An inequality for solutions of the
Navier-Stokes equations in $\mathbb{R}^n$  

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Abstract
We obtain a new inequality that holds for general Leray solutions of the incompressible Navier-Stokes equations in $\mathbb{R}^n$ ($n \leq 4$). This recovers important results previously obtained by other authors regarding the time decay of solution derivatives (of arbitrary order).

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1. Introduction
In this note we derive a fundamental new inequality for general Leray solutions of the incompressible Navier-Stokes equations (in dimension $n \leq 4$), that is, global solutions $u(\cdot, t) \in L^\infty((0, \infty), L^2(\mathbb{R}^n)) \cap L^2((0, \infty), \mathcal{H}^1(\mathbb{R}^n)) \cap C_w([0, \infty), L^2(\mathbb{R}^n))$ of the fluid flow system

\begin{align*}
  u_t + u \cdot \nabla u + \nabla p &= \nu \Delta u, \quad (1.1a) \\
  \nabla \cdot u(\cdot, t) &= 0, \quad (1.1b) \\
  u(\cdot, 0) &= u_0 \in L^2(\mathbb{R}^n), \quad (1.1c)
\end{align*}

that satisfy the generalized energy inequality
\[ \| u(\cdot, t) \|_{L^2(\mathbb{R}^n)}^2 + 2 \nu \int_s^t \| Du(\cdot, \tau) \|_{L^2(\mathbb{R}^n)}^2 d\tau \leq \| u(\cdot, s) \|_{L^2(\mathbb{R}^n)}^2, \quad \forall \ t \geq s \]  

(1.2)

for a.e. \( s \geq 0 \), including \( s = 0 \). Such solutions were first constructed by Leray [9, 10] for \( n \leq 3 \), and later by other authors with different methods and more general space dimension, see e.g. [3, 4, 8, 16, 17, 18]. In (1.1) above, \( \nu > 0 \) is a given constant, \( u = u(x, t) \) and \( p = p(x, t) \) are the unknowns (the flow velocity and pressure, respectively), with condition (1.1c) satisfied in \( L^2(\mathbb{R}^n) \), i.e., \( \| u(\cdot, t) - u_0 \|_{L^2(\mathbb{R}^n)} \to 0 \) as \( t \to 0 \). In the present work, we always assume \( 2 \leq n \leq 4 \).

A well known property of Leray solutions is that they are eventually very regular: there is always some \( t^* \geq 0 \) such that one has \( u \in C(\mathbb{R}^n \times (t^*, \infty)) \) and, moreover,

\[ u(\cdot, t) \in C(((t^*, \infty), H^m(\mathbb{R}^n)), \quad \forall \ m \geq 0, \]  

(1.3)

see e.g. [4, 7, 9, 10, 15, 17]. It is also well established that \( \lim_{t \to 0} \| u(\cdot, t) \|_{L^2(\mathbb{R}^n)} = 0 \) and, more generally,

\[ \lim_{t \to 0} t^{m/2} \| D^m u(\cdot, t) \|_{L^2(\mathbb{R}^n)} = 0 \]  

(1.4)

for every \( m \geq 1 \), and for all Leray solutions to the system (1.1) [11, 12, 13, 15]. Furthermore, suitable stronger assumptions on the initial data have led to interesting finer estimates for the solutions and their derivatives, see e.g. [6, 12, 15, 19]. An important shortcut for many of these results (including (1.4) and the Schonbek-Wiegner estimates [15]) is provided by the following fundamental inequality recently discovered by the authors, which has eluded previous studies.\(^2\)

\[ \limsup_{t \to \infty} t^{\alpha + m/2} \| D^m u(\cdot, t) \|_{L^2(\mathbb{R}^n)} \leq K(\alpha, m) \nu^{-m/2} \limsup_{t \to \infty} t^\alpha \| u(\cdot, t) \|_{L^2(\mathbb{R}^n)} \]  

(1.5)

for every \( m \geq 1 \), where

\[ K(\alpha, m) = \min_{\delta > 0} \{ \delta^{-1/2} \prod_{j=0}^{m} (\alpha + j/2 + \delta)^{1/2} \}. \]

\(^1\) It is known that \( t^* = 0 \) if \( n = 2 \), \( t^* \leq \nu^{-5/2} \| u_0 \|^4_{L^2(\mathbb{R}^3)} \) if \( n = 3 \), \( t^* \leq \nu^{-3} \| u_0 \|^2_{L^2(\mathbb{R}^4)} \) if \( n = 4 \).

\(^2\) For the definition of \( \| u(\cdot, t) \|_{L^2(\mathbb{R}^n)}, \| D^m u(\cdot, t) \|_{L^2(\mathbb{R}^n)} \) and other similar norms, see (1.6). 

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**Main Theorem.** Let \( n \leq 4 \), \( u_0 \in L^2_0(\mathbb{R}^n) \), and let \( u(\cdot, t) \) be any particular Leray solution to the Navier-Stokes equations (1.1). Then we have, for every \( \alpha \geq 0 \):

\[ \limsup_{t \to \infty} t^{\alpha + m/2} \| D^m u(\cdot, t) \|_{L^2(\mathbb{R}^n)} \leq K(\alpha, m) \nu^{-m/2} \limsup_{t \to \infty} t^\alpha \| u(\cdot, t) \|_{L^2(\mathbb{R}^n)} \]  

(1.5)
In Section 2 we present our original derivation of (1.5), which was based in part on some previous ideas in [2 5 6 20]. Alternative proofs could also be developed (using e.g. Schonbek’s Fourier splitting method [13 14]), but we prefer to follow the very way in which (1.5) was first revealed.

**Notation.** As already shown, boldface letters are used for vector quantities, as in \( \mathbf{u}(x, t) = (u_1(x, t), ..., u_n(x, t)) \). Also, \( \nabla p = \nabla p(\cdot, t) \) denotes the spatial gradient of \( p(\cdot, t) \); \( D_j = \partial/\partial x_j \); \( \nabla \cdot \mathbf{u} = D_1 u_1 + ... + D_n u_n \) is the (spatial) divergence of \( \mathbf{u}(\cdot, t) \). \( L^2(\mathbb{R}^n) \) denotes the space of functions \( \mathbf{v} = (v_1, ..., v_n) \in L^2(\mathbb{R}^n) \equiv L^2(\mathbb{R}^n) \) with \( \nabla \cdot \mathbf{v} = 0 \) in the distributional sense; \( \mathcal{H}^1(\mathbb{R}^n) = \mathcal{H}^1(\mathbb{R}^n) \) with \( \mathcal{H}^1(\mathbb{R}^n) \) being the homogeneous \( L^2 \) Sobolev space of order 1; \( \mathcal{H}^m(\mathbb{R}^n) = \mathcal{H}^m(\mathbb{R}^n) \), where \( \mathcal{H}^m(\mathbb{R}^n) \) is the space of functions \( v \in L^2(\mathbb{R}^n) \) whose \( m \)-th order derivatives are also square integrable. \( C_w(I, L^2(\mathbb{R}^n)) \) denotes the set of mappings from a given interval \( I \subseteq \mathbb{R} \) to \( L^2(\mathbb{R}^n) \) that are \( L^2 \)-weakly continuous at each \( t \in I \). \( \| \cdot \|_{L^q(\mathbb{R}^n)} \), \( 1 \leq q \leq \infty \), are the standard norms of the Lebesgue spaces \( L^q(\mathbb{R}^n) \), with the vector counterparts

\[
\| \mathbf{u}(\cdot, t) \|_{L^q(\mathbb{R}^n)} = \left\{ \sum_{i=1}^{n} \int_{\mathbb{R}^n} |u_i(x, t)|^q \, dx \right\}^{1/q} \tag{1.6a}
\]

\[
\| \mathbf{D}^m \mathbf{u}(\cdot, t) \|_{L^q(\mathbb{R}^n)} = \left\{ \sum_{i,j_1,...,j_m=1}^{n} \int_{\mathbb{R}^n} |D_{j_1} \cdots D_{j_m} u_i(x, t)|^q \, dx \right\}^{1/q} \tag{1.6b}
\]

if \( 1 \leq q < \infty \); if \( q = \infty \), then \( \| \mathbf{u}(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} = \max \{ \| u_i(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} \; : \; 1 \leq i \leq n \} \), \( \| \mathbf{D} \mathbf{u}(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} = \max \{ \| D_j u_i(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} \; : \; 1 \leq i, j \leq n \} \), and so forth.

**2. Proof of (1.5)**

The derivation of (1.5) below takes advantage of the regularity property (1.3) and proceeds by induction in \( m \). It combines standard techniques (energy inequalities and related interpolation estimates) with well known properties of Leray solutions (namely, that \( \| \mathbf{u}(\cdot, t) \|_{L^2(\mathbb{R}^n)} \to 0 \) as \( t \to \infty \)), or that

\[
\lim_{t \to \infty} \| \mathbf{D} \mathbf{u}(\cdot, t) \|_{L^2(\mathbb{R}^n)} = 0, \tag{2.1}
\]

which are easy to obtain directly). As the proofs for \( n = 2, 3, 4 \) are entirely similar, we will present the details for one case only — say, \( n = 4 \). Let then \( \mathbf{u}(\cdot, t) \) be any given Leray solution to (1.1), in \( \mathbb{R}^4 \), such that we have, for some \( \alpha \geq 0 \),

\[
\limsup_{t \to \infty} t^\alpha \| \mathbf{u}(\cdot, t) \|_{L^2(\mathbb{R}^4)} =: \lambda_0(\alpha) < \infty. \tag{2.2}
\]
Let $\delta > 0$, $0 < \epsilon < 2$ be given, and let $t_*$ be the solution’s regularity time as defined in (1.3). Recalling the basic estimate

$$\| u \|_{L^2(\mathbb{R}^4)} \leq \| D u \|_{L^2(\mathbb{R}^4)}, \tag{2.3}$$

from which we get

$$\| D^\ell u \|_{L^4(\mathbb{R}^4)} \leq \| D^m u \|_{L^4(\mathbb{R}^4)} \leq \| D^m u \|_{L^4(\mathbb{R}^4)}$$

for arbitrary $m \geq 0$, $0 \leq \ell \leq m$, we may proceed along the lines of [2, 20] as follows.

Taking the dot product of (1.1a) with $(t - t_0)^{2\alpha + \delta} u(x, t)$ and integrating the result on $\mathbb{R}^4 \times [t_0, t]$, for $t \geq t_0 > t_*$, we obtain, because of (1.1b),

$$(t - t_0)^{2\alpha + \delta} \| u(\cdot, t) \|_{L^2(\mathbb{R}^4)}^2 + 2\nu \int_{t_0}^{t} (\tau - t_0)^{2\alpha + \delta} \| D u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}^2 \, d\tau$$

$$= (2\alpha + \delta) \int_{t_0}^{t} (\tau - t_0)^{2\alpha + \delta - 1} \| u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}^2 \, d\tau$$

for $t \geq t_0 \geq t_*$. This promptly gives, by (2.2), that

$$\int_{t_0}^{t} (\tau - t_0)^{2\alpha + \delta} \| D u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}^2 \, d\tau \leq \frac{1}{2\nu} \frac{2\alpha + \delta}{\delta} (\lambda_0(\alpha) + \epsilon)^2 (t - t_0)^{\delta} \tag{2.5}$$

for all $t \geq t_0$ (choosing $t_0 \geq t_*$ sufficiently large). Next, for $m = 1$, we similarly have

$$(t - t_0)^{2\alpha + 1}\| D u(\cdot, t) \|_{L^2(\mathbb{R}^4)}^2 + 2\nu \int_{t_0}^{t} (\tau - t_0)^{2\alpha + 1}\| D^2 u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}^2 \, d\tau$$

$$\leq (2\alpha + 1 + \delta) \int_{t_0}^{t} (\tau - t_0)^{2\alpha + \delta}\| D u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}^2 \, d\tau +$$

$$K_1 \int_{t_0}^{t} (\tau - t_0)^{2\alpha + 1 + \delta}\| u(\cdot, \tau) \|_{L^4(\mathbb{R}^4)}\| D u(\cdot, \tau) \|_{L^4(\mathbb{R}^4)}\| D^2 u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)} \, d\tau$$

(where $K_1 = 8 \sqrt{2}$), which gives, by (2.3):

$$(t - t_0)^{2\alpha + 1}\| D u(\cdot, t) \|_{L^2(\mathbb{R}^4)}^2 + 2\nu \int_{t_0}^{t} (\tau - t_0)^{2\alpha + 1}\| D^2 u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}^2 \, d\tau$$

$$\leq (2\alpha + 1 + \delta) \int_{t_0}^{t} (\tau - t_0)^{2\alpha + \delta}\| D u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}^2 \, d\tau +$$

$$K_1 \int_{t_0}^{t} (\tau - t_0)^{2\alpha + 1 + \delta}\| D u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}\| D^2 u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)} \, d\tau$$

for $t \geq t_0$. By (2.1) and (2.5), we then get (increasing $t_0$ if necessary):
\[(t - t_0)^{2\alpha + 1} \| D u(\cdot, t) \|_{L^2(\mathbb{R}^4)}^2 \leq \frac{1}{2\nu} (2\alpha + 1 + \delta) \frac{2\alpha + \delta}{\delta} (\lambda_0(\alpha) + \epsilon)^2 \quad (2.6a)\]

and

\[
\int_{t_0}^t (\tau - t_0)^{2\alpha + 1 + \delta} \| D^2 u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}^2 d\tau \leq \frac{(2\alpha + 1 + \delta)(2\alpha + \delta)}{\delta \cdot [(2 - \epsilon)\nu]^2} (\lambda_0(\alpha) + \epsilon)^2 (t - t_0)^\delta \quad (2.6b)
\]

for all \( t \geq t_0 \). Proceeding in this way \( (m = 2, 3, \ldots) \) we obtain at the \( m \)th step

\[
(t - t_0)^{2\alpha + m + \delta} \| D^m u(\cdot, t) \|_{L^2(\mathbb{R}^4)}^2 + 2 \nu \int_{t_0}^t (\tau - t_0)^{2\alpha + m + \delta} \| D^{m+1} u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}^2 d\tau
\]

\[
\leq (2\alpha + m + \delta) \int_{t_0}^t (\tau - t_0)^{2\alpha + m - 1 + \delta} \| D^m u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}^2 d\tau + K_m \int_{t_0}^t (\tau - t_0)^{2\alpha + m + \delta} \| D^{m+1} u(\cdot, \tau) \|_{L^4(\mathbb{R}^4)} \| D^{m+\ell} u(\cdot, \tau) \|_{L^4(\mathbb{R}^4)} d\tau \quad (2.7)
\]

for \( t \geq t_0 \), and some constant \( K_m > 0 \), where \([m/2]\) denotes the integer part of \( m/2 \).

This gives, by (2.4):

\[
(t - t_0)^{2\alpha + m + \delta} \| D^m u(\cdot, t) \|_{L^2(\mathbb{R}^4)}^2 + 2 \nu \int_{t_0}^t (\tau - t_0)^{2\alpha + m + \delta} \| D^{m+1} u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}^2 d\tau
\]

\[
\leq (2\alpha + m + \delta) \int_{t_0}^t (\tau - t_0)^{2\alpha + m - 1 + \delta} \| D^m u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}^2 d\tau + \left(1 + \left[\frac{m}{2}\right]\right) \cdot K_m \int_{t_0}^t (\tau - t_0)^{2\alpha + m + \delta} \| D^{m+1} u(\cdot, \tau) \|_{L^4(\mathbb{R}^4)} \| D^{m+\ell} u(\cdot, \tau) \|_{L^4(\mathbb{R}^4)} d\tau.
\]

At this stage, we would already know from the previous steps that

\[
(t - t_0)^{2\alpha + k} \| D^k u(\cdot, t) \|_{L^2(\mathbb{R}^4)}^2 \leq \frac{1}{\delta \cdot [(2 - \epsilon)\nu]^k} \left\{ \prod_{j=0}^k (2\alpha + j + \delta) \right\} (\lambda_0(\alpha) + \epsilon)^2 \quad (2.8a)
\]

and

\[
\int_{t_0}^t (\tau - t_0)^{2\alpha + k + \delta} \| D^{k+1} u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}^2 d\tau \leq \frac{\delta^{-1}}{[(2 - \epsilon)\nu]^{k+1}} \left\{ \prod_{j=0}^k (2\alpha + j + \delta) \right\} \times (\lambda_0(\alpha) + \epsilon)^2 \cdot (t - t_0)^\delta \quad (2.8b)
\]

for all \( t \geq t_0 \), and each \( 0 \leq k < m \). By (2.1) and (2.7), and increasing \( t_0 \) if necessary, we would then obtain (2.8) for \( k = m \) as well, completing the induction step.
The argument above established that, for each \( m \geq 1 \), we have
\[
(t - t_0)^{2\alpha + m} \| D^m \mathbf{u}(\cdot, t) \|_{L^2(\mathbb{R}^4)}^2 \leq \frac{1}{\delta \cdot (2 - \epsilon) \nu}^{m} \left\{ \prod_{j=0}^{m} (2\alpha + j + \delta) \right\} \left( \lambda_0(\alpha) + \epsilon \right)^2
\]
for all \( t \) sufficiently large. Since \( \delta > 0 \), \( 0 < \epsilon < 2 \) are arbitrary, this gives the result.

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