Spherical-vectors and geometric interpretation of unit quaternions

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Abstract. In this article, we introduce and study the concept of spherical-vectors, which can be perceived as a natural extension of the arguments of complex numbers in the context of quaternions. We initially establish foundational properties of these distinct vectors, followed by a demonstration, through the transfer of structure, that spherical-vectors constitute a non-abelian additive group, isomorphic to the group of unit quaternions. This identification facilitates the presentation of a novel polar form of quaternions, highlighting its algebraic properties, as well as the algebraic properties of the exponential writing. Furthermore, it enables the depiction of unit quaternions on the unit sphere of $\mathbb{R}^3$, allowing for a geometric interpretation of their multiplication.

1. Introduction

The algebraic properties of the additive group of oriented angles between two vectors in the Euclidean plane have allowed for the definition of the argument of complex numbers and the utilization of the algebraic properties of their exponential form to their full advantage. However, if we move to the three-dimensional Euclidean space, most of the preceding results collapse. Thus, there is no way to define the measure of an oriented angle between two vectors (in the case where we consider this measure to be a real number), and the arguments of quaternions are non-oriented and unstructured angles that no longer follow the Chasles relation. As T.Y. Lam wrote in his book [8] “The failure of the equality $e^p \cdot e^q = e^{p+q}$ on $\mathbb{H}$ is a serious drawback, and may have been the principal roadblock to the development of a really useful theory of functions on the quaternions”. For more details on the argument and the polar form of quaternions, please refer to [7, 8, 1].

The present work aims to generalize, in a way naturally compatible with complex numbers, the notions of argument, polar form and exponential writing to quaternions. Thus, we obtain a manageable tool allowing to establish algebraically and geometrically all the properties related to the multiplication of quaternions in their exponential forms, namely the fundamental equation $e^{i\alpha}e^{i\beta} = e^{i(\beta+\alpha)}$. To achieve this, we first need to define and study the notion of spherical-vectors. This goes

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1We will see later that $\alpha$ and $\beta$ represent the new concept of argument for quaternions, and that their addition is non-commutative. Calculations will also demonstrate that, in the multiplication of quaternions, it is generally necessary to commute the addition of their new arguments.
back to Hamilton’s model for the group of unit quaternions (Hamilton [5], Book II, Chapter I, Section 9) and ([10], Subsection 6.3, page 22). In his model, Hamilton interpreted unit quaternions as vector-arcs on the unit sphere $S^2$ of the three euclidean space. In this paper, we reinterpret these vector-arcs from an alternative perspective to endow them with an additive algebraic structure. Consequently, we construe each unit quaternion $q$ as an ordered pair $(u, v)$ of unit vectors in $\mathbb{R}^3$, which we designate as spherical-vectors and which serve as a novel representation of the argument of $q$. Following Hamilton’s approach, the quaternion $q = x + yi + zj + tk$, with $(x, y, z, t) \in \mathbb{R}^4$, is expressed in the form $q = x + U$, where $U(y, z, t)$ is the vector component of $q$ depicted by the pure quaternion $U = yi + zj + tk$. From our standpoint, we represent $q$ in the slightly different way $q = x + iV$, which we call spherical form of $q$, where $V$ is the vector $(t, -z, y)$ represented by the quaternion $V = tj - zk + y$. Throughout this article, we have adopted this novel vector convention to render the form of $q$ more coherent and in alignment with the algebraic and exponential forms of complex numbers, albeit with the potential to initially perplex the reader. Given that $x^2 + \|V\|^2 = 1$, it is straightforward to identify a spherical-vector $(u, v)$, representing our novel argument of $q$, such that $q = u \cdot v + i(u \times v)$, with $u \cdot v = x$ and $u \times V = V$. Consequently, we introduce the exponential notation $q = e^{i(u,v)}$ with arg($q$) = $(u, v)$, and we demonstrate the fundamental algebraic property of the argument and exponential notation as follows. Given two unit quaternions $p$ and $q$ with arguments represented by spherical-vectors $\alpha$ and $\beta$, respectively, we show that it is possible to find three unit vectors $u$, $v$, and $w$ such that $\alpha = (v, w)$ and $\beta = (u, v)$. We then establish the result:

$$pq = [v \cdot w + i(v \times w)] [u \cdot v + i(u \times v)] = u \cdot w + i(u \times w),$$

which can be expressed as

$$e^{i(v,w)} e^{i(u,v)} = e^{i[(u,v)+(v,w)]} = e^{i(u,w)}.$$

In other words, arg($pq$) = arg($q$) + arg($p$). In this case, arguments possess an additive algebraic structure on $S^2$ that is non-commutative (Figure 1).

**Figure 1.** The multiplication $pq$ of two unit quaternions $p = v \cdot w + i(v \times w)$ and $q = u \cdot v + i(u \times v)$ can be geometrically interpreted through the elementary Chasles relation $(u, v) + (v, w) = (u, w)$. 
In this article, we propose an innovative approach compared to previous studies\cite{1,2,3}, which all address the issue of the geometric interpretation of quaternion multiplication based on rotations. The two main contributions of our article lie in the visualization of unit quaternions and the algebraic properties of quaternion multiplication in their exponential forms.

Our original method allows for the apprehension of unit quaternions and their multiplication through the use of spherical-vectors and their addition represented on $S^2$, without resorting to rotations. This approach highlights the algebraic properties of quaternion multiplication in their exponential forms, thus providing a new perspective on the understanding of these mathematical objects.

Our contribution is all the more significant as it extends the current knowledge of quaternions and their algebraic properties, thereby offering new perspectives for researchers and practitioners in this field.

We briefly outline the structure of the article. Section 2 is dedicated to the discussion of spherical-vectors. Initially, we present the definition and the primary properties associated with this concept. Subsequently, we offer a geometric representation of a spherical-vector on $S^2$. To establish a connection between spherical-vectors and unit quaternions, we introduce the spherical form of quaternions in Section 3. In Section 4, we construct a bijective map $\mu$ from the set $S$ of spherical-vectors to the multiplicative group $\mathbb{H}_1$ of unit quaternions. This bijection enables us to transport the structure of $\mathbb{H}_1$ into $S$. Consequently, we define the following additive internal composition law in $S$: $\alpha + \beta = \mu^{-1}(\mu(\beta)\mu(\alpha))$. Equipped with this operation, $S$ forms a non-commutative additive group, and $\mu$ serves as an anti-isomorphism of groups. Utilizing spherical-vectors, we propose in Section 5 a new reformulation of the definitions of argument and polar form of quaternions. We then demonstrate the algebraic properties of the argument and the exponential form. Lastly, Section 6 provides several examples of applications.

In the sequel the space $\mathbb{R}^3$ is endowed with its standard oriented Euclidean structure and its standard basis $(e_x, e_y, e_z)$. We designate respectively by “·”, “×” and “∥·∥”, the dot product, the cross product and the Euclidean norm.

2. Spherical-Vectors

In this section, we introduce the concept of a spherical-vector, detailing its components and support. We then proceed to illustrate its geometric representation on the unit sphere $S^2$.

2.1. Definitions.

Definition 2.1. [Components of an ordered pair of non-zero vectors]
Let $(u, v)$ be an ordered pair of non-zero vectors in $\mathbb{R}^3$. We call scalar component and vector component of $(u, v)$, the real number $\lambda = \frac{u \cdot v}{\|u\|\|v\|}$ and the vector $n = \frac{u \times v}{\|u\|\|v\|}$, respectively.

Definition 2.2. [Spherical-vector] A spherical-vector of $\mathbb{R}^3$ is an equivalence class of ordered pairs of non-zero vectors. Two ordered pairs represent the same spherical-vector if and only if they have the same scalar component and the same vector component.
Definition 2.3. [Support of a spherical-vector] Let \( \alpha \) be a spherical-vector with a non-zero vector component \( n \). We define the support of \( \alpha \), denoted as \( P_\alpha \), to be the vector plane for which \( n \) serves as a normal vector.

Remark 2.1. Let \( \alpha = (u, v) \) be a spherical-vector with a non-zero vector component \( n \). It is important to note that both vectors \( u \) and \( v \) belong to the support \( P_\alpha \) of \( \alpha \).

The following theorem provides a characteristic property of the components of a spherical-vector, which holds significant importance. Indeed, this will allow us to establish an anti-isomorphism between the set of spherical-vectors and the set of unit quaternions in subsequent stages.

Theorem 2.1. Let \( \lambda \) be a real number and \( n \) be a vector in \( \mathbb{R}^3 \). We can assert that \( \lambda^2 + ||n||^2 = 1 \) if, and only if, there exists a unique spherical-vector characterized by components \( \lambda \) and \( n \).

Proof. – Assume the existence of a spherical-vector \( (u, v) \) with components \( \lambda \) and \( n \). Let \( \theta \in [0, \pi] \) represent the non-oriented angle between the two vectors \( u \) and \( v \). According to Definition 2.1, \( \lambda = \cos \theta \) and \( ||n|| = \sin \theta \). Consequently, \( \lambda^2 + ||n||^2 = 1 \).

– Conversely, assume that \( \lambda^2 + ||n||^2 = 1 \) and demonstrate the existence of a unique spherical-vector \( (u, v) \) with components \( \lambda \) and \( n \). According to Definition 2.2, the uniqueness arises from the condition that two spherical-vectors are equal if and only if they share identical components. It remains now to show the existence. To do this, choose any non-zero vector \( u \) orthogonal to \( n \), and let \( v = \lambda u + n \times u \). We begin by verifying that \( v \) is non-zero:

\[
||v||^2 = \lambda^2 ||u||^2 + ||n \times u||^2 = \lambda^2 ||u||^2 + ||n||^2 ||u||^2 = ||u||^2,
\]

where \( u \) and \( n \times u \) are orthogonal. Consequently, the ordered pair \( (u, v) \) represents a spherical-vector with components:

\[
\frac{u \cdot v}{||u|| ||v||} = \frac{u \cdot (\lambda u + n \times u)}{||v||^2} = \lambda
\]

and

\[
\frac{u \times v}{||u|| ||v||} = \frac{u \times (\lambda u + n \times u)}{||u||^2} = \frac{u \times (n \times u)}{||u||^2} = n,
\]

which can be deduced from the formula,

\[
u \times (n \times u) = (u \cdot u)n - (u \cdot n)u = ||u||^2 n.
\]

2.2. Geometric interpretation of a spherical-vector. Let \( u \) and \( v \) be two non-zero vectors in \( \mathbb{R}^3 \). According to Definition 2.1, the ordered pairs \((u, v)\) and \((u/||u||, v/||v||)\) share the same components. Consequently, as per Definition 2.2, they represent identical spherical-vectors. Given that the vectors in the second ordered pair are unitary, there exists a unique ordered pair \((A, B)\) of points on the unit sphere such that \( u/||u|| = A \), \( v/||v|| = B \), and \((u, v) = (A, B)\). This observation leads to the following remark.
Remark 2.2. 

(i) A spherical-vector can be represented by an ordered pair \((u, v)\) of unit vectors in \(\mathbb{R}^3\). In this case, based on Definition 2.1, the components of \(\alpha\) are \(\lambda = u \cdot v\) and \(n = u \times v\).

(ii) A spherical-vector can be denoted by an ordered pair \((A, B)\), represented as \(\widehat{AB}\), consisting of points on the unit sphere \(S^2\) (refer to Figure 2).

![Figure 2. Geometric representation of a spherical-vector \(\alpha\). The components of \(\alpha\) are \(\lambda = u \cdot v = \cos \theta\) and \(n = u \times v\) where \(\|n\| = \sin \theta\) and \(\theta\) is the non-oriented angle between the vectors \(u\) and \(v\).](image)

3. Spherical Form of a Quaternion

In this section, a novel quaternion form is introduced, which facilitates, in Section 4, the establishment of a relationship between unit quaternions and spherical-vectors. This relationship unveils intriguing results. For a more comprehensive understanding of quaternion algebra, readers are encouraged to consult (\[7\], Chapter 5) and \[9\].

Definition 3.1. We denote by \(V\) the set of quaternions written in the form \(aj + bk + c\), where \((a, b, c) \in \mathbb{R}^3\).

Remark 3.1. Hamilton’s approach represents a vector \((a, b, c)\) in \(\mathbb{R}^3\) as the pure quaternion \(ai + bj + ck\). For further details on this concept, readers are directed to Chapter 5 of \[7\] or \[9\]. To prevent confusion with Hamilton’s convention, it is important to clarify that the following discussion opts to identify each vector \(V = (a, b, c)\) with the quaternion \(aj + bk + c \in \mathbb{V}\), which is also denoted by \(V\) (with \(\mathbb{V}\) defined in Definition 3.1). This deliberate choice, which will be elucidated in Section 5, allows for the reformulation of the exponential form of quaternions in a manner that adheres to algebraic properties analogous to those of complex numbers’ exponential form.

Let \(q = x + yi + zj + tk\), with \((x, y, z, t) \in \mathbb{R}^4\), be a quaternion. We can factorize its imaginary part with respect to \(i\) as follows: \(q = x + i(tj - zk + y)\). Consequently, we may express \(q\) as \(x + iV\), where \(V\) represents the quaternion \(jt - kz + y \in \mathbb{V}\), which is identified with the vector \((t, -z, y)\) according to Remark 3.1.
Definition 3.2. [Spherical form of a quaternion] Every quaternion $q$ can be uniquely expressed in the form $x + iV$, where $x \in \mathbb{R}$ and $V \in \mathbb{V}$. This representation is referred to as the spherical form of $q$. The spherical components of $q$ consist of the real number $x$ (called the scalar component) and the vector $V$ (referred to as the vector component).

Proposition 3.1. (i) Two quaternions are equal if and only if their spherical components are the same.

(ii) Given a quaternion $q$ with spherical form $x + iV$, the squared magnitude of $q$ is given by $|q|^2 = x^2 + ||V||^2$.

Proof. Immediate. □

The subsequent lemma presents several valuable properties of the spherical form, which will be utilized later in this article.

Lemma 3.1. For all vectors $u$ and $v$, the following relationships holds.

(i) $iu = \overline{v}$,
(ii) $uv = v \cdot \overline{u} + i(v \times \overline{u})$,
(iii) If $v$ is a unit vector then $v^{-1}u = u \cdot v + i(u \times v)$.

Proof. (i) Let $u = (x, y, z)$ be a vector in $\mathbb{R}^3$. In accordance with Remark 3.1, we identify $u$ with the quaternion $jx + ky + z$. Thus, we obtain $iu = i(jx + ky + z) = kx - jy + iz = (-jx - ky + z)i = \overline{v}$.

(ii) Let $u = (x, y, z)$ and $v = (x', y', z')$ be two vectors in $\mathbb{R}^3$. We have

$uv = (jx + ky + z)(jx' + ky' + z')$
$= (-xx' - yy' + zz') + i \left[ j(yz' + y'z) - k(xz' + x'z) + (xy' - x'y) \right]$
$= (-xx' - yy' + zz') + i \begin{pmatrix} yz' + y'z \\ -xz' - x'z \\ xy' - x'y \end{pmatrix}$
$= v \cdot \overline{u} + i(v \times \overline{u})$.

(iii) The result follows from (ii) and the fact that the inverse of unit vector is equal to its conjugate. □

4. Algebraic structure of spherical-vectors

Throughout the subsequent discussion, let $S$ denote the set of spherical-vectors and $\mathbb{H}_1$ represent the multiplicative group of unit quaternions. In this section, we define and examine an additive group structure on $S$, facilitated by a bijective map from $S$ to $\mathbb{H}_1$.

4.1. Additive group of spherical-vectors. Let $\alpha$ be a spherical-vector with components $\lambda$ and $n$, denoted as $\alpha(\lambda, n)$. It is straightforward to associate $\alpha$ with the quaternion $q_\alpha$, defined by its spherical form $\lambda + in$. According to Theorem 2.1 and Proposition 3.1(ii), the quaternion $q_\alpha$ is unitary.

Theorem 4.1. The mapping $\mu$ that associates each spherical-vector $\alpha(\lambda, n)$ with the unit quaternion $\lambda + in$ is bijective.

(4.1) $\mu : S \rightarrow \mathbb{H}_1$, $\alpha(\lambda, n) \mapsto \lambda + in$. 
Proof. Let \( q = x + iw \) be a unit quaternion expressed in its spherical form. According to Theorem 2.1, there exists a unique spherical-vector \( \alpha \) with components \( x \) and \( w \). Thus, \( \mu(\alpha) = q \). □

Proposition 4.1. Let \( \alpha = (u, v) \) be a spherical-vector represented by an ordered pair of unit vectors. We obtain

\[
\mu(\alpha) = u \cdot v + i(u \times v) = v^{-1}u.
\]

Proof. The first expression of \( \mu(\alpha) \) is derived from the definition of \( \mu \) in (4.1) and the fact that \( u \cdot v \) and \( u \times v \) are the components of \( \alpha \) according to Definition 2.1. The second expression is a consequence of Lemma 3.1.(iii). □

The mapping \( \mu \) enables us to transfer the structure of \( \mathbb{H}_1 \) to \( S \). We can then define an additive internal composition law on \( S \) as \( \alpha + \beta = \mu^{-1}(\mu(\beta)\mu(\alpha)) \), which implies that \( \mu(\alpha + \beta) = \mu(\beta)\mu(\alpha) \). Consequently, \( \mu \) serves as an anti-isomorphism of groups, leading to the following theorem.

Theorem 4.2. The mapping \( \mu \) is an anti-isomorphism of groups, and the set \( S \) of spherical-vectors is a non-commutative additive group isomorphic to the group \( \mathbb{H}_1 \) of unit quaternions.

4.2. Algebraic properties of spherical-vectors. In this section, using the anti-isomorphism \( \mu \) established in Section 4.1, we transfer the algebraic properties of the multiplicative group \( \mathbb{H}_1 \) of unit quaternions into the additive group \( S \) of spherical-vectors.

4.2.1. Zero spherical-vector.

Proposition 4.2. The neutral element of \( S \) which we call zero spherical-vector and denote by \( \bowtie 0 \) is represented by any ordered pair \( (u, u) \) where \( u \) is a unit vector.

Proof. Suppose the zero spherical-vector \( \bowtie 0 \) is represented by an ordered pair \( (u, v) \) of unit vectors. Given that \( \mu \) is a group anti-isomorphism, we have \( \mu((u, v)) = 1 \). According to (4.2), \( \mu((u, v)) = v^{-1}u \). Consequently, \( \mu((u, v)) = 1 \) holds true if and only if \( u = v \). □

4.2.2. Opposite of a spherical-vector.

Proposition 4.3. (i) Let \( u \) and \( v \) be two unit vectors. The opposite of the spherical-vector \( (u, v) \), which we denote by \( -(u, v) \), is the spherical-vector \( (v, u) \).

(ii) If \( \lambda \) and \( n \) are the components of a spherical-vector \( \alpha \), then the components of \( -\alpha \) are \( \lambda \) and \( -n \).

Proof. (i) Given that \( \mu \) represents an anti-isomorphism of groups, according to (4.2), the proof stems from the identity

\[
\mu(-(u, v)) = (\mu((u, v)))^{-1} = (v^{-1}u)^{-1} = u^{-1}v = \mu((v, u)).
\]

(ii) This is an easy consequence of Definition 2.1. □
4.2.3. Chasles relation.

**Proposition 4.4.** Let $u$, $v$ and $w$ be three unit vectors. We have

$$(u, v) + (v, w) = (u, w).$$

**Proof.** As $\mu$ represents an anti-isomorphism of groups and according to (4.2), the Chasles relation can be readily derived from the following equation

$$\mu((u, v) + (v, w)) = \mu((v, w)) \mu((u, v)) = w^{-1}u = \mu((u, w)).$$

\qed

4.2.4. Subtraction operation on $S$.

**Definition 4.1.** Let $\alpha$ and $\beta$ be two spherical-vectors. We define the subtraction operation on $S$ by,

$$\alpha - \beta = \alpha + (-\beta).$$

4.2.5. Straight spherical-vector. Let $u$ be a unit vector. According to Definition 2.1, the components of the spherical-vector $(u, -u)$ are $-1$ and $\vec{0}$, and they are independent of $u$. This observation leads to the following definition.

**Definition 4.2.** We define the straight spherical-vector, denoted by $\pi \mapsto \pi$, as the spherical-vector with components $-1$ and $\vec{0}$, represented by any ordered pair $(u, -u)$, where $u$ is a unit vector.

Let $\alpha$ be a spherical-vector represented by an ordered pair $(u, v)$ of unit vectors. According to Definition 2.1, the vector component of $\alpha$ is $n_\alpha = u \times v$. Given that $u$ and $v$ are unit vectors, the vector $n_\alpha$ is zero if and only if $v = u$ or $v = -u$. Consequently, by invoking Proposition 4.2 and Definition 4.2, we arrive at the following remark.

**Remark 4.1.** The vector component of a spherical-vector is zero if and only if this spherical-vector is zero or straight.

4.3. Sum of two spherical-vectors. In this section we return with more details to the sum of two spherical-vectors defined by their components. To do so, we will need the following lemmas.

**Lemma 4.1.** Let $\alpha$ be a spherical-vector with components $\lambda$ and $n$, where $n \neq \vec{0}$. Let $u$, $v$ and $w$ be three unit vectors (Figure 3). We have

(i) The equality $(u, v) = \alpha$ is satisfied if and only if $v = u(\lambda - in)$, which holds if and only if $u$ is orthogonal to $n$ and $v$ can be expressed as $v = \lambda u + n \times u$.

(ii) The equality $(u, v) = \alpha$ is satisfied if and only if $u = v(\lambda + in)$, which holds if and only if $v$ is orthogonal to $n$ and $u = \lambda v - n \times v$. 

Figure 3. Spherical-vector $\alpha(\lambda, n)$ can be decomposed into a pair of unit vectors $(u, v)$, revealing the relationship between the spherical components of $\alpha$ and its corresponding unit vectors.

Proof. (i) Based on (4.1) and (4.2), we have the following equivalence:

$$(u, v) = \alpha \iff \mu((u, v)) = \mu(\alpha) \iff v^{-1}u = \lambda + in \iff v = u(\lambda - in).$$

According to Remark 2.1, if $(u, v) = \alpha$, it follows that $u \perp n$. Therefore,

$$(u, v) = \alpha \iff u \perp n \text{ and } v = u(\lambda - in).$$

Finally, using Lemma 3.1(i) and Lemma 3.1(ii), when $u \perp n$, we can deduce that

$$u(\lambda - in) = \lambda u - \vec{n} = \lambda u + n \times u.$$ 

Thus, the result is obtained.

(ii) This outcome can be readily derived from part (i) by noting that, according to Proposition 4.3, $(v, u) = -(u, v)$ and the components of $-\alpha$ are $\lambda$ and $-n$.

Lemma 4.2. Let $\alpha$ and $\beta$ be two spherical-vectors with respective supports $P_\alpha$ and $P_\beta$, and components $(\lambda_\alpha, n_\alpha)$ and $(\lambda_\beta, n_\beta)$, where $n_\alpha \neq \vec{0}$ and $n_\beta \neq \vec{0}$. We proceed by considering two distinct cases.

(i) In the case where $n_\alpha \times n_\beta \neq \vec{0}$, there exist precisely two sets of three unit vectors $u, v,$ and $w$ satisfying the conditions $\alpha = (u, v)$ and $\beta = (v, w)$, as illustrated in Figure 4. These sets are as follows,

$$v = \frac{n_\alpha \times n_\beta}{\|n_\alpha \times n_\beta\|}, \quad u = \lambda_\alpha v - n_\alpha \times v \quad \text{and} \quad w = \lambda_\beta v + n_\beta \times v,$$

$$v' = -v, \quad u' = -u \quad \text{and} \quad w' = -w.$$

(ii) If $n_\alpha \times n_\beta = \vec{0}$, the two spherical-vectors $\alpha$ and $\beta$ have the same support $P$. In this case, for any unit vector $v$ in $P$, there exist exactly two unique vectors $u = \lambda_\alpha v - n_\alpha \times v$ and $w = \lambda_\beta v + n_\beta \times v$ that satisfy the conditions $\alpha = (u, v)$ and $\beta = (v, w)$, as illustrated in Figure 5.
Proof. (i) The objective is to determine the solution to the following system of equations, the unknowns being the unit vectors $u$, $v$ and $w$.

\[
\begin{align*}
(u,v) &= \alpha \\
(v,w) &= \beta,
\end{align*}
\]

As stated in Lemma 4.1 we can rewrite this system as follows:

\[
\begin{align*}
v &\perp n_\alpha \text{ and } u = \lambda_\alpha v - n_\alpha \times v \\
v &\perp n_\beta \text{ and } w = \lambda_\beta v + n_\beta \times v.
\end{align*}
\]

If we assume that $v$ is orthogonal to both $n_\alpha$ and $n_\beta$, and since $v$ is a unit vector, we can write $v$ as $v = \pm \frac{n_\alpha \times n_\beta}{\|n_\alpha \times n_\beta\|}$. Consequently, we can proceed to solve the system with the knowledge of $v$, and then use the expressions for $u$ and $w$ given above. As a result, we are left with only two possible
solutions:
\[
\begin{align*}
  v &= \frac{n_\alpha \times n_\beta}{\|n_\alpha \times n_\beta\|} \\
  u &= \lambda_\alpha v - n_\alpha \times v \\
  w &= \lambda_\beta v + n_\beta \times v
\end{align*}
\]

(ii) This is an immediate consequence of Lemma 4.1.

We now possess the essential tools to construct the sum of two spherical-vectors \(\alpha\) and \(\beta\), with respective vector components \(n_\alpha\) and \(n_\beta\). Two cases must be considered:

- If both \(n_\alpha\) and \(n_\beta\) are non-zero (refer to Figure 6), then according to Lemma 4.2, there exist three unit vectors \(u\), \(v\), and \(w\) such that \(\alpha = (u,v)\) and \(\beta = (v,w)\). Therefore, based on the Chasles relation in Proposition 4.4, we obtain:
\[
\alpha + \beta = (u,v) + (v,w) = (u,w).
\]

- If either \(n_\alpha\) or \(n_\beta\) is zero, let’s assume \(n_\beta\) (the other case follows a similar logic), then by Remark 4.1, either \(\beta = 0\) or \(\beta = \pi\). Let \(u\) and \(v\) be two unit vectors such that \(\alpha = (u,v)\). If \(\beta = 0\), then \(\alpha + \beta = \alpha\). If \(\beta = \pi\), then according to Definition 4.2, the straight spherical-vector \(\pi\) can be represented by \((v,-v)\). Consequently,
\[
\alpha + \beta = (u,v) + (v,-v) = (u,-v).
\]

\[\text{Figure 6. Addition of spherical-vectors on } S^2: \hat{A}B + \hat{B}C = \hat{A}C.\]

5. ARGUMENT AND POLAR FORM OF A QUATERNION

In this section, we reformulate the definition of the argument of a quaternion through the concept of spherical-vectors, enabling us to rewrite the polar form of a quaternion and establish all algebraic properties of the exponential representation (refer to Chapter 5 of [7], [8], and [1] for further details on the polar and exponential forms of quaternions). In the oriented complex plane, each oriented angle
(u, v) between two vectors u and v is identified with its measure θ in ] − π, π]. Consequently, the fundamental algebraic property of the exponential form of unit complex numbers can be interpreted in two equivalent ways: either by utilizing the geometric aspect of oriented angles between two vectors (i.e., the spherical-vectors they represent) or by their numerical aspect (i.e., their measures). For instance, we have \( i = e^{i(e_x, e_y)} = e^{\pi/2} \) and \( -1 = e^{i(e_y, -e_y)} = e^{i\pi} \) (refer to Figure 7). Therefore, the product \( i \times (-1) \) can be represented by the following expressions (See [2], Chapter 2)

\[
i \times (-1) = e^{i(e_x, e_y)} \times e^{i(e_y, -e_y)} = e^{i((e_x, e_y)+(e_y, -e_y))} = e^{i(e_x, -e_y)}
\]

or

\[
i \times (-1) = e^{i\pi/2} \times e^{i\pi} = e^{i(\pi/2+\pi)} = e^{i\frac{3\pi}{2}} = e^{-i\frac{\pi}{2}}.
\]

Later on, we will demonstrate that the first formula presented above, involving \( i \times (-1) \), can be naturally extended to the set \( \mathbb{H}_1 \) of unit quaternions.

\[\text{Figure 7. Spherical-vectors generalize oriented angles between two vectors to } \mathbb{R}^3 \text{ space.}\]

5.1. Argument of a quaternion.

Definition 5.1. Let \( q \) be a non-zero quaternion. According to Theorem 4.1, the mapping \( \mu \) is bijective from \( S \) to \( \mathbb{H}_1 \). We define the argument of \( q \) as the spherical-vector given by:

\[
\arg(q) = \mu^{-1}\left(\frac{q}{|q|}\right).
\]

In other words, given a spherical-vector \( \alpha \),

\[
\arg(q) = \alpha \text{ if and only if } \mu(\alpha) = \frac{q}{|q|}.
\]

Remark 5.1. Let \( q \) be a unit quaternion and \( \alpha \) a spherical-vector whose components are \( \lambda \) and \( n \).

(i) By (5.2) and (4.1), \( \arg(q) = \alpha \) if and only if \( q = \mu(\alpha) = \lambda + in \). Therefore, the spherical components of \( q \) and the components of its argument \( \alpha \) are the same.

(ii) If \( \alpha \) is represented by an ordered pair \((u, v)\) of unit vectors, then by (i) and (4.2), \( \arg(q) = \alpha \) if and only if \( q = u.v + i(u \times v) \).
5.2. Cosine and sine of a spherical-vector. We have seen in this paper that spherical-vectors are order pairs of non-zero vectors in the oriented space \( \mathbb{R}^3 \). Consequently, the oriented angles between two vectors in the oriented plane \( \mathbb{R}^2 \), represent particular instances of spherical-vectors. We would therefore like to define mappings that extend the cosine and sine functions of oriented angles to the set \( S \) of spherical-vectors. We illustrate the idea behind this extension through the subsequent example.

Let \( z = a + ib \) denote a unit complex number, represented by a unit vector \( v(a, b) \) in the complex plane, with \( a, b \in \mathbb{R} \). The argument of \( z \) can be expressed either as the oriented angle \( \alpha = (e_x, v) \), which constitutes a spherical-vector, or as its measure \( \theta \in ] - \pi, \pi \] (refer to Figure 8). Consequently, we arrive at the following equation:

\[
(5.3) \quad z = \cos \theta + i \sin \theta = \cos \alpha + i \sin \alpha.
\]

On the other hand, according to Definition 3.2 the expression \( a + ib \) represents the spherical form of \( z \). This can be written as:

\[
(5.4) \quad z = a + i(j \times 0 + k \times 0 + b) = a + i(0, 0, b) = a + iV,
\]

where \( V = (0, 0, b) = be_z \) denotes the vector component of \( z \) associated with the real number \( b \in \mathbb{V} \), as per Remark 3.1. By employing Equations (5.3) and (5.4), we obtain:

\[
\begin{align*}
\cos \alpha &= a \in \mathbb{R} \\
\sin \alpha &= V \in \mathbb{V},
\end{align*}
\]

where \( a \) and \( V \) also serve as the components of \( \alpha \), according to Remark 5.1(i). This leads us to the next definition.

**Definition 5.2.** Let \( \alpha \) represent a spherical-vector with components \( \lambda \) and \( n \).

- The **cosine** of \( \alpha \), denoted as \( \cos \alpha \), refers to the scalar component \( \lambda \).
- The **sine** of \( \alpha \), denoted as \( \sin \alpha \), refers to the vector component \( n \), which is identified with a quaternion in \( \mathbb{V} \) according to Remark 3.1.
Consequently, we establish two mappings on $S$ as follows:

$$\cos : S \rightarrow \mathbb{R} \quad \text{and} \quad \sin : S \rightarrow \mathbb{V} \quad \alpha \mapsto \lambda \quad \text{and} \quad \alpha \mapsto \mathbf{n}.$$

**Remark 5.2.** Let $\alpha$ be a spherical-vector represented by an ordered pair $(u, v)$ of unit vectors.

(i) By Definition 2.1, the components of $\alpha$ are $u \cdot v$ and $u \times v$. So, by Definition 5.2 above,

\[
\begin{aligned}
\cos \alpha &= u \cdot v \\
\sin \alpha &= u \times v \in \mathbb{V}.
\end{aligned}
\]

Thus by (4.2),

\[
\mu(\alpha) = \cos \alpha + i \sin \alpha.
\]

(ii) Let $\theta$ denote the non-oriented angle between the two vectors $u$ and $v$, where $\theta \in [0, \pi]$. Based on (5.5), we observe that $\cos \alpha = \cos \theta$ and $|\sin \alpha| = |u \times v| = \sin \theta$ (Figure 2). Consequently, the following relationship is established:

\[
\cos^2 \alpha + |\sin \alpha|^2 = 1,
\]

**Example 5.1.** Consider the unit vectors $u = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$ and $v = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$. The components of the spherical-vector $\alpha$, represented by the ordered pair $(u, v)$, can be expressed as:

\[
\begin{aligned}
\lambda &= u \cdot v = \frac{\sqrt{3}}{3} \\
n &= u \times v = \left(\frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{6}, 0\right).
\end{aligned}
\]

According to Remark 3.1, $n = \frac{\sqrt{3}}{6} j - \frac{\sqrt{3}}{6} k \in \mathbb{V}$. Therefore, based on (5.5),

\[
\begin{aligned}
\cos \alpha &= \frac{\sqrt{3}}{3} \\
\sin \alpha &= \frac{\sqrt{3}}{6} j - \frac{\sqrt{3}}{6} k.
\end{aligned}
\]

5.3. **Polar form of a quaternion.**

**Theorem 5.1.** Let $q$ denote a non-zero quaternion, $r > 0$ represent a positive real number, and $\alpha$ be a spherical-vector. The equality $q = r(\cos \alpha + i \sin \alpha)$ holds if and only if $|q| = r$ and $\arg(q) = \alpha$. The expression $r(\cos \alpha + i \sin \alpha)$ is referred to as the polar form of $q$.

**Proof.**

- Assume that $q = r(\cos \alpha + i \sin \alpha)$. According to Proposition 3.1(ii) and (5.7), we derive that $|q|^2 = r^2(\cos^2(\alpha) + |\sin(\alpha)|^2) = r^2$. Furthermore, based on (5.6), we have $\mu(\alpha) = \frac{q}{|q|}$. Consequently, by referring to (5.2), it follows that $\arg(q) = \alpha$.

- Conversely, assume that $|q| = r$ and $\arg(q) = \alpha$. By (5.2) and (5.6), we obtain the relation $q/r = \mu(\alpha) = \cos(\alpha) + i \sin(\alpha)$.
5.4. Exponential form of a quaternion.

Definition 5.3. Let \(\alpha\) denote a spherical-vector. The expression \(e^{i\alpha}\) refers to the unit quaternion corresponding to the argument \(\alpha\). Based on Theorem 5.1, we can express this relationship as:

\[
e^{i\alpha} = \cos \alpha + i \sin \alpha.
\]

Hence, if \(q\) is a quaternion with a modulus \(r > 0\) and an argument \(\alpha\), by applying Theorem 5.1 and (5.8), we derive:

\[
q = r(e^{i\alpha} = re^{i\alpha}).
\]

Remark 5.3. Assuming \(\alpha\) is a spherical-vector, equations (5.6) and (5.8) yield the expression

\[
\mu(\alpha) = e^{i\alpha}.
\]

5.5. Algebraic properties of the exponential form.

Proposition 5.1. Let \(\alpha\) and \(\beta\) represent two spherical-vectors, and let \(m\) be an integer. The following properties hold:

\begin{align*}
1) \quad e^{i(\alpha+\beta)} & = e^{i\beta}e^{i\alpha} & 2) \quad (e^{i\alpha})^{-1} & = \overline{e^{i\alpha}} = e^{-i\alpha} \\
3) \quad e^{i(\alpha-\beta)} & = (e^{i\beta})^{-1}e^{i\alpha} & 4) \quad e^{i2\pi} & = -1 \\
5) \quad e^{i\alpha} & = e^{i(\pi+\alpha)} = e^{i(\alpha+\pi)} & 6) \quad (e^{i\alpha})^m & = e^{ima}.
\end{align*}

Proof. (1) Based on Theorem 4.2, \(\mu\) is an anti-isomorphism of groups. Consequently, \(\mu(\alpha + \beta) = \mu(\beta)\mu(\alpha)\). According to (5.10), we derive that \(e^{i(\alpha+\beta)} = e^{i\beta}e^{i\alpha}\).

(2) Similarly, we have \(\mu(-\alpha) = \mu(\alpha)^{-1}\). Therefore, by (5.10), \(e^{i(\alpha)} = (e^{i\alpha})^{-1}\).

As \(e^{i\alpha}\) is a unit quaternion, it follows that \(\overline{e^{i\alpha}} = (e^{i\alpha})^{-1}\).

(3) To establish this property, one can simply combine (1) and (2).

(4) According to Definition 4.2, the components of \(\widehat{\pi}\) are \(-1\) and \(0\). Thus, by (4.1) and (5.10), we obtain \(\mu(\widehat{\pi}) = -1 = e^{i\pi}\).

(5) To prove this property, one can combine (1) and (4).

(6) This result can be easily demonstrated through recursion on \(m \in \mathbb{N}\), using (1). To extend it to \(\mathbb{Z}\), we apply (2).

\qed

We can restate the aforementioned properties as follows.

Proposition 5.2. Let \(p\) and \(q\) be two non-zero quaternions, and let \(m\) be an integer. The following properties hold.

- \(\arg(pq) = \arg(q) + \arg(p)\)
- \(\arg(p^{-1}q) = \arg(q) - \arg(p)\)
- \(\arg(-q) = \pi + \arg(q) = \arg(q) + \pi\)
- \(\arg(q^{-1}) = \arg(q)\)
- \(\arg(q)^{-1} = \arg(q)\)
- \(\arg(-1) = \pi\)
- \(\arg(q^m) = m \arg(q)\).
6. Examples of applications

As observed in previous sections, the anti-isomorphism $\mu$ enables us to associate every unit quaternion $q$ with its argument, which is a spherical-vector. Consequently, we can conveniently represent unit quaternions and their multiplication geometrically through spherical-vectors.

6.1. Practical method for determining the argument of a unit quaternion.

Let $q = \lambda + in$ be a unit quaternion expressed in its spherical form, with $\alpha$ denoting its argument. According to Remark [5.1](i), the components of $\alpha$ are $\lambda$ and $n$. The approach discussed here involves representing the spherical-vector $\alpha$ as an ordered pair $(u, v)$ of unit vectors.

- Assume that $n = \vec{0}$. According to Remark [4.1], $\alpha$ can be represented as $\alpha = (u, u)$ or $\alpha = \pi = (u, -u)$, where $u$ is an arbitrary unit vector.

- Now, let us consider the case where $n \neq \vec{0}$. According to Lemma [4.1](i), $(6.1) (u, v) = (u, w)$ if and only if $(u \perp n$ and $v = \lambda u + n \times u = u \vec{q})$.

To determine an ordered pair $(u, v)$ of unit vectors representing the argument $\alpha$ of $q$, we simply need to select any unit vector $u$ orthogonal to $n$, and then determine $v$ using one of the two formulas $v = u \vec{q} = \lambda u + n \times u$.

example 6.1 (Argument of quaternions $i$, $j$ and $k$).

- **Argument of $i$.** Using Definition [3.2] and Remark [3.1] the spherical form of $i$ is given by

  \[ i = i(0j + 0k + 1) = i(0, 1) = ie_z. \]

Consequently, the spherical components of $i$ are $x = 0$ and $w = e_z$. From $\alpha = (u, u)$, to represent the argument of $i$ as an ordered pair $(u, v)$ of unit vectors, we observe that $e_x$ is orthogonal to $e_z$. Therefore, we can select $u = e_x$ and $v = e_z \times e_x = e_y$. As a result, the argument of $i$ corresponds to the spherical-vector $\alpha_i = (e_x, e_y)$ (Figure 9). In accordance with Definition [5.3],

  \[ i = e^{i(e_x, e_y)} \]  \[ \text{with arg}(i) = (e_x, e_y). \]  \[ (6.2) \]

- **Argument of $j$.** Given that

  \[ j = -ik = -i(0j + 1k + 0) = -i(1, 0) = -ie_y, \]

the spherical components of $j$ are $0$ and $-e_y$. An ordered pair $(u, v)$ of unit vectors represents the argument of $j$ if and only if $u \perp e_y$ and $v = -e_y \times u$. As $e_x \perp e_y$, we can conveniently choose $u = e_x$ and $v = -e_y \times e_x = e_z$. Consequently, the argument of $j$ corresponds to the spherical-vector $\alpha_j = (e_x, e_z)$ (Figure 9). Thus,

  \[ j = e^{i(e_x, e_z)} \]  \[ \text{with arg}(j) = (e_x, e_z). \]  \[ (6.3) \]

- **Argument of $k$.** We have

  \[ k = ij = i(1j + 0k + 0) = i(1, 0) = ie_x. \]

Consequently, the spherical components of $k$ are $0$ and $e_x$. An ordered pair $(u, v)$ of unit vectors represents the argument of $k$ if and only if $u \perp e_x$ and
Given that \( e_y \perp e_x \), we select \( u = e_y \) and \( v = e_x \times e_y = e_z \). Thus, \( \alpha_k = (e_y, e_z) \) (Figure 9). Therefore,

\[
(6.4) \quad k = e^{i(e_y, e_z)} \text{ with } \arg(k) = (e_y, e_z).
\]

6.2. Geometric interpretation of the formula \( ki = j \). In Example 6.1, we determined the arguments of the unit quaternions \( i, j, \) and \( k \). Utilizing (6.2), (6.3), (6.4), and Proposition 5.1(1), we obtain the equivalent formulations,

\[
ki = j \iff e^{i\alpha_i}e^{i\alpha_i} = e^{i\alpha_j} \iff e^{i(\alpha_i + \alpha_k)} = e^{i\alpha_j}.
\]

Applying (5.10), this implies that \( \mu(\alpha_i + \alpha_k) = \mu(\alpha_j) \). Consequently, \( \alpha_i + \alpha_k = \alpha_j \), leading to the relation \( (e_x, e_y) + (e_y, e_z) = (e_x, e_z) \).

As illustrated in Figure 9, the multiplication relation \( ki = j \) can be geometrically interpreted through the straightforward Chasles relation,

\[
(e_x, e_y) + (e_y, e_z) = (e_x, e_z).
\]

6.3. Geometric interpretation of the non-commutativity of unit quaternion multiplication. Let \( p \) and \( q \) be two unit quaternions with respective arguments \( \alpha \) and \( \beta \), where \( \alpha \) and \( \beta \) are spherical-vectors. According to Proposition 5.2, we obtain \( \arg(pq) = \beta + \alpha \) and \( \arg(qp) = \alpha + \beta \). Consequently, the non-commutativity of the multiplication of \( p \) and \( q \) can be geometrically interpreted through the non-commutativity of the addition of their arguments \( \alpha \) and \( \beta \), as depicted in Figure 10 and further demonstrated in the following numerical Example 6.2.
example 6.2. Consider the following two unit quaternions \( p = \frac{\sqrt{6}}{3}(2 - j - k) \) and \( q = \frac{\sqrt{2}}{2} (1 + i) \). In this final example, we employ spherical-vectors to provide a geometric representation for the quaternions \( p \) and \( q \), as well as their products \( qp = \frac{\sqrt{3}}{3}(1 - i - k) \) and \( pq = \frac{\sqrt{3}}{3}(1 + i - j) \). Let us denote the respective arguments of the quaternions \( p \), \( q \), \( qp \), and \( pq \) as \( \alpha_p \), \( \alpha_q \), \( \alpha_{qp} \), and \( \alpha_{pq} \). According to Proposition 5.2, \( \alpha_{qp} = \alpha_p + \alpha_q \) and \( \alpha_{pq} = \alpha_q + \alpha_p \). To represent these four spherical-vectors on the unit sphere \( S^2 \), we first need to express \( p \) and \( q \) in their spherical forms. To do so, we utilize Definition 3.2 and Remark 3.1:

\[
p = \frac{\sqrt{6}}{3} + i \frac{\sqrt{6}}{6} (-j + k + 0) = \frac{\sqrt{6}}{3} + i \left( -\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, 0 \right)
\]

and

\[
q = \frac{\sqrt{2}}{2} + i \left( 0j + 0k + \frac{\sqrt{2}}{2} \right) = \frac{\sqrt{2}}{2} + i \left( 0, 0, \frac{\sqrt{2}}{2} \right).
\]

The spherical components of \( p \) are given by

\[
\lambda_p = \frac{\sqrt{6}}{3} \quad \text{and} \quad n_p = \left( -\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, 0 \right),
\]

while the spherical components of \( q \) are

\[
\lambda_q = \frac{\sqrt{2}}{2} \quad \text{and} \quad n_q = \left( 0, 0, \frac{\sqrt{2}}{2} \right).
\]

According to Remark 5.1(i), the components of \( \alpha_p \) are the spherical components of \( p \), and the components of \( \alpha_q \) are the spherical components of \( q \). Then, as per Lemma 4.2(i), given that \( n_p \times n_q \) is non-zero, there are only two possible combinations of three unit vectors \( u \), \( v \), and \( w \) such that \( \alpha_p = (u, v) \) and \( \alpha_q = (v, w) \). We may choose, for instance, the combination

\[
\begin{align*}
v &= \frac{n_p \times n_q}{||n_p \times n_q||} \\
u &= \lambda_p v - n_p \times v \\
w &= \lambda_q v + n_q \times v.
\end{align*}
\]
Upon calculation, we obtain the following results:

\[
\begin{align*}
  v &= \left( \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, 0 \right) \\
  u &= \left( \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right) \\
  w &= (0, 1, 0) = e_y.
\end{align*}
\]

Consequently, we can represent the product \( qp \) through the sum \( \alpha_q + \alpha_p = \alpha_{qp} \), as depicted in Figure [1](1).

Similarly, we observe that \( n_q \times n_p \neq 0 \), indicating that there are two combinations of three unit vectors \( u', v' \), and \( w' \) such that \( \alpha_q = (u', v') \) and \( \alpha_p = (v', w') \). We choose, for instance, the combination:

\[
\begin{align*}
  v' &= \frac{n_q \times n_p}{\|n_q \times n_p\|} = v \\
  u' &= \lambda_q v - n_q \times v \\
  w' &= \lambda_p v + n_p \times v.
\end{align*}
\]

Which results in:

\[
\begin{align*}
  v' &= v = \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right) \\
  u' &= (1, 0, 0) = e_x \\
  w' &= \left( \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3} \right).
\end{align*}
\]

We can now represent the product \( pq \) through the sum \( \alpha_q + \alpha_p = \alpha_{pq} \), as depicted in Figure [1](1).

Due to the identification established throughout this paper between unit quaternions and spherical-vectors, we now possess a robust geometric approach for representing unit quaternions (which belong to a 4-dimensional space) and their products on a surface of a 3-dimensional space. As observed in (Figure [1](2)), we adopt the notation \( p = \alpha_p, q = \alpha_q, qp = \alpha_p + \alpha_q, \) and \( pq = \alpha_q + \alpha_p \).
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