorphism of $L$. Let $d = (\beta, 0, 0)$ and $e = (\gamma, \alpha, 0)$ in $V$. Then $q(d) = q(e) = \beta$ and $f(d, e) = 1$, so $L$ is also the splitting field of $q(d)x^2 + x + q(e)$ over $K$. Applying Theorem 5.16 with $\xi = (1, 0, 0) \in \hat{\hat{V}}$, we conclude that $L$ is a splitting field of $q$. Thus the Tits endomorphism $\theta$ of $K$ has an extension to a Tits endomorphism of some of the splitting fields of $q$ but there are also splitting fields of $q$ to which $\theta$ does not have an extension to a Tits endomorphism.

Proposition 10.4. — Let $S = (E/K, F, \alpha, \beta)$ be an $F_4$-datum, let $\Xi = \mathbb{Q}(S)$ and suppose that $\theta$ is a Tits endomorphism of $K$ such that $F = K^\theta$ and $\alpha = \beta^{-\theta}$. Choose $\lambda \in K$ such that $E$ is the splitting field of the polynomial $x^2 + x + \lambda$ over $K$. Then $\Xi$ has a polarity if and only if there exists $u \in K$ such that

\[\lambda + \alpha u^2\]

is a Tits trace with respect to $\theta$.

Proof. — Let $q = q_S$, let $f = \partial q$ and let $\gamma \in E$ be a root of $x^2 + x + \lambda$. Let $V$, $D$ and $\hat{\hat{V}}$ be as in Notation 5.6, let $d = (1, 0, 0)$ and $e = (\beta \gamma, 0, 0)$ in $V$ and let $\xi = (1, 0, 0) \in \hat{\hat{V}}$. Then $q(d) = \beta^{-1}$ and $q(e) = \beta \lambda$. Hence $\omega := \beta \gamma$ is a root of $q(d)x^2 + x + q(e)$.

We suppose now that $u$ is an element of $K$ such that $\lambda + \alpha u^2$ is a Tits trace and let $E'$ be the splitting field of $x^2 + x + \lambda + \alpha u^2$ over $K$. By 8.3(1) and the choice of $u$, we can choose a Tits endomorphism $\theta_1$ of $E'$ extending $\theta$. As in Remark 9.4, we have $(E')^{\theta_1} = (E')^2 F$. Since $\alpha u^2 \in F$, we can set $e' = e + (0, 0, \alpha u^2)$. Thus $f(d, e') = 1$ and, by (F0) and (F6), $f(d, e' \xi) = 0$. Applying Theorem 5.16 with $e'$ in place of $e$, it follows that we can assume that $E' = E$. Now let $\varphi : V \to \hat{\hat{V}}$ be given by the formula in Proposition 9.5 and let $\rho$ be defined by the equations in Notation 6.4 with $\varphi_1 = \varphi$ and
\[\phi = \psi_1 = \psi^{-1}.\] Then \(\rho\) is an automorphism of \(U_+\) of order 2 mapping \(U_i\) to \(U_{5-i}\) for each \(i \in [1, 4]\). Thus \(\Xi\) has a polarity.

Suppose, conversely, that \(\Xi\) has a polarity \(\rho\). Our goal is to find an element \(u \in K\) such that \(\lambda + au^2\) is a Tits trace. By Proposition 9.1, we can choose \(e' \in V\) and \(\xi' \in \hat{V}\) such that \(f(d, e') = 1\), \(f(d, e'\xi') = 0\) and \(\hat{q}(\xi') = \alpha\) such that \(\theta\) has an extension to the splitting field of \(x^2 + x + q(d)q(e')\) over \(K\). Thus \(q(d)q(e')\) is a Tits trace. Since \(f(d, e') = 1\), we have \(e' = (t + \beta \gamma, y + z \gamma, s)\) for some \(t, y, z \in K\) and some \(s \in F\). Let \(e'' = e' + (t, 0, 0)\). By [2, Lemma 2.1], we have \(f(d, e''\xi') = 0\). We also have \(q(e'') = q(e') + \beta^{-1}t^2 + t\) and hence \(q(d)q(e'') + q(d)q(e') = \beta^{-2}t^2 + \beta^{-1}t\).

By 8.2(5), this expression is a Tits trace. It follows that we can assume that

\[(10.5)\]
\[e' = (\beta \gamma, y + z \gamma, s).\]

Hence

\[q(d)q(e') = \beta^{-2}(N(\beta \gamma) + \alpha(y^2 + yz + \lambda z^2)) + \beta^{-1}s\]
\[= \lambda + \alpha \beta^{-2}(y^2 + yz + \lambda z^2) + \beta^{-1}s.\]

We have \(s = x^\theta\) for some \(x \in K\) and thus

\[(\beta^{-1}s)^\theta = \alpha s^\theta = \alpha x^2.\]

By 8.2(1), therefore,

\[(10.6)\]
\[p : = \lambda + \alpha \beta^{-2}(y^2 + yz + \lambda z^2) + \alpha x^2\]
\[= \lambda + \alpha \beta^{-2}((y + \beta x)^2 + z(y + \lambda z))\]

is a Tits trace.

By (F12) and the choice of \(e'\) and \(\xi'\), we have

\[(10.7)\]
\[f(d\xi', e') = f(d, e'\xi') = 0.\]

By (F12), we also have \(f(d\xi', d) = 0\), from which it follows that there exist \(w, u, v, r \in K\) such that

\[(10.8)\]
\[d\xi' = (w, u + v \gamma, r^{\theta}).\]

By [3, 8.95], we have \(q(d\xi') = q(d)\hat{q}(\xi') = \beta^{-1}\alpha\). Hence

\[(10.9)\]
\[w^2 + \alpha(u^2 + uv + \lambda v^2 + 1) + \beta r^{\theta} = 0.\]

By (10.5), (10.7) and (10.8), we have

\[(10.10)\]
\[\beta w + \alpha(zu + yv) = 0.\]

Suppose that \(v = 0\). Then \(w^2 + \alpha(u^2 + 1) + \beta r^{\theta} = 0\) by (10.9), hence \(\beta r^{\theta} \in F\) and therefore, \(r = 0\) since \(\beta \notin F\). Hence \(\alpha(u + 1)^2 \in K^2\) and
therefore $u = 1$ since $\alpha \not\in K^2$. Hence $w = 0$. By (10.10), therefore, $z = 0$. Thus by (10.6), we have $p = \lambda + \alpha a^2$ for $a = \beta^{-1}(y + \beta x)$. Hence $w = 0$. By (10.10), therefore, $z = 0$. Thus by (10.6), we have $p = \lambda + \alpha a$ for $a = \beta^{-1}[(y + \beta x) + (\alpha^{-1} \beta w + zu)]$ by (10.10).

Suppose, finally, that $v \neq 0$. Then $y = v^{-1}(\alpha^{-1} \beta w + zu)$ by (10.10). Hence

\[
\alpha \beta^{-2} z(y + \lambda z) = \alpha \beta^{-2} z v^{-1}(\alpha^{-1} \beta w + zu + \lambda vz) = \beta^{-1} z v^{-1} w + \beta^{-2} z^2 v^{-2} \cdot \alpha(w + \lambda v^2)
\]

\[
= \beta^{-1} z v^{-1} w + \beta^{-2} z^2 v^{-2}(\alpha(u^2 + 1) + w^2 + \beta r^\theta) \quad \text{by (10.9)}
\]

\[
= \alpha(\beta^{-1} z v^{-1}(u + 1))^2 + \beta^{-1} z v^{-1} w + (\beta^{-1} z v^{-1} w)^2 + \beta^{-1} z^2 v^{-2} r^\theta.
\]

By 8.2(5) and (10.6), it follows that

(10.11) $\lambda + \alpha b^2 + \beta^{-1} z^2 v^{-2} r^\theta$

is a Tits trace for $b = \beta^{-1}(z v^{-1}(u + 1) + (y + \beta x))$. Adding the Tits trace

$\beta^{-1} z^2 v^{-2} r^\theta + (\beta^{-1} z^2 v^{-2} r^\theta)^\theta$

to the expression (10.11), we conclude that

$\lambda + \alpha b^2 + (\beta^{-1} z^2 v^{-2} r^\theta)^\theta$

is also a Tits trace. Finally, we observe that

$(\beta^{-1} z^2 v^{-2} r^\theta)^\theta = \alpha c^2$

for $c = z^\theta v^{-\theta} r$. Thus $\lambda + \alpha(b + c)^2$ is a Tits trace. $\square$

**Example 10.12.** — Let $K = F_2(\alpha, \beta)$ be a purely transcendental extension of the field $F_2$, let $E$ be the splitting field of the polynomial $p(x) = x^2 + x + 1$ over $K$, let $\gamma \in E$ be a root of $p(x)$, let $\theta$ denote the unique Tits endomorphism of $K$ that maps $\beta$ to $\alpha$ and let $F = K^\theta$. By [23, 14.25], $S := (E/K, F, \alpha^{-1}, \beta)$ is an $F_4$-datum, so we can set $\Xi = Q(S)$. There are exactly three elements of $E^*$ of finite order. Let $\hat{\theta}$ be the unique extension of $\theta$ to an endomorphism of $E$ which acts trivially on these three elements and let $\chi$ denote the non-trivial element of $\text{Gal}(E/K)$. The endomorphism $\hat{\theta}$ is, of course, not a Tits endomorphism of $E$. (By 8.2(6) and 8.3(1), $\theta$ does not have an extension to a Tits endomorphism of $E$.) Let $V$, $\hat{V}$,
\[ \Omega := (U_+, U_1, \ldots, U_4) \text{ and } x_1, \ldots, x_4 \text{ be as in Notation 5.7, let } \varphi \text{ denote the map from } V \text{ to } \hat{V} \text{ given by} \]
\[ \varphi(u, v, s) = (u^\theta, \beta^{-2} v^\theta, s^{\theta^{-1}}) \]
for all \((u, v, s) \in V\) and let \(\psi\) denote the automorphism of \(V\) given by
\[ \psi(u, v, s) = (u^\chi, v^\chi, s) \]
for all \((u, v, s) \in V\). There is a unique anti-automorphism \(\kappa\) of \(\Omega\) extending the maps \(x_i(b) \mapsto x_5 - i(\varphi(\psi(b)))\) for \(i = 2\) and \(4\) and \(x_i(a) \mapsto x_5 - i(\varphi^{-1}(a))\) for \(i = 1\) and \(3\). The square of \(\kappa\) is an involution. By [23, 7.5], therefore, \(\kappa\) gives rise to a non-type-preserving automorphism of \(\Xi\) of order 4.

We claim that \(\Xi\) does not, however, have any polarities. Let \(\Gamma\) be the additive group \(\{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}\), let \(k\) denote the field of Hahn series \(\mathbb{F}_2(t^\Gamma)\) and let \(\nu: k^* \to \Gamma\) be the canonical valuation on \(k\) (as described, for example, in [8, 3.5.6]). There is a unique embedding \(\pi\) from \(K\) to \(k\) which sends \(\beta\) to \(t\) and \(\alpha\) to \(t^{\sqrt{2}}\). We identify \(K\) with its image under \(\pi\). Modulo this identification, there is a unique extension of \(\theta\) to a Tits endomorphism of \(k\) which we also denote by \(\theta\). If \(u, v \in k\), then the constant coefficient of \(t^{\sqrt{2}}u^2\) is 0 and the constant coefficient of \(v\) is the same as the constant coefficient of \(v^\theta\). It follows that there do not exist \(u, v \in k\) such that
\[ 1 + t^{\sqrt{2}}u^2 = v + v^\theta. \]
By Proposition 10.4 with \(\alpha^{-1}\) in place of \(\alpha\), it follows that our Moufang quadrangle \(\Xi\) does not have a polarity, as claimed.

11. Buildings of Type \(F_4\)

The main results of this section are Theorems 11.10 and 11.11.

Notation 11.1. — Let \(L/E\) be a field extension such that \(\text{char}(E) = 2\) and \(L^2 \subset E\) and let \(\Delta = F_4(L, E)\) as defined in [26, 30.15]. Let \(\Phi\) be a root system of type \(F_4\), let \(\Sigma\) be an apartment of \(\Delta\), let \(c\) be a chamber of \(\Sigma\) and for each \(\alpha \in \Phi\), let \(s_\alpha\) denote the corresponding reflection. Let \(\alpha_1, \ldots, \alpha_4\) be a basis of \(\Phi\) ordered so that \(\alpha_1\) and \(\alpha_2\) are long and \(|s_{\alpha_2}s_{\alpha_3}| = 4\), let \(S\) be the set of reflections \(s_{\alpha_i}\) for \(i \in [1, 4]\) and let \(W = \langle S \rangle\) be the Weyl group of \(\Phi\). We think of the map \(i \mapsto \alpha_i\) as a bijection from the vertex set of the Coxeter diagram \(F_4\) to \(S\). There is a unique action of \(W\) on \(\Sigma\) with respect to which \(s_{\alpha_i}\) interchanges \(c\) with the unique chamber of
Σ that is $i$-adjacent to $c$, there is a unique chamber $C$ of $\Phi$ contained in the half-space determined by $\alpha_i$ for all $i \in [1, 4]$ and there is a unique $W$-equivariant bijection $\iota$ from the set of chambers of $\Sigma$ to the set of chambers of $\Phi$ mapping $c$ to $C$. The bijection $\iota$ induces a bijection from the set of roots of $\Sigma$ to $\Phi$ and its inverse induces an injection from $\Aut(\Phi)$ to $\Aut(\Sigma)$. From now on, we identify $\Aut(\Phi)$ with its image under this injection and we identify the roots of $\Sigma$ with the corresponding elements of $\Phi$. Thus for each $\beta \in \Phi$, we have a root group $U_{\beta}$ of $\Delta$.

**Theorem 11.2.** — There exists a collection of isomorphisms $x_{\beta}: E \to U_{\beta}$, one for each long root $\beta$ of $\Phi$, and a collection of isomorphisms $x_{\beta}: L \to U_{\beta}$, one for each short root $\beta$ of $\Phi$, such that for all $\alpha, \beta \in \Phi$ at an angle $\omega < 180^\circ$ to each other and for all $s$ in the domain of $x_{\alpha}$ and all $t$ in the domain of $x_{\beta}$, the following hold:

1. If $\omega = 120^\circ$, then $\alpha + \beta \in \Phi$ and $[x_{\alpha}(s), x_{\beta}(t)] = x_{\alpha+\beta}(st)$.
2. If $\omega = 135^\circ$, then $\alpha$ and $\beta$ have different lengths; if $\alpha$ is long, then $\alpha + \beta \in \Phi, \alpha + 2\beta \in \Phi$ and $[x_{\alpha}(s), x_{\beta}(t)] = x_{\alpha+\beta}(st)x_{\alpha+2\beta}(st^2)$.
3. $[x_{\alpha}(s), x_{\beta}(t)] = 1$ if $\omega$ is neither $120^\circ$ nor $135^\circ$.

**Proof.** — This holds by [1, 5.2.2] and [19, 10.3.2].

**Definition 11.3.** — We call a set $\{x_{\beta}\}_{\beta \in \Phi}$ satisfying the three conditions in Theorem 11.2 a coordinate system for $\Delta$.

**Theorem 11.4.** — Let $\{x_{\beta}\}_{\beta \in \Phi}$ be a coordinate system for $\Delta$, let $\gamma \in \Aut(\Phi)$, let $\lambda_1, \lambda_2$ be non-zero elements of $E$, let $\lambda_3, \lambda_4$ be non-zero elements of $L$ and let $\chi$ be an element of $\Aut(L)$ stabilizing $E$. Then the following hold:

1. There exists a unique automorphism
   
   $$g = g_{\gamma, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \chi}$$

   of $\Delta$ that stabilizes the apartment $\Sigma$ such that
   
   $$x_{\alpha_i}(t)^g = x_{\gamma(\alpha_i)}(\lambda_i t^\chi)$$

   for all $t$ in the domain of $x_{\alpha_i}$ and for all $i \in [1, 4]$.
2. If
   
   $$\beta = \sum_{i=1}^4 c_i \alpha_i \in \Phi,$$

   then
   
   $$x_{\beta}(t)^g = x_{\gamma(\beta)}(\lambda_\beta t^\chi)$$
for all \( t \) in the domain of \( x_\beta \), where
\[
\lambda_\beta = \prod_{i=1}^{4} \lambda_i^{c_i}.
\]

**Proof.** — Inserting \( \chi \) into [16, Lemma 58] and restricting scalars to \( E \) in the long root groups, we obtain the existence assertion in (1); (see also [16, Theorem 29]). The uniqueness assertion holds by [24, 9.7]. By 11.2(1)–(3), [9, §10.2, Lemma A] and induction, it follows that (2) holds for all \( \beta \in \Phi^+ \) (i.e. for all \( \beta \in \Phi \) that are positive with respect to the basis \( \{\alpha_1, \ldots, \alpha_4\} \)). For each \( i \in [1, 4] \), there exists a unique \( j \in [1, 4] \) such that the angle between \( \alpha_i \) and \( \alpha_j \) is 120°. By 11.2(1), \( \beta := \alpha_i + \alpha_j \in \Phi \) and \( [x_\beta(t), x_{-\alpha_j}(t)] = x_{\alpha_i}(t) \) for all \( t \) in the domain of \( x_{-\alpha_j} \) (which is the same as the domain of \( x_{\alpha_j} \)). Conjugating by \( g \) and applying (1), we conclude that \( x_{\alpha_j}(t)^g = x_{\gamma(-\alpha_i)}(\lambda_i^{-1}t) \) for all \( t \) in the domain of \( x_{\alpha_i} \). Thus by 11.2(1)–(3), [9, §10.2, Lemma A] and induction again, (2) holds for all \( \beta \in \Phi^- \). □

**Proposition 11.5.** — Every type-preserving automorphism of \( \Delta \) that stabilizes \( \Sigma \) is of the form \( g_{\gamma, \lambda_1, \ldots, \lambda_4, \chi} \) for some \( \gamma \in \text{Aut}(\Phi) \), some \( \lambda_1, \lambda_2 \in E \), some \( \lambda_3, \lambda_4 \in L \) and some \( \chi \in \text{Aut}(L,E) \).

**Proof.** — By 11.4(1), it suffices to show that every type-preserving automorphism of \( \Delta \) that stabilizes \( \Sigma \) pointwise is of the desired form. Let \( g \) be such an element. By [24, 9.7], \( g \) is uniquely determined by its restrictions to the irreducible rank 2 residues containing \( c \). These are isomorphic to \( A_2(E) \), \( B_2^P(\Lambda) \) and \( A_2(L) \), where \( \Lambda \) is the indifferent set \( (L, L, E) \). By [23, 37.13], it follows that there exist \( \lambda_1, \lambda_2 \in E^*, \lambda_3, \lambda_4 \in L \), \( \chi_E \in \text{Aut}(E) \) and \( \chi_L \in \text{Aut}(L) \) such that \( x_{\alpha_i}(t)^g = x_{\alpha_i}(\lambda_i t^{\chi_E}) \) for all \( t \in E \) if \( i = 1 \) or 2 and \( x_{\alpha_i}(t)^g = x_{\alpha_i}(\lambda_i t^{\chi_L}) \) for all \( t \in L \) if \( i = 3 \) or 4. By [23, 37.32] applied to the indifferent set \( (L, L, E) \), \( \chi_L \in \text{Aut}(L,E) \) and the restriction of \( \chi_L \) to \( E \) equals \( \chi_E \). Thus \( g = g_{\text{id}, \lambda_1, \ldots, \lambda_4, \chi} \) for \( \chi = \chi_L \). □

**Remark 11.6.** — By [11, 28.8], \( \{L/E, E/L\} \) is the pair of defining extensions of \( \Delta \); see Notation 3.5. Let \( G^0 \) and \( G^\dagger \) be as in Notation 3.2. By [24, 2.8 and 11.12], the stabilizer \( G_{\Sigma}^\dagger \) induces the same group as the stabilizer \( G_{\Sigma}^0 \) on \( \Sigma \). Thus every element in \( G_{\Sigma}^0 \) is conjugate by an element in \( G^\dagger \) to one which fixes the chamber \( c \) of \( \Sigma \). By 3.8(1)–(2), therefore, we can choose a Galois map \( \psi \) of \( \Delta \) such that
\[
\psi(g_{\gamma, \lambda_1, \ldots, \lambda_4, \chi}) = \chi
\]
for all $\gamma \in \text{Aut}(\Phi)$, for all $\lambda_1, \lambda_2 \in E$, for all $\lambda_3, \lambda_4 \in L$ and for all $\chi \in \text{Aut}(L, E)$.

**Notation 11.7.** — Let $w_1 = (s_{\alpha_2} s_{\alpha_3})^2 \in \text{Aut}(\Phi)$, where $s_{\alpha_2}$ and $s_{\alpha_3}$ are as in Notation 11.1.

**Notation 11.8.** — Let $\chi$ be an involution in the group $\text{Aut}(L, E)$ defined as in Notation 3.5, let $F_0 = \text{Fix}_L(\chi)$, let $K = F_0 \cap E$ and let $N$ be the norm of the extension $L/F_0$. Thus $F_0/K$ is a purely inseparable extension such that $F_0^2 \subset K$, the restriction of $N$ to $E$ is the norm of the extension $E/K$ and $L$ is the composite $EF_0$.

**Notation 11.9.** — Let $\chi$, $F_0$ and $K$ be as in Notation 11.8, let $F = F_0^2$ and suppose that

$$S = (E/K, F, \alpha, \beta)$$

is an $F_4$-datum for some $\alpha \in F$ and some $\beta \in K$. Let $\lambda_1 = \alpha \beta^{-1}$, let $\lambda_2 = \alpha^{-1}$, let $\lambda_3 = \beta$, let $\lambda_4$ be the unique element of $F_0$ such that $\lambda_2^2 = \beta^{-2} \alpha$ and let

$$\xi = g_{w_1, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \chi},$$

where $w_1$ is as in Notation 11.7.

In the next two results, we use the term “$\chi$-involution” (as defined in 3.11) with respect to the Galois map $\psi$ chosen in Remark 11.6.

**Theorem 11.10.** — Let $\Delta$ be as in Notation 11.1, let $S$ and $\xi$ be as in Notation 11.9 and let $\Gamma = \langle \xi \rangle$. Then $\xi$ is a type-preserving isotropic $\chi$-involution of $\Delta$, $\Gamma$-chambers are residues of type $\{2, 3\}$ and

$$\Delta^{\Gamma} \cong Q(S).$$

**Proof.** — This holds by [12, p. 368 at the bottom]. See [14, 17.14] for a shorter proof. See also Remark 12.14. \(\square\)

**Theorem 11.11.** — Let $\Delta$ be as in Notation 11.1, let $\xi$ be an arbitrary type-preserving $\chi$-involution of $\Delta$ for some $\chi \in \text{Aut}(L, E)$, let $\Gamma = \langle \xi \rangle$ and suppose that $\Gamma$-chambers are residues of type $\{2, 3\}$. Then the following hold:

1. There exist $\alpha \in F$ and $\beta \in K$ such that $\xi$ is conjugate by an element in $G^\dagger$ to

$$g_{w_1, \alpha \beta^{-1}, \alpha^{-1}, \beta, \beta^{-1} \sqrt{\alpha}, \chi}.$$

2. $\Delta^{\Gamma}$ is a Moufang quadrangle of type $F_4$. 

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Proof. — By [12, Lemma 3.2], for every \( \Gamma \)-chamber \( R \), there exists an apartment that is stabilized by \( \xi \) and contains chambers of \( R \). By [24, 11.12], there exists an element \( \delta \) in the group \( G^\Gamma \) such that \( \xi^\delta \) stabilizes the apartment \( \Sigma \) and the unique \( \{2, 3\} \)-residue containing \( c \), where \( \Sigma \) and \( c \) are as in Notation 11.1. By [11, 25.17], \( \xi^\delta \) induces the automorphism \( w_1 \) on \( \Sigma \) and by 3.8(1), \( \xi^\delta \) is also a \( \chi \)-involution. By Proposition 11.5 and [12, Lemma 4.3], it follows that there exist \( \alpha \in F \) and \( \beta \in K \) such that \( \xi^\delta = g_{w_1, \alpha \beta^{-1}, \alpha^{-1}, \beta, \beta^{-1} \sqrt{\alpha \chi}} \). Thus (1) holds. By Theorem 11.10, (2) follows from (1).

\[ \square \]

12. \( F_4 \)-Buildings with Polarity

The goal of this section is to prove Proposition 12.15.

Notation 12.1. — Suppose now that \( \Xi, \rho, S = (E/K, F, \alpha, \beta), \theta \) and the identification of \( \Xi \) with \( \mathbb{Q}(S) \) are as in Theorem 9.12. Let \( F_0 = F^{1/2} \) in the algebraic closure of \( E \). Thus \( K \subset F_0 \) and \( F_0^2 = F \). Let \( L \) be the composite field \( EF_\theta \). Choose \( \gamma \in K \) such that \( E = K(\gamma) \). Then \( L = F_0(\gamma) \). In particular, \( L/F_0 \) is a separable quadratic extension. Let \( \chi \) be the generator of \( \text{Gal}(L/F_0) \). The map \( x \mapsto ((x^2)^\theta)^{1/2} \) is the unique extension of \( \theta \) to a Tits endomorphism of \( L \). We denote this extension by the same letter \( \theta \).

Since \( K^\theta = F \), we have \( F_0^\theta = K \). Thus \( K = F_0^\theta \neq L^\theta = K(\gamma^\theta) \). Since \( E^\theta \subset E \), it follows that \( E = K(\gamma^\theta) \). Hence \( L^\theta = E \). By 8.3(2), \( \theta \) commutes with \( \chi \). We set \( \Delta = F_4(L, E) \).

Let \( c, \Sigma, \Phi, \{\alpha_1, \ldots, \alpha_4\} \), the identification of \( \Phi \) with the set of roots of \( \Sigma \), etc., be as in Notation 11.1 applied to \( \Delta = F_4(L, E) \), let \( \{x_\alpha\}_{\alpha \in \Phi} \) be as in Theorem 11.2, let

\[ |\Phi| = \{\alpha/|\alpha| \mid \alpha \in \Phi\} \]

and let \( \pi \) be denote the bijection \( \alpha \mapsto \alpha/|\alpha| \) from \( \Phi \) to \( |\Phi| \). We now identify the set of roots of \( \Sigma \) with \( |\Phi| \) via \( \pi \).

Notation 12.2. — Let \( \hat{x}_{\pi(\alpha)}(t) = x_\alpha(t) \) for all \( t \in E \) and all long \( \alpha \in \Phi \), let \( \hat{x}_{\pi(\alpha)}(t) = x_\alpha(t^{\theta^{-1}}) \) for all \( t \in E \) and all short \( \alpha \in \Phi \) and let \( U_{\pi(\alpha)} = U_\alpha \) for all \( \alpha \in \Phi \). Thus \( \hat{x}_{\alpha} \) is an isomorphism from the additive group of \( E \) to \( U_\alpha \) for each \( \alpha \in |\Phi| \) and by Theorem 11.2, if \( s, t \in E \) and \( \alpha \) and \( \beta \) are elements of \( |\Phi| \) with an angle \( \omega < 180^\circ \) between them, then the following hold:

1. If \( \omega = 120^\circ \), then \( \alpha + \beta \in |\Phi| \) and \( [\hat{x}_\alpha(s), \hat{x}_\beta(t)] = \hat{x}_{\alpha+\beta}(st) \)
(2) If \( \omega = 135^\circ \), then \( \sqrt{2}\alpha + \beta \in |\Phi| \), \( \alpha + \sqrt{2}\beta \in |\Phi| \) and \( \hat{x}_\alpha(s), \hat{x}_\beta(t) = \hat{x}_{\sqrt{2}\alpha + \beta}(s^\theta t)\hat{x}_{\alpha + \sqrt{2}\beta}(s^\theta) \).

(3) \( \hat{x}_\alpha(s), \hat{x}_\beta(t) = 1 \) if \( \omega \) is neither \( 120^\circ \) nor \( 135^\circ \).

Let
\[
B := \{\eta_1, \ldots, \eta_4\}
\]
be the image of the basis \( \{\alpha_1, \ldots, \alpha_4\} \) of \( \Phi \) under \( \pi \). We set \( m' = \sqrt{2}m \) for each positive integer \( m \) and
\[
abcd = a\eta_1 + b\eta_2 + c\eta_3 + d\eta_4
\]
for all \( a, b, c, d \in \mathbb{N} \cup \sqrt{2}\mathbb{N} \). Thus, for example,
\[
1'2'21 = \sqrt{2}\eta_1 + 2\sqrt{2}\eta_2 + 2\eta_3 + \eta_4.
\]

We then set
\[
W_0 = \{0100, 0010, 011'0, 01'10\},
W_1 = \{0001, 0011, 011'1', 01'11, 01'21\},
W_2 = \{111'1', 121'1', 1'2'32, 122'1', 132'1'\},
W_3 = \{1'1'11, 1'1'21, 232'1', 1'2'21, 1'2'31\},
W_4 = \{1000, 1100, 1'1'10, 111'0, 121'0\}.
\]

Let \( |\Phi^+| \) denote the image under \( \pi \) of the set of positive roots of \( \Phi \) with respect to the basis \( \{\alpha_1, \ldots, \alpha_4\} \). Then
\[
|\Phi^+| = W_0 \cup W_1 \cup W_2 \cup W_3 \cup W_4.
\]

**Notation 12.3.** — Let \( R_1 \) be the unique \( \{2, 3, 4\} \)-residue of \( \Delta \) containing \( c \), let \( R_4 \) be the unique \( \{1, 2, 3\} \)-residue containing \( c \), let \( R = R_1 \cap R_4 \) and for \( i = 1 \) and \( 4 \), let \( R_i' \) be the unique residue such that \( R_i' \cap \Sigma \) is opposite \( R \cap \Sigma \) in \( R_i \cap \Sigma \). Then \( W_i \) is the set of roots of \( \Sigma \) that contain \( R \cap \Sigma \) but are disjoint from \( R_i' \cap \Sigma \) for \( i = 1 \) and \( 4 \).

**Notation 12.4.** — There exists a unique set \( X \) of \( \{2, 3\} \)-residues of \( \Sigma \) containing \( R \cap \Sigma \) with the property that there exists a bijection \( i \mapsto T_i \) from \( \mathbb{Z}_8 \) to \( X \) such that for each \( i \in \mathbb{Z}_8 , T_i - 1 \) and \( T_i \) are opposite residues of a residue of rank 3 of \( \Sigma \). We denote by \( \Lambda \) the graph with vertex set \( X \), where \( T_i \) is adjacent to \( T_j \) whenever \( i - j = \pm 1 \). Thus the residues \( R_i' \cap \Sigma \) and \( R_i' \cap \Sigma \) are the two vertices adjacent to \( R \cap \Sigma \) in \( \Lambda \).

**Notation 12.5.** — Let \( \tilde{X} \) be the graph obtained from the set \( X \) in Notation 12.4 by replacing each vertex \( T_i \) by the unique residue \( \tilde{T}_i \) of \( \Delta \) such that \( \tilde{T}_i \cap \Sigma = T_i \). Let \( \tilde{\Sigma} \) be the graph with vertex set \( \tilde{X} \), where \( \tilde{T}_i \) is adjacent to \( \tilde{T}_j \) whenever \( i - j = \pm 1 \).

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Notation 12.6. — Let $\kappa$ denote the unique involutory permutation of $|\Phi|$ which interchanges $abcd$ with $dcba$ for all $abcd \in |\Phi|$. Note that $W_0^\kappa = W_0$ and $W_i^\kappa = W_{5-i}$ for each $i \in [1, 4]$. By [20, 1.2], there is a unique polarity of $\Delta$ stabilizing $c$ and $\Sigma$ and interchanging $\dot{x}_{\alpha}(t)$ and $\dot{x}_{\kappa(\alpha)}(t)$ for all $\alpha \in |\Phi|$ and all $t \in E$. We denote this polarity by $\sigma$.

Notation 12.7. — Let $[abcd]$ denote the reflection associated with the vector $abcd$ for all $abcd \in |\Phi|$. Let $r_1 = [011'1']$ and $r_4 = [1'1'10]$ and let $R, R_1, R_4, R'_1$ and $R'_4$ be as in Notation 12.3. Then $|r_1 r_4| = 4$. The reflection $r_1$ stabilizes $R \cap \Sigma$ and interchanges $R \cap \Sigma$ with $R'_1 \cap \Sigma$ as well as $W_2$ and $W_4$. The reflection $r_4$ stabilizes $R_4 \cap \Sigma$ and interchanges $R \cap \Sigma$ with $R'_4 \cap \Sigma$ as well as $W_1$ and $W_3$. In particular, $r_1$ induces the reflection on the graph $\Lambda$ defined in Notation 12.4 that interchanges $R \cap \Sigma$ and $R'_1 \cap \Sigma$ and $r_4$ induces the reflection that interchanges $R \cap \Sigma$ and $R'_4 \cap \Sigma$.

Notation 12.8. — We denote by $r$ the square of the product

$$[0100] \cdot [0010].$$

The element $r$ is an involution commuting with $\kappa$ and with $r_1$ and $r_4$. It stabilizes the residue $R$ and hence acts trivially on the graph $\Lambda$. It stabilizes the four sets $W_1, \ldots, W_4$ and fixes the vectors $01'1' \in W_1, 1'2'32 \in W_2, 232'1' \in W_3$ and $1'1'10 \in W_4$, but does not fix any other elements of $W_1 \cup W_2 \cup W_3 \cup W_4$.

By 11.4(1), there exists a unique automorphism $\zeta$ of $\Delta$ stabilizing $\Sigma$ such that

$$\dot{x}_v(t)^{\zeta} = \dot{x}_{r(v)}(t)$$

for all $t \in E$.

We set

$$\lambda^{m'} = \lambda^{m\theta}$$

for all $\lambda \in K$ and all $m \in \mathbb{N}$ and let

$$h_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} = g_{1, \lambda_1, \lambda_2, \lambda_3^{a-1}, \lambda_4^{a-1}, 1}$$

for all $\lambda_1, \ldots, \lambda_4 \in E^*$. Let $h = h_{\lambda_1, \ldots, \lambda_4}$ for some choice of $\lambda_1, \ldots, \lambda_4 \in E^*$. By 11.4(2), we have

$$\dot{x}_{abcd}(t)^{h} = \dot{x}_{abcd}(\lambda t)$$

for all $abcd \in |\Phi|$ and all $t \in E$, where

$$\lambda = \lambda_1^a \lambda_2^b \lambda_3^c \lambda_4^d.$$
Thus, for example,

$$\dot{x}_{1'2'21}(t)^h = \dot{x}_{1'2'21}(\lambda t)$$

for all $t \in L$, where

$$\lambda = \lambda_1^\theta \lambda_2^2 \lambda_3^3 \lambda_4.$$  

**Notation 12.11.** — We set

$$\xi = g_{w_1, \beta-(\theta+1), \beta, \beta-(\theta+1), \chi,}$$

where $w_1$ is as in Notation 11.7. By 9.12(2) and Notation 12.1, we have $\alpha = \beta^{-\theta}$; thus $\xi$ is the same as the element $\xi$ in Notation 11.9. Note that

$$\dot{x}_{abcd}(t)^\xi = (\dot{x}_{abcd}(t^h)^h)$$

for all $abcd \in |\Phi|$, where $\zeta$ is as in (12.9) and

$$h = h_{\beta-(\theta+1), \beta, \beta-(\theta+1), \chi,}.$$  

**Notation 12.12.** — By Theorem 11.10, we already know that the automorphism $\xi$ is a type-preserving $\chi$-involution of $\Delta$ and that $\tilde{\Xi} := \Delta(\xi)$ is isomorphic to $\Xi$. The polarity $\sigma$ defined in Notation 12.6 commutes with $\xi$ and thus induces a polarity of $\tilde{\Xi}$ which we denote by $\tilde{\rho}$. Our goal in Proposition 12.15 is to show that there is an isomorphism from $\tilde{\Xi}$ to $\Xi$ which carries $\tilde{\rho}$ to $\rho$.

**Remark 12.13.** — Since (by Theorem 11.10) the minimal residues stabilized by $\xi$ are of type $\{2, 3\}$, the residue $R$ in Notation 12.3 is a chamber of $\tilde{\Xi}$.

Since $\xi$ stabilizes $\Sigma$, the graph $\tilde{\Sigma}$ defined in Notation 12.5 is an apartment of $\tilde{\Xi}$ containing $R$. The polarity $\tilde{\rho}$ stabilizes both $R$ and $\tilde{\Sigma}$.

**Remark 12.14.** — It might appear that we are giving a new proof of Theorem 11.10 in Proposition 12.15. In fact, however, the proof of Proposition 12.15 we give relies on Remark 12.13 which, in turn, relies on the fact that the minimal residues stabilized by $\xi$ are of type $\{2, 3\}$. It is exactly in the proof of this fact that the proof of Theorem 11.10 in [14] differs from the proof in [12].

**Proposition 12.15.** — Let $\Xi$, $\Sigma$, $c$ and $\rho$ be as Notation 9.14 and let $\tilde{\Xi}$, $\tilde{\Sigma}$, $R$ and $\tilde{\rho}$ be as in Notation 12.3, Notation 12.12 and Remark 12.13. Then there is an isomorphism from $\tilde{\Xi}$ to $\Xi$ mapping the pair $(\tilde{\Sigma}, R)$ to the pair $(\Sigma, c)$ that carries the polarity $\tilde{\rho}$ to $\rho$.  

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Proof. — We define maps $X_1, \ldots, X_4$ from $V = E \oplus E \oplus [K]$ to $\text{Aut}(\Delta)$ as follows:

$$
X_1(u, v, t) = \dot{x}_{0011}(u) \dot{x}_{0111}(\beta^{-1} \bar{u}) \cdot \dot{x}_{0001}(v) \dot{x}_{0121}(\beta^{-(\theta+1)} \bar{v}) \cdot \dot{x}_{0111}(t)
$$

$$
X_2(u, v, t) = \dot{x}_{1211}(u) \dot{x}_{1221}(\beta^{-1} \bar{u}) \cdot \dot{x}_{1111}(v) \dot{x}_{1321}(\beta^{-(\theta+1)} \bar{v}) \cdot \dot{x}_{1232}(t)
$$

$$
X_3(u, v, t) = \dot{x}_{1121}(u) \dot{x}_{1221}(\beta^{-1} \bar{u}) \cdot \dot{x}_{1111}(v) \dot{x}_{1231}(\beta^{-(\theta+1)} \bar{v}) \cdot \dot{x}_{2321}(t)
$$

$$
X_4(u, v, t) = \dot{x}_{1100}(u) \dot{x}_{1110}(\beta^{-1} \bar{u}) \cdot \dot{x}_{1000}(v) \dot{x}_{1210}(\beta^{-(\theta+1)} \bar{v}) \cdot \dot{x}_{1110}(t)
$$

for all $(u, v, t) \in V$, where $\bar{x} = x^\chi$ for all $x \in E$. Note that

$$(12.16) \quad X_i(u, v, t) = X_{5-i}(u, v, t)$$

for all $(u, v, t) \in V$. Let $M_i = X_i(V)$ for all $i \in [1, 4]$, let $M_+$ denote the subgroup generated by $M_1, \ldots, M_4$ and let

$$
\dot{\Psi} := (M_+, M_1, M_2, M_3, M_4).
$$

We have $M_i = C_{U_s}(\alpha \in W_s)(\xi)$ for each $i \in [1, 4]$. By Notation 12.3, Remark 12.13 and [11, 24.32], $M_1$ and $M_4$ are root groups of $\Delta$ corresponding to the two roots of the apartment $\bar{\Sigma}$ containing $R$ but not some chamber of $\bar{\Sigma}$ adjacent to $R$. By Notation 12.7, we conclude that $\dot{\Psi}$ is a root group sequence of the Moufang quadrangle $\bar{\Xi}$.

It follows from 12.2(1)–(3) that the map $X_i$ is additive and thus $M_i$ is abelian for all $i \in [1, 4]$, that

$$[M_1, M_2] = [M_2, M_3] = [M_3, M_4] = [M_1, M_3] = 1,$$

and that

$$[X_2(a, b, s), X_4(u, v, t)] = X_3(0, 0, \beta^{-1}(u\bar{a} + a\bar{u}) + \beta^{-(\theta+1)}(v\bar{b} + b\bar{v}))$$

$$= X_3(0, 0, f((a, b, s), (u, v, t))$$

for all $(a, b, s), (u, v, t) \in V$, where $f = \partial q$. Applying also the identities (1.3), we find that

$$[X_1(a, 0, 0), X_4(u, 0, 0)] = X_2(0, a^\theta u, 0)X_3(0, u^\theta a, 0)$$

$$[X_1(a, 0, 0), X_4(v, 0, 0)] = X_2(\beta^{-\theta}a^\theta v, 0, 0)X_3(0, \beta^{-1}v^\theta a, 0)$$

$$[X_1(a, 0, 0), X_4(0, s, 0)] = X_2(0, 0, s\beta^{-1}N(a))X_3(sa, 0, 0)$$

$$[X_1(0, b, 0), X_4(v, 0, 0)] = X_2(\beta^{-(\theta+1)}b^\theta v, 0, 0)X_3(\beta^{-(\theta+1)}v^\theta \bar{v}, 0, 0)$$

$$[X_1(0, b, 0), X_4(0, s, 0)] = X_2(0, 0, s\beta^{-(\theta+1)}N(b))X_3(0, sb, 0)$$

$$[X_1(0, r, 0), X_4(0, s, 0)] = X_2(0, 0, r^\theta s)X_3(0, 0, s^\theta r)$$

for all $(a, b, r), (u, v, s) \in V$. The commutator of an arbitrary element of $M_i$ with an arbitrary element of $M_j$ for $(i, j) = (1, 3)$ and $(1, 4)$ is uniquely
determined by the identities (1.3), (12.16) and the identities above. It follows that there is an isomorphism $\omega$ from $\tilde{\Psi}$ to the root group sequence $\Psi := (U_+, U_1, \ldots, U_4)$ defined by the commutator relations in Notation 9.14 sending $X_i(u, v, t)$ to $x_i(u, v, t)$ for all $(u, v, t) \in V$ and all $i \in [1, 4]$. By (9.15) and (12.16), $\omega$ carries the anti-automorphism of $\tilde{\Psi}$ induced by $\tilde{\rho}$ to the anti-automorphism of $\Psi$ induced by $\rho$. By [23, 7.5], there exists a unique isomorphism $\omega_1$ from $\tilde{\Xi}$ to $\Xi$ mapping the pair $(\tilde{\Sigma}, R)$ to the pair $(\Sigma, c)$ and inducing the map $\omega$ from $\tilde{\Psi}$ to $\Psi$. By [23, 3.7], $\omega_1$ carries $\tilde{\rho}$ to $\rho$. □

13. Moufang Octagons

Let $\Omega$ be a Moufang octagon. By [23, 17.7], $\Omega = \mathcal{O}(E, \theta)$ for some octagonal set $(E, \theta)$ as defined in Definition 2.1. Let $\Delta = F_4(E, \theta) = F_4(E, E^\theta)$ as defined in Notation 2.3.

Notation 13.1. — Let $L/E$ and the extension of $\theta$ to $L$ be as in Notation 12.1. Then $\theta$ maps the pair $(L, E)$ to the pair $(E, E^\theta)$ and hence induces an isomorphism from $F_4(L, E)$ to $\Delta$. Let $c, \Sigma, \Phi$ and the identification of $\Phi$ with the set of roots of $\Sigma$ be as in Notation 11.1, let $\{x_\alpha\}_{\alpha \in \Phi}$ be as in Theorem 11.2 and let $\sigma$ be the polarity of $F_4(L, E)$ defined in Notation 12.6. We identify $F_4(L, E)$ with $\Delta$ via the isomorphism induced by $\theta$.

Proposition 13.2. — $\Omega \cong \Delta^{(\sigma)}$.

Proof. — This holds by [20, Theorem (on p. 540)]. □

For the rest of this section, we will simply identify $\Omega$ with $\Delta^{(\sigma)}$.

Proposition 13.3. — Every automorphism of $\Omega$ has a unique extension to a type-preserving automorphism of $\Delta$, and all of these extensions commute with $\sigma$.

Proof. — Let $G^\circ$ denote the group of type-preserving automorphisms of $\Delta$ (as in Notation 3.2 applied to $\Delta$). By [20, 1.6 and 1.13.1 (ii)], every automorphism of $\Omega$ can be extended to an element in the centralizer $C_{G^\circ}(\sigma)$. Suppose that $g$ is a type-preserving automorphism of $\Delta$ that acts trivially on $\Omega$. It remains only to show that $g$ is trivial. Opposite chambers of $\Omega$ are opposite chambers of $\Delta$ and opposite chambers of $\Delta$ are contained in a unique apartment of $\Delta$. We can thus assume that $g$ fixes the apartment $\Sigma$ and chamber $c$ in Notation 13.1. Since $g$ is type-preserving, it acts trivially.
on $\Sigma$ and hence normalizes the root group $U_\alpha$ for all $\alpha \in \Phi$. By [20, 1.5], the map from the additive group of $E$ to $\text{Aut}(\Omega)$ which sends $t \in E$ to the element of $\text{Aut}(\Omega)$ induced by $x_\alpha(t^\theta)x_\alpha(t)$ is injective as is the map from the additive group of $E$ to $\text{Aut}(\Omega)$ which sends $t \in E$ to the element of $\text{Aut}(\Omega)$ induced by $x_\alpha(t^\theta)x_\alpha(t)x_{2\alpha_1}(t^{\theta+2})$. It follows that $g$ centralizes $U_\alpha$ for each $i \in [1,4]$. By 11.4(1), therefore, $g = 1$.

By [11, 28.8], $E$ is the defining field of $\Omega$ and \{E/E^\theta, E^\theta/E\} is the pair of defining extensions of $\Delta$.

**Proposition 13.4.** — Let $A = \text{Aut}(E, E^\theta)$ be as in Notation 3.6, let $\iota$ denote the inclusion map from $A$ to $\text{Aut}(E)$ and let $\psi_\Delta$ denote the Galois map of $\Delta$ in Remark 11.6. Then there is a unique Galois map $\psi_\Omega$ of $\Omega$ such that

$$(13.5) \quad \psi_\Omega(\kappa) = \iota(\psi_\Delta(\zeta))$$

for all pairs $(\kappa, \zeta)$, where $\kappa \in \text{Aut}(\Omega)$ and $\zeta$ is the unique extension of $\kappa$ to a type-preserving automorphism of $\Delta$.

**Proof.** — Let $G^o$ and $G^\dagger$ be as in Notation 3.2 applied to $\Delta$ and let $H = \text{Aut}(\Omega)$. By Proposition 13.3, there is a unique homomorphism $\psi = \psi_\Omega$ from $H$ to $\text{Aut}(E)$ such that (13.5) holds. Let $\kappa \in \text{Aut}(\Omega)$ and let $\zeta$ be its unique extension to an element of $G^o$. If $\kappa$ lies in a root group of $\Omega$, then by [11, 24.32], $\zeta \in G^\dagger$ and hence $\psi(\kappa) = 1$. Thus $\psi$ satisfies 3.8(1). Let $d$ be the chamber opposite $c$ in $\Sigma$. Then $d$ is a chamber of $\Omega$ opposite $c$. Hence there exists a unique apartment $\Sigma_{\Omega}$ containing $c$ and $d$. Suppose that $\kappa$ acts trivially on $\Sigma_{\Omega}$. Then $\zeta$ acts trivially on $\Sigma$ since it is type-preserving. By Proposition 11.5, therefore,

$$\zeta = g(\gamma, \lambda_1, ..., \lambda_4, \chi)$$

for some $\gamma \in \text{Aut}(\Phi)$, some $\lambda_1, \lambda_2 \in E^\theta$, some $\lambda_3, \lambda_4 \in E$ and some $\chi \in \text{Aut}(E, E^\theta)$. By Remark 11.6, $\psi_\Delta(\zeta) = \chi$. Let $B = B_{\Pi}$ be as in Notation 3.3 for $\Pi = I_2(8)$, let $(s,t)$ be the standard element of $B$ as defined in Remark 3.4 and let $\Theta = (\hat{U}_+, \hat{U}_1, ..., \hat{U}_8)$ be the root group sequence and $\hat{x}_1, ..., \hat{x}_8$ the isomorphisms obtained by applying [23, 16.9] to the octagonal set $(E, \theta)$. By [20, 1.5–1.7], there exists an isomorphism $\varphi: \Omega_{st} \to \Theta$ such that there exist $\delta_1, ..., \delta_8 \in E^*$ so that for each $i \in [1,8]$, $\hat{x}_i(u^h) = \hat{x}_i(\delta_i u^\chi)$ for all $u \in E$ if $i$ is odd and $\hat{x}_i(u, v)^h = \hat{x}_i(\delta_i u^\chi, \lambda_i^{\theta+1} u^\chi)$ for all $u, v \in E$ if $i$ is even, where $h := \varphi^{-1} \kappa \varphi$. Thus $\chi = \psi_\Omega(\kappa)$ equals the element called $\lambda(\kappa)$ in [11, 29.5] with $\Omega = \Theta$. By Notation 3.8, $\psi_\Omega$ is the unique Galois map of $\Omega$ determined by $\varphi$. □
Remark 13.6. — Let $\kappa \in \text{Aut}(\Omega)$ and let $\zeta$ be the unique extension $\kappa$ to a type-preserving automorphism of $\Delta$. By Proposition 13.3, $\kappa$ is an involution if and only if $\zeta$ is. If we choose Galois maps as in Proposition 13.4, it follows that $\zeta$ is, in fact, a $\chi$-involution for some $\chi \in A$ if and only if $\kappa$ is a $\iota(\chi)$-involution, where $A$ and $\iota$ are as in Proposition 13.4; see Notation 3.11.

Proposition 13.7. — Let $\kappa$ be a Galois involution of $\Omega$ that fixes panels of one type but none of the other type. Then $\kappa$ has a unique extension to a type-preserving Galois involution $\zeta$ of $\Delta$, $\langle \zeta \rangle$ is a descent group of $\Delta$ and $\langle \zeta \rangle$-chambers are residues of type $\{2, 3\}$.

Proof. — By Remark 13.6, $\kappa$ has a unique extension to a type-preserving Galois involution $\zeta$ of $\Delta$. By Theorem 3.10, $\langle \zeta \rangle$ is a descent group of $\Delta$. By Proposition 13.2, some panels of $\Omega$ are $\{1, 4\}$-residues of $\Delta$ and the others are $\{2, 3\}$-residues (with respect to the standard numbering of the vertex set of the Coxeter diagram $F_4$). Since $\kappa$ fixes panels of $\Omega$, we can choose a $J$-residue $R$ of $\Delta$ stabilized by $\langle \kappa, \sigma \rangle$, where $J$ is either $\{1, 4\}$ or $\{2, 3\}$. Let $R_1$ be a minimal $\langle \zeta \rangle$-residue contained in $R$ and let $J_1$ be its type. Since $\zeta$ commutes with $\sigma$, $R_1 \cap R_1^\sigma$ is also stabilized by $\zeta$. By the choice of $R_1$, it follows that $R_1$ is stabilized by $\sigma$. Thus $J_1$ is a subset of $J$ invariant under the non-trivial automorphism of the Coxeter diagram of $\Delta$, so either $J_1 = \emptyset$ or $J_1 = J$. Suppose that $J_1 = \emptyset$. Then $R_1$ is contained in a unique $J'$-residue $R_2$, where $J'$ denotes the complement of $J$ in the vertex set of the Coxeter diagram $F_4$. Since $\langle \sigma, \zeta \rangle$ stabilizes $R_1$, it must stabilize $R_2$ as well. This contradicts the assumption, however, that $\kappa$ does not fix panels of $\Omega$ of both types. We conclude that $J_1 = J$ and hence $R_1 = R$. Thus $R$ is a $\langle \zeta \rangle$-chamber.

Let $\Pi$ be the Coxeter diagram of type $F_4$ and let $\Theta$ denote the trivial subgroup of $\text{Aut}(\Pi)$. By Definitions 2.14 and 2.15, the triple $(\Pi, \Theta, \{1, 4\})$ is not a Tits index. By 2.18(3), we conclude that $J = \{2, 3\}$. □

Proposition 13.8. — Let $\chi = \psi_\Delta(\xi)$, where $\psi_\Delta$ is as in Proposition 13.4 and $\xi$ is as in Proposition 13.7. Then the following hold:

1. $\chi \theta = \theta \chi$ and $\chi \theta$ is a Tits endomorphism of $E$.
2. $\Omega_\xi := \Delta^{\langle \xi \sigma \rangle}$ is isomorphic to $\mathcal{O}(E, \chi \theta)$.
3. Proposition 13.3 holds with $\Omega_\xi$ and $\xi \sigma$ in place of $\Omega$ and $\sigma$.
4. There exists a Galois map $\psi_{\Omega_\xi}$ of $\Omega_\xi$ such that (13.5) holds with $\psi_{\Omega_\xi}$ in place of $\psi_{\Omega}$.

Proof. — By 11.11(1) and the choice of $\psi_\Delta$ in Proposition 13.4, we can assume that

$$\xi = gw_1, \lambda_1, \ldots, \lambda_4, \chi$$
for some $\lambda_1, \lambda_2 \in E^\theta$ and some $\lambda_3, \lambda_4 \in E$. For each $\alpha \in \Phi$, we denote by $s_\alpha$ the corresponding reflection of $\Phi$ (as in Notation 11.1). Let $s = s_{\alpha_3}s_{\alpha_2+2\alpha_3}$, let $\beta_i = \alpha_i^s$ and let $d = e^s$ with respect to the action of $W$ on $\Sigma$ described in Notation 11.1. Then $\beta_1, \ldots, \beta_4$ is a basis of $\Phi$ and $\beta_1 = \alpha_1 + \alpha_2 + 2\alpha_3$, $\beta_2 = -\alpha_2 - 2\alpha_3$, $\beta_3 = \alpha_2 + \alpha_3$ and $\beta_4 = \alpha_3 + \alpha_4$. The restriction of $\sigma\xi$ to $\Sigma$ induces the unique automorphism of $\Phi$ that interchanges $\beta_i$ and $\beta_{5-i}$ for all $i \in \{1,\ldots,4\}$ (via the identification of the roots of $\Sigma$ with $\Phi$ in Notation 11.1). By 11.4(2), there exist non-zero $\varepsilon_3, \varepsilon_4, \varepsilon_4 \in E^\theta$ such that
\begin{equation}
 x_{\beta_i}(t)^{\xi_\sigma} = x_{\beta_{5-i}}(\varepsilon_i t^{\chi}) \tag{13.9}
\end{equation}
for $i = 3$ and $4$ and all $t \in E^\theta$ and
\begin{align*}
x_{\beta_3}(t)^{\sigma_\xi} &= x_{\beta_2}(\varepsilon_3 t^{\theta\chi})
\end{align*}
for all $t \in E^\theta$. Since $\sigma$ and $\xi$ commute, we conclude that $\varepsilon_3 = \varepsilon'_3$ and $\theta\chi = \chi\theta$. Thus (1) holds.

Let $h = g_{1,\varepsilon_3\varepsilon_4^{-1},\varepsilon_3,\varepsilon_4^{-1},\varepsilon_4,1}$. By 11.4(2) again, $h$ centralizes $U_{\beta_3}$ and $U_{\beta_4}$ and $x_{\beta_1}(t)^h = x_{\beta_1}(\varepsilon_4^{-1} t)$ and $x_{\beta_2}(t)^h = x_{\beta_2}(\varepsilon_3^{-1} t)$ for all $t \in F$. Let $\tilde{x}_\alpha = x_\alpha \cdot h_\alpha$ for all $\alpha \in \Phi$, where $h_\alpha$ denotes the automorphism $a \mapsto a^h$ of $U_\alpha$. Then $\{\tilde{x}_\alpha\}_{\alpha \in \Phi}$ is a coordinate system for $\Delta$ (as defined in Definition 11.3), $\tilde{x}_{\beta_i} = x_{\beta_i}$ for $i = 3$ and $4$ and $\tilde{x}_{\beta_1}(\varepsilon_4 t) = x_{\beta_1}(t)$ and $\tilde{x}_{\beta_2}(\varepsilon_3 t) = x_{\beta_2}(t)$ for all $t \in E$. By (13.9), therefore, $\tilde{x}_{\beta_i}(t)^{\sigma_\xi} = \tilde{x}_{\beta_{5-i}}(t^{\chi})$ for all $i \in \{1,\ldots,4\}$ and for all $t \in E$. We can thus apply Propositions 13.2–13.4 with $\xi \sigma$ and $\{\tilde{x}_\alpha\}_{\alpha \in \Phi}$ in place of $\sigma$ and $\{x_\alpha\}_{\alpha \in \Phi}$ to conclude that (2)-(4) hold. 

\[\square\]

14. Proofs of Theorems 4.1 and 4.2

We first prove Theorem 4.1. Suppose that $\Xi$ and $\rho$ satisfy the hypotheses, let
\[S = (E/K, F, \alpha, \beta)\]
and $\theta$ be as in Theorem 9.12 and let $\Delta = F_4(L, E)$ and $\chi$ be as in Notation 12.1. The Tits endomorphism $\theta$ commutes with $\chi$; it also maps the pair $(L, E)$ to the pair $(E, E^\theta)$ and hence induces an isomorphism from $\Delta$ to $F_4(E, \theta)$. Let $\sigma$ be as in Notation 12.6 and let $\xi$ be as in Notation 12.11. By Notation 12.12, $[\sigma, \xi] = 1$, $\xi$ is a type-preserving $\chi$-involution of $\Delta$ and $\sigma$ induces a polarity on $\Xi := \Delta^{(\xi)}$. By Proposition 3.1, the restriction of $\langle \sigma \rangle$ to $\Xi$ is a descent group of relative rank 1. It follows that $\langle \xi, \sigma \rangle$ is a descent group of $\Delta$. Thus (1) holds. By Proposition 12.15, (2) holds. By Proposition 3.1 again, $\Delta^{(\sigma)}$ and $\Delta^{(\sigma\xi)}$ are Moufang octagons. By Notation 13.1...
and Proposition 13.2, the first of these octagons is isomorphic to $O(E, \theta)$ and by 13.8(2), the second is isomorphic to $O(E, \chi \theta)$. Thus (3) holds.

By Proposition 3.1, $\Delta^\Gamma$, $(\Delta^{(\xi)})^{(\sigma)}$, $(\Delta^{(\sigma)})^{(\xi)}$ and $(\Delta^{(\sigma \xi)})^{(\xi)}$ are all Moufang sets. The underlying set of each of them is the set $X$ of all $\Gamma$-chambers and by 2.18(5) and [11, 24.32], the root group corresponding to a $\Gamma$-chamber $R$ is the permutation group induced by $C_{\Gamma}(U_R)$ on $X$, where $U_R$ is the unipotent radical of $R$ in $\Delta$. Thus (4) holds. By 2.18(2) and Theorem 11.10, there are $\{2, 3\}$-residues of $\Delta$ stabilized by $\xi$ but none of type $\{1, 4\}$. By Proposition 13.4 and 13.8(4), therefore, (5) holds. This concludes the proof of Theorem 4.1.

We turn now to Theorem 4.2. Suppose that $\chi, (E, \theta), \Delta, \Omega$ and $\kappa$ satisfy the hypotheses, let $\sigma$ be as in Notation 13.1 and let $\xi$ be the type-preserving automorphism of $\Delta$ obtained by applying Proposition 13.3 to $\kappa$. Then $\xi$ and $\sigma$ commute and by Remark 13.6, $\xi$ is a $\chi$-involution. Let $\Gamma = \langle \xi, \sigma \rangle$. Then $\Delta^\Gamma = \Omega^{(\kappa)}$. By Proposition 3.13, it follows that $\Gamma$ is a descent group of $\Delta$. Thus (1) holds. Assertion (2) holds by Proposition 13.2 and the choice of $\xi$. By Proposition 13.7, $\langle \xi \rangle$-chambers are of type $\{2, 3\}$ and by Proposition 3.12, $\Xi := \Delta^{(\sigma)}$ is a Moufang quadrangle of type $F_4$. Since $\xi$ and $\sigma$ commute, $\sigma$ induces a polarity on $\Xi$. Thus (3) holds. Assertion (4) holds for the same reason that 4.1(4) holds and assertion (5) holds by Proposition 13.8. This concludes the proof of Theorem 4.2.

15. Moufang Sets of Outer $F_4$-Type

Our goal in the remaining sections is to determine a few essential properties of the Moufang sets of outer $F_4$-type defined in Definition 4.5.

Notation 15.1. — Let $S, V = E \oplus E \oplus [K], U_+, x_1, \ldots, x_4$ and $\theta$ be as in Notation 9.14, let $\theta_K$ be the restriction of $\theta$ to $K$, let $\Xi = Q(S)$, $\Sigma$ and $c$ be as in Notation 5.7, let $q = q_S$, let $f = \partial q$, let $g$ be as in Notation 6.21, let $\rho$ be as in (9.15) and let $M = (X, \{U_x\}_{x \in X})$ be the Moufang set $\Xi^{(\rho)}$ obtained by applying 2.18(5) with $\langle \rho \rangle$ in place of $\Gamma$. Note that $c \in X$.

Notation 15.2. — We have $c \in X$ and by 2.18(5), the centralizer $C_{U_+}(\rho)$ equals the root group $U_c$ of the Moufang set $M$. We set $U = U_c$ and write $U$ additively even though it is not, as we will see, abelian. In this section we use (6.20) and Proposition 6.23 to compute a few basic properties of $U$.

Let $\eta \in U_+$. Thus

$$\eta = x_1(b)x_2(w)x_3(v)x_4(u)$$
for some $b, w, u, v \in V$. Note that $f(a, g(y, z)) = 0$ for all $a, y, z \in V$ by Notations 5.17 and 6.21 (and, of course, that the characteristic of $K$ is 2). Thus

$$
\eta^\rho = x_4(b)x_3(w)x_2(v)x_1(u)
= x_4(b)x_1(u)x_3(w)x_2(v + g(u, w))
= x_1(u)x_2(bu)x_3(ub)x_4(b)x_3(w)x_2(v + g(u, w))
= x_1(u)x_2(bu)x_3(ub)x_3(w)x_3(g(b, v))x_2(v + g(u, w))x_4(b)
= x_1(u)x_2(bu + v + g(u, w))x_3(ub + w + g(b, v))x_4(b),
$$

so $\eta \in U$ if and only if $b = u$, $w = bu + v + g(u, w)$ and $v = ub + w + g(b, v)$. Note that $g(u, w) = g(u, uu) + g(u, v) = g(u, v)$ by 7.4(1). It follows that

$$
U := \{ x_1(u)x_2(w)x_3(uu + w + g(u, w))x_4(u) \mid u, w \in V \}
$$

Let

$$
\{ u, w \} = x_1(u)x_2(w)x_3(uu + w + g(u, w))x_4(u)
$$

for all $u, w \in V$. Then

$$
\{ u, w \} + \{ a, b \} = x_1(u)x_2(w)x_3(uu + w + g(u, w))x_4(u)
\cdot x_1(a)x_2(b)x_3(aa + b + g(a, b))x_4(a)
\in x_1(u + a)x_2(w + b + ua + g(a, uu + w + g(u, w)))U_{[3, 4]}
$$

and thus

$$
\{ u, w \} + \{ a, b \} = \{ u + a, w + b + ua + g(a, w) + g(a, uu) \}
$$

for all $u, w, a, b \in V$. It follows that

$$
-\{ u, w \} = \{ u, w + uu + g(u, w) \}
$$

and the commutator $-\{ u, w \} - \{ a, b \} + \{ u, w \} + \{ a, b \}$ equals

$$
\{ 0, ua + au + g(u, b) + g(a, w) + g(a, aa) + g(a, uu) \}
$$

for all $u, w, a, b \in V$.

**Proposition 15.7.** — $U' = \{ 0, V \}$ and $[U, U'] = Z(U) = \{ 0, [K] \}$. In particular, $U$ is nilpotent and has nilpotency class 3.

**Proof.** — Setting $a = [1]$ in (15.6), we obtain

$$
\{ 0, u + [q(u) + f(u, b)] \} \in U'
$$

for all $u, b \in V$. For each $u \in V \setminus [K]$, there exists $b$ such that $q(u) = f(u, b)$. Therefore $\{ 0, u \}$ is in the commutator group $U'$ of $U$ for all $u \in V \setminus [K]$. Hence $U' = \{ 0, V \}$. If we set $u = 0$ in (15.6), we are left with only
\{0, g(a, w)\}. It follows that \([U', U] = \{0, [K]\} \subset Z(U), Z(U) \subset \{[K], V\}\) and

\[(15.8)\quad Z(U) \cap \{0, V\} = \{0, [K]\}.

Let \(t \in K^\ast\). By Remark 7.3, we can choose \(s \in K\) with \(s \neq 0\) and \(s \neq t\).

Since \((x^\theta t)^{\theta + 1} = x\) for all \(x \in K^\ast\), it follows that \(s^{\theta - 1} \neq t^{\theta - 1}\) and hence \(s^t x + st^\theta \neq 0\). Setting \(u = [s]\) and \(a = [t]\) in (15.6), we obtain \(\{0, [s^t x + st^\theta]\}\). It follows that \(Z(U) \subset \{0, V\}\). By (15.8), we conclude that \(Z(U) = \{0, [K]\}\).

\[\square\]

16. The Element \(\tau\)

Let \(\Xi, \Sigma, c, U_+, x_1, \ldots, x_4, \rho, X, \) etc., be as in Notation 15.1 and let \(\phi\) and \(\Omega = G(\Theta, U_+, \phi)\) be as in [23, 7.2] with \(n = 4\), where \(\Theta\) is a circuit of length 8 whose vertex set \(V(\Theta)\) has been numbered by the integers modulo 8 so that the vertex \(x\) is adjacent to the vertex \(x - 1\) for all \(x\). The vertex set of \(\Omega\) consists of pairs \((x, B)\), where \(x \in V(\Theta)\) and \(B\) is a right coset in \(U_+\) of the subgroup \(\phi(x)\). The vertices \((x, \phi(x))\) span an apartment of \(\Omega\) which we identify with \(\Theta\) via the map \(x \mapsto (x, \phi(x))\). We set \(\bullet = (4, \phi(4))\) and \(\ast = (5, \phi(5))\). Thus \(e := \{\bullet, \ast\}\) is an edge of \(\Omega\). For all vertices \((x, B)\) of \(\Omega\) other than \(\bullet\) and \(\ast\), the vertex \(x\) of \(\Theta\) is uniquely determined by \(B\) and we can denote the vertex \((x, B)\) simply by \(B\). The elements of \(U_+\) fix \(\bullet\) and \(\ast\) and acts on all other vertices by right multiplication.

By [23, 8.11], \(\Omega\) is a Moufang quadrangle. We identify \(\Xi\) with the corresponding bipartite graph as described in [24, 1.8] and let \(\pi\) be an isomorphism \(\Sigma \to \Theta\) mapping the chamber \(c\) to the edge \(\{\bullet, \ast\}\). By [23, 7.5], \(\pi\) extends to a unique \(U_+\)-equivariant isomorphism from \(\Xi\) to \(\Omega\). We identify \(\Xi\) with \(\Omega\) via this extension, so that \(\Sigma = \Theta, c = e\) and the polarity \(\rho\) is an element of \(\text{Aut}(\Xi)\) stabilizing \(\Sigma\) and interchanging the vertices \(\bullet\) and \(\ast\).

In particular, \(c\) and \(d\) are in \(X\), where \(d = \{U_1, U_4\}\) is the chamber of \(\Sigma\) opposite \(e = \{\bullet, \ast\}\). Let \(U = U_e = U_c\) be as in Notation 15.2.

Let \(m_1 = \mu_\Sigma(x_1(0, 0, 1))\) and \(m_4 = \mu_\Sigma(x_4(0, 0, 1))\) be as in [24, 11.22]. By (9.15), conjugation by the polarity \(\rho\) interchanges \(x_1(0, 0, 1)\) and \(x_4(0, 0, 1)\). By [24, 11.23], therefore, conjugation by \(\rho\) interchanges \(m_1\) and \(m_4\). By the identities in [23, 14.18 and 32.11], \(m_1\) and \(m_4\) both have order 2. By [23, 6.9], therefore, \((m_1, m_4)\) is a dihedral group of order 8. In particular, \((m_1 m_4)^2 = (m_4 m_1)^2\) and hence

\[(16.1)\quad \nu := (m_1 m_4)^2\]
is centralized by $\rho$. The element $m_1$ fixes the vertices $\star$ and $U_1$ and reflects $\Sigma$ onto itself and the element $m_4$ fixes the vertices $\bullet$ and $U_4$ and reflects $\Sigma$ onto itself. Thus, in particular, $d = c''$. Hence $U_d = \nu U_c \nu = \nu U \nu$ and thus $\nu U \nu$ acts sharply transitively on $X \setminus \{d\}$. The map $u \mapsto d^u$ is a bijection from the root group $U$ of $M$ to the set $X \setminus \{e\}$. Hence there exists a unique permutation $\tau$ of $U$ such that

$$d(u)\tau = c^{\nu u \nu}$$

for all $u \in U^\ast$.

The Moufang set $M$ is isomorphic to the Moufang set $M(U, \tau)$ defined in [6, §3], where $\tau$ is as in (16.2). As the results [6, Theorems 3.1 and 3.2] indicate, $\tau$ is an essential structural feature of the Moufang set $M$. Our goal in this section is to compute the formula for $\tau$ in Theorem 16.9.

Using the definition of the graph $\Omega$ in [23, 7.1], one can check that the permutation of the vertex set of $\Omega$ given in Table 1 (where $U_{ij} := U_{[i,j]}$ is as in [23, 5.1]) is an automorphism of $\Omega$ which, like $m_1$, fixes the vertices $\star$ and $U_1$ and reflects $\Sigma$ onto itself. It follows from [23, 32.11] (even though we have reparametrized $U_+^+$) that $m_1$ centralizes $U_3$ and $x_4(u)^{m_1} = x_2(u)$ for all $u \in V$. Since $m_1$ maps the vertex $U_{13}$ to the vertex $U_{34}$, it maps the image $U_{13}x_4(u)$ of the vertex $U_{13}$ under the element $x_4(u) \in U_+$ to the image $U_{34}x_2(u)$ of the vertex $U_{34}$ under the element $x_4(u)^{m_1} = x_2(u)$ of $U_+$ for all $u \in V$. Similarly, it maps $U_{12}x_3(u)$ to $U_4x_3(u)$ for all $u \in V$. It follows by [23, 3.7] that the automorphism in Table 1 is $m_1$. By similar arguments, the action of $m_4$ on the vertex set of $\Omega$ is as in Table 2.

In Table 16.3, which is derived from Tables 1 and 2, we have displayed the action of the product $m_1m_4$. We consider the vertex

$$U_1\{v, u\} = U_1x_2(u)x_3(ww + u + g(u, w))x_4(w)$$

with $u, w \in V$, where $\{v, u\}$ is as in (15.5), and compute the image of this vertex under $\nu = (m_1m_4)^2$ in Theorem 16.9 using Table 16.3. First, though, we need to make a few preparations.

**Lemma 16.3.** — Let $u, w \in V$ with $u \neq 0$, and let $v = ww + u + g(u, w) \in V$. Then $u^{-1}v + w \neq 0$.

**Proof.** — Suppose by contradiction that $w = u^{-1}v$. By Proposition 7.4(1),

$$g(u, w) = g(u, q(u)^{-1}uv) = 0,$$
\[ \begin{align*}
U_1 x_2(u) x_3(v) x_4(w) & \longleftrightarrow U_1 x_2(w) x_3(v + g(u, w)) x_4(u) \\
U_{12} x_3(v) x_4(w) & \longleftrightarrow U_4 x_2(w) x_3(v) \\
U_{13} x_4(w) & \longleftrightarrow U_{34} x_2(w)
\end{align*} \]

\[ \begin{align*}
U_4 x_1(u) x_2(v) x_3(w) & \xrightarrow{\not \equiv} U_4 x_1(u^{-1}) x_2(vu^{-1}) x_3(u^{-1}v + w) \\
U_4 x_2(v) x_3(w) & \longleftrightarrow U_{12} x_3(w) x_4(v) \\
U_{34} x_1(u) x_2(v) & \xrightarrow{\not \equiv} U_{34} x_1(u^{-1}) x_2(vu^{-1}) \\
U_{34} x_2(v) & \longleftrightarrow U_{13} x_4(v) \\
U_{24} x_1(u) & \xrightarrow{\not \equiv} U_{24} x_1(u^{-1})
\end{align*} \]

**Table 16.1. The Involution \( m_1 \)**

\[ \begin{align*}
* & \longleftrightarrow * \\
\bullet & \longleftrightarrow U_{24}
\end{align*} \]

\[ \begin{align*}
U_1 x_2(u) x_3(v) x_4(w) & \xrightarrow{\equiv} U_1 x_2(w^{-1}v + u + w^{-1}g(u, w)) x_3(vw^{-1}) x_4(w^{-1}) \\
U_1 x_2(u) x_3(v) & \longleftrightarrow U_{34} x_1(v) x_2(u) \\
U_{12} x_3(v) x_4(w) & \xrightarrow{\equiv} U_{12} x_3(vw^{-1}) x_4(w^{-1}) \\
U_{12} x_3(v) & \longleftrightarrow U_{24} x_1(v) \\
U_{13} x_4(w) & \xrightarrow{\equiv} U_{13} x_4(w^{-1}) \\
U_4 x_1(u) x_2(v) x_3(w) & \longleftrightarrow U_4 x_1(w) x_2(v + g(u, w)) x_3(u) \\
U_{34} x_1(u) x_2(v) & \longleftrightarrow U_1 x_2(v) x_3(u) \\
U_{24} x_1(u) & \longleftrightarrow U_{12} x_3(u)
\end{align*} \]

**Table 16.2. The Involution \( m_4 \)**

so \( v = ww + u \). Then \( u^{-1} = vw^{-1} \) by 7.12(2), so by 7.11(3), \( u = w^{-1}v = w^{-1}(ww + u) \). By \( (R7), 7.12(1) \) and \( (16.4), \)

\[ \begin{align*}
u &= w^{-1} \cdot ww + w^{-1}u + g(ww \cdot w^{-1}, u) \\
&= ww + w^{-1}u + g(w, u) = ww + w^{-1}u.
\end{align*} \]

Hence \( v = ww + u = w^{-1}u \), so 7.12(1) implies \( w = u^{-1}v = u^{-1} \cdot w^{-1}u = uw^{-1} \). Then \( u = ww \), and hence \( v = 0 \), so \( w = u^{-1}v = 0 \), and then \( u = ww = 0 \), a contradiction. We conclude that \( u^{-1}v + w \neq 0 \). \qed
\[
U_1 x_2(u) x_3(v) x_4(w) \mapsto U_1 x_2(u^{-1} v + w) x_3(vu^{-1} + g(u^{-1}, w)) x_4(u^{-1}) \\
U_1 x_3(v) x_4(w) \mapsto U_{34} x_1(v) x_2(w) \\
U_{12} x_3(v) x_4(w) \mapsto U_4 x_1(v) x_2(w) \\
U_{13} x_4(w) \mapsto U_1 x_2(w)
\]

\[
U_4 x_1(u) x_2(v) x_3(w) \mapsto U_4 x_1(u^{-1} v + w) x_2(vu^{-1} + g(u^{-1}, w)) x_3(u^{-1}) \\
U_4 x_2(v) x_3(w) \mapsto U_12 x_3(wv^{-1}) x_4(v^{-1}) \\
U_4 x_3(w) \mapsto U_{24} x_1(w) \\
U_{34} x_1(u) x_2(v) \mapsto U_1 x_2(vu^{-1}) x_3(u^{-1}) \\
U_{34} x_2(v) \mapsto U_{13} x_4(v^{-1}) \\
U_{24} x_1(u) \mapsto U_{12} x_3(u^{-1})
\]

**Table 16.3. The Product \(m_1m_4\)**

**Notation 16.5.** — We set

\[
N(\{w, u\}) := \begin{cases} q(u)q(u^{-1}(ww + u + g(u, w)) + w) & \text{if } u \neq 0, \\
q(w)^{\theta+2} & \text{if } u = 0
\end{cases}
\]

for all \(\{w, u\} \in U\). By Lemma 16.3, \(N(\{w, u\}) = 0\) only if \(w = u = 0\). We call \(N\) the norm of \(M\).

**Lemma 16.6.** — Let \(\{w, u\} \in U\). Then

\[
N(\{w, u\}) = q(u)^{\theta} + q(u)q(w) + q(w)^{\theta+2} + f(u, ww)^{\theta} + f(u, wu) + q(w)f(u, ww).
\]

**Proof.** — This is obvious if \(u = 0\), so assume that \(u \neq 0\). Let \(v = ww + u + g(u, w)\). Then

\[
N(\{w, u\}) = q(u)q(u^{-1}v + w) = q(v)^{\theta} + f(uv, w) + q(u)q(w)
\]

by Proposition 7.9. We have

\[
q(v) = q(ww) + q(u) + f(u, w)^{\theta} + f(ww, u)
\]

\[
= q(w)^{\theta+1} + q(u) + f(u, w)^{\theta} + f(ww, u),
\]
and hence

\[(16.8) \quad q(v)^\theta = q(w)^{\theta+2} + q(u)^\theta + f(u,w)^2 + f(ww,u)^\theta.\]

We also have

\[f(uv,w) = f(u,wv) = f(u,ww+wu+g(u,w)) = q(w)f(u,ww)+f(u,wu)+f(u,w)^2.\]

Combining this with (16.7) and (16.8), we obtain the required formula. □

**Theorem 16.9.** — Let \( \{w,u\} \in U^* \). Then

\[\{w,u\}^\tau = \begin{cases} \frac{q(u)w+f(u,w)u+u(ww+u)}{N(\{w,u\})}, & q(w)u+w(wu+u) \end{cases} \]

Proof. — Assume first that \( u = 0 \). Then

\[U_1 \{w,u\} = U_1 \{w,0\} = U_1 x_3(ww)x_4(w).\]

Since \( \{w,u\} \in U^* \), we have \( w \neq 0 \) and hence \( ww \neq 0 \). Using Table 16.3, we obtain

\[U_1 x_3(ww)x_4(w) \xrightarrow{m1m4} U_3 4 x_1(ww)x_2(w) \xrightarrow{m1m4} U_1 x_2(w \cdot (ww)^{-1})x_3((ww)^{-1}).\]

By 7.12(1), we have \( w \cdot (ww)^{-1} = w^{-1}w^{-1} = w^{-1}w^{-1} = (ww)^{-1} \) and hence

\[U_1 x_2(w \cdot (ww)^{-1})x_3((ww)^{-1}) = U_1 x_2((ww)^{-1})x_3((ww)^{-1}) = U_1 \{0,(ww)^{-1}\} \]

By (16.2), therefore,

\[\{w,0\}^\tau = \{0,(ww)^{-1}\}.\]

Since

\[\frac{w \cdot ww}{N(\{w,0\})} = \frac{q(w)ww}{q(w)^{\theta+2}} = (ww)^{-1},\]

we obtain the required formula.

Assume now that \( u \neq 0 \), and let \( v := ww+u+g(u,w) \). By Lemma 16.3, \( u^{-1}v+w \neq 0 \). Let

\[a = u^{-1}v+w, \quad b = vu^{-1}+g(u^{-1},w) \text{ and } c = u^{-1},\]
so that
\[ U_1\{w, u\} = U_1 x_2(u) x_3(v) x_4(w) \xrightarrow{m_1 m_4} U_1 x_2(a) x_3(b) x_4(c), \]
and hence
\[ U_1\{w, u\} \xrightarrow{(m_1 m_4)^2} U_1\{a^{-1}, a^{-1} b + c\}. \]
Thus
\[ \{w, u\}^\tau = \{a^{-1}, a^{-1} b + c\} \]
by (16.2). Observe that
\[ (16.10) \quad a = u^{-1}(ww + u + g(u, w)) + w = w + f(u, w)u^{-1} + u^{-1}(ww + u) \]
by (R2) and
\[ (16.11) \quad b = (ww + u)u^{-1} + g(u, w)u^{-1} + g(u^{-1}, w) = (ww + u)u^{-1} \]
by 7.4(3). Also notice that
\[ (16.12) \quad q(a) = q(u^{-1}v + w) = q(u)^{-1}N(\{w, u\}) \]
by Notation 16.5, and hence
\[ (16.13) \quad a^{-1} = \frac{q(u)a}{N(\{w, u\})} = \frac{q(u)w + f(u, w)u + u(ww + u)}{N(\{w, u\})}. \]
To compute \(a^{-1}b + c\), we first notice that, by Proposition 7.5,
\[ w \cdot (ww + u)u^{-1} = f(u^{-1}, w(ww + u))u^{-1} + f(u^{-1}, w)u^{-1}(ww + u) + q(u^{-1})w(ww + u), \]
and hence
\[ (16.14) \quad q(u)w \cdot (ww + u)u^{-1} = f(u^{-1}, w(ww + u))u + f(u, w)u^{-1}(ww + u) + w(ww + u). \]
Next, by 7.12(1),
\[ (16.15) \quad f(u, w)u \cdot (ww + u)u^{-1} = f(u, w)u^{-1}(ww + u). \]
Finally, by 7.14(4),
\[ (16.16) \quad u(ww + u) \cdot (ww + u)u^{-1} = q(u)u^{-1}(ww + u) \cdot (ww + u)u^{-1} = q(u)q(u^{-1}(ww + u))u^{-1} = q(u^{-1}(ww + u))u. \]
Also observe that
\[ (16.17) \quad c = u^{-1} = q(u)^{-1}u = q(a)u/N(\{w, u\}) \]
by (16.12). By (16.11), (16.13), (16.14), (16.15), (16.16) and (16.17), we obtain
\[ a^{-1}b + c = \frac{f(u^{-1}, w(w + u))u + w(w + u) + q(u^{-1}(w + u))u + q(a)u}{N\{w, u\}}. \]

It remains to show that
\[ f(u^{-1}, w(w + u)) + q(u^{-1}(w + u)) + q(a) = q(w). \]

By (16.10), however, we have
\[ q(a) = q(w) + f(u, w)^2q(u^{-1}) + q(u^{-1}(w + u)) + f(u, w)f(w, u^{-1}) \]
\[ + f(w, u^{-1}(w + u)) + f(u, w)f(u^{-1}, w^2(w + u)) \]

Notice that the last term is 0 by 7.4(1) and that
\[ f(u, w)^2q(u^{-1}) = f(u, w)f(u, u^{-1}). \]

We conclude that
\[ q(a) = q(w) + q(u^{-1}(w + u)) + f(w, u^{-1}(w + u)) \]
\[ = q(w) + q(u^{-1}(w + u)) + f(u^{-1}, w(w + u)) \]
(by 7.4(2)). Thus (16.18) holds.

\[ \square \]

17. Moufang Subsets

Our next goal is to identify three Moufang subsets of the Moufang set $M$. We continue with the notation in Notation 15.1. Let $G^\dagger$ be as in Definition 2.4 applied to $M$.

Remark 17.1. — Let $U$ and $\tau$ be as in Section 16 and suppose that $R$ is a subgroup of $U$ such that $R^\ast$ is $\tau$-invariant. Let $\tau_R$ denote the restriction of $\tau$ to $R$. By [4, 6.2.2(1)], $M_R := \mathbb{M}(R, \tau_R)$ (as defined in [6, §3]) is a Moufang set. Let $G^\dagger_R$ be as in Definition 2.4 applied to $M_R$. Let $c$, $d$ and $\nu$ be as in (16.2), let $X_R = \{c\} \cup d^R$ and let $N = (R, R^\ast)$. By [4, 6.2.6], $X_R$ is an $N$-orbit, $R$ acts faithfully on $X_R$ and the map from the underlying set $\{\infty\} \cup R$ of $M_R$ to $X_R$ sending $\infty$ to $c$ and $u$ to $d^u$ for all $u \in R$ is a bijection which induces an isomorphism from the group induced by $N$ on $X_R$ to $G^\dagger_R$ with kernel $\mathbb{Z}(N)$.

Notation 17.2. — Let $\Lambda = (L, \kappa)$ be an arbitrary octagonal set as defined in Definition 2.1. We denote by MouSu($\Lambda$) the Moufang set corresponding to the group Suz($\Lambda$). The root groups of MouSu($\Lambda$) are the root
groups of $\text{Suz}(\Lambda)$. Each of them is isomorphic to the group $P_\Lambda$ with underlying set $L \times L$, where
\begin{equation}
(a, b) \cdot (u, v) = (a + u, b + v + au^\kappa)
\end{equation}
for all $(a, b), (u, v) \in L \times L$. For each $z \in L^*$ and for each automorphism $\sigma$ of $L$ commuting with $\kappa$ (possibly trivial), let

$$(a, b)^{\tau_\kappa, \sigma} = \left( z \left( \frac{b}{a^\kappa + ab + b^\kappa} \right)^\sigma, z^{\kappa + 1} \left( \frac{a}{a^\kappa + ab + b^\kappa} \right)^\sigma \right)$$

for all $(a, b) \in P_\Lambda^\kappa$. By [22, Exemple 2], we have $\text{MouSu}(\Lambda) \cong \text{M}(P_\Lambda, \tau_\kappa)$ for all $z \in L^*$ and all $\sigma \in \text{Aut}(L)$ that commute with $\kappa$.

**Remark 17.4.** Let $\Lambda = (L, \kappa)$ be as in Notation 17.2 and suppose that $|L| > 2$. Let $\text{MouSu}(\Lambda) = (X, \{U_x\})_{x \in X}$ and let $B^+$ be the group obtained by applying Definition 2.4 to $\text{MouSu}(\Lambda)$. By [23, 33.17], we have $U_x = [B^+_x, U_x]$ for all $x \in X$.

**Notation 17.5.** Let $\chi, \theta$ and $\theta_K$ be as in Notations 9.3 and 15.1, let $\theta_1 = \theta$ and let $\theta_2 = \chi \theta_1$. Thus $\chi$ commutes with $\theta_1$, and $\theta_2$ is also a Tits endomorphism of $E$.

**Proposition 17.6.** Let $\theta_K, \theta_1, \theta_2$ and $\chi$ be as in Notation 17.5.

\begin{align*}
R_0 & := \{((0, 0, s), (0, 0, t)) \mid s, t \in K\}, \\
R_1 & := \{((a, 0, 0), (0, b, 0)) \mid a, b \in E\} \quad \text{and} \\
R_2 & := \{((0, b, 0), (a, 0, 0)) \mid a, b \in E\}
\end{align*}

and let $\tau_i$ denote the restriction of $\tau$ to $R_i$ for $i \in [0, 2]$. Then $\theta_2$ is a Tits endomorphism of $E$, $R_0$, $R_1$ and $R_2$ are $\tau$-invariant subgroups of $U$, $\text{M}(R_0, \tau_0) \cong \text{MouSu}(K, \theta_K)$ and $\text{M}(R_i, \tau_i) \cong \text{MouSu}(E, \theta_i)$ for $i \in [1, 2]$.

**Proof.** We calculate using Proposition 9.16. First note that

$$\{[s], [t]\} \cdot \{[a], [b]\} = \{[s] + [a], [t] + [b] + [s] \cdot [a]\}$$

$$= \{[s + a], [t + b + sa^\theta]\}$$

for all $s, t, a, b \in K$ by (15.5). Let $w = [s]$ and $u = [t]$ for some $s, t \in K$. Then $f(u, w) = 0, q(u) = t^\theta$, and $q(w) = s^\theta$, $ww + u = [s^\theta + 1 + t]$. Hence $N([w, u]) = (s^{\theta + 2} + st + t^\theta)^\theta$ by Lemma 16.6. Thus

$$\{[s], [t]\}^\tau = \left[ \frac{t}{s^{\theta + 2} + st + t^\theta}, \frac{s}{s^{\theta + 2} + st + t^\theta} \right]$$

by Theorem 16.9. Therefore $R_0$ is $\tau$-invariant subgroup of $U$ and by Notation 17.2, the map $(s, t) \mapsto \{[s], [t]\}$ is an isomorphism from $P_{(K, \theta_K)}$ to $R_0$ which carries the map $\tau_{1, 1}$ to $\tau_0$. Hence $\text{M}(R_0, \tau_0) \cong \text{MouSu}(K, \theta_K)$.
Let \( w = (a,0,0) \in V \) and \( u = (0,b,0) \in V \) for some \( a, b \in E \). Then \( q(w) = \beta^{-1}N(a) \) and \( q(u) = \beta^{-\theta_1-1}N(b) \), where \( N \) is the norm of the extension \( E/K \), and \( v := ww + u = (0, a^{\theta_1+1} + b, 0) \). Hence

\[
q(u)w + u(ww + u) = \left( \beta^{-\theta_1-1}(a^{\theta_1+2} + ab + b^{\theta_1})b^\chi, 0, 0 \right)
\]
and

\[
q(w)u + w(ww + u) = (0, \beta^{-1}(a^{\theta_1+2} + ab + b^{\theta_1})a^\chi, 0).
\]
By (16.7), we also have

\[
N(\{w, u\}) = q(u)^{-1}q(ww + q(u)w) = \beta^{-\theta_1-2}N(a^{\theta_1+2} + ab + b^{\theta_1}).
\]
Setting \([a, b] = \{(a,0,0),(0,b,0)\}\) for all \( a, b \in E \), we obtain

\[
[a, b]^\tau = \left[ \beta\left( \frac{b}{a^{\theta_1+2} + ab + b^{\theta_1}} \right)^\chi, \beta^{\theta_1+1}\left( \frac{a}{a^{\theta_1+2} + ab + b^{\theta_1}} \right)^\chi \right]
\]
for all \( a, b \in E \) by Theorem 16.9. By (15.5), we have

\[
[a, b] \cdot [u, v] = [a + u, b + v + au^{\theta_1}]
\]
for all \( a, b, u, v \in E \). Hence \( R_1 \) is a \( \tau \)-invariant subgroup of \( U \) and by Notation 17.2, \( (a, b) \mapsto [a, b] \) is an isomorphism from \( P(E, \theta_1) \) to \( R_1 \) that carries the map \( \tau_{\beta, \chi} \) to \( \tau_1 \). Hence \( \mathcal{M}(R_1, \tau_1) \cong \text{MouSu}(E, \theta_1) \).

The computations for \( R_2 \) are similar. Setting

\[
[a, b] = \{(0, \beta a^\chi, 0), (b,0,0)\},
\]
we find that

\[
[a, b] \cdot [u, v] = [a + u, b + v + au^{\theta_2+1}]
\]
for all \( a, b, u, v \in E \) and

\[
[a, b]^\tau = \left[ \beta^{\theta_2-1}\left( \frac{b}{a^{\theta_2+2} + ab + b^{\theta_2}} \right)^\chi, \beta\left( \frac{a}{a^{\theta_2+2} + ab + b^{\theta_2}} \right)^\chi \right]
\]
for all \( a, b \in E \). Hence \( R_2 \) is a \( \tau \)-invariant subgroup of \( U \) and by Notation 17.2, the map \( (a, b) \mapsto [a, b] \) is an isomorphism from \( P(E, \theta_2) \) to \( R_2 \) that carries the map \( \tau_{\beta^{\theta_2-1}, \chi} \) to \( \tau_2 \). Hence \( \mathcal{M}(R_2, \tau_2) \cong \text{MouSu}(E, \theta_2) \). \( \square \)

18. Simplicity

We can now deduce Theorem 4.7 as a corollary of Proposition 17.6. Let \( M, X, c, \Sigma \) and \( U_c \) be as in Notation 15.1, let \( G^\dagger \) be as in Definition 2.4 applied to \( M \) and let \( R_0, R_1 \) and \( R_2 \) be as in Proposition 17.6. Let \( i \in [0,2] \) and let \( N \) be as in with \( R_i \) in place of \( R \). By Remarks 7.3, 17.1, 17.4 and Proposition 17.6, we have \( R_i \subset [N_c, R_i] \subset [G^\dagger, G^\dagger] \). Thus \( \langle R_0, R_1, R_2 \rangle \subset \)
[G^\dagger, G^{\dagger\dagger}]. By Proposition 9.16 and (15.5) and some calculation, on the other hand, \( U_c = \langle R_0, R_1, R_2 \rangle \). Since \( G^\dagger \) is generated by conjugates of \( U_c \), it follows that \( G^\dagger \) is perfect. To finish the proof, we proceed with a standard argument which goes back to [10]: Let \( I \) be a non-trivial normal subgroup of \( G^\dagger \). Since \( I \) is normal, the product \( IU_c \) is a subgroup of \( G^\dagger \). Since \( G^\dagger \) acts 2-transitively on \( X \), the subgroup \( I \) acts transitively. Hence the subgroup \( IU_c \) contains all the root groups of \( G^\dagger \). Therefore, \( G^\dagger = IU_c \). Thus
\[
G^\dagger/I \cong IU_c/I \cong U_c/U_c \cap I.
\]
Since \( U_c \) is nilpotent, it follows that \( G^\dagger/I \) is nilpotent. Since \( G^\dagger \) is perfect, the quotient \( G^\dagger/I \) is also perfect. A perfect nilpotent group must be trivial. It follows that \( I = G^\dagger \). Thus \( G^\dagger \) is simple. This concludes the proof of Theorem 4.7.

19. Invariants

In this last section, we show that \( q \) is an invariant of \( M \) (where \( q \) and \( M \) are as in Notation 15.1).

Theorem 19.1. — Let \( (K, V, q, \theta, t \mapsto [t], \cdot) \) and \( (\tilde{K}, \tilde{V}, \tilde{q}, \tilde{\theta}, t \mapsto [t], *) \) be two polarity algebras as defined in Definition 7.1 and assume that the corresponding Moufang sets \( M \) and \( \tilde{M} \) are isomorphic. Then there is a field isomorphism \( \psi: K \rightarrow \tilde{K} \), an additive bijection \( \zeta: V \rightarrow \tilde{V} \) and an element \( e \in \tilde{K}^\times \) such that

1. \( \zeta(v \cdot w) = e^{-1} \zeta(v) * \zeta(w) \) for all \( v, w \in V \);
2. \( \zeta(tv) = \psi(t) \zeta(v) \) for all \( t \in K \) and \( v \in V \);
3. \( \tilde{q}(\zeta(v)) = e^\theta \psi(q(v)) \) for all \( v \in V \); and
4. \( \psi(t^\theta) = \psi(t)^\tilde{\theta} \) for all \( t \in K \).

In particular, the quadratic forms \( q \) and \( \tilde{q} \) are similar.

Proof. — Let \( \pi \) be an isomorphism from \( M = (X, \{U_x\}_{x \in X}) \) to \( \tilde{M} = (\tilde{X}, \{\tilde{U}_x\}_{x \in \tilde{X}}) \), i.e. a bijection from \( X \) to \( \tilde{X} \) such that \( \pi^{-1} U_z \pi = \tilde{U}_{(z)\pi} \) for all \( z \in X \). We can assume that \( M, c \) and \( \Sigma \) are as in Notation 15.1. Thus by Notation 15.2, \( U_x \) is the group \( U \) described in Section 16. Let \( d \in \Sigma \) be the unique chamber of \( \Sigma \) opposite \( c \) (as in Section 16) and let \( \tilde{c} \) and \( \tilde{d} \) be the images of \( c \) and \( d \) under \( \pi \). Let \( H \) denote the pointwise stabilizer of \( \{c, d\} \) in \( M \) and let \( \tilde{H} \) denote the pointwise stabilizer of \( \{\tilde{c}, \tilde{d}\} \) in \( \tilde{M} \). For each \( b \in U_c^* \), we denote by \( m_b \) the unique element in \( U_d b U_d \) interchanging \( c \) and \( d \) and by \( \mu_b \) be the unique permutation of \( U_c^* \) such that

\[
\mu_b = c m_b^{-1} a m_b.
\]
for each $a \in U_c^*$. We define $\tilde{\mu}_{\tilde{b}}$ for each $\tilde{b} \in \tilde{U}_c$ analogously. Let $\varphi: U_c \to \tilde{U}_c$ be the isomorphism induced by $\pi$. Then

$$(19.3) \quad \varphi((a)\mu_b) = (\varphi(a))\tilde{\mu}_{\varphi(b)} \text{ for all } a, b \in U \text{ with } b \neq 1.$$  

Let $\{u, v\}$ for $u, v \in V$ be as in (15.4); we define $\{\tilde{u}, \tilde{v}\}$ for $\tilde{u}, \tilde{v} \in \tilde{V}$ analogously. Thus $U = \{V, V\}$ and $\tilde{U} = \{\tilde{V}, \tilde{V}\}$. Recall from Proposition 15.7 that $U' = \{0, V\}$ and $\tilde{U}' = \{0, \tilde{V}\}$. Therefore

$$(19.4) \quad \varphi(\{0, V\}) = \{0, \tilde{V}\}$$  

and

$$(19.5) \quad \{0, \tilde{V}\} \text{ is } H-\text{invariant.}$$  

Let $\tilde{c} = \varphi([0, [1]]), \tau_1 = \mu_{\{0, [1]\}}, \nu$ be as in (16.1), let $\tau$ be as in (16.2) and let $\tilde{\tau} = \varphi^{-1}\tau\varphi$. The product $m_{\{0, [1]\}}\nu$ fixes $c$ and $d$ and hence lies in $H$. By (16.2) and (19.2), it follows that $\tau_1 \in H^*\tau$, where $H^*$ denotes the permutation group $\{u \mapsto h^{-1}uh \mid h \in H\}$ on $U_c^*$. Similarly, $\tilde{\mu}_c \in \tilde{H}^*\tilde{\tau}$, where $\tilde{H}^*$ is defined analogously. By Theorem 16.9 and (19.5), it follows that $\{0, V^*\}\tau_1 = \{V^*, 0\}$ and $\{0, \tilde{V}^*\}\tilde{\mu}_c = \{\tilde{V}^*, 0\}$. By (19.3), (19.4) and (19.5), therefore,

$$(19.6) \quad \varphi(\{V^*, 0\}) = \varphi(\{0, V^*\}\tau_1) = \varphi(\{0, V^*\})\tilde{\mu}_c = \{\tilde{V}^*, 0\}.$$  

By (19.4) and (19.6), we conclude that there exist maps $\zeta: V \to \tilde{V}$ and $\gamma: V \to \tilde{V}$ such that $\varphi(\{u, 0\}) = \{\zeta(u), 0\}$ and $\varphi(\{0, v\}) = \{0, \gamma(v)\}$ for all $u, v \in V$. Since $\{u, v\} = \{u, 0\} + \{0, v\}$ by (15.5), we have

$\varphi(\{u, v\}) = \{\zeta(u), \gamma(v)\}$

for all $u, v \in V$. Therefore $\zeta$ is additive and

$$(19.7) \quad \gamma(w + b + ua + g(a, w) + g(a, uu))$$

$$= \gamma(w) + \gamma(b) + \zeta(u) * \zeta(a)$$

$$+ \tilde{g}(\zeta(a), \gamma(w)) + \tilde{g}(\zeta(a), \zeta(u) * \zeta(u))$$

for all $u, w, a, b \in V$ by (15.5) since $\phi$ is a homomorphism. Setting $a = 0$ in (19.7), we see that also $\gamma$ is additive. By (19.7), therefore,

$\gamma(ua) + \gamma(g(a, w) + g(a, uu))$

$$= \zeta(u) * \zeta(a) + \tilde{g}(\zeta(a), \gamma(w)) + \tilde{g}(\zeta(a), \zeta(u) * \zeta(u))$$

for all $u, w, a \in V$. Substituting $uu$ for $w$ in this identity, we obtain

$$(19.8) \quad \gamma(ua) = \zeta(u) * \zeta(a) + \tilde{g}(\zeta(a), \gamma(uu)) + \tilde{g}(\zeta(a), \zeta(u) * \zeta(u))$$
and if we set $a = u$ in this identity, we have
\[ \gamma(uu) = \zeta(u) * \zeta(u) + \tilde{g}(\zeta(u), \gamma(uu)) + \tilde{g}(\zeta(u), \zeta(u) * \zeta(u)). \]
Applying the map $\tilde{x} \mapsto \tilde{g}(\zeta(a), \tilde{x})$ to this last identity, we obtain
\[ \tilde{g}(\zeta(a), \gamma(uu)) = \tilde{g}(\zeta(a), \zeta(u) * \zeta(u)). \]
Substituting this back into (19.8) now yields
\[ (19.9) \quad \gamma(ua) = \zeta(u) * \zeta(a) \]
for all $u, a \in V$. In particular,
\[ (19.10) \quad \gamma(u) = \zeta(u) * \zeta([1]). \]

Now observe that by Proposition 15.7 again, $\gamma$ maps $[K]$ onto $[\tilde{K}]$, so by (19.10) and (R4), also $\zeta([K]) = [\tilde{K}]$. In particular, there exists an $e \in K^\times$ such that $\zeta([1]) = [e]$, and hence $\gamma(u) = e\zeta(u)$ for all $u \in V$ by (R2).
Substituting this back into (19.9) now yields (1).

Since $\zeta([K]) = [\tilde{K}]$, there is a unique map $\psi: K \to \tilde{K}$ such that
\[ (19.11) \quad \zeta([t]) = [e\psi(t)] \]
for all $t \in K$. Substituting $[t]$ for $w$ in (1) and applying (R2), we obtain $\zeta(tv) = \zeta(v[t]) = e^{-1}\zeta(v)e\psi(t)[\psi(t)] = \psi(t)\zeta(v)$. Thus (2) holds.

Since $\zeta$ is additive, so is $\psi$. By (2), we have
\[ \psi(st)\zeta(u) = \zeta(stu) = \psi(s)\zeta(tu) = \psi(s)\psi(t)\zeta(u) \]
for all $s, t \in K$ and all $u \in V$, so $\psi$ is multiplicative. By the definition of $e$, we have $\psi(1) = 1$. Thus $\psi$ is a field isomorphism.

We have $[1][v] = [q(v)]$ for all $v \in V$ by (R1). Applying $\zeta$, we obtain using (1) that $[e] * \zeta(v) = e\zeta([q(v)])$. By (7.2), (19.11) and (R1), this implies that $[e\tilde{g}(\zeta(v))] = e[\psi(q(v))] = e^{\theta+1}[\psi(q(v))]$ for all $v \in V$. Thus (3) holds.

Finally, we have $t[1] = [t^\theta]$ for all $t \in K$ by (7.2). Applying $\zeta$ again, we obtain using (2) that $\psi(t)\zeta([1]) = \zeta([t^\theta])$ and hence, by (19.11), $\psi(t)[e] = [e\psi(t^\theta)]$. Another application of (7.2) now yields $[\psi(t)^\theta e] = [e\psi(t^\theta)]$ for all $t \in K$. We conclude that (4) holds.

\[ \Box \]

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