Fokker-Planck Kinetic description of small-scale fluid turbulence for classical incompressible fluids

M. Tessarotto\textsuperscript{a,b}, M. Ellero\textsuperscript{c}, D. Sarmah\textsuperscript{b} and P. Nicolini\textsuperscript{a,b}

\textsuperscript{a}Department of Mathematics and Informatics, University of Trieste, Italy, \textsuperscript{b}Consortium of Magneto-fluid-dynamics, University of Trieste, Italy, \textsuperscript{c}Department of Aerodynamics, Technical University of Munich, Munich, Germany

Abstract

Extending the statistical approach proposed in a parallel paper \cite{1}, purpose of this work is to propose a stochastic inverse kinetic theory for small-scale hydrodynamic turbulence based on the introduction of a suitable local phase-space probability density function (pdf). In particular, we pose the problem of the construction of Fokker-Planck kinetic models of hydrodynamic turbulence. The approach here adopted is based on the so-called IKT approach (inverse kinetic theory), developed by Ellero et al. (2004-2008) which permits an exact phase-space description of incompressible fluids based on the adoption of a local pdf. We intend to show that for prescribed models of stochasticity the present approach permits to determine uniquely the time evolution of the stochastic fluid fields. The stochastic-averaged local pdf is shown to obey a kinetic equation which, although generally non-Markovian, locally in velocity-space can be approximated by means of a suitable Fokker-planck kinetic equation. As a side result, the same pdf is proven to have generally a non-Gaussian behavior.

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I. INTRODUCTION

In this paper - extending the results of Ref. [1] - we intend to formulate an IKT (inverse kinetic theory) approach for the full set of fluid equations describing the phenomenon of turbulence in an incompressible fluid, here identified with the stochastic incompressible Navier-Stokes equations. Unlike Ref. [1], where the set of stochastic-averaged fluid equations were considered, we intend to show that the kinetic description, for the stochastic local probability distribution function (stochastic local pdf), is generally non-Markovian when the complete set of fluid equations is considered. The theory is shown to satisfy an H-theorem assuring the strict positivity of the local pdf as well of its stochastic average. In particular, the stochastic-averaged pdf is shown to satisfy an approximate Fokker-Planck-type kinetic equation. The explicit representation of the leading Fokker-Planck coefficients is provided. The result, which holds in principle for arbitrary prescribed stochasticity of the fluid fields, is achieved by means of an IKT which delivers the complete set of fluid fields, all expressed in terms of appropriate moments of the stochastic local pdf.

A. Background and open problems

The investigation hydrodynamic turbulence in incompressible fluids is nowadays playing a major role in fluid dynamics research. In fact the phenomenon of turbulence is essentially ubiquitous, being related to its statistical character. Indeed in many cases the fluid fields which define an incompressible isothermal fluid, i.e., the fluid velocity and pressure \( Z(\mathbf{r}, t) \equiv \{\mathbf{V}, p\} \) are actually not known deterministically but only in a statistical sense. This means that the fluid fields must contain some kind of parameter-dependence \( Z(\mathbf{r}, t, \alpha) \), where \( \alpha \equiv (\alpha_1, \ldots, \alpha_n) \in V_\alpha \subseteq \mathbb{R}^n \) denotes a suitable stochastic real vector independent of \( (\mathbf{r}, t) \) to which a stochastic probability density \( g(\alpha) \) can be attached, so that \( \int_{V_\alpha} d^n \alpha g(\alpha) = 1 \) and furthermore the stochastic average \( \langle Z(\mathbf{r}, t, \alpha) \rangle \equiv \int_{V_\alpha} d^n \alpha g(\alpha) Z(\mathbf{r}, t, \alpha) \) exists. As a consequence, the fluid fields can be represented in terms of the stochastic decomposition \( Z(\mathbf{r}, t, \alpha) = \langle Z(\mathbf{r}, t) \rangle + \delta Z(\mathbf{r}, t, \alpha) \), where \( \delta Z(\mathbf{r}, t, \alpha) \equiv \{\delta \mathbf{V}, \delta p\} \) are suitable stochastic fluctuations of the fluid fields. The precise form which these fluctuations may take defines what is usually denoted as the stochastic behavior (or stochasticity) of the fluid. The vector \( \alpha \), which spans a suitable subset \( V_\alpha \) of \( \mathbb{R}^n \), can in principle be assumed either continuous or discrete.
Its possible definition, as well the identification of the related probability density appearing in the stochastic-averaging operator $\langle \rangle$, is however manifestly non-unique. In fact, these definitions are closely related to the types of stochastic behavior which may appear in the fluid. In the mathematical theory of turbulence one can distinguish in principle two possible types of stochasticity: either intrinsic or numerical. The first type (intrinsic stochasticity) arises when, leaving unchanged the functional form of the fluid equations, the fluid equations are intended as stochastic pde’s. This happens if at least one of the following sources of stochasticity is introduced: 1) Stochastic initial conditions: in this case the initial fluid fields $Z(r,t_0) \equiv Z_0(r)$ are assumed stochastic, i.e. of the form, $Z_0(r,\alpha) = \langle Z_0(r,\alpha) \rangle + \delta Z_0(r,\alpha)$, being $\langle Z_0(r,\alpha) \rangle$ and $\delta Z_0(r,\alpha)$ suitable vector fields. 2) Stochastic boundary conditions: this happens if the boundary fluid fields $Z_w(r,t)|_{\delta\Omega}$ are prescribed in terms of a suitable stochastic vector field of the form $Z_w(r,t,\alpha)|_{\delta\Omega} = \langle Z_w(r,t,\alpha)|_{\delta\Omega} \rangle + \delta Z_w(r,t,\alpha)|_{\delta\Omega}$. Here, Dirichlet (no-slip) boundary conditions have been imposed (for the fluid fields) on the boundary set $\delta\Omega$ of the fluid domain $\Omega \subseteq \mathbb{R}^3$ by letting $Z(r,t)|_{\delta\Omega} = Z_w(r,t)|_{\delta\Omega}$. 3) Stochastic forcing. In this case the volume force density acting on the fluid is assumed stochastic, i.e., of the form $f(r,t,\alpha) = \langle f(r,t,\alpha) \rangle + \delta f(r,t,\alpha)$, being $\langle f(r,t,\alpha) \rangle$ and $\delta f(r,t,\alpha)$ suitable vector fields. It is obvious that the parameters $\alpha \equiv (\alpha_1, \ldots, \alpha_n)$ and the related probability density $g(\alpha)$ can be set, in principle, arbitrarily. In fact, no information on them can be gathered from the deterministic fluid equations. For the same reason, also the definition of the stochastic fluctuations appearing in the previous equations, namely $\delta Z_0(r,\alpha)$, $\delta Z_w(r,t,\alpha)|_{\delta\Omega}$ and $\delta f(r,t,\alpha)$, remains unspecified. As a consequence, each of the stochastic vector fields $Z_0(r,\alpha)$, $Z_w(r,t,\alpha)|_{\delta\Omega}$ and $f(r,t,\alpha)$ may in principle be characterized by different stochastic parameters $\alpha$ and probability densities $g(\alpha)$. Regarding, in particular, the definition of the boundary conditions, we remark that the stochasticity of the boundary fluid fields $Z_w(r,t,\alpha)|_{\delta\Omega}$ may be simply a result of the choice of the prescribed boundary $\delta\Omega$. This happens if $\delta\Omega$ is identified, for example, with a moving surface in which each point (of the surface) moves with random motion. The second type (numerical stochasticity) arises - instead - as a result of the approximate numerical solution methods adopted. All numerical methods, in fact, involve in some sense the introduction of appropriate approximations for the relevant differential operators, based on suitable time and space discretizations. As a consequence, the numerical solutions obtained for the fluid fields become inaccurate on scale lengths comparable or smaller than the spatial discretization (or grid) scales, thus producing...
stochastic error fields. As a result, even if the fluid equations are treated as deterministic (which means ignoring all possible sources of intrinsic turbulence indicated above), numerical errors produce fluid fields which behave effectively as stochastic, giving rise to the phenomenon of numerical turbulence, also known as small-scale or sub-grid turbulence. This means that in principle it is possible to treat the two problems in a similar way. However, numerical stochasticity, as opposite to intrinsic stochasticity, is expected - in principle - to allow a well-defined statistical description. Nevertheless, a consistent theoretical formulation of small-scale turbulence of general validity seems still far away. In particular, still missing is a consistent statistical description, based on the proper definition of the stochastic probability density $g(\alpha)$, able to recover from first principles the correct form of the pdf appropriate for arbitrary turbulence regimes. A widespread picture (of turbulence) consists both of an ensemble of finite-amplitude waves with random phase. However, there is an increasing evidence that this picture is an oversimplification. In fact, it is well known that turbulence may include fluctuations whose phase-coherence characteristics are incompatible with wave-like properties. These are so-called coherent structures, like shocks, vortices and convective cells. In fluid turbulence the signature of the presence of coherent structures is provided by the existence of non-Gaussian features in the probability density. This is usually identified with the velocity-difference probability density function (pdf), traditionally adopted for the description of homogeneous turbulence. This explains why in the past the treatment of hydrodynamic turbulence was based on stochastic models of various nature. These models, which are based on tools borrowed from the study of random dynamical systems, typically rely - however - on experimental verification rather than on first principles. An example is provided by stochastic models - based on Markovian Fokker-Planck (F-P) models of small-scale fluid turbulence recently investigated in the literature by several authors (including: Naert et al., 1997 [2]; Friedrich and Peinke et al., 1999 [3]; Luck et al., 1999 [4]; Cleve et al., 2000 [5]; Ragwitz and Kantz, 2001 [6]; Renner et al., 2001, 2002 [7, 8]; Hosokawa, 2002 [9]). The validity of phenomenological statistical Markovian Fokker-Planck (F-P) models of small-scale fluid turbulence indicate that they are capable of reproducing correctly, at least in some approximate sense, key features of the basic phenomenology of turbulent flows. Their approach is based on the assumption that the probability density associated to the velocity increments should obey a stationary generalized F-P equation. Experimental evidence [2] shows reasonable agreement both with the Markovian assum-
tion and the F-P approximation, at least in a limited subset of parameter space. However, several aspects of the theory need further investigations. In particular, still missing is a consistent statistical description following uniquely from the fluid equations. The theory should be able, specifically, to recover correctly the structure functions, characteristic for the appropriate turbulence regime, but also - possibly - to apply to non-Gaussian probability densities. The latter is, in fact, a typical feature suggested by experimental observations, performed at sufficiently short scale-lengths in the inertial range\[10\]. Based on a recently proposed inverse kinetic theory for classical and quantum fluids (Ellero and Tessarotto, 2004-2008 [12, 13]), a statistical model of small-scale hydrodynamic turbulence is proposed which holds for a generic form of the stochastic probability density \(g(\alpha)\). The approach is intended to determine the local pdf’s (i.e., the local position-velocity joint probability density functions), both for the stochastic-averaged fluid fields \(\langle Z(r,t) \rangle\) and - unlike Ref. [1] - also their stochastic fluctuations \(\delta Z(r,t,\alpha)\), respectively denoted as \(\langle f \rangle\) and \(\delta f\). In particular, it is proven that \(\langle f \rangle\) and \(\delta f\) uniquely determine, by means of suitable velocity-moments, the fluid fields \(\langle Z(r,t) \rangle\) and \(\delta Z(r,t,\alpha)\). Key feature of the approach concerns the construction of the statistical evolution equations for \(\langle f \rangle\) and \(\delta f\). In particular, it is shown that \(\langle f \rangle\) is generally non-Gaussian and obeys an H-theorem. Finally, \(\langle f \rangle\) it is shown to obey - under suitable asymptotic assumptions - to an approximate Fokker-Planck kinetic equation which hold in principle even in the case on non-stationary, non-isotropic and non-homogenous turbulence.

II. STOCHASTIC INSE AND STOCHASTIC IKT

It is convenient first to recall the basic equations for the average and stochastic fluid fields and the corresponding initial-boundary value problem. Starting from the incompressible Navier-Stokes (NS) equations (INSE) and invoking the stochastic decomposition given above the relevant stochastic fluid equations read

\[
\langle NV \rangle = 0, \tag{1}
\]

\[
\nabla \cdot \langle V \rangle = 0, \tag{2}
\]

\[
NV - \langle NV \rangle = 0, \tag{3}
\]

\[
\nabla \cdot \delta V = 0, \tag{4}
\]
to be denoted as stochastic incompressible NS equations (*stochastic INSE*). In particular the first ones (1)-(2) are hereon denoted as *stochastic-averaged INSE*. These equations are assumed to be satisfied pointwise in a set \( \Omega \times I \), being \( \Omega \) an open subset of \( \mathbb{R}^3 \) and \( I \) a finite time interval. Assuming that the fluid fields and \( f \) are sufficiently smooth, the conditions of isochoricity and incompressibility imply the validity of the Poisson equations respectively for \( \langle p \rangle \) and \( \delta p \).

Let us now adopt the statistical approach developed in Ref. [12], which allows us to cast the stochastic INSE problem in terms of a so-called inverse kinetic theory (IKT) [13]. This is based on the identification of the fluid fields \( Z = \{ V, p \} \) with the moments of a suitable probability density \( f(x, t; Z) \) defined in the extended phase space \( \Gamma \times I \) [being \( x \) the state vector \( x = (r, v) \in \Gamma \) spanning the phase-space \( \Gamma = \Omega \times V \), with \( \Omega \) the fluid domain and \( V = \mathbb{R}^3 \) the corresponding velocity space]. In particular, it follows that \( f(x, t; Z) \) must obey an inverse kinetic equation (IKE), which can be identified with the Vlasov-type equation

\[
L(Z)f = 0. 
\]  

Here the notation is standard. Thus \( L \) is the streaming operator \( L(Z) \equiv \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \cdot \{ X(Z) \} \) and \( X(Z) = \{ v, F(Z) \} \), with \( F(Z) \equiv F(x, t; f, Z) \) a suitable vector field (mean field force). Subject only to the requirement of suitable smoothness for the fluid fields, several important consequences follow [12], in particular:

- The mean-field force reads:

\[
F(x, t; f, Z) = \frac{1}{\rho_o} \left[ \nabla \Pi - \nabla p_1 - f \right] + \frac{1}{2} u \cdot \nabla V + \frac{1}{2} \nabla V \cdot u + \nu \nabla^2 V + \frac{1}{2} u \left[ \frac{\partial \ln p_1}{\partial t} + V \cdot \nabla \ln p_1 + \frac{1}{p_1} \left( \nabla \cdot Q - \left[ \nabla \cdot \Pi \right] \cdot Q \right) \right] + \frac{1}{\rho_o} \nabla \cdot \left\{ \frac{u^2}{v^2_{th}} - \frac{3}{2} \right\}, \tag{6}
\]

where \( Q \) and \( \Pi \) are the velocity moments \( \int d^3v G f \) for \( G = u^2 \frac{1}{3}, uu \) and \( u \equiv v - V(r,t) \) is the relative velocity.

- \( \{ V, p \} \) can be identified in the whole fluid domain \( \Omega \) with the velocity moments \( G = \frac{1}{3} u^2 \) of \( f(x, t; Z) \), i.e., respectively, \( V(r,t) = \int d^3v f(x, t; Z) \) and \( p(r,t) = p_1(r,t) - P_0 \), where \( P_0(t) \) (pseudo-pressure) is an arbitrary strictly positive real function of time defined so that the physical realizability condition \( p(r,t) \geq 0 \) is satisfied everywhere in \( \Omega \times I \) and \( p_1(r,t) = \rho_o \int dV \frac{1}{3} u^2 f(x, t; Z) \) is the kinetic scalar pressure.
• If \( f(x,t;Z) \) is a strictly positive and summable phase-function in \( \Gamma \), the Shannon entropy functional \( S(f(t)) = -\int_{\Gamma} dxf(x,t;Z) \ln f(x,t;Z) \) exists \( \forall t \in I \) and can be required to fulfill a constant H-Theorem (see Ref.[14]), i.e., \( \frac{\partial}{\partial t} S(f(t)) = 0 \).

• Introducing the notation \( x^2 = \frac{u^2}{v_{th}^2}, v_{th}^2 = 2p_1/\rho_o \), the pdf

\[
f_M(x,t;Z) = \frac{\rho^{3/2}}{(2\pi)^{3/2} p_1^{3/2}} \exp \left\{ -x^2 \right\}
\]

(local Maxwellian kinetic equilibrium) is a particular solution of the inverse kinetic equation (5) if and only if \( \{V,p\} \) satisfy INSE.

It is now immediate to obtain an inverse kinetic theory for the previous stochastic fluid equations. In fact, let us assume that the operator \( \langle \rangle \) is taken at constant \( r,v \) and \( t \) and introduce the stochastic decompositions \( f(x,t;Z) = \langle f(x,t;Z) \rangle + \delta f(x,t;Z), L(Z) = \langle L(Z) \rangle + \delta L(Z) \) and \( F(x,t;f,Z) = \langle F \rangle + \delta F \), where \( \langle L(Z) \rangle = \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial v} \cdot \{ \langle F(Z) \rangle \} \) and respectively \( \delta L(Z) = \frac{\partial}{\partial v} \cdot \{ \langle F(Z) \rangle \} \). Then the following theorem holds:

**Theorem 1 - Stochastic IKT for the stochastic INSE problem**

If \( f(x,t;Z) \) is a particular solution of the IKE [Eq.(5)] then it follows that: B1) \( \langle f(x,t;Z) \rangle \)
and \( \delta f(x,t;Z) \) obey the coupled system of stochastic kinetic equations

\[
\langle L(Z) \rangle \langle f \rangle = -\langle \delta L(Z) \delta f \rangle \equiv \langle C \rangle ,
\]

\[
\langle L(Z) \rangle \delta f = -\delta L(Z) \{ \langle f \rangle + \delta f \} + \langle \delta L(Z) \delta f \};
\]

B2) A particular solution is provided by \( \langle f \rangle = \langle f_M(x,t;Z) \rangle, \delta f = \delta f_M(x,t;Z) ; \) B3) Eq. (8) can also be written in the form

\[
\langle L(Z) \rangle \langle \Delta f(x,t;Z) \rangle + \langle \Delta L \rangle f(x,t;Z) = -\langle \delta \Delta L \delta f(x,t;Z) \rangle ,
\]

where \( \Delta L = L(Z) - L(\langle Z \rangle), \Delta f(x,t;Z) = f(x,t;Z) - f(x,t;\langle Z \rangle) \) and \( L(\langle Z \rangle), f(x,t;\langle Z \rangle) \) are respectively the stochastic-averaged streaming operator and pdf given in Ref.[14] [see Eq.(5)] and there results \( \langle C \rangle = -\langle \delta \Delta L \delta f(x,t;\langle Z \rangle) \rangle ; \) B4) If \( P_0(t) \) is defined so that the constant H-theorem \( \frac{\partial}{\partial t} S(f(t)) = 0 \) is fulfilled identically in \( I \), it follows that \( \langle f(x,t;Z) \rangle \)
satisfies the weak H-theorem \( \frac{\partial}{\partial t} S(\langle f(t) \rangle) \geq 0 \), hence both \( f(x,t;Z) \) and \( \langle f(x,t;Z) \rangle \) are probability densities.
PROOF - The proof is immediate. In fact B1) follows invoking the inverse kinetic Eq.(5) and the stochastic decompositions for \( f(x, t; Z), L(F) \) and \( F(x, t; f, Z) \). B2) is manifestly fulfilled since by construction \( f_M(x, t; Z) \) is a particular solution of the inverse kinetic equation \( [5] \). Proposition B3) follows by noting that \( f(x, t; Z) \) can be represented as

\[
\int f(x, t; Z) \Delta f(x, t; Z) = \int f(x, t; (Z)) + \Delta f(x, t; Z), \text{ where } f(x, t; (Z)) \text{ is the stochastic-averaged pdf solution of the IKE of the form } L((Z))f(x, t; (Z)) = 0, \text{ where } L((Z)) \text{ is a suitable streaming operator (see Eq.(5) in Ref.[1]). Finally for B4) we notice that by construction }

\[
\int d\Gamma f(x, t; Z) = 1.
\]

It follows \( \int d\Gamma f(x, t; Z) \ln f(x, t; Z) \leq \int d\Gamma f(x, t; Z) \ln f(x, t; Z) \) and hence \( S(f(t)) \leq S((f(t))) \). Hence, requiring since \( P_0(t) \) can be determined so that

\[
\frac{\partial}{\partial t} S(f(t)) = 0 \tag{14}.
\]

This proves the H-theorem.

An obvious implication (of this result) is the manifest non-Gaussian behavior of the stochastic-averaged pdf \( \langle f(x, t; Z) \rangle \). In fact, even if \( f(x, t; Z) \) is identified with the local Maxwellian distribution \( f_M(x, t; Z) \), for arbitrary prescribed choices of the stochastic probability density \( g(\alpha) \) its stochastic average is generally non-Gaussian since \( \langle f_M(x, t; Z) \rangle \neq f_M(x, t; (Z)) \equiv \frac{\rho_{3/2}^{\beta/2}}{(2\pi)^{3/2}p_1^{3/2}} \exp \left\{ -\frac{\langle u^2 \rangle_{p_0}^{\beta/2} p_1}{2\langle p_1 \rangle} \right\} \).

III. FOKKER–PLANCK APPROXIMATION

It is interesting to stress that Eqs. (8) and (9) are formally similar to the Vlasov equation arising in the kinetic theory of quasi-linear and strong turbulence for Vlasov-Poisson plasmas \([15, 17, 18]\) and related renormalized kinetic theory \([19]\), which are known to lead generally to a non-Markovian kinetic equation for \( \langle f \rangle \) alone. Nevertheless, the stochastic-averaged kinetic equation [i.e., Eq. (8)] is known to be amenable, under suitable assumptions, to an approximate Fokker-Planck kinetic equation advancing in time \( \langle f \rangle \) alone. This is achieved by formally constructing a perturbative solution of the equation (9) for the stochastic perturbation \( \delta f \). To obtain a convergent perturbative theory, however, this usually requires the adoption of a suitable renormalization scheme in order to obtain a consistent kinetic equation for \( \langle f \rangle \). An analogous suggestion is posed by the phenomenological Fokker-Planck models of small-scale fluid turbulence recently investigated in the literature. This suggests to seek for a possible approximate representation of this type holding for the stochastic-averaged kinetic equation (8) which should hold for generally non-Gaussian pdf’s (in fact it is obvious that generally the average distribution function \( \langle f(x, t; Z) \rangle \), even in the case in which it
coincides with \( \langle f_M(x, t; Z) \rangle \) results generally non-Gaussian. Let us now first assume that \( \langle f \rangle \equiv \langle f_M(x, t; Z) \rangle \). In such a case the pseudo-pressure \( P_0 \) can be defined - consistent with Eq. (13) - so that locally in phase-space the following asymptotic orderings

\[
\frac{p}{P_0} \ll 1 \sim o(\zeta),
\]

\[
X \equiv o(\zeta),
\]

are satisfied, being \( \zeta \) a dimensionless infinitesimal parameter. In such a case, without additional assumptions on the amplitude of the stochastic fields and in the sub-domain of velocity space in which (12) holds, the ”collision operator” \( \langle C \rangle \) in Eq. (1) can be approximated by the generalized Fokker-Planck (F-P) operator of the form

\[
\langle C \rangle \cong C_{FP} \equiv \sum \partial \partial v \cdot \left[ C_{j+1,m} \cdot \frac{\partial^n}{\partial v^m} f(x, t; \langle Z \rangle) \right]
\]

with summation carried out on \( j, m \) from 0 to \( \infty \) or to finite \( N \geq 3 \) and with \( n = j + m > 1 \), being \( C_{FP} \) a F-P operator and \( C_{j,m} \) suitable F-P coefficients. In the previous equation \( f(x, t; \langle Z \rangle) \equiv f_M(x, t; \langle Z \rangle) \) which denotes the local averaged-Maxwellian

\[
f_M(x, t; \langle Z \rangle) = \frac{1}{(2\pi)^{3/2} \rho_0 v_{th}^3} \exp \left\{ -\bar{X}^2 \right\},
\]

with \( \bar{X}^2 = \frac{u^2}{\sigma^2 v_{th}}, v_{th}^2 = 2 \langle p_1 \rangle / \rho_0 \) and \( \langle u \rangle = v - \langle V(r, t) \rangle \).

An analogous result holds also in the case \( f \neq f_M(x, t; Z) \). In particular, the following theorem holds:

**Theorem 2 - Approximate Fokker-Planck IKE**

Let us assume that \( f(x, t; Z) \) is a particular solution of the IKE [Eq. (5)], then provided \( f(x, t; Z) \) is a function of \( (u, p, r, t) \) which depends slowly both on the relative velocity \( u = v - V \) and fluid pressure \( p \), in the sense that, in validity of the asymptotic orderings (11) and (12), the local pdf is taken of the form \( f = f(\zeta u, \zeta p, r, t) \) (smoothness assumption).

Then in follows that:

B1) Eq. (13) holds also for a generic pdf \( \langle f \rangle \);

B2) there exists a minimal representation for the F-P operator (13) obtained by retaining only the following F-P coefficients: \( C_{1,1} = -\langle \delta F \delta p \rangle \) and \( C_{i,0} = \delta \delta A_i \) with \( i = 2, 3, 4 \) where respectively \( \delta A_2 = \delta V, \delta A_3 = -\delta V \delta V \) and \( \delta A_4 = \frac{1}{2} \delta V \langle \delta V \delta V \rangle - \left[ \frac{1}{6} \delta V \delta V \delta V - \langle \delta V \delta V \delta V \rangle \right] \), such that the stochastic-averaged kinetic equation Eqs. (8) recovers the exact stochastic-averaged fluid equations (1)-(2).

PROOF - The proof (of proposition B1) follows by explicitly evaluating \( \delta \Delta f \) in terms of \( f(x, t; \langle Z \rangle) \). This can be achieved formally by introducing a suitable Taylor expansion
for $\delta \Delta f$ in the neighborhood of $f(x, t; \langle Z \rangle)$. This is permitted if $f$ satisfies the smoothness assumption [2] indicated above, which implies in particular that the orderings (11), (12) must be satisfied. This requirement is manifestly satisfied, in particular, by $f \equiv f_M$. For the general case in which $f \neq f_M$ this permits to estimate $\delta \Delta f$ in terms of $f(x, t; \langle Z \rangle)$ by a perturbative (Taylor) expansion. In particular, to leading-order in $O(\zeta)$, $\delta \Delta f$ reads $\delta \Delta f \equiv -\left(\delta V \cdot \frac{\partial}{\partial x} + \delta p \frac{\partial}{\partial P_0}\right)f(x, t; \langle Z \rangle) \left[1 + o(\zeta)\right]$. Since the $n$-th order terms in Eq.(13) result by construction of order $o(\zeta^n)$, this implies the asymptotic convergence of the expansion for $\delta f$ in the velocity sub-domain in which the ordering (12) holds. Furthermore, one can prove that in order that the stochastic kinetic equations (8) and (9) yield a consistent inverse kinetic theory for the stochastic-averaged N-S equations (including the corresponding energy equation), it is sufficient to retain only a finite number of F-P coefficients (as indicated in proposition B2). The proof by explicit calculation of the relevant moment equations of IKE in which the truncated F-P collision operator is invoked.

A basic consequence of THM.2 is that in the velocity-space sub-domain defined by the inequality $|X| \sim o(\zeta)$, the stochastic-averaged kinetic equation (1) can be approximated by a time and spatially-dependent F-P equation containing generally both velocity and pressure perturbations. In particular, due to the asymptotic ordering (11), (12), to leading order in $o(\zeta)$ pressure perturbations appear only through the explicit contributions carried by $\delta f$ and $C_{1,1}$. Finally, the form of the F-P operator is independent of the specific choice of the pdf, provided the above smoothness assumptions are satisfied.

IV. CONCLUSIONS

In this paper a statistical model of hydrodynamic turbulence has been formulated, based on the IKT approach earlier developed by Ellero et al. [11, 12, 13, 14], which holds for a generic form of the stochastic probability density $g(\alpha)$. Basic feature of the new theory is that it satisfies exactly the full set of stochastic fluid equations while permitting, at the same time, the construction of the stochastic pdf which - in difference with Ref.[1] - advances in time the full set of stochastic fluid fields. Unlike customary statistical approaches, this is identified with the local position-velocity joint probability density function, rather than the two-point correlation function (velocity-difference pdf). The present theory displays several interesting features. In particular, the stochastic-averaged pdf has been shown to
satisfy an H-theorem, assuring its strict positivity. The corresponding kinetic equation is formally similar to the Vlasov equation arising in the strong turbulence theory of Vlasov-Poisson plasmas [15, 17], for which a renormalized kinetic theory can be in principle applied [19]. However, unlike plasmas, the stochastic IKT here considered admits exact local kinetic equilibria for \( \langle f(x, t; Z) \rangle \), which are expressed by the stochastic-averaged Maxwellian distribution \( \langle f_M(x, t; Z) \rangle \). More generally, for pdf’s which are suitably smooth, in the sense of Thm.2, the stochastic-averaged kinetic equation can be approximated in terms of Fokker-Planck kinetic equation. This result suggests a possible new interesting viewpoint for the investigation of turbulence theory in neutral fluids, which includes, in particular, the analysis of previous F-P models of turbulence [2]. This topic will be the subject of a forthcoming investigation.

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Notice

\footnote{contributed paper at RGD26 (Kyoto, Japan, July 2008).}
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