Probing the Hall voltage in synthetic quantum systems

Maximilian Buser,1 Sebastian Greschner,2 Ulrich Schollwöck,1 and Thierry Giamarchi2

1Department of Physics, Arnold Sommerfeld Center for Theoretical Physics (ASC), Munich Center for Quantum Science and Technology (MCQST), Fakultät für Physik, Ludwig-Maximilians-Universität München, D-80333 München, Germany
2Department of Quantum Matter Physics, University of Geneva, 1211 Geneva, Switzerland

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In the context of experimental advances in the realization of artificial magnetic fields in quantum gases, we discuss feasible schemes to extend measurements of the Hall polarization to a study of the Hall voltage, allowing for direct comparison with solid state systems. Specifically, for the paradigmatic example of interacting flux ladders, we report on characteristic zero crossings and a remarkable robustness of the Hall voltage with respect to interaction strengths, particle fillings, and ladder geometries, which is unobservable in the Hall polarization. Moreover, we investigate the site-resolved Hall response in spatially inhomogeneous quantum phases.

In the age of synthetic quantum systems, the realization of artificial gauge fields in ultracold gases [1–12] opens up an exciting path for the study of interacting particles in the presence of large magnetic fluxes. In these platforms, the Hall-like response of a particle current constitutes a typical fingerprint of the presence of an emulated magnetic field: pioneering experiments measured the transverse polarization $p_y$ in synthetic few-leg flux ladders after inducing a transient longitudinal current $j_x$ [13–17], readily giving rise to the Hall polarization $P_H = p_y/j_x$.

Above and beyond that, there is the prospect of quantum gases probing the Hall response in the strongly interacting regime. As theoretical calculations remain challenging therein [18–26], quantum gases might help address open questions concerning the Hall effect in strongly correlated quantum phases in solid state systems [27]. Complementarily to recent efforts in nanodevices [28, 29], they might open a new window to study ballistic magneto-transport [30, 31].

While quantum gas experiments typically focus on the measurement of the Hall polarization, the central quantity of interest in solid state systems is the Hall voltage $V_H$ or the closely related Hall coefficient $R_H$. In semiclassical approaches, the latter is often interpreted as a measure of the inverse carrier density $1/\nu$ [32]. For certain cases, such as noninteracting Chern-insulating states [16], the Hall polarization $P_H$ can be directly related to $V_H$ or $R_H$. However, in general, this relation is nontrivial. Thus, it is desirable to generically access $V_H$ in quantum gas experiments, paving the way for a direct comparison with solid state systems.

In this paper, for finite systems with open boundaries, we show that the Hall voltage $V_H$ as well as the microscopically resolved Hall polarization $P_H$ can be probed in the transient dynamics induced by suitable quantum quenches, leading to a complementary characterization of the Hall response in the interacting regime. For the paradigmatic example of bosonic flux ladders, extensive matrix-product-state (MPS) based simulations, as well as a weak-coupling approach, reveal a remarkable robustness and zero crossings of $V_H$ in different quantum phases.

**Hall voltage** – We specify our approach for the case of synthetic flux ladders as realized in Refs. [6, 14–16] and described by the Hamiltonian

$$H = -t_x \sum_{m=0}^{M-1} \sum_{r=0}^{L-1} \left( \epsilon^{\Theta_m} a_{r,m}^{\dagger} a_{r+1,m} + \text{h.c.} \right) - t_y \sum_{m=0}^{M-2} \sum_{r=0}^{L-1} \left( a_{r,m}^{\dagger} a_{r,m+1} + \text{h.c.} \right) + H_{\text{int}}, \quad (1)$$

with $\Theta_m = (m - (M - 1)/2) \chi + \Phi/L$. The bosonic or fermionic annihilation (creation) operator $a_{r,m}^{(\dagger)}$ acts on the $r$-th rung and $m$-th leg of a ladder comprising a total number of $M$ legs and $L$ rungs. Particle hopping along the legs and rungs is parametrized by $t_x$ and $t_y$, respectively. We typically consider site-local interactions,

$$H_{\text{int}} = \frac{U}{2} \sum_{m,r} n_{r,m} (n_{r,m} - 1) \text{ with } n_{r,m} = a_{r,m}^{\dagger} a_{r,m} \text{ and note that } \chi \text{ accounts for the magnetic flux piercing each plaquette.}$$

The flux ladder Hamiltonian (1) hosts a panoply of emergent quantum phases [33–47], among them Meissner phases [48–51], with particle currents encircling the ladder along its boundaries, and vortex-lattice (VL$p/q$) phases, resembling regular crystals with $p$ vortices per $(Mq)$-site unit cell [52, 53].

In ring-ladder systems with periodic boundary conditions (PBC), as shown in Fig. 1(a), the theoretically ap-

![FIG. 1. (a) Sketch of the flux-ladder ring. (b) Statically tilted ladder with open boundaries. (c) Linear ramp scheme for the calculation of the Hall voltage $V_H$; see text.](image-url)
pealing definition of a (reactive) ground-state DC Hall response employed in Refs. [20, 21, 26, 31] is based on a current-inducing Aharonov-Bohm flux $\Phi$ piercing the ring. In general, a finite value of $\Phi$ induces a current $j_x = \frac{\Phi}{ML} \sum_{m,r} e^{i\Theta_m} \langle a_{r,m}^\dagger a_{r+1,m} \rangle + \text{h.c.}$ and a polarization $p_y = \langle P_y \rangle / (ML)$, with $P_y = \sum_{m,r} (m - M^{-1}) n_{r,m}$, giving rise to the Hall polarization $P_H = p_y / j_x$. On the other hand, the induced polarization $p_y$ might be compensated by means of an external potential term $\mu_y P_y$ in the Hamiltonian (1), enabling the definition of the Hall voltage $V_H$. Generalizing an idea by Prelovšek et al. [20], in which a Hall coefficient was determined in the limit $\chi \to 0$, $V_H$ is here defined for finite values of the magnetic flux $\chi$ by the requirement that $p_y$ vanishes for suitably chosen values of $\Phi$ and $\mu_y$.

$$V_H = \mu_y / j_x. \quad (2)$$

However, despite their theoretical appeal, PBC and persistent currents $j_x$ are hardly accessible in experiments. Hence, in the following, we propose alternative routes to compute the Hall voltage.

**Measuring the Hall voltage** – In a system with open boundary conditions (OBC), the Hall voltage $V_H$ can be efficiently computed within the transient dynamics induced by a linear ramp or a static tilt.

(i) Linear ramp. Starting off with the ground state of the Hamiltonian (1), the instantaneous turning on of a static potential $V_i = \mu_x \sum_{r,m} r n_{r,m}$ at time $\tau = 0$, see Fig. 1(b), induces a transient current $j_x(\tau)$, which, in the presence of a magnetic flux $\chi$, typically polarizes the system. However, by means of an additional time-dependent potential $\tau \mu_y P_y$, as shown in Fig 1(c), the induced polarization might be compensated. Adjusting $\mu_y$ such that the time-average of $p_y$ vanishes, $\langle p_y(\tau) \rangle_{\tau} = 0$, the Hall voltage can be computed as $V_H = (\mu_y / j_x(\tau))_{\tau}$, where $\langle \cdot \rangle_{\tau} = \int_{\tau_i}^{\tau_f} \frac{d\tau}{\tau_i - \tau_f}$ for a suitable time interval $[\tau_i, \tau_f]$.

(ii) Static tilt. By neglecting the dual Hall effect, referring to the current induced by the polarization itself, the Hall voltage $V_H$ can be effectively calculated using a simplified protocol. First, by instantaneously tilting the ladder by means of $V_i$, the Hall polarization $P_H$ can be computed by time-averaging $P_H = (p_y(\tau) / j_x(\tau))_{\tau}$ in the transient dynamics. Moreover, the Hall voltage $V_H$ is approximated by means of $V_H = (\mu_y / p_y)$, where $(\mu_y / p_y)$ is obtained for OBC and in the limit $\mu_y \to 0$.

The consistency of both protocols with the ring-ladder setup is exemplified for a noninteracting fermionic two-leg ladder in Fig. 2; see below and the Supplemental Material [54] for further comparisons. Figure 2(a) and Fig. 2(b) show transient dynamics in $p_y$, $j_x$, and $P_H = p_y / j_x$ induced by the tilt potential $V_i$. The time-averaged results for $P_H$ perfectly agree with the analytic results for PBC for $\chi \in [0, \pi]$, as shown in Fig. 2(c). The Hall voltage $V_H$, shown in Fig. 2(d), as well as $P_H$ exhibit a nonanalyticity at the transition from a weak-flux Meissner-like region to a vortex-liquid phase [50] found for large values of $\chi$. Moreover, as shown in Fig. 2(d), $V_H$ as obtained from the linear ramp protocol perfectly agrees with the analytic results for PBC, while $V_H$ as obtained from the static tilt approximation merely deviates in the immediate proximity to the quantum phase transition. Note that the time-dependent protocols discussed above are realistic in state-of-the art experiments [13-17] and can analogously be applied to continuum systems with spin-orbit coupling [1, 2, 17].

**Interacting systems** – In the following, we examine the Hall voltage in bosonic flux ladders in the interacting regime. Employing extensive MPS based simulations, performed by means of the SyTen toolkit [55, 56], we calculate the Hall voltage in quantum quenches as well as in ring-ladder setups, providing evidence for the consistency of both approaches in the strongly correlated regime. Specifically, for ground-state calculations, we employ the single-site variant [57] of the density-matrix renormalization-group method [58–60]. For quench simulations, we employ the time-dependent variational-principle algorithm [54, 61, 62]. We detail on the MPS based simulations in the Supplemental Material [54].

Figure 3 shows the Hall voltage $V_H$ for a system of strongly correlated particles ($U/t_x = 2$, $t_y/t_x = 1.6$) as a function of the magnetic flux $\chi$, considering an incommensurate particle filling $\nu = 0.8$, where $\nu = N/(ML)$ and $N$ denotes the particle number. Specifically, $V_H$ is shown in the Meissner phase, in the VL$_{1/2}$ phase, and in the VL$_{1/3}$ phase, noting that intermediate regions of vortex-liquid phases are omitted [43]. We stress that...
the MPS based results obtained by simulating tilt dynamics show excellent agreement with the ones obtained from ground-state calculations in ring ladders with PBC. Moreover, our results shown in Fig. 3 reveal a remarkable interaction-driven effect: a series of zero crossings of $\nu V_H$ in different VL phases.

In order to approach the Hall response in the VL phases from a different angle, we extend a weak-coupling (Josephson array) description [63–66], substituting in the expectation value of the Hamiltonian (1) $a_{j,m}$ with $\sqrt{\nu} \theta_{r,m}$ and introducing the classical Josephson phase $\theta_{r,m}$ and density $\nu_{r,m}$. In the limit $t_y/t_x \to 0$ and for a homogeneous density $\nu_{r,m} = \nu$, a complete devil’s staircase of such VL phases VL$_{p/q}$ at each commensurate vortex density $p/q$ is predicted. Finite values of $t_y/t_x$ and interactions gradually destabilize the VL$_{p/q}$ phases with largest $q$ [33]. By employing the semi-classical ansatz and minimizing the energy in the presence of a current-inducing Aharonov-Bohm flux $\Phi$, we obtain the Hall voltage $V_H = -\frac{2}{\gamma} \tan(\chi/2)$ in the Meissner phase. Moreover, in the VL$_{1/2}$ phase, we analytically find that $V_H$ is independent of $U$ and proportional to $1/\nu$; see the Supplemental Material [54], which details on the weak-coupling approach. In Fig. 3 the weak-coupling results are depicted by the blue solid line. Noteworthy, they show good agreement with the MPS based results, noting that Fig. 3(b) shows deviations in the VL$_{1/3}$ phase.

Within the weak-coupling framework, the analysis of $V_H$ generically reveals a zero crossing in the center of each VL$_{p/q}$ phase at a certain value of flux $\chi_{p/q}$. Thus, we define generalized Hall coefficients $R_{H}^{p/q} = \partial_{\chi} V_H |_{\chi=\chi_{p/q}}$ in analogy to the Hall coefficient obtained in the limit $\chi \to 0$ [20]. Specifically, the weak-coupling approach yields

$$R_{H}^{p/q} \approx -\frac{1}{\nu} \left(1 + \gamma_{p/q} \left(\frac{t_y}{t_x}\right)^2\right).$$

We emphasize that in the Meissner phase ($\chi_0 = 0$) and in the VL$_{1/2}$ ($\chi_{1/2} = \pi$), Eq. (3) holds exactly with $\gamma_{0} = 0$, which is in accordance with Ref. [26], and $\gamma_{1/2} = 1/4$. In the VL$_{1/3}$ phase, we find $\gamma_{1/3} \approx 0.51$ and higher order corrections in $t_y/t_x$. The lines in Figure 3(c) depict $R_{H}^{p/q}$ in the Meissner, in the VL$_{1/2}$, and in the VL$_{1/3}$ phases as obtained from the weak-coupling approach. They are in accordance with the values calculated from the MPS based data for $t_y/t_x = 1.6$.

Local Hall response - Microscopic features, such as the rung-resolved polarization $p_y$, provide additional insight into the Hall response in spatially inhomogeneous VL phases. Using MPS based simulations of the static tilt scheme introduced above, we examine the site-resolved Hall response. Figure 4(a) depicts the local configuration of a tilted state in the VL$_{1/3}$ phase, where vortices with currents circulating counter-clockwise are surrounded by Meissner-like regions of opposite chirality. Figure 4(b) and Fig. 4(c) show the transient dynamics in the rung-resolved polarization. Interestingly, the Hall response is
strongly inhomogeneous, following the crystalline structure of the underlying VL phase. In particular, we observe a positive Hall polarization of the vortices, while the Meissner-like rungs exhibit a negative Hall polarization. Thus, we are able to attribute to the different regions an effective local charge reflecting their Hall response. The vortices behave hole-like, while the Meissner-like regions behave particle-like. At a certain value of the magnetic flux, $\chi_{p/q}$ in each VL$_{p/q}$ phase, the competing contributions from hole-like and particle-like regions cancel out, leading to a vanishing macroscopic Hall response. The structure of the local Hall response may also be understood as a signature of the vortex-hole duality, meaning that vortices in a weakly interacting ladder may be identified with holes in a strongly interacting one-dimensional chain with a staggered potential, related to thin-torus-limit states of the fractional quantum Hall effect [47].

Moreover, the spatially inhomogeneous Hall response following the structure of the underlying VL phases can be recovered in the weak-coupling framework, which is discussed in detail in the Supplemental Material [54]. Indeed, numerical solutions confirm a direct relation between the rung-resolved polarization $p_y$ and the chirality of the local currents in the vortex-like and Meissner-like rungs, which has been tested for various VL$_{1/q}$ phases up to $q = 20$. Thus, quantum gas microscopy [11, 67] might open a new window in the study of the Hall response of coherent quantum systems, addressing microscopic features of the Hall response and effective local charge distributions.

**Robustness** — The remarkable overlap between the MPS based results for the Hall voltage $V_H$ in the strongly correlated regime and the results obtained from the weak-coupling approach, as discussed in the context of Fig. 3, indicates a robustness of $V_H$ with respect to the interaction strength $U$. In Fig. 5 we examine this robustness in more detail, considering different values of $U$ and different particle fillings $\nu$ for various values of the magnetic flux $\chi$. In contrast to the Hall polarization $P_H$, which depends non-universally on the values $U$ and $\nu$, the scaled Hall voltage $\nu V_H$ collapses to one curve for a broad regime of parameters in the Meissner phase and in the VL$_{1/2}$ phase. Moreover, in the Meissner phase, up to $M = 4$ legs are considered within the adiabatic ring-ladder framework and in the static tilt approach, revealing an additional robustness of $V_H$ with respect to the ladder geometry. For strong interactions and particle fillings close to the transition to a vortex-liquid phase, we observe deviations from the robust behavior. We emphasize that the robustness described here is different from the universal behavior of the Hall imbalance occurring for SU(M)-symmetric interactions and small magnetic fluxes [26], and in certain quench scenarios [31].

In summary, we have shown that the Hall voltage $V_H$ can be consistently calculated in ladder systems for finite values of the magnetic flux, employing time-dependent quench protocols with longitudinal and transverse potential gradients. The quench protocols are realistic in state-of-the-art experiments with synthetic quantum matter and a study of $V_H$ in ultracold quantum gases might demonstrate its remarkable robustness with respect to the interaction strength $U$, the particle filling $\nu$, and the ladder geometry in different ground-state phases. Furthermore, they open the exciting possibility to study $V_H$ in clean and highly tunable optical lattice systems and allow for direct comparison with the Hall voltage measured in solid state devices. A site-resolved analysis of the Hall response in vortex-lattice VL$_{p/q}$ phases provided insight into characteristic zero crossings of $V_H$ at certain values of the magnetic flux $\chi_{p/q}$, where competing contributions from particle-like Meissner regions and hole-like vortices cancel out.

Our schemes might prove useful in future studies of the Hall response in interesting quantum states, such as biased-ladder states [68] and Laughlin-like states [41, 46].

![Image](image.png)
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In this supplemental material, we provide further details on the flux-ladder model, the matrix-product-state based simulations, and the weak-coupling calculations. We exemplify the static tilt protocol and the linear ramp protocol in a fermionic four-leg ladder, and, moreover, we show how the spatially inhomogeneous Hall response follows the structure of underlying vortex-lattice phases in the weak-coupling framework.

I. FLUX-LADDER MODEL

It is recapped that in our work, we study the Hall response in the paradigmatic $M$-leg flux-ladder model, governed by the Hamiltonian

$$H = -t_x \sum_{m=0}^{M-1} \sum_{r=0}^{L-1} \left( e^{i \Theta_m} a_{r,m}^\dagger a_{r+1,m} + \text{h.c.} \right)$$

$$-t_y \sum_{m=0}^{M-2} \sum_{r=0}^{L-1} \left( a_{r,m}^\dagger a_{r+1,m} + \text{h.c.} \right)$$

$$+ U \sum_{m=0}^{M-1} \sum_{r=0}^{L-1} n_{r,m} (n_{r,m} - 1) + \mu_y P_y,$$  \hspace{1cm} (1)

where $\Theta_m = (m - (M - 1)/2) \chi + \Phi/L$ accounts for the current-inducing Aharonov-Bohm flux $\Phi$ as well as the magnetic flux $\chi$. In the case of periodic boundary conditions, the latter needs to be quantized, $\chi = m 2\pi/L$, with an even integer $m$ for an even number of bosons. The bosonic or fermionic annihilation (creation) operator $a_{r,m}$ acts on the $r$-th rung and $m$-th leg of the ladder and $n_{r,m}$ is given by $n_{r,m} = a_{r,m}^\dagger a_{r,m}$. Particle hopping along the legs and rungs of the ladder is parametrized by $t_x$ and $t_y$, respectively, and $U$ quantifies the inter-particle interaction strength. Moreover, in Eq. (1), we explicitly account for the external potential $\mu_y P_y$, with

$$P_y = \sum_{m,r} \left( m - \frac{M - 1}{2} \right) n_{r,m},$$  \hspace{1cm} (2)

as it plays a crucial role for the definition of the Hall voltage $V_H$. The polarization $P_y$ is defined to be an intensive quantity, given by $p_y = \langle P_y \rangle / (ML)$, noting that throughout our work, angled brackets denote expectation values.

Operators representing local particle currents can be derived from the continuity equation for the occupation of local lattice sites,

$$\frac{d}{dt} \langle n_{r,m} \rangle = -i \langle [n_{r,m}, H] \rangle.$$  \hspace{1cm} (3)

Thus, operators $j_{r,m}^{\parallel}$ and $j_{r,m}^{\perp}$ representing the local particle flow from site $(r, m)$ to $(r+1, m)$ and from site $(r, m)$ to $(r, m+1)$, respectively, take the form

$$j_{r,m}^{\parallel} = -it_x e^{i \Theta_r} (m, \chi, \phi) a_{r,m}^\dagger a_{r+1,m} + \text{h.c.},$$  \hspace{1cm} (4)

$$j_{r,m}^{\perp} = -it_y a_{r,m}^\dagger a_{r,m+1} + \text{h.c.}.$$  \hspace{1cm} (5)

Moreover, the persistent current $j_x$ measures the unidirectional particle transport in the (ring) ladder, while the chirality $j_c(r)$ accounts for the rung-local particle flow along the outer legs in opposite directions. They are defined by means of

$$\hat{j}_x = \frac{1}{ML} \sum_{m=0}^{M-1} \sum_{r=0}^{L-1} j_{r,m}^{\parallel},$$  \hspace{1cm} (6)

$$\hat{j}_c(r) = \frac{1}{2} \sum_{r'=0}^{r} \left( j_{r',0}^{\parallel} - j_{r',M}^{\parallel} \right).$$  \hspace{1cm} (7)

The Hall response is complementarily described by the Hall polarization $P_H$ and the Hall voltage $V_H$. The former is defined in the absence of an external potential, $\mu_y = 0$,

$$P_H = p_y / j_x,$$  \hspace{1cm} (8)

while the latter is defined for a vanishing polarization, $p_y = 0$, as realized by means of a suitably chosen $\mu_y$,

$$V_H = \mu_y / j_x.$$  \hspace{1cm} (9)

Concerning open boundary conditions, $P_H$ and $V_H$ can be obtained from the static tilt and linear ramp protocols introduced in the main text.

II. TRANSIENT DYNAMICS IN THE FOUR-LEG LADDER

In analogy to Fig. 2 of the main text, in Fig. 1, we exemplify the different quench protocols for the case of noninteracting spinless fermions in a four-leg flux ladder. Again, the results for the Hall polarization $P_H$ and the Hall voltage $V_H$ which are obtained from the linear ramp protocol are in perfect accordance with the exact results obtained from the ground states in setups with periodic boundaries for $\chi \in [0, \pi]$. The results obtained from the static tilt protocol overlap well with the exact curves except for a small window of parameters in the immediate proximity to a phase transition.
III. WEAK-COUPLING APPROACH

Putting the focus on two-leg ladders, we detail on the semi-classical weak-coupling approach [1–4].

A. Semi-classical description

Assuming a local coherent Josephson phase $\theta_{r,m}$ and a classical density $\nu_{r,m}$, we employ a coherent-state description of the ring ladder, replacing $a_{r,m}$ with $\sqrt{\nu_{r,m}}e^{i\theta_{r,m}}$ in the expectation value of the Hamiltonian (1). Thus, the starting point of the weak-coupling approach is

$$\langle H \rangle = -2t_x \sum_{r,m} \sqrt{\nu_{r,m}} \nu_{r+1,m} \cos(\theta_{r+1,m} - \theta_{r,m} + \Theta_m)$$

$$- 2t_y \sum_j \sqrt{\nu_{r,m} \nu_{r,m+1}} \cos(\theta_{r,m} - \theta_{r,m+1})$$

$$+ \frac{U}{2} \sum_{r,m} \nu_{r,m} (\nu_{r,m} - 1)$$

$$+ \mu_y \sum_{m,r} (m - \frac{M - 1}{2}) n_{r,m}.$$  \hspace{1cm} (10)

This ansatz, which generically addresses the regime of large particle fillings $\nu$ and weak but finite interaction strengths $U$ (similar to Josephson junction arrays), is well suited for the description of vortex-lattice (VL$_{p/q}$) phases. Typical low-energy configurations of the Josephson phase $\theta_{r,m}$ are shown in Fig. 2. They exhibit a regular series of localized vortices where $\theta_{r,m}$ slips by $\pi$, while in the intermediate regions the phases $\theta_{r,0}$ and $\theta_{r,1}$ are aligned, similar to a small Meissner phases. Moreover, the vortices delocalize as $t_y/t_x$ decreases and typical low-energy configurations satisfy $\langle \theta_{r+1,0} - \theta_{r,0} \rangle \approx - \langle \theta_{r+1,1} - \theta_{r,1} \rangle$.

For the practical calculation of the Hall response in the weak-coupling approach, as well as for the calculation of the configurations shown in Fig. 2, we generically consider a homogeneous particle density per rung, employing the follow parametrization

$$\nu_{r,0} = 2\nu \cos^2(\alpha_r) \quad \nu_{r,1} = 2\nu \sin^2(\alpha_r),$$  \hspace{1cm} (11)

and minimize $\langle H \rangle$, as given in Eq. (10), with respect to the parameters $\alpha_r$ and $\theta_{r,m}$, noting that $r \in \{0, 1, \ldots, L - 1\}$ and $m \in \{0, 1\}$. Concerning the Hall voltage $V_H$, $\mu_y$ is considered as a Lagrange multiplier, while the Hall polarization $P_H$ is obtained for $\mu_y = 0$. Thus, in the Meissner phase, which can also be understood as the VL$_0$ phase, we find

$$V_H = -\frac{2}{\nu} \tan \left( \frac{\chi}{2} \right);$$  \hspace{1cm} (12)

as stated in the main text. Moreover, the weak-coupling result for $V_H$ in the VL$_{1/2}$ phase, as discussed in the context of Fig. 3 of the main text, is explicitly given by

$$V_H = -\frac{1}{\nu} \frac{2 \sin(\chi) \left( \frac{1}{t_x^2} - 2 \cos(\chi) + 2 \right)}{\left( \frac{t_x^2}{t_y^2} + 4 \right) \cos(\chi) + \frac{t_y^2}{t_x^2} - \cos(2\chi) - 3}.$$  \hspace{1cm} (13)

Concerning VL$_{p/q}$ phases with $q > 2$, we numerically find (local) minima of $\langle H \rangle$ in the vicinity to an initial configuration given by $\alpha_r = \pi/4$ and $\theta_{r,m} = \pi \lfloor r/q \rfloor$ for $r \in \{0, 1, \ldots, L - 1\}$ and $m \in \{0, 1\}$. Thus, the VL$_{p/q}$ configurations shown in Fig. 2 do not necessarily correspond to the true ground state in the weak-coupling approach.
model (10) (for the considered model parameters) but resemble metastable configurations for a fixed vortex filling with $p = 1$ vortices in $q = 20$ rungs.

**B. Rung-resolved Hall polarization**

In accordance with our matrix-product-state based results addressing the strongly interacting regime, the rung-resolved Hall polarization $P_H$, as obtained from the weak-coupling approach, follows the local structure, and, specifically the chirality $j_c(r)$, of the underlying vortex-lattice phases. Figure 3(a) and Fig. 3(b) show $P_H$ and $j_c(r)$ in the VL$_{1/20}$ phase for different values of the magnetic flux $\chi$. The vortices exhibit a hole-like Hall response with $V_H > 0$ and $j_c > 0$, while in the surrounding particle-like Meissner regions one finds $V_H < 0$ and $j_c < 0$.

**IV. MATRIX-PRODUCT-STATE BASED CALCULATIONS**

Here, we provide additional information on the matrix-product-state based approaches employed for the calculation of the Hall response in the strongly correlated regime, acknowledging that both, ground-state calculations as well as time-dependent calculations, are performed by means of the SyTen toolkit [5, 6]. Throughout our work, the $U(1)$ symmetry of the flux-ladder Hamiltonian (1) corresponding to the conservation of the total particle number is enforced on the level of the matrix-product-state tensors. Moreover, a cutoff to at most six bosons per lattice site is sufficient for the model parameters considered in our work.

**A. Ground states in the ring-ladder setup**

The ground states in the ring-ladder setup are calculated by means of the density-matrix renormalization-group method [7–9]. Specifically, we employ the single-site variant of the algorithm using subspace expansion [10]. In the course of these calculations, we consider bond dimensions up to typically 3000.

In general, periodic boundary conditions complicate the variational ground-state optimization, and, in practice, they require an increased amount of sweeping, as compared to the optimization in analogous systems with open boundary conditions. Moreover, the quantization of the magnetic flux $\chi$, as discussed in the context of the Hamiltonian (1), manifests itself as a challenging constraint. Especially for ground-state phases appearing in a narrow window of $\chi$, large systems may need to be considered. Specifically, in order to resolve the VL$_{1/3}$ phase in Fig. 3 of the main text, we consider ladders with $L = 60, 75$, and, $90$ rungs in the ring-shaped setup. For the calculation of ground states in the presence of a finite current-inducing Aharonov-Bohm flux $\Phi$, we generically employ the ground states attained at $\Phi = 0$ as an initial state.

Convergence of the variationally optimized ground states is ensured by means of a comparison of the energies $\langle H \rangle$, the energetic fluctuations $\langle H^2 \rangle - \langle H \rangle^2$, as well as all relevant global and local observables for different bond dimensions and different values of the site-local bosonic cutoff. Additionally, it is ensured that the ground states in the Meissner phase and in the vortex-lattice phases exhibit regular patterns of well-defined unit cells.

**B. Time-dependent simulations**

The static tilt protocol introduced in the main text is simulated using the two-site variant of the time-dependent variational-principle algorithm [11, 12] after obtaining the ground state in a ladder with open boundaries from a preliminary density-matrix renormalization-group calculation as described above. For the propagation in time, we employ bond dimensions up to 500 and ensure convergence in all relevant observables by varying the time-step size and the maximum bond dimension independently. Finally, the consistent Hall response, which is independently obtained from either time-dependent quench simulations or ground-state calculations in ring-ladder setups, confirms our results and the feasibility of both approaches.

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