RELATIVE \((\varphi, \Gamma)\)-MODULES AND PRISMATIC \(F\)-CRYSTALS

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Abstract. In this paper, we prove that for any \(p\)-adic smooth separated formal scheme \(X\), the category of prismatic \(F\)-crystals over \(\mathcal{O}_{\Delta}[^1-instagram]\) is equivalent to the category of étale \(\mathbb{Z}_p\)-local systems on the generic fiber of \(X\). We then compare the cohomology of the corresponding coefficients.

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1. Introduction

In rational \(p\)-adic Hodge theory, there is the classical theorem due to Colmez–Fontaine that the category \(\text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(G_K)\) of crystalline representations of the absolute Galois group \(G_K\) of a \(p\)-adic field \(K\), is equivalent to the category \(\text{MF}_{\mathbb{Q}_p}^{\varphi}\) of weakly admissible filtered \(\varphi\)-modules. It is tempting to ask for an integral version of this result. In fact, one also hopes to use some semilinear algebras to describe crystalline \(\mathbb{Z}_p\)-representations. There are two ways to deal with this question. One is the cyclotomic case: there is the category \(\text{Rep}_{\mathbb{Z}_p}^{\text{cris}}(G_K)\). The other is the Kummer case: there are the theories of Breuil modules and Breuil–Kisin modules which however can not describe the whole...
category $\text{Rep}^{\text{cris}}_{\mathbb{Z}_p}(G_K)$ in general. Combining these two points of view, there is the theory of $(\varphi, \tilde{\Gamma})$-modules which also describes the whole category $\text{Rep}^{\text{cris}}_{\mathbb{Z}_p}(G_K)$.

On the other hand, integral $p$-adic Hodge theory has seen great development recently. In [BS19], Bhatt and Scholze introduced the theory of prismatic cohomology, which unifies its predecessors: the $\kappa_{\text{inf}}$-cohomology theory in [BMS18] and the Breuil–Kisin cohomology in [BMS19]. The latter two originate from the seeking of a cohomological construction of the Breuil–Kisin modules. It turns out that the prismatic formalism contains finer integral information than the Breuil–Kisin theory. In [BS21], Bhatt and Scholze proved that the category of prismatic $F$-crystals over $\mathcal{O}_\Delta$ (resp. $\mathcal{O}_\Delta[1/p]$) on the absolute prismatic site $(\mathcal{O}_K)_{\Delta}$, where $\mathcal{O}_K$ is the ring of integers in a $p$-adic field $K$, is equivalent to the category $\text{Rep}^{\text{cris}}_{\mathbb{Z}_p}(G_K)$ (resp. $\text{Rep}_{\mathbb{Z}_p}(G_K)$). The equivalence between the category of prismatic $F$-crystals and the category $\text{Rep}^{\text{cris}}_{\mathbb{Z}_p}(G_K)$ has also been proved in [Wu21].

In this paper, we want to generalize the work of Wu to the relative case, i.e. compare prismatic $F$-crystals and relative $(\varphi, \Gamma)$-modules. More precisely, we prove the following theorem.

**Theorem 1.1** (Theorem 3.1). Let $\mathcal{X}$ be a separated $p$-adic smooth formal scheme over $\mathcal{O}_K$ and $X$ be its adic generic fiber. Then there is an equivalence of categories

$$\text{Crys}(\mathcal{O}_\Delta[1/p, \varphi]) \cong \text{LS}(X_{\text{ét}}, \mathbb{Z}_p),$$

where the right one is the category of étale $\mathbb{Z}_p$-local systems on $X$.

The proof of this theorem uses relative $(\varphi, \Gamma)$-modules as intermediate objects, which are certain pro-étale sheaves on the generic fiber by the work of Kedlaya–Liu [KL15]. We then give an equivalence between the category of prismatic $F$-crystals and the category of relative $(\varphi, \Gamma)$-modules following the idea of Wu.

**Remark 1.1.** In fact, Bhatt and Scholze proved a more general result than the above theorem in [BS21]. Their results apply to crystals in perfect complex and $p$-adic formal schemes, which are not necessarily smooth. Compared to their proof, the proof of our theorem involves a concrete construction of the equivalence. We hope the readers can still find this concrete description useful.

**Remark 1.2.** It seems natural to hope for a relative version of the result of Bhatt and Scholze concerning the crystalline representations. Namely, we hope there is an equivalence between the category of prismatic $F$-crystals over $\mathcal{O}_\Delta$ on the absolute...
prismatic site $\mathcal{X}_\Delta$ and the category of “crystalline $\mathbb{Z}_p$-local systems” on the rigid generic fiber of $\mathcal{X}$. Note that the category of “crystalline $\mathbb{Z}_p$-local systems” are only well-defined in the unramified case at the moment. Based on the idea of Bhatt–Scholze, one might need a description of crystalline $\mathbb{Q}_p$-local systems via certain relative weakly admissible filtered $\varphi$-modules. But there seems still no satisfying relative theory of “weakly admissible implies admissible”, cf. [Moon18].

Our second main theorem is an étale comparison with coefficients.

**Theorem 1.2** (Theorem [4.1]) Let $(A, I)$ be a perfect prism such that $A/I$ contains all the $p$-power roots of unity and $\mathcal{X}$ be a separated $p$-adic smooth formal scheme over $A/I$. For any $\mathcal{M}$ which is a prismatic $F$-crystal over $\mathcal{O}_\Delta[\frac{1}{I}]^\wedge_p$ with corresponding $\mathbb{Z}_p$-local system $\mathcal{L}$ on $X_{\text{ét}}$, there is a quasi-isomorphism

$$R\Gamma((X/A)_\Delta, \mathcal{M})^{\varphi=1} \simeq R\Gamma(X_{\text{ét}}, \mathcal{L}).$$

The proof of this theorem also depends on an intermediate object, the perfect prismatic site, which appears already in the étale comparison with constant coefficients in [BS19]. By the same argument, one can also get the following theorem.

**Theorem 1.3** (Theorem [4.11]) Let $\mathcal{X}$ be a separated $p$-adic smooth formal scheme over $\mathcal{O}_K$ and $\mathcal{M}$ be a prismatic $F$-crystal over $\mathcal{O}_\Delta[\frac{1}{I}]^\wedge_p$ on $\mathcal{X}_\Delta$ with corresponding $\mathbb{Z}_p$-local system $\mathcal{L}$ on $X_{\text{ét}}$. Then there is a quasi-isomorphism

$$R\Gamma((\mathcal{X})_\Delta, \mathcal{M})^{\varphi=1} \simeq R\Gamma(X_{\text{ét}}, \mathcal{L}).$$

Note that this theorem is a generalization of the étale comparison in [BS19] even in the case of constant coefficients. In particular, it shows that we need to consider cohomology of absolute prismatic site instead of relative one when we study the case of imperfect prisms.

**Remark 1.3.** The above theorems should be regarded as a part of the whole comparison theory with coefficients as in [BS19]. In [MT20], Morrow and Tsuji have proved an equivalence between the category of relative Breuil–Kisin–Fargues modules and the category of prismatic $F$-crystals with the base prism $(A_{\text{inf}}, (\text{Ker}(\theta)))$. In particular, they have constructed explicitly the $\mathbb{Z}_p$-local systems, the vector bundles with integrable connections and the crystalline $F$-crystals out of relative Breuil–Kisin–Fargues modules (see also [BS21] for some similar statements). They have also got some comparison results with these coefficients. Note that these coefficient objects
are not constructed directly out of prismatic $F$-crystals and it seems that Morrow–Tsuji’s arguments could not deal with imperfect prisms like the Breuil–Kisin prism $(\mathcal{G}, (E))$. We plan to study the case of imperfect prisms in a future work.

ACKNOWLEDGMENTS

We would like to thank Zhiyou Wu for answering our questions. We also want to thank Heng Du, Shizhang Li, Ruochuan Liu, Matthew Morrow and Takeshi Tsuji for their valuable comments on the earlier drafts of this work.

2. LOCAL CONSTRUCTION

Let $A_{\inf}$ be the period ring of infinitesimal deformation due to Fontaine and $\theta : A_{\inf} \to \mathcal{O} := \mathcal{O}_{p}$ be the canonical surjection. Fix a compatible system of $p$-power roots $\{\zeta_{p}^{n}\}_{n}$. Then $\text{Ker}(\theta)$ is a principal ideal generated by $\xi = (\mu - 1)$ where $\mu = [\epsilon] - 1$ and $\epsilon = (1, \zeta_{p}, \zeta_{p}^{2}, \ldots)$.

Let $\text{Spf}(R)$ be a $p$-adic smooth formal scheme over $\mathcal{O}$ with a framing $\square : \mathcal{O}(\mathcal{T}^{\pm 1}) \to R$, i.e. an étale morphism over $\mathcal{O}$. Let $(R/A_{\inf})_{A}$ be the prismatic site of $\text{Spf}(R)$ over $(A_{\inf}, (\xi))$ of bounded prisms with the structure sheaf $\mathcal{O}_{A}$ and $(R/A_{\inf})_{A}^{\text{perf}}$ be the site of perfect prisms over $(A_{\inf}, (\xi))$. Denote $\mathcal{O}_{A}[\frac{1}{\xi}]^{\bullet}$ the $p$-adic completion of $\mathcal{O}_{A}[\frac{1}{\xi}]$ and then for any $\mathcal{B} = (\text{Spf}(R) \leftarrow \text{Spf}(B/\xi B) \to \text{Spf}(B))$ in $(R/A_{\inf})_{A}$, we have $\mathcal{O}_{A}[\frac{1}{\xi}]^{\bullet}(\mathcal{B}) = B[\frac{1}{\xi}]$ (cf. [Wu21]).

Remark 2.1. The site $(R/A_{\inf})_{A}$ can be viewed as the absolute prismatic site $(R)_{A}$ over $R$ of bounded prisms ([BST19 Remark 4.7]). In fact, for any prism $(A, I)$ endowed with a morphism $R \to A/I$, the composition $\mathcal{O} \to R \to A/I$ lifts uniquely to a morphism of prisms $(A_{\inf}, (\xi)) \to (A, I)$ by the deformation theory as $\mathcal{O}$ is perfectoid.

If we fix an $A_{\inf}$-lifting $A_{\inf}(\mathcal{T}^{\pm 1})$ (with the $(p, \xi)$-adic topology) of $\mathcal{O}(\mathcal{T}^{\pm 1})$, then $R$ also admits an $A_{\inf}$-lifting $A_{\inf}^{\square}(R)$ together with a formally étale morphism $A_{\inf}(\mathcal{T}^{\pm 1}) \to A_{\inf}^{\square}(R)$ lifting $\square$, which is uniquely determined by the given framing $\square$. Now, we equip $A_{\inf}(\mathcal{T}^{\pm 1})$ with the $\varphi$-action given by $\varphi(T_{i}) = T_{i}^{p}$ for $1 \leq i \leq d$ compatible with the usual $\varphi$-action on $A_{\inf}$, and then such a $\varphi$-action extends uniquely to $A_{\inf}^{\square}(R)$.

Example 2.1. (1) The prism $(A_{\inf}^{\square}(R), (\xi))$ is an object in $(R/A_{\inf})_{A}$, since $A_{\inf}^{\square}(R)$ is formally smooth over $A_{\inf}$.
(2) Let \((A^\square_{\text{inf}}(R), (\xi))^{\text{perf}}\) be the perfection of \((A^\square_{\text{inf}}(R), (\xi))\) ([BS19 Lemma 3.9]). Then \((A^\square_{\text{inf}}(R), (\xi))^{\text{perf}}\) is a cover of the final object of the topos \(\text{Shv}((R/A^\square_{\text{inf}})_{\Delta})\) (see Lemma 2.5); the same holds for \((A^\square_{\text{inf}}(R), (\xi))\).

**Definition 2.2.** (1) By a prismatic F-crystal over \(O_A\) on \((R/A)_{\Delta}\), we mean a sheaf of \(O_A\)-module \(\mathcal{M}\) endowed with an isomorphism \((\varphi^*\mathcal{M})[\frac{1}{p}] \xrightarrow{\sim} \mathcal{M}[\frac{1}{p}]\), such that for any object \(\mathfrak{B} = (\text{Spf}(R) \leftarrow \text{Spf}(B/\xi B) \rightarrow \text{Spf}(B))\), \(\mathcal{M}(\mathfrak{B})\) is a finite projective \(B = O_A(\mathfrak{B})\)-module and such that for any morphism

\[\mathfrak{B}' = (\text{Spf}(R) \leftarrow \text{Spf}(B'/\xi B') \rightarrow \text{Spf}(B')) \rightarrow \mathfrak{B}\]

in \((R/A_{\text{inf}})_{\Delta}\), the induced morphism \(\mathcal{M}(\mathfrak{B}) \otimes_B B' \rightarrow \mathcal{M}(\mathfrak{B}')\) is an isomorphism of \(\varphi\)-modules.

(2) By a prismatic F-crystal over \(O_A[\frac{1}{\xi}]\) on \((R/A)_{\Delta}\), we mean a sheaf of \(O_A[\frac{1}{\xi}]_{\Delta}\)-module \(\mathcal{M}\) endowed with an isomorphism \(\varphi_M: (\varphi^*\mathcal{M}) \xrightarrow{\sim} \mathcal{M}\), such that for any object \(\mathfrak{B}\), \(\mathcal{M}(\mathfrak{B})\) is a finite projective \(\mathcal{M}(\mathfrak{B}) = O_A[\frac{1}{\xi}]_{\Delta}(\mathfrak{B})\)-module such that for any morphism \(\mathfrak{B}' \rightarrow \mathfrak{B}\) in \((R/A_{\text{inf}})_{\Delta}\), the induced morphism \(\mathcal{M}(\mathfrak{B}) \otimes_{\mathcal{M}(\mathfrak{B})} \mathcal{M}(\mathfrak{B}')\) is an isomorphism of \(\varphi\)-modules.

**Definition 2.3.** Let \(R\) be a ring equipped with a continuous endomorphism \(\varphi: R \rightarrow R\) and \(\Gamma\) be a topological group.

(1) By a \(\varphi\)-module over \(R\), we mean an \(R\)-module \(M\) together with a \(\varphi\)-semi-linear continuous morphism \(\varphi_M: M \rightarrow M\). A \(\varphi\)-module \(M\) is called étale, if it is finite projective and the linearization \(\varphi^* M \rightarrow M\) of \(\varphi_M\) is an isomorphism of \(R\)-modules. Let \(\Phi M(R)\) and \(\text{Ét}\Phi M(R)\) be the categories of \(\varphi\)-modules and étale \(\varphi\)-modules over \(R\), respectively.

(2) Assume, moreover, \(R\) is endowed with a continuous \(\Gamma\)-action commuting with \(\varphi\). By a \((\varphi, \Gamma)\)-module over \(R\), we mean a \(\varphi\)-module \(M\) together with a semi-linear continuous \(\Gamma\)-action commuting with \(\varphi\). A \((\varphi, \Gamma)\)-module \(M\) is call étale, if it is an étale \(\varphi\)-module. Let \(\Phi \Gamma M(R)\) and \(\text{Ét}\Phi \Gamma M(R)\) be the categories of \((\varphi, \Gamma)\)-modules and étale \((\varphi, \Gamma)\)-modules over \(R\), respectively.

Now, we consider the category of prismatic F-crystals over \(O_A[\frac{1}{\xi}]_{\Delta}\) on \((R/A)_{\Delta}\).

We first give another description of \((A^\square_{\text{inf}}(R), (\xi))^{\text{perf}}\). Let \(R_\infty = R \otimes_{O_{(\mathbb{Z}/p)^{\text{perf}}}} O_{(\mathbb{Z}/p^{\infty})}\), which is a perfectoid \(O\)-algebra. Thus, there is a unique \(A_{\text{inf}}\)-lifting \(A_{\text{inf}}(R_\infty)\) of \(R_\infty\) together with a unique morphism \((A_{\text{inf}}, (\xi)) \rightarrow (A_{\text{inf}}(R_\infty), (\xi))\) of prisms lifting the composition \(O \rightarrow R \rightarrow R_\infty\). One can easily check that \(A_{\text{inf}}(R_\infty)\) is the \((p, \xi)\)-adic completion of \(A_{\text{inf}}(\mathbb{T}^{(\xi)}) \otimes_{A_{\text{inf}}(\mathbb{Z}^{\text{perf}})} A_{\text{inf}}(R)\), where \(T_i\) is identified with \([T_i^{\text{perf}}]\)
for $T_i = (T_i, T_i^2, \cdots) \in R^\infty_{\mathbb{C}}$ and that the morphism $A^{\square}_{\inf}(R) \to A_{\inf}(R_{\infty})$ carrying $T_i$ to $T_i$ of $A_{\inf}$-algebras preserves $\delta$-structures and induces a morphism of prisms

$$(A^{\square}_{\inf}(R), (\xi)) \to (A_{\inf}(R_{\infty}), (\xi)).$$

**Lemma 2.4.** The above morphism induces an isomorphism

$$(A^{\square}_{\inf}(R), (\xi))_{\text{perf}} \to (A_{\inf}(R_{\infty}), (\xi)).$$

**Proof.** Denote $(A_{\infty}, (\xi)) := (A^{\square}_{\inf}(R), (\xi))_{\text{perf}}$. By [BST19, Lemma 3.9], $A_{\infty}$ is the $(p, \xi)$-adic completion of colim$_{\varphi} A^{\square}_{\inf}(R)$ and is the initial object among all perfect prisms over $(A^{\square}_{\inf}(R), (\xi))$. In particular, there is a unique morphism $i : (A_{\infty}, (\xi)) \to (A_{\inf}(R_{\infty}), (\xi))$. We shall construct the inverse of $i$ to complete the proof.

When $R = O(\mathbb{T}^{\pm1})$, it is straightforward to check $A_{\infty} = A_{\inf}(\mathbb{T}^{\pm1})$ and in particular, the lemma holds in this case. For the general case, as $R$ is étale over $O(\mathbb{T}^{\pm1})$, there is a unique morphism $j : A^{\square}_{\inf}(R) \hat{\otimes}_{A^{\square}_{\inf}(\mathbb{T}^{\pm1})} A_{\inf}(\mathbb{T}^{\pm1}) =: A'_{\infty} \to A_{\infty}$ once one regards $(A_{\infty}, (\xi))$ as a perfect prism over $(A_{\inf}(\mathbb{T}^{\pm1}), (\xi))$. According to the $(p, \xi)$-étaleness of $A^{\square}_{\inf}(R)$ over $A_{\inf}(\mathbb{T}^{\pm1})$, $(A'_{\infty}, (\xi))$ is a prism. By construction of $A'_{\infty}$, its reduction modulo $\xi$ is the $p$-adic completion of $R \otimes_{O(\mathbb{T}^{\pm1})} O(\mathbb{T}^{\pm1}) = R_{\infty}$. By the equivalence in [BST19, Theorem 3.10], we deduce $(A'_{\infty}, \xi)$ is a perfect prism and is isomorphic to $(A_{\inf}(R_{\infty}), (\xi))$.

Thanks to the universal property of $(A_{\infty}, (\xi))$, the composite $j \circ i$ is the identity morphism. It remains to check $i \circ j$ is also the identity morphism. By [BST19, Theorem 3.10], it is enough to verify this modulo $\xi$. However, by construction, we have the following commutative diagram of $A_{\inf}$-algebras

$$
\begin{array}{ccc}
A_{\inf}(\mathbb{T}^{\pm1}) & \longrightarrow & A_{\inf}(\mathbb{T}^{\pm1}) \\
\downarrow & & \downarrow \\
A'_{\infty} & \overset{j}{\longrightarrow} & A_{\infty} \\
& \overset{i}{\longrightarrow} & A'_{\infty}
\end{array}
$$

Modulo $\xi$, we see that the morphism $i \circ j : R_{\infty} \to R_{\infty}$ of $R$-algebras preserve $T_i^{\pm1}$ for all $1 \leq i \leq d$ and all $n \geq 0$. By definition of $R_{\infty}$, we deduce $i \circ j$ is the identity morphism modulo $\xi$. This shows that $j$ is the inverse of $i$ as desired. \qed
Remark 2.2 (\(\Gamma\)-action on \(A_{\inf}(R_\infty)\)). During the proof of Lemma 2.4, we have shown that

\begin{equation}
A_{\inf}(R_\infty) = \hat{\bigoplus}_{\alpha \in (\mathbb{Z}/[1/p]\cap [0,1])^d} A_{\inf}^\alpha(R) T^\alpha,
\end{equation}

where \(\hat{\bigoplus}\) means the topological direct sum with respect to the \((p, \xi)\)-adic topology and for any \(\alpha = (\alpha_1, \ldots, \alpha_d) \in (\mathbb{Z}/[1/p]\cap [0,1])^d\), \(T^\alpha = T_1^{\alpha_1} \cdots T_d^{\alpha_d}\).

We know \(\text{Spa}(R_\infty[\frac{1}{p}], R_\infty)\) is a Galois cover of \(\text{Spa}(R[\frac{1}{p}], R)\) with Galois group \(\Gamma = \oplus_{i=1}^d \mathbb{Z}_p \gamma_i\), where \(\gamma_i(T_j^\xi) = e^{\delta_{ij}} T_j^\xi\) for \(1 \leq i, j \leq d\). As \(R_\infty\) is perfectoid, by deformation theory, the above \(\Gamma\)-action lifts uniquely to a continuous action on \(A_{\inf}(R_\infty)\) determined by \(\gamma_i(T_j^\xi) = e^{\delta_{ij}} T_j^\xi\) for all \(1 \leq i, j \leq d\). From this, one can easily check that the equation (2.1) is a topological decomposition of \(\Gamma\)-modules. In particular, \(A_{\inf}^\alpha(R)\) is equipped with the restricted \(\Gamma\)-action.

Clearly, all above \(\Gamma\)-actions on \(A_{\inf}(R_\infty)\) and \(A_{\inf}^\alpha(R)\) commute with the actions of \(\varphi\).

Lemma 2.5. The prism \((A_{\inf}(R_\infty), (\xi)) \in (R/A_{\inf})_\Delta\) is a cover of the final object of the topos \(\text{Shv}((R/A_{\inf})_\Delta)\). In particular, the prism \((A_{\inf}^\alpha(R), (\xi))\) is also a cover of the final object of the topos \(\text{Shv}((R/A_{\inf})_\Delta)\).

Proof. It can be checked directly that \(R_\infty\) is a quasi-syntomic cover of \(R\) in the sense of [BMS19, Definition 4.10(3)]. So for any bounded prism \((A, I) \in (R/A_{\inf})_\Delta\), \(A/I \hat{\otimes}_R R_\infty\) is also a quasi-syntomic cover of \(A/I\). By [BS19, Proposition 7.11], there exists a prism \((B, IB)\) over \((A, I)\) (hence over \((A_{\inf}, (\xi))\)) together with a \(p\)-faithfully flat morphism \(A/I \hat{\otimes}_R R_\infty \to B/IB\). In particular, \((B, IB)\) is a cover of \((A, I)\). Now, the composition \(R_\infty \to A/I \hat{\otimes}_R R_\infty \to B/IB\) lifts uniquely to a morphism \((A_{\inf}(R_\infty), (\xi)) \to (B, IB)\) by deformation theory as \(R_\infty\) is perfectoid. Since \((B, IB)\) is a cover of \((A, I)\), we complete the proof.

Let \(\widehat{A_{\inf}^\alpha(R)}[\frac{1}{\xi}]\) be the \(p\)-adic completion of \(A_{\inf}^\alpha(R)[\frac{1}{\xi}]\). As \(\varphi(\xi) = \xi^p + p\delta(\xi)\) with \(p\delta(\xi)\) topologically nilpotent (since \(\delta(\xi) \in A_{\inf}^\alpha\)), the \(\varphi\)-action on \(A_{\inf}^\alpha(R)\) extends (uniquely) to \(A_{\inf}^\alpha(R)[\frac{1}{\xi}]\). Similarly, if we denote by \(A_{\inf}(R_\infty)[\frac{1}{\xi}]\) the \(p\)-adic completion of \(A_{\inf}(R_\infty)[\frac{1}{\xi}]\), then it also admits an induced \(\varphi\)-action. Moreover, as \(\Gamma\) fixes the base ring \(A_{\inf}\), both \(A_{\inf}^\alpha(R)[\frac{1}{\xi}]\) and \(A_{\inf}(R_\infty)[\frac{1}{\xi}]\) are endowed with (continuous) \(\Gamma\)-actions as well.

We recall the following proposition in [Wu21].
Proposition 2.6 ([Wu21, Theorem 4.6]). Let \((A, I)\) be a bounded prism such that \(\varphi(I) \mod p\) is generated by a non-zero divisor in \(A/p\). Let \((A_\infty, IA_\infty)\) be the perfection of the prism \((A, I)\), then we have an equivalence of categories

\[
\check{\text{Et}}\Phi M(A[I]) \rightarrow \check{\text{Et}}\Phi M(A_\infty[I])
\]

induced by base change.

Remark 2.3. Keep notations as in Proposition 2.6. If moreover \(I = (d)\) such that \((p, d)\) is regular in \(A\), then the equivalence in Proposition 2.6 preserves tensor products, dualities and extensions. The first two assertions are obvious. To see the third, one needs to check the proof of [Wu21, Theorem 4.6] carefully. We split the sketch of the proof in steps.

(1) Let \(R\) be as in [Wu21, Proposition 4.2]. Then the equivalence there preserves extensions. Denote \(R_\infty = \colim R\). It suffices to check short exact sequences of \(\text{etale} \ \varphi\)-modules over \(R_\infty\) lift to short exact sequences of \(\text{etale} \ \varphi\)-modules over \(R\). Since all modules and morphisms lift to \(R_n = R\) for some \(n\), this follows from the proof of [Wu21 Proposition 4.2].

(2) Let \(R\) be as in [Wu21 Proposition 4.3]. Let \(\hat{R}_\infty\) be the \(p\)-adic completion of \(R_\infty\). Assume \(R\) is \(p\)-torsion free (and thus so are \(R_\infty\) and \(\hat{R}_\infty\)). Then the equivalence there preserves extensions. Since all rings involved are \(p\)-torsion free, this can be checked modulo \(p^n\) and then reduces to the above case.

(3) Under the assumption on \((A, (d))\), the equivalence in [Wu21 Theorem 4.6] preserves extensions. By the argument in the proof of [Wu21 Theorem 4.4], it is enough to check that for a perfect \(F_p\)-algebra \(R\), the equivalence between the category of \(\text{etale} \ \varphi\)-modules over \(W(R)\) and the category of \(\text{lisse} \ \mathbb{Z}_p\)-sheaves on \(X = \text{Spec}(R)\) preserves extensions. It suffices to check this modulo \(p^n\). We regard \(\text{lisse} \ \mathbb{Z}_p/p^n\)-sheaves as \(\mathbb{Z}_p/p^n\)-representations of the algebraic fundamental group \(G\) of \(X\). For a short exact sequence of \(\mathbb{Z}_p/p^n\)-representations

\[
0 \rightarrow L_1 \rightarrow L_2 \rightarrow L_3 \rightarrow 0,
\]

choose a finite \(\text{etale} \ \text{Galois} \) cover \(Y = \text{Spec}(S)\) of \(X\) trivializing all \(L_i\)'s. Denote by \(G_S\) the corresponding Galois group. Since the \(\text{etale} \ \varphi\)-modules corresponding to \(L_i\) are \((W_n(S) \otimes L_i)^{G_S}\), the resulting sequence of \(\text{etale} \ \varphi\)-modules is exact by Hilbert’s theorem 90. Conversely, for any short exact sequence of \(\text{etale} \ \varphi\)-modules

\[
0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0,
\]
denote $L_i$ the corresponding representations for all $i$ and choose $S$ as above. Then $L_i = (M_i \otimes_{W_n(R)} W_n(S))^{\varphi=1}$. Since taking $\varphi$-invariants is a left exact functor, we get the desired exactness by counting dimensions.

Proposition 2.7. (1) The functor $\mathcal{E}t\Phi M(A^{\varphi\infty}(R)([1]) \to \mathcal{E}t\Phi M(A_{\inf}(R_{\infty})([1]))$ induced by the base change $M \mapsto M \otimes_{A^{\varphi\infty}(R)([1])} A_{\inf}(R_{\infty})([1])$ is a tensor equivalence.

(2) The functor $\mathcal{E}t\Phi \Gamma M(A^{\varphi\infty}(R)([1]) \to \mathcal{E}t\Phi \Gamma M(A_{\inf}(R_{\infty})([1]))$ induced by the base change $M \mapsto M \otimes_{A^{\varphi\infty}(R)([1])} A_{\inf}(R_{\infty})([1])$ is a tensor equivalence.

Proof. (1) This is a special case of Proposition 2.6 (in which $(A, I) = (A^{\varphi\infty}(R), (\xi))$).

(2) The fully faithful part follows from (1). It is enough to show the essential surjectivity. For simplicity, put $A^{\varphi\infty} := A^{\varphi\infty}(R)([1])$ and $A_{\infty} := A_{\inf}(R_{\infty})([1])$.

Let $M_{\infty} \in \mathcal{E}t\Phi \Gamma(A_{\infty})$ and $M$ be the corresponding étale $\varphi$-module over $A^{\varphi\infty}$ via the equivalence in (1). Then we have $\text{End}_{\varphi}(M) \simeq \text{End}_{\varphi}(M_{\infty})$. As $M_{\infty}$ is a $(\varphi, \Gamma)$-module, we get a continuous homomorphism $\Gamma \to \text{End}_{\varphi}(M_{\infty})^\times \simeq \text{End}_{\varphi}(M)^\times$. Therefore, $M$ is endowed with a continuous $\Gamma$-action commuting with $\varphi$ such that the isomorphism $M \otimes_{A^{\varphi\infty}} A_{\infty} \simeq M_{\infty}$ is $\Gamma$-equivariant. As a consequence, if we regard $M$ as an étale $(\varphi, \Gamma)$-module over $A^{\varphi\infty}$, then $M \otimes_{A^{\varphi\infty}} A_{\infty}$ is isomorphic to $M_{\infty}$ in $\mathcal{E}t\Phi \Gamma M(A_{\infty})$. This gives the essential surjectivity as desired.

We have established the equivalence parts of both (1) and (2). It follows from some standard linear algebra-theoretic arguments that these are both tensor equivalences. \qed

Let $\text{Crys}(\mathcal{O}_{A_{\frac{1}{\xi}}}[\xi], \varphi)$ be the category of prismatic $F$-crystals over $\mathcal{O}_{A_{\frac{1}{\xi}}}[\xi]$ on $(R/A_{\infty})_{\Delta}$. For any $M \in \text{Crys}(\mathcal{O}_{A_{\frac{1}{\xi}}}[\xi], \varphi)$, we get an étale $\varphi$-module $\text{ev}^{\varphi}(M)$ by evaluating it at $(A_{\inf}(R) \to R = R) \in (R/A_{\inf})_{\Delta}$. This gives a functor

$$\text{ev}^{\varphi} : \text{Crys}(\mathcal{O}_{A_{\frac{1}{\xi}}}[\xi], \varphi) \to \mathcal{E}t\Phi M(A_{\inf}(R)([1])).$$  

Similarly, the evaluation at $(A_{\inf}(R_{\infty}) \to R_{\infty} \leftarrow R) \in (R/A_{\inf})_{\Delta}$ gives a functor

$$\text{ev}_{\infty} : \text{Crys}(\mathcal{O}_{A_{\frac{1}{\xi}}}[\xi], \varphi) \to \mathcal{E}t\Phi M(A_{\inf}(R_{\infty})([1])).$$
Therefore, we get a commutative diagram as follows:

\[
\begin{array}{ccc}
\text{Crys}(O_{\Delta}[\frac{1}{\xi}]^\wedge_p, \varphi) & \xrightarrow{\text{ev}^\triangledown} & \text{Ét}^\Phi M(A_{\text{inf}}(R)[\frac{1}{\xi}]) \\
\text{Ét}^\Phi M(A_{\text{inf}}(R)[\frac{1}{\xi}]) & \xrightarrow{-\otimes_{A_{\text{inf}}(R)[\frac{1}{\xi}]} A_{\text{inf}}(R\infty)[\frac{1}{\xi}]} & \text{Ét}^\Phi M(A_{\text{inf}}(R_{\infty})[\frac{1}{\xi}])
\end{array}
\]

**Lemma 2.8.** The evaluation functor \( \text{ev}^\triangledown \) can be upgraded to a functor from the category \( \text{Crys}(O_{\Delta}[\frac{1}{\xi}]^\wedge_p, \varphi) \) to the category \( \text{ét}^\Phi \Gamma M(A_{\text{inf}}(R)[\frac{1}{\xi}]) \).

**Proof.** For any given \( M \in \text{Crys}(O_{\Delta}[\frac{1}{\xi}]^\wedge_p, \varphi) \), we only need to specify the \( \Gamma \)-action on \( \text{ev}^\triangledown(M) \). For simplicity, we denote \( A_{\text{inf}}(R)[\frac{1}{\xi}]^\triangledown \) by \( A^\triangledown \).

Let \( M := \text{ev}^\triangledown(M) \). As we have seen in Remark 2.2, \( A_{\text{inf}}(R) \) admits a continuous \( \Gamma \)-action and hence one can regard elements in \( \Gamma \) as automorphisms of the prism \((A_{\text{inf}}(R), (\xi))\). As \( M \) is a crystal, for any \( \gamma \in \Gamma \), we get an isomorphism of \( \varphi \)-modules \( \gamma^* M = M \otimes_A \gamma \to M \) over \( A^\triangledown \). In other words, we get a \( \gamma \)-action on \( M \), which commutes with \( \varphi \). As this holds for any \( \gamma \in \Gamma \), we see \( M \) is an \( \text{ét} \)-(\( \varphi, \Gamma \))-module as desired. \( \square \)

**Remark 2.4.** By the same argument, we see that \( \text{ev}_{\infty} \) can be upgraded to a functor from \( \text{Crys}(O_{\Delta}[\frac{1}{\xi}]^\wedge_p, \varphi) \) to \( \text{ét}^\Phi \Gamma M(A_{\text{inf}}(R_{\infty})[\frac{1}{\xi}]) \). By construction, the \( \Gamma \)-action on \( ev_{\infty}(M) \) induced by \(-\otimes_{A_{\text{inf}}(R_{\infty})[\frac{1}{\xi}]} A_{\text{inf}}(R\infty)[\frac{1}{\xi}]) \) is compatible with the action induced by viewing \( \Gamma \) as automorphisms of the prism \((A_{\text{inf}}(R_{\infty}), (\xi))\) and taking pull-backs as in the proof of Lemma 2.8. We still denote the resulting functor

\[
\text{Crys}(O_{\Delta}[\frac{1}{\xi}]^\wedge_p, \varphi) \to \text{ét}^\Phi \Gamma M(A_{\text{inf}}(R_{\infty})[\frac{1}{\xi}])
\]

by \( \text{ev}^\triangledown \) when the contexts are clear for the sake that we shall give another description of \( \Gamma \)-actions on evaluations of \( \text{ev}_{\infty} \) in a moment.

**Remark 2.5.** We will see \( \text{ev}^\triangledown \) is indeed an equivalence (See Proposition 2.15 and Theorem 2.16).

Now, we are going to study \( \text{ev}_{\infty} \). As \( (A_{\text{inf}}(R_{\infty}) \to R_{\infty} \leftarrow R) \) is a cover of the final object of the topos \( \text{Shv}(R/A_{\text{inf}}) \), the evaluation of an \( F \)-crystal on the prism \((A_{\text{inf}}(R_{\infty}) \to R_{\infty} \leftarrow R)\) gives an object in \( \text{ét} \Phi M(A_{\text{inf}}(R_{\infty})) \) with stratifications as what Morrow-Tsuji did in [MT20, § 3.1]. More precisely, let \((A^\triangledown, (\xi))\) be the Čech nerve of \((A_{\text{inf}}(R_{\infty}), (\xi))\) in \( (R/A_{\text{inf}}) \). Denote \( p_0, p_1 : A^0_{\infty} \to A^1_{\infty} \) and \( p_{01}, p_{02}, p_{12} :
$A^1_{\infty} \to A^2_{\infty}$ the face maps in low degrees and denote $\Delta : A^1_{\infty} \to A^0_{\infty}$ the degeneracy map in degree 1. Then we make the following definition as an analogue of [MT20 Definition 3.14].

**Definition 2.9.** An stratification on an étale $\varphi$-module $M$ over $\widehat{A^0_{\infty}[\frac{1}{\xi}]} = A\inf(R_{\infty})[\frac{1}{\xi}]$ with respect to $A^\bullet_{\infty}[\frac{1}{\xi}]$ is an isomorphism

$$\varepsilon : M \otimes A^0_{\infty}[\frac{1}{\xi}], p_1 A^1_{\infty}[\frac{1}{\xi}] \to M \otimes A^0_{\infty}[\frac{1}{\xi}], p_0 A^1_{\infty}[\frac{1}{\xi}]$$

of $\varphi$-modules over $\widehat{A^0_{\infty}[\frac{1}{\xi}]}$ satisfying the cocycle condition:

$$p^0_1(\varepsilon) \circ p^0_2(\varepsilon) = p^0_2(\varepsilon) : M \otimes A^0_{\infty}[\frac{1}{\xi}], q_1 A^2_{\infty}[\frac{1}{\xi}] \to M \otimes A^0_{\infty}[\frac{1}{\xi}], q_0 A^2_{\infty}[\frac{1}{\xi}],$$

We denote $\text{Strat}(A^\bullet_{\infty}[\frac{1}{\xi}])$ the category of étale $\varphi$-modules over $\widehat{A^0_{\infty}[\frac{1}{\xi}]}$ with stratifications with respect to $A^\bullet_{\infty}[\frac{1}{\xi}]$.

**Lemma 2.10.** The evaluation functor $ev_{\infty}$ induces an equivalence between the category $\text{Crys}(\mathcal{O}_\Delta[\frac{1}{\xi}], \varphi)$ and the category $\text{Strat}(A^\bullet_{\infty}[\frac{1}{\xi}])$.

**Proof.** This follows from [Wu21 Proposition 3.2]. □

Although $(A^0_{\infty}, (\xi))$ is a perfect prism, for any $i \geq 1$, $(A^i_{\infty}, (\xi))$ is not perfect.

We will replace the cosimplicial object $(A^\bullet_{\infty}, (\xi))$ by another one $B^\bullet_{\infty}$, of which each term is perfect. The $(B^\bullet_{\infty}, (\xi))$ is constructed as follows.

By the same argument as in the proof of Lemma 2.3, we see $(A\inf(R_{\infty}), (\xi))$ is also a cover of the final object in $\text{Shv}((R/A\inf)^{\text{perf}}_\Delta)$. Let $(B^\bullet_{\infty}, (\xi))$ be the Čech nerve of $(A\inf(R_{\infty}), (\xi))$ in $(R/A\inf)^{\text{perf}}_\Delta$.

**Lemma 2.11.** For any $i \geq 0$, $(B^i_{\infty}, (\xi))$ is the perfection of $(A^i_{\infty}, (\xi))$.

**Proof.** Clearly, $A^0_{\infty} = A\inf(R_{\infty}) = B^0_{\infty}$. For $i \geq 1$, by definition of $(A^i_{\infty}, (\xi))$, it is the initial object among the category of prisms $(A, I) \in (R/A\inf)^{\text{perf}}_\Delta$ together with $(i+1)$-morphisms from $(A^0_{\infty}, (\xi))$ to $(A, I)$. Similarly, $(B^i_{\infty}, (\xi))$ is the initial object among the category of perfect prisms $(B, J) \in (R/A\inf)^{\text{perf}}_\Delta$ together with $(i+1)$-morphisms from $(A^0_{\infty}, (\xi))$ to $(B, J)$. In particular, we get a unique morphism $(A^i_{\infty}, (\xi)) \to (B^i_{\infty}, (\xi))$ of prisms, which factors uniquely through the perfection $(A^i_{\infty}, (\xi))^{\text{perf}}$ of the source. Denote this morphism by $f : (A^i_{\infty}, (\xi))^{\text{perf}} \to (B^i_{\infty}, (\xi))$. On the other hand, the $(i+1)$ arrows from $(A^0_{\infty}, (\xi))$ to $(A^i_{\infty}, (\xi))$ composed with the canonical morphism $(A^i_{\infty}, (\xi)) \to (A^i_{\infty}, (\xi))^{\text{perf}}$ induces a unique morphism
g : (B_{\infty}^{\bullet}, (\xi)) \to (A_{\infty}^{\bullet}, (\xi))^{\text{perf}}. By construction, both f \circ g and g \circ f are the identity morphisms on their domains, which completes the proof. \hfill \Box

Similar to Definition 2.9, one can define stratifications on étale \varphi-modules in \Et\PhiM(A_{\infty}^{0}([1/\ell])) with respect to \widehat{B}_{\infty}^{\bullet}([1/\ell]) and we denote Strat(\widehat{B}_{\infty}^{\bullet}([1/\ell])) the corresponding category. Then we have the following lemma.

**Lemma 2.12.** Let \( M \in \Et\PhiM(A_{\infty}^{0}([1/\ell])) \). Then \( M \) admits a stratification with respect to \( \widehat{A}_{\infty}^{\bullet}([1/\ell]) \) if and only if it admits a stratification with respect to \( \widehat{B}_{\infty}^{\bullet}([1/\ell]) \).

As a consequence, the evaluation functor \( ev_{\infty} \) induces an equivalence between the category \( \text{Crys}(\mathcal{O}_{\Delta}([1/\ell]), \varphi) \) and the category \( \text{Strat}(\widehat{B}_{\infty}^{\bullet}([1/\ell])) \).

**Proof.** As all morphisms \( \varepsilon, \Delta^{\bullet}(\varepsilon), p_{\bullet}^{\bullet}(\varepsilon) \) for \( \bullet \in \{01, 02, 12\} \) involved in the definition of stratifications (Definition 2.9) are isomorphisms of étale \varphi-modules. The lemma follows from Proposition 2.6 and Lemma 2.10 immediately. \hfill \Box

**Remark 2.6.** As \( (A_{\text{inf}}(R_{\infty}), (\xi)) \) is a perfect prism, by deformation theory, to give \((i+1)\) morphisms from \((A_{\text{inf}}^{0}, (\xi))\) to \((A, (\xi))\) in \((R/A_{\text{inf}})_{\Delta}\) is equivalently to give \((i+1)\) \(R\)-morphisms from \(R_{\infty}\) to \(A/I\), which is also equivalent to give an \(R\)-morphism \(S^{i} \to A/I\) where \(S^{i} = R_{\infty} \hat{\otimes}_{R} \cdots \hat{\otimes}_{R} R_{\infty}\) is the \((i+1)\)-folds \(p\)-complete tensor product of \(R_{\infty}\) over \(R\). It is easy to check that \(S_{i}\) is a quasi-regular semiperfectoid ring in the sense of [BS19, Notation 7.1]. It follows from [BS19, Proposition 7.2] that \((A_{\text{inf}}^{i}, (\xi)) = (\Delta_{S_{i}}^{\text{inf}}, (\xi)).\)

It remains to consider stratifications with respect to \(\widehat{B}_{\infty}^{\bullet}([1/\ell])\) on étale \varphi-modules over \(A_{\text{inf}}(R_{\infty})[1/\ell]\). To do so, we need to describe the \(\check{\text{C}}\)ech nerve \(\widehat{B}_{\infty}^{\bullet}([1/\ell])\) in a more explicit way.

**Lemma 2.13.** (1) For \(i = 0\), we have \(\widehat{B}_{\infty}^{0}([1/\ell]) = W((R_{\infty}[1/p])^{\text{perf}})\). The ring \(A_{\text{inf}}(R_{\infty})\) with \((p, \xi)\)-adic topology is an open bounded subring of the both sides.

(2) In general, for any \(i \geq 1\), \(\widehat{B}_{\infty}^{i}([1/\ell]) = C(\Gamma^{i}, W((R_{\infty}[1/p])^{\text{perf}})\) is the ring of continuous functions on \(\Gamma^{i}\). Moreover, \(\widehat{B}_{\infty}^{i}([1/\ell]) \simeq C(\Gamma^{i}, W((R_{\infty}[1/p])^{\text{perf}})\) is an isomorphism of cosimplicial rings.

**Proof.** (1) Since both sides are \(p\)-complete \(\mathbb{Z}_{p}\)-algebras on which \(\varphi\) acts as automorphisms, it is enough to verify the equality modulo \(p\), which holds obviously.

(2) We follow the idea of the proof of [Wu21, Lemma 5.3]. Let \(S_{\infty}^{i} = B_{\infty}^{i}/\xi\), which is a perfectoid algebra, for any \(i \geq 0\). We claim that for any \(i \geq 0\),

\[(2.3) \quad S_{\infty}^{i} \left[\frac{1}{p}\right] = C(\Gamma^{i}, R_{\infty}) \left[\frac{1}{p}\right].\]
By [BS10] Theorem 3.10, it follows from the construction of $B^\bullet_{\infty}$ that $S^\bullet_{\infty}$ is the Čech nerve associated to $R_{\infty}$ in the category of perfectoid $R$-algebras and $S^i_{\infty}$ is the initial object in the category of perfectoid $R$-algebras as the target of $(i + 1)$ morphisms from $R_{\infty}$. As a consequence, $S^i_{\infty}[\frac{1}{p}]$ is the initial object in the category of perfectoid $R[\frac{1}{p}]$-algebras as the target of $(i + 1)$ morphisms from $R_{\infty}$.

Let $S^i_{\infty}[\frac{1}{p}]^+$ be the $p$-adic completion of the integral closure of $S^i_{\infty}$ in $S^i_{\infty}[\frac{1}{p}]$. Then it is almost isomorphic to the quotient of $S^i_{\infty}$ by its $p$-torsion part, which is again a perfectoid algebra for all $i \geq 0$. More precisely, the natural map

$$S^i_{\infty} \rightarrow S^i_{\infty}[\frac{1}{p}]^+$$

has kernel and cokernel which are both killed by $m_{C_p}$. From this, we see that $V_i := \text{Spa}(S^i_{\infty}[\frac{1}{p}], S^i_{\infty}[\frac{1}{p}]^+)$ is the final object in the category of affinoid perfectoid spaces over $U = \text{Spa}(R[\frac{1}{p}], R)$ together with $(i + 1)$ morphisms to $V_0 = \text{Spa}(R_{\infty}[\frac{1}{p}], R_{\infty})$.

We regard $U$ and $V_i$ for $i \geq 0$ as diamonds. Then the above argument shows that $V_i^\circ$ is the $(i + 1)$-folds fibre products of $V_0^\circ$ over $U^\circ$. As $V_1$ is a Galois pro-étale cover of $U$ with the Galois group $\Gamma$, we know that $V_1^\circ / \Gamma = U^\circ$ and that $V_1^\circ \rightarrow U^\circ$ is a $\Gamma$-torsor. Therefore, we deduce that $V_i^\circ \simeq V_1^\circ \times \Gamma^i$. Since all spaces involved are defined over $\text{Spa}(C_p, O_{C_p})$, by [Sch17, Example 11.12], we see that $V_1^\circ \times \Gamma^i \simeq V_i^\circ \times \text{Spa}(C_p, O_{C_p}) \simeq \text{Spa}(C(\Gamma^i, C_p), C(\Gamma^i, O_{C_p}))^\circ$. By [SW] Proposition 10.2.3, we have

$$V_i^\circ \times \text{Spa}(C_p, O_{C_p})^\circ \simeq \text{Spa}(C(\Gamma^i, C_p), C(\Gamma^i, O_{C_p}))^\circ.$$  

Combining all isomorphisms above, we deduce that

$$V_i^\circ \simeq \text{Spa}(C(\Gamma^i, R_{\infty})[\frac{1}{p}], C(\Gamma^i, R_{\infty}))^\circ.$$  

By [SW] Proposition 10.2.3] again, we have $S^i_{\infty}[\frac{1}{p}] = C(\Gamma^i, R_{\infty})[\frac{1}{p}]$ for all $i \geq 0$ as desired.

Similar to the proof of (1), in order to prove (2), we are reduced to verify

$$\overline{B^\bullet_{\infty}[\frac{1}{\xi}] / p} = B^\bullet_{\infty}[\frac{1}{\xi}] / p \simeq C(\Gamma^i, (R_{\infty}[\frac{1}{p}]))^\circ.$$

Combining the claim (23) with the tilting equivalence, we get $(S^i_{\infty}[\frac{1}{p}]^+)^\circ \simeq C(\Gamma^i, R_{\infty}^p)$. On the other hand, since the map $S^i_{\infty} \rightarrow S^i_{\infty}[\frac{1}{p}]^+$ has kernel and cokernel which are
annihilated by $m_{O^\infty}$, the composition $(S^i_\infty)^\flat \to (S^i_\infty[\frac{1}{p}]^+)\simeq C(\Gamma^i, R^i_\infty)$ has kernel and cokernel which are annihilated by $m_{O^\infty}$, which implies that

$$B^\bullet_\infty[\frac{1}{\xi}] / p \simeq (S^i_\infty[\frac{1}{p}]^+) \simeq C(\Gamma^i, (R^i_\infty[\frac{1}{p}]^+))$$

Finally, by chasing isomorphisms above, we see that

$$\widehat{B}^\bullet_\infty[\frac{1}{\xi}] \simeq C(\Gamma^\bullet, W((R^\infty[\frac{1}{p}]^+)^\flat))$$

is an isomorphism of cosimplicial rings and complete the proof. □

**Example 2.14.** We identify $\widehat{B}^\bullet_\infty[\frac{1}{\xi}]$ with $C(\Gamma^i, W((R^\infty[\frac{1}{p}]^+)^\flat))$ for $i \in \{0, 1, 2\}$ via the isomorphisms in Lemma 2.11 (2). Then

$$p_j : W((R^\infty[\frac{1}{p}]^+)^\flat) = C(\Gamma^0, W((R^\infty[\frac{1}{p}]^+)^\flat)) \to C(\Gamma^1, W((R^\infty[\frac{1}{p}]^+)^\flat))$$

for $j \in \{0, 1\}$ is given by

$$p_j(x) = \begin{cases} 
  x, & \text{if } j = 0 \\
  \gamma(x), & \text{if } j = 1
\end{cases}$$

for any $x \in W((R^\infty[\frac{1}{p}]^+)^\flat)$ and $\gamma \in \Gamma$. The degeneracy morphism

$$\Delta : C(\Gamma^1, W((R^\infty[\frac{1}{p}]^+)^\flat)) \to W((R^\infty[\frac{1}{p}]^+)^\flat)$$

is given by $\Delta(f) = f(1)$, for any continuous function $f : \Gamma \to W((R^\infty[\frac{1}{p}]^+)^\flat)$. For $j \in \{01, 02, 12\}$, for any $f \in C(\Gamma^1, W((R^\infty[\frac{1}{p}]^+)^\flat))$ and $\gamma_0, \gamma_1 \in \Gamma$,

$$p_j(f)(\gamma_0, \gamma_1) = \begin{cases} 
  f(\gamma_0), & \text{if } j = 01 \\
  f(\gamma_0 \gamma_1), & \text{if } j = 02 \\
  \gamma_0(f(\gamma_1)), & \text{if } j = 12
\end{cases}$$

As an application of Lemma 2.13 we give the description of $ev_\infty$.

**Proposition 2.15.** The functor $ev_\infty$ induces an equivalence of categories

$$\text{Crys}(O^\infty[\frac{1}{\xi}]_p^\flat, \varphi) \xrightarrow{ev_\infty} \text{Ét} \Phi \Gamma M(\hat{A}_{\text{inf}}(R^\infty[\frac{1}{\xi}]))$$

**Proof.** It follows from Lemma 2.13 combined with the Galois descent that the evaluation functor $ev_\infty$ can be upgraded to a functor to $\text{Ét} \Phi \Gamma M(\hat{A}_{\text{inf}}(R^\infty[\frac{1}{\xi}]))$. By Lemma 2.12 $ev_\infty$ is an equivalence. □
Remark 2.7. For any $i \geq 0$, denote $C^i := C(\Gamma^i, (R_{\infty}^{\frac{1}{2}})^\gamma) \simeq B_{\infty}^{\frac{1}{2}}$. Let $M \in \hat{\Phi}M(A_{\inf}(R_{\infty}^{\frac{1}{2}}))$ with a stratification $\varepsilon : M \otimes_{C^0, p_1} C_1 \to M \otimes_{C^0, p_0} C_1$ with respect to $C^\bullet$. We describe the induced $\Gamma$-action on $M$ explicitly as follows.

For any $\gamma \in \Gamma$ and $m \in M$, define $\gamma(m) := \varepsilon(m \otimes_{p_1} 1)(\gamma)$; that is, if $\varepsilon(m \otimes_{p_1} 1) = \sum_{i=1}^d m_i \otimes_{p_0} f_i$ for $m_i \in M$ and $f_i \in C^1$, then $\gamma(m) = \sum_{i=1}^d f_i(\gamma)m_i$. One can easily check this “action” is semi-linear. As $\Delta^*(\varepsilon) = \text{id}_M$ is the identity morphism on $M$, we see $m = \sum_{i=1}^d f_i(1)m_i$, which shows that $1 \in \Gamma$ acts identically on $M$.

Finally, for any $\gamma_0, \gamma_1 \in \Gamma$, one can check that $p_{02}^\gamma(\varepsilon)(m \otimes_{q_2} 1)(\gamma_0, \gamma_1) = (\gamma_0 \gamma_1)(m)$ and that $(p_{01}^\gamma \circ p_{12}^\gamma)(\varepsilon)(\gamma_0, \gamma_1) = \gamma_0(\gamma_1(m))$. So we get a $\Gamma$-action on $M$.

Remark 2.8. We have seen $\Gamma$ acts on $A_{\infty}^0 = A_{\inf}(R_{\infty})$ as automorphisms. By the construction of $A_{\infty}^0$, for any $i \geq 0$, $A_{\infty}^i$ is equipped with an action of $\Gamma^{i+1}$ such that the face and degeneracy morphisms are compatible with these actions. These actions extend to $B_{\infty}^{\frac{1}{2}}$ and thus for any $M \in \hat{\Phi}M(A_{\inf}(R_{\infty}^{\frac{1}{2}}))$ with a stratification, one can define a $\Gamma$-action on $M$ as what Morrow-Tsuji did in the paragraph below [MT20, Remark 3.15]. One can check this $\Gamma$-action coincides with the one given in Proposition 2.15.

Combining Proposition 2.15 with Lemma 2.8 we get a diagram

\[
\begin{array}{ccc}
\text{Crys}(O_{\Delta}^{\frac{1}{2}}[\xi_{p}], \varphi) & & \\
\hat{\Phi}\Gamma M(A_{\inf}^{\frac{1}{2}}(R^{\frac{1}{2}})) & & \hat{\Phi}\Gamma M(A_{\inf}(R_{\infty})^{\frac{1}{2}}), \\
\end{array}
\]

which is commutative after forgetting $\Gamma$-actions. The main theorem says that this diagram is indeed commutative.

Theorem 2.16. The above diagram (2.4) is commutative such that all arrows involved are equivalences of categories.

Proof. By Remark 2.4 to check the commutativity, it suffices to see the two functors $\text{ev}^\Delta, \text{ev}_\infty : \text{Crys}(O_{\Delta}^{\frac{1}{2}}[\xi_{p}], \varphi) \to \hat{\Phi}\Gamma M(A_{\inf}(R_{\infty})^{\frac{1}{2}})$ coincide.

For any given $M \in \text{Crys}(O_{\Delta}^{\frac{1}{2}}[\xi_{p}], \varphi)$, let $\text{ev}_{\varphi}(M)$ be the evaluation of $M$ at $(B_{\infty}^1, (\xi))$ as étale $\varphi$-modules in $\hat{\Phi}M(C(\Gamma^i, W((R_{\infty}^{\frac{1}{2}}(\varphi)})$. For any $\gamma \in \Gamma$, we
claim the following diagram

\[ (2.5) \]

\[
p_1^* \gamma^* \text{ev}_{\mathcal{C}^0}(\mathcal{M}) \longrightarrow (1, \gamma)^* p_1^* \text{ev}_{\mathcal{C}^0}(\mathcal{M}) \longrightarrow (1, \gamma)^* p_0^* \text{ev}_{\mathcal{C}^0}(\mathcal{M})
\]

commutes with all arrows being isomorphisms of étale \( \varphi \)-modules over \( C(\Gamma, W((R_\infty[p])^\flat)) \), where \((1, \gamma)\) acts on \( \hat{B}_1 \infty^{\flat} \) as explained in Remark 2.8 and for any continuous function \( f : \Gamma \to C(\Gamma, W((R_\infty[p])^\flat)) \), \((1, \gamma)(f)(\bullet) = f(\gamma \bullet)\).

In fact, by Remark 2.8, we have the following commutative diagram of prisms

\[
(B_\infty^0(\xi)) \xrightarrow{\gamma} (B_\infty^0(\xi)).
\]

Considering the evaluations of \( \mathcal{M} \) along this diagram, we see the left square of (2.5) is commutative. Similarly, one can prove the right square commutes by considering the diagram

\[
(B_\infty^0(\xi)) \xrightarrow{1} (B_\infty^0(\xi)).
\]

We explain how the claim implies the compatibility of ev^\square and ev^\infty. In fact, for any \( m \in \text{ev}_\infty(\mathcal{M}) \) (just as a \( \varphi \)-module and so is equal to ev^\square(\mathcal{M}))), consider \( m \otimes_1 1 \otimes_{p_1} 1 \in p_1^* \gamma^* \text{ev}_{\mathcal{C}^0}(\mathcal{M}) \), its image in \( p_0^* \text{ev}_{\mathcal{C}^0}(\mathcal{M}) \) via the left-bottom arrows is \( \varepsilon(\gamma(m) \otimes_{p_1} 1) \) with \( \gamma \)-action being induced via ev^\square and the image via the top-right arrows in \( \varepsilon(m \otimes_{p_1} 1) \otimes (1, \gamma) 1 \). In other words, we have

\[
\varepsilon(\gamma(m) \otimes_{p_1} 1) = \varepsilon(m \otimes_{p_1} 1) \otimes (1, \gamma) 1.
\]

Assume \( \varepsilon(m \otimes_{p_1} 1) = \sum_{i=1}^d m_i \otimes_{p_0} f_i \) for \( m_i \in M \) and \( f_i \in C(\Gamma, W((R_\infty[p])^\flat)) \).

Evaluating both sides at \( 1 \in \Gamma \), we get

\[
\gamma(m) = \sum_{i=1}^d f_i(\gamma)m_i.
\]

According to Remark 2.8, the right hand side is the \( \gamma \)-action on \( m \), which shows the compatibility as desired.
The last assertion follows from Proposition 2.7 (2), Proposition 2.15 and the commutativity of the diagram (2.4). □

Remark 2.9. One can replace \( R_\infty \) by \( \widehat{R} \) the \( p \)-adic completion of the normalization of \( R \) in the filtered colimit of all finite étale Galois extensions of \( R \) which are unramified outside \( p \), that is, Spa(\( \widehat{R} \frac{1}{(p)} \), \( \widehat{R} \)) is the “universal cover” of Spa(\( R \frac{1}{(p)} \), \( R \)) ((MT20 Example 5.5 (ii))). It is easy to see \( \widehat{R} \) is a perfectoid \( R \)-algebra and is also a quasi-syntomic cover of \( R \). Let \( \tilde{A} = \text{A}_{\text{inf}}(\widehat{R} \frac{1}{(ξ)} \) \). Then, by the same argument as above, one can show that the evaluation at \( (\tilde{A}, (ξ)) \) induces an equivalence from the category Crys(\( \mathcal{O}_{\tilde{A}} \frac{1}{(ξ)} \), \( ϕ \)) to the category of étale \( (ϕ, G) \)-modules over \( \tilde{A} \).

Until now we have assumed \( R \) lives over \( \mathcal{O} \). By similar arguments as above, we have the following corollary.

Corollary 2.17. Let Spf(\( R \)) be a \( p \)-adic smooth formal scheme over \( \mathcal{O}_K \) with a framing \( □ : \mathcal{O}_K(\mathcal{T}^{±1}) \to R \), where \( \mathcal{O}_K \) is the ring of integers in a \( p \)-adic field \( K \). Then there is an equivalence between the category of F-crystals over \( \mathcal{O}_{\tilde{A}} \frac{1}{(ξ)} \) on the absolute prismatic site \( (R)_{\tilde{A}} \) and the category of étale \( (ϕ, G) \)-modules over \( \text{A}_{\text{inf}}(\widehat{R} \frac{1}{(ξ)} \) \). \( G = \pi^t(\text{Spa}(R \frac{1}{(p)} , R)) \).

3. Prismatic crystals and relative \( (ϕ, Γ) \)-modules

Let \( \mathfrak{X} \) be a separated \( p \)-adic smooth formal scheme over \( \mathcal{O}_K \) and \( X \) be its adic generic fiber. Let Crys(\( \mathcal{O}_{\tilde{A}} \frac{1}{(ξ)} \), \( ϕ \)) denote the category of F-crystals over \( \mathcal{O}_{\tilde{A}} \frac{1}{(ξ)} \) on the absolute prismatic site \( (\mathfrak{X})_{\tilde{A}} \).

The main result of this section is the following theorem.

Theorem 3.1. There is an equivalence of categories

\[ \text{Crys}(\mathcal{O}_{\tilde{A}} \frac{1}{(ξ)}, ϕ) \cong \mathcal{L}(X_\text{ét}, \mathbb{Z}_p), \]

where the right one is the category of étale \( \mathbb{Z}_p \)-local systems on \( X \).

Proof. By [KL13 Theorem 9.3.7], there is a natural equivalence between the category \( \text{ÉtΦ}(W(\widehat{O}_{\mathfrak{X}^+})) \) of étale \( ϕ \)-modules over \( W(\widehat{O}_{\mathfrak{X}^+}) \) and the category \( \mathcal{L}(X_\text{ét}, \mathbb{Z}_p) \). So it reduces to proving a natural equivalence between the category Crys(\( \mathcal{O}_{\tilde{A}} \frac{1}{(ξ)}, ϕ \) and the category \( \text{ÉtΦ}(W(\widehat{O}_{\mathfrak{X}^+ })) \). This follows from the next theorem.

□
Note that the above equivalence can be checked locally on small affine opens and then glued up globally. In fact, the association \( X \mapsto \text{Crys}((O_X^{\Delta \cup [1]} \wedge \varphi)) \) is a stack for the Zariski topology (for example, see [MT20, Theorem 5.17]). So we assume \( X = \text{Spf}(R) \) such that there exists a framing \( \square : O_K \langle T^{-1} \rangle \to R \).

**Theorem 3.2.** There exists a commutative diagram of categories

\[
\begin{array}{ccc}
\text{Crys}(O^{\Delta \cup [1]} \wedge \varphi) & \xrightarrow{\alpha} & \text{Et}(W(\hat{O}_{X^\dag})) \\
\alpha & \downarrow & \downarrow \text{ev}_\infty \\
\text{Et}(W(\hat{O}_{X^\dag})) & \xrightarrow{\Gamma(U_\infty, -)} & \text{Et}(G(\hat{A}_{\text{inf}}(\hat{R}[1], (\xi))))
\end{array}
\]

where all the functors are equivalences. Here \( U_\infty \) is the affinoid perfectoid object corresponding to \( (\hat{R}[1], \hat{R}) \) as in Remark 2.9. The functor \( \text{ev}_\infty \) is the evaluation on the object \( (\hat{A}_{\text{inf}}(\hat{R}[1], (\xi))) \).

**Proof.** We first construct the functor \( \alpha \) in a natural way and prove it is an equivalence by studying this commutative diagram.

Let \( M \) be a prismatic \( F \)-crystal in \( \text{Crys}(O^{\Delta \cup [1]} \wedge \varphi) \). Note that the site \( X_{\text{pro\acute{e}t}} \) admits a basis consisting of affinoid perfectoid objects. To define a sheaf on \( X_{\text{pro\acute{e}t}} \) then is equivalent to defining a sheaf on the site consisting of affinoid perfectoid objects with the induced topology. We can define a pro-\( \acute{e} \)tale presheaf \( \alpha (M) \) on \( X_{\text{pro\acute{e}t}} \) corresponding to \( M \) as follows: let \( V = \lim_{\rightarrow} V_i \) be affinoid perfectoid objects and \( S^+ = \lim_{\rightarrow} S_i^+ \) being a perfectoid ring and \( \hat{V} \) be the associated perfectoid space. We can define

\[
\alpha (M)(V) := M((W((S^+)\wedge), (\text{Ker}(\theta)))).
\]

By the fact that there is an equivalence between the two sites \( \hat{V}_{\text{pro\acute{e}t}} \) and \( X_{\text{pro\acute{e}t}} / V \) and [KL15, Corollary 9.3.8], we see that for any \( \acute{e} \)tale \( W(\hat{O}_{X^\dag}) \)-module \( E \) on \( X_{\text{pro\acute{e}t}} \), its restriction on \( X_{\text{pro\acute{e}t}} / V \) is isomorphic to \( E(V) \otimes_{W(\hat{O}_{X^\dag}(V))} W(\hat{O}_{X^\dag})|_V \). This means \( \alpha (M)|_V \) is already a sheaf on \( X_{\text{pro\acute{e}t}} / V \) for any affinoid perfectoid object \( V \) by the definition of prismatic crystals.

Then it is easy to see that \( \alpha (M) \) is in \( \text{Et}(W(\hat{O}_{X^\dag})) \) and the diagram in the theorem is commutative by the construction. Now by Corollary 2.17 the functor \( \text{ev}_\infty \) is an equivalence. Then \( \text{ev}_\infty^{-1} \circ \Gamma(U_\infty, -) \) is a left inverse to the functor \( \alpha \).
To show $\alpha$ is an equivalence, it remains to show that $\alpha(\ev^{-1}_\infty(\Gamma(U_\infty, \mathcal{E}))) \cong \mathcal{E}$ for any $\mathcal{E} \in \Et(W(\hat{O}_X^\flat))$. Let $\mathcal{M} = \ev^{-1}_\infty(\Gamma(U_\infty, \mathcal{E}))$. For any affinoid perfectoid objects $V \in X_{\pro\acute{e}t}$, write $V_\infty := U_\infty \times_X V$. Then $\mathcal{E}(V) = \ker(\mathcal{E}(V_\infty) \to \mathcal{E}(V_\infty \times_V V_\infty))$. By abuse of notation, let $M(V) = \mathcal{E}(V_\infty)$. For any affinoid perfectoid objects $V \in X_{\pro\acute{e}t}$, write $V_\infty := U_\infty \times_X V$. Then $E(V) = \ker(E(V_\infty) \to E(V_\infty \times_V V_\infty))$. By abuse of notation, let $M(V)$ denote the value of $M$ on the prism corresponding to the affinoid perfectoid space $\hat{V}$. Then we also have $\mathcal{M}(V) = \ker(\mathcal{M}(V_\infty) \to \mathcal{M}(V_\infty \times_V V_\infty))$. Since both $V_\infty$ and $V_\infty \times_V V_\infty$ live over $U_\infty$, we get $E(V) = \alpha(M)(V) = M(V)$. We are done. □

Remark 3.1. By similar arguments, one can also prove that there is an equivalence between the category $\text{Crys}(\mathcal{O}_A[\frac{1}{p}])$ of rational Hodge–Tate crystals over $(X)_\Delta$ and the category $\text{Vect}(X_{\pro\acute{e}t}, \hat{O}_X)$ of generalized representations over $X_{\pro\acute{e}t}$.

4. Étale comparison

In this section, we fix a perfect prism $(A, I = (d))$ such that $A/I$ contains all $p$-power roots of unity, which ensures that the arguments in Section 2 can be safely applied. Note that for any $p$-adic formal scheme $X$ over $A/I$ we have $(X/A)_\Delta \simeq X_{\acute{e}t}.$ Under the equivalence of categories in Section 3 we want to study the relationship between the prismatic cohomology of crystals and the pro-étale cohomology of the corresponding $\mathbb{Z}_p$-local systems. The main theorem of this section is the following.

Theorem 4.1. Let $X$ be a separated $p$-adic smooth formal scheme over $A/I$ and $M$ be a prismatic $F$-crystal over $\mathcal{O}_A[\frac{1}{p}]^\wedge$ with corresponding $\mathbb{Z}_p$-local system $L$ on $X_{\acute{e}t}$. Then there is a quasi-isomorphism

$$R\Gamma((\mathcal{O}/A)_\Delta, M)^{\geq 1} \simeq R\Gamma(X_{\acute{e}t}, L).$$

We check this theorem locally in a good functorial way. Assume $X = \text{Spf}(R)$ where $R$ admits an étale map from $A/I(T^{\pm 1}_1, \cdots, T^{\pm 1}_n)$ for some $n$. Our strategy is to relate prismatic cohomology on $(R/A)_\Delta$ to that on $(R/A)_{\text{perf}}$. The latter is more closely related to the étale cohomology of the generic fiber $\text{Spa}(R(\frac{1}{p}), R)$.

We first prove some lemmas showing that the prismatic cohomology of crystals can also be calculated by some Čech-Alexander complexes. The following lemma is an analogue of [BST19 Corollary 3.12].

Lemma 4.2. Let $(B, (d))$ be a transversal prism in $(R/A)_\Delta$, i.e. $(p, d)$ is a regular sequence in $B$. Then the higher cohomology of $\mathcal{O}_B[\frac{1}{p}]^\wedge$ on $(B, (d))$ vanishes.

Proof. Let $(B, (d)) \to (C, (d))$ be a flat cover. Then $\mathcal{C} \otimes^\wedge_B \mathcal{B}/(p, d)$ is concentrated in degree 0 by definition. Since $(p, d)$ is a regular sequence in $B$, $(p, d)$ is also a
regular sequence in $C$. In particular, $C$ is $p$-torsion free. Let $(C^i, (d))$ be $(i+1)$-fold fiber product $(C, (d)) \times_{(B, (d))} \cdots \times_{(B, (d))} (C, (d))$ of $(C, (d))$ over $(B, (d))$. By the same argument, $C^i$ is also $p$-torsion free for any $i \geq 0$.

On the other hand, we have $B \simeq R \lim \leftarrow \bigwedge^{\infty} C \cdot$. Inverting $I$, we have $B[1/\mathfrak{p}] \simeq R \lim \leftarrow \bigwedge^{\infty} C \cdot$. By taking derived $p$-adic completion, we have

$$B[1/\mathfrak{p}] \simeq R \lim \leftarrow \bigwedge^{\infty} C \cdot.$$

Since $B[1/\mathfrak{p}]$ and the $C \cdot$’s are all $p$-torsion free, their derived $p$-adic completion is the same as their classical $p$-adic completion. So we have

$$B[1/\mathfrak{p}] \simeq R \lim \leftarrow \bigwedge^{\infty} C \cdot.$$

This means all the higher Čech cohomology groups vanish, which implies

$$R \Gamma((B, (d)), \mathcal{O}_{\Delta[1/\mathfrak{p}]}) = \Gamma((B, (d)), \mathcal{O}_{\Delta[1/\mathfrak{p}]})$$

$\square$

**Corollary 4.3.** Let $(B, (d))$ be a transversal prism in $(R/A)_{\Delta}$. Then for any prismatic $F$-crystal $\mathcal{M}$ over $\mathcal{O}_{\Delta[1/\mathfrak{p}]}$, we have

$$R \Gamma((B, (d)), \mathcal{M}|_{(B, (d))}) = \Gamma((B, (d)), \mathcal{M}).$$

**Proof.** By definition, we see that $\mathcal{M}|_{(B, (d))}$ is a finite projective module over $B[1/\mathfrak{p}]$ and $\mathcal{M}|_{(B, (d))} = \mathcal{M}(B, (d)) \otimes_{B[1/\mathfrak{p}]} \mathcal{O}_{\Delta[1/\mathfrak{p}]}$. By Lemma 4.2 and the projection formula

$$R \Gamma((B, (d)), \mathcal{M}|_{(B, (d))}) \otimes_{B[1/\mathfrak{p}]} \mathcal{O}_{\Delta[1/\mathfrak{p}]} \simeq R \Gamma((B, (d)), \mathcal{O}_{\Delta[1/\mathfrak{p}]}) \otimes_{B[1/\mathfrak{p}]} \mathcal{M}(B, (d)),$$

we are done. $\square$

Let’s choose a framing, i.e. an étale map $\square : A/I(T_1^{\pm 1}, \cdots, T_n^{\pm 1}) \to R$. There is a unique lift $A\square(R)$ of $R$ to $A$. Lemma 2.5 tells us that $(A\square(R), (d))$ is a cover of the final object of the topos $\text{Shv}((R/A)_{\Delta})$.

**Proposition 4.4.** For any prismatic $F$-crystal $\mathcal{M}$ over $\mathcal{O}_{\Delta[1/\mathfrak{p}]}$, there is a quasi-isomorphism

$$R \Gamma((R/A)_{\Delta}, \mathcal{M}) \simeq R \lim \leftarrow \bigwedge^{\infty} \Gamma((A\square(R)^i, (d)), \mathcal{M})$$

where $(A\square(R)^i, (d))$ is the $(i+1)$-fold self-product $(A\square(R), (d)) \times \cdots \times (A\square(R), (d))$ of $(A\square(R), (d))$. 


Proof. Note that all $A^\square (R)^i$’s are $p$-torsion free. Since $(A^\square (R), (d))$ is a cover of the final object in the topos $\text{Shv}((R/A)_\triangle)$, we have

$$R\Gamma((R/A)_\triangle, \mathcal{M}) \simeq \varprojlim_i R\Gamma((A^\square (R)^i, (d)), \mathcal{M}).$$

Then this proposition follows from Corollary 4.3. □

We have similar results for the perfect prismatic site. Recall that the prism $(A_{\text{inf}}(R^\infty), (d))$, which is a cover of the final object of the topos $\text{Shv}((R/A)^{\text{perf}}_\Delta)$, is the perfection of $(A^\square (R), (d))$.

**Proposition 4.5.** For any prismatic $F$-crystal $\mathcal{M}$ over $O_\Delta[\frac{1}{I}] \wedge p$, there is a quasi-isomorphism

$$R\Gamma((R/A)^{\text{perf}}_\Delta, \mathcal{M}) \simeq \varprojlim_i R\Gamma((A_{\text{inf}}(R^\infty)^i, (d)), \mathcal{M}),$$

where $(A_{\text{inf}}(R^\infty)^i, (d))$ is the $(i+1)$-fold self-product of $(A_{\text{inf}}(R^\infty), (d))$ in $(R/A)^{\text{perf}}_\Delta$.

Proof. This follows from the same argument as in Proposition 4.4. □

Next we compare $R\Gamma((R/A)_\triangle, \mathcal{M})$ to $R\Gamma((A/R)^{\text{perf}}_\Delta, \mathcal{M})$.

**Theorem 4.6.** There is a quasi-isomorphism

$$R\Gamma((R/A)_\triangle, \mathcal{M})^{\varphi=1} \simeq R\Gamma((R/A)^{\text{perf}}_\Delta, \mathcal{M})^{\varphi=1}.$$

Proof. Note that $R\Gamma((R/A)_\triangle, \mathcal{M})^{\varphi=1}$ is calculated by the total complex of the following bicomplex

$$
\begin{array}{c}
\Gamma((A^\square (R), (d)), \mathcal{M}) \\
\varphi^{-1}
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow
\end{array}
\begin{array}{c}
\Gamma((A^\square (R)^1, (d)), \mathcal{M}) \\
\varphi^{-1}
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow
\end{array}
\cdots
$$

in the first quadrant. We denote this bicomplex by $C^{\bullet, \bullet}$. There is a spectral sequence $E^{i,j}_1$ associated with this bicomplex, i.e. $E^{i,j}_0 = C^{i,j}$. In particular, we have

$$E^{1,1}_1 = \Gamma((A^\square (R)^i, (d)), \mathcal{M})/\text{Im}(\varphi - 1),$$

$$E^{1,0}_1 = \Gamma((A^\square (R)^i, (d)), \mathcal{M})^{\varphi=1}.$$

For the perfect prismatic site, we have a similar bicomplex $C^{\bullet, \bullet}_{\text{perf}}$, whose totalization calculates $R\Gamma((R/A)^{\text{perf}}_\Delta, \mathcal{M})^{\varphi=1}$, and a spectral sequence $E^{i,j}_{0, \text{perf}} = C^{i,j}_{\text{perf}}$. Clearly, there is a natural map from $C^{\bullet, \bullet}$ to $C^{\bullet, \bullet}_{\text{perf}}$. Since in the both cases, the two
spectral sequences converge to the cohomologies of the totalizations of corresponding bicomplexes. This means that we just need to compare the $E_1$-page $E_1^{i,j}$ and $E_1^{i,j,\text{perf}}$. Then this theorem follows from the next lemma.

Lemma 4.7. For any $i$, we have

$$\Gamma((A^\square(R)^i,(d)),\mathcal{M})/\text{Im}(\varphi-1) \cong \Gamma((A_{\text{inf}}(R^\infty)^i,(d)),\mathcal{M})/\text{Im}(\varphi-1)$$

and

$$\Gamma((A^\square(R)^i,(d)),\mathcal{M})_{\varphi=1} \cong \Gamma((A_{\text{inf}}(R^\infty)^i,(d)),\mathcal{M})_{\varphi=1}.$$

Proof. Since $(p,(d))$ is a regular sequence in $A^\square(R)$, by [BS19, Proposition 3.13] it is also regular in $A^\square(R)^i$ for any $i$ (in fact, $A^\square(R)^i/(d)$ is the $p$-adic completion of a free PD-polynomial ring by [Tian21]). As mentioned in Remark 2.3, the equivalence

$$\text{Ét}\Phi M(A^\square(R)^i[\frac{1}{\mathfrak{I}^\wedge}]_p) \to \text{Ét}\Phi M(A_{\text{inf}}(R^\infty)^i[\frac{1}{\mathfrak{I}^\wedge}]_p)$$

preserves extensions.

Let $E_1$ and $E_2$ be extensions of $N$ by $M$ for $\varphi$-modules $N$ and $M$ over some ring $A$. In other words, for $i = 1, 2$, we have

$$0 \to M \xrightarrow{\alpha_i} E_i \xrightarrow{\pi_i} N \to 0.$$

Then a morphism $f : E_1 \to E_2$ is an isomorphism of extensions if and only if

$$f \circ \alpha_1 = \alpha_2 : M \to E_2$$

and

$$\pi_1 = \pi_2 \circ f : E_1 \to N.$$

As a consequence, let $N, M \in \text{Ét}\Phi M(A^\square(R)^i[\frac{1}{\mathfrak{I}^\wedge}]_p)$ with associated $N_\infty, M_\infty \in \text{Ét}\Phi M(A_{\text{inf}}(R^\infty)^i[\frac{1}{\mathfrak{I}^\wedge}]_p)$. Then the above arguments imply that the set of isomorphism classes of extensions of $N$ by $M$ coincides with the one of $N_\infty$ by $M_\infty$. For the same reason, fix an extension $E$ of $N$ by $M$ and denote by $E_\infty$ the corresponding extension of $N_\infty$ by $M_\infty$, then the group of automorphisms of $E$ (as an extension) coincides with the one of $E_\infty$.

Now the result follows from Lemma 4.8. \square

The following lemma on $\varphi$-modules was used above.
Lemma 4.8. Let $A$ be a $\delta$-ring and $M$ be a $\varphi$-module over $A$. Then the set of isomorphic classes of extensions of $A$ by $M$ forms an $H^1_{\varphi}(M)$-torsor and the group of automorphisms of such a fixed extension is $H^0_{\varphi}(M)$.

Proof. Let $E$ be an extension of $A$ by $M$ with the underlying $A$-module $E = M \oplus A$. Then $E$ is uniquely determined by an element $m \in M$ satisfying $\varphi(0,1) = (m,1)$. For this reason, we denote $E$ by $E_m$. Let $f : E_m \to E_m'$ be an isomorphism of extensions. Then $f$ is uniquely determined by some $n \in M$ such that $f(0,1) = (n,1)$. Since $\varphi \circ f = f \circ \varphi$, we deduce that $m = m' + (\varphi - 1)(n)$. This implies the first assertion. If moreover $m = m'$, that is, $f$ is an automorphism of $E_m$ (as an extension), then $n \in H^0_{\varphi}(M)$. This implies the second assertion. \hfill \qed

Remark 4.1. The description of $\varphi$-coinvariants in terms of extensions also appears in [FF18, Chapter 11].

Next we relate perfect prismatic cohomology to pro-étale cohomology. Let $X = \text{Spa}(R[\frac{1}{p}], R)$ and $U = \text{lim}_{\leftarrow i} \text{Spa}(R[\frac{1}{p}], R_i) \to X$ be the usual pro-étale covering of $X$, where $R_i = R \otimes_{A/I} A/I(\mathbb{T}^{+1})^i A/I(\mathbb{T}^{+1})^{i+1}$. Then this theorem follows from $\Gamma((A\text{inf}(R_{\infty}))^i, (d))$ and Lemma 2.5. Then by Proposition 4.5, we have

$$R\Gamma((R/A)_{\Delta}^\text{perf}, \mathcal{M}) \simeq R\Gamma(X_{\text{pro\text{-}ét}}, \mathcal{E}).$$

Proof. Let $(A\text{inf}(R_{\infty}), (d))$ be the usual perfect prism which is a cover of the final object of the topos $\text{Shv}((R/A)_{\Delta}^\text{perf})$ by Lemma 2.5. Then by Proposition 4.5, we have

$$R\Gamma((R/A)_{\Delta}^\text{perf}, \mathcal{M}) \simeq R\lim_{\leftarrow i} \Gamma((A\text{inf}(R_{\infty}))^i, (d)), \mathcal{M}).$$

On the other hand, we also have

$$R\Gamma((X_{\text{pro\text{-}ét}}, \mathcal{E}) \simeq R\lim_{\leftarrow i} R\Gamma(U^i, \mathcal{E}).$$

where $U^i$ is the $(i+1)$-fold self-product of $U$ in $X_{\text{pro\text{-}ét}}$. By [KL15, Lemma 9.3.4] and the fact that $\mathcal{E}|_{U^i} = \mathcal{E}(U^i) \otimes_{W(\tilde{O}_{X^i})} W(\tilde{O}_{X^i})|_{U^i}$, we have $R\Gamma(U^i, \mathcal{E}) \simeq \Gamma(U^i, \mathcal{E})$. Then this theorem follows from $\Gamma((A\text{inf}(R_{\infty}))^i, (d)), \mathcal{M}) \cong \Gamma(U^i, \mathcal{E})$, which is proved in the next paragraph.

On one hand, let $\hat{U}^i$ be the affinoid perfectoid space associated with $U^i$. Then $\hat{U}^i = \hat{U} \times \mathbb{Z}_p^{-i}$. On the other hand, as the subcategory of perfect prisms in $\text{Perfd}_{R}$ is equivalent to the category $\text{Perfd}_{R}$ of perfectoid rings over $R$, we see that
\( R_{\infty}^i := A_{\inf}(R_{\infty})^i / (d) \) is isomorphic to the \((i+1)\)-fold self-product of \(R_{\infty}\) in the category \( \text{Perfd}_{R} \). By checking the universal property, we see that \( \text{Spa}(R_{\infty}^i[\frac{1}{p}], R_{\infty}^i[\frac{1}{p}]) \) is the \((i+1)\)-fold self-product of \( \text{Spa}(R_{\infty}[\frac{1}{p}], R_{\infty}) \) in the category \( \text{Perfd}_{X} \) of affinoid perfectoid spaces over \( X \), which is equivalent to the category \( \text{Perfd}_{X^o} \). Here \( R_{\infty}^i[\frac{1}{p}]^+ \) is the \( p \)-adic completion of the integral closure of the image of \( S_{\infty}^i \) in \( R_{\infty}^i[\frac{1}{p}] \). Since \( \text{Spa}(R_{\infty}[\frac{1}{p}], R_{\infty})^p \) is a \( \mathbb{Z}_p \)-torsor over \( R_{\infty}[\frac{1}{p}], R_{\infty}^p \), we see that the \((i+1)\)-fold fiber product \( \text{Spa}(R_{\infty}[\frac{1}{p}], R_{\infty})^p \times_{\text{Spa}(R_{\infty}[\frac{1}{p}], R_{\infty})^p} \cdots \times_{\text{Spa}(R_{\infty}[\frac{1}{p}], R_{\infty})^p} \text{Spa}(R_{\infty}[\frac{1}{p}], R_{\infty})^p \) is representable by an affinoid perfectoid space \( \text{Spa}(R_{\infty}[\frac{1}{p}], R_{\infty})^p \times \mathbb{Z}_p^i \). This implies \( \hat{U}^i \cong \text{Spa}(R_{\infty}[\frac{1}{p}], R_{\infty})^p \). Since \( W((R_{\infty}^i)^p)[\frac{1}{p}]^\wedge \cong W((R_{\infty}^i[\frac{1}{p}])^p)[\frac{1}{p}]^\wedge \), we have \( \Gamma((A_{\inf}(R_{\infty})^i, (d)), \mathcal{M}) \cong \Gamma(U^i, \mathcal{E}) \). 

**Theorem 4.10.** There is a natural quasi-isomorphism

\[
R\Gamma((R/A)_{\Delta}, \mathcal{M})^{e=1} \cong R\Gamma(\text{Spa}(R[\frac{1}{p}], R), \mathcal{L})
\]

where \( \mathcal{L} \) is the étale \( \mathbb{Z}_p \)-local system associated with the prismatic \( F \)-crystal \( \mathcal{M} \).

**Proof.** For each framing \( \square : A/I(\mathbb{Z}_p^{\pm 1}) \to R \), we have quasi-isomorphisms

\[
R\Gamma((R/A)_{\Delta}^{\text{perf}}, \mathcal{M}) \cong \Gamma((A_{\inf}(R_{\infty}^i)^\bullet, (d)), \mathcal{M}) \cong R\Gamma(X_{\text{pro\-ét}}, \mathcal{E})
\]

Consider the following index set

\[
J = \{ S \mid S \text{ is a finite set of framings of } R \}
\]

For each \( S = \{ \square_1, \cdots, \square_n \} \), we set

\[
(A_{\inf}(R_{\infty}^{\square}), (d)) = (A_{\inf}(R_{\infty}^1), (d)) \times \cdots \times (A_{\inf}(R_{\infty}^n), (d))
\]

in \( (R/A)^{\text{perf}}_{\Delta} \).

Since \((A_{\inf}(R_{\infty}^{\square}), (d)) \) also covers the final object of \( \text{Shv}((R/A)_{\Delta}) \), we have

\[
R\Gamma((R/A)_{\Delta}^{\text{perf}}, \mathcal{M}) \cong \Gamma((A_{\inf}(R_{\infty}^{\square})^\bullet, (d)), \mathcal{M}) \cong R\Gamma(X_{\text{pro\-ét}}, \mathcal{E}).
\]

If \( S_1 \subset S_2 \), then \( \Gamma((A_{\inf}(R_{\infty}^{S_1})^\bullet, (d)), \mathcal{M}) \cong \Gamma((A_{\inf}(R_{\infty}^{S_2})^\bullet, (d)), \mathcal{M}) \). Now we consider the filtered colimit

\[
M := \lim_{\longleftarrow S \in J} \Gamma((A_{\inf}(R_{\infty}^{\square})^\bullet, (d)), \mathcal{M})
\]

which then does not depend on any framing. Then we get natural quasi-isomorphisms

\[
R\Gamma((R/A)_{\Delta}^{\text{perf}}, \mathcal{M}) \cong M \cong R\Gamma(X_{\text{pro\-ét}}, \mathcal{E}).
\]
Since the quasi-isomorphisms $R\Gamma((R/A)_{\Delta}, \mathcal{M})^{\varphi=1} \simeq R\Gamma((R/A)_{\Delta}^{\text{perf}}, \mathcal{M})^{\varphi=1}$ and $R\Gamma(X_{\text{proet}}, \mathcal{E})^{\varphi=1} \simeq R\Gamma(S\text{pa}(R[\frac{1}{p}], R), \mathcal{L})$ are already natural, we get the desired natural quasi-isomorphism

$$R\Gamma((R/A)_{\Delta}, \mathcal{M})^{\varphi=1} \simeq R\Gamma(S\text{pa}(R[\frac{1}{p}], R), \mathcal{L}).$$

□

Now the Theorem 4.1 is just the globalization of the affine case via these natural quasi-isomorphisms.

**Remark 4.2.** Our proof can not apply to the case of arbitrary $p$-adic formal scheme as in [BS19]. In fact, our local arguments highly depend on a good cover of the final object of the topos $\text{Shv}((X/A)_{\Delta})$, which enables us to use Čech-Alexander complex to calculate the prismatic cohomology of crystals. One might also try to study arc-descent results in the non-constant coefficient case.

By the same arguments, one can also deduce the following theorem.

**Theorem 4.11.** Let $\mathcal{X}$ be a separated $p$-adic smooth formal scheme over $\mathcal{O}_K$ and $\mathcal{M}$ be a prismatic $F$-crystal over $\mathcal{O}_\Delta[\frac{1}{p}]$ on $\mathcal{X}_\Delta$ with corresponding $\mathbb{Z}_p$-local system $\mathcal{L}$ on $X_{\text{ét}}$. Then there is a quasi-isomorphism

$$R\Gamma((\mathcal{X}_\Delta, \mathcal{M})^{\varphi=1} \simeq R\Gamma(X_{\text{ét}}, \mathcal{L}).$$

**Proof.** The proof is the same as that of Theorem 4.1. □

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