The CFT dual of AdS gravity with torsion

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Abstract
We consider the Mielke–Baekler model of three-dimensional AdS gravity with torsion, which has gravitational and translational Chern–Simons terms in addition to the usual Einstein–Hilbert action with cosmological constant. It is shown that the topological nature of the model leads to a finite Fefferman–Graham expansion. We derive the holographic stress tensor and the associated Ward identities and show that, due to the asymmetry of the left- and right-moving central charges, a Lorentz anomaly appears in the dual conformal field theory. Both the consistent and the covariant Weyl and Lorentz anomaly are determined, and the Wess–Zumino consistency conditions for the former are verified. Moreover we consider the most general solution with flat boundary geometry, which describes left- and right-moving gravitational waves on AdS3 with torsion, and show that in this case the holographic energy–momentum tensor is given by the wave profiles. The anomalous transformation laws of the wave profiles under diffeomorphisms preserving the asymptotic form of the bulk solution yield the central charges of the dual CFT and confirm the results that appeared earlier on in the literature. Finally we comment on some points concerning the microstate counting for the Riemann–Cartan black hole.

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1. Introduction

According to the AdS/CFT correspondence (cf [1] for a review), any theory of gravity on a \((d + 1)\)-dimensional asymptotically anti-de Sitter space is dual to a conformal field theory living on the \(d\)-dimensional boundary of AdS. This allows us to compute CFT correlation functions of operators \(\mathcal{O}\) by considering fields \(\phi\) propagating in the \((d + 1)\)-dimensional bulk spacetime. The boundary value \(\phi_0\) of \(\phi\) represents a source for the associated operator \(\mathcal{O}\). By turning on various bulk fields one can deform the corresponding CFT, and break symmetry explicitly or spontaneously, depending on the boundary condition on \(\phi\).
A generalization that has not been investigated very much up to now is to admit torsion in the gravity theory, and addressing this point is the purpose of the present paper. We will study the effects of torsion in a simple setting, represented by a topological model of three-dimensional gravity, whose equations of motion imply both constant curvature and constant torsion. What makes this model particularly appealing is the fact that, similar to ordinary three-dimensional general relativity with negative cosmological constant, it can be written as a sum of two $SL(2, \mathbb{R})$ Chern–Simons theories, but with unequal coupling constants.

We derive the central charges of the dual CFT, the holographic energy–momentum tensor and the associated (anomalous) Ward identities. In particular, there is a Lorentz anomaly, which comes from the presence of a gravitational Chern–Simons term in the bulk action, invariant under local Lorentz transformations only up to a boundary term. The holographic description of diffeomorphism and Lorentz anomalies by gravitational Chern–Simons terms was explored in [6]. We find that bulk torsion modifies the trace anomaly, but the Lorentz anomaly is given by the prefactor of the gravitational Chern–Simons term alone.

This paper is organized as follows. In section 2, we briefly review the Mielke–Baekler model of three-dimensional gravity with torsion, and its formulation as a Chern–Simons theory. In the following section, we work out the Fefferman–Graham expansion for the dreibein and the spin connection and show that it is finite. In section 4, the holographic stress tensor and the associated anomalous Ward identities are obtained. We determine both the consistent and the covariant anomalies, as well as the Bardeen–Zumino polynomial relating them. It is furthermore shown that no diffeomorphism (Einstein) anomaly appears. We then consider the most general bulk solution with flat boundary, which represents left- and right-moving gravitational waves on AdS$_3$ with torsion. In this case, the CFT energy–momentum tensor reduces to the wave profiles, and transforms anomalously under diffeomorphisms preserving the asymptotic form of the solution. From the transformation laws one can read off the central charges, and confirm the results of [7]. Finally, in section 5 we discuss some points related to the microstate counting for the Riemann–Cartan black hole. In the appendix, we check that our anomalies satisfy the Wess–Zumino consistency conditions.

### 2. Three-dimensional gravity with torsion

A simple three-dimensional model that yields nonvanishing torsion was proposed by Mielke and Baekler (MB) [3] and further analysed by Baekler, Mielke and Hehl [8]. The action reads [3]

\[
I = aI_1 + \Lambda I_2 + \alpha_3 I_3 + \alpha_4 I_4,
\]

where $a$, $\Lambda$, $\alpha_3$ and $\alpha_4$ are constants,

\[
I_1 = 2 \int \hat{e}_A \wedge \hat{R}^A,
\]

\[
I_2 = -\frac{1}{3} \int \epsilon_{ABC} \hat{e}^A \wedge \hat{e}^B \wedge \hat{e}^C,
\]

\[
I_3 = \int \hat{\omega}_A \wedge d \hat{\omega}^A + \frac{1}{3} \epsilon_{ABC} \hat{\omega}^A \wedge \hat{\omega}^B \wedge \hat{\omega}^C,
\]

3 The holographic currents associated with five-dimensional Chern–Simons gravity with nonvanishing torsion were studied in [2].

4 Our conventions are as follows: $A, B, \ldots$ are 3D Lorentz indices, while $\mu, \nu, \ldots$ are 3D spacetime indices. Two-dimensional Lorentz and world indices on the boundary of AdS$_3$ are denoted by $a, b, \ldots$ and $i, j, \ldots$, respectively. The signature is mostly plus, and hatted fields are objects in three dimensions.
\[ I_4 = \int \hat{e}_A \wedge \hat{T}^A, \]

and

\[ \hat{R}^A = d\hat{\omega}^A + \frac{1}{2} \epsilon^A_{\ BC}\hat{\omega}^B \wedge \hat{\omega}^C, \quad \hat{T}^A = d\hat{\epsilon}^A + \epsilon^A_{\ BC}\hat{\omega}^B \wedge \hat{\epsilon}^C, \qquad (2.2) \]

denote the curvature and torsion 2-forms, respectively. \( \hat{\omega}^A \) is defined by \( \hat{\omega}^A = \frac{1}{2} \epsilon^{AB\ C}\hat{\omega}_{BC} \) with \( \epsilon_{012} = 1 \). \( I_4 \) yields the Einstein–Hilbert action, \( I_2 \) is a cosmological constant, \( I_3 \) is a Chern–Simons term for the spin connection and \( I_4 \) represents a translational Chern–Simons term. Note that, in order to obtain the topologically massive gravity of Deser, Jackiw and Templeton (DJT) \[ (2.1) \] from (2.1), one has to add a Lagrange multiplier term that ensures vanishing torsion. The field equations following from (2.1) take the form

\[ 2a\hat{R}^A - \Lambda\epsilon^A_{\ BC}\hat{e}^B \wedge \hat{e}^C + 2\alpha_3\hat{T}^A = 0, \]
\[ 2a\hat{T}^A + 2\alpha_3\hat{R}^A + \alpha_4\epsilon^A_{\ BC}\hat{e}^B \wedge \hat{e}^C = 0. \]

In what follows, we assume \( \alpha_3\alpha_4 - a^2 \neq 0 \). Then the equations of motion can be rewritten as

\[ 2\hat{T}^A = A\epsilon^A_{\ BC}\hat{e}^B \wedge \hat{e}^C, \quad 2\hat{R}^A = B\epsilon^A_{\ BC}\hat{e}^B \wedge \hat{e}^C, \qquad (2.3) \]

where

\[ A = \frac{\alpha_3\Lambda + \alpha_4a}{\alpha_3\alpha_4 - a^2}, \quad B = -\frac{a\Lambda + \alpha_4^2}{\alpha_3\alpha_4 - a^2}. \]

Thus, the field configurations are characterized by constant curvature and constant torsion. From (2.2) one obtains

\[ \hat{\omega}^A = \hat{\omega}^{(0)A} - \hat{\hat{K}}^A, \qquad (2.4) \]

where \( \hat{\omega}^{(0)A} \) denotes the Christoffel connection and \( \hat{\hat{K}}^A \) is the contorsion 1-form given by

\[ \hat{\hat{K}}^A_{\ \mu} = \frac{1}{2} \epsilon_A^\ B\ C\hat{\hat{e}}^B \hat{\hat{e}}^C \hat{e}^\gamma \hat{K}_{\ B\ C\ \mu}, \]

with the contorsion tensor

\[ \hat{K}_{\ B\ C\ \mu} = \frac{1}{2} (\hat{T}_{\ B\ C\ \mu} - \hat{T}_{\ C\ B\ \mu} - \hat{T}_{\ B\ C\ \mu}), \]

and \( \hat{T}_{\ B\ C\ \mu} = \hat{e}_A\hat{T}^A_{\ B\ C\ \mu} \). Equation (2.4) allows us to express the curvature \( \hat{R}^A \) of a Riemann–Cartan spacetime in terms of its Riemannian part \( \hat{R}^{(0)A} \) and \( \hat{K}^A \),

\[ \hat{R}^A = \hat{R}^{(0)A} - d\hat{K}^A - \epsilon^A_{\ BC}\hat{e}^B \wedge \hat{K}^C - \frac{1}{2}\epsilon^A_{\ BC}\hat{e}^B \wedge \hat{K}^C. \quad (2.5) \]

Using the equations of motion (2.3) into (2.5), one obtains for the Riemannian part

\[ 2\hat{R}^{(0)A} = \Lambda_{\ eff}\epsilon^A_{\ BC}\hat{e}^B \wedge \hat{e}^C, \qquad (2.6) \]

with the effective cosmological constant

\[ \Lambda_{\ eff} = B - \frac{A^2}{4}. \]

This means that locally the metric is given by the (anti-)de Sitter or Minkowski solution, depending on whether \( \Lambda_{\ eff} \) is negative, positive or zero. It is interesting to note that \( \Lambda_{\ eff} \) can be nonvanishing even if the bare cosmological constant \( \Lambda \) is zero [8]. In this simple model, dark energy (i.e., \( \Lambda_{\ eff} \)) would then be generated by the translational Chern–Simons term \( I_4 \).

In [4], it was shown that for \( \Lambda_{\ eff} < 0 \), the Mielke–Baekler model (2.1) can be written as a sum of two \( SL(2,\ \mathbb{R}) \) Chern–Simons theories. This was then generalized in [5] to the case of arbitrary effective cosmological constant. In what follows we shall be interested in the case \( \Lambda_{\ eff} < 0 \), so we briefly summarize the results of [4]. For \( \Lambda_{\ eff} < 0 \) the geometry is locally

\[ \epsilon A_{\ B C} = 1 \] Some aspects of three-dimensional gravity with gravitational Chern–Simons term were studied in [9].

\[ \alpha A_{\ B C} \neq 0 \] the theory becomes singular [8].
AdS$_3$, which has the isometry group $\text{SO}(2, 2) \cong \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$, so if the MB model is equivalent to a Chern–Simons theory, one expects a gauge group $\text{SO}(2, 2)$. Indeed, if one defines the $\text{SL}(2, \mathbb{R})$ connections
\[ A^A = \hat{\omega}^A + q \hat{\epsilon}^A, \quad \tilde{A}^A = \hat{\omega}^A + \tilde{q} \hat{\epsilon}^A, \]
then the $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ Chern–Simons action$^7$
\[ I_{\text{CS}} = \frac{t}{8\pi} \int \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) - \frac{\tilde{t}}{8\pi} \int \left( \tilde{A} \wedge d\tilde{A} + \frac{2}{3} \tilde{A} \wedge \tilde{A} \wedge \tilde{A} \right) \]
(2.7)
coinsides (up to boundary terms) with $I$ in (2.1), if the parameters $q, \tilde{q}$ and the coupling constants $t, \tilde{t}$ are given by
\[ q = -\frac{A}{2} + \sqrt{-\Lambda_{\text{eff}}}, \quad \tilde{q} = -\frac{A}{2} \]
and
\[ \frac{t}{2\pi} = 2\alpha_3 + \frac{2a + \alpha_3 A}{\sqrt{-\Lambda_{\text{eff}}}}, \quad \frac{\tilde{t}}{2\pi} = -2\alpha_3 + \frac{2a + \alpha_3 A}{\sqrt{-\Lambda_{\text{eff}}}}. \]
(2.8)
(2.9)
We see that $q, \tilde{q}$, and thus the connections $A^A, \tilde{A}^A$ are real for negative $\Lambda_{\text{eff}}$. The coupling constants $t, \tilde{t}$ are also real, but in general different from each other due to the presence of $I_3$.

3. Finite Fefferman–Graham expansion

Let us now determine the Fefferman–Graham (FG) expansion$^{11}$ for the dreibein $\hat{e}^A$ and the spin connection $\hat{\omega}^A$, which will turn out to be finite$^8$. To this end, we proceed in a similar way to $[2, 13]$, using the CS formulation of the MB model. First of all, one assumes that the manifold is diffeomorphic to $M_2 \times \mathbb{R}$ asymptotically and that it is parametrized by the local coordinates $x^\mu = (x^i, \rho)$, with $\rho$ denoting the radial coordinate and $M_2$ being the spacetime on which the dual CFT resides. The corresponding Lorentz indices are split as $A = (a, 2)$. The field equations $F = \tilde{F} = 0$ following from (2.7) imply
\[ \partial_{\rho} A_i - \partial_i A_{\rho} + [A_{\rho}, A_i] = 0, \]
(3.1)
and an analogous equation for $\tilde{A}$. Note that the simplest gauge choice $A_{\rho} = \tilde{A}_{\rho} = 0$ is not allowed, as this would lead to a degenerate dreibein. A nondegenerate choice is to take $A_{\rho}$ and $\tilde{A}_{\rho}$ to be constant Lie algebra elements. The general solution of (3.1) is then given by
\[ A_i(\rho, x^j) = e^{-\rho A_i} A_i(0, x^j)e^{\rho A_i}. \]
(3.2)
As in [13] we choose $A_{\rho} = \tau_2, \tilde{A}_{\rho} = -\tau_2$, so that (3.2) leads to
\[ A_i(\rho, x^j) = A_i^0(0, x)(\tau_0 \cosh \rho - \tau_1 \sinh \rho) + A_i^1(0, x)(\tau_1 \cosh \rho - \tau_0 \sinh \rho) + A_i^2(0, x) \tau_2, \]
\[ \tilde{A}_i(\rho, x^j) = \tilde{A}_i^0(0, x)(\tau_0 \cosh \rho + \tau_1 \sinh \rho) + \tilde{A}_i^1(0, x)(\tau_1 \cosh \rho + \tau_0 \sinh \rho) + \tilde{A}_i^2(0, x) \tau_2. \]

Next, we shall impose one extra condition on the vielbein, namely $\hat{e}^2_i = 0$, or equivalently $A_i^2(0, x) = 0$. This breaks three-dimensional Lorentz symmetry down to a two-dimensional one, and leaves a 2D tetrad as a gravitational source. Moreover, it ensures that

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$^7$ In (2.7), $(\tau_A, \tau_B) = 2(\tau_{x_A}, \tau_{x_B}) = \epsilon_{AB}^C \tau_C$, and the $\text{SL}(2, \mathbb{R})$ generators $\tau_A$ satisfy $[\tau_A, \tau_B] = \epsilon_{AB}^C \tau_C$.

$^8$ The fact that three-dimensional Einstein spaces with negative curvature have a finite FG expansion was first shown in [12].
the boundary metric is torsion free \([13]\). One obtains then the finite FG expansion
\[
\hat{e}^a(\rho, x) = e^a e^\beta(x) + e^{-\rho} k^a(x),
\]
\[
\hat{e}^2(\rho, x) = \ell d\rho,
\]
\[
\hat{\omega}^a(\rho, x) = e^\rho \left\{ \frac{A}{2} e^a(x) + \frac{1}{\ell} e^a e^b(x) \right\} + e^{-\rho} \left\{ \frac{A}{2} k^a(x) - \frac{1}{\ell} e^a k^b(x) \right\},
\]
\[
\hat{\omega}^2(\rho, x) = \omega(x) + \frac{A\ell}{2} d\rho,
\]
for the dreibein and the spin connection, with \(\ell\) defined by
\[
\Lambda_{\text{eff}}^{-1} = -\frac{1}{\ell^2}, \quad \epsilon_{01} = 1, \quad \omega_i(x) = A^i_2(0, x)
\]
and
\[
e^a_i = \frac{\ell}{4} \left( A^a_i(0, x) - \tilde{A}^a_i(0, x) \right) + \frac{\ell}{4} e^a e^b (A^b_i(0, x) + \tilde{A}^b_i(0, x)),
\]
\[
k^a_i = \frac{\ell}{4} \left( A^a_i(0, x) - \tilde{A}^a_i(0, x) \right) - \frac{\ell}{4} e^a k^b (A^b_i(0, x) + \tilde{A}^b_i(0, x)).
\]
\(e^a\) and \(\omega^{ab} = -e^{ab}\omega\) represent the tetrad and the spin connection on the CFT manifold \(M_2\). Finally, the FG expansion of the three-dimensional line element \(\hat{e}^{A\mu} \hat{e}^{A\nu} d\xi^\mu d\xi^\nu\) is given by
\[
d\hat{s}^2 = \left[ e^{2\rho} g_{ij} + 2 k_{ij} + e^{-2\rho} \epsilon_{ab} k^a_i k^b_j \right] dx^i dx^j + \ell^2 d\rho^2,
\]
where \(g_{ij} = \eta_{ab} e^a_i e^b_j\) and \(k_{ij} = e_{ai} k^a_j\). Note that the equations of motion \((2.3)\) for \(\hat{T}^a\) imply
\[
dk^a - e^a_{\ b\omega} \wedge k^b = 0,
\]
as well as
\[
de^a - e^a_{\ b\omega} \wedge e^b = 0,
\]
i.e. the boundary torsion indeed vanishes. Equation \((2.3)\) for \(\hat{R}^2\) gives furthermore \(k_{ij} = 0\), whereas \(\hat{R}^2\) yields
\[
d\omega + \frac{2}{\ell^2} \epsilon_{ab} e^a \wedge k^b = 0,
\]
and the field equation for \(\hat{R}^a\) is identically satisfied.

4. **Holographic stress tensor**

In order to find the holographic energy–momentum tensor, we vary the action \((2.1)\) on-shell, to obtain
\[
\delta I = \int_{M_2} \left\{ -2 \alpha \hat{\varepsilon}^A \wedge \delta \hat{\omega}_A - \alpha_3 \hat{\omega}^A \wedge \delta \hat{\omega}_A - \alpha_4 \hat{e}^A \wedge \delta \hat{e}_A \right\}.
\]
Next, we evaluate this variation on the asymptotic solution \((3.3)\). One finds that the only divergent term in the limit \(\rho \to \infty\) is given by
\[
\delta I_{\text{div}} = -\frac{2\alpha}{\ell} \int e^{2\rho} \epsilon_{ab} e^a \wedge \delta e^b.
\]
This can be removed by adding to the action a local counterterm
\[
I_{\text{ct}} = \frac{\alpha}{\ell} \int \epsilon_{ab} \hat{\varepsilon}^a \wedge \hat{\varepsilon}^b,
\]
which is the usual counterterm needed to regularize AdS$_3$ gravity \cite{14}. Up to terms that cancel in the limit $\rho \to \infty$ one obtains then

$$\delta(I + I_\text{ct}) = -\frac{2\alpha_3}{\ell^2} \int e_a \wedge \delta k^a + \left(\frac{4a}{\ell} + \frac{A}{\ell}\right) \int \epsilon_{abc} e^a \wedge \delta k^b$$

$$-\frac{2\alpha_3}{\ell^2} \int k_a \wedge \delta e^a - \alpha_3 \frac{A}{\ell} \int \epsilon_{abc} k^a \wedge \delta e^b - \alpha_3 \int \omega \wedge \delta \omega.$$ 

The next step is to transform variations of $k^a$ into variations of $e^a$. Up to finite boundary terms, that we are free to add, one has

$$e_a \wedge \delta k^a = k_a \wedge \delta e^a,$$

and a similar expression for $\epsilon_{abc} k^a \wedge \delta e^b$. In this way, we finally arrive at

$$\delta I_{\text{tot}} = -\frac{4\alpha_3}{\ell^2} \int k_a \wedge \delta e^a - \left(\frac{4a}{\ell} + 2\alpha_3 A\right) \int \epsilon_{abc} k^a \wedge \delta e^b - \alpha_3 \int \omega \wedge \delta \omega,$$ \hspace{1cm} (4.2)

where $I_{\text{tot}} = I + I_\text{ct} + I_{\text{fin}}$. One can now define the holographic energy–momentum tensor by\textsuperscript{10}

$$T^i_a = \frac{2\pi}{|e|} \frac{\delta I_{\text{tot}}}{\delta e^{ai}} = \frac{2\pi e^{ij}}{|e|} \left[ -\frac{4\alpha_3}{\ell^2} k_{aj} + \frac{2}{\ell} \left[ 2a + a_3 A \right] \epsilon_{abc} k^b_j + \alpha_3 \epsilon_{am} \nabla_j (\ast \omega)^m \right].$$ \hspace{1cm} (4.3)

As was said earlier, the boundary torsion is zero, and thus the spin connection $\omega$ is determined completely by $e^a$. This means that $\delta \omega$ in (4.2) has to be expressed in terms of $\delta e^a$, and contributes to the stress tensor\textsuperscript{11}. Note also that $T^i_a$ is the Hodge dual of the energy–momentum 1-form $\tau_a$, $T^i_a = |e|^{-1} e^{ij} \tau_j$.

### 4.1. Anomalies

Let us now consider the Ward identities satisfied by the stress tensor (4.3). First of all, its trace is given by

$$T = e^a T^i_a = \pi \ell \left[ 2a + a_3 A \right] R - 2\pi \alpha_3 \nabla_i \omega^i,$$ \hspace{1cm} (4.4)

where $R$ denotes the scalar curvature of the boundary. To obtain (4.4), we used $k_{(i)j} = 0$ and equation (3.7), which implies

$$R = \frac{4}{\ell^2 |e|} \epsilon^{ij} \epsilon_{ab} e^a k^b_j.$$

Using the central charges

$$c_L = 24\pi \left[ a \ell + a_3 \left( A \ell - 1 \right) \right], \quad c_R = 24\pi \left[ a \ell + a_3 \left( A \ell + 1 \right) \right],$$ \hspace{1cm} (4.5)

of the dual conformal field theory, obtained in \cite{7} by computing the Poisson bracket algebra of the asymptotic symmetry generators, (4.4) can be rewritten as

$$T = \frac{c_L + c_R}{24} R - 2\pi \alpha_3 \nabla_i \omega^i.$$ \hspace{1cm} (4.6)

\textsuperscript{9} Note that $a = 1/16\pi G$.

\textsuperscript{10} In (4.3), $e^{ij}$ is defined by $e^{i*} = -1$, if $t, x$ are local coordinates on $M_2$. The orientation is such that $dx^i \wedge dx^j = -e^{ij} dx^x$, and the Hodge dual is defined by $(\ast \omega)^i = |e|^{-1} e^{ij} \omega_j$. $\nabla_j$ denotes the covariant derivative on $M_2$.

\textsuperscript{11} If the tetrad and the spin connection were independent, the last term in (4.2) would not contribute to the stress tensor, but would give rise to a spin current $\sigma^i = |e|^{-1} \delta I_{\text{tot}}/\delta \omega^i$. In five dimensions, such a scenario was considered in \cite{2}. 


The first piece is the usual covariant expression for the trace anomaly, whereas the second one transforms non-covariantly under local Lorentz transformations. We will come back to this point later.

The energy–momentum tensor \( T_{ab} \) is not symmetric,

\[
T_{ab} - T_{ba} = 2\pi \alpha^*_R R_{ab} = \frac{CR - CL}{24} R_{ab},
\]

(4.7)

where \( T_{ab} = e_a T^i_i T^j_j \) and \(* R_{ab} = (2|e|)^{-1}e^{ij}R_{ab}^{ij} \) is the Hodge dual of the Riemann tensor. Equation (4.7) means that there is a Lorentz anomaly in the dual field theory \([15–18]\): under an infinitesimal local Lorentz transformation the zweibein transforms as

\[
\delta \alpha e^a_l = -\alpha^{ab} e^b_l,
\]

so the variation of the quantum effective action is

\[
\delta \alpha / \Gamma_1 \text{eff} = \frac{1}{2\pi} \int d^2x |e| \alpha^{ab} T_{ba}.
\]

Since \( \alpha^{ab} \) is antisymmetric, it follows that the non-invariance of the effective action under local Lorentz transformations is equivalent to asymmetry of \( T_{ab} \).

Let us finally compute the divergence of (4.3). Making use of (3.5), one obtains

\[
\nabla_i T^i_a = \alpha_0 \alpha^a_j \omega^b_{ij},
\]

(4.8)

where \( \nabla_i \) denotes the covariant derivative with respect to both local Lorentz transformations and diffeomorphisms, i.e.

\[
\nabla_i T^i_a = \partial_i T^i_a + \Gamma^i_{ij} T^j_a - \omega^b_{ia} T^i_b.
\]

To see that (4.8) is the correct Ward identity, observe that under an infinitesimal coordinate transformation \( x^i \mapsto x^i - \xi^i \), the zweibein varies as

\[
\delta \xi e^a_l = e^a_j \tilde{\nabla}_j \xi^i + \xi^i \tilde{\nabla}_j e^a_l,
\]

with \( \tilde{\nabla}_j e^a_l = \partial_j e^a_l - \Gamma^k_{ji} e^a_k \) being the covariant derivative w.r.t. diffeomorphisms. Using \( \delta \Gamma_1 \text{eff} / \delta e^a_l = |e| T^i_a / 2\pi \), the variation of the effective action becomes

\[
\delta \xi / \Gamma_1 \text{eff} = \frac{1}{2\pi} \int d^2x |e| \alpha^{ab} \left[ -\nabla_i T^i_j + \omega^b_{ij} T_{ab} \right].
\]

(4.9)

Integrating by parts the first term, using \( T^i_j = T^i_j e^a_j \) and \( \tilde{\nabla}_j e^a_l = -\omega^b_{jk} e^b_l \) (which follows from \( \nabla_j e^a_l = 0 \)), one finally obtains

\[
\delta \xi / \Gamma_1 \text{eff} = \frac{1}{2\pi} \int d^2x |e| \xi^j \left[ -\nabla_j T^i_j + \omega^b_{ij} T_{ab} \right].
\]

(4.10)

If Lorentz symmetry is preserved so that \( T_{ab} \) is symmetric, the term on the r.h.s. vanishes due to the antisymmetry of the spin connection, and one has the usual conservation law \( \nabla_i T^i_j = 0 \). In our case, however, Lorentz symmetry is broken, and the antisymmetric part of \( T_{ab} \) is given by (4.7). Plugging this into (4.10) yields exactly (4.8). This means that in the field theory dual of (2.1), diffeomorphism invariance is preserved. Note that, by adding local counterterms, it is always possible to shift the Lorentz anomaly into a diffeomorphism anomaly and vice versa \([16]\).

As we said earlier, the trace (4.6) of the stress tensor is not covariant. This is a general feature of anomalies: there are consistent and covariant anomalies \([16]\). The former satisfy the Wess–Zumino consistency conditions \([19]\) and the corresponding currents are obtained by varying the vacuum functional with respect to the gauge potential, whereas the latter are obtained by adding to the corresponding consistent anomaly a local function of the gauge
potential (the so-called Bardeen–Zumino polynomial). The resulting current is covariant under local gauge transformations. In our case, by adding to the energy–momentum tensor (4.3) the Bardeen–Zumino polynomial

\[ \mathcal{P}^i_a = \frac{2\pi \alpha_3}{|\epsilon|} \epsilon^{ij} \epsilon_{am} \nabla_j (\omega)^m, \]  

we get the covariantly transforming stress tensor \( \tilde{T}^i_a = T^i_a + \mathcal{P}^i_a \), whose trace and divergence are given, respectively, by

\[ \tilde{T}^a = c_L + c_R \frac{24}{\ell^2} R, \quad \nabla_i \tilde{T}^i_a = 0. \]  

For the antisymmetric part of \( \tilde{T}^a_{ab} \) one obtains

\[ \tilde{T}^a_{ab} - \tilde{T}^b_{a} = 4\pi \alpha_3^2 R_{ab}, \]  

which is twice the right-hand side of (4.7). Observe that \( \tilde{T}^i_a \) is exactly the result we would have obtained by dropping the contribution of the last piece in (4.2), i.e., by considering the zweibein and the spin connection as independent fields.

4.2. Chern–Simons gauge transformations

A particular example resolving the constraints (3.5), (3.6) and (3.7) is given by

\[ e^0 = \frac{\ell}{2} (du - dv), \quad e^1 = \frac{\ell}{2} (du + dv), \quad \omega = 0, \]  

\[ k^0 = 2G [ -\tilde{L}(u) du + L(v) dv ], \quad k^1 = 2G [ \tilde{L}(u) du + L(v) dv ], \]  

where \( u = (x + t)/\ell, v = (x - t)/\ell \) are light-cone coordinates on the boundary, \( L(v) \) and \( \tilde{L}(u) \) denote arbitrary functions, and \( G \) is the 3D Newton constant. The corresponding three-dimensional line element reads

\[ ds^2 = 4G \ell (\tilde{L} du^2 + L dv^2) + (\ell^2 \epsilon^{2\rho} + 16G^2 L \tilde{L} e^{-2\rho}) du dv + \ell^2 d\rho^2. \]  

Equation (4.14) represents a generalization to nonvanishing torsion of the general solution with flat boundary geometry obtained in [20]. \( L \) and \( \tilde{L} \) describe right- and left-moving gravitational waves on AdS3, respectively. Using (4.14) in (4.3) yields the holographic stress tensor

\[ T_{vv} = \frac{2Gc_L}{3\ell} \tilde{L}(u), \quad T_{uu} = \frac{2Gc_R}{3\ell} \tilde{L}(u), \quad T_{uv} = T_{vu} = 0. \]  

In the case \( \alpha_3 = \alpha_4 = 0 \), when \( c_L = c_R = 3\ell/2G \) [21], this reduces to \( T_{vv} = L, T_{uu} = \tilde{L} \), as it must be [20].

The Chern–Simons connections corresponding to the solution (4.14) are

\[ A^0_u = -e^\rho + e^{-\rho} \frac{4GL}{\ell}, \quad A^1_v = e^\rho + e^{-\rho} \frac{4GL}{\ell}, \quad A^2_\rho = 1, \]  

\[ \tilde{A}^0_u = -e^\rho + e^{-\rho} \frac{4G\tilde{L}}{\ell}, \quad \tilde{A}^1_v = -e^\rho - e^{-\rho} \frac{4G\tilde{L}}{\ell}, \quad \tilde{A}^2_\rho = -1, \]  

and all other components vanishing. We may now ask which gauge transformations preserve this form of the connection. Under an infinitesimal gauge transformation the connection \( A \) changes according to

\[ \delta A = -du - [A, u]. \]
where \( u = u^A \tau_A \) is an \( sl(2, \mathbb{R}) \)-valued scalar. One finds that the form (4.17) is preserved iff

\[
\begin{align*}
\mathbf{u}^0 &= -\alpha(v) e^\rho + \left[ \frac{4GL}{\ell} \alpha(v) - \frac{\alpha'(v)}{2} \right] e^{-\rho}, \\
\mathbf{u}^1 &= \alpha(v) e^\rho + \left[ \frac{4GL}{\ell} \alpha(v) - \frac{\alpha'(v)}{2} \right] e^{-\rho}, \\
\mathbf{u}^2 &= -\alpha'(v),
\end{align*}
\]

where \( \alpha(v) \) denotes an arbitrary function. The variation of \( L \) is

\[
\delta L = -2\alpha'(v) L - \alpha(v) L' + \frac{\ell}{8G} \alpha'''(v),
\]

which implies

\[
\delta T_{vv} = -2\alpha'(v) T_{vv} - \alpha(v) T_{vv}' + \frac{c_R}{12} \alpha'''(v) \tag{4.18}
\]

for the component \( T_{vv} \) of the stress tensor. Equation (4.18) is the correct transformation law under conformal transformations, and confirms that \( c_R \) is the central charge of the right-moving sector. An analogous calculation for \( \tilde{\mathbf{A}} \) yields the transformation law for \( T_{uu} \) with anomaly proportional to \( c_L \). Note that one has \( c_R = 6\ell, c_L = 6\ell \), where \( \ell \) and \( \ell' \) denote the Chern–Simons coupling constants (2.9).

5. Entropy of the Riemann–Cartan black hole

If we choose

\[
\begin{align*}
L(v) &= \frac{m\ell - j}{2}, \\
\bar{L}(u) &= \frac{m\ell + j}{2},
\end{align*}
\]

where \( m \) and \( j \) are constants, and change the coordinates according to

\[
e^{2\rho} = \frac{1}{2} \left[ \sqrt{\frac{r^4}{\ell^2} - 8Gm + \frac{16G^2 j^2}{\ell^2} + \frac{r^2}{\ell^2} - 4Gm} \right], \quad u = \phi + \frac{t}{\ell}, \quad v = \phi - \frac{t}{\ell},
\]

equation (4.14) reduces to the so-called Riemann–Cartan (RC) black hole [22], whose metric is identical to that of the BTZ solution,

\[
d\hat{s}^2 = -N^2 dt^2 + \frac{dr^2}{N^2} + r^2 (d\phi + N^\phi dt)^2, \tag{5.1}
\]

with

\[
N^2 = -8Gm + \frac{r^2}{\ell^2} + \frac{16G^2 j^2}{r^2}, \quad N^\phi = \frac{4Gj}{r^2}.
\]

Note that the spin connection is different from the Christoffel connection due to nonvanishing torsion.

The holographic stress tensor (4.16) corresponding to the RC black hole is given by

\[
T_{vv} = \frac{Gc_R}{3\ell} (m\ell - j) \equiv T_0, \quad T_{uu} = \frac{Gc_L}{3\ell} (m\ell + j) \equiv \tilde{T}_0. \tag{5.2}
\]

\( T_0 \) and \( \tilde{T}_0 \) are the zero-modes in a Fourier expansion of the energy–momentum tensor. The mass and angular momentum of the solution are

\[
\begin{align*}
M &= \frac{1}{\ell} (T_0 + \tilde{T}_0) = m + \frac{\alpha_3}{a} \left( \frac{Am}{2} - \frac{j}{\ell^2} \right), \\
J &= \tilde{T}_0 - T_0 = j + \frac{\alpha_3}{a} \left( \frac{Aj}{2} - m \right). \tag{5.3}
\end{align*}
\]
The conserved charges (5.3) coincide with the ones computed in [22, 23]. For AdS$_3$ in global coordinates, which represents the ground state and corresponds to $j = 0$, $8Gm = -1$, one obtains

$$M_{AdS_3} = -2\pi \ell \left[ a + \alpha_3 \frac{A}{2} \right] = -\frac{c_R + c_L}{24}, \quad J_{AdS_3} = 2\pi \alpha_3 = \frac{c_R - c_L}{24}.$$ 

The nonvanishing ground-state angular momentum is due to the asymmetry of the central charges, which prevents the left- and right-moving zero point momenta from cancelling each other [6].

The entropy of the RC black hole was obtained in [24] by calculating the Euclidean action, with the result

$$S = \frac{2\pi r_+}{4G} + 4\pi^2 \alpha_3 \left( A r_+ - 2\frac{r_+}{\ell} \right),$$

where

$$r_\pm = 4Gm\ell^2 \left[ 1 \pm \sqrt{1 - \frac{j^2}{m^2\ell^2}} \right]$$

are the locations of the outer and inner horizon. The first term in (5.4) is the standard Bekenstein–Hawking result, proportional to the area of the event horizon, whereas the second term represents a correction due to the other terms in the action (2.1). The quantities $S, M, J$ satisfy the first law of thermodynamics [24]

$$dM = T dS + \Omega dJ,$$

with the Hawking temperature $T$ and the angular velocity of the horizon $\Omega$ given by

$$T = \frac{r_+^2 - r_-^2}{2\pi \ell^2 r_+}, \quad \Omega = \frac{4Gj}{r_+^2}.$$ 

Using the central charges (4.5) and the conformal weights $T_0, \tilde{T}_0$ in the Cardy formula yields the microscopic entropy

$$S_{\text{micro}} = 2\pi \sqrt{\frac{c_R T_0}{6} + 2\pi \sqrt{\frac{c_L \tilde{T}_0}{6}}},$$

which agrees exactly with the thermodynamic entropy (5.4). This was first shown in [25]. Note that the derivation of the Cardy formula uses modular invariance of the CFT partition function (see, e.g., [26]), which requires $c_R - c_L$ to be a multiple of 24 [27], i.e., one must have

$$2\pi \alpha_3 \in \mathbb{Z}.$$ 

Note in this context that in the Euclidean signature, the gauge group in the CS formulation of the MB model becomes $SL(2, \mathbb{C})$, with maximal compact subgroup $SU(2)$, so that $\alpha_3$ is subject to a topological quantization condition [5, 28], which might be related to (5.6).

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Appendix. Wess–Zumino consistency conditions

In this appendix, we showed that the anomalies (4.6) and (4.7) satisfy the Wess–Zumino consistency conditions [19]. It was shown in section 4.1 that under an infinitesimal local Lorentz transformation $\alpha_{ab}$, the vacuum functional changes as

$$\delta_\alpha \Gamma_{\text{eff}} = \frac{1}{2\pi} \int d^2x |e| \alpha^{ab} T_{ab}. \quad (A.1)$$

Let us assume that the Lorentz anomaly takes the form $\epsilon_{ab} T_{ab} = \beta R$ for some constant $\beta$ (in our case $\beta = -\pi \alpha_3$). Under an infinitesimal local Weyl transformation $\delta \varphi = \varphi$, \( (A.1) \) varies as

$$\delta_\varphi \delta_\alpha \Gamma_{\text{eff}} = -\beta \pi \int d^2x |e| \nabla^2 \varphi, \quad (A.2)$$

where the function $\alpha$ is defined by $\alpha_{ab} = \alpha \epsilon_{ab}$. On the other hand, applying first a Weyl transformation yields

$$\delta_\varphi \Gamma_{\text{eff}} = \frac{1}{2\pi} \int d^2x |e| T \varphi. \quad (A.3)$$

Under the assumption $T = \gamma R + \tilde{\gamma} \nabla_i \omega^i$, with $\gamma, \tilde{\gamma}$ constants (in our case $\gamma = (c_L + c_R)/24$, $\tilde{\gamma} = -2\pi \alpha_3$), \( (A.3) \) splits into two pieces, the first of which being Lorentz invariant, whereas the second gives the variation

$$\delta_\varphi \delta_\alpha \Gamma_{\text{eff}} = -\tilde{\gamma} \frac{2\pi}{\pi} \int d^2x |e| \varphi \nabla^2 \alpha, \quad (A.4)$$

and we used $\delta \omega = -d\alpha$. As Weyl and Lorentz transformations commute, \( (A.4) \) and \( (A.2) \) must be the same. Integrating by parts twice then yields the relation

$$\tilde{\gamma} = 2\beta, \quad (A.5)$$

which is indeed satisfied by the anomalies (4.6) and (4.7).

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