DOOB–MEYER FOR ROUGH PATHS

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ABSTRACT. Recently, Hairer–Pillai proposed the notion of \( \theta \)-roughness of a path which leads to a deterministic Norris lemma. In the Gubinelli framework (Hölder, level 2) of rough paths, they were then able to prove a Hörmander type result (SDEs driven by fractional Brownian motion, \( H > 1/3 \)). We take a step back and propose a natural "roughness" condition relative to a given \( p \)-rough path in the sense of Lyons; the aim being a Doob-Meyer result for rough integrals in the sense of Lyons. The interest in our (weaker) condition is that it is immediately verified for large classes of Gaussian processes, also in infinite dimensions. We conclude with an application to non-Markovian system under Hörmander’s condition.

1. Introduction

Recently, Hairer–Pillai [11] proposed the notion of \( \theta \)-roughness of a path which leads to a deterministic Norris lemma, i.e. some sort of quantitative Doob-Meyer decomposition, for (level-2, Hölder) rough integrals in the sense of Gubinelli. It is possible to check that this roughness condition holds for fractional Brownian motion (fBm); indeed in [11] the author show \( \theta \)-roughness for any \( \theta > H \) where \( H \) denotes the Hurst parameter. (Recall that Brownian motion corresponds to \( H = 1/2 \); in comparison, the regime \( H < 1/2 \) should be thought of as "rougher".) All this turns out to be a key ingredient in their Hörmander type result for stochastic differential equations driven by fBm, any \( H > 1/3 \), solutions of which are in general non-Markovian.

In the present note we take a step back and propose a natural "roughness" condition relative to a given \( p \)-rough path (of arbitrary level \( [p] = 1, 2, \ldots \)) in the sense of Lyons; the aim being a Doob-Meyer result for (general) rough integrals in the sense of Lyons. The interest in our (weaker) condition is that it is immediately verified for large classes of Gaussian processes, also in infinite dimensions. (In essence one only needs a Khintchine law of iterated logarithms for 1-dimensional projections.)

We conclude with an application to non-Markovian systems under Hörmander’s condition, in the spirit of [2].

2. Truely "rough" paths and a deterministic Doob-Meyer result

Let \( V \) be a Banach-space. Let \( p \geq 1 \). Assume \( f \in Lip^\gamma (V, L(V, W)) \), \( \gamma > p - 1 \), and \( X : [0, T] \to V \) to be a \( p \)-rough path in the sense of T. Lyons [12] [13] controlled by \( \omega \).

Recall that such a rough path consists of a underlying path \( X : [0, T] \to V \), together with higher order information which somewhat prescribes the iterated integrals \( \int_0^t dX_s \otimes \ldots \otimes dX_{t_k} \) for \( 1 < k \leq [p] \).

Definition 1. For fixed \( s \in [0, T] \) we call \( X \) "rough at time \( s \)" if (convention \( 0/0 := 0 \))

\[
(*) : \forall v^* \in V^* \setminus \{0\} : \limsup_{t \downarrow s} \left( \frac{|\langle v^*, X_{s,t} \rangle|}{\omega (s, t)^{2/p}} \right) = +\infty.
\]
If $X$ is rough on some dense set of $[0,T]$, we call it truly rough.

**Theorem 1.** (i) Assume $X$ is rough at time $s$. Then

$$
\int_s^t f(X) \, dX = O\left(\omega(s,t)^{2/p}\right) \text{ as } t \downarrow s \implies f(X_s) = 0.
$$

(i') As a consequence, if $X$ is truly rough, then

$$
\int_0^t f(X) \, dX \equiv 0 \text{ on } [0,T] \implies f(X) \equiv 0 \text{ on } [0,T].
$$

(i'') As another consequence, assume $g \in C(V,W)$ and $|t-s| = O(\omega(s,t)^{2/p})$, satisfied e.g. when $\omega(s,t) \approx t-s$ and $p \geq 2$ (the "rough" regime of usual interest) then

$$
\int_0^t f(X) \, dX + \int_0^t g(X) \, dt \equiv 0 \text{ on } [0,T] \implies f(X) + g(X) \equiv 0 \text{ on } [0,T].
$$

(ii) Assume $Z := X \oplus Y$ lifts to a rough path and set, with $\tilde{f}(z)(x,y) := f(z)x$,

$$
\int f(Z) \, dX := \int \tilde{f}(Z) \, dZ.
$$

Then the conclusions from (i), (i') and (i''), with $g = g(Z)$, remain valid.

**Remark 1.** Solutions of rough differential equations $dY = V(Y) \, dX$ in the sense of Lyons are understood in the integral sense, based on the integral defined in (ii) above. This is our interest in this (immediate) extension of part (i).

**Proof.** (i) A basic estimate (e.g. [3]) for the $W$-valued rough integral is

$$
\int_s^t f(X) \, dX = f(X_s)X_{s,t} + O\left(\omega(s,t)^{2/p}\right).
$$

By assumption, for fixed $s \in [0,T)$, we have

$$
0 = \frac{f(X_s)X_{s,t}}{\omega(s,t)^{2/p}} + O(1) \text{ as } t \downarrow s
$$

and thus, for any $w^* \in W^*$,

$$
\left| \frac{\langle v^*, X_{s,t}\rangle}{\omega(s,t)^{2/p}} \right| := \left| \frac{\langle w^*, f(X_s)X_{s,t}\rangle}{\omega(s,t)^{2/p}} \right| = O(1) \text{ as } t \downarrow s;
$$

where $v^* \in V^*$ is given by $V \ni v \mapsto \langle w^*, f(X_s)v \rangle$ recalling that $f(X_s) \in L(V,W)$. Unless $v^* = 0$, the assumption (*) implies that, along some sequence $t_n \downarrow s$, we have the divergent behaviour $|\langle v^*, X_{s,t_n}\rangle|/\omega(s,t_n)^{2/p} \to \infty$, which contradicts that the same expression is $O(1)$ as $t_n \downarrow s$. We thus conclude that $v^* = 0$. In other words,

$$
\forall w^* \in W^*, v \in V : \langle w^*, f(X_s)v \rangle = 0.
$$

and this clearly implies $f(X_s) = 0$. (Indeed, assume otherwise i.e. $\exists v : w := f(X_s)v \neq 0$. Then define $\langle w^*, \lambda w \rangle := \lambda$ and extend, using Hahn-Banach if necessary, $w^*$ from span($w$) $\subset W$ to the entire space, such as to obtain the contradiction $\langle w^*, f(X_s)v \rangle = 1$.)

(i'') From the assumptions, $\int_s^t g(X_r) \, dr \leq |g|_\infty |t-s| = O\left(\omega(s,t)^{2/p}\right)$. We may thus use (i) to conclude $f(X_s) = 0$ on $s \in [0,T)$. It follows that $\int_0^t g(X_r) \, dr \equiv 0$ and by differentiation,
g(X) \equiv 0 \text{ on } [0,T].

(ii) By definition of \( \int f(Z) \, dX \) and \( \tilde{f} \) respectively,

\[
\int_s^t f(Z) \, dX := \int_s^t \tilde{f}(Z) \, dZ
\]

\[
= \tilde{f}(Z_s) Z_{s,t} + O\left(\omega(s,t)^{2/p}\right)
\]

\[
= f(Z_s) X_{s,t} + O\left(\omega(s,t)^{2/p}\right)
\]

and the identical proof (for (i'), then (i'')) goes through, concluding \( f(Z_s) = 0. \)

\[\square\]

**Remark 2.** The reader may wonder about the restriction to \( p \geq 2 \) in (i'') for Hölder type controls \( \omega(s,t) \asymp t - s \). Typically, when \( p < 2 \), one uses Young theory, thereby avoiding the full body of rough path theory. That said, one can always view a path of finite \( p \)-variation, \( p < 2 \), as rough path of finite \( 2 \)-variation (iterated integrals are well-defined as Young integrals). Moreover, by a basic consistency result, the respective integrals (Young, rough) coincide. In the context of fBM with Hurst parameter \( H \in (1/2,1) \), for instance, we can take \( p = 2 \) and note that in this setting fBM is truely rough (cf. below for a general argument based on the law of iterated logarithm). By the afore-mentioned consistency, the Doob–Meyer decomposition of (i'') then becomes a statement about Young integrals. Such a decomposition was previously used in [1].

**Remark 3.** The argument is immediately adapted to the Gubinelli setting of “controlled” paths and would (in that context) yield uniqueness of the derivative process.

**Remark 4.** In definition [1] one could replace the denominator \( \omega(s,t)^{2/p} \) by \( \omega(s,t)^\theta \), say for \( \frac{1}{p} < \theta \leq \frac{2}{p} \). Unlike [11], where \( 2/p - \theta \) affects the quantitative estimates, there seems to be no benefit of such a stronger condition in the present context.

## 3. True roughness of stochastic processes

Fix \( \rho \in [1,2) \) and \( p \in (2\rho,4) \). We assume that the \( V \)-valued stochastic process \( X \) lifts to a random \( p \)-rough path. We assume \( V^* \) separable which implies separability of the unit sphere in \( V^* \) and also (by a standard theorem) separability of \( V \). (Separability of the dual unit sphere in the weak-* topology, guaranteed when \( V \) is assumed to be separable, seems not enough for our argument below.)

The following 2 conditions should be thought of as a weak form of a LIL lower bound, and a fairly robust form of a LIL upper bound. As will be explained below, they are easily checked for large classes of Gaussian processes, also in infinite dimensions.

**Condition 1.** Set \( \psi(h) = h^{\frac{\rho}{2}} (\ln \ln 1/h)^{1/2} \). Assume (i) there exists \( c > 0 \) such that for every fixed dual unit vector \( \varphi \in V^* \) and \( s \in [0,T) \)

\[
P \left[ \limsup_{t \downarrow s} \frac{|\varphi(X_{s,t})|}{\psi(t-s)} \geq c \right] = 1
\]

and (ii) for every fixed \( s \in [0,T) \),

\[
P \left[ \limsup_{t \downarrow s} \frac{|X_{s,t}|_{V^*}}{\psi(t-s)} < \infty \right] = 1
\]
Theorem 2. Assume $X$ satisfies the above condition. Then $X$ is a.s. truly rough.

Proof. Take a dense, countable set of dual unit vectors, say $K \subset V^*$. Since $K$ is countable, the set on which condition (i) holds simultaneously for all $\varphi \in K$ has full measure,

$$\mathbb{P} \left[ \forall \varphi \in K : \limsup_{t \downarrow s} |\varphi (X_{s,t})| / \psi (t-s) \geq c \right] = 1$$

On the other hand, every unit dual vector $\varphi \in V^*$ is the limit of some $(\varphi_n) \subset K$. Then

$$\frac{|\langle \varphi_n, X_{s,t} \rangle|}{\psi (t-s)} \leq |\langle \varphi, X_{s,t} \rangle| / \psi (t-s) + |\varphi_n - \varphi|_{V^*} \frac{|X_{s,t}|_V}{\psi (t-s)}$$

so that, using $\limsup (|a| + |b|) \leq \limsup (|a|) + \limsup (|b|)$, and restricting to the above set of full measure,

$$c \leq \limsup_{t \downarrow s} \frac{|\langle \varphi_n, X_{s,t} \rangle|}{\psi (t-s)} \leq \limsup_{t \downarrow s} \frac{|\langle \varphi, X_{s,t} \rangle|}{\psi (t-s)} + |\varphi_n - \varphi|_{V^*} \limsup_{t \downarrow s} \frac{|X_{s,t}|_V}{\psi (t-s)}.$$

Sending $n \to \infty$ gives, with probability one,

$$c \leq \limsup_{t \downarrow s} \frac{|\langle \varphi, X_{s,t} \rangle|}{\psi (t-s)}.$$

Hence, for a.e. sample $X = X_\omega$ we can pick a sequence $(t_n)$ converging to $s$ such that $|\langle \varphi, X_{s,t_n} \rangle| / \psi (t_n - s) \geq c - 1/n$. On the other hand, for any $\theta \geq 1/(2\rho)$

$$\frac{|\langle \varphi, X_{s,t_n} (\omega) \rangle|}{|t_n - s|^\theta} = \frac{|\langle \varphi, X_{s,t_n} (\omega) \rangle| \psi (t_n - s)}{\psi (t_n - s) |t_n - s|^\theta} \geq (c - 1/n) |t_n - s|^{1 - \theta} L (t_n - s) \to \infty$$

since $c > 0$ and $\theta \geq 1/(2\rho)$ and slowly varying $L (\tau) := (\ln \ln 1/\tau)^{1/2}$ (in the extreme case $\theta = 1/(2\rho)$ the divergence is due to the (very slow) divergence $L (\tau) \to \infty$ as $\tau = t_n - s \to 0$.) \hfill \square

3.1. Gaussian processes. The conditions put forward here are typical for Gaussian process (so that the pairing $\langle \varphi, X \rangle$ is automatically a scalar Gaussian process). Sufficient conditions for (i), in fact, a law of iterated logarithm, with equality and $c = 1$ are e.g. found in [16, Thm 7.2.15]. These conditions cover immediately - and from general principles - many Gaussian (rough paths) examples, including fractional Brownian motion ($\rho = 1/(2H)$, lifted to a rough path [5,9]) and the stationary solution to the stochastic heat equation on the torus, viewed as as Gaussian processes parametrized by $x \in [0, 2\pi]$; here $\rho = 1$, the fruitful lift to a ”spatial” Gaussian rough path is due to Hairer [10].

As for condition (ii), it holds under a very general condition [9, Thm A.22]

$$\exists \eta > 0 : \sup_{0 \leq s,t \leq T} E \exp \left( \eta \frac{|X_{s,t}|_V^2}{|t-s|^{2\rho}} \right) < \infty.$$

In presence of some scaling, this condition is immediately verifed by Fernique’s theorem.

Example 1. $d$-dimensional fBM is a.s. truly rough (in fact, $H$-rough)
In order to apply this in the context of (random) rough integration, we need to intersect the class of truly rough Gaussian processes with the classes of Gaussian processes which admit a rough path lift. To this end, we recall the following standard setup \[9\]. Consider a continuous \(\phi\)-rough Gaussian process, say \(X\), realized as coordinate process on the (not-too abstract) Wiener space \((E, H, \mu)\) where \(E = C([0, T], \mathbb{R}^d)\) equipped with \(\mu\) is a Gaussian measure s.t. \(X\) has zero-mean, independent components and that \(V_{\rho, \text{var}} \left( R, [0, T]^2 \right)\), the \(\rho\)-variation in 2D sense of the covariance \(R\) of \(X\), is finite for \(\rho \in [1, 2)\). (In the fBM case, this condition translates to \(H > 1/4\).) From \[9\] Theorem 15.33 it follows that we can lift the sample paths of \(X\) to \(p\)-rough paths for any \(p > 2\rho\) and we denote this process by \(\overline{X}\), called the enhanced Gaussian process. In this context, modulo a deterministic time-change, condition (ii) will always be satisfied (with the same \(\rho\)). The non-degeneracy condition (i), of course, cannot be expected to hold true in this generality; but, as already noted, conditions are readily available \[10\].

**Example 2.** \(Q\)-Wiener processes are a.s. truly rough. More precisely, consider a separable Hilbert space \(H\) with ONB \((e_k)\), \((\lambda_k) \in l^1, \lambda_k > 0 \) for all \(k\), and a countable sequence \(\left( \beta^k \right)\) of independent standard Brownians. Then the limit

\[
X_t := \sum_{k=1}^{\infty} \lambda_k^{1/2} \beta^k_t e_k
\]

exists a.s. and in \(L^2\), uniformly on compacts and defines a \(Q\)-Wiener process, where \(Q = \sum \lambda_k \langle e_k, \cdot \rangle\) is symmetric, non-negative and trace-class. (Conversely, any such operator \(Q\) on \(H\) can be written in this form and thus gives rise to a \(Q\)-Wiener process.) By Brownian scaling and Fernique, condition (ii) is obvious. As for condition (i), let \(\varphi\) be an arbitrary unit dual vector and note that \(\varphi(X_\cdot)/\sigma_\varphi\) is standard Brownian provided we set

\[
\sigma_\varphi^2 := \sum \lambda_k \langle \varphi, e_k \rangle^2 > 0.
\]

By Khintchine’s law of iterated logarithms for standard Brownian motion, for fixed \(\varphi\) and \(s\), with probability one,

\[
\limsup_{t \downarrow s} \frac{\varphi(X_{s,t})}{\psi(t-s)} \geq \sqrt{2} \sigma_\varphi.
\]

Since \(\varphi \mapsto \sigma_\varphi^2\) is weakly continuous (this follows from \((\lambda) \in l^1\) and dominated convergence) and compactness of the unit sphere in the weak topology, \(c := \inf \sigma_\varphi > 0\), and so condition (ii) is verified.

Let us quickly note that \(Q\)-Wiener processes can be naturally enhanced to rough paths. Indeed, it suffices to define the \(H \otimes H\)-valued ”second level” increments as

\[
(s, t) \mapsto X_{s,t} := \sum_{i,j} \lambda_i^{1/2} \lambda_j^{1/2} \int_s^t \beta^i_s \circ d \beta^j_s \otimes e_i \otimes e_j.
\]

which essentially reduces the construction of the ”area-process” to the Lévy area of a 2-dimensional standard Brownian motion. (Alternatively, one could use integration against \(Q\)-Wiener processes.) Rough path regularity, \(|X_{s,t}|_{H \otimes H} = O \left( (t-s)^{2\alpha} \right)\) for some \(\alpha \in (1/3, 1/2]\) (in fact: any \(\alpha < 1/2\)), is immediate from a suitable Kolmogorov-type or GRR criterion (e.g. \[8 \#\]). Variations of the scheme are possible of course, it is rather immediate to define \(Q\)-Gaussian processes in which \(\left( \beta^k \right)\) are replaced by \(\left( X^k \right)\), a sequence of independent Gaussian processes, continuous each with covariance uniformly of finite \(\rho\)-variation, \(\rho < 2\).
Let us insist that the (random) rough integration against Brownian, or \( Q \)-Wiener processes) is well-known to be consistent with Stratonovitch stochastic integration (e.g. \([13, 9, 8]\)). In fact, one can also construct a rough path lift via Itô-integration, in this case (random) rough integration (now against a ”non-geometric” rough path) coincides with Itô-integration.

4. An Application

Let \( X \) be a continuous \( d \)-dimensional Gaussian process which admits a rough path lift in the sense described at the end of the previous section. Assume in addition that the Cameron-Martin space \( \mathcal{H} \) has complementary Young regularity in the sense that \( \mathcal{H} \) embeds continuously in \( C^{p,q}\var (([0,T], \mathbb{R}^d) \) with \( \frac{1}{p} + \frac{1}{q} > 1 \). Note \( q \leq p \) for \( \mu \) is supported on the paths of finite \( p \)-variation. This is true in great generality with \( q = \rho \) whenever \( \rho < 3/2 \) and also for fBM (and variations thereof) for all \( H > 1/4 \). Complementary Young regularity of the Cameron-Martin space is a natural condition, in particular complementarity with \( \rho = \frac{1}{3} \) and also for fBM (and variations thereof) for all \( H > 1/3 \). We give a quick proof of existence of density, with drift, with general non-degenerate Gaussian driving noise (including fBM \( H > 1/4 \)). To this end, consider the rough differential equation

\[
dY = V_0 (Y) \, dt + V (Y) \, dX
\]

subject to a weak Hörmder condition at the starting point. (Vector fields, on \( \mathbb{R}^c \), say are assumed to be bounded, with bounded derivatives of all orders.) In the drift free case, \( V_0 = 0 \), conditions on the Gaussian driving signal \( X \) where given in \([2]\) which guarantee existence of a density. With no need of going into full detail here, the proof (by contradiction) follows a classical pattern which involves a deterministic, non-zero vector \( z \) s.t. \( z^T J^{X (\omega)}_0 (V_k (Y (\omega))) = 0 \) on \([0, \Theta (\omega))\) for some a.s. positive random time \( \Theta \). (This follows from a global non-degeneracy condition, which, for instance, rules out Brownian bridge type behaviour, and a 0-1 law, see conditions 3.4 in \([2]\). From this

\[
\int_0^T z^T J^{X}_{0,\omega-t} ([V, V_k] (Y_1)) \, dX + \int_0^T z^T J^{X}_{0,\omega-t} ([V_0, V_k] (Y_1)) \, dt \equiv 0
\]

on \([0, \Theta (\omega))\): here \( V = (V_1, \ldots, V_d) \) and \( V_0 \) denote smooth vector fields on \( \mathbb{R}^c \) along which the RDEs under consideration do not explode. Now we assume the driving (rough) path to be truely rough, at least on a positive neighbourhood of 0. Since \( Z := (X, Y, J) \) can be constructed simultaneously as rough path, say \( Z \), we conclude with Theorem\([11]\) (iii):

\[
z^T J^{X}_{0,\omega-t} ([V_i, V_k] (Y_1)) \equiv 0 \equiv z^T J^{X}_{0,\omega-t} ([V_0, V_k] (Y_1)).
\]

Usual iteration of this argument shows that \( z \) is orthogonal to \( V_1, \ldots, V_d \) and then all Lie-brackets (also allowing \( V_0 \), always at \( y_0 \). Since the weak-Hörmander condition asserts precisely that all these vector fields span the tangent space (at starting point \( y_0 \)) we then find \( z = 0 \) which is the desired contradiction. We note that the true roughness condition on the driving (rough) path replaces the support type condition put forward in \([2]\). Let us also note that this argument allows a painfree handling of a drift vector field (not including in \([2]\)); examples include immediately fBM with \( H > 1/4 \) but we have explained above that far more general driving signals can be treated. In fact, it transpires true roughness of \( Q \)-Wiener processes (and then, suitable generalizations to \( Q \)-Gaussian processes) on a seperable Hilbert space \( \mathbb{H} \) allows to obtain a Hörmder type result where the \( Q \)-process ”drives” countably many vectorfields given by \( V : \mathbb{R}^c \rightarrow Lin (\mathbb{H}, \mathbb{R}^c) \).
The Norris type lemma put forward in [11] suggests that the argument can be made quantitative, at least in finite dimensions, thus allowing for a Hörmander type theory (existence of smooth densities) for RDE driven by general non-degenerate Gaussian signals. (In [11] the authors obtain this result for fBM, $H > 1/3$.)

Acknowledgement 1. P.K. Friz has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013) \( / \) ERC grant agreement nr. 258237. A. Shekhar is supported by Berlin Mathematical School (BMS).

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