The influence of imprecise quantum measurement on quantum dense coding

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Abstract. In this paper, the influence of imprecise quantum measurement on quantum dense coding would be analysed. The realization processes of dense coding under the condition of imprecise measurement are given in detail. The performance of dense coding is analysed from the successful probability and average classical information. The successful probability and the classical information are immune to the phase parameter error of quantum measurement. However, with the increase of amplitude factor error, both probability and classical information will decrease. The results of this paper would be useful in the field of quantum communication.

1. Introduction

Quantum dense coding is a simple yet surprising application of quantum information theory[1]. The original theoretical scheme of quantum dense coding was proposed by Bennett et al[2], and was experimentally realized by Mattle et al[3]. People can use quantum dense coding to transmit two-bit classical information via one-quit communication with the help of quantum entanglement.

Due to its potential applications of quantum communication, quantum dense coding has been explored, and some theoretical and experimental schemes[4-10] have been presented. Pati et al[4] analytically construct deterministic dense coding schemes for certain classes of no-maximally entangled states, and numerically obtain schemes in the general case. Yang et al[5] proposed a simultaneous dense coding protocol with genuine four-particle entangled state, χ-type entangled state, in which two receivers can simultaneously obtain their respective classical information sent by a sender. Li[6] presented efficient dense coding schemes with cluster states via local measurements. Huang et al[7] presented dense coding schemes by using a extended four-qubit GHZ-W state, where the supervisors can control the average amount of information transmitted from the sender to the receiver. Yi et al[8] presented controlled dense coding schemes with extended GHZ-type states. Wei et al[9] presented an efficient scheme for controlled dense coding is presented with the aid of the introduction of auxiliary particles, appropriate local unitary operations and measurement basis. Nevertheless, there are also some important and open subjects to be investigated in the quantum dense coding region.

It should be emphasized that quantum measurement is the essential process for quantum dense coding. However, quantum measurement would be influenced by human beings and instruments inevitably for quantum dense coding. This paper would explore the performance of dense coding under the condition of imprecise quantum measurement. The realization processes of dense coding
with imprecise measurement are presented. The performance of dense coding would be analyzed from the successful probability and average classical information. The research results could be made use for quantum communication network.

This paper is structured as follows: In Section 2, a previous typical scheme for quantum dense coding proposal would be stated. In Section 3, the concrete realization processes for quantum dense coding under the condition of imprecise measurement are presented. In Section 4, the performance of dense coding would be analyzed from the successful probability and average classical information. At last, we close the paper with a brief summary in Section 5.

2. Typical quantum dense coding scheme

First of all, we will give a brief statement of typical dense coding scheme using one two-qubit maximally entangled state, which can be expressed as

$$|\text{Bell}_{12}\rangle = \frac{\sqrt{2}}{2}(|00\rangle_{12} + |11\rangle_{12})$$

(1)

For this entangled state, the sender Alice has particle 1, and the receiver Bob has particle 2. Suppose that Alice need to transmit some two-bit number strings to Bob. These strings are composed of this set \{00, 01, 10, 11\}. In order to realize the task, Alice would perform the Pauli matrices on her particle, and these matrices would be given by

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(2)

The corresponding system state after Pauli matrices can be described as

$$00 \rightarrow I \otimes I \frac{\sqrt{2}}{2}(|00\rangle_{12} + |11\rangle_{12})$$

$$01 \rightarrow \sigma_x \otimes I \frac{\sqrt{2}}{2}(|00\rangle_{12} + |11\rangle_{12}) = \frac{\sqrt{2}}{2}(|00\rangle_{12} - |11\rangle_{12})$$

$$10 \rightarrow \sigma_y \otimes I \frac{\sqrt{2}}{2}(|00\rangle_{12} + |11\rangle_{12}) = \frac{\sqrt{2}}{2}(|10\rangle_{12} + |01\rangle_{12})$$

$$11 \rightarrow i\sigma_y \otimes I \frac{\sqrt{2}}{2}(|00\rangle_{12} + |11\rangle_{12}) = \frac{\sqrt{2}}{2}(|10\rangle_{12} - |01\rangle_{12})$$

(3)

Furthermore, Alice sends the first particle to Bob, and he has particle 1 and 2. Then, one can get four kinds of quantum states, which are

$$\frac{\sqrt{2}}{2}(|00\rangle_{12} \pm |11\rangle_{12}) \quad \frac{\sqrt{2}}{2}(|10\rangle_{12} \pm |01\rangle_{12})$$

(4)

It can be found that they are mutually orthogonal. Thus, these states are composed of the projective measurement bases, and can be reliably distinguished. So that, one can get the relative classical bit if and only if the states of particles 1 and 2 are determined. The relationship between Bob’s measurement results and Alice’ number strings are shown in Table 1. For example, if particles 1 and 2 are \(\frac{\sqrt{2}}{2}(|00\rangle_{12} + |11\rangle_{12})\), the two-bit number string transmitted from Alice to Bob is ‘00’.

| Bob’s measurement results | Pauli matrices | Alice’s number strings |
|--------------------------|---------------|-----------------------|
| \(\frac{\sqrt{2}}{2}(|00\rangle_{12} + |11\rangle_{12})\) | \(I \otimes I\) | 00 |
| \(\frac{\sqrt{2}}{2}(|00\rangle_{12} - |11\rangle_{12})\) | \(\sigma_x \otimes I\) | 01 |
| \(\frac{\sqrt{2}}{2}(|10\rangle_{12} + |01\rangle_{12})\) | \(\sigma_y \otimes I\) | 10 |
| \(\frac{\sqrt{2}}{2}(|10\rangle_{12} - |01\rangle_{12})\) | \(i\sigma_y \otimes I\) | 11 |

Table 1 The relationship between Bob’s measurement results and Alice’s number strings.
It should be emphasized that one can take advantages of typical quantum dense coding scheme to transmit one two-bit number string by sending single-qubit via the Pauli matrices and quantum measurement. The quantum entanglement is essential resources.

3. Quantum dense coding Scheme with imprecise quantum measurement

According to the above analysis, one can find that quantum measurement plays an important role on quantum dense coding. It is natural to ask these questions: If the measurement is imperfect, how can we realize quantum dense coding? How about the performance of dense coding under the condition of imprecise quantum measurement. This section would discuss about these question.

For quantum measurement, the single-qubit projective measurement is a simple but the most important measurement protocol. The general measurement could be performed based on the single-qubit projective measurement and unitary operations. For instance, the Bell states measurement can be accomplished in terms of single-qubit projective measurement \{0,1\} and the controlled-NOT operation \(U_{\text{CNOT}}\) and the Hadamard gate \(H\), which can be expressed as

\[
U_{\text{CNOT}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}
\]

(5)

In quantum communication, we always use the orthogonal projective measurement bases \{0,1\} to distinguish quantum states. However, quantum measurement would be influenced by human beings and instruments inevitably in the quantum dense coding process. Suppose that the projective measurement bases \{0,1\} in real world would become

\[
|\varphi\rangle = \cos \frac{\theta}{2}|0\rangle + e^{i\phi} \sin \frac{\theta}{2}|1\rangle
\]

\[
|\varphi_\perp\rangle = e^{-i\phi} \sin \frac{\theta}{2}|0\rangle - \cos \frac{\theta}{2}|1\rangle
\]

Here \(\theta \in [0, \pi]\) and \(\phi \in [0, 2\pi]\), and \(\theta\) and \(\phi\) can be considered as errors of amplitude factor and phase parameter in measurement basis, respectively. When \(\theta = \phi = 0\), the measurement states \{\langle \psi |, | \psi_\perp \rangle\} shown as Eq. (6) would reduce to \{0,1\}. The states \{\langle \psi |, | \psi_\perp \rangle\} can be considered as the imprecise measurement protocol.

This section will present the realization processes of quantum dense coding under the imprecise measurement. The corresponding processes can be expressed as follow

**Step 1:** Suppose that the sender Alice and the receiver Bob previously share a pair of qubits in the entangled state shown as Eq. (1). According to the classical bit string \{00,01,10,11\}, which would be transmitted from Alice to Bob, Alice performs the relative Pauli matrices \(\{I, \sigma_x, \sigma_z, i\sigma_y\}\) on her particle. Then, Alice send her particle to Bob using quantum communication channel.

**Step 2:** After that, Bob has two particles, of which the states can be expressed as follow

\[
\begin{align*}
00 & \rightarrow I \rightarrow \frac{\sqrt{2}}{2}(|00\rangle_{12} + |11\rangle_{12}) \\
01 & \rightarrow \sigma_x \rightarrow \frac{\sqrt{2}}{2}(|00\rangle_{12} - |11\rangle_{12}) \\
10 & \rightarrow \sigma_z \rightarrow \frac{\sqrt{2}}{2}(|10\rangle_{12} + |01\rangle_{12}) \\
11 & \rightarrow i\sigma_y \rightarrow \frac{\sqrt{2}}{2}(|10\rangle_{12} - |01\rangle_{12})
\end{align*}
\]

(7)
In order to accomplish quantum dense coding, Bob perform the two-qubit controlled-NOT operation $U_{CNOT}$ on particles 1 and 2. The state of particle 1 is the control qubit, and the state of particle 2 is the target qubit. After this unitary operation, the states in Eq. (7) could be transformed into

$$
\begin{align*}
00 & \rightarrow U_{CNOT} \frac{\sqrt{2}}{2} (|00\rangle_{12} + |11\rangle_{12}) = \frac{\sqrt{2}}{2} (|0\rangle_1 + |1\rangle_1) \otimes |0\rangle_2 \\
01 & \rightarrow U_{CNOT} \frac{\sqrt{2}}{2} (|00\rangle_{12} - |11\rangle_{12}) = \frac{\sqrt{2}}{2} (|0\rangle_1 - |1\rangle_1) \otimes |0\rangle_2 \\
10 & \rightarrow U_{CNOT} \frac{\sqrt{2}}{2} (|10\rangle_{12} + |01\rangle_{12}) = \frac{\sqrt{2}}{2} (|1\rangle_1 + |0\rangle_1) \otimes |1\rangle_2 \\
11 & \rightarrow U_{CNOT} \frac{\sqrt{2}}{2} (|10\rangle_{12} - |01\rangle_{12}) = \frac{\sqrt{2}}{2} (|1\rangle_1 - |0\rangle_1) \otimes |1\rangle_2
\end{align*}
$$

(8)

Furthermore, the Hadamard gate $H$ need to be performed by Bob on particle 1. The Hadamard gate $H$ can change $|0\rangle + |1\rangle$ into $|0\rangle$. Using the gate $H$, the state $|0\rangle - |1\rangle$ would become to $|1\rangle$. Therefore, the states in Eq. (8) should be rewritten as

$$
\begin{align*}
00 & \rightarrow H \otimes I \frac{\sqrt{2}}{2} (|00\rangle_{12} + |11\rangle_{12}) = |00\rangle_{12} \\
01 & \rightarrow H \otimes I \frac{\sqrt{2}}{2} (|00\rangle_{12} - |11\rangle_{12}) = |10\rangle_{12} \\
10 & \rightarrow H \otimes I \frac{\sqrt{2}}{2} (|11\rangle_{12} + |00\rangle_{12}) = |01\rangle_{12} \\
11 & \rightarrow H \otimes I \frac{\sqrt{2}}{2} (|11\rangle_{12} - |00\rangle_{12}) = |11\rangle_{12}
\end{align*}
$$

(9)

**Step 3:** Based on Eq. (6), the common orthogonal projective measurement bases $\{|0\rangle, |1\rangle\}$ could be re-expressed as

$$
|0\rangle = \cos \frac{\theta}{2} |\varphi\rangle + e^{i\theta} \sin \frac{\theta}{2} |\varphi\rangle \\
|1\rangle = e^{-i\theta} \sin \frac{\theta}{2} |\varphi\rangle - \cos \frac{\theta}{2} |\varphi\rangle
$$

(10)

According to Eq. (10), one can present Eq. (9) as follow

$$
\begin{align*}
00 & \rightarrow |00\rangle_{12} \rightarrow \left(\cos \frac{\theta}{2} |\varphi\rangle + e^{i\theta} \sin \frac{\theta}{2} |\varphi\rangle\right) \left(\cos \frac{\theta}{2} |\varphi\rangle + e^{i\theta} \sin \frac{\theta}{2} |\varphi\rangle\right) \\
01 & \rightarrow |01\rangle_{12} \rightarrow \left(e^{i\theta} \sin \frac{\theta}{2} |\varphi\rangle - \cos \frac{\theta}{2} |\varphi\rangle\right) \left(\cos \frac{\theta}{2} |\varphi\rangle + e^{i\theta} \sin \frac{\theta}{2} |\varphi\rangle\right) \\
10 & \rightarrow |10\rangle_{12} \rightarrow \left(\cos \frac{\theta}{2} |\varphi\rangle + e^{i\theta} \sin \frac{\theta}{2} |\varphi\rangle\right) \left(e^{i\theta} \sin \frac{\theta}{2} |\varphi\rangle - \cos \frac{\theta}{2} |\varphi\rangle\right) \\
11 & \rightarrow |11\rangle_{12} \rightarrow \left(e^{i\theta} \sin \frac{\theta}{2} |\varphi\rangle - \cos \frac{\theta}{2} |\varphi\rangle\right) \left(e^{i\theta} \sin \frac{\theta}{2} |\varphi\rangle - \cos \frac{\theta}{2} |\varphi\rangle\right)
\end{align*}
$$

(11)

**Step 4:** On the precise measurement, Bob would measure firstly particle 2, and then measure particle 1. The measurement bases for these are both $\{|0\rangle, |1\rangle\}$. However, influenced by human beings and instruments inevitably in the real world, the measurement bases would become to $\{|\psi\rangle, |\psi_{\perp}\rangle\}$ shown as Eq. (6). The relationship between Bob’ measurement results with classical communication information of Alice is shown as Table 2 in From Table 2, one can find that the quantum dense coding scheme is successful when the measurement outcomes are on Cases 1, 6, 11 and 16.
Table 2 The relationship between Bob’s measurement results with Alice’s classical communication information

| Case | Classical information of Alice | Particle 2 | Particle 1 | Probability | Classical information of Bob |
|------|--------------------------------|-----------|------------|-------------|----------------------------|
| 1    |                                 | $|\varphi\rangle$ | $|\varphi\rangle$ | $\cos^4(\theta/2)$ | $00$ |
| 2    | 00                              | $|\varphi\rangle$ | $|\varphi\rangle$ | $\cos^2(\theta/2)\sin^2(\theta/2)$ | $01$ |
| 3    |                                 | $|\varphi\rangle$ | $|\varphi\rangle$ | $\cos^2(\theta/2)\sin^2(\theta/2)$ | $10$ |
| 4    |                                 | $|\varphi\rangle$ | $|\varphi\rangle$ | $\sin^4(\theta/2)$ | $11$ |
| 5    |                                 | $|\varphi\rangle$ | $|\varphi\rangle$ | $\cos^2(\theta/2)\sin^2(\theta/2)$ | $00$ |
| 6    | 01                              | $|\varphi\rangle$ | $|\varphi\rangle$ | $\cos^4(\theta/2)$ | $01$ |
| 7    |                                 | $|\varphi\rangle$ | $|\varphi\rangle$ | $\sin^4(\theta/2)$ | $10$ |
| 8    |                                 | $|\varphi\rangle$ | $|\varphi\rangle$ | $\cos^2(\theta/2)\sin^2(\theta/2)$ | $11$ |
| 9    |                                 | $|\varphi\rangle$ | $|\varphi\rangle$ | $\cos^2(\theta/2)\sin^2(\theta/2)$ | $00$ |
| 10   | 10                              | $|\varphi\rangle$ | $|\varphi\rangle$ | $\sin^4(\theta/2)$ | $11$ |
| 11   |                                 | $|\varphi\rangle$ | $|\varphi\rangle$ | $\cos^4(\theta/2)$ | $10$ |
| 12   |                                 | $|\varphi\rangle$ | $|\varphi\rangle$ | $\cos^2(\theta/2)\sin^2(\theta/2)$ | $11$ |
| 13   |                                 | $|\varphi\rangle$ | $|\varphi\rangle$ | $\sin^4(\theta/2)$ | $00$ |
| 14   |                                 | $|\varphi\rangle$ | $|\varphi\rangle$ | $\cos^2(\theta/2)\sin^2(\theta/2)$ | $01$ |
| 15   |                                 | $|\varphi\rangle$ | $|\varphi\rangle$ | $\cos^2(\theta/2)\sin^2(\theta/2)$ | $10$ |
| 16   | 11                              | $|\varphi\rangle$ | $|\varphi\rangle$ | $\cos^4(\theta/2)$ | $11$ |

4. The scheme performance with imprecise quantum measurement

The successful probability and the average classical information are always considered as ones of the most factors for the schemes of quantum dense coding.

If and only if the classical communication information relative to the receiver Bob’ measurement outcome is equal to the number string transmitted by the sender Alice, the quantum dense coding scheme is successful. Based on Table 2, it could be obtained that the total successful probability is

$$P = \left(\cos^4\frac{\theta}{2} + \cos^4\frac{\theta}{2} + \cos^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right)/4 = \cos^4\frac{\theta}{2}$$

(12)

From Eq. (12), it should be emphasized that the error $\phi$ of phase parameter is irrelevant with the total successful probability of quantum dense coding with imprecise measurement. On the other hand, the successful probability is 100% if and only if the error $\theta$ of amplitude factor is equal to zero, this means that the receiver Bob performs precise quantum measurements. Meanwhile, the probability is zero when $\theta = \pi$.

For the average classical communication information from Alice to Bob, three conditions need to be discussed as follow

Condition 1: If and only if the measurement outcomes are on Cases 1, 6, 11 and 16, the dense coding scheme is realized with the probability of $\cos^4(\theta/2)$, two-bit classical information could be transmitted.

Condition 2: When the measurement outcomes are on Cases 2, 5, 12 and 15, only one-bit classical information could be transmitted. The probability on this condition is $\cos^2(\theta/2)\sin^2(\theta/2)$.
Condition 3: If and only if the measurement outcomes are on Case 3, 4, 7, 8, 9, 10, 13 and 14, any classical information cannot be transmitted.

Based on the above discussion, one can obtain that the average classical communication information from Alice to Bob can be expressed as

$$I_{\text{total}} = 2\cos^2 \frac{\theta}{2} + 1 + \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} = \cos \frac{\theta}{2} \left( \cos \frac{\theta}{2} + 1 \right)$$

(13)

The average classical communication information from Alice to Bob is two bits if and only if the error $\theta$ of amplitude factor is equal to zero, this means that the receiver Bob performs precise quantum measurements. This result about classical information is equal to the original quantum dense coding protocol.

5. Conclusion
In summary, this paper discussed the influence of imprecise quantum measurement on quantum dense coding. The dense coding scheme under imprecise measurement is presented in detail. The performance of quantum dense coding with imprecise quantum measurement is analysed from the total successful probability and the average classical information. It should be emphasized that the probability and the classical information are irrelevant with the error of phase parameter. The probability and the classical information would change with the error of amplitude factor. If and only if the error of amplitude factor is equal to zero, i.e., the quantum measurement is precise, the results about the total successful probability and the average classical information are agreement with the original dense coding proposal. From the application point of quantum dense coding scheme, the results of this paper would be useful in the field of quantum communication.

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References
[1] Nielsen, M.A., Chuang, I.L.: Quantum Computation and Quantum Information. Cambridge University Press, Cambridge (2000)
[2] Bennett, C.H., Wiesner, S.J.: Communication via one-and two-particle operators on Einstein-Podolsky-Rosen states. Phys. Rev. Lett. 69, 2881 (1992)
[3] K. Mattle, H. Weinfurter, P.G. Kwiat and A. Zeilinger, Phys. Rev. Lett. 76, 4656 (1996)
[4] Pati, A.K., Parashar, P., Agrawal, P.: Probabilistic superdense coding. Phys. Rev. A 72, 012329 (2005)
[5] Yang, X., Bai, M., Zuo, Z. et al. Secure simultaneous dense coding using $\chi$-type entangled state. Quantum Inf Process 17, 261 (2018).
[6] Li, S. Dense Coding with Cluster State Via Local Measurements. Int J Theor Phys 51, 724–730 (2012).
[7] Huang, J., Huang, G. Dense Coding with Extended GHZ-W State via Local Measurements. Int J Theor Phys 50, 2842–2849 (2011).
[8] Yi, X., Wang, J. Huang, G. Controlled Dense Coding using Generalized GHZ-type State. Int J Theor Phys 49, 376–383 (2010)
[9] Wei, J., Shi, L., Zhao, S. et al. Multi-parties Controlled Dense Coding via Maximal Slice States and the Physical Realization Using the Optical Elements. Int J Theor Phys 57, 1479–1485 (2018)
[10] Wei, J., Shi, L., Zhao, S. et al. Controlled Dense Coding via Partially Entangled States and its Quantum Circuits. Int J Theor Phys 57, 1376–1383 (2018)