Four-dimensional Bloch sphere representation of qudits using Heisenberg-Weyl Operators

Gautam Sharma* and Sibasish Ghosh†
Optics and Quantum Information Group, Institute of Mathematical Sciences,
HBNI, CIT Campus, Taramani, Chennai 600113, India

In the Bloch sphere based representation of qudits with dimensions greater than two, the Heisenberg-Weyl operator basis is not preferred because of presence of complex Bloch vector components. We try to address this issue and parametrize a qudit using the Heisenberg-Weyl operators by identifying eight real parameters and separate them as four weight and four angular parameters each. The four weight parameters correspond to the weights in front of the four mutually unbiased bases sets formed by the eigenbases of Heisenberg-Weyl observables and they form a four-dimensional unit radius Bloch sphere. The analysis of purity, rank, orthogonality, mutual unbiasedness conditions and the distance between two qudit states inside the sphere indicates that it is a natural extension of the qubit Bloch sphere. However, unlike the qubit Bloch sphere, the four-dimensional sphere is not a solid sphere. We also analyze the two and three-dimensional sections which gives a non-convex but closed structure for physical qudit states inside the sphere. Significantly, we have applied our representation to find mutually unbiased bases (MUBs), to characterize the unital maps in three dimensions and also to characterize the ensemble generated using Hilbert-Schmidt and Bures metrics. Lastly, we also give a basic idea of how to extend this idea in higher dimensions.

I. INTRODUCTION

A density matrix representing the state of a finite-dimensional quantum system of dimension $d$, is a $d \times d$ matrix, which satisfies the conditions of positive semi-definiteness, hermiticity and trace one. In general, it is difficult to study the properties of a density matrix directly. Parametrizations of the density matrix provide a simple method to study the properties of the density matrix and to use the density matrix in solving various problems of physics.

One of the most famous representations of density matrices is by using the Bloch vector parametrization. The Bloch vector parametrization is extremely popular because of its simplicity in representing a qubit and its various applications, see refs [1–3]. A qubit can be uniquely represented by a three-dimensional vector in the Bloch sphere so that every point inside the Bloch sphere corresponds to a physical qubit state. This lends a simple method to not only represent the qubit states but also to identify the dynamics of the qubit. For example, all rotations of the Bloch sphere correspond to a unitary operation. However, such an extension of all the beautiful properties of qubit Bloch sphere is not completely possible for higher dimensional states.

It is known that $d^2 − 1$ parameters are needed to characterize a $d$-level density matrix $\rho$ [1]. Most of the works till now have used the Gell-Mann operator basis to characterize the qudits as they admit real numbers as the Bloch vector elements. However, the structure of the $d^2 − 1$ dimensional solid corresponding to valid quantum states is extremely complex, even in three dimensions [1, 4–8, 11]. Moreover, all the points inside the $d^2 − 1$ dimensional sphere do not represent physical quantum states. A shortcoming of this feature is that all the rotations in the sphere do not represent a unitary operation, which is a prominent feature in the qubit Bloch sphere. To resolve this issue, there have been efforts to develop a qubit based parametrization in higher dimensions [9, 10], but without much success. Moreover, the structure corresponding to physical states using the Gell-Mann operator based representation, is asymmetric with respect to different axes. Also, in the Gell-Mann operator based representation the relation between Bloch vectors corresponding to orthonormal kets is not very useful geometrically. That is, it is not very useful for constructing an orthonormal basis starting from a pure qutrit state, by using the eight-D Bloch sphere. Therefore, these features which are very prominent and useful in the qubit Bloch sphere are not present for qudits, when we use Gell-mann basis based representation.

On contrary, the Heisenberg-Weyl(HW) operators have received much less attention because they are not hermitian and thereby require complex numbers as Bloch vector components [13, 14]. As such it becomes difficult to study the parameters and put them to use. However, as the HW operators do provide an alternative way to represent a quantum state, it is worthwhile to study them despite the presence of complex coefficients as there can be certain tasks where the HW operator based representation could be more suitable.

In this work, we try to address the issue of complex Bloch vector components while using the Weyl operator basis to parametrize a qutrit. We do so by identifying 4 weight and 4 angular parameters and construct a four-dimensional(four-D) Bloch sphere from the weight parameters. We also obtain the constraints on the weight and angular parameters, which give a physical qutrit state density matrix. It is found that not all the points inside the four-D sphere correspond to a positive semidefinite matrix. But still, the new Bloch sphere has several features that looks like an extension of the qubit Bloch sphere. Namely, 1) the length of the Bloch vector determines the purity of the state, 2) the rank of a randomly chosen qutrit state can be guessed to a certain extent 3) the conditions for two orthogonal or mutually unbiased vectors are quite similar to qubit Bloch sphere under some restriction and 4) the Hilbert-Schmidt distance between qutrit states is equivalent to a factor time the Euclidean distance in the Bloch sphere for

---

* gautam.oct@gmail.com
† sibasis@imsc.res.in
some states. It is also conjectured that the orthonormal basis kets lie on the same Bloch vector or on the antipodal points, similar to what we observe in the qubit Bloch sphere. Further, we apply our representation to 1) identify mutually unbiased bases (MUBs) in 3 dimensions from the geometry of four-D Bloch sphere 2) to characterize the unital map acting on qudit states and 3) to find the representation of ensembles generated from Hilbert-Schmidt and Bures metric. We also suggest a method to find a similar Bloch sphere representation in higher dimensions than three.

The paper is organized as follows. First, we review the HW operators has been in the construction of the discrete Wigner technique. For convenience, it is helpful to write the HW operators from Hilbert-Schmidt and Bures metric. We also suggest a hermitian generalization of the HW operators to find a similar Bloch sphere representation in higher dimensions than three. Finally, we conclude in sec.VIII with a summary and future works possible based on our work.

II. EXPANDING A QUDIT IN THE HEISENBERG WEYL OPERATOR BASIS

HW operators are unitary operators with several useful properties which makes them useful in several applications [15–18]. One of the most significant applications of the HW operators has been in the construction of the discrete Wigner function in finite dimensions [19, 20]. These operators are constructed from the generalized Pauli operators X and Z, which are also referred to as boost and shift operators respectively. They can be defined by their action on a pure state in the computational basis as

\[ X|i\rangle = |i + 1 \mod d\rangle, \]
\[ Z|i\rangle = \omega^i|i\rangle. \]

where \( \omega = e^{\frac{2\pi i}{d}} \) is the \( d \)th root of unity. Now, the HW displacement operators are defined as

\[ U_{pq} = \omega^{\frac{2\pi i}{d}} Z^p X^q. \]

where \( p \) and \( q \) are integers modulo \( d \). It is known [1, 14] that a qudit can be represented using the HW operator basis. These operators satisfy the following properties that are necessary to parametrize a physical state.

1. There are \( d^2 \) matrices including the Identity matrix and \( d^2 - 1 \) other traceless matrices.
2. The \( d^2 \) matrices are unitary and form an orthonormal basis, i.e., \( \text{Tr}(U_{pq}^* U_{p'q'}) = d \delta_{pp'} \delta_{qq'} \).

Therefore, we can decompose a bounded density matrix operator using the HW operator basis in the following form

\[ \rho = \frac{1}{d} \sum_{p=0,q=0}^{d-1} b_{pq} U_{pq} = \frac{1}{d} (d + \sum_{p=0,q=0, p\neq 0 \neq q}^{d-1} b_{pq} U_{pq}), \]

where \( b_{00} = 1 \) and \( b_{pq} = \text{Tr}(\rho U_{pq}^\dagger) \) form the Bloch vector components. However, the \( b_{pq} \)'s are complex in general because of \( U_{pq} \) not being hermitian. Hence, we need to find \( d^2 - 1 \) complex numbers to completely characterize a state. Moreover, it is not possible to define what is a “Bloch” sphere for these complex components. A solution was suggested in [14] by introducing a hermitian generalization of the HW operators to make the Bloch vector components real. In the next section we suggest an alternate approach to address this issue.

III. FOUR-D BLOCH SPHERE REPRESENTATION OF A QUTRIT

In this section, we propose a four-D unit radius Bloch sphere representation of a qutrit state. We will now demonstrate the procedure to represent a qutrit state by our technique. For convenience, it is helpful to write the HW operators in three dimensions using Eq.(1), as following

\[ U_{00} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad U_{01} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad U_{10} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad U_{11} = \begin{bmatrix} 0 & \omega^{-\frac{\pi}{d}} & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}, \quad U_{02} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad U_{20} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{bmatrix}, \quad U_{12} = \begin{bmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{bmatrix}, \quad U_{21} = \begin{bmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{bmatrix}, \quad U_{22} = \begin{bmatrix} 0 & 0 & \omega^2 \\ 0 & \omega & 0 \end{bmatrix}. \]

From Eq.(2) and Eq.(3) , a qutrit can be expanded as

\[ \rho = \frac{1}{3} (U_{00} + b_{01} U_{01} + b_{10} U_{10} + b_{11} U_{11} + b_{02} U_{02} + b_{20} U_{20} + b_{12} U_{12} + b_{21} U_{21} + b_{22} U_{22}). \]

Using the form of HW matrices in Eq.(3), we find that the coefficients \( b_{pq} = \text{Tr}(\rho U_{pq}^\dagger) \) must obey following relations

\[ b_{01} = n_1 e^{i\theta_1}, \quad b_{02} = n_1 e^{-i\theta_1}, \]
\[ b_{10} = n_2 e^{i\theta_2}, \quad b_{20} = n_2 e^{-i\theta_2}, \]
\[ b_{11} = n_3 e^{i\theta_3}, \quad b_{21} = n_3 e^{-i\theta_3}, \]
\[ b_{22} = n_4 e^{i\theta_4}, \quad b_{12} = n_4 e^{-i\theta_4}. \]

Thus, we can rewrite the expansion of \( \rho \) as

\[ \rho = \frac{1}{3} \left( U_{00} + n_1 (e^{i\theta_1} U_{01} + e^{-i\theta_1} U_{02}) + n_2 (e^{i\theta_2} U_{10} + e^{-i\theta_2} U_{20}) + n_3 (e^{i\theta_3} U_{12} + e^{-i\theta_3} \frac{\pi}{d} U_{21}) + n_4 (e^{i\theta_4} U_{11} + e^{-i\theta_4} \frac{\pi}{d} U_{22}) \right). \]

Therefore, we have obtained eight real parameters that can completely characterize a qutrit. The unique property of these parameters is that they consist of four weight parameters \( n_i \)'s and and four angular parameters \( \theta_j \)'s, with \(-1 \leq n_i \leq 1 \) and \( 0 \leq \theta_j \leq \pi \). We define \( \vec{n} = \{n_1, n_2, n_3, n_4\} \) as the four-D Bloch
vector. The \(n_i's\) are the weight elements corresponding to each commuting pair of HW operators. We will now obtain the constraints on the Bloch vector parameters \(n_i\) and the angular parameters \(\theta_i\). Note 1- An implication of using the four-D Bloch vector representation is that more than one state lie at the same point in the sphere. The states lying on the same point are distinguished only by the angular parameters \(\theta_i\). These states are equivalent under the action of some unitary operators. It would be interesting to identify these unitary operators.

Note 2- One can also characterize a qutrit state by allowing \(0 \leq n_i \leq 1\) and \(0 \leq \theta_i \leq 2\pi\) range of values, but then we will not obtain a sphere structure.

Such a structure might also be worth studying because the weight parameters have proper meaning when they are positive valued. However, we do not study this in the current work.

IV. CONSTRAINTS ON THE BLOCH VECTOR AND ANGULAR PARAMETERS

The matrix form of a qutrit can be written in terms of \(n_i\) and \(\theta_i's\) as following

\[
\rho = \frac{1}{3} \begin{pmatrix}
1 + 2n_2 \cos(\theta_2) & n_1 e^{\theta_1} + n_3 e^{-\theta_1} + n_4 e^{-\frac{3i}{2} \theta_1} & n_1 e^{-\theta_1} + n_3 e^{-\theta_1 - \frac{3i}{2}} + n_4 e^{\frac{3i}{2} - \theta_1} \\
-2n_2 \cos(\theta_2) & 1 - n_2 \cos(\theta_2) - \sqrt{3}n_2 \sin(\theta_2) & n_1 e^{\theta_1} + n_3 e^{\theta_1 + \frac{3i}{2}} + n_4 e^{-\frac{3i}{2} - \theta_1} \\
-n_1 e^{\theta_1} + n_3 e^{-\theta_1 + \frac{3i}{2}} + n_4 e^{\frac{3i}{2} + \theta_1} & n_1 e^{\theta_1 + \frac{3i}{2}} + n_3 e^{\theta_1 - \frac{3i}{2}} + n_4 e^{-\theta_1} & 1 - n_2 \cos(\theta_2) + \sqrt{3}n_2 \sin(\theta_2)
\end{pmatrix}
\]

(7)

It is clear that, \(\rho\) is hermitian, which is guaranteed by the choice of expansion coefficients. Moreover, \(\text{Tr}[\rho] = 1\) as the HW matrices are traceless except for \(U_{00} = I\). The only condition that remains to be satisfied is the positive semi-definiteness of \(\rho\), i.e. \(x_i \geq 0\), where \(x_i's\) are the eigenvalues of \(\rho\). In order to do this, we construct the characteristic polynomial \(\text{Det}(\chi \mathbb{I} - \rho)\), of the density matrix \(\rho\). The necessary and sufficient condition for the eigenvalues \(x_i\) to be positive semi-definite is that the coefficients \(a_i's\) of the characteristic polynomial are also positive semi-definite [4]. The characteristic polynomial has the following form

\[
\text{Det}(\chi \mathbb{I} - \rho) = \prod_{i=1}^{N}(x - x_i) = \sum_{j=0}^{N}(-1)^j a_j x^{N-j} = 0.
\]

(8)

Notice that \(a_0 = 1\) by definition. Now, we apply Newton’s formulas to find the values of other coefficients \(a_i's\) (for details please see ref. [4]). Newton’s formulas relate the coefficients \(a_i\) and the eigenvalues \(x_i\) as

\[
a_l = \sum_{k=1}^{l} C_{N,k} a_{l-k}, (1 \leq l \leq N),
\]

where \(C_{N,k} = \sum_{l=1}^{N} x^k\). Using the results directly from ref. [4], we get the following expressions for \(a_i's\) in terms of \(\rho\)

\[
\begin{align*}
a_0 &= 1 \\
a_1 &= \text{Tr} \rho \\
a_2 &= \frac{(1 - \text{Tr} \rho^2)}{2!} \\
a_3 &= \frac{1 - 3 \text{Tr} \rho^2 + 2 \text{Tr} \rho^3}{3!}.
\end{align*}
\]

(9)

Ensuring that the terms in Eq.(9) are non-negative results in positive semi-definite density matrix \(\rho\). \(a_1 = \text{Tr} \rho = 1 > 0\) is trivially satisfied. Where as, \(a_2 = \frac{(1 - \text{Tr} \rho^2)}{2!} \geq 0\) imposes the following constraint

\[
2a_2 = 1 - \text{Tr} \rho^2 = 1 - \frac{1}{3} \left(1 + 2(n_1^2 + n_2^2 + n_3^2 + n_4^2)\right) \geq 0 \\
\Rightarrow n_1^2 + n_2^2 + n_3^2 + n_4^2 \leq 1.
\]

(10)

This constraint simply states that the physical states must lie inside a four-D sphere of radius one. The only condition remaining to be satisfied now is \(a_3 \geq 0\), which has the following form
\[ 3!a_3 = 1 - 3 \text{Tr} \rho^2 + 2 \text{Tr} \rho^3 \geq 0 \]
\[ \implies 2 \left( 1 - 3 \left( n_1^2 + n_2^2 + n_3^2 + n_4^2 \right) + 2 \left( n_1^3 \cos(3\theta_1) + n_2^3 \cos(3\theta_2) + n_3^3 \cos(3\theta_3) + n_4^3 \cos(3\theta_4) \right) - 6n_1n_3n_4 \cos(\theta_1 - \theta_3 - \theta_4) \right. \]
\[ + 6n_1n_2n_3 \cos(\theta_1 - \theta_2 + \theta_3 - \frac{\pi}{3}) + 6n_2n_3n_4 \cos(\theta_2 + \theta_3 - \theta_4 + \frac{\pi}{3}) + 6n_1n_2n_4 \cos(\theta_1 + \theta_2 + \theta_4 + \frac{\pi}{3}) \left. \right) \geq 0. \quad (11) \]

In the above form, it is difficult to picture the set of valid states inside the sphere. We take the one, two, and three-dimensional sections passing through the centre to get a better understanding of the allowed space inside the four-D sphere.

### A. One-dimensional sections

One-dimensional sections (one sections) passing through center can be obtained by setting three out of four \( n_i \)'s as zero, in Eq. (11). We find that the expressions of one sections of \( a_3 \) are the same with respect to all \( n_i \)'s with the following form

\[ 3!a_3^{one} = \frac{2}{9} \left( 1 - 3n_i^2 + 2n_i^3 \cos[\theta_i] \right) \geq 0. \quad (12) \]

Therefore, these one-dimensional sections are symmetric with respect to the four axes. The non-negativity constraints are:

- \( a_3 \geq 0 \) ∀ \( \theta_i \) for \(- \frac{1}{2} \leq n_i \leq \frac{1}{2} \).

- For \( 0.5 < n_2 \leq 1 \), \( \theta \in [0, \zeta] \cup [\frac{2\pi}{3} - \zeta, \frac{2\pi}{3} + \zeta] \).

- For \( -1 \leq n_2 < -0.5 \), \( \theta \in [\frac{\pi}{2} - \zeta, \frac{\pi}{2} + \zeta] \cup [\pi - \zeta, \pi] \).

where \( \zeta = \arccos \left( \frac{1}{2n_i} \right) - \frac{2\pi}{3} \). It can be also observed (see Fig. 1) that the range of allowed values of \( \theta_i \) is gradually reducing as we move away from the origin along the \( n_i \) axis after \( |n_i| \geq 0.5 \).

### B. Two-dimensional sections

A two-dimensional section (two section) centered at the origin can be obtained by setting two out of four \( n_i \)'s to be zero in Eq. (11). There are 6 possible two sections which can be constructed in the four-D sphere. The positivity constraint for all the two-dimensional sections have the following same form

\[ 3!a_3^{two} = \frac{2}{9} \left( 1 - 3n_i^2 - 3n_j^2 + 2n_i^3 \cos(3\theta_i) + 2n_j^3 \cos(3\theta_j) \right) \geq 0. \quad (13) \]

Similar to the one-dimensional sections, we see that the two-dimensional sections are also symmetric with respect to the four axes. We point out that this is unlike the Gell-Mann basis based Bloch vector representation of a qutrit, where there are 4 different types of such two sections [4], which are asymmetric with respect to the axes.

Now, we are interested in obtaining the region in the two-dimensional section which corresponds to physical qutrit states, i.e. there exist values of \( \theta_i \) and \( \theta_j \) so that the inequality in Eq. (13) is satisfied. This can be easily found by maximizing \( a_3^{two} \) with respect \( \theta_i \) and \( \theta_j \), which gives the following inequal-
FIG. 2: (Color online) The shaded region depicts the allowed values of $n_i$ and $n_j$ for a physical state lying on the two-dimensional section built on $n_i$ and $n_j$ axes.

FIG. 3: (Color online) The solid region in the plots gives the allowed values of $\theta_i$ and $\theta_j$ against $r$. Here $\phi$ is the angle the direction makes with $n_i$.

Further, it is informative to see what are the allowed values of $\theta_i$ and $\theta_j$ in different directions in the two-dimensional section as we move away from the center in the four-D sphere. To do this, we replace with $n_i = r \cos(\phi)$ and $n_j = r \sin(\phi)$ in Eq. (13), so that

$$3! a_i^{two} \geq 0 \implies \frac{2}{9} \left(1 - 3n_i^2 - 3n_j^2 + 2|n_i^3| + 2|n_j^3|\right) \geq 0. \quad (14)$$

We note that when $r \leq 0.5$, the above inequality is satisfied for all values of $\theta_i$ and $\theta_j \in [0, \pi]$. Moreover, to find what are the allowed values of $\theta_i$ and $\theta_j$ as we move away from the origin in a radial direction, we do a 3D plot of the allowed region between $r$, $\theta_i$, and $\theta_j$ by fixing the directions determined by $\phi$ in fig. 3, where $\phi$ is the angle the direction makes with $n_i$.

C. Three-dimensional sections

Next, we consider the three-dimensional sections (three sections) centered at the origin inside the four-D sphere. There are four such three-dimensional sections possible which can be obtained by setting one of the $n_i$'s as zero in Eq. (11). However, unlike the one and two-dimensional sections the three-dimensional sections are all different, with following expressions.

$$3! a_i^{three} = \frac{2}{9} \left(1 - 3n_i^2 - n_i^2 + 2n_i^3 \cos(3\theta_i) + n_i^3 \cos(3\theta_2) + n_i^3 \cos(3\theta_3)\right) + 6n_i n_2 n_3 \cos(\theta_1 - \theta_2 + \theta_3 - \frac{\pi}{3}).$$
\[
3! \alpha_3^{\text{three}2} = \frac{2}{9} \left( 1 - 3 \left( n_1^2 + n_2^2 + n_4^2 \right) + 2 \left( n_1^3 \cos(3\theta_1) + n_2^3 \cos(3\theta_2) + n_4^3 \cos(3\theta_4) \right) + 6n_1n_2n_4 \cos(\theta_1 + \theta_2 + \theta_4 + \frac{\pi}{3}) \right). 
\]

\[
3! \alpha_3^{\text{three}3} = \frac{2}{9} \left( 1 - 3 \left( n_1^2 + n_3^2 + n_4^2 \right) + 2 \left( n_1^3 \cos(3\theta_1) + n_3^3 \cos(3\theta_3) + n_4^3 \cos(3\theta_4) \right) - 6n_1n_3n_4 \cos(\theta_1 - \theta_3 - \theta_4) \right). 
\]

\[
3! \alpha_3^{\text{three}4} = \frac{2}{9} \left( 1 - 3 \left( n_2^2 + n_3^2 + n_4^2 \right) + 2 \left( n_2^3 \cos(3\theta_2) + n_3^3 \cos(3\theta_3) + n_4^3 \cos(3\theta_4) \right) + 6n_2n_3n_4 \cos(\theta_2 + \theta_3 - \theta_4 + \frac{\pi}{3}) \right). 
\]

(15)

FIG. 4: (Color online) Two different views of the three section \(\alpha_3^{\text{three}}\) showing that the three section is closely approximated by an octahedron. The other three sections are also approximated by an octahedron, which we have not shown here for brevity. The colourscale used is only for a better clarity of the plots.

It seems that these three-dimensional sections are not symmetric with respect to the axes as they have different forms in Eq.(15). Therefore we need to find the regions for which the expressions in Eq.(15) are non-negative. To find the non-negative regions of the three-dimensional sections means to find out whether for a given triple of \(n_i, n_j, \) and \(n_k\), a corresponding \(\theta_i, \theta_j, \) and \(\theta_k\) exists which gives a non-negative value of terms in Eq.(15). It is difficult to do so analytically, i.e., to maximize the expressions of \(\alpha_3^{\text{three}}\) in Eq.(15) with respect to \(\theta\) parameters only. Instead we numerically plot the regions which satisfy \(\alpha_3^{\text{three}} \geq 0\) only. As we can see from the fig.4 that the different faces have different forms, we conclude that the three dimensional sections are not symmetric with respect to the axes.

It can be seen from fig.4 that the three section \(\alpha_3^{\text{three}}\) is of the form with bulges on the faces of an octahedron. The other three sections also are of a similar form but not the same, which we have not put in the paper for lack of space and brevity.

A few remarks from the study of the one, two and three sections are in order.

1. It is possible to approximately construct the three sections from the knowledge of the two sections, which is not the case in the representation using Gell-Mann operator based representation. This is because by rotating the two sections along one of the axes, one will obtain an octahedron which is a very good approximation to all the three sections.

2. It looks like from the numerical plots, that the three sections structure is not convex. This could be because of the presence of complex coefficients.

3. It is also clearly visible how the one section arise from the two sections and the two sections from the three sections.

Adding to the remarks from the study of one, two and three dimensional sections is the following result.

**Lemma 1** - The four dimensional Bloch sphere is not a solid sphere.

An implication of this result is that a rotation in the Bloch sphere does not always correspond to a unitary operation, unlike the qubit Bloch sphere. However, a very interesting finding is that inside the Bloch sphere of radius
$r \leq 0.5, a_3$ is positive for all the values of angular parameters $\theta_i$. This can be proven by using the polar coordinate forms of $n_i$’s in Eq.(11), i.e. we replace with $n1 = r \cos \alpha_1, n2 = r \sin \alpha_1, n3 = r \sin \alpha_1 \cos \alpha_3, n4 = r \sin \alpha_1 \sin \alpha_3 \cos \alpha_4$ in Eq.(11), where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the polar angles. Then $a_3$ can be written in the following simple form

$$3!a_3 = 1 - 3r^2 + 2r^3 F(\theta_1, \theta_2, \theta_3, \theta_4, \alpha_1, \alpha_2, \alpha_3, \alpha_4). \quad (16)$$

where $F$ is a function of $\theta_i$’s and $\alpha_i$’s. It can be easily shown that $F$ can have values in the range [-1,1]. And, for all values of $F$, $a_3$ is monotonically decreasing with $r$. If $a_3 \geq 0$ at a certain point $r = r_{\max}$, for some theta values $[\theta_1, \theta_2, \theta_3, \theta_4]$, then $a_3 \geq 0$ for all points on $r < r_{\max}$ for the same theta values. We state the following result from this analysis

**Lemma 2:** If in a certain direction the point at distance $r = r_{\max}$ represents a physical state, then all the points in the same direction with $r \leq r_{\max}$ represent a physical state.

Now, we set $r = 0.5$ in Eq.(16), and then minimize it with respect to all the angles $\alpha_i$’s and $\theta_i$’s, which gives a minimum value of zero. Therefore, for $r < 0.5$, $a_3$ must have values greater than zero for all values of $\theta_i$’s. Therefore, we can present the following Lemma.

**Lemma 3:** All the points inside sphere of radius $r \leq 0.5$ are physical states for all the angular parameter values of $\theta_i$’s.

This result is significant, as all the points lying inside the sphere of radius 0.5 correspond to physical states.

**V. FEATURES OF THE BLOCH SPHERE**

In this section, we discuss several features of the Bloch sphere which have surprising resemblance with the qubit Bloch sphere representation.

### A. Mixed and Pure states

$\text{Tr}[\rho^2]$ is the measure of the purity of a density matrix operator. Using Eq.(7) we can write

$$\text{Tr}[\rho^2] = \frac{1}{3} \left(1 + 2(n_1^2 + n_2^2 + n_3^2 + n_4^2)\right) \quad (17)$$

Thus, we find that the length of the Bloch vector determines the purity of the qutrit state. Further, $\text{Tr}[\rho^2] = 1$ for $n_1^2 + n_2^2 + n_3^2 + n_4^2 = 1$, i.e., the pure states lie on the surface of the unit sphere. Also, $\text{Tr}[\rho^2] = 0$ only when $n_1^2 + n_2^2 + n_3^2 + n_4^2 = 0$, i.e., the maximally mixed state lies at the center of the sphere. Also, the purity increases as we move away from the center of the sphere.

### B. Rank of a Qutrit state

A closely related concept to purity/mixedness is the rank of a physical state. From the qubit Bloch sphere it is very easy to determine that the rank 1 states lie on the surface while all the remaining states are of rank 2. In our representation of qutrit states, all the pure states lie on the surface, hence they have rank 1. The challenging task is to determine where the rank 2 and 3 states lie inside the four-D sphere. To do this we note that if the determinant of a qutrit is greater than zero, i.e., $\text{Det}(\rho) > 0$, then the state must be of rank 3. From the Eq.(7), we find that $\text{Det}(\rho) = a_3$. But we saw in Lemma 3 that for $r < 0.5, a_3 = \text{Det}(\rho) > 0$. Hence all the states inside the four-D sphere with radius $r = 0.5$ are rank 3 states. However, the states on the surface of this sphere with $r = 0.5$ can be both rank 2 and 3. Further, when $0.5 \leq r < 1$ the states can be both of rank 2 and rank 3 depending on the choice of the angular parameters $\theta_i$. To summarize

1. $r = 1 \implies \text{rank}(\rho) = 1$.
2. $0.5 \leq r < 1 \implies \text{rank}(\rho) = 2 \text{ and } 3$.
3. $r < 0.5 \implies \text{rank}(\rho) = 3$.

A representative figure of the qutrit state space depicting the location of rank 1, 2 and 3 states can be seen in Fig.(5). This, helps us to identify the rank of the qutrit states by simply looking at where they lie inside the four-D sphere.

![FIG. 5](Color online) A representative figure depicting the position of the Rank-1, 2, and 3 states and the qutrit state space containing a solid ball of radius $1/2$. [Color online]}
C. Orthogonal states

Let us consider two pure states $\rho_a$ and $\rho_b$ and expand it in the form of Eq.(6)

$$\rho_a = \frac{1}{3} \left( U_{00} + n_{a1} e^{i\theta_a} U_{01} + e^{-i\theta_a} U_{02} + n_{a2} e^{i\theta_a} U_{10} + e^{-i\theta_a} U_{20} + n_{a3} e^{i\theta_a} U_{12} + e^{-i\theta_a} \frac{2}{3} U_{21} + n_{a4} e^{i\theta_a} U_{11} + e^{-i\theta_a} \frac{2}{3} U_{22} \right).$$

$$\rho_b = \frac{1}{3} \left( U_{00} + n_{b1} e^{i\theta_b} U_{01} + e^{-i\theta_b} U_{02} + n_{b2} e^{i\theta_b} U_{10} + e^{-i\theta_b} U_{20} + n_{b3} e^{i\theta_b} U_{12} + e^{-i\theta_b} \frac{2}{3} U_{21} + n_{b4} e^{i\theta_b} U_{11} + e^{-i\theta_b} \frac{2}{3} U_{22} \right).$$

The states $\rho_a$ and $\rho_b$ have Bloch vectors $n_a = \{n_{a1}, n_{a2}, n_{a3}, n_{a4}\}$ and $n_b = \{n_{b1}, n_{b2}, n_{b3}, n_{b4}\}$ respectively. The orthogonality condition can simply be checked by $\text{Tr}[\rho_a \rho_b] = 0$, which reduces to

$$\text{Tr}[\rho_a \rho_b] = \frac{1}{3} \left( 1 + 2n_{a1} n_{b1} \cos(\theta_{a1} - \theta_{b1}) + 2n_{a2} n_{b2} \cos(\theta_{a2} - \theta_{b2}) + 2n_{a3} n_{b3} \cos(\theta_{a3} - \theta_{b3}) + 2n_{a4} n_{b4} \cos(\theta_{a4} - \theta_{b4}) \right) = 0. \quad (18)$$

This condition is richer than the orthogonality condition that we know for the two orthogonal qubit states. We know that Bloch vectors for two orthogonal qubit states obey, $\hat{b}_1 \cdot \hat{b}_2 = -1 = \cos(\pi/2)$. The orthogonality condition in Eq.(18) reduces to $n_{a1} n_{b1} = \cos(\pi/2)$, whenever $\theta_{ai} = \theta_{bi}$. However, it is also needed that the points corresponding to $n_{a1}$ and $n_{b1}$ with an overlap of $\cos(\pi/2)$ correspond to a positive semidefinite matrix for $\theta_{ai} = \theta_{bi}$, which we were unable to identify. Thus, it is similar to the orthogonality constraint for qubit Bloch vectors. For $\theta_{ai} \neq \theta_{bi}$, the orthogonality condition is far more complex.

It can be easily shown that eigenbases of the four sets of non-commuting HW operators lie at antipodal points on the Bloch sphere. For example, the eigenkets of $Z$ lies on points with Bloch vectors $\{n_1, n_2, n_3, n_4\} : \{0, 1, 0, 0\}$ and $\{0, -1, 0, 0\}$. This is very similar to what we observe in the qubit Bloch sphere, where the orthonormal basis kets lie at the antipodal points. Although, all the orthonormal kets do not lie on the antipodal points in the four-D sphere. We conjecture the following regarding the Bloch vectors belonging to an orthonormal basis.

**Conjecture 1** - The kets belonging to an orthonormal basis lie at the antipodal points or at the same point on the surface of the Bloch sphere.

The motivation of this conjecture comes from the observation (see Appendix A) that the orthonormal basis kets which are mutually unbiased to the computational basis, lie on the same point in the four-D sphere.

D. Mutually unbiased states

By doing a similar analysis as for orthogonal states, we can also obtain the relation between Bloch vectors corresponding to two mutually unbiased state vectors. For two mutually unbiased states in 3 dimension, $\rho_a$ and $\rho_b$, $\text{Tr}[\rho_a \rho_b] = \frac{1}{3}$. By using the result from previous subsection we get

$$\frac{1}{3} \left( 1 + 2n_{a1} n_{b1} \cos(\theta_{a1} - \theta_{b1}) + 2n_{a2} n_{b2} \cos(\theta_{a2} - \theta_{b2}) + 2n_{a3} n_{b3} \cos(\theta_{a3} - \theta_{b3}) + 2n_{a4} n_{b4} \cos(\theta_{a4} - \theta_{b4}) \right) = \frac{1}{3} \left( n_{a1} n_{b1} \cos(\theta_{a1} - \theta_{b1}) + n_{a2} n_{b2} \cos(\theta_{a2} - \theta_{b2}) + n_{a3} n_{b3} \cos(\theta_{a3} - \theta_{b3}) + n_{a4} n_{b4} \cos(\theta_{a4} - \theta_{b4}) \right) = 0. \quad (19)$$

Whenever $\cos(\theta_{a1} - \theta_{b1}) = \text{const}$, Eq.(19) reduces to $n_{a1} n_{b1} = 0$, i.e. the Bloch vectors corresponding to mutually unbiased state vectors are orthogonal to each other, exactly what we get for mutually unbiased qubits. Here we require that for orthogonal Bloch vectors the angular parameters obey the relation $\cos(\theta_{ai} - \theta_{bi}) = \text{const}$. For $\cos(\theta_{ai} - \theta_{bi}) \neq \text{const}$, the condition is more complex.

E. Distance between density matrices

Let us consider two states $\rho_1 : n_1, n_2, n_3, n_4, \theta_1, \theta_2, \theta_3,$ and $\rho_2 : m_1, m_2, m_3, m_4, \phi_1, \phi_2, \phi_3, \phi_4$. By using the form of Eq.(7), we obtain the Hilbert-Schmidt distance between them as [21]

$$D_{HS}(\rho_1, \rho_2) = \left( \text{Tr}(\rho_1 - \rho_2)^2 \right)^{\frac{1}{2}} \quad (20)$$

$$= \sqrt{\frac{2}{3} \left( n_1^2 + n_2^2 + n_3^2 + m_1^2 + m_2^2 + m_3^2 + m_4^2 - 2n_1 m_1 \cos(\theta_1 - \phi_1) - 2n_2 m_2 \cos(\theta_2 - \phi_2) - 2n_3 m_3 \cos(\theta_3 - \phi_3) - 2n_4 m_4 \cos(\theta_4 - \phi_4) \right)}^{\frac{1}{2}}. \quad (21)$$

When $\theta_1 = \phi_1$, the Hilbert-Schmidt distance reduces to the euclidean distance(times a const.) in the four-D sphere, i.e. $D_{HS}(\rho_1, \rho_2) = \sqrt{\frac{2}{3} \sum (n_i - m_i)^2}$. This is analogous to the fact that the Hilbert-Schmidt distance between two density matrices is proportional to the euclidean distance between them in the qubit Bloch sphere [21].

VI. APPLICATIONS

A. Employing the Bloch sphere geometry to find MUBs in three dimensions

It is known that in prime or power of prime dimension $d = p^n$, where $p$ is a prime number and $n$ is an integer greater than zero, there exist a maximum of $d + 1$ MUBs. For the qubit, the existence of 3 MUBs can be very easily explained through qubit Bloch sphere, but such an explanation is difficult in higher dimensions. In this section, we show that the qutrit Bloch sphere geometry restricts the maximum number of MUBs to four.
**MUBs in 2 dimensions**- The qubit Bloch sphere is a three dimensional sphere, in which the Bloch vectors corresponding to orthonormal basis kets lie on the antipodal points on the sphere, i.e. they lie along the line passing through the center. Also, the Bloch vectors corresponding to mutually unbiased kets are orthogonal to each other [22]. As, there can be only three such orthogonal lines passing through the center, which explains why there are only three possible mutually unbiased bases in dimension 2.

**MUBs in 3 dimensions**- To find the qutrit MUBs, we first fix one of the orthonormal basis to be the eigenbasis of HW operator Z or the computational basis. The eigenkets of Z have the following Bloch vector and angular parameters

\[ \{0\} : n_2 = 1, \theta_2 = 0, n_1 = n_3 = n_4 = 0, \{\theta_1, \theta_3, \theta_4\} - \text{any value}, \]

\[ \{1\} : n_2 = 1, \theta_2 = \frac{\pi}{3}, n_1 = n_3 = n_4 = 0, \{\theta_1, \theta_3, \theta_4\} - \text{any value}, \]

\[ \{2\} : n_2 = 1, \theta_2 = \frac{2\pi}{3}, n_1 = n_3 = n_4 = 0, \{\theta_1, \theta_3, \theta_4\} - \text{any value}. \]

Now any qutrit ket which is mutually unbiased to all the computational basis kets must have \( n_2 = 0 \) and \( \theta_2 \) can have any arbitrary value, which can be also deduced from Eq.(19). Hence, we need to find the remaining mutually unbiased kets using the remaining weight parameters \( n_1, n_3 \) and \( n_4 \) and the angular parameters \( \theta_1, \theta_3, \theta_4 \).

We fix one of the ket \( n^0_0 \), parametrized as

\[ n^0_0 : n_1 = 1, n_3 = m_3, n_4 = m_4, \theta_1 = \phi_1, \theta_3 = \phi_3 \quad \text{and} \quad \theta_4 = \phi_4. \]

As shown in Appendix A, we find that an orthonormal basis \( N = \{n^0_0, n^0_1, n^0_2\} \) including \( n^0_0 \) are parametrized as

\[ n^0_0 : n_1 = m_1, n_3 = m_3, n_4 = m_4, \theta_1 = \phi_1, \theta_3 = \phi_3 \quad \text{and} \quad \theta_4 = \phi_4, \]

\[ n^0_1 : n_1 = m_1, n_3 = m_3, n_4 = m_4, \theta_1 = \phi'_1, \theta_3 = \phi'_3 \quad \text{and} \quad \theta_4 = \phi'_4, \]

\[ n^0_2 : n_1 = m_1, n_3 = m_3, n_4 = m_4, \theta_1 = \phi''_1, \theta_3 = \phi''_3 \quad \text{and} \quad \theta_4 = \phi''_4. \]

We have derived the values of angles \( \phi'_1, \phi'_3, \phi''_1, \phi''_3 \) and \( \phi'_4 \) relative to \( \phi_1, \phi_3 \) and \( \phi_4 \) in the Appendix A. It should be noted that all the kets of the orthonormal basis lie on the same point \( \vec{n} = \{m_1, 0, m_3, m_4\} \) in the four-D sphere. Moreover, from Appendix A we note that the vectors mutually unbiased to the kets of orthonormal basis \( N \) lie on the points \( \vec{p} = \{m_1, m_4, m_1\} \) and \( \vec{q} = \{m_4, m_1, m_3\} \). For points \( \vec{p} \) and \( \vec{q} \) again we can form an orthonormal bases \( B \) and \( Q \), respectively by varying the angular parameters only. The three orthonormal bases lie at the end points of the Bloch vectors, with Bloch vectors which can be obtained by cyclic permutations of \( \{n_1, n_3, n_4\} \), which we write down for brevity.

\[ N : n_1 = m_1, n_3 = m_3 \quad \text{and} \quad n_4 = m_4, \]

\[ P : n_1 = m_4, n_3 = m_1 \quad \text{and} \quad n_4 = m_3, \]

\[ Q : n_1 = m_3, n_3 = m_4 \quad \text{and} \quad n_4 = m_1. \number{22} \]

Therefore, there can be only three orthonormal bases are mutually unbiased to each other, so that all of them are mutually unbiased to the computational basis. Hence, we have obtained from the geometry of the Bloch hypersphere that there are maximum four mutually unbiased bases in three dimension.

**B. Characterization of Unital Maps**

In this section, we characterize the unital maps acting on the qutrit states. Unital maps are quantum operations that preserve the identity matrix or the maximally mixed density matrix. It is known that the unital maps acting on a qutrit density matrix is characterized by a convex tetrahedron[23, 24].

To analyze the unital channels acting on a qutrit density matrix \( \rho = \frac{1}{3} \sum_{p,q} b_{pq} U_{pq} \) with Bloch vector \( \vec{b}_{pq} \) (see Eq.(2)), we note that a linear quantum map can be written in the form of an affine transformation acting on the \( d^2 - 1 = 8 \) dimensional Bloch vector. Thus, every linear qutrit quantum map \( \Phi : \mathbb{C}^{3 \times 3} \rightarrow \mathbb{C}^{3 \times 3} \) can be represented using a \( 9 \times 9 \) matrix \( \mathcal{L} = \begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix} \) acting on the column vector \( \{1, \vec{b}^* \} \), where \( L \) is an \( 8 \times 8 \) matrix and \( L \) is a column vector containing eight elements. The action of the quantum channel \( \rho \rightarrow \Phi(\rho) = \frac{1}{3} \sum_{p,q} b'_{pq} U_{pq} \) can be written as

\[ \vec{b} \rightarrow \vec{b}' = L \vec{b} + \vec{l}. \]

By observing Eq.(5), it can be seen that to make sure that \( \vec{b}' \) corresponds to a hermitian density matrix \( \mathcal{E}(\rho) \), it is necessary that 1) \( L \) is a diagonal matrix with eigenvalues \( \{\lambda_0, \lambda_2, ..., \lambda_2\} \) and 2) the eigenvalues must be of the following form

\[ \lambda_0 = \lambda_1 e^{i \phi_1}, \lambda_2 = \lambda_1 e^{-i \phi_1}, \]

\[ \lambda_10 = \lambda_2 e^{i \phi_2}, \lambda_20 = \lambda_2 e^{-i \phi_2}, \]

\[ \lambda_{12} = \lambda_3 e^{i \phi_3}, \lambda_{21} = \lambda_3 e^{-i \phi_3}, \]

\[ \lambda_{22} = \lambda_4 e^{i \phi_4}, \lambda_{11} = \lambda_4 e^{-i \phi_4}. \]

Next, we note that to preserve the identity matrix, \( \vec{l} = \vec{0} \), so that \( \mathcal{L} = \begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix} \). Now, to do the complete characterization of the map \( \mathcal{L} \) we impose the complete positivity requirement via Choi’s theorem which requires that Choi Matrix \( \mathcal{C} = (\mathcal{I} \otimes \Phi)(\sum_{i,j} E_i \otimes E_j) \) is positive semidefinite. To simplify the problem, we find the eigenvalues only when the angles \( \phi_i = 0, \forall i \in \{0, 3\} \). The constraints on the parameters \( \{\lambda_i\} \)'s are given by

\[ 1 + 2\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 \geq 0, \]

\[ 1 - \lambda_1 + 2\lambda_2 - \lambda_3 - \lambda_4 \geq 0, \]

\[ 1 - \lambda_1 - 2\lambda_2 + 2\lambda_3 - 4\lambda_4 \geq 0, \]

\[ 1 - \lambda_1 - \lambda_2 - 2\lambda_3 + 2\lambda_4 \geq 0, \]

\[ 1 + 2\lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4 \geq 0. \number{23} \]

The above constraint give a convex polygon space with five vertices

\[ v_1 = \{1, 1, 1, 1\}, \]

\[ v_2 = \{1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\}, \]

\[ v_3 = \{1, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\}, \]

\[ v_4 = \{1, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\}, \]

\[ v_5 = \{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\}. \]
It is an irregular polygon with 8 edges, out of which 4 edges have euclidean length $\sqrt{\frac{9}{2}}$ and 4 other edges have euclidean length $\sqrt{\frac{27}{4}}$.

It is insightful to visualize the effect of the action of the channel on a state in the four-D sphere. The parameters $\{|\lambda_i|\}$ reduce the length of each Bloch vector component from $n_i$ to $\lambda_i n_i$, thus bringing the state closer to the origin. Now notice that, we have obtained the constraints in Eq.(23) assuming that all $\phi_i = 0$, i.e., there is no change in the angular parameters $\theta_i$. The allowed values of $\phi$'s will therefore depend on how much the lengths of the Bloch vector components have been reduced by $\lambda_i$.

C. Characterization of Randomly Generated Density Matrices

In this section, we characterize the structure of the state space of randomly generated density matrices, using the four-D Bloch sphere. Specifically, we show the representation of ensembles generated by Hilbert-Schmidt and Bures metrics [25, 26]. The infinitesimal Hilbert-Schmidt(Eq.(20)) distance between $\rho$ and $\delta \rho$, has a very simple form given as $d^2_{\text{HS}} = \text{Tr}[(\delta \rho)^2]$.

In $n$-dimensions, the probability distribution induced by this metric, derived by Hall [27] is given by

$$P_{\text{HS}}(\lambda_1, ..., \lambda_n) = C_N \delta(1 - \sum_{i=1}^{n} \lambda_i) \prod_{j<k} (\lambda_j - \lambda_k)^2,$$  \hspace{1cm} (24)

where $\lambda$'s are the eigenvalues of $\rho$ and $C_N$ is determined by the normalization.

For mixed quantum states there is another useful distance measure known as the Bures distance [28, 29].

$$D_{\text{Bures}}(\rho_1, \rho_2) = \sqrt{2(1 - \text{Tr}(\sqrt{\rho_1 \rho_2} \sqrt{\rho_1 \rho_2}))}.$$  \hspace{1cm} (25)

where again $\lambda_k$ and $|k\rangle$ are respectively the eigenvalues and eigenvectors of $\rho$. For this metric also, the probability distribution was derived by Hall [27], which is given by

$$P_{\text{Bures}}(\lambda_1, ..., \lambda_n) = C'_{N} \delta(1 - \sum_{i=1}^{n} \lambda_i) \prod_{j<k} (\lambda_j - \lambda_k)^2,$$  \hspace{1cm} (26)

where $C'_{N}$ is again determined by the normalization. In Eqs.(24) and (25), we have the probability distributions defined on the simplex of eigenvalues. However, we want to see how this probability distribution picks out the states from the bloch sphere. For a two-dimensional state $\rho = \frac{1}{2}(I + \vec{\sigma} \cdot \vec{\rho})$, we can translate the eigenvalues to Bloch sphere parameters using the simple formulas $\lambda_1 = \frac{1}{2}r$ and $\lambda_2 = \frac{1}{2}r$, where $\lambda_1, \lambda_2$ are the two eigenvalues of $\rho$. By substituting these in Eqns.(24) and (25), we get the following probability distributions in terms of Bloch sphere parameters [27]

$$P_{\text{HS}}(r) = \frac{3}{4\pi},$$  \hspace{1cm} (27)

$$P_{\text{Bures}}(r) = \frac{4}{\pi \sqrt{1 - r^2}}.$$  \hspace{1cm} (28)

We can see that both probability distributions are dependent only on the radial parameter $r$. While the HS distribution is uniform over the Bloch sphere while the Bures distribution is sharply peaked at the surface of the Bloch sphere.

Next, we derive the form of these probability distributions with respect to our representation of qutrit states. For a qutrit state $\rho_3$, its eigenvalues $\lambda_1, \lambda_2$ and $\lambda_3$ can be written directly in terms of the Bloch sphere parameters $n_i$'s and angular parameters $\theta_i$'s. However, a direct approach will lead to cumbersome calculations. Instead, we write the eigenvalues $\lambda$'s in terms of the characteristic equation coefficients $a_i$'s from Eq.(8) and substitute in the Eqns.(24) and (25) which gives us the following

$$P_{\text{HS}}(r, \xi, \theta_i) = C'_{N} \frac{((r - 1)^2(2r + 1) - 27\text{Det}(\rho))((r + 1)^2(2r - 1) + 27\text{Det}(\rho))}{27r^3},$$  \hspace{1cm} (29)

where we have switched to polar representation with $n_1 = r \cos(\xi)$, $n_2 = r \sin(\xi) \cos(\theta_i)$, $n_3 = r \sin(\xi) \sin(\theta_i) \cos(\theta_i)$ and $n_4 = r \sin(\xi) \sin(\theta_i) \sin(\theta_i)$. Also, $C'_{HS}$ and $C_B$ are constants and are determined by the normalization. In this form these probability distributions don’t give much information about the states in the Bloch sphere because of dependence on the angular parameters $\theta_i$'s which are not a part of the four-D sphere. We can obtain a distribution for a subset of states by fixing the $\theta_i$ values(say all zero), we get $\text{Det}(\rho) = f(r, \xi, \theta_2, \theta_3)$. The distributions in Eqns.(28) and (29) are not invariant with respect to unitary operations unlike in the qubit scenario. This is a signature of the fact that all points inside the four-D Bloch sphere don’t represent physical states.

After some algebraic calculations, it is found that the HS distribution in Eq.(28) is always positive irrespective of $\text{Det}(\rho)$ being positive or negative. Where as the Bures distribution in Eq.(29) is positive if $\text{Det}(\rho) > 0$, hence picks out the closed structure of the qutrit states inside the Bloch sphere. Moreover, the HS distribution in non-decreasing with respect to the radial parameter $r$, everywhere. Where as, the Bures distribution is non-decreasing with respect to $r$ in the re-
gion where the $\text{Det(}\rho\text{)} \geq 0$. It can also be seen that the Bures distribution is sharply peaked whenever the denominator $(\frac{1}{\tau} - \text{Det(}\rho\text{)})\text{Det(}\rho\text{)}$ vanishes. While $\text{Det(}\rho\text{)} = 0$ for rank-2 or rank-1 states, the $\frac{1}{\tau} - \text{Det(}\rho\text{)}$ term can vanish only at the surface of the Bloch sphere or beyond.

Thus if we fix the $\theta_i$’s, both these distributions are localized closer to the surface of the Bloch sphere. For the HS distribution this is unlike what happens in the qubit scenario where it is uniform all over the sphere. Whereas, the Bures distribution is sharply peaked near or at the surface of the Bloch sphere. It is similar to the behavior of the Bures distribution in the qubit scenario, where the Bures distribution is sharply peaked on the surface. These results are matching with the plots presented in Fig. 2 of Ref. [26], which depicts the plots in the simplex of eigenvalues.

As an example, we fix all the radial parameter $\zeta_i$’s as $\zeta_1 = \pi/3, \zeta_2 = 0, \zeta_3 = \pi/7$, to see the dependence on the radial parameter $r$, and obtain the following

$$P_{\text{HS}}(r) = C_{\text{HS}} \frac{6 - \sqrt{3}}{72} r^3$$

$$P_B(r) = C_B \frac{162(6 - \sqrt{3})r^3}{(\sqrt{4 - 12r^2 + 6.19r^2})(-32 + 24r^2 + 6.19r^2)}.$$  \tag{30}

We see that in the chosen direction, HS distribution is peaked on the surface of the Bloch sphere and it is everywhere positive. While the Bures distribution is sharply peaked at $r \approx 0.73$ and while is negative for $r > 0.73$. It simply tells that for the chosen $\theta_i$’s there are no more physical states beyond $r \approx 0.73$ in the chosen direction and also that there is a rank 2 state at $r \approx 0.73$. The other singularity of the Bures distribution lies at $r \approx 1.02$, but $P_B(r)$ is negative after $r \approx 0.73$ and hence we ignore it.

In section B of the Appendix, we also do the analysis of HS and Bures distributions when the qutrit states are represented using Gell-Mann operators. There also we make similar observations, i.e.,

1) The HS distributions is always positive where as the Bures distribution is positive iff $\text{Det(}\rho\text{)} \geq 0$.
2) HS distribution is non-decreasing with respect to the radial parameter and hence the states are localized on the surface of the convex structure of the states and Bures distribution is non-decreasing for $\text{Det(}\rho\text{)} \geq 0$ and it also blows up at the surface of the Bloch sphere or for the rank-2 states.

VII. BLOCH VECTOR FOR QUDITS

This approach of separating the weight parameters and angular parameters, in the HW operator based representation can also be extended in higher dimensions. If we consider a qudit of dimension $d$, the number of HW operators are $d^2 - 1$ (excluding the identity matrix), among them there will be $d + 1$ sets of commuting HW operators containing $d - 1$ HW operators each. Each set will have one weight parameter associated with it as $w_i$ and $d - 2$ angular parameters, so that there are total $d^2 - 1$ real parameters. In this way one can create a $d + 1$ dimensional Bloch sphere built from $d + 1$ weight parameters. However, as dimensions increase this analysis will only become more complex.

VIII. CONCLUSION

To conclude, we have used the HW operator basis to represent a qutrit state. In doing so, we identified eight independent parameters consisting of four weight and four angular parameters and constructed a four-D Bloch sphere representation of the qutrit. We have obtained the constraints which must be satisfied for the parametrization to represent a physical qutrit. This representation seems like a natural extension of the qubit Bloch sphere because of the following properties.

1. The one and two-dimensional sections are symmetric with respect to the axes.
2. Purity of a state depends on the length of the Bloch vector.
3. Rank of the states can be identified to some extent by looking at the distance from the origin of the four-D sphere.
4. The conditions of orthogonality and mutual unbiasedness of two Bloch vectors has lot of similarity with conditions for the qubit Bloch vectors.
5. Hilbert-Schmidt distance between two qutrit states(with same angular parameter values $\theta_i$’s) is proportional to the Euclidean distance in the four-D sphere.

We have applied our Bloch vector representation to show that there can be a maximum of four MUBs in three dimensions. The characterization of unital maps acting on qutrits is also demonstrated using our representation. We also did a characterization of randomly generated density matrices, when the probability distributions are induced by Hilbert-Schmidt and Bures distances. Lastly, we have mentioned the basic steps required to extend this representation in dimensions greater than three.

One of the future works based on extending this work could be to identify the structure of the allowed set of points which represent a physical qutrit state. Another significant work would be to generalize this representation in higher dimensions than three.

As we have shown in this paper that the geometry of the Bloch sphere limits the existence of the number of MUBs in qubits and qutrits. This approach can be used to study the existence of MUBs in 6 dimensions, where the maximum number of MUBs is not known yet [31–33]. An extension to the characterization of unital maps would be to characterize qutrit entanglement breaking channels similar to qubit entanglement breaking channels [19]. Similar to characterization of ensembles generated by HS and Bures metric, another interesting study could be to identify the form of Fubini-Study metric and the corresponding volume element [34]. Such an analysis
could be useful for sampling of pure qudit states and averaging over them.

Our four-D sphere representation could also have significant applications in studying the dynamics of qudit states and to find the constants of motion in d-level systems. It can also be used to detect entanglement of bipartite systems and identifying the reachable states in open system dynamics. We hope that this approach leads to better insight in the study of qudit systems and their dynamics.

1. R. A. Bertlmann and P. Krammer, *Journal of Physics A: Mathematical and Theoretical* 41, 235303 (2008).
2. U. Fano, Rev. Mod. Phys. 55, 855 (1983).
3. E. Brüning, H. Mäkelä, A. Messina, and F. Petruccione, *Journal of Modern Optics* 59, 1 (2012).
4. G. Kimura, *Physics Letters A* 314, 339 (2003).
5. G. Kimura and A. Kossakowski, “The bloch-vector space for n-level systems – the spherical-coordinate point of view,” (2004), arXiv:quant-ph/0408014 [quant-ph].
6. S. Kryszewski and M. Zachcial, arXiv e-prints, quant-ph/0602065 (2006), arXiv:quant-ph/0602065 [quant-ph].
7. I. P. Mendaš, *Journal of Physics A: Mathematical and General* 39, 11313 (2006).
8. S. K. Goyal, B. N. Simon, R. Singh, and S. Simon, *Journal of Physics A: Mathematical and Theoretical* 49, 165203 (2016).
9. P. Kurzyński, Quantum Info. Comput. 11, 361–373 (2011).
10. P. Kurzyński, A. Kołodziejski, W. Laskowski, and M. Markiewicz, *Phys. Rev. A* 93, 062126 (2016).
11. C. Eltschka, M. Huber, S. Morelli, and J. Siewert, *Quantum* 5, 485 (2021).
12. I. Bengtsson, S. Weis, and K. Życzkowski, in *Geometric Methods in Physics*, edited by P. Kielanowski, S. T. Ali, A. Odzijewicz, M. Schlichenmaier, and T. Voronov (Springer Basel, Basel, 2013) pp. 175–197.
13. A. Vourdas, *Reports on Progress in Physics* 67, 267 (2004).
14. A. Asadian, P. Erker, M. Huber, and C. Klöckl, *Phys. Rev. A* 94, 010301 (2016).
15. N. Cotfas and D. Dragoman, *Journal of Physics A: Mathematical and Theoretical* 45, 425305 (2012).
16. B. Baumgartner, B. C. Hiesmayr, and H. Narnhofer, *Phys. Rev. A* 74, 032327 (2006).
17. A. Asadian, C. Budroni, F. E. S. Steinhoff, P. Rabl, and O. Gühne, *Phys. Rev. Lett.* 114, 250403 (2015).
18. C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, *Phys. Rev. Lett.* 70, 1895 (1993).
19. M. B. Ruskai, *Reviews in Mathematical Physics* 15, 643 (2003), https://doi.org/10.1142/S0129055X03001710.
20. K. S. Gibbons, M. J. Hoffman, and W. K. Wootters, *Phys. Rev. A* 70, 062101 (2004).
21. M. Wilde, *Quantum Information Theory* (Cambridge University Press, 2013).
22. G. Sharma, S. Sazim, and S. Mal, *Phys. Rev. A* 104, 032424 (2021).
23. C. King and M. B. Ruskai, *IEEE Transactions on Information Theory* 47, 192 (2001).
24. M. Beth Ruskai, S. Szarek, and E. Werner, *Linear Algebra and its Applications* 347, 159 (2002).
25. K. Życzkowski, K. A. Penson, I. Nechita, and B. Collins, *Journal of Mathematical Physics* 52, 062201 (2011).
26. K. Życzkowski and H.-J. Sommers, *Journal of Physics A: Mathematical and General* 34, 7111 (2001).
27. M. J. Hall, *Physics Letters A* 242, 123 (1998).
28. D. Bures, *Transactions of the American Mathematical Society* 135, 199 (1969).
29. A. Uhlmann, in *Groups and related Topics* (Springer, 1992) pp. 267–274.
30. M. Hübner, *Physics Letters A* 163, 239 (1992).
31. M. Grassl, “On sic-povms and mubs in dimension 6,” (2004), arXiv:quant-ph/0406175 [quant-ph].
32. P. Raynal, X. Lü, and B.-G. Englert, *Phys. Rev. A* 83, 062303 (2011).
33. I. Bengtsson, W. Bruzda, Å. Ericsson, J.-Å. Larsson, W. Tadej, and K. Życzkowski, *Journal of mathematical physics* 48, 052106 (2007).
34. I. Bengtsson, J. Brännlund, and K. Życzkowski, *International Journal of Modern Physics A* 17, 4675 (2002).

Appendix A: Finding the orthonormal bases mutually unbiased to the computational basis

Here we will show, how to construct an orthonormal basis given a qudit ket parametrized by \( n_{\alpha} : n_1 = m_1, n_2 = 0, n_3 = m_3, n_4 = m_4, \theta_1 = \phi_1, \theta_2 = \text{any number}, \theta_3 = \phi_3 \) and \( \theta_4 = \phi_4 \), so that the orthonormal basis is mutually unbiased with respect to the computational basis. Since, we are looking for pure states mutually unbiased to the computational basis, we can also represent \( n_{\alpha} : \{\alpha = \delta, \beta = \gamma\} \) using only two parameters as

\[
  n_{\alpha}^\gamma = \frac{1}{\sqrt{3}} \begin{pmatrix}
    e^{i\delta} \\
    e^{i\gamma} \\
    e^{i\phi}
  \end{pmatrix} \rightarrow \frac{1}{3} \begin{pmatrix}
    1 \\
    e^{i\delta} \\
    e^{i(\gamma-\delta)}
  \end{pmatrix}.
\]

The corresponding Bloch vector parameters can be obtained on comparing the Bloch vector representation with the above two parameter representation.
where \( \omega = e^{i \frac{\pi}{3}} \) is the cube root of unity. In this representation, a ket orthogonal to \( n_0 \) must have parameters \( n_i([\alpha = \delta + \frac{2\pi}{3}, \beta = \gamma - \frac{2\pi}{3}] \) or \( n_i([\alpha = \delta - \frac{2\pi}{3}, \beta = \gamma + \frac{2\pi}{3}] \). It is straightforward to observe from Eq. (A1) that for orthonormal vectors the weight parameters \( n_i \) are the same, only the angular parameters \( \theta_i \) vary.

Moreover, the orthonormal bases \( P \) and \( Q \) which are mutually unbiased to basis \( N \) (and to each other) are obtained by changing the angular parameters to \( \alpha = \delta + \frac{2\pi}{3}, \beta = \gamma \) and \( \alpha = \delta + \frac{2\pi}{3}, \beta = \gamma \) respectively, and then forming an orthonormal basis from them. It can be seen that on moving between the MUBs \( N, P \) and \( Q \), the weight parameters permute cyclically. Thus, concluding the proof.

**Appendix B: Random Density Matrices in Gell-Mann operator representation**

Using the Gell-Mann operator basis also one can write a qutrit state in the following way [1]

$$\rho = \frac{1}{3}(1 + \sum_{i=1}^{d^2-1} g_i \Lambda_i),$$

(B1)

where \( \Lambda_i \) are the Gell-Mann operators in three dimensions and \( g_i = \text{Tr}(\Lambda_i \rho) \) form the components of the eight-dimensional(eight-D) Bloch vector \( \vec{g} \). The eight Gell-Mann operators in three dimensions contain diagonal, symmetric and anti-symmetric matrices, but for simplicity we denote all of them with \( \Lambda_i \). Using the similar trick as in the case of Weyl operator representation we can get the HS and Bures distribution in terms of the Bloch vector parameters \( g_i \) as following

$$P_{HS}(\vec{r}, \gamma) = \frac{C_{HS}}{r^7} \left( \frac{1}{729}(r^2 - 3)^2(4r^2 - 3) + (2 - 2r^2 - 27\text{Det}(\rho))\text{Det}(\rho) \right),$$

(B2)

$$P_{BG}(\vec{r}, \gamma) = \frac{C_{BG}}{r^7(3 - r^2 - 9\text{Det}(\rho))\sqrt{\text{Det}(\rho)}} \left( \frac{1}{729}(r^2 - 3)^2(4r^2 - 3) + (2 - 2r^2 - 27\text{Det}(\rho))\text{Det}(\rho) \right),$$

(B3)

where we have switched to polar representation with \( r \) being the radial distance in the eight-D Bloch sphere and \( \gamma \)'s are the seven polar angles. \( C_{HS} \) and \( C_{BG} \) are constants determined by the normalization.

As in the Weyl representation, here also the HS distribution is always positive inside the eight-D Bloch sphere irrespective of \( \text{Det}(\rho) \) being positive of negative. Also, it is non-decreasing with respect to \( r \). Thus the states chosen are localized at the surface of the Bloch sphere.

The Bures distribution also behaves similar to the Weyl representation. It is positive iff \( \text{Det}(\rho) \geq 0 \) and also it is non-decreasing for \( \text{Det}(\rho) \geq 0 \). The singularity in \( P_{BG}(\vec{r}, \gamma) \) occurs either at \( (3 - r^2 - 9\text{Det}(\rho)) = 0 \) or when \( \sqrt{\text{Det}(\rho)} = 0 \). The first condition is only possible at or beyond the surface of the eight-D sphere. Where as, \( \text{Det}(\rho) = 0 \) can happen for rank-1 or rank-2 states, i.e., at the surface of the structure formed by the qutrit states. Thus, \( H_{BG} \) is sharply localized at the surface of the convex structure formed by the qutrit states.