Comprehensive understanding of parity-time transitions in $\mathcal{PT}$ symmetric photonic crystals with an antiunitary group theory

Adam Mock

School of Engineering and Technology, Central Michigan University, ET 100, Mount Pleasant, MI 48859, USA

and Science of Advanced Materials Program Central Michigan University, Mount Pleasant, MI 48859, USA

(Dated: January 13, 2017)

Electromagnetic materials possessing parity-time symmetry have received significant attention since it was discovered that the eigenmodes of these materials possess either real-frequency eigenvalues or the eigenfrequencies appear in complex-conjugate pairs. Interestingly, some eigenstates of these systems show thresholdless $\mathcal{PT}$ transitions to the complex-conjugate regime, some exhibit a transition as a function of the degree of non-Hermiticity and some show no $\mathcal{PT}$ transition at all. While previous work has provided some insight on the nature of $\mathcal{PT}$ transitions, this work lays out a general and rigorous mathematical framework that is able to predict, based on symmetry alone, whether an eigenmode will exhibit a thresholdless $\mathcal{PT}$ transition or no $\mathcal{PT}$ transition at all. Developed within the context of ferromagnetic solids, Heesh-Shubnikov group theory is an extension of classical group theory that is applicable to antiunitary operators. This work illustrates the Heesh-Shubnikov approach by categorizing the modes of a two-dimensionally periodic photonic lattice that possesses $\mathcal{PT}$ symmetry.

I. INTRODUCTION

Electromagnetic (EM) systems described by non-Hermitian wave equations but invariant under the combined operation of parity $\mathcal{P}$ and time-inversion $\mathcal{T}$ have been the subject of intense investigation in recent years. Such systems have been shown to possess either real-frequency eigenvalues or paired complex-conjugate eigenvalues. In the latter case, the modes separate into gain or loss modes depending on the sign of the complex part of the frequency. And the spatial field distribution of the gain and loss modes show a preferential overlap with the amplifying and absorbing portions of the geometry, respectively. This work provides a theoretical foundation for understanding $\mathcal{PT}$ transitions in two-dimensionally periodic systems with $\mathcal{PT}$ symmetry. The modes in these systems exhibit a rich variety of behavior much of which lacks a rigorous mathematical basis.

Preliminary studies of $\mathcal{PT}$ symmetric EM systems concerned two identical parallel waveguides, one of which provided gain and the other was absorbing. More complicated spatial arrangements were studied using optical fiber, fiber amplifiers and fiber splitters/combiners. And fascinating recent work has exploited $\mathcal{PT}$ symmetry for laser mode control, optical isolation and vortex beam generation in ring resonator devices. However, periodic systems (especially with periodicity in dimensions higher than one) with $\mathcal{PT}$ symmetry have only begun to be studied. This work shows how a group theory approach originally developed to classify the irreducible representations of ferromagnetic solids can be used to predict $\mathcal{PT}$ transitions and degeneracies in the photonic band diagram.

Materials whose optical properties are periodically modulated in two dimensions (also known as two-dimensional photonic crystals) are critical in numerous areas of optics including integrated photonics, photonic metasurfaces, optical sensing, and studies of quantum electrodynamics and embedded eigenvalues. In each of these areas, understanding the nature of the electromagnetic modes is a critical first step. In relatively simple structures, modes can be characterized according to their parity about some axis of the system, and this information is typically sufficient to estimate critical design metrics such as coupling and radiation efficiencies. However, in two-dimensional (2D) photonic crystals (PCs), modes exhibit a richer variety of behavior and a more rigorous mode classification scheme is required. Group theory has been extremely useful in the classification of modes in 2D PCs. Additionally, group theory can predict mode degeneracies. However, application of “classical” group theory to $\mathcal{PT}$ symmetric systems is hampered by the fact that the time-inversion operation is antiunitary, and classical representation theory assumes all operations can be represented by unitary matrices (or matrices that are equivalent under unitary transformations). Fortunately, Heesh-Shubnikov group theory was developed to handle such situations with its associated corepresentations. Heesh-Shubnikov group theory was developed to analyze magnetic ordering in ferromagnetic materials in the middle decades of the twentieth century, and here we show that it continues to
be instrumental in the understanding of $\mathcal{PT}$ symmetric photonics.

II. HEESH-SHUBNIKOV GROUP THEORY

Formally, $\mathcal{PT}$ symmetry concerns electromagnetic systems whose refractive index obeys $\mathcal{P}n(r) = n(r)^*$ where the parity operator $\mathcal{P}$ inverts one or more spatial coordinates. However, the interesting consequences of $\mathcal{PT}$ symmetry hold even when the parity operator is generalized to be any spatial symmetry operation while keeping the total amount of gain and loss balanced. In this paper, we concern ourselves with the geometry depicted in Fig. 1 with $d_x = d_y = \Lambda/2$ where $\Lambda_x = \Lambda_y = \Lambda$ is the lattice constant for a square lattice. With the origin centered on a gain rod, it is apparent that the structure is invariant under point symmetry operations in the $C_{4v}(4nm)$ point group. Further, the lattice is invariant under the combination of a spatial shift of $\tau = \frac{1}{2}(x + y)$ and complex-conjugation (or $\mathcal{T}$). In this sense the generalized $\mathcal{PT}$ invariance is $\{E|x, y\}(x, y) = (x, y)^*$ where $E$ is the identity operator and the Seitz notation is used to represent the combined point symmetry operation ($E$) and spatial shift ($\tau$). After the lattice has been shifted and complex conjugated, one can again apply the point symmetry operations in $C_{4v}$, and the geometry will remain invariant. Ultimately, the lattice possesses two $C_{4v}$ point group symmetry centers centered on the gain and loss rods.

The central question this paper addresses is whether thresholdless $\mathcal{PT}$ transitions can be predicted from symmetry. Group theory has been instrumental in understanding the qualitative features of both electronic and photonic band structures. Notably, it allows the determination of band degeneracies at both high and low symmetry points in reciprocal space from the irreducible representations of the groups. The presence of time-inversion introduces complex-conjugation into the symmetry operations considered here which makes the operators antiunitary. Because the irreducible representations of “classical” groups (those without complex-conjugation) assume the symmetry operators are unitary, an alternative approach for constructing the irreducible representations is needed. While the Heesh-Shubnikov group theory was pioneered by their namesake, their exposition by Wigner played a significant role in their application to practical problems [45–50]. Heesh-Shubnikov groups are also referred to as a magnetic groups or black and white groups. In the context of periodic structures they apply to crystals whose lattice sites have a property that can take on one of two values.

The Heesh-Shubnikov group theory apparatus actually provides two critical pieces of information. In addition to providing the irreducible representations (more commonly referred to as “corepresentations”) of the antiunitary operators, it also provides a categorization system that can be used to determine whether thresholdless $\mathcal{PT}$ transitions will occur. The categorization is based on a sum-rule test developed by Dimmock and Wheeler [51–52]. They discovered that corepresentations of $\mathcal{M}$ fall into three categories [48, 53], and the categorization is determined by summing the classical characters of the elements resulting from squaring all symmetry elements that include the antiunitary operator $\mathcal{T}$:

$$
\sum_{B \in W} \chi(B^2) = \begin{cases} 
  n & \text{Type (a)}, \\
  -n & \text{Type (b)}, \\
  0 & \text{Type (c)},
\end{cases}
$$

(1)

where $\chi(R)$ is the character of the classical representation of $R$, $n$ is the order of the unitary subgroup (to be explained below) and $W$ is the set of antiunitary operators (those containing $\mathcal{T}$). In previous work, we discovered that Type (a) corepresentations correspond to a single representation of the unitary subgroup, and no new $\mathcal{PT}$ degeneracy is introduced. The eigenfrequencies of Type (a) modes remain real. Type (c) corepresentations contain two inequivalent representations of the unitary subgroup, and new $\mathcal{PT}$ degeneracy is introduced [48, 53]. Modes with Type (c) corepresentations exhibit thresholdless $\mathcal{PT}$ symmetry transitions, so their eigenfrequencies come in complex-conjugate pairs even in the limit of infinitesimal non-Hermiticity (that is, C_{4v} [that is, let $C_{4v} = \{E, C_2, C_4, C_4^{-1}, \sigma_x, \sigma_y, \sigma_d, \sigma_d'\}$] and $\{T|\frac{\Lambda}{2} \} \equiv \{T|\frac{\Lambda}{2}(x + y)\}$.)
infinitesimal amounts of gain and loss). Type (b) corepresentations contain the same single representation of the unitary subgroup twice, and new $\mathcal{PT}$ degeneracy may appear.

Returning to the space group $\mathcal{M}$ defined above, we note that the $C_{4v}$ part forms a unitary subgroup of index 2, and the antiunitary elements form a coset of $C_{4v}$. Generally, $\mathcal{PT}$ symmetric geometries will have symmetry groups of the form $\mathcal{M} = \mathcal{N} + \mathcal{A}\mathcal{N}$ where $\mathcal{N}$ is a unitary subgroup, and $\mathcal{A}$ is an antiunitary symmetry operator. Cracknell classifies Heesch-Shubnikov groups of this form as Type IV [49, 50].

Type (a) corepresentations maintain the dimensionality of the irreducible representations of the corresponding unitary subgroup $\mathcal{N}$. The dimensionalities of Type (b) and (c) corepresentations are twice that of the corresponding unitary subgroup. This is consistent with the observation that eigenstates with complex-conjugate eigenfrequencies associated with thresholdless $\mathcal{PT}$ transitions transform into each other under antiunitary symmetry operations, whereas those with real frequencies transform only into themselves.

In the following we illustrate these concepts at high-symmetry points in the photonic band structure shown in Fig. 2. Immediately one sees a variety of interesting behavior including thresholdless $\mathcal{PT}$ symmetry transitions at the high-symmetry $X$ and $M$ points. Furthermore, the structure also supports “classical” degeneracy or points where bands coalesce but do not yield a $\mathcal{PT}$ symmetry transition (eigenfrequencies remain real). This paper provides a general framework that explains where in the photonic band diagram $\mathcal{PT}$ transitions are expected to occur. Our analysis will use only the TE polarization $\{E_x, E_y, H_z\}$ to illustrate the concept. The approach applies to the TM polarization as well.

III. RESULTS

Here we show how the Heesch-Shubnikov group theory approach can predict the nature of $\mathcal{PT}$ transitions at the high symmetry points $\Gamma$ [$k = 0$], $X$ [$k = \frac{1}{2}x$] and $M$ [$k = \frac{1}{4}(x + y)$]. In each case the points of interest will be band crossings occurring in the empty lattice (a lattice with infinitesimally small periodic perturbation) [57]. What we find ultimately is that every empty lattice band crossing exhibits either a thresholdless $\mathcal{PT}$ transition or no $\mathcal{PT}$ transition at all. The group theory approach described here shows how symmetry can be used to distinguish and predict which points will or will not exhibit a $\mathcal{PT}$ transition.

III.A Absence of $\mathcal{PT}$ transitions at the $\Gamma$ point [$k = 0$]

Symmetry analysis in periodic structures is facilitated by identifying the group of $k$ or little group. In classical group theory, the little group consists of elements of the full space group which send a particular $k$ into $k + K$ where $K$ is a reciprocal lattice vector [43, 55]. However, for Heesch-Shubnikov little groups (HSLG) that include antiunitary operators, such a group includes (i) unitary elements of the space group that send $k$ into $k + K$ (as before) and (ii) antiunitary elements of the space group that send $k$ into $-k + K$ [50].

Based on the Bloch form for modes of periodic systems, one can use a representation of the space group, $\exp[i k \cdot (m x + n y)\Lambda]$, where $\{m, n\} \in \mathbb{Z}$ [50, 60]. At $k = 0$ (the $\Gamma$ point), the analysis is simplified since the space group representation $\exp[i k \cdot (m x + n y)] = 1$ for all $m$ and $n$. In this case, the HSLG is the same as the full space group $\mathcal{M}^{k=0} = \mathcal{M} = C_{4v} + \{T\frac{\Lambda}{2} \frac{\Lambda}{2}\}C_{4v}$. More details on the specific symmetry operators are provided in Ref. [50].

Squaring each element in the group $\{T\frac{\Lambda}{2} \frac{\Lambda}{2}\}C_{4v}$ yields the elements $\{E, E, E, E, E, C_2, C_2\}$. Summing the characters of these elements according to the Dimmock-Wheeler test yields $\sum_{B \in \mathcal{W}} \chi(B^2) = 8$ for all of the classical representations of $C_{4v}$. Therefore, the modes of the $\mathcal{PT}$ symmetry lattice shown in Fig. 2 at $k = 0$ are of Type (a), and no thresholdless $\mathcal{PT}$ transitions are expected there. The group $C_{4v}$ contains four one-dimensional (1D) representations and one two-dimensional (2D) representation, so one anticipates non-degenerate and 2D degenerate modes at $k = 0$.

The components of the $i$th Type (a) corepresenta-
Fig. 3. Left side shows a detailed view of the bandstructure at the Γ point near the frequency Λ/λ₀ = 0.5. Bands labeled (1,2) are degenerate at k = 0. The band labeled 3 crosses k = 0 at a frequency slightly above the (1,2) bands and is not degenerate with them. Right side shows Hᵢ(x,y) for the modes labeled 1, 2 and 3. For modes 1 and 2, the amplitude scale of the real part is eight times smaller than that of the imaginary part. For mode 3, the amplitude scale of the imaginary part is twenty times smaller than that of the real part.

Fig. 4. Left side shows a detailed view of the bandstructure at the X point near the (real) frequency Λ/λ₀ = 0.25. Right side shows Hᵢ(x,y) for the gain and loss modes associated with the PT transition at the X point. For mode 1 (2), the amplitude scale of the real (imaginary) part is 0.053 times that of the imaginary (real) part. Bottom right figure depicts the imaginary part of the frequency as a function of the imaginary part of the refractive index. The trend is characteristic of a thresholdless PT transition.

III.B Thresholdless PT transitions at the X point

[k = \pi /x]

The symmetry elements in the HSLG at k = \pi /x include \{E, C₂, σₓ, σᵧ\} and \{T|\frac{\pi}{2}, \frac{\pi}{2}\}\{E, C₂, σₓ, σᵧ\}. Because \exp[i(k \cdot (xm + yn)\Lambda)] = \exp[i\pi mn] which is 1 for m even and -1 for m odd, these two space group representations must be explicitly included in the group. This doubles the size of the HSLG, and the resulting unitary subgroup is isomorphic to D₂h(mmm).

More details on the symmetry operators are provided in Ref. [56]. Ultimately the HSLG at k = \pi /x is Mₖ=π /x = D₂h + \{T|\frac{\pi}{2}, \frac{\pi}{2}\}D₂h. The group D₂h has eight 1D representations. Performing the Dimmock-Wheeler test yields four Type (a) and four Type (c) corepresentations. Enforcing the physical constraint that the character of \exp[i\pi m] switch sign for even and odd m, leaves only the four Type (c) corepresentations. Therefore, we anticipate that thresholdless PT transitions are expected at every empty-lattice band crossing at k = \pi /x. In this case only pairwise coupled modes with complex-conjugate frequencies are expected; no higher order degeneracy (nor nondegeneracy) is anticipated. Figures 4 and 5 show detailed views of the lowest three bands at the X point. Indeed in each of these cases a PT transition is seen near k = \pi /x. Moreover, the bottom right panel of Fig. 4 shows that the exceptional point approaches k = \pi /x as n₋₁ → 0 indicating that the PT transition is thresholdless.
The components of the \(i\)th Type (c) corepresentation \(\Gamma_i\) for the unitary elements \(R \in \mathcal{N}\) are given by \[\Gamma_i(R) = \begin{pmatrix} \Delta(R) & \mathbf{0} \\ \mathbf{0} & \Delta^*(S^{-1}RS) \end{pmatrix}\] (2)

where \(A = ST\). The components of the \(i\)th Type (c) corepresentation \(\Gamma_i\) for the antiunitary elements \(R \in \mathcal{W}\) are given by

\[\Gamma_i(R) = \begin{pmatrix} \mathbf{0} & \Delta^*(A^{-1}R) \\ \Delta(RA) & \mathbf{0} \end{pmatrix}\] (3)

where these matrices operate on the loss/gain mode pair

\[H = \begin{pmatrix} H_t \\ H_g \end{pmatrix}\] (4)

where \(l\) \((g)\) refers to loss \((gain)\).

These equations show precisely how two-dimensional corepresentations representing the coupled complex-conjugate eigenfrequency pairs are constructed from nominally 1D irreducible representations of the unitary subgroup. Further, one sees that the corepresentations of the unitary operators are diagonal, so that they simply transform the modes making up the complex-conjugate pair into themselves. In contrast, the corepresentations of the antiunitary operators are antidiagonal which means the antiunitary operators transform these modes into each other. A critical aspect of the antiunitary operators is that the shift by \((x + y)\) almost always, the gain mode and loss mode can be transformed into each other (with an appropriate constant multiplier) with this shift. This can be seen in the field profiles displayed in Figs. 4 and 5 where there is an additional factor of \(\pm 1\) involved in the shift. The specific transformation matrices are provided in tables in Ref. [56].

III.C Behavior at the M point \([k = (x+y)\frac{\pi}{a}]\)

Lastly, consider \(k = (x+y)\frac{\pi}{a}\) which is the M point. At the M point, the applicable point symmetry operations include all of those in \(C_{4v}\). Because \(\exp[ik \cdot (x m + y n)A] = \exp[i\pi(m + n)]\) which is 1 for \(m + n\) even and \(-1\) for \(m + n\) odd, these two space group representations must be explicitly included in the group. This doubles the size of the HSLG, and the resulting 16-element unitary subgroup is isomorphic to \(D_{4h}(4/mmm)\). Altogether the HSLG at \(k = (x+y)\frac{\pi}{a}\) is \(\mathcal{M}(k=(x+y)\frac{\pi}{a}) = D_{4h} + \{T \frac{\pi}{2} \frac{\pi}{2}\}D_{4h}\). More details on the particular symmetry operations are provided in Ref. [56]. The group \(D_{4h}\) contains four applicable 1D Type (c) representations that produce two-dimensional corepresentations associated with thresholdless \(PT\) transitions. It also contains one applicable 2D Type (a) representation, so one expects
III.D Unexpected Behavior at the M point

\[ \mathbf{k} = (x + y) \frac{\pi}{2} \]

We show in Figure 7 a detailed view of the band structure at the M point, this time near the real frequency \( \Lambda/\lambda_0 = 0.79 \). The empty lattice bandstructure possesses eight-fold degeneracy at this point. When the \( \mathcal{PT} \) symmetric lattice is introduced, the eight-fold degeneracy is lifted. The view afforded by Fig. 7 depicts 4 bands all of which are in the complex-conjugate frequency regime, so each band consists of two overlapping bands with the same real frequency. The \( \mathcal{PT} \) transition point can be seen in Fig. 2 and occurs around \( k\Lambda/(\sqrt{2}\pi) = 0.8 \).

This point is highlighted because as these four bands approach the M point, the bottom two coalesce resulting in a doubly degenerate set of modes in the complex-conjugate frequency regime (point labeled Q). That is, the four modes have two sets of equal complex-conjugate frequencies. This is a surprising observation since the Heesch-Shubnikov analysis predicts either non-degenerate Type (c) representations (two coupled modes with complex-conjugate frequencies) or doubly degenerate Type (a) representations that remain in the real-frequency domain. The lowest frequency modes shown in Fig. 7 appear to exhibit properties of a doubly degenerate Type (c) representation even though one does not exist in the present analysis.

However, inspection of these bottom two coalescing bands indicates that they join only at the M point which is different from the other \( \mathcal{PT} \) degeneracies whose overlapping occurs for a range of wavevectors and depends on the non-Hermiticity factor \( n_i \). Therefore, this phenomenon is likely not directly related to the prediction of thresholdless \( \mathcal{PT} \) transitions via the Heesch-Shubnikov theory. Instead it results from the allowed symmetries at high-symmetry points determined by a reduction procedure. The procedure begins by identifying the number of equivalent high-symmetry points in reciprocal space that are invariant under symmetry operations. Fig. 8 displays a reciprocal lattice diagram with high symmetry points labeled. It is sufficient to work with the nominal unitary subgroup without the space group contribution, which here is just \( C_{4v} \). For example there is only one \( \Gamma^{(1)} \) point (at the origin), and it is invariant under all symmetry operations in \( C_{4v} \). There are four \( \Gamma^{(2)} \) points all of which are invariant under \( E \), two of which are invariant under \( \sigma_d \) and \( \sigma'_{d} \) (each), and none of which are invariant under \( C_2, C_{4}, C_{4}^{-1}, \sigma_x \) and \( \sigma_y \). The point of interest in the present discussion is \( M^{(2)} \), and there are eight equivalent points that are invariant under \( E \). \( M^{(2)} \) is not invariant under any other transformations in \( C_{4v} \). The “character profile” of \( M^{(2)} \) will then be \( P = (8, 0, 0, 0, 0, 0, 0, 0) \) where the ordered set represents the instances of the elements of \( C_{4v} \) in the order \( (E, C_2, C_{4}, C_{4}^{-1}, \sigma_x, \sigma_y, \sigma_d, \sigma'_{d}) \).

One then calculates the projection of the possible irreducible representations onto the character profile of \( M^{(2)} \). This is done using the formula

\[
\lambda_i = \frac{1}{h} \sum_{R \in \text{Rec} C_{4v}} P(R) \chi_i(R) \tag{5}
\]

where \( h \) is the order of the group \( C_{4v} \) \( (h = 8) \), \( P(R) \) represents the character profile of the element \( R \) discussed above, \( \chi_i(R) \) is the character of operation \( R \) in
the $i$th irreducible representation of $C_{4v}$, and $\lambda_i$ counts the instances of the $i$th representation associated with $M^{(2)}$. Performing this calculation for the 5 irreducible representations of $C_{4v}$ results in one each of the four 1D representations and two instances of the lone 2D representation. Because the 1D representations pair up to form the 2D corepresentations in the complex-conjugate frequency regime, one expects two lone Type (c) corepresentations, and two 2D “classically degenerate” representations. And this behavior is exhibited in Fig. 7 where one sees the two Type (c) bands labeled $\Gamma_1$ and $\Gamma_3'$. The bottom two bands are also essentially Type (c) modes since they are in the complex-conjugate regime, but due to the reduction procedure, they also exhibit behavior associated with “classically” doubly degenerate modes exactly at the M point.

This observation raises an interesting question: if the reduction procedure requires that two 2D classically degenerate modes appear at $M^{(2)}$, then shouldn’t these appear as pure 2D degenerate modes similar to modes (1,2) at $M^{(1)}$? It appears that there are two competing effects at play: on the one hand, the reduction procedure provides information based purely on the lattice (square in this case); whereas, the formation of gain and loss modes occurs within the lattice sites. Apparently, the system favors the formation of gain/loss modes (Type (c) corepresentations) even though the lattice symmetry demands that two 2D classically degenerate modes appear. Ultimately, the phenomenon at $M^{(2)}$ is the result of satisfying both constraints.

Based on observing the transformation properties of the fields in Fig. 7 we can write down the matrix representations of the operators associated with the degenerate set of complex-conjugate modes. The unitary operators $R$ transform according to

$$ R \begin{pmatrix} H_1 \\ H_3 \\ H_2 \\ H_4 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} H_1 \\ H_3 \\ H_2 \\ H_4 \end{pmatrix} \tag{6} $$

where the fields are numbered according to the format in Fig. 7. This $4 \times 4$ matrix consists of two identical $2 \times 2$ matrices on the diagonal. These $2 \times 2$ matrices are diagonal for $E, C_2, \sigma_d, \sigma'_d$ and antidiagonal for $C_4, C_4^{-1}, \sigma_x, \sigma_y$.

The antiunitary operators $R$ transform according to

$$ R \begin{pmatrix} H_1 \\ H_3 \\ H_2 \\ H_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ a_{13} & a_{14} & 0 & 0 \\ a_{23} & a_{24} & 0 & 0 \end{pmatrix} \begin{pmatrix} H_1 \\ H_3 \\ H_2 \\ H_4 \end{pmatrix} . \tag{7} $$

Similarly, this $4 \times 4$ matrix consists of two identical $2 \times 2$ matrices but now on the antidiagonal. These $2 \times 2$ matrices are diagonal for $\xi, \zeta, \mu_d, \mu'_d$ and antidiagonal for $\gamma, \gamma^{-1}, \mu_x, \mu_y$. These operators correspond to the same elements of $C_{4v}$ listed above but combined with $\{T \frac{1}{2} \frac{1}{2} \}$ (see Ref. 56 for more details).

IV. DISCUSSION AND CONCLUSIONS

In this work we have shown how Heesch-Shubnikov group theory can be used to predict thresholdless $\mathcal{PT}$ transitions in 2D periodic photonic lattices that possess $\mathcal{PT}$ symmetry. While the theoretical framework was developed decades ago, we have found it instrumental in understanding the complex and varied modal behavior of these modern materials. Notably, we focussed on high symmetry points in the reciprocal lattice and found that at the $\Gamma$ point, one expects all modes to remain in the real-frequency regime; at the $X$ point, one expects all modes to be in the complex-conjugate frequency regime; and at the $M$ point, one expects modes in the complex-conjugate frequency regime and doubly degenerate modes in the real-frequency regime. Interesting behavior was observed at the $M$ point where modes possessed a combination of 2D degeneracy and complex-conjugate frequency formation.

This work lays a theoretical foundation for important future studies. Theoretically, a similar analysis applied to other lattice types as well as three-dimensional photonic crystals will likely uncover even more interesting behavior. From an engineering perspective, the real frequency states at the $\Gamma$ point shows that these materials could be designed to create photonic bandgap devices such as waveguides and cavities. The large density of
states associated with the high-order degeneracy discovered at the M point could be useful for bandedge lasers. And the broad understanding of the band structure in general will prove critical for designing superprism-type devices and metasurfaces.

*mocklap@cmich.edu*

[1] R. El-Ganainy, K. G. Makris, D. N. Christodoulides, and Ziad H. Musslimani, “Theory of coupled optical $PT$-symmetric structures,” Optics Letters 32, 2632–2634 (2007).

[2] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, “Observation of $PT$-symmetry breaking in complex dielectric materials,” Physical Review Letters 103, 093902 (2009).

[3] Ali Mostafazadeh, “Spectral singularities of complex scattering potentials and infinite reflection and transmission coefficients at real energies,” Physical Review Letters 102, 220402 (2009).

[4] Ali Mostafazadeh, “Resonance phenomenon related to spectral singularities, complex barrier potential, and resonating waveguides,” Physical Review A 80, 032711 (2009).

[5] Christian E. Rüter, Konstantinos G. Makris, Ramy El-Ganainy, Demetrios N. Christodoulides, Mordechai Segev, and Detlef Kip, “Observation of parity-time symmetry in optics,” Nature Physics 6, 192–195 (2010).

[6] Stefano Longhi, “$PT$-symmetric laser absorber,” Physical Review A 82, 031801(R) (2010).

[7] Jiří Čtyroký, Kuzmiak, Vladimír, and Sergey Eyderman, “Waveguide structures with antisymmetric gain/loss profile,” Optics Express 18, 21585–21593 (2010).

[8] Henri Benisty, Aloyse Degeiron, Anatole Lupu, André De Lustrac, Sébastien Chénais, Sébastien Forget, Monder Besbes, Grégoery Barbillon, Suredien Bruyant, Blaize, Sylvain, and Gilles Léondel, “Implementation of $PT$ symmetric devices using plasmonics: principle and applications,” Optics Express 19, 18004–18019 (2011).

[9] Li Ge, Y. D. Chong, and A. D. Stone, “Conservation relations and anisotropic transmission resonances in one-dimensional $PT$-symmetric optical heterostructures,” Physical Review A 85, 023802 (2012).

[10] Hossein Hodaei, Mohammad-Ali Miri, Matthias Heinrich, Demetrios N. Christodoulides, and Mercedeh Khajavikhan, “Parity-time-symmetric microring lasers,” Science 346, 975–978 (2014).

[11] Stefano Longhi and Liang Feng, “$PT$-symmetric microring laser-absorber,” Optics Letters 39, 5026–5029 (2014).

[12] Long Chang, Xiaoshun Jiang, Shiyue Hua, Chao Yang, Jianming Wen, Liang Jiang, Guanyu Li, Guanzhong Liu, and Min Xiao, “Parity-time symmetry and variable optical isolation in active-passive-coupled microring lasers,” Nature Photonics 8, 524–529 (2014).

[13] Bo Peng, Şahin Kaya Özdemir, Fuchuan Lei, Faraz Monifi, Mariagiovanna Gianfreda, Gui Lu Long, Shanhui Fan, Franco Nori, Carl M. Bender, and Lan Yang, “Parity-time-symmetric whispering-gallery micro cavities,” Nature Physics 10, 394–398 (2014).

[14] Sendy Phang, Ana Vukovic, Stephen C. Creagh, Trevor M. Benson, Philip D. Sewell, and Gabriele Gradoni, “Parity-time symmetric coupled micro resonators with a dispersive gain/loss,” Optics Express 23, 11493–11507 (2015).

[15] Pei Miao, Zhifeng Zhang, Jingbo Sun, Wiktorof Walsak, Stefano Longhi, Natalia M. Litchinitser, and Liang Feng, “Orbital angular momentum microlaser,” Science 353, 464–467 (2016).

[16] Carl M. Bender and Stefan Boettcher, “Real spectra in non-Hermitian Hamiltonians having $PT$ symmetry,” Physical Review Letters 80, 5243–5246 (1998).

[17] Carl M. Bender, Stefan Boettcher, and Peter N. Meisinger, “$PT$-symmetric quantum mechanics,” Journal of Mathematical Physics 40, 2201–2229 (1999).

[18] Carl M. Bender, Dorie C. Brody, and Hugh F. Jones, “Complex extension of quantum mechanics,” Physical Review Letters 89, 270401 (2002).

[19] Adam Mock, “Parity-time symmetry breaking in two-dimensional photonic crystals: square lattice,” Physical Review A 93, 063812 (2016).

[20] Alexander Cerjan, Aaswath Raman, and Shanhui Fan, “Exceptional contours and band structure design in parity-time symmetric photonic crystals,” Physical Review Letters 116, 203902 (2016).

[21] K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. H. Musslimani, “Beam dynamics in $PT$ symmetric optical lattices,” Physical Review Letters 100, 103904 (2008).

[22] Alois Regensburger, Christoph Bersch, Mohammad-Ali Miri, Georgy Onishchukov, Demetrios N. Christodoulides, and Ulf Peschel, “Exceptional and band structure design in parity-time symmetric photonic crystals,” Physical Review Letters 118, 203902 (2016).

[23] Alois Regensburger, Mohammad-Ali Miri, Christoph Bersch, Jakob Næger, Georgy Onishchukov, Demetrios N. Christodoulides, and Ulf Peschel, “Observation of defect states in $PT$-symmetric optical lattices,” Physical Review Letters 110, 223902 (2013).

[24] Liang Feng, Zi Jing Wang, Ren-Min Ma, Yuan Wang, and Xiang Zhang, “Single-mode laser by parity-time symmetry breaking,” Science 346, 972–975 (2014).

[25] Z. H. Musslimani, K. G. Makris, R. El-Ganainy, and D. N. Christodoulides, “Optical solitons in $PT$ periodic potentials,” Physical Review Letters 100, 030402 (2008).

[26] Keya Zhou, Zhongyi Guo, Jicheng Wang, and Shuttian Liu, “Defect modes in defective parity-time symmetric periodic complex potentials,” Optics Letters 35, 2928–2930 (2010).

[27] Zin Lin, Hamidreza Ramezani, Toni Eichelkraut, Tsampikos Kottos, Hui Cao, and Demetrios N. Christodoulides, “Parity-time symmetric photonic crystals,” Physical Review Letters 106, 213901 (2011).

[28] Alexander Szameit, Mikael C. Rechtsman, Omri Bahat-Treidel, and Mordechai Segev, “$PT$-symmetry in honeycomb photonic lattices,” Physical Review A 84, 021806 (2011).

[29] Liang Feng, Ye-Long Xu, William S. Fegadolli, Ming-Hui Lu, José E. B. Oliveira, Vilson R. Almeida, Yan-Feng Chen, and Axel Scherer, “Experimental demonstration of a unidirectional reflectionless parity-time metamaterial at optical frequencies,” Nature Materials 12, 108–113 (2013).

[30] Xue-Feng Zhu, Yu-Gui Peng, and De-Gang Zhao, “Anisotropic reflection oscillation in periodic multilayer
structures of parity-time symmetry,” Optics Express 22, 18401–18411 (2014).

[31] Hadiseh Alaeian and Jennifer A. Dionne, “Parity-time-symmetric plasmonic metamaterials,” Physical Review A 89, 033829 (2014).

[32] Jianing Xie, Zhikun Su, Weicheng Chen, Guojie Chen, Jiantao Lv, Dumitru Mihalache, and Yingji He, “Defect solitons in two-dimensional photonic lattices with parity-time symmetry,” Optics Communications 313, 139–145 (2014).

[33] Vassilios Yannopapas, “Spontaneous PT-symmetry breaking in complex frequency band structures,” Physical Review A 89, 013808 (2014).

[34] Mykola Kulishov, Bernard Kress, and H. F. Jones, “Novel optical characteristics of a Fabry-Perot resonator with embedded PT-symmetrical grating,” Optics Express 22, 23164–23181 (2014).

[35] Li Ge, Konstantinos G. Makris, Demetrios N. Christodoulides, and Liang Feng, “Scattering in PT- and RT-symmetric multimode waveguides: Generalized conservation laws and spontaneous symmetry breaking beyond one dimension,” Physical Review A 92, 062135 (2015).

[36] Hong Wang, Shuang Shi, Xiaoping Ren, Xing Zhu, Boris A. Malomed, Dumitru Mihalache, and Yingji He, “Two-dimensional solitons in triangular photonic lattices with parity-time symmetry,” Optics Communications 335, 146–152 (2015).

[37] Xue-Feng Zhu, “Defect states and exceptional point splitting in the band gaps of one-dimensional parity-time lattices,” Optics Express 23, 22274–22284 (2015).

[38] Kaustubh S. Agarwal, Rajeev K. Pathak, and Yogesh N. Joglekar, “Exactly solvable-symmetric models in two dimensions,” Physical Review A 92, 052103 (2015).

[39] Li Ge, “Parity-time symmetry in a flat-band system,” Physical Review A 92, 052103 (2015).

[40] Kun Ding, Z. Q. Zhang, and C. T. Chan, “Coalescence of exceptional points and phase diagrams for one-dimensional PT-symmetric photonic crystals,” Physical Review B 92, 235310 (2015).

[41] M. Turdeuev, M. Botey, I. Giden, R. Herrero, H. Kurt, E. Ozbay, and K Stalnunas, “Two-dimensional complex parity-time-symmetric photonic structures,” Physical Review A 91, 023825 (2015).

[42] Alexander Cerjan and Shanhui Fan, “Effects of non-uniform distributions of gain and loss in photonic crystals,” New Journal of Physics 18, 125007 (2016).

[43] Kazuaki Sakoda, Optical Properties of Photonic Crystals (Springer, Germany, 2001).

[44] K. Srinivasan and O. Painter, “Momentum space design of high-Q photonic crystal optical cavities,” Optics Express 10, 670–684 (2002).

[45] H. Heesh, “Zur Strukturbeweizung der ebenen Symmetrietriggruppen,” Zeitschrift für Kristallographie-Crystalline Materials 71, 95–102 (1929).

[46] A. V. Shubnikov, Symmetry and Antisymmetry of Finite Figures (Izd-vo Akademii nauk SSSR, Moscow, 1951).