A LOWER BOUND FOR $K^2_S$

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Abstract. Let $(S, \mathcal{L})$ be a smooth, irreducible, projective, complex surface, polarized by a very ample line bundle $\mathcal{L}$ of degree $d > 35$. In this paper we prove that $K^2_S \geq -d(d - 6)$. The bound is sharp, and $K^2_S = -d(d - 6)$ if and only if $d$ is even, the linear system $|H^0(S, \mathcal{L})|$ embeds $S$ in a smooth rational normal scroll $T \subset \mathbb{P}^5$ of dimension 3, and here, as a divisor, $S$ is linearly equivalent to $\frac{d}{2}Q$, where $Q$ is a quadric on $T$.

Keywords: Projective surface, Castelnuovo-Halphen’s Theory, Rational normal scroll.

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Dedicated to Philippe Ellia on his sixtieth birthday.

1. Introduction

The study of numerical invariants of projective varieties, and of the relations between them, is a classical subject in Algebraic Geometry. We refer to [16] and [17] for an overview on this argument. In this paper we turn our attention to the self-intersection of the canonical bundle of a smooth projective surface $S$. One already knows an upper bound in terms of the degree of $S$ and of the dimension of the space where $S$ is embedded [8]. Now we are going to prove the following lower bound:

**Theorem 1.1.** Let $(S, \mathcal{L})$ be a smooth, irreducible, projective, complex surface, polarized by a very ample line bundle $\mathcal{L}$ of degree $d > 35$. Then:

$$K^2_S \geq -d(d - 6).$$

The bound is sharp, and the following properties are equivalent.

(i) $K^2_S = -d(d - 6)$;

(ii) $h^0(S, \mathcal{L}) = 6$, and the linear system $|H^0(S, \mathcal{L})|$ embeds $S$ in $\mathbb{P}^5$ as a scroll with sectional genus $g = \frac{d^2}{8} - \frac{3d}{4} + 1$;

(iii) $h^0(S, \mathcal{L}) = 6$, $d$ is even, and the linear system $|H^0(S, \mathcal{L})|$ embeds $S$ in a smooth rational normal scroll $T \subset \mathbb{P}^5$ of dimension 3, and here $S$ is linearly equivalent to $\frac{d}{2}(H_T - W)$, where $H_T$ is the hyperplane class of $T$, and $W$ the ruling (i.e. $S$ is linearly equivalent to an integer multiple of a smooth quadric $Q \subset T$).

This is an inequality in the same vein of the classical Plücker-Clebsch formula

$$g \leq \frac{1}{2}(d - 1)(d - 2).$$
for the genus $g$ of a projective curve of degree $d$. Unfortunately, the argument we developed does not enable us to state a sharp lower bound depending on the embedding dimension, like Castelnuovo’s bound, neither to examine the case $d \leq 35$.

2. Proof of Theorem 1.1

Put $r + 1 := h^0(S, \mathcal{L})$. Therefore $|H^0(S, \mathcal{L})|$ embeds $S$ in $\mathbb{P}^r$. Let $H \subseteq \mathbb{P}^{r-1}$ be the general hyperplane section of $S$, so that $\mathcal{L} \cong \mathcal{O}_S(H)$. We denote by $g$ the genus of $H$. If $r = 2$ then $d = 1$ and $K_S^2 = 9 > 5$. If $r = 3$ then $K_S^2 = d(d-4)^2 > -d(d-6)$ for $d > 5$. Therefore we may assume $r \geq 4$.

The case $r = 4$.

First we examine the case $r = 4$. In this case we only have to prove that, for $d > 35$, one has $K_S^2 > -d(d-6)$.

When $r = 4$ we have the double point formula ([12], p. 433-434, Example 4.1.3):

\begin{equation}
(1) \quad d(d-5) - 10(g-1) + 12 \chi(\mathcal{O}_S) - 2K_S^2 = 0
\end{equation}

(use the adjunction formula $2g-2 = H \cdot (H + K_S)$). Moreover by Lefschetz Hyperplane Theorem we know that the restriction map $H^1(S, \mathcal{O}_S) \to H^1(H, \mathcal{O}_H)$ is injective. So, taking into account that

\begin{equation}
\chi(\mathcal{O}_S) = h^0(\mathcal{O}_S) - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S),
\end{equation}

we get

\begin{equation}
(2) \quad \chi(\mathcal{O}_S) \geq 1 - g.
\end{equation}

By (1) and (2) we deduce:

\begin{equation}
2K_S^2 \geq d(d-5) - 22(g-1).
\end{equation}

Therefore to prove that $K_S^2 > -d(d-6)$, it is enough to prove that

\begin{equation}
(3) \quad 22(g-1) < 3d^2 - 17d.
\end{equation}

First assume that $S$ is not contained in a hypersurface of degree $s < 5$. In this case, since $d > 14$, then by Roth’s Lemma [14], we know that $H$ is not contained in a surface of degree $< 5$ in $\mathbb{P}^3$. Recall that the arithmetic genus of an irreducible, reduced, nondegenerate space curve of degree $d > s^2 - s$, not contained in a surface of degree $< s$, is bounded from above by the Halphen’s bound [10]:

\begin{equation}
G(3; d, s) := \frac{d^2}{2s} + \frac{d}{2}(s-4) + 1 - \frac{(s-1-\epsilon)(\epsilon+1)(s-1)}{2s},
\end{equation}

where $\epsilon$ is defined by dividing $d-1 = ms + \epsilon$, $0 \leq \epsilon \leq s-1$. Since $d > 20$, we may apply this bound with $s = 5$, and we have:

\begin{equation}
g \leq \frac{d^2}{10} + \frac{d}{2} + 1.
\end{equation}

It follows (3) as soon as $d > 35$.

So we may assume that $S$ is contained in an irreducible and reduced hypersurface of degree $s \leq 4$. First assume $s \in \{2, 3\}$. In this case one knows that for $d > 12
then \( S \) is of general type \((\text{[2]}, \text{p. 213})\), therefore \( \chi(O_S) \geq 1 \) \((\text{[1]}, \text{Théorème X.4, p. 154})\). Using this and \((\text{[1]})\), we see that a sufficient condition for \( K_S^2 > -d(d - 6) \) is:

\[
10(g - 1) < 3d^2 - 17d + 12.
\]

If \( s = 2 \) then by Halphen’s bound we have \( g \leq \frac{d^2}{8} - \frac{9d}{8} + 1 \). It follows \((\text{[1]})\) for \( d > 12 \). If \( s = 3 \) then by Halphen’s bound we have \( g \leq \frac{d^2}{8} - \frac{4d}{8} + 1 \), from which it follows \((\text{[1]})\) for \( d > 7 \).

It remains to consider the case \( S \) is contained in an irreducible and reduced hypersurface of degree \( s = 4 \). In this case we need to refine previous analysis (in fact when \( s = 4 \) one knows that \( S \) is of general type only for \( d > 97 \) \((\text{[2]}, \text{p. 213})\); moreover if one simply inserts Halphen’s bound \( g \leq \frac{d^2}{8} + 1 \) into \((\text{[3]})\), the inequality \((\text{[3]})\) is satisfied only for \( d > 68 \)). Now first recall that by \((\text{[8]}, \text{Lemme 1})\) one has

\[
\frac{d^2}{8} - \frac{9d}{8} + 1 \leq g \leq \frac{d^2}{8} + 1.
\]

Hence there exists a rational number \( 0 \leq x \leq 9 \) such that

\[
g = \frac{d^2}{8} + d \left( \frac{x - 9}{8} \right) + 1.
\]

If \( 0 \leq x \leq 6 \) then \( g \leq \frac{d^2}{8} - \frac{4d}{8} + 1 \), and \((\text{[3]})\) is satisfied for \( d > 35 \). So we may assume \( 6 < x \leq 9 \). By \((\text{[5]}, \text{Proposition 2, (2.2), (2.3) and proof})\) we have

\[
\chi(O_S) \geq \frac{d^3}{96} - \frac{d^2}{16} - \frac{5d}{3} - \frac{333}{16} - (d - 3)d \left( \frac{9 - x}{8} \right)
\]

\[
> \frac{d^3}{96} - \frac{d^2}{16} - \frac{5d}{3} - \frac{333}{16} - \frac{3d(d - 3)}{8} = \frac{d^3}{96} - \frac{7d^2}{24} - \frac{13d}{24} - \frac{333}{16}.
\]

From \((\text{[1]})\) it follows that in order to prove that \( K_S^2 > -d(d - 6) \), it is enough that

\[
10(g - 1) \leq 3d^2 - 17d + 12 \left( \frac{d^3}{96} - \frac{7d^2}{24} - \frac{13d}{24} - \frac{333}{16} \right),
\]

i.e. it is enough that

\[
10(g - 1) \leq \frac{d^3}{8} - \frac{9d^2}{4} - \frac{47d}{2} - \frac{999}{4}.
\]

Taking into account that \( g \leq \frac{d^2}{8} + 1 \), one sees that previous inequality holds true for \( d > 35 \).

This concludes the proof of Theorem \((\text{[1]}, \text{Theorem 1.1})\) in the case \( r = 4 \).

**The case** \( r \geq 6 \).

Now we are going to examine the case \( r \geq 6 \). Also in this case, we only have to prove that \( K_S^2 > -d(d - 6) \). We distinguish two cases, according that the line bundle \( O_S(K_S + H) \) is spanned or not.

If \( O_S(K_S + H) \) is spanned then \((K_S + H)^2 \geq 0 \), therefore, taking into account the adjunction formula \( 2g - 2 = H \cdot (H + K_S) \), we get

\[
K_S^2 \geq d - 4(g - 1).
\]

Let

\[
G(r - 1; d) = \frac{d^2}{2(r - 2)} - \frac{rd}{2(r - 2)} + \frac{(r - 1 - \epsilon)(1 + \epsilon)}{2(r - 2)}
\]
be the Castelnuovo’s bound for the genus of a nondegenerate integral curve of degree $d$ in $\mathbb{P}^{r-1}$, that we may apply to $g$ (here $\epsilon$ is defined by dividing $d-1 = m(r-2)+\epsilon$, $0 \leq \epsilon \leq r-3$) ([17], Theorem (3.7), p. 87). So we deduce
\[
K_S^2 + d(d-6) \geq d-4(G(r-1; d) - 1) + d(d-6)
= \frac{1}{r-2} \left[ (r-4)d^2 - (3r-10)d + 2(r+\epsilon^2 - \epsilon r + 2\epsilon - 3) \right].
\]
Since $r \geq 3$ and $\epsilon \geq 0$, we may write:
\[
(r-4)d^2 - (3r-10)d + 2(r+\epsilon^2 - \epsilon r + 2\epsilon - 3) \geq (r-4)d^2 - (3r-10)d - 2\epsilon r = d^2(r-4) - (5r-10)d + 2rd - 2\epsilon r.
\]
Observe that we have $d \geq r-1$ for $S$ is nondegenerate in $\mathbb{P}^r$. It follows $2rd - 2\epsilon r > 0$ because $\epsilon \leq r-3 < d$. Hence, in order to prove that $K_S^2 > -d(d-6)$ it suffices to prove that $(r-4)d^2 - (5r-10)d \geq 0$, i.e. that
\[
d \geq \frac{5r-10}{r-4}.
\]
Since $d \geq r-1$, this certainly holds for $r \geq 9$. On the other hand, an elementary direct computation shows that
\[
(r-4)d^2 - (3r-10)d + 2(r+\epsilon^2 - \epsilon r + 2\epsilon - 3) > 0
\]
holds true also for $6 \leq r \leq 8$, $0 \leq \epsilon \leq r-3$ and $d \geq r-1$, and for $r = 5$ and $d > 5$. Summing up, previous argument shows that
\[
(5) \quad \text{if } O_S(K_S + H) \text{ is spanned, } r \geq 5 \text{ and } d > 5, \text{ then } K_S^2 > -d(d-6).
\]
Now we assume that $O_S(K_S + H)$ is not spanned. In this case one knows that $S$ is a scroll ([15], Theorem (0.1)), i.e. $S$ is a $\mathbb{P}^1$-bundle over a smooth curve $C$, and the restriction of $O_S(1)$ to a fibre is $O_{\mathbb{P}^1}(1)$ (either $S$ is isomorphic to $\mathbb{P}^2$, but in this case $K_S^2 = 9$). In particular one has that $g$ is equal to the genus of $C$, and so we have ([12], Corollary 2.11, p. 374)
\[
K_S^2 = 8(1-g).
\]
Let
\[
G(r; d) = \frac{d^2}{2(r-1)} - \frac{(r+1)d}{2(r-1)} + \frac{(r-\epsilon)(1+\epsilon)}{2(r-1)}
\]
be the Castelnuovo’s bound for the genus of a nondegenerate integral curve of degree $d$ in $\mathbb{P}^r$ (now $\epsilon$ is defined by dividing $d-1 = m(r-1)+\epsilon$, $0 \leq \epsilon \leq r-2$) ([17], Theorem (3.7), p.87). Since $g$ is equal to the genus of $C$, hence to the irregularity of $S$, by ([9], Lemma 4) we have:
\[
g \leq G(r; d).
\]
Hence we deduce:
\[
K_S^2 = 8(1-g) \geq 8(1 - G(r; d)),
\]
and
\[
K_S^2 + d(d-6) \geq 8(1 - G(r; d)) + d(d-6) =: \psi(r; d),
\]
with
\[
\psi(r, d) = \left( \frac{r-5}{r-1} \right) (d^2 - 2d) - \frac{4}{r-1} (-r + 2 - \epsilon - \epsilon^2 + \epsilon r).
\]
A LOWER BOUND FOR $K_S^2$

Taking into account that the function $d \to d^2 - 2d$ is increasing for $d \geq 1$, and that $d \geq r - 1$, we have:

$$\psi(r,d) \geq \psi(r,r-1) = \frac{1}{r-1} \left( r^3 - 9r^2 + 27r - 23 + 4\epsilon + 4\epsilon^2 - 4\epsilon r \right).$$

Now we notice:

$$r^3 - 9r^2 + 27r - 23 + 4\epsilon + 4\epsilon^2 - 4\epsilon r \geq r^3 - 9r^2 + 27r - 23 + 4\epsilon^2 - 4\epsilon r$$

which is $> 0$ for $r \geq 7$. An elementary direct computation proves that $\psi(r,d) > 0$ also for $r = 6$ (and $d > 4$). This concludes the proof of Theorem 1.1 in the case $r \geq 6$.

**Remark 2.1.** We also remark that for $r = 5$ we have $\psi(5,d) = \epsilon^2 - 4\epsilon + 3$. Since

$$\epsilon^2 - 4\epsilon + 3 = \begin{cases} 3 & \text{if } \epsilon = 0 \\ 0 & \text{if } \epsilon \in \{1,3\} \\ -1 & \text{if } \epsilon = 2, \end{cases}$$

taking into account (5), it follows that $K_S^2 > -d(d - 6)$ holds true also for $r = 5$ and $d > 5$, unless $S \subset \mathbb{P}^5$ is a scroll, $K_S^2 = 8(1 - g)$, and

$$g = G(5; d) = \frac{1}{8} d^2 - \frac{3}{4} d + \frac{(5 - \epsilon)(\epsilon + 1)}{8},$$

with $d - 1 = 4m + \epsilon, 0 < \epsilon \leq 3$. We will use this fact in the analysis of the case $r = 6$ below.

**The last case: $r = 5$.**

In this section we examine the case $r = 5, S \subset \mathbb{P}^5$.

By previous remark, we know that for $d > 5$ one has $K_S^2 > -d(d - 6)$, except when the surface $S$ satisfies the condition $g = G(5; d)$. Now we are going to prove that these exceptions are necessarily contained in a smooth rational normal scroll of dimension 3. As an intermediate step we prove that such surfaces are contained in a threefold of degree $\leq 4$ (when $d > 30$).

To this purpose, assume that $S$ is as before, and that it is not contained in a threefold of degree $< 5$. By ([3], Theorem (0.2)) we know that if $d > 24$ then $H$ is not contained in a surface of degree $< 5$ in $\mathbb{P}^4$. Then by ([7], Theorem (3.22), p. 117) we deduce that for $d > 143$ one has

$$g \leq G(4; d, 5) := \frac{1}{10} d^2 - \frac{3}{10} d + \frac{1}{5} v - \frac{1}{10} v^2 + w,$$

where $v$ is defined by dividing $d - 1 = 5n + v, 0 \leq v \leq 4$, and $w := \max\{0, \lceil \frac{v}{5} \rceil\}$ (with the notation of [7] we have $\pi_2(d, 4) = G(4; d, 5)$). An elementary computation proves that

$$(7) \quad G(4; d, 5) - G(5; d) < 0$$
for \( d > 18 \). This is absurd, therefore if \( K_S^2 \leq -d(d - 6) \) and \( d > 143 \), then \( S \) is contained in a threefold of degree \( \leq 4 \). In order to prove this also for \( 30 < d < 144 \) we have to refine previous analysis. To this aim, first recall that

\[
G(4; d, 5) = \sum_{i=1}^{+\infty} (d - h(i)),
\]

where

\[
h(i) = \begin{cases} 
5i - 1 & \text{if } 1 \leq i \leq n \\
d - w & \text{if } i = n + 1 \\
d & \text{if } i \geq n + 2
\end{cases}
\]

([7], p. 119). Let \( \Gamma \subset \mathbb{P}^3 \) be the general hyperplane section of \( H \), and let \( h_\Gamma \) be its Hilbert function.

Assume first that \( h^0(\mathbb{P}^3, I_\Gamma(2)) \geq 2 \). Then, if \( d > 4 \), by monodromy ([4], Proposition 2.1), \( \Gamma \) is contained in a reduced and irreducible space curve of degree \( \leq 4 \). By ([3], Theorem (0.2)) we deduce that, for \( d > 20 \), \( S \) is contained in a threefold of degree \( \leq 4 \). Hence we may assume \( h^0(\mathbb{P}^3, I_\Gamma(2)) \leq 1 \).

Assume now \( h^0(\mathbb{P}^3, I_\Gamma(2)) = 1 \), and \( h^0(\mathbb{P}^3, I_\Gamma(3)) > 4 \). As before, if \( d > 6 \), by monodromy ([4], Proposition 2.1), \( \Gamma \) is contained in a reduced and irreducible space curve \( X \) of degree \( \deg(X) \leq 4 \). Again as before, if \( \deg(X) \leq 4 \), then \( S \) is contained in a threefold of degree \( \leq 4 \). So we may assume \( 5 \leq \deg(X) \leq 6 \). By ([4], Proposition 4.1) we know that, when \( d > 30 \),

\[
h_\Gamma(i) \geq h(i) \quad \text{for any } i \geq 0.
\]

Hence we have ([7], Corollary (3.2), p. 84):

\[
g \leq \sum_{i=1}^{+\infty} (d - h_\Gamma(i)) \leq \sum_{i=1}^{+\infty} (d - h(i)) = G(4; d, 5).
\]

Since \( g = G(5; d) \), this is absurd for \( d > 18 \) (compare with ([7]). If \( h^0(\mathbb{P}^3, I_\Gamma(2)) = 1 \) and \( h^0(\mathbb{P}^3, I_\Gamma(3)) = 4 \), then we have

\[
h_\Gamma(1) = 4, \quad h_\Gamma(2) = 9, \quad h_\Gamma(3) = 16.
\]

Using induction and ([7], Corollary (3.5), p. 86) we get for any \( i \geq 4 \):

\[
h_\Gamma(i) \geq \min\{d, h_\Gamma(i - 3) + h_\Gamma(3) - 1\} \geq \min\{d, h(i - 3) + 15\} \geq h(i).
\]

As before, this leads to \( g \leq G(4; d, 5) \), which is absurd for \( d > 18 \).

Next assume \( h^0(\mathbb{P}^3, I_\Gamma(2)) = 0 \), and that \( h^0(\mathbb{P}^3, I_\Gamma(3)) \leq 1 \). Then we have:

\[
h_\Gamma(1) = 4, \quad h_\Gamma(2) = 10, \quad h_\Gamma(3) \geq 19.\]

Then a similar computation as before leads to a contradiction if \( d > 18 \).

Finally assume \( h^0(\mathbb{P}^3, I_\Gamma(2)) = 0 \), and \( h^0(\mathbb{P}^3, I_\Gamma(3)) \geq 2 \). Then, by monodromy ([4], Proposition 2.1), \( \Gamma \) is contained in a reduced and irreducible curve \( X \subset \mathbb{P}^3 \) of degree \( \deg(X) \leq 9 \). By ([4], Proposition 4.1) we may also assume \( \deg(X) \geq 7 \). Let \( X' \subset \mathbb{P}^2 \) be the general hyperplane section of \( X \). By Castelnuovo’s Theory ([7],
Lemma (3.1), p. 83) we know that:

\[ h_X(i) \geq \sum_{j=0}^{i} h_X(j). \]

Therefore, taking into account ([7], Corollary (3.6), p. 87), we have \( h_X(1) \geq 4, h_X(2) \geq 9, h_X(3) \geq 16. \) On the other hand, since \( d > 27, \) by Bezout’s Theorem we have \( h_T(i) = h_X(i) \) for any \( 1 \leq i \leq 3. \) Hence we may repeat the same argument as in [5], obtaining \( g \leq G(4; d, 5), \) which is absurd.

Summing up, we proved that if \( r = 5, d > 30 \) and \( K_S^2 \leq -d(d - 6), \) then \( S \) is a scroll, \( K_S^2 = 8(1 - g), \) \( g = G(5; d), \) \( d \not\equiv 1 \pmod{4}, \) and \( S \) is contained in a threefold \( T \subset \mathbb{P}^5 \) of degree \( \leq 4. \) Unfortunately, assuming \( S \) is not contained in a threefold of degree \( < 4, \) previous argument does not work. Therefore we need a different argument to prove that \( S \) cannot lie in a threefold of degree 4.

To this aim, assume by contradiction that \( S \) is contained in a threefold of degree 4. Recall that we are assuming that \( S \) is a scroll, \( K_S^2 = 8(1 - g), \) \( g = G(5; d), \) \( d \not\equiv 1 \pmod{4}, \) and that \( d > 30. \) In particular we have (compare with (6)):

\[ g \geq \frac{1}{8} d^2 - \frac{3}{4} d + 1. \]

On the other hand, by ([7], p. 98-99) we know that

\[ h_T(i) \geq k(i) := \begin{cases} 
4i & \text{if } 1 \leq i \leq p \\
 d - 1 & \text{if } i = p + 1 \text{ and } q = 3 \\
 d & \text{if } i = p + 1 \text{ and } q < 3 \text{ or } i \geq p + 2,
\end{cases} \]

where \( p \) is defined by dividing \( d - 1 = 4p + q, \) \( 0 \leq q \leq 3. \) It follows that

\[ g \leq \sum_{i=1}^{\infty} (d - h_T(i)) \leq \sum_{i=1}^{\infty} (d - k(i)) = G(4; d, 4), \]

with

\[ G(4; d, 4) = \frac{1}{8} d^2 - \frac{1}{2} d + \frac{3}{8} + \frac{1}{4} q - \frac{1}{8} q^2 + t, \]

where \( t = 0 \) if \( 0 \leq q \leq 2, \) and \( t = 1 \) if \( q = 3 \) (with the notation as in ([7], p. 99) we have \( G(4; d, 4) = \pi_1(d, 4). \) Moreover, since \( S \) is a scroll, we also have

\[ \chi(\mathcal{O}_S) = 1 - g. \]

And using the same argument as in the proof of ([5], Proposition 1, (1.2)), we get:

\[ \chi(\mathcal{O}_S) = 1 - g \geq 1 + \sum_{i=1}^{d-4} (i - 1)(d - k(i)) - (d - 4) \left( \sum_{i=1}^{d-4} (d - k(i)) - g \right) \]

\[ = 1 + \sum_{i=1}^{d-4} (i - 1)(d - k(i)) - (d - 4) (G(4; d, 4) - g). \]

Hence we have

\[ (d - 3)g \leq - \sum_{i=1}^{d-4} (i - 1)(d - k(i)) + (d - 4) G(4; d, 4). \]
Using (9) we get:
\[
(d - 3) \left( \frac{1}{8}d^2 - \frac{3}{4}d + 1 \right) \leq - \sum_{i=1}^{d-4} (i - 1)(d - k(i)) + (d - 4)G(4; d, 4).
\]

Taking into account that
\[
\sum_{i=1}^{d-4} (i - 1)(d - k(i)) = \left( \frac{p}{2} \right)d - 8 \left( \frac{p + 1}{3} \right) + tp,
\]
previous inequality is equivalent to:
\[
-d^3 + 24d^2 + \left( -9q^2 + 18q - 125 + 72t \right)d - 2q^3 + 42q^2 - 70q + 174 - 360t + 24tq \geq 0.
\]
This is impossible if \(d > 24\) (recall that \(d - 1 = 4p + q\), \(0 \leq q \leq 3\), and that \(t = 0\) for \(0 \leq q \leq 2\), and that \(t = 1\) for \(q = 3\)).

So we proved that if \(d > 30\) and \(K_S^2 \leq -d(d - 6)\), then \(S\) is a scroll, \(g = G(5; d)\), and it is contained in a threefold \(T \subset \mathbb{P}^5\) of minimal degree 3, i.e. in a rational normal scroll \(T \subset \mathbb{P}^5\) of dimension 3 and degree 3 ([11], p. 51).

First we prove that \(T\) is necessarily nonsingular. Suppose not. Let \(L\) be a general hyperplane passing through a singular point of \(T\). Then \(H \subset L\) is a curve contained in the surface \(T' := T \cap L\), which is a singular rational normal scroll. Put \(d - 1 = 3p + q\), \(0 \leq q \leq 2\). Since the divisor class group of \(T'\) is generated by a line of the ruling, then \(H\) is residual to \(2 - q\) lines of the ruling of \(T'\), in a complete intersection of \(T'\) with a hypersurface of degree \(p + 1\). Therefore \(H\) is a.C.M., and so also \(S\) is. In particular the arithmetic genus of \(S\) is equal to the geometric genus, therefore \(\chi(O_S) = 1 - g \geq 1\), i.e. \(g = 0\), which is impossible in view of the inequality \(g \geq \frac{1}{5}d^2 - \frac{3}{4}d + 1\).

To conclude the proof of the Theorem it suffices to prove the following:

**Proposition 2.2.** Let \(S \subset \mathbb{P}^5\) be a nondegenerate, smooth, irreducible, projective, complex surface of degree \(d \geq 18\), contained in a smooth rational normal scroll \(T\) of dimension 3. Then \(K_S^2 \geq -d(d - 6)\). The bound is sharp, and the following properties are equivalent.

(i) \(K_S^2 = -d(d - 6)\);

(ii) \(S\) is a scroll with sectional genus \(g = \frac{d^2}{8} - \frac{3d}{4} + 1\);

(iii) \(S\) is linearly equivalent to \(\frac{d}{2}(H_T - W)\), where \(H_T\) is the hyperplane class of \(T\), and \(W\) the ruling.

Before proving this, we need the following lemma:

**Lemma 2.3.** Let \(T \subset \mathbb{P}^5\) be a nonsingular rational normal scroll of dimension 3. Let \(H_T\) be a hyperplane section of \(T\), and \(W\) a plane of the ruling. Let \(\alpha\) and \(\beta\) be integer numbers. Then the linear system \(|\alpha H_T + \beta W|\) contains an irreducible, nonsingular, and nondegenerate surface if and only if \(\alpha > 0\), \(\alpha + \beta \geq 0\), and \(3\alpha + \beta \geq 4\).
Proof of Lemma 2.3. First assume that $|\alpha H_T + \beta W|$ contains an irreducible, non-singular, and nondegenerate surface $S$. Let $T' := T \cap \mathbb{P}^4$ be a general hyperplane section of $T$, which is a rational normal scroll surface in $\mathbb{P}^4$. Let $H_{T'}$ be a hyperplane section of $T'$, and $W'$ a line of the ruling of $T'$. Using the same notation of (\cite{12}, Notation 2.8.1, p. 373, Example 2.19.1, p. 381), we have $C_0 = H_{T'} - 2W'$, $C_0^2 = -e = -1$. Therefore the general hyperplane section of $S$ belongs to the linear system $|\alpha H_{T'} + \beta W'| = |\alpha C_0 + (2\alpha + \beta)W'|$. Taking into account that $S$ is nondegenerate, then by (\cite{12}, Corollary 2.18, p. 380) we get $\alpha > 0$, $\alpha + \beta \geq 0$, and $\deg(S) = 3\alpha + \beta \geq 4$.

Conversely, assume $\alpha > 0$ and $\alpha + \beta \geq 0$. Using the same argument as in the proof of (\cite{6}, Proposition 2.3), we see that the linear system $|\alpha H_T + \beta W|$ is non empty, and base point free. By Bertini’s Theorem it follows that its general member is nonsingular. As for the irreducibility, consider the exact sequence:

$$0 \to \mathcal{O}_T((\alpha - 1)H_T + \beta W) \to \mathcal{O}_T(\alpha H_T + \beta W) \to \mathcal{O}_T(\alpha H_T + \beta W) \otimes \mathcal{O}_{T'} = \mathcal{O}_{T'}(\alpha H_{T'} + \beta W') \to 0,$$

Since $K_T \sim -3H_T + W$ then we may write:

$$(\alpha - 1)H_T + \beta W = K_T + (\alpha + 2)H_T + (\beta - 1)W.$$

As before, by (\cite{6}), we know that the line bundle $\mathcal{O}_T((\alpha + 2)H_T + (\beta - 1)W)$ is spanned, hence nef. On the other hand we have

$$(\alpha + 2)H_T + (\beta - 1)W)^3 = 3(\alpha + 2)^2(\alpha + \beta + 1) > 0.$$

Therefore $\mathcal{O}_T((\alpha + 2)H_T + (\beta - 1)W)$ is big and nef. Then by Kawamata-Viehweg Theorem we deduce

$$H^1(\mathcal{O}_T((\alpha - 1)H_T + \beta W)) = 0.$$

This implies that the linear system $|\alpha H_T + \beta W|$ cut on $T'$ the complete linear system $|\mathcal{O}_{T'}(\alpha H_{T'} + \beta W')|$, whose general member is irreducible by (\cite{12}, Corollary 2.18, p. 380). A fortiori the general member of $|\alpha H_T + \beta W|$ is irreducible.

Finally we notice that the general $S \in |\alpha H_T + \beta W|$ is nondegenerate. In fact otherwise we would have $S = H_T$, which is in contrast with our assumption $\deg(\alpha H_T + \beta W) = 3\alpha + \beta \geq 4$. \hfill $\square$

Proof of Proposition 2.4. Define $m$ and $\epsilon$ by diving

$$d - 1 = 3m + \epsilon, \quad 0 \leq \epsilon \leq 3.$$

Since the Picard group of $T$ is freely generated by the hyperplane class $H_T$ of $T$ and by the plane $W$ of the ruling, then there exists an unique integer $a \in \mathbb{Z}$ such that

$$S \sim (m + 1 + a)H_T + (\epsilon + 1 - 3(a + 1))W.$$

By previous lemma, we may restrict our analysis to the range

$$-m \leq a \leq \frac{1}{2}(m + \epsilon - 1).$$

Taking into account that

$$K_T \sim -3H_T + W,$$
from the adjunction formula we get (compare with [6], (0.4) and p. 149)
\[ K_S^2 = \phi(a) = -6a^3 + a^2(-9m + 5 + 3\epsilon) + a(2m(3\epsilon - 4) - 6\epsilon + 10) + 3m^3 + m^2(3\epsilon - 13) + m(10 - 6\epsilon) + 8. \]
In the given range this function takes its minimum exactly when
\[ a = a^* := \frac{1}{2}(m + \epsilon - 1) \]
(see Appendix below). Since \( \phi(a^*) = -d(d - 6) \), it follows \( K_S^2 > -d(d - 6) \), except when \( d \) is even and
\[ \frac{d}{2} = m + 1 + a^* = -(\epsilon + 1 - 3(a^* + 1)). \]
In this case we already know that \( S \) is a scroll with \( g = \frac{a^2}{3} - \frac{3d}{4} + 1. \)

Appendix.

With the notation as in the proof of Proposition 2.2 consider the function
\[ \phi(a) := -6a^3 + a^2(-9m + 5 + 3\epsilon) + a(2m(3\epsilon - 4) - 6\epsilon + 10) + 3m^3 + m^2(3\epsilon - 13) + m(10 - 6\epsilon) + 8. \]
We are going to prove that if \( d \geq 18 \) and \( -m \leq a \leq \frac{1}{2}(m + \epsilon - 1) \), then \( \phi(a) \geq -d(d - 6) \), and \( \phi(a) = -d(d - 6) \) if and only if \( a = a^* \). To this purpose we derive with respect to \( a \):
\[ \phi'(a) = -18a^2 + 2a(-9m + 5 + 3\epsilon) + 2m(3\epsilon - 4) - 6\epsilon + 10. \]
This is a degree 2 polynomial in the variable \( a \), whose discriminant is:
\[ \Delta = 324m^2 - 936m + 216m\epsilon + 110 + 114\epsilon + 36\epsilon^2, \]
which is > 0 when \( m \geq 3 \), hence when \( d \geq 12 \) (compare with (10)). Denote by \( a_1 \) and \( a_2 \) the real roots of the equation \( \phi'(a) = 0 \), with \( a_1 < a_2 \), and let \( I \) be the open interval \( I = (a_1, a_2) \). Then \( \phi'(a) > 0 \) if and only if \( a \in I \). In particular \( \phi(a) \) is strictly increasing for \( a \in I \), and strictly decreasing for \( a \notin I \). Now observe that
\[ \phi(-m) = 8, \quad \phi(-m + 1) = -9m + 17 - 3\epsilon, \quad \phi(-m + 2) = 0. \]
Notice that \(-9m + 17 - 3\epsilon \geq -9m + 11 \) because \( 0 \leq \epsilon \leq 2, -9m + 11 \geq -9\frac{m}{3} + 11 \) since \( \frac{d-1}{3} \leq m \leq \frac{d+1}{3} \), and \(-3(d - 1) + 11 > -d(d - 6) \) if \( d > 7 \). So for \( a \in \{-m, -m + 1, -m + 2\} \) we have \( K_S^2 > -d(d - 6) \). Moreover we have \( \phi'(-m + 2) = 18m + 6\epsilon - 42 > 0 \) if \( d \geq 12 \), and \( \phi'(-1) = 10m + 6m\epsilon - 12\epsilon - 18 > 0 \) if \( d \geq 9 \). Therefore \( [-m + 2, -1] \subset I \) and so \( \phi(a) \geq \phi(-m + 2) = 0 > -d(d - 6) \) for \(-m \leq a \leq -1 \) and \( d \geq 12 \). We also have:
\[ \phi(0) = (m - 2)(3m^2 - 7m + 3m\epsilon - 4), \]
which is \( \geq 0 \) for \( m \geq 3 \), hence for \( d \geq 12 \). And
\[ \phi(1) = (m - 1)(3m^2 - 10m + 3m\epsilon + 3\epsilon - 17), \]
which is $\geq 0$ for $m \geq 5$, hence for $d \geq 18$. Moreover we have:

$$\phi'(1) = 2 - 26m + 6m \epsilon < 0.$$ 

Therefore $\phi(a)$ is strictly decreasing for $a \geq 1$. It follows that in the range $1 \leq a \leq a^* := \left\lceil \frac{m+\epsilon-1}{2} \right\rceil$, the function $\phi$ takes its minimum exactly when $a = a^*$. Define $p$ and $q$ by dividing:

$$m + \epsilon - 1 = 2p + q, \quad 0 \leq q \leq 1,$$

so that $p = a^*$. Notice that $d$ is even if and only if $q = 0$. We have:

$$\phi(a^*) = \begin{cases} -d(d-6) & \text{if } q = 0 \\ -\frac{1}{4}d^2 + \frac{1}{2}d + \frac{35}{4} & \text{if } q = 1. \end{cases}$$

Since when $d > 5$ we have

$$-\frac{1}{4}d^2 + \frac{1}{2}d + \frac{35}{4} > -d(d-6),$$

by previous analysis it follows that, for any integer $-m \leq a \leq \frac{m+\epsilon-1}{2}$, one has $\phi(a) \geq -d(d-6)$, and $\phi(a) = -d(d-6)$ if and only if $d$ is even and $a = \frac{m+\epsilon-1}{2}$.

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