Quantum correlations dynamics of quasi-Bell cat states

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Abstract

A model of dynamics of quantum correlations of two modes quasi-Bell cat states, based on Glauber coherent states, is considered. The analytic expressions of pairwise entanglement of formation, quantum discord and its geometrized variant are explicitly derived. We analyze the distribution of quantum correlations between the two modes and the environment. We show that, in contrast with squared concurrence, entanglement of formation, quantum discord and geometric quantum discord do not follow the property of monogamy except in some particular situations that we discuss.

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1 Introduction

The idea to encode information in multi-photon coherent states constitutes a promising tool in quantum information. Indeed, the coherent states superpositions have been employed as resource to implement many quantum tasks including quantum teleportation \cite{1,2}, quantum computation \cite{3,4,5}, entanglement purification \cite{6} and errors correction \cite{7}. These potential applications explain the special attention paid, during the last years, to the identification, characterization and quantification of quantum correlations in bipartite coherent states systems (see for instance the papers \cite{8,9,10} and references therein). The bipartite treatment was extended to superpositions of multimode coherent states \cite{12,13,14,15,16} which exhibit multipartite entanglement as for instance in GHZ (Greenberger-Horne-Zeilinger), W (Werner) states \cite{17,18} and entangled coherent state versions of cluster states \cite{19,20,21}. To quantify quantum correlations beyond entanglement in coherent states systems, measures such as bipartite quantum discord \cite{22,23} and its geometric variant \cite{24} were used. Explicit results were derived for quantum discord \cite{25,26,27,28,29,30,31} and geometric quantum discord \cite{32,33,34,35} for some special sets of coherent states.

In other hand, decoherence is a crucial process to understand the emergence of classicality in quantum systems. It describes the inevitable degradation of quantum correlations due to experimental and environmental noise. Various decoherence models were investigated and in particular the phenomenon of entanglement sudden death was considered in a number of distinct contexts (see for instance \cite{36} and reference therein). For optical qubits based on coherent states, the influence of the environment, is mainly due to energy loss or photon absorption. The photon loss or equivalently amplitude damping in a noisy environment can be modeled by assuming that some of field energy and information is lost after transmission through a beam splitter \cite{30,37}. Another important issue in analyzing the decoherence process concerns the distribution of quantum correlations between the bipartite coherent states and the environment. Accordingly, the study of the distribution of quantum correlations in a quantum system among its different parts constitutes an important issue. In fact, the free shareability of classical correlations is no longer valid in the quantum case and the distribution of quantum correlations obeys to severe restrictions. These restrictions are known in the literature as monogamy constraints. The concept of monogamy of entanglement for qubits was first proved by Coffman, Kundu and Wootters in 2001 \cite{38} and since then it was extended to other measures of quantum correlations \cite{39,40,41,42,43,44}. For a tripartite system $ABE$, the monogamy relation can be presented as follows. Let $Q_{A|B}$ (resp. $Q_{A|E}$ ) denote the shared correlation $Q$ between $A$ and $B$ (resp. $A$ and $E$) and $Q_{A|BE}$ the correlation shared between $A$ and the composite subsystem $BE$ comprising $B$ and $E$. The quantum correlation measured by $Q$ is monogamous if

$$Q_{A|BE} \geq Q_{A|B} + Q_{A|E}.$$  

(1)

In this paper, the focus will be maintained strictly on the evolution quantum correlations present in two modes quasi-Bell cat states based on Glauber coherent states. We study the monogamy relation to understand the distribution of quantum correlations between the two modes of quasi-Bell cat states and the environment. To approach this question, we use the bipartite measures: entanglement of formation, quantum discord or geometric quantum discord. This approach has the advantage relying upon bipartite measures of entanglement of formation and quantum discord that are physically
motivated and analytically computable.

This paper is organized as follows. In section 2, we introduce two modes quasi-Bell cat states based on Glauber coherent states. We discuss their evolution under amplitude damping modeled by the action of a beam splitter. We give the density matrices describing the evolution of the two modes as well as ones describing each mode coupled to the environment. In section 3, we explicitly derive the entanglement of formation for each bipartite subsystem. We also consider the distribution of entanglement between the system and the environment. The explicit expressions of pairwise quantum discord are derived in section 4. We also consider the monogamy relation of this measure which goes beyond entanglement of formation. Similar analysis are presented in the section 5 when bipartite correlations are measured by means of the geometric discord. Concluding remarks close this paper.

2 Evolution of quasi-Bell states under amplitude damping

2.1 Quasi-Bell states

Usually the standard Bell states are constructed as balanced superpositions of orthogonal states. Here, we consider superpositions involving non orthogonal states. In particular, we consider quasi-Bell states based on Glauber coherent states

\[ |\alpha\rangle = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \]

where \(|n\rangle\) is a Fock state and \(\alpha\) is the complex amplitude of the coherent state. The four quasi-Bell cat states are defined as

\[ |\alpha, \pm \alpha; m\rangle = N_m^{-\frac{1}{2}} (|\alpha, \pm \alpha\rangle + e^{im\pi} | - \alpha, \mp \alpha\rangle) \]

where the normalization factor \(N_m\) is

\[ N_m = (2 + 2e^{-4|\alpha|^2} \cos m\pi) \]

and the integer \(m = 0, 1 \text{ (mod 2)}\). Notice that the two modes quasi-Bell states can be converted in a state describing two logical qubits. This can be realized by means of even and odd coherent states

\[ |\pm\rangle = N_{\pm} (|\alpha\rangle \pm | - \alpha\rangle) \]

representing the two superpositions of Glauber states of same amplitude and opposite phase and \(N_{\pm}\) being the normalization factors. The vectors \(+\rangle\) (even cat state) and \(-\rangle\) (odd cat state) form an orthogonal basis of two dimensional Hilbert space and then can be viewed as two logical qubits. Furthermore, the even and odd Glauber coherent states constitute an important resource in implementing experimentally non orthogonal quasi-Bell states. For instance the state \(|\alpha, \alpha; 0\rangle\) can be produced by sending a coherent cat state of the form \(|\sqrt{2}\alpha\rangle + | - \sqrt{2}\alpha\rangle\) and the vacuum into the two input ports of a 50/50 beam splitter. Clearly, the generation of quasi-Bell states requires a source of coherent cat states. Some experimental achievements in this sense were obtained recently (see for instance [46, 47, 48, 49] and references therein). It is interesting to note that the quasi-Bell states could be successfully employed for quantum teleportation and many others quantum information processing
From this perspective quantum optical states, including the quasi-Bell coherent states, are expected to be useful in the context of quantum information science, especially for communications using qubits over long distance. This is mainly motivated by the fact that coherent states are more robust against photon absorption (see [7]) and subsequently presents an advantageous alternative to reduce the decoherence effects.

2.2 Photon loss mechanism of quasi-Bell cat states

Coherent fields, traveling through a long optical channel, interact inevitably with the environment. The coupling field-environment changes the quantum correlations and consequently causes the decoherence of the system. To characterize the environmental effects, we suppose that absorption of the transmitted photons is the dominant source of the decoherence mechanism. The description of the photon loss mechanism, also termed amplitude damping, can be modeled by the action of a beam splitter. In other words, we assume that some of the coherent field is lost in transit via a beam splitter. The coherent states enters one port of the beam splitter and the vacuum \( |0\rangle_E \), representing the environment, enters the second port. After transmission some information encoded in the coherent states is transferred and the remaining amount of information is lost to the noisy channel. Let us consider the system \( AB \) (the two mode quasi-Bell states) and its environment \( E \) in initial state

\[
\rho_{ABE}(0) = \rho_{AB}(0) \otimes \rho_E(0)
\]

where

\[
\rho_{AB}(0) = |\alpha, \pm \alpha; m\rangle \langle \alpha, \pm \alpha; m| \quad \rho_E(0) = |0\rangle_E \langle 0|.
\]

The dynamics of the whole closed system is unitary, i.e.,

\[
\rho_{ABE} = U \rho_{ABE}(0) U^\dagger.
\]

Two cases can be distinguished: the case that the two qubits only interact with their local environments and the case in which only one qubit is affected by a local environment. The first case (resp. the second) is called two-qubits (resp. one qubit) local amplitude damping channel. Here, we shall consider the situation where only the second mode of quasi-Bell cat states interact with the environment. In this scheme, we write the unitary operator describing the dynamical evolution of the whole system as

\[
U = \mathbb{I} \otimes B(\theta)
\]

where \( \mathbb{I} \) is the identity and the beam splitter operator, describing the interaction between the subsystem \( B \) and the environment \( E \), is given

\[
B(\theta) = \exp \left[ \frac{\theta}{2} (a_B^+ a_E^+ - a_B^- a_E^-) \right]. \tag{4}
\]

The objects \( a_L^+ \) and \( a_L^- \) \((L = B, E)\) are the usual harmonic oscillator ladder operators acting on the Fock modes of the subsystems \( B \) and \( E \). The reflection and transmission coefficients are

\[
t = \cos \frac{\theta}{2}, \quad r = \sin \frac{\theta}{2} \tag{5}
\]
in terms of the angle $\theta$ of the equation (11). The beam splitter transmissivity describes the decoherence behavior of the transmitted states. It can be related to the exponential energy loss of an optical fiber used in the transmission process as $t = e^{-\lambda L}$ where $\lambda$ is a parameter characterizing the energy loss of the fiber over a distance $L$. The dynamical evolution of the initial state under the action of the beam splitter writes as

$$|Q\rangle_{ABE} = (\mathbb{1} \otimes \mathcal{B}(\theta))|\alpha, \pm \alpha; m\rangle_{AB} \otimes |0\rangle_E.$$ 

It is simple to check that

$$|Q\rangle_{ABE} = \frac{1}{\sqrt{N_m}}(|\alpha, \pm \alpha t, \pm \alpha r| + e^{im\pi}| - \alpha, \mp \alpha t, \mp \alpha r\rangle).$$

(6)

The whole system is then represented by the density matrix

$$\rho_{ABE} = |Q\rangle_{ABE}\langle Q| = \frac{1}{N_m}\left(|\alpha, \pm \alpha t, \pm \alpha r\rangle\langle \alpha, \pm \alpha t, \pm \alpha r| + e^{im\pi}| - \alpha, \mp \alpha t, \mp \alpha r\rangle\langle -\alpha, \mp \alpha t, \mp \alpha r| + e^{-im\pi}|\alpha, \pm \alpha t, \pm \alpha r\rangle\langle -\alpha, \mp \alpha t, \mp \alpha r|\right)$$

(7)

It is important to emphasize that the environment is constituted by the universe minus the subsystems $A$ and $B$. The density matrix $\rho_{ABE}$ is pure. As we shall be concerned with the distribution of quantum correlations in this pure tripartite system, we denote by $\rho_{AB}$ the reduced states for the subsystems $A$ and $B$ and analogously for $\rho_{AE}$ and $\rho_{AE}$. After tracing out all the modes of the environment, one gets

$$\rho_{AB} = \frac{N_m(t)}{N_m}\left[\frac{1}{2}(1 + c_r)|\alpha, \pm \alpha t; m\rangle\langle \alpha, \pm \alpha t; m| + \frac{1}{2}(1 - c_r)Z|\alpha, \pm \alpha t; m\rangle\langle -\alpha, \mp \alpha t; m|Z\right]$$

(8)

where the $r$-dependant quantity $c_r$ is

$$c_r = e^{-2r^2|\alpha|^2}$$

and the states $|\alpha, \pm \alpha t; m\rangle$ are given by

$$|\alpha, \pm \alpha t; m\rangle = N_m(t)^{-\frac{1}{2}}(|\alpha, \pm \alpha t| + e^{im\pi}| - \alpha, \mp \alpha t\rangle)$$

with

$$N_m(t) = \left(2 + 2e^{-2(1+r^2)|\alpha|^2}\cos(m\pi)\right).$$

The third Pauli operator $Z$ in (5) is defined by

$$Z|\alpha, \pm \alpha t; m\rangle = N_m(t)^{-\frac{1}{2}}(|\alpha, \pm \alpha t| - e^{im\pi}| - \alpha, \mp \alpha t\rangle).$$

Similarly, tracing out the degrees of freedom of the subsystem $B$, one finds

$$\rho_{AE} = \frac{N_m(r)}{N_m}\left[\frac{1}{2}(1 + c_t)|\alpha, \pm \alpha r; m\rangle\langle \alpha, \pm \alpha r; m| + \frac{1}{2}(1 - c_t)Z|\alpha, \pm \alpha r; m\rangle\langle -\alpha, \mp \alpha r; m|Z\right]$$

(9)

where $N_m(r)$, $c_t$ and the operation $Z$ are defined as above modulo the obvious substitution $r \leftrightarrow t$.

It is also simply verified that the reduced state $\rho_{BE} = \text{Tr}_A \rho_{ABE}$ is given by

$$\rho_{BE} = \frac{N_m(0)}{N_m}\left[\frac{1}{2}(1 + c_1)|\alpha t, \pm \alpha r; m\rangle\langle \alpha t, \pm \alpha r; m| + \frac{1}{2}(1 - c_1)Z|\alpha t, \pm \alpha r; m\rangle\langle \alpha t, \pm \alpha r; m|Z\right].$$

(10)
where $N_m(0) = N_m(t = 0)$ and

$$\ket{\alpha t, \pm \alpha r; m} = N_m^{-1/2} (\ket{\alpha t, \pm \alpha r} + e^{i\pi\alpha} \ket{-\alpha t, \mp \alpha r})$$

and $c_1 = e^{-2|\alpha|^2}$.

Having expressed the reduced density matrices of the different subcomponents of the system quasi-Bell cat states coupled to its surroundings, we shall consider, in the following sections, the explicit evaluation of pairwise quantum correlations measured by entanglement of formation, usual quantum discord and geometric quantum discord. A special attention will be paid to the monogamy relation of each of these measures.

3 Entanglement of formation

To begin our task, we first derive the explicit expressions of entanglement of formation measuring the bipartite correlations present in the states $\rho_{AB}$, $\rho_{AE}$ and $\rho_{|A|BE}$ to discuss the entanglement monogamy measured by the concurrence and entanglement of formation [51]. For this, we shall map each of these bipartite subsystems in a pair of two logical qubits. This mapping is based on the fact, as mentioned above, that Shr"{o}dinger cat (even and odd) coherent states can be identified with two orthogonal qubits.

3.1 Concurrence and entanglement of formation

For the state $\rho_{AB}$, a qubit mapping can be introduced as follows. For the first mode $A$, we introduce a two dimensional basis spanned by the vectors $\ket{u_\alpha}$ and $\ket{v_\alpha}$ defined by

$$\ket{\alpha} = a_\alpha \ket{u_\alpha} + b_\alpha \ket{v_\alpha} \quad \ket{-\alpha} = a_\alpha \ket{u_\alpha} - b_\alpha \ket{v_\alpha} \quad (11)$$

where

$$|a_\alpha|^2 + |b_\alpha|^2 = 1 \quad |a_\alpha|^2 - |b_\alpha|^2 = \langle -\alpha | \alpha \rangle.$$ 

To simplify our purpose, we take $a_\alpha$ and $b_\alpha$ reals such as

$$a_\alpha = \frac{\sqrt{1 + p}}{\sqrt{2}} \quad b_\alpha = \frac{\sqrt{1 - p}}{\sqrt{2}} \quad \text{with} \quad p = \langle -\alpha | \alpha \rangle = e^{-2|\alpha|^2}.$$ 

Similarly, for the second mode $B$, a two-dimensional basis generated by the vectors $\ket{u_{at}}$ and $\ket{v_{at}}$ is defined as

$$\ket{\alpha t} = a_{at} \ket{u_{at}} + b_{at} \ket{v_{at}} \quad \ket{-\alpha t} = a_{at} \ket{u_{at}} - b_{at} \ket{v_{at}} \quad (12)$$

where

$$a_{at} = \frac{\sqrt{1 + p^{at}}}{\sqrt{2}} \quad b_{at} = \frac{\sqrt{1 - p^{at}}}{\sqrt{2}}.$$ 

The density matrix $\rho_{AB}$ can be cast in the following matrix form

$$\rho_{AB} = \frac{2}{N_m} \begin{pmatrix}
(1 + q_r)a^2_{at}a^2_{at} & 0 & 0 & (1 + q_r)a_\alpha a_{at}b_{at}b_{at} \\
0 & (1 - q_r)a^2_{at}a^2_{at} & (1 - q_r)a_\alpha a_{at}b_{at}b_{at} & 0 \\
0 & (1 - q_r)a_\alpha a_{at}b_{at}b_{at} & (1 - q_r)b^2_{at}a^2_{at} & 0 \\
(1 + q_r)a_\alpha a_{at}b_{at}b_{at} & 0 & 0 & (1 + q_r)b^2_{at}b^2_{at}
\end{pmatrix} \quad (13)$$
in the representation spanned by two-qubit product states

$$|1\rangle = |u\rangle_A \otimes |u\rangle_B \quad |2\rangle = |u\rangle_A \otimes |v\rangle_B \quad |3\rangle = |v\rangle_A \otimes |u\rangle_B \quad |4\rangle = |v\rangle_A \otimes |v\rangle_B.$$ 

In (13), the quantity $q_r$ is defined by

$$q_r = c_r \cos(m\pi).$$

It is simple to check that the Wootters concurrence \[51\] writes

$$C(\rho_{AB}) = p^2 \frac{\sqrt{1 - p^2} \sqrt{1 - p^{2r^2}}}{1 + p^2 \cos m\pi} \quad (14)$$

which coincides with the concurrence of the quasi-Bell cat states $|\alpha, \pm\alpha; m\rangle$ when $t = 1$. It follows that the bipartite quantum entanglement of formation in the state $\rho_{AB}$ is

$$E(\rho_{AB}) = H\left(\frac{1}{2} + \frac{1}{2} \frac{\sqrt{1 + 2p^2 \cos m\pi + p^{2r^2}(p^2 - 1)}}{1 + p^2 \cos m\pi}\right) \quad (15)$$

where $H(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)$. Remark that the reduced density $\rho_{AE}$ \[9\] can be obtained from the density $\rho_{AB}$ \[8\] by interchanging the roles of the transmission and reflection parameters $r$ and $t$. Accordingly, the state $\rho_{AE}$ can be converted in a two qubits system analogously to qubit mapping realized for the system $AB$. Then, it is easy to see that the concurrence is

$$C(\rho_{AE}) = p^2 \frac{\sqrt{1 - p^2} \sqrt{1 - p^{2r^2}}}{1 + p^2 \cos m\pi}, \quad (16)$$

and the entanglement of formation writes as

$$E(\rho_{AE}) = H\left(\frac{1}{2} + \frac{1}{2} \frac{\sqrt{1 + 2p^2 \cos m\pi + p^{2r^2}(p^2 - 1)}}{1 + p^2 \cos m\pi}\right). \quad (17)$$

Finally, the pure system $ABE$ can be partitioned into two qubits subsystems $A$ and $BE$. For the first mode $A$, we consider the two dimensional basis spanned by the vectors $|u\rangle$ and $|v\rangle$ defined by \[11\]. For the subsystem $BE$, we introduce two logical qubits $|0\rangle$ and $|1\rangle$ as follows

$$|\pm \alpha t, \pm\alpha r\rangle = a_\alpha |0\rangle + b_\alpha |1\rangle \quad |\mp \alpha t, \mp\alpha r\rangle = a_\alpha |0\rangle - b_\alpha |1\rangle \quad (18)$$

where $a_\alpha$ and $b_\alpha$ are given by

$$a_\alpha = \frac{\sqrt{1 + p}}{\sqrt{2}} \quad b_\alpha = \frac{\sqrt{1 - p}}{\sqrt{2}}.$$ 

It follows that, for $m = 0 \mod 2$, we have

$$\rho_{A|BE} = \frac{4}{N_0} \begin{pmatrix} a_\alpha^4 & 0 & 0 & a_\alpha^2 b_\alpha^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_\alpha^2 b_\alpha^2 & 0 & 0 & b_\alpha^4 \end{pmatrix}, \quad (19)$$

and for $m = 1 \mod 2$, we have

$$\rho_{A|BE} = \frac{4a_\alpha^2 b_\alpha^2}{N_1} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (20)$$

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in the basis \{\left|u_\alpha,0\right\rangle, \left|u_\alpha,1\right\rangle, \left|v_\alpha,0\right\rangle, \left|v_\alpha,1\right\rangle\}. In this representation, the concurrence writes

\[ C(\rho_{A|BE}) = \frac{1 - p^2}{1 + p^2 \cos m\pi}, \tag{21} \]

from which one derives the entanglement of formation

\[ E(\rho_{A|BE}) = H\left(\frac{1}{2} + \frac{1}{2} \frac{p \cos \frac{m\pi}{2}}{1 + p^2 \cos m\pi}\right). \tag{22} \]

As expected, it is completely independent of the reflection and transmission parameters \(r\) and \(t\).

### 3.2 Monogamy of concurrence and entanglement of formation

To examine the monogamy relation of entanglement measured by the concurrence in quantum systems involving three qubits, Coffman et al. introduced the so-called three tangle. It is defined from the bipartite concurrences as

\[ \tau_{A,B,E} = C^2(\rho_{A|BE}) - C^2(\rho_{AB}) - C^2(\rho_{AE}). \tag{23} \]

From equations (14), (16) and (21), we obtain

\[ \tau_{A,B,E} = (1 - p^2) \frac{(1 + p^2) - (p^2 r^2 + p^2 t^2)}{(1 + p^2 \cos m\pi)^2}. \tag{24} \]

In the figures 1 and 2, corresponding respectively to symmetric and antisymmetric quasi-Bell cat states, we plot the three tangle \(\tau_{A,B,E}\) as a function of \(p\) and \(t^2\). As it can be easily seen, \(\tau_{A,B,E}\) is always positive. The inequality given by (11) is then satisfied. This indicates that the squared concurrence is a monogamous measure.

![Figure 1](https://example.com/figure1.png)

**Figure 1.** The three tangle \(\tau_{A,B,E}\) versus the overlapping \(p\) and the transmission \(t^2\) for \(m = 0\).
Figure 2. The three tangle $\tau_{A,B,E}$ versus the overlapping $p$ and the transmission $t^2$ for $m = 1$.

In particular when the decoherence effect is modeled by a 50:50 beam splitter (i.e. $\tau^2 = t^2 = \frac{1}{2}$), one obtains

$$\tau_{A,B,E}(m=0) = \frac{(1-p^2)(1-p^2)}{(1+p^2)^2}$$

for symmetric quasi-Bell cat states and for the antisymmetric ones one has

$$\tau_{A,B,E}(m=1) = \frac{1-p}{1+p}$$

Clearly, for both cases, $\tau_{A,B,E}$ is positive reflecting the monogamous property of the squared concurrence in agreement with the results reported in the figures 1 and 2. In the limiting case, $p \rightarrow 0$ (resp. $p \rightarrow 1$), using the equations (14), (16) and (21), one can check that $\tau_{A,B,E} = 1$ (resp. $\tau_{A,B,E} = 0$).

Similarly, to decide about the monogamy of entanglement of formation, we examine the positivity of the following quantity

$$E_{A,B,E} \equiv E_{A,B,E}(t^2,p) = E(\rho_{A|BE}) - E(\rho_{AB}) - E(\rho_{AE})$$

defined in terms of the bipartite entanglement of formation given by the equations (15), (17) and (22). Noticing that

$$E_{A,B,E}(t^2,p) = E_{A,B,E}(r^2 = 1-t^2, p),$$

we shall restrict our discussion in what follows to the interval $0 \leq t^2 \leq 0.5$. The behavior of the function $E = E_{A,B,E}$ defined by (25) versus the overlapping $p$ and the transmission coefficient $t$ is plotted in the figures 3 for even quasi-Bell cat states ($m = 0$). It is symmetric with respect to the $t^2 = \frac{1}{2}$-axis as expected. The figure 3.a shows that the function $E_{A,B,E}$ is not always positive for symmetric quasi-Bell cat states and the entanglement of formation does not satisfy the monogamy relation (11) for small values of $t^2$. To see clearly this feature, we plot in the figure 3.b, the quantity $E_{A,B,E}$ for transmission $t^2$ ranging from 0.0125 to 0.2. The figure 3.b reveals that for $t^2 \leq 0.025$, the monogamy relation is violated for quasi-Bell cat states involving Glauber coherent states with overlap such that $0 \leq p \leq 0.4$. 

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Figure 3.a. $E = E_{A,B,E}$ versus the overlapping $p$ and the transmission $t^2$ for $m = 0$.

Figure 3.b $E = E_{A,B,E}$ versus the overlapping $p$ for small values of $t^2$ when $m = 0$.

For odd quasi-Bell cat states ($m = 1$), the function $E = E_{A,B,E}$ vs the transmission coefficient $t^2$ and the overlap $p$ is reported in the figures 4. The plot shows that the function $E = E_{A,B,E}$ decreases quickly from the unity to vanishes when $p \simeq 0.33$ for any value of the transmission parameter $t^2$. It follows that for $0 \leq p \leq 0.33$ the entanglement of formation is monogamous. The function $E = E_{A,B,E}$ becomes negative and the monogamy inequality cease to be satisfied for $0.33 \leq p \leq 1$.

Figure 4. $E = E_{A,B,E}$ versus the overlapping $p$ and the transmission $t^2$ for $m = 1$. 
4 Quantum discord

4.1 Definition and Koashi-Winter relation

For a state $\rho_{AB}$ of a bipartite quantum system composed of two particles or modes $A$ and $B$, the quantum discord is defined as the difference between total correlation $I(\rho_{AB})$ and classical correlation $J(\rho_{AB})$. The total correlation is usually quantified by the mutual information:

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}),$$

where $\rho_{A(B)} = \text{Tr}_{B(A)}(\rho_{AB})$ is the reduced state of $A(B)$, and $S(\rho)$ is the von Neumann entropy of a quantum state $\rho$. Suppose that a positive operator valued measure (POVM) measurement is performed on particle $A$. The set of POVM elements is denoted by $\mathcal{M} = \{M_k\}$ with $M_k \geq 0$ and $\sum_k M_k = I$. We remind that the generalized positive operator valued measure is not required. Indeed, it has been shown in [52] that the optimal measurement of the conditional entropy is ensured by projective one. Thus, a projective measurement on the subsystem $A$ project the system into a statistical ensemble $\{p_k^B, \rho_{Bk}\}$, such that

$$\rho_{AB} \rightarrow \rho_{Bk} = \frac{(M_k \otimes I)\rho_{AB}(M_k \otimes I)}{p_k^B},$$

where the von Neumann measurement for subsystem $A$ writes as

$$M_k = U \Pi_k U^\dagger : \quad k = 0, 1,$$

with $\Pi_k = |k\rangle\langle k|$ is the projector for subsystem $A$ along the computational base $|k\rangle$, $U \in SU(2)$ is a unitary operator with unit determinant, and

$$p_k^B = \text{tr}\left[(M_k \otimes I)\rho_{AB}(M_k \otimes I)\right].$$

The classical correlation is then obtained by performing the maximization over all the measurements. This gives

$$J(\rho_{AB}) = \max_{\mathcal{M}} \left[ S(\rho_B) - \sum_k p_k^B S(\rho_{Bk}) \right] = S(\rho_B) - \tilde{S}_{\text{min}},$$

where $\tilde{S}_{\text{min}}$ denotes the minimal value of the conditional entropy

$$\tilde{S} = \sum_k p_k^B S(\rho_{Bk}).$$

Then, the difference between $I(\rho_{AB})$ and $J(\rho_{AB})$ gives the amount of quantum discord present in the bipartite system $AB$

$$D(\rho_{AB}) = I(\rho_{AB}) - J(\rho_{AB}) = S(\rho_A) + \tilde{S}_{\text{min}} - S(\rho_{AB}).$$

The minimal value of the conditional entropy is related to the entanglement of formation of $E(\rho_{BC})$ of the state $\rho_{BC}$ which the complement of the density $\rho_{AB}$. This relation is the so-called Koashi-Winter relation [45]. It is given by

$$\tilde{S}_{\text{min}} = E(\rho_{BC}) = H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - |C(\rho_{BC})|^2}\right),$$

where $H(x)$ is the binary entropy function.
and establishes the connection between the classical correlation of a bipartite state $\rho_{AB}$ and the entanglement of formation of its complement $\rho_{BC}$. This connection requires a purification of the state $\rho_{AB}$ by an ancilla qubit $C$ and provides an explicit algorithm to determine the quantum discord especially for rank-two states.

### 4.2 Analytical computation of quantum discord

To evaluate the quantum discord present in the state (8), we first calculate the mutual information $I(\rho_{AB})$. The density $\rho_{AB}$ is a two qubit state of rank two. The corresponding non vanishing eigenvalues are given by

$$
\lambda_{\pm}^{AB} = \frac{(1 \pm p^2 \cos m\pi)(1 \pm p^{(2+1)})}{2 + 2p^2 \cos m\pi}.
$$

(33)

It follows that the joint entropy is

$$
S(\rho_{AB}) = -\lambda_{+}^{AB} \log_2 \lambda_{+}^{AB} - \lambda_{-}^{AB} \log_2 \lambda_{-}^{AB}.
$$

(34)

The quantum mutual information is then given by

$$
I(\rho_{AB}) = S(\rho_A) + S(\rho_B) + \sum_{j=+,-} \lambda_{j}^{AB} \log_2 \lambda_{j}^{AB}
$$

(35)

where $\rho_A$ and $\rho_B$ are the marginal states of $\rho_{AB}$, and

$$
S(\rho_A) = -\lambda_{+}^{A} \log_2 \lambda_{+}^{A} - \lambda_{-}^{A} \log_2 \lambda_{-}^{A} \quad S(\rho_B) = -\lambda_{+}^{B} \log_2 \lambda_{+}^{B} - \lambda_{-}^{B} \log_2 \lambda_{-}^{B}
$$

(36)

with

$$
\lambda_{+}^{A} = \frac{1}{2}(1 \pm p^2) \frac{1 \pm p \cos m\pi}{1 + p^2 \cos m\pi} \quad \lambda_{+}^{B} = \frac{1}{2}(1 \pm p^2) \frac{1 \pm p^{2+1} \cos m\pi}{2 + 2p^2 \cos m\pi}.
$$

(37)

Reporting (36) into (35), the quantum mutual information reads

$$
I(\rho_{AB}) = H(\lambda_{+}^{A}) + H(\lambda_{+}^{B}) - H(\lambda_{+}^{AB})
$$

(38)

To derive the explicit form of the classical correlation $J(\rho_{AB})$, we decompose the state $\rho_{AB}$ as

$$
\rho_{AB} = \lambda_{+}^{AB} |\psi_{+}\rangle \langle \psi_{+}| + \lambda_{-}^{AB} |\psi_{-}\rangle \langle \psi_{-}|
$$

(39)

where the eigenvalues $\lambda_{\pm}^{AB}$ are given by (33) and the eigenstates $|\psi_{\pm}\rangle$ are

$$
|\psi_{+}\rangle = \frac{1}{\sqrt{a_a^2 a_{at}^2 + b_a^2 b_{at}^2}} (a_a a_{at} |u_a, u_{at}\rangle + b_a b_{at} |v_a, v_{at}\rangle)
$$

$$
|\psi_{-}\rangle = \frac{1}{\sqrt{a_a^2 b_{at}^2 + b_a^2 a_{at}^2}} (a_a b_{at} |u_a, v_{at}\rangle + b_a a_{at} |v_a, v_{at}\rangle).
$$

(40)

Attaching a qubit $C$ to the bipartite system $AB$, we write the purification of $\rho_{AB}$ as

$$
|\psi\rangle = \sqrt{\lambda_{+}^{AB}} |\psi_{+}\rangle \otimes |u_a\rangle + \sqrt{\lambda_{-}^{AB}} |\psi_{-}\rangle \otimes |v_a\rangle
$$

(41)

such that the whole system $ABC$ is described by the pure density state $\rho_{ABC} = |\psi\rangle \langle \psi|$ so that $\rho_{AB} = \text{Tr}_C \rho_{ABC}$ and $\rho_{BC} = \text{Tr}_A \rho_{ABC}$. As mentioned above, The Koachi and Winter relation [45]
simplifies drastically the minimization process of the conditional entropy and the minimal amount of conditional entropy coincides with the entanglement of formation of $\rho_{BC}$. Therefore, employing the prescription presented in [51], the entanglement of formation in the state $\rho_{BC}$ is

$$\tilde{S}_{\min} = E(\rho_{BC}) = H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - |C(\rho_{BC})|^2}\right)$$  \hspace{1cm} (41)$$

with

$$C(\rho_{BC}) = \frac{\sqrt{p^2(1-p^2)(1-p^{2t^2})}}{(1+p^2 \cos m\pi)}.$$  

It must be noticed that this result can be alternatively obtained using the minimization procedure presented in [26] (see also [53]). According to the equation (29), the classical correlation is

$$J(\rho_{AB}) = H\left(\frac{1}{4}(1 + p^2 + 1 + p^{r^2 + 1} \cos m\pi)\right) - H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - p^2(1-p^{2r^2})(1-p^{2t^2})}/(1+p^2 \cos m\pi)^2\right)$$  \hspace{1cm} (42)$$

and using the definition (31), the explicit expression of quantum discord reads

$$D(\rho_{AB}) = H\left(\frac{1+p}{2} \frac{1+p \cos m\pi}{1+p^2 \cos m\pi}\right) - H\left(\frac{1+p^2 \cos m\pi}{2+2p^2 \cos m\pi}(1+p^{r^2+1})\right) + H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - p^2(1-p^{2r^2})(1-p^{2t^2})}/(1+p^2 \cos m\pi)^2\right)$$  \hspace{1cm} (43)$$

Note that for $r = 0$, the density state $\rho_{AB}$ [5] reduces to the pure density of quasi-Bell cat states [3] and the the quantum discord (43) gives

$$D(|\alpha, \pm \alpha; m\rangle) = H\left(\frac{1+p}{2} \frac{1+p \cos m\pi}{1+p^2 \cos m\pi}\right)$$  \hspace{1cm} (44)$$

which coincides with the entanglement of formation of quasi-Bell cat states given by

$$E(|\alpha, \pm \alpha; m\rangle) = H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - |C(|\alpha, \pm \alpha; m\rangle)|^2}\right)$$

where the concurrence is given by

$$C(|\alpha, \pm \alpha; m\rangle) = \frac{1-p^2}{1+p^2 \cos m\pi}.$$  

The quantum discord present in the bipartite state $\rho_{AE}$ can be simply obtained from the equation (43) by interchanging $r$ and $t$. So, we have

$$D(\rho_{AE}) = H\left(\frac{1+p}{2} \frac{1+p \cos m\pi}{1+p^2 \cos m\pi}\right) - H\left(\frac{1+p^2 \cos m\pi}{2+2p^2 \cos m\pi}(1+p^{r^2+1})\right) + H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - p^2(1-p^{2r^2})(1-p^{2t^2})}/(1+p^2 \cos m\pi)^2\right)$$  \hspace{1cm} (45)$$

The state $\rho_{A|BE}$ is pure and the quantum discord coincides with the entanglement of formation

$$D(\rho_{A|BE}) = E(\rho_{A|BE})$$

given by the expression (22).
4.3 Monogamy of quantum discord

Analogously to the treatment of squared concurrence and entanglement of formation presented in the previous section, we define the quantity

\[ D_{A,B,E} = D(\rho_{A|BE}) - D(\rho_{AB}) - D(\rho_{AE}) \]

as the difference between the quantum discord \( D(\rho_{A|BE}) \) and the sum \( D(\rho_{AB}) + D(\rho_{AE}) \).

For symmetric quasi-Bell cat states \((m = 0)\), the numerical results reported in the figures 5 and 6 show that the quantum discord is monogamous for any value of the reflection parameter \( r \) and the overlap \( p \). For a 50:50 beam splitter, the behavior of the quantity \( D_{A,B,E} \) is given in the figure 6. It reveals that \( D_{A,B,E} \), which is maximal for \( p = 0 \), decreases to reach a minimal value for \( p \approx 0.5 \) and increases after slowly.

\[ \text{Figure 5. } D = D_{A,B,E} \text{ versus the overlapping } p \text{ and the transmission } r^2 \text{ for } m = 0. \]

For antisymmetric quasi-Bell cat states \((m = 1)\), the monogamy becomes violated when the overlap \( p \) approaches the unity (see the figures 7 and 8). This feature is clearly illustrated in the figure 8 corresponding to the situation where \( t^2 = 1/2 \). The function \( D = D_{A,B,E} \) becomes negative for \( 0.85 \leq p \leq 1 \).

\[ \text{Figure 6. } D = D_{A,B,E} \text{ versus the overlapping } p \text{ for } t^2 = \frac{1}{2} \text{ and } m = 0. \]
Figure 7. $D = D_{A,B,E}$ versus the overlapping $p$ and the transmission $r^2$ for $m = 1$.

Figure 8. $D = D_{A,B,E}$ versus the overlapping $p$ for $i^2 = \frac{1}{2}$ and $m = 1$.

5 Geometric measure of quantum discord

5.1 Geometric quantum discord: Generalities

The geometrized version of quantum discord, introduced by Dakic et al [24], measures the distance between a state $\rho$ of a bipartite system $AB$ and the closest classical-quantum state presenting zero discord. It is defined by

$$D_g(\rho) := \min_{\chi} ||\rho - \chi||^2$$

(46)

where the minimum is over the set of zero-discord states $\chi$ and the distance is the square norm in the Hilbert-Schmidt space:

$$||\rho - \chi||^2 := \text{Tr}((\rho - \chi)^2).$$

When the measurement is taken on the subsystem $A$, the zero-discord state $\chi$ can be represented as [22]

$$\chi = \sum_{i=1,2} p_i |\psi_i\rangle \langle \psi_i| \otimes \rho_i$$
where \( p_i \) is a probability distribution, \( \rho_i \) is the marginal density matrix of \( B \) and \( \{|\psi_1\rangle, |\psi_2\rangle\} \) is an arbitrary orthonormal vector set. A general two qubit state writes in Bloch representation as

\[
\rho = \frac{1}{4} \left[ \sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} \left( x_i \sigma_i \otimes \sigma_0 + y_i \sigma_0 \otimes \sigma_i + \sum_{i,j=1}^{3} R_{ij} \sigma_i \otimes \sigma_j \right) \right]
\]  

(47)

where \( x_i = \text{Tr} \rho (\sigma_i \otimes \sigma_0) \), \( y_i = \text{Tr} \rho (\sigma_0 \otimes \sigma_i) \) are components of local Bloch vectors and \( R_{ij} = \text{Tr} \rho (\sigma_i \otimes \sigma_j) \) are components of the correlation tensor. The operators \( \sigma_i \) (\( i = 1, 2, 3 \)) stand for the three Pauli matrices and \( \sigma_0 \) is the identity matrix. The explicit expression of the geometric measure of quantum discord is given by\ [24]:

\[
D_g(\rho) = \frac{1}{4} \left( ||x||^2 + ||R||^2 - k_{\text{max}} \right)
\]  

(48)

where \( x = (x_1, x_2, x_3)^T \), \( R \) is the matrix with elements \( R_{ij} \), and \( k_{\text{max}} \) is the largest eigenvalue of matrix defined by

\[
K := xx^T + RR^T.
\]  

(49)

Denoting the eigenvalues of the \( 3 \times 3 \) matrix \( K \) by \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) and considering \( ||x||^2 + ||R||^2 = \text{Tr} K \), we get an alternative compact form of the geometric measure of quantum discord

\[
D_g(\rho) = \frac{1}{4} \min\{\lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3\}.
\]  

(50)

### 5.2 Explicit expressions

The density \( \rho^{AB} \) \[5\] writes, in the Bloch representation, as

\[
\rho_{AB} = \frac{1}{4} \left( \sigma_0 \otimes \sigma_0 + R_{30} \sigma_3 \otimes \sigma_0 + R_{03} \sigma_0 \otimes \sigma_3 + \sum_{i=1}^{3} R_{ii} \sigma_i \otimes \sigma_i \right)
\]  

(51)

where the correlation matrix elements are given by

\[
R_{03} = \frac{p^2 + p^2 - t^2 \cos m\pi}{1 + p^2 \cos m\pi} \quad R_{30} = \frac{p(1 + \cos m\pi)}{1 + p^2 \cos m\pi}
\]  

(52)

\[
R_{11} = \frac{\sqrt{(1 - p^2)(1 - p^2 t^2)}}{1 + p^2 \cos m\pi} \quad R_{22} = -p^2 \cos m\pi \sqrt{(1 - p^2)(1 - p^2 t^2)} \quad R_{33} = \frac{p^{1-t^2} \cos m\pi + p^{1+t^2}}{1 + p^2 \cos m\pi}.
\]  

(53)

The eigenvalues of the matrix \( K \), defined by\ [24], are thus given

\[
\lambda_1 = R_{30}^2 + R_{33}^2 \quad \lambda_2 = R_{11}^2 \quad \lambda_3 = R_{22}^2
\]

in terms of the elements of the matrix correlation. They also rewrite as

\[
\lambda_1 = p^2 \frac{p^{2t^2} + p^{-2t^2} + 4 \cos m\pi + 2}{(1 + p^2 \cos m\pi)^2}, \quad \lambda_2 = \frac{(1 - p^2)(1 - p^2 t^2)}{(1 + p^2 \cos m\pi)^2}, \quad \lambda_3 = p^{2t^2} \frac{(1 - p^2)(1 - p^2 t^2)}{(1 + p^2 \cos m\pi)^2}
\]  

(54)

It is clear that \( \lambda_3 \leq \lambda_2 \) and we have

\[
D_g(\rho_{AB}) = \frac{1}{4} \min\{\lambda_1 + \lambda_3, \lambda_2 + \lambda_3\}
\]  

(55)
For $\lambda_1 \geq \lambda_2$, the geometric measure of quantum discord gives
\[ D_g = \frac{\lambda_2 + \lambda_3}{4}. \]  
(56)
Alternatively, for $\lambda_1 \leq \lambda_2$, one obtains
\[ D_g = \frac{\lambda_1 + \lambda_3}{4}. \]  
(57)
Explicitly, the condition $\lambda_1 \geq \lambda_2$ writes as
\[ p^{2r^2} + p^{2t^2} + p^2(4 \cos m \pi + 3) - 1 \geq 0. \]  
(58)
A condition that we shall discuss separately for the symmetric and anti-symmetric cases. We first consider the situation where $m = 0$. In this case, the condition (58) becomes
\[ p^{2r^2} + p^{2t^2} + 7p^2 - 1 \geq 0. \]  
(59)
which is satisfied when $\frac{2\sqrt{2} - 1}{7} \leq p \leq 1$ for all possible values of $t$ ranging between 0 and 1. It follows that, for $\frac{2\sqrt{2} - 1}{7} \leq p \leq 1$, the geometric discord is given by
\[ D_g(\rho_{AB}) = \frac{\lambda_2 + \lambda_3}{4} = \frac{1 + p^{2r^2} (1 - p^2)(1 - p^{2t^2})}{(1 + p^2)^2}. \]  
(60)
For $0 \leq p \leq \frac{2\sqrt{2} - 1}{7}$, the condition (59) is satisfied for
\[ 0 \leq t^2 \leq t^2_- \quad t^2_- \leq t^2 \leq 1 \]
where
\[ t^2_- = \frac{1}{2} + \frac{1}{2} \frac{\ln \left[ \frac{1 - 7p^2}{4p} \pm \sqrt{\left( \frac{1 - 7p^2}{4p} \right)^2 - 1} \right]}{\ln p}. \]
In this situation, the geometric quantum discord is
\[ D_g(\rho_{AB}) = \frac{\lambda_2 + \lambda_3}{4} = \frac{1 + p^{2r^2} (1 - p^2)(1 - p^{2t^2})}{(1 + p^2)^2}. \]  
(61)
However, for coherent states with overlapping $p$ such that $0 \leq p \leq \frac{2\sqrt{2} - 1}{7}$ and when the transmission parameter $t$ satisfies
\[ t^2_- \leq t^2 \leq t^2_+ , \]
we have
\[ p^{2r^2} + p^{2t^2} + 7p^2 - 1 \leq 0, \]  
(62)
and the geometric quantum discord is given by
\[ D_g(\rho_{AB}) = \frac{\lambda_1 + \lambda_3}{4} = \frac{1}{4} p^2 p^{2(1 - t^2)} (1 - p^2)(1 - p^{2t^2}) \quad (1 + p^2)^2. \]  
(63)
For antisymmetric quasi-Bell states associated with $m = 1 \pmod{2}$, the condition $\lambda_1 \leq \lambda_2$ is always satisfied and in this case the geometric discord takes the simpie form
\[ D_g(\rho_{AB}) = \frac{\lambda_1 + \lambda_3}{4} = \frac{p^{2r^2}(2 - p^{2t^2} - p^2)}{4} \frac{1 - p^{2t^2}}{(1 - p^2)^2}. \]  
(64)
Here also, the geometric measure of quantum discord in the state $\rho_{AE}$ is simply obtained from $D_g(\rho_{AB})$ modulo the substitution $r \leftrightarrow s$.

In the pure bi-partitioning scheme $A|BE$, it is easy to check, using the method presented in the previous subsection, that the geometric discord is related to the concurrence of the state $\rho_{A|BE}$ as follows

$$D_g(\rho_{A|BE}) = \frac{1}{2} C^2(\rho_{A|BE})$$

which can be written as

$$D_g(\rho_{A|BE}) = \frac{1}{2} \frac{(1-p)^2}{(1+p^2 \cos m \pi)^2}.$$  \hspace{1cm} (65) \hspace{1cm} (66)

## 5.3 Monogamy of geometric discord

To illustrate the above analysis, we shall consider the special case where the decoherence of quasi-Bell cat states is simulated by the action of a 50:50 beam splitter. We treat first the evolution of the geometric quantum discord for symmetric quasi-Bell cat states ($m = 0$). In this case, using the results obtained in the previous subsection, it is simply verified that for $0 \leq p \leq \frac{2 \sqrt{2} - 1}{7}$

$$D_g(\rho_{AB}) = D_g(\rho_{AE}) = \frac{p}{4} \frac{p^3 + 5p + 2}{(1+p^2)^2}$$

and for $\frac{2 \sqrt{2} - 1}{7} \leq p \leq 1$

$$D_g(\rho_{AB}) = D_g(\rho_{AE}) = \frac{1}{4} \frac{(1-p^2)^2}{(1+p^2)^2}.$$  \hspace{1cm} (71)

We have also

$$D_g(\rho_{A|BE}) = \frac{1}{2} \frac{(1-p)^2}{(1+p^2)^2}$$

for $0 \leq p \leq 1$. The behavior of the quantity

$$D_g(A, B, E) = D_g(\rho_{A|BE}) - D_g(\rho_{AB}) - D_g(\rho_{AE}),$$

as function of the overlap $p$, is plotted in the figure 9.

![Figure 9](image)

**Figure 9.** $D_g = D_g(A, B, E)$ versus the overlapping $p$ for $t^2 = \frac{1}{2}$ and $m = 0$.

Clearly, the geometric quantum discord is monogamous for quasi-Bell cat states with $p$ such that $0 \leq p \leq 0.206783$, but does not follow the monogamy property elsewhere.

For antisymmetric quasi-Bell states ($m = 1$), we have

$$D_g(\rho_{AB}) = D_g(\rho_{AE}) = \frac{1}{4} \frac{p^2 + 2p}{(1+p)^2}$$
and

\[ D_g(\rho_{A|BE}) = \frac{1}{2} \frac{1}{(1 + p)^2}. \]

In this case, the quantity defined by \( D_g(A, B, E) \) is positive for \( 0 \leq p \leq \sqrt{2} - 1 \) and the geometric quantum discord is monogamous. However, the monogamy is violated when \( \sqrt{2} - 1 \leq p \leq 1 \).

6 Concluding remarks

To summarize, we have studied the decoherence properties of quasi-Bell cat states based on Glauber coherent states. The decoherence effects are qualitatively modeled by the action of a beam splitter. This effect is parameterized by a transmission coefficient \( t \) to take into account the loss of the information and subsequently the inevitable degradation of the quantum correlations present in the initial system. We used a qubit mapping to convert the continuous variables (even and odd Glauber coherent states) to a discrete qubit setting. Through concurrence, entanglement of formation, quantum discord and its geometrized version, we characterized the quantum correlations between the two modes of quasi-Bell cat states and the noisy channel. The explicit analytic expressions of these measures were obtained. Finally, we have investigated the distribution of entanglement of formation, quantum discord and geometric discord between quasi-Bell cat states and the environment. We have demonstrated that the quantum correlations measured by squared concurrence satisfy the monogamy relation. However, when the correlations are measured by means of based-entropy measure like entanglement of formation and quantum discord or distance-based measure as the geometric quantum discord, the monogamy is satisfied in some particular cases depending on the strength of the coupling to the environment which is characterized by the parameter \( t \) and the overlapping \( p \) of the Glauber coherent associated with the quasi-Bell cat states under consideration. Especially, for each of above mentioned measures, we determined the critical values of transmission parameter \( t \) and overlap \( p \) under or below which the monogamy relation is satisfied or violated.

The analysis presented here can be extended in many ways. For instance, it is readily generalizable to quasi-Bell cat states based on spin coherent states as well as coherent states associated with other Lie algebras. It will be also an important issue to extend these results to others mechanisms inducing decoherence effects. Further thought in this direction might be worthwhile.

References

[1] S.J. van Enk and O. Hirota, Phys. Rev. A 64, 022313 (2001).
[2] H. Jeong, M.S. Kim and J. Lee, Phys. Rev. A 64, 052308 (2001).
[3] H. Jeong and M.S. Kim, Phys. Rev. A 65, 042305 (2002).
[4] T.C. Ralph, W.J. Munro, and G.J. Milburn, Proceedings of SPIE 4917, 1 (2002); quant-ph/0110115.
[5] T.C. Ralph et al., Phys. Rev. A 68, 042319 (2003).
[6] H. Jeong and M.S. Kim, Quantum Information and Computation 2, 208 (2002); J. Clausen, L. Knöll, and D.-G. Welsch, Phys. Rev. A 66, 062303 (2002).

[7] P.T. Cochrane, G.J. Milburn and W.J. Munro, Phys. Rev. A 59, 2631 (1999); S. Glancy, H. Vasconcelos, and T.C. Ralph, quant-ph/0311093.

[8] B.C Sanders, J. Phys. A: Math. Theor. 45 (2012) 244002.

[9] B.C. Sanders, Phys. Rev. A 45 (1992) 6811.

[10] X. Wang, B.C. Sanders and S.H. Pan, J. Phys. A 33 (2000) 7451.

[11] B.C. Sanders, Phys. Rev. A 46 (1992) 2966.

[12] I. Jex, P. Törmä and S. Stenholm, J. Mod. Opt. 42 (1995) 1377.

[13] S.-B. Zheng, Quant. Semiclass. Opt. B: J. European Opt. Soc. B 10 (1998) 691.

[14] X. Wang and B.C. Sanders, Phys. Rev. A 65 (2001) 012303.

[15] M. Daoud, A. Jellal, E.B. Choubabi and E.H. El Kinani, J. Phys. A: Math. Theor. 44 (2011) 325301.

[16] M. Daoud and E.B. Choubabi, International Journal of Quantum Information 10 (2012) 1250009.

[17] H. Jeong and N.B. An, Phys. Rev. A 74 (2006) 022104.

[18] H.-M. Li, H.-C. Yuan and H.-Y. Fan, Int. J. Theor. Phys. 48 (2009) 2849.

[19] P.P. Munhoz, F.L. Semião and Vidiello, Phys. Lett. A 372 (2008) 3580.

[20] W.-F. Wang, X.-Y. Sun and X.-B. Luo, Chin. Phys. Lett. 25(2008) 839.

[21] E.M. Becerra-Castro, W.B. Cardoso, A.T. Avelar and B. Baseia, J. Phys. B: At. Mol. Opt. Phys. 41 (2008) 085505.

[22] H. Ollivier and W.H. Zurek, Phys. Rev. Lett. 88 (2001) 017901.

[23] L. Henderson and V. Vedral, J. Phys. A 34(2001) 6899; V. Vedral, Phys. Rev. Lett. 90 (2003) 050401; J. Maziero, L. C. Celéria, R.M. Serra and V. Vedral, Phys. Rev A 80 (2009) 044102.

[24] B. Dakic, V. Vedral and C. Brukner, phys. Rev. Lett. 105 (2010) 190502.

[25] S. Luo, Phys. Rev. A 77 (2008) 042303; Phys. Rev. A 77 (2008) 022301.

[26] M. Ali, A.R.P. Rau and G. Alber, Phys. Rev. A 81 (2010) 042105.

[27] M. Shi, W. Yang, F. Jiang and J. Du, J. Phys. A: Mathematical and Theoretical 44 (2011) 415304.

[28] D. Girolami and G. Adesso, Phys. Rev. A 83 (2011) 052108.

[29] M. Shi, F. Jiang, C. Sun and J. Du, New Journal of Physics 13 (2011) 073016.
[30] M. Daoud and R. Ahl Laamara, J. Phys. A: Math. Theor. 45 (2012) 325302.

[31] M. Daoud and R. Ahl Laamara, International Journal of Quantum Information 10 (2012) 1250060.

[32] G. Adesso and A. Datta, Phys. Rev. Lett. 105 (2010) 030501; G. Adesso and D. Girolami, Int. J. Quant. Info. 9 (2011) 1773.

[33] P. Giorda and M.G.A. Paris, Phys. Rev. Lett. 105 (2010) 020503.

[34] X. Yin, Z. Xi, X-M Lu, Z. Sun and X. Wang, J. Phys. B: At. Mol. Opt. Phys. 44 (2011) 245502.

[35] M. Daoud and R. Ahl Laamara, Phys. Lett. A 376 (2012) 2361.

[36] T. Yu and J.H. Eberly, Quantum Inform. Comput. 7 (2007) 459.

[37] R. Wickert, N.K. Bernardes, P. van Loock, Phys. Rev. A 81 (2010) 062344.

[38] V. Coffman, J. Kundu and W.K. Wootters, Phys. Rev. A 61 (2000) 052306.

[39] G.L. Giorgi, Phys. Rev. A 84 (2011) 054301.

[40] R. Prabhu, A. K. Pati, A.S. De and U. Sen, Phys. Rev. A 86 (2012) 052337.

[41] Sudha, A.R. Usha Devi and A.K. Rajagopal, Phys. Rev. A 85 (2012) 012103.

[42] M. Allegra, P. Giorda and A. Montorsi, Phys. Rev. B 84 (2011) 245133.

[43] X.-J. Ren and H. Fan, Quant. Inf. Comp. Vol. 13 (2013) 0469.

[44] A. Streltsov, G. Adesso, M. Piani and D. Bruss, Phys. Rev. Lett. 109 (2012) 050503.

[45] M. Koachi and A. Winter, Phys. Rev. A 69 (2004) 022309.

[46] B. Yurke and D. Stoler, Phys. Rev. Lett. 57 (1986) 13.

[47] T.C. Ralph, A. Gilchrist, G.J. Milburn, W.J. Munro and S. Glancy, Phys. Rev. A 68 (2003) 042319.

[48] S. Song, C.M. Caves and B. Yurke, Phys. Rev. A 41 (1990) 5261.

[49] A.P. Lund, H. Jeong, T.C. Ralph, M.S. Kim, Phys. Rev. A 70 (2004) 020101(R).

[50] H. Jeong and T.C. Ralph in ”Quantum Information with Continuous Variables of Atoms and Light” (Imperial College Press (2007) 159.

[51] W.K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).

[52] S. Hamieh, R. Kobes and H. Zaraket, Phys. Rev. A 70 (2004) 052325.

[53] X-M Lu, Jian Ma, Z. Xi and X. Wang, Phys. Rev. A 83 (2011) 012327.