THE QUANTUM WAVE PACKET AND THE FEYNMAN DE BROGLIE BOHM PROPAGATOR OF THE LINEARIZED SUSSMANN HASSE ALBRECHT KOSTIN NASSAR EQUATION ALONG A CLASSICAL TRAJECTORY

J. M. F. Bassalo¹, P. T. S. Alencar², D. G. da Silva³, A. B. Nassar⁴ and M. Cattani⁵

¹ Fundação Minerva, Avenida Governador José Malcher 629 - CEP 66035-100, Belém, Pará, Brasil E-mail: jmfbassalo@gmail.com

² Universidade Federal do Pará - CEP 66075-900, Guamá, Belém, Pará, Brasil E-mail: tarso@ufpa.br

³ Escola Munguba do Jari, Vitória do Jari - CEP 68924-000, Amapá, Brasil E-mail: danielgemaque@yahoo.com.br

⁴ Extension Program-Department of Sciences, University of California, Los Angeles, California 90024, USA E-mail: nassar@ucla.edu

⁵ Instituto de Física da Universidade de São Paulo. C. P. 66318, CEP 05315-970, São Paulo, SP, Brasil E-mail: mcattani@if.usp.br

Abstract: In this paper we study the quantum wave packet and the Feynman-de Broglie-Bohm propagator of the linearized Süssmann-Hasse-Albrecht-Kostin-Nassar equation along a classical trajectory.

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1. Introduction

In the present work we investigate the quantum wave packet and the Feynman-de Broglie-Bohm propagator of the linearized Süssmann-Hasse-Albrecht-Kostin-Nassar equation along a classical trajectory by using the Quantum Mechanical of the de Broglie-Bohm.[¹]

2. The Süssmann-Hasse-Albrecht-Kostin-Nassar Equation

In 1973,[²] D. Süssmann, and in 1975, R. W. Hasse,[³] K. Albrecht[⁴] and M. D. Kostin[⁵] proposed a non-linear Schrödinger Equation, that was generalized by A. B. Nassar, in 1986,[⁶] to represent time dependent physical systems, given by:

\[
\dot{\psi}(x, t) = \frac{i}{\hbar} \left[ \frac{\partial^2 \psi(x, t)}{\partial x^2} + V(x, t) + \nu \left( [x - q(t)] [c \hat{p} + (1 - c) \langle \hat{p} \rangle] - \frac{i}{2} \frac{\hbar c}{m} \right) \right] \psi(x, t), \tag{2.1}
\]
where $\hat{p}$ is the operator of linear momentum:
\[
\hat{p} = -i \hbar \frac{\partial}{\partial x}, \quad (2.2)
\]
and $c$ is a constant, where: $c = 1$, for Süssmann; $c = 1/2$, for Hasse; and $c = 0$, for Albrecht and Kostin. Besides, $\psi(x, t)$ and $V(x, t)$ are, respectively, the wavefunction and the time dependent potential of the physical system in study, $q(t) = <x>$, and $\nu$ is a constant.

Writing the wavefunction $\psi(x, t)$ in the polar form, defined by the Madelung-Bohm:[7, 8]
\[
\psi(x, t) = \phi(x, t) \exp [i S(x, t)], \quad (2.3)
\]
where $S(x, t)$ is the classical action and $\phi(x, t)$ will be defined in what follows. Now, using the eq. (2.2) into eq. (2.3), we get:[1]
\[
\hat{p} \psi = -i \hbar \left( \phi \frac{\partial}{\partial t} + i \phi \frac{\partial S}{\partial t} \right) = -i \hbar \left( \frac{1}{\phi} \frac{\partial \phi}{\partial x} + i \frac{\partial S}{\partial x} \right) \psi \rightarrow \\
\hat{p} \psi = \hbar \left( \frac{\partial S}{\partial x} - i \frac{\partial \phi}{\partial x} \right) \psi. \quad (2.4)
\]

Inserting the eqs. (2.3,4) into eq. (2.1), will be (remember that $e^{i S}$ is a common factor):
\[
i \hbar \left( \frac{\partial \phi}{\partial t} + i \phi \frac{\partial S}{\partial t} \right) = -\frac{\hbar^2}{2m} \left[ \frac{\partial^2 \phi}{\partial x^2} + \\
+ 2i \frac{\partial \phi}{\partial x} \frac{\partial S}{\partial x} + i \phi \frac{\partial^2 S}{\partial x^2} - \phi \left( \frac{\partial S}{\partial x} \right)^2 \right] + \\
+ \left[ V(x, t) + \nu \left( [x - q(t)] [c \hbar \left( \frac{\partial S}{\partial x} - i \frac{\partial \phi}{\partial x} \right) + \\
+ \left( 1 - c \right) \langle \hat{p} \rangle - \frac{1}{2} i \hbar c \right] \right) \phi. \quad (2.5)
\]

Taking the real and imaginary parts of eq. (2.5), we obtain (remember that: $\langle \hat{p} \rangle = m \langle \dot{v}_{xu} \rangle = m \langle v_{xu} \rangle =$ real):

a) imaginary part
\[
\frac{\hbar^2}{\phi} \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m} \left( \frac{\partial^2 S}{\partial x^2} + 2 \frac{1}{\phi} \frac{\partial \phi}{\partial x} \frac{\partial S}{\partial x} \right) - \\
- \nu [x - q(t)] c \frac{\hbar}{\phi} \frac{\partial \phi}{\partial x} - \frac{\nu}{2} \hbar c, \quad (2.6)
\]
b) real part

\[-\hbar \frac{\partial S}{\partial t} = - \frac{\hbar^2}{2m} \left[ \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} - \left( \frac{\partial S}{\partial x} \right)^2 \right] + \]

\[+ \nu [x - q(t)] c \hbar \frac{\partial S}{\partial x} + V(x, t) + \]

\[+ \nu [x - q(t)] (1 - c) m < v_{qu} > . \quad (2.7)\]

2.1 Dynamics of the Süssmann-Hasse-Albrecht-Kostion-Nassar Equation

Now, let us to study the dynamics of the Süssmann-Hasse-Albrecht-Kostin-Nassar equation. To do is let us perform the following correspondences:\[9\]

\[\rho(x, t) = \phi^2(x, t) , \quad (2.8) \quad \text{(quantum mass density)}\]

\[v_{qu}(x, t) = \frac{\hbar}{m} \frac{\partial S(x, t)}{\partial x} , \quad (2.9) \quad \text{(quantum velocity)}\]

\[V_{qu}(x, t) = - \frac{\hbar^2}{2m} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2} = - \frac{\hbar^2}{2m \phi} \frac{\partial^2 \phi}{\partial x^2} . \quad (2.10a,b) \quad \text{(Bohm quantum potential)}\]

Putting the eqs. (2.8,9) into eq. (2.6) we get [remember that \(\frac{\partial}{\partial x} (\ell n u) = \frac{1}{u} \frac{\partial u}{\partial x}\) and \(\ell n (u^n) = n \ell n u\):]

\[\frac{\partial}{\partial t} (2 \ell n \phi) = - \frac{\hbar}{m} \left[ \frac{\partial^2 S}{\partial x^2} + \frac{\partial S}{\partial x} \frac{\partial}{\partial x} (2 \ell n \phi) \right] - \]

\[- \nu [x - q(t)] c \frac{\partial}{\partial x} (2 \ell n \phi) - \nu c \rightarrow \]

\[\frac{\partial}{\partial t} (\ell n \phi^2) = - \frac{\hbar}{m} \left[ \frac{\partial^2 S}{\partial x^2} + \frac{\partial S}{\partial x} \frac{\partial}{\partial x} (\ell n \phi^2) \right] - \]

\[- \nu [x - q(t)] c \frac{\partial}{\partial x} (\ell n \phi^2) - \nu c \rightarrow \]

\[\frac{\partial}{\partial t} (\ell n \rho) = - \frac{\hbar}{m} \left[ \frac{\partial^2 S}{\partial x^2} + \frac{\partial S}{\partial x} \frac{\partial}{\partial x} (\ell n \rho) \right] - \]

\[- \nu [x - q(t)] c \frac{\partial}{\partial x} (\ell n \rho) - \nu c \rightarrow \]

\[\frac{1}{\rho} \frac{\partial \rho}{\partial t} = - \frac{\hbar}{m} \left( \frac{\partial^2 S}{\partial x^2} + \frac{\partial S}{\partial x} \frac{1}{\rho} \frac{\partial \rho}{\partial x} \right) - \]

\[- \nu [x - q(t)] c \frac{1}{\rho} \frac{\partial \rho}{\partial x} - \nu c \rightarrow \]
\[
\frac{1}{\rho} \frac{\partial \rho}{\partial t} + \frac{\partial v_{qu}}{\partial x} + \frac{v_{qu}}{\rho} \frac{\partial \rho}{\partial x} = - \nu c - \nu c [x - q(t)] \frac{1}{\rho} \frac{\partial \rho}{\partial x} \rightarrow \\
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_{qu})}{\partial x} = \\
= - \nu c \rho - \nu c [x - q(t)] \frac{\partial \rho}{\partial x}.
\] (2.11)

We must note that the presence of the second member in expression (2.11), indicates descoerence of the considered physical system represented by (2.1).

Now, taking the eq. (2.7) and using the eqs. (2.9,10b), will be:

\[
- \hbar \frac{\partial S}{\partial t} = - \frac{\hbar^2}{2 m} \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{2} m \left( \frac{\hbar}{m} \frac{\partial \phi}{\partial x} \right)^2 + \\
+ m \nu [x - q(t)] \frac{\hbar}{m} \frac{\partial \phi}{\partial x} + V(x, t) + \\
+ \nu [x - q(t)] (1 - c) m < v_{qu} > \rightarrow \\
- \hbar \frac{\partial S}{\partial t} = V_{qu}(x, t) + \frac{1}{2} m v_{qu}^2 + m \nu [x - q(t)] c v_{qu} + V(x, t) + \\
+ \nu [x - q(t)] (1 - c) m < v_{qu} > \rightarrow \\
\hbar \frac{\partial S}{\partial t} + \nu m [x - q(t)] [c v_{qu} + (1 - c) < v_{qu}>] + \\
+ \frac{1}{2} m v_{qu}^2 + V_{qu}(x, t) + V(x, t) = 0.
\] (2.12)

Considering that:

\[
< f(x, t) > = \int_{-\infty}^{+\infty} \rho(x, t) f(x, t) dx = g(t),
\] (2.13)

where \(\rho(x, t)\) is given by:\cite{10}

\[
\rho(x, t) = [2\pi a^2(t)]^{-1/2} exp \left( - \frac{|x - q(t)|^2}{2 a^2(t)} \right),
\] (2.14)

where \(a(t)\) and \(q(t) = < x >\) are auxiliary functions of time, to be determined in what follows; they represent the width and center of mass of wave packet, respectively.

Then, using the eqs. (2.13,14) and remembering that \(\int_{-\infty}^{+\infty} exp(- z^2) dz = \sqrt{\pi},\) will be:

\[
< q(t) > = \int_{-\infty}^{+\infty} \rho(x, t) q(t) dx = q(t),
\] (2.15)
\[
\frac{\partial< v_{nu} >}{\partial x} = \frac{\partial}{\partial x} \int_{-\infty}^{+ \infty} \rho(x, t) v_{qu}(x, t) dx = \frac{\partial g(t)}{\partial x} = 0, \quad (2.16)
\]

and (remember that \(x\) and \(t\) are variables independents):

\[
\frac{\partial g(t)}{\partial x} = 0. \quad (2.17)
\]

Defining,[11]

\[
\vartheta_{qnc} = v_{qu} + \nu c [x - q(t)], \quad (2.18)
\]

the **quantum velocity non-conservative**, we have [by using the eq. (2.17)]:

\[
\frac{\partial}{\partial x} (\rho \vartheta_{qnc}) = \frac{\partial}{\partial x} \left[ \rho (v_{qu} + \nu c [x - q(t)]) \right] = \frac{\partial(\rho v_{qu})}{\partial x} + \rho \nu c [x - q(t)] \frac{\partial \rho}{\partial x} + \rho \nu c \frac{\partial [x - q(t)]}{\partial x} = \frac{\partial(\rho v_{qu})}{\partial x} + \nu c [x - q(t)] \frac{\partial \rho}{\partial x} + \nu c \rho \rightarrow -\nu c \rho - \nu c [x - q(t)] \frac{\partial \rho}{\partial x} = \frac{\partial(\rho v_{qu})}{\partial x} - \frac{\partial(\rho \vartheta_{qnc})}{\partial x}. \quad (2.19)
\]

Insering the eq. (2.19) into eq. (2.12), results:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial(\rho \vartheta_{qnc})}{\partial x} = 0. \quad (2.20)
\]

The eq. (2.20) indicates that, considering the **quantum velocity non-conservative** \(\vartheta_{qnc}\), there are coherence of the considered physical system represented by (2.1).

Now, differentiating the eq. (2.7) with respect \(x\), and using the eqs. (2.9,10b,16-18), we have (remember that \(x\) and \(t\) are variables independents):

\[
-\hbar \frac{\partial^2 S}{\partial x \partial t} = -\frac{\hbar^2}{2 m} \frac{\partial}{\partial x} \left[ \frac{1}{2} \frac{\partial^2 \phi}{\partial x^2} - \left( \frac{\partial S}{\partial x} \right)^2 \right] + \frac{\partial}{\partial x} \left( \nu [x - q(t)] c \hbar \frac{\partial S}{\partial x} + \nu [x - q(t)] (1 - c) m < v_{qu} > + V(x, t) \right) \rightarrow
\]
\[
\frac{\partial}{\partial t} \left( \frac{h}{m} \frac{\partial S}{\partial x} \right) = \frac{1}{m} \frac{\partial}{\partial x} \left( \frac{h^2}{2m} \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} \right) - \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{h}{m} \frac{\partial S}{\partial x} \right)^2 - \\
- \frac{\partial}{\partial x} \left( \nu [x - q(t)] c \left( \frac{h}{m} \frac{\partial S}{\partial x} \right) \right) - \\
- \frac{\partial}{\partial x} \left( \nu [x - q(t)] (1 - c) <v_{qu}> \right) - \frac{1}{m} \frac{\partial V}{\partial x} \rightarrow \\
\frac{\partial v_{qu}}{\partial t} = - \frac{1}{m} \frac{\partial V}{\partial x} - \frac{1}{2} \frac{\partial}{\partial x} \left( v_{qu}^2 \right) - \frac{\partial}{\partial x} \left( \nu c v_{qu} [x - q(t)] \right) - \\
- \frac{\partial}{\partial x} \left( \nu [x - q(t)] (1 - c) <v_{qu}> \right) - \frac{1}{m} \frac{\partial V}{\partial x} \rightarrow \\
\frac{\partial v_{qu}}{\partial t} + v_{qu} \frac{\partial v_{qu}}{\partial x} + v_{qu} \frac{\partial}{\partial x} \left( \nu c [x - q(t)] \right) + \\
+ \nu c [x - q(t)] \frac{\partial v_{qu}}{\partial x} + \\
+ \nu (1 - c) <v_{qu}> \frac{\partial}{\partial x} [x - q(t)] = \\
= - \frac{1}{m} \frac{\partial}{\partial x} (V + V_{qu}) \rightarrow \\
\frac{\partial v_{qu}}{\partial t} + \frac{\partial v_{qu}}{\partial x} \left( v_{qu} + \nu c [x - q(t)] \right) + \\
+ [\nu v_{qu} c + \nu (1 - c) <v_{qu}>] = \\
= - \frac{1}{m} \frac{\partial}{\partial x} (V + V_{qu}) \rightarrow \\
\frac{\partial v_{qu}}{\partial t} + \partial_{qnc} \frac{\partial v_{qu}}{\partial x} + \frac{1}{m} \frac{\partial}{\partial x} (V + V_{qu}) = \\
= - \nu [c v_{qu} + (1 - c) <v_{qu}>] . \quad (2.21)
\]

Taking the eqs. (2.15,17,18) and considering that <> is a linear operation, results [remember that < x > = q(t)]:

\[
<\partial_{qnc}> = <v_{qu}> + \nu c (<x> - <q(t)>) \rightarrow \\
<\partial_{qnc}> = <v_{qu}> . \quad (2.22)
\]

Differentiating the eq. (2.18) with respect t, considering that \(<v_{qu}> = \frac{\partial<x>}{\partial t} = \dot{q}(t), <x> = q(t)\) and the eq. (2.22) (remember that x and t are variables independents), we have:
\[ \frac{\partial \vartheta_{qnc}}{\partial t} = \frac{\partial v_{qu}}{\partial t} + \nu \ c \left( \frac{\partial x}{\partial t} - \frac{\partial <x>}{\partial t} \right) \rightarrow \]

\[ \frac{\partial v_{qu}}{\partial t} = \frac{\partial \vartheta_{qnc}}{\partial t} + \nu \ c \ <\vartheta_{qnc}> . \quad (2.23) \]

Now, differentiating the eq. (2.18) with respect \( x \) and considering the eq. (2.17), results:

\[ \frac{\partial \vartheta_{qnc}}{\partial x} = \frac{\partial v_{qu}}{\partial x} + \nu \ c = \rightarrow \frac{\partial v_{qu}}{\partial x} = \frac{\partial \vartheta_{qnc}}{\partial x} - \nu \ c . \quad (2.24) \]

Substituting the eqs. (2.22-24) into eq. (2.21), we have:

\[ \frac{\partial \vartheta_{qnc}}{\partial t} + \nu \ c <\vartheta_{qnc}> + \]

\[ + \vartheta_{qnc} \left( \frac{\partial \vartheta_{qnc}}{\partial x} - \nu \ c \right) + \frac{1}{m} \frac{\partial}{\partial x} (V + V_{qu}) = \]

\[ = - \nu \left[ c \ v_{qu} + (1 - c) \ <\vartheta_{qnc}> \right] = \]

\[ = - \nu \ c \ v_{qu} - \nu \ <\vartheta_{qnc}> + \nu \ c <\vartheta_{qnc}> \rightarrow \]

\[ \frac{\partial \vartheta_{qnc}}{\partial t} \]

\[ + \vartheta_{qnc} \left( \frac{\partial \vartheta_{qnc}}{\partial x} - \nu \ c \right) + \frac{1}{m} \frac{\partial}{\partial x} (V + V_{qu}) = \]

\[ = - \nu \ c \ v_{qu} - \nu <\vartheta_{qnc}> + \nu c <\vartheta_{qnc}> \rightarrow \]

\[ \frac{\partial \vartheta_{qnc}}{\partial t} + \vartheta_{qnc} \frac{\partial \vartheta_{qnc}}{\partial x} + \frac{1}{m} \frac{\partial}{\partial x} (V + V_{qu}) = \]

\[ = - \nu \left[ c (v_{qu} - \vartheta_{qnc}) + <\vartheta_{qnc}> \right] . \quad (2.25) \]

Considering the "substantive differentiation" (local plus convective) or "hidrodynamic differentiation": \( d/dt = \partial/\partial t + \vartheta_{qnc} \partial/\partial x \) and that \( \vartheta_{qnc} = dx_{qnc}/dt \), the eq. (2.25) could be written as:

\[ m \frac{d^2 x_{qnc}}{dt^2} = - \nu \ m \left[ c (v_{qu} - \vartheta_{qnc}) + <\vartheta_{qnc}> \right] - \frac{\partial}{\partial x} (V + V_{qu}) , \quad (2.26) \]

that has a form of the Second Newton Law.

3. The Quantum Wave Packet of the Linearized Süssmann-Hasse-Albrecht-Kostin-Nassar Equation along a Classical Trajectory

In order to find the quantum wave packet of the Süssmann-Hasse-Albrecht-Kostin-Nassar equation, let us considerer the eq. (2.13):
\[ \rho (x, t) = [2\pi a^2(t)]^{-1/2} \exp \left( - \frac{|x - q(t)|^2}{2 a^2(t)} \right). \] \hspace{1cm} (3.1)

Using the eq. (3.1), the integration of the eq. (2.20), is given by:\cite{1}

\[ \vartheta_{qu} (x, t) = \frac{\dot{a}(t)}{a(t)} [x - q(t)] + \ddot{\vartheta}(t), \] \hspace{1cm} (3.2a)

where \( \ddot{\vartheta}(t) = \frac{d\vartheta}{dt} \).

Now, using the eq. (3.2a) into eq. (2.18), we have:

\[ v_{qu} (x, t) = \left[ \frac{\dot{a}(t)}{a(t)} - \nu c \right] [x - q(t)] + \ddot{q}(t). \] \hspace{1cm} (3.2b)

To obtain the quantum wave packet of the linear Süssmann-Hasse-Albrecht-Kostin-Nassar equation along a classical trajectory given by (2.1), let us expand the functions \( S(x, t) \), \( V(x, t) \) and \( V_{qu}(x, t) \) around of \( q(t) \) up to second Taylor order. In this way we have:

\[ S(x, t) = S[q(t), t] + S'[q(t), t] [x - q(t)] + \frac{S''[q(t), t]}{2} [x - q(t)]^2, \] \hspace{1cm} (3.3)

\[ V(x, t) = V[q(t), t] + V'[q(t), t] [x - q(t)] + \frac{V''[q(t), t]}{2} [x - q(t)]^2. \] \hspace{1cm} (3.4)

\[ V_{qu}(x, t) = V_{qu}[q(t), t] + V'_{qu}[q(t), t] [x - q(t)] + \frac{V''_{qu}[q(t), t]}{2} [x - q(t)]^2. \] \hspace{1cm} (3.5)

Differentiating (3.3) in the variable \( x \), multiplying the result by \( \frac{\hbar}{m} \), using the eqs. (2.9) and (3.2b), taking into account the polynomial identity property and also considering the second Taylor order, we obtain:

\[ \frac{\hbar}{m} \frac{\partial S(x, t)}{\partial x} = \frac{\hbar}{m} \left( S'[q(t), t] + S''[q(t), t] [x - q(t)] \right) = \]

\[ = v_{qu}(x, t) = \left[ \frac{\dot{a}(t)}{a(t)} - \nu c \right] [x - q(t)] + \ddot{\vartheta}(t) \rightarrow \]

\[ S'[q(t), t] = \frac{m}{\hbar} \frac{\dot{q}(t)}{h}, \quad S''[q(t), t] = \frac{m}{\hbar} \left[ \frac{\dot{a}(t)}{a(t)} - \nu c \right], \] \hspace{1cm} (3.6a,b)

Substituting (3.6a,b) into (3.3), results:

\[ S(x, t) = S_o(t) + \frac{m}{\hbar} \frac{\dot{q}(t)}{h} [x - q(t)] + \frac{m}{2 \hbar} \left[ \frac{\dot{a}(t)}{a(t)} - \nu c \right] [x - q(t)]^2, \] \hspace{1cm} (3.7a)

where:

\[ S_o(t) \equiv S[q(t), t], \] \hspace{1cm} (3.7b)
are the classical actions.

Differentiating the (3.7a) with respect to \( t \), we obtain (remembering that \( \frac{\partial x}{\partial t} = 0 \)):

\[
\frac{\partial S (x, t)}{\partial t} = \dot{S}_0(t) + \frac{\partial}{\partial t} \left[ \frac{m \dot{q}(t)}{\hbar} [x - q(t)] \right] + \frac{\partial}{\partial t} \left[ \frac{m}{2 \hbar} \left( \frac{\dot{a}(t)}{a(t)} - \nu c \right) [x - q(t)]^2 \right]
\]

\[
+ \frac{m}{2 \hbar} \left( \frac{\dot{a}(t)}{a(t)} - \frac{\dot{a}^2(t)}{a^2(t)} \right) [x - q(t)]^2 - \frac{m \dot{q}(t)}{\hbar} \left( \frac{\dot{a}(t)}{a(t)} - \nu c \right) [x - q(t)] . \quad (3.8)
\]

Considering the eqs. (2.8) and (3.1), let us write \( V_{q u} \) given by (2.10a,b) in terms of potencies of \([x - q(t)]\). Before, we calculate the following derivations:

\[
\frac{\partial \phi (x, t)}{\partial x} = \frac{\partial}{\partial x} \left( [2 \pi \, a^2(t)]^{-1/4} e^{-\frac{[x - q(t)]^2}{4 \, a^2(t)}} \right) =
\]

\[
= [2 \pi \, a^2(t)]^{-1/4} e^{-\frac{[x - q(t)]^2}{4 \, a^2(t)}} \frac{\partial}{\partial x} \left( - \frac{[x - q(t)]^2}{4 \, a^2(t)} \right) \rightarrow
\]

\[
\frac{\partial \phi (x, t)}{\partial x} = - [2 \pi \, a^2(t)]^{-1/4} e^{-\frac{[x - q(t)]^2}{4 \, a^2(t)}} \frac{[x - q(t)]}{2 \, a^2(t)} ,
\]

\[
\frac{\partial^2 \phi (x, t)}{\partial x^2} = \frac{\partial}{\partial x} \left( [2 \pi \, a^2(t)]^{-1/4} e^{-\frac{[x - q(t)]^2}{4 \, a^2(t)}} \frac{[x - q(t)]}{2 \, a^2(t)} \right) =
\]

\[
= - [2 \pi \, a^2(t)]^{-1/4} e^{-\frac{[x - q(t)]^2}{4 \, a^2(t)}} \frac{[x - q(t)]}{2 \, a^2(t)} \frac{\partial}{\partial x} \left( - \frac{[x - q(t)]^2}{4 \, a^2(t)} \right) \rightarrow
\]

\[
\frac{\partial^2 \phi (x, t)}{\partial x^2} = - [2 \pi \, a^2(t)]^{-1/4} e^{-\frac{[x - q(t)]^2}{4 \, a^2(t)}} \frac{[x - q(t)]}{2 \, a^2(t)} \frac{1}{2 \, a^2(t)} + [2 \pi \, a^2(t)]^{-1/4} e^{-\frac{[x - q(t)]^2}{4 \, a^2(t)}} \frac{x - q(t)^2}{4 \, a^4(t)} =
\]

\[
= - \phi (x, t) \frac{1}{2 \, a^2(t)} + \phi (x, t) \frac{[x - q(t)]^2}{4 \, a^4(t)} \rightarrow
\]

\[
\frac{1}{\phi (x, t)} \frac{\partial^2 \phi (x, t)}{\partial x^2} = \frac{[x - q(t)]^2}{4 \, a^4(t)} - \frac{1}{2 \, a^2(t)} . \quad (3.9)
\]

Substituting (3.9) into (2.10b) and taking into account (3.4), results:

\[
V_{q u} (x, t) = \frac{\hbar^2}{4 \, m \, a^2(t)} - \frac{\hbar^2}{8 \, m \, a^4(t)} [x - q(t)]^2 . \quad (3.10)
\]
\[ V_{qu}(q(t), t) = \frac{\hbar^2}{4 m a^2(t)}, \quad (3.11a) \]

\[ V'_{qu}(q(t), t) = 0, \quad V''_{qu}(q(t), t) = -\frac{\hbar^2}{4 m a^4(t)}. \quad (3.11b,c) \]

Inserting the eqs. (3.2b,3.4) and (3.7a,8,10), into (2.12), we obtain [remembering that \( S_o(t), a(t), q(t) \) and \(<v_{qu}> = \frac{d<\dot{x}>}{dt} = \dot{q}(t)>]:

\[
\begin{align*}
\hbar \frac{\dot{S}}{m} + \nu m [x - q(t)] [c v_{qu} + (1 - c) <v_{qu}>] + \frac{1}{2} m v^2_{qu} + V + V_{qu} &= \\
= \hbar \dot{S}_o(t) + m \ddot{q}(t) [x - q(t)] - m \dot{q}^2(t) + \frac{m}{2} \left[ \frac{\ddot{a}(t)}{a(t)} - \frac{a^2(t)}{a^2(t)} \right] [x - q(t)]^2 - \\
- m \dot{q}(t) \left[ \frac{\dot{a}(t)}{a(t)} - \nu c \right] [x - q(t)] + \nu m [x - q(t)] \times \\
\times \left[ c \left( \frac{\ddot{a}(t)}{a(t)} - \nu c \right) [x - q(t)] + \dot{q}(t) \right] + (1 - c) \dot{q}(t) + \\
+ \frac{m}{2} \left[ \frac{\ddot{a}(t)}{a(t)} - \nu c \right]^2 [x - q(t)]^2 + m \dot{q}(t) \left[ \frac{\ddot{a}(t)}{a(t)} - \nu c \right] [x - q(t)] + \frac{m \dot{q}^2(t)}{2} + \\
+ V[q(t), t] + V'[q(t), t] [x - q(t)] + \frac{1}{2} V''[q(t), t] [x - q(t)]^2 + \\
+ \frac{\hbar^2}{4 m a^2(t)} - \frac{\hbar^2}{8 m a^4(t)} [x - q(t)]^2 &= \\
= \hbar \dot{S}_o(t) + m \ddot{q}(t) [x - q(t)] - m \dot{q}^2(t) + \frac{m}{2} \left[ \frac{\ddot{a}(t)}{a(t)} - \frac{a^2(t)}{a^2(t)} \right] [x - q(t)]^2 - \\
- m \dot{q}(t) \left[ \frac{\ddot{a}(t)}{a(t)} - \nu c \right] [x - q(t)] + \nu m c \left[ \frac{\dot{a}(t)}{a(t)} - \nu c \right] [x - q(t)]^2 + \\
+ \nu m c \dot{q}(t) [x - q(t)] + \nu m (1 - c) \dot{q}(t) \right) + \\
+ \frac{m}{2} \left[ \frac{\ddot{a}(t)}{a(t)} - \nu c \right]^2 [x - q(t)]^2 + m \dot{q}(t) \left[ \frac{\ddot{a}(t)}{a(t)} - \nu c \right] [x - q(t)] + \frac{m \dot{q}^2(t)}{2} + \\
+ V[q(t), t] + V'[q(t), t] [x - q(t)] + \frac{1}{2} V''[q(t), t] [x - q(t)]^2 + \\
+ \frac{\hbar^2}{4 m a^2(t)} - \frac{\hbar^2}{8 m a^4(t)} [x - q(t)]^2 = 0. \quad (3.12)
\end{align*}
\]

Expanding the eq. (3.12) in potencies of \([x - q(t)], \text{we obtain} \text{remember that} [x - q(t)]^0 = 1):
\[
\left( \hbar \dot{S}_o(t) - \frac{m \dot{q}^2(t)}{2} + \nu \frac{m}{} (1 - c) \ddot{q}(t) + V[q(t), t] + \frac{\hbar^2}{4m a^2(t)} \right) [x - q(t)]^o + \\
\left( m \ddot{q}(t) + \nu m c \dot{q}(t) + V'[q(t), t] \right) [x - q(t)] + \\
\left( \frac{m}{2} \frac{\dddot{a}(t)}{a(t)} - \frac{m \nu^2 c^2}{2} + \frac{1}{2} V''[q(t), t] - \frac{\hbar^2}{8m a^4(t)} \right) [x - q(t)]^2 = 0. \quad (3.13)
\]

As (3.13) is an identically null polynomium, all coefficients of the potencies must be all equal to zero, that is:

\[
\dot{S}_o(t) = \frac{1}{\hbar} \left[ \frac{1}{2} m \dot{q}^2(t) - \nu \frac{m}{} (1 - c) \ddot{q}(t) - V[q(t), t] - \frac{\hbar^2}{4m a^2(t)} \right], \quad (3.14)
\]

\[
\ddot{q}(t) + \nu c \dot{q}(t) + \frac{1}{m} V'[q(t), t] = 0, \quad (3.15)
\]

\[
\dddot{a}(t) + a(t) \left( \frac{1}{m} V''[q(t), t] - \nu^2 c^2 \right) = \frac{\hbar^2}{4m^2 a^4(t)} \cdot \quad (3.16)
\]

Assuming that the following initial conditions are obeyed:

\[
q(0) = x_o, \quad \dot{q}(0) = v_o, \quad a(0) = a_o, \quad \dot{a}(0) = b_o, \quad (3.17a-d)
\]

and that:

\[
S_o(0) = \frac{m v_o x_o}{\hbar}, \quad (3.18)
\]

the integration of (3.14) gives:

\[
S_o(t) = \frac{1}{\hbar} \int_0^t dt' \left[ \frac{1}{2} m \dot{q}^2(t') - \nu \frac{m}{} (1 - c) \ddot{q}(t') - V[q(t'), t'] - \frac{\hbar^2}{4m a^2(t')} \right] + \frac{m v_o x_o}{\hbar}. \quad (3.19)
\]

Taking the eq. (3.19) in the eq. (3.7a) results:

\[
S(x, t) = \frac{1}{\hbar} \int_0^t dt' \left[ \frac{1}{2} m \dot{q}^2(t') - \nu \frac{m}{} (1 - c) \ddot{q}(t') - V[q(t'), t'] - \frac{\hbar^2}{4m a^2(t')} \right] + \\
\frac{m v_o x_o}{\hbar} + \frac{m \dddot{a}(t)}{\hbar} [x - q(t)] + \frac{m}{2 \hbar} \left[ \frac{\dddot{a}(t)}{a(t)} - \nu c \right] [x - q(t)]^2. \quad (3.20)
\]

The above result permit us, finally, to obtain the wave packet for the linearized Süssmann-Hasse-Albrecht-Kostin-Nassar equation along a classical trajectory. Indeed, considering the eqs. (2.3, 8) and (3.1, 20), we get.
\( \psi(x, t) = [2\pi a^2(t)]^{-1/4} \exp \left( \left( \frac{i m}{2\hbar} \left[ \frac{\dot{a}(t)}{a(t)} - \nu c \right] - \frac{1}{4a^2(t)} \right) [x - q(t)]^2 \right) \times \)
\[ \times \exp \left[ \frac{i m}{\hbar} [x - q(t)] + \frac{i m v_0 x_o}{\hbar} \right] \times \]
\[ \times \exp \left[ \frac{i}{\hbar} \int_{t_o}^{t} dt' \left( \frac{1}{2} m \dot{q}^2(t') - m \nu (1 - c) \dot{q}(t) - V[q(t'), t'] - \frac{\hbar^2}{4m a^2(t')} \right) \right] . \] (3.21)

Note that putting \( \nu = 0 \) into (3.21) we obtain the quantum wave packet of the Schrödinger equation with the potential \( V(x, t) \).[1]

4. The Feynman-de Broglie-Bohm Propagator of the Linearized Schuch-Chung-Hartmann Equation along a Classical Trajectory

4.1. Introduction

In 1948,[12] Feynman formulated the following principle of minimum action for the Quantum Mechanics:

The transition amplitude between the states \( |a> \) and \( |b> \) of a quantum-mechanical system is given by the sum of the elementary contributions, one for each trajectory passing by \( |a> \) at the time \( t_a \) and by \( |b> \) at the time \( t_b \). Each one of these contributions have the same modulus, but its phase is the classical action \( S_{cl} \) for each trajectory.

This principle is represented by the following expression known as the ”Feynman propagator”:

\[ K(b, a) = \int_{a}^{b} e^{i\int_{x_o}^{x} S_{cl}(b, a) \ D x(t)} , \] (4.1)

with:

\[ S_{cl}(b, a) = \int_{t_a}^{t_b} L(x, \dot{x}, t) \ dt , \] (4.2)

where \( L(x, \dot{x}, t) \) is the Lagrangean and \( D x(t) \) is the Feynman’s Measurement. It indicates that we must perform the integration taking into account all the ways connecting the states \( |a> \) and \( |b> \).

Note that the integral which defines \( K(b, a) \) is called ”path integral” or ”Feynman integral” and that the Schrödinger wavefunction \( \psi(x, t) \) of any physical system is given by (we indicate the initial position and initial time by \( x_o \) and \( t_o \), respectively):[13]

\[ \psi(x, t) = \int_{-\infty}^{+\infty} K(x, x_o, t, t_o) \psi(x_o, t_o) \ dx_o , \] (4.3)

with the quantum causality condition:

\[ \lim_{t, t_o \to 0} K(x, x_o, t, t_o) = \delta(x - x_o) . \] (4.4)
4.2. Calculation of the Feynman-de Broglie-Bohm Propagator for the Linearized Süssmann-Hasse-Albrecht-Kostin-Nassar equation along a Classical Trajectory

According to Section 3, the wavefunction $\psi(x, t)$ that was named wave packet of the linearized Süssmann-Hasse-Albrecht-Kostin-Nassar equation along a classical trajectory, can be written as [see (3.21)]:

$$\psi(x, t) = [2\pi^2 a^2(t)]^{-1/4} \exp \left[ \left( \frac{im}{2\hbar} \left( \frac{\dot{a}(t)}{a(t)} - \nu c \right) - \frac{1}{4a^2(t)} \right) [x - q(t)]^2 \right] \times$$

$$\times \exp \left[ i \frac{m}{\hbar} \dot{q}(t) [x - q(t)] + i \frac{m v_o x_o}{\hbar} \right] \times$$

$$\times \exp \left[ i \int_0^t dt' \left( \frac{1}{2} m \dot{q}^2(t') - m \nu (1 - c) \dot{q}(t') - V[q(t'), t'] - \frac{\hbar^2}{4m a^2(t')} \right) \right].$$

(4.5)

where [see (3.15,16)]:

$$\ddot{q}(t) + \nu c \dot{q}(t) + \frac{1}{m} V'[q(t), t] = 0, \quad (4.6)$$

$$\ddot{a}(t) + a(t) \left( \frac{1}{m} V'[q(t), t] - \nu^2 c^2 \right) = \frac{\hbar^2}{4m^2 a^3(t)}.$$

(4.7)

where the following initial conditions were obeyed [see (3.17a-d)]:

$$q(0) = x_o, \quad \dot{q}(0) = v_o, \quad a(0) = a_o, \quad \dot{a}(0) = b_o.$$ (4.8a-d)

Therefore, considering (4.3), the Feynman-de Broglie-Bohm propagator will be calculated using (4.5), in which we will put with no loss of generality, $t_o = 0$. Thus:

$$\psi(x, t) = \int_{-\infty}^{+\infty} K(x, x_o, t) \psi(x_o, 0) dx_o.$$ (4.9)

Let us initially define the normalized quantity:

$$\Phi(v_o, x, t) = (2\pi a^2_o)^{1/4} \psi(v_o, x, t),$$ (4.10)

which satisfies the following completeness relation:[13]

$$\int_{-\infty}^{+\infty} dv_o \Phi^*(v_o, x, t) \Phi(v_o, x', t) = (2\pi \frac{\hbar}{m}) \delta(x - x').$$ (4.11)

Taking the eqs. (2.3,8), we have:

$$\psi^*(x, t) \psi(x, t) = \rho^2 = \rho(x, t).$$ (4.12)
Now, using the eqs. (4.10,12), we get:

\[ \Phi^*(v_o, x, t) \psi(v_o, x, t) = (2 \pi a_o^2)^{1/4} \psi(v_o, x, t) \psi(v_o, x, t) = (2 \pi a_o^2)^{1/4} \rho(v_o, x, t) \rightarrow \]

\[ \rho(v_o, x, t) = (2 \pi a_o^2)^{-1/4} \Phi^*(v_o, x, t) \psi(v_o, x, t) . \quad (4.13) \]

On the other side, substituting (4.13) into (2.20), integrating the result and using (3.1) and (4.10) results [remembering that \( \psi^* \psi(\pm \infty) \rightarrow 0 \)]:

\[ \frac{\partial(\Phi^* \psi)}{\partial t} + \frac{\partial(\Phi^* \psi \vartheta)}{\partial x} = 0 \rightarrow \]

\[ \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} dx \Phi^* \psi + (\Phi^* \psi \vartheta)|_{-\infty}^{+\infty} = \]

\[ = \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} dx \Phi^* \psi = 0 . \quad (4.14) \]

The eq. (4.14) shows that the integration is time independent. Consequently:

\[ \int_{-\infty}^{+\infty} dx' \Phi^*(v_o, x', t) \psi(x', t) = \int_{-\infty}^{+\infty} dx_o \Phi^*(v_o, x_o, 0) \psi(x_o, 0) . \quad (4.15) \]

Multiplying (4.15) by \( \Phi(v_o, x, t) \) and integrating over \( v_o \) and using (4.11), we obtain [remembering that \( \int_{-\infty}^{+\infty} dx' f(x') \delta(x' - x) = f(x) \)]:

\[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dv_o \int_{-\infty}^{+\infty} dx \Phi(v_o, x, t) \Phi^*(v_o, x', t) \psi(x', t) = \]

\[ = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dv_o \int_{-\infty}^{+\infty} dx_o \Phi(v_o, x, t) \Phi^*(v_o, x_o, 0) \psi(x_o, 0) \rightarrow \]

\[ \int_{-\infty}^{+\infty} dx' \left( \frac{2 \pi h}{m} \right) \delta(x' - x) \psi(x', t) = \left( \frac{2 \pi h}{m} \right) \psi(x, t) = \]

\[ = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dv_o \int_{-\infty}^{+\infty} dx_o \Phi(v_o, x, t) \Phi^*(v_o, x_o, 0) \psi(x_o, 0) \rightarrow \]

\[ \psi(x, t) = \int_{-\infty}^{+\infty} \left[ \left( \frac{m}{2 \pi h} \right) \int_{-\infty}^{+\infty} dv_o \Phi(v_o, x, t) \times \right. \]

\[ \left. \times \Phi^*(v_o, x_o, 0) \right] \psi(x_o, 0) \ dx_o . \quad (4.16) \]

Comparing (4.9) and (4.16), we have:
\[ K(x, x_o, t) = \frac{m}{2 \pi \hbar} \int_{-\infty}^{+\infty} dv_o \Phi(v_o, x, t) \Phi^*(v_o, x_o, 0). \quad (4.17) \]

Substituting (4.5) and (4.10) into (4.17), we finally obtain the Feynman-de Broglie-Bohm Propagator of the linearized S(\text{us})mann-Hasse-Albrecht-Kostin-Nassar equation along a classical trajectory [remembering that \( \Phi^*(v_o, x_o, 0) = \exp\left(-\frac{i m v_o x_o}{\hbar}\right) \)]:

\[ K(x, x_o; t) = \frac{m}{2 \pi \hbar} \int_{-\infty}^{+\infty} dv_o \sqrt{a_o(t)} \times \]

\[ \times \exp \left[ \left( \frac{i m}{\hbar} \left[ \frac{\dot{a}(t)}{a(t)} - \nu c \right] - \frac{1}{4 a^2(t)} \right) [x - q(t)]^2 + \frac{i m}{\hbar} \frac{\dot{a}(t)}{a(t)} [x - q(t)] \right] \times \]

\[ \times \exp \left[ \frac{i}{\hbar} \int_{0}^{t} dt' \left( \frac{1}{2} m \ddot{q}(t') - m \nu (1 - c) \dot{q}(t') - V[q(t'), t'] - \frac{\hbar^2}{4 m a^2(t')} \right) \right], \quad (4.18) \]

where \( q(t) \) and \( a(t) \) are solutions of the (4.6, 7) differential equations.

Finally, it is important to note that putting \( \nu = 0 \) and \( V[q(t'), t'] = 0 \) into (4.6), (4.7) and (4.18) we obtain the free particle Feynman propagator.\[1,14\]

\begin{center}
\textbf{NOTES AND REFERENCES}
\end{center}

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