Abstract

In this paper, we prove the existence of global solutions in $H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$ to the Fokas-Lenells (FL) equation on the line when the initial data includes solitons. A key tool in proving this result is a newly modified Darboux transformation, which adds or subtracts a soliton with given spectral and scattering parameters. In this way the inverse scattering transform technique is then applied to establish the global well-posedness of initial value problem with a finite number of solitons based on our previous results on the global well-posedness of the FL equation.

Keywords: Fokas-Lenells equation, Cauchy problem, weighted Sobolev space, modified Darboux transformation, global well-posedness.

Mathematics Subject Classification: 35P25; 35Q51; 35Q15; 35A01; 35G25.

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1. Introduction

The present paper is concerned with the existence of global solutions to the Cauchy problem for the Fokas-Lenells (FL) equation on the line

\[
\begin{align*}
  u_{tx} + u - 2iu_x - u_{xx} - |u|^2u_x &= 0, \\
  u(x, t)|_{t=0} &= u_0(x).
\end{align*}
\]

The FL equation is an integrable generalization of the nonlinear Schrödinger (NLS) equation [1]. It was also used as a model for the femtosecond pulse propagation through single mode optical silica fiber [2, 3, 4]. In recent years, some interesting mathematical structure and exact solutions to the FL equation have been studied by using various methods [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. For the Schwartz initial data \(u_0(x) \in S(\mathbb{R})\), we obtained the long-time asymptotics for the solution to the Cauchy problem (1)-(2) via Deift-Zhou steepest descent method [17]. For the weighted Sobolev initial data \(u_0(x) \in H^{3,3}(\mathbb{R})\), we presented long-time behaviors of the solution to the Cauchy problem (1)-(2) by using \(\bar{\partial}\)-steepest descent method [18]. It is well-known that the existence of a global solutions or the well-posedness of the initial value problem of a partial differential equation is the theoretical guarantee to study the long-time asymptotics. The global well-posedness of the periodic initial value problem for the FL equation was proved by Fokas and Himonas [19]. A natural question is whether a global solution to the FL equation on the line for appropriate initial data \(u_0(x)\) exists.

Recently, for the weighted Sobolev initial value \(u_0 \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})\) including no eigenvalues or resonances, we have proved that there exists a unique global solution to the Cauchy problem (1)-(2) of the FL equation [20]

\[ u \in C \left([0, \infty); H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})\right). \]
Furthermore, the map

$$H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}) \ni u_0 \mapsto u \in C\left([0, \infty); H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})\right)$$

is Lipschitz continuous. The assumption on the initial data $u_0(x)$ is reasonable due to the fact that if the norm

$$2\|u_{0,x}\|_{L^2}^2 + \|u_{0,x}\|_{L^1}^3 + 2\|u_{0,xx}\|_{L^1} + \|u_{0,x}\|_{L^1} < 1,$$

then the scattering coefficient $a(k)$ has no eigenvalues or resonances. Our main technical tool is the inverse scattering transform method based on the representation of a Riemann-Hilbert (RH) problem associated with the above Cauchy problem. This technique was first applied to prove the global well-posedness of the derivative NLS equation [21]. Recently, we also used it to prove the existence of global solutions to the nonlocal Schrödinger equation on the line [22].

The goal of our present work is to extend our previous results on existence of global solutions [20] to the case that the initial data $u_0 \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$ may allow the appearance of zeros of the scattering coefficient $a(k)$. The main results are enunciated in the following theorem.

**Theorem 1.1.** For every $u_0 \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$ such that the spectral problem (6) admits no resonances, there exists a unique global solution $u(t, \cdot) \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$ of the Cauchy problem (1)-(2) for every $t \in \mathbb{R}$.

This result is achieved by construction of a new invertible Darboux transformation (DT) which removes these zeros of $a(k)$ so that we can apply the
previous results [20]. The proof scheme can be described as the diagram in Figure 1, in which we denote the space

$$Z_N = \{ u \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}), a(k_j) = 0, \ k_j \in \mathbb{C}, j = 1, \cdots, N \} \quad (5)$$

The Darboux and Bäcklund transformations have been successfully applied in the NLS equation and derivative NLS equation to prove existence of global solutions with solitons or asymptotic stability of solitons [23, 24, 25].

Indeed there is some work on the Darboux transformations of FL equation to construct soliton solutions or rogue wave solutions [26, 27, 28, 29]. However, these classical Darboux transformations do not involve the key components of inverse scattering theory such as the analyticity of Jost functions, the distribution of zeros of the scattering data $a(k)$. Moreover, considering our special requirement, our desired Darboux transformations should be invertible and has no singularity with respect to the variable $x \in \mathbb{R}$, which can therefore be used to add or subtract zeros of scattering data $a(k)$, and preserves the original and the new potentials in the same Sobolev space $H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$. For this purpose, we construct a kind of modified two-fold Darboux transformation. A little different from those [26, 27, 28, 29], our Darboux matrix is in the term of the negative power for the spectral parameter $k$. The invertibility of the Darboux transformation helps us establish a bijection between the 0-soliton solution and the 1-soliton solution. Therefore, according to the procedure of Figure 1, we obtain the existence of global solutions to the Cauchy problem (1)-(2) with finite solitons.

The structure of the paper is as follows. In Section 2, we quickly review some main results on the direct scattering transform detailed in [20]. In Section 3, we present a new two-fold Darboux transformation which is related to direct scattering transform and suitable for our subsequent analysis. The invertibility of the Darboux transformation is further shown. In Section 4, we concentrate on the properties of scattering data and Jost functions under the action of the Darboux transformation. We prove that both original and new potentials belongs to the same Sobolev space $H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$. In Section 5, we consider the time evolution of the Darboux transformation and obtain the existence of global solutions in the space $H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$ to FL equation on the line when the initial data $u_0$ includes solitons.
2. Review of the direct scattering transform

In this paper, we define and use the following weighted Sobolev spaces

\[ L^{p,s}(\mathbb{R}) := \left\{ u \in L^p(\mathbb{R}) : \langle x \rangle^s u \in L^p(\mathbb{R}) \right\}, \]

\[ H^{k,s}(\mathbb{R}) = \left\{ u \in L^2(\mathbb{R}) : \partial_x^j u \in L^2(\mathbb{R}), \quad j = 1, \ldots, k \right\}, \]

where \( \langle x \rangle := \sqrt{1 + |x|^2} \).

2.1. Existence of Jost functions

We review some main results on the direct scattering transform associated with the Cauchy problem (1)-(2) and the existence of its Jost solutions. For details, please see [20]. The FL equation (1) admits a Lax pair [5, 6]

\[ \phi_x + ik^2 \sigma_3 \phi = k P_x \phi, \quad \phi_t + i \eta^2 \sigma_3 \phi = H \phi, \]

where

\[ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}, \]

\[ \eta = \left( k - \frac{1}{2k} \right), \quad H = kU_x + \frac{i}{2} \sigma_3 \left( \frac{1}{k} P - P^2 \right). \]

By making a transformation

\[ \psi(x, t; k) = \phi(x, t; k)e^{(k^2 x + \eta^2 t)\sigma_3}, \]

the Lax pair (6)-(7) is changed to

\[ \psi_x + ik^2 [\sigma_3, \psi] = k P_x \psi, \quad \psi_t + i \eta^2 [\sigma_3, \psi] = H \psi. \]

This Lax pair admits the Jost functions with asymptotics

\[ \psi_\pm(x, t; k) \sim I, \quad x \to \pm \infty, \]

which satisfy Volterra integral equations

\[ \psi_\pm(x, t; k) = I + k \int_{\pm \infty}^x e^{-2ik^2(x-y)\bar{\sigma}_3} P_y(y) \psi_\pm(y, t; k) dy. \]
Proposition 2.1. Let \( u_0(x) \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}) \), and denote \( \psi_{\pm}(x, t; k) = (\psi_{\pm,1}(x, t; k), \psi_{\pm,2}(x, t; k)) \) with the scripts 1 and 2 denoting the first and second columns of \( \psi_{\pm}(x, t; k) \). Then we have

- **Analyticity:** The integral equation (13) would admit a unique solution \( \psi_{\pm}(x, t; k) \). Moreover, \( \psi_{-,1}(x, t; k), \psi_{+,2}(x, t; k) \) are analytical in the domain \( D_+ \); and \( \psi_{+,1}(x, t; k) \) and \( \psi_{-,2}(x, t; k) \) are analytical in the domain \( D_- \), where

  \[
  D_+ = \{ k : \text{Im}k^2 > 0 \}, \quad D_- = \{ k : \text{Im}k^2 < 0 \};
  \]

- **Symmetry:** \( \psi_{\pm}(x, t; k) \) satisfy the symmetry relations

  \[
  \psi_{\pm}(x, t; k) = \sigma_2 \psi_{\pm}(x, t; k) \sigma_2, \quad \psi_{\pm}(x, t; k) = \sigma_3 \psi_{\pm}(x, t; -k) \sigma_3,
  \]

- **Asymptotics:** \( \psi_{\pm}(x, t; k) \) have asymptotic properties

  \[
  \psi_{\pm}(x, t; k) = e^{-ic_{\pm}(x)\sigma_3} + O(k^{-1}), \quad k \to \infty,
  \]

  where

  \[
  c_{\pm}(x) = \frac{1}{2} \int_{\pm\infty}^{x} |u_y(y, t)|^2 dy.
  \]

- **Boundedness:** For every \( k \) with \( \text{Im}(k^2) > 0 \) and for all \( u \) satisfying

  \[
  \|u_x\|_{L^1 \cap L^\infty} + \|\partial_x^2 u\|_{L^1} \leq M \]

  there exists a constant \( C_M \) which does not depend on \( u \), such that

  \[
  \|\psi_{\pm}(\cdot; k)\|_{L^\infty} \leq C_M.
  \]

In the following sections, we first consider the partial spectral problem (6) with \( t \) being a parameter, so we omit the variable \( t \) as usual. For example, \( \psi(x, t; k) \) is just written as \( \psi(x; k) \). We will discuss the time evolution of scattering data and reconstruct the potential \( u(x, t) \) with \( t \) in Section 5.1.

Since \( \psi_{\pm}(x; k)e^{-ik^2x\sigma_3} \) are two solutions to the spectral problem (6), they are linearly dependent and satisfy the scattering relation

\[
\psi_-(x; k) = \psi_+(x; k)e^{-ik^2\sigma_3}S(k),
\]

where \( S(k) \) is a scattering matrix given by

\[
S(k) = \begin{pmatrix}
a(k) & b(k) \\
-b(k) & a(k)
\end{pmatrix}, \quad \det S(k) = 1.
\]

6
By Proposition 2.1 and (19), $S(k)$ admits the following symmetries

$$S(k) = \sigma_2 S(\overline{k}) \sigma_2, \quad S(k) = \sigma_3 S(-k) \sigma_3,$$

and asymptotics

$$S(k) = I + \mathcal{O}(k^{-1}), \quad k \to \infty. \quad (22)$$

It is worth to mention that we would use the vector form $\varphi_\pm(x; k)$ and $\phi_\pm(x; k)$ which are the first and second columns respectively of the Jost function $\psi_\pm(x; k)$ in the following calculation and analysis, as analysis on the vector form preserves the same properties as on the matrix form and can make the calculation more concise. We need to convert the matrix results of the relevant Jost functions and scattering coefficients into the vector cases for use.

**Proposition 2.2.** Let $u \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$, for every $k \in \mathbb{R} \cup i \mathbb{R}$, there exist unique solutions $\varphi_\pm(x; k)e^{-ik^2x}$ and $\phi_\pm(x; k)e^{ik^2x}$ to the spectral problem (10) with $\varphi_\pm(\cdot; k) \in L^\infty(\mathbb{R})$ and $\phi_\pm(\cdot; k) \in L^\infty(\mathbb{R})$ such that

**Asymptotics:**

$$\varphi_\pm(x; k) \to e_1, \quad \phi_\pm(x; k) \to e_2, \quad \text{as} \quad x \to \pm \infty; \quad (23)$$

**Scattering Data:** The scattering relation (19), the scattering data $a(k)$ and $b(k)$ can be expressed in term of the determinant

$$a(k) = \det(\varphi_-(x; k)e^{-ik^2x}, \phi_+(x; k)e^{ik^2x}), \quad (24)$$

$$b(k) = \det(\varphi_+(x; k)e^{-ik^2x}, \varphi_-(x; k)e^{ik^2x}). \quad (25)$$

Then, we have

**Proposition 2.3.** The scattering data $a(k)$ and $b(k)$ are even and odd functions respectively and admit the following properties

**Symmetries:**

$$a(-k) = a(k), \quad k \in \mathcal{D}_+, \quad b(-k) = -b(k), \quad \text{Im} k^2 = 0; \quad (26)$$
Asymptotics:

\[ a(k) = e^{-ic} + O\left(k^{-1}\right), \quad b(k) = O\left(k^{-1}\right), \quad k \to \infty, \]
\[ a(k) = e^{-ic} \left(1 + O\left(k^2\right)\right), \quad b(k) = O\left(k^3\right), \quad k \to 0, \tag{27} \]

where

\[ c = c_-(x) - c_+(x) = \frac{1}{2} \int_{-\infty}^{\infty} |u_x|^2 \, dx, \]

and \( c_\pm(x) \) are defined by (17).

2.2. Regularity of Jost functions

**Proposition 2.4.** For every \( u \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}) \) satisfying \( \|u\|_{H^3(\mathbb{R})} \cap H^{2,1}(\mathbb{R}) \leq M \) for some \( M > 0 \), let \( \varphi_\pm(x; k)e^{-ik^2x} \) and \( \phi_\pm(x; k)e^{ik^2x} \) be Jost functions of the spectral problem (10). Fix \( k_1 \in \mathbb{C} \) satisfying \( \text{Im} \left(k_1^2\right) > 0 \) and denote \( \varphi_- := \varphi_- (\cdot; k_1) = (\varphi_{-1}, \varphi_{-2})^T \) and \( \phi_+ := \phi_+ (\cdot; k_1) = (\phi_{+1}, \phi_{+2})^T \). Then,

\[
\| \langle x \rangle \varphi_-, 2 \|_{L^2(\mathbb{R})} + \| \langle x \rangle \partial_x \varphi_- \|_{L^2(\mathbb{R})} + \| \langle x \rangle \partial_x^2 \varphi_- \|_{L^2(\mathbb{R})} \\
+ \| \partial_x^3 \varphi_- \|_{L^2(\mathbb{R})} \leq C_M, \tag{28} \\
\| \langle x \rangle \phi_+, 1 \|_{L^2(\mathbb{R})} + \| \langle x \rangle \partial_x \phi_+ \|_{L^2(\mathbb{R})} + \| \langle x \rangle \partial_x^2 \phi_+ \|_{L^2(\mathbb{R})} \\
+ \| \partial_x^3 \phi_+ \|_{L^2(\mathbb{R})} \leq C_M, \tag{29} 
\]

where the constant \( C_M \) does not depend on \( u \).

**Proof.** It is suffice to prove the statement for \( \varphi_- \), as the procedure for \( \phi_+ \) is analogous. From (18), \( \varphi_- \in L^\infty(\mathbb{R}) \) is obvious. First, our aim is to prove \( \varphi_- \) belongs to \( L^2(\mathbb{R}) \). Under the result that the existence of Jost functions is uniformly in \( k \) (for details, please see in [20, Proposition 3.1]), it is convenient to prove the case for \( k = k_1 \). Using the integral equation for \( \varphi_- \):

\[ \varphi_- = e_1 + K\varphi_-, \tag{30} \]

where the operator \( K \) is given by

\[ K\varphi_- = k_1 \int_{-\infty}^{x} \begin{pmatrix} 1 & 0 \\ e^{2ik_1(x-y)} & 0 \end{pmatrix} P_y(y)\varphi_-(y; k) \, dy, \tag{31} \]
which means

\[ \varphi_{-2}(x; k) = k_1 \int_{-\infty}^{x} -\bar{u}_x e^{2ik_1^2(x-y)} \varphi_{-1}(y; k) dy, \quad (32) \]

we deduce

\[
\|\varphi_{-2}(x; k)\|_{L^2(-\infty, x_0)} \leq \frac{|k_1|}{4 \text{ Im}(k_1^2)} \|\varphi_{-1}\|_{L^\infty(-\infty, x_0)}. \quad (33)
\]

As \( u_x \in L^2(\mathbb{R}) \), we can divide \( \mathbb{R} \) into finite sub-intervals such that \( K \) is a contraction as shown above within each sub-interval. By gluing solutions together, we have \( \varphi_{-2} \in L^2(\mathbb{R}) \) and

\[
\|\varphi_{-2}\|_{L^2(\mathbb{R})} \leq C_M \|u_x\|_{L^2(\mathbb{R})},
\]

where \( C_M \) does not depend on \( u_x \). Next, it follows directly from the (10) that

\[
\partial_x \varphi_{-1} = k_1 u_x \varphi_{-2} \implies \partial_x \varphi_{-1} \in L^2(\mathbb{R}), \quad (34)
\]

and

\[
\partial_x \varphi_{-2} = -k_1 \bar{u}_x \varphi_{-1} + 2ik_1^2 \varphi_{-2} \implies \partial_x \varphi_{-2} \in L^2(\mathbb{R}). \quad (35)
\]

Differentiating (10) once and twice, we also obtain \( \partial_x^2 \varphi_{-}, \partial_x^3 \varphi_{-} \in L^2(\mathbb{R}) \).

To show \( x\varphi_{-2} \in L^2(\mathbb{R}) \), we write

\[
\partial_x (x \varphi_{-2}) = \varphi_{-2} + x \partial_x \varphi_{-2}. \quad (36)
\]

Also, with the second component of (10), we obtain

\[
\partial_x (x \varphi_{-2}) = 2ik_1^2 x \varphi_{-2} + \varphi_{-2} - kx \bar{u}_x \varphi_{-2}, \quad (37)
\]

the integral expression of which is

\[
x \varphi_{-2}(x) = \int_{-\infty}^{x} e^{2ik_1^2(x-y)} \varphi_{-2}(y) dy - k_1 \int_{-\infty}^{x} e^{2ik_1^2(x-y)} y \bar{u}_y(y) \varphi_{-2}(y) dy.
\]

As each component is bounded in \( L^2(\mathbb{R}) \) in the right-hand side, we have \( x\varphi_{-2} \in L^2(\mathbb{R}) \). Then, it follows from the system (10) and its derivative that \( x \partial_x \varphi_{-}, x \partial_x^2 \varphi_{-} \in L^2(\mathbb{R}) \). Combining the above results, we have the bounds (28) for \( \varphi_{-} \) .
3. A modified Darboux transformation

3.1. Construction of Darboux transformation

We need to construct a kind of two-fold Darboux transformation to obtain a 0-soliton solution $u^{(1)}$ from a 1-soliton solution $u$ to remove the zeros of $a(k)$.

According to the characteristic of the Lax pair (10)-(11) for the FL equation, we consider a general two-fold Darboux transformation expressed by

$$
\psi^{(1)}(x; k) = T(k)\psi(x; k),
$$

where

$$
T(k) = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} k^{-2} + \begin{pmatrix} 0 & g_1 \\ g_2 & 0 \end{pmatrix} k^{-1} + \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix},
$$

and $f_1, f_2, g_1, g_2, h_1, h_2$ are unknown functions to be determined.

Let $\eta = (\eta_1, \eta_2)^T$ and $\xi = (\xi_1, \xi_2)^T$ be two vector solutions of the Lax pair (10)-(11) associated with the spectral parameters $k_1, k_2 \in \mathbb{C}\{0\}$ respectively. The two-fold Darboux transformation is characterized by a two-dimensional kernel

$$
T(k_1)\eta = 0, \ T(k_2)\xi = 0,
$$

which lead to two linear systems

$$
\begin{pmatrix} k_1^{-2} & k_1^{-1} \\ k_2^{-2} & k_2^{-1} \end{pmatrix} \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} = \begin{pmatrix} -h_1 \eta_2 \\ -h_2 \xi_2 \end{pmatrix},
$$

$$
\begin{pmatrix} k_1^{-2} & k_2^{-1} \\ k_2^{-2} & k_2^{-1} \end{pmatrix} \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} = \begin{pmatrix} -h_1 \eta_1 \\ -h_2 \xi_1 \end{pmatrix}.
$$

Specially, taking $h_1 = h_2 = -1$, we obtain

$$
f_1 = \frac{k_1^2 k_2 \eta_2 \xi_1 - k_1 k_2^2 \eta_1 \xi_2}{k_2 \eta_2 \xi_1 - k_1 \eta_1 \xi_2}, \quad f_2 = \frac{k_1^2 k_2 \eta_1 \xi_2 - k_1 k_2^2 \eta_2 \xi_1}{k_2 \eta_1 \xi_2 - k_1 \eta_2 \xi_1},
$$

and

$$
g_1 = \frac{k_2^3 \eta_2 \xi_2 - k_2^2 \eta_2 \xi_2}{k_2 \eta_2 \xi_1 - k_1 \eta_1 \xi_2}, \quad g_2 = \frac{k_2^2 \eta_1 \xi_2 - k_1^2 \eta_1 \xi_1}{k_2 \eta_1 \xi_2 - k_1 \eta_2 \xi_1}.
$$
Substituting (38) into the Lax pair (10)-(11) yields
\[
\begin{align*}
    u^{(1)}(x, t) &= \frac{f_1}{f_2} u(x, t) + \frac{g_1}{f_2} := D(\psi; k) u(x, t), \\
    u_x^{(1)}(x, t) &= u_x(x, t).
\end{align*}
\] (42)

Therefore, (38) together with (42)-(43) give the desired two-fold Darboux transformation, in which the relation between old and new components is given by
\[
\{u, \psi\} \rightarrow \{u^{(1)}, \psi^{(1)}\}. \quad (44)
\]

To make above Darboux transformation (44) be able to use for us, we need to make some modification to it.

3.2. Modification to the Darboux transformation

(A) odd or subtract zeros:

For a fixed parameter \(k \in \mathbb{C}\), and \(\eta = (\eta_1, \eta_2)^T, \xi = (\xi_1, \xi_2)^T \in \mathbb{C}^2\), we define a bilinear form
\[
m_k(\eta, \xi) := k\eta_1\xi_1 + k\eta_2\xi_2.
\]

According to the symmetry (21) of scattering data, if \(\pm k_1\) is a zero of \(a(k)\) in the first and third quadrants, then \(\pm \bar{k}_1\) are also zeros of \(\overline{a(k)}\) in the second and forth quadrants. To remove these zeros, by specially taking \(k_2 = \bar{k}_1\), \(\xi = \bar{\eta}\) in (40)-(41), we rewrite them as
\[
\begin{align*}
    f_1 &= |k_1|^2 A_{k_1}(\eta), \quad f_2 = |k_1|^2 \overline{A_{k_1}(\eta)}, \\
    g_1 &= C_{k_1}(\eta), \quad g_1 = \overline{C_{k_1}(\eta)},
\end{align*}
\] (45)

where
\[
A_{k_1}(\eta) := \frac{m_{k_1}(\eta, \eta)}{m_{k_1}(\eta, \eta)}, \quad C_{k_1}(\eta) := (k_1^2 - \bar{k}_1^2) \frac{\eta_1\bar{\eta}_2}{m_{k_1}(\eta, \eta)}.
\] (47)

The Darboux transformation (38) and (42) become
\[
\begin{align*}
    \psi^{(1)}(x; k) &= T(\eta, k; k_1)\psi(x; k), \\
    u^{(1)} &= A_{k_1}(\eta)(-A_{k_1}(\eta)u + C_{k_1}(\eta)/|k_1|^2),
\end{align*}
\] (48) (49)
where the Darboux matrix is given by

\[
T(\eta, k, k_1) := \begin{pmatrix}
|k_1|^2 A_{k_1}(\eta) k^{-2} - 1 & C_{k_1}(\eta) k^{-1} \\
C_{k_1}(\eta) k^{-1} & |k_1|^2 A_{k_1}(\eta) k^{-2} - 1
\end{pmatrix}.
\]  

(50)

(B) retain the same asymptotics:

We need to make some modification to the Darboux matrix (50) such that the boundary conditions

\[
\varphi^{(1)}(x; k) \to e_1, \quad \phi^{(1)}(x; k) \to e_2, \quad \text{as } x \to \infty
\]

are satisfied.

We choose \( \eta = \varphi_-(x; k_1) e^{-ik_1 x} \), then

\[
C_{k_1}(\eta) \to 0, \quad A_{k_1}(\eta) = A_{k_1}(\varphi_-) \to \bar{k}_1 / k_1, \quad \text{as } x \to -\infty,
\]

and we thus have

\[
T(\eta, k, k_1) \to T_{-\infty}(e_1, k, k_1) = \text{diag}\left(-\frac{k^2 - k_1^2}{k^2}, -\frac{k^2 - \bar{k}_1^2}{\bar{k}_1^2}\right), \quad x \to -\infty.
\]

The definition of the new Jost functions \( \varphi^{(1)}_\pm \) and \( \phi^{(1)}_\pm \) are as follows

\[
(\varphi^{(1)}_-, \phi^{(1)}_-) = T(\varphi_-, \phi_-) T^{(1),\prime}_{-\infty}^{-1}
= (T^{(1)} \varphi_-, T \phi_-) \text{diag}\left(-\frac{k^2}{k^2 - k_1^2}, -\frac{k^2}{k^2 - \bar{k}_1^2}\right).
\]

Based on the asymptotics (23), it is clear that

\[
\varphi^{(1)}_- = -\frac{k^2}{k^2 - k_1^2} T \varphi_- \sim e_1, \quad x \to -\infty,
\]

\[
\phi^{(1)}_- = -\frac{k^2}{k^2 - \bar{k}_1^2} T \phi_- \sim e_2, \quad x \to -\infty.
\]

On the other hand, if we choose \( \eta = \varphi_+(x, k) e^{ik_1 x} \), we notice that

\[
C_{k_1}(\eta) \to 0, \quad A_{k_1}(\eta) = A_{k_1}(\varphi_-) \to k_1 / \bar{k}_1, \quad x \to +\infty,
\]

thus we have

\[
T(k) \to T_{+\infty}(k) = \text{diag}\left(-\frac{k^2 - k_1^2}{k^2}, -\frac{k^2 - \bar{k}_1^2}{k^2}\right), \quad x \to +\infty.
\]
Then
\[
(\varphi_+^{(1)}, \phi_+^{(1)}) = T (\varphi_+, \phi_+) T_+^{(1),-1}
\]
\[
= (T \varphi_+, T \phi_+) \text{diag} \left( -\frac{k^2}{k^2 - k_1^2}, -\frac{k^2}{k^2 - k_1^2} \right).
\]

Under the asymptotics (23), we also have
\[
\varphi_+^{(1)} = -\frac{k^2}{k^2 - k_1^2} T \varphi_+ \sim e_1, \quad x \to +\infty, \quad (56)
\]
\[
\phi_+^{(1)} = -\frac{k^2}{k^2 - k_1^2} T \phi_+ \sim e_2, \quad x \to +\infty. \quad (57)
\]

Based on the above results, we finally modify the Darboux matrix (50) as follows
\[
T(\eta, k, k_1) = -\frac{1}{k^2 - k_1^2} \begin{pmatrix}
|k_1|^2 A_{k_1}(\eta) - k^2 & C_{k_1}(\eta)k \\
C_{k_1}(\eta)k & |k_1|^2 A_{k_1}(\eta) - k^2
\end{pmatrix}. \quad (58)
\]

As $A_{k_1}(\eta)$ and $C_{k_1}(\eta)$ are bounded in $x$ for all considered choices of $\eta$, the functions $\varphi_\pm^{(1)}(x; k)$ and $\phi_\pm^{(1)}(x; k)$ are bounded functions of $x$ for every $k \in \mathbb{C}\setminus\{\pm k_1, \pm k_1\}$. Also, the asymptotics of $\varphi_\pm^{(1)}(x; k)$ and $\phi_\pm^{(1)}(x; k)$ are
\[
\lim_{x \to \pm \infty} \varphi_\pm^{(1)}(x; k) = T(e_1, k, k_1)e_1 = e_1, \quad (59)
\]
\[
\lim_{x \to \pm \infty} \phi_\pm^{(1)}(x; k) = T(e_2, k, k_1)e_2 = e_2. \quad (60)
\]

The invariance of $A_k$ and $C_k$ under a multiplication by a nonzero complex number is shown in the following proposition.

**Proposition 3.1.** As for $x \to \pm \infty$, we note that $A_k(\eta) = A_k(\varphi_-)$ and $C_k(\eta) = C_k(\varphi_-)$ by
\[
A_k(e_1) = \bar{k}/k, \quad A_k(e_2) = k/\bar{k}, \quad (61)
\]
\[
A_k(\eta) = A_k(\eta), \quad A_k(\alpha \eta) = A_k(\eta), \quad (62)
\]
\[
C_k(e_1) = C_k(e_2) = 0, \quad C_k(\alpha \eta) = C_k(\eta), \quad (63)
\]
\[
A_k(\sigma_3 \eta) = A_k(\eta), \quad A_k(\sigma_1 \eta) = A_k(\eta), \quad (64)
\]
\[
C_k(\sigma_3 \eta) = -C_k(\eta), \quad C_k(\sigma_1 \eta) = C_k(\eta). \quad (65)
\]

We will show the mapping $D(\eta, k_1)$ is independent of the choice of the fundamental Jost function $\varphi_-(x; k_1)$, $\phi_+(x; k_1)$, $\varphi_+(x; \bar{k}_1)$ and $\phi_-(x; \bar{k}_1).$
Proposition 3.2. Assume that $k_1 \in \mathbb{C}$ satisfies $a(k_1) = 0$. Given a potential $u \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$, it can be seen that

$$u^{(1)}(x) = D(\varphi_-(x;k_1)e^{-ik_1^2x},k_1)u(x) = D(\phi_+(x;k_1)e^{ik_1^2x},k_1)u(x) = D(\varphi_+(x;\bar{k}_1)e^{-ik_1^2x},\bar{k}_1)u(x) = D(\phi_-(x;\bar{k}_1)e^{ik_1^2x},\bar{k}_1)u(x), \quad (66)$$

where the four Jost functions are the solutions to the spectral problem (10).

Proof. We mainly focus on prove the identity of the four equations in (66). The first one is obtained by setting $\eta$ as $\varphi_-(x;k_1)e^{-ik_1^2x}$. Next, combining (62), (63) and $a(k_1) = 0$, we apply the linear relation between $\varphi_-(x;k_1)e^{-ik_1^2x}$ and $\phi_+(x;k_1)e^{ik_1^2x}$ to obtain the second equation. Then, we use the symmetry relation (15) and properties (64)-(65) to obtain

$$A_{k_1}(\varphi_-(x;k_1)) = A_{\bar{k}_1}(\sigma_1\sigma_3\varphi_-(x;k_1)) = A_{\bar{k}_1}(\phi_-(x;\bar{k}_1)) = A_{k_1}(\phi_-(x;\bar{k}_1)), \quad (69)$$

and

$$C_{k_1}(\varphi_-(x;k_1)) = -C_{k_1}(\sigma_1\sigma_3\varphi_-(x;k_1)) = -C_{k_1}(\phi_-(x;\bar{k}_1)) = C_{k_1}(\phi_-(x;\bar{k}_1)). \quad (70)$$

By invoking the transformation formula (49), we prove the third equation. Finally, we derive the fourth equation by

$$\varphi_+(x;\bar{k}_1)e^{-ik_1^2x} = \bar{\varphi}_-(x;\bar{k}_1)e^{ik_1^2x} \quad (67)$$

as $a(k_1) = 0$. \hfill $\square$

After imposing the transformation (58) on the Jost functions $\varphi_-(x;k)$, $\phi_-(x;k)$, $\phi_+(x;k)$, and $\phi_+(x;k)$, for $k \in \mathbb{C} \setminus \{\pm k_1, \pm \bar{k}_1\}$, the new Jost functions becomes

$$\varphi^{(1)}_-(x;k) = T(\varphi_-(x;k_1)e^{-ik_1^2x},k_1)\varphi_-(x;k), \quad (68)$$

$$\varphi^{(1)}_+(x;k) = T(\phi_+(x;\bar{k}_1)e^{-ik_1^2x},\bar{k}_1)\phi_+(x;k), \quad (69)$$

$$\phi^{(1)}_+(x;k) = T(\phi_+(x;k_1)e^{ik_1^2x},k_1)\phi_+(x;k), \quad (70)$$

$$\phi^{(1)}_-(x;k) = T(\phi_-(x;\bar{k}_1)e^{ik_1^2x},\bar{k}_1)\phi_-(x;k). \quad (71)$$

Therefore, $\{\varphi^{(1)}_\pm(x;k)e^{-ik_1^2x}, \phi^{(1)}_\pm(x;k)e^{ik_1^2x}\}$ are Jost functions of the spectral problem (10) associated with the potential $u^{(1)} = D(\eta;k_1)u$. 

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3.3. Invertibility of the Darboux transformation

Next, we construct the left inverse of the transformation \( D(\eta, k_1) \), which proves the invertibility of the Darboux transformation and finally builds the one-to-one correspondence between the 1-soliton solution and the 0-soliton solution.

**Proposition 3.3.** The left inverse for the mapping \( D(\eta; k_1) \) exists.

**Proof.** If we assume the left inverse of \( D(\eta; k_1) \) is \( \widetilde{D}(\eta; k_1) \), then we can construct the following equation

\[
\widetilde{u} = D(\widetilde{\eta}; k_1)D(\eta; k_1)u = A_{k_1}(\widetilde{\eta})^2A_{k_1}(\eta)^2u + A_{k_1}(\widetilde{\eta})[-A_{k_1}(\widetilde{\eta})A_{k_1}(\eta)C_{k_1}(\eta) + C_{k_1}(\widetilde{\eta})],
\]

where \( \widetilde{\eta} \) satisfies

\[
A_{k_1}(\widetilde{\eta})^2A_{k_1}(\eta)^2 = 1,
\]

and

\[-A_{k_1}(\widetilde{\eta})A_{k_1}(\eta)C_{k_1}(\eta) + C_{k_1}(\widetilde{\eta}) = 0.\]

With further analysis, (73) and (74) are decomposed into one of the following forms:

\[
\overline{A_{k_1}(\widetilde{\eta})} = A_{k_1}(\eta), \quad C_{k_1}(\widetilde{\eta}) = C_{k_1}(\eta),
\]

or

\[
\overline{A_{k_1}(\widetilde{\eta})} = -A_{k_1}(\eta), \quad C_{k_1}(\widetilde{\eta}) = -C_{k_1}(\eta).
\]

Next, we proceed to decide which of the two forms would yield the existence of the left inverse. Based on (51), we realize that

\[
\lim_{x \to -\infty} |\widetilde{\eta}_1|/|\widetilde{\eta}_2| \neq 0,
\]

as it contradicts with the first equation in (74) with \( k_1 \neq 0 \). According to the second term in (74), we derive \( C_{k_1}(\widetilde{\eta}) \to 0 \) as \( x \to -\infty \) because \( C_{k_1}(\eta) \to 0 \) as \( x \to -\infty \). Also, from (77), we have

\[
\lim_{x \to -\infty} C_{k_1}(\widetilde{\eta}) = 0 \Rightarrow \lim_{x \to -\infty} |\widetilde{\eta}_2|/|\widetilde{\eta}_1| = 0,
\]

which implies

\[
\lim_{x \to -\infty} \overline{A_{k_1}(\eta)} = k_1/\bar{k}_1.
\]
Meanwhile, from the first equation in (74), as \( k_1 \in \mathbb{C} \), we obtain \( \text{Re}(k_1^2) = 0 \) with \( \text{arg}(k_1) = \pi/4 \). Thus, we have \( k_1 = |k_1|e^{i\pi/4} \) and use the first equation in (74) to obtain

\[
|\tilde{\eta}_1|^2|\eta_2|^2 + |\tilde{\eta}_2|^2|\eta_1|^2 = 0.
\]

This indicates \( \tilde{\eta} = 0 \) which contradicts with \( \tilde{\eta} \neq 0 \). Consequently, we conclude that the choice (75) is reasonable.

Therefore, we use the choice (76) to define \( \tilde{\eta} \) and satisfy the system (73)-(74). As \( k_1 \in \mathbb{C} \), the condition \( A_{k_1}(\tilde{\eta}) = A_{k_1}(\eta) \) is equivalently written as

\[
|\eta_1|^2|\tilde{\eta}_1|^2 = |\eta_2|^2|\tilde{\eta}_2|^2.
\]

(79)

Hence, there exists a positive number \( n \) such that

\[
|\tilde{\eta}_1| = n|\eta_2|, \quad |\tilde{\eta}_2| = n|\eta_1|.
\]

(80)

In addition, the condition \( C_{k_1}(\tilde{\eta}) = C_{k_1}(\eta) \) yields

\[
\frac{\eta_1\tilde{\eta}_2}{\tilde{\eta}_1\eta_2} = \frac{k_1|\eta_1|^2 + \bar{k}_1|\eta_2|^2}{k_1|\tilde{\eta}_1|^2 + \bar{k}_1|\tilde{\eta}_2|^2}.
\]

(81)

By substituting (80) into (81), it is reformulated as

\[
n^2 \frac{\eta_1\tilde{\eta}_2}{\tilde{\eta}_1\eta_2} = \frac{k_1|\eta_1|^2 + \bar{k}_1|\eta_2|^2}{k_1|\tilde{\eta}_1|^2 + k_1|\tilde{\eta}_2|^2},
\]

(82)

where the right-hand side is of modulus one. Combining (80) and (82), the most general solution of (75) is as follows

\[
\tilde{\eta}_1 = n_1\tilde{\eta}_2, \quad \tilde{\eta}_2 = n_2\tilde{\eta}_1,
\]

(83)

where \( n_1, n_2 \in \mathbb{C} \) meet the condition \( |n_1| = |n_2| \). Therefore, \( D(\tilde{\eta}, k_1) \) with \( \tilde{\eta} \) given by (83) is the left inverse of the transformation \( D(\eta, k_1) \).

Then, we will prove a unique choice for the function \( \tilde{\eta} \) can be given by \( \tilde{\eta} \). As \( \eta = (\eta_1, \eta_2)^T \) is the Jost function of the FL spectral problem (10) for \( k = k_1 \), \( \tilde{\eta} = (\tilde{\eta}_2, \tilde{\eta}_1)^T \) is accordingly the Jost function of the FL spectral problem (10) for \( k = -\bar{k}_1 \). Thus, we have the following equation of \( \tilde{\eta} \)

\[
\partial_x \tilde{\eta} = -(ik_1^2\sigma_3 + \bar{k}_1P_x)\tilde{\eta}.
\]

(84)
Rewriting the equation as elements of the $\eta$, we have
\[
\begin{align*}
\partial_x \bar{\eta}_1 &= i\bar{k}_1^2 \bar{\eta}_1 + \bar{k}_1 \bar{u}_x \bar{\eta}_2, \\
\partial_x \bar{\eta}_2 &= -i\bar{k}_1^2 \bar{\eta}_2 - \bar{k}_1 \bar{u}_x \bar{\eta}_1.
\end{align*}
\]

With (83), we deduce the equation of $\tilde{\eta}$ as follows
\[
\partial_x \tilde{\eta}_1 = \partial_x n_1 \bar{\eta}_2 + n_1 \partial_x \bar{\eta}_2 = (\partial_x n_1 - i\bar{k}_1^2 n_1) \bar{\eta}_2 - \bar{k}_1 u_x n_1 \bar{\eta}_1. \tag{85}
\]

On the other hand, if $\tilde{\eta}$ is the solution of the FL spectral problem (10) associated with the potential $u_x^{(1)} = u_x$ and $k = k_1$, we have
\[
\partial_x \tilde{\eta}_1 = -ik_1^2 n_1 \bar{\eta}_2 + k_1 u_x^{(1)} n_2 \bar{\eta}_1. \tag{86}
\]

Comparing (85) with (86), we finally reach the specific values of $n_{1,2}$
\[
n_1 = e^{i(k_1^2 - k_1^2)x}, \quad n_2 = \frac{k_1}{k_1} e^{i(k_1^2 - k_1^2)x},
\]
which give
\[
\bar{\eta}_1 = e^{i(k_1^2 - k_1^2)x} \bar{\eta}_2, \quad \bar{\eta}_2 = \frac{k_1}{k_1} e^{i(k_1^2 - k_1^2)x} \bar{\eta}_1. \tag{87}
\]

Combining the above conditions, we finally obtain $\tilde{\eta}$, which satisfies the FL spectral problem (10) with the potential $u_x^{(1)}$ and $k = k_1$.

4. Action of the Darboux transformation

4.1. Zeros of the new scattering data

Let $Z_N$ be a space defined by (5). If there exists $k_1 \in C_I$ satisfies $a(k_1) = 0$, then $u$ belongs to $Z_1 \subset H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$. In the following, we prove, under the Darboux transformation, $u^{(1)}$ belongs to $Z_0 \subset H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$ with $a^{(1)}(k_1) \neq 0$.

**Proposition 4.1.** Let $u \in Z_1$ and $k_1 \in C_I$ such that $a(k_1) = 0$. If $\eta(x) = \varphi_-(x; k_1)e^{-ik_1^2x}$, where $\varphi_-$ is the Jost function of the spectral problem (10), then $u^{(1)} = D(\eta, k_1)u$ belongs to $Z_0$. 

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Proposition 4.2. Smoothness of new Jost functions

Proof. We show that if \( a(k) \) has a simple zero \( k = k_1 \), then after the Darboux transformation,

\[
a^{(1)}(k) := \det \left( \varphi_{-}^{(1)}(\cdot; k), \varphi_{+}^{(1)}(\cdot; k) \right)
\]

has no zero in \( \mathbb{C}_I \), where \( \varphi_{-}^{(1)} \) and \( \varphi_{+}^{(1)} \) are given by (68) and (70). Explicit calculation gives

\[
a^{(1)}(k) = \det \left( \varphi_{-}^{(1)}(x; k), \varphi_{+}^{(1)}(x; k) \right)
= \det \left( \left( T \left[ \varphi_{-} (x; k_1) e^{-ik_1^2 x}, k_1 \right] \varphi_{-} - T \left[ \varphi_{+} (x; k_1) e^{ik_1^2 x}, k_1 \right] \varphi_{+} \right) - a(k) \right)
= \det \left( \left( T \left[ \varphi_{-} (x; k_1), k, k_1 \right] \varphi_{-} (x; k) - T \left[ \varphi_{-} (x; k_1), k, k_1 \right] \varphi_{+} (x; k) \right) - a(k) \right)
= -a(k) \det \left( \left( T \left[ \varphi_{-} (x; k_1), k, k_1 \right] \varphi_{-} (x; k) \right) - a(k) \right)
= \frac{k^2 - k_1^2}{k^2} a(k).
\]

As \( k_1 \) is the only simple zero of \( a(k) \) in \( \mathbb{C}_I \), \( a^{(1)}(k) \) has no zeros for \( k \) in \( \mathbb{C}_I \).

We also calculate the transformation from \( b(k) \) to \( b^{(1)}(k) \) as follows:

\[
b^{(1)}(k) = \det \left( e^{-2ik^2 x} \varphi_{+}^{(1)}(x; k), \varphi_{-}^{(1)}(x; k) \right)
= \det \left( \left( T \left[ \varphi_{-} (x; k_1) e^{-ik_1^2 x}, k_1 \right] \varphi_{-} - T \left[ \varphi_{+} (x; k_1) e^{-ik_1^2 x}, k_1 \right] \varphi_{+} \right) \right)
= b(k) \det \left( \left( T \left[ \varphi_{-} (x; k_1), k, k_1 \right] \varphi_{-} (x; k) \right) \right)
= -b(k).
\]

\( b(k) \) adds no new zeros and singularities. \( \square \)

4.2. Smoothness of new Jost functions

Proposition 4.2. Let \( \varphi_{\pm}^{(1)}(x; k) \) and \( \varphi_{\pm}^{(1)}(x; k) \) as defined by (68)-(71). Then, \( k = \pm k_1 \) and \( k = \pm \bar{k}_1 \) are removable singularities in the corresponding domains of analyticity of \( \varphi_{\pm}^{(1)}(x; k) \) and \( \varphi_{\pm}^{(1)}(x; k) \).

Proof. It is suffice to consider the first Jost function \( \varphi_{-}^{(1)}(x; k) \) represent by (68) and the other cases follow similarly. We denote \( \varphi_{-} = (\varphi_{-,1}, \varphi_{-,2})^T \) and \( \varphi_{-}^{(1)} = (\varphi_{-,1}^{(1)}, \varphi_{-,2}^{(1)})^T \) for the 2-vectors, and obtain for \( k \in \mathbb{C}_I \cup \mathbb{C}_{III} \setminus \{ \pm k_1 \} \)

\[
\varphi_{-}^{(1)}(x; k) = -\frac{k^2}{k^2 - k_1^2} \left\{ \left( \frac{|k|^2 m_{k_1}(\varphi_{-,1}^{(1)}, \varphi_{-}^{(1)})}{k^2 m_{k_1}(\varphi_{-,1}, \varphi_{-})} - 1 \right) \varphi_{-,1} + \frac{(k^2 - k_1^2) \varphi_{-,1}^{(1)} - 2|k|^2 \varphi_{-,1}^{(1)} | \varphi_{-,2}^{(1)} |^2}{k m_{k_1}(\varphi_{-,1}, \varphi_{-})} \right\}
= \frac{(k^2 - k_1^2) | \varphi_{-,1}(x; k) |^2 \varphi_{-,1}(x; k) + G(x; k)}{(k^2 - k_1^2) m_{k_1}(\varphi_{-,1}, \varphi_{-})},
\]

(88)
where
\[ G(x; k) := (k^2 - \bar{k}^2)k_1 |\varphi_{-2}(x; k)|^2 \varphi_{-1}(k) - k(k^2 - \bar{k}^2)|\varphi_{-2}(x; k)|^2 \varphi_{-1}(x; k). \] (89)

It is clear \( G(x; k_1) = G(x; -k_1) = 0 \), as \( \varphi_{-1}(x; k) \) is even in \( k \) and \( \varphi_{-2}(x; k) \) is odd in [20]. \( G \) is analytic in \( \mathbb{C}_I \cup \mathbb{C}_{III} \) on the basis of Proposition 3.1 [20]. Hence, we have \( G(x; k) = (k^2 - \bar{k}^2) \tilde{G}(x; k) \), where \( \tilde{G} \) is analytic in \( \mathbb{C}_I \cup \mathbb{C}_{III} \).

Furthermore, through the calculation
\[ \varphi_{-1}^{(1)}(x; k) = \frac{\bar{\kappa}_1 |\varphi_{-1}(x; k)|^2 \varphi_{-1}(x; k) + \tilde{G}(x; k)}{m_k(\varphi_-, \varphi_-)}, \] (90)

we deduce \( \pm k_1 \) are removable singularities of \( \varphi_{-1}^{(1)}(x; k) \). It is easy to know \( \pm k_1 \) are also removable singularities of \( \varphi_{-2}^{(1)}(x; k) \). \( \square \)

Next, we characterize the relation between \( \tilde{\eta} \) and the new Jost functions in the following proposition.

**Proposition 4.3.** Fix \( k_1 \in \mathbb{C}_I \) such that \( a(k_1) = 0 \) and \( a'(k_1) \neq 0 \). By (87), \( \tilde{\eta} \) can be expressed as
\[ \tilde{\eta}(x) = \frac{\gamma \bar{\kappa}_1}{k_1 a^{(1)}(k_1)} e^{-ik^2 x} \varphi_{-1}^{(1)}(x; k_1) + \frac{\bar{\kappa}_1}{k_1 a^{(1)}(k_1)} e^{ik^2 x} \phi_+^{(1)}(x; k_1), \] (91)

where the new Jost functions \( \varphi_{-1}^{(1)} \) and \( \phi_+^{(1)} \) are constructed in (68) and (70), \( \gamma \neq 0 \) is the norming constant under \( a(k_1) = 0 \), and \( a^{(1)}(k_1) \neq 0 \) as in Lemma 4.1.

**Proof.** We denote \( \tilde{\eta} = (\tilde{\eta}_1, \tilde{\eta}_2)^T \) and \( \varphi_- = (\varphi_{-1}, \varphi_{-2})^T \). Components of \( \tilde{\eta} \) given by (87) are explicitly expressed by
\[ \tilde{\eta}_1(x) = e^{ik^2 x} \varphi_{-2}(x; k_1), \] (92)
\[ \tilde{\eta}_2(x) = e^{ik^2 x} \varphi_{-1}(x; k_1). \] (93)

As
\[ \lim_{x \to -\infty} \varphi_{-1}(x; k_1) = e_1, \]

we have
\[ \lim_{x \to -\infty} e^{-ik^2 x} \tilde{\eta}(x) = \frac{\bar{\kappa}_1}{k_1} e_2. \] (94)
When \( a(k_1) = 0 \), the relation
\[
\varphi_-(x; k_1)e^{-ik_1^2x} = \gamma \phi_+(x; k_1)e^{ik_1^2x} \quad x \in \mathbb{R}
\] (95)
follows. Substituting it into (92) and (93), \( \tilde{\eta} \) can be rewritten as:
\[
\tilde{\eta}_1(x) = \gamma e^{i(k_1^2 - 2k_1^2)x} \phi_{+,2}(x; k_1),
\]
\[
\tilde{\eta}_2(x) = \frac{\gamma k_1 e^{i(k_1^2 - 2k_1^2)x} \phi_{+,1}(x; k_1)}{k_1}.
\] (96) (97)
Since
\[
\lim_{x \to +\infty} \phi_+(x; k_1) = e_2,
\]
then
\[
\lim_{x \to +\infty} e^{ik_1^2x} \tilde{\eta}(x) = \frac{\gamma k_1}{k_1} e_1,
\] (98)
follows. We know that \( \tilde{\eta} \) is a solution of the FL spectral problem (10) associated with the new potential \( u^{(1)} \) for \( k = k_1 \) and \( \varphi^{(1)}(x; k) \) and \( \phi_+^{(1)}(x; k) \) are analytic at \( k_1 \). Any solution of the second-order system can be expressed by
\[
\tilde{\eta}(x) = c_1 \varphi_-^{(1)}(x; k_1)e^{-ik_1^2x} + c_2 \phi_+^{(1)}(x; k_1)e^{ik_1^2x},
\] (99)
where \( c_1, c_2 \) are independent of \( x \). With the boundary conditions (23) and the representation (24), we obtain the boundary conditions
\[
\lim_{x \to -\infty} e^{-ik_1^2x} \tilde{\eta}(x) = c_2 a^{(1)}(k_1)e_2, \quad \lim_{x \to +\infty} e^{ik_1^2x} \tilde{\eta}(x) = c_1 a^{(1)}(k_1)e_1,
\] (100)
where \( k_1 \in \mathbb{C}_I \). As \( a^{(1)}(k_1) \neq 0 \) by Lemma 4.1, from (94)-(100), \( c_1 \) and \( c_2 \) can be expressed as
\[
c_1 = \frac{\gamma \bar{k}_1}{k_1 a^{(1)}(k_1)}, \quad c_2 = \frac{\bar{k}_1}{k_1 a^{(1)}(k_1)},
\] (101)
which yield the equation (91).

Instead of the decomposition (91), we can write
\[
\tilde{\eta}(x) := \varphi_-^{(1)}(x; k_1)e^{-ik_1^2x} + \alpha_1 \phi_+^{(1)}(x; k_1)e^{ik_1^2x}
\] (102)
because the Darboux transformation (49) is invariant if \( \tilde{\eta} \) is multiplied by a nonzero constant. □
4.3. Regularity of new potentials

**Proposition 4.4.** Under the same conditions as in Lemma 4.3, for every $u^{(1)} \in H^3(\mathbb{R}) \cap H^{2.1}(\mathbb{R})$ satisfying $\|u^{(1)}\|_{H^3 \cap H^{2.1}} \leq M$ for some $M > 0$, the transformation

$$D(\tilde{\eta}, k_1) u^{(1)} \in H^3(\mathbb{R}) \cap H^{2.1}(\mathbb{R})$$

satisfies

$$\|D(\tilde{\eta}, k_1) u^{(1)}\|_{H^3 \cap H^{2.1}} \leq C_M,$$  \hspace{1cm} (104)

where the constant $C_M$ does not depend on $u^{(1)}$.

**Proof.** From the representation (24), we have

$$|a^{(1)}(k_1)| = \left| \left( \varphi_{+1}^{(1)}(x; k_1)e^{-ik_1^2 x} + \alpha_1 \varphi_{+2}^{(1)}(x; k_1)e^{ik_1^2 x} \right) \varphi_{+1}^{(1)}(x; k_1)e^{ik_1^2 x} - \left( \varphi_{-2}^{(1)}(x; k_1)e^{-ik_1^2 x} + \alpha_1 \varphi_{-2}^{(1)}(x; k_1)e^{ik_1^2 x} \right) \varphi_{+1}^{(1)}(x; k_1)e^{ik_1^2 x} \right| \leq \left\| \varphi_{+1}^{(1)}(\cdot; k_1) \right\|_{L^\infty} \left( |e^{ik_1^2 x}\tilde{\eta}_1(x)| + |e^{ik_1^2 x}\tilde{\eta}_2(x)| \right).$$

As $a^{(1)}(k_1) \neq 0$ by Lemma 4.1 and $|m_k(\eta, \eta)| \geq |\text{Re} (k_1)| (|\eta_1|^2 + |\eta_2|^2)$, there is a constant $C_M > 0$ independently of $u^{(1)}$ which yields

$$\frac{1}{m_k \left(e^{ik_1^2 x}\tilde{\eta}(x), e^{ik_1^2 x}\tilde{\eta}(x)\right)} \leq C_M \quad \text{for all} \quad x \in \mathbb{R}.$$  \hspace{1cm} (105)

By using the same argument, we also obtain

$$|a^{(1)}(k_1)| \leq |\alpha_1|^{-1} \left\| \varphi_{+1}^{(1)}(\cdot; k_1) \right\|_{L^\infty} \left( |e^{-ik_1^2 x}\tilde{\eta}_1(x)| + |e^{-ik_1^2 x}\tilde{\eta}_2(x)| \right),$$

such that

$$\frac{1}{m_k \left(e^{-ik_1^2 x}\tilde{\eta}(x), e^{-ik_1^2 x}\tilde{\eta}(x)\right)} \leq C_M \quad \text{for all} \quad x \in \mathbb{R}.$$  \hspace{1cm} (106)

With the bound

$$\left| \frac{\tilde{\eta}_1 \tilde{\eta}_2}{m_k(\tilde{\eta}, \tilde{\eta})} \right| \leq \left| \varphi_{+1}^{(-1)}(x; k_1) \varphi_{+2}^{(1)}(x; k_1) \right| \left| m_k \left(e^{ik_1^2 x}\tilde{\eta}, e^{ik_1^2 x}\tilde{\eta}\right) \right| + |\alpha_1|^2 \left| \varphi_{+1}^{(1)}(x; k_1) \varphi_{-2}^{(1)}(x; k_1) \right| \left| m_k \left(e^{ik_1^2 x}\tilde{\eta}(1), e^{-ik_1^2 x}\tilde{\eta}(1)\right) \right|$$

$$+ \left| \varphi_{+1}^{(-1)}(x; k_1) \varphi_{+2}^{(1)}(x; k_1) \right| \left| \alpha_1 \right| \left| \varphi_{+1}^{(-1)}(x; k_1) \varphi_{-2}^{(-2)}(x; k_1) \right|$$

$$\leq \left| \alpha_1 \right| \left| \varphi_{+1}^{(-1)}(x; k_1) \varphi_{+2}^{(1)}(x; k_1) \right| \left| \varphi_{+1}^{(1)}(x; k_1) \varphi_{-2}^{(1)}(x; k_1) \right| \left| \alpha_1 \right| \left| \varphi_{+1}^{(-1)}(x; k_1) \varphi_{+2}^{(-2)}(x; k_1) \right|$$

$$+ \left| \alpha_1 \right| \left| \varphi_{+1}^{(-1)}(x; k_1) \varphi_{+2}^{(1)}(x; k_1) \right| \left| \varphi_{+1}^{(-1)}(x; k_1) \varphi_{-2}^{(-2)}(x; k_1) \right|.$$  \hspace{1cm} (107)
and the bounds (28)-(29), we obtain

$$\left\| C_{k_1}(\tilde{\eta})u^{(1)} \right\|_{L^2,1} \leq C_M. \quad (108)$$

By the same proof of Lemma 4.6, it shows that

$$\left\| D(\tilde{\eta}, k_1)u^{(1)} \right\|_{L^2,1} \leq C_M. \quad (109)$$

The conclusion of $$\left\| \partial_x (D(\tilde{\eta}, k_1)u^{(1)}) \right\|_{L^2,1}$$ and $$\left\| \partial_x^2 (D(\tilde{\eta}, k_1)u^{(1)}) \right\|_{L^2}$$ are also from (28)-(29). Finally, the bound (104) is obtained immediately. □

**Lemma 4.5.** Consider the initial value problem of a system of linear ordinary differential equations

$$\frac{dY}{dx} = A(x)Y, \quad (110)$$

$$Y(x_0) = Y_0, \quad x_0 \in [\alpha, \beta], \quad (111)$$

where $$Y = (y_1(x), \cdots, y_n(x))^T$$, $$A(x)$$ is an $$n \times n$$ continuous matrix-valued function, then the system (110)-(111) has a unique zero solution.

Based on the regularity of $$\varphi(x; k)$$ and $$\phi(x; k)$$ given by Proposition 2.4, we immediately obtain the following proposition that the transformation (58) can be defined as an operator from $$u \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$$ to $$u^{(1)} \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$$.

**Proposition 4.6.** Fix $$k_1 \in \mathbb{C}_I$$. Given a potential $$u \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$$, define $$\eta(x) := \varphi_-(x; k_1)e^{-ik_2^2x}$$, where $$\varphi_-$$ is the Jost function for the spectral problem (10). Then, $$u^{(1)}$$ belongs to $$H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$$.

**Proof.** The proof is in two parts:

- $$u^{(1)}$$ is another smooth solution of the FL equation.
- $$\left\| u^{(1)} \right\|_{H^3 \cup H^{2,1}} < \infty$$.

We see

$$A_k(\eta) = -1 \quad \text{and} \quad C_k(\eta) = 0 \quad \text{if} \quad k \in \mathbb{R} \cup i\mathbb{R},$$

which implies $$u^{(1)} = -u$$ in this case. Therefore, the transformation has no sense to a value of $$k$$ on the continuous spectrum. Thus, our analysis focuses on the case when $$k$$ is outside the continuous spectrum, i.e., for $$k \in \mathbb{C}_I$$.

With Proposition 4.5 and Re($$k_1$$) > 0, we note that

$$m_{k_1}(\eta, \eta) = 0 \iff \eta = 0. \quad (112)$$

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\( \eta'(x_0) = 0 \) follows \( \eta(x_0) = 0 \) at \( x_0 \in \mathbb{R} \), which implies \( \eta(x) = 0 \) for every \( x \in \mathbb{R} \). As \( \varphi_-(x; k_1) \) satisfies the nonzero asymptotic limit (12) as \( x \to -\infty \), then \( \eta(x) = \varphi_-(x; k_1) e^{-ik_1^2 x} \neq 0 \) and \( m_{k_1}(\eta, \eta) \neq 0 \) for every finite \( x \in \mathbb{R} \). It is suffice to replace \( m_{k_1}(\eta, \eta) \) by \( m_{k_1}(\varphi_-, \varphi_-) \).

According to (62) and (63), if \( a(k_1) \neq 0 \), there exists \( a > 0 \) such that
\[
|m_{k_1}(\varphi_-, \varphi_-)| \geq a, \tag{113}
\]
for all \( x \in \mathbb{R} \). In fact, as
\[
\lim_{x \to -\infty} m_{k_1}(\varphi_-, \varphi_-) = k_1, \tag{114}
\]
and with (113), \( m_{k_1}(\varphi_-, \varphi_-) \) may only tend to zero when \( x \to +\infty \). However, it follows from the representation (12) that
\[
\lim_{x \to +\infty} \phi_+(x; k_1) = e_2, \tag{115}
\]
and the fact that \( \varphi_-(\cdot; k_1) \in L^\infty(\mathbb{R}) \) imply that
\[
\lim_{x \to +\infty} \varphi_{-1}(x; k_1) = a(k_1),
\]
so that
\[
\lim_{x \to +\infty} m_{k_1}(\varphi_-, \varphi_-) \neq 0. \tag{116}
\]
Therefore, (113) is true. With the triangle inequality, the bounds (28)-(29) of Proposition 2.4, the bound (113), and \( |A_{k_1}(\eta)| = 1 \), we obtain
\[
\|u^{(1)}\|_{L^2,1} \leq \|u\|_{L^2,1} + \|C_{k_1}(\varphi_-)\|_{L^2,1} \leq \|u\|_{L^2,1} + 2a^{-1} |k_1^2 - k_1^2| \|\varphi_{-1}(\cdot, k_1)\|_{L^2,1} \quad < \infty. \tag{117}
\]
The norms \( \|\partial_x u^{(1)}\|_{L^2,1}, \|\partial_x^2 u^{(1)}\|_{L^2}, \text{ and } \|\partial_x^3 u^{(1)}\|_{L^2} \) are estimated similarly with the bounds (28)-(29) and (113).

If \( a(k_1) = 0 \), the uniform bound (113) is no longer valid since
\[
\lim_{x \to +\infty} m_{k_1}(\varphi_-, \varphi_-) = 0. \tag{118}
\]
The proof on the estimate (117) is done on the interval \( (-\infty, R) \) with arbitrary \( R > 0 \). To extend the estimate (117) on the interval \( (R, \infty) \), we use
\[
\varphi_-(x; k_j)e^{-ik_j^2 x} = \gamma \phi_+(x; k_j) e^{ik_j^2 x}, \quad x \in \mathbb{R} \quad \tag{119}
\]
and write $\eta(x) = \varphi_-(x; k_1)e^{-ik_1^2x} = \gamma\varphi_+(x; k_1)e^{ik_1^2x}$.

Therefore, $u^{(1)} = D(\varphi_-, k_1)u$ can be rewritten as the second equation in \((66)\). As
\[ \lim_{x \to +\infty} m_{k_1}(\phi_+, \phi_+) = \bar{k}_1, \]
we use the same method on the interval \((R, \infty)\) by using the equivalent representation of $u^{(1)}$.

5. Global well-posedness with solitons

5.1. Time-evolution of Darboux transformation

We first explain the idea for the 1-soliton and then the statement on the case of finitely many solitons is proved iteratively.

Let $u(t, \cdot) \in Z_1 \subset H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$ be a local solution of the Cauchy problem \((2)\) on $(-T, T)$ for some $T > 0$. For every fixed time $t \in (-T, T)$, we obtain a new potential $u^{(1)}(t, \cdot) = D(\eta(t, \cdot), k_1)u(t, \cdot)$ of the spectral problem \((10)\) by means of the Darboux transformation. If $k_1 \in \mathbb{C}I$ is taken such that $a(k_1) = 0$, then $u^{(1)}(t, \cdot) \in Z_0$. On the other hand, let $\hat{u}(t, \cdot) \in Z_0$ be a solution to the Cauchy problem \((2)\) which satisfies the initial condition $\hat{u}(0, \cdot) = u^{(1)}(0, \cdot) \in Z_0$, then the solution $\hat{u}(t, \cdot) \in Z_0$ exists for every $t \in \mathbb{R}$, in particular, for $t \in (-T, T)$ \([18, 20]\).

Therefore, the core is to prove $\hat{u}(t, \cdot) = u^{(1)}(t, \cdot)$ for every $t \in (-T, T)$, which first requires the following lemma on the identity of the scattering data for the two potentials.

**Lemma 5.1.** For every $t \in (-T, T)$, the potentials $\hat{u}(t, \cdot)$ and $u^{(1)}(t, \cdot)$ produce the same scattering data.

**Proof.** As both potentials $\hat{u}(t, \cdot)$ and $u^{(1)}(t, \cdot)$ remain in $Z_0$ for every $t \in (-T, T)$, then the scattering data consist only of the reflection coefficient in \([20]\). For the potential $u(t, \cdot) \in Z_1$ with $t \in (-T, T)$, we have $r(t; k) = b(t; k)/a(t; k)$ for $k \in \mathbb{R} \cup i\mathbb{R}$. By denoting
\[ r^{(1)}(t; k) = b^{(1)}(t; k)/a^{(1)}(t; k), \]
we have the reflection coefficient of $u^{(1)}(t, \cdot) \in Z_0$ for $t \in (-T, T)$. Lemma 4.1 characterizes how the two reflection coefficients are related with one another:
\[ r^{(1)}(t; k) = -r(t; k)\frac{k_1^2 k_2^2 - \bar{k}_1^2}{k_1^2 k^2 - k_1^2}, \quad k \in \mathbb{R} \cup i\mathbb{R}, \quad t \in (-T, T). \]
When \( u(x,t) \) is a solution to the FL equation in \( \mathcal{Z}_1 \), we may derive the time evolution of the reflection coefficient \( r(t;k) \) as

\[
r(t;k) = r(0;k)e^{2i\eta^2t}, \quad t \in (-T,T), \tag{123}
\]

which, with the aid of (122), implies that

\[
r^{(1)}(t;k) = r^{(1)}(0;k)e^{2i\eta^2t}, \quad t \in (-T,T). \tag{124}
\]

Note that the equation (123) coincides with the case without solitons in [20].

For the reflection coefficient \( \hat{r} \) of the potential \( \hat{u} \), we know \( r^{(1)}(0,k) = \hat{r}(0,k) \) since \( u^{(1)}(0,\cdot) = \hat{u}(0,\cdot) \). By using the time evolution of the reflection coefficient from [20] and the expression (124), we obtain

\[
\hat{r}(t;k) = \hat{r}(0;k)e^{2i\eta^2t} = r^{(1)}(0;k)e^{2i\eta^2t} = r^{(1)}(t;k), \quad t \in (-T,T), \tag{125}
\]

which proves the lemma.

\[\Box\]

**Proposition 5.2.** The potential \( u^{(1)}(t,\cdot) = D(\eta(t,\cdot),k_1)u(t,\cdot) \) is a new solution of the FL equation for \( t \in (-T,T) \).

**Proof.** In [20], the existence and the Lipschitz continuity of the mapping \( L^{2,1}(\mathbb{R} \cup i\mathbb{R}) \supset U \ni r \mapsto u \in \mathcal{Z}_0 \subset H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}) \) are established by means of the solvability of the associated Riemann-Hilbert problem. Therefore, the same reflection coefficients is mapped to the same solution and \( \hat{u}(t,\cdot) = u^{(1)}(t,\cdot) \) for every \( t \in (-T,T) \) with the result in Lemma 5.1. As \( \hat{u} \) is a solution of the FL equation, so does \( u^{(1)} \).

\[\Box\]

### 5.2. Existence of global solution

Combing Lemma 5.1 and 5.2, we immediately obtain the following conclusion.

**Proposition 5.3.** Fix \( k_1 \in \mathbb{C}_I \). Given a local solution \( u(t,\cdot) \in H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}), t \in (-T,T) \) to the Cauchy problem (2) for some \( T > 0 \), we define

\[
\eta(t,x) := \varphi_-(t,x;k_1)e^{-i(k_1^2x + \eta^2(k_1)t)}, \tag{126}
\]

where \( \varphi_- \) is the Jost function of the linear system (6) and (7). Then, \( u^{(1)}(t,\cdot) = D(\eta(t,\cdot),k_1)u(t,\cdot) \) belongs to \( H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}) \) for every \( t \in [0,T] \) and satisfies the Cauchy problem (2) for \( u^{(1)}(0,\cdot) = D(\eta(0,\cdot),k_1)u(0,\cdot) \).
Proof. The proof of Theorem 5.3 in the case of finitely many solitons relies on the iterative use of the argument above. For a given $u \in Z_N$, $N \in \mathbb{N}$, we remove the distinct eigenvalues $\{k_1, \ldots, k_N\}$ in $C_I$ by iterating the Darboux transformation $N$ times. We set $u^{(0)} = u$ and

$$u^{(l)} = D\left(\tilde{\eta}^{(l-1)}, k_1\right)u^{(l-1)}, \quad (1 \leq l \leq N),$$

which eventually constructs $u^{(N)} \in Z_0$. The arguments of Lemma 5.1 and 5.2 apply to the last potential $u^{(N)}$. As a result, the $N$-fold iteration of the Darboux transformation of a solution $u(t, \cdot) \in Z_N$ of the Cauchy problem (1)-(2) for $t \in [0, T)$ produces a new solution $u^{(N)}(t, \cdot) \in Z_0$ of the Cauchy problem (1)-(2). Figure 1 gives the proof diagram. \hfill \square

Proof of the Theorem 1.1

Proof. Let $u_0 \in Z_1 \subset H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$ and $k_1 \in C_I$ be the only root of $a(k)$ in $C_I$. By Lemma 4.1, if $\eta(x) = \varphi_-(x; k_1)e^{-ik_1x}$, where $\varphi_-$ is the Jost function of the spectral problem (6) associated with $u_0$, then $u_0^{(1)} = D(\eta, k_1)u_0$ belongs to $Z_0 \subset H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$. Also, the mapping is invertible with $u_0 = D(\bar{\eta}, k_1)u_0^{(1)}$, where $\bar{\eta}$ is expressed in term of the new Jost functions $\varphi_-^{(1)}$ and $\varphi_+^{(1)}$ by the decomposition formula (91).

Let $T > 0$ be the maximal existence time for the solution $u(t, \cdot) \in Z_1, t \in (-T, T)$ to the Cauchy problem (1)-(2) with the initial data $u_0 \in Z_1$ and the eigenvalue $k_1$. Then, for every fixed $t \in (-T, T)$, the solution $u(t, \cdot) \in Z_1$ admits the Jost functions $\{\varphi_{pm}(t, x; k), \phi_\pm(t, x; k)\}$. For every $t \in (-T, T)$, we define $u^{(1)}$ by the Darboux transformation

$$u^{(1)} := D(\eta, k_1)u, \quad \eta(t, x; k) := \varphi_-(t, x; k_1)e^{-i(k_1^2x + \eta^2(k_1)t)}$$

(127)

where the boundary conditions (12) are used in the definition of $\varphi_-(t, x; k_1)$ for every $t \in (-T, T)$.

By Lemma 5.2, $u^{(1)}(t, \cdot) \in Z_0, t \in (-T, T)$ is a solution of the Cauchy problem (1)-(2) with the initial data $u_0^{(1)} \in Z_0$. By the existence and uniqueness results [18, 20], the solution $u^{(1)}(t, \cdot) \in Z_0$ is uniquely continued for every $t \in \mathbb{R}$. Let $\{\varphi_{\pm}^{(1)}(t, x; k), \phi_{\pm}^{(1)}(t, x; k)\}$ be the Jost functions for $u^{(1)}(t, x)$. For every $t \in (-T, T)$, we have $u = D(\eta^{(1)}, k_1)u^{(1)}$ with

$$\bar{\eta}(t, x) = \frac{\gamma e^{-i(k_1^2x + \eta^2t)}}{k_1a^{(1)}(k_1)} \varphi_-^{(1)}(t, x; k_1) + \frac{e^{i(k_1^2x + 2\eta(k_1)^2t)}}{k_1a^{(1)}(k_1)} \phi_+^{(1)}(t, x; k_1),$$

(128)

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where $a^{(1)}(k_1) \neq 0$ thanks to Lemma 4.1.

On the other hand, as $u^{(1)}(t, \cdot) \in \mathcal{Z}_0$ exists for every $t \in \mathbb{R}$, the associated Jost functions $\{\varphi^{(1)}_{\pm}(t, x; k), \phi^{(1)}_{\pm}(t, x; k)\}$ exist for every $t \in \mathbb{R}$. Therefore, we can define $\tilde{u} = D(\tilde{\eta}, k_1)u^{(1)}$, $t \in \mathbb{R}$. As $u(t, \cdot) = \hat{u}(t, \cdot) \in \mathcal{Z}_1$ for every $t \in (-T, T)$ by uniqueness, the extended function $\tilde{u}$ is unique to the solution $u$ of the same Cauchy problem (1)-(2), which exists globally in time thanks to the bound (104) which is proved in Lemma 4.4. Indeed, by [20] we have $\|u^{(1)}(t, \cdot)\|_{H^3 \cap H^{2,1}} \leq M_T$ for every $t \in (-T, T)$, where $T > 0$ is arbitrary and $M_T$ depends on $T$. Next, with the bound in Lemma 4.6, we have $\|u(t, \cdot)\|_{H^3 \cap H^{2,1}} \leq C_{M_T}$ for every $t \in (-T, T)$. Thus, the solution will not blow up in a finite time and hence there exists a unique global solution $u(t, \cdot) \in \mathcal{Z}_1, t \in \mathbb{R}$ to the Cauchy problem (1)-(2) for every $u_0 \in \mathcal{Z}_1 \subset H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R})$.

By iterating the Darboux transformation $N$ times and by the same argument as above, we prove the global existence of $u(t, \cdot) \in \mathcal{Z}_N \subset H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}), t \in \mathbb{R}$ from the global existence of $u^{(N)}(t, \cdot) \in \mathcal{Z}_0 \subset H^3(\mathbb{R}) \cap H^{2,1}(\mathbb{R}), t \in \mathbb{R}$. This completes the proof of Theorem 1.1.

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\section*{Data Availability Statements}

The data which supports the findings of this study is available within the article.

\section*{Conflict of Interest}

The authors have no conflicts to disclose.

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