Noether’s first theorem in Hamiltonian mechanics

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Abstract

Non-autonomous non-relativistic mechanics is formulated as Lagrangian and Hamiltonian theory on fibre bundles over the time axis $\mathbb{R}$. Hamiltonian mechanics herewith can be reformulated as particular Lagrangian theory on a momentum phase space. This facts enable one to apply Noether’s first theorem both to Lagrangian and Hamiltonian mechanics. By virtue of Noether’s first theorem, any symmetry defines a symmetry current which is an integral of motion in Lagrangian and Hamiltonian mechanics. The converse is not true in Lagrangian mechanics where integrals of motion need not come from symmetries. We show that, in Hamiltonian mechanics, any integral of motion is a symmetry current. In particular, an energy function relative to a reference frame is a symmetry current along a connection on a configuration bundle which is this reference frame. An example of the global Kepler problem is analyzed in detail.

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1 Introduction

Noether’s theorems are well known to treat symmetries of Lagrangian systems. Noether’s first theorem associates to a Lagrangian symmetry the conserved current whose total differential vanishes on-shell. We refer the reader to the brilliant book of Yvette Kosmann-Schwarzbach for the history and references on the subject.
By mechanics throughout the work is meant classical non-autonomous non-relativistic mechanics subject to time-dependent coordinate and reference frame transformations. This mechanics is formulated adequately as Lagrangian and Hamiltonian theory on fibre bundles $Q \rightarrow \mathbb{R}$ over the time axis $\mathbb{R}$.

Since equations of motion of mechanics almost always are of first and second order, we restrict our consideration to first order Lagrangian and Hamiltonian theory. Its velocity space is the first order jet manifold $J^1Q$ of sections of a configuration bundle $Q \rightarrow \mathbb{R}$, and its phase space is the vertical cotangent bundle $V^*Q$ of $Q \rightarrow \mathbb{R}$.

This formulation of mechanics is similar to that of classical field theory on fibre bundles over a smooth manifold $X$ of dimension $n > 1$ [9, 23, 27, 30]. A difference between mechanics and field theory however lies in the fact that fibre bundles over $\mathbb{R}$ always are trivial, and that all connections on these fibre bundles are flat. Consequently, they are not dynamic variables, but characterize non-relativistic reference frames (Definition 2.3).

In Lagrangian mechanics, Noether’s first theorem (Theorem 4.2) is formulated as a straightforward corollary of the global variational formula (3.8). It associates to any classical Lagrangian symmetry (Definition 4.1) the conserved current (4.7) whose total differential vanishes on-shell.

In particular, an energy function relative to a reference frame is the symmetry current (4.10) along a connection $\Gamma$ on a configuration bundle $Q \rightarrow \mathbb{R}$ which characterizes this reference frame (Definition 2.3).

A key point is that, in Lagrangian mechanics, any conserved current is an integral of motion (Theorem 4.3), but the converse need not be true (e.g., the Runge-Lenz vector [9, 6] in a Lagrangian Kepler model).

Hamiltonian formulation of non-autonomous non-relativistic mechanics is similar to covariant Hamiltonian field theory on fibre bundles [9, 30, 32] in the particular case of fibre bundles over $\mathbb{R}$ [10, 22, 31]. In accordance with the Legendre map (3.5) and the homogeneous Legendre map (3.10), a phase space and a homogeneous phase space of mechanics on a configuration bundle $Q \rightarrow \mathbb{R}$ are the vertical cotangent bundle $V^*Q$ and the cotangent bundle $T^*Q$ of $Q$, respectively.

It should be emphasized that this is not the most general case of a phase space of non-autonomous non-relativistic mechanics which is defined as a fibred manifold $\Pi \rightarrow \mathbb{R}$ provided with a Poisson structure such that the corresponding symplectic foliation belongs to the fibration $\Pi \rightarrow \mathbb{R}$ [13]. Putting $\Pi = V^*Q$, we in fact restrict our consideration to Hamiltonian systems which admit the Lagrangian counterparts on a configuration space $Q$.

A key point is that a non-autonomous Hamiltonian system of $k$ degrees of freedom on a phase space $V^*Q$ is equivalent both to some autonomous symplectic Hamiltonian system of $k + 1$ degrees of freedom on a homogeneous phase space $T^*Q$ (Theorem 6.1) and a particular first order Lagrangian system with the characteristic Lagrangian (6.22) on $V^*Q$ as a configuration space.

This facts enable one to apply Noether’s first theorem both to study symmetries in Hamiltonian mechanics (Section 7). In particular, we show that, since Hamiltonian symmetries are vector fields on a phase space $V^*Q$ (Definition 7.5), any integral of motion in Hamiltonian mechanics (Definition 7.1) is some conserved symmetry current (Theorem 7.7).

Therefore, it may happen that symmetries and the corresponding integrals of motion define a Hamiltonian system in full. This is the case of commutative and noncommutative completely integrable systems (Section 8).
In Section 9, we provide the global analysis of the Kepler problem as an example of a mechanical system which entirely is characterized by its symmetries. It falls into two distinct global noncommutative completely integrable systems on different open subsets of a phase space. Their integrals of motion form the Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{so}(2,1)$ with compact and noncompact invariant submanifolds, respectively [10, 25, 31].

2 Geometry of fibre bundles over $\mathbb{R}$

This Section summarizes peculiarities of geometry of fibre bundles over $\mathbb{R}$ [10, 16].

Let

$$\pi : Q \to \mathbb{R}$$  \hspace{1cm} (2.1)

be a fibred manifold whose base is regarded as the time axis $\mathbb{R}$ parameterized by the Cartesian coordinate $t$ with transition functions $t' = t + \text{const}$. Relative to the Cartesian coordinate $t$, the time axis $\mathbb{R}$ is provided with the global standard vector field $\partial_t$ and the global standard one-form $dt$ which also is a global volume form on $\mathbb{R}$. The symbol $dt$ also stands for any pull-back of the standard one-form $dt$ onto a fibre bundle over $\mathbb{R}$.

**Remark 2.1.** Point out one-to-one correspondence between the vector fields $f \partial_t$, the densities $f dt$ and the real functions $f$ on $\mathbb{R}$. Roughly speaking, we can neglect the contribution of $T\mathbb{R}$ and $T^*\mathbb{R}$ to some expressions. In particular, there is the canonical imbedding (2.7) of $J^1Q$.

In order that the dynamics of a mechanical system can be defined at any instant $t \in \mathbb{R}$, we further assume that a fibred manifold $Q \to \mathbb{R}$ is a fibre bundle with a typical fibre $M$.

**Remark 2.2.** A fibred manifold $Q \to \mathbb{R}$ is a fibre bundle if and only if it admits an Ehresmann connection $\Gamma$, i.e., the horizontal lift $\Gamma \partial_t$ onto $Q$ of the standard vector field $\partial_t$ on $\mathbb{R}$ is complete.

Given bundle coordinates $(t, q^i)$ on the fibre bundle $Q \to \mathbb{R}$ (2.1), the first order jet manifold $J^1Q$ of $Q \to \mathbb{R}$ is provided with the adapted coordinates $(t, q^i, q^i_t)$ possessing transition functions

$$t' = t + \text{const}, \quad q^n = q^i(t, q^j), \quad q^i_t = (\partial_t + q^i_j \partial_j)q^i.$$  \hspace{1cm} (2.2)

In mechanics on a configuration space $Q \to \mathbb{R}$, the jet manifold $J^1Q$ plays the role of a velocity space.

**Remark 2.3.** Any fibre bundle $Q \to \mathbb{R}$ is trivial. Its different trivializations

$$\psi : Q = \mathbb{R} \times M$$  \hspace{1cm} (2.3)

differ from each other in fibrations $Q \to M$. Given the trivialization (2.3) coordinated by $(t, \tilde{q}^i)$, there is a canonical isomorphism

$$J^1(\mathbb{R} \times M) = \mathbb{R} \times TM, \quad \tilde{q}^i_t = \dot{\tilde{q}}^i,$$  \hspace{1cm} (2.4)

that one can justify by inspection of transition functions of coordinates $\tilde{q}^i$ and $\dot{\tilde{q}}^i$ when transition functions of $q^i$ are time-independent. Due to the isomorphism (2.4), every trivialization (2.3) yields the corresponding trivialization of the jet manifold

$$J^1Q = \mathbb{R} \times TM.$$  \hspace{1cm} (2.5)
As a palliative variant, one develops non-relativistic mechanics on the configuration space and the velocity space. Its phase space is provided with the presymplectic form
\[ \text{pr}^*_T \Omega_T = dp_i \wedge dq^i \] (2.6)
which is the pull-back of the canonical symplectic form on \( T^*M \). A problem is that the presymplectic form (2.6) is broken by time-dependent transformations.

With respect to the bundle coordinates, the canonical imbedding of \( J^1Q \) takes a form
\[ \lambda_{(1)} : J^1Q \ni (t, q^i, \dot{q}^i) \rightarrow (t, q^i, \dot{q}^i) \in TQ, \] (2.7)
\[ \lambda_{(1)} = dt = \partial_t + q^i \partial_{q^i}. \] (2.8)

From now on, the jet manifold \( J^1Q \) is identified with its image in \( TQ \) which is an affine subbundle of \( TX \) modelled over the vertical tangent bundle \( VQ \) of a fibre bundle \( Q \rightarrow \mathbb{R} \). Using the morphism (2.7), one can define the contraction
\[ J^1Q \times T^*Q \rightarrow Q \times \mathbb{R}, \quad (q^i; \dot{i}, \dot{q}_i) \rightarrow \lambda_{(1)}((idt + \dot{q}_i dq^i) = \dot{i} + q^i \dot{q}_i, \]
where \((t, q^i, \dot{t}, \dot{q}_i)\) are holonomic coordinates on the cotangent bundle \( T^*Q \).

In view of the morphism \( \lambda_{(1)} \) (2.7), any connection \( \Gamma = dt \otimes (\partial_t + \Gamma^i \partial_i) \) (2.9) on a fibre bundle \( Q \rightarrow \mathbb{R} \) can be identified with a nowhere vanishing horizontal vector field \( \Gamma = \partial_t + \Gamma^i \partial_i \) (2.10) on \( Q \) which is the horizontal lift \( \Gamma \partial_t \) of the standard vector field \( \partial_t \) on \( \mathbb{R} \) by means of the connection (2.9). Conversely, any vector field \( \Gamma \) on \( Q \) such that \( dt | \Gamma = 1 \) defines a connection on \( Q \rightarrow \mathbb{R} \). Therefore, the connections (2.9) further are identified with the vector fields (2.10). The integral curves of the vector field (2.10) coincide with the integral sections for the connection (2.9).

Connections on a fibre bundle \( Q \rightarrow \mathbb{R} \) constitute an affine space modelled over a vector space of vertical vector fields on \( Q \rightarrow \mathbb{R} \). Accordingly, the covariant differential, associated to a connection \( \Gamma \) on \( Q \rightarrow \mathbb{R} \), takes its values into the vertical tangent bundle \( VQ \) of \( Q \rightarrow \mathbb{R} \):
\[ D_\Gamma : J^1Q \rightarrow VQ, \quad \nabla^i D_\Gamma = q^i_l - \Gamma^i. \] (2.11)

A connection \( \Gamma \) on a fibre bundle \( Q \rightarrow \mathbb{R} \) is obviously flat.

**Theorem 2.1.** Being a flat, every connection \( \Gamma \) on a fibre bundle \( Q \rightarrow \mathbb{R} \) defines an atlas of local constant trivializations of \( Q \rightarrow \mathbb{R} \) such that the associated bundle coordinates \((t, q^i)\) on \( Q \) possess transition functions independent of \( t \), and
\[ \Gamma = \partial_t \] (2.12)
with respect to these coordinates. Conversely, every atlas of local constant trivializations of a fibre bundle \( Q \rightarrow \mathbb{R} \) determines a connection on \( Q \rightarrow \mathbb{R} \) which is equal to (2.12) relative to this atlas.
A connection $\Gamma$ on a fibre bundle $Q \to \mathbb{R}$ is said to be complete if the horizontal vector field $(2.10)$ is complete. By the well known theorem, a connection on a fibre bundle $Q \to \mathbb{R}$ is complete if and only if it is an Ehresmann connection. The following holds [15].

**Theorem 2.2.** Every trivialization of a fibre bundle $Q \to \mathbb{R}$ yields a complete connection on this fibre bundle. Conversely, every complete connection $\Gamma$ on $Q \to \mathbb{R}$ defines its trivialization $(2.3)$ such that the horizontal vector field $(2.10)$ equals $\partial_t$ relative to the bundle coordinates associated to this trivialization.

It follows from Theorem 2.1 that, in mechanics unlike field theory, connections $\Gamma$ $(2.10)$ on a configuration bundle $(2.1)$ fail to be dynamic variables. They characterize reference frames as follows.

From the physical viewpoint, a reference frame in mechanics determines a tangent vector at each point of a configuration space $Q$, which characterizes the velocity of an observer at this point. This speculation leads to the following mathematical definition of a reference frame in mechanics [10, 16, 22].

**Definition 2.3.** In non-relativistic mechanics, a reference frame is the connection $\Gamma$ $(2.10)$ on a configuration bundle $Q \to \mathbb{R}$, i.e., a section of the velocity bundle $J^1Q \to Q$.

By virtue of this definition, one can think of the horizontal vector field $(2.10)$ associated to a connection $\Gamma$ on $Q \to \mathbb{R}$ as being a family of observers, while the corresponding covariant differential $(2.11)$:

$$\overline{\partial^\Gamma_i} = D^\Gamma(q^i_t) = q^i_t - \Gamma^i,$$

(2.13)

determines the relative velocity with respect to a reference frame $\Gamma$. Accordingly, $q^i_t$ are regarded as the absolute velocities.

In accordance with Theorem 2.1, any reference frame $\Gamma$ on a configuration bundle $Q \to \mathbb{R}$ is associated to an atlas of local constant trivializations, and vice versa. A connection $\Gamma$ takes the form $\Gamma = \partial_t$ $(2.12)$ with respect to the corresponding coordinates $(t, q^i_t)$, whose transition functions are independent of time. One can think of these coordinates as also being a reference frame, corresponding to the connection $(2.12)$. They are called the adapted coordinates to a reference frame $\Gamma$. Thus, we come to the following definition, equivalent to Definition 2.3.

**Definition 2.4.** In mechanics, a reference frame is an atlas of local constant trivializations of a configuration bundle $Q \to \mathbb{R}$.

In particular, with respect to the coordinates $q^i_t$ adapted to a reference frame $\Gamma$, the velocities relative to this reference frame $(2.13)$ coincide with the absolute ones

$$\overline{\partial^\Gamma_i} = D^\Gamma(q^i_t) = q^i_t.$$

A reference frame is said to be complete if the associated connection $\Gamma$ is complete. By virtue of Theorem 2.2, every complete reference frame defines a trivialization of a bundle $Q \to \mathbb{R}$, and vice versa.

### 3 Lagrangian mechanics. Integrals of motion

As was mentioned above, our exposition is restricted to first order Lagrangian theory on a fibre bundle $Q \to \mathbb{R}$ [10, 16, 28]. This is a standard case of Lagrangian mechanics.
In mechanics, a first order Lagrangian is defined as a horizontal density

\[ L = \mathcal{L} dt, \quad \mathcal{L} : J^1Q \to \mathbb{R}, \quad (3.1) \]

on a velocity space \( J^1Q \).

The corresponding second order Euler–Lagrange operator, termed the Lagrange operator, reads

\[ \delta L = (\partial_i \mathcal{L} - d_i \partial^j \mathcal{L}) \theta^i \wedge dt, \quad (3.2) \]

\[ d_i = \partial_i + q_i^j \partial_t + q_i^{ij} \partial^j_t. \]

Let us further use the notation

\[ \pi_i = \partial_i \mathcal{L}, \quad \pi_{ji} = \partial^j_t \pi_i. \quad (3.3) \]

The kernel \( \mathcal{E}_L = \text{Ker} \delta L \subset J^2Q \) of the Lagrange operator \( 3.2 \) defines a second order Lagrange equation

\[ (\partial_i - d_i \partial^j_t) \mathcal{L} = 0 \quad (3.4) \]

on \( Q \). Its classical solutions are (local) sections \( c \) of a fibre bundle \( Q \to \mathbb{R} \) whose second order jet prolongations \( J^2c = \partial_t^2c \) live in \( \mathcal{E}_L \) \( 3.4 \).

Every first order Lagrangian \( L \) \( 3.1 \) yields the Legendre map

\[ \hat{L} : J^1Q \rightarrow V^*Q, \quad p_i \circ \hat{L} = \pi_i, \quad (3.5) \]

where the Legendre bundle \( \Pi = V^*Q \) is the vertical cotangent bundle \( V^*Q \to Q \to \mathbb{R} \) provided with holonomic coordinates \((t, q^i, p_i)\). As was mentioned above, it plays the role of a phase space of mechanics on a configuration space \( Q \to \mathbb{R} \). The corresponding Lagrangian constraint space is

\[ N_L = \hat{L}(J^1Q) \quad (3.6) \]

**Definition 3.1.** A Lagrangian \( L \) is said to be:

- hyperregular if the Legendre map \( \hat{L} \) is a diffeomorphism;
- regular if \( \hat{L} \) is a local diffeomorphism, i.e., \( \det(\pi_{ij}) \neq 0 \);
- almost regular if the Lagrangian constraint space \( N_L \) \( 3.6 \) is a closed imbedded subbundle \( i_N : N_L \rightarrow V^*Q \) of the Legendre bundle \( V^*Q \to Q \) and the Legendre map

\[ \hat{L} : J^1Q \rightarrow N_L \quad (3.7) \]

is a fibred manifold with connected fibres.

Given a first order Lagrangian \( L \), there is the global decomposition (called the variational formula)

\[ dL = \delta L - d_i \Xi_L \quad (3.8) \]

where the Lepage equivalent \( \Xi_L \) of \( L \) is the Poincaré–Cartan form

\[ H_L = \pi_i dq^i - (\pi_i q^i_t - \mathcal{L}) dt \quad (3.9) \]
(see the notation \((3.3)\)). This form takes its values into a subbundle

\[ J^1Q \times T^*Q \subset T^*J^1Q. \]

Hence, it defines the homogeneous Legendre map

\[ \hat{H}_L : J^1Q \to Z_Y = T^*Q, \quad (3.10) \]

whose range is an imbedded subbundle

\[ Z_L = \hat{H}_L(J^1Q) \subset T^*Q \quad (3.11) \]

of the homogeneous Legendre bundle \( Z_Y = T^*Q \) \((3.10)\). Let \((t, q^i, p_0 = \dot{t}, p_i = \dot{q}_i)\) denote holonomic coordinates on \( T^*Q \) which possess transition functions

\[ p'_i = \frac{\partial q^j}{\partial q^i} p_j, \quad p'_0 = \left(p_0 + \frac{\partial q^j}{\partial t} p_j\right). \quad (3.12) \]

With respect to these coordinates, the homogeneous Legendre map \( \hat{H}_L \) \((3.10)\) reads

\[ (p_0, p_i) \circ \hat{H}_L = (L - q^i_i \dot{q}_i, \dot{q}_i). \]

In view of the morphism \((3.10)\), the cotangent bundle \( T^*Q \) plays the role of a homogeneous phase space of mechanics.

A glance at the transition functions \((3.12)\) shows that the canonical map

\[ \zeta : T^*Q \to V^*Q, \quad (3.13) \]

is a one-dimensional affine bundle over the vertical cotangent bundle \( V^*Q \). Herewith, the Legendre map \( \hat{L} \) \((3.5)\) is exactly the composition of morphisms

\[ \hat{L} = \zeta \circ H_L : J^1Q \to V^*Q. \]

**Remark 3.1.** The Poincaré–Cartan form \( H_L \) \((3.9)\) also is the Poincaré–Cartan form \( H_L = H^*_L \) of a first order Lagrangian

\[ \bar{L} = \tilde{h}_0(H_L) = (L + (q^i_{\dot{i}} - q^i_i) \pi_i) dt, \quad \tilde{h}_0(dq^i) = q^i_{\dot{i}} dt, \quad (3.14) \]

on the repeated jet manifold \( J^1J^1Q \). The Lagrange operator \((3.2)\) for \( \bar{L} \) \((3.14)\) reads

\[ \delta \bar{L} = [(\partial_i L - \dot{q}^i_i \pi_i + \partial_i \pi_j (q^j_{\dot{i}} - q^j_i)) dq^i + \partial_i^j \pi_j (q^j_{\dot{i}} - q^j_i) dq^i] \wedge dt, \]

\[ \dot{\pi}_i = \partial_i + \dot{q}^i_i \partial_i + q^i_{\dot{i}} \partial_i. \]

Its kernel \( \text{Ker} \delta \bar{L} \subset J^1J^1Q \) defines the Cartan equation

\[ \partial_i^j \pi_j (q^j_{\dot{i}} - q^j_i) = 0, \quad \partial_i L - \dot{q}^i_i \pi_i + \partial_i \pi_j (q^j_{\dot{i}} - q^j_i) = 0 \quad (3.15) \]

on a velocity space \( J^1Q \).

In mechanics, the Lagrange equation \((3.4)\) as like as the Hamilton one \((6.17)\) is an ordinary differential equation. One can think of its classical solutions \( s(t) \) as being a motion in a configuration space \( Q \). In this case, the notion of integrals of motion can be introduced as follows [10 31].
In a general setting, let an equation of motion of a mechanical system is an \( r \)-order differential equation \( E \) on a fibre bundle \( Y \to \mathbb{R} \) given by a closed subbundle of the jet bundle \( J^r Y \to \mathbb{R} \).

**Definition 3.2.** An integral of motion of this mechanical system is defined as a \((k < r)\)-order differential operator \( \Phi \) on \( Y \) such that \( E \) belongs to the kernel of an \( r \)-order jet prolongation of a differential operator \( d_t \Phi \), i.e.,

\[
J^{r-k-1}(d_t \Phi)\big|_E = J^{r-k} \Phi \big|_E = 0, \quad (3.16)
\]

\[
d_t = \partial_t + y^a \partial_a + y^a \partial_t \partial_a + \cdots.
\]

It follows that an integral of motion \( \Phi \) is constant on classical solutions \( s \) of a differential equation \( E \), i.e., there is the differential conservation law

\[
(J^k s)^* \Phi = \text{const.}, \quad (J^{k+1} s)^* d_t \Phi = 0. \quad (3.17)
\]

We agree to write the condition \((3.16)\) as a weak equality

\[
J^{r-k-1}(d_t \Phi) \approx 0, \quad (3.18)
\]

which holds on-shell, i.e., on solutions of a differential equation \( E \) by the formula \((3.17)\).

In mechanics, we restrict our consideration to integrals of motion \( \Phi \) which are functions on \( J^k Y \). As was mentioned above, equations of motion of mechanics mainly are either of first or second order. Accordingly, their integrals of motion are functions on \( Y = J^0 Y \) or \( J^1 Y \). In this case, the corresponding weak equality \((3.16)\) takes a form

\[
d_t \Phi \approx 0 \quad (3.19)
\]

of a weak conservation law of an integral of motion.

Integrals of motion can come from symmetries. This is the case both of Lagrangian mechanics on a configuration space \( Y = Q \) (Theorems 4.3 – 4.4) and Hamiltonian mechanics on a phase space \( Y = V^* Q \) (Theorem 7.2).

**Definition 3.3.** Let an equation of motion of a mechanical system be an \( r \)-order differential equation \( E \subset J^r Y \). Its infinitesimal symmetry (or, shortly, a symmetry) is defined as a vector field on \( J^r Y \) whose restriction to \( E \) is tangent to \( E \).

Following Definition 3.3, let us introduce a notion of the symmetry of differential operators in the following relevant case. Let us consider an \( r \)-order differential operator on a fibre bundle \( Y \to \mathbb{R} \) which is represented by an exterior form \( E \) on \( J^r Y \). Let its kernel \( \text{Ker} E \) be an \( r \)-order differential equation on \( Y \to \mathbb{R} \).

**Theorem 3.4.** It is readily justified that a vector field \( \vartheta \) on \( J^r Y \) is a symmetry of the equation \( \text{Ker} E \) in accordance with Definition 3.3 if and only if \( L_\vartheta E \approx 0 \).

Motivated by Theorem 3.4 we come to the following notion.

**Definition 3.5.** Let \( E \) be the above mentioned differential operator. A vector field \( \vartheta \) on \( J^r Y \) is termed the symmetry of a differential operator \( E \) if the Lie derivative \( L_\vartheta E \) vanishes.
By virtue of Theorem 3.4, a symmetry of a differential operator \( \mathcal{E} \) also is a symmetry of a differential equation \( \text{Ker} \mathcal{E} \).

Note that there exist integrals of motion which are not associated with symmetries of an equation of motion, e.g., the Rug–Lenz vector \([9.6]\) in a Lagrangian Kepler system (Section 9).

4 Noether’s first theorem: Energy conservation laws

In Lagrangian mechanics, integrals of motion come from symmetries of a Lagrangian (Theorem 4.3) in accordance with Noether’s first theorem (Theorem 4.2). However as was mentioned above, not all integrals of motion are of this type.

In the framework of first order Lagrangian formalism, we restrict our consideration to classical symmetries of a Lagrangian system represented by vector fields \( \upsilon \) on a configuration bundle \( Q \rightarrow \mathbb{R} \). Moreover, not concerned with time-reparametrization, we deal with vector fields

\[
\upsilon = \upsilon^j \partial_j + \upsilon^t (t, q^i) \partial_t, \quad \upsilon^t = 0, 1.
\] (4.1)

Their first order jet prolongation onto the velocity space \( J^1Q \) read

\[
J^1 \upsilon = \upsilon^t \partial_t + \upsilon^i \partial_i + d_t \upsilon^t \partial_t\].
\] (4.2)

Let \( L \) be the Lagrangian \([3.1]\) on a velocity space \( J^1Q \). Due to the variational formula \([3.8]\), its Lie derivative \( L_{J^1 \upsilon}L \) along the \( J^1 \upsilon \) \((4.2)\) obeys the relation (called the first variational formula)

\[
L_{J^1 \upsilon}L = v^i \partial_i + d_t v^i \partial_t + \delta L + d_t (v^i H_L),
\] (4.3)

where \( H_L \) is the Poincaré–Cartan form \([3.3]\). Its coordinate expression reads

\[
[v^t \partial_t + v^i \partial_i + d_t v^i \partial_t] L = (v^i - q^i_t v^t) E_i + d_t [\pi_i (v^i - v^t q^i_t) + v^i L].
\] (4.4)

Definition 4.1. The vector field \( \upsilon \) \((4.1)\) on \( Q \) is called the Lagrangian symmetry (or, shortly, the symmetry) of a Lagrangian \( L \) if the Lie derivative \( L_{J^1 \upsilon}L \) of \( L \) is \( d_t \)-exact, i.e.,

\[
L_{J^1 \upsilon}L = d_t \sigma dt
\] (4.5)

where \( \sigma \) is a function on \( J^1Q \).

Then Noether’s first theorem is formulated as follows.

Theorem 4.2. If the vector field \( \upsilon \) \((4.1)\) is a symmetry of a Lagrangian \( L \), a corollary of the first variational formula \((4.3)\) on-shell is the weak Lagrangian conservation law

\[
0 \approx d_t (v^i H_L - \sigma) dt \approx d_t (\pi_i (v^i - v^t q^i_t) + v^i L - \sigma) dt \approx -d_t J_{\upsilon} dt
\] (4.6)

of a symmetry current

\[
J_{\upsilon} = -(v^i H_L - \sigma) = -(\pi_i (v^i - v^t q^i_t) + v^i L - \sigma)
\] (4.7)

along \( \upsilon \). The symmetry current \((4.7)\) obviously is defined with the accuracy to a constant summand.
Theorem 4.3. It is readily observed that the conserved current \( J_\upsilon \) (4.7) along a classical symmetry is a function on a velocity space \( J^1Q \), and it is an integral of motion of a Lagrangian system in accordance with Definition 3.2.

Theorem 4.4. If a symmetry \( \upsilon \) of a Lagrangian \( L \) is classical, this is a symmetry of the Lagrange operator \( \delta L \) (3.2) and, as a consequence, an infinitesimal symmetry of the Lagrange equation \( E_L \) (3.4) (Theorem 3.4).

Remark 4.1. Given a Lagrangian \( L \), let \( \tilde{L} \) be its partner (3.14) on the repeated jet manifold \( J^1J^1Q \). Since \( H_L \) (3.5) is the Poincaré–Cartan form both for \( L \) and \( \tilde{L} \), a Lagrangian \( \tilde{L} \) does not lead to new conserved currents.

If a symmetry \( \upsilon \) of a Lagrangian \( L \) is exact, i.e,

\[
L_{\upsilon,\upsilon}L = 0,
\]

the first variational formula (4.3) takes a form

\[
0 = \upsilon V \delta L + \delta_L H_L.
\]

(4.8)

It leads to the weak conservation law (4.6):

\[
0 \approx -\frac{d}{dt} J_\upsilon,
\]

(4.9)

of the symmetry current

\[
J_\upsilon = -\upsilon H_L = -\pi_\upsilon (v^i - q^i_t) - v^i L
\]

(4.10)

along a vector field \( \upsilon \).

Remark 4.2. The first variational formula (4.8) also can be utilized when a Lagrangian possesses exact symmetries, but an equation of motion is a sum

\[
(\partial_t - d_{ij} \partial_i^j) L + f_i(t, q^i, q^j_t) = 0
\]

(4.11)

of a Lagrange equation and an additional non-Lagrangian external force. Let us substitute \( E_i = -f_i \) from this equality in the first variational formula (4.8). Then we have the weak transformation law

\[
(v^i - q^i_t v^j) f_i \approx d_i J_\upsilon
\]

of the symmetry current \( J_\upsilon \) (4.10) on the shell (4.11).

It is readily observed that the first variational formula (4.4) is linear in a vector field \( \upsilon \). Therefore, one can consider superposition of the equalities (4.4) for different vector fields.

For instance, if \( \upsilon \) and \( \upsilon' \) are projectable vector fields (4.1), they are projected onto the standard vector field \( \partial_t \) on \( \mathbb{R} \), and the difference of the corresponding equalities (4.4) results in the first variational formula (4.4) for a vertical vector field \( \upsilon - \upsilon' \).

Conversely, every vector field \( \upsilon \) (4.11), projected onto \( \partial_t \), can be written as a sum

\[
\upsilon = \Gamma + \upsilon
\]

(4.12)
of some reference frame (2.10):

\[ \Gamma = \partial_t + \Gamma^i \partial_i, \]  

(4.13)

and a vertical vector field \( v \) on \( Q \to \mathbb{R} \).

It follows that the first variational formula (4.4) for the vector field \( v \) (4.1) can be represented as a superposition of those for the reference frame \( \Gamma \) (4.13) and a vertical vector field \( v \).

If \( v = v \) is a vertical vector field, the first variational formula (4.4) reads

\[ (v^i \partial_i + d_t v^i \partial_i^t) \mathcal{L} = v^i \mathcal{E}_i + d_t (\pi_i v^i). \]

If \( v \) is an exact symmetry of \( L \), we obtain from (4.9) the Noether conservation law

\[ 0 \approx d_t (\pi_i v^i) \]

of the Noether current

\[ \mathcal{J}_v = -\pi_i v^i, \]  

(4.14)

which is a Lagrangian integral of motion by virtue of Theorem 4.3.

**Remark 4.3.** Let us assume that, given a trivialization \( Q = \mathbb{R} \times M \) in bundle coordinates \((t, q^i)\), a Lagrangian \( L \) is independent of some coordinate \( q^a \). Then a vertical vector field \( v = \partial_t \) is an exact symmetry of \( L \), and we have the conserved Noether current \( \mathcal{J}_v = -\pi_i \) (4.14) which is an integral of motion.

In the case of the reference frame \( \Gamma \) (4.13), where \( v^t = 1 \), the first variational formula (4.4) reads

\[ (\partial_t + \Gamma^i \partial_i + d_t \Gamma^i \partial_i^t) \mathcal{L} = (\Gamma^i - q_i^t) \mathcal{E}_i - d_t (\pi_i (q_i^t - \Gamma^i)) - \mathcal{L}, \]  

(4.15)

where

\[ E_\Gamma = \mathcal{J}_\Gamma = \pi_i (q_i^t - \Gamma^i) - \mathcal{L} \]  

(4.16)

is called the energy function relative to a reference frame \( \Gamma \) [10, 22].

With respect to the coordinates \( q_i^t \) adapted to a reference frame \( \Gamma \), the first variational formula (4.15) takes a form

\[ \partial_t \mathcal{L} = -q_i^t \mathcal{E}_i - d_t (\pi_i q_i^t - \mathcal{L}), \]  

(4.17)

and the \( E_\Gamma \) (4.16) coincides with the canonical energy function

\[ E_L = \pi_i q_i^t - \mathcal{L}. \]  

(4.18)

A glance at the expression (4.17) shows that the vector field \( \Gamma \) (4.13) is an exact symmetry of a Lagrangian \( L \) if and only if, written with respect to coordinates adapted to \( \Gamma \), this Lagrangian is independent on the time \( t \). In this case, the energy function \( E_\Gamma \) (4.17) relative to a reference frame \( \Gamma \) is conserved:

\[ 0 \approx -d_t E_\Gamma. \]  

(4.19)

It is a Lagrangian integral of motion in accordance with Theorem 4.3.

Since any vector field \( v \) (4.1) can be represented as the sum (4.12) of the reference frame \( \Gamma \) (4.13) and a vertical vector field \( v \), the symmetry current (4.10) along the vector field \( v \) (4.1) is a sum

\[ \mathcal{J}_v = E_\Gamma + \mathcal{J}_v \]
of the Noether current $\mathcal{J}_v$ (4.14) along a vertical vector field $v$ and the energy function $E_\Gamma$ (4.16) relative to a reference frame $\Gamma$. Conversely, energy functions relative to different reference frames $\Gamma$ and $\Gamma'$ differ from each other in the Noether current (4.14) along a vertical vector field $\Gamma' - \Gamma$:

\[ E_\Gamma - E_{\Gamma'} = \pi_i (\Gamma'^i - \Gamma^i) = \mathcal{J}_{\Gamma - \Gamma'}. \] (4.20)

**Example 4.4.** Given a configuration space $Q$ of a mechanical system and the connection $\Gamma$ (4.13) on $Q \to \mathbb{R}$, let us consider a quadratic Lagrangian

\[ L = \frac{1}{2} m_{ij}(t, q^k)(\dot{q}_i^\Gamma - \Gamma^i)(\dot{q}_j^\Gamma - \Gamma^j) dt, \] (4.21)

where $m_{ij}$ is a non-degenerate positive-definite fibre metric in the vertical tangent bundle $VQ \to Q$. It is called the mass tensor. Such a Lagrangian is globally defined owing to the linear transformation laws of the relative velocities $\dot{q}_i^\Gamma$ (2.13). Let $q_i^\Gamma$ be fibre coordinates adapted to a reference frame $\Gamma$. Then the Lagrangian (4.21) reads

\[ L = \frac{1}{2} m_{ij}(q^k)q_i^\Gamma \dot{q}_j^\Gamma dt. \] (4.22)

Since coordinates $q_i^\Gamma$ possess time-independent transition functions, let us assume that a mass tensor $\pi_{ij}$ is independent of time. It is readily observed that, in this case, a horizontal vector field $\Gamma \partial_t = \partial_t = \partial_t$ is an exact symmetry of the Lagrangian $L$ (4.22) that leads to a weak conservation law of the canonical energy function (4.18):

\[ E_L = \frac{1}{2} m_{ij}(q^k)q_i^\Gamma \dot{q}_j^\Gamma dt. \] (4.23)

With respect to arbitrary bundle coordinates $(t, q^i)$ on $Q$, the energy function (4.23) takes a form

\[ E_\Gamma = \frac{1}{2} m_{ij}(t, q^k)(\dot{q}_i^\Gamma - \Gamma^i)(\dot{q}_j^\Gamma - \Gamma^j). \]

This is an energy function relative to a reference frame $\Gamma$.

**Example 4.5.** Let us consider a one-dimensional motion of a point mass $m_0$ subject to friction. It is described by the dynamic equation

\[ m_0 \ddot{q}_t = -k \dot{q}_t, \quad k > 0, \]

on a configuration space $\mathbb{R}^2 \to \mathbb{R}$ coordinated by $(t, q)$. This equation is a Lagrange equation of a Lagrangian

\[ L = \frac{1}{2} m_0 \exp \left[ \frac{k}{m_0} t \right] q_t^2 dt, \]

termed the Havas Lagrangian [10, 21]. It is readily observed that the Lie derivative of this Lagrangian along a vector field

\[ \Gamma = \partial_t - \frac{1}{2} \frac{k}{m_0} q \partial_q \] (4.24)

vanishes. Consequently, we have the conserved energy function (4.16) with respect to the reference frame $\Gamma$ (4.24). This energy function reads

\[ E_{\Gamma} = \frac{1}{2} m_0 \exp \left[ \frac{k}{m_0} t \right] q_t \left( q_t + \frac{k}{m_0} q \right). \]

In Section 9, an example of the global Kepler problem is analyzed in details.
5 Noether’s third theorem: Gauge symmetries

In mechanics, we follow general definition of gauge symmetries of Lagrangian theory on fibre bundles \([9, 24, 29]\). It is given by a vector field

\[
u = \left( \sum_{0 \leq |\lambda| \leq m} u^{i A}_{\lambda} (t, q^i) \chi^a_{\lambda} \right) \partial_i.
\] (5.1)
on a configuration space \(Q\) which depends on sections \(\chi\) of some vector bundle \(E \to \mathbb{R}\).

If a Lagrangian \(L\) admits the gauge symmetry \(\nu\), the weak conservation law (4.4) of the corresponding symmetry current \(\mathcal{J}_u\) holds. Because gauge symmetries depend on derivatives of gauge parameters, all gauge conservation laws in first order Lagrangian mechanics possess the following peculiarity.

**Theorem 5.1.** If \(u\) is a gauge symmetry of a first order Lagrangian \(L\), the corresponding symmetry current \(\mathcal{J}_u\) vanishes on-shell, i.e., \(\mathcal{J} \approx 0\).

**Proof.** Let a gauge symmetry \(u\) be at most of jet order \(N\) in gauge parameters. Then the current \(\mathcal{J}_u\) is decomposed into a sum

\[
\mathcal{J}_u = \sum_{1 \leq |\lambda| \leq N} J^A_{\lambda} \chi^a_{\lambda} + J^I_{\lambda} \chi^a_{\lambda} + J_a \chi^a.
\] (5.2)

The first variational formula (4.4) takes a form

\[
0 = \left[ \sum_{|\lambda| = 1}^N u^{i A}_{\lambda} \chi^a_{\lambda} + u^i \chi^a \right] \mathcal{E}_i - \left( \sum_{|\lambda| = 1}^N J^A_{\lambda} \chi^a + J_a \chi^a \right).
\]

It falls into a set of equalities for each \(\chi^a_{\lambda}, \chi^a, |\lambda| = 1, \ldots, N\), and \(\chi^a\) as follows:

\[
\begin{align*}
0 &= J^A_{\lambda}, \quad |\lambda| = N, \\
0 &= -u^i u^i \mathcal{E}_i + J^A_{\lambda} + dt J^A_{\lambda}, \quad 1 \leq |\lambda| < N, \\
0 &= -u^i \mathcal{E}_i + J_a + dt J_a, \\
0 &= -u^i \mathcal{E}_i + dt J_a.
\end{align*}
\] (5.3) - (5.6)

With the equalities (5.3) - (5.6), the decomposition (5.2) takes a form

\[
\mathcal{J}_u = \sum_{1 \leq |\lambda| < N} \left[ (u^i u^i \mathcal{E}_i - dt J^A_{\lambda}) \chi^a_{\lambda} + (u^i u^i \mathcal{E}_i - dt J^A_{\lambda}) \chi^a_{\lambda} + (u^i u^i \mathcal{E}_i - dt J^A_{\lambda}) \chi^a_{\lambda} \right].
\]

A direct computation leads to the expression

\[
\mathcal{J}_u = \left( \sum_{|\lambda| \leq N} v^i u^i \chi^a_{\lambda} + v^i u^i \chi^a_{\lambda} \right) \mathcal{E}_i - \left( \sum_{|\lambda| \leq N} dt J^A_{\lambda} \chi^a_{\lambda} + dt J^A_{\lambda} \chi^a_{\lambda} \right).
\] (5.7)

The first summand of this expression vanishes on-shell. Its second one contains the terms \(dt J^A_{\lambda}, |\lambda| = 1, \ldots, N\). By virtue of the equalities (5.4), every \(dt J^A_{\lambda}, |\lambda| < N\), is expressed in the
terms vanishing on-shell and the term $d_i d_t J^{\Lambda}_a$. Iterating the procedure and bearing in mind the equality (5.3), one can easily show that the second summand of the expression (5.7) also vanishes on-shell. Thus, a current $J_a$ vanishes on-shell. □

Let us note that the statement of Theorem 5.1 is a particular case of so-called Noether’s third theorem that currents of gauge symmetries in Lagrangian theory are reduced to a superpotential [6, 9, 11, 24] because a superpotential equals zero on $X = \mathbb{R}$.

6 Non-autonomous Hamiltonian mechanics

As was mentioned above, a Hamiltonian formulation of non-autonomous non-relativistic mechanics is similar to covariant Hamiltonian field theory on fibre bundles [9, 30, 32] in the particular case of fibre bundles over $\mathbb{R}$ [10, 22, 28, 31].

In accordance with the Legendre map (3.5) and the homogeneous Legendre map (3.10), a phase space and a homogeneous phase space of mechanics on a configuration bundle $Q \to \mathbb{R}$ are the vertical cotangent bundle $V^*Q$ and the cotangent bundle $T^*Q$ of $Q$, respectively.

A key point is that a non-autonomous Hamiltonian system of $k$ degrees of freedom on a phase space $V^*Q$ is equivalent both to some autonomous symplectic Hamiltonian system of $k + 1$ degrees of freedom on a homogeneous phase space $T^*Q$ (Theorem 6.1) and to a particular first order Lagrangian system with the characteristic Lagrangian (6.22) on a configuration space $V^*Q$.

The cotangent bundle $T^*Q$ is endowed with holonomic coordinates $(t, q^i, p_0, p_i)$, possessing the transition functions (3.12). It admits the canonical Liouville form (8.8):

$$\Xi_T = p_0 dt + p_i dq^i,$$

the canonical symplectic form (8.9):

$$\Omega_T = d\Xi_T = dp_0 \wedge dt + dp_i \wedge dq^i,$$

and the corresponding canonical Poisson bracket (8.11):

$$\{f, g\}_T = \partial^0 f \partial g - \partial^0 g \partial f + \partial^i f \partial_i g - \partial^i g \partial_i f, \quad f, g \in C^\infty(T^*Q).$$

There is the canonical one-dimensional affine bundle (3.13):

$$\zeta : T^*Q \to V^*Q.$$ (6.4)

A glance at the transformation law (3.12) shows that it is a trivial affine bundle. Indeed, given a global section $h$ of $\zeta$, one can equip $T^*Q$ with a global fibre coordinate

$$I_0 = p_0 - h, \quad I_0 \circ h = 0,$$

possessing the identity transition functions. With respect to coordinates

$$(t, q^i, I_0, p_i), \quad i = 1, \ldots, m,$$ (6.6)

the fibration (6.4) reads

$$\zeta : \mathbb{R} \times V^*Q \ni (t, q^i, I_0, p_i) \to (t, q^i, p_i) \in V^*Q,$$
where \((t, q^i, p_i)\) are holonomic coordinates on the vertical cotangent bundle \(V^*Q\).

Let us consider a subring of \(C^\infty(T^*Q)\) which comprises the pull-back \(\zeta^* f\) onto \(T^*Q\) of functions \(f\) on the vertical cotangent bundle \(V^*Q\) by the fibration \(\zeta\) (6.4). This subring is closed under the Poisson bracket (6.3). Then by virtue of the well known theorem \([10, 34]\), there exists a degenerate coinduced Poisson bracket
\[
\{ f, g \}_V = \partial_i f \partial_i g - \partial_i g \partial_i f, \quad f, g \in C^\infty(V^*Q),
\]
on a phase space \(V^*Q\) such that
\[
\zeta^* \{ f, g \} = \{ \zeta^* f, \zeta^* g \}_T.
\]
Holonomic coordinates on \(V^*Q\) are canonical for the Poisson structure (6.7).

With respect to the Poisson bracket (6.7), the Hamiltonian vector fields of functions on \(V^*Q\) read
\[
\vartheta_f = \partial_i f \partial_i - \partial_i f \partial_i, \quad f \in C^\infty(V^*Q),
\]
\[
\{ f, f' \}_V = \vartheta_f df', \quad [\vartheta_f, \vartheta_{f'}] = \vartheta_{\{ f, f' \}}_V.
\]
They are vertical vector fields on \(V^*Q \to \mathbb{R}\). Accordingly, the characteristic distribution of the Poisson structure (6.7) is the vertical tangent bundle \(VV^*Q \subset TV^*Q\) of a fibre bundle \(V^*Q \to \mathbb{R}\). The corresponding symplectic foliation on a phase space \(V^*Q\) coincides with the fibration \(V^*Q \to \mathbb{R}\).

However, the Poisson structure (6.7) fails to provide any dynamic equation on a phase space \(V^*Q \to \mathbb{R}\) because Hamiltonian vector fields (6.9) of functions on \(V^*Q\) are vertical vector fields. Hamiltonian dynamics on \(V^*Q\) is described as a particular Hamiltonian dynamics on fibre bundles \([10, 22, 31]\).

A Hamiltonian on a phase space \(V^*Q \to \mathbb{R}\) is defined as a global section
\[
h : V^*Q \to T^*Q, \quad p_0 \circ h = \mathcal{H}(t, q^i, p_i),
\]
of the affine bundle \(\zeta\) (6.4). Given the Liouville form \(\Xi_T\) (6.1) on \(T^*Q\), this section yields the pull-back Hamiltonian form
\[
H = (-h)^* \Xi_T = p_k dq^k - \mathcal{H} dt
\]
on \(V^*Q\). This is the well-known invariant of Poincaré–Cartan \([10]\).

It should be emphasized that, in contrast with a Hamiltonian in autonomous mechanics, the Hamiltonian \(\mathcal{H}\) (6.11) is not a function on \(V^*Q\), but it obeys the transformation law
\[
\mathcal{H}'(t, q^n, p'_i) = \mathcal{H}(t, q^i, p_i) + p'_i \partial_i q^n.
\]

**Remark 6.1.** Any connection \(\Gamma\) (2.10) on a configuration bundle \(Q \to \mathbb{R}\) defines the global section \(h_\Gamma = p_i \Gamma^i\) (6.11) of the affine bundle \(\zeta\) (6.4) and the corresponding Hamiltonian form
\[
H_\Gamma = p_k dq^k - \mathcal{H}_\Gamma dt = p_k dq^k - p_i \Gamma^i dt.
\]
Furthermore, given a connection \(\Gamma\), any Hamiltonian form (6.12) admits a splitting
\[
H = H_\Gamma - \mathcal{E}_\Gamma dt,
\]
where
\[ E_\Gamma = \mathcal{H} - H_\Gamma = \mathcal{H} - p_i \Gamma^i \]  \hspace{1cm} (6.15)
is a function on \( V^*Q \). It is called the Hamiltonian function relative to a reference frame \( \Gamma \). With respect to the coordinates adapted to a reference frame \( \Gamma \), we have \( E_\Gamma = \mathcal{H} \). Given different reference frames \( \Gamma \) and \( \Gamma' \), the decomposition (6.14) leads at once to a relation
\[ E_{\Gamma'} = E_\Gamma + \mathcal{H}_{\Gamma} - \mathcal{H}_{\Gamma'} = E_\Gamma + (\Gamma^i - \Gamma'^i)p_i \]  \hspace{1cm} (cf. (4.20))
between the Hamiltonian functions with respect to different reference frames.

Given the Hamiltonian form \( H \) (6.12), there exists a unique Hamiltonian connection
\[ \gamma_H = \partial_t + \partial^k \mathcal{H} \partial_k - \partial_k \mathcal{H} \partial^k \]  \hspace{1cm} (6.16)
on \( V^*Q \rightarrow \mathbb{R} \) such that \( \gamma_H \rceil \text{d}H = 0 \). It yields a first order dynamic Hamilton equation
\[ q^k_t = \partial^k \mathcal{H}, \quad p_k = -\partial_k \mathcal{H} \]  \hspace{1cm} (6.17)
on \( V^*Q \rightarrow \mathbb{R} \), where \((t, q^k, p_k, q^k, p_k)\) are adapted coordinates on the first order jet manifold \( J^1V^*Q \) of \( V^*Q \rightarrow \mathbb{R} \).

A classical solution of the Hamilton equation (6.17) is an integral section \( r \) for the connection \( \gamma_H \) (6.16).

We agree to call \((V^*Q, H)\) the Hamiltonian system of \( k = \dim Q - 1 \) degrees of freedom.

In order to describe evolution of a Hamiltonian system at any instant, the Hamilton connection \( \gamma_H \) (6.16) is assumed to be complete, i.e., it is an Ehresmann connection (Remark 2.2). In this case, the Hamilton equation (6.17) admits a unique global classical solution through each point of a phase space \( V^*Q \). By virtue of Theorem 2.2 there exists a trivialization of a fibre bundle \( V^*Q \rightarrow \mathbb{R} \) (not necessarily compatible with its fibration \( V^*Q \rightarrow Q \)) such that
\[ \gamma_H = \partial_t, \quad H = \bar{p}_i d\bar{q}^i \]with respect to the associated bundle coordinates \((t, \bar{q}^i, \bar{p}_i)\). A direct computation shows that the Hamilton vector field \( \gamma_H \) (6.16) is an infinitesimal generator of a one-parameter group of automorphisms of a Poisson manifold \((V^*Q, \{,\}_V)\). Then one can show that \((t, \bar{q}^i, \bar{p}_i)\) are canonical coordinates for the Poisson bracket \( \{,\}_V \) [16]. Since \( \mathcal{H} = 0 \), the Hamilton equation (6.17) in these coordinates takes a form
\[ \bar{q}^i_t = 0, \quad \bar{p}_ti = 0, \]i.e., \((t, \bar{q}^i, \bar{p}_i)\) are the initial data coordinates.

**Remark 6.2.** In applications, the condition of the Hamilton connection \( \gamma_H \) (6.16) to be complete need not holds on the entire phase space (Section 9). In this case, one consider its subsets, and sometimes we have different Hamiltonian systems on different subsets of \( V^*Q \).

As was mentioned above, one can associate to any non-autonomous Hamiltonian system on a phase space \( V^*Q \) an equivalent autonomous symplectic Hamiltonian system on the cotangent bundle \( T^*Q \) as follows (Theorem 6.1).
Given a Hamiltonian system \((V^\ast Q, H)\), its Hamiltonian \(H\) \((6.11)\) defines a function
\[
H^* = \partial_j (\Xi_T - \zeta^*(\langle -h \rangle^* \Xi_T)) = p_0 + h = p_0 + H
\]
on \(T^\ast Q\). Let us regard \(H^*\) \((6.18)\) as a Hamiltonian of an autonomous Hamiltonian system on a symplectic manifold \((T^\ast Q, \Omega_T)\). The corresponding autonomous Hamilton equation on \(T^\ast Q\) takes a form
\[
i = 1, \quad \dot{p}_0 = -\partial_t H, \quad \dot{q}^i = \partial^i H, \quad \dot{p}_i = -\partial_i H. \quad (6.19)
\]

**Remark 6.3.** Let us note that the splitting \(H^* = p_0 + H\) \((6.18)\) is ill defined. At the same time, any reference frame \(\Gamma\) yields the decomposition
\[
H^* = (p_0 + H_\Gamma) + (H - H_\Gamma) = H^*_\Gamma + \mathcal{E}_\Gamma,
\]
where \(H_\Gamma\) is the Hamiltonian \((6.13)\) and \(\mathcal{E}_\Gamma\) \((6.15)\) is the Hamiltonian function relative to a reference frame \(\Gamma\).

A Hamiltonian vector field \(\vartheta_{H^*}\) of the function \(H^*\) \((6.18)\) on \(T^\ast Q\) is
\[
\vartheta_{H^*} = \partial_t - \partial_i H \partial^0 + \partial^i H \partial_i - \partial_i H \partial^i, \quad \vartheta_{H^*} \Omega_T = -dH^*.
\]
Written relative to the coordinates \((6.6)\), this vector field reads
\[
\vartheta_{H^*} = \partial_t + \partial^i H \partial_i - \partial_i H \partial^i. \quad (6.20)
\]
It is identically projected onto the Hamiltonian connection \(\gamma_H\) \((6.16)\) on \(V^\ast Q\) such that
\[
\zeta^*(L_{\gamma_H} f) = \{H^*, \zeta^* f\}_T, \quad f \in C^\infty(V^\ast Q). \quad (6.21)
\]
Therefore, the Hamilton equation \((6.17)\) is equivalent to the autonomous Hamilton equation \((6.19)\).

Obviously, the Hamiltonian vector field \(\vartheta_{H^*}\) \((6.20)\) is complete if the Hamilton vector field \(\gamma_H\) \((6.16)\) is so.

Thus, the following has been proved \([3, 10, 17]\).

**Theorem 6.1.** A non-autonomous Hamiltonian system \((V^\ast Q, H)\) of \(k\) degrees of freedom is equivalent to an autonomous Hamiltonian system \((T^\ast Q, H^*)\) of \(k + 1\) degrees of freedom on a symplectic manifold \((T^\ast Q, \Omega_T)\) whose Hamiltonian is the function \(H^*\) \((6.18)\).

We agree to call \((T^\ast Q, H^*)\) the homogeneous Hamiltonian system and \(H^*\) \((6.18)\) the homogeneous Hamiltonian.

It is readily observed that the Hamiltonian form \(H\) \((6.12)\) also is the Poincaré–Cartan form \([8, 9]\) of the characteristic Lagrangian
\[
L_H = h_0(H) = (p_i q^i - H)dt \quad (6.22)
\]
on the jet manifold \(J^1 V^\ast Q\) of \(V^\ast Q \to \mathbb{R}\).

**Remark 6.4.** In fact, the Lagrangian \((6.22)\) is the pull-back onto \(J^1 V^\ast Q\) of an exterior form \(L_H\) on a product \(V^\ast Q \times Q J^1 Q\).
The Lagrange operator \( \mathcal{E}_H = \delta L_H \) associated to the characteristic Lagrangian \( L_H \) reads

\[
\mathcal{E}_H = \delta L_H = \left[(q_i - \partial^i H)dp_i - (p_i + \partial_i H)dq_i\right] \wedge dt.
\]

The corresponding Lagrange equation (3.4) is of first order, and it coincides with the Hamilton equation \( (6.17) \) on \( V^*Q \).

Due to this fact, Hamiltonian mechanics can be formulated as a specific Lagrangian mechanics on a configuration space \( V^*Q \).

In particular, let

\[
u = u^i \partial_t + u^i \partial_q^i, \quad u^t = 0,\]

be a vector field on a configuration space \( Q \). Its canonical functorial lift onto the cotangent bundle \( T^*Q \) is

\[
\tilde{\nu} = u^t \partial_t + u^i \partial_i - p_j \partial_q^i u^j.
\]

This vector field is identically projected onto a vector field, also given by the expression \( (6.24) \), on a phase space \( V^*Q \) as a base of the trivial fibre bundle \( (6.4) \). Then we have the equality

\[
L_{\tilde{\nu}}H = L_{\tilde{\nu}H} = (\partial_t H + p_i \partial_i u^t - u^i \partial_i H + p_i \partial_j u^i \partial^j H)dt.
\]

This equality enables us to study conservation laws in Hamiltonian mechanics similarly to those in Lagrangian mechanics (Section 7).

Lagrangian and Hamiltonian formulations of mechanics as like as those of field theory fail to be equivalent, unless Lagrangians are hyperregular \([9, 30, 32]\). The comprehensive relations between Lagrangian and Hamiltonian systems can be established in the case of almost regular Lagrangians \([10, 17, 22]\). This is a particular case of the relations between Lagrangian and covariant Hamiltonian theories on fibre bundles \([9, 30]\).

If the Lagrangian \( L \) (Definition 3.1) is hyperregular, it admits a unique associated Hamiltonian form

\[
H = p_i dq^i - (p_i \hat{L}^{-1i} - L(t, q^j, \hat{L}^{-1j}))dt.
\]

Let \( s \) be a classical solution of the Lagrange equation (3.4) for a Lagrangian \( L \). A direct computation shows that \( \hat{H} \circ J^1s \) is a classical solution of the Hamilton equation \( (6.17) \) for the Hamiltonian form \( H \). Conversely, if \( r \) is a classical solution of the Hamilton equation \( (6.17) \) for the Hamiltonian form \( H \), then \( s = \pi_H \circ r \) is a solution of the Lagrange equation (3.4) for \( L \).

Let us restrict our consideration to almost regular Lagrangians \( L \) (Definition 3.1).

**Theorem 6.2.** Let a section \( r \) of \( V^*Q \to \mathbb{R} \) be a classical solution of the Hamilton equation \( (6.17) \) for a Hamiltonian form \( H \) weakly associated to an almost regular Lagrangian \( L \). If \( r \) lives in the Lagrangian constraint space \( N_L \), a section \( s = \pi_H \circ r \) of \( \pi : Q \to \mathbb{R} \) satisfies the Lagrange equation (3.4), while \( \pi = \hat{H} \circ r \), where

\[
\hat{H} : V^*Q \to J^1Q, \quad q_i \circ \hat{H} = \partial^i H
\]

is a Hamiltonian map, obeys the Cartan equation (3.15).

**Theorem 6.3.** Given an almost regular Lagrangian \( L \), let a section \( \pi \) of the jet bundle \( J^1Q \to \mathbb{R} \) be a solution of the Cartan equation (3.15). Let \( H \) be a Hamiltonian form weakly associated to \( L \), and let \( H \) satisfy a relation

\[
\hat{H} \circ \hat{L} \circ \pi = J^1s,
\]

for any Hamiltonian form \( H \). This equality enables us to study conservation laws in Hamiltonian mechanics similarly to those in Lagrangian mechanics (Section 7).
where $s$ is the projection of $\pi$ onto $Q$. Then a section $r = \hat{L} \circ \pi$ of a fibre bundle $V^*Q \to \mathbb{R}$ is a classical solution of the Hamilton equation (6.17) for $H$.

A set of Hamiltonian forms $H$ weakly associated to an almost regular Lagrangian $L$ is said to be complete if, for each classical solution $s$ of a Lagrange equation, there exists a classical solution $r$ of a Hamilton equation for a Hamiltonian form $H$ from this set such that $s = \pi_H \circ r$. By virtue of Theorem 6.3, a set of weakly associated Hamiltonian forms is complete if, for every classical solution $s$ of a Lagrange equation for $L$, there exists a Hamiltonian form $H$ from this set which fulfills the relation \( \pi = J^1 s \), i.e.,
\[
\hat{H} \circ \hat{L} \circ J^1 s = J^1 s.
\]

(6.27)

7 Hamiltonian conservation laws: Noether’s inverse first theorem

As was mentioned above, integrals of motion in Lagrangian mechanics can come from Lagrangian symmetries (Theorem 4.3), but not any integral of motion is of this type. In Hamiltonian mechanics, all integrals of motion are conserved currents (Theorem 7.7). One can think of this fact as being Noether’s inverse first theorem.

**Definition 7.1.** An integral of motion of a Hamiltonian system \((V^*Q, H)\) is defined as a smooth real function $\Phi$ on $V^*Q$ which is an integral of motion of the Hamilton equation (6.17) in accordance with Definition 3.2, i.e., it satisfies the relation (3.19).

Since the Hamilton equation (6.17) is the kernel of the covariant differential $D_{\gamma H}$, this relation $d_t \Phi \approx 0$ is equivalent to the equality
\[
L_{\gamma_H} \Phi = (\partial_i + \gamma_i^j \partial_i + \gamma_H \partial^j) \Phi = \partial_i \Phi + \{H, \Phi\}_V = 0,
\]
(7.1)
i.e., the Lie derivative of $\Phi$ along the Hamilton connection $\gamma_H$ (6.16) vanishes.

At the same time, it follows from Theorem 3.4 that a vector field $u$ on $V^*Q$ is a symmetry of the Hamilton equation (6.17) in accordance with Definition 3.3 if and only if $[\gamma_H, u] = 0$. Given the Hamiltonian vector field $\vartheta_\Phi$ (6.9) of $\Phi$ with respect to the Poisson bracket (6.7), it is easily justified that
\[
[\gamma_H, \vartheta_\Phi] = \vartheta_{L_{\gamma_H} \Phi}.
\]
(7.2)

Thus, we can conclude the following.

**Theorem 7.2.** The Hamiltonian vector field of an integral of motion is a symmetry of the Hamilton equation (6.17).

Given a Hamiltonian system \((V^*Q, H)\), let \((T^*Q, \mathcal{H}^*)\) be an equivalent homogeneous Hamiltonian system. It follows from the equality (6.21) that
\[
\zeta^*(L_{\gamma_H} \Phi) = \{\mathcal{H}^*, \zeta^* \Phi\}_T = \zeta^*(\partial_i \Phi + \{\mathcal{H}, \Phi\}_V)
\]
(7.3)
for any function $\Phi \in C^\infty(V^*Q)$. This formula is equivalent to that (7.1).

**Theorem 7.3.** A function $\Phi \in C^\infty(V^*Q)$ is an integral of motion of a Hamiltonian system \((V^*Q, H)\) if and only if its pull-back $\zeta^* \Phi$ onto $T^*Q$ is an integral of motion of a homogeneous Hamiltonian system \((T^*Q, \mathcal{H}^*)\).
Proof. The result follows from the equality (7.3):

\[ \{ \mathcal{H}^\ast, \zeta^\ast \Phi \}_T = \zeta^\ast (L_H \Phi) = 0. \] (7.4)

\[ \square \]

**Theorem 7.4.** If \( \Phi \) and \( \Phi' \) are integrals of motion of a Hamiltonian system, their Poisson bracket \( \{ \Phi, \Phi' \}_V \) also is an integral of motion.

**Proof.** This fact results from the equalities (6.10) and (7.4).

\[ \square \]

Consequently, integrals of motion of a Hamiltonian system \((V^*Q, H)\) constitute a real Lie subalgebra of a Poisson algebra \(C^\infty(V^*Q)\).

Let us turn to Hamiltonian conservation laws. We are based on the fact that the Hamilton equation (6.17) also is a Lagrange equation of the characteristic Lagrangian \(L_H (6.22)\). Therefore one can study conservation laws in Hamiltonian mechanics on a phase space \(V^*Q\) similarly to those in Lagrangian mechanics on a configuration space \(V^*Q\) [10, 18, 32].

Since the Hamilton equation (6.17) is of first order, we restrict our consideration to classical symmetries, i.e., vector fields on \(V^*Q\).

**Definition 7.5.** A vector field on a phase space \(V^*Q\) of a Hamiltonian system \((V^*Q, H)\) is said to be its Hamiltonian symmetry if it is a Lagrangian symmetry of the characteristic Lagrangian \(L_H\).

Let \(\upsilon = \upsilon^t \partial_t + \upsilon^i \partial_i + \upsilon^t H, \upsilon^t = 0, 1,\) \(7.5)\) be a vector field on a phase space \(V^*Q\). Its prolongation onto \(V^*Q \times Q J^1Q\) (Remark 6.1) reads

\[ J^1\upsilon = \upsilon^t \partial_t + \upsilon^i \partial_i + \upsilon^i \partial^t + dt \upsilon^t \partial_t. \]

Then the first variational formula (4.4) for the characteristic Lagrangian \(L_H (6.22)\) takes a form

\[ -\upsilon^t \partial_t H - \upsilon^i \partial_i H + \upsilon_i (q^i_t - \partial^t H) + p_i dt \upsilon^t = (q^t v^t - v^t) (p_i + \partial_i H) + (v_i - p_i v^t) (q^t_i - \partial^i H) + dt (p_i v^i - v^i H). \] (7.6)

If \(\upsilon\) \((7.5)\) is a symmetry of \(L_H\), i.e.,

\[ L_{J^1\upsilon}L_H = d_0 \sigma dt, \]

we obtain the weak Hamiltonian conservation law \(4.6)\):

\[ 0 \approx -d_0 J \] (7.7)

of the Hamiltonian symmetry current \(167)\):

\[ J_\upsilon = -p_i v^i + v^t H + \sigma. \] (7.8)

The vector field \(\upsilon\) \((7.5)\) on \(V^*Q\) is a symmetry of the characteristic Lagrangian \(L_H (6.22)\) if and only if

\[ v^t (p_i + \partial_i H) - \upsilon_i (q^t_i - \partial^t H) + v^i \partial_i H = dt (-J_\upsilon + v^t H). \] (7.9)
A glance at this equality shows the following.

**Theorem 7.6.** The vector field \( v \) (7.5) is a Hamiltonian symmetry in accordance with Definition 7.5 only if

\[
\partial^i v_i = -\partial^i v^i. \tag{7.10}
\]

**Remark 7.1.** It is readily observed that the Hamiltonian connection \( \gamma_H \) (6.16) is a symmetry of the characteristic Lagrangian \( L_H \) whose conserved Hamiltonian current (7.8) equals zero. It follows that, given a non-vertical Hamiltonian symmetry \( v, v^t = 1 \), there exists a vertical Hamiltonian symmetry \( \gamma_H \) with the same conserved Hamiltonian current as \( v \).

By virtue of Theorem 4.4, any Hamiltonian symmetry, being classical symmetry of the characteristic Lagrangian \( L_H \) (6.22), also is symmetry of the Hamilton equation (6.17). In accordance with Theorem 4.3, the corresponding conserved Hamiltonian current (7.8) is an integral of motion of a Hamiltonian system which, thus, comes from its Hamiltonian symmetry.

The converse also is true.

**Theorem 7.7.** Any integral of motion \( \Phi \) of a Hamiltonian system \((V^*Q, H)\) is the conserved Hamiltonian current \( J_{-\partial_\Phi} \) (7.8) along the Hamiltonian vector field \(-\partial_\Phi\) (6.9) of \(-\Phi\).

**Proof.** It follows from the relations (7.1) and (7.6) that

\[
L_{-J_{\partial_\Phi}} = d_t(\Phi - p_i \partial_i \Phi). \tag{7.11}
\]

Then the equality (7.8) results in a desired relation \( \Phi = J_{-\partial_\Phi} \). \(\square\)

This assertion can be regarded as above mentioned Noether’s inverse first theorem.

For instance, if the Hamiltonian symmetry \( v \) (7.5) is projectable onto \( Q \) (i.e., its components \( v^i = u^i \) are independent of momenta \( p_i \)), then we \( v_i = -p_j \partial_j u^i \) in accordance with the equality (7.10). Consequently, \( v \) is the canonical lift \( \tilde{u} \) (6.24) onto \( V^*Q \) of the vector field \( u \) (6.23) on \( Q \). If \( \tilde{u} \) is a symmetry of the characteristic Lagrangian \( L_H \), it follows at once from the equality (7.9) that \( \tilde{u} \) is an exact symmetry of \( L_H \). The corresponding conserved Hamiltonian symmetry current (7.8) reads

\[
\tilde{J}_{\tilde{u}} = J_{\tilde{u}} = -p_i u^i + u^t \mathcal{H}. \tag{7.12}
\]

**Definition 7.8.** The vector field \( u \) (6.23) on a configuration space \( Q \) is said to be the basic Hamiltonian symmetry if its canonical lift \( \tilde{u} \) (6.24) onto \( V^*Q \) is a Hamiltonian symmetry.

If a basic Hamiltonian symmetry \( u \) is vertical, the corresponding conserved Hamiltonian symmetry current (7.11):

\[
\tilde{J}_{\tilde{u}} = \tilde{J}_u = -p_i u^i, \tag{7.12}
\]

is a Noether current.

Now let \( \Gamma \) be the connection (2.10) on \( Q \). The corresponding symmetry current (7.11) is the Hamiltonian function (6.15):

\[
\tilde{J}_{\tilde{\Gamma}} = \tilde{J}_\Gamma = \mathcal{E}_\Gamma = \mathcal{H} - p_i \Gamma^i, \tag{7.13}
\]
relative to a reference frame $\Gamma$. Given bundle coordinates adapted to $\Gamma$, we obtain the Lie derivative

$$L_{\tilde{J}}L_H = -\partial_t H.$$  

It follows that a connection $\Gamma$ is a basic Hamiltonian symmetry if and only if the Hamiltonian $H$ (6.11), written with respect to the coordinates adapted to $\Gamma$, is time-independent. In this case, the Hamiltonian function (7.13) is an integral of motion of a Hamiltonian system.

There is the following relation between Lagrangian symmetries and basic Hamiltonian symmetries if they are the same vector fields on a configuration space $Q$.

**Theorem 7.9.** Let a Hamiltonian form $H$ be associated with an almost regular Lagrangian $L$. Let $r$ be a solution of the Hamilton equation (6.17) for $H$ which lives in the Lagrangian constraint space $N_L$ (3.9). Let $s = \pi_{\Pi} \circ r$ be the corresponding solution of a Lagrange equation for $L$ so that the relation (6.27) holds. Then, for any vector field $u$ (6.25) on a fibre bundle $Q \to \mathbb{R}$, we have

$$\tilde{J}_u(r) = J_u(\pi_{\Pi} \circ r), \quad \tilde{J}_u(\hat{L} \circ J^1 s) = J_u(s),$$  

(7.14)

where $J_u$ is the symmetry current (4.10) on $J^1 Y$ and $\tilde{J}_u = J_{\tilde{u}}$ is the symmetry current (7.11) on $V^* Q$.

By virtue of Theorems 6.2–6.3 it follows that:

- if $J_u$ in Theorem 7.9 is a conserved symmetry current, then the symmetry current $\tilde{J}_u$ (7.14) is conserved on solutions of a Hamilton equation which live in the Lagrangian constraint space;
- if $\tilde{J}_u$ in Theorem 7.9 is a conserved symmetry current, then the symmetry current $J_u$ (7.14) is conserved on solutions $s$ of a Lagrange equation which obey the condition (6.27).

In particular, let $u = \Gamma$ be a connection and $E_{\Gamma}$ the energy function (4.16). Then the relations (7.14):

$$E_{\Gamma}(r) = \tilde{J}_{\Gamma}(r) = J_{\Gamma}(\pi_{\Pi} \circ r) = E_{\Gamma}(\pi_{\Pi} \circ r),$$  

$$E_{\Gamma}(\hat{L} \circ J^1 s) = \tilde{J}_{\Gamma}(\hat{L} \circ J^1 s) = J_{\Gamma}(s) = E_{\Gamma}(s),$$

show that the Hamiltonian function $E_{\Gamma}$ (7.13) can be treated as a Hamiltonian energy function relative to a reference frame $\Gamma$.

8 Completely integrable Hamiltonian systems

It may happen that symmetries and the corresponding integrals of motion define a Hamiltonian system in full. This is the case of commutative and noncommutative completely integrable systems (henceforth, CISs) (Definition 8.1).

In view of Remark 7.1 we can restrict our consideration to vertical symmetries $v^t$ (7.5) where $v^t = 0$.

**Definition 8.1.** A non-autonomous Hamiltonian system $(V^* Q, H)$ of $n = \dim Q - 1$ degrees of freedom is said to be completely integrable if it admits $n \leq k < 2n$ vertical classical symmetries $v_{\alpha}$ which obey the following conditions.

(i) Symmetries $v_{\alpha}$ everywhere are linearly independent.
(ii) They form a $k$-dimensional real Lie algebra $\mathfrak{g}$ of corank $m = 2n - k$ with commutation relations

$$[v_\alpha, v_\beta] = c^\nu_{\alpha\beta} v_\nu.$$  \hfill (8.1)

If $k = n$, then a Lie algebra $\mathfrak{g}$ is commutative, and we are in the case of a commutative CIS. If $n < k$, the Lie algebra (8.1) is noncommutative, and a CIS is called noncommutative or superintegrable.

The conditions of Definition 8.1 can be reformulated in terms of integrals of motion $\Phi_\alpha = -J_{v_\alpha}$ corresponding to symmetries $v_\alpha$. By virtue of Noether’s inverse first Theorem 7.7, $v_\alpha = \partial_{\Phi_\alpha}$ are the Hamiltonian vector fields (6.9) of integrals of motion $\Phi_\alpha$. In accordance with the relation (6.10), integrals of motion obey the commutation relations

$$\{\Phi_\alpha, \Phi_\beta\}_V = c^\nu_{\alpha\beta} \Phi_\nu.$$  \hfill (8.2)

Then we come to an equivalent definition of a CISs [10, 26, 31].

**Definition 8.2.** A non-autonomous Hamiltonian system $(V^*Q, H)$ of $n = \dim Q - 1$ degrees of freedom is a CIS if it possesses $n \leq k < 2n$ integrals of motion $\Phi_1, \ldots, \Phi_k$, obeying the following conditions.

(i) All the functions $\Phi_\alpha$ are independent, i.e., a $k$-form $d\Phi_1 \wedge \cdots \wedge d\Phi_k$ nowhere vanishes on $V^*Q$. It follows that a map

$$\Phi : V^*Q \to N = (\Phi_1(V^*Q), \ldots, \Phi_k(V^*Q)) \subset \mathbb{R}^k$$  \hfill (8.3)

is a fibred manifold over a connected open subset $N \subset \mathbb{R}^k$.

(ii) The commutation relations (8.2) are satisfied.

Given a non-autonomous CIS in accordance with Definition 8.2, the equivalent autonomous Hamiltonian system on a homogeneous phase space $T^*Q$ (Theorem 6.1) possesses $k + 1$ integrals of motion

$$(\mathcal{H}^*, \zeta^*\Phi_1, \ldots, \zeta^*\Phi_k)$$ \hfill (8.4)

with the following properties (Theorem 7.3).

(i) The integrals of motion (8.4) are mutually independent, and a map

$$\tilde{\Phi} : T^*Q \to (\mathcal{H}^*(T^*Q), \zeta^*\Phi_1(T^*Q), \ldots, \zeta^*\Phi_k(T^*Q)) = (I_0, \Phi_1(V^*Q), \ldots, \Phi_k(V^*Q)) = \mathbb{R} \times N = N'$$  \hfill (8.5)

is a fibred manifold.

(ii) The integrals of motion (8.4) obey the commutation relations

$$\{\zeta^*\Phi_\alpha, \zeta^*\Phi_\beta\} = c_{\alpha\beta}^{\nu} \zeta^*\Phi_\nu, \quad \{\mathcal{H}^*, \zeta^*\Phi_\alpha\} = 0.$$  \hfill (8.6)

They generate a real $(k + 1)$ dimensional Lie algebra of corank $2n + 1 - k$.

As a result, integrals of motion (8.4) form an autonomous CIS on a symplectic manifold $(T^*Q, \Omega_T)$ in accordance with Definition 8.3. In order to describe it, one then can follow the Mishchenko–Fomenko theorem [1, 5, 19] extended to the case of noncompact invariant submanifolds [7, 25, 31].
Therefore, let us turn to CISs (superintegrable systems) on a symplectic manifold.

**Remark 8.1.** Let \( Z \) be a manifold. Any exterior two-form \( \Omega \) on \( Z \) yields a linear bundle morphism

\[
\Omega^b : TZ \to T^*Z, \quad \Omega^b : v \to -v|\Omega(z), \quad v \in T_zZ, \quad z \in Z.
\]  

(8.6)

One says that a two-form \( \Omega \) is non-degenerate if \( \text{Ker} \Omega^b = 0 \). A closed non-degenerate form is called the symplectic form. Accordingly, a manifold \( Z \) equipped with a symplectic form is said to be the symplectic manifold. A symplectic manifold necessarily is even-dimensional. A closed two-form on \( Z \) is called presymplectic if it is not necessarily degenerate. A vector field \( u \) on a symplectic manifold \( (Z, \Omega) \) is said to be Hamiltonian if a one-form \( u|\Omega \) is exact. Any smooth function \( f \in C^\infty(Z) \) on \( Z \) defines a unique Hamiltonian vector field \( \vartheta_f \), called the Hamiltonian vector field of a function \( f \), such that

\[
\vartheta_f|\Omega = -df, \quad \vartheta_f = \Omega^b(df),
\]

(8.7)

where \( \Omega^b \) is the inverse isomorphism to \( \Omega^b \). Given an \( m \)-dimensional manifold \( M \) coordinated by \((q^i)\), let \( T^*M \) be its cotangent bundle equipped with the holonomic coordinates \((q^i, \dot{q}^i)\). It is endowed with the canonical Liouville form

\[
\Xi_T = \dot{q}^i dq^i
\]

(8.8)

and the canonical symplectic form

\[
\Omega_T = d\Xi = dq^i \wedge d\dot{q}^i.
\]

(8.9)

The Hamiltonian vector field \( \vartheta_f \) [8.7] with respect to the canonical symplectic form [8.9] reads

\[
\vartheta_f = \partial^i f \partial \dot{q}^i - \partial_i f \partial_i.
\]

(8.10)

A symplectic form \( \Omega \) on a manifold \( Z \) defines a Poisson bracket

\[
\{f, g\} = \vartheta_g|\vartheta_f|\Omega, \quad f, g \in C^\infty(Z).
\]

The canonical symplectic form \( \Omega_T \) [8.9] on \( T^*M \) yields the canonical Poisson bracket

\[
\{f, g\}_T = \frac{\partial f}{\partial \dot{q}^i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial \dot{q}^i}.
\]

(8.11)

**Definition 8.3.** Let \( (Z, \Omega) \) be a \( 2n \)-dimensional connected symplectic manifold, and let \((C^\infty(Z), \{,\})\) be a Poisson algebra of smooth real functions on \( Z \). A subset

\[
F = (F_1, \ldots, F_k), \quad n \leq k < 2n,
\]

(8.12)

of a Poisson algebra \( C^\infty(Z) \) is called the CIS or the superintegrable system if the following conditions hold.

(i) All the functions \( F_i \) (called the generating functions of a CIS) are independent, i.e., a \( k \)-form \( \wedge dF_i \) nowhere vanishes on \( Z \). It follows that a map \( F : Z \to \mathbb{R}^k \) is a submersion, i.e.,

\[
F : Z \to N = F(Z)
\]

(8.13)
is a fibred manifold over a domain $N \subset \mathbb{R}^k$ endowed with the coordinates $(x_i)$ such that $x_i \circ F = F_i$.

(ii) There exist smooth real functions $s_{ij}$ on $N$ such that
\[
\{F_i, F_j\} = s_{ij} \circ F, \quad i, j = 1, \ldots, k.
\] (8.14)

(iii) The $(k \times k)$-matrix function $s$ with the entries $s_{ij}$ (8.14) is of constant corank $m = 2n - k$ at all points of $N$.

**Remark 8.2.** If $k = n$, then $s = 0$, and we are in the case of commutative CISs when $F_1, \ldots, F - n$ are independent functions in involution.

If $k > n$, the matrix $s$ is necessarily nonzero. If $k = 2n - 1$, a CIS is called maximally integrable.

The following two assertions clarify a structure of CISs [5, 7, 31].

**Theorem 8.4.** Given a symplectic manifold $(Z, \Omega)$, let $F : Z \to N$ be a fibred manifold such that, for any two functions $f, f'$ constant on fibres of $F$, their Poisson bracket $\{f, f'\}$ is so. By virtue of the well known theorem [10, 34], $N$ is provided with an unique coinduced Poisson structure $\{\cdot, \cdot\}_N$ such that $F$ is a Poisson morphism.

Since any function constant on fibres of $F$ is the pull-back of some function on $N$, the CIS (8.12) satisfies the condition of Theorem 8.4 due to item (ii) of Definition 8.3. Thus, a base $N$ of the fibration (8.13) is endowed with a coinduced Poisson structure of corank $m$. With respect to coordinates $x_i$ in item (i) of Definition 8.3 its bivector field reads
\[
w = s_{ij}(x_k) \partial^i \wedge \partial^j.
\] (8.15)

**Theorem 8.5.** Given a fibred manifold $F : Z \to N$ in Theorem 8.4, the following conditions are equivalent [3]:

(i) a rank of the coinduced Poisson structure $\{\cdot, \cdot\}_N$ on $N$ equals $2\text{dim} N - \text{dim} Z$,

(ii) the fibres of $F$ are isotropic,

(iii) the fibres of $F$ are maximal integral manifolds of the involutive distribution spanned by the Hamiltonian vector fields of the pull-back $F^*C$ of Casimir functions $C$ of the coinduced Poisson structure (8.13) on $N$.

It is readily observed that the fibred manifold $F$ (8.13) obeys condition (i) of Theorem 8.5 due to item (iii) of Definition 8.3, namely, $k - m = 2(k - n)$.

Fibres of the fibred manifold $F$ (8.13) are called the invariant submanifolds.

**Remark 8.3.** In practice, condition (i) of Definition 8.3 fails to hold everywhere. It can be replaced with that a subset $Z_R \subset Z$ of regular points (where $\wedge dF_i \neq 0$) is open and dense. Let $M$ be an invariant submanifold through a regular point $z \in Z_R \subset Z$. Then it is regular, i.e., $M \subset Z_R$. Let $M$ admit a regular open saturated neighborhood $U_M$ (i.e., a fibre of $F$ through a point of $U_M$ belongs to $U_M$). For instance, any compact invariant submanifold $M$ has such a neighborhood $U_M$. The restriction of functions $F_i$ to $U_M$ defines a CIS on $U_M$ which obeys Definition 8.3. In this case, one says that a CIS is considered around its invariant submanifold $M$. 25
Let \((Z, \Omega)\) be a 2\(n\)-dimensional connected symplectic manifold. Given the CIS \((F_i)\) on \((Z, \Omega)\), the well known Mishchenko–Fomenko theorem (Theorem 8.7) states the existence of action-angle coordinates around its connected compact invariant submanifold \([1, 5, 19]\). This theorem has been extended to CISs with noncompact invariant submanifolds (Theorem 8.6) \([7, 25, 31]\). These submanifolds are diffeomorphic to a toroidal cylinder
\[ \mathbb{R}^{m-r} \times T^r, \quad m = 2n - k, \quad 0 \leq r \leq m. \] (8.16)

**Theorem 8.6.** Let the Hamiltonian vector fields \(\vartheta_i\) of the functions \(F_i\) be complete, and let the fibres of the fibred manifold \(F\) (8.13) be connected and mutually diffeomorphic. Then the following hold.

(I) The fibres of \(F\) (8.13) are diffeomorphic to the toroidal cylinder (8.16).

(II) Given a fibre \(M\) of \(F\) (8.13), there exists its open saturated neighborhood \(U_M\) which is a trivial principal bundle \(U_M = N_M \times \mathbb{R}^{m-r} \times T^r \xrightarrow{F} N_M\) (8.17) with the structure group (8.16).

(III) A neighborhood \(U_M\) is provided with the bundle action-angle coordinates \((I_\lambda, p_s, q^s, y^\lambda)\), \(\lambda = 1, \ldots, m, s = 1, \ldots, n - m\), such that: (i) the angle coordinates \((y^\lambda)\) are those on a toroidal cylinder, i.e., fibre coordinates on the fibre bundle (8.17), (ii) \((I_\lambda, p_s, q^s)\) are coordinates on its base \(N_M\) where the action coordinates \((I_\lambda)\) are values of Casimir functions of the coinduced Poisson structure \(\{ , \}_N\) on \(N_M\), and (iii) a symplectic form \(\Omega\) on \(U_M\) reads
\[ \Omega = dI_\lambda \wedge dy^\lambda + dp_s \wedge dq^s. \] (8.18)

**Remark 8.4.** The condition of the completeness of Hamiltonian vector fields of the generating functions \(F_i\) in Theorem 8.6 is rather restrictive. One can replace this condition with that the Hamiltonian vector fields of the pull-back onto \(Z\) of Casimir functions on \(N\) are complete.

If the conditions of Theorem 8.6 are replaced with that fibres of the fibred manifold \(F\) (8.13) are compact and connected, this theorem restarts the Mishchenko–Fomenko theorem as follows.

**Theorem 8.7.** Let the fibres of the fibred manifold \(F\) (8.13) be connected and compact. Then they are diffeomorphic to a torus \(T^m\), and statements (II) – (III) of Theorem 8.6 hold.

**Remark 8.5.** In Theorem 8.7, the Hamiltonian vector fields \(\upsilon_\lambda\) are complete because fibres of the fibred manifold \(F\) (8.13) are compact. As well known, any vector field on a compact manifold is complete.

To study a CIS, one conventionally considers it with respect to action-angle coordinates. A problem is that an action-angle coordinate chart on an open subbundle \(U\) of the fibred manifold \(Z \rightarrow N\) (8.13) in Theorem 8.6 is local. The following generalizes this theorem to the case of global action-angle coordinates.

**Definition 8.8.** The CIS \(F\) (8.13) on a symplectic manifold \((Z, \Omega)\) in Definition 8.3 is called globally integrable (or, shortly, global) if there exist global action-angle coordinates
\[ (I_\lambda, x^A, y^\lambda), \quad \lambda = 1, \ldots, m, \quad A = 1, \ldots, 2(n - m), \] (8.19)
such that: (i) the action coordinates \((I_\lambda)\) are expressed in values of some Casimir functions \(C_\lambda\) on a Poisson manifold \((N, \{,\}_N)\), (ii) the angle coordinates \((y^\lambda)\) are coordinates on the toroidal cylinder \(\mathbb{R}^{m-r} \times T^r, 0 \leq r \leq m\), and (iii) a symplectic form \(\Omega\) on \(Z\) reads
\[
\Omega = dI_\lambda \wedge dy^\lambda + \Omega_{AB}(I_\mu, x^C)dx^A \wedge dx^B.
\] (8.20)

It is readily observed that the action-angle coordinates on \(U\) in Theorem 8.6 are global on \(U\) in accordance with Definition 8.8.

Forthcoming Theorem 8.9 provides the sufficient conditions of the existence of global action-angle coordinates of a CIS on a symplectic manifold \((Z, \Omega)\) \([10, 18, 25, 31]\). It generalizes the well-known result for the case of compact invariant submanifolds \([2, 5]\).

**Theorem 8.9.** A CIS \(F\) on a symplectic manifold \((Z, \Omega)\) is globally integrable if the following conditions hold.

(i) Hamiltonian vector fields \(\vartheta_i\) of the generating functions \(F_i\) are complete.
(ii) The fibred manifold \(F\) (8.13) is a fibre bundle with connected fibres.
(iii) Its base \(N\) is simply connected and the cohomology \(H^2(N; \mathbb{Z})\) with coefficients in the constant sheaf \(\mathbb{Z}\) is trivial.
(iv) The coinduced Poisson structure \(\{,\}_N\) on a base \(N\) admits \(m\) independent Casimir functions \(C_\lambda\).

Theorem 8.9 restarts Theorem 8.6 if one considers an open subset \(V\) of \(N\) admitting the Darboux coordinates \(x^A\) on symplectic leaves of \(U\). If invariant submanifolds of a CIS are assumed to be compact, condition (i) of Theorem 8.9 is unnecessary since vector fields \(\vartheta_\lambda\) on compact fibres of \(F\) are complete. In this case, Theorem 8.9 reproduces the well known result in [2].

Furthermore, one can show that condition (iii) of Theorem 8.9 guarantee that fibre bundles \(F\) in conditions (ii) of these theorems are trivial [31]. Therefore, Theorem 8.9 can be reformulated as follows.

**Theorem 8.10.** A CIS \(F\) on a symplectic manifold \((Z, \Omega)\) is global if and only if the following conditions hold.

(i) The fibred manifold \(F\) (8.13) is a trivial fibre bundle.
(ii) The coinduced Poisson structure \(\{,\}_N\) on a base \(N\) admits \(m\) independent Casimir functions \(C_\lambda\) such that Hamiltonian vector fields of their pull-back \(F^*C_\lambda\) are complete.

Bearing in mind the autonomous CIS (8.4), let us turn to autonomous CISs whose generating functions are integrals of motion, i.e., they are in involution with a Hamiltonian \(H\), and the functions \((\mathcal{H}, F_1, \ldots, F_k)\) are nowhere independent, i.e.,

\[
\{\mathcal{H}, F_i\} = 0, \quad (8.21)
\]
\[
d\mathcal{H} \wedge (\wedge dF_i) = 0. \quad (8.22)
\]

Let us note that, in accordance with item (i) of Theorem 8.10 and forthcoming Theorem 8.11 the Hamiltonian vector field of a Hamiltonian \(\mathcal{H}\) of a CIS always is complete.

**Theorem 8.11.** It follows from the equality (8.22) that a Hamiltonian \(\mathcal{H}\) is constant on the
invariant submanifolds. Therefore, it is the pull-back of a function on \( N \) which is a Casimir function of the Poisson structure (8.19) because of the conditions (8.21).

Theorem 8.11 leads to the following.

**Theorem 8.12.** Let \( H \) be a Hamiltonian of an autonomous global CIS provided with the action-angle coordinates \((I_\lambda, x^A, y^\lambda)\) (8.19). Then a Hamiltonian \( H \) depends only on the action coordinates \( I_\lambda \). Consequently, the Hamilton equation of a global CIS takes a form

\[
\dot{y}^\lambda = \frac{\partial H}{\partial I_\lambda}, \quad I_\lambda = \text{const.}, \quad x^A = \text{const}.
\]

**Remark 8.6.** Given a Hamiltonian \( H \) of a Hamiltonian system on a symplectic manifold \( Z \), it may happen that we have different CISs on different open subsets of \( Z \). For instance, this is the case of the global Kepler problem (Section 9).

**Remark 8.7.** Bearing in mind again the autonomous CIS (8.4), let us also consider CISs whose generating functions \( \{F_1, \ldots, F_k\} \) form a \( k \)-dimensional real Lie algebra \( g \) of corank \( m \) with commutation relations

\[
\{F_i, F_j\} = c^h_{ij} F_h, \quad c^h_{ij} = \text{const}. \tag{8.23}
\]

Then \( F \) (8.13) is a momentum mapping of \( Z \) to the Lie coalgebra \( g^* \) provided with the coordinates \( x_i \) in item (i) of Definition 8.3 [8, 12]. In this case, the coinduced Poisson structure \( \{,\}_N \) coincides with the canonical Lie–Poisson structure on \( g^* \) given by the Poisson bivector field

\[
w = \frac{1}{2} c^h_{ij} x_h \partial^i \wedge \partial^j.
\]

Let \( V \) be an open subset of \( g^* \) such that conditions (i) and (ii) of Theorem 8.10 are satisfied. Then an open subset \( F^{-1}(V) \subset Z \) is provided with the action-angle coordinates. Let Hamiltonian vector fields \( \vartheta_i \) of the generating functions \( F_i \) which form a Lie algebra \( g \) be complete. Then they define a locally free Hamiltonian action on \( Z \) of some simply connected Lie group \( G \) whose Lie algebra is isomorphic to \( g \) [20]. Orbits of \( G \) coincide with \( k \)-dimensional maximal integral manifolds of the regular distribution \( V \) on \( Z \) spanned by Hamiltonian vector fields \( \vartheta_i \) [33]. Furthermore, Casimir functions of the Lie–Poisson structure on \( g^* \) are exactly the coadjoint invariant functions on \( g^* \). They are constant on orbits of the coadjoint action of \( G \) on \( g^* \) which coincide with leaves of the symplectic foliation of \( g^* \).

Now, let us return to the autonomous CIS (8.4) on homogeneous phase space of non-autonomous mechanics.

There is the commutative diagram

\[
\begin{array}{ccc}
T^*Q & \xrightarrow{\zeta} & V^*Q \\
\Phi \downarrow & & \downarrow \Phi \\
N' & \xrightarrow{\xi} & N
\end{array}
\]

where \( \zeta \) (3.13) and \( \xi : N' = \mathbb{R} \times N \to N \) are trivial bundles. It follows that the fibred manifold (8.5) is the pull-back \( \Phi = \xi^* \Phi \) of the fibred manifold \( \Phi \) (8.3) onto \( N' \).
Let the conditions of Theorem 8.6 hold. If the Hamiltonian vector fields
\[ (\gamma_H, \partial_{\Phi_1}, \ldots, \partial_{\Phi_k}), \quad \partial_{\Phi_\alpha} = \partial^i \Phi_\alpha \partial_i - \partial_i \Phi_\alpha \partial^i, \]
of integrals of motion \( \Phi_\alpha \) on \( V^*Q \) are complete, the Hamiltonian vector fields
\[ (u_H, u_{\zeta} \Phi_1, \ldots, u_{\zeta} \Phi_k), \quad u_{\zeta} \Phi_\alpha = \partial^i \Phi_\alpha \partial_i - \partial_i \Phi_\alpha \partial^i, \]
on \( T^*Q \) are complete. If fibres of the fibred manifold \( \Phi \) are connected and mutually diffeomorphic, the fibres of the fibred manifold \( \tilde{\Phi} \) also are well.

Let \( M \) be a fibre of \( \Phi \) and \( h(M) \) the corresponding fibre of \( \tilde{\Phi} \). In accordance with Theorem 8.6, there exists an open neighborhood \( U' \) of \( h(M) \) which is a trivial principal bundle with the structure group
\[ \mathbb{R}^{1+m-r} \times T^r \]
whose bundle coordinates are the action-angle coordinates
\[ (I_0, I_\lambda, t, y^\lambda, p_A, q^A), \quad A = 1, \ldots, n-m, \quad \lambda = 1, \ldots, k, \]
such that:
(i) \((t, y^\lambda)\) are coordinates on the toroidal cylinder \(8.24\),
(ii) the symplectic form \( \Omega_T \) on \( U' \) reads
\[ \Omega_T = dI_0 \wedge dt + dI_\alpha \wedge dy^\alpha + dp_A \wedge dq^A, \]
(iii) the action coordinates \((I_0, I_\lambda)\) are expressed in values of the Casimir functions \( C_0 = I_0, C_\alpha \) of the coinduced Poisson structure \( w = \partial^A \wedge \partial_A \) on \( N' \),
(iv) a homogeneous Hamiltonian \( H^* \) depends on action coordinates, namely, \( H^* = I_0 \),
(iv) the integrals of motion \( \zeta^* \Phi_1, \ldots, \zeta^* \Phi_k \) are independent of coordinates \((t, y^\lambda)\).

Provided with the action-angle coordinates \(8.25\), the above mentioned neighborhood \( U' \) is a trivial bundle \( U' = \mathbb{R} \times U_M \) where \( U_M = \zeta(U') \) is an open neighborhood of the fibre \( M \) of the fibre bundle \( \Phi \). As a result, we come to the following.

**Theorem 8.13.** Let symmetries \( \nu_\alpha \) in Definition 8.7 be complete, and let fibres of the fibred manifold \( \Phi \) defined by the corresponding conserved integrals of motion be connected and mutually diffeomorphic. Then there exists an open neighborhood \( U_M \) of a fibre \( M \) of \( \Phi \) which is a trivial principal bundle with a structure group \(8.24\) whose bundle coordinates are the action-angle coordinates
\[ (p_A, q^A, I_\lambda, t, y^\lambda), \quad A = 1, \ldots, k-n, \quad \lambda = 1, \ldots, m, \]
such that:
(i) \((t, y^\lambda)\) are coordinates on the toroidal cylinder \(8.24\),
(ii) the Poisson bracket \( \{, \}_V \) on \( U_M \) reads
\[ \{f, g\}_V = \partial^A f \partial_A g - \partial^A g \partial_A f + \partial^\lambda f \partial_\lambda g - \partial^\lambda g \partial_\lambda f, \]
(iii) a Hamiltonian \( H \) depends only on action coordinates \( I_\lambda \),
(iv) the integrals of motion \( \Phi_1, \ldots, \Phi_k \) are independent of coordinates \((t, y^\lambda)\).
9 Global Kepler problem

We provide a global analysis of the Kepler problem as an example of a mechanical system which is characterized by its symmetries in full. It falls into two distinct global CISs on different open subsets of a phase space. Their integrals of motion form the Lie algebras $so(3)$ and $so(2, 1)$ with compact and noncompact invariant submanifolds, respectively [10, 25, 31].

Let us consider a mechanical system of a point mass in the presence of a central potential. Its configuration space is

$$Q = \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$$

(9.1)

endowed with the Cartesian coordinates $(t, q^i), i = 1, 2, 3$.

A Lagrangian of this mechanical system reads

$$\mathcal{L} = \frac{1}{2} \left( \sum_i (q_i^t)^2 \right) - V(r), \quad r = \left( \sum_i (q_i^r)^2 \right)^{1/2}. \quad (9.2)$$

The vertical vector fields

$$v_a^b = q^b \partial_a - q^a \partial_b \quad (9.3)$$
on $Q$ (9.1) are infinitesimal generators of a group $SO(3)$ acting on $\mathbb{R}^3$. Their jet prolongations (4.2) read

$$J^1 v_a^b = q^b \partial_a - q^a \partial_b + q^b \dot{q}^a_t - q^a \dot{q}^b_t \quad (9.4)$$

It is easily justified that the vector fields (9.3) are exact symmetries of the Lagrangian (9.2). In accordance with Noether’s first theorem, the corresponding conserved Noether currents (4.14) are orbital momenta

$$M^a_b = J_{v_a^b} = (q^a \pi_b - q^b \pi_a) = q^a q^b_t - q^b q^a_t. \quad (9.5)$$

They are integrals of motion, which however fail to be independent.

Let us consider the Lagrangian system (9.2) where

$$V(r) = -\frac{1}{r} \quad (9.5)$$
is the Kepler potential. This Lagrangian system possesses additional integrals of motion

$$A^a = \sum_b (q^b q^a_t - q^a q^b_t)q^b_t - \frac{q^a}{r}, \quad (9.6)$$

besides the orbital momenta (9.4). They are components of the Rung–Lenz vector.

However, there is no Lagrangian symmetry on $Q$ (9.1) whose symmetry currents are $A^a$ (9.6).

Let us consider a Hamiltonian Kepler system on the configuration space $Q$ (9.1). Its phase space is $V^*Q = \mathbb{R} \times \mathbb{R}^6$ coordinated by $(t, q^i, p_i)$. It is readily observed that the Lagrangian (9.2) with the Kepler potential (9.5) of a Kepler system is hyperregular. The associated Hamiltonian form reads

$$H = p_i dq^i - \left[ \frac{1}{2} \left( \sum_i (p_i)^2 \right) - \frac{1}{r} \right] dt. \quad (9.7)$$
The corresponding characteristic Lagrangian $L_H$ (6.22) is

$$L_H = \left[ p_i q_i - \frac{1}{2} \left( \sum_i (p_i)^2 \right) + \frac{1}{r} \right] dt. \quad (9.8)$$

Then a Hamiltonian Kepler system possesses the following integrals of motion:

- an energy function $\mathcal{E} = H$;
- orbital momenta
  $$M^a_b = q^a p_b - q^b p_a; \quad (9.9)$$
- components of the Rung–Lenz vector
  $$A^a = \sum_b \left( q^a p_b - q^b p_a \right) \left( q^a \right) / r. \quad (9.10)$$

By virtue of the Noether’s inverse first Theorem 7.7, these integrals of motion are the conserved currents of the following Hamiltonian symmetries:

- the exact symmetry $\partial_t$,
- the exact vertical symmetries
  $$v^a_b = q^b \partial_a - q^a \partial_b - p_a \partial^b + p_b \partial^a, \quad (9.11)$$
- the vertical symmetries
  $$v^a = \sum_b \left[ p_b v^a_b + \left( q^b p_a - q^a p_b \right) \partial_b \right] - \partial_b \left( \frac{q^a}{r} \right) \partial^b. \quad (9.12)$$

Note that the Hamiltonian symmetries $v^a_b$ (9.11) are the canonical lift (6.24) onto $V^*Q$ of the vector fields $v^a_b$ (9.3) on $Q$, which thus are basic Hamiltonian symmetries, and integrals of motion $M^a_b$ (9.9) are the Noether currents (7.12).

At the same time, the Hamiltonian symmetries (9.12) do not come from any vector fields on a configuration space $Q$. Therefore, in contrast with the Rung–Lenz vector (9.12) in Hamiltonian mechanics, the Rung–Lenz vector (9.6) in Lagrangian mechanics fails to be a conserved current of a Lagrangian symmetry.

As was mentioned above, the Hamiltonian symmetries of the Kepler problem make up CISs. To analyze them, we further consider the Kepler problem on a configuration space $\mathbb{R}^2$ without a loss of generality.

Its phase space is $T^*\mathbb{R}^2 = \mathbb{R}^4$ provided with the Cartesian coordinates $(q_i, p_i), i = 1, 2,$ and the canonical symplectic form

$$\Omega_F = \sum_i dp_i \wedge dq_i. \quad (9.13)$$

Let us denote

$$p = \left( \sum_i (p_i)^2 \right)^{1/2}, \quad r = \left( \sum_i (q^i)^2 \right)^{1/2}, \quad (p, q) = \sum_i p_i q_i.$$

An autonomous Hamiltonian of the Kepler system reads

$$\mathcal{H} = \frac{1}{2} p^2 - \frac{1}{r} \quad (9.14)$$
The Kepler system is a Hamiltonian system on a symplectic manifold
\[ Z = \mathbb{R}^4 \setminus \{0\} \]
endowed with the symplectic form \( \Omega \).

Let us consider functions
\[ M_{12} = -M_{21} = q_1 p_2 - q_2 p_1, \]
\[ A_i = \sum_j M_{ij} p_j - \frac{q_i}{r} = q_i p^2 - p_i(p,q) - \frac{q_i}{r}, \quad i = 1, 2, \]
on the symplectic manifold \( Z \). As was mentioned above, they are integrals of motion of
the Hamiltonian \( H \) where \( M_{12} \) is an angular momentum and \( (A_i) \) is the Rung–Lenz vector.
Let us denote
\[ M^2 = (M_{12})^2, \quad A^2 = (A_1)^2 + (A_2)^2 = 2M^2H + 1. \]

Let \( Z_0 \subset Z \) be a closed subset of points where \( M_{12} = 0 \). A direct computation shows that
the functions \((M_{12}, A_i)\) are independent on an open submanifold
\[ U = Z \setminus Z_0 \]
of \( Z \). At the same time, the functions \((H, M_{12}, A_i)\) are independent nowhere on \( U \) because it follows from the expression \( 9.18 \) that
\[ H = \frac{A^2 - 1}{2M^2} \]
on \( U \). The well known dynamics of the Kepler system shows that the Hamiltonian vector
field of its Hamiltonian is complete on \( U \) (but not on \( Z \)).

The Poisson bracket of integrals of motion \( M_{12} \) and \( A_i \) obeys relations
\[ \{M_{12}, A_i\} = \eta_{2i}A_1 - \eta_{1i}A_2, \]
\[ \{A_1, A_2\} = 2HM_{12} = \frac{A^2 - 1}{M_{12}}, \]
where \( \eta_{ij} \) is an Euclidean metric on \( \mathbb{R}^2 \). It is readily observed that these relations take the form \( 8.14 \). However, the matrix function \( s \) of the relations \( 9.21 \) – \( 9.22 \) fails to be of constant rank at points where \( H = 0 \). Therefore, let us consider the open submanifolds \( U_- \subset U \) where \( H < 0 \) and \( U_+ \) where \( H > 0 \). Then we observe that the Kepler system with the Hamiltonian \( H \) and the integrals of motion \((M_{ij}, A_i)\) on \( U_- \) and the Kepler system with the Hamiltonian \( H \) and the integrals of motion \((M_{ij}, A_i)\) on \( U_+ \) are noncommutative CISs. Moreover, these CISs can be brought into the form \( 8.23 \) as follows.

Let us replace the integrals of motions \( A_i \) with the integrals of motion
\[ L_i = \frac{A_i}{\sqrt{-2H}} \]
on \( U_- \), and with the integrals of motion
\[ K_i = \frac{A_i}{\sqrt{2H}} \]
on \( U_+ \).
The CIS \((M_{12}, L_i)\) on \(U_-\) obeys relations
\[
\{M_{12}, L_i\} = \eta_{2i}L_1 - \eta_{1i}L_2, \quad \{L_1, L_2\} = -M_{12}. \tag{9.25}
\]
Let us denote \(M_{i\beta} = -L_i\) and put the indexes \(\mu, \nu, \alpha, \beta = 1, 2, 3\). Then the relations (9.25) are brought into a form
\[
\{M_{\mu\nu}, M_{\alpha\beta}\} = \eta_{\mu\beta}M_{\nu\alpha} + \eta_{\nu\alpha}M_{\mu\beta} - \eta_{\mu\alpha}M_{\nu\beta} - \eta_{\nu\beta}M_{\mu\alpha} \tag{9.26}
\]
where \(\eta_{\mu\nu}\) is an Euclidean metric on \(\mathbb{R}^3\). A glance at the expression (9.26) shows that the integrals of motion \(M_{12}\) (9.16) and \(L_i\) (9.23) constitute a Lie algebra \(\mathfrak{so}(3)\). Its corank equals 1. Therefore the CIS \((M_{12}, L_i)\) on \(U_-\) is maximally integrable. The equality (9.20) takes a form
\[
M^2 + L^2 = -\frac{1}{2\mathcal{H}}. \tag{9.27}
\]

The CIS \((M_{12}, K_i)\) on \(U_+\) obeys relations
\[
\{M_{12}, K_i\} = \eta_{2i}K_1 - \eta_{1i}K_2, \quad \{K_1, K_2\} = M_{12}. \tag{9.28}
\]
Let us denote \(M_{i\beta} = -K_i\) and put the indexes \(\mu, \nu, \alpha, \beta = 1, 2, 3\). Then the relations (9.28) are brought into a form
\[
\{M_{\mu\nu}, M_{\alpha\beta}\} = \rho_{\mu\beta}M_{\nu\alpha} + \rho_{\nu\alpha}M_{\mu\beta} - \rho_{\mu\alpha}M_{\nu\beta} - \rho_{\nu\beta}M_{\mu\alpha} \tag{9.29}
\]
where \(\rho_{\mu\nu}\) is a pseudo-Euclidean metric of signature \((+, +, -)\) on \(\mathbb{R}^3\). A glance at the expression (9.29) shows that the integrals of motion \(M_{12}\) (9.16) and \(K_i\) (9.24) constitute a Lie algebra \(\mathfrak{so}(2,1)\). Its corank equals 1. Therefore the CIS \((M_{12}, K_i)\) on \(U_+\) is maximally integrable. The equality (9.20) takes a form
\[
K^2 - M^2 = \frac{1}{2\mathcal{H}}. \tag{9.30}
\]

Thus, the Kepler problem on a phase space \(\mathbb{R}^4\) falls into two different maximally integrable systems on open submanifolds \(U_-\) and \(U_+\) of \(\mathbb{R}^3\). We agree to call them the Kepler CISs on \(U_-\) and \(U_+\), respectively.

Let us study the first one, and let us put
\[
F_1 = -L_1, \quad F_2 = -L_2, \quad F_3 = -M_{12}, \quad \{F_1, F_2\} = F_3, \quad \{F_2, F_3\} = F_1, \quad \{F_3, F_1\} = F_2. \tag{9.31}
\]
We have a fibred manifold
\[
F : U_- \rightarrow N \subset \mathfrak{g}^*, \tag{9.32}
\]
which is the momentum mapping to a Lie coalgebra \(\mathfrak{g}^* = \mathfrak{so}(3)^*\), endowed with the coordinates \((x_i)\) such that integrals of motion \(F_i\) on \(\mathfrak{g}^*\) read \(F_i = x_i\) (Remark 8.7). A base \(N\) of the fibred manifold (9.32) is an open submanifold of \(\mathfrak{g}^*\) given by a coordinate condition \(x_3 \neq 0\). It is a union of two contractible components defined by conditions \(x_3 > 0\) and \(x_3 < 0\). The coinduced Lie–Poisson structure on \(N\) takes a form
\[
w = x_2 \partial^3 \wedge \partial^1 + x_3 \partial^1 \wedge \partial^2 + x_1 \partial^2 \wedge \partial^3. \tag{9.33}
\]
The coadjoint action of $so(3)$ on $N$ reads
\begin{equation}
\varepsilon_1 = x_3 \partial^2 - x_2 \partial^3, \quad \varepsilon_2 = x_1 \partial^3 - x_3 \partial^1, \quad \varepsilon_3 = x_2 \partial^1 - x_1 \partial^2.
\end{equation}
(9.34)
Orbits of this coadjoint action are given by an equation
\begin{equation}
x_1^2 + x_2^2 + x_3^2 = \text{const.}
\end{equation}
(9.35)
They are level surfaces of a Casimir function
\begin{equation}
C = x_1^2 + x_2^2 + x_3^2
\end{equation}
and, consequently, the Casimir function
\begin{equation}
h = -\frac{1}{2} \left( x_1^2 + x_2^2 + x_3^2 \right)^{-1}.
\end{equation}
(9.36)
A glance at the expression (9.27) shows that the pull-back $F^* h$ of this Casimir function (9.36) onto $U_-$ is the Hamiltonian $\mathcal{H}$ (9.11) of the Kepler system on $U_-$. As was mentioned above, the Hamiltonian vector field of $F^* h$ is complete. Furthermore, it is known that invariant submanifolds of the Kepler CIS on $U_-$ are compact. Therefore, the fibred manifold $F$ (9.32) is a fibre bundle. Moreover, this fibre bundle is trivial because $N$ is a disjoint union of two contractible manifolds. Consequently, it follows from Theorem 8.10 that the Kepler CIS on $U_-$ is global, i.e., it admits global action-angle coordinates as follows.

The Poisson manifold $N$ (9.32) can be endowed with the coordinates
\begin{equation}
(I, x_1, \gamma), \quad I < 0, \quad \gamma \neq \frac{\pi}{2}, \frac{3\pi}{2},
\end{equation}
(9.37)
defined by the equalities
\begin{equation}
I = -\frac{1}{2} \left( x_1^2 + x_2^2 + x_3^2 \right)^{-1},
\end{equation}
(9.38)
\begin{equation}
x_2 = \left( -\frac{1}{2I} - x_1^2 \right)^{1/2} \sin \gamma, \quad x_3 = \left( -\frac{1}{2I} - x_1^2 \right)^{1/2} \cos \gamma.
\end{equation}
It is readily observed that the coordinates (9.37) are Darboux coordinates of the Lie–Poisson structure (9.33) on $U_-$, namely,
\begin{equation}
w = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial \gamma}.
\end{equation}
(9.39)

Let $\vartheta_I$ be the Hamiltonian vector field of the Casimir function $I$ (9.38). Its flows are invariant submanifolds of the Kepler CIS on $U_-$ (Remark 8.7). Let $\alpha$ be a parameter along the flow of this vector field, i.e.,
\begin{equation}
\vartheta_I = \frac{\partial}{\partial \alpha}.
\end{equation}
(9.40)
Then $U_-$ is provided with the action-angle coordinates $(I, x_1, \gamma, \alpha)$ such that the Poisson bivector associated to the symplectic form $\Omega_T$ on $U_-$ reads
\begin{equation}
W = \frac{\partial}{\partial I} \wedge \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial \gamma}.
\end{equation}
(9.41)
Accordingly, Hamiltonian vector fields of integrals of motion $F_i$ (9.31) take a form

$$
\vartheta_1 = \frac{\partial}{\partial \gamma},
$$

$$
\vartheta_2 = \frac{1}{4I^2} \left( \frac{1}{2I} - x_1^2 \right)^{-1/2} \sin \gamma \frac{\partial}{\partial \alpha} - x_1 \left( \frac{1}{2I} - x_1^2 \right)^{-1/2} \sin \gamma \frac{\partial}{\partial \gamma} - \left( \frac{1}{2I} - x_1^2 \right)^{1/2} \cos \gamma \frac{\partial}{\partial x_1},
$$

$$
\vartheta_3 = \frac{1}{4I^2} \left( \frac{1}{2I} - x_1^2 \right)^{-1/2} \cos \gamma \frac{\partial}{\partial \alpha} - x_1 \left( \frac{1}{2I} - x_1^2 \right)^{-1/2} \cos \gamma \frac{\partial}{\partial \gamma} + \left( \frac{1}{2I} - x_1^2 \right)^{1/2} \sin \gamma \frac{\partial}{\partial x_1}.
$$

A glance at these expressions shows that the vector fields $\vartheta_1$ and $\vartheta_2$ fail to be complete on $U_-$ (Remark 8.4).

One can say something more about the angle coordinate $\alpha$. The vector field $\vartheta_1$ (9.40) reads

$$
\frac{\partial}{\partial \alpha} = \sum_i \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right).
$$

This equality leads to relations

$$
\frac{\partial q_i}{\partial \alpha} = \frac{\partial H}{\partial p_i}, \quad \frac{\partial p_i}{\partial \alpha} = -\frac{\partial H}{\partial q_i},
$$

which take a form of the Hamilton equation. Therefore, the coordinate $\alpha$ is a cyclic time $\alpha = t \mod 2\pi$ given by the well-known expression

$$
\alpha = \phi - a^{3/2} e \sin(a^{-3/2} \phi), \quad r = a(1 - e \cos(a^{-3/2} \phi)),
$$

$$
a = -\frac{1}{2I}, \quad e = (1 + 2IM^2)^{1/2}.
$$

Now let us turn to the Kepler CIS on $U_+$. It is a globally integrable system with noncompact invariant submanifolds as follows.

Let us put

$$
S_1 = -K_1, \quad S_2 = -K_2, \quad S_3 = -M_{12},
$$

$$
\{S_1, S_2\} = -S_3, \quad \{S_2, S_3\} = S_1, \quad \{S_3, S_1\} = S_2.
$$

We have a fibred manifold

$$
S : U_+ \to N \subset \mathfrak{g}^*,
$$

which is the momentum mapping to a Lie coalgebra $\mathfrak{g}^* = so(2, 1)^*$, endowed with the coordinates $(x_i)$ such that integrals of motion $S_i$ on $\mathfrak{g}^*$ read $S_i = x_i$. A base $N$ of the fibred manifold (9.43) is an open submanifold of $\mathfrak{g}^*$ given by a coordinate condition $x_3 \neq 0$. It is a union of two contractible components defined by conditions $x_3 > 0$ and $x_3 < 0$. The coinduced Lie–Poisson structure on $N$ takes a form

$$
w = x_2 \partial^3 \wedge \partial^1 - x_3 \partial^1 \wedge \partial^2 + x_1 \partial^2 \wedge \partial^3.
$$

The coadjoint action of $so(2, 1)$ on $N$ reads

$$
\varepsilon_1 = -x_3 \partial^2 - x_2 \partial^3, \quad \varepsilon_2 = x_1 \partial^3 + x_3 \partial^1, \quad \varepsilon_3 = x_2 \partial^1 - x_1 \partial^2.
$$
The orbits of this coadjoint action are given by an equation
\[ x_1^2 + x_2^2 - x_3^2 = \text{const}. \]
They are the level surfaces of the Casimir function
\[ C = x_1^2 + x_2^2 - x_3^2 \]
and, consequently, the Casimir function
\[ h = \frac{1}{2} (x_1^2 + x_2^2 - x_3^2)^{-1}. \] (9.45)

A glance at the expression (9.30) shows that the pull-back $S^* h$ of this Casimir function (9.45)
on to $U_+$ is the Hamiltonian $\mathcal{H}$ (9.14) of the Kepler system on $U_+$.

As was mentioned above, the Hamiltonian vector field of $S^* h$ is complete. Furthermore, it is
known that invariant submanifolds of the Kepler CIS on $U_+$ are diffeomorphic to $\mathbb{R}$. Therefore,
the fibred manifold $S$ (9.43) is a fibre bundle. Moreover, this fibre bundle is trivial because $N$
is a disjoint union of two contractible manifolds. Consequently, it follows from Theorem 8.10
that the Kepler CIS on $U_+$ is globally integrable, i.e., it admits global action-angle coordinates
as follows.

The Poisson manifold $N$ (9.43) can be endowed with the coordinates
\[ (I, x_1, \lambda), \quad I > 0, \quad \lambda \neq 0, \]
defined by the equalities
\[ I = \frac{1}{2} (x_1^2 + x_2^2 - x_3^2)^{-1}, \]
\[ x_2 = \left( \frac{1}{2I} - x_1^2 \right)^{1/2} \cosh \lambda, \quad x_3 = \left( \frac{1}{2I} - x_1^2 \right)^{1/2} \sinh \lambda. \]
These coordinates are Darboux coordinates of the Lie–Poisson structure (9.44) on $N$, namely,
\[ w = \frac{\partial}{\partial \lambda} \wedge \frac{\partial}{\partial x_1}. \] (9.46)

Let $\vartheta_I$ be the Hamiltonian vector field of the Casimir function $I$ (9.38). Its flows are invariant
submanifolds of the Kepler CIS on $U_+$ (Remark 8.7). Let $\tau$ be a parameter along the flows of
this vector field, i.e.,
\[ \vartheta_I = \frac{\partial}{\partial \tau}. \] (9.47)
Then $U_+$ (9.43) is provided with the action-angle coordinates $(I, x_1, \lambda, \tau)$ such that the Poisson
bivector associated to the symplectic form $\Omega_T$ on $U_+$ reads
\[ W = \frac{\partial}{\partial I} \wedge \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \lambda} \wedge \frac{\partial}{\partial x_1}. \] (9.48)
Accordingly, Hamiltonian vector fields of integrals of motion $S_i$ (9.42) take a form

$$
\vartheta_1 = -\frac{\partial}{\partial \lambda},
$$

$$
\vartheta_2 = \frac{1}{4I^2} \left( \frac{1}{2I} - x_1^2 \right)^{-1/2} \cosh \lambda \frac{\partial}{\partial \tau} + x_1 \left( \frac{1}{2I} - x_1^2 \right)^{-1/2} \cosh \lambda \frac{\partial}{\partial \lambda} + \left( \frac{1}{2I} - x_1^2 \right)^{1/2} \sinh \lambda \frac{\partial}{\partial x_1},
$$

$$
\vartheta_3 = \frac{1}{4I^2} \left( \frac{1}{2I} - x_1^2 \right)^{-1/2} \sinh \lambda \frac{\partial}{\partial \tau} + x_1 \left( \frac{1}{2I} - x_1^2 \right)^{-1/2} \sinh \lambda \frac{\partial}{\partial \lambda} + \left( \frac{1}{2I} - x_1^2 \right)^{1/2} \cosh \lambda \frac{\partial}{\partial x_1}.
$$

Similarly to the angle coordinate $\alpha$ (9.40), the angle coordinate $\tau$ (9.47) obeys the Hamilton equation

$$
\frac{\partial q_i}{\partial \tau} = \frac{\partial H}{\partial p_i}, \quad \frac{\partial p_i}{\partial \tau} = -\frac{\partial H}{\partial q_i}.
$$

Therefore, it is the time $\tau = t$ given by the well-known expression

$$
\tau = s - a^{3/2} e \sinh(a^{-3/2} s), \quad r = a(e \cosh(a^{-3/2} s) - 1),
$$

$$
a = \frac{1}{2I}, \quad e = (1 + 2IM^2)^{1/2}.
$$

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